HALF CONFORMALLY FLAT
GENERALIZED QUASI-EINSTEIN MANIFOLDS

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Abstract. We provide classification results for and examples of half conformally flat generalized quasi Einstein manifolds of signature $(2, 2)$. This analysis leads to a natural equation in affine geometry called the affine quasi-Einstein equation that we explore in further detail.

1. Introduction

The analytical study of differential equations often focuses on the existence and uniqueness of solutions on a given domain. From a geometric point of view, the converse question is also of interest. Given a differential equation, one may look for a manifold that supports a non-trivial solution and one might ask about the local/global geometry of the manifold, thus leading to an analytical characterization of a manifold structure by a differential equation if this manifold corresponds to a unique domain where the given equation has a non-trivial solution. Clearly one may not expect any positive answer for arbitrary equations but there are important examples when the equation has some geometrical/physical meaning. The equation of Obata [43] is a typical example; see also the discussion in [27, 44, 47].

Let $\rho$ be the Ricci tensor of a pseudo-Riemannian manifold $M = (M, g)$. If $f$ is a smooth function on $M$, let $\text{Hes}_f$ be the Hessian. Both $\rho$ and $\text{Hes}_f$ are $(0, 2)$-tensor fields on $M$; we refer to Section 1.5 for a precise definition. The generalized quasi-Einstein equation links these two objects with the metric tensor in a very natural fashion. This single equation extends equations studied previously such as the equation of Obata [43], the Möbius equation [49], the Einstein equation, and the gradient Ricci soliton equation as we shall see in the discussion given below. In this paper, we examine the generalized quasi-Einstein equation (see Equation (1) below) in the setting of half conformally flat manifolds of signature $(2, 2)$.

Definition 1. A quadruple $(M, g, f, \mu)$, where $(M, g)$ is a pseudo-Riemannian manifold of dimension $n$, $f$ is a smooth function on $M$, and $\mu \in \mathbb{R}$, is said to be a generalized quasi-Einstein manifold if the tensor $\text{Hes}_f + \rho - \mu df \otimes df$ is a multiple of the metric, i.e. if the following equation (which is called the generalized quasi-Einstein equation) is satisfied:

\begin{equation}
H_{ef} + \rho - \mu df \otimes df = \lambda g \quad \text{for some } \lambda \in C^\infty(M).
\end{equation}

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There are several interesting families of generalized quasi-Einstein manifolds that have been considered in the literature previously:

**Example 2 (Einstein manifolds).** One can recover the Einstein equation by letting $f$ be constant in Equation (1). Consequently any Einstein manifold is in fact a generalized quasi-Einstein manifold. Suppose on the other hand that $\mathcal{M}$ is Einstein. We consider Equation (1) for $\mu \neq 0$. The change of variable $h = e^{\mu f}$ provides the equivalent equation $\frac{1}{\mu h} \text{Hes}_h + \rho = \lambda g$. Let $\tau$ be the scalar curvature; as $\mathcal{M}$ is Einstein, $\rho = \frac{\tau}{n} g$. Multiplying by $\mu h$ converts the relation $\frac{1}{\mu h} \text{Hes}_h + \rho = \lambda g$ into the equation

$$
(2) \quad \text{Hes}_h + \mu h (\frac{\tau}{n} - \lambda) g = 0.
$$

This is precisely the Equation of Möbius, where $\Delta h = \mu (\tau - n \lambda) h$ (see, for example, [49]). Moreover, if $\lambda$ is constant, then Equation (2) resembles the equation of Obata $\text{Hes}_h + \kappa h g = 0$ since $\kappa = \mu (\frac{\tau}{n} - \lambda)$ is a constant (see [49]).

**Example 3 (Gradient Ricci almost solitons).** For $\mu = 0$, Equation (1) corresponds to the gradient Ricci almost soliton equation (see, for example, [2] [3] [46]). In particular, if $\lambda$ is constant, then one obtains the gradient Ricci soliton equation (see [14] [17] [26] [41] and references therein), which identifies self-similar solutions of the Ricci flow: $\frac{\partial}{\partial t} g(t) = -2\rho(t)$. Although gradient Ricci solitons are a special case of quasi-Einstein metrics, they exhibit quite different properties (see [18]). We emphasize that the gradient Ricci almost soliton equation is not just a formal generalization of the Ricci soliton equation, but includes families of self-similar solutions of other geometric flows such as the Ricci-Bourguignon flow [21]. This flow is defined for a $\kappa \in \mathbb{R}$ by the evolution equation $\partial_t g(t) = -2 (\rho(t) - \kappa \tau(t) g(t))$. The self-similar solutions of this flow are gradient Ricci almost solitons with soliton function $\lambda = \kappa \tau + \nu$ (for some $\nu \in \mathbb{R}$) and are called $\kappa$-Einstein solitons (see [23] [24] for further details).

**Example 4 (Conformally Einstein manifolds).** For $n \geq 3$, $(M, g, f, -\frac{1}{n-2})$ is a generalized quasi-Einstein manifold if and only if $(M, e^{-\frac{1}{n-2}} f g)$ is Einstein. Consequently, the parameter $\mu = -\frac{1}{n-2}$ is a distinguished value which is often exceptional, see Theorem 8, Theorem 12 and Example 14 for example. We refer to [39] [36] for more detailed information on conformally Einstein manifolds.

**Example 5 (Static space-times).** For $\mu = 1$, the change of variable $h = e^{-f}$ transforms Equation (1) into the equation $\text{Hes}_h - \lambda h = -h \lambda g$. If $\lambda = -\frac{\Delta h}{h}$, then this equation becomes $\text{Hes}_h - \lambda h = \Delta h g$. This is the defining equation of the so-called static manifolds that arise in the study of static space-times (we refer to [39] [40] for further details).

**Example 6 (Quasi-Einstein manifolds and Einstein warped products).** A solution Equation (1) with $\lambda$ constant is said to be quasi-Einstein and the resulting equation is called the quasi-Einstein equation. Let $B \times_\varphi F$ be an Einstein warped product with $\text{dim } F = r$ and warping function $\varphi = e^{-\frac{r}{2}}$. The structure on the base $(B, g, f, \frac{1}{r})$ is then quasi-Einstein. Conversely, starting with a quasi-Einstein manifold $(B, g, f, \mu)$ where $\mu = \frac{1}{r}$ for $r$ a positive integer, there exist appropriate Einstein fibers $F$ so that $B \times_\varphi F$ is Einstein, see, for example, [38].

**Remark 7.** Even more general classes exist in the literature [20] [25] [35] [42].
1.1. Motivation. Equation (1) provides information on the curvature of the manifold since it involves the associated Ricci tensor. We shall impose various conditions on the Weyl tensor to obtain related families of generalized quasi-Einstein manifolds. One could assume, for example, that $M$ is locally conformally flat; this condition turns out to be quite restrictive. We refer, for example, to the discussion in [14, 17] in relation to gradient Ricci solitons and to the discussion in [7, 22] for quasi-Einstein manifolds. Other weaker conditions were considered in [20] for a slightly more general class of manifolds than the one we consider here. Suppose that $\mu$ is not assumed to be constant. It is known that 4-dimensional generalized quasi-Einstein manifolds with harmonic Weyl tensor and zero radial Weyl curvature are indeed locally conformally flat in Riemannian signature. Associated rigidity results are available. See, for example, [3, 18, 37] and the references therein.

Other natural conditions on the conformal curvature were previously considered for 4-dimensional manifolds and particular families of generalized quasi-Einstein manifolds. One says that $M = (M, g)$ is half conformally flat if $M$ is either self-dual or anti-self-dual. The notation is chosen to avoid specifying the orientation. One has that half conformally flat quasi-Einstein manifolds are locally conformally flat in the Riemannian setting [19, 28]. We refer to [26] for the gradient Ricci soliton case and to [42] for related work.

The key point in this analysis is that, in definite signature, the level hypersurfaces of the potential function are non-degenerate and have constant sectional curvature. However, this need no longer hold true if the signature is indefinite. In this setting, the metric may be degenerate on the level hypersurfaces of the potential function. This gives rise to null parallel distributions (Walker structures) and to examples which are not locally conformally flat (see [6, 9]).

In this paper, we shall examine 4-dimensional generalized quasi-Einstein manifolds in neutral signature $(2, 2)$. We wish to find examples which are half conformally flat, but not locally conformally flat. The analysis depends to a large extent on the nature of the vector field $\nabla f$. If $\|\nabla f\| \neq 0$, then $M$ is said to be non-isotropic while if $\|\nabla f\| = 0$ but $\nabla f \neq 0$, then $M$ is said to be isotropic. We shall see that solutions of Equation (1) in the non-isotropic setting behave very much like solutions of Equation (1) in Riemannian signature. The isotropic setting has genuinely new phenomena not present in the Riemannian setting and Walker structures play a fundamental role. We are interested in the local theory and can restrict to an arbitrarily small open neighborhood $O$ of the point $P$ of $M$ in question. We shall assume $\nabla f$ does not vanish on $O$. We shall also assume either $\|\nabla f\|$ never vanishes on $O$ or that $\|\nabla f\|$ vanishes identically on $O$. We shall not treat the mixed case where the type of $\nabla f$ changes.

1.2. Walker manifolds. We now summarize the basic facts we shall need about Walker geometry and introduce some important families of Walker manifolds. Following the seminal work of Walker [48] (see also [31]), a pseudo-Riemannian manifold $M = (M, gw)$ is said to be a Walker manifold if $M$ admits a null parallel distribution $D$. We shall work in signature $(2, 2)$ and assume that $D$ is 2-dimensional. There are then the canonical local coordinates $(x^1, x^2, x^1', x^2')$ of Walker. To simplify the notation, let $\partial_{x^i} := \frac{\partial}{\partial x^i}$ and $\partial_{x^i'} := \frac{\partial}{\partial x^i'}$ for $i = 1, 2$. Let $\circ$ denote the symmetric product. We adopt the Einstein convention and sum over repeated indices. There are smooth functions $a_{ij} = a_{ij}(x^1, x^2, x^1', x^2')$ which are defined
locally on $M$ so that the metric $g_W$ and the distribution $\mathcal{D}$ take the form

$$g_W = 2dx^i \circ dx_i' + a_{ij}dx^i \circ dx^j$$

and $\mathcal{D} = \text{span}\{\partial_{x_i'},\partial_{x_j'}\}$.

Any Walker manifold has a canonical orientation [29, 30] which is linked to the orientation of the null distribution $\mathcal{D}$. If $\ast$ is the Hodge operator, we require that $\ast\mathcal{D} = \mathcal{D}$ so $\mathcal{D}$ is self-dual or, equivalently, $\ast(dx_1 \wedge dx_2) = dx_1 \wedge dx_2$. We fix this orientation henceforth.

Let $D$ be a torsion free connection on a surface $\Sigma$. If $(x^1, x^2)$ are local coordinates on $\Sigma$, let $(x_1', x_2')$ be the corresponding dual coordinates on the cotangent bundle $T^*\Sigma$; if $\omega$ is a 1-form, we can express $\omega = x_1'dx^1 + x_2'dx^2$. Let $\Phi$ be an auxiliary symmetric $(0, 2)$-tensor field. Let $\Gamma_{ij}^k$ be the Christoffel symbols of the connection $D$. The **deformed Riemannian extension** is defined by setting:

$$g_{D,\Phi} = 2dx^i \circ dx_i' + \left\{-2x_k'D\Gamma_{ij}^k + \Phi_{ij}\right\}dx^i \circ dx^j.$$  

This is an invariantly defined neutral signature metric on the cotangent bundle. Deformed Riemannian extensions were used in [6] to describe self-dual gradient Ricci solitons which are not locally conformally flat. More generally, let $T = (T_i^j)$ and $S = (S_i^j)$ be endomorphisms of the tangent bundle of $\Sigma$. The **modified Riemannian extension** is defined [12] by setting:

$$g_{D,\Phi,T,S} = 2dx^i \circ dx_i' + \left\{\frac{1}{2}x_k'(T^i_jS^s_j + T^s_jS^i_j) - 2x_k'D\Gamma_{ij}^k + \Phi_{ij}\right\}dx^i \circ dx^j.$$  

Modified Riemannian extensions were used in [12] to describe Walker manifolds which are self-dual. This metric and other related metrics appear in many contexts; see, for example, [11, 11, 12].

Although it is possible to show directly that the metrics of Equation (4) and of Equation (5) are defined invariantly, it is worth introducing a coordinate free formalism as we shall need the requisite notation subsequently in any event. Let $\pi : T^*\Sigma \to \Sigma$ be the natural projection. The geometries of the affine surface $\Sigma$ and of the cotangent bundle $T^*\Sigma$ are linked through evaluation maps and complete lifts. Given a vector field $X$ on $\Sigma$, the evaluation map $\iota X$ is the function on $T^*\Sigma$ which is characterized by the identity

$$(\iota X)(p, \omega) := \omega_p(X(p)) \text{ for } (p, \omega) \in T^*\Sigma.$$  

Vector fields on $T^*\Sigma$ are determined by their action on evaluation maps (see [50]). For a vector field $X$ on $\Sigma$, the complete lift of $X$, which is denoted by $X^C$, is the vector field on $T^*\Sigma$ that satisfies $X^C(\iota Z) = \iota[X, Z]$ for any vector field $Z$ on $\Sigma$. The deformed Riemannian extension of Equation (4) is characterized invariantly by its action on complete lifts:

$$g_{D,\Phi}(X^C, Y^C) = -\iota(D_XY + D_YX) + \Phi(X, Y).$$  

Similarly, if $T = (T_i^j)$ is an endomorphism of $T\Sigma$, then the evaluation $\iota T$ is a 1-form on $T^*\Sigma$ which is characterized by the property $\iota(T)(X^C) = \iota(T(X))$. The metric of Equation (5) is given invariantly by the equation:

$$g_{D,\Phi,T,S} = \iota T \circ \iota S + g_{D,\Phi}.$$
1.3. Main results. Let $\mathcal{M}$ be a 4-dimensional half conformally flat generalized quasi-Einstein manifold. If $\mathcal{M}$ is Riemannian, under fairly mild assumptions, one can show that $\mathcal{M}$ is locally conformally flat; see, for example, the discussion in [19, 20, 26]. By contrast, in the signature $(2, 2)$ setting, there are examples which are half conformally flat, but not locally conformally flat (see Remark 11 below). We work purely locally and shall replace the original manifold by an arbitrarily small neighborhood of the point in question. As noted above, we shall either assume that $\|\nabla f\| \neq 0$ or that $\|\nabla f\|$ vanishes identically but $\nabla f \neq 0$; we shall not consider the "mixed" case since we are especially interested in describing self-dual generalized quasi-Einstein metrics that are not locally conformally flat. We shall establish the following results in Section 2 and in Section 3, respectively.

Theorem 8. Let $(M, g, f, \mu)$ be a half conformally flat generalized quasi-Einstein manifold of signature $(2, 2)$ with $\mu \neq -\frac{1}{2}$ and $\|\nabla f\| \neq 0$. Then $(M, g)$ is conformally flat and is locally isometric to a warped product of the form $I \times_{\varphi} N$, where $I \subset \mathbb{R}$ and $N$ is of constant sectional curvature.

Theorem 9. Let $(M, g, f, \mu)$ be a half conformally flat generalized quasi-Einstein manifold of signature $(2, 2)$ with $\mu \neq -\frac{1}{2}$, with $\nabla f \neq 0$, and with $\|\nabla f\| = 0$. Then $(M, g)$ is a Walker manifold with a 2-dimensional null parallel distribution so the metric $g$ has the form of Equation (3) in some suitable system of local coordinates.

These two results are not sensitive to the choice of the orientation. However, as noted above, the Walker manifolds we shall be considering come equipped with natural orientations. Adopt the notation of Equation (4) and of Equation (5). We will establish the following result in Section 4.

Theorem 10. Let $(\Sigma, D)$ be an affine surface, let $\hat{f} \in C^\infty(\Sigma)$ and let $f = \pi^* \hat{f}$.

(1) Let $\Phi$ be arbitrary. Suppose $\hat{f}$ satisfies

$$
\text{Hes}_f^D + 2\rho_s^D - \mu \hat{f} \otimes d\hat{f} = 0 \text{ for some } \mu \in \mathbb{R}.
$$

Then $(T^* \Sigma, g_{D, \Phi, f, \mu})$ is a self-dual isotropic quasi-Einstein Walker manifold with $\lambda = 0$.

(2) Let $\Phi = \frac{1}{C} e^f (\text{Hes}_f^D + 2\rho_s^D - \mu \hat{f} \otimes d\hat{f})$ and let $T = Ce^{-f} \text{Id}$ for $\mu \in \mathbb{R}$ and for $0 \neq C \in \mathbb{R}$. Then $(T^* \Sigma, g_{D, \Phi, T, \mu})$ is a self-dual isotropic generalized quasi-Einstein Walker manifold with $\lambda = \frac{3}{4} Ce^{-f}$.

Remark 11. The manifolds described in Assertion (1) are not locally conformally flat in general. If $(\Sigma, D)$ is not projectively flat (see Definition 20), then the deformed Riemannian extension $(T^* \Sigma, g_{D, \Phi})$ is not locally conformally flat. However even if $(\Sigma, D)$ is projectively flat, one can choose $\Phi$ so that $(T^* \Sigma, g_{D, \Phi})$ is not locally conformally flat. The metrics of Assertion (2) are self-dual but never anti-self-dual. And if $\hat{f}$ is non-constant, then $\lambda$ is non-constant so $(T^* \Sigma, g_{D, \Phi, T, \mu})$ is not quasi-Einstein. Anti-self-dual modified Riemannian extensions $(T^* \Sigma, g_{D, \Phi, T, \mu})$ have zero scalar curvature. This does not happen for these examples since $\lambda = \frac{7}{4}$, as we will see in Section 4. Moreover, notice that these manifolds are $\frac{1}{4}$-Einstein solitons if $\mu = 0$ (see Example 3).

The following result is a partial converse to Theorem 11 and describes the possible local forms of self-dual isotropic generalized quasi-Einstein metrics.
Theorem 12. Let \((M, g, f, \mu)\) be a self-dual generalized quasi-Einstein manifold of signature \((2, 2)\) with \(\mu \neq -\frac{1}{2}\) and \(\|\nabla f\| = 0\) which is not Ricci flat.

1. If \(\lambda\) is constant, then \(\lambda = 0\) and \((M, g, f, \mu)\) is locally isometric to a manifold which has the form given in Assertion (1) of Theorem 10.

2. If \(\lambda\) is non-constant, then \((M, g, f, \mu)\) is locally isometric to a manifold which has the form given in Assertion (2) of Theorem 10.

1.4. Outline of the paper. In Section 2 we provide some general results concerning generalized quasi-Einstein manifolds. In Section 3 we examine the non-isotropic setting and establish Theorem 6. In Section 4 we examine the isotropic setting and establish Theorem 9. We continue our analysis of the isotropic setting in Section 5 and establish Theorem 10 and Theorem 12. Let \((M, D)\) be an affine manifold of arbitrary dimension. In Section 6 we linearize Equation (7) to define an equivalent partial equation which is natural in the context of affine geometry and is important in its own right. The remaining part of the paper deals with examples that illustrate important aspects of the equation. In Section 7 we give solutions to Equation (8) which are based on homogeneous affine surface geometries; this gives rise to purely algebraic considerations. In Section 8 we use the Cauchy-Kovalevskaya Theorem to construct inhomogeneous surface geometries solving Equation (8). In Section 9 we give examples which are anti-self-dual but not self-dual and consequently do not fit into the hypothesis of Theorem 12.

1.5. Notational conventions. Let \(\nabla\) be the Levi-Civita connection of a pseudo-Riemannian manifold \(\mathcal{M} = (M, g)\) of dimension \(n\). Let \(\mathcal{R}\) be the curvature operator, \(R\) be the curvature tensor, \(\rho\) be the Ricci tensor, \(\text{Ric}\) be the Ricci operator, \(\tau\) be the scalar curvature, \(W\) be the Weyl tensor, and \(C\) be the Cotton tensor:

\[
\mathcal{R}(X, Y)Z = \nabla_{[X, Y]}Z - [\nabla_X, \nabla_Y]Z, \quad R(X, Y, Z, T) = g(\mathcal{R}(X, Y)Z, T),
\]

\[
\rho(X, Y) = \text{Tr}(Z \rightarrow \mathcal{R}(X, Z)Y), \quad \text{Ric}(X, Y) = \rho(X, Y), \quad \tau = \text{Tr} \text{Ric},
\]

\[
W(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{(n-1)(n-2)}\{g(X, Z)g(Y, T) - g(X, T)g(Y, Z)\} - \frac{1}{(n-2)}\{\rho(X, T)g(Y, Z) - \rho(X, Z)g(Y, T) + \rho(Y, Z)g(X, T) - \rho(Y, T)g(X, Z)\},
\]

\[
C(X, Y, Z) = -\frac{n-3}{n-2} \text{div}_4 W(X, Y, Z).
\]

The Hessian tensor \(\text{Hes}_f(X, Y)\), Hessian operator \(\text{hes}_f\), and the Laplacian \(\Delta f\) of a smooth function \(f\) are given by:

\[
\text{Hes}_f(X, Y) = (\nabla_X df)(Y) = XY(f) - (\nabla_X Y)(f),
\]

\[
\text{g}(\text{hes}_f(X, Y)) = \text{Hes}_f(X, Y), \quad \Delta f = \text{Tr}(\text{hes}_f).
\]

Note that \(\mathcal{R}, \rho,\) and \(\text{Hes}_f\) are well defined in the context of affine geometry; the other tensors and operators are not. Since we are assuming the connection \(D\) is torsion free, the Hessian is symmetric. However, even with this assumption, the Ricci tensor need not be symmetric. Consequently, we decompose \(\rho^D = \rho^D_s + \rho^D_a\) into the symmetric and alternating Ricci tensors where

\[
\rho^D_s(X, Y) := \frac{1}{2}\{\rho^D(X, Y) + \rho^D(Y, X)\} \quad \text{and} \quad \rho^D_a(X, Y) := \frac{1}{2}\{\rho^D(X, Y) - \rho^D(Y, X)\}. 
\]
2. Generalized quasi-Einstein manifolds

We now establish some general results concerning generalized quasi-Einstein manifolds that we will use subsequently.

Lemma 13. Let \((M, g, f, \mu)\) be a generalized quasi-Einstein manifold. Then

1. \(\tau + \Delta f - \mu \|\nabla f\|^2 = n\lambda\).
2. \(\nabla \tau + \nabla \Delta f - 2\mu \text{hes}_f(\nabla f) = n\nabla \lambda\).
3. \(\nabla \tau + 2\mu(\lambda(n-1) - \tau)\nabla f + 2(\mu - 1)\text{Ric}(\nabla f) = 2(n-1)\nabla \lambda\).
4. \(R(X, Y, Z, \nabla f) = d\lambda(X)g(Y, Z) - d\lambda(Y)g(X, Z) + (\nabla_Y \rho)(X, Z) + 2\mu \{df(Y) \text{Hes}_f(X, Z) - df(X) \text{Hes}_f(Y, Z)\} \).
5. Let \(\eta = \mu(n-2) + 1\). Then
   \[W(X, Y, Z, \nabla f) = -C(X, Y, Z) + \frac{\tau n\{df(Y)g(X, Z) - df(X)g(Y, Z)\}}{(n-1)(n-2)} + \frac{n\{\rho(\nabla_Y \rho)g(X, Z) - \rho(\nabla_X \rho)g(Y, Z)\}}{(n-2)} + \frac{n\{\rho(\nabla_Y \rho)g(X) - \rho(\nabla_X \rho)f(Y)\}}{(n-2)}.\]

Proof. As one may use the generalized quasi-Einstein equation and the Bochner formula to establish Assertions (1)–(3) in exactly the same fashion that analogous formulas for gradient Ricci almost solitons were established in \[7, 46\]; we omit details in the interests of brevity. One covariantly differentiates Equation (11) and uses the definition to establish Assertion (4). One can express the Cotton tensor in the form:

\[C(X, Y, Z) = (\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z) - \frac{1}{2(n-1)}(X(\tau)g(Y, Z) - Y(\tau)g(X, Z)).\]

We substitute the curvature tensor term into the Weyl tensor and use Assertion (4). Using Equation (10) we may then make a direct computation to establish Assertion (5).

\[\square\]

Remark 14. Note that for \(\eta = 0\) many of the terms in Assertion (5) of Lemma 13 vanish. This is precisely the case in which the manifold is conformally Einstein as described previously in Example 13.

3. The non-isotropic setting: the proof of Theorem 8

Let \(\Lambda^{\pm} = \{\omega \in \Lambda^2 : \ast \omega = \pm \omega\}\) be the spaces of self-dual \((\Lambda^+)\) and anti-self-dual \((\Lambda^-)\) 2-forms for a 4-dimensional pseudo-Riemannian manifold \(\mathcal{M} = (M, g)\). Let \(\{E_1, E_2, E_3, E_4\}\) be an orthonormal local frame for the tangent bundle, let \(\varepsilon_i = g(E_i, E_i) = \pm 1\) for \(i \in \{2, 3, 4\}\). One has

\[\Lambda^{\pm} = \text{span}\{E^1 \wedge E^2 \pm \varepsilon_3 \varepsilon_4 E^3 \wedge E^4, E^1 \wedge E^3 \mp \varepsilon_2 \varepsilon_4 E^2 \wedge E^4, E^1 \wedge E^4 \pm \varepsilon_2 \varepsilon_3 E^2 \wedge E^3\}.\]

We say that \((M, g)\) is self-dual if \(W^- = 0\). Let \(\{i, j, k\}\) be a re-ordering of the indices \(\{2, 3, 4\}\) and let \(\sigma_{ijk}\) be the sign of the associated permutation. Then \(\mathcal{M}\) is self-dual if and only if the following identity is satisfied:

\[W(E_1, E_i, X, Y) = \sigma_{ijk} \varepsilon_j \varepsilon_k W(E_j, E_k, X, Y) \text{ for any vector fields } X \text{ and } Y.\]
We use Assertion (5) of Lemma 13. Let \( \eta = 2\mu + 1 \). We use Equation (11) to see that if a quasi-Einstein manifold is self-dual then
\[
\begin{align*}
\frac{\partial}{\partial z} \{ df(E_i)g(E_i, Z) - df(E_i)g(E_i, Z) \} \\
+ \frac{\partial}{\partial z} \{ \rho(E_i, \nabla f)g(E_i, Z) - \rho(E_i, \nabla f)g(E_i, Z) \} \\
+ \frac{\partial}{\partial z} \{ \rho(E_i, Z)df(E_i) - \rho(E_i, Z)df(E_i) \} \\
= \sigma_{ijk} \varepsilon_j \varepsilon_k \frac{\partial}{\partial z} \{ df(E_k)g(E_j, Z) - df(E_j)g(E_k, Z) \} \\
+ \sigma_{ijk} \varepsilon_j \varepsilon_k \frac{\partial}{\partial z} \{ \rho(E_j, \nabla f)g(E_k, Z) - \rho(E_k, \nabla f)g(E_j, Z) \} \\
+ \sigma_{ijk} \varepsilon_j \varepsilon_k \frac{\partial}{\partial z} \{ \rho(E_k, Z)df(E_i) - \rho(E_i, Z)df(E_k) \} .
\end{align*}
\]

Since \( \| \nabla f \| \neq 0 \), we may choose the local orthonormal frame so \( E_i \) is a non-zero multiple of \( \nabla f \). We may then use Equation (11) with \( Z = E_2, Z = E_3, \) or \( Z = E_4 \) to see that \( \rho \) is diagonal with respect to this basis. It now follows that \( 3\varepsilon_i \rho(E_i, E_i) = \tau - \varepsilon_i \rho(E_i, E_1) \). The orientation plays no role and the same conclusion follows if \( \mathcal{M} \) is anti-self-dual. We use Equation (11) to see that for \( 2 \leq i \leq 4 \) we have:
\[
\text{Hes}_f(E_i, E_i) = \lambda g(E_i, E_i) - \rho(E_i, E_i) = \left( \lambda - \frac{\tau - \varepsilon_i \rho(E_i, E_1)}{3} \right) g(E_i, E_i).
\]

Hence, the level hypersurfaces of \( f \) are totally umbilical. Since \( \text{span}\{\nabla f\} \) is a 1-dimensional totally geodesic distribution, \( \mathcal{M} \) decomposes locally as a twisted product \( I \times \varphi N \). Since the mixed terms \( \rho(E_1, E_i) \) vanish, the twisted product reduces to a warped product. And, since \( I \times \varphi N \) is self-dual, the warped product is locally conformally flat and the fiber \( N \) has constant sectional curvature (see [9] for a more detailed exposition).

4. The isotropic setting I: the proof of Theorem 9

In this section we study isotropic generalized quasi-Einstein manifolds which are half conformally flat and which have neutral signature \((2, 2)\). We fix the orientation so that the manifold is self-dual to simplify the arguments of this section. We use the fact that \( \nabla f \) is a null vector field to choose a local orthonormal frame so that
\[
- g(E_1, E_1) = g(E_2, E_2) = 1, \quad - g(E_3, E_3) = g(E_4, E_4) = 1, \\
g(E_i, E_j) = 0 \text{ for } i \neq j, \\
\nabla f = \frac{1}{\sqrt{2}} (E_1 + E_2).
\]

We also introduce a corresponding frame of null vector fields
\[
(12) \quad \mathcal{B} = \left\{ \nabla f = \frac{E_1 + E_2}{\sqrt{2}}, U = \frac{E_4 - E_3}{\sqrt{2}}, V = \frac{E_2 - E_1}{\sqrt{2}}, T = \frac{E_3 + E_4}{\sqrt{2}} \right\} .
\]

This is a hyperbolic frame; the only nonzero components of the metric tensor relative to the local frame \( \mathcal{B} \) are
\[
(13) \quad g(\nabla f, V) = g(U, T) = 1 .
\]

We use Equation (11) to see that the metric is self-dual if and only if we have the following identities for any \( X \) and \( Y \):
\[
(14) \quad W(\nabla f, V, X, Y) = W(U, T, X, Y) , \\
(15) \quad W(U, V, X, Y) = 0 , \\
(16) \quad W(\nabla f, T, X, Y) = 0 .
\]
Lemma 15. Let \((M, g, f, \mu)\) be an isotropic self-dual generalized quasi-Einstein manifold. Then \(\lambda = \frac{1}{4}\) and the Ricci operator has the form:

\[
\text{Ric} = \begin{pmatrix}
\frac{1}{4} & 0 & a & c \\
0 & \frac{1}{4} & c & b \\
0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{4}
\end{pmatrix}.
\]

Proof. Since \(g(\nabla f, \nabla f) = 0\),

\[
0 = \nabla_X g(\nabla f, \nabla f) = 2g(\nabla_X \nabla f, \nabla f) = 2g(\nabla f, \nabla f, X) \text{ for any } X.
\]

Consequently, \(\text{hes}_f(\nabla f) = 0\). Thus by Equation (14), \(\text{Ric}(\nabla f) = \lambda \nabla f\). Since \((M, g)\) is self-dual, we take \(Y = \nabla f\) in Equation (14) to see that

\[
W(\nabla f, V, X, \nabla f) = W(U, T, X, \nabla f).
\]

We then use Assertion (5) of Lemma 13 to conclude that \(\tau = \frac{\tau}{4} g(U, X)\) for any \(X\). Consequently, \(\text{Ric}(U) = \frac{\tau}{4} U\). We set \(X = V\) in Equation (16) to see that \(\text{Ric}(U, T, V, Y) = 0\) for all \(Y\). Thus:

\[
0 = W(Y, T, V, \nabla f) = \frac{1}{2} \{ \frac{\tau}{4} g(Y, T) - \rho(Y, T) + \rho(V, T) g(Y, \nabla f) \} \text{ for all } Y.
\]

We set \(Y = T\) to see that \(\rho(T, T) = 0\) so the Ricci tensor has the form given. □

Proof of Theorem 9. We have already seen that \(g(\nabla X \nabla f, \nabla f) = 0\). Moreover, a similar argument using the fact that \(g(U, U) = 0\) shows that \(g(\nabla_X U, U) = 0\) for all \(X\). On the other hand, since \(\text{Ric}(U) = \lambda U\), we may use Equation (11) to see that \(\text{hes}_f(U) = 0\). Now, since \(g(U, \nabla f) = 0\), we have that

\[
g(\nabla_X U, \nabla f) = -g(U, \nabla_X \nabla f) = -\text{hes}_f(U, X) = 0 \text{ for all } X.
\]

We have shown that

\[
\begin{align*}
g(\nabla_X \nabla f, \nabla f) &= 0, \\
g(\nabla_X U, \nabla f) &= 0, \\
g(\nabla_X \nabla f, U) &= 0, \\
g(\nabla_X \nabla f, U) &= 0.
\end{align*}
\]

Let \(\mathcal{D} = \text{span}\{\nabla f, U\}\). By Equation (13), \(\mathcal{D}\) is a null distribution. Furthermore, Equation (13) implies that \(\nabla \mathcal{D} \subset \mathcal{D}\). Consequently, \(\mathcal{D}\) is a 2-dimensional null parallel distribution and \((M, g)\) is locally a Walker manifold. □

5. The isotropic setting II: the proof of Theorem 10 and Theorem 12

We continue our study of half conformally flat isotropic generalized quasi-Einstein manifolds by examining the Walker setting. The orientation of the manifold, which has not played a role previously, now plays as a role since, as we saw earlier, Walker structures have a canonical orientation determined by the null parallel distribution. Adopt the notation of Equation (6).

Lemma 16. Let \((M, g)\) be a 4-dimensional Walker manifold of neutral signature \((2, 2)\) which is not Ricci-flat. If \((M, g, f, \mu)\) is a self-dual isotropic generalized quasi-Einstein manifold with \(\mu \neq -\frac{1}{2}\), then \((M, g)\) is locally isometric to a modified Riemannian extension \((T^* \Sigma, g_{D, \pi, T, \lambda})\) of an affine surface \((\Sigma, D)\) and \(f = \pi^* \hat{f}\), where \(\hat{f} \in C^\infty(\Sigma)\).
Hence setting a contradiction. Choose local coordinates on \( \Sigma \) so
\[ g = \iota X (\iota \text{Id} \circ \text{Id}) + (T \circ \iota \text{Id} + g_{D, \Phi}) . \]
As case \( \mu = 0 \) was considered previously in [14], we shall assume that \( \mu \neq 0 \). We first show that \( f = \pi^* \tilde{f} \) for some \( \tilde{f} \in C^\infty (\Sigma) \). We set \( h = e^{-\eta f} \) in Equation (11) to obtain the equivalent equation
\[ -\operatorname{Hess}_h + \mu h \rho = \mu h \lambda g . \]
A similar change of variable will play a central role in the discussion of Section 6.

Proof. We generalize the metric of Equation (6) slightly. Results of [12] show that there exists an affine surface \( (\Sigma, D) \), an endomorphism \( T \) of the tangent bundle of \( \Sigma \), a symmetric bilinear form \( \Phi \) on the tangent bundle of \( \Sigma \), and a vector field \( X \) on \( \Sigma \) so that \( \mathcal{M} \) is locally isometric to \( (T^* \Sigma, g) \) where
\[ (19) \quad g = \iota X (\iota \text{Id} \circ \text{Id}) + (T \circ \iota \text{Id} + g_{D, \Phi}) . \]
Let \( \mathbf{T} = (T_i^j) \). Adopt the notation of Equation (19). Expand
\[ \frac{\partial}{\partial x} \circ \iota \text{Id} + (1 + 2 \mu) X_1 J, \]
and correspondingly
\[ \frac{\partial}{\partial x} \circ \iota \text{Id} + (1 + 2 \mu) X_2 J. \]
We have \( \mathbf{O}(h) = 0 \). Since \( \mu \neq -\frac{1}{2} \), we may show that \( X = 0 \) by computing:
\[
\text{Coef}(\mathbf{O}(h)(\partial_{x^1}, \partial_{x^j})); x^j, x^k) = (1 + 2 \mu) X_1, \\
\text{Coef}(\mathbf{O}(h)(\partial_{x^2}, \partial_{x^j})); x^2, x^k) = \frac{1}{2} X_2 .
\]
Let \( T = (T_i^j) \). Since \( \mu \neq 0 \) and \( \mu \neq -\frac{1}{2} \), we show similarly that \( T = 0 \) by computing:
\[
\text{Coef}(\mathbf{O}(h)(\partial_{x^1}, \partial_{x^j})); x^2) = \frac{1}{2} T_1^2 , \\
\text{Coef}(\mathbf{O}(h)(\partial_{x^2}, \partial_{x^j})); x^1) = \frac{1}{2} (1 + 2 \mu) T_1^2 , \\
\text{Coef}(\mathbf{O}(h)(\partial_{x^2}, \partial_{x^j})); x^2) = \frac{1}{2} (T_1^2 + T_2^2) , \\
\text{Coef}(\mathbf{O}(h)(\partial_{x^2}, \partial_{x^j})); x^1) = \frac{1}{4} \left\{ (1 - 2 \mu) T_1^2 + (1 + 2 \mu) T_2^2 \right\} .
\]
Because \( \lambda = \frac{\iota T}{\iota} = \frac{1}{2} (T_1^1 + T_2^2 + 4X_1 x_1' + 4X_2 x_2') \), we have \( \lambda = 0 \). We compute:
\[
\mathbf{O}(h)(\partial_{x^1}, \partial_{x^j}) = -\Gamma_{11}^{1} , \quad \mathbf{O}(h)(\partial_{x^1}, \partial_{x^j}) = -\Gamma_{11}^{2} , \\
\mathbf{O}(h)(\partial_{x^2}, \partial_{x^j}) = -\Gamma_{12}^{1} , \quad \mathbf{O}(h)(\partial_{x^2}, \partial_{x^j}) = -\Gamma_{12}^{2} , \\
\text{Coef}(\mathbf{O}(h)(\partial_{x^2}, \partial_{x^j})); x^1) = (1 + 2 \mu) \partial_{x^2} \Gamma_{12}^{1} , \\
\text{Coef}(\mathbf{O}(h)(\partial_{x^2}, \partial_{x^j})); x^2) = \partial_{x^2} \Gamma_{12}^{2} .
\]
Hence setting \( \mathbf{O}(h) = 0 \),
\[
\partial_{x^1} \Gamma_{22}^{1} = \partial_{x^1} \Gamma_{22}^{2} = 0 \quad \text{and} \quad \Gamma_{11}^{1} = \Gamma_{11}^{2} = \Gamma_{12}^{1} = \Gamma_{12}^{2} = 0 .
\]
This implies that $\rho = 0$ which is contrary to our assumption. Consequently, as desired

$$\xi = 0 \text{ so } h = \pi^* \hat{h}.$$ 

We expand $X = X^1 \partial_{x^1} + X^2 \partial_{x^2}$ and compute:

$$\text{Coef}(\Omega(h)(\partial_{x^1}, \partial_{x^1}); x_1') = \mu h X^1 h,$$

$$\text{Coef}(\Omega(h)(\partial_{x^2}, \partial_{x^2}); x_1') = \mu h X^2 h.$$ 

This shows that $X = 0$ and, as desired, $g = \iota T \circ \iota \text{Id} + g_{D, \Phi}$. □

The following is an example where $(T^* \Sigma, g_{D, \Phi, T, \text{Id}})$ is a conformally Einstein modified Riemannian extension (i.e. a generalized quasi-Einstein manifold with $\mu = -\frac{1}{2}$) where the conformal function is not the pull-back of a function on the surface $(\Sigma, D)$. Consequently the assumption that $\mu \neq -\frac{1}{2}$ in Lemma 16 is essential.

**Example 17.** Let $(x^1, x^2)$ be the usual coordinates on $\Sigma = \mathbb{R}^2$. Suppose given smooth functions $\alpha(x^1, x^2)$, $\beta(x^1, x^2)$, and $\gamma(x^1, x^2)$ where $\gamma \neq 0$. Also suppose given smooth functions $\psi_1(x^2)$ and $\psi_2(x^2)$. We consider the following structures defining $D, T, f$, and $\Phi$:

$$\Gamma_{12}^1 = \alpha, \quad \Gamma_{22}^1 = \beta$$

$$\Gamma_{22}^2 = \alpha + \psi_1(x^2), \quad T = \gamma dx^2 \otimes \partial_{x^1},$$

$$f(x^1, x^2, x_1', x_2') = x_1' - 2 \frac{\gamma}{\alpha}, \quad \Phi_{11} = -4 \partial_{x_1} \left( \frac{\alpha}{\gamma} \right),$$

$$\Phi_{12} = \Phi_{21} = 2 \frac{\alpha^2}{\gamma} - \partial_{x^2} \left( \frac{2 \alpha}{\gamma} \right), \quad \Phi_{22} = 4 \frac{\alpha^2}{\gamma} + \psi_2(x^2).$$

One then has that $(T^* \Sigma, g_{D, \Phi, T, \text{Id}}; f, -\frac{1}{2})$ is generalized quasi-Einstein. The Ricci tensor is always nonzero and two-step nilpotent and the conformal function has a null gradient.

**The proof of Assertion (1) of Theorem 10 and Theorem 12.** Let $(M, g, f, \mu)$ be a self-dual generalized quasi-Einstein manifold of signature $(2, 2)$ with $\mu \neq -\frac{1}{2}$ and $||\nabla f|| = 0$ which is not Ricci flat. Suppose that $\lambda$ is constant. By Lemma 14 $\tau = 4\lambda$. We may now use Assertion (3) of Lemma 13 to see that $\lambda = 0$.

Lemma 16 shows that $g = g_{D, \Phi, T, \text{Id}}$ is locally isometric to a modified Riemannian extension. Adopt the notation of the proof of Lemma 16. We compute that

$$\Omega(h)(\partial_{x^1}, \partial_{x^1'}) = \mu h T_1^1, \quad \Omega(h)(\partial_{x^2}, \partial_{x^2'}) = \mu h T_2^2,$$

$$\Omega(h)(\partial_{x^1}, \partial_{x^1'}) = \mu h T_1^2, \quad \Omega(h)(\partial_{x^2}, \partial_{x^2'}) = \mu h T_2^1.$$ 

Since $\mu \neq 0$, since $h \neq 0$, and since $\Omega(h) = 0$, we have $T = 0$. This shows that $g$ is indeed locally isometric to a deformed Riemannian extension $g_{D, \Phi}$ as introduced in Subsection 1.2.

We know from Lemma 16 that $f = \pi^* \hat{f}$. Since $\lambda = 0$, the generalized quasi-Einstein equation reduces to the quasi-Einstein equation $\text{Hes}_f + \rho - \mu df \otimes df = 0$. A direct computation shows that the only non-vanishing terms of this equation are

$$(\text{Hes}_f + \rho - \mu df \otimes df)(\partial_{x^i}, \partial_{x^i'}) = (\text{Hes}_f^D + 2 \rho_D^f - \mu df \otimes df)(\partial_{x^i}, \partial_{x^i'}),$$

for $i, j = 1, 2$. Thus the manifolds of Assertion (1) of Theorem 10 are indeed quasi-Einstein. Furthermore, as stated in Assertion (1) of Theorem 12 these are the only examples. □
The proof of Assertion (2) of Theorem 10 and Theorem 12. Let \((M, g, f, \mu)\) be a self-dual generalized quasi-Einstein manifold of signature \((2, 2)\) with \(\mu \neq -\frac{1}{3}\) and \(\|\nabla f\| = 0\) which is not Ricci flat. Suppose that \(\lambda\) is non-constant. By Lemma 10, \(M\) is locally isometric to a modified Riemannian extension of the form \(g_{D, \Phi, T, Id}\) with \(f = \pi^* \hat{f}\). Since the case \(\mu = 0\) was already studied in [9], we suppose \(\mu \neq 0\) as well. Once again, we make the change of variable \(h = e^{-\mu f}\) and work with the symmetric bilinear form \(\Omega(h)\) of (20). We compute:

\[
\Omega(h)(\partial_{x^1}, \partial_{x^1}) = \frac{1}{2} \mu h (T_1^1 - T_2^2),
\]

\[
\Omega(h)(\partial_{x^2}, \partial_{x^2}) = \mu h T_2^1, \quad \text{and} \quad \Omega(h)(\partial_{x^1}, \partial_{x^2}) = \mu h T_1^2.
\]

Since \(\Omega(h) = 0\), \(T = t \text{Id}\) is a multiple of the identity and consequently

\[
g = \pi^*(t) \text{Id} \circ \text{Id} + g_{D, \Phi} \text{ for some } t \in C^\infty(\Sigma).
\]

We once again work with the generalized quasi-Einstein Equation (11) to compute

\[
\text{Coef}(\text{Hess}_f + \rho - \mu \text{df} \otimes \text{df} - \lambda g)(\partial_{x^1}, \partial_{x^2}); x^1) = t \partial_{x^1} f + \partial_{x^2} t,
\]

\[
\text{Coef}(\text{Hess}_f + \rho - \mu \text{df} \otimes \text{df} - \lambda g)(\partial_{x^2}, \partial_{x^2}); x^2) = t \partial_{x^2} f + \partial_{x^1} t.
\]

Setting these terms to zero yields \(t = Ce^{-f}\). One verifies that

\[
(\text{Hess}_f + \rho - \mu \text{df} \otimes \text{df} - \lambda g)(\partial_{x^1}, \partial_{x^2}) = C e^{-f} \Phi_{ij} - (\text{Hess}_f^D + 2\rho_s^D - \mu \text{df} \otimes \text{df}) \delta_{ij}(\partial_{x^1}, \partial_{x^2}) \text{ for } i, j = 1, 2.
\]

This shows that the construction of Assertion (2) of Theorem 10 provides examples of generalize quasi-Einstein manifolds; Assertion (2) of Theorem 12 follows. \(\square\)

Remark 18. Let \((M, g)\) be a Walker manifold of signature \((2, 2)\). If \((M, g, f, \mu)\) is an anti-self-dual isotropic generalized quasi-Einstein manifold then one still has that \(\lambda = \frac{\tau}{3}\). To see this, one can proceed as in the proof of Lemma 15 and use the anti-self-dual relation \(W(V, V, X, Y) = -W(U, T, X, Y)\) instead of (14). It was shown in [32] that if the self-dual Weyl curvature \(W^+\) of a Walker manifold vanishes, then \(\tau = 0\), so \(\lambda = 0\) and there are not generalized quasi-Einstein examples with non-constant \(\lambda\). We refer to Section 8 for some explicit examples of anti-self-dual quasi-Einstein manifolds which are not realized as a deformed Riemannian extension as in Theorem 12 (1). These differences between the self-dual and the anti-self-dual conditions illustrate the fact that the Walker structure determines the orientation and, hence, self-duality and anti-self-duality are not interchangeable conditions in Walker geometry.

6. The affine quasi-Einstein equation

Let \((M, g, f, \mu)\) be a self-dual generalized quasi-Einstein manifold of signature \((2, 2)\) with \(\lambda\) constant which is not Ricci flat. Assume \(\mu \neq -\frac{1}{3}\) and \(\|\nabla f\| = 0\). By Theorem 12 \(\lambda = 0\) and \((M, g, f, \mu)\) is locally isomorphic to \((T^* \Sigma, g_{D, \Phi}, f, \mu)\) where \(f = \pi^* \hat{f}\) and \(\hat{f}\) satisfies \(\text{Hess}_{\hat{f}}^D + 2\rho_s^D - \mu \text{df} \otimes \text{df} = 0\). Conversely, by Theorem 10 every such example is a self-dual isotropic quasi-Einstein Walker manifold with \(\lambda = 0\). Thus it is natural to consider this equation in its own right on affine surfaces. If \(\mu = 0\), then \((\Sigma, D, \hat{f})\) is an affine gradient Ricci soliton as discussed in [6]. Thus we shall suppose that \(\mu \neq 0\) and consider the change of variables \(\hat{h} = e^{-\frac{\mu}{2}f}\); a similar change of variables played a crucial role in the analysis of
Section 5. Equation (11) then becomes $-\frac{2}{\mu} \text{Hes}_h^D + 2\rho_s^D = 0$. This leads to the affine quasi-Einstein equation

$$\text{Hes}_h^D = \mu \hat{h} \rho_s^D \text{ for } \mu \in \mathbb{R}.$$  \hspace{1cm} (23)

Let $\mathcal{M} := (M, D)$ be an affine manifold. In Equation (23), the eigenvalue $\mu$ is a parameter that needs to be determined. Let $E_S(\mu) = E(\mu)$ be the vector space of smooth solutions to the linear partial differential Equation (23):

$$E(\mu) := \{ \hat{h} \in C^\infty(M) : \text{Hes}_h^D = \mu \hat{h} \rho_s^D \}.$$  

Similarly, if $p$ is a point of $M$, let $E(p, \mu)$ be the linear space of all germs of solutions to Equation (23) based at $p$. Let $\mathfrak{A}(p)$ be the Lie algebra of germs of affine Killing vector fields based at $p$. We summarize as follows some results concerning this equation and refer to a subsequent paper [10] for the proof. We work in complete generality and do not restrict to the case of surfaces for the moment.

**Theorem 19.** Let $\mathcal{M}$ be an affine manifold of dimension $n$. Let $p \in M$.

1. If $\hat{h}$ is a $C^2$ solution to Equation (23), then $\hat{h}$ is in fact smooth. If $(M, D)$ is real analytic, then $\hat{h}$ is real analytic.
2. Let $\hat{h} \in E(p, \mu)$. If $\hat{h}(p) = 0$ and if $d\hat{h}(p) = 0$, then $\hat{h} \equiv 0$.
3. $\dim\{ E(p, \mu) \} \leq n + 1$.
4. If $X \in \mathfrak{A}(p)$ and if $\hat{h} \in E(p, \mu)$, then $X\hat{h} \in E(p, \mu)$.
5. If $\Sigma$ is simply connected and if $\dim\{ E(p, \mu) \}$ is constant on $\Sigma$, then any element $\hat{h} \in E(p, \mu)$ extends uniquely to an element of $E(\mu)$.

The extremal case where $\dim\{ E(p, \mu) \} = n + 1$ merits additional attention.

**Definition 20.** We say that $D$ is projectively flat if there exists a 1-form $\omega$ and a flat connection $\hat{D}$ so that $D_X Y = \hat{D}_X Y + \omega(X)Y + \omega(Y)X$ for all $X$ and $Y$: $D$ is projectively flat if and only if it is possible to choose a coordinate system so that the unparametrized geodesics of $D$ are straight lines. We say that $D$ is strongly projectively flat if in addition $\omega$ can be chosen to be closed.

The eigenvalue $\mu_n = -\frac{1}{n-1}$ is distinguished in this subject. It appears, for example, in Example [4]. The following result relates the analytic properties of the affine quasi-Einstein equation to the underlying affine geometry.

**Theorem 21.** Let $\mathcal{M}$ be an affine manifold of dimension $n$. Let $\mu_n := -\frac{1}{n-1}$.

1. $\mathcal{M}$ is strongly projectively flat if and only if $\dim\{ E(\mu_n) \} = n + 1$.
2. If $\dim\{ E(\mu) \} = n + 1$ for any $\mu$, then $\mathcal{M}$ is strongly projectively flat.
3. If $\dim\{ E(\mu) \} = n + 1$ for $\mu \neq \mu_n$, then $\mathcal{M}$ is Ricci flat.
4. Suppose $\dim\{ E(p, \mu_n) \} = n + 1$. One may choose a basis $\{ \phi_0, \ldots, \phi_n \}$ for $E(\mu_n)$ so that $\phi_0(p) \neq 0$ and $\phi_i(p) = 0$ for $i > 0$. Set $z^i := \phi_i/\phi_0$. Then $\tilde{z} = (z^1, \ldots, z^n)$ is a system of coordinates defined near $p$ such that the unparametrized geodesics of $\mathcal{M}$ are straight lines.

We remark that if $\rho_s^D = 0$, then $E(\mu) = E(0)$ for any $\mu$. The space $E(0)$ is the space of Yamabe solitons.
In this section, we examine solutions to the affine quasi-Einstein equation in the context of homogeneous affine surfaces. Since an affine surface is flat if and only if the Ricci tensor vanishes, all the geometry is encoded in the Ricci tensor. In particular, a geometry is symmetric if and only if $D\rho^D = 0$.

We say that $\mathcal{S} = (\mathbb{R}^2, D)$ is a Type $A$ surface model if the Christoffel symbols $\Gamma_{ij}^k$ of the connection $D$ are constant. Similarly, we say that $\mathcal{S} = (\mathbb{R}^+ \times \mathbb{R}, D)$ is a Type $B$ surface model if the Christoffel symbols of the connection $D$ have the form $\Gamma_{ij}^k = (x^1)^{-1}C_{ij}^k$ for $C_{ij}^k$ constant. The Ricci tensor of any Type $A$ surface model is symmetric; this can fail for Type $B$ surface models. Any Type $A$ surface model is projectively flat; this can fail for Type $B$ surface models. These geometries are not disjoint. A Type $B$ model is projectively flat; this can fail for Type $A$ surface models. Since an affine surface is flat if and only if the Christoffel symbols $\Gamma_{ij}^k$ of the connection $D$ are homogeneous; the translations $(\partial^1, \partial^2)$ act transitively on $\mathbb{R}^2$ and preserve any Type $A$ connection. Similarly, the coordinate transformations $(x^1, x^2) \rightarrow (ax^1, ax^2 + b)$ for $a > 0$ act transitively on $\mathbb{R}^+ \times \mathbb{R}$ and preserve any Type $B$ connection. Since the geometries are homogeneous, $\dim\{E(p, \mu)\}$ is constant. Since the underlying topological space is simply connected, we may use Theorem 19 (5) to identify the global solutions $E(p, \mu)$ with the germs of local solutions $E(p, \mu)$ for any point $p$. Thus for these geometries, there is no difference between the global and the local theory.

The importance of these two geometries lies in the fact that Opozda [15] showed that any locally homogeneous affine surface which is not flat is modeled on a Type $A$ geometry, on a Type $B$ geometry, or on the Levi-Civita connection of the sphere. These categories are not disjoint. A Type $B$ surface model $\mathcal{S}$ is locally isomorphic to a Type $A$ surface model if and only if $C_{12} = C_{22} = 0$; we refer to [8] for details. For surfaces, the critical eigenvalue is $\mu_2 = -\frac{1}{n-1}|n=2 = -1$. Since any Type $A$ surface model is strongly projectively flat, $\dim\{E(-1)\} = 3$ by Theorem 24. As we assumed that $\mathcal{S}$ is not flat, $\mathcal{S}$ is not Ricci flat and consequently $\dim\{E(\mu)\} \leq 2$ for $\mu \neq -1$.

If $\mathcal{S}$ is a Type $A$ surface model, then $\partial_{x^1}$ and $\partial_{x^2}$ are affine Killing vector fields and thus $E(\mu)$ is a Span$\{\partial_{x^1}, \partial_{x^2}\}$ module by Theorem 19 (4). In the Type $B$ setting, $x^1\partial_{x^1} + x^2\partial_{x^2}$ and $\partial_{x^2}$ are affine Killing vector fields and thus $E(\mu)$ is a Span$\{x^1\partial_{x^1}, x^2\partial_{x^2}, \partial_{x^2}\}$ module. These module structures play a crucial role in the proof given in [10] of the following results:

**Theorem 22.** Let $\mathcal{S}$ be a Type $A$ surface model. Let $\mu \neq 0$.

$$\dim\{E(\mu)\} = \begin{cases} 3 & \text{if } \mu = -1 \\ 2 & \text{if } \mu \neq -1 \text{ and } \text{Rank}\{\rho^D\} = 1 \\ 0 & \text{if } \mu \neq -1 \text{ and } \text{Rank}\{\rho^D\} = 2 \end{cases}.$$\n
**Theorem 23.** Let $\mathcal{S}$ be a Type $B$ surface model which is not also Type $A$. If $E(\mu) \neq \{0\}$, then up to linear equivalence, one of the following holds:

1. $C_{22} = \pm 1$, $C_{12} = 0$, $C_{22} = \pm 2C_{11}$, $\Delta := -C_{11} + C_{12}^2 + 1 \neq 0$, $\mu = \Delta^{-2}\{-(C_{11})^2 + 2C_{11}C_{12}^2 + 2(C_{11}^2)^2 - (C_{12}^2)^2 + 2(C_{12}^2 + 1)\}$.
2. $C_{22} = 0$, $C_{12} = C_{22} \neq 0$, $\mu = -1$.

Undoing the transformation from Equation 7 to Equation 23, we see that $\mu = -1$ corresponds to the conformally Einstein case of Example 14. We give the following example to illustrate Theorem 23 and we refer to [10] for an explicit
We consider the following family of Type $B$ surface models. It is convenient to change notation slightly to simplify the computations:

**Definition 24.** Let $S := \{(x^1, x^2) \in \mathbb{R}^2 : x^1 + x^2 > 0\}$. For $0 \neq \kappa \in \mathbb{R}$, let $S_\kappa = (S, D_\kappa)$ be a Type $B$ surface model where the nonzero Christoffel symbols of $D_\kappa$ are given by taking $\Gamma^{11}_1 = \Gamma^{22}_2 = \kappa (x^1 + x^2)^{-1}$. Let $E_\kappa(\mu)$ be the associated eigenspace of the affine quasi-Einstein equation on $S_\kappa$.

**Theorem 25.** The Ricci tensor of $S_\kappa$ is recurrent. The affine surface $S_\kappa$ is a symmetric space if and only if $\kappa = -2$. One has:

1. $E_\kappa(0) = \mathbb{R}$.
2. If $\kappa = -2$ and $\mu = -1$, then $E_\kappa(\mu) = (x^1 + x^2)^{-1} \text{Span}\{1, x^1 - x^2\}$.
3. If $\kappa \neq -2$ and $\mu = \kappa + 1$, then $E_\kappa(\mu) = (x^1 + x^2)^{\kappa + 1} \cdot \mathbb{R}$.
4. If $\mu \neq 0$ and $\mu \neq \kappa + 1$, then $E_\kappa(\mu) = \{0\}$.

**Proof.** We note that $S_{-2}$ is isomorphic to the Lorentzian hyperbolic plane. We show that the Ricci tensor is recurrent and that $S_\kappa$ is symmetric if and only if $\kappa = -2$ by computing:

$$\rho^D = \frac{\kappa}{(x^1 + x^2)^2} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad D\rho^D = -\frac{2 + \kappa}{x^1 + x^2} (dx^1 + dx^2) \otimes \rho^D.$$ 

One can use the structure of $E_\kappa(\mu)$ as a module over the Lie algebra of affine Killing vector fields to show that if $E_\kappa(\mu)$ is non-trivial, then there exists $\alpha \in \mathbb{C}$ so that $\hat{h} := (x^1 + x^2)^\alpha \in E_\kappa(\mu)$. We compute

$$\text{Hes}_\hat{h}^D - \mu \hat{h} \rho^D = (x^1 + x^2)^{\alpha - 2} \left( \begin{array}{ccc} \alpha(\alpha - \kappa - 1) & \alpha^2 - \alpha - \mu \kappa & \alpha(\alpha - \kappa - 1) \\ \alpha^2 - \alpha - \mu \kappa & \alpha(\alpha - \kappa - 1) \end{array} \right).$$

Since $\kappa \neq 0$, if $\mu \neq 0$ then $\alpha \neq 0$ and thus $\alpha = \mu = \kappa + 1$. So $E_\kappa(0)$ and $E_\kappa(\kappa + 1)$ are the only non-trivial eigenspaces; this is in agreement with Theorem 23 where there is at most 1 non-trivial eigenvalue $\mu \neq 0$. The module structure is used in [11] to find the general form of an element of $E(\mu)$ for Type $A$ and Type $B$ surface models. When those results are applied to the setting at hand, we may conclude that $E_\kappa(\mu)$ is spanned by elements of the form:

$$\hat{h}(x^1, x^2) = (x^1 + x^2)^\beta \{c_1(x^1 + x^2) + c_2(x^1 - x^2) + c_3(x^1 + x^2) \log(x^1 + x^2)\}.$$

A computer aided calculation yields that $c_3 = 0$. Furthermore, $\beta = \kappa + 1$, if $c_2 \neq 0$, and $\beta = \kappa$, if $c_2 = 0$. A careful analysis of the different cases then yields the remainder of Theorem 25.

8. INHOMOGENEOUS EXAMPLES

In Section 7, we exhibited a number of homogeneous examples illustrating different phenomena related to the affine quasi-Einstein equation (Equation 23). In this section, we exhibit some inhomogeneous examples. We begin with a useful ansatz; we shall suppose $n = 2$ but it is valid for general $n$. Let $\phi_i := \partial_{x^i} \phi$, $\phi_{ij} := \partial_{x^i} \phi_j$, etc. We omit the details of the following computation as it is entirely elementary:

**Lemma 26.** Let $\mathcal{O}$ be a simply connected open subset of $\mathbb{R}^2$. Let $\phi \in C^\infty(\mathcal{O})$ and let $\mathcal{M}_\phi = (\mathcal{O}, D^\phi)$ where $D^\phi$ is the affine connection on $\mathcal{O}$ with $\Gamma_{ii}^{i} = \phi_{ii}/\phi_i$ and
\( \Gamma_{ij}^k = 0. \) Then \( \phi \in E(\mu) \) if and only if \( \phi \) satisfies the non-linear partial differential equation:

\[
0 = \frac{1}{2} \mu \phi \left( \frac{\phi_{122}}{\phi_2} + \frac{\phi_{112}}{\phi_1} \right) + \phi_{12} \left( 1 - \frac{1}{2} \mu \phi \left( \frac{\phi_{22}}{\phi_2^2} + \frac{\phi_{11}}{\phi_1^2} \right) \right).
\]

We will use the Cauchy-Kovalevskaya Theorem (see, for example, Evans [34]) to construct solutions to Equation (24) and thereby show Lemma 26 is non-trivial. Fix a point \( p \in \mathbb{R}^2 \). Let \( c = \phi(p), c_i = \phi_i(p) \), and so forth. Let \( \tilde{c} := J_k(\phi)(p) \) be the \( k \)-jet of \( \phi \) at \( p \). For example

\[ J_3 = \{ \tilde{c} = (c, c_1, c_2, c_{11}, c_{12}, c_{22}, c_{111}, c_{112}, c_{122}, c_{222}) \} \subset \mathbb{R}^{10}. \]

Let \( \tilde{J}_k \) be the subset of \( J_k \) where \( c \neq 0 \), \( c_1 \neq 0 \), \( c_2 \neq 0 \), and where the relations imposed by differentiating Equation (24) are satisfied. For example, \( J_2 \) is the open dense subset of \( \mathbb{R}^8 \) with 8 path components obtained by deleting 3 hyperplanes. If we fix an element \( \xi_2 \in \tilde{J}_2 \), then the relation of Equation (24) is linear in the third derivatives of \( \phi \) and thus the natural projection \( \tilde{J}_3 \to \tilde{J}_2 \) is a real analytic 3-dimensional vector bundle. If we fix an element \( \xi_3 \in \tilde{J}_3 \), there are two linearly independent relations in the 4-jets of \( \phi \) which arise from differentiating Equation (24) and the natural projection \( \tilde{J}_4 \to \tilde{J}_3 \) is again a 3-dimensional real analytic vector bundle. Arguing similarly, we see that \( \tilde{J}_k \) is a real analytic manifold of dimension \( 3k \). Assume that the symmetric Ricci tensor is non-degenerate. Let \( \rho_{ij} \) be the components of \( \rho^k_i \), let \( \rho_{ij} \) be the components of the inverse matrix, and let \( \rho_{ijk} \) be the components of the covariant derivatives of the symmetric Ricci tensor. We define a scalar invariant \( E \) of such a geometry by setting

\[ E := \rho^{ia} \rho^{jb} \rho^{kc} \rho_{ijk} \rho_{abc}. \]

**Theorem 27.**

1. Given \( \xi \in \tilde{J}_k \), there exists the germ of a function \( \phi \) with \( J_k(\phi) = \xi \) solving Equation (24).
2. If \( k \geq 3 \), there is an open dense subset \( O_k \) of \( \tilde{J}_k \) so that if \( J_k(\phi) \in O_k \), then the Ricci tensor is not symmetric.
3. Let \( \mu \neq -1 \). If \( k \geq 5 \), there is an open dense subset \( O_k \) of \( \tilde{J}_k \) so that if \( J_k(\phi) \in O_k \), then \( \rho^k_i \) is non degenerate, \( dE \neq 0 \), and the geometry is not homogeneous.

**Proof.** We recall the classical Cauchy-Kovalevskaya Theorem. Suppose we are given a relation in the 3-jets of \( \phi \) which is linear once the 2-jets have been fixed and has the form:

\[
0 = a_{111}(J_2(\phi))\phi_{111} + a_{112}(J_2(\phi))\phi_{112} + a_{122}(J_2(\phi))\phi_{122}
+ a_{222}(J_2(\phi))\phi_{222} + a(\phi) + a(\phi).
\]

Assume the coefficient functions are real analytic on some suitable open set of \( J_2 \). Suppose that \( a_{111} \neq 0 \). Then given Cauchy data \( f_0(x^2), f_1(x^2), \) and \( f_2(x^2) \), there is a unique real analytic solution to Equation (25) with \( \partial_i^2 \phi(0, x^2) = f_i(x^2) \) for \( i = 0, 1, 2 \). If one expands \( \phi \) in a Taylor series, the derivatives \( \partial_i^2 \phi(0, 0) \) can be specified arbitrarily for \( i \leq 2 \) and the remaining Taylor series coefficients are then determined by Equation (26). Reinterpreting this in the language we have introduced, this means that if \( \xi \in \tilde{J}_k \) is given, there exists a solution \( \phi \) with \( J_k(\phi) = \xi \). This observation is not directly applicable to the setting at hand since
either it vanishes identically or it does not vanish on an open dense set. We compute to ensure \( \tilde{\phi} \) this connection is recurrent. We impose the identity of Equation (26) and compute:

\[
\begin{align*}
\rho^a_{\gamma}(\partial_{x^1}, \partial_{x^2}) &= \frac{1}{2} \left( \frac{\phi_{22}\phi_{12} - \phi_{2}\phi_{122}}{\phi_2^2} + \frac{\phi_1\phi_{112} - \phi_{11}\phi_{12}}{\phi_1^2} \right).
\end{align*}
\]

If we take \( \phi(0) = 1, \phi_1(0) = \phi_2(0) = 1, \phi_{11}(0) = \phi_{12}(0) = \phi_{22}(0) = 0, \phi_{122} = -\phi_{112} = 1, \)
then Equation (24) is satisfied and \( \rho^a_{\gamma}(\partial_{x^1}, \partial_{x^2})(0) \neq 0 \). Assertion (2) follows.

Similarly, either \( d\mathcal{E} \) vanishes identically or \( d\mathcal{E} \) is non-zero on an open dense subset of \( \tilde{\mathcal{J}}_k \) for any \( k \geq 5 \). We take \( \phi(x^1, x^2) = \gamma(x^1 + x^2) \); Equation (24) becomes:

\[
\gamma^{(3)}(t) = \frac{\gamma'(t)^2 \gamma''(t) - \mu \gamma(t) \gamma'''(t)^2}{\mu \gamma(t) \gamma'(t)}.
\]

This ODE can be solved with arbitrary initial conditions \( \{\gamma(0), \gamma'(0), \gamma''(0)\} \). We have \( \phi_2(0) = \phi_1(0) \) so we obtain 4 of the 8 components of \( \tilde{\mathcal{J}}_5 \); the other 4 components can be obtained by considering \( \gamma(x^1 - x^2) \). A direct calculation shows that this connection is recurrent. We impose the identity of Equation (26) and compute:

\[
\rho^D = \frac{\gamma''}{\mu \gamma}(0 \ 1 \ 1), \quad D\rho^D = -\frac{1 \mu \gamma'}{\mu \gamma}(dx^1 + dx^2) \otimes \rho^D,
\]

\[
\mathcal{E} = 4(1+\mu)^2(\gamma')^2, \quad \dot{\mathcal{E}} = 4(\mu+1)^2(\mu \gamma \gamma' \gamma'' - (\mu-1)(\gamma')^3).
\]

Since \( \mu \neq -1 \), \( \dot{\mathcal{E}} \) is non-zero for generic values of \( \{\gamma(0), \gamma'(0), \gamma''(0)\} \). Assertion (3) follows.

\[\square\]

**Remark 28.** Note that \( \gamma(t) := t^{\mu} \) solves Equation (26). For this choice of the defining function, \( \Gamma_{ii} = (\mu - 1)(x^1 + x^2)^{-1} \) and one obtains the example in Definition 24.

9. **Affine surfaces supporting a parallel nilpotent \((1,1)\)-tensor field**

In this section, we give examples of anti-self-dual quasi-Einstein manifolds which do not fit into the classification of Theorem 12 thus emphasizing the role of the Walker orientation. We will examine a special family of affine surfaces \((\Sigma, D)\) which admit a parallel nilpotent \((1,1)\)-tensor field \( T (DT = 0, T^2 = 0) \); we refer to [13] for details. We assume the system of local coordinates \((x^1, x^2)\) is chosen so that:

\[
T \partial_{x^1} = \partial_{x^1} \quad \text{and} \quad T \partial_{x^2} = 0.
\]

Then \( DT = 0 \) if and only if we have the relations:

\[
\Gamma_{12}^1 = 0, \quad \Gamma_{12}^2 = \Gamma_{21}^1, \quad \Gamma_{22}^1 = 0, \quad \Gamma_{22}^2 = 0.
\]

Since \( \text{ker} T = \text{span} \{\partial_{x^1}\} \) is parallel, \( \partial_{x^2} \) is a geodesic vector field. We have:

\[
\rho^a_{\gamma}(\partial_{x^1}, \partial_{x^1}) = \partial_{x^2} \Gamma_{11}^1 \partial_{x^1} \partial_{x^1}, \quad \rho^D_{\gamma} = \partial_{x^2} \Gamma_{11}^1 dx^1 \wedge dx^2.
\]

We shall suppose that \( \partial_{x^1} \Gamma_{11}^1 = 0 \) to ensure that \( \rho^D \) is symmetric. In this situation the Ricci tensor is recurrent and of rank one. Work of Wong [51] shows that the local
coordinates \((x^1, x^2)\) can be further specialized so that the only nonzero Christoffel symbol is \(\Gamma_{11}^2\) and hence
\[
\rho^D = \partial_{x^2} \Gamma_{11}^2 \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right),
\]
\[
D\rho^D = (\partial_{x^1}(\log \partial_{x^2} \Gamma_{11}^2) dx^1 + \partial_{x^2}(\log \partial_{x^2} \Gamma_{11}^2) dx^2) \otimes \rho^D.
\]
Thus, with this choice of local coordinates the recurrence 1-form \(\omega\) is given by
\[
(28) \quad \omega = \partial_{x^1}(\log \partial_{x^2} \Gamma_{11}^2) dx^1 + \partial_{x^2}(\log \partial_{x^2} \Gamma_{11}^2) dx^2.
\]
We adopt the notation of Equation (6). The modified Riemannian extension \(g_{D,0,T,T} = iT \circ iT + g_D\) is never self-dual, but it is anti-self-dual if \(\omega\) satisfies \(\omega(\ker T) = 0\) (see [13] for details). These affine surfaces are strongly projectively flat. Although Riemannian extensions of projectively flat connections are locally conformally flat, deformed Riemannian extensions \(g_{D,\Phi}\) or modified Riemannian extensions \(g_{D,T,S}\) are not for generic tensors \(\Phi, T\) and \(S\). Consequently, the following result will show that there exist examples of anti-self-dual quasi-Einstein Walker metrics that are never self-dual and hence not covered by the classification of Theorem 12.

**Theorem 29.** Let \((\Sigma, D)\) be an affine surface. Assume that \(\rho^D\) is symmetric, that \(\text{Rank}\{\rho^D\} = 1\), and that \(D\rho^D = \omega \otimes \rho^D\). Let \(T\) be a parallel nilpotent \((1,1)\)-tensor field on \(\Sigma\), let \(\hat{f} \in C^\infty(\Sigma)\), and let \(f = \pi^* \hat{f}\). Assume that \(df(\ker T) = 0\).

1. If \(\hat{f}\) satisfies Equation (7), then \((T^* \Sigma, g_{D,0,T,T}, f, \mu)\) is an isotropic quasi-Einstein manifold with \(\lambda = 0\).

2. If \(\omega(\ker T) = 0\), then there exist local coordinates \((x^1, x^2)\) on \(\Sigma\) so that the only nonzero Christoffel symbol is given by \(\Gamma_{11}^2 = u(x^1) + x^2 v(x^1)\). If \(\hat{f}(x^1)\) satisfies \(f^\mu(x^1) - \mu \hat{f}(x^1)^2 + 2v(x^1) = 0\), then \((T^* \Sigma, g_{D,0,T,T}, \hat{f}, \mu)\) is an anti-self-dual quasi-Einstein manifold which is not locally conformally flat.

**Proof.** Choose local coordinates \((x^1, x^2)\) on \(\Sigma\) so that \(T\partial_{x^1} = \partial_{x^2}, T\partial_{x^2} = 0\) and so that the only nonzero Christoffel symbol is \(\Gamma_{11}^2\). Let \((x^1, x^2, x_1, x_2)\) be the induced coordinates on \(T^* \Sigma\). Let \(\hat{f} \in C^\infty(\Sigma)\) satisfy \(d\hat{f}(\ker T) = 0\). Then:

\[
(\text{Hes}_f + \rho - \mu df \otimes df)(\partial_{x^1}, \partial_{x^1}) = (\text{Hes}^D_f + 2\rho^D_x - \mu \hat{f} \otimes d\hat{f})(\partial_{x^1}, \partial_{x^1}),
\]

the other components being identically zero. This proves Assertion (1).

To prove Assertion (2), we assume that \(\omega(\ker T) = 0\). By Equation (28) we have \(\omega(\ker T) = 0\) if and only if \(\partial_{x^2} \Gamma_{11}^2 = 0\). Consequently, \(\Gamma_{11}^2\) is a linear function of \(x^2\), i.e. \(\Gamma_{11}^2 = u(x^1) + x^2 v(x^1)\). Finally, a direct computation shows that the only nonzero term in Equation (7) is

\[
(\text{Hes}^D_f + 2\rho^D_x - \mu \hat{f} \otimes d\hat{f})(\partial_{x^1}, \partial_{x^1}) = \hat{f}''(x^1) - \mu \hat{f}'(x^1)^2 + 2v(x^1)\).
\]

This shows that the solutions of the quasi-Einstein equation are determined by the ODE \(\hat{f}''(x^1) - \mu \hat{f}'(x^1)^2 + 2v(x^1) = 0\). 

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