Self-dual projective algebraic varieties
associated with symmetric spaces

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Abstract
We discover a class of projective self-dual algebraic varieties. Namely, we consider
actions of isotropy groups of complex symmetric spaces on the projectivized nilpotent
varieties of isotropy modules. For them, we classify all orbit closures \( X \) such that
\( X = \check{X} \) where \( \check{X} \) is the projective dual of \( X \). We give algebraic criteria of projective
self-duality for the considered orbit closures.

1. Introduction
Under different guises dual varieties of projective algebraic varieties have been considered
in various branches of mathematics for over a hundred years. In fact, the dual variety is
the generalization to algebraic geometry of the Legendre transform in classical mechanics,
and the Biduality Theorem essentially rephrases the duality between the Lagrange and
Hamilton–Jacobi approaches in classical mechanics.

Let \( X \) be an \( n \)-dimensional projective subvariety of an \( N \)-dimensional projective space
\( \mathbf{P} \), and let \( \check{X} \) be the dual variety of \( X \) in the dual projective space \( \mathbf{P}^* \). Since various kinds of
geometrically meaningful unusual behavior of hyperplane sections are manifested more
explicitly in terms of dual varieties, it makes sense to consider their natural invariants. The
simplest invariant of \( \check{X} \) is its dimension \( \check{n} \). “Typically”, \( \check{n} = N - 1 \), i.e., \( \check{X} \) is a hypersurface.
The deviation from the “typical” behavior admits a geometric interpretation: if \( \check{X} \) is not a
hypersurface and, say, \( \text{codim} \check{X} = s + 1 \), then \( X \) is uniruled by \( s \)-planes.

Assume that \( X \) is a smooth variety not contained in a hyperplane. Then \( n \leq \check{n} \), by [Z, Chapter 1]. If the extremal case \( n = \check{n} \) holds and \( n = \check{n} \leq 2N/3 \), then, by [Ei1], [Ei2], such
\( X \) are classified by the following list:

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(i) hypersurfaces,
(ii) $\mathbf{P}^1 \times \mathbf{P}^{n-1}$ embedded in $\mathbf{P}^{2n-1}$ by the Segre embedding,
(iii) the Grassmannian of lines in $\mathbf{P}^4$ embedded in $\mathbf{P}^9$ by the Plücker map, or
(iv) the 10-dimensional spinor variety of 4-dimensional linear subspaces on a nonsingular 8-dimensional quadric in $\mathbf{P}^{15}$.

According to Hartshorne’s famous conjecture, if $n > 2N/3$, then $X$ is a complete intersection, and hence, by [Ei2], $\tilde{X}$ is a hypersurface. Therefore it is plausible that the above-stated list contains every smooth $X$ such that $n = \tilde{n}$. Furthermore, in cases (ii), (iii) and (iv), the variety $X$ is self-dual in the sense that, as an embedded variety, $\tilde{X}$ is isomorphic to $X$. In case (i), it is self-dual if and only if it is a quadric. Thus, modulo Hartshorne’s conjecture, the above-stated list gives the complete classification of all smooth self-dual projective algebraic varieties. In particular it shows that there are not many of them.

In [P2], it was found a method for constructing many nonsmooth self-dual projective algebraic varieties. It is related to algebraic transformation groups theory: the self-dual projective algebraic varieties appearing in this construction are certain projectivized orbit closures of some linear actions of reductive algebraic groups. We recall this construction.

Let $G$ be a connected reductive algebraic group, let $V$ be a finite dimensional algebraic $G$-module and let $B$ be a nondegenerate symmetric $G$-invariant bilinear form on $V$. We assume that the ground field is $\mathbb{C}$. For a subset $S$ of $V$, put $S^\perp := \{v \in V \mid B(v, s) = 0 \ \forall \ s \in S\}$. We identify $V$ and $V^*$ by means of $B$ and denote by $\mathbf{P}$ the associated projective space of $V = V^*$. Thereby the projective dual $\tilde{X}$ of a Zariski closed irreducible subset $X$ of $\mathbf{P}$ is a Zariski closed subset of $\mathbf{P}$ as well. Let $\mathcal{N}(V)$ be the null-cone of $V$, i.e.,

$$\mathcal{N}(V) := \{v \in V \mid 0 \in \overline{G \cdot v}\},$$

where bar stands for Zariski closure in $V$.

**Theorem 1.** ([P2, Theorem 1]) Assume that there are only finitely many $G$-orbits in $\mathcal{N}(V)$. Let $v \in \mathcal{N}(V)$ be a nonzero vector and let $X := \mathbf{P}(\overline{G \cdot v}) \subseteq \mathbf{P}$ be the projectivization of its orbit closure. Then the following properties are equivalent:

(i) $X = \tilde{X}$.
(ii) $(\text{Lie}(G) \cdot v)^\perp \subseteq \mathcal{N}(V)$.

Among the modules covered by this method, there are two naturally allocated classes, namely, that of the adjoint modules and that of the isotropy modules of symmetric spaces. In fact the first class is a subclass of the second one; this subclass has especially nice geometric properties and is studied in more details. Projective self-dual algebraic varieties associated with the adjoint modules by means of Theorem 1 were explicitly classified and studied in [P2]. In particular, according to [P2], these varieties are precisely the projectivized orbit closures of nilpotent elements in the Lie algebra of $G$ that are distinguished in the sense of Bala and Carter (see below Theorem 2). This purely algebraic notion plays an important role in the Bala–Carter classification of nilpotent elements, [BaCa].

For the isotropy modules of symmetric spaces, in [P2] it was introduced the notion of $(-1)$-distinguished element of a $\mathbb{Z}_2$-graded semisimple Lie algebra and shown that the projective self-dual algebraic varieties associated with such modules by means of Theorem 1
are precisely the projectivized orbit closures of \((-1)\)-distinguished elements. Thereby classification of such varieties was reduced to the problem of classifying \((-1)\)-distinguished elements. In [P2], it was announced that the latter problem will be addressed in a separate publication.

The goal of the present paper is to give the announced classification: here we explicitly classify \((-1)\)-distinguished elements of all \(\mathbb{Z}_2\)-graded complex semisimple Lie algebras; this yields the classification of all projective self-dual algebraic varieties associated with the isotropy modules of symmetric spaces by means of the above-stated construction from [P2]. In the last section we briefly discuss some geometric properties of these varieties.

Notice that there are examples of singular projective self-dual algebraic varieties constructed in a different way. For instance, the Kummer surface in \(\mathbb{P}^3\), see [GH], or the Coble quartic hypersurface in \(\mathbb{P}^7\), see [Pa], are projective self-dual. A series of examples is given by means of “Pyasetskii pairing”, [T] (e.g., projectivization of the cone of \(n \times m\)-matrices of rank \(\leq n/2\) is projective self-dual for \(n \leq m\), \(n\) even).

Given a projective variety, in general it may be difficult to explicitly identify its dual variety. So our classification contributes to the problem of finding projective varieties \(X\) for which \(\check{X}\) can be explicitly identified. Another application concerns the problem of explicit describing the projective dual varieties of the projectivized nilpotent orbit closures in the isotropy modules of symmetric spaces: our classification yields its solution for all \((-1)\)-distinguished orbits. To the best of our knowledge, at this writing this problem is largely open; even finding the dimensions of these projective dual varieties would be interesting (for the minimal nilpotent orbit closures in the adjoint modules, a solution was found in [KM]; cf. [Sn] for a short conceptual proof).

2. Main reductions

Let \(g\) be a semisimple complex Lie algebra, let \(G\) be the adjoint group of \(g\), and let \(\theta \in \text{Aut} \ g\) be an element of order 2. We set

\[
\mathfrak{k} := \{x \in g \mid \theta(x) = x\}, \quad \mathfrak{p} := \{x \in g \mid \theta(x) = -x\}.
\]

Then \(\mathfrak{k}\) and \(\mathfrak{p} \neq 0\), the subalgebra \(\mathfrak{k}\) of \(g\) is reductive, and \(g = \mathfrak{k} \oplus \mathfrak{p}\) is a \(\mathbb{Z}_2\)-grading of the Lie algebra \(g\), cf., e.g., [OV]. Denote by \(G\) the adjoint group of \(g\). Let \(K\) be the connected reductive subgroup of \(G\) with the Lie algebra \(\mathfrak{k}\); it is the adjoint group of \(\mathfrak{k}\).

Consider the adjoint \(G\)-module \(g\). Its null-cone \(\mathcal{N}(g)\) is the cone of all nilpotent elements of \(g\); it contains only finitely many \(G\)-orbits, [D], [K1]. The space \(\mathfrak{p}\) is \(K\)-stable, and we have

\[
\mathcal{N}(\mathfrak{p}) = \mathcal{N}(g) \cap \mathfrak{p}.
\]

By [KR], there are only finitely many \(K\)-orbits in \(\mathcal{N}(\mathfrak{p})\) and

\[
2 \dim_C K \cdot x = \dim_C G \cdot x \text{ for any } x \in \mathcal{N}(\mathfrak{p}).
\]

The Killing form \(B_g\) of \(g\) (resp., its restriction \(B_{g|\mathfrak{p}}\) to \(\mathfrak{p}\)) is symmetric nondegenerate and \(G\)-invariant (resp., \(K\)-invariant). We identify the linear spaces \(g\) and \(g^*\) (resp., \(\mathfrak{p}\) and \(\mathfrak{p}^*\)) by means of \(B_g\) (resp., \(B_{g|\mathfrak{p}}\)).
Recall from [BaCa], [Ca] that a nilpotent element \( x \in \mathcal{N}(g) \) or its \( G \)-orbit \( G \cdot x \) is called \textit{distinguished} if the centralizer \( z_g(x) \) of \( x \) in \( g \) contains no nonzero semisimple elements. According to [P2], distinguished orbits admit the following geometric characterization:

**Theorem 2.** ([P2, Theorem 2]) Let \( O \) be a nonzero \( G \)-orbit in \( \mathcal{N}(g) \) and \( X := P(O) \). The following properties are equivalent:

(i) \( X = \bar{X} \).

(ii) \( O \) is distinguished.

Hence the classification of distinguished elements obtained in [BaCa] yields the classification of projective self-dual orbit closures in \( P(\mathcal{N}(g)) \). This, in turn, helps studying geometric properties of these projective self-dual projective varieties, [P2].

There is a counterpart of Theorem 2 for the action of \( K \) on \( p \). Namely, if \( x \in p \), then \( z_g(x) \) is \( \theta \)-stable, hence it is a graded Lie subalgebra of \( g \),

\[
\mathfrak{z}_g(x) = \mathfrak{z}_\mathfrak{k}(x) \oplus \mathfrak{z}_p(x), \quad \text{where} \quad \mathfrak{z}_\mathfrak{k}(x) := z_g(x) \cap \mathfrak{k}, \quad \mathfrak{z}_p(x) := z_g(x) \cap \mathfrak{p}.
\]

**Definition 1.** ([P2]) An element \( x \in \mathcal{N}(p) \) and its \( K \)-orbit are called \((-1)\)-distinguished if \( \mathfrak{z}_p(x) \) contains no nonzero semisimple elements.

**Remark.** It is tempting to use simpler terminology and replace “\((-1)\)-distinguished” with merely “distinguished”. However this might lead to the ambiguity since, as it is shown below, there are the cases where \( x \) is \((-1)\)-distinguished with respect to the action of \( K \) on \( p \) but \( x \) is not distinguished with respect to the action of \( G \) on \( g \).

Notice that since \( p \) contains nonzero semisimple elements (see below the proof of Theorem 7), every \((-1)\)-distinguished element is nonzero.

**Theorem 3.** ([P2, Theorem 5]) Let \( O \) be a nonzero \( K \)-orbit in \( \mathcal{N}(p) \) and \( X := P(O) \). The following properties are equivalent:

(i) \( X = \bar{X} \).

(ii) \( O \) is \((-1)\)-distinguished.

By virtue of Theorem 3, our goal is classifying \((-1)\)-distinguished orbits.

**Theorem 4.** Let \( O \) be a \( K \)-orbit in \( \mathcal{N}(p) \) and let \( x \) be an element of \( O \). The following properties are equivalent:

(i) \( O \) is \((-1)\)-distinguished.

(ii) The reductive Levi factors of \( \mathfrak{z}_\mathfrak{k}(x) \) and \( \mathfrak{z}_g(x) \) have the same dimension.

(iii) The reductive Levi factors of \( \mathfrak{z}_\mathfrak{k}(x) \) and \( \mathfrak{z}_g(x) \) are isomorphic.

If \( g = h \oplus h \) where \( h \) is a semisimple complex Lie algebra, and \( \theta((y,z)) = (z,y) \), then

\[
\mathfrak{k} = \{(y,y) \mid y \in h\}, \quad \mathfrak{p} = \{(y,-y) \mid y \in h\}.
\]

If \( H \) is the adjoint group of \( h \), then (3) implies that \( K \) is isomorphic to \( H \) and the \( K \)-module \( p \) is isomorphic to the adjoint \( H \)-module \( h \). Therefore, for \( y \in \mathcal{N}(h) \), the variety \( P(K \cdot y) \) is projective self-dual if and only if the variety \( P(\mathbf{1} \cdot x) \) for \( x := (y,-y) \in \mathcal{N}(p) \) is projective.
self-dual. This is consistent with Theorems 2 and 3: the definitions clearly imply that \( x \) is \((-1)\)-distinguished if and only \( y \) is distinguished. So Theorem 2 follows from Theorem 3.

In [N], it was made an attempt to develop an analogue of the Bala–Carter theory for nilpotent orbits in real semisimple Lie algebras. The Kostant–Sekiguchi bijection (recalled below in this section) reduces this to finding an analogue of the Bala–Carter theory for \( K \)-orbits in \( \mathcal{N}(p) \). Given the above discussion, we believe that if such a natural analogue exists, \((-1)\)-distinguished elements should play a key role in it, analogous to that of distinguished elements in the original Bala–Carter theory (in [N], the so-called noticed elements, different from \((-1)\)-distinguished ones, play a central role).

Theorems 3, 4 and their proofs are valid over any algebraically closed ground field of characteristic zero. They give a key to explicit classifying \((-1)\)-distinguished elements since there is a method (see the end of this section) for finding reductive Levi factors of \( z_{k}(x) \) and \( z_{g}(x) \).

However, over the complex numbers, there is another nice characterization of \((-1)\)-distinguished orbits. It is given below in Theorem 5. For classical \( g \), our approach to explicit classifying \((-1)\)-distinguished orbits is based on this result. The very formulation of this characterization uses complex topology. However remark that, by the Lefschetz’s principle, the final explicit classification of \((-1)\)-distinguished orbits for classical \( g \) given by combining Theorem 5 with Theorems 8–13 is valid over any algebraically closed ground field of characteristic zero.

Namely, according to the classical theory (cf., e.g., [OV]), there is a \( \theta \)-stable real form \( g_{R} \) of \( g \) such that

\[
(4) \quad g_{R} = \mathfrak{k}_{R} \oplus p_{R}, \quad \text{where} \quad \mathfrak{k}_{R} := g_{R} \cap \mathfrak{k}, \quad p_{R} := g_{R} \cap p,
\]

is a Cartan decomposition of \( g_{R} \). The semisimple real algebra \( g_{R} \) is noncompact since \( p \neq 0 \). Assigning \( g_{R} \) to \( \theta \) induces well defined bijection from the set of all conjugacy classes of elements of order 2 in \( \text{Aut} g \) to the set of conjugacy classes of noncompact real forms of \( g \). If \( x \) is an element of \( g_{R} \), we put

\[
\mathfrak{z}_{g_{R}}(x) := g_{R} \cap \mathfrak{z}(x);
\]

\( \mathfrak{z}_{g_{R}}(x) \) is the centralizer of \( x \) in \( g_{R} \). The identity component of the Lie group of real points of \( G \) is the adjoint group \( \text{Ad}(g_{R}) \) of \( g_{R} \). We set

\[
\mathcal{N}(g_{R}) := \mathcal{N}(g) \cap g_{R}.
\]

**Definition 2.** An element \( x \in \mathcal{N}(g_{R}) \) is called **compact** if the reductive Levi factor of its centralizer \( \mathfrak{z}_{g_{R}}(x) \) is a compact Lie algebra.

Recall that there is a bijection between the sets of nonzero \( K \)-orbits in \( \mathcal{N}(p) \) and nonzero \( \text{Ad}(g_{R}) \)-orbits in \( \mathcal{N}(g_{R}) \), cf. [CM], [M]. Namely, let \( \sigma \) be the complex conjugation of \( g \) defined by \( g_{R} \), viz.,

\[
\sigma(a + ib) = a - ib, \quad a, b \in g_{R}.
\]

Let \( \{e, h, f\} \) be an \( \mathfrak{sl}_{2} \)-triple in \( g \), i.e., an ordered triple of elements of \( g \) spanning a three-dimensional simple subalgebra of \( g \) and satisfying the bracket relations

\[
[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.
\]
It is called a **complex Cayley triple** if \( e, f \in \mathfrak{p} \) (hence \( h \in \mathfrak{k} \)) and \( \sigma(e) = -f \). Given a complex Cayley triple \( \{e, h, f\} \), set
\[
e' := i(-h + e + f)/2, \quad h' := e - f, \quad f' := -i(h + e + f)/2.
\]
Then \( \{e', h', f'\} \) is an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g}_\mathbb{R} \) such that \( \theta(e') = f' \) (and hence \( \theta(f') = e' \), \( \theta(h') = -h' \)). The \( \mathfrak{sl}_2 \)-triples in \( \mathfrak{g}_\mathbb{R} \) satisfying the last property are called the **real Cayley triples**. Given \( \{e', h', f'\} \), the following formulas restore \( \{e, h, f\} \):
\[
e := (h' + if' - ie')/2, \quad h := i(e' + f'), \quad f := (-h' + if' - ie')/2.
\]
The map \( \{e, h, f\} \mapsto \{e', h', f'\} \) is a bijection from the set of complex to the set of real Cayley triples. The triple \( \{e, h, f\} \) is called the **Cayley transform** of \( \{e', h', f'\} \).

Now let \( \mathcal{O} \) be a nonzero \( K \)-orbit in \( \mathcal{N}(\mathfrak{p}) \). Then, according to [KR], there exists a complex Cayley triple \( \{e, h, f\} \) in \( \mathfrak{g} \) such that \( e \in \mathcal{O} \). Let \( \{e', h', f'\} \) be the real Cayley triple in \( \mathfrak{g}_\mathbb{R} \) such that \( \{e, h, f\} \) is its Cayley transform. Let \( \mathcal{O}' = \text{Ad}(\mathfrak{g}_\mathbb{R}) \cdot e' \). Then the map assigning \( \mathcal{O}' \) to \( \mathcal{O} \) is well defined and establishes a bijection, called the **Kostant–Sekiguchi bijection**, between the set of nonzero \( K \)-orbits in \( \mathcal{N}(\mathfrak{p}) \) and the set of nonzero \( \text{Ad}(\mathfrak{g}_\mathbb{R}) \)-orbits in \( \mathcal{N}(\mathfrak{g}_\mathbb{R}) \). We say that the orbits \( \mathcal{O} \) and \( \mathcal{O}' \) **correspond** one another via the Kostant–Sekiguchi bijection. One can show, [KR], cf. [CM], that
\[
G \cdot e' = G \cdot e,
\]
\[
dim_{\mathbb{R}} \text{Ad}(\mathfrak{g}_\mathbb{R}) \cdot e' = \dim_{\mathbb{C}} G \cdot e'.
\]

**Theorem 5.** Let \( \mathcal{O} \) be a nonzero \( K \)-orbit in \( \mathcal{N}(\mathfrak{p}) \) and let \( x \) be an element of the \( \text{Ad}(\mathfrak{g}_\mathbb{R}) \)-orbit in \( \mathcal{N}(\mathfrak{g}_\mathbb{R}) \) corresponding to \( \mathcal{O} \) via the Kostant–Sekiguchi bijection. The following properties are equivalent:

(i) \( \mathcal{O} \) is \((-1)\)-distinguished.

(ii) \( x \) is compact.

Next result, Theorem 6, reduces studying projective self-dual varieties associated with symmetric spaces to the case of simple Lie algebras \( \mathfrak{g} \). Namely, since the Lie algebra \( \mathfrak{g} \) is semisimple, it is the direct sum of all its simple ideals. As this set of ideals is \( \theta \)-stable, \( \mathfrak{g} \) is the direct sum of its \( \theta \)-stable ideals, \( \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_d \), where each \( \mathfrak{g}_i \) is either

(a) simple or

(b) direct sum of two isomorphic simple ideals permuted by \( \theta \).

Let \( G_\mathfrak{t}, \mathfrak{t}_1, \mathfrak{p}_1, K_1 \) and \( \mathcal{N}(\mathfrak{p}_1) \) have the same meaning for \( \mathfrak{g}_1 \) with respect to \( \theta_1 := \theta|_{\mathfrak{g}_1} \) as resp., \( G, \mathfrak{t}, \mathfrak{p}, K \) and \( \mathcal{N}(\mathfrak{p}) \) have for \( \mathfrak{g} \) with respect to \( \theta \). Then \( G = G_1 \times \ldots \times G_d \), \( \mathfrak{t} = \mathfrak{t}_1 \oplus \ldots \oplus \mathfrak{t}_d \), \( \mathfrak{p} = \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_d \), \( K = K_1 \times \ldots \times K_d \) and \( \mathcal{N}(\mathfrak{p}) = \mathcal{N}(\mathfrak{p}_1) \times \ldots \times \mathcal{N}(\mathfrak{p}_d) \).

In Theorem 6, we use the following notation. Let \( X_1, \ldots, X_s \) be the closed subvarieties of a projective space \( \mathbb{P} \), and let \( s \geq 2 \). Consider the variety
\[
\{(x_1, \ldots, x_s, y) \in X_1 \times \ldots \times X_s \times \mathbb{P} \mid \dim(x_1, \ldots, x_s) = s - 1, \ y \in \langle x_1, \ldots, x_s \rangle \},
\]
where bar denotes Zariski closure in \( X_1 \times \ldots \times X_s \times \mathbb{P} \) and \( \langle S \rangle \) denotes the linear span of \( S \) in \( \mathbb{P} \). Consider the projection of variety (9) to \( \mathbb{P} \). Its image is denoted by
\[
\text{Join}(X_1, \ldots, X_s)
\]
and called the join of $X_1, \ldots, X_s$. If $s = 2$, then (10) is the usual join of $X_1$ and $X_2$, cf. [Ha]. If $s > 2$, then (10) is the usual join of $\text{Join}(X_1, \ldots, X_{s-1})$ and $X_s$.

**Theorem 6.** Let $x = x_1 + \ldots + x_d$ where $x_l \in \mathcal{N}(p_l)$, $l = 1, \ldots, d$. Consider in $\mathbf{P}(g)$ the projective subvarieties $X := \mathbf{P}(K \cdot x)$ and $X_l := \mathbf{P}(K_l \cdot x_l)$. Then

$$X = \text{Join}(X_1, \ldots, X_d)$$

and the following properties are equivalent:

(i) $X = \tilde{X}$.

(ii) $X_l = \tilde{X}_l$ for all $l$.

**Remark.** This is a specific geometric property of the considered varieties. In general setting, projective self-duality of $\text{Join}(Z_1, \ldots, Z_m)$ is not equivalent to projective self-duality of all $Z_1, \ldots, Z_m$.

If $g_l$ is of type (b), viz., $g_l$ is isomorphic to the direct sum of algebras $s \oplus s$, where $s$ is a simple algebra, and $\theta_l$ acts by $\theta_l((x, y)) = (y, x)$, then, as explained above (see (3)), classifying $(-1)$-distinguished orbits in $\mathcal{N}(p_l)$ amounts to classifying distinguished orbits in $\mathcal{N}(s)$. Since the last classification is known, [BaCa], [Ca], this and Theorem 6 reduce classifying $(-1)$-distinguished orbits in $g$ to the case where $g$ is a simple Lie algebra.

In this case, for explicit classifying $(-1)$-distinguished orbits, one can apply Theorem 4 since there are algorithms, [Ka], [Vi], yielding, in principle, a classification of $K$-orbits in $\mathcal{N}(p)$ and a description of the reductive Levi factors of $\mathfrak{z}_g(x)$ and $\mathfrak{z}_l(x)$ for $x \in \mathcal{N}(p)$. Moreover, there is a computer program, [L], implementing these algorithms and yielding, for concrete pairs $(g, \theta)$, the explicit representatives $x$ of $K$-orbits in $\mathcal{N}(p)$ and the reductive Levi factors of $\mathfrak{z}_g(x)$ and $\mathfrak{z}_l(x)$. There is also another way for finding representatives and dimensions of $K$-orbits in $\mathcal{N}(p)$: they may be obtained using the general algorithm of finding Hesselink strata for linear reductive group actions given in [P3], since in our case these strata coincide with $K$-orbits, see [P3, Proposition 4].

Fortunately, for exceptional simple $g$, explicit finding of the aforementioned classification and description already has been performed by D. Ž. Đoković in [D3], [D4], cf. [CM]. In these papers, the answers are given in terms of the so called characteristics in the sense of Dynkin (in Section 5, we recall this classical approach, [D], [K1], [KR], cf. [CM] and [Vi]). Combining either of Theorems 4 and 5 with these results, we obtain, for all exceptional simple $g$ and all $\theta$, the explicit classification of $(-1)$-distinguished $K$-orbits in $\mathcal{N}(p)$ in terms of their characteristics (see Theorem 14).

For classical $g$, we use another approach and obtain the classification of $(-1)$-distinguished $K$-orbits in $\mathcal{N}(p)$ by means of the Kostant–Sekiguchi bijection, Theorem 5 and elementary representation theory of $\mathfrak{sl}_2$ (see Theorems 8–13).

Theorems 4, 5, 6 are proved in Section 3. For simple Lie algebras $g$, the classifications of $(-1)$-distinguished $K$-orbits in $\mathcal{N}(p)$ are obtained in the next two sections: in Section 4, we consider the case of classical $g$, and in Section 5, that of exceptional simple $g$. In Section 6, we briefly discuss some geometric properties of the constructed self-dual projective algebraic varieties.
Proof of Theorem 4. Theorem 4 immediately follows from Theorem 7 proved below.

Let $k$ be a field of characteristic 0. If $c$ is a finite dimensional algebraic Lie algebra over $k$, we denote by $\text{rad } c$ (resp., $\text{rad}_u c$) the radical (resp., the unipotent radical, i.e., the maximal ideal whose elements are nilpotent) of $c$. We call $c/\text{rad } c$ (resp., $c/\text{rad}_u c$) the Levi factor (resp., the reductive Levi factor) of $c$. According to [Ch], there is a semisimple (resp., reductive) subalgebra $l$ in $c$ such that $c$ is the semidirect sum of $l$ and $\text{rad } c$ (resp., $\text{rad}_u c$). Every such $l$ is called a Levi subalgebra (resp., a reductive Levi subalgebra) of $c$.

Theorem 7. Assume that $k$ is algebraically closed and let $a$ be a finite dimensional algebraic Lie algebra over $k$. Let $\theta \in \text{Aut } a$ be an element of order 2. For any $\theta$-stable linear subspace $l$ of $a$, set

$$ l^\pm = \{ x \in l \mid \theta(x) = \pm x \}. $$

The following properties are equivalent:

(i) $a^-$ contains no nonzero semisimple elements.

(ii) $a^- = (\text{rad}_u a)^-$.

(iii) $\dim a^- = \dim (\text{rad}_u a)^-$.

(iv) $a^- \subseteq \text{rad}_u a$.

(v) The reductive Levi factors of $a^+$ and $a$ have the same dimension.

(vi) The reductive Levi factors of $a^+$ and $a$ are isomorphic.

(vii) The set of all reductive Levi subalgebras of $a^+$ coincides with the set of all $\theta$-stable reductive Levi subalgebras of $a$.

Proof. It is known that $a$ contains a $\theta$-stable Levi subalgebra $s$ (e.g., see [KN, Appendix 9]), and $\text{rad } a$ contains a $\theta$-stable maximal torus $t$, see [St]. Thus we have the following $\theta$-stable direct sum decompositions of vector spaces:

$$ a = s \oplus \text{rad } a, \quad \text{rad } a = t \oplus \text{rad}_u a, $$

$$ s = s^+ \oplus s^-, \quad t = t^+ \oplus t^-, \quad \text{rad}_u a = (\text{rad}_u a)^+ \oplus (\text{rad}_u a)^-, $$

$$ a^\pm = s^\pm \oplus t^\pm \oplus (\text{rad}_u a)^\pm. $$

(i) $\Rightarrow$ (ii): Assume that (i) holds. Then, by (11), we have $t^- = 0$. If $s^- \neq 0$, then $\theta|_s \in \text{Aut } a$ is an element of order 2. By [Vu], this implies that $s^-$ contains a nonzero $\theta$-stable algebraic torus. By (11), this contradicts (i). Thus $s^- = 0$. Whence (ii) by (11).

(ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i): This is because all elements of $\text{rad}_u a$ are nilpotent.

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv): This is evident.

(iii) $\Rightarrow$ (ii): This follows from (11).

(ii) $\Rightarrow$ (v): It follows from (11) that (ii) is equivalent to

$$ s \oplus t \subseteq a^+. $$

Assume that (12) holds. Let $\pi : a \rightarrow a/\text{rad}_u a$ be the natural homomorphism. By (11), (12), we have $\pi(a^+) = a/\text{rad}_u a$. Since the algebra $a/\text{rad}_u a$ is reductive, this implies

$$ \dim a/\text{rad}_u a \leq \text{dimension of reductive Levi subalgebras of } a^+ $$

$$ = \dim a^+ / \text{rad}_u (a^+). $$
On the other hand, since reductive Levi subalgebras of \( \mathfrak{a} \) and \( \mathfrak{a}^+ \) are precisely their maximal reductive subalgebras, cf., e.g., \([OV]\), the inclusion \( \mathfrak{a}^+ \subseteq \mathfrak{a} \) implies

\[
\dim \mathfrak{a} / \text{rad}\mathfrak{a} = \dim \text{dimension of reductive Levi subalgebras of } \mathfrak{a} \geq \dim \text{dimension of reductive Levi subalgebras of } \mathfrak{a}^+.
\]

Now (v) follows from (13), (14).

(v) \( \Rightarrow \) (ii): Assume that (v) holds. Let \( \mathfrak{l} \) be a reductive Levi subalgebra of \( \mathfrak{a}^+ \). Since reductive Levi subalgebras of \( \mathfrak{a} \) are precisely its maximal reductive subalgebras, it follows from (v) that \( \mathfrak{l} \) is a reductive Levi subalgebra of \( \mathfrak{a} \) as well. Let \( \mathfrak{s} \) and \( \mathfrak{t} \) be resp. the derived subalgebra and the center of \( \mathfrak{l} \); so we have \( \mathfrak{l} = \mathfrak{s} \oplus \mathfrak{t} \). Then \( \mathfrak{s} \) is a \( \theta \)-stable Levi subalgebra of \( \mathfrak{a} \), and \( \mathfrak{t} \) is a \( \theta \)-stable maximal torus of \( \text{rad} \mathfrak{a} \). Hence, as above, (ii) is equivalent to the inclusion (12). As the latter holds by the very construction of \( \mathfrak{l} \), we conclude that (ii) holds as well.

(vii) \( \Rightarrow \) (v): This is clear.

(i) \( \Rightarrow \) (vii): Let \( \mathfrak{l} \) be a \( \theta \)-stable reductive Levi subalgebra of \( \mathfrak{a} \). If \( \mathfrak{I}^- \neq 0 \), then \( \theta |_{\mathfrak{l}} \in \text{Aut} \mathfrak{l} \) is an element of order 2. Hence, by \([Vu]\), there is a nonzero \( \theta \)-stable algebraic torus in \( \mathfrak{I}^- \). This contradicts (i). Whence \( \mathfrak{I}^- = 0 \), i.e., \( \mathfrak{l} \) lies in \( \mathfrak{a}^+ \).

(vi) \( \Rightarrow \) (v) and (vii) \( \Rightarrow \) (vi): This is clear. \( \square \)

**Proof of Theorem 5.** According to \([KR]\), there is a complex Cayley triple \( \{e, h, f\} \) in \( \mathfrak{g} \) such that \( \mathcal{O} = K \cdot e \). Let \( \{e', h', f'\} \) be the real Cayley triple in \( \mathfrak{g}_{\mathbb{R}} \) whose Cayley transform is \( \{e, h, f\} \). Since \( \mathcal{O}' = \text{Ad}(\mathfrak{g}_{\mathbb{R}}) \cdot e' \), we may, and will, assume that \( x = e' \).

Let \( \mathfrak{s} \) (resp., \( \mathfrak{s}_{\mathbb{R}} \)) be the simple three-dimensional subalgebra of \( \mathfrak{g} \) (resp., \( \mathfrak{g}_{\mathbb{R}} \)) spanned by \( \{e, h, f\} \) (resp., \( \{e', h', f'\} \)),

\[
\mathfrak{s} = \mathbb{C}e + \mathbb{C}h + \mathbb{C}f, \quad \mathfrak{s}_{\mathbb{R}} = \mathbb{R}e' + \mathbb{R}h' + \mathbb{R}f'.
\]

Denote by \( \mathfrak{z}_\mathfrak{g}(\mathfrak{s}) \) (resp., \( \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \)) the centralizer of \( \mathfrak{s} \) (resp., \( \mathfrak{s}_{\mathbb{R}} \)) in \( \mathfrak{g} \) (resp., \( \mathfrak{g}_{\mathbb{R}} \)).

It follows from (6), (7), (15) that \( \mathfrak{s}_{\mathbb{R}} \) is a real form of \( \mathfrak{s} \). Since \( \mathfrak{g}_{\mathbb{R}} \) is a real form of \( \mathfrak{g} \), this yields that \( \mathfrak{z}_\mathfrak{g}(\mathfrak{s}) \) is a real form of \( \mathfrak{z}_\mathfrak{g}(\mathfrak{g}) \). The definitions of complex and real Cayley triples imply that \( \mathfrak{s} \) and \( \mathfrak{s}_{\mathbb{R}} \) are \( \theta \)-stable subalgebras. Therefore \( \mathfrak{z}_\mathfrak{g}(\mathfrak{s}) \) and \( \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \) are \( \theta \)-stable as well.

Whence

\[
\begin{align*}
\mathfrak{z}_\mathfrak{g}(\mathfrak{s}) &= (\mathfrak{z}_\mathfrak{g}(\mathfrak{s}) \cap \mathfrak{t}) \oplus (\mathfrak{z}_\mathfrak{g}(\mathfrak{s}) \cap \mathfrak{p}), & \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) &= (\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{t}_{\mathbb{R}}) \oplus (\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{p}_{\mathbb{R}}), \\
\mathfrak{z}_\mathfrak{g}(\mathfrak{s}) \cap \mathfrak{t} &= \mathbb{C}(\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{t}_{\mathbb{R}}), & \mathfrak{z}_\mathfrak{g}(\mathfrak{s}) \cap \mathfrak{p} &= \mathbb{C}(\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{p}_{\mathbb{R}}).
\end{align*}
\]

Now take into account that \( \mathfrak{z}_\mathfrak{g}(\mathfrak{s}) \) (resp., \( \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \)) is the reductive Levi subalgebra of \( \mathfrak{z}_\mathfrak{g}(\mathfrak{e}) \) (resp., \( \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{e}') \)): regarding \( \mathfrak{z}_\mathfrak{g}(\mathfrak{s}) \), see, e.g., \([\text{SpSt}]\); for \( \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \), the arguments are the same. Hence, by Theorem 7, the orbit \( \mathcal{O} \) is \((-1)\)-distinguished if and only if \( \mathfrak{z}_\mathfrak{g}(\mathfrak{s}) \cap \mathfrak{p} = 0 \). By (17), the latter condition is equivalent to \( \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{p}_{\mathbb{R}} = 0 \). On the other hand, since (4) is the Cartan decomposition of \( \mathfrak{g}_{\mathbb{R}} \), it follows from (16) that \( \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{p}_{\mathbb{R}} = 0 \) if and only if the Lie algebra \( \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \) is compact. This completes the proof of Theorem 5. \( \square \)

**Proof of Theorem 6.** Clearly the following proposition implies the first statement.
Proposition 1. ([P2, Proposition 2]) Let $H_t$ be an algebraic group and let $L_t$ be a finite dimensional algebraic $H_t$-module, $t = 1, 2$. Let $v_t \in L_t$ be a nonzero vector such that the orbit $H_t \cdot v_t$ is stable with respect to scalar multiplications. Put $H := H_1 \times H_2$, $L := L_1 \oplus L_2$. Identify $L_t$ with the linear subspace of $L$ and set $v := v_1 + v_2$. Then

$$
P(\overline{H} \cdot v) = \text{Join}(P(\overline{H_1} \cdot v_1), P(\overline{H_2} \cdot v_2)).$$

Regarding the second statement, notice that the centralizers $\mathfrak{z}_\theta(x_t)$ and $\mathfrak{z}_\theta(x)$ are resp. $\theta_t$- and $\theta$-stable and $\mathfrak{z}_\theta(x) = \mathfrak{z}_{\Phi_1}(x_1) \oplus \ldots \oplus \mathfrak{z}_{\Phi_d}(x_d)$. Now the claim follows from this decomposition, Definition 1 and Theorem 3. □

4. Classification for classical $\mathfrak{g}$

If $\mathfrak{g}$ is classical, our approach to classifying $(-1)$-distinguished nilpotent $K$-orbits in $\mathfrak{p}$ is based on the Kostant–Sekiguchi bijection and Theorem 5. Namely, for every noncompact real form $\mathfrak{g}_R$ of $\mathfrak{g}$, we will find in $\mathcal{N}(\mathfrak{g}_R)$ all $\text{Ad}(\mathfrak{g}_R)$-orbits $O$ whose elements are compact. If this is done, then for the complexification of a Cartan decomposition of $\mathfrak{g}_R$, the $K$-orbits in $\mathcal{N}(\mathfrak{p})$ corresponding to all such $O$ via the Kostant–Sekiguchi bijection are precisely all $(-1)$-distinquished orbits.

First we recall the classification of real forms $\mathfrak{g}_R$ of classical complex Lie algebras $\mathfrak{g}$. Let $D$ be either real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or quaternions $\mathbb{H}$. If $a \in D$, we denote by $\overline{a}$ the conjugate of $a$. Let $V$ be a finite dimensional vector space over $D$ (left $D$-module). Denote by $\text{GL}_D(V)$ the Lie group of all $D$-linear automorphisms of $V$. Its Lie algebra $\mathfrak{gl}_D(V)$ is identified with the real Lie algebra of all $D$-linear endomorphisms of $V$. The derived algebra of $\mathfrak{gl}_D(V)$ is denoted by $\mathfrak{sl}_D(V)$.

Let a map $\Phi : V \times V \to D$ be $D$-linear with respect to the first argument. It is called

- symmetric bilinear form on $V$ if $\Phi(x, y) = \Phi(y, x)$ for all $x, y \in V$,
- skew-symmetric bilinear form on $V$ if $\Phi(x, y) = -\Phi(y, x)$ for all $x, y \in V$,
- Hermitian form on $V$ if $\Phi(x, y) = \overline{\Phi(y, x)}$ for all $x, y \in V$,
- skew-Hermitian form on $V$ if $\Phi(x, y) = -\overline{\Phi(y, x)}$ for all $x, y \in V$.

By “form on $V$” we always mean form of one of these four types. A form $\Phi$ on $V$ is called nondegenerate if, for any $x \in V$, there is $y \in V$ such that $\Phi(x, y) \neq 0$. If $\Phi_s$, $s = 1, 2$, is a form on a finite dimensional vector space $V_s$ over $D$, then $\Phi_1$ and $\Phi_2$ are called equivalent forms if there exists an isometry $\psi : V_1 \to V_2$ with respect to $\Phi_1$ and $\Phi_2$, i.e., an isomorphism of vector spaces over $D$ such that $\Phi_2(\psi(x), \psi(y)) = \Phi_1(x, y)$ for all $x, y \in V_1$.

The automorphism group of a form $\Phi$ on $V$,

$$\text{GL}_D^\Phi(V) := \{g \in \text{GL}_D(V) \mid \Phi(g \cdot x, g \cdot y) = \Phi(x, y) \text{ for all } x, y \in V\},$$

is a Lie subgroup of $\text{GL}_D(V)$ whose Lie algebra is

$$\mathfrak{gl}_D^\Phi(V) := \{A \in \mathfrak{gl}_D(V) \mid \Phi(A \cdot x, y) + \Phi(x, A \cdot y) = 0 \text{ for all } x, y \in V\}.$$  

For $\Phi = 0$ (zero form), we have $\text{GL}_D^0(V) = \text{GL}_D(V)$ and $\mathfrak{gl}_D^0(V) = \mathfrak{gl}_D(V)$.

Let $n := \dim_D V$ and let $(e) := \{e_1, \ldots, e_n\}$ be a basis of $V$ over $D$. Identifying $D$-linear endomorphisms of $V$ with their matrices with respect to $(e)$, we identify $\mathfrak{gl}_D(V), \mathfrak{sl}_D(V)$ and
that are denoted respectively by \( \mathfrak{gl}_n \) and \( \mathfrak{sl}_n \), and for \( D = \mathbb{C} \), by \( \mathfrak{gl}_n(\mathbb{C}) \) and \( \mathfrak{sl}_n(\mathbb{C}) \). The last algebras are endowed with the natural structures of complex Lie algebras that are denoted respectively by \( \mathfrak{gl}_n(\mathbb{C}) \) and \( \mathfrak{sl}_n(\mathbb{C}) \). If \( D = \mathbb{H} \), then \( V \) is a 2\( n \)-dimensional vector space over the subfield \( \mathbb{C} \) of \( \mathbb{H} \), and elements of \( \mathfrak{gl}_n(\mathbb{H}) \) are its \( \mathbb{C} \)-linear endomorphisms. Identifying them with their matrices with respect to the basis \( e_1, \ldots, e_n, je_1, \ldots, je_n \), we identify \( \mathfrak{gl}_n(\mathbb{H}) \) (resp., \( \mathfrak{sl}_n(\mathbb{H}) \)) with the corresponding Lie subalgebra of \( \mathfrak{gl}_{2n}(\mathbb{C}) \) (resp., \( \mathfrak{sl}_{2n}(\mathbb{C}) \)).

If \( \Phi \) is a form on \( V \), then the matrix \( (\Phi_{st}) \) of \( \Phi \) with respect to \((e)\) defines \( \Phi \) by

\[
\Phi \left( \sum_{s=1}^{n} x_s e_s, \sum_{t=1}^{n} y_t e_t \right) = \left\{ \begin{array}{ll}
\sum_{s,t=1}^{n} x_s y_t & \text{if } \Phi \text{ is symmetric or skew-symmetric,} \\
\sum_{s,t=1}^{n} x_s y_t \Phi_{st} & \text{if } \Phi \text{ is Hermitian or skew-Hermitian.}
\end{array} \right.
\]

The map \( \Phi \mapsto (\Phi_{st}) \) is a bijection between all symmetric (resp., skew-symmetric, Hermitian, skew-Hermitian) forms on \( V \) and all symmetric (resp., skew-symmetric, Hermitian, skew-Hermitian) \( n \times n \)-matrices over \( D \). \( \Phi \) is nondegenerate if and only if \( (\Phi_{st}) \) is nonsingular.

We denote by \( I_d \) the unit matrix of size \( d \) and put \( I_{p,q} := \text{diag}(1, \ldots, 1, -1, \ldots, -1) \) where \( p \) (resp., \( q \)) is the number of 1’s (resp., -1’s).

Now we summarize the results from linear algebra about classification of forms and fix some notation and terminology. In the sequel, \( \mathbb{N} \) denotes the set of all nonnegative integers.

**Symmetric bilinear forms**

\( D = \mathbb{R} \). There are exactly \( n + 1 \) equivalence classes of nondegenerate symmetric bilinear forms on \( V \). They are represented by the forms \( \Phi \) with \( (\Phi_{st}) = I_{p,q}, p + q = n, p = 0, \ldots, n \). If \( (\Phi_{st}) = I_{p,q} \), then \( \text{sgn } \Phi := p - q \) is called the signature of \( \Phi \), and the real Lie algebra \( \mathfrak{gl}_n^D(V) \) is denoted by \( \mathfrak{so}_{p,q} \).

\( D = \mathbb{C} \). There is exactly one equivalence class of nondegenerate symmetric bilinear forms on \( V \). It is represented by the form \( \Phi \) with \( (\Phi_{st}) = I_n \). The corresponding real Lie algebra \( \mathfrak{gl}_n^D(V) \) is denoted by \( \mathfrak{so}_n(\mathbb{C}) \). It has a natural structure of complex Lie algebra that is denoted by \( \mathfrak{so}_n(\mathbb{C}) \).

\( D = \mathbb{H} \). There are no nonzero symmetric bilinear forms on \( V \).

**Skew-symmetric bilinear forms**

\( D = \mathbb{R} \) and \( \mathbb{C} \). Nondegenerate skew-symmetric bilinear forms on \( V \) exist if and only if \( n \) is even. In this case, there is exactly one equivalence class of such forms. It is represented by the form \( \Phi \) with \( (\Phi_{st}) = \begin{bmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{bmatrix} \). Corresponding real Lie algebra \( \mathfrak{gl}_n^D(V) \) is denoted by \( \mathfrak{sp}_{n}(\mathbb{R}) \) for \( D = \mathbb{R} \), and by \( \mathfrak{sp}_n(\mathbb{C}) \) for \( D = \mathbb{C} \). The algebra \( \mathfrak{sp}_n(\mathbb{C}) \) has a natural structure of complex Lie algebra that is denoted by \( \mathfrak{sp}_n(\mathbb{C}) \).

\( D = \mathbb{H} \). There are no nonzero skew-symmetric bilinear forms on \( V \).

**Hermitian forms**

\( D = \mathbb{R} \). Hermitian forms on \( V \) coincide with symmetric bilinear forms.

\( D = \mathbb{C} \) and \( \mathbb{H} \). There are exactly \( n + 1 \) equivalence classes of nondegenerate Hermitian forms on \( V \). They are represented by the forms \( \Phi \) with \( (\Phi_{st}) = I_{p,q}, p + q = n, p = 0, \ldots, n \).
If \((\Phi_{st}) = I_{p,q}\), then \(\text{sgn} \Phi := p-q\) is called the \textit{signature} of \(\Phi\), and the real Lie algebra \(\mathfrak{gl}^\Phi_D(V)\) is denoted by \(\mathfrak{su}_{p,q}\) for \(D = \mathbb{C}\), and by \(\mathfrak{sp}_{p,q}\) for \(D = \mathbb{H}\).

\textit{Skew-Hermitean forms}

\(D = \mathbb{R}\). Skew-Hermitean forms on \(V\) coincide with skew-symmetric forms.

\(D = \mathbb{C}\). The map \(\Phi \mapsto i\Phi\) is a bijection between all nondegenerate Hermitean forms on \(V\) and all nondegenerate skew-Hermitean forms on \(V\).

\(D = \mathbb{H}\). There is exactly one equivalence class of nondegenerate skew-Hermitean forms on \(V\). It is represented by the form \(\Phi\) with \((\Phi_{st}) = jI_n\). The corresponding real Lie algebra \(\mathfrak{gl}^\Phi_D(V)\) is denoted by \(u_n^*(\mathbb{H})\).

Some of the above-defined Lie algebras are isomorphic to each other. Obviously \(\mathfrak{su}_{p,q} = \mathfrak{su}_{q,p}\), \(\mathfrak{so}_{p,q} = \mathfrak{so}_{q,p}\), \(\mathfrak{sp}_{p,q} = \mathfrak{sp}_{q,p}\), and we also have

\[
\begin{align*}
\mathfrak{so}_2 &= \mathfrak{u}_1^*(\mathbb{H}), \quad \mathfrak{so}_2(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sp}_2(\mathbb{C}), \quad \mathfrak{so}_3 \cong \mathfrak{su}_2 = \mathfrak{sp}_{1,0} = \mathfrak{sl}_1(\mathbb{H}), \\
\mathfrak{so}_{1,2} &\cong \mathfrak{su}_{1,1} \cong \mathfrak{sl}_2(\mathbb{R}) = \mathfrak{sp}_2(\mathbb{R}), \quad \mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}), \quad \mathfrak{so}_4 \cong \mathfrak{su}_2 \oplus \mathfrak{su}_2, \\
\mathfrak{so}_{1,3} &\cong \mathfrak{sl}_2(\mathbb{C})^\mathbb{R}, \mathfrak{so}_{2,2} \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R}), \mathfrak{u}_2^*(\mathbb{H}) \cong \mathfrak{su}_2 \oplus \mathfrak{sl}_2(\mathbb{R}), \mathfrak{so}_5(\mathbb{C}) \cong \mathfrak{sp}_4(\mathbb{C}), \\
\mathfrak{so}_5 &\cong \mathfrak{sp}_2, \mathfrak{so}_{1,4} \cong \mathfrak{sp}_{1,1}, \quad \mathfrak{so}_{2,3} \cong \mathfrak{sp}_4(\mathbb{R}), \quad \mathfrak{so}_6(\mathbb{C}) \cong \mathfrak{sl}_4(\mathbb{C}), \quad \mathfrak{so}_6 \cong \mathfrak{su}_4, \\
\mathfrak{so}_{1,5} &\cong \mathfrak{sl}_2(\mathbb{H}), \mathfrak{so}_{2,4} \cong \mathfrak{su}_{2,2}, \mathfrak{so}_{3,3} \cong \mathfrak{sl}_4(\mathbb{R}), \mathfrak{u}_3(\mathbb{H}) \cong \mathfrak{su}_{1,3}, \quad \mathfrak{u}_4(\mathbb{H}) \cong \mathfrak{so}_{2,6}.
\end{align*}
\]

By definition, \textit{classical complex Lie algebras} \(\mathfrak{g}\) are the algebras \(\mathfrak{sl}_n(\mathbb{C})\), \(\mathfrak{so}_n(\mathbb{C})\) and \(\mathfrak{sp}_n(\mathbb{C})\). According to E. Cartan’s classification (cf., e.g., [OV]), up to isomorphism, all their real forms \(\mathfrak{g}_\mathbb{R}\) are listed in Table 1.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\(\mathfrak{g}\) & \(\mathfrak{g}_\mathbb{R}\) & compact \(\mathfrak{g}_\mathbb{R}\) \\
\hline
\(\mathfrak{sl}_n(\mathbb{C})\), \(n \geq 2\) & \(\mathfrak{sl}_n(\mathbb{R})\), & \(\mathfrak{su}_n := \mathfrak{su}_{n,0}\) \\
 & \(\mathfrak{sl}_n(\mathbb{H})\), \(n = 2l\), & \\
 & \(\mathfrak{su}_{n,q}, q = 0, \ldots, [n/2]\) & \\
\hline
\(\mathfrak{so}_n(\mathbb{C})\), \(n = 3\) or \(n \geq 5\) & \(\mathfrak{so}_{n,q}, q = 0, \ldots, [n/2]\), & \(\mathfrak{so}_n := \mathfrak{so}_{n,0}\) \\
 & \(\mathfrak{u}_l^*(\mathbb{H})\), \(n = 2l\) & \\
\hline
\(\mathfrak{sp}_n(\mathbb{C})\), \(n = 2l \geq 2\) & \(\mathfrak{sp}_n(\mathbb{R})\), & \(\mathfrak{sp}_{l,0}\) \\
 & \(\mathfrak{sp}_{l,q}, q = 0, \ldots, [l/2]\) & \\
\hline
\end{tabular}
\caption{Table 1}
\end{table}

There are the isomorphisms between some of the \(\mathfrak{g}_\mathbb{R}\)'s in Table 1 given by (18).

Now we restate the classification of nilpotent \(\text{Ad}(\mathfrak{g}_\mathbb{R})\)-orbits in all real forms \(\mathfrak{g}_\mathbb{R}\) of classical complex Lie algebras \(\mathfrak{g}\) (cf. [BoCu], [CM], [SpSt], [M], [W]) in the form adapted to our goal of classifying compact nilpotent elements in \(\mathfrak{g}_\mathbb{R}\).

If a subalgebra of \(\mathfrak{g}_\mathbb{R}\) is isomorphic to \(\mathfrak{sl}_2(\mathbb{R})\), we call it an \(\mathfrak{sl}_2(\mathbb{R})\)-subalgebra. We use the following basic facts, cf. [CM], [KR], [M], [SpSt]:

(F1) For any nonzero nilpotent element \(x \in \mathfrak{g}_\mathbb{R}\), there is an \(\mathfrak{sl}_2(\mathbb{R})\)-subalgebra of \(\mathfrak{g}_\mathbb{R}\) containing \(x\).

(F2) If \(a_1\) and \(a_2\) are the \(\mathfrak{sl}_2(\mathbb{R})\)-subalgebras of \(\mathfrak{g}_\mathbb{R}\), and a nonzero nilpotent \(\text{Ad}(\mathfrak{g}_\mathbb{R})\)-orbit intersects both \(a_1\) and \(a_2\), then \(a_2 = g \cdot a_1\) for some \(g \in \text{Ad}(\mathfrak{g}_\mathbb{R})\).
(F3) There are exactly two nonzero nilpotent $\text{Ad}(\mathfrak{sl}_2(\mathbb{R}))$-orbits in $\mathfrak{sl}_2(\mathbb{R})$. Scalar multiplication by $-1$ maps one of them to the other.

(F4) Any finite dimensional $\mathfrak{sl}_2(\mathbb{R})$-$D$-module is completely reducible. For any integer $d \in \mathbb{N}$, there is a unique, up to isomorphism, $d$-dimensional simple $\mathfrak{sl}_2(\mathbb{R})$-$D$-module $S_d$. The module $S_1$ is trivial. Let $T_m$ be the direct sum of $m$ copies of $S_1$, and $T_0 := 0$.

(F5) If $a$ is an $\mathfrak{sl}_2(\mathbb{R})$-subalgebra of $\mathfrak{g}_\mathbb{R}$ and $x \in a$ is a nonzero nilpotent element, then $z_{\mathfrak{g}_\mathbb{R}}(a)$ is the reductive Levi subalgebra of $z_{\mathfrak{g}_\mathbb{R}}(x)$.

(F6) If there is a nonzero $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $S_d$ of a given type (symmetric, skew-symmetric, Hermitian or skew-Hermitian), it is nondegenerate and unique up to proportionality. We fix such a form. Table 2 contains information about its existence and the notation for the fixed forms.

| $D$ | $d$ | symmetric | sgn | skew-symmetric | Hermitian | sgn | skew-Hermitian |
|-----|-----|-----------|-----|---------------|-----------|-----|----------------|
| $\mathbb{R}$ | even | $\Delta_d^s$ | 1 | $\Delta_d^{ss}$ |           |     |                |
|       | odd  |           |    |               |           |     |                |
| $\mathbb{C}$ | even | $\Delta_d^s$ |   | $\Delta_d^{ss}$ | $\Delta_d^H$ | 0   | $i\Delta_d^H$  |
|       | odd  |           |    |               |           |     |                |
| $\mathbb{H}$ | even |           |    | $\Delta_d^H$ |           |     | $\Delta_d^{HH}$|
|       | odd  |           |    |               |           |     |                |

**Partitions, $\mathfrak{sl}_2(\mathbb{R})$-$D$-modules, and $\mathfrak{sl}_2(\mathbb{R})$-invariant forms**

We call any vector

$$m = (m_1, \ldots, m_p) \in \mathbb{N}^p, \text{ where } m_p \neq 0,$$

a partition of the number

$$|m| := \sum_d dm_d$$

(this is nontraditional usage of the term “partition” but it is convenient for our purposes). If $p = 1$, then $m$ is called a trivial partition.

The Young diagram of $m$ is a left-justified array $Y(m)$ of empty boxes with $p$ boxes in each of the first $m_p$ rows, $p - 1$ boxes in each of the next $m_{p-1}$ rows, and so on. The partition

$$\tilde{m} = \{\tilde{m}_1, \ldots, \tilde{m}_q\}$$

is called the transpose partition to $m$ if $Y(\tilde{m})$ is the transpose of $Y(m)$, i.e., the rows of $Y(\tilde{m})$ are the columns of $Y(m)$ from left to right. We have $|\tilde{m}| = |m|$.

Let $m$ be a nontrivial partition. It is called a symmetric (resp., skew-symmetric) partition of $|m|$ if $m_d$ in (19) is even for every even (resp., odd) $d$.

Let $m$ be a sequence obtained from $m$ by replacing $m_d$ in (19) for every $d$ (resp., every odd $d$, every even $d$) with a pair $(p_d, q_d) \in \mathbb{N}^2$ such that $p_d + q_d = m_d$. Such $m$ is called a fine (resp., fine Hermitian, fine skew-Hermitian) partition of $|m|$ associated with $m$. If $m$ is fine or fine Hermitian,

$$\text{sgn } m := \sum_d \text{odd } (p_d - q_d)$$
is called the signature of $m$. If $m$ is fine Hermitian (resp., fine skew-Hermitian) and $m$ is symmetric (resp., skew-symmetric), then $m$ is called a fine symmetric (resp., fine skew-symmetric) partition of $|m|$.

Any partition (19) defines the $|m|$-dimensional $\mathfrak{sl}_2(\mathbb{R})$-$D$-module

$$V_m := \bigoplus_{d \geq 1} (T_{m_d} \otimes S_d)$$

(tensor product is taken over $D$, with respect to the canonical $D$-bimodule structure on $D$-vector spaces). By (F4), any nonzero finite dimensional $\mathfrak{sl}_2(\mathbb{R})$-$D$-module is isomorphic to $V_m$ for a unique $m$. It is trivial if and only if $m$ is trivial.

Identifying every $\varphi \in \mathfrak{gl}_D(T_{m_d})$ with the transformation of $V_m$ acting as $\varphi \otimes \text{id}$ on the summand $T_{m_d} \otimes S_d$ and as $0$ on the other summands in the right-hand side of (21), we identify $\bigoplus_{d \geq 1} \mathfrak{gl}_D(T_{m_d})$ with the subalgebra of $\mathfrak{gl}_D(V_m)$.

Two $\mathfrak{sl}_2(\mathbb{R})$-invariant forms on $V_m$ are called $\mathfrak{sl}_2(\mathbb{R})$-equivalent if there is an $\mathfrak{sl}_2(\mathbb{R})$-equivariant isometry $V_m \to V_m$ with respect to them. To describe the $\mathfrak{sl}_2(\mathbb{R})$-equivality classes of $\mathfrak{sl}_2(\mathbb{R})$-invariant forms on $V_m$, we fix, for every positive integer $r$ and pair $(p,q) \in \mathbb{N}^2$ with $p + q = r$, a nondegenerate form on $T_r$ whose notation, type and signature (if applicable) are specified in Table 3.

### Table 3

| $D$ | symmetric | sgn | skew-symmetric | Hermitian | sgn | skew-Hermitian |
|-----|-----------|-----|----------------|-----------|-----|----------------|
| $\mathbb{R}$ | $\Theta^s_{p,q}$ | $p-q$ | $\Theta^s_{r}$, $r$ even | $\Theta^s_{r}$ | $p-q$ | $\Theta^s_{r}$ |
| $\mathbb{C}$ | $\Theta^u_{r}$ | $\Theta^u_{r}$, $r$ even | $\Theta^u_{p,q}$ | $p-q$ | $\Theta^u_{p,q}$ |
| $\mathbb{H}$ | $\Theta^u_{p,q}$ | $p-q$ | $\Theta^u_{r}$ |

According to the above discussion, any nondegenerate form on $T_r$ is equivalent to a unique form from Table 3.

If $\Psi$ is a nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_m$, then $\Psi|_d$, its restriction to the summand $T_{m_d} \otimes S_d$ in (21), is a nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant form of the same type (i.e., symmetric, skew-symmetric, Hermitian or skew-Hermitian) as $\Psi$, and different summands in (21) are orthogonal with respect to $\Psi$. If $\Psi'$ is another nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_m$, then $\Psi$ and $\Psi'$ are $\mathfrak{sl}_2(\mathbb{R})$-equivalent if and only if $\Psi|_d$ and $\Psi'|_d$ are $\mathfrak{sl}_2(\mathbb{R})$-equivalent for every $d$. If, for every $d$, a nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant form $\Phi_d$ on $T_{m_d} \otimes S_d$ is fixed, and all $\Phi_d$’s are of the same type, then there is a nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant form $\Psi$ on $V_m$ such that $\Psi|_d = \Phi_d$ for all $d$. This reduces describing the $\mathfrak{sl}_2(\mathbb{R})$-equivalency classes of nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant forms on $V_m$ to describing that on $T_{m_d} \otimes S_d$ for all $d$. For any positive integers $r$ and $d$, Table 4 describes all, up to $\mathfrak{sl}_2(\mathbb{R})$-equivalency, nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant forms on $T_r \otimes S_d$ (in these tables, $p + q = r$).
Returning back to the \( n \)-dimensional vector space \( V \) over \( D \), fix a form \( \Phi \) (not necessarily nondegenerate) on \( V \). Let \( m \) be a nontrivial partition of \( n \) and let

\[
\alpha_m : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathfrak{gl}_D(V_m)
\]

be the injection defining the \( \mathfrak{sl}_2(\mathbb{R}) \)-module structure on \( V_m \). Let \( \Psi \) be an \( \mathfrak{sl}_2(\mathbb{R}) \)-invariant form on \( V_m \). If \( \Psi \) and \( \Phi \) are equivalent and \( \iota : V_m \rightarrow V \) is an isometry with respect to \( \Psi \) and \( \Phi \), then the image of the homomorphism

\[
\mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathfrak{gl}_D(V), \quad \varphi \mapsto \iota \circ \alpha_m(\varphi) \circ \iota^{-1},
\]

is an \( \mathfrak{sl}_2(\mathbb{R}) \)-subalgebra of \( \mathfrak{gl}_D^p(V) \). The above discussion shows that any \( \mathfrak{sl}_2(\mathbb{R}) \)-subalgebra of \( \mathfrak{gl}_D^p(V) \) is obtained in this way from some pair \( \{m, \Psi\} \). The \( \mathfrak{sl}_2(\mathbb{R}) \)-subalgebras of \( \mathfrak{gl}_D^p(V) \) obtained in this way from the pairs \( \{m, \Psi\}, \{m', \Psi'\} \) are \( \text{GL}_D^p(V) \)-conjugate if and only if \( m = m' \), and \( \Psi \) and \( \Psi' \) are \( \mathfrak{sl}_2(\mathbb{R}) \)-equivalent. This yields a bijection between the union of \( \mathfrak{sl}_2(\mathbb{R}) \)-equivalency classes of \( \mathfrak{sl}_2(\mathbb{R}) \)-invariant forms equivalent to \( \Phi \) on \( V_m \), where \( m \) ranges over all nontrivial partitions of \( n \), and the set of all \( \text{GL}_D^p(V) \)-conjugacy classes of \( \mathfrak{sl}_2(\mathbb{R}) \)-subalgebras in \( \mathfrak{gl}_D^p(V) \). Denote this bijection by \( * \). If \( \Psi \) is either zero or nondegenerate, considering \( * \) leads to the classification of nilpotent \( \text{Ad}(_D^p(V)) \)-orbits in \( \mathfrak{gl}_D^p(V) \) described below. In each case, we give the formulas for the orbit dimensions (one obtains them using (8) and [CM], [M]); then (2), (8) yield the dimensions of the corresponding \( K \)-orbits in \( N(p) \).

- **Nilpotent orbits in \( \mathfrak{sl}_D(V) \)**

  Take \( \Phi = 0 \). Then \( \text{GL}_D^p(V) = \text{GL}_D(V) \) and \( \mathfrak{gl}_D^p(V) = \mathfrak{gl}_D(V) \). Any \( \mathfrak{sl}_2(\mathbb{R}) \)-subalgebra of \( \mathfrak{gl}_D(V) \) is contained in \( \mathfrak{sl}_D(V) \). For any nontrivial partition \( m \) of \( n \), consider the \( \text{GL}_D^p(V) \)-conjugacy class of \( \mathfrak{sl}_2(\mathbb{R}) \)-subalgebras in \( \mathfrak{sl}_D(V) \) corresponding under \( * \) to the \( \mathfrak{sl}_2(\mathbb{R}) \)-equivalency class of zero form on \( V_m \). Then by (F1)–(F3), [KR], the subset of \( \mathfrak{gl}_D^p(V) \) that is the union of all \( \mathfrak{sl}_2(\mathbb{R}) \)-subalgebras from this class contains a unique nonzero nilpotent \( \text{GL}_D(V) \)-orbit \( O_m \).

  \( \text{Ad}(\mathfrak{sl}_D(V)) \)-orbits in \( N(\mathfrak{sl}_D(V)) \):

  \( O_m \) is an \( \text{Ad}(\mathfrak{sl}_D(V)) \)-orbit, except that \( O_m \) is the union of two \( \text{Ad}(\mathfrak{sl}_D(V)) \)-orbits \( O_1 \) and \( O_2 \) (that are the connected components of \( O_m \)) if \( D = \mathbb{R} \) and \( m_d = 0 \) for every odd \( d \). Such \( \text{Ad}(\mathfrak{sl}_D(V)) \)-orbits, taken over all partitions \( m \) of \( n \), are pairwise different and exhaust all nonzero \( \text{Ad}(\mathfrak{sl}_D(V)) \)-orbits in \( N(\mathfrak{sl}_D(V)) \).
Orbit dimensions:

\[ n^2 - \sum_d d^2 \tilde{m}_d = \begin{cases} 
\dim_{\mathbb{R}} \mathcal{O}_m & \text{for } D = \mathbb{R}, \\
\dim_{\mathbb{C}} \mathcal{O}_m & \text{for } D = \mathbb{C}, \\
\dim_{\mathbb{R}} \mathcal{O}_m / 4 & \text{for } D = \mathbb{H}. 
\end{cases} \]

- Nilpotent orbits in $\mathfrak{gl}^\Phi_D(V)$ for $D = \mathbb{R}$ and nondegenerate symmetric $\Phi$

Let $\mathbf{m}$ be a fine symmetric partition of $n$ associated with (19). Let $\Psi_{\mathbf{m}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\mathbf{m}}$ such that for all $d$,

\[ \Psi_{\mathbf{m}}|_d = \begin{cases} 
\Theta^{s}_{p_d, q_d} \otimes \Delta^s_d & \text{if } d \text{ is odd}, \\
\Theta^{ss}_{m_d} \otimes \Delta^{ss}_d & \text{if } d \text{ is even}. 
\end{cases} \]

Then $\Psi_{\mathbf{m}}$ is equivalent to $\Phi$ if and only if

\[ \text{sgn } \mathbf{m} = \text{sgn } \Phi. \]

If (24) holds, consider the $\text{GL}^\Phi_D(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}^\Phi_D(V)$ corresponding under $\star$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\mathbf{m}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $\text{GL}^\Phi_D(V)$-orbit $\mathcal{O}_m$.

- Nilpotent orbits in $\mathfrak{gl}^\Phi_D(V)$ for $D = \mathbb{C}$ and nondegenerate symmetric $\Phi$

Let $\mathbf{m}$ be a symmetric partition of $n$. Let $\Psi_{\mathbf{m}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\mathbf{m}}$ such that for all $d$,

\[ \Psi_{\mathbf{m}}|_d = \begin{cases} 
\Theta^{s}_{m_d} \otimes \Delta^s_d & \text{if } d \text{ is odd}, \\
\Theta^{ss}_{m_d} \otimes \Delta^{ss}_d & \text{if } d \text{ is even}. 
\end{cases} \]
Then $\Psi_m$ is equivalent to $\Phi$. Consider the $GL_D^\Phi(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^\Phi(V)$ corresponding under $\ast$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_m$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $GL_D^\Phi(V)$-orbit $O_m$.

$Ad(\mathfrak{gl}_D^\Phi(V))$-orbits in $N(\mathfrak{gl}_D^\Phi(V))$:

$O_m$ is an $Ad(\mathfrak{gl}_D^\Phi(V))$-orbit, except that if $m_d = 0$ for all odd $d$, then $O_m$ is the union of two $Ad(\mathfrak{gl}_D^\Phi(V))$-orbits $O_m^1$ and $O_m^2$ that are the connected components of $O_m$. Such $Ad(\mathfrak{gl}_D^\Phi(V))$-orbits, taken over all symmetric partitions $m$ of $n$, are pairwise different and exhaust all nonzero $Ad(\mathfrak{gl}_D^\Phi(V))$-orbits in $N(\mathfrak{gl}_D^\Phi(V))$.

Orbit dimensions:

$$\dim_C O_m = (n^2 - n - \sum_d d^2 \tilde{m}_d + \sum_d \text{odd } m_d)/2.$$  

- Nilpotent orbits in $\mathfrak{gl}_D^\Phi(V)$ for $D = \mathbb{R}$ and nondegenerate skew-symmetric $\Phi$

  Let $\underline{m}$ be a fine skew-symmetric partition of $n$ associated with (19). Let $\Psi_{\underline{m}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\underline{m}}$ such that for all $d$,

$$\Psi_{\underline{m},d} = \begin{cases} 
\Theta_{\ast,d,qd} \otimes \Delta_{s_{\ast,d}} & \text{if } d \text{ is even,} \\
\Theta_{s_{\ast,d},d} \otimes \Delta_{s_{\ast,d}} & \text{if } d \text{ is odd.}
\end{cases}$$  

Then $\Psi_{\underline{m}}$ is equivalent to $\Phi$. Consider the $GL_D^\Phi(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^\Phi(V)$ corresponding under $\ast$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\underline{m}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $GL_D^\Phi(V)$-orbit $O_{\underline{m}}$.

$Ad(\mathfrak{gl}_D^\Phi(V))$-orbits in $N(\mathfrak{gl}_D^\Phi(V))$:

$O_{\underline{m}}$ is an $Ad(\mathfrak{gl}_D^\Phi(V))$-orbit. Such $Ad(\mathfrak{gl}_D^\Phi(V))$-orbits, taken over all fine skew-symmetric partitions $\underline{m}$ of $n$, are pairwise different and exhaust all nonzero $Ad(\mathfrak{gl}_D^\Phi(V))$-orbits in $N(\mathfrak{gl}_D^\Phi(V))$.

Orbit dimensions:

$$\dim_{\mathbb{R}} O_{\underline{m}} = (n^2 + n - \sum_d d^2 \tilde{m}_d - \sum_d \text{odd } m_d)/2.$$  

- Nilpotent orbits in $\mathfrak{gl}_D^\Phi(V)$ for $D = \mathbb{C}$ and nondegenerate skew-symmetric $\Phi$

  Let $\underline{m}$ be a skew-symmetric partition of $n$. Let $\Psi_{\underline{m}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\underline{m}}$ such that for all $d$,

$$\Psi_{\underline{m},d} = \begin{cases} 
\Theta_{\ast,d,qd} \otimes \Delta_{s_{\ast,d}} & \text{if } d \text{ is even,} \\
\Theta_{s_{\ast,d},d} \otimes \Delta_{s_{\ast,d}} & \text{if } d \text{ is odd.} 
\end{cases}$$  

Then $\Psi_{\underline{m}}$ is equivalent to $\Phi$. Consider the $GL_D^\Phi(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^\Phi(V)$ corresponding under $\ast$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\underline{m}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $GL_D^\Phi(V)$-orbit $O_{\underline{m}}$.

$Ad(\mathfrak{gl}_D^\Phi(V))$-orbits in $N(\mathfrak{gl}_D^\Phi(V))$:...
\(O_m\) is an \(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbit. Such \(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbits, taken over all skew-symmetric partitions \(m\) of \(n\), are pairwise different and exhaust all nonzero \(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbits in \(N(\mathfrak{gl}_D^\Phi(V))\).

Orbit dimensions:

\[
\dim C O_m = \left( n^2 + n - \sum_d d^2 \bar{m}_d - \sum_{d, \text{odd}} m_d \right) / 2.
\]

- Nilpotent orbits in \(\mathfrak{gl}_D^\Phi(V)\) for \(D = \mathbb{C}\) and nondegenerate Hermitian \(\Phi\)

Let \(m\) be a fine partition of \(n\) associated with (19). Let \(\Psi_m\) be the \(\mathfrak{sl}_2(\mathbb{R})\)-invariant form on \(V_m\) such that for all \(d\),

\[
(26) \quad \Psi_m|_d = \Theta_{p_d,q_d}^{H} \otimes \Delta^H_d.
\]

Then \(\Psi_m\) is equivalent to \(\Phi\) if and only if (24) holds. In the last case, consider the \(\text{GL}_D^\Phi(V)\)-conjugacy class of \(\mathfrak{sl}_2(\mathbb{R})\)-subalgebras in \(\mathfrak{gl}_D^\Phi(V)\) corresponding under \(\star\) to the \(\mathfrak{sl}_2(\mathbb{R})\)-equivalency class of \(\Psi_m\). Then by (F1)–(F3), [KR], the union of all \(\mathfrak{sl}_2(\mathbb{R})\)-subalgebras from this class contains a unique nilpotent \(\text{GL}_D^\Phi(V)\)-orbit \(O_m\).

\(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbits in \(N(\mathfrak{gl}_D^\Phi(V))\):

If (24) holds, \(O_m\) is an \(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbit. Such \(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbits, taken over all fine partitions \(m\) of \(n\) satisfying (24), are pairwise different and exhaust all nonzero \(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbits in \(N(\mathfrak{gl}_D^\Phi(V))\).

Orbit dimensions:

\[
\dim R O_m = n^2 - \sum_d d^2 \bar{m}_d.
\]

- Nilpotent orbits in \(\mathfrak{gl}_D^\Phi(V)\) for \(D = \mathbb{H}\) and nondegenerate Hermitian \(\Phi\)

Let \(m\) be a fine Hermitian partition of \(n\) associated with (19). Let \(\Psi_m\) be the \(\mathfrak{sl}_2(\mathbb{R})\)-invariant form on \(V_m\) such that for all \(d\),

\[
(27) \quad \Psi_m|_d = \begin{cases} 
\Theta_{m_d}^{sH} \otimes \Delta_{d}^{sH} & \text{if } d \text{ is even,} \\
\Theta_{p_d,q_d}^{H} \otimes \Delta^H_d & \text{if } d \text{ is odd.}
\end{cases}
\]

Then \(\Psi_m\) is equivalent to \(\Phi\) if and only if (24) holds. In the last case, consider the \(\text{GL}_D^\Phi(V)\)-conjugacy class of \(\mathfrak{sl}_2(\mathbb{R})\)-subalgebras in \(\mathfrak{gl}_D^\Phi(V)\) corresponding under \(\star\) to the \(\mathfrak{sl}_2(\mathbb{R})\)-equivalency class of \(\Psi_m\). Then by (F1)–(F3), [KR], the union of all \(\mathfrak{sl}_2(\mathbb{R})\)-subalgebras from this class contains a unique nilpotent \(\text{GL}_D^\Phi(V)\)-orbit \(O_m\).

\(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbits in \(N(\mathfrak{gl}_D^\Phi(V))\):

If (24) holds, \(O_m\) is an \(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbit. Such \(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbits, taken over all fine Hermitian partitions \(m\) of \(n\) satisfying (24), are pairwise different and exhaust all nonzero \(\text{Ad}(\mathfrak{gl}_D^\Phi(V))\)-orbits in \(N(\mathfrak{gl}_D^\Phi(V))\).

Orbit dimensions:

\[
\dim R O_m = 2n^2 + n - 2 \sum_d d^2 \bar{m}_d - \sum_{d, \text{odd}} m_d.
\]
• Nilpotent orbits in $\mathfrak{gl}_D^\Phi(V)$ for $D = \mathbb{H}$ and skew-Hermitian $\Phi$

Let $\underline{m}$ be a fine skew-Hermitian partition of $n$ associated with (19). Let $\Psi_{\underline{m}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\underline{m}}$ such that for all $d$,

\begin{equation}
\Psi_{\underline{m}}|_d = \begin{cases} 
\Theta^H_{p_d,q_d} \otimes \Delta_d^H & \text{if } d \text{ is even}, \\
\Theta_{s_d}^H \otimes \Delta_d^H & \text{if } d \text{ is odd}.
\end{cases}
\end{equation}

Then $\Psi_{\underline{m}}$ is equivalent to $\Phi$. Consider the $\text{GL}_\Phi D(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^\Phi(V)$ corresponding under $\ast$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\underline{m}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $\text{GL}_\Phi D(V)$-orbit $O_{\underline{m}}$.

\begin{itemize}
\item Orbit dimensions:
\begin{equation}
\dim_{\mathbb{R}} O_{\underline{m}} = 2n^2 - n - 2 \sum_{d \geq 1} d^2 m_d + \sum_{d \text{ odd}} m_d.
\end{equation}
\end{itemize}

Now we are ready to classify compact nilpotent elements in the real forms of complex classical Lie algebras. Recall that $n = \dim_D V$.

• Compact nilpotent elements in $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{sl}_n(\mathbb{H})$

**Theorem 8.** Let $\mathcal{O}$ be the $\text{Ad}(\mathfrak{sl}_D(V))$-orbit of a nonzero element $x \in \mathcal{N}(\mathfrak{sl}_D(V))$ where $D$ is $\mathbb{R}$ or $\mathbb{H}$. Let $\mathfrak{a}$ be an $\mathfrak{sl}_D(\mathbb{R})$-subalgebra of $\mathfrak{sl}_D(V)$ containing $x$. The following properties are equivalent:

(i) $x$ is compact.

(ii) $\mathfrak{a}$-module $V$ is simple.

(iii) If $D = \mathbb{R}$, then $\mathcal{O} = \mathcal{O}_{(0,\ldots,0,1)}$ for odd $n$, and $\mathcal{O}_{1(0,\ldots,0,1)}$ or $\mathcal{O}_{2(0,\ldots,0,1)}$ for even $n$. If $D = \mathbb{H}$, then $\mathcal{O} = \mathcal{O}_{(0,\ldots,0,1)}$.

**Proof.** According to the previous discussion, we may, and will, assume that $V = V_{\underline{m}}$ and $\mathfrak{a} = \alpha \underline{m}(\mathfrak{sl}_2(\mathbb{R}))$ (see (22)) for some nontrivial partition $\underline{m}$ of $n$. The Double Centralizer Theorem implies that

\begin{equation}
\mathfrak{z}_{\mathfrak{gl}_D(V)}(\mathfrak{a}) = \bigoplus_{d \geq 1} \mathfrak{gl}_D(T_{m_d}).
\end{equation}

Since $\mathfrak{gl}_D(T_{m_d})$ (resp., $\mathfrak{sl}_D(T_{m_d})$) is compact if and only if $m_d = 0$ (resp., 0 or 1), the claim follows from (29) and (F5).

• Compact nilpotent elements in $\mathfrak{so}_{n-q,q}$

**Theorem 9.** Let $\mathcal{O}$ be the $\text{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbit of a nonzero element $x \in \mathcal{N}(\mathfrak{gl}_D^\Phi(V))$ for a nondegenerate symmetric form $\Phi$. The following properties are equivalent:

(i) $x$ is compact.
(ii) $O = O^1_m$ or $O^2_m$ where $m$ is a fine symmetric partition of $n$ such that (24) holds, $p_d q_d = 0$ for all odd $d$, and $m_d = 0$ for all even $d$.

Proof. We may, and will, assume that $V = V_m$. $\Phi$ is $\Psi_m$ defined by (23), and $x \in a := \alpha m(\mathfrak{sl}_2(\mathbb{R}))$ for some fine symmetric partition $m$ of $n$ such that (24) holds. Then (29) holds, whence by (23) $z^{gl}_{\Phi}(a) = \bigoplus_{d \geq 1} gl_{R}(T_{m_d})$ where $\Theta_d = \begin{cases} \Theta^s_{p_d q_d} & \text{if } d \text{ is odd}, \\ \Theta^s_{m_d} & \text{if } d \text{ is even}. \end{cases}$

Since $gl_{R}(T_{m_d})$ for $\Theta_d$ given by (30) is compact if and only if $p_d q_d = 0$ for odd $d$ and $m_d = 0$ for even $d$ (see Table 1 and (18)), the claim follows from (30) and (F5).

• Compact nilpotent elements in $\mathfrak{sp}_n(\mathbb{R})$

Theorem 10. Let $O$ be the $\text{Ad}(gl_{2}(V))$-orbit of a nonzero element $x \in N(gl_{2}(V))$ for a nondegenerate skew-symmetric form $\Phi$. The following properties are equivalent:

(i) $x$ is compact.
(ii) $O = O_m$ where $m$ is a fine skew-symmetric partition of $n$ such that $p_d q_d = 0$ for all even $d$, and $m_d = 0$ for all odd $d$.

Proof. The arguments are similar to that in the proof of Theorem 9 with (23) replaced with (25).

• Compact nilpotent elements in $\mathfrak{su}_{n-q,q}$

Theorem 11. Let $O$ be the $\text{Ad}(gl_{2}(V))$-orbit of a nonzero element $x \in N(gl_{2}(V))$ for a nondegenerate Hermitian form $\Phi$. The following properties are equivalent:

(i) $x$ is compact.
(ii) $O = O_m$ where $m$ is a fine partition of $n$ such that (24) holds and $p_d q_d = 0$ for all $d$.

Proof. By Table 1, the algebra $gl_{2}(W)$ for $\Psi = \Theta^H_{p_d q_d}$ and $W = T_{m_d}$ is compact if and only if $p_d q_d = 0$. Now one completes the proof using the arguments similar to that in the proof of Theorem 9 with (23) replaced with (26).

• Compact nilpotent elements in $\mathfrak{sp}_{n/2-q,q}$

Theorem 12. Let $O$ be the $\text{Ad}(gl_{2}(V))$-orbit of a nonzero element $x \in N(gl_{2}(V))$ for a nondegenerate Hermitian form $\Phi$. The following properties are equivalent:

(i) $x$ is compact.
(ii) $O = O_m$ where $m$ is a fine Hermitian partition of $n$ such that (24) holds, $p_d q_d = 0$ for all odd $d$, and $m_d = 0$ or 1 for all even $d$.

Proof. It follows from Table 1 and (18) that the algebra $gl_{2}(W)$ for $\Psi = \Theta^H_{p_d q_d}$ (resp., $\Theta^s_{m_d}$) and $W = T_{m_d}$ is compact if and only if $p_d q_d = 0$ (resp., $m_d = 0$ or 1). Now one completes the proof using the arguments similar to that in the proof of Theorem 9 with (23) replaced with (27).
• Compact nilpotent elements in \( u^*_n(\mathbb{H}) \)

**Theorem 13.** Let \( \mathcal{O} \) be the Ad\((\text{gl}\Phi)\)-orbit of a nonzero element \( x \in \mathcal{N}(\text{gl}\Phi(V)) \) for a nondegenerate skew-Hermitian form \( \Phi \). The following properties are equivalent:

(i) \( x \) is compact.

(ii) \( \mathcal{O} = \mathcal{O}_m \) where \( m \) is a fine skew-Hermitian partition of \( n \) such that \( p_d q_d = 0 \) for all even \( d \), and \( m_d = 0 \) or 1 for all odd \( d \).

**Proof.** The arguments are similar to that in the proof of Theorem 12 with (27) replaced with (28). \( \square \)

5. **Classification for exceptional simple \( \mathfrak{g} \)**

In this section, we assume that \( \mathfrak{g} \) is an exceptional simple complex Lie algebra and \( \mathfrak{g}_\mathbb{R} \) is its noncompact real form. According to E. Cartan’s classification (cf., e.g., [OV]), up to isomorphism there are, in total, twelve possibilities listed in Table 5. Its second column gives both types of E. Cartan’s notation for \( \mathfrak{g}_\mathbb{R} \); the notation \( X_{s(t)} \) means that \( X_s \) is the type of \( \mathfrak{g} \) and \( t \) is the signature of the Killing form of \( \mathfrak{g}_\mathbb{R} \), i.e., \( t = \dim_{\mathbb{C}} p - \dim_{\mathbb{C}} \mathfrak{k} \).

| \( \mathfrak{g} \) | \( \mathfrak{g}_\mathbb{R} \) | type of \( \mathfrak{k}_R \) | \( \dim_{\mathbb{C}} \mathfrak{k} \) | \( \dim_{\mathbb{C}} p \) |
|---|---|---|---|---|
| \( E_6 \) | EI = \( E_6(6) \) | \( \mathfrak{sp}_4 \) | 36 | 42 |
| \( E_6 \) | EII = \( E_6(2) \) | \( \mathfrak{su}_2 \oplus \mathfrak{su}_6 \) | 38 | 40 |
| \( E_6 \) | EIII = \( E_6(-14) \) | \( \mathfrak{so}_{10} \oplus \mathbb{R} \) | 46 | 32 |
| \( E_6 \) | EIV = \( E_6(-26) \) | \( \mathfrak{F}_4(-52) \) | 52 | 26 |
| \( E_7 \) | EV = \( E_7(7) \) | \( \mathfrak{su}_8 \) | 63 | 70 |
| \( E_7 \) | EVI = \( E_7(-5) \) | \( \mathfrak{su}_2 \oplus \mathfrak{so}_{12} \) | 69 | 64 |
| \( E_7 \) | EVII = \( E_7(-25) \) | \( \mathfrak{E}_6(-78) \oplus \mathbb{R} \) | 79 | 54 |
| \( E_8 \) | EVIII = \( E_8(8) \) | \( \mathfrak{so}_{16} \) | 120 | 128 |
| \( E_8 \) | EIX = \( E_8(-24) \) | \( \mathfrak{su}_2 \oplus \mathfrak{E}_7(-133) \) | 136 | 112 |
| \( F_4 \) | FI = \( F_4(4) \) | \( \mathfrak{su}_2 \oplus \mathfrak{sp}_3 \) | 24 | 28 |
| \( F_4 \) | FII = \( F_4(-20) \) | \( \mathfrak{so}_9 \) | 36 | 16 |
| \( G_2 \) | GI = \( G_2(2) \) | \( \mathfrak{so}_3 \oplus \mathfrak{so}_3 \) | 6 | 8 |

Recall the classical approach to classifying nilpotent orbits by means of their characteristics and weighted Dynkin diagrams, cf. [CM], [M]. Fix a \( \theta \)-stable Cartan subalgebra \( \mathfrak{t} \) of \( \mathfrak{g} \) such that \( \mathfrak{t} \cap \mathfrak{t} \) is a Cartan subalgebra of \( \mathfrak{t} \). Denote

\[ l := \dim_{\mathbb{C}} \mathfrak{t}, \quad s := \dim_{\mathbb{C}} (\mathfrak{t} \cap \mathfrak{t}). \]

Let \( \Delta_{(\mathfrak{g}, \mathfrak{t})} \) and \( \Delta_{(\mathfrak{t}, \mathfrak{t} \cap \mathfrak{t})} \) be the root systems of resp. \( (\mathfrak{g}, \mathfrak{t}) \) and \( (\mathfrak{t}, \mathfrak{t} \cap \mathfrak{t}) \). Given a nonzero element \( e \in \mathcal{N}(\mathfrak{g}) \) (resp., \( e \in \mathcal{N}(\mathfrak{p}) \)), there are the elements \( h, f \in \mathfrak{g} \) (resp., \( h \in \mathfrak{t}, f \in \mathfrak{p} \))
such that \( \{e, h, f\} \) is an \( \mathfrak{sl}_2 \)-triple. The intersection of \( t \) (resp., \( t \cap \mathfrak{t} \)) with the orbit \( O = G \cdot e \) (resp., \( K \cdot e \)) contains a unique element \( h_0 \) lying in a fixed Weyl chamber of \( \Delta_{(g, t)} \) (resp., \( \Delta_{(t, \mathfrak{t} \cap \mathfrak{t})} \)). The mapping \( O \mapsto h_0 \) is a well defined injection of the set of all nonzero \( G \)-orbits in \( \mathcal{N}(g) \) (resp., \( K \)-orbits in \( \mathcal{N}(p) \)) into \( t \) (resp., \( t \cap \mathfrak{t} \)). The image \( h_0 \) of the orbit \( O \) under this injection is called the characteristic of \( O \) (and \( e \)). Thus the \( G \)-orbits in \( \mathcal{N}(g) \) (resp., \( K \)-orbits in \( \mathcal{N}(p) \)) are defined by their characteristics. In turn, the characteristics are defined by the numerical data, namely, the system of values \( \{ \beta_j(h_0) \} \) where \( \{ \beta_j \} \) is a fixed basis of \( t^* \) (resp., \( (t \cap \mathfrak{t})^* \)). In practice, \( \{ \beta_j \} \) is always chosen to be a base of some root system. Then assigning the integer \( \beta_j(h_0) \) to each node \( \beta_j \) of the Dynkin diagram of this base yields the weighted Dynkin diagram of the orbit \( O \), denoted by \( \text{Dyn} O \). It uniquely defines \( O \). This choice of the base \( \{ \beta_j \} \) is the following.

For \( t^* \), since \( g \) is semisimple, it is natural to take \( \{ \beta_j \} \) to be the base \( \alpha_1, \ldots, \alpha_l \) of \( \Delta_{(g, t)} \) defining the fixed Weyl chamber that is used in the definition of characteristics. Below the Dynkin diagrams of \( G \)-orbits in \( \mathcal{N}(g) \) are considered with respect to this base \( \alpha_1, \ldots, \alpha_l \).

To describe \( \{ \beta_j \} \) for \( (t \cap \mathfrak{t})^* \), denote by \( \alpha_0 \) the lowest root of \( \Delta_{(g, t)} \). The extended Dynkin diagrams of \( \Delta_{(g, t)} \) indicating the location of the \( \alpha_i \)'s are given in Table 6.

The real form \( g_\mathbb{R} \) of \( g \) is of inner type (i.e., \( s = l \), \( t \subset \mathfrak{t} \), and thereby \( \Delta_{(t \cap \mathfrak{t} \cap \mathfrak{t})} \subset \Delta_{(g, t)} \)) if and only if \( g_\mathbb{R} \neq E_6(6), E_6(-26) \). If \( g_\mathbb{R} \) is of inner type, then \( t \) is semisimple if and only if \( g_\mathbb{R} \neq E_6(-14), E_7(-25) \); in the last two cases the center of \( t \) is one-dimensional. If \( g_\mathbb{R} \) is of inner type, then the set \( \{ \alpha_0, \ldots, \alpha_l \} \) contains a unique base \( \beta_1, \ldots, \beta_r \) of \( \Delta_{(t \cap \mathfrak{t} \cap \mathfrak{t})} \). Moreover, \( \alpha_0 \) is one of the \( \beta_j \)'s \( \iff \) \( t \) is semisimple \( \iff \) \( r = l \). If \( g_\mathbb{R} \) is of inner type and \( t \) is not semisimple, then \( r = l - 1 \) and \( \beta_j \in \{ \alpha_1, \ldots, \alpha_l \} \) for all \( j = 1, \ldots, r \); in this case we set \( \beta_1 := \alpha_1 \).

Thus in all cases where \( g_\mathbb{R} \) is of inner type, we have a basis \( \{ \beta_j \} \) of \( (t \cap \mathfrak{t})^* \) that is the system of simple roots of some root system. The location of the \( \beta_j \)'s is given in Table 6.

**Table 6**

| GI | FI | FII | EII | EIII |
|---|---|---|---|---|
| \( \beta_j \) | \( \beta_j \) | \( \beta_j \) | \( \alpha_0 \beta_j \) | \( \beta_j \) |

| EV | EVI | EVII | EVIII | EIX |
|---|---|---|---|---|
| \( \beta_j \) | \( \beta_j \) | \( \beta_j \) | \( \beta_j \) | \( \beta_j \) |


If $g_{\mathbb{R}}$ is of outer type, then $\mathfrak{k}$ is semisimple, so we may, and will, take $\{\beta_j\}$ to be a base of $\Delta(\mathfrak{k}, \mathfrak{t} \cap \mathfrak{k})$. We take the following base. If $g_{\mathbb{R}} = E_6(-26)$, then

$$\beta_1 = \alpha_1 |_{\mathfrak{k} \cap \mathfrak{t}} = \alpha_6 |_{\mathfrak{k} \cap \mathfrak{t}}, \quad \beta_3 = \alpha_3 |_{\mathfrak{k} \cap \mathfrak{t}} = \alpha_5 |_{\mathfrak{k} \cap \mathfrak{t}}, \quad \beta_2 = \alpha_4 |_{\mathfrak{k} \cap \mathfrak{t}}, \quad \beta_1 = \alpha_2 |_{\mathfrak{k} \cap \mathfrak{t}},$$

with the Dynkin diagram

$$\begin{array}{c}
\circ_{\beta_1} \rightarrow \circ_{\beta_2} \rightarrow \circ_{\beta_3} \\
\end{array}$$

and if $g_{\mathbb{R}} = E_6(6)$, then

$$\beta_1 = -2\beta_2 - 3\beta_3 - 2\beta_4 - \alpha_2 |_{\mathfrak{k} \cap \mathfrak{t}}, \quad \beta_2 = \alpha_1 |_{\mathfrak{k} \cap \mathfrak{t}} = \alpha_6 |_{\mathfrak{k} \cap \mathfrak{t}}, \quad \beta_3 = \alpha_3 |_{\mathfrak{k} \cap \mathfrak{t}} = \alpha_5 |_{\mathfrak{k} \cap \mathfrak{t}}, \quad \beta_4 = \alpha_4 |_{\mathfrak{k} \cap \mathfrak{t}}$$

with the Dynkin diagram

$$\begin{array}{c}
\circ_{\beta_1} \rightarrow \circ_{\beta_2} \rightarrow \circ_{\beta_3} \rightarrow \circ_{\beta_4} \\
\end{array}$$

Below the weighted Dynkin diagrams of $K$-orbits in $\mathcal{N}(\mathfrak{g})$ are considered with respect to the described base $\{\beta_j\}$.

In [D3], [D4] one finds:

- (D1) the weighted Dynkin diagrams of all nonzero orbits $G \cdot x$ and $K \cdot x$ for $x \in \mathcal{N}(\mathfrak{p})$,
- (D2) the type and dimension of the reductive Levi factor of $z_k(x)$ and $\dim_{\mathbb{C}} z_k(x)$,
- (D3) the type of the reductive Levi factor of $z_{g_{\mathbb{R}}}(y)$ for an element $y$ of the $\text{Ad}(g_{\mathbb{R}})$-orbit in $\mathcal{N}(g_{\mathbb{R}})$ corresponding to $K \cdot x$ via the Kostant–Sekiguchi bijection.

Given the weighted Dynkin diagram of $G \cdot x$, one finds the type of reductive Levi factor of $z_{\mathfrak{g}}(x)$ in [El], [Ca]. So using (D1), one can apply Theorem 4 for explicit classifying $(-1)$-distinguished $K$-orbits in $\mathcal{N}(\mathfrak{p})$. On the other hand, given (D3), Theorem 5 can be applied for this purpose as well. So following either of these ways, one obtains, for every exceptional simple $\mathfrak{g}$, the explicit classification of $(-1)$-distinguished $K$-orbits in $\mathcal{N}(\mathfrak{p})$ in terms of their weighted Dynkin diagrams. The final result is the following.

**Theorem 14.** For all exceptional simple Lie algebras $\mathfrak{g}$ and all conjugacy classes of elements $\theta \in \text{Aut} \mathfrak{g}$ of order 2, all $(-1)$-distinguished $K$-orbits in $\mathcal{N}(\mathfrak{p})$ are listed in Tables 7–18 at the end of this paper.

Tables 7–18 contain further information as summarized below.

- (T1) The conjugacy class of $\theta$ is defined by specifying the type of noncompact real form $g_{\mathbb{R}}$ of $\mathfrak{g}$ canonically corresponding to this class.
- (T2) Column 2 gives the weights $\{\beta_j(h)\}$ (listed in the order of increasing of $j$) of the weighted Dynkin diagram $\text{Dyn} K \cdot x$ of the $(-1)$-distinguished $K$-orbit $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$, where $h$ is the characteristic of $K \cdot x$. 

(T3) Column 3 gives the weights $\{\alpha_j(H)\}$ (listed in the order of increasing of $j$) of the weighted Dynkin diagram $\text{Dyn}_G\cdot x$ of the $G$-orbit $G\cdot x$ in $\mathcal{N}(\mathfrak{g})$, where $H$ is the characteristic of $G\cdot x$.

(T4) Column 4 gives $\dim_{\mathbb{C}} K\cdot x$, and hence, by (2), also $\dim_{\mathbb{C}} G\cdot x$.

(T5) Column 5 gives the number of $K$-orbits (not necessarily $(-1)$-distinguished) in $G\cdot x \cap \mathfrak{p}$. One of them is $K\cdot x$. Since this number is equal to the number of $K$-orbits $K\cdot x'$ in $\mathcal{N}(\mathfrak{p})$ such that $\text{Dyn}_G\cdot x' = \text{Dyn}_G\cdot x$, one finds it using the tables in [D3], [D4].

(T6) Columns 6 gives the type of the reductive Levi factor of $\mathfrak{z}_k(x)$. By $T_m$ is denoted the Lie algebra of an $m$-dimensional torus.

(T7) Column 7 gives the complex dimension of the unipotent radical of $\mathfrak{z}_k(x)$.

Remark 1. For $E_6(6)$ and $E_6(-26)$, our numeration of $\{\alpha_j\}$ is the same as in [D3]; it differs from that in [D4]. For $E_6(-26)$, our numeration of $\{\beta_j\}$ differs from that in [D3].

Remark 2. One finds in [D5], [D6], [D7] the explicit classification of all Cayley triples in $\mathfrak{g}$. This yields the explicit representatives of all $K$-orbits in $\mathcal{N}(\mathfrak{p})$ and $\text{Ad}(\mathfrak{g}_R)$-orbits in $\mathcal{N}(\mathfrak{g}_R)$.

6. Geometric properties

Geometric properties of the varieties $\mathbf{P}(\overline{O})$ where $\mathcal{O}$ is a nilpotent $K$-orbit in $\mathcal{N}(\mathfrak{p})$ have been studied by several authors. Thereby their results provide some information on the geometry of the projective self-dual varieties that we associated with symmetric spaces. However these results are less complete than in the adjoint case (where a rather detailed information about singular loci and normality of the projective self-dual varieties $\mathbf{P}(\overline{O})$ is available, see [P2]). Some phenomena valid for the projective self-dual $\mathbf{P}(\overline{O})$’s in the adjoint case fail in general. Below we briefly summarize some facts about the geometry of the projective self-dual varieties that we associated with symmetric spaces.

Intersections of $G$-orbits in $\mathbf{P}(\mathcal{N}(\mathfrak{g}))$ with the linear subspace $\mathbf{P}(\mathfrak{p})$

Let $\mathcal{O}$ be a nonzero $G$-orbit in $\mathcal{N}(\mathfrak{g})$, and $X = \mathbf{P}(\overline{\mathcal{O}})$. If $X \cap \mathbf{P}(\mathfrak{p}) \neq \emptyset$, then, by (2), all irreducible components $Y_1, \ldots, Y_s$ of the variety $X \cap \mathbf{P}(\mathfrak{p})$ have dimension $\frac{1}{2} \dim X$, and every $Y_j$ is the closure of a $K$-orbit in $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$. Theorems 2, 3 and Definition 1 imply that

$$X \text{ is self-dual } \implies \text{ all } Y_1, \ldots, Y_s \text{ are self-dual.}$$

There are many examples showing that the converse is not true. For instance, one deduces from Table 8 that $\mathfrak{g} = E_6$, $\mathfrak{g}_R = E_6(2)$ and $\text{Dyn}\mathcal{O} = 111011, 121011$ or $220202$ are such cases. There are also many examples where some of the $Y_j$’s are self-dual and some are not: for instance, this is so if $\mathfrak{g} = E_6$, $\mathfrak{g}_R = E_6(2)$ and $\text{Dyn}\mathcal{O} = 020000, 001000, 002000, 020200$ or $22002$. Finally, there are many instances where all $Y_j$ are not self-dual (to obtain them, e.g., compare Tables 7–18 with tables in [D3], [D4]).

Affine $(-1)$-distinguished orbits

Recall that by Matsushima’s criterion, an orbit of a reductive algebraic group acting on an affine algebraic variety is affine if and only if the stabilizer of a point of this orbit is reductive, cf., e.g. [PV, Theorem 4.17]. Therefore distinguished $G$-orbits in $\mathcal{N}(\mathfrak{g})$ are never
affine. However in general there exist affine \((-1)\)-distinguished \(K\)-orbits in \(N(p)\). For instance, if \(g\) is exceptional simple, we immediately obtain their classification from Tables 7–18: these are precisely the orbits for which the number in the last column is 0.

If \(K \cdot x\) is affine, then each irreducible component of the boundary \(P(K \cdot x) \setminus P(K \cdot x)\) has codimension 1 in \(P(K \cdot x)\), cf. [P1, Lemma 3]. Therefore if a point of the open \(K\)-orbit of such an irreducible component lies in the singular locus of \(P(K \cdot x)\), then \(P(K \cdot x)\) is not normal.

**Orbit closure ordering and orbit decomposition of the orbit boundary \(P(K \cdot x) \setminus P(K \cdot x)\)**

The closure ordering on the set of \(K\)-orbits in \(N(p)\) (resp., \(Ad(g)\)-orbits in \(N(g)\)) is defined by the condition that \(O_1 > O_2\) if and only if \(O_2\) is contained in the closure of \(O_1\) and \(O_1 \neq O_2\). According to [BS], the Kostant–Sekiguchi bijection preserves the closure ordering. Clearly its describing can be reduced to the case of simple \(g\). In this case, the explicit description of the closure ordering is obtained in [D1], [D2], [D8], [D9], [D10], [D11], [D12], [D13] (see also [O], [Se]).

This information and Theorems 8–14 yield the orbit decomposition of every irreducible component of the orbit boundary \(P(K \cdot x) \setminus P(K \cdot x)\) for any \((-1)\)-distinguished \(K\)-orbit \(K \cdot x\). In particular, this gives the dimensions of these components and their intersection configurations.

**Singular locus of a self-dual variety \(P(K \cdot x)\)**

Apart from a rather detailed information on the singular loci of the self-dual projectivized nilpotent orbit closures in the adjoint case (see [P2]), in general case only a partial information is available. In \([O]\), for \((g, \mathfrak{t}) = (\mathfrak{sl}_n(C), \mathfrak{so}_n(C))\) and \((\mathfrak{sl}_2n(C), \mathfrak{sp}_n(C))\), the normality of the nilpotent orbit closures \(K \cdot x\) (hence that of \(P(K \cdot x)\)) is studied. In particular, it is shown there that for \((g, \mathfrak{t}) = (\mathfrak{sl}_2n(C), \mathfrak{sp}_n(C))\), normality holds for any \(K \cdot x\), but if \((g, \mathfrak{t}) = (\mathfrak{sl}_n(C), \mathfrak{so}_n(C))\), this is not the case. See also [Se], [O], [SeSh], where the local equations of the generic singularities of some orbit closures in \(P(N(p))\) are found. By [P3, Proposition 4], Hesselink’s desingularization of the closures of Hesselink strata, [He], cf. [PV], yields a desingularization of any orbit closure \(P(K \cdot x)\) in \(P(N(p))\) (in [R], another approach to desingularization of such orbit closed is considered).
TABLE 7

$(-1)$-distinguished $K$-orbits $K \cdot x$ in $\mathcal{N}(p)$ for $g_R = E_6(6)$

| No. | Dyn $K \cdot x$ | Dyn $G \cdot x$ | $\dim_{\mathbb{C}} K \cdot x$ | $\sharp(G \cdot x \cap p)/K$ | $\delta(x)/\text{rad}_{u,\delta}(x)$ | $\dim_{\mathbb{C}} \text{rad}_{u,\delta}(x)$ |
|-----|----------------|----------------|-------------------------------|--------------------------|-----------------------------|-----------------------------|
| 1   | 2222          | 202222         | 35                           | 1                        | 0                           | 1                           |
| 2   | 2202          | 220002         | 33                           | 2                        | 0                           | 3                           |
| 3   | 0220          | 220002         | 33                           | 2                        | 0                           | 3                           |
| 4   | 4224          | 222222         | 36                           | 1                        | 0                           | 0                           |

TABLE 8

$(-1)$-distinguished $K$-orbits $K \cdot x$ in $\mathcal{N}(p)$ for $g_R = E_6(2)$

| No. | Dyn $K \cdot x$ | Dyn $G \cdot x$ | $\dim_{\mathbb{C}} K \cdot x$ | $\sharp(G \cdot x \cap p)/K$ | $\delta(x)/\text{rad}_{u,\delta}(x)$ | $\dim_{\mathbb{C}} \text{rad}_{u,\delta}(x)$ |
|-----|----------------|----------------|-------------------------------|--------------------------|-----------------------------|-----------------------------|
| 1   | 000004         | 020000         | 21                           | 3                        | $2A_2$                      | 1                           |
| 2   | 301000         | 001010         | 25                           | 3                        | $A_1 + T_1$                 | 9                           |
| 3   | 001030         | 001010         | 25                           | 3                        | $A_1 + T_1$                 | 9                           |
| 4   | 004000         | 000200         | 29                           | 3                        | $T_2$                       | 7                           |
| 5   | 020204         | 000200         | 29                           | 3                        | $T_2$                       | 7                           |
| 6   | 004008         | 020200         | 30                           | 2                        | $A_2$                       | 0                           |
| 7   | 400044         | 220002         | 30                           | 2                        | $A_1 + T_1$                 | 4                           |
| 8   | 121131         | 111011         | 31                           | 2                        | $T_1$                       | 6                           |
| 9   | 311211         | 111011         | 31                           | 2                        | $T_1$                       | 6                           |
| 10  | 313104         | 121011         | 32                           | 2                        | $T_1$                       | 5                           |
| 11  | 013134         | 121011         | 32                           | 2                        | $T_1$                       | 5                           |
| 12  | 222222         | 200202         | 33                           | 2                        | 0                           | 5                           |
| 13  | 040040         | 200202         | 33                           | 2                        | 0                           | 5                           |
| 14  | 224224         | 220202         | 34                           | 2                        | $T_1$                       | 3                           |
| 15  | 404048         | 220202         | 34                           | 2                        | $T_1$                       | 3                           |
| 16  | 440444         | 222022         | 35                           | 1                        | 0                           | 3                           |
| 17  | 444448         | 222222         | 36                           | 1                        | 0                           | 2                           |

TABLE 9

$(-1)$-distinguished $K$-orbits $K \cdot x$ in $\mathcal{N}(p)$ for $g_R = E_6(-26)$

| No. | Dyn $K \cdot x$ | Dyn $G \cdot x$ | $\dim_{\mathbb{C}} K \cdot x$ | $\sharp(G \cdot x \cap p)/K$ | $\delta(x)/\text{rad}_{u,\delta}(x)$ | $\dim_{\mathbb{C}} \text{rad}_{u,\delta}(x)$ |
|-----|----------------|----------------|-------------------------------|--------------------------|-----------------------------|-----------------------------|
| 1   | 0001           | 100001         | 16                           | 1                        | $B_3$                       | 15                          |
| 2   | 0002           | 200002         | 24                           | 1                        | $G_2$                       | 14                          |
TABLE 10

(−1)-distinguished $K$-orbits $K \cdot x$ in $\mathcal{N}(p)$ for $g_\mathbb{R} = E_6(-14)$

| No. | $\text{Dyn } K \cdot x$ | $\text{Dyn } G \cdot x$ | $\text{dim}_C K \cdot x$ | $\sharp (G \cdot x \cap p)/K$ | $\delta G(x)/\text{rad}_G \delta G(x)$ | $\text{dim}_C \text{rad}_G \delta G(x)$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | 100001          | 100001          | 16              | 3               | $B_3 + T_1$     | 8               |
| 2   | 10000−2         | 100001          | 16              | 3               | $B_3 + T_1$     | 8               |
| 3   | 10101−2         | 110001          | 23              | 2               | $A_2 + T_1$     | 14              |
| 4   | 11100−3         | 110001          | 23              | 2               | $A_2 + T_1$     | 14              |
| 5   | 40000−2         | 200002          | 24              | 1               | $G_2$           | 8               |
| 6   | 03001−2         | 120001          | 26              | 2               | $B_2 + T_1$     | 9               |
| 7   | 01003−6         | 120001          | 26              | 2               | $B_2 + T_1$     | 9               |
| 8   | 02202−6         | 220002          | 30              | 1               | $A_1 + T_1$     | 12              |

TABLE 11

(−1)-distinguished $K$-orbits $K \cdot x$ in $\mathcal{N}(p)$ for $g_\mathbb{R} = E_7(-5)$

| No. | $\text{Dyn } K \cdot x$ | $\text{Dyn } G \cdot x$ | $\text{dim}_C K \cdot x$ | $\sharp (G \cdot x \cap p)/K$ | $\delta G(x)/\text{rad}_G \delta G(x)$ | $\text{dim}_C \text{rad}_G \delta G(x)$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | 00000004        | 20000000        | 33              | 3               | $A_5$           | 1               |
| 2   | 4000000        | 00000020        | 42              | 2               | $G_2 + A_1$     | 10              |
| 3   | 00004000       | 00200000        | 47              | 3               | $3A_1$          | 13              |
| 4   | 00020004       | 00200000        | 47              | 3               | $3A_1$          | 13              |
| 5   | 0000408        | 20200000        | 48              | 2               | $C_3$           | 0               |
| 6   | 2010112        | 0001010         | 49              | 1               | $A_1 + T_1$     | 16              |
| 7   | 04000004       | 20000020        | 50              | 2               | $A_2 + T_1$     | 10              |
| 8   | 1111101        | 1001010         | 52              | 1               | $T_2$           | 15              |
| 9   | 2010314        | 2001010         | 53              | 1               | $A_1 + T_1$     | 12              |
| 10  | 0040000        | 00020000        | 53              | 1               | $A_1$           | 13              |
| 11  | 0202202        | 0020020         | 55              | 2               | $A_1$           | 11              |
| 12  | 0004004        | 0020020         | 55              | 2               | $A_1$           | 11              |
| 13  | 0202404        | 2020020         | 56              | 2               | $2A_1$          | 7               |
| 14  | 0400408        | 2020020         | 56              | 2               | $2A_1$          | 7               |
| 15  | 4004000        | 0002020         | 57              | 1               | $A_1$           | 9               |
| 16  | 0404004        | 2002020         | 59              | 1               | $T_1$           | 9               |
| 17  | 0404408        | 2022020         | 60              | 1               | $A_1$           | 6               |
TABLE 12

\((-1)\)-distinguished \(K\)-orbits \(K \cdot x\) in \(\mathcal{N}(p)\) for \(g_\mathbb{R} = E_7(7)\)

| No. | Dyn \(K \cdot x\) | Dyn \(G \cdot x\) | dim\(_C\) \(K \cdot x\) | \(\mathfrak{z}(G \cdot x \cap p)/K\) | \(\mathfrak{z}_T(x)/\operatorname{rad}_{\mathfrak{z}_T}(x)\) | dim\(_C\) \(\operatorname{rad}_{\mathfrak{z}_T}(x)\) |
|-----|------------------|------------------|------------------|------------------|------------------|------------------|
| 1   | 4000000          | 0200000          | 42               | 4                | \(G_2\)           | 7                |
| 2   | 0000004          | 0200000          | 42               | 4                | \(G_2\)           | 7                |
| 3   | 0040000          | 0000200          | 50               | 4                | \(A_1\)           | 10               |
| 4   | 0000400          | 0000200          | 50               | 4                | \(A_1\)           | 10               |
| 5   | 3101021          | 1001010          | 52               | 3                | \(T_2\)           | 9                |
| 6   | 1201013          | 1001010          | 52               | 3                | \(T_2\)           | 9                |
| 7   | 4004000          | 2000200          | 54               | 4                | \(A_1\)           | 6                |
| 8   | 0004004          | 2000200          | 54               | 4                | \(A_1\)           | 6                |
| 9   | 2220202          | 0002002          | 56               | 4                | 0                | 7                |
| 10  | 2020222          | 0002002          | 56               | 4                | 0                | 7                |
| 11  | 0400400          | 0002002          | 56               | 4                | 0                | 7                |
| 12  | 0040040          | 0002002          | 56               | 4                | 0                | 7                |
| 13  | 2222202          | 2002002          | 58               | 4                | 0                | 5                |
| 14  | 2022222          | 2002002          | 58               | 4                | 0                | 5                |
| 15  | 4004040          | 2002002          | 58               | 4                | 0                | 5                |
| 16  | 0404004          | 2002002          | 58               | 4                | 0                | 5                |
| 17  | 4220224          | 2002020          | 59               | 2                | \(T_1\)           | 3                |
| 18  | 2422222          | 2002022          | 60               | 4                | 0                | 3                |
| 19  | 2222242          | 2002022          | 60               | 4                | 0                | 3                |
| 20  | 4404040          | 2002022          | 60               | 4                | 0                | 3                |
| 21  | 0404044          | 2002022          | 60               | 4                | 0                | 3                |
| 22  | 4404040          | 2220202          | 61               | 2                | 0                | 2                |
| 23  | 4044044          | 2220202          | 61               | 2                | 0                | 2                |
| 24  | 4444044          | 2220222          | 62               | 2                | 0                | 1                |
| 25  | 4404444          | 2220222          | 62               | 2                | 0                | 1                |
| 26  | 8444444          | 2222222          | 63               | 2                | 0                | 0                |
| 27  | 4444448          | 2222222          | 63               | 2                | 0                | 0                |
TABLE 13

(−1)-distinguished \(K\)-orbits \(K \cdot x\) in \(N(p)\) for \(g_R = E_7(-25)\)

| No. | \(\text{Dyn } K \cdot x\) | \(\text{Dyn } G \cdot x\) | \(\text{dim}_C K \cdot x\) | \(\sharp(G \cdot x \cap p)/K\) | \(\chi(x)/\text{rad}_u\chi(x)\) | \(\text{dim}_C \text{rad}_u\chi(x)\) |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | 00000002        | 00000002        | 27              | 4               | \(F_4\)         | 0               |
| 2   | 00000002−2      | 00000002        | 27              | 4               | \(F_4\)         | 0               |
| 3   | 010010−2        | 1000010         | 38              | 2               | \(A_3 + T_1\)   | 25              |
| 4   | 011000−3        | 100010          | 38              | 2               | \(A_3 + T_1\)   | 25              |
| 5   | 200002−2        | 2000002         | 43              | 4               | \(B_3\)         | 15              |
| 6   | 400000−2        | 2000002         | 43              | 4               | \(B_3\)         | 15              |
| 7   | 000004−6        | 2000002         | 43              | 4               | \(B_3\)         | 15              |
| 8   | 200002−6        | 2000002         | 43              | 4               | \(B_3\)         | 15              |
| 9   | 220002−6        | 2000020         | 50              | 1               | \(A_2 + T_1\)   | 20              |
| 10  | 400004−6        | 2000022         | 51              | 2               | \(G_2\)         | 14              |
| 11  | 400004−10       | 2000022         | 51              | 2               | \(G_2\)         | 14              |

TABLE 14

(−1)-distinguished \(K\)-orbits \(K \cdot x\) in \(N(p)\) for \(g_R = E_8(8)\)

| No. | \(\text{Dyn } K \cdot x\) | \(\text{Dyn } G \cdot x\) | \(\text{dim}_C K \cdot x\) | \(\sharp(G \cdot x \cap p)/K\) | \(\chi(x)/\text{rad}_u\chi(x)\) | \(\text{dim}_C \text{rad}_u\chi(x)\) |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | 40000000        | 20000000        | 78              | 3               | \(2G_2\)        | 14              |
| 2   | 00000004        | 02000000        | 92              | 3               | \(A_2\)         | 20              |
| 3   | 21010100        | 00010001        | 96              | 3               | \(A_1 + T_1\)   | 20              |
| 4   | 00400000        | 00002000        | 97              | 2               | \(2A_1\)        | 17              |
| 5   | 40000040        | 02000002        | 99              | 3               | \(A_2\)         | 13              |
| 6   | 20200200        | 00002000        | 104             | 3               | \(0\)           | 16              |
| 7   | 00004000        | 00002000        | 104             | 3               | \(0\)           | 16              |
| 8   | 02002002        | 00002000        | 104             | 3               | \(0\)           | 16              |
| 9   | 40040000        | 20002000        | 105             | 2               | \(2A_1\)        | 9               |
| 10  | 02002022        | 00020002        | 107             | 3               | \(T_1\)         | 12              |
| 11  | 00400040        | 00020020        | 107             | 3               | \(T_1\)         | 12              |
| 12  | 31010211        | 10010101        | 108             | 2               | \(T_1\)         | 11              |
| 13  | 31111101        | 10010102        | 109             | 2               | \(T_1\)         | 10              |
| 14  | 20202022        | 00022000        | 110             | 2               | \(T_1\)         | 10              |
| 15  | 04004000        | 00020002        | 110             | 2               | \(T_1\)         | 10              |
| 16  | 02022022        | 20002002        | 111             | 3               | \(T_1\)         | 8               |
| 17  | 40040040        | 20002002        | 111             | 3               | \(T_1\)         | 8               |
### TABLE 14 (continued)

$(-1)$-distinguished $K$-orbits $K \cdot x$ in $\mathcal{N}(p)$ for $g_R = E_{6(8)}$

| No. | Dyn $K \cdot x$ | Dyn $G \cdot x$ | dim$_K K \cdot x$ | $|G \cdot x \cap p|/K$ | $\sharp x/K$ | $\text{rad}_u \text{rad}_x(x)$ |
|-----|-----------------|-----------------|------------------|------------------|---------------|-----------------------------|
| 18  | 004000040       | 000200020       | 112              | 2                | 0             | 8                           |
| 19  | 222200222       | 000200020       | 112              | 2                | 0             | 8                           |
| 20  | 222200222       | 000200022       | 113              | 2                | 0             | 7                           |
| 21  | 000040040       | 000200022       | 113              | 2                | 0             | 7                           |
| 22  | 222220222       | 200200202       | 114              | 2                | 0             | 6                           |
| 23  | 400400040       | 200200202       | 114              | 2                | 0             | 6                           |
| 24  | 222220242       | 200200202       | 115              | 2                | 0             | 5                           |
| 25  | 000400044       | 200200202       | 115              | 2                | 0             | 5                           |
| 26  | 222220222       | 200200202       | 116              | 2                | 0             | 4                           |
| 27  | 440404000       | 200200202       | 116              | 2                | 0             | 4                           |
| 28  | 242220242       | 200200222       | 117              | 2                | 0             | 3                           |
| 29  | 440404400       | 200200222       | 117              | 2                | 0             | 3                           |
| 30  | 444404444       | 222200222       | 118              | 1                | 0             | 2                           |
| 31  | 444404444       | 222022222       | 119              | 1                | 0             | 1                           |
| 32  | 844444444       | 222222222       | 120              | 1                | 0             | 0                           |

### TABLE 15

$(-1)$-distinguished $K$-orbits $K \cdot x$ in $\mathcal{N}(p)$ for $g_R = E_{8(-24)}$

| No. | Dyn $K \cdot x$ | Dyn $G \cdot x$ | dim$_K K \cdot x$ | $|G \cdot x \cap p|/K$ | $\sharp x/K$ | $\text{rad}_u \text{rad}_x(x)$ |
|-----|-----------------|-----------------|------------------|------------------|---------------|-----------------------------|
| 1   | 000000004       | 000000002       | 57               | 3                | $E_6$         | 1                           |
| 2   | 000000204       | 000000020       | 83               | 3                | $D_4$         | 25                          |
| 3   | 000000040       | 000000020       | 83               | 3                | $D_4$         | 25                          |
| 4   | 000000048       | 000000022       | 84               | 2                | $F_4$         | 0                           |
| 5   | 01000012        | 100000100       | 89               | 1                | $B_2 + T_1$   | 36                          |
| 6   | 400000004       | 20000002        | 90               | 2                | $A_4$         | 22                          |
| 7   | 10100111        | 100000101       | 94               | 1                | $A_2 + T_1$   | 33                          |
| 8   | 01100034        | 100000102       | 95               | 1                | $A_3$         | 26                          |
| 9   | 000200000       | 000002000       | 97               | 1                | $2A_1$        | 33                          |
| 10  | 20000222        | 200000202       | 99               | 2                | $G_2$         | 23                          |
| 11  | 00000404        | 200000202       | 99               | 2                | $G_2$         | 23                          |
| 12  | 20000244        | 200000222       | 100              | 2                | $B_3$         | 15                          |
| 13  | 40000048        | 200000222       | 100              | 2                | $B_3$         | 15                          |
| No. | Dyn $K \cdot x$ | Dyn $G \cdot x$ | dim$_K K \cdot x$ | $\sharp(G \cdot x \cap p)/K$ | $\mathfrak{g}_t(x)/\text{rad}_u \mathfrak{g}_t(x)$ | dim$_C \text{rad}_u \mathfrak{g}_t(x)$ |
|-----|----------------|----------------|-----------------|-----------------|-----------------|-----------------|
| 14  | 00020200       | 20000200       | 105             | 1               | $2A_1$          | 25              |
| 15  | 40000404       | 20000202       | 107             | 1               | $A_2$           | 21              |
| 16  | 40000448       | 20000222       | 108             | 1               | $G_2$           | 14              |

TABLE 16

$(-1)$-distinguished $K$-orbits $K \cdot x$ in $\mathcal{N}(p)$ for $g_R = F_4(4)$

| No. | Dyn $K \cdot x$ | Dyn $G \cdot x$ | dim$_K K \cdot x$ | $\sharp(G \cdot x \cap p)/K$ | $\mathfrak{g}_t(x)/\text{rad}_u \mathfrak{g}_t(x)$ | dim$_C \text{rad}_u \mathfrak{g}_t(x)$ |
|-----|----------------|----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | 0004           | 2000           | 15              | 3               | $A_2$           | 1               |
| 2   | 0040           | 0200           | 20              | 3               | 0               | 4               |
| 3   | 0204           | 0200           | 20              | 3               | 0               | 4               |
| 4   | 2022           | 0200           | 20              | 3               | 0               | 4               |
| 5   | 2200           | 0048           | 21              | 2               | $A_1$           | 0               |
| 6   | 0404           | 0202           | 22              | 2               | 0               | 2               |
| 7   | 2222           | 0202           | 22              | 2               | 0               | 2               |
| 8   | 2244           | 2202           | 23              | 2               | 0               | 1               |
| 9   | 4048           | 2202           | 23              | 2               | 0               | 1               |
| 10  | 4448           | 2222           | 24              | 1               | 0               | 0               |

TABLE 17

$(-1)$-distinguished $K$-orbits $K \cdot x$ in $\mathcal{N}(p)$ for $g_R = F_4(-20)$

| No. | Dyn $K \cdot x$ | Dyn $G \cdot x$ | dim$_K K \cdot x$ | $\sharp(G \cdot x \cap p)/K$ | $\mathfrak{g}_t(x)/\text{rad}_u \mathfrak{g}_t(x)$ | dim$_C \text{rad}_u \mathfrak{g}_t(x)$ |
|-----|----------------|----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | 0001           | 0001           | 11              | 1               | $A_3$           | 10              |
| 2   | 4000           | 0002           | 15              | 1               | $G_2$           | 7               |

TABLE 18

$(-1)$-distinguished $K$-orbits $K \cdot x$ in $\mathcal{N}(p)$ for $g_R = G_2(2)$

| No. | Dyn $K \cdot x$ | Dyn $G \cdot x$ | dim$_K K \cdot x$ | $\sharp(G \cdot x \cap p)/K$ | $\mathfrak{g}_t(x)/\text{rad}_u \mathfrak{g}_t(x)$ | dim$_C \text{rad}_u \mathfrak{g}_t(x)$ |
|-----|----------------|----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | 22             | 02             | 5               | 2               | 0               | 1               |
| 2   | 04             | 02             | 5               | 2               | 0               | 1               |
| 3   | 48             | 22             | 6               | 1               | 0               | 0               |
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