On a simple verification test of codes for modelling of magnetohydrodynamic turbulence

Igor Kulikov, Igor Chernykh and Marina Boronina
Institute of Computational Mathematics and Mathematical Geophysics SB RAS, Lavrentieva ave., 6, Novosibirsk, 630090 Russia
E-mail: kulikov@ssd.sscc.ru

Abstract. In the paper results for modelling a characteristic pattern of the solar wind flow and its turbulent structure are presented. We use a gravitational magnetic hydrodynamic model, and for the model we have created an effective numerical method based on Godunov’s method and piecewise parabolic reconstruction of the solution. We consider the evolution of the MHD turbulence of the supersonic gas flow, where random strong transversal velocity disturbances are set, the problem is considered in the vertical magnetic field. For the formation of a reference test of the supersonic MHD turbulence evolution a detailed analysis of the flow is carried out.

1. Introduction
The MHD turbulence evolution is important in a large number of astrophysical applications. The turbulence in the solar wind was examined [1], one-dimensional MHD model of interaction of the solar wind with the comet 67P/Churyumov-Gerasimenko [2] and with the Halley’s comet [3] are built, and problems of the interaction for the gas planet with the solar wind are solved [4, 5]. Similar problems on the interaction of the interstellar wind with the stars should also be noted [6].

Among the many codes for simulating magnetohydrodynamic flows let us make a special emphasis on the following three codes:

(i) Code ATHENA [7], orientated to the modelling of the magnetized cosmic plasma without taking into account the gravitation. In the code sufficiently high number of schemes for the solution of the Riemann problem and different methods of piecewise polynomial reconstruction are used. Stokes’ theorem is also used to ensure the non-divergence of the magnetic field.

(ii) Code PENCIL CODE [8], based on a finite-difference scheme of a high order of accuracy with artificial viscosity. In order to ensure the non-divergence of the magnetic field the method of the reconstruction of the electrostatic potential based on Poisson’s equation is used.

(iii) Code ZEUS [9], built on the base of the operator splitting approach, where the finite-difference scheme is used for the work of forces approximation and the advective transfer takes place due to the piecewise parabolic representation of the solution.

Of course, this list is far from exhaustive, however, these three codes are the most significant for the three-dimensional modelling of the magnetized cosmic plasma.
In the second section of the paper the computational model and a short instruction of the numerical method are described. The third section is devoted to the test description and the results of the numerical experiments. In the fourth section the flow analysis is presented. In the fifth section the conclusion is formulated.

2. Computational model

Let us write the overdetermined system of MHD equations in the operator form in three-dimensional Cartesian coordinates:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}
\]

\[
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{vv} - \mathbf{BB} + p^*) = -\rho \nabla \Phi, \tag{2}
\]

\[
\frac{\partial \rho \mathbf{E}}{\partial t} + \nabla \cdot ((\rho \mathbf{E} + p^*) \mathbf{v} - \mathbf{B} (\mathbf{B} \cdot \mathbf{v})) = - (\rho \nabla \Phi, \mathbf{v}), \tag{3}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{4}
\]

\[
\nabla \cdot \mathbf{B} = 0, \tag{5}
\]

where \(\rho\) is the density, \(\mathbf{v}\) the velocity, \(\mathbf{B}\) the magnetic field, \(p = \rho e (\gamma - 1)\) the gas pressure, \(p^* = p + \mathbf{B}^2/2\) the total pressure, \(\rho E = \frac{p}{\gamma - 1} + \rho v^2/2 + \mathbf{B}^2/2\) the total mechanical energy, \(\gamma\) the adiabatic index, \(\Phi\) the gravitational potential. The equations of the magnetic hydrodynamics are supplemented with Poisson’s equation for the relative gravitational potential.

\[
\Delta \Phi = 4\pi G (\rho - \rho_0), \tag{6}
\]

where \(G\) is the gravity constant and \(\rho_0\) the mean density. We will use the periodic boundary conditions. For the equation solution we will use the combination of the Roe-like schemes [10] and Rusanov-like schemes [11] based on the piecewise parabolic representation of the solution [12] with the approach presented in [13]. The choice of the scheme weights is based on the cosine of the angles between the vectors of the velocity and the magnetic field.

For the computation of the time step \(\tau\) in every cell of the computational domain the sound speed \(c = \sqrt{\frac{\gamma p}{\rho}}\), the Alfvén sound speed \(c_a = \frac{\mathbf{B}}{\sqrt{\rho}}\), the fast \(c_f\) and slow \(c_s\) magnetic speeds must be determined:

\[
c_f = \sqrt{\frac{(c^2 + b^2) + \sqrt{(c^2 + b^2)^2 - 4c^2c_a^2}}{2}},
\]

\[
c_s = \sqrt{\frac{(c^2 + b^2) - \sqrt{(c^2 + b^2)^2 - 4c^2c_a^2}}{2}},
\]

where \(b = \sqrt{B_x^2 + B_y^2 + B_z^2}\). After that the time step is calculated from the equation:

\[
\tau = \min \left( \frac{CFL \times h}{v + b + c + c_a + c_s + c_f} \right)
\]

where \(v = \sqrt{v_x^2 + v_y^2 + v_z^2}\) is the absolute value of the velocity vector, \(h\) is the time step, \(CFL = 0.2\) is the Courant number. Note, that these velocities are used in some part of the Roe and Rusanov schemes.
For the construction of the Roe and Rusanov schemes we need to formulate the equations of the magnetohydrodynamics in form of a quasilinear hyperbolic equation system, written for the physical variables:

\[
\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = 0,
\]

where the vector \( u \) and the matrix \( A \) have the following form:

\[
u = \begin{pmatrix}
\rho \\
v_x \\
v_z \\
p \\
B_y \\
B_z
\end{pmatrix}, \quad A = \begin{pmatrix}
\rho & 0 & 0 & 0 & 0 & 0 \\
v_x & \rho & 0 & 0 & \rho^{-1} & \frac{B_y}{\rho} \\
v_z & 0 & v_x & 0 & 0 & -\frac{B_z}{\rho} \\
p & 0 & \rho c^2 & 0 & v_x & 0 \\
B_y & 0 & -B_x & 0 & v_x & 0 \\
B_z & 0 & 0 & B_x & 0 & v_x
\end{pmatrix}.
\]

The eigenvalue vector \( \Lambda \) and the matrices of the right \( R \) and left \( L \) eigenvectors have the following form:

\[
\Lambda = (v_x - c_f, v_x - c_s, v_x - c_s, v_x + c_s, v_x + c_s, v_x + c_f)^T, \]

\[
R = \begin{pmatrix}
\rho \alpha_f & 0 & \rho \alpha_s & 1 & \rho \alpha_s & 0 \\
-c_f & 0 & -c_s & 0 & c_s & 0 \\
Q_s \beta_y & -\beta_y & -Q_f \beta_y & 0 & Q_f \beta_y & \beta_y & -Q_s \beta_y \\
Q_s \beta_z & \beta_y & -Q_f \beta_z & 0 & Q_f \beta_z & -\beta_y & -Q_s \beta_z \\
\rho \alpha_c \alpha_f & 0 & \rho \alpha_c \alpha_s & 0 & \rho \alpha_c \alpha_s & 0 \\
A_s \beta_y - sgn(B_x) \sqrt{\rho} \beta_z & -A_f \beta_y & 0 & -A_f \beta_y & -sgn(B_x) \sqrt{\rho} \beta_z & A_s \beta_y & A_s \beta_z
\end{pmatrix},
\]

\[
L = \begin{pmatrix}
0 & -c_f & \frac{Q_s \beta_y}{2 c^2} & \frac{Q_s \beta_z}{2 c^2} & \alpha_f & \frac{A_s \beta_y}{2 c^2 \rho} & \frac{A_s \beta_z}{2 c^2 \rho} \\
0 & 0 & -\beta_y/2 & \beta_y/2 & 0 & -\frac{A_f \beta_y}{2 c^2 \rho} & -\frac{A_f \beta_z}{2 c^2 \rho} \\
1 & 0 & 0 & 0 & -1/2 & 0 & 0 \\
0 & c_s & \frac{Q_s \beta_y}{2 c^2} & \frac{Q_s \beta_z}{2 c^2} & \alpha_s & \frac{A_s \beta_y}{2 c^2 \rho} & \frac{A_s \beta_z}{2 c^2 \rho} \\
0 & 0 & 0 & \beta_y/2 & -\beta_y/2 & 0 & -\frac{A_f \beta_y}{2 c^2 \rho} & -\frac{A_f \beta_z}{2 c^2 \rho} \\
0 & c_f & \frac{Q_s \beta_y}{2 c^2} & \frac{Q_s \beta_z}{2 c^2} & \alpha_f & \frac{A_s \beta_y}{2 c^2 \rho} & \frac{A_s \beta_z}{2 c^2 \rho}
\end{pmatrix},
\]

where \( c_{ff} = c_f \alpha_f, \ c_{ss} = c_s \alpha_s, \ Q_f = sgn(B_x) c_f \alpha_f, \ Q_s = sgn(B_x) c_s \alpha_s, \ A_f = \alpha_f \sqrt{\rho}, \ A_s = \alpha_s c \sqrt{\rho}. \) We need to complete the parameter definition:

\[
(\alpha_f, \alpha_s) = \begin{cases}
\left( \frac{\sqrt{c_f^2 - c_s^2} / \sqrt{c_f^2 - c_s^2} \sqrt{c_f^2 - c_s^2}}{\sqrt{c_f^2 - c_s^2} \sqrt{c_f^2 - c_s^2}} \right) & B_y^2 + B_z^2 > 0, \\
\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) & B_y^2 + B_z^2 = 0.
\end{cases}
\]

\[
(\beta_f, \beta_s) = \begin{cases}
\left( \frac{B_y B_z}{\sqrt{B_y^2 + B_z^2}} \right) & B_y^2 + B_z^2 > 0, \\
\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) & B_y^2 + B_z^2 = 0.
\end{cases}
\]

This decomposition is sufficient for the construction of the Roe flux, where we use a classical approach similar to [10], and the Rusanov flux from [11]. Let us denote the corresponding fluxes as \( F_{Roe} \) and \( F_{Rusanov} \), the resultant flux we find from the equation:

\[
F = k^2 \times F_{Roe} + (1 - k^2) \times F_{Rusanov},
\]
where \( k \) is calculated as the cosine of the angle between the velocity vector and the magnetic field vector:

\[
k = \frac{(\vec{v}, \vec{B})}{|\vec{v}| \times |\vec{B}|}.
\]

For the fulfilment of condition \( \nabla \cdot (\vec{B}) = 0 \) we use a scheme, based on Stokes' theorem:

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})
\]

For this, by solving the Riemann problem, the values of the electric field vector \( \vec{E} = \vec{v} \times \vec{B} \) are determined at the nodes of the computational grid \( E_{i, k, l}^{x,y,z} \). The magnetic field values, located in the centre of the cells, are recalculated from the previous time step using the finite-difference scheme of the equation

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E},
\]

Thus, we obtain the corresponding values on the faces of the cells \( B_{i, k, l}^{x}, B_{i, k, l}^{y}, \) and \( B_{i, k, l}^{z} \) for the nondivergent magnetic field. For the transition to the values of the vector \( \vec{B} \) in cell, we average in each direction the values of the corresponding components of the magnetic field taken from the cell edges.

3. Test description

Let us consider the evolution of the gas with the unit density in the dimensionless units in the cubic domain with length \( L = 2.56 \) until the time moment \( T = 5 \). The initial gas speed is \( v_x = 1 \) and magnetic field is \( B_x = 0.2 \), the gas pressure is \( p = 0.5 \). The disturbance is set as random value for the transversal gas speed \( v_x, v_y \). The results of the modelling are presented in the Fig. 1. From the figures the clusterization process, leading to the cloud formation, can be observed.

![Figure 1](image_url)

Figure 1. For the numerical experiment the grid with \( 256^3 \) cells is used.
4. Test analysis
The dependence of the Alfven speed on the gas density (see Fig. 2) and the cosine of the angle between the velocity vector and the magnetic field vector on the gas density (see Fig. 3) were analyzed. From the figures it can be seen, that the bigger part of the dense cloud is located in super-Alfven domain (see. Fig. 2). The reason for such regime occurrence is connected with the self-organization in the magnetized turbulent medium in the trans-Alfven regime $M \sim 1$. For such densities (see. Fig. 3) the contours of the cosine of the angle between the vectors of the velocity and the magnetic field form a saddle-like structure, which indicates that compression occurs along the lines of force of the magnetic field. Then due to the self-gravitation influence the further gain of the mass and the density of the clouds occur. In turn, in the resulted dense clouds the turbulence is only super-Alfven with the Mach number $M > 100$.

![Figure 2](image2.png)

**Figure 2.** The dependence of the Alfven speed on the gas density.

![Figure 3](image3.png)

**Figure 3.** The dependence of the cosine of the angle between the vectors of the velocity and the magnetic field and the magnetic field on the gas density.
5. Conclusion
A test for the verification of the numerical methods for the solution of the gravitational magnetic hydrodynamics and their program realizations adapted for the modelling of the solar wind flow is proposed. The detailed analysis of the flow for the formation of the reference test of the supersonic MHD turbulence evolution is carried out.

Acknowledgements
This work is supported by the Russian Foundation for the Basic Research grant 20-47-540002 and grant 20-41-000001 of the Government of Novosibirsk region of the Russian Federation.

References
[1] Galtier S., Buchlin E. Multiscale Hall-Magnetohydrodynamic Turbulence in the Solar Wind // The Astrophysical Journal. – 2007. – V. 656. – P. 560-566.
[2] Mendis D., Horanyi M. The Global Morphology of the Solar Wind Interaction with Comet Churyumov-Gerasimenko // The Astrophysical Journal. – 2014. – V. 794, I. 1. – Article Number 14.
[3] Ogino T., Walker R.J., Ashour-Abdalla M. A Three-Dimensional MHD Simulation of the Interaction of the Solar Wind With Comet Halley // Journal of Geophysical Research. – 1988. – V. 93, I. A9. – P. 9568-9576.
[4] Johnstone C.P., et al. Colliding winds in low-mass binary star systems: wind interactions and implications for habitable planets // Astronomy & Astrophysics. – 2015. – V. 577, A122.
[5] Shematovich V.I., Bisikalo D.V., Barabash S., Stenberg G. Monte Carlo study of interaction between solar wind plasma and Venusian upper atmosphere // Solar System Research. – 2014. – V. 48, I. 5. – P. 317-323.
[6] Villaver E., Manchado A., Garcia-Segura G. The interaction of asymptotic giant branch stars with the interstellar medium // The Astrophysical Journal. – 2012. – V. 748, I. 2. – Article Number 94.
[7] Stone J., et al. Athena: A New Code for Astrophysical MHD // The Astrophysical Journal Supplement Series. – 2008. – V. 178. – P. 137-177.
[8] Brandenburg A., Dobler W. Hydromagnetic turbulence in computer simulations // Computer Physics Communications. – 2002. – V. 147. – P. 471-475.
[9] Hayes J., et al. Simulating Radiating and Magnetized Flows in Multiple Dimensions with ZEUS-MP // The Astrophysical Journal Supplement Series. – 2006. – V. 165. – P. 188-228.
[10] Kulikov I. A new code for the numerical simulation of relativistic flows on supercomputers by means of a low-dissipation scheme // Computer Physics Communications. – 2020. – V. 257. – Article Number 107532.
[11] Kulikov I., Chernykh I., Tutukov A. A New Hydrodynamic Code with Explicit Vectorization Instructions Optimizations that Is Dedicated to the Numerical Simulation of Astrophysical Gas Flow. I. Numerical Method, Tests, and Model Problems // The Astrophysical Journal Supplement Series. – 2019. – V. 243. – Article Number 4.
[12] Kulikov I., Vorobyov E. Using the PPML approach for constructing a low-dissipation, operator-splitting scheme for numerical simulations of hydrodynamic flows // Journal of Computational Physics. – 2016. – V. 317. – P. 318-346.
[13] Nishikawa H., Kitamura K. Very simple, carbuncle-free, boundary-layer-resolving, rotated-hybrid Riemann solvers // Journal of Computational Physics. – 2008. – V. 227. – P. 2560-2581.