Proximal-like algorithms for equilibrium seeking in mixed-integer Nash equilibrium problems

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Abstract—We consider potential games with mixed-integer variables, for which we propose two distributed, proximal-like equilibrium seeking algorithms. Specifically, we focus on two scenarios: i) the underlying game is generalized ordinal and the agents update through iterations by choosing an exact optimal strategy; ii) the game admits an exact potential and the agents adopt approximated optimal responses. By exploiting the properties of integer-compatible regularization functions used as penalty terms, we show that both algorithms converge to either an exact or an ϵ-approximate equilibrium. We corroborate our findings on a numerical instance of a Cournot oligopoly model.

I. INTRODUCTION

Tracing back from the seminal work by Monderer and Shapley [1], potential games represent a broad class of noncooperative games characterized by the existence of a real-valued function, the potential function, such that any collective strategy profile minimizing the underlying function coincides with a Nash equilibrium of the game. Potential games hence provide a means to naturally model many control-theoretic applications [2] such as routing [3], complex social networks [4] and Cournot competition [5].

We consider generalized ordinal potential games in which part of the decision variables of the agents are constrained to assume integer values. Such mixed-integer (MI) games have been recently proposed as strategic models for the distributed coordination of autonomous vehicles [6], [7], transportation and traffic control [8], and smart grids [9], [10]. In addition, MI restrictions are often encountered in market games [11], [12] and combinatorial congestion games [13], [14] as well.

For this practically relevant, yet intrinsically nonconvex, MI game-theoretic setting the existence of equilibria follows by assuming that a certain master problem admits a solution, and we hence present two distributed, proximal-like equilibrium seeking algorithms. In particular, we consider the following scenarios: i) the underlying game is generalized ordinal and the agents update their control variables iteratively by choosing an exact proximal best-response (BR) strategy; ii) the game admits an exact potential function but we allow agents to choose an inexact proximal BR for their updates.

Similar to Bregman-versions of proximal algorithms [15], we formulate the proximal best-response function using a class of norm-like regularizers, known in the MI optimization community as integer-compatible regularization functions (ICRFs). We choose ICRFs as penalty terms in place of standard quadratic regularizations as we believe they could provide us with a mean to include continuous reformulations of the MI optimization subproblems. We leave this topic for future research. Thus, by exploiting the properties of the ICRFs, acting as penalty terms in the individual agent’s BR problems, we prove that both proposed algorithms enjoy convergence guarantees to an equilibrium of the mixed-integer Nash equilibrium problem (MI-NEP).

Since MI-NEPs constitute a rather new class of strategic optimization problems, there are not many solution techniques available. To the best of our knowledge, this work represents a first attempt proposing proximal-like distributed algorithms for a MI game setting. The only alternative applicable algorithm for MI-NEPs is the Gauss–Southwell method designed in [16]. Given the practical relevance of MI-NEPs, this is rather surprising, and completely diametric to continuous Nash equilibrium problems (NEPs), for which a whole arsenal of numerical solution techniques is available [17], [18]. In fact, proximal BR-based algorithms have been extensively studied in both stochastic and deterministic Nash games [19], [20]. All these schemes, however, leverage the variational inequality (VI) reformulation of NEPs [21], and thus require strong (or strict) monotonicity of the VI, assumptions that cannot be structurally satisfied in MI-NEPs.

The algorithms we develop, in particular our adaptive update of the penalty parameter regulating the proximal BR, are inspired by the decomposition method proposed in [22]. As main contributions we show i) how a proximal-like BR-scheme with ICRFs can be used to compute Nash equilibria satisfying MI restrictions, and ii) we show convergence to an approximate equilibrium even under inexact computations of BR strategies.

The rest of the paper is organized as follows. In §II we introduce the problem addressed, along with some preliminaries, while in §III we present the two methods and related convergence analysis. Finally, in §IV, we test our findings.
on a MI Cournot oligopoly model and we conclude in §V.

II. PROBLEM FORMULATION AND PRELIMINARIES

Let \( I := \{1,\ldots,N\} \) be the set indexing the \( N \) agents taking part in the noncooperative game \( \Gamma := (\mathcal{I}, (J_i)_{i \in \mathcal{I}}, (\mathcal{X}_i)_{i \in \mathcal{I}}) \). Each agent controls MI variables \( x_i \) belonging to a compact, nonempty set \( \mathcal{X}_i \subseteq \mathbb{R}^{n_i} \times \mathbb{Z}^{n_i'} \), and aims at minimizing a given cost function \( J_i : \mathcal{X} \to \mathbb{R} \) with \( \mathcal{X} := \prod_{i \in \mathcal{I}} \mathcal{X}_i \subseteq \mathbb{R}^n \) and \( n := \sum_{i \in \mathcal{I}} n_i = \sum_{i \in \mathcal{I}} (n_i^e + n_i^d) \).

The resulting MI-NEP thus reads

\[
\forall i \in \mathcal{I} : \min_{x_i \in \mathcal{X}_i} J_i(x_i, x_{-i}),
\]

where \( x_{-i} := \text{col}((x_j)_{j \in \mathcal{T}(i)}) \). Given the strategies of the other agents, \( x_{-i} \), the MI BR of agent \( i \in \mathcal{I} \) is defined as

\[
B_i(x_{-i}) := \arg\min_{x_i \in \mathcal{X}_i} J_i(x_i, x_{-i}).
\]

Our goal is to design distributed algorithms, able to drive the set of agents to an MI-NE of the game \( \Gamma \), according to the definition given next.

**Definition 1**: (Mixed-integer \( \epsilon \)-Nash equilibrium) Given some \( \epsilon \geq 0 \), a strategy profile \( x^* \in \mathcal{X} \) is an \( \epsilon \)-approximate MI-NE (or \( \epsilon \)-MI-NE) of the game \( \Gamma \) if, for all \( i \in \mathcal{I} \),

\[
J_i(x^*_i, x^*_{-i}) \leq J_i(y_i, x^*_{-i}) + \epsilon \text{ for all } y_i \in \mathcal{X}_i. \tag{3}
\]

If \( \epsilon = 0 \), then we call \( x^* \) an exact MI-NE. □

Definition 1 points out that a MI-NE of the game (if it exists) is achieved when all the agents adopt a BR strategy.

**A. Generalized ordinal and exact potential games**

Existence theorems for NEPs typically require continuity of the agents’ cost functions as well as compactness and convexity of the feasible sets [21]. Since the NEP in (1) is nonconvex, existence of Nash equilibria is, in principle, not guaranteed. We thus focus on the classes of *exact* and *generalized ordinal potential games*, for which existence of solutions can be guaranteed under certain assumptions.

**Definition 2**: (Potential game) A game \( \Gamma \) is called

- **exact potential game** [1] if there exists a continuous function \( P : \mathbb{R}^n \to \mathbb{R} \) such that, for all \( i \in \mathcal{I} \),

\[
P(x_i, x_{-i}) - P(y_i, x_{-i}) = J_i(x_i, x_{-i}) - J_i(y_i, x_{-i}),
\]

for all \( x_{-i} \) and \( y_i \in \mathcal{X}_i \);

- **generalized ordinal** potential game [22] if, there exists a **forcing function** \( \phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) such that, for all \( i \in \mathcal{I} \), \( x_{-i} \), and \( y_i, x_i \in \mathcal{X}_i \),

\[
J_i(y_i, x_{-i}) - J_i(x_i, x_{-i}) > 0 \text{ implies } P(y_i, x_{-i}) - P(x_i, x_{-i}) \geq \phi(J_i(y_i, x_{-i}) - J_i(x_i, x_{-i}))
\]

where \( \lim_{t \to 0} \phi(t) = 0 \).

Note that any exact potential game is a generalized ordinal potential game. By exploiting the tight relation between first-order information of the potential function and the local cost functions of the agents, it is well-known that potential functions can be employed in the construction of a suitable master problem facilitating the computation of equilibria.

**Proposition 1**: [16, Th. 2] Let \( P \) be a generalized ordinal potential function for the game \( \Gamma \). Given some \( \epsilon \geq 0 \), any \( \epsilon \)-approximate solution of the optimization problem

\[
\min_{x \in \mathcal{X}} P(x) \tag{4}
\]

yields an \( \epsilon \)-approximate MI-NE of \( \Gamma \). □

**Assumption 1**: Problem (4) is solvable, i.e., there exists an \( x^* \in \mathcal{X} \) with \( P(x) \geq P(x^*) \) for all \( x \in \mathcal{X} \). □

This assumption guarantees that the game \( \Gamma \) admits at least one Nash equilibrium in the nonconvex domain \( \mathcal{X} \). Clearly, the solutions to the master problem (4) may not contain all possible Nash equilibria of the game \( \Gamma \) [26, Ex. 1].

**B. Integer-compatible regularization functions**

Standard regularization techniques in NEPs are based on the proximal BR function obtained from (2) by adding a quadratic penalty term. Motivated by the proximal point interpretation of MI optimization heuristics, we propose a regularization strategy of the individual agents’ cost functions via *integer-compatible regularization functions* (ICRFs) [23]. This family of functions has been introduced in the mixed-integer optimization community to control the duality gap [25]. They are also related to penalty methods for mixed-integer optimization [24].

**Definition 3**: A continuous function \( \rho : \mathbb{R}^n \to \mathbb{R} \) is an *integer-compatible regularization function* (ICRF) if

- i) \( \rho(t) \geq 0 \) for all \( t \in \mathbb{R}^n \) and \( \rho(t) = 0 \iff t = 0 \);

- ii) for \( \gamma \in (0,1) \), we have \( \rho(\gamma t) < \rho(t) \) for all \( t \neq 0 \);

- iii) there exists a continuous and strictly increasing function \( s : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) and some \( K \in \mathbb{N} \) such that, for all \( K < \bar{K} \), \( \rho(t) \leq K \Rightarrow ||t||_1 \leq s^{-1}(K) \), where \( \cdot \) is the \( \ell_1 \) norm in \( \mathbb{R}^n \).

Note that any norm defined in \( \mathbb{R}^n \) is an ICRF. A constructive way to design ICRFs is to consider decomposable penalties of the form \( \rho(t) = \sum_{i=1}^{n} P(|t_i|) \), where \( p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is a concave and strictly increasing function. Prominent examples are \( p(t) = \log(\alpha + t^2) \) and \( p(t) = -t + a^\alpha - q + \alpha^{-q} \), \( p(t) = 1-e^{-\alpha t} \), or \( p(t) = (1 + e^{-\alpha t})^{-1} - \frac{1}{2} \), for some \( \alpha, q > 0 \) (see e.g., [23], [24]). With these choices, \( \rho(\cdot) \) amounts to an ICRF [23, Prop. 3.2]. In particular, for the special case of binary constraints, a sensible formulation of an ICRF is \( \rho(t) = \sum_{i=1}^{n} \min\{p(|t_i|), p(1-|t_i|)\} \).

III. PROXIMAL-LIKE ALGORITHMS FOR MI-NEPS

We now propose two MI-NE seeking algorithms. Algorithm 1 assumes that agents are able to compute an exact proximal BR at each iteration. Algorithm 2 relaxes this and instead allows for inexact BR computations. To show convergence, in the former case we rely on the fact that the MI-NEP in (1) is generalized ordinal, whereas in the latter case we require the existence of an exact potential function.

Let \( \rho_i \) denote the ICRF employed by agent \( i \in \mathcal{I} \) and let \( \tau > 0 \) be a positive regularization parameter. We introduce the *proximal augmented local cost function* as a regularized version of the local cost in (1), which is given by

\[
\tilde{J}_{i,\tau}(y_i, x_{-i}) := J_i(y_i, x_{-i}) + \tau \rho_i(y_i - x_i). \tag{5}
\]
Set $k := 0$, choose $x^k \in \mathcal{X}$, $\tau^k > 0$ and $\omega \in (0,1)$

**Algorithm 1: Proximal-like method for MI-NEPs with generalized ordinal potential and exact optimization**

Set $k := 0$, ... the proof, we obtain $\lim_{k \to \infty} (J_i(\hat{x}_i(k)) - J_i(\hat{x}_i+1(k))) = 0$ and $\lim_{k \to \infty} \|x_{k+1} - x_k\|_1 = 0$ for all $i \in I$. Next, we show by

Successively, the next internal state $k$ to perform an update, i.e., it computes the new strategy as a strongly depends on $k$ convergence of Algorithm 1 under the following assumption.

This corresponds to the collective vector of strategies at the beginning of round $k + 1$. For an arbitrary agent $i \in I$, we also define the (local) population state as $\hat{x}_i(k) := \text{col}(x_1^{k+1}, \ldots, x_i^{k+1}, x_{i+1}^{k}, \ldots, x_N^{k})$.

We stress that the proposed algorithms leverage the adaptive update of the regularization parameter $\tau$ in (7), which produces a monotonically decreasing sequence $\{\tau^k\}_{k \geq 0}$, i.e., $\tau^{k+1} \leq \tau^k$ for all $k \geq 0$ [22]. Note that the rate of decrease strongly depends on $d_\rho(x^{k+1}, x^k) := \max_{i \in I} \rho_i(x^{k+1} - x^k) \geq 0$, which measures the progress the method is making in the agents’ proximal steps at the $k$-th iteration. As it will be clear from the convergence analysis, this quantity decreases over time, thus inducing a step towards an MI-NE.

### A. Exact BR computation in generalized ordinal potential games

Next, by focusing on MI-NEPs as in (1), we study the convergence of Algorithm 1 under the following assumption.

**Assumption 2:** The game $\Gamma = (\mathcal{I}, (J_i)_{i \in \mathcal{I}}, (X_i)_{i \in \mathcal{I}})$ is a generalized ordinal potential game with potential function $P(\cdot)$.

Thus, the Gauss–Seidel sequence of iterations in Algorithm 1 has the following convergence property. We stress that, in our framework, an accumulation point for the sequence $\{x^k\}_{k \geq 0}$ exists in view of Assumption 1.

**Theorem 1:** Under Assumption 2, any accumulation point of the sequence of strategy profiles generated by Algorithm 1, $\{x^k\}_{k \geq 0}$, is an MI-NE of the game $\Gamma$ in (1). □

**Proof:** We first show that, in case the sequence $\{x^k\}_{k \geq 0}$ generated by Algorithm 1 admits a limit point $\bar{x} \in \mathcal{X}$, then the regularization parameter, adaptively updated via (7), satisfies $\lim_{k \to \infty} \tau^k = 0$ and there exists an infinite index set $K$ such that $\tau^{k+1} < \tau^k$ for all $k \in K$. Then, we prove that $\bar{x} \in \mathcal{X}$ is actually an MI-NE of the MI-NEP in (1).

By construction of the regularization parameter sequence defined by (7), we have $\tau^{k+1} \leq \tau^k$ for all $k \geq 0$. Then, for the sake of contradiction, assume that there exists some $\bar{\tau} > 0$ such that $\tau^k \geq \bar{\tau}$ for all $k \geq 0$. In view of the updating rule at the $k$-th iteration, for all $i \in I$ we have

$$J_i(\hat{x}_i(k)) - J_i(\hat{x}_{i+1}(k)) \geq \tau^k \rho_i(x_{i+1}^{k+1} - x_i^k) \geq \bar{\tau} \rho_i(x_{i+1}^{k+1} - x_i^k) \geq 0.$$ (8)

By exploiting the definition of a generalized ordinal potential game provided in Definition 2, we hence deduce

$$P(\hat{x}_i(k)) - P(\hat{x}_{i+1}(k)) \geq \phi(J_i(\hat{x}_i(k)) - J_i(\hat{x}_{i+1}(k))) \geq 0.$$ (9)

Therefore, $P(x^{k+1}) = P(\hat{x}_{N+1}(k)) \leq P(\hat{x}(k)) \leq \cdots \leq P(\hat{x}_1(k)) = P(x^k)$, and hence the sequence $\{P(x^k)\}_{k \geq 0}$ is monotonically non-increasing. By the continuity of $P$, it follows that the full sequence $\{P(x^k)\}_{k \geq 0}$ is convergent to a finite value $\bar{P}$. Moreover, it follows from (9) that $\lim_{k \to \infty} \phi(J_i(\hat{x}_i(k)) - J_i(\hat{x}_{i+1}(k))) = 0$. By definition of the forcing function, we also have

$$\lim_{k \to \infty} (J_i(\hat{x}_i(k)) - J_i(\hat{x}_{i+1}(k))) = 0.$$ (10)

From (10), we obtain $\lim_{k \to \infty} \rho_i(x_{i+1}^{k+1} - x_i^k) = 0$. Let $r^k \to 0$ be such that $\rho_i(x_{i+1}^{k+1} - x_i^k) \leq r^k$ for all $k$ sufficiently large. By definition of an ICRF, we deduce $\|x_i^{k+1} - x_i^k\|_1 \leq s_i^{-1}(r^k)$. We recall that the function $s_i^{-1}(\cdot)$ is monotonically increasing, and therefore $\lim_{k \to \infty} \|x_i^{k+1} - x_i^k\|_1 = 0$. Consequently, for all $k$ sufficiently large, it holds that $d_\rho(x^{k+1}, x^k) < \bar{\tau}$. From (7), we have $\tau^{k+1} = \max \{\omega \tau^k, d_\rho(x^{k+1}, x^k)\}$, and hence we need $\tau^{k+1} = \omega \tau^k$ for all $k$. However, this implies that $\tau^k \to 0$ at a geometric rate, thus contradicting the hypothesis that $\tau^k \geq \bar{\tau} > 0$ for all $k \geq 0$ and concluding the first part of the proof.

Now, let $\{x^k\}_{k \geq 0}$ be a convergent subsequence with accumulation point $\bar{x} \in \mathcal{X}$. The existence of such a convergent subsequence is guaranteed by the compactness of $\mathcal{X}$. By invoking the same arguments as in the first part of the proof, we obtain $\lim_{k \to \infty} (J_i(\hat{x}_i(k)) - J_i(\hat{x}_{i+1}(k))) = 0$ and $\lim_{k \to \infty} \|x_i^{k+1} - x_i^k\|_1 = 0$ for all $i \in I$. Next, we show by
Algorithm 2: Proximal-like method for MI-NEPs with exact potential and inexact optimization

Set $k := 0$, choose $x^k \in X$, $\tau^k > 0$, $\omega \in (0, 1)$ and error tolerance sequence $\{\delta^k\}_{k \geq 0}$

While $x^k$ is not satisfying a stopping criterion do
  For all $i \in I$ do
    Obtain $x_i^k(k)$
    If $x_i^k \in \beta_{i,\tau^k}(\hat{x}_i(k); \delta^k)$ then
      Set $x_i^{k+1} = x_i^k$
    Else
      Set $x_i^{k+1} \in \beta_{i,\tau^k}(\hat{x}_i(k); \delta^k)$
    End
    Update $x_i^{k+1}(k) = (x_i^{k+1}, \hat{x}_{i-1}(k))$
  End
  Update $\tau^{k+1}$ as in (7) and set $k := k + 1$
End

Definition 4: ($\delta$-proximal BR) Given any $x \in X$ and tolerance $\delta \geq 0$, $y_i \in X_i$ is a $\delta$-optimal response to $x_{-i}$ if

$$J_i(y_i, x_i; x_{-i}) - J_i(z_i, x_i; x_{-i}) \leq J_i,\tau(z_i, x_i; x_{-i}) + \delta$$

for all $z_i \in X_i$.

For each agent $i \in I$, we define $\hat{\beta}_{i,\tau}(x; \delta) := \{y_i \in X_i \mid J_i(y_i, x_i; x_{-i}) \leq J_i,\tau(z_i, x_i; x_{-i}) + \delta$ for all $z_i \in X_i\}$ as the set of $\delta$-optimal responses, given some collective vector of strategies $x$. For the theoretical developments of this subsection, we then make the following assumption.

Assumption 3: The game $\Gamma = (I, (J_i)_{i \in I}, (X_i)_{i \in I})$ is an exact potential game with potential function $P(\cdot)$. □

Algorithm 2 summarizes the main steps of the resulting distributed, Gauss–Seidel type sequence of iterations. For the considered instance, after choosing an initial strategy $x^0 \in X$ and a sequence of penalty parameters $\{\tau^k\}_{k \geq 0}$, the preliminary step requires to further define a certain error tolerance in computing a $\delta$-optimal response. To this end, we let $\{\delta^k\}_{k \geq 0}$ be a given sequence of positive numbers such that $\delta^k \to \epsilon$ for some $\epsilon > 0$. In Algorithm 2 we stick to the sequential update architecture, but instead of requiring that agents pursue an exact BR, we allow the updating agent to choose an inexact BR, $x_i^{k+1} \in \beta_{i,\tau^k}(\hat{x}_i(k); \delta^k)$, only in case the inexact BR computed at the previous step, i.e., the one obtained by considering $\hat{x}_i(k-1)$, $\tau^{k-1}$ and $\delta^{k-1}$, does not belong to $\beta_{i,\tau^k}(\hat{x}_i(k); \delta^k)$. The updated strategy is then sent to the agent down the line, and the procedure repeats as long as some stopping criterion is not met.

Theorem 2: Let Assumption 3 holds true. Consider an error sequence $\{\delta^k\}_{k \geq 0}$ satisfying $\delta^k \to \epsilon$ with $\epsilon > 0$. Then any accumulation point of the sequence $\{x^k\}_{k \geq 0}$ generated by Algorithm 2 is an $\epsilon$-MI-NE of the game $\Gamma$ in (1).

Proof: The proof makes use of similar arguments as the one of Theorem 1. As a starting point, by definition of the update $x_i^{k+1}$, we have

$$J_i(\hat{x}_i(k)) - J_i(\hat{x}_{i+1}(k)) \geq \tau^k \rho_i(x_i^{k+1} - x_i^k).$$

Since in this case $P$ is an exact potential function, we have

$$P(\hat{x}_i(k)) - P(\hat{x}_{i+1}(k)) \geq \tau^k \rho_i(x_i^{k+1} - x_i^k) \geq 0,$$

where the last inequality uses the non-negativity of the ICRF. By summing from $i = 1$ to $N$, we hence obtain

$$P(x^k) - P(x^{k+1}) \geq 0.$$

In view of [28, Lemma 3.4], it follows that $\lim_{k \to \infty}(P(x^k) - \min_{x \in X} P(x))$ exists and is finite. Therefore, since $X = \prod_{i \in I} X_i$ is compact, the sequence $\{x^k\}_{k \geq 0}$ admits a convergent subsequence $\{x^k\}_{k \in K}$ with indices contained in some countable and infinite set $K$, which has a limit point $x \in X$. Therefore, in view of the continuity of the potential function, we have that $\lim_{k \to \infty} P(x^k) = P(x)$ and $\lim_{k \to \infty}(P(x^k) - P(x)) = 0$. Thus, for all $i \in I$,

$$\lim_{k \to \infty}(P(\hat{x}_{i+1}(k)) - P(\hat{x}_i(k))) = 0.$$

By definition of the potential function, it follows that

$$\lim_{k \to \infty}(J_i(\hat{x}_i(k)) - J_i(\hat{x}_{i+1}(k))) = 0.$$
| Symbol | Unit | Description | Value |
|--------|------|-------------|-------|
| $p_i$  | E/good | Selling price | ~ $U(10, 20) \times 10^3$ |
| $m_i$  | E/good | Marginal cost | ~ $U(7, 12) \times 10^3$ |
| $u_i^d$ | Upper bound (discrete) | ~ $U(200, 400)$ |
| $u_i^c$ | Upper bound (continuous) | ~ $U(200, 400)$ |
| $\tau^0$ | Regularization parameter | 5000 |
| $x_i^0$ | Initial goods vector | $10^{10}$ |
| $\{\delta_k\}$ | Error tolerance sequence | $10^{-5} \times (k^2 + 1) \times 10^{-6}$ |
| $\epsilon$ | Approximation tolerance | $10^{-6}$ |

where the main parameters are described in Table I. Also, we define matrices $C_{ij} \in \mathbb{R}^{100 \times 100}$, typically related with the customers’ inverse demand, according to the procedure described in [26, §4] for all $j \geq i$, and then we impose $C_{ij} = C_{ij}^T$. This degree of symmetry ensures the existence of a potential function for the MI-NEP in (10), see, e.g., [9], which has the form

$$P(x) = \sum_{i \in \mathcal{I}} h_i(x_i) + \sum_{j \in \mathcal{I}, j < i} g(x_i, x_j)$$

with $h_i(x_i) := (m_i - p_i)^T x_i + x_i^T C_{ii} x_i$ and $g(x_i, x_j) := x_i^T C_{ij} x_j$. In case the $(l, q)$-entry of matrix $C_{ij}$ is nonnegative, the $q$-th product of the firm $j$ is a substitute for the $l$-th product of firm $i$. On the other hand, if that entry is negative then the $q$-th product of the firm $j$ is a complement for the $l$-th product of firm $i$. This framework resembles the 2-groups partitionable class in [26]. As an ICRF for each agent, we adopted a piecewise affine approximation of the function $\rho_i(t_i) = \sum_{p=1}^{100} (1 - e^{0.9(t_i,p)})$ applied componentwise ($t_{i,p}$ denotes the $p$-th element of $t_i := y_i - x_i$). This is in order to handle each problem in (10) with the solver used.

The numerical results reported in Figure 1 are obtained in Matlab by using Gurobi [29] as a solver on a laptop with a Quad-Core Intel Core i5-2 4.7 GHz CPU and 8 GB RAM. Specifically, we generate 50 random instances of the considered MI-NEP in (10), and test the behavior of Algorithm 1 and 2, where each agent takes, on average, 0.0788 s to compute a BR strategy. While Fig. 1(a) shows the averaged convergence behavior of the sequence of MI strategy profiles generated by Algorithm 1, which actually converge in less than 30 iterations, Fig. 1(b), instead, illustrates the averaged behavior of the sequence of sub-optimal MI strategy profiles generated by Algorithm 2, which converges to an $10^{-6}$-approximate MI-NE of the Cournot model in (10). Note that the if-condition in the procedure generates a typical staircase behavior and the convergence, in general, requires few more iterations. Finally, Fig. 1(c) illustrates the averaged (over the agents) evolution of $\delta^k := \col((\delta^k_{i,j})_{i,j} \in \mathcal{I})$, which corresponds to the approximation error actually made in computing an exact BR. Each $\delta^k_{i,j}$, indeed, denotes the distance returned by the solver between the inexact and exact BR solutions made by agent $i$ at the $k$-th iteration. The quantity $1^T \delta^k / N$ is always upper bounded by the error sequence $\{\delta_k\}_{k \geq 0}$ reported in Table I, as expected.

V. CONCLUSION

We have presented two proximal-like equilibrium seeking algorithms for NEPs with mixed-integer variables admitting either generalized ordinal or exact potential functions. Exploiting the properties of integer-compatible regularization functions used as penalty terms in the agents’ cost functions is key to prove convergence both in case the agents pursue an exact optimal strategy or an approximated one.

Future research directions include, but are not limited to, the extension of the proposed algorithms to generalized MI-NEPs as well as developing their stochastic counterparts. There are also interesting computational questions to investigate. From the proximal-point interpretation of the feasibility
argue that large penalty parameters enforce integer restrictions. One possibility to approach this within the proposed equilibrium seeking algorithms is to design a double loop procedure with large penalties at the beginning of the scheme.

REFERENCES

[1] D. Monderer and L. S. Shapley, “Potential games,” Games and Economic Behavior, vol. 14, no. 1, pp. 124–143, 1996.
[2] J. R. Marden, G. Arslan, and J. S. Shamma, “Cooperative control and potential games,” IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics), vol. 39, no. 6, pp. 1393–1407, 2009.
[3] A. Orda, R. Rom, and N. Shimkin, “Competitive routing in multi-user communication networks,” in IEEE INFOCOM ’93 The Conference on Computer Communications, Proceedings, 1993, pp. 964–971 vol.3.
[4] M. Staudigl, “Potential games in volatile environments,” Games and Economic Behavior, vol. 72, no. 1, pp. 271–287, 2011.
[5] J. Conteras, M. Klusch, and J. Krawczyk, “Numerical solutions to Nash-Cournot equilibria in coupled constraint electricity markets,” IEEE Transactions on Power Systems, vol. 19, no. 1, pp. 195–206, 2004.
[6] F. Facchinei and S. Grammatico, “A mixed-logical-dynamical model for automated driving on highways,” in 2018 IEEE Conference on Decision and Control (CDC). IEEE, 2018, pp. 1011–1015.
[7] ——, “Multi-vehicle automated driving as a generalized mixed-integer potential game,” IEEE Transactions on Intelligent Transportation Systems, vol. 21, no. 3, pp. 1064–1073, 2019.
[8] C. Cenedese, M. Cucuzzella, J. Scherpen, S. Grammatico, and M. Cao, “Highway traffic control via smart e-mobility–Part I: Theory,” arXiv preprint arXiv:2102.09354, 2021.
[9] C. Cenedese, F. Facchinei, M. Cucuzzella, J. M. Scherpen, M. Cao, and S. Grammatico, “Charging plug-in electric vehicles as a mixed-integer aggregative game,” in 2019 IEEE 58th Conference on Decision and Control (CDC). IEEE, 2019, pp. 4904–4909.
[10] R. Vujanic, P. Mohajerin Esfahani, P. J. Goulart, S. Mariéthoz, and M. Morari, “A decomposition method for large scale MILPs, with performance guarantees and a power system application,” Automatica, vol. 67, pp. 144–156, 2016.
[11] S. A. Gabriel, S. A. Siddiqui, A. J. Conejo, and C. Ruiz, “Solving discretely-constrained Nash-Cournot games with an application to power markets,” Networks and Spatial Economics, vol. 13, no. 3, pp. 307–326, 2013.
[12] S. Sagratella, “Computing equilibria of Cournot oligopoly models with mixed-integer quantities,” Mathematical Methods of Operations Research, vol. 86, no. 3, pp. 549–565, 2017.
[13] R. W. Rosenthal, “The network equilibrium problem in integers,” Networks: An International Journal, vol. 3, 1973.
[14] P. Kleer and G. Schäfer, “Computation and efficiency of potential function minimizers of combinatorial congestion games,” Mathematical Programming, vol. 190, no. 1, pp. 523–560, 2021.
[15] P. Dvurechensky, S. Shtern, and M. Staudigl, “First-order methods for convex optimization,” EURO Journal on Computational Optimization, vol. 9, p. 100015, 2021.
[16] S. Sagratella, “Algorithms for generalized potential games with mixed-integer variables,” Computational Optimization and Applications, vol. 68, no. 3, pp. 689–717, 2017.
[17] A. Dreves, F. Facchinei, C. Kanzow, and S. Sagratella, “On the solution of the KKT conditions of generalized Nash equilibrium problems,” SIAM Journal on Optimization, vol. 21, no. 3, pp. 1082–1108, 2011.
[18] P. Mertikopoulos and M. Staudigl, “Convergence to Nash equilibrium in continuous games with noisy first-order feedback,” in 2017 IEEE 56th Annual Conference on Decision and Control (CDC). IEEE, 2017, pp. 5609–5614.
[19] G. Scutari, F. Facchinei, J.-S. Pang, and D. P. Palomar, “Real and complex monotone communication games,” IEEE Transactions on Information Theory, vol. 60, no. 7, pp. 4197–4231, 2014.
[20] J. Lei, U. V. Shanbhag, J.-S. Pang, and S. Sen, “On synchronous, asynchronous, and randomized BR schemes for stochastic Nash games,” Mathematics of Operations Research, vol. 45, no. 1, pp. 157–190, 2020.
[21] F. Facchinei and C. Kanzow, “Generalized Nash equilibrium problems,” AOR, vol. 5, no. 3, pp. 173–210, 2007.
[22] F. Facchinei, V. Piccialli, and M. Sciandrone, “Decomposition algorithms for generalized potential games,” Computational Optimization and Applications, vol. 50, no. 2, pp. 237–262, 2011.
[23] N. L. Boland, A. C. Eberhard, F. Engineer, and A. Tsoukalas, “A new approach to the feasibility pump in mixed-integer programming,” SIAM Journal on Optimization, vol. 22, no. 3, pp. 831–861, 2012.
[24] S. Lucidi and F. Rinaldi, “Exact penalty functions for nonlinear integer programming problems,” Journal of Optimization Theory and Applications, vol. 145, no. 3, pp. 479–488, 2010.
[25] M.J. Feizollahi, A. Shabbir and A. Sun, “Exact augmented Lagrangian duality for mixed integer linear programming”, Mathematical Programming, vol. 161, no. 1, pp. 365-387, 2017.
[26] S. Sagratella, “Computing all solutions of Nash equilibrium problems with discrete strategy sets,” SIAM Journal on Optimization, vol. 26, no. 4, pp. 2190–2218, 2016.
[27] A. Simonetto, E. Dell’Anese, J. Monteil, and A. Bernstein, “Personalized optimization with user’s feedback,” Automatica, vol. 131, p. 109767, 2021.
[28] B. Franci and S. Grammatico, “Convergence of sequences: A survey,” Annual Reviews in Control, 2022.
[29] Gurobi Optimization, LLC. Gurobi Optimizer Reference Manual, 2021. https://www.gurobi.com

Fig. 1: (a) Distance between the strategy profile $x^{k}$ generated by Algorithm 1 and an MI-NE, averaged over the considered 50 randomly generated instances of (10); (b) Distance between the strategy profile $x^{k}$ generated by Algorithm 2 and an MI-NE of the Cournot model in (10), averaged over the considered 50 randomly generated instances. The sequence of strategy profiles $\{x^{k}\}_{k \geq 0}$ converges to a $10^{-6}$-approximate MI-NE; (c) Averaged evolution over $k$ of the approximation error measured by the solver in computing an inexact BR, $q^{k}$ (red line) – Algorithm 2. This quantity is upper bounded by the error tolerance sequence $\{\delta^{k}\}_{k \geq 0}$ in Table I (blue line).