Numerical Analysis and Simulation of an Adhesive Contact Problem with Damage and Long Memory

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Abstract. This paper studies an adhesive contact model which also takes into account the damage and long memory term. The deformable body is composed of a viscoelastic material and the process is taken as quasistatic. The damage of the material caused by the compression or the tension is involved in the constitutive law and the damage function is modelled through a nonlinear parabolic inclusion. Meanwhile, the adhesion process is modelled by a bonding field on the contact surface while the contact is described by a nonmonotone normal compliance condition. The variational formulation of the model is governed by a coupled system which consists of a history-dependent hemivariational inequality for the displacement field, a nonlinear parabolic variational inequality for the damage field and an ordinary differential equation for the adhesion field. We first consider a fully discrete scheme of this system and then focus on deriving error estimates for numerical solutions. Under appropriate solution regularity assumptions, an optimal order error estimate is derived. At the end of this paper, we report some two-dimensional numerical simulation results for the contact problem in order to provide numerical evidence of the theoretical results.

1. Introduction. In this paper, our objective is to study an adhesive contact problem between a viscoelastic body and a foundation while the effect due to the damage of the material is taken into consideration as well as the long memory effect. This problem is meaningful since it appears in every aspect of real life. First, processes of adhesion are important in many industrial settings where parts, usually nonmetallic, are glued together. In recent years, composite materials reached prominence due to their high strength and light weight, and as a result, have been widely used in various kinds of fields. However, composite materials may undergo delamination due to their high strength and light weight, and as a result, have been widely used in various kinds of fields. In order to model such process, it is especially important to add the adhesion to the description of the contact process. The number of the literature on contact problems with adhesion is increasing and we refer to [10, 9, 13, 22] for more information. In the mean time, in many materials, such as concrete, as the internal cracks develop and...
grow, a decrease in the load-bearing capacity is observed over time. We can find a large number of engineering work on it now that it has affected deeply the useful life-span of the designed structure or component. For example, we can see the corresponding papers [2, 16, 26]. Recently, some literature on the contact problems has considered the effect of the internal damage, see [23, 28]. However, only few documents considering the effect of the internal damage on the contact process have considered the numerical solutions, such references include [8, 7, 6, 18].

This paper is dedicated to the study on numerical approximation of a system formulated as a general hemivariational inequality involving a history-dependent operator, a nonlinear parabolic variational inequality and an ordinary differential equation which models a quasistatic adhesive contact problem with damage and long memory. As far as we know, there is no paper that studies the numerical solutions of a hemivariational inequality arising in the contact problem with adhesion and damage. In this paper, we will fill this gap. For this system, we first give the existence and uniqueness result for it and then employ the numerical method to solve it. Under appropriate regularity assumption, optimal error estimates for the numerical solutions, such references include [8, 7, 6, 18].

In the following, we will state the mathematical model of the contact problem. Let $\Omega$ be a domain in $\mathbb{R}^d$ with $d = 2, 3$ representing the reference configuration of a viscoelastic body and it is in contact with a foundation. The boundary $\Gamma$ of $\Omega$ is supposed to be Lipschitz continuous and composed of three mutually disjoint parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$. Assume that the measure of $\Gamma_D$, denoted $m(\Gamma_D)$, is positive. The body is fixed on $\Gamma_D$, thus there is no displacement field there. The surface tractions $f_N$ act on $\Gamma_N$ and the volume forces $f_D$ act in $\Omega$. The process of the mechanical state of the body in the time interval $(0,T)$ with $T > 0$ is studied.

We first introduce and discuss the following contact problem. Let $\Omega$ be a domain in $\mathbb{R}^d$ with $d = 2, 3$ representing the reference configuration of a viscoelastic body and it is in contact with a foundation. The boundary $\Gamma$ of $\Omega$ is supposed to be Lipschitz continuous and composed of three mutually disjoint parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$. Assume that the measure of $\Gamma_D$, denoted $m(\Gamma_D)$, is positive. The body is fixed on $\Gamma_D$, thus there is no displacement field there. The surface tractions $f_N$ act on $\Gamma_N$ and the volume forces $f_D$ act in $\Omega$. The process of the mechanical state of the body in the time interval $(0,T)$ with $T > 0$ is studied.

We use the notation $u = (u_i)$, $\sigma = (\sigma_{ij})$ and $\varepsilon(u) = (\varepsilon_{ij}(u))$ for the displacement vector, the stress tensor, and linearized strain tensor, respectively. For the sake of simplicity, we have not explicitly indicated the dependence of the variables on the spatial variable $x$. Recall that the components of the linearized strain tensor $\varepsilon(u)$ are $\varepsilon_{ij}(u) = \frac{1}{2}(u_{ij} + u_{ji})$, where $u_{ij} = \frac{\partial u_i}{\partial x_j}$. The indices $i,j,k,l$ run between 1 and $d$, without any statement, the summation convention over repeated indices is used. An index following a comma stands for a partial derivative with respect to the corresponding component of the spatial variable $x$. A superscript prime of a variable indicates the time derivative of the variable. The outward unit normal on $\partial \Omega$ is denoted by $\nu$. In addition, we use the notation $v_\nu$ and $v_\tau$ for the normal and tangential components of $v$ on $\partial \Omega$ given by $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu \nu$. The normal and tangential components of the stress field $\sigma$ on the boundary are defined by $\sigma_\nu = (\sigma v) \cdot \nu$ and $\sigma_\tau = \sigma v - \sigma_\nu \nu$, respectively. The symbol $\mathbb{S}^d$ is used for the space of second order symmetric tensors on $\mathbb{R}^d$.

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**Problem 1.1.** Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times (0,T) \to \mathbb{S}^d$, a damage field $\xi : \Omega \times (0,T) \to \mathbb{R}$ and a bonding field $\beta : \Gamma_C \times (0,T) \to \mathbb{R}$ such that for all $t \in (0,T)$,

\begin{align}
\sigma(t) &= A \varepsilon(u(t)) + \int_0^t G(t-s, \varepsilon(u(s)), \xi(s)) \, ds & \text{in } \Omega, \tag{1.1} \\
\text{Div } \sigma(t) + f_D(t) &= 0 & \text{in } \Omega, \tag{1.2} \\
\xi'(t) - \kappa \Delta \xi(t) + \partial \psi_{[0,1]}(\xi(t)) &\ni \phi(\varepsilon(u(t)), \xi(t)) & \text{in } \Omega, \tag{1.3}
\end{align}
\[
\begin{align*}
\frac{\partial \zeta(t)}{\partial n} &= 0 & \text{on } \Gamma, & (1.4) \\
u(t) &= 0 & \text{on } \Gamma_D, & (1.5) \\
\sigma(t) \nu &= f_N(t) & \text{on } \Gamma_N, & (1.6) \\
-\sigma\tau(t) &= p(\beta(t))R^*(u_\tau(t)) & \text{on } \Gamma_C, & (1.7) \\
\beta'(t) &= -(\gamma_\nu(t)R(u_\nu(t))^2 - \epsilon_a)_+ & \text{on } \Gamma_C, & (1.9) \\
u(0) = u_0, \quad \zeta(0) = \zeta_0, \quad \beta(0) = \beta_0 & \text{in } \Omega. & (1.10)
\end{align*}
\]

Eq. (1.1) is the constitutive law for viscoelastic materials with damage. Here, \( A \) represents the elastic operator, \( G \) represents the viscoelastic long memory operator and \( \zeta \) represents the material damage. When \( \zeta = 1 \), the material is taken as undamaged, when \( \zeta = 0 \), the material is completely damaged and when \( 0 < \zeta < 1 \), a part of material is damaged. Eq. (1.2) is the equation of equilibrium, here, we neglect the inertial term. The parabolic nonlinear differential inclusion (1.3) is used to model the evolution of the damage. Here \( \Delta \) is the Laplacian, \( \kappa > 0 \) represents the damage diffusion constant, \( \phi \) is the mechanical source function of damage and \( \partial \psi_{[0,1]} \) represents the subdifferential of the indicator function \( \psi_{[0,1]} \) of the interval \([0,1]\). We assume that there is no damage influx throughout the boundary \( \Gamma \), thus Eq. (1.4) is used. (1.5) on \( \Gamma_D \) is the clamped boundary condition and (1.6) represents the surface traction boundary condition.

Relation (1.7) describes a normal compliance contact condition with adhesion, where \( j_\nu \) is a given function and \( \partial j_\nu \) denotes the Clarke’s subdifferential of \( j_\nu \) with respect to the last variable. Here, we omit the friction since we assume that the frictional tangential traction on the contact surface is far smaller than the adhesive one. And then, the shear on the contact surface is generated only by the glue, and is assumed to depend on the adhesion field and on the tangential displacement, but only up to the bond length \( L \). Thus (1.8) is used. The function \( \beta \), a surface internal variable which is called the bonding field or the adhesion field. It is a variable which is used to describe the fractional density of active bonds on the contact surface. Following [14, 15], the evolution of the adhesion field is governed by the differential equation (1.9) depending on the displacement, where \( \gamma_\nu \) is a positive adhesion coefficient and \( \epsilon_a \) is the debonding threshold energy. The truncation operator \( R : \mathbb{R} \to \mathbb{R} \) is defined by

\[
R(s) = \begin{cases} 
L & s \geq L, \\
\frac{s}{\|s\|} & |s| \leq L, \\
-L & s \leq -L,
\end{cases}
\]

where \( L > 0 \) is the characteristic length of the bond, beyond which it offers no additional resistance. When \( \beta = 1 \), the adhesion is complete and all the bonds are active, when \( \beta = 0 \) all bonds are inactive and there is no adhesion, when \( 0 < \beta < 1 \) the adhesion is partial and a fracture \( \beta \) of the bonds is active. The truncation operator \( R^* : \mathbb{R} \to \mathbb{R} \) is defined by

\[
R^*(v) = R^*_L(v) = \begin{cases} 
v & \|v\| \geq L, \\
\frac{L v}{\|v\|} & \|v\| \leq L.
\end{cases}
\]

\( u_0, \zeta_0 \) and \( \beta_0 \) represent the initial values of the displacement, damage and adhesion fields, respectively.
Hemivariational inequalities are a particularly useful generalization of variational inequalities and were first studied in early 1980s. For the analysis of hemivariational inequalities, it takes properties of the subdifferential in the sense of Clarke’s defined for locally Lipschitz functions as main ingredient and allows nonconvex functionals in the formula. For the representative references on the mathematical theory of hemivariational inequalities, we refer to [24, 25]. Recently, more and more researchers focus on numerical analysis of hemivariational inequalities, see [11, 19, 29]. More recently, there have been increasing researches on optimal error estimate for numerical solution of hemivariational inequalities, such as [17, 21, 30].

The rest of the paper is organized as follows. In Section 2, we first introduce preliminary materials as well as some assumptions on the data. And then, we establish a history-dependent hemivariational inequality, a parabolic variational inequality and an ordinary differential equation corresponding to the contact model. In Section 3, we introduce a fully discrete problem and then derive an optimal order error estimate for finite element method. In the last section, we report the numerical simulations results of some two-dimensional contact models for providing the numerical evidence of the theoretical results.

2. Notation and assumptions. In this section, we recall some notation, definitions and basic materials. We derive the variational form of the contact model and give a unique solvability result of the coupled system. We start with the definitions of Clarke’s directional derivative and Clarke’s subdifferential. Let $X$ be a Banach space, $X^*$ represents its dual. We denote $\langle \cdot, \cdot \rangle_{X^* \times X}$ the duality pairing between $X^*$ and $X$.

**Definition 2.1.** Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative, in the sense of Clarke’s, of $\psi$ at $x \in X$ in the direction $v \in X$, denoted by $\psi^0(x; v)$, is defined by

$$\psi^0(x; v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{\psi(y + \lambda v) - \psi(y)}{\lambda},$$

and the Clarke’s subdifferential of $\psi$ at $x$, denoted by $\partial \psi(x)$, is a subset of a dual space $X^*$ given by

$$\partial \psi(x) = \{ \zeta \in X^* \mid \psi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X}, \forall v \in X \}.$$

We recall the standard notation for Lebesgue and Sobolev spaces. For $v \in H^1(\Omega; \mathbb{R}^d)$, we use the same symbol $v$ for the trace of $v$ on $\partial \Omega$ and the notation $v_\nu$ and $v_\tau$ are taken as its normal and tangential traces, respectively. In addition, the spaces $V$ and $Q$ are as follows:

$$V = \{ v = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D \},$$

$$Q = L^2(\Omega; \mathbb{S}^d), \ H = L^2(\Omega; \mathbb{R}^d).$$

These are real Hilbert spaces with the canonical inner products in $Q$ and $H$, and the inner product

$$\langle u, v \rangle_V = \langle \varepsilon(u), \varepsilon(v) \rangle_Q$$

in $V$. The associated norms are $\| \cdot \|_V$, $\| \cdot \|_Q$ and $\| \cdot \|_H$. With the Sobolev trace theorem, we have

$$\| v \|_{L^2(\Gamma_{C; \mathbb{R}^d})} \leq \| \gamma \|_Q \| v \|_V \quad \forall v \in V,$$

(2.1)
where \( \| \gamma \| \) represents the norm of the trace operator \( \gamma : V \to L^2(\Gamma_C; \mathbb{R}^d) \). In addition, we employ \( Z_0 \) and \( Z_1 \) for the spaces \( L^2(\Omega) \) and \( H^1(\Omega) \), respectively. Moreover, we use the \( L^2(\Omega) \) norm and inner product
\[
\| \cdot \|_{Z_0} = \| \cdot \|_{L^2(\Omega)}, \quad \langle \cdot, \cdot \rangle_{Z_0} = \langle \cdot, \cdot \rangle_{L^2(\Omega)},
\]
respectively as well as the \( H^1(\Omega) \) seminorm
\[
\| \cdot \|_{Z_1} = \| \cdot \|_{H^1(\Omega)}.
\]
At the same time, \( B \) will be used for the space \( L^2(\Gamma_C) \) whenever it is used for the adhesion function \( \beta(t) \). The corresponding norm and inner product are
\[
\| \cdot \|_B = \| \cdot \|_{L^2(\Gamma_C)}, \quad \langle \cdot, \cdot \rangle_B = \langle \cdot, \cdot \rangle_{L^2(\Gamma_C)}.
\]
The damage function \( \zeta(t) \) will be sought in the subset \( \mathcal{K} \subset Z_1 \), which is defined by
\[
\mathcal{K} = \{ \xi \in Z_1 : \xi \in [0, 1] \text{ a.e. in } \Omega \}.
\]
As for the bonding field \( \beta \), we will define the set \( Q \) as
\[
Q = \{ \theta : [0, T] \to L^2(\Gamma_C) \mid 0 \leq \theta(t) \leq 1, \quad t \in [0, T] \text{ a.e. on } \Gamma_C \}.
\]
Now we introduce some assumptions on the data in the study of Problem 1.1.

For the elasticity operator \( A : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \), we assume
\[
\begin{cases}
(a) \text{ there exists } L_A > 0 \text{ such that for all } \epsilon_1, \epsilon_2 \in \mathbb{S}^d \text{ and a.e. } x \in \Omega, \\
\quad \| A(x, \epsilon_1) - A(x, \epsilon_2) \| \leq L_A \| \epsilon_1 - \epsilon_2 \|; \\
(b) \text{ there exists } m_A > 0 \text{ such that for all } \epsilon_1, \epsilon_2 \in \mathbb{S}^d \text{ and a.e. } x \in \Omega, \\
\quad (A(x, \epsilon_1) - A(x, \epsilon_2)) \cdot (\epsilon_1 - \epsilon_2) \geq m_A \| \epsilon_1 - \epsilon_2 \|^2; \\
(c) \text{ the mapping } x \mapsto A(x, \epsilon) \text{ is measurable on } \Omega, \text{ for all } \epsilon \in \mathbb{S}^d; \\
(d) A(x, 0) = 0 \text{ a.e. } x \in \Omega.
\end{cases}
\tag{2.2}
\]

For the viscoelastic long memory operator \( \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d \), we assume
\[
\begin{cases}
(a) \text{ there exists } L_G > 0 \text{ such that for all } \epsilon_1, \epsilon_2 \in \mathbb{S}^d, \zeta_1, \zeta_2 \in \mathbb{R} \text{ and a.e. } x \in \Omega, \\
\quad \| \mathcal{G}(x, \epsilon_1, \zeta_1) - \mathcal{G}(x, \epsilon_2, \zeta_2) \| \leq L_G (\| \epsilon_1 - \epsilon_2 \| + |\zeta_1 - \zeta_2|); \\
(b) \text{ the mapping } x \mapsto \mathcal{G}(x, \epsilon, \zeta) \text{ is measurable in } \Omega, \\
\quad \text{for all } \epsilon \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}; \\
(c) \mathcal{G}(x, 0, 0) = 0 \text{ a.e. } x \in \Omega.
\end{cases}
\tag{2.3}
\]

For the damage source function \( \phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R} \), we assume
\[
\begin{cases}
(a) \text{ there exists } L_\phi > 0 \text{ such that for all } \\
\quad \epsilon_1, \epsilon_2 \in \mathbb{S}^d, \zeta_1, \zeta_2 \in \mathbb{R} \text{ and a.e. } x \in \Omega, \\
\quad |\phi(x, \epsilon_1, \zeta_1) - \phi(x, \epsilon_2, \zeta_2)| \leq L_\phi (\| \epsilon_1 - \epsilon_2 \| + |\zeta_1 - \zeta_2|); \\
(b) \text{ the mapping } x \mapsto \phi(x, \epsilon, \zeta) \text{ is measurable on } \Omega, \\
\quad \text{for all } \epsilon \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}; \\
(c) \phi(x, 0, 0) \in L^2(\Omega).
\end{cases}
\tag{2.4}
Moreover, from the decomposition formula of the Green formula, we deduce that
\[ (1.1) - (1.10). \]
Let \( v \) do this, we suppose in the following that \( (u, \sigma) \) are smooth functions which satisfy (1.1)–(1.10). Let \( v \in V \). Multiplying the equilibrium equation (1.2) by \( v \), and use the Green formula, we deduce that
\[
(\sigma(t), v) = \langle f(t), v \rangle_{V^* \times V} + \int_{\Gamma_C} \sigma(t) \nu \cdot v \, d\Gamma \quad \text{for a.e. } t \in (0, T). \tag{2.14}
\]
Moreover, from the decomposition formula of \( \sigma(t) \nu \cdot v \), we have
\[
\int_{\Gamma_C} \sigma(t) \nu \cdot v \, d\Gamma = \int_{\Gamma_C} (\sigma_{\nu}(t)v_{\nu} + \sigma_{\tau}(t)v_{\tau}) \, d\Gamma. \tag{2.15}
\]
By the definition of the Clarke’s subdifferential and the boundary condition (1.7), we deduce
\[
\int_{\Gamma_C} -\sigma_\nu(t)v_\nu \, d\Gamma \leq \int_{\Gamma_C} j^0_\nu(\beta(t), u_\nu(t); v_\nu) \, d\Gamma, \quad \forall v \in V.
\]
(2.16)
Hence, we can obtain the following variational formulation of the problem (1.1)–(1.10).

**Problem 2.2.** Find a displacement field \( u : (0, T) \to V \), a stress field \( \sigma : (0, T) \to Q \), a damage field \( \zeta : (0, T) \to Z_1 \) and a bonding field \( \beta : (0, T) \to B \) such that for a.e. \( t \in (0, T) \),
\[
\sigma(t) = A \varepsilon(u(t)) + \int_0^t G(t-s, \varepsilon(u(s)), \zeta(s)) \, ds,
\]
(2.17)
\[
(\sigma(t), \varepsilon(v))_Q + \int_{\Gamma_C} p_r(\beta(t)) R^*(u_r(t)) \cdot v_r \, d\Gamma + \int_{\Gamma_C} j^0_\nu(\beta(t), u_\nu(t); v_\nu) \, d\Gamma \geq \langle f(t), v \rangle_{V^* \times V} \quad \forall v \in V,
\]
(2.18)
\[
\zeta(t) \in K, \quad (\zeta'(t), \xi - \zeta(t))_{Z_0} + a(\zeta(t), \xi - \zeta(t)) \geq (\phi(\varepsilon(u(t))), \zeta(t)), \xi - \zeta(t))_{Z_0} \quad \forall \xi \in K,
\]
(2.19)
\[
\beta'(t) = -\gamma_{\nu}\beta(t)\bar{R}(u_\nu(t)^2 - \epsilon_a)_+ \quad \text{a.e. } t \in (0, T),
\]
(2.20)
and
\[
u(0) = u_0, \quad \zeta(0) = \zeta_0, \quad \beta(0) = \beta_0.
\]
(2.21)

The unique solvability of Problem 2.2 is provided in the following result.

**Theorem 2.3.** Assume (2.2)–(2.10). If
\[
m_A > \bar{\alpha} \|\gamma\|^2,
\]
(2.22)
then Problem 2.2 has a unique solution \((u, \zeta, \beta)\) with regularity
\[
u \in C([0, T]; V), \quad \zeta \in H^1(0, T; Z_0) \cap L^2(0, T; Z_1), \quad \beta \in W^{1, \infty}(0, T; B) \cap Q.
\]
(2.23)

Similar to [1, Theorem 3.3], we can prove the above result and the proof is based on arguments of hemivariational inequalities, a classical existence and uniqueness result on parabolic inequalities, and Banach fixed point theorem.

3. **A fully discrete scheme and error estimate.** In this section, we introduce a fully discrete scheme for the coupled system of hemivariational inequality, variational inequality and ordinary differential equation formulated in Problem 2.2 and provide a result on error estimate. First, we recall a discrete Gronwall inequality (cf. [20, Lemma 7.26]).

**Lemma 3.1.** Let \( T > 0 \) be given. For a positive integer \( N \), define \( k = T/N \). Assume that \( \{g_n\}_{n=1}^N \) and \( \{e_n\}_{n=1}^N \) are two sequences of nonnegative numbers satisfying
\[
e_n \leq \bar{c}g_n + \bar{c} \sum_{j=1}^n ke_j, \quad n = 1, \ldots, N
\]
for a positive constant $\bar{c}$ independent of $N$ or $k$. Then, if $k$ is sufficiently small, there exists a positive constant $c$, independent of $N$ or $k$, such that

$$\max_{1 \leq n \leq N} e_n \leq c \max_{1 \leq n \leq N} g_n.$$ 

The following modified Cauchy-Schwarz inequality with an arbitrary $\epsilon > 0$ will also be used in the later numerical analysis:

$$a b \leq \epsilon a^2 + c b^2; \quad a, b \in \mathbb{R}, \quad (3.1)$$

where the constant $c > 0$ depends on $\epsilon$.

Let $V^h$ be a finite dimensional subspace of $V$, $Z_1^h$ be a finite dimensional subspace of $Z_1$ and $B^h$ be a finite dimensional subspace of $B$ where $h > 0$ denotes a spatial discretization parameter. The space in which we seek the stress and the strain fields is $Q$ and it will be approximated by

$$Q^h = \{ \tau^h \in Q : \tau^h |_{K} \in \mathbb{R}^d \forall K \in T^h \}. \quad (3.2)$$

Let $P_{Q^h} : Q \rightarrow Q^h$ be the orthogonal projection operator defined by

$$(P_{Q^h} \mathbf{q}, \mathbf{q}^h)_Q = (\mathbf{q}, \mathbf{q}^h)_Q \quad \forall \mathbf{q} \in Q, \mathbf{q}^h \in Q^h. \quad (3.3)$$

Similarly, we define $P_{B^h} : B \rightarrow B^h$ to be the orthogonal projection operator in $B$, i.e.

$$(P_{B^h} \theta, \theta^h)_B = (\theta, \theta^h)_B \quad \forall \theta \in B, \theta^h \in B^h. \quad (3.4)$$

Let $K^h = K \cap Z_1^h$. In addition, we consider an equidistant time grid with abscissae $t_n = nk$, where $n = 0, 1, \cdots, N, N \in \mathbb{N}$, and the constant step-size $k = T/N$. For a time continuous function $z = z(t)$, we write $z_n = z(t_n)$ for $n = 0, 1, \cdots, N$. For a sequence $\{z_n\}_{n=0}^N$, we define $\delta z_n = \frac{z_{n+1} - z_n}{h}$ as its corresponding divided differences. Let $u_0^h, \beta_0^h$ and $\xi_0^h$ be the appropriate approximation of the initial conditions $u_0$, $\beta_0$ and $\xi_0$, respectively. And then, we have the following discrete problem.

**Problem 3.2.** Find a discrete displacement field $u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset U^h$, a discrete stress field $\sigma^{hk} = \{\sigma_n^{hk}\}_{n=0}^N \subset Q^h$, a discrete damage field $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset \mathcal{K}^h$, a discrete bonding field $\beta^{hk} = \{\beta_n^{hk}\}_{n=0}^N \subset \mathcal{B}^h$ such that for $1 \leq n \leq N$,

$$\sigma_n^{hk} = P_{Q^h} \mathcal{A} \mathcal{E}(u_n^{hk}) + \sum_{j=1}^n kP_{Q^h} G(t_n - t_{j-1}, \mathcal{E}(u_{j-1}^{hk}), \zeta_{j-1}^{hk}), \quad (3.5)$$

$$(\sigma_n^{hk}, \mathcal{E}(v^h))_Q + \int_{T^h} p_T(\beta_n^{hk} R^*(u_{n,T}^{hk}) \cdot v^h_T) d\Gamma \quad \forall v^h \in V^h,$$

$$+ \int_{T^h} j_0^{\beta}(\beta_n^{hk}, u_{n,v}^{hk}, v^h_T) d\Gamma \geq (f_{n}, v^h)_{V \times V} \quad \forall u^h \in V^h, \quad (3.6)$$

$$(\delta_n^{\beta^{hk}}, \zeta^h - \zeta^{hk})z_0 + \mathcal{A}(\zeta_n^{hk}, \zeta^h - \zeta^{hk})z_0 \geq (\phi(\mathcal{E}(u_{n-1}^{hk}), \zeta_{n-1}^{hk}), \zeta^h - \zeta^{hk})z_0 \quad \forall \xi^h \in \mathcal{K}^h, \quad (3.7)$$

$$\delta_{\beta_n}^{\beta^{hk}} = -P_{B^h}(\gamma_{n-1}^{\beta^{hk}} R(\mathcal{L}^{hk}_{n-1})^2 - \epsilon_a)_+$$

and

$$u_0^{hk} = u_0^h, \quad \zeta_0^{hk} = \zeta_0^h, \quad \beta_0^{hk} = \beta_0^h. \quad (3.9)$$

In the rest of the paper, $c$ will be used for a generic positive constant whose value may change from time to time and it is independent of $h$ and $k$. 
Theorem 3.3. Assume that (2.2)–(2.10) hold. Then, there exists a unique solution $(u_n^{hk}, \sigma_n^{hk}, \zeta_n^{hk}, \beta_n^{hk})$ of Problem 3.2.

For $n = 1, \ldots, N$, suppose that $u_{n-1}^{hk}, \zeta_{n-1}^{hk}, \beta_{n-1}^{hk}$ are given, then by using the induction argument, the solution existence and uniqueness of Problem 3.2 can be proved. Our interest in this paper lies in the error estimates for $\|u_n - u_n^{hk}\|_V, \|\sigma_n - \sigma_n^{hk}\|_Q, \|\zeta_n - \zeta_n^{hk}\|_{Z_0}$ and $\|\beta_n - \beta_n^{hk}\|_B$. For an error analysis of the numerical method, we will give the additional smoothness assumptions

\[
u \in H^1(0, T; V),
\]
\[
\sigma \in C([0, T]; H^1(\Omega; \mathbb{R}^d)),
\]
\[
\beta \in W^{2,1}(0, T; B),
\]
\[
\zeta \in C([0, T]; H^2(\Omega)) \cap H^2(0, T; Z_0), \quad \zeta' \in L^2(0, T; H^2(\Omega)).
\]

We first set $t = t_n$ in (2.17) to obtain

\[
\sigma_n = A\varepsilon(u_n) + \int_0^{t_n} G(t_n - s, \varepsilon(u(s)), \zeta(s)) \, ds.
\]

Combining it with (3.5), we have

\[
\sigma_n - \sigma_n^{hk} = (I - P_{Q^h})\sigma_n + P_{Q^h}A\varepsilon(u_n) - P_{Q^h}A\varepsilon(u_n^{hk})
\]
\[
+ P_{Q^h} \left( \int_0^{t_n} G(t_n - s, \varepsilon(u(s)), \zeta(s)) \, ds - \sum_{j=1}^n kG(t_n - t_{j-1}, \varepsilon(u_{j-1}), \zeta_{j-1}) \right)
\]
\[
+ P_{Q^h} \sum_{j=1}^n k[G(t_n - t_{j-1}, \varepsilon(u_{j-1}), \zeta_{j-1}) - G(t_n - t_{j-1}, \varepsilon(u_{j-1}^{hk}), \zeta_{j-1}^{hk})],
\]

where $I$ denotes the identity operator on $Q$.

By using (2.3)(a), we have

\[
\|G(t_n - t_{j-1}, \varepsilon(u_{j-1}), \zeta_{j-1}) - G(t_n - t_{j-1}, \varepsilon(u_{j-1}^{hk}), \zeta_{j-1}^{hk})\|_Q
\]
\[
\leq c (\|u_{j-1} - u_{j-1}^{hk}\|_V + \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_{Z_0}).
\]

Moreover, we can easily obtain that

\[
\left\| \int_0^{t_n} G(t_n - s, \varepsilon(u(s)), \zeta(s)) \, ds - \sum_{j=1}^n kG(t_n - t_{j-1}, \varepsilon(u_{j-1}), \zeta_{j-1}) \right\|_Q
\]
\[
= \left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left( G(t_n - s, \varepsilon(u(s)), \zeta(s)) - G(t_n - t_{j-1}, \varepsilon(u_{j-1}), \zeta_{j-1}) \right) \, ds \right\|_Q \, dt
\]
\[
\leq c k (\|u\|_{H^1(0, T; V)} + \|\zeta'\|_{C([0, T]; Z_0)}).
\]

By the regularity assumption (3.11), we deduce

\[
\|(I - P_{Q^h})\sigma_n\|_Q \leq ch.
\]

Finally, we deduce the following error estimate by using the nonexpansiveness of the projection operator $P_{Q^h}$, (3.15), (3.18) and (2.2)(a),

\[
\|\sigma_n - \sigma_n^{hk}\|_Q \leq c k \sum_{j=1}^n (\|u_{j-1} - u_{j-1}^{hk}\|_V + \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|_{Z_0})
\]
\[
+ c k (\|u\|_{H^1(0, T; V)} + \|\zeta'\|_{C([0, T]; Z_0)}) + c(h + \|u_n - u_n^{hk}\|_V),
\]
which implies

$$
\|\sigma_n - \sigma_n^{hk}\|_Q^2 \leq c k \sum_{j=1}^{n} (\|u_{j-1} - u_{j-1}^{hk}\|^2_Q + \|\zeta_{j-1} - \zeta_{j-1}^{hk}\|^2_{\nu}) 
+ c k^2 (\|u\|_{H^1(0,T;V)}^2 + \|\zeta\|_{C^0([0,T];Z_0)}^2) + c(h^2 + \|u_n - u_n^{hk}\|^2_Q).
$$

(3.20)

The next, we substitute (2.17) into (2.18) at $t = t_n$ and take $v = u_n^{hk} - u_n^{h}$ to obtain

$$
(\mathcal{A}\varepsilon(u_n), \varepsilon(u_n^{hk} - u_n^{h}))_Q + \int_{\Gamma_C} p_{\tau}(\beta_n) R^*(u_n, \varepsilon(u_n^{hk} - u_n^{h})) d\Gamma 
+ \left( \int_0^{t_n} \mathcal{G}(t_n - s, \varepsilon(u(s)), \zeta(s)) ds, \varepsilon(u_n^{hk} - u_n^{h}) \right)_Q 
+ \int_{\Gamma_C} j_0^C(\beta_n, u_n; u_n^{hk} - u_n^{h}) d\Gamma \geq (f_n, u_n^{hk} - u_n^{h})_{V^* \times V}.
$$

(3.21)

At the same time, we substitute (3.5) into (3.6) and take $v = u_n^{h} - u_n^{hk}$ to obtain

$$
(P_{Q^n}. \mathcal{A}\varepsilon(u_n^{hk}), \varepsilon(v_n - u_n^{hk}))_Q + \int_{\Gamma_C} p_{\tau}(\beta_n) R^*(u_n^{hk} - u_n^{h}) d\Gamma 
+ \left( \sum_{j=1}^{n} k P_{Q^n} \mathcal{G}(t_n - t_{j-1}, \varepsilon(u_{j-1}^{hk}, \zeta^{hk}_{j-1}), \varepsilon(v_n^{h} - u_n^{hk})) \right)_Q 
+ \int_{\Gamma_C} j_0^C(\beta_n, u_n; v_n - u_n^{hk}) d\Gamma \geq (f_n, v_n^{h} - u_n^{hk})_{V^* \times V} \quad \forall v_n^{h} \in V^h.
$$

(3.22)

By (2.2)(b), we obtain

$$
m_{\lambda} \|u_n - u_n^{hk}\|_V^2 \leq (\mathcal{A}\varepsilon(u_n^{hk}), \varepsilon(u_n - u_n^{hk}))_Q.
$$

(3.23)

Adding (3.21) and (3.22), through (3.3) we deduce that

$$
(\mathcal{A}\varepsilon(u_n), \varepsilon(u_n^{hk} - u_n^{h}))_Q + I_1 + I_2 + I_3,
$$

(3.24)

where

$$
I_1 = \left( \int_0^{t_n} \mathcal{G}(t_n - s, \varepsilon(u(s)), \zeta(s)) ds, \varepsilon(u_n^{hk} - u_n^{h}) \right)_Q 
+ \left( \sum_{j=1}^{n} k \mathcal{G}(t_n - t_{j-1}, \varepsilon(u_{j-1}^{hk}, \zeta_{j-1}^{hk}), \varepsilon(v_n^{h} - u_n^{hk})) \right)_Q,
$$

$$
I_2 = \left( \int_{\Gamma_C} p_{\tau}(\beta_n) R^*(u_n, \varepsilon(u_n^{hk} - u_n^{h})) d\Gamma - \int_{\Gamma_C} p_{\tau}(\beta_n) R^*(u_n^{hk} - u_n^{h}) d\Gamma, \varepsilon(u_n^{hk} - u_n^{h}) \right)_Q,
$$

$$
I_3 = \int_{\Gamma_C} j_0^C(\beta_n, u_n; u_n^{hk} - u_n^{h}) d\Gamma + \int_{\Gamma_C} j_0^C(\beta_n, u_n; v_n - u_n^{hk}) d\Gamma.
$$

Now, let us give the estimate for $I_1, I_2, I_3$. For $I_1$, we have

$$
\| \int_0^{t_n} \mathcal{G}(t_n - s, \varepsilon(u(s)), \zeta(s)) ds - \sum_{j=1}^{n} k \mathcal{G}(t_n - t_{j-1}, \varepsilon(u_{j-1}^{hk}, \zeta_{j-1}^{hk})) \|_Q 
\leq \| \int_0^{t_n} \mathcal{G}(t_n - s, \varepsilon(u(s)), \zeta(s)) ds - \sum_{j=1}^{n} k \mathcal{G}(t_n - t_{j-1}, \varepsilon(u_{j-1}, \zeta_{j-1})) \|_Q
$$

(3.25)
+ \left\| \sum_{j=1}^{n} kG(t_n - t_{j-1}, \varepsilon(u_{j-1}, \zeta_{j-1}) - \sum_{j=1}^{n} kG(t_n - t_{j-1}, \varepsilon(u_{j}^{hk}, \zeta_{j-1})) \right\|_{Q} \\
\leq c k(||u||_{H^{1}(0,T;V)} + ||\zeta'||_{C([0,T];Z_0)}) \\
+ c \sum_{j=1}^{n} k(||u_{j-1} - u_{j-1}^{hk}||_{V} + ||\zeta_{j-1} - \varepsilon_{j-1}^{hk}||_{Z_0}).

Next, we turn to bound $I_2$. By using 2.6(a)(b) and [10, Lemma 5.2] we have

$$\|p_{r}(\beta)R^{*}(u_{n,\tau}) - \|p_{r}(\beta)R^{*}(u_{n,\tau}^{hk})\|_{L^2(\Gamma_C)}$$

$$\leq \|p_{r}(\beta)R^{*}(u_{n,\tau}) - \|p_{r}(\beta)R^{*}(u_{n,\tau}^{hk})\|_{L^2(\Gamma_C)}$$

$$+ \|p_{r}(\beta)R^{*}(u_{n,\tau}^{hk}) - \|p_{r}(\beta)R^{*}(u_{n,\tau}^{hk})\|_{L^2(\Gamma_C)}$$

$$\leq M_{\tau}||u_{n,\tau} - u_{n,\tau}^{hk}||_{L^2(\Gamma_C)} + LL_{\tau}||\beta_n - \beta_n^{hk}||_{B}$$

$$\leq M_{\tau}||\gamma||||u_{n,\tau} - u_{n,\tau}^{hk}||_{V} + LL_{\tau}||\beta_n - \beta_n^{hk}||_{B}.$$ 

Finally, we bound $I_3$, which is expressed as

$$I_3 = \int_{\Gamma_C} J^{0}_{\beta}(\beta_n, u_{n,\nu}; u_{n,\nu}^{hk}; u_{n,\nu}^{hk} - u_{n,\nu}^{hk}) d\Gamma + \int_{\Gamma_C} J^{0}_{\beta}(\beta_n^{hk}, u_{n,\nu}; u_{n,\nu}^{hk}; u_{n,\nu}^{hk} - u_{n,\nu}^{hk}) d\Gamma$$

By (2.5) (c), $J^{0}_{\beta}(x, t, r, \xi; \eta) \leq C_0 ||\eta||$. Using the subadditivity of the generalized directional derivative, (2.5) (d) and Cauchy-Schwarz inequality, it follows that

$$\int_{\Gamma_C} J^{0}_{\beta}(\beta_n, u_{n,\nu}; u_{n,\nu}^{hk}; u_{n,\nu}^{hk} - u_{n,\nu}^{hk}) d\Gamma + \int_{\Gamma_C} J^{0}_{\beta}(\beta_n^{hk}, u_{n,\nu}; u_{n,\nu}^{hk}; u_{n,\nu}^{hk} - u_{n,\nu}^{hk}) d\Gamma$$

$$\leq \int_{\Gamma_C} [J^{0}_{\beta}(\beta_n, u_{n,\nu}; u_{n,\nu}^{hk}; u_{n,\nu}^{hk} - u_{n,\nu}^{hk}) + J^{0}_{\beta}(\beta_n, u_{n,\nu}; u_{n,\nu}^{hk}; u_{n,\nu}^{hk} - u_{n,\nu}^{hk})] d\Gamma$$

$$+ \int_{\Gamma_C} [J^{0}_{\beta}(\beta_n^{hk}, u_{n,\nu}; u_{n,\nu}^{hk}; u_{n,\nu}^{hk} - u_{n,\nu}^{hk}) + J^{0}_{\beta}(\beta_n^{hk}, u_{n,\nu}; u_{n,\nu}^{hk}; u_{n,\nu}^{hk} - u_{n,\nu}^{hk})] d\Gamma$$

$$\leq \tilde{c}(||\gamma||_{V}||u_{n,\tau} - u_{n,\tau}^{hk}||_{V}^{2} + ||\gamma||||\beta_n - \beta_n^{hk}||_{B}||u_{n,\tau} - u_{n,\tau}^{hk}||_{V})$$

$$+ 2\tilde{c} \sqrt{m(\Gamma_C)}||u_{n,\tau} - u_{n,\tau}^{hk}||_{L^2(\Gamma_C;R^d)}.$$ (3.27)

Now, we are ready to bound the left term on the right side of (3.24). Using (2.2)(a) and applying the modified Cauchy-Schwarz inequality (3.1), we have

$$(A\varepsilon(u_{n}) - A\varepsilon(u_{n}^{hk}), \varepsilon(u_{n} - v_{n}^{hk}))_{Q} \leq L_{A}||u_{n} - u_{n}^{hk}||_{V}||u_{n} - v_{n}^{hk}||_{V}$$

$$\leq \epsilon ||u_{n} - u_{n}^{hk}||_{V}^{2} + c ||u_{n} - v_{n}^{hk}||_{V}^{2}. \quad (3.28)$$

Then, by using inequalities (3.25)–(3.28) on the right side of the inequality (3.24) and taking $\epsilon > 0$ sufficiently small, under assumption $m_{A} > \tilde{c}||\gamma||^{2}$, we obtain the following result.

$$\|u_{n} - u_{n}^{hk}\|_{V}^{2} \leq c(||u_{n} - v_{n}^{hk})||_{V}^{2} + ||u_{n} - u_{n}^{hk}||_{L^2(\Gamma_C;R^d)}$$

$$+ c \sum_{j=1}^{n} k(||u_{j-1} - u_{j-1}^{hk}||_{V} + ||\zeta_{j-1} - \varepsilon_{j-1}^{hk}||_{Z_0})||u_{n} - u_{n}^{hk}||_{V}$$

$$+ c ||u_{n} - u_{n}^{hk}||_{V} + ||\beta_{n} - \beta_{n}^{hk}||_{B}||v_{n} - u_{n}^{hk}||_{V}$$

$$+ c k||u_{n}^{hk} - v^{hk}_{n}||_{V}(||u||_{H^{1}(0,T;V)} + ||\gamma'||_{C([0,T];Z_0)}). \quad (3.29)$$
+ c \| \beta_n - \beta_{jh}^h \|_B \| u_n - u_n^{hjk} \|_V.

Since \( \| u_n^{hjk} - v_n^h \|_V^2 \leq 2 (\| u_n^{hjk} - u_n^h \|_V^2 + \| u_n - v_n^h \|_V^2) \), it follows that
\[
\| u_n - u_n^{hjk} \|_V^2 \leq c (\| u_n - v_n^h \|_V^2 + \| u_n - v_n^h \|_{L^2(G;R^d)}) + c \sum_{j=1}^n k (\| u_{j-1} - u_j^{hjk} \|_V^2 + \| \zeta_{j-1} - \zeta_{j-1}^{hjk} \|_{Z_0}^2)
\]
\[
+ c k^2 (\| u \|_{H^1(0,T;V)} + \| \zeta' \|_{L^2([0,T];Z_0)}) + \| \beta_n - \beta_n^{hjk} \|_B^2.
\]

Next, we turn to bound the error \( \| \beta_n - \beta_{jh}^h \|_B \). We first write
\[
\beta_n - \beta_{jh}^h = \beta_0 - \beta_0^h + \sum_{j=1}^n k \| \delta(\beta_j - \beta_j^{hjk}) \|,
\]
and then
\[
\| \beta_n - \beta_{jh}^h \|_B \leq \| \beta_0 - \beta_0^h \|_B + \sum_{j=1}^n k \| \delta(\beta_j - \beta_j^{hjk}) \|_B.
\]

Let us bound \( \| \delta(\beta_j - \beta_j^{hjk}) \|_B \). We have
\[
\delta(\beta_j - \beta_j^{hjk}) = \delta \beta_j - \beta_j^j + (I - \mathcal{P}_{B^h}) \beta_j^j + \mathcal{P}_{B^h} \beta_j^j - \delta \beta_j^j.
\]
Thus,
\[
\| \delta(\beta_j - \beta_j^{hjk}) \|_B \leq \| \delta \beta_j - \beta_j^j \|_B + \| (I - \mathcal{P}_{B^h}) \beta_j^j \|_B + \| \mathcal{P}_{B^h} \beta_j^j - \delta \beta_j^j \|_B,
\]
where \( I : B \to B \) is the identity operator. Combining (2.20) and (3.8), we obtain
\[
\mathcal{P}_{B^h} \beta_j^j - \delta \beta_j^j = - \mathcal{P}_{B^h} [\gamma_{\nu} \beta_j \bar{R}(u_{j,\nu})^2 - \epsilon_a] - (\gamma_{\nu} \beta_j^{hjk} \bar{R}(u_{j-1,\nu}^{hjk})^2 - \epsilon_a). + \]
\[
\| \mathcal{P}_{B^h} \beta_j^j - \delta \beta_j^j \|_B \leq \| \gamma_{\nu} \beta_j \bar{R}(u_{j,\nu})^2 - \beta_j^{hjk} \bar{R}(u_{j-1,\nu}^{hjk})^2 \|_B.
\]

Since that
\[
\beta_j \bar{R}(u_{j,\nu})^2 - \beta_j^{hjk} \bar{R}(u_{j-1,\nu}^{hjk})^2
\]
\[
= (\beta_j - \beta_j^{hjk}) \bar{R}(u_{j,\nu})^2 + \beta_j^{hjk} (\bar{R}(u_{j,\nu})^2 - \bar{R}(u_{j-1,\nu}^{hjk})^2),
\]
by using the uniform boundedness of \( \| \bar{R}(u_{j,\nu})^2 \|_{L^\infty(G;\mathbb{C})} \) and \( \| \beta_j^{hjk} \|_{L^\infty(G;\mathbb{C})} \), we have
\[
\| \beta_j \bar{R}(u_{j,\nu})^2 - \beta_j^{hjk} \bar{R}(u_{j-1,\nu}^{hjk})^2 \|_B \leq c (\| \beta_j - \beta_j^{hjk} \|_B + \| \bar{R}(u_{j,\nu})^2 - \bar{R}(u_{j-1,\nu}^{hjk})^2 \|_B).
\]
Thus,
\[
\| \mathcal{P}_{B^h} \beta_j^j - \delta \beta_j^j \|_B \leq c (\| \beta_j - \beta_j^{hjk} \|_B + \| \bar{R}(u_{j,\nu})^2 - \bar{R}(u_{j-1,\nu}^{hjk})^2 \|_B).
\]

Summarizing the above results, yields
\[
\| \delta(\beta_j - \beta_j^{hjk}) \|_B \leq c (\| \beta_j - \beta_j^{hjk} \|_B + \| (I - \mathcal{P}_{B^h}) \beta_j^j \|_B
\]
\[
+ c (\| \beta_j - \beta_j^{hjk} \|_B + \| \bar{R}(u_{j,\nu})^2 - \bar{R}(u_{j-1,\nu}^{hjk})^2 \|_B),
\]
where \( \| \beta_j - \beta^{hk}_{j-1} \|_B = \| \beta_j - \beta_{j-1} \|_B + \| \beta_{j-1} - \beta^{hk}_{j-1} \|_B \). Using these relations in (3.32), the following inequality holds:

\[
\| \beta_n - \beta^{hk}_n \|_B \leq \| \beta_0 - \beta^h \|_B + c \sum_{j=1}^n k(\| \beta_{j-1} - \beta^{hk}_{j-1} \|_B + \| \tilde{R}(u_{j,\nu})^2 - \tilde{R}(u^{hk}_{j-1,\nu})^2 \|_B) + \sum_{j=1}^n k(\| \delta \beta_j - \beta^{hk}_j \|_B + \| (I - \mathcal{P}_{B^h}) \beta^{hk}_j \|_B + \| \beta_j - \beta_{j-1} \|_B). \quad (3.38)
\]

The next, we will derive a bound on \( \| \tilde{R}(u_{j,\nu})^2 - \tilde{R}(u^{hk}_{j-1,\nu})^2 \|_B \). First, we write

\[
\tilde{R}(u_{j,\nu})^2 - \tilde{R}(u^{hk}_{j-1,\nu})^2 = (\tilde{R}(u_{j,\nu}) + \tilde{R}(u^{hk}_{j-1,\nu}))(\tilde{R}(u_{j,\nu}) - \tilde{R}(u^{hk}_{j-1,\nu})), \quad (3.39)
\]

and then

\[
\tilde{R}(u_{j,\nu})^2 - \tilde{R}(u^{hk}_{j-1,\nu})^2 \leq 2L \| \tilde{R}(u_{j,\nu}) - \tilde{R}(u^{hk}_{j-1,\nu}) \| \leq 2L \| u_{j,\nu} - u^{hk}_{j-1,\nu} \|. \quad (3.40)
\]

Therefore,

\[
\| \tilde{R}(u_{j,\nu})^2 - \tilde{R}(u^{hk}_{j-1,\nu})^2 \|_B \leq c \| u_{j,\nu} - u^{hk}_{j-1,\nu} \| \leq c \| u_j - u^{hk}_{j-1} \|_V. \quad (3.41)
\]

From the above inequalities, we deduce that

\[
\| \beta_n - \beta^{hk}_n \|_B \leq \| \beta_0 - \beta^h \|_B + c \sum_{j=1}^n k(\| \beta_{j-1} - \beta^{hk}_{j-1} \|_B + \| u_j - u^{hk}_{j-1} \|_V) + \sum_{j=1}^n k(\| \delta \beta_j - \beta^{hk}_j \|_B + \| (I - \mathcal{P}_{B^h}) \beta^{hk}_j \|_B + \| \beta_j - \beta_{j-1} \|_B). \quad (3.42)
\]

For the term \( \| u_j - u^{hk}_{j-1} \|_V \), we have

\[
\| u_j - u^{hk}_{j-1} \|_V = \| u_j - u_{j-1} \|_V + \| u_{j-1} - u^{hk}_{j-1} \|_V, \quad (3.43)
\]

and

\[
\| u_j - u_{j-1} \|_V \leq c k \| u \|_{H^1(0,T;V)}. \quad (3.44)
\]

Similarly, for the term \( \| \beta_j - \beta_{j-1} \|_B \), we have

\[
\| \beta_j - \beta_{j-1} \|_B \leq c k \| \beta \|_{W^{1,1}(0,T;B)}, \quad (3.45)
\]

which implies

\[
\| \beta_n - \beta^{hk}_n \|_B^2 \leq c \sum_{j=1}^n k(\| \beta_{j-1} - \beta^{hk}_{j-1} \|_B^2 + \| u_{j-1} - u^{hk}_{j-1} \|_V^2) + \sum_{j=1}^n k(\| \delta \beta_j - \beta^{hk}_j \|_B^2 + \| (I - \mathcal{P}_{B^h}) \beta^{hk}_j \|_B^2) + \| \beta_0 - \beta^h \|_B^2 + c k^2 \| \beta \|_{W^{1,1}(0,T;B)}^2 + c k^2 \| \beta \|_{H^1(0,T;V)}^2. \quad (3.46)
\]

The next, we estimate \( |\zeta_n - \zeta^{hk}_n| \|_{Z_0} \). Now, we choose \( \xi = \zeta^{hk}_n \) in (2.19) at \( t = t_n \) and find

\[
(\zeta^t_n, \zeta^{hk}_n)_{\Sigma} + a(\zeta_n, \zeta^{hk}_n - \zeta_n) \geq (\phi(\mathbf{e}(u_n), \zeta_n), \zeta^t_n)_{\Delta_0}. \quad (3.47)
\]

Adding (3.47) and (3.7), with \( \xi = \zeta^{hk}_n \in K^h \), yields

\[
(\delta(\zeta_n - \zeta^{hk}_n), \zeta_n - \zeta^{hk}_n)_{\Sigma} + a(\zeta_n - \zeta^{hk}_n, \zeta_n - \zeta^{hk}_n)_{\Sigma} \leq (\delta(\zeta_n - \zeta^{hk}_n), \zeta_n - \zeta^{hk}_n)_{\Sigma} + (\delta(\zeta_n - \zeta^t_n), \zeta_n - \zeta^{hk}_n)_{\Sigma}.
\]
The first term on the left-hand side of (3.48) can be bounded from below
\[ a(\zeta_n - \zeta_n^h, \zeta_n - \zeta_n^h) - (\delta \zeta_n, \zeta_n^h)_{Z_0} \]
\[ - a(\zeta_n, \zeta_n - \zeta_n^h) + (\varphi(\varepsilon(u_n), \zeta_n), \zeta_n - \zeta_n^h)_{Z_0} \]
\[ + (\varphi(\varepsilon(u_n), \zeta_n) - \varphi(\varepsilon(u_{n-1}), \zeta_n^h - \zeta_n^h)_{Z_0}). \] (3.48)

And then, under the assumption on the bilinear form \( a(\cdot, \cdot) \), we obtain
\[ a(\zeta_n - \zeta_n^h, \zeta_n - \zeta_n^h) \geq c_0|\zeta_n - \zeta_n^h|_{Z_1}^2. \] (3.49)

The first term on the left-hand side of (3.48) can be bounded from below
\[ (\delta(\zeta_n - \zeta_n^h), \zeta_n - \zeta_n^h)_{Z_0} \geq \frac{1}{2k}(\|\zeta_n - \zeta_n^h\|_{Z_0}^2 - \|\zeta_{n-1} - \zeta_n^h\|_{Z_0}^2). \] (3.50)

Now we use (3.49) and (3.50) in (3.48), after some elementary estimates just like [27], we achieve
\[ \|\zeta_n - \zeta_n^h\|_{Z_0}^2 - \|\zeta_{n-1} - \zeta_n^h\|_{Z_0}^2 + k|\zeta_n - \zeta_n^h|_{Z_1}^2 \]
\[ \leq c k(\|\zeta_n - \zeta_n^h\|_{Z_0}^2 + \|\zeta_{n-1} - \zeta_n^h\|_{Z_0}^2)
+ c k(\|\delta(\zeta_n - \zeta_n^h), \zeta_n - \zeta_n^h\|_{Z_0} + \|\zeta_n - \zeta_n^h\|_{Z_1}^2)
+ c k(\|\zeta_n - \zeta_n^h\|_{Z_0} + \|\zeta_n - \zeta_{n-1}\|_{Z_0} + \|u_n - u_{n-1}\|_{V})
+ c k|\varphi(\varepsilon(u_n), \zeta_n) - \varphi(\varepsilon(u_{n-1}), \zeta_n^h - \zeta_n^h)_{Z_0}|. \] (3.51)

In this inequality, we replace \( n \) by \( j \), sum over \( j = 1, \ldots, n \) and deduce
\[ \|\zeta_n - \zeta_n^h\|_{Z_0}^2 - \|\zeta_0 - \zeta_0^h\|_{Z_0}^2 + \sum_{j=1}^n k|\zeta_j - \zeta_j^h|_{Z_1}^2 \]
\[ \leq c \sum_{j=1}^n k|\zeta_j - \zeta_j^h|_{Z_0}^2 + c \sum_{j=1}^n k(\delta(\zeta_j - \zeta_j^h), \zeta_j - \zeta_j^h)_{Z_0}
+ c \sum_{j=1}^n k|\zeta_j - \zeta_j^h|_{Z_1}^2
+ c \sum_{j=1}^n k(\delta(\zeta_j - \zeta_j^h), \zeta_j - \zeta_{j-1})_{Z_0} + k\|u_j - u_{j-1}\|_{V}
+ c \sum_{j=1}^n k|\varphi(\varepsilon(u_j), \zeta_j) - \delta(\zeta_j - \zeta_j^h)_{Z_0}|. \] (3.52)

As for the second term on the right side of (3.52), we have
\[ \sum_{j=1}^n k(\delta(\zeta_j - \zeta_j^h), \zeta_j - \zeta_j^h)_{Z_0} \]
\[ = \sum_{j=1}^n ((\zeta_j - \zeta_j^h) - (\zeta_j - \zeta_j^h)_{Z_0})
= (\zeta_n - \zeta_n^h, \zeta_n - \zeta_n^h)_{Z_0} - (\zeta_0 - \zeta_0^h, \zeta_0 - \zeta_0^h)_{Z_0}
+ \sum_{j=1}^{n-1} ((\zeta_j - \zeta_j^h) - (\zeta_j - \zeta_j^h)_{Z_0} - (\zeta_{j+1} - \zeta_{j+1}^h))_{Z_0}. \] (3.53)
Combining (3.52)–(3.54), we have
\[
\|\zeta_n - \zeta^h_n\|_{Z_0}^2 + c\|\zeta_n - \zeta^h_n\|_{Z_0}^2 + c\|\zeta_0 - \zeta^h_0\|_{Z_0}^2 + c\|\zeta_1 - \zeta^h_1\|_{Z_0}^2 \\
+ k\sum_{j=1}^{n-1}\|\zeta_j - \zeta^h_j\|_{Z_0}^2 + \frac{1}{k}\sum_{j=1}^{n-1}\|\zeta_j - \zeta^h_j - (\zeta_{j+1} - \zeta^h_{j+1})\|_{Z_0}^2.
\]
Moreover,
\[
\|\zeta_j - \zeta_{j-1}\|_{Z_0} \leq c k^2\|\zeta'\|^2_{C([0,T];Z_0)}.
\] (3.54)
Combining (3.52)–(3.54), we have
\[
\|\zeta_n - \zeta^h_n\|_{Z_0}^2 + k\sum_{j=1}^{n}\|\zeta_j - \zeta^h_j\|_{Z_0}^2 \\
\leq c\left\{|\zeta_0 - \zeta^h_0\|_{Z_0}^2 + \|\zeta_1 - \zeta^h_1\|_{Z_0}^2 + k^2(\|u\|_{H^1(0,T;V)}^2 + \|\zeta'\|_{C([0,T];Z_0)}^2)\\
+ \|\zeta_n - \zeta^h_n\|_{Z_0}^2 + k\sum_{j=1}^{n}\|\phi(e(u_j), \zeta_j) - \zeta_j + k\Delta \zeta_j\|_{Z_0}\|\zeta_j - \zeta^h_j\|_{Z_0}^2 \\
+ k\sum_{j=1}^{n}\|\zeta_j - \zeta^h_j\|_{Z_0}^2 + k\sum_{j=1}^{n}\|u_{j-1} - u^h_{j-1}\|_{V}^2 + k\sum_{j=1}^{n}\|\delta \zeta_j - \zeta^h_j\|_{Z_0}^2 \right\}.
\] (3.55)
Combining this inequality with (3.20), (3.30) and (3.42), we obtain
\[
\|u_n - u^h_n\|_V^2 + \|\sigma_n - \sigma^h_n\|_Q^2 + \|\beta_n - \beta^h_n\|_B^2 + \|\zeta_n - \zeta^h_n\|_{Z_0}^2 \\
+ k\sum_{j=1}^{n}\|\zeta_j - \zeta^h_j\|_{Z_0}^2 + k\sum_{j=1}^{n}\|\phi(e(u_j), \zeta_j) - \zeta_j + k\Delta \zeta_j\|_{Z_0}\|\zeta_j - \zeta^h_j\|_{Z_0}^2 \\
+ k\sum_{j=1}^{n}\|\zeta_j - \zeta^h_j\|_{Z_0}^2 + k\sum_{j=1}^{n}\|u_{j-1} - u^h_{j-1}\|_{V}^2 + k\sum_{j=1}^{n}\|\beta_j - \beta^h_j\|_B^2
\]
Applying the Gronwall inequality, we have
\[
\max_{1 \leq n \leq N}(\|u_n - u^h_n\|_V^2 + \|\sigma_n - \sigma^h_n\|_Q^2 + \|\beta_n - \beta^h_n\|_B^2)
\]
\[
\leq c k^2(\|u\|_{H^1(0,T;V)}^2 + \|\beta\|_{W^{1,1}(0,T;B)}^2 + \|\zeta'\|_{C([0,T];Z_0)}^2)
\]
\[
+ c(\|u_0 - u^h_0\|_V^2 + \|\zeta_0 - \zeta^h_0\|_{Z_0}^2 + \|\beta_0 - \beta^h_0\|_B^2) + c\max_{1 \leq n \leq N}\tilde{R}_n
\]
Problem 3.2, respectively. Assume (2.2)–(2.10), regularity assumption in (3.10)–(3.13), we have

\[ T \subseteq \Omega \] be a polygonal or polyhedral domain and let no more than one in

\[ T \text{ continuous affine functions} \]

Theorem 3.4. Let \((u, \zeta, \beta)\) and \((u^{hk}, \zeta^{hk}, \beta^{hk})\) be solutions to Problem 2.2 and Problem 3.2, respectively. Assume (2.2)–(2.10), \(m_\Lambda > \alpha \| \gamma \|^2\). Then under the regularity assumption in (3.10)–(3.13), we have

\[
\max_{1 \leq n \leq N} \left( \| u_n - u_n^{hk} \|_V^2 + \| \sigma_n - \sigma_n^{hk} \|_Q^2 + \| \zeta_n - \zeta_n^{hk} \|_{Z_0}^2 + \| \beta_n - \beta_n^{hk} \|_{B}^2 \right) \leq c k^2 \left( \| u \|_{H^1(0,T;V)}^2 + \| \beta \|_{W^{1,1}(0,T;B)}^2 + \| \zeta \|_{C(0,T;Z_0)}^2 \right) + c \| u_0 - u_0^{hk} \|_V^2 + \| \zeta_0 - \zeta_0^{hk} \|_{Z_0}^2 + \| \beta_0 - \beta_0^{hk} \|_{B}^2 \]  

(3.56)

where

\[
\tilde{R}_n = \| u_n - u_n^{hk} \|_V^2 + \| u_n - u_n^{h} \|_{L^2(\Gamma_C;\mathbb{R}^d)}
+ \| \zeta_n - \zeta_n^{hk} \|_{Z_0}^2 + k \sum_{j=1}^{n} \| \delta \zeta_j - \zeta_j^{hk} \|_{Z_0}^2
+ \sum_{j=1}^{n} k \| \phi(\epsilon(u_j), \zeta_j) - \delta \zeta_j + k \Delta \zeta_j \|_{Z_0} \| \zeta_j - \zeta_j^{hk} \|_{Z_0}
+ k \sum_{j=1}^{n} \| \delta \beta_j - \beta_j^{hk} \|_{B}^2 + k \sum_{j=1}^{n} \| (I - P_{B^h}) \beta_j^{hk} \|_{B}^2 + k^2 \]

Theorem 3.4 is the basis of analyzing the optimal error estimate. As an example, let \( \Omega \) be a polygonal or polyhedral domain and let \( T^h \) be a regular triangulation of \( \Omega \) compatible with the partition of the boundary \( \Gamma = \partial \Omega \) into \( \Gamma_D, \Gamma_N \) and \( \Gamma_C \). For an element \( T \in T^h, P_1(T;\mathbb{R}^d) \) is used as the space of polynomials of a total degree no more than one in \( T \). Then we can use the linear element space of piecewise continuous affine functions

\[ V^h = \{ v^h \in C(\Omega;\mathbb{R}^d) : v^h|_T \in P_1(T;\mathbb{R}^d) \ \forall \ T \in T^h, v^h = 0 \text{ on } \Gamma_D \} \]

(3.57)

\[ k_h^h = \{ \zeta^h \in C(\Omega) ; \zeta^h|_T \in P_1(T) , \forall T \in T^h \} \]

(3.58)
If $\mathcal{T}_C^h$ is the partition of $\Gamma_C$ induced by triangulation $\mathcal{T}^h$, then the space $B$ is approximated by the space of piecewise constant function

$$B^h = \{ \theta^h \in B : \theta^h |_{\Gamma} \in \mathbb{R}, \forall \gamma \in \mathcal{T}_C^h \}. \quad (3.59)$$

**Corollary 3.5.** Under the assumptions stated in Theorem 3.4. Assume $\Omega$ is a polygonal or polyhedral domain, and let $\{V^h\}$, $\{K^h\}$ be the family of linear element spaces made of continuous and piecewise affine functions defined by (3.57) and (3.58), and $\{B^h\}$ the family of piecewise constant function defined by (3.59) corresponding to a regular family of finite element triangulations of $\overline{\Omega}$ into triangles or tetrahedrons. Assume further that

$$\begin{align*}
\mathbf{u} &\in C([0,T]; H^2(\Omega; \mathbb{R}^d)), \quad \mathbf{u}|_{\Gamma_C} \in C([0,T]; H^2(\Gamma_C; \mathbb{R}^d)), \\
\zeta &\in C([0,T]; H^2(\Omega)) \cap H^2(0,T; Z_0) \cap H^1(0,T; Z_1).
\end{align*}$$

Then we have the following optimal order error estimate:

$$\max_{1 \leq n \leq N} \left\{ \| \mathbf{u}_n - \mathbf{u}_h^k \|_V^2 + \| \mathbf{\sigma}_n - \mathbf{\sigma}_h^k \|_B^2 + \| \beta_n - \beta_h^k \|_B^2 + \| \zeta - \zeta_h \|_B^2 \right\} \leq c(k^2 + h^2).$$

**Proof.** In the following, we may apply the standard finite element interpolation error estimates, see [3, 5, 12]. Since $\mathbf{u} \in C([0,T]; H^2(\Omega; \mathbb{R}^d))$, we have

$$\max_{1 \leq n \leq N} \inf_{v_0^h \in V^h} \| \mathbf{u}_n - v_0^h \|_V \leq c h \| \mathbf{u} \|_{C([0,T]; H^2(\Omega; \mathbb{R}^d))}. \quad (3.60)$$

By the finite element interpolation error estimates, take $\xi_j^h$ to be the interpolant of $\zeta_j$ for $j = 1, \ldots, N$, then

$$\begin{align*}
\| \zeta_j - \xi_j^h \|_{Z_0} &\leq c h^2 \| \zeta_j \|_{H^2(\Omega)}, \\
\| \zeta_j - \xi_j^h \|_{Z_1} &\leq c h \| \zeta_j \|_{H^2(\Omega)}.
\end{align*} \quad (3.61, 3.62)$$

Since $\mathbf{u}_0^h$ is the finite element interpolant of $\mathbf{u}_0$, $\zeta_0^h$ be the finite element orthogonal projection of $\zeta_0$ in $L^2(\Omega)$ and $\beta_0^h$ is the finite element interpolant of $\beta_0$. Then we have the following error estimate for the discrete initial values,

$$\| \mathbf{u}_0 - \mathbf{u}_0^h \|_V + \| \zeta_0 - \zeta_0^h \|_{Z_0} + \| \beta_0 - \beta_0^h \|_B \leq c h.$$ \quad (3.63)

Since $\zeta \in H^2(0,T; Z_0)$, it follows that

$$\kappa \sum_{j=1}^{n} \| \delta \zeta_j - \zeta_j^h \|_{Z_0}^2 \leq c k^2 \| \zeta \|_{H^2(0,T; Z_0)}^2, \quad 1 \leq n \leq N. \quad (3.64)$$

From [18], we have

$$\frac{1}{k} \sum_{j=1}^{n-1} \| \zeta_j - \zeta_j^h - (\zeta_j+1 - \zeta_j^h) \|_{Z_0}^2 \leq c h^2 \| \zeta \|_{H^1(0,T; Z_1)}^2, \quad 1 \leq n \leq N. \quad (3.65)$$

Similarly, the fact that

$$\delta \beta_j - \delta \beta_j' = \frac{1}{k} \int_{t_{j-1}}^{t_j} (\beta'(s) - \beta'(t_j)) \, ds = \frac{1}{k} \int_{t_{j-1}}^{t_j} \left( \int_{t_j}^{s} \beta''(\tau) \, d\tau \right) \, ds$$

implies

$$\kappa \sum_{j=1}^{n} \| \delta \beta_j - \delta \beta_j' \|_B^2 \leq c k^2 \| \beta \|_{W^{2,1}(0,T; B)}^2. \quad (3.66)$$
Also, it is obvious that
\[
| \sum_{j=1}^{n} ((I - \mathcal{P}_{h^n})\beta_j^|_B \| B^2 \leq ch^2.
\] (3.67)

Finally, we have
\[
\max_{1 \leq n \leq N} \| u_n - u_n^h \|_{L^2(\Gamma_{C}; \mathbb{R}^4)} \leq c h^2 \| u \|_{C([0,T]; H^2(\Gamma_{C}; \mathbb{R}^4))}.
\] (3.68)

Combining (3.60)–(3.68) and (3.56), the following optimal order error estimate is obtained:
\[
\max_{1 \leq n \leq N} \{ \| u_n - u_n^h \|_V^2 + \| \sigma_n - \sigma_n^h \|_Q^2 + \| \zeta_n - \zeta_n^h \|_Z^2 + \| \beta_n - \beta_n^h \|_B^2 \} \leq c (k^2 + h^2).
\]

This concludes the proof of Corollary 3.5.

\[\square\]

4. Numerical example. In this section, we give three numerical examples and present some numerical results to illustrate the behavior of the solution of the history-dependent adhesive contact problem Problem 1.1. In these examples, the long memory operator has the form
\[
G(t-s, \varepsilon(u(s)), \zeta(s)) = \int_0^t \eta_\nu(\zeta) A\varepsilon(u(s)) \, ds.
\]
Here, the elasticity tensor \( A \) is given by
\[
(A\tau)_{\alpha\beta} = \frac{E}{1 + \kappa} \tau_{\alpha\beta} + \frac{E\kappa}{(1 - \kappa)(1 - 2\kappa)} (\tau_{11} + \tau_{22}) \delta_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2, \quad \forall \tau \in \mathbb{S}^2,
\]
where \( E \) and \( \kappa \) are Young’s modulus and Poisson’s ratio of the material, and \( \delta_{\alpha\beta} \) denotes the Kronecker symbol. We take \( E = 1000N/m \) and \( \kappa = 0.3 \) in the numerical examples.

The damage source function used here has the form
\[
\phi(\varepsilon(u), \zeta) = -\lambda_1 \left( \frac{1 - \eta_\nu(\zeta)}{\eta_\nu(\zeta)} \right) - \frac{1}{2} \lambda_2 \phi_{q^*}(\varepsilon(u)) + \lambda_3
\]
where \( \phi_{q^*}(\varepsilon(u)) = \min\{\varepsilon(u) \cdot \varepsilon(u), q^*\} \) for some constant \( q^* > 0 \) and
\[
\eta_\nu(\zeta) = \begin{cases} 
\zeta, & \text{if } \zeta \leq 1, \\
1, & \text{if } \zeta > 1.
\end{cases}
\]

We assume that the function \( j_\nu : \Gamma_C \times (0, T) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is of the form \( j_\nu(x, t, r, s) = \varphi_\nu(x, t, r) \psi_\nu(x, t, s) \), so the condition (1.7) can be formulated as follows
\[
-\sigma_\nu(t) \in \varphi_\nu(x, t, \beta(t)) \partial \psi_\nu(x, t, u_\nu(t)) \quad \text{on } \Gamma_C \times (0, T).
\] (4.1)

In the model, we take \( \varphi_\nu(x, t, \beta(t)) = \beta(t) \) and the function \( \psi_\nu : \Gamma_C \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R} \) is defined by
\[
\psi_\nu(x, t, s) = \begin{cases} 
\frac{k^2 s^2}{2} & \text{if } s > 0, \\
\frac{k_3 s^2}{2} & \text{if } s \in [-L, 0], \\
\frac{k_3 s^2}{2} & \text{if } s < -L.
\end{cases}
\]
for a.e. \((x,t) \in \Gamma_C \times (0,T)\), where \(\varepsilon > 0\) is a small positive constant, \(k_1 > 0\). The Clarke’s subdifferential of this function is given by

\[
\partial \psi(x,t,s) = \begin{cases} 
\frac{s}{\varepsilon} & \text{if } s \geq 0, \\
k_1s & \text{if } s \in (-L, 0), \\
[-k_1L, 0] & \text{if } s = -L, \\
0 & \text{if } s < -L.
\end{cases}
\]

for a.e. \((x,t) \in \Gamma_C \times (0,T)\). Moreover, \(\tilde{R}(s)^2\) is defined by

\[
\tilde{R}(s)^2 = |\tilde{R}(s)|^2 = \begin{cases} 
L^2 & \text{if } s < -L, \\
s^2 & \text{if } s \in [-L, 0], \\
0 & \text{if } s > 0.
\end{cases}
\]

4.1. First example

We consider the physical setting shown in Figure 1. Here, the domain \(\Omega = (0, 2) \times (0, 1)\), and its boundary is split into:
\(\Gamma_D = \{0\} \times (0, 1)\), \(\Gamma_N = ((0, 2) \times \{1\}) \cup \{(2) \times (0, 1)\}\) and \(\Gamma_C = (0, 2) \times \{0\}\),
\(\mathbf{f}_0 = (0, 0)N/m^2\) in \(\Omega\), \(\mathbf{f}_N = (0, 50t)N/m\) on \((0, 2) \times \{1\}\), \(\zeta_0 = 0.01\).
\(L = 1, \lambda_1 = 0.1, \lambda_2 = 2, \lambda_3 = 0, \beta_0 = 1, u_0 = 0, \zeta_0(x) = 1, \forall x \in \Omega\).
\(\varepsilon_a = 0.00001, \varepsilon = 0.1, k_1 = 1, \gamma_0 = 0.8, T = 1s\).

In Figure 2, we plot the body’s shape after \(t = 0.125s, t = 0.5s, t = 0.75s, t = 1s\), respectively during the compression process. As the time goes, the displacement is bigger. In Figure 3, we plot the damage field at time \(t = 0.5s\) and \(t = 1s\), respectively. We observe that in both cases the damage is most concentrated on the left boundary due to the clamping condition. As the time goes, the stress is bigger and the damage is more serious. In Figure 4, we drawn the adhesion field at times \(t = 0.125, 0.5, 0.75, 1(s)\). We notice that both the values of the adhesion field and the damage field are between 0 and 1, which corresponds to the theoretical results.
4.2. Second example

As the second example, the physical setting is similar to the above test. Now we assume that there is no damage effects and $f_N$ is changed to be periodic in time and $f_N = (0, -30\sin50t - \sin50t)N/m$ on $(0, 2) \times \{1\}$.

In Figure 5, we plot the body’s shape after $t = 0.125s$, $t = 0.5s$, $t = 0.75s$, $t = 1s$, respectively during the compression process. We find that as the external force turns to be periodic, the displacement is also periodic. In Figure 6, we drawn the adhesion field at times $t=0.125, 0.5, 0.75, 1s$. 

Figure 2. the deformed configuration at $t=0.125s$, $t=0.5s$, $t=0.75s$ and $t=1s$

Figure 3. the damage field at $t=0.5s$ and $t=1s$
Figure 4. the adhesion field at several times

(a) $t=0.125s$  
(b) $t=0.5s$

Figure 5. the deformed configuration at several times

(c) $t=0.75s$  
(d) $t=1s$
4.3. The third example

As the third example, the physical setting is also similar to the first test. However, we let $\Gamma_D = (\{0\} \times (0, 1)) \cup (\{2\} \times (0, 1))$, $\Gamma_N = (0, 2) \times \{1\}$ and $\Gamma_C = (0, 2) \times \{0\}$. Moreover, now $f_N$ is made to be $(0,0)N/m$ on $\Gamma_N$ and $f_0 = (0, 100t)N/m^2$ in $\Omega$.

In Figure 7, we plot the body’s shape at time $t = 0.5s$ and $t = 1s$, respectively. We observe that in both cases the damage is concentrated on the left boundary and the right boundary due to the clamping condition. As the time goes, the stress is bigger and the damage is more serious. In Figure 9, we plot the body’s shape at $\gamma_\nu = 1$ and $\gamma_\nu = 10$, we observe that for stronger adhesion, it is more difficult for the body to leave the foundation.
From the above examples, we conclude that the damage is more concentrate on the clamping boundaries, the contact boundaries, i.e. the most stressed fields, for stronger adhesion, it is more difficult for the body to leave the foundation.

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