Infrared behavior of dipolar Bose systems at low temperatures

Volodymyr Pastukhov

Department for Theoretical Physics, Ivan Franko National University of Lviv, 12 Drahomanov Street, Lviv-5, 79005, Ukraine

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We rigorously discuss the infrared behavior of the uniform three dimensional dipolar Bose systems. In particular, it is shown that low-temperature physics of the system is controlled by two parameters, namely isothermal compressibility and intensity of the dipole-dipole interaction. By using a hydrodynamic approach we calculate the spectrum and damping of low-lying excitations and analyze the infrared behavior of the one-particle Green’s function. The low-temperature corrections to the anisotropic superfluid density as well as condensate depletion are found. Additionally we derive equations of the two-fluid hydrodynamics for dipolar Bose systems and calculate velocities of first and second sound.

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I. INTRODUCTION

The experimental realization of the Bose-Einstein condensation of atoms with large magnetic moments such as chromium (52Cr) [1] and more recently dysprosium (164Dy) [2] and erbium (168Er) [3] stimulated extensive theoretical studies of these systems [4–6]. The presence of the comparatively strong dipole-dipole interaction gives rise to exciting properties of the dipolar Bose condensates, namely the direction-dependent spectrum of elementary excitations, anisotropic superfluidity [7] at finite temperatures, non-typical soliton formation [8, 9], and two types of the ground state [10]. In some configurations of the trapping potential the roton character [11, 12] of the spectrum of collective modes is intrinsic for dipolar condensates providing the remarkable phenomena like spatial roton confinement [13] and anomalous atom-number fluctuations [14].

Despite the trapped case where even a fully dipolar Bose gas can be realized by using a strong harmonic potential that confines the system in the plane perpendicular to external magnetic field [15] for the stabilization of three-dimensional homogeneous condensates with dipole-dipole interaction the short-range repulsion between particles is required. Fortunately, the strength of this repulsive term can be tuned by Feshbach resonance techniques allowing to study the collapse dynamics and expansion of dipolar condensates [16]. It was usually believed [17, 18] that the properties of these systems can be understood on the mean-field level but the recent observation of droplet formation in dysprosium condensates [19] breaks this notion providing the influential role of quantum fluctuations [20–22]. Recent path-integral Monte Carlo simulations [23] confirm these findings. It is interesting that the mean-field impact of three-body repulsion forces also leads to stable droplets [24] in dipolar Bose condensates. At finite temperatures the calculations with dipolar condensates are complicated but still tractable [26], where the role of the dipole-dipole interaction and trap geometry on the thermodynamics of dipolar Bose gases was discussed.

The purpose of the present paper is to explore exact low-energy and, in turn, low-temperature properties of dipolar Bose systems.

II. FORMULATION OF THE MODEL

We consider a system of $N$ spinless particles immersed in volume $V$ with the second-quantized Hamiltonian

$$H = H_0 + \Phi,$$  \hspace{1cm} (2.1)

were the first term is the kinetic energy operator

$$H_0 = -\frac{\hbar^2}{2m} \int d\mathbf{r} \psi^+(\mathbf{r}) \nabla^2 \psi(\mathbf{r})$$  \hspace{1cm} (2.2)

and $\Phi$ takes into account the pairwise interaction between particles

$$\Phi = \frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' \Phi(\mathbf{r} - \mathbf{r}') \psi^+(\mathbf{r}) \psi^+(\mathbf{r}') \psi(\mathbf{r}) \psi(\mathbf{r}).$$  \hspace{1cm} (2.3)

The field operators satisfy the usual bosonic commutation relations $[\psi(\mathbf{r}), \psi^+(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}')$, $[\psi(\mathbf{r}), \psi(\mathbf{r}')] = 0$. The two-particle potential $\Phi(\mathbf{r})$ necessarily contains dipole-dipole interaction

$$\Phi_d(\mathbf{r}) = \frac{3g_d}{4\pi} \left\{ \frac{1}{r^3} - \frac{3z^2}{r^5} \right\}.$$  \hspace{1cm} (2.4)

and any additional term to stabilize the system. Actually, the results obtained below are not dependent on the specific form of this part of the two-body interaction, except it must be repulsive enough at small inter-particle separations.
Avoiding the problem with infrared divergences \cite{27} we adopt the scheme normally used for low-dimensional systems \cite{28}

\[
\psi(r) = e^{i\varphi(r)}\sqrt{\rho(r)}, \quad \psi^+(r) = \sqrt{\rho(r)} e^{-i\varphi(r)}, \quad (2.5)
\]

where we introduce phase \(\varphi(r)\) and density operators \(\rho(r) = \psi^+(r)\psi(r)\) with commutator \([\rho(r'), \varphi(r)] = i\delta(r - r')\). After this substitution the Hamiltonian of the dipolar Bose gas reads

\[
H[\varphi(r), \rho(r)] = \frac{1}{2} \int dr \int dr' \Phi(r - r') \rho(r) \rho(r') \left( + \frac{\hbar^2}{2m} \int dr \left\{ \frac{(\nabla \rho(r))^2}{4\rho(r)} + \rho(r)(\nabla \varphi(r))^2 \right\} \right), (2.6)
\]

where the inconsequential constant term that shifts the ground-state energy of the system is omitted. Actually, Eq. (2.6) is the Hamiltonian obtained for the first time in Ref. \cite{29} and written here explicitly in the Hermitian form. In order to derive the original Bogoliubov-Zubarev result we have to use the following representation of field operators: \(\psi(r) \sim e^{i\varphi(r)}, \quad \psi^+(r) \sim \rho(r) e^{-i\varphi(r)}\). In context of liquid \(^4\)He theory the Hamiltonian (2.6) was studied extensively in Refs. \cite{30}.

To proceed we pass to the path-integral representation \cite{31}. In doing so we have to introduce new fields with explicit imaginary time \(\tau \in [0, \beta]\) dependence (where \(\beta = 1/T\) is the inverse temperature of the system). Then the partition function reads

\[
Z = \int D\varphi D\rho e^S, (2.7)
\]

where the functional integral is carried out over \(\beta\) periodic fields \(\varphi(x), \rho(x) (x = (\tau, r))\) with action

\[
S = \int_0^\beta d\tau \int dr \left\{ \rho(x) i\partial_\tau \varphi(x) + \mu \rho(x) \right\} - \int_0^\beta d\tau H[\varphi(x), \rho(x)]. (2.8)
\]

Here we made the following replacement in the Hamiltonian \(H[\varphi(r), \rho(r)] \rightarrow H[\varphi(x), \rho(x)]\) and introduced the chemical potential \(\mu\) of the system. For the spatially uniform system we can use the following decomposition of the density and phase fields

\[
\rho(x) = \rho(\tau) + \frac{1}{\sqrt{V}} \sum_{k \neq 0} e^{ikr} \rho_k(\tau), \quad \varphi(x) = \varphi(\tau) + \frac{1}{\sqrt{V}} \sum_{k \neq 0} e^{ikr} \varphi_k(\tau). (2.9)
\]

From the structure of action (2.8) it is clearly seen that the only term containing zero-momentum \(\varphi\)-variables is

\[
V \int_0^\beta d\tau \rho(\tau) i\partial_\tau \varphi(\tau). \]

The integration over \(\varphi(\tau)\) results in delta-functions that require \(\int \rho(\tau)\) to be a constant (independent of \(\tau\)) integer positive number \cite{32}, so for the partition function we have

\[
Z = \sum_{\mathcal{N} \geq 0} \exp \{ \beta V [\mu \mathcal{N} / V - f(\mathcal{N} / V)] \}, (2.10)
\]

where \(e^{-\beta V f(\mathcal{N} / V)}\) denotes the part of Eq. (2.7) with \(\rho_k(\tau), \varphi_k(\tau)\) integrated out and \(\rho(\tau)\) replaced by \(\mathcal{N} / V\).

In the thermodynamic limit when the number of particles \(N\) together with volume \(V\) of the system tend to infinity (but \(N/V = \text{const} = \rho\)), the summation in the above formula can be easily changed by integration, which we perform using the steepest descent method. Finally, the asymptotically exact expression for the partition function is

\[
Z = \exp \{ \beta V [\mu \rho - f(\rho)] \}, \quad \mu = \left( \frac{\partial f}{\partial \rho} \right)_T, (2.11)
\]

and therefore we can proceed our consideration in the canonical ensemble.

### III. LOW-TEMPERATURE BEHAVIOR

#### A. Low-energy excitations

In the previous section it was argued that considering only part of action (2.8) with non-zero momentum modes we can calculate the free energy of the system \(V f(\rho)\) identifying \(\rho\) as the equilibrium (uniform) density. Now we are in position to formulate the theory in terms of density and phase fluctuations. The corresponding action after series expansion of \(\rho(x)\)-fields near \(\rho\) reads

\[
S = S_0 + S_{\text{int}}, (3.12)
\]

where \(S_0\) is the Gaussian term

\[
S_0 = \text{const} - \frac{1}{2} \sum_k \left\{ \omega_k \varphi_k \rho_{-k} - \omega_k \varphi_{-k} \rho_k \right\} + \frac{\hbar^2 k^2}{2m} \rho \varphi_k \varphi_{-k} + \left[ \frac{\hbar^2 k^2}{4m^2} + \nu(k) \right] \rho_{-k} \rho_k, (3.13)
\]

here we introduce the notation for the four-momentum \(K = (\omega_k, k)\), where \(\omega_k\) is the bosonic Matsubara frequency; \(\nu(k)\) is the Fourier transform of \(\Phi(r)\); \text{const} = \((\beta V)^2 \nu(0)/2 + \beta \frac{1}{2} \sum_{k \neq 0} \nu(k) + \hbar^2 k^2 / 2m\) and \(\nu(0)\) should be treated as a direction-averaged value of \(\nu(k)\) in the \(k \rightarrow 0\) limit (see discussion in Ref. \cite{33}). Thus, due to the homogeneity of the system the dipolar interaction does not affect on the mean-field ground-state energy. On the other hand, the presence of external non-uniform potential that breaks continuous translation invariance and causes a small deviation \(\delta \rho(r)\) in the density distribution shifts the thermodynamic potentials by \(\frac{1}{2} \int dr \int dr' \delta \rho(r') \Phi_d(r' - r) \delta \rho(r)\).
The last term in Eq. (3.12) takes into account the simplest collision processes between elementary excitations

\[ S_{int} = -\frac{1}{2\sqrt{3}V} \sum_{K,Q} D^{(0)}_{\varphi \rho \rho}(K,Q) - K - Q)\varphi_K \varphi_Q \rho_K - Q \rho_Q + \frac{1}{3!\sqrt{3}V} \sum_{K+Q+P=0} D^{(0)}_{\rho \rho \rho}(K,Q,P)\rho_K \rho_Q \rho_P, \quad (3.14) \]

where bare vertices \( D^{(0)}_{\varphi \rho \rho}(K,Q) - K - Q) = -\frac{\hbar^2}{m} k q \) and

\[ D^{(0)}_{\rho \rho \rho}(K,Q,P) = -\frac{\hbar^2}{m^2} m \rho (k^2 + q^2 + p^2). \]

In fact, \( S_{int} \) contains an infinite series, but we present only the relevant terms for our low-temperature description.

Recently [34] we showed how to relate the infrared asymptotics of the matrix correlation function

\[ D^{-1}(K) = \left( \frac{\varphi_K \varphi_Q - K}{\varphi_Q \varphi_K - Q} \right), \quad (3.15) \]

with macroscopic parameters of the system. For later convenience we introduce the notation for the second-order vertices \( D(K) = D_0(K) - \Pi(K) \), where \( \Pi(K) \) is the self-energy matrix and \( D_0(K) \) is given by Eq. (3.13).

particularly, it was demonstrated that the sound velocity of low-lying excitations can be uniquely written in terms of the superfluid density and inverse susceptibility. It should be noted that the same behavior of the low-energy spin-wave excitations is observed in the Heisenberg and easy-plane antiferromagnets [35, 36]. This analysis can be naturally extended on the Bose systems with dipole-dipole interaction. Moreover, for the anisotropic interaction the situation is even more interesting. In particular, for a vertex with two external \( \rho \)-lines \( D_{\rho \rho}(K) \) we obtain

\[ D_{\rho \rho}(K \rightarrow 0) = \left( \frac{\partial \mu}{\partial \rho} \right) T \left\{ 1 + \epsilon(3k^2/k^2 - 1) \right\}, \quad (3.16) \]

where \( (\partial \mu/\partial \rho)_T \) is the inverse isothermal compressibility of the system and for convenience the notation \( \epsilon = g_d^f (\partial \mu/\partial \rho)_T \) is used. For the mechanical stability it is crucial that \( \epsilon < 1 \). A key moment in obtaining of this exact identity is very similar to the original derivation of the Hugenholtz-Pines relation [31, 37]. Taking into account the fact that action (3.12) is the Taylor series expansion of the initial Eq. (2.8) near equilibrium value of density it immediately follows that exact vertices in the low-energy limit

\[ D_{\rho \rho}(0,0,0) = \left( \frac{\partial^3 f}{\partial \rho^3} \right)_T \left( \frac{\partial^2 \mu}{\partial \rho^2} \right) T, \ldots (3.17) \]

For the same reason we conclude that differentiation of every exact vertex function with respect to \( \rho \) gives the vertex with one more zero-momentum \( \rho(K) \) line.

To derive the second class of identities suppose that our system is moving as a whole with velocity \( v \). In the field-theoretic language it is equivalent to the following local gauge transformation \( \varphi(x) \rightarrow \varphi(x) - m v \rho v / \hbar \) of the initial action (2.8). Rotational invariance in the \( x - y \) plane ensures that the thermodynamic quantities of a moving Bose system are functions of \( v_1^2 \) and \( v_2^2 \) only, so for the free energy density we have

\[ f_s(\rho) = f(\rho) + \frac{m}{2} \left\{ \rho_1^2 v_1^2 + \rho_2^2 v_2^2 \right\} + o(v_1^2, v_2^2). \quad (3.18) \]

At zero temperature limit and in the absence of disorder [35, 39] the whole system is superfluid, i.e., \( \rho_\perp = \rho_\parallel = \rho \) and the last term in (3.18) is equal zero identically, restoring Galilean invariant form of the function \( f_s(\rho) \).

Although we do not present the explicit formula for gauge transformed action, it is easy to argue that the differential operator \( \frac{k_{\perp}}{\eta_{\perp}} \frac{\partial}{\partial \rho} \) acting on the bare vertex function adds \( \varphi(K) \) line with zero frequency and vanishingly small \( k \). Of course, the same conclusions can be drawn for the exact vertices. Thus despite of the complexity of action (3.12), the latter observation leads to (formally treating the Matsubara frequency as a continuous variable)

\[ D_{\varphi \varphi}(K \rightarrow 0) = \frac{\hbar^2}{m} \left\{ \rho_\perp k_\perp^2 + \rho_\parallel k_\parallel^2 \right\}, \]

\[ D_{\varphi \rho}(K \rightarrow 0) = \omega_k, \quad (3.19) \]

and allows to shed light on the infrared structure of the perturbation theory.

First of all, let us briefly discuss the long-length behavior of third-order vertex functions. From the previous analysis we have already found out that \( D_{\rho \rho \rho}(K,Q,P) \) tends to constant at small values of its arguments. Combining two above-mentioned differentiation rules with respect to the density and velocity of a moving Bose system we arrive with asymptotically exact result

\[ D_{\varphi \varphi \rho}(K,Q) - K - Q)|_{K,Q \rightarrow 0} = \frac{\hbar^2}{m} \left\{ \left( \frac{\partial \rho_\parallel}{\partial \rho} \right) T q_\parallel + \left( \frac{\partial \rho_\perp}{\partial \rho} \right) T q_\perp \right\}. \quad (3.20) \]

The presence of even powers in the series expansion (3.18) over velocity provides the following estimation for \( D_{\rho \rho \rho}(K,Q,P)|_{K,Q \rightarrow 0} \propto \omega_k k_\perp^2 \) when \( k_\parallel \gg k_\perp \) and \( D_{\rho \rho \rho}(K,Q,P)|_{K,Q \rightarrow 0} \propto \omega_k k_\perp^2 \) in the opposite limit. The main advantage of the hydrodynamic description is absence of infrared divergences in perturbative calculations. Therefore, it means that the behavior of the corresponding exact correlation function is qualitatively reproduced even on the one-loop level. The appropriate analysis leads to \( D_{\varphi \varphi \varphi}(K,Q,P)|_{K,Q,P \rightarrow 0} \propto k_\parallel q_\parallel + perm \).

The above results can be easily extended to higher-order vertices. In general, the presence of two external \( \varphi \)-lines adjoins to the appropriate vertex function \( D_{\varphi \varphi \varphi}(K,Q,P) \) a factor \( k_\parallel q_\parallel \) (or \( k_\parallel q_\perp \)); for the vertex with four lines we find \( D_{\varphi \varphi \varphi \varphi}(K,Q,P,S) \propto (k_\parallel q_\parallel + perm \) and so on for any vertex with even number of \( \varphi \)-lines at finite temperatures. The low-energy result for functions with odd number of external phase-field lines can be immediately obtained from the limiting behavior of \( D_{\varphi \varphi \varphi}(K,Q,P) \) and \( D_{\varphi \rho \rho \rho}(K,Q,P) \).

Summarizing the results of this subsection, it should be noted that in the long-length limit the quasiparticle spectrum exhibits acoustic behavior \( E_k = \hbar k c_k \) with
anisotropic sound velocity

\[ c_k^2 = \frac{1}{m} \left( \frac{\partial \mu}{\partial \rho} \right)_T \left\{ \rho_n^+ + (\rho_n^+ - \rho_n^-)k_z^2/k^2 \right\} \times \left\{ 1 + \epsilon(3k_z^2/k^2 - 1) \right\}, \] (3.21)

and vertices given by Eqs. (3.17), (3.20) describe the effective interaction between phonons at low temperatures.

**B. Anisotropic superfluid density**

Let us calculate the leading-order contribution to the superfluid density at low temperatures. Nevertheless it is well-known result \[10\] for Bose systems with isotropic interaction and the original Landau prescription is an easier one, but in order to prove correctness of our approach we will perform these calculations diagrammatically. The task is rather simple because we have to find the long-length behavior of \( \Pi_{\varphi\varphi}(K) \) at zero Matsubara frequency. In this limit the only relevant vertex is \( D_{\varphi\varphi\rho}(K, Q|P) \), so the result is given by two diagrams (see Fig. 1). Moreover, at the low-temperature region we can neglect the difference between superfluid densities and the total one, i.e., the following temperature region we can neglect the difference between given by two diagrams (see Fig. 1). Moreover, at the low-

**C. Phonon damping**

The quasiparticle picture of the low-energy description provides that the damping has to be small compared to the energy of elementary excitations. In general, the inverse life-time of low-lying excitations written in terms of the imaginary \( \Im \Pi_{\varphi\varphi}(\omega, k) \), \( \Im \Pi_{\varphi\rho}(\omega, k) \) parts and real \( \Re \Pi_{\varphi\varphi}(\omega, k) \) part of appropriate self-energies (after analytical continuation in the upper complex half-plane \( \Im \omega \rightarrow \omega + i0 \)) reads

\[ \Gamma_k = \frac{\rho \hbar k}{2mc_k} \Im \Pi_{\varphi\rho}(E_k, k) + \frac{mc_k}{2\hbar k} \Re \Pi_{\varphi\varphi}(E_k, k) - \Re \Pi_{\varphi\rho}(E_k, k), \] (3.24)

Similarly to systems with isotropic interaction, the damping of spectrum of a dipolar Bose gas is small in the long-length limit and at very low temperatures. Actually, this fact enables to calculate the leading-order contribution to the damping of phonon mode exactly. Applying unitarity conditions at finite temperatures \[41\] and using our estimations for the infrared structure of the effective action from the previous section we evaluated the imaginary parts of various analytically continued second-order vertices (see Appendix for details).

In zero-temperature limit the considerable contribution to the damping is caused by the quasiparticle decay into two excitations with lower energies. This mechanism which was originally proposed in Ref. \[42\] and usually called the Beliaev damping is also responsible for the

FIG. 1: Diagrams contributing to the superfluid density. Dashed and solid lines denote phase and density fields, respectively. Light dot is \( D_{\varphi\varphi\rho}(0) \) and black dot stands for the exact vertex function.

FIG. 2: Diagrammatic representation of the leading-order contribution to \( \Re \Pi_{\varphi\rho}(\omega, k) \). Crosses denote the spectral weights of the appropriate pair correlation functions (see Ref. \[11\] for details).

FIG. 3: Exact low-energy asymptotics of \( \Im \Pi_{\varphi\rho}(\omega, k) \).
decay of anisotropic phonons in a dipolar Bose system

\[ \Gamma_k^B = \frac{3}{64\pi} \frac{\hbar^2 k^5}{m \rho} \frac{(1 - \epsilon + 3\epsilon k^2/k^2)^{3/2}}{(1 - \epsilon)\sqrt{1 + 2\epsilon}} \times \left\{ 1 + \frac{\rho^2}{3k} \frac{\partial^2}{\partial \rho^2} \right\}^2. \]  

(3.25)

Of course, the processes when a given excitation decays into three, four, and larger number of quasiparticles are also possible, but their contribution to the damping is negligible in the long-length limit \( \hbar k/mc \ll 1 \). Indeed, using a dimensional analysis it is easy to argue that in the leading order the three-phonon collisions give rise to the asymptotics of order \( k^3 \) for the damping.

The situation changes at finite temperatures where the damping is controlled by the Landau mechanism of quasi-particle decay. At the low-temperature limit the inverse damping is controlled by the Landau mechanism of quasi-vector and taking into account only binary quasi-particle collisions we obtain

\[ \Gamma_k^L = \frac{3\pi^3}{40} \frac{\hbar k T^4}{mc (\hbar c)^3 \rho} \frac{(1 - \epsilon + 3\epsilon k^2/k^2)^{-1/2}}{(1 - \epsilon)\sqrt{1 + 2\epsilon}} \times \left\{ 1 + \frac{\rho^2}{3k} \frac{\partial^2}{\partial \rho^2} \right\}^2. \]  

(3.26)

It should be mentioned that the above formula is valid only in the limit \( k \ll mc/\hbar \) and for temperatures \( T \ll mc^2 \). In order to explore the damping at higher temperatures and finite momenta one has to take into account the contribution of all third-order vertex functions \[43\]. Moreover in contrast to the zero-temperature result, formula (3.26) is not an exact low-temperature estimation of the decay rate because the \( T^4 \) terms are also present in the damping caused by three-phonon, four-phonon, etc. scattering processes. Therefore, equation (3.26) is accurate only in the dilute limit where it generalizes the well-known result \([44]\) on the Bose systems with dipole-dipole interaction.

### IV. CONNECTION TO A MODEL WITH CONDENSATE

The model with condensate is crucial for understanding properties of interacting Bose particles. In fact, starting from the original Bogoliubov theory \[45\] very important results concerning the field-theoretical description of these systems \[37\] very important results concerning the field-theoretical description of the excitation spectrum \[37\] \[47\] were obtained within this approach. The purpose of this section is to show how the long-length limit of the one-particle Green’s function can be reproduced using a hydrodynamic formulation. Therefore, our objective is to study the properties of the following function

\[ G(x - x') = -\langle e^{i\varphi(x)\sqrt{\rho(x)}\rho(x')e^{-i\varphi(x')}} \rangle. \]  

(4.27)

In general, these calculations cannot be done to the end, but for our consideration it is enough to know the long-range behavior of this function. At equal time-arguments \( (\tau \to \tau') \) \[4.27\] up to a sign is equal to the one-body density-matrix \( F(\mathbf{r} - \mathbf{r}') \). The latter is the Fourier transform of the particle number distribution in coordinate space and \( F(\infty) \) coincides with condensate density \( \rho_0 \). It is intuitive that Eq. (4.27) suggests the condensate density to have an exponential form. This very important feature of the one-particle density matrix was mentioned for the first time in Ref. \[48\]. Due to non-commutativity of phase and density operators (see Eq. (2.5)) the condensate fraction in the hydrodynamic approach can be calculated as follows

\[ \rho_0 = \lim_{\tau' \to \tau - 0} \langle \varphi(x)\sqrt{\rho(x)}\rangle_{|x' = x}, \]  

(4.28)

or, equivalently \( \lim_{\tau' \to \tau - 0} \langle \rho(x)e^{-i\varphi(x')}\rangle_{|x' = x} \). Of course, the number of particles with zero momentum is a model-dependent quantity, but the leading-order low-temperature condensate depletion demonstrates universal power-law behavior \[49\]. Taking into account the above estimation for the vertices one may show that

\[ \frac{\rho_0|_{T \to 0}}{\rho_0|_{T = 0}} = 1 - \frac{1}{V} \sum_{k \neq 0} \left\{ \frac{1}{\beta} \sum_{\omega_k} - \int_{-\infty}^{\infty} \frac{d\omega_k}{2\pi} \right\} \times \langle \varphi_k \varphi_{-k} \rangle + \ldots. \]  

(4.29)

At very low temperatures it is enough to substitute only the infrared asymptotic of \( \langle \varphi_k \varphi_{-k} \rangle \) in Eq. (4.29). After calculation of simple integrals we arrive at the asymptotically exact result

\[ \frac{\rho_0|_{T \to 0}}{\rho_0|_{T = 0}} = 1 - \frac{mT^2}{24\hbar^4 c^3 \rho_0} \frac{1}{\sqrt{\epsilon}} \ln \frac{\sqrt{1 + 2\epsilon} + \sqrt{3\epsilon}}{\sqrt{1 + 2\epsilon} - \sqrt{3\epsilon}}. \]  

(4.30)

Making use of transformation to the four-momentum space for the one-particle Green’s function we find

\[ G(P) = \int dx e^{-iPx} \{ \rho_0 + G(x) \}, \]  

(4.31)

and taking into account equations (4.27) and (4.28) we have

\[ G(P \to 0) = -\rho_0 \langle \varphi P \varphi_{-P} \rangle. \]  

(4.32)

Actually, the above equation generalizes the well-known Gavoret-Nozières result \[47\], which was also obtained in Ref. \[27\] with taking into consideration correct infrared properties of self-energies. More recently the low-energy behavior of the one-particle Green’s function was analyzed using renormalization group techniques \[51\] \[62\] and two-fluid hydrodynamics \[53\]. At finite temperatures the zero-frequency limit of the formula (4.32) is the extension of the Josephson result \[60\] on systems with anisotropic interaction. It is also clear that due to the presence of an external field that aligns all dipoles along one direction,
equation [4.31] possesses an anisotropic dependence on the wave-vector, which for the two-dimensional system leads to very intriguing non-diagonal long-range behavior of the one-body density matrix \( F(r) \).

Finally, the same analysis can be easily applied to the anomalous one-particle Green’s function. Without going into details we note that the infrared asymptotic behavior of this function up to a sign coincides with [4.32].

V. PROPAGATION OF SOUND MODES

Now we briefly discuss peculiarities of macroscopic two-fluid hydrodynamics in Bose systems with anisotropic interaction. In particular we are interested in describing of the small-amplitude oscillations of various physical quantities near their equilibrium values. Thus, local thermodynamic equilibrium is assumed. Moreover, it is also supposed that all observables are smooth functions of spatial coordinates \( r \) and time \( t \) although we will not write this explicit dependence. Our further consideration with minor changes is adopted from [54]. The local density should satisfy the continuity equation

\[ \partial_t \rho + \text{div} \mathbf{j} = 0, \quad (5.33) \]

where the density current consists of two terms \( \mathbf{j} = \mathbf{j}_s + \mathbf{j}_n \).

The first one is the current of the superfluid component \( \mathbf{j}_s = \left( \rho_s^+ \mathbf{v}_s^+, \rho_s^+ \mathbf{v}_s^+ \right) \), which correctly reproduces the anisotropic behavior of the system, and the second \( \mathbf{j}_n = \left( \rho_n^+ \mathbf{v}_n^+, \rho_n^+ \mathbf{v}_n^+ \right) \) corresponds to the normal component of the dipolar Bose system. The evolution of the superfluid velocity is governed by the following equation

\[ \partial_t \mathbf{v}_s + \frac{1}{m} \nabla \mu_d = 0, \quad (5.34) \]

where the chemical potential of a slightly non-uniform system contains the additional term \( \mu_d(r,t) = \mu(\rho(r,t)) + \int d^2 r \Phi_d(r-r')\rho(r',t) \). The same concerns the local pressure \( p_d \) in the equation for the density current

\[ \partial_t \mathbf{j} + \frac{1}{m} \nabla p_d = 0. \quad (5.35) \]

At last we have to write down the equation for the entropy per unit volume \( s \) which together with thermodynamic identity

\[ \rho_d p_d = -sdT + dp_d, \quad (5.36) \]

close the system of equations for sound propagation. From the general arguments it is clear that the entropy \( s \) is proportional to the number of quasiparticles \( n_T = \frac{1}{\beta} \sum_{\alpha} \rho_\alpha (e^{\beta E_\alpha} - 1)^{-1} \) in a unit volume of the system. On the other hand, by using kinetic equation for the quasiparticle distribution function [55] (in the frame, where superfluid component is at rest \( \mathbf{v}_s = 0 \)) it is easy to show that \( n_T \) should satisfy the continuity equation, which can be immediately rewritten for the entropy density

\[ \partial_t s + \text{div}(s \mathbf{v}_n) = 0. \quad (5.37) \]

Note that instead of the density current (5.35) the entropy transport is isotropic. The further derivation of sound velocities is almost standard. Combining equations (5.34) and (5.35) one gets

\[ \partial_t^2 \rho = \frac{1}{m} \Delta p_d. \quad (5.38) \]

Introducing the entropy per particle \( \tilde{s} = s/\rho \) and making use of Eqs. (5.34), (5.37) we finally obtain

\[ \partial_t^2 \tilde{s} = \frac{\tilde{s}^2}{m} \left\{ \left( \rho_s^+ / \rho_n^+ \right) \Delta_\perp + \left( \rho_s^z / \rho_n^z \right) \partial^2_\perp \right\} T. \quad (5.39) \]

Taking the local density and temperature as independent variables and after substitution of the plane-wave solution \( \rho, T \propto e^{i(kr-\omega t)} \) the obtained secular equation that determines sound velocities reads

\[ \left\{ u^2 - \frac{1}{m} \left( \frac{\partial^2 s}{\partial p} \right)_T \right\} \left[ 1 + \epsilon(3\cos^2 \theta - 1) \right] \left\{ \frac{\tilde{c}_p}{T} u^2 - \frac{\tilde{s}^2}{m} \left[ \frac{\rho_s^+}{\rho_n^+} \sin^2 \theta + \frac{\rho_s^z}{\rho_n^z} \cos^2 \theta \right] \right\} - \frac{u^2}{m} \left[ 1 - \frac{1}{\rho} \frac{\partial (\rho \partial T)}{\partial T} \right] = 0, \quad (5.40) \]

perturbations (first sound) is

\[ u_1^2 = c^2 (1 + \epsilon(3\cos^2 \theta - 1)), \quad (5.41) \]

and taking into account Eq. (3.23) as well as limiting behavior of various thermodynamic quantities for the ve-
loicity of temperature waves (second sound) we obtain
\[ u_2^2 = u_1^2 / 3. \]  \hspace{1cm} (5.42)

In the normal phase Eq. (5.40) has a unique solution
\[ u^2 = \frac{1}{m} \left( \frac{\partial \rho}{\partial \rho} \right) + \frac{\rho g_4}{m} (3 \cos^2 \theta - 1) \]  \hspace{1cm} (5.43)

that recovers the usual velocity of adiabatic pressure waves in the fully polarized dipolar system. In the magnetic field of finite magnitude the above hydrodynamic equations should be complemented with the equation for magnetization [50].

VI. CONCLUSIONS

In conclusion, we have studied the properties of Bose systems with dipole-dipole interaction. Particularly, by means of hydrodynamic approach we analyzed the exact infrared structure of low-energy action that describes density and phase fluctuations. Within these results we calculate the spectrum and damping of collective modes in terms of macroscopic quantities of a dipolar Bose system, find out the long-length behavior of the one-particle Green’s functions and perform calculations of low-temperature corrections to the anisotropic superfluid density. The leading-order temperature dependence of the condensate fraction is also obtained. Additionally we considered the linearized macroscopic hydrodynamics of dipolar superfluids and explored peculiarities of first and second sounds. These findings clearly demonstrate the impact of dipole-dipole interaction on the anisotropic low-energy properties of the system.

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VII. APPENDIX

In this section we present the explicit expressions for the imaginary parts of the exact second-order vertices in the low-length limit. The leading-order contribution is determined by all diagrams with only two vertices, namely \( D_{\varphi \varphi}(K, Q, P) \) and \( D_{\rho \rho}(K, Q, P) \). Feynman diagrams contributing to \( \Im \Pi_{\varphi \varphi}(\omega, k) \) and \( \Im \Pi_{\rho \rho}(\omega, k) \) are given in Fig. 2 and Fig. 3, respectively. In the same manner the low-energy behavior of \( \Re \Pi_{\varphi \varphi}(\omega, k) \) can be easily obtained from Fig. 1. For convenience terms contributing to the Beliaev damping

\[
\Im \Pi^B_{\varphi \varphi}(\omega, k) = \frac{\pi}{4V} \sum_{q \neq 0} \frac{\hbar^2 \mathbf{kq}}{m^2} \left[ \frac{\mathbf{kq} \cdot \mathbf{c}_q \cdot (\mathbf{q} + \mathbf{k})}{q c_{q+k}} - k(\mathbf{q} + \mathbf{k}) \right] \delta(E_{q+k} + E_q - \omega),
\]

\[
\Im \Pi^B_{\rho \rho}(\omega, k) = \frac{\pi}{8V} \sum_{q \neq 0} \frac{\hbar^2 \mathbf{q} \cdot (\mathbf{q} + \mathbf{k})}{\rho^2 c_q c_{q+k}} \left[ \frac{\mathbf{q} \cdot (\mathbf{q} + \mathbf{k})}{\mathbf{q} c_{q+k}} c_{q+k} - \frac{\rho^2}{c_q} \frac{\partial}{\partial \rho} \right] \delta(E_{q+k} + E_q - \omega),
\]

\[
\Re \Pi^B_{\varphi \varphi}(\omega, k) = -\frac{\pi}{4V} \sum_{q \neq 0} \frac{\hbar^2 \mathbf{kq}}{m \rho} \left[ \frac{\mathbf{q} \cdot (\mathbf{q} + \mathbf{k})}{\mathbf{q}} c_q - \frac{\mathbf{q} \cdot (\mathbf{q} + \mathbf{k})}{\mathbf{q}} \frac{c_{q+k}}{c_q} \rho^2 \frac{\partial}{\partial \rho} \right] \delta(E_{q+k} + E_q - \omega),
\]

and the Landau damping

\[
\Im \Pi^L_{\varphi \varphi}(\omega, k) = -\omega \frac{\pi}{V} \sum_{q \neq 0} \left[ \frac{\hbar^2 \mathbf{kq}}{m} \right]^2 \left\{ \frac{\partial}{\partial E_q} n(\beta E_q) \right\} \delta(E_{q+k} - E_q - \omega),
\]

\[
\Im \Pi^L_{\rho \rho}(\omega, k) = -\omega \frac{\pi}{4V} \sum_{q \neq 0} \left( \frac{E_q}{\rho} \right)^2 \left[ 1 + \rho^2 \frac{\partial}{\partial \rho} \left( \frac{c_q^2}{\rho} \right) \right]^2 \left\{ \frac{\partial}{\partial E_q} n(\beta E_q) \right\} \delta(E_{q+k} - E_q - \omega),
\]

\[
\Re \Pi^L_{\varphi \varphi}(\omega, k) = \omega \frac{\pi}{2V} \sum_{q \neq 0} \frac{\hbar^2 \mathbf{kq} E_q}{m \rho} \left[ 1 + \rho^2 \frac{\partial}{\partial \rho} \left( \frac{c_q^2}{\rho} \right) \right] \left\{ \frac{\partial}{\partial E_q} n(\beta E_q) \right\} \delta(E_{q+k} - E_q - \omega),
\]

are written separately.
