Algorithms and Complexity for some Multivariate Problems

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Zusammenfassung

Auch mit den modernsten Computern können wir in endlicher Zeit nur endlich viele Informationen über unsere Welt sammeln und verarbeiten. Das macht das Finden exakter Lösungen für viele numerische Probleme unmöglich. Beispiele hierfür sind die Frage nach der Abhängigkeit einer beliebigen Größe von verschiedenen Parametern (ein sogenanntes Approximationsproblem) sowie die Berechnung eines Durchschnitts unter unendlich vielen Werten (ein sogenanntes Integrationsproblem). In diesen Fällen müssen wir uns mit Näherungslösungen begnügen, die wir auch mit endlich vielen Informationen bestimmen können.

Üblicherweise gibt es zwei Arten von Informationen: das a priori Wissen und die empirischen Daten. Das a priori Wissen ist bereits in der Problemstellung enthalten und basiert in der Regel auf Modellannahmen. Zum Beispiel wissen wir möglicherweise aus theoretischen Vorbetrachtungen schon etwas über die Regularität der Funktion, die wir approximieren wollen. Die empirischen Daten müssen wir dagegen erst durch Messungen, Umfragen, Programme oder andere Interaktionen mit der Probleminstanz gewinnen. Sie werden anschließend zu unserer Näherungslösung verarbeitet.

Dieser Vorgang, also das Sammeln und das Verarbeiten der Daten, kann durch einen Algorithmus beschrieben werden. Jeder Algorithmus $A$ hat bestimmte Kosten und einen bestimmten Fehler, die wir mit $\text{cost}(A)$ und $\text{err}(A)$ bezeichnen. Die Kosten messen den Aufwand, der zur Gewinnung der Näherungslösung nötig ist. Dieser ist oft proportional zu der Anzahl der gesammelten Informationen. Der Fehler misst die zu erwartende Abweichung der Näherungslösung von der exakten Lösung. Für ein gegebenes Problem $\mathcal{P}$ stellen wir uns nun die Frage, wie klein der Fehler eines Algorithmus mit vorgegebenen maximalen Kosten $n$ im besten Fall sein kann. Wir fragen also nach dem $n$-ten minimalen Fehler

$$e(n, \mathcal{P}) = \inf \{ \text{err}(A) \mid \text{cost}(A) \leq n \}.$$  

Umgekehrt fragen wir nach den minimal nötigen Kosten, mit denen ein vorgegebener Fehler von höchstens $\varepsilon > 0$ erreicht werden kann. Es geht also um die Größe

$$n(\varepsilon, \mathcal{P}) = \min \{ \text{cost}(A) \mid \text{err}(A) \leq \varepsilon \},$$

die sogenannte $\varepsilon$-Komplexität des Problems. Diese Größen sind invers zueinander und messen die Schwierigkeit des Problems. Zahlreiche klassische Untersuchungen beschäftigen sich mit der Abfallgeschwindigkeit des $n$-ten minimalen Fehlers, wenn $n$ gegen unendlich strebt. Dies entspricht der Frage nach dem Verhalten der $\varepsilon$-Komplexität des Problems, wenn $\varepsilon$ gegen null strebt.
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In vielen Fällen hat das Problem eine intrinsische Dimension \( d \in \mathbb{N} \). Beispielsweise ist die Probleminstanz häufig eine Funktion von \( d \) Variablen. Wir sprechen dann von einem multivariatn Problem, das wir auch mit \( P_d \) bezeichnen. In diesem Fall interessieren wir uns für das Verhalten der Komplexität \( n(\varepsilon, P_d) \) als Funktion in beiden Parametern \( \varepsilon \) und \( d \). Viele Probleme unterliegen dem Fluch der Dimension. Die Komplexität wächst exponentiell mit der Dimension. Solche Probleme sind für große Dimensionen praktisch unlösbar. Man hofft also, dass die Komplexität nicht exponentiell von \( d \) oder \( \varepsilon^{-1} \) abhängt. In diesem Fall spricht man von Tractability. Noch besser ist es, wenn die Komplexität durch ein Polynom in \( \varepsilon^{-1} \) und \( d \) beschränkt ist. Man spricht dann von polynomialer Tractability.

Für praktische Anwendungen reicht es allerdings nicht aus, die Komplexität des Problems zu studieren. Diese gibt nur Auskunft darüber, was der beste Algorithmus leisten kann. Sie liefert uns nicht den besten Algorithmus. Diesen zu finden, ist im Allgemeinen eine unrealistische Hoffnung. Es ist in der Regel bereits eine große Herausforderung, einen Algorithmus zu finden, der den Fehler \( \varepsilon \) erreicht und dessen Kosten sich ähnlich wie die Komplexität des Problems verhalten.

In dieser Dissertation tragen wir Ergebnisse zu verschiedenen multivariaten Problemen bei. Wir studieren die numerische Integration und Approximation mit verschiedenen Arten von a priori Wissen. Außerdem betrachten wir das Problem der globalen Optimierung und das Dispersionsproblem. In manchen Fällen erhalten wir neue Ergebnisse zur Konvergenzordnung des Fehlers \( e(n, P_d) \). In anderen Fällen beweisen wir Ergebnisse bezüglich der Tractability des Problems. Aus der jeweiligen Sicht präsentieren wir optimale Algorithmen für die meisten dieser Probleme. Diese Resultate finden sich in den Kapiteln 2–4. In Kapitel 1 stellen wir zunächst einige Grundlagen und Begrifflichkeiten zur Verfügung.

Zu Kapitel 2: Integration und Approximation von Funktionen gemischter Glattheit

Dieses Kapitel beschäftigt sich mit der Integration und der Approximation von Funktionen mit beschränkten gemischten Ableitungen, wie sie beispielsweise im Zusammenhang mit der elektronischen Schrödingergleichung und verschiedenen Integralgleichungen auftreten [DTU18, Sec.9.1]. Wir betrachten Funktionen aus der Klasse

\[
F^r_d = \left\{ f \in L^2([0,1]^d) \left| \sum_{\alpha \in \{0,\ldots,r\}^d} \|D^\alpha f\|^2_2 \leq 1 \right\} \right. 
\]

Wir beginnen mit dem Integrationsproblem. Sei also \( P[\text{INT}, F^r_d, \text{det}] \) das Problem, Funktionen aus \( F^r_d \) auf Basis von Funktionswerten mithilfe deterministischer Algorithmen zu integrieren. Die Konvergenzordnung des \( n \)-ten minimalen Fehlers ist für dieses Problem bekannt. Ein optimaler Algorithmus wurde 1976 von Frolov vorgestellt [Fro76]. Es gilt

\[
e(n, P[\text{INT}, F^r_d, \text{det}]) \asymp n^{-r} \ln \frac{d-1}{r} n.
\]

Mithilfe randomisierter Algorithmen lässt sich diese Konvergenzordnung verbessern.
Das Problem $P[\text{INT}, F_d^r, \text{ran}]$, Funktionen aus $F_d^r$ auf Basis von Funktionswerten mit-hilfe randomisierter Algorithmen zu integrieren, erfüllt die asymptotische Äquivalenz
\[ e(n, P[\text{INT}, F_d^r, \text{ran}]) \asymp n^{-r/2}. \]

Insbesondere ist die Konvergenzordnung für letzteres Problem unabhängig von der Dimension $d$ des Gebietes. Dies ist eine Konsequenz von Satz 2.1, welcher außerdem aufzeigt, dass Frolovs Algorithmus in Kombination mit einer zufälligen Verschiebung und Streckung der Menge der Knotenpunkte optimal ist, siehe auch [KN17, Ull17].

Die folgenden Abschnitte beschäftigen sich mit dem Problem der $L^2$-Approximation. Für dieses Problem ist es sinnvoll, sowohl Algorithmen zu betrachten, deren Information durch Funktionswerte gegeben ist, als auch solche, die beliebige lineare Information nutzen. In Abschnitt 2.2 widmen wir uns dem Fall der linearen Information. In diesem Fall sind deterministische Algorithmen praktisch genauso gut wie randomisierte Algorithmen [Nov92]. Wir studieren daher nur das Problem $P[\text{APP}, F_d^r, \Lambda^\text{all}, \text{det}]$, die $L^2$-Approximation von Funktionen aus $F_d^r$ auf Basis linearer Information mithilfe deterministischer Algorithmen. Es ist bereits sein 1960 bekannt, dass
\[ e(n, P[\text{APP}, F_d^r, \Lambda^\text{all}, \text{det}]) \asymp n^{-r\ln(d-1)} n \]
im Sinne der schwachen asymptotischen Äquivalenz gilt [Bab60]. Ein optimaler Algorithmus ist anhand der Singularwertzerlegung der assoziierten Einbettung gegeben. Wir wollen diese Fehlerzahlen hier jedoch etwas genauer betrachten. Im Hinblick auf [KSU15], zeigen wir die starke asymptotische Äquivalenz
\[ e(n, P[\text{APP}, F_d^r, \Lambda^\text{all}, \text{det}]) \sim (\pi^d(d-1)!n)^{-r\ln(d-1)} n, \]
siehe Korollar 2.36. Dies bedeutet, dass die Fehlerzahlen für großes $n$ sehr gut durch die rechte Seite der Gleichung beschrieben werden können. Da diese Ergebnisse nur für sehr große $n$ relevant sind, stellen wir auch präasymptotische Abschätzungen bereit. In Korollar 2.38 beweisen wir die obere Schranke
\[ e(n, P[\text{APP}, F_d^r, \Lambda^\text{all}, \text{det}]) \leq 2n^{-c(d)} \quad \text{mit} \quad c(d) = \frac{1.1929}{2 + \ln d} \]
für alle $n \in \mathbb{N}$. Weiter zeigen wir, dass diese Abschätzung für $n < 2^d$ nicht wesentlich verbessert werden kann, siehe Korollar 2.37.

In Abschnitt 2.3 wenden wir uns dem Fall zu, dass die Informationen durch Funktionswerte gegeben sind. Wir betrachten randomisierte Algorithmen. Für das entsprechende Problem $P[\text{APP}, F_d^r, \Lambda^\text{std}, \text{ran}]$ beweisen wir die asymptotische Äquivalenz
\[ e(n, P[\text{APP}, F_d^r, \Lambda^\text{std}, \text{ran}]) \asymp n^{-r\ln(d-1)} n, \]
siehe Korollar 2.53. Wir geben einen Algorithmus an, dessen Fehler sich in dieser Hinsicht optimal verhält, siehe Algorithmus 2.47. Außerdem beweisen wir die präasymptotische Abschätzung
\[ e(n, P[\text{APP}, F_d^r, \Lambda^\text{std}, \text{ran}]) \leq 8n^{-c(d)} \]

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für alle $n \in \mathbb{N}$ mit $c(d)$ wie oben, siehe (2.39). Diese Abschätzungen zeigen, dass richtig gewählte Funktionswerte für das Approximationsproblem eine genauso große Aussagekraft haben wie beliebige lineare Information, insofern randomisierte Algorithmen erlaubt sind. Es ist ein ungelöstes Problem, ob dieser Sachverhalt bestehen bleibt, wenn wir nur deterministische Algorithmen betrachten.

An dieser Stelle wollen wir noch anmerken, dass die oben genannten Ergebnisse jeweils für allgemeinere Fragestellungen formuliert werden können:

- Abschnitt 2.1: Frolov’s Algorithmus und seine Randomisierung sind optimal für viele Klassen glatter Funktionen.
- Abschnitt 2.2: Wir studieren optimale Algorithmen für beliebige Tensorproduktden Probleme zwischen Hilberträumen.
- Abschnitt 2.3: Wir präsentieren optimale Algorithmen für die $L^2$-Approximation von Funktionen aus der Einheitskugel von Hilberträumen, die kompakt in einen $L^2$-Raum eingebettet sind, vorausgesetzt die Singularwerte dieser Einbettung erfüllen eine gewisse Abfallbedingung.

Zu Kapitel 3: Tractability des Problems der gleichmäßigen Approximation

In diesem Kapitel studieren wir die Leistungsfähigkeit deterministischer Algorithmen für das Problem, eine Funktion $f : [0, 1]^d \to \mathbb{R}$ gleichmäßig anhand endlich vieler Funktionswerte zu approximieren. Um hier überhaupt etwas erreichen zu können, ist a priori Wissen über die Funktion $f$ vornötig, sagen wir $f \in F_d$ für eine Menge $F_d$ von beschränkten Funktionen. Sei $\mathcal{P}[F_d]$ das Problem der gleichmäßigen Approximation mit a priori Wissen $F_d$. Wir interessieren uns für die Tractability dieses Problems. Insbesondere würden wir gerne mehr darüber wissen, welche Art von a priori Wissen zu positiven Ergebnissen in Hinblick auf die Tractability und damit zur praktischen Lösbarkeit des Problems in hohen Dimensionen führt.

Es ist bekannt, dass Glattheit alleine nicht ausreicht. Selbst mit dem a priori Wissen $F_d = \{ f \in C^\infty([0,1]^d) \mid \|D^\alpha f\|_\infty \leq 1 \text{ für alle } \alpha \in \mathbb{N}_0^d \}$, unterliegt das Problem dem Fluch der Dimension [NW09]. Selbstverständlich überträgt sich dieser Umstand auf den Fall endlicher Glattheit $r \in \mathbb{N}$, das heißt, auf den Fall von a priori Wissen $C^r_d = \{ f \in C^r([0,1]^d) \mid \|D^\alpha f\|_\infty \leq 1 \text{ für alle } \alpha \in \mathbb{N}_0^d \text{ mit } |\alpha| \leq r \}$.

Aber wie schlimm genau ist dieser Fluch? Ab welcher Dimension hat man mit der Unlösbarkeit des Problems zu rechnen? Um diese Fragen dreht sich Abschnitt 3.1. Für gerade Zahlen $r$ stellen wir fest, dass es positive Konstanten $c_r$, $C_r$ und $\varepsilon_r$ gibt, sodass

$$(c_r \sqrt{d} \varepsilon^{-1/r})^d \leq n(\varepsilon, \mathcal{P}[C^r_d]) \leq (C_r \sqrt{d} \varepsilon^{-1/r})^d$$

für alle $d \in \mathbb{N}$ und $\varepsilon \in (0, \varepsilon_r)$ gilt, siehe Satz 3.1. Aus Ergebnissen von [Was84] folgt, dass selbige Abschätzungen auch für das Problem der globalen Optimierung gelten, da
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die Klasse $C^r$ konvex und symmetrisch ist, siehe Abschnitt 3.3. Insbesondere wächst die Komplexität beider Probleme im Fall $r \geq 2$ für eine vorgegebene Fehlerschranke $\varepsilon > 0$ wie $d^{r/2}$ und damit superexponentiell.

Andererseits wissen wir, dass zusätzliches Wissen über die Struktur der Funktion $f$ durchaus zu Tractability führen kann. Beispiele hierfür sind folgende Annahmen:

- Die Funktion ist eine Ridge-Funktion [MUV15]. Das heißt, sie hat die Gestalt $f = g(\langle \cdot, x_0 \rangle)$ für ein $x_0 \in \mathbb{R}^d$ und ein $g : \mathbb{R} \to \mathbb{R}$.

- Die Funktion ist separierbar [NR97, WW04]. Sie lässt sich als Summe von Funktionen in $m$ Variablen schreiben, wobei die Ordnung $m$ unabhängig von der Dimension ist. Man beachte, dass sich obige Paper nicht mit gleichmäßiger Approximation, sondern mit $L^2$-Approximation und Integration beschäftigen.

- Die Funktion ist symmetrisch [Wei12]. Das heißt, $f(x)$ ist invariant bezüglich Umordnungen der Koordinaten von $x \in [0, 1]^d$. Man beachte allerdings, dass die Funktionen in [Wei12] nicht anhand von Funktionswerten, sondern anhand von anderen linearen Informationen approximiert werden.

Ein weiteres Beispiel studieren wir in Abschnitt 3.2. Hier stellen wir uns vor, dass $f$ ein Rank-1-Tensor ist. Das bedeutet, die $d$-dimensionale Funktion kann als Produkt von $d$ eindimensionalen Funktionen geschrieben werden. Genauer gesagt nehmen wir an, dass $f$ ein Element der Klasse

$$F_{r,M}^d = \left\{ \bigotimes_{i=1}^d f_i \left| f_i : [0, 1] \to [-1, 1], \|f_i^{(r)}\|_\infty \leq M \right. \right\}$$

ist, wobei die Parameter $r \in \mathbb{N}$ und $M > 0$ die Glattheit der Funktion beschreiben. Die Funktion $\bigotimes_{i=1}^d f_i$ heißt Tensorprodukt der Funktionen $f_i$ und bildet $x \in [0, 1]^d$ auf das Produkt der Funktionswerte $f_i(x_i)$ ab. In Satz 3.19 stellen wir fest, dass das Problem der gleichmäßigen Approximation mit a priori Wissen $F_{r,M}^d$ genau dann am Fluch der Dimension leidet, wenn $M \geq 2^r r!$. Gilt dagegen $M < 2^r r!$, so wächst die Komplexität nur polynomial mit der Dimension. Falls $M \leq r!$ ist der Grad dieser polynomialen Abhängigkeit sogar unabhängig von der Fehlertoleranz $\varepsilon$ und wir erhalten polynomiale Tractability. Andernfalls wächst der Exponent logarithmisch mit $\varepsilon^{-1}$. In allen drei Fällen stellen wir einen Algorithmus vor, dessen Kosten genau dieses Verhalten aufzeigen. Der Algorithmus ist daher optimal im Hinblick auf die Tractability des Problems. In Abschnitt 3.3 beweisen wir außerdem, dass die Komplexität des Problems der globalen Optimierung auf $F_{r,M}^d$ dasselbe Verhalten aufweist. Dies gilt, obwohl die Klasse $F_{r,M}^d$ nicht konvex ist.

Im Verlauf von Abschnitt 3.2 wird klar, dass das Problem der Approximation von Rank-1-Tensoren eng mit dem geometrischen Problem der Dispersion zusammenhängt. Die Dispersion einer Menge von Punkten im $d$-dimensionalen Einheitswürfel ist das Volumen der größten achsenparallelen Box, die keinen dieser Punkte enthält. Diese Größe ist auch unabhängig von der vorgegebenen Dispersion $\varepsilon$ erreichen oder unterbieten. In Abschnitt 3.4 geben wir eine solche Punktmenge für...
alle $\varepsilon > 0$ und jede Dimension $d \in \mathbb{N}$ an. Die Punktmenge ist ein dünnes Gitter und hat daher eine besonders einfache Struktur. Für viele Parameter $(\varepsilon, d)$ ist uns keine kleinere Punktmenge mit der gewünschten Dispersion bekannt.

Zu Kapitel 4: Optimale Information versus zufällige Information

Das letzte Kapitel unterscheidet sich wesentlich von den beiden vorigen Kapiteln. Bisher haben wir danach gestrebt, optimale Algorithmen zu finden, welche optimale Information über die Probleminstanz sammeln. In Wirklichkeit haben wir jedoch oft keinen Zugriff auf optimale Information. Das kann zum Beispiel daran liegen, dass wir nicht wissen, wie wir die Parameter wählen müssen, um möglichst aussagekräftige Messergebnisse zu erhalten. Es kann auch sein, dass wir die Parameter für unsere Messung nicht frei bestimmen können. In diesem Kapitel nehmen wir an, dass die Parameter dem Zufall unterliegen. Wir erhalten also zufällige Information und stellen uns die folgende Frage.

Was ist die typische Güte von zufälliger Information?

Selbstverständlich ist die zufällige Information niemals besser als optimale Information, aber es kann passieren, dass zufällige Information nur unwesentlich schlechter ist. In diesem Fall macht es wenig Sinn, mühsam nach optimaler Information zu streben.

Um unsere Frage präzise formulieren zu können, müssen wir klären, wie wir die Güte der Information messen und welchem Zufall die Information unterliegt. Die Güte der Information messen wir anhand ihres Radius. Dies hat zur Folge, dass wir die Güte zufälliger Information im Fall $\ell^p$ asymptotisch genauso verhält wie die Güte optimaler Information, siehe Kapitel 1.2.2. Unsere Information soll aus unabhängigen zufälligen Messungen stammen, die alle derselben Verteilung genügen. Sicher gibt es hier viele Verteilungen, die es zu studieren wert sind. Wir werden die obige Frage allerdings für zwei Klassen von Beispielen betrachten, bei denen wir jeweils eine Verteilung für besonders natürlich und daher für besonders interessant halten.

Das erste Beispiel ist das Problem der $L^p$-Approximation periodischer Lipschitz-Funktionen von $d$ Variablen mithilfe von $n$ Funktionswerten. Hier ist die optimale Information durch Funktionswerte auf einem regulären Gitter gegeben. Zufällige Information soll dagegen durch Funktionswerte an $n$ unabhängigen, gleichverteilten Punkten gegeben sein. Es stellt sich heraus, dass sich die Güte zufälliger Information im Fall $p < \infty$ asymptotisch genauso verhält wie die Güte optimaler Information, siehe Korollar 1.5. Der Fall $p = \infty$ ist das Problem der gleichmäßigen Approximation von Lipschitz-Funktionen. Hier ist zufällige Information asymptotisch etwas schlechter als optimale Information, jedoch nur wenig, siehe Korollar 1.9 sowie [BDKKW17].

Das zweite Beispiel ist das Problem der $\ell^2$-Approximation von Punkten aus einem $m$-dimensionalen Ellipsoid mithilfe von $n$ linearen Messungen, wobei wir uns vorstellen, dass $m$ viel größer als $n$ ist, beispielsweise $m = 2^n$. Optimale Information ist hier durch die Koordinaten in Richtung der $n$ größten Halbachsen des Ellipsoids gegeben. Zufällige Information ist dagegen durch Koordinaten in $n$ zufällige Richtungen gegeben, die unabhängig und gleichverteilt auf der Sphäre in $\mathbb{R}^m$ sind. Abhängig von der Dicke des Ellipsoids erhalten wir sehr verschiedene Ergebnisse.
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über die Güte zufälliger Information: Wenn die geordnete Folge der Halbachsen des Ellipsoids schnell genug abfällt, so ist zufällige Information fast genauso gut wie optimale Information. Fällt die Folge zu langsam, so ist zufällige Information beinahe völlig nutzlos, siehe Satz 4.17. Wir werden auch eine Version dieses Problems im Fall \( m = \infty \) besprechen. Dieser Fall entspricht dem Problem der \( L^2 \)-Approximation von Funktionen aus einem kompakt eingebetteten Hilbertraum.

Veröffentlichungen

Die meisten Ergebnisse dieser Dissertation wurden bereits veröffentlicht. Es folgt eine Liste der relevanten Veröffentlichungen des Autors. Die Reihenfolge entspricht der Reihenfolge der Abschnitte dieser Arbeit. Der zweite Punkt der Liste ist die Masterarbeit des Autors.

1. mit E. Novak. A universal algorithm for multivariate integration. *Foundation of Computational Mathematics*, 17(4):895–916, 2017, siehe Abschnitt 2.1.

2. On the randomization of Frolov’s algorithm for multivariate integration. Masterarbeit, Friedrich-Schiller-Universität Jena, 2016, verfügbar als arXiv:1603.04637 [math.NA], siehe Abschnitt 2.1.

3. Tensor power sequences and the approximation of tensor product operators. *Journal of Complexity*, 44:30–51, 2018, siehe Abschnitt 2.2.

4. Optimal Monte Carlo methods for \( L^2 \)-approximation. *Constructive Approximation*, 2018, https://doi.org/10.1007/s00365-018-9428-4, siehe Abschnitt 2.3.

5. Uniform recovery of high-dimensional \( C^r \)-functions. *Journal of Complexity*, 50:116–126, 2019, siehe Abschnitt 3.1.

6. mit D. Rudolf. Recovery algorithms for high-dimensional rank one tensors. *Journal of Approximation Theory*, 237:17–29, 2019, siehe Abschnitt 3.2.

7. On the dispersion of sparse grids. *Journal of Complexity*, 45:115–119, 2018, siehe Abschnitt 3.4.

8. mit A. Hinrichs, E. Novak, J. Prochno, and M. Ullrich. Random Ab schnitte of ellipsoids and the power of random information. Preprint, verfügbar als arXiv:1901.06639 [math.FA], siehe Abschnitt 4.2.
Introduction and Results

Even with the help of modern computers, we cannot hope to collect or process more than a finite amount of information about the world in finite time. This makes it impossible to find exact solutions to many numerical problems, such as the question for the dependence of a certain quantity upon several parameters (a so-called approximation problem) or the computation of some average of infinitely many values (a so-called integration problem). It is then necessary to settle for approximate solutions which may be obtained from a finite amount of information.

The information usually consists of two parts: the a priori knowledge and the empirical data. The a priori knowledge is inherent to the problem or simply assumed by our model. For example, we might have some knowledge about the regularity of the function that we want to approximate. The data has to be gained from measurements, surveys, programs, etc. We process the data to generate the approximate solution.

The whole procedure is described by an algorithm. Each algorithm $A$ has a certain cost, denoted by $\text{cost}(A)$, and a certain error, denoted by $\text{err}(A)$. The cost measures the effort that is needed to obtain the approximate solution. It is often proportional to the amount of collected data. The error measures the possible disparity of the approximate and the exact solution. Given a problem $\mathcal{P}$, the question is how small the error of an algorithm with maximal cost $n$ can possibly be. We ask for the $n^{th}$ minimal error

$$e(n, \mathcal{P}) = \inf \{ \text{err}(A) \mid \text{cost}(A) \leq n \}.$$ 

Conversely, we ask for the minimal cost that is needed to achieve an error of at most $\varepsilon > 0$, that is,

$$n(\varepsilon, \mathcal{P}) = \min \{ \text{cost}(A) \mid \text{err}(A) \leq \varepsilon \}.$$ 

This quantity is called the $\varepsilon$-complexity of the problem. We also talk about the $\varepsilon$-information complexity if $\text{cost}(A)$ is given by the amount of information that is required by the algorithm. The $n^{th}$ minimal error and the $\varepsilon$-complexity are inverse to one another and measure the difficulty of the problem.

Many classical investigations are concerned with the speed of decay of the $n^{th}$ minimal error as $n$ tends to infinity, or equivalently, with the behavior of the $\varepsilon$-complexity as $\varepsilon$ tends to zero. But quite often, the problem has some intrinsic dimension $d \in \mathbb{N}$. For example, the problem instance may be a function of $d$ variables. We then talk about a multivariate problem, which we denote by $\mathcal{P}_d$. In this case, we are interested in the behavior of $n(\varepsilon, \mathcal{P}_d)$ as a function of both $\varepsilon$ and $d$. At best, we hope that the problem is polynomially tractable, that is, the complexity depends at most polynomially on both $\varepsilon^{-1}$ and $d$. However, many problems suffer
from the curse of dimensionality: the $\varepsilon$-complexity increases exponentially with the dimension for some $\varepsilon$. Of course, there are many shades of tractability in between these extremes. For instance, the problem is called quasi-polynomially tractable if the $\varepsilon$-complexity increases at most polynomially with the dimension for any fixed $\varepsilon$ and the polynomial order increases at most logarithmically with $\varepsilon^{-1}$.

For practical purposes, however, it is not enough to know how much an algorithm can possibly achieve. One actually wants to get hold of optimal algorithms. These are algorithms that achieve an error of at most $\varepsilon$ with (almost) minimal cost.

In this thesis, we contribute to a collection of several multivariate problems. We study numerical integration and approximation, global optimization and the problem of dispersion. In some cases, we present new results on the speed of decay of $e(n, \mathcal{P}_d)$. In other cases, we give tractability results. From the respective points of view, we provide optimal algorithms for most of the problems. These results can be found in Chapters 2–4. The theoretical foundations are discussed in Chapter 1.

On Chapter 2: Integration and Approximation of Functions with Mixed Smoothness

This chapter is centered around the integration and approximation problem for multivariate functions having bounded mixed derivatives. Such functions appear, for example, in the context of the electronic Schrödinger equation or certain integral equations \cite[Sec. 9.1]{DTU18}. More precisely, we consider functions from the class

$$F_r^d = \left\{ f \in L_2([0,1]^d) \mid \sum_{\alpha \in \{0,\ldots,r\}^d} \| D^\alpha f \|_2^2 \leq 1 \right\}.$$ 

The first section is concerned with the integration problem. Let $\mathcal{P}[\text{INT}, F_r^d, \text{det}]$ be the problem of integrating functions from $F_r^d$ with deterministic algorithms that use function values as information. For this problem, it is known that

$$e(n, \mathcal{P}[\text{INT}, F_r^d, \text{det}]) \asymp n^{-r} \ln^{\frac{d-1}{2}} n.$$ 

An optimal algorithm was given by Frolov in 1976 \cite{Fro76}. This order of convergence may be improved by randomized algorithms. The problem $\mathcal{P}[\text{INT}, F_r^d, \text{ran}]$ of integrating such functions with randomized algorithms satisfies

$$e(n, \mathcal{P}[\text{INT}, F_r^d, \text{ran}]) \asymp n^{-r-1/2}.$$ 

In particular, the order is independent of the dimension $d$. This is a consequence of Theorem 2.1 which states that a randomly shifted and dilated version of Frolov’s algorithm is optimal for this problem, see also \cite{KN17, Ull17}.

The remaining sections are concerned with the problem of $L^2$-approximation. For this problem it makes sense to study algorithms that use function values as information as well as algorithms that use arbitrary pieces of linear information.

Section 2.2 is concerned with the case of linear information. In this case, deterministic algorithms are practically as powerful as randomized algorithms \cite{Nov92}. This
means that it is enough to study the problem \(\mathcal{P}[\text{APP}, F^r_d, \Lambda^{\text{all}}, \text{det}]\) of approximating such functions in \(L^2\) with deterministic algorithms that use linear information. It is known since 1960 [Bab60] that

\[
e(n, \mathcal{P}[\text{APP}, F^r_d, \Lambda^{\text{all}}, \text{det}]) \asymp n^{-r} \ln^{r(d-1)} n.
\]

An optimal algorithm is given by the singular value decomposition of the associated embedding. Here, we go a little more into detail. In the spirit of [KSU15], we show that

\[
e(n, \mathcal{P}[\text{APP}, F^r_d, \Lambda^{\text{all}}, \text{det}]) \sim (\pi^d (d-1)! n)^{-r} \ln^{r(d-1)} n
\]

in the sense of strong equivalence of sequences, see Corollary 2.36. Since these results are only relevant for very large \(n\), we also provide explicit estimates for small \(n\), preasymptotic estimates. In Corollary 2.37 and Corollary 2.38, we prove that

\[
e(n, \mathcal{P}[\text{APP}, F^r_d, \Lambda^{\text{all}}, \text{det}]) \leq 2 n^{-c(d)} \quad \text{with} \quad c(d) = \frac{1.1929}{2 + \ln d}
\]

for all \(n \in \mathbb{N}\) and that this bound cannot be improved much for \(n < 2^d\).

In Section 2.3, we turn to the case of function values as information. We provide an optimal randomized algorithm for the respective problem \(\mathcal{P}[\text{APP}, F^r_d, \Lambda^{\text{std}}, \text{ran}]\) and show that

\[
e(n, \mathcal{P}[\text{APP}, F^r_d, \Lambda^{\text{std}}, \text{ran}]) \asymp n^{-r} \ln^{r(d-1)} n,
\]

see Corollary 2.53. Therefore, function values are as powerful as arbitrary linear information, as long as randomized algorithms are allowed. Also the preasymptotic estimates are similar. We get

\[
e(n, \mathcal{P}[\text{APP}, F^r_d, \Lambda^{\text{std}}, \text{ran}]) \leq 8 n^{-c(d)}
\]

for all \(n \in \mathbb{N}\) with \(c(d)\) as above, see (2.39). Note that the question for optimal algorithms and the order of convergence is still unsolved for deterministic algorithms that use function values as information. We remark that each section will cover a more general setting:

- **Section 2.1** Frolov’s algorithm and its randomization are optimal for many classes of smooth functions.

- **Section 2.2** We study optimal algorithms for the \(L^2\)-approximation of functions from the unit ball of any tensor product Hilbert space.

- **Section 2.3** We provide optimal randomized algorithms for the \(L^2\)-approximation of functions from the unit ball of any Hilbert space that is compactly embedded in the respective \(L^2\)-space, provided that the singular values of this embedding satisfy a certain decay condition.
On Chapter 3: Tractability of the Uniform Approximation Problem

In this chapter, we study the power of deterministic algorithms for the problem of recovering a function \( f : [0, 1]^d \rightarrow \mathbb{R} \) from a finite number of function values in the uniform norm. In order to achieve anything at all, it is necessary to have some a priori knowledge about the function, say \( f \in F_d \subset L^\infty([0, 1]^d) \). Let \( \mathcal{P}[F_d] \) be the problem of uniform approximation with a priori knowledge \( F_d \). We are interested in the tractability of this problem. In particular, we want to know what kind of a priori knowledge leads to positive tractability results.

It is well known that smoothness alone is not enough. Even if we have the a priori knowledge \( F_d = \{ f \in C^\infty([0, 1]^d) \mid \|D^\alpha f\|_\infty \leq 1 \text{ for all } \alpha \in \mathbb{N}_0^d \} \), the problem suffers from the curse of dimensionality [NW09]. Of course, the curse stays present if we only know about finite smoothness \( r \in \mathbb{N} \), that is, if we have the a priori knowledge \( C^r_d = \{ f \in C^r([0, 1]^d) \mid \|D^\alpha f\|_\infty \leq 1 \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq r \} \).

But how bad is the situation exactly? This question is studied in Section 3.1. For even numbers \( r \), we find that there are positive constants \( c_r, C_r \) and \( \varepsilon_r \) such that

\[
(c_r \sqrt{d \varepsilon}^{-1/r})^d \leq n(\varepsilon, \mathcal{P}[C^r_d]) \leq (C_r \sqrt{d \varepsilon}^{-1/r})^d
\]

for all \( d \in \mathbb{N} \) and \( \varepsilon \in (0, \varepsilon_r) \), see Theorem 3.1. It follows from [Was84] that the same estimates hold for the problem of global optimization on \( C^r_d \) since this class is convex and symmetric, see Section 3.3. In particular, the complexity of both problems grows like \( d^{d/2} \) for any fixed \( \varepsilon > 0 \) and \( r \geq 2 \). For odd numbers \( r \), the precise behavior of the complexity as a function of both \( \varepsilon \) and \( d \) is still unclear.

On the other hand, it is known that additional knowledge about the structure of \( f \) may lead to tractability of the uniform approximation problem. For example, we may assume that \( f \) is

- a ridge function [MUV15]. That is, it can be written in the form \( g(\langle \cdot, x_0 \rangle) \) for some \( x_0 \in \mathbb{R}^d \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \).

- separable [NR97, WW04]. It can be written as a sum of \( m \)-variate functions, where \( m \) is independent of \( d \). Note that the above papers are not concerned with uniform approximation but with \( L^2 \)-approximation and integration.

- symmetric [Wei12]. That is, \( f(x) \) is invariant under a reordering of the coordinates of \( x \in [0, 1]^d \). Note that the functions in [Wei12] are to be recovered from arbitrary linear information and not exclusively from function values.

Another example is studied in Section 3.2. Here, we assume that \( f \) is a rank one tensor. That is, it can be written as a product of \( d \) univariate functions. More precisely, we assume that \( f \) is contained in the class

\[
F^d_{r,M} = \left\{ \bigotimes_{i=1}^d f_i \mid f_i : [0, 1] \rightarrow [-1, 1], \|f_i^{(r)}\|_\infty \leq M \right\}
\]
for some smoothness parameters $r \in \mathbb{N}$ and $M > 0$. The function $\bigotimes_{i=1}^{d} f_i$ is called the tensor product of the functions $f_i$ over $i \leq d$ and maps $x \in [0,1]^d$ to the product of all $f_i(x_i)$. In Theorem 3.19 we find that uniform approximation with a priori knowledge $F_{r,M}^d$ suffers from the curse of dimensionality iff $M \geq 2^r r!$. It is quasi-polynomially tractable iff $M < 2^r r!$ and even polynomially tractable iff $M \leq r!$. In every case we provide an optimal algorithm. Moreover, we show that the same tractability results hold for the problem of global optimization on the class $F_{r,M}^d$ which is symmetric but not convex, see Section 3.3.

It will become apparent that the uniform approximation of rank one tensors is closely related to the problem of dispersion. The dispersion of a finite point set in $[0,1]^d$ is the volume of the largest empty axis-aligned box amidst the point set. This quantity is also of independent interest. One asks for the minimal cardinality that is necessary to achieve a dispersion of at most $\varepsilon$ in dimension $d$, but also for explicit point sets with this property. In Section 3.4 we provide such a point set for every $\varepsilon > 0$ and every $d \in \mathbb{N}$. In a vast range of the parameters $(\varepsilon,d)$, we do not know any smaller point set with this property. The point set is an instance of a sparse grid and hence easy to handle. It may be used for the algorithms from Section 3.2.

On Chapter 4: Optimal Information versus Random Information

The last chapter is somewhat different. In the previous chapters, we aimed at finding optimal algorithms that use optimal information about the problem instance. However, quite often we do not have access to optimal information. The reason may be that we do not know which kind of measurements lead to optimal information or that we do not even get to choose our measurements. In this chapter, we assume that the information comes in randomly and ask the following question:

What is the typical quality of random information?

Of course, random information cannot be better than optimal information, but it may turn out that typical random information is only slightly worse. In this case, searching for optimal information is rather pointless.

To make this more precise, we need to clarify how we measure the quality of our information and what we mean by random. The first is done with the so-called radius of information, which is the worst case error of the best algorithm that uses nothing but the given information and the a priori knowledge about the problem instance, see Section 1.2.2. The random information, on the other hand, shall be obtained from a certain number of independent measurements that all follow the same law. In general, there is no right or wrong in the choice of the distribution that we want to investigate. However, we study this question for two basic examples for which there seems to be a natural choice for this distribution.

The first example is the problem of $L^p$-approximation of periodic Lipschitz functions on the $d$-dimensional unit cube using $n$ function values. While optimal information is given by function values on a regular grid, random information shall be given by function values at $n$ random points that are chosen independently and uniformly from the domain. It turns out that typical random information is asymptotically
just as good as optimal information if \( p < \infty \), see Corollary 4.5. For \( p = \infty \), it is only slightly worse, see Corollary 4.9 and [BDKKW17].

The second example is the problem of \( \ell^2 \)-approximation of a point from an \( m \)-dimensional ellipsoid by means of \( n \) linear measurements, where we imagine that \( m \) is much larger than \( n \). While optimal information is given by \( n \) scalar products in direction of the largest semi-axes, random information shall be given by scalar products in \( n \) directions taken independently from the uniform distribution on the sphere in \( \mathbb{R}^m \). We obtain very different results depending on the shape of the ellipsoid: If the ordered sequence of semi-axes decays fast enough, typical random information is almost as good as optimal information. If it decays too slowly, typical random information is practically useless, see Theorem 4.17. We shall also present a variant of these results for \( m = \infty \), which corresponds to the problem of \( L^2 \)-approximation in a Hilbert space.

**Publications**

Most of the results in this thesis are already published. Below, the relevant papers are listed in order of the corresponding sections. The second item is the author’s master thesis.

1. with E. Novak. A universal algorithm for multivariate integration. *Foundation of Computational Mathematics*, 17(4):895–916, 2017, see Section 2.1.

2. On the randomization of Frolov’s algorithm for multivariate integration. Master thesis, Friedrich-Schiller-Universität Jena, 2016, available on arXiv:1603.04637 [math.NA], see Section 2.1.

3. Tensor power sequences and the approximation of tensor product operators. *Journal of Complexity*, 44:30–51, 2018, see Section 2.2.

4. Optimal Monte Carlo methods for \( L^2 \)-approximation. *Constructive Approximation*, 2018, https://doi.org/10.1007/s00365-018-9428-4, see Section 2.3.

5. Uniform recovery of high-dimensional \( C^r \)-functions. *Journal of Complexity*, 50:116–126, 2019, see Section 3.1.

6. with D. Rudolf. Recovery algorithms for high-dimensional rank one tensors. *Journal of Approximation Theory*, 237:17–29, 2019, see Section 3.2.

7. On the dispersion of sparse grids. *Journal of Complexity*, 45:115–119, 2018, see Section 3.4.

8. with A. Hinrichs, E. Novak, J. Prochno, and M. Ullrich. Random sections of ellipsoids and the power of random information. Preprint, available on arXiv:1901.06639 [math.FA], see Section 4.2.

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Chapter 1

Problems and Algorithms

In most cases, a numerical problem is associated with a solution operator $S : F \to G$. For example, we may think of the computation of integrals

$$S : F \to \mathbb{R}, \quad S(f) = \int_0^1 f(x) \, dx$$

for some input set $F$ of integrable functions on $[0, 1]$. Then a (deterministic) algorithm is just a particular mapping $A : F \to G$, computing some output $A(f) \in G$ for every input $f \in F$. For example, a quadrature rule is a mapping

$$A : F \to \mathbb{R}, \quad A(f) = \sum_{i=1}^n a_i f(x_i)$$

for some number $n \in \mathbb{N}$, weights $a_i \in \mathbb{R}$ and nodes $x_i \in [0, 1]$. Each algorithm is assigned a cost and an error. In one way or another, the error measures the distance between the output $A(f)$ and the solution $S(f)$, while the cost measures the effort for computing the output. In the above example, one could define

$$\text{cost}(A) = n \quad \text{and} \quad \text{err}(A) = \sup_{f \in F} |S(f) - A(f)|.$$  

We shall discuss various types of problems that are defined via a solution operator in Section 1.2. However, we also want to study the problem of dispersion and the problem of finding a global maximizer, which are not associated with a solution operator. For this reason, we first introduce an abstract notion of a problem.

1.1 General Notions

**Definition 1.1.** A problem $\mathcal{P}$ is a triple $(\mathcal{A}, \text{err}, \text{cost})$ consisting of a set $\mathcal{A}$ and two functions

$$\text{err} : \mathcal{A} \to [0, \infty], \quad \text{cost} : \mathcal{A} \to \{0, 1, 2, \ldots, \infty\}.$$  

For $A \in \mathcal{A}$ the numbers $\text{err}(A)$ and $\text{cost}(A)$ are called the error and the cost of $A$. For every $n \in \mathbb{N}$ the $n^{\text{th}}$ minimal error of $\mathcal{P}$ is defined by

$$e(n, \mathcal{P}) = \inf \{ \text{err}(A) \mid A \in \mathcal{A}, \text{cost}(A) \leq n \}.$$
For every $\varepsilon \geq 0$ the $\varepsilon$-complexity of $P$ is defined by

$$n(\varepsilon, P) = \min \{\text{cost}(A) \mid A \in \mathcal{A}, \text{err}(A) \leq \varepsilon\}.$$  

Many problems are inherited from a solution operator. In this case, the set $\mathcal{A}$ consists of algorithms. Before we turn to such problems, let us have a look at an example of a geometric problem which is not defined via a solution operator, the problem of dispersion. A second example, the problem of finding a global maximizer, will be described in Section 3.3.

**Example 1.2** (The problem of dispersion, Part 1 of 3). For every $d \in \mathbb{N}$ let $\mathcal{S}_d$ be the set of all finite subsets of $[0, 1]^d$. Let $\mathcal{B}_d$ be the set of all boxes in $[0, 1]^d$, that is,

$$\mathcal{B}_d = \left\{ \prod_{j=1}^d I_j \mid I_j \subset [0, 1] \text{ interval} \right\}.$$  

The dispersion of a point set $P \in \mathcal{S}_d$ is the volume of the largest empty box amidst the point set, that is,

$$\text{disp}(P) = \sup \left\{ \lambda^d(B) \mid B \in \mathcal{B}_d, B \cap P = \emptyset \right\}.$$  

We consider the problem $\mathcal{P}_d = (\mathcal{S}_d, \text{disp}, \text{card})$. In this case, $e(n, \mathcal{P}_d)$ is the minimal dispersion of $n$ points in $[0, 1]^d$. The complexity $n(\varepsilon, \mathcal{P}_d)$ is the minimal cardinality of a $d$-dimensional point set achieving a dispersion of at most $\varepsilon$.

A problem is called solvable if the $n^{th}$ minimal error tends to 0 as $n$ tends to infinity. Numerous classical investigations ask for the speed of this convergence.

**Example 1.2** (Part 2 of 3). By dividing the unit cube into $(n + 1)$ boxes of equal volume, we immediately see that the dispersion of $n$ points is at least $1/(n + 1)$. On the other hand, Rote and Tichy [RT96] showed in 1996 that the dispersion of the first $n$ points of the Halton-Hammersely sequence is at most $2^{d-1} \pi_d/n$, where $\pi_d$ is the product of the first $(d - 1)$ primes. Hence,

$$e(n, \mathcal{P}_d) \asymp n^{-1},$$  

that is, the $n^{th}$ minimal error of the problem of dispersion decays with polynomial order 1 for all fixed $d \in \mathbb{N}$. In particular, the problem of dispersion is solvable.

The question for the speed of decay translates into the question for the dependence of the complexity on $\varepsilon$. Many problems, like the dispersion problem, have some intrinsic dimension $d \in \mathbb{N}$. There is growing interest in the $d$-dependence of the complexity. Tractability asks for the behavior of the complexity as a function of both $\varepsilon$ and $d$. We give some examples of tractability notions. Note that the following list is far from complete.

**Definition 1.3.** Consider a family of problems $\mathcal{P}_d$ with index $d \in \mathbb{N}$. The family

- is *strongly polynomially tractable* if there are constants $c, p > 0$ such that $n(\varepsilon, \mathcal{P}_d) \leq c \varepsilon^{-p}$ for all $\varepsilon \in (0, 1)$ and all $d \in \mathbb{N}$;
1.2. Important Types of Problems

- is *polynomially tractable* if there are constants $c, q, p > 0$ such that $n(\varepsilon, P_d) \leq c \varepsilon^{-p} d^q$ for all $\varepsilon \in (0, 1)$ and all $d \in \mathbb{N}$;

- is *quasi-polynomially tractable* if there are constants $c, t > 0$ such that
  \[
  n(\varepsilon, P_d) \leq c \exp \left( t \left( 1 + \ln(\varepsilon^{-1}) \right) \left( 1 + \ln d \right) \right)
  \]
  for all $\varepsilon \in (0, 1)$ and all $d \in \mathbb{N}$;

- suffers from the *curse of dimensionality* if there is some $\varepsilon > 0$, $c > 0$ and $\alpha > 1$ such that $n(\varepsilon, P_d) \geq c \alpha^d$ for all $d \in \mathbb{N}$.

Note that the term problem often refers to a whole family of problems.

**Example 1.2** (Part 3 of 3). The interest in the $d$-dependence of the complexity of the problem of dispersion started much later. Aistleitner, Hinrichs, and Rudolf [AHR17] were the first to show that the complexity increases with the dimension. In 2017, they proved
\[
\frac{\varepsilon}{1 - 4 \varepsilon} \log_2 d
\]
for all $d \in \mathbb{N}$ and $\varepsilon \leq 1/4$. In 2018, Sosnovec [Sos18] showed that this logarithmic dependence on $d$ is already optimal. Not much later, Ullrich and Vybíral [UV18] proved that
\[
\left\lceil 2^7 \varepsilon^{-2} \left( 1 + \log_2 \left( \varepsilon^{-1} \right) \right)^2 \log_2 d \right\rceil
\]
for all $d \geq 2$ and $\varepsilon < 1/2$. In particular, the problem of dispersion is polynomially tractable, but not strongly polynomially tractable.

1.2 Important Types of Problems

We now turn to problems that are inherited from a solution operator. In the following, let $F$ be a set and let $(G, \text{dist})$ be a metric space. We consider a mapping
\[
S : F \to G,
\]
which we call the *solution operator*. The set $F$ is called the *input set*, $f \in F$ is called the *input* or *problem instance* and $S(f)$ is called the *solution*.

In this section, we discuss important types of algorithms, error functions and cost functions that are associated with $S$, thereby defining various problems in the sense of Definition 1.1. We introduce basic concepts of information-based complexity. For a detailed discussion and a variety of further problems, we refer the reader to [TWW88] and the monographs [NW08, NW10, NW12].

1.2.1 Algorithms

In this thesis, a (deterministic) algorithm is nothing but a particular mapping
\[
A : F \to G.
\]
It is described by the *output* $A(f)$ belonging to each *input* $f \in F$. 

3
Remark 1.4. One may rightfully object that an actual algorithm is not fully determined by its outputs. However, we are only interested in the error of the algorithm in comparison with its information cost. These characteristics are already given by the input-output mapping \( A \) itself, see Section 1.2.2 and Section 1.2.3. If we wanted to talk about computational cost, then we would have to describe an algorithm by all the computational steps it performs.

We assume that an algorithm can be decomposed into two parts. The first is a mapping \( N : F \to c_{00} \) that collects a finite amount of information about the input. Here \( c_{00} \) is the union of all \( \mathbb{R}^n \) over \( n \in \mathbb{N}_0 \). The second is a mapping \( \varphi : N(F) \to G \) that uses this information to produce an output. We now discuss these two parts.

The information mapping \( N \) collects the information by taking several measurements of the problem instance. For different problems, different types of measurements may be executable. Let \( \Lambda \) be a class of real-valued functions \( L : F \to \mathbb{R} \). A functional \( L \in \Lambda \) is called a measurement, the number \( L(f) \in \mathbb{R} \) is called a piece of information about \( f \).

We give two popular examples:

- If \( F \) consists of real-valued functions on a common domain \( D \), we often consider the class \( \Lambda = \Lambda^{\text{std}} \) of function evaluations \( L(f) = f(x) \) for all \( x \in D \), the class of standard information.
- If \( F \) is a subset of a normed space, we may allow the class \( \Lambda = \Lambda^{\text{all}} \) of all continuous linear functionals, the class of linear information.

A nonadaptive information mapping based on \( \Lambda \) is a mapping of the form

\[
N_n : F \to \mathbb{R}^n, \quad N_n(f) = (L_1(f), \ldots, L_n(f))
\]

for some \( n \in \mathbb{N} \) and measurements \( L_1, \ldots, L_n \in \Lambda \). That is, \( N_n \) collects \( n \) pieces of information about the problem instance. We take the same measurements for every input. In contrast, an adaptive information mapping may use the already collected pieces of information after each measurement to decide whether and how to take another measurement. In general, a mapping \( N : F \to c_{00} \) is called an information mapping based on \( \Lambda \) if there are

- functionals \( L_i : F \times \mathbb{R}^{i-1} \to \mathbb{R} \) such that \( L_i(\cdot, y) \in \Lambda \) for all \( y \in \mathbb{R}^{i-1}, i \in \mathbb{N} \);
- a function \( T : c_{00} \to \{0, 1\} \), which we call the termination function;

such that for every \( f \in F \) we have \( N(f) = (y_1, \ldots, y_n(f)) \) with

\[
y_i = L_i(f, y_1, \ldots, y_{i-1}) \quad \text{and} \quad n(f) = \min \{ n \in \mathbb{N} \mid T(y_1, \ldots, y_n) = 0 \}.
\]

The family \(( (L_i)_{i \in \mathbb{N}}, T) \) is called a representation of the information mapping. The information is called adaptive if it is not nonadaptive.

\(^1\)Analogously, we may consider functionals \( L : F \to \mathbb{C} \) such that one piece of of information is given by one complex number \( L(f) \in \mathbb{C} \). For simplicity, we only discuss the \( \mathbb{R} \)-valued case.
To generate an output from the collected information, we allow any function \( \varphi : N(F) \to \mathbb{R} \). Of course, this means that the computational cost to obtain \( \varphi(y) \) for \( y \in N(F) \) may be arbitrarily high. For concrete algorithms, the function \( \varphi \) should be as simple as possible.

A mapping \( A : F \to G \) is called a deterministic algorithm based on \( \Lambda \) if there is an information mapping \( N : F \to c_{00} \) and a function \( \varphi : N(F) \to \mathbb{R} \) such that \( A = \varphi \circ N \). The pair \((\varphi, N)\) is called a representation of the algorithm \( A \). It is said to be nonadaptive if the information mapping \( N \) can be chosen to be nonadaptive. Else, it is called adaptive. The class of all deterministic algorithms based on \( \Lambda \) is denoted by

\[
\mathcal{A}[F,G,\Lambda,\text{det}].
\]

The class of all nonadaptive deterministic algorithms based on \( \Lambda \) is denoted by

\[
\mathcal{A}[F,G,\Lambda,\text{det},\text{nonada}].
\]

If \( F \) and \( G \) are subsets of linear spaces over \( \mathbb{R} \), the algorithm may be linear. The class of all linear deterministic algorithms based on \( \Lambda \) is denoted by

\[
\mathcal{A}[F,G,\Lambda,\text{det},\text{lin}].
\]

Note that every linear algorithm is nonadaptive and hence

\[
\mathcal{A}[F,G,\Lambda,\text{det},\text{lin}] \subset \mathcal{A}[F,G,\Lambda,\text{det},\text{nonada}] \subset \mathcal{A}[F,G,\Lambda,\text{det}].
\]

Let us turn to randomized algorithms. Here, we assume that \( F \) is equipped with a topology. A randomized algorithm based on \( \Lambda \) is a family \((A^\omega)_{\omega \in \Omega}\) of deterministic algorithms based on \( \Lambda \) which is indexed by a probability space \((\Omega, \mathcal{F}, P)\) such that the mapping

\[
F \times \Omega \to \mathbb{R}, \quad (f, \omega) \mapsto \text{dist}(S(f), A^\omega(f))
\]

is measurable. The class of all such algorithms is denoted by

\[
\mathcal{A}[F,G,\Lambda,\text{ran}].
\]

The randomized algorithm is called nonadaptive or linear if \( A^\omega \) is nonadaptive or linear for almost every \( \omega \in \Omega \). A randomized algorithm is also referred to as a Monte Carlo method. We use these terms interchangeably. Randomized algorithms can be regarded as a generalization of deterministic algorithms since any deterministic algorithm may be viewed as a family of deterministic algorithms \( A^\omega \) that is independent of \( \omega \).

**Remark 1.5.** The definition of a randomized algorithm would still make sense if we skipped the property of measurability. We prefer this definition since it simplifies the notion of the error of a randomized algorithm. Moreover, Theorem 1.18 has only been proven for measurable algorithms.

We now introduce different ways to measure the error and cost of such algorithms.
1.2.2 Errors

We introduce the error criteria that are used in this thesis. The \textit{worst case error} of a deterministic algorithm $A : F \to G$ is defined as

$$\text{err}(A, S, F, G, \text{wc}) = \sup_{f \in F} \text{dist}(S(f), A(f)).$$

It measures the maximal distance between the output and the solution. One may weaken this error criterion by considering the average distance instead. Given a probability measure $\mu$ on the input class $F$ we define the \textit{average case error}

$$\text{err}(A, S, F, G, \mu) = \int_{F} \text{dist}(S(f), A(f))^2 \, d\mu(f).$$

provided that the error functional $f \mapsto \text{dist}(S(f), A(f))$ is $\mu$-measurable.

The \textit{worst case error} of a randomized algorithm $(A_\omega)_{\omega \in \Omega}$ is defined as

$$\text{err}((A_\omega)_{\omega \in \Omega}, S, F, G, \text{wc}) = \sup_{f \in F} \sqrt{\mathbb{E} \left[ \text{dist}(S(f), A(f))^2 \right]}.$$ 

That is, it measures the maximal root mean square of the distance of the output and the solution. Given a Borel probability measure $\mu$ on $F$ we also define the \textit{average case error} of randomized algorithms as

$$\text{err}((A_\omega)_{\omega \in \Omega}, S, F, G, \mu) = \int_{F} \mathbb{E} \left[ \text{dist}(S(f), A(f))^2 \right] \, d\mu(f).$$

One could easily replace the root mean square in the above definitions by every other normalized moment of the distance. Note that these errors coincide with the respective error of a deterministic algorithm if the algorithm is independent of $\omega$.

To measure the quality of an information mapping, we introduce the notion of the \textit{radius of information}. The radius of an information mapping is the smallest error which can be achieved with algorithms that use this information mapping. It can be defined for each of the above error criteria. For instance, the radius of a deterministic information mapping $N : F \to c_{00}$ in the worst case setting is given by

$$\text{rad}(N, S, F, G) = \inf_{\varphi : c_{00} \to G} \text{err}(\varphi \circ N, S, F, G, \text{wc}).$$

Proposition 1.6 below explains why we call this quantity a radius. Note that the radius of a subset $M$ of $G$ is given by

$$\text{rad}(M) = \inf_{g \in G} \sup_{m \in M} \text{dist}(g, m). \tag{1.1}$$

An algorithm based on $N$ cannot distinguish inputs with the same information. Thus the optimal algorithm based on $N$ maps $f \in F$ to the center of the set $S(N^{-1}(y))$ of all solutions that are possible for the information $y = N(f)$. The radius of this set is called the \textit{radius of information at $y$} and denoted by

$$r_y(N, S, F, G) = \text{rad} \left( S(N^{-1}(y)) \right).$$

For given information $y$, we cannot guarantee an error less than $r_y(N, S, F, G)$. This leads to the following result.
Proposition 1.6 ([TW80]). Let $S : F \to G$ be a solution operator from a set $F$ to a metric space $G$ and let $N : F \to c_{00}$ be an information mapping. Then

$$\text{rad}(N,S,F,G) = \sup_{y \in N(F)} r_y(N,S,F,G).$$

### 1.2.3 Cost

In this thesis, the cost of an algorithm is given by the amount of information that the algorithm uses about the problem instance, that is, we study the information cost of an algorithm. We do not study their computational cost or other cost models. This is based on the assumption that collecting information usually consumes much more time than processing it: while the information may be obtained from complicated subroutines, physical measurements or even surveys, it is usually processed by basic arithmetic operations. Although this assumption is fulfilled in many examples, it may not always be adequate. Then we would have to define problems $(A, \text{err}, \text{cost})$ with other cost functions.

We first define the cost of information mappings. The cost of a nonadaptive information mapping $N$ based on $\Lambda$ is simply given by the number $n$ of measurements. The definition of the cost of adaptive information mappings is not quite as indisputable, since it may take a different number $n(f)$ of measurements for different inputs $f \in F$. We study the worst case cost of an information mapping. That is, given an information mapping $N : F \to c_{00}$ as defined in Section 1.2.1, we take the maximum of the number $n(f)$ of measurements over all possible inputs $f \in F$. Note that the number $n(f)$ does not depend on the representation $((L_i)_{i \in N}, T)$ of the information mapping $N$. Hence, we define

$$\text{cost}(N,F,\Lambda,\text{wc}) = \sup_{f \in F} n(f).$$

Another approach would be to consider the average number $n(f)$ of measurements with respect to some measure $\mu$ on $F$, the average case cost of $N$.

We now define the cost of algorithms. The worst case cost of a deterministic algorithm $A \in \mathcal{A}[F,G,\Lambda,\text{det}]$ is the worst case cost of the information mapping in an optimal representation of $A$, that is,

$$\text{cost}(A,F,\Lambda,\text{wc}) = \min \{ \text{cost}(N,F,\Lambda,\text{wc}) \mid (\varphi,N) \text{ representation of } A \}. $$

Moreover, we define the worst case cost of a randomized algorithm $(A^\omega)_{\omega \in \Omega}$ by

$$\text{cost}((A^\omega)_{\omega \in \Omega},F,\Lambda,\text{wc}) = \sup_{\omega \in \Omega} \text{cost}(A^\omega,F,\Lambda,\text{wc}).$$

This is the cost of computing $A^\omega(f)$ for the worst input $f \in F$ and the worst realization $A^\omega$ of $A$. Note that it is also common to consider the expectation over all realizations instead of the maximum.
1.2.4 Resulting Problems

We may now formally define the problems of our interest that are inherited from a solution operator $S$.

**Definition 1.7.** Let $S : F \to G$ be an operator from a topological space $F$ to a metric space $G$ and let $\Lambda$ be a class of real-valued functions on $F$. Let $\star \in \{\det, \ran\}$, $\circ \in \{\emptyset, \text{nonada}, \text{lin}\}$ and $\triangle \in \{\mu, \text{wc}\}$, where $\mu$ is some probability measure on $F$. Then we define the problem

$$\mathcal{P}[S, F, G, \Lambda, \star, \circ, \triangle] = (\mathcal{A}, \text{err}, \text{cost})$$

of approximating $S$ with (nonadaptive/linear) deterministic/randomized algorithms based on $\Lambda$ in the worst/average case setting by

$$\mathcal{A} = \mathcal{A}[F, G, \Lambda, \star, \circ],$$

$$\text{err} = \text{err}(\cdot, S, F, G, \triangle),$$

$$\text{cost} = \text{cost}(\cdot, F, \Lambda, \text{wc}).$$

**Remark 1.8.** Note that we always consider the worst case cost. The term average case only refers to the error criterion.

**Remark 1.9.** The setting is determined by the parameters $S$, $F$, $G$, $\Lambda$, $\det$ or $\ran$, $\text{wc}$ or $\mu$, and possibly nonada or lin. So far, we put all relevant parameters in the definition of the problems, the classes of algorithms, and the error and cost functions. In what follows, a part of the setting will often be clear from the context. For instance, a whole chapter may be concerned with the same solution operator $S$. We usually skip the respective parameters in this case.

Let us discuss some basic relations between the minimal errors in the different settings. Obviously, we have the relation

$$e(n, \mathcal{P}[S, F, G, \Lambda, \star, \circ, \mu]) \leq e(n, \mathcal{P}[S, F, G, \Lambda, \star, \circ, \text{wc}])$$

since the worst case error of an algorithm is always at least as large as the average case error. Moreover, we have

$$e(n, \mathcal{P}[S, F, G, \Lambda, \ran, \circ, \triangle]) \leq e(n, \mathcal{P}[S, F, G, \Lambda, \det, \circ, \triangle])$$

since the class of randomized algorithms is larger than the class of deterministic algorithms. In fact, we even have equality in the average case setting, that is, if $\mu$ is a Borel probability measure on the topological space $F$, we have

$$e(n, \mathcal{P}[S, F, G, \Lambda, \ran, \mu]) = e(n, \mathcal{P}[S, F, G, \Lambda, \det, \mu]).$$

(1.2)

This means that randomization has no effect in the average case setting. This is a simple consequence of Tonelli’s theorem: if $(A^\omega)_{\omega \in \Omega}$ is a randomized algorithm with worst case cost $n$ or less, we have

$$\text{err}((A^\omega)_{\omega \in \Omega}, \mu)^2 = \int_F \mathbb{E} \left( \text{dist}(S(f), A^\omega(f))^2 \right) \ d\mu(f)$$

$$= \mathbb{E} \left( \int_F \text{dist}(S(f), A^\omega(f))^2 \ d\mu(f) \right) = \mathbb{E} \left( \text{err}(A^\omega, \mu)^2 \right).$$
This means that there is a realization $A^\omega$ of the randomized algorithm such that
\[ \text{err} (A^\omega, \mu) \leq \text{err} ((A^\omega)_{\omega \in \Omega}, \mu) . \]

Since $A^\omega$ is a deterministic algorithm with cost $n$ or less, this proves (1.2).

In particular, we obtain the following theorem, which links the worst case error of randomized algorithms and the average case error of deterministic algorithms. It is called Bakhvalov’s technique and is essential for proving lower bounds for the error of randomized algorithms. We refer to [NW08, Section 4.3.3] for more details.

**Theorem 1.10** ([NW08]). Let $S : F \to G$ be an operator from a topological space $F$ to a metric space $G$ and let $\Lambda$ be a class of real-valued functions on $F$. For any Borel probability measure $\mu$ on $F$ and any $n \in \mathbb{N}_0$, we have
\[ e(n, \mathcal{P}[S, F, G, \Lambda, \text{ran}, \text{wc}]) \geq e(n, \mathcal{P}[S, F, G, \Lambda, \text{det}, \mu]) . \]

We finish this section with an example. Note that many other examples are provided throughout this thesis.

**Example 1.11** (An integration problem, Part 1 of 2). Assume that the function $f : [0, 1] \to \mathbb{R}$ is known to be in the Lipschitz class
\[ F = \left\{ f : [0, 1] \to \mathbb{R} \mid \forall (x, y) \in [0, 1]^2 : |f(x) - f(y)| \leq |x - y| \right\} . \]

The function itself, however, is unknown. We want to approximate the integral
\[ S(f) = \int_0^1 f(x) \, dx \]
of the function up to a guaranteed error. To do so, we may request a finite number of function values using any deterministic scheme. The cost of an algorithm $A : F \to \mathbb{R}$ is the maximal number of requested function values and its error is
\[ \text{err}(A) = \sup_{f \in F} |S(f) - A(f)| . \]

In the above terms, we study the problem $\mathcal{P}[S, F, \mathbb{R}, \Lambda^{\text{std}}, \text{det}, \text{wc}]$.

We continue this example after gathering some results on so-called linear problems.

### 1.2.5 Linear Problems

We consider deterministic problems in the worst case setting. Many of these problems are linear in the sense of the following definition.

**Definition 1.12.** The problem $\mathcal{P}[S, F, G, \Lambda, \text{det}, \text{wc}]$ is called a *linear problem* if
\begin{itemize}
  \item $F$ is a nonempty, convex and symmetric subset of a normed space $\tilde{F}$;
  \item $G$ is a normed space;
\end{itemize}
• $S : \tilde{F} \rightarrow G$ is linear;
• $\Lambda$ is a class of continuous linear functionals.

We present some basic results on linear problems without proof. We refer the reader to [NW08, Section 4.2] for further details, proofs and references. The first result says that the radius of a nonadaptive information mapping $N : F \rightarrow \mathbb{R}^n$ is already (almost) determined by its radius at zero.

**Theorem 1.13 ([NW08]).** Let $S, F, G$, and $\Lambda$ describe a linear problem and let $N$ be a nonadaptive information mapping based on $\Lambda$. Then

$$r_0(N, S, F, G) \leq \text{rad}(N, S, F, G) \leq 2r_0(N, S, F, G).$$

It is easy to check that the radius of $N$ at zero satisfies

$$r_0(N, S, F, G) = \sup_{f \in F : N(f) = 0} \|Sf\|_G.$$

An important consequence of the previous theorem is that adaption is not necessary for linear problems. If $N : F \rightarrow c_{00}$ is an adaptive information mapping as defined in Section 1.2.1, we define a corresponding nonadaptive information mapping $N^\text{non} : F \rightarrow \mathbb{R}^n$ by setting $n = n(0)$ and

$$N^\text{non}(f) = (L_1(f), L_2(f, 0), \ldots, L_n(f, 0, \ldots, 0)).$$

This means that the nonadaptive information $N^\text{non}$ takes the same measurements for every input and these measurements are the same as for the adaptive information $N$ for the input zero. By Proposition 1.6 and Theorem 1.13, we obtain

$$\text{rad}(N^\text{non}, S, F, G) \leq 2r_0(N^\text{non}, S, F, G) = 2r_0(N, S, F, G) \leq 2 \text{rad}(N, S, F, G).$$

Clearly the worst case cost of $N^\text{non}$ is bounded above by the worst case cost of $N$. In particular, we may loose a factor of at most 2 if we study the error of nonadaptive algorithms in comparison to arbitrary algorithms [NW08, Section 4.2.1].

**Corollary 1.14 ([NW08]).** Let $S, F, G$ and $\Lambda$ describe a linear problem. For every information mapping $N : F \rightarrow c_{00}$ the nonadaptive information mapping $N^\text{non} : F \rightarrow \mathbb{R}^n$ satisfies

$$\text{rad}(N^\text{non}, S, F, G) \leq 2 \text{rad}(N, S, F, G).$$

In particular, for all $n \in \mathbb{N}$, we have

$$e(n, \mathcal{P}[S, F, G, \Lambda, \text{det, wc, nonada}]) \leq 2 e(n, \mathcal{P}[S, F, G, \Lambda, \text{det, wc}]).$$

In many cases we do not even lose the factor 2. In addition, it turns out that linear algorithms are optimal in the very same cases [NW08, Section 4.2.2]. In the following theorem, $\mathcal{B}(X)$ and $\mathcal{C}(X)$ are the spaces of bounded respectively continuous real valued functions on $X$. 

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1.2. Important Types of Problems

Theorem 1.15 ([NW08]). Let $S, F, G$ and $\Lambda$ describe a linear problem. Assume that one of the following conditions is satisfied:

- $G = \mathbb{R}$ or $G = B(X)$ for some set $X$ or $G$ is some $L^\infty$-space;
- $F$ is the unit ball of a pre-Hilbert space $\tilde{F}$;
- $G = \mathcal{C}(X)$ for some compact Hausdorff space $X$ and $S$ is compact.

Then every information mapping $N : F \to c_{00}$ yields a nonadaptive information mapping $N_{\text{non}} : F \to \mathbb{R}^n$ with

$$\text{rad}(N_{\text{non}}, S, F, G) \leq \text{rad}(N, S, F, G).$$

Moreover, the nonadaptive information satisfies

$$\text{rad}(N_{\text{non}}, S, F, G) = r_0(N_{\text{non}}, S, F, G) = \inf_{\varphi \text{ linear}} \text{err } (\varphi \circ N_{\text{non}}).$$

This leads to a very useful formula for the $n^{\text{th}}$ minimal worst case error.

Theorem 1.16 ([NW08]). Let $\mathcal{P} = (A, \text{err}, \text{cost})$ be a linear problem given by $S, F, G$ and $\Lambda$ such that one of the conditions in Theorem 1.15 holds. Then, for every $n \in \mathbb{N}$,

$$e(n, \mathcal{P}) = \inf_{\text{cost}(A) \leq n} \text{err}(A) = \inf_{N \in \Lambda^n} \sup_{f \in F : N(f) = 0} \|Sf\|_G.$$

Example 1.11 (Part 2 of 2). The problem of integrating Lipschitz-functions is linear. The target space is $\mathbb{R}$. Hence, linear algorithms are optimal and we only need to consider algorithms of the form

$$A_n : F \to \mathbb{R}, \quad A_n(f) = \sum_{i=1}^{n} a_i f(x_i)$$

with some $n \in \mathbb{N}$, weights $a_i \in \mathbb{R}$ and nodes $x_i \in [0, 1]$. With the help of Theorem 1.16 it is easily verified that

$$e(n, \mathcal{P}) = \frac{1}{4n}$$

and that the minimal error is achieved by the algorithm $A_n$ if we choose constant weights $a_i = 1/n$ and equidistant nodes $x_i = \frac{2i-1}{2n}$ for $i = 1, \ldots, n$.

Linear Problems over Hilbert Spaces

We finish this section with linear problems over Hilbert spaces based on $\Lambda^{\text{all}}$. We assume that $F$ is the unit ball of a Hilbert space $H$ and that $G$ is another Hilbert space. Let $S : H \to G$ be a compact linear operator.

The operator $W = S^*S : H \to H$ is positive and compact. Hence, it admits a finite or countable orthonormal basis $\mathcal{B}$ of $\ker(S)^\perp$ consisting of eigenvectors $b \in \mathcal{B}$ to eigenvalues

$$\lambda(b) = \langle Wb, b \rangle_H = \|Sb\|_G^2 > 0.$$
For any $f \in H$ we have the relation

$$S(f) = \sum_{b \in \mathcal{B}} \langle f, b \rangle_H Sb.$$ 

The square-roots of the eigenvalues of $W$ are called singular values of $S$. Let $\sigma_n$ be the $n^{th}$ largest singular value if $n \leq |\mathcal{B}|$. Else, let $\sigma_n = 0$. We consider the linear algorithm

$$A_n : F \to G, \quad A_n(f) = \sum_{b \in \mathcal{B}(n)} \langle f, b \rangle_H Sb,$$

where $\mathcal{B}(n)$ consists of all $b \in \mathcal{B}$ that satisfy $\|Sb\|_G > \sigma_{n+1}$. This algorithm is optimal among all algorithms with cost $n$ or less [NW08, Section 4.2.3].

**Theorem 1.17 (NW08).** The algorithm $A_n$ satisfies $\text{cost}(A_n) \leq n$ and

$$\text{err}(A_n) = e(n, \mathcal{P}[S, F, G, \Lambda^{all}, \det, wc]) = \sigma_{n+1}.$$ 

It is known from [Nov92] that randomized algorithms cannot be much better than deterministic algorithms in this setting: up to a factor of at most $\sqrt{2}$, the algorithm $A_{2n-1}$ is as good as any deterministic or randomized algorithm with cost $n$ or less.

**Theorem 1.18 (NW08).** Let $H$ and $G$ be Hilbert spaces, let $F$ be the unit ball of $H$, and let $S : H \to G$ be compact. For any $n \in \mathbb{N}$, we have

$$e(n, \mathcal{P}[S, F, G, \Lambda^{all}, \text{ran}, wc]) \geq \frac{1}{\sqrt{2}} e(2n - 1, \mathcal{P}[S, F, G, \Lambda^{all}, \det, wc]).$$
Chapter 2
Integration and Approximation of Functions with Mixed Smoothness

In this chapter, we study the following multivariate problems.

- Section 2.1: The integration of multivariate functions from different smoothness classes. We allow randomized algorithms based on $\Lambda^{\text{std}}$. This section is based on [Kri16, KN17, Ull17].

- Section 2.2: The approximation of a tensor product operator between Hilbert spaces. We allow deterministic algorithms based on $\Lambda^{\text{all}}$. This section is based on [Kri18a].

- Section 2.3: The $L^2$-approximation of functions from a Hilbert space that is compactly embedded into $L^2$. We allow randomized algorithms based on $\Lambda^{\text{std}}$. This section is based on [Kri18c].

We will focus on the rate of convergence of the $n^{\text{th}}$ minimal error and provide algorithms that achieve the optimal error rate. In Section 2.2 and 2.3, we will also discuss the error of these algorithms for small $n$. All results can be applied for multivariate functions with mixed smoothness.

2.1 A Universal Algorithm for Integration

We want to approximate the integral

$$S_d(f) = \int_{[0,1]^d} f(x) \, dx$$

of a multivariate function $f : [0,1]^d \to \mathbb{R}$. To compute an approximation, we may request a certain amount $n$ of function values. The function $f$ itself is not known. We do, however, have some a priori knowledge about the function. We assume that the function is smooth in the sense that certain weak derivatives $D^\alpha f$ exist and are square-integrable. Which derivatives are known to be existent and square-integrable is different in different applications.
Chapter 2. Integration and Approximation of Functions with Mixed Smoothness

In several applications, $\alpha$ covers the range of all multi-indices with $|\alpha| \leq r$ for some $r \in \mathbb{N}$. We say that $f$ has isotropic smoothness $r$. For example, the solutions of elliptic partial differential equations in general and Poisson’s equation in particular have this type of smoothness [GT01, HT08]. They typically appear in electrostatics or continuum mechanics. With deterministic algorithms, the integral of such functions can be computed up to an error of order $n^{-r/d}$, but not with higher accuracy [Bak59, Nov88]. The expected error may be smaller, if randomness can be used. With randomized algorithms, we may achieve an expected error of order $n^{-r/d-1/2}$ [Bak59, Bak62, Nov88].

In other applications, $\alpha$ covers the range of all multi-indices with $\|\alpha\|_\infty \leq r$. We say that $f$ has mixed smoothness $r$. This is a stronger smoothness condition. For example, solutions of the electronic Schrödinger equation have this type of smoothness [Yse10]. With deterministic algorithms, the integral of such functions can be computed up to an error of order $n^{-r/\ln n}$ [Pro76]. Using randomness, we may achieve an expected error of order $n^{-r-1/2}$ [Bak62, Ull17]. These rates are much better than the rates in the isotropic case if the number $d$ of variables is large.

In most applications, however, we do not really know how smooth our integrand is. Thus, we would like to have an algorithm which can be applied to any integrable function and automatically detects its smoothness. That is, whenever $f$ has isotropic or mixed smoothness $r$ for some $r \in \mathbb{N}$, the expected and guaranteed error should decay with the above mentioned error rates. We say that the algorithm is universal. In this section, we will present a universal algorithm for multivariate integration.

Let us formulate the main result of this section. For every $r \in \mathbb{N}$, let $H^r([0,1]^d)$ be the linear space of functions with isotropic smoothness $r$ and let $H^r_{\text{mix}}([0,1]^d)$ be the linear space of functions with mixed smoothness $r$. We define norms on these spaces via the relations

$$\|f\|_{H^r([0,1]^d)}^2 = \sum_{|\alpha| \leq r} \|D^\alpha f\|_{L^2([0,1]^d)}^2$$

$$\|f\|_{H^r_{\text{mix}}([0,1]^d)}^2 = \sum_{\|\alpha\|_\infty \leq r} \|D^\alpha f\|_{L^2([0,1]^d)}^2$$

for $f \in H^r([0,1]^d)$ and $f \in H^r_{\text{mix}}([0,1]^d)$, respectively.

For each $n \in \mathbb{N}$, we define a randomized algorithm $(A^\omega_n)_{\omega \in \Omega}$ of the form

$$A^\omega_n(f) = \sum_{j=1}^n a_j(\omega) f(x^{(j)}(\omega))$$

for $f \in L^1([0,1]^d)$ and $\omega \in \Omega$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $x^{(j)} : \Omega \to [0,1]^d$ and $a_j : \Omega \to \mathbb{R}$ are random variables for each $j \leq n$, see Algorithm 2.21. These algorithms have the following properties.

**Theorem 2.1 (KN17, Ull17).** There are positive constants $c, c_1, c_2, \ldots$ such that the following holds for all $n \in \mathbb{N}$ with $n \geq c$ and $f \in L^1([0,1]^d)$.

- $\mathbb{E}(A_n(f)) = S_d(f)$.

If $f$ has mixed smoothness $r \in \mathbb{N}$, then
2.1. A Universal Algorithm for Integration

- \( \sqrt{\mathbb{E} \left| S_d(f) - A_n(f) \right|^2} \leq c_r n^{-r-1/2} \| f \|_{H^{r}_{\text{mix}}([0,1]^d)} \),
- \( \mathbb{P} \left( |S_d(f) - A_n(f)| \leq c_r n^{-r} (\ln n)^{(d-1)/2} \| f \|_{H^{r}_{\text{mix}}([0,1]^d)} \right) = 1. \)

If \( f \) has isotropic smoothness \( r \in \mathbb{N} \) with \( r > d/2 \), then

- \( \sqrt{\mathbb{E} \left| S_d(f) - A_n(f) \right|^2} \leq c_r n^{-r/d-1/2} \| f \|_{H^{r}_{\text{mix}}([0,1]^d)} \),
- \( \mathbb{P} \left( |S_d(f) - A_n(f)| \leq c_r n^{-r/d} \| f \|_{H^{r}_{\text{mix}}([0,1]^d)} \right) = 1. \)

We remark that these constants may depend on \( d \). The condition \( r > d/2 \) ensures that the functions in \( H^{r}_{\text{mix}}([0,1]^d) \) are continuous. The algorithm is a randomization of Frolov’s algorithm \([Fro76]\). It was first proposed in \([KN17]\). The order of the expected error for functions with mixed smoothness was proven in \([Ull17]\).

In particular, we obtain the following result on the order of convergence. Let \( F^r_d \) be the unit ball of \( H^{r}_{\text{mix}}([0,1]^d) \) and let

\[ \mathcal{P}_d^r = \mathcal{P}[S_d, F^r_d, \mathbb{R}, \Lambda_{\text{std}}, \text{ran}, \text{wc}] \]

be the problem of integrating a function from \( F^r_d \) with randomized algorithms based on \( \Lambda_{\text{std}} \) in the worst case setting.

**Corollary 2.2** \([Bak59, Ull17]\). For any \( r \in \mathbb{N} \) and \( d \in \mathbb{N} \), we have

\[ e(n, \mathcal{P}_d^r) \leq n^{-r-1/2}. \]

Section 2.1 is organized as follows. In Section 2.1.1 we define and characterize the function classes of our interest. In Section 2.1.2 we introduce Frolov’s deterministic algorithm for the integration of functions with compact support. In Section 2.1.3 and Section 2.1.4 we discuss how this algorithm can be improved by introducing a random dilation and a random shift to the set of nodes. Section 2.1.5 shows how we can integrate functions without compact support using a transformation of the unit cube. Here we also give a proof of Theorem 2.1 and Corollary 2.2. We remark that our algorithm is optimal for many other classes of smooth functions in terms of the order of convergence of its error, see \([Ull17]\).

### 2.1.1 The Function Classes

Let \( r \in \mathbb{N} \) and \( d \in \mathbb{N} \). The Sobolev space of mixed smoothness \( r \) is the vector space

\[ H^{r}_{\text{mix}}(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \mid D^\alpha f \in L^2(\mathbb{R}^d) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } \| \alpha \|_\infty \leq r \right\} \]

of \( d \)-variate, real-valued functions, equipped with the scalar product

\[ \langle f, g \rangle_{H^{r}_{\text{mix}}(\mathbb{R}^d)} = \sum_{\| \alpha \|_\infty \leq r} \langle D^\alpha f, D^\alpha g \rangle_{L^2(\mathbb{R}^d)}. \]
It is known that \( H^r_{\text{mix}}(\mathbb{R}^d) \) is a Hilbert space and its elements \( f \in L^2(\mathbb{R}^d) \) have continuous representatives. The Fourier transform is the unique continuous linear operator \( \mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) satisfying
\[
\mathcal{F}f(y) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i (x,y)} \, dx
\]
for integrable \( f : \mathbb{R}^d \to \mathbb{R} \) and almost all \( y \in \mathbb{R}^d \). The space \( H^r_{\text{mix}}(\mathbb{R}^d) \) contains exactly those functions \( f \in L^2(\mathbb{R}^d) \) with \( \mathcal{F}f \cdot h^l_{\alpha} \in L^2(\mathbb{R}^d) \) for the weight function
\[
h^l_{\alpha} : \mathbb{R}^d \to \mathbb{R}^+, \quad h^l_{\alpha}(x) = \sum_{\|\alpha\| \leq l} d \prod_{j=1}^{d} |2\pi x_j|^{2\alpha_j} = \prod_{j=1}^{d} \sum_{k=0}^{r} |2\pi x_j|^{2k}.
\]
In terms of the Fourier transform, the scalar product in \( H^r_{\text{mix}}(\mathbb{R}^d) \) is given by
\[
\langle f, g \rangle_{H^r_{\text{mix}}(\mathbb{R}^d)} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2(\mathbb{R}^d, h^r)} ,
\]
where \( L^2(\mathbb{R}^d, h^r) \) is the weighted \( L^2 \)-space with weight \( h^r \). Analogously, the Sobolev space of isotropic smoothness \( r \) is
\[
H^r(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \mid D^\alpha f \in L^2(\mathbb{R}^d) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq r \right\},
\]
equipped with the scalar product
\[
\langle f, g \rangle_{H^r(\mathbb{R}^d)} = \sum_{|\alpha| \leq r} \langle D^\alpha f, D^\alpha g \rangle_{L^2(\mathbb{R}^d)}.\]
This is again a Hilbert space. If \( r \) is greater than \( d/2 \), then \( H^r(\mathbb{R}^d) \) also consists of continuous functions. The space contains exactly those functions \( f \in L^2(\mathbb{R}^d) \) with \( \mathcal{F}f \cdot v^l_{\alpha} \in L^2(\mathbb{R}^d) \) for the weight function
\[
v^l_{\alpha} : \mathbb{R}^d \to \mathbb{R}^+, \quad v^l_{\alpha}(x) = \sum_{|\alpha| \leq l} d \prod_{j=1}^{d} |2\pi x_j|^{2\alpha_j}.
\]
In terms of its Fourier transform, the scalar product in \( H^r(\mathbb{R}^d) \) is given by
\[
\langle f, g \rangle_{H^r(\mathbb{R}^d)} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2(\mathbb{R}^d, v^r)} ,
\]
where \( L^2(\mathbb{R}^d, v^r) \) is the weighted \( L^2 \)-space with weight \( v^r \). We refer to [SU09] for an overview regarding these spaces of mixed and isotropic smoothness.

Furthermore, let \( C_c(\mathbb{R}^d) \) be the real vector space of all continuous real valued functions with compact support in \( \mathbb{R}^d \). The spaces \( \hat{H}^r_{\text{mix}}([0,1]^d) \) and \( H^r([0,1]^d) \) of functions in \( H^r_{\text{mix}}(\mathbb{R}^d) \) or \( H^r(\mathbb{R}^d) \) with compact support in the unit cube are subspaces of \( C_c(\mathbb{R}^d) \). They can also be considered as subspaces of the Hilbert space
\[
H^r_{\text{mix}}([0,1]^d) = \left\{ f \in L^2([0,1]^d) \mid D^\alpha f \in L^2([0,1]^d) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq r \right\},
\]
equipped with the scalar product
\[
\langle f, g \rangle_{H^r_{\text{mix}}([0,1]^d)} = \sum_{|\alpha| \leq r} \langle D^\alpha f, D^\alpha g \rangle_{L^2([0,1]^d)} ,
\]
2.1. A Universal Algorithm for Integration

or the Hilbert space
\[ H^r([0,1]^d) = \{ f \in L^2([0,1]^d) \mid D^\alpha f \in L^2([0,1]^d) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq r \}, \]
with scalar product
\[ \langle f, g \rangle_{H^r([0,1]^d)} = \sum_{|\alpha| \leq r} \langle D^\alpha f, D^\alpha g \rangle_{L^2([0,1]^d)}. \]

2.1.2 Frolov’s Deterministic Algorithm

Our methods are based on the following family of deterministic linear algorithms.

**Algorithm 2.3.** Let \( B \in \mathbb{R}^{d \times d} \) be invertible and \( v \in \mathbb{R}^d \). We define
\[ Q_B^v(f) = \frac{1}{|\det B|} \sum_{m \in \mathbb{Z}^d} f \left( B^{-\top}(m + v) \right) \]
for any \( f : \mathbb{R}^d \to \mathbb{R} \) such that the right hand side converges absolutely. The vector \( v \) is called shift parameter. We write \( Q_B = Q_B^0 \).

**Remark 2.4.** The value \( Q_B^v(f) \) can be thought of as a Riemann sum: The nodes of the algorithm are the lower left corners of the parallelepipeds
\[ B^{-\top}(m + [0,1]^d), \quad m \in \mathbb{Z}^d, \]
and the weight \( |\det B|^{-1} \) is the volume of this parallelepiped.

The algorithm is well defined for functions with compact support. To integrate these functions, the algorithm \( Q_B^v \) only uses the nodes \( B^{-\top}(m + v) \) for all
\[ m \in \mathbb{Z}^d \cap (B^\top (\text{supp } f) - v). \]
The number of these nodes should be close to the volume of the latter set. In particular, the number of nodes of \( Q_B^v \) should behave like \( a^d \) as \( a \) tends to infinity. The following lemma gives an exact upper bound, see [Skr94] for other bounds.

**Lemma 2.5.** Assume that \( f : \mathbb{R}^d \to \mathbb{R} \) is supported in an axis-parallel cube of edge length \( l > 0 \). For any invertible matrix \( B \in \mathbb{R}^{d \times d} \), \( v \in \mathbb{R}^d \), and \( a \geq 1 \), the algorithm \( Q_B^v \) uses at most \( (l \|B\|_1 + 1)^d a^d \) function values of \( f \).

**Proof.** The number of computed function values is given by the cardinality of
\[ M = \{ m \in \mathbb{Z}^d \mid (aB)^{-\top}(m + v) \in \text{supp } f \}. \]
By assumption, \( f \) has support in \([-l/2, l/2]^d + z \) for some \( z \in \mathbb{R}^d \). Thus, any \( m \in M \) satisfies
\[ m + \left( v - aB^\top z \right) \in \frac{al}{2} B^\top([-1,1]^d). \]
Since \( \|B^\top x\|_\infty \leq \|B^\top\|_\infty = \|B\|_1 \) for \( x \in [-1,1]^d \), we obtain
\[ M \subset \left\{ m \in \mathbb{Z}^d \mid m + \left( v - aB^\top z \right) \in \frac{al}{2} [-\|B\|_1, \|B\|_1]^d \right\} \]
and \( \text{card}(M) \leq (al\|B\|_1 + 1)^d. \) Since \( 1 \leq a \), we get the desired estimate. \[ \square \]
Chapter 2. Integration and Approximation of Functions with Mixed Smoothness

The error of this algorithm for integration on $C_c(\mathbb{R}^d)$ can be expressed in terms of the Fourier transform.

**Lemma 2.6.** For any invertible matrix $B \in \mathbb{R}^{d \times d}$, $v \in \mathbb{R}^d$, and $f \in C_c(\mathbb{R}^d)$

$$\left| Q^v_B(f) - \int_{\mathbb{R}^d} f(x) \, dx \right| \leq \sum_{m \in \mathbb{Z}^d \setminus \{0\}} |F f(Bm)|.$$  

**Proof.** The function $g = f \circ B^{-\top}(\cdot + v)$ is continuous with compact support. Hence, the Poisson summation formula and an affine linear substitution $x = B^\top y - v$ yield

$$Q^v_B(f) = \frac{1}{|\det B|} \sum_{m \in \mathbb{Z}^d} g(m) = \frac{1}{|\det B|} \sum_{m \in \mathbb{Z}^d} F g(m)$$

$$= \sum_{m \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(B^{-\top}(x + v)) \cdot e^{-2\pi i (x,m)} \, dx$$

$$= \sum_{m \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(y) \cdot e^{-2\pi i (B y - v,m)} \, dy$$

$$= \sum_{m \in \mathbb{Z}^d} F f(Bm) \cdot e^{2\pi i (v,m)},$$

if the latter series converges absolutely, see [Koc00, pp.356]. If not, the stated inequality is obvious. This proves the statement, since $F f(Bm)e^{2\pi i (v,m)}$ yields the integral of $f$ for $m = 0$. \qed

It is known how to choose the matrix $B$ in the rule $Q^v_B$ to get a good deterministic quadrature rule on $\mathcal{H}_{mix}^r([0,1]^d)$.

**Definition 2.7.** We say that $B \in \mathbb{R}^{d \times d}$ is a **Frolov matrix** if the following holds:

- $B$ is invertible.
- $\prod_{j=1}^d (Bm)_j \geq 1$ for any $m \in \mathbb{Z}^d \setminus \{0\}$.
- Any axis-aligned box of volume $c > 0$ contains at most $c + 1$ points of the lattice $B\mathbb{Z}^d$.

If $B$ is a Frolov matrix, then the algorithm $Q_{n^{1/d}B}$ for $n \in \mathbb{N}$ (see Algorithm 2.3) is referred to as **Frolov’s algorithm.**

We first note that the number of nodes of the Frolov algorithm is of order $n$. To be precise, Lemma 2.5 says that $Q_{n^{1/d}B}$ uses at most $(\|B\|_1 + 1)^d n$ function values if the input function is supported in $[0,1]^d$.

It is known that one can construct a Frolov matrix $B$ in the following way. Let $p \in \mathbb{Z}[x]$ be a polynomial of degree $d$ with leading coefficient 1 which is irreducible over $\mathbb{Q}$ and has $d$ different real roots $\zeta_1, \ldots, \zeta_d$. Then the matrix

$$B = (\zeta_j^{i-1})_{i,j=1}^d$$
has the desired properties, as shown in [Tem93, p.364] and [Ull16]. In arbitrary
dimension $d$ we can choose $p(x) = (x - 1)(x - 3) \ldots (x - 2d + 1) - 1$, see [Pro76]
or [Ull16]. In particular, there exists a $d$-dimensional Frolov matrix for any $d \in \mathbb{N}$.
If $d$ is a power of two, we can also choose $p(x) = 2 \cos(d \cdot \arccos(x/2)) = 2T_d(x/2)$,
where $T_d$ is the Chebyshev polynomial of degree $d$, see [Tem93, p.365]. Then the
roots of $p$ are explicitly given by $\zeta_j = 2 \cos\left(\frac{2j-1}{2d}\pi\right)$ for $j = 1, \ldots, d$ and the lattice
$BZ^d$ is orthogonal [KOU17]. We remark that an invertible matrix $B$ is a Frolov
matrix iff there is some $c > 0$ such that $cB^{-\top}$ is a Frolov matrix [Skr94].

Geometrically speaking, the second property of Definition 2.7 says that every point
of the Frolov lattice $BZ^d$ but zero is contained in the complement of a hyperbolic
cross. We denote these sets by

$$D_t = \left\{ x \in \mathbb{R}^d \mid \prod_{j=1}^{d} |x_j| \geq t \right\} \quad \text{for} \quad t > 0.$$ 

This property is illustrated in Figure 2.1.

![Figure 2.1](image)

This figure shows the lattice $3BZ^d$ for $d = 2$ and the Frolov matrix

$$B = \begin{pmatrix} 1 & 2 - \sqrt{2} \\ 1 & 2 + \sqrt{2} \end{pmatrix}. $$

Except the origin, every lattice point is contained in $D_9$.

In 1976, Frolov showed that this deterministic algorithm has the optimal order of
convergence on $H_{\text{mix}}^r([0, 1]^d)$ and that it satisfies the error bound below. Note that
the constant in this error bound depends on the choice of the Frolov matrix.

**Theorem 2.8 ([Pro76]).** Let $B \in \mathbb{R}^{d \times d}$ be a Frolov matrix and $r \in \mathbb{N}$. There is some $c_r > 0$
such that for every $n \geq 2$ and $f \in H_{\text{mix}}^r([0, 1]^d)$

$$|Q_{n^{1/d}B}(f) - S_d(f)| \leq c_r n^{-r} (\ln n)^{d-1} \|f\|_{H_{\text{mix}}^r([0, 1]^d)}.$$ 

For a proof of this error bound and its optimality, we refer the reader to [Ull16]. In
fact, this error bound holds uniformly for $Q_n^{v^\top UB}$ for any diagonal matrix $U \in \mathbb{R}^{d \times d}$
with diagonal entries in $[1, 2^{1/d}]$ and $v \in \mathbb{R}^d$, which is the statement of Theorem 2.9.
We note that Frolov’s algorithm also has the optimal rate of convergence on $H^r([0, 1]^d)$.
This is implied by Theorem 2.10.
2.1.3 Random Dilation

We study the impact of random dilations on Frolov’s algorithm. We consider the method \(Q_{n^{1/d} UB}^\nu\) (see Algorithm 2.3) for a Frolov matrix \(B \in \mathbb{R}^{d \times d}\), \(n \in \mathbb{N}\), shift parameter \(\nu \in \mathbb{R}^d\) and a random diagonal matrix \(U \in \mathbb{R}^{d \times d}\) whose diagonal entries are independent and uniformly distributed in \([1, 2^{1/d}]\). This method computes at most \(2 (\|B\|_1 + 1)^d n\) function values, see Lemma 2.5.

Guaranteed Errors

With probability 1, the error has the same rate of convergence as Frolov’s algorithm.

**Theorem 2.9** ([KN17]). Let \(B \in \mathbb{R}^{d \times d}\) be a Frolov matrix and \(r \in \mathbb{N}\). There is some \(c_r > 0\) such that for any \(n \geq 2\) and \(f \in H_{mix}^r([0, 1]^d)\),

\[
\sup_{U, \nu} |Q_{n^{1/d} UB}^\nu(f) - S_d(f)| \leq c_r n^{-r} (\ln n)^{\frac{d+1}{2}} \|f\|_{H_{mix}^r([0, 1]^d)},
\]

where the supremum is taken over all diagonal matrices \(U \in \mathbb{R}^{d \times d}\) with diagonal entries in \([1, 2^{1/d}]\) and \(\nu \in \mathbb{R}^d\).

**Proof.** Let us fix \(U\) and \(\nu\) as above. By Lemma 2.6 and Hölder’s inequality,

\[
|Q_{n^{1/d} UB}^\nu(f) - S_d(f)| \leq \left( \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \left| F f(n^{1/d} UBm) \right| \right)^2 \leq \left( \sum_{m \in \mathbb{Z}^d \setminus \{0\}} h_r(n^{1/d} UBm)^{-1} \left( \sum_{m \in \mathbb{Z}^d \setminus \{0\}} h_r(n^{1/d} UBm) \cdot |F f(n^{1/d} UBm)| \right)^2 \right)^{1/2}.
\]

We first prove that the first factor in this product is bounded above by a constant multiple of \(n^{-2r} (\ln n)^{d-1}\), where the constant is independent of \(\nu\) and \(U\). To that end, we consider the auxiliary set

\[N(\beta) = \left\{ x \in \mathbb{R}^d \mid \forall 1 \leq j \leq d : |2^{\beta_j} - 1| \leq |x_j| < 2^{\beta_j} \right\}\]

for \(\beta \in \mathbb{N}_0^d\) and

\[G_n^\beta = \left\{ m \in \mathbb{Z}^d \setminus \{0\} \mid n^{1/d} UBm \in N(\beta) \right\}.\]

The domain \(\mathbb{Z}^d \setminus \{0\}\) of summation is the disjoint union of all \(G_n^\beta\) over \(\beta \in \mathbb{N}_0^d\).

For \(|\beta| \leq \log_2 n\), the points \(x\) in \(N(\beta)\) satisfy \(\prod_{j=1}^d |x_j| < 2^{\beta_j} \leq n\). But the second property of the Frolov matrix \(B\) yields \(\prod_{j=1}^d |n^{1/d}(UBm)_j| \geq n\) for any \(m \in \mathbb{Z}^d \setminus \{0\}\). Hence, \(G_n^\beta\) is empty for \(|\beta| \leq \log_2 n\). For \(|\beta| > \log_2 n\), any \(m \in G_n^\beta\) satisfies

\[h_r(n^{1/d} UBm) \geq \prod_{j=1}^d \left( 1 + |2^{\beta_j} - 1|^{2r} \right) \geq \prod_{j=1}^d 2^{2r(\beta_j - 1)} = 2^{2r(|\beta| - d)}\]
The latter cardinality is bounded by $h_r(n^{1/d}UBm)^{-1} \leq 2^{2r(d-|\beta|)}$. Because of the third property of the Frolov matrix, we obtain that the cardinality of $G_n^d$ is bounded above by

$$\text{card} \left( \left\{ m \in \mathbb{Z}^d \setminus \{0\} \mid |(Bm)_j| < 2^{\beta_+ n^{-1/d}} \right\} \right) \leq 2^{d+|\beta|} n^{-1} + 1 \leq 2^{d+1+|\beta|} n^{-1}.$$  

This shows that the first factor of (2.1) satisfies

$$\sum_{|\beta| > \log_2 n} \sum_{m \in G_n^d} h_r(n^{1/d}UBm)^{-1} \leq \sum_{|\beta| > \log_2 n} 2^{2r(d-|\beta|)} \cdot 2^{d+1+|\beta|} n^{-1} \leq \sum_{k = [\log_2 n]}^{\infty} 2^{2r(d-k)} \cdot 2^{d+1+k} n^{-1} \cdot \text{card} \left( \left\{ \beta \in \mathbb{N}_0^d \mid |\beta| = k \right\} \right).$$

The latter cardinality is bounded by $(k + 1)^{d-1}$. This yields the upper bound

$$2^{2rd+d+1} n^{-1} \sum_{k = [\log_2 n]}^{\infty} 2^{(1-2r)k} (k + 1)^{d-1}$$

$$= 2^{2rd+d+1} n^{-1} \sum_{k = 0}^{\infty} 2^{(1-2r)(k + [\log_2 n])} (k + 1 + [\log_2 n])^{d-1}$$

$$\leq 2^{2rd+d+1} n^{-1} \cdot n^{1-2r} \cdot \sum_{k = 0}^{\infty} 2^{(1-2r)k} \cdot 2^{d-1} \cdot (k + 1)^{d-1} \cdot [\log_2 n]^{d-1}$$

$$\leq 2^{2rd+2d} \cdot n^{-2r} \cdot \sum_{k = 0}^{\infty} 2^{(1-2r)k} (k + 1)^{d-1} \left(2 \cdot \frac{\ln n}{\ln 2}\right)^{d-1}$$

$$= \left(2^{2rd+3d-1} (\ln 2)^{1-d} \sum_{k = 0}^{\infty} (2^{1-2r})^k (k + 1)^{d-1}\right) \cdot n^{-2r} (\ln n)^{d-1},$$

which is the desired estimate since $2^{1-2r} < 1$.

We now show that the second factor in the above inequality is bounded above by a constant multiple of $\|f\|_{H_n^\alpha([0,1]^d)}^2$, where the constant is independent of $v$ and $U$. This will prove the theorem. For $x \in \mathbb{R}^d$ we have

$$h_r(x) \cdot |Ff(x)|^2 = \sum_{|\alpha| \leq r} |\mathcal{F}D^\alpha f(x)|^2.$$  

The function $g_\alpha = D^\alpha f \circ (n^{1/d}UB)^{-1}$ has compact support in $(n^{1/d}UB)^{-1} [0,1]^d$. Consider the set $J_n$ of all $k \in \mathbb{Z}^d$ for which $(k + [0,1]^d)$ has nonempty intersection with $(n^{1/d}UB)^{-1} [0,1]^d$. The transformation $y = (n^{1/d}UB)^{-1} x$ yields

$$|\mathcal{F}D^\alpha f \left(n^{1/d}UBm\right)|^2 = \left| \int_{\mathbb{R}^d} D^\alpha f(y) \cdot e^{-2\pi i (n^{1/d}UBm)\cdot y} dy \right|^2$$

$$= \left| \frac{1}{\det(n^{1/d}UB)} \int_{\mathbb{R}^d} g_\alpha(x) \cdot e^{-2\pi i (m\cdot x)} dx \right|^2$$

$$= \left| \frac{1}{\det(n^{1/d}UB)} \sum_{k \in J_n} \left\langle g_\alpha, e^{2\pi i (m\cdot \cdot)} \right\rangle_{L^2([0,1]^d)}^2 \right|^2$$

$$\leq \frac{\text{card}(J_n)}{|\det(n^{1/d}UB)|^2} \sum_{k \in J_n} \left| \left\langle g_\alpha, e^{2\pi i (m\cdot \cdot)} \right\rangle_{L^2([0,1]^d)}^2 \right|^2.$$  

(2.3)
Thus we obtain

\[
\sum_{m \in \mathbb{Z}^d \setminus \{0\}} h_r(n^{1/d}UBm) \cdot \|\mathcal{F}f(n^{1/d}UBm)\|^2 \leq \sum_{m \in \mathbb{Z}^d} \sum_{\|\alpha\|_{\infty} \leq r} \|\mathcal{F}D^\alpha f(n^{1/d}UBm)\|^2
\]

\[
\leq \frac{\text{card}(J_n)}{|\det(n^{1/d}UB)|^2} \sum_{m \in \mathbb{Z}^d} \sum_{\|\alpha\|_{\infty} \leq r} \sum_{k \in J_n} \left| g_\alpha \cdot e^{2\pi i \langle m, \cdot \rangle} \right|_{L^2(k+[0,1]^d)}^2
\]

\[
= \frac{\text{card}(J_n)}{|\det(n^{1/d}UB)|^2} \sum_{\|\alpha\|_{\infty} \leq r} \sum_{k \in J_n} \|g_\alpha\|_{L^2(k+[0,1]^d)}^2
\]

\[
= \frac{\text{card}(J_n)}{|\det(n^{1/d}UB)|^2} \sum_{\|\alpha\|_{\infty} \leq r} \|g_\alpha\|_{L^2(\mathbb{R}^d)}^2 = \frac{\text{card}(J_n)}{|\det(n^{1/d}UB)|} \sum_{\|\alpha\|_{\infty} \leq r} \|D^\alpha f\|_{L^2(\mathbb{R}^d)}^2.
\]

Since both \text{card}(J_n) and \|\text{det}(n^{1/d}UB)\| are of order \(n\), their ratio is bounded by a constant and the above inequality yields the statement. \qedhere

\[\text{Theorem 2.10 (KN17)}\] Let \(B \in \mathbb{R}^{d \times d}\) be any invertible matrix and \(r > d/2\). There is some \(c_r > 0\) such that, for any \(n \in \mathbb{N}\) and \(f \in \dot{H}^r([0,1]^d),\)

\[
\sup_{U,\nu} |Q_{n^{1/d}UB}(f) - S_d(f)| \leq c_r n^{-r/d} \|f\|_{\dot{H}^r([0,1]^d)},
\]

where the supremum is taken over all diagonal matrices \(U \in \mathbb{R}^{d \times d}\) with diagonal entries in \([1,2^{1/d}]\) and \(\nu \in \mathbb{R}^d\).

\[\text{Proof.}\] Let \(U\) and \(\nu\) be as above. By Lemma 2.6 and Hölder’s inequality,

\[
|Q_{n^{1/d}UB}(f) - S_d(f)|^2 \leq \left( \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \|\mathcal{F} f\left(n^{1/d}UBm\right)\| \right)^2
\]

\[
\leq \left( \sum_{m \in \mathbb{Z}^d \setminus \{0\}} v_r \left(n^{1/d}UBm\right)^{-1} \right) \left( \sum_{m \in \mathbb{Z}^d \setminus \{0\}} v_r \left(n^{1/d}UBm\right) \|\mathcal{F} f\left(n^{1/d}UBm\right)\|^2 \right).
\]

(2.4)

The first factor in this product is bounded by a constant multiple of \(n^{-2r/d}\): since

\[
v_r \left(n^{1/d}UBm\right) \geq \|n^{1/d}UBm\|_{\mathbb{R}^d}^{2r/2} \geq n^{2r/d} \|Bm\|_{\mathbb{R}^d}^{2r/2} \geq n^{2r/d} \|B^{-1}\|_{\mathbb{R}^{d \times d}}^{-2r/2} \|m\|_{\mathbb{R}^d}^{2r},
\]

we have

\[
\sum_{m \in \mathbb{Z}^d \setminus \{0\}} v_r \left(n^{1/d}UBm\right)^{-1} \leq n^{-2r/d} \|B^{-1}\|_{\mathbb{R}^{d \times d}}^{-2r} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \|m\|_{\mathbb{R}^d}^{-2r},
\]

where this last series converges for \(2r > d\).

We show that the second factor in \(\text{(2.4)}\) is bounded above by a constant multiple of \(\|f\|_{\dot{H}^r([0,1]^d)}^2\). This will prove the theorem. For any \(x \in \mathbb{R}^d\) we have

\[
v_r(x) \cdot \|\mathcal{F} f(x)\|^2 = \sum_{\|\alpha\| \leq r} \|\mathcal{F}D^\alpha f(x)\|^2.
\]

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The function \( g_\alpha = D^\alpha f \circ (n^{1/d}UB)^{-\top} \) has compact support in the parallelepiped \((n^{1/d}UB)^{-\top}[0,1]^d\). Again consider the set \( J_n \) of all \( k \in \mathbb{Z}^d \) for which \((k + [0,1]^d)\) has a nonempty intersection with \((n^{1/d}UB)^{-\top}[0,1]^d\). With (2.3), we obtain
\[
\sum_{m \in \mathbb{Z}^d \setminus \{0\}} v_r(n^{1/d}UBm) \cdot |\mathcal{F}f(n^{1/d}UBm)|^2 \leq \sum_{m \in \mathbb{Z}^d} \sum_{|\alpha| \leq m} |\mathcal{F}D^\alpha f(n^{1/d}UBm)|^2 \leq \frac{\text{card}(J_n)}{|\det(n^{1/d}UB)|^2} \sum_{|\alpha| \leq m} \sum_{k \in J_n} \left| \langle g_\alpha, e^{2\pi i (m \cdot k)} \rangle \right|^2_{L^2([0,1]^d)} \leq \frac{\text{card}(J_n)}{|\det(n^{1/d}UB)|^2} \sum_{|\alpha| \leq m} \|g_\alpha\|^2_{L^2([0,1]^d)} = \frac{\text{card}(J_n)}{|\det(n^{1/d}UB)|} \sum_{|\alpha| \leq m} \|D^\alpha f\|^2_{L^2(\mathbb{R}^d)}.
\]
Since both \(\text{card}(J_n)\) and \(|\det(n^{1/d}UB)|\) are of order \(n\), their ratio is bounded by a constant and the above inequality yields the statement.

**Expected Errors**

In expectation, the random dilations improve the order of the error of Frolov’s algorithm by \(1/2\) for both \(\hat{H}_{\text{mix}}([0,1]^d)\) and \(\hat{H}^r([0,1]^d)\). These results are based on the following general error bound for continuous functions with compact support. Recall that \(D_n\) is the set of all \(x \in \mathbb{R}^d\) with \(\prod_{j=1}^d |x_j| \geq n\).

**Theorem 2.11** ([KN17]). Let \(B \in \mathbb{R}^{d \times d}\) be a Frolov matrix and let \(U \in \mathbb{R}^{d \times d}\) be a diagonal matrix whose diagonal entries are independent and uniformly distributed in \([1,2^{1/d}]\). There is a constant \(c > 0\) such that, for every \(n \in \mathbb{N}\), shift parameter \(v \in \mathbb{R}^d\) and \(f \in C_c(\mathbb{R}^d)\),
\[
\mathbb{E} \left| Q^v_{n^{1/d}UB}(f) - \int_{\mathbb{R}^d} f(x) \, dx \right| \leq c n^{-1} \int_{D_n} |\mathcal{F}f(x)| \, dx.
\]

**Proof.** Thanks to Lemma 2.6 and the monotone convergence theorem we have
\[
\mathbb{E} \left| Q^v_{n^{1/d}UB}(f) - \int_{\mathbb{R}^d} f(x) \, dx \right| \leq \mathbb{E} \left( \sum_{m \in \mathbb{Z}^d \setminus \{0\}} |\mathcal{F}f(n^{1/d}UBm)| \right) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} |\mathcal{F}f(n^{1/d}UBm)|.
\]
Since each \(n^{1/d}UBm\) is uniformly distributed in the box \([n^{1/d}BM, (2n)^{1/d}BM]\) of volume \(c_d \prod_{j=1}^d n^{1/d}(BM)_j\) with \(c_d = (2^{1/d} - 1)^d\), this series equals
\[
\frac{1}{c_d} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \int_{[n^{1/d}BM,(2n)^{1/d}BM]} \frac{|\mathcal{F}f(x)|}{\prod_{j=1}^d n^{1/d}(BM)_j} \, dx \leq \frac{1}{c_d} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \int_{[n^{1/d}BM,(2n)^{1/d}BM]} \frac{|\mathcal{F}f(x)|}{\prod_{j=1}^d 2^{-1/d} |x_j|} \, dx.
\]

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Thus, we arrive at

\[ \frac{2}{c_d} \cdot \int_{\mathbb{R}^d} \left| \mathcal{F} f(x) \right| \cdot \left\{ m \in \mathbb{Z}^d \setminus \{0\} \mid x \in [n^{1/d} B m, (2n)^{1/d} B m] \right\} \, dx \]

\[ = \frac{2}{c_d} \cdot \int_{\mathbb{R}^d} \frac{|\mathcal{F} f(x)|}{\prod_{j=1}^d |x_j|} \cdot \left\{ m \in \mathbb{Z}^d \setminus \{0\} \mid B m \in \left[ \frac{x}{(2n)^{1/d} \cdot n^{1/d}} \right] \right\} \, dx. \]

Thanks to the properties of the Frolov matrix, if \( \prod_{j=1}^d |x_j| < n \), the latter set is empty and otherwise contains no more than \( \prod_{j=1}^d |x_j|^\beta + 1 \leq 2n^{-1} \prod_{j=1}^d |x_j| \) points. Thus, we arrive at

\[ \mathbb{E} \left| Q_n^{1/d UB}(f) - \int_{\mathbb{R}^d} f(x) \, dx \right| \leq \frac{4}{c_d} \cdot n^{-1} \int_{D_n} |\mathcal{F} f(x)| \, dx. \]

Additional differentiability properties of the function \( f \in C_c(\mathbb{R}^d) \) result in decay properties of its Fourier transform \( \mathcal{F} f \). This leads to estimates of the integral \( \int_{D_n} |\mathcal{F} f(x)| \, dx \). Hence, the general upper bound for the error of \( Q_n^{1/d UB}(f) \) in Theorem 2.11 adjusts to the differentiability of \( f \). Two such examples are functions from \( H_{\text{mix}}^r([0,1]^d) \) and \( \hat{H}^r([0,1]^d) \).

**Lemma 2.12.** There is some \( c_r > 0 \) such that, for each \( n \geq 2 \) and \( f \in \hat{H}_{\text{mix}}^r([0,1]^d) \),

\[ \int_{D_n} |\mathcal{F} f(x)| \, dx \leq c_r n^{-r+1/2} (\ln n)^{d-1} \| f \|_{H_{\text{mix}}^r([0,1]^d)}. \]

**Proof.** Applying Hölder’s inequality and a linear substitution \( x = n^{1/d} B y \) to the above integral, we get

\[ \left( \int_{D_n} |\mathcal{F} f(x)| \, dx \right)^2 \leq \left( \int_{D_n} h_r(x)^{-1} \, dx \right) \| f \|_{H_{\text{mix}}^r([0,1]^d)}^2 \]

\[ = n |\det B| \left( \int_G h_r(n^{1/d} B y)^{-1} \, dy \right) \| f \|_{H_{\text{mix}}^r([0,1]^d)}^2 \]

with \( G = B^{-1} D_1 \) being the set of all \( y \in \mathbb{R}^d \) with \( \prod_{j=1}^d |(B y)_j| \geq 1 \). Hence, it is sufficient to prove that the integral \( \int_G h_r(n^{1/d} B y)^{-1} \, dy \) is bounded by a constant multiple of \( n^{-2r} \ln n \) for every \( y \in G_n^\beta \). We again consider the auxiliary set

\[ N(\beta) = \left\{ x \in \mathbb{R}^d \mid |2^{\beta_j} - 1| \leq |x_j| < 2^{\beta_j}, 1 \leq j \leq d \right\} \]

for \( \beta \in \mathbb{N}_0^d \) and

\[ G_n^\beta = \left\{ y \in G \mid n^{1/d} B y \in N(\beta) \right\}. \]

Similar to the proof of Theorem 2.9, the domain \( G \) of integration is the disjoint union of \( G_n^\beta \) over \( \beta \in \mathbb{N}_0^d \), where \( G_n^\beta = \emptyset \) if \( |\beta| \leq \log_2 n \), and otherwise the integrand is bounded above by \( 2^{2r(d-|\beta|)} \) for \( y \in G_n^\beta \). On the other hand,

\[ \lambda^d (G_n^\beta) \leq \lambda^d \left( (n^{1/d} B)^{-1} N(\beta) \right) = n^{-1} |\det B|^{-1} \lambda^d (N(\beta)) \]

\[ = n^{-1} |\det B|^{-1} 2^d \prod_{j=1}^d \left( 2^{\beta_j} - |2^{\beta_j} - 1| \right) \leq n^{-1} |\det B|^{-1} 2^{d+|\beta|}. \]

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Like in the proof of Theorem 2.9, we obtain
\[
\int_G h_r(n^{1/d} By)^{-1} \, dy = \sum_{|\beta| > \log_2 n} \int_{G_n^{d}} h_r(n^{1/d} By)^{-1} \, dy
\leq \sum_{|\beta| > \log_2 n} 2^{2r(d-|\beta|) n^{-1}} |\det B|^{-1} 2^{d+|\beta|} |\det B|^{-1} 2^{-1} \sum_{|\beta| > \log_2 n} 2^{2r(d-|\beta|) n^{-1} 2^{d+|\beta|}}
\leq \left(2^{2rd+3d-2}\right) |\det B|^{-1} \left(\sum_{k=0}^{\infty} \left(2^{1-2r}\right)^k (k+1)^{d-1}\right) n^{-2r} (\ln n)^{d-1},
\]
where the constant is finite since $2^{1-2r} < 1$.

Combining Theorem 2.11 and Lemma 2.12 yields the following.

**Theorem 2.13 ([KN17]).** Let $B \in \mathbb{R}^{d \times d}$ be a Frolov matrix and let $U \in \mathbb{R}^{d \times d}$ be a diagonal matrix whose diagonal entries are independent and uniformly distributed in $[1, 2^{1/d}]$. For all $r \in \mathbb{N}$, there is a constant $c_r > 0$ such that, for every $n \geq 2$, shift parameter $v \in \mathbb{R}^d$, and $f \in H_{\text{mix}}^r([0, 1]^d)$,
\[
\mathbb{E} |Q_{n^1/4UB}^*(f) - S_d(f)| \leq c_r n^{-r-1/2} (\ln n)^{\frac{d-1}{2}} \|f\|_{H_{\text{mix}}^r([0, 1]^d)}.
\]

If the integrand is from the space $\hat{H}^r([0, 1]^d)$, the following lemma holds.

**Lemma 2.14.** For $r > d/2$, there is some $c_r > 0$ such that, for all $n \in \mathbb{N}$ and $f \in \hat{H}^r([0, 1]^d)$,
\[
\int_{D_n} |\mathcal{F} f(x)| \, dx \leq c_r n^{-r/d+1/2} \|f\|_{\hat{H}^r([0, 1]^d)}.
\]

**Proof.** Like in Lemma 2.12, we apply Hölder’s inequality and get
\[
\left(\int_{D_n} |\mathcal{F} f(x)| \, dx\right)^2 \leq \left(\int_{D_n} v_r(x)^{-1} \, dx\right) \|f\|_{\hat{H}^r(\mathbb{R}^d)}^2 \leq \left(\int_{D_n} \|x\|^{-2r} \, dx\right) \|f\|_{\hat{H}^r([0, 1]^d)}^2.
\]
Since $\|x\|_2 \geq \|x\|_\infty \geq n^{1/d}$ for $x \in D_n$, the latter integral in the above relation is bounded above by
\[
\int_{\|x\|_2 \geq n^{1/d}} \|x\|^{-2r} \, dx = \int_{n^{1/d}}^{\infty} \int_{S_{d-1}} R^{-2r+d-1} \, d\sigma(y) \, dR = \frac{\sigma(S_{d-1})}{2r-d} n^{-2r/d+1}.
\]
Here, $\sigma$ is the surface measure on $S_{d-1}$. 

In this case, combining Theorem 2.11 and Lemma 2.14 yields the following, where we recall that $\hat{H}^r([0, 1]^d) \subset C_c(\mathbb{R}^d)$ for $r > d/2$.

**Theorem 2.15 ([KN17]).** Let $B \in \mathbb{R}^{d \times d}$ be a Frolov matrix and let $U \in \mathbb{R}^{d \times d}$ be a diagonal matrix whose diagonal entries are independent and uniformly distributed in $[1, 2^{1/d}]$. For all $r \in \mathbb{N}$ with $r > d/2$, there is a constant $c_r > 0$ such that, for every $n \in \mathbb{N}$, shift parameter $v \in \mathbb{R}^d$, and $f \in \hat{H}^r([0, 1]^d)$,
\[
\mathbb{E} |Q_{n^1/4UB}^*(f) - S_d(f)| \leq c_r n^{-r/d-1/2} \|f\|_{H_{\text{mix}}^r([0, 1]^d)}.
\]
We remark that the Frolov properties of the matrix \( B \) are not needed to get this estimate on \( \hat{H}_r^r([0,1]^d) \), although they are essential for the upper bound on \( \hat{H}_\text{mix}^r([0,1]^d) \) from Theorem 2.13. For example, also the identity matrix would do. But if \( B \) is a Frolov matrix, \( Q_{n^{1/4}UB}^\nu \) works universally for \( \hat{H}_\text{mix}^r([0,1]^d) \) and \( \hat{H}_r^r([0,1]^d) \). Furthermore, the Frolov properties of \( B \) prevent large jumps in the number of nodes of \( Q_{n^{1/4}UB}^\nu \) for small changes of the dilation matrix \( U \).

### 2.1.4 Random Shift

Now we also choose the shift parameter \( v \) in \( Q_{n^{1/4}UB}^\nu \) randomly. We choose it uniformly distributed in \([0,1]^d\). Note that the number of function values the algorithm uses for functions with support in \([0,1]^d\) is still of order \( n \). The first advantage of this method is its unbiasedness.

**Proposition 2.16 (KlN17).** Let \( B \in \mathbb{R}^{d \times d} \) be a random matrix which is almost surely invertible. Let \( v \) be uniformly distributed in \([0,1]^d\) and independent of \( B \). For any \( f \in L^1(\mathbb{R}^d) \), the series \( Q_B^\nu(f) \) converges absolutely almost surely and

\[
\mathbb{E}(Q_B^\nu(f)) = \int_{\mathbb{R}^d} f(y) \, dy.
\]

**Proof.** Let us first fix an invertible realization of \( B \). By the monotone convergence theorem, we obtain

\[
\mathbb{E} \left( \sum_{m \in \mathbb{Z}^d} \frac{1}{|\det B|} \left| f \left( B^{-T}(m + v) \right) \right| \right) = \sum_{m \in \mathbb{Z}^d} \mathbb{E} \left( \frac{1}{|\det B|} \left| f \left( B^{-T}(m + v) \right) \right| \right) \\
= \sum_{m \in \mathbb{Z}^d} \frac{1}{|\det B|} \int_{[0,1]^d} \left| f \left( B^{-T}(m + x) \right) \right| \, dx \\
= \sum_{m \in \mathbb{Z}^d} \int_{B^{-T}(m+[0,1]^d)} |f(y)| \, dy = \int_{\mathbb{R}^d} |f(y)| \, dy < \infty.
\]

In particular, the series \( Q_B^\nu(f) \) converges absolutely almost surely and is dominated by an integrable function. Lebesgue’s dominated convergence theorem yields

\[
\mathbb{E}(Q_B^\nu(f)) = \sum_{m \in \mathbb{Z}^d} \frac{1}{|\det B|} \int_{[0,1]^d} f \left( B^{-T}(m + x) \right) \, dx = \int_{\mathbb{R}^d} f(y) \, dy.
\]

Fubini’s theorem implies that the same equalities hold if \( B \) is a random matrix which is independent of \( v \) and almost surely invertible. In particular, \( Q_B^\nu(f) \) still converges absolutely almost surely. \( \square \)

The second advantage of this method is the slight improvement in the order of convergence of the expected error on \( \hat{H}_\text{mix}^r([0,1]^d) \). If only \( U \) is random, the expected error is of order \( n^{-r-1/2}(\ln n)^{d-1} \), see Theorem 2.13. If both \( U \) and \( v \) are random, the expected error is of order \( n^{-r-1/2} \), as proven in [Ull17]. The proof even shows that the quantity

\[
\left( \mathbb{E} \left| Q_{n^{1/4}UB}^\nu(f) - S_d(f) \right|^2 \right)^{1/2}
\]
satisfies this bound. This is a stronger statement, as implied by Hölder’s inequality. We now turn to the proof. Similar to Lemma 2.6, the expected error of the randomized algorithm for integration on \( C_c(\mathbb{R}^d) \) can be expressed in terms of the Fourier transform.

**Lemma 2.17.** Let \( B \in \mathbb{R}^{d \times d} \) be invertible and \( \mathbf{v} \) be uniformly distributed in \([0, 1]^d\). For any \( f \in C_c(\mathbb{R}^d) \), we have

\[
\mathbb{E} \left| Q_B^*(f) - \int_{\mathbb{R}^d} f(x) \, dx \right|^2 = \sum_{\mathbf{m} \in \mathbb{Z}^d \setminus \{0\}} |\mathcal{F}(B\mathbf{m})|^2.
\]

**Proof.** We first recall that

\[
\mathbb{E} Q_B^*(f) = \int_{\mathbb{R}^d} f(x) \, dx = \mathcal{F}(0).
\]

In particular, we obtain

\[
\mathbb{E} \left| Q_B^*(f) - \int_{\mathbb{R}^d} f(x) \, dx \right|^2 = \text{Var} \left( Q_B^*(f) \right) = \mathbb{E} |Q_B^*(f)|^2 - |\mathbb{E} Q_B^*(f)|^2.
\]

The algorithm \( Q_B^*(f) \) considered as a function of \( \mathbf{v} \in [0, 1]^d \) is a finite sum of square-integrable functions and hence square-integrable. Parseval’s identity states

\[
\mathbb{E} |Q_B^*(f)|^2 = \|Q_B^*(f)\|^2_{L^2([0,1]^d)} = \sum_{\mathbf{m} \in \mathbb{Z}^d} \left| \langle Q_B^*(f), e^{2\pi i (\mathbf{m} \cdot \cdot \cdot \cdot)} \rangle_{L^2([0,1]^d)} \right|^2.
\]

For each index \( \mathbf{m} \in \mathbb{Z}^d \) we have the equality

\[
\langle Q_B^*(f), e^{2\pi i (\mathbf{m} \cdot \cdot \cdot \cdot)} \rangle_{L^2([0,1]^d)} = |\det B|^{-1} \sum_{\mathbf{k} \in \mathbb{Z}^d} \int_{[0,1]^d} f \left( B^{-\top} (\mathbf{k} + \mathbf{v}) \right) e^{-2\pi i (\mathbf{m} \cdot \mathbf{v})} \, d\mathbf{v}
\]

\[
= |\det B|^{-1} \int_{\mathbb{R}^d} f \left( B^{-\top} \mathbf{v} \right) e^{-2\pi i (\mathbf{m} \cdot \mathbf{v})} \, d\mathbf{v} = \int_{\mathbb{R}^d} f \left( \mathbf{v} \right) e^{-2\pi i (B\mathbf{m} \cdot \mathbf{v})} \, d\mathbf{v} = \mathcal{F}(B\mathbf{m}).
\]

Putting everything together, we obtain the stated identity. \( \square \)

Now follows an analogue of Theorem 2.11 for expected quadratic errors.

**Theorem 2.18 (Ull17).** Let \( B \) be a Frolov matrix, let \( U \) be a random diagonal matrix whose diagonal entries are independent and uniformly distributed in \([1, 2^{1/d}]\), and let \( \mathbf{v} \) be independent of \( U \) and uniformly distributed in \([0, 1]^d\). There is a constant \( c > 0 \) such that, for every \( n \in \mathbb{N} \) and \( f \in C_c(\mathbb{R}^d) \),

\[
\mathbb{E} \left| Q_{n^{1/d}UB}^*(f) - \int_{\mathbb{R}^d} f(x) \, dx \right|^2 \leq c n^{-1} \| \mathcal{F} f \|_{L^2(D_n)}^2.
\]

**Proof.** By Lemma 2.17 and the monotone convergence theorem, we have

\[
\mathbb{E} \left| Q_{n^{1/d}UB}^*(f) - \int_{\mathbb{R}^d} f(x) \, dx \right|^2 = \sum_{\mathbf{m} \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}_U \left| \mathcal{F} f(n^{1/d}UB\mathbf{m}) \right|^2.
\]
Since each $n^{1/d} U B \mathbf{m}$ is uniformly distributed in the box $[n^{1/d} B \mathbf{m}, (2n)^{1/d} B \mathbf{m}]$ of volume $c_d \prod_{j=1}^{d} n^{1/d} (B \mathbf{m})_j$ with $c_d = (2^{1/d} - 1)^d$, this series equals
\[
\frac{1}{c_d} \sum_{\mathbf{m} \in \mathbb{Z}^d \setminus \{0\}} \int_{[n^{1/d} B \mathbf{m}, (2n)^{1/d} B \mathbf{m}]} \frac{\left| \mathcal{F} f(x) \right|^2}{\prod_{j=1}^{d} n^{1/d} (B \mathbf{m})_j} \, dx
\]
\[
\leq \frac{1}{c_d} \sum_{\mathbf{m} \in \mathbb{Z}^d \setminus \{0\}} \int_{[n^{1/d} B \mathbf{m}, (2n)^{1/d} B \mathbf{m}]} \frac{\left| \mathcal{F} f(x) \right|^2}{\prod_{j=1}^{d} n^{1/d} (B \mathbf{m})_j} \, dx = \frac{2}{c_d} \int_{\mathbb{R}^d} \frac{\left| \mathcal{F} f(x) \right|^2}{\prod_{j=1}^{d} |x_j|} N(x) \, dx,
\]
where
\[
N(x) = \text{card} \left\{ \mathbf{m} \in \mathbb{Z}^d \setminus \{0\} \mid x \in [n^{1/d} B \mathbf{m}, (2n)^{1/d} B \mathbf{m}] \right\}
\]
\[
= \text{card} \left\{ \mathbf{m} \in \mathbb{Z}^d \setminus \{0\} \mid B \mathbf{m} \in \left[ \frac{x}{(2n)^{1/d}}, \frac{x}{n^{1/d}} \right] \right\}.
\]
Thanks to the properties of the Frolov matrix $B$, if $\prod_{j=1}^{d} |x_j| < n$, the latter set is empty and otherwise contains no more than
\[
\prod_{j=1}^{d} \left| \frac{x_j}{n^{1/d}} \right| + 1 \leq 2n^{-1} \prod_{j=1}^{d} |x_j|
\]
points. Thus, we arrive at the upper bound
\[
\frac{4}{c_d} n^{-1} \int_{D_n} \left| \mathcal{F} f(x) \right|^2 \, dx
\]
and the theorem is proven. \[\square\]

Like the upper bound of Theorem \ref{thm:ub_hmix}, the upper bound of Theorem \ref{thm:ub_hmix} adjusts to the smoothness of the function. This leads to the previously mentioned result on the rate of convergence on $H_{\text{mix}}^r([0, 1]^{d})$.

**Theorem 2.19 ([UL17]).** Let $B$ be a Frolov matrix, let $U$ be a random diagonal matrix whose diagonal entries are independent and uniformly distributed in $[1, 2^{1/d}]$, and let $\mathbf{v}$ be independent of $U$ and uniformly distributed in $[0, 1]^{d}$. For every $r \in \mathbb{N}$, there is some $c_r > 0$ such that, for every $n \geq 2$ and $f \in H_{\text{mix}}^r([0, 1]^{d})$,
\[
\left( \mathbb{E} \left| Q^\mathbf{v}_{n^{1/d} U B}(f) - S_d(f) \right|^2 \right)^{1/2} \leq c_r n^{-r-1/2} \| f \|_{H_{\text{mix}}^r([0, 1]^{d})}.
\]

**Proof.** If $c$ is the constant of Theorem \ref{thm:ub_hmix}, we have the upper bound
\[
\mathbb{E} \left| Q^\mathbf{v}_{n^{1/d} U B}(f) - S_d(f) \right|^2 \leq c n^{-1} \| \mathcal{F} f \|_{L^2(D_n)}^2
\]
\[
= c n^{-1} \int_{D_n} h_r(x)^{-1} \left| \mathcal{F} f(x) \right|^2 h_r(x) \, dx
\]
\[
\leq c n^{-1} \left\| h_r^{-1} \right\|_{L^\infty(D_n)} \int_{\mathbb{R}^d} \left| \mathcal{F} f(x) \right|^2 h_r(x) \, dx
\]
for the expected quadratic error. Since $h_r(x) \geq n^{2r}$ for $x \in D_n$, we get the estimate
\[
\mathbb{E} \left| Q^\mathbf{v}_{n^{1/d} U B}(f) - S_d(f) \right|^2 \leq c n^{-2r-1} \| f \|_{H_{\text{mix}}^r([0, 1]^{d})}^2,
\]
which proves the theorem. \[\square\]
The error of the algorithm in Theorem \ref{thm:universal_algorithm} also has the optimal order of convergence for $\mathcal{H}_r([0, 1]^d)$. This can be derived from Theorem \ref{thm:optimal_order} using the same argument as in the proof of Theorem \ref{thm:universal_algorithm}.

**Theorem 2.20 (\cite{KN17}).** Let $B$ be a Frolov matrix, let $U$ be a random diagonal matrix whose diagonal entries are independent and uniformly distributed in $[1, 2^{1/d}]$, and let $v$ be independent of $U$ and uniformly distributed in $[0, 1]^d$. For every $r \in \mathbb{N}$ with $r > d/2$, there is some $c_r > 0$ such that, for every $n \in \mathbb{N}$ and $f \in \mathcal{H}_r([0, 1]^d)$,

$$
\left(\mathbb{E} |Q_{n^1/nUB}^v(f) - S_d(f)|^2\right)^{1/2} \leq c_r n^{-r/d-1/2} \|f\|_{\mathcal{H}_r([0, 1]^d)}.
$$

Note that the corresponding upper bound for the expected absolute error (instead of the expected mean square error) is a direct consequence of either Theorem \ref{thm:finite_d} or Theorem \ref{thm:universal_algorithm}.  

### 2.1.5 Functions Without Boundary Conditions

We can transform the algorithm from Section \ref{sec:boundary_conditions} such that its error satisfies the same upper bounds for every function in $\mathcal{H}_r^\text{mix}([0, 1]^d)$ and $\mathcal{H}_r([0, 1]^d)$, not only for those vanishing at the boundary. This is done by a standard method, which was already used in \cite[pp.359]{Tem03} to transform Frolov’s deterministic algorithm. The transformation is independent of $r$ and preserves the unbiasedness of the algorithm.

To that end, let $\psi : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function that is a diffeomorphism of $(0, 1)$, vanishes on $(-\infty, 0)$, and equals 1 on $(1, \infty)$. An example is given by the following definition for $x \in \mathbb{R}$:

$$
\psi(x) = \frac{\int_{-\infty}^{x} h(t) \, dt}{\int_{-\infty}^{\infty} h(t) \, dt},
$$

like $h$ also $\psi$ is infinitely differentiable and vanishes on $(-\infty, 0)$ and equals 1 on $(1, \infty)$. Since the derivative of $\psi$ is strictly positive on $(0, 1)$, it is strictly increasing and a bijection of $(0, 1)$ and its inverse function is smooth.

Given such a function, the mapping

$$
\Psi : \mathbb{R}^d \to \mathbb{R}^d, \quad \Psi(x) = (\psi(x_1), \ldots, \psi(x_d))
$$

is a diffeomorphism of $(0, 1)^d$ with inverse

$$
\Psi^{-1}(x) = (\psi^{-1}(x_1), \ldots, \psi^{-1}(x_d))^\top.
$$
and Jacobian
\[ |D\Psi(x)| = \prod_{i=1}^{d} \psi'(x_i). \]

If \(A_n\) is any linear and deterministic quadrature formula with nodes \(x^{(j)} \in [0, 1]^d\) and weights \(a_j \in \mathbb{R}\) for \(j = 1, \ldots, n\) we define the transformed quadrature formula \(\tilde{A}_n\) by choosing the new nodes and weights
\[ \tilde{x}^{(j)} = \Psi(x^{(j)}) \quad \text{and} \quad \tilde{a}_j = a_j |D\Psi(x^{(j)})|. \]

Thus, \(\tilde{Q}_B^v\) for \(v \in \mathbb{R}^d\) and invertible \(B \in \mathbb{R}^{d \times d}\) takes the form
\[
\tilde{Q}_B^v(f) = \frac{1}{|\det B|} \sum_{m \in \mathbb{Z}^d} f \left( \Psi \left( B^{-\top}(m + v) \right) \right) |D\Psi \left( B^{-\top}(m + v) \right)|
\]
for any input function \(f : [0, 1]^d \to \mathbb{R}\). Note that the Jacobian is zero for any \(m \in \mathbb{Z}^d\) with \(B^{-\top}(m + v) \notin [0, 1]^d\). We now define the algorithm from Theorem 2.1 in the introduction of this section.

**Algorithm 2.21.** Let \(B \in \mathbb{R}^{d \times d}\) be a Frolov matrix and let \(c = 2 (\|B\|_1 + 1)^d\). For any \(n \in \mathbb{N}\) with \(n \geq c\), we consider the randomized algorithm
\[ A_n = \tilde{Q}_{(n/c)^{1/d}UB}^v, \]
see Algorithm 2.3 where \(U\) is a diagonal matrix whose diagonal entries are independent and uniformly distributed in \([1, 2^{1/d}]\) and \(v\) is independent of \(U\) and uniformly distributed in \([0, 1]^d\).

By Lemma 2.5, the number of function values that this algorithm uses is bounded by \(n\) for any \(f : [0, 1]^d \to \mathbb{R}\). We turn to the proof of Theorem 2.1.

**Proof of Theorem 2.1.** For any \(f \in L^1([0, 1]^d)\), we define \(f_0 = (f \circ \Psi)|D\Psi|\). By the change of variables theorem, this function is integrable on \([0, 1]^d\). It satisfies
\[ S_d(f) = S_d(f_0) \quad \text{and} \quad \tilde{Q}_{(n/c)^{1/d}UB}^v(f) = Q_{(n/c)^{1/d}UB}^v(f_0). \]

**Part 1.** Proposition 2.16 yields, for any \(f \in L^1([0, 1]^d)\),
\[ \mathbb{E} \left( \tilde{Q}_{(n/c)^{1/d}UB}^v(f) \right) = \mathbb{E} \left( Q_{(n/c)^{1/d}UB}^v(f_0) \right) = S_d(f_0) = S_d(f). \]

**Part 2.** Since \(\psi'\) vanishes outside \((0, 1)\), all derivatives of \(f_0\) vanish at the boundary of \([0, 1]^d\). This implies \(f_0 \in H_{\text{mix}}^r([0, 1]^d)\) for all \(f \in H_{\text{mix}}^r([0, 1]^d)\). By Theorem 2.19
\[
\mathbb{E} \left| \tilde{Q}_{(n/c)^{1/d}UB}^v(f) - S_d(f) \right|^2 = \mathbb{E} \left| Q_{(n/c)^{1/d}UB}^v(f_0) - S_d(f_0) \right|^2 \leq c_r^2 n^{-2r-1} \|f_0\|_{H_{\text{mix}}^r([0, 1]^d)}^2
\]
and Theorem 2.9 yields that
\[
\sup_{U, v} \left| \tilde{Q}_{(n/c)^{1/d}UB}^v(f) - S_d(f) \right| = \sup_{U, v} \left| Q_{(n/c)^{1/d}UB}^v(f_0) - S_d(f_0) \right| \leq c_r n^{-r} (\ln n)^{d-1} \|f_0\|_{H_{\text{mix}}^r([0, 1]^d)}.
\]

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if $c_r > 0$ is the maximum of the constants of these theorems. It remains to show that there is some $C_r > 0$ such that every $f \in H^r_{\text{mix}}([0, 1]^d)$ satisfies

$$\|f_0\|_{H^r_{\text{mix}}([0, 1]^d)} \leq C_r \|f\|_{H^r_{\text{mix}}([0, 1]^d)}.$$  

This is proven as follows. The partial derivatives of $f_0$ take the form

$$D^\alpha f_0(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f(\Psi(x)) \prod_{i=1}^d \psi'(x_i) = \sum_{\substack{\alpha_1, \ldots, \alpha_d \\ \beta_1, \ldots, \beta_d = 0}}^{\alpha_1, \ldots, \alpha_d} D^\beta f(\Psi(x)) R_{\alpha, \beta}(x)$$

for $\alpha \in \{0, 1, \ldots, r\}^d$, where $R_{\alpha, \beta}(x)$ is a finite sum of finite products of terms $\psi^{(j)}(x_i)$ with $i \in \{1, \ldots, d\}$, $j \in \{1, \ldots, rd + 1\}$ and does not depend on $f$. It is therefore continuous and bounded by some $c_{\alpha, \beta} > 0$. We get

$$\|D^\alpha f_0\|^2_{L^2([0, 1]^d)} \leq \left( \sum_{\substack{\alpha_1, \ldots, \alpha_d \\ \beta_1, \ldots, \beta_d = 0}}^{\alpha_1, \ldots, \alpha_d} \left\|D^\beta f \circ \Psi \right\|_{L^2([0, 1]^d)} \right)^2 \leq \left( \sum_{\substack{\alpha_1, \ldots, \alpha_d \\ \beta_1, \ldots, \beta_d = 0}}^{\alpha_1, \ldots, \alpha_d} c_{\alpha, \beta} \left\|D^\beta f \circ \Psi\right\|_{L^2([0, 1]^d)} \right)^2.$$  

We proceed with Hölder’s inequality and the change of variables theorem for the diffeomorphism $\Psi^{-1}$ of $(0, 1)^d$ and obtain the upper bound

$$(r + 1)^d c_{\alpha, \beta}^2 \left\|D^\beta f \circ \Psi\right\|_{L^2([0, 1]^d)}^2 \leq (r + 1)^d \sum_{\substack{\alpha_1, \ldots, \alpha_d \\ \beta_1, \ldots, \beta_d = 0}}^{\alpha_1, \ldots, \alpha_d} c_{\alpha, \beta}^2 \int_{(0,1)^d} |D^\beta f(x)|^2 |D\Psi^{-1}(x)| \, d\mathbf{x} \leq (r + 1)^d \sup_{x \in (0,1)^d} |D\Psi^{-1}(x)| \sum_{\substack{\alpha_1, \ldots, \alpha_d \\ \beta_1, \ldots, \beta_d = 0}}^{\alpha_1, \ldots, \alpha_d} c_{\alpha, \beta}^2 \left\|D^\beta f\right\|^2_{L^2([0, 1]^d)}.$$  

Summing over all $\alpha \in \{0, \ldots, r\}^d$ yields the desired estimate.

Part 3. This is proven in the exact same manner, where we use Theorem 2.20 and Theorem 2.10 instead of Theorem 2.19 and Theorem 2.9.

We finish this section by showing how to arrive at Corollary 2.2. Obviously, Theorem 2.1 implies that

$$e(n, \mathcal{P}_d^r) \leq n^{-r-1/2}.$$  

On the other hand, it is proven in [Bak59] that

$$e(n, \mathcal{P}_1^r) \geq n^{-r-1/2}.$$  

Moreover, we note that the function $f_d : [0, 1]^d \to \mathbb{R}$ with $f_d(x) = f_1(x_1)$ is contained in $F_d^r$ for any $f_1 \in F_1^r$ and has the same integral. If $A^d$ is a randomized algorithm on $F_d^r$, we can define a randomized algorithm $A^1$ on $F_1^r$ via $A^1(f_1) = A^d(f_d)$. The cost and error of $A^1$ are bounded above by the cost and error of $A^d$. This yields the relation

$$e(n, \mathcal{P}_1^r) \leq e(n, \mathcal{P}_d^r),$$  

which proves the corollary.
2.2 Tensor Product Problems

Let $H$ and $G$ be Hilbert spaces and let $S : H \to G$ be a compact linear operator. Let $F$ be the unit ball of $H$. The problem

$$\mathcal{P} = \mathcal{P}[S, F, G, \Lambda^{all}, \text{det}, \text{wc}]$$

of approximating $S$ with deterministic algorithms based on $\Lambda^{all}$ in the worst case setting is linear and was discussed in Section 1.2.5. We study the corresponding problem

$$\mathcal{P}_d = \mathcal{P}[S_d, F_d, G_d, \Lambda^{all}, \text{det}, \text{wc}],$$

of approximating the $d$th tensor product operator $S_d$. This problem is linear as well. See Section 2.2.1 for a more detailed description of the problem.

The difficulty of the $d$-dimensional problem $\mathcal{P}_d$ is completely determined by the difficulty of the 1-dimensional problem $\mathcal{P}$. In Section 2.2.2 we study the asymptotic behavior of the $n$th minimal error $e(n, \mathcal{P}_d)$ for $n \to \infty$ based on the respective behavior of $e(n, \mathcal{P})$. In Section 2.2.3 we do likewise for the preasymptotic behavior of the minimal error. Section 2.2.4 contains several examples. The preasymptotic estimates also lead to a tractability result in Section 2.2.5.

Remark 2.22. It follows from Theorem 1.17 that the $n$th minimal error satisfies

$$e(n, \mathcal{P}_d) = \inf \{ \| S_d - A_n \| \mid A_n : H_d \to G_d \text{ linear, rank}(A_n) \leq n \}.$$ 

This means that the error coincides with the $(n + 1)$st approximation number and all other $s$-numbers of the solution operator, see also [Pie78, Section 11.3].

2.2.1 The Setting

Let $H$ and $G$ be Hilbert spaces and let $S : H \to G$ be a compact linear operator. Let $F$ be the unit ball of $H$. We consider the problem

$$\mathcal{P} = \mathcal{P}[S, F, G, \Lambda^{all}, \text{det}, \text{wc}]$$

of approximating $S$ with deterministic algorithms based on $\Lambda^{all}$ in the worst case setting. From Section 1.2.5 we know that linear algorithms are optimal for this problem and that optimal linear algorithms are given by the singular value decomposition of $S$ in the following way, see also [NW08, Section 5.2].

Since $W = S^*S \in \mathcal{L}(H)$ is positive and compact, it admits a finite or countable orthonormal basis $B$ of $\ker(S)^\perp$ consisting of eigenvectors $b \in B$ to eigenvalues

$$\lambda(b) = \langle Wb, b \rangle_H = \|Sb\|^2_G > 0.$$ 

We will refer to $B$ as the orthonormal basis associated with $S$. It can be characterized as the orthonormal basis of $\ker(S)^\perp$ whose image is an orthogonal basis of $\overline{S(H)}$. It is unique up to the choice of orthonormal bases in the finite-dimensional eigenspaces of $W$. We have

$$Sf = \sum_{b \in B} \langle f, b \rangle_H Sb$$
2.2. Tensor Product Problems

for all $f \in H$. This representation is called the singular value decomposition or Schmidt decomposition of $S$. The square-roots of the eigenvalues of $W$ are called singular values of $S$. Let $\sigma_n$ be the $n^{th}$ largest singular value of $S$ for all $n \leq |\mathcal{B}|$. For $n > |\mathcal{B}|$, let $\sigma_n = 0$. We consider the linear algorithm $A_n : F \to G$, $A_n(f) = \sum_{b \in \mathcal{B}(n)} \langle f, b \rangle_H Sb$,

where $\mathcal{B}(n)$ consists of all $b \in \mathcal{B}$ that satisfy $\|Sb\|_G > \sigma_{n+1}$. We know that $A_n$ is optimal among all algorithms with cost $n$ or less, see Theorem 1.17. It satisfies $\text{err}(A_n) = e(n, \mathcal{P}) = \sigma_{n+1}$.

Moreover, we can easily verify the relation

$$\sigma_{n+1} = \min_{V \subset H} \max_{\dim(V) \leq n} \|Sf\|_G,$$

where equality is obtained for $V = \text{span}(\mathcal{B}(n))$ and $f = b_{n+1}$.

We are concerned with tensor product problems, defined as follows. Let $D$ be a set and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $D_d$ be the $d$-fold Cartesian product of $D$. The tensor product of $\mathbb{K}$-valued functions $f_1, \ldots, f_d$ on $D$ is the function $f_1 \otimes \ldots \otimes f_d : D_d \to \mathbb{K}, \ x \mapsto f_1(x_1) \cdot \ldots \cdot f_d(x_d)$.

If $H$ is a Hilbert space of $\mathbb{K}$-valued functions on $D$, its $d^{th}$ tensor product $H_d$ is the smallest Hilbert space of $\mathbb{K}$-valued functions on $D_d$ that contains any tensor product of functions in $H$ and satisfies

$$\langle f_1 \otimes \ldots \otimes f_d, g_1 \otimes \ldots \otimes g_d \rangle = \langle f_1, g_1 \rangle \cdot \ldots \cdot \langle f_d, g_d \rangle$$

for any choice of functions $f_1, \ldots, f_d$ and $g_1, \ldots, g_d$ in $H$. Let $G$ be another Hilbert space of $\mathbb{K}$-valued functions with tensor product $G_d$ and let $S \in \mathcal{L}(H, G)$. The $d^{th}$ tensor product of $S$ is the unique operator $S_d \in \mathcal{L}(H_d, G_d)$ that satisfies

$$S_d(f_1 \otimes \ldots \otimes f_d) = Sf_1 \otimes \ldots \otimes Sf_d$$

for any choice of functions $f_1, \ldots, f_d$ in $H$. If $S$ is compact, then so is $S_d$. Finally, the $d^{th}$ tensor product problem is the problem

$$\mathcal{P}_d = \mathcal{P}[S_d, F_d, G_d, \Lambda^{\text{all}}, \det, \text{wc}],$$

where $F_d$ is the unit ball of $H_d$.

Just like for the 1-dimensional problem $\mathcal{P}$, optimal algorithms for $\mathcal{P}_d$ are linear and given by the singular value decomposition of $S_d$. Based on the singular value decomposition of $S$, we easily obtain the singular value decomposition of $S_d$. If $\mathcal{B}$ is the orthonormal basis associated with $S$, then

$$\mathcal{B}_d = \{b_1 \otimes \ldots \otimes b_d \mid b_1, \ldots, b_d \in \mathcal{B}\}$$
is the orthonormal basis associated with the tensor product $S_d$. In particular, the family of singular values of $S_d$ is given by

$$\sigma(n) = \sigma_{n_1} \cdot \ldots \cdot \sigma_{n_d} \quad \text{for} \quad n \in \mathbb{N}^d.$$  

Recall that $e(n, P_d)$ coincides with the $(n + 1)^{st}$ largest singular value of $S_d$. In particular,

$$n(\varepsilon, P_d) = \text{card} \left\{ n \in \mathbb{N}^d \mid \sigma(n) > \varepsilon \right\}$$

for all $\varepsilon \geq 0$. The question for the difficulty of the tensor product problem is thus of combinatorial nature.

### 2.2.2 Asymptotic Behavior

A classical result of Babenko [Bab60] and Mityagin [Mit62] is concerned with the speed of decay of the $n^{th}$ minimal error:

**Theorem 2.23** ([Bab60 Mit62]). Let $P_d$ be a tensor product problem as defined in Section 2.2.1. For any $r > 0$ the following holds:

(i) If $e(n, P) \leq n^{-r}$, then $e(n, P_d) \leq n^{-r} (\ln n)^{r(d-1)}$.

(ii) If $e(n, P) \geq n^{-r}$, then $e(n, P_d) \geq n^{-r} (\ln n)^{r(d-1)}$.

Of course, other decay assumptions on $e(n, P)$ may be of interest. For instance, Pietsch [Pie82] and König [Koe84] study the decay of $e(n, P_d)$ if $e(n, P)$ lies in the Lorentz sequence space $\ell_{p,q}$ for positive indices $p$ and $q$, which is a stronger assumption than (i) for $r = 1/p$, but weaker than (i) for any $r > 1/p$. However, we are motivated by the example of Sobolev embeddings, see Section 2.2.4. We will hence stick to the assumptions of Theorem 2.23. However, this theorem does not provide explicit estimates for $e(n, P_d)$, even if $n$ is huge. This is because of the constants hidden in the notation. But Theorem 2.23 can be sharpened.

**Theorem 2.24** ([KSU15 Kri18a]). Let $P_d$ be a tensor product problem as defined in Section 2.2.1. For any $c > 0$ and $r > 0$, the following holds:

(i) If $e(n, P) \leq c n^{-r}$, then $e(n, P_d) \leq \frac{c}{(d-1)!} n^{-r} (\ln n)^{r(d-1)}$.

(ii) If $e(n, P) \geq c n^{-r}$, then $e(n, P_d) \geq \frac{c}{(d-1)!} n^{-r} (\ln n)^{r(d-1)}$.

In particular, we obtain that asymptotic equality $e(n, P) \sim c n^{-r}$ implies asymptotic equality $e(n, P_d) \sim \frac{c}{(d-1)!} n^{-r} (\ln n)^{r(d-1)}$ for the tensor product problem. Theorem 2.24 is due to Theorem 4.3 in [KSU15]. There, Kühn, Sickel, and Ullrich prove this asymptotic equality in the special case where $S_d$ is the embedding of the mixed order Sobolev space $H_{\text{mix}}^r(T_d)$ on the $d$-torus $T_d = [0, 2\pi]^d$. The general statement can be deduced from this special case with the help of their Lemma 4.14. However, we prefer to give a direct proof by generalizing the proof of Theorem 4.3 in [KSU15].
For the proof, it will be essential to study the asymptotics of the cardinalities

\[ K_N(R, l) = \text{card} \left\{ n \in \{N, N + 1, \ldots\}^l \mid \prod_{j=1}^l n_j \leq R \right\} \quad (2.6) \]

for \( l \in \{1, \ldots, d\} \) and \( N \in \mathbb{N} \) as \( R \to \infty \). In [KSU15, Lemma 3.2] it is shown that

\[ R \left( \frac{(\ln \frac{R}{l})^{l-1}}{(l-1)!} - \frac{(\ln \frac{R}{l})^{l-2}}{(l-2)!} \right) \leq K_2(R, l) \leq R \frac{(\ln R)^{l-1}}{(l-1)!} \quad (2.7) \]

for \( l \geq 2 \) and \( R \in \{4^l, 4^l + 1, \ldots\} \), see also [CD16, Theorem 3.4]. Consequently we have

\[ \lim_{R \to \infty} \frac{K_N(R, l)}{R(\ln R)^{l-1}} = \frac{1}{(l-1)!} \quad (2.8) \]

for \( N = 2 \). In fact, (2.8) holds true for any \( N \in \mathbb{N} \). This can be derived from the case \( N = 2 \), but for the reader’s convenience we give a complete proof.

**Lemma 2.25.**

\[ \lim_{R \to \infty} \frac{K_N(R, l)}{R(\ln R)^{l-1}} = \frac{1}{(l-1)!} \]

**Proof.** Note that for all values of the parameters,

\[ K_N(R, l + 1) = \sum_{k=N}^{\infty} K_N \left( \frac{R}{k}, l \right) , \]

where \( K_N \left( \frac{R}{k}, l \right) = 0 \) for \( k > \frac{R}{N} \). This allows a proof by induction on \( l \in \mathbb{N} \). Like in estimate (2.7), we first show that

\[ K_2(R, l) \leq R \frac{(\ln R)^{l-1}}{(l-1)!} \quad (2.9) \]

for any \( l \in \mathbb{N} \) and \( R \geq 1 \). This is obviously true for \( l = 1 \). On the other hand, if this relation holds for some \( l \in \mathbb{N} \) and if \( R \geq 1 \), then

\[
K_2(R, l + 1) = \sum_{k=2}^{\lfloor R/k \rfloor} K_2 \left( \frac{R}{k}, l \right) & \leq \sum_{k=2}^{\lfloor R/k \rfloor} R \frac{(\ln R)^{l-1}}{k(l-1)!} \\
& \leq \frac{R}{(l-1)!} \int_1^R \frac{(\ln x)^{l-1}}{x} \, dx = \frac{R}{(l-1)!} \left[ -\frac{1}{l} \left( \frac{R}{x} \right)^l \right]_1^R = \frac{R(\ln R)^{l-1}}{l!} 
\]

and (2.9) is proven. In particular, we have

\[ \limsup_{R \to \infty} \frac{K_N(R, l)}{R(\ln R)^{l-1}} \leq \frac{1}{(l-1)!} \quad (2.10) \]
for \( l \in \mathbb{N} \) and \( N = 2 \). Clearly, the same holds for \( N \geq 2 \), since \( K_N(R, l) \) is decreasing in \( N \). Relation (2.10) for \( N = 1 \) follows from the case \( N = 2 \) by the identity

\[
K_1(R, l) = \sum_{m=0}^{l} \text{card} \left\{ n \in \mathbb{N}^l \mid \text{card} \{1 \leq j \leq l \mid n_j \neq 1\} = m \wedge \prod_{j=1}^{l} n_j \leq R \right\}
= 1_{R \geq 1} + \sum_{m=1}^{l} \binom{l}{m} \cdot K_2(R, m).
\]

It remains to prove

\[
\liminf_{R \to \infty} \frac{K_N(R, l)}{R (\ln R)^{l-1}} \geq \frac{1}{(l-1)!}
\]  

(2.11)

for \( N \in \mathbb{N} \) and \( l \in \mathbb{N} \). Again, this is obvious for \( l = 1 \). Suppose, (2.11) holds for some \( l \in \mathbb{N} \) and let \( b < 1 \). Then there is some \( R_0 \geq 1 \) such that

\[
K_N(R, l) \geq b R \frac{(\ln R)^{l-1}}{(l-1)!}
\]

for all \( R \geq R_0 \) and hence

\[
K_N(R, l + 1) \geq \sum_{k=N}^{|R/R_0|} K_N \left( \frac{R}{k}, l \right) \geq \sum_{k=N}^{|R/R_0|} b R \frac{(\ln \frac{R}{k})^{l-1}}{k (l-1)!}
\]

\[
\geq b R \frac{(\ln \frac{R}{R_0})^{l-1}}{(l-1)!} \int_{N}^{R/R_0} \frac{(\ln \frac{x}{R_0})^{l-1}}{x} \, dx = b R \frac{(\ln \frac{R}{R_0})^{l}}{l!} \left( (\ln \frac{R}{N})^{l} - (\ln R_0)^{l} \right) \geq b^2 R \frac{(\ln R)^{l}}{l!}
\]

for large \( R \). Since this is true for any \( b < 1 \), the induction step is complete.

We turn to the proof of Theorem 2.24.

**Proof of Theorem 2.24.** We first realize that changing the singular numbers \( \sigma_n \) by a multiplicative constant \( c \) for all \( n \in \mathbb{N} \) changes \( e(n, \mathcal{P}_d) \) by the the factor \( c^d \). Moreover, raising the singular numbers to some fixed power changes \( e(n, \mathcal{P}_d) \) by the same power. We can hence assume without loss of generality that \( \sigma_1 = 1 \) and \( r = 1 \).

Proof of (i): Let \( c_3 > c_2 > c_1 > c \). Since \( \sigma_n \preceq c n^{-r} \), there is some \( N \in \mathbb{N} \) such that for any \( n \geq N \) we have

\[
\sigma_n \leq c_1 n^{-1}.
\]

(2.12)

We want to prove

\[
\limsup_{n \to \infty} \frac{e(n, \mathcal{P}_d) n}{(\ln n)^{d-1}} \leq \frac{c^d}{(d-1)!}.
\]

(2.13)

Since \( n/(\ln n)^{d-1} \) is eventually increasing, instead of giving an upper bound for \( e(n, \mathcal{P}_d) \) in terms of \( n \), we can just as well give an upper bound for \( n \) in terms of \( e(n, \mathcal{P}_d) \) to obtain (2.13). Clearly, there are at least \( n + 1 \) singular values of \( S_d \) greater.
We want to prove that the fraction in brackets tends to one as $n \to \infty$ for large values of $n$. For every $n$, let $\frac{d}{d-1} n_j / \ln n_j$ be greater than $d$ and hence $\frac{d}{d-1} n_j / \ln n_j \geq 1$. Since this is true for any $n_j$, we have

$$\lim_{n \to \infty} \frac{e(n, P_d)n}{(\ln n)^{d-1}} \leq \frac{c_3^d}{(d-1)!} \cdot \frac{\ln \left( \frac{c_3^d e(n, P_d)^{-1}}{(d-1)!} \right)}{\ln \left( \frac{c_3^d e(n, P_d)^{-1}}{(d-1)!} \right) \cdot \ln \left( \frac{c_3^d e(n, P_d)^{-1}}{(d-1)!} \right)}. $$

The fraction in brackets tends to one as $n$ and hence $e(n, P_d)^{-1}$ tends to infinity and thus

$$\lim_{n \to \infty} \frac{e(n, P_d)n}{(\ln n)^{d-1}} \leq \frac{c_3^d}{(d-1)!}. $$

Since this is true for any $c_3 > c$, the proof of (2.13) is complete.

Proof of (ii): Let $0 < c_3 < c_2 < c_1 < c$. Since $\sigma_n \geq c n^{-\tau}$, there is some $N \in N$ such that for $n \geq N$, we have

$$\sigma_n \geq c_1 n^{-1}. $$

We want to prove

$$\liminf_{n \to \infty} \frac{e(n, P_d)n}{(\ln n)^{d-1}} \geq \frac{c^d}{(d-1)!} $$

for any $d \in N$. Clearly, there are at most $n$ singular values of $S_d$ greater than $e(n, P_d)$ and hence

$$n \geq \card \left\{ n \in \mathbb{N}^d \mid \sigma(n) > e(n, P_d) \right\}$$

$$\geq \card \left\{ n \in \{N, N+1, \ldots\}^d \mid \sigma(n) > e(n, P_d) \right\}. $$
Relation (2.14) implies that every \( n \in \{ N, N + 1, \ldots \}^d \) with \( \prod_{j=1}^d n_j < c_1^d e(n, \mathcal{P}_d)^{-1} \) is contained in the last set. This observation and Lemma 2.25 yield that

\[
n \geq K_N \left( c_2^d e(n, \mathcal{P}_d)^{-1}, d \right) \geq \frac{c_3^d e(n, \mathcal{P}_d)^{-1}}{(d-1)!} \left( \ln \left( c_3^d e(n, \mathcal{P}_d)^{-1} \right) \right)^{d-1}
\]

for sufficiently large \( n \). By the monotonicity of \( n / (\ln n)^{d-1} \) for large \( n \) we obtain

\[
\frac{e(n, \mathcal{P}_d)}{(\ln n)^{d-1}} \geq \frac{c_3^d}{(d-1)!} \cdot \left( \frac{\ln \left( c_3^d e(n, \mathcal{P}_d)^{-1} \right)}{\ln \left( \ln \left( c_3^d e(n, \mathcal{P}_d)^{-1} \right) \right)^{d-1}} \right)^{d-1}.
\]

The fraction in brackets tends to 1 as \( n \) and hence \( e(n, \mathcal{P}_d)^{-1} \) tends to infinity and thus

\[
\lim_{n \to \infty} \inf \frac{e(n, \mathcal{P}_d)}{(\ln n)^{d-1}} \geq \frac{c_3^d}{(d-1)!}.
\]

Since this is true for any \( c_3 < c \), the proof of (2.15) is complete.

We give an interpretation of Theorem 2.24. For \( r > 0 \) let us consider the quantities

\[
C_1 = \limsup_{n \to \infty} e(n, \mathcal{P}) n^r, \quad c_1 = \liminf_{n \to \infty} e(n, \mathcal{P}) n^r,
\]

\[
C_d = \limsup_{n \to \infty} \frac{e(n, \mathcal{P}_d) \cdot n^r}{(\ln n)^{r(d-1)}}, \quad c_d = \liminf_{n \to \infty} \frac{e(n, \mathcal{P}_d) \cdot n^r}{(\ln n)^{r(d-1)}}.
\]

These limits may be both infinite or zero. They can be interpreted as asymptotic or optimal constants for the bounds

\[
e(n, \mathcal{P}_d) \leq C \cdot n^{-r} (\ln n)^{r(d-1)} \quad \text{and} \quad (2.16)
\]

\[
e(n, \mathcal{P}_d) \geq c \cdot n^{-r} (\ln n)^{r(d-1)}. \quad (2.17)
\]

For any \( C > C_d \) respectively \( c < c_d \) there is a threshold \( n_0 \in \mathbb{N} \) such that (2.16) respectively (2.17) holds for all \( n \geq n_0 \), whereas for any \( C < C_d \) respectively \( c > c_d \) there is no such threshold. Note that our proof provides a possibility to track down admissible thresholds \( n_0 \) for any \( C > \frac{C_d}{(d-1)!} \) respectively any \( c < \frac{c_d}{(d-1)!} \). Theorem 2.23 states that \( C_d \) is finite, whenever \( C_1 \) is finite, whereas \( c_d \) is positive, whenever \( c_1 \) is positive. Theorem 2.24 is more precise. It states that

\[
\frac{c_3^d}{(d-1)!} \leq c_d \leq C_d \leq \frac{C_1^d}{(d-1)!}. \quad (2.18)
\]

Obviously, there must be equality in all the relations of (2.18), if the limit of the sequence \( e(n, \mathcal{P}) n^r \) for \( n \to \infty \) exists, that is, if \( C_1 = c_1 \). It is natural to ask, whether any of these equalities always holds true. The answer is no, as shown by the following example.
Example 2.26. Consider a solution operator $S$ with singular values $\sigma_n = 2^{-k}$ for $n \in \{2^k, \ldots, 2^{k+1} - 1\}$ and $k \in \mathbb{N}_0$. That is, the singular values decay linearly in $n$, but are constant on segments of length $2^k$. They satisfy

$$C_1 = \limsup_{n \to \infty} \frac{\sigma_{n+1} n}{2^{-k} \cdot \left(2^{k+1} - 2\right)} = 2$$

and

$$c_1 = \liminf_{n \to \infty} \frac{\sigma_{n+1} n}{2^{-k} \cdot \left(2^k - 1\right)} = 1.$$

Also the singular values $\sigma(n)$ of the tensor product operator $S_d$ are of the form $2^{-k}$ for some $k \in \mathbb{N}_0$, where

$$\text{card}\left\{ n \in \mathbb{N}^d \mid \sigma(n) = 2^{-k} \right\} = \sum_{|k|=k} \text{card}\left\{ n \in \mathbb{N}^d \mid \sigma_{n_j} = 2^{-k_j} \text{ for } j = 1 \ldots d \right\}$$

$$= \sum_{|k|=k} 2^k = 2^k \cdot \binom{k + d - 1}{d - 1} = \frac{2^k}{(d-1)!} \cdot (k + 1) \cdot \ldots \cdot (k + d - 1).$$

Hence, $e(n, P_d) = 2^{-k}$ for $N(k - 1, d) \leq n < N(k, d)$ with $N(-1, d) = 0$ and

$$N(k, d) = \sum_{j=0}^{k} \frac{2^j}{(d-1)!} \cdot (j + 1) \cdot \ldots \cdot (j + d - 1)$$

for $k \in \mathbb{N}_0$. The monotonicity of $n / (\ln n)^{d-1}$ for large $n$ implies

$$C_d = \limsup_{n \to \infty} \frac{e(n, P_d) \cdot n}{(\ln n)^{d-1}} = \lim_{k \to \infty} \frac{2^{-k} \cdot N(k, d)}{(\ln N(k, d))^{d-1}}$$

and

$$c_d = \liminf_{n \to \infty} \frac{e(n, P_d) \cdot n}{(\ln n)^{d-1}} = \lim_{k \to \infty} \frac{2^{-k} \cdot N(k - 1, d)}{(\ln N(k - 1, d))^{d-1}}. \tag{2.19}$$

We insert the relations

$$N(k, d) \leq \frac{(k + d)^{d-1}}{(d-1)!} \sum_{j=0}^{k} 2^j \leq \frac{2^{k+1} \cdot (k + d)^{d-1}}{(d-1)!}$$

and

$$N(k, d) \geq \frac{(k - l)^{d-1}}{(d-1)!} \sum_{j=k-l+1}^{k} 2^j = \frac{2^{k+1} (k - l)^{d-1}}{(d-1)!} \left(1 - 2^{-l}\right)$$

for arbitrary $l \in \mathbb{N}$ in (2.19) and (2.20) to obtain

$$C_d = 2 \cdot \frac{(\log_2 e)^{d-1}}{(d-1)!} \quad \text{and} \quad c_d = \frac{\left(\log_2 e\right)^{d-1}}{(d-1)!}.$$ 

In particular,

$$\frac{c_1^d}{(d-1)!} < c_d < C_d < \frac{C_1^d}{(d-1)!} \quad \text{for } d \neq 1.$$
More generally, one can define the tensor product $S_d$ of $d$ different compact operators $S^{(j)}$ between Hilbert spaces. If the singular numbers of $S^{(j)}$ are given by $\sigma_n^{(j)}$ for $n \in \mathbb{N}$, then the singular numbers of $S_d$ are given by

$$\sigma(n) = \prod_{j=1}^{d} \sigma_n^{(j)} \text{ for } n \in \mathbb{N}^d.$$ 

An example for $S_d$ is given by the $L^2$-embedding of Sobolev functions on the $d$-torus with mixed smoothness $(r_1, \ldots, r_d)$, where $r_j > 0$ for all $j \leq n$. It is the tensor product of the $L^2$-embeddings of the univariate Sobolev spaces with smoothness $r_j$, whose singular numbers are of order $n^{-r_j}$. It is known that $e(n, \mathcal{P}_d)$ has the order $n^{-r} (\ln n)^{r(l-1)}$ in this case where $r$ is the minimum among all numbers $r_j$ and $l$ is its multiplicity. This was proven by Mityagin [Mit62] for integer vectors $(r_1, \ldots, r_d)$ and by Nikol’skaya [Nik74] in the general case. See [Tem86, pp.32, 36, 72] and [DTU18] for more details.

**Example 2.27.** Consider solution operators $S$, $T$ and $\tilde{T}$ with singular numbers

$$\sigma_n = n^{-1}, \quad \mu(n) = n^{-2}, \quad \tilde{\mu}(n) = \begin{cases} 1, & \text{for } n \leq N, \\ n^{-2}, & \text{for } n > N, \end{cases}$$

for some $N \in \mathbb{N}$ and all $n \in \mathbb{N}$. The tensor product $S_2$ of $S$ and $T$ has the singular values

$$\sigma(n) = n_1^{-1} n_2^{-2} \text{ for } n \in \mathbb{N}^2$$

which yields for the respective problem $\mathcal{P}_2$ and all $n \in \mathbb{N}$ that

$$n \leq \text{card} \left\{ n \in \mathbb{N}^2 \mid \sigma(n) \geq e(n, \mathcal{P}_2) \right\} = \text{card} \left\{ n \in \mathbb{N}^2 \mid n_1 n_2^2 \leq e(n, \mathcal{P}_2)^{-1} \right\}$$

$$\leq \sum_{n_2=1}^{\infty} \text{card} \left\{ n_1 \in \mathbb{N} \mid n_1 \leq e(n, \mathcal{P}_2)^{-1} n_2^{-2} \right\} \leq e(n, \mathcal{P}_2)^{-1} \sum_{n_2=1}^{\infty} n_2^{-2} \leq 2 e(n, \mathcal{P}_2)^{-1},$$

and hence

$$\limsup_{n \to \infty} e(n, \mathcal{P}_2)n \leq 2.$$ 

The tensor product $\tilde{S}_2$ of $S$ and $\tilde{T}$ has the singular values

$$\tilde{\sigma}(n) = \begin{cases} n_1^{-1}, & \text{if } n_2 \leq N, \\ n_1^{-1} n_2^{-2}, & \text{else,} \end{cases} \text{ for } n \in \mathbb{N}^2.$$ 

For the respective problem $\tilde{\mathcal{P}}_2$ and $n \in \mathbb{N}$ we obtain

$$n \geq \text{card} \left\{ n \in \mathbb{N}^2 \mid \tilde{\sigma}(n) > e(n, \tilde{\mathcal{P}}_2) \right\} \geq N \text{ card} \left\{ n_1 \in \mathbb{N} \mid n_1^{-1} > e(n, \tilde{\mathcal{P}}_2) \right\}$$

$$\geq N \left( e(n, \tilde{\mathcal{P}}_2)^{-1} - 1 \right)$$

and thus

$$\liminf_{n \to \infty} e(n, \tilde{\mathcal{P}}_2)n \geq N.$$ 

Hence, matching asymptotic constants of the factor problems do not necessarily lead to matching asymptotic constants of the tensor product problems.
2.2.3 Preasymptotic Behavior

Theorem 2.24 leads to a good understanding of the asymptotic behavior of the \(n\)th minimal error of \(P_d\) if the \(n\)th minimal error of \(P\) is of polynomial decay. If \(e(n, P)\) is roughly \(c n^{-r}\) for large \(n\), then \(e(n, P_d)\) is roughly \(c^d(d-1)!^{-r} n^{-r} (\ln n)^{r(d-1)}\) for \(n\) larger than a certain threshold. But even for modest dimensions, the size of this threshold may go far beyond the scope of computational capabilities. Indeed, while the minimal error decreases, the function \(n^{-r} (\ln n)^{r(d-1)}\) grows rapidly as \(n\) goes from 1 to \(e^{d-1}\). For this function to become less than 1, the number \(n\) even has to be super-exponentially large with respect to the dimension. Thus, any estimate for the minimal error in terms of this function is useless to describe its behavior in the range \(n \leq 2^d\), its so called preasymptotic behavior. As a replacement we present the following estimate.

**Theorem 2.28 ([Kri18a]).** Let \(P_d\) be a tensor product problem as defined in Section 2.2.1. Let \(\sigma_1 = 1\) and \(\sigma_2 \in (0, 1)\) and assume \(\sigma_n \leq C n^{-r}\) for some \(r, C > 0\) and all \(n \geq 2\). Then

\[
\sigma_2 \left(\frac{1}{n+1}\right)^{\ln \left(\frac{\sigma_2^{-1}}{\ln \left(\frac{\sigma_2^{-1}}{\ln (\sigma_2^{-1}/2)}\right)}\right)} \leq e(n, P_d) \leq \left(\frac{\exp \left(C^2/r\right)}{n+1}\right)^{\ln \left(\frac{\sigma_2^{-1}}{\ln \left(\frac{\sigma_2^{-1}}{\ln (\sigma_2^{-1}/2)}\right)}\right)}
\]

for any \(n \in \{1, \ldots, 2^d - 1\}\).

Let us assume that the dimension \(d\) is large. Then the \(n\)th minimal error, which roughly decays like \(n^{-r}\) for huge values of \(n\), roughly decays like \(n^{-t_d}\) with

\[
t_d = \frac{\ln \left(\frac{\sigma_2^{-1}}{\ln (\sigma_2^{-1}/2)}\right)}{\ln d}
\]

for small values of \(n\). This is why we refer to \(t_d\) as preasymptotic rate of the tensor product sequence. The preasymptotic rate is much worse than the asymptotic rate. This is not an unusual phenomenon for high-dimensional problems. Comparable estimates for the case of \(S_d\) being the \(L^2\)-embedding of the mixed order Sobolev space on the \(d\)-torus are established in Theorems 4.9, 4.10, 4.17 and 4.20 of [KSU15]. See [CW17a, CW17b, KMU16] for other examples.

In order to obtain bounds on the \(n\)th minimal error for small values of \(n\), we give explicit estimates for \(K_2(R, l)\) from (2.6) for \(l \leq d\) and small values of \(R\). The right asymptotic behavior of these estimates is less important in this case. Note that \(K_2(R, l) = 0\) for \(R < 2^l\).

**Lemma 2.29.** Let \(R \geq 0\) and \(l \in \mathbb{N}\). For any \(\delta > 0\) we have

\[
K_2(R, l) \leq \frac{R^{1+\delta}}{\delta^{l-1}} \quad \text{and} \quad K_2(R, l) \geq \frac{R}{3 \cdot 2^{l-1}} \quad \text{for} \ R \geq 2^l.
\]
We have thus proven Lemma 2.29 by induction.

**Theorem 2.30 (Kri18a).** Consider $\mathcal{P}_d$ as defined in Section 2.2.1 with $\sigma_1 > \sigma_2 > 0$.

(i) Suppose that $\sigma_n \leq C n^{-r}$ for some $r, C > 0$ and all $n \geq 2$ and let $\delta \in (0, 1]$. For any $n \in \mathbb{N}_0$,

$$e(n, \mathcal{P}_d) \leq \sigma_1^d \left( \frac{\tilde{C}(\delta)}{n+1} \right)^{\alpha(d, \delta)}$$

with

$$\tilde{C}(\delta) = \exp \left( \frac{(C/\sigma_1)^{(1+\delta)/r}}{\delta} \right) \quad \text{and} \quad \alpha(d, \delta) = \frac{\ln(\sigma_1/\sigma_2)}{\ln(d(\sigma_1/\sigma_2)^{(1+\delta)/r})} > 0.$$

(ii) Let $v = \text{card} \{ n \geq 2 \mid \sigma_n = \sigma_2 \}$. For any $n \in \{ 1, \ldots, (1+v)^d - 1 \}$,

$$e(n, \mathcal{P}_d) \geq \sigma_1^{d-1} \sigma_2 \left( \frac{1}{n+1} \right)^{\beta(d,n+1)}$$

with

$$\beta(d, n) = \frac{\ln(\sigma_1/\sigma_2)}{\ln \left( 1 + \frac{v}{\log_{1+n} d} \right)} > 0.$$

We have thus proven Lemma 2.29 by induction. \(\square\)

We give a slight refinement of Theorem 2.28.

**Theorem 2.30 (Kri18a).** Consider $\mathcal{P}_d$ as defined in Section 2.2.1 with $\sigma_1 > \sigma_2 > 0$.

(i) Suppose that $\sigma_n \leq C n^{-r}$ for some $r, C > 0$ and all $n \geq 2$ and let $\delta \in (0, 1]$. For any $n \in \mathbb{N}_0$,

$$e(n, \mathcal{P}_d) \leq \sigma_1^d \left( \frac{\tilde{C}(\delta)}{n+1} \right)^{\alpha(d, \delta)}$$

with

$$\tilde{C}(\delta) = \exp \left( \frac{(C/\sigma_1)^{(1+\delta)/r}}{\delta} \right) \quad \text{and} \quad \alpha(d, \delta) = \frac{\ln(\sigma_1/\sigma_2)}{\ln(d(\sigma_1/\sigma_2)^{(1+\delta)/r})} > 0.$$

(ii) Let $v = \text{card} \{ n \geq 2 \mid \sigma_n = \sigma_2 \}$. For any $n \in \{ 1, \ldots, (1+v)^d - 1 \}$,

$$e(n, \mathcal{P}_d) \geq \sigma_1^{d-1} \sigma_2 \left( \frac{1}{n+1} \right)^{\beta(d,n+1)}$$

with

$$\beta(d, n) = \frac{\ln(\sigma_1/\sigma_2)}{\ln \left( 1 + \frac{v}{\log_{1+n} d} \right)} > 0.$$

Note that the assumption $\sigma_1 > \sigma_2 > 0$ is in fact the only interesting case. If $\sigma_2 = \sigma_1$, we have $e(n, \mathcal{P}_d) = \sigma_1^d$ for every $n < (1+v)^d$. On the other hand, $\sigma_2 = 0$ implies $e(n, \mathcal{P}_d) = 0$ for every $n \in \mathbb{N}$.

**Proof.** Recall that multiplying the singular numbers with a constant factor $c$ scales the minimal errors of $\mathcal{P}_d$ with the factor $c^d$. Hence we may assume that $\sigma_1 = 1$ without loss of generality.

Part (i): Let $n \in \mathbb{N}$. There is some $L \geq 0$ with $e(n-1, \mathcal{P}_d) = \sigma_2^L$. If $\sigma(n) \geq e(n-1, \mathcal{P}_d)$, the number $l$ of components of $n \in \mathbb{N}^d$ that are not equal to 1 is at most $\lfloor L \rfloor$ and hence

$$n \leq \text{card} \left\{ n \in \mathbb{N}^d \mid \sigma(n) \geq e(n-1, \mathcal{P}_d) \right\}$$

$$= \sum_{l=0}^{\lfloor L \rfloor} \text{card} \left\{ n \in \mathbb{N}^d \mid \text{card} \left\{ 1 \leq j \leq d \mid n_j \neq 1 \right\} = l \wedge \sigma(n) \geq e(n-1, \mathcal{P}_d) \right\}$$

$$= 1 + \sum_{l=1}^{\lfloor L \rfloor} \binom{d}{l} \text{card} \left\{ n \in \{ 2, 3, \ldots \}^d \mid \sigma(n) \geq e(n-1, \mathcal{P}_d) \right\}.$$
A combination of these bounds yields

\[ n \leq \sum_{l=0}^{\min\{L,d\}} \binom{d}{l} \cdot C^{(1+\delta)/r} e(n-1, \mathcal{P}_d)^{-1/(\delta r) - \delta - l} \]

\[ \leq e(n-1, \mathcal{P}_d)^{-1/(\delta r)} \sum_{l=0}^{\min\{L,d\}} \frac{d^l}{l!} C^{(1+\delta)/r} \delta - l \]

\[ \leq \sigma_2^{-(1+\delta)/r} d^L \sum_{l=0}^{\min\{L,d\}} \binom{C^{(1+\delta)/r}}{l} \frac{\delta^{-l}}{l!} \]

\[ \leq \left( \sigma_2^{-(1+\delta)/r} \cdot d \right)^L \exp \left( \frac{C^{(1+\delta)/r}}{\delta} \right) \]

and hence

\[ L \geq \frac{\ln n - \frac{C^{(1+\delta)/r}}{\delta}}{\ln \left( \sigma_2^{-(1+\delta)/r} \cdot d \right)}. \]

Thus

\[ e(n-1, \mathcal{P}_d) = \sigma_2^L \leq \exp \left( \frac{(C^{(1+\delta)/r} - \ln n) \ln \sigma_2^{-1}}{\ln \left( \sigma_2^{-(1+\delta)/r} \cdot d \right)} \right) = \left( \frac{\exp \left( \frac{C^{(1+\delta)/r}}{\delta} \right)}{n} \right)^{\alpha(d,\delta)} \]

with

\[ \alpha(d, \delta) = \frac{\ln \sigma_2^{-1}}{\ln \left( \sigma_2^{-(1+\delta)/r} \cdot d \right)}. \]

Part (ii): Let \( n \in \{2, \ldots, (1 + v)^d\} \). Then \( \sigma_2^d \leq e(n-1, \mathcal{P}_d) \leq \sigma_2 \). If \( e(n-1, \mathcal{P}_d) \) equals \( \sigma_2 \), the lower bound is trivial. Else, there is some \( L \in \{1, \ldots, d-1\} \) such that \( e(n-1, \mathcal{P}_d) \in [\sigma_2^{L+1}, \sigma_2^L] \). Clearly,

\[ n > \text{card} \left\{ \mathbf{n} \in \mathbb{N}^d \mid \sigma(\mathbf{n}) > e(n-1, \mathcal{P}_d) \right\} \]

\[ \geq \sum_{l=1}^{L} \binom{d}{l} \text{card} \left\{ \mathbf{n} \in \{2, 3, \ldots\}^l \mid \sigma(\mathbf{n}) > e(n-1, \mathcal{P}_d) \right\}. \]

If \( l \leq L \), we have \( \sigma(\mathbf{n}) > e(n-1, \mathcal{P}_d) \) for every \( \mathbf{n} \in \{2, \ldots, 1 + v\}^l \) and hence

\[ n \geq \sum_{l=0}^{L} \binom{d}{l} v^l \geq \sum_{l=0}^{L} \binom{L}{l} \binom{d}{L}^l v^l = \left( 1 + \frac{vd}{L} \right)^L. \]
Since $d/L$ is bigger than 1, this yields in particular that $L \leq \log_{1+v} n$. We insert this auxiliary estimate on $L$ in (2.23) and get

$$n \geq \left(1 + \frac{vd}{\log_{1+v} n}\right)^L,$$

or equivalently

$$L \leq \frac{\ln n}{\ln \left(1 + \frac{vd}{\log_{1+v} n}\right)}.$$

We recall that $e(n-1, P_d) \geq \sigma_2^{L+1}$ and realize that the proof is finished.

The bounds of Theorem 2.30 are completely explicit, but complex. One might be bothered by the dependence of the exponent in the lower bound on $n$. This can be overcome, if we restrict the lower bound to the case $n < (1 + v)^a$ for some $0 < a < 1$ and replace $\beta(d, n)$ by

$$\tilde{\beta}(d) = \frac{\ln \sigma_2^{-1}}{\ln (1 + v \cdot d^{1-a})}.$$

Of course we throw away information this way. Similarly we get a worse but still valid estimate, if we replace $v$ by 1. Note that the lower bounds are valid for any zero sequence $\sigma$, independent of its rate of convergence. The additional parameter $\delta$ in the upper bound was introduced to maximize the exponent $\alpha(d, \delta)$. If $\delta$ tends to zero, $\alpha(d, \delta)$ gets bigger, but also the constant $\tilde{C}(\delta)$ explodes.

These kinds of estimates are also closely related to those in [GW11, Section 3]. Using the language of generalized tractability, it is shown in [GW11] that the supremum of all $p > 0$ such that there is a constant $c > 0$ with

$$e(n, P_d) \leq e\sigma_1^d \left(\frac{c}{n + 1}\right)^{1+\ln d}$$

for all $n \in \mathbb{N}_0$ and $d \in \mathbb{N}$ is the minimum of $r$ and $\ln(\sigma_2^{-1})$.

### 2.2.4 Examples

We apply our results to three different solution operators $S_d$. The solution operators are embeddings of Hilbert spaces of $d$-variate functions into $L^2$. That is, we study the problem of $L^2$-approximation of certain $d$-variate functions using deterministic algorithms based on $\Lambda^{\text{all}}$.

#### Approximation of Mixed Order Sobolev Functions on the Torus

Let $\mathbb{T}$ be the 1-torus, the circle, represented by the interval $[a, b]$, where the two end points $a < b$ are identified. By $L^2(\mathbb{T})$ we denote the Hilbert space of square-integrable, complex-valued functions on $\mathbb{T}$, equipped with the scalar product

$$\langle f, g \rangle = \frac{1}{L} \int_{\mathbb{T}} f(x)g(x) \, dx$$

where

$$L = \int_{\mathbb{T}} 1 \, dx.$$
2.2. Tensor Product Problems

and the induced norm \( \| \cdot \| \) for some \( L > 0 \). Typical normalizations are \( [a, b] \in \{ [0, 1], [-1, 1], [0, 2\pi] \} \) and \( L \in \{ 1, b - a \} \). The family \((b_k)_{k \in \mathbb{Z}}\) with

\[
b_k(x) = \sqrt{\frac{L}{b-a}} \exp \left( 2\pi i k \frac{x-a}{b-a} \right)
\]

is an orthonormal basis of \( L^2(\mathbb{T}) \), its Fourier basis, and

\[
\hat{f}(k) = \langle f, b_k \rangle
\]

is the \( k^{\text{th}} \) Fourier coefficient of \( f \in L^2(\mathbb{T}) \). By Parseval’s identity,

\[
\|f\|^2 = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 \quad \text{and} \quad \langle f, g \rangle = \sum_{k \in \mathbb{Z}} \hat{f}(k) \cdot \hat{g}(k).
\]

Let \( w = (w_k)_{k \in \mathbb{N}} \) be a nondecreasing sequence of nonnegative numbers with \( w_0 = 1 \) and let \( w_{-k} = w_k \) for \( k \in \mathbb{N} \) and so let \( \tilde{w} \). The univariate Sobolev space \( H^w(\mathbb{T}) \) is the Hilbert space of functions \( f \in L^2(\mathbb{T}) \) for which

\[
\|f\|_w^2 = \sum_{k \in \mathbb{Z}} w_k^2 \cdot |\hat{f}(k)|^2
\]

is finite, equipped with the scalar product

\[
\langle f, g \rangle_w = \sum_{k \in \mathbb{Z}} w_k \hat{f}(k) \cdot \overline{w_k \hat{g}(k)}.
\]

Note that \( H^w(\mathbb{T}) \) and \( H^{\tilde{w}}(\mathbb{T}) \) coincide and their norms are equivalent, if and only if \( w_k \asymp \tilde{w}_k \). In case \( w_k \asymp k^r \) for some \( r \geq 0 \), the space \( H^w(\mathbb{T}) \) is the classical Sobolev space of periodic univariate functions with fractional smoothness \( r \), also denoted by \( H^r(\mathbb{T}) \). In particular, \( H^w(\mathbb{T}) = L^2(\mathbb{T}) \) for \( w_k \asymp 1 \).

In accordance with previous notation, let \( H = H^w(\mathbb{T}) \) and \( G = H^{\tilde{w}}(\mathbb{T}) \). The embedding \( S \) of \( H \) into \( G \) is compact, if and only if \( w_k / \tilde{w}_k \) tends to infinity as \( k \) tends to infinity. The Fourier basis \((b_k)_{k \in \mathbb{Z}}\) is an orthogonal basis of \( H \) consisting of eigenfunctions of \( W = S^* S \) with corresponding eigenvalues

\[
\lambda(b_k) = \frac{\|b_k\|^2_G}{\|b_k\|^2_H} = \frac{\tilde{w}_k^2}{w_k^2}.
\]

The \( n^{\text{th}} \) singular value \( \sigma_n \) of this embedding is the square root of the \( n^{\text{th}} \) biggest eigenvalue. Hence, replacing the Fourier weight sequences \( w \) and \( \tilde{w} \) by equivalent sequences does not affect the order of convergence of the corresponding singular values, but it may drastically affect their asymptotic constants and preasymptotic behavior. If \( G = L^2(\mathbb{T}) \), we obtain

\[
\sigma_n = w_k^{-1}, \quad \text{where} \quad k_n = (-1)^n \lfloor n/2 \rfloor.
\]

Note that \( \sigma_1 \), the norm of the embedding \( S \), is always 1.

The \( d^{\text{th}} \) tensor product \( H_d = H^{w_{\text{mix}}}(\mathbb{T}^d) \) of \( H \) is a space of mixed order Sobolev functions on the \( d \)-torus. If \( w_k \asymp k^r \) for some \( r \geq 0 \), this is the space \( H^{r_{\text{mix}}}(\mathbb{T}^d) \) of
functions with mixed smoothness $r$. If $r \in \mathbb{N}_0$, this space consists of all real-valued functions on the $d$-torus, which have a weak (or distributional) derivative of order $\alpha$ in $L^2(\mathbb{T}^d)$ for any $\alpha \in \{0, 1, \ldots, r\}^d$. The same holds for the $d$th tensor product $G_d = H_{\text{mix}}^\infty(\mathbb{T}^d)$ of $G$. The tensor product operator $S_d : H_d \to G_d$ is the compact embedding of $H_{\text{mix}}^w(\mathbb{T}^d)$ into $H_{\text{mix}}^w(\mathbb{T}^d)$.

If $\tilde{w}_k/w_k$ is of polynomial decay for $k \to \infty$, Theorem 2.24 and Theorem 2.30 can be applied. We formulate the results for the embedding of $H_{\text{mix}}^w(\mathbb{T}^d)$ into $L^\gamma(\mathbb{T}^d)$, which we denote by $\text{APP}$. The space $H_{\text{mix}}^w(\mathbb{T}^d)$ is equipped with different equivalent norms. To indicate the norm, we write $H_{\text{mix}}^{r,\gamma}(\mathbb{T}^d)$ with $\square \in \{\circ, *, +, \#\}$ and $\gamma > 0$. The norms are given by the following weights.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\square & \circ & * & + & \# \\
\hline
w_k^2 & \sum_{k=0}^{l-2} \frac{2^k}{\gamma(b-a)} & 2l & 1 + \left(1 + \frac{2^k}{\gamma(b-a)}\right)^{2r} & \left(1 + \frac{2^k}{\gamma(b-a)}\right)^{2r} \\
\hline
\end{array}
\]

The last three norms are due to Kühn, Sickel and Ullrich [KSU13], who study all these norms for $\gamma = 1$, $L = 1$ and $[a, b] = [0, 2\pi]$. The last norm is also studied by Chernov and Dung in [CD16] for $L = 2\pi$, $[a, b] = [-\pi, \pi]$ and arbitrary values of $\gamma$. If $r$ is a natural number, the first two scalar products take the form

\[
\langle f, g \rangle_{H_{\text{mix}}^{r,\gamma}(\mathbb{T}^d)} = \sum_{\alpha \in \{0, \ldots, r\}^d} \gamma^{-2r|\alpha|} \langle D^\alpha f, D^\alpha g \rangle,
\]

\[
\langle f, g \rangle_{H_{\text{mix}}^{r*,\gamma}(\mathbb{T}^d)} = \sum_{\alpha \in \{0, \ldots, r\}^d} \gamma^{-2r|\alpha|} \langle D^\alpha f, D^\alpha g \rangle.
\]

This is why $H_{\text{mix}}^{r,\gamma}(\mathbb{T}^d)$ and $H_{\text{mix}}^{r*,\gamma}(\mathbb{T}^d)$ might be considered the most natural choice. We now translate Theorems 2.24 and 2.30 for the approximation problem

\[P^r_{\text{d,\gamma}} = \text{APP}[F^r_{\text{d,\gamma}}, L^2, \Lambda^{\text{all}}, \text{det}, \text{wc}],\]

where $F^r_{\text{d,\gamma}}$ is the unit ball of $H_{\text{mix}}^{r,\gamma}(\mathbb{T}^d)$.

**Corollary 2.31** ([Kri18a]). For any $r > 0$, $\gamma > 0$ and $\square \in \{\circ, *, +, \#\}$, we have

\[e \left(n, P^r_{\text{d,\gamma}}\right) \sim \left(\frac{(\gamma(b-a))^d}{\pi^d (d-1)!}\right)^r n^{-r} (\ln n)^{r(d-1)}.\]

This agrees with the limits that are computed in [KSU13] for the case $\gamma(b-a)/\pi = 2$. The limit for $\square = +$, $[a, b] = [-\pi, \pi]$ and $L = 2\pi$ can also be derived from [CD16] Theorem 4.6]. The preasymptotic estimates take the following form.

**Corollary 2.32** ([Kri18a]). For any $r, \gamma > 0$, $\square \in \{\circ, *, +, \#\}$, and $n < 3^d$ we have

\[\sigma^2_\square \left(\frac{1}{n+1}\right)^{\beta_\square(d,n+1)} \leq e \left(n, P^r_{\text{d,\gamma}}\right) \leq \left(\frac{\tilde{C}(\delta)}{\gamma(b-a)}\right)^{\alpha_\square(d,d)}.
\]

The parameter $\delta \in (0, 1]$ is arbitrary, $\tilde{C}(\delta) = \exp \left(\left(3/\eta\right)^{1/\delta}\right)$ for $\eta = \frac{2\pi}{\gamma(b-a)}$ and the values $\sigma^\square$, $\alpha_\square$ and $\beta_\square$ are listed below. The upper bound holds for all $n \in \mathbb{N}_0$. 46
2.2. Tensor Product Problems

| □ | $\sigma_2$ | $\alpha_\square(d, \delta)$ | $\beta_\square(d, n)$ |
|---|---|---|---|
| • | $(\sum_{l=0}^{r} \eta^2)^{-1/2}$ | $r \ln \left( \frac{\sum_{l=0}^{r} \eta^2}{2} \right)$ | $\frac{\ln \left( \sum_{l=0}^{r} \eta^2 \right)}{2} \frac{1}{(1 + 2d/\log_3 n)}$ |
| * | $(1 + \eta^2)^{-1/2}$ | $\frac{r \ln (1 + \eta^2)}{2r \ln d + (1 + \delta) \ln (1 + \eta^2)}$ | $\ln (1 + \eta^2) \frac{1}{2} \frac{1}{(1 + 2d/\log_3 n)}$ |
| + | $(1 + \eta^2)^{-r/2}$ | $\frac{r \ln (1 + \eta^2)}{2r \ln d + (1 + \delta) \ln (1 + \eta^2)}$ | $\frac{r \ln (1 + \eta^2)}{2r \ln d + (1 + \delta) \ln (1 + \eta^2)}$ |
| # | $(1 + \eta)^{-r}$ | $\frac{r \ln (1 + \eta)}{\ln d + (1 + \delta) \ln (1 + \eta)}$ | $\frac{r \ln (1 + \eta)}{\ln d + (1 + \delta) \ln (1 + \eta)}$ |

Let us discuss the setting of [KSU15], where $\gamma = 1$ and $b - a = 2\pi$ and hence $\eta = 1$. The exponents $\alpha_\#(d, \delta) = r(\log_2 d + 1 + \delta)^{-1}$ and $\alpha_+(d, \delta) = r(2 \log_2 d + 1 + \delta)^{-1}$ in our upper bounds are slightly better than the exponents $r(\log_2 d + 2)^{-1}$ and $r(2 \log_2 d + 4)^{-1}$ in Theorem 4.9, 4.10 and Theorem 4.17 of [KSU15], but almost the same. Also the lower bounds basically coincide. Regarding $H_{mix,1}^{\gamma,1}(T^d)$, Kühn, Sickel and Ullrich only studied the case $1/2 \leq r \leq 1$ in Theorem 4.20. As we see now, there is a major difference between this natural norm and the last two norms: For large $d$, the preasymptotic behavior of the singular values is roughly $n^{-t_{d,\square}}$, where

$$t_{d,o} = \frac{\log_2 (r + 1)}{2 \log_2 d}, \quad t_{d,*} = \frac{1}{2 \log_2 d}, \quad t_{d,+} = \frac{r}{2 \log_2 d}, \quad t_{d,#} = \frac{2r}{2 \log_2 d}.$$

This means that the smoothness of the space only has a minor or even no impact on the preasymptotic decay of the singular values for $H_{mix,1}^{\gamma,1}(T^d)$ and for $H_{mix,1}^{\gamma,1}(T^d)$. This changes, however, if the value of $\eta$ changes. If we have $\eta > 1$, then also the exponents $t_{d,o}$ and $t_{d,*}$ get linear in $r$. For the other two families of norms, the smoothness does show and the value of $\eta$ is less important.

**Approximation of Mixed Order Jacobi Functions on the Cube**

The above results also apply to problem of the $L^2$-approximation of mixed order Jacobi functions on the $d$-cube as considered in [CD16, Section 5]. For fixed parameters $\alpha, \beta > -1$ with $a := \frac{\alpha + \beta + 1}{2} > 0$, the weighted $L^2$-space $G = L^2([-1, 1], w)$ is the Hilbert space of measurable, real-valued functions on $[-1, 1]$ with

$$\int_{-1}^{1} f(x)^2 w(x) \, dx < \infty,$$

with scalar product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)w(x) \, dx$$

and the induced norm $\| \cdot \|$, where $w : [-1, 1] \to \mathbb{R}$ is the Jacobi weight

$$w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}.$$
This reduces to the classical space of square-integrable functions, if both parameters are zero. As \( \alpha \) respectively \( \beta \) increases, the space grows, since we allow for stronger singularities on the right respectively left endpoint of the interval, and vice versa.

The family of Jacobi polynomials \( (P_k)_{k \in \mathbb{N}_0} \) is an orthogonal basis of \( G \). These polynomials can be defined as the unique solutions of the differential equations

\[
\mathcal{L} P_k = k(k+2a) P_k
\]

for the second order differential operator

\[
\mathcal{L} = -w(x)^{-1} \frac{d}{dx} \left( (1-x^2) w(x) \frac{d}{dx} \right)
\]

that satisfy

\[
P_k(1) = \binom{k+\alpha}{k} \quad \text{and} \quad P_k(-1) = (-1)^k \binom{k+\beta}{k}.
\]

For more details on Jacobi polynomials we refer the reader to [Sze39, Chapter 4]. We denote the \( k \)th Fourier coefficient of \( f \) with respect to the normalized Jacobi basis by \( f_k \). The scalar product in \( G \) hence admits the representation

\[
\langle f, g \rangle = \sum_{k=0}^{\infty} f_k g_k.
\]

For \( r > 0 \) let \( H = K^r([-1,1], w) \) be the Hilbert space of functions \( f \in G \) with

\[
\sum_{k=0}^{\infty} \left( 1 + a^{-1} k \right)^{2r} f_k^2 < \infty,
\]

equipped with the scalar product

\[
\langle f, g \rangle_r = \sum_{k=0}^{\infty} \left( 1 + a^{-1} k \right)^{2r} f_k g_k
\]

and the induced norm \( \| \cdot \|_r \). Obviously, \( (P_k)_{k \in \mathbb{N}_0} \) is an orthogonal basis of \( H \), too. In case \( r \) is an even integer, this is the space of all functions \( f \in L^2([-1,1], w) \) such that \( \mathcal{L}^j f \in L^2([-1,1], w) \) for \( j = 1, \ldots, r/2 \) and the scalar product

\[
\langle f, g \rangle_{r,*} = \sum_{j=0}^{r/2} \langle \mathcal{L}^j f, \mathcal{L}^j g \rangle
\]

is equivalent to the one above. Hence the parameter \( r \) can be interpreted as smoothness of the functions in \( K^r([-1,1], w) \). The embedding \( S \) of \( H \) into \( G \) is compact and its \( n \)th singular value is given by

\[
\sigma_n = \frac{\| P_{n-1} \|}{\| P_{n-1} \|_r} = \left( 1 + a^{-1} (n-1) \right)^{-r}.
\]

We can apply our theorems to study the singular values of the \( d \)th tensor product \( S_d \) of \( S \). This is the embedding of \( H_d = K^r([-1,1]^d, w_d) \) into \( G_d = L^2([-1,1]^d, w_d) \), where \( G_d \) is the weighted \( L^2 \)-space on the \( d \)-cube with respect to the Jacobi weight \( w_d = w \otimes \ldots \otimes w \) and \( H_d \) is the subspace of Jacobi functions of mixed order \( r \). Like in the univariate case, \( H_d \) can be described via differentials of mixed order \( r \) and less, if \( r \) is an even integer. Let \( \mathcal{P}^r_d \) be the respective approximation problem.
Corollary 2.33 ([CD16, Kri18a]). For any $d \in \mathbb{N}$ and $r > 0$ we have

$$e(n, P_d^r) \sim \left( \frac{a^d}{(d-1)!} \right)^r n^{-r} (\ln n)^{-r(d-1)}.$$  

This result can also be derived from [CD16, Theorem 5.5]. In addition we get the following preasymptotic estimates.

Corollary 2.34 ([Kri18a]). For any $\delta \in (0, 1], r > 0, d \in \mathbb{N}$ and $n < 2^d$ we have

$$\left( \frac{a}{a+1} \right)^r \left( \frac{1}{n+1} \right)^{p_{r,a,d,n+1}} \leq e(n, P_d^r) \leq \left( \frac{\exp \left( \frac{(2a)^{1+\delta}}{\delta} \right)}{n+1} \right)^{q_{r,a,d,\delta}}$$

with $p_{r,a,d,n} = \frac{r \ln \frac{a+1}{a}}{\ln \left( 1 + \frac{d}{\log_2 n} \right)}$ and $q_{r,a,d,\delta} = \frac{r \ln \frac{a+1}{a}}{\ln d + (1 + \delta) \ln \frac{a+1}{a}}$.

The upper bound holds for all $n \in \mathbb{N}_0$.

This means that for large dimension $d$, a preasymptotic decay of approximate order $t_d = r \ln \frac{a+1}{a} / \ln d$ in $n$ can be observed.

**Approximation of Mixed Order Sobolev Functions on the Cube**

Another example is the problem of approximating mixed order Sobolev functions on the $d$-cube in $L^2$ with $n$ pieces of linear information. We want to compare the difficulty of this problem with the difficulty of the respective problem for the subspace of periodic functions as considered in the first example. Of course, the nonperiodic problem can only be harder than the periodic problem. In fact we find that it is much harder if $n$ is small but just about as hard if $n$ is large.

We consider an interval $[a, b]$ and the circle $T$. The latter shall also be represented by $[a, b]$, where $a$ and $b$ are identified. For any $r \in \mathbb{N}_0$, the vector space

$$H^r([a, b]) = \left\{ f \in L^2([a, b]) \mid f^{(l)} \in L^2([a, b]) \text{ for } 1 \leq l \leq r \right\},$$

equipped with the scalar product

$$\langle f, g \rangle_r = \sum_{l=0}^{r} \int_{a}^{b} f^{(l)}(x) \cdot \overline{g^{(l)}(x)} \, dx \tag{2.24}$$

and induced norm $\|\cdot\|_r$, is a Hilbert space, the Sobolev space of order $r$ on $[a, b]$. In case $r = 0$, it coincides with $L^2([a, b])$. The subset

$$H^r(T) = \left\{ f \in H^r([a, b]) \mid f^{(l)}(a) = f^{(l)}(b) \text{ for } l = 0, 1, \ldots, r-1 \right\}$$
of periodic functions is a closed subspace with codimension $r$, the Sobolev space of order $r$ on $T$. By means of Parseval’s identity and integration by parts, the above norm can be rearranged to

$$\|f\|^2_r = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 \sum_{l=0}^{r} \left| \frac{2 \pi k}{b-a} \right|^{2l} \quad \text{for } f \in H^r(T), \tag{2.25}$$
where
\[ \hat{f}(k) = \sqrt{\frac{1}{b-a}} \int_a^b f(x) \cdot \exp \left(-2\pi ik \frac{x-a}{b-a} \right) \, dx \]

is the \( k \)th Fourier coefficient of \( f \). In the limiting case \( r = \infty \), the Sobolev space \( H^\infty ([a, b]) \) shall be defined as the Hilbert space
\[ H^\infty ([a, b]) = \left\{ f \in C^\infty ([a, b]) \mid \sum_{l=0}^{\infty} \| f^{(l)} \|_0^2 < \infty \right\}, \]
equipped with the scalar product \( (2.24) \) for \( r = \infty \). It contains all polynomials and is hence infinite-dimensional. The space \( H^\infty (\mathbb{T}) \) shall be the closed subspace of periodic functions, i.e.
\[ H^\infty (\mathbb{T}) = \left\{ f \in H^\infty ([a, b]) \mid f^{(l)}(a) = f^{(l)}(b) \text{ for any } l \in \mathbb{N}_0 \right\}. \]

Note that \((2.25)\) also holds for \( r = \infty \). Hence,
\[ H^\infty (\mathbb{T}) = \text{span} \left\{ \exp \left(2\pi ik \frac{x-a}{b-a} \right) \mid k \in \mathbb{Z} \text{ with } \left| \frac{2\pi k}{b-a} \right| < 1 \right\} \]
is finite-dimensional with dimension \( 2 \lceil \frac{b-a}{2\pi} \rceil - 1 \). In case \( b-a \leq 2\pi \), it consists of constant functions only. If \( r \) is positive, \( H^r ([a, b]) \) is compactly embedded into \( L^2 ([a, b]) \). Let \( \sigma_n^{(r)} \) be the \( n \)th singular value of this embedding and let \( \tilde{\sigma}_n^{(r)} \) be the \( n \)th singular value of the embedding of the subspace \( H^r (\mathbb{T}) \) into \( L^2 (\mathbb{T}) \). Recall from the first example of this subsection that
\[ \tilde{\sigma}_n^{(r)} = \left( \sum_{k=0}^{r} \left| \frac{2\pi k}{b-a} \right| \right)^{-1/2} \quad \text{for } n \in \mathbb{N} \text{ and } r \in \mathbb{N}. \]

The singular values \( \sigma_n^{(r)} \) for nonperiodic functions are not known explicitly. However, \( \sigma_n^{(r)} \) and \( \tilde{\sigma}_n^{(r)} \) interrelate as follows.

**Lemma 2.35.** For any \( n \in \mathbb{N} \) and \( r \in \mathbb{N} \), it holds that \( \sigma_n^{(r)} \leq \tilde{\sigma}_n^{(r)} \leq \sigma_n^{(r)} \).

**Proof.** The second inequality is obvious, since \( H^r (\mathbb{T}) \) is a subspace of \( H^r ([a, b]) \). The first inequality is true, since the codimension of this subspace is \( r \). Let \( U \) be the orthogonal complement of \( H^r (\mathbb{T}) \) in \( H^r ([a, b]) \). By relation \( (2.5) \),
\[ \sigma_n^{(r)} = \min_{V \subset H^r ([a, b])} \max_{\dim(V) \leq n-r-1} \| f \|_0 \leq \min_{V \subset H^r (\mathbb{T})} \max_{\dim(V) \leq n-1} \| f \|_0 = \tilde{\sigma}_n^{(r)}, \]
as it was to be proven. \( \square \)
Lemma 2.35 implies that the asymptotic constants of the singular values for the periodic and the nonperiodic functions coincide in the univariate case:

\[
\lim_{n \to \infty} n^{r} \sigma_{n+1}^{(r)} = \lim_{n \to \infty} n^{r} \tilde{\sigma}_{n+1}^{(r)} = \pi^{-r} (b - a)^{r}.
\]

Let \( H_{\text{mix}}^{r}([a, b]^d) \) be the \( d \)th tensor product space of \( H^{r}([a, b]) \). Note that this space satisfies the identity

\[
H_{\text{mix}}^{r}([a, b]^d) = \left\{ f \in L^2([a, b]^d) \mid D^{\alpha} f \in L^2([a, b]^d) \text{ for all } \alpha \in \{0, \ldots, r\}^d \right\}
\]

in the case that \( r \) is finite, and the scalar product is given by

\[
\langle f, g \rangle_{r} = \sum_{\alpha \in \{0, \ldots, r\}^d} \int_{[a,b]^d} D^{\alpha} f(x) \cdot \overline{D^{\alpha} g(x)} \, dx. \tag{2.26}
\]

We want to study the tensor product problem

\[
P^{r}_{d} = P [\text{APP}, F^{r}_{d}, L^2, \Lambda^{\text{all}}, \text{det}, \text{wc}],
\]

where APP is the embedding of \( H_{\text{mix}}^{r}([a, b]^d) \) into \( L^2([a, b]^d) \), and \( F^{r}_{d} \) is the unit ball of \( H_{\text{mix}}^{r}([a, b]^d) \). Since we know the asymptotic behavior of the \( n \)th minimal error for the case \( d = 1 \), Theorem 2.24 yields the asymptotic behavior in the general case.

**Corollary 2.36 ([Kri18a]).** For any \( d \in \mathbb{N} \) and \( r \in \mathbb{N} \) we have

\[
e(n, P^{r}_{d}) \sim \left( \frac{(b - a)^{d}}{\pi^{d} (d - 1)!} \right)^{r} n^{-r} (\ln n)^{(r(d-1))}.
\]

In particular, the \( n \)th minimal errors for the nonperiodic and the periodic problem are strongly equivalent as \( n \to \infty \).

We turn to preasymptotic estimates. As depicted in Section 2.2.3, the singular values show a preasymptotic decay of approximate order \( \ln(1/\sigma_{2}^{(r)})/\ln d \). Lemma 2.35 gives no information on \( \sigma_{2}^{(r)} \). However, relation (2.5) implies that

\[
\sigma_{2}^{(\infty)} = \max_{f \neq 1, f \neq 0} \frac{\|f\|_{0}}{\|f\|_{\infty}} \geq \frac{\|2x - a-b\|_{0}}{\|2x - a-b\|_{\infty}} = \sqrt{\frac{(b-a)^{2}}{12 + (b-a)^{2}}},
\]

If, for example, the length of the interval \([a, b]\) is 1, we obtain

\[
\sigma_{2}^{(\infty)} \geq 0.27735.
\]

Since any lower bound on the singular values for \( r = \infty \) is a lower bound for \( r \in \mathbb{N} \), Theorem 2.30 yields the following corollary.

**Corollary 2.37 ([Kri18a]).** Let \( b - a = 1 \). For any \( d \in \mathbb{N} \), any \( r \in \mathbb{N} \cup \{\infty\} \) and \( d \leq n < 2^{d} \), we have

\[
e(n, P^{r}_{d}) \geq 0.27 \cdot (n + 1)^{-c(d,n+1)}, \text{ where } c(d, n) = \frac{1.2825}{\ln \left(1 + \frac{2d}{\log_{2} n}\right)} \leq 1.17.
\]
Chapter 2. Integration and Approximation of Functions with Mixed Smoothness

On the other hand, any upper bound on the singular values for \( r = 1 \) is an upper bound for \( r \geq 1 \). The singular values \( \sigma_n^{(r)} \) for \( r = 1 \) are known. It is shown in [Tho96] that the family \( (b_k)_{k \in \mathbb{N}_0} \) is a complete orthogonal system in \( H^1([a,b]) \), where the function \( b_k : [a,b] \rightarrow \mathbb{R} \) with

\[
b_k(x) = \cos \left( k\pi \cdot \frac{x-a}{b-a} \right) \quad \text{for} \quad k \in \mathbb{N}_0
\]

is an eigenfunction of \( W = S^*S \) for \( r = 1 \) with respective eigenvalue

\[
\lambda_k = \left( 1 + \left( \frac{k\pi}{b-a} \right)^2 \right)^{-1}.
\]

In case \( b-a = 1 \),

\[
\sigma_2^{(1)} = \left( \sqrt{1 + \pi^2} \right)^{-1} \leq 0.30332
\]

and

\[
\sigma_n^{(1)} \leq 0.607 \cdot n^{-1}
\]

for \( n \geq 2 \). Theorem 2.30 for \( \delta = 0.65 \) yields the following upper bound.

**Corollary 2.38 ([Kri18a]).** Let \( b-a = 1 \). For any \( d \in \mathbb{N} \), \( r \in \mathbb{N} \cup \{\infty\} \) and \( n \in \mathbb{N}_0 \), we have

\[
e(n, P_r^d) \leq \left( \frac{2}{n+1} \right)^{c(d)} \quad \text{with} \quad c(d) = \frac{1.1929}{2 + \ln d}.
\]

Apparently, the upper bound for \( r = 1 \) and the lower bound for \( r = \infty \) are already close if \( d \) is large. The gap between the cases \( r = 2 \) and \( r = \infty \) is even smaller.

Let \( c \) be the midpoint of \([a,b]\) and let \( l \) be its radius. Moreover, let \( \hat{\omega} = \sqrt{1 + \omega^2} \) for \( \omega \in \mathbb{R} \) and consider the countable sets

\[
I_1 = \left\{ \omega \geq 0 \mid \hat{\omega}^3 \cosh(\hat{\omega}l) \sin(\omega l) + \omega^3 \sinh(\hat{\omega}l) \cos(\omega l) = 0 \right\},
\]

\[
I_2 = \left\{ \omega > 0 \mid \hat{\omega}^3 \sinh(\hat{\omega}l) \cos(\omega l) - \omega^3 \cosh(\hat{\omega}l) \sin(\omega l) = 0 \right\}.
\]

It can be shown (with some effort) that the family \( (b_\omega)_{\omega \in I_1 \cup I_2} \) is a complete orthogonal system in \( H^2([a,b]) \), where the function \( b_\omega : [a,b] \rightarrow \mathbb{R} \) with

\[
b_\omega(x) = \omega^2 \cdot \frac{\cosh(\hat{\omega}(x-c))}{\cosh(\hat{\omega}l)} + \hat{\omega}^2 \cdot \frac{\cos(\omega(x-c))}{\cos(\omega l)}, \quad \text{if} \quad \omega \in I_1,
\]

\[
b_\omega(x) = \omega^2 \cdot \frac{\sinh(\hat{\omega}(x-c))}{\sinh(\hat{\omega}l)} + \hat{\omega}^2 \cdot \frac{\sin(\omega(x-c))}{\sin(\omega l)}, \quad \text{if} \quad \omega \in I_2,
\]

is an eigenfunction of \( W = S^*S \) with respective eigenvalue

\[
\lambda_\omega = \left( 1 + \omega^2 + \omega^4 \right)^{-1}.
\]

In particular,

\[
\sigma_2^{(2)} = \left( \sqrt{1 + \omega_0^2 + \omega_0^4} \right)^{-1},
\]

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where \( \omega_0 \) is the smallest nonzero element of \( I_1 \cup I_2 \). If, for example, the interval \([a, b]\) has length 1, we obtain
\[
\sigma^{(2)}_2 \leq 0.27795
\]
and like before,
\[
\sigma^{(2)}_n \leq 0.607 \cdot n^{-1}
\]
for \( n \geq 2 \). Theorem \ref{thm:spec} for \( \delta = 0.65 \) yields the following upper bound.

**Corollary 2.39** (\cite{Kri18a}). Let \( b - a = 1, d \in \mathbb{N}, n \in \mathbb{N}_0 \) and \( r \geq 2 \). Then
\[
e(n, \mathcal{P}_d^r) \leq \left( \frac{2}{n + 1} \right)^{c(d)} \text{ with } c(d) = \frac{1.2803}{2 + \ln d}
\]

In short, the preasymptotic rate of the \( n^{th} \) minimal error is \( 1.1929/\ln d \) for \( r = 1 \), and in between \( 1.2803/\ln d \) and \( 1.2825/\ln d \) for any other \( r \in \mathbb{N} \cup \{ \infty \} \). In contrast, the preasymptotic rate for the periodic problem is roughly \( 1.8379 r/\ln d \). Thus, the nonperiodic problem is much harder than the periodic problem if \( n \) is small compared to \( 2^d \) and the smoothness \( r \) is large compared to 1.

### 2.2.5 A Tractability Result

A consequence of the preasymptotic estimates from Section \ref{sec:preas} is the following tractability result. For each \( d \in \mathbb{N} \), let \( S^{(d)} \) be a compact norm-one operator between two Hilbert spaces with singular values \( \sigma^{(d)}_n \). Let \( \mathcal{P}^{(d)} \) be the respective approximation problem with deterministic algorithms based on \( \Lambda^{all} \). With \( \mathcal{P}^{(d)}_d \) we denote the \( d^{th} \) tensor product problem that belongs to the \( d^{th} \) tensor product operator \( S^{(d)}_d \) of \( S^{(d)} \).

Note that now the univariate problem \( \mathcal{P}^{(d)} \) may be different for every \( d \in \mathbb{N} \). In fact, it is shown in [NW08, Theorem 5.5] that the family of multivariate problems \( \mathcal{P}^{(d)}_d \) is not polynomially tractable if the univariate problem \( \mathcal{P}^{(d)} \) does not depend on \( d \). However, we may hope for tractability, if the univariate problem gets easier as \( d \) increases, that is, if \( \sigma^{(d)}_n \) is decreasing in \( d \). It turns out that we obtain polynomial tractability and even strong polynomial tractability if the second singular value \( \sigma^{(d)}_2 \) of \( S^{(d)} \) decreases polynomially in \( d \). This condition is also necessary.

**Theorem 2.40** (\cite{Kri18a}). Let \( \sigma^{(d)}_n \) be nonincreasing in \( d \) and polynomially decreasing in \( n \) for \( d = 1 \). The family of multivariate problems \( \mathcal{P}^{(d)}_d \) is strongly polynomially tractable, iff it is polynomially tractable, iff \( \sigma^{(d)}_2 \) decays polynomially in \( d \).

**Proof.** Clearly, strong polynomial tractability implies polynomial tractability.

Let the problem be polynomially tractable and choose nonnegative numbers \( C, p \) and \( q \) such that
\[
n(\varepsilon, \mathcal{P}^{(d)}_d) \leq C \varepsilon^{-q} d^p
\]
for all \( \varepsilon > 0 \) and \( d \in \mathbb{N} \). In particular, there is some \( r \in \mathbb{N} \) with
\[
n(d^{-1}, \mathcal{P}^{(d)}_d) \leq d^r - 1
\]
for every \( d \geq 2 \). If \( d \) is large enough, we can apply the second part of Theorem 2.30 for \( n = d^r \) and the estimate

\[
\beta(d, d^r) = \frac{\ln(1/\sigma_2^{(d)})}{\ln \left(1 + \frac{e^d}{r \log_{1+e} d}\right)} \leq \frac{2 \ln(1/\sigma_2^{(d)})}{\ln d}
\]

to obtain

\[
d^{-1} \geq e^{d^r - 1, P_{d^r}(d)} \geq \sigma_2^{(d)} \cdot d^{-r \beta(d, d^r)} \geq (\sigma_2^{(d)})^{2r+1}.
\]

Consequently, \( \sigma_2^{(d)} \) decays polynomially in \( d \).

Now let \( \sigma_2^{(d)} \) be of polynomial decay. Then there are \( p > 0 \) and \( d_0 \in \mathbb{N} \) such that \( \sigma_2^{(d)} \) is bounded above by \( d^{-p} \) for any \( d \geq d_0 \). On the other hand, there are positive constants \( C \) and \( r \) such that

\[
\sigma_n^{(d)} \leq \sigma_n^{(1)} \leq C n^{-r}.
\]

We apply the first part of Theorem 2.30 and the estimate

\[
\alpha(d, 1) = \frac{\ln(1/\sigma_2^{(d)})}{\ln d + \frac{2}{r} \ln(1/\sigma_2^{(d)})} \geq \frac{p}{1 + \frac{2p}{r}} =: s > 0
\]

to obtain

\[
e\left(n, P_{d^r}(d)\right) \leq \left(\frac{\exp\left(C^{2/r}\right)}{n+1}\right)^s
\]

for any \( n \in \mathbb{N} \) and \( d \geq d_0 \). Consequently,

\[
n\left(\varepsilon, P_{d^r}(d)\right) \leq \exp\left(C^{2/r}\right) \cdot \varepsilon^{-1/s}
\]

for any \( d \geq d_0 \) and \( \varepsilon > 0 \) and the problem is strongly polynomially tractable. \(\square\)

As an example we consider the embeddings

\[
S_d^{(i)} : H_{\text{mix}}^{r_d}(\mathbb{N}, b]^d) \hookrightarrow L^2(\mathbb{N}, b]^d), \quad S_d^{(i)} : H_{\text{mix}}^{r_d}(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d),
\]

where the mixed order Sobolev spaces with smoothness \( r_d \in \mathbb{N} \) are equipped with the scalar product (2.26), see Section 2.2.4. Let \( P_{d^r}^{r_d} \) and \( \tilde{P}_{d^r}^{r_d} \) be the respective approximation problems. If the smoothness \( r_d \) is independent of \( d \) these problems are not polynomially tractable. Can we achieve polynomial tractability by increasing the smoothness with the dimension? We obtain the following result.

**Corollary 2.41** ([Kri18a]). The problem \( P_{d^r}^{r_d} \) is not polynomially tractable for any choice of natural numbers \( r_d \). The problem \( \tilde{P}_{d^r}^{r_d} \) is strongly polynomially tractable, iff it is polynomially tractable, iff \( b - a < 2\pi \) and \( r_d \) grows at least logarithmically in \( d \) or \( b - a = 2\pi \) and \( r_d \) grows at least polynomially in \( d \).

With regard to tractability, the \( L^2 \)-approximation of mixed order Sobolev functions is hence much harder for nonperiodic than for periodic functions. The negative tractability result for nonperiodic functions can be explained by the difficulty of approximating \( d \)-variate polynomials with degree 1 or less in each variable and \( H_{\text{mix}}^{1} \)-norm less than 1. The corresponding set of functions is contained in the unit ball of the nonperiodic space \( H_{\text{mix}}^{r} \) for every \( r \in \mathbb{N} \cup \{\infty\} \).
Remark 2.42. Corollary 2.41 for cubes of unit length is in accordance with the results of [PW10], where Papageorgiou and Woźniakowski prove the corresponding statement for the $L^2$-approximation in Sobolev spaces of mixed smoothness $(r_1, \ldots, r_d)$ on the unit cube. The smoothness of such functions increases from variable to variable, but the smoothness with respect to a fixed variable does not increase with the dimension. There, the authors raise the question for a characterization of spaces and their norms for which increasing smoothness yields polynomial tractability. Theorem 2.40 says that in the setting of uniformly increasing mixed smoothness, polynomial tractability is achieved, if and only if it leads to a polynomial decay of the second singular value of the univariate problem. It would be interesting to verify whether the same holds in the case of variable-wise increasing smoothness and to compute the exponents of strong polynomial tractability.

Remark 2.43 (Impact of the interval representation). The reason for the great sensibility of the tractability results for the periodic spaces to the length of the interval can be seen in the difficulty of approximating trigonometric polynomials with frequencies in $\{\frac{2\pi}{b-a} -1, 0, 1\}^d$ that are contained in the unit ball of $H_{\text{mix}}^\infty(\mathbb{T}^d)$. The corresponding set of functions is nontrivial, if and only if $\frac{2\pi}{b-a}$ is smaller than 1.

It may yet seem unnatural that the singular values are so sensible to the representation $[a, b]^d$ of the $d$-torus or the $d$-cube. This can only happen, since the above and common scalar products (2.26) do not define a homogeneous family of norms on $H_{\text{mix}}^r([a, b]^d)$. To see that, let $S$ be the embedding of $H_{\text{mix}}^r([a, b]^d)$ into $L^2([a, b]^d)$ and let $S_0$ be the embedding in the case $[a, b] = [0, 1]$. The dilation operation $Mf = f(a + (b - a)\cdot)$ defines a linear homeomorphism both from $L^2([a, b]^d)$ into $L^2([0, 1]^d)$ and from $H_{\text{mix}}^r([a, b]^d)$ into $H_{\text{mix}}^r([0, 1]^d)$ and we have $S_0 = MSM^{-1}$. The $L^2$-spaces satisfy the homogeneity relation

$$\|Mf\|_{L^2([0, 1]^d)} = (b - a)^{-d/2} \cdot \|f\|_{L^2([a, b]^d)} \quad \text{for} \quad f \in L^2([a, b]^d).$$

If the chosen family of norms on $H_{\text{mix}}^r([a, b]^d)$ is also homogeneous, i.e.

$$\|Mf\|_{H_{\text{mix}}^r([0, 1]^d)} = (b - a)^{-d/2} \cdot \|f\|_{H_{\text{mix}}^r([a, b]^d)} \quad \text{for} \quad f \in H_{\text{mix}}^r([a, b]^d),$$

the singular values of $S$ and $S_0$ clearly must coincide. The above scalar products do not yield a homogeneous family of norms. An example of an equivalent and homogeneous family of norms on $H_{\text{mix}}^r([a, b]^d)$ is given by the scalar products

$$\langle f, g \rangle = \sum_{\alpha \in \{0, \ldots, r \}^d} (b - a)^{|\alpha|} \langle D^\alpha f, D^\alpha g \rangle_{L^2([a, b]^d)}.$$

Hence, the singular values and tractability results with respect to this scalar product do not depend on $a$ and $b$ at all, both in the periodic and the nonperiodic case. They coincide with the singular values with respect to the scalar product (2.26) for the case $[a, b] = [0, 1]$.

2.3 Randomized Approximation in $L^2$

We want to approximate an unknown real or complex valued function $f$ on a set $D$ based on a finite number $n$ of function values which may be evaluated at randomly
and adaptively chosen points. In general, we cannot avoid to make an approximation error. We measure this error in the space $L^2(D, A, \mu)$ of square integrable functions on $D$ with respect to some measure $\mu$.

If we want to say anything about this error, we need to have some a priori knowledge of the function. For example, $D$ might be a compact manifold and we might know that $f$ is bounded with respect to some Sobolev norm on $D$. More generally, we may assume that the function can be approximated well with respect to some orthonormal system $B = \{b_1, b_2, \ldots\}$ in $L^2$. That is, there is a nonincreasing zero sequence $\varepsilon : \mathbb{N}_0 \to (0, \infty)$ such that the function is contained in

$$F_\varepsilon^c = \left\{ f \in L^2 \left| \left\| f - \sum_{j=1}^{m} \langle f, b_j \rangle b_j \right\|_2^2 \leq \varepsilon(m) \right. \text{ for all } m \in \mathbb{N}_0 \right\}.$$

The approximation is described by a random mapping $A_n : F_\varepsilon^c \to L^2$, which we call algorithm. The error of the algorithm $A_n$ is defined by

$$\text{err} \left( A_n, F_\varepsilon^c \right) = \sup_{f \in F_\varepsilon^c} \left( \mathbb{E} \left\| f - A_n(f) \right\|_2^2 \right)^{1/2}.$$

This is the worst mean squared error that can occur for the given a priori knowledge. The algorithm is called $n^{th}$ optimal, we write $A_n^*$ instead of $A_n$, if it satisfies

$$\text{err} \left( A_n^*, F_\varepsilon^c \right) = \inf_{A_n} \text{err} \left( A_n, F_\varepsilon^c \right),$$

where the infimum is taken over all algorithms $A_n$ that require at most $n$ function values of the unknown function.

It seems to be an unrealistic hope to find such algorithms $A_n^*$. Things look better if $n$ arbitrary pieces of linear information are allowed. The optimal deterministic algorithm that requires at most $n$ pieces of linear information is given by the orthogonal projection $P_n$ onto the span of the first $n$ functions in $B$. Its worst case error is the square-root of $\varepsilon(n)$. The algorithm $P_n$ asks for the first $n$ coefficients of $f$ with respect to the orthonormal system $B$.

In most applications, however, it is not possible to sample these coefficients and we may only make use of function values. This leads to the following questions:

- How does the error of $A_n^*$ compare to the error of $P_n$?
- Find an algorithm $A_n$ whose error is close to the error of $A_n^*$.

Note that $A_n^*$ cannot be much better than $P_n$. In 1992, Novak [Nov92] proved that

$$\text{err} \left( A_n^*, F_\varepsilon^c \right) \geq \frac{1}{\sqrt{2}} \text{err} \left( P_{2n-1}, F_\varepsilon^c \right),$$

see also Theorem 1.18. On the other hand, there are various examples where the error of the algorithm $A_n^*$ behaves similarly to the error of $P_n$, see for instance [CDL13, CM17, Hei94, Mat91, TWW88]. In 2006, Wasilkowski and Woźniakowski [WW06] proved for the general case that

$$\text{err} \left( P_n, F_\varepsilon^c \right) \approx n^{-p}(\ln n)^q \quad \Rightarrow \quad \text{err} \left( A_n^*, F_\varepsilon^c \right) \approx n^{-p}(\ln n)^q(\ln \ln n)^{p+1/2}.$$
for all $p > 0$ and $q \geq 0$. In that sense, $A_n^*$ is almost as good as $P_n$. The proof is constructive. Of course, we immediately wonder whether the additional power of the double logarithm is necessary. In 2012, Novak and Woźniakowski showed that this is not the case for $q = 0$, that is,

$$\text{err} \left( P_n, F_\mathcal{B}^\varepsilon \right) \lesssim n^{-p} \Rightarrow \text{err} \left( A_n^*, F_\mathcal{B}^\varepsilon \right) \lesssim n^{-p}$$

for all $p > 0$. The proof of this result is not constructive. Both proofs can be found in [NW12, Chapter 22]. In this section we prove the corresponding statement for $q > 0$. This solves Open Problem 99 as posed in [NW12].

More generally, we consider sequences with the property

$$\varepsilon(2^n) \asymp \varepsilon(n). \tag{2.28}$$

For any such sequence and any orthonormal system $\mathcal{B}$, we provide an algorithm $A_n$ and a constant $c_\varepsilon > 0$ such that, for all $n \in \mathbb{N}$, we have

$$\text{err} \left( A_n, F_\mathcal{B}^\varepsilon \right) \leq c_\varepsilon \text{err} \left( P_n, F_\mathcal{B}^\varepsilon \right),$$

see Theorem 2.48. Together with (2.27), this answers both questions from above: The errors of $A_n$ and $A_n^*$ only differ by a constant and we have

$$\text{err} \left( A_n^*, F_\mathcal{B}^\varepsilon \right) \asymp \text{err} \left( P_n, F_\mathcal{B}^\varepsilon \right).$$

The algorithm is a refinement of the algorithm proposed in [WW06]. Note that the constant $c_\varepsilon$ only depends on the equivalence constant of (2.28). This constant is usually independent of the dimension of the domain $D$.

These results are presented in Section 2.3.1. In Section 2.3.2 we consider several examples. In particular, we study the problem of approximating mixed order Sobolev functions in $L^2$ with randomized algorithms based on $\Lambda^{\text{std}}$, and obtain the optimal order of convergence for the $n$th minimal error, see Corollary 2.53.

In Section 2.3.3, we use these algorithms for the integration of functions $f$ in $F_\mathcal{B}^\varepsilon$ with respect to probability measures $\mu$. We simply exploit the relation

$$\int_D f \, d\mu = \int_D A_n f \, d\mu + \int_D (f - A_n f) \, d\mu.$$

We compute the integral of $A_n f$ precisely and use a direct simulation to approximate the integral of $f - A_n f$, which has a small variance. This technique is called control variates or separation of the main part and is widely used for Monte Carlo integration, see [Hei94, Theorem 5.3] for another example. The error of the resulting algorithm significantly improves on the error of a sole direct simulation, even if the number of samples is small and $D$ is a high-dimensional domain.

### 2.3.1 The Algorithm and its Error

Let $(D, \mathcal{A}, \mu)$ be a measure space and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The space $L^2 = L^2(D, \mathcal{A}, \mu)$ is the space of square integrable $\mathbb{K}$-valued functions on $(D, \mathcal{A}, \mu)$, equipped with the scalar product

$$\langle f, g \rangle_2 = \int_D f \, \overline{g} \, d\mu.$$
Let $\mathcal{B} = \{b_1, b_2, \ldots\}$ be an orthonormal system in $L^2$ and let $\mathcal{B} = \{b_1, b_2, \ldots\}$ be a nonincreasing zero-sequence. We consider the set

$$F_\mathcal{B}^\varepsilon = \left\{ f \in L^2 \left| \| f - \sum_{j=1}^{m} \langle f, b_j \rangle_2 b_j \|_2^2 \leq \varepsilon(m) \right. \right\}.$$

The functions in this set can be approximated well with respect to $\mathcal{B}$. In other words, they can be approximated well based on $m$ pieces of linear information. The goal is to show that they can be approximated just as well based on $n$ randomly chosen function values, where $n$ is not much larger than $m$.

We introduce some further notation. Let $S$ be the identity on $L^2$. For $m \in \mathbb{N}_0$, let $V_m$ be the span of the first $m$ elements of $\mathcal{B}$. By $P_m$ we denote the orthogonal projection onto $V_m$ in $L^2$, that is,

$$P_m : L^2 \to L^2, \quad P_m(f) = \sum_{j=1}^{m} \langle f, b_j \rangle_2 b_j.$$

The orthogonal projection onto the orthogonal complement of $V_m$ is denoted by $Q_m$. Note that $Q_m + P_m = S$. Moreover, we define the function

$$u_m = \frac{1}{m} \sum_{j=1}^{m} |b_j|^2.$$

This is a probability density with respect to $\mu$. We consider the probability measures

$$\mu_m : \mathcal{A} \to [0, 1], \quad \mu_m(E) = \int_E u_m \, d\mu.$$

on $(D, \mathcal{A})$. We now define a family of randomized algorithms based on function evaluations. Using the notions from Section 1.2, we study algorithms

$$A \in \mathcal{A}[L^2, L^2, \Lambda^{\text{std}}, \text{ran}].$$

Recall that the worst-case mean-square error of the randomized algorithm $A$ for the $L^2$-approximation of a function from $F \subset L^2$ is defined by

$$\text{err} (A, F)^2 = \text{err} \left( A, S, F, L^2, \text{wc} \right)^2 = \sup_{f \in F} \mathbb{E} \| f - A(f) \|_2^2$$

and that $\text{cost}(A, F) = \text{cost}(A, F, \Lambda^{\text{std}}, \text{wc})$ is the maximal number of function values that the algorithm requests about a problem instance $f \in F$.

**Algorithm 2.44.** Let $n$ and $m$ be sequences of nonnegative integers such that $m$ is nondecreasing. For every nonnegative integer $k$, we define

$$M_{n,m}^{(k)} : L^2 \to L^2,$$

by the following recursive scheme.

- For $f \in L^2$, we set $M_{n,m}^{(0)}(f) = 0$. 

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For \( k \geq 1 \) and \( f \in L^2 \), let \( X^{(k)}_1, \ldots, X^{(k)}_n \) be random variables with distribution \( \mu_{m_k} \) that are each independent of all the other random variables and set

\[
M^{(k)}_{n,m} f = M^{(k-1)}_{n,m} f + \sum_{j=1}^{m_k} \left[ \frac{1}{n_k} \sum_{i=1}^{n_k} \frac{(f - M^{(k-1)}_{n,m} f) b_j}{u_{m_k}} (X^{(k)}_i) \right] b_j.
\]

Note that the expectation of each term in the inner sum is

\[
\langle f - M^{(k-1)}_{n,m} f, b_j \rangle_2.
\]

The algorithm hence approximates \( f \) in \( k \) steps. In the first step, \( n_1 \) function values of \( f \) are used for standard Monte Carlo type approximations of its \( m_1 \) leading coefficients with respect to the orthonormal system \( \mathcal{B} \). In the second step, \( n_2 \) values of the residue are used for standard Monte Carlo type approximations of its \( m_2 \) leading coefficients and so on. In total, the algorithm uses

\[
\text{cost} \left( M^{(k)}_{n,m}, L^2 \right) = \sum_{j=1}^{k} n_j
\]

function values of \( f \). The total number of approximated coefficients is \( m_k \).

Algorithms of this type have already been studied by Wasilkowski and Woźniakowski in [WW06]. The simple but crucial difference with the above algorithms is the variable number \( n_j \) of nodes in each approximation step. Note that this stepwise approximation is similar to several multilevel Monte Carlo methods as introduced by Heinrich in 1998, see [Hei01].

The benefit from the \( k \)th step is controlled by \( m_k \) and \( n_k \) as shown by the following lemma. The lemma corresponds to Theorem 22.14 in [NW12]. The setting here is slightly more general, but the proof is almost the same.

**Lemma 2.45.** Algorithm 2.44 satisfies for all \( k \in \mathbb{N} \) that

\[
\varepsilon(m_k) \leq \text{err} \left( M^{(k)}_{n,m}, F^\varepsilon_\mathcal{B} \right)^2 \leq \frac{m_k}{n_k} \text{err} \left( M^{(k-1)}_{n,m}, F^\varepsilon_\mathcal{B} \right)^2 + \varepsilon(m_k).
\]

**Proof.** We start with the lower bound. We consider the function

\[
f = \sqrt{\varepsilon(m_k)} \cdot b_{m_k+1} \in F^\varepsilon_\mathcal{B}.
\]

Note that \( M^{(k)}_{n,m}(f) \) is contained in \( V_{m_k} \) and hence

\[
\|f - M^{(k)}_{n,m}(f)\|_2^2 \geq \|f - P_{m_k} f\|_2^2 = \|f\|_2^2 = \varepsilon(m_k)
\]

for any realization of \( M^{(k)}_{n,m} \). This yields the lower bound.

We turn to the upper bound. Let \( f \in F^\varepsilon_\mathcal{B} \). Let us first fix a realization of \( M^{(k)}_{n,m} \). We have

\[
\|f - M^{(k)}_{n,m}(f)\|_2^2 = \|P_{m_k}(f) - M^{(k)}_{n,m}(f)\|_2^2 + \|Q_{m_k}(f)\|_2^2.
\]
Recall that the second term is bounded by $\varepsilon(m_k)$. The first term satisfies

$$\| P_{m_k}(f) - M^{(k)}_{n,m}(f) \|^2 = \sum_{j=1}^{m_k} \left\| f - M^{(k)}_{n,m} f, b_j \right\|^2.$$ 

We turn back to the randomized setting. For $j \leq m_k$, we use the abbreviation

$$g_j = \frac{1}{u_{m_k}} \left( f - M^{(k-1)}_{n,m} f \right) b_j.$$

Note that $u_{m_k} = 0$ implies $b_j = 0$ and we set $g_j = 0$ in this case. Let $E_k$ denote the expectation with respect to the random variables $X_i^{(k)}$ for $i \leq n_k$. We obtain

$$E_k \left| \left( f - M^{(k)}_{n,m} f, b_j \right) \right|^2 = E_k \left| \left( f - M^{(k-1)}_{n,m} f, b_j \right) - \frac{1}{n_k} \sum_{i=1}^{n_k} g_j \left( X_i^{(k)} \right) \right|^2$$

$$= E_k \left| \int_D g_j(x) \, d\mu_{m_k}(x) - \frac{1}{n_k} \sum_{i=1}^{n_k} g_j \left( X_i^{(k)} \right) \right|^2 \leq \frac{1}{n_k} \int_D |g_j(x)|^2 \, d\mu_{m_k}(x)$$

and hence

$$E_k \sum_{j=1}^{m_k} \left| \left( f - M^{(k)}_{n,m} f, b_j \right) \right|^2 \leq \frac{1}{n_k} \int_D \sum_{j=1}^{m_k} |g_j(x)|^2 \, d\mu_{m_k}(x) \, d\mu(x)$$

$$= \frac{m_k}{n_k} \int_D \left| \left( f - M^{(k-1)}_{n,m} f \right)(x) \right|^2 \, d\mu(x) = \frac{m_k}{n_k} \left\| f - M^{(k-1)}_{n,m} f \right\|^2.$$ 

With Fubini’s theorem this yields that

$$E \left\| f - M^{(k)}_{n,m} f \right\|^2 \leq \frac{m_k}{n_k} E \left\| f - M^{(k-1)}_{n,m} f \right\|^2 + \varepsilon(m_k)$$

and the upper bound is proven. \(\square\)

Based on this error formula, we now tune the parameters of Algorithm 2.44.

**Proposition 2.46 (Kri18c).** Let $m_j = 2^{j-1}$ and $n_j = 2^j \lceil \varepsilon(2^j - 1) / \varepsilon(2^{j-1}) \rceil$ for all $j \in \mathbb{N}$. Then Algorithm 2.44 satisfies for all $k \in \mathbb{N}_0$ that

- $\text{err} \left( M^{(k)}_{n,m}, F^e_B \right)^2 \leq 2 \varepsilon \left( \left\lfloor 2^{k-1} \right\rfloor \right)$.

- $\text{cost} \left( M^{(k)}_{n,m}, L^2 \right) \leq 2^{k+1} \max_{0 \leq j < k} \left\lfloor \frac{\varepsilon \left( \left\lfloor 2^{j-1} \right\rfloor \right)}{\varepsilon(2^j)} \right\rfloor$.

**Proof.** The second estimate is obvious from (2.29). The first estimate follows from Lemma 2.45 by induction on $k$. For $k = 0$, we have $M^{(k)}_{n,m} = 0$ and hence

$$\text{err} \left( M^{(k)}_{n,m}, F^e_B \right)^2 = \sup_{f \in F^e_B} \| f \|^2 = \varepsilon(0).$$

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If the statement holds for all \( k < k' \) with some \( k' \in \mathbb{N} \), Lemma 2.45 yields

\[
\text{err} \left( M_{n,m}^{(k')} F_{B}^{\varepsilon} \right)^{2} \leq \frac{m_{k'}}{n_{k'}} \text{err} \left( M_{n,m}^{(k'-1)}, F_{B}^{\varepsilon} \right)^{2} + \varepsilon(m_{k'}) \\
\leq \frac{\varepsilon \left( 2^{k'-1} \right)}{2 \varepsilon \left( \left\lfloor 2^{k'-2} \right\rfloor \right)^{2}} \left( \left\lfloor 2^{k'-2} \right\rfloor \right) + \varepsilon \left( 2^{k'-1} \right) = 2 \varepsilon \left( 2^{k'-1} \right) = 2 \varepsilon \left( \left\lfloor 2^{k'-1} \right\rfloor \right)
\]

and the proof by induction is complete.

For many sequences the maximum in the cost bound of Proposition 2.46 is bounded by a constant, that is,

\[
\sup_{j \in \mathbb{N}_{0}} \frac{\varepsilon \left( \left\lfloor 2^{j-1} \right\rfloor \right)}{\varepsilon \left( 2^{j} \right)} < \infty.
\] (2.30)

In this case, Proposition 2.46 says that we may achieve an error of order \( \varepsilon(n) \) for every \( n \in \mathbb{N} \) using only \( n \) function values of the target function. To make this precise, we define the following instance of Algorithm 2.44.

**Algorithm 2.47.** Let \( \mathcal{B} \) be an orthonormal system in \( L^2 \) and let \( \varepsilon : \mathbb{N}_{0} \to (0, \infty) \) be a nonincreasing zero sequence. For any \( n \in \mathbb{N}_{0} \), we consider the algorithm \( A_{n} = M_{n,m}^{(k)} \) as defined in Algorithm 2.44 for the following parameters:

- \( m_{j} = 2^{j-1} \) for all \( j \in \mathbb{N} \).
- \( n_{j} = 2^{j} \left\lceil \frac{\varepsilon(\lfloor 2^{j-2} \rfloor)}{\varepsilon(2^{j-1})} \right\rceil \) for all \( j \in \mathbb{N} \).
- \( k \in \mathbb{N}_{0} \) maximal such that \( \sum_{j=1}^{k} n_{j} \leq n \).

Note that the randomized algorithm \( A_{n} \) requires at most \( n \) function values of every input, see (2.29). We obtain the following.

**Theorem 2.48 \([\text{Kri18c}]\).** Let \( \varepsilon : \mathbb{N}_{0} \to (0, \infty) \) be a nonincreasing zero sequence that satisfies (2.30) and let \( \mathcal{B} \) be an orthonormal system in \( L^2 \). We put

\[
C = 2^{\ell+3} + 1,
\]

with \( \ell = \min \left\{ r \in \mathbb{N}_{0} \mid \frac{\varepsilon(\lfloor 2^{j-1} \rfloor)}{\varepsilon(2^{j})} \leq 2^{r} \text{ for all } j \in \mathbb{N}_{0} \right\} \).

Then Algorithm 2.47 satisfies for all \( n \in \mathbb{N} \) that cost \( (A_{n}) \leq n \) and

\[
\text{err} \left( A_{n}, F_{B}^{\varepsilon} \right)^{2} \leq C \varepsilon(n).
\]

**Proof.** Proposition 2.46 and our assumption yield that

\[
\text{err} \left( A_{n}, F_{B}^{\varepsilon} \right)^{2} \leq 2 \cdot \varepsilon \left( \left\lfloor 2^{k-1} \right\rfloor \right) \leq 2 \cdot 2^{\ell+3} \varepsilon \left( 2^{k+\ell+2} \right) \leq C \varepsilon(n),
\]

where the last inequality follows from \( n < \sum_{j=1}^{k+1} n_{j} \leq 2^{k+\ell+2} \).

Note that the constant \( C \) in Theorem 2.48 is usually rather harmless. We will consider several examples in Section 2.3.2 In all these examples, \( F_{B}^{\varepsilon} \) will be a class of \( d \)-variate functions and the constant will be independent of \( d \). Let us consider two sequences \( \varepsilon \) that satisfy the assumption of Theorem 2.48. 

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Example 2.49. Let $c \geq 1$, $p > 0$ and $q \geq 0$. Property (2.30) is satisfied by the sequence $\varepsilon : \mathbb{N}_0 \to (0, \infty)$ with $\varepsilon(0) = 1$ and

$$\varepsilon(n) = \min \left\{ 1, c n^{-p} (1 + \log_2 n)^q \right\}, \quad n \geq 1.$$ 

Another example is given by $\varepsilon(0) = 1$ and

$$\varepsilon(n) = \min \left\{ 1, c (1 + \log_2 n)^{-p} \right\}, \quad n \geq 1.$$ 

In both cases we have the estimate

$$\sup_{j \in \mathbb{N}_0} \varepsilon \left( \lfloor 2^j - 1 \rfloor \right) \leq 2^p \leq 2^\lceil p \rceil.$$ 

Thus Theorem 2.48 can be applied with $\ell = \lceil p \rceil$. Property (2.30) is not satisfied if the sequence decays exponentially or if it has big jumps.

Remark 2.50 (Less a priori knowledge). We assumed that our target function $f$ satisfies

$$\| f - P_n f \|_2^2 \leq \varepsilon(n) \quad (2.31)$$

for all $n \in \mathbb{N}_0$ for some $\varepsilon : \mathbb{N}_0 \to (0, \infty)$ with $\varepsilon(2n) \asymp \varepsilon(n)$ and proved that randomized algorithms can achieve a squared error of order $\varepsilon(n)$ with a sample size of order $n$. Our algorithm depends on $\varepsilon$. If we do not know an admissible upper bound $\varepsilon$, we can still achieve a squared error of order $\| f - P_n f \|_2^2$ using a weighted least squares method, see [CM17, Theorem 2.1 (iv)]. The sample size of this method, however, is at least of order $n \ln n$. Here it is assumed that $D$ is a Borel subset of $\mathbb{R}^d$ with positive Lebesgue measure, $\mathcal{A}$ is the Borel sigma algebra on $D$ and $\mu$ is a probability measure on $(D, \mathcal{A})$. Again, the involved proportionality constants are independent of the dimension of the domain $D$.

### 2.3.2 Approximation of Functions from a Hilbert Space

An important application of Theorem 2.48 is the $L^2$-approximation of functions from a Hilbert space. Let $\tilde{F}$ be an infinite-dimensional Hilbert space that is compactly embedded into $L^2 = L^2(D, \mathcal{A}, \mu)$. From Section 1.2.5, we know that there is a countable orthogonal basis

$$\mathcal{B} = \{ b_1, b_2, \ldots \}$$

of $\tilde{F}$ such that $\mathcal{B}$ is orthonormal in $L^2$ and the sequence $\varepsilon : \mathbb{N}_0 \to (0, \infty)$ with

$$\varepsilon(n) = \| b_{n+1} \|_{\tilde{F}}^{-2}$$

is a nonincreasing zero sequence. Let $F$ be the unit ball of $\tilde{F}$. For every $n \in \mathbb{N}_0$, the linear algorithm

$$A_n : F \to L^2, \quad A_n(f) = \sum_{j=1}^{n} \langle f, b_j \rangle_2 b_j,$$

1We point to the fact that the elements of $\mathcal{B}$ are normalized in $L^2$. In Section 1.2.5, we normalized the functions in $\tilde{F}$. 62
is the optimal deterministic algorithm for the problem of approximating functions from \( F \) with \( n \) pieces of linear information. It satisfies

\[
\text{err} (A_n, F)^2 = \varepsilon(n) .
\]

(2.32)

In other words, we have

\[
F \subset F^\varepsilon_B ,
\]

(2.33)

and we can apply Algorithm 2.47. In particular, we obtain the following result on the order of convergence.

**Theorem 2.51** ([Kri18c]). Let \( F \) be the unit ball of an infinite-dimensional Hilbert space that is compactly embedded into some \( L^2 \)-space. If the singular values \( \sigma : \mathbb{N} \to (0, \infty) \) of the embedding APP satisfy \( \sigma(2n) \asymp \sigma(n) \) then

\[
e(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^\text{all}, \text{ran}, \text{wc}]) \asymp e(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{std}}, \text{ran}, \text{wc}]) .
\]

**Proof.** The first relation follows from Theorem 1.18. The second relation follows from Theorem 2.48. Note that we use the condition \( \sigma(2n) \asymp \sigma(n) \) for both these relations. \( \square \)

This means that for the problem of approximating functions from a Hilbert space in \( L^2 \), randomized algorithms based on function values are just as powerful as randomized or deterministic algorithms based on arbitrary linear information. Note that deterministic algorithms based on function values may perform much worse, as shown by Hinrichs, Novak and Vybíral [HNV08], see also [NW12, Section 26.6.1].

We do not know whether the condition on the decay of the singular values can be relaxed.

**Remark 2.52.** We point to the fact that the error bounds of Proposition 2.46 and Theorem 2.48 do not only hold for the class \( F \) but for the whole class \( F^\varepsilon_B \). In general, the second class is strictly larger than the first. For example, if \( \varepsilon(m) = 1/(m + 1)^2 \) for \( m \in \mathbb{N}_0 \), then

\[
f = \sum_{m \in \mathbb{N}} (\varepsilon(m - 1) - \varepsilon(m))^{1/2} b_m
\]

belongs to \( F^\varepsilon_B \) but is not even contained in the space \( \tilde{F} \).

We now consider several examples. In each example, we first determine the order of convergence of the \( n^{\text{th}} \) minimal error. We then discuss explicit upper bounds.

**Functions with Mixed Smoothness on the Torus**

Let \( D \) be the \( d \)-dimensional torus \( \mathbb{T}^d \), represented by the unit cube \([0,1]^d\), where opposite faces are identified. Let \( \mathcal{A} \) be the Borel \( \sigma \)-algebra on \( \mathbb{T}^d \) and \( \mu \) the Lebesgue measure. Let \( \tilde{F} \) be the Sobolev space of complex valued functions on \( D \) with mixed smoothness \( r \in \mathbb{N} \), equipped with the scalar product

\[
\langle f, g \rangle_{\tilde{F}} = \sum_{\|\alpha\|_\infty \leq r} \langle D^\alpha f, D^\alpha g \rangle_2 .
\]

(2.34)
A classical result by Babenko \cite{Bab60} and Mityagin \cite{Mit62} states that
\[
e(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{all}}, \det, \text{wc}]) \asymp n^{-r} \ln^{r(d-1)} n.
\] (2.35)
We remark that the same can be proven for fractional smoothness \(r > 0\), see \cite{Mit62}.

Theorem 2.51 yields that
\[
e(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{std}}, \text{ran}, \text{wc}]) \asymp n^{-r} \ln^{r(d-1)} n.
\]

This result is new. The optimal order is achieved by Algorithm 2.47 with \(\varepsilon(0) = 1\) and
\[
\varepsilon(n) = \min \left\{1, c^2 n^{-2r} (1 + \log_2 n)^{2r(d-1)} \right\}
\]
for \(n \geq 1\), where \(c\) is the constant in the upper bound of (2.35). We do not know any other algorithm with this property. It is still an open problem whether the same rate can be achieved with deterministic algorithms based on function values. So far, it is only known that
\[
n^{-r} \ln^{r(d-1)} n \asymp e(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{std}}, \det, \text{wc}]) \asymp n^{-r} \ln^{(r+1/2)(d-1)} n.
\]
The upper bound is achieved by Smolyak’s algorithm, see \cite{SU10}.

We now turn to explicit estimates. We know that there is some \(C_{r,d} > 0\) such that
\[
e(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{std}}, \text{ran}, \text{wc}]) \leq C_{r,d} n^{-r} \ln^{r(d-1)} n
\] (2.36)
for all \(n \geq 2\). This upper bound is optimal as \(n\) tends to infinity. However, it is not useful to describe the error numbers for small values of \(n\). Simple calculus shows that the right hand side in (2.36) is increasing for \(n \leq e^{d-1}\). The error numbers, on the other hand, are decreasing. Moreover, the right hand side attains its minimum for \(n = 2\) if restricted to \(n \leq (d-1)^{d-1}\) and is hence larger than the error for \(n = 2\). This means that the trivial upper bound
\[
e(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{std}}, \text{ran}, \text{wc}]) \leq e(2, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{std}}, \text{ran}, \text{wc}])
\] (2.37)
is better than (2.36) for all \(n \in \{2, \ldots, (d-1)^{d-1}\}\) for any valid constant \(C_{r,d}\). For these reasons, it is important to consider different error bounds, if the dimension \(d\) is large. Based on \cite{KSU15}, we already proved that the upper bound
\[
e(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{std}}, \text{ran}, \text{wc}]) \leq (2/n)^p \quad \text{with} \quad p = \frac{r}{2 + (\ln d)/(\ln 2\pi)}
\]
holds for all \(n \in \mathbb{N}\). See Corollary 2.32 for the parameters \([a, b] = [0, 1], \gamma = 1, \square = \circ\) and \(\delta = 1\). By Theorem 2.48, Algorithm 2.47 with \(\varepsilon(n) = \min \{1, 2^{2r} n^{-2r}\}\) satisfies
\[
\text{err}(A_n, F) \leq 2^{(\ell^2 + 4d + 1)/2} \cdot n^{-p}
\] (2.38)
for all \(n \in \mathbb{N}\), where \(\ell = \lceil 2p \rceil\) is nonincreasing in \(d\). See Example 2.49 for details on the constant. For instance, let \(r = 6\) and \(d = 2000\). Then (2.36) is useless up to \(n = 10^{6598}\), whereas (2.38) yields the upper bound
\[
\text{err}(A_n, F) \leq 91 \cdot n^{-0.97},
\]
which is useful for all \(n \geq 105\).
2.3. Randomized Approximation in $L^2$

Functions with Mixed Smoothness on the Cube

Let $D$ be the $d$-dimensional unit cube $[0,1]^d$ equipped with the Borel $\sigma$-algebra $\mathcal{A}$ and the Lebesgue measure $\mu$. Let $\hat{F}$ be the Sobolev space of complex valued functions on $[0,1]^d$ with mixed smoothness $r \in \mathbb{N}$, equipped with the scalar product (2.34). Just like on the torus, we have

$$e\left(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{all}}, \det, \text{wc}]\right) \asymp n^{-r} \ln^{(d-1)} n$$

and Theorem 2.48 leads to the following result.

**Corollary 2.53** ([Kri18c]). Let $F$ be the unit ball of the Sobolev space of mixed smoothness $r$ on the $d$-torus or on the $d$-cube. Then

$$e\left(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{std}}, \text{ran}, \text{wc}]\right) \asymp n^{-r} \ln^{(d-1)} n.$$ 

Again, the optimal rate can be achieved with Algorithm 2.47. Also in this case, the corresponding upper bounds are bad for $n \leq (d-1)^{d-1}$. In this range, we need different estimates for the error. By Corollary 2.38 we know that

$$e\left(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{all}}, \det, \text{wc}]\right) \leq (2/n)^p \quad \text{with} \quad p = \frac{1.1929}{2 + \ln d}$$

for all $n \in \mathbb{N}$. This estimate cannot be improved significantly for $n \leq 2^d$. By Theorem 2.48 Algorithm 2.47 with $\varepsilon(n) = \min\{1, 2^p n^{-2p}\}$ satisfies for all $n \in \mathbb{N}$ and $d \geq 2$ that

$$\text{err} (A_n, F) \leq 8 n^{-p}. \quad (2.39)$$

Functions from Tensor Product Spaces

This example is more general than the previous examples. By $H_1 \otimes H_2$ we denote the tensor product of two Hilbert spaces $H_1$ and $H_2$. For $j = 1, \ldots, d$ let $(D_j, \mathcal{A}_j, \nu_j)$ be a $\sigma$-finite measure space and let $\hat{F}_j$ be a Hilbert space of $\mathbb{K}$-valued functions with unit ball $F_j$ such that the embedding $\text{APP}_j$ of $\hat{F}_j$ into $L^2(D_j, \mathcal{A}_j, \nu_j)$ is compact. The $\sigma$-finiteness of the measure spaces ensures that

$$L^2(D_1, \mathcal{A}_1, \nu_1) \otimes \cdots \otimes L^2(D_d, \mathcal{A}_d, \nu_d) = L^2(D, \mathcal{A}, \mu),$$

where $D$ is the Cartesian product of the sets $D_j$ and $\mu$ is the unique product measure of the measures $\nu_j$ on the tensor product $\mathcal{A}$ of the $\sigma$-algebras $\mathcal{A}_j$. The embedding $\text{APP}$ of the tensor product space $\hat{F} = \hat{F}_1 \otimes \cdots \otimes \hat{F}_d$ into $L^2(D, \mathcal{A}, \mu)$ is compact, too. Assuming that the approximation numbers of the univariate embeddings $\text{APP}_j$ are of polynomial decay, that is,

$$e\left(n, \mathcal{P}[\text{APP}_j, F_j, L^2(D_j, \mathcal{A}_j, \nu_j), \Lambda^{\text{all}}, \det, \text{wc}]\right) \asymp n^{-r_j}$$

for some $r_j > 0$, it can be derived from [Mit62, Nik74] that

$$e\left(n, \mathcal{P}[\text{APP}, F, L^2(D, \mathcal{A}, \mu), \Lambda^{\text{all}}, \det, \text{wc}]\right) \asymp n^{-r} \ln^{(d_0-1)} n,$$
where $F$ is the unit ball of $\mathcal{F}$, $r$ is the minimum among all numbers $r_j$ and $d_0$ is its multiplicity. Theorem 2.51 implies

$$e\left(n, \mathcal{P}[\text{APP}, F, L^2(D, \mathcal{A}, \mu), \Lambda^{std}, \text{ran}, \text{wc}] \right) \asymp n^{-r} \ln^{r(d_0-1)} n,$$

where the optimal order can be achieved with Algorithm 2.47. We do not discuss explicit estimates in this general setting.

Functions with Isotropic Smoothness on the Torus

Our algorithm may also be used for functions with isotropic smoothness. Let $D$ again be the $d$-torus, this time represented by $[0, 2\pi]^d$. Let $\mathcal{F}$ be the Sobolev space of complex valued functions on $D$ with isotropic smoothness $r \in \mathbb{N}$, equipped with the scalar product

$$\langle f, g \rangle_{\mathcal{F}} = \sum_{\|\alpha\| \leq r} \langle D^\alpha f, D^\alpha g \rangle_2.$$ 

This is not a tensor product problem. For this classical problem, it is known that

$$e\left(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{std}, \text{det}, \text{wc}] \right) \asymp e\left(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{all}, \text{det}, \text{wc}] \right) \asymp e\left(n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{all}, \text{ran}, \text{wc}] \right) \asymp n^{-r/d}$$

for $r > d/2$. In the case $r \leq d/2$, where function values are only defined almost everywhere, the last three relations stay valid. We refer to [Hei08, Jer67, Mat91, Tri05]. In the range $n \leq 2^d$, however, the function $n^{-r/d}$ is not suited to describe the behavior of the errors. It has been proven by Kühn, Mayer and Ullrich [KMU16] that there are positive constants $b_r$ and $B_r$ that do not depend on $d$ such that

$$b_r \left( \frac{\log_2 (1 + d/ \log_2 n)}{\log_2 n} \right)^{r/2} \leq e\left(n - 1, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{all}, \text{det}, \text{wc}] \right) \leq B_r \left( \frac{\log_2 (1 + d/ \log_2 n)}{\log_2 n} \right)^{r/2} \quad (2.40)$$

for all $d > 1$ and $n \in \mathbb{N}$ with $d \leq n \leq 2^d$. If we apply Theorem 1.18 and Theorem 2.48 we obtain the existence of $d$-independent positive constants $\tilde{b}_r$ and $\tilde{B}_r$ such that

$$\tilde{b}_r \left( \frac{\log_2 (1 + d/ \log_2 n)}{\log_2 n} \right)^{r/2} \leq e\left(n - 1, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{std}, \text{ran}, \text{wc}] \right) \leq \tilde{B}_r \left( \frac{\log_2 (1 + d/ \log_2 n)}{\log_2 n} \right)^{r/2}$$

for all $d > 1$ and $n \in \mathbb{N}$ with $d \leq n \leq 2^{d-1}$. 

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Implementation of these algorithms

We are able to implement Algorithm 2.47 if we know the basis $\mathcal{B}$ that is associated with the embedding of $\mathcal{F}$ into $L^2$ and if we can sample from the probability measures $\mu_m$. These tasks may be very hard. In the case of Sobolev functions on the torus, however, it is not. Here, $\mathcal{B}$ is the Fourier basis of $L^2$ and all the random variables are independent and uniformly distributed on the unit cube. Also the case of general tensor product spaces can be handled if the orthonormal bases $\mathcal{B}_j$ that are associated with the univariate embeddings are known. Then the basis $\mathcal{B}$ is given by

$$\mathcal{B} = \{ b^{(1)} \otimes \cdots \otimes b^{(d)} \mid b^{(j)} \in \mathcal{B}_j \text{ for } j = 1 \ldots d \}$$

and the probability measure $\mu_m$ is the average of $m$ product measures, that is,

$$\mu_m = \frac{1}{m} \sum_{i=1}^{m} \bigotimes_{j=1}^{d} \eta_{i,j},$$

where $d\eta_{i,j} = |b_{i,j}|^2 d\nu_j$ with some $b_{i,j} \in \mathcal{B}_j$. A random sample $x$ from this distribution can be obtained as follows:

1. Get $i$ from the uniform distribution on $\{1, \ldots, m\}$.
2. Get $x_1, \ldots, x_d$ independently from the probability distributions $\eta_{i,1}, \ldots, \eta_{i,d}$.

The second step can for example be done by rejection sampling, if the measures $\eta_{i,j}$ have a bounded Lebesgue density. This way, the total sampling costs are linear in $d$.

Another method of sampling from $\mu_m$ is proposed in [CM17, Section 5].

2.3.3 Integration via Separation of the Main Part

We use the notation of Section 2.3.1. In this section, we require the measure $\mu$ to be finite. This ensures that the integral operator

$$\text{INT} : L^2 \to \mathbb{K}, \quad \text{INT}(f) = \int_D f \, d\mu$$

is well defined and continuous on $L^2$. Let us assume that $\mu$ is a probability measure. We want to approximate $\text{INT}(f)$ for an unknown function $f \in F_\varepsilon^\varepsilon$ by a randomized algorithm $Q_n$, which evaluates at most $n$ function values of $f$. Recall that the worst case error of $Q_n$ is the quantity

$$\text{err} \left( Q_n, \text{INT}, F_\varepsilon^\varepsilon \right) = \sup_{f \in F_\varepsilon^\varepsilon} \left( \mathbb{E} |\text{INT}(f) - Q_n(f)|^2 \right)^{1/2}.$$ 

The minimal worst case error among such algorithms is denoted by

$$e \left( n, \mathcal{P}[\text{INT}, F_\varepsilon^\varepsilon, \Lambda^{\text{std}}, \text{ran}, \text{wc}] \right) = \inf_{Q_n} \text{err} \left( Q_n, \text{INT}, F_\varepsilon^\varepsilon \right).$$

Like any approximation method, Algorithm 2.47 can also be used for integration.
Algorithm 2.54. Let $\mathcal{B}$ be an orthonormal system in $L^2$ and let $\varepsilon : \mathbb{N}_0 \to (0, \infty)$ be a nonincreasing zero sequence. For all $n \in \mathbb{N}$ and $f \in L^2$, let

$$Q_{2n}(f) = \text{INT}(A_n f) + \frac{1}{n} \sum_{j=1}^{n} (f - A_n f) (X_j),$$

where $A_n$ is defined in Algorithm 2.47 and $X_1, \ldots, X_n$ are random variables with distribution $\mu$ which are independent of each other and the random variables in $A_n$.

It is easy to verify that $Q_{2n}$ is unbiased, evaluates at most $2n$ function values of $f$ and satisfies

$$\mathbb{E} |\text{INT}(f) - Q_{2n}(f)|^2 \leq \frac{1}{n} \mathbb{E} \|f - A_n f\|^2_2$$

for each $f$ in $L^2$. Thus we obtain the following corollary.

Corollary 2.55 ([Kri18c]). Let $\mathcal{B}$ be an orthonormal system in $L^2$ and let $\varepsilon : \mathbb{N}_0 \to (0, \infty)$ be a nonincreasing zero sequence. We assume that

$$\sup_{j \in \mathbb{N}_0} \left[ \frac{\varepsilon(\lfloor 2^j - 1 \rfloor)}{\varepsilon(2^j)} \right] \leq 2^\ell$$

for some $\ell \in \mathbb{N}_0$. For any $n \in \mathbb{N}$, Algorithm 2.54 satisfies

$$\text{err} (Q_{2n}, \text{INT}, F^\varepsilon_{\mathcal{B}})^2 \leq 2^{\ell+3\ell+1} \varepsilon(n)n^{-1}.$$

In particular, if $F$ is the unit ball of a Hilbert space that is compactly embedded in $L^2$, we obtain for all $p > 0$ and $q \geq 0$ that

$$e \left( n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{all}}, \text{det}, \text{wc}] \right) \precsim n^{-p} \ln^q n$$

$$\Rightarrow e \left( n, \mathcal{P}[\text{INT}, F, \Lambda^{\text{std}}, \text{ran}, \text{wc}] \right) \precsim n^{-p-1/2} \ln^q n.$$

The result on the order of convergence is quite general but not always optimal. An example is given by integration with respect to the Lebesgue measure $\mu$ on the Sobolev space $\overline{F}$ with mixed smoothness $r$ on the $d$-dimensional unit cube, as treated in Section 2.1. In this case, we have

$$e \left( n, \mathcal{P}[\text{APP}, F, L^2, \Lambda^{\text{all}}, \text{det}, \text{wc}] \right) \precsim n^{-r} \ln^{r(d-1)} n,$$

$$e \left( n, \mathcal{P}[\text{INT}, F, \Lambda^{\text{std}}, \text{ran}, \text{wc}] \right) \precsim n^{-r-1/2},$$

see [Bab60, Mit62, Ull17], respectively (2.35) and Corollary 2.2. The main strength of Corollary 2.55 is that it provides an unbiased algorithm achieving a reasonable integration error with a modest number of function values in high dimensions.

Example 2.56 (Functions with mixed smoothness on the torus). Like in the first example of Section 2.3.2, let $\overline{F}$ be the Sobolev space of mixed smoothness $r$ on the $d$-torus and let $\mu$ be the Lebesgue measure. Among all randomized algorithms for multivariate integration in $\overline{F}$ the randomized Frolov algorithm $Q_n^*$ is known to have
the optimal error rate, see Theorem 2.1. It is shown by Ullrich [Ull17] that there is some constant $c > 2^d$ such that

$$\text{err} \left( Q^*_n, \text{INT}, F \right) \leq cn^{-r-1/2} \quad (2.41)$$

for all $n \in \mathbb{N}$. However, this estimate is trivial for $n \leq 2^{d/(r+1/2)}$. In this range, an error less than one is guaranteed by the direct simulation

$$S_n(f) = \frac{1}{n} \sum_{j=1}^{n} f(X_j),$$

with independent and uniformly distributed random variables $X_j$. It satisfies

$$\text{err} \left( S_n, \text{INT}, F \right) \leq n^{-1/2} \quad (2.42)$$

for all $n \in \mathbb{N}$. However, this error bound converges only slowly, as $n$ tends to infinity. It does not reflect the smoothness of the integrands at all. Algorithm 2.54 also guarantees nontrivial error bounds for smaller values of $n$, but converges faster than $S_n$. Relation (2.38) immediately yields that

$$\text{err} \left( Q_{2n}, \text{INT}, F \right) \leq 2^{(\ell^2+4\ell+1)/2} \cdot n^{-p-1/2} \quad (2.43)$$

for all $n \in \mathbb{N}$, where $p = r/(2 + (\ln d)/((\ln 2\pi)))$ and $\ell = \lceil 2p \rceil$. For the example $d = 2000$ and $r = 6$, we obtain

$$\text{err}(Q_{2n}, \text{INT}, F) \leq 91 \cdot n^{-1.47}.$$ 

For one million samples, the estimate (2.41) for Frolov’s algorithm is larger than one, the estimate (2.42) for the direct simulation gives the error $10^{-3}$ and the estimate (2.43) for our new algorithm gives an error smaller than $4 \times 10^{-7}$.

**Remark 2.57 (Implementation).** We are able to implement Algorithm 2.54 under the following assumptions:

- We can implement Algorithm 2.47. This issue is discussed in Section 2.3.2.
- We know the integrals $\text{INT}(b_j)$ of the eigenfunctions $b_j \in B$ for all $j \leq n$.
- We can sample from the probability distribution $\mu$.

In the above example, the implementation is comparably easy, since $B$ is the Fourier basis and all the random variables are independent and uniformly distributed on the unit cube.
Chapter 3

Tractability of the Uniform Approximation Problem

We study the task of approximating a function \( f : [0,1]^d \to \mathbb{R} \) in the uniform norm with a deterministic scheme based on a finite number of function values. As a priori knowledge, we assume that the function is contained in some class \( F \) of smooth functions. In the notation of Chapter 1 we consider the problem

\[ \mathcal{P}[\text{APP}, F] = \mathcal{P}[\text{APP}, F, B([0,1]^d), \Lambda^{std}, \text{det}, \text{wc}], \]

where APP : \( F \to B([0,1]^d) \) is given by APP\((f) = f\) and \( B([0,1]^d) \) is the set of bounded real valued functions on \([0,1]^d\). We first study the classes \( F = \mathcal{C}^r_{\sigma} \) of real-valued functions on \([0,1]^d\) whose partial derivatives up to order \( r \in \mathbb{N} \) are continuous and bounded by 1, see Section 3.1. We derive new results on the complexity of the approximation problem and compare with known results on the complexity of the corresponding integration problem

\[ \mathcal{P}[\text{INT}, F] = \mathcal{P}[\text{INT}, F, \mathbb{R}, \Lambda^{std}, \text{det}, \text{wc}], \]

where INT : \( F \to \mathbb{R} \) is given by INT\((f) = \int_{[0,1]^d} f(x) \, dx\). For both problems, the complexity grows super-exponentially with the dimension. In particular, the problems suffer from the curse of dimensionality. This section is based on \([\text{Kri19}]\). Section 3.2 is based on \([\text{KR19}]\). We show that the curse of dimensionality can be avoided if \( F \) is a class of rank one tensors. The same observations hold for the problem of global optimization on \( F \), as explained in Section 3.3. Finally, Section 3.4 is concerned with the problem of dispersion, which is closely related to the uniform approximation of rank one tensors. This section is based on \([\text{Kri18b}]\).

### 3.1 Smooth Functions

It is known that the integration of functions from the class

\[ \mathcal{C}^r_{\sigma} = \{ f \in \mathcal{C}^r ([0,1]^d) \mid \| D^\beta f \|_\infty \leq 1 \text{ for all } \beta \in \mathbb{N}_0^d \text{ with } |\beta| \leq r \} \]

suffers from the curse of dimensionality. In fact, the minimal number \( n(\varepsilon, \mathcal{P}[\text{INT}, \mathcal{C}^r_{\sigma}]) \) of function values that is needed to guarantee an integration error \( \varepsilon \in (0,1/2) \) for
3.1. Smooth Functions

any function from the class $\mathcal{C}_d^r$ grows super-exponentially with the dimension. It is proven in [HNUW17] that there are positive constants $c_r$ and $C_r$ such that

$$(c_r d^{d/r} \varepsilon^{-1/r})^d \leq n(\varepsilon, \mathcal{P}[\text{INT}, \mathcal{C}_d^r]) \leq (C_r d^{d/r} \varepsilon^{-1/r})^d$$

for all $\varepsilon \in (0, 1/2)$ and $d \in \mathbb{N}$. Roughly speaking $n(\varepsilon, \mathcal{P}[\text{INT}, \mathcal{C}_d^r])$ is of order $(d/\varepsilon)^{d/r}$. Since an $\varepsilon$-approximation of the function immediately yields an $\varepsilon$-approximation of its integral, the uniform recovery of functions from $\mathcal{C}_d^r$ can only be harder. But how hard is the uniform recovery problem? Is it significantly harder than the integration problem? These questions were recently posed in [Woz18, Section 6].

In the case $r = 1$ the answer is known. In this case the minimal number $n(\varepsilon, \mathcal{P}[\text{APP}, \mathcal{C}_d^r])$ of function values that is needed to guarantee an approximation error $\varepsilon > 0$ for any function from $\mathcal{C}_d^r$ in the uniform norm behaves similarly to $n(\varepsilon, \mathcal{P}[\text{INT}, \mathcal{C}_d^r])$. There are positive constants $c$ and $C$ such that

$$(c d \varepsilon^{-1})^d \leq n(\varepsilon, \mathcal{P}[\text{APP}, \mathcal{C}_d^r]) \leq (C d \varepsilon^{-1})^d$$

for all $\varepsilon \in (0, 1/2)$ and $d \in \mathbb{N}$. This result is basically contained in [Suk78]. Nonetheless, we will present its proof. If $r \geq 2$ is even, we obtain the following result.

**Theorem 3.1** ([Kri19]). Let $r \in \mathbb{N}$ be even. Then there are constants $c_r, C_r, \varepsilon_r > 0$ such that

$$(c_r \sqrt{d} \varepsilon^{-1/r})^d \leq n(\varepsilon, \mathcal{P}[\text{APP}, \mathcal{C}_d^r]) \leq (C_r \sqrt{d} \varepsilon^{-1/r})^d$$

for all $d \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon_r)$. The upper bound holds for all $\varepsilon > 0$.

Roughly speaking $n(\varepsilon, \mathcal{P}[\text{APP}, \mathcal{C}_d^r])$ is of order $(d^{r/2}/\varepsilon)^{d/r}$. If the error tolerance $\varepsilon$ is fixed, the complexity grows like $d^{d/2}$. This is in contrast to the case $r = 1$, where we have a growth of order $d^d$. If $r \geq 3$ is odd, we only have a partial result.

**Theorem 3.2** ([Kri19]). Let $r \geq 3$ be odd. Then there are constants $c_r, C_r, \varepsilon_r > 0$ such that

$$(c_r \sqrt{d} \varepsilon^{-1/r})^d \leq n(\varepsilon, \mathcal{P}[\text{APP}, \mathcal{C}_d^r]) \leq (C_r d^{r+1}/\varepsilon^{1/r})^d$$

for all $d \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon_r)$. The upper bound holds for all $\varepsilon > 0$.

We point to the fact that $n(\varepsilon, \mathcal{P}[\text{APP}, \mathcal{C}_d^r]) \leq n(\varepsilon, \mathcal{P}[\text{APP}, \mathcal{C}_d^{r-1}])$ since the upper bound resulting from Theorem 3.1 may improve on the upper bound of Theorem 3.2 for $d \gg \varepsilon^{2/(r-1)}$ if $r \geq 3$ is odd. In this case, we do not know the exact behavior of $n(\varepsilon, \mathcal{P}[\text{APP}, \mathcal{C}_d^r])$ as a function of both $d$ and $\varepsilon$. If regarded as a function of $\varepsilon$, the complexity is of order $\varepsilon^{-d/r}$. If regarded as a function of $d$, it is of order $d^{d/2}$.

Altogether, our results justify the following comparison.

**Corollary 3.3** ([Kri19]). The uniform recovery problem on the class $\mathcal{C}_d^r$ is significantly harder than the integration problem if and only if $r \geq 3$.

Aside from the case $r = 1$, the lower bounds in Theorem 3.1 and Theorem 3.2 even hold for the smaller class

$$\tilde{\mathcal{C}}_d^r = \left\{ f \in \mathcal{C}^r \left([0, 1]^d\right) \mid \|\partial_{\theta_1} \cdots \partial_{\theta_\ell} f\|_\infty \leq 1 \text{ for all } \ell \leq r \text{ and } \theta_i \in S_{d-1} \right\}$$

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of functions whose directional derivatives up to order \( r \in \mathbb{N} \) are bounded by one. For this class, we obtain sharp bounds on the \( \varepsilon \)-complexity of the uniform recovery problem for any \( r \in \mathbb{N} \). The minimal number \( n(\varepsilon, \mathcal{P}[\text{APP}, \tilde{C}_d^r]) \) of function values that is needed to guarantee an approximation error \( \varepsilon \) for every function from \( \tilde{C}_d^r \) in the uniform norm satisfies the following.

**Theorem 3.4 ([Kri19]).** Let \( r \in \mathbb{N} \). There are constants \( c_r, C_r, \varepsilon_r > 0 \) such that

\[
(c_r \sqrt{d} \varepsilon^{-1/r})^d \leq n(\varepsilon, \mathcal{P}[\text{APP}, \tilde{C}_d^r]) \leq (C_r \sqrt{d} \varepsilon^{-1/r})^d
\]

for all \( d \in \mathbb{N} \) and \( \varepsilon \in (0, \varepsilon_r) \). The upper bound holds for all \( \varepsilon > 0 \).

These theorems also imply new results on the complexity of global optimization. We shortly discuss this problem in Section 3.3. In Sections 3.1.1 and 3.1.2 we prove the upper and lower bounds of Theorems 3.1, 3.2 and 3.4. Before we turn to the proofs, we comment on some related problems.

**Remark 3.5 (Infinite smoothness).** It is proven in [NW09] that even the uniform recovery of functions from \( C_\infty^d = \{ f \in C_\infty([0,1]^d) \mid \|D^\beta f\|_\infty \leq 1 \text{ for all } \beta \in \mathbb{N}_0^d \} \) suffers from the curse of dimensionality. This cannot be avoided even if we allow randomized algorithms that may evaluate arbitrary continuous linear functionals [Kun17, Section 2.4.2]. In fact, we have seen that the complexity \( n(\varepsilon, \mathcal{P}[\text{APP}, C_\infty^d]) \) depends super-exponentially on the dimension for \( r \in \mathbb{N} \). It would be interesting to verify whether this is also true for \( r = \infty \) and randomized algorithms. We remark that the uniform recovery problem does not suffer from the curse if the target function lies within the modified class

\[
\tilde{C}_d^\infty = \{ f \in C_\infty([0,1]^d) \mid \sum_{|\beta|=k} \frac{\|D^\beta f\|_\infty}{\beta!} \leq 1 \text{ for all } k \in \mathbb{N}_0 \} \nonumber\]

of smooth functions. This is proven in [Vyb14].

**Remark 3.6 (Algorithms).** This section is not concerned with explicit algorithms. Nonetheless, our proof shows that there are optimal algorithms in the sense of Theorem 3.1, 3.2 and 3.3 whose information is given by function values at a regular grid and small clouds around the grid points. This information can be used for a subcubewise Taylor approximation of the target function around the grid points, where the partial derivatives of order less than \( r \) are replaced by divided differences. The resulting algorithm is indeed optimal for the class \( C_r^d \). However, the author does not know whether it is also optimal for \( C_\infty^d \).

**Remark 3.7 (Other domains).** Our lower bounds are still valid, if the domains \([0,1]^d\) are replaced by any other sequence of domains \( D_d \subset \mathbb{R}^d \) that satisfies \( \lambda^d(D_d) \geq a^d \) for some \( a > 0 \) and all \( d \in \mathbb{N} \). The upper bounds, however, heavily exploit the geometry of the unit cube. The curse of dimensionality for the integration problem on general domains is studied in the recent paper [HPU18].

**Remark 3.8 (Integration on the class \( \tilde{C}_d^r \)).** The precise behavior of the \( \varepsilon \)-complexity of the integration problem on \( \tilde{C}_d^r \) as a function of both \( d \) and \( \varepsilon \) is still open.
3.1. Smooth Functions

3.1.1 Upper bounds

Let \( F \in \{ C^r_d, \tilde{C}^r_d \} \). These classes are convex and symmetric and \( \mathcal{P}[\text{APP}, F] \) is a linear problem. Since we measure the error in \( B([0,1]^d) \), Theorem 1.16 yields that

\[
e(n, \mathcal{P}[\text{APP}, F]) = \inf_{P \subset [0,1]^d} \sup_{f \in F, \text{card}(P) \leq n} \| f \|_\infty.
\]  

(3.1)

Therefore, if we want to derive an upper bound on the \( n \)th minimal error, we may choose any point set \( P \) with cardinality at most \( n \) and give an upper bound on the maximal value of a function \( f \in F \) that vanishes on \( P \). In fact, we can choose any point set \( Q \) with cardinality at most \( n/(d+1)^{r-1} \) and assume that not only \( f \) but all its derivatives of order less than \( r \) are arbitrarily small on \( Q \). We start with the case \( F = C^r_d \). More precisely, for any \( \delta > 0 \), any \( r \in \mathbb{N}_0 \), \( d \in \mathbb{N} \) and \( Q \subset [0,1]^d \), we define the subclasses

\[ C^r_d(Q, \delta) = \left\{ f \in C^r_d \mid |D^\alpha f(x)| \leq \delta^{2r-|\alpha|-1} \text{ for all } x \in Q \text{ and } |\alpha| < r \right\} \]

and the auxiliary quantities

\[ E(Q, C^r_d, \delta) = \sup_{f \in C^r_d(Q, \delta)} \| f \|_\infty \quad \text{and} \quad E(Q, C^r_d) = \lim_{\delta \downarrow 0} E(Q, C^r_d, \delta) \]

and obtain the following.

**Lemma 3.9.** Let \( d \in \mathbb{N}, r \in \mathbb{N} \) and \( n \in \mathbb{N}_0 \). If the cardinality of \( Q \subset [0,1]^d \) is at most \( n/(d+1)^{r-1} \), then

\[ e(n, \mathcal{P}[\text{APP}, C^r_d]) \leq E(Q, C^r_d). \]

**Proof.** Let \( \delta \in (0,1) \). We will construct a point set \( P \subset [0,1]^d \) with cardinality at most \( n \) such that any \( f \in C^r_d \) with \( f|_P = 0 \) is contained in \( C^r_d(Q, \delta) \). Then equation (3.1) yields

\[ e(n, \mathcal{P}[\text{APP}, C^r_d]) \leq \sup_{f \in C^r_d; f|_P = 0} \| f \|_\infty \leq \sup_{f \in C^r_d(Q, \delta)} \| f \|_\infty = E(Q, C^r_d, \delta). \]

Taking the limit for \( \delta \to 0 \) yields the statement.

If \( r = 1 \), we can choose \( P = Q \). Let us start with the case \( r = 2 \). Given a set \( M \subset [0,1]^d \) and \( h \in (0,1/2) \), we define

\[
M[h] = M \cup \bigcup_{(x,j) \in M \times \{1, \ldots, d\}} \{ x + he_j \} \cup \bigcup_{(x,j) \in M \times \{1, \ldots, d\}} \{ x - he_j \}.
\]

Obviously, the cardinality of \( M[h] \) is at most \( (d+1) |M| \). Furthermore, we have

\[ f \in C^2_d \text{ with } |f| \leq h^2 \text{ on } M[h] \Rightarrow \left| \frac{\partial f}{\partial x_j} \right| \leq 3h \text{ on } M \text{ for } j = 1, \ldots, d. \]

(3.2)
This is a simple consequence of the mean value theorem: For any \( j \in \{1, \ldots, d\} \) and \( x \in M \) with \( x + he_j \in [0, 1]^d \) there is some \( \eta \in (0, h) \) with
\[
\left| \frac{\partial f}{\partial x_j} (x + \eta e_j) \right| = \left| \frac{f(x + he_j) - f(x)}{h} \right| \leq 2h.
\]
The same estimate holds for some \( \eta \in (-h, 0) \), if \( x + he_j \not\in [0, 1]^d \). The fundamental theorem of calculus yields
\[
\left| \frac{\partial f}{\partial x_j} (x) \right| \leq \left| \frac{\partial f}{\partial x_j} (x + \eta e_j) \right| + |\eta| \cdot \max_{|\eta| \leq h} \left| \frac{\partial^2 f}{\partial x_j^2} (x + te_j) \right| \leq 3h.
\]
This means that we can choose \( P = Q[\delta/3] \).

For \( r > 2 \) we repeat this procedure \( r - 1 \) times. We use the notation
\[
M[h_1, \ldots, h_i] = M[h_1, \ldots, h_{i-1}] [h_i]
\]
for \( i > 1 \). We choose the point set
\[
P = Q[h_1, \ldots, h_{r-1}], \text{ where } h_i = 3(\delta/9)^{2^{i-1}}
\]
for \( i = 1, \ldots, r-1 \). Note that \( 3h_i = h_{i-1}^2 \) for each \( i \geq 2 \). Clearly, the cardinality of \( P \) is at most \((d + 1)^{r-1} |Q| \) and hence bounded by \( n \). Let \( f \in C^r_d \) vanish on \( P \) and let \( \frac{\partial^r f}{\partial x_{i_1} \cdots \partial x_{i_r}} \) be any derivative of order \( \ell < r \). Fact (3.2) yields:
\[
\begin{align*}
& f \in C^r_d \quad \text{with} \quad |f| = 0 \leq h_{r-1}^2 \quad \text{on} \quad Q[h_1, \ldots, h_{r-1}] \\
\Rightarrow & \frac{\partial f}{\partial x_{i_1}} \in C^{r-1}_d \quad \text{with} \quad \left| \frac{\partial f}{\partial x_{i_1}} \right| \leq 3h_{r-1} = h_{r-2}^2 \quad \text{on} \quad Q[h_1, \ldots, h_{r-2}] \\
\Rightarrow & \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}} \in C^{r-2}_d \quad \text{with} \quad \left| \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}} \right| \leq 3h_{r-2} \quad \text{on} \quad Q[h_1, \ldots, h_{r-3}] \\
\Rightarrow & \quad \ldots \\
\Rightarrow & \frac{\partial^\ell f}{\partial x_{i_1} \cdots \partial x_{i_\ell}} \in C^{r-\ell}_d \quad \text{with} \quad \left| \frac{\partial^\ell f}{\partial x_{i_1} \cdots \partial x_{i_\ell}} \right| \leq 3h_{r-\ell} \quad \text{on} \quad Q[h_1, \ldots, h_{r-\ell-1}].
\end{align*}
\]
Since \( Q \subset Q[h_1, \ldots, h_{r-\ell-1}] \) and \( 3h_{r-\ell} \leq \delta^{2^{\ell-1}} \), the lemma is proven.

We can prove the desired upper bounds on \( e(n, C_d^n) \) by choosing \( Q \) as a regular grid. We set
\[
Q_m^d = \{0, 1/m, 2/m, \ldots, 1\}^d
\]
for \( m \in \mathbb{N} \). The following recursive formula is crucial.

**Lemma 3.10.** Let \( m \in \mathbb{N} \), \( d \geq 2 \) and \( r \geq 2 \). Then
\[
E\left( Q_m^d, C_d^n \right) \leq E\left( Q_m^{d-1}, C_{d-1}^r \right) + \frac{1}{8m^2} E\left( Q_m^d, C_{d-2}^{r-2} \right).
\]
3.1. Smooth Functions

Proof. We will prove for any \( \delta > 0 \) that

\[
E \left( Q^d_m, C^r_d, \delta \right) \leq E \left( Q^{d-1}_m, C^{r-1}_{d-1}, \delta \right) + \frac{1}{8m^2} E \left( Q^d_m, C^{r-2}_d, \delta \right).
\]

(3.3)

Taking the limit for \( \delta \to 0 \) yields the statement.

Let \( f \in C^r_d \left( Q^d_m, \delta \right) \). We need to show that \( \| f \|_\infty \) is bounded by the right hand side of (3.3). Since \( f \) is continuous, there is some \( z \in [0,1]^d \) such that \( |f(z)| = \| f \|_\infty \).

We distinguish two cases.

If \( z_d \in \{0,1\} \), the restriction \( f|_H \) of \( f \) to the hyperplane

\[
H = \{ x \in [0,1]^d \mid x_d = z_d \}
\]

is contained in \( C^{r-1}_{d-1} \left( Q^{d-1}_m, \delta \right) \). This implies that

\[
|f(z)| = \| f|_H \|_\infty \leq E \left( Q^{d-1}_m, C^{r-1}_{d-1}, \delta \right)
\]

and the statement is proven.

Let us now assume that \( z_d \in (0,1) \). Then we have \( \frac{\partial f}{\partial x_d} (z) = 0 \). We choose \( y \in [0,1]^d \) such that \( y_j = z_j \) for \( j < d \) and \( y_d \in Q_m \) with \( |y_d - z_d| \leq 1/(2m) \). The restriction \( f|_{H'} \) of \( f \) to the hyperplane

\[
H' = \{ x \in [0,1]^d \mid x_d = y_d \}
\]

is contained in \( C^{r-1}_{d-1} \left( Q^{d-1}_m, \delta \right) \). This implies that

\[
|f(y)| = \| f|_{H'} \|_\infty \leq E \left( Q^{d-1}_m, C^{r-1}_{d-1}, \delta \right).
\]

Moreover, the second derivative \( \frac{\partial^2 f}{\partial x_d^2} \) is contained in \( C^{r-2}_d \left( Q^d_m, \delta \right) \) and hence

\[
\left\| \frac{\partial^2 f}{\partial x_d^2} \right\|_\infty \leq E \left( Q^d_m, C^{r-2}_d, \delta \right).
\]

By Taylor’s theorem, there is some \( a \) on the line segment between \( y \) and \( z \) such that

\[
f(y) = f(z) + \frac{1}{2 \partial x_d^2} (a) \cdot (y_d - z_d)^2.
\]

We obtain

\[
|f(z)| \leq |f(y)| + \frac{(y_d - z_d)^2}{2} \cdot \left\| \frac{\partial^2 f}{\partial x_d^2} \right\|_\infty
\]

\[
\leq E \left( Q^{d-1}_m, C^{r-1}_{d-1}, \delta \right) + \frac{1}{8m^2} E \left( Q^d_m, C^{r-2}_d, \delta \right),
\]

as it was to be proven.

By a double induction on \( r \) and \( d \) we obtain the following result for even \( r \).
Lemma 3.11. Let \( d \in \mathbb{N}, m \in \mathbb{N} \) and \( r \in \mathbb{N}_0 \) be even. Then

\[
E \left( Q_m^d, C_r^d \right) \leq \frac{ed^{r/2}}{(2m)^r}.
\]

Proof. We give a proof by induction on \( d \). Let \( \delta > 0 \) and \( f \in C^r(Q_m, \delta) \) for some even number \( r \). Since \( f \) is continuous, there is some \( z \in [0, 1] \) such that \( |f(z)| = \|f\|_\infty \). Let \( y \in Q_m \) with \( |y - z| < 1/(2m) \). By Taylor’s theorem, there is some \( \xi \) between \( y \) and \( z \) such that

\[
f(z) = \sum_{k=0}^{r-1} \frac{f^{(k)}(y)}{k!} (z - y)^k + \frac{f^{(r)}(\xi)}{r!} (z - y)^r.
\]

Using that \( |f^{(k)}(y)| \leq \delta^{2^{r-k-1}} \leq \delta^{r-k} \), we obtain for \( \delta \leq 1/(2m) \) that

\[
\|f\|_\infty \leq \sum_{k=0}^{r} \frac{\delta^{r-k}}{k!} \left( \frac{1}{2m} \right)^k \leq \left( \frac{1}{2m} \right)^r \sum_{k=0}^{r} \frac{1}{k!} \leq \frac{e}{(2m)^r}.
\]

Since this is true for any such \( f \) and any \( \delta \leq 1/(2m) \), this proves the case \( d = 1 \).

Now let \( d \geq 2 \). We assume that the statement holds for every dimension smaller than \( d \). To show that it also holds in dimension \( d \), we use induction on \( r \). For \( r = 0 \) the statement is trivial since \( E(Q_m^d, C_0^d) = 1 \). Let \( r \geq 2 \) be even and assume that the statement holds in dimension \( d \) for any even smoothness smaller than \( r \). Lemma 3.10 yields

\[
E \left( Q_m^d, C_r^d \right) \leq \frac{ed^{r/2}}{(2m)^r} \left( 1 - \frac{1}{d} \right)^{r/2} + \frac{1}{2d} \leq \frac{ed^{r/2}}{(2m)^r},
\]

which completes the inner and therefore the outer induction.

This immediately yields the upper bound of Theorem 3.1.

Proof of Theorem 3.1 (Upper Bound). Let \( d \in \mathbb{N}, r \in \mathbb{N} \) be even and \( \varepsilon > 0 \). We set

\[
n = (d + 1)^{r-1}(m + 1)^d,
\]

where \( m = \left\lfloor \frac{e^{1/r}}{2} \sqrt{d \varepsilon^{-1/r}} \right\rfloor \).

Lemmas 3.9 and 3.11 yield

\[
e(n, \mathcal{P}[\text{APP}, C_d^d]) \leq E \left( Q_m^d, C_d^d \right) \leq \frac{ed^{r/2}}{(2m)^r} \leq \varepsilon.
\]

Hence,

\[
n(\varepsilon, \mathcal{P}[\text{APP}, C_d^d]) \leq n
\]

and this implies the result.

To derive the upper bounds for odd \( r \), we use the following recursive formula.
Lemma 3.12. Let $m \in \mathbb{N}$, $d \in \mathbb{N}$ and $r \in \mathbb{N}$. Then

$$E\left(Q^d_m, C^r_d\right) \leq \frac{d}{2m} E\left(Q^d_m, C^{r-1}_d\right).$$

Proof. It suffices to show for any $\delta > 0$ that

$$E\left(Q^d_m, C^r_d, \delta\right) \leq \delta^{2^{r-1}} + \frac{d}{2m} E\left(Q^d_m, C^{r-1}_d, \delta\right).$$

Taking the limit for $\delta \to 0$ yields the statement. Let $f \in C^r_d(\mathbb{Q}^d_m, \delta)$ and let $z \in [0,1]^d$ such that $|f(z)| = \|f\|_{\infty}$. There is some $y \in \mathbb{Q}_m$ such that $y$ and $z$ are connected by an axis-parallel polygonal chain of length at most $d/(2m)$. For every $j \in \{1, \ldots, d\}$, the partial derivative $\partial f/\partial x_j$ is contained in $C^{r-1}_d(\mathbb{Q}^d_m, \delta)$. Integrating along the curve yields

$$|f(z)| \leq |f(y)| + \frac{d}{2m} \max_{j=1 \ldots d} \left\|\frac{\partial f}{\partial x_j}\right\|_{\infty} \leq \delta^{2^{r-1}} + \frac{d}{2m} E\left(Q^d_m, C^{r-1}_d, \delta\right).$$

This proves the lemma.

Now the upper bounds of Theorem 3.2 follow from the results for even $r$. Note that the upper bound for $r=1$ is included.

Proof of Theorem 3.2 (Upper Bound). Let $d \in \mathbb{N}$, $r \in \mathbb{N}$ be odd and $\varepsilon > 0$. For any $m \in \mathbb{N}$, Lemma 3.11 and 3.12 yield

$$E\left(Q^d_m, C^r_d\right) \leq \epsilon d^{(r+1)/2}/(2m)^r.$$

We set

$$n = (d+1)^{r-1}(m+1)^d, \quad \text{where} \quad m = \left[\frac{\epsilon^{1/r}}{2d^{\frac{1}{2r}} \varepsilon^{1/r}}\right].$$

We obtain

$$e(n, \mathcal{P}[\text{APP}, C^r_d]) \leq E\left(Q^d_m, C^r_d\right) \leq \varepsilon$$

and hence

$$n(\varepsilon, \mathcal{P}[\text{APP}, C^r_d]) \leq n,$$

as it was to be proven.

We proceed similarly to prove of the upper bound of Theorem 3.4. For any $\delta > 0$, any $r \in \mathbb{N}_0$, $d \in \mathbb{N}$ and $Q \subset [0,1]^d$, we define the subclasses

$$\tilde{C}^r_d(Q, \delta) = \left\{ f \in \tilde{C}^r_d \mid |\partial_{\theta_1} \cdots \partial_{\theta_{\ell}} f(x)| \leq \delta^{2^{r-\ell-1}} \text{ for } x \in Q, \ell < r, \theta_1 \ldots \theta_{\ell} \in S_{d-1}\right\}$$

and the auxiliary quantities

$$E\left(Q, \tilde{C}^r_d, \delta\right) = \sup_{f \in \tilde{C}^r_d(Q, \delta)} \|f\|_{\infty}$$

and obtain the following.
Lemma 3.13. Let $d, r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. If the cardinality of $Q \subset [0, 1]^d$ is at most $n/(d + 1)^{r-1}$, then

$$e(n, \mathcal{P}[\text{APP}, C_d^n]) \leq E \left( Q, \tilde{C}_d^r \right).$$

Proof. Let $\delta \in (0, 1)$. In the proof of Lemma 3.9, we constructed a point set $P$ with cardinality at most $n$ such that any $f \in C_d^n$ with $|f|_P = 0$ is contained in $C_d^n(Q, \delta)$. In particular, any $f \in \tilde{C}_d^n$ with $|f|_P = 0$ satisfies $|D^\alpha f(x)| \leq \delta^{2^{r-|\alpha|} - 1}$ for all $x \in Q$ and $|\alpha| < r$. Taking into account that for $x \in [0, 1]^d$ and $r < r$, we have

$$|\partial_{\alpha_1} \cdots \partial_{\alpha_r} f(x)| \leq d^{\ell/2} \max_{|\alpha|=\ell} |D^\alpha f(x)|,$$

we obtain that $f \in \tilde{C}_d^n(Q, d^{-1/2} \delta)$ and hence

$$e(n, \mathcal{P}[\text{APP}, C_d^n]) \leq \sup_{f \in \tilde{C}_d^n} \|f\|_\infty \leq \sup_{f \in \tilde{C}_d^n(Q, d^{(r-1)/2} \delta)} \|f\|_\infty = E \left( Q, \tilde{C}_d^r, d^{-1/2} \delta \right).$$

Taking the limit for $\delta \to 0$ yields the statement. \hfill \Box

For these classes, it is enough to consider the following single-step recursion.

Lemma 3.14. Let $m \in \mathbb{N}$, $d \in \mathbb{N}$ and $r \in \mathbb{N}$. Then

$$E \left( Q_m^d, \tilde{C}_d^r \right) \leq \sqrt{d} \frac{2m}{E \left( Q_m^d, \tilde{C}_d^{r-1} \right)}.$$}

Proof. It suffices to show for any $\delta > 0$ that

$$E \left( Q_m^d, \tilde{C}_d^r, \delta \right) \leq \delta^{2^{r-1}} + \frac{\sqrt{d}}{2m} E \left( Q_m^d, \tilde{C}_d^{r-1}, \delta \right).$$

To this end, let $f \in \tilde{C}_d^n(Q_m^d, \delta)$ and let $z \in [0, 1]^d$ such that $|f(z)| = \|f\|_\infty$. There is some $y \in Q_m^d$ such that $y$ and $z$ are connected by a line segment of length at most $\sqrt{d}/(2m)$. Let $\theta = z - y/\|z - y\|_2$. Then we have $\partial_\theta f \in \tilde{C}_d^{r-1}(Q_m^d, \delta)$. Integrating along the line yields

$$|f(z)| \leq |f(y)| + \frac{\sqrt{d}}{2m} \|\partial_\theta f\|_\infty \leq \delta^{2^{r-1}} + \frac{\sqrt{d}}{2m} E \left( Q_m^d, \tilde{C}_d^{r-1}, \delta \right).$$

Taking the limit for $\delta \to 0$ yields the statement. \hfill \Box

The upper bound of Theorem 3.4 can now be proven by induction on $r$.

Proof of Theorem 3.4 (Upper Bound). Lemma 3.14 and $E(Q_m^d, \tilde{C}_d^0) = 1$ yield

$$E \left( Q_m^d, \tilde{C}_d^r \right) \leq \left( \frac{\sqrt{d}}{2m} \right)^r$$

for any $m \in \mathbb{N}$, $d \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Now let $d \in \mathbb{N}$, $r \in \mathbb{N}$ and $\epsilon > 0$. We set

$$n = (d + 1)^{r-1}(m + 1)^d, \quad \text{where} \quad m = \left\lfloor \frac{1}{2} \sqrt{d} \epsilon^{-1/r} \right\rfloor.$$}

Lemma 3.13 yields

$$e(n, \mathcal{P}[\text{APP}, C_d^n]) \leq E \left( Q_m^d, \tilde{C}_d^r \right) \leq \epsilon$$

and hence

$$n(\epsilon, \mathcal{P}[\text{APP}, \tilde{C}_d^n]) \leq n,$$

as it was to be proven. \hfill \Box

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3.1. Smooth Functions

3.1.2 Lower bounds

By equation (3.1), we can estimate \( e(n, \mathcal{P}[\text{APP}, \mathcal{C}^d]) \) from below as follows. For any point set \( P \) with cardinality at most \( n \), we construct a function \( f \in \mathcal{C}^d \) that vanishes on \( P \) but has a large maximum in \([0, 1]^d\), a so-called fooling function. We will use the following lemma. Note that

\[
\|f\|_{r,d} = \sup_{\ell \leq r, \theta_i \in S_{d-1}} \|\partial_{\theta_1} \cdots \partial_{\theta_r} f\|_\infty
\]
defines a norm on the space of smooth functions \( f : \mathbb{R}^d \to \mathbb{R} \) with compact support.

Lemma 3.15. There exists a sequence \((g_d)_{d \in \mathbb{N}}\) of infinitely differentiable functions \( g_d : \mathbb{R}^d \to \mathbb{R} \) with support in the Euclidean unit ball that satisfy \( g_d(0) = 1 \) and

\[
\sup_{d \in \mathbb{N}} \|g_d\|_{r,d} < \infty \quad \text{for all} \quad r \in \mathbb{N}_0.
\]

Proof. Take any function \( h \in C^\infty(\mathbb{R}) \) which equals 1 on \((-\infty, 0]\) and 0 on \([1, \infty)\). Then the radial functions

\[
g_d : \mathbb{R}^d \to \mathbb{R}, \quad g_d(x) = h \left( \|x\|_2^2 \right)
\]

for \( d \in \mathbb{N} \) have the desired properties. This follows from the fact that the directional derivative \( \partial_{\theta_1} \cdots \partial_{\theta_r} g_d(x) \) only depends on the length of \( x \) and the angles between each pair of vectors \( \theta_1, \ldots, \theta_r \in S_{d-1} \) and \( x \in \mathbb{R}^d \). As soon as \( d \) is large enough such that all constellations of lengths and angles are possible, the norm \( \|g_d\|_{r,d} \) is independent of the dimension \( d \).

To obtain a suitable fooling function for a given point set \( P \), it is enough to shrink and shift the support of \( g_d \) to the largest euclidean ball that does not intersect with \( P \). The radius of this ball can be estimated by a simple volume argument.

Lemma 3.16. Let \( P \subset [0, 1]^d \) be of cardinality \( n \in \mathbb{N} \). There exists \( z \in [0, 1]^d \) such that for all \( x \in P \) we have

\[
\|z - x\|_2 \geq \frac{\sqrt{d}}{5n^{1/d}}.
\]

Proof. The set

\[
B^2_R(P) = \bigcup_{x \in P} B^2_R(x)
\]

of points within a distance \( R > 0 \) of \( P \) has the volume

\[
\lambda^d \left( B^2_R(P) \right) \leq n R^d \lambda^d \left( B^2_R(0) \right) = \frac{n R^d \pi^{d/2}}{\Gamma \left( \frac{d}{2} + 1 \right)}.
\]

By Stirling’s Formula, this can be estimated from above by

\[
\lambda^d \left( B^2_R(P) \right) \leq \frac{n R^d \pi^{d/2}}{\sqrt{2\pi} \left( \frac{d}{2e} \right)^{d/2}} \leq \left( \frac{n^{1/4} e^{3/2}}{\sqrt{d}} R \right)^d.
\]

If \( R = \sqrt{d/(5n^{1/d})} \), the volume is less than 1 and \([0, 1]^d \setminus B^2_R(P) \) must be nonempty. \(\square\)
We are ready to prove the lower bound of Theorem 3.4.

Proof of Theorem 3.4 (Lower Bound). Let $r \in \mathbb{N}$, $d \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $P$ be any subset of $[0,1]^d$ with cardinality at most $n$. Let $g_d$ be like in Lemma 3.15 and set

$$K_r = \sup_{d \in \mathbb{N}} \|g_d\|_{r,d} \quad \text{and} \quad R = \min \left\{1, \frac{\sqrt{d}}{5n^{1/d}} \right\}.$$ 

By Lemma 3.16 there is a point $z \in [0,1]^d$ such that $B^2_R(z)$ does not contain any element of $P$. Hence, the function

$$f_\ast : [0,1]^d \to \mathbb{R}, \quad f_\ast(x) = \frac{R^r}{K_r} g_d \left( \frac{x - z}{R} \right)$$

is an element of $C_d^r$ and vanishes on $P$. We obtain

$$\sup_{f \in C_d^r; f|_{r^*} = 0} \|f\|_\infty \geq \|f_\ast\|_\infty \geq f_\ast(z) = \frac{R^r}{K_r} = \min \left\{1, \frac{d^{r/2}}{5^r K_r n^{r/d}} \right\}.$$ 

Since this is true for any such $P$, equation (3.1) yields

$$e(n, P[APP, C_d^r]) \geq \min \left\{1, \frac{d^{r/2}}{5^r K_r n^{r/d}} \right\}. \quad (3.4)$$

We set $\varepsilon_r = 1/K_r$. Given $\varepsilon \in (0,\varepsilon_r)$, the right hand side in (3.4) is larger than $\varepsilon$ for any $n$ smaller than $d^{r/2}/(5^r K_r^{d/r} \varepsilon^{d/r})$. This yields

$$n(\varepsilon, P[APP, \tilde{C}_d^r]) \geq \left((5^r K_r)^{1/r} \sqrt{d} \varepsilon^{-1/r}\right)^d$$

as it was to be proven.

In the same way, we obtain lower bounds for the case that the domains $[0,1]^d$ are replaced by other domains $D_d \subset \mathbb{R}^d$ that satisfy $\lambda^d(D_d) \geq a^d$ for some $a > 0$ and all $d \in \mathbb{N}$. We simply have to multiply the radii in the previous proofs by $a$.

We now turn to the lower bounds of Theorem 3.1 and 3.2.

Proof of Theorem 3.1 and 3.2 (Lower Bounds). Note that $O_d^r$ contains $\tilde{C}_d^r$ and hence

$$n(\varepsilon, P[APP, O_d^r]) \geq n(\varepsilon, P[APP, \tilde{C}_d^r]).$$

Furthermore, any $\varepsilon$-approximation of a function on $[0,1]^d$ immediately yields an $\varepsilon$-approximation of its integral and hence

$$n(\varepsilon, P[APP, C_d^r]) \geq n(\varepsilon, P[INT, C_d^r]).$$

With these relations at hand, the desired lower bounds for $r \geq 2$ immediately follow from Theorem 3.4. The lower bound for $r = 1$ follows from the complexity of numerical integration as studied in [HNUW17].

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3.2 Rank One Tensors

The uniform approximation of smooth functions \( f : [0, 1]^d \rightarrow \mathbb{R} \) suffers from the curse of dimensionality [NW09]. The number of function values that we need to capture \( f \) up to some error \( \varepsilon \in (0, 1) \) in the uniform norm grows exponentially with the dimension. But suppose we know that \( f \) is the tensor product of \( d \) univariate functions. How many function values do we need then? This question has first been posed and investigated in the recent work of Bachmayr, Dahmen, DeVore, and Grasedyck [BDDG14]. More precisely, it is assumed that \( f \) is contained in a class of rank one tensors that is given by

\[
F_{r,M}^d = \left\{ \bigotimes_{i=1}^d f_i : [0,1] \rightarrow [-1,1], \|f_i^{(r)}\|_\infty \leq M \right\}
\]

for smoothness parameters \( r \in \mathbb{N} \) and \( M \geq 0 \), where the function

\[
\bigotimes_{i=1}^d f_i : [0,1]^d \rightarrow \mathbb{R}, \quad f(x) = \prod_{i=1}^d f_i(x_i)
\]

is called a rank one tensor. Note that \( f_i^{(r)} \) denotes the \( r \)th weak derivative of \( f_i \). In particular, it is assumed that \( f_i \) is contained in the Sobolev class \( W^r_\infty([0,1]) \) of univariate functions that have \( r \) weak derivatives in \( L^\infty([0,1]) \).

It is proven in [NR16] that for \( M \geq 2^r r! \) also this problem suffers from the curse of dimensionality. Even for randomized methods, the curse is present. For \( M < 2^r r! \) however, a randomized algorithm is constructed that does not require exponentially many function values with respect to the dimension \( d \). We show that the same is possible with a deterministic algorithm. In fact, we construct algorithms for every constellation of the smoothness parameters that are optimal in terms of tractability.

3.2.1 Results

A deterministic algorithm for the uniform recovery of rank one tensors is already constructed in [BDDG14]. It achieves the worst case error \( \varepsilon \) while using at most

\[
C_{r,d} M^{d/r} \varepsilon^{-1/r}
\]

function values of \( f \), see [BDDG14, Theorem 5.1]. This number behaves optimally as a function of \( \varepsilon \). However, the constant \( C_{r,d} \) and hence the number of function values grows super-exponentially with \( d \) for any \( M > 0 \) and \( r \in \mathbb{N} \). The algorithm uses the following observation. If we know a nonzero \( z \) of \( f \), we can essentially recover every factor \( f_i \) separately by sampling along the line

\[
E_i = \left\{ x \in [0,1]^d \mid \forall j \in \{1,\ldots,d\} \setminus \{i\} : x_j = z_j \right\}.
\]

This results in a deterministic algorithm \( I_m(z, \cdot) \) that requests function values at

\[
m = \left\lfloor C_{r,M} d^{1+1/r} \varepsilon^{-1/r} \right\rfloor
\]
points and satisfies
\[ \| I_m(z, f) - f \|_\infty \leq \varepsilon \]  \hspace{1cm} (3.6)
for any \( f \in F_{r,M}^d \) with \( f(z) \neq 0 \). Here, the constant \( C_{r,M} \) is positive and depends only on \( r \) and \( M \). See [BDDG14] for details. Roughly speaking, the knowledge of a nonzero of \( f \) allows us to reduce the problem to \( d \) univariate approximation problems which can, for example, be treated by the use of polynomial interpolation. With this observation at hand, the authors of [BDDG14] use an approximation scheme of the following type:

**Algorithm 3.17.** Given \( m \in \mathbb{N} \), a finite point set \( P \subset [0,1]^d \) and a function \( f \in F_{r,M}^d \), obtain \( A_{P,m}(f) \) as follows:

1. For any \( x \in P \) check whether \( f(x) = 0 \).
2. If we found some \( z \in P \) with \( f(z) \neq 0 \) then call \( I_m(z, f) \) from (3.6). If \( f|_P = 0 \), then return the zero function.

The idea behind this algorithm is to choose a point set \( P \) such that every \( f \) that vanishes on \( P \) must also be small on the whole domain, and thus the zero function is a good approximation of \( f \). This property is characterized by the notion of detectors. We call a finite point set \( P \) in \([0,1]^d\) an \( \varepsilon \)-detector for the class \( F_{r,M}^d \) if it contains (detects) a nonzero of every function \( f \in F_{r,M}^d \) with uniform norm greater than \( \varepsilon \). If \( P \) is an \( \varepsilon \)-detector and \( m \) is chosen as in (3.5), it is easy to see that Algorithm 3.17 satisfies

\[
\text{err}(A_{P,m}, F_{r,M}^d) \leq \varepsilon \quad \text{and} \quad \text{cost}(A_{P,m}, F_{r,M}^d) \leq \text{card}(P) + m,
\]
see also Lemma 3.20. The authors of [BDDG14] use a point set \( P \) that contains a finite Halton sequence \( H \). They obtain that \( P \) is an \( \varepsilon \)-detector if

\[
\text{card}(H) \geq 2^{d+d/r} M^{d/r} \varepsilon^{-d/r} \pi_d,
\]
where \( \pi_d \) is the product of the first \( d \) primes. However, this number increases super-exponentially with the dimension for all parameters \( M \) and \( r \). Here, we want to construct smaller \( \varepsilon \)-detectors for \( F_{r,M}^d \).

In the range \( M \geq 2^4 r! \) we know that the problem suffers from the curse of dimensionality such that we cannot expect to find an \( \varepsilon \)-detector with small cardinality. However, we provide a detector whose cardinality depends merely exponentially on the dimension and not super-exponentially. In the range \( M < 2^4 r! \) we give a detector whose cardinality only grows polynomially with the dimension. The order of growth is proportional to \( \ln(\varepsilon^{-1}) \). There even exists a detector whose cardinality grows at most quadratically with the dimension for all \( \varepsilon \) if we have \( M \leq r! \). Altogether, this yields the following theorem.

**Theorem 3.18 ([KR19]).** For any \( r \in \mathbb{N} \) and \( M > 0 \), there are positive constants \( c_1, \ldots, c_4 \) such that the following holds. For any \( d \in \mathbb{N} \) and \( \varepsilon \in (0,1) \), there is a finite point set \( P \subset [0,1]^d \) and a natural number \( m \) such that \( \text{err}(A_{P,m}, F_{r,M}^d) \leq \varepsilon \) and

\[
\text{cost}(A_{P,m}, F_{r,M}^d) \leq \begin{cases} 
   c_1 d^2 \varepsilon^{-1/r} \ln(\varepsilon^{-1/r}) & \text{if } M \in (0, \infty), \\
   c_2 \exp(c_3 (1 + \ln(\varepsilon^{-1}))(1 + \ln d)) & \text{if } M \in (0, 2^4 r!), \\
   c_4 d^2 \varepsilon^{-1/r} \ln(\varepsilon^{-1/r}) & \text{if } M \in (0, r!].
\end{cases}
\]
3.2. Rank One Tensors

We always choose $m$ as in (3.5). The point sets $P$ and the constants $c_i$ can be found in Section 3.2.2. In each of these ranges we also give a lower bound on the complexity of the problem, which is the reason for us to call the resulting algorithms optimal. In particular, we obtain the following tractability results [KR19].

Theorem 3.19 ([KR19]). The problem $\mathcal{P}[\text{APP}, F_{r,M}^d]$ of uniform approximation on $F_{r,M}^d$ with deterministic standard information in the worst case setting

- suffers from the curse of dimensionality iff $M \geq 2^r r!$.
- is quasi-polynomially tractable iff $M < 2^r r!$.
- is polynomially tractable iff $M \leq r!$.
- is strongly polynomially tractable iff $M = 0$ and $r = 1$.

We also show that the first three statements of Theorem 3.19 do not change for randomized algorithms. In this sense, randomization does not help for the problem of recovering high-dimensional rank one tensors. However, we do not know whether the last statement has to be modified for randomized algorithms.

Before we turn to the proofs, let us introduce some further notation. For any $k \in \mathbb{N}$ we write $[k] = \{1, \ldots, k\}$. If $x_i \in \mathbb{R}$ for each $i \in J$ with some finite index set $J$, we set $x_J = (x_i)_{i \in J}$. If $I_i$ is an interval for each $i \in J$, then $I_J$ denotes the Cartesian product of these intervals. The term box will always refer to a product of nonempty subintervals of $[0, 1]$. If we are given functions $f_i : I_i \to \mathbb{R}$ for each $i \in J$, their tensor product is denoted by $f_J : I_J \to \mathbb{R}$. We recall that the dispersion of a finite subset $P$ of $[0, 1]^d$ is the minimal number $\eta > 0$ such that $P$ has non-empty intersection with every box of volume greater than $\eta$, see also Example 1.2. Throughout this section, we always assume that $r, d \in \mathbb{N}$, $\varepsilon \in (0, 1)$ and $M > 0$.

3.2.2 Algorithms

This section contains the proof of Theorem 3.18. Here we always assume that $M > 0$. We start with the observation that the construction of an $\varepsilon$-detector is sufficient to achieve the worst case error $\varepsilon$ with the algorithm $A_{P,m}$. Recall that a point set $P$ in $[0, 1]^d$ is called an $\varepsilon$-detector for $F_{r,M}^d$, if it contains a nonzero of any function $f \in F_{r,M}^d$ with $\|f\|_{\infty} > \varepsilon$. Any such function is of the following form:

$$f = \bigotimes_{i=1}^d f_i, \quad \text{where } f_i : [0, 1] \to [-1, 1] \quad \text{with } \|f_i^{(r)}\|_{\infty} \leq M$$

\begin{equation}
(3.7)
\end{equation}

and $\|f\|_{\infty} = \prod_{i=1}^d \|f_i\|_{\infty} > \varepsilon$.

Note that this representation of $f \in F_{r,M}^d$ is usually not unique, since we may rescale the factors $f_i$ without changing the product.
Lemma 3.20. Let \( r \in \mathbb{N}, d \in \mathbb{N} \) and \( M > 0 \). If \( P \) is an \( \varepsilon \)-detector for \( F_{r,M}^d \) and \( m \) is chosen as in (3.5), then Algorithm 3.17 satisfies

\[
\text{err}(A_{P,m}, F_{r,M}^d) \leq \varepsilon \quad \text{and} \quad \text{cost}(A_{P,m}, F_{r,M}^d) \leq \text{card}(P) + m.
\]

Proof. Let \( f \in F_{r,M}^d \). If \( P \) contains a nonzero of \( f \), Algorithm 3.17 returns an \( \varepsilon \)-approximation of \( f \) due to relation (3.6). If not, the output is zero. But since \( P \) is a detector, we necessarily have \( \|f\|_{\infty} \leq \varepsilon \) and zero is an \( \varepsilon \)-approximation of \( f \), as well. The second statement is obvious. \( \square \)

Furthermore, we will use the following formula for polynomial interpolation.

Lemma 3.21. Let \( a < b \), \( r \in \mathbb{N} \) and \( g \in W_r^\infty([a,b]) \). Let \( x_1, \ldots, x_r \in [a,b] \) be distinct and \( p \) be the unique polynomial with degree less than \( r \) such that \( p(x_i) = g(x_i) \) for all \( i \in [r] \). For every \( x \in [a,b] \), there exist \( \xi_1, \xi_2 \in [a,b] \) such that

\[
g(x) - p(x) = \frac{1}{r!} \cdot \frac{g^{(r-1)}(\xi_2) - g^{(r-1)}(\xi_1)}{\xi_2 - \xi_1} \cdot \prod_{i=1}^{r} (x - x_i).
\]

Lemma 3.21 is well known for \( g \in C^r([a,b]) \). In this case, the second fraction can be replaced by \( g^{(r)}(\xi) \) for some \( \xi \in [a,b] \). We refer to [CK91, Theorem 2, Section 6.1]. Under the more general assumption that \( g \in W_r^\infty([a,b]) \), we have to modify the proof of the mentioned theorem.

Proof. If \( x \) coincides with one of the nodes, the statement is trivial. Hence, let \( x \) be distinct from all the nodes. We consider

\[
w : [a,b] \to \mathbb{R}, \quad w(y) = \prod_{i=1}^{r} (y - x_i)
\]

and set

\[
\lambda = \frac{g(x) - p(x)}{w(x)}.
\]

The function \( \varphi = g - p - \lambda w \) vanishes at the points \( x_1, \ldots, x_r \) and \( x \). Since \( g \) and \( \varphi \) are elements of \( W_r^\infty([a,b]) \), their \((r-1)\)st derivatives are absolutely continuous. If we apply Rolle’s Theorem \((r-1)\) times, we obtain that \( \varphi^{(r-1)} \) has at least 2 distinct zeros \( \xi_1 \) and \( \xi_2 \) in \([a,b]\) and hence

\[
0 = \int_{\xi_1}^{\xi_2} \varphi^{(r)}(y) \, dy = \int_{\xi_1}^{\xi_2} g^{(r)}(y) - \lambda r! \, dy = g^{(r-1)}(\xi_2) - g^{(r-1)}(\xi_1) - \lambda r! (\xi_2 - \xi_1).
\]

This is the stated identity in disguise. \( \square \)

If \( g \in W_r^\infty([0,1]) \) has \( r \) distinct zeros \( x_1, \ldots, x_r \in [0,1] \), and \( x \) is a maximum point of \( |g| \), we get

\[
\|g\|_{\infty} \leq \frac{\|g^{(r)}\|_{\infty}}{r!} \prod_{i=1}^{r} |x - x_i|.
\]
3.2. Rank One Tensors

This follows from Lemma 3.21 since the unique polynomial \( p \) with degree less than \( r \) and \( p(x_i) = g(x_i) \) for \( i \in [r] \) is the zero polynomial and

\[
\left| g^{(r-1)}(\xi_2) - g^{(r-1)}(\xi_1) \right| = \left| \int_{\xi_1}^{\xi_2} g^{(r)}(y) \, dy \right| \leq \|g^{(r)}\|_{\infty} \cdot |\xi_2 - \xi_1|.
\]

The rest of this section is devoted to the construction of small \( \varepsilon \)-detectors for \( F^d_{r,M} \). Thanks to Lemma 3.20, this is sufficient to prove Theorem 3.18. We will use three different strategies for three different ranges of the parameter \( M \).

Detectors for large derivatives

In this section, the smoothness parameter \( M \) can be arbitrarily large. It is shown in [NR16] that the cost of any algorithm with worst case error smaller than \( \varepsilon \) is at least \( 2^d \) if \( M \geq 2^r r! \). In particular, the cardinality of any \( \varepsilon \)-detector must grow exponentially with the dimension. Yet, it does not get any worse: We construct an \( \varepsilon \)-detector whose cardinality “only” grows exponentially with the dimension but not super-exponentially. We use the following lemma.

**Lemma 3.22.** For each \( g \in W^r_{\infty}([0, 1]) \) with \( \|g^{(r)}\|_{\infty} \leq M \) there is a subinterval of \([0, 1]\) with length

\[
L(g) = \min \left\{ \frac{1}{r}, \left( \frac{\|g\|_{\infty}}{M} \right)^{1/r} \right\}
\]

that does not contain any zero of \( g \).

**Proof.** The function \( |g| \) attains its maximum, say for \( x \in [0, 1] \). We choose an interval \( I \subset [0, 1] \) of length \( rL(g) \) that contains \( x \). There are \( r \) open and disjoint subintervals of \( I \) with length \( L(g) \). We label these intervals \( I_1, \ldots, I_r \) such that the distance of \( x \) and \( I_i \) is increasing with \( i \). Assume that every interval \( I_i \) contains a zero \( x_i \) of \( g \). Then we have \( |x - x_i| < iL(g) \) for all \( i \in [r] \) and (3.8) leads to

\[
\|g\|_{\infty} \leq \frac{M}{r!} \prod_{i=1}^{r} |x - x_i| < ML(g)^r \leq \|g\|_{\infty}.
\]

This is a contradiction and the assertion is proven. \( \square \)

If, in addition, the uniform norm of \( g \) is bounded by 1, we have

\[
L(g) \geq g^{-1} \|g\|_{\infty}^{1/r} \quad \text{for} \quad g = \max \left\{ r, M^{1/r} \right\}.
\]

Hence, for every \( f \) satisfying (3.7) there is a box \( B \) in \([0, 1]^d\) with volume

\[
\prod_{i \in [d]} L(f_i) \geq g^{-d} \prod_{i \in [d]} \|f_i\|_{\infty}^{1/r} = g^{-d} \|f\|_{\infty}^{1/r} > g^{-d} \varepsilon^{1/r}
\]

such that \( f \) does not vanish anywhere on \( B \). Hence, any point set \( P \) in \([0, 1]^d\) with dispersion \( g^{-d} \varepsilon^{1/r} \) or less is an \( \varepsilon \)-detector for \( F^d_{r,M} \). We know from the estimate of Larcher, see [AHRI17], that we can choose \( P \) as a \((t, s, d)\)-net with cardinality

\[
\text{card}(P) = \left[ 2^{t+1} g^d \varepsilon^{-1/r} \right].
\]
By Lemma 3.20 the resulting algorithm achieves the worst case error $\varepsilon$ with the cost
\[
\text{cost} \left( A_{P,m}, F_{r,M}^d \right) \leq \left[ 2^{7d+1} g^d \varepsilon^{-1/r} \right] + C_{r,M} d^{1+1/r} \varepsilon^{-1/r}.
\]
This proves the first statement of Theorem 3.18 with $c_1 = 2^8 \rho + C_{r,M}$. Note that the cost of this algorithm has the minimal order of growth with respect to $\varepsilon$. It grows like $\varepsilon^{-1/r}$ if $d$ is fixed and $\varepsilon$ tends to zero.

Detectors for moderately large derivatives

In this section, we assume that $M < 2^r r!$. In this case, we construct detectors $P$ with a cardinality that only grows polynomially with $d$ for any fixed $\varepsilon$. The construction of $P$ is based on the observation that for any function $f$ from (3.7) only some of the factors $f_i$ can have more than $(r-1)$ zeros close to 1/2. This is an essential difference to the case $M \in [2^r r!, \infty)$, where all factors $f_i$ may have infinitely many zeros in any neighborhood of 1/2. We are going to specify this statement in Lemma 3.24, but first we need the following observation. For $\delta \in (0,1/2]$, we consider the interval $I_\delta := [1/2 - \delta, 1/2 + \delta]$.

Lemma 3.23. Let $g \in W_\infty^r([0,1])$ with $\|g^{(r)}\|_\infty \leq M$. Assume that $g$ has $r$ distinct zeros in $I_\delta$. Then
\[
\|g\|_\infty \leq C_\delta := \frac{M (1 + 2\delta)^r}{2^r r!}.
\]

Proof. Let $x_1, \ldots, x_r$ be those zeros. The function $|g|$ attains its maximum, say for $x \in [0,1]$. By (3.8) we have
\[
\|g\|_\infty \leq \frac{\|g^{(r)}\|_\infty}{r!} \prod_{i=1}^r |x - x_i|.
\]
This yields the desired inequality since $|x - x_i| \leq 1/2 + \delta$ for each $i \in [r]$. \hfill \Box

Since $M < 2^r r!$, we can choose $\delta \in (0,1/2]$ such that $C_\delta < 1$. We define the pseudo-dimension $d_0$ as the largest number in $[d] \cup \{0\}$ that satisfies $C_\delta^{d_0} > \varepsilon$, that is,
\[
d_0 := \min \left\{ \left\lfloor \frac{\ln \varepsilon}{\ln C_\delta} \right\rfloor - 1, d \right\}.
\]
Obviously, the pseudo-dimension is bounded above independently of $d$. We can now specify the statement from the beginning of this section.

Lemma 3.24. Let $f$ be given as in (3.7). Then there are at most $d_0$ coordinates $i \in [d]$ such that $f_i$ has more than $(r-1)$ zeros in $I_\delta$.

Proof. Let $k$ be the number of coordinates $i \in [d]$ for which $f_i$ has more than $(r-1)$ zeros in $I_\delta$. Lemma 3.23 yields that $\varepsilon < \|f\|_\infty \leq C_\delta^k$. The maximality of $d_0$ yields that $k \leq d_0$. \hfill \Box

This means that there is a subset $J^*$ of $[d]$ with cardinality $d_0$ such that $f_i$ has at most $(r-1)$ zeros in $I_\delta$ for all $i \in [d] \setminus J^*$. Let us suppose for the moment that we know this set $J^*$. Then we can find a nonzero of $f$ by solving the following tasks:
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1. Find a nonzero of \( f_{J^*} \).
2. Find a nonzero of \( f_{[d] \setminus J^*} \).

We can easily solve the first task since this problem is only \( d_0 \)-dimensional. By Lemma 3.22, there is a box \( B \) in \([0,1]^{d_0}\) with volume

\[
\prod_{i \in J^*} L(f_i) \geq \prod_{i \in J^*} \varepsilon^{-1} \|f_i\|^{1/r} \geq \varepsilon^{-d_0} \prod_{i \in [d]} \|f_i\|^{1/r} = \varepsilon^{-d_0} \|f\|^{1/r} > \varepsilon^{-d_0} \varepsilon^{1/r}
\]

such that \( f_{J^*} \) does not vanish on \( B \). Hence, any point set \( P_1 \) in \([0,1]^{d_0}\) with dispersion \( \varepsilon^{-d_0} \varepsilon^{1/r} \) or less contains a nonzero of \( f_{J^*} \). Again by the result of Larcher, see [AHR17], we know that we can choose \( P_1 \) as a \((t,s,d)\)-net of cardinality \( 2^{7d_0+1}d_0^d \varepsilon^{-1/r} \).

We can also cope with the second task since \( f_i \) has at most \((r-1)\) zeros in \( I_\delta \) for all \( i \in [d] \setminus J^* \). We use the following observation.

Lemma 3.25. Let \( J \) be a subset of \([d]\) with \( \ell \) elements and, for all \( i \in J \), let \( f_i \) be a function with at most \( k \) zeros on some interval \( I_i \). Then every set in \( I_J \) with \((\ell k + 1)\) elements that are pairwise distinct in every coordinate contains a nonzero of \( f_J \).

Proof. Let \( P \) be a set in \( I_J \) with \((\ell k + 1)\) elements that are pairwise distinct in every coordinate and suppose that \( f_J \) vanishes everywhere on \( P \). For each \( i \in J \) let \( P_i = \{ x_j \in P \mid f_i(x_j) = 0 \} \). Since \( f_J(x_j) = 0 \) implies that there is some \( i \in J \) with \( f_i(x_j) = 0 \), we have \( P = \bigcup_{i \in J} P_i \). This can only be true, if one of the sets \( P_i \) has more than \( k \) elements. But since \( x_i \) is different for every \( x_j \in P_i \), this means that the corresponding function \( f_i \) has more than \( k \) zeros, a contradiction.

Applying this lemma for the functions \( f_i \) in (3.7), for the index set \( J = [d] \setminus J^* \) and for \( k = r-1 \), we obtain that the diagonal set

\[
P_2 = \left\{ \left( \frac{1}{2} - \delta + \frac{2\delta j}{(r-1)(d-d_0)} \right) \cdot 1 \mid j \in \mathbb{N}_0 \text{ with } j \leq (r-1)(d-d_0) \right\}
\]

in \([0,1]^{d-d_0}\) contains a nonzero of \( f_{[d] \setminus J^*} \). Together, this yields that there must be at least one nonzero of \( f \) in the set

\[
\left\{ x \in [0,1]^d \mid x_{J^*} \in P_1, x_{[d] \setminus J^*} \in P_2 \right\}.
\]

This would solve our problem if we knew the set \( J^* \). Since we do not know this set, we simply try all sets \( J \subset [d] \) of cardinality \( d_0 \). This is OK since number of such sets only depends polynomially on \( d \). Altogether, we obtain the \( \varepsilon \)-detector

\[
P = \bigcup_{J \subset [d]; \text{card}(J) = d_0} \left\{ x \in [0,1]^d \mid x_J \in P_1, x_{[d] \setminus J} \in P_2 \right\}.
\]

In fact, we have seen that for any \( f \) as in (3.7) there must be some \( J^* \subset [d] \) with cardinality \( d_0 \), a nonzero \( y \in P_1 \) of \( f_{J^*} \) and a nonzero \( z \in P_2 \) of \( f_{[d] \setminus J^*} \). The point \( x \in [0,1]^d \) with \( x_{J^*} = y \) and \( x_{[d] \setminus J^*} = z \) is contained in the set \( P \) and a nonzero of \( f \). The cardinality of the detector is given by

\[
\text{card}(P) = \binom{d}{d_0} \text{card}(P_1) \text{card}(P_2) = \binom{d}{d_0} [ (r-1)(d-d_0) + 1 ] 2^{7d_0+1} d_0^d \varepsilon^{-1/r}.
\]
This number grows like $d_{d_{0}+1}$ if $\varepsilon$ is fixed and $d$ tends to infinity. Together with Lemma 3.20, this proves the second statement of Theorem 3.18 with

$$
c_2 = 2r + C_{r,M}, \quad \text{and} \quad c_3 = \ln(2^r \rho) \left(1 + 1/\ln(C_{\delta}^{-1})\right).
$$

Note that $d_0$ equals $d$ if $\varepsilon$ is small enough. Hence, the cardinality of $P$ and the cost of the algorithm grows like $\varepsilon^{-1/r}$ if $d$ is fixed and $\varepsilon$ tends to zero, which is optimal.

**Detectors for small derivatives**

In this section, we assume that $M \leq r!$. In this case, each function $f$ satisfying (3.7) does not vanish almost everywhere on a box whose size is independent of $d$. This is due to the following fact.

**Lemma 3.26.** For each $g \in W^{r}_{c}(\mathbb{R})$ with $\|g^{(r)}\|_{\infty} \leq r!$ there is an interval in $[0, 1]$ with length $\min\{1, \|g\|_{\infty}^{1/r}\}$ that contains at most $(r-1)$ zeros of $g$.

**Proof.** The function $|g|$ attains its maximum, say for $x \in [0, 1]$. We choose an open interval $I \subset [0, 1]$ of length $\min\{1, \|g\|_{\infty}^{1/r}\}$ whose closure contains $x$. Assume that $I$ contains $r$ distinct zeros $x_1, \ldots, x_r$ of $g$. Then $|x - x_i| < \|g\|_{\infty}^{1/r}$ for all $i \in [r]$. (3.8) yields

$$
\|g\|_{\infty} \leq \frac{\|g^{(r)}\|_{\infty}}{r!} \prod_{j=1}^{r} |x - x_j| \leq \prod_{j=1}^{r} |x - x_j| < \|g\|_{\infty}.
$$

This is a contradiction and the assertion is proven.

We now construct an $\varepsilon$-detector for any $\varepsilon \in (0, 1)$. To this end, let

$$
\gamma = (1 - 2^{-1/d}) \varepsilon^{1/r}.
$$

Note that $\gamma$ is smaller than $1/2$. We choose a point set $P_0$ in $[0, 1]^d$ whose dispersion is at most $\varepsilon^{1/r}/2$ and consider the point set $P$ in $[0, 1]^d$, given by

$$
P = \left\{ (1 - \gamma) \cdot x + \frac{\gamma j}{(r-1)d} \cdot 1 \mid x \in P_0 \text{ and } j \in \mathbb{N}_0 \text{ with } j \leq (r-1)d \right\}.
$$

**Lemma 3.27.** The point set $P$ is an $\varepsilon$-detector for $P_{r,M}$.

**Proof.** Let $f$ be given as in (3.7). By Lemma 3.26, there are intervals $(a_i, b_i]$ in $[0, 1]$ with length $\|f_i\|_{\infty}^{1/r}$ containing at most $(r-1)$ zeros of $f_i$. By the choice of $\gamma$, we have

$$
\gamma \leq (1 - 2^{-1/d}) \|f_i\|_{\infty}^{1/r}.
$$

In particular, the box

$$
\tilde{B} = \prod_{i \in [d]} (a_i, b_i - \gamma)
$$

is well defined. In fact, the volume of this box satisfies

$$
|\tilde{B}| = \prod_{i \in [d]} (\|f_i\|_{\infty}^{1/r} - \gamma) \geq \prod_{i \in [d]} \left(2^{-1/d} \|f_i\|_{\infty}^{1/r}\right) = \frac{\|f\|_{\infty}^{1/r}}{2} > \frac{\varepsilon^{1/r}}{2}.
$$
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The box $\tilde{B}/(1 - \gamma)$ is contained in $[0, 1]^d$ and even larger than $\tilde{B}$. It hence contains some $x \in P_0$. Consequently, we have $(1 - \gamma)x \in \tilde{B}$ and all the points

$$x^{(j)} = (1 - \gamma) \cdot x + \frac{\gamma j}{(r - 1)d} \cdot 1 \quad \text{for } j \in \mathbb{N}_0 \text{ with } j \leq (r - 1)d$$

are elements of $P$. These are $(r - 1)d + 1$ points that are pairwise distinct in every coordinate and that are all contained in the larger box

$$B = \prod_{i \in [d]} (a_i, b_i).$$

Recall that each function $f_i$ has at most $(r - 1)$ zeros in $(a_i, b_i)$. By Lemma 3.25, one of the points $x^{(j)}$ must be a nonzero of $f$. As an example, Figure 3.1 illustrates the case $d = 2$ and $r = 3$.

This means that we have an $\varepsilon$-detector for $F_{r,M}^d$ with the cardinality

$$\text{card}(P) = ((r - 1)d + 1) \text{card}(P_0),$$

where $P_0$ is a point set with dispersion $\varepsilon^{1/r}/2$ or less. For example, we know from [Rud18] that this can be achieved with

$$\text{card}(P_0) = \left[2^d d \varepsilon^{-1/r} \ln \left(66\varepsilon^{-1/r}\right)\right]$$

points. In particular, Lemma 3.27 and Lemma 3.20 give the last statement of Theorem 3.18 with the constant $c_4 = 85r + C_{r,M}$.
Remark 3.28 (Alternative choices of \( P_0 \)). If follows from [UV18] that the point set \( P_0 \) with dispersion \( \varepsilon^{1/r}/2 \) can also be chosen such that

\[
\text{card}(P_0) \leq C \log_2(d) \left( 1 + \log_2 \left( \varepsilon^{-1/r} \right) \right)^2 \varepsilon^{-2/r} \tag{3.10}
\]

with an absolute constant \( C > 0 \). This number is smaller than (3.9) if \( d \) is large, but the dependence on \( \varepsilon \) is worse. In both cases, however, the resulting algorithm \( A_{P,m} \) is not completely explicit since we do not know how to construct the point sets \( P_0 \) from (3.9) and (3.10). We only know that they exist. What we do know from [UV19] is how to construct \( P_0 \) such that

\[
\text{card}(P_0) \leq C \varepsilon^{-6/r} \left( 1 + \log_2 \left( \varepsilon^{-1/r} \right) \right)^6 \log_2 d^*
\]

with \( d^* = \max\{d, 4\varepsilon^{-1/r}\} \), see also Section 3.4.2. Using this construction of \( P_0 \), the algorithm \( A_{P,m} \) from above is completely explicit and although the \( \varepsilon \)-dependence is far from optimal, its information cost is still polynomial in both \( d \) and \( \varepsilon^{-1} \).

3.2.3 Lower bounds

This section contains the proof of Theorem 3.19. Of course, the positive tractability results are implied by Theorem 3.18, where the case \( M = 0 \) follows from the case \( M > 0 \). The only exception is the case \( M = 0 \) and \( r = 1 \). Here, the functions in \( F_{d,r,M} \) are constant and can be recovered from a single function value.

We now provide lower bounds on the complexity of the uniform approximation problem which imply the negative tractability results. Note that the first result of the following lemma is already contained in [NR16, Theorem 2].

Theorem 3.29 ([KR19]). Let \( r, d \in \mathbb{N}, \varepsilon \geq 0 \) and \( M \geq 0 \).

- If \( M \geq 2^r r! \), then \( n(\varepsilon, \mathcal{P}[\text{APP}, F_{r,M}^d]) \geq 2^d \) for any \( \varepsilon < 1 \).
- If \( M > r! \), then the problem \( \mathcal{P}[\text{APP}, F_{r,M}^d] \) is not polynomially tractable.
- If \( r \geq 2 \), then \( n(\varepsilon, \mathcal{P}[\text{APP}, F_{r,M}^d]) > d \) for any \( \varepsilon < 1 \).
- If \( r = 1 \) and \( M > 0 \), then \( n(\varepsilon, \mathcal{P}[\text{APP}, F_{r,M}^d]) > \lfloor \log_2 d \rfloor \) for any \( \varepsilon < M^2/4 \).

Proof. Part 1. Let \( M \geq 2^r r! \). The function

\[
g(x) = 2^r |x - 1/2|^r \cdot 1_{[0,1/2]}(x), \quad x \in [0,1]
\]

is \( r \)-times differentiable with \( \|g\|_\infty = 1 \) and \( \|g^{(r)}\|_\infty \leq M \). The same holds for the function

\[
h(x) = 2^r |x - 1/2|^r \cdot 1_{[1/2,1]}(x), \quad x \in [0,1].
\]

Hence, all functions \( f = f_i[d] \) with \( f_i \in \{g, h\} \) for \( i \in [d] \) are contained in \( F_{r,M}^d \) and satisfy \( \|f\|_\infty = 1 \). We obtain a set \( E \) of \( 2^d \) functions with pairwise disjoint support.

Let \( A \) be an algorithm and let \( x_1, \ldots, x_n \) be the sample points the algorithm uses for the input \( f_0 = 0 \). If \( n < 2^d \), there is at least one function \( f \in E \) that vanishes at
all these points. The algorithm cannot distinguish this function $f$ from $-f$ and $f_0$ such that

$$A(f) = A(f_0) = A(-f)$$

and we obtain the error bound

$$\text{err}(A, F^d_{r,M}) \geq \max \{ \|A(f_0) - f\|_\infty, \|A(f_0) + f\|_\infty \} \geq \|f\|_\infty = 1.$$ 

**Part 2.** Let $M > r!$ and $\varepsilon \in (0, 1)$. Note that the point $x_0 = (r!/M)^{1/r}$ is contained in $(1/2, 1)$. The function

$$g(x) = \frac{M|x - x_0|^{r}}{r!} \cdot 1_{[0,x_0]}(x), \quad x \in [0, 1]$$

is $r$-times differentiable with $\|g\|_\infty = 1$ and $\|g^{(r)}\|_\infty \leq M$. The function

$$h(x) = \frac{M|x - x_0|^{r}}{r!} \cdot 1_{[x_0,1]}(x), \quad x \in [0, 1]$$

is also $r$-times differentiable with $\|h^{(r)}\|_\infty \leq M$ and $\|h\|_\infty = h(1)$ is in $(0, 1)$. Let $k(\varepsilon, d)$ be the largest number in $\{0, 1, \ldots, d\}$ such that $h(1)^{k(\varepsilon,d)} > \varepsilon$. Namely, let

$$k(\varepsilon, d) = \min \{ \kappa(\varepsilon), d \} \quad \text{with} \quad \kappa(\varepsilon) := \left\lceil \frac{\ln(\varepsilon^{-1})}{\ln(h(1)^{-1})} \right\rceil - 1.$$

For every subset $J$ of $[d]$ with cardinality $k(\varepsilon, d)$, the function $f = f_{\{i\}}$ with $f_i = g$ for $i \in J$ and $f_i = h$ for $i \in [d] \setminus J$ is contained in $F^d_{r,M}$ and satisfies $\|f\|_\infty > \varepsilon$. We obtain a set $E$ of

$$N(\varepsilon, d) = \binom{d}{k(\varepsilon, d)}$$

functions with pairwise disjoint support.

Let $A$ be an algorithm and let $x_1, \ldots, x_n$ be the sample points the algorithm uses for the input $f_0 = 0$. If $n < N(\varepsilon, d)$, there is at least one function $f \in E$ that vanishes at all these points. The algorithm cannot distinguish this function $f$ from $-f$ and $f_0$, such that its error satisfies

$$\text{err}(A, F^d_{r,M}) \geq \max \{ \|A(f_0) - f\|_\infty, \|A(f_0) + f\|_\infty \} \geq \|f\|_\infty > \varepsilon.$$ 

We obtain

$$n(\varepsilon, \mathcal{P}[\text{APP}, F^d_{r,M}]) \geq N(\varepsilon, d) \geq \binom{d}{k(\varepsilon, d)}^{k(\varepsilon,d)}.$$ 

This implies that the problem is not polynomially tractable. In fact, let us assume that the problem is polynomially tractable. Then there are $c, q, p > 0$ such that

$$n(\varepsilon, \mathcal{P}[\text{APP}, F^d_{r,M}]) \leq c \varepsilon^{-p} d^n \quad (3.11)$$

for all $\varepsilon \in (0, 1)$ and all $d \in \mathbb{N}$. We can, however, choose $\varepsilon \in (0, 1)$ such that $\kappa(\varepsilon) > q$ and hence

$$\lim_{d \to \infty} \frac{n(\varepsilon, d)}{d^n} \geq \lim_{d \to \infty} \frac{d^{\kappa(\varepsilon)-q}}{\kappa(\varepsilon)^{\kappa(\varepsilon)}} = \infty.$$ 

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which contradicts the assumption \((3.11)\).

**Part 3.** Let \(r \geq 2\). Let \(A\) be an algorithm and let \(x_1, \ldots, x_n\) be the sample points the algorithm uses for the input \(f_0 = 0\). Let us assume that \(n \leq d\). For each \(i \in [n]\), there is a linear function \(f_i\) on \([0, 1]\) that vanishes at the \(i\)th coordinate of \(x_i\) and satisfies \(\|f_i\|_\infty = 1\). For \(i \in [d] \setminus [n]\) we set \(f_i = 1\). The function \(f = f_\delta\) is in \(F_{rM}^d\) and vanishes at all sample points. Hence, \(f\) and \(- f\) cannot be distinguished from \(f_0\) and the error of \(A\) is at least \(\|f\|_\infty = 1\).

**Part 4.** Let \(r = 1\) and \(M > 0\). The previous argument does not work in this case, since the first derivative of \(f_i\) is not necessarily bounded by \(M\). Here, we assume that the number of sample points of the algorithm \(A\) for the input \(f_0 = 0\) is at most \([\log_2 d]\). By the proof of \([AHR17, \text{Lemma 2}]\), we know that there are two distinct coordinates \(j, \ell \in [d]\) such that the box \(I_{\delta j}^\ell\) does not contain any of these points, where \(I_j = [0, 1/2), I_\ell = (1/2, 1]\) and \(I_i = [0, 1]\) otherwise. The function \(f = f_\delta\) with

\[
f_i(x) = M(x - 1/2) \cdot 1_{I_i}(x), \quad x \in [0, 1]
\]

for \(i \in \{j, \ell\}\) and \(f_i = 1\) otherwise, is contained in \(F_{rM}^d\) and vanishes at all sample points. Therefore, the algorithm cannot distinguish \(f\) and \(- f\) from \(f_0\) such that its error is at least \(\|f\|_\infty = M^2/4\). \(\Box\)

The first two statements of the previous lemma can be extended to randomized algorithms based on \(\Lambda^{\text{std}}\). We use Bakhvalov’s proof technique, see Theorem \(1.10\).

The first statement of the following theorem is again contained in \([NR16, \text{Theorem 3}]\).

**Theorem 3.30.** Let \(r, d \in \mathbb{N}, \varepsilon \geq 0\) and \(M \geq 0\).

- If \(M \geq 2^r r!\), then \(e(n, \mathcal{P}^{\text{APP}}, F_{rM}^d, \text{ran}) > 2^{d - 1}\) for all \(\varepsilon < 2^{-1/2}\).
- If \(M > r!\), then the problem \(\mathcal{P}^{\text{APP}}, F_{rM}^d, \text{ran}\) is not polynomially tractable.

**Proof.** **Part 1.** Let \(M \geq 2^r r!\). In the first part of the proof of Lemma \(3.29\) we defined a set \(E\) consisting of \(2^d\) functions in \(F_{rM}^d\). Let \(\mu\) be the uniform distribution on \(E \cup (-E)\). Let \(A\) be a deterministic algorithm and let \(x_1, \ldots, x_n\) be the sample points the algorithm uses for the input \(f_0 = 0\). If \(n \leq 2^{d - 1}\), there is a subset \(E_0\) of \(E\) with cardinality at least \(2^{d - 1}\) such that any \(f \in E_0\) vanishes at all the sample points. The algorithm cannot distinguish \(f\) from \(- f\) and the triangle inequality yields

\[
\|A(f) - f\|_\infty + \|A(-f) - (-f)\|_\infty \geq 2 \|f\|_\infty = 2,
\]

and hence

\[
\|A(f) - f\|_\infty^2 + \|A(-f) - (-f)\|_\infty^2 \geq 2.
\]

We obtain the error bound

\[
\text{err}(A, \mu)^2 = \frac{1}{2^{d+1}} \sum_{f \in E} \left(\|A(f) - f\|_\infty^2 + \|A(-f) - (-f)\|_\infty^2\right) \geq \frac{2 \text{card}(E_0)}{2^{d+1}} \geq \frac{1}{2}.
\]

Together with Theorem \(1.10\) this yields the statement.

**Part 2.** Let \(M > r!\) and \(\varepsilon \in (0, 1)\). In the second part of the proof of Lemma \(3.29\)
we defined a set $E$ consisting of $N(\varepsilon, d)$ functions in $F^d_{r,M}$. Let $\mu$ be the uniform distribution on $E \cup (-E)$. Let $A$ be a deterministic algorithm and let $x_1, \ldots, x_n$ be the sample points the algorithm uses for the input $f_0 = 0$. If $n \leq N(\varepsilon, d)/2$, there is a subset $E_0$ of $E$ with cardinality at least $N(\varepsilon, d)/2$ such that any $f \in E_0$ vanishes at all the sample points. The algorithm cannot distinguish $f$ from $-f$ and the triangle inequality yields

$$\|A(f) - f\|_{\infty} + \|A(-f) - (-f)\|_{\infty} \geq 2\|f\|_{\infty} > 2\varepsilon,$$

and hence

$$\|A(f) - f\|_{\infty}^2 + \|A(-f) - (-f)\|_{\infty}^2 > 2\varepsilon^2.$$

We obtain the error bound

$$\text{err}(A, \mu)^2 = \frac{1}{2N(\varepsilon, d)} \sum_{f \in E} \left(\|A(f) - f\|_{\infty}^2 + \|A(-f) - (-f)\|_{\infty}^2\right) > \frac{\varepsilon^2}{2}.$$

Together with Theorem 1.10, we obtain that

$$n(\varepsilon, \mathcal{P}[^{\text{APP}}, F^d_{r,M}, \text{ran}]) > \frac{N(\sqrt{2}\varepsilon, d)}{2}.$$

Like above, this implies that the problem is not polynomially tractable. \hfill \Box

### 3.3 Global Optimization

Let $F$ be a class of bounded real-valued functions on $[0,1]^d$. We study the problem $\mathcal{P}[\text{OPT}, F]$ of global optimization on $F$ in the worst case setting. That is, given any function $f \in F$, we want to find a point $x^* \in [0,1]^d$ such that $f(x^*)$ is almost maximal. We emphasize that we want want to find the maximizer and not just the maximum. In order to find $x^*$, we may request $n$ function values of the unknown function at adaptively and deterministically chosen points.

In the sense of Definition 1.1, we define $\mathcal{P}[\text{OPT}, F] = (A, \text{err}, \text{cost})$, where

$$A = \mathcal{A}[F, [0,1]^d, \Lambda^{\text{std}}, \text{det}]$$

is the class of all deterministic algorithms based on standard information with input $f$ in $F$ and output $x^* = A(f)$ in $[0,1]^d$ (see Section 1.2.1) and each algorithm $A \in A$ is assigned the cost

$$\text{cost}(A) = \text{cost}(A, F, \Lambda^{\text{std}}, \text{wc}),$$

which is the maximal number of function values that $A$ requests of the input function (see Section 1.2.3), and the error

$$\text{err}(A) = \sup_{f \in F} \left( \sup_{x \in [0,1]^d} f(x) - f(x^*) \right),$$

which is the residual error in the worst case. Note that this problem is not described by solution operator $\text{OPT} : F \to [0,1]^d$. The reason is that we usually cannot assign a unique maximizer to every function $f \in F$. \hfill 93
We now show that the results from Section 3.1 and Section 3.2 for the problem $\mathcal{P}[\text{APP}, F]$ of uniform approximation in the class $F \in \{C_d^r, \bar{C}_d^\epsilon, F_{r,d}^d\}$ also hold for the problem $\mathcal{P}[\text{OPT}, F]$ of global optimization. On the one hand, it is easy to see that global optimization is never harder than uniform approximation. On the other hand, it is known that global optimization is practically as hard as uniform approximation if $F$ is convex and symmetric [Was84, Nov88]. Note that the classes $C_d^r$ and $\bar{C}_d^\epsilon$ are convex and symmetric but the class $F_{r,M}^d$ is not convex. In this case, we may still apply the following comparison statement, which is implicitly contained in the proof of [Nov88 Proposition 1.3.2].

**Proposition 3.31.** Let $F \subset \mathcal{B}([0,1]^d)$ be symmetric with $0 \in F$. Then

$$\inf_{P \subset [0,1]^d} \sup_{f \in F} \|f\|_\infty \leq e(n, \mathcal{P}[\text{OPT}, F]) \leq 2e(n, \mathcal{P}[\text{APP}, F]).$$

**Proof.** Upper Bound. Let $A$ be an algorithm for uniform approximation. For every $\delta > 0$, we define an algorithm

$$Q_\delta : F \to [0,1]^d, \quad f \mapsto \mathbf{x}^* = Q_\delta(f)$$

for global optimization as follows. Let $g = A(f)$ be our approximation of $f \in F$. We choose $\mathbf{x}^* \in [0,1]^d$ such that $g(\mathbf{x}^*) \geq \sup g - \delta$. Then

$$\sup f - f(\mathbf{x}^*) \leq \sup f - \sup g + g(\mathbf{x}^*) - f(\mathbf{x}^*) + \delta \leq 2\|f - g\|_\infty + \delta.$$

We obtain

$$\text{cost}(Q_\delta) \leq \text{cost}(A), \quad \text{err}(Q_\delta) \leq 2\text{err}(A) + \delta.$$

The statement is obtained as $\delta$ tends to zero.

Lower Bound. Let $Q$ be an algorithm for global optimization with cost $n$ or less. Then there is a point set $P \subset [0,1]^d$ with cardinality $n + 1$ that contains all nodes of the algorithm for the input $f_0 = 0$ and the point $Q(f_0)$. If $f \in F$ vanishes on $P$, the algorithm cannot distinguish $f$ from $f_0$ and hence $Q(f) = Q(f_0)$. This yields $f(Q(f)) = 0$ and hence the error of the algorithm $Q$ is at least $\sup f$. With the symmetry of $F$, we obtain

$$\text{err}(Q) \geq \sup_{f \in F} \sup_{f_{|P}=0} \|f\|_\infty \geq \inf_{f \in F} \sup_{f_{|P}=0} \|f\|_\infty,$$

as it was to be proven. \qed

Theorem 1.16 implies that the lower bound of Proposition 3.31 coincides with the $(n + 1)^{\text{st}}$ minimal error of the approximation problem if $F$ is convex and symmetric. Thus optimization is just as hard as uniform approximation in this case. Since $C_d^r$ and $\bar{C}_d^\epsilon$ are convex and symmetric, we can translate Theorems 3.1, 3.2 and 3.4 for the problem of global optimization. For example, we obtain the following result.

**Corollary 3.32.** Let $r \in \mathbb{N}$ be even. Then there are positive constants $c_r$, $C_r$ and $\epsilon_r$ such that

$$\left(c_r \sqrt{d} \epsilon^{-1/r}\right)^d \leq n(\epsilon, \mathcal{P}[\text{OPT}, C_d^r]) \leq \left(C_r \sqrt{d} \epsilon^{-1/\epsilon}\right)^d$$

for all $d \in \mathbb{N}$ and $\epsilon \in (0, \epsilon_r)$. The upper bound holds for all $\epsilon > 0$.  

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3.4 Dispersion

The class \( F_{r,M}^d \) of rank one tensors is not convex. However, the lower bounds of Theorem \[3.29\] were in fact proven for the left hand side in Proposition \[3.31\]. This yields the following.

**Corollary 3.33.** The problem \( \mathcal{P}[\text{OPT}, F_{r,M}^d] \) of global optimization on \( F_{r,M}^d \) with deterministic standard information in the worst case setting

- suffers from the curse of dimensionality iff \( M \geq 2^r r! \).
- is quasi-polynomially tractable iff \( M < 2^r r! \).
- is polynomially tractable iff \( M \leq r! \).
- is strongly polynomially tractable iff \( M = 0 \) and \( r = 1 \).

Thus, in the sense of tractability, global optimization is just as hard as uniform approximation also for the non-convex class \( F = F_{r,M} \).

3.4 Dispersion

Let \( d \in \mathbb{N} \) and let \( \mathcal{S}_d \) be the set of all finite subsets of \([0, 1]^d\). The dispersion of a point set \( P \in \mathcal{S}_d \) is the volume of the largest empty box amidst the point set, that is,

\[
\text{disp}(P) = \sup \{ \lambda^d(B) \mid B \in \mathcal{B}_d, B \cap P = \emptyset \},
\]

where \( \mathcal{B}_d \) is the set of all axis-aligned boxes inside \([0, 1]^d\). Point sets with small dispersion already proved to be useful for the uniform recovery of rank one tensors, see Section \[3.4\] and for the discretization of the uniform norm of trigonometric polynomials [Tem17]. Recently great progress has been made in the question for the minimal cardinality for which there exists a point set whose dispersion is at most \( \varepsilon \),

\[
n(\varepsilon, \mathcal{P}_d) = \min \{ \text{card}(P) \mid P \in \mathcal{S}_d, \text{disp}(P) \leq \varepsilon \},
\]

see [DJ13, AHR17, Rud18, Sos18, UV18]. This is the \( \varepsilon \)-complexity of the problem \( \mathcal{P}_d = (\mathcal{S}_d, \text{disp, card}) \) as already considered in Example \[1.2\]. Here we shall provide a point set with small cardinality that achieves the desired dispersion. This point set has a simple geometric structure. It is generated by the one-dimensional sets

\[
M_j = \left\{ \frac{1}{2^{j+1}}, \frac{3}{2^{j+1}}, \ldots, \frac{2^{j+1} - 1}{2^{j+1}} \right\}
\]

for \( j \in \mathbb{N}_0 \). The \( d \)-dimensional point set of order \( k \in \mathbb{N}_0 \) is defined as

\[
P(k, d) = \bigcup_{|j|=k} M_{j_1} \times \cdots \times M_{j_d}.
\]

These point sets are particular instances of a sparse grid as widely used for high-dimensional numerical integration and approximation. We refer to Novak and Woźniakowski [NW10] and the references therein. A picture of the set of order 3 in dimension 2 can be found in Figure \[3.2\]. Here we prove the following result.
Chapter 3. Tractability of the Uniform Approximation Problem

Figure 3.2: The Point Set \( P(3, 2) \).

This figure shows the set \( P(k, d) \) of order 3 in dimension 2. The largest empty box has the volume 1/16, the size of 16 of the little squares. If any of the 32 points is removed, an empty box of volume 1/8 emerges.

Theorem 3.34 (Kri18a). Let \( \varepsilon \in (0, 1) \) and \( k(\varepsilon) = \lceil \log_2 (\varepsilon^{-1}) \rceil - 1 \). For any \( d \geq 2 \) the dispersion of the set \( P(k(\varepsilon), d) \) is at most \( \varepsilon \) and its cardinality is given by

\[
\text{card} \left(P(k(\varepsilon), d)\right) = 2^{k(\varepsilon)} \left(\frac{d + k(\varepsilon) - 1}{d - 1}\right).
\]

The formula for the cardinality of \( P(k(\varepsilon), d) \) may be simplified. On the one hand, we have

\[
\text{card} \left(P(k(\varepsilon), d)\right) \leq \varepsilon^{-1} \left\lfloor \log_2 (\varepsilon^{-1}) \right\rfloor^{d-1},
\]

which shows that the size roughly grows linearly in \( 1/\varepsilon \) for a fixed dimension \( d \). On the other hand,

\[
\text{card} \left(P(k(\varepsilon), d)\right) \leq (2d)^{k(\varepsilon)},
\]

which shows that the size grows at most polynomially in \( d \) for a fixed error tolerance \( \varepsilon \). Although very simple, \( P(k(\varepsilon), d) \) is the smallest explicitly known point set in \([0, 1]^d\) with dispersion at most \( \varepsilon \) for many instances of \( \varepsilon \) and \( d \), see Section 3.4.2.

3.4.1 Proof of Theorem 3.34

In the following, we write \([d] = \{1, \ldots, d\}\) for each \( d \in \mathbb{N} \). The vector \( \mathbf{e}_\ell \in \mathbb{R}^d \) has entry 1 in the \( \ell \)-th and 0 in all other coordinates. We start with computing the number of elements in \( P(k, d) \) for \( k \in \mathbb{N}_0 \) and \( d \in \mathbb{N} \).

Lemma 3.35.

\[
\text{card}(P(k, d)) = 2^k \left(\frac{d + k - 1}{d - 1}\right).
\]
3.4. Dispersion

Proof. Note that the cardinality of \( M_j \) is \( 2^j \) for all \( j \in \mathbb{N}_0 \). The identity

\[
\text{card}(P(k,d)) = \sum_{|j|=k} \text{card}(M_{j_1} \times \cdots \times M_{j_d})
= \sum_{|j|=k} 2^{j_1 + \cdots + j_d} = 2^k \text{card} \{ j \in \mathbb{N}^d_0 \mid |j| = k \}
\]

yields the statement of the lemma.

It follows from [Tem17, Theorem 2.3] that the dispersion of \( P(k,d) \) decays with order \( 2^{-k} \) if \( d \) is fixed and \( k \) tends to infinity. For our purposes, however, we need to study the dependence of the dispersion of \( P(k,d) \) on both \( k \) and \( d \). In turns out that the dispersion can be computed precisely. In dimension \( d = 1 \), it is readily checked that the dispersion equals \( 2^{-k} \) for \( k \geq 1 \) and \( 1/2 \) for \( k = 0 \). In any other case, we obtain the following.

Lemma 3.36. For any \( k \in \mathbb{N}_0 \) and \( d \geq 2 \), we have

\[
\text{disp}(P(k,d)) = 2^{-(k+1)}.
\]

Proof. We first observe that there are many boxes of volume \( 2^{-(k+1)} \) which do not intersect with \( P(k,d) \). For instance, the box

\[
(0,2^{-(k+1)}) \times (0,1) \times \cdots \times (0,1)
\]

has these properties. This yields \( \text{disp}(P(k,d)) \geq 2^{-(k+1)} \). To prove the upper bound, let \( B = I_1 \times \cdots \times I_d \) be any box in \([0,1]^d\) whose intersection with \( P(k,d) \) is empty. The set

\[
P = \bigcup_{m \in \mathbb{N}} P(m,d) = \left\{ \frac{\alpha}{2^\beta} \mid \beta \in \mathbb{N} \text{ and } \alpha \in \left[2^\beta - 1\right] \right\}
\]

is dense in \([0,1]^d\). Without loss of generality, we assume that the interior of \( B \) is nonempty. Therefore, \( B \) has nonempty intersection with \( P \) and hence with \( P(m,d) \) for some \( m \in \mathbb{N} \). Let \( m \) be minimal with this property. Since \( B \) has empty intersection with \( P(k,d) \), we either have \( m > k \) or \( m < k \). Let \( x \in P(m,d) \cap B \). This means that there is some \( j \in \mathbb{N}^d_0 \) with \( |j| = m \) and

\[
x_\ell \in M_{j_\ell} \cap I_\ell
\]

for all \( \ell \in [d] \). We observe that the numbers \( x_\ell \pm \frac{1}{2^{m+1}} \) are either contained in \( \{0,1\} \) or in \( M_j \) for some \( j < j_\ell \). Hence, they are not contained in \( I_\ell \), because \( I_\ell \) is a subset of \((0,1)\) and \( m \) is minimal. We obtain that

\[
I_\ell \subset \left( x_\ell - \frac{1}{2^{m+1}}, x_\ell + \frac{1}{2^{m+1}} \right),
\]

and hence

\[
\chi^d(B) \leq \prod_{\ell \in [d]} 2^{-j_\ell} = 2^{-m}.
\]
In the case $m > k$, this yields the statement. In the case $m < k$, we observe that the numbers $x_\ell \pm \frac{1}{2^{k-m+j_\ell+1}}$ cannot be contained in $I_\ell$ for any $\ell \in [d]$, since otherwise the points

$$x \pm \frac{\mathbf{e}_\ell}{2^{k-m+j_\ell+1}}$$

would be both in $B$ and in $P(k,d)$. This means that

$$I_\ell \subset \left( x_\ell - \frac{1}{2^{k-m+j_\ell+1}}, x_\ell + \frac{1}{2^{k-m+j_\ell+1}} \right).$$

We obtain

$$\lambda^d(B) \leq \prod_{\ell \in [d]} 2^{m-k-\ell} = 2^{dm-dk-m} \leq 2^{-(k+1)},$$

where we used that $d \geq 2$. This yields $\text{disp}(P(k,d)) \leq 2^{-(k+1)}$. \hfill \qed

Note that the smallest number $k \in \mathbb{N}_0$ that satisfies $2^{-(k+1)} \leq \varepsilon$ for some fixed $\varepsilon \in (0,1)$ is given by

$$k(\varepsilon) = \left\lceil \log_2 \left( \frac{1}{\varepsilon} - 1 \right) \right\rceil - 1.$$

This yields the statement of Theorem 3.34.

3.4.2 A Comparison with Known Results

Let $d \geq 2$ be an integer and let $\varepsilon \leq 1/4$ be positive. Let us call $P \in S_d$ admissible if the dispersion of $P$ is at most $\varepsilon$. In 2017, Aistleitner, Hinrichs and Rudolf [AHR17] proved that any admissible point set satisfies

$$\text{card}(P) \geq (4\varepsilon)^{-1}(1 - 4\varepsilon) \log_2 d.$$  \hfill (3.12)

At that time, the smallest known admissible point set was a finite Halton-Hammersley sequence $H$ of size

$$\text{card}(H) \leq \left\lceil 2^{d-1}\pi_d \varepsilon^{-1} \right\rceil,$$  \hfill (3.13)

where $\pi_d$ is the product of the first $(d - 1)$ primes. This was proven by Rote and Tichy [RT96], see also Dumitrescu an Jiang [DJ13] for more details. The cardinality of this set grows as slowly as possible as $\varepsilon$ tends to zero and $d$ is fixed. However, it grows super-exponentially with $d$. Larcher realized that there is a $(t,m,d)$-net $N$ which is admissible and satisfies

$$\text{card}(N) \leq \left\lceil 2^{7d+1}\varepsilon^{-1} \right\rceil.$$  \hfill (3.14)

The proof is included in [AHR17]. This number is smaller than (3.13) for $d \geq 54$. However, its exponential growth with respect to $d$ for fixed $\varepsilon$ is still far away from the logarithmic growth of the lower bound (3.12). In the beginning of 2017, Rudolf [Rud18] significantly narrowed this gap. Based on results of Blumer, Ehrenfeucht, Haussler and Warmuth [BEHW89], he obtained the existence of an admissible point set with

$$\text{card}(P) \leq \left\lfloor 8d \varepsilon^{-1} \ln(33\varepsilon^{-1}) \right\rfloor.$$  \hfill (3.15)
3.4. Dispersion

Quite recently, the remaining gap was closed by Sosnovec \cite{Sos18}, who proved the existence of an admissible point set with

$$\text{card}(P) \leq \left\lceil q^{2+2(1 + 4 \ln q) \cdot \ln d} \right\rceil, \quad q = \lceil 1/\varepsilon \rceil.$$  \hspace{1cm} (3.16)

This shows that the logarithmic dependence on the dimension in (3.12) is sharp.

On the other hand, the upper bound (3.16) depends super-exponentially on $1/\varepsilon$. This was improved by Ullrich and Vybiral \cite{UV18} who proved the existence of an admissible point set with

$$\text{card}(P) \leq \left\lceil 2^{7 \varepsilon^{-2}} \left(1 + \log_2 \left(\varepsilon^{-1}\right)\right)^2 \log_2 d^* \right\rceil.$$ \hspace{1cm} (3.17)

Up to now, this is the best known upper bound for the minimal cardinality $n(\varepsilon, P_d)$ of admissible point sets for many parameters $\varepsilon$ and $d$. We point to the fact that the upper bounds (3.15), (3.16) and (3.17) are based on the probabilistic method and only yield the existence of the point set $P$. On the other hand, it is shown by Ullrich and Vybiral in \cite{UV19} that one can construct an admissible point set $P$ with

$$\text{card}(P) \leq \left\lceil C \varepsilon^{-6} \left(1 + \log_2 \left(\varepsilon^{-1}\right)\right)^6 \log_2 d^* \right\rceil$$ \hspace{1cm} (3.18)

with $d^* = \max\{d, 2/\varepsilon\}$ and an absolute constant $C$. The construction takes a running time which is polynomial in $d$ but super-exponential in $\varepsilon^{-1}$.

Here, we provided an admissible point set $P(k(\varepsilon), d)$ with

$$\text{card}(P(k(\varepsilon), d)) = 2^{k(\varepsilon)} \binom{d + k(\varepsilon) - 1}{d - 1}, \quad k(\varepsilon) = \left\lfloor \log_2 \left(\varepsilon^{-1}\right) \right\rfloor - 1.$$  \hspace{1cm} (3.19)

It can be constructed in a running time which is linear in the cardinality. For many parameters $(\varepsilon, d)$ this cardinality is much smaller than the cardinalities of the point sets from (3.13), (3.14) and (3.18). In some cases, it is even smaller than the cardinalities resulting from the nonconstructive results (3.15), (3.16) and (3.17). To illustrate these facts, we consider the dimension $d \in \{2, \ldots, 100\}$ and error tolerance $\varepsilon \in \{1/4, 1/5, \ldots, 1/100\}$ in Figure 3.3.\footnote{Figure 3.3}
Figure 3.3: Cardinalities of admissible point sets. For the parameters $(\varepsilon^{-1}, d)$ in the dark gray area, the author does not know an admissible set which is smaller than the sparse grid $P(k(\varepsilon), d)$ although the existence of such a point set follows from relation (3.15). In the black area, it is not even clear whether such a point set exists. In the light gray area, the Halton-Hammersley set from (3.13) is a smaller admissible set.
Chapter 4

Optimal Information versus Random Information

In complexity theory, we often want to approximate the solution of a linear problem based on \( n \) pieces of information about the unknown problem instance. The power of the information is measured by the minimal worst case error that can be achieved with the given information. Usually, we assume that some kind of oracle is available which grants us this information at our request. We call the oracle \( n \) times to get \( n \) pieces of information. Of course, we try to choose clever questions such that the information is most powerful. We hope to obtain optimal information.

Often, however, this model does not match reality. There is no oracle which we can call at our will. The information rather comes random and we simply have to get along with the information at hand. Note that this is a standard assumption in learning theory and uncertainty quantification. It may also happen that an oracle is available but we just do not know which questions to ask to obtain optimal information from the oracle. Also in this case, we may simply ask random questions. What we obtain is random information.

In this chapter, we want to compare the power of optimal information with the expected power of random information. It is clear that random information cannot be better than optimal information. But how much do we loose? We study this question for some basic examples. Depending on the problem, the answers will range from almost nothing over a little up to almost everything. But before we turn to these examples, let us state the general question a little more precisely.

A linear problem is given by a linear solution operator \( S \) that maps from a convex and symmetric subset \( F \) of a normed space to a normed space \( G \) and a class \( \Lambda \) of continuous linear functionals on \( F \), the class of admissible measurements. We may think of an integration problem, where \( S(f) \) is the integral of a function \( f \), or a recovery problem, where \( S \) is an embedding. One wants to approximate the solution \( S(f) \) for unknown \( f \in F \) based on \( n \) of these measurements such that we can guarantee a small error with respect to the norm in \( G \). We refer the reader to Section 1.2.5 for more details. We consider a random family of information mappings \( N_n : F \to \mathbb{R}^n \), \( N_n(f) = (L_1(f), \ldots , L_n(f)) \), where the random functionals \( L_i \in \Lambda \) are independent and identically distributed.
The power or quality of the information mapping is measured by the radius of information $\text{rad}(N_n, F, S, G)$. This is the worst case error of the best algorithm $A_n = \varphi \circ N_n$ based on the information $N_n$, see Section 1.2.2. The goal is to compare

$$\inf_{N_n} \text{rad}(N_n, F, S, G) \quad \text{vs.} \quad \mathbb{E}(\text{rad}(N_n, F, S, G)),$$

the radius of optimal information and the expected radius of random information.

If the infimum and the expected value are comparable, this means that there are many good algorithms based on many different information mappings. In this case, optimal information and therefore optimal algorithms are not very special. On the other hand, if the infimum is significantly smaller than the expected value, this means that optimal information is very special. It seems to be an interesting characteristic of a problem whether optimal information is special or not.

Of course, the answer to this question heavily depends on the distribution of our measurements. While the question may be interesting for many distributions, we feel that there often is a natural choice. Often, the distribution only depends on the class $\Lambda$ of admissible measurements. In this case, collecting random information might even be a good idea if optimal information is available. It may happen that we do not loose much in terms of the radius but gain the following nice properties.

- Since the distribution is independent of $n$, it is easy to increase the number of measurements if our current approximation is not yet satisfactory.

- The information can be used for many different input classes $F$, solution operators $S$ and target spaces $G$. It is universal.

We note that the second property does not mean that the corresponding algorithm $A_n = \varphi \circ N_n$ is universal. The optimal choice of $\varphi$ usually depends on $F$, $S$ and $G$.

We will study this question for two linear problems. In both cases, there is a rather canonical choice for the distribution of the measurements. The first problem is the $L^p$-approximation of $d$-variate Lipschitz functions from standard information. We assume that random information is given by function values at $n$ random points that are independent and uniformly distributed on the domain. This problem is studied in Section 4.1. The second problem is the $\ell^2$-approximation of a point from an $m$-dimensional ellipsoid by means of $n$ linear measurements, where we imagine that $m$ is much larger than $n$. We assume that random information is given by scalar products in $n$ directions taken independently from the uniform distribution on the sphere in $\mathbb{R}^m$. This is studied in Section 4.2, which is based on [HKNPU19].

We point to the fact that several examples of the sort random information is good can be deduced from various papers that use the probabilistic method. We refer to [CG84, SW98, HNWW01, UV18]. In these papers, the authors introduce a random family of algorithms or point sets and show that the expected worst case error (respectively discrepancy or dispersion) is small. This is used to obtain the existence of good algorithms. However, it actually implies that most of the algorithms in that family are good. Therefore, the expected radius of the random information that lies on the bottom of these algorithms must also be small.

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Let \( \text{dist} \) denote the maximum metric on the \( d \)-torus, that is, 
\[
\text{dist}(x, y) = \min_{k \in \mathbb{Z}^d} \|x + k - y\|_\infty \text{ for } x, y \in [0, 1]^d.
\]
We study the problem of \( L^p \)-approximation for \( 1 \leq p \leq \infty \) on the class 
\[
F_d = \left\{ f : [0, 1]^d \to \mathbb{R} \mid \forall x, y \in [0, 1]^d : |f(x) - f(y)| \leq \text{dist}(x, y) \right\}
\] of Lipschitz continuous functions on the \( d \)-torus with deterministic algorithms based on \( n \) pieces of standard information in the worst case setting. We note that this is a linear problem and thus it is enough to consider nonadaptive information 
\[
N_n : F_d \to \mathbb{R}^n, \quad N_n(f) = (f(x))_{x \in P_n}
\]
for point sets \( P_n \subset [0, 1]^d \) of cardinality \( n \), see Corollary 1.14. The quality of the information mapping \( N_n \) is measured by its radius 
\[
\text{rad}(N_n, \text{APP}, F_d, L^p) = \inf_{\varphi : \mathbb{R}^n \to L^p} \sup_{f \in F_d} \|\varphi(N_n(f)) - f\|_p,
\]
which is the worst case error of the best algorithm based on \( N_n \), or alternatively, by its radius at zero 
\[
r_0(N_n, \text{APP}, F_d, L^p) = \sup_{f \in F_d : f|_{P_n} = 0} \|f\|_p.
\]
By Theorem 1.13 the radius at zero coincides with the overall radius up to a factor of at most 2. In this case, we even know that the additional factor is not necessary and that the optimal algorithm based on \( N_n \) works as follows.

**Algorithm 4.1.** Given \( N_n \) as above and \( f \in F_d \), let 
\[
f^+ = \min_{x \in P_n} (f(x) + \text{dist}(\cdot, x)) \quad \text{and} \quad f^- = \max_{x \in P_n} (f(x) - \text{dist}(\cdot, x)).
\]
We define \( A_n(f) = (f^+ + f^-)/2 \).

Note that \( f^+ \) and \( f^- \) are the maximal and the minimal function in \( F_d \) that interpolate \( f \) at the points of \( P_n \).

**Lemma 4.2.** For any nonadaptive information \( N_n \), Algorithm 4.1 satisfies 
\[
\sup_{f \in F_d} \|f - A_n(f)\|_p = \text{rad}(N_n, \text{APP}, F_d, L^p) = r_0(N_n, \text{APP}, F_d, L^p).
\]

**Proof.** Clearly \( A_n \) is of the form \( \varphi \circ N_n \) for some mapping \( \varphi : \mathbb{R}^n \to L^p \). The definitions of the radii easily yield the first two inequalities of 
\[
\sup_{f \in F_d} \|f - A_n(f)\|_p \geq \text{rad}(N_n) \geq r_0(N_n) \geq \sup_{f \in F_d} \|f - A_n(f)\|_p.
\]
On the other hand, any \( f \in F_d \) satisfies the pointwise estimate 
\[
|f - A_n(f)| \leq \frac{f^+ - f^-}{2}.
\]
This implies the remaining inequality since the right hand side is an element of \( F_d \) that vanish on \( P_n \).  \( \square \)
Chapter 4. Optimal Information versus Random Information

It is known that optimal information satisfies
\[
\inf_N \text{rad}(N_n, \text{APP}, F_d, L^p) \asymp n^{-1/d}
\]
for all \(1 \leq p \leq \infty\). This follows from the upper bound on the complexity of uniform approximation as studied in [Suk78] and the lower bound on the complexity of numerical integration as studied in [Suk79]. Using the proof technique of the latter, we even obtain a precise formula for the minimal radius if \(n = m^d\).

**Proposition 4.3.** Let \(n = m^d\) for some \(m \in \mathbb{N}\). Then
\[
\inf_N \text{rad}(N_n, \text{APP}, F_d, L^p) = \begin{cases} \frac{1}{2} \left( \frac{d}{d+p} \right)^{n^{-1/d}} & \text{if } 1 \leq p < \infty, \\ \frac{1}{2} n^{-1/d} & \text{if } p = \infty. \end{cases}
\]
The infima are attained for \(P_n = \{i/m \mid 0 \leq i < m\}^d\).

**Proof.** We realize that the function \(\text{dist}(\cdot, P_n)\) is contained in \(F_d\) and vanishes on \(P_n\). On the other hand, every other function \(f \in F_d\) that vanishes on \(P_n\) must satisfy
\[
|f(x)| \leq \text{dist}(x, P_n)
\]
for all \(x \in [0,1]^d\). This yields
\[
\text{rad}(N_n, \text{APP}, F_d, L^p) = \sup_{f \in F_d : f|_{P_n} = 0} \|f\|_p = \|\text{dist}(\cdot, P_n)\|_p.
\]
Let us first consider the case \(p = \infty\). Since the volume of the union of the balls \(B_r^\infty(x)\) over \(x \in P_n\) is smaller than 1 for all \(r < 1/(2m)\), there must be some \(x \in [0,1]^d\) with \(\text{dist}(x, P_n) \geq r\) and thus
\[
\|\text{dist}(\cdot, P_n)\|_\infty \geq \frac{1}{2m}.
\]
It is easy to see that equality is satisfied for \(P_n = \{i/m \mid 0 \leq i < m\}^d\). Let us now turn to the case \(p < \infty\). We have the formula
\[
\text{rad}(N_n, \text{APP}, F_d, L^p)^p = \int_{[0,1]^d} \text{dist}(x, P_n)^p \, dx = \int_0^\infty \lambda^d(\text{dist}(x, P_n)^p \geq t) \, dt.
\]
We note that
\[
\lambda^d(\text{dist}(x, P_n)^p \geq t) = 1 - \lambda^d(B_{1/t^{1/p}}^\infty(P_n)) \geq 1 - n(2t^{1/p})^d,
\]
where equality holds if the sets \(B_{1/t^{1/p}}^\infty(y)\) for \(y \in P_n\) are pairwise disjoint. This yields
\[
\text{rad}(N_n, \text{APP}, F_d, L^p)^p \geq \int_0^{(1/2m)^p} \lambda^d(\text{dist}(x, P_n)^p \geq t) \, dt \\
\geq \frac{1}{(2m)^p} - 2^d n \int_0^{(1/2m)^p} t^{d/p} \, dt = \frac{1}{(2m)^p} \frac{d}{d+p}
\]
with equality for \(P_n = \{i/m \mid 0 \leq i < m\}^d\). This proves the statement. \(\square\)
In the following, we want to study the quality of an average information mapping with cost $n$. That is, we ask for the expected radius of the random information $N_n : F_d \to \mathbb{R}^n$, $N_n(f) = \left( f\left( x^{(1)} \right), \ldots, f\left( x^{(n)} \right) \right)$, where the points $x^{(i)}$ are independent and uniformly distributed in $[0, 1]^d$. If $p$ is finite, the $p^{th}$ moment of the radius at zero can be computed precisely.

**Theorem 4.4.** Let $p > 0$ and $n \in \mathbb{N}$. Then

$$
\mathbb{E}(\text{rad}(N_n, \text{APP}, F_d, L^p))^p = \frac{1}{2^p} \frac{n!}{(p/d + 1) \cdots (p/d + n)}.
$$

In particular, the following sequences are strongly equivalent as $n$ tends to infinity:

$$
\sqrt[p]{\mathbb{E}(\text{rad}(N_n, \text{APP}, F_d, L^p))^p} \sim \frac{1}{2} \sqrt[n]{\Gamma(p/d + 1) \cdot n^{-1/d}}.
$$

**Proof.** Let $P_n = \{x^{(1)}, \ldots, x^{(n)}\}$. Recall that

$$
\text{rad}(N_n, \text{APP}, F_d, L^p)^p = \int_{[0,1]^d} \text{dist}(x, P_n)^p \, dx.
$$

Using Tonelli’s theorem, we obtain

$$
\mathbb{E}(\text{rad}(N_n, \text{APP}, F_d, L^p))^p = \int_{[0,1]^d} \mathbb{E}(\text{dist}(x, P_n))^p \, dx.
$$

We will show that the integrand of the latter integral is constant. Let us fix $x \in [0, 1]^d$ and note that $\text{dist}(x, P_n) \in [0, 1/2]$. For any $t \in [0, 1/2]$ we have

$$
\text{dist}(x, P_n) \geq t \iff \forall i \in \{1, \ldots, n\} : x^{(i)} \notin B_t^\infty(x)
$$

and thus

$$
\mathbb{P}(\text{dist}(x, P_n) \geq t) = \left( 1 - (2t)^d \right)^n.
$$

The substitution $s = 1 - (2t^{1/p})^d$ and integration by parts yields

$$
\mathbb{E}(\text{dist}(x, P_n))^p = \int_0^{2^{-p}} \mathbb{P}(\text{dist}(x, P_n)^p \geq t) \, dt = \int_0^{2^{-p}} \left( 1 - (2t^{1/p})^d \right)^n \, dt
$$

$$
= \frac{p/d}{2^p} \int_0^1 s^n (1 - s)^{p/d - 1} \, ds = \frac{1}{2^p} \frac{n!}{(p/d + 1) \cdots (p/d + n)},
$$

which implies the statement of our theorem. \hfill \Box

For $p \geq 1$, the expected radius is bounded above by its $p^{th}$ moment and bounded below by the radius of optimal information. This leads to the following corollary.

**Corollary 4.5.** Let $1 \leq p < \infty$. Then

$$
\mathbb{E}(\text{rad}(N_n, \text{APP}, F_d, L^p)) \asymp \inf_{N_n} \text{rad}(N_n, \text{APP}, F_d, L^p) \asymp n^{-1/d}.
$$
Thus, in the sense of order of convergence, an average information mapping is already optimal for the problem of $L^p$-approximation on $F_d$.

**Remark 4.6 (Modifications of $F_d$).** The rates of convergence of the average and the optimal radius do not change if we replace the maximum metric on the torus by some equivalent metric. The same holds true if we change the Lipschitz constant or if we switch to the nonperiodic setting.

We now turn to the case $p = \infty$, the problem of uniform approximation. In this case, the expected radius of information is closely related to the so called coupon collector’s problem. This is the question for the random number $\tau_\ell$ of coupons that a coupon collector has to collect to obtain a complete set of $\ell$ distinct coupons. The following facts on the distribution of $\tau_\ell$ are well known. We refer to [LPW09]. Here $H_\ell = \sum_{k=1}^\ell 1/k$ is the $\ell$th harmonic number. Note that $H_\ell \sim \ln \ell$ as $\ell \to \infty$.

**Proposition 4.7.** Let $(Y_i)_{i=1}^\infty$ be a sequence of random variables that are uniformly distributed in the set $\{1, \ldots, \ell\}$ and let

$$\tau_\ell = \min \{n \in \mathbb{N} \mid \{Y_1, \ldots, Y_n\} = \{1, \ldots, \ell\}\}.$$  

Then

$$\mathbb{E} \tau_\ell = \ell H_\ell \quad \text{and} \quad \text{Var} \tau_\ell \leq \ell^2 \sum_{k=1}^\ell 1/k^2$$

and for any $c > 0$,

$$\mathbb{P}(\tau_\ell > \lceil c \ell \ln \ell \rceil) \leq \ell^{-c+1}.$$  

**Proof.** For $1 \leq i \leq \ell$, let $\nu_i$ be the number of coupons that have to be collected to get the $i$th distinct coupon after having collected $i-1$ distinct coupons. These are independent geometric random variables with

$$\mathbb{E} \nu_i = \frac{\ell}{\ell - i + 1} \quad \text{and} \quad \text{Var} \nu_i = \frac{\ell(i-1)}{(\ell - i + 1)^2} \leq \frac{\ell^2}{(\ell - i + 1)^2}.$$  

Now the first two statements follow from $\tau_\ell = \sum_{i=1}^\ell \nu_i$. To obtain the tail bound, we consider the events $A_i$ that the coupon with number $i$ was not collected during the first $\lceil c \ell \ln \ell \rceil$ trials. Then

$$\mathbb{P}(A_i) = (1 - 1/\ell)^{\lceil c \ell \ln \ell \rceil} \leq \exp(-c \ell \ln \ell) = \ell^{-c}.$$  

This yields

$$\mathbb{P}(\tau_\ell > \lceil c \ell \ln \ell \rceil) = \mathbb{P} \left( \bigcup_{i=1}^\ell A_i \right) \leq \sum_{i=1}^\ell \mathbb{P}(A_i) \leq \ell^{-c+1},$$

as stated in the proposition. □

This leads to the following estimates of the expected radius for $p = \infty$. 

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4.1. Standard Information for Lipschitz Functions

**Theorem 4.8.** Let \( n \in \mathbb{N} \) and let
\[
m_1 = \min \{ m \in \mathbb{N} \mid m^d (H_{m^d} - 2) \geq n \}, \quad m_2 = \max \{ m \in \mathbb{N} \mid 2m^d \ln(m^d) \leq n \}.
\]
Then
\[
\frac{1}{4m_1} \leq \mathbb{E}(\text{rad}(N_n, \text{APP}, F_d, L^\infty)) \leq \frac{2}{m_2}.
\]

**Proof.** We decompose \([0, 1]^d\) into \( \ell = m^d \) subcubes
\[
B_k = \prod_{i=1}^{d} \left[ \frac{k_i - 1}{m}, \frac{k_i}{m} \right], \quad k \in \{1, 2, \ldots, m\}^d
\]
of equal volume for some \( m \in \mathbb{N} \). Recall that the radius at zero is given by
\[
\text{rad}(N_n, \text{APP}, F_d, L^\infty) = \max_{x \in [0, 1]^d} \text{dist}(x, P_n),
\]
and therefore bounded above by \( 1/m \) if every box contains a point \( x^{(i)} \in P_n \), and bounded below by \( 1/(2m) \) if one of the boxes does not contain a point. Let \( A \) be the event that every box contains a point. Note that the number of random points \( x^{(i)} \) that it takes to hit all the boxes follows the distribution of the coupon collector’s variable \( \tau_\ell \) as defined in Proposition 4.7. Thus
\[
\mathbb{P}(A) = \mathbb{P}(\tau_\ell \leq n).
\]
For the upper bound, we choose \( m = m_2 \). Proposition 4.7 yields
\[
\mathbb{P}(A^c) = \mathbb{P}(\tau_\ell > n) \leq \frac{1}{\ell}
\]
and hence
\[
\mathbb{E}(\text{rad}(N_n, \text{APP}, F_d, L^\infty)) \leq \mathbb{P}(A) \cdot \frac{1}{m} + \mathbb{P}(A^c) \cdot 1 \leq \frac{2}{m}.
\]
For the lower bound, we choose \( m = m_1 \). Chebyshev’s inequality yields
\[
\mathbb{P}(A) = \mathbb{P}(\tau_\ell \leq n) \leq \mathbb{P}(\tau_\ell \leq \ell H_\ell - 2\ell) \leq \frac{\text{Var} \tau_\ell}{4\ell^2} \leq \frac{1}{2}.
\]
We obtain
\[
\mathbb{E}(\text{rad}(N_n, \text{APP}, F_d, L^\infty)) \geq \mathbb{P}(A^c) \frac{1}{2m} \geq \frac{1}{4m},
\]
as it was to be proven. \( \square \)

Note that both \( m_1^d \) and \( m_2^d \) are of order \( n/\ln n \). This yields the following corollary. This corollary is already known from [BDKKW17], where the authors study the uniform approximation of functions on \([0, 1]^d\) with bounded \( r \)th derivative. The upper bound for Sobolev spaces on closed manifolds can also be found in [EGO18].

**Corollary 4.9 ([BDKKW17]).** For all \( n \in \mathbb{N} \) let \( \ell = [n/\ln(n+1)] \). Then
\[
\mathbb{E}(\text{rad}(N_n, \text{APP}, F_d, L^\infty)) \asymp \inf_{N_\ell} \text{rad}(N_\ell, \text{APP}, F_d, L^\infty) \asymp \ell^{-1/d}.
\]

Thus, for the problem of uniform approximation on \( F_d \), an average information mapping with cost \( n \) is as good as an optimal information mapping with cost \( n/\ln n \).
4.2 Linear Information for $\ell^2$-Approximation

We study random information for $\ell^2$-approximation of points from a high or infinite dimensional ellipsoid and compare it to optimal information. The radius of optimal information with cost $n$ is given by the length $\sigma_{n+1}$ of the $(n+1)$th largest semi-axis of the ellipsoid. The sequence $\sigma$ of semi-axes also determines the distribution of the radius $R_n$ of Gaussian random information. We find that random information behaves very differently depending on whether $\sigma \in \ell^2$ or not. For $\sigma \notin \ell^2$ random information is completely useless and we have $\mathbb{E}[R_n] = \sigma_1$. For $\sigma \in \ell^2$ the expected radius of random information tends to zero at least at rate $o(1/\sqrt{n})$ as $n \to \infty$. The case

$$\sigma_n \asymp n^{-\alpha} \ln^{-\beta}(n + 1),$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$, is very interesting for applications. Here we prove

$$\mathbb{E}[R_n] \asymp \begin{cases} 
\sigma_1 & \text{if } \alpha < 1/2 \text{ or } \beta \leq \alpha = 1/2, \\
\sigma_{n+1} \sqrt{\ln(n + 1)} & \text{if } \beta > \alpha = 1/2, \\
\sigma_{n+1} & \text{if } \alpha > 1/2.
\end{cases}$$

For the proof we use a comparison result for Gaussian processes à la Gordon, exponential estimates for sums of chi-squared random variables, and estimates for the extreme singular values of (structured) Gaussian random matrices. This section is based on [HKNPU19].

4.2.1 The Problem

Let $\sigma$ be a sequence of nonnegative numbers $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$. We consider the ellipsoid

$$F(\sigma) = \left\{ x \in \ell^2 \mid \sum_{j=1}^{\infty} \frac{x_j^2}{\sigma_j^2} \leq 1 \right\},$$

where we require that $x_j = 0$ whenever $\sigma_j = 0$. For all $n \in \mathbb{N}$ let $G_n \in \mathbb{R}^{n \times \infty}$ be a random matrix with independent standard Gaussian entries $g_{ij}$. We want to study the distribution of the random variable

$$\mathcal{R}_n(\sigma) = \sup \{ \|x\|_2 \mid x \in F(\sigma), G_n(x) = 0 \}. \quad (4.1)$$

Of course, the equation $G_n(x) = 0$ requires that the involved series converge at all. We now give several interpretations of the random variable $\mathcal{R}_n(\sigma)$. We start with the case that

$$\exists m \in \mathbb{N} : \sigma_j = 0 \iff j > m. \quad (4.2)$$

Then $F(\sigma)$ can be regarded as an ellipsoid in $\mathbb{R}^m$ and $G_n$ can be regarded as an $n \times m$-matrix. In this case we assume that $n < m$. In fact, our main interest lies in the case that $n$ is much smaller than $m$. 

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4.2. Linear Information for $\ell^2$-Approximation

Version 1

Let $E_n$ be uniformly distributed on the Grassmannian manifold of $n$-codimensional hyperplanes in $\mathbb{R}^m$. The intersection of $E_n$ and $F$ is an $(m-n)$-dimensional ellipsoid. We study the circumradius of the random intersection, that is,

$$\mathcal{R}_n^{(1)}(\sigma) = \text{rad} (F(\sigma) \cap E_n).$$

This is the radius of the smallest Euclidean ball that contains the intersection ellipsoid, or equivalently the length of its largest semi-axis. It is easy to see that the radius is maximal if $E_n$ contains $e_1$. In this case, it takes the value $\sigma_1$. The minimal radius on the other hand, is attained if $E_n$ is the span of the vectors $e_i$ for $i > n$. It is given by $\sigma_{n+1}$. Thus, we always have

$$\mathcal{R}_n^{(1)}(\sigma) \in [\sigma_{n+1}, \sigma_1].$$

But how large is the radius of a typical intersection? Is it comparable to the minimal or the maximal radius or does it behave completely different?

Version 2

We study the problem of recovering $x \in F(\sigma)$ from $n$ pieces of information, where we want to guarantee a small error in the Euclidean norm. The information about $x \in F(\sigma)$ is given by coordinates in $n$ directions $y^{(i)} \in S_{m-1}$. This is described by the mapping

$$N_n : F(\sigma) \to \mathbb{R}^n, \quad N_n(x) = (\langle x, y^{(i)} \rangle)_{i=1}^n.$$

The quality of the information mapping is measured by its radius, which is the worst case error of the best recovery algorithm based on the information $N_n$, that is,

$$\text{rad}(N_n, \text{APP}, F(\sigma), \ell^2_m) = \inf_{\varphi : \mathbb{R}^n \to \mathbb{R}^m} \sup_{x \in F(\sigma)} \|\varphi(N_n(x)) - x\|_2.$$

This is a linear problem over Hilbert spaces as described in Section 1.2.5 since $F(\sigma)$ is the unit ball of the Hilbert space $H(\sigma) = \mathbb{R}^m$ equipped with the scalar product

$$\langle x, y \rangle_{H(\sigma)} = \sum_{j \leq m} \frac{x_j y_j}{\sigma_j^2}.$$

The numbers $\sigma_j$ are the singular values of the embedding of $H(\sigma)$ into $\ell^2_m$. It is well known that the information is optimal (its radius is minimal) if the directions $y^{(i)}$ coincide with the $n$ largest semi-axes of the ellipsoid, see Theorem 1.17. The quality of optimal information is given by

$$\min_{y^{(1)}, \ldots, y^{(n)} \in S_{m-1}} \text{rad}(N_n, \text{APP}, F(\sigma), \ell^2_m) = \sigma_{n+1}.$$ 

Here we want to study the typical quality of random information in comparison to optimal information. We ask for the radius

$$\mathcal{R}_n^{(2)}(\sigma) = \text{rad}(N_n, \text{APP}, F(\sigma), \ell^2_m),$$

of the random information mapping $N_n$, where the points $y^{(i)}$ are independent and uniformly distributed on the Euclidean sphere $S_{m-1}$. Is typical random information much worse than optimal information?
Chapter 4. Optimal Information versus Random Information

Version 3

Like in the previous version, we study the radius of a random information mapping. This time we consider the Gaussian information \( G_n \) from above. We denote the radius of information by

\[
R_n^{(3)}(\sigma) = \inf_{\varphi: \mathbb{R}^n \to \mathbb{R}^m} \sup_{x \in F(\sigma)} \| \varphi(G_n(x)) - x \|_2. \tag{4.3}
\]

The following lemma says that these are indeed merely three versions of \( R_n(\sigma) \). Moreover, the alignment of the ellipsoid with the standard axes of \( \mathbb{R}^m \) is not a relevant assumption.

**Lemma 4.10.** Under the assumption (4.2), the random variables \( R_n^{(1)}(\sigma), R_n^{(2)}(\sigma) \) and \( R_n^{(3)}(\sigma) \) have the same distribution as \( R_n(\sigma) \), which does not change if \( F(\sigma) \) is replaced by \( QF(\sigma) \) for some orthogonal matrix \( Q \in O(m) \) in any of their definitions.

**Proof.** The orthogonal invariance immediately follows from the fact that the distributions of the hyperplane \( E_n \), the matrix \( N_n \) and the matrix \( G_n \) are invariant under orthogonal transformations. To see that the variables \( R_n^{(i)}(\sigma) \) for \( i \leq 3 \) are interchangeable, we need the fact that

\[
\text{rad}(A, \text{APP}, F(\sigma), \ell_m^2) = \sup \{ \| x \|_2 \mid x \in F(\sigma) \cap \ker(A) \} = \text{rad}(F(\sigma) \cap \ker(A))
\]

for any matrix \( A \in \mathbb{R}^{n \times m} \), which follows from Theorem 1.15. Now we only need to notice that the kernels of \( N_n \) and \( G_n \) are uniformly distributed on the Grassmannian which follows from the orthogonal invariance of both distributions and the uniqueness of the normalized Haar measure on compact groups. \( \Box \)

**Remark 4.11.** The radius of the section of a symmetric convex body with a random lower-dimensional subspace has already been studied in \([GM97, GM98] \) and subsequently in many other papers such as \([LT00, GMT05, LPT06] \). However, one cannot expect these bounds to be sharp for the whole class of symmetric convex bodies, as has already been pointed out in \([GM97, \text{Example } 2.2] \) for the example of ellipsoids with highly incomparable semi-axes. Moreover, the focus in these papers was on subspaces of proportional codimension, while we focus on subspaces with comparably small codimension such as \( m = 2^n \).

In the infinite-dimensional case, that is, if (4.2) does not hold, the interpretations according to Versions 1 and 2 fail. There is no uniform distribution on the sphere in \( \ell^2 \) and the Grassmannian for \( m = \infty \). However, \( R_n(\sigma) \) may still be interpreted as the radius of Gaussian random information:

- Let \( \sigma \in \ell^2 \). Then the matrix \( G_n \) almost surely defines a bounded operator from the Hilbert space

\[
H(\sigma) = \left\{ x \in \ell^2 \mid \sum_{j=1}^{\infty} \frac{x_j^2}{\sigma_j^2} < \infty \right\}, \quad \langle x, y \rangle_{H(\sigma)} = \sum_{j=1}^{\infty} \frac{x_j y_j}{\sigma_j^2}
\]
4.2. Linear Information for $\ell^2$-Approximation

to $\ell_n^2$. This follows for example from [BVY16, Theorem 3.1], see also Lemma 4.21. Since $H(\sigma)$ is a Hilbert space and $F(\sigma)$ is its unit ball, we have

$$R_n(\sigma) = \text{rad}(G_n, \text{APP}, F(\sigma), \ell^2_m)$$

almost surely, see Theorem 1.15.

- Let $\sigma \not\in \ell^2$. Then the matrix $G_n$ almost surely defines an unbounded operator from $H(\sigma)$ to $\ell_n^2$. This follows for example from [LVY18, Corollary 4.1], see also Lemma 4.27. The mapping $G_n$ need not even be defined for all $x \in F(\sigma)$. Thus, the definition of the radius $\text{rad}(G_n, \text{APP}, F(\sigma), \ell^2_m)$ according to equation (4.3) makes no sense and we need to define the radius in some other way. Recall that the radius is supposed to reflect the worst case error of the best recovery algorithm based on $G_n$. On the one hand, the zero algorithm has the worst case error $\sigma_1$. On the other hand, any algorithm based on $G_n$ cannot distinguish the elements $x \in F(\sigma)$ for which $G_n(x) = 0$. Thus, we must have

$$R_n(\sigma) \leq \text{rad}(G_n, \text{APP}, F(\sigma), \ell^2_m) \leq \sigma_1$$

for any reasonable definition of the radius. It will turn out that $R_n(\sigma) = \sigma_1$ almost surely, which is why the precise definition of the radius does not matter.

Remark 4.12. Instead of $\ell^2$ we may also consider a separable $L^2$-space since both spaces are isometrically isomorphic. Then we may study a compact embedding of a Hilbert space $H$ into $L^2$ and denote the unit ball of $H$ by $F(\sigma)$, where $\sigma$ is the sequence of singular values of the embedding. An important case are Sobolev embeddings, where $H$ is a Sobolev space of functions that are defined on a bounded domain in $\mathbb{R}^d$. It is well known that then the singular values behave as $\sigma_n \asymp n^{-\alpha} \ln^{-\beta}(n + 1)$, where $\alpha$ and $\beta$ depend on the smoothness and the dimension $d$.

4.2.2 Results

We prove the following bounds on the random variable $R_n(\sigma)$ which hold with high probability. We start with upper bounds.

Theorem 4.13 ([HKNPU19]). Let $\sigma \in \ell^2$ be nonincreasing. Then, for all $n \in \mathbb{N}$ and $c, s \in [1, \infty)$, we have

$$\mathbb{P} \left[ R_n(\sigma) \geq \frac{221}{\sqrt{n}} \left( \sum_{j \geq \lfloor n/4 \rfloor} \sigma_j^2 \right)^{1/2} \right] \leq 2e^{-n/100}$$

and

$$\mathbb{P} \left[ R_n(\sigma) \geq 14sn \left( \sum_{j > n} \sigma_j^2 \right)^{1/2} \right] \leq e^{-c^2n} + \frac{c\sqrt{2e}}{s}.$$
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**Theorem 4.14** ([HKNPU19]). Let $\sigma \in \ell^2$ be nonincreasing, $\varepsilon \in (0, 1)$, and $n, k \in \mathbb{N}$ be such that $\sigma_k \neq 0$ and

$$\sum_{j > k} \sigma_j^2 \geq \frac{3n\sigma_k^2}{\varepsilon^2}. $$

Then

$$\mathbb{P}\left[ R_n(\sigma) \leq \sigma_k(1 - \varepsilon) \right] \leq 5e^{-n/64}. $$

As will become apparent in the proof, the lower bound of Theorem 4.14 already holds for the easier problem of recovering just the $k$th coordinate of $x \in F(\sigma)$. As a consequence of the previous theorems, we obtain that random information is useful if and only if $\sigma \in \ell^2$.

**Corollary 4.15** ([HKNPU19]). If $\sigma \notin \ell^2$, then $R_n(\sigma) = \|\sigma\|_\infty$ holds almost surely for all $n \in \mathbb{N}$. If $\sigma \in \ell^2$, then

$$\lim_{n \to \infty} \sqrt{n} \mathbb{E}[R_n(\sigma)] = 0. $$

**Remark 4.16.** The phenomenon that the results very much depend on whether $\sigma \in \ell^2$ or not is known from a related problem that was studied earlier in several papers. There $F$ is the unit ball of a reproducing kernel Hilbert space $H$, that is, $H \subset L^2(D)$ consists of functions on a common domain $D$ and function evaluation $f \mapsto f(x)$ is a continuous functional on $H$ for every $x \in D$. The optimal linear information $N_n$ for the $L^2$-approximation problem is given by the singular value decomposition and has radius $\sigma_{n+1}$. This information might be difficult to implement and hence one might allow only standard information $N_n$ of the form

$$N_n(f) = \begin{pmatrix} f(x_1), \ldots, f(x_n) \end{pmatrix}, \quad x_i \in D. $$

The goal is to relate the power of function evaluations to the power of all continuous linear functionals. Ideally one would like to prove that their power is roughly the same. Unfortunately, in general this is not true. In the case $\sigma \notin \ell^2$ the convergence of optimal algorithms that may only use standard information can be arbitrarily slow [HNV08]. The situation is much better if we assume that $\sigma \in \ell^2$. It was shown in [WW01] and [KWW09] that function values are almost as good as general linear information. We refer to [NW12, Chapter 26] for a presentation of these results. We must say that we do not fully understand the analogy of the two different problems.

Before we present the proofs, let us provide some of the results on the expected radius that follow from our main results for special sequences. We start with the case of polynomial decay.

**Corollary 4.17** ([HKNPU19]). Let $\sigma$ be a nonincreasing sequence such that

$$\sigma_n \propto n^{-\alpha} \ln^{-\beta}(n + 1) $$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Then

$$\mathbb{E}[R_n(\sigma^m)] \asymp \begin{cases} 1 & \text{if } \alpha < 1/2 \text{ or } \beta \leq \alpha = 1/2, \\ n^{-\alpha} \ln^{\beta+1/2}(n + 1) & \text{if } \beta > \alpha = 1/2, \\ n^{-\alpha} \ln^{-\beta}(n + 1) & \text{if } \alpha > 1/2. \end{cases} $$
The very same estimates hold in the finite-dimensional case, that is, if $\sigma_j$ is replaced by 0 for all $j > m$, provided that $n$ is small enough in comparison with $m$, see Corollaries 4.31, 4.32 and 4.33. This means that random information is just as good as optimal information if the singular values decay with a polynomial rate greater than $1/2$. The size of a typical intersection ellipsoid is comparable to the size of the smallest intersection. On the other hand, if the singular values decay too slowly, random information is completely useless. A typical intersection ellipsoid is almost as large as the largest. There is also an intermediate case where random information is worse than optimal information, but only slightly. Moreover, we discuss sequences of exponential decay and obtain the following.

**Corollary 4.18** ([HKNPU19]). Let $\sigma$ be a nonincreasing sequence that satisfies $\sigma_n \asymp a^n$ for some $a \in (0, 1)$. Then

$$a^n \preceq \mathbb{E}[R_n(\sigma)] \preceq n^2 a^n.$$ 

**Remark 4.19.** We have seen that $\mathbb{E}[R_n(\sigma)] \asymp \sigma_{n+1}$ holds for sequences with sufficiently fast polynomial decay. It remains open whether the same holds for sequences of exponential decay. We note that, despite the gap, the result of Corollary 4.18 is even stronger than the result of Corollary 4.17 if considered from the complexity point of view. Corollary 4.17 states that there is a constant $c$ such that $cn$ pieces of random information are at least as good as $n$ pieces of optimal information. Corollary 4.18 states that there is a constant $c$ such that $n + c \ln n$ pieces of random information are at least as good as $n$ pieces of optimal information.

### 4.2.3 Proofs

We now present the proofs of the results that were presented in the previous section. But first we repeat and extend some of our notation. Let $\sigma = (\sigma_j)_{j=1}^{\infty}$ be a nonincreasing sequence of nonnegative numbers. We consider the Hilbert space

$$H(\sigma) = \{ x \in \ell^2 \mid x_j = 0 \text{ if } \sigma_j = 0, \sum_{j=1}^{\infty} \frac{x_j^2}{\sigma_j^2} < \infty \}$$

with scalar product

$$\langle x, y \rangle_{H(\sigma)} = \sum_{j=1}^{\infty} \frac{x_j y_j}{\sigma_j^2}.$$ 

Note that we write $\sum_{j=1}^{\infty}$ but only take the sum over all $j \in \mathbb{N}$ for which $\sigma_j$ is positive. The unit ball of $H(\sigma)$ is denoted by $F(\sigma)$. The numbers $g_{ij}$ shall be independent real standard Gaussian variables for all $i, j \in \mathbb{N}$. For index sets $I \subset \mathbb{N}$ and $J \subset \mathbb{N}$, we consider the (structured) Gaussian $I \times J$-matrices

$$G_{I,J} = (g_{ij})_{i \in I, j \in J} \quad \text{and} \quad \Sigma_{I,J} = (\sigma_j g_{ij})_{i \in I, j \in J}.$$ 

Recall that $[k]$ denotes the set of integers from 1 to $k$ and note that $G_n = G_{[n],\mathbb{N}}$. Moreover, we consider

$$H_I(\sigma) = \{ x \in H(\sigma) \mid x_j = 0 \text{ for all } j \in \mathbb{N} \setminus I \}$$

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as a closed subspace of the Hilbert space $H(\sigma)$ and denote its unit ball by $F_I(\sigma)$. The projection of $x \in H(\sigma)$ onto $H_I(\sigma)$ is denoted by $x_I$. We want to study the distributions of the random variables $R_n(\sigma)$ from (4.1).

As mentioned earlier, a crucial role in our proofs is played by estimates for the extreme singular values of random matrices. So let us recall some basic facts about singular values. Let $A$ be a real $r \times k$-matrix, where we allow that $r = \infty$ or $k = \infty$ provided that $A$ describes a compact operator from $\ell^2_k$ to $\ell^2_r$. For every $j \leq k$, the $j^{th}$ singular value $s_j(A)$ of this matrix can be defined as the square-root of the $j^{th}$ largest eigenvalue of the symmetric matrix $A^T A$, which describes a positive operator on $\ell^2_k$. Note that $s_j(A) = s_j(A^T)$ if we have $j \leq \min\{r,k\}$. Our interest lies in the extreme singular values of $A$. The largest singular value of $A$ is given by

$$s_1(A) = \sup_{x \in \ell^2_k \setminus \{0\}} \frac{\|A x\|_2}{\|x\|_2} = \|A : \ell^2_k \to \ell^2_r\|.$$  

This number is also called the spectral norm of $A$. The smallest singular value is given by

$$s_k(A) = \inf_{x \in \ell^2_k \setminus \{0\}} \frac{\|A x\|_2}{\|x\|_2}.$$  

Clearly, we have $s_k(A) = 0$ whenever $k > r$. If $r \leq k$, it also makes sense to talk about the $r^{th}$ singular value of $A$. This number equals the radius of the largest Euclidean ball that is contained in the image of the unit ball of $\ell^2_k$ under $A$, that is,

$$s_r(A) = \sup\left\{ \varrho \geq 0 \mid B_\varrho^2(0) \subset A(B_1^2(0)) \right\}.$$  

These extreme singular values are also defined for noncompact operators $A$, where $A$ is restricted to its domain if necessary.

**Proof of Theorem 4.13**

We give upper bounds on the radius $R_n(\sigma)$ in terms of $\sigma$. Here we always assume that $\sigma \in \ell^2$. As shown in Corollary 4.30, this is no real restriction. We start with a pointwise upper bound in terms of the extreme singular values of the corresponding (structured) Gaussian matrices.

**Proposition 4.20** [HKNPU19]. Let $\sigma \in \ell^2$ be nonincreasing and let $k \leq n$. If $G_{[n],[k]} \in \mathbb{R}^{n \times k}$ has full rank, then

$$R_n(\sigma) \leq \sigma_{k+1} + \frac{s_1 \left( \Sigma_{[n],[n\setminus[k]]} \right)}{s_k \left( G_{[n],[k]} \right)}.$$  

**Proof.** We first note that $s_k(G_{[n],[k]})$ is positive if $G_{[n],[k]}$ has full rank. Moreover, we may assume that $R_n(\sigma) > 0$ without loss of generality. Let $\varrho > 0$ such that $\varrho < R_n(\sigma)$. By the very definition of $R_n(\sigma)$ there exists some $y \in F(\sigma)$ such that $\|y\|_2 = \varrho$ and $G_n(y) = 0$. The triangle inequality yields

$$\varrho = \|y\|_2 \leq \|y - y_{[k]}\|_2 + \|y_{[k]}\|_2. \quad (4.4)$$  

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The first summand in (4.4) can be bounded by \( \sigma_{k+1} \) since
\[
\left\| y-y[k]\right\|_2^2 = \sum_{j>k} y_j^2 = \sum_{j>k} \sigma_j^2 \left( \frac{y_j}{\sigma_j} \right)^2 \leq \sigma_{k+1}^2 \left\| y \right\|_{H(\sigma)}^2 \leq \sigma_{k+1}^2.
\]

On the other hand, the definition of \( s_k(G[n],k) \) yields
\[
s_k(G[n],k) : \left\| y[k]\right\|_2 \leq \left\| G[n],[k] \left( y[k]\right) \right\|_2 = \left\| G_n \left( y - y[k]\right) \right\|_2 \\
\leq \left\| G_n : H_{N\setminus[k]}(\sigma) \to \ell^2_n \right\| \left\| y - y[k]\right\|_{H(\sigma)} \leq \left\| G_n : H_{N\setminus[k]}(\sigma) \to \ell^2_n \right\|.
\]

Note that we have \( G_n = \Sigma[n],N\setminus[k] D_k \) as mappings on \( H_{N\setminus[k]}(\sigma) \), where
\[
D_k : H_{N\setminus[k]}(\sigma) \to \ell^2, \quad (x_j)_{j=1}^\infty \mapsto (x_{k+j}/\sigma_{k+j})_{j=1}^\infty.
\]

Since \( D_k \) is an isometry, we get
\[
\left\| G_n : H_{N\setminus[k]}(\sigma) \to \ell^2_n \right\| = \left\| \Sigma[n],N\setminus[k] : \ell^2 \to \ell^2_n \right\| = s_1(\Sigma[n],N\setminus[k]).
\]

This means that the second summand in (4.4) can be bounded by
\[
\left\| y[k]\right\|_2 \leq \frac{s_1(\Sigma[n],N\setminus[k])}{s_k(G[n],k)}.
\]

Since these bounds hold for all \( \varrho < \mathcal{R}_n(\sigma) \), we obtain the stated inequality. \( \square \)

Now the task is to bound the \( k \)th singular value of the Gaussian matrix \( G[n],k \) from below and the largest singular value of the structured Gaussian matrix \( \Sigma[n],N\setminus[k] \) from above. We start with the largest singular value of the latter. We note that the question for the order of the expected value of the largest singular value of a structured Gaussian matrix has recently been settled by Latala, Van Handel, and Youssef [LVY18]. The result we use here is due to Bandeira and Van Handel [BVI16].

**Lemma 4.21.** Let \( \sigma \in \ell^2 \) be nonincreasing. For every \( c \geq 1 \) and \( n, k \in \mathbb{N} \), we have
\[
\mathbb{P}\left[ s_1(\Sigma[n],N\setminus[k]) \geq \frac{3}{2} \sqrt{\sum_{j>k} \sigma_j^2} + 11 c \sigma_{k+1} \sqrt{n} \right] \leq e^{-cn^2}.
\]

**Proof.** Without loss of generality, we may assume that \( \sigma_{k+1} \neq 0 \). Let us first consider the finite matrix
\[
A_m = \Sigma[n],[m+k]\setminus[k] \in \mathbb{R}^{n \times m} \quad \text{for} \quad m \in \mathbb{N},
\]
and set
\[
C_m = \frac{3}{2} \left( \sum_{j=k+1}^{k+m} \sigma_j^2 \right)^{1/2} + \frac{103c}{10} \sigma_{k+1} \sqrt{n},
\]
where \( A \) and \( C \) denote their infinite dimensional variants. It is proven in [BVI16 Corollary 3.11] that, for every \( t \geq 0 \) (and \( \varepsilon = 1/2 \)), we have
\[
\mathbb{P}\left[ s_1(A_m) \geq 3 \left( \sum_{j=k+1}^{k+m} \sigma_j^2 \right)^{1/2} + \sigma_{k+1} \sqrt{n} + \frac{5}{\sqrt{5}} \frac{\ln(n)}{\sqrt{\ln(3/2)}} \sigma_{k+1} \right] + t \leq e^{-t^2/2\sigma_{k+1}^2}.
\]
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By setting \( t = \sqrt{2c\sigma_{k+1}\sqrt{n}} \), it follows that

\[
P[s_1(A_m) \geq C_m] \leq e^{-c^2n}.
\]

Turning to the infinite dimensional case, we note that we have \( s_1(A) > C \) if and only if there is some \( m \in \mathbb{N} \) such that \( s_1(A_m) > C \). This yields

\[
P[s_1(A) > C] = \mathbb{P}[\exists m \in \mathbb{N} : s_1(A_m) > C] = \lim_{m \to \infty} \mathbb{P}[s_1(A_m) > C] \leq e^{-c^2n}
\]

since \( s_1(A_m) \) is increasing in \( m \) and \( C \geq C_m \).

Together with Proposition 4.20 this means that the estimate

\[
R_n(\sigma) \leq \sigma_{k+1} + \frac{3}{2} \sqrt{\sum_{j>k} \sigma_j^2 + 11c \sigma_{k+1} \sqrt{n}} \quad \frac{3}{2}s_k\left(G_{[n],[k]}\right)
\]

holds with probability at least \( 1 - e^{-c^2n} \) for all \( k \leq n \) and \( c \geq 1 \). It remains to bound the \( k^{th} \) singular value of the Gaussian matrix \( G_{[n],[k]} \) from below. It is known from [RV09, Theorem 1.1] that this number typically is of order \( \sqrt{n} - \sqrt{k} - 1 \) for all \( n \in \mathbb{N} \) and \( k \leq n \). To exploit our upper bound to the full extend, the number \( k \leq n \) may be chosen such that the right hand side of (4.5) becomes minimal. We realize that the term \( 1/s_k(G_{[n],[k]}) \) increases with \( k \), whereas all other terms decrease with \( k \). However, the inverse singular number achieves its minimal order \( n^{-1/2} \) already for \( k = cn \) with some \( c < 1 \). If \( \sigma \) does not decay extremely fast, this does not lead to a loss regarding the other terms of (4.5). For instance, we may choose \( k = \lfloor n/2 \rfloor \) and use the following special case of [DS01, Theorem II.13].

**Lemma 4.22.** Let \( n \in \mathbb{N} \) and \( k = \lfloor n/2 \rfloor \). Then

\[
P\left[s_k\left(G_{[n],[k]}\right) \leq \sqrt{n}/7\right] \leq e^{-n/100}.
\]

*Proof.* It is shown in [DS01] Theorem II.13 that, for all \( k \leq n \) and \( t > 0 \), we have

\[
P\left[s_k\left(G_{[n],[k]}\right) \leq \sqrt{n} \left(1 - \sqrt{k/n} - t\right)\right] \leq e^{-nt^2/2}.
\]

The statement follows by putting \( k = \lfloor n/2 \rfloor \) and \( t^{-1} = \sqrt{50} \).

If \( \sigma \) decays very fast, \( k = \lfloor n/2 \rfloor \) might not be the best choice. The term \( \sigma_{k+1} \) in estimate (4.5) may be much smaller for \( k = n \) than for \( k = \lfloor n/2 \rfloor \). It is better to choose \( k = n \). In this case, the inverse singular number is of order \( \sqrt{n} \). We state a result of [Sza91, Theorem 1.2].

**Lemma 4.23.** Let \( n \in \mathbb{N} \) and \( t \geq 0 \). Then

\[
P\left[s_n\left(G_{[n],[n]}\right) \leq \frac{t}{\sqrt{n}}\right] \leq t\sqrt{2e}.
\]
4.2. Linear Information for $\ell^2$-Approximation

This leads to the two different probabilistic estimates of the radius $R_n(\sigma)$ as presented in Theorem 4.13. The first is optimized for sequences $\sigma$ with moderate decay, whereas the second is optimized for sequences with rapid decay.

Proof of Theorem 4.13 For the first part, let $k = \lfloor n/2 \rfloor$. We combine Lemma 4.22 and Lemma 4.21 for $c = 1$ with Proposition 4.20 and obtain that

$$R_n(\sigma) \leq 78 \sigma_{k+1} + \frac{21}{2\sqrt{n}} \left( \sum_{j > \lfloor n/2 \rfloor} \sigma_j^2 \right)^{1/2}$$

with probability at least $1 - e^{-\eta} - e^{-n/100}$. The statement follows if we take into account that $\sigma_{k+1}^2 \leq \frac{4}{n} \sum_{j = \lfloor n/4 \rfloor}^{\lfloor n/2 \rfloor} \sigma_j^2$.

For the second part, set $t = c/s$. We combine Lemma 4.23 and Lemma 4.21 with Proposition 4.20 and obtain that

$$R_n(\sigma) \leq \sigma_{n+1} + \frac{1}{t} \left( \frac{3\sqrt{n}}{2} \left( \sum_{j > n} \sigma_j^2 \right)^{1/2} + 11cn \sigma_{n+1} \right)$$

with probability at least $1 - e^{-c^2n} - t\sqrt{2e}$. The rough estimates $\sigma_{n+1}^2 \leq \sum_{j > n} \sigma_j^2$ and $3\sqrt{n}/2 < 2cn$ and $1 \leq sn$ yield the statement. $\square$

Proof of Theorem 4.14

We want to give lower bounds on the radius of information

$$R_n(\sigma) = \sup \{ \|x\|_2 \mid x \in F(\sigma), G_n(x) = 0 \},$$

which corresponds to the difficulty of recovering an unknown element $x \in F(\sigma)$ from the information $G_n(x)$ in $\ell^2$. In fact, our lower bounds already hold for the smaller quantity

$$R_{n,k}(\sigma) = \sup \{ |x_k| \mid x \in F(\sigma), G_n(x) = 0 \},$$

which corresponds to the difficulty of recovering just the $k^{\text{th}}$ coordinate of $x$. Before we come to our bound which holds with high probability, we shall prove the following pointwise estimate.

Proposition 4.24 ([HKNPU19]). Let $\sigma \in \ell^2$ be nonincreasing. For all $n, k \in \mathbb{N}$ with $\sigma_k \neq 0$ we almost surely have

$$R_{n,k}(\sigma) \geq \sigma_k \left( 1 - \frac{\|g_{i,k}^{n}\|_2}{\sigma_k^{-1} s_n \left( \Sigma_{[n],N \setminus \{k\}} \right) + \|g_{i,k}^{n}\|_2} \right).$$

Proof. We may assume that $g = (g_{i,k})_{i=1}^n$ is nonzero since this happens almost surely. Let

$$s_n := s_n \left( \Sigma_{[n],N \setminus \{k\}} \right) = \sup \{ \varrho \geq 0 \mid B_\varrho^2(0) \subset \Sigma_{[n],N \setminus \{k\}} \left( B_1^2(0) \right) \}.$$
Since we have
\[ \Sigma_{[n],\mathbb{N}\setminus\{k\}} \left( B_1^2(0) \right) = G_n \left( F_{\mathbb{N}\setminus\{k\}}(\sigma) \right), \]
the image of \( F_{\mathbb{N}\setminus\{k\}}(\sigma) \) under \( G_n \) contains a Euclidean ball of radius \( s_n \). Let \( e_k \) be the \( k^{th} \) standard unit vector in \( \ell^2 \). We find an element \( \bar{y} \) of \( F_{\mathbb{N}\setminus\{k\}}(\sigma) \) such that
\[ G_n \bar{y} = \frac{s_n \cdot G_n e_k}{\|G_n e_k\|_2}. \]
Our statement is trivial if \( s_n = 0 \), so let \( s_n > 0 \). For \( y = s_n^{-1} \|G_n e_k\|_2 \bar{y} \) we obtain
\[ G_n y = G_n e_k = g \]
and
\[ \|y\|_{H(\sigma)} = s_n^{-1} \|G_n e_k\|_2 \|\bar{y}\|_{H(\sigma)} \leq s_n^{-1} \|g\|_2. \]
Then the vector \( z := e_k - y \) satisfies \( G_n z = 0 \) and \( z_k = 1 \) as well as
\[ \|z\|_{H(\sigma)} \leq \|e_k\|_{H(\sigma)} + \|y\|_{H(\sigma)} \leq \sigma_k^{-1} + s_n^{-1} \|g\|_2. \]
The statement is obtained if we insert the \( H(\sigma) \)-normalization of \( z \) into the very definition of \( \mathcal{R}_{n,k}(\sigma) \).

It remains to bound the \( n^{th} \) singular value of \( \Sigma_{[n],\mathbb{N}\setminus\{k\}} \) and the norm of the Gaussian vector \( (g_k)_{j=1}^n \) with high probability. For both estimates, we use the following concentration result from [LM00] Lemma 1.

**Lemma 4.25.** Let \( u_j \) be independent centered Gaussian variables with variance \( a_j \) for \( 1 \leq j \leq m \). Then, for any \( 0 < \delta \leq 1 \), we have
\[ \mathbb{P} \left[ \sum_{j=1}^m u_j^2 \leq (1 - \delta) \sum_{j=1}^m a_j \right] \leq \exp \left( -\frac{\delta^2 \|a\|_1}{4 \|a\|_\infty} \right), \]
\[ \mathbb{P} \left[ \sum_{j=1}^m u_j^2 \geq (1 + \delta) \sum_{j=1}^m a_j \right] \leq \exp \left( -\frac{\delta^2 \|a\|_1}{16 \|a\|_\infty} \right). \]

**Proof.** By [LM00] Lemma 1 we have for all \( t > 0 \) that
\[ \mathbb{P} \left[ \sum_{j=1}^m u_j^2 \leq \|a\|_1 - 2 \|a\|_2 t \right] \leq e^{-t^2}, \]
\[ \mathbb{P} \left[ \sum_{j=1}^m u_j^2 \geq \|a\|_1 + 2 \|a\|_2 t + 2 \|a\|_\infty t^2 \right] \leq e^{-t^2}. \]
The formulation of Lemma 4.25 follows if we put
\[ t = \frac{\delta \|a\|_1}{2 \|a\|_2}, \]
respectively \( t = \min \left\{ \frac{\delta \|a\|_1}{4 \|a\|_2}, \frac{\delta \|a\|_1}{4 \|a\|_\infty} \right\} \).
The desired probability estimate then follows by using \( \|a\|_3^2 \leq \|a\|_1 \|a\|_\infty \). \( \square \)
4.2. Linear Information for $\ell^2$-Approximation

In particular, the norm of the Gaussian vector $(g_{ik})_{i=1}^n$ concentrates around $\sqrt{n}$. In order to bound the $n^{th}$ singular value of $\Sigma_{[n],[N]\{k\}}$ we shall use Gordon’s min-max theorem. Let us state Gordon’s theorem [Gor88, Lemma 3.1] in a form that can be found in [HOT15].

**Theorem 4.26** ([HOT15]). Let $n, m \in \mathbb{N}$ and let $S_1 \subset \mathbb{R}^n$, $S_2 \subset \mathbb{R}^m$ be compact sets. Assume that $\psi : S_1 \times S_2 \to \mathbb{R}$ is a continuous mapping. Let $G \in \mathbb{R}^{m \times n}$, $u \in \mathbb{R}^m$, and $v \in \mathbb{R}^n$ be independent random objects with independent standard Gaussian entries. Moreover, define

$$
\Phi_1(G) := \min_{x \in S_1} \max_{y \in S_2} \left( \langle y, G\cdot x \rangle_2 + \psi(x, y) \right),
$$

$$
\Phi_2(u, v) := \min_{x \in S_1} \max_{y \in S_2} \left( \|x\|_2 \langle u, y \rangle_2 + \|y\|_2 \langle v, x \rangle_2 + \psi(x, y) \right).
$$

Then, for all $c \in \mathbb{R}$, we have

$$
P\left[ \Phi_1(G) < c \right] \leq 2 \cdot P\left[ \Phi_2(u, v) \leq c \right].
$$

This yields the following lower bound on the smallest singular value of structured Gaussian matrices. Note that this is a generalization of Lemma 4.22.

**Lemma 4.27.** Let $A \in \mathbb{R}^{m \times n}$ be a random matrix with $m \geq n$ whose entries $a_{ij}$ are centered Gaussian variables with variance $a_i$ for all $i \leq m$ and $j \leq n$. Then, for all $0 < \delta < 1$, we have

$$
P\left[ s_n(A) \leq \sqrt{(1 - \delta) \|a\|_1} - \sqrt{(1 + \delta) n \|a\|_\infty} \right] \leq 4 \exp \left( -\frac{\delta^2}{16} \min \left\{ n, \frac{\|a\|_1}{\|a\|_\infty} \right\} \right).
$$

**Proof.** Note that the statement is trivial if $m \leq n$. We may assume that the $a_i$ are positive since an additional row of zeros does neither change $s_n(A)$ nor the norms of the vector $a$. We have the identity $A = DG$ where $G \in \mathbb{R}^{m \times n}$ is a random matrix with independent standard Gaussian entries and $D \in \mathbb{R}^{m \times m}$ is the diagonal matrix

$$
D = \text{diag} \left( \sqrt{a_1}, \ldots, \sqrt{a_m} \right).
$$

We want to apply Gordon’s theorem for the matrix $G$ and $\psi = 0$, where $S_1$ is the sphere in $\ell^2_n$ and $S_2$ is the image of the sphere in $\ell^2_m$ under $D$. Then we have

$$
\Phi_1(G) = \min_{x \in S_1} \max_{y \in S_2} \langle y, G\cdot x \rangle_2 = \min_{\|x\|_2 = 1} \max_{\|y\|_2 = 1} \langle Dz, G\cdot x \rangle_2
$$

$$
= \min_{\|x\|_2 = 1} \max_{\|z\|_2 = 1} \langle Az, x \rangle_2 = \min_{\|x\|_2 = 1} \|Ax\|_2 = s_n(A).
$$

On the other hand, if $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ are standard Gaussian vectors, the choice
of \( z = Du/\|Du\|_2 \) yields
\[
\Phi_2(u, v) = \min_{x \in S^1} \max_{y \in S^2} \left( \langle u, y \rangle_2 + \|y\|_2 \langle v, x \rangle_2 \right)
\]
\[
= \min_{\|x\|_2 = 1} \max_{\|y\|_2 = 1} \left( \langle u, Dz \rangle_2 + \|Dz\|_2 \langle v, x \rangle_2 \right)
\]
\[
\geq \min_{\|x\|_2 = 1} \left( \|Du\|_2 + \frac{\|D^2u\|_2}{\|Du\|_2} \langle v, x \rangle_2 \right)
\]
\[
= \|Du\|_2 - \frac{\sum_{n \geq \ell} a_n}{\|Du\|_2} \|v\|_2 \geq \|Du\|_2 - \sqrt{\|a\|_\infty} \|v\|_2.
\]

Theorem 4.26 implies for all \( c \in \mathbb{R} \) that
\[
P \left[ s_n(A) < c \right] \leq 2P \left[ \Phi_2(u, v) \leq c \right] \leq 2P \left[ \|Du\|_2 - \sqrt{\|a\|_\infty} \|v\|_2 \leq c \right].
\]

To obtain the statement of our lemma, we set \( c = \sqrt{(1 - \delta)\|a\|_1} - \sqrt{(1 + \delta)n\|a\|_\infty}. \)

By Lemma 4.25 we have
\[
P \left[ \|Du\|_2 \leq \sqrt{(1 - \delta)\|a\|_1} \right] \leq \exp \left( -\frac{\delta^2}{4} \|a\|_1 \right)
\]
and
\[
P \left[ \|v\|_2 \geq \sqrt{(1 + \delta)n} \right] \leq \exp \left( -\frac{\delta^2 n}{16} \right).
\]

Now the statement is obtained from a union bound.

We need the statement of Lemma 4.27 for matrices with infinitely many rows, which is obtained from a simple limit argument.

**Lemma 4.28.** The estimate in Lemma 4.27 also holds for \( m = \infty \) if \( a \in \ell^1 \).

**Proof.** Again, we may assume that \( a \) is strictly positive. For \( m \in \mathbb{N} \) let \( A_m \) be the sub-matrix consisting of the first \( m \) rows of \( A \) and let \( a^{(m)} \) be the sub-vector consisting of the first \( m \) entries of \( a \). We use the notation
\[
c_m(\delta) = \sqrt{(1 - \delta)\|a^{(m)}\|_1} - \sqrt{(1 + \delta)n\|a^{(m)}\|_\infty},
\]
\[
p_m(\delta) = 4 \exp \left( -\frac{\delta^2}{16} \min \left\{ n, \frac{\|a^{(m)}\|_1}{\|a^{(m)}\|_\infty} \right\} \right).
\]

where \( c(\delta) \) and \( p(\delta) \) correspond to the case \( m = \infty \). For any \( \varepsilon > 0 \) with \( \varepsilon < \delta/2 \) we can choose \( m \geq n \) such that \( c(\delta) \leq c_m(\delta - \varepsilon) \) and \( p_m(\delta - \varepsilon) \leq p(\delta - 2\varepsilon) \). Note that we have \( s_n(A) \geq s_n(A_m) \) and thus
\[
P [s_n(A) \leq c(\delta)] \leq P [s_n(A_m) \leq c(\delta)] \leq P [s_n(A_m) \leq c_m(\delta - \varepsilon)] \leq p_m(\delta - \varepsilon) \leq p(\delta - 2\varepsilon).
\]

Letting \( \varepsilon \) tend to zero yields the statement.  \( \square \)
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We arrive at our main lower bound.

**Lemma 4.29.** Let $\sigma \in \ell^2$ be nonincreasing and let $n, k \in \mathbb{N}$ be such that $\sigma_k \neq 0$. Define

$$C_k := C_k(\sigma) = \sigma_k^2 \sum_{j > k} \sigma_j^2.$$

Then, for all $\delta \in (0, 1)$, we have

$$\mathbb{P} \left[ R_n^{(k)}(\sigma) \leq \sigma_k \left( 1 - \sqrt{\frac{(1 + \delta)n}{(1 - \delta)C_k^2}} \right) \right] \leq 5 \exp \left( -\frac{\delta}{4} \min \{n, C_k\} \right).$$

**Proof.** First note that, in the setting of Proposition 4.24, the matrix $\Sigma_{[n], N \setminus \{k\}}^T$ and the vector $(g_{ik})_{i=1}^n$ are independent. Lemma 4.25 and Lemma 4.28 yield

$$\| (g_{ik})_{i=1}^n \|_2 \leq \sqrt{1 + \delta \sqrt{n}} \quad \text{and} \quad s_n \left( \Sigma_{[n], N \setminus \{k\}}^T \right) \geq \sqrt{1 - \delta} \sigma_k \sqrt{C_k} - \sqrt{1 + \delta} \sigma_{k+1} \sqrt{n}$$

with probability at least $1 - 5 \exp(-\frac{\delta}{4} \min\{n, C_k\})$. Note that we have

$$s_n \left( \Sigma_{[n], N \setminus \{k\}} \right) = s_n \left( \Sigma_{[n], N \setminus \{k\}}^T \right) \geq s_n \left( \Sigma_{[n], N \setminus \{k\}} \right)$$

since erasing rows can only shrink the smallest singular value. In this case, we have

$$\frac{\| (g_{ik})_{i=1}^n \|_2}{\sigma_k^{-1} s_n \left( \Sigma_{[n], N \setminus \{k\}} \right)} + \| (g_{ik})_{i=1}^n \|_2 \leq \sqrt{1 + \delta \sqrt{n}} \cdot \frac{\sqrt{1 - \delta \sqrt{C_k}} - (\sigma_{k+1}/\sigma_k) \sqrt{1 + \delta \sqrt{n}} + \sqrt{1 + \delta \sqrt{n}}}{\sqrt{1 - \delta \sqrt{C_k}}} \leq \frac{\sqrt{1 + \delta \sqrt{n}}}{\sqrt{1 - \delta \sqrt{C_k}}}.$$

Now the statement is obtained from Proposition 4.24.

The proof of Theorem 4.14 is completed by choosing $\delta = 1/2$. \hfill \Box

**Proofs of Corollaries 4.15, 4.17 and 4.18**

In order to optimize the lower bound of Theorem 4.14, we may choose $k \in \mathbb{N}$ such that the right-hand side of our lower bound becomes maximal. If the Euclidean norm of $\sigma$ is large, we simply choose $k = 1$. Taking into account that $\mathcal{R}_n(\sigma)$ is decreasing in $n$, we immediately arrive at the following result.

**Lemma 4.30.** Let $\sigma \in \ell^2$ be a nonincreasing sequence of nonnegative numbers and let

$$n_0 = \left[ \frac{\varepsilon^2}{3\sigma_1^2} \sum_{j=2}^\infty \sigma_j^2 \right], \quad \varepsilon \in (0, 1).$$

Then $\mathcal{R}_n(\sigma) \geq \sigma_1 (1 - \varepsilon)$ for all $n \leq n_0$ with probability at least $1 - 5e^{-n_0/64}$.

We can now prove that random information is useful if and only if $\sigma \in \ell^2$.
Proof of Corollary 4.15. We first consider the case that $\sigma \in \ell^2$. Since $R_n(\sigma) \leq \sigma_1$, Theorem 4.13 yields

$$E[R_n(\sigma)] \leq 2e^{-n/100} \cdot \sigma_1 + \frac{156}{\sqrt{n}} \left( \sum_{j \geq \lfloor n/4 \rfloor} \sigma_j^2 \right)^{1/2}.$$ 

Now the statement is implied by the fact that $\sigma \in \ell^2$.

For the case that $\sigma \notin \ell^2$, let $0 < \varepsilon < 1$. For $m \in \mathbb{N}$ let $\sigma^{(m)}$ be the sequence obtained from $\sigma$ by replacing $\sigma_j$ with zero for all $j > m$. For any $N \geq n$, we can choose $m \in \mathbb{N}$ such that

$$\frac{\varepsilon^2}{3\sigma_1^2} \sum_{j=2}^m \sigma_j^2 \geq N$$

since $\sigma \notin \ell^2$. Lemma 4.30 yields that

$$P[R_n(\sigma) \geq \sigma_1(1 - \varepsilon)] \geq P[R_n(\sigma^{(m)}) \geq \sigma_1(1 - \varepsilon)] \geq 1 - 5 \exp\left(-N/64\right).$$

Since this holds for any $N \geq n$, we get that the event $R_n(\sigma) \geq \sigma_1(1 - \varepsilon)$ happens with probability 1 for any $\varepsilon \in (0, 1)$. This yields the statement since the event $R_n(\sigma) \geq \sigma_1$ is the intersection of countably many such events.

We now apply our general estimates for $R_n(\sigma)$ to specific sequences $\sigma$ and give a proof of Corollaries 4.17 and 4.18. Note that the first part of Corollary 4.17 which is concerned with slowly decaying sequences is already proven by Corollary 4.15. We add a finite dimensional version of this statement.

Corollary 4.31 (HKNPU19). Let $m, n \in \mathbb{N}$ and consider the sequence $\sigma$ with

$$\sigma_j = \begin{cases} \min\{1, j^{-\alpha}(1 + \ln j)^{-\beta}\} & \text{for } j \leq m, \\ 0 & \text{for } j > m. \end{cases}$$

where $0 \leq \alpha \leq 1/2$ and $\beta \in \mathbb{R}$ with $\beta > 0$ for $\alpha = 0$ and $\beta \leq 1/2$ for $\alpha = 1/2$. Then, for any $0 < \varepsilon < 1$, we have with probability at least $1 - 5 \exp(-n_0/64)$ for all $n \leq n_0$ that

$$1 - \varepsilon \leq R_n(\sigma) \leq 1$$

if we put

$$n_0 = \begin{cases} \frac{\varepsilon^2(m - 2)m^{-2\alpha}}{3(1 + \ln m)^{\max\{2, 0\}}} & \text{for } \alpha < 1/2, \\ \frac{\varepsilon^2(\ln m - 1)}{3(1 + \ln m)^{\max\{2, 0\}}} & \text{for } \alpha = 1/2, \beta < 1/2, \\ \frac{\varepsilon^2(\ln \ln m - 1)}{3} & \text{for } \alpha = \beta = 1/2. \end{cases}$$

We now present a result for sequences on the edge of $\ell^2$. This result shows that random information may be worse than optimal information even if $\sigma \in \ell^2$. 122
Corollary 4.32 ([HKNPU19]). Let $\beta > 1/2$ and consider the sequence $\sigma$ with

$$\sigma_j = \begin{cases} j^{-1/2}(1 + \ln j)^{-\beta} & \text{for } j \leq m, \\ 0 & \text{for } j > m. \end{cases}$$

Then there exist constants $c_\beta, C_\beta > 0$ such that for all $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$ with $m > n^2$ we have with probability at least $1 - 7e^{-n/100}$ that

$$c_\beta n^{-1/2}(1 + \ln n)^{1/2-\beta} \leq R_n(\sigma) \leq C_\beta n^{-1/2}(1 + \ln n)^{1/2-\beta}.$$  

Proof. Note that we have for any $1 < k < m < \infty$ that

$$\sum_{j=k+1}^{m} \sigma_j^2 = \sum_{j=k+1}^{m} j^{-1}(1 + \ln j)^{-2\beta} \asymp \ln^{1-2\beta}(k) - \ln^{1-2\beta}(m),$$

where $\asymp$ means that the both sides of the equation are bounded by a constant multiple of the other side, where the constant depends only on $\beta$. Now the upper bound follows from the first part of Theorem 4.13 and the lower bound follows from the second part of Theorem 4.14 with $k = \lceil c_\beta n/(1 + \ln n) \rceil$ for some $c_\beta > 0$. \hfill \Box

If $\sigma$ decays with a polynomial rate strictly larger than $1/2$, then random information is up to a constant as good as optimal information.

Corollary 4.33 ([HKNPU19]). Let $\alpha > 1/2$ and $\beta \in \mathbb{R}$ and consider the sequence $\sigma$ with

$$\sigma_j = \begin{cases} \min\{1, j^{-\alpha}(1 + \ln j)^{-\beta}\} & \text{for } j \leq m, \\ 0 & \text{for } j > m. \end{cases}$$

Then there exists a constant $C_{\alpha, \beta} > 0$ such that for all $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$ with $n < m$ we have with probability at least $1 - 2e^{-n/100}$ that

$$\sigma_{n+1} \leq R_n(\sigma) \leq C_{\alpha, \beta} \sigma_{n+1}.$$  

Proof. The lower bound is trivial, it holds for every realization of $R_n(\sigma)$. The upper bound is a consequence of Theorem 4.13 since for large $n$ we have

$$\sum_{j \geq \lceil n/4 \rceil} \sigma_j^2 = \sum_{j \geq \lceil n/4 \rceil} j^{-2\alpha}(1 + \ln j)^{-2\beta} \leq Cn^{1-2\alpha}(1 + \ln n)^{-2\beta}$$

with a constant $C$ depending only on $\alpha$ and $\beta$. \hfill \Box

Corollaries 4.31, 4.32 and 4.33 form a proof of Corollary 4.17.

Proof of Corollary 4.17. It suffices to consider the sequences from Corollaries 4.31, 4.32 and 4.33 since $\sigma \leq C\sigma'$ implies $R_n(\sigma) \leq C R_n(\sigma')$ for all $n$. Since we have $0 \leq R_n(\sigma) \leq \sigma_1$ almost surely, the statements for the expected value hold if the corresponding lower bounds hold at least with a constant positive probability and if the corresponding upper bounds hold with probability at least $1 - c\sigma_{n+1}$ for some constant $c > 0$. This is shown in the corollaries. \hfill \Box
Remark 4.34. The case $\sigma_n \asymp n^{-\alpha} \ln^{-\beta}(n + 1)$ with $\alpha > 1/2$ can be extended to
$\sigma_n \asymp n^{-\alpha}\varphi(n)$ for any slowly varying function $\varphi$. In this case, random information is
up to a constant as powerful as optimal information, i.e., $\mathbb{E}[R_n(\sigma)] \asymp \sigma_{n+1}$.

We turn to the case of exponentially decaying singular values

Proof of Corollary 4.18. The lower bound is implied by the trivial estimate $R_n(\sigma) \geq \sigma_{n+1}$. To prove the upper bound, we use the second part of Theorem 4.13. Without
loss of generality, we may assume that $\sigma_j = a^{j-1}$ for all $j \in \mathbb{N}$. The general case
follows from the fact that $\sigma \leq C\sigma'$ implies $R_n(\sigma) \leq C R_n(\sigma')$ for all $n$. We choose
$c \geq 1$ such that $e^{-c^2} \leq a$. Note that there is some $b > 0$ such that

$$\left( \sum_{j > n} \sigma_j^2 \right)^{1/2} = \frac{b a^n}{14}$$

for all $n \in \mathbb{N}$. The theorem yields for all $t \geq bna^n$ that

$$P[R_n(\sigma) \geq t] \leq a^n + \frac{bna^n c\sqrt{2e}}{t}.$$ 

This yields that

$$\mathbb{E}[R_n(\sigma)] = \int_0^1 P[R_n(\sigma) \geq t] \, dt \leq a^n + bna^n + na^n \int_{bna^n}^1 \frac{bc\sqrt{2e}}{t} \, dt \lesssim n^2 a^n,$$

as it was to be proven. \qed
Symbols

General

\( \mathbb{N}_0, \mathbb{N} \) the set of natural numbers with and without zero
\( [k] \) the set of natural numbers from 1 to \( k \in \mathbb{N} \)
\( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) the sets of integers, rational, real and complex numbers
\( \lfloor a \rfloor \) the largest integer smaller than or equal to \( a \in \mathbb{R} \)
\( \lceil a \rceil \) the smallest integer larger than or equal to \( a \in \mathbb{R} \)
\( \ln(x) \) natural logarithm of \( x > 0 \)
\( \log_a(x) \) logarithm of \( x > 0 \) in base \( a > 0 \)
\( \text{card}(A) \) cardinality of a set \( A \); number of elements if \( A \) is finite
\( A \subset B \) set inclusion, equality allowed
\( \text{dist}(f,g) \) distance of \( f \) and \( g \) in a metric space
\( \|f\|_G \) norm of \( f \) in a normed space \( G \)
\( \langle f,g \rangle_H \) scalar product of \( f \) and \( g \) in a pre-Hilbert space \( H \)
\( \text{rad}(M) \) radius of a set \( M \) in a metric space, see (1.1)
\( T \) a circle, usually represented by \([0, 1]\) where 0 and 1 are identified
\( T^d \) the \( d \)-torus, usually represented by \([0, 1]^d\)

Vectors and Sequences of Real Numbers

\( \mathbf{x} = (x_1, \ldots, x_m) \) vector in \( \mathbb{C}^m \) with entries \( x_i \)
\( \mathbf{x} = (x_1, x_2, \ldots) \) vector in \( \mathbb{C}^\mathbb{N} \) with entries \( x_i \)
\( \mathbf{x}_J \) sub-vector \((x_i)_{i \in J}\) of \( \mathbf{x} \) for some index set \( J \)
\( \mathbf{0} \) vector with all the entries set to 0
\( \mathbf{1} \) vector with all the entries set to 1
\( \mathbf{e}_i \) vector with the \( i \)-th entry set to 1 and all other entries set to 0
\( [\mathbf{x}, \mathbf{y}] \) set of vectors \( \mathbf{z} \) with entries \( z_i \) between \( x_i \in \mathbb{R} \) and \( y_i \in \mathbb{R} \)
\( I_J \) Cartesian product of intervals \( I_j \) over \( j \in J \)
\( \langle \mathbf{x}, \mathbf{y} \rangle \) Euclidean scalar product, that is, \( \langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i \)
\( \|\mathbf{x}\|_p \) \( p \)-norm of a vector, that is, \( \|\mathbf{x}\|_p = (\sum_i |x_i|^p)^{1/p} \) for \( 1 \leq p < \infty \)
and \( \|\mathbf{x}\|_\infty = \sup_i |x_i| \) for \( p = \infty \)
\( |\mathbf{x}| \) sometimes used instead of \( \|\mathbf{x}\|_1 \), mainly if \( \mathbf{x} \in \mathbb{Z}^d \)
\( \ell^p_m \) \( \mathbb{R}^m \) equipped with the \( p \)-norm; in some contexts \( \mathbb{C}^m \)
\( \ell^p \) space of all vectors in \( \mathbb{R}^\mathbb{N} \) with finite \( p \)-norm equipped with the \( p \)-norm; in some contexts \( \mathbb{C}^\mathbb{N} \)
Comparison of Sequences of Positive Numbers

\[ x_n \preceq y_n \]  
for every constant \( c > 0 \) there is a threshold \( n_0 \in \mathbb{N} \) such that \( x_n \leq cy_n \) for all \( n \geq n_0 \)

\[ x_n \succeq y_n \]  
for every constant \( c > 0 \) there is a threshold \( n_0 \in \mathbb{N} \) such that \( x_n \geq cy_n \) for all \( n \geq n_0 \)

\[ x_n \precsim y_n \]  
for every constant \( 0 < c < 1 \) there is a threshold \( n_0 \in \mathbb{N} \) such that \( x_n \geq cy_n \) for all \( n \geq n_0 \); equivalent to \( y_n \preceq x_n \)

\[ x_n \succsim y_n \]  
for every constant \( 0 < c < 1 \) there is a threshold \( n_0 \in \mathbb{N} \) such that \( x_n \leq cy_n \) for all \( n \geq n_0 \); equivalent to \( y_n \succeq x_n \)

Matrices and Operators

\[ \text{diag}(x) \]  
square matrix with main diagonal \( x \in \mathbb{R}^m \) and all other entries set to 0

\[ A^{-1} \]  
inverse of a square matrix

\[ A^\top \]  
transpose of a matrix

\[ (A^\top)^{-1} \]  
transpose of the inverse of a square matrix

\[ \det(A) \]  
determinant of a square matrix

\[ \ker(A) \]  
kernel of a matrix

\[ \| T : X \to Y \| \]  
operator norm of a bounded linear operator \( T \) between normed spaces \( X \) and \( Y \), that is, \( \sup \{ \| Tx \|_Y \mid x \in X, \| x \|_X = 1 \} \)

\[ \mathcal{L}(X,Y) \]  
space of bounded linear operators between \( X \) and \( Y \) equipped with the operator norm

\[ \| A \|_p \]  
operator norm of the matrix \( A \in \mathbb{R}^{n \times m} \) in \( \mathcal{L}(\ell_p^n, \ell_p^m) \)

\[ X \hookrightarrow Y \]  
embedding, \( X \) is identified with a subset of \( Y \), \( f \mapsto f \)

Functions and Derivatives

\[ f : D \to \mathbb{R} \]  
real valued function on a domain \( D \subset \mathbb{R}^d \), mapping a point \( x \in D \) to a number \( f(x) \in \mathbb{R} \)

\[ \sup f \]  
supremum of \( f \), that is, \( \sup f = \sup \{ f(x) \mid x \in D \} \)

\[ \text{supp } f \]  
support of \( f \); closure of the set \( \{ x \in D \mid f(x) \neq 0 \} \)

\[ f|_P \]  
restriction of \( f \) to the set \( P \subset D \)

\[ f^{(r)} \]  
the \( r^{th} \) weak derivative of \( f \) in the case \( D \subset \mathbb{R} \); if possible, \( f^{(r)} \) is identified with a continuous function

\[ \partial_{\theta} f \]  
directional (weak) derivative of \( f \) in the direction \( \theta \in \mathbb{S}_{d-1} \)
\[ \frac{\partial f}{\partial x_i} \] partial (weak) derivative with respect to \( x_i \); equivalently \( \partial_{x_i} f \)

\[ D^\alpha f \] partial (weak) derivative \( \frac{\partial^{\alpha_1} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \) of order \( \alpha \in \mathbb{N}^d \)

\[ D\Psi \] Jacobian matrix of a function \( \Psi : D \to \mathbb{R}^d \)

\[ |D\Psi| \] absolute value of the determinant of \( D\Psi \)

\[ f_1 \otimes \cdots \otimes f_d \] tensor product of the functions \( f_i : D_i \to \mathbb{C} \); maps \( x \in \prod_{i=1}^d D_i \) to \( \prod_{i=1}^d f_i(x_i) \in \mathbb{C} \)

\[ f_J \] tensor product of the functions \( f_i \) over \( i \in J \)

**Measures and Function Spaces**

\((D, \mathcal{A}, \mu)\) measure space

\[ \|f\|_p \] \( p \)-norm of a measurable function \( f : D \to \mathbb{C} \) (with respect to \( \mathcal{A} \) and the Borel \( \sigma \)-algebra on \( \mathbb{C} \) ), that is, \( \|f\|_p = (\int |f|^p \ d\mu)^{1/p} \) for \( 1 \leq p < \infty \) and \( \|f\|_\infty = \text{ess sup}_{x \in D} |f(x)| \) for \( p = \infty \);

\[ L^p(D, \mathcal{A}, \mu) \] the space of measurable functions \( f : D \to \mathbb{R} \) with finite \( p \)-norm; functions that are equal \( \mu \)-almost everywhere are identified; sometimes \( \mathbb{C} \) instead of \( \mathbb{R} \)

\[ \langle f, g \rangle \] scalar product in \( L^2(D, \mathcal{A}, \mu) \), that is, \( \langle f, g \rangle = \int f \bar{g} \ d\mu \);

\[ L^p(D) \] short for \( L^p(D, \mathcal{A}, \mu) \) if \( D \) is a domain in \( \mathbb{R}^d \), \( \mathcal{A} \) is the Borel \( \sigma \)-algebra and \( \mu \) is the \( d \)-dimensional Lebesgue measure

\[ \lambda^d \] \( d \)-dimensional Lebesgue measure

\[ \mu(\text{Statement}) \] measure of the set of all \( x \in D \) for which \( \text{Statement} \) is true

\((\Omega, \mathcal{F}, \mathbb{P})\) usually used instead of \( (D, \mathcal{A}, \mu) \) if \( \mu(D) = 1 \); probability space

\[ \mathbb{E} \] expectation, that is, \( \mathbb{E}X = \int X \ d\mathbb{P} \) for \( X \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \)

\[ A^c \] complement of \( A \subset \Omega \), that is, \( A^c = \Omega \setminus A \)

\[ \mathcal{B}(D) \] bounded real valued functions on a set \( D \)

\[ \mathcal{C}(D) \] continuous real valued functions on a topological space \( D \)

\[ \mathcal{C}_c(D) \] continuous real valued functions on \( D \) with compact support

\[ \mathcal{C}^r(D) \] \( r \) times continuously differentiable real valued functions on a domain \( D \subset \mathbb{R}^d \)

\[ \mathcal{C}^\infty(D) \] infinitely differentiable real valued functions on \( D \)

\[ W^r_p(D) \] Sobolev space of functions \( f : D \to \mathbb{R} \) whose weak derivatives \( D^\alpha f \) exist and are in \( L^p(D) \) for all \( \alpha \in \mathbb{N}^d \) with \(|\alpha| \leq r \)

\[ H^r(D) \] equal to \( W^r_2(D) \)

\[ W^r_{p,\text{mix}}(D) \] Sobolev space of functions \( f : D \to \mathbb{R} \) whose weak derivatives \( D^\alpha f \) exist and are in \( L^p(D) \) for all \( \alpha \in \mathbb{N}^d \) with \( \|\alpha\|_{\infty} \leq r \)

\[ H^r_{\text{mix}}(D) \] equal to \( W^r_{2,\text{mix}}(D) \)
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Jena, 05. Februar 2019

David Krieg