INHOMOGENEOUS MINIMIZATION PROBLEMS FOR THE $p(x)$-LAPLACIAN

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Abstract. This paper is devoted to the study of inhomogeneous minimization problems associated to the $p(x)$-Laplacian. We make a thorough analysis of the essential properties of their minimizers and we establish a relationship with a suitable free boundary problem.

On the one hand, we study the problem of minimizing the functional $J(v) = \int_{\Omega} (\frac{\nabla u|^{p(x)}}{p(x)} + \lambda(x)\chi_{\{v > 0\}} + f v) \, dx$. We show that nonnegative local minimizers $u$ are solutions to the free boundary problem: $u \geq 0$ and

\[ (P(f, p, \lambda^*)) \begin{cases} \Delta_{p(x)} u := \text{div}(\nabla u|^{p(x)-2} \nabla u) = f & \text{in } \{u > 0\} \\ u = 0, |\nabla u| = \lambda^*(x) & \text{on } \partial \{u > 0\} \end{cases} \]

with $\lambda^*(x) = \left(\frac{p'(x)}{p(x)} \lambda(x)\right)^{1/p(x)}$ and that the free boundary is a $C^{1,\alpha}$ surface with the exception of a subset of $H^{N-1}$-measure zero.

On the other hand, we study the problem of minimizing the functional $J_*(u) = \int_{\Omega} (\frac{|\nabla u|^{p(x)}}{p(x)} + B_*(v) + f^* v) \, dx$, where $B_*(s) = \int_0^s \beta_*(\tau) \, d\tau$, $\beta_*(s) = \frac{1}{2} \beta(\frac{1}{s})$, with $\beta$ a Lipschitz function satisfying $\beta > 0$ in $(0, 1)$, $\beta \equiv 0$ outside $(0, 1)$. We prove that if $u^*$ are nonnegative local minimizers, then $u^*$ are solutions to

\[ (P_*(f^*, p_*)) \Delta_{p_*(x)} u^* = \beta_*(u^*) + f^*, \quad u^* \geq 0. \]

Moreover, if the functions $u^*, f^*$ and $p_*$ are uniformly bounded, we show that limit functions $u (\varepsilon \to 0)$ are solutions to the free boundary problem $P(f, p, \lambda^*)$ with $\lambda^*(x) = \left(\frac{p'(x)}{p(x)} M\right)^{1/p(x)}$, $M = \int \beta(s) \, ds$, $p = \lim p_*$, $f = \lim f^*$, and that the free boundary is a $C^{1,\alpha}$ surface with the exception of a subset of $H^{N-1}$-measure zero.

In order to obtain our results we need to overcome deep technical difficulties and develop new strategies, not present in the previous literature for this type of problems.

1. Introduction

This paper is devoted to the study of inhomogeneous minimization problems associated to the $p(x)$-Laplacian. We make a thorough analysis of the essential properties of their minimizers and we establish a relationship with a suitable free boundary problem.

The first minimization problem under consideration corresponds to the functional

\[ J(v) = \int_{\Omega} \left( \frac{\nabla u|^{p(x)}}{p(x)} + \lambda(x)\chi_{\{v > 0\}} + f v \right) \, dx. \]
In the particular case in which \( p(x) \equiv 2 \) and \( f(x) \equiv 0 \), the functional becomes
\[
\int_{\Omega} \left( \frac{|\nabla u|^2}{2} + \lambda(x) \chi_{\{u > 0\}} \right) \, dx.
\]
The corresponding minimization problem in \( H^1(\Omega) \) with prescribed nonnegative values on \( \partial \Omega \) was first treated by Alt and Caffarelli in the seminal paper [2] motivated by the study of flow problems of jets and cavities. In [2] it was shown that local minimizers are solutions of the following free boundary problem: \( u \geq 0 \) and
\[
\begin{cases}
\Delta u = 0 & \text{in } \{u > 0\} \\
u = 0, \ |\nabla u| = \lambda^*(x) & \text{on } \partial\{u > 0\},
\end{cases}
\]
with \( \lambda^*(x) = (2\lambda(x))^{1/2} \) and that the free boundary \( \partial\{u > 0\} \) is a \( C^{1,\alpha} \) surface with the exception of a subset of \( \mathcal{H}^{N-1} \)-measure zero.

In the present work we prove that nonnegative local minimizers of functional (1.1) are solutions to the inhomogeneous free boundary problem for the \( p(x) \)-Laplacian: \( u \geq 0 \) and
\[
\begin{cases}
\Delta_{p(x)} u := \text{div}(\nabla u(x)|p(x)-2\nabla u) = f & \text{in } \{u > 0\} \\
u = 0, |\nabla u| = \lambda^*(x) & \text{on } \partial\{u > 0\},
\end{cases}
\]
with \( \lambda^*(x) = \left( \frac{p(x)}{p(x)-1} \lambda(x) \right)^{1/p(x)} \).

The \( p(x) \)-Laplacian serves as a model for a stationary non-newtonian fluid with properties depending on the point in the region where it moves. For example, such a situation corresponds to an electrorheological fluid. These are fluids such that their properties depend on the magnitude of the electric field applied to it. In some cases, fluid and Maxwell’s equations become uncoupled and a single equation for the \( p(x) \)-Laplacian appears (see [33]).

The second minimization problem we deal with corresponds to the functional (1.2)
\[
J_\varepsilon(v) = \int_{\Omega} \left( \frac{|\nabla v|^{p(x)}}{p_\varepsilon(x)} + B_\varepsilon(v) + f^\varepsilon v \right) \, dx,
\]
where \( B_\varepsilon(s) = \int_0^s \beta_\varepsilon(\tau) \, d\tau, \varepsilon > 0, \beta_\varepsilon(s) = \frac{1}{\varepsilon} \beta(\frac{s}{\varepsilon}) \), with \( \beta \) a Lipschitz function satisfying \( \beta > 0 \) in \((0,1), \beta \equiv 0 \) outside \((0,1)\).

The minimization problem for functional (1.2) is a regularization of the one corresponding to functional (1.1). The primary purpose in studying a regularized problem is to obtain uniform properties and establish results which carry over in the limit. In fact, we prove that if \( u^\varepsilon \) are nonnegative local minimizers to (1.2), then \( u^\varepsilon \) are solutions to
\[
(P_\varepsilon(f^\varepsilon,p_\varepsilon)) \quad \Delta_{p_\varepsilon(x)} u^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon, \quad u^\varepsilon \geq 0
\]
and moreover, if the functions \( u^\varepsilon, f^\varepsilon \) and \( p_\varepsilon \) are uniformly bounded, we show that limit functions \( u (\varepsilon \to 0) \) are solutions to the free boundary problem \( P(f,p,\lambda^*) \) with \( \lambda^*(x) = \left( \frac{p(x)}{p(x)-1} M \right)^{1/p(x)} \), \( M = \int \beta(s) \, ds, \lim p_\varepsilon = p, f = \lim f^\varepsilon \).

Problem \( P_\varepsilon(f^\varepsilon,p_\varepsilon) \), when \( p_\varepsilon(x) \equiv 2 \) and \( f^\varepsilon \equiv 0 \), arises in combustion theory to describe the propagation of curved premixed equi-diffusional deflagration flames. The study of the limit \( (\varepsilon \to 0) \) was proposed in the 1930s and was first rigorously studied in [4]. The inhomogeneous case, \( f^\varepsilon \neq 0 \), allows the treatment of more general combustion models with nonlocal diffusion and/or transport. In the case of the \( p_\varepsilon(x) \)-Laplacian, this singular perturbation problem may model flame propagation in a fluid with electromagnetic sensitivity.
Our work here, for both minimization problems, consists in an exhaustive analysis of the properties of nonnegative local minimizers, namely, global regularity and behavior close to the free boundary. This analysis allows us to prove that nonnegative local minimizers $u$ of (1.1), and functions $u^\varepsilon$ (as $\varepsilon \to 0$), with $u^\varepsilon$ nonnegative local minimizers of (1.2), are weak solutions to the free boundary problem $P(f,p,\lambda^*)$ (Theorems 3.3 and 5.2).

In order to obtain our results we need to overcome deep technical difficulties and develop new strategies, not present in the previous literature for this type of problems.

One of the results we would like to highlight is the proof of the Lipschitz continuity of nonnegative local minimizers of functional (1.1) (Theorem 6.3 and Corollary 6.2). Our proof relies on a careful rescaling argument, which transforms the problem into a minimization problem for a more general local minimizers of functional (1.1) (Theorem 3.3) and Corollary 3.2). Our proof relies on a careful strategies, not present in the previous literature for this type of problems.

Let us also emphasize that a key ingredient in many of our proofs is the use of rescaling arguments which, in particular, involve the handling of sequences of functions exhibiting nonuniform integrability. Thus, the use of these kind of arguments for functional (1.1) requires the introduction of the new concept of mild minimizers (see Definition 3.2). Similar subtle ideas are also required when dealing with functional (1.2) (see Theorems 4.2 and 4.4).

Once we achieve our goal, namely, once we prove the fundamental properties of nonnegative local minimizers described above, we are able to apply results for solutions to the singular perturbation problem $P_\varepsilon(f^\varepsilon,p_\varepsilon)$ and for weak solutions to the free boundary problem $P(f,p,\lambda^*)$ we recently obtained in our works [24] and [25], respectively.

As a consequence we derive the smoothness of the free boundary for nonnegative local minimizers $u$ of (1.1). More precisely, we prove that the free boundary $\partial\{u > 0\}$ is a $C^{1,\alpha}$ surface with the exception of a subset of $\mathcal{H}^{N-1}$-measure zero (Theorem 5.2).

In an analogous way, we get the smoothness of the free boundary for limit functions $u (\varepsilon \to 0)$ of nonnegative local minimizers $u^\varepsilon$ of (1.2), i.e., the free boundary $\partial\{u > 0\}$ is a $C^{1,\alpha}$ surface with the exception of a subset of $\mathcal{H}^{N-1}$-measure zero (Theorem 5.3).

We also obtain further regularity results on the free boundary, for both minimization problems, under further regularity assumptions on the data (Corollaries 5.1 and 5.2). In particular, if the data are analytic, the free boundary is an analytic surface with the exception of a subset of $\mathcal{H}^{N-1}$-measure zero.

As stated above, the minimization problem with the functional in (1.1) was first studied by Alt and Caffarelli in [2] with $p(x) \equiv 2$ and $\lambda \equiv 0$. Still in the homogeneous case $\lambda \equiv 0$, the problem was studied by Alt, Caffarelli and Friedman in [3] for a quasilinear equation in the uniformly elliptic case, then the $p$-Laplacian ($p(x) \equiv p$) was treated in [11], an operator with power-like growth was studied in [27], and the case of a variable power $p(x) \equiv p$ was considered in [5]. The linear inhomogeneous case was treated in [16] and [19].

We remark that the inhomogeneous minimization problem for functional (1.1) with $\lambda \not\equiv 0$ we consider here had not been treated in previous literature even in the case of $p(x) \equiv p \not\equiv 2$.

On the other hand, as pointed out above, problem $P_\varepsilon(f^\varepsilon,p_\varepsilon)$ —arising in combustion theory— was first rigorously studied in [4] when $p_\varepsilon(x) \equiv 2$ and $f^\varepsilon \equiv 0$. Since then, much research has been done on this problem, see [1, 7, 9, 12, 20, 21, 28, 32, 34]. For the inhomogeneous case we refer to [22, 23, 29, 30]. Preliminary results for the $p_\varepsilon(x)$-Laplacian were obtained in [24].
We also remark that the inhomogeneous minimization problem for functional (1.2) with \( f^e \neq 0 \) we consider here had not been treated in previous literature even in the case of \( p_e(x) \equiv p_e \neq 2 \). When \( f^e \equiv 0 \) our results are also new when \( p_e(x) \neq p_e \).

An outline of the paper is as follows: In Section 2 we define the notion of weak solution to the free boundary problem \( P(f, p, \lambda^e) \) and include some related definitions and results. In Section 3 we prove existence of minimizers of the energy functional (1.1) and develop an exhaustive analysis of the essential properties of functions \( u \) which are nonnegative local minimizers of that energy. In Section 4 we prove existence of minimizers of the energy functional (1.2) and develop an analogous analysis of the properties of functions \( u^e \) which are nonnegative local minimizers of that energy and moreover, we get results for their limit functions \( u \). Finally, in Section 5 we study the regularity of the free boundary for both minimization problems. We conclude the paper with an Appendix where we collect some results on variable exponent Sobolev spaces as well as some other results that are used in the paper.

### 1.1. Preliminaries on Lebesgue and Sobolev spaces with variable exponent.

Let \( p : \Omega \to [1, \infty) \) be a measurable bounded function, called a variable exponent on \( \Omega \) and denote \( p_{min} = \text{esssup} p(x) \) and \( p_{max} = \text{essinf} p(x) \). We define the variable exponent Lebesgue space \( L^{p(\cdot)}(\Omega) \) to consist of all measurable functions \( u : \Omega \to \mathbb{R} \) for which the modular \( q_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} \, dx \) is finite. We define the Luxemburg norm on this space by

\[
\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{\lambda > 0 : q_{p(\cdot)}(u/\lambda) \le 1\}.
\]

This norm makes \( L^{p(\cdot)}(\Omega) \) a Banach space.

There holds the following relation between \( q_{p(\cdot)}(u) \) and \( \|u\|_{L^{p(\cdot)}} \):

\[
\min \left\{ \left( \int_\Omega |u|^{p(x)} \, dx \right)^{1/p_{min}}, \left( \int_\Omega |u|^{p(x)} \, dx \right)^{1/p_{max}} \right\} \le \|u\|_{L^{p(\cdot)}(\Omega)} \le \max \left\{ \left( \int_\Omega |u|^{p(x)} \, dx \right)^{1/p_{min}}, \left( \int_\Omega |u|^{p(x)} \, dx \right)^{1/p_{max}} \right\}.
\]

Moreover, the dual of \( L^{p(\cdot)}(\Omega) \) is \( L^{p'\, (\cdot)}(\Omega) \) with \( \frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \).

Let \( W^{1,p(\cdot)}(\Omega) \) denote the space of measurable functions \( u \) such that \( u \) and the distributional derivative \( \nabla u \) are in \( L^{p(\cdot)}(\Omega) \). The norm

\[
\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}
\]

makes \( W^{1,p(\cdot)}(\Omega) \) a Banach space.

The space \( W^{1,p(\cdot)}_0(\Omega) \) is defined as the closure of the \( C_0^\infty(\Omega) \) in \( W^{1,p(\cdot)}(\Omega) \).

For the sake of completeness we include in an Appendix at the end of the paper some additional results on these spaces that are used throughout the paper.

### 1.2. Preliminaries on solutions to \( p(x) \)-Laplacian.

Let \( p(x) \) be as above, \( g \in L^{\infty}(\Omega) \) and \( a \in L^{\infty}(\Omega), a(x) \ge a_0 > 0 \) in \( \Omega \). We say that \( u \) is a solution to

\[
(1.3) \quad \text{div}(a(x)|\nabla u(x)|^{p(x)-2}\nabla u) = g(x) \quad \text{in} \quad \Omega
\]

if \( u \in W^{1,p(\cdot)}(\Omega) \) and, for every \( \varphi \in C_0^\infty(\Omega) \), there holds that

\[
\int_\Omega a(x)|\nabla u(x)|^{p(x)-2}\nabla u \cdot \nabla \varphi \, dx = -\int_\Omega \varphi \, g(x) \, dx.
\]
Under the assumptions of the present paper (see 1.3 below) it follows as in Remark 3.2 in [35] that $u \in L^\infty_{\text{loc}}(\Omega)$.

Moreover, for any $x \in \Omega$, $\xi, \eta \in \mathbb{R}^N$ fixed we have the following inequalities

\[
|\eta - \xi|^p(x) - 2 \eta \cdot (\eta - \xi) \leq C(\eta^p(x) - 2 \eta - |\xi|^p(x) - 2 \xi) \cdot (\eta - \xi)
\]

if $p(x) \geq 2$,

\[
|\eta - \xi|^2(\eta + |\xi|)^{p(x) - 2} - 2 \eta \cdot (\eta - \xi) \leq C(|\eta|^p(x) - 2 \eta - |\xi|^p(x) - 2 \xi) \cdot (\eta - \xi)
\]

if $p(x) < 2$,

with $C = C(N, p_{\text{min}}, p_{\text{max}})$. These inequalities imply that the function $A(x, \xi) = a(x)|\xi|^p(x) - 2 \xi$ is strictly monotone. Then, the comparison principle for equation (1.3) holds on bounded domains since it follows from the monotonicity of $A(x, \xi)$.

1.3. Assumptions. Throughout the paper we let $\Omega \subset \mathbb{R}^N$ be a domain.

**Assumptions on $p_\varepsilon(x)$ and $p(x)$.** We assume that the functions $p_\varepsilon(x)$ are measurable and verify

\[
1 < p_{\text{min}} \leq p_\varepsilon(x) \leq p_{\text{max}} < \infty, \quad x \in \Omega.
\]

For our main results we need to assume further that $p_\varepsilon(x)$ are uniformly Lipschitz continuous in $\Omega$. In that case, we denote by $L$ the Lipschitz constant of $p_\varepsilon(x)$, namely, $\|\nabla p_\varepsilon\|_{L^\infty(\Omega)} \leq L$.

Unless otherwise stated, the same assumptions above will be made on the function $p(x)$.

When we are restricted to a ball $B_r$ we use $p_- = p_-(B_r)$ and $p_+ = p_+(B_r)$ to denote the infimum and the supremum of $p(x)$ over $B_r$.

In some results we assume further that $p \in W^{1,\infty}(\Omega) \cap W^{2,q}(\Omega)$, for some $q > 1$.

**Assumptions on $\lambda(x)$.** We assume that the function $\lambda(x)$ is measurable in $\Omega$ and verifies

\[
0 < \lambda_{\text{min}} \leq \lambda(x) \leq \lambda_{\text{max}} < \infty, \quad x \in \Omega.
\]

In some results we assume that $\lambda(x)$ is continuous in $\Omega$ and in our main results we assume further that $\lambda(x)$ is Hölder continuous in $\Omega$.

**Assumptions on $f_\varepsilon(x)$ and $f(x)$.** We assume that $f_\varepsilon, f \in L^\infty(\Omega)$. In some results we assume further that $f \in W^{1,q}(\Omega)$, for some $q > 1$.

**Assumptions on $\beta_\varepsilon$.** We assume that the functions $\beta_\varepsilon$ are defined by scaling of a single function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying:

i) $\beta$ is a Lipschitz continuous function,

ii) $\beta > 0$ in $(0, 1)$ and $\beta \equiv 0$ otherwise,

iii) $\int_0^1 \beta(s) \, ds = M$.

And then $\beta_\varepsilon(s) := \frac{1}{\varepsilon} \beta \left( \frac{s}{\varepsilon} \right)$.
1.4. Notation.

- $N$ spatial dimension
- $\Omega \cap \partial \{ u > 0 \}$ free boundary
- $|S|$ $N$-dimensional Lebesgue measure of the set $S$
- $\mathcal{H}^{N-1}$ $(N-1)$-dimensional Hausdorff measure
- $B_r(x_0)$ open ball of radius $r$ and center $x_0$
- $B_r^+ = B_r \cap \{ x_N > 0 \}$, $B_r^- = B_r \cap \{ x_N < 0 \}$
- $B_r'(x_0)$ open ball of radius $r$ and center $x_0$ in $\mathbb{R}^{N-1}$
- $B_r(s) = \int_0^s \beta_c(\tau) d\tau$

2. Weak solutions to the free boundary problem $P(f,p,\lambda^*)$

In this section, for the sake of completeness, we define the notion of weak solution to the free boundary problem $P(f,p,\lambda^*)$ and we give other related definitions and results that we are going to employ in the paper.

We point out that in [25] we derived some properties of the weak solutions to problem $P(f,p,\lambda^*)$ and we developed a theory for the regularity of the free boundary for weak solutions.

In this section $p(x)$ will be a Lipschitz continuous function.

We first need

**Definition 2.1.** Let $u$ be a continuous and nonnegative function in a domain $\Omega \subset \mathbb{R}^N$. We say that $\nu$ is the exterior unit normal to the free boundary $\Omega \cap \partial \{ u > 0 \}$ at a point $x_0 \in \Omega \cap \partial \{ u > 0 \}$ in the measure theoretic sense, if $\nu \in \mathbb{R}^N$, $|\nu| = 1$ and

$$\lim_{r \to 0} \frac{1}{r^N} \int_{B_r(x_0)} |\chi_{\{ u > 0 \}} - \chi_{\{ x / \langle x-x_0,\nu \rangle < 0 \}}| dx = 0.$$  

Then we have

**Definition 2.2.** Let $\Omega \subset \mathbb{R}^N$ be a domain. Let $p$ be a measurable function in $\Omega$ with $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$, $\lambda^*$ continuous in $\Omega$ with $0 < \lambda_{\min} \leq \lambda^*(x) \leq \lambda_{\max} < \infty$ and $f \in L^\infty(\Omega)$. We call $u$ a weak solution of $P(f,p,\lambda^*)$ in $\Omega$ if

1. $u$ is continuous and nonnegative in $\Omega$, $u \in W^{1,p(\cdot)}_{\text{loc}}(\Omega)$ and $\Delta_{p(x)} u = f$ in $\Omega \cap \{ u > 0 \}$.
2. For $D \subset \subset \Omega$ there are constants $c_{\min} = c_{\min}(D)$, $C_{\max} = C_{\max}(D)$, $r_0 = r_0(D)$, $0 < c_{\min} \leq C_{\max}$, $r_0 > 0$, such that for balls $B_r(x) \subset D$ with $x \in \partial \{ u > 0 \}$ and $0 < r \leq r_0$

   $$c_{\min} \leq \frac{1}{r} \sup_{B_r(x)} u \leq C_{\max}.$$  

3. For $\mathcal{H}^{N-1}$ a.e. $x_0 \in \partial_{\text{red}} \{ u > 0 \}$ (that is, for $\mathcal{H}^{N-1}$-almost every point $x_0 \in \partial \{ u > 0 \}$ such that $\Omega \cap \partial \{ u > 0 \}$ has an exterior unit normal $\nu(x_0)$ in the measure theoretic sense)
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$u$ has the asymptotic development

$$u(x) = \lambda^*(x_0)(x - x_0, \nu(x_0))^- + o(|x - x_0|).$$

(4) For every $x_0 \in \Omega \cap \partial\{u > 0\}$,

$$\limsup_{x \to x_0} |\nabla u(x)| \leq \lambda^*(x_0).$$

If there is a ball $B \subset \{u = 0\}$ touching $\Omega \cap \partial\{u > 0\}$ at $x_0$, then

$$\limsup_{x \to x_0} \frac{u(x)}{\text{dist}(x, B)} \geq \lambda^*(x_0).$$

**Definition 2.3.** Let $v$ be a continuous nonnegative function in a domain $\Omega \subset \mathbb{R}^N$. We say that $v$ is nondegenerate at a point $x_0 \in \Omega \cap \{v = 0\}$ if there exist $c > 0, \bar{r}_0 > 0$ such that one of the following conditions holds:

1. $\int_{B_r(x_0)} v dx \geq cr$ for $0 < r \leq \bar{r}_0$,
2. $\int_{\partial B_r(x_0)} v dx \geq cr$ for $0 < r \leq \bar{r}_0$,
3. $\sup_{B_r(x_0)} v \geq cr$ for $0 < r \leq \bar{r}_0$.

We say that $v$ is uniformly nondegenerate on a set $\Gamma \subset \Omega \cap \{v = 0\}$ in the sense of (2.1) (resp. (2.2), (2.3)) if the constants $c$ and $\bar{r}_0$ in (2.1) (resp. (2.2), (2.3)) can be taken independent of the point $x_0 \in \Gamma$.

**Remark 2.1.** Assume that $v \geq 0$ is locally Lipschitz continuous in a domain $\Omega \subset \mathbb{R}^N$, $v \in W^{1,p(x)}(\Omega)$ with $\Delta_{p(x)}v \geq f\chi_{\{v > 0\}}$, where $f \in L^\infty(\Omega)$, $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ and $p(x)$ is Lipschitz continuous. Then the three concepts of nondegeneracy in Definition 2.3 are equivalent (for the idea of the proof, see Remark 3.1 in [21], where the case $p(x) \equiv 2$ and $f \equiv 0$ is treated).

3. **Energy minimizers of energy functional (1.1)**

In this section we prove existence of minimizers of the energy functional (1.1) and we develop an exhaustive analysis of the essential properties of functions $u$ which are nonnegative local minimizers of that energy.

We start with a definition and some related remarks

**Definition 3.1.** Let $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$, $f \in L^\infty(\Omega)$ and $\lambda(x)$ measurable with $0 < \lambda_{\min} \leq \lambda(x) \leq \lambda_{\max} < \infty$. We say that $u \in W^{1,p(x)}(\Omega)$ is a local minimizer in $\Omega$ of

$$J(v) = J_\Omega(v) = \int_{\Omega} \left(\frac{|\nabla v|^{p(x)}}{p(x)} + \lambda(x)\chi_{\{v > 0\}} + fv\right) dx$$

if for every $\Omega' \subset \subset \Omega$ and for every $v \in W^{1,p(x)}$ such that $v = u$ in $\Omega \setminus \Omega'$ there holds that $J(v) \geq J(u)$. 

Remark 3.1. Let $u$ be as in Definition 3.1. Let $\Omega' \subset \subset \Omega$ and $w - u \in W_0^{1,p(\cdot)}(\Omega')$. If we define

$$\tilde{w} = \begin{cases} w & \text{in } \Omega', \\ u & \text{in } \Omega \setminus \Omega', \end{cases}$$

then $\tilde{w} \in W^{1,p(\cdot)}(\Omega)$ and therefore $J(\tilde{w}) \geq J(u)$. If we now let

$$J_{\Omega'}(v) = \int_{\Omega'} \left( \frac{\|\nabla v\|^{p(x)}}{p(x)} + \lambda(x) \chi_{\{v > 0\}} + fv \right) dx,$$

it follows that $J_{\Omega'}(\tilde{w}) \geq J_{\Omega'}(u)$.

Remark 3.2. Let $J$ be as in Definition 3.1. If $u \in W^{1,p(\cdot)}(\Omega)$ is a minimizer of $J$ among the functions $v \in u + W_0^{1,p(\cdot)}(\Omega)$, then $u$ is a local minimizer of $J$ in $\Omega$.

We first prove

Theorem 3.1. Assume that $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ with $\|\nabla p\|_{L^\infty} \leq L$, $f \in L^\infty(\Omega)$ and $\lambda(x)$ is measurable with $0 < \lambda_{\min} \leq \lambda(x) \leq \lambda_{\max} < \infty$. Let $\phi \in W^{1,p(\cdot)}(\Omega)$ and assume that $\Omega$ is a bounded domain. There exists $u \in W^{1,p(\cdot)}(\Omega)$ that minimizes the energy

$$J(v) = \int_{\Omega} \left( \frac{\|\nabla v\|^{p(x)}}{p(x)} + \lambda(x) \chi_{\{v > 0\}} + fv \right) dx,$$

among functions $v \in W^{1,p(\cdot)}(\Omega)$ such that $v - \phi \in W_0^{1,p(\cdot)}(\Omega)$. Then, for every $\Omega' \subset \subset \Omega$ there exists $C = C(\Omega', \|\phi\|_{1,p(\cdot)}, \|f\|_{L^\infty(\Omega)}, p_{\min}, p_{\max}, \lambda_{\max}, L)$ such that

$$\sup_{\Omega'} u \leq C.$$

Proof. Let us prove first that a minimizer exists. In fact, let

$$\mathcal{K} = \left\{ v \in W^{1,p(\cdot)}(\Omega) : v - \phi \in W_0^{1,p(\cdot)}(\Omega) \right\}.$$

In order to prove that $J$ is bounded from below in $\mathcal{K}$, we observe that if $v \in \mathcal{K}$, then

$$J(v) \geq \frac{1}{p_{\max}} \int_{\Omega} \|\nabla v\|^{p(x)} dx + \int_{\Omega} fv dx,$$

and we have, by Theorem A.3 and Theorem A.4

$$\int_{\Omega} |fv| dx \leq 2\|f\|_{p'(\cdot)} \|v\|_{p(\cdot)} \leq 2\|f\|_{p'(\cdot)} (\|v - \phi\|_{p(\cdot)} + \|\phi\|_{p(\cdot)}) \leq C_0\|\nabla v - \nabla \phi\|_{p(\cdot)} + C_1 \leq C_0\|\nabla v\|_{p(\cdot)} + C_2.$$

If $\left( \int_{\Omega} \|\nabla v\|^{p(x)} dx \right)^{1/p_{\min}} \geq \left( \int_{\Omega} \|\nabla v\|^{p(x)} dx \right)^{1/p_{\max}}$ we get, by Proposition A.1

$$\int_{\Omega} |fv| dx \leq C_0 \left( \int_{\Omega} \|\nabla v\|^{p(x)} dx \right)^{1/p_{\min}} + C_2 \leq C_3 + \frac{1}{2p_{\max}} \int_{\Omega} \|\nabla v\|^{p(x)} dx.$$

If, on the other hand, $\left( \int_{\Omega} \|\nabla v\|^{p(x)} dx \right)^{1/p_{\min}} < \left( \int_{\Omega} \|\nabla v\|^{p(x)} dx \right)^{1/p_{\max}}$, we get in an analogous way

$$\int_{\Omega} |fv| dx \leq C_0 \left( \int_{\Omega} \|\nabla v\|^{p(x)} dx \right)^{1/p_{\max}} + C_2 \leq C_4 + \frac{1}{2p_{\max}} \int_{\Omega} \|\nabla v\|^{p(x)} dx.$$
Taking $C_5 = \max\{C_3, C_4\}$, we get
\begin{equation}
J(u) \geq -C_5 + \frac{1}{2p_{\max}} \int_{\Omega} |\nabla v|^p dx \geq -C_5,
\end{equation}
which shows that $J$ is bounded from below in $\mathcal{K}$.

At this point we want to remark that the constants $C_0, \ldots, C_5$ above can be taken depending only on $\|\phi\|_{1,p(\cdot)}, \|f\|_{L^\infty(\Omega), p_{\min}, p_{\max}}$ and $L$.

We now take a minimizing sequence $\{u_n\} \subset \mathcal{K}$. Without loss of generality we can assume that $J(u_n) \leq J(\phi)$, so by (3.2), $\int_{\Omega} |\nabla u_n|^p dx \leq C_6$. By Proposition A.1 $\|\nabla u_n - \nabla \phi\|_{p(\cdot)} \leq C_7$ and, as $u_n - \phi \in W^{1,p(\cdot)}(\Omega)$, by Theorem A.1 we have $\|u_n - \phi\|_{p(\cdot)} \leq C_8$. Therefore, by Theorem A.1 there exist a subsequence (that we still call $u_n$) and a function $u \in W^{1,p(\cdot)}(\Omega)$ such that
\begin{equation}
\|u\|_{W^{1,p(\cdot)}(\Omega)} \leq \tilde{C},
\end{equation}
with $\tilde{C} = \tilde{C}(\|\phi\|_{1,p(\cdot)}, \|f\|_{L^\infty(\Omega), p_{\min}, p_{\max}, \lambda_{\max}, L})$.

and, by Theorem A.2
$$u_n \rightharpoonup u \text{ weakly in } W^{1,p_{\min}}(\Omega).$$

Now, by the compactness of the immersion $W^{1,p_{\min}}(\Omega) \hookrightarrow L^{p_{\min}}(\Omega)$ we have that, for a subsequence that we still denote by $u_n$,
$$u_n \rightarrow u \text{ in } L^{p_{\min}}(\Omega),$$
$$u_n \rightarrow u \text{ a.e. } \Omega.$$

As $\mathcal{K}$ is convex and closed, it is weakly closed, so $u \in \mathcal{K}$.

It follows that
\begin{equation}
\lambda(x)\chi_{\{u > 0\}} \leq \liminf_{n \to \infty} \lambda(x)\chi_{\{u_n > 0\}},
\end{equation}
\begin{equation}
\int_{\Omega} \lambda(x)\chi_{\{u > 0\}} dx \leq \liminf_{n \to \infty} \int_{\Omega} \lambda(x)\chi_{\{u_n > 0\}} dx,
\end{equation}
\begin{equation}
\lim_{n \to \infty} \int_{\Omega} fu_n dx = \int_{\Omega} fu dx,
\end{equation}
\begin{equation}
\int_{\Omega} \frac{|\nabla u_n|^p}{p(x)} dx \leq \liminf_{n \to \infty} \int_{\Omega} \frac{|\nabla u_n|^p}{p(x)} dx.
\end{equation}

In order to prove the last inequality we observe that there holds
\begin{equation}
\int_{\Omega} \frac{|\nabla u_n|^p}{p(x)} dx \geq \int_{\Omega} \frac{|\nabla u|^p}{p(x)} dx + \int_{\Omega} |\nabla u|^p - 2\nabla u \cdot (\nabla u_n - \nabla u) dx.
\end{equation}

Recall that $\nabla u_n$ converges weakly to $\nabla u$ in $L^p(\Omega)$. Now, since $|\nabla u|^p - 1 \in L^p(\Omega)$, by Theorem A.1 and passing to the limit in (3.4) we get
\begin{equation}
\liminf_{n \to \infty} \int_{\Omega} \frac{|\nabla u_n|^p}{p(x)} dx \geq \int_{\Omega} \frac{|\nabla u|^p}{p(x)} dx.
\end{equation}

Hence
$$J(u) \leq \liminf_{n \to \infty} J(u_n) = \inf_{v \in \mathcal{K}} J(v).$$

Therefore, $u$ is a minimizer of $J$ in $\mathcal{K}$.

Finally, in order to prove (3.1), we observe that, from Proposition A.1 and estimate (3.3), we have that $\int_{\Omega} |u|^p dx \leq C_1(\|\phi\|_{1,p(\cdot)}, \|f\|_{L^\infty(\Omega), p_{\min}, p_{\max}, \lambda_{\max}, L})$. Thus, the desired estimate follows
from the application of Proposition 2.1 in [35], since, by Lemma 3.1, \( \Delta_{p(x)} u \geq f \geq -\|f\|_{L^\infty(\Omega)} \) in \( \Omega \).

For local minimizers we first have

**Lemma 3.1.** Let \( p, f \) and \( \lambda \) be as in Theorem 3.1. Let \( u \in W^{1,p(x)}(\Omega) \) be a local minimizer of

\[
J(v) = \int_{\Omega} \left( \frac{|\nabla u|^{p(x)}}{p(x)} + \lambda(x) \chi_{\{u > 0\}} + fv \right) dx.
\]

Then

\[
(3.5) \quad \Delta_{p(x)} u \geq f \quad \text{in} \ \Omega.
\]

**Proof.** In fact, let \( t > 0 \) and \( 0 \leq \xi \in C_0^\infty(\Omega) \). Using the minimality of \( u \) we have

\[
0 \leq \frac{1}{t} (J(u - t \xi) - J(u)) \leq \frac{1}{t} \int_{\Omega} \left( \frac{|\nabla u - t \nabla \xi|^{p(x)}}{p(x)} - \frac{|\nabla u|^{p(x)}}{p(x)} \right) dx - \int_{\Omega} f \xi dx
\]

\[
\leq - \int_{\Omega} |\nabla u - t \nabla \xi|^{p(x)-2} (\nabla u - t \nabla \xi) \cdot \nabla \xi dx - \int_{\Omega} f \xi dx
\]

and if we take \( t \to 0 \), we obtain

\[
(3.6) \quad 0 \leq - \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \xi dx - \int_{\Omega} f \xi dx,
\]

which gives (3.5). \( \square \)

**Remark 3.3.** We are interested in studying the behavior of nonnegative local minimizers of the energy functional (1.1).

If \( u \) is as in Theorem 3.1 and we have, for instance, \( \phi \geq 0 \) in \( \Omega \) and \( f \leq 0 \) in \( \Omega \), then we have \( u \geq 0 \) in \( \Omega \). In fact, the result follows by observing that \( \xi = \min(u, 0) \in W^{1,p(x)}_0(\Omega) \) so, for every \( 0 < t < 1 \), \( u - t \xi \in \phi + W^{1,p(x)}_0(\Omega) \), with \( \chi_{(u-t\xi>0)} = \chi_{(u>0)} \). Then, in a similar way as in Lemma 3.1, we get (3.6) and using that \( f \leq 0 \) \( \text{in} \) \( \Omega \)

On the other hand, if \( u \) is any local minimizer of (1.1), the same argument employed in Theorem 3.1 gives \( \sup_{\Omega'} u \leq C_{\Omega'} \), for any \( \Omega' \subset \subset \Omega \). Therefore, if \( u \) is any nonnegative local minimizer of (1.1), then \( u \in L^\infty_{\text{loc}}(\Omega) \).

From now on we will deal with nonnegative local minimizers. Next we will prove that they are locally Lipschitz continuous.

First we need

**Lemma 3.2.** Let \( p \) and \( f \) be as in Theorem 3.1. Let \( \Omega \subset (0,d) \times \mathbb{R}^{N-1} \) be a bounded domain. Assume \( a \in L^\infty(\Omega) \), \( a(x) \geq a_0 > 0 \), with \( \|a\|_{L^\infty} \leq L_1 \). Let \( u \in W^{1,p(x)}(\Omega) \) be a solution to

\[
\text{div}(a(x)|\nabla u|^{p(x)-2} \nabla u) = f \quad \text{in} \ \Omega \quad \text{with} \quad |u| \leq M \quad \text{on} \ \partial \Omega.
\]

Assume moreover that \( Ld < p_{\text{min}} - 1 \).

Then, there exists \( C = C(M, p_{\text{min}}, \|f\|_{L^\infty(\Omega)}, d, a_0, L_1) \) such that \( |u| \leq C \) in \( \Omega \).

**Proof.** We consider, for \( \alpha > 1 \), the function \( w(x) = M + e^{\alpha d} - e^{\alpha x_1} \). Computing, we have

\[
w_{x_i} = -ae^{\alpha x_1} \delta_{i1}, \quad w_{x_1} = -ae^{\alpha x_1} \delta_{i1} \delta_{j1}, \quad |\nabla w| = ae^{\alpha x_1}.
\]
Therefore we obtain
\[
\text{div}(a(x)|\nabla w|^{p(x)} - 2\nabla w) = |\nabla w|^{p(x) - 2} \left[ a(x)\Delta w + a(x)(\nabla w, \nabla p) \log |\nabla w| + a(x) \frac{(p(x) - 2)}{|\nabla w|^2} \sum_{i,j} w_{x_i} w_{x_j} w_{x_i x_j} + \langle \nabla w, \nabla a \rangle \right]
\]
\[
= a(x)(\alpha e^{\alpha x_1})^{p(x) - 1} \left[ - (p(x) - 1)\alpha - p_{x_1}(x) \log(\alpha e^{\alpha x_1}) - \frac{a_{x_1}(x)}{a(x)} \right]
\]
\[
\leq a(x)(\alpha e^{\alpha x_1})^{p(x) - 1} \left[ - (p_{\min} - 1)\alpha + L \log \alpha + L\alpha x_1 + \frac{|a_{x_1}(x)|}{a(x)} \right]
\]
\[
\leq a(x)(\alpha e^{\alpha x_1})^{p(x) - 1} \left[ - (p_{\min} - 1) + L\alpha \right].
\]
If we let \( \alpha \geq \alpha_0 = a_0(p_{\min}, d, a_0, L, L_1) \) so that \( - (p_{\min} - 1) + L\alpha \leq 0, \) we get
\[
\text{div}(a(x)|\nabla w|^{p(x)} - 2\nabla w)
\]
\[
\leq a_0 \alpha^{p_{\min} - 1} \left[ - (p_{\min} - 1) + L\alpha \right]
\]
\[
\leq - \|f\|_{L^\infty(\Omega)},
\]
where the last inequality holds if we choose \( \alpha \geq \alpha_1 = \alpha_1(\|f\|_{L^\infty(\Omega)}, p_{\min}, d, a_0, L, L_1). \)

It follows that for \( \alpha = \max\{\alpha_0, \alpha_1, 1\} \) the corresponding function \( w \) satisfies
\[
\text{div}(a(x)|\nabla w|^{p(x)} - 2\nabla w) \leq - \|f\|_{L^\infty(\Omega)} \leq \pm f \quad \text{in} \quad \Omega.
\]
Since \( \pm u \leq w \leq M + e^{\alpha d} \) in \( \Omega. \) This concludes the proof. \( \square \)

**Remark 3.4.** Let \( u \) be as in Lemma 3.2 in a domain \( \Omega \subset (-r, r) \times \mathbb{R}^{N-1}. \) Then, defining \( \bar{u}(x) = u(x - re_1), \bar{a}(x) = a(x - re_1), \bar{p}(x) = p(x - re_1), \bar{f}(x) = f(x - re_1) \) and \( \bar{\Omega} = \Omega + re_1, \) we have
\[
\text{div}(\bar{a}(x)|\nabla \bar{u}|^{\bar{p}(x)} - 2\nabla \bar{u}) = \bar{f} \quad \text{in} \quad \bar{\Omega}. \]
Then, the invariance by translations of the problem allows us to apply Lemma 3.2 to \( \bar{u} \) and conclude that, if \( L2r < p_{\min} - 1, \) then \( |u| \leq C \) in \( \bar{\Omega}, \) for a constant \( C = C(M, p_{\min}, \|f\|_{L^\infty(\Omega)}, r, a_0, L, L_1). \)

Next, we prove that nonnegative local minimizers —of a more general functional than (1.1)— are locally Hölder continuous.

**Theorem 3.2.** Let \( p, f \) and \( \lambda \) be as in Theorem 3.1. Assume that \( 0 < a_0 \leq a(x) \leq a_1 < \infty, \) with \( \|\nabla a\|_{L^\infty} \leq L_1. \) Let \( u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega) \) be a nonnegative local minimizer of
\[
J^a(v) = \int_\Omega \left( a(x) \frac{|\nabla v|^{p(x)}}{p(x)} + \lambda(x) \chi_{\{v > 0\}} + f v \right) dx
\]
and let \( B_{\hat{r}_0}(x_0) \subset \Omega. \) Then, there exist \( 0 < \gamma < 1 \) and \( 0 < \hat{\rho}_0 < \hat{r}_0, \hat{\rho}_0 = \hat{\rho}_0(\hat{r}_0, N, p_{\min}, L) \) and \( \gamma = \gamma(N, p_{\min}), \) such that \( u \in C^{\gamma}(B_{\hat{r}_0}(x_0)). \) Moreover, \( \|u\|_{C^{\gamma}(B_{\hat{r}_0}(x_0))} \leq C \) with \( C \) depending only on \( N, \hat{r}_0, p_{\min}, p_{\max}, L, \lambda_{\max}, \|u\|_{L^\infty(B_{\hat{r}_0}(x_0))}, \|f\|_{L^\infty(B_{\hat{r}_0}(x_0))}, a_0, a_1 \) and \( L_1. \)

**Proof.** We will prove that there exist \( 0 < \gamma < 1 \) and \( 0 < \rho_0 < \hat{r}_0 \) such that, if \( B_{\rho_0}(y) \subset B_{\hat{r}_0}(x_0) \) and \( \rho \leq \rho_0, \) then
\[
\left( \int_{B_{\rho}(y)} |\nabla u|^{p-} dx \right)^{1/p-} \leq C \rho^{\gamma-1},
\]
where \( p- = p-(B_{\rho_0}(y)). \) Without loss of generality we will assume that \( y = 0. \)
In fact, let $0 < r_0 \leq \min \{ \frac{r_0}{2}, 1 \}$, $0 < r \leq r_0$ and $v$ the solution of
\begin{equation}
\text{div} (a(x) |\nabla v|^{p(x)} - 2 \nabla v) = f \quad \text{in } B_r, \quad v - u \in W^{1,p(r)}_0(B_r).
\end{equation}
If $r_0 \leq \frac{1}{17} (p_{\min} - 1)$, it follows from Lemma 3.8 and Remark 3.4 that
\begin{equation}
|v|_{L^\infty(B_r)} \leq \tilde{C} \quad \text{with } \tilde{C} = \tilde{C}(L, p_{\min}, \|u\|_{L^\infty(B_{r_0}(x_0))}, \|f\|_{L^\infty(B_{r_0}(x_0))}, a_0, L_1).
\end{equation}

Let $u^s(x) = su(x) + (1 - s)v(x)$. By using (3.8) and the inequalities in (1.4), we get
\begin{align*}
&\int_{B_r} a(x) \frac{\nabla u^s}{p(x)} - a(x) \frac{\nabla v}{p(x)} + \int_{B_r} f(u - v) = \\
&\int_0^1 \frac{ds}{s} \int_{B_r} a(x) \left( |\nabla u^s|^{p(x)} - |\nabla v|^{p(x)} - 2 \nabla u^s \cdot \nabla v \right) - \nabla (u^s - v) \geq \\
&\quad C \left( \int_{B_r \cap \{p \geq 2\}} a(x) |\nabla u - v|^{p(x)} + \int_{B_r \cap \{p < 2\}} a(x) |\nabla u - v|^2 \left( |\nabla u| + |\nabla v| \right)^{p(x) - 2} \right),
\end{align*}
where $C = C(p_{\min}, p_{\max}, N)$.

Therefore, by the minimality of $u$, we have (if $A_1 = B_r \cap \{ p(x) < 2 \}$ and $A_2 = B_r \cap \{ p(x) \geq 2 \}$)
\begin{align*}
&\int_{A_2} |\nabla u - v|^{p(x)} \ dx \leq C r N, \\
&\int_{A_1} |\nabla u - v|^2 \left( |\nabla u| + |\nabla v| \right)^{p(x) - 2} \ dx \leq C r N,
\end{align*}
where $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, a_0)$.

Let $\varepsilon > 0$. Take $\rho = r^{1+\varepsilon}$ and suppose that $r^\varepsilon \leq 1/2$. Take $0 < \eta < 1$ to be chosen later. Then, by Young’s inequality, the definition of $A_1$ and (3.12), we obtain
\begin{align*}
&\int_{A_1 \cap B_\rho} |\nabla u - v|^{p(x)} \ dx \leq \frac{C}{\eta^{2/p_{\min}}} \int_{A_1 \cap B_r} \left( |\nabla u| + |\nabla v| \right)^{p(x) - 2} |\nabla u - v|^2 \ dx \\
&\quad + C \eta \int_{B_r \cap A_1} \left( |\nabla u| + |\nabla v| \right)^{p(x)} \ dx \\
&\leq \frac{C}{\eta^{2/p_{\min}} r N} + C \eta \int_{B_r \cap A_1} \left( |\nabla u| + |\nabla v| \right)^{p(x)} \ dx.
\end{align*}
Therefore, by (3.11) and (3.13), we get
\begin{equation}
\int_{B_\rho} |\nabla u - v|^{p(x)} \ dx \leq C \frac{r N}{\eta^{2/p_{\min}}} + C \eta \int_{B_r \cap A_1} \left( |\nabla u| + |\nabla v| \right)^{p(x)} \ dx,
\end{equation}
where $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, a_0)$.

Since, $|\nabla u|^q \leq C (|\nabla u - v|^q + |\nabla v|^q)$, for any $q > 1$, with $C = C(q)$, we have, by (3.14), choosing $\eta$ small, that
\begin{equation}
\int_{B_\rho} |\nabla u|^{p(x)} \ dx \leq C r^N + C \int_{B_r} |\nabla v|^{p(x)} \ dx,
\end{equation}
where $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, a_0)$.

Now let $M \geq 1$ such that $\|v\|_{L^\infty(B_r)} \leq M$ and define
\[ w(x) = \frac{v(x)}{M} \quad \text{in } B_1. \]
Then, there holds that
\[
\text{div}(\tilde{a}(x)|\nabla w|^{p(x)-2}\nabla w) = \left(\frac{r}{M}\right)^{p(x)-1} r f(rx) + r \log \left(\frac{r}{M}\right) a(rx) \nabla p(rx) \cdot \nabla w(x)|\nabla w(x)|^{p(x)-2}
\]
in \(B_1\), with \(\tilde{p}(x) = p(rx)\) and \(\tilde{a}(x) = a(rx)\). That is,
\[
\text{div}(\tilde{a}(x)|\nabla w|^{p(x)-2}\nabla w) = B(x, \nabla w(x)) \quad \text{in } B_1,
\]
with
\[
|B(x, \nabla w(x))| \leq C \left(1 + |\nabla w(x)|^{p(x)}\right) \quad \text{in } B_1,
\]
where \(C = C(L, M, \|f\|_{L^\infty(B_{r_0}(x_0))}, a_1)\).

From Theorem 1.1 in [9], it follows that \(w \in C^1_{\text{loc}}(B_1)\) for some \(0 < \alpha < 1\) and that
\[
\sup_{B_{1/2}} |\nabla w| \leq C(L, M, \|f\|_{L^\infty(B_{r_0}(x_0))}, \rho_{\text{max}}, p_{\text{min}}, p_{\text{max}}, N, a_0, a_1, L_1),
\]
which implies
\[
(3.16) \quad \sup_{B_{r/2}} |\nabla v| \leq \frac{CM}{r}.
\]

Therefore, from (3.15) and (3.16), we deduce that
\[
(3.17) \quad \int_{B_\rho} |\nabla u|^{p(x)} \, dx \leq C r^N + C r^N r^{-p_+},
\]
with \(p_+ = p_+(B_{r_0})\) and \(C = C(L, \|f\|_{L^\infty(B_{r_0}(x_0))}, \|f\|_{L^\infty(B_{r_0}(x_0))}, \rho_{\text{max}}, p_{\text{min}}, p_{\text{max}}, N, a_0, a_1, L_1)\).

Here we have used the bound in (3.9).

Then, if we take \(\varepsilon \leq \frac{p_{\text{min}}}{N}\), we have by (3.17) and by our election of \(\rho\), that
\[
\int_{B_\rho} |\nabla u|^{p(x)} \, dx \leq \int_{B_\rho} |\nabla u|^{p(x)} \, dx + \frac{1}{|B_\rho|} \int_{B_\rho \cap \{|\nabla u| < 1\}} |\nabla u|^{p(x)} \, dx
\]
\[
\leq \int_{B_\rho} |\nabla u|^{p(x)} \, dx + 1
\]
\[
\leq 1 + C \left(\frac{r}{\rho}\right)^N + C r^{-p_+}
\]
\[
\leq 1 + C r^{-\varepsilon N} + C r^{-p_+}
\]
\[
\leq C r^{-p_+} = C \rho^{-\frac{p_+}{1 + \varepsilon}}.
\]

Now let \(r_0 \leq r_0(\varepsilon, p_{\text{min}}, L)\) so that
\[
\frac{p_+}{p_-} = \frac{p_+(B_{r_0})}{p_-(B_{r_0})} \leq 1 + \frac{\varepsilon}{2},
\]
and small enough so that, in addition, \(r_0^\gamma \leq 1/2\). Then, if \(\rho \leq \rho_0 = r_0^{1+\varepsilon},\)
\[
\int_{B_\rho} |\nabla u|^{p_-} \, dx \leq C \rho^{-\frac{(1+\varepsilon)\gamma}{1+\varepsilon}} \leq C \rho^{-1-\gamma},
\]
where \(\gamma = \frac{\varepsilon}{(1+\varepsilon)} = \gamma(N, p_{\text{min}})\). That is, if \(\rho \leq \rho_0 = r_0^{1+\varepsilon}\)
\[
\left(\int_{B_\rho} |\nabla u|^{p_-} \, dx\right)^{1/p_-} \leq C \rho^{\gamma-1}.
\]
Thus (3.7) holds, with \( C = C(L, \|
abla\|_{L^{\infty}(B_{\rho_0}(x_0))}, \|f\|_{L^{\infty}(B_{\rho_0}(x_0))}, \lambda_{\max}, p_{\min}, p_{\max}, N, a_0, a_1, L_1) \).

Applying Morrey’s Theorem, see e.g. [29], Theorem 1.53, we conclude that \( u \in C^{\gamma}(B_{\rho_0}(x_0)) \) and \( \|u\|_{C^{\gamma}(B_{\rho_0}(x_0))} \leq C \) for \( C = C(\rho_0, L, \|
abla\|_{L^{\infty}(B_{\rho_0}(x_0))}, \|f\|_{L^{\infty}(B_{\rho_0}(x_0))}, \lambda_{\max}, p_{\min}, p_{\max}, N, a_0, a_1, L_1) \).

As a corollary we obtain

**Corollary 3.1.** Let \( u \) be as in Theorem 3.1. Then \( u \in C^{\gamma}(\Omega) \) for some \( 0 < \gamma < 1 \), \( \gamma = \gamma(N, p_{\min}) \). Moreover, if \( \Omega' \subset \subset \Omega \), then \( \|u\|_{C^{\gamma}(\Omega)} \leq C \) with \( C \) depending only on \( N \), \( \text{dist}(\Omega', \partial\Omega) \), \( p_{\min}, p_{\max}, L, \lambda_{\max}, \|u\|_{L^{\infty}(\Omega)} \), \( \|f\|_{L^{\infty}(\Omega)} \), \( a_0 \), \( a_1 \) and \( L_1 \).

Then, under the assumptions of the previous corollary we have that \( u \) is continuous in \( \Omega \) and therefore, \( \{u > 0\} \) is open. We can now prove the following property for nonnegative local minimizers of (1.1)

**Lemma 3.3.** Let \( p, f \) and \( \lambda \) be as in Theorem 3.1. Let \( u \in W^{1,p_1}(\Omega) \cap L^{\infty}(\Omega) \) be a nonnegative local minimizer of

\[
J(v) = \int_{\Omega} \left( \frac{|\nabla u|^{p(x)}}{p(x)} + \lambda(x) \chi_{\{u > 0\}} + fv \right) dx.
\]

Then

\[
\Delta_{p(x)} u = f \quad \text{in} \ \{u > 0\}.
\]

**Proof.** From Lemma 3.1 we already know that (3.5) holds. In order to obtain the opposite inequality in \( \{u > 0\} \), we let \( 0 \leq \xi \in C^{\infty}_c(\{u > 0\}) \) and consider \( u - t\xi \), for \( t < 0 \), with \( |t| \) small.

Using the minimality of \( u \) we have

\[
0 \geq \frac{1}{t}(J(u - t\xi) - J(u)) = \frac{1}{t} \int_{\Omega} \left( \frac{|\nabla u - t\nabla \xi|^{p(x)}}{p(x)} - \frac{|\nabla u|^{p(x)}}{p(x)} \right) dx - \int_{\Omega} f\xi dx
\]

\[
\geq -\int_{\Omega} |\nabla u - t\nabla \xi|^{p(x) - 2}(\nabla u - t\nabla \xi) \cdot \nabla \xi dx - \int_{\Omega} f\xi dx
\]

and if we take \( t \to 0 \), we obtain

\[
0 \geq -\int_{\Omega} |\nabla u|^{p(x) - 2}\nabla u \cdot \nabla \xi dx - \int_{\Omega} f\xi dx,
\]

which gives the desired inequality, so (3.18) follows.

We will make use of the following version of Harnack’s inequality

**Proposition 3.1.** Let \( x_0 \in \mathbb{R}^N \) and \( 0 < \delta \leq 1 \). Let \( 1 < p_{\min} \leq p(x) \leq p_{\max} < \infty \) in \( B_{\delta}(x_0) \), with \( \|
abla p\|_{L^{\infty}(B_{\delta}(x_0))} \leq L \) and \( f \in L^{\infty}(B_{\delta}(x_0)) \). There exists a constant \( C > 0 \) such that, if \( u \in W^{1,p}(B_{\delta}(x_0)) \cap L^{\infty}(B_{\delta}(x_0)) \) is a nonnegative solution of

\[
\Delta_{p(x)} u = f \quad \text{in} \ B_{\delta}(x_0),
\]

then,

\[
\sup_{B_{\delta}(x_0)} u \leq C \left[ \inf_{B_{\delta}(x_0)} u + \delta \right].
\]

The constant \( C \) depends only on \( N, p_{\min}, p_{\max}, L, \|f\|_{L^{\infty}(B_{\delta}(x_0))} \) and \( \|u\|_{L^{\infty}(B_{\delta}(x_0))} \), where \( p_+ = \sup_{B_{\delta}(x_0)} p(x) \) and \( p_- = \inf_{B_{\delta}(x_0)} p(x) \).
Let \( y_0 \in B_{3/4}(0) \). Since \( \Delta_p(x)u = f \) in \( B_1(0) \), by Theorem 2.1 in \( \text{[35]} \), applied in \( B_{1/8}(y_0) \), we get
\[
\sup_{B_{1/8}(y_0)} u \leq C \left[ \inf_{B_{1/8}(y_0)} u + 1 \right],
\]
where \( C \) is a positive constant that can be chosen so that \( C > 1 \) and so that it depends only on \( N, p_{\min}, p_{\max}, L, \|f\|_{L^\infty(B_1(0))} \) and \( \|u\|_{L^\infty(B_1(0))} \), where \( p_+ = \sup_{B_1(0)} p(x) \) and \( p_- = \inf_{B_1(0)} p(x) \).

We will first assume that \( x_0 = x \in B_1, x_i \in B_i \cap B_{i+1} \) and \( x_j = y \in B_j \), for \( 1 \leq i \leq j - 1 \) and \( j \leq k \).

It follows from (3.20) that \( u(x_i) \leq C [u(x_{i+1}) + 1], \) \( i = 0, \ldots, j - 1, \)
which gives \( u(x) \leq C^k [u(y) + k] \). Therefore,
\[
\Delta_p(x)u = f \quad \text{in} \ B_1(0),
\]
implies
\[
\sup_{B_{1/4}(0)} u \leq C \left[ \inf_{B_{1/4}(0)} u + 1 \right],
\]
for a constant \( C > 0 \) depending only on \( N, p_{\min}, p_{\max}, L, \|f\|_{L^\infty(B_1(0))} \) and \( \|u\|_{L^\infty(B_1(0))} \).

For general \( x_0 \in \mathbb{R}^N \) and \( 0 < \delta \leq 1 \), we take \( \bar{u}(x) = \frac{1}{\delta} u(x_0 + \delta x) \). Then, as
\[
\Delta_p(x) \bar{u} = \bar{f} \quad \text{in} \ B_1(0),
\]
with \( \bar{p}(x) = p(x_0 + \delta x) \) and \( \bar{f}(x) = \delta f(x_0 + \delta x) \), there holds that \( \bar{u} \) satisfies (3.21). Finally, observing that \( p_{\min} \leq \bar{p}(x) \leq p_{\max} \) in \( B_1(0) \), \( \|\nabla p\|_{L^\infty(B_1(0))} \leq L \), \( \|\bar{f}\|_{L^\infty(B_1(0))} \leq \|f\|_{L^\infty(B_{3/4}(x_0))} \)
\[
\|\bar{u}\|_{L^\infty(B_1(0))} = \left( \frac{1}{\delta} \right)^{\frac{p_- - p_+}{2}} \|u\|_{L^\infty(B_{3/4}(x_0))} \left( \frac{1}{\delta} \right)^{2L_\delta} \left( \frac{1}{\delta} \right)^{2L_\delta} \leq C(L),
\]
and
we obtain the desired result. \( \square \)

We will next prove the Lipschitz continuity of nonnegative local minimizers of (1.1). In the case in which \( f \equiv 0 \) and \( p(x) \geq 2 \) this result was proven in [5]. In order to deal with the general case we will employ a different strategy than the one in [5].

Before getting the Lipschitz continuity we prove the following result

**Theorem 3.3.** Let \( p, f, \lambda, u \) be as in Lemma 3.3. Let \( \Omega' \subset \subset \Omega \). There exist constants \( C > 0, r_0 > 0 \) such that if \( x_0 \in \Omega' \cap \partial \{ u > 0 \} \) and \( r \leq r_0 \) then
\[
\sup_{B_r(x_0)} u \leq C r.
\]
The constants depend only on \( N, p_{\min}, p_{\max}, L, \|f\|_{L^\infty(\Omega)}, \lambda_{\min}, \lambda_{\max}, \|u\|_{L^\infty(\Omega)} \) and \( \text{dist}(\Omega', \partial \Omega) \).
Proof. Let us suppose by contradiction that there exist a sequence of nonnegative local minimizers $u_k$ corresponding to functionals $J_k$ given by functions $p_k$, $f_k$ and $\lambda_k$, with $u_k \in W^{1,p_k(\cdot)}(\Omega) \cap L^\infty(\Omega)$, $p_{\min} \leq p_k(x) \leq p_{\max}$, $\|\nabla p_k\|_{L^\infty} \leq L$, $\|f_k\|_{L^\infty(\Omega)} \leq M_0$, $\lambda_{\min} \leq \lambda_k(x) \leq \lambda_{\max}$, $\|u_k\|_{L^\infty(\Omega)} \leq M$ and points $\bar{x}_k \in \Omega \cap \partial\{u_k > 0\}$, such that

$$\sup_{B_{r_k/4}(\bar{x}_k)} u_k \geq kr_k \quad \text{and} \quad r_k \leq \frac{1}{k}.$$  

Without loss of generality we will assume that $\bar{x}_k = 0$.

Let us define in $B_1$, for $k$ large, $\bar{u}_k(x) = \frac{1}{r_k} u_k(r_k x)$, $\bar{p}_k(x) = p_k(r_k x)$, $\bar{f}_k(x) = r_k f_k(r_k x)$ and $\bar{\lambda}_k(x) = \lambda_k(r_k x)$. Then $p_{\min} \leq \bar{p}_k(x) \leq p_{\max}$, $\|\nabla \bar{p}_k\|_{L^\infty(B_1)} \leq L r_k$, $\lambda_{\min} \leq \bar{\lambda}_k(x) \leq \lambda_{\max}$ and $\|\bar{f}_k\|_{L^\infty(B_1)} \leq M_0 r_k$. Moreover, $\bar{u}_k$ is a nonnegative minimizer in $\bar{u}_k + W^{1,\bar{p}_k(\cdot)}(B_1)$ of the functional

$$(3.22) \quad \bar{J}_k(v) = \int_{B_1} \left( \frac{\nabla v|\bar{p}_k(x)}{|\bar{p}_k(x)} + \bar{\lambda}_k(x) \chi_{\{v > 0\}} \right) + \bar{f}_k v \, dx$$

with

$$\bar{u}_k(0) = 0 \quad \text{and} \quad \max_{B_{1/4}} \bar{u}_k(x) > k.$$

Let $d_k(x) = \text{dist}(x, \{\bar{u}_k = 0\})$ and $O_k = \{ x \in B_1 : d_k(x) \leq \frac{1 - |x|}{3} \}$. Since $\bar{u}_k(0) = 0$ then $B_{1/4} \subset O_k$, therefore

$$m_k := \sup_{O_k} (1 - |x|) \bar{u}_k(x) \geq \max_{B_{1/4}} (1 - |x|) \bar{u}_k(x) \geq \frac{3}{4} \max_{B_{1/4}} \bar{u}_k(x) > \frac{3}{4} k.$$  

For each fix $k$, $\bar{u}_k$ is bounded, then $(1 - |x|) \bar{u}_k(x) \to 0$ when $|x| \to 1$ which means that there exists $x_k \in O_k$ such that $(1 - |x_k|) \bar{u}_k(x_k) = \sup_{O_k} (1 - |x|) \bar{u}_k(x)$, and then

$$(3.23) \quad \bar{u}_k(x_k) = \frac{m_k}{1 - |x_k|} \geq m_k > \frac{3}{4} k$$

as $x_k \in O_k$, and $\delta_k := d_k(x_k) \leq \frac{1 - |x_k|}{3}$. Let $y_k \in \partial\{\bar{u}_k > 0\} \cap B_1$ such that $|y_k - x_k| = \delta_k$. Then,

1. $B_{2\delta_k}(y_k) \subset B_1$,

   since if $y \in B_{2\delta_k}(y_k) \Rightarrow |y| < 3\delta_k + |x_k| \leq 1$,

2. $B_{\frac{\delta_k}{2}}(y_k) \subset O_k$,

   since if $y \in B_{\frac{\delta_k}{2}}(y_k) \Rightarrow |y| \leq \frac{3}{2} \delta_k + |x_k| \leq 1 - \frac{3}{2} \delta_k \Rightarrow d_k(y) \leq \frac{\delta_k}{2} \leq \frac{1 - |y|}{3}$ and

3. if $z \in B_{\frac{\delta_k}{2}}(y_k) \Rightarrow 1 - |z| \geq 1 - |x_k| - |x_k - z| \geq 1 - |x_k| - \frac{3}{2} \delta_k \geq \frac{1 - |x_k|}{2}$.

By (2) we have

$$\max_{O_k} (1 - |x|) \bar{u}_k(x) \geq \max_{B_{\frac{\delta_k}{2}}(y_k)} (1 - |x|) \bar{u}_k(x) \geq \max_{B_{\frac{\delta_k}{2}}(y_k)} \left( \frac{1 - |x_k|}{2} \bar{u}_k(x) \right).$$
where in the last inequality we are using (3). Then,

\[(3.24) \quad 2\bar{u}_k(x_k) \geq \max_{\{\bar{u}_k > 0\}} \bar{u}_k(x).
\]

As \(B_{\delta_k}(x_k) \subset \{\bar{u}_k > 0\}\) then \(\Delta \bar{p}_k(x_k) \bar{u}_k = \tilde{f}_k\) in \(B_{\delta_k}(x_k)\), and by Harnack’s inequality (Proposition 3.1) we have

\[(3.25) \quad \max_{\bar{u}_k(x_k)} \bar{u}_k(x) \leq \min_{\{\bar{u}_k \neq 0\}} \bar{u}_k(x) + \delta_k,
\]

with \(C\) a positive constant depending only on \(N, p_{\min}, p_{\max}, L, M_0\) and \(M\). We point out that, in order to get this uniform constant \(C\) in (3.25), we have used, while applying Proposition 3.1, that

\[
\gamma_k := \sup_{B_{\delta_k}(x_k)} \bar{p}_k - \inf_{B_{\delta_k}(x_k)} \bar{p}_k \leq 2Lr_k \delta_k \leq 2Lr_k,
\]

so that

\[
||\bar{u}_k||_{L^p(B_{\delta_k}(x_k))} \leq (M/r_k)^{2Lr_k} \leq C_0(L, M).
\]

Recalling (3.23), we get from (3.25), for \(k\) large,

\[(3.26) \quad \min_{\bar{u}_k(x_k)} \bar{u}_k(x) \geq c\bar{u}_k(x_k),
\]

with \(c\) a positive constant depending only on \(N, p_{\min}, p_{\max}, L, M_0\) and \(M\). As \(\bar{u}_k(x_k) \cap B_{\delta_k}(y_k) \neq \emptyset\) we have by (3.26)

\[(3.27) \quad \max_{\bar{u}_k(y_k)} \bar{u}_k(x) \geq c\bar{u}_k(x_k).
\]

Let \(w_k(x) = \frac{\bar{u}_k(y_k + \frac{\delta_k}{2}x)}{\bar{u}_k(x_k)}\). Then, \(w_k(0) = 0\) and, by (3.24) and (3.27), we have

\[(3.28) \quad \max_{B_1} w_k \leq 2 \quad \max_{B_{1/2}} w_k \geq c > 0.
\]

Now, recalling that \(\bar{u}_k\) is a nonnegative minimizer in \(\bar{u}_k + W_0^{1,\bar{p}_k}(B_1)\) of the functional \(\bar{J}_k\) in (3.22) and that \(B_{\delta_k}(y_k) \subset B_1\), we see that \(w_k\) is a nonnegative minimizer of \(\tilde{J}_k\) in \(w_k + W_0^{1,\bar{p}_k}(y_k + \frac{\delta_k}{2}x)(B_1)\), where

\[
\tilde{J}_k(v) = \int_{B_1} \left( c_k \frac{\bar{p}_k(y_k + \frac{\delta_k}{2}x)}{\bar{u}_k(y_k + \frac{\delta_k}{2}x)} \right) \left( \frac{\nabla \tilde{p}_k(y_k + \frac{\delta_k}{2}x)}{\bar{u}_k(y_k + \frac{\delta_k}{2}x)} + \tilde{\lambda}_k(y_k + \frac{\delta_k}{2}x) \chi_{\{v > 0\}} + \tilde{f}_k(y_k + \frac{\delta_k}{2}x) \bar{u}_k(x_k) v \right) dx,
\]

and \(c_k = \frac{2\bar{u}_k(x_k)}{\delta_k}\).

We now notice that \(c_k \to \infty\). So we define \(\bar{p}_k(x) = \bar{p}_k(y_k + \frac{\delta_k}{2}x)\) and divide the functional \(\tilde{J}_k\) by \(c_k \bar{p}_k\), with \(\bar{p}_k = \inf_{B_1} \bar{p}_k\). Then, it follows that \(w_k\) is a nonnegative minimizer of \(\tilde{J}_k\) in \(\tilde{J}_k + W_0^{1,\bar{p}_k}(B_1)\), where

\[
\tilde{J}_k(v) = \int_{B_1} \left( \tilde{a}_k(x) \frac{\nabla \tilde{p}_k(x)}{\tilde{p}_k(x)} + \tilde{\lambda}_k(x) \chi_{\{v > 0\}} + \tilde{f}_k v \right) dx,
\]

\[
\tilde{a}_k(x) = c_k \bar{p}_k(x) - \bar{p}^-_k, \quad \tilde{\lambda}_k(x) = \bar{\lambda}_k(y_k + \frac{\delta_k}{2}x)c_k \bar{p}_k(x), \quad \text{and} \quad \tilde{f}_k(x) = \tilde{f}_k(y_k + \frac{\delta_k}{2}x) \bar{u}_k(x_k)c_k \bar{p}_k(x).
\]
We claim that
\begin{equation}
\|\tilde{f}_k\|_{L^\infty} \leq \tilde{M}_0 \quad \text{and} \quad \tilde{f}_k \to 0 \quad \text{uniformly in } B_1,
\end{equation}
\begin{equation}
\tilde{\lambda}_k \to 0 \quad \text{uniformly in } B_1,
\end{equation}
\begin{equation}
\tilde{a}_k \to 1 \quad \text{uniformly,} \quad 1 \leq \tilde{a}_k \leq M_1 \quad \text{and} \quad \|\nabla \tilde{a}_k\|_{L^\infty} \leq L_1 \quad \text{in } B_1,
\end{equation}
\begin{equation}
\tilde{p}_k \to p_0 \quad \text{uniformly and} \quad p_{\min} \leq p_0 \leq p_{\max} \quad \text{in } B_1,
\end{equation}
up to a subsequence, for some constants \(\tilde{M}_0, M_1, L_1\) and \(p_0\).

In fact, (3.29) follows since \(|\tilde{f}_k(x)| = |r_k f_k(r_k(y_k + \frac{\tilde{a}_k}{2} x))\frac{\|\nabla \tilde{a}_k\|_{L^\infty} \log c_k}{r_k} \tilde{p}_k| \leq M_0 M c_k^{-1} \to 0\). On the other hand, \(0 < \tilde{\lambda}_k(x) \leq \lambda_{\max} c_k^{-1} \to 0\) gives (3.30).

In addition, in \(B_1\) there holds, for \(k\) large, that \(1 \leq \tilde{a}_k(x) \leq e^{2 \|\nabla \tilde{p}_k\|_{L^\infty} \log c_k}\) and \(\|\nabla \tilde{a}_k\|_{L^\infty} \leq \|\nabla \tilde{p}_k\|_{L^\infty} \log c_k \leq \tilde{M}_0 \tilde{M} \log (\frac{\tilde{M}}{\tilde{M}}) \to 0\), which implies (3.31).

Finally, to see (3.32) we observe that \(p_{\min} \leq p_k(x) \leq p_{\max}\) and \(\|\nabla p_k\|_{L^\infty(\Omega)} \leq L\) and then, for a subsequence, \(p_k \to p\) uniformly on compacts of \(\Omega\), so \(\tilde{p}_k(x) = p_k(r_k(y_k + \frac{\tilde{a}_k}{2} x)) \to p_0 = p(0)\) uniformly in \(B_1\).

We now take \(v_k\) the solution of
\begin{equation}
\text{div} \left( \tilde{a}_k(x) |\nabla v_k| \tilde{p}_k(x) - 2 \nabla v_k \right) = \tilde{f}_k \quad \text{in } B_{3/4}, \quad v_k - w_k \in W_0^{1, \tilde{p}_k(\cdot)}(B_{3/4}).
\end{equation}

From Lemma 3.2, Remark 3.4 and the bounds in (3.29), (3.29) and (3.31), it follows that if \(k\) is large enough
\begin{equation}
\|v_k\|_{L^\infty(B_{3/4})} \leq C \quad \text{with} \quad C = C(p_{\min}, \tilde{M}_0, L_1).
\end{equation}

Here we have used that \(\|\nabla \tilde{p}_k\|_{L^\infty} \leq L r_k \frac{\tilde{a}_k}{2}\) so \(\|\nabla \tilde{p}_k\|_{L^\infty(\Omega)} \leq p_{\min} - 1\) for \(k\) large.

Then, applying Theorem 1.1 in [14] we obtain that, for \(k\) large,
\begin{equation}
\|v_k\|_{C^{1,\alpha}(\overline{B_{1/2}})} \leq \tilde{C} \quad \text{with} \quad \tilde{C} = \tilde{C}(p_{\min}, p_{\max}, \tilde{M}_0, L, L_1, M_1, N),
\end{equation}
for some \(0 < \alpha < 1\). Therefore, there is a function \(v_0 \in C^{1,\alpha}(\overline{B_{1/2}})\) such that, for a subsequence,
\begin{equation}
v_k \to v_0 \quad \text{and} \quad \nabla v_k \to \nabla v_0 \quad \text{uniformly in } \overline{B_{1/2}}.
\end{equation}

Moreover, (3.29), (3.31) and (3.32) imply that
\begin{equation}
\Delta_{p_0} v_0 = 0 \quad \text{in } B_{1/2}.
\end{equation}

Let us now show that
\begin{equation}
w_k - v_k \to 0 \quad \text{in } L^{p_{\min}}(B_{3/4}).
\end{equation}

From the minimality of \(w_k\) we have
\begin{equation}
\int_{B_{3/4}} \tilde{a}_k(x) \frac{\nabla w_k}{\tilde{p}_k(x)} - \tilde{a}_k(x) \frac{\nabla v_k}{\tilde{p}_k(x)} + \int_{B_{3/4}} \tilde{f}_k(w_k - v_k) \leq C(N) \|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}.
\end{equation}
Then, we can argue as in the proof of Theorem 3.2 and get estimate (3.10) for $u = w_k$, $v = v_k$, $a(x) = \tilde{a}_k(x)$, $p(x) = \tilde{p}_k(x)$, $f = \tilde{f}_k$ and $r = 3/4$, which together with (3.39), gives

\begin{align}
\int_{A_2^k} |\nabla w_k - \nabla v_k| \tilde{p}_k(x) \, dx \leq C \|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}, \\
\int_{A_1^k} |\nabla w_k - \nabla v_k|^2 (|\nabla w_k| + |\nabla v_k|) \tilde{p}_k(x)^{-2} \, dx \leq C \|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})},
\end{align}

where $A_1^k = B_{3/4} \cap \{\tilde{p}_k(x) < 2\}$, $A_2^k = B_{3/4} \cap \{\tilde{p}_k(x) \geq 2\}$ and $C = C(p_{\min}, p_{\max}, N)$.

Applying Hölder’s inequality (Theorem A.3) with exponents $\frac{2}{\tilde{p}_k(x)}$ and $\frac{2}{2-\tilde{p}_k(x)}$, we get

\begin{align}
\int_{A_1^k} |\nabla w_k - \nabla v_k|^2 \tilde{p}_k(x) \, dx \leq 2 \|F_k\|_{L^{2/\tilde{p}_k()}(A_1^k)} \|G_k\|_{L^{2/(2-\tilde{p}_k())}(A_1^k)},
\end{align}

where

$$F_k = |\nabla w_k - \nabla v_k| \tilde{p}_k (|\nabla w_k| + |\nabla v_k|) (\tilde{p}_k - 2) \tilde{p}_k / 2$$

$$G_k = (|\nabla w_k| + |\nabla v_k|)^{(2-\tilde{p}_k)\tilde{p}_k / 2}.$$

Since

$$\int_{A_1^k} \frac{1}{\tilde{p}_k^2} \, dx = \int_{A_1^k} |\nabla w_k - \nabla v_k|^2 (|\nabla w_k| + |\nabla v_k|) \tilde{p}_k(x)^{-2} \, dx,$$

then, from (3.39), (3.40) and Proposition A.1, we get, for $k$ large,

\begin{align}
\|F_k\|_{L^{2/\tilde{p}_k()}(A_1^k)} \leq C \|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}^{p_{\min}/2},
\end{align}

$C = C(p_{\min}, p_{\max}, N)$. On the other hand, (3.33) and the bounds in (3.29), (3.31) and (3.34) give

\begin{align}
\frac{1}{p_{\max}} \int_{B_{3/4}} |\nabla v_k| \tilde{p}_k(x) \, dx \leq \int_{B_{3/4}} \tilde{\lambda}_k(x) \frac{|\nabla v_k| \tilde{p}_k(x)}{\tilde{p}_k(x)} \, dx \\
\leq \int_{B_{3/4}} \tilde{\lambda}_k(x) \frac{|\nabla w_k| \tilde{p}_k(x)}{\tilde{p}_k(x)} \, dx + \int_{B_{3/4}} \tilde{f}_k(w_k - v_k) \\
\leq C \left( 1 + \int_{B_{3/4}} |\nabla w_k| \tilde{p}_k(x) \right).
\end{align}

This implies

\begin{align}
\int_{A_1^k} |G_k|^{2/(2-\tilde{p}_k(x))} \, dx \leq C \int_{B_{3/4}} (|\nabla w_k| \tilde{p}_k(x) + |\nabla v_k| \tilde{p}_k(x)) \, dx \leq \bar{C} \left( 1 + \int_{B_{3/4}} |\nabla w_k| \tilde{p}_k(x) \right),
\end{align}

for some $\bar{C} = \bar{C}(p_{\min}, p_{\max}, \tilde{M}_0, M_1, L_1) \geq 1$. Now (3.44) and Proposition A.1 give

\begin{align}
\|G_k\|_{L^{2/(2-\tilde{p}_k())}(A_1^k)} \leq \bar{C} \left( 1 + \int_{B_{3/4}} |\nabla w_k| \tilde{p}_k(x) \right),
\end{align}

Let us show that the right hand side in (3.45) can be bounded independently of $k$.

In fact, let $\tilde{v}_k$ be the solution of

\begin{align}
\text{div}(\tilde{\lambda}_k(x)|\nabla \tilde{v}_k| \tilde{p}_k(x)^{-2} \nabla \tilde{v}_k) = \tilde{f}_k \quad \text{in } B_{7/8}, \quad \tilde{v}_k - w_k \in W^{1,\tilde{p}_k()}(B_{7/8}).
\end{align}

Then, similar arguments to those leading to (3.34) and (3.35), give, for $k$ large enough,

\begin{align}
||\tilde{v}_k||_{L^\infty(B_{7/8})} \leq \bar{C} \quad \text{with } \bar{C} = \bar{C}(p_{\min}, \tilde{M}_0, L_1),
\end{align}

where $\bar{C} = \bar{C}(p_{\min}, \tilde{M}_0, L_1)$.
Thus, using Poincaré’s inequality (Theorem A.4) and Theorem A.2, we get (3.38).

Let \( \bar{\gamma} \) for the right hand side in (3.45). That is,

\[
\|w(3.52) \|_{\mathcal{L}^1} \leq C \text{ with } C = \bar{C}(p_{\min}, p_{\max}, M_0, L_1, L, M_1, N),
\]

for some \( 0 < \alpha < 1 \).

Since \( w_k \) is a nonnegative minimizer of \( \tilde{J}_k \) in \( B_1 \), then we can argue as in the proof of Theorem 3.2 and get estimate (3.15) for \( u = w_k \), \( v = \tilde{v}_k \), \( a(x) = \tilde{a}_k(x) \), \( p(x) = \tilde{p}_k(x) \), \( \lambda(x) = \tilde{\lambda}_k(x) \), \( f = \tilde{f}_k \), \( r = 7/8 \) and \( \rho = 3/4 \). That is,

\[
\int_{B_{3/4}} |\nabla w_k| \bar{p}_k(x) \, dx \leq C + C \int_{B_{3/4}} |\nabla \tilde{v}_k| \tilde{p}_k(x) \, dx,
\]

where \( C = C(p_{\min}, p_{\max}, N, \lambda_{\max}) \). Therefore (3.49) and (3.48) give, for \( k \) large, a uniform bound for the right hand side in (3.45). That is,

\[
\|G_k\|_{L^2((\tilde{p}_k(\cdot)))(A_k)} \leq \bar{C} \text{ with } \bar{C} = \bar{C}(p_{\min}, p_{\max}, M_0, L_1, L, M_1, N, \lambda_{\max}),
\]

with \( C = \bar{C}(p_{\min}, p_{\max}, M_0, L_1, L, M_1, N, \lambda_{\max}) \).

Now, putting together (3.40), (3.42), (3.43), (3.50) and (3.30), we obtain

\[
\int_{B_{3/4}} |\nabla w_k - \nabla \tilde{v}_k| \bar{p}_k(x) \, dx \to 0.
\]

Thus, using Poincaré’s inequality (Theorem A.1) and Theorem A.2 we get (3.38).

In order to conclude the proof, we now observe that, by Corollary 3.1 there exists \( 0 < \gamma < 1 \), \( \gamma = \gamma(N, p_{\min}) \), such that

\[
\|w_k\|_{C^\gamma(B_{1/2})} \leq C \text{ with } C = C(p_{\min}, p_{\max}, M_0, L_1, L, M_1, N, \lambda_{\max})
\]

(recall that \( \|w_k\|_{L^\infty(B_1)} \leq 2 \)).

Therefore, there is a function \( w_0 \in C^\gamma(B_{1/2}) \) such that, for a subsequence,

\[
w_k \to w_0 \text{ uniformly in } B_{1/2}.
\]

In addition, recalling (3.35), (3.37) and (3.38), we get \( v_0 = w_0 \) in \( B_{1/2} \) and \( \Delta p_w = 0 = 0 \) in \( B_{1/2} \). Finally, since there holds that \( w_k \geq 0 \), \( w_k(0) = 0 \) and (3.28), now (3.52) implies

\[
w_0 \geq 0, \quad w_0(0) = 0, \quad \max_{B_{1/2}} w_0 \geq c > 0,
\]

which contradicts the strong minimum principle and concludes the proof.

We can now prove the Lipschitz continuity of nonnegative local minimizers

**Corollary 3.2.** Let \( p, f, \lambda \) and \( u \) be as in Lemma 3.3. Then \( u \) is locally Lipschitz continuous in \( \Omega \). Moreover, for any \( \Omega' \subset \subset \Omega \) the Lipschitz constant of \( u \) in \( \Omega' \) can be estimated by a constant \( C \) depending only on \( N, p_{\min}, p_{\max}, L_1, \lambda_{\max} \), \( \|u\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)} \) and \( \text{dist}(\Omega', \partial \Omega) \).

**Proof.** The result is a consequence of Corollary 3.1 Lemma 3.3 and Theorem 3.3 above, and Proposition 2.1 in [25].

Next we have

**Theorem 3.4.** Let \( p, f, \lambda \) and \( u \) be as in Lemma 3.3. Assume moreover that \( \nabla u \in L^\infty(\Omega) \). There exist positive constants \( c_0 \) and \( \rho \) such that, for every \( x \in \Omega' \),

\[
u(x) \geq c_0 \text{dist}(x, \{u \equiv 0\}), \quad \text{if } \text{dist}(x, \{u \equiv 0\}) \leq \rho.
\]

The constants depend only on \( p_{\min}, p_{\max}, L, \|f\|_{L^\infty(\Omega)}, \lambda_{\min}, \lambda_{\max}, \|\nabla u\|_{L^\infty(\Omega)} \) and \( \text{dist}(\Omega', \partial \Omega) \).
We will prove the statement for $x \in \Omega'$ such that $u(x) > 0$ (otherwise there is nothing to prove). Let us suppose by contradiction that there exist a sequence of nonnegative local minimizers $u_k \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ corresponding to functionals $J_k$ given by functions $p_k, f_k$ and $\lambda_k$, with $p_{\min} \leq p_k(x) \leq p_{\max}$, $\|\nabla p_k\|_{L^\infty} \leq L$, $\|f_k\|_{L^\infty(\Omega)} \leq L_1$, $\lambda_{\min} \leq \lambda_k(x) \leq \lambda_{\max}$, $\|\nabla u_k\|_{L^\infty(\Omega)} \leq L_2$ and points $x_k \in \Omega'$, with $u_k(x_k) > 0$, such that
\[
d_k = \text{dist}(x_k, \{u_k = 0\}) \to 0 \quad \text{and} \quad \frac{u_k(x_k)}{d_k} \to 0.
\]

Let us define in $B_1$, for $d_k$ small, $w_k(x) = \frac{1}{d_k} u_k(x_k + d_k x)$, $\bar{p}_k(x) = p_k(x_k + d_k x)$, $\bar{f}_k(x) = d_k f_k(x_k + d_k x)$ and $\bar{\lambda}_k(x) = \lambda_k(x_k + d_k x)$. Then $p_{\min} \leq \bar{p}_k(x) \leq p_{\max}$, $\|\nabla \bar{p}_k\|_{L^\infty(B_1)} \leq L\delta_k$, $\lambda_{\min} \leq \bar{\lambda}_k(x) \leq \lambda_{\max}$ and $\|\bar{f}_k\|_{L^\infty(B_1)} \leq L_1\delta_k$. Moreover, $w_k$ is a nonnegative local minimizer of the functional
\[
\bar{J}_k(v) = \int_{B_1} \left( \frac{\nabla \|\bar{p}_k(x)\| \|\nabla v\|}{\bar{p}_k(x)} + \bar{\lambda}_k(x) \chi_{\{v > 0\}} + \bar{f}_k v \right) dx.
\]

Since $w_k > 0$ in $B_1$, we have $\Delta \bar{p}_k w_k = \bar{f}_k$ in $B_1$ (see (3.18)). In addition, $w_k(0) = \frac{u_k(x_k)}{d_k} \to 0$ and $\|\nabla w_k\|_{L^\infty(B_1)} \leq L_2$. Then, by interior Hölder gradient estimates it follows that, for a subsequence, $w_k \to w_0$ and $\nabla w_k \to \nabla w_0$ uniformly on compact subsets of $B_1$. Moreover, for a subsequence, $\bar{f}_k \to 0$ and $p_k \to p_0$ uniformly on compact subsets of $B_1$, with $p_0$ constant. This implies that $\Delta_{p_0} w_0 = 0$ in $B_1$.

By Harnack’s inequality there exists a constant $\tau > 0$, depending on $N$ and $p_0$, such that
\[
\sup_{B_1/2} w_0 \leq \tau \inf_{B_1/2} w_0
\]
and therefore, given $\delta > 0$, there exists $k_0$ such that for $k \geq k_0$
\[
\sup_{B_1/2} w_k \leq \tau \inf_{B_1/2} w_k + C_0\delta,
\]
for a constant $C_0$ depending on $N$ and $p_0$. In particular we have, for $k$ large,
\[
w_k(x) \leq \tau w_k(0) + C_0\delta \quad \text{in} \ B_1/2.
\]

Let $\alpha_k > 0$ be such that $u_k(x_k) = \alpha_k d_k$, this is, $\alpha_k = w_k(0)$. Let $\psi \in C^\infty(\overline{B_1})$ such that $\psi \equiv 0$ in $B_1/4$, $\psi \equiv 1$ in $\overline{B_1} \setminus B_1/2$, $0 \leq \psi \leq 1$ and let
\[
z_k(x) = \min\left( \frac{w_k(x)}{\alpha_k + C_0\delta}, \psi \right) \quad \text{in} \ B_1/2,
\]
outside $B_1/2$.

Then, $z_k \in W^{1,p_k}(B_1)$ and $z_k$ coincides with $w_k$ on $\partial B_1$ so that there holds that $\bar{J}_k(z_k) \geq \bar{J}_k(w_k)$.

Let $D_k = B_1/2 \cap \{w_k > (\tau\alpha_k + C_0\delta)\psi\}$. Observe that $z_k \leq w_k$, so that $\chi_{\{z_k > 0\}} \leq \chi_{\{w_k > 0\}}$. In addition, $w_k > 0$ in $B_1/4$, $z_k = 0$ in $B_1/4$ and $B_1/4 \subset D_k$. Therefore, if $C_0\delta \leq \frac{1}{2}$ and $k$ is large enough so that $\tau\alpha_k \leq \frac{1}{2}$, we get
\[
\lambda_{\min|B_1/4} \leq \int_{D_k} \bar{\lambda}_k(x) \{\chi_{\{w_k > 0\}} - \chi_{\{z_k > 0\}}\} \, dx
\]
\[
\leq \int_{D_k} \frac{(\tau\alpha_k + C_0\delta) p_{\min}}{p_{\min}} \|\nabla \psi\| \|\nabla w_k\| + L_1 d_k \int_{D_k} \left[ (\tau\alpha_k + C_0\delta) \psi + w_k \right] \, dx \leq C(\tau\alpha_k + C_0\delta),
\]
with $C = C(\psi, p_{\min}, p_{\max}, L_1)$. So that
\[
\lambda_{\min|B_1/4} \leq C(\tau\alpha_k + C_0\delta),
\]
and, if \( CC_0 \delta \leq \frac{1}{2} \lambda_{\min}|B_{1/4}| \), it follows that
\[
\frac{1}{2} \lambda_{\min}|B_{1/4}| \leq C_{\alpha_k} = C \frac{u_k(x_k)}{d_k} \rightarrow 0,
\]
which is a contradiction. \( \square \)

We also have

**Lemma 3.4.** Let \( p \) and \( f \) be as in Theorem 3.4. Let \( \Omega' \subset \subset \Omega \) and \( u \in C(\Omega) \), \( u \geq 0 \), \( \nabla u \in L^\infty(\Omega) \) with \( \Delta_p(x)u = f \) in \( \{ u > 0 \} \) be such that there exist positive constants \( c_0 \) and \( \rho \) such that, for every \( x \in \Omega' \), there holds that \( u(x) \geq c_0 \text{dist}(x, \{ u \equiv 0 \}) \) if \( \text{dist}(x, \{ u \equiv 0 \}) \leq \rho \). Then, there exist positive constants \( \delta_0 \) and \( \rho_0 \) such that for every \( x \in \Omega' \cap \{ u > 0 \} \) with \( d(x) = \text{dist}(x, \{ u \equiv 0 \}) \leq \rho_0 \), we have
\[
\sup_{B_{\rho_0}(x)} u \geq (1 + \delta_0)u(x).
\]
The constants depend only on \( p_{\min}, p_{\max}, L, ||f||_{L^\infty(\Omega)}, ||\nabla u||_{L^\infty(\Omega)}, c_0, \rho \) and \( \text{dist}(\Omega', \partial \Omega) \).

**Proof.** Suppose by contradiction that there exist functions \( u_k, p_k, f_k \), with \( 1 < p_{\min} \leq p_k(x) \leq p_{\max} < \infty \), \( ||\nabla p_k||_{L^\infty(\Omega)} \leq L \), \( ||f_k||_{L^\infty(\Omega)} \leq L_1 \), \( u_k \in C(\Omega) \), \( u_k \geq 0 \), \( ||\nabla u_k||_{L^\infty(\Omega)} \leq L_2 \), with \( \Delta_{p_k(x)}u_k = f_k \) in \( \{ u_k > 0 \} \) and \( u_k(x) \geq c_0 \text{dist}(x, \{ u_k \equiv 0 \}) \) if \( \text{dist}(x, \{ u_k \equiv 0 \}) \leq \rho \) and \( x \in \Omega' \), and sequences \( \delta_k \rightarrow 0 \), \( \rho_k \rightarrow 0 \) and \( x_k \in \Omega' \cap \{ u_k > 0 \} \) with \( d_k = \text{dist}(x_k, \{ u_k \equiv 0 \}) \leq \rho_k \) such that
\[
\sup_{B_{\rho_k}(x_k)} u_k \leq (1 + \delta_k)u_k(x_k).
\]

Take \( w_k(x) = \frac{u_k(x_k + d_k x)}{u_k(x_k)} \). Then, \( w_k(0) = 1 \) and
\[
\max_{B_{\frac{1}{2}}(0)} w_k \leq (1 + \delta_k), \quad w_k > 0 \quad \text{and} \quad \text{div}\left(\left(\frac{u_k(x_k)}{d_k}\right)^{\bar{p}_k(x)-1}\nabla w_k \frac{\bar{p}_k(x)-2}{\nabla w_k}\right) = \tilde{f}_k \quad \text{in} \quad B_1,
\]
where \( \bar{p}_k(x) = p_k(x_k + d_k x) \) and \( \tilde{f}_k(x) = d_k f_k(x_k + d_k x) \). On the other hand, we have
\[
c_0 \leq \frac{u_k(x_k)}{d_k} \leq L_2, \quad ||\nabla w_k||_{L^\infty(B_1)} \leq L_2 \frac{d_k}{u_k(x_k)} \leq \frac{L_2}{c_0}.
\]

Then, using the gradient estimates in [14], we deduce that, for a subsequence, \( \frac{u_k(x_k)}{d_k} \rightarrow a \in [c_0, L_2] \), \( w_k \rightarrow \overline{w} \) and \( \bar{p}_k \rightarrow p_0 \in \mathbb{R} \) uniformly in \( B_1 \) and \( \nabla w_k \rightarrow \nabla \overline{w} \) uniformly on compact subsets of \( B_1 \).

There holds that \( \Delta_{p_0} \overline{w} = 0 \) in \( B_1 \), \( \overline{w}(0) = 1 \) and \( \overline{w} \leq 1 \) in \( B_1 \). Therefore \( \overline{w} \equiv 1 \) in \( B_1 \).

Let \( y_k \in \partial\{ u_k > 0 \} \) with \( |x_k - y_k| = d_k \). Then, if \( z_k = \frac{y_k - x_k}{d_k} \), we have
\[
w_k(z_k) = \frac{u_k(y_k)}{u_k(x_k)} = 0
\]
and we may assume that \( z_k \rightarrow \bar{z} \in \partial B_1 \). Thus, \( 1 = \overline{w}(\bar{z}) = 0 \). This is a contradiction, and the lemma is proved. \( \square \)

As a consequence of the previous results, we obtain

**Theorem 3.5.** Let \( p, f, \lambda \) and \( u \) be as in Theorem 3.4. Let \( \Omega' \subset \subset \Omega \). There exist constants \( c > 0 \), \( r_0 > 0 \) such that if \( x_0 \in \Omega' \cap \partial\{ u > 0 \} \) and \( r \leq r_0 \) then
\[
\sup_{B_r(x_0)} u \geq cr.
\]
The constants depend only on $p_{\min}, p_{\max}, L, ||f||_{L^\infty(\Omega)}, \lambda_{\min}, \lambda_{\max}, ||\nabla u||_{L^\infty(\Omega)}$ and $\text{dist}(\Omega', \partial\Omega)$. 

Proof. We will follow the ideas of Theorem 1.9 in [8].

Step 1. We will prove that there exist positive constants $\bar{c}, \bar{r}$ and $\bar{\rho}$ such that if $x_0 \in \Omega' \cap \{u > 0\}$, 
$$\text{dist}(x_0, \{u \equiv 0\}) \leq \bar{\rho} \text{ and } r \leq \bar{r},$$
then
$$\sup_{B_r(x_0)} u \geq \bar{c}r.$$

In fact, let $\rho_1 = \text{dist}(\Omega', \partial\Omega)$ and $\tilde{\Omega} = B_{\rho_1/2}(\Omega')$, so $\Omega' \subset\subset \tilde{\Omega} \subset\subset \Omega$. 

By Theorem 3.2 and Lemma 3.4 (applied to points in $\tilde{\Omega}$), there exist positive constants $c_0$ and $\rho$ such that, for every $x \in \tilde{\Omega}$ with $\text{dist}(x, \{u \equiv 0\}) \leq \rho$,
$$u(x) \geq c_0 \text{dist}(x, \{u \equiv 0\}),$$
and positive constants $\delta_0$ and $\rho_0$ such that for every $x \in \tilde{\Omega} \cap \{u > 0\}$ with $d(x) = \text{dist}(x, \{u \equiv 0\}) \leq \rho_0$, we have
$$\sup_{B_{d(x)}(x)} u \geq (1 + \delta_0)u(x).$$

The constants depend only on $p_{\min}, p_{\max}, L, ||f||_{L^\infty(\Omega)}, ||\nabla u||_{L^\infty(\Omega)}$, $\text{dist}(\tilde{\Omega}, \partial\Omega)$, $\lambda_{\min}$ and $\lambda_{\max}$.

Let $\bar{r} = \min\{\frac{1}{2}\text{dist}(\Omega', \partial\tilde{\Omega}), \rho, \rho_0\}$, $\bar{\rho} = \rho$ and $r \leq \bar{r}$. Let $x_0 \in \Omega' \cap \{u > 0\}$ such that $d_0 = \text{dist}(x_0, \{u \equiv 0\}) \leq \bar{\rho}$, then
$$u(x_0) \geq c_0d_0.$$

There are two possibilities:

i) $d_0 \geq \frac{\bar{r}}{2}$.

In this case $u(x_0) \geq c_0\frac{\bar{r}}{2}$ and the result follows.

ii) $d_0 < \frac{\bar{r}}{2}$.

In this case, proceeding as in [8], we construct a polygonal that never leaves $B_r(x_0)$, starting at $x_0$ and finishing at $\tilde{x} \in B_r(x_0)$, such that $u(\tilde{x}) \geq \bar{c}r$, with an explicit $\bar{c} > 0$ depending on the constants mentioned above. We refer to [8] for the details. In the present situation, the mean value argument employed in [8] is replaced by the argument in Lemma 3.4.

Step 2. Now let $\bar{r}$ and $\bar{\rho}$ as above, $r \leq \bar{r}$ and $x_0 \in \Omega' \cap \partial\{u > 0\}$. We take $x_1 \in B_{\bar{r}}(x_0) \cap \{u > 0\} \cap \Omega'$. Then
$$\text{dist}(x_1, \{u \equiv 0\}) \leq |x_1 - x_0| \leq \bar{\rho} \text{ and thus, from the result in Step 1,}$$
$$\sup_{B_r(x_0)} u \geq \sup_{B_{\bar{r}}(x_1)} u \geq \frac{\bar{c}r}{2}.$$  

This completes the proof. \qed

The following result in the section is

**Theorem 3.6.** Let $p, f, \lambda$ and $u$ be as in Theorem 3.4. Let $\Omega' \subset\subset \Omega$. There exist constants $ar{c} \in (0, 1)$ and $\bar{r}_0 > 0$ such that, if $x_0 \in \Omega' \cap \partial\{u > 0\}$ with $B_r(x_0) \subset\subset \Omega'$ and $r \leq \bar{r}_0$, there holds
$$\frac{|B_r(x_0) \cap \{u > 0\}|}{|B_r(x_0)|} \leq 1 - \bar{c}.$$

The constants depend only on $p_{\min}, p_{\max}, L, ||f||_{L^\infty(\Omega)}, \lambda_{\min}, \lambda_{\max}, ||\nabla u||_{L^\infty(\Omega)}$ and $\text{dist}(\Omega', \partial\Omega)$. 

**Proof.** Let us suppose by contradiction that there exist a sequence of nonnegative local minimizers $u_k \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ corresponding to functionals $J_k$ given by functions $p_k$, $f_k$ and $\lambda_k$, with
We have

\[ \max_{\partial \{ \bar{u}_k > 0 \}} u_k \geq c \sigma , \quad \text{for } 0 < \sigma < 1, \]

where \( c \) is the positive constant given by Theorem 3.5.

Let \( \bar{u}_k(x) = \frac{u_k(x_k + r_k x)}{r_k} \), \( \bar{\rho}_k(x) = p_k(x_k + r_k x) \) and \( f_k(x) = r_k f_k(x_k + r_k x) \). Then \( p_{\min} \leq \bar{\rho}_k(x) \leq p_{\max}, \| \bar{\rho}_k \|_{L^\infty(B_1)} \leq L r_k, \| f_k \|_{L^\infty(B_1)} \leq L_1 r_k, 0 \in \partial \{ \bar{u}_k > 0 \}, \)

\[ |B_1 \cap \{ \bar{u}_k = 0 \}| = \varepsilon_k \to 0, \]

and

\[ \Delta \bar{\rho}_k(x) \bar{u}_k \geq \bar{f}_k \quad \text{in } B_{1/2}. \]

Let us take \( v_k \in W^1 \bar{\rho}_k^{-1}(B_{1/2}), \) such that

\[ \Delta \bar{\rho}_k(x) v_k = \bar{f}_k \quad \text{in } B_{1/2}, \quad v_k - \bar{u}_k \in W^1_{0} \bar{\rho}_k^{-1}(B_{1/2}). \]

Observe that there holds that \( \| \bar{u}_k \|_{L^\infty(B_{1/2})} \leq L_2/2 \) implying that

\[ \text{this estimate follows from Lemma 3.2 and Remark 3.4 if } k \text{ is large enough}. \]

Since \( v_k \geq \bar{u}_k \) then \( 0 \leq \chi_{\{ \bar{v}_k > 0 \}} - \chi_{\{ \bar{u}_k > 0 \}} \leq \chi_{\{ \bar{u}_k = 0 \}} \) and therefore, using that \( \bar{u}_k \) are nonnegative local minimizers, we get

\[ \left( \frac{\nabla \bar{u}_k}{\bar{\rho}_k(x)} - \frac{\nabla v_k}{\bar{\rho}_k(x)} \right) \leq \lambda_{\max} |B_1 \cap \{ \bar{u}_k = 0 \}| + L_1 r_k \int_{B_{1/2}} |\bar{u}_k - v_k|. \]

Applying (3.52), we now obtain

\[ \int_{B_{1/2}} \left( \frac{\nabla \bar{u}_k}{\bar{\rho}_k(x)} - \frac{\nabla v_k}{\bar{\rho}_k(x)} \right) \leq C(\varepsilon_k + L_1 r_k). \]

We claim that

\[ \int_{B_{1/2}} |\nabla \bar{u}_k - \nabla v_k| \bar{\rho}_k(x) dx \to 0. \]

In fact, let \( u^*(x) = s \bar{u}_k(x) + (1 - s) v_k(x) \). By using (3.54) and the inequalities in (1.4), we get

\[ \int_{B_{1/2}} \frac{\nabla \bar{u}_k}{\bar{\rho}_k(x)} - \frac{\nabla v_k}{\bar{\rho}_k(x)} + \int_{B_{1/2}} \bar{f}_k(\bar{u}_k - v_k) = \]

\[ \int^{1}_{0} \frac{ds}{s} \int_{B_{1/2}} \left( \nabla u^s \bar{\rho}_k(x) - \nabla v_k \bar{\rho}_k(x) \right) \cdot \nabla (u^s - v_k) \geq \]

\[ C \left( \int_{B_{1/2} \cap \{ \bar{\rho}_k \geq 2 \}} \nabla \bar{u}_k - \nabla v_k \bar{\rho}_k(x) + \int_{B_{1/2} \cap \{ \bar{\rho}_k < 2 \}} |\nabla \bar{u}_k - \nabla v_k| \left( |\nabla \bar{u}_k| + |\nabla v_k| \right) \bar{\rho}_k(x)^{-2} \right). \]
Now (3.57) implies
\[ \int_{\{ \bar{p}_k \geq 2 \} \cap B_{1/2}} |\nabla \bar{u}_k - \nabla v_k|^{\bar{p}_k(x)} \, dx \leq \tilde{C}(\varepsilon_k + L_1 r_k) \quad \text{and} \]
\[ \int_{\{ \bar{p}_k < 2 \} \cap B_{1/2}} \left( |\nabla \bar{u}_k| + |\nabla v_k| \right)^{\bar{p}_k(x)-2} |\nabla \bar{u}_k - \nabla v_k|^2 \, dx \leq \tilde{C}(\varepsilon_k + L_1 r_k). \]
From these inequalities we obtain, reasoning as in the proof of Theorem 5.1 in [5],
\[ \int_{B_{1/2}} |\nabla \bar{u}_k - \nabla v_k|^{\bar{p}_k(x)} \, dx \leq C \max \{ \varepsilon_k + L_1 r_k, (\varepsilon_k + L_1 r_k)^{\bar{p}_{\text{min}}/2} \} \]
and thus, (3.58) follows.

On the other hand, by interior Hölder gradient estimates, there holds that, for a subsequence, \( v_k \to v_0 \) and \( \nabla v_k \to \nabla v_0 \) uniformly on compact subsets of \( B_{1/2} \). Since \( \| \nabla \bar{p}_k \|_{L^\infty(B_1)} \leq L r_k \), there exists a constant \( p_0 \) such that (for a subsequence) \( \bar{p}_k \to p_0 \) uniformly in \( B_{1/2} \).

Finally, since \( \| \nabla \bar{u}_k \|_{L^\infty(B_{1/2})} \leq L_2 \) we have, for a subsequence, \( \bar{u}_k \to u_0 \) uniformly in \( B_{1/2} \).

Let \( w_k = \bar{u}_k - v_k \). Then, \( w_k \to u_0 - v_0 \) uniformly on compact subsets of \( B_{1/2} \). By (3.58) we have that \( \| \nabla w_k \|_{L^{\bar{p}_k}(B_{1/2})} \to 0 \). Since \( u_k \in W^1_0, \bar{p}_k(B_{1/2}) \), by Poincaré’s inequality (Theorem A.4) we get that \( \| u_k \|_{L^{p_0}(B_{1/2})} \to \| u_0 - v_0 \|_{L^{p_0}(B_{1/2})} = 0 \). Thus, \( u_0 = v_0 \).

Now, using that \( v_k \to u_0 \) locally in \( C^1(B_{1/2}) \) and \( \tilde{f}_k \to 0 \) uniformly in \( B_{1/2} \), we deduce that \( \Delta_{p_0} u_0 = \Delta_{p_0} v_0 = 0 \) in \( B_{1/2} \).

As \( \bar{u}_k \to u_0 \) uniformly in \( B_{1/2} \) we get, by (3.58), that \( \sup_{B_{1/4}} u_0 \geq \frac{\varepsilon}{2} \). But \( u_0(0) = \lim \bar{u}_k(0) = 0 \) and \( u_0 \geq 0 \). By the strong maximum principle we arrive at a contradiction and the result follows. \( \square \)

We devote the last part of the section to discuss the fulfillment of properties (3) and (4) in the definition of weak solution for nonnegative local minimizers of (1.1).

We need

**Definition 3.2.** Let \( p, f \) and \( \lambda \) be as in Definition 3.1 and let \( u \in W^{1,p} + \delta_0(\Omega) \), for some \( \delta_0 > 0 \). For an open set \( D \subset \Omega \) let
\[ J_D^{p,\lambda, f}(u) = J_D(v) = \int_D \left( |\nabla v|^{p(x)} \frac{p(x)}{p(x)} + \lambda(x) \chi_{\{v > 0\}} + f v \right) \, dx. \]
We say that \( u \) is a mild minimizer of \( J \) in \( \Omega \) if for every \( B_r(x_0) \subset \subset \Omega \) and \( v \in W^{1,p} + \delta_0(\Omega) \) with \( v - u \in W^1_0, p(x) + \delta(B_r(x_0)) \), for some \( 0 < \delta < \delta_0 \),
\[ J_{B_r(x_0)}(u) \leq J_{B_r(x_0)}(v). \]

We have the following results for mild minimizers

**Proposition 3.2.** Let \( p, f \) and \( \lambda \) be as in Theorem 3.1. Assume moreover that \( \lambda \in C(\Omega) \). Let \( u \) be a nonnegative Lipschitz mild minimizer of \( J \) in \( \Omega \). Let \( x_k \in \Omega \cap \partial \{ u > 0 \} \), \( x_k \to x_0 \in \Omega \), \( p_k \to 0 \) and \( u_k(x) = \frac{u(x + p_k x)}{p_k} \). Assume that \( u_k \to u_0 \) uniformly on compact sets of \( \mathbb{R}^N \). Then \( u_0 \) is a nonnegative Lipschitz mild minimizer of \( J \) in \( \mathbb{R}^N \), with \( p(x) \equiv p(x_0), \lambda(x) \equiv \lambda(x_0) \) and \( f \equiv 0 \).
Proof. Let \( B_r = B_r(x_0) \) be any ball in \( \mathbb{R}^N \) and assume for simplicity that \( x_0 = 0 \). Denote \( p_k(x) = p(x_k + p_k x), \) \( p_0 = p(x_0), \) \( \lambda_k(x) = \lambda(x_k + p_k x), \) \( \lambda_0 = \lambda(x_0), \) \( f_k(x) = p_k f(x_k + p_k x) \) and

\[
J_{r,k}(v) = \int_{B_r} \left( \frac{\nabla v|p_k(x)}{p_k(x)} + \lambda_k(x) \chi_{\{v > 0\}} + f_k v \right) dx,
\]

\[
J_{r,0}(v) = \int_{B_r} \left( \frac{\nabla v|p_0}{p_0} + \lambda_0 \chi_{\{v > 0\}} \right) dx.
\]

Let \( v \in W^{1,p_0+\delta}(B_r) \) with \( v - u_0 \in W^{1,\tilde{p}_0+\delta}(B_r) \) for some \( \delta > 0 \). We want to show that

(3.59) \[
J_{r,0}(u_0) \leq J_{r,0}(v).
\]

For \( h > 0 \) small, we define

\[
v_{h,k} = \begin{cases} v & \text{in } B_r, \\ u_0 + \frac{|x| - r}{h}(u_k - u_0) & \text{in } B_{r+h} \setminus B_r. \end{cases}
\]

Then, since \( p_k \leq p_0 + \delta/2 \) in \( B_{r+h} \) for \( k \) large, it follows that \( v_{h,k} \in W^{1,\tilde{p}_0+\delta/2}(B_{r+h}), v_{h,k} - u_k \in W^{1,\tilde{p}_0+\delta/2}(B_{r+h}), \) for \( k \) large, and there holds

\[
J_{r+h,k}(v_{h,k}) = \int_{B_{r+h}} \left( \frac{\nabla v_{h,k}|p_k(x)}{p_k(x)} + \lambda_k(x) \chi_{\{v_{h,k} > 0\}} + f_k v_{h,k} \right) dx
\]

\[
J_{r,0}(v) + \int_{B_{r+h} \setminus B_r} \left( \frac{\nabla v_{h,k}|p_k(x)}{p_k(x)} + \lambda_k(x) \chi_{\{v_{h,k} > 0\}} + f_k v_{h,k} \right) dx
\]

\[
\int_{B_r} \left( \frac{\nabla v|p_k(x)}{p_k(x)} - \frac{\nabla v|p_0}{p_0} + (\lambda_k(x) - \lambda_0) \chi_{\{v > 0\}} + f_k v \right) dx \leq J_{r,0}(v) + C_0 hr^{N-1} +
\]

\[
C_1 \int_{B_{r+h} \setminus B_r} \left( u_k - u_0 \right) \frac{|p_k(x)|}{p_k(x)} + \int_{B_r} \left( \frac{\nabla v|p_k(x)}{p_k(x)} - \frac{\nabla v|p_0}{p_0} + (\lambda_k(x) - \lambda_0) \chi_{\{v > 0\}} + f_k v \right).
\]

Therefore,

(3.60) \[
\limsup_{k \to \infty} J_{r+h,k}(v_{h,k}) \leq J_{r,0}(v) + C_0 hr^{N-1}.
\]

On the other hand, \( \lambda_0 \chi_{\{u_0 > 0\}} \leq \liminf_{k \to \infty} \lambda_k(x) \chi_{\{u_k > 0\}} \), which implies

(3.61) \[
\int_{B_r} \lambda_0 \chi_{\{u_0 > 0\}} dx \leq \liminf_{k \to \infty} \int_{B_r} \lambda_k(x) \chi_{\{u_k > 0\}} dx.
\]

In addition, since \( \nabla u_k \rightharpoonup \nabla u_0 \) weakly in \( L^{p_0}(B_r) \), arguing in a similar way as in Theorem 3.1, we get

(3.62) \[
\int_{B_r} |\nabla u_0| dx \leq \liminf_{k \to \infty} \int_{B_r} |\nabla u_k| dx = \liminf_{k \to \infty} \int_{B_r} \frac{|\nabla u_k|}{p_0} dx.
\]

Now, using (3.61) and (3.62), and the fact that \( u_k \) are nonnegative Lipschitz mild minimizers of \( J \) with \( p(x) = p_k(x), \lambda(x) = \lambda_k(x) \) and \( f(x) = f_k(x) \) we obtain

\[
J_{r,0}(u_0) \leq \liminf_{k \to \infty} J_{r,k}(u_k) \leq \liminf_{k \to \infty} J_{r+h,k}(u_k) + C_2 hr^{N-1} \leq \liminf_{k \to \infty} J_{r+h,k}(v_{h,k}) + C_2 hr^{N-1},
\]
which in combination with (3.60) gives
\[ J_{\tau,0}(u_0) \leq J_{\tau,0}(v) + C_3 h r^{N-1}. \]
Therefore, letting \( h \to 0 \) we obtain (3.59).

We will need

**Proposition 3.3.** Let \( 1 < p_0 \) and \( \lambda_0, \alpha \) be positive constants. Let \( u \) be a Lipschitz mild minimizer of \( J \) in \( \mathbb{R}^N \), with \( p(x) \equiv p_0 \), \( \lambda(x) \equiv \lambda_0 \) and \( f \equiv 0 \). Assume that \( u = \alpha x_1^+ \) in \( B_{r_0} \), for some \( r_0 > 0 \). Then,
\[
\alpha = \left( \frac{p_0}{p_0 - 1} \lambda_0 \right)^{1/p_0}.
\]

**Proof.** Let \( \varepsilon > 0 \) small, let \( \tau_{\varepsilon}(x) = x + \varepsilon \phi(|x|) e_1 \) with \( \phi \in C_0^\infty(-r_0, r_0) \), and let \( u_{\varepsilon}(x) = u(\tau_{\varepsilon}^{-1}(x)) \).

Then, \( u_{\varepsilon} \in W^{1,p_0+\delta}(B_{r_0}) \) with \( u_{\varepsilon} - u \in W_0^{1,p_0+\delta}(B_{r_0}) \), for some \( \delta > 0 \), which implies that
\[
0 \leq J_{\tau,0}(u_{\varepsilon}) - J_{\tau,0}(u),
\]
for
\[
J_{\tau,0}(v) = \int_{B_{r_0}} \left( \frac{\nabla v|_{p_0}}{p_0} + \lambda_0 \chi_{\{v>0\}} \right) dx.
\]

We now proceed as in Lemma 7.3 in [27]. In fact, there it is proved an analogous result with \( J_{\tau,0} \) replaced by
\[
J(v) = \int_{B_{r_0}} \left( G(|\nabla v|) + \lambda \chi_{\{v>0\}} \right) dx,
\]
for a general \( G \) and a positive constant \( \lambda \), and it is shown that
\[
G'(\alpha) \alpha - G(\alpha) = \lambda.
\]
Since in our case we have \( G(t) = \frac{tp_0}{p_0} \) and \( \lambda = \lambda_0 \), [27] applies and thus (3.63) yields
\[
\alpha^{p_0} - \frac{\alpha^{p_0}}{p_0} = \lambda_0,
\]
which gives the desired result. \( \square \)

Next we prove

**Theorem 3.7.** Let \( p, f, \lambda \) and \( u \) be as in Lemma 3.3. Assume moreover that \( \lambda \in C(\Omega) \). Let \( x_0 \in \Omega \cap \partial\{u > 0\} \). Then,
\[
\limsup_{x \to x_0 \atop u(x) > 0} |\nabla u(x)| = \lambda^*(x_0),
\]
where \( \lambda^*(x) = \left( \frac{p(x)}{p(x)-1} \lambda(x) \right)^{1/p(x)} \).

**Proof.** Let
\[
\alpha := \limsup_{x \to x_0 \atop u(x) > 0} |\nabla u(x)|.
\]
Since \( u \in Lip_{loc}(\Omega), 0 \leq \alpha < \infty \). By the definition of \( \alpha \) there exists a sequence \( z_k \to x_0 \) such that
\[
u(z_k) > 0, \quad |\nabla u(z_k)| \to \alpha.
\]
Let \( y_k \) be the nearest point from \( z_k \) to \( \Omega \cap \partial\{u > 0\} \) and let \( d_k = |z_k - y_k| \).

Consider the blow up sequence \( u_{d_k} \) with respect to \( B_{d_k}(y_k) \). That is, \( u_{d_k}(x) = \frac{1}{d_k} u(y_k + d_k x) \).
Since \( u \) is locally Lipschitz, and \( u_{d_k}(0) = 0 \) for every \( k \), there exists \( u_0 \), with \( u_0(0) = 0 \), such that
Let $\lambda$ be contained in $\partial B$. Choose a subsequence with blow up limit $\lambda_0$, such that $(\lambda_0)_\tau(x) = \frac{1}{\gamma} u_0(\gamma x)$. There exist a sequence $\gamma_n \to 0$ and $u_0 \in Lip(\mathbb{R}^N)$ such that $(\lambda_0)_\tau \to u_0$ uniformly on compact sets of $\mathbb{R}^N$.  

Next, define for $\gamma > 0$, $(u_0)_\gamma(x) = \frac{1}{\gamma} u_0(\gamma x)$. There exist a sequence $\gamma_n \to 0$ and $u_0 \in Lip(\mathbb{R}^N)$ such that $(\lambda_0)_\tau \to u_0$ uniformly on compact sets of $\mathbb{R}^N$.  

Now, since $u$ is a nonnegative local minimizer of functional $J$ in $\Omega$, then $u$ is locally Lipschitz and it is a nonnegative mild minimizer of $J$ in $\Omega$. Thus, applying Proposition 3.2 to $u$ and to the blow up sequence $u_k$, we get that $u_0$ is a nonnegative Lipschitz mild minimizer of $J$ in $\mathbb{R}^N$, with $p(x) \equiv p(x_0)$, $\lambda(x) \equiv \lambda(x_0)$ and $f \equiv 0$.  

Then, applying again Proposition 3.2 now to $u_0$ and to the blow up sequence $(u_0)_\gamma$, we also get that $u_0(x) = \alpha x_1^+$ is a nonnegative Lipschitz mild minimizer of $J$ in $\mathbb{R}^N$, with $p(x) \equiv p(x_0)$, $\lambda(x) \equiv \lambda(x_0)$ and $f \equiv 0$.  

Thus, using Proposition 3.3 we get that $\alpha = \lambda^*(x_0)$. 

Our next result is

**Theorem 3.8.** Let $p, f, \lambda$ and $u$ be as in Theorem 3.7. Let $x_0 \in \Omega \cap \partial \{ u > 0 \}$. Assume there is a ball $B$ contained in $\{ u = 0 \}$ touching $x_0$, then

\begin{equation}
\limsup_{x \to x_0 \atop \{ u(x) > 0 \}} \frac{u(x)}{\text{dist}(x, B)} = \lambda^*(x_0),
\end{equation}

where $\lambda^*(x) = \left( \frac{p(x)}{p(x)} \right)^{1/p(x)} \lambda(x)$. 

**Proof.** Let $\ell$ be the finite limit on the left hand side of (3.64) and let $y_k \to x_0$ with $u(y_k) > 0$ be such that

\[
\frac{u(y_k)}{d_k} \to \ell, \quad d_k = \text{dist}(y_k, B).
\]

Consider the blow up sequence $u_k$ with respect to $B_{d_k}(x_k)$, where $x_k \in \partial B$ are points with $|x_k - y_k| = d_k$, that is, $u_k(x) = \frac{u(x_k + d_k e_k)}{d_k}$. Choose a subsequence with blow up limit $u_0$, such that there exists

\[
e := \lim_{k \to \infty} \frac{y_k - x_k}{d_k}.
\]

Using Lemma 3.3 and Theorem 3.5 and proceeding as in the proof of Theorem 5.2 in [24] we have that $u_0(x) = \ell(x, e)^+$. Thus, applying Propositions 3.2 and 3.3 we get that $\ell = \lambda^*(x_0)$. 

The last result in this section is

**Theorem 3.9.** Let $p, f, \lambda$ and $u$ be as in Theorem 3.7. Let $x_0 \in \Omega \cap \partial \{ u > 0 \}$ be such that $\partial \{ u > 0 \}$ has at $x_0$ an inward unit normal $\nu$ in the measure theoretic sense. Then,

\[u(x) = \lambda^*(x_0) \langle x - x_0, \nu \rangle^+ + o(|x - x_0|),\]
where $\lambda^*(x) = \left( \frac{p(x)}{p(x)-1} \lambda(x) \right)^{1/(p(x))}$. 

**Proof.** Take $u_\lambda(x) = \frac{1}{\lambda} u(x_0 + \lambda x)$. Let $\rho > 0$ such that $B_\rho(x_0) \subset \subset \Omega$. Since $u_\lambda \in \text{Lip}(B_{\rho/(\lambda)})$ uniformly in $x$, $u_\lambda(0) = 0$, there exist $\lambda_j \to 0$ and $U$ such that $u_{\lambda_j} \to U$ uniformly on compact sets of $\mathbb{R}^N$. Since $|\nabla u(x)| \leq L_0$ in $B_{\rho_j}(x_0)$ for some positive $L_0$ and $r_0$ then, for any $M > 0$, $|\nabla u_{\lambda_j}(x)| \leq L_0$ in $B_M(0)$ for $j$ large. Therefore, $|\nabla U(x)| \leq L_0$ in $\mathbb{R}^N$ and $U \in \text{Lip}(\mathbb{R}^N)$.

Without loss of generality we assume that $x_0 = 0$, and $\nu = e_1$. From Lemma 3.3, $\Delta_{p(\lambda x)} u_\lambda = \lambda f(\lambda x)$ in $\{u_\lambda > 0\}$. Using the fact that $e_1$ is the inward normal in the measure theoretic sense, we have, for fixed $k$,

$$\{(u_\lambda > 0) \cap \{x_1 < 0\} \cap B_k \to 0 \quad \text{as} \quad \lambda \to 0.$$ 

Hence, $U = 0$ in $\{x_1 < 0\}$. Moreover, $U$ is nonnegative in $\{x_1 > 0\}$, $\Delta_{p_0} U = 0$ in $\{U > 0\}$ with $p_0 = p(x_0)$ and $U$ vanishes in $\{x_1 \leq 0\}$. Then, by Lemma A.1 we have that there exists $\alpha \geq 0$ such that

$$U(x) = \alpha x_1^+ + o(|x|).$$ 

Define $U_\lambda(x) = \frac{1}{\lambda} U(\lambda x)$, then $U_\lambda \to \alpha x_1^+$ uniformly on compact sets of $\mathbb{R}^N$. 

Now, by Theorem 3.3 and Remark 2.1, we have, for some $c > 0$ and $0 < r < r_0$,

$$\frac{1}{r^N} \int_{B_r} u_{\lambda_j} \, dx \geq cr$$

and then

$$\frac{1}{r^N} \int_{B_r} U_\lambda \, dx \geq cr.$$

Therefore $\alpha > 0$. Now, since $u$ is a nonnegative local minimizer of functional $J$ in $\Omega$, then $u$ is locally Lipschitz and it is a nonnegative mild minimizer of $J$ in $\Omega$. Thus, by Proposition 3.2, $U$ is a nonnegative Lipschitz mild minimizer of $J$ in $\mathbb{R}^N$ with $p(x) \equiv p(x_0)$, $\lambda(x) \equiv \lambda(x_0)$ and $f \equiv 0$. Then, applying Proposition 3.2 to $U$ we get that $U_0 = \alpha x_1^+$ is also a nonnegative Lipschitz mild minimizer of $J$ in $\mathbb{R}^N$ with $p(x) \equiv p(x_0)$, $\lambda(x) \equiv \lambda(x_0)$ and $f \equiv 0$.

Now, by Proposition 3.3 $\alpha = \lambda^*(x_0)$.

We have shown that

$$U(x) = \begin{cases} \lambda^*(x_0)x_1 + o(|x|) & x_1 > 0 \\ 0 & x_1 \leq 0. \end{cases}$$

Then, using that $\Delta_{p(\lambda x)} u_\lambda = \lambda f(\lambda x)$ in $\{u_\lambda > 0\}$, by interior Hölder gradient estimates we have $\nabla u_{\lambda_j} \to \nabla U$ uniformly on compact subsets of $\{U > 0\}$. Then, by Theorem 3.7, $|\nabla U| \leq \lambda^*(x_0)$ in $\mathbb{R}^N$. As $U = 0$ on $\{x_1 = 0\}$ we have, $U \leq \lambda^*(x_0)x_1$ in $\{x_1 > 0\}$.

Now, proceeding as in the proof of Theorem 5.3 in [21], we conclude that $U \equiv \lambda^*(x_0)x_1^+$ and the result follows. 

### 4. Energy Minimizers of Energy Functional (1.2)

In this section we prove existence of minimizers of the energy functional (1.2) and, in the spirit of the previous section, we develop an exhaustive analysis of the essential properties of functions $u^\varepsilon$ which are nonnegative local minimizers of that energy. As a consequence we obtain results for solutions $u^\varepsilon$ to the singular perturbation problem $P_\varepsilon(f^\varepsilon, p_\varepsilon)$ which are nonnegative local energy minimizers and moreover, we get results for their limit functions $u$.

We start by pointing out that the same considerations in Definition 3.1 and Remarks 3.1 and 3.2 for functional (1.1) apply to functional (1.2) in the present section.
We first obtain

**Theorem 4.1.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and let \( \phi_\varepsilon \in W^{1,p_\varepsilon} (\Omega) \) be such that \( \| \phi_\varepsilon \|_{1,p_\varepsilon} \leq A_1 \), with \( 1 < p_{\text{min}} \leq p_\varepsilon (x) \leq p_{\text{max}} < \infty \) and \( \| \nabla p_\varepsilon \|_{L^\infty} \leq L \). Let \( f_\varepsilon \in L^\infty (\Omega) \) such that \( \| f_\varepsilon \|_{L^\infty (\Omega)} \leq A_2 \). There exists \( u_\varepsilon \in W^{1,p_\varepsilon} (\Omega) \) that minimizes the energy

\[
J_\varepsilon (v) = \int_{\Omega} \left( \frac{\nabla v}{p_\varepsilon (x)} + B_\varepsilon (v) + f_\varepsilon v \right) dx
\]

among functions \( v \in W^{1,p_\varepsilon} (\Omega) \) such that \( v - \phi_\varepsilon \in W^{1,p_\varepsilon} (\Omega) \). Here \( B_\varepsilon (s) = \int_{\varepsilon}^{\varepsilon} \beta_\varepsilon (\tau) d\tau \).

Then, the function \( u_\varepsilon \) satisfies

\[
\Delta_{p_\varepsilon (x)} u_\varepsilon = \beta_\varepsilon (u_\varepsilon) + f_\varepsilon \quad \text{in} \quad \Omega
\]

and for every \( \Omega' \subset \subset \Omega \) there exists \( C = C (\Omega', A_1, A_2, p_{\text{min}}, p_{\text{max}}, L) \) such that

\[
\sup_{\Omega'} u_\varepsilon \leq C.
\]

**Proof.** Let us prove first that a minimizer exists. In fact, let

\[
K^\varepsilon = \left\{ v \in W^{1,p_\varepsilon} (\Omega) : v - \phi_\varepsilon \in W^{1,p_\varepsilon} (\Omega) \right\}.
\]

In order to prove that \( J_\varepsilon \) is bounded from below in \( K^\varepsilon \), we observe that if \( v \in K^\varepsilon \), then

\[
J_\varepsilon (v) \geq \frac{1}{p_{\text{max}}} \int_{\Omega} \| \nabla v \|_{p_\varepsilon (x)} + \int_{\Omega} f_\varepsilon v \, dx,
\]

and we have, by Theorem [A.3] and Theorem [A.4]

\[
\int_{\Omega} |f_\varepsilon v| \, dx \leq 2 \| f_\varepsilon \|_{p_\varepsilon (\cdot) \| v \|_{p_\varepsilon (\cdot)}} \leq 2 \| f_\varepsilon \|_{p_\varepsilon (\cdot)} (\| v - \phi_\varepsilon \|_{p_\varepsilon (\cdot)} + \| \phi_\varepsilon \|_{p_\varepsilon (\cdot)})
\]

\[
\leq C_0 \| \nabla v - \nabla \phi_\varepsilon \|_{p_\varepsilon (\cdot)} + C_1 \leq C_0 \| \nabla v \|_{p_\varepsilon (\cdot)} + C_2.
\]

If \( \left( \int_{\Omega} |\nabla v|^{p_\varepsilon (x)} \, dx \right)^{1/p_{\text{min}}} \geq \left( \int_{\Omega} |\nabla v|^{p_\varepsilon (x)} \, dx \right)^{1/p_{\text{max}}} \) we get, by Proposition [A.1]

\[
\int_{\Omega} |f_\varepsilon v| \, dx \leq C_0 \left( \int_{\Omega} |\nabla v|^{p_\varepsilon (x)} \, dx \right)^{1/p_{\text{min}}} + C_2 \leq C_3 + \frac{1}{2p_{\text{max}}} \int_{\Omega} |\nabla v|^{p_\varepsilon (x)} \, dx.
\]

If, on the other hand, \( \left( \int_{\Omega} |\nabla v|^{p_\varepsilon (x)} \, dx \right)^{1/p_{\text{min}}} < \left( \int_{\Omega} |\nabla v|^{p_\varepsilon (x)} \, dx \right)^{1/p_{\text{max}}} \), we get in an analogous way

\[
\int_{\Omega} |f_\varepsilon v| \, dx \leq C_0 \left( \int_{\Omega} |\nabla v|^{p_\varepsilon (x)} \, dx \right)^{1/p_{\text{max}}} + C_2 \leq C_4 + \frac{1}{2p_{\text{max}}} \int_{\Omega} |\nabla v|^{p_\varepsilon (x)} \, dx.
\]

Taking \( C_5 = \max \{ C_3, C_4 \} \), we get

\[
J_\varepsilon (v) \geq -C_5 + \frac{1}{2p_{\text{max}}} \int_{\Omega} |\nabla v|^{p_\varepsilon (x)} \, dx \geq -C_5,
\]

which shows that \( J_\varepsilon \) is bounded from below in \( K^\varepsilon \).

At this point we want to remark that the constants \( C_0, ..., C_5 \) above can be taken depending only on \( A_1, A_2, p_{\text{min}}, p_{\text{max}} \) and \( L \).

We now take a minimizing sequence \( \{ u_n \} \subset K^\varepsilon \). Without loss of generality we can assume that \( J_\varepsilon (u_n) \leq J_\varepsilon (\phi_\varepsilon) \), so by (4.4), \( \int_{\Omega} |\nabla u_n|^{p_\varepsilon (x)} \leq C_6 \). By Proposition [A.1] \( \| \nabla u_n - \nabla \phi_\varepsilon \|_{p_\varepsilon (\cdot)} \leq C_7 \) and,
as \( u_n - \phi \in W_0^{1,p(\cdot)}(\Omega) \), by Theorem A.4 we have \( \|u_n - \phi\|_{p(\cdot)} \leq C_8 \). Therefore, by Theorem A.1 there exist a subsequence (that we still call \( u_n \)) and a function \( u^\varepsilon \in W_0^{1,p(\cdot)}(\Omega) \) such that

\[
|u^\varepsilon|_{W^{1,p(\cdot)}(\Omega)} \leq \tilde{C}, \quad \text{with } \tilde{C} = \tilde{C}(A_1, A_2, p_{\min}, p_{\max}, L),
\]

and, by Theorem A.2

\[
u_n \rightharpoonup u^\varepsilon \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega),
\]

and

\[
u_n \rightarrow u^\varepsilon \quad \text{weakly in } W^{1,p_{\min}}(\Omega).
\]

Now, by the compactness of the immersion \( W^{1,p_{\min}}(\Omega) \hookrightarrow L^{p_{\min}}(\Omega) \) we have that, for a subsequence that we still denote by \( u_n \),

\[
u_n \rightarrow u^\varepsilon \quad \text{in } L^{p_{\min}}(\Omega),
\]

\[
u_n \rightarrow u^\varepsilon \quad \text{a.e. } \Omega.
\]

As \( K^\varepsilon \) is convex and closed, it is weakly closed, so \( u^\varepsilon \in K^\varepsilon \).

It follows that

\[
\lim_{n \to \infty} \int_\Omega B_\varepsilon(u_n) \, dx = \int_\Omega B_\varepsilon(u^\varepsilon) \, dx,
\]

\[
\lim_{n \to \infty} \int_\Omega f^\varepsilon u_n \, dx = \int_\Omega f^\varepsilon u^\varepsilon \, dx,
\]

\[
\int_\Omega |\nabla u^\varepsilon|_{p_\varepsilon(x)} \, dx \leq \liminf_{n \to \infty} \int_\Omega |\nabla u_n|_{p_\varepsilon(x)} \, dx.
\]

In order to prove the last inequality we proceed as in (3.4) in Theorem 3.1.

Hence

\[
J_\varepsilon(u^\varepsilon) \leq \liminf_{n \to \infty} J_\varepsilon(u_n) = \inf_{v \in K^\varepsilon} J_\varepsilon(v).
\]

Therefore, \( u^\varepsilon \) is a minimizer of \( J_\varepsilon \) in \( K^\varepsilon \).

Let us now prove that there holds (4.2). Let \( t > 0 \) and \( \xi \in C_0^\infty(\Omega) \). Using the minimality of \( u^\varepsilon \) we have

\[
0 \leq \frac{1}{t}(J_\varepsilon(u^\varepsilon - t\xi) - J_\varepsilon(u^\varepsilon)) = \frac{1}{t} \int_\Omega \left( \frac{\nabla u^\varepsilon - t\nabla \xi|_{p_\varepsilon(x)}}{p_\varepsilon(x)} \cdot \frac{\nabla u^\varepsilon}{p_\varepsilon(x)} - \frac{|\nabla u^\varepsilon|_{p_\varepsilon(x)}}{p_\varepsilon(x)} \right) \, dx + \frac{1}{t} \int_\Omega \left( B_\varepsilon(u^\varepsilon - t\xi) - B_\varepsilon(u^\varepsilon) \right) \, dx + \frac{1}{t} \int_\Omega \left( f^\varepsilon(u^\varepsilon - t\xi) - f^\varepsilon u^\varepsilon \right) \, dx
\]

\[
\leq -\int_\Omega |\nabla u^\varepsilon|_{p_\varepsilon(x)} \cdot \nabla \xi \, dx + \beta_\varepsilon(u^\varepsilon) |\nabla u^\varepsilon|_{p_\varepsilon(x)} \, dx - \int_\Omega f^\varepsilon \xi \, dx.
\]

If we take \( t \to 0 \), we obtain

\[
0 \leq -\int_\Omega |\nabla u^\varepsilon|_{p_\varepsilon(x)} \cdot \nabla \xi \, dx + \beta_\varepsilon(u^\varepsilon) |\nabla u^\varepsilon|_{p_\varepsilon(x)} \, dx - \int_\Omega f^\varepsilon \xi \, dx.
\]

If we now take \( t < 0 \), and proceed in a similar way, we obtain the opposite sign in (4.6) and (4.2) follows.

Finally, in order to prove (1.3), we observe that, from Proposition A.1 and estimate (4.5), we have that \( \int_\Omega |u^\varepsilon|_{p_\varepsilon(x)} \, dx \leq \tilde{C}(A_1, A_2, p_{\min}, p_{\max}, L) \). Thus, the desired estimate follows from the application of Proposition 2.1 in [35], since \( \Delta_{p_\varepsilon(x)} u^\varepsilon \geq f^\varepsilon \geq -A_2 \) in \( \Omega \).
Remark 4.1. We are interested in studying the behavior of a family \( u^\varepsilon \) of nonnegative local minimizers of the energy \( J_\varepsilon \) defined in (4.1).

If \( u_\varepsilon \) are as in Theorem 4.1 then \( u_\varepsilon \) satisfy (4.2) and it follows from Proposition 2.1 in [24] that \( u_\varepsilon \in L^{\infty}_{\text{loc}}(\Omega) \). Moreover, by Theorem 1.1 in [13] \( u_\varepsilon \in C^1(\Omega) \) and \( \nabla u_\varepsilon \) are locally Hölder continuous in \( \Omega \).

We first observe that the estimates of Theorem 2.1 in [24] apply, as well as the results in Proof.

We will follow the ideas in Theorem 1.16 in [8]. In fact, let \( B_r = B_r(x_0) \subset \subset \Omega \), for simplicity assume \( x_0 = 0 \), and denote

\[
J_{r,j}(v) = \int_{B_r} \left( \frac{|\nabla v|^{p_j}(x)}{p_j(x)} + B_{\varepsilon_j}(v) + f^{\varepsilon_j}v \right) dx,
\]

\[
J_{r,0}(v) = \int_{B_r} \left( \min \{0, \nabla v\} \chi_{\{v > 0\}} + f v \right) dx,
\]

where \( M = \int \beta(s) ds \).

i) If \( v \in W^{1,p_j(\cdot)}(B_r) \) for some \( \delta > 0 \) and \( v - u \in W^{1,\cdot}_{0,p_j(\cdot)}(B_r) \), then \( J_{r,0}(u) \leq J_{r,0}(v) \).

ii) If there holds that \( p_{\varepsilon_j} \leq p \in \Omega \) and \( u \in W^{1,p(\cdot)}(\Omega) \), then \( u \) is a nonnegative local minimizer of functional (4.8).

Proof. We first observe that the estimates of Theorem 2.1 in [24] apply, as well as the results in Lemma 3.1 in [24]. In particular, \( u^{\varepsilon_j} \) are locally uniformly Lipschitz and therefore \( u \) is locally Lipschitz in \( \Omega \).

We will follow the ideas in Theorem 1.16 in [8]. In fact, let \( B_r = B_r(x_0) \subset \subset \Omega \), for simplicity assume \( x_0 = 0 \), and denote

\[
J_{r,j}(v) = \int_{B_r} \left( \frac{|\nabla v|^{p_j}(x)}{p_j(x)} + B_{\varepsilon_j}(v) + f^{\varepsilon_j}v \right) dx,
\]

\[
J_{r,0}(v) = \int_{B_r} \left( \min \{0, \nabla v\} \chi_{\{v > 0\}} + f v \right) dx,
\]
Let us first assume that ii) holds.

Given \( v \in W^{1,p(x)}(B_r) \) such that \( v-u \in W^{1,p(x)}_0(B_r) \), we want to show that

(4.10) \[ J_{r,0}(u) \leq J_{r,0}(v). \]

For \( h > 0 \) small, we define

\[
v_{h,j} = \begin{cases} 
  v & \text{in } B_r, \\
  u + \frac{|x|-r}{h}(u^{\varepsilon_j} - u) & \text{in } B_{r+h} \setminus B_r.
\end{cases}
\]

Then, since \( p_{\varepsilon_j} \leq p \), it follows that \( v_{h,j} \in W^{1,p_{\varepsilon_j}}(B_{r+h}) \), \( v_{h,j} - u^{\varepsilon_j} \in W^{1,p_{\varepsilon_j}}_0(B_{r+h}) \) and there holds

\[
J_{r+h,0}(v_{h,j}) = \int_{B_{r+h}} \frac{|\nabla v_{h,j}|^{p(x)}_{p_{\varepsilon_j}(x)}}{p_{\varepsilon_j}(x)} + B_{\varepsilon_j}(v_{h,j}) + f^{\varepsilon_j} v_{h,j} \leq J_{r,0}(v)
\]

\[
+ \int_{B_{r+h} \setminus B_r} \frac{|\nabla v_{h,j}|^{p(x)}_{p_{\varepsilon_j}(x)}}{p_{\varepsilon_j}(x)} + B_{\varepsilon_j}(v_{h,j}) + f^{\varepsilon_j} v_{h,j} + \int_{B_r} \frac{|\nabla v|^{p(x)}_{p_{\varepsilon_j}(x)}}{p(x)} - \frac{|\nabla v|^{p(x)}_{p(x)}}{p(x)} + (f^{\varepsilon_j} - f)v
\]

\[
\leq J_{r,0}(v) + C_0 h r^{N-1} + C_1 \int_{B_{r+h} \setminus B_r} \frac{|u^{\varepsilon_j} - u^{p_{\varepsilon_j}}(x)|}{h^{p_{\varepsilon_j}}(x)} + \int_{B_r} \frac{|\nabla v|^{p(x)}_{p_{\varepsilon_j}(x)}}{p(x)} + (f^{\varepsilon_j} - f)v.
\]

Therefore,

(4.11) \[ \limsup_{j \to \infty} J_{r+h,0}(v_{h,j}) \leq J_{r,0}(v) + C_0 h r^{N-1}. \]

On the other hand,

\[ M \chi_{\{u>0\}} \leq \liminf_{j \to \infty} B_{\varepsilon_j}(u^{\varepsilon_j}), \]

which implies

(4.12) \[ \int_{B_r} M \chi_{\{u>0\}} \, dx \leq \liminf_{j \to \infty} \int_{B_r} B_{\varepsilon_j}(u^{\varepsilon_j}) \, dx. \]

In addition, since \( \nabla u^{\varepsilon_j} \to \nabla u \) weakly in \( L^{p(x)}(B_r) \), arguing in a similar way as in Theorem 4.1, we get

(4.13) \[ \int_{B_r} \frac{|\nabla u|^{p(x)}_{p(x)}}{p(x)} \, dx \leq \liminf_{j \to \infty} \int_{B_r} \frac{|\nabla u|^{p(x)}_{p(x)}}{p(x)} \, dx = \liminf_{j \to \infty} \int_{B_r} \frac{|\nabla u|^{p(x)}_{p_{\varepsilon_j}(x)}}{p_{\varepsilon_j}(x)} \, dx. \]

Now, using (4.12) and (4.13), and the fact that \( u^{\varepsilon_j} \) are nonnegative local minimizers of \( J_{\varepsilon_j} \), we obtain

\[ J_{r,0}(u) \leq \liminf_{j \to \infty} J_{r,0}(u^{\varepsilon_j}) \leq \liminf_{j \to \infty} J_{r+h,0}(u^{\varepsilon_j}) + C_2 h r^{N-1} \leq \liminf_{j \to \infty} J_{r+h,0}(v_{h,j}) + C_2 h r^{N-1}, \]

which in combination with (4.11) gives

\[ J_{r,0}(u) \leq J_{r,0}(v) + C_3 h r^{N-1}. \]

Therefore, letting \( h \to 0 \) we obtain (4.10).

Finally, if there holds i) we can proceed exactly as above to prove that (4.10) holds, using that in this case we also have \( v_{h,j} \in W^{1,p_{\varepsilon_j}}(B_{r+h}) \), \( v_{h,j} - u^{\varepsilon_j} \in W^{1,p_{\varepsilon_j}}_0(B_{r+h}) \) for large \( j \). \( \square \)
Remark 4.3. Let \( u^\epsilon \) be a family of nonnegative local minimizers of \( J_\epsilon(v) = \int_{\Omega} \left( \frac{|\nabla u^\epsilon|^{p_s}(x)}{p_s(x)} + B_\epsilon(v) + f^\epsilon v \right) dx \), with \( 1 < p_{\min} \leq p_s(x) \leq p_{\max} < \infty \), \( \|\nabla p_s\|_{L^\infty(\Omega)} \leq L_1 \), \( \|u^\epsilon\|_{L^\infty(\Omega)} \leq L_1 \) and \( \|\nabla f^\epsilon\|_{L^\infty(\Omega)} \leq L_2 \). Then, with a minor modification of the proof of Theorem 4.1, we can prove that, given \( \Omega' \subset \subset \Omega \), there exist positive constants \( c_0 \) and \( \rho \) such that, for every \( x_0 \in \Omega' \),

\[
u^\epsilon > \epsilon \quad \text{in } B_{d_0}(x_0) \quad \text{with} \quad 0 < d_0 \leq \rho, \quad \text{implies} \quad u^\epsilon(x_0) \geq c_0 d_0,
\]

and, in particular,

\[
u^\epsilon(x_0) \geq c_0 \text{dist}(x_0, \{u^\epsilon \leq \epsilon\}), \quad \text{if} \quad \text{dist}(x_0, \{u^\epsilon \leq \epsilon\}) \leq \rho,
\]

with \( c_0 \) and \( \rho \) depending only on \( p_{\min}, p_{\max}, L, L_1, L_2, M = \int \beta(s) ds \) and \( \text{dist}(\Omega', \partial \Omega) \).

As a consequence it follows that, if \( u = \lim_{\epsilon \to 0} u^\epsilon \) as \( \epsilon_j \to 0 \) then, for every \( x_0 \in \Omega' \),

\[
u(x_0) \geq c_0 \text{dist}(x_0, \{u \equiv 0\}), \quad \text{if} \quad \text{dist}(x_0, \{u \equiv 0\}) \leq \rho.
\]

As in the case of minimizers of the energy (1.1), for minimizers of the singular perturbation problem we have

Theorem 4.3. Let \( p_{\epsilon_j}, f_{\epsilon_j}, u_{\epsilon_j}, \epsilon_j, p, f \) and \( u \) be as in Theorem 4.2. Let \( \Omega' \subset \subset \Omega \). There exist constants \( c > 0 \), \( r_0 > 0 \) such that if \( x_0 \in \Omega' \cap \partial \{u > 0\} \) and \( r \leq r_0 \) then

\[\sup_{B_r(x_0)} u \geq cr.\]

The constants depend only on \( N, p_{\min}, p_{\max}, L, L_1, L_2, M, ||\beta||_{L^\infty} \) and \( \text{dist}(\Omega', \partial \Omega) \).

Proof. The proof follows as that of Theorem 3.6, replacing Theorem 3.4 by Remark 4.3. \( \square \)

In an analogous way as we obtained for minimizers of functional (1.1), for minimizers of the singular perturbation problem we have

Theorem 4.4. Let \( p_{\epsilon_j}, f_{\epsilon_j}, u_{\epsilon_j}, \epsilon_j, p, f \) and \( u \) be as in Theorem 4.2. Let \( \Omega' \subset \subset \Omega \). There exist constants \( \epsilon \in (0, 1) \) and \( r_0 > 0 \) such that, if \( x_0 \in \Omega' \cap \partial \{u > 0\} \) with \( B_r(x_0) \subset \Omega' \) and \( r \leq r_0 \), there holds

\[
\frac{|B_r(x_0) \cap \{u > 0\}|}{|B_r(x_0)|} \leq 1 - \epsilon.
\]

The constants depend only on \( N, p_{\min}, p_{\max}, L, L_1, L_2, M, ||\beta||_{L^\infty} \) and \( \text{dist}(\Omega', \partial \Omega) \).

Proof. The proof follows as that of Theorem 3.6. In this case we obtain estimate (3.56) by using part i) in Theorem 4.2, since \( v_k \in W^{1, p_s(1+\delta_k)}(B_1/2) \), for some \( \delta_k > 0 \) (see, for instance, Lemma 4.1 in [14]). \( \square \)

5. Regularity of the Free Boundary

In this section, we first consider nonnegative local minimizers to the energy functional (1.1) and we obtain results on the regularity of the free boundary for these functions, which are a consequence of the results in Section 4 and the results in our work [24].

In addition, we consider any family \( u^\epsilon \) of nonnegative local minimizers to the energy functional (1.1) which are uniformly bounded, with \( f^\epsilon \) and \( p_s \) uniformly bounded (like, for instance, the one constructed in Theorem 4.1 and Remark 4.1). Then (recall Remark 4.2), all the results in our previous paper [24] apply to such a family. Hence, as a consequence of the results in Section 4 and in our work [25], we obtain results on the regularity of the free boundary for limit functions of this family.

First, for nonnegative local minimizers to the energy functional (1.1), we get
Theorem 5.1. Assume that
\((1.1)\)

\[ \text{Let } u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega) \text{ be a nonnegative local minimizer of } \|\nabla p\|_{L^\infty} \leq L, \text{ if } f \in L^\infty(\Omega) \text{ and } \|
abla u\|_{L^p} \leq L, f \in L^\infty(\Omega) \text{ and } \|
abla \lambda\|_{L^\infty} \leq L, \text{ if } \lambda \in C(\Omega). \]

Then, \(u\) is a weak solution to the free boundary problem: \(u \geq 0\) and
\[
(P(f,p,\lambda^*)) \quad \begin{cases} 
\Delta_{p(x)} u = f & \text{in } \{u > 0\} \\
u = 0, |\nabla u| = \lambda^*(x) & \text{on } \partial\{u > 0\}
\end{cases}
\]

with \(\lambda^*(x) = \left( \frac{p(x)}{p(x)-1} \lambda(x) \right)^{1/p(x)}.\)

Proof. The result follows by applying Lemma 3.3, Corollary 3.2 and Theorems 3.3, 3.5, 3.7, 3.8 and 3.9.

Now, we can apply the results in [25] and deduce

Theorem 5.2. Let \(p, f, \lambda \text{ and } u \) be as in Theorem 5.1. Assume moreover that \(f \in W^{1,q}(\Omega), p \in W^{2,q}(\Omega) \) with \(q > \max\{1, N/2\} \) and \(\lambda \) is Hölder continuous in \(\Omega\).

Then, there is a subset \(\mathcal{R} \) of the free boundary \(\Omega \cap \partial\{u > 0\} \) \((\mathcal{R} = \partial_{\text{red}}\{u > 0\})\) which is locally a \(C^{1,\alpha}\) surface, for some \(0 < \alpha < 1\), and the free boundary condition is satisfied in the classical sense in a neighborhood of \(\mathcal{R}\). Moreover, \(\mathcal{R}\) is open and dense in \(\Omega \cap \partial\{u > 0\} \) and the remainder of the free boundary has \((N-1)\)-dimensional Hausdorff measure zero.

If moreover \(\nabla p \) and \(f\) are Hölder continuous in \(\Omega\), then the equation is satisfied in the classical sense in a neighborhood of \(\mathcal{R}\).

Proof. We first observe that, by Theorem 5.1, Theorem 4.4 in [25] applies at every \(x_0 \in \Omega \cap \partial_{\text{red}}\{u > 0\}.\)

Finally we observe that, since \(u\) is a weak solution to \(P(f,p,\lambda^*),\) Theorem 2.1 in [25] and Lemma 2.3 in [24] apply to \(u\). Therefore, recalling Theorem 3.6 we deduce, from Theorem 4.5.6(3) in [15], that \(H^{N-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0.\)

We also obtain higher regularity from the application of Corollary 4.1 in [25].

Corollary 5.1. Let \(p, f, \lambda \text{ and } u \) be as in Theorem 5.2. Assume moreover that \(p \in C^2(\Omega), f \in C^1(\Omega) \text{ and } \lambda \in C^2(\Omega) \) then \(\partial_{\text{red}}\{u > 0\} \subset C^{2,\mu}\) for every \(0 < \mu < 1.\)

If \(p \in C^{m+2,\mu}(\Omega), \) \(f \in C^{m+1,\mu}(\Omega)\) and \(\lambda \in C^{m+1,\mu}(\Omega)\) for some \(0 < \mu < 1\) and \(m \geq 1,\) then \(\partial_{\text{red}}\{u > 0\} \subset C^{m+2,\mu}\).

Finally, if \(p, f \) and \(\lambda\) are analytic, then \(\partial_{\text{red}}\{u > 0\} \) is analytic.

Next, for minimizers of the energy functional \((1.2)\) we obtain, as a consequence of the results in Section 4 and the results in [24].

Theorem 5.3. Assume that \(1 < p_{\min} \leq p_{\epsilon_j}(x) \leq p_{\max} < \infty\) and \(\|\nabla p_{\epsilon_j}\|_{L^\infty} \leq L.\) Let \(u^{\epsilon_j} \in W^{1,p_{\epsilon_j}(\cdot)}(\Omega) \) be a family of nonnegative local minimizers of \((1.7)\) in a domain \(\Omega \subset \mathbb{R}^N\) such that \(u^{\epsilon_j} \to u \) uniformly on compact subsets of \(\Omega, f^{\epsilon_j} \to f \) \*–weakly in \(L^\infty(\Omega), p_{\epsilon_j} \to p \) uniformly on compact subsets of \(\Omega \) and \(\epsilon_j \to 0.\)

Then, \(u\) is a weak solution to the free boundary problem: \(u \geq 0\) and
\[
(P(f,p,\lambda^*)) \quad \begin{cases} 
\Delta_{p(x)} u = f & \text{in } \{u > 0\} \\
u = 0, |\nabla u| = \lambda^*(x) & \text{on } \partial\{u > 0\}
\end{cases}
\]

with \(\lambda^*(x) = \left( \frac{p(x)}{p(x)-1} M \right)^{1/p(x)} \) and \(M = \int \beta(s) \, ds.\)
Proof. The result follows by applying first Remark 4.2 and Theorems 4.3 and 4.4 and then, Theorem 6.1 in [24].

We can now apply the results in [25] and deduce

Theorem 5.4. Let $\rho_{\varepsilon_j}, f^{\varepsilon_j}, \varepsilon_j, p, f$ and $u$ be as in Theorem 5.3. Assume moreover that $f \in W^{1,q}(\Omega)$ and $p \in W^{2,q}(\Omega)$ with $q > \max\{1, N/2\}$.

Then, there is a subset $\mathcal{R}$ of the free boundary $\Omega \cap \partial\{u > 0\} (\mathcal{R} = \partial_{\text{red}}\{u > 0\})$ which is locally a $C^{1,\alpha}$ surface, for some $0 < \alpha < 1$, and the free boundary condition is satisfied in the classical sense in a neighborhood of $\mathcal{R}$. Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u > 0\}$ and the remainder of the free boundary has $(N - 1)$--dimensional Hausdorff measure zero.

If moreover $\nabla p$ and $f$ are Hölder continuous in $\Omega$, then the equation is satisfied in the classical sense in a neighborhood of $\mathcal{R}$.

Proof. We first observe that, by Theorem 5.3, Theorem 4.4 in [25] applies at every $x_0 \in \Omega \cap \partial_{\text{red}}\{u > 0\}$.

Finally we observe that, since $u$ is a weak solution to $P(f, p, \lambda^*)$, Theorem 2.1 in [25] and Lemma 2.3 in [25] apply to $u$. Therefore, recalling Theorem 4.4 we deduce, from Theorem 4.5.6(3) in [15], that $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$.

We also obtain higher regularity from the application of Corollary 4.1 in [25].

Corollary 5.2. Let $p, f$ and $u$ be as in Theorem 5.4. Assume moreover that $p \in C^2(\Omega)$ and $f \in C^1(\Omega)$, then $\partial_{\text{red}}\{u > 0\} \subset C^{2,\mu}$ for every $0 < \mu < 1$.

If $p \in C^{m+1,\mu}(\Omega)$ and $f \in C^{m,\mu}(\Omega)$ for some $0 < \mu < 1$ and $m \geq 1$, then $\partial_{\text{red}}\{u > 0\} \subset C^{m+2,\mu}$.

Finally, if $p$ and $f$ are analytic, then $\partial_{\text{red}}\{u > 0\}$ is analytic.

Appendix A.

In Section 1 we included some preliminaries on Lebesgue and Sobolev spaces with variable exponent. For the sake of completeness we collect here some additional results on these spaces as well as some other results that are used throughout the paper.

Proposition A.1. There holds

$$\min\left\{\left(\int_{\Omega} |u|^{p(x)} \, dx\right)^{1/p_{\min}}, \left(\int_{\Omega} |u|^{p(x)} \, dx\right)^{1/p_{\max}}\right\} \leq \|u\|_{L^p(\cdot)(\Omega)} \leq \max\left\{\left(\int_{\Omega} |u|^{p(x)} \, dx\right)^{1/p_{\min}}, \left(\int_{\Omega} |u|^{p(x)} \, dx\right)^{1/p_{\max}}\right\}.$$

Some important results for these spaces are

Theorem A.1. Let $p'(x)$ such that

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Then $L^{p'(\cdot)}(\Omega)$ is the dual of $L^{p(\cdot)}(\Omega)$. Moreover, if $p_{\min} > 1$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive.

Theorem A.2. Let $q(x) \leq p(x)$. If $\Omega$ has finite measure, then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ continuously.

We also have the following Hölder’s inequality
Theorem A.3. Let \( p'(x) \) be as in Theorem A.1. Then there holds

\[
\int_{\Omega} |f||g| \, dx \leq 2 \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)},
\]

for all \( f \in L^p(\cdot)(\Omega) \) and \( g \in L^{p'(\cdot)}(\Omega) \).

The following version of Poincare’s inequality holds

Theorem A.4. Let \( \Omega \) be bounded. Assume that \( p(x) \) is log-Hölder continuous in \( \Omega \) (that is, \( p \) has a modulus of continuity \( \omega(r) = C(\log \frac{1}{r})^{-1} \)). For every \( u \in W^{1,p(x)}_0(\Omega) \), the inequality

\[
\|u\|_{L^p(\cdot)(\Omega)} \leq C \|
abla u\|_{L^{p(x)}(\Omega)}
\]

holds with a constant \( C \) depending only on \( N, \text{diam}(\Omega) \) and the log-Hölder modulus of continuity of \( p(x) \).

For the proof of these results and more about these spaces, see [13], [18], [31], [17] and the references therein.

We will also need

Lemma A.1. Let \( 1 < p_0 < +\infty \). Let \( u \) be Lipschitz continuous in \( \overline{B}^+_1 \), \( u \geq 0 \) in \( B^+_1 \), \( \Delta p_0 u = 0 \) in \( \{u > 0\} \) and \( u = 0 \) on \( \{x_N = 0\} \). Then, in \( B^+_1 \) \( u \) has the asymptotic development

\[
u(x) = \alpha x_N + o(|x|),
\]

with \( \alpha \geq 0 \).

Proof. See [6] for \( p_0 = 2 \), [12] for \( 1 < p_0 < +\infty \) and [28] for a more general operator. \( \square \)

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