Covariance Properties of Reflection Equation Algebras

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The reflection equations (RE) are a consistent extension of the Yang-Baxter equations (YBE) with an addition of one element, the so-called reflection matrix or \( K \)-matrix. For example, they describe the conditions for factorizable scattering on a half line just like the YBE give the conditions for factorizable scattering on an entire line. The YBE were generalized to define quadratic algebras, 'Yang-Baxter algebras' (YBA) which were used intensively for the discussion of quantum groups. Similarly, the RE define quadratic algebras, 'the reflection equation algebras' (REA), which enjoy various remarkable properties both new and inherited from the YBA. Here we focus on the various properties of the REA, in particular, the quantum group comodule properties, generation of a series of new solutions by composing known solutions, the extended REA and the central elements, etc.

§ 1. Introduction

Many remarkable properties of the Yang-Baxter equations (YBE) and Yang-Baxter Algebras (YBA), in particular, the fruitful application of the \( R \)-matrix approach to the quantum group theory,\(^1\) have prompted the study of natural extensions of these concepts. The reflection equations (RE) appeared in the factorized scattering on a half line,\(^2\) and its algebraic structure, the reflection equation algebras (REA) and various \( c \)-number solutions are studied in our previous papers.\(^3\),\(^4\) The RE and REA appeared in papers dealing with quite different topics: in quantum groups and quantum algebras, in generalizing the concept of commuting transfer matrix to non-periodic boundary conditions on lattice models,\(^5\),\(^6\) in modified Knizhnik-Zamolodchikov equations,\(^7\) in non-commutative differential geometry,\(^8\),\(^9\) in quantum Liouville theory,\(^10\) in the lattice Kac-Moody algebras,\(^11\)\(^13\) in quantum homogeneous spaces,\(^14\) in integrable finite dimensional systems,\(^15\) in the generalization of the braid group to manifolds of non-trivial topology\(^16\),\(^4\) and in "braided groups",\(^17\) etc.

In this paper we discuss various properties of the quadratic algebras related with reflection equations (REA). The main purpose is an extension of REA by generalizing the coincidence of RE with the additional relation of the braid group of solid handlebody (2·23). Namely the \( K \)-matrix of RE corresponds to \( \tau \) (2·24), an additional generator of the braid group of solid handlebody which indicates the path of the first string going around the "hole". For a manifold of higher genus \( (g > 1) \) we have braid group generators \( \tau, \tau', \) etc., corresponding to different "holes". They, in turn,
correspond to $K, K'$, etc. which generate two (many) copies of REA's. Many copies of REA's thus generated do not commute with each other because of the consistency conditions under the quantum group coaction. The consistent relationships among $K$ and $K'$, etc. together with the construction of new solutions satisfying the REA by composing various known solutions constitute the main content of the extended REA. The central elements of the (extended) REA are derived in full generality and their properties are discussed. The basic idea is the covariance under the associated quantum group coaction. In other words we treat the REA as typical examples of the “quantum group tensors”. This makes it easy to generalize the concepts and applications of REA to a wider class of objects and algebras, which will be discussed in some detail.

The present paper is organized as follows. In § 2 we give a short description of the fields in which the YBE and YBA are useful. It is pointed out that the RE and related quadratic algebras have the same range of applications. This section is for setting a proper stage for the subsequent discussion and for introducing appropriate notation. In § 3 we treat the “extended REA”, namely the multiple copies of the REA as mentioned above, from the viewpoint of the braid group for higher genus manifolds. In § 4 we discuss the quantum group tensors and give various examples. We show that many properties of REA and its related algebras can be lucidly understood in this framework, in particular those discussed in § 3. In § 5 the central elements of the REA and the “extended REA” are discussed. In § 6 we discuss the “dual” of the RE, which helps to generalize the concept of the commuting transfer matrix for non-periodic boundary conditions on lattice statistical models. The “quantum group tensor” viewpoint again gives a lucid understanding of this concept and its generalization. The role played by non-trivial $c$-number solutions of RE is emphasized here. Section 7 contains a summary and comments.

§ 2. Summary of Yang-Baxter equations (algebras) and reflection equation (algebras)

The reflection equations (RE) and reflection equation algebras (REA) are extension of the Yang-Baxter equation (YBE) and Yang-Baxter algebras (YBA), which play very important roles in solving various problems in many branches of theoretical and mathematical physics.

2.1. YBE and YBA: Summary

Let us briefly review these well known subjects for the purpose of setting a suitable stage for the discussion of RE and REA. We will mainly follow the notation of our previous paper.4)

The YBE and YBA appeared in
i) One dimensional quantum chains.
For example the Heisenberg spin chain as investigated by Bethe18) and the “Bethe ansatz” for determining the energy eigenvalues.

ii) Factorizable scattering in $1+1$ dimensions.
Here the YBE appears as the conditions for factorizability on an entire line. Many
explici examples are known.\textsuperscript{19)}

iii) Statistical lattice models in 2 dimensions. For example, the Ising model and various vertex models and others.\textsuperscript{20)} The existence of the commuting transfer matrices is one of the key points in this context.

iv) Braid groups.

The interaction among YBA, conformal field theory, braid group and knot theory has produced quite fruitful results.\textsuperscript{21)-24)}

Let us introduce various notions by using the language of factorized scattering. By the assumption of the existence of an infinite set of quantum conserved quantities all inelastic processes are forbidden. The set of outgoing momenta is the same as the set of incoming momenta. The $S$-matrix element for an $M$ particle elastic scattering is factorized into a product of $M(M-1)/2$ two-particle $S$-matrix elements. In particular, for a three body $S$-matrix, there are two ways of factorization (Fig. 1) which should be equal. This equality gives rise to the Yang-Baxter equation. The incoming or outgoing particles are in one of the $N$-different internal states specified by $i$ and $j$, $1 \leq i, j \leq N$. The energy-momenta of the relativistic particles are conveniently parameterized by the 'rapidity' $p_a = m(\cosh \theta_a, \sinh \theta_a)$ thanks to the two dimensionality. Here $\theta_a$ is the rapidity of the $a$-th particle ($a=1, 2, 3$). The mass $m$ is assumed to be the same for all the particles. The internal state index $i$ and $j$ can change after the scattering but the rapidities cannot due to the elasticity. The $S$-matrix $R_{ab}$, $a, b=1, 2, 3$, being an $N^2 \times N^2$ matrix of the internal indices, depends only on the difference of the rapidities $\theta_a - \theta_b$ because of the Lorentz invariance. The Yang-Baxter equation corresponding to Fig. 1 can be expressed symbolically as

$$R(\theta_1 - \theta_2)_{12}R(\theta_1 - \theta_3)_{13}R(\theta_2 - \theta_3)_{23} = R(\theta_2 - \theta_3)_{23}R(\theta_1 - \theta_3)_{13}R(\theta_1 - \theta_2)_{12} . \quad (2.1)$$

Namely YBE is a tensor equation acting on the tensor product of three linear spaces $V_1 \otimes V_2 \otimes V_3$ ($V_1 = V_2 = V_3 = C^N$) and the suffices indicate the spaces on which each $S$-matrix works. If we write the matrix indices explicitly (suppressing the rapidities) it reads

$$R_{i_1i_2k_1k_212}R_{k_1i_3j_1j_313}R_{k_2k_3j_2j_323} = R_{i_3i_2k_3k_223}R_{k_2k_31213}R_{k_1i_2j_1j_212} , \quad (2.2)$$

in which the repeated indices $(k_1, k_2, k_3)$ are summed, as usual.

Let us introduce another form of the YBE which is frequently used in connection

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Fig. 1.}
\end{figure}
with the braid group
\[ \tilde{R}_{12} \tilde{R}_{23} \tilde{R}_{12} = \tilde{R}_{23} \tilde{R}_{12} \tilde{R}_{23}, \]  \hspace{1cm} (2.3)

in which \( \tilde{R}_{12} = \mathcal{P}_{12} R_{12} \) and \( \mathcal{P}_{12} \) is the permutation operator of the first and the second spaces. Namely \( \mathcal{P} u \otimes v = v \otimes u \). In terms of the indices it reads \( \mathcal{P}_{ijk} = \delta_{ik} \delta_{jk} \).

There is only one step between the YBE and the YBA. We only have to consider the third space as a 'quantum' space or treat the indices in the third space as 'invisible'
\[ R_{12} T_1 T_2 = T_2 T_1 R_{12}, \]  \hspace{1cm} (2.4)

in which \( T_1 (T_2) \) is a matrix working on the first (second) space and a linear operator working on the quantum space. This defines a quadratic algebra generated by \( T_\mu \), each is a linear operator on the quantum space. The YBA has an important algebra homomorphism, called comultiplication
\[ \Delta T_\mu = \sum_k T_{ik} \otimes T_{kj}, \]  \hspace{1cm} (2.5)

in which \( \otimes \) means that the two \( T \)'s belong to different quantum spaces or that they are 'independent' physical operators thus commuting with each other. Alternatively one can express this by introducing a different symbol for the second (first) quantum space operator
\[ \Delta T_\mu = \sum_k T_{ik} T'_k , \quad [T_\mu , T_{kl}] = 0. \]

Obviously the coproduct can be applied an arbitrary number of times. Let us apply it \( L \) times, \( L \) being the lattice size of a lattice problem, then (2.4) reads
\[ R(\theta_1 - \theta_2)_{12} \{ T T' \cdots T^{(L)}(\theta_1) \} \{ T T' \cdots T^{(L)}(\theta_2) \}_2 \]
\[ = \{ T T' \cdots T^{(L)}(\theta_2) \}_2 \{ T T' \cdots T^{(L)}(\theta_1) \} \{ R(\theta_1 - \theta_2)_{12} \}, \]  \hspace{1cm} (2.6)

in which the 'rapidity' dependence (in this context the 'spectral parameter' \( \lambda \) might be preferred to the 'rapidity' but we stick to the convention) is shown explicitly. By multiplying \( R_{12}' \) from the left and by taking the trace in the first and the second spaces we get
\[ [t(\theta_1), t(\theta_2)] = 0 , \quad t(\theta) = \text{Tr}((T T' \cdots T^{(L)}(\theta))). \]  \hspace{1cm} (2.7)

This gives the (one parameter \( \theta \) family of) commuting transfer matrix, which is the corner stone for the solvable lattice models. It should be remarked that the trace obviously corresponds to the periodic boundary condition in one direction of the lattice. The generalization of the commuting transfer matrix for non-periodic boundary conditions can be achieved in terms of REA and will be discussed in § 6.

For most of the discussion in this paper we deal with the 'rapidity independent' Yang-Baxter equations and Yang-Baxter algebras. Here is the simplest example of the solution of the 'rapidity independent YBE associated with \( GL_q(2) \) in the fundamental representation,
in which $q$ is a complex deformation parameter. We adopt the convention that any matrix elements not written explicitly are zeros. Then the YBA (2.4) determines a quadratic algebra generated by a $2 \times 2$ matrix $T$, which is the well-known $GL_q(2)$ quantum group:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}: \quad ab = qba, \quad bd = qdb, \quad ad - qbc = da - q^{-1}bc,$$

$$ac = qca, \quad cd = qdc, \quad bc = cb.$$  \hspace{1cm} (2.9)

At the end of the review of the YBE and YBA let us discuss the braid group. The braid group with $n$ strands is generated by $n - 1$ generators $\{\sigma_i\}$ with the following set of relations (Fig. 2):

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2,$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \hspace{1cm} (2.10)$$

These are the relations implied by the YBE (2.3) by the identification $\sigma_i = \tilde{R}_{ii+1}$.

2.2. RE and REA: Summary

Here we briefly summarize the basic properties of the Reflection Equations (RE) and the Reflection Equation Algebras (REA) in connection with YBE and YBA.

The most natural stage to extend the YBE to the RE is the factorizable scattering
with reflection at the end of a line, namely the factorizable scattering on a half line. In this case the factorizability conditions consist of two parts, the first part is the YBE itself (2.1) and the second part contains an additional element, called \( K \)-matrix, describing the reflection at the endpoint of the half line (Fig. 3).\(^2\)

\[
R^{(0)}_1 K_1 R^{(0)}_2 K_2 = K_2 R^{(0)}_1 K_1 R^{(0)}_2 .
\]  

(2.11)

Here \( K \) is an \( N \times N \) matrix and the various (rapidity dependent) \( R \)-matrices \( R^{(1)}, \ldots, R^{(4)} \) are related to the \( R \)-matrix in the YBE and will be specified later. For notational simplicity the rapidity dependence is suppressed. (The suffixes of the \( R \)-matrices like \( R = R_{12} \) indicating the base space \( V_1 \otimes V_2 \) will be suppressed in most cases.) As in the YBE case we mainly discuss the RE and REA in the rapidity independent form.

The above equation can be considered as a quadratic equation for \( N \times N \) \( c \)-number entries of \( K \) for a given solution \( R \) of YBE. On the other hand they can be considered as the defining relations of a quadratic algebra (Reflection Equation Algebra) generated by \( K \) just like the YBA. Then the \( c \)-number solutions can be considered as the one-dimensional representations of the algebra.

We require the REA to be closely related with the YBA, namely \( K \) to be a “quantum group comodule”. In other words the REA should be invariant under either of the following transformations:

\[
K \rightarrow \delta(K) = \begin{cases} 
TKT^t & [K_{ti}, T_{ki}] = 0 .
\end{cases}
\]  

(2.12)

This requirement leads to two groups of RE, called RE1 and RE2\(^3,4\) for which we write the standard forms. For RE1, namely for \( \delta(K) = TKT^t \):

\[
R\tilde{K}_1 R^\dagger_1 = \tilde{K}_2 R^\dagger_1 K_1 R^{t_1 t_2},
\]  

(2.13)

and for RE2, namely for \( \delta(K) = TKT^{-1} \):

\[
RK_1 \tilde{K} K_2 = K_2 RK_1 \tilde{K}.
\]  

(2.14)

In order to distinguish these two groups of RE's we put a bar on the \( K \)-matrix appearing in the RE1. Here \( t_i \) means transposition in the first space and \( \tilde{A} = \mathcal{P} A \mathcal{P} \) for any matrix on \( V_1 \otimes V_2 \). The other members of the group are obtained by simply replacing any \( R \) by \( \tilde{R}^{-1} \).\(^3\) However, we mainly discuss these forms of the RE's.

Let us give the explicit forms of the REA's for the \( G_{L_q(2)} \) \( R \)-matrix given above (2.8). For the RE1:

\[
\tilde{K} = \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix}
\]  

(2.15)

which satisfy the relations\(^3,4\)

\[
[a, \beta] = \omega a \gamma , \quad [a, \delta] = \omega (q \beta \gamma + \gamma^2) , \quad [\beta, \delta] = \omega \gamma \delta , \quad a \gamma = q^2 \gamma a , \quad [\beta, \gamma] = 0 , \quad \gamma \delta = q^2 \delta \gamma .
\]  

(2.16)

This algebra has two central elements, linear and quadratic in \( \tilde{K} \).
The \( c \)-number solutions are
\[
\vec{K} = \varepsilon_q = \begin{pmatrix} 1 & q \\ -q & 1 \end{pmatrix}, \quad \vec{K} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\] (2.18)

For the RE2:
\[
\vec{K} = \begin{pmatrix} u & x \\ y & z \end{pmatrix},
\] (2.19)
we find that the component form of the algebra reads\(^{(17,4)}\)
\[
ux = q^{-2} xu, \quad [u, z] = 0, \quad [x, z] = -q^{-1} \omega ux, \quad uy = q^2 yu, \quad [x, y] = q^{-1} \omega (uz - u^2), \quad [y, z] = q^{-1} \omega yu.
\] (2.20)

The central elements of the above algebra are also known\(^{(41,8)}\) and they are invariant under the \( GL_q(2) \) coaction
\[
c_1 = u + q^2 z, \quad c_2 = uz - q^2 yx.
\] (2.21)

The corresponding \( c \)-number solutions are
\[
\vec{K} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \vec{K} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\] (2.22)

It is interesting to note that the RE2, a generalization of the YBE, has also an interpretation of the braid group for solid handlebody (i.e., genus 1)\(^{(16,17,27,4)}\). Here we need to introduce one additional generator \( \tau \), which corresponds to a single turn around a hole (Fig. 4). As is shown in Fig. 5 it satisfies a relation
\[
\sigma_1 \tau \sigma_1 \tau = \tau \sigma_1 \tau \sigma_1.
\] (2.23)

This is just relation (2.14) if we identify
\[
\vec{K}_1 = \tau, \quad \vec{R} = \sigma_1.
\] (2.24)
§ 3. Extended reflection equation algebras

The above relationship between the REA and the braid group for solid hand­
lebody \(g=1\) (2·24) can be readily generalized to a braid group \(B_n^g\) for a manifold of an arbitrary genus \(g\). The corresponding REA, to be called “extended REA” consists of \(g\)-copies of REA, \(K^{(1)}, \ldots, K^{(g)},\)

\[
RK_1^{(j)} \bar{R}K_2^{(j)} = K_2^{(j)} RK_1^{(j)} \bar{R}, \quad j=1, \ldots, g.
\] (3·1)

The mutual relationship among various copies should also be covariant under the “quantum group comodule” \(K \rightarrow \delta(K) = TKT^{-1}\). This induces non-trivial relations among them, since the trivial ones (i.e., simply commuting) are not preserved by the comodule

\[
[\delta(K_1^{(j)}), \delta(K_2^{(j)})] \neq 0, \quad \text{even if } [K_1^{(j)}, K_2^{(j)}] = 0.
\]

We are led to the following relations between \(K^{(i)}\) and \(K^{(j)}\) which are “linearly ordered” (i.e., non-symmetric in \(i\) and \(j\)),

\[
K^{(i)} > K^{(j)}, \quad i > j \Leftrightarrow R^{-1} K_2^{(j)} RK_1^{(i)} = K_1^{(i)} R^{-1} K_2^{(j)} R.
\] (3·2)

By keeping the “order” in the matrix multiplication we can “compose” \(K^{(i,j)} = K^{(i)} K^{(j)}, \quad i > j\) which also satisfies the REA,

\[
RK_1^{(i,j)} \bar{R}K_2^{(i,j)} = K_2^{(i,j)} RK_1^{(i,j)} \bar{R}.
\] (3·3)

It should be remarked that the unit element \(K=1\) always satisfies both (3·1) and (3·2). The above “composition rule” and the “linear order” and most of the composition rules below are easily understood pictorially by the following identification \(K^{(j)} \Leftrightarrow \gamma_j\) (Figs. 6 ~ 8) of the extended REA with \(B_n^g\) \((g \geq 1)\). The above relation (3·2) can be rewritten in an equivalent form

\[
RK_1^{(i)} R^{-1} K_2^{(j)} = K_2^{(j)} RK_1^{(i)} R^{-1}.
\] (3·4)

The extended REA has several interesting algebraic properties. After the composition the linear order is preserved. Namely,

\[
K^{(i,j)} > K^{(k)} \quad \text{for } i > j > k.
\] (3·5)

This implies the second (and further) step of composition is possible. Namely

\[
K^{(i,j,k)} = K^{(i,j)} K^{(k)}
\]

\[
= (K^{(i)} K^{(j)}) K^{(k)}
\] (3·6)

also satisfies the REA. The linear order is preserved also in the other direction

\[
K^{(i)} > K^{(j,k)} \quad \text{for } i > j > k.
\] (3·7)
which results in another composition
\[ K^{(i,j,h)} = K^{(i)} K^{(j,h)} = K^{(i)} (K^{(j)} K^{(h)}) , \tag{3.8} \]
By the associativity we have simply
\[ K^{(i,j,h)} = K^{(i,j,k)} = K^{(i)} K^{(j)} K^{(k)} , \quad i > j > k , \tag{3.9} \]
satisfying the REA.
A different type of composition is possible. Let us define a notation
\[ K^{(-i)} = (K^{(i)})^{-1} . \tag{3.10} \]
Then
\[ K^{(i,j,-i)} = K^{(i)} K^{(j)} K^{(-i)} \quad \text{for} \quad i > j \tag{3.11} \]
also satisfies the REA. The linear order is preserved also for this composition. Namely
\[ K^{(i,j,-i)} > K^{(k)} \quad \text{for} \quad i > j > k \tag{3.12} \]
and
\[ K^{(i)} > K^{(j,k,-j)} \quad \text{for} \quad i > j > k. \] (3.13)

That is \( K^{(i)} K^{(j)} K^{(-i)} K^{(k)} \) and \( K^{(i)} K^{(j)} K^{(k)} K^{(-j)} \) satisfy the REA.

Yet another type of composition is possible. Namely
\[ K^{(-j,i,j)} = K^{(-j)} K^{(i)} K^{(j)} \quad \text{for} \quad i > j \] (3.14)
also satisfies the REA. Correspondingly, the linear order is preserved also for this composition,
\[ K^{(-j,i,j)} > K^{(k)} \quad \text{for} \quad i > j > k \] (3.15)

and
\[ K^{(i)} > K^{(-j,k,j)} \quad \text{for} \quad i > j > k. \] (3.16)

That is \( K^{(-j)} K^{(i)} K^{(j)} K^{(-k)} \) and \( K^{(i)} K^{(-k)} K^{(j)} K^{(k)} \), \( i > j > k \) satisfy the REA.

Here we have demonstrated various explicit examples of the composition rules for the REA in the framework of the extended REA. One could discuss them from a slightly different point of view. Namely in terms of "corepresentation" and "twisted (braided) coproduct"\(^{(17)}\) for REA. Let \( V \) be a linear space. We introduce a "corepresentation" of REA \( \tilde{\delta}: V \rightarrow K \otimes V \) in such a way that the "twisted (braided) coproduct \( \tilde{\Delta} \)" for \( K \) is consistent with \( \tilde{\delta} \),
\[ (\tilde{\Delta} \otimes \text{id}) \circ \tilde{\delta} = (\text{id} \otimes \tilde{\delta}) \circ \tilde{\delta}, \] (3.17)
in which
\[ \tilde{\Delta}(K) = KK', \quad \tilde{\delta}(X) = KX. \quad (X \in V) \] (3.18)
Namely both sides of (3.17) give \((KK')X = K(K'X) = KK'X\). However, there seem to be many other cases of compositions than is naively suggested by the "twisted (braided) coproducts" (3.18).

Before closing this section, let us give a brief remark that the "extension" and "composition" of the REA can be generalized to a class of associative algebras proposed in Ref. 13). Let us take, for simplicity, the case corresponding to \( g=2 \). Namely we have two copies \( K \) and \( K' \) satisfying
\[ AK_1 \tilde{C} K_2 = K_2 C K_1 \tilde{A}, \]
\[ AK'_1 \tilde{C} K'_2 = K'_2 C K'_1 \tilde{A}. \] (3.19)

Here \( A \) and \( C \) should satisfy the YBE and its related equations.\(^{(13)}\) The special case \( A = C = R \) reduces to the REA and to the extension arguments given above. The relation between \( K \) and \( K' \) reads
\[ K > K' \quad \tilde{C}^{-1} K_1 \tilde{C} K_2 = K_2 \tilde{A}^{-1} K'_1 \tilde{C}, \] (3.20)
which can be rewritten as
\[ K_1 A^{-1} K'_2 C = C^{-1} K'_2 C K_1. \] (3.21)
Then as before \( KK' \) satisfies the algebra relations
\[
A(KK')_1 \tilde{C}(KK')_2 = (KK')_2 C(KK')_1 \tilde{A}.
\] (3.22)
It is interesting to check whether the other compositions of the extended REA work.

§ 4. Quantum group tensors

At first sight the relations among \( K(i), K(l) \) and their composition rules discussed in the previous section look quite strange algebraically and/or from the point of view of a physical model. Here we show that such relations are rather common among the "quantum group tensors" or "\( q \)-tensors" for short. This is quite a general and useful concept and it contains various relations of \( q \)-differential geometry (non-commutative differential geometry) as a particular subclass.

In general terms tensors \( A^{ij}_{...} \) are covariant and contravariant objects which transform
\[
A^{ij}_{...} \to (l)_{i'l'}(l)_{j'l'} ... (l^{-1})_{i'i''} (l^{-1})_{j'j''} ... A^{ij}_{...}.
\]
In this sense the \( K \)-matrices of REA are \( q \)-tensors owing to the comodule properties. Namely the \( \tilde{K} \) of RE1 is a rank \((2,0)\) \( q \)-tensor
\[
\tilde{K}_{ij} \to (T \tilde{K} T^{-1})_{ij} = T^{i'}_i T^{j'}_j \tilde{K}_{i'j'},
\] (4·1)
and \( K \) for RE2 is a rank \((1,1)\) \( q \)-tensor
\[
K_{ij} \to (T \tilde{K} T^{-1})_{ij} = T^{i'}_i T^{j'}_j K_{i'j'}.
\] (4·2)

An obvious difference from the ordinary tensors is that the transformation coefficients \( T_{ij} \) are not \( c \)-numbers but satisfy the YBA (2·4).

Let us begin with some explicit examples of the consistent set of \( q \)-tensors. The first example is the well known "\( q \)-hyperplanes" \( \{x_i\}, x_i x_j = q x_j x_i \) for \( 1 \leq i < j \leq N \), which can be written neatly in terms of the \( GL_q(N) \) \( R \)-matrix
\[
X = \{x_i\}, \quad RX_i X_j = q X_j X_i \quad \text{or} \quad \tilde{R} X_i X_j = q X_j X_i,
\] (4·3)
where \( \tilde{R} = \mathcal{D} R \). Another example is the "one-forms" on the \( q \)-hyperplanes \( \{\xi_i\}, \xi_i \xi_j = -(1/q) \xi_j \xi_i \) for \( 1 \leq i < j \leq N \). They can also be written by using the same \( R \)-matrix as above
\[
\mathcal{E} = \{\xi_i\}, \quad R \mathcal{E}_i \mathcal{E}_j = -(1/q) \mathcal{E}_j \mathcal{E}_i \quad \text{or} \quad \tilde{R} \mathcal{E}_i \mathcal{E}_j = -(1/q) \mathcal{E}_j \mathcal{E}_i.
\] (4·4)
In these examples \( q \) and \( -1/q \) on the right hand side are the eigenvalues of \( \tilde{R} \). These relations are invariant under the \( GL_q(N) \) group coaction:
\[
\delta: x_i \to T_{i'i} x_{i'}, \quad \xi_i \to T_{i'i} \xi_{i'}
\] (4·5)
for which they are assumed to commute with the elements of the quantum group matrix \( T_{ij}, [T_{ij}, x_k]=[T_{ij}, \xi_k]=0 \). They are rank one \( q \)-tensors or "\( q \)-vectors", for short. Now it is easy to give a general definition of the \( q \)-vector \( V^{(i)} \)
\[ V^{(1)} = \{ v_i \}, \quad RV^{(1)}V_2^{(1)} = a V_1^{(1)}V_2^{(1)} \quad \text{or} \quad \tilde{R}V^{(1)}V_2^{(1)} = a V_1^{(1)}V_2^{(1)}, \quad (4\cdot6) \]
in which \( R \) is an arbitrary \( R \)-matrix and \( a \) is one of the eigenvalues of \( \tilde{R} \). In other words \( V^{(1)}V_2^{(1)} \) lies in the eigenspace of \( \tilde{R} \) belonging to the eigenvalue \( a \). As is clear from the above examples, we get different algebras (having the same \( q \)-tensor properties) for different choices of eigenvalues of \( \tilde{R} \). The dimension of the eigenspace with respect to the total space determines the number of quadratic relations imposed by the condition \((4\cdot6)\). The relations \((4\cdot6)\) are invariant under the quantum group coaction specified by the \( R \)-matrix,

\[ \delta: v_i \rightarrow T_{ij}v_j. \quad \text{(4\cdot7)} \]
The relationships among various \( q \)-tensors are severely restricted by the requirement of consistency with the quantum group coaction. The situation is similar to the case of the extension of the reflection equation algebras in which many copies of \( K \) appear in \((3\cdot1)\) with nontrivial exchange relations \((3\cdot2)\) among them. One of the simplest such examples are those among the two \( q \)-vectors \( X \) and \( \Xi \) above,

\[ R\Xi X_2 = (1/q)X_2\Xi, \quad \text{or} \quad \tilde{R}\Xi X_2 = (1/q)X_1\Xi_2. \quad \text{(4\cdot8)} \]

These are the relations among the coordinates of the \( q \)-hyperplane and the corresponding one-forms, see, for example, Ref. 30).

One consistent exchange relation among two \( q \)-vectors, \( V^{(1)} \) and \( V'^{(1)} \), is obtained by generalizing the above example:

\[ \tilde{R}V'^{(1)}V_2^{(1)} = \beta V_1^{(1)}V_2^{(1)}, \quad \text{(4\cdot9)} \]
in which \( \beta \) is an arbitrary constant (to be determined by some requirements other than the quantum group covariance). Take, for example, \( N \) different copies of \( q \)-hyperplanes \( X'^{(j)}(1 \leq j \leq N), \quad X'^{(i)} = \{ x'^{(j)}, 1 \leq i \leq N \} \) and assume that they satisfy the above consistency relations with \( \beta = 1^{31,33} \).

\[ \tilde{R}X'^{(i)}X_2^{(j)} = X'^{(i)}X_2^{(j)}, \quad 1 \leq i < j \leq N. \quad \text{(4\cdot10)} \]

Then \( U_{ij} = x'^{(i)} \) satisfy exactly the same exchange relations as \( T_{ij} \in GL_q(N) \) and the coaction \( \delta U_{ij} \) is equivalent to the quantum group coproduct \( \Delta T_{ij} \). This is another characterization of the quantum group itself in terms of many copies of \( q \)-tensors.

Another example of the exchange relations for two (many) copies of \( q \)-vectors occurs in the following problem (see also Ref. 32)): to construct a \( q \)-plane as a sum of two \( q \)-planes, \( X'^{(1)} = (x^1, x^2), \quad X'^{(2)} = (x^3, x^4) \), namely,

\[ (x + w)(y + z) = q(y + z)(x + w). \]

This is achieved by taking \( \beta = 1/q \) (which is one of the eigenvalues of the corresponding \( \tilde{R} \) with a minus sign) in \((4\cdot9)\),

\[ \tilde{R}X'^{(2)}X_2^{(1)} = (1/q)X'^{(1)}X_2^{(2)}, \quad \text{(4\cdot11)} \]
or, more explicitly,

\[ xw = q^2wx, \quad xz = qx + wwy, \]
yw = qwy, \quad yz = q^2 yz.

The same problem for the one-forms is also conceivable, namely to form a one-form by a sum of two copies, $\mathcal{E}^{(1)} = (\xi, \zeta)$ and $\mathcal{E}^{(2)} = (\eta, \mu)$,

$$(\xi + \xi)^2 = (\eta + \mu)^2 = 0, \quad q(\xi + \xi)(\eta + \mu) + (\eta + \mu)(\xi + \xi) = 0.$$ 

This can be achieved by taking $\beta = -q$ (again one of the eigenvalues of $\hat{R}$ with a minus sign) in (4.9),

$$\hat{R}\mathcal{E}^{(2)}, \mathcal{E}^{(1)} = -q \mathcal{E}^{(1)} \mathcal{E}^{(2)}.$$

It is straightforward to generalize the above examples to the $GL_q(N)$ case.

It is straightforward to get a rank two $q$-tensor by taking a tensor product of two arbitrary rank one $q$-tensors. If we define $\hat{K}_{ij} = \nu_i \nu_j^\alpha$ then $\hat{K}$ is a rank two $q$-tensor whose coaction is given by

$$\tilde{\mathcal{V}}^{(2)} = \{\hat{K}_{ij}\} \text{ for RE1}, \quad \delta: \hat{K}_{ij} \rightarrow T_{ii} T_{jm} \hat{K}_{lm} = (T \hat{K} T^\dagger)_{ij}. \quad (4.12)$$

By combining (4.6) (both copies are assumed to have the same $a$) and (4.9) (for arbitrary $\beta$) it is easy to show that $\hat{K}$ satisfies the REI

$$R \hat{K}_{ij} R_i^{\gamma} \hat{K}_{\gamma j} = \hat{K}_s \hat{R}_s a \hat{R}_a a.$$

In the $GL_q(N)$ example above, if we choose $\hat{K} = X \otimes \mathcal{E}$ or $\hat{K}_{ij} = x_i \xi_j$, then it satisfies the RE1 with an extra minus sign on the right hand side.

For rank two $q$-tensors (that is for $(2, 0)$ $q$-tensors) one can introduce the notion of 'q-symmetry' and 'q-antisymmetry' analogous to the ordinary tensor calculus. For concreteness, let us discuss $GL_q(2)$ (the fundamental representation) case. A $q$-symmetric tensor $\hat{K}_s$ and a $q$-antisymmetric tensor $\hat{K}_a$ are defined as follows:

$$(\hat{K}_s)_{ij} = q(\hat{K}_s)_{ji}, \quad 1 \leq i < j \leq N, \quad (4.14)$$

$$q(\hat{K}_a)_{ij} + (\hat{K}_a)_{ji} = 0, \quad 1 \leq i < j \leq N, \quad (\hat{K}_a)_{ii} = 0. \quad (4.15)$$

It is straightforward to show that the above definitions of $\hat{K}_s$ and $\hat{K}_a$ are consistent with the $GL_q(N)$ coaction. The simplest examples are given by the $q$-hyperplanes and the one-forms,

$$(\hat{K}_s)_{ij} = x_i x_j, \quad (\hat{K}_a)_{ij} = \xi_i \xi_j.$$

An arbitrary rank two tensor $\hat{K}$ can be uniquely decomposed into a sum of a $q$-symmetric and a $q$-antisymmetric parts. This simply corresponds to the fact that the tensor product of two fundamental representations can be decomposed into a direct sum of the symmetric and the antisymmetric representations. For example, $\hat{K}$ for $GL_q(2)$ (see (2.16)) is decomposed as

$$\hat{K} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \hat{K}_s + \hat{K}_a,$$
\[ \tilde{K}_S = \begin{pmatrix} a & qS \\ S & \delta \end{pmatrix}, \quad \tilde{K}_A = \begin{pmatrix} 1 & \varepsilon_0A \\ -\varepsilon_0 & 1 \end{pmatrix} \]

\[ S = \frac{1}{1 + q^\beta} (q\beta + \gamma), \quad A = \frac{1}{1 + q^\beta} (\beta - q\gamma) = \frac{1}{1 + q^2} c_1, \]

in which \( c_1 \) is the central element of the REA (2.17). Thus in this simplest example, the antisymmetric part commutes with the symmetric part. Under the requirement of the vanishing antisymmetric part \((A = c_1 = 0)\) the symmetric part is related with the quantum homogeneous space.

As remarked above the \( K \)-matrices of REA are rank two (to be more precise a \((2,0)\) and a \((1,1)\) rank) \( q \)-tensors since their comodule properties are given by (4.1) and (4.2). It is interesting to note that the REA itself can be expressed in a similar form to (4.3), (4.4) and (4.6), \( ^{18,17,3} \)

\[ \mathcal{R} K_1 K_2 = K_2 K_1, \quad (4.16) \]

in which \( \mathcal{R} \) is a matrix acting on \( C^N \times C^N \) which can be expressed by a product of four ordinary \( R \)-matrices. This form of \( \mathcal{R} \) will be important for the commuting transfer matrix of lattice models under non-periodic boundary conditions to be discussed in § 6. For concreteness let us write (4.16) with full indices

\[ \mathcal{R}_{ijab; ij' uv'} K_{ij' uv'} = \mathcal{R}_{ijab; ij' uv'}. \]

For the RE1 the explicit form of \( \mathcal{R} \) is

\[ \mathcal{R}_{ijab; ij' uv'} = \tilde{R}_{ij' ab}^{ij} R_{ij' uv'} R_{ij' uv'} \]

and for the RE2

\[ \mathcal{R}_{ijab; ij' uv'} = (R_{ij' ab}^{ij})^{-1} R_{ij' uv'} R_{ij' uv'} \tilde{R}_{ij' uv'}. \]

One can also write down in a similar fashion the relations among \( K^{(i)} \) and \( V^{(i)} \) of the extended REA.

Now one can require the consistency among various kinds of \( q \)-tensors. For example an exchange relation between \( K_0 \) and \( V^{(1)} = \{ v_h \} \), a \( q \)-vector, reads

\[ \mathcal{R}^{(1,2)} V^{(i)}_1 K_2 = K_2 V^{(i)}_1. \]

The consistency with the quantum group coaction requires

\[ \mathcal{R}^{(1,2)} T_1 T_2 (T^{-1})_{ij} = T_2 (T^{-1})_{ij} T_1 \mathcal{R}^{(1,2)}. \]

An explicit form of \( \mathcal{R}^{(1,2)} \) satisfying (4.20) is given by

\[ \mathcal{R}^{(1,2)} = ((\tilde{R}_{ij' ab}^{ij})^{-1}) (\tilde{R}_{ij}^{ij})'^{-1}. \]

\[ (\tilde{R})^{-1} V^{(i)}_1 K_2 = K_2 (\tilde{R})^{-1} V^{(i)}_1. \]

Moreover there is a natural "coaction" of \( K \) on \( V^{(i)} \):

\[ (\tilde{R})^{-1} V^{(i)}_1 K_2 = K_2 (\tilde{R})^{-1} V^{(i)}_1. \]
\[ \delta: V^{(1)} \to KV^{(1)} = V^{(1)}. \] (4.23)

It is straightforward to verify that \( V^{(1)} \) satisfies (4.6) by using RE2 and (4.22). Namely they are again \( q \)-vectors. We might call this a “contraction” of a \((1,1)\) tensor with a \((1,0)\) tensor (a \( q \)-vector).

One can require the consistency between the two kinds of rank two \( q \)-tensors, \( K \) for RE2 and \( \bar{K} \) for RE1, which reads
\[ \bar{R}^{-1} \bar{K} R^{\varepsilon_1} K_2 = K_2 \bar{R}^{-1} \bar{K} R^{\varepsilon_1}. \] (4.24)

Similarly to the above case (4.23), there is a natural “coaction” \( K \) on \( \bar{V}^{(2)} \sim \bar{K} \):
\[ \delta: \bar{V}^{(2)} \to K \bar{V}^{(2)} = K \bar{K}. \] (4.25)

It is easy to show that \( K \bar{K} \) satisfies RE1 by using (4.24). As before, for the trivial element of the RE2 algebra, i.e., \( K = 1 \), the above consistency conditions among the \( q \)-tensors, (4.22) and (4.24), are trivially satisfied. The relation (4.25) may be considered again as the “contraction” of \( K \), a \((1,1)\) tensor, and \( \bar{K} \), a \((2,0)\) tensor, to form another \((2,0)\) tensor.

One can decompose a \((1,1)\) tensor (i.e., the \( K \)-matrix of RE2) into a matrix product of \((1,0)\) tensor say \( U^+ \) and \((0,1)\) tensor say \( U^- \),
\[ K = U^+ U^- . \] (4.26)

The component \( q \)-tensors should satisfy
\[ R U^+_1 U^+_2 = a U^+_2 U^+_1 , \quad U^-_1 \bar{R} = a U^-_1 U^-_2 , \] (4.27)
in which \( a \) is one of the eigenvalues of \( \bar{R} \) as before. The quantum group coaction is given by
\[ \delta: U^+ \to TU^+ , \quad U^- \to U^- T^{-1} . \] (4.28)

The consistency condition among them is
\[ U^-_1 \bar{R} U^+_2 = U^+_2 U^-_1 . \] (4.29)

This decomposition was also introduced in Ref. 13 for the ultra-localization of the lattice Kac-Moody algebra. The above decomposition (4.26) is definitely ‘invariant’ under the redefinition \( U^+ \to U^+ S \) and \( U^- \to S^{-1} U^- \). One can also rewrite (4.26) as in Ref. 13 \( K = U^+ K_0 U^- \), in which \( K_0 \) is an arbitrary \( c \)-number matrix. Then the above ‘gauge’ transformation should be augmented by \( K_0 \to S^{-1} K_0 S \).

Generalization of these examples and discussion to higher rank tensors are straightforward.

§ 5. Central elements of REA

In this section we will discuss the central elements of (extended) REA and/or of the general \( q \)-tensors. For concrete examples, see (2.17) and (2.21) in § 2. In the language of the \( q \)-tensors, the central elements correspond to various invariants of the \( q \)-tensors. In general, the central elements of an arbitrary YBA (2.4) are not easily
known. The best known examples are the so-called quantum determinants for certain YBA's. Therefore the central elements for general $q$-tensors are not easily calculable, except for the $(n, n)$ $q$-tensors. In these cases, for which the (extended) REA is the best example, the dependence on the YBA (or the quantum group) can be cancelled by choosing certain combination of the covariant and contravariant indices.

Let us consider an $N \times N$ matrix $\mathcal{D}$ which enjoys the following property:

$$\text{Tr}(\mathcal{D} M) = \text{Tr}(\mathcal{D} TMT^{-1}),$$

in which $M$ is an arbitrary $N \times N$ matrix and $T$ is the matrix of quantum group generators associated with an $R$-matrix (2.4). In other words, $\mathcal{D}$ defines an invariant trace with respect to the $(1, 1)$ $q$-tensor (or the comodule transformation of the RE2). In terms of $\mathcal{D}$ we can express the central elements of the REA

$$c_n(K) = \text{Tr}(\mathcal{D} K^n),$$

or more generally for the extended REA system

$$c_n(K^{ij}) = \text{Tr}(\mathcal{D} (K^{ij})^n), \quad 1 \leq j \leq g, \quad n: \text{integer}.$$  \hspace{1cm} (5·3)

Namely they satisfy

$$[c_n(K^{ij}), K^{ij}] = 0, \quad [c_n(K^{ij}), K^{kl}] = 0 \quad \text{for} \quad \forall i, \forall j.$$  \hspace{1cm} (5·4)

Similarly we can define the central elements for $K^{ij}$ as $c_n(K^{ij})$ in which $I$ is an ordered set of indices, for example, $I = (i, j, k), \quad i > j > k > l$,

$$[c_n(K^{ij}), K^{ij}] = 0, \quad [c_n(K^{ij}), K^{kl}] = 0.$$  \hspace{1cm} (5·5)

The explicit forms of $\mathcal{D}$ for some particular $R$-matrices have been known for sometime.\textsuperscript{1) Here we give a simple derivation of the explicit form of $\mathcal{D}$ for an arbitrarily given $R$-matrix, namely,

$$\mathcal{D} = \text{Tr}_2[\mathcal{P}((R^{i'})^{-1})^{i'}].$$

In order that (5·1) should hold for any matrix $M$, $\mathcal{D}$ should satisfy

$$\mathcal{D}^t = T^t \mathcal{D}^t (T^{-1})^t.$$  \hspace{1cm} (5·7)

Starting from (2.4) it is easy to get

$$(R^{i'})^{-1} T_2^{-1} T_1^t = T_1^t T_2^{-1} (R^{i'})^{-1}.$$  \hspace{1cm} (5·8)

By taking a sum over the second indices of the first and second spaces, $T_2^{-1}$ and $T_1^t$ on the left hand side cancel to each other to give

$$\mathcal{D}^t = T^t \mathcal{D}^t (T^{-1})^t,$$

in which

$$\mathcal{D}^t = \text{Tr}_2[\mathcal{P}((R^{i'})^{-1})^{i'}].$$

Namely we arrive at

$$\mathcal{D} = \text{Tr}_2[\mathcal{P}((R^{i'})^{-1})^{i'}].$$

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Namely we arrive at

$$\mathcal{D} = \text{Tr}_2[\mathcal{P}((R^{i'})^{-1})^{i'}].$$
Next we give a sketch of the proof that $c_n(K)$ in (5·2) are central in the RE2 algebra. By multiplying $K_2$ successively from the right to the RE2,

$RK_1 R K_2 = K_2 R K_1 R$,

we get

$RK_1 R K_2^n = K_2^n R K_1 R$, \hspace{1cm} n: integer

which is rewritten as

$	ilde{R} K_2 R K_1^n = K_1^n R K_2$, \hspace{1cm} (5·10)

and further as

$K_2 R K_1^n R^{-1} = R^{-1} K_1^n R K_2$. \hspace{1cm} (5·11)

Since $T = R$ and $T = R^{-1}$ satisfy the FRT relations (2·4) we see that (5·7) implies, in particular,

$\text{Tr}_1 (D_1 R M_1 R^{-1}) = \text{Tr}(D M) l_2$, \hspace{1cm} $\text{Tr}_1 (D_1 R^{-1} M_1 \tilde{R}) = \text{Tr}(D M) l_2$,

in which, as before, $M$ is an arbitrary matrix and $l_2$ is a unit matrix in the second space. Thus, by multiplying $D_1$ on both sides of (5·11) and taking the trace in the first space we obtain

$K_2 c_n(K) = c_n(K) K_2$

or

$[K, c_n(K)] = 0$. \hspace{1cm} (5·12)

It is straightforward to generalize the above centrality arguments to the extended REA. One only has to note that (3·4) implies

$RK_1^{(i)} R^{-1} (K^{(i)})_2^n = (K^{(i)})_2^n R K_1^{(i)} R^{-1}$. \hspace{1cm} (5·13)

This corresponds to (5·10) above. It should be remarked that for higher $n$, $c_n(K)$ is linearly dependent on lower $c_n(K)$'s owing to the (generalized) Cayley-Hamilton theorem for the matrix $K$ (see Ref. 31)).

 Suppose we have a consistent set of a (1, 1) and a (2, 0) tensor (i.e., $K$ of RE2 and $\tilde{K}$ of RE1) satisfying the condition (4·24). Then the central elements of the (1, 1) tensor (RE2 algebra) commute with the (2, 0) tensor ($\tilde{K}$ of RE1)

$[c_n(K), \tilde{K}] = 0$. \hspace{1cm} (5·14)

Here the following formula is useful:

$\tilde{R}^{-1} K_1 R^{t_1} K_2^n = K_2^n \tilde{R}^{-1} K_1 R^{t_1}$.

In a similar fashion we can show that the central elements of a (1, 1) tensor ($K$ of RE2) commute with the $q$-vector $V^{(i)}$ if (4·22) is satisfied.

A certain class of YBA is known to have some additional quadratic relations besides (2·4). For example, for the YBA based on the orthogonal and symplectic
$R$-matrix there exists a $c$-number matrix $C$ satisfying the relation

\[ TC T^t = C \quad \text{and} \quad T^t C T = C, \quad C^2 = \varepsilon I, \]  

in which $\varepsilon = 1$ for $B_n$ and $D_n$ whereas $\varepsilon = -1$ for $C_n$. In these cases the distinction between the covariant and the contravariant indices becomes inessential and we can express the central element of the RE1 algebra, which is $n$-th order in $K$, as

\[ \bar{c}_n(K) = \text{Tr}(C^t(K)^n K), \quad K' = K C. \]  

This indicates that $C$ and $D$ are related. In fact, we should have

\[ D = C^{-1} C^t \quad \text{or} \quad D = C C^t. \]

In the simplest case of the RE1 algebra for the $GL_q(2)$ $R$-matrix discussed in § 2 (2·16), $C = \varepsilon_q$ in (2·18) and $\bar{c}_1$ and $\bar{c}_2$ above are essentially the same as $c_1$ and $c_2$ given in (2·17).

§ 6. Commuting transfer matrices for non-periodic lattice

In this section we discuss another important application of the REA, the construction of commuting transfer matrices for non-periodic boundary conditions on lattice statistical models, which was proposed in Ref. 5) and further developed in Refs. 6) and 13). We have seen in § 2.1 that the usual ‘rapidity’-dependent YBA (in this context the ‘spectral parameter’ $\lambda$ might be preferred to the ‘rapidity’ $\theta$ but we stick to the convention) with the comultiplication (2·5) gives a one parameter family of commuting transfer matrices associated with periodic boundary conditions. Thus our starting point is the general ‘rapidity’-dependent RE (2·11) with the quantum group comodule property (2·12). We need another ingredient, a “dual” reflection equation or a “dual” $q$-tensor. The RE and its dual RE could be interpreted as describing the ‘reflection’ at the two ends of a line segment corresponding to the finite lattice.

The essential ideas can be most easily explained in the case of $q$-vectors instead of the rank two $q$-tensors (the REA). Let us start from the ‘rapidity’-dependent $q$-vector equation (4·6) (which could be interpreted as the defining equations of a Zamolodchikov algebra\(^{19}\))

\[ R(\theta_1 - \theta_2) v(\theta_1) v(\theta_2) = a v(\theta_2) v(\theta_1). \]  

By applying the ‘quantum group coaction’ $L$ times, $L$ being the lattice size of a lattice problem, we arrive at

\[ R(\theta_1 - \theta_2)[(TT'\cdots T^{(L)}) v(\theta_1)][(TT'\cdots T^{(L)}) v(\theta_2)]_2 \]

\[ = a[(TT'\cdots T^{(L)}) v(\theta_2)]_2[(TT'\cdots T^{(L)}) v(\theta_1)]_1. \]  

Next we introduce the “dual” $q$-vector by

\[ u^t(\theta_2) u^t(\theta_1) R(\theta_1 - \theta_2) = a u^t(\theta_1) u^t(\theta_2), \]  

which can be rewritten as
Covariance Properties of Reflection Equation Algebras

\[ au^t(\theta_1)u^t(\theta_2)J(\theta_1, \theta_2)=u^t(\theta_2)u^t(\theta_1). \] (6.4)

The $R$-matrix dependence is cancelled by multiplying (6.2) and (6.4) and we obtain the commuting transfer matrix $t(\theta)$

\[ [t(\theta_1), t(\theta_2)]=0, \quad t(\theta)=u^t(\theta)(TT''\cdots T^{(L)})u(\theta), \] (6.5)

provided that $u(\theta)$ commutes with $(TT''\cdots T^{(L)})u(\theta)$, component-wise. Obviously this condition is satisfied if the dual equation has a $c$-number solution. However, to the best of our knowledge, the algebra of rank one $q$-tensors does not usually admit non-trivial $c$-number solutions in contradistinction to the rank two $q$-tensors (the REA), which have a variety of $c$-number solutions as exemplified in Ref. 4. This is why the REA is used for the derivation of commuting transfer matrices for non-periodic lattices.\(^{5,6,13}\)

The actual derivation of the commuting transfer matrix for the rank two $q$-tensors (the REA) runs almost parallel with the rank one case if we start from the rewritten form of the RE (4.16)

\[ \mathcal{R}(\theta_1, \theta_2)\mathcal{K}(\theta_1)\mathcal{K}(\theta_2)=\mathcal{K}(\theta_2)\mathcal{K}(\theta_1). \] (6.6)

As is clear from the rank one case, the detailed form of $\mathcal{R}$ is inessential so long as (6.6) has the covariance under the necessary coaction

\[ \mathcal{K}(\theta)\rightarrow \mathcal{K}_{T}(\theta)=(TT''\cdots T^{(L)})K(\theta)(TT''\cdots T^{(L)})^\sigma, \] (6.7)

in which the anti-automorphism $\sigma$ is either the inverse or the transpose according to the specific problem. It should be noted that the 'rapidity' of the $T$-matrix in the second group (the $T^\sigma$ part) can be different from $\theta$. In most studied cases, it is $-\theta$ corresponding to the change of the 'rapidity' after mirror reflection. The dual rank two tensor is introduced in a similar way to the rank one case and is written symbolically as

\[ J^t(\theta_2)J^t(\theta_1)\mathcal{R}(\theta_1, \theta_2)=J^t(\theta_1)J^t(\theta_2). \] (6.8)

We get a one parameter family of commuting transfer matrix $t(\theta)$

\[ [t(\theta_1), t(\theta_2)]=0, \quad t(\theta)=\text{Tr}[J^t(\theta)(TT''\cdots T^{(L)})K(\theta)(TT''\cdots T^{(L)})^\sigma]. \] (6.9)

Namely, various $c$-number solutions of the (dual) RE specify the possible non-periodic boundary conditions compatible with the integrability.

\section{7. Summary and comments}

Many remarkable properties of the reflection equation algebras and their extensions are elucidated from the viewpoint of the covariance under the quantum group transformations or "quantum group tensors". They include the generation of a series of new solutions by composing known ones for the REA and for other $q$-tensors, the central elements of REA, construction of integrable lattice models with non-periodic boundary conditions, etc. We believe that the general framework introduced here will be quite useful for actual problems related with the quantum groups or YBA. To
name some of them, the non-commutative differential geometry, deformation of conformal field theory, quantum gravity and lattice version of Kac-Moody algebra.

After finishing the manuscript we were informed that Majid\cite{majid} had also come across with some of the examples of $q$-tensors discussed in § 4. We thank him for the information. We also thank E. K. Sklyanin and E. Corrigan for useful comments. P. K. thanks Yukawa Institute for Theoretical Physics, Kyoto University, in which this work started. R. S. thanks the Department of Mathematical Sciences, Durham for hospitality.

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