Simplifying the axiomatization for the order affine geometry
Dafa Li
Dept of mathematical sciences
Tsinghua University, Beijing 100084, China

Abstract
Based on an ordering with directed lines and using constructions instead of
existential axioms, von Plato proposed a constructive axiomatization of ordered
affine geometry. There are 22 axioms for the ordered affine geometry, of which
the axiom I.7 is about the convergence of three lines (ignoring their directions).
In this paper, we indicate that the axiom I.7 includes much redundancy, and
demonstrate that the complicated axiom I.7 can be replaced with a simpler
and more intuitive new axiom (called ODO) which describes the properties of
oppositely and equally directed lines. We also investigate a possibility to replace
the axiom I.6 with ODO.

Keywords: the order affine geometry, axiomatization, natural deduction,
automated theorem proving, the first order logic.

1 Introduction
Heyting proposed the constructive axiomatization for the elementary geometry
[1]. He adopted the concepts of distinct points and distinct lines rather than the
concepts of equal points and lines. von Plato proposed the axioms of construc-
tive geometry for which he introduced the concepts of convergent lines instead
of parallel lines and of apartness of a point from a line instead of incidence of a
point with a line [2]. Recently, he presented a constructive theory of ordered
affine geometry [3].

Automated theorem proving makes a great progress [5, 6, 7, 8]. McCune
proved that Robbins algebras are Boolean with the theorem prover EQP [8].
Since the problem was posed by Herbert Robbins in the 1930s, it was conjectured
that all Robbins algebras are Boolean algebras.

By using theorem prover ANDP [9, 10], we found a natural deduction of
Halting problem [11], and simplified von Plato’s axiomatization of constructive
apartness geometry and orthogonality [12, 13], and indicated that the equality
axioms are not independent [14].

In this paper, we simplify the axiomatization for ordered affine geometry by
using theorem prover. We indicate that the axiom 1.7 includes some redundancy
and show that the complicated axiom I.7 can be replaced with a simpler and
more intuitive new axiom.

2 Simplifying the axiomatization for von Plato’s
order affine geometry
The five basic relations DiPt, DiLn, Undir, L-Apt, and L-Con and four con-
structions ln(a, b), pt(a, b), par(l, m), and rev(l) are used to describe the axiom-
atization for the order affine geometry [4]. In total, von Plato’s axiomatization has 22 axioms [4].

In this paper, we want to replace the axiom I.7 with a simpler and more intuitive new axiom. Only four axioms I.5 to I.8 are concerned in this paper. Only a basic relation Undir(l, m) and only a construction rev(l) appear in the axioms I.5 to I.8.

Undir(l, m) means l and m are unequally directed lines.

rev(l) stands for the reverse of line l.

2.1 List the four axioms I.5 to I.8 as follows.

The axiom I.5,
\[ \sim \text{Undir}(l, l). \] (1)

The axiom I.6:
\[ \text{Undir}(l, m) \rightarrow \text{Undir}(l, n) \lor \text{Undir}(m, n) \] (2)

The axiom I.7:
\[ \text{Undir}(l, m) \land \text{Undir}(l, \text{rev}(m)) \rightarrow \text{Undir}(l, n) \land \text{Undir}(l, \text{rev}(n)) \lor \text{Undir}(m, n) \land \text{Undir}(m, \text{rev}(n)). \] (3)

The convergence Con(l, m) is defined by means of the basic relation Undir as Undir(l, m) and Undir(l, rev(m)) [4]. Thus, Con(l, m) means that l and m are convergent lines. Note that directions of lines l and m are ignored for Con(l, m).

And then by means of the definition of Con(l, m), the axiom I.7 can be shortened as
\[ \text{Con}(l, m) \rightarrow \text{Con}(l, n) \lor \text{Con}(m, n). \] (4)

From Eq. (4), one can see that the axiom I.7 describes the convergence of three lines (the direction is ignored).

The axiom I.8,
\[ \text{Undir}(l, m) \lor \text{Undir}(l, \text{rev}(m)) \] (5)

In [4], the new relation Inopp(l, m) was defined as undir(l, rev(m)). By means of the definition of Inopp, the axiom I.8 can be written as Undir(l, m) \lor Inopp(l, m).

2.2 The axiom I.7 is complicated and includes much redundancy

The axiom I.7 can be rewritten equivalently as W1\&W2\&W3\&W4, where
oppositely directed lines. One can know that it is impossible for both

\[ W_3 = \sim Undir(l, m) \lor \sim Undir(l, rev(m)) \]  
\[ \lor Undir(l, n) \lor Undir(m, n), \]  
\[ W_4 = \sim Undir(l, m) \lor \sim Undir(l, rev(m)) \]  
\[ \lor Undir(l, rev(n)) \lor Undir(m, n). \]  

The classical concepts Opp and Dir are defined by means of the basic relation Undir. Opp(x, y) is defined as \( \sim Undir(x, rev(y)) \). Thus, Opp(x, y) means that the lines x and y are oppositely directed lines. Dir(x, y) is defined as \( \sim Undir(x, y) \). Thus, Dir(x, y) means that x and y are equally directed lines.

By means of the definitions of Dir and Opp, W1, W2, W3, and W4 can be rewritten as

\[ W_1 = Dir(l, n) \land Dir(m, n) \implies Dir(l, m) \lor Opp(l, m), \]  
\[ W_2 = Dir(l, n) \land Opp(m, n) \implies Dir(l, m) \lor Opp(l, m), \]  
\[ W_3 = Opp(l, n) \land Dir(m, n) \implies Dir(l, m) \lor Opp(l, m), \]  
\[ W_4 = Opp(l, n) \land Opp(m, n) \implies Dir(l, m) \lor Opp(l, m). \]

One can see that W1 in Eq. (10) means that if l and n are equally directed line and m and n are equally directed lines, then lines l and m are equally directed lines or oppositely directed lines. One can know that it is impossible for both Dir(l, m) and Opp(l, m) happen. By means of the axiom I.6, one can know that lines l and m are equally directed lines. Furthermore, from Eqs. (2) and (3), clearly the axiom I.6 implies W1. Thus, W1 is redundant.

W2 in Eq. (11) means that if l and n are equally directed line and m and n are oppositely directed lines, then lines l and m are equally directed lines or oppositely directed lines. One can know that it is impossible for both Dir(l, m) and Opp(l, m) happen. Factually, lines l and m should be oppositely directed lines. It means that W2 includes the redundancy.

W3 in Eq. (12) means that if l and n are oppositely directed line and m and n are equally directed lines, then lines l and m are equally directed lines or oppositely directed lines. It is impossible for both Dir(l, m) and Opp(l, m) happen. Factually, lines l and m should be oppositely directed lines. It means that W3 includes the redundancy.

W4 in Eq. (13) means that if l and n are oppositely directed line and m and n are oppositely directed lines, then lines l and m are equally directed lines or oppositely directed lines. It is impossible for both Dir(l, m) and Opp(l, m) happen. Factually, lines l and m should be equally directed lines.
By other hand, the theorem prover ANDP finds a natural deduction of W4 from only the axiom I-6. It means that the axiom I.6 implies W4 also. Thus, W4 is redundant.

2.3 The axiom I.7 can be replaced with the following simpler and more intuitive new axiom called ODO.

ODO:

\[ \sim \text{Undir}(l, rev(m)) \& \sim \text{Undir}(l, n) \rightarrow \sim \text{Undir}(m, rev(n)). \]  

(14)

By means of the definitions of Opp and Dir, ODO becomes

\[ \text{Opp}(l, m) \& \text{Dir}(l, n) \rightarrow \text{Opp}(m, n), \]  

(15)

which is just Theorem 3.5 [4]. Thus, ODO means that if \( l \) and \( m \) are oppositely directed lines and \( l \) and \( n \) are equally directed lines, then \( m \) and \( n \) are oppositely directed lines.

Theorem 1. Let the new axiomatization 1 be obtained from von Plato’s one by replacing axiom I.7 with ODO. Then, the new axiomatization 1 is equivalent to von Plato’s one.

Proof. It is known that ODO can be derived from von Plato’s axiomatization. Conversely, we can derive the axiom I.7 from the new axiomatization 1 as follows.

To reduce the difficulty to derive the axiom I.7, we consider W1, W2, W3, and W4 in Eqs. (6, 7, 8, 9). Instead of finding a deduction of the axiom I.7, we only need to find deductions of W1, W2, W3, and W4, respectively. After lots of trials, the natural deductions of W1, W2, W3, and W4 are obtained as follows.

W1 is derived from only the axiom I.6 without using other axioms. Ref. Appendix A.

W2 is derived from the axioms I.5, I.6, and ODO without using other axioms. Ref. Appendix B.

W3 is derived from the axioms I.5, I.6, ODO without using other axioms. Ref. Appendix C.

W4 is derived from only the axiom I.6 without using other axioms. Ref. Appendix D.

3 Automated natural deduction

The natural deduction is adapted from Gentzen system. We push quantifiers inside as possible as we can in our theorem prover ANDP. Thus, we can have more chances to apply rules for propositional logic. ANDP uses the following rules [9].

The rules for propositional logic:

MP (modus ponens), MT (modus tollens), IMP (\( \sim A \lor B \) from \( A \rightarrow B \)), LDS (\( B \) from \( A \) and \( \sim A \lor B \)), RDS (\( B \) from \( A \) and \( B \lor \sim A \)), CP (conditional proof), SIMPlication (\( A \) or \( B \) from \( A \land B \)), CASES (or dilemma).
The rules for quantifiers:
US (universal specialization), UG (universal generalization), EG (existential quantifier), EE (eliminate existential quantifier), SUB (substitute a term \( t \) for an individual variable \( v \)).

The wffs of the axioms I.5, I.6, I.7, I.8, and ODO are listed as follows.
The axiom I.5,
\[(\forall x) \sim Undir(x, x)\] (16)
The axiom I.6
\[(\forall x)(\forall y)[Undir(x, y) \rightarrow (\forall z)[Undir(x, z) \lor Undir(y, z)]]\] (17)

For the axiom I.7, the wffs of W1, W2, W3, and W4 are given as follows.
\[W_1 = (\forall x)(\forall y)(\forall z)[\sim Undir(x, y) \lor Undir(x, rev(y)) \lor Undir(x, z) \lor Undir(y, z)]\] (18)
\[W_2 = (\forall x)(\forall y)(\forall z)[\sim Undir(x, y) \lor Undir(x, rev(y)) \lor Undir(x, z) \lor Undir(y, rev(z)))]\] (19)
\[W_3 = (\forall x)(\forall y)(\forall z)[\sim Undir(x, y) \lor Undir(x, rev(y)) \lor Undir(x, rev(z)) \lor Undir(y, z)]\] (20)
\[W_4 = (\forall x)(\forall y)(\forall z)[\sim Undir(x, y) \lor Undir(x, rev(y)) \lor Undir(x, rev(z)) \lor Undir(y, rev(z))]\] (21)

The axiom I.8,
\[(\forall x)(\forall y)[Undir(x, y) \lor Undir(x, rev(y)))]\] (22)

ODO:
\[(\forall x)(\forall y)(\forall z)[\sim Undir(x, rev(y)) \land \sim Undir(x, z) \rightarrow \sim Undir(y, rev(z))]\] (23)

Note that only one predicate (i.e. Undir(\( l, m \))) and only one function symbol (i.e. rev(\( l \))) appear in the axioms I.5 to I.8, and ODO. For our automated prover, Undir(\( l, m \)) is written as Undir \( l \ m \), and the function symbol rev(\( l \)) is written as \[rev \ l \]. The quantifiers (\( \forall x \)) and (\( \exists x \)) are written as \( Ax \) and \( Ex \), respectively. The connectives \( \rightarrow \), \( \lor \), and \( \land \) are written as \( \rightarrow \), \( \lor \), and \( \land \), respectively. “\( \sim \)” stands for “\( \sim \)” (i.e. “not”).

4 Replacing the axiom I.6 with ODO

It is also interesting to replace the axiom I.6 with ODO.

Theorem 2. Let the axiomatization 2 be obtained from von Plato’s axiomatization by replacing axiom I.6 with ODO. Then, the axiomatization 2 is equivalent to von Plato’s axiomatization.

Proof. It is known that ODO can be derived from von Plato’s axiomatization.

Thing left to do is to derive the axiom I.6 from the axiomatization 2. Our theorem proving ANDP finds a natural deduction of the axiom I.6 from the axioms I.7, I.8, and ODO. Ref. Appendix E.
5 Appendix A. The axiom I.6 implies W1

Solution
1. (Ax)(Ay)[UNDIR x y -> (Az)[UNDIR x z | UNDIR y z]] ASSUMED-PREMISE
2. UNDIR v1 v2 ASSUMED-PREMISE
3. UNDIR v1 v3 ASSUMED-PREMISE
4. (Ay)[UNDIR v1 y -> (Az)[UNDIR v1 z | UNDIR y z]] US (v1 x) 1
5. UNDIR v1 v2 -> (Az)[UNDIR v1 z | UNDIR v2 z] US (v2 y) 4
6. (Az)[UNDIR v1 z | UNDIR v2 z] MP 5 2
7. UNDIR v1 v3 | UNDIR v2 v3 US (v3 z) 6
8. UNDIR v2 v3 LDS 7 3
9. UNDIR v2 v3 SAME 8
10. UNDIR v1 v3 -> UNDIR v2 v3 CP 9
11. UNDIR v1 v3 | UNDIR v2 v3 IMP 10
12. UNDIR v1 [rev v2]-> UNDIR v1 v3 | UNDIR v2 v3 CP 11
13. UNDIR v1 [rev v2] | UNDIR v1 v3 | UNDIR v2 v3 IMP 12
14. UNDIR v1 v2 -> [UNDIR v1 [rev v2] | UNDIR v1 v3 | UNDIR v2 v3] CP 13
15. UNDIR v1 v2 | UNDIR v1 v3 | UNDIR v2 v3 IMP 14
16. (Ax)(Ay)(Az)[[UNDIR x y | ~UNDIR x [rev y]] | UNDIR x z | UNDIR y z] UG 15
17. (Ax)(Ay)[UNDIR x y -> (Az)[UNDIR x z | UNDIR y z]]
   -> (Ax)(Ay)(Az)[[UNDIR x y | ~UNDIR x [rev y]] | UNDIR x z | UNDIR y z] CP 16

6 Appendix B. Derive W2 from the axioms I.5, I.6, and ODO

To reduce the difficulty to derive W2, first we derive the following result from the axiom I.5 and ODO. Ref. Appendix B.1.

\((\forall x)(\forall y)[Undir(x, rev(y)) \rightarrow Undir(y, rev(x))]\)  

(24)

Then, W2 can be derived from I.5, I.6, and the above result. Ref. Appendix B.2.

6.1 B.1. Derive Eq. (24) from the axiom I.5 and ODO

Show that
(\(Ax)(Ay)[\text{UNDIR } x \ [rev \ y] -> \text{UNDIR } y \ [rev \ x]\)
from the following premises:
(\(Ax)(Ay)(Az)[\sim \text{UNDIR } x \ [rev \ y] \ & \sim \text{UNDIR } x \ z -> \sim \text{UNDIR } y \ [rev \ z]]; \sim (Ex)\text{UNDIR } x \ x\).
Solution

1. \( (Ax)(Ay)(Az)[\neg UNDIR x [rev y] \land \neg UNDIR x z] \rightarrow \neg UNDIR y [rev z] \) \hspace{1cm} \text{PREMISE}
2. \( \neg (Ex)UNDIR x x \) \hspace{1cm} \text{PREMISE}
3. \( UNDIR v1 [rev v2] \) \hspace{1cm} \text{ASSUMED-PREMISE}
4. \( UNDIR v3 v3 \) \hspace{1cm} \text{US (v3 x) 2}
5. \( (Ay)(Az)[\neg UNDIR v4 [rev y] \land \neg UNDIR v4 z] \rightarrow \neg UNDIR y [rev z] \) \hspace{1cm} \text{US (v4 x) 1}

6. \( (Az)[\neg UNDIR v4 [rev v5] \land \neg UNDIR v4 z] \rightarrow \neg UNDIR v5 [rev z] \) \hspace{1cm} \text{US (v5 y) 5}
7. \( UNDIR v4 [rev v5] \land \neg UNDIR v4 v6 \)
8. \( \neg UNDIR v4 [rev v1] \land \neg UNDIR v4 v2 \) \hspace{1cm} \text{SUB 7}
9. \( \neg UNDIR v4 [rev v1] \land \neg UNDIR v4 v2 \) \hspace{1cm} \text{MT 3 8}
10. \( UNDIR v4 [rev v1] \land UNDIR v4 v2 \) \hspace{1cm} \text{DE.MORGAN 9}
11. \( UNDIR v2 v2 \) \hspace{1cm} \text{SUB 4}
12. \( UNDIR v2 [rev v1] \land UNDIR v2 v2 \) \hspace{1cm} \text{SUB 10}
13. \( UNDIR v2 [rev v1] \) \hspace{1cm} \text{RDS 11 12}
14. \( UNDIR v2 [rev v1] \) \hspace{1cm} \text{SAME 13}
15. \( UNDIR v1 [rev v2] \rightarrow UNDIR v2 [rev v1] \) \hspace{1cm} \text{CP 14}
16. \( (Ax)(Ay)[UNDIR x [rev y] \rightarrow UNDIR y [rev x]] \) \hspace{1cm} \text{UG 15}

6.2 B.2. Derive W2 from the axioms I.5, I.6, and Eq. (24):

Show that
\[
[(Ax)\neg UNDIR x x \\
& (Ax)(Ay)[UNDIR x [rev y] \rightarrow UNDIR y [rev x]]] \\
& (Ax)(Ay)[UNDIR x y \rightarrow (Az)[UNDIR x z \land UNDIR y z]] \\
\rightarrow (Ax)(Ay)[(Az)[\neg UNDIR x y \land \neg UNDIR x [rev y]]] \\
| UNDIR x z | \neg UNDIR y [rev z] \\
\]

Solution

1. \( [(Ax)\neg UNDIR x x \land (Ax)(Ay)[UNDIR x [rev y] \rightarrow UNDIR y [rev x]]] \) \hspace{1cm} \text{ASSUMED-PREMISE}
2. \( \neg (Ex)UNDIR x x \) \hspace{1cm} \text{SIMP 1}
3. \( (Ax)(Ay)[UNDIR x [rev y] \rightarrow UNDIR y [rev x]] \) \hspace{1cm} \text{SIMP 1}
4. \( (Ax)(Ay)[UNDIR x y \rightarrow (Az)[UNDIR x z \land UNDIR y z]] \) \hspace{1cm} \text{SIMP 1}
5. \( UNDIR v1 [rev v2] \) \hspace{1cm} \text{ASSUMED-PREMISE}
6. \( \neg UNDIR v1 v3 \) \hspace{1cm} \text{ASSUMED-PREMISE}
7. \( (Ay)[UNDIR v1 y \rightarrow (Az)[UNDIR v1 z \land UNDIR y z]] \) \hspace{1cm} \text{US (v1 x) 4}
8. \( UNDIR v1 [rev v2] \) \hspace{1cm} \text{US (v2 v2) 7}

7
9. (Az)UNDIR v1 z | UNDIR [rev v2] z
10. UNDIR v1 v3 | UNDIR [rev v2] v3
11. UNDIR [rev v2] v3
12. (Ay)UNDIR [rev v2] y
   -> (Az)UNDIR [rev v2] z | UNDIR y z
13. UNDIR [rev v2] v3
   -> (Az)UNDIR [rev v2] z | UNDIR v3 z
14. (Az)[UNDIR [rev v2] z | UNDIR v3 z]
15. UNDIR [rev v2] [rev v2] | UNDIR v3 [rev v2]
16. (Ev11)UNDIR v11 v11 | UNDIR v3 [rev v2]
17. UNDIR v3 [rev v2]
18. (Ay)UNDIR v3 [rev y] -> UNDIR y [rev v3]
19. UNDIR v3 [rev v2] -> UNDIR v2 [rev v3]
20. UNDIR v2 [rev v3]
21. UNDIR v2 [rev v3]
22. ¬UNDIR v1 v3 -> UNDIR v2 [rev v3]
23. UNDIR v1 v3 | UNDIR v2 [rev v3]
24. UNDIR v1 [rev v2]
   -> UNDIR v1 v3 | UNDIR v2 [rev v3]
25. ¬UNDIR v1 [rev v2] | UNDIR v1 v3 | UNDIR v2 [rev v3]
26. UNDIR v1 v2
   -> ¬UNDIR v1 [rev v2] | UNDIR v1 v3 | UNDIR v2 [rev v3]
27. [[¬UNDIR v1 v2 | ¬UNDIR v1 [rev v2]] | UNDIR v1 v3] | UNDIR v2 [rev v3]
28. (Ax)(Ay)(Az)[[¬UNDIR x y | ¬UNDIR x [rev y] | UNDIR x z] | UNDIR y [rev z]]
29. [(Ax)¬UNDIR x x
    & (Ax)(Ay)[UNDIR x y -> (Az)[UNDIR x z | UNDIR y z]]
    & (Ax)(Ay)[¬UNDIR x [rev y] & ¬UNDIR x z -> ¬UNDIR y [rev z]]

7 Appendix C. Derive W3 from the axioms I.5, I.6, and ODO.

Solution
1. [(Ax)¬UNDIR x x
   & (Ax)(Ay)[UNDIR x y -> (Az)[UNDIR x z | UNDIR y z]]
   & (Ax)(Ay)[¬UNDIR x [rev y] & ¬UNDIR x z -> ¬UNDIR y [rev z]]
   ASSUMED-PREMISE
2. ¬(Ex)UNDIR x x
   SIMP 1
3. (Ax)(Ay)[UNDIR x y -> (Az)[UNDIR x z | UNDIR y z]]
   SIMP 1
4. (Ax)(Ay)(Az)[¬UNDIR x [rev y] & ¬UNDIR x z
   -> ¬UNDIR y [rev z]]
   SIMP 1
5. UNDIR v1 [rev v2] ASSUMED-PREMISE
6. UNDIR v1 [rev v3] ASSUMED-PREMISE
7. (Ay)(Az)(¬UNDIR v1 [rev y] & ¬UNDIR v1 z) US (v1 x) 4
8. (Ay)UNDIR v1 y

- > (Az)[UNDIR v1 z | UNDIR y z] US (v1 x) 3
9. (Az)(¬UNDIR v1 [rev v3] & ¬UNDIR v1 z)
- > ¬UNDIR v3 [rev z] US (v3 y) 7
10. UNDIR v1 [rev v2]

- > (Az)[UNDIR v1 z | UNDIR [rev v2] z] US (rev(v2) y) 8
11. (Az)UNDIR v1 z | UNDIR [rev v2] z MP 10 5
12. UNDIR v1 [rev v3] | UNDIR [rev v2] [rev v3] US (rev(v3) z) 11
13. UNDIR [rev v2] [rev v3] LDS 12 6
14. (Ay)UNDIR [rev v2] y

- > (Az)[UNDIR [rev v2] z | UNDIR y z] US (rev(v2) x) 3
15. UNDIR [rev v2] [rev v3]

- > (Az)[UNDIR [rev v2] z | UNDIR [rev v3] z] US (rev(v3) y) 14
16. (Az)UNDIR [rev v2] z | UNDIR [rev v3] z MP 15 13
17. UNDIR [rev v2] [rev v2] | UNDIR [rev v3] [rev v2] US (rev(v2) z) 16
18. (Ev7)UNDIR v7 v7 | UNDIR [rev v3] [rev v2] EG 17
19. UNDIR [rev v3] [rev v2] LDS 12 18
20. (Az)(¬UNDIR v3 [rev y] & ¬UNDIR v3 z)

- > ¬UNDIR v3 [rev z] US (v3 x) 9
21. UNDIR v1 [rev v3] | [UNDIR v1 [rev v3]

| ¬UNDIR v3 [rev [rev v3]]] IMP 20
22. UNDIR v1 [rev v3] | ¬UNDIR v3 [rev [rev v3]] LDS 21 6
23. ¬UNDIR v3 [rev [rev v3]] LDS 22 6
24. (Ay)(Az)(¬UNDIR v3 [rev y] & ¬UNDIR v3 z)

- > ¬UNDIR y [rev z] US (v3 x) 4
25. (Ay)UNDIR v3 y

- > (Az)[UNDIR v3 z | UNDIR y z] US (v3 x) 3
26. (Az)(¬UNDIR v3 [rev v3] & ¬UNDIR v3 z)

- > ¬UNDIR [rev v3] [rev z] US (rev(v3) y) 24
27. ¬UNDIR v3 [rev v3] | ¬UNDIR v3 v2

- > ¬UNDIR [rev v3] [rev v2] US (v2 z) 26
28. UNDIR v3 [rev v2] | [UNDIR v3 [rev v2] | ¬UNDIR v3 [rev v2]] IMP 27
29. UNDIR v3 v2 | ¬UNDIR [rev v3] [rev v2] LDS 28 23
30. UNDIR v3 v2 RDS 29 19
31. UNDIR v3 v2

- > (Az)[UNDIR v3 z | UNDIR v2 z] US (v2 y) 25
32. (Az)[UNDIR v3 z | UNDIR v2 z] MP 31 30
33. UNDIR v3 v3 | UNDIR v2 v3 US (v3 z) 32
34. (Ev15)UNDIR v15 v15 | UNDIR v2 v3 EG 33
35. UNDIR v2 v3 LDS 2 34
36. UNDIR v2 v3 SAME 35
8 Appendix D. The axiom I.6 implies W4

Show that

\[(Ax)(Ay)(Az)[\neg\text{UNDIR} x \land \neg\text{UNDIR} x z \land \text{UNDIR} y z]\]

Solution

1. \[(Ax)(Ay)[\text{UNDIR} x y \rightarrow (Az)[\text{UNDIR} x z \land \text{UNDIR} y z]]\]  \hspace{1cm} \text{ASSUMED-PREMISE}

2. \[\text{UNDIR} v1 v2\]  \hspace{1cm} \text{ASSUMED-PREMISE}

3. \[\neg\text{UNDIR} v1 [\text{rev v3}]\]  \hspace{1cm} \text{ASSUMED-PREMISE}

4. \[(Ay)[\text{UNDIR} v1 y \rightarrow (Az)[\text{UNDIR} v1 z \land \text{UNDIR} y z]]\]  \hspace{1cm} \text{US (v1 x)}

5. \[\text{UNDIR} v1 v2\]  \hspace{1cm} \text{US (v2 y)}

6. \[(Az)[\text{UNDIR} v1 z \land \text{UNDIR} v2 z]\]  \hspace{1cm} \text{MP 5 2}

7. \[\text{UNDIR} v2 [\text{rev v3}] \land \text{UNDIR} v2 [\text{rev v3}]\]  \hspace{1cm} \text{US (rev(v3) z) 6}

8. \[\text{UNDIR} v2 [\text{rev v3}]\]  \hspace{1cm} \text{LDS 7 3}

9. \[\text{UNDIR} v2 [\text{rev v3}]\]  \hspace{1cm} \text{SAME 8}

10. \[\neg\text{UNDIR} v1 [\text{rev v3}] \rightarrow \text{UNDIR} v2 [\text{rev v3}]\]  \hspace{1cm} \text{CP 9}

11. \[\text{UNDIR} v1 [\text{rev v3}] \land \text{UNDIR} v2 [\text{rev v3}]\]  \hspace{1cm} \text{IMP 10}

12. \[\text{UNDIR} v1 [\text{rev v2}]\]  \hspace{1cm} \text{IMP 11}

13. \[\neg\text{UNDIR} v1 [\text{rev v2}] \lor \text{UNDIR} v1 [\text{rev v3}]\]  \hspace{1cm} \text{IMP 12}

14. \[\text{UNDIR} v1 v2\]  \hspace{1cm} \text{CP 13}

15. \[\neg\text{UNDIR} v1 [\text{rev v2}] \lor \text{UNDIR} v1 [\text{rev v3}] \lor \text{UNDIR} v2 [\text{rev v3}]\]  \hspace{1cm} \text{CP 13}
9 Appendix E Derive the axiom I.6 from the axioms I.7, I.8, and ODO

Solution

1. [(Ax)(Ay)[UNDIR x y | UNDIR x [rev y]] &
   (Ax)(Ay)(Az)[~UNDIR x [rev y] & ~UNDIR x z
   -> ~UNDIR y [rev z]] & (Az)[UNDIR x z & UNDIR x [rev z] | UNDIR y [rev z] & UNDIR y [rev z]]]      ASSUMED-PREMISE

2. (Ax)(Ay)[UNDIR x y | UNDIR x [rev y]]              SIMP 1

3. (Ax)(Ay)(Az)[~UNDIR x [rev y] & ~UNDIR x z -~ UNDIR y [rev z]]            SIMP 1

4. (Ax)(Ay)[UNDIR x y & UNDIR x [rev y]
   ->(Az)[UNDIR x z & UNDIR x [rev z] | UNDIR y z & UNDIR y [rev z]]]            ASSUMED-PREMISE

5. UNDIR v1 v2                                                  ASSUMED-PREMISE

6. ~UNDIR v1 v3                                               ASSUMED-PREMISE

7. (Ay)[UNDIR v1 y & UNDIR v1 [rev y]
   ->(Az)[UNDIR v1 z & UNDIR v1 [rev z] | UNDIR y z & UNDIR y [rev z]]]      US (v1 x) 4

8. UNDIR v1 v2 & UNDIR v1 [rev v2]
   ->(Az)[UNDIR v1 z & UNDIR v1 [rev z]
   | UNDIR v2 z & UNDIR v2 [rev z]]                           US (v2 y) 7

9. ~UNDIR v1 v2 | ~UNDIR v1 [rev v2] |
   (Az)[UNDIR v1 z & UNDIR v1 [rev z]
   | UNDIR v2 z & UNDIR v2 [rev z]]                        IMP 8

10. ~UNDIR v1 [rev v2] |
    (Az)[UNDIR v1 z & UNDIR v1 [rev z] | UNDIR v2 z &
     UNDIR v2 [rev z]]                                      LDS 9 5

11. ~UNDIR v1 [rev v2]                                        CASE2 10

12. (Az)[UNDIR v1 z & UNDIR v1 [rev z] | UNDIR v2 z &
    UNDIR v2 [rev z]]                                     CASE1 10

13. UNDIR v1 v3 & UNDIR v1 [rev v3] |
    UNDIR v2 v3 & UNDIR v2 [rev v3]                    US (v3 z) 12

14. [UNDIR v1 v3 & UNDIR v1 [rev v3] |
    UNDIR v2 v3 & UNDIR v1]                                US (v3 v3) 12
[rev v3] | UNDIR v2 [rev v3]  DISTRIBUTIVE-LAW 13
15. UNDIR v1 v3 & UNDIR v1 [rev v3] | UNDIR v2 v3  SIMP 14
16. [UNDIR v1 v3 | UNDIR v2 v3] & [UNDIR v1 [rev v3] | UNDIR v2 v3]  DISTRIBUTIVE-LAW 15
17. UNDIR v1 v3 | UNDIR v2 v3  SIMP 16
18. UNDIR v2 v3  LDS 17 6
19. (Ay)(Az)[¬UNDIR v1 [rev y] & ¬UNDIR v1 z
     -> ¬UNDIR y [rev z]]  US (v1 x) 3
20. (Az)[¬UNDIR v1 [rev v2] & ¬UNDIR v1 z
     -> ¬UNDIR v2 [rev z]]  US (v2 y) 19
21. ¬UNDIR v1 [rev v2] & ¬UNDIR v1 v3
     -> ¬UNDIR v2 [rev v3]  US (v3 z) 20
22. UNDIR v1 [rev v2] | [UNDIR v1 v3 | ¬UNDIR v2 [rev v3]]  IMP 21
23. UNDIR v1 v3 | ¬UNDIR v2 [rev v3]  LDS 22 11
24. ¬UNDIR v2 [rev v3]  LDS 23 6
25. (Ay)(Az)[¬UNDIR v2 [rev y] & ¬UNDIR v2 z
     -> ¬UNDIR y [rev z]]  US (v2 x) 3
26. (Az)[¬UNDIR v2 [rev v3] & ¬UNDIR v2 z
     -> ¬UNDIR v3 [rev z]]  US (v3 x) 25
27. ¬UNDIR v2 [rev v3] & ¬UNDIR v2 [rev v3]
     -> ¬UNDIR v3 [rev v3]  US (rev(v3) z) 26
28. UNDIR v2 [rev v3] | [UNDIR v2 [rev v3] | ¬UNDIR v3 [rev v3]]  IMP 27
29. UNDIR v2 [rev v3] | ¬UNDIR v3 [rev [rev v3]]  LDS 28 24
30. ¬UNDIR v3 [rev [rev v3]]  LDS 29 24
31. (Ay)[UNDIR v3 y | UNDIR v3 [rev y]]  US (v3 x) 2
32. (Ay)(Az)[¬UNDIR v3 [rev y] & ¬UNDIR v3 z
     -> ¬UNDIR y [rev z]]  US (v3 x) 3
33. UNDIR v3 [rev [rev v3]] | UNDIR v3 [rev [rev [rev v3]]]  US (rev(rev(v3)) y) 31
34. UNDIR v3 [rev [rev [rev v3]]]  LDS 33 30
35. (Az)[¬UNDIR v3 [rev v3] & ¬UNDIR v3 z
     -> ¬UNDIR v3 [rev z]]  US (v3 y) 32
36. ¬UNDIR v3 [rev v3] & ¬UNDIR v3 [rev v3]
     -> ¬UNDIR v3 [rev [rev [rev v3]]]  US (rev(rev(v3)) z) 35
37. ¬[¬UNDIR v3 [rev v3] & ¬UNDIR v3 [rev [rev v3]]]  MT 36 34
38. UNDIR v3 [rev v3] | UNDIR v3 [rev [rev v3]]  DE.MORGAN 37
39. UNDIR v3 [rev v3]  RDS 38 30
40. ¬UNDIR v2 [rev v3] & ¬UNDIR v2 v3
     -> ¬UNDIR v3 [rev v3]  US (v3 z) 26
41. UNDIR v2 [rev v3] | [UNDIR v2 v3 | ¬UNDIR v3 [rev v3]]  IMP 40
42. UNDIR v2 v3 | ¬UNDIR v3 [rev v3]  LDS 41 24
43. UNDIR v2 v3  RDS 42 39
44. UNDIR v2 v3
45. UNDIR v2 v3
46. UNDIR v2 v3
47. ˜UNDIR v1 v3 -> UNDIR v2 v3
48. UNDIR v1 v3 | UNDIR v2 v3
49. (Az)[UNDIR v1 z | UNDIR v2 z]
50. UNDIR v1 v2
51. (Ax)(Ay)[UNDIR x y -> (Az)[UNDIR x z | UNDIR y z]]
52. [(Ax)(Ay)]UNDIR x y | UNDIR x [rev y]
& (Ax)(Ay)(Az)[˜UNDIR x [rev y] & ˜UNDIR x z -> ˜UNDIR y [rev z]]
& (Ax)(Ay)[UNDIR x y & UNDIR x [rev y]
-> (Az)[UNDIR x z & UNDIR x [rev z] | UNDIR y z & UNDIR y [rev z]]
-> (Ax)(Ay)[UNDIR x y -> (Az)[UNDIR x z | UNDIR y z]]

References

[1] Heyting, A.: Axioms for intuitionistic plane affine geometry In L. Henkin et al., eds.. The Axiomatic Method, North-Holland, Amsterdam, 160-173 (1959).
[2] von Plato, J.: The axioms of constructive geometry. Annals of Pure and Applied Logic, vol. 76, 169-200 (1995).
[3] van Dalen, D.: ‘Outside’ as a primitive notion in constructive projective geometry. Geometriae Dedicata 60, 107-111 (1996).
[4] von Plato, J.: A constructive theory of ordered affine geometry. Indag. Mathem., N.S. 9 (4), 549-562 (1998).
[5] McCune, W., Wos, L.: Application of automated deduction to the search for single axioms for exponent groups. Proc. of LPAR-92, St. Petersburg, Russia, July 15–20, 1992, pp. 131-136.
[6] Padmanabhan, R., McCune, W.: Single identities for ternary Boolean algebra. Comput. Math. Appl., 29 (2), 13-16 (1993).
[7] McCune, W., Padmanabhan, R.: Single identities for lattice theory and weakly associative lattices. Algebra Universalis, 36 (4), 436-449(1996).
[8] McCune, W.: Solution of the Robbins Problem. J. of automated reasoning 19(3), 263–276 (1997).
[9] Li, D.: Unification Algorithms for Eliminating and Introducing Quantifiers in Natural Deduction Automated Theorem Proving. J. of Automated Reasoning 18(1), 105-134(1997).
[10] Li, D.: Natural Deduction Prover and Experiments, Tableaux 97, Nancy, France, pp153-157, May 13-16, (1997).
[11] Li, D.: A Mechanical Proof of the Halting Problem in Natural Deduction Style, AAR Newsletter No.23, June 1993.

[12] Li, D., Jia, P., Li, X.: Simplifying von Plato’s axiomatization of constructive apartness geometry. Annals of pure and applied logic, vol. 102, No1-2, (2000).

[13] Li, D.: Using Prover ANDP to simplify orthogonality. Annals of pure and applied logic, 124, 49-70 (2003).

[14] Li, D.: The equality axioms are not independent. ACM SIGACT NEWS, Vol. 35, Issue 3, 98-101 (2004).