Relations between $N$ and $\Delta$ electromagnetic form factors $^a$

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The inclusion of two-body exchange currents in the constituent quark model leads to several new relations between the electromagnetic form factors of nucleon and $\Delta(1232)$. These are: (i) the neutron charge form factor can be expressed as the difference between proton and $\Delta^+$ charge form factors, and (ii) the $N \rightarrow \Delta$ charge quadrupole ($C_2$) transition form factor is connected to the charge monopole ($C_0$) form factor of the neutron. The latter relation is used to estimate the charge radius of constituent quarks. Furthermore, we find that exchange currents do not modify the $SU(6)$ relation between the magnetic $N \rightarrow \Delta$ and the magnetic neutron form factor. Consequently, after including exchange currents, the $C_2/M_1$ ratio in the $N \rightarrow \Delta$ transition can be expressed as a ratio of the elastic charge and magnetic neutron form factors as follows

$$\frac{C_2}{M_1}(q^2) = \frac{M_N}{2q^2} \frac{G_C^N(q^2)}{G_M^N(q^2)}.$$

1 Introduction

Baryons are complicated many-particle systems composed of valence quarks, which carry the quantum numbers, and nonvalence quark degrees of freedom, such as quark-antiquark ($q\bar{q}$) pairs and gluons. The constituent quark model with two-body exchange currents describes both these aspects baryon structure. One-body currents describe the interaction of the photon with one valence quark at a time. Two-body exchange currents are connected with the exchange particles and with $q\bar{q}$ pairs. Baryon properties which are dominated by two-body exchange currents show their common dynamical origin in analytical interrelations between them.

A new quark model relation between the neutron charge form factor $G_C^N$ and the quadrupole transition form factor $G_{C_2}^{N \rightarrow \Delta}$ is used in order to predict $G_{C_2}^{N \rightarrow \Delta}$ and the $C_2/M_1$ ratio from the elastic neutron form factor data. An astonishingly good agreement with the direct pion electroproduction data is found $^2$.

2 $N$ and $\Delta$ charge monopole form factors

In a quark potential model with gluon and pion exchange currents the baryon charge consists of a sum of one- and two-quark pieces: $\rho(q) = \rho_{[1]}(q) + \rho_{[2]}(q)$. After a multipole expansion up to quadrupole terms, the one- and two-body quark operators corresponding to Fig.1 can be schematically written as

$$\rho_{[1]}(q) \approx [Y^0(r_i) \times Y^0(q)]^0 - \sqrt{2}[Y^2(r_i) \times Y^2(q)]^0$$

$$\rho_{[2]}(q) \approx [[\sigma_i \times \sigma_j]^0 \times Y^0(q)]^0 + \frac{1}{\sqrt{2}}[[\sigma_i \times \sigma_j]^2 \times Y^2(q)]^0. \quad (1)$$

$^a$Excited Nucleons and Hadronic Structure, Proceedings of the NSTAR 2000 conference, Eds. V. D. Burkert, L. Elouadrhiri, J.J. Kelly, R. C. Minehart, World Scientific, Singapore, 2001, pg. 59
where \( r_i \) is the spatial, and \( \sigma_i \) the spin operator of a single quark, \( q \) is the three-momentum transfer of the photon, and \( Y^l \) a spherical harmonic of rank \( l \). The spin-dependent two-body terms come from the exchange current diagrams in Fig. 1(b-d).

Figure 1: Feynman diagrams of the four vector current \( J^\mu = (\rho, J) \): photon (\( \gamma \)) coupling to (a) one-body current \( J^\mu_{[1]} \), and to (b-d) two-body gluon and pion exchange currents \( J^\mu_{[2]} \). Diagrams (b-d) must be taken into account in order to satisfy the continuity equation \( q_\mu J^\mu = 0 \) for the electromagnetic current \( J_\mu \). They represent the nonvalence (gluon and pion) degrees of freedom in the nucleon in the presence of an external electromagnetic field.

Evaluating these operators between three-quark proton wave functions, one obtains for the proton charge radius \( r_p^2 \)

\[
    r_p^2 = b^2 + r_{\gamma q}^2 + \frac{b^2}{2m_q} (\delta_{g}(b) - \delta_{\pi}(b)).
\]

Here, \( b \) is the quark core (matter) radius of the nucleon, \( r_{\gamma q} \) the finite charge radius of the constituent quark, and \( m_q \) the constituent quark mass satisfying \( M_N = 3m_q \).

The terms proportional to \( \delta_{g}(b) \) and \( \delta_{\pi}(b) \) describe the gluon and pion exchange current contributions to the charge radius. These functions also express the gluon and pion contributions to the \( N-\Delta \) mass splitting: \( M_\Delta - M_N = \delta_{g}(b) + \delta_{\pi}(b) \).

The proton charge radius is mainly determined by the valence quark terms \( b^2 \) and \( r_{\gamma q}^2 \); i.e., the one-body current depicted in Fig. 1(a); the exchange currents of Fig. 1(b-c) provide only a small correction to \( r_p^2 \).

In contrast, the Sachs charge form factor of the neutron \( G_C^N(q^2) \) (see Fig. 2) and the corresponding charge radius \( r_n^2 = -6(d/dq^2)G_C^N(q^2) \big|_{q^2=0} \) are dominated by the quark-antiquark pair exchange currents, and one obtains

\[
    r_n^2 = -\frac{b^2}{3m_q} (\delta_{g}(b) + \delta_{\pi}(b)) = -b^2 \left( \frac{M_\Delta - M_N}{M_N} \right).
\]

The above relations show that the spin-dependent gluon and pion exchange potentials, which generate the \( N-\Delta \) mass splitting, are also responsible for the nonvanishing neutron charge radius via the corresponding spin-spin term in Eq. (1). The valence quark terms \( b^2 \) and \( r_{\gamma q}^2 \) do not appear in Eq. (3). The reasons for this will become clear soon.
The charge radii of all charged $\Delta(1232)$ states are calculated in the same $N_c = 3$ quark model as

$$r_{\Delta}^2 = b^2 + r_{7q}^2 + \frac{b^2}{6m_q}(5\delta_g(b) - \delta_\pi(b)), \quad r_{\Delta^0}^2 = 0. \quad (4)$$

The charge form factor of the $\Delta^0$ and the corresponding charge radius are zero (for $N_c = 3$) as it should be on general grounds. Subtracting Eq.\((4)\) from Eq.\((2)\) and Eq.\((3)\) the valence quark contributions $b^2$ and $r_{7q}^2$ cancel, and one finds

$$r_{\Delta}^2 - r_{\Delta^+}^2 = r_{\Delta^0}^2, \quad r_{\Delta^+}^2 - r_{\Delta^0}^2 = 0. \quad (5)$$

Eqs.\((5)\) are the first moments of the more general relations between the charge form factors of the $N$ and $\Delta$

$$G_C^N(q^2) - G_C^{\Delta^+}(q^2) = G_C^{\Delta^0}(q^2), \quad G_C^p(q^2) - G_C^{\Delta^+}(q^2) = G_C^{\Delta^0}(q^2). \quad (6)$$

Including the $\Delta^{++}$ and the $\Delta^{-}$ charge states this can be written in closed form as

$$G_C^\Delta(q^2) = (G_C^p(q^2) - G_C^{\Delta^0}(q^2)) e_\Delta, \quad (7)$$
where \( \epsilon_\Delta = (1 + 2T_3)/2 \) is the \( \Delta \) charge. These relations are not equivalent to the \( SU(6) \) result \( G^\Delta_\alpha = (G^p_\alpha + G^n_\alpha)/2 + (G^p_\alpha - G^n_\alpha)T_3 \), where \( T_3 \) is the third component of the \( \Delta \) isospin. Eq.(7) contains the important symmetry breaking effect coming from the spin-spin term in Eq.(5).

Dillon and Morpurgo\(^7\) have recently shown that Eq.(5) is a direct consequence of the underlying \( SU(6) \) spin-flavor symmetry and the quark-gluon dynamics of quantum chromodynamics. They have also shown that three-body currents slightly modify, but do not invalidate the general relationship between the proton, neutron, and \( \Delta \) charge radii. The work of Dillon and Morpurgo makes it clear that Eq.(5) originally found in the quark model with exchange currents, is a general relation if three-body operators and strange quark loops are neglected.

Considering constituent quarks with an arbitrary number of colors \( N_c \), one can generalize these findings\(^8\). Eq.(5) then no longer holds for arbitrary \( N_c \), but the relation

\[
r^2_\Delta - r^2_p = r^2_n - r^2_\Delta^0
\]

is valid for any \( N_c \). It is broken only by three-body \( O(1/N_c^2) \) terms, which are suppressed compared to the two-body terms included. For \( N_c = 3 \), \( r^2_\Delta^0 = 0 \) and we reobtain Eq.(5). The general \( N_c \) analysis gives the \( N - \Delta \) charge radius relationship a rigorous theoretical foundation\(^8\).

The following discussion suggests a connection between the \( N - \Delta \) mass and charge radius difference. We recall that the spin-spin structure \( \sigma_i \cdot \sigma_j \) in Eq.(1) is responsible for the splitting between \( N \) and \( \Delta \) charge radii. It leads to a \( \Delta \) charge radius that is larger than the proton charge radius by an amount that is equal to the negative neutron charge radius. The neutron charge radius is nonzero, because the spin-spin term in Eq.(5) gives different matrix elements for quark pairs in spin 0 and spin 1 states. This splitting of the \( N \) and \( \Delta \) charge radii is of the same generality as, and closely connected with the \( N - \Delta \) mass splitting due to the spin-spin interaction in the Hamiltonian. The latter is repulsive in quark pairs with spin 1 and makes the \( \Delta \) heavier than the nucleon. Combining Eq.(3) and Eq.(5) and the fact that baryon charge radii and masses have the same large \( N_c \) operator expansion\(^8\) we conjecture that

\[
\frac{r^2_\Delta - r^2_p}{r^2_\Delta^0 - r^2_n} = \frac{M_{\Delta^+} - M_p}{M_{\Delta^0} - M_n}
\]

contains some of the three-body corrections not included in Eq.(8).

### 3 \( N \) and \( \Delta \) magnetic dipole form factors

The quark model with two-body exchange currents also relates the magnetic form factors of the \( N \) and \( \Delta \)

\[
G^\Delta_M(q^2) = 3 \left( G^p_M(q^2) + G^n_M(q^2) \right) \epsilon_\Delta.
\]

Eq.(10) differs from the \( SU(6) \) relation\(^3\): \( G^\Delta_M = G^p_M \epsilon_\Delta \). There is no difference between the quark model with exchange currents and the \( SU(6) \) result if the additional \( SU(6) \) relation \( G^n_M = -2G^p_M/3 \) is used in Eq.(10). Our predictions for the
Table 1: ∆(1232) magnetic moments based on the relations suggested in this paper. As input the experimental proton and neutron magnetic moments are used. The experimental range for the ∆++ magnetic moment is $\mu_{\Delta^{++}} = (3.7 - 7.5) \mu_N$. For the $N \rightarrow \Delta$ transition magnetic moment experimental values lie between $\mu_{p \rightarrow \Delta^+} = (3.5 - 4.2) \mu_N$. All entries are in given in units of nuclear magnetons $\mu_N = e/(2M_p)$.

| Baryon | Quark Model | $SU(6)$ |
|--------|-------------|---------|
| $\Delta^{++}$ | 5.28 | 5.58 |
| $\Delta^+$ | 2.64 | 2.79 |
| $\Delta^0$ | 0.000 | 0.000 |
| $\Delta^-$ | -2.64 | -2.79 |
| $p \rightarrow \Delta^+$ | 2.70 | 2.70 |
| $n \rightarrow \Delta^0$ | 2.70 | 2.70 |

Δ magnetic moments based on Eq. (10) are given in Table 1. We observe that the $\Delta^+$ magnetic moment is only slightly smaller than the proton magnetic moment.

4 $N \rightarrow \Delta$ charge quadrupole transition form factor

In the constituent quark model with exchange currents a connection between the neutron charge form factor $G_n^p(q^2)$ and the $N \rightarrow \Delta$ quadrupole transition form factor $G_{C2}^{p \rightarrow \Delta^+}(q^2)$ emerges:

$$G_{C2}^{p \rightarrow \Delta^+}(q^2) = -\frac{3\sqrt{2}}{q^2} G_n^p(q^2) = -\frac{3\sqrt{2}}{q^2} \left( G_C^p(q^2) - G_{\Delta^+}^C(q^2) \right),$$  

which is plotted in Fig. 2 (lower curve).

In the low-momentum transfer limit we derive from Eq. (11) that the $N \rightarrow \Delta$ transition quadrupole moment $Q_{p \rightarrow \Delta^+}$ is determined by the neutron charge radius $r_n^2$:

$$Q_{p \rightarrow \Delta^+} = \frac{r_n^2}{\sqrt{2}}.$$

We recall that $Q_{p \rightarrow \Delta^+}$ is in combination with the $\Delta^+$ quadrupole moment $Q_{\Delta^+}$ a measure of the intrinsic deformation of the nucleon and $\Delta$. The quantities that determine the intrinsic deformation are the intrinsic quadrupole moments $Q_0^p$ and $Q_0^{\Delta}$. The connection between the observable (spectroscopic) $Q_{p \rightarrow \Delta^+}$ and the $Q_{\Delta^+}$ and corresponding intrinsic quadrupole moments has recently been evaluated in different models. In the quark model we find using the empirical neutron charge radius $r_n = 0.729$ fm the relation $Q_0^p = -Q_0^{\Delta^+} = -Q_{\Delta^+} = -\sqrt{2} Q_{p \rightarrow \Delta^+} = +0.113$ fm$^2$. A negative $C2/M1$ ratio therefore implies a prolate (cigar-shaped) intrinsic deformation of the nucleon and an oblate (pancake-shaped) intrinsic deformation of the $\Delta$.

The quark model with exchange currents explains $Q_{p \rightarrow \Delta^+}$ as a double spin flip of two quarks, with all valence quarks remaining in the dominant, spherically symmetric $L = 0$ state. The spin-flip of two quarks comes from the tensor structure in Eq. (1). The latter is closely related to the tensor term in the Hamiltonian, which via the $D$ waves in the $N$ and $\Delta$ also contributes to $Q_{p \rightarrow \Delta^+}$. This orbital excitation
of a valence quark amounts to about 20% (due to the smallness of the $D$ wave amplitudes) of the double spin flip amplitude. We conclude that the collective $q\bar{q}$ degrees of freedom are mainly responsible for the deformation of the $N$ and $\Delta$. The importance of the spin tensor in Eq. (3) for a complete explanation of the $N \to \Delta$ quadrupole transition moment in the quark model was anticipated by Morpurgo.

We have also calculated the radius of the $N \to \Delta$ transition quadrupole form factor, and obtained

$$r^2_{Q,p\to\Delta^+} = \frac{11}{20} b^2 + r_{\gamma q}^2.$$  \hspace{1cm} (13)

Unlike in Eq. (2), there is no correction from two-body exchange currents in Eq. (13), which makes the quadrupole transition radius an ideal observable to experimentally determine the quark charge radius $r_{\gamma q}$. With the help of Eq. (11), the quadrupole transition radius can be expressed as

$$r^2_{Q,p\to\Delta^+} = \left(\frac{18}{r_n^4}\right) \frac{d}{d\alpha^2} G_C^n(q^2)|_{q^2=0} = \frac{3}{10} \frac{r_n^4}{r_n^2},$$  \hspace{1cm} (14)

where $r_n^4$ is the fourth moment of the neutron charge distribution. Because the quark core radius $b$ is fixed by Eq. (3), one can extract the charge radius of the light constituent quarks from the $G_C^n(q^2)$ data. A recent fit to the $G_C^n$ data determines $r_n^4 = -0.32$ fm$^4$ and the transition quadrupole radius as $r_{Q,p\to\Delta^+} = 0.84(21)$ fm$^2$. An additional data point of $G_C^n$ at $Q^2 = 0.9$ GeV$^{-2}$ would reduce the error by a factor of three. From Eq. (13) and Eq. (3) we obtain $r_{\gamma q} = 0.64$ fm$^2$, a rather large constituent quark charge radius. This implies a proton charge radius of about 1 fm.

5 $N \to \Delta$ magnetic dipole transition form factor

After including the gauge-invariant two-body exchange currents of Fig. 1(b-d), the $SU(6)$ Beg-Lee-Pais relation between the magnetic $N \to \Delta$ transition and the neutron magnetic moments $\mu_{p\to\Delta^+} = -\sqrt{2} \mu_n$ remains unchanged, and holds even at finite momentum transfers

$$G_{M\to\Delta^+}^p(q^2) = -\sqrt{2} G_M^n(q^2), \quad \mu_{p\to\Delta^+} = -\sqrt{2} \mu_n.$$  \hspace{1cm} (15)

The $N \to \Delta$ transition magnetic moment predicted by Eq. (15) underestimates the empirical value $\mu^{exp}_{p\to\Delta^+} = (3.5 - 4.0) \mu_N$ by about (30 - 50)%.

On the other hand, replacing $G_M^n$ with the help of Eq. (10) allows to consider the problem from a different perspective

$$G_{M\to\Delta^+}^p(q^2) = \sqrt{2} \left( G_M^n(q^2) - \frac{1}{3} G_M^\Delta(q^2) \right), \quad \mu_{p\to\Delta^+} = \sqrt{2} \left( \mu_p - \frac{1}{3} \mu_\Delta \right).$$  \hspace{1cm} (16)

Eq. (16), which describes the transition magnetic form factor in terms of the magnetic form factors of the two baryons involved in the transition, constrains $\mu_{p\to\Delta^+}$ to values below $3 \mu_N$. In order to make $\mu_{p\to\Delta^+}$ larger one needs an unacceptably

\footnote{The term $\frac{1}{4} b^2$ in Eq. (53) of Ref. 5 should be replaced by $\frac{1}{20} b^2$.}
small or even a negative value for the $\Delta^+$ magnetic moment. Thus, $\mu_{\pi\Delta}^{\exp}$ most likely includes diagrams that should not be included in the definition of the proper strength of the electromagnetic $\gamma N \Delta$ vertex. An analogous observation has been made for the $\pi N \Delta$ coupling strength.

For the radius of the $N \to \Delta$ magnetic transition form factor, $r^2_{M,p\to\Delta^+}$ we find

$$r^2_{M,p\to\Delta^+} = \frac{-\sqrt{2} \mu_n}{\mu_{p\to\Delta^+}} r^2_{M,n},$$  \hspace{1cm} (17)$$

where $r^2_{M,n}$ is the magnetic radius of the neutron. With the $SU(6)$ result of Eq.(15), which is equivalent to the quark model result with two-body exchange currents, we obtain $r^2_{M,p\to\Delta^+} = r^2_{M,n}$. If $\mu_{p\to\Delta^+} > -\sqrt{2} \mu_n$ we get $r^2_{M,p\to\Delta^+} < r^2_{M,n}$, contradicting the experimental observation that the $N \to \Delta$ magnetic transition form factor drops off faster than the magnetic neutron form factor.

### 6 C2/M1 ratio in the $N \to \Delta$ transition

Combining Eq. (11) and Eq. (15) we find that the ratio of the charge quadrupole and magnetic dipole $N \to \Delta$ transition form factors can be expressed in terms of the elastic neutron form factors

$$\frac{C^2}{M^1}(q^2) = \frac{M_N}{2\sqrt{q^2}} \frac{G^n_C(q^2)}{G^n_M(q^2)}, \quad \frac{C^2}{M^1} = -\frac{M_N \omega_{cm}}{6} \frac{r^2_n}{2\mu_n},$$  \hspace{1cm} (18)$$

where the last expression is obtained in the zero momentum transfer limit. Here, $r^2_n$ is the neutron charge radius (in fm$^2$), $\mu_n$ the neutron magnetic moment (divided by $\mu_N = e/(2M_N)$), and $\omega_{cm}$ is the center of mass energy of the photon-nucleon system at the $\Delta$ resonance. Fig. 3 shows the predictions of $C2/M1$ based on the elastic neutron form factor data and Eq. (18). Sign and magnitude of the $C2/M1$ ratio calculated in this way are in astonishingly good agreement with direct pion electroproduction data. This supports our finding that the neutron and $N \to \Delta$ quadrupole transition form factors are related as suggested by Eq. (18).

### 7 Summary

By including two-body currents in the constituent quark model we have found a number of new relations between the elastic electromagnetic form factors of the $N$ and $\Delta$. These relations contain the effect of the $SU(6)$ symmetry breaking spin-spin term in the charge operator of Eq. (4). The latter lifts the degeneracy of the nucleon and $\Delta$ charge form factors and shows that their difference is equal to the neutron charge form factor. Dillon and Morpurgo have recently proven that the ensuing Eq. (5), which relates proton, neutron, and $\Delta$ charge radii is generally valid if the (numerically small) three-quark operators and strange quark loops are neglected. Furthermore, in a large $N_c$ approach we find that the relation $G^p_C(q^2) - G^n_C(q^2) = G^n_C(q^2) - G^\Delta_C(q^2)$ holds for any $N_c$, and that it is broken only by small three-body $O(1/N^2_c)$ terms, which also underlines its generality.
The $N \rightarrow \Delta$ quadrupole transition form factor $G_{C_2}^{p\rightarrow\Delta}$ is found to be expressable in terms of the neutron charge form factor $G_n^p$. Eq. (11) displays the underlying SU(6) symmetry and its breaking due to the spin-dependent two-body exchange currents. The SU(6) relation between the $N \rightarrow \Delta$ magnetic dipole transition form factor and the magnetic neutron form factor is not changed by the two-body exchange currents of Fig. 1. Even after the inclusion of spatial two-body currents the $N \rightarrow \Delta$ transition magnetic moment is given by the Beg-Lee-Pais relation.

In summary, our theory leads to a number of new relations between the electromagnetic form factors of the nucleon and the $\Delta$. In particular, the $C_2/M_1$ ratio in the electromagnetic $N \rightarrow \Delta$ transition is given by the ratio of the elastic neutron charge and magnetic form factors. The good agreement between our prediction for the $C_2/M_1$ ratio and the pion electroproduction data supports our analysis. Our work makes it clear that the deformation of the nucleon and the neutron charge radius are related phenomena. They are different manifestations of the $q\bar{q}$ degrees of freedom in the nucleon, which are in leading order expressable by two-body exchange currents.

**Acknowledgement:** This work is supported by the DFG BU813/2-1 and Jefferson Lab.

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