The distance domination of generalized de Bruijn and Kautz digraphs

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Abstract

Let \(G = (V, A)\) be a digraph and \(k \geq 1\) an integer. For \(u, v \in V\), we say that the vertex \(u\) distance \(k\)-dominate \(v\) if the distance from \(u\) to \(v\) at most \(k\). A set \(D\) of vertices in \(G\) is a distance \(k\)-dominating set if for each vertex of \(V \setminus D\) is distance \(k\)-dominated by some vertex of \(D\). The distance \(k\)-domination number of \(G\), denoted by \(\gamma_k(G)\), is the minimum cardinality of a distance \(k\)-dominating set of \(G\). Generalized de Bruijn digraphs \(G_B(n, d)\) and generalized Kautz digraphs \(G_K(n, d)\) are good candidates for interconnection networks. Tian and Xu showed that \(\lceil n/\sum_{j=0}^{k} d^j \rceil \leq \gamma_k(G_B(n, d)) \leq \lceil n/d^k \rceil\) and \(\lceil n/\sum_{j=0}^{k} d^j \rceil \leq \gamma_k(G_K(n, d)) \leq \lceil n/d^k \rceil\). In this paper we prove that every generalized de Bruijn digraph \(G_B(n, d)\) has the distance \(k\)-domination number \(\lceil n/\sum_{j=0}^{k} d^j \rceil\) or \(\lceil n/\sum_{j=0}^{k} d^j \rceil + 1\), and the distance \(k\)-domination number of every generalized Kautz digraph \(G_K(n, d)\) bounded above by \(\lceil n/(d^{k-1} + d^k) \rceil\). Additionally, we present various sufficient conditions for \(\gamma_k(G_B(n, d)) = \lceil n/\sum_{j=0}^{k} d^j \rceil\) and \(\gamma_k(G_K(n, d)) = \lceil n/\sum_{j=0}^{k} d^j \rceil\).

Keywords: Combinatorial problems; generalized de Bruijn digraph; generalized Kautz digraph; distance dominating set; dominating set

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1 Introduction

In this paper we deal with directed graphs (or digraphs) which admit self-loops but no multiple arcs. Unless otherwise defined, we follow [3, 10] for terminology and definitions. Let $G$ be a digraph with vertex set $V(G)$ and arc set $A(G)$. If there is an arc from $u$ to $v$, i.e., $(u, v) \in A(G)$, then $v$ is called an out-neighbor of $u$; we also say that $u$ dominates $v$. The out-neighborhood $O(u)$ of a vertex $u$ is the set $\{v: (u, v) \in A(G)\}$. For $S \subseteq V(G)$, its out-neighborhood $O(S)$ is the set $\cup_{u \in S} O(u)$. The closed out-neighborhood $O[u]$ of $u$ is the set $O(u) \cup \{u\}$. The closed out-neighborhood $O[S]$ is defined analogously.

For $x, y \in V(G)$, the distance $d_G(x, y)$ from $x$ to $y$ is the length of an shortest $(x, y)$-directed path in $G$. Let $k$ be a positive integer. A subset $D \subseteq V(G)$ is called a distance $k$-dominating set of $G$ if for every vertex $v$ of $V(G) \setminus D$, there is a vertex $u \in D$ such that $d_G(u, v) \leq k$, i.e., $\cup_{i=0}^{k} O_i(D) = V(G)$. The distance $k$-domination number of $G$, denoted by $\gamma_k(G)$, is the minimum cardinality of a distance $k$-dominating set of $G$. In particular, the distance 1-dominating set is the ordinary dominating set, which has been well studied [11].

Slater [11] termed a distance $k$-dominating set as a $k$-basis and also gave an interpretation for a $k$-basis in terms of communication networks. Since then many researchers pay much attention to this subject, for example [9, 19, 23]. The concept of distance domination in graphs finds applications in many structures and situations which give rise to graphs. A minimum distance $k$-dominating set of $G$ may be used locate a minimum number of facilities (such as utilities, police stations, hospitals, transmission towers, blood banks, waste disposal dump) such that every intersection is within $k$ city block of a facility. Barkauskas and Host [1] showed that the problem of determining $\gamma(G)$ is NP-hard for a general graph.

The network topology has a great impact on the system performance and reliability [26]. There are some well-known networks with good properties such as de Bruijn networks, Kautz networks and their generalizations (see, for example, [2, 4, 5, 13, 26]). Generalized de Bruijn and Kautz networks, denoted by $G_B(n, d)$ and $G_K(n, d)$ respectively, were introduced by Imase and Itoh [14]. The generalization removes the restriction on the cardinality of vertex set and make the network more general and valuable as a network model. A lot of features make it suitable for implementation of reliable networks. The most important feature such as small diameter [14], high connectivity [15], easy routing, and high reliability.
The generalized de Bruijn digraph $G_B(n, d)$ is defined by congruence equations as follows:

\[
\begin{align*}
V(G_B(n, d)) &= \{0, 1, 2, \ldots, n-1\} \\
A(G_B(n, d)) &= \{(x, y) \mid y \equiv dx + i \pmod{n}, 0 \leq i \leq d - 1\}.
\end{align*}
\]

In particular, if $n = d^m$, then $G_B(n, d)$ is the de Bruijn digraph $B(d, m)$. The generalized Kautz digraph $G_K(n, d)$ is defined by following congruence equation:

\[
\begin{align*}
V(G_K(n, d)) &= \{0, 1, 2, \ldots, n-1\} \\
A(G_K(n, d)) &= \{(x, y) \mid y \equiv -dx - i \pmod{n}, 1 \leq i \leq d\}.
\end{align*}
\]

In particular, if $n = d^m + d^{m-1}$, then $G_K(n, d)$ is the Kautz digraph $K(d, m)$. The graphs $G_B(6, 3)$ and $G_K(9, 2)$ are exhibited in Fig. 1.

![Figure 1 (a): $G_B(6, 3)$](image1)

![Figure 1 (b): $G_K(9, 2)$](image2)

The structure properties of the generalized de Bruijn and Kautz digraphs receive more attention. Du et al. [6] studied the hamiltonian property of generalized de Bruijn and Kautz networks. Also, several structural objects such as spanning trees, Eulerian tours [17], closed walks [24] and small cycles [12] have been counted. Shan et al. [20, 21, 22] studied the absorbants and twin domination of generalized de Bruijn digraphs. Recently, Dong et al. [7] completely determined the domination number of generalized de Bruijn digraphs. Wang [24] showed that there is an efficient twin dominating set in $G_B(n, d)$ with $n = c(d + 1)$ if and only if $d$ is even and relatively prime to $c$. More studied progress on the generalized de Bruijn and Kautz networks can be found in [8, 25, 26].

In order to make our arguments easier to follow we introduce the modulo interval so as to represent the out-neighborhood of each vertex in $G_B(n, d)$ and $G_K(n, d)$. Let $I = \{0, 1, \ldots, n - 1\}$ denote the vertex set of $G_B(n, d)$. For any integers $i, j$ satisfying $i \not\equiv j \pmod{n}$, a modulo
interval \([i, j] \pmod{n}\), with respect to modulo \(n\), is defined by

\[
[i, j] \pmod{n} = \begin{cases} 
\{i, i+1, \ldots, j\} \pmod{n} & \text{if } i \pmod{n} < j \pmod{n}, \\
\{i, \ldots, n-1, 0, \ldots, j\} \pmod{n} & \text{if } i \pmod{n} > j \pmod{n}.
\end{cases}
\]

By the definitions, \(I = [0, n-1]\), and for each \(j \in [0, n-1]\), clearly \(O(j) = \left[jd, jd + (d-1)\right] \pmod{n}\) in \(G_B(n, d)\) and \(O(j) = [-jd - d, -jd - 1] \pmod{n}\) in \(G_K(n, d)\).

Notice that if \(d = 1\) then the graph \(G_B(n, 1)\) (or \(G_K(n, 1)\)) has \(n\) self-loops. Throughout this paper, we always assume \(d \geq 2\) and \(n \geq d\). If the set \(D = \{x, x+1, \ldots, x+k\} \pmod{n}\) is a dominating set or a distance \(k\)-dominating set of \(G_B(n, d)\) (or \(G_K(n, d)\)), then \(D\) is called a consecutive dominating set or a consecutive distance \(k\)-dominating set of \(G_B(n, d)\) (or \(G_K(n, d)\)).

A consecutive minimum dominating set of \(G_B(n, d)\) (or \(G_K(n, d)\)) is a consecutive dominating set with cardinality \(\gamma(G_B(n, d))\) (or \(\gamma(G_K(n, d))\)) and a consecutive distance \(k\)-dominating set of \(G_B(n, d)\) (or \(G_K(n, d)\)) is a consecutive distance \(k\)-dominating set with cardinality \(\gamma_k(G_B(n, d))\) (or \(\gamma_k(G_K(n, d))\)).

Tian and Xu [25] established the upper and lower bounds on the distance \(k\)-domination number of \(G_B(n, d)\) and \(G_K(n, d)\). This paper continues to study distance \(k\)-domination in generalized de Bruijn and Kautz digraphs. In Subsection 2.1, we show that every generalized de Bruijn digraph \(G_B(n, d)\) has the distance \(k\)-domination number either \(\lceil n/\sum_{j=0}^{k} d^j \rceil\) or \(\lceil n/\sum_{j=0}^{k} d^j \rceil + 1\). In Subsection 2.2, we derive various sufficient conditions for \(\gamma_k(G_B(n, d)) = \lceil n/\sum_{j=0}^{k} d^j \rceil\). In Section 3, we gives a sharp upper bound of \(\gamma_k(G_K(n, d))\), which improves the previous upper bound of \(\gamma_k(G_K(n, d))\), due to Tian and Xu [25]. In closing section, we pose two open problems.

2 The minimum distance \(k\)-dominating sets in \(G_B(n, d)\)

In the first subsection of this section, by constructing a distance \(k\)-dominating set of an arbitrary generalized de Bruijn digraph \(G_B(n, d)\), we show that the distance \(k\)-domination number of \(G_B(n, d)\) has exactly two values. In next subsection, we describe various sufficient conditions for the distance \(k\)-domination number equal to one of two values.

2.1 The distance \(k\)-domination number of \(G_B(n, d)\)

Tian and Xu [25] observed the following upper and lower bounds on \(\gamma_k(G_B(n, d))\).
Lemma 2.1. \((\ref{lem:gamma-bound})\) For every generalized de Bruijn digraph \(G_B(n, d)\),

\[
\left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \leq \gamma_k(G_B(n, d)) \leq \left\lceil \frac{n}{d^k} \right\rceil .
\]

We are ready to improve the above upper bound on \(\gamma_k(G_B(n, d))\) by directly constructing a (consecutive) distance \(k\)-dominating set of \(G_B(n, d)\) with cardinality \(\left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil + 1\). The following lemma plays a key role in constructing such a distance \(k\)-dominating set of \(G_B(n, d)\).

Lemma 2.2. Every generalized de Bruijn digraph \(G_B(n, d)\) contains a vertex \(x\) satisfying the following inequality:

\[
x + \left\lceil \frac{n}{k \sum_{j=0}^{k} d^j} \right\rceil - (d - 2) \leq dx \leq x + \left\lceil \frac{n}{k \sum_{j=0}^{k} d^j} \right\rceil \pmod{n}.
\]

Proof. We choose an arbitrary vertex \(x \in V(G_B(n, d))\). If \(x\) satisfies \((\ref{eq:gamma-bound})\), we are done. Otherwise, the vertex \(x\) clearly satisfies either

\[
0 \leq dx \leq x + \left\lceil \frac{n}{k \sum_{j=0}^{k} d^j} \right\rceil - (d - 1) \pmod{n}
\]

or

\[
x + \left\lceil \frac{n}{k \sum_{j=0}^{k} d^j} \right\rceil + 1 \leq dx \leq n - 1 \pmod{n}.
\]

We find the desired vertex by distinguishing the following two cases.

Case 1. \(0 \leq dx \leq x + \left\lceil \frac{n}{k \sum_{j=0}^{k} d^j} \right\rceil - (d - 1) \pmod{n}\). Note that if \(x\) increases by integer \(i\), then the value of \(dx\) is increased to \(d(x + i) = dx + di\). In this case, we find the desired vertex by increasing the value of \(x\). Since \(dx \leq x + \left\lceil \frac{n}{k \sum_{j=0}^{k} d^j} \right\rceil - (d - 1) \pmod{n}\), there exists an integer \(i (\geq 0)\) such that \(x\) and \(i\) satisfy the following inequality

\[
d(x + i) \leq x + \left\lceil \frac{n}{k \sum_{j=0}^{k} d^j} \right\rceil - (d - 2) \pmod{n},
\]

since \(i = 0\) satisfies the inequality. Let \(i\) be the maximal integer satisfying \((\ref{eq:gamma-bound})\). We claim that

\[
d(x + i) \geq (x + i) + \left\lceil \frac{n}{k \sum_{j=0}^{k} d^j} \right\rceil - 2(d - 2) \pmod{n}.
\]
Indeed, if \(d(x + i) \leq (x + i) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - 2(d - 2) - 1 \pmod{n}\), then

\[
d(x + i + 1) \leq (x + i + 1) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - (d - 2) \pmod{n}.
\]

So \(i + 1\) satisfies (2), too, this contradicts the maximality of \(i\). Hence (3) follows. If the equality holds in (2), that is,

\[
d(x + i) = x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - (d - 2) \pmod{n},
\]

then \(x + i\) satisfies (1). So we replace \(x\) by \(x + i\), and obtain the desired vertex. Otherwise, by (3), we have

\[
(x + i) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - 2(d - 2) \leq d(x + i) \leq (x + i) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - (d - 1) \pmod{n}.
\]

Hence,

\[
(x + i + 1) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - (d - 3) \leq d(x + i + 1) \leq (x + i + 1) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \pmod{n}.
\]

Clearly, \(x + i + 1\) satisfies (1). Thus we replace \(x\) by \(x + i + 1\) and obtain the desired vertex.

Case 2. \(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil + 1 \leq dx \leq n - 1 \pmod{n}\). We can obtain the desired vertex by decreasing the value of \(x\). Clearly, there exists an integer \(i \geq 0\) such that \(x\) and \(i\) satisfy the following inequality

\[
d(x - i) \geq (x - i) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \pmod{n}, \quad (4)
\]

since the inequality \(dx \geq x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil + 1\) implies that \(i = 0\) satisfies (1). Let \(i\) be the maximal integer satisfying (1). We claim that

\[
d(x - i) \leq (x - i) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil + d - 2 \pmod{n}.
\]

(5)
Suppose, to the contrary, that \( d(x - i) \geq (x - i) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil + d - 1 \mod n \). Equivalently,

\[
d(x - (i + 1)) \geq (x - (i + 1)) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \mod n.
\]

But then \( i + 1 \) satisfies (1). This contradicts the maximality of \( i \). Thus (5) holds. If the equality holds in (4), then the vertex \( x - i \) satisfies (1). So we obtain the desired vertex by replacing \( x \) by \( x - i \). Otherwise, by (5), we have

\[
(x - i) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil + 1 \leq d(x - i) \leq (x - i) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil + d - 2 \mod n.
\]

Hence,

\[
(x - (i + 1)) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - (d - 2) \leq d(x - (i + 1))
\]

\[
\leq (x - (i + 1)) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - 1 \mod n.
\]

Hence \( x - (i + 1) \) satisfies (1). We obtain the desired vertex by replacing \( x \) by \( x - (i + 1) \). \( \square \)

**Theorem 2.1.** For every generalized de Bruijn digraph \( G_B(n, d) \),

\[
\gamma_k(G_B(n, d)) = \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \text{ or } \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil + 1.
\]

**Proof.** By Lemma 2.1, it suffices to show that \( \gamma(G_B(n, d)) \leq \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil + 1 \). The proof is by directly constructing a (consecutive) distance \( k \)-dominating set of \( G_B(n, d) \) with cardinality \( \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil + 1 \). By Lemma 2.2, there is a vertex \( x \) in \( G_B(n, d) \) that satisfies (1). Let

\[
D = \{ x, x + 1, \ldots, x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \}. \quad \text{We show that } D \text{ is a distance } k \text{-dominating set of } G_B(n, d). \text{ By the definition, we need to prove that } \bigcup_{i=0}^{k} O_i(D) = V(G_B(n, d)).
\]

First, we show that the vertices of \( O_{i-1} \cup O_i(D) \) are consecutive for all \( i, 1 \leq i \leq k \). The out-neighborhoods of vertices in \( D \) are given as follows.

\[
O(x) = \{ dx, dx + 1, \ldots, dx + d - 1 \} \mod n,
\]

\[
O(x + 1) = \{ d(x + 1), d(x + 1) + 1, \ldots, d(x + 1) + d - 1 \} \mod n,
\]

\[
O(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil) = \{ d(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil), \ldots, d(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil) + d - 1 \} \mod n.
\]
Then \( O(D) = \left[ dx, d(x + \left[ n/\sum_{j=0}^{k} d^j \right]) + d - 1 \right] \pmod{n} \). Similarly, the \( i \)-th out-neighborhoods \( O_i(D) = \left[ d^i x, d^i(x + \left[ n/\sum_{j=0}^{k} d^j \right]) + (d - 1) \sum_{j=0}^{i} d^j \right] \pmod{n} \) for each \( i, 1 \leq i \leq k \). Since \( x \) satisfying the inequality (1), there exists an integer \( h, 0 \leq h \leq d - 2 \), such that \( dx = x + \left[ n/\sum_{j=0}^{k} d^j \right] - h \pmod{n} \), so we have

\[
d^2 x = d\left( x + \left[ n/\sum_{j=0}^{k} d^j \right] \right) - dh \pmod{n},
\]
\[
d^3 x = d^2\left( x + \left[ n/\sum_{j=0}^{k} d^j \right] \right) - d^2 h \pmod{n},
\]
\[
\vdots
\]
\[
d^k x = d^{k-1}\left( x + \left[ n/\sum_{j=0}^{k} d^j \right] \right) - d^{k-1} h \pmod{n}.
\]

Thus \( O_{i-1}(D) \cap O_i(D) \neq \emptyset \) for all \( i, 1 \leq i \leq k \). This implies that the vertices of \( O_{i-1}(D) \cup O_i(D) \) are consecutive, since the vertices of \( O_i(D) \) are consecutive for each \( i, 0 \leq i \leq k \). Therefore, the vertices of \( \bigcup_{i=0}^{k} O_i(D) \) are consecutive.

Next we show that \( \bigcup_{i=0}^{k} O_i(D) \) contains all the vertices of \( G_B(n, d) \). Note that \( O_1(D) \cap D \neq \emptyset \). Thus it suffices to show that \( O_k(D) \cap D \neq \emptyset \). For the last vertex in \( O_k(D) \), since \( x \) satisfies (1), we have

\[
d^k\left( x + \left[ n/\sum_{j=0}^{k} d^j \right] \right) + (d - 1) \sum_{j=0}^{k} d^j
\]
\[
= d^{k-1}\left( x + \left[ n/\sum_{j=0}^{k} d^j \right] - h \right) + d^k\left[ n/\sum_{j=0}^{k} d^j \right] + (d - 1) \sum_{j=0}^{k} d^j
\]
\[
= d^{k-1} x + (d^k + d^{k-1})\left[ n/\sum_{j=0}^{k} d^j \right] + (d - 1)d^k - hd^{k-1} + (d - 1) \sum_{j=0}^{k} d^j
\]
\[
\vdots
\]
\[
= x + \left[ n/\sum_{j=0}^{k} d^j \right] \sum_{j=0}^{k} d^j - h \sum_{j=0}^{k-1} d^j + (d - 1) \sum_{j=0}^{k} d^j
\]
\[
= x + (d - 1) + \left[ n/\sum_{j=0}^{k} d^j \right] \sum_{j=0}^{k} d^j + (d(d - 1) - h) \sum_{j=0}^{k-1} d^j
\]
\[
\geq x \pmod{n}
\]
The last inequality holds, since \( d \geq 2 \) and \( 0 \leq h \leq d - 2 \). Hence \( O_k(D) \cap D \neq \emptyset \), and so

\[
\bigcup_{i=1}^{k} O_i(D) \supseteq \{ x + \lceil n/ \sum_{j=0}^{k} d^j \rceil, \ldots, n - 1, 0, 1, \ldots, x \}.
\]

This implies that \( \bigcup_{i=0}^{k} O_i(D) = V(G_B(n, d)) \), that is, \( D \) is a (consecutive) distance \( k \)-dominating set of \( G_B(n, d) \). Consequently, \( \gamma_k(G_B(n, d)) \leq |D| = \lceil n/ \sum_{j=0}^{k} d^j \rceil + 1 \). ☐

For distance \( k = 1 \) we obtain the following result.

**Corollary 2.1.** ([7]) For every generalized de Bruijn digraph \( G_B(n, d) \), either \( \gamma(G_B(n, d)) = \left\lceil \frac{n}{d+1} \right\rceil \) or \( \gamma(G_B(n, d)) = \left\lceil \frac{n}{d+1} \right\rceil + 1 \).

### 2.2 The generalized de Bruijn digraphs \( G_B(n, d) \) with \( \gamma(G_B(n, d)) = \left\lceil \frac{n}{d+1} \right\rceil \)

In the next subsection, we derive various sufficient conditions for the distance \( k \)-domination number to achieve the value \( \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil \) in a generalized de Bruijn digraph \( G_B(n, d) \).

**Theorem 2.2.** If there exists a vertex \( x \in V(G_B(n, d)) \) satisfying the following congruence equation:

\[
(d-1)x \equiv \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil - h \pmod{n},
\]

for some \( h \) where \( 0 \leq (\sum_{j=0}^{k-1} d^j)h \leq (\sum_{j=0}^{k} d^j)[\left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil - n] \), then \( \gamma_k(G_B(n, d)) = \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil \), and \( D = \{ x, x+1, x+2, \ldots, x + \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil - 1 \} \) is a consecutive minimum distance \( k \)-dominating set of \( G_B(n, d) \).

**Proof.** Let \( x \) be a vertex of \( G_B(n, d) \) satisfying Eq. (6). Note that \( |D| = \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil \).

By Theorem 2.1, it is sufficient to show that \( D = \{ x, x+1, x+2, \ldots, x + \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil - 1 \} \) is a distance \( k \)-dominating set of \( G_B(n, d) \). For this purpose, we show that \( \bigcup_{i=1}^{k} O_i(D) = V(G_B(n, d)) \).

We first prove that the vertices of \( O_{i-1}(D) \cup O_i(D) \) are consecutive for all \( i, 1 \leq i \leq k \). By
the definition of $G_B(n,d)$, the out-neighborhoods $O(D)$ of $D$ are given as follows.

$$O(x) = \{dx, dx+1, \ldots, dx+d-1\} \pmod{n},$$

$$O(x+1) = \{d(x+1), d(x+1)+1, \ldots, d(x+1)+d-1\} \pmod{n},$$

$$\vdots$$

$$O\left(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - 1 \right) = \left \{ d\left(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - d, d\left(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - d+1, \ldots, d\left(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - 1 \right \} \pmod{n}.$$  

Then $O(D) = \left\{ dx, dx+d\left[n/\sum_{j=0}^{k} d^j \right] - 1 \right\} \pmod{n}$. Similarly, we have $O_i(D) = \left\{ d^i x, d^i \left(x + \left[n/\sum_{j=0}^{k} d^j \right] \right) - 1 \right\} \pmod{n}$. Clearly, $|O_i(D)| = d^i \left[n/\sum_{j=0}^{k} d^j \right]$ for all $i, 0 \leq i \leq k$. Since $x$ satisfies Eq. (6), we have

$$O(D) = \left\{ x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - h, d\left(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - 1 \right\} \pmod{n},$$

$$O_2(D) = \left\{ d\left(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - dh, d^2\left(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - 1 \right\} \pmod{n},$$

$$\vdots$$

$$O_k(D) = \left\{ d^{k-1}\left(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - d^{k-1}h, d^k\left(x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - 1 \right\} \pmod{n}.$$  

Hence it can be seen that $|O_{i-1}(D) \cap O_i(D)| = d^{i-1}h$ for all $i, 1 \leq i \leq k$. Note that the vertices of each $O_i(D)$ ($i \geq 0$) are consecutive. By the above observations, if $h = 0$, then the last vertex of $O_{i-1}(D)$ and the first vertex of $O_i(D)$ are consecutive; while if $h > 0$, then $O_{i-1}(D) \cap O_i(D) \neq \emptyset$. Thus the vertices of $O_{i-1}(D) \cup O_i(D)$ are consecutive for all $i, 1 \leq i \leq k$.

We next show that $\bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d))$. As observed above, we see that the vertices of $\bigcup_{i=0}^{k} O_i(D)$ are consecutive. In particular, the vertices of $D \cup O_1(D)$ are consecutive. Thus it suffices to show that the vertices $O_k(D) \cup D$ are consecutive. For the last vertex in $O_k(D)$,
because $0 \leq \left( \sum_{j=0}^{k-1} d^j \right) h \leq \left( \sum_{j=0}^{k} d^j \right) \left[ n / \sum_{j=0}^{k} d^j \right] - n$, we have

$$d^k \left( x + \left[ n / \sum_{j=0}^{k} d^j \right] \right) - 1 \pmod{n}$$

$$= x + \left( \sum_{j=0}^{k} d^j \right) \left[ n / \sum_{j=0}^{k} d^j \right] - \left( \sum_{j=0}^{k-1} d^j \right) h - 1 \pmod{n} \quad (\text{by } (6))$$

$$\geq x - 1 \pmod{n}.$$  

This implies that the vertices of $O_k(D) \cup D$ are consecutive, so

$$\bigcup_{i=1}^{k} O_i(D) \supseteq \{ x + \left[ n / \sum_{j=0}^{k} d^j \right], \ldots, n - 1, 0, 1, \ldots, x - 1 \}.$$  

This implies that $\bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d))$, hence $D$ is a distance $k$-dominating set of $G_B(n,d)$. This complete the proof of Theorem 2.2.  

As a special case of Theorem 2.2, we immediately have the following corollary.

**Corollary 2.2.** Let $\sum_{j=0}^{k} d^j \mid n$. If there is a vertex $x \in V(G_B(n,d))$ satisfying congruence equation:

$$(d - 1)x \equiv n / \sum_{j=0}^{k} d^j \pmod{n}, \quad (7)$$

then $\gamma_k(G_B(n,d)) = n / \sum_{j=0}^{k} d^j$ and $D = \{ x, x + 1, \ldots, x + n / \sum_{j=0}^{k} d^j - 1 \}$ is a consecutive minimum distance $k$-dominating set of $G_B(n,d)$.

**Remark 2.1.** If $G_B(n,d)$ contains no vertex $x$ satisfying (6) in Theorem 2.2, it is possible to encounter $\gamma_k(G_B(n,d)) = \left[ n / \sum_{j=0}^{k} d^j \right] + 1$. For example, let $G_B(40,3)$ and $k = 3$. The congruence equation $(d - 1)x \equiv \left[ n / \sum_{j=0}^{k} d^j \right] - h \pmod{n}$ is $2x \equiv 1 \pmod{40}$ where $h = 0$, since $40 / \sum_{j=0}^{3} 3^j = 1$. Clearly, there is no vertex satisfying $2x \equiv 1 \pmod{40}$. We can deduce that $\gamma_3(G_B(40,3)) = \left[ 40 / \sum_{j=0}^{3} 3^j \right] + 1 = 2$. Indeed, for each $x$ of $G_B(40,3)$, it can be verify that $\{ x \}$ is not a distance $3$-dominating set of $G_B(40,3)$ by simply enumeration.

Recalling that $G_B(d^m, d) = B(d, m)$ when $n = d^m$. For cases $k = 1$ and $k = 2$, the distance $k$-domination numbers of a de Bruijn digraph $B(d, m)$ were proved by Araki [1] and Tian [25], respectively. As an application of Theorem 2.2 we can determine the distance $k$-domination number of a de Bruijn digraph for all $k \geq 1$.  

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Corollary 2.3. For $d \geq 2$, $\gamma_k(B(d, m)) = \left\lceil \frac{d^m}{\sum_{j=0}^{k} d^j} \right\rceil$.

Proof. If $m \leq k$, then clearly $\gamma_k(B(d, m)) = \gamma_k(G_B(d^m, d)) = 1 = \left\lceil \frac{d^m}{\sum_{j=0}^{k} d^j} \right\rceil$ by Theorem 2.2, so the assertion holds. We may therefore assume $m > k$. Let $m = ik + l$, where $i \geq 1$ and $0 \leq l \leq k - 1$. Note that $d^m = (\sum_{j=0}^{k} d^j)(d^{m-k} - d^{m-k-1}) + d^{m-k-1}, \ d^{m-k-1} = (\sum_{j=0}^{k} d^j)(d^{m-2k-1} - d^{m-2k-2}) + d^{m-2k-2}, \ldots$, then we have

$$d^m = \begin{cases} (\sum_{j=0}^{k} d^j)(d^{m-k} - d^{m-k-1}) + (d^{m-2k-1} - d^{m-2k-2}) \\ + \ldots + (d^{m-(i-1)k-(i-2)} - d^{m-(i-1)k-(i-1)}) + d^{m-(i-1)k-(i-1)}, \text{ if } l < i, \\ (\sum_{j=0}^{k} d^j)(d^{m-k} - d^{m-k-1}) + (d^{m-2k-1} - d^{m-2k-2}) \\ + \ldots + (d^{m-ik-(i-1)} - d^{m-ik-i}) + d^{m-ik-i}, \text{ if } l \geq i. \end{cases}$$

Because $m = ik + l$ and $0 \leq l \leq k - 1$, if $l < i$, then $d^{m-(i-1)k-(i-1)} = d^{l+k-(i-1)} \leq d^k$; and if $l \geq i$, then $d^{m-ik-i} = d^{l-i} < d^k$. Thus

$$\left\lceil \frac{d^m}{\sum_{j=0}^{k} d^j} \right\rceil = \begin{cases} (d-1)(d^{m-k-1} + d^{m-2k-2} + \ldots + d^{m-(i-1)k-(i-1)}) + 1, \text{ if } l < i, \\ (d-1)(d^{m-k-1} + d^{m-2k-2} + \ldots + d^{m-ik-i}) + 1, \text{ if } l \geq i. \end{cases}$$

Hence either $x = d^{m-k-1} + d^{m-2k-2} + \ldots + d^{m-(i-1)k-(i-1)}$ or $x = d^{m-k-1} + d^{m-2k-2} + \ldots + d^{m-ik-i}$ in $B(d, m)$ satisfies the congruence equation $(d-1)x \equiv \left\lceil \frac{d^m}{\sum_{j=0}^{k} d^j} \right\rceil - h \pmod{n}$ where $h = 1$ and $0 \leq h \sum_{j=0}^{k} d^j \leq (\sum_{j=0}^{k} d^j)(d^m/\sum_{j=0}^{k} d^j) - d^m$. Therefore, $\gamma_k(B(d, m)) = \left\lceil \frac{d^m}{\sum_{j=0}^{k} d^j} \right\rceil$ by Theorem 2.2. \hfill $\Box$

As an application of Corollary 2.2, we provide a new sufficient condition for $\gamma_k(G_B(n, d))$ equal to $\left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil$. For this purpose, we need the following result in elementary number theory.

For notational convenience, $m \mid n$ means that $m$ divides $n$ and $m \nmid n$ means that $m$ does not divide $n$ where $m, n$ are integers. For integers $a_1, a_2, \ldots, a_n$, the greatest common divisor of $a_1, a_2, \ldots, a_n$ is denoted by $(a_1, a_2, \ldots, a_n)$.

Lemma 2.3. \((13)\) For integers $a_1, a_2, \ldots, a_m \ (m \geq 1)$, $b$ and $n$, the congruence equation $\sum_{i=1}^{m} a_i x_i \equiv b \pmod{n}$ has at least a solution if and only if $(a_1, a_2, \ldots, a_m, n) \mid b$.

Theorem 2.3. For every generalized de Bruijn digraph $G_B(n, d)$, if both $n$ and $d$ satisfy one of the following conditions:

(i) $\sum_{j=0}^{k} d^j \mid n$ and $(d-1, n) \mid \sum_{j=0}^{k} d^j$,
As already observed in Theorem 2.2, we have\n
\[
O(\gamma)\quad \text{and}\quad |x| \text{ be such a vertex and let}
\]

then \(\gamma_k(G_B(n,d)) = \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil\) and there is a vertex \(x \in V(G_B(n,d))\) such that \(D = \{x, x+1, \cdots, x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - 1\}\) is a consecutive minimum distance \(k\)-dominating set of \(G_B(n,d)\).

**Proof.** Let \(n\) and \(d\) satisfy one of the conditions (i)-(ii). We show that \(G_B(n,d)\) contains a vertex \(x\) such that \(D = \{x, x+1, \cdots, x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - 1\}\) is a consecutive minimum distance \(k\)-dominating set of \(G_B(n,d)\). By Theorem 2.1 and Theorem 2.2, it suffices to show that there exists a vertex \(x \in V(G_B(n,d))\) satisfying \((d-1)x = \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - h \pmod{n}\) (Eq. 2.4) for some \(h\) where \(0 \leq (\sum_{j=0}^{k-1} d^j)h \leq \left(\sum_{j=0}^{k} d^j\right)\left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - n\).

(i) Suppose that \(\sum_{j=0}^{k} d^j | n\) and \((d-1,n) | \sum_{j=0}^{k} d^j\). By Lemma 2.3, there is a vertex \(x \in V(G_B(n,d))\) satisfying \((d-1)x \equiv \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - h \pmod{n}\), so the assertion follows directly from Corollary 2.2.

(ii) Suppose that \(\left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \equiv q \pmod{(d-1,n)}\), where \(q\) satisfies the inequality \(0 \leq q(\sum_{j=0}^{k-1} d^j) \leq (\sum_{j=0}^{k} d^j)\left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - n\). Let \((d-1,n) = r\) and \(\left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil = pr + q\) where \(p \geq 0\) and \(0 \leq q \leq r - 1\). Set \(q = h\). Since \((d-1,n)|pr\), the equation \((d-1)x \equiv pr \pmod{n}\) has a solution by Lemma 2.3. Hence, there exists a vertex \(x \in V(G_B(n,d))\) satisfying \((d-1)x \equiv \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - h \pmod{n}\), as desired. \(\square\)

By applying Theorems 2.1 and 2.2 we obtain the following sufficient condition for \(\gamma_k(G_B(n,d))\) equal to \(\left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil\).

**Theorem 2.4.** If \(n = p(\sum_{j=0}^{k} d^j) + q\), where \(p \geq 1\) and \(1 \leq q \leq \min\{1 + 2\sum_{j=0}^{k-1} d^j, \sum_{j=1}^{k} d^j\}\), then \(\gamma_k(G_B(n,d)) = \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil\).

**Proof.** By Theorem 2.1 we have known that \(G_B(n,d)\) contains a vertex satisfying (1). Let \(x\) be such a vertex and let \(D = \{x, x+1, \cdots, x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - 1\}\). We claim that \(D\) is a distance \(k\)-dominating set of \(G_B(n,d)\). By the definition, it suffices to show that \(\bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d))\).

As before, we first show the vertices of \(O_{i-1}(D) \cup O_i(D)\) are consecutive for all \(i, 1 \leq i \leq k\). As already observed in Theorem 2.2, we have \(O_k(D) = |d^i x, d^i (x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - 1)| \pmod{n}\) and \(|O_i(D)| = d^i \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil\) for all \(i, 0 \leq i \leq k\). Since \(x\) satisfies the inequality (1), there
exists an integer $h$, $0 \leq h \leq d - 2$ such that $dx = x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - h \pmod{n}$.

\[ d^2 x = d \left( x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - dh \pmod{n}, \]

\[ d^3 x = d^2 \left( x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - d^2 h \pmod{n}, \]

\[ \vdots \]

\[ d^k x = d^{k-1} \left( x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - d^{k-1} h \pmod{n}. \]

Since $O_i(D) = [d^i x, d^i \left( x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \right) - 1] \pmod{n}$ for all $i, 0 \leq i \leq k$, the vertices of $O_{i-1}(D) \cap O_i(D) \neq \emptyset$ are consecutive for all $i, 1 \leq i \leq k$.

By the above fact, we show that $\bigcup_{i=1}^{k} O_i(D)$ contains all the vertices of $G_B(n,d) \setminus D$ by showing the vertices of $O_k(D) \cup D$ are consecutive. We consider the last vertex in $O_k(D)$. Since $n = p \left( \sum_{j=0}^{k} d^j \right) + q$, $\left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \sum_{j=0}^{k} d^j = n - q + \sum_{j=0}^{k} d^j$. Hence, by $dx = x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - h \pmod{n}$ where $0 \leq h \leq d - 2$, we have

\[ d^k x + d^k \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - 1 = d^{k-1} \left( x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - h \right) + d^k \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - 1 \]

\[ = d^{k-1} x + (d^k + d^{k-1}) \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil - d^{k-1} h - 1 \]

\[ = \ldots \]

\[ = (x - 1) + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil \sum_{j=0}^{k} d^j - h \sum_{j=0}^{k} d^j \pmod{n} \]

\[ = (x - 1) + 1 + (d - h) \sum_{j=0}^{k-1} d^j - q \pmod{n} \]

\[ \geq (x - 1) + 1 + 2 \sum_{j=0}^{k-1} d^j - q \pmod{n} \]

\[ \geq x - 1, \]

The last inequality holds, since $1 \leq q \leq \min\{1 + 2 \sum_{j=0}^{k-1} d^j, \sum_{j=1}^{k} d^j \}$. Note that the vertices of $O_i(D)$ are consecutive for all $i, 0 \leq i \leq k$, so $\bigcup_{i=1}^{k} O_i(D) \supseteq \{x + \left\lceil \frac{n}{\sum_{j=0}^{k} d^j} \right\rceil, \ldots, n - 1, 0, 1, \ldots, x - 1\}$. This implies that $\bigcup_{i=1}^{k} O_i(D) \supseteq V(G_B(n,d)) \setminus D$, hence $D = \{x, x + 1, x + 14$
Lemma 3.1. Tian and Xu [25] observed the following upper and lower bounds on $\gamma_k(G_B(n,d))$. Thus $\gamma_k(G_B(n,d)) \leq |D| = \left\lceil n/\sum_{j=0}^{k} d^j \right\rceil$. By Theorem 2.1 $\gamma_k(G_B(n,d)) = \left\lceil n/\sum_{j=0}^{k} d^j \right\rceil$. \hfill $\square$

3 The minimum distance $k$-dominating sets in $G_K(n,d)$

Tian and Xu [25] observed the following upper and lower bounds on $\gamma_k(G_K(n,d))$.

Lemma 3.1. ([25]) For any generalized Kautz digraph $G_K(n,d)$,

$$\left\lceil n/\sum_{j=0}^{k} d^j \right\rceil \leq \gamma_k(G_K(n,d)) \leq \left\lceil \frac{n}{d^k} \right\rceil.$$

In this section, we shall improve the above upper bound on $\gamma_k(G_K(n,d))$ by constructing a consecutive distance $k$-dominating set of $G_K(n,d)$.

Theorem 3.1. Let $G_K(n,d)$ be a generalized Kautz digraph. Then $D = \{0,1,\ldots,\left\lceil n/(d^k + d^{k-1}) \right\rceil - 1\}$ is a distance $k$-dominating set of $G_K(n,d)$, and so

$$\gamma_k(G_K(n,d)) \leq \left\lceil \frac{n}{d^k + d^{k-1}} \right\rceil.$$

Proof. We show that $D$ is a distance $k$-dominating set of $G_K(n,d)$. By the definitions of $G_K(n,d)$ and $i$-th out-neighborhood, if $k$ is odd, then we obtain

$$O_{k-1}(D) = \{0,1,\ldots,d^{k-1} \left\lceil n/(d^k + d^{k-1}) \right\rceil - 1\},$$

$$O_k(D) = \{n-1,n-2,\ldots,n-d^k \left\lceil n/(d^k + d^{k-1}) \right\rceil \};$$

if $k$ is even, then

$$O_{k-1}(D) = \{n-1,n-2,\ldots,n-d^{k-1} \left\lceil n/(d^k + d^{k-1}) \right\rceil \},$$

$$O_k(D) = \{0,1,\ldots,d^k \left\lceil n/(d^k + d^{k-1}) \right\rceil - 1\}.$$

In both cases, we have $|O_{k-1}(D)| = d^{k-1} \left\lceil n/(d^k + d^{k-1}) \right\rceil$ and $|O_k(D)| = d^k \left\lceil n/(d^k + d^{k-1}) \right\rceil$. Note that the vertices of $O_{k-1}(D)$ and $O_k(D)$ are consecutive, and $(d^k + d^{k-1}) \left\lceil n/(d^k + d^{k-1}) \right\rceil \geq n$, so $O_{k-1}(D) \cup O_k(D) = V(G_K(n,d))$. Hence $D$ is a distance $k$-dominating set of $G_K(n,d)$. Therefore, $\gamma_k(G_K(n,d)) \leq |D| = \left\lceil n/(d^k + d^{k-1}) \right\rceil$. \hfill $\square$
**Remark 3.1.** The upper bound on the distance $k$-domination number given in Theorem 3.1 is sharp. For example, we consider the digraph $G_K(7, 2)$. We claim that $\gamma_2(G_K(7, 2)) = 2 = \left\lceil \frac{7}{2+1} \right\rceil$. Suppose not, we have $\gamma_2(G_K(7, 2)) = 1$ by Lemma 3.1. Let $\{x_0\}$ be a minimum distance 2-dominating set of $G_K(7, 2)$. Since $|O_i(x)| = d = 2$ for each $x \in V(G_K(7, 2))$, we have $O_i(x_0) \cap O_j(x_0) = \emptyset$ for all $0 \leq i \neq j \leq 2$. On the other hand, it can be verified that for each $x \in V(G_K(7, 2))$, there exist integers $i, j$, $0 \leq i \neq j \leq 2$, such that $O_i(x) \cap O_j(x) \neq \emptyset$ by the simply enumeration. Thus each vertex $x$ of $G_K(7, 2)$ can not form a distance 2-dominating set of $G_K(7, 2)$, as claimed. By Theorem 3.1, $D = \{0, 1\}$ must be a minimum distance 2-dominating set of $G_K(7, 2)$.

The following result on the domination number of $G_K(n, d)$, due to Kikuchi and Shibata [16], is an immediate consequence of Lemma 3.1 and Theorem 3.1.

**Corollary 3.1.** ([16]) For every generalized Kautz digraph $G_K(n, d)$, $\gamma(G_K(n, d)) = \left\lceil \frac{n}{d+1} \right\rceil$.

It seems to be difficult to determine the minimum distance $k$-dominating set for general generalized Kautz digraphs $G_K(n, d)$. Now we present a sufficient condition for the distance $k$-domination number of $G_K(n, d)$ to be the lower bound $\left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil$ in Theorem 3.1.

**Theorem 3.2.** For every generalized Kautz digraph $G_K(n, d)$, if $(d^{k-1} + d^k) \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil \geq n$ or $d^{k-1} \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil \geq \left\lceil \frac{n}{d+1} \right\rceil$ then $\gamma_k(G_K(n, d)) = \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil$.

**Proof.** The proof is by directly constructing a (consecutive) distance $k$-dominating set of $G_K(n, d)$ with cardinality $\left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil$. Let $D = \{0, 1, \cdots, \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil - 1\}$. We claim that $D$ is a distance $k$-dominating set of $G_K(n, d)$. As we have observed, if $k$ is odd, then

$$O_{k-1}(D) = \{0, 1, \cdots, d^{k-1} \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil - 1\};$$

$$O_k(D) = \{n - 1, n - 2, \cdots, n - d^k \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil \};$$

if $k$ is even, then

$$O_{k-1}(D) = \{n - 1, n - 2, \cdots, n - d^{k-1} \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil \};$$

$$O_k(D) = \{0, 1, \cdots, d^k \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil - 1\}.$$

Clearly, $|O_{k-1}(D)| = d^{k-1} \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil$ and $|O_k(D)| = d^k \left\lceil n/ \sum_{j=0}^{k} d^j \right\rceil$. 

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Suppose that \((d^{k-1} + d^k) \left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor \geq n\). Note that the vertices of \(O_{k-1}(D)\) and \(O_k(D)\) are consecutive, so \(O_{k-1}(D) \cup O_k(D) = V(G_K(n,d))\). Thus \(D = \{0,1,\cdots, \left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor - 1\}\) is a distance \(k\)-dominating set of \(G_K(n,d)\).

Suppose that \(d^{k-1} \left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor \geq \left\lceil \frac{n}{d+1} \right\rceil\). By Lemma 3.1 and Theorem 3.1 \(D_1 = \{0,1,\cdots, \left\lceil \frac{n}{d+1} \right\rceil - 1\}\) is a minimum dominating set of \(G_K(n,d)\). Let \(D'_1 = \{n-1,n-2,\cdots,n-\left\lceil \frac{n}{d+1} \right\rceil\}\). By the definition of \(G_K(n,d)\), we have \(O(D'_1) = \{0,1,\cdots,d\left\lceil \frac{n}{d+1} \right\rceil - 1\}\). Because \(|D'_1 \cup O(D'_1)| = (d+1)\left\lceil \frac{n}{d+1} \right\rceil \geq n\), then \(D'_1\) is also a minimum dominating set of \(G_K(n,d)\). Since the vertices of \(D\) are consecutive and \(d^{k-1} \left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor \geq \left\lceil \frac{n}{d+1} \right\rceil\), we have either \(O_{k-1}(D) \supseteq D_1\) or \(O_{k-1}(D) \supseteq D'_1\). Hence \(D = \{0,1,\cdots, \left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor - 1\}\) is a distance \(k\)-dominating set of \(G_K(n,d)\). \(\square\)

4 Closing remarks

In this paper, we prove that the distance \(k\)-domination number of \(G_B(n,d)\) takes on exactly one of two values \(\left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor\) and \(\left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor + 1\). In Theorems 2.2, 2.4 we provide various sufficient conditions for \(\gamma_k(G_B(n,d))\) equal to \(\left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor\). It is of interest to determine the necessary and sufficient condition for \(\gamma_k(G_B(n,d))\) equal to \(\left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor\). In Theorem 3.1 we establish the sharp upper bound on \(\gamma_k(G_B(n,d))\). Furthermore, we provide a sufficient conditions for \(\gamma_k(G_K(n,d))\) equal to \(\left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor\) in Theorem 3.2. We propose the following open problems.

**Problem 4.1.** The sufficient condition in Theorem 2.3 is also necessary for \(\gamma_k(G_B(n,d))\) equal to \(\left\lfloor \frac{n}{\sum_{j=0}^{k} d^j} \right\rfloor\).

For Problem 4.1 Dong, Shan and Kang [7] proved that the assertion is true for the case when \(k = 1\).

**Problem 4.2.** If \(G_K(n,d)\) does not satisfy the conditions in Theorem 3.2, then \(\gamma_k(G_K(n,d)) = \left\lfloor \frac{n}{(d^{k-1} + d^k)} \right\rfloor\).

For Problem 4.2 if \(k = 1\), Corollary 3.1 due to Kikuchi and Shibata [15], implies that the assertion is true.
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