M -theory on $AdS_4 \times M^{111}$: 
the complete $Osp(2|4) \times SU(3) \times SU(2)$ spectrum from harmonic analysis*

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Abstract

We reconsider the Kaluza Klein compactifications of $D = 11$ supergravity on $AdS_4 \times (G/H)_7$ manifolds that were classified in the eighties, in the modern perspective of $AdS_4/CFT_3$ correspondence. We focus on one of the three $\mathcal{N} = 2$ cases: $(G/H)_7 = M^{111} = SU(3) \times SU(2) \times U(1)/SU(2) \times U(1)' \times U(1)''$. Relying on the systematic use of the harmonic analysis techniques developed in the eighties by one of us (P. Fré) with R. D’Auria, we derive the complete spectrum of long, short and massless $Osp(2|4) \times SU(3) \times SU(2)$ unitary irreducible representations obtained in this compactification. Our result also provides a general scheme for the other $\mathcal{N} = 2$ compactifications. Furthermore it is a necessary comparison term in the $AdS_4/CFT_3$ correspondence: the complete $AdS/CFT$ match of the spectra that we obtain will provide a much more stringent proof of the $AdS/CFT$ correspondence than in the $S^7$ case, since the structure of the superconformal field theory on the $M2$–brane world volume must be such as to reproduce, at the level of composite operators, the flavor group representations, the conformal dimensions and the hypercharges that we obtain in the present article. The investigation of this match is left to future publications. Here we provide an exhaustive construction of the Kaluza Klein side of our spectroscopy.

*Supported in part by EEC under TMR contract ERBFMRX-CT96-0045
1 Introduction to $M^{111}$: old and new viewpoints

In this paper we present the complete Kaluza Klein spectrum of M-theory compactified on the seven–manifold, $M^{111}$ that has $\mathcal{N} = 2$ supersymmetry and $SU(3) \times SU(2) \times U(1)$ as isometry group. This is part of a wider project [1], [2] that aims at constructing the Kaluza Klein spectra for all the supersymmetric homogeneous seven–manifolds $G/H$ classified in 1984 by Castellani Romans and Warner [3].

As we illustrate in this introductory section, this is the final completion of a research programme that was actively ongoing sixteen years ago [4, 5, 6, 7, 8]. Actually the results presented here supersede all previous partial results and provide, in our opinion, a comprehensive and complete understanding of the involved problem. Sixteen years ago the project was suddenly stopped for two reasons: a) failure to obtain the goals that in the 1984 perspective were considered the main motivations, b) occurrence of the first string revolution that provided alternative perspectives to pursue the same goals, that is the derivation of the standard model from a higher–dimensional, unified, locally supersymmetric theory.

The motivation to resume the project is provided by the current interest in the $AdS/CFT$ correspondence and by the occurrence of the second string revolution. This latter has shown that all ten-dimensional string theories are, together with D=11 Supergravity, perturbative limits of the same quantum theory, named M–theory. Hence all compactified solutions of D=11 supergravity are relevant and highlight different aspects of M–theory. In particular, as shown in [9], each supersymmetric $AdS_4 \times (G/H)_7$ vacuum of the Castellani et al classification has a corresponding supersymmetric $M2$–brane solution which interpolates between such a vacuum at the horizon and a vacuum $M_3 \times C_8$ at infinity, $M_3$ being the three–dimensional minkowskian world volume of the M2–brane and $C_8$ being a suitable Kählerian cone constructed over the Sasakian manifold $(G/H)_7$. Following the ideas explained in [10] and the suggestions coming from the recent literature [11, 12] the final goal of our investigation is to construct the three–dimensional superconformal field theory that is dual to each $(G/H)_7$ compactification of M–theory. In this direction an essential constructive step and comparison term is the derivation of the corresponding Kaluza Klein spectrum, which is our present aim.

In order to introduce the specific example of Kaluza Klein spectrum we shall calculate and the harmonic analysis techniques we shall employ we feel it necessary to make a step back and provide a brief historical perspective for our problem.

1.1 Old Kaluza Klein supergravity: the search for a realistic model based on $M^{pqrs}$ spaces and harmonic analysis

In the beginning of the eighties, after $D = 11$ supergravity had been discovered [13] and also reformulated geometrically [14], a lot of interest was attracted by the revival of the Kaluza Klein idea [13]. It was conceived that the gauge symmetries of fundamental particle interactions might be interpreted as isometries of the seven compactified extra dimensions. In particular it was realized by Freund and Rubin [10] that $D = 11$ supergravity admits classical vacua where the eleven–dimensional space is the product of four–dimensional anti de Sitter space with a
compact seven-dimensional Einstein space:

\[ \mathcal{M}_{11} = AdS_4 \times M_7 \]  

The Killing vectors \( k^m_I(y) \) \( (I = 1, \ldots, \dim \mathcal{G}) \) admitted by the Einstein metric on \( M_7 \) close the Lie algebra of the gauge group \( \mathcal{G} \), while the number \( \mathcal{N} \) of Killing spinors \( \eta^A(y) \) \( (A = 1, \ldots, \mathcal{N}) \) existing on such a geometry coincides with the number of supersymmetries preserved by the four–dimensional compactified theory. Initially, Kaluza Klein supergravity \[17\] focused on the case where the compact manifold \( M_7 \) is topologically a sphere, endowed either with the \( SO(8) \) invariant Einstein metric (round \( S^7 \)) \[18\] or with a second Einstein metric which is \( SO(5) \times SO(3) \) invariant (squashed \( S^7 \)) and preserves \( \mathcal{N} = 1 \) rather than \( \mathcal{N} = 8 \) supersymmetry, as it was shown by Awada, Duff and Pope \[19\]. Although the two gauge groups \( \mathcal{G} \) emerging from such compactifications are not realistic and cannot embed the standard model, an in–depth analysis \[21\], \[22\], \[23\] of these cases revealed many subtle features of Kaluza Klein supergravity and of its relation with the gauged \( \mathcal{N} = 8 \) supergravity of de Wit and Nicolai \[24\].

Furthermore, after D’Auria and Fré proved \[24\] that \( \mathcal{O}sp(8|4) \) is the isometry superalgebra of the round \( S^7 \) background, then the mass spectrum of such a compactification could be organized by Gümaydin and Warner \[26\] into unitary irreducible representations (UIR) of this non–compact superalgebra. This result of 1985, later extended by Gümaydin and Marcus to the spectrum of \( SU(2,2|4) \) supermultiplets that arise in the compactification of type IIB superstring on \( AdS_5 \times S^5 \) \[27\], is one of the cornerstones for the recent reinterpretation of Kaluza Klein supergravity within the framework of an \( AdS/CFT \) correspondence. Indeed as a consequence of the Maldacena conjecture \[28\] that classical supergravity on \( AdS_{p+2} \times M_{D-p-2} \) is dual to a quantum superconformal theory on the \( (p+1) \)–dimensional boundary of \( AdS_{p+2} \) it follows that the Kaluza Klein spectrum of \( AdS_{p+2} \) supermultiplets corresponds to the spectrum of composite primary operators in the conformal field theory \[29\], \[30\], \[31\], \[32\], \[33\], \[34\]. Yet from the Kaluza Klein point of view the seven sphere was not very satisfactory since, as already observed, neither \( SO(8) \) nor \( SO(5) \times SO(3) \) contain the gauge group of the standard model:

\[ \mathcal{G}_s = SU(3)^c \times SU(2)^w \times U(1)^Y \]  

(1.2)

However, already in 1981 Witten had observed \[33\] that seven is the smallest number of dimensions where the group (1.2) can be realized as a regular isometry group. Indeed one can introduce the seven–dimensional homogeneous spaces:

\[ M_{pqr}^{spr} = \frac{\mathcal{G}_s}{\mathcal{H}} \equiv \frac{SU(3)^c \times SU(2)^w \times U(1)^Y}{SU(2)^c \times U(1)^r \times U(1)^w} \]  

(1.3)

where the integer numbers \( p, q, r \in \mathbb{Z} \) characterize the topology of the manifold and are introduced through the embedding of the isotropy subgroup \( \mathcal{H} = SU(2)^c \times U(1)^r \times U(1)^w \) into \( \mathcal{G}_s \) as explained in section 4. In order to show that \( AdS_4 \times M_{pqr}^{spr} \) are exact vacua of \( D = 11 \) supergravity that produce the standard model gauge group through the Kaluza Klein mechanism it was necessary to construct an \( SU(3)^c \times SU(2) \times U(1) \) invariant Einstein metric on each of these spaces. This was done by Castellani, D’Auria and Fré in \[4\]. In the same paper the properties of these spaces with respect to supersymmetry breaking were discussed. It was established that
for $p \neq q$ there are no Killing spinors while in the case $p = q$ the resulting four–dimensional theory has $\mathcal{N} = 2$ supersymmetry. Since the spectrum does not depend on the actual value of $p = q$ or of $r \neq 0$, we can just focus on the space $M^{111}$. Hence an early conclusion reached in [4] was that by compactifying $D = 11$ supergravity on $AdS_4 \times M^{111}$ one obtains a low energy effective action containing the $\mathcal{N} = 2$ supergravity multiplet plus, at least, the vector multiplets of the group $SU(3) \times SU(2)$. The factor $U(1)^Y$ in the isometry group is the R–symmetry of $\mathcal{N} = 2$ supergravity and the associated gauge boson is the graviphoton.

In order to establish the precise content of the theory, both at the massless and the massive level more information on the spectrum was necessary. In this respect the case of $M^{111}$ was (and it is) quite different with respect to the case of the round $S^7$. In the latter case it suffices to know that the isometry superalgebra is $Osp(8|4)$ to deduce the entire spectrum from a purely algebraic construction. The reason is the following. As shown by G¨unaydin and Warner [20] the complete Kaluza Klein spectrum of the round $S^7$ compactification is composed of short $Osp(8|4)$ supermultiplets characterized by a quantization of masses (or better anti de Sitter energy labels) in terms of the $SO(8)$ R-symmetry representation in which their Clifford vacuum falls. In modern language all the Kaluza Klein states are BPS states and their spectrum can be derived by constructing the short UIR of $Osp(8|4)$ with highest spin limited to be two. From the perspective of the three dimensional superconformal theory this means that all the composite primary operators have conformal weights equal to their naive dimensions, no anomalous dimensions being generated.

The same results can also be derived from harmonic analysis on the compact space $S^7$ but no new information is obtained with respect to the algebraic construction.

On the contrary in the case of anti de Sitter vacua with $\mathcal{N} = 1, 2, 3$ supersymmetry the Kaluza Klein states have to be organized in supermultiplets of $Osp(\mathcal{N}|4) \times G'$ where $G'$ is the factor in the isometry group of the seven–manifold that commutes with supersymmetry and with the $R$–symmetry factor $O(N)$. In this case the Kaluza Klein states do not necessarily fall into short multiplets of $Osp(\mathcal{N}|4)$ and do not necessarily have quantized energy labels (or masses). Indeed their masses depend not only on the $R$–symmetry representation but also on the $G'$ representation. In the modern perspective of the dual superconformal field theory this means that anomalous dimensions are generated, yet if the $AdS/CFT$ correspondence is really true then these anomalous dimensions can be exactly calculated from the supergravity side by harmonic analysis on the compact seven–manifold. Because of the interest in the $D = 11$ compactification on $M^{111}$ that displayed the standard model group (1.2) as gauge group, the oldest of us (P.Fré), in collaboration with R. D’Auria, developed in the years 1983-1984 a systematic approach to the harmonic analysis on coset manifolds with Killing spinors and in particular on $M^{pq r}$ spaces [4, 5, 6]. The formalism and techniques developed in those papers are the basis for the present investigation where we provide the completion of the classification programme started fifteen years ago in [8]. The results obtained in [2, 3, 4, 5, 8] showed that the massless sector of M-theory on $AdS_4 \times M^{111}$ is given by the $\mathcal{N} = 2$ graviton multiplet plus the vector multiplets of $SU(3) \times SU(2) \times U(1)$. Indeed, although $U(1)^Y$ is the R–symmetry and its gauge-boson is the graviphoton, an additional massless vector multiplet, at the time named the Betti multiplet, comes from the three–form $A_{\mu ij}$ with a space–time index and two internal indices. This happens because the $M^{111}$ space has a closed non trivial two–form. At the
classical level there are no states charged with respect to such a $U(1)$ but in non perturbative string theory there might be. Indeed this is like a Ramond-Ramond multiplet. It is still an open problem to determine the twelve-dimensional special Kähler manifold:

$$SK_{12}^{111}$$

(1.4)

whose geometry determines the $\mathcal{N} = 2$ low energy supergravity action for the $AdS_4 \times M_{111}$ compactification.

1.2 The results of this paper in the modern perspective of AdS/CFT correspondence

The programme of deriving the complete $Osp(2|4) \times SU(3) \times SU(2)$ spectrum was left unachieved in 1984 since the hopes to obtain a realistic Kaluza Klein model were doomed. Indeed not only was the theory plagued by a too large cosmological constant (anti de Sitter space) and by an intrinsic non chiral nature, but also the quark and lepton representations could not be found in the hypermultiplet spectrum [30]. The harmonic analysis had been carried through to the point of determining the spectrum of graviton and gravitino multiplets (see [8]) but in order to derive the spectrum of vector multiplets and hypermultiplets a major effort was still necessary. Indeed one needed to find the eigenvalues of the Laplace Beltrami operators:

$$M_{(1)(0)}^2, \quad M_{(1)^2(0)}^2, \quad M_{(1)^3}$$

(1.5)

acting respectively on 1–forms, 2–forms and 3–forms.

The spectra of these eigenvalues is precisely what we determine in the present paper and this information combined with the old results allows us to organize all the Kaluza Klein states in $Osp(2|4)$ supermultiplets. Indeed as a by product of our analysis we also find the structure of short and long UIR representations of the superalgebra $Osp(2|4)$ limited by highest spin less or equal to two. In other words the path we follow here is somehow the reciprocal path to that followed by Günaydin and Warner in the case of the round $S^7$. There representation theory of the non compact superalgebra $Osp(8|4)$ could be used to retrieve all the data inherent to harmonic analysis on $S^7$. Here full fledged harmonic analysis on $M_{111}$, besides giving detailed information on the energy labels of the supermultiplets, determines also the structure of the UIR of the non–compact superalgebra $Osp(2|4)$. Obviously such a structure is universal and must be the same in the other $\mathcal{N} = 2$ compactifications. What will be different in the other cases is the $G'$ group and the $G''$ representation assignment of the supermultiplets. Similarly, for long multiplets, but not for short multiplets we will have different values of the energy labels (anomalous conformal dimensions in the three dimensional SCFT interpretation). According to the Castellani et al classification (see [3] for a summary in modern perspective) there are just three $\mathcal{N} = 2$ theories in this setup and they correspond to the cases,

$$M^{ppp} ; \quad G' = SU(3) \times SU(2)$$

$$V_{5,2} ; \quad G' = SO(5)$$

$$Q^{ppp} ; \quad G' = SU(2)^3$$

(1.6)
The first manifold is treated in the present paper. The spectrum of the Stiefel manifold $V_{5,2}$ is presently under construction and will appear in a forthcoming publication [1]. As for the last case $Q_{111}$, originally introduced by R. D’Auria, P. Fré and Van Nieuwenhuizen [37], in this case there already exists a proposal for the dual superconformal field theory [11]. If the $Osp(2|4) \times SU(2)^3$ spectrum had been determined the conjectured AdS/CFT correspondence could be tested. Unfortunately the harmonic analysis on $Q_{111}$ is not yet available and has still to be planned.

Hence the present paper is a first essential step along a path that we outline in the outlook at the end of the paper.

1.3 Structure of the paper

Our paper is divided in two parts. In the first we give a summary of our result, namely we present the full spectrum of $Osp(2|4) \times SU(3) \times SU(2)$ supermultiplets while in the second part we give all the details of its derivation from systematic harmonic analysis on $M^{111}$. The first part is written in such a way as to be self-consistent. The reader interested only in the spectrum but not in harmonic analysis can completely skip part two. More specifically the content of the sections is as follows.

Section two describes the structure of $Osp(2|4)$ supermultiplets from a general standpoint.

Section three presents the spectrum of $Osp(2|4)$ supermultiplets that occur in the $M^{111}$ compactifications giving their $SU(3) \times SU(2)$-quantum numbers. The multiplets are divided in three classes: long, short and massless and for each class we have graviton, gravitino and vector multiplets. The hypermultiplets are always short. There are no massless gravitino multiplets because this would mean that supersymmetry is enhanced from $\mathcal{N} = 2$ to higher $\mathcal{N}$, which is not the case.

Section four describes the geometry of the $M^{111}$ space.

Section five introduces the basic items of harmonic analysis and applies them to the space under consideration.

Section six explains how to derive the eigenvalues of the Laplace-Beltrami operators relevant to our problem from harmonic analysis. The section is divided in subsections dealing with the Laplace-Beltrami operators acting on zero-forms, one-forms, two-forms, three-forms and spinors respectively.

Section seven is devoted to filling the various states found by harmonic analysis into $Osp(2|4)$ supermultiplets.

Section eight gives an outlook on the continuation of our research project and future developments.

Appendix A contains our conventions while appendix B summarizes, for the reader convenience, the main formulas on harmonic analysis and mass relations that we use throughout the paper.
PART ONE: THE RESULT

In this part of the article we present our result while in the second part we give all the details of its derivation.

Specifically, in part two we shall give a summary of $M^{111}$ geometry and a full fledged discussion of harmonic analysis on such a manifold. In that part the interested reader can find all the details concerning the calculation of the eigenvalue spectra for the operators $M_{(1)(0)^2}$, $M_{(1)^2(0)}$ and $M_{(1)^3}$ [6, 39] which are the key tool to obtain the complete spectrum of supermultiplets. Yet the reader whose interest is limited to the actual form of this spectrum can just confine his reading to part one.

Here we give the result in two steps. In the first step we present the structure of long and short multiplets of the $Osp(2|4)$ superalgebra that are limited by highest spin less or equal to two. Such a structure could have been derived either by means of the Freedman Nicolai method of vanishing norms [40], or by the Güneydin et Warner oscillator method [26]. We bypass such calculations obtaining the result as a byproduct of harmonic analysis on the specific manifold $M^{111}$. Yet we emphasize (as we did in the introduction) that this part of the result is universal and applies to all other $\mathcal{N} = 2$ compactifications on $AdS_4$. In the second step we exactly specify how many long and short $Osp(2|4)$ multiplets occur in the $M^{111}$ compactification and for each of them we give both the $SU(3) \times SU(2)$ representation assignment and the value of the energy and hypercharge labels $(E_0, y_0)$ of the corresponding Clifford vacuum.

2 Structure of the $Osp(2|4)$ multiplets

The structure of the relevant $\mathcal{N} = 2$ supermultiplets in $AdS_4$ superspace is summarized in tables 1, 2, 3, 4, 5, 6, 7, 8, 9 whose content we discuss in the present section. To describe the organization of these tables we recall the basic facts about $Osp(2|4)$ unitary irreducible representations (UIR).

The even subalgebra of the superalgebra:

$$G_{\text{even}} = Sp(4, \mathbb{R}) \oplus SO(2) \subset Osp(2|4)$$  \hspace{1cm} (2.1)

is a direct sum of subalgebras, where $Sp(4, \mathbb{R}) \sim SO(2, 3)$ is the isometry algebra of $AdS_4$ while the compact subalgebra $SO(2)$ generates $R$-symmetry. The maximally compact subalgebra of $G_{\text{even}}$ is

$$G_{\text{compact}} = SO(2)_E \oplus SO(3)_S \oplus SO(2)_R \subset G_{\text{even}}.$$  \hspace{1cm} (2.2)

The generator of $SO(2)_E$ is interpreted as the hamiltonian of the system when $Osp(2|4)$ acts as the isometry group of anti de Sitter superspace. Consequently its eigenvalues $E$ are the energy levels of possible states for the system. The group $SO(3)_S$ is the ordinary rotation group and similarly its representation labels $s$ describe the possible spin states of the system. Finally the eigenvalue $y$ of the generator of $SO(2)_R$ is the hypercharge of a state.

A supermultiplet, namely a UIR of the superalgebra $Osp(2|4)$ is composed of a finite number of UIR of the even subalgebra $G_{\text{even}}$ (2.1), each of them being what in physical language we
call a particle state, characterized by a spin “$s$”, a mass “$m$” and a hypercharge “$y$”. From a mathematical viewpoint each UIR representation of the non-compact even subalgebra $G_{\text{even}}$ is an infinite tower of finite dimensional UIR representations of the compact subalgebra $G_{\text{compact}}$ (2.2). The lowest lying representation of such a tower $|E, s, y>$ is the Clifford vacuum. The mass, spin and hypercharge of the corresponding particle are read from the labels of the Clifford vacuum by use of the relations between mass and energy [8] that we have recalled in appendix B eq. (B.7).

In the same way there is a Clifford vacuum $|E_0, s_0, y_0>$ for the entire supermultiplet out of which we not only construct the corresponding particle state but also, through the action of the $SUSY$ charges, we construct the Clifford vacua $|E_0 + \ldots, s_0 + \ldots, y_0 + \ldots>$ of the other members of the same supermultiplet. Hence the structure of a supermultiplet is conveniently described by listing the energy $E$, the spin $s$ and the hypercharge $y$ of all the Clifford vacua of the multiplet.

In tables 1, 2, 3, 4, 5, 6, 7, 8, 9 we provide such information in the first three columns. The remaining columns provide additional information concerning the way the abstract $Osp(2|4)$ supermultiplets are actually realized in Kaluza Klein supergravity. Indeed for each particle state appearing in the supermultiplet we write the name of the corresponding field in the Kaluza Klein expansion of $D = 11$ supergravity to which such a particle state contributes. The standard expansion of linearized $D = 11$ supergravity and the conventions for the names of the $D = 4$ fields are given in appendix B, eq. (B.1).

Given these preliminaries let us discuss our result for the structure of the $Osp(2|4)$ supermultiplets with $s \leq 2$.

In $\mathcal{N}$–extended $AdS_4$ superspace there are three kinds of supermultiplets:

- the long multiplets
- the short multiplets
- the massless multiplets

For $\mathcal{N} = 2$ the long multiplets satisfy the following unitarity relation, without saturation:

$$E_0 > |y_0| + s_0 + 1. \quad (2.3)$$

Furthermore we distinguish three kinds of long multiplets depending on the highest spin state they contain: $s_{\text{max}} = 2, \frac{3}{2}, 1$. These multiplets are respectively named long graviton, long gravitino and long vector multiplets.

The long graviton multiplet, satisfying $E_0 > |y_0| + 2$, has the structure displayed in table 1.

The long gravitino multiplet, satisfying $E_0 > |y_0| + \frac{3}{2}$, has the structure displayed in table 2. As we see this type of multiplet is realized in Kaluza Klein supergravity in two different ways. The two ways correspond to a resolution of the ambiguity inherent to the quadratic form of the mass/energy relations (B.7). One realization of the multiplets chooses one branch of the relation, the second realization chooses the second branch. The two realizations of the multiplets are implemented by different fields of the Kaluza Klein expansion. For instance in one realization the vector fields ($Z$) arise from the expansion of the three–form $A_{\mu ij}$ (with one
\begin{table}
\centering
\begin{tabular}{c|c|c|c|c}
Spin & Energy & Hypercharge & Mass (\textsuperscript{2}) & Name \\
\hline
2 & \(E_0 + 1\) & \(y_0\) & \(16(E_0 + 1)(E_0 - 2)\) & \(h\) \\
\frac{1}{2} & \(E_0 + \frac{3}{2}\) & \(y_0 - 1\) & \(-4E_0 - 4\) & \(\chi^-\) \\
\frac{1}{2} & \(E_0 + \frac{1}{2}\) & \(y_0 + 1\) & \(-4E_0 - 4\) & \(\chi^-\) \\
\frac{1}{2} & \(E_0 + \frac{1}{2}\) & \(y_0 - 1\) & \(4E_0 - 8\) & \(\chi^+\) \\
\frac{1}{2} & \(E_0 + \frac{1}{2}\) & \(y_0 + 1\) & \(4E_0 - 8\) & \(\chi^+\) \\
1 & \(E_0 + 2\) & \(y_0\) & \(16E_0(E_0 + 1)\) & \(W\) \\
1 & \(E_0 + 1\) & \(y_0 - 2\) & \(16E_0(E_0 - 1)\) & \(Z\) \\
1 & \(E_0 + 1\) & \(y_0 + 2\) & \(16E_0(E_0 - 1)\) & \(Z\) \\
1 & \(E_0 + 1\) & \(y_0\) & \(16E_0(E_0 - 1)\) & \(Z\) \\
1 & \(E_0\) & \(y_0\) & \(16(E_0 - 1)(E_0 - 2)\) & \(A\) \\
\frac{1}{2} & \(E_0 + \frac{3}{2}\) & \(y_0 - 1\) & \(4E_0\) & \(\lambda_T\) \\
\frac{1}{2} & \(E_0 + \frac{1}{2}\) & \(y_0 + 1\) & \(4E_0\) & \(\lambda_T\) \\
\frac{1}{2} & \(E_0 + \frac{1}{2}\) & \(y_0 - 1\) & \(-4E_0 + 4\) & \(\lambda_T\) \\
\frac{1}{2} & \(E_0 + \frac{1}{2}\) & \(y_0 + 1\) & \(-4E_0 + 4\) & \(\lambda_T\) \\
0 & \(E_0 + 1\) & \(y_0\) & \(16E_0(E_0 - 1)\) & \(\phi\)
\end{tabular}
\caption{\(\mathcal{N} = 2\) long graviton multiplet}
\end{table}

\begin{table}
\centering
\begin{tabular}{c|c|c|c|c|c|c}
Spin & Energy & Hypercharge & Mass (\textsuperscript{2}) & Name & Mass (\textsuperscript{2}) & Name \\
\hline
\frac{1}{2} & \(E_0 + 1\) & \(y_0\) & \(4E_0 - 6\) & \(\chi^+\) & \(-4E_0 - 2\) & \(\chi^-\) \\
1 & \(E_0 + \frac{3}{2}\) & \(y_0 - 1\) & \(16(E_0 - \frac{1}{2})(E_0 + \frac{1}{2})\) & \(Z\) & \(16(E_0 - \frac{1}{2})(E_0 + \frac{1}{2})\) & \(W\) \\
1 & \(E_0 + \frac{1}{2}\) & \(y_0 + 1\) & \(16E_0 - \frac{1}{2}(E_0 + \frac{1}{2})\) & \(Z\) & \(16E_0 - \frac{1}{2}(E_0 + \frac{1}{2})\) & \(W\) \\
1 & \(E_0 + \frac{1}{2}\) & \(y_0 - 1\) & \(16(E_0 - \frac{1}{2})(E_0 - \frac{1}{2})\) & \(A\) & \(16(E_0 - \frac{1}{2})(E_0 - \frac{1}{2})\) & \(Z\) \\
1 & \(E_0 + \frac{1}{2}\) & \(y_0 + 1\) & \(16E_0 - \frac{1}{2}(E_0 - \frac{1}{2})\) & \(A\) & \(16E_0 - \frac{1}{2}(E_0 - \frac{1}{2})\) & \(Z\) \\
\frac{1}{2} & \(E_0 + 2\) & \(y_0\) & \(4E_0 + 2\) & \(\lambda_T\) & \(-4E_0 - 2\) & \(\lambda_T\) \\
\frac{1}{2} & \(E_0 + 1\) & \(y_0 - 2\) & \(-4E_0 + 2\) & \(\lambda_T\) & \(-4E_0 - 2\) & \(\lambda_T\) \\
\frac{1}{2} & \(E_0 + 1\) & \(y_0\) & \(-4E_0 + 2\) & \(\lambda_T\) & \(4E_0 - 2\) & \(\lambda_T\) \\
\frac{1}{2} & \(E_0 + 1\) & \(y_0 + 2\) & \(-4E_0 + 2\) & \(\lambda_T\) & \(4E_0 - 2\) & \(\lambda_T\) \\
\frac{1}{2} & \(E_0 + 1\) & \(y_0\) & \(-4E_0 + 2\) & \(\lambda_T\) & \(4E_0 - 2\) & \(\lambda_T\) \\
\frac{1}{2} & \(E_0\) & \(y_0\) & \(4E_0 - 6\) & \(\lambda_L\) & \(-4E_0 + 6\) & \(\lambda_T\) \\
0 & \(E_0 + \frac{3}{2}\) & \(y_0 - 1\) & \(16(E_0 - \frac{1}{2})(E_0 + \frac{1}{2})\) & \(\phi\) & \(16(E_0 - \frac{1}{2})(E_0 + \frac{1}{2})\) & \(\pi\) \\
0 & \(E_0 + \frac{3}{2}\) & \(y_0 + 1\) & \(16(E_0 - \frac{1}{2})(E_0 + \frac{1}{2})\) & \(\phi\) & \(16(E_0 - \frac{1}{2})(E_0 + \frac{1}{2})\) & \(\pi\) \\
0 & \(E_0 + \frac{1}{2}\) & \(y_0 - 1\) & \(16(E_0 - \frac{1}{2})(E_0 - \frac{1}{2})\) & \(\pi\) & \(16(E_0 - \frac{1}{2})(E_0 - \frac{1}{2})\) & \(\phi\) \\
0 & \(E_0 + \frac{1}{2}\) & \(y_0 + 1\) & \(16(E_0 - \frac{1}{2})(E_0 - \frac{1}{2})\) & \(\pi\) & \(16(E_0 - \frac{1}{2})(E_0 - \frac{1}{2})\) & \(\phi\)
\end{tabular}
\caption{\(\mathcal{N} = 2\) long gravitino multiplets \(\chi^+\) and \(\chi^-\)}
\end{table}
Table 3: $\mathcal{N} = 2$ long vector multiplets $A, W$ and $Z$

| Spin | Energy | Hypercharge | Mass ($^2$) | Name | Name | Mass ($^2$) | Name |
|------|--------|-------------|-------------|------|------|-------------|------|
| 1    | $E_0 + 1$ | $y_0$ | $16E_0(E_0 - 1)$ | $A$ | $W$ | $16E_0(E_0 - 1)$ | $Z$ |
| $1/2$ | $E_0 + 3/2$ | $y_0 - 1$ | $-4E_0$ $\lambda_T$ | $\lambda_L$ | $4E_0$ $\lambda_T$ |
| $1/2$ | $E_0 + 1/2$ | $y_0 + 1$ | $-4E_0$ $\lambda_T$ | $\lambda_L$ | $4E_0$ $\lambda_T$ |
| $1/2$ | $E_0 + 1/2$ | $y_0 - 1$ | $4E_0 - 4$ $\lambda_L$ | $\lambda_T$ | $-4E_0 + 4$ $\lambda_T$ |
| $1/2$ | $E_0 + 1/2$ | $y_0 + 1$ | $4E_0 - 4$ $\lambda_L$ | $\lambda_T$ | $-4E_0 + 4$ $\lambda_T$ |
| 0    | $E_0 + 2$ | $y_0$ | $16E_0(E_0 + 1)$ | $\phi$ | $\Sigma$ | $16E_0(E_0 + 1)$ | $\pi$ |
| 0    | $E_0 + 1$ | $y_0 - 2$ | $16E_0(E_0 - 1)$ | $\pi$ | $\pi$ | $16E_0(E_0 - 1)$ | $\phi$ |
| 0    | $E_0 + 1$ | $y_0 + 2$ | $16E_0(E_0 - 1)$ | $\pi$ | $\pi$ | $16E_0(E_0 - 1)$ | $\phi$ |
| 0    | $E_0 + 1$ | $y_0$ | $16(E_0 - 2)(E_0 - 1)$ | $S$ | $\phi$ | $16(E_0 - 2)(E_0 - 1)$ | $\pi$ |

Table 4: $\mathcal{N} = 2$ short graviton multiplet with positive hypercharge $y_0 > 0$

| Spin | Energy | Hypercharge | Mass ($^2$) | Name |
|------|--------|-------------|-------------|------|
| $2$  | $y_0 + 3$ | $y_0$ | $16y_0(y_0 + 3)$ | $h$ |
| $3/2$ | $y_0 + 1$ | $y_0 - 1$ | $-4y_0 - 12$ $\chi^-$ | $\chi^+$ |
| $3/2$ | $y_0 + 1$ | $y_0 + 1$ | $4y_0$ $\chi^+$ | $\chi^+$ |
| $1$  | $y_0 + 3$ | $y_0 - 2$ | $16(y_0 + 2)(y_0 + 1)$ | $Z$ |
| $1$  | $y_0 + 3$ | $y_0$ | $16(y_0 + 2)(y_0 + 1)$ | $Z$ |
| $1$  | $y_0 + 2$ | $y_0$ | $16y_0(y_0 + 1)$ | $A$ |
| $1/2$ | $y_0 + 3/2$ | $y_0 - 1$ | $-4y_0 - 4$ $\lambda_T$ |

space–time index and two internal indices), in the other realization the vector fields $(A/W)$ arise both from the metric $g_{\mu i}$ and from the three–form $A_{\mu\nu i}$ with two legs on space–time and one internal leg.

The long vector multiplet, satisfying $E_0 > |y_0| + 1$, has the structure displayed in table 3. Also the long vector multiplet has different realizations from the Kaluza Klein viewpoint.

The short multiplets are of two kinds: the short graviton, gravitino and vector multiplets, that saturate the bound

$$E_0 = |y_0| + s_0 + 1 \quad (2.4)$$

and the hypermultiplets (spin 1/2 multiplets), that saturate the other bound

$$E_0 = |y_0| \quad \text{with} \quad |y_0| \geq \frac{1}{2}. \quad (2.5)$$

The short graviton multiplet, satisfying $E_0 = |y_0| + 2$, has the structure displayed in table 4.

The short gravitino multiplet, satisfying $E_0 = |y_0| + \frac{3}{2}$, has the structure displayed in table 5.

The short vector multiplet, satisfying $E_0 = |y_0| + 1$, has the structure displayed in table 6.

\footnote{In this table there is a ±. The actual sign can be calculated looking at the norms of a three-creation operator state.}
We must stress that the multiplets displayed in tables 4, 5, 6 are only half of the story, since they can be viewed as the BPS states where \( E_0 = y_0 + s_0 + 1 \) and \( y_0 > 0 \). In addition one has also the anti BPS states. These are the short multiplets where \( E_0 = -y_0 + s_0 + 1 \) with \( y_0 < 0 \). The structure of these anti short multiplets can be easily read off from tables 4, 5, 6 by reversing the sign of all hypercharges.

The hypermultiplet, satisfying \( E_0 = |y_0| \geq \frac{1}{2} \), has the structure displayed in table 7. This structure is different from the others because this multiplet is complex. This means that for each field there is another field with same energy and spin but opposite hypercharge. So it is built with two \( N = 1 \) Wess-Zumino multiplets. The four real scalar field can be arranged into a quaternionic complex form.

The massless multiplets are either short graviton or short vector multiplets satisfying the

| Spin | Energy | Hypercharge | Mass (\( \Lambda^2 \)) | Name |
|------|--------|-------------|----------------------|------|
| \(\frac{1}{2}\) | \(y_0 + \frac{1}{2}\) | \(y_0\) | \(4y_0\) | \(\lambda_L^+\) |
| 0 | \(y_0\) | \(y_0 - 1\) | \(4y_0 - 4\) | \(\pi\) |
| \(\frac{1}{2}\) | \(y_0 + \frac{1}{2}\) | \(y_0\) | \(16y_0(y_0 - 1)\) | \(\pi\) |
| 0 | \(y_0\) | \(-y_0 + 1\) | \(4y_0 - 4\) | \(S\) |

Table 5: \( N = 2 \) short gravitino multiplet \( \chi^+ \) with positive hypercharge \( y_0 > 0 \)

| Spin | Energy | Hypercharge | Mass (\( \Lambda^2 \)) | Name |
|------|--------|-------------|----------------------|------|
| 1 | \(y_0 + 2\) | \(y_0\) | \(16y_0(y_0 + 1)\) | \(A\) |
| \(\frac{1}{2}\) | \(y_0 + \frac{1}{2}\) | \(y_0\) | \(-4y_0 - 4\) | \(\lambda_T\) |
| \(\frac{1}{2}\) | \(y_0 + \frac{1}{2}\) | \(y_0\) | \(4y_0\) | \(\lambda_L\) |
| 0 | \(y_0 + 2\) | \(y_0 - 1\) | \(16y_0(y_0 + 1)\) | \(\pi\) |
| 0 | \(y_0 + 2\) | \(y_0\) | \(16y_0(y_0 + 1)\) | \(\pi\) |
| 0 | \(y_0 + 1\) | \(y_0\) | \(16y_0(y_0 - 1)\) | \(S\) |

Table 6: \( N = 2 \) short vector multiplet \( A \) with positive hypercharge \( y_0 > 0 \)

| Spin | Energy | Hypercharge | Mass (\( \Lambda^2 \)) | Name |
|------|--------|-------------|----------------------|------|
| \(\frac{1}{2}\) | \(y_0 + \frac{1}{2}\) | \(-y_0 + 1\) | \(4y_0 - 4\) | \(\lambda_L\) |
| 0 | \(y_0\) | \(-y_0 + 2\) | \(16y_0(y_0 - 1)\) | \(\pi\) |
| \(\frac{1}{2}\) | \(y_0 + \frac{1}{2}\) | \(-y_0 + 1\) | \(4y_0 - 4\) | \(\lambda_L\) |
| 0 | \(y_0\) | \(-y_0 + 2\) | \(16y_0(y_0 - 1)\) | \(\pi\) |
| 0 | \(y_0\) | \(-y_0 \) | \(16(y_0 - 1)(y_0 - 2)\) | \(S\) |

Table 7: \( N = 2 \) hypermultiplet, \( y_0 > 0 \)
| Spin | Energy | Hypercharge | Mass (2) | Name |
|------|--------|-------------|----------|------|
| 2    | 3      | 0           | 0        | $h$  |
| $\frac{1}{2}$ | 5     | $-1$        | 0        | $\chi^+$ |
| $\frac{3}{2}$ | 2     | $+1$        | 0        | $\chi^+$ |
| 1    | 2      | 0           | 0        | $A$  |

Table 8: $\mathcal{N} = 2$ massless graviton multiplet

| Spin | Energy | Hypercharge | Mass (2) | Name | Mass (2) | Name |
|------|--------|-------------|----------|------|----------|------|
| 1    | 2      | 0           | 0        | $A$  | 0        | $Z$  |
| $\frac{1}{2}$ | 2     | $-1$        | 0        | $\lambda_L$ | 0        | $\lambda_T$ |
| $\frac{3}{2}$ | 2     | $+1$        | 0        | $\lambda_L$ | 0        | $\lambda_T$ |
| 0    | 2      | 0           | 0        | $\pi$ | 0        | $\phi$ |
| 0    | 1      | 0           | 0        | $S$  | 0        | $\pi$ |

Table 9: $\mathcal{N} = 2$ massless vector multiplets $A$ and $Z$

The further condition

$$E_0 = s_0 + 1 \quad \text{equivalent to} \quad y_0 = 0.$$  \hfill (2.6)

The massless graviton multiplet, satisfying $E_0 = 2 \quad y_0 = 0$, has the structure displayed in table 8.

The massless vector multiplet, satisfying $E_0 = 1 \quad y_0 = 0$, has the structure displayed in table 9. These are the $\mathcal{N} = 2$ supermultiplets in anti de Sitter space that can occur in Kaluza Klein supergravity.

The structure of the long multiplets was derived in the eighties (see [8]), whereas the structure of the short and massless multiplets we have given here, is, to the best of our knowledge, a new result that we have obtained as a byproduct of harmonic analysis on $M^{11}$. In establishing our result we have also used as a tool the necessary decomposition of the $\mathcal{N} = 2$ multiplets into $\mathcal{N} = 1$ multiplets (see Figures 1, 2, 3, 4, 5, 6, 7) for long and short multiplets. Finally we note that the structure of massless multiplets is identical in the anti de Sitter and in the Poincaré case, as it is well known.

This is all the information that we can have on the multiplets from superalgebra theory. The determination of the actual spectrum by harmonic analysis teaches us a lot more: indeed we find out which multiplets are present for each representation of $G'$, how many they are, and the exact value of the $E_0$ label for each multiplet. So, at the end we know the energy of every field in the multiplets. In the dual superconformal field theory on the two–brane these are the conformal dimensions of the primary operators. In addition we find also the hypercharge of each multiplet in the spectrum.
Figure 1: $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ decomposition of the long graviton multiplet

Figure 2: $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ decomposition of the long gravitino multiplet

Figure 3: $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ decomposition of the long vector multiplet
Figure 4: $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ decomposition of the short graviton multiplet

Figure 5: $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ decomposition of the short gravitino multiplet

Figure 6: $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ decomposition of the short vector multiplet

Figure 7: $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ decomposition of the hypermultiplet
3 The complete spectrum of $Osp(2|4) \times SU(3) \times SU(2)$ supermultiplets

As stressed in the introduction the irreducible supermultiplets of $Osp(2|4)$ occur into irreducible representations of the bosonic group

$$G' = SU(3) \times SU(2),$$

whose interpretation is the flavor group in the conformal field theory side of the correspondence and it is the gauge group in the supergravity side. In any case the crucial information one needs to extract from harmonic analysis is precisely the $G'$ representation assignment of the supermultiplets and the actual value of $E_0$ and $y_0$.

To present our result, we first need to fix our conventions for labelling the irreducible representations of $G'$. It has rank three, so that its irreducible representations are labeled by three integer numbers. A representation of $SU(3)$ can be identified by a Young diagram of the following type

```
  · · ·
  · · ·
  ·   ·
```

while an irreducible representation of $SU(2)$ can be described by a Young diagram as follows

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  · · ·
  ·   ·
```

Hence we can take the nonnegative integers $M_1$, $M_2$, $2J$, as the labels of a $G'$ irreducible representation.

Relying on the procedures explained in the following sections and originally introduced in [5], we have found the following result.

Not every $G'$ representation is actually present, but only those representations that satisfy the following relations

$$M_2 - M_1 \in 3\mathbb{Z} \quad ; \quad J \in \mathbb{N}. \quad (3.1)$$

In the following pages, for each type of $\mathcal{N} = 2$ multiplet we list the $G'$ representations through which it occurs in the spectrum. We do this by writing bounds on the range of values for the $M_1$, $M_2$, $2J$ labels. The reader should take into account that, case by case, in addition to the specific bounds we write, also the general restriction (3.1) is to be imposed.

Furthermore for every multiplet, we give the energy and hypercharge values $E_0$ and $y_0$ of the Clifford vacuum. From the tables 1, 2, 3, 4, 5, 6, 7, 8, 9 it is straightforward to get the energies and the hypercharges of all other fields in each multiplet.

As a short-hand notation let us name $H_0$ the following quadratic form in the representation labels:

$$H_0 \equiv \frac{64}{3} (M_2 + M_1 + M_2 M_1) + 32J (J + 1) + \frac{32}{9} (M_2 - M_1)^2. \quad (3.2)$$
Up to multiplicative constants, the first two addends $M_2 + M_1 + M_2 M_1$ and $J(J+1)$ are the Casimirs of $G' = SU(3) \times SU(2)$. The last addendum is contributed by the square of the hypercharge through its relation with the $SU(3)$ representation implied by the geometry of the space.

LONG MULTIPLETS

1. **Long graviton multiplets** $(1 \, 2), 4 \left( \frac{3}{2} \right), 6 \, (1), 4 \left( \frac{1}{2} \right), 1$

   One long graviton multiplet (table [II]) in each representation of the series

   \[
   \left\{ M_1 \geq 0, \ M_2 \geq 0, \ J > \frac{1}{3} (M_2 - M_1) \right\} \cup \\
   \left\{ M_1 > 0, \ M_2 > 0, \ J = \frac{1}{3} (M_2 - M_1) \right\}
   \]

   (3.3)

   with

   \[
   h : \quad E_0 = \frac{1}{2} + \frac{1}{4} \sqrt{H_0 + 36}, \quad y_0 = \frac{2}{3} (M_2 - M_1)
   \]

   (3.4)

2. **Long gravitino multiplets** $(1 \left( \frac{3}{2} \right), 4 \, (1), 6 \left( \frac{1}{2} \right), 4 \, (0))$

   - Four long gravitino multiplets (two $\chi^+$ and two $\chi^-$, table [III]) in each representation of the series

   \[
   \left\{ M_2 > 0, \ M_1 > 0, \ J > \frac{1}{3} (M_2 - M_1) + 1 \right\} \cup \\
   \left\{ M_2 \geq M_1 > 1, \ J = \frac{1}{3} (M_2 - M_1) + 1 \right\} \cup \\
   \left\{ M_1 \geq M_2 > 1, \ J = -\frac{1}{3} (M_2 - M_1) + 1 \right\}
   \]

   (3.5)

   with

   \[
   \chi^+ : \quad E_0 = \frac{3}{2} + \frac{1}{4} \sqrt{H_0 + \frac{32}{3} (M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3} (M_2 - M_1) - 1 \quad (3.6)
   \]

   \[
   \chi^+ : \quad E_0 = \frac{3}{2} + \frac{1}{4} \sqrt{H_0 - \frac{32}{3} (M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3} (M_2 - M_1) + 1 \quad (3.7)
   \]

   \[
   \chi^- : \quad E_0 = \frac{3}{2} + \frac{1}{4} \sqrt{H_0 + \frac{32}{3} (M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3} (M_2 - M_1) - 1 \quad (3.8)
   \]

   \[
   \chi^- : \quad E_0 = \frac{3}{2} + \frac{1}{4} \sqrt{H_0 - \frac{32}{3} (M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3} (M_2 - M_1) + 1 \quad (3.9)
   \]

   - Three long gravitino multiplets (one $\chi^+$ and two $\chi^-$, table [IV]), in each representation of the series

   \[
   \left\{ M_2 \geq M_1 = 1, \ J = \frac{1}{3} (M_2 - M_1) + 1 \right\}
   \]

   (3.10)
with
\[
\chi^+ : \quad E_0 = -\frac{1}{2} + \frac{1}{4} \sqrt{H_0 + \frac{32}{3} (M_2 - M_1)} + 16, \quad y_0 = \frac{2}{3} (M_2 - M_1) - 1
\] (3.11)
\[
\chi^- : \quad E_0 = \frac{3}{2} + \frac{1}{4} \sqrt{H_0 - \frac{32}{3} (M_2 - M_1)} + 16, \quad y_0 = \frac{2}{3} (M_2 - M_1) + 1
\] (3.12)
\[
\chi^- : \quad E_0 = \frac{3}{2} + \frac{1}{4} \sqrt{H_0 + \frac{32}{3} (M_2 - M_1)} + 16, \quad y_0 = \frac{2}{3} (M_2 - M_1) - 1
\] (3.13)

and the conjugate multiplets in the complex conjugate representations of \( G' \).

- Two long gravitino multiplets (one \( \chi^+ \) and one \( \chi^- \), table 2) in each representation of the series
\[
\left\{ M_2 > M_1 > 0, \ J = \frac{1}{3} (M_2 - M_1) \right\} \cup \\
\left\{ M_2 > M_1 > 0, \ J = \frac{1}{3} (M_2 - M_1) - 1 \right\} \cup \\
\left\{ M_2 > M_1 = 0, \ J \geq \frac{1}{3} (M_2 - M_1) \right\}
\] (3.14)

with
\[
\chi^+ : \quad E_0 = -\frac{1}{2} + \frac{1}{4} \sqrt{H_0 + \frac{32}{3} (M_2 - M_1)} + 16, \quad y_0 = \frac{2}{3} (M_2 - M_1) - 1
\] (3.15)
\[
\chi^- : \quad E_0 = \frac{3}{2} + \frac{1}{4} \sqrt{H_0 + \frac{32}{3} (M_2 - M_1)} + 16, \quad y_0 = \frac{2}{3} (M_2 - M_1) - 1
\] (3.16)

and the conjugate multiplets in the complex conjugate representations of \( G' \).

- One long gravitino multiplet (a \( \chi^- \), table 2), in each representation of the series
\[
\left\{ M_2 > M_1 = 0, \ J = \frac{1}{3} (M_2 - M_1) - 1 \right\}
\] (3.17)

with
\[
\chi^- : \quad E_0 = \frac{3}{2} + \frac{1}{4} \sqrt{H_0 + \frac{32}{3} (M_2 - M_1)} + 16, \quad y_0 = \frac{2}{3} (M_2 - M_1) - 1
\] (3.18)

and the conjugate multiplet in the complex conjugate representations of \( G' \).

3. **Long vector multiplets** \( (1 (1), 4 \left( \frac{1}{2} \right), 5 (0)) \)

As already stressed there are different realizations of the long vector multiplet arising from different fields of the \( D = 11 \) theory. We have the \( W \) vector multiplets, the \( A \) vector multiplets, and the \( Z \) vector multiplets.

- One \( W \) long vector multiplet (table 3) in each representation of the series
\[
\left\{ M_2 \geq 0, \ M_1 \geq 0, \ J \geq \frac{1}{3} (M_2 - M_1) \right\}
\] (3.19)

with
\[
W : \quad E_0 = \frac{5}{2} + \frac{1}{4} \sqrt{H_0 + 36}, \quad y_0 = \frac{2}{3} (M_2 - M_1)
\] (3.20)
• One $A$ long vector multiplet (table 3) in each representation of the series

$$\left\{ M_2 \geq M_1 = 0, \; J > \frac{1}{3} (M_2 - M_1) + 1 \right\} \cup$$

$$\left\{ M_2 \geq M_1 = 1, \; J > \frac{1}{3} (M_2 - M_1) \right\} \cup$$

$$\left\{ M_2 \geq M_1 > 1, \; J \geq \frac{1}{3} (M_2 - M_1) \right\}$$

(3.21)

with

$$A : \quad E_0 = -\frac{3}{2} + \frac{1}{4} \sqrt{H_0 + 36}, \; y_0 = \frac{2}{3} (M_2 - M_1)$$

(3.22)

and the conjugate multiplet in the complex conjugate representations of $G'$.

• One $Z$ long vector multiplet (table 3) in each representation of the series

$$\left\{ M_2 > M_1 > 0, \; J \geq \frac{1}{3} (M_2 - M_1) \right\}$$

(3.23)

with

$$Z : \quad E_0 = \frac{1}{2} + \frac{1}{4} \sqrt{H_0 + 4}, \; y_0 = \frac{2}{3} (M_2 - M_1)$$

(3.24)

• One $Z$ long vector multiplet (table 3) in each representation of the series

$$\left\{ M_2 > M_1 + 3, \; J \geq \frac{1}{3} (M_2 - M_1) - 2 \right\} \cup$$

$$\left\{ M_1 + 3 \geq M_2 > 1, \; J > -\frac{1}{3} (M_2 - M_1) + 1 \right\}$$

(3.25)

with

$$Z : \quad E_0 = \frac{1}{2} + \frac{1}{4} \sqrt{H_0 + \frac{64}{3} (M_2 - M_1) - 28}, \; y_0 = \frac{2}{3} (M_2 - M_1) - 2$$

(3.26)

and the conjugate multiplet in the complex conjugate representations of $G'$.

**SHORT MULTIPLETS**

1. **Short graviton multiplets** $\left( 1(2), 3 \left( \frac{3}{2} \right), 3(1), \frac{1}{2} \right)$

• One short graviton multiplet (table 4) in each representation of the series

$$\left\{ M_2 = 3k, \quad M_1 = 0, \quad k > 0 \text{ integer} \right\}$$

(3.27)

with

$$E_0 = 2k + 2, \; y_0 = 2k$$

(3.28)

and the conjugate multiplet in the complex conjugate representations of $G'$. 
2. **Short gravitino multiplets** \((1\left(\frac{3}{2}\right), 3(1), 3\left(\frac{1}{2}\right), 0)\)

- One short gravitino multiplet \((\chi^+, \text{table} \[\])\) in each representation of the series
  \[
  \begin{align*}
  M_2 &= 3k + 1 \\
  M_1 &= 1 \\
  J &= k + 1 \\
  k &\geq 0 \text{ integer (3.29)}
  \end{align*}
  \]
  with
  \[
  E_0 = 2k + \frac{1}{2}, \ y_0 = 2k + 1 \quad (3.30)
  \]
  and the conjugate multiplet in the complex conjugate representations of \(G'\).

- One short gravitino multiplet \((\chi^+, \text{table} \[\])\) in each representation of the series
  \[
  \begin{align*}
  M_2 &= 3k \\
  M_1 &= 0 \\
  J &= k - 1 \\
  k &> 0 \text{ integer (3.31)}
  \end{align*}
  \]
  with
  \[
  E_0 = 2k + \frac{1}{2}, \ y_0 = 2k - 1 \quad (3.32)
  \]
  and the conjugate multiplet in the complex conjugate representations of \(G'\).

3. **Short vector multiplets** \((1(1), 3\left(\frac{1}{2}\right), 3(0))\)

- One short vector multiplet \((A, \text{table} \[\])\), in each representation of the series
  \[
  \begin{align*}
  M_2 &= 3k + 1 \\
  M_1 &= 1 \\
  J &= k \\
  k &> 0 \text{ integer (3.33)}
  \end{align*}
  \]
  \[
  \begin{align*}
  M_2 &= 3k \\
  M_1 &= 0 \\
  J &= k + 1 \\
  k &> 0 \text{ integer (3.34)}
  \end{align*}
  \]
  with
  \[
  E_0 = 2k + 1, \ y_0 = 2k \quad (3.35)
  \]
  and the conjugate multiplet in the complex conjugate representations of \(G'\).

4. **Hypermultiplets** \((2\left(\frac{1}{2}\right), 4(0))\)

- One hypermultiplet (table \[\]) in each representation of the series
  \[
  \begin{align*}
  M_2 &= 3k \\
  M_1 &= 0 \\
  J &= k \\
  k &> 0 \text{ integer (3.36)}
  \end{align*}
  \]
  \[
  E_0 = |y_0| = 2k \quad (3.37)
  \]
  
  18
We mean that a half of the hypermultiplet is in a representation among the $[3,36]$, the other half is in a representation among their conjugates. Indeed the hypermultiplet is complex, so half of it transforms in a representation $\rho$ of $G'$, half of it in the conjugate representation $\rho^*$.

**MASSLESS MULTIPLETS**

1. **The massless graviton multiplet** (table 8) $(1 \ (2), \ 2 \ (3), \ 1)$ in the singlet representation

   \[ M_2 = M_1 = J = 0 \]  \hspace{1cm} (3.38)

   with

   \[ E_0 = 2, \ y_0 = 0. \]  \hspace{1cm} (3.39)

   In this multiplet the graviphoton is associated with the Killing vector of the $R$–symmetry group $U(1)_R$.

2. **The massless vector multiplet** (table 9) $(1 \ (1), \ 2 \ (1), \ 2 \ (0))$ in the adjoint representation of the $G'$ group

   \[ M_2 = M_1 = 1, \ J = 0 \]  \hspace{1cm} (3.40)

   \[ M_2 = M_1 = 0, \ J = 1 \]  \hspace{1cm} (3.41)

   with

   \[ E_0 = 1, \ y_0 = 0. \]  \hspace{1cm} (3.42)

3. **An additional massless vector multiplet** in the singlet representation of the gauge group

   \[ M_2 = M_1 = J = 0 \]  \hspace{1cm} (3.43)

   with the same energy and hypercharges as in (3.42) that arises from the three–form $A_{\mu ij}$ and is due to the existence of one closed cohomology two–form on the $M^{111}$ manifold. This multiplet is named the Betti multiplet.

Summarizing, the massless spectrum, besides the supergravity multiplet contains twelve vector multiplets: so the total number of massless gauge bosons is thirteen, one of them being the graviphoton. In the low energy effective lagrangian we just couple to supergravity these twelve vector multiplets. However we expect the gauging of a thirteen–parameter group:

\[ SU(3) \times SU(2) \times U(1)_R \times U(1)' \]  \hspace{1cm} (3.44)

the further $U(1)'$ being associated with the Betti multiplet. All Kaluza Klein states are neutral under $U(1)'$ yet non perturbative states might carry $U(1)'$ charges. This is a completely open problem.
PART TWO: THE DERIVATION

In the second part of the paper we give the detailed derivation of the results presented in the first part. Our main tools in this derivation are:

- harmonic analysis on $G/H$ seven–dimensional coset manifolds as developed in the eighties by D’Auria and Frè [3, 4]
- the general diagonalization of linearized $D = 11$ supergravity field equations and the resulting mass–formulae derived by D’Auria and Frè [5] and by Castellani et al [41] also in the eighties
- the mass relations following from the existence of Killing spinors on $G/H$ derived by D’Auria and Frè in [3]
- The structure of $\mathcal{N} = 2$ long multiplets that were obtained in [40, 8].

We begin with a review of $M^{111}$ differential geometry.

4 The differential geometry of $M^{111}$

Let us briefly review the essential features of $M^{111}$ geometry. For notations and basic lore we mainly refer to [4, 5, 39].

The homogeneous space $M^{111}$ is the quotient of $G = SU(3) \times SU(2) \times U(1)$ by the action of its subgroup $H = SU(2)^c \times U(1)' \times U(1)''$, where the embedding of $H$ in $G$ is defined as follows. $SU(2)^c$ is the isospin $SU(2)$ subgroup of $SU(3)$ with respect to which the fundamental triplet representation branches as follows:

$$3 \rightarrow 2 \oplus 1.$$  

The additional $U(1)'$ and $U(1)''$ factors of the $H$ subgroup are generated by $Z'$ and $Z''$ being two independent linear combinations of the three remaining abelian generators of $SU(3) \times SU(2) \times U(1)$ commuting with $SU(2)^c$. Using the Gell–Mann matrices for $SU(3)$ and the Pauli matrices for $SU(2)$ (see appendix A for conventions) these three generators are $\frac{\sqrt{3}}{2}i\lambda_8$, $\frac{1}{2}i\sigma_3$, and $iY_3$. The two linear combinations $Z'$ and $Z''$ are defined as a basis for the orthogonal complement of the generator

$$Z = p\frac{\sqrt{3}}{2}i\lambda_8 + \frac{1}{2}qi\sigma_3 + riY_3$$

with $p = q = r = 1$

The superscript $^c$ stands for color. This nomenclature is due to the original interest in the group $G$ as the gauge group of the GSW standard model. Even if this perspective has faded, the terminology is still useful. So we will label quantities referring to $SU(3)$ with the index $^c$, and quantities referring to the weak isospin $SU(2) \not\subset SU(3)$ with $^w$. 

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which belongs to $\mathbb{K}$ rather than $\mathbb{H}$ in the decomposition of the $SU(3) \times SU(2) \times U(1)$ Lie algebra

$$
\begin{align*}
\mathbb{G} &= \mathbb{H} \oplus \mathbb{K} \\
\mathbb{G} &= SU(3) \times SU(2) \times U(1) \\
\mathbb{H} &= SU(2) \times U(1)' \times U(1)''
\end{align*}
\tag{4.3}
$$

In the above equations $i\sqrt{3}\lambda_8$ is the hypercharge generator of $SU(3)$, $\frac{1}{2}i\sigma_3$ is the third component of the weak isospin and $iY_3$ is the weak $U(1)$ generator.

An explicit representation of the group $G$ is given by the following $6 \times 6$ block–diagonal matrix:

$$
G \ni g = \begin{pmatrix} SU(3) & 0 & 0 \\
0 & SU(2) & 0 \\
0 & 0 & U(1) \end{pmatrix},
\tag{4.4}
$$

where the diagonal blocks contain the fundamental representation of $SU(3)$, $SU(2)$ and $U(1)$ respectively. The whole set of generators of $G$ is given by:

$$
T_\Lambda \equiv \left(\frac{1}{2}i\lambda_A, \frac{1}{2}i\lambda_m, \frac{1}{2}i\lambda_8, \frac{1}{2}i\sigma_\perp, \frac{1}{2}i\sigma_3, iY_3\right),
\tag{4.5}
$$

where $\lambda_i$ stands for the $i$-th Gell-Mann matrix trivially extended to a $6 \times 6$ matrix:

$$
\lambda_i \rightarrow \begin{pmatrix} \lambda_i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}
\tag{4.6}
$$

Similarly $\sigma_i$ denotes the following extension of the Pauli matrices:

$$
\sigma_i \rightarrow \begin{pmatrix} 0 & 0 & 0 \\
0 & \sigma_i & 0 \\
0 & 0 & 0 \end{pmatrix},
\tag{4.7}
$$

and $Y_3$ is given by:

$$
Y_3 = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \end{pmatrix}
\tag{4.8}
$$

A basis for the two abelian generators of $H$ can is given by

$$
Z' = \sqrt{3}i\lambda_8 + i\sigma_3 - 4iY_3, \\
Z'' = -\sqrt{3}\frac{3}{2}i\lambda_8 + \frac{3}{2}i\sigma_3,
\tag{4.9}
$$

which are orthogonal among themselves and with $Z$:

$$
Tr(ZZ') = Tr(ZZ'') = Tr(Z'Z'') = 0.
\tag{4.11}
$$
An explicit parametrization of the coset $G/H$ is given by the seven coordinates $(y^A, y^m, y^3)$:

$$L(y^A, y^m, y^3) = \exp(\frac{i}{2} \lambda_A y^A) \exp(\frac{i}{2} \sigma_m y^m) \exp(Zy^3).$$  \hspace{1cm} (4.12)

From this we can construct the left-invariant one-forms on $G/H$ as:

$$\Omega(y) = L^{-1}(y) dL(y) = \Omega^A(y) T_A,$$  \hspace{1cm} (4.13)

which satisfy the Maurer-Cartan equations

$$d\Omega^A + \frac{1}{2} C^A_{\Sigma \Pi} \Omega^\Sigma \wedge \Omega^\Pi = 0$$  \hspace{1cm} (4.14)

with the structure constants of $G$:

$$[T_S, T_H] = C^A_{\Sigma \Pi} T_A.$$  \hspace{1cm} (4.15)

The one-forms $\Omega^A$ can be separated into a set $\{\Omega^H\}$ corresponding to the generators of the subalgebra $\mathfrak{H}$ and a set $\{\Omega^\alpha\}$ corresponding to the coset generators. These latter can be identified with the $SU(3) \times SU(2) \times U(1)$ invariant seven-vielbeins on $G/H$:

$$\mathcal{B}^\alpha \equiv (\mathcal{B}^A, \mathcal{B}^m, \mathcal{B}^3),$$

$$\begin{cases}
\mathcal{B}^A = \frac{\sqrt{3}}{8} \Omega^A, \\
\mathcal{B}^m = \frac{\sqrt{2}}{8} \Omega^m, \\
\mathcal{B}^3 = \frac{1}{8}(\sqrt{3}\Omega^8 + \Omega^3 + 2\Omega^Z) = \frac{3}{2} \Omega^Z,
\end{cases}$$  \hspace{1cm} (4.16)

where the multiplicative coefficients in front of the vielbeins have been properly chosen to let the metric on $G/H$ be Einstein. The invariant forms $\Omega^H$ are:

$$\begin{cases}
\Omega^m, \\
\Omega^2 = \frac{1}{24}(\sqrt{3}\Omega^8 + \Omega^3 - 4\Omega^Z), \\
\Omega^Z = \frac{1}{12}(3\Omega^3 - \sqrt{3}\Omega^8).
\end{cases}$$  \hspace{1cm} (4.17)

The spin-connection $\mathcal{B}^\alpha_{\beta}$ is easily determined from the vielbeins $\mathcal{B}^\alpha$ by imposing vanishing torsion:

$$d\mathcal{B}^\alpha - \mathcal{B}^\alpha_{\beta} \wedge \mathcal{B}^\beta = 0,$$  \hspace{1cm} (4.18)

$$\begin{cases}
\mathcal{B}^{mn} = \epsilon^{mn}(\Omega^3 - 2\mathcal{B}^3), \\
\mathcal{B}^3m = -2\epsilon^{mn} \mathcal{B}_n, \\
\mathcal{B}^{mA} = 0, \\
\mathcal{B}^{A3} = -\frac{1}{\sqrt{3}} f^{8AB} B_B, \\
\mathcal{B}^{AB} = f^{mAB} \Omega^m + f^{8AB} \Omega_8 - \frac{4}{\sqrt{3}} f^{8AB} \mathcal{B}^3.
\end{cases}$$  \hspace{1cm} (4.19)
5 Harmonic analysis on $M^{111}$

In this section we summarize the essential ideas concerning the techniques of harmonic analysis on homogeneous seven–manifolds originally developed in [3, 4]. These techniques are the basic ingredient of our calculations and in the present summary we present them already applied to the specific case of the $M^{111}$ manifold.

The essential goal of harmonic analysis, is that of translating a differential equation problem into a linear algebraic one, by means of group theory. In the present case, the differential equations to solve are the linearized field equations of Kaluza Klein supergravity, whose typical form is:

$$\left( \Box^{[J_1J_2]}_x + \Box^{[\lambda_1\lambda_2\lambda_3]}_y \right) \Phi^{[J_1J_2]}_x(y) = 0,$$

where $\Phi^{[J_1J_2]}_x(y)$ is a field transforming in the irreducible representations $[J_1J_2]$ of $SO(3,2)$ and $[\lambda_1\lambda_2\lambda_3]$ of $SO(7)$, and depends both on the coordinates $x$ of Anti-de Sitter space and on the coordinates $y$ of $G/H$. $\Box^{[J_1J_2]}_x$ is the kinetic operator for a field of spin $[J_1J_2]$ in four dimension while $\Box^{[\lambda_1\lambda_2\lambda_3]}_y$ is the kinetic operator for a field of spin $[\lambda_1\lambda_2\lambda_3]$ in seven dimensions.

Now, the harmonics constitute a complete set of functions for the expansion of any $SO(7)$-irreducible field over $G/H$, $\mathcal{Y}^{n}_{[\lambda_1\lambda_2\lambda_3]}(y)$. But their most important property is that they transform irreducibly under $G$, the group of isometries of the coset space. This group acts on $\mathcal{Y}^{n}_{[\lambda_1\lambda_2\lambda_3]}(y)$ through the so-called covariant Lie derivative (see eq.(2.25) of [3]):

$$\delta_A \mathcal{Y}^{n}_{[\lambda_1\lambda_2\lambda_3]}(y) := \mathcal{L}_A \mathcal{Y}^{n}_{[\lambda_1\lambda_2\lambda_3]}(y),$$

which satisfy the Lie algebra of the group $G$:

$$[\mathcal{L}_A, \mathcal{L}_B] = C^{\Pi}_{AB} \mathcal{L}_\Pi,$$

Moreover, the operators $\mathcal{L}_A$ commute with the $SO(7)$ covariant derivative:

$$\mathcal{L}_A D \mathcal{Y}^i = D \mathcal{L}_A \mathcal{Y}^i,$$

where $D$ is defined by

$$D = d + B^{\alpha\beta} t_{\alpha\beta},$$

and $(t_{\alpha\beta})^i_j$ are the generators of the $SO(7)$ irreducible representation $[\lambda_1\lambda_2\lambda_3]$ of $\mathcal{Y}^i$ (e.g. $(t_{\alpha\beta})^{\gamma\delta} = -\delta^{\gamma\delta}_{\alpha\beta}$ for the vector representation).

An important thing to note is that $H = SU(2) \times U(1) \times U(1)$ is necessarily a subgroup of $SO(7)$, whose natural embedding is given in terms of the following embedding of the algebra $\mathfrak{H}$ into the adjoint representation of $SO(7)$:

$$(T_H)^\alpha_\beta = C^{\alpha}_{H \beta},$$

$$(T_Z^\prime)^\alpha_\beta = \begin{pmatrix} 2\sqrt{3} f^{8AB} & 0 & 0 \\ 0 & 2\epsilon^{mn} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

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\[
(T_{Z''})^\alpha_\beta = \begin{pmatrix}
-\sqrt{3}f_{SAB}^{\alpha} & 0 & 0 \\
0 & 3\epsilon^{mn} & 0 \\
0 & 0 & 0 
\end{pmatrix},
\]
(5.7)

\[
(T_m)^\alpha_\beta = \begin{pmatrix}
f_{mAB}^{\alpha} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix},
\]
(5.8)

This means that the $SO(7)$-indices of the various $n$-forms can be split in the following subsets, each one transforming into an irreducible representation of $H$:

\[
Y_\alpha = \{Y_A, Y_m, Y_3\}
\]

\[
Y^{[\alpha\beta]} = \{Y_{AB}, Y_{Am}, Y_{mn}, Y_{A3}, Y_{m3}\}
\]

\[
Y^{[\alpha\beta\gamma]} = \{Y_{ABC}, Y_{ABm}, Y_{AB3}, Y_{Amn}, Y_{A3m}, Y_{mn3}\}
\]

(5.9)

and the $SO(7)$ irreducible representations $[\lambda_1\lambda_2\lambda_3]$ break into the direct sum of $H$ irreducible representations. Let us then introduce some notation for the harmonics of $M^{111}$, which will be denoted by $H$.

A generic $SO(3,2) \times H$-irreducible field can be expanded as follows:

\[
\Phi_{[J^c Z'|Z'']}_{i_1\ldots i_{2Jc}}(x,y) = \sum_{[M_1M_2JY]} \sum_{[J^c Z'|Z'']} \sum_{m} H_{M_1M_2JY}^{M_1M_2JY} m \mu \cdot \Phi_{[M_1M_2JY]} m(x) \tag{5.10}
\]

The coefficients $\varphi(x)$ of the expansion will become the space–time fields of the theory in $AdS_4$. The first sum is over all the $G$ irreducible representations $[M_1M_2JY]$ which break into the given $H$-one. We call $\sum'$ the sum over this subset of the possible representations of $G$. The subscripts $i_1,\ldots,i_{2Jc}$ span the representation space of $[J^c Z'|Z'']$, while $m$ is a collective index which spans the representation space of $[M_1M_2JY]$. Finally $\mu$ accounts for the fact that the same $H$ irreducible representation can be embedded in $G$ in different ways. The cases of interest for us will be the following:

\[
J^c = \frac{1}{2}:
\]

\[
\left\{
\begin{array}{ll}
\mu = (a) & \mapsto \begin{array}{c}
1 \\
2 \\
3 \\
i
\end{array}
\end{array}
\right.
\]

\[
\left\{
\begin{array}{ll}
\mu = (b) & \mapsto \begin{array}{c}
i \\
1 \\
3 \\
2 \\
\end{array}
\end{array}
\right.
\]

\[
\left\{
\begin{array}{ll}
\mu = (c) & \mapsto \frac{1}{2} \begin{pmatrix}
1 & \ldots & 1 & \ldots & 1 \\
2 & \ldots & 3 \\
i & j & 3 & \ldots & 3 \\
\end{pmatrix} + (i \leftrightarrow j)
\end{array}
\right.
\]

\[
J^c = 1:
\]

\[
\left\{
\begin{array}{ll}
\mu = (d) & \mapsto \begin{array}{c}
i & j \\
3 & 3 \\
2 & \ldots & 2
\end{array}
\end{array}
\right.
\]

\[
\left\{
\begin{array}{ll}
\mu = (e) & \mapsto \begin{array}{c}
1 \\
3 \\
2 \\
i & j
\end{array}
\end{array}
\right.
\]

(5.12)
where a double-row Yang tableau diagrammatically represents the $SU(3)$ irreducible representation labeled by $M_1$ and $M_2$, the number of boxes in the two rows of the diagram:

\[ \begin{array}{c}
\vdots \\
\vdots
\end{array} \]

$J$ is the flavor isospin characterizing the representation of $SU(2)^w$, and $Y$ is the hypercharge of the generator $U(1) \not\subset SU(3)^c \times SU(2)^w$.

### 5.1 The constraints on the irreducible representations

As we have seen in eq. (5.10), the expansion of a generic field contains only the harmonics whose $H$- and $G$-quantum numbers are such that the $G$ representation, decomposed under $H$, contain the $H$ representation of the field. This fact poses some constraints on the $G$-quantum numbers.

Depending on which constraints are satisfied by a certain $G$ representation, only part of the harmonics is present, and only their corresponding four-dimensional fields appear in the spectrum. Then, in the $G$ representations in which such field disappear, there is multiplet shortening. In the modern perspective of Kaluza Klein theory, the exact spectrum of the short multiplets is crucial. Hence the importance of analyzing this disappearance of harmonics with care.

Every harmonic is defined by its $SU(2) \times U(1)' \times U(1)''$ representation, identified by the labels $[J^c \ Z' \ Z'']$. Substituting these values in equations (4.9), (4.10), (5.11), (5.12), we can determine the constraints on the $G$ representations.

The first constraint gives the value of $Y$ in terms of $M_1$ and $M_2$. We have five possible expressions of $Y$, identifying five families of $G$ representations which we will denote with the superscripts $^0$, $^+$, $^-$, $^{++}$ and $^{--}$:

\[
\begin{align*}
0 : & \quad Y = \frac{2}{3}(M_2 - M_1) \\
^{++} : & \quad Y = \frac{2}{3}(M_2 - M_1) - 2 \\
^{--} : & \quad Y = \frac{2}{3}(M_2 - M_1) + 2 \\
^+ : & \quad Y = \frac{2}{3}(M_2 - M_1) - 1 \\
^- : & \quad Y = \frac{2}{3}(M_2 - M_1) + 1
\end{align*}
\]

\text{for bosonic fields}

\text{for fermionic fields}

(5.13)

It is worth noting that the value of $Y$ identifies a $U(1)_R$ representation, so these five families of representations correspond to the five possible representations of $U(1)_R$.

A second constraint is the lower bound on the quantum number $J$, since the third component of the weak isospin, $J_3$, is linked to $Y$. We have three possibilities:

\[
J \geq \begin{cases} 
|Y/2| \\
|Y/2 + 1| \\
|Y/2 - 1|
\end{cases}
\]

(5.14)
The last kind of constraint refers to $M_1$ and $M_2$:

\[
\begin{array}{|l|c|c|}
\hline
\text{constraints} & J^c & \mu \\
\hline
M_1 \geq 0 & 0 & - \\
M_2 \geq 0 & & \\
M_1 \geq 1 & \frac{1}{2} & (a) \\
M_2 \geq 0 & (b) \\
M_1 \geq 1 & 1 & (c) \\
M_2 \geq 1 & (d) \\
M_1 \geq 0 & 1 & (e) \\
M_2 \geq 2 & & \\
M_1 \geq 2 & & \\
M_2 \geq 0 & & \\
\hline
\end{array}
\]

\((5.15)\)

When some of the \((5.13)\) constraints are not satisfied, the Young tableau (see \((5.11), (5.12)\)) corresponding to $\mu$ in \((5.15)\) does not exist.

We organize the series of the $G = G' \times U(1)_R$ representations in the following way. The constraints \((5.14)\) and \((5.15)\), with the five values of $Y$ in terms of $M_1$, $M_2$ given by \((5.13)\), define the series of $G'$ representations that we list in table \(\underline{10}\). Every $G'$ representation, together with a superscript $0$, $+$, $-$, $++$ or $--$ that define the value of $Y$, is a $G$ representation. So the series of $G'$ representations defined in table \(\underline{10}\) with such a superscript are series of representations of the whole $G$ group.

For each family of representations ($0$, $+$, $-$, $++$, and $--$) we will call a series regular if it contains the maximum number of harmonics. The regular series cover all the representations with $M_1$, $M_2$ and $J$ sufficiently high to satisfy all the inequality constraints. When some of these inequalities are not satisfied instead, some of the harmonics may be absent in the expansion.

In tables \(\underline{11}, \underline{12}, \underline{13}, \underline{14}, \underline{15}\), we show which harmonics are present for the different series of $G$ representations. The first column contains the name of each series. The other columns contain the possible harmonics, each labeled by its $H$-quantum numbers. An asterisk denotes the presence of a given harmonic. To obtain the constraints on the conjugate series it suffices to exchange $M_1$ and $M_2$, as explained in \(\underline{39}\).
### Table 10: Series of $G''$ representations

| $J$' constraints | $M_1 > M_2$ | $M_2 > M_1$ |
|------------------|-------------|-------------|
| $J > (M_2 - M_1)/3$ | $M_2 > 0, M_1 > 0$ | $J > (M_2 - M_1)/3$ |
| $J > (M_2 - M_1)/3 - 1$ | $M_2 > M_1 > 0$ | $J = (M_2 - M_1)/3$ |
| $J > (M_2 - M_1)/3 - 2$ | $M_2 > M_1 = 0$ | $J = (M_2 - M_1)/3$ |
| $J > (M_2 - M_1)/3 - 1$ | $M_2 = M_1 > 0$ | $J = 0$ |
| $J > (M_2 - M_1)/3 + 1$ | $M_2 = M_1 = 0$ | $J > 0$ |
| $J > (M_2 - M_1)/3 + 1$ | $M_2 = M_1 = 0$ | $J = 0$ |

### Table 11: Harmonics content for the series of type $^0$

| $G'$-name | $M_1, M_2$ | $J$ constraints |
|------------|------------|-----------------|
| $A_1$ | $M_2 > 0, M_1 > 0$ | $J > (M_2 - M_1)/3$ |
| $A_2$ | $M_2 > M_1 > 0$ | $J = (M_2 - M_1)/3$ |
| $A_3$ | $M_2 > M_1 = 0$ | $J = (M_2 - M_1)/3$ |
| $A_4$ | $M_2 = M_1 = 0$ | $J = 0$ |

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| $J^c$ | $Z'$ | $Z''$ | $\mu$ | 0 | 0 | $1/2$ | $1/2$ |
|-------|------|-------|-------|---|---|-------|-------|
|       | $2i$ | $-3i$ |       | 0 | $4i$ | $-i$ | $i$   |
|       | $4i$ | $0$   |       | $-i$ | $-3i/2$ | $3i/2$ |
| $Z''$ | $-3i$ | 0     |       |    |     |       |       |

Table 12: Harmonics content for the series of type $^+$

| $J^c$ | $Z'$ | $Z''$ | $\mu$ | 0 | 0 | $1/2$ | $1/2$ |
|-------|------|-------|-------|---|---|-------|-------|
|       | $-2i$| $3i$  |       | 0 | $-4i$| $i$   | $-i$  |
|       | $-4i$| $0$   |       | $i$| $3i/2$| $-3i/2$|
| $Z''$ | $3i$ | 0     |       |    |     |       |       |

Table 13: Harmonics content for the series of type $^-$
### Table 14: Harmonics content for the series of type $++$  

| $J^c$ | $Z'$ | $Z''$ | $\mu$ | $B_{R^+}^{++}$ | $B_{1}^{++}$ | $B_{2}^{++}$ | $B_{3}^{++}$ | $B_{4}^{++}$ | $B_{5}^{++}$ | $B_{6}^{++}$ | $B_{7}^{++}$ | $B_{8}^{++}$ | $B_{9}^{++}$ | $B_{10}^{++}$ | $B_{11}^{++}$ |
|------|------|------|------|----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0    | 0    | 0    | 0    | *             | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           |
| 0    | 0    | 0    | 0    | *             | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           |
| 1/2  | 1/2  | 1/2  | 1/2  | *             | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           |
| 1/2  | 1/2  | 1/2  | 1/2  | *             | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           |
| 1    | 1    | 1    | 1    | *             | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           |

### Table 15: Harmonics content for the series of type $--$  

| $J^c$ | $Z'$ | $Z''$ | $\mu$ | $B_{R^-}^{--}$ | $B_{1}^{--}$ | $B_{2}^{--}$ | $B_{3}^{--}$ | $B_{4}^{--}$ | $B_{5}^{--}$ | $B_{6}^{--}$ | $B_{7}^{--}$ | $B_{8}^{--}$ | $B_{9}^{--}$ | $B_{10}^{--}$ | $B_{11}^{--}$ |
|------|------|------|------|----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0    | 0    | 0    | 0    | *             | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           |
| 0    | 0    | 0    | 0    | *             | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           |
| 1/2  | 1/2  | 1/2  | 1/2  | *             | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           |
| 1/2  | 1/2  | 1/2  | 1/2  | *             | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           |
| 1    | 1    | 1    | 1    | *             | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           | *           |

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6 Differential calculus via harmonic analysis

The Kaluza Klein kinetic operators $\mathcal{X}_y^{[\lambda_1\lambda_2\lambda_3]}$ are covariant differential operators. Eq. (5.4) then implies that they are $SO(7)$-invariant:

$$\mathcal{L}_\Lambda \mathcal{X}_y^{[\lambda_1\lambda_2\lambda_3]} = \mathcal{X}_y^{[\lambda_1\lambda_2\lambda_3]} \mathcal{L}_\Lambda.$$  

(6.1)

Using this fact and Schur’s lemma we conclude that they act irreducibly on the harmonics. In other words they act as (finite dimensional) matrices on the harmonic subspaces of fixed $G$-quantum numbers:

$$\mathcal{X}_y^{[\lambda_1\lambda_2\lambda_3]} \mathcal{H}_\zeta^{[M_1M_2JY]}(y) = \mathcal{M} (\mathcal{H}_\zeta^{[M_1M_2JY]}(y)),$$

(6.2)

where, for short, we have summarized with $\zeta$ the whole set of indices labelling the $H$ representations of $\mathcal{H}$.

Let us now consider the explicit action of the covariant derivative (5.5) on the harmonics. Following to the standard procedures of harmonic analysis [5, 6], $H$ is a subgroup of $SO(7)$. The $SO(7)$-covariant derivative (5.5) can then be decomposed as:

$$D = d + \Omega^H t_H + \mathcal{B}^\gamma \mathcal{M}_\gamma \equiv D^H + \mathcal{B}^\gamma \mathcal{M}_\gamma,$$

(6.3)

where $t_H$ are the generators of $H$ and $\mathcal{M}_\gamma$ the part of the $SO(7)$-connection not belonging to $H$. The $\mathcal{M}$ generators in the fundamental $SO(7)$ representation acquire the following form:

$$\begin{align*}
\mathcal{M}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{M}_2 &= \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{M}_3 &= \frac{1}{3} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{M}_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathcal{M}_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}$$
\[
M_6 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad , \quad
M_7 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The harmonics transform as the inverse \( L^{-1}(y) \) of the fundamental elements of \( G \) (see [3]). Eq. (4.14) can be restated as
\[
\Omega L^{-1} = L^{-1} d L L^{-1} = -d L^{-1}
\]
from which we deduce that
\[
D^H H = (d + \Omega^H t_H) H = -\Omega^a t_\alpha H.
\]

By means of eq. (4.16) we can calculate the explicit components of \( D^H \), i.e. its projection along the vielbeins:
\[
D^H = B_\alpha D^H_\alpha = -\Omega^a t_\alpha,
\]
\[
\begin{cases}
D_A = -\frac{4}{\sqrt{3}} i \lambda_A, \\
D_m = -\frac{1}{\sqrt{2}} i \sigma_m, \\
D_3 = -\frac{1}{3} Z,
\end{cases}
\]
where the coset generators \( t_\alpha \) act on \( H \) as follows. \( \lambda_A \) acts on the \( SU(3) \) part of the \( G \) representation of the harmonic. The fundamental representation of \( \lambda_A \) is given by the Gell–Mann matrices (see Appendix A). On a generic Young tableau \( \lambda_A \) acts as the tensor representation. To give an example, consider a component of an harmonic with \( SU(3) \) indices given by
\[
\begin{array}{ccc}
1 & 1 & 3 \\
2 & 3 & 3
\end{array}
\]

Then \( \lambda_4 \), which exchanges 1 with 3, acts as follows:
\[
\lambda_4 \begin{array}{ccc}
1 & 1 & 3 \\
2 & 3 & 3
\end{array} =
\begin{array}{ccc}
3 & 1 & 3 \\
2 & 3 & 3
\end{array} + \begin{array}{ccc}
1 & 3 & 3 \\
2 & 3 & 3
\end{array} + 2 \begin{array}{ccc}
1 & 1 & 3 \\
2 & 1 & 3
\end{array}
\]

Similarly, \( \sigma^m \) \( (m = \{1, 2\}) \) acts as the \( m \)-th Pauli matrix on the fundamental representation of \( SU(2) \), and as its \( n \)-th tensor power on the \( n \)-boxes \( SU(2) \) Young tableau:
\[
\sigma^1 \begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 2
\end{array} = 3 \begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 2
\end{array} + 2 \begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 2
\end{array}.
\]
Finally, $Z$ acts trivially, multiplying $H$ by its $Z$-charge:

$$Z \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 2 & 2 \end{bmatrix} = \left( -\frac{3}{2}i - \frac{1}{2}i + iY \right) \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 2 & 2 \end{bmatrix}.$$  

In the course of the calculations, one often encounters the $H$-covariant Laplace-Beltrami operator on $G/H$:

$$\eta^{\alpha\beta}D^H_\alpha D^H_\beta = \frac{16}{3}\lambda_A\lambda_A + \frac{16}{2}\sigma^m\sigma^m - \frac{16}{9}Z^2  \tag{6.7}$$

The eigenvalues of the first operator, $\lambda_A\lambda_A$, are listed in the following table:

| $J^c$ | $\mu$ | $\lambda^A\lambda^A$ eigenvalues |
|-------|-------|----------------------------------|
| 0     |      | $4(M_1 + M_2 + M_1M_2)$          |
| 1/2   | (a)  | $2(4M_1 + 2M_1M_2 - 3)$          |
| 1/2   | (b)  | $2(4M_2 + 2M_1M_2 - 3)$          |
| 1     | (c)  | $4(M_1 + M_2 + M_1M_2 - 2)$      |
| 1     | (d)  | $4(3M_2 - M_1 + M_1M_2 - 5)$     |
| 1     | (e)  | $4(3M_1 - M_2 + M_1M_2 - 5)$     |

while the eigenvalues of $\sigma^m\sigma^m$ depend on the flavor isospin quantum numbers $J$ and $J^3$:

$$\sigma^m\sigma^m \begin{bmatrix} 1 & \cdots & 1 & 2 & \cdots & 2 \end{bmatrix}_{m_1} \begin{bmatrix} 1 & \cdots & 1 & 2 & \cdots & 2 \end{bmatrix}_{m_2} = 4 \left[ J(J + 1) - J_3^2 \right] \begin{bmatrix} 1 & \cdots & 1 & 2 & \cdots & 2 \end{bmatrix}_{m_1} \begin{bmatrix} 1 & \cdots & 1 & 2 & \cdots & 2 \end{bmatrix}_{m_2}.  \tag{6.8}$$

where $2J = m_1 + m_2$ and $2J^3 = m_1 - m_2$. The complete Kaluza Klein mass operator heavily depends on the kind of field it acts on and will be analyzed in details in the next sections.

### 6.1 The zero-form

The only representation into which the [0, 0, 0] (i.e. the scalar) of $SO(7)$ breaks under $H$, is obviously the $H$-scalar representation. The question now is: which $G$-irreducible representations contain the $H$-scalar? From equations (4.9),(4.10) we see that $Z' = Z'' = 0$ implies

$$2J_3 = Y = \frac{2}{3}(M_2 - M_1).  \tag{6.9}$$

This means that

- $M_2 - M_1 \in 3\mathbb{Z}$
- $J \in \mathbb{N}$
- $J \geq \left\lfloor \frac{1}{3}(M_2 - M_1) \right\rfloor$
\[ Y = \frac{2}{3} (M_2 - M_1). \]

We will denote the scalar as
\[ \mathcal{Y} = [0|I] \equiv \sum'_{[M_1 M_2 J Y]} H^{[M_1 M_2 J Y]}(y) \cdot S_{[M_1 M_2 J Y]}(x). \] (6.10)

The Kaluza Klein mass operator for the zero-form \( \mathcal{Y} \) is given by
\[ \mathfrak{M}^{[00]} \mathcal{Y} \equiv D_\beta D^\beta \mathcal{Y} = D^H_\beta D^H H^\beta \mathcal{Y}. \] (6.11)

For the scalar, there are no \( \mathfrak{M} \)–connection terms. So, by means of eq. (6.7), the computation of its eigenvalues, on the \( G \)–representations as listed above, is immediate:
\[ \mathfrak{M}^{[00]} \mathcal{Y} \equiv \mathcal{Y} = \left[ \frac{4\pi}{3} (M_1 + M_2 + M_1 M_2) + 32J(J + 1) + \frac{32}{9} (M_2 - M_1)^2 \right] \mathcal{Y} = H_0 \mathcal{Y} \] (6.12)

where \( H_0 \) is the same quantity defined in eq. (3.2).

As we see from the Kaluza Klein expansion (3.1), the eigenvalues of the zero–form harmonic allow us to determine (see [1]) the masses of the AdS_4 graviton field \( h \) and the scalar fields \( S, \Sigma \).

### 6.2 The one-form

The decomposition under \( H \) of the vector representation of \( SO(7) \) is the following [7]:
\[ [1, 0, 0] \to [0, 0, 0] \oplus [0, -2i, -3i] \oplus [0, 2i, 3i] \oplus [\frac{1}{2}, 3i, -\frac{3}{2}i] \oplus [\frac{1}{2}, -3i, \frac{3}{2}i]. \] (6.13)

Concretely, the decomposition of the one–form in \( H \)–irreducible fragments is done as follows (see also [7]):
\[ \mathcal{Y}^A = \lambda_A^d (1|I)_i + \lambda_A^d (1|I)_i^* \] (6.14)
\[ \mathcal{Y}^m = \sigma_m^d (1|I)_i + \sigma_m^d (1|I)_i^* \] (6.15)
\[ \mathcal{Y}^3 = [1|I] \] (6.16)

These \( H \)–irreducible fragments can be expanded as in (6.10) \( ^4 \) (summation over the \( G \)-quantum numbers is intended):

For type \( ^0 \):

\[ \langle 1|I \rangle_i = \mathcal{H}^{(1/2, -3i, -3i/2)}_i \cdot W \langle \frac{1}{2}, 1 \rangle, \]
\[ \langle 1|I \rangle_i^* = \varepsilon^{-1} \mathcal{H}^{(1/2, -3i, 3i/2)}_i \cdot \overline{W} \langle \frac{1}{2}, 1 \rangle, \]
\[ \langle 1|I \rangle_s = \mathcal{H}^{(0, -2i, -3i)}_s \cdot W \langle 0, 1 \rangle, \]
\[ \langle 1|I \rangle_s^* = \mathcal{H}^{(0, 2i, -3i)}_s \cdot \overline{W} \langle 0, 1 \rangle, \]
\[ [1|I] = \mathcal{H}^{(0, 0, 0)} \cdot W [0, 1], \] (6.17)

---

3 When we write a pair of complex conjugate representations we assume a conjugation relation between them. For example, by writing \([0, -2i, -3i] \oplus [0, 2i, 3i]\) we intend a complex representation of complex dimension one or real dimension two.

4 Using the same conventions as in [7], the reader might notice that there appears a sign \((-1)^{J_3 - J_5}\) upon taking the complex conjugate of the fragments \( \langle \ldots | \ldots \rangle_x \). In order to reduce the notation we have absorbed this sign in the \( x \)-space fields \( \overline{W} \langle \ldots, \ldots \rangle_x \). This will be done for all the complex conjugates henceforth.

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For type $++$:

$$\langle 1|I \rangle_i = \mathcal{H}_i^{[1/2, 3i, -3/2]}(b) \cdot W\langle \frac{1}{2}, II \rangle,$$

For type $--$:

$$\langle 1|I \rangle_i^* = -\varepsilon^{ij} \mathcal{H}_j^{[1/2, -3i, 3/2]}(a) \cdot \tilde{W}\langle \frac{1}{2}, II \rangle.$$

As we see, there are five different $AdS_4$ fields ($W, \tilde{W}$) in the case of the $0$ series, and one field in the case of the $++$ and $--$ series. So, for the regular $0$ series the Laplace Beltrami operator acts on the $AdS_4$ fields as a $5 \times 5$ matrix. For the exceptional series it acts as a matrix of lower dimension.

The Laplace Beltrami operator for the transverse one-form field $\mathcal{Y}^\alpha$, is given by

$$\mathfrak{L}^{[100]} \mathcal{Y}^\alpha \equiv M(1)(0)^2 \mathcal{Y}^\alpha = 2D_\beta D^\beta \mathcal{Y}^\alpha = (D^\beta D_\beta + 24)\mathcal{Y}^\alpha,$$  \hspace{1cm} (6.18)

where transversality of $\mathcal{Y}^\alpha$ means that $D_\alpha \mathcal{Y}^\alpha = 0$. From the decomposition $D_\alpha = D_\alpha^H + M_\alpha$ we obtain:

$$\mathfrak{L}^{[100]} \mathcal{Y}^\alpha = (D^H_\beta D^H_\beta + 24)\mathcal{Y}^\alpha + \eta^{\gamma\delta} \left(2(M_\gamma)^{\alpha}_{\beta} D^H_\delta + (M_\gamma)^{\alpha}_{\epsilon} (M_\delta)^\epsilon_{\beta}\right)\mathcal{Y}^\beta.$$  \hspace{1cm} (6.19)

The matrix of this operator on the $AdS_4$ fields is given by

| $M(1)(0)^2$ | $W\langle \frac{1}{2}, I \rangle$ | $W\langle \frac{1}{2}, I \rangle^*$ | $W\langle 0, I \rangle$ | $W\langle 0, I \rangle^*$ | $W\langle 0, 0 \rangle$ |
|--------------|---------------------------------|---------------------------------|------------------------|------------------------|------------------------|
| $W\langle \frac{1}{2}, I \rangle$ | $H_0 - \frac{32(M_2 - M_1)}{3}$ | $0$ | $H_0 + \frac{32(M_2 - M_1)}{3}$ | $0$ | $0$ | $\frac{16M_1}{\sqrt{3}}$ |
| $\tilde{W}\langle \frac{1}{2}, I \rangle$ | $0$ | $H_0 + \frac{32(M_2 - M_1)}{3}$ | $0$ | $0$ | $\frac{16M_1}{\sqrt{3}}$ | $\frac{16M_1}{\sqrt{3}}$ |
| $W\langle 0, I \rangle$ | $0$ | $0$ | $H_0 - \frac{32(M_2 - M_1)}{3}$ | $0$ | $\frac{32(2+M_2)}{\sqrt{3}}$ | $\frac{16(2+2J+Y)}{\sqrt{2}}$ |
| $\tilde{W}\langle 0, I \rangle$ | $0$ | $0$ | $0$ | $H_0 - \frac{32(M_2 - M_1)}{3}$ | $\frac{32(2+M_2)}{\sqrt{3}}$ | $\frac{32(2+2J+Y)}{\sqrt{2}}$ |
| $W\langle 0, 0 \rangle$ | $\frac{32(2+M_2)}{\sqrt{3}}$ | $\frac{32(2+M_2)}{\sqrt{3}}$ | $-16(2+2J-Y)$ | $\frac{24(2+2J+Y)}{\sqrt{2}}$ | $H_0 + 48$ |

Its eigenvalues are:

$$\lambda_1 = H_0 + \frac{32}{3}(M_2 - M_1),$$
$$\lambda_2 = H_0 - \frac{32}{3}(M_2 - M_1),$$
$$\lambda_3 = H_0,$$
$$\lambda_4 = H_0 + 24 + 4\sqrt{H_0 + 36},$$
$$\lambda_5 = H_0 + 24 - 4\sqrt{H_0 + 36}.$$  \hspace{1cm} (6.21)

Actually, what we have just calculated are the eigenvalues of

$$M(1)(0)^2 \mathcal{Y}^\alpha + D^\alpha D_\beta \mathcal{Y}^\beta.$$  \hspace{1cm} (6.22)

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It coincides with \(M_{(1)(0)^2}\) when acting on a transverse one-form. But on a generic \(\mathcal{Y}^\alpha\), which possibly contains a longitudinal term, the second part of (6.22), \(D^\alpha D^\beta \mathcal{Y}^\beta\), is not inert. Indeed, let us suppose
\[
\mathcal{Y}^\alpha = D^\alpha \mathcal{Y}
\]
for some scalar function \(\mathcal{Y}\). Then
\[
D^\alpha D^\beta \mathcal{Y}^\beta = D^\alpha D^\beta D^\beta \mathcal{Y}^\beta = D^\alpha M_{(0)^2} \mathcal{Y}^\alpha = M_{(0)^3} \mathcal{Y}^\alpha.
\]
(6.23)

So, our actual operator (6.20) contains the eigenvalues of \(M_{(0)^3}\), which are longitudinal (hence non-physical) for the one–form. This fact is true also for the two-form.

The eigenvalue \(\lambda_3\) in (6.21) is the longitudinal one, equal to the zero–form eigenvalue \(H_0\). The other four, instead, are transverse physical eigenvalues.

The matrices corresponding to the exceptional series are easily obtained from (6.20) by removing the rows and the columns of the fields that disappear in the expansions (6.17), as we read from table 11. We list the mass eigenvalues of each series:

| \(A_R\) | \(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\) |
| \(A_1\) | \(\lambda_1, \lambda_3, \lambda_4, \lambda_5\) |
| \(A_1^*\) | \(\lambda_2, \lambda_3, \lambda_4, \lambda_5\) |
| \(A_2\) | \(\lambda_1\) |
| \(A_2^*\) | \(\lambda_2\) |
| \(A_3\) | \(\lambda_1, \lambda_3, \lambda_4, \lambda_5\) |
| \(A_3^*\) | \(\lambda_2, \lambda_3, \lambda_4, \lambda_5\) |
| \(A_4\) | \(\lambda_1, \lambda_3, \lambda_4\) |
| \(A_4^*\) | \(\lambda_2, \lambda_3, \lambda_4\) |
| \(A_5\) | \(\lambda_1\) |
| \(A_5^*\) | \(\lambda_2\) |
| \(A_6\) | \(\lambda_3, \lambda_4, \lambda_5\) |
| \(A_7\) | \(\lambda_3, \lambda_4, \lambda_5\) |
| \(A_8\) | \(\lambda_4\) |

(6.24)

For the series of type \(++\) the operator \(M_{(1)(0)^2}\) acts as a \(1 \times 1\) matrix on the AdS\(_4\) fields and has eigenvalue:
\[
H_0 + \frac{32}{9} (M_2 - M_1)
\]
(6.25)
for the series \(B_R, B_1, B_3, B_4, B_6\) and \(B_7\). For the type \(--\)-series the eigenvalue is the conjugate one \((M_2 \leftrightarrow M_1)\) in the conjugate series.

We can use the eigenvalues of the one–form harmonic to determine (see [6]) the masses of the AdS\(_4\) vector field \(A, W\).

### 6.3 The two-form

Under the action of \(H = SU(2) \times U(1)' \times U(1)''\) the 21 components of the \(SO(7)\) two-form transform into the completely reducible representation:
\[
[1, 1, 0] \rightarrow [1, 0, 0] \oplus [0, 0, 0] \oplus [0, 0, 0] \oplus [0, 6i, -3i] \oplus [0, -6i, 3i] \oplus [1/2, i, -9/2i] \oplus [1/2, -i, 9/2i] \oplus [1/2, 5i, 3/2i] \oplus [1/2, -5i, -3/2i] \oplus [1/2, 3i, -3/2i] \oplus [1/2, -3i, 3/2i] \oplus [0, -2i, -3i] \oplus [0, 2i, 3i].
\]
(6.26)
The decomposition of the two–form in $H$–irreducible fragments is as follows:

\[ \mathcal{Y}^{AB} = -i\lambda^{[\alpha}_{\iota_3} \lambda^{B]}_{\iota_3} [2|I], -i\lambda^{[\alpha}_{\iota_3} \lambda^{B]}_{\iota_3} \varepsilon^{ik}[2|I]_{jk} + \lambda^{[\alpha}_{\iota_3} \lambda^{B]}_{\iota_3} \varepsilon^{ij}[2|I]_{jk} + \lambda^{[\alpha}_{\iota_3} \lambda^{B]}_{\iota_3} \varepsilon^{ik}[2|I]_{jk} + \lambda^{[\alpha}_{\iota_3} \lambda^{B]}_{\iota_3} \varepsilon^{ij}[2|I]_{jk} \]

\[ \mathcal{Y}^{Am} = \lambda^{A}_\iota \sigma^m_{21} [2|II]_{i} + \lambda^{A}_\iota \sigma^m_{12} [2|II]^*_{i} + \lambda^{A}_\iota \sigma^m_{21} [2|III]_{i} + \lambda^{A}_\iota \sigma^m_{21} [2|III]^*_{i} \]

\[ \mathcal{Y}^{mn} = \varepsilon^{mn}[2|II]. \]

\[ \mathcal{Y}^{m3} = \sigma^m_{21} [2|II]_{i} + \sigma^m_{12} [2|II]^*_{i} \]

\[ \mathcal{Y}^{A3} = \lambda^{A}_\iota \sigma^m_{21} [2|I]_{i} + \lambda^{A}_\iota \sigma^m_{21} [2|I]^*_{i} \]

where:

\[ [2|I]_{i} = \mathcal{H}^{0,[0,0]} Z[0, I|\rho] \]

\[ [2|II]_{i} = \mathcal{H}^{0,[0,0]} Z[0, II|\rho] \]

\[ \langle 2|I \rangle_{i} = \mathcal{H}^{0,[0,-3i]} Z \langle 0, I|\rho \rangle \]

\[ \langle 2|I \rangle^*_{i} = \mathcal{H}^{0,[0,-3i]} Z \langle 0, I|\rho \rangle \]

\[ \langle 2|II \rangle_{i} = \mathcal{H}^{0,[0,-3i]} Z \langle 0, II|\rho \rangle \]

\[ \langle 2|II \rangle^*_{i} = \mathcal{H}^{0,[0,-3i]} Z \langle 0, II|\rho \rangle \]

\[ \langle 2|III \rangle_{i} = \mathcal{H}^{0,[0,-3i]} Z \langle 0, III|\rho \rangle \]

\[ \langle 2|III \rangle^*_{i} = \mathcal{H}^{0,[0,-3i]} Z \langle 0, III|\rho \rangle \]

\[ \langle 2|I \rangle_{ij} = \mathcal{H}^{1,0,0} Z[1, I|\rho] \]

\[ \langle 2|II \rangle_{ij} = \mathcal{H}^{1,0,0} Z[2, II|\rho] \]

\[ \langle 2|III \rangle_{ij} = \mathcal{H}^{1,0,0} Z[3, III|\rho] \]

The Laplace Beltrami operator for the transverse two–form field $\mathcal{Y}^{\alpha\beta}$, is given by

\[ \mathcal{M}^{[110]} \mathcal{Y}^{[\alpha\beta]} \equiv M_{(110)} \mathcal{Y}^{[\alpha\beta]} = 3 D^{\gamma}_{\alpha \gamma}[\gamma \mathcal{Y}^{\alpha\beta}] = (D^\gamma D_\gamma + 48) \mathcal{Y}^{[\alpha\beta]} - 4 \mathcal{R}^{[\alpha \beta]}_{\gamma \delta} \mathcal{Y}^{[\gamma \delta]} \]

\[ \text{(6.27)} \]

From the decomposition $D_\alpha \mathcal{Y}^{\beta\gamma} = D_\alpha^H \mathcal{Y}^{\beta\gamma} + (M_\alpha)_{\beta \delta} \mathcal{Y}^{\delta\gamma} + (M_\alpha)_{\gamma \delta} \mathcal{Y}^{\beta\delta} \quad \text{we obtain:} \]

\[ \mathcal{M}^{[110]} \mathcal{Y}^{[\alpha\beta]} = \{ 48 \delta^{[\alpha \beta]}_{[\gamma \delta]} - 4 \mathcal{R}^{[\alpha \beta]}_{[\gamma \delta]} + 2 \eta^{\mu\nu} (M_\mu)_{[\gamma [\alpha} \delta^{\beta]}_{\delta]} + 2 \eta^{\mu\nu} (M_\mu M_\nu)_{[\gamma [\alpha} \delta^{\beta]}_{\delta]} + 4 \eta^{\mu\nu} (M_\mu)_{[\gamma [\alpha} \delta^{\beta]}_{\delta]} D_\nu^H \} \mathcal{Y}^{[\gamma \delta]} \]

\[ \text{(6.28)} \]
For the regular $G$ representations of type $^0$ this operators acts on $AdS_4$ fields as the following $11 \times 11$ matrix:

**Column one to three:**

| $M_{(1/2,0)}$ | $Z[0, I]$ | $Z[0, II]$ | $Z[1, I]$ |
|------------|-----------|-----------|---------|
| $Z[0, I]$  | $H_0 + 32$ | $-16$ | $\frac{16}{\sqrt{3}}i(M_2 + 2)$ |
| $Z[0, II]$ | $-32$ | $H_0 + 16$ | $0$ |
| $Z[1, I]$  | $0$ | $0$ | $H_0$ |
| $Z[1/2, I]$ | $-\frac{16}{\sqrt{3}}iM_1$ | $0$ | $\frac{16}{\sqrt{3}}i(M_1 + 2)$ |
| $\tilde{Z}[1/2, I]$ | $\frac{16}{\sqrt{3}}iM_2$ | $0$ | $\frac{16}{\sqrt{3}}i(M_2 + 2)$ |
| $Z[0, II]$ | $0$ | $\frac{16}{3\sqrt{2}}i(M_2 - M_1 + 3J)$ | $0$ |
| $\tilde{Z}[0, II]$ | $0$ | $-\frac{16}{3\sqrt{2}}i(M_2 - M_1 - 3J)$ | $0$ |
| $Z[0, I]$ | $0$ | $0$ | $0$ |
| $\tilde{Z}[0, I]$ | $0$ | $0$ | $0$ |
| $Z[1/2, III]$ | $0$ | $0$ | $0$ |
| $\tilde{Z}[1/2, III]$ | $0$ | $0$ | $0$ |

**Column four to seven:**

| $M_{(1/2,0)}$ | $Z[1/2, I]$ | $Z[3/2, I]$ | $Z[0, II]$ | $Z[0, II]$ |
|------------|-----------|-----------|---------|---------|
| $Z[0, I]$  | $-\frac{8}{3\sqrt{2}}i(M_1 + 2)$ | $0$ | $0$ | $0$ |
| $Z[0, II]$ | $0$ | $\frac{32}{\sqrt{2}}i(M_2 - M_1 - 3J)$ | $0$ | $0$ |
| $Z[1, I]$  | $-\frac{16}{\sqrt{3}}iM_2$ | $\frac{16}{\sqrt{3}}iM_1$ | $0$ | $0$ |
| $Z[1/2, I]$ | $H_0 + 32 - \frac{32}{3}(M_2 - M_1)$ | $0$ | $0$ | $0$ |
| $\tilde{Z}[1/2, I]$ | $0$ | $H_0 + 32 + \frac{32}{3}(M_2 - M_1)$ | $0$ | $0$ |
| $Z[0, II]$ | $0$ | $0$ | $H_0 + 32 - \frac{32}{3}(M_2 - M_1)$ | $0$ |
| $\tilde{Z}[0, II]$ | $0$ | $-\frac{16}{\sqrt{3}}i(M_2 - M_1 - 3J)$ | $0$ | $-\frac{16}{\sqrt{3}}iM_2$ |
| $Z[0, I]$ | $0$ | $0$ | $-\frac{16}{\sqrt{3}}i(M_2 - M_1 + 3J)$ | $0$ |
| $\tilde{Z}[0, I]$ | $0$ | $0$ | $-\frac{16}{\sqrt{3}}iM_2$ | $0$ |
| $Z[1/2, III]$ | $0$ | $0$ | $0$ | $-\frac{16}{\sqrt{3}}iM_1$ |
| $\tilde{Z}[1/2, III]$ | $0$ | $0$ | $0$ | $0$ |

**Column eight to eleven:**

| $M_{(1/2,0)}$ | $Z[3/2, II]$ | $Z[5/2, II]$ | $Z[3/2, III]$ | $Z[5/2, III]$ |
|------------|-----------|-----------|---------|---------|
| $Z[0, I]$  | $0$ | $0$ | $0$ | $0$ |
| $Z[0, II]$ | $0$ | $0$ | $0$ | $0$ |
| $Z[1, I]$  | $0$ | $0$ | $0$ | $0$ |
| $Z[1/2, I]$ | $-\frac{32}{\sqrt{3}}(M_2 - M_1 - 3J)$ | $0$ | $0$ | $0$ |
| $\tilde{Z}[1/2, I]$ | $0$ | $\frac{32}{\sqrt{3}}(M_2 - M_1 + 3J)$ | $0$ | $0$ |
| $Z[0, II]$ | $-\frac{32}{\sqrt{3}}(M_2 + 2)$ | $0$ | $0$ | $0$ |
| $\tilde{Z}[0, II]$ | $0$ | $-\frac{32}{\sqrt{3}}(M_1 + 2)$ | $0$ | $0$ |
| $Z[0, I]$ | $H_0$ | $0$ | $0$ | $0$ |
| $\tilde{Z}[0, I]$ | $H_0$ | $0$ | $0$ | $0$ |
| $Z[1/2, III]$ | $0$ | $H_0 - \frac{64}{3}(M_2 - M_1)$ | $0$ | $H_0 + \frac{64}{3}(M_2 - M_1)$ |
| $\tilde{Z}[1/2, III]$ | $0$ | $0$ | $0$ | $0$ |

This matrix has the following eigenvalues:

$$\lambda_1 = H_0 + \frac{32}{3}(M_2 - M_1),$$

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\[\lambda_2 = H_0 - \frac{32}{3}(M_2 - M_1),\]
\[\lambda_3 = H_0,\]
\[\lambda_4 = H_0 + 24 + 4\sqrt{H_0 + 36},\]
\[\lambda_5 = H_0 + 24 - 4\sqrt{H_0 + 36},\]
\[\lambda_6 = H_0 + \frac{32}{3}(M_2 - M_1) + 16 + 4\sqrt{H_0 + \frac{32}{3}(M_2 - M_1) + 16},\]
\[\lambda_7 = H_0 + \frac{32}{3}(M_2 - M_1) + 16 - 4\sqrt{H_0 + \frac{32}{3}(M_2 - M_1) + 16},\]
\[\lambda_8 = H_0 - \frac{32}{3}(M_2 - M_1) + 16 + 4\sqrt{H_0 - \frac{32}{3}(M_2 - M_1) + 16},\]
\[\lambda_9 = H_0 - \frac{32}{3}(M_2 - M_1) + 16 - 4\sqrt{H_0 - \frac{32}{3}(M_2 - M_1) + 16},\]
\[\lambda_{10} = \lambda_{11} = H_0 + 32.\]

The eigenvalues \(\lambda_1, \lambda_2, \lambda_4, \lambda_5\), equal to the one–form physical ones, are the longitudinal eigenvalues. The other seven are the physical two-form eigenvalues.

As in the case of the one–form, by removing rows and columns we find the matrix of each exceptional series, and the corresponding eigenvalues:

\[
\begin{array}{c|cccccccccccc}
A_R & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11} \\
A_1 & \lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_{10}, \lambda_{11} \\
A_1^* & \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11} \\
A_2 & \lambda_1, \lambda_6, \lambda_7 \\
A_2^* & \lambda_2, \lambda_8, \lambda_9 \\
A_3 & \lambda_1, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_{10}, \lambda_{11} \\
A_3^* & \lambda_2, \lambda_4, \lambda_5, \lambda_8, \lambda_{10}, \lambda_{11} \\
A_4 & \lambda_1, \lambda_4, \lambda_6, \lambda_7, \lambda_{10} \\
A_4^* & \lambda_2, \lambda_4, \lambda_8, \lambda_9, \lambda_{10} \\
A_5 & \lambda_1, \lambda_6 \\
A_5^* & \lambda_2, \lambda_8 \\
A_6 & \lambda_3, \lambda_4, \lambda_5, \lambda_{10}, \lambda_{11} \\
A_6^* & \lambda_4, \lambda_5, \lambda_{10}, \lambda_{11} \\
A_7 & \lambda_3, \lambda_4 \\
A_7^* & \lambda_3, \lambda_4
\end{array}
\]

The two–form operator matrix in the representations \(++\) is the following 5 \times 5 matrix:

Columns one to two:

| \(M_{(1)^2(0)}\) | \(Z(0, I)\) | \(Z(\frac{2}{3}, I)\) |
|------------------|------------|------------------|
| \(Z(0, I)\)     | \(H_0 - \frac{32}{3}(M_2 - M_1)\) | \(\frac{16}{\sqrt{3}}(M_1 + 2)\) |
| \(Z(\frac{1}{2}, I)\) | \(\sqrt{3} M_2\) | \(H_0 + 32\) |
| \(Z(\frac{1}{2}, \Pi)\) | 0 | \(-\frac{16}{3\sqrt{2}}(M_2 - M_1 + 3J - 3)\) |
| \(Z(\frac{1}{2}, \Pi)\) | 0 | \(-\frac{16}{\sqrt{3}}(M_2 - M_1 + 3J - 3)\) |
| \(Z(1, I)\)     | 0 | \(\frac{32}{3}(M_2 - 1)\) |
The other four are the physical eigenvalues. The eigenvalue $\lambda_1$ for the $Z(1, I)$ is longitudinal. The other four are the physical eigenvalues.

\[
\begin{array}{c|cc}
B_R & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \\
B_1 & \lambda_1, \lambda_2, \lambda_3 \\
B_2 & \lambda_5 \\
B_3 & \lambda_1, \lambda_2, \lambda_3 \\
B_4 & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \\
B_5 & \lambda_4 \\
B_6 & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \\
B_7 & \lambda_1, \lambda_2, \lambda_4 \\
B_8 & \lambda_4 \\
B_9 & \lambda_4 \\
B_{10} & \\
\end{array}
\]

For the $-\epsilon$ representations, the eigenvalues are the conjugates ($M_2 \leftrightarrow M_1$) of the ones in (6.32). We can use the eigenvalues of the two-form harmonic to determine (see [4]) the masses of the $AdS_4$ vector field $Z$.

### 6.4 The three-form

The $H$ decomposition of the three-form in $H$-irreducible fragments has been done in [4]:

\[
\begin{align*}
\gamma^{ABC} &= \varepsilon^{ABCD} \{\lambda_{3i} D \langle 3|I| i \rangle + \lambda_{3i} D \langle 3|I| i \rangle^* \}, \\
\gamma^{Abm} &= \lambda_{3i} A \lambda_{3j} B \varepsilon^{ij} \{\sigma_{21} \langle 3|II| \rangle + \sigma_{12} \langle 3|III| \rangle \} + \lambda_{3i} A \lambda_{3j} B \varepsilon^{ij} \{\sigma_{21} \langle 3|II| \rangle^* + \sigma_{21} \langle 3|III| \rangle^* \} + i \lambda_{3i} A \lambda_{3j} B \varepsilon^{ij} \{\sigma_{12} \langle 3|IV| \rangle + \sigma_{12} \langle 3|IV| \rangle^* \} + \lambda_{3i} A \lambda_{3j} B \varepsilon^{ij} \{\sigma_{21} \langle 3|II| \rangle k - \sigma_{12} \langle 3|IV| \rangle ik \}, \\
\gamma^{A3} &= \lambda_{3i} A \lambda_{3j} B \varepsilon^{ij} \{\sigma_{21} \langle 3|II| \rangle + i \lambda_{3i} A \lambda_{3j} B \varepsilon^{ij} \langle 3|III| \rangle^* \} + i \lambda_{3i} A \lambda_{3j} B \varepsilon^{ij} \{\sigma_{21} \langle 3|IV| \rangle + \lambda_{3i} A \lambda_{3j} B \varepsilon^{ij} \langle 3|III| \rangle \}, \\
\gamma^{Amn} &= \varepsilon^{mn} \{\lambda_{3i} A \langle 3|II| \rangle + \lambda_{3i} A \langle 3|III| \rangle \}, \\
\gamma^{Am3} &= \lambda_{3i} A \{\sigma_{21} \langle 3|IV| \rangle + \sigma_{21} \langle 3|III| \rangle + \lambda_{3i} A \{\sigma_{21} \langle 3|IV| \rangle^* + \sigma_{21} \langle 3|III| \rangle^* \}, \\
\gamma^{mn3} &= \varepsilon^{mn} \langle 3|II| \rangle, 
\end{align*}
\]
where the fragments of type $^0$ are:

\[
\begin{align*}
\langle 3 | I \rangle_{ij} &= \mathcal{H}^{[1,-2i,-3i]}_{ij}(c) \cdot \pi(1, I), \\
\langle 3 | I \rangle^*_{ij} &= -\varepsilon^k \varepsilon^l \mathcal{H}^{[1,2i,3i]}_{kl}(c) \cdot \bar{\pi}(1, I), \\
\langle 3 | I \rangle_{ij} &= \mathcal{H}^{[1,0,0]}_{ij}(c) \cdot \pi[1, I], \\
\langle 3 | I \rangle^*_{ij} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | II \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | III \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | IV \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | V \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | VI \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | VII \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | VIII \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | IX \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I),
\end{align*}
\]

while the fragments of type $^{++}$ are:

\[
\begin{align*}
\langle 3 | I \rangle_{ij} &= \mathcal{H}^{[1,-2i,-3i]}_{ij}(d) \cdot \pi(1, I), \\
\langle 3 | I \rangle^*_{ij} &= \varepsilon^k \varepsilon^l \mathcal{H}^{[1,2i,3i]}_{kl}(d) \cdot \bar{\pi}(1, I), \\
\langle 3 | I \rangle_{ij} &= \mathcal{H}^{[1,0,0]}_{ij}(d) \cdot \pi[1, I], \\
\langle 3 | I \rangle^*_{ij} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | II \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | III \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | IV \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | V \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | VI \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | VII \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | VIII \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I), \\
\langle 3 | IX \rangle_{i} &= \mathcal{H}^{[1,2,3i,-3i/2]}_{i}(a) \cdot \pi(1/2, I),
\end{align*}
\]

The fragments that are present the type $^{--}$ series are the complex conjugates of the fragments above. The Laplace Beltrami operator for the transverse three-form $\mathcal{Y}^{[\alpha\beta\gamma]}$, is a first-order differential operator, given by

\[
\mathcal{M}^{[111]} \mathcal{Y}^{[\alpha\beta\gamma]} \equiv M(1)^{\alpha\beta\gamma} \mathcal{Y}^{[\alpha\beta\gamma]} = \frac{1}{24} \varepsilon^{\alpha\beta\gamma\delta}_{\mu\nu\rho} D_\delta \mathcal{Y}^{\mu\nu\rho} = \frac{1}{24} \varepsilon^{\alpha\beta\gamma\delta}_{\mu\nu\rho} \left[ D_\delta \mathcal{Y}^{\mu\nu\rho} + (\mathcal{M}_\delta)^{\mu}_{\nu} \mathcal{Y}^{\sigma\mu\rho} + (\mathcal{M}_\delta)^{\nu}_{\sigma} \mathcal{Y}^{\mu\sigma\rho} + (\mathcal{M}_\delta)^{\rho}_{\sigma} \mathcal{Y}^{\mu\nu\sigma} \right].
\]

(6.33)
For the regular series of type $0$ this operator acts on the $AdS_4$ fields as a $15 \times 15$ matrix:

Column one to five:

| $M_{(1)}$ | $\pi(1, I)$ | $\bar{\pi}(1, I)$ | $\pi[1, I]$ | $\bar{\pi}[1, I]$ | $\bar{\pi}[2, I]$ | $\bar{\pi}[2, II]$ |
|-----------|-------------|-------------------|------------|----------------|----------------|-------------|
| $\pi(1, I)$ | $Y$ | $0$ | $-\frac{(2 J + Y)}{\sqrt{2}}$ | $0$ | $0$ | $0$ |
| $\bar{\pi}(1, I)$ | $0$ | $-Y$ | $\frac{2 J - Y}{\sqrt{2}}$ | $0$ | $0$ | $0$ |
| $\pi[1, I]$ | $\frac{-2 - 2 J + Y}{\sqrt{2}}$ | $\frac{-2 + 2 J + Y}{\sqrt{2}}$ | $1$ | $0$ | $0$ | $0$ |
| $\bar{\pi}[1, I]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\bar{\pi}[1, II]$ | $0$ | $0$ | $\frac{\sqrt{3}}{3} (2 + M_1)$ | $-i Y$ | $0$ | $0$ |
| $\bar{\pi}[1, III]$ | $0$ | $0$ | $\frac{\sqrt{3}}{3} (2 + M_2)$ | $0$ | $-i Y$ | $0$ |
| $\bar{\pi}[1, IV]$ | $0$ | $0$ | $\frac{2 J - Y}{\sqrt{2}}$ | $0$ | $0$ | $0$ |
| $\bar{\pi}[1, V]$ | $0$ | $0$ | $\frac{2 J - Y}{\sqrt{2}}$ | $0$ | $0$ | $0$ |
| $\pi[0, IV]$ | $0$ | $0$ | $\sqrt{3}$ | $0$ | $0$ | $0$ |
| $\bar{\pi}[0, IV]$ | $0$ | $0$ | $\sqrt{3}$ | $0$ | $0$ | $0$ |
| $\bar{\pi}[0, I]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\bar{\pi}[0, II]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |

Column six to ten:

| $\bar{\pi}[2, I]$ | $\pi[2, I]$ | $\bar{\pi}[2, II]$ | $\pi[2, III]$ | $\bar{\pi}[2, III]$ | $\bar{\pi}[2, IV]$ |
|-------------------|-------------|-------------------|------------|----------------|-------------|
| $\pi(1, I)$ | $0$ | $0$ | $\frac{2 M_1}{\sqrt{3}}$ | $0$ | $0$ |
| $\bar{\pi}(1, I)$ | $0$ | $0$ | $\frac{2 M_1}{\sqrt{3}}$ | $\frac{2 M_1}{\sqrt{3}}$ | $0$ |
| $\pi[1, I]$ | $\frac{2 - 2 J + Y}{\sqrt{3}}$ | $\frac{-2 J + Y}{\sqrt{3}}$ | $0$ | $0$ | $0$ |
| $\bar{\pi}[1, I]$ | $0$ | $i Y$ | $\frac{-2 - 2 J + Y}{\sqrt{3}}$ | $0$ | $0$ |
| $\bar{\pi}[1, II]$ | $0$ | $i Y$ | $0$ | $\frac{2 J - Y}{\sqrt{2}}$ | $0$ |
| $\bar{\pi}[1, III]$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\bar{\pi}[1, IV]$ | $0$ | $0$ | $0$ | $0$ | $1$ |
| $\bar{\pi}[1, V]$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\pi[0, IV]$ | $0$ | $0$ | $\frac{-i (2 + M_1)}{\sqrt{3}}$ | $0$ | $0$ |
| $\bar{\pi}[0, IV]$ | $0$ | $0$ | $\frac{i (2 + M_1)}{\sqrt{3}}$ | $0$ | $0$ |
| $\pi[0, I]$ | $\frac{-2 + M_2}{\sqrt{3}}$ | $\frac{-2 + M_1}{\sqrt{3}}$ | $0$ | $0$ | $0$ |
| $\bar{\pi}[0, I]$ | $0$ | $0$ | $0$ | $0$ | $0$ |

(6.34)

(6.35)
This matrix has the following eigenvalues:

\[
\begin{align*}
\lambda_1 &= \frac{1}{4} \sqrt{H_0 + \frac{32}{3} (M_2 - M_1) + 16}, \\
\lambda_2 &= \frac{1}{4} \sqrt{H_0 - \frac{32}{3} (M_2 - M_1) + 16}, \\
\lambda_3 &= -\frac{1}{4} \sqrt{H_0 + \frac{32}{3} (M_2 - M_1) + 16}, \\
\lambda_4 &= -\frac{1}{4} \sqrt{H_0 - \frac{32}{3} (M_2 - M_1) + 16}, \\
\lambda_5 &= \frac{1}{4} \sqrt{H_0 + 36 - \frac{1}{2}}, \\
\lambda_6 &= -\frac{1}{4} \sqrt{H_0 + 36 - \frac{1}{2}}, \\
\lambda_7 &= -\frac{1}{4} \sqrt{H_0 + 4 + \frac{1}{2}}, \\
\lambda_8 &= \frac{1}{4} \sqrt{H_0 + 4 + \frac{1}{2}}, \\
\lambda_9 &= \ldots = \lambda_{15} = 0.
\end{align*}
\]

We note that seven eigenvalues are 0. They correspond to the longitudinal three-forms \( \mathcal{Y}^{(3)} = D \wedge \mathcal{Y}^{(2)} \), which are annihilated by \( \mathcal{H}^{[11]} \) \( = \ast D \wedge \).

As in the cases of the one–form and of the two–form, by removing rows and columns we
find the matrix for each exceptional series, and the corresponding eigenvalues:

\[
\begin{array}{c|cccc}
& A_R & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \\
A_1 & \lambda_1, \lambda_3, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \\
A_1^2 & \lambda_2, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \\
A_2 & \lambda_2, \lambda_4 \\
A_3 & \lambda_1, \lambda_3, \lambda_5, \lambda_6 \\
A_3^2 & \lambda_2, \lambda_4, \lambda_5, \lambda_6 \\
A_4 & \lambda_1, \lambda_3, \lambda_6 \\
A_5 & \lambda_2, \lambda_4, \lambda_6 \\
A_6 & \lambda_1, \lambda_3, \lambda_6 \\
A_7 & \lambda_2, \lambda_4, \lambda_6 \\
A_8 & \lambda_6, \lambda_8
\end{array}
\]

(6.38)

The two–form operator matrix for the regular series of type ++ is the following 10 × 10 matrix:

| Column one to five: | \(\pi(1,1)\) | \(\pi(1,1)\) | \(\pi[1,1]\) | \(\pi(\frac{1}{2},1)\) | \(\pi(\frac{1}{2},2)\) |
|---------------------|--------------|--------------|-------------|----------------|----------------|
| \(\pi(1,1)\)       | \(Y\)        | 0            | \(-\frac{(2J+Y)}{\sqrt{2}}\) | 0              | 0              |
| \(\pi(1,1)\)       | 0            | \(-Y\)       | \(-\frac{(2J+Y)}{\sqrt{2}}\) | 0              | 0              |
| \(\pi[1,1]\)       | \(-2-2J+Y\) | \(2+2J+Y\)  | 1           | 0              | \(-2i(-1+M_1)\) |
| \(\pi(\frac{1}{2},1)\) | \(\sqrt{2}\) | \(\sqrt{2}\) | 0           | 0              | \(iY\)         |
| \(\pi(\frac{1}{2},2)\) | 0            | 0            | 0           | 0              | \(\frac{2\sqrt{2}}{\sqrt{3}}\) |
| \(\pi(\frac{1}{2},3)\) | \(\frac{3+M_1}{\sqrt{3}}\) | 0            | 0           | \(-2J+Y\)     | 0              |
| \(\pi(\frac{1}{2},4)\) | 0            | \(-\frac{3+M_1}{\sqrt{3}}\) | 0           | \(\frac{2\sqrt{2}}{\sqrt{3}}\) | 0              |
| \(\pi(0,1)\)       | 0            | 0            | 0           | 0              | \(\frac{i(2+M_1)}{\sqrt{3}}\) |
| \(\pi(0,2)\)       | 0            | 0            | 0           | 0              | 0              |
| \(\pi(0,3)\)       | 0            | 0            | 0           | 0              | 0              |
| \(\pi(0,4)\)       | 0            | 0            | 0           | 0              | 0              |
| \(\pi(0,5)\)       | 0            | 0            | 0           | 0              | 0              |
| \(\pi(0,6)\)       | 0            | 0            | 0           | 0              | 0              |
| \(\pi(0,7)\)       | 0            | 0            | 0           | 0              | 0              |
| \(\pi(0,8)\)       | 0            | 0            | 0           | 0              | 0              |
| \(\pi(0,9)\)       | 0            | 0            | 0           | 0              | 0              |
| \(\pi(0,10)\)      | 0            | 0            | 0           | 0              | 0              |

(6.39)

Column six to ten:

| Column six to ten: | \(\pi(\frac{1}{2},3)\) | \(\pi(\frac{1}{2},4)\) | \(\pi(0,1)\) | \(\pi(0,2)\) | \(\pi(0,3)\) |
|---------------------|----------------|----------------|-------------|-------------|-------------|
| \(\pi(1,1)\)       | \(2(1+\frac{M_2}{\sqrt{2}})\) | 0              | 0           | 0           | 0           |
| \(\pi(1,1)\)       | 0              | \(-2(1+\frac{M_2}{\sqrt{2}})\) | 0           | 0           | 0           |
| \(\pi[1,1]\)       | 0              | 0              | \((-2J+Y)\) | 0           | 0           |
| \(\pi(\frac{1}{2},1)\) | \(-2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) | 0           | 0           | 0           |
| \(\pi(\frac{1}{2},2)\) | 0              | 0              | \(-2\frac{J+Y}{\sqrt{2}}\) | 0           | 0           |
| \(\pi(\frac{1}{2},3)\) | 0              | 0              | 0           | \(-2\frac{J+Y}{\sqrt{2}}\) | 0           |
| \(\pi(\frac{1}{2},4)\) | 0              | 0              | 0           | \(2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |
| \(\pi(0,1)\)       | 0              | 0              | 0           | \(-2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |
| \(\pi(0,2)\)       | 0              | 0              | 0           | \(2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |
| \(\pi(0,3)\)       | 0              | 0              | 0           | \(-2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |
| \(\pi(0,4)\)       | 0              | 0              | 0           | \(2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |
| \(\pi(0,5)\)       | 0              | 0              | 0           | \(2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |
| \(\pi(0,6)\)       | 0              | 0              | 0           | \(2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |
| \(\pi(0,7)\)       | 0              | 0              | 0           | \(2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |
| \(\pi(0,8)\)       | 0              | 0              | 0           | \(2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |
| \(\pi(0,9)\)       | 0              | 0              | 0           | \(2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |
| \(\pi(0,10)\)      | 0              | 0              | 0           | \(2\frac{J+Y}{\sqrt{2}}\) | \(2\frac{J+Y}{\sqrt{2}}\) |

(6.40)

It has eigenvalues:

\[
\lambda_1 = \frac{1}{4} \sqrt{H_0 + \frac{32}{9} (M_2 - M_1) + 16},
\]

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\[\begin{align*}
\lambda_2 &= -\frac{1}{4}\sqrt{H_0 + \frac{32}{3}(M_2 - M_1) + 16}, \\
\lambda_3 &= \frac{1}{4}\sqrt{H_0 + 36 - \frac{1}{2}}, \\
\lambda_4 &= -\frac{1}{4}\sqrt{H_0 + 36 - \frac{1}{2}}, \\
\lambda_5 &= -\frac{1}{4}\sqrt{H_0 + \frac{64}{9}(M_2 - M_1) - 28 + \frac{1}{2}}, \\
\lambda_6 &= \frac{1}{4}\sqrt{H_0 + \frac{64}{9}(M_2 - M_1) - 28 + \frac{1}{2}}, \\
\lambda_7 &= \ldots = \lambda_{10} = 0. 
\end{align*}\]

The complete table of eigenvalues for the type \(++\) series is:

| \(B_R\) | \(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\) |
|---------|-------------------------------------------------|
| \(B_1\) | \(\lambda_1, \lambda_2, \lambda_5, \lambda_6\)     |
| \(B_2\) | \(\lambda_5, \lambda_6\)                        |
| \(B_3\) | \(\lambda_1, \lambda_2\)                       |
| \(B_4\) | \(\lambda_1, \lambda_2, \lambda_3, \lambda_4\) |
| \(B_5\) | \(\lambda_3, \lambda_4\)                       |
| \(B_6\) | \(\lambda_1, \lambda_2, \lambda_3, \lambda_4\) |
| \(B_7\) | \(\lambda_2, \lambda_3, \lambda_4\)            |
| \(B_8\) | \(\lambda_4\)                                  |
| \(B_9\) | \(\lambda_3, \lambda_4\)                       |
| \(B_{10}\) | \(\lambda_4\)                                    |
| \(B_{11}\) | \(\lambda_4\)                                    |

For the representations of the \(-\) series, the eigenvalues are the conjugates of the one in (6.42).

### 6.5 The spinor

The harmonic analysis of the eight–component Majorana spinor has been completely worked out in [5]. We reformulate these results in our framework, in order to facilitate the matching of the spectrum with the \(\mathcal{N} = 2\) multiplets.

The decomposition of the spinor in its \(H\)–irreducible components is

\[\eta = \begin{pmatrix} \langle \frac{1}{2}|I\rangle_i \\ \langle \frac{1}{2}|I\rangle_1 \\ \langle \frac{1}{2}|I\rangle_2 \\ -i\sigma_2 \langle \frac{1}{2}|I\rangle_3^* \\ \langle \frac{1}{2}|\Pi\rangle_1^* \\ -\langle \frac{1}{2}|\Pi\rangle_2^* \end{pmatrix}\]  

where

\[
\begin{align*}
\langle \frac{1}{2}|I\rangle_i &= \mathcal{H}_i^{[1/2,-i,-3i/2]} \chi \langle \frac{1}{2}, I \rangle, \\
\langle \frac{1}{2}|I\rangle_1 &= \mathcal{H}_i^{[0,2i,-3]} \chi \langle 0, I \rangle, \\
\langle \frac{1}{2}|I\rangle_2 &= \mathcal{H}_i^{[0,-4i,0]} \chi \langle 0, \Pi \rangle, \\
\langle \frac{1}{2}|I\rangle_3^* &= \pm \epsilon_{ij} \mathcal{H}_j^{[1/2,i,3i/2]} \chi \langle \frac{1}{2}, I \rangle,
\end{align*}\]
\[ \langle \frac{1}{2} | I \rangle^* = \mathcal{H}^{[0, -2i, 3i]} \cdot \bar{\chi} \langle 0, I \rangle, \]
\[ \langle \frac{1}{2} | II \rangle^* = \mathcal{H}^{[0, -4i, 0]} \cdot \bar{\chi} \langle 0, II \rangle. \] (6.44)

The fragments of type ++ are
\[ \langle \frac{1}{2} | I \rangle_i = \mathcal{H}^{[1/2, -i, -3i/2]}(a) \cdot \chi \langle \frac{1}{2}, I \rangle, \]
\[ \langle \frac{1}{2} | II \rangle = \mathcal{H}^{[0, 2i, -3i]} \cdot \chi \langle 0, I \rangle, \]
\[ \langle \frac{1}{2} | I \rangle^*_\epsilon = \varepsilon^{ij} \mathcal{H}^{[1/2, i, 3i/2]}(a) \cdot \bar{\chi} \langle \frac{1}{2}, I \rangle, \]
\[ \langle \frac{1}{2} | II \rangle^*_\epsilon = \mathcal{H}^{[0, 4i, 0]} \cdot \bar{\chi} \langle 0, II \rangle. \] (6.45)

For the regular series ++ the spinor operator acts on the AdS$_4$ fields as a 4 × 4 matrix, whose eigenvalues are:
\[ \lambda_1 = -6 + \sqrt{H_0 + 36}, \]
\[ \lambda_2 = -6 - \sqrt{H_0 + 36}, \]
\[ \lambda_3 = -8 + \sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)}, \]
\[ \lambda_4 = -8 - \sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)}. \] (6.46)

The eigenvalues for each exceptional series are

\[
\begin{array}{|c|c|}
\hline
A_1^+, A_2^+, A_3^+, A_4^+ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \\
A_1^-, A_2^- & \lambda_3, \lambda_4 \\
A_3^+, A_4^+ & \lambda_1, \lambda_2 \\
A_4^+, A_5^+ & \lambda_1 \\
\hline
\end{array}
\] (6.47)

The fragments of type -- are
\[ \langle \frac{1}{2} | I \rangle_i = \mathcal{H}^{[1/2, -i, -3i/2]}(a) \cdot \chi \langle \frac{1}{2}, I \rangle, \]
\[ \langle \frac{1}{2} | II \rangle = \mathcal{H}^{[0, -4i, 0]} \cdot \chi \langle 0, I \rangle, \]
\[ \langle \frac{1}{2} | I \rangle^*_\epsilon = \varepsilon^{ij} \mathcal{H}^{[1/2, i, 3i/2]}(a) \cdot \bar{\chi} \langle \frac{1}{2}, I \rangle, \]
\[ \langle \frac{1}{2} | II \rangle^*_\epsilon = \mathcal{H}^{[0, -2i, 3i]} \cdot \bar{\chi} \langle 0, II \rangle. \] (6.48)

For the regular series -- the spinor operator acts on the AdS$_4$ fields as a 4 × 4 matrix, whose eigenvalues are:
\[ \lambda_1 = -6 + \sqrt{H_0 + 36}, \]
\[ \lambda_2 = -6 - \sqrt{H_0 + 36}, \]
\[ \lambda_3 = -8 + \sqrt{H_0 + 16 - \frac{32}{3}(M_2 - M_1)}, \]
\[ \lambda_4 = -8 - \sqrt{H_0 + 16 - \frac{32}{3}(M_2 - M_1)}. \] (6.49)

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The eigenvalues for each exceptional series are

| $A_{II}, A_1^\pm, A_3^\pm, A_4^\pm$ | $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ |
| $A_2^\pm, A_5^\pm$ | $\lambda_3, \lambda_4$ |
| $A_1^\pm, A_6^\pm$ | $\lambda_1, \lambda_2$ |
| $A_3, A_7$ | $\lambda_1, \lambda_2$ |
| $A_4, A_8$ | $\lambda_1$ |

(6.50)

7 Matching the spectrum with the $Osp(2|4)$ multiplets

As already mentioned, the structure of the long multiplets that arise from $\mathcal{N} = 2$ compactifications of eleven–dimensional supergravity in $AdS_4$ has already been presented in [8]. The structure and the $G'$ representations of the long graviton multiplet, the long gravitino multiplets and the massless multiplets are known since the eighties [8]. However this is not the case for the long vector multiplets and for the short multiplets, due to the fact that the crucial knowledge of the eigenvalues of the operators (1.5) was lacking. In this section we match the complete spectrum that we have calculated with the multiplets as they are already known from old literature. Furthermore, while doing this we also find the short $\mathcal{N} = 2$ multiplets as truncations of the long multiplets that were already known.

In order to achieve this we use a procedure of exhaustion, i.e. one starts with one of the four different types of multiplets for which all the masses of a certain field component are most easily retrieved (this is for instance the case for the graviton field of the graviton multiplet) and using the mass relations of [8] (see also appendix [3]), one calculates all the masses of the other types of fields present in the multiplet. One uses also the information that all the fields in a multiplet are in the same irreducible $G' = SU(3) \times SU(2)$ representation and that their hypercharges are related according to the group theoretical structure of the multiplets shown in tables [1, 2, 3, 4, 5, 6, 7]. So one knows in which $G'$ representation to find the other fields of the multiplet, whose masses have been determined. Then, upon using the relations (B.3), these masses are compared with the eigenvalues of the invariant operators on the spinor, the one–form, the two–form or the three–form depending on the type of field one is considering. The upshot of this is that some of these eigenvalues yield all the masses obtained from the mass relations. However, the remaining eigenvalues signal the existence of some extra masses which then pertain to other fields that are to be found in other multiplets. In this way one establishes the existence of new unknown multiplets and determines their structure by filling out their field content. After repeatedly applying this procedure one will have filled out all the existing multiplets in the spectrum.

We should remark here that we did not calculate the eigenvalues of the Lichnerowicz and Rarita–Schwinger operators $M_{(2)(0)^2}$ and $M_{(3/2)(1/2)^2}$. However we succeeded in finding the complete multiplet structure without making use of this. The $AdS_4$ fields whose spectrum is determined by $M_{(2)(0)^2}$ and $M_{(3/2)(1/2)^2}$ are the scalar field $\phi$ and the transverse spinor field $\lambda_T$ (see (B.3)). We can fill the multiplets without knowing the spectrum of these two fields with the help of the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ decompositions (see pictures 1, 2, 3, 4, 5, 6, 7). Every theory with $\mathcal{N} = 2$ supersymmetry also has $\mathcal{N} = 1$ supersymmetry, so every $\mathcal{N} = 2$ multiplet has to
be decomposable in $\mathcal{N} = 1$ multiplets. If we know every field of a multiplet except for $\phi$ and $\lambda_T$, we can deduce which $\phi$ and $\lambda_T$ are present by trying to organize the $\mathcal{N} = 2$ multiplet in $\mathcal{N} = 1$ multiplets. There is no ambiguity, because no $\mathcal{N} = 1$ multiplet is built using $\phi$ and $\lambda_T$ fields alone. In particular, a Wess Zumino multiplet with one $\lambda_T$ and two $\phi$’s is not allowed, since it has to contain both a scalar and a pseudoscalar.

In practice one starts with the graviton multiplet since the masses of the graviton field in the different representations are immediate to derive, being the eigenvalues of the scalar operator $M_{(0)3}$. By means of the above procedure, one exhausts all the spin–$\frac{3}{2}$ fields in the graviton multiplet comparing the masses of the spin–$\frac{3}{2}$ fields in the graviton multiplet with the eigenvalues of the operator $M_{(1/2)3}$. The spin–$\frac{3}{2}$ fields that provide the remaining eigenvalues of the operator $M_{(1/2)3}$, can only be the highest–spin component gravitino fields of the gravitino multiplet and hence we know all the masses of the gravitini in the gravitino multiplet. At this stage we can repeat the same procedure. We use the eigenvalues of the one–form operator $M_{(1)(0)2}^1$ to identify the vector field $A$ and $W$ and we use the eigenvalues of the two–form operator $M_{(1)2(0)}^1$ to identify the vector fields $Z$ in the graviton and the gravitino multiplet. The remaining vector fields constitute the highest–component vector fields of the vector multiplet. Then we determine the masses of the longitudinal spinors, provided by the eigenvalues of the operator $M_{(1/2)3}$, and we find the longitudinal spinors of the gravitino and vector multiplet. The remaining longitudinal spinors belong to hypermultiplets. At the end we determine the masses of the scalars $S, \Sigma$, that are provided by the eigenvalues of $M_{(0)3}$, and of the pseudoscalar $\pi$, provided by the eigenvalues of the three–form operator $M_{(1)3}$. At this point, the matching of the spectrum with the multiplets will be complete.

Since we are in particular interested in multiplet shortening, it is of utmost importance to pay attention to what happens with the eigenvalues in the exceptional series. As it is clear from tables (6.24), (6.38), (6.42) of the eigenvalues, there are always less eigenvalues present when the operators act on the harmonics in the exceptional series. This is reflected into the fact that certain field components are not present in the multiplets, thus multiplet shortening.

In the next sections we give a detailed discussion of the matching of the multiplets. Doing so we show that the information that we collected about the invariant operators on the zero form, the one–form, the two–form, the three–form and the spinor is in perfect agreement with the group theoretical information that was already known in [8].

7.1 The graviton multiplet

As pointed out above, the graviton multiplet is the appropriate multiplet to start with. In particular we look at the spin–two graviton field. The mass of the graviton is given by the eigenvalue of the scalar operator (see eq.s (B.3)):

$$m_h^2 = M_{(0)3} \equiv H_0.$$  (7.1)

Using table (I1) we find that its harmonics can sit in all the $G$ representations of the series

$$A_R^0, A_1^0, A_1^{*0}, A_3^0, A_3^{*0}, A_4^0, A_4^{*0}, A_6^0, A_7^0, A_8^0,$$  (7.2)

Remember that the superscripts $^0$ mean that the hypercharge is $Y = \frac{2}{3} (M_2 - M_1)$. 47
Using the group–theoretical information of the long graviton multiplet (see table 1) we find the energy and hypercharge \( (E_0, y_0) \) of the graviton multiplet:

\[
E_0 = \frac{1}{4} \sqrt{H_0 + 36 + \frac{1}{2}} \\
y_0 = \frac{2}{3} (M_2 - M_1),
\]

and using table 1 we find the energies and hypercharges of all the fields in the multiplet. In particular, we see that the gravitini are in \( U(1)_R \) representations, +, −, the \( A,W \) vectors in \( U(1)_R \) representations, 0, the \( Z \) vectors in \( U(1)_R \) representations, 0, ++, −−. From the mass of the graviton we deduce, using the mass relations in appendix B, the masses of the gravitini and vectors present in the graviton multiplet:

\[
m_{\chi^\pm} = -6 \pm \sqrt{H_0 + 36}, \\
m_A^2 = H_0 + 48 - 8 \sqrt{H_0 + 36}, \\
m_W^2 = H_0 + 48 + 8 \sqrt{H_0 + 36}, \\
m_Z^2 = H_0 + 32.
\]

From equations (B.3), we predict the presence of the eigenvalues \( M_{(1/2)^2} = m_{\chi^\pm} \) for the spinor. Indeed, looking at (6.46), we see that the two eigenvalues \( \lambda_1 \) and \( \lambda_2 \) come from spin–\( \frac{3}{2} \) fields that belong to the graviton multiplet. To find out whether there are some short graviton multiplets present in the spectrum, we now use table (6.47). The absence of these eigenvalues \( \lambda_1 \) or \( \lambda_2 \) in some of the exceptional series implies the existence of a short graviton multiplet in that particular \( G' \) series. Let us look at it more closely. For instance, for \( A_4^+ \) and \( A_5^+ \), there is none of the eigenvalues \( \lambda_1 \) or \( \lambda_2 \). This would imply a graviton multiplet without gravitino fields. But fortunately, the series \( A_2 \) and \( A_5 \) do not contain representations of \( G' \) in which there is a graviton field, see (7.2). Considering the rest of table (6.47) and also table (6.50), we find three types of graviton multiplets: a long graviton multiplet and two types of short graviton multiplets. The long graviton multiplet contains four spinors \( \chi \): \( \chi^+ \) with hypercharge \( y_0 \pm 1 \) and \( \chi^- \) with hypercharge \( y_0 \pm 1 \). They are found in the \( G' \) representations of \( A_R, A_1, A_1^+, A_3, A_3^+, A_6, A_7 \). Then there is a short graviton multiplet in the series \( A_4 \) and \( A_4^+ \). From tables (6.47) and (6.50), one sees that they contain the two \( \chi^+ \) with hypercharge \( y_0 \pm 1 \), but only one \( \chi^- \), i.e. for \( A_4 \) we have one \( \chi^- \) with \( y_0 - 1 \), and for \( A_4^+ \) we have one \( \chi^- \) with \( y_0 + 1 \). We also find the massless multiplet in \( A_8 \) for which none of the spin–\( \frac{3}{2} \) fields \( \chi^- \) are present.

At this stage, we know that the spin–\( \frac{3}{2} \) fields that correspond to the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) in (6.46) and (5.49) sit in the graviton multiplets. However, there are also spin–\( \frac{3}{2} \) fields that yield the eigenvalues \( \lambda_3 \) and \( \lambda_4 \) in (6.46) and (5.49). They can only be gravitini of the gravitino multiplets in the spectrum. So now we know the highest components of gravitino multiplets, their energies, hypercharges and \( G' \) representations. But before we continue with the gravitino multiplet, let us look at the vectors of the graviton multiplet.

\footnote{Remember that \( E_0, y_0 \) denote the energy and hypercharge of the Clifford vacuum of the multiplet}
Let us consider \(A\) and \(W\) first. We know that, if present, they should be in the series (7.2). Using equations (B.3) we see that their \(M^{(1)(0)^2}\) eigenvalues would then be

\[
M_A^{(1)(0)^2} = H_0 + 24 + 4\sqrt{H_0 + 36}, \\
M_W^{(1)(0)^2} = H_0 + 24 - 4\sqrt{H_0 + 36}.
\]

Indeed, these eigenvalues are present, namely for \(A\) we find \(\lambda_4\) and for \(W\) we find \(\lambda_5\) of eq. (6.24). To determine whether, in the exceptional series, the vector \(A\) or the vector \(W\) is present we use table (6.24). The absence of one of the vectors will imply shortening of the graviton multiplet. Studying the spin \(3/2\) fields, we have found that there are long graviton multiplets in the series \(A_R, A_1, A_3, A_4, A_6, A_7\) and short graviton multiplets in the series \(A_4, A_1^*\). This is confirmed here: in the former series both the \(A\) and \(W\) fields are present, in the latter only the field \(A\) is present. For the massless multiplet of \(A_8\) we also see that only the vector \(A\) is present.

Let us look at the vector \(Z\) in the graviton multiplet. We know that the \(Z\) vectors should be in the same \(G'\) representations of the graviton:

\[
A_R, A_1, A_3, A_4, A_6, A_7, A_8
\]

and that two \(Z\) vectors should be in the series \(0\), one in the series \(++\) and one in the series \(--\). For the operator \(M^{(1)^2(0)}\) on the two–form we predict, using eq.s (B.3), the presence the eigenvalue,

\[
M_Z^{(1)^2(0)} = H_0 + 32.
\]

Indeed, it corresponds to \(\lambda_{10}\) and \(\lambda_{11}\) in (3.29) for the series \(0\), and \(\lambda_4\) in (6.31) for series \(++\) (and \(--\), which are the series of the conjugate representations of \(++\) \((M_2 \leftrightarrow M_1)\)). So we see that for the long graviton multiplets all the vectors \(Z\) are present. Using the fact that

\[
B_R \cup B_4 \cup B_5 \cup B_6 \cup B_7 \cup B_8 \cup B_9 \cup B_{10} = A_R \cup A_1 \cup A_1^* \cup A_3 \cup A_3^* \cup A_4 \cup A_4^* \cup A_6 \cup A_7, \\
B_R^* \cup B_4^* \cup B_5^* \cup B_6^* \cup B_7^* \cup B_8^* \cup B_9^* \cup B_{10}^* = A_R \cup A_1 \cup A_1^* \cup A_3 \cup A_3^* \cup A_4 \cup A_4^* \cup A_6 \cup A_7,
\]

and tables (6.30) and (6.32) we find that for the short graviton multiplets of \(A_4\) we have two \(Z\)’s, one with hypercharge \(y\) and one with hypercharge \(y - 2\); for the short graviton multiplets of \(A_4^*\) we have two \(Z\)’s, one with hypercharge \(y\) and one with hypercharge \(y + 2\); for the massless graviton multiplet we have no vectors \(Z\).

To determine which \(\lambda_T\) fields and scalar fields \(\phi\) are present, we use the \(\mathcal{N} = 2 \rightarrow \mathcal{N} = 1\) decomposition of the multiplets. We already know where \(\lambda_T\) and \(\phi\) are located in the long graviton multiplet from table [1]. From figure [1] we see that the long \(\mathcal{N} = 2\) graviton multiplet is made by four \(\mathcal{N} = 1\) massive multiplets: one graviton, two gravitino and a vector multiplet. Harmonic analysis teaches us that in the short graviton multiplet there are three gravitino fields and three vector fields. The only possible structure of the short graviton multiplet is then the one displayed in figure [1].
The multiplet that we have found in the representation of series $A_8$ is in fact the massless graviton multiplet. In this case the field $A$ becomes the graviphoton. The final structure of the short graviton multiplet and the massless graviton multiplet is displayed in tables 4 and 8 respectively.

### 7.2 The gravitino multiplet

As already previously explained, we know the $M_{1/2}^3$ eigenvalues and the $G$ representations of the spin-$\frac{3}{2}$ in the gravitino multiplet from the matching of the graviton multiplet. Their masses are given by equations (7.10),

\[
m_{\chi^+} = -8 + \sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)} = \lambda_3
\]
\[
m_{\chi^-} = -8 - \sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)} = \lambda_4
\]  

(7.10)

for series of type $+$ and

\[
m_{\chi^+} = -8 + \sqrt{H_0 + 16 - \frac{32}{3}(M_2 - M_1)} = \lambda_3
\]
\[
m_{\chi^-} = -8 - \sqrt{H_0 + 16 - \frac{32}{3}(M_2 - M_1)} = \lambda_4
\]  

(7.11)

for series of type $-$. Each of the above four different eigenvalues gives rise to gravitino multiplets of different types and/or in different $G'$ representations. Now we look at tables (6.47) and (6.50) and see that we have gravitino multiplets for the series $A^\pm_R$ and $A^\pm_1$. We consider the gravitino multiplets in the series of type $+$ only. The gravitino multiplets in the series of type $-$ coming from (7.11) can be obtained by taking the conjugates of the gravitino multiplets in the series of type $+$.

We start with $\chi^+$ in the series of type $+$. The energy and hypercharge $(E_0, y_0)$ of the gravitino multiplets are given by,

\[
E_0 = \frac{1}{4}\sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)} - \frac{1}{2}
\]
\[
y_0 = \frac{2}{3}(M_2 - M_1) - 1
\]  

(7.12)

Let us look at the vectors in the gravitino multiplets. As we know from group theory (see table 6, 8) we should find a vector with hypercharge $y_0 + 1$ and energy $E_0 + \frac{1}{2}$, in the series $^0$. However group theory does not tell us whether it is the vector $A$ or the vector $W$. But since we know that in series of type $+$ we have $m_{\chi^+} \geq -8$, we can use the mass relations of appendix 4 to derive,

\[
m_A^2 = H_0 + \frac{32}{3}(M_2 - M_1) + 48 - 12\sqrt{H_0 + \frac{32}{3}(M_2 - M_1)} + 16
\]  

(7.13)

or

\[
m_W^2 = m_{\chi^+}^2 + 2m_{\chi^+} + 192
\]  

(7.14)
We see from table 2 that it is the $A$ vector which is present in the $\chi^+$ gravitino multiplet and not $W$. Hence, comparing with the formula (B.3) in order to find $A$, we expect the following eigenvalue

$$M^A_{(1)(0)^2} = H_0 + \frac{32}{3}(M_2 - M_1)$$

(7.15)

for the $M_{(1)(0)^2}$ operator. Looking at table (B.21) we see that it is indeed present: $\lambda_1$. Looking at table (B.24) we see that it appears in the series $A^0_{F^R}, A^1_{1}, A^2_{0}, A^3_{0}, A^4_{1}, A^5_{0}$. We also find a vector $A$ with hypercharge $y_0 - 1$ in series $\mathbf{+}$. Indeed, using

$$B_R \cup B_1 \cup B_3 \cup B_4 \cup B_6 \cup B_7 = A_R \cup A_0 \cup A_2 \cup A_3 \cup A_4 \cup A_5,$$

(7.16)

we see that $\lambda_1$ is an eigenvalue of the one–form operator $M_{(1)(0)^2}$ in series $\mathbf{+}$. Both the spin–1 fields $A$ with $y_0 - 1$ and $y_0 + 1$ of the gravitino multiplet for $\chi^+$ are present and there are no other left with eigenvalue (7.15). For the vector $Z$ with hypercharge $y_0 - 1$, we expect the presence of two states with mass (using the table 1)

$$m^2_Z = H_0 + 16 + \frac{32}{3}(M_2 - M_1) - 4\sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)},$$

(7.17)

one in the $G$ representations of type $^0$, the other in the representations $\mathbf{+}$ or $\mathbf{−}$ (depending on the $G$ representation of the gravitino). The mass (7.17) corresponds to $\lambda_{7}$ in (6.29) and $\lambda_{3}$ in (6.31). From this we see that $Z$ is present except for series $A_5$, and series $B_7$. The series $A_5$ and $B_7$ have no overlap. So we conclude that we have long gravitino multiplets except if the multiplet sits in a representation of $A_5$ or $B_7$. For the gravitino multiplet with $\chi^+$ in the series $\mathbf{+}$, we now look at the mass of the scalar $\pi$,

$$m^2_\pi = 16 \left( \frac{1}{4}\sqrt{H_0 + 16 + \frac{32}{3}} - 1 \right) \left( \frac{1}{4}\sqrt{H_0 + 16 + \frac{32}{3}} - 2 \right)$$

(7.18)

From eq.s (B.3) we predict the eigenvalue

$$M^\pi_{(1)^2} = \frac{1}{4}\sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)}$$

(7.19)

which we do find as $\lambda_1$ in (6.37) in series $A^0_{F^R}, A^0_{R}, A^2_{0}, A^3_{0}, A^4_{0}$ (see (6.38)) and as $\lambda_1$ in (6.41) in the series $B^0_{R}, B^1_{1}, B^3_{0}, B^4_{1}, B^5_{0}$ (see (6.42)). So none of the fields $\pi$ with $y_0 - 1$ and $y_0 + 1$ is present in the short gravitino multiplets with $\chi^+$ in the series $\mathbf{+}$. Let us now consider the spin–$\frac{1}{2}$ field $\lambda^\xi_+$. Looking at the expansion (B.4), we see that $\lambda_{L}$ appears in the expansion of the spinor. So we can check whether it is present in the gravitino multiplet with $\chi^+$ in the series $\mathbf{+}$. We know its mass from group theory [8],

$$m_{\lambda^\xi_+} = -8 + \sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)},$$

(7.20)

so, from eq.s (B.3) we expect the eigenvalue

$$M^\lambda_{(1/2)^2} = -8 - \sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)}$$

(7.21)

which we do find as $\lambda_4$ in (6.46) in $A^0_{F^R}, A^1_{+}, A^2_{+}, A^3_{+}, A^4_{+}$ (see (6.47)). So the field $\lambda^\xi_+$ is present in both long and short gravitino multiplets with hypercharge $y_0$. In fact it has to
be there since it provides the Clifford vacuum of the representation. For the short gravitino multiplets we have found which of the fields $\phi$ and $\lambda_T$ are present by using the $N = 2 \rightarrow N = 1$ decomposition (see figures 2, 3) and by calculating the norms of the states using creation and annihilation operators [40, 8]. The result is displayed in table 5.

Let us consider $\chi^-$ for the series of type $\pm$. It has mass $m_{\chi^-}$ from (7.10). The energy and hypercharge ($E_0$, $y_0$) of the multiplet are

\[
E_0 = \frac{1}{4}\sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)} + \frac{3}{2},
\]

\[
y_0 = \frac{2}{3}(M_2 - M_1) - 1. \tag{7.22}
\]

We now have $m_{\chi^-} \leq -8$. So, using the mass relations for $W$ we find

\[
m^2_{W} = H_0 + \frac{32}{3}(M_2 - M_1) + 48 + 12\sqrt{H_0 + \frac{32}{3}(M_2 - M_1) + 16}. \tag{7.23}
\]

Thus in this case it is $W$ that is present and not $A$. We find the same eigenvalue (7.15), so we conclude that $W$ is present in all types of gravitino multiplets with $\chi^-$ in series of type $\pm$. For $Z$ we have

\[
m^2_{Z} = H_0 + 16 + \frac{32}{3}(M_2 - M_1) + 4\sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)} \tag{7.24}
\]

which, according to eq.s (3.3), has to be an eigenvalue of the two–form mass operator. Indeed, for series of type $0$ it corresponds to $\lambda_6$, which is present in series $A_R, A_1, A_2, A_3, A_4, A_5$ (see (5.30)). Notice that these are the same series of representations as the ones in which we found $\chi^+$. For the series $++$ we find $\lambda_2$, which is present in the series $B_R, B_1, B_3, B_4, B_6, B_7$ (see (5.32)), which are again the same series of representations as for $\chi^+$. The fields $\pi$ present have mass,

\[
m^2_{\pi} = 16(-\frac{1}{4}\sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)} - 1)(-\frac{1}{4}\sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)} - 2) \tag{7.25}
\]

So we predict the eigenvalue

\[
M^\pi_{(1)3} = -\frac{1}{4}\sqrt{H_0 + 16 + \frac{32}{3}(M_2 - M_1)} \tag{7.26}
\]

Indeed it is $\lambda_3$ in (5.37), present in the series $A_R, A_1, A_2, A_3, A_4, A_5$ (5.38) and $\lambda_2$ in (6.42), present in the series $B_R, B_1, B_3, B_4, B_6, B_7$ (6.41). We conclude that all the gravitino multiplets with $\chi^-$ are long gravitino multiplets.

### 7.3 The vector multiplet

What are the vector field we have been left with? They have to be the highest components of the vector multiplets. Well, we have a multiplet with highest component vector $A$ with eigenvalue $\lambda_5$ in (6.21). We have a vector multiplet with highest vector component $W$ with eigenvalue $\lambda_4$ in (6.21). We have some vector multiplets with highest vector component $Z$ with
eigenvalues $\lambda_3$ in (6.29), $\lambda_5$ in (6.31) and $\lambda_5^*$ in the series $^-$. All these eigenvalues give rise to the existence of different types of vector multiplets in different representations of $G'$.

Let us start with $A$. We call this the $A$–vector multiplet. It has eigenvalue $\lambda_5$ in (6.21). Its energy and hypercharge are

$$\begin{align*}
E_0 &= \frac{1}{4}\sqrt{H_0 + 36} - \frac{3}{2} \\
y_0 &= \frac{2}{3}(M_2 - M_1)
\end{align*}$$

and the mass of the field component $A$ is

$$m_A^2 = H_0 + 96 - 16\sqrt{H_0 + 36}.$$  

This eigenvalue is present in the series $A^0_R, A^0_A, A^0_1, A^0_2, A^0_3, A^0_4, A^0_6, A^0_7$. We now figure out which of these there is shortening. From the table in [8] we see that $\pi$ has the same mass as $A$ (7.28), and using eq.s (3.3) we conclude that we should find the eigenvalues

$$M_\pi^{(1)} = \frac{1}{4}\sqrt{H_0 + 36} - \frac{1}{2},$$

which is present: $\lambda_5$ in $A^0_R, A^0_A, A^0_1, A^0_2, A^0_3, A^0_4, A^0_6, A^0_7$ (6.37) (6.38). It is also present as $\lambda_3$ in $B^+_R, B^+_4, B^+_5, B^+_6, B^+_7, B^+_8, B^+_9, B^+_10$ (6.41) (6.42). Considering (7.9) this seems strange at first sight. However, what happens is that here we discover a scalar $\pi$ in the series $A_4$ of a hypermultiplet. We can see this as follows. Suppose the eigenvalue were also present in series $B_8$ and series $B_{10}$. Then the eigenvalue $\lambda_3$ would appear in the representations of $B$ that are on the right–hand side of (7.9). So we would find the field $\pi$ in the $G'$ representations $A_R, A_1, A_2, A_3, A_4, A_6, A_7$ and in $A_4$, with $Y = \frac{2}{3}(M_2 - M_1) - 2$. The series $A_4$ and $B_8$ and $B_{10}$ have no overlap. Consequently, the $\pi$ in $A_4$ can not belong to the $A$–vector multiplet and thus has to be a scalar of a hypermultiplet. Similarly, we find $\pi$ in $B^{-*-}_R, B^{-*-}_4, B^{-*-}_5, B^{-*-}_6, B^{-*-}_7, B^{-*-}_8, B^{-*-}_9, B^{-*-}_10$. With the same reasoning, we conclude that $\pi$ in $A^*_1$ with $Y = \frac{2}{3}(M_2 - M_1) + 2$ has to be a scalar of some hypermultiplet. However, $\lambda_3$ does not sit in the series $B_8, B_9, B_{10}, B_{10}^*$.

So we conclude that that we get shortening in these series. Now we get different types of short vector multiplets. This is due to fact the $B_8$ and $B_8^*$ have overlap, namely if $M_1 = M_2 = 1, J = 0$ and that also $B_{10}$ and $B_{10}^*$ have overlap, namely for the representation $M_1 = M_2 = 0, J = 1$. For the representations in the series $B_8$ and $B_{10}$ with $M_1 > M_2 = 1$, we find that the field $\pi$ with hypercharge $y - 2$ in the long vector multiplet decouples. The representations

$$\begin{align*}
M_1 &= M_2, \quad J = 1 \\
M_1 &= M_2 = 1, \quad J = 0
\end{align*}$$

yield massless vector multiplets. They contain the vectors that gauge $SU(2)$ and $SU(3)$ respectively.

Let us now figure out whether we can learn something about the presence of $\phi$, $S$ and $\Sigma$ in the $A$–vector multiplet. The table in [8] gives the mass,

$$m_{\phi,S/\Sigma}^2 = 16E_0(E_0 + 1) = H_0 + 48 - 4\sqrt{H_0 + 36}.$$  

(7.31)
Looking at eq.s (7.3), we see that the entry in the table can not be $S$ or $\Sigma$, but has to be $\phi$. If we look at the other $\phi, S/\Sigma$ in the table with mass

$$m_{\phi,S/\Sigma}^2 = 16 (E_0 - 2)(E_0 - 1) = H_0 + 176 - 24\sqrt{H_0 + 36}, \quad (7.32)$$

we see that it is the mass for the field $S$. So at this place in the table we find the field $S$. The field $S$ is found in the series $A_0^0, A_1^0, A_1^0, A_0^0, A_3^0, A_4^0, A_4^0, A_4^0, A_4^0, A_4^0, A_5^0, A_5^0, A_6^0, A_7^0, A_8^0$. So it is always present in the $A$–vector multiplets. Besides, we get some extra $S$–fields that are to be put in the hypermultiplets in the series $A_4, A_4, A_8$.

To conclude the discussion of the $A$–vector multiplet, there is shortening of $A$–vector multiplets in series $B_8, B_8^*$ and $B_{10}, B_{10}^*$. In the representation (7.30) there are massless vector multiplets, in the other $B_8, B_8^*, B_{10}, B_{10}^*$ representations there are short vector multiplets. The $\phi$ and $\lambda_T$ contents of the short vector multiplets can be determined by using the $N = 2 \rightarrow N = 1$ decomposition (see pictures 3,8). The structure of the long vector multiplet and the short vector multiplet is displayed in table 8 and 9 respectively.

Let us now consider the vector multiplet with highest vector component $W$. We will call this the $W$–vector multiplet. We expect eigenvalue $\lambda_5$ in (6.21) and (6.22), which we find in series $A_0^0, A_1^0, A_1^0, A_3^0, A_3^0, A_4^0, A_4^0, A_6^0, A_7^0, A_8^0$. This multiplet has energy and hypercharge,

$$E_0 = \frac{1}{4}\sqrt{H_0 + 36} + \frac{5}{2},$$
$$y_0 = \frac{2}{3}(M_2 - M_1), \quad (7.33)$$

the $W$ field has mass

$$m_W^2 = H_0 + 96 + 16\sqrt{H_0 + 36}. \quad (7.34)$$

For the fields $\pi$, we expect to find the eigenvalues $\lambda_6$ in series $A_0^0, A_1^0, A_1^0, A_3^0, A_3^0, A_4^0, A_4^0, A_6^0, A_7^0, A_8^0$ (7.37) (7.38), and $\lambda_4$ in series $B_4^{++}, B_4^{++}, B_5^{++}, B_6^{++}, B_7^{++}, B_8^{++}, B_9^{++}, B_{10}^{++}, B_{11}^{++}$ (6.41), and $\lambda_4^*$ in series $B_4^{--}, B_4^{--}, B_5^{--}, B_6^{--}, B_7^{--}, B_8^{--}, B_9^{--}, B_{10}^{--}, B_{11}^{--}$. Using

$$B_{11} = A_4 \cup A_8,$$
$$B_{11}^* = A_4^* \cup A_8, \quad (7.35)$$

and (7.3), we see that all these $0, ++,$ and $--$ series coincide. Thus all the fields $\pi$ in the table of $8$ are always present and we find no fields $\pi$ that have to be put in other multiplets. So the $W$–vector multiplet is always long. Which of the fields $\phi, S/\Sigma$ are present? Let us look at $\phi, S/\Sigma$ with mass

$$m_{\phi,S/\Sigma}^2 = 16 E_0(E_0 + 1) = H_0 + 176 + 24\sqrt{H_0 + 36}. \quad (7.36)$$

From eq.s (3.3) we see that it is the field $\Sigma$ that is present in the series $A_0^0, A_1^0, A_1^0, A_3^0, A_3^0, A_4^0, A_4^0, A_4^0, A_4^0, A_4^0, A_6^0, A_7^0, A_8^0$. So this confirms that there is no shortening and we do not find any extra fields $\Sigma$ that are to be put in the hypermultiplets. Let us look at $\phi, S/\Sigma$ with mass

$$m_{\phi,S/\Sigma}^2 = 16 (E_0 - 2)(E_0 - 1) = H_0 + 48 + 8\sqrt{H_0 + 36}. \quad (7.37)$$

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This can only be the field $\phi$. So we conclude that the $W$–vector multiplets are always long vector multiplets. And there are no scalar left that have to be put in hypermultiplets. Its structure is displayed in table 3.

Let us now look at the $Z$–vector multiplet with eigenvalue $\lambda_3$ in series $A_R, A_1, A_1^*, A_6, A_8$ (6.29) (6.30). The multiplet has energy and hypercharge

$$E_0 = \frac{1}{2} \sqrt{H_0 + 4 + \frac{1}{2}},$$
$$y_0 = \frac{2}{3} (M_2 - M_1),$$

the field $Z$ has mass

$$m_Z^2 = H_0.$$  

What about the two fields $\pi$? Let us look at $\pi$ with mass

$$m_\pi^2 = 16 \left(E_0 - 2 \right)(E_0 - 1),$$

From eq.s (B.3) we expect there to be $\lambda_7$ in (6.37). Indeed, it is present in series $A_8$. For $\pi$ with mass

$$m_\pi^2 = 16 \left(E_0 - 2 \right)(E_0 - 1),$$

we find $\lambda_8$ in series $A_R, A_1^0, A_1^{0*}, A_6^0, A_8^0$. So finally, we conclude that for this type of $Z$–vector multiplet (with $\lambda_3$ in (6.29)) there is shortening in series $A_8$. The structure of the long $Z$–vector multiplet and the massless Betti multiplet is displayed in table 3 and 9 respectively.

Let us now look at the $Z$–vector multiplet with $\lambda_5$ in (6.31). It appears in series $B_R, B_1, B_2$ (6.32). The multiplet has energy and hypercharge

$$E_0 = \frac{1}{4} \sqrt{H_0 + \frac{64}{3} (M_2 - M_1) - 28 + \frac{1}{2}}$$
$$y_0 = \frac{2}{3} (M_2 - M_1) - 2,$$

the field $Z$ has mass

$$m_Z^2 = H_0 + \frac{64}{3} (M_2 - M_1) - 32.$$  

What about the presence of the fields $\pi$? For $\pi$ with mass

$$m_\pi^2 = 16 \left(E_0 - 2 \right)(E_0 - 1),$$

we expect the eigenvalue $\lambda_5$ in (1.41), which is found in the series $B_{R}^{++}, B_{1}^{++}, B_{2}^{++}$ (6.42). For $\pi$ with mass

$$m_\pi^2 = 16 E_0 (E_0 + 1),$$

we expect $\lambda_6$ in (6.41), which is found in the series $B_{R}^{++}, B_{1}^{++}, B_{2}^{++}$ (6.42). So we conclude that for the $Z$–vector multiplet (with vector $Z$ with eigenvalue $\lambda_5$ in (6.31)), there is never shortening. We do not find extra scalars that are to be put in hypermultiplets either. The structure of this long $Z$ vector multiplet is displayed in table 3.

For the $Z$–vector multiplet with $\lambda_5$ in series $B_R, B_1, B_2$, one just takes the conjugate of the previous results.
7.4 The hypermultiplet

After having put the scalars $\pi$ in the right places in the graviton, the gravitino and the vector multiplet, we are only left with scalars $\pi$ in series $A^0_4$ and $A^{0*}_4$ and $S$ in series $A^0_4 A^{0*}_4 A^0_8$.

So for each representation of $A_4$ we find a hypermultiplet with energy

$$E_0 = \frac{1}{4} \sqrt{H_0 + 36 - \frac{3}{2}}$$

(7.46)

containing the field $\pi$ with hypercharge $Y = \frac{2}{3} (M_2 - M_1) - 2$ and mass

$$m^2_\pi = H_0 + 96 - 16 \sqrt{H_0 + 36}$$

(7.47)

and the field $S$ with $Y = \frac{2}{3} (M_2 - M_1)$ and mass

$$m^2_S = H_0 + 176 - 24 \sqrt{H_0 + 36}$$

(7.48)

The scalars of this hypermultiplet are complete if we add the scalars $\pi$ and $S$ of $A^*_4$, which are in fact the complex conjugates of the scalars in $A_4$. From the eigenvalues of the operator $M_{(1/2)}^3$ we find the $\lambda_L$ necessary to fill all the hypermultiplets. The structure of the hypermultiplets is displayed in the table [1].

In order to correctly match the fields with the multiplets, it is important to note that in the singlet $G$ representation $M_1 = M_2 = J = Y = 0$ the scalar $S$ is absent. This is due to the fact that, from the Kaluza Klein expansion of the eleven-dimensional field $h_{\mu\nu}(x, y)$, the scalar $S$ appears in the expressions $(6 - \sqrt{M_{(0)}^3 + 36}) S^I(x)$ and $D_m D_n (2 + \sqrt{M_{(0)}^3 + 36}) S^I(x)$. The coefficient of the former, $6 - \sqrt{M_{(0)}^3 + 36}$, disappears in the singlet representation. The latter become a pure gauge term, due to the freedom of coordinate reparametrization, being the graviton in the singlet $G$ representation the massless graviton.

At this point we have done the complete matching of the multiplets with the spectrum of Laplace Beltrami operators. It is reassuring that all the fields we have found have been organized in $\mathcal{N} = 2$ $AdS_4$ multiplets. An important result is that we have established the existence of short multiplets. From the expressions of the energies and hypercharges ($E_0, y_0$) we have found, we can easily derive that

- for all the long multiplets
  $$E_0 > |y_0| + s_0 + 1$$

- for all the short graviton, gravitino and vector multiplets
  $$E_0 = |y_0| + s_0 + 1$$

- for all the hypermultiplets
  $$E_0 = |y_0| \geq \frac{1}{2}$$
\begin{itemize}
  \item for all the massless multiplets
  \[ E_0 = s_0 + 1 \quad y_0 = 0. \]
\end{itemize}

This confirms the algebraic lore on multiplet shortening from the literature \[40, 8\].

8 Outlook and plan for future development

In this paper we have constructed the complete Kaluza Klein spectrum of the \( AdS_4 \times M^{111} \) compactification, organizing it into \( Osp(2|4) \) supermultiplets. As stressed in the introduction this must be regarded both as the completion of an outstanding problem and as a first essential step along a further research plan that can be outlined as follows:

\begin{itemize}
  \item Construction of the three \( \mathcal{N} = 2 \) multiplet spectra:
    \begin{enumerate}
      \item \( Osp(2|4) \times SU(3) \times SU(2) \) for \( M^{111} \)
      \item \( Osp(2|4) \times SO(5) \) for \( V_{5,2} \)
      \item \( Osp(2|4) \times SU(2)^3 \) for \( Q^{111} \)
    \end{enumerate}
  \item Construction of the single \( \mathcal{N} = 3 \) multiplet spectrum, corresponding to the \( N^{010} \) space. In this case the Kaluza Klein states are organized in \( Osp(3|4) \times SU(3) \) multiplets.
  \item Search of the appropriate Kählerian conifold that leads to the three dimensional conformal field theory corresponding to each of the above cases.
  \item Test of the \( AdS/CFT \) conjecture at the level of both the long and the short multiplets.
  \item Determination of the effective low energy \( \mathcal{N} = 2 \) lagrangians, namely search of the appropriate Special Kähler manifold and of its geometrical interpretation as moduli space of an appropriate structure.
  \item As for the \( \mathcal{N} = 3 \) theory, here supersymmetry is already strong enough to predict the structure of the effective Lagrangian. As shown in \[38\] the \( 3 \times 8 \) complex scalars of the \( N^{010} \) compactification will fill the coset manifold:
    \begin{equation}
    \frac{SU(3, 8)}{SU(3) \times SU(8) \times U(1)} \tag{8.1}
    \end{equation}
\end{itemize}

The final goal of our research plan is to provide new non trivial checks of the \( AdS/CFT \) correspondence in cases where the spectrum of primary conformal operators is not too much restricted by supersymmetry. In particular we are interested in exploring properties of non trivial three–dimensional conformal field theories by means of classical supergravity on anti de Sitter compactifications \( AdS_4 \times X_7 \) where \( X_7 \) are suitable 7-manifolds with interesting geometrical structures and to establish the relation between the quantum aspects of the 3D field theory and the geometry of \( X_7 \).
A  Conventions used in the calculations

The Gell–Mann matrices are:

\[
\lambda_1 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda_2 = \begin{pmatrix}
0 & -i & 0 \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix},
\lambda_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda_4 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
\lambda_5 = \begin{pmatrix}
0 & 0 & -i \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda_6 = \begin{pmatrix}
0 & 0 & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda_7 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -i \\
i & 0 & 0
\end{pmatrix},
\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix}.
\]

(A.1)

The Pauli matrices are:

\[
\sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\sigma_2 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix},
\sigma_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

(A.2)

Furthermore, in order to follow the notations of [5] we define

\[
\sigma_3 \equiv -\sigma_3.
\]

(A.3)

The generators of \( G = SU(3)^c \times SU(2)^w \times U(1) \) are:

\[
SU(3)^c : \quad \frac{i}{2} \lambda_1, \ldots, \frac{i}{2} \lambda_8
\]

\[
SU(2)^w : \quad \frac{i}{2} \sigma_1, \ldots, \frac{i}{2} \sigma_3
\]

\[
U(1) : \quad iY_3
\]

The orthogonal decomposition gives

\[
G = \mathbb{H} \oplus \mathbb{K}
\]

(A.4)

where \( \mathbb{H} \) is a subalgebra of \( G \), and \( \mathbb{K} \) is a representation of \( \mathbb{H} \).

The generators of \( H = SU(2)^c \times U(1)' \times U(1)'' \) are:

\[
SU(2)^c : \quad \frac{i}{2} \lambda_m = \frac{i}{2} \lambda_1, \ldots, \frac{i}{2} \lambda_3
\]

\[
U(1)' : \quad Z' = \sqrt{3} i \lambda_8 + i \sigma_3 - 4 i Y_3
\]

\[
U(1)'' : \quad Z'' = -\frac{\sqrt{3}}{2} i \lambda_8 + \frac{3}{2} i \sigma_3
\]

\( ^6 \)the subfix \( _i \) has been given in order to follow the notations of [5]
so the generators of the orthogonal space $\mathbf{K}$ are
\[
\frac{i}{2} \lambda_A = \frac{i}{2} \lambda_1 \ldots \frac{i}{2} \lambda_7, \\
\sigma_m = \frac{i}{2} \sigma_1, \frac{i}{2} \sigma_2 \\
Z = \frac{\sqrt{3}}{2} i \lambda_8 + \frac{1}{2} i \sigma_3 + i Y_3.
\] (A.5)

Due to this decomposition we divide the indices into six groups:
\[
\hat{m}, \hat{n} = 1, 2, 3, \\
m, n = 1, 2, 3, \\
A, B, C = 4, 5, 6, 7, \\
8
\] (A.6)

Other indices used in this paper are:
\[
\Sigma, \Lambda : \text{indices of the adjoint representation of } G \\
\alpha, \beta : \text{indices of the vector representation of } SO(7) \\
i, j : \text{indices of the vector representation of } SU(2)^c
\] (A.7)

Out conventions for the $\varepsilon$ tensors are the following:
\[
SU(2)^W : \varepsilon^{mn} = 1, \varepsilon^{12} = -1 \\
SU(3)^c : \varepsilon^{\hat{m}\hat{n}\hat{r}} = 1, \varepsilon^{123} = 1 \\
SU(2)^c : \varepsilon^m = 1 \\
SO(7)^c : \varepsilon^{\alpha\beta\gamma\delta\mu\nu} = 1, \varepsilon^{1234567} = -1
\] (A.8)

**B  Main formulas of harmonic analysis**

The fields of the four dimensional Kaluza Klein theory are defined in the expansion the eleven dimensional fields in $M^{111}$ harmonics [33, 34]:
\[
h_{mn} (x, y) = \left( h_{mn}^I (x) - \frac{3}{M_{(0)}^3 + 32} D_{(m} D_{n)} \right) \left[ (2 + \sqrt{M_{(0)}^3 + 36}) \Sigma^I (x) + (2 - \sqrt{M_{(0)}^3 + 36}) \Sigma^I (x) \right] + \frac{2}{3} \delta_{mn} \left[ (6 + \sqrt{M_{(0)}^3 + 36}) S^I (x) + (6 - \sqrt{M_{(0)}^3 + 36}) S^I (x) \right] Y^I (y),
\]
\[
h_{ma} (x, y) = \left[ (\sqrt{M_{(1)}^2 + 16} - 4) A_m^I (x) + (\sqrt{M_{(1)}^2 + 16} + 4) W_m^I (x) \right] Y_a^I (y),
\]
\[
h_{ab} (x, y) = \phi^I (x) Y_{(ab)}^I (y) - \delta_{ab} \left[ (6 - \sqrt{M_{(0)}^3 + 36}) S^I (x) + \right.
\]

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\[ (6 + \sqrt{M(0)^3 + 36}) \Sigma^I(x) Y^I(y) , \]
\[ a_{mnr}(x, y) = 2 \epsilon_{mnrp} D_p (S^I(x) + \Sigma^I(x)) Y^I(y) , \]
\[ a_{mna}(x, y) = \frac{2}{3} \epsilon_{mnrs} (D_r A^I_s(x) + D_r W^I_s(x)) Y^I_a(y) , \]
\[ a_{mab}(x, y) = Z^I_m(x) Y^I_{[ab]}(y) , \]
\[ a_{abc}(x, y) = \pi^I(x) Y^I_{[abc]}(y) , \]
\[ \psi_m(x, y) = (\lambda^I_m(x) + \frac{1}{2} M(1/2)^3 + 8 [D_m \lambda^I_L(x)]_{3/2} - \\
(6 + \frac{2}{3} M(1/2)^3) \gamma_5 \gamma_m \lambda^I_L(x)) \Xi^I(y) , \]
\[ \psi_a = \lambda^I_T(x) \Xi^I_a(y) + \lambda^I_L(x) [\nabla_a \Xi^I(y)]_{3/2} , \]

(B.1)

where we call \( x \) the coordinates of the four dimensional space, and \( y \) the coordinates of the internal compact space.

The convention for the names of the eigenvalues of the transverse harmonics are the following:

| Harmonic  | Eigenvalue         |
|-----------|--------------------|
| \( Y^I \) | \( M(0)^3 \)      |
| \( Y^I_a \) | \( M(1)(0)^2 \) | (B.2)
| \( Y^I_{[ab]} \) | \( M(1)^2(0) \)  |
| \( Y^I_{[abc]} \) | \( M(1)^3 \)     |
| \( Y^I_{(ab)} \) | \( M(2)(0)^2 \)  |
| \( \Xi^I \) | \( M(\frac{1}{2})^3 \) |
| \( \Xi^I_a \) | \( M(\frac{1}{2})(\frac{1}{2})^2 \) |

In the previous expansion, to each eigenvalue of a harmonic does correspond an \( AdS_4 \) field of mass:

\[ m_h = M(0)^3 , \]
\[ m^2_{s'} = M(0)^3 + 176 + 24 \sqrt{M(0)^3 + 36} , \]
\[ m^2_s = M(0)^3 + 176 - 24 \sqrt{M(0)^3 + 36} , \]
\[ m^2_{\phi} = M(2)(0)^2 , \]
\[ m^2_{\pi} = 16 \left( M(1)^3 - 2 \right) \left( M(1)^3 - 1 \right) , \]
\[ m^2_W = M(1)(0)^2 + 48 + 12 \sqrt{M(1)(0)^2 + 16} , \]
\[ m^2_A = M(1)(0)^2 + 48 - 12 \sqrt{M(1)(0)^2 + 16} , \]
\[ m^2_Z = M(1)^2(0) , \]
\[ m_{\lambda_L} = - \left( M(\frac{1}{2})^3 + 16 \right) , \]
\[ m_{\lambda_T} = M(\frac{1}{2})(\frac{1}{2})^2 + 8 \]
\[ m_{\chi} = M^{(1/2)^3}. \] (B.3)

In order to determine the matching of the spectrum with the \( \mathcal{N} = 2 \) multiplets and the structure of the \( \mathcal{N} = 2 \) multiplets themselves the following mass relations \[6\], \[39\], due to the supersymmetry relations between the various fields, have been used

\[
\begin{align*}
m^2_\pi & = m_{\chi}(m_{\chi} + 12), \\
m^2_3 & = m_{\chi}(m_{\chi} + 4) \quad \text{if} \quad m_{\chi} \geq -8, \\
m^2_A & = m^2_{\chi} + 2m_{\chi} + 192 \quad \text{if} \quad m_{\chi} \leq -8, \\
m^2_W & = m^2_{\chi} + 2m_{\chi} + 192 \quad \text{if} \quad m_{\chi} \geq -8, \\
m^2_W & = m_{\chi}(m_{\chi} + 4) \quad \text{if} \quad m_{\chi} \leq -8, \\
m^2_Z & = (m_{\chi} + 8)(m_{\chi} + 4),
\end{align*}
\] (B.4)

\[
\begin{align*}
m^2_{\pi} & = m_{\lambda_T}(m_{\lambda_T} + 4), \\
m^2_{\phi} & = m_{\lambda_T}(m_{\lambda_T} - 4), \\
m^2_A & = m^2_{\lambda_T} - 20 m_{\lambda_T} + 96 \quad \text{if} \quad m_{\lambda_T} \geq 4, \\
m^2_A & = m_{\lambda_T}(m_{\lambda_T} + 4) \quad \text{if} \quad m_{\lambda_T} < 4, \\
m^2_W & = m_{\lambda_T}(m_{\lambda_T} + 4) \quad \text{if} \quad m_{\lambda_T} \geq 4, \\
m^2_W & = m^2_{\lambda_T} - 20 m_{\lambda_T} + 96 \quad \text{if} \quad m_{\lambda_T} < 4, \\
m^2_Z & = m_{\lambda_T}(m_{\lambda_T} - 4),
\end{align*}
\] (B.5)

\[
\begin{align*}
m^2_{\pi} & = m_{\lambda_L}(m_{\lambda_L} + 4), \\
m^2_S & = (m_{\lambda_L} + 24)(m_{\lambda_L} + 20) \quad \text{if} \quad m_{\lambda_L} < -10, \\
m^2_S & = m_{\lambda_L}(m_{\lambda_L} - 4) \quad \text{if} \quad m_{\lambda_L} \geq -10, \\
m^2_S & = m_{\lambda_L}(m_{\lambda_L} - 4) \quad \text{if} \quad m_{\lambda_L} < -10, \\
m^2_Z & = (m_{\lambda_L} + 24)(m_{\lambda_L} + 20) \quad \text{if} \quad m_{\lambda_L} \geq -10, \\
m^2_A & = m^2_{\lambda_L} - 2 m_{\lambda_L} + 192 \quad \text{if} \quad m_{\lambda_L} < -8, \\
m^2_A & = m_{\lambda_L}(m_{\lambda_L} + 4) \quad \text{if} \quad m_{\lambda_L} \geq -8, \\
m^2_W & = m_{\lambda_L}(m_{\lambda_L} + 4) \quad \text{if} \quad m_{\lambda_L} < -8, \\
m^2_W & = m^2_{\lambda_L} - 2 m_{\lambda_L} + 192 \quad \text{if} \quad m_{\lambda_L} \geq -8.
\end{align*}
\] (B.6)

These supersymmetry relations are pictorially represented in Figure B.
Figure 8: Supersymmetry relations between the Kaluza Klein fields: for every couple of fields linked by an arrow there is a mass relation descending by supersymmetry.

In the construction of the \( \mathcal{N} = 2 \) multiplets, the following equations (see \([3]\)), that give the masses of the \( Osp(4|2) \) fields in terms of their energies, has been used

\[
\begin{align*}
m_{(0)}^2 &= 16 \left( E_{(0)} - 2 \right) \left( E_{(0)} - 1 \right), \\
m_{(\frac{1}{2})}^2 &= 4E_{(\frac{1}{2})} - 6, \\
m_{(1)}^2 &= 16 \left( E_{(1)} - 2 \right) \left( E_{(1)} - 1 \right), \\
m_{(\frac{3}{2})}^2 + 4 &= 4E_{(\frac{3}{2})} - 6 .
\end{align*}
\]

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