A finite volume method for the simulation of elastoviscoplastic flows and its application to the lid-driven cavity case

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Abstract

We propose a Finite Volume Method for the simulation of elastoviscoplastic flows, modelled after the extension to the Herschel-Bulkley model by Saramito [J. Non-Newton. Fluid Mech. 158 (2009) 154–161]. The method is akin to methods for viscoelastic flows. It is applicable to cell-centred grids, both structured and unstructured, and includes a novel pressure stabilisation technique of the “momentum interpolation” type. Stabilisation of the velocity and stresses is achieved through a “both sides diffusion” technique and the CUBISTA convection scheme, respectively. A second-order accurate temporal discretisation scheme with adaptive time step is employed. The method is used to obtain benchmark results of lid-driven cavity flow, with the model parameters chosen so as to represent Carbopol. The results are compared against those obtained with the classic Herschel-Bulkley model. Simulations are performed for various lid velocities, with slip and no-slip boundary conditions, and with different initial conditions for stress. Furthermore, we investigate the cessation of the flow, once the lid is suddenly halted.

Keywords: elastoviscoplastic flow; finite volume method; carbopol; lid-driven cavity; benchmark problem

1. Introduction

Viscoplastic (VP) fluids, also referred to as yield stress fluids, are a class of materials whose distinctive property is that they flow as fluids if subjected to large enough stresses but behave as solids if the applied stress is below a critical value, termed the yield stress. Although solids can also undergo plastic deformation, the materials classified as viscoplastic fluids are characterised by complete reversibility of the structural changes caused during plastic flow once the flow ceases [1, 2]. The structure recovery can occur either immediately or gradually; in the latter case the fluids are called thixotropic. The class of VP fluids includes a variety of materials such as foams, emulsions, colloids and physical gels, with possibly different microscopic mechanisms being responsible for the emergence of a yield stress in each case [3]. Nevertheless, in every case the emergence of a yield stress is associated with the complex structure of the fluid. Viscoplastic flows are of major relevance in many industries (oil, construction, cosmetics, foodstuffs, etc.) while also many natural flows can be classified as such (mud and lava flows, landslides, avalanches etc.).

Viscoplastic fluids were first studied in depth by Eugene Bingham [1] who proposed the renowned constitutive equation named after him to describe their behaviour [4]. A short time later, Herschel and Bulkley [5] extended the Bingham model to describe also the shear-thinning (or shear-thickening) post-yield behaviour that most of these materials exhibit, by assuming a viscosity dependence on the shear rate according to a power law. These models were originally proposed in scalar form, but it was not long until a full tensorial form was proposed, employing the von Mises yield criterion [6]. The empirical Herschel-Bulkley (HB) constitutive equation has been found to represent well the behaviour of yield stress fluids under steady shear flow, and is arguably the most popular model for VP fluids. It is commonly given in the following form:

\[
\begin{align*}
\tau & \leq \tau_y \Rightarrow \dot{\gamma} = 0 \\
\tau & > \tau_y \Rightarrow \tau = \left( \frac{\tau_y}{\dot{\gamma}} + k \dot{\gamma}^{n-1} \right) \dot{\gamma} 
\end{align*}
\] (1)

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where \( \tau \) is the extra-stress tensor; \( \dot{\gamma} = \nabla \mathbf{u} + (\nabla \mathbf{u})^T \) is the rate-of-strain tensor, \( \mathbf{u} \) being the fluid velocity vector and \( T \) denoting the tensor transpose; and \( \tau \equiv \left( \frac{1}{2} \tau : \tau \right)^{\frac{1}{2}} \) and \( \dot{\gamma} \equiv \left( \frac{1}{2} \tau : \tau \right)^{\frac{1}{2}} \) are their magnitudes. The HB model includes three parameters, the yield stress \( \tau_y \), the consistency \( k \), and the exponent \( n \); the latter usually assumes values in the range \( n = 0.2 - 0.8 \) [3, 2], which represents a shear-thinning behaviour (the special case \( n = 1 \) corresponds to the Bingham model). According to Eq. (1), the material behaves either as a rigid solid (\( \dot{\gamma} = 0 \)) or as a generalised Newtonian fluid (\( \tau = \eta(\dot{\gamma})\dot{\gamma} \), where \( \eta(\dot{\gamma}) \) is the term in brackets in (1)), with the criterion of flow being that the magnitude of the stress exceeds the value \( \tau_y \). Thus the flow domain consists of “unyielded” (solid) and “yielded” (fluid) regions, separated by a sharp interface, the yield surface, where \( \tau = \tau_y \).

A subtle issue concerning Eq. (1) requires clarification for it to provide a complete description of the HB fluid. Equation (1) has two branches, one applying to the solid state and one to the fluid state. For fluids, the total stress tensor \( \sigma \) is conveniently decomposed into a part which is related to the fluid deformation \( \dot{\gamma} \), and an isotropic pressure part, as

\[
\sigma = \tau - pI
\]

(2)

where \( I \) is the identity tensor. The component \( \tau \) called the extra-stress tensor (in the present work we also refer to it as the VP or EVP stress tensor), expresses the forces that arise in the fluid due to the deformation of its particles, and for each class of fluid it is described by a suitable constitutive equation, such as (1). For incompressible fluids, the pressure \( p \) then arises so as to enforce the continuity equation (mass conservation) through the momentum equations. However, when it comes to the unyielded branch of Eq. (1), it is apparent that the decomposition (2) is not relevant: on one hand, that branch does not precisely define \( \tau \) in terms of \( \dot{\gamma} \) (or its history), which is zero anyway. On the other hand, the meaning of “pressure” is also ambiguous, because the incompressible continuity equation, \( \nabla \cdot \mathbf{u} = 0 \), is already implied by the rigidity condition \( \dot{\gamma} = 0 \) and does not require a pressure to be enforced.

For solids it is more convenient to decompose the total stress into deviatoric (traceless) and isotropic parts:

\[
\sigma = \sigma_d + \sigma_i = \tau - \frac{1}{3} \sigma : I + \frac{1}{3} \sigma : I = \frac{1}{3} \tau - \frac{1}{3} \sigma : I + \frac{1}{3} \sigma : I
\]

(3)

where \( \sigma : I \equiv \sum_{ij} \sigma_{ji} \) is the trace of \( \sigma \). The popular von Mises yield criterion for solids, employed also by the HB constitutive model [6, 7], assumes that the isotropic component \( \sigma_i \) does not contribute to yielding and materials yield when the magnitude of the deviatoric component \( \sigma_d \) equals the yield stress.

It turns out that in the fluid branch of the HB model the two decompositions are equivalent. This can be seen by decomposing \( \tau \) in the right-hand side of Eq. (3) according to the decomposition (2):

\[
\tau = \tau - \frac{1}{3} \sigma : I + \left( \frac{1}{3} \tau - pI \right) I
\]

(4)

For a generalised Newtonian fluid, such as the HB material in its fluid state, \( \tau = \eta(\dot{\gamma}) \dot{\gamma} \Rightarrow \tau = \eta(\dot{\gamma}) \dot{\gamma} = 2\eta(\dot{\gamma}) \nabla \cdot \mathbf{u} = 0 \) by virtue of the incompressible continuity equation. Thus, Eq. (4) reveals that \( \sigma_d = \tau \) and \( \sigma_i = -pI \); decomposing the total stress into extra stress and pressure is equivalent to decomposing it into deviatoric and isotropic parts.

Since the two decompositions are equivalent in the fluid branch, it is convenient to impose the same also for the solid branch by defining that \( \tau \equiv \sigma_d \) and \( -pI \equiv \sigma_i \Rightarrow p = -(1/3) \tau \) there. We are allowed to do so because the meanings of \( \tau \) and \( p \) are ambiguous in that branch. With these definitions, the HB equation can be written in the form (1) where \( \tau \) is both the extra stress tensor and the deviatoric stress tensor in both branches, so that the yield condition is precisely the von Mises criterion. It should be kept in mind that in the unyielded regions \( \tau \) is merely the deviatoric part of the total stress and \( p \) is its isotropic part.

When elasticity is introduced into the model (Sec. 2), \( \tau \) will no longer be necessarily zero and decompositions (2) and (3) will no longer be equivalent.

With some manipulations, the fluid branch of Eq. (1) can be solved for \( \dot{\gamma} \), and then combined with the solid branch [8]. In the fluid branch, the stress tensor and the rate-of-strain tensor are parallel (as in
generalised Newtonian flow) and thus the unit tensors $\dot{\gamma}/\gamma$ and $\tau/\tau$ are equal. Therefore, the rate-of-strain tensor can be written as $\dot{\gamma} = \dot{\gamma}(\dot{\gamma}/\gamma) = \dot{\gamma}(\tau/\tau)$. It remains to express $\dot{\gamma}$ as a function of $\tau$, which can be done by taking the magnitudes of both sides of the fluid branch: $\tau = \tau_y + k\dot{\gamma}^n \Rightarrow \dot{\gamma} = [(\tau - \tau_y)/k]^{1/n}$.

Substituting this into $\dot{\gamma} = \dot{\gamma}(\tau/\tau)$, and combining the result with the solid branch of Eq. (1), we get the following convenient equivalent form of the HB law [8] which is valid for all stress magnitudes:

$$\dot{\gamma} = \left(\max(0, \frac{\tau - \tau_y}{k})\right)^{\frac{1}{n}} \frac{1}{\tau} \tau$$

(5)

Analytical solutions of VP flows are possible only for simple flow configurations [9]. However, the assessment of how accurately VP models represent real materials requires complex two- and three-dimensional simulations complemented by corresponding accurate experimental measurements. For example, there is some experimental and theoretical evidence that the von Mises criterion is appropriate for VP fluids [10] but the issue is not completely resolved. On the other hand, it is reported that for some materials the nonlinear (shear-thinning) viscous term of the HB model is appropriate only for shear flows [10]. For complex configurations one has to resort to numerical solution, which is a challenging task due to the discontinuity of the constitutive equations at the yield surfaces and the indeterminacy of the stress tensor within the unyielded regions which is inherent in the traditional VP models. A popular way to circumvent these challenges is to regularise the constitutive equation, replacing it with a single-branched equation that treats the solid phase as an extremely viscous fluid. Alternatively, the discontinuous constitutive equation can be solved directly using techniques that are more mathematically involved, such as the augmented Lagrangian (AL) algorithm. More information on regularisation and direct methods, together with examples of applications of VP simulations, can be found in the review papers [11, 12] (see also [13] for recent developments). Both of these classes of methods, direct and regularised, can be implemented in combination with various discretisation methods. The Finite Element Method (FEM) has been the most popular, but the Finite Volume Method (FVM), which is very popular in Newtonian fluid dynamics, has also been used. FVMs for VP flows in the regularisation and AL frameworks are described in [14] and [15], respectively.

Apart from the aforementioned difficulties in solving VP constitutive equations, there is the question of whether their assumptions of a completely rigid (inelastic) solid phase and a purely viscous fluid phase are physically realistic. Allowing for some deformation of the solid phase under stress seems more natural and indeed experimental studies on Carbopol, a prototypical material often used in experimental studies on viscoelasticity, have shown that prior to yielding it exhibits elastic deformation under stress [16, 17]. Furthermore, elastic effects can be observed also in the fluid phase. For example, bubbles rising in Carbopol solutions usually acquire the shape of an inverted teardrop, with a cusp at their leeward side [18, 19], but the classical Bingham and HB viscoplastic models fail to predict such behaviour [20, 21]; on the other hand, such shapes are observed also in viscoelastic fluids, and are correctly captured by viscoelastic constitutive equations [22], which suggests fluid elasticity as their cause. Similarly, for the settling of spherical particles in Carbopol, classical VP models cannot predict phenomena such as the loss of fore-aft symmetry under creeping flow conditions and the formation of a negative wake behind the sphere, but these phenomena are predicted if elasticity is incorporated into the constitutive modelling [23].

Therefore, recently the focus has been shifting towards constitutive equations that incorporate both plasticity and elasticity, usually called elastoviscoplastic (EVP) constitutive equations. Actually, EVP constitutive modelling dates back to the beginning of the previous century – a nice historical overview can be found in [24]. Although several EVP models have been proposed (see e.g. the literature reviews in [24, 25]), many of them appear only in scalar form. To be applicable in complex two- and three-dimensional flow simulations (2D/3D), a full tensorial form is required; models that have this form include those proposed in [24, 26]. Some of these models are compared in [27].

Complex simulations also require that these models be accompanied by appropriate numerical solvers characterised by accuracy, robustness and efficiency. Usually, FEM solvers are employed in EVP flow simulations (e.g. [28, 29, 27]). An alternative, very popular discretisation / solution method in Computational Fluid Dynamics is the Finite Volume Method; it has been successfully applied to viscoplastic (e.g. [14, 30]) and viscoelastic (e.g. [31–34]) flows individually, but not to EVP flows, to the best of our knowledge (a hybrid FE/FV method was used in [35]). In this paper a FVM for the simulation of EVP flows is described.
The EVP constitutive equation chosen is that of Saramito [26] which introduces elasticity into the classic HB model, to which it reduces in the limit of inelastic behaviour. This model shall be referred to as the Saramito-Herschel-Bulkley (SHB) model. We chose this model because of its simplicity, its positive potential as revealed in a number of recent studies on materials such as foams [36] and Carbopol [37], and because of the popularity of the classic HB model. Nevertheless, the general framework of the presented FVM should be applicable to a range of other EVP models, particularly those that can be regarded as modifications/extensions of viscoelastic constitutive equations.

The presentation of the method in Sections 3 and 4 and its validation in Sec. 5 are followed by application of the method to simulate EVP flow in a lid-driven cavity. The lid-driven cavity test case is arguably the most popular benchmark test case for new numerical methods for flow simulations. As such, it has been used also as a benchmark problem for viscoplastic [14, 38] and viscoelastic [39] flows; there exist also the EVP flow studies of [40, 29], but with a different EVP model that incorporates a kind of regularisation. In the present study the parameters of the SHB model are chosen so as to represent Carbopol, which is regarded as a simple VP fluid (more complex behaviour such as thixotropy [41] and kinematic hardening [42] are not considered, but may be incorporated into the model in the future). The lid-driven cavity problem constitutes a convenient “playground” for testing the numerical method, testing the behaviour of the SHB model under conditions of complex 2D flow, comparing its predictions against those of the classic HB model, and providing benchmark results. The tests include varying the lid velocity to vary the flow character as quantified by dimensionless numbers, flow cessation (which occurs in finite time for VP flows), and varying the initial conditions to investigate the issue of multiplicity of solutions of the SHB model.

2. Governing equations

The flow is governed by the continuity, momentum, and constitutive equations. The first two of these are, respectively,

\[ \nabla \cdot (\rho \mathbf{u}) = 0 \quad (6) \]

\[ \frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla p + \nabla \cdot \mathbf{\tau} \quad (7) \]

where \( t \) is time and \( \rho \) is the density of the material; the rest of the variables have been defined in Sec. 1. The right-hand side of Eq. (7) can be written collectively as \( \nabla \cdot \sigma \). The material will be assumed incompressible, so \( \rho \) is constant (temperature variations, which may affect \( \rho \), will also be neglected in this work).

Closure of the system of governing equations requires that the extra stress tensor be related to the flow kinematics through a constitutive equation. In the present work we use the EVP constitutive equation proposed by Saramito [26], which is based on the Herschel-Bulkley viscoplastic model. This equation assumes that the total deformation of the material is equal to the sum of an elastic strain \( \gamma_e \) (applicable in both the solid and fluid phases) and the provisional (if the yield stress is exceeded) viscous deformation \( \gamma_v \) predicted by the HB model (5). This behaviour is depicted schematically in Fig. 1 by a mechanical analogue. In terms of the rate-of-strain this is written as:

\[ \frac{1}{G} \dot{\gamma_e} + \left( \frac{\max(0, \tau_d - \tau_y)}{k} \right) \frac{1}{\tau_d} \dot{\tau_d} = \dot{\gamma}_v \quad (8) \]

where \( G \) is the elastic modulus, the triangle denotes the upper-convected derivative

\[ \dot{\tau_d} \equiv \frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau - (\nabla \mathbf{u})^T \cdot \tau - \tau \cdot \nabla \mathbf{u} \quad (9) \]

(a form of time derivative obtained on a coordinate system that travels, rotates and stretches with the flow) and \( \tau_d \equiv (\frac{1}{2} \tau_d : \tau_d)^{1/2} \) is the magnitude of the deviatoric part of the EVP stress tensor,

\[ \tau_d = \tau - \frac{1}{3} \text{tr}(\tau) I \quad (10) \]
Figure 1: A mechanical analogue of the Saramito-Herschel-Bulkley (SHB) model [26].

Note that because $\tau$ and $\sigma$ differ only by an isotropic quantity ($-pI$) it holds that $\tau_d = \sigma_d$ (as noted on Eq. (4)). An important difference with the classic HB model (5) is that, due to the last two terms of the upper convected derivative (9), the trace of $\tau$ is now not necessarily zero and thus, in general, $\tau_d \neq \tau$, necessitating explicit use of $\tau_d$ inside the “max” term of Eq. (8) in order to express the von Mises yield criterion. The full 3D formula (10) must be used for the deviatoric stress even in 1D or 2D simulations: a 2D stress state where all $\tau_{ij}$ are zero except $\tau_{11} = \tau_{22} \neq 0$ is not isotropic ($\tau_{33} = 0 \neq \tau_{11} , \tau_{22}$) and $\tau_d$ is not zero.

Other qualitative differences with the HB model exist. For example, the SHB model allows non-zero rate of strain in the unyielded regions, arising from elastic deformations of the solid phase. Conversely, it is theoretically possible that the rate of strain is zero at some point and yet the material is yielded there because the stresses have not had enough time to relax. In particular, in the SHB model, contrary to the HB model, the stresses do not respond instantaneously to changes in the rate of strain; rather, due to the presence of its time derivative, the stress tensor is also a function of all past states of $\dot{\gamma}$, as in viscoelastic fluids (the fluid has “memory”). Indeed, familiarity with viscoelasticity is a strong aid for an in-depth understanding of the SHB model.

Furthermore, as noted in [43, 36], the combination of plastic and elastic terms in the SHB Eq. (8) can produce complex behaviour not observed in either purely viscoplastic or purely viscoelastic flows. One such feature of the SHB model is that it allows the extra stress tensor and the velocity gradient to vary discontinuously across yield surfaces [43]. Another such feature is that flows are inherently transient, in the sense that there exist an infinitude of steady states which depend in a continuous manner on the initial conditions. Even if it is just the steady state that is sought, one cannot simply discard the time derivatives in Eqs. (7) and (8) but that steady state must be obtained by a transient simulation with appropriate initial conditions. It is not only the steady state stresses that depend on the initial conditions, but also the steady state velocity. This issue was studied in [36] in the context of cylindrical Couette flow. In fact, actual EVP materials do behave this way in experiments, with the steady state depending on the residual stresses present in the unyielded, stationary material at the start of the experiment [36]. The residual stresses are stresses that are “trapped” inside unyielded material because there is no relaxation mechanism there. E.g. for stationary ($u = 0$) unyielded material, Eq. (8) predicts $\partial\tau/\partial t = 0$. In experiments, residual stresses do develop during the preparation of the material and are difficult to eliminate.

This contrasts the behaviour of classic VP models such as the HB, where there is a stress indeterminacy in the unyielded regions: there exist infinite $\sigma$ fields within these regions that have the same divergence $\nabla \cdot \sigma$ which yields $\dot{\gamma} = 0$ when substituted in the momentum equation (7). Each of these fields is a valid solution. The HB steady-state does not depend on the initial conditions, and can be obtained directly, by dropping the time derivative from Eq. (7). The HB model makes no connection between this indeterminacy and the initial conditions, and in fact it even allows that stresses in the unyielded regions vary discontinuously in time. No such indeterminacy is exhibited by the SHB model, the stresses in both the yielded and unyielded regions being precisely determined for given initial conditions.

In the limit of very high elastic modulus, $G \rightarrow \infty$, the material becomes so stiff that it behaves as an inelastic fluid (when yielded) or inelastic solid (when unyielded); the importance of the first term on the left-hand side of the SHB constitutive equation (8) diminishes and that equation tends to become identical to the classic HB equation (5). However, it must be kept in mind that the difference in character between the SHB and HB models is in some respects retained no matter how large (but finite) the value of $G$ is.
This concerns mostly the stress state in the unyielded regions, which for the SHB model remains uniquely
determined by the initial conditions whereas that of the HB model is indeterminate. Also, as discussed in
Appendix A, in the HB model \( \tau = \sigma d \) whereas for the SHB model the identification of \( \tau \)
with \( \sigma d \) as \( G \to \infty \) is not guaranteed.

Figure 1 shows that the complete SHB model contains an additional viscous component labelled \( \kappa \) not
discussed thus far. This is a Newtonian component, of viscosity \( \kappa \), which makes the unyielded phase behave
as a Kelvin-Voigt solid. The entire extra stress tensor is then
\[
\tau_e = \tau + \kappa \dot{\gamma}
\]
The SHB model then reduces to the Oldroyd-B viscoelastic model when \( \tau_y = 0 \) and \( n = 1 \). However, in the
main results of Sec. 6 we will use \( \kappa = 0 \).

The governing equations include the boundary conditions. Here we consider solid wall boundaries.
We will mostly employ the no-slip boundary condition, but since (elasto)viscoplastic materials are usually
slippery we will also employ a Navier slip boundary condition, according to which the relative velocity
between the fluid and the wall, in the tangential direction, is proportional to the tangential stress. For
two-dimensional flows this is expressed as follows: Let \( \mathbf{n} \) be the unit vector normal to the wall, and \( \mathbf{s} \) be the
unit vector tangential to the wall within the plane in which the equations are solved. Let also \( \mathbf{u} \) and \( \mathbf{u}_w \) be
the fluid and wall velocities, respectively. Then,
\[
(u - u_w) \cdot \mathbf{s} = \beta (n \cdot \tau) \cdot \mathbf{s}
\]
where the parameter \( \beta \) is called the slip coefficient.

2.1. De-dimensionalisation of the governing equations

Although in the present work we will solve the governing equations in their dimensional form, it is useful
to express them also in dimensionless form so as to expose a number of dimensionless parameters that
characterise the flow. Selecting characteristic scales of length, \( L \), time, \( T \), speed, \( U \), and stress, \( S \), we define
the dimensionless variables, denoted with a tilde (\( \tilde{\cdot} \)), as \( \tilde{x}_i = \hat{x}_i L \) (spatial coordinates), \( t = \hat{t} T \), \( u = \hat{u} U \),
\( p = \hat{p} S \), and \( \tau = \hat{\tau} S \). Substituting these into the governing equations (6), (7), and (8) combined with (9),
we obtain their dimensionless forms:
\[
\tilde{\nabla} \cdot \tilde{u} = 0
\]
\[
Re \left( \frac{1}{Sr} \frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{u} \tilde{u}) \right) = -\nabla \tilde{p} + \tilde{\nabla} \cdot \tilde{\tau}
\]
\[
Wi \left( \frac{1}{Sr} \frac{\partial \tilde{\tau}}{\partial \tilde{t}} + \tilde{u} \cdot \tilde{\nabla} \tilde{\tau} - (\tilde{\nabla} \tilde{u})^T \cdot \tilde{\tau} - \tilde{\tau} \cdot \tilde{\nabla} \tilde{u} \right) + \left( \frac{S}{k(U/L)^n} \right)^\frac{1}{n} \max(0, \tilde{\tau}_d - Bn)^\frac{1}{n} \frac{1}{\tilde{\tau}_d^{\frac{1}{2}}} = \tilde{\gamma}
\]
where \( \tilde{\nabla} \equiv \epsilon_j \partial / \partial \tilde{x}_i = (1/L) \nabla \) and \( \tilde{\gamma} \equiv \tilde{\nabla} \tilde{u} + (\tilde{\nabla} \tilde{u})^T = \dot{\gamma} / (U/L) \). The following dimensionless numbers appear:

Reynolds number \( Re \equiv \rho U^2 / S \) (16)

Strouhal number \( Sr \equiv T / L/U \) (17)

Weissenberg number \( Wi \equiv S / G \) (18)

Bingham number \( Bn \equiv \tau_y / S \) (19)

The physical significance of the above dimensionless numbers depends on the choice of the reference stress
\( S \), which is usually chosen to be a typical value of the extra stress tensor for the given flow. In the present
work, we assign to \( S \) the value
\[
S \equiv \tau_y + k \left( \frac{U}{L} \right)^n
\]

(20)
which is expected to be such a typical value (the influence of the Newtonian viscosity $\kappa$, Eq. (11) is omitted, but as mentioned we will set $\kappa = 0$ in the main numerical experiments of Sec. 6). With this definition of $S$, the extra dimensionless number $S(U/L)^{-n}/k$ appearing in Eq. (15), which has not been assigned a name, reduces to $1 - Bn$ and the non-dimensional constitutive equation becomes:

$$Wi \left( \frac{1}{Sr} \frac{\partial \tilde{\tau}}{\partial t} + \tilde{u} \cdot \nabla \tilde{\tau} - (\nabla \tilde{u})^T \cdot \tilde{\tau} - \tilde{\tau} \cdot \nabla \tilde{u} \right) + (1 - Bn) \tilde{\tau} \max \left( 0, \frac{\tilde{\tau}_d - Bn \tilde{\tau}}{\tilde{\tau}_d} \right) \tilde{\tau}_d = \frac{\tilde{\tau}}{\tilde{\tau}_d}$$ (21)

A more standard choice for $S$ would be one that accounts only for a typical viscous stress, omitting any plastic contributions: $S = k(U/L)^n$, which would lead to more standard definitions of the Reynolds and Bingham numbers:

$$Re' \equiv \frac{\rho U^{2-n} L^n}{k} = \frac{Re}{1 - Bn}$$ (22)

$$Bn' \equiv \frac{\tau_y L^n}{k U^n} = \frac{Bn}{1 - Bn}$$ (23)

However, it was shown in [44] (see also the discussion in [45]) that in viscoplastic flows $Re$ suffices as a standalone indicator of inertial effects in the flow, whereas $Re'$ does not (the inertial character must be inferred from the values of $Re'$ and $Bn'$ combined). Hence the definition (20) was preferred.

Concerning the Bingham number (19), the chosen variant that results from the choice (20) assumes values in the range $Bn \in [0, 1]$ and is an indicator of the fraction of the extra stress tensor that is due to plastic effects. On the other hand, the standard Bingham number (23) compares the plastic part ($\tau_y$) only against a typical viscous part of the extra stress tensor and thus $Bn' \in [0, \infty)$. Both $Bn$ and $Bn'$ carry exactly the same information and are simply related through Eq. (23); because of familiarity with $Bn'$ ($0, \infty$) that is almost universally used in the literature, we will use both $Bn$ and $Bn'$ in this study.

The Weissenberg number $Wi$ is an indicator of elastic effects in the flow. Unlike the $Re$ and $Bn$ numbers, it is not a ratio between stresses of different nature or momentum fluxes. In fact, as seen in the mechanical analogue of Fig. 1, omitting the Newtonian component $\kappa$, the viscoplastic and elastic components are connected in series and thus carry the same load. Therefore, the representative stress $S$, although defined by considerations pertaining to the viscoplastic component of the material behaviour (Eq. (20)), is borne also by the elastically behaving component of the material structure and thus $Wi \equiv S/G$ is a typical elastic deformation. In the literature, the standard form for the Weissenberg number is $Wi = \lambda U/L$, where $\lambda$ is a relaxation time of the material. This becomes equivalent to our definition if we define

$$\lambda \equiv \frac{\eta}{G} \quad \text{where} \quad \eta \equiv \frac{S}{U/L}$$ (24)

In the above, $\eta$ is an apparent viscosity (the ratio of a characteristic stress $S$ by a characteristic shear rate $U/L$). The relaxation time $\lambda$ is proportional to this viscosity (the more viscous the fluid is, the slower it recovers from elastic deformations) and inversely proportional to the elastic modulus $G$ (the stiffer the material is, the faster it recovers). The same meaning of $Wi$ is implied by this definition as well: multiplying the strain rate $U/L$ by the time period $\lambda$ during which the material has not yet had time to relax, we get the total elastic strain $\lambda(U/L) \equiv Wi$.

The Strouhal number compares the time scale $T$ to $L/U$. In steady state problems one can simply define $T = L/U$ and $Sr$ becomes irrelevant. It becomes relevant in transient flows that are driven by an external time-varying cause (e.g. an oscillatory driving mechanism), characterised by a time scale $T$ different than $L/U$. A related number is the Deborah number,

$$De \equiv \frac{\lambda}{T} = \frac{Wi}{Sr}$$ (25)

which compares the relaxation time $\lambda$ of the material to the time scale $T$.

Finally, an important dimensionless number is the yield strain,

$$\gamma_y = \frac{\tau_y}{G} = Bn \cdot Wi$$ (26)
Figure 2: Cell $P$ and its neighbouring cells, each having a single common face with $P$. Its faces and neighbours are numbered in anticlockwise order, with face $f$ separating $P$ from its neighbour $N_f$. The shaded area lies outside the domain, so face 5 is a boundary face. The geometric characteristics of face 1 are displayed. The position vectors of the centroids of cells $P$ and $N_f$ are denoted as $P$ and $N_f$; $c_f$ is the centroid of face $f$ and $c'_f$ is its closest point on the line connecting $P$ and $N_f$; $m_f$ is the midpoint between $P$ and $N_f$; $n_f$ is the unit vector normal to face $f$, pointing outwards of $P$, and $d_f$ is the unit vector pointing from $P$ towards $N_f$.

Obviously, the yield strain depends only on the material parameters and not on kinematic or geometric parameters of the flow such as $U$ and $L$. The fact that the product of the $Bn$ and $Wi$ numbers equals the yield strain means that these two numbers are not independent but their product is constant for a given material. Since $Bn \in [0, 1]$, it follows that $Wi \in [\gamma_y, \infty)$. That $Wi$ is bounded from below by $\gamma_y$ follows also directly from the definitions (18) and (20), and should be construed as a statement of the fact that the characteristic stress definition (20) is based on an assumption that the material is yielded (at least in some parts of the domain); if there are unyielded regions then the strain there, of which $Wi$ is a measure, is higher than the yield strain $\gamma_y$.

3. Discretisation of the governing equations

3.1. Preliminary considerations

In this section we propose a Finite Volume Method (FVM) for the discretisation of the governing equations. The method will be described for two-dimensional problems, but extension to three dimensions is straightforward. The first step is the tessellation of the domain $\Omega$ into a number of non-overlapping volumes, or cells. Each cell is bounded by a number of straight faces, each of which separates it from another single cell or from the exterior of the domain (the latter are called boundary faces). Figure 2 shows such a cell $P$ along with its faces and neighbours, and the associated nomenclature. Our intention is to develop a FVM applicable to grids of arbitrary polygonal cells (unstructured grids), although for the results of Sec. 6 we will employ Cartesian grids due to the regularity of the problem geometry. Grids will be labelled after a characteristic cell length $h$; for example, grid $h/2$ has twice the resolution of grid $h$, and four (2D) or eight (3D) times as many cells.

The discretisation procedure will produce from the governing equations (6), (7) and (8), and the boundary conditions, a large set of algebraic equations involving only (approximate) values of the dependent variables at the cell centroids. These values will be denoted using the cell index as a subscript, i.e. $\phi_P$ is the value of the variable $\phi$ at the centroid of cell $P$; if the variable name already includes a subscript then the cell index will be separated by a comma (e.g. $\tau_{d,P}$ is the deviatoric stress at cell $P$). We aim for a second-order accurate method, i.e. one whose discretisation error (the difference between the exact and approximate solutions) scales as $h^2$. With the present governing equations, a difficulty arises from the fact that they all contain only first spatial derivatives. When, according to the standard FVM methodology, these derivatives are integrated over each cell and the Gauss theorem is applied (see below for more details), one ends up with integrals of the differentiated variable over the cell faces, eventually discretised using the values at the face centres (the points marked as ($\circ$) and ($\bullet$) in Fig. 3). The values at points ($\circ$) are in turn approximated by interpolation between the cell centres (the points marked as ($\triangle$) and ($\blacktriangle$) in Fig. 3). So, imagine a scenario
Figure 3: A checkerboard distribution of values on a Cartesian grid. Different values are stored at points ▲ than at points ▼.

where, with reference to Fig. 3, a quantity $\phi$ has the value zero at the boundary points (●), the value +1 at points (▲), and the value −1 at points (▼). If we use linear interpolation to approximate the values of $\phi$ at the face centroids (○) from the values at the cell centres on either side of each face, then we obtain $\phi = 0$ at every face centre (○). This ultimately results in the integral of any first derivative of $\phi$ being calculated as zero over any cell. Thus, interpolation between cell centres to find the values at the face centres filters out the oscillations and leaves a smooth field over the face centres, which in turn produces an image of the discrete operator that varies smoothly from one cell to the next.

Eventually, the FVM discretisation on grid $h$ of the following PDE

$$f(\phi) = b$$

(27)

where $b$ is a known function, gives rise to a system of algebraic equations that can be written as

$$F_h(\phi_h) = B_h$$

(28)

where $\phi_h$ is the vector of unknown values of $\phi$ at the cell centroids, $F_h$ is the discrete operator representing the integral of the differential operator $f$ over each cell, and $B_h$ is the vector of integrals of the known function $b$ over each cell. The aforementioned filtering action of the face interpolation scheme means that the image $F_h(\phi_h)$ can vary smoothly across the cells even if the vector $\phi_h$ itself is oscillatory; therefore, the solution $\phi_h$ to the system (28) can be oscillatory even if the right-hand side $B_h$ is smooth. This is not just a remote possibility, but it does occur in practice, a notorious example being the spurious pressure oscillations produced by the FVM solution of the Navier-Stokes equations (see [46] for a demonstration). In order to suppress such spurious oscillations, the discretisation schemes incorporated in the operator $F_h$ must be modified so that any oscillations in the vector $\phi_h$ will not be filtered out but they will be reflected in the image $F_h(\phi_h)$. Thus, if its right-hand side $B_h$ is smooth, the system (28) can only have a smooth solution $\phi_h$. When solving the Navier-Stokes equations, the main concern is the pressure oscillations because any velocity oscillations will be reflected by the second derivatives of velocity present in the momentum equations and are therefore inhibited. But in our case, all three of the governing equations, (6), (7) and (8)–(9), only contain first derivatives and thus we have to be concerned about the possibility of spurious oscillations in all three variables $u$, $p$, and $\tau$. In the following subsections the adopted measures will be described.

It will be useful to define at this point a “characteristic viscosity”, a quantity with units of viscosity somewhat characteristic of the flow’s viscous character, for use with the discretisation schemes whose descriptions will follow. Our first choice is the following:

$$\eta_a \equiv \kappa + \frac{S}{U/L}$$

(29)

where $S$ is given by (20). This value is constant throughout the domain. A second option we tried is similar to the coefficient of the DAVSS-G technique proposed in [47] and varies throughout the domain, depending on the ratio between the magnitudes of the stress and rate-of-strain tensors:

$$\eta_a \equiv \kappa + \frac{1}{2} \frac{S}{U/L} + \frac{1}{2} \frac{S + \tau p + \gamma_{Nf}}{U/L + \gamma p + \gamma_{Nf}}$$

(30)
The above formula gives the characteristic viscosity at face $f$ of cell $P$ that separates it from its neighbour $N_f$. It can be seen that $\eta_a$ as given by (30) never falls below $\kappa + 0.5S/(U/L)$, tends to $\kappa + S/(U/L)$ when both $\tau$ and $\dot{\gamma}$ are small, and tends to $\kappa + 0.5S/(U/L) + \tau/\dot{\gamma}$ when both $\tau$ and $\dot{\gamma}$ are large. Also, it does not tend to infinity when $\dot{\gamma} \to 0$ (which is a concern in viscoplastic flows), although it has no upper bound.

The discretisation of several terms (e.g. the calculation of $\dot{\gamma} \equiv \nabla u + (\nabla u)^T$ and its magnitude) requires the use of a discrete gradient operator which approximates $\nabla \phi$ at the cell centres. We employ the least-squares operator described in detail in [48], denoted here as $\nabla_s^q$, the subscript $h$ indicating that it is a discrete version of the gradient operator, and the superscript $q$ being the exponent employed: in [48] it was shown that on smooth structured grids the choice $q = 1.5$ engenders second-order accuracy ($\nabla_s^{1.5} \phi = \nabla \phi + O(h^2)$) while with other $q$ values the accuracy degrades to first-order at boundary cells. On irregular (unstructured) grids all choices of $q$ result in first order accuracy everywhere. Nevertheless, first-order accurate gradients suffice to obtain second-order accuracy of the differentiated variables, i.e. are compatible with second-order accurate FVMs [48].

Finally, we can extend the simple linear interpolation scheme discussed in relation to Fig. 3 for calculating face centre values at unstructured grids as

$$\bar{c}_f = \bar{c}_j + \nabla_s^q \phi \cdot (c_f - c_f')$$

(31)

where the overbar denotes an interpolated value, and in the right-hand side both $\phi$ and its gradient are interpolated linearly to point $c_f'$ (the closest point to the face centre lying on the line joining $P$ to $N_f$, Fig. 2) according to the formula

$$\bar{c}_f = \frac{\|c_f - N_f\|}{\|N_f - P\|} \phi_P + \frac{\|c_f' - P\|}{\|N_f - P\|} \phi_{N_f}$$

(32)

The gradient term in scheme (31) accounts for grid skewness, i.e. for the discrepancy between points $c_f$ and $c_f'$ and recovers second-order accuracy on skewed grids.

### 3.2. Discretisation of the continuity equation

Integrating Eq. (6) over cell $P$ and applying the divergence theorem, we get the mass flux balance for that cell:

$$\sum_f \int_{s_f} n \cdot (\rho u) \, ds = 0$$

(33)

where $s_f$ is the surface of face $f$, $n$ is the outward (of $P$) unit vector normal to that face, and $ds$ is an infinitesimal element of that surface. The integrals summed in Eq. (33) are the outward mass fluxes through the respective faces of cell $P$. They are approximated using midpoint rule integration with additional stabilising terms:

$$\int_{s_f} n \cdot (\rho u) \, ds \approx \rho s_f \left[ \bar{u}_{c_f} \cdot n_f + u_f^+ - u_f^- \right] = \dot{M}_f$$

(34)

where

$$u_f^+ = a_f^{mi} (p_P - p_{N_f})$$

(35)

$$u_f^- = a_f^{mi} \nabla_h^{q_{pmf}} \cdot (P - N_f), \quad \nabla_h^{q_{pmf}} = \frac{1}{2} \left( \nabla_h^{q_{pp}} + \nabla_h^{q_{pp_{N_f}}} \right)$$

(36)

$$a_f^{mi} = \frac{1}{\rho \left( \|u_{N_f} - u_P\| + h_f/\Delta t \right) + 2\eta_a/h_f}$$

(37)

$$h_f = \frac{1}{2} \left( \Omega_P + \Omega_{N_f} \right)^{1/2}$$

(38)

where $\Delta t$ is the current time step (see Sec. 3.5), and $D$ equals either 2 or 3, for 2D and 3D problems, respectively. With the definition (34), the discrete version of the continuity equation for a cell $P$ is

$$\sum_f \dot{M}_f = 0$$
The above scheme must be expounded. First of all, we note that if $u_j^{+}$ and $u_j^{-}$ are omitted from Eq. (34) then the scheme simply reverts to midpoint rule integration ($\bar{u}_j$ is obtained using the scheme (31)). Therefore, the part of $\dot{M}_f$ due to $u_j^{+}$ and $u_j^{-}$ can be viewed as entirely belonging to the truncation error of the continuity equation of cell $P$. The contribution of this stabilising term to this error is therefore

$$
\frac{1}{\Omega_P \rho s_f} (u_j^{+} - u_j^{-}) = \frac{\rho s_f}{\Omega_P} a_{ji}^{mi} \left[ (p_P - p_{Nj}) - \nabla q | h_{pmj} \cdot (P - N_f) \right] (39)
$$

(the truncation error is defined per unit volume, hence we divide by the cell volume $\Omega_P$). It is easy to show, by expanding the pressure in a Taylor series about point $m_f$ (Fig. 2), that $(p_P - p_{Nj}) - \nabla p(m_f) \cdot (P - N_f) = O(h^3)$. However, here we use only an approximation to the pressure gradient, $\nabla h_p = \nabla p + O(h)$, so that the term in square brackets in Eq. (39) is $O(h^2)$ (because $O(h) \cdot (P - N_f) = O(h^2)$). Given also that $\Omega_P = O(h^2)$, $s_f = O(h)$ and $a_{ji}^{mi} = O(h)$, it turns out that the truncation error component (39) is $O(h^2)$, which is compatible with a second-order accurate method such as the present one. On smooth structured grids $\nabla q _p = \nabla p + O(h^2)$ [48] and the component (39) is $O(h^3)$.

Secondly, we can consider why the term (39) inhibits spurious pressure oscillations. It does so by reflecting them on the image of the discrete continuity operator (see Eq. (28) and associated discussion). If pressure oscillations are present, then the pressure difference across the face, and the associated term $u_j^{+}$, will oscillate from face to face and thus the term $u_j^{+}$ causes a reflection of these oscillations on the image. On the other hand, the gradient operator $\nabla h_q$ is insensitive to such oscillations, so that the term $u_j^{-}$ varies smoothly from face to face (it does not pass the oscillations on to the image). The $u_j^{-}$ term is used simply for counterbalancing $u_j^{+}$ and making the discretisation consistent, with a $O(h^2)$ truncation error.

It remains to explain the choice of the coefficient $a_{ji}^{mi}$. What we want is that, in the mass flux expression (34), the quantities $u_j^{+}$ and $u_j^{-}$, which have units of velocity, are neither too small to have a stabilising effect nor so large that they dominate the mass flux. Since the “velocities” $u_j^{+}$ and $u_j^{-}$ are functions of local pressure variations, a connection can be made between these pressure variations and velocity differences that they would produce. Pressure forces are related to velocities through the momentum equation, so that equation is our starting point. We use the following non-conservative form, where we assume that the stress tensor can be approximated through the use of a characteristic viscosity such as (29) or (30) as $\tau \approx \eta (\nabla u + (\nabla u)^T)$:

$$
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + \nabla \cdot (\eta \nabla u) (40)
$$

where we have neglected the term $\nabla \cdot (\eta (\nabla u)^T)$, assuming that it is small due to the continuity equation (it is zero if $\eta$ is constant). We are free to make these sorts of approximations because all we want is a rough estimate of the effect that the pressure gradient has on velocity.

So, consider the simple uniform grid, of spacing $h$, shown in Fig. 4, where $u$ denotes the velocity component normal to the face separating cells $P$ and $N$. We will employ a simple FV discretisation of Eq. (40) in order to relate the velocity $u_c$ at the face centre $c$ to the pressures at the centres of the adjacent cells $P$ and $N$. The momentum conservation, Eq. (40), in the direction $x$ normal to the face, for the imaginary cell drawn in dashed line surrounding that face, can be discretised as

$$
\rho \frac{u_c - u^{old}_c}{\Delta t} h^2 + \rho u_c \frac{du}{dx} |_{c} h^2 = -\frac{PN - PP}{h} h^2 + \frac{1}{h} \left( \eta_N \frac{u_{cN} - u_c}{h} - \eta_P \frac{u_c - u_{cP}}{h} \right) h^2 (41)
$$
where \( u_c^{\text{old}} \) is the velocity at the previous time step. Assuming that \((\eta_N + \eta_P)u_c \approx 2\eta_cu_c\) we can solve the above equation for \( u_c \) to obtain

\[
 u_c = \frac{1}{\rho \left( \frac{du}{dx} \right)_c \left( h + \frac{h}{\Delta t} \right) + \frac{2\eta_c}{h}} (p_P - p_N) + \cdots
\]  

(42)

where the dots (\( \cdots \)) denote the terms that are not related to pressure. The above equation provides a quantification of the local effect of pressure gradient on velocity. It was derived from a simplistic one-dimensional consideration; for more general flows, the coefficient \( a_f^{\text{mi}} \) (37) can be seen to be a generalisation of the coefficient multiplying the pressure difference in Eq. (42). It should be noted that the one-dimensional momentum equation (41) accounts for the viscous force due to the velocity variation in the direction perpendicular to the face, but omits that due to the velocity variation in the direction parallel to the face; had the latter also been accounted for, the viscous term in the denominator of (42), and of \( a_f^{\text{mi}} \) (37), would have been \( 4\eta/h \) (or \( 6\eta/h \) in 3D) instead of \( 2\eta/h \), which would have reduced the magnitude of the stabilisation terms in the mass flux scheme (34), but would also likely somewhat increase the accuracy, according to the results of Sec. 5. We did not investigate this issue further, but the choice is not crucial as it affects neither the stabilisation ability of the technique nor the \( O(h^2) \) magnitude of the error (39) that it introduces.

This scheme is a variant of the popular technique known as \textit{momentum interpolation}, which was originally proposed in [49]. Ever since, many variants of this technique have been proposed (see e.g. [50] and references therein) but the large majority of them are intertwined with the SIMPLE algebraic solver (exceptions include [51, 46]). Although this connection with SIMPLE can be useful in some respects [52], and SIMPLE is the algebraic solver employed in the present work, we prefer an independent method such as the presently proposed because it is more general, transparent, and easily adaptable. It allows calculation of the image of the continuity operator without the need to assemble any SIMPLE matrices. For example, it allows us to apply multigrid in the standard way, where immediately after restriction (when the SIMPLE matrices on the coarse grid have not yet been assembled) the coarse grid continuity image is calculated using the same scheme (34) as on the fine grid rather than setting the coarse grid mass fluxes equal to the sum of the mass fluxes through the respective child faces of the fine grid [53].

The explicit form of the scheme (34) – (37) makes it easier to analyse it further. It follows from the definition of \( a_f^{\text{mi}} \) (37) that as the grid is refined \((h_f \to 0)\) the viscous term in the denominator dominates over the inertial one: at fine grids \( a_f^{\text{mi}} \approx h_f/(2\kappa + 2\eta_h) \). The reason behind this can be seen more clearly in the simple one-dimensional example used for deriving the scheme. From Eq. (42), if we want to locally perturb a velocity field at point \( c \) by \( u'_c \) (Fig. 5) by locally perturbing the pressure gradient by \(-dp'/dx|_c \approx (p'_P - p'_N)/h\) then these perturbations are related by

\[
 u'_c = \frac{1}{\rho \left( \frac{du}{dx} \right)_c + \frac{1}{\Delta t} + \frac{2\eta_c}{h^2}} \cdot \frac{p'_P - p'_N}{h} = O(h^2) \cdot \frac{dp'}{dx}|_c \Rightarrow \frac{dp'}{dx}|_c = O(h^{-2}) \cdot u'_c
\]

(43)

Thus, in order to effect a local velocity perturbation of wavelength equal to the grid spacing, as shown in Fig. 5, the pressure gradient must be adjusted locally by an amount that scales as \( (h^{-2}) \), i.e. it must become larger and larger as the grid is refined. This is because it has to overcome the viscous resistance which scales as \( d^2u'/dx^2|_c = O(h^{-2}) \); grid refinement not only increases the velocity slopes (and viscous stresses) but also changes these slopes (second derivative) over a shorter distance. On the contrary, the part of the pressure gradient that is needed to balance the inertial contribution of \( u'_c \) to the momentum balance is independent of the grid spacing. Conversely, Eq. (43) shows that fixed localised perturbations of the pressure gradient of wavelength equal to the grid spacing cause smaller and smaller local velocity perturbations because of the increased viscous resistance.

The pseudo-velocities \( u_f^{p+} \) and \( u_f^{p-} \) defined by Eqs. (35) and (36), have magnitudes of \( O(h^2) \) (because \( a_f^{p+} = O(h) \) and, in smooth pressure fields, both \( p_P - p_N \) and \( \nabla p \cdot (P - N) \) are \( O(h) \)). On the other hand, the actual velocity \( \overline{u}_{cf} \) in the mass flux approximation (34) does not diminish with refinement but tends to a finite value, the exact velocity at the given point. Therefore, as the grid is refined the mass flux scheme becomes more and more dominated by the interpolated velocity \( \overline{u}_{cf} \) while the stabilisation pseudo-velocities \( u_f^{p+} \) and \( u_f^{p-} \) diminish. This observation may give rise to concerns that the ability of the scheme to suppress
spurious pressure oscillations may also diminish with grid refinement. However, this is not the case. Suppose that spurious pressure oscillations do arise, of amplitude $\Delta p^o$ (in the absence of preventive measures, this amplitude is unaffected by grid refinement [46]). The total pressure can be decomposed into a smooth part, which is close to the exact solution, and a spurious oscillatory part: $p = p^s + p^o$. Then the sum of pseudo-velocities at point $c$ of the grid shown in Fig. 4 is equal to

$$ u_c^{p+} - u_c^{p-} = a_c^{m1} [(p_P^s + p_P^o) - (p_N^s + p_N^o) - \nabla_h^q (p^s + p^o)c \cdot (P - N)] = a_c^{m1} \left[ \Delta p^o + (p_P^s - p_N^s) - \nabla_h^q p_c^s \cdot (P - N) \right] \approx a_c^{m1} \Delta p^o \quad (44) $$

because $p_P^s - p_N^s = \Delta p^o$ and $\nabla_h^q p_c^s = 0$ (the $\nabla_h^q$ operator is insensitive to oscillations). Also, according to the discussion of Eq. (39), the underlined terms add up to $O(h^2)$ (or $O(h^3)$ on smooth structured grids), which is small compared to $\Delta p^o$. In other words, $u_f^{p+}$ cancels out with the smooth part of $u_f^{p-}$ but leaves out the oscillatory part. A similar consideration for point $c_P$ (Fig. 4) leads to

$$ u_{c_P}^{p+} - u_{c_P}^{p-} \approx -a_{c_P}^{m1} \Delta p^o \quad (45) $$

because if the oscillatory pressure component decreases by $\Delta p^o$ from $P$ to $N$, then it increases by $\Delta p^o$ from $PP$ to $P$, according to the nature of the spurious oscillation. Finally, we can examine the contributions of faces $c$ and $c_P$ to the image of the continuity operator for cell $P$, the left-hand side of Eq. (38), divided by the cell volume. Their combined contribution is

$$ \frac{1}{\Omega_P} \left[ \dot{M}_c + \dot{M}_{c_P} \right] = \frac{1}{\rho h^2} \rho h \left[ (\bar{u}_c + u_c^{p+} - u_c^{p-}) - (\bar{u}_{c_P} + u_{c_P}^{p+} - u_{c_P}^{p-}) \right] $$

which, considering Eqs. (44) and (45), and also assuming that $a_{c_P}^{m1} \approx a_{c_P}^{m1} \approx C\delta$ for some constant $C$ ($C = 2(\kappa + \eta_h)$ if the grid is fine enough, from Eq. (37)), becomes

$$ \frac{1}{\Omega_P} \left[ \dot{M}_c + \dot{M}_{c_P} \right] = \frac{\bar{u}_c - \bar{u}_{c_P}}{h} + 2\rho C \Delta p^o \quad (46) $$

The first term on the right-hand side is the contribution of the actual velocities to the continuity image, and can be seen to tend to the $h$-independent value $\rho \partial u / \partial x |_P$ with grid refinement. The second term on the right-hand side is the contribution of the pseudo-velocities, which can also be seen to be $h$-independent, i.e. it does not diminish with grid refinement, although the pseudo-velocities themselves do diminish compared to the actual velocities, as mentioned above. If we take into account also the other two faces of cell $P$ (the horizontal ones) then the total contribution of the pseudo-velocities to the continuity image is $4\rho C \Delta p^o$. If we repeat this analysis for the neighbouring cells $N$ and $PP$, then it will turn out that the contributions of the pseudo-velocities to the continuity images of those cells are $-4\rho C \Delta p^o$, i.e. they have opposite sign to that of cell $P$. Overall, in the presence of pressure oscillations the contributions of the pseudo-velocities to
the continuity image over the entire grid have a checkerboard (oscillatory) pattern like the one shown in Fig. 3, with amplitude proportional to the amplitude of the pressure oscillations, $\Delta p^\omega$. As discussed in relation to Eq. (28), when solving the system of all continuity equations (38), because the right-hand side is smooth (it is zero), spurious pressure oscillations that would produce an oscillatory right-hand side are excluded.

Thus, the effectiveness of momentum interpolation in suppressing spurious pressure oscillations does not degrade with grid refinement. The amplitude of the oscillations produced by momentum interpolation to the continuity image is proportional to the amplitude of the spurious pressure oscillations themselves, and independent of the grid spacing $h$.

### 3.3. Discretisation of the momentum equation

As for the continuity equation, the FVM discretisation of the momentum equation (7) begins by integrating it over a cell $P$ and applying the divergence theorem, to get

$$
\int_{\Omega_P} \frac{\partial (\rho u)}{\partial t} \, d\Omega + \sum_f \int_{s_f} n \cdot (\rho u) \, ds = - \sum_f \int_{s_f} p \, ds + \sum_f \int_{s_f} n \cdot \tau \, ds
$$

(47)

Using midpoint rule integration, the above equation is approximated by

$$
\left. \frac{\partial (\rho u)}{\partial t} \right|_{\partial \Omega_P} + \sum_f \hat{M}_f \bar{u}_f = - \sum_f \bar{p}_{c_f} s_f u_f + \sum_f F_f^\tau
$$

(48)

where $\partial/\partial t|_{\partial P}$, the approximate time derivative at point $P$, will be defined in Sec. 3.5, $\hat{M}_f$ is the mass outflux through face $f$ defined by Eq. (34), $\bar{u}_f$, and $\bar{p}_{c_f}$ are approximated from cell-centre values with the linear interpolation scheme (31), and $F_f^\tau$ is the approximation of the force on face $f$ due to the EVP stress tensor $\tau$. Note that Eq. (48) is a vector equation, decomposed into two (in 2D) or three (in 3D) scalar components. The force $F_f^\tau$ is calculated as

$$
\int_{s_f} n \cdot \tau \, ds \approx F_f^\tau \equiv s_f \left[ n_f \cdot \bar{\tau}_{f} + D_f^{\tau+} - D_f^{\tau-} \right]
$$

(49)

Again, the stress tensor at the face centroid, $\bar{\tau}_{f}$, is calculated via the linear interpolation scheme (31) (applied component-wise). The viscous pseudo-stresses $D_f^{\tau+}$ and $D_f^{\tau-}$ are stabilisation terms employed to suppress spurious velocity oscillations. They are vectors, whose $i$-th components are

$$
D_f^{\tau+} = -\eta_a \frac{u_i,N_f - u_i,P}{\|N_f - P\|}
$$

(50)

$$
D_f^{\tau-} = -\eta_a \nabla_h^q u_{i,m_f} \cdot d_f, \quad \nabla_h^q u_{i,m_f} \equiv \frac{1}{2} \left( \nabla_h^q u_i,P + \nabla_h^q u_i,N_f \right)
$$

(51)

where $u_i$ is the $i$-th velocity component ($u = (u_1, u_2)$ in 2D), and $d_f = (N_f - P)/\|N_f - P\|$ is the unit vector pointing from $P$ to $N_f$ (Fig. 2). It can be seen that both $D_f^{\tau+}$ and $D_f^{\tau-}$ equal a characteristic viscosity, given by (29) or (30), times the velocity gradient in the direction $d_f$, albeit calculated differently: the gradient as computed in $D_f^{\tau+}$ is sensitive to velocity oscillations whereas that computed in $D_f^{\tau-}$ is not. The mechanism of oscillation suppression is similar to that for momentum interpolation: in the presence of spurious velocity oscillations, the smooth part of $D_f^{\tau+}$ is cancelled out by $D_f^{\tau-}$, but the oscillatory part produces oscillations in the operator image (in $F_h(\phi_h)$, in the terminology of Eq. (28)).

In the context of colocated FVMs for viscoelastic flows, this technique was first proposed in [31], inspired from the corresponding “momentum interpolation” technique for pressure. A question concerning the method is the appropriate choice of the parameter $\eta_a$. The original method [31] as well as some subsequent variants (e.g. [54, 55]) followed the approach of [49] for momentum interpolation, in that they derived the coefficient from the constitutive equation, arriving at complicated expressions. The present simpler approach is essentially equivalent to that adopted by [56, 57] who, for viscoelastic flows, set the coefficient $\eta_a$ equal to the polymeric viscosity. The aim is that $D_f^{\tau+}$ and $D_f^{\tau-}$ are of the same order of magnitude as the EVP stress acting on the cell face, and this can be achieved using an characteristic viscosity $\eta_a$ defined as the ratio of a typical stress to a typical rate of strain for the given problem.
The present technique can also be interpreted as a “both-sides diffusion” technique [57] where the momentum equation discretised by the FVM is not (7) but an equivalent equation where the same diffusion term $\nabla \cdot (\eta_h \nabla \mathbf{u})$ has been subtracted from both sides:

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \nabla \cdot (\eta_h \nabla \mathbf{u}) = -\nabla p + \nabla \cdot \mathbf{f} - \nabla \cdot (\eta_h \nabla \mathbf{u})$$ (52)

The left-hand side such term is not discretised in the same way as the right-hand side term; in particular, the component $D_f^+$ in (49) comes from the discretisation of the left-hand side term, whereas the component $D_f^-$ comes from the discretisation of the right-hand side term. The other components of these diffusion terms are discretised in exactly the same way for both of them and cancel out, leaving only $D_f^+$ and $D_f^-$. Comparing the expressions (50) and (51), and expanding $u_i$ in Taylor series, it is easy to show that $D_f^+ - D_f^-$ is either $O(h)$, on unstructured grids, or $O(h^2)$, on smooth structured grids (see the discussion of Eq. (39)). Therefore, the contribution of the stabilising terms of the scheme (49) to the truncation error is $(s_f/\Omega_P)(D_f^+ - D_f^-) = O(1)$ on unstructured grids, or $O(h)$ on smooth structured grids. When it comes to diffusion terms, such truncation errors are compatible with second-order accuracy with respect to the discretisation error [48]. This can be demonstrated more easily by considering $D_f^+$ and $D_f^-$ separately, as in Eq. (52). So, consider that Eq. (52) corresponds to Eq. (27), and its FVM discretisation with the present schemes leads to the algebraic system (28), with the dependent variable $\phi \equiv \mathbf{u}$ being the velocity. If $\phi^e$ is the exact velocity (the solution of Eq. (27)) and $\phi$ is the approximate velocity obtained by solving the system (28), then the discretisation error is $\varepsilon = \phi^e - \phi$. The latter is related to the truncation error $\alpha$ by $F^e_h(\phi_h) \cdot \varepsilon_h \approx -\alpha_h$, where $F^e_h(\phi_h)$ is the Jacobian matrix of the operator $F_h$ evaluated at the approximate solution $\phi_h$ (see e.g. [58] for a derivation). Now, we can focus on the stabilising diffusive components of $f$, e.g. on the $\nabla \cdot (\eta_h \nabla \mathbf{u})$ term of the left-hand side of (52), which is discretised with a compact stencil giving rise to the $D_f^+$ term of the scheme (49) (a similar analysis of the term on the right-hand side can be made). Because the diffusion operator is linear, its contribution to the truncation error can be calculated by replacing the velocities by their discretisation errors in (50) (we neglect the rest of the contributions, which cancel out with the corresponding ones of the diffusion term of the right side of Eq. (52)). Thus, plugging $\varepsilon$ into Eq. (50) instead of $u_i$ we get a contribution to the truncation error of cell $P$ of $(s_f/\Omega_P)\eta_h(\varepsilon_{N_i} - \varepsilon_P)/|\mathbf{N}_j - \mathbf{P}| = O(h^{-2})(\varepsilon_{N_i} - \varepsilon_P)$. If our method is second-order accurate then $\varepsilon = O(h^2)$. On unstructured grids, the variation of discretisation error in space has a random component [58] so that $\varepsilon_{N_i} - \varepsilon_P = O(h^2)$ and each face contributes $O(h)$ to the truncation error (on such grids there is also a cancellation between opposite faces for each cell which makes the net contribution to the truncation error to be $O(h^2)$). Thus in any case the truncation errors contributed by the stabilisation terms are compatible with a second-order accurate method.

Another benefit that comes from the incorporation of the stabilisation terms in the scheme (49) concerns the iterative solution algorithm employed in the present work, namely SIMPLE [59]. Within each SIMPLE iteration a succession of linearised systems of equations are solved, one for each dependent variable. The systems for the velocity components $u_i$ are derived from linearisation of the (discretised) momentum equation (7). The latter, unlike in (generalised) Newtonian flows, contains no diffusion terms and the velocity appears directly only in the convection terms of the left-hand side. In low Reynolds number flows, these terms play only a minor role and the momentum balance involves mainly the pressure and stress. Constructing the linear systems for the velocity components only from these inertial terms would lead to bad convergence of the SIMPLE algorithm. Nevertheless, velocity plays an important indirect role in the momentum equation through the EVP stress tensor, to which it is related via the constitutive equation (8). Therefore, either the momentum and constitutive equations have to be solved in a coupled manner, or the effect of velocity on the stress tensor has to somehow be directly approximated in the momentum equation, which is precisely what the stabilisation term $D_f^+$ achieves; in particular, the diffusion term on the left-hand side of Eq. (52) contributes to the matrix of coefficients of the linear systems for $u_i$, making this matrix very similar to those for (generalised) Newtonian flows.

Finally, the momentum balance for cells that lie at the wall boundaries involves the pressure and the stress tensor at these boundaries. The pressure is linearly extrapolated from the interior, as is a common
practice [60]. For stress two possible options are linear extrapolation, as for pressure, and the imposition of a zero-gradient condition which is roughly equivalent to setting the stress at the boundary face equal to that at the owner cell centre. In [61], the former approach was found, as expected, to be more accurate, while the latter was found to be more robust. In the present work we employed either linear extrapolation or a modified extrapolation which is akin to the scheme (49). So, if \( b \) is a boundary face of cell \( P \), the linearly extrapolated value of stress at it centre is calculated as

\[
\tau_{cb} = \tau_P + \nabla^T h_P \cdot (\kappa_b - P)
\]  

where the least-squares stress gradient \( \nabla^T h_P \) is calculated with \( q = 1 \) and using only the stress values at the centres of \( P \) and its neighbour cells (i.e. excluding extrapolated values at wall faces of \( P \)). This scheme is used also for extrapolation of pressure, but when it comes to stress we can go a step further: the force due to the EVP stress tensor on boundary face \( b \) is calculated by a scheme that has precisely the form (49) only that the interpolated stress \( \tau_{cb} \) (31) is replaced by the extrapolated value \( \tau_{cb} \) (53) and the \( D \) terms are replaced by

\[
D^+_{b,i} \equiv -\eta_s u_{i,c_b} - u_{i,P}
\]

\[
D^-_{b,i} \equiv -\eta_s \nabla^T h_{i,m_b} \cdot d_b
\]

where \( d_b \) is the unit vector pointing from \( P \) to \( \kappa_b \) and the gradient in (55) is extrapolated to point \( m_b \equiv (P + \kappa_b)/2 \) via the scheme (53) with \( \kappa_b \) replaced by \( m_b \). The terms (54) – (55) do not have an apparent stabilising effect, and in fact our experience has shown that they sometimes can cause convergence difficulties to the algebraic solver. Such is the case with wall slip examined in Sec. 6.4, where we had to set these terms to zero and use simple linear extrapolation of stresses at the walls. Nevertheless, we also found that these terms can increase the accuracy of the solution (see Sec. 5) and therefore we employed them whenever possible (they were used for all the main results except those of Sec. 6.4).

3.4. Discretisation of the constitutive equation

Again, by integrating the SHB equation (8) – (9) and applying the divergence theorem we obtain

\[
\int_{\Omega_P} \frac{\partial \tau}{\partial t} \, d\Omega + \sum_f \int_{s_f} n \cdot u \tau \, ds =
\]

\[
G \int_{\Omega_P} \dot{\gamma} \, d\Omega - G \int_{\Omega_P} \left( \frac{\max(0, \tau_d - \tau_y)}{k} \right)^{\frac{1}{2}} \frac{1}{\tau_d} \tau \, d\Omega + \int_{\Omega_P} ((\nabla u)^T \cdot \tau + \tau \cdot \nabla u) \, d\Omega
\]  

(56)

where the surface integral on the left-hand side has been obtained by application of the divergence theorem to the volume integral of the second term on the right-hand side of Eq. (9), after use was made of the identity \( u \cdot \nabla \tau = \nabla \cdot (u\tau) - (\nabla \cdot u)\tau \) and the fact that the flow is incompressible (\( \nabla \cdot u = 0 \)). FVMs are usually applied to transport equations, which when integrated over a finite volume express the conservation of a quantity. In the case of the constitutive equation, at first glance this intuitive meaning of the FVM procedure becomes blurred; however, apart from the fact that from a mathematical standpoint the FVM procedure is perfectly valid for any differential equation irrespective of its physical meaning, equation (56) has the form of a conservation equation for stress. On the left-hand side, there is the rate of change of stress in the volume plus the outflux of stress (this sum equals the rate of change of stress in a fixed mass of fluid moving with the flow, in a Lagrangian frame of reference). On the right-hand side, there are three source terms which express, respectively, stress generation (the \( \dot{\gamma} \) term), relaxation (the viscous term), and transformation (the upper convective derivative term). Equation (56) is then discretised as follows:

\[
\left. \frac{\partial \tau^a}{\partial t} \right|_{P} = \Omega_P + \frac{1}{\rho} \sum_f \Omega_f \tau_{T_f}
\]

\[
G \Omega_P \dot{\gamma}_P - G \Omega_P \left( \frac{\max(0, \tau_d,P - \tau_y)}{k} \right)^{\frac{1}{2}} \frac{1}{\tau_d,P} \tau_P + \Omega_P \left( (\nabla^T h_{P}) \cdot \tau_P + \tau_P \cdot \nabla^T h_{P} \right)
\]  

(57)
Figure 6: On a uniform structured grid, CUBISTA interpolates the value at face centre \( \xi \) from the values at the downwind (\( D \)), upwind (\( P \)), and farther-upwind (\( U \)) cells (assuming the mass flux is directed from cell \( P \) to cell \( D \)).

where, again, \( \partial/\partial t \bigg|_{P} \) is the approximate time derivative at \( P \) (Sec. 3.5); \( \bar{M}_f \) is the mass outflux through face \( f \) (Eq. (34)); \( \Omega_P \) is the volume of cell \( P \); \( \nabla^q f \) is the discrete (least-squares) gradient of the velocity at point \( P \) and the superscript T denotes its transpose; \( \dot{\gamma}_P = \nabla^q f \dot{y}_P + (\nabla^q f)_P \) is the discrete value of \( \dot{\gamma} \) at point \( P \), calculated from the discrete velocity gradients there; \( \tau_{P} \) is the EVP stress tensor at point \( P \); and \( \tau_{d,P} \) is the value of \( \tau_d \) at point \( P \), calculated from \( \tau_P \) via Eq. (10).

In the convection term of Eq. (57) (the second term on the left-hand side), \( \bar{c}_f \) is the value of the stress tensor interpolated at the face centre \( \bar{c}_f \), but the interpolation scheme is not the simple linear extrapolation (31). Since the constitutive equation also lacks diffusion terms, there is a danger of spurious stress oscillations, and a usual preventive measure is to use so-called high resolution schemes [62] for the discretisation of the convection terms. In viscoelastic flows, the CUBISTA scheme [63] has proved to be effective and is widely adopted (e.g. [64–66, 34]). In the present work, we adapted the CUBISTA scheme for use on unstructured grids as follows. To account for skewness, a scheme similar to (31) is used, but the value at point \( \bar{c}_f \) is interpolated with CUBISTA:

\[
\tilde{\phi}_{c_f} = \tilde{\phi}_{c_f} + \nabla^q \bar{c}_f \cdot (c_f - c_f)
\]  

(58)

Note that the second term on the right-hand side is exactly the same as for the scheme (31) (the gradient is interpolated linearly). The first term on the right-hand side is the value at point \( \bar{c}_f \), interpolated via CUBISTA. CUBISTA is a high-resolution scheme based on the Normalised Variable Formulation [67], and on structured grids such schemes interpolate the value at a face from the values at two upwind cell centres and one downwind cell centre; these are the two cells on either side of the face, labelled \( P \) and \( D \) in Fig. 6, and a farther cell \( U \). But on an unstructured grid the farther-upwind cell \( U \) is not straightforward (or even not possible) to identify. To overcome this hurdle, we follow an approach [68, 62] which is based on the observation that, on a uniform structured grid such as that shown in Fig. 6, it holds that

\[
\phi_U = \phi_D - \nabla^q \phi_P \cdot (D - U)
\]  

(59)

because common gradient schemes (including the presently used least-squares) are such that \( \nabla^q \phi_P \cdot (D - U) = \phi_D - \phi_U \). On unstructured grids, following [68, 62] we define the location \( \bar{U} = P - (D - P) \) which lies at the diametrically opposite position of \( D \) relative to \( P \); since the grid is unstructured, it is unlikely that the location \( \bar{U} \) coincides with an actual cell centre, but still we can use Eq. (59) to estimate a value there, \( \phi_U \). Information from the upwind side of cell \( P \) is implicitly incorporated into \( \phi_U \) through the gradient \( \nabla^q \phi_P \), and if the scheme is used on a uniform structured grid then it automatically retrieves the value at the centroid of the actual cell \( U \).

So now we have a set of three co-linear and equidistant points, \( \bar{U}, P \) and \( D \), and three corresponding values, \( \phi_U, \phi_P \) and \( \phi_D \), so that we can employ CUBISTA to calculate the value at point \( \bar{c}_f \), which lies on the same line as these three points. CUBISTA is a blend of quadratic interpolation between the tree points (QUICK) and first-order upwind differencing (UDS), depicted in the Normalised Variable Diagram (NVD) of Fig. 7. The normalised variables are defined as

\[
\xi = \frac{\|c' - P\|}{\|D - P\|}
\]  

(60)
Figure 7: Normalised Variable Diagram of the CUBISTA scheme (continuous red line); is is to scale for $\xi = 0.5$.

\[ \phi_P = \frac{\phi_P - \phi_U}{\phi_D - \phi_U} \quad (61) \]

\[ \tilde{\phi}_{c_f}(\xi, \tilde{\phi}_P) = \frac{\tilde{\phi}_c - \phi_U}{\phi_D - \phi_U} \quad (62) \]

Note that $\xi$ is just the linear interpolation factor multiplying $\phi_{N_f}$ in Eq. (32). CUBISTA defines the normalised interpolated value at the face, $\tilde{\phi}_{c_f}$ (Eq. (62)), as a function of $\xi$ and $\tilde{\phi}_P$, according to the NVD of Fig. 7:

\[
\tilde{\phi}_{c_f}(\xi, \tilde{\phi}_P) = \begin{cases} 
\tilde{\phi}_P & \text{if } \tilde{\phi}_P \leq 0 \text{ (UDS)} \\
\frac{1}{3}(1 + \xi)(3 + \xi)\tilde{\phi}_P & \text{if } 0 < \tilde{\phi}_P < \frac{3}{8} \text{ (Transition 1)} \\
(1 - \xi^2)\tilde{\phi}_P + \frac{1}{2}\xi(1 + \xi) & \text{if } \frac{3}{8} < \tilde{\phi}_P \leq \frac{3}{4} \text{ (QUICK)} \\
(1 - \xi^2)\tilde{\phi}_P + \xi(2 - \xi) & \text{if } \frac{3}{4} < \tilde{\phi}_P \leq 1 \text{ (Transition 2)} \\
\tilde{\phi}_P & \text{if } \tilde{\phi}_P > 1 \text{ (UDS)}
\end{cases}
\]

(63)

From $\tilde{\phi}_{c_f}$, we can recover the un-normalised value $\bar{\phi}_{c_f}$ from the definition (62) and substitute it in Eq. (58) to obtain the value of $\phi$ at the face centre. If $\phi_D = \phi_U$ then we set $\bar{\phi}_{c_f} = \phi_P$ (UDS). To avoid excessive use of IF statements, we can use the median function\(^1\) as suggested in [69], to reformulate the scheme in the following two successive steps:

\[ \bar{\phi}_{c_f}^{*} = \text{median} \left( \phi_P, \phi_U + \frac{(1 + \xi)(3 + \xi)}{3}(\phi_P - \phi_U), \phi_D - (1 - \xi)^2(\phi_D - \phi_P) \right) \quad (64) \]

\[ \bar{\phi}_{c_f} = \text{median} \left( \phi_P, \bar{\phi}_{c_f}^{*}, \phi_P + \frac{\phi_D - \phi_U}{2}\xi + \frac{1}{2}(\phi_D - 2\phi_P + \phi_U)\xi^2 \right) \quad (65) \]

It will be useful to make some remarks concerning the scheme (63) at this point. It is based on quadratic interpolation (QUICK), but switches to the UDS when there is a local minimum or maximum at $\phi_P$ (if $\phi_P < 0$ or $\phi_P > 1$), which could indicate the formation of a spurious oscillation. The region $\tilde{\phi}_P \in [3/8, 3/4]$, where the variation between the three values $\phi_U, \phi_P$ and $\phi_D$ is not far from linear, is considered to be “safe” and QUICK is applied unreservedly there. The upper bound of this region, which for non-uniform structured grids is given to be a $\xi$-dependent value in [63], is chosen here to be the fixed number $\tilde{\phi}_P = 3/4$ based on the same criterion as in [63], namely the condition that the quadratic profile passing through the three points $U, P$ and $D$ is monotonic (no overshoots / undershoots). In terms of normalised variables, this condition

\(^1\)It can be implemented as $\text{median}(a, b, c) = \max(\min(a, b), \min(\max(a, b), c))$.  

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requires that $\tilde{\phi}_c(\xi, \tilde{\phi}_P)$, as given by the QUICK branch of Eq. (63), increases monotonically towards the value 1 as $\xi \to 1$, which is ensured by the condition $\partial \tilde{\phi}_c(\xi, \tilde{\phi}_P)/\partial \xi|_{\xi=1} \geq 0 \Rightarrow \tilde{\phi}_P \leq 3/4$.

We also note that, if CUBISTA achieves its goal of producing smooth $\tilde{\phi}$ distributions, then grid refinement, which brings the points $U_i$, $P$ and $D_i$ closer together, will cause the variation of $\tilde{\phi}$ in the neighbourhood of the three points to become more linear, as a Taylor series expansion of $\tilde{\phi}$ shows. This means that $\tilde{\phi}_P \to 0.5$ as the grid is refined. On fine grids, therefore, the QUICK branch of CUBISTA (63) is the dominant one throughout the domain. QUICK is a third-order accurate interpolation, but degrades to second-order accuracy on unstructured grids if $\tilde{\phi}_P$ is defined via Eq. (59) using a first-order accurate gradient $\nabla_i q$. This is not a major drawback, however, because even on structured grids the accuracy of CUBISTA is limited to second-order by the midpoint rule integration step that follows that of interpolation.

Finally, we turn our attention to another issue. In viscoelastic flows, it is well known that, with common discretisations of the constitutive equation, solutions can be obtained only for relatively small Weissenberg numbers (the notorious high-$Wi$ problem). This has been attributed to exponential spatial stress variations that are due to the additional terms of the upper convected derivative. The usual remedy is to express the constitutive equation, prior to discretisation, in terms of a more weakly varying function of the stress tensor, the first and most commonly used of which is the logarithm the conformation tensor [70–72] (this is referred to, for brevity, as the “log-conformation” technique). This has been applied also to the Saramito model with $n = 1$ [73]. Other functions have also been proposed and used [74, 75].

Recently, a simple discretisation scheme was proposed in [76] that can be used with the primitive stress formulation and allows the attainment of solutions at higher Weissenberg numbers than the common schemes. It expresses some cell-centre stress component values $\tau_{ij}(P)$ in the upper convected derivative as weighted averages of the values $\tau_{ij,P}$ stored at these cell centres and of the values $\tau_{ij,N_k}$ stored at the neighbouring cell centres (instead of just using the stored values $\tau_{ij,P}$). We employed a modified version of it in the present work, hence an interpolated value $\bar{\tau}_{ij}$ appears in the upper convected derivative terms in the right-hand side of Eq. (57). In the original scheme of Zhou et al. [76], the weighting is based on the mass fluxes through the cell’s faces. However, we noticed that this causes a noticeable accuracy degradation compared to choosing the weights on a purely geometrical basis in order to achieve second-order accuracy. More specifically, here the last term of Eq. (57) is expanded as (for 2D problems):

$$
(\nabla_i u_j)^T P \cdot \bar{\tau}_{ij} + \bar{\tau}_{ij} \cdot \nabla_i u_j P = \begin{bmatrix}
2 \tau_{11,P} \frac{\partial u_1}{\partial x_1} + 2 \tau_{12,P} \frac{\partial u_1}{\partial x_2} + \tau_{22,P} \frac{\partial u_1}{\partial x_2} \\
2 \tau_{11,P} \frac{\partial u_2}{\partial x_1} + 2 \tau_{12,P} \frac{\partial u_2}{\partial x_2} + \tau_{22,P} \frac{\partial u_2}{\partial x_2}
\end{bmatrix}
$$

where the velocity derivatives are the components of $\nabla_i u_j P$. In the (1,2) component of (66), the middle term, although equal to zero in the limit of infinite grid fineness due to the continuity equation, has been retained for numerical stability reasons. In each component $(i,j)$, the corresponding stress component $\tau_{ij}$ is printed overlined as it is possibly an interpolated value. Two choices for its evaluation will be examined. The obvious choice is to set $\bar{\tau}_{ij} = \tau_{ij,P}$; the alternative choice is akin to the scheme of Zhou et al. [76]: $\bar{\tau}_{ij} = \tau_{ij,P}$ is a weighted average of the values already computed at the faces of cell $P$ (using CUBISTA). A combination of these face values is effectively a weighted average of the values at the centroids of cell $P$ and of its neighbours. We employ the following second-order accurate (see Appendix B for a proof) averaging scheme which exploits the fact that $P$ is the centroid of cell $P$ and avoids evaluation of the stress gradient:

$$
\tau_{ij,P} = \frac{1}{D \Omega_P} \sum_{j=1}^{F} \tau_{ij,c_j} s_j (c_j - P) \cdot n_j
$$

where $D$ is the dimensionality of the problem (= 2 or 3, for 2D or 3D problems, respectively). Compared to simply setting $\bar{\tau}_{ij} = \tau_{ij,P}$, this scheme was found, in the numerical tests of Sec. 5, to allow an increase of about 40% in the $Wi$ number and to provide a slight increase of accuracy; therefore it was adopted in the present study. For higher $Wi$ numbers the log-conformation technique should be used, but in EVP flows the Weissenberg number is not usually high enough to trigger the high-$Wi$ problem.

A complication arises at wall boundaries, where we found that using linearly extrapolated stress values in (67) leads to convergence difficulties. Therefore, wall boundary values were omitted in the calculation,
by weighting the values at the remaining faces by their \( s_f (\mathbf{e}_f - \mathbf{P}) \cdot \mathbf{n}_f \) factors, i.e. by replacing the \( D \Omega_P \) product in the denominator of (67) by

\[
D \Omega'_P = \sum_{f \notin \{W_P\}} s_f (\mathbf{e}_f - \mathbf{P}) \cdot \mathbf{n}_f = D \Omega_P - \sum_{f \notin \{W_P\}} s_f (\mathbf{e}_f - \mathbf{P}) \cdot \mathbf{n}_f
\]

where \( \{W_P\} \) is the set of wall boundary faces of cell \( P \). Unfortunately the resulting interpolation in only first-order accuracy at boundary cells.

### 3.5. Temporal discretisation scheme

The approximate time derivatives in the momentum and constitutive equations are calculated with a three-time level implicit scheme that allows a variable time step. Suppose we are at time step \( i \) and let \( \phi_P^i \) be the current, unknown, value of \( \phi \) at cell \( P \), and \( \phi_P^{i-1} \) and \( \phi_P^{i-2} \) be the already calculated values at the previous two time steps. For simplicity, we use a local time coordinate system where the current time is \( t_i = 0 \), and therefore \( t_{i-1} = -\Delta t_i, t_{i-2} = -(\Delta t_i + \Delta t_{i-1}) \) are the previous two time instances, where \( \Delta t_i \) and \( \Delta t_{i-1} \) are the current and previous time step sizes, respectively. Fitting a quadratic polynomial, in Lagrange form, between the three points \((t_i, \phi_P^i), (t_{i-1}, \phi_P^{i-1})\) and \((t_{i-2}, \phi_P^{i-2})\) gives the following variation of \( \phi_P \) in the vicinity of the three points:

\[
\phi_P(t) = l_{i,i}(t)\phi_P^i + l_{i,i-1}(t)\phi_P^{i-1} + l_{i,i-2}(t)\phi_P^{i-2} + O(\Delta t^3)
\]

where \( \Delta t \) is the general order of magnitude of the time steps, representative of all of \( \Delta t_i, \Delta t_{i-1} \) and \( t \), and

\[
l_{i,i}(t) = \frac{(t-t_{i-1})(t-t_{i-2})}{(t_i-t_{i-1})(t_i-t_{i-2})}, \quad l_{i,i-1}(t) = \frac{(t-t_i)(t-t_{i-2})}{(t_{i-1}-t_i)(t_{i-1}-t_{i-2})}, \quad l_{i,i-2}(t) = \frac{(t-t_i)(t-t_{i-1})}{(t_{i-2}-t_i)(t_{i-2}-t_{i-1})}
\]

Differentiation of Eq. (69) gives

\[
\frac{d\phi_P}{dt}(t) = l'_{i,i}(t)\phi_P^i + l'_{i,i-1}(t)\phi_P^{i-1} + l'_{i,i-2}(t)\phi_P^{i-2} + O(\Delta t^2)
\]

where

\[
l'_{i,i}(t) = \frac{2t-t_{i-1}-t_{i-2}}{(t_i-t_{i-1})(t_i-t_{i-2})}, \quad l'_{i,i-1}(t) = \frac{2t-t_i-t_{i-2}}{(t_{i-1}-t_i)(t_{i-1}-t_{i-2})}, \quad l'_{i,i-2}(t) = \frac{2t-t_i-t_{i-1}}{(t_{i-2}-t_i)(t_{i-2}-t_{i-1})}
\]

Our approximate derivative \( \partial / \partial t \big|_P \) appearing in Eqs. (48) and (57) comes from Eq. (71) by evaluating the \( l' \) coefficients at the current time \( t = t_i \) and dropping the unknown \( O(\Delta t^2) \) term. All other terms appearing in the governing equations are evaluated at the current time, so that the scheme is fully implicit.

After solving the equations at the current time \( t_i \) to obtain \( \phi_P^i \), and deciding (in a manner that will be described shortly) the next time step size \( \Delta t_{i+1} \), in order to facilitate the solution at the new time level \( t_{i+1} \) we can make a prediction by substituting \( t = t_{i+1} \) in Eq. (69) (omitting the \( O(\Delta t^3) \) term) to obtain a provisional value \( \phi_P^{i+1*} \). This value serves as an initial guess for solving the equations at the new time step \( i + 1 \), offering significant acceleration. We perform such a prediction even for pressure, even though the governing equations do not contain its time derivative (this was found to accelerate the solution of the continuity equation).

The subsequent solution at time \( t_{i+1} \) will give us a value \( \phi_P^{i+1} \neq \phi_P^{i+1*} \), in general. In order to obtain an estimate of the \( O(h^2) \) truncation error of Eq. (71) we can augment our set of three points by the addition of the point \((t_{i+1}, \phi_P^{i+1})\) and fit a cubic polynomial through the four points; the difference between \( \partial \phi / \partial t \big|_P \) and \( \partial \phi / \partial t \big|_P^{i+1} \), the derivatives predicted at time \( t_i \) by the quadratic and cubic polynomials, respectively, is a measure of the truncation error associated with \( \partial \phi / \partial t \big|_P \). Omitting the details, it turns out that

\[
\frac{\partial \phi}{\partial t} \bigg|_P (t_i) - \frac{\partial \phi}{\partial t} \bigg|_P^{i+1} (t_i) = c \frac{\phi_P^{i+1} - \phi_P^{i+1*}}{\Delta t_{i+1}}
\]

where the factor \( c = O(1) \) depends on the relative magnitudes of \( \Delta t_{i-1}, \Delta t_i \) and \( \Delta t_{i+1} \) (\( c = 1/3 \) for \( \Delta t_{i-1} = \Delta t_i = \Delta t_{i+1} \)).
The above result allows us to keep an approximately constant level of truncation error by choosing the successive time steps so as to keep the right-hand side of Eq. (73) constant. To this end, we set the following goal, which accommodates also for the fact that all grid cells have to be accounted for:

\[
g^i_t \equiv \left[ \frac{1}{\Omega} \sum_P \Omega_P \left( \frac{\phi^i_P - \phi^{i+1}_P}{\Delta t_i} \right)^2 \right]^{1/2} = \bar{g}^0_t \Phi / T^2
\]  

(74)

I.e. we want \( g^i_t \), the \( L_2 \) norm of \( (\phi^i_P - \phi^{i+1}_P)/\Delta t_i \), for any time step \( i \), to equal a pre-selected non-dimensional target value \( \bar{g}^0_t \) times \( \Phi / T \) where \( \Phi \) is either \( U \) (for the momentum equation, where \( \phi \equiv u_j \)) or \( S \) (for the constitutive equation, where \( \phi \equiv \tau_{jk} \)). Suppose then that at some time step \( i \) we are somewhat off target, i.e. Eq. (74) is not satisfied; we can try to amend this at the new time step \( i+1 \) by noting that the truncation error, and the associated metric \( g^i_t \), are of order \( O(\Delta t^2) \) (Eq. (71)), which means that \( g^{i+1}_t/g^i_t = (\Delta t_{i+1}/\Delta t_i)^2 \). This relation allows us to choose \( \Delta t_{i+1} \) so that \( g^{i+1}_t \) will equal \( \bar{g}^0_t \Phi / T \) (Eq. (74)):

\[
\Delta t_{i+1} = r^i_t \Delta t_i \quad \text{where} \quad r^i_t = \sqrt{\bar{g}^0_t/\bar{g}^i_t}, \quad \bar{g}^i_t = \frac{g^i_t}{\Phi / T}.
\]  

(75)

In practice, we calculate the adjustment ratios \( r^i_t \) for all variables whose time derivatives appear in the governing equations (one per velocity component and per stress component) and choose the smallest among them. We also limit the allowable values of \( r^i_t \) to a range \( r^i_t \in [r^i_t^{\min}, r^i_t^{\max}] \), and also the overall time step size to \( \Delta t_i \in [\Delta t_{i}\min, \Delta t_{i}\max] \). Typical values used for these parameters in the simulations of Sec. 6 are \( \bar{g}^0_t = 10^{-4} \), \( r^i_t^{\min} = 0.95 \), \( r^i_t^{\max} = 1.001 \), \( \Delta t_{i}\min = 5 \times 10^{-5} \) s, \( \Delta t_{i}\max = 2 \times 10^{-3} \) s (these may vary slightly between simulations). The scheme automatically chooses small time steps when the flow evolves rapidly, such as during the early stages of the flow, and large time steps when it evolves slowly, such as when the flow is nearing its steady state. A similar scheme is used in [77].

4. Solution of the algebraic system

The discretisation procedure described in the previous Section produces one large nonlinear algebraic system for each time step. These systems are solved using the SIMPLE algorithm, used either as a stand-alone solver or in a multigrid context, where the algebraic equations are arranged into groups, one for each momentum and constitutive equation component, and through linearisation each such group produces a linear system for one of the dependent variables. The algorithm is applied in the same manner as for viscoelastic flows [31, 33]: within each outer SIMPLE iteration, we solve successively the linear systems for each stress component, followed by the linear systems for the velocity components, ending with the solution of the pressure correction system which enforces continuity. The systems are solved with preconditioned conjugate gradient (pressure correction) or GMRES (all other variables) solvers.

In the velocity systems, derived from the components of the momentum eq. (48), the matrix of coefficients is constructed via a contribution from the time derivative, a UDS discretisation of the convective term, and the \( D^+_f \) (Eq. (50)) part of the EVP force \( F_f^e \). The remaining terms are evaluated from their currently estimated values and moved to the right-hand side vector, as is the difference between the UDS and CDS convection schemes (deferred correction).

For the stress component systems we follow the established practice in viscoelastic flows of constructing the matrix of coefficients with contributions only from the time derivative term and the convection term (left-hand side of Eq. (57)), with the latter treated again via deferred correction, breaking the CUBISTA flux into a UDS component and a remainder which is moved to the right-hand side vector. This limited representation of the constitutive equation by the matrix of coefficients may be responsible for the observed fact that SIMPLE converges slowly or fails to converge for large time steps; for small time steps the role of the time derivative in the constitutive equation becomes dominant and convergence is fast.

It was noticed that if the residual reduction within each time step is not at least a couple orders of magnitude, then the temporal prediction step (Sec. 3.5) may not produce good enough initial guesses and the solution may not be smooth in time. To avoid this possibility we set a target effort per time step of about 20 single-grid SIMPLE iterations followed by about 5 W(4,4) multigrid cycles [78, 14], and if further iterations are needed to accomplish the required residual reduction then the time step is reduced by a factor of \( r^i_t^{\min} \) (Sec. 3.5). Setting \( \Delta t_{\max} \) to an appropriate value avoids the need for this.
Table 1: $\tilde{x}$– and $\tilde{y}$– coordinates ($\tilde{x}_c$, $\tilde{y}_c$) of the centre of the main vortex for Oldroyd-B flow in a square cavity of side $L$, and associated value of the streamfunction there, $\tilde{\psi}_c$. The top wall moves in the positive $x$–direction with variable velocity $u(x) = 16U(x/L)^2(1 - x/L)^2$. The coordinates are normalised by the cavity side $L$ and the streamfunction by $UL$. The flow is steady, the Reynolds number is zero, and the solvent and polymeric viscosities are equal ($\kappa = k$).

|                | $Wi = 0.5$ | $Wi = 1.0$ |
|----------------|------------|------------|
|                | $\tilde{x}_c$ | $\tilde{y}_c$ | $\tilde{\psi}_c$ | $\tilde{x}_c$ | $\tilde{y}_c$ | $\tilde{\psi}_c$ |
| Pan et al. [79]| 0.469      | 0.798      | −0.0700        | 0.439         | 0.816       | −0.0638       |
| Saramito [80]  | 0.429      |            |                |              |             |         |
| Sousa et al. [39]| 0.467    | 0.801      | −0.0698        | 0.434         | 0.814       | −0.0619       |
| Zhou et al. [76]| 0.468      | 0.798      |                | 0.430         | 0.818       |         |
| Present results| 0.468      | 0.799      | −0.0698        | 0.434         | 0.818       | −0.0619       |

5. Validation and testing of the method in Oldroyd-B flow

For $\tau_y = 0$ and $n = 1$, the SHB model reduces to the Oldroyd-B viscoelastic model, for which several benchmark results for square lid-driven cavity flow can be found in the literature. We apply our method to this problem in order to validate it as well as to compare the performances of alternative options for the FVM components. The problem is solved as steady-state, omitting the time derivatives from the governing equations. Table 1 lists the computed location and strength of the main vortex, along with corresponding values reported in the literature, with which there is good agreement.

The test case is used to assess the various spatial discretisation schemes of Sec. 3. To this end, we obtained accurate estimates of the $Wi = 0.5$ and $Wi = 1.0$ solutions by Richardson extrapolation of the distorted grids obtained from the CU ones by randomly perturbing their vertices as described in [48]. In particular, from a $N \times N$ Cartesian grid (Fig. 7a) a $N \times N$ distorted grid is constructed (Fig. 7c) by moving each interior node $(x, y)$ to a location $(x', y') = (x + \delta x, y + \delta y)$ where $\delta x$ and $\delta y$ are randomly selected for each node under the restriction that $|\delta x|, |\delta y| < h/4$. Because of this restriction, it is ensured that all grid cells of the distorted grid are simple convex quadrilaterals. This procedure results in a series of grids whose skewness, unevenness and non-orthogonality do not diminish with refinement [48], as is typically the case for unstructured grids. Checking the convergence of the method on this sort of grids is important because in [48] it was shown...
that there exist popular FVM discretisations that, although widely believed to be second-order accurate, actually do not converge to the exact solution with refinement on unstructured grids.

So, we employed three series of grids, one for each of the grid types shown in Fig. 8, having $32 \times 32$, $64 \times 64$, $128 \times 128$, $256 \times 256$ and $512 \times 512$ cells. The CN grids were constructed as follows: in the $512 \times 512$ such grid, the grid spacing at the walls equals $L/1024$ and grows under a constant ratio towards the cavity centre. Then, by removing every second grid line we obtain the $256 \times 256$ grid, and so on for coarser grids.

The numerical tests led to the following observations. Concerning the oscillations issue, we noticed that the stabilisation strategies achieved their goal as all dependent variables varied smoothly across the domain without spurious oscillations; Fig. 10 shows smooth pressure and $\tau_{12}$ fields and streamlines obtained on the $512 \times 512$ distorted grid. Concerning the accuracy of the method, Fig. 9 shows that, with all the discretisation schemes tested, our FVM converges to the exact solution with grid refinement on any type of grid, including the randomly distorted type. The order of accuracy is close to 1 on coarse grids and increases towards 2 (the design value) as the grid is refined, but for the $Wi = 1$ case (Figs. 9b, 9e) it is still quite far from 2 even on the finest ($512 \times 512$) grids. Evidently, the convergence rates deteriorate at higher elasticity (compare Figs. 9a, 9d and 9b, 9e). The substandard accuracy performance of the method at high elasticity is most likely associated with the exponential stress growth near the lid. The Oldroyd-B predictions can be unrealistic, such as predicting unlimited extension of the material at finite extension rates; in such cases the accuracy of numerical methods can degrade significantly [81]. The SHB model suffers from these same issues as the Oldroyd-B model for $n \geq 1$, but not for $n < 1$ [26].

Typical EVP materials are such that usually $Wi < 1$, which, combined with the fact that their behaviour is described by $n < 1$, avoids the “high-$Wi$” problem. Interestingly, Fig. 9 shows that the log-conformation technique can be beneficial even at low $Wi$ as it provides enhanced accuracy. Accuracy improvement results also from use of the interpolation (67) compared to simply setting $\tau_{ij,P} = \tau_{ij,P}$, although rate of convergence of the latter method appears to be higher for $Wi = 0.5$ so that it would become more accurate through further grid refinement. Use of (67) was observed to enable obtaining a solution at slightly higher $Wi$ compared to setting $\tau_{ij,P} = \tau_{ij,P}$ (up to $Wi \approx 1.4$ compared to $Wi = 1$), so that for the present work the scheme (67) was selected for performing the experiments of Sec. 6, given also that it is cheaper than the log-conformation method.

The scheme (30) can be seen to perform poorly in terms of accuracy (Figs. 9a, 9d and 9b, 9e, dash-dot lines) compared to the simpler scheme (29). Concerning the latter, it is noteworthy that the larger the value of $\eta_a$ (adjusted by replacing $S$ with larger / smaller values in (29)) the less accurate the results (Figs. 9a, 9d, lines with long or short dashes). Thus, stabilisation should be exercised with parsimony. Henceforth we abandon the scheme (30) and employ scheme (29).

Fig. 9 shows that packing the grid lines close to the walls – i.e. using the CN grids – achieves a very significant increase of accuracy. This is likely related to the aforementioned singular behaviour near the lid. As expected, the discretisation errors are largest on the distorted grids (Figs. 9c, 9f, red lines), but they are not dramatically larger than those of the uniform Cartesian grids, while their rate of convergence towards zero is not different than those of the undistorted grids. Another disadvantage of the distorted grids is that

![Figure 8: 32 x 32 grids of different kinds, used for the Oldroyd-B benchmark problems.](Image)
Figure 9: Velocity (top row) and stress (bottom row) discretisation errors (Eqs. (76) and (77)) of Oldroyd-B flow as a function of the grid spacing $h$, for various cases. Unless otherwise stated, the $\nabla^{1.5} h$ gradient is used and the $D$’s at boundary faces are given by (54) – (55). **Figs. (a), (d):** $W_i = 0.5$, CU grids (Fig. 8a). Green, red and blue lines correspond to setting $\tau_{ij,P} = 0$ in (66), using the log-conformation technique, and defining $\tau_{ij,P}$ by (67) in (66), respectively. In the latter case, the solid and dash-dot lines correspond to the use of Eqs. (29) and (30), respectively. Dashed lines with short or long dashes correspond, respectively, to replacing $S$ by $S/2$ or $2S$ in (29). **Figs. (b), (e):** as for Figs. (a), (d), but for $W_i = 1.0$. The dashed line now corresponds to CN grids (Fig. 8b), using (29) and (20). **Figs. (c), (f):** $W_i = 0.5$, use of (29) and (20). Green, blue, and red lines: CU (Fig. 8a), CN (Fig. 8b), and D (Fig. 8c) grids, respectively. Dashed lines correspond to use of $\nabla^1 h$ and simple linear extrapolation of stresses to the walls, respectively. The high-$W_i$ problem is intensified: for $W_i = 1$, we only managed to obtain a solution up to grid $64 \times 64$ with the scheme (29), and up to grid $128 \times 128$ with the scheme (30).

Finally, we note that Figs. 9c,9f show that the stress extrapolation scheme at the boundaries that makes use of (54) – (55), in combination with the gradient $\nabla^{1.5} h$, which has greater accuracy at the boundaries\footnote{Note that $\nabla^{1.5} h$ is more accurate (second-order) for the velocity components only, whose boundary values are set by the Dirichlet conditions; for pressure and stress, whose values are extrapolated from the interior, $\nabla^1 h$ would also deteriorate to first-order accuracy and hence we use the more standard gradient $\nabla^1 h$ in the extrapolation scheme (53).}, compared to the more common $\nabla^1 h$ [48], leads to a noticeable improvement of the accuracy compared to simple linear extrapolation of stresses.

6. EVP flow in a lid-driven cavity

We now turn our attention to EVP flow in a lid-driven cavity. The SHB model parameters were chosen so as to represent the behaviour of Carbopel. Carbopel gels are used very frequently as prototypical viscoplastic fluids in experimental studies [16, 82]. In [37], the SHB model was fitted to experimental data for a Carbopel gel of concentration 0.2% in weight, and we rounded those parameters to arrive at our own
chosen parameters, listed in Table 2. This material has a yield strain (Eq. (26)) of \( \gamma_y = 0.175 \). The material is enclosed in a square cavity of side \( L = 0.1 \) m and the flow is driven by horizontal motion of the top wall (lid), towards the right. We employed a grid of \( 384 \times 384 \) cells, of uniform size in the \( x \) direction but packed near the lid so that the vertical size of the cells touching the lid is \( L/625 \), and of those touching the bottom wall is about \( L/252 \). A coarse \( 48 \times 48 \) grid of similar packing is shown in Fig. 11.

6.1. The base case

We start with a case where the enclosed material is initially at rest and fully relaxed (\( \tau = 0 \) everywhere) and we impose a lid motion towards the right with velocity \( U = 0.1 \) m/s. The lid starts from rest and accelerates gradually, as

\[
\dot{u}_{\text{lid}} = U \sin \left( \frac{\min(t,T) \pi}{T} \right)
\]

The maximum lid velocity, \( U \), is reached after time \( T = L/U = 1 \) s, and remains constant thereafter. With this choice of time scale \( T \), the Strouhal number, Eq. (17), is equal to 1, while the values of other dimensionless numbers are listed in the corresponding column (\( U = 0.100 \) m/s) of Table 3.

Figure 12 shows the evolution in time of the normalised kinetic energy of the fluid and of the normalised average absolute value of the trace of the stress tensor, calculated as

\[
\text{N.K.E.} = \frac{1}{U^2 \Omega} \sum_P \Omega_P \| \hat{u}_P \|^2
\]

\[
\text{tr}(\tau)_{\text{avg}} = \frac{1}{S \Omega} \sum_P \Omega_P |\text{tr}(\tau_P)|
\]

where \( \Omega = L^2 \) is the volume of the cavity. It is immediately obvious from Fig. 12 that the kinetic energy assumes a constant value very quickly, but the average stress trace evolves much more slowly. To investigate this, the simulation was carried on until a time of \( t = 210 \) s.

Figure 10: The Oldroyd-B, \( Wi = 0.5 \) flow field calculated on the 512 × 512 distorted grid. (a) Pressure contours (colour, \( \delta p = 4S \) with \( S \) given by Eq. (20)) and streamlines (white lines). (b) Contours of \( \tau_{12} (\delta \tau_{12} = 2S) \) and streamlines. (c) Close-up view of (b) near the upper-right corner.

Table 2: Properties of the fluid used in the EVP lid-driven cavity simulations.

| \( \rho \) | 1000 kg/m\(^3\) |
| \( \tau_y \) | 70 Pa |
| \( G \) | 400 Pa |
| \( k \) | 20 Pa s\(^n\) |
| \( n \) | 0.40 |
Figure 11: A coarse, $48 \times 48$ cell grid, showing the packing of the cells near the lid (a coarse grid is shown for clarity; the EVP lid-driven cavity simulations were performed on a similar grid of $384 \times 384$ cells).

Table 3: Values of some dimensionless numbers, and of the nominal relaxation time (24), corresponding to different lid velocities.

| $U$ [m/s] | 0.025 | 0.100 | 0.400 |
|-----------|-------|-------|-------|
| $Bn'$     | 6.094 | 3.500 | 2.010 |
| $Bn$      | 0.859 | 0.778 | 0.668 |
| $Wi$      | 0.204 | 0.225 | 0.262 |
| $Re$      | 0.008 | 0.111 | 1.526 |
| $\lambda$ [s] | 0.815 | 0.255 | 0.066 |

The flow field is visualised in Fig. 13. It resembles that for pure viscoplastic flow [14, 38], in that there are two unyielded zones ($\tau_d < \tau_y$), one at the bottom of the cavity touching the walls, where the fluid is stationary, and one near the lid which is rotating with the flow and does not touch the walls (called a plug zone). The condition that determines whether the material at a point is in a yielded (liquefied) or unyielded (solidified) state is, of course, whether $\tau_d$ is larger or smaller, respectively, than the yield stress $\tau_y$ (Eq. (8)). Figure 13b shows that the streamlines do not cross into the lower unyielded zone, which therefore always consists of the same material, but they cross into and out of the upper (rotating) unyielded zone. Thus, at every instant in time, liquefied particles are entering the plug zone, solidifying upon entry, while other, solidified particles, are exiting the zone, liquefying upon exit.

The differences between SHB and classic HB flow will be examined in detail in Sec. 6.3, but one major difference is the slowness with which the stationary unyielded zone tends to obtain its final shape. Figure 13a shows that although the plug zone has reached its steady state already from $t = 30$ s, the stationary zone continues to expand even at $t = 210$ s. The shape of the yield line that separates this zone from the yielded fluid is quite irregular. Furthermore, Fig. 13b shows that this zone is surrounded by an amount of fluid that is practically stationary and yet yielded. It is likely that this fluid will eventually become part of the stationary unyielded zone as $t \to \infty$, i.e. that the “steady state” boundary of that zone will lie somewhere close to the nearby streamline drawn in Fig. 13b, which separates the fluid with near-zero velocity from that whose velocity is larger. However, to ascertain this the simulation would have to be prolonged to prohibitively long times; already the expansion of the lower unyielded zone from $t = 180$ s to $t = 210$ s, marked in black colour in Fig. 13a, is very small.

A related feature is that the magnitude of the deviatoric stress tensor, $\tau_d$, is very close to the yield stress throughout the aforementioned near-zero velocity region into which the lower unyielded zone is expanding. Thus this region appears as an almost completely flat surface in the three-dimensional plot of $\tau_d$ in Fig. 14 (the surface outlined in dashed line contains fluid where $\tau_d$ is within $\pm1$ Pa of $\tau_y$). Due to these distinctive

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features of this region, we will refer to it as the “transition zone”. In it, the fluid practically behaves as unyielded, although some of it can be formally yielded. The plug zone possesses no transition zone.

In order to explain the transition zone behaviour, we consider the constitutive equation (8) – (9) in the case that the fluid velocity is zero and \( \tau_d \) is slightly above \( \tau_y \). It becomes:

\[
\frac{\partial \tau}{\partial t} = -\frac{G}{k^{\frac{1}{n}}} \left( \tau_d - \tau_y \right) \frac{1}{\tau_d} \tau \equiv -r(\tau_d) \tau
\] (81)

Since the function \( r(\tau_d) \) assumes only positive values, the minus sign on the right-hand side of Eq. (81) drives the components of \( \tau \) towards zero as time passes, with a rate proportional to \( r(\tau_d) \) (and to the magnitude of those components themselves). Hence \( \tau_d \) decreases towards zero as well (Eq. (10)). However, the rate at which they decrease, \( r(\tau_d) \), diminishes as \( \tau_d \to \tau_y \) and in fact \( r(\tau_y) = 0 \). Therefore, \( \tau_d \) actually converges towards \( \tau_y \) and not to zero. This is what we observe happening inside the transition zone.

It is useful to consider also a scalar version of Eq. (81) (\( \tau_d \equiv \tau \) in this case)\(^3\):

\[
\frac{d\tau}{dt} = -\frac{G}{k^{\frac{1}{n}}} (\tau - \tau_y) \frac{1}{\tau_y} \Rightarrow \frac{d(\tau - \tau_y)}{dt} = -\frac{1}{\lambda'} (\tau - \tau_y) \frac{1}{n}
\] (82)

where \( \lambda' \equiv k^{\frac{1}{n}}/G \) is a relaxation “time” (it actually has units of \( \text{s}/\text{Pa}^{1-1/n} \)). For \( n = 1 \), the solution to the above equation is

\[
\tau - \tau_y = (\tau_0 - \tau_y) e^{-t/\lambda'}
\] (83)

where \( \tau_0 \) is the value of \( \tau \) at \( t = 0 \). This behaviour is similar to that of a Maxwell viscoelastic fluid, only that now \( \tau \) decays exponentially towards \( \tau_y \) instead of towards zero. For \( n \neq 1 \), Eq. (82) can be written as

\[
\frac{d(\tau - \tau_y)}{dt} = -\frac{1}{\lambda'} \frac{(\tau - \tau_y)(\tau - \tau_y)^{1-n}}{n}
\]

so that the rate of decay of \( \tau \) towards \( \tau_y \) equals that for \( n = 1 \) multiplied by a factor of \( (\tau - \tau_y)^{(1-n)/n} \). If \( n < 1 \), as is the present choice but also the most common case, then \( (1-n)/n > 0 \) and as time progresses

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\(^3\)Eq. (82) can be viewed as describing the behaviour of the mechanical system of Fig. 1 (with \( \kappa = 0 \)) in the case that at some stress \( \tau = \tau_0 > \tau_y \) we pin the right end of the spring to a fixed location, so that the total length \( \gamma_v + \gamma_e \) remains constant henceforth, and we leave the system to relax. The spring will try to recover its equilibrium length by contracting or expanding \( \gamma_e \), which will cause an opposite expansion or contraction of \( \gamma_v \), resisted by the viscous and friction elements. Once the tension in the spring drops to \( \tau_y \), the spring cannot overcome the friction \( \tau_y \) of the friction element and motion stops, without the spring having attained its equilibrium length.
and $\tau \to \tau_y$, the extra factor $(\tau - \tau_y)^{(1-n)/n}$ tends to zero, and the rate of decay becomes progressively smaller than that of the $n = 1$ case, eventually becoming infinitesimal compared to it. This behaviour is explained physically by the fact that the material relaxation time is proportional to the fluid viscosity, which resists recovery from deformation (relaxation), and inversely proportional to the elastic modulus, which drives towards such recovery (Eq. (24)). For shear-thinning fluids ($n < 1$), the viscosity increases as $\tau$ becomes smaller and, in the SHB and HB models, tends to infinity as $\tau \to \tau_y^4$. Thus, for these fluids the relaxation time tends to infinity as $\tau \to \tau_y$. The opposite happens in the less common case of $n > 1$.

The above discussion helps to interpret the observation that in Fig. 12 the velocity field appears to reach a steady state very quickly, while the stress field does so more gradually. Since the lid velocity is constant for $t > T = 1$ s, the temporal variation of the fluid velocity is caused either directly by its time derivative appearing in the momentum equation, or indirectly through the evolution of stress. Concerning the first cause, by dividing both sides of Eq. (14) by $Re$, because $Re$ is very small (Table 3), it becomes apparent that the fluid particle accelerations, the left-hand side of that equation, are large and the steady-state is reached very quickly (in other words, the inertial time scales are very small). Concerning the second cause, since the velocity and stress are interrelated by the governing equations, at first sight it seems odd that the evolution of stress is slower than that of velocity. However, a closer inspection of the flow field reveals that the stress actually also evolves quickly over most of the domain (the relaxation time $\lambda = 0.255$ s, Table 3, is quite small compared to the time scale of stress evolution in Fig. 12b) except in the transition zone where the definition (24) is not representative and the actual relaxation time tends to infinity as time passes. There, the stress magnitude is of the order of $\tau_y$, and therefore its local slow evolution has a noticeable impact on the average stress trace plotted in Fig. 12. On the other hand, the velocity is almost zero in this zone and hence any local velocity variations caused by the local stress evolution have a negligible impact on the overall kinetic energy plotted in Fig. 12. We noticed that between $t = 30$ s and $t = 210$ s the change in the velocity components at any point in the domain is of the order of $10^{-6}$ m/s, except in the transition zone where it can be about five times larger.

Figure 15 shows vertical profiles of most flow variables along the centreline ($x = L/2$, Figs. 15a and 15b) and close to the right wall ($x = 0.95L$, Figs. 15c and 15d). Two profiles are drawn for each variable, one at time $t = 30$ s and one at $t = 210$ s. These profiles are identical inside the yielded and plug zones, and only deviate very slightly inside the bottom unyielded zone. Thus the steady state has largely been reached

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In the language of the mechanical analogue of Fig. 1 this means that the damper component resisting the relaxation of the spring becomes stiffer as this relaxation proceeds.
already at $t = 30$ s. The vertical line at $x = 0.95L$ cuts through a significant width of the transition zone, which can be recognised by the vertical line segment where $\tau_d \approx \tau_y$ in Fig. 15c (approximately from $y = 0.041$ m to $y = 0.051$ m). A close-up view of the variation of $\tau_d$ in the transition zone is shown in the inset of the same figure, where one can see that as time passes, $\tau_d$ decreases towards $\tau_y$ throughout the zone.

An interesting observation that can be made in Fig. 15a is that the variation of the normal stress component $\tau_{11}$ across the boundary of the lower unyielded zone tends to become discontinuous as time passes. The possibility of the SHB model producing solutions with discontinuous stress components and/or velocity gradients was noted and discussed in [43, 28], where it was attributed to the combination of the upper convective derivative and the viscoplastic “max” term. Nevertheless, as noted in [43], stress discontinuities must be such that the components of the force $\nabla \cdot \sigma = \nabla \cdot (\tau - pI)$ appearing in the momentum equation remain bounded (discontinuities in $\tau$ that lead to infinite derivatives in $\nabla \cdot \tau$ can exist if they are counterbalanced by opposite discontinuities in $-pI$). Otherwise, at a point of discontinuity there will result a finite (i.e. non-zero) force acting on an infinitesimal mass, producing infinite acceleration of that fluid particle. Consequently, the velocity field will be discontinuous, but this violates the continuity equation for an incompressible medium. In our case though, the variation of $\tau_{11}$ in the $x_2$ direction does not cause any force on the fluid (there is no $\partial \tau_{11}/\partial x_2$ derivative in $\nabla \cdot \tau$) and is therefore allowed to be discontinuous.

6.2. Varying the lid velocity

Next we examine the flow driven by higher ($U = 0.4$ m/s) and lower ($U = 0.025$ m/s) lid velocities, which affects the dimensionless numbers as shown in Table 3. Lowering the velocity increases the Bingham number and decreases the Weissenberg number, i.e. it accentuates the plastic character of the flow at the expense of its elastic character. The opposite happens when $U$ is increased ($Bn$ and $Wi$ are inversely proportional, according to Eq. (26)). The $Re$ values listed in Table 3 suggest that inertia effects should be negligible for $U = 0.025$ and 0.100 m/s, while they should be slightly noticeable for $U = 0.400$ m/s. The Table shows also that the nominal relaxation time increases as $U$ decreases; at lower velocities the apparent viscosity $\eta$, Eq. (24), is higher and therefore the stresses relax more slowly. Thus, we expect the flow to evolve more slowly as $U$ is reduced.

The lid velocity is again increased gradually, according to Eq. (78), with $T = L/U$. Figure 12a shows that the KE of the fluid builds up in an oscillatory manner, usually with an overshoot, due to elastic effects. The larger $U$ is, the more prominent and more persistent the oscillations. Figure 12b confirms that the stress evolution is slower at lower $U$, as discussed above. The top row of Fig. 16 shows the corresponding

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Figure 14: 3D contour plot of the base case results (Sec. 6.1) at $t = 210$ s. Both the colour contours and the plot height (z axis) represent $\tau_d$. In addition, the dashed line encloses an area where $\tau_d \in [69, 71]$ Pa.
near-steady-state flow fields. As expected, increasing the lid velocity leads to smaller percentage of unyielded material in the cavity, and the vortex has more free space to move away from the lid (see the $y_c$ coordinate in Table 4). In each case there is a transition zone between the bottom unyielded zone and the yielded material, with the former not having yet expanded throughout the near-zero velocity region. The number and density of streamlines in Fig. 16 indicates that the higher $Bn$ is, the weaker the flow and the more it is confined to a thin layer below the lid while circulation is very weak in the rest of the domain. This is reflected also in the normalised KE diagram 12a, and also in the normalised vortex strengths listed in Table 4. That Table also shows that the vortex lies slightly to the left of the centreline, as is typical of viscoelastic lid-driven cavity flows, although this shift is not as pronounced as for the Oldroyd-B flows of Table 1. Interestingly, as $U$ is increased the vortex centre moves towards the centreline despite $Wi$ increasing; this could be due to inertia effects that will be discussed in Sec. 6.3.
Table 4: Dimensionless coordinates of the centre of the vortex \((\tilde{x}_c, \tilde{y}_c) \equiv (x_c/L, y_c/L)\) and value of the streamfunction there \(\tilde{\psi}_c \equiv \psi_c/(LU)\), for various cases, at steady-state.

|          | SHB          | HB           |
|----------|--------------|--------------|
|          | \(\tilde{x}_c\) | \(\tilde{y}_c\) | \(\tilde{\psi}_c\) | \(\tilde{x}_c\) | \(\tilde{y}_c\) | \(\tilde{\psi}_c\) |
| \(U = 0.025\) m/s | 0.495 | 0.935 | -0.0211 | 0.500 | 0.933 | -0.0214 |
| \(U = 0.100\) m/s | 0.495 | 0.917 | -0.0270 | 0.500 | 0.915 | -0.0281 |
| \(U = 0.400\) m/s | 0.500 | 0.899 | -0.0333 | 0.505 | 0.897 | -0.0352 |
| \(U = 0.100\) m/s, slip | 0.482 | 0.846 | -0.0125 | 0.500 | 0.853 | -0.0112 |

Figure 16: Top row: SHB flow snapshots at the times indicated, for different lid velocities. Bottom row: corresponding HB steady-state flowfields. The unyielded regions are shown shaded; the lines are instantaneous streamlines, plotted at streamfunction values \(\psi/(UL) = 0.04, 0.08, 0.12\) etc., plus a \(\psi = 5 \times 10^{-6}\) m\(^3\)/s streamline just above the lower unyielded region.

6.3. Comparison with the classic HB model

Since the HB equation is used extensively to model viscoplastic flows, we compare its predictions against those of the SHB equation in order to get a feel of the error involved in neglecting elastic effects. The HB simulations were carried out by performing Papanastasiou regularisation [83], implemented as [84]:

\[
\tau = \eta \dot{\gamma}, \quad \eta = \frac{\tau}{\dot{\gamma}} \approx \frac{(\tau_0 + k\dot{\gamma}^n)(1 - e^{-\dot{\gamma}/\epsilon})}{\dot{\gamma}}
\]

(85)

where the regularisation parameter \(\epsilon\) determines the accuracy of the approximation. This method was used for simulating lid-driven cavity Bingham flows in [14, 38], where it was noticed that the equations became
too stiff to solve for $\epsilon < 1/400$. However, with the present method, and using a continuation procedure where $\epsilon$ is progressively halved, solutions for $\epsilon = 1/128,000$ were obtained. We performed steady-state simulations (since the steady-state of classic viscoplastic flows does not depend on the initial conditions) where the HB parameters $\tau_y$, $k$, and $n$ have the same values as for the SHB fluid (Table 2).

The flowfields, depicted in the second row of Fig. 16, are much more symmetric than their SHB counterparts. In HB lid-driven cavity flow, the only source of asymmetry is inertia. Inertial effects are imperceptible for $U = 0.025$ and 0.100 m/s (Figs. 16d and 16e) and only slightly noticeable for $U = 0.400$ m/s (Fig. 16d), which is explained by the corresponding Re values listed in Table 3. So, for $U = 0.4$ m/s the vortex centre is shifted slightly towards the right (Table 4), whereas at the lower $U$’s it is almost exactly on the centreline. Also, in Fig. 16f one can see that the upper unyielded (plug) region is somewhat stretched towards the left. These features are opposite to those of the SHB case, where elasticity causes the vortex centre to move towards the left and the plug region to stretch towards the right. The opposite effects of inertia and elasticity have been noted also in [40, 85, 34].

Figure 16 and Table 4 show that the union of the lower unyielded and transition zones in SHB flow is slightly larger than the corresponding HB stationary regions, pushing the SHB vortex upwards and lowering its strength compared to the HB case. Figures 17a and 17b show that the SHB and HB velocity profiles along the vertical centreline are very similar; the $\tau_d$ profiles are somewhat more dissimilar but still not far apart, especially in the yielded regions. In the unyielded regions they cannot be expected to be similar due to the stress indeterminacy of the HB model (the currently predicted HB stress field in the unyielded regions is one of infinite possibilities, the one conforming to the Papanastasiou regularisation). Profiles of $\tau_{11}$ are plotted in Fig. 17c; for HB flow, due to the symmetric flowfield and $\tau_{11}$ being proportional to $\partial u_1/\partial x_1$, this stress component is nearly zero, except inside the plug region for $U = 0.400$ m/s, which is somewhat asymmetric (Fig. 16f). The SHB stresses, on the other hand, are significant due to elasticity, especially in the higher $U$ case which corresponds to higher $Wi$. Finally, we note that in Fig. 17 two SHB profiles are plotted for each lid velocity: at $t = 60$ and 90 s for $U = 0.025$ m/s, and at $t = 30$ and 60 s for $U = 0.400$ m/s. These profiles are hardly distinguishable, indicating that the steady state for $U = 0.025$ m/s has been practically reached at $t = 60$ s, and for $U = 0.400$ m/s at $t = 30$ s.

Next, we performed a couple of simulations similar to the base case, only that we artificially increased the elastic modulus to 10 and 100 times the original value listed in Table 2. This reduces the relaxation times by factors of 10 and 100, respectively, so that the SHB material becomes more inelastic and its predictions should tend to match those of the HB model. Due to the smaller relaxation times, we also used smaller time steps: starting from an initial time step of $5 \times 10^{-5}$ s in both cases, we allowed the time step to increase up to $\Delta t_{\text{max}} = 1.5 \times 10^{-3}$ s and $10^{-4}$ s in the $G = 4,000$ Pa and $G = 40,000$ Pa cases, respectively. The simulations were stopped at $t = 50$ s and 10 s respectively, and in the $G = 40,000$ Pa case the lid...
acceleration period (Eq. (78)) was reduced to 0.01 s. Figure 18 shows that the SHB and HB streamlines converge rapidly with increasing \( G \), while Fig. 19 shows that convergence is not as rapid for the stresses. Of course, for \( \tau_d < \tau_y \) (top row of Fig. 19) we cannot expect the SHB and HB stresses to ever match, due to the aforementioned indeterminacy of the HB stress tensor in the unyielded regions. The yield lines (second row of Fig. 19) appear to converge, but at a slow pace. Despite the shorter duration of the \( G = 40,000 \) Pa simulation, Fig. 19f shows that during this time the transition zone has evolved further than in the lower \( G \) cases, due to the much shorter relaxation times. The same figure reveals some slight spurious stress oscillations on the lower yield line (can also be seen in Fig. 19d). They could be due to the near-zero velocities there, which complicates the application of upwinding within CUBISTA. Finally, the last row of Fig. 19 shows that within the yielded region the \( \tau_d \) predictions of the two models are quite close, even for \( G = 400 \) Pa. A persistent discrepancy between the two models at the sides of the plug region seen in Fig. 19i may be due to spatial or temporal discretisation errors, regularisation, etc.

### 6.4. A case with slip

Carbopol gels are quite slippery [16, 86, 82]; the previous, no-slip results can be considered to correspond to roughened walls, like those used in rheological measurements to avoid slip [37]. However, if the walls are smooth then noticeable slip is expected. The slip behaviour of viscoplastic or viscoelastic fluids can be quite complex, where, for example, the slip velocity can depend on the wall stress through a power law [87, 88], it can depend on pressure [89], or there may exist a “slip yield stress”, i.e. a limit value that the wall stress must exceed in order for slip to occur [90, 91]. In [86] a power-law slip behaviour of the form

\[
    u_s = 5.151 \times 10^{-4} \tau_w^{0.876}
\]

is reported for a 0.2% (by weight) Carbopol gel, where \( u_s \) is the slip velocity (the left-hand side of Eq. (12)) and \( \tau_w \) is the wall tangential stress (the term \((n \cdot \tau) \cdot s\) of the right-hand side of Eq. (12)). Similarly, in [82] a power-law slip behaviour of

\[
    u_s = 2.3 \times 10^{-4} \tau_w^{1.32}
\]

is reported for a 0.08% (by weight) Carbopol solution. Both of these exponents (0.876 and 1.32) are close to 1, i.e. the observed slip is of approximately Navier type, Eq. (12). In [82] it is conjectured that this may be due to the formation of a thin layer of Newtonian fluid (solvent) that separates the wall surface from the gel micro-particles. Since the Carbopol concentration in [86] is the same as that assumed here\(^5\), in the present section we choose to impose a Navier slip condition, Eq. (12), on all walls, with a slip coefficient \( \beta = 5 \times 10^{-4} \) m/Pa.s. For this case, concerning stress extrapolation to the walls, we could not get SIMPLE to converge with the \( D \) coefficients given by (54) – (55) and we set them equal to zero.

Figure 20a shows the flowfield. A distinguishing feature is that the two unyielded zones touch the cavity walls and yet contain moving material which is sliding over the wall. This feature is common also to the

\(^5\)In [86] glass particles were added to the Carbopol gel to enable PIV measurements, and this resulted in a modification of the final pH and an associated decrease of the yield stress (21.6 Pa) and viscosity; the effect on slip was not examined.
Figure 19: Continuous lines: contours of τ_d predicted by the SHB model (top row: τ_d = 40 Pa; middle row: τ_d = τ_y = 70 Pa; bottom row: τ_d = 80 Pa) for various values of the elastic modulus G (left column: G = 400 Pa; middle column: G = 4,000 Pa; right column: G = 40,000 Pa), the rest of the parameters having the values listed in Table 2, and the lid velocity being U = 0.1 m/s. Dashed lines: corresponding contours predicted by the HB model.

HB flowfield, shown in Fig. 20b, which is, again, much more symmetric. Figure 20c shows that SHB and HB contours of τ_d are more similar for τ_d > τ_y. Due to slip, the flow induced by the lid is much weaker than in the no-slip case, as can be seen from the lower kinetic energy in Fig. 12, the smaller vortex strength in Table 4, and the wider streamline spacing in Fig. 21b than in Fig. 21a. On the other hand, the streamlines in Fig. 21b are more evenly spaced near the bottom of the cavity compared to Fig. 21a as slip allows the circulation to extend all the way down to the bottom wall, where now there is no stationary unyielded zone. Consequently, there is also no transition zone. Indeed, comparing in Fig. 20a the yield lines at t = 30 s (dashed lines) and t = 60 s (the boundary of the shaded regions) one sees that there is very little change,
Figure 20: (a) SHB flowfield for the slip case; shaded region: unyielded material at $t = 60$ s; dashed lines: yield lines at $t = 30$ s; continuous lines: streamlines (they are drawn at equal streamfunction intervals). (b) Corresponding steady-state HB flowfield (the streamlines are drawn at the same values of streamfunction as in (a)). (c) The $\tau_d = 75$ Pa contour of the SHB (continuous line) and HB (dashed line) flowfields.

Figure 21: Contours of $\dot{\gamma}$ [$s^{-1}$] for $U = 0.100$ m/s without wall slip (a) (base case) or with wall slip (b). The thick continuous lines are yield lines and the dashed lines are streamlines. The latter are plotted at equal streamfunction intervals (the same intervals in both figures).

and the bottom unyielded zone even slightly contracts with time. The absence of a transition zone can be explained by examining the $\dot{\gamma}$ distributions of Fig. 21; whereas in the no-slip case (Fig. 21a) the velocity gradients are practically zero inside the bottom unyielded zone, in the slip case (Fig. 21b) they are non-zero throughout the domain, thus excluding the transition zone situation where the constitutive equation reduces to Eq. (81). A comparison between Figs. 21a and 21b also shows that in the upper part of the cavity $\dot{\gamma}$ is much lower in the presence of slip. Due to the shear-thinning nature of the fluid, this results in higher viscosities and associated relaxation times which may explain why the stress evolution is slower in the slip case than in the no-slip one as seen in Fig. 12b (and also in Figs. 20a and 22, where there are slight changes between $t = 30$ and $60$ s). A rather surprising observation in Fig. 12b is that $\text{tr}(\tau)_{\text{avg}}$ is actually higher in the slip case, which may be attributed to the larger circulation area, which includes the bottom of the cavity. For benchmarking purposes we plot profiles of some dependent variables along the vertical centreline in Fig. 22.

6.5. Multiplicity of solutions

It was mentioned in Sec. 2 that Cheddadi et al. [36] found that the steady state of a SHB flow, including both the stress and velocity fields, can depend in a continuous manner on the initial conditions. That ob-
Observation was made in simple cylindrical Couette flow, where the shear stress is determined by the geometry and kinematics, but the circumferential normal stress depends also on its initial value, which affects the final extent of the yield zone and therefore also the final velocity field. We performed analogous experiments to see if this can be observed also in the present, more complex flow. So, we solved two more cases that are identical to the “base case” of Sec. 6.1, except for the initial stress conditions: \( \tau_{11} = \tau_{22} = +\sqrt{3}\tau_y \) (first case) or \( -\sqrt{3}\tau_y \) (second case) throughout, the remaining stress components being zero. These initial conditions correspond, respectively, to tension and compression states that are isotropic in the \( xy \) plane; from Eq. (10) it follows that \( \tau_d(t = 0) = \tau_y \) everywhere, i.e. the material is on the verge of yielding. The material is initially at rest (\( \mathbf{u} = 0 \)). The simulations were carried on until time \( t = 60 \) s.

Interestingly, the velocity fields of both of these new cases arrive at practically the same steady state; this can be seen in Fig. 23, where the kinetic energies of both cases converge, and more clearly in Fig. 24a where the respective streamlines can be seen to be identical. However, this steady-state is not the same as that arrived at in the base case (\( \tau_{11} = \tau_{22} = 0 \)); Fig. 23 shows that unlike in the two new cases, the kinetic energy of the base case does not exhibit an overshoot and eventually increases to a value which is about 2.25\% larger than that of the other two cases. In Fig. 24a the base case streamlines are also not identical to those of the other two cases. As far as the main vortex is concerned, the new cases have it located at \((\tilde{x}_c, \tilde{y}_c) = (0.497, 0.917)\) with a strength of \( \tilde{\psi}_c = -0.0266 \), which is located slightly to the right and is slightly

### Figures

**Figure 22:** Profiles of some dependent variables along the vertical centreline, for the slip case. SHB results at \( t = 60 \) and \( 30 \) s are plotted in continuous and dashes lines, respectively, and steady state HB results in dash-dot line.

**Figure 23:** Histories of N.K.E. (Eq. (79)) and \( \text{tr}(\tilde{\mathbf{t}})_{\text{avg}} \) (Eq. (80)) for the cases with initial conditions \( \tau_{11} = \tau_{22} = -\sqrt{3}\tau_y \) (continuous lines), \( \tau_{11} = \tau_{22} = +\sqrt{3}\tau_y \) (dashed lines) and \( \tau_{11} = \tau_{22} = 0 \) (dash-dot lines, the base case).
Concerning the stress fields, we can note the following. Firstly, an obvious difference between the two new cases and the base case is that in the former the bottom unyielded zone is largely absent and is replaced by a large transition zone; this is shown in Fig. 24c where the transition zone is approximately the green region, throughout which \( \tau_d \) is slightly above \( \tau_y \). Inside the transition region the \( \tau_d \) fields of the two new cases are similar (Fig. 24c), but the individual \( \tau_{11} \) and \( \tau_{22} \) stress components have opposite signs which they have inherited from the initial conditions, as seen in Fig. 25; in fact, near the bottom wall these stress components retain values close to the initial ones, \( \pm \sqrt{3} \tau_y \approx 121 \text{ Pa} \). On the other hand, the base case stresses are close to zero in that region, again an inheritance of the initial conditions. This reflects on the much lower stress trace values seen in Fig. 23. Outside of the transition region, where the rates of deformation are non-zero (including the plug zone), the two new cases have identical steady-state stress fields, as seen in Figs. 24c and 25, whereas those of the base case deviate slightly. This includes the yield line that forms the boundary of the plug zone: Fig. 24b shows that the plug zones of the two new cases are identical, but that of the base case is slightly smaller, which is likely the cause of the slightly different velocity field.

### 6.6. Cessation

Our final investigation concerns the sudden stopping of the lid and the study of the subsequent flow decay. Classic viscoplastic models are known to predict flow cessation in finite time after the driving agent is removed [92]. This can be proved rigorously using variational inequalities (e.g. upper bounds for the
Figure 26: (a): Fluid kinetic energy (Eq. (79)), normalised by its value at the instant the lid is halted, versus time elapsed (the lid is halted at time $t = 0$) for the no-slip and slip cases. (b): The top diagram is a close-up view of a portion of the no-slip curve in (a). The bottom diagram plots the corresponding values of the force exerted by the fluid on the lid, normalised by $\tau_y L$.

cessation times of some one-dimensional flows are derived in [93–95]), but roughly speaking it is due to the effect of the yield stress on the rate of energy dissipation: it keeps it high enough during the flow decay such that all of the fluid’s kinetic energy (KE) is dissipated in finite time – a rough explanation is sketched in Appendix C. The cessation of viscoplastic (Bingham) lid-driven cavity flow was studied in [44], according to the findings of which we would expect HB flow to cease completely at some time $t_c \ll T_c \equiv \rho UL/S \approx 0.11 \text{s}$ ($T_c$ is the time needed for a force of magnitude $SL^2$ to bring a mass of momentum $\rho UL$ to rest).

The situation concerning SHB flow is expected to be somewhat different. Firstly, even though the energy conversion rate $\int_\Omega \tau \cdot \nabla u \, d\Omega$ is still expected to be large enough to convert all the KE of the material in finite time, nevertheless this rate now includes not only energy dissipation, but also energy storage in the form of elastic (potential) energy, which can later be converted back into KE [96]. In fact, once all of the material becomes unyielded there is no mechanism for energy dissipation and it is expected that the remaining energy will perpetually change form from elastic to kinetic and vice versa, resulting in oscillatory motions. Furthermore, the formation of transition regions in previous simulations makes it uncertain whether all of the material will become unyielded in finite time. To investigate these issues, using the “steady-state” base flow (Sec. 6.1) as initial condition, we suddenly stopped the lid motion and carried out a simulation to see how the flow evolves. We repeated the procedure also with the slip case of Sec. 6.4. Figure 26a shows the evolution of the KE with time ($t = 0$ is the instant the lid is halted). In neither case does the flow cease in finite time.

In the no-slip case, the KE history is highly oscillatory, confirming the anticipated perpetual back-and-forth conversion between kinetic and elastic energies. The top diagram of Fig. 26b shows a clearer picture over a narrower time window. The KE peaks 29 times within that 2 s window, i.e. a single oscillation lasts about 69 ms; the time step is kept to about $\Delta t = 2 \times 10^{-4} \text{s}$ by the adjustable time step scheme, so that each KE oscillation period is resolved into about 350 time steps. The diagram shows that the KE variation does not consist of a single harmonic component, and Fig. 27 shows that the flow is indeed quite complex. Usually at any given instant there appears one main vortex rotating in either the clockwise or anticlockwise sense, but its location and shape are not fixed, while smaller vortices may also appear. The material is not completely unyielded, and the yielded zones are transported along as the material oscillates while their size varies with time. The existence of these yielded zones allows some energy dissipation, and hence the mean KE in Fig. 26 very slowly drops. The bottom diagram of Fig. 26b shows the force $F_{\text{lid}}$ exerted by the fluid on the lid. This force is negative, pushing the lid towards the left, i.e. in the opposite direction than it
Figure 27: (a)-(g): Snapshots of the no-slip cessation flow field, with colour contours of $\tau_d$, of different base colour for yielded (yellow) and unyielded (blue) regions (see the colour map (h)), and instantaneous streamlines plotted at streamfunction intervals of $\delta \psi / LU = 0.002$, with $\psi = 0$ as one of the contours. (i): Part of the history of the force exerted by the fluid on the lid, normalised by $\tau_y L$, with the instants corresponding to snapshots (a)-(g) marked on the plot. (j)-(k): Snapshots of the slip cessation flow field.

was moving prior to its halting. The magnitude of the force oscillates slightly above the value $\tau_y L$, and an inspection shows that this is because the magnitude of $\tau_{12}$ on the lid slowly drops towards $\tau_y$ (the lid is in touch with a transition zone). Figure 27i shows that the magnitude of $F_{\text{lid}}$ drops when the lid touches a clockwise vortex (Figs. 27b, 27d, 27g) and it increases when it touches a counter-clockwise vortex (Figs. 27c, 27e). A comparison between the KE and $F_{\text{lid}}$ diagrams in Fig. 26b shows that $F_{\text{lid}}$ oscillates at a frequency that is roughly half of that of the KE, which is explained by the fact that during a single $F_{\text{lid}}$ period both the clockwise and anticlockwise vortex velocities are maximised, which leads to two KE peaks.

In the slip case, the KE peaks immediately after the lid halt (Fig. 26a) but then the oscillations die out relatively quickly and the KE diminishes. The KE peak is due to the material recoiling right after the lid halts (Fig. 27j), which is possible as the walls provide limited resistance, due to slip. Then, the KE is dissipated through a mechanism that is absent in the no-slip case: as the material oscillates inside the cavity, it slides past the walls due to slip, and the friction at the wall / material interface dissipates the kinetic energy of the latter. Even in this case, however, transition zones exit long after the lid has halted (Fig. 27k).
7. Conclusions

This work presented a FVM for elastoviscoplastic flows described by the SHB model. The method is applicable to collocated structured and unstructured meshes; it incorporates a new variant of momentum interpolation, a variant of “two sides diffusion”, and the CUBISTA scheme, in order to suppress spurious pressure, velocity, and stress oscillations, respectively. The method was shown to achieve this goal, and also to be consistent on both smooth and irregular meshes. An implicit temporal discretisation with adaptive time step is employed.

The method was applied to the simulation of EVP flow in a lid-driven cavity, with the SHB parameters chosen so as to represent Carbopol. The results can serve as benchmark solutions, but furthermore the simulations were designed so as to allow investigation of several aspects of the SHB model behaviour. Complementary simulations with the classic HB model were also performed for comparison. It was noticed that the SHB model can predict “transition zones”, regions where the velocity is near-zero and it takes a very long, or infinite, time to transition from a formally yielded to an unyielded state. The differences between the SHB and HB velocity fields are rather small for the lid velocities tested, although some elastic effects are noticeable in the SHB case, such as a slight upstream displacement of the vortex, a downstream stretching of the plug zone, and a small swelling of the bottom unyielded zone, compared to the HB results. These differences diminish by increasing the elastic modulus $G$, although the convergence of the SHB and HB yield lines is rather slow and requires very large values of $G$. The velocity fields converge much faster.

We also applied slip at the walls, as EVP materials are slippery, in which case no transition zones developed.

Motivated by the observation of Cheddadi et al. [36] that different initial conditions of stress can lead to different steady states even as concerns the velocity, we performed two additional simulations where the initial state of the material was stationary but on the verge of yielding, with either compressive or tensile residual stresses, respectively. Indeed, although both of these simulations converged to the same steady-state velocity field, that field was slightly but noticeably different than that obtained when the initial stresses were zero. Due to this property of the SHB model, an accurate calculation of a steady state requires not only sufficient spatial resolution (grid spacing) but also sufficient temporal resolution (time step size), which is facilitated by the use of an adaptive time discretisation scheme such as the one proposed herein.

Finally, simulations of the cessation of the flow after the lid is suddenly halted were performed, which showed that, unlike what is predicted by the HB model, the flow does not cease in finite time but there is a perpetual back-and-forth conversion between kinetic and elastic energies. In the no-slip case, a very slow net energy dissipation was observed, which is possible due to the persisting existence of yielded regions. The energy dissipation is much faster under wall slip, due to friction between the material and the walls.

Incorporation of additional rheological phenomena such as thixotropy is planned for the future.

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Appendix A The SHB extra stress tensor in the limit of infinite elastic modulus

The SHB constitutive equation (8) distinguishes between $\tau$ and $\tau_d$, unlike the HB equation (5). For the two models to become equivalent as $G \to \infty$ it is necessary that $\tau \to \tau_d$ or, equivalently, tr($\tau$) $\to$ 0.

In the limiting case $G \to \infty$ the first term on the left-hand side of Eq. (8) diminishes. This makes yielded material (where the “max” term is non-zero) tend to behave as a generalised Newtonian fluid so that tr($\tau$) $\to$ 0 $\Rightarrow$ $\tau \to \tau_d$ and the two models do become equivalent in yielded regions. On the other hand, in unyielded regions (where the “max” term is zero) Eq. (8) tends to reduce to $\dot{\gamma} = 0$, which does not, at first glance, imply that $\tau \to \tau_d$. However, the following observation can be made. In an unyielded region the SHB model can be written as (substituting Eq. (9) into Eq. (8))

$$\frac{D\tau}{Dt} \equiv \frac{\partial\tau}{\partial t} + u \cdot \nabla \tau = G \dot{\gamma} + G (\nabla u)^T \cdot \tau + \tau \cdot \nabla \dot{u}$$  \hspace{1cm} (A.1)

The material derivative $D\tau/Dt$ is the rate of change of the stress tensor with respect to time in a fluid particle moving with the flow. This derivative is applied component-wise to the stress tensor, and therefore
the rate of change of \( \text{tr}(\mathbf{r}) \) is

\[
\frac{D(\text{tr}(\mathbf{r}))}{Dt} = \text{tr}\left( \frac{D\mathbf{r}}{Dt} \right) \tag{A.2}
\]

or, using Eq. (A.1),

\[
\frac{D(\text{tr}(\mathbf{r}))}{Dt} = G \text{tr}(\dot{\gamma}) + G \text{tr}\left( (\nabla \mathbf{u})^T \cdot \mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u} \right) \tag{A.3}
\]

In the right-hand side, \( \text{tr}(\dot{\gamma}) \to 0 \) because in the unyielded regions \( G \to \infty \) forces \( \dot{\gamma} \to 0 \). For the second term in the right-hand side, in index notation, we have

\[
(\nabla \mathbf{u})^T \cdot \mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u} = \sum_{i=1}^{3} \sum_{j=1}^{3} \left( \sum_{k=1}^{3} \frac{\partial u_j}{\partial x_k} r_{ik} + \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k} r_{ik} \right) \mathbf{e}_i \mathbf{e}_j \tag{A.4}
\]

where \( \mathbf{e}_i \) is the unit vector in the direction \( i \). The trace of the above tensor is obtained by replacing the dyadic product \( \mathbf{e}_i \mathbf{e}_j \) by the Kronecker delta, \( \delta_{ij} \). After some manipulation and index renaming, we obtain

\[
\text{tr}\left( (\nabla \mathbf{u})^T \cdot \mathbf{r} + \mathbf{r} \cdot \nabla \mathbf{u} \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) r_{ij} \tag{A.5}
\]

The terms in parentheses in the sum (A.5) are just the components of \( \dot{\gamma} \), and so they tend to zero as \( G \to \infty \). Thus the whole right-hand side of Eq. (A.3) tends to zero in the unyielded regions as \( G \to \infty \). Therefore, under these conditions, \( D(\text{tr}(\mathbf{r})) / Dt \to 0 \), i.e. \( \text{tr}(\mathbf{r}) \) remains constant in any particle moving in an unyielded region. Given that \( \text{tr}(\mathbf{r}) = 0 \) for yielded particles, as mentioned above, there appears to be no mechanism for \( \text{tr}(\mathbf{r}) \) to acquire any value different than zero. Thus we expect that \( \mathbf{r} \to \mathbf{r}_f \) also in the unyielded regions. But even if \( \text{tr}(\mathbf{r}) \neq 0 \Rightarrow \mathbf{r} \neq \mathbf{r}_f \) in such a region, for practical purposes the stress field would still be equivalent to a HB stress field where the isotropic part of \( \mathbf{r} \) is incorporated into the pressure.

**Appendix B Second order accurate reconstruction of cell centre value from face values**

Suppose that the exact (or approximated to at least second-order accuracy) values \( \phi_f \) of a quantity \( \phi \) are known at the face centres of a cell \( P \) (Fig. 2), and from these we want to approximate the value of \( \phi \) at the cell centre, \( \bar{\phi}_P \). One way to proceed is the following. The fact that the known values are located at the cell boundary provides an incentive to try to derive a scheme based on the divergence theorem. Let \( \mathbf{r}(x) = x - P \) be the vector function that returns the relative position of location \( x \) relative to the centroid \( P \). Then \( \nabla \cdot \mathbf{r} = \nabla \cdot x = D \) where \( D = 2 \) or \( 3 \) in two- and three-dimensional space, respectively. We can then use the product rule to the divergence of the product \( \phi \mathbf{r} \):

\[
\nabla \cdot (\phi \mathbf{r}) = \nabla \phi \cdot \mathbf{r} + \phi \nabla \cdot \mathbf{r} = \nabla \phi \cdot \mathbf{r} + D \phi
\]

Integrating both sides over cell \( P \) and applying the divergence theorem on the left-hand side we get

\[
\sum_{f=1}^{F} \int_{s_f} \phi \mathbf{r} \cdot \mathbf{n}_f \, ds = \int_{\Omega_P} \nabla \phi \cdot \mathbf{r} \, d\Omega + D \int_{\Omega_P} \phi \, d\Omega \tag{B.1}
\]

This is an exact equation, but now we will approximate each of the integrals by the midpoint rule:

\[
\int_{s_f} \phi \mathbf{r} \cdot \mathbf{n}_f \, ds = \phi(\mathbf{r}_f) (\mathbf{r}(\mathbf{r}_f) \cdot \mathbf{n}_f) s_f + O(h^2) s_f \tag{B.2}
\]

\[
\int_{\Omega_P} \nabla \phi \cdot \mathbf{r} \, d\Omega = \nabla \phi(P) \cdot \mathbf{r}(P) \Omega_P + O(h^2) \Omega_P = 0 + O(h^2) \Omega_P \tag{B.3}
\]

\[
\int_{\Omega_P} \phi \, d\Omega = \phi(P) \Omega_P + O(h^2) \Omega_P \tag{B.4}
\]
where in (B.3) we have used that \( r(P) = P - P = 0 \). Substituting these into (B.1) we get

\[
\phi(P) = \frac{1}{D\Omega_P} \sum_{f=1}^{F} \left[ \phi(\zeta_f) s_f (\zeta_f) \cdot \nu_f \right] + \sum_{f=1}^{F} \frac{O(h^2) s_f}{\Omega_P} = O(h)
\]

(B.5)

The first term on the right hand side is the approximation \( \phi_P \), as can be seen by comparing to Eq. (67). The second term on the right hand side is the difference between this approximation and the exact value \( \phi(P) \), i.e. the error, which is \( O(h) \) because \( s_f / \Omega_P = O(h^{-1}) \). Therefore, the approximation (67) appears to be only first-order accurate, of the same order as simply setting \( \phi_P = \phi(\zeta_f) \) for any arbitrary face \( f \).

Is it possible that this error estimation is too pessimistic and does not account for some error cancellation that occurs when adding the truncation errors of the midpoint rule approximations (B.2) – (B.4)? This is indeed the case\(^6\); the approximation (67) turns out to be second-order accurate, as can be proved by showing that it is exact for linear functions. Indeed, if it is exact for linear functions then expanding \( \phi \) in a Taylor series about \( P \) we find that the error is

\[
\phi(P) = \phi(\zeta_f) + \nabla \phi(\zeta_f) \cdot r(\zeta_f) + O(h^2) \Rightarrow \phi_P = \phi(\zeta_f) + \nabla \phi(\zeta_f) \cdot r(\zeta_f) + O(h^2)
\]

where we have used the facts that our interpolation operator (67) is linear\(^7\) (\( [\phi + \psi]_P = \phi_P + \psi_P \), that it is exact for linear functions (and \( \phi(\zeta_f) + \nabla \phi(\zeta_f) \cdot r(\zeta_f) \) is a linear function), that \( r(P) = 0 \), and finally that \( O(h^2)P = O(h^2) \) (which can be seen by replacing \( \tau_{ij} \) with \( O(h^2) \) in the formula (67) and noting that both \( s_f (\zeta_f - P) \cdot \nu_f \) and \( \Omega_P \) are of the same order, \( O(h^2) \) in 2D and \( O(h^3) \) in 3D).

It remains to show that the interpolation operator is exact for linear functions. Suppose a linear function \( \phi(x) = \phi(P) + g \cdot r(P) \) where \( g \) is a constant vector (it is the gradient of \( \phi \)). We want to show that \( \phi_P = \phi(\zeta_f) \).

Applying the approximation (67) to \( \phi \) we get

\[
\phi_P = \frac{1}{D\Omega_P} \sum_{f=1}^{F} \left[ (\phi(\zeta_f) + g \cdot r(\zeta_f)) (\zeta_f) \cdot \nu_f \right] = \phi(\zeta_f) + \frac{1}{D\Omega_P} g \cdot \sum_{f=1}^{F} \left[ r(\zeta_f) (\zeta_f) \cdot \nu_f \right] s_f
\]

(B.6)

To proceed further we need a couple of geometrical results. Figure 28 shows cell \( P \) divided into \( F \) triangles by the dashed lines which connect the centroid \( P \) to the vertices \( \zeta_f \). Consider the triangle \( (P, \zeta_f, \zeta_{f+1}) \), a side of which is face \( f \). The product \( r(\zeta_f) \cdot \nu_f \) is the perpendicular distance from \( P \) to face \( f \), and therefore the product \( (r(\zeta_f) \cdot \nu_f) s_f \) is equal to the shaded rectangular area of Fig. 28. The area of the triangle \( (P, \zeta_f, \zeta_{f+1}) \) is half that of this rectangle (in three dimensions, the volume of the cone-like shape obtained by joining \( P \) with the edges of face \( f \) is one third of the volume of the shaded rectangular box). By adding the areas of all triangles we get the total area of cell \( P \)^8:

\[
\frac{1}{D} \sum_{f=1}^{F} (r(\zeta_f) \cdot \nu_f) s_f = \Omega_P
\]

(B.7)

---

\(^6\)A calculation involving Taylor expansions along the faces shows that, in the sum of the truncation errors of Eq. (B.2) over all cell faces, their leading order terms cancel out.

\(^7\)The CUBISTA scheme is nonlinear, but it reduces to the linear QUICK scheme if the grid is fine enough (Sec. 3.4).

\(^8\)Result (B.7) is also obtainable by setting \( \phi = 1 \) in Eq. (B.1).
A second geometric result that will be needed is the following. The centroid of the triangle \((P_f, v_f, P_{f+1})\) is \((P + v_f + v_{f+1})/3 = (P + 2c_f)/3\). The centroid of the whole cell, \(P\), is equal to the sum of the individual centroids of the triangles, \(P_f\) say, weighted by their areas, \(\Omega_f = (1/3)(r(c_f) \cdot n_f)s_f\):

\[
P = \frac{1}{\Omega_P} \sum_{f=1}^{F} P_f \Omega_f \Rightarrow P\Omega_P = \sum_{f=1}^{F} \frac{1}{3} (P + 2c_f) \frac{1}{2} (r(c_f) \cdot n_f) s_f
\]

Substituting for \(\Omega_P\) from (B.7) (with \(D = 2\)), moving everything to the left hand side, and merging the two sums, we arrive at

\[
\sum_{f=1}^{F} r(c_f) (r(c_f) \cdot n_f) s_f = 0
\]  

(B.8)

where we have used also that \(c_f - P = r(c_f)\). In three dimensions, the exact same equation (B.8) holds, but is derived by noting that the volume of the cone- or pyramid-like shape whose base is face \(f\) and its apex is at \(P\) is \(\Omega_f = (1/3)(r(c_f) \cdot n_f)s_f\) (\(D = 3\) in Eq. (B.7)) while its centroid is \(P_f = (P + 3c_f)/4\).

Finally, substituting Eqs. (B.7) and (B.8) in the first and second terms, respectively, of the right-hand side of Eq. (B.6), we arrive at \(\bar{\varphi}_P = \phi(P)\): the interpolation scheme (67) is exact for linear functions, and is therefore second-order accurate.

**Appendix C  Energy dissipation and flow cessation**

This appendix sketches an explanation for the finite cessation times of HB fluids, but for rigorous proofs the reader is referred to the literature cited in Sec. 6.6. For inelastic fluids, once the lid motion ceases, the rate of energy dissipation equals the rate of decrease of the kinetic energy of the fluid [96].

\[
\frac{d}{dt} \int_\Omega \frac{1}{2} \rho u \cdot u \, d\Omega = - \int_\Omega \varpi : \nabla u \, d\Omega
\]  

(C.1)

For generalised Newtonian fluids \((\varpi = \eta(\gamma) \hat{\gamma})\) we have

\[
\varpi : \nabla u = \eta(\gamma) \frac{\hat{\gamma}}{\gamma} : \nabla u = \eta(\gamma) \hat{\gamma}^2
\]  

(C.2)

because

\[
\hat{\gamma} \equiv \sqrt{\frac{1}{2} \gamma} \Rightarrow \gamma^2 = \frac{1}{2} \gamma = \frac{1}{2} \hat{\gamma} = \frac{1}{2} \hat{\gamma} : (\nabla u + (\nabla u)^T) = \frac{1}{2} \hat{\gamma} : \nabla u + \frac{1}{2} \hat{\gamma} : (\nabla u)^T = \hat{\gamma} : \nabla u
\]

since \(\hat{\gamma} : (\nabla u)^T = \hat{\gamma}^T : (\nabla u)^T = \hat{\gamma} : \nabla u\), as \(\hat{\gamma}^T = \hat{\gamma}\). For a HB fluid, \(\eta(\gamma) = (\tau_y + k\hat{\gamma}^n)/\gamma\) and the energy dissipation rate (C.2) becomes \((\tau_y + k\hat{\gamma}^n) \hat{\gamma}\); in fact, this expression holds even in unyielded regions as it predicts zero energy dissipation there\(^9\) due to \(\hat{\gamma} = 0\). So, for HB flow the energy balance (C.1) becomes

\[
\frac{d}{dt} \int_\Omega \frac{1}{2} \rho u \cdot u \, d\Omega = - \int_\Omega (\tau_y + k\hat{\gamma}^n) \hat{\gamma} \, d\Omega
\]  

(C.3)

---

\(^9\)In HB unyielded regions the energy dissipation is zero due to \(\hat{\gamma} = 0\) since, by symmetry of the stress tensor, we have \(\varpi : \nabla u = \varpi : (\frac{1}{2} \nabla u + \frac{3}{2} \nabla u) = \varpi : (\frac{1}{2} \nabla u + \frac{1}{2} \nabla u^T) = \varpi : \gamma = 0\).
Next we will make an approximation. We will assume that the velocity at any point \( x \) at any instant \( t \) can be expressed as the product of a function of time, \( \chi \), and a function of space, \( u_0 \), as

\[
\mathbf{u}(x,t) = \chi(t-t_0) u_0(x)
\]  
(C.4)

where \( u_0(x) \) is the velocity at time \( t_0 \), if we set \( \chi(0) = 1 \). We therefore assume that the velocity field retains its shape in time, but just downscales by a factor of \( \chi(t-t_0) \) at time \( t \) relative to time \( t_0 \). This assumption is correct for the cessation of Newtonian flow if \( t_0 \) is large enough [44], but obviously involves an error in the viscoplastic case, where the unyielded regions expand in time; nevertheless, we will assume this error to be acceptable, not invalidating the final conclusion. Then, let \( V(t) \) be a measure of the velocity in the domain, e.g. the mean or maximum velocity magnitude. Whatever the precise choice of \( V \), it follows from (C.4) that \( V(t) = \chi(t-t_0) V(0) \). We can then normalise the velocity as

\[
\frac{\mathbf{u}(x,t)}{V(t)} = \frac{\chi(t-t_0) u_0(x)}{V(0)} = \frac{u_0(x)}{V(0)} \equiv \tilde{u}_0(x) \Rightarrow \mathbf{u}(x,t) = \tilde{u}_0(x) V(t)
\]  
(C.5)

Substituting for \( \mathbf{u} \) from (C.5) into the left and right side of Eq. (C.3) we get, respectively,

\[
\int_{\Omega} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \, d\Omega = V^2 \int_{\Omega} \frac{1}{2} \rho \tilde{u}_0 \cdot \tilde{u}_0 \, d\Omega = C_k V^2
\]  
(C.6)

\[
\int_{\Omega} (\tau_y + k \dot{\gamma}^n) \dot{\gamma} \, d\Omega = \tau_y V \int_{\Omega} \tilde{\gamma}_0 \, d\Omega + k V^{n+1} \int_{\Omega} \tilde{\gamma}_0^{n+1} \, d\Omega = C_p \tau_y V + C_v k V^{n+1}
\]  
(C.7)

where \( \tilde{\gamma}_0 \) is the magnitude of \( \tilde{\gamma}_0 \equiv \nabla \tilde{u}_0 + (\nabla \tilde{u}_0)^T \) (it is dimensional). Since \( \tilde{u}_0 \) is not a function of \( t \), neither are the constants \( C_k, C_p \) and \( \tilde{C}_v \), and substituting (C.6) and (C.7) into (C.3) we are left with the following ordinary differential equation:

\[
\frac{d(V^2)}{dt} = -C_p' \tau_y V - C_v' k V^{n+1} \leq -C_p' \tau_y V
\]  
(C.8)

where \( C_p' = C_p/C_k \) and \( C_v' = C_v/C_k \) are positive constants. The above inequality means that the flow will decay at least as fast as if the plastic term \( -C_p' \tau_y V \) was the only one present. In that case, solving the equation would give

\[
V(t) = V(t_0) - \frac{C_p' \tau_y}{2} (t - t_0)
\]  
(C.9)

so that the flow reaches complete cessation in finite time: \( V(t_c) = 0 \Rightarrow t_c = t_0 + 2V(t_0)/(C_p' \tau_y) \). Thus, the plastic dissipation (due to \( \tau_y \)) alone suffices to bring the whole material to rest in finite time \( t_c \); including also the viscous dissipation term, \( -C_v' k V^{n+1} \), will only make this process faster.

It is interesting to briefly look at the case \( \tau_y = 0 \), when the plastic term is absent from (C.8). If \( n = 1 \), the flow is Newtonian and the solution of the equation is

\[
V(t) = V(t_0) e^{-C_p' k (t-t_0)}/2
\]  
(C.10)

This result is identical with what was found in [44], and means that the flow continuously decays but never completely ceases. If \( n \neq 1 \) (power-law fluid) the solution is

\[
V(t) = \left[ V(t_0)^{1-n} - (1-n) \frac{C_v' k}{2} (t-t_0) \right]^{\frac{1}{1-n}}
\]  
(C.11)

For \( n > 1 \), the exponent \( 1/(1-n) \) is negative, the flow decays more slowly than in the Newtonian case as the viscosity drops with flow decay, and complete cessation is never reached. However, in the \( n < 1 \) case the viscosity rises to infinity as the flow decays towards cessation; the exponent \( 1/(1-n) \) is positive, and cessation is reached in finite time, \( V(t_c) = 0 \Rightarrow t_c = t_0 + 2V(t_0)^{1-n}/((1-n)C_v' k) \). Thus, the existence of a yield stress is not a prerequisite for finite cessation time.
References

[1] P. Coussot, “Bingham’s heritage,” *Rheologica Acta*, vol. 56, pp. 163–176, 2017.

[2] P. Coussot, “Slow flows of yield stress fluids: yielding liquids or flowing solids?,” *Rheologica Acta*, vol. 57, pp. 1–14, 2018.

[3] D. Bonn, M. M. Denn, L. Berthier, T. Divoux, and S. Manneville, “Yield stress materials in soft condensed matter,” *Reviews of Modern Physics*, vol. 89, 2017.

[4] E. C. Bingham, *Fluidity and Plasticity*. McGraw-Hill, 1922.

[5] W. H. Herschel and R. Bulkley, “Konsistenzmessungen von gummi-benzollsungen,” *Kolloid-Zeitschrift*, vol. 39, pp. 291–300, 1926.

[6] K. v. Hohenemser and W. Prager, “Über die ansätze der mechanik isotroper kontinua,” *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 12, pp. 216–226, 1932.

[7] J. G. Oldroyd, “A rational formulation of the equations of plastic flow for a Bingham solid,” in *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 43, pp. 100–105, Cambridge University Press, 1947.

[8] P. Saramito, “A damped newton algorithm for computing viscoplastic fluid flows,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 238, pp. 6–15, 2016.

[9] R. B. Bird, G. Dai, and B. J. Yarusso, “The rheology and flow of viscoplastic materials,” *Reviews in Chemical Engineering*, vol. 1, pp. 1–70, 1982.

[10] N. J. Balmforth, I. A. Frigaard, and G. Ovarlez, “Yielding to stress: Recent developments in viscoplastic fluid mechanics,” *Annual Review of Fluid Mechanics*, vol. 46, pp. 121–146, 2014.

[11] E. Mitsoulis and J. Tsamopoulos, “Numerical simulations of complex yield-stress fluid flows,” *Rheologica Acta*, vol. 56, pp. 231–258, 2017.

[12] P. Saramito and A. Wachs, “Progress in numerical simulation of yield stress fluid flows,” *Rheologica Acta*, vol. 56, pp. 211–230, 2017.

[13] Y. Dimakopoulos, G. Makrigiorgos, G. Georgiou, and J. Tsamopoulos, “The PAL (Penalized Augmented Lagrangian) method for computing viscoplastic flows: A new fast converging scheme,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 256, pp. 23–41, 2018.

[14] A. Syrakos, G. Georgiou, and A. Alexandrou, “Solution of the square lid-driven cavity flow of a Bingham plastic using the finite volume method,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 195, pp. 19–31, 2013.

[15] R. Glowinski and A. Wachs, “On the numerical simulation of viscoplastic fluid flow,” in *Numerical Methods for Non-Newtonian Fluids*, pp. 483–717, Elsevier, 2011.

[16] J. Piau, “Carbopol gels: Elastoviscoplastic and slippery glasses made of individual swollen sponges: Meso- and macroscopic properties, constitutive equations and scaling laws,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 144, pp. 1–29, 2007.

[17] M. Dinkgreve, M. M. Denn, and D. Bonn, “‘everything flows?’: elastic effects on startup flows of yield-stress fluids,” *Rheologica Acta*, vol. 56, pp. 189–194, 2017.

[18] N. Dubash and I. Frigaard, “Propagation and stopping of air bubbles in Carbopol solutions,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 142, pp. 123–134, 2007.

[19] W. F. Lopez, M. F. Naccache, and P. R. de Souza Mendes, “Rising bubbles in yield stress materials,” *Journal of Rheology*, vol. 62, pp. 209–219, 2018.
[20] J. Tsamopoulos, Y. Dimakopoulos, N. Chatzidai, G. Karapetsas, and M. Pavlidis, “Steady bubble rise and deformation in Newtonian and viscoplastic fluids and conditions for bubble entrapment,” *Journal of Fluid Mechanics*, vol. 601, 2008.

[21] Y. Dimakopoulos, M. Pavlidis, and J. Tsamopoulos, “Steady bubble rise in Herschel–Bulkley fluids and comparison of predictions via the Augmented Lagrangian Method with those via the Papanastasiou model,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 200, pp. 34–51, 2013.

[22] D. Fraggedakis, M. Pavlidis, Y. Dimakopoulos, and J. Tsamopoulos, “On the velocity discontinuity at a critical volume of a bubble rising in a viscoelastic fluid,” *Journal of Fluid Mechanics*, vol. 789, pp. 310–346, 2016.

[23] D. Fraggedakis, Y. Dimakopoulos, and J. Tsamopoulos, “Yielding the yield-stress analysis: a study focused on the effects of elasticity on the settling of a single spherical particle in simple yield-stress fluids,” *Soft Matter*, vol. 12, pp. 5378–5401, 2016.

[24] P. Saramito, “A new constitutive equation for elastoviscoplastic fluid flows,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 145, pp. 1–14, 2007.

[25] P. R. de Souza Mendes and R. L. Thompson, “A critical overview of elasto-viscoplastic thixotropic modeling,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 187-188, pp. 8–15, 2012.

[26] P. Saramito, “A new elastoviscoplastic model based on the Herschel–Bulkley viscoplastic model,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 158, pp. 154–161, 2009.

[27] D. Fraggedakis, Y. Dimakopoulos, and J. Tsamopoulos, “Yielding the yield-stress analysis: A thorough comparison of recently proposed elasto-visco-plastic (EVP) fluid models,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 238, pp. 170–188, 2016.

[28] I. Cheddadi and P. Saramito, “A new operator splitting algorithm for elastoviscoplastic flow problems,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 202, pp. 13–21, 2013.

[29] S. L. Frey, M. F. Naccache, P. R. de Souza Mendes, R. L. Thompson, D. D. dos Santos, F. B. Link, and C. Fonseca, “Performance of an elasto-viscoplastic model in some benchmark problems,” *Mechanics of Time-Dependent Materials*, vol. 19, pp. 419–438, 2015.

[30] A. Syrakos, Y. Dimakopoulos, G. C. Georgiou, and J. Tsamopoulos, “Viscoplastic flow in an extrusion damper,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 232, pp. 102–124, 2016.

[31] P. J. Oliveira, F. T. Pinho, and G. A. Pinto, “Numerical simulation of non-linear elastic flows with a general collocated finite-volume method,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 79, pp. 1–43, 1998.

[32] J. L. Favero, A. R. Secchi, N. S. M. Cardozo, and H. Jasak, “Viscoelastic flow analysis using the software OpenFOAM and differential constitutive equations,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 165, pp. 1625–1636, 2010.

[33] A. Afonso, M. Oliveira, P. Oliveira, M. Alves, and F. Pinho, “The finite-volume method in computational rheology,” in *Finite-Volume Methods - Powerful Means of Engineering Design*, ch. 7, pp. 141–170, In-Tech Open Publishers, 2012.

[34] A. Syrakos, Y. Dimakopoulos, and J. Tsamopoulos, “Theoretical study of the flow in a fluid damper containing high viscosity silicone oil: Effects of shear-thinning and viscoelasticity,” *Physics of Fluids*, vol. 30, p. 030708, 2018.

[35] F. Belblidia, H. R. Tamaddon-Jahromi, M. F. Webster, and K. Walters, “Computations with viscoplastic and viscoelastoplastic fluids,” *Rheologica Acta*, vol. 50, pp. 343–360, 2011.

[36] I. Cheddadi, P. Saramito, and F. Graner, “Steady Couette flows of elastoviscoplastic fluids are nonunique,” *Journal of Rheology*, vol. 56, pp. 213–239, 2012.
[37] L. Lacaze, A. Filella, and O. Thual, “Steady and unsteady shear flows of a viscoplastic fluid in a cylindrical Couette cell,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 220, pp. 126–136, 2015.

[38] A. Syrakos, G. C. Georgiou, and A. N. Alexandrou, “Performance of the finite volume method in solving regularised Bingham flows: Inertia effects in the lid-driven cavity flow,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 208–209, pp. 88–107, 2014.

[39] R. Sousa, R. Poole, A. Afonso, F. Pinho, P. Oliveira, A. Morozov, and M. Alves, “Lid-driven cavity flow of viscoelastic liquids,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 234, pp. 129–138, 2016.

[40] R. da R. Martins, G. M. Furtado, D. D. dos Santos, S. Frey, M. F. Naccache, and P. R. de Souza Mendes, “Elastic and viscous effects on flow pattern of elasto-viscoplastic fluids in a cavity,” *Mechanics Research Communications*, vol. 53, pp. 36–42, 2013.

[41] A. Syrakos, G. C. Georgiou, and A. N. Alexandrou, “Thixotropic flow past a cylinder,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 220, pp. 44–56, 2015.

[42] C. J. Dimitriou and G. H. McKinley, “A comprehensive constitutive law for waxy crude oil: a thixotropic yield stress fluid,” *Soft Matter*, vol. 10, pp. 6619–6644, 2014.

[43] I. Cheddadi, P. Saramito, C. Raufaste, P. Marmottant, and F. Graner, “Numerical modelling of foam Couette flows,” *The European Physical Journal E*, vol. 27, 2008.

[44] A. Syrakos, G. C. Georgiou, and A. N. Alexandrou, “Cessation of the lid-driven cavity flow of Newtonian and Bingham fluids,” *Rheologica Acta*, vol. 55, pp. 51–66, 2016.

[45] R. L. Thompson and E. J. Soares, “Viscoplastic dimensionless numbers,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 238, pp. 57–64, 2016.

[46] A. Syrakos and A. Goulas, “Estimate of the truncation error of finite volume discretization of the Navier-Stokes equations on colocated grids,” *International Journal for Numerical Methods in Fluids*, vol. 50, pp. 103–130, 2006.

[47] J. Sun, M. Smith, R. Armstrong, and R. Brown, “Finite element method for viscoelastic flows based on the discrete adaptive viscoelastic stress splitting and the discontinuous Galerkin method: DAVSS-G/DG,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 86, pp. 281–307, 1999.

[48] A. Syrakos, S. Varchanis, Y. Dimakopoulos, A. Goulas, and J. Tsamopoulos, “A critical analysis of some popular methods for the discretisation of the gradient operator in finite volume methods,” *Physics of Fluids*, vol. 29, p. 127103, 2017.

[49] C. M. Rhie and W. L. Chow, “Numerical study of the turbulent flow past an airfoil with trailing edge separation,” *AIAA Journal*, vol. 21, pp. 1525–1532, 1983.

[50] S. Zhang, X. Zhao, and S. Bayyuk, “Generalized formulations for the Rhie–Chow interpolation,” *Journal of Computational Physics*, vol. 258, pp. 880–914, 2014.

[51] G. B. Deng, J. Piquet, P. Queutey, and M. Visoneau, “Incompressible flow calculations with a consistent physical interpolation finite volume approach,” *Computers & fluids*, vol. 23, pp. 1029–1047, 1994.

[52] C. Klaij, “On the stabilization of finite volume methods with co-located variables for incompressible flow,” *Journal of Computational Physics*, vol. 297, pp. 84–89, 2015.

[53] M. Hortmann, M. Perić, and G. Scheneberger, “Finite volume multigrid prediction of laminar natural convection: Bench-mark solutions,” *International Journal for Numerical Methods in Fluids*, vol. 11, pp. 189–207, 1990.
[54] H. M. Matos, M. A. Alves, and P. J. Oliveira, “New formulation for stress calculation: Application to viscoelastic flow in a T-junction,” *Numerical Heat Transfer, Part B: Fundamentals*, vol. 56, pp. 351–371, 2009.

[55] M. Niethammer, H. Marschall, C. Kunkelmann, and D. Bothe, “A numerical stabilization framework for viscoelastic fluid flow using the finite volume method on general unstructured meshes,” *International Journal for Numerical Methods in Fluids*, vol. 86, pp. 131–166.

[56] F. Pimenta and M. Alves, “Stabilization of an open-source finite-volume solver for viscoelastic fluid flows,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 239, pp. 85–104, 2017.

[57] C. Fernandes, M. S. B. Araujo, L. L. Ferras, and J. M. Nobrega, “Improved both sides diffusion (iBSD): A new and straightforward stabilization approach for viscoelastic fluid flows,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 249, pp. 63–78, 2017.

[58] A. Syrakos, G. Efthimiou, J. G. Bartzis, and A. Goulas, “Numerical experiments on the efficiency of local grid refinement based on truncation error estimates,” *Journal of Computational Physics*, vol. 231, pp. 6725–6753, 2012.

[59] S. Patankar, *Numerical heat transfer and fluid flow*. CRC press, 1980.

[60] J. H. Ferziger and M. Peric, *Computational methods for fluid dynamics*. Springer, 3rd ed., 2002.

[61] F. Habla, A. Woitalka, S. Neuner, and O. Hinrichsen, “Development of a methodology for numerical simulation of non-isothermal viscoelastic fluid flows with application to axisymmetric 4:1 contraction flows,” *Chemical Engineering Journal*, vol. 207-208, pp. 772–784, 2012.

[62] F. Moukalled, L. Mangani, and M. Darwish, *The Finite Volume Method in Computational Fluid Dynamics*. Springer, 2016.

[63] M. A. Alves, P. J. Oliveira, and F. T. Pinho, “A convergent and universally bounded interpolation scheme for the treatment of advection,” *International Journal for Numerical Methods in Fluids*, vol. 41, pp. 47–75, 2003.

[64] X. Chen, H. Marschall, M. Schfer, and D. Bothe, “A comparison of stabilisation approaches for finite-volume simulation of viscoelastic fluid flow,” *International Journal of Computational Fluid Dynamics*, vol. 27, pp. 229–250, 2013.

[65] F. Habla, M. W. Tan, J. Haßlberger, and O. Hinrichsen, “Numerical simulation of the viscoelastic flow in a three-dimensional lid-driven cavity using the log-conformation reformulation in OpenFOAM®,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 212, pp. 47–62, 2014.

[66] R. Comminal, J. Spangenberg, and J. H. Hattel, “Robust simulations of viscoelastic flows at high weissenberg numbers with the streamfunction/log-conformation formulation,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 223, pp. 37–61, 2015.

[67] P. H. Gaskell and A. K. C. Lau, “Curvature-compensated convective transport: SMART, a new boundedness-preserving transport algorithm,” *International Journal for Numerical Methods in Fluids*, vol. 8, pp. 617–641.

[68] H. Jasak, H. Weller, and A. Gosman, “High resolution NVD differencing scheme for arbitrarily unstructured meshes,” *International Journal for Numerical Methods in Fluids*, vol. 31, pp. 431–449.

[69] B. P. Leonard, “Bounded high-order upwind multidimensional finite-volume convection-diffusion algorithms,” in *Advances in Numerical Heat Transfer* (W. J. Minkowycz and E. M. Sparrow, eds.), vol. 1, pp. 1–57, Taylor and Francis, 1996.

[70] R. Fattal and R. Kupferman, “Constitutive laws for the matrix-logarithm of the conformation tensor,” *Journal of Non-Newtonian Fluid Mechanics*, vol. 123, pp. 281–285, 2004.
[71] R. Fattal and R. Kupferman, “Time-dependent simulation of viscoelastic flows at high weissenberg number using the log-conformation representation,” Journal of Non-Newtonian Fluid Mechanics, vol. 126, pp. 23–37, 2005.

[72] A. Afonso, P. J. Oliveira, F. T. Pinho, and M. A. Alves, “The log-conformation tensor approach in the finite-volume method framework,” Journal of Non-Newtonian Fluid Mechanics, vol. 157, no. 1-2, pp. 55–65, 2009.

[73] F. D. Vita, M. Rosti, D. Izbassarov, L. Duffo, O. Tammisola, S. Hormozi, and L. Brandt, “Elasto-viscoplastic flows in porous media,” Journal of Non-Newtonian Fluid Mechanics, vol. 258, pp. 10–21, 2018.

[74] N. Balci, B. Thomases, M. Renardy, and C. R. Doering, “Symmetric factorization of the conformation tensor in viscoelastic fluid models,” Journal of Non-Newtonian Fluid Mechanics, vol. 166, pp. 546–553, 2011.

[75] A. Afonso, F. Pinho, and M. Alves, “The kernel-conformation constitutive laws,” Journal of Non-Newtonian Fluid Mechanics, vol. 167-168, pp. 30–37, oct 2012.

[76] W. Zhou, J. Ouyang, X. Wang, J. Su, and B. Yang, “Numerical simulation of viscoelastic fluid flows using a robust FVM framework on triangular grid,” Journal of Non-Newtonian Fluid Mechanics, vol. 236, pp. 18–34, 2016.

[77] P. M. Gresho, R. L. Lee, R. L. Sani, and T. Stullich, “Time-dependent fem solution of the incompressible navier–stokes equations in two-and three-dimensions,” tech. rep., Lawrence Livermore Laboratory, University of California, 1978.

[78] A. Syrakos and A. Goulas, “Finite volume adaptive solutions using SIMPLE as smoother,” International Journal for Numerical Methods in Fluids, vol. 52, pp. 1215–1245, 2006.

[79] T.-W. Pan, J. Hao, and R. Glowinski, “On the simulation of a time-dependent cavity flow of an Oldroyd-B fluid,” International Journal for Numerical Methods in Fluids, vol. 60, pp. 791–808, 2009.

[80] P. Saramito, “On a modified non-singular log-conformation formulation for Johnson–Segalman viscoelastic fluids,” Journal of Non-Newtonian Fluid Mechanics, vol. 211, pp. 16–39, 2014.

[81] M. A. Hulsen, R. Fattal, and R. Kupferman, “Flow of viscoelastic fluids past a cylinder at high weissenberg number: Stabilized simulations using matrix logarithms,” Journal of Non-Newtonian Fluid Mechanics, vol. 127, pp. 27–39, 2005.

[82] A. Poumaere, M. Moyers-Gonzalez, C. Castelain, and T. Burghela, “Unsteady laminar flows of a Carbopol gel in the presence of wall slip,” Journal of Non-Newtonian Fluid Mechanics, vol. 205, pp. 28–40, 2014.

[83] T. C. Papanastasiou, “Flows of materials with yield,” Journal of Rheology, vol. 31, pp. 385–404, 1987.

[84] K. Sverdrup, N. Nikiforakis, and A. Almgren, “Highly parallelisable simulations of time-dependent viscoplastic fluid flow with structured adaptive mesh refinement,” Physics of Fluids, vol. 30, p. 093102, 2018.

[85] M. R. Hashemi, M. T. Manzari, and R. Fatehi, “Non-linear stress response of non-gap-spanning magnetic chains suspended in a newtonian fluid under oscillatory shear test: A direct numerical simulation,” Physics of Fluids, vol. 29, p. 107106, 2017.

[86] J. Pérez-González, J. J. López-Durán, B. M. Marín-Santibáñez, and F. Rodríguez-González, “Rheo-PIV of a yield-stress fluid in a capillary with slip at the wall,” Rheologica Acta, vol. 51, pp. 937–946, 2012.

[87] D. M. Kalyon, “Apparent slip and viscoplasticity of concentrated suspensions,” Journal of Rheology, vol. 49, pp. 621–640, 2005.
[88] P. Panaseti, A.-L. Vayssade, G. C. Georgiou, and M. Cloitre, “Confined viscoplastic flows with heterogeneous wall slip,” *Rheologica Acta*, vol. 56, pp. 539–553, 2017.

[89] P. Panaseti, K. D. Housiadas, and G. C. Georgiou, “Newtonian Poiseuille flows with pressure-dependent wall slip,” *Journal of Rheology*, vol. 57, pp. 315–332, 2013.

[90] S. G. Hatzikiriakos, “Wall slip of molten polymers,” *Progress in Polymer Science*, vol. 37, pp. 624–643, 2012.

[91] M. Philippou, Y. Damianou, X. Miscouridou, and G. C. Georgiou, “Cessation of newtonian circular and plane couette flows with wall slip and non-zero slip yield stress,” *Meccanica*, vol. 52, pp. 2081–2099, 2017.

[92] I. A. Frigaard, *Background Lectures on Ideal Visco-Plastic Fluid Flows*, pp. 1–40. Springer International Publishing, 2019.

[93] R. Glowinski, *Numerical methods for nonlinear variational problems*. Springer-Verlag, 1984.

[94] R. Huilgol, B. Mena, and J. Piau, “Finite stopping time problems and rheometry of Bingham fluids,” *Journal of non-Newtonian Fluid Mechanics*, vol. 102, pp. 97–107, 2002.

[95] L. Muravleva, E. Muravleva, G. C. Georgiou, and E. Mitsoulis, “Unsteady circular Couette flow of a Bingham plastic with the Augmented Lagrangian Method,” *Rheologica Acta*, vol. 49, pp. 1197–1206, 2010.

[96] H. H. Winter, “Viscous dissipation term in energy equations,” in *Modular Instruction Series C: Calculation and Measurement Techniques for Momentum, Energy and Mass Transfer, Vol. 7*, pp. 27–34, American Institute of Chemical Engineers: New York, 1987.