ON EXPONENTIAL POLYNOMIALS AND QUANTUM COMPUTING

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Abstract. We calculate the zeros of an exponential polynomial of three variables by
a classical algorithm and quantum algorithms which are based on the method of van
Dam and Shparlinski, they treated the case of two variables, and compare with the time
complexity of those cases. Further we compare the case of van Dam and Shparlinski
with our case by considering the ratio (classical/quantum) of the time complexity. Then
we can observe the ratio decreases.

1. Introduction

For a prime number p, we put q = pν, where ν is a certain positive integer. Then we
denote the finite field by Fq which has q − 1 elements. Namely, Fq forms an additive
group and Fq× := Fq\{0} forms a multiplicative group, where 0 is the zero element in Fq.
Any element of α ∈ Fq× have a periodicity, that is there exits a smallest natural number
s such that αs = 1. We call such s the “multiplicative order” of α. It is known that the
multiplicative order is a divisor of #Fq× = q − 1.

To calculate the number of the zeros of a polynomial

F(x1, . . . , xm) = \sum_{(n1, . . . , nm) \in \mathbb{N}_0^m} a_{n1, . . . , nm} x_1^{n1} \cdots x_m^{nm}

is a very important problem in mathematics. Here, \mathbb{N}_0 := \mathbb{N} \cup \{0\} and a_{n1, . . . , nm} ∈ Fq. In
[3], van Dam and Shparlinski treated the following exponential polynomial

(1.1) f(x, y) = a_1 x_1^x + a_2 y_2^y - b

and calculated the zeros of (1.1) by quantum algorithms. Further they compared the time
complexity due to a classical algorithm with that due to a quantum algorithm. Then the
“cubic” speed-up was observed.

In this article, we treat the following exponential polynomial

(1.2) f_b(x_1, x_2, x_3) := a_1 x_1^x_1 + a_2 x_2^x_2 + a_3 x_3^x_3 - b

and calculate the solutions of f_b(x_1, x_2, x_3) = 0 by using quantum algorithms which are
natural generalizations of the method of van Dam and Shparlinski. Here, a_i, g_j ∈ Fq×
(i, j = 1, 2, 3) and b ∈ \mathbb{F}_q. Further we also compare the time complexity due to a classical algorithm with that due to a quantum algorithm. Then exponentially “5/2 times” speed-up is observed.

In the next section, we introduce some notation and give the considerable lemma which supports whether there exit the zeros of (1.2). In Section 3, we evaluate the time complexity due to a classical algorithm. Further in Section 4, we evaluate the time complexity due to a quantum algorithm.

2. The number of solutions of equation

In this section, we give an important formula with respect to the density of solutions of

\[(2.1) \quad f_b(x_1, x_2, x_3) := a_1 g_1^{x_1} + a_2 g_2^{x_2} + a_3 g_3^{x_3} - b = 0 \]

as Lemma 2.1, below. To state it, we introduce some notation.

Let each s_i be the multiplicative order of g_i (i = 1, 2, 3) in (2.1). We put

\[ X_i := \{0, 1, \ldots, s_i - 1\} \cong \mathbb{Z}/s_i \mathbb{Z}, \quad (i = 1, 2, 3), \]

\[ X_3(r) := \{0, 1, \ldots, r - 1\} \subseteq X_3 \quad (r = 1, 2, \ldots, s_3), \]

\[ X^3(r) := X_1 \times X_2 \times X_3(r) \]

and

\[ X^3 := X^3(s_3) = X_1 \times X_2 \times X_3. \]

Then we define

\[ S_{f_b}(r) := \{(x_1, x_2, x_3) \in X^3(r) \mid f_b(x_1, x_2, x_3) = 0\}, \]

\[ N_{f_b}(r) := \#S_{f_b}(r) \]

for r = 1, \ldots, s_3.

By using above notation, we can state the following result:

**Lemma 2.1.** Let \( \delta \) be a parameter satisfying \( \delta = o(q) \). For \( r > \delta^2 q^3 (s_1 s_2)^{-2} \), we have

\[(2.2) \quad N_{f_b}(r) = \frac{s_1 s_2 r}{q} + O(\delta \sqrt{rq}), \]

except for at most \( q/\delta^2 \) exceptional \( b \)'s. Further \( O \)-constant can be taken 1.

Choosing \( \delta = (\log q)^{1/2} \) in Lemma 2.1, we have

**Corollary 2.2.** If \( q^3 (s_1 s_2)^{-2} \log q < r \leq s_3 \), then we see that \( S_{f_b}(r) \neq \phi \) holds except for at most \( q/\log q \) exceptional \( b \)'s.
Remark 2.3. The above lemma and corollary make the point that the solutions of (2.1) exit only when
\[ \frac{s_1 s_2}{q} \geq \left( \frac{q}{s_3 - 2 \log q} \right)^{1/2} (> 1). \]
This inequality implies that the multiplicative orders \( s_1 \) and \( s_2 \) are somewhat large.

Remark 2.4. The exponent \( 1/2 \) of \( \delta = (\log q)^{1/2} \) is not necessary. In fact, \( \delta = (\log q)^{\varepsilon} \) with any \( \varepsilon > 0 \) is sufficient.

Proof of Lemma 2.1. Let \( \psi \) be a non-trivial additive character over \( \mathbb{F}_q \), in fact, any additive character over \( \mathbb{F}_q \) can be given as a map \( \mathbb{F}_q \to \mathbb{C}_1^* \), where \( \mathbb{C}_1^* := \{ z \in \mathbb{C} | |z| = 1 \} \) (see [5, Theorem 5.7]). To evaluate \( N_{f_b}(v) \), we use the following formula which plays as a counting function:

\[ \frac{1}{q} \sum_{\mu \in \mathbb{F}_q} \psi(u \mu) = \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{otherwise.} \end{cases} \]

Then we have

\[ N_{f_b}(r) = \sum_{(x_1, x_2, x_3) \in X^3(r)} \frac{1}{q} \sum_{\mu \in \mathbb{F}_q} \psi(f_b(x_1, x_2, x_3)) \]
\[ = \frac{s_1 s_2 r}{q} + \frac{1}{q} \sum_{\mu \in \mathbb{F}_q^*} \sum_{(x_1, x_2, x_3) \in X^3(r)} \psi(f_b(x_1, x_2, x_3)) \]
\[ =: \frac{s_1 s_2 r}{q} + \Delta_b(r). \]

If the contribution from the second term on the right-hand side of the above formula can be estimated by \( o(s_1 s_2 r / q) \), the above formula tells us the existence of the solution of \( f_b(x_1, x_2, x_3) \). To consider it, we evaluate the mean value of the second term on the right-hand side of (2.4) with respect to \( b \). Namely, we evaluate

\[ E(r) := \sum_{b \in \mathbb{F}_q} |\Delta_b(r)|^2. \]
From (2.3) and some properties of the additive character over $\mathbb{F}_q$, we obtain

$$ E(r) = \frac{1}{q^2} \sum_{\mu, \mu' \in \mathbb{F}_q^*} \left( \prod_{j=1}^{2} \left( \sum_{x_j, x'_j \in X_j} \psi(a_j \mu g_{j}^{x_j} - \mu' g_{j}^{x'_j}) \right) \right) \sum_{x_3, x'_3 \in X_3(r)} \psi(a_3 (\mu g_{3}^{x_3} - \mu' g_{3}^{x'_3})) $$

$$ \times \sum_{b \in \mathbb{F}_q} \psi(b(\mu' - \mu)) $$

$$ = \frac{1}{q} \sum_{\mu \in \mathbb{F}_q^*} \left( \prod_{j=1}^{2} \left( \sum_{x_j, x'_j \in X_j} \psi(a_j \mu g_{j}^{x_j} - g_{j}^{x'_j}) \right) \right) \sum_{x_3, x'_3 \in X_3(r)} \psi(a_3 \mu (g_{3}^{x_3} - g_{3}^{x'_3})) $$

$$ = \frac{1}{q} \sum_{\mu \in \mathbb{F}_q^*} \left( \prod_{j=1}^{2} \left( \sum_{x_j \in X_j} \psi(a_j \mu g_{j}^{x_j}) \right)^2 \right) \sum_{x_3 \in X_3(r)} \psi(a_3 \mu g_{3}^{x_3})^2. $$

It is known that

$$ \left| \sum_{x_j \in X_j} \psi(a_j \mu g_{j}^{x_j}) \right| \leq \sqrt{q} \quad \text{for} \ j = 1, 2 \ \text{and any} \ \mu \in \mathbb{F}_q^* $$

(see Theorem 8.78 in [5]). Hence we have

$$ E(r) < q \sum_{\mu \in \mathbb{F}_q^*} \left| \sum_{x_3 \in X_3(r)} \psi(a_3 \mu f^{x_3}) \right|^2 = q^2 r. $$

Therefore, if we put $\delta = o(q)$, then we can see that there exit at most $q/\delta^2$ exceptional $b$'s such that

$$ (2.5) \quad \left| \frac{1}{q} \sum_{\mu \in \mathbb{F}_q^*} \sum_{(x_1, x_2, x_3) \in X_3(r)} \psi((f_0(x_1, x_2, x_3))) \right| \geq \delta \sqrt{rq}. $$

Hence we obtain

$$ N_{f_0}(r) = \frac{s_1 s_2 r}{q} + O(\delta \sqrt{qr}) $$

for other $b$'s. Now, the proof of Lemma 2.1 is completed. \hfill $\Box$

### 3. Calculation of the deterministic time for a classical algorithm

We follow the method of van Dam and Shparlinski [3]. Then we have

**Theorem 3.1.** Except for at most $q/\log q$ exceptional $b$'s, we can either find a solution $(x_1, x_2, x_3) \in \mathbb{X}^3$ of the equation (2.1) or decide that it does not have a solution in deterministic time $q^{3/2}(\log q)^{O(1)}$ as a classical computer.

**Proof.** Using a standard deterministic factorization algorithm, we factorize $q - 1$ and find the orders $s_j$ ($j = 1, 2, 3$) of $g_j$ in time $q^{1/2}(\log q)^{O(1)}$. We may assume without loss of generality that $s_1 \geq s_2 \geq s_3$. For calculated orders $s_1$ and $s_2$, we put

$$ r = \lceil q^2(s_1 s_2)^{-2} \log q \rceil. $$
Then we see that the solution of (2.1) certainly exists when \( r \leq s_3 \). However, when \( r > s_3 \), we do not know whether such solutions exits. Therefore we have to consider those two cases.

For each \((x_2, x_3) \in X_2 \times X_3(r)\), we calculate the deterministic time of the discrete logarithm \( x_1 \) such that \( g_1^{x_1} = a_1^{-1}(b - a_2g_2^{x_2} - a_3g_3^{x_3}) \). It is known that the deterministic time for this case is \( s_1^{1/2} \left( \log q \right)^{O(1)} \) (see Section 5.3 in [2]).

(i) The case \( r \leq s_3 \). We have

\[
(s_2r)s_1^{1/2} \left( \log q \right)^{O(1)} \ll q^{3/2} \left( \log q \right)^{O(1)},
\]

since \( s_1^{1/2} s_2 r < (s_1^2 s_2^2 r)^{1/2} \).

(ii) The case \( r > s_3 \). Similarly, we see that the deterministic time is

\[
(s_2 s_3 s_1^{1/2}) \left( \log q \right)^{O(1)} \ll q^{3/2} \left( \log q \right)^{O(1)},
\]

since \( s_1^{1/2} s_2 s_3 < (s_1^2 s_2^2 s_3)^{1/2} < (s_1^2 s_2^2 r)^{1/2} \).

\[\square\]

4. Calculation of the Time Complexity for a Quantum Algorithm

In this section, we describe quantum algorithms which are based on the method of [3].

**Theorem 4.1.** Except for at most \( q/ \log q \) exceptional \( b \)'s, we can either find a solution \((x_1, x_2, x_3) \in X^3\) of the equation (2.1) or decide that it does not have a solution in time \( q^{3/5} \left( \log q \right)^{O(1)} \) as a quantum computer.

**Proof.** Using Shor's algorithm [6], we can obtain the multiplicative orders \( s_j \)'s \((j = 1, 2, 3)\) in polynomial time. We may assume without loss of generality that \( s_1 \geq s_2 \geq s_3 \). As in the proof of Theorem 3.1, we put \( r \) as (3.1). Further, we consider a polynomial time quantum subroutine \( S(x_2, x_3) \) which either finds and returns \( x_1 \in X_1 \) with

\[g_1^{x_1} = a_1^{-1}(b - a_2g_2^{x_2} - a_3g_3^{x_3})\]

or reports that no such \( x_1 \) exists for a given \((x_2, x_3) \in X_2 \times X_3(r)\) by using Shor's discrete logarithm algorithm.

(i) The case \( r \leq s_3 \). Using Grover's search algorithm [4], we search the subroutine \( S(x_2, x_3) \) for all \((x_2, x_3) \in X_2 \times X_3(r)\) in time

\[
(x_2 r)^{1/2} \left( \log q \right)^{O(1)} \ll q^{3/5} \left( \log q \right)^{O(1)},
\]

since \( x_2 r \leq (s_1^2 s_2^2 r)^{2/5} \).

(ii) The case \( r > s_3 \). Similarly, we search the \( S(x_2, x_3) \) for all \((x_2, x_3) \in X_2 \times X_3\) in time

\[
(x_2 x_3)^{1/2} \left( \log q \right)^{O(1)} \ll q^{3/5} \left( \log q \right)^{O(1)},
\]

since \( x_2 x_3 \leq (s_1^2 s_2^2 s_3)^{2/5} < (s_1^2 s_2^2 r)^{2/5} \).
In [3], van Dam and Shparlinski mentioned when the multiplicative orders are large, there is a more efficient quantum algorithm. Similarly, we can also consider a more efficient quantum algorithm.

**Theorem 4.2.** If we assume

\[(s_1s_2)^2s_3 > q^3 \log q,\]

then we can either find a solution \((x_1, x_2, x_3) \in X^3\) of the equation (2.1) or decide that it does not have a solution in time \(q^2(s_1^2s_2^2s_3^2)^{-1/10}(\log q)^{O(1)}\) as a quantum computer, except for at most \(q/\log q\) exceptional \(b\)'s.

**Remark 4.3.** The upper bound of the running time of the algorithm of Theorem 4.2 is

\[O(q^{1/5}(\log q)^{O(1)}).\]

**Proof of Theorem 4.2.** We may assume without loss of generality that \(s_1 \geq s_2 \geq s_3\). We put

\[(4.1) \quad r = \lfloor q^3(s_1s_2)^{-2}\log q \rfloor\]

Then from the assumption of the theorem we see that \(r \leq s_3\). Hence there are some solutions of (2.1) in \(X^3(r)\) and we denote the number of the solutions of (2.1) by \(M\). Note that \(M \geq (s_1s_2r)/q\).

As in the case of [3], we use the version of Grover’s algorithm as described in [1] that finds one out of \(m\) matching items in a set of size \(t\) by using only \(O(\sqrt{t/m})\) queries. We search the subroutine \(S(x_2, x_3)\) for all \((x_2, x_3) \in X_2 \times X_3(r)\). Then the time complexity is

\[
\left(\frac{s_2r}{M}\right)^{1/2}(\log q)^{O(1)} \leq q^{1/2}(s_1^2s_2^2s_3^2)^{-1/10}(\log q)^{O(1)}.
\]

\[\square\]

5. Concluding remarks

At the end of this article, we compare the case of van Dam and Shparlinski with our case. See the following list.

| # of variables | Classical | Quantum | ratio (C/Q) |
|----------------|----------|---------|-------------|
| 2              | 1        | 1/3     | 3           |
| 3              | 3/2      | 3/5     | 5/2         |

The case of two variables is that of van Dam and Shparlinski and the case of three variables is our case. We notice that the ratio decreases. Does the ratio decrease to 1 when the dimension increase? We can apply the method used in [3] and this paper to the case of any variables. By roughly calculating, the ratio seems to converge 2, when the number of the variables increases. It seems to come from the effect of Grover’s algorithm.
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