Detecting invariant expanding cones for generating word sets to identify chaos in piecewise-linear maps

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ABSTRACT

We show how to formally identify chaotic attractors in continuous, piecewise-linear maps on $\mathbb{R}^N$. For such a map $f$, this is achieved by constructing three objects. First, $\Omega_{\text{trap}} \subset \mathbb{R}^N$ is trapping region for $f$. Second, $W$ is a finite set of words that encodes the forward orbits of all points in $\Omega_{\text{trap}}$. Finally, $C \subset T \mathbb{R}^N$ is an invariant expanding cone for derivatives of compositions of $f$ formed by the words in $W$. The existence of $\Omega_{\text{trap}}$, $W$, and $C$ implies $f$ has a topological attractor with a positive Lyapunov exponent. We develop an algorithm that identifies these objects for two-dimensional homeomorphisms comprised of two affine pieces. The main effort is in the explicit construction of $\Omega_{\text{trap}}$ and $C$. Their existence is equated to a set of computable conditions in a general way. This results in a computer-assisted proof of chaos throughout a relatively large region of parameter space. We also observe how the failure of $C$ to be expanding can coincide with a bifurcation of $f$. Lyapunov exponents are evaluated using one-sided directional derivatives so that forward orbits that intersect a switching manifold (where $f$ is not differentiable) can be included in the analysis.

1. Introduction

Piecewise-linear maps form canonical representations of nonlinear dynamics and provide effective models of diverse physical systems [10]. Much work has been done to identify properties that imply a piecewise-linear map has a chaotic attractor. These include Markov partitions [15], homoclinic connections [26], and situations where the dynamics is effectively one-dimensional [21]. In the context of ergodic theory, an attractor with an absolutely continuous invariant measure exists for piecewise-expanding maps [7,35] and piecewise-smooth maps with certain expansion properties [31,38]. However, when applied to the two-dimensional border-collision normal form (2d BCNF) such properties have only been verified over regions of parameter space that are small in comparison to where numerical simulations suggest chaotic attractors actually exist [12,13]. To address this issue we use invariant expanding cones to bound Lyapunov exponents. We show that
this appears to be a highly effective method for identifying chaotic attractors in a formal way.

For a continuous map \( f \) on \( \mathbb{R}^N \), the Lyapunov exponent \( \lambda(x, v) \) characterizes the asymptotic rate of separation of the forward orbits of arbitrarily close points \( x \) and \( x + \delta v \):

\[
\| f^n(x + \delta v) - f^n(x) \| \sim e^{\lambda n} \| \delta v \|.
\]

A positive Lyapunov exponent for bounded orbits is a standard indicator of chaos [24].

Now suppose there exist \( N \times N \) matrices \( A_i \) such that

\[
f^n(x + \delta v) - f^n(x) = \delta A_{n-1} \cdots A_1 A_0 v + o(\delta),
\]

where \( o(\delta) \) vanishes faster than \( \delta \) as \( \delta \to 0 \). These matrices exist if \( f \) is \( C^1 \): each \( A_i \) is the Jacobian matrix \( Df \) evaluated at \( f^i(x) \). If \( M \) denotes the set of all \( A_i \) and there exists \( c > 1 \) and a cone \( C \subset T\mathbb{R}^N \) with the property

\[
Mv \in C \text{ and } \|Mv\| \geq c\|v\|, \text{ for all } v \in C \text{ and all } M \in M,
\]

then immediately we have \( \lambda(x, v) > 0 \) for any \( v \in C \). This idea is attributed to Alekseev [1] (see [5, 37]) and is useful for establishing splitting and hyperbolicity in smooth dynamical systems [8, 27, 34].

Condition (3) is rather strong as it requires \( C \) to be invariant and expanding for every matrix in the possibly infinite set \( M \). But if \( f \) is piecewise-linear then \( M \) contains only as many matrices as pieces in the map. For this reason invariant expanding cones are perfectly suited, and perhaps under-utilized, for analysing piecewise-linear maps. Invariant cones were central to Misiurewicz’s strategy for establishing hyperbolicity and transitivity in the Lozi map [26]. More recently in [17] an invariant expanding cone was constructed for the 2d BCNF to finally prove the widely held conjecture that a chaotic attractor exists throughout a physically-important parameter regime \( \mathcal{R}_{\text{BYG}} \) first highlighted in [4].

In this paper we present a simple but powerful generalization of the above approach and use it to show that chaotic attractors persist beyond \( \mathcal{R}_{\text{BYG}} \), in fact in some places right up to where there exist stable low-period orbits. The idea is to let \( M \) consist of certain products of the \( A_i \), rather than of the \( A_i \) themselves. Each product \( M \in M \) is characterized by a word \( W \) connecting its constitute matrices to the pieces of \( f \). As long as the set of all such words \( W \) generates the symbolic itinerary of the forward orbit of \( x \), then again (3) implies \( \lambda(x, v) > 0 \). This approach is quite flexible because there is a large amount of freedom in our choice of the set of words \( W \).

To prove \( f \) has a chaotic attractor one needs to identify a trapping region \( \Omega_{\text{trap}} \subset \mathbb{R}^N \) (which ensures there exists a topological attractor), a set of words \( W \) that generates the symbolic itineraries for all \( x \in \Omega_{\text{trap}} \), and an invariant expanding cone \( C \) for the matrices corresponding to the words in \( W \). Below we find these objects for the 2d BCNF. More precisely, we propose a way by which \( \Omega_{\text{trap}} \) and \( C \) can be constructed for a particular word set \( W \) and prove that all three objects have the required properties if a certain set of computable conditions are met. While, for a given combination of parameter values, these conditions could be checked by hand, it is more appropriate to check them numerically. Below we formulate this as an algorithm (Algorithm 7.1).

The remainder of this paper is organized as follows. We start in Section 2 by showing where Algorithm 7.1 detects a chaotic attractor in a typical two-dimensional slice of
the parameter space of the 2d BCNF. Then in Section 3 we clarify technical features mentioned above (cones, words, trapping regions, etc) and express $\lambda(x, v)$ in terms of one-sided directional derivatives in order to accommodate points whose forward orbits intersect a switching manifold (where $Df$ is not defined). The result $\lambda(x, v) > 0$ is formalized by Theorem 3.9 for $N$-dimensional, continuous, piecewise-linear maps with two pieces. The theorem is framed in terms of $W$-recurrent sets (these are sets to which forward orbits return following one or more words in $W$). Such sets provide a practical way by which the approach can be applied to concrete examples, and in Section 4 we show they imply that $W$ generates symbolic itineraries as needed. Here we also characterize the matrices $A_i$ in the expression (2). This is quite subtle because if $f^i(x)$ lies on a switching manifold then $A_i$ depends on $v$ as well as $x$. Section 4 concludes with a proof of Theorem 3.9.

In subsequent sections we work to apply this methodology to the 2d BCNF. In Section 5 we consider sets of $2 \times 2$ matrices and devise a set of conditions for the existence of an invariant expanding cone. In Section 6 we connect consecutive points of an orbit to construct a polygon $\Omega$. We then identify conditions implying $\Omega$ is forward invariant and can be perturbed into a trapping region $\Omega_{\text{trap}}$. In Section 7 we state Algorithm 7.1 and prove its validity. Here we comment further on the application of the algorithm to the 2d BCNF and discuss instances in which failure of the algorithm coincides with a bifurcation of $f$ at which the chaotic attractor is destroyed. Concluding remarks are provided in Section 8.

2. Chaotic attractors in the two-dimensional border-collision normal form

Let $f$ be a continuous map on $\mathbb{R}^2$ that is affine on each side of a line $\Sigma$. Assume coordinates $x = (x_1, x_2) \in \mathbb{R}^2$ are chosen so that $\Sigma = \{x | x_1 = 0\}$. If $f$ is generic in the sense that $f(\Sigma)$ intersects $\Sigma$ at exactly one point, and this point is not a fixed point of $f$, then there exists an affine coordinate change that puts $f$ into the form

$$f(x) = \begin{cases} 
\left[ \begin{array}{cc} \tau_L & 1 \\
-\delta_L & 0 \end{array} \right] \left[ \begin{array}{c} x_1 \\
 x_2 \end{array} \right] + \left[ \begin{array}{c} 1 \\
 0 \end{array} \right], & x_1 \leq 0, \\
\left[ \begin{array}{cc} \tau_R & 1 \\
-\delta_R & 0 \end{array} \right] \left[ \begin{array}{c} x_1 \\
 x_2 \end{array} \right] + \left[ \begin{array}{c} 1 \\
 0 \end{array} \right], & x_1 \geq 0,
\end{cases}$$

(4)

where $\tau_L, \tau_R, \delta_L, \delta_R \in \mathbb{R}$. The coordinate change required to arrive at (4) is provided in Appendix A.

The four parameter family (4) is the 2d BCNF of [28] except the value of the border-collision bifurcation parameter (usually denoted $\mu$) is fixed at 1. The family (4) has been studied extensively to understand border-collision bifurcations (where a fixed point collides with a switching manifold) arising in many applications, particularly vibrating mechanical systems with impacts or friction [10,32]. If $\tau_L = -\tau_R$ and $\delta_L = \delta_R$ then (4) reduces to the Lozi map [23].

Figure 1 shows a two-dimensional slice of the parameter space of (4) defined by fixing $\delta_L = 0.3, \delta_R = 0.3$.

(5)

In the blue regions (4) has a stable period-$n$ orbit for one value $1 \leq n \leq 5$ as indicated. To the right of the curve labelled HC, where the fixed point in $x_1 < 0$ attains a homoclinic
Figure 1. A two-parameter bifurcation diagram of the 2d BCNF (4) with (5). In the regions labelled $n = 1, n = 2, \ldots$ (4) has a stable period-$n$ orbit for the indicated value of $n$ (in places where the $n = 3$ and $n = 4$ regions overlap only the $n = 4$ region is shown). In the regions labelled $p_{\text{max}} = 1, p_{\text{max}} = 2, \ldots$ Algorithm 7.1 establishes the existence of a chaotic attractor by using (6) with the indicated value of $p_{\text{max}}$. The three black dots indicate the parameter combinations examined in Figures 15–17 [see also Figure 2(b)]. The boundaries $B_1$ to $B_5$ are discussed in Section 7.

connection, there is usually no attractor because here the stable and unstable manifolds of the fixed point form a homoclinic tangle that provides a path by which orbits can escape. Numerical explorations suggest that in all other areas of Figure 1 (i.e. left of the HC curve and not inside a blue region) (4) has a chaotic attractor [3]. Different values of the determinants $\delta_L, \delta_R \in (0, 1)$ correspond to different rates of contraction for the action of the map, so seem to produce qualitatively similar results in the sense that, as the value of $\tau_L$ is increased, the dynamics broadly changes from low-period orbits, to chaos, and then to the absence of an attractor. Note that $\delta_L, \delta_R \in (0, 1)$ implies (4) is orientation-preserving (determinants positive) and dissipative (determinants less than 1) thus can arise as a local approximation to Poincaré map of a ODE model of a dissipative system.

The two striped regions of Figure 1 are of particular interest. In the central striped region numerical simulations reveal that (4) has two chaotic attractors [16]. This shows that the problem of the uniqueness of the chaotic attractor (not addressed in this paper) is not trivial. The region $\mathcal{R}_{\text{BYG}}$ (bounded by the HC curve and the lines $\tau_L = \delta_L + 1$ and $\tau_R = -\delta_R - 1$) is the ‘robust chaos’ parameter regime of [4] [restricted to (5)]. This parameter regime is exactly where (4) has two saddle fixed points, one with positive eigenvalues and one with negative eigenvalues. In $\mathcal{R}_{\text{BYG}}$ (4) has a quasi-one-dimensional attractor, Figure 2(a), because the attractor can be characterized as the closure of a one-dimensional unstable manifold [17].
In [17] it was shown that throughout $\mathcal{R}_{BYG}$ (4) has an attractor with a positive Lyapunov exponent. This was achieved by constructing a trapping region and an invariant expanding cone for both matrices in (4). That is, the methodology of this paper was used with $W = \{L,R\}$. In [17] it was not necessary to show that the symbolic itineraries of orbits are generated by $W$ because in this case $W$ generates all symbolic itineraries.

The approach of [17] fails to find a chaotic attractor in $\tau_L < \delta_L + 1$ because here both eigenvalues of $A_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix}$ have modulus less than 1, so there does not exist an invariant expanding cone for $A_L$. Therefore, if we are to show that (4) has a chaotic attractor with $\tau_L < \delta_L + 1$, we cannot include $L$ in our word set $W$. We instead use word sets of the form

$$W = \{RL^p \; | \; 0 \leq p \leq p_{\text{max}}\}. \quad (6)$$

This is clarified in Section 3.5. The red regions of Figure 1 show where Algorithm 7.1 finds an attractor with a positive Lyapunov exponent by using (6) with some $p_{\text{max}} \geq 1$ over a $1024 \times 512$ grid of $(\tau_L, \tau_R)$-values [Figure 2(b) shows an example]. The algorithm is highly effective in that it establishes chaos over about 90% of parameter space between the blue regions and $\tau_L = \delta_L + 1$ (and also succeeds in part of $\mathcal{R}_{BYG}$). In fact in some places there is no gap between the blue and red regions meaning that the algorithm finds a chaotic attractor right up to the bifurcation at which the attractor is destroyed (this is discussed further in Section 7).

3. Main definitions and a bound on the Lyapunov exponent

In this section we define the main objects and state the main theoretical result, Theorem 3.9. In order to arrive at Theorem 3.9 quickly some discussion is deferred to Section 4.

3.1. Trapping regions and topological attractors

Here we provide topological definitions for a continuous map $f$ on $\mathbb{R}^N$, see for instance [18,29] for further details.

**Definition 3.1:** A set $\Omega \subseteq \mathbb{R}^N$ is forward invariant if $f(\Omega) \subseteq \Omega$. 

**Figure 2.** Chaotic attractors of (4) with (5). More precisely, these plots show 5000 iterates of the forward orbit of the origin with some transient points removed.
Figure 3. A sketch of an invariant expanding cone $C$ and its image $MC = \{Mv \mid v \in C\}$.

Definition 3.2: A non-empty compact set $\Omega_{\text{trap}} \subset \mathbb{R}^N$ is a trapping region if $f(\Omega_{\text{trap}}) \subseteq \text{int}(\Omega_{\text{trap}})$ (where $\text{int}(\cdot)$ denotes interior).

Definition 3.3: An attracting set is $\Lambda = \cap_{i=0}^\infty f^i(\Omega_{\text{trap}})$ for some trapping region $\Omega_{\text{trap}} \subset \mathbb{R}^N$.

Topological attractors are invariant subsets of an attracting set that satisfy some kind of indivisibility condition (e.g. they contain a dense orbit) [24]. In this paper it is not necessary to consider such conditions as we only seek to show that $f$ has an attractor and this is achieved by showing that $f$ has a trapping region.

3.2. Cones

We denote the tangent space to a point $x \in \mathbb{R}^N$ by $T\mathbb{R}^N$. This is the usual tangent space – it is not affected by the piecewise-linear nature of the maps considered below. The tangent space is isomorphic to $\mathbb{R}^N$ and indeed we often treat tangent vectors $v \in T\mathbb{R}^N$ as elements of $\mathbb{R}^N$, as in the expression $x + \delta v$.

Definition 3.4: A non-empty set $C \subseteq T\mathbb{R}^N$ is said to be a cone if $tv \in C$ for all $v \in C$ and all $t \in \mathbb{R}$.

Definition 3.5: Let $M$ be an $N \times N$ matrix. A cone $C \subseteq T\mathbb{R}^N$ is said to be invariant under $M$ if

$$Mv \in C, \quad \text{for all } v \in C. \quad (7)$$

The cone is said to be expanding under $M$ if there exists $c > 1$ such that

$$\|Mv\| \geq c\|v\|, \quad \text{for all } v \in C, \quad (8)$$

see Figure 3. For a finite set of real-valued $N \times N$ matrices $M$, we say $C$ is invariant [resp. expanding] if it is invariant [resp. expanding] under every $M \in M$.

3.3. One-sided directional derivatives and Lyapunov exponents

The one-sided directional derivative of a function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ at $x \in \mathbb{R}^N$ in a direction $v \in T\mathbb{R}^N$ is defined as

$$D_v^+ \phi(x) = \lim_{\delta \to 0^+} \frac{\phi(x + \delta v) - \phi(x)}{\delta}, \quad (9)$$
Figure 4. A sketch of the one-dimensional map (11) with $0 < a_L < 1 < a_R$.

if this limit exists \[9,30\]. If $D_v^+ f^n(x)$ exists for all $n \geq 1$, then taking $\delta \to 0^+$ in (1) gives $\|D_v^+ f^n(x)\| \sim e^{\lambda n} \|v\|$.

By further taking $n \to \infty$ we arrive at

$$\lambda(x, v) = \limsup_{n \to \infty} \frac{1}{n} \ln \left( \|D_v^+ f^n(x)\| \right), \quad (10)$$

where, following the usual convention, the supremum limit is taken because from the point of view of ascertaining stability one wants to record the largest possible fluctuations.

If $f$ is $C^1$ then the Jacobian matrix $Df$ exists everywhere and (10) becomes

$$\lambda(x, v) = \limsup_{n \to \infty} \frac{1}{n} \ln \left( \|D f^n(x) v\| \right).$$

Oseledet’s theorem gives conditions under which $\lambda(x, v)$ takes at most $N$ values for almost all $x$ in an invariant set \[6,36\]. But this is often not the case for piecewise-linear maps. As a minimal example, consider the one-dimensional map

$$f(x) = \begin{cases} a_L x, & x \leq 0, \\ a_R x, & x \geq 0, \end{cases} \quad (11)$$

shown in Figure 4. If $a_L, a_R > 0$ and $a_L \neq a_R$ then the fixed point $x = 0$ has two different Lyapunov exponents: $\lambda(0, -1) = \ln(a_L)$ and $\lambda(0, 1) = \ln(a_R)$.

3.4. Two-piece, piecewise-linear, continuous maps

For ease of explanation we develop our methodology for piecewise-linear maps comprised of only two pieces. The extension to maps with more pieces is expected to be reasonably straight-forward. Specifically we consider maps of the form

$$f(x) = \begin{cases} A_L x + b, & x_1 \leq 0, \\ A_R x + b, & x_1 \geq 0, \end{cases} \quad (12)$$

where $A_L$ and $A_R$ are $N \times N$ matrices and $b \in \mathbb{R}^N$. The assumption that $f$ is continuous on the switching manifold $\Sigma = \{x \mid x_1 = 0\}$ implies that $A_L$ and $A_R$ differ in only their first columns. That is,

$$A_R - A_L = \xi e_1^T,$$

for some $\xi \in \mathbb{R}^N$ and where $e_1$ is the first standard basis vector of $\mathbb{R}^N$. 
If \( \det(A_L) \neq 0 \) and \( \det(A_R) \neq 0 \) then \( f \) maps the half-spaces \( x_1 \leq 0 \) and \( x_1 \geq 0 \) in a one-to-one fashion to half-spaces with boundary \( f(\Sigma) \). If \( \det(A_L) \det(A_R) < 0 \) then \( f \) is not invertible because \( x_1 \leq 0 \) and \( x_1 \geq 0 \) are mapped to the same half-space, see [25]. If \( \det(A_L) \det(A_R) > 0 \) then \( f \) is invertible (see [32] for an explicit expression for \( f^{-1} \)) and so we have the following result.

**Lemma 3.6:** The map (12) is invertible if and only if \( \det(A_L) \det(A_R) > 0 \).

### 3.5. Words as symbolic representations of finite parts of orbits

To describe the symbolic itineraries of orbits of (12) relative to \( \Sigma_1 \) we use words (defined here) and symbol sequences (defined in Section 4.1) on the alphabet \{L, R\}.

**Definition 3.7:** A word of length \( n \geq 1 \) is a function \( \mathcal{W} : \{0, 1, \ldots, n-1\} \to \{L, R\} \) and we write \( \mathcal{W} = \mathcal{W}_0 \mathcal{W}_1 \cdots \mathcal{W}_{n-1} \).

Sometimes we abbreviate \( k \) consecutive instances of a symbol by putting \( k \) as a superscript. For example \( \mathcal{W} = RL^3 = RLLL \) is a word of length four with \( \mathcal{W}_0 = R, \mathcal{W}_1 = L, \mathcal{W}_2 = L, \mathcal{W}_3 = L \).

In order to obtain words from orbits we first define the following set-valued function on \( \mathbb{R}^N \):

\[
\gamma_{\text{set}}(x) = \begin{cases} 
\{L\}, & x_1 < 0, \\
\{L, R\}, & x_1 = 0, \\
\{R\}, & x_1 > 0. 
\end{cases}
\tag{14}
\]

Then, given \( x \in \mathbb{R}^N \) and \( n \geq 1 \), we define

\[
\Gamma(x; n) = \left\{ \mathcal{W} : \{0, 1, \ldots, n-1\} \to \{L, R\} \mid \mathcal{W}_i \in \gamma_{\text{set}}(f^i(x)) \right\}
\]

for all \( i = 0, 1, \ldots, n-1 \).

Notice that if \( f^i(x) \in \Sigma \) for \( m \) values of \( i \in \{0, 1, \ldots, n-1\} \), then \( \Gamma(x; n) \) contains \( 2^m \) words. For example for Figure 5 we have \( \Gamma(x; 3) = \{RLL, RRL\} \).

Symbolic representations are often instead defined in a way that produces a unique word for every \( x \) and \( n \) [19]. In Section 4.2 we will see how the above formulation is particularly convenient for describing \( D^+_v f^n(x) \).
3.6. Sufficient conditions for a positive Lyapunov exponent

Here we state our main result for obtaining $\lambda(x, v) > 0$. To do this we first provide a few more definitions that are discussed further in Section 4.

Given a finite set of words $W$, let

$$M = \left\{ \Phi(W) \mid W \in W \right\},$$

where

$$\Phi(W) = A_W \cdots A_1 A_0.$$ (17)

Roughly speaking, $\Phi(W)$ is equal to $Df^n$ for orbits that map under $f$ following the word $W$.

**Definition 3.8:** A set $\Omega_{\text{rec}} \subset \mathbb{R}^N$ is said to be $W$-recurrent if for all $x \in \Omega_{\text{rec}}$ there exists $n \geq 1$ such that $f^n(x) \in \Omega_{\text{rec}}$ and every $W \in \Gamma(x; n)$ can be written as a concatenation of words in $W$.

**Theorem 3.9:** Suppose $\Omega_{\text{rec}}$ is $W$-recurrent. Suppose there exists an invariant expanding cone for $M$ given by (16). Then there exists $\lambda_{\text{bound}} > 0$ such that for all $x \in \Omega_{\text{rec}}$,  

$$\liminf_{n \to \infty} \frac{1}{n} \ln \left( \| D_v f^n(x) \| \right) \geq \lambda_{\text{bound}}, \quad \text{for some } v \in T\mathbb{R}^N.$$ (18)

Moreover, if $f$ is invertible and $\Omega_{\text{trap}} \subset \bigcup_{i=\infty}^{-\infty} f(i)(\Omega_{\text{rec}})$ is a trapping region for $f$, then $f$ has an attractor $\Lambda \subset \Omega_{\text{trap}}$ and (18) holds for all $x \in \Lambda$.

Observe (18) implies $\lambda(x, v) > 0$ because the supremum limit is greater than or equal to the infimum limit. Theorem 3.9 is proved in the next section.

4. Relationships between words, recurrent sets, and directional derivatives

Here we look deeper at the concepts introduced in the previous section for maps of the form (12). First in Section 4.1 we take the limit $n \to \infty$ in (15) to obtain a set of symbol sequences for the forward orbit of a point $x \in \mathbb{R}^N$. The key observation is that these sequences are generated by $W$ whenever $x$ belongs to a $W$-recurrent set. Then in Section 4.2 we show that $D_v^+ f^n(x)$ exists for all $x$, $v$, and $n$, and describe it in terms of the product (17). Finally in Section 4.3 we prove Theorem 3.9.

4.1. Symbol sequences and generating word sets

**Definition 4.1:** A symbol sequence is a function $S : \mathbb{N} \to \{L, R\}$ and we write $S = S_0 S_1 S_2 \cdots$.

**Definition 4.2:** Let $W = \{W^{(1)}, \ldots, W^{(k)}\}$ be a finite set of words. We say that $W$ generates a symbol sequence $S$ if there exists a sequence $a : \mathbb{N} \to \{1, \ldots, k\}$ such that $S = W^{(a_0)} W^{(a_1)} W^{(a_2)} \cdots$. 
For example the set $W = \{R, RL, RLL\}$ generates $S$ if and only if $S_0 = R$ and $S$ does not contain the word $LLL$. In the language of coding $LLL$ is a forbidden word and the set of sequences generated by $W$ is a run-length limited shift [22].

Next define

$$
\Gamma(x) = \left\{ S : \mathbb{N} \to \{L, R\} \big| S_i \in \gamma_{set}(f^i(x)) \text{ for all } i \geq 0 \right\},
$$

(19)

which represents the $n \to \infty$ limit of (15).

**Lemma 4.3:** Let $\Omega_{rec}$ be $W$-recurrent. Then $W$ generates every $S \in \Gamma(x)$ for all $x \in \Omega_{rec}$.

**Proof:** Choose any $x \in \Omega_{rec}$ and $S \in \Gamma(x)$. Let $x_0 = x$ and $S^{(0)} = S$. By induction we have that for all $j \geq 0$ there exists $n_j \geq 1$ such that $x_{j+1} = f^{n_j}(x_j) \in \Omega_{rec}$ and $S^{(j)} = \mathcal{X}^{(j)}S^{(j+1)}$, where $S^{(j+1)} \in \Gamma(x_{j+1})$ and $\mathcal{X}^{(j)}$ is a word of length $n_j$ that can be written as a concatenation of words in $W$. Then $S = \mathcal{X}^{(0)}\mathcal{X}^{(1)}\mathcal{X}^{(2)} \cdots$ as required. □

### 4.2. A characterization of one-sided directional derivatives

So far we have constructed sets of words $\Gamma(x; n)$ and sets of symbol sequences $\Gamma(x)$ to describe the forward orbit of a point $x$. These have the advantage that they only depend on $x$, so they are relatively simple to describe and analyse. However, in order to identify the matrices $A_i$ in the expression (2) (which we use to evaluate $D^+_vf^n(x)$), we also require knowledge of $n$ (albeit only for forward orbits that intersect $\Sigma$). To this end we define

$$
\gamma((x, v)) = \begin{cases} 
L, & x_1 < 0, \text{ or } x_1 = 0 \text{ and } v_1 < 0, \\
R, & x_1 > 0, \text{ or } x_1 = 0 \text{ and } v_1 \geq 0,
\end{cases}
$$

(20)

where $(x, v)$ is an element of the tangent bundle $\mathbb{R}^N \times T\mathbb{R}^N$. The choice of the symbol $R$ when $x_1 = v_1 = 0$ is not important to the results below because in this case $A_Lv = A_Rv$ by (13). We then have the following result (given also in [33]).

**Lemma 4.4:** For any $x \in \mathbb{R}^N$ and $v \in T\mathbb{R}^N$,

$$
D^+_vf(x) = A_{\gamma((x, v))}v.
$$

(21)

**Proof:** If $x_1 > 0$ then $(x + \delta v)_1 > 0$ for sufficiently small values of $\delta > 0$. In this case $f(x + \delta v) - f(x) = f_R(x + \delta v) - f_R(x) = \delta A_Rv$, and so $D^+_vf(x) = A_Rv$. The same calculation occurs in the case $x_1 = 0$ and $v_1 \geq 0$ because we may take $f(x) = f_R(x)$ (by the continuity of $f$ on $\Sigma$) and $(x + \delta v)_1 \geq 0$ so we may similarly take $f(x + \delta v) = f_R(x + \delta v)$. The same arguments apply to $f^L$ if $x_1 < 0$, or $x_1 = 0$ and $v_1 < 0$, and result in $D^+_vf(x) = A_Lv$. □
Higher directional derivatives are dictated by the evolution of tangent vectors. For this reason we define the following map on $\mathbb{R}^N \times T\mathbb{R}^N$:

$$h((x, v)) = (f(x), D^+_v f(x)). \quad (22)$$

Since $D^+$ satisfies composition rule

$$D^+_v f^{i+j}(x) = D^+_v D^+_v f^i f^j(x), \quad \text{for any } i, j \geq 1, \quad (23)$$

the $n^{th}$ iterate of $h$ is

$$h^n((x, v)) = (f^n(x), D^+_v f^n(x)). \quad (24)$$

Then Lemma 4.4 implies

$$D^+_v f^n(x) = A\gamma(h^{n-1}((x, v)))) \cdots A\gamma(h((x, v)))A\gamma((x, v))v. \quad (25)$$

Finally we can use (17) to write this as

$$D^+_v f^n(x) = \Phi(\mathcal{W})v, \quad \text{where } \mathcal{W}_i = \gamma(h^i((x, v))) \text{ for each } i \in \{0, \ldots, n-1\}. \quad (25)$$

We have thus proved the following result.

**Proposition 4.5:** The derivative $D^+_v f^n(x)$ exists for all $x \in \mathbb{R}^N$, $v \in T\mathbb{R}^N$, and $n \geq 1$, and is given by (25).

### 4.3. A lower bound on the Lyapunov exponent

Here we work towards a proof of Theorem 3.9. Note that in Lemma 4.7 the bound on the Lyapunov exponent does not require that the cone is expanding, but gives $\lambda_{\text{bound}} > 0$ if $c > 1$.

**Lemma 4.6:** The map $h$ is invertible if and only if $f$ is invertible.

**Proof:** The first component of $h$ is $f$. If $\det(A_L) \det(A_R) \leq 0$ then $f^{-1}$ is not well-defined by Lemma 3.6 and thus $h^{-1}$ is also not well-defined.

Now suppose $\det(A_L) \det(A_R) > 0$. By Lemma 3.6 the first component of $h$ is well-defined. By Lemma 4.4 the second component of $h$ is $h_2(x, v) = A\gamma((x, v))v$. If $x_1 < 0$ then $h_2(x, v) = A_L v$ is invertible because $\det(A_L) \neq 0$. Similarly if $x_1 > 0$ then $h_2(x, v) = A_R v$ is invertible because $\det(A_R) \neq 0$. If $x_1 = 0$ then

$$h_2(x, v) = \begin{cases} 
A_L v, & v_1 < 0, \\
A_R v, & v_1 \geq 0, 
\end{cases}$$

which is equal to (12) with $b = 0$ (the zero vector) and so is invertible by Lemma 3.6.

**Lemma 4.7:** Let $W$ be a finite set of words and $M$ be given by (16). Suppose there exists an invariant cone $C$ satisfying (8) with some $c > 0$ for all $M \in M$. Let $x \in \mathbb{R}^N$ and suppose $W$
generates every \( S \in \Gamma(x) \). Then (18) is satisfied with \( \lambda_{\text{bound}} = \frac{\ln(c)}{L_{\text{max}}} \), where \( L_{\text{max}} \) is the length of the longest word(s) in \( W \). If \( f \) is invertible the same bound applies to \( f^i(x) \) for any \( i \in \mathbb{Z} \).

**Proof:** Let \( W^{(1)}, \ldots, W^{(k)} \) be the words in \( W \) and let \( L_1, \ldots, L_k \) be their lengths. Let \( v \in C \) with \( v \neq 0 \). Define the sequence

\[
S = \gamma((x, v)) \gamma(h((x, v))) \gamma(h^2((x, v))) \cdots.
\]

Then \( S \in \Gamma(x) \) thus there exists a sequence \( \alpha_j \) such that

\[
S = W^{(\alpha_0)} W^{(\alpha_1)} W^{(\alpha_2)} \cdots.
\]

For each \( j \geq 1 \), let \( n_j = L_{\alpha_0} + L_{\alpha_1} + \cdots + L_{\alpha_{j-1}} \) and let

\[
v_j = D_v^+ f^{n_j}(x).
\]

By (25) we have \( v_j = \Phi(W^{(\alpha_0)} W^{(\alpha_1)} \cdots W^{(\alpha_{j-1})}) v \). By (17) we have

\[
\Phi(W^{(\alpha_0)} W^{(\alpha_1)} \cdots W^{(\alpha_{j-1})}) = \Phi(W^{(\alpha_{j-1})}) \Phi(W^{(\alpha_0)} W^{(\alpha_1)} \cdots W^{(\alpha_{j-2})})
\]

and thus, for all \( j \geq 1 \),

\[
v_j = \Phi(W^{(\alpha_{j-1})}) v_{j-1},
\]

where \( v_0 = v \). Since \( v_0 \in C \) and \( C \) is invariant under each \( \Phi(W^{(\alpha_i)}) \), we have \( v_j \in C \) for all \( j \geq 1 \). Moreover \( \|v_j\| \geq c\|v_{j-1}\| \) and so \( \|v_j\| \geq c^j\|v_0\| \). Then by (26)

\[
\liminf_{n \to \infty} \frac{1}{n} \ln \left( \|D_v^+ f^n(x)\| \right) = \liminf_{j \to \infty} \frac{1}{n_j} \ln (\|v_j\|)
\]

\[
\geq \liminf_{j \to \infty} \frac{1}{jL_{\text{max}}} \ln (c^j\|v\|)
\]

\[
= \lambda_{\text{bound}}.
\]

Finally suppose \( f \) is invertible and choose any \( i \in \mathbb{Z} \). Write \( (\tilde{x}, \tilde{v}) = h^i(x, v) \). In the case \( i < 0 \) this is well-defined by Lemma 4.6. By the composition rule (23) we have

\[
D_v^+ f^n(\tilde{x}) = D_v^+ f^{n+i}(x),
\]

for any \( n \geq 0 \) for which \( n + i \geq 0 \). Thus by taking \( n \to \infty \) we obtain at the same bound for \( \tilde{x} = f^i(x) \).

**Proof of Theorem 3.9:** Choose any \( x \in \Omega_{\text{rec}} \). By Lemma 4.3, \( W \) generates every \( S \in \Gamma(x) \). Thus by Lemma 4.7, inequality (18) holds for some \( \lambda_{\text{bound}} > 0 \). For the second part of the theorem, an attractor \( \Lambda \subset \Omega_{\text{trap}} \) exists because \( \Omega_{\text{trap}} \) is a trapping region. Choose any \( y \in \Lambda \). We can write \( y = f^i(x) \) for some \( i \in \mathbb{R} \), thus (18) holds for \( y \) by the second part of Lemma 4.7.
5. Cones for sets of $2 \times 2$ matrices

For the remainder of the paper we apply the above methodology to maps on $\mathbb{R}^2$ with the Euclidean norm $\| \cdot \|$.

Let $M$ be a real-valued $2 \times 2$ matrix. We are interested in the behaviour of $v \mapsto Mv$, where $v \in \mathbb{R}^2$, in regards to cones. We start in Section 5.1 by deriving properties of this map for vectors of the form $v = (1, m)$, where $m \in \mathbb{R}$ is the slope of $v$. The behaviour of scalar multiples of $(1, m)$ follows trivially by linearity. The behaviour of $e_2 = (0, 1)$ can be inferred by considering the limit $m \to \infty$. However, this vector will not be of interest to us because if $W$ is given by (6), then it contains $R$. Notice $\|A_2v\| = \|v\|$ so $e_2$ cannot belong to an invariant expanding cone for $M$ given by (16).

In Section 5.2 we consider several matrices $M^{(i)}$. We use fixed points of $v \mapsto M^{(i)}v$ to construct a cone and derive conditions sufficient for the cone to be invariant and expanding.

5.1. Results for a single matrix $M$

Write $v = (1, m)$ and

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad (28)$$

We first decompose $v \mapsto Mv$ into two real-valued functions $G$ and $H$. Let

$$H(m) = \|Mv\|^2 - \|v\|^2. \quad (29)$$

This function is particularly amenable to analysis because it is quadratic in $m$:

$$H(m) = (b^2 + d^2 - 1) m^2 + 2(ab + cd)m + a^2 + c^2 - 1. \quad (30)$$

The factor $\|Mv\|/\|v\|$ by which the norm of $v$ changes under multiplication by $M$ is less than 1 if $H(m) < 0$, and greater than 1 if $H(m) > 0$.

The slope of $Mv$ is

$$G(m) = \frac{c + dm}{a + bm}, \quad (31)$$

assuming $a + bm \neq 0$. That is, $Mv$ is a scalar multiple of $(1, G(m))$. From (31),

$$\frac{dG}{dm} = \frac{\det(M)}{(a + bm)^2}, \quad (32)$$

and so if $\det(M) > 0$ (which is the case below where $M$ is a product instances of $A_L$ and $A_R$ with $\det(A_L) > 0$ and $\det(A_R) > 0$) then $G$ is increasing on any interval for which $a + bm \neq 0$. Fixed points of $G$ satisfy

$$bm^2 + (a - d)m - c = 0. \quad (33)$$

Note that $m \in \mathbb{R}$ is a fixed point of $G$ if and only if $v = (1, m)$ is an eigenvector of $M$. 
Lemma 5.1: Suppose
\[ 0 < \det(M) < \frac{1}{4} \text{trace}(M)^2 \quad \text{and} \quad b \neq 0. \tag{34} \]

Then \( G \) has exactly two fixed points. At one fixed point, call it \( m_{\text{stab}} \), we have \( \frac{dG}{dm} = \eta \), for some \( 0 < \eta < 1 \), and at the other fixed point, call it \( m_{\text{unstab}} \), we have \( \frac{dG}{dm} = \frac{1}{\eta} \). Moreover, \( m_{\text{unstab}} \) lies between \( m_{\text{stab}} \) and \( m_{\text{blow-up}} = -\frac{a}{b} \) (see for example Figure 6).

**Proof:** By (34) \( M \) has distinct eigenvalues \( \lambda_1, \lambda_2 \in \mathbb{R} \) with \( \lambda_1 \lambda_2 = \det(M) > 0 \). Without loss of generality assume \( |\lambda_1| > |\lambda_2| \). Since \( \lambda_1 \) and \( \lambda_2 \) satisfy the characteristic equation for \( M \), we find that \( m_{\text{stab}} = \frac{\lambda_1-a}{b} \) and \( m_{\text{unstab}} = \frac{\lambda_2-a}{b} \) satisfy the fixed point equation (33). These are only fixed points of \( G(m) \) because (33) is quadratic. By evaluating (32) at \( m = m_{\text{stab}} \) we obtain \( \frac{dG}{dm} = \frac{\det(M)}{\lambda_1} = \frac{\lambda_2}{\lambda_1} \). So \( \eta = \frac{\lambda_2}{\lambda_1} \) and indeed \( \eta \in (0,1) \). Similarly at \( m = m_2 \) we have \( \frac{dG}{dm} = \frac{\lambda_2}{\lambda_2} = 1 \).

Now suppose \( m_{\text{stab}} < m_{\text{blow-up}} \). By the intermediate value theorem, \( G(m) \) has a fixed point between \( m_{\text{stab}} \) and \( m_{\text{blow-up}} \) because \( \frac{dG}{dm} < 1 \) at \( m_{\text{stab}} \) whereas \( G(m) \to \infty \) as \( m \) converges to \( m_{\text{blow-up}} \) from the right. This fixed point must be \( m_{\text{unstab}} \), thus we have \( m_{\text{stab}} < m_{\text{unstab}} < m_{\text{blow-up}} \). If instead \( m_{\text{stab}} > m_{\text{blow-up}} \), an analogous argument produces \( m_{\text{blow-up}} < m_{\text{unstab}} < m_{\text{stab}} \). 

5.2. Results for a set of matrices \( M \)

Let \( M = \{ M^{(1)}, \ldots, M^{(k)} \} \) be a set of real-valued \( 2 \times 2 \) matrices. Write
\[
M^{(j)} = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix},
\]
for each \( j \in \{1, \ldots, k\} \), and let
\[
G_j(m) = \frac{c_j + djm}{a_j + bjm}, \tag{36}
\]
\[
H_j(m) = \left( b_j^2 + b_j^2 - 1 \right) m^2 + 2(a_j b_j + c_j d_j)m + a_j^2 + c_j^2 - 1. \tag{37}
\]
Figure 7. A sketch of five slope maps \((36)\). Here condition \((40)\) is satisfied, thus \(G_j(J) \subset J\), for each \(j\), and so the cone \(C_J\) \((39)\) is invariant under \(M\), by Proposition 5.2.

denote (30) and (31) applied to (35). Assume

\[
0 < \det \left( M^{(j)} \right) < \frac{1}{4} \text{trace} \left( M^{(j)} \right)^2 \quad \text{and} \quad b_j \neq 0,
\]

for each \(j\) so that each \(M^{(j)}\) satisfies the assumptions of Lemma 5.1. Let \(m^{(j)}_{\text{stab}}\) and \(m^{(j)}_{\text{unstab}}\) denote the stable and unstable fixed points of \((36)\) and let \(m_{\text{stab},\min}\) and \(m_{\text{stab},\max}\) denote the minimum and maximum values of \(\{m^{(1)}_{\text{stab}}, \ldots, m^{(k)}_{\text{stab}}\}\). Define the interval \(J = [m_{\text{stab},\min}, m_{\text{stab},\max}]\), see Figure 7, and the cone

\[
C_J = \{ t(1,m) \mid t \in \mathbb{R}, m \in J \}.
\]  

**Proposition 5.2:** If

\[
m^{(j)}_{\text{unstab}} \notin J, \quad \text{for each } j \in \{1, \ldots, k\},
\]

then \(C_J\) is invariant under \(M\). If also

\[
H_j(m) > 0, \quad \text{for each } j \in \{1, \ldots, k\} \text{ and all } m \in J,
\]

then \(C_J\) is expanding under \(M\).

**Proof:** Choose any \(j \in \{1, \ldots, k\}\). Let \(m^{(j)}_{\text{blow-up}}\) be the value of \(m\) at which \((36)\) is undefined. By assumption \(m^{(j)}_{\text{unstab}} \notin J\), so by Lemma 5.1 if \(m^{(j)}_{\text{unstab}} < m_{\text{stab},\min}\), then \(m^{(j)}_{\text{blow-up}} < m^{(j)}_{\text{unstab}}\), while if \(m^{(j)}_{\text{unstab}} > m_{\text{stab},\max}\), then \(m^{(j)}_{\text{blow-up}} > m^{(j)}_{\text{unstab}}\). In either case \(m^{(j)}_{\text{blow-up}} \notin J\), thus \(G_j|_J\) (the restriction of \(G_j\) to \(J\)) is continuous.

We now show that

\[
\min_{m \in J} G_j(m) \geq m_{\text{stab},\min}.
\]  

By (32) \(G_j|_J\) is increasing because \(\det(M^{(j)}) > 0\). Thus \(G_j|_J\) achieves its minimum at \(m = m_{\text{stab},\min}\). If \(m^{(j)}_{\text{stab}} \neq m_{\text{stab},\min}\) (otherwise (42) is trivial) then, since \(\frac{dG_j}{dm}|_{m = m^{(j)}_{\text{stab}}} < 1\) by Lemma 5.1, we have \(G_j(m) > m\) for some values of \(m < m^{(j)}_{\text{stab}}\) close to \(m^{(j)}_{\text{stab}}\). Thus \(G_j(m_{\text{stab},\min}) > m_{\text{stab},\min}\), for otherwise \(G_j\) would have another fixed point in \(J\) by the
immediate value theorem and this is not possible because \( m_{\text{stab}}^{(j)} \) is unique fixed point of \( G_j \). This verifies (42). We similarly have \( \max_{m \in J} G_j(m) \leq m_{\text{stab,max}}^{(j)} \). Thus \( G_j(m) \in J \) for all \( m \in J \). Thus \( M^{(j)} u \in C_J \) for all \( u \in C_J \). Since \( j \) is arbitrary, \( C_J \) is forward invariant under \( M \).

Finally, with \( v = (1, m) \) we have

\[
\min_{u \in C_J \setminus \{0\}} \frac{\|M^{(j)} u\|}{\|u\|} = \min_{m \in J} \frac{\|M^{(j)} v\|}{\|v\|} = \min_{m \in J} \sqrt{H_j(m)}\frac{\|v\|}{\|v\|^2} + 1 = c_j,
\]

where the minimum value \( c_j \) is achieved because \( J \) is compact. If (41) is satisfied then \( c_j > 1 \). Thus with \( c = \min_j c_j \), \( C_J \) is expanding under \( M \). ■

We complete this section by showing how the expansion condition (41) can be checked with a finite set of calculations based on the fact that each \( H_j \) is quadratic in \( m \). If \( H_j \) has two distinct real roots, then \( H_j \) is increasing at one root, call it \( m_{\text{incr}}^{(j)} \), and decreasing at the other root, call it \( m_{\text{decr}}^{(j)} \).

**Proposition 5.3:** Suppose \( H_j \) has two distinct real roots. Then

\[ H_j(m) > 0, \quad \text{for all } m \in J, \tag{43} \]

if and only if two of the following inequalities are satisfied

\[
\begin{align*}
    m_{\text{stab,max}} &= m_{\text{decr}}^{(j)}, \\
    m_{\text{decr}}^{(j)} &= m_{\text{incr}}^{(j)}, \\
    m_{\text{incr}}^{(j)} &= m_{\text{stab,min}}.
\end{align*}
\]

Notice (44)–(46) cannot all be satisfied because \( m_{\text{stab,min}} < m_{\text{stab,max}} \).

**Proof:** The proof is achieved by brute-force; there are six cases to consider.

If (44) and (45) are true, then \( H_j(m) > 0 \) for all \( m < m_{\text{decr}}^{(j)} \), which includes all \( m \in J \), thus (43) is true. If (44) and (46) are true, then \( H_j(m) > 0 \) for all \( m \in (m_{\text{decr}}^{(j)}, m_{\text{stab,max}}^{(j)}) \) so (43) is true. If (45) and (46) are true, then \( H_j(m) > 0 \) for all \( m > m_{\text{decr}}^{(j)} \) so (43) is true.

If (44) and (45) are false, then \( H_j(m) < 0 \) at \( m = m_{\text{stab,max}}^{(j)} \) so (43) is false. If (44) and (46) are false then \( H_j(m) \leq 0 \) at \( m = \min(m_{\text{decr}}^{(j)}, m_{\text{stab,max}}^{(j)}) \), which belongs to \( J \) thus (43) is false. If (45) and (46) are false, then \( H_j(m) < 0 \) at \( m = m_{\text{stab,min}}^{(j)} \) so (43) is false. ■

**Remark 5.1:** In the special case that \( b_j^2 + d_j^2 - 1 = 0 \) and \( a_j b_j + c_j d_j \neq 0 \), the function \( H_j \) has exactly one root:

\[ m_{\text{root}}^{(j)} = -\frac{a_j^2 + c_j^2 - 1}{2(a_j b_j + c_j d_j)}. \tag{47} \]

If \( a_j b_j + c_j d_j < 0 \) [resp. \( a_j b_j + c_j d_j > 0 \)] then \( H_j(m) > 0 \) for all \( m \in J \) if and only if \( m_{\text{root}} > m_{\text{stab,max}}^{(j)} \) [resp. \( m_{\text{root}} < m_{\text{stab,min}}^{(j)} \)]. This case arises in the implementation below because with \( M^{(1)} = A_R \) we have \( b_1 = 1 \) and \( d_1 = 0 \).
Figure 8. The action of the map (4) between the quadrants of \( \mathbb{R}^2 \) in the orientation-preserving setting, \( \delta_L > 0 \) and \( \delta_R > 0 \). Here \( Q_i \) denotes the closure of the \( i \)th quadrant of \( \mathbb{R}^2 \). For example if \( x \in Q_1 \) then \( f(x) \in Q_3 \cup Q_4 \).

6. The construction of a trapping region

In this section we study the 2d BCNF (4) in the orientation-preserving case: \( \delta_L > 0 \) and \( \delta_R > 0 \). In this case the sign of \( f(x)_2 \) is opposite to the sign of \( x_1 \). This implies points map between the quadrants of \( \mathbb{R}^2 \) as shown in Figure 8. For example if \( x \) belongs to the first quadrant, then \( f(x) \) belongs to either the third quadrant or the fourth quadrant.

Let

\[
\Pi_L = \{ x \in \mathbb{R}^2 \mid x_1 \leq 0 \} ,
\]

\[
\Pi_R = \{ x \in \mathbb{R}^2 \mid x_1 \geq 0 \} ,
\]

denote the closed left and right half-planes. Also write

\[
f_L(x) = A_Lx + b,
\]

\[
f_R(x) = A_Rx + b,
\]

for the left and right half-maps of (4).

6.1. Preimages of the switching manifold under \( f_L \)

With \( \delta_L \neq 0 \) the set \( f_L^{-p}(\Sigma) \) is a line for all \( p \geq 1 \). Below we use these lines to partition \( \Pi_L \) into regions \( D_p \) whose points escape \( \Pi_L \) after exactly \( p \) iterations of \( f \), see already Figure 11.

If \( f_L^{-p}(\Sigma) \) is not vertical, let \( m_p \) denote its slope and let \( c_p \) denote its \( x_2 \)-intercept with \( \Sigma \). That is,

\[
f_L^{-p}(\Sigma) = \{ x \in \mathbb{R}^2 \mid x_2 = m_px_1 + c_p \} . \tag{48}
\]

By iterating backwards under \( f_L \) we obtain \( m_1 = -\tau_L, c_1 = -1 \), and for all \( p \geq 1 \),

\[
m_{p+1} = -\frac{\delta_L}{m_p} - \tau_L, \tag{49}
\]

\[
c_{p+1} = -\frac{c_p}{m_p} - 1, \tag{50}
\]

whenever \( m_p \neq 0 \). As an example Figure 9 shows (49) with \( p^* = 3 \).
**Figure 9.** A typical plot of (49). Here $p^* = 3$ (see Definition 6.1).

**Definition 6.1:** Let $p^*$ be the smallest $p \geq 1$ for which $m_p \geq 0$, with $p^* = \infty$ if $m_p < 0$ for all $p \geq 1$.

Next we establish monotonicity of $f_{\Sigma_1}^{-p}(\Sigma)$ in $\Pi_L$ and provide an explicit expression for $p^*$.

**Lemma 6.2:** The sequence $\{m_p\}_{p=1}^{p^*}$ is increasing; the sequence $\{c_p\}_{p=1}^{p^*}$ is decreasing.

With $p^*$ finite, this can be more suitably written as $m_1 < m_2 < \cdots < m_{p^*}$ and $c_1 > c_2 > \cdots c_{p^*}$.

**Proof:** Since $m_1 = -\tau_L$, if $\tau_L \leq 0$ then $p^* = 1$ and the result is trivial. Suppose $\tau_L > 0$ (for which $p^* \geq 2$). Let $g(m) = -\frac{\delta_L}{m} - \tau_L$ denote the map (49). Since $g(m_1) - m_1 = \frac{\delta_L}{\tau_L} > 0$ and $\frac{dg}{dm} = \frac{\delta_L}{m^2} > 0$ for all $m < 0$, the forward orbit of $m_1$ under $g$ is increasing while $m < 0$.

That is, $\{m_p\}_{p=1}^{p^*}$ is increasing.

We now prove $\{c_p\}_{p=1}^{p^*}$ is decreasing by induction. Observe $c_1 > c_2$ because $c_1 = -1$ and $c_2 = -\frac{1}{\tau_L} - 1$ (where $\tau_L > 0$). Thus it remains to consider $p^* \geq 3$. Suppose $c_p > c_{p+1}$ for some $p \in \{1, \ldots, p^* - 2\}$ (this is our induction hypothesis). It remains to show $c_{p+1} > c_{p+2}$. Rearranging (50) produces

$$\frac{c_{p+1} + 1}{-c_p} = \frac{1}{m_p}.$$  

But $c_{p+1} < c_p < 0$ and $m_p < m_{p+1} < 0$ thus

$$\frac{c_{p+1} + 1}{-c_{p+1}} > \frac{1}{m_{p+1}},$$

which is equivalent to $c_{p+1} > -\frac{c_{p+1} + 1}{m_{p+1}} - 1 = c_{p+2}$. \[\square\]
Figure 10. The value $p^*$ of Definition 6.1 as given by Lemma 6.2.

Figure 11. An example of the first few preimages of the switching manifold $\Sigma$ under the left half-map $f_L$. These lines are described by (48) and bound the regions $D_p$ of Definition 6.4.

Proposition 6.3: The value $p^*$ of Definition 6.1 is given by

$$p^* = \begin{cases} 
1, & \tau_L \leq 0, \\
\left\lceil \frac{\pi}{\phi} - 1 \right\rceil, & 0 < \tau_L < 2\sqrt{\delta_L}, \\
\infty, & \tau_L \geq 2\sqrt{\delta_L}, 
\end{cases}$$

(51)

where $\phi = \cos^{-1}\left(\frac{\tau_L}{2\sqrt{\delta_L}}\right) \in (0, \frac{\pi}{2})$, see Figure 10.

Proof: Since $m_1 = -\tau_L$, if $\tau_L \leq 0$ then $p^* = 1$. If $\tau_L \geq 2\sqrt{\delta_L}$ then $m_\infty = \frac{1}{2}(-\tau_L - \sqrt{\tau_L^2 - 4\delta_L})$ is a fixed point of (49), call this map $g(m)$. Moreover $m_1 < m_\infty < 0$ and $g(m)$ is $m$ for all $m_1 \leq m < m_\infty$ so $m_p \to m_\infty$ as $p \to \infty$. Thus $p^* = \infty$ in this case.

If $0 < \tau_L < 2\sqrt{\delta_L}$ then the eigenvalues of $A_L$ are $\lambda_1 = \sqrt{\delta_L} e^{i\phi}$ and $\lambda_2 = \sqrt{\delta_L} e^{-i\phi}$. Then $m_1 = -(\lambda_1 + \lambda_2)$ and (49) can be written as $m_{p+1} = -\frac{\lambda_1 + \lambda_2}{m_p} - (\lambda_1 + \lambda_2)$. From these we obtain, by induction on $p$, that while $\lambda_1 \neq \lambda_2$ we have

$$m_p = \frac{\lambda_1^{p+1} - \lambda_2^{p+1}}{\lambda_2^p - \lambda_1^p},$$
Definition 6.4: Let 
By (52), if \( \frac{\pi}{p+1} \leq \phi < \frac{\pi}{p} \), then \( m_p \geq 0 \) and \( m_j < 0 \) for all \( j \in \{1, \ldots, p-1\} \), so \( p^* = p \). Since \( \frac{\pi}{p+1} \leq \phi < \frac{\pi}{p} \) is equivalent to \( p = \lfloor \frac{\pi}{\phi} - 1 \rfloor \), the proof is completed. □

6.2. Partitioning the left half-plane by the number of iterations required to escape

By Lemma 6.2, in \( \Pi_L \) each \( f^p_L (\Sigma) \) is located below \( f^{(p-1)}_L (\Sigma) \), for \( p = 2, \ldots, p^* \), see Figure 11. This implies that the regions \( D_p \subset \Pi_L \), defined below, are disjoint.

Definition 6.5: For all finite \( p \in \{2, \ldots, p^*\} \) let

\[
D_p = \{ x \in \Pi_L \mid m_p x_1 + c_p < x_2 \leq m_{p-1} x_1 + c_{p-1} \}. \tag{54}
\]

If \( p^* < \infty \) also let

\[
D_{p^*+1} = \{ x \in \Pi_L \mid x_2 \leq m_p x_1 + c_{p^*} \}. \tag{55}
\]

Notice that if \( p^* < \infty \) then \( D_1, \ldots, D_{p^*+1} \) partition \( \Pi_L \). We now look the number of iterations of \( f_L \) required for \( x \in \Pi_L \) to escape \( \Pi_L \).

Definition 6.6: Given \( x \in \Pi_L \) let \( \chi_L(x) \) be the smallest \( p \geq 1 \) for which \( f^p_L(x) \notin \Pi_L \), with \( \chi_L(x) = \infty \) if there exists no such \( p \).

Proposition 6.6: Let \( x \in \Pi_L \) and \( p \in \{1, \ldots, p^* + 1\} \) be finite. Then \( x \in D_p \) if and only if \( \chi_L(x) = p \).

Proof: We first show that \( x \in D_p \) implies \( \chi_L(x) = p \) for all \( p \in \{1, \ldots, p^* + 1\} \) by induction on \( p \). If \( x \in D_1 \), then \( x_2 > -\tau_L x_1 - 1 \) (recalling \( m_1 = -\tau_L \) and \( c_1 = -1 \)) and so \( f_L(x)_1 = \tau_L x_1 + x_2 + 1 > 0 \), hence \( \chi_L(x) = 1 \). Suppose \( x \in D_p \) implies \( \chi_L(x) = p \) for some \( p \in \{1, \ldots, p^*\} \) (this is our induction hypothesis). Choose any \( x \in D_{p+1} \). Then \( m_{p+1} x_1 + c_{p+1} < x_2 \leq m_p x_1 + c_p \). By using (4), (49), and (50) we obtain \( m_p f_L(x)_1 + c_p < f_L(x)_2 \leq m_{p-1} f_L(x)_1 + c_{p-1} \), except if \( p = 1 \) the latter inequality is absent. Thus \( f_L(x) \in D_p \) and so \( \chi_L(f_L(x)) = p \) by the induction hypothesis. Also \( f_L(x)_1 \leq 0 \) (because \( x \notin D_1 \)), therefore \( \chi_L(x) = p + 1 \).

If \( p^* < \infty \) the converse is true because \( D_1, \ldots, D_{p^*+1} \) partition \( \Pi_L \). It remains show in the case \( p^* = \infty \) that if \( x \in E = \Pi_L \setminus \bigcup_{p=1}^{\infty} D_p \), then \( \chi_L(x) = \infty \). We have

\[
E = \{ x \in \Pi_L \mid x_2 \leq m_\infty x_1 + c_\infty \},
\]

where \( m_\infty = \lim_{p \to \infty} m_p \) is given in the proof of Proposition 6.3 and \( c_\infty = \lim_{p \to \infty} c_p = -\frac{m_\infty}{m_\infty + 1} \). If \( x \in E \) then \( f_L(x)_2 \leq m_\infty f_L(x)_1 + c_\infty \) (obtained by repeating the calculations in the above induction step) and \( f_L(x)_1 = \tau_L x_1 + x_2 + 1 \leq c_\infty + 1 \) (obtained by using also \( x_1 \leq 0 \)). Thus \( f_L(x)_1 < 0 \) (because \( c_\infty < -1 \) by Lemma 6.2). Therefore \( f_L(x) \in E \) which shows that \( E \) is forward invariant under \( f_L \). Therefore \( \chi_L(x) = \infty \) for any \( x \in E \). □
Figure 12. A plot of \( \Omega \) (shaded) and \( f(\Omega) \) (striped). The values in Definition 6.7 are \( r = 4 \) (so \( Z = f^4(X) \)) and \( \ell = 5 \) (so \( V = f^{-4}(X) \)). Here (58)–(60) are satisfied so \( f(\Omega) \subseteq \Omega \) by Proposition 6.10.

### 6.3. A forward invariant region and a trapping region

Let \( \beta > 0 \) and \( X = (0, \beta) \). Here we use the first few images and preimages of \( X \) to form a polygon \( \Omega \). To do this we require the following assumption on \( X \):

\[
\text{There exist } i, j \geq 1 \text{ such that } f^i(X) \in \Pi_L \text{ and } f^{-j}(X) \in \Pi_R. \tag{56}
\]

**Definition 6.7:** Let \( r \) and \( \ell \) be the smallest values of \( i \) and \( j \) satisfying (56), respectively.

That is, \( f^i(X)_1 > 0 \) for all \( i = 1, \ldots, r - 1 \) and \( f^r(X)_1 \leq 0 \). Also \( f^{-j}(X)_1 < 0 \) for all \( j = 1, \ldots, \ell - 1 \) and \( f^{-\ell}(X)_1 \geq 0 \). Notice \( r \geq 2 \) because \( f(X)_1 = \beta + 1 > 0 \). Also \( \ell \geq 2 \) because \( f^{-1}(X)_1 = -\frac{\beta}{\delta_L} < 0 \).

In what follows \( \overrightarrow{PQ} \) denotes the line through distinct points \( P, Q \in \mathbb{R}^2 \). Proofs of the next three results are deferred to the end of this section.

**Lemma 6.8:** Suppose (56) is satisfied. Let \( Z = f^r(X) \) and let \( Y \) denote the intersection of \( Z f^{-1}(Z) \) with \( \Sigma \). Let \( V = f^{-(\ell-1)}(X) \) and let \( U \) denote the intersection of \( V f(V) \) with \( f(\Sigma) \), see Figure 12. The closed polygonal chain formed by connecting the points

\[
U, f^{-(\ell-2)}(X), \ldots, f^{-1}(X), X, f(X), \ldots, f^{r-1}(X), Z, \tag{57}
\]

in order, and then from \( Z \) back to \( U \), has no self-intersections (it is a Jordan curve).

**Definition 6.9:** Let \( \Omega \) be the polygonal chain of Lemma 6.8 and its interior.

Now we show that if three conditions are satisfied then \( \Omega \) is forward invariant and we can shrink it by an arbitrarily small amount to obtain a trapping region, \( \Omega_{\text{trap}} \).
Proposition 6.10: Suppose (56) is satisfied and
\[
Y \text{ lies above } f^{-1}(U),
\]
\[
Z \text{ lies above } f^{-1}(U)V,
\]
\[
Z \text{ lies to the right of } \overrightarrow{Vf(V)}.
\]

Then \( \Omega \) is forward invariant under \( f \). Moreover, for all \( \varepsilon > 0 \) there exists a trapping region \( \Omega_{\text{trap}} \subset \Omega \) for \( f \) with \( d_H(\Omega, \Omega_{\text{trap}}) \leq \varepsilon \), where \( d_H \) denotes Hausdorff distance.²

The next result allows us to apply Theorem 3.9. The actual application to Theorem 3.9 is detailed in the next section.

Proposition 6.11: Suppose (56), (58), (59), and (60) are satisfied. Let
\[
\Omega_{\text{rec}} = \{ x \in \Omega \mid x_1 > 0 \},
\]
and \( W \) be given by (6) with
\[
p_{\text{max}} = \max [\chi_L(Y), \chi_L(Z)].
\]

Then \( \Omega_{\text{rec}} \) is \( W \)-recurrent and \( \Omega \subset \bigcup_{i=-\ell}^{0} f^i(\Omega_{\text{rec}}) \).

Proof of Lemma 6.8: Here we use the notation \( [A,B] \) to denote the line segment \( \{(1-s)A + sB \mid 0 \leq s < 1\} \) for points \( A,B \in \mathbb{R}^2 \).

The points \( X \) and \( f(X) \) belong to \( \Pi_R \), thus the line segment \( [X,f(X)] \), call it \( \zeta \), is contained in \( \Pi_R \). The points \( f^{-(\ell-1)}(X), \ldots, f^{-2}(X), f^{-1}(X) \) all belong to \( \text{int}(\Pi_L) \), thus for each \( i \in \{1, \ldots, \ell - 1\} \) the line segment \( L_i = [f^{-i}(X), f^{-(i-1)}(X)] \) is contained in \( \text{int}(\Pi_L) \). These line segments are mutually disjoint because if \( L_i \) and \( L_j \), with \( i < j \), intersect at some point \( P \), then \( P \in L_i \) implies \( f^i(P) \in \zeta \) and so \( f^i(P)_1 \geq 0 \), while \( P \in L_j \) implies \( f^j(P) \in L_{j-i} \) and so \( f^j(P)_1 < 0 \), which is a contradiction. By a similar argument the line segments \( [f^i(X), f^{i+1}(X)] \), for \( i \in \{1, \ldots, r - 1\} \), are mutually disjoint and contained in \( x_2 \leq 0 \) (whereas \( \zeta \) is contained in \( x_2 > 0 \)). This shows that the chain formed by connecting the points (57) has no self-intersections. The addition of \([Z,U]\) introduces no intersections because \([Z,U]\) and part of \([f_R^{r-1}(X), Z]\) are the only components of the chain that belong to the third quadrant. \( \blacksquare \)

Proof of Proposition 6.10: Step 1 — Interior angles of \( \Omega \).

Define \( \theta : \mathbb{R}^2 \to [0, 2\pi) \) as follows: \( \theta(x) \) is the angle at \( x \) from \([f_L^{-1}(x), x]\) anticlockwise to \([x,f_L(x)]\), see Figure 13. Also define \( g(x) = f_L(x) - x \) and
\[
S(x) = g(x) \land g(f_L^{-1}(x)),
\]
where we have defined the wedge product \( A \land B = A_1B_2 - A_2B_1 \). Through a little algebraic manipulation we can obtain
\[
\sin(\theta(x)) = \frac{S(x)}{\|g(x)\| \|g(f_L^{-1}(x))\|}.
\]
From \( f_L(x) = A_Lx + b \) we obtain the formula \( g(f_L(x)) = A_Lg(x) \). From this and (63) we obtain

\[
S(f_L(x)) = \det(A_L)S(x). \tag{65}
\]

Since \( \det(A_L) > 0 \) we can conclude that sign of \( S \) (and thus also the sign of \( \sin(\theta) \)) is constant along orbits of \( f_L \). In particular, \( \sin(\theta(f_L^{-i}(X))) \) has the same sign for each \( i \in \{0, 1, \ldots, \ell - 2\} \). Thus the angles \( \theta(f_L^{-i}(X)) \) must be all less than \( \pi \), all equal to \( \pi \), or all greater than \( \pi \). But the path connecting \( X, f^{-1}(X), \ldots, f^{-\ell}(X) \) (where \( f = f_L \)) includes \( X \) on the positive \( x_2 \)-axis, \( U \) on the negative \( x_1 \)-axis, and \( f_L^{-1}(U) \) on the negative \( x_2 \)-axis, and has no self-intersections (Lemma 6.8). Therefore the angles are all less than \( \pi \). By applying a similar argument to \( f_R \) we conclude that all interior angles of \( \Omega \) are less than \( \pi \), except possibly at the points \( U \) and \( Z \).

**Step 2** — Convex subsets of \( \Omega \).

We now define two convex subsets of \( \Omega \) (one in \( x_2 \leq 0 \) and one in \( x_2 \geq 0 \). Let \( \Omega_{\text{upper}} \) be the polygon formed by connecting the points

\[
U, f^{-(\ell-2)}(X), \ldots, f^{-1}(X), X, f(X),
\]

in order, and from \( f(X) \) back to \( U \). Let \( \Omega_{\text{lower}} \) be the polygon formed by connecting the points

\[
f(X), f^2(X), \ldots, f^{\ell-1}(X), Z, f(Y),
\]

in order, and from \( f(Y) \) back to \( f(X) \). Since \( U \) and \( f(X) \) lie on \( x_2 = 0 \) while all other vertices of \( \Omega_{\text{upper}} \) lie in \( x_2 \geq 0 \), the interior angles of \( \Omega_{\text{upper}} \) at \( U \) and \( f_R(x) \) are less than \( \pi \). Thus by the previous result \( \Omega_{\text{upper}} \) is convex. For similar reasons, \( \Omega_{\text{lower}} \) is also convex.

**Step 3** — Consequences of assumptions (58)–(60).

All points between \( U = (U_1, 0) \), where \( U_1 < 0 \), and \( f_R(X) = (\beta + 1, 0) \), where \( \beta > 0 \), belong to \( \text{int}(\Omega) \). This includes \( O = (0, 0) \) and \( f(O) = (1, 0) \). Also \( f(Y) \in \text{int}(\Omega) \) because (58) implies that \( f(Y) \) lies between \( U \) and \( f(O) \). We now show \( f(Z) \in \text{int}(\Omega) \). Assumptions (59) and (60) imply that either \( Z = Y \) (in which case \( f(Z) \in \text{int}(\Omega) \) is immediate) or \( Z \) belongs to the interior of the quadrilateral \( Q = f^{-1}(U)VUO \). Each vertex of \( Q \) belongs to \( \Pi_L \) where \( f = f_L \) is affine, thus \( f(Q) \) is the quadrilateral \( Uf(V)f(U)f(O) \). Each vertex of \( f(Q) \) belongs to \( \Omega_{\text{upper}} \), which is convex, thus \( f(Q) \cap \Omega_{\text{upper}} \). Since \( Z \in \text{int}(Q) \) we have that \( f(Z) \in \text{int}(\Omega_{\text{upper}}) \subset \text{int}(\Omega) \).

**Step 4** — Forward invariance of \( \Omega \).
Write $\Omega = \Omega_L \cup \Omega_R$ where

$$\Omega_L = \Omega \cap \Pi_L,$$ 
$$\Omega_R = \Omega \cap \Pi_R.$$ 

Observe $f(\Omega) = f_L(\Omega_L) \cup f_R(\Omega_R)$. Since $f_L$ is affine, $f_L(\Omega_L)$ is a polygon. Evidently its vertices all belong to $\Omega_{upper}$. Since $\Omega_{upper}$ is convex, $f_L(\Omega_L) \subset \Omega_{upper} \subset \Omega$. By a similar argument, $f_R(\Omega_R) \subset \Omega_{lower} \subset \Omega$, thus $\Omega$ is forward invariant.

**Step 5 —** Define $\Omega_{trap}$.

Let $P^{(1)}, \ldots, P^{(\ell + r)}$ denote the points (57) in order. That is, $P^{(1)} = U$ around to $P^{(\ell + r)} = Z$. For each $j \in \{1, \ldots, \ell + r\}$ define

$$P^{(j)}_\varepsilon = \left(1 - \frac{\varepsilon^j}{\|P^{(j)}\|}\right)P^{(j)}, \quad (68)$$

and assume $\varepsilon$ is small enough that $1 - \frac{\varepsilon^j}{\|P^{(j)}\|} > 0$ for each $j$. Each $P^{(j)}_\varepsilon$ is the result of moving from $P^{(j)}$ a distance $\varepsilon^j$ towards $O$, see Figure 14. Let $\Omega_{trap}$ be the polygon with vertices $P^{(j)}_\varepsilon$ (connected in the same order as for $\Omega$). Immediately we have $d_H(\Omega, \Omega_{trap}) \leq \varepsilon$. Also $\Omega_{trap} \subset \Omega$ because $[P^{(j)}_\varepsilon, O] \subset \Omega$ for each $j$.

**Step 6 —** Convex subsets of $\Omega_{trap}$.

Analogous to $\Omega_{upper}$ and $\Omega_{lower}$, let $\Omega_{trap,upper} \subset \Omega_{trap}$ be the polygon in $x_2 \geq 0$ formed by connecting the points

$$P^{(1)}_\varepsilon, P^{(2)}_\varepsilon, \ldots, P^{(\ell + 1)}_\varepsilon, \quad (69)$$

in order, and from $P^{(\ell + 1)}_\varepsilon$ back to $P^{(1)}_\varepsilon$. Notice $P^{(1)}_\varepsilon$ and $P^{(\ell + 1)}_\varepsilon$ lie on $x_2 = 0$. Also let $Y_\varepsilon \in \Sigma$ denote the intersection of $[P^{(\ell + r)}_\varepsilon, P^{(\ell + r - 1)}_\varepsilon]$ with $\Sigma$ and let $\Omega_{trap,lower} \subset \Omega_{trap}$ be the

![Figure 14. A plot of $\Omega$ (unshaded), $\Omega_{trap}$ (shaded), and $f(\Omega_{trap})$ (striped). Here $r = 4$ and $\ell = 5$ as in Figure 12. The set $\Omega_{trap}$ is plotted by using $\varepsilon = 0.5$ in (68).](image-url)
Figure 15. Constructive elements produced by Algorithm 7.1 for (4) with (5) and \((\tau_L, \tau_R) = (0.7, -1.4)\). Here the algorithm obtains \(\beta = 0.25\) and returns \(\chi_{\text{chaos}} = \text{true}\). Panel (a) shows the forward invariant region \(\Omega\) (see Figure 12), the regions \(D_p\) (see Figure 11), and a numerically computed attractor. Panel (b) shows the slope maps \(G_j\) (36) and the functions \(H_j\) (37) for \(j = 1, 2\).

Polygon in \(x_2 \leq 0\) formed by connecting the points

\[
p^{(\ell+1)} , p^{(\ell+2)} , \ldots , p^{(\ell+r)} , f(Y),
\]

in order, and from \(f(Y)\) back to \(p^{(\ell+1)}\). Notice \(f(Y)\) lies on \(x_2 = 0\) close to \(f(Y)\). Each interior angle of \(\Omega_{\text{trap,upper}}\) and \(\Omega_{\text{trap,lower}}\) is at most an order-\(\varepsilon\) perturbation of the corresponding interior angle of \(\Omega_{\text{upper}}\) or \(\Omega_{\text{lower}}\). All interior angles of \(\Omega_{\text{upper}}\) and \(\Omega_{\text{lower}}\) are less than \(\pi\), so the same is true for \(\Omega_{\text{trap,upper}}\) and \(\Omega_{\text{trap,lower}}\) assuming \(\varepsilon\) is sufficiently small. That is, \(\Omega_{\text{trap,upper}}\) and \(\Omega_{\text{trap,lower}}\) are convex.

**Step 7** — The set \(\Omega_{\text{trap}}\) is a trapping region.

Write \(\Omega_{\text{trap}} = \Omega_{\text{trap,L}} \cup \Omega_{\text{trap,R}}\) where

\[
\Omega_{\text{trap,L}} = \Omega_{\text{trap}} \cap \Pi_L,
\]

\[
\Omega_{\text{trap,R}} = \Omega_{\text{trap}} \cap \Pi_R,
\]

and observe \(f(\Omega_{\text{trap}}) = f_L(\Omega_{\text{trap,L}}) \cup f_R(\Omega_{\text{trap,R}})\).

The vertices of \(\Omega_{\text{trap,L}}\) are \(Y, p^{(\ell)}\), and \(p^{(j)}\) for \(j = 1, \ldots, \ell\). We now show that, if \(\varepsilon\) is sufficiently small, then these vertices all map under \(f = f_L\) to either \(\text{int}(\Omega_{\text{trap,upper}})\) or to a point on \(x_2 = 0\) in the open line segment \(I = (p^{(1)} , p^{(\ell+1)})\). Since \(\Omega_{\text{trap,upper}}\) is convex this implies \(f_L(\Omega_{\text{trap,L}}) \subset \text{int}(\Omega_{\text{trap,upper}}) \cup I\). Consequently \(f_L(\Omega_{\text{trap,L}}) \subset \text{int}(\Omega_{\text{trap}})\) because \(I \subset \text{int}(\Omega_{\text{trap}})\).

Certainly \(f(Y), f(p^{(\ell)}) \in I\), assuming \(\varepsilon\) is sufficiently small. If \(Z \neq Y\) then \(f(p^{(\ell+r)}) \in \text{int}(\Omega_{\text{trap,upper}})\), assuming \(\varepsilon\) is sufficiently small, because \(f(Z) \in \text{int}(\Omega)\). If \(Z = Y\) then \(f(p^{(\ell+r)}) \in I\). Now choose any \(j = 1, \ldots, \ell - 1\). By definition, \(p^{(j)} = (1-s)P^{(j)} + sO\), with \(s = \frac{\parallel P_j \parallel}{\parallel P_j \parallel}\). In \(\Pi_L, f = f_L\) is affine, so we have

\[
f(p^{(j)}) = \begin{cases} 
(1-s)f(U) + sf(O), & j = 1, \\
(1-s)P^{(j+1)} + sf(O), & j = 2, \ldots, \ell - 1,
\end{cases}
\]
Therefore, for \( j \neq 1 \), \( f(P^{(j)}_{\varepsilon}) \) is the result of moving from \( P^{(j+1)} \) a distance \( \varepsilon \|P^{(j+1)} - f(O)\|/\|P^{(j)}\| \) towards \( f(O) \). Thus \( f(P^{(j)}_{\varepsilon}) \) belongs to the triangle \( P^{(j+1)}P^{(j+2)}O \), assuming \( \varepsilon \) is sufficiently small, because \( P^{(j+1)} \) and \( P^{(j+2)} \) lie in \( x_2 \geq 0 \) with \( P^{(j+2)} \) located clockwise (with respect to \( O \)) from \( P^{(j+1)} \). This is true in the case \( j = 1 \) also. The distance from \( f(P^{(j)}_{\varepsilon}) \) to \( [P^{(j+1)}, P^{(j+2)}] \) is proportional to \( \varepsilon \), while \( d_H([P^{(j+1)}_{\varepsilon}, P^{(j+2)}_{\varepsilon}], [P^{(j+1)}, P^{(j+2)}]) \) is proportional to \( \varepsilon^{j+1} \). Therefore, assuming \( \varepsilon \) is sufficiently small, \( f(P^{(j)}_{\varepsilon}) \) lies in the triangle \( P^{(j+1)}P^{(j+2)}O \) and not on the line segment \([P^{(j+1)}_{\varepsilon}, P^{(j+2)}_{\varepsilon}] \). Thus \( f(P^{(j)}_{\varepsilon}) \in \text{int}(\Omega_{\text{trap,upper}}) \) and this completes our demonstration that \( f_{\varepsilon}(\Omega_{\text{trap,upper}}) \subset \text{int}(\Omega_{\text{trap}}) \).

From similar arguments it follows that \( f_{\varepsilon}(\Omega_{\text{trap,lower}}) \subset \text{int}(\Omega_{\text{trap,lower}}) \cup I \), assuming \( \varepsilon \) is sufficiently small, and so \( f_{\varepsilon}(\Omega_{\text{trap,lower}}) \subset \text{int}(\Omega_{\text{trap}}) \). Hence \( \Omega_{\text{trap}} \) is a trapping region for \( f \).

**Proof of Proposition 6.11:** We first show \( \Omega \subset \bigcup_{i=-\ell}^{0} f^i(\Omega_{\text{rec}}) \). Choose any \( x \in \Omega \). If \( x_1 > 0 \) then \( x \in \Omega_{\text{rec}} \). If \( x_1 \leq 0 \) then \( x \in D_p \), for some \( i \in \{1, \ldots, \ell \} \). The upper bound \( i = \ell \) is a consequence of (58)–(60) because \( V \) lies above \( f_{\ell}^{(i)}(\Sigma) \) and \( f^{(-k)}(U) \) lies on or above \( f_{\ell}^{(-k)}(\Sigma) \). Thus by Proposition 6.6, \( f^{(i)}(x)_1 > 0 \), and so \( f^{(i)}(x) \in \Omega_{\text{rec}} \) because \( \Omega \) is forward invariant.

Now choose any \( y \in \Omega_{\text{rec}} \). If \( f(y) \in \Omega_{\text{rec}} \), let \( n = 1 \) and observe that the first symbol of any \( S \in \Gamma(y) \) is \( R \), which belongs to \( W \). So now suppose \( f(y) \notin \Omega_{\text{rec}} \). Also suppose \( f(Y)_1 < 0 \), then \( f(y) \) belongs to the quadrilateral \( YZf(Y)O \) (if instead \( f(Y)_1 \geq 0 \) then the following arguments can be applied to the part of \( YZf(Y)O \) that belongs to \( \Pi_L \)). We have \( \chi_L(f(y)) \leq \max[\chi_L(Y), \chi_L(Z), \chi_L(f(Y)), \chi_L(O)] \) (this follows from the linear ordering of the regions \( D_p \)). But \( \chi_L(f(Y)) = \chi_L(Y) - 1 \) and \( \chi_L(O) = 1 \), thus \( \chi_L(f(Y)) \leq p_{\max} \).

Let \( n = \chi_L(f(y)) + 1 \). Then \( f^{n}(y)_1 > 0 \) and so \( f^{n}(y) \in \Omega_{\text{rec}} \) because \( \Omega \) is forward invariant. Also \( f^j(y)_1 \leq 0 \) for all \( j = 1, \ldots, n - 1 \) with \( f^j(y)_1 = 0 \) only possible for \( j = 1 \) and \( j = n-1 \). In summary, \( y_1 > 0, f(y)_1 \leq 0, f^j(y)_1 < 0 \) for all \( j = 2, \ldots, n - 2, f^{n-1}(y)_1 \leq 0 \), and \( f^n(y) \in \Omega_{\text{rec}} \). Thus there are four possibilities for the first \( n \) symbols of \( S \in \Gamma(y) \): \( RL^{n-1}, RRL^{n-2}, RLL^{n-2}R \), and \( RRL^{n-3}R \) (the last possibility can only arise if \( f(y)_1 = 0 \) and \( f^{n-1}(y)_1 = 0 \)). All four words can be expressed as a concatenation of words in \( W \) (because \( n - 1 \leq p_{\max} \)). Thus \( \Omega_{\text{rec}} \) is \( W \)-recurrent.

**7. An algorithm for detecting a chaotic attractor**

In the previous two sections we obtained sufficient conditions for the assumptions of Theorem 3.9 to hold with a trapping region for the 2d BCNF (4). In Section 7.1 we summarize these conditions and state Algorithm 7.1 (in pseudo-code) for testing their validity. In Section 7.2 we further discuss the application of the algorithm to the slice of parameter space shown in Figure 1.

**7.1. Statement and proof of the algorithm**

The polygon \( \Omega \) constructed in Section 6.3 typically satisfies (58)–(60) for some interval of \( \beta \)-values (where \( X = (0, \beta) \)). Within this interval, smaller values of \( \beta \) tend to correspond to smaller values of \( \chi_L(Y) \) and \( \chi_L(Z) \) and so produce a smaller value for \( p_{\max} \), (62). Smaller values of \( p_{\max} \) are more favourable for the cone \( C_1 \) (39) to be well-defined, invariant, and
expanding. This is because with a smaller value of $p_{\text{max}}$ there are less matrices in $M$ and therefore fewer inequalities that need to be satisfied.

For these reasons we search for a suitable value of $\beta$ by iteratively increasing its value in steps of size $\beta_{\text{step}}$ from $\beta_{\text{min}}$ up to (at most) $\beta_{\text{max}}$. To produce Figure 1 we used

$$
\beta_{\text{step}} = 0.01, \quad \beta_{\text{min}} = 0.01, \quad \beta_{\text{max}} = 5.
$$

For a given value of $\beta$ there are five groups of conditions that need to be checked. These are labelled (C1)–(C5) in Algorithm 7.1. First we require $\Omega$ to be well-defined. This is established by showing that $r$ and $\ell$ of Definition 6.7 exist. To produce Figure 1 this was implemented by iterating $X$ backwards and forwards up to maximum allowed values

$$
r_{\text{max}} = 15, \quad \ell_{\text{max}} = 15.
$$

Second we check conditions (58)–(60). If these are satisfied then $\Omega$ is forward invariant and in Algorithm 7.1 this fixes the value of $\beta$. We then evaluate $p_{\text{max}}$ by iterating $Y$ and $Z$ under $f$, (62). The remaining three conditions are that the cone $C_{j}$ is well-defined, that $C_{j}$ is invariant, and that $C_{j}$ is expanding. The computations involved in the last two steps are elementary because $G_{j}(m)$ and $H_{j}(m)$ are polynomials of degree two or less. Algorithm 7.1 registers its success or failure by the termination value of the Boolean variable $\chi_{\text{chaos}}$.

The theorem below assumes calculations are done exactly. For Figure 1 calculations were performed with rounding at 16 digits.

**Theorem 7.1:** Let $f$ be a map of the form (4) with $\delta_{L}, \delta_{R} > 0$. If Algorithm 7.1 outputs $\chi_{\text{chaos}} = \text{true}$ then $f$ has an attractor with a positive Lyapunov exponent.

**Proof:** Suppose Algorithm 7.1 outputs $\chi_{\text{chaos}} = \text{true}$. Then (C1)–(C5) all hold for some fixed $\beta > 0$. Since (C1) holds, $\Omega$ is well-defined by Lemma 6.8. Since (C2) holds, $f$ has a trapping region $\Omega_{\text{trap}} \subset \Omega$ by Proposition 6.10. Thus $f$ has an attractor $\Lambda \subset \Omega_{\text{trap}}$. Let $W$ be given by (6) and $\Omega_{\text{rec}}$ be given by (61). Then, by Proposition 6.11, $\Lambda \subset \bigcup_{i=-\infty}^{\infty} f^{i}(\Omega_{\text{rec}})$ and $W$ generates $\Gamma(y)$ for all $y \in \Omega_{\text{rec}}$.

Since (C3) holds, the cone $C_{j}$ is well-defined. Since (C4) holds, $C_{j}$ is forward invariant under $M = \{ \Phi(W) \mid W \in W \}$ by Proposition 5.2. Since (C5) holds, $C_{j}$ is also expanding under $M$ by Propositions 5.2 and 5.3. Then by Theorem 3.9 for all $x \in \Lambda$ there exists $v \in TR^{2}$ such that $\lambda(x, v) > 0$.

7.2. Comments on the results of Algorithm 7.1

As mentioned in Section 2, for (4) with $\delta_{L} = \delta_{R} = 0.3$, Algorithm 7.1 outputs $\chi_{\text{chaos}} = \text{true}$ throughout the red regions of Figure 1. Here we examine three sample parameter combinations in detail. For each of the three black dots in Figure 1, the polygon $\Omega$ is well-defined and forward invariant. Figures 15(a)–17(a) show $\Omega$ using the value of $\beta > 0$ generated by Algorithm 7.1.

In Figure 15(a) we have $Y, Z \in D_{1}$, so $p_{\text{max}} = 1$ and $W = \{ R, RL \}$. Figure 15(b) shows how (C5) is satisfied. With $j = 1$ we have $W = R$, so $H_{1}(m)$ is linear, see Remark 5.1, and $m_{\text{root}} > m_{\text{stab, max}}$. With $j = 2$ we have $W = RL$ with which $H_{2}(m)$ is quadratic and (44) and (46) are satisfied. Numerical simulations suggest that at these parameter values $f$ has
a unique two-piece chaotic attractor with one piece intersecting $f(\Sigma)$ [as shown in the magnification of Figure 15(a)], and its image intersecting $\Sigma$.

In Figure 16(a) we have $Y \in D_1$ and $Z \in D_2$, so $p_{\text{max}} = 2$ and $W = \{R, RL, RL^2\}$. Here Algorithm 7.1 returns $\chi_{\text{chaos}} = \text{false}$ because (C5) is not satisfied. This is because $H_3(m)$ has no real roots, and this is evident in Figure 16(b). Indeed at these parameter values $f$ has a stable period-3 orbit corresponding to the word $RL^2$. Nevertheless, $f$ does appear to have a chaotic attractor contained in $D_1 \cup \Pi_R$. It may be possible to prove this attractor has a positive Lyapunov exponent by constructing a trapping region in $D_1 \cup \Pi_R$. For the parameter values of Figure 17 we again have $p_{\text{max}} = 2$ but now $H_3(m)$ has real roots satisfying (44) and (46) and Algorithm 7.1 terminates with $\chi_{\text{chaos}} = \text{true}$.
Figure 16. Constructive elements produced by Algorithm 7.1 for (4) with (5) and \((\tau_L, \tau_R) = (0.7, -1.8)\). Here the algorithm obtains \(\beta = 0.65\) and returns \(\chi_{\text{chaos}} = \text{false}\). Panel (a) shows the forward invariant region \(\Omega\), the regions \(D_p\), a numerically computed attractor, and a stable period-3 orbit (indicated with circles). Panel (b) shows \(G_j\) and \(H_j\) for \(j = 1, 2, 3\).

Figure 17. Constructive elements produced by Algorithm 7.1 for (4) with (5) and \((\tau_L, \tau_R) = (1, -2)\). Here the algorithm obtains \(\beta = 0.49\) and returns \(\chi_{\text{chaos}} = \text{true}\). Panel (a) shows the forward invariant region \(\Omega\), the regions \(D_p\), and a numerically computed attractor (shown also in Figure 2(b)). Panel (b) shows \(G_j\) and \(H_j\) for \(j = 1, 2, 3\).

We now discuss the region boundaries in Figure 1 labelled \(B_1\) to \(B_5\). Boundary \(B_1\) is the horizontal line \(\tau_R = -\delta_R - 1\). Below \(B_1\), and for \(\tau_L > 0.7\) approximately, Algorithm 7.1 terminates with \(\chi_{\text{chaos}} = \text{true}\). On \(B_1\) (C5) is not satisfied because \(A_R\) has an eigenvalue of \(-1\) so \(H_1(m) = 0\) at \(m = m^{(1)}_{\text{stab}}\). Indeed above \(B_1\) the map \(f\) has an asymptotically stable fixed point in \(x_1 > 0\). In this way Algorithm 7.1 detects a true bifurcation boundary between chaotic and non-chaotic dynamics.

On \(B_2\) (C5) is not satisfied because \(H_2(m) = 0\) at \(m = m^{(1)}_{\text{stab}}\). Thus to the left of \(B_2\), Algorithm 7.1 terminates with \(\chi_{\text{chaos}} = \text{false}\) because some \(v \in C_j\) do not expand when multiplied by \(M = A_L A_R\). Nevertheless numerical results suggest \(f\) has a chaotic attractor here. It may be possible to prove this by using a different word set \(W\).

Boundary \(B_3\) is the upper boundary of the blue region in which there exists a stable period-3 orbit of period \(n = 3\) [corresponding to the word \(RL^2\), see Figure 16(a)].
this boundary the periodic orbit is destroyed in a border-collision bifurcation by having one of its points collide with $\Sigma$. Algorithm 7.1 does not detect this boundary exactly as evident in Figure 1 by the presence of white pixels immediately above $B_3$. At these pixels Algorithm 7.1 obtains $p_{\text{max}} = 2$ with which (C5) is not satisfied because $H_3(m)$ has no real roots. In nearby red pixels Algorithm 7.1 obtains $p_{\text{max}} = 1$ with which the behaviour of $H_3(m)$ is irrelevant. The number of white pixels appears to tend to zero in the limit $\beta \text{step} \to 0$ because the size of the interval of $\beta$-values for which $\Omega_1$ is forward invariant with $p_{\text{max}} = 1$ vanishes as we approach $B_3$ from above. In a similar way as we approach the homoclinic bifurcation HC from above the size of the interval of $\beta$-values for which (C1) and (C2) are satisfied approaches zero (in [17] a different approach was used to construct a trapping region).

On $B_4$ the period-three orbit loses stability by attaining an eigenvalue of $-1$. For $\tau_R < -2.6$, approximately, Algorithm 7.1 detects this boundary exactly. On $B_4$ (C5) is not satisfied because $H_3(m) = 0$ at $m = m_{\text{stab}}^{(3)}$. That is, $\|Mv\| = \|v\|$ for the eigenvector $v = (1, m_{\text{stab}}^{(3)})$ of $M = A_2^3A_R$ corresponding to the eigenvalue $-1$. Finally, boundary $B_5$ is analogous to boundary $B_2$. On $B_5$ we have $H_3(m) = 0$ at $m = m_{\text{stab}}^{(1)}$.

8. Discussion

We have presented a general method by which one can prove, possibly with computer assistance, that a piecewise-linear map has a chaotic attractor. We applied the method to the 2d BCNF and found a chaotic attractor throughout a parameter regime that, unlike the logistic family for example, does not contain periodic windows. Such robust chaos is typical for piecewise-linear maps and for this reason piecewise-linear maps are desirable in applications that use chaos such as chaos-based cryptography [20].

In our implementation we considered only one approach for the construction of $\Omega_{\text{trap}}$ and only word sets of the form (6). There is considerable room to generalize these, such as by defining $\Omega_{\text{trap}}$ be to the union of a polygon and its images under $f$ [33].

A major next step would be the application of this method to families of higher-dimensional maps, such the $N$-dimensional border-collision normal form. Results of this nature have already been achieved in [11,14]. To construct a trapping region and a cone it may be helpful to work with convex polytopes [2]. It would also be useful to obtain a converse to Theorem 3.9: if $f$ has a topological attractor with a positive Lyapunov exponent, must some $\Omega_{\text{trap}}$, $W$, and $C$ (satisfying the required properties) exist?

Notes

1. To the left of $\tau_L = \delta_L + 1$, Figure 1 contains 173779 red pixels and 21462 white pixels, giving 89.01%.
2. The Hausdorff distance between sets $\Omega_1$ and $\Omega_2$ is defined as

   $$d_H(\Omega_1, \Omega_2) = \max \left[ \sup_{x \in \Omega_1} \inf_{y \in \Omega_2} \| x - y \|, \sup_{y \in \Omega_2} \inf_{x \in \Omega_1} \| x - y \| \right].$$

3. Equation (64) is a version of the well-known formula $\sin(\theta) = \frac{\|u \times v\|}{\|u\| \|v\|}$ for the sine of the angle between $u, v \in \mathbb{R}^3$. 
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References

[1] V.M. Alekseev, Quasirandom dynamical systems. I. Quasirandom diffeomorphisms, Math. USSR Sb. 5 (1968), pp. 73–128.
[2] N. Athanasopoulos and M. Lazar, Alternative stability conditions for switched discrete time linear systems, IFAC Proceedings Volumes 47 (2014), pp. 6007–6012.
[3] S. Banerjee and C. Grebogi, Border collision bifurcations in two-dimensional piecewise smooth maps, Phys. Rev. E 59(4) (1999), pp. 4052–4061.
[4] S. Banerjee, J.A. Yorke, and C. Grebogi, Robust chaos, Phys. Rev. Lett. 80(14) (1998), pp. 3049–3052.
[5] L. Barreira, D. Dragičević, and C. Valls, Positive top Lyapunov exponents via invariant cones: Single trajectories, J. Math. Anal. Appl. 423 (2015), pp. 480–496.
[6] L. Barreira and Y. Pesin, Introduction to Smooth Ergodic Theory, Vol. 148, American Mathematical Society, Providence, RI, 2013.
[7] J. Buzzi, Absolutely continuous invariant measures for generic multi-dimensional piecewise affine expanding maps, Int. J. Bifurcation Chaos 9(9) (2011), pp. 1743–1750.
[8] S. Das and J.A. Yorke, Multichaos from quasiperiodicity, SIAM J. Appl. Dyn. Syst. 16(4) (2017), pp. 2196–2212.
[9] A. Dhara and J. Dutta, Optimality Conditions in Convex Optimization. A Finite-Dimensional View, CRC Press, Boca Raton, FL, 2012.
[10] M. di Bernardo, C.J. Budd, A.R. Champneys, and P. Kowalczyk, Piecewise-Smooth Dynamical Systems. Theory and Applications, Springer-Verlag, New York, 2008.
[11] P. Glendinning, Bifurcation from stable fixed point to N-dimensional attractor in the border collision normal form, Nonlinearity 28 (2015), pp. 3457–3464.
[12] P. Glendinning, Bifurcation from stable fixed point to 2D attractor in the border collision normal form, IMA J. Appl. Math. 81(4) (2016), pp. 699–710.
[13] P. Glendinning, Robust chaos revisited, Eur. Phys. J. Spec. Top. 226(9) (2017), pp. 1721–1738.
[14] P. Glendinning and M.R. Jeffrey, Grazing-sliding bifurcations, border collision maps and the curse of dimensionality for piecewise smooth bifurcation theory, Nonlinearity 28 (2015), pp. 263–283.
[15] P. Glendinning and C.H. Wong, Two dimensional attractors in the border collision normal form, Nonlinearity 24 (2011), pp. 995–1010.
[16] P.A. Glendinning and D.J.W. Simpson, Robust chaos and the continuity of attractors, Trans. Math. Appl. 4(1) (2020), pp. tnaa002.
[17] P.A. Glendinning and D.J.W. Simpson, A constructive approach to robust chaos using invariant manifolds and expanding cones, Discrete Contin. Dyn. Syst. 41(7) (2021), pp. 3367–3387.
Appendix. The significance of the 2d BCNF

Let $f$ be a continuous map on $\mathbb{R}^2$ that is affine on each side of $\Sigma = \{ x \mid x_1 = 0 \}$. Then $f$ has the form

\[
f(x) = \begin{cases} 
    \begin{bmatrix} a_L & b \\ c_L & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix}, & x_1 \leq 0, \\
    \begin{bmatrix} a_R & b \\ c_R & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix}, & x_1 \geq 0,
\end{cases}
\]  

(A1)

for some $a_L, a_R, b, c_L, c_R, d, p, q \in \mathbb{R}$. By computing the image of $\Sigma$ under $f$ we find that $f(\Sigma)$ intersects $\Sigma$ at a unique point if and only if $b \neq 0$. Moreover, if $b \neq 0$ then this point is not a fixed point of (A1) if and only if $\xi = (1 - d)p + bq \neq 0$. 

Now suppose $b \neq 0$ and $\xi \neq 0$. Then the coordinate change
\[
\tilde{x} = \frac{1}{\xi} \left( \begin{bmatrix} 1 & 0 \\ -d & b \end{bmatrix} x + \begin{bmatrix} 0 \\ dp - bq \end{bmatrix} \right),
\]
(A2)
is well-defined and invertible. Also notice it leaves $\Sigma$ unchanged. By directly applying (A2) to (A1) we find that if $\xi > 0$ then $f$ is transformed to (4) with $\tilde{x}$ in place of $x$ and $\tau_L = a_L + d, \delta_L = a_L d - bc_L, \tau_R = a_R + d$, and $\delta_R = a_R d - bc_R$. If instead $\xi < 0$ then $\tau_L = a_R + d, \delta_L = a_R d - bc_R, \tau_R = a_L + d$, and $\delta_R = a_L d - bc_L$. 