Dirac type relativistic quantum mechanics for massive photons

Soon-Tae Hong
Center for Quantum Spacetime and Department of Physics, Sogang University, Seoul 04107, Korea
(Dated: June 9, 2022)

Constructing a relativistic quantum mechanics (RQM) for a massive photon, without appealing to the quantum field theoretical approach such as the massive Proca model, we find a new theoretical particle solution which allows the massive photon having either positive or negative energy as solutions. In particular, we predict the existence of the so-called anti-photon corresponding to the negative energy solution, similar to the positron in the Dirac RQM for an electron. Note that the anti-photons could be an intense radiation flare of the gamma ray burst. In this RQM for the massive photon, we construct a positive definite probability density and a nontrivial diagonal Hamiltonian, and also discuss a massless photon. Moreover we confirm the covariances of the relativistic equation of motion for the massive photon and the corresponding probability continuity equation under the Lorentz transformation.

Keywords: relativistic quantum mechanics; massive photon; anti-photon; gamma ray burst; Lorentz transformation

I. INTRODUCTION

Generalizing the Hawking-Penrose singularity theorem (HPST) [1], the stringy cosmology in a higher dimensional total spacetime has been investigated [2, 3] with a success that one can describe precisely the motion types of stringy congruence in terms of the universe expansion rate after the Big Bang. Moreover, in the stringy HPST described in the higher dimensions, we can have an advantage that the degrees of freedom (DOF) of the rotation and shear of the stringy congruence are introduced naturally in the early universe. In the stringy HPST with the higher dimensional spacetime, assuming that the smallest particle is the photon, one can find that in the universe there exists a massive photon possessing the finite size which is filled with mass. Note that the higher dimensional theories have been exploited in the higher dimensional de Sitter cosmology [4].

Next, the gamma ray burst (GRB) in the universe has been detected in 1967, and the GRB discovery has been published in 1973 [5]. Since then, to explain the GRB, there have been extremely lots of theoretical models. For a recent detection of the GRB associated with the magnetar, see Ref. [6] for instance. If the GRB sources were from within the Milky Way, they would be strongly concentrated near the galactic plane. The absence of any such pattern in the GRB has suggested a strong evidence that the GRB can come from beyond our own galaxy. The intense radiation flare of most detected GRB has been then assumed to emerge from a supernova, during a high-mass star implodes to construct a neutron star or a black hole.

On the other hand, Dirac has formulated the relativistic quantum mechanics (RQM) for an electron [7]. In his theory, a new particle solution has been proposed to allow the electron having either positive or negative energy solutions. In particular, the positron corresponding to the negative energy solution has been theoretically predicted and later has been experimentally confirmed [8]. Note that the Dirac equation in condensed matter systems has been investigated in terms of bound state in continuum like solutions, in addition to discrete energy bound state solutions [9].

Recently making use of an open string which performs both rotational and pulsating motions, we have predicted the intrinsic frequency \( \omega_r = 9.00 \times 10^{23} \text{ sec}^{-1} \) [10] for the bosonic photon with spin one which is comparable to the intrinsic frequencies \( \omega_N = 9.87 \times 10^{23} \text{ sec}^{-1} \) and \( \omega_\Delta = 1.74 \times 10^{23} \text{ sec}^{-1} \) [11] of the nucleon and delta baryon with spin 1/2, respectively. Next we have calculated the finite photon size \( \langle r^2 \rangle^{1/2} \text{(photon)} = 0.17 \text{ fm} \) in the phenomenological stringy photon model [10].

In this paper, we will propose a new RQM for a massive photon with finite size, without appealing to the quantum field theoretical approach such as the massive Proca model. In this RQM for the massive photon, we will theoretically predict a new particle solution which allows the massive photon having either positive or negative energy as solutions. In the massless limit we will also recover the RQM for the photon with transverse polarizations only. Moreover, we will investigate the intense radiation flare of the GRB in terms of the negative energy solution of the massive photon.
Next we will discuss the covariances of the relativistic equation of motion for the massive photon and its corresponding probability continuity equation under the Lorentz transformation.

In Sec. II, we will construct the Hamiltonian for a massive photon in our model. In Sec. III, we will investigate the phenomenology of the RQM for the massive photon. Explicitly we will show that the probability density for the massive photon is positive definite in our model. The negative energy solution of the massive photon will be also discussed together with the GRB. In Sec. IV, we will study the massless limit of the photon. We will also compare the RQM for the massive photon, with the Proca model. In Sec. V, we will investigate the Lorentz transformation for the massive photon to discuss the covariances of its relativistic equation of motion and probability continuity equation. Sec. VI includes conclusions.

II. HAMILTONIAN CONSTRUCTION IN RQM FOR A MASSIVE PHOTON

In this section, we will construct the Hamiltonian in the RQM for a massive photon. To do this, we assume that the photon trajectory is a straight line along the $z$ direction, for simplicity. We then have a relativistic relation $E^2 = m^2 + p^2$ where $p = p_z = |\vec{p}|$. The RQM equation of motion for the massive photon is then given by

$$H \phi^a = i \partial_0 \phi^a. \quad (2.1)$$

Here $\phi^a$ denotes the wave function for the massive photon, explicitly given by

$$\phi^a = (\phi_1^a, \phi_2^a)^t, \quad (2.2)$$

where the superscript $t$ stands for the transpose of the wave function components. Here the spin index $a (a = 0, 1, 2, 3)$ denotes the spin DOF for the massive photon with spin one. The component index $A (A = 1, 2)$ stands for the two DOF which have the same DOF of the positive and negative energy solutions with the energy index $\pm$ in (3.2), since the positive and negative energy solutions are given by linear combinations of the two wave functions with the component indices. Note that the wave function $\phi^a_A$ is described in terms of a $1 \times (2_{\text{energy}} \otimes 2_{\text{spin}}) = 1 \times 8$ column vector. From now on we will drop the index $A$ in the wave functions except (3.9) and (3.24) below, for simplicity.

Note that, in the quantum field theoretical Proca model for the massive photon, we need a $1 \times 1$ matrix Hamiltonian as in (1.5) below. Moreover, in this model, we cannot have a negative energy solution. Now we include a possibility of a negative energy solution for the massive photon in our model, by following Dirac idea for the positron. To do this, for the corresponding Hamiltonian for the bosonic massive photon, we introduce a minimal $2 \times 2$ matrix associated with the positive and negative energy solutions.\(^1\)

Next, complying with the Dirac algorithm for the RQM for the positron, we proceed to find $H$ in (2.1) for the case of the RQM for the massive photon. The Hamiltonian $H$ is then given by a $2 \times 2$ matrix acting on the component index $A$ only

$$H = \vec{A} \cdot \vec{p} + Bm. \quad (2.3)$$

where $A_i (i = 1, 2, 3)$ and $B$ are $2 \times 2$ matrices. Using the relation $E^2 = m^2 + p^2$, we obtain the algebra among $A_i$ and $B$ as follows

$$\{A_i, A_j\} = 2\delta_{ij}I, \quad B^2 = I, \quad \{A_i, B\} = 0, \quad A_i^\dagger = A_i, \quad B^\dagger = B. \quad (2.4)$$

where $I$ is a $2 \times 2$ unit matrix. Note that eigenvalues of $A_i$ or $B$ are $\pm 1$ and $\text{tr}A_i = \text{tr}B = 0$. In our construction, the photon spin DOF is included in the wave function in (2.2), as in the Proca model wave function in (4.4).

As in the Dirac relativistic formalism for the positron, exploiting the above relations in (2.4) together with the massive photon Hamiltonian in (2.3), we obtain the representations for $A_i$ and $B$ given by $A_1 = A_2 = 0, A_3 = \sigma_1$ and $B = \sigma_3$, with $\sigma_i$ being the Pauli matrices. Inserting the above representations for $A_i$ and $B$ into (2.3), we arrive at the desired $2 \times 2$ Hamiltonian of the form

$$H = \begin{pmatrix} m & -i\partial_3 \\ -i\partial_3 & -m \end{pmatrix}. \quad (2.5)$$

\(^1\) This feature will be applied to the Lorentz covariance of the relativistic equation of motion for the massive photon in Sec. V. Note that, in the RQM for a fermionic electron, the wave function is described by $1 \times (2_{\text{energy}} \otimes 2_{\text{spin}}) = 1 \times 4$ column vector in (3.16) below originated from the positive and negative energy solutions in addition to $1/2$ and $-1/2$ spin states, while the corresponding Hamiltonian is given by a minimal $4 \times 4$ matrix in (3.13).
Making use of (2.1) and (2.5), we find the relativistic equation of motion for the massive photon as follows

\[(i\Gamma^\mu\partial_\mu - m)\phi^a(x) = 0, \quad (2.6)\]

where \(\phi^a(x)\) is a function of \(x^\mu\) and \(\Gamma^\mu\) is given by

\[\Gamma^\mu \equiv (\Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3) = (\sigma_3, 0, 0, i\sigma_2). \quad (2.7)\]

### III. ANTI-PHOTON IN RQM FOR A MASSIVE PHOTON

In this section, we will investigate the phenomenological aspects of the RQM for the massive photon. To do this, we start with the equation in (2.6), which describes the motion of the massive photon. Since, for the relativistic massive photon satisfying the relation \(E^2 = m^2 + p^2\), we have two kinds of solutions corresponding to \(E = \pm (m^2 + p^2)^{1/2} = \pm p_0\), we introduce an ansatz for the wave function \(\phi^a\) as follows

\[\phi^a(x) = \phi^a(p^\mu)e^{\mp ip_\sigma x^\sigma}, \quad (3.1)\]

for a positive (negative) solution with an upper (lower) sign.

Next we find the positive energy solution \(\phi^a_+(x)\) with \(E > 0\) and the negative energy solution \(\phi^a_-(x)\) with \(E = -|E| < 0\), respectively,

\[\phi^a_+(x) = u^a(p^\mu)e^{-ip_\sigma x^\sigma}, \quad \phi^a_-(x) = v^a(p^\mu)e^{ip_\sigma x^\sigma}, \quad (3.2)\]

where \(u^a(p^\mu)\) and \(v^a(p^\mu)\) are given by the nontrivial forms,

\[u^a(p^\mu) = \epsilon^a \left( \frac{E + m}{2m} \right)^{1/2} \left( \frac{1}{E + m} \right), \quad v^a(p^\mu) = \epsilon^a \left( \frac{|E| + m}{2m} \right)^{1/2} \left( \frac{p}{|E| + m} \right). \quad (3.3)\]

Here \(\epsilon^a\) is a unit polarization vector possessing the spacetime index \(\alpha\) (\(\alpha = 0, 1, 2, 3\)) which is the same as the spin index and is needed to incorporate minimally the spin DOF for the massive photon. Here we have considered the Lorentz frame where \(\epsilon^a\) is purely space-like so that we can readily find that \(\epsilon^a\epsilon_\alpha = \epsilon \cdot \epsilon = 1\), or \(\epsilon \epsilon^a = -\epsilon \cdot \epsilon = -1\). Moreover we have the relation \(\epsilon_{\alpha\beta}p^\alpha \neq 0\), since for the massive photon we have longitudinal component in addition to transverse ones, similar to the phonon associated with massive particle lattice vibrations [12]. Next we have the normalization relations: \(\bar{u}^a u^a \equiv u^a \sigma_3 u^a = 1\), \(u^a u_\alpha = \frac{E}{m}\), \(\bar{v}^a v^a \equiv v^a \sigma_3 v^a = -1\) and \(v^a v_\alpha = \frac{|E|}{m}\). Inserting (3.2) into (2.6), we obtain the equations of motion in the momentum space

\[(\not{p} - m)u^a(p^\mu) = 0, \quad (\not{p} + m)v^a(p^\mu) = 0, \quad (3.4)\]

where \(\not{p} \equiv \Gamma^\mu p_\mu\). It is well known that, in the Dirac RQM for the positron, we have the electron and positron corresponding to the positive and negative energy solutions, respectively. It is now interesting to note that, similar to the Dirac theory for the positron, there exists the massive photon with negative energy solution.

Reshuffling the equation of motion in (2.6) we obtain

\[i\sigma_3 \partial_0 \phi^a - \sigma_2 \partial_3 \phi^a - m \phi^a = 0. \quad (3.5)\]

Taking Hermitian conjugate of the equation in (3.5), we next construct

\[i\partial_0 \bar{\phi}^a \sigma_3 - \partial_3 \bar{\phi}^a \sigma_2 + m \bar{\phi}^a = 0, \quad (3.6)\]

where

\[\bar{\phi}^a \equiv \phi^{a\dagger} \Gamma^0. \quad (3.7)\]

Exploiting (3.5) and (3.6), we find the probability continuity equation

\[\partial_0 \rho + \nabla \cdot \bar{J} = 0, \quad (3.8)\]

where the probability density \(\rho\) and probability current \(\bar{J}\), respectively, are given by

\[\rho \equiv \bar{\phi}^{a\dagger} \sigma_3 \phi^a = \phi^{a\dagger} \phi^a = \phi_1^a \phi_1 + \phi_2^a \phi_2, \quad (3.9)\]

\[\bar{J} \equiv \bar{\phi}^{a\dagger} (i\sigma_2) \phi^a \hat{z} = \phi^{a\dagger} \sigma_1 \phi^a \hat{z} = (\phi^{a\dagger}_1 \phi_2 + \phi^{a\dagger}_2 \phi_1) \hat{z}. \]
where, for a given relativistic four momentum variable 

\[ p \]

corresponding to spin \[ \pm \]

we have used the four probability current \[ J^\mu \] in (3.9) are physically well defined to yield a good quantization, similar to the Dirac RQM for the electron where the corresponding probability density \( \rho_D \) is also positive definite. However, in the Klein-Gordon model, the probability density \( \rho_{KG} \) for a relativistic spinless boson is not positive definite \[ 13 \]. Note that the probability continuity equation in (3.8) can be rewritten in the covariant form as follows

\[
\partial_\mu J^\mu = 0, \quad (3.10)
\]

where we have used the four probability current \( J^\mu = (\rho, \vec{J}) \). For the positive energy solution with \( E > 0 \), inserting \( \phi^+(x) \) in (3.2) into \( \rho \) and \( \vec{J} \) in (3.9), we obtain

\[
\rho = \frac{E}{m}, \quad \vec{J} = \frac{\vec{p}}{m}. \quad (3.11)
\]

Next, for the negative energy solution with \( E = -|E| < 0 \) where \( |E| = (m^2 + p^2)^{1/2} \), inserting \( \phi^-(x) \) in (3.2) into \( \rho \) and \( \vec{J} \), we find

\[
\rho = \frac{|E|}{m}, \quad \vec{J} = \frac{\vec{p}}{m}. \quad (3.12)
\]

It seems appropriate to comment on the four probability current \( J^\mu_D = (\rho_D, \vec{J}_D) \) in the Dirac RQM for an electron with mass \( m_e \). The relativistic equation of motion for the electron is given by \[ 13 \]

\[(i\gamma^\mu \partial_\mu - m_e)\psi(x) = 0, \quad (3.13)\]

with \( \gamma^\mu \) (\( \mu = 0, 1, 2, 3 \)) being given by \( 4 \times 4 \) matrices \( \gamma^0 = \beta \) and \( \gamma^i = \beta \alpha_i \) where

\[
\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \quad (3.14)
\]

The electron wave equation is then given by

\[
\psi(x) = \psi(p^\mu)e^{\mp ip_\sigma x^\sigma}, \quad (3.15)
\]

for a positive (negative) solution with an upper (lower) sign. Now we find the positive energy solution \( \psi_+(x) \) with \( E_e > 0 \) and the negative energy solution \( \psi_-(x) \) with \( E_e = -|E_e| < 0 \), respectively. For a given index \( I \) \((I = 1, 2)\) corresponding to spin \( \pm 1/2 \) states we construct

\[
\psi_+(x) = u^I(p^\mu)e^{-ip_\sigma x^\sigma}, \quad \psi_-(x) = v^I(p^\mu)e^{+ip_\sigma x^\sigma}, \quad (3.16)
\]

where, for a given relativistic four momentum variable \( p^\mu_e = (E_e, \vec{p}_e) \), \( u^I(p^\mu) \) and \( v^I(p^\mu) \) are given by

\[
u^I(p^\mu) = \left( \frac{E_e + m_e}{2m_e} \right)^{1/2} \left( \frac{\chi^I}{E_e + m_e} \chi^I \right), \quad v^I(p^\mu) = \left( \frac{|E_e| + m_e}{2m_e} \right)^{1/2} \left( \frac{\sigma_e \chi^I}{|E_e| + m_e} \chi^I \right). \quad (3.17)
\]

where \( \chi^1 = (1, 0)^t \) and \( \chi^2 = (0, 1)^t \). In the Dirac RQM for the electron, the four probability current \( J^\mu_D = (\rho_D, \vec{J}_D) \) is given by \[ 13 \]

\[
\rho_D = \psi^+ \psi, \quad \vec{J}_D = \psi^\dagger \alpha_i \psi. \quad (3.18)
\]

Making use of (3.15)–(3.17), we find

\[
\rho_D = \frac{2E_e}{m_e}, \quad \vec{J}_D = \frac{2\vec{p}_e}{m_e}. \quad (3.19)
\]

for the positive energy solution with \( E_e > 0 \), and

\[
\rho_D = \frac{2|E_e|}{m_e}, \quad \vec{J}_D = \frac{2\vec{p}_e}{m_e}. \quad (3.20)
\]
for the negative energy solution with \( E_c = -|E_c| < 0 \). Here we notice that the four probability current \( J^\mu_D \) satisfies the probability continuity equation both for the positive and negative solutions \[ \partial_\mu J^\mu_D = 0. \] (3.21)

Note that there exists the similarity such as \( J^\mu \) in (3.11) and (3.12), and \( J^\mu_D \) in (3.19) and (3.20) between the RQM for the massive photon and the Dirac RQM for the electron.

Now we have some comments to address on the negative solution \( \phi_\alpha^a \) in the RQM for the massive photon. First, for the negative energy solution of the massive photon, since we have \( \rho = |\phi_1|^2 + |\phi_2|^2 > 0 \), \( \rho = |E|/m \) implies that \( m \) is positive. Even in this negative energy solution case, the positive mass \( m \) moves with the probability current \( \vec{J} \) along the \( z \) direction. From now on, we will name the photon possessing the characteristic that the particle has the positive mass \( m \) and positive energy \( |E| \) and is associated with the negative energy solution, an anti-photon. To be more specific, we find that the anti-photon possessing the positive mass \( m \) has a positive definite probability density \( \rho \) and propagates with a probability current \( \vec{J} \) along the direction of \( \vec{p} \).

Second, we propose that the \textit{anti}-photon related with the negative energy solution is defined to interact repulsively with the ordinary massive photon, \textit{oppositely} to the ordinary massive photon-electron attractive gravitational interaction pattern. Next, the positron and electron can annihilate each other via the particle and anti-particle pair annihilation mechanism. However, the uncharged anti-photon scatters away from the ordinary massive particle including the photon, in the repulsive gravitational interaction between the anti-photon and the ordinary massive particle.

Third, since the anti-photon is repulsive against charged or uncharged ordinary massive matters, the anti-photon does not adhere to the ordinary massive matters, so that the anti-photon can yield an intense radiation flare. Now we propose that the anti-photons with positive masses could be the intense radiation flare of the GRB released from a supernova, during a high-mass star implodes and then forms a neutron star or a black hole.

Next we diagonalize the Hamiltonian in (2.5) by introducing a unitary operator matrix and its inverse one, respectively,

\[
U = \left( \frac{m + (m^2 - \Delta)^{1/2}}{2(m^2 - \Delta)^{1/2}} \right)^{1/2} \begin{pmatrix} 1 & i\partial_3 \\ -i\partial_1 & 1 \end{pmatrix},
\]

\[
U^{-1} = \left( \frac{m + (m^2 - \Delta)^{1/2}}{2(m^2 - \Delta)^{1/2}} \right)^{1/2} \begin{pmatrix} 1 & i\partial_3 \\ -i\partial_1 & 1 \end{pmatrix},
\]

(3.22)

where \( \Delta = \partial_3 \partial_3 \) is the Laplacian operator. Acting the unitary operator and its inverse one on the Hamiltonian \( H \) in (2.5), we construct the diagonal Hamiltonian matrix \( H_D \) of the form

\[
H_D = U^{-1}HU = \begin{pmatrix} (m^2 - \Delta)^{1/2} & 0 \\ 0 & -(m^2 - \Delta)^{1/2} \end{pmatrix}.
\]

(3.23)

Here one can readily check that the eigenfunctions for the diagonal Hamiltonian matrix (3.23) are given by the positive and negative energy solutions in (3.2).

Finally we give couple of comments on the wave function \( \phi_\alpha^a \) in (2.6). Note that, after some manipulations using (2.6), we find the equation of motion in terms of \( \phi_\alpha^a \)

\[
(\Box + m^2)\phi_\alpha^a (x) = 0.
\]

(3.24)

Even though the equation of motion in (3.24) seems to be simpler than that in (2.6), in investigating the phenomenology of the massive photon, the wave function \( \phi_\alpha^a (x) \) is not so physically meaningful since it is described in terms of the rather non-physical component index \( A \), instead of the physical energy index \( \pm \) in (3.2). The equation of motion in (3.24) will be discussed in the next section, to investigate the differences among the RQM for the massless photon, the massive Proca model and the RQM for the massive photon.

\footnote{Note that, in the Dirac RQM, the positron (or \textit{anti}-electron) associated with the negative energy solution is defined to interact attractively with the electron, \textit{oppositely} to the ordinary charged electron-electron repulsive electromagnetic interaction pattern. The same logic can be applied to the gravitational interaction case related with the anti-photon.}
IV. RQM FOR MASSLESS PHOTONS

In this section, we will formulate wave functions for massless photons. To do this, we revisit the RQM equation of motion in (2.6) and the solutions in (3.2) and (3.3) from which, in the massless limit, we obtain the following positive and negative energy solutions

$$\phi^{a(m=0)}_+ (x) = N e^a \left( \frac{1}{i} \right) e^{-ip_\sigma x^\sigma}, \quad \phi^{a(m=0)}_- (x) = N e^a \left( \frac{1}{i} \right) e^{ip_\sigma x^\sigma},$$

where a normalization factor $N$ will be fixed later. Note that the above solutions in (4.1) satisfy the following equation of motion

$$\Box \phi^{a(m=0)} (x) = 0,$$

which can be readily obtained from the massless limit of the RQM equation of motion in (3.2) for the massive photon.

Note also that the positive and negative solutions $\phi^a_\pm (x)$ in (3.2) describing the massive photon and anti-photon, respectively, are given by the nontrivial linear combinations of the wave functions $\phi^a_{A=1}$ and $\phi^a_{A=2}$. However, in the massless photon, $\phi^{a(m=0)}_\pm (x)$ are given by the trivial linear combinations of $\phi^{a(m=0)}_{A=1} (x)$ and $\phi^{a(m=0)}_{A=2} (x)$ as shown in (4.1). Moreover, since the neutral massless photon is equal to its massless anti-particle, we construct the massless photon wave functions in terms of the wave functions possessing the spin index only. In other words, by exploiting $\phi^{a(m=0)}_\pm (x)$ in (4.1) and the unit polarization vector $\varepsilon^a$, we find $\phi^{a(m=0)} (x)$ corresponding to the massless photon wave function possessing the spin index $a$ only without the component index $A$

$$\phi^{a(m=0)} (x) = \frac{\varepsilon^a}{\sqrt{2p_\sigma V}} (e^{-ip_\sigma x^\sigma} + e^{+ip_\sigma x^\sigma}).$$

Here the normalization factor $N \equiv \frac{1}{\sqrt{2p_\sigma V}}$ associated with $p_\sigma$ and space volume $V$ is now fixed so that the massless photon energy $\omega$ in the electromagnetic wave can be given by $\omega = p_\sigma = |\varepsilon|$ [13]. Note that $\phi^{a(m=0)} (x) \equiv A^a (x)$ is the four-vector potential in the electromagnetism for the massless photon.

It seems appropriate to comment on the normalization factor $N$. In the Bose-Einstein statistics [14], we can find the massive photon statistics where the total number of the photons is countable and fixed so that we can have the constraint of the form $\sum_r n_r = N$. Here $n_r$ is the number of photons in quantum state $r$ and $N$ is the total photon number. However in treating the massless photon we have a puzzle that, the total number of the massless photons is not fixed so that we cannot have the above constraint on the total number of the photons. Now, this puzzle can be solved by the intrinsic property that the massless photon is a point-like particle without any size, differently from the massive photon with finite size. After quantization of the light we thus cannot count the number of quantized massless photons. Without resorting to the above constraint on the total number of the photons, the quantum statistics for the massless photon is then known to yield the Planck distribution [14] $n_s = \frac{1}{e^\beta E_s - 1}$, where $\beta = 1/kT$ with $k$ and $T$ being the Boltzmann constant and temperature, respectively, and $E_s \ (s = 1, 2, \ldots)$ is the energy in state $s$. For the massless point-like photons, we thus have the normalization factor in (4.3), which is different from those in (3.3) for the massive photon with finite size.

Note that in constructing $\phi^{a(m=0)} (x)$, we have reduced the eight components of the wave function for the massive photon into the four components of the wave function for the massless photon. Here we have included only the DOF originated from the spin index $a$ in $\phi^{a(m=0)} (x)$ for the massless photon. In other words, we do not have the DOF associated with the positive and negative energy solutions, and thus $\phi^{a(m=0)} (x)$ do not have the energy index. This construction is consistent with the traditional photon relativistic representation. Note also that the polarization vector $\varepsilon^a$ satisfies the transversality condition $\varepsilon_\sigma p^\sigma = 0$, which is needed since the massless photon has the transverse components only. Moreover we can readily check that the above wavefunction in (4.3) fulfills the equation of motion for the massless photon

$$\Box \phi^{a(m=0)} (x) = 0.$$  

The above transversality condition $\varepsilon_a p^a = 0$ then yields $\varepsilon \cdot \vec{p} = 0$, so that we can define two space-like polarization vectors $\vec{\varepsilon}_I \ (I = 1, 2)$ satisfying $\varepsilon_I \cdot \varepsilon_J = \delta_{IJ}$. Note that $\varepsilon_I$ and $\vec{p}$ form a three dimensional orthogonal basis system as desired [13].

Next, in the massive Proca model described by a wave function $\varphi^a (x)$ with spin index $a$ only, we find the relativistic equation of motion for a massive photon

$$\Box + m^2 \varphi^a (x) = 0,$$
where $S$ is the spin index. Similar to the scheme exploited in the Dirac theory for the electron \[13\], we now make an ansatz for $\phi^a(x)$ in (2.6). Now we introduce an equation which takes the form of (2.6) in the primed system shown to be Lorentz covariant.

Moreover the equation for the massive photon in (2.6) should be of the probability interpretation, and the Dirac equation for the electron in (3.13) must be shown to be Lorentz covariant. These characteristics have been well established in the Dirac RQM for the electron \[13\]. Similar relativistic physics arguments can be applied to the massive photon, and one can find that a massive photon analog of the Dirac wave equation in (3.15)–(3.17) for a massive electron exists, as shown in \[34\]–\[33\].

V. LORENTZ TRANSFORMATION FOR A MASSIVE PHOTON

In this section we will investigate the Lorentz transformation for a massive photon. To do this, we first consider the infinitesimal Lorentz transformation given by

$$x'^\mu = a^\mu_\nu x^\nu \equiv (g^\mu_\nu + e^\mu_\nu)x^\nu \quad \text{(5.1)}$$

where $e^\mu_\nu$ are the full Lorentz group transformation parameters. Note that for the positive and negative energy solutions for a given relativistic four momentum variable $p^\mu = (E, \vec{p})$ in the RQM for the massive photon, we obtain the four probability current $J^\mu$ for finding the photon in (3.11) and (3.12). Next the probability density and current in (3.8) must form a four vector under the Lorentz transformation in order to confirm the covariance of the continuity equation and of the probability interpretation associated with the Born’s rule in a space-plus-time split of spacetime manifold, as in the Dirac RQM for the electron. Moreover the equation for the massive photon in (2.6) should be shown to be Lorentz covariant.3

Exploiting (5.1), we will show the Lorentz covariance of the relativistic equation of motion for the massive photon in (2.6). Now we introduce an equation which takes the form of (2.6) in the primed system

$$i\Gamma^\mu \partial'_\mu - m)\phi^a(x') = 0. \quad \text{(5.2)}$$

Similar to the scheme exploited in the Dirac theory for the electron \[13\], we now make an ansatz for $\phi^a(x')$ as follows

$$\phi^a(x') = S(a)\phi^a(x). \quad \text{(5.3)}$$

Here $S(a)$ is a function of $a^\mu\nu$ and a $2 \times 2$ matrix acting on the $2_{\text{energy}}$-component column vector $\phi^a(x)$ for a given spin index $a$. In order to find $S(a)$ satisfying (5.3), we manipulate (5.2) to yield

$$[iS^{-1}(a)\Gamma^\mu a^\nu S(a)\partial_\nu - m]\phi^a(x) = 0, \quad \text{(5.4)}$$

where $S^{-1}(a)$ is an inverse matrix of $S(a)$. Making use of (5.2) and (5.4), we arrive at

$$\Gamma^\mu a^\nu = S(a)\Gamma^\nu S^{-1}(a). \quad \text{(5.5)}$$

Expanding $S(a)$ in powers of $\epsilon^\mu\nu$ given by $a^\mu\nu$ in (5.1) and keeping only the linear term in the infinitesimal generators, we make an ansatz

$$S = I - \frac{i}{4} \sigma_{\mu\nu} \epsilon^{\mu\nu}, \quad \text{(5.6)}$$

where $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$ are $2 \times 2$ matrices in the RQM for the massive photon.

Inserting (5.1) and (5.6) into (5.5), we are left with

$$\Gamma^\mu_{\nu} = \frac{i}{4} [\Gamma^{\nu}, \sigma_{\rho\sigma}] \epsilon^{\rho\sigma} \quad \text{(5.7)}$$

from which we obtain the covariance condition of the form

$$[\Gamma^\mu, \sigma_{\rho\sigma}] = 2i(g^\mu_\rho \Gamma_\sigma - g^\mu_\sigma \Gamma_\rho). \quad \text{(5.8)}$$

3 Note that we find the four probability current $J^\mu_\rho$ for finding the electron in (3.19) and (3.20), for the positive and negative energy solutions with a given relativistic four momentum variable $p^\mu = (E, \vec{p})$ in the Dirac RQM for the electron. The probability density and current in (3.21) should form a four vector under the Lorentz transformation in order to ensure the covariance of the continuity equation and of the probability interpretation, and the Dirac equation for the electron in (3.13) must be shown to be Lorentz covariant. These characteristics have been well established in the Dirac RQM for the electron \[13\]. Similar relativistic physics arguments can be applied to the massive photon, and one can find that a massive photon analog of the Dirac wave equation in (3.15)–(3.17) for a massive electron exists, as shown in \[34\]–\[33\].
The problem of finding the Lorentz covariance of the relativistic equation of motion for the massive photon in (2.6) under the full Lorentz group transformation is now reduced to that of constructing matrices $\sigma_{\rho\sigma}$ satisfying (5.8). Similar to the algorithm used in the Dirac theory for the electron [13], the simplest guess to make is an anti-symmetric product of two matrices, and we find that

$$\sigma_{\rho\sigma} = \frac{i}{2}[\Gamma_{\rho}, \Gamma_{\sigma}]$$

(5.9)

is the desired matrix which satisfies the covariance condition in (5.8). Here we have used (2.7) and the ensuing results for $\sigma_{\rho\sigma}$ in the RQM for the massive photon, given by

$$\sigma_{03} = -\sigma_{30} = -i\sigma_1, \quad \sigma_{\rho\sigma} = 0, \text{ otherwise.}$$

(5.10)

We thus prove that (5.2) also holds covariantly in the primed system, and we finally show the covariance of the relativistic equation of motion for the massive photon in (2.6) under the full Lorentz group transformation. Note that, in the RQM for the massive photon, the generators of the full Lorentz group $K_{i} = \frac{1}{2}\sigma_{0i}$ and $N_{i} = \frac{1}{2}\epsilon_{ijk}\sigma_{jk}$ ($i = 1, 2, 3$) satisfy the commutation relations

$$[K_{i}, K_{j}] = -i\epsilon_{ijk}N_{k}, \quad [N_{i}, N_{j}] = i\epsilon_{ijk}N_{k}, \quad [N_{i}, K_{j}] = i\epsilon_{ijk}K_{k},$$

(5.11)

similar to the corresponding relations in the RQM for the electron [15].

Next we consider explicitly the full Lorentz group transformation for a massive photon whose trajectory is a straight line along the z axis. To do this, we introduce a rotation around z axis and a boost along z axis, which are physically of interest in the RQM for the massive photon

$$
\begin{pmatrix}
\begin{array}{c}
\xi^{0} \\
\xi^{1} \\
\xi^{2} \\
\xi^{3}
\end{array}
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
\xi^{0} \\
\xi^{1} \\
\xi^{2} \\
\xi^{3}
\end{array}
\end{pmatrix},
\begin{pmatrix}
\begin{array}{c}
\xi^{0} \\
\xi^{1} \\
\xi^{2} \\
\xi^{3}
\end{array}
\end{pmatrix}
= 
\begin{pmatrix}
\cosh \omega & 0 & 0 & -\sinh \omega \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\sinh \omega & 0 & 0 & \cosh \omega
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
\xi^{0} \\
\xi^{1} \\
\xi^{2} \\
\xi^{3}
\end{array}
\end{pmatrix},
\end{equation}

(5.12)

where $\theta$ and $\omega$ are the rotation and boost parameters, respectively. Exploiting (5.1) and (5.12) we obtain the non-vanishing Lorentz transformation parameters $\epsilon_{\rho\sigma}^{\mu}$ and $\epsilon_{\rho\sigma}^{\mu}$ for the rotation and boost with the conditions $\theta \ll 1$ and $\omega \ll 1$ given by

$$\epsilon_{\rho\sigma}^{12} = -\epsilon_{\rho\sigma}^{21} = -\theta, \quad \epsilon_{\rho\sigma}^{03} = -\epsilon_{\rho\sigma}^{30} = \omega.$$ 

(5.13)

Now, by making use of $\sigma_{\rho\sigma}$ in (5.11), we explicitly find the corresponding matrix $S$ in (5.6) for the rotation and boost associated with $\epsilon_{\rho\sigma}^{\mu}$ in (5.13). First, for the case of the rotation around z axis, we readily obtain

$$S_{\text{rot}}^{z} = I,$$

(5.14)

regardless of the non-vanishing rotation parameters $\epsilon_{\rho\sigma}^{12} = -\epsilon_{\rho\sigma}^{21} = -\theta$ in (5.13), since we have $\sigma_{12} = \sigma_{21} = 0$ in (5.10). In particular, for the case of a $2\pi$ radian rotation, we still find $S_{\text{rot}}^{z} = I$ implying a bosonic photon property that it takes a rotation of $2\pi$ radian to return $\phi^a(x)$ to its original value. Note that for the fermionic electron case it takes a $4\pi$ radian rotation to return $\psi(x)$ to its original value [13] and this characteristic is also discussed in terms of the Möbius strip structure of the hypersphere manifold in the hypersphere soliton model [16]. Second, for the case of the boost along z axis related with the non-vanishing boost parameters $\epsilon_{\rho\sigma}^{03} = -\epsilon_{\rho\sigma}^{30} = \omega$, we find

$$S_{\text{boost}}^{z} = I - \frac{1}{2}\omega\sigma_1,$$

(5.15)

Next we investigate the covariance of the probability continuity equation in (3.10) under the full Lorentz group transformation. To do this, we first rewrite the four probability current $J^{\mu}$ in (3.10) in terms of $\Gamma^{\mu}$

$$J^{\mu} = \bar{\phi}^{\alpha} \Gamma^{\mu} \phi^{\alpha}.$$ 

(5.16)

Exploiting (3.7), (5.3) and (5.16), we construct

$$J^{\mu\nu} = \bar{\phi}^{\alpha} \Gamma^{0} \Gamma^{\mu} \Gamma^{\nu} \phi^{\alpha}.$$ 

(5.17)

For the rotation around z axis associated with $S_{\text{rot}}^{z} = I$, we readily find

$$J^{\mu}_{\text{rot}} = J^{\mu},$$

(5.18)
to produce
\[ \partial'_\mu J''_\mu = a_\mu^\alpha\partial_\alpha J'_\mu = \partial'_\mu J''_\mu = 0. \] (5.19)

For the boost along z axis related with \( S^z_{\text{boost}} = I - \frac{1}{2}\omega_1 \), keeping only the linear term in the infinitesimal generators we obtain the non-vanishing components of \( J''_\mu^{\text{boost}} \)
\[ J''_\mu^{\text{boost}} = \bar{\phi}^a\Gamma^0(I - \omega_1)\phi^a, \quad J''^3_{\text{boost}} = \bar{\phi}^a\Gamma^3(I - \omega_1)\phi^a. \] (5.20)

Note that, from (5.18) and (5.20), the probability density \( \rho \) transforms like the time component of the four probability current \( J^\mu = (\rho, \vec{J}) \) of a rank one tensor. Exploiting (5.20) we obtain
\[ \partial'_\mu J''_\mu^{\text{boost}} = a_\mu^\alpha\partial_\alpha J''_\mu^{\text{boost}} = (1 - \omega_1^2)\partial'_\mu J''_\mu^{\text{boost}} = 0. \] (5.21)

Making use of (5.19) and (5.21), we thus show that in the primed system
\[ \partial'_\mu J''_\mu = 0, \] (5.22)

to confirm the covariance of the probability continuity equation in (3.10) under the full Lorentz group transformation.

VI. CONCLUSIONS

In summary, making use of the RQM for the electron, Dirac predicted the existence of the positron before the quantum field theory had not been developed. Similarly, exploiting the RQM for the massive photon, we have predicted the existence of the anti-photon without resorting to the quantum field theory. To do this, following the Dirac formalism for the RQM for the electron, we have developed a physics algorithm for the RQM for a massive photon, to formulate a \( 2\times2 \) Hamiltonian matrix for the massive photon. Exploiting the Hamiltonian, we have found a new theoretical particle solution which allows the massive photon possessing either positive or negative energy solutions. In particular, we have proposed theoretically the anti-photon corresponding to the negative energy solution, similar to the positron in the Dirac RQM for an electron. In our model, the massive photon has been shown to possess the longitudinal polarization in addition to the transverse ones as in the case of a phonon. Moreover we have formulated a nontrivial diagonal Hamiltonian for the massive photon. We also have investigated the Lorentz transformation associated with the RQM for the massive photon, to ensure the covariances of the relativistic equation of motion and the corresponding probability continuity equation. One of the main points of this paper is that, the anti-photons could be the candidate for the intense radiation flare of the mysterious GRB released from a supernova which is located far from the Earth. Note that the RQM for the anti-photon possessing positive mass is a new phenomenology, which could be consistent with the well-established Dirac positron theory and related with a fundamental prediction of the GRB.

Acknowledgments

The author would like to thank the anonymous referee for helpful comments. He was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, NRF-2019R1I1A1A01058449.

[1] S.W. Hawking and R. Penrose, *The singularities of gravitational collapse and cosmology*, Proc. Roy. Soc. Lond. A 314, 529 (1970).
[2] Y.S. Cho and S.T. Hong, *Singularities in geodesic surface congruence*, Phys. Rev. D 78, 067301 (2008), arXiv:0806.2061.
[3] Y.S. Cho and S.T. Hong, *Dynamics of stringy congruence in early universe*, Phys. Rev. D 83, 104040 (2011), arXiv:1103.0300.
[4] K.A. Bronnikov, A.A. Popov and S.G. Rubin, *Inhomogeneous compact extra dimensions and de Sitter cosmology*, Eur. Phys. J. C 80, 970 (2020), arXiv:2004.03277.
[5] R.W. Klebesadel, I.B. Strong and R.A. Olson, *Observations of gamma-ray bursts of cosmic origin*, Astrophys. J. Lett. 182, L85 (1973).
[6] D. Svinkin, D. Frederiks, K. Hurley, R. Aptekar and S. Golenetskii et al., *A bright gamma-ray flare interpreted as a giant magnetar flare in NGC 253*, Nature **589**, 211 (2021). [arXiv:2101.08104]

[7] P.A.M. Dirac, *The quantum theory of the electron*, Proc. Royal Soc. A. **117**, 610 (1928).

[8] C.D. Anderson, *The positive electron*, Phys. Rev. **43**, 491 (1933).

[9] O. Panella and P. Roy, *Bound state in continuum like solutions in one dimensional hetero-structures*, Phys. Lett. A **376**, 2580 (2012). [arXiv:1207.3639]

[10] S.T. Hong, *Photon intrinsic frequency and size in stringy photon model*, Nucl. Phys. B **976**, 115720 (2022). [arXiv:2111.11852]

[11] S.T. Hong, *Dirac quantization and baryon intrinsic frequencies in hypersphere soliton model*, Nucl. Phys. B **973**, 115611 (2021). [arXiv:2105.11456]

[12] N.W. Ashcroft and N.D. Mermin, *Solid State Physics* (Brooks/Cole, London, 1976).

[13] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[14] F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, London, 1965).

[15] S. Pokorski, *Gauge Field Theories* (Cambridge University Press, Cambridge, 1987).

[16] S.T. Hong, *Baryon topology in hypersphere soliton model*, [arXiv:2001.08120].