Tunable frequency-bin multi-mode squeezed states of light

Christian Drago$^{1,2,*}$ and Agata M. Brańczyk$^3$

$^1$Department of Physics and Astronomy, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, Canada N2L 3G1
$^2$Department of Physics, University of Toronto, 60 St. George Street, Toronto, Ontario, Canada, M5S 1A7
$^3$Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5, Canada

Squeezed states are a versatile class of quantum states with applications ranging from quantum computing to high-precision detection. We propose a method for generating tunable squeezed states of light with multiple modes encoded in frequency bins. Our method uses custom-engineered spontaneous parametric downconversion pumped by a pulse-shaped pump field. The multi-mode squeezed states are generated in a single pass and can be tuned in real time by adjusting the properties of the pump field. Exploring new quantum states of light, encoded in new degrees of freedom, can be a fruitful path toward discovering new quantum applications.

I. INTRODUCTION

Quantum light is a key ingredient in emerging quantum technologies such as quantum communication [1], quantum cryptography [2], quantum computing [3], quantum imaging [4] and quantum metrology [5]. Its broad application results from the wide variety of available degrees of freedom (e.g. polarization, frequency, temporal mode), which in turn can encode a wide variety of quantum states (e.g. qubits, Fock states, cat states). Exploring new quantum states of light, encoded in new degrees of freedom, can thus be a fruitful path toward discovering new quantum applications.

A versatile class of quantum states are known as squeezed vacuum states [6], or simply squeezed states$^1$. They are typically classified by the number of modes they populate, where different kinds of squeezed states are better suited for different applications. Single-mode squeezed states have reduced quantum noise in one degree of freedom, making them most useful for quantum cryptography [2] and quantum metrology [5]. Two-mode squeezed states, on the other hand, possess entanglement between the modes of the electromagnetic field, and can be used for quantum teleportation [1]. Squeezed states can also be distributed across multiple modes. These multi-mode squeezed states can be used to generalize various quantum information protocols, e.g. multi-parameter quantum metrology [7], multi-channel quantum imaging [8], multi-partite teleportation [9] or be used for quantum computation [10].

The generalization of single- and two-mode squeezed states was first proposed by Yeoman and Barnett in 1993 [11]. They considered states (originally called two-mode squeezed gaussons) that had both single- and two-mode squeezing properties and proposed a method to generate them with two single-mode squeezed states and a beam splitter. In 2000 this method was generalized to include multi-mode squeezed states by Van Loock and Braunstein [12] who proposed using a sequence of $N$ beam splitters to prepare a multimode squeezed state entangled across $N$ spatial modes. This method requires interferometric stabilization of the optical paths and indistinguishability between frequency and polarization modes [13]. To overcome these challenges, another approach was proposed [14, 15] in which an optical parametric oscillator (OPO) and a single periodically poled ferroelectric crystal is used to generate $N$-mode entangled states between the cavity modes of the OPO. This method has been further expanded upon by multiplexing the light in time allowing an unlimited number of entangled modes [16, 17]. Further, one can encode squeezed state modes into Schmidt modes as was done in [18]. These newer methods are much more compact and stable, and benefit from the intrinsic compatibility of frequency and time encodings with waveguides and fiber transmission, and we expand on them here.

We introduce a method for generating tunable multi-mode squeezed states of light encoded in frequency bins. Our proposal builds on the idea that frequency-bin entanglement can be generated by domain-engineered down-conversion, which was recently demonstrated by Morrison et al. [19]. We show how the addition of a frequency-shaped pump can yield grid states similar to those introduced by Fabre et al. [10], but with more control over the

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$^*$ christian.drago@mail.utoronto.ca

$^1$ For a brief review of squeezed states, refer to Appendix A.
peaks, and without the need for a cavity. We further show how, with the addition of frequency filtering, one can create a squeezed state that can be tuned from single-mode to two-mode in real time.

Our method for generating multimode squeezed states differs from previous proposals in several ways: (i) unlike \cite{10,14–17}, our method does not require a cavity to generate multiple modes, (ii) unlike \cite{11,12}, our method does not require increasingly more beam splitters to generate more modes, and (iii) unlike the Schmidt modes in \cite{18}, which are difficult to distinguish experimentally due to their complex spectral shape, our modes are encoded in discrete frequency bins and can be distinguished by their central frequencies. Furthermore, our method allows the squeezing parameters of the state to be tuned in real time. Depending on the application, these properties may make this method advantageous over other methods.

II. FREQUENCY BIN ENCODING

A desirable property of quantum light is localization in space and time, which makes it possible to deliver wavepackets of quantum light in a clocked manner \cite{20}. A Gaussian-like frequency encoding satisfies these properties. As with their CW counterparts \cite{14,15}, such an encoding is intrinsically compatible with waveguides and fiber transmission.

To encode our modes we define a multi-frequency field operator:

\[
A_n = \int d\omega G_n(\omega) a(\omega),
\]

where \(\{G_n(\omega)\}\) describe non-overlapping Gaussian-like pulses, such that to good approximation

\[
\int d\omega G_n(\omega) G_m(\omega) = \delta_{nm},
\]

and thus

\[
[A_n, A_m^\dagger] = \delta_{nm}.
\]

In terms of these operators, the multi-mode squeezed state takes the form

\[
|\psi\rangle = e^{i \sum_{n,m} \gamma_{nm} A_n^\dagger A_m^\dagger} |\text{vac}\rangle,
\]

which can be rewritten as

\[
|\psi\rangle = e^{\frac{1}{2} \int d\omega_s d\omega_i h(\omega_s, \omega_i) a^\dagger(\omega_s) a^\dagger(\omega_i) - H.c.} |\text{vac}\rangle,
\]

where

\[
h(\omega_s, \omega_i) = \sum_{n,m} \gamma_{nm} G_n(\omega_i) G_m(\omega_s),
\]

and \(\gamma_{nm}\) is in general complex and taken to be symmetric. The state in Eq. (5) could in principle be generated by Type-I spontaneous parametric downconversion (where both idler and signal modes have the same polarization), if one had means of engineering a joint spectral amplitude of the form given by (6). To date, most joint-spectral engineering methods have been developed for Type-II parametric downconversion (where both idler and signal have orthogonal polarization), so we will focus on a Type-II setup in this paper. In the next section, we show how to use spectrally engineered Type-II downconversion to generate a state that, once sent through a Mach-Zehnder interferometer with a half-wave plate (HWP) set to 45° in one arm, is equal to the multi-mode squeezed state in (4).

III. IMPLEMENTATION WITH SPONTANEOUS PARAMETRIC DOWNCONVERSION

We consider Type-II spontaneous parametric downconversion \cite{21}, under the following conditions: (i) rotating wave approximation, (ii) undepleted pump approximation, (iii) ignoring time-ordering effects, (iv) symmetric group-velocity-matching, and (v) linear phase-mismatch (see Appendix B for discussion of the assumptions). In this regime, the state generated by this process is:

\[
|\psi\rangle = e^{i \gamma \int d\omega_s d\omega_i f(\omega_s, \omega_i) a^\dagger(\omega_s) a^\dagger(\omega_i) - H.c.} |\text{vac}\rangle,
\]

where \(\gamma\) is the squeezing parameter and

\[
f(\omega_s, \omega_i) = \alpha(\omega_s + \omega_i) \phi(\omega_s - \omega_i),
\]

is the joint spectral amplitude (JSA). The JSA is given by the spectral profile of the pump \(\alpha(\omega_s + \omega_i)\) and the normalized phase-matching function \(\phi(\omega_s - \omega_i)\) which depends on the properties of the nonlinear material. Following the normalization of the pump and phase-matching function in Appendix B, the joint spectral amplitude is normalized according to

\[
\int d\omega_s d\omega_i |f(\omega_s, \omega_i)|^2 = 1.
\]
A. Designing the joint spectral amplitude

Our goal is to design the pump-envelope function $\alpha(\omega_s + \omega_i)$ and the phase matching function $\phi(\omega_s - \omega_i)$ such that the joint spectral amplitude $f(\omega_s, \omega_i)$ in Eq. (8) matches the target joint spectral amplitude $b(\omega_s, \omega_i)$ in Eq. (6).

To achieve this, the pump envelope function should be prepared as a superposition of Gaussian-like functions centred at different frequencies:

$$\alpha(\omega_s + \omega_i) = \sum_{n=-N}^{N} a_n \frac{1}{(4\pi\sigma^2)^{\frac{3}{4}}} e^{-\frac{(\omega_s + \omega_i - \Omega_n - \delta\omega)^2}{8\sigma^2}},$$

(10)

for some set of dimensionless constants $a_n$ and integers $n$. The pump envelope function can be customized with various pulse shaping techniques; for a review of pulse shaping in various regimes see [22–24]. Since the pump function is square normalized, the coefficients satisfy $\sum_n a_n^2 = 1$.

The phase-matching function should also be a superposition of Gaussian-like functions, centred at different frequencies:

$$\phi(\omega_s - \omega_i) = \sum_{m=-N}^{N} b_m \frac{1}{(4\pi\sigma^2)^{\frac{3}{4}}} e^{-\frac{(\omega_s - \omega_i - \Omega_n - \delta\omega)^2}{8\sigma^2}},$$

(11)

where $b_m$ are some set of dimensionless amplitude coefficients that satisfy $\sum_m b_m^2 = 1$, and $m$ is an integer.

We will show how to customize the phase-matching function using custom-poling in Section IIIB. In the mean time, we note that if the width of each term in the PMF and pump are the same, and we insert Eqs. (10) and (11) into Eq. (8) we yield the joint spectral amplitude

$$f(\omega_s, \omega_i) = \frac{1}{\sqrt{2}} \sum_{n,m} a_{n-m} b_{n+m} G_n(\omega_s) G_m(\omega_i),$$

(12)

where for the remainder of this section, the summations are over $n$ and $m$ such that $n - m = \pm 1, \pm 2, \ldots, \pm N$, and

$$G_n(\omega_i) = e^{-\frac{(\omega_i - \Omega_n - \delta\omega)^2}{8\sigma^2}} \left(\frac{2\pi\sigma^2}{4\pi\sigma^2}\right)^{\frac{3}{4}},$$

(13)

with $\Omega_n \equiv n\delta\omega/2$ and $J = s, i$ runs over the two signal and idler frequencies. When absolute-value-squared, these functions yield Gaussian intensity distributions, which are optimal for such decompositions [25]. Finally, if $|\Omega_n - \Omega_m| \gg \sigma$, then to good approximation

$$\int d\omega G_n(\omega) G_m(\omega) = \delta_{nm}. \quad (14)$$

Inserting the decomposition of the JSA in Eq. (12) into the downconverted state in Eq. (7), defining $\gamma_{nm} \equiv \gamma a_{n-m} b_{n+m}/\sqrt{2}$ and using the definition of the mode $A_n$ in Eq. (1), yields

$$|\psi\rangle = e^{\sum_{n,m} \gamma_{nm} A_n^\dagger A_m^{\dagger}} |\text{vac}\rangle.$$  \quad (15)

This a multi-mode squeezed state in the Gaussian-mode degree of freedom and a two-mode squeezed state in the polarization degree of freedom (indicated by the subscripts $H$ and $V$). To eliminate the polarization degree of freedom, we pass the state through a Mach-Zehnder interferometer with a HWP set to 45° in one arm (see Appendix C). The output state is

$$|\psi'\rangle = \bigotimes_{j=1,2} e^{\frac{i\gamma_{nm} A_n^\dagger A_m^{\dagger}}{2}} |\text{vac}\rangle.$$  \quad (16)

which consists of two copies of a multi-mode squeezed state, all in the same polarization, where the subscript $j = 1, 2$ labels the output modes of the interferometer. Alternatively, on can pass the

B. Customizing the phase-matching function

We now turn to designing an appropriately shaped phase-matching function. Consider a nonlinear material whose nonlinearity can vary along the longitudinal direction $z$; this variation can be captured by a dimensionless function $g(z)$ (defined in Eq. (B3)). The function $g(z)$ can be transformed as follows:

$$\Phi(\Delta k(\omega_s, \omega_i)) = \frac{1}{L} \int_{-L/2}^{L/2} dz g(z) e^{i\Delta k(\omega_s, \omega_i) z}, \quad (17)$$

which we can think of as an unnormalized phase matching function. Here, $L$ is the length of the nonlinear material and $\Delta k(\omega_s, \omega_i) = k_p(\omega_s + \omega_i) - k_i(\omega_i) - k_s(\omega_s)$ is the phase mismatch, where $k_j(\omega_j) = \omega_j n_j(\omega_j)/c$ and $n_j(\omega_j)$ is the refractive index for the mode $J$, $c$ is the speed of light and $J = s, i$. The unnormalized phase matching function $\Phi(\Delta k(\omega_s, \omega_i))$ is related to the normalized phase
matching function $\phi(\omega_s - \omega_i)$ via Eq. (B16). When modelling custom-poled materials, it’s easier to work with the unnormalized function $\Phi(\Delta k(\omega_s, \omega_i))$, and then to normalize the function numerically at the end. The target phase-matching function can then be written, in unnormalized form, as

$$\Phi(\Delta k) = \sum_{m=-N}^{N} c_m e^{-\frac{(\Delta k - \Delta k_0 - m\delta k)^2}{8\sigma_k^2}}, \quad (18)$$

where

$$\sigma_k = \frac{\alpha}{2}(k_s' - k_i') \quad (19)$$

$$\delta k = \frac{\delta \omega}{2}(k_s' - k_i'), \quad (20)$$

and together with the Taylor expansion of $\Delta k(\omega_s, \omega_i)$ in Eq. (B14) is equivalent to Eq. (11).

To generate the desired phase matching function, we must determine the right form of $g(z)$. In principle, one can imagine varying $g(z)$ continuously. Such methods, however, don’t exist for nonlinear crystals—in practice, for a given crystal, $g(z)$ is constrained to take on values of ±1 [26]. Experimentally, the sign of $g(z)$ can be alternated using a technique known as ferroelectric poling [27–30], giving rise to individual domains. In $\Delta k$-space, each domain contributes a sinc function with a phase that depends on the domain position and orientation. By carefully arranging the positive and negative domains in $g(z)$, it is possible to interfere the sinc functions to construct phase-matching functions with almost arbitrary shapes. As with all quasi-phase-matching techniques, such as periodic poling, the resulting amplitude will necessarily be reduced when compared with intrinsically phase-matched materials.

Several methods for designing appropriate domain configurations have been developed [31–39]. Here, we focus on a variation of the algorithm originally proposed in [36] and further developed in [39, 40] (effects of experimental imperfections in this approach were recently examined by Graffitti et al. [41]). In this approach, one computes an amplitude function—defined as the PMF, evaluated at a specific value of $\Delta k$, along the length of the crystal—that selects domains (one at a time from left to right) that bring the customized crystal’s amplitude function closer to the target amplitude function. When the customized crystal’s amplitude function closely approximates the target amplitude function at all points within the crystal, the customized crystal’s PMF will also closely approximate the target PMF. This approach was recently demonstrated for an eight-peak PMF in KTP [19].

In Appendix E we derive the following constraints on $c_m$:

$$c_0 + 2 \sum_{m=1}^{N} c_m \leq \sqrt{\frac{2}{\pi L\sigma_k}}. \quad (21)$$

For the simple case where $c_m$ are all equal, the condition reduces to

$$c_m = \sqrt{\frac{2}{\pi L\sigma_k N_G}}, \quad -N \leq m \leq N, \quad (22)$$

where $N_G$ is the number of Gaussian amplitudes. It’s desirable to maximize the conversion efficiency, and thus to maximize $c_m$ within these constraints.

Our implementation of the algorithm uses domain widths equal to the crystal’s coherence length. As a result, the phase-matching function is constrained to be real and the coefficients should satisfy $c_m \approx c_{-m}$ (the approximation in this equality comes from a slight bias in the PMF discussed in Appendix D). These restrictions would be lifted if a sub-coherence-length version of the algorithm was implemented [39]. In the next section, we demonstrate how to apply this technique to a specific example.

**IV. EXAMPLE: 15-MODE SQUEEZED STATES**

As an example, we customize a 2cm Potassium tita

tyl phosphate (KTP) crystal to generate a PMF with five Gaussian peaks centered at $\Delta k_0 - m\delta k$, where $m = (0, \pm 1, \pm 2)$ and $\delta k = 24\sigma_k$. We set $\sigma_k = 2.5/L$ to ensure that the target nonlinearity profile fits within the length of the crystal. The target PMF is given by Eq. (18) with $N = 2$. Figure 1 shows the generated PMF compared to the target PMF, as well as the resulting domain configuration $g(z)$. Notice the expected slight bias in the generated PMF discussed in Appendix D.

We design a pump function with five peaks centred at $\omega = \Omega_p - n\delta \omega$, where $n = (0, \pm 1, \pm 2)$. As with the PMF, we take all coefficients $c_m$ to be real, equal and given by the restriction in Eq. (22). We take $\sigma = 2\sigma_k/(k_s' - k_i')$ and to ensure that the spacing is the same as for the PMF, we take $\delta \omega = 24\pi$. Then using Eq. (19) and $\sigma_k = 2.5/L$, the pump bandwidth in frequency is $\sigma/2\pi = 0.127THz$ and in time is on the order of 7ps. The resulting JSA with corresponding PMF and pump is shown in Figure 2. Each peak in the JSA corresponds to a term in
Eq. (15). Notice that there is a slight bend in the PMF due to dispersion (for these plots, we used the full Sellmeier equations rather than the first-order approximation of $\Delta k$). Too much dispersion will reduce the effectiveness of this technique, but in the example shown here, the effect on the JSA is negligible.

It’s possible to tune the amplitudes $\gamma_{nm}$ to some extent by tuning the height of the peaks in the pump function (the height of the peaks in the PMF are fixed for a given crystal). Tuning the peak of the pump function is equivalent to scaling the height of the modes that lie along the same anti-diagonal in the JSA.

The JSA in Fig 2 has 25 peaks, but it corresponds to a 15-mode squeezed state because the PMF is symmetric in its amplitudes. The JSA is also symmetric along the line $\omega_s = \omega_i$. There are 10 amplitudes above the line $\omega_s = \omega_i$ which are all centered at different frequencies and thus correspond to two-mode squeezing terms. Each amplitude along the diagonal is centered at the same center frequency and thus corresponds to each of the 5 single-mode squeezing terms.

V. Extension: Tunable Hybrid Squeezed States

In this section, we introduce, and describe the generation of hybrid squeezed states, which have features of single-mode and two-mode squeezed vacuum states. Consider the special case of multi-mode

![Figure 1](image1.png)

**FIG. 1.** (a) Five-peak target phase-matching function as defined in Eq. (18) (with $c_m = \sqrt{2/(\sqrt{\pi}L\sigma_k^5)}$, $\sigma_k = 2.5/L$ and $\delta k = 24\sigma_k$) and corresponding custom phase-matching function generated by the custom-poled crystal shown in (b). We took $\sigma_k = 2.5/L$ to ensure that the target nonlinearity profile fits within the length of the crystal. The custom-poled crystal has $N = 1073$ domains of width $w = 18.63 \mu m$. The domain width was chosen to match the phase-matching conditions of Type-II KTP in the symmetric group-velocity-matched regime.

![Figure 2](image2.png)

**FIG. 2.** (a) Five-peak phasematching function (PMF) as defined by Eq. (11) with $\sigma$ and $\delta \omega$ related to $\sigma_k$ and $\delta k$ in Fig 1 via Eqs. (19) and (20). (b) Five-peak pump function as defined in Eq. (10) with the same $\sigma$ and $\delta \omega$ as the PMF. (c) The resulting joint spectral amplitude (JSA) as defined in Eq. (8).
squeezed state (Eq. (A3)) restricted to two modes:

$$|\text{THSS}\rangle = e^{\frac{\beta_{12}}{2}a_1^\dagger a_1^\dagger + \beta_{21}a_2^\dagger a_2^\dagger + \frac{\beta_{12}}{2}a_2^\dagger a_1^\dagger} |\text{vac}\rangle,$$

(23)

where we restricted $\beta_{12} = \beta_{21}$. We can tune the constants to continuously move between a single-mode squeezed state ($\beta_{12} = \beta_{22} = 0$), a two-mode squeezed state ($\beta_{11} = \beta_{22} = 0$) or a product of two single-mode squeezed states ($\beta_{12} = 0$). We call this a tuneable hybrid squeezed state (THSS).

Using the pulse-mode encoding introduced in Section II, a JSA corresponding to a THSS has four amplitudes located at, say, $(\Omega_{-1}, \Omega_{-1}), (\Omega_{-1}, \Omega_{1}), (\Omega_{1}, \Omega_{-1})$ and $(\Omega_{1}, \Omega_{1})$, each corresponding to a squeezing term in Eq. (23).

To generate such a JSA, we create a three-peak PMF and a three-peak pump function, to produce a 9-peak JSA show in figure 3. The JSA in Figure 3 has extra squeezing terms located at $\Omega_0$. However, the full state generated from the JSA in Figure 3 can be written as a product of two squeezing operators acting on the vacuum state given by

$$|\psi\rangle = \left( e^{\frac{\gamma_{-20}}{2}A_{-2}^\dagger A_{-2}^\dagger + \gamma_{-20}A_{-2}A_{-2}^\dagger + \gamma_{20}A_0A_0^\dagger} \right) |\text{vac}\rangle.$$  

(24)

We can always decompose the state in this way because the modes are orthogonal and thus $[A_n, A_m^\dagger] = \delta_{nm}$. This is crucial because it means the state corresponding to Figure 3 can be written as $|\psi\rangle = |\text{THSS}\rangle \otimes |\phi\rangle$ and thus $|\phi\rangle$ can be traced out without degrading the purity of the desired modes. To trace out the state $|\phi\rangle$ in practice, we can pass the 9-peak JSA state through a spectral filter that blocks frequencies around $\Omega_0$. The resulting JSA is shown in Figure 3 (d).

This hybrid squeezed state can be tuned in real time by tuning the amplitude of each term in the pump function. This is because each term in the pump function uniquely corresponds to terms in the squeezing operator; specifically, each peak centred at $(\Omega_n, \Omega_m)$ corresponds to the squeezing term associated with the $A_n A_m$ operators in (24). The middle peak in the pump amplitude generates two JSA amplitudes at $(\Omega_{-1}, \Omega_{1})$ and $(\Omega_{1}, \Omega_{-1})$ but these both contribute to the same amplitude in the squeezed state. Since the pump is a function of three centre frequencies with no overlap we can independently vary the amplitude of all three and thus tune the squeezing parameters of the final squeezed state independently.

We proposed a method for generating multi-mode squeezed states of light encoded in Gaussian-like frequency modes. This method differs from related methods [10–12, 14–18] in several ways that might make it advantageous, depending on the application.

VI. CONCLUSION

FIG. 3. (a) Three-peak target phase-matching function as defined in Eq. (18) (with $c_0 = \sqrt{2}/(\sqrt{\pi}L_{s}(5))$, $c_1 = c_{1-3} = 3c_0/2$) and corresponding custom phase-matching function generated by the custom-poled crystal shown in (b). Parameters $\sigma_k$, $\delta k$, $N$ and $w$ are the same as in Figure 1. (c) The resulting joint spectral amplitude (JSA) as defined in Eq. (8) for a 3-peak phasematching function (PMF) and a 3-peak pump function with $\sigma$ and $\delta \omega$ the same as in Figure (d) The JSA in (c) with a filter applied to both the signal and idler modes, with a transmission function $(1 - \exp(-(\omega_2 - \Omega_2)^2/4\sigma^2_j)) \times (1 - \exp(-(\omega_i - \Omega_i)^2/4\sigma^2_j))$ where $\sigma_j = 2\sigma$. 

We proposed a method for generating multi-mode squeezed states of light encoded in Gaussian-like frequency modes. This method differs from related methods [10–12, 14–18] in several ways that might make it advantageous, depending on the application.
The proposed method relies on customizing the joint spectral properties of light generated via spontaneous parametric down conversion, which requires two independent ingredients. The first is the spectral engineering of light incident on the crystal, which we took as a given. The second is the engineering of the nonlinear crystal to have desired phasematching properties (captured by the phasematching function), for which we used an algorithm [39] that takes as an input, the target phasematching function, and outputs a binary string that defines poles in a ferroelectrically-poled crystal [42].

We used this method to design two kinds of squeezed states. The first is an N-mode squeezed state, which could generalize various quantum information protocols [7–9]. The second is a tunable hybrid squeezed state, which could make good resource states for the generation of non-Gaussian states via post-selection, particularly in situations where mode tunability is desired, e.g., to compensate for loss.

Future work could investigate the effects of time-ordering on the JSA [43], suppression of modes within the crystal using photonic stop bands [44], and possible applications in other areas of research, such as, multi-parameter quantum metrology and multi-channel quantum imaging. We expect that exploring new quantum states of light, encoded in new degrees of freedom, such as those proposed here will be a fruitful path toward discovering new quantum applications.

The Jupyter notebook used to generate the results in this paper can be found at https://github.com/abranczyk/custom-poling.

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Appendix A: Review of Squeezed states

Here we briefly review single-mode, two-mode and multi-mode squeezed states. A squeezed state prepared in a single mode $a_0$, can be written as [45]

$$|\text{SMSV}\rangle = e^{\frac{1}{2}\beta_{00}a_0^\dagger a_0^\dagger - \text{H.c.}} |\text{vac}\rangle,$$  \hspace{1cm} (A1)

where $\beta_{00}$ is the complex single mode squeezing parameter. If we define the uncertainty of an operator $A$ by $(\Delta A)^2 = \langle \text{SMSV}\rangle (A - \langle A \rangle)^2 |\text{SMSV}\rangle$, the single-mode squeezed state has the property that it minimizes the uncertainty relation below the quantum noise limit. This is taken advantage of in quantum sensing experiments, where one decreases the phase uncertainty to measure changes in distance beyond the quantum noise limit [46, 47].

A two-mode squeezed state prepared in modes $a_0$ and $a_1$, can be written as [45]

$$|\text{TMSV}\rangle = e^{\beta_{01}a_0^\dagger a_1^\dagger - \text{H.c.}} |\text{vac}\rangle,$$  \hspace{1cm} (A2)

where $\beta_{01}$ is the complex two-mode squeezing parameter. If one considers the effect that the squeezing operator has on the variance of the sum and difference of each modes conjugate variables, one finds a similar squeezing effect as for the single-mode state. For the two operators $A$ and $B$, the two-mode squeezed state minimizes the uncertainty relation between the conjugate variables $A - B$ and $A + B$ below the quantum noise limit. The reduction in noise between $A - B$ generates a high degree of correlation, and in the limit when $|\beta_{01}| \to \infty$, two-mode squeezed states are the continuous variable extension of maximally entangled Bell states [48]. The entanglement properties of two-mode squeezed state can be used for various quantum information protocols, such as, quantum teleportation and quantum cryptography [1, 2].

The squeezed states above can be generalized to $N$-partite entangled states as follows:

$$|\text{MMSV}\rangle = e^{\frac{1}{2}\sum_{n,m} \beta_{nm}a_n^\dagger a_m^\dagger - \text{H.c.}} |\text{vac}\rangle,$$  \hspace{1cm} (A3)

where the double sum in the exponential ranges from $n, m = -N, -N + 1, ..., 0, ..., N - 1, N$ and due to symmetry we take $\beta_{nm} = \beta_{mn}$. These multi-mode squeezed states can be used to generalize various quantum information protocols [49, 50].
Appendix B: Spontaneous Parametric Downconversion

In this section we review the nonlinear-optical process known as spontaneous parametric downconversion (SPDC). In a SPDC process the input photons, typically called pump photons, are each “downconverted” into two daughter photons, called the signal and idler. The signal and idler photons satisfy energy and momentum conservation with the pump photon given by

\[ \omega_p = \omega_s + \omega_i, \quad k_p(\omega_p) = k_s(\omega_s) + k_i(\omega_i) \]  

where \( \omega_j \) and \( k_j(\omega_j) \) are the frequency and wave vectors. The three fields of interest are labeled by \( J = p, s, i \), denoting the pump, signal and idler respectively. A detailed derivation of SPDC was given in [51] which we follow.

For a Type II downconversion process—in the rotating wave approximation, assuming the material is in an effective 1D structure where the field doesn’t vary in the orthogonal direction of area \( A \), with a non-linear coefficient \( \chi^{(2)}(z) \) that varies along the longitudinal direction \( z \)—the nonlinear Hamiltonian in the interaction picture is

\[
H_I(t) = -\hbar \int_0^\infty d\omega_p d\omega_i d\omega_s c_V(\omega_p) a_V^\dagger(\omega_i) a_H(\omega_s) e^{i(\omega_s + \omega_i - \omega_p)t} 2L \sqrt{\frac{\hbar \omega_p \omega_i \omega_s}{(4\pi)^3 \epsilon_0 A c^3 n_p(\omega_p)n_s(\omega_s)n_i(\omega_i)}} 
\times \frac{1}{L} \int_{-L/2}^{L/2} dz \chi^{(2)}(z) e^{i(k_p(\omega_p) - k_i(\omega_i) - k_s(\omega_s))z} + H.c.,
\]

For nonlinear materials we will be considering, \( \chi^{(2)}(z) \) is constant over a specified domain length and given by \( \chi^{(2)}(z) = \pm \chi^{(2)} \). We find it useful to define

\[
g(z) = \frac{\chi^{(2)}(z)}{\chi^{(2)}}
\]

to be the scaled nonlinearity function which has two values given by \( g(z) = \pm 1 \). If we take the initial ket \( |\psi(-\infty)\rangle \) to be a coherent state in the pump mode and the signal and idler vacuum then in the undepleted pump approximation (where we assume the pump light is unchanged with the removal of a photon) we can make the substitution that

\[
c_V(\omega_p) \rightarrow \sqrt{\alpha_p} |\alpha_p(\omega_p)\rangle,
\]

where \( \alpha(\omega_p) \) is the normalized frequency distribution of the pump laser, which satisfies

\[
\int |\alpha(\omega)|^2 d\omega = 1,
\]

and \( |\alpha_p|^2 \) is the number of photons in the pump. Then the Hamiltonian is given by

\[
H_I(t) = -\hbar \int_0^\infty d\omega_p d\omega_i d\omega_s A(\omega_i, \omega_s, \omega_p) \alpha(\omega_p) \Phi(\omega_s, \omega_i, \omega_p) a_V^\dagger(\omega_i) a_H(\omega_s) e^{i(\omega_s + \omega_i - \omega_p)t} + H.c.,
\]

where we set

\[
A(\omega_s, \omega_i, \omega_p) = 2L \chi^{(2)}(z) \sqrt{\frac{\alpha_p}{(4\pi)^3 \epsilon_0 A c^3 n_p(\omega_p)n_s(\omega_s)n_i(\omega_i)}},
\]

and define

\[
\Phi(\Delta k(\omega_i, \omega_s, \omega_p)) = \frac{1}{L} \int_{-L/2}^{L/2} dz g(z) e^{i\Delta k(\omega_i, \omega_s, \omega_p)z},
\]
to be the phasematching function (PMF) and set $\Delta k(\omega_p, \omega_s, \omega_i) = k_p(\omega_p) - k_i(\omega_i) - k_s(\omega_s)$ to be the phase mismatch.

In the interaction picture the states evolves according to interaction Hamiltonian given by [52]

$$i\hbar \frac{d|\psi(t)|}{dt} = H_I(t) |\psi(t)|,$$  \hspace{1cm} (B9)

which has a formal solution of

$$|\psi(\infty)\rangle = \mathcal{T} \left[ e^{-i \int_{-\infty}^{\infty} dt H_I(t)} \right] |\psi(-\infty)\rangle,$$  \hspace{1cm} (B10)

where $\mathcal{T}$ is the time-ordering operator and $|\psi(\infty)\rangle$ is the final state. Since the interaction Hamiltonian does not commute with itself at different times, we cannot in general drop the time ordering operator. It was shown in [43], that the time ordering leads to non trivial effects but only in the high pump power regime. For this work, we will assume low pump powers and not worry about these time ordering effects, i.e. we drop the time-ordering operator. Then integrating the interaction Hamiltonian with respect to $t$ introduces an energy conserving delta function $\delta(\omega_s + \omega_i - \omega_p)$ which we use to evaluate the $\omega_p$ integral. Then the state is given by

$$|\psi\rangle = e^{i \int d\omega_i d\omega_s A(\omega_s, \omega_i, \omega_i + \omega_s) \Phi(\Delta k(\omega_i, \omega_s))} \alpha_{\omega_i}(\omega_i) \alpha_{\omega_s}(\omega_s) + H.c. |\text{vac}\rangle,$$  \hspace{1cm} (B11)

where we made the simplification $A(\omega_s, \omega_i, \omega_i + \omega_s) \rightarrow A(\omega_s, \omega_i)$ and $\Phi(\Delta k(\omega_s, \omega_i)) \rightarrow \Phi(\Delta k(\omega_i, \omega_s))$ with

$$\Delta k(\omega_i, \omega_s) = k_p(\omega_i + \omega_s) - k_i(\omega_i) - k_s(\omega_s),$$  \hspace{1cm} (B12)

being the phase mismatch and $k_j(\omega_i) = \omega_j n_j(\omega_i)/c$. Lastly, for the frequency range of interest, to good approximation we can evaluate the frequency dependent term $A(\omega_s, \omega_i) \approx A(\Omega_s, \Omega_i) = A$, where $\Omega_s$ and $\Omega_i$ are the center frequencies of the signal and idler.

To push the equations further, we expand the phase mismatch $\Delta k$ to first order around the center frequencies $\Omega_s, \Omega_i, \Omega_p = \Omega_s + \Omega_i$ such that

$$\Delta k(\omega_i, \omega_s) = \Delta k_0 + k'_p(\omega_p - \Omega_p) - k'_i(\omega_i - \Omega_i) - k'_s(\omega_s - \Omega_s)$$  \hspace{1cm} (B13)

where we set $\Delta k_0 = k_p(\Omega_s + \Omega_i) - k_i(\Omega_i) - k_s(\Omega_s)$ and the first-order derivatives are $k'_j = \partial k_j(\omega_i)/\partial \omega_j|_{\omega_i=\Omega_i}$, and energy conservation ensures that $\omega_p = \omega_i + \omega_s$ and $\Omega_p = \Omega_i + \Omega_s$. Next, we work in the symmetric group-velocity matching regime, such that $k'_p = (k'_s + k'_i)/2$, then with these choices, we are left with

$$\Delta k(\omega_i, \omega_s) = \Delta k_0 + \frac{k'_i - k'_s}{2} (2(\omega_i - \Omega_i) - (\omega_s - \Omega_s)).$$  \hspace{1cm} (B14)

We note that $\Delta k(\omega_i, \omega_s)$ is now a function of the variable $\omega_i - \omega_s$ so we take $\Delta k(\omega_i, \omega_s) \rightarrow \Delta k(\omega_i - \omega_s)$ and therefore $\Phi(\Delta k(\omega_i, \omega_s)) \rightarrow \Phi(\omega_i - \omega_s)$ which for typical phase matching functions will be peaked along the diagonal. Calculating the square integral of $\Phi(\omega)$ we find it is not yet normalized and set its value to be $\mathcal{N}^2$ for some $\mathcal{N}$, therefore

$$\int d\omega |\Phi(\omega)|^2 = \mathcal{N}^2.$$  \hspace{1cm} (B15)

To normalize $\Phi(\omega)$ we define

$$\phi(\omega) = \frac{2}{\mathcal{N}} \Phi(\omega),$$  \hspace{1cm} (B16)

then the squeezed state is given by

$$|\psi\rangle = e^{i \gamma \int d\omega_i d\omega_s f(\omega_s, \omega_i) \alpha_{\omega_i}(\omega_i) \alpha_{\omega_s}(\omega_s) + H.c.} |\text{vac}\rangle,$$  \hspace{1cm} (B17)
where we defined the squeezing parameter $\gamma$ to be $\gamma = AN$ and defined the join spectral amplitude (JSA) as

$$f(\omega_s, \omega_i) = \alpha(\omega_i + \omega_s)\phi(\omega_i - \omega_s),$$  \hspace{1cm} (B18)

which due to the normalization in Eq. (B5) and (B16) satisfies the normalization

$$\int d\omega_s d\omega_i |f(\omega_s, \omega_i)|^2 = 1.$$  \hspace{1cm} (B19)

**Appendix C: Eliminating the Polarization Degree of Freedom for the Multi-mode State**

In Eq. (12) we decomposed the JSA into a set of frequency modes with a frequency distribution given by $G_n(\omega)$. Although we have provided a decomposition there is still the polarization degree of freedom which we need to remove. To eliminate the polarization degree of freedom we apply the following set of transformation. We begin by applying a polarization beam splitter which has the transformation property

$$U_{PBS}^\dagger a_H(1)(\omega)U_{PBS} = a_H(1')(\omega),$$
$$U_{PBS}^\dagger a_V(1)(\omega)U_{PBS} = a_V(1')(\omega),$$
$$U_{PBS}^\dagger a_H(2)(\omega)U_{PBS} = a_H(2')(\omega),$$
$$U_{PBS}^\dagger a_V(2)(\omega)U_{PBS} = a_V(2')(\omega),$$

where the superscripts denote the input and output spatial modes. Next we apply a half-wave-plate to one of the spatial modes set to 45° using the following transformation property

$$U_{HWP}^\dagger(45°)a_H(\omega)U_{HWP}(45°) = ia_V(\omega)$$
$$U_{HWP}^\dagger(45°)a_V(\omega)U_{HWP}(45°) = ia_H(\omega).$$  \hspace{1cm} (C2)

Finally, we recombine the modes using a 50:50 beam splitter set to 0° with the transformation given by

$$U_{BS}^\dagger(0°)a(1)(\omega)U_{BS}(0°) = \frac{1}{\sqrt{2}}(a(1)(\omega) + a(2)(\omega))$$
$$U_{BS}^\dagger(0°)a(2)(\omega)U_{BS}(0°) = \frac{1}{\sqrt{2}}(-a(1)(\omega) + a(2)(\omega)).$$  \hspace{1cm} (C3)

After applying these three transformations and using the property that $\gamma_{nm} = \gamma_{mn}$ and dropping the polarization degree of freedom since they are all the same the final state is given by

$$|\psi'\rangle = \frac{1}{2} \sum_{n,m} \gamma_{nm} A_n^{(1)} A_m^{(1)} - H.c. |\text{vac}\rangle \otimes e^{-\frac{i}{2} \sum_{n,m} \gamma_{nm} A_n^{(2)} A_m^{(2)} - H.c.} |\text{vac}\rangle,$$  \hspace{1cm} (C4)

which is two copies of a multi-mode squeezed state as defined in Eq. (4) in two different spatial degrees of freedom.

**Appendix D: Bias in the PMF**

For a crystal of length $L$ the approximate phasematching function is given by

$$\Phi_a(\Delta k) = \frac{1}{L} \int_{-L/2}^{L/2} dz g_a(z)e^{i\Delta k z},$$  \hspace{1cm} (D1)
where \( g_a(z) \) is the approximate nonlinearity function specified by the poling algorithm and is either \( \pm 1 \). The phasematching function can be written as a coherent sum by expanding the integral over \( z \) into each domain by

\[
\Phi_a(\Delta k) = \frac{1}{L} \sum_{n=0}^{N} g_n \int_{-\frac{L}{2} + nl_c}^{\frac{L}{2} + nl_c} e^{i\Delta k z} dz, \tag{D2}
\]

where \( l_c \) is the coherence length, \( Nl_c = L \), and \( g_n \equiv g_a(-\frac{L}{2} + nl_c \leq z \leq \frac{L}{2} + (n+1)l_c) \) is the nonlinearity within each domain. Then evaluating the integral and simplifying the approximate phasematching function is given by

\[
\Phi_a(\Delta k) = \frac{l_c}{L} \text{sinc} \left( \frac{\Delta kl_c}{2} \right) e^{i\Delta kl_c/2} e^{-i\Delta kL/2} \sum_{n=0}^{n=N} g_n e^{i\Delta kl_c n}. \tag{D3}
\]

Since the algorithm we are using to determine \( g_n \) is only for real phasematching functions we know the imaginary part will sum to zero and the nonzero contribution is only from the real part which is given by

\[
\text{Re}\{\Phi_a(\Delta k)\} = \frac{l_c}{L} \sum_{n=0}^{n=N} g_n \cos \left( \frac{\Delta kl_c}{2} (2n + 1 - N) \right), \tag{D4}
\]

which is manifestly symmetric about \( \Delta k = 0 \). However, the phasematching function we are designing has peaks at \( \Delta k = \Delta k_0 \pm m\delta k \) which is centered at \( \Delta k_0 \) ensuring the nonlinear generation is phasematched. Evaluating the approximate phasematching function in this vicinity leads to a \( 1/(\Delta k_0 \pm m\delta k) \) dependence which is no longer symmetric and leads to a bias on the left and right sides of \( \Delta k_0 \) shown in Figure 1.

Although the phasematching function bias for large or smaller values of \( \Delta k \) leads to less accurate PMFs; by decreasing the coherence length the affect of the bias is lessened, and the PMF is better approximated. Why can understand why this necessarily follows in two ways: As we decrease the coherence length the \( \text{sinc} \) function prefactor in Eq. (D4) becomes more broad as is therefore constant for larger values of \( \Delta k \); minimizing the bias. Secondly, by decreasing the coherence length we are increasing the “resolution” of our tracking algorithm which increases the accuracy of the approximated PMF. To further increase the accuracy of the generated PMFs we can move to sub-coherence domain engineering which was discussed in detail by Graffitti et al. in [39].

Appendix E: Design Considerations

Quasi-phase-matched periodically-poled crystals are known to generate amplitudes reduced by a factor of \( 2/\pi \) when compared with their phase-matched counterparts [36]. For example, if the unnormalized phase-matching function for a phase-matched crystal of length \( L \) is \( \Phi(\Delta k) = \frac{1}{L} \int_{-L/2}^{L/2} e^{-i\Delta k z} dz = \text{sinc}(\Delta kL/2) \), then for a periodically-poled crystal, it is \( \Phi(\Delta k) \approx (2/\pi)\text{sinc}(\Delta kL/2) \). This is due to the interference behind the quasi-phase-matching effect. The phase-matching function is at its maximum value of \( (2/\pi) \) at the phase-matching condition \( (\Delta k = 0) \). The amplitude along the longitudinal direction \( z \) is \( (2z/\pi L) \), which defines the maximum slope for the amplitude inside the crystal.

We can use the maximum slope restriction to put bounds on the coefficients \( c_n \) that scale each Gaussian peak in the target PMFs by making sure that the gradient of the amplitude function satisfies:

\[
\frac{d}{dz} A_t(z, \Delta k_0) \leq \frac{d}{dz} \left( \frac{2z}{\pi L} \right) = \frac{2}{\pi L}. \tag{E1}
\]

The amplitude of the PMF for a given \( \Delta k_0 \) throughout the crystal is given by

\[
A_t(z, \Delta k_0) = \frac{1}{L} \int_{-L/2}^{z} dz' g_t(z') e^{i\Delta k_0 z'}. \tag{E2}
\]
then the target nonlinearity function $g_t(z)$ can be found by inverting Eq. (17):

$$g_t(z) = \frac{L}{2\pi} \int_{-\infty}^{\infty} d(\Delta k) \Phi_t(\Delta k) e^{-i\Delta k z}. \quad (E3)$$

We now pick a specific form for the target PMF $\Phi_t(\Delta k)$ with $N_G = 2N + 1$ Gaussian amplitudes,

$$\Phi_t(\Delta k) = \sum_{m=-N}^{N} c_m e^{-\frac{(\Delta k - \Delta k_0 - mk)^2}{8\sigma_k^2}}, \quad (E4)$$

and insert it into (E3) to give

$$g_t(z) \approx \frac{2L\sigma_k}{\sqrt{2\pi}} e^{-2z^2\sigma_k^2} e^{-i\Delta k_0 z} \left( c_0 + 2 \sum_{m=1}^{N} c_m \cos(m\delta kz) \right), \quad (E5)$$

where we used the approximation that the amplitudes of the PMF are symmetric ($c_m = c_{-m}$). Now inserting (E5) into (E2) and taking the derivative with respect to $z$ we find that the coefficients must satisfy

$$\frac{2L\sigma_k}{\sqrt{2\pi}} e^{-2z^2\sigma_k^2/2} \left( c_0 + 2 \sum_{m=1}^{N} c_m \cos(m\delta kz) \right) \leq \frac{2L}{\pi}. \quad (E6)$$

If the inequality holds for $z = 0$, it holds for all $z$. Then taking the inequality at $z = 0$ the amplitude coefficients $c_m$ must satisfy

$$c_0 + 2 \sum_{m=1}^{N} c_m \leq \frac{1}{\sqrt{\frac{2}{\pi} L\sigma_k}}. \quad (E7)$$

If we ensure the prefactor coefficients satisfy the above inequality we guarantee that the target PMF amplitude can always be tracked by changing the domains of the crystal. More sophisticated treatments can be made to determine the optimal choice of constants $c_m$ by considering different choices of $z$ but as a first considering we stop with the inequality in Eq. (E7).
FIG. 4. Real part (a) and imaginary part (b) of five-peak target phase-matching function as defined in Eq. (18) (with $c_m = \sqrt{2}/(\sqrt{\pi}L\sigma_k)$, $\sigma_k = 2.5/L$ and $\delta k = 24\sigma_k$) and real part (c) and imaginary part (d) of corresponding amplitude function. We took $\sigma_k = 2.5/L$ to ensure that the target nonlinearity profile fits within the length of the crystal. The custom-poled crystal has $N = 1073$ domains of width $w = 18.63\mu$m. The domain width was chosen to match the phase-matching conditions of Type-II KTP in the symmetric group-velocity-matched regime.