Abstract

The $C^*$-algebra of continuous functions on the quantum quaternion sphere $H_q^{2n}$ can be identified with the quotient algebra $C(SP_q(2n)/SP_q(2n-2))$. In commutative case i.e. for $q = 1$, the topological space $SP(2n)/SP(2n-2)$ is homeomorphic to the odd dimensional sphere $S^{4n-1}$. In this paper, we prove the noncommutative analogue of this result. Using homogeneous $C^*$-extension theory, we prove that the $C^*$-algebra $C(H_q^{2n})$ is isomorphic to the $C^*$-algebra $C(S_q^{4n-1})$. This further implies that for different values of $q \in [0,1)$, the $C^*$-algebras underlying the noncommutative space $H_q^{2n}$ are isomorphic.

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1 Introduction

Quantization of Lie groups and their homogeneous spaces have played an important role in linking the theory of compact quantum group with noncommutative geometry. Many authors (see [16], [12], [3], [10]) have studied different aspects of the theory of quantum homogeneous spaces. However, in these papers, main examples have been the quotient spaces of the compact quantum group $SU_q(n)$. Neshveyev & Tuset ([9]) studied quantum homogeneous spaces in a more general set up and gave a complete classification of the irreducible representations of the $C^*$-algebra $C(G_q/H_q)$ where $G_q$ is the $q$-deformation of a simply connected semisimple compact Lie group and $H_q$ is the $q$-deformation of a closed Poisson-Lie subgroup $H$ of $G$. Moreover, Neshveyev & Tuset ([9]) proved that $C(G_q/H_q)$ is $KK$-equivalent to the classical counterpart $C(G/H)$. Quantum symplectic group $SP_q(2n)$ and its homogeneous space $C(SP_q(2n)/SP_q(2n-2))$ have been studied.
by the author in [14] and $K$-groups of quotient space $C(SP_q(2n)/SP_q(2n - 2))$ with explicit generators were obtained.

The $C^*$-algebra $C(H^{2n}_q)$ of continuous functions on the quantum quaternion sphere is defined as the universal $C^*$-algebra given by a finite set of generators and relations (see [14]). In [14], the isomorphism between the quotient algebra $C(SP_q(2n)/SP_q(2n - 2))$ and $C(H^{2n}_q)$ has been established. Now several questions arise about this noncommutative space $H^{2n}_q$.

1. Is $H^{2n}_q$ topologically same as $S^{4n-1}_{q}$, i.e. are the $C^*$-algebras $C(H^{2n}_q)$ and $C(S^{4n-1}_{q})$ isomorphic?

2. Are the $C^*$-algebras $C(H^{2n}_q)$ isomorphic for different values of $q$?

3. Does the quantum quaternion sphere admit a good spectral triple equivariant under the $SP_q(2n)$-group action?

We attempt the first two questions in this paper. In commutative case i.e. for $q = 1$, the quotient space $SP(2n)/SP(2n - 2)$ can be realized as the quaternion sphere $H^{2n}_q$. It can be easily verified that the quaternion sphere $H^{2n}_q$ is homeomorphic to the odd dimensional sphere $S^{4n-1}_{q}$. One can now expect the quotient algebra $C(SP_q(2n)/SP_q(2n - 2))$ or equivalently the $C^*$-algebra $C(H^{2n}_q)$ to be isomorphic to the $C^*$-algebra underlying the odd dimensional quantum sphere $S^{4n-1}_{q}$. In this paper, using homogeneous $C^*$-extension theory, we show that this is indeed the case.

The remarkable work done by L. G. Brown, R. G. Douglas and P. A. Fillmore ([2]) on extensions of commutative $C^*$-algebras by compact operators has led many authors to extend this theory further in order to provide a tool for analysing the structure of $C^*$-algebras. For a nuclear, separable $C^*$-algebra $A$ and a separable $C^*$-algebra $B$, G. G. Kasparov ([8]) constructed the group $Ext(A, B)$ consisting “stable equivalence classes” of $C^*$-algebra extensions of the form

$$0 \rightarrow B \otimes \mathcal{K} \rightarrow E \rightarrow A \rightarrow 0$$

Here $E$ will be called the middle $C^*$-algebra. One of the important features of this construction is that the group $Ext(A, B)$ coincides with the group $KK^1(A, B)$. Another important aspect is that it does not demand much. It does not require the extensions to be unital or essential. But at the same time, it does not provide much informations about the middle $C^*$-algebras. Since elements of the group $Ext(A, B)$ are stable equivalence classes and not unitary equivalence classes of extensions, two elements in the same class may have nonisomorphic middle $C^*$-algebras. For a nuclear $C^*$-algebra $A$ and a finite dimensional compact metric space $Y$ (i.e. a closed subset of $S^n$ for
some $n \in \mathbb{N}$, M. Pimsner, S. Popa and D. Voiculescu ([11]) constructed another group $Ext_{PPV}(Y, A)$ consisting of unitary equivalence classes of unital homogeneous extensions of $A$ by $C(Y) \otimes \mathcal{K}$. For $y_0 \in Y$, the subgroup $Ext_{PPV}(Y, y_0, A)$ consists of those elements of $Ext_{PPV}(Y, A)$ that split at $y_0$. For a commutative $C^*$-algebra $A$, the group $Ext_{PPV}(Y, A)$ was computed by Schochet in [15]. Further Rosenberg & Schochet ([13]) showed that $Ext_{PPV}(Y, A^+) = Ext(A, C(Y))$ and $Ext_{PPV}(Y^+, +, A^+) = Ext(A, C(Y))$ where $Y$ is a finite dimensional locally compact Hausdorff space, $+$ is the point at infinity and $A^+$ is the $C^*$-algebra obtained by adjoining unity to $A$.

To show that the $C^*$-algebra $C(H^2_q)$ is isomorphic to $C(S^4_q - 1)$, we first exhibit an isomorphism between the group $Ext_{PPV}(Y, y_0, A)$ and the group $Ext_{PPV}(Y, y_0, \Sigma^2 A)$ under certain assumptions on the topological space $Y$ where $\Sigma^2 A$ is the quantum double suspension of $A$ and $y_0 \in Y$. Using this, we describe all elements of the group $Ext_{PPV}(\mathbb{T}, C(S^2_q - 1))$ explicitly. We then prove that all nonisomorphic middle $C^*$-algebras that occur in all the extensions of the group $Ext_{PPV}(\mathbb{T}, C(S^2_q - 1))$ have different $K$-groups. Then using representation theory of $C(H^2_q)$, we show that the following extension

$$0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C(H^2_q) \to C(S^4_q - 3) \to 0$$

is unital and homogeneous. Now by comparing the $K$-groups, we prove that the above extension is unitarily equivalent to either the following extension

$$0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C(S^4_q - 3) \to C(S^4_q - 3) \to 0$$

or its inverse in the group $Ext_{PPV}(\mathbb{T}, C(S^2_q - 1))$. This proves that the $C^*$-algebras $C(H^2_q)$ and $C(S^4_q - 1)$ are isomorphic. For $q = 0$, it follows directly from the defining relations. In [10], it was proved that for different values of $q \in [0, 1)$ the $C^*$-algebras $C(S^4_q - 1)$ are isomorphic. As a consequence, the $C^*$-algebras $C(H^2_q)$ and $C(S^4_q - 1)$ are isomorphic for all $q \in [0, 1)$. Also, this shows that the $C^*$-algebras $C(H^2_q)$ are isomorphic for different values of $q$ which establishes $q$-invariance of the quantum quaternion spheres.

We now set up some notations. The standard bases of the Hilbert spaces $L_2(\mathbb{N})$ and $L_2(\mathbb{Z})$ will be denoted by \{e_n : n \in \mathbb{N}\} and \{e_n : n \in \mathbb{Z}\} respectively. We denote the left shift operator on $L_2(\mathbb{N})$ and $L_2(\mathbb{Z})$ by the same notation $S$. For $m < 0$, $(S^*)^m$ denotes the operator $S^{-m}$. Let $p_i$ denote the rank one projection sending $e_i$ to $e_i$ and $p$ denote the operator $p_0$. We write $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ for the sets of all bounded linear operators on $\mathcal{H}$ and compact operators on $\mathcal{H}$ respectively. We denote by $\mathcal{K}$ the $C^*$-algebra of compact operators. For a $C^*$-algebra $A$, $\Sigma^2 A$ and $M(A)$ are used to denote the quantum double
suspension of $A$ and multiplier algebra of $A$ respectively. The map $\pi$ will denote the canonical homomorphism from $M(A)$ to $Q(A) := M(A)/A$ and for $a \in M(A)$, $[a]$ stands for the image of $a$ under the map $\pi$. For a locally compact Hausdorff space $Y$, we write $Y^+$ to denote one point compactification of $Y$. For a $C^*$-algebra $A$, $A^+$ denotes the $C^*$-algebra obtained by adjoining unity to $A$. Both the symbols $S^n$ and $T^n$ will denote the $n$-dimensional sphere. Unless otherwise stated, $q$ will denote a real number in the interval $(0, 1)$.

## 2 $C^*$-algebra extensions

We first recall some notions related to the $C^*$-extension theory. Let $A$ be a unital separable nuclear $C^*$-algebra. Let $B$ be a stable $C^*$-algebra. An extension of $A$ by $B$ is a short exact sequence $0 \to B \overset{i}{\to} E \overset{j}{\to} A \to 0$. In such case, there exists a unique homomorphism $\sigma : E \to M(B)$ such that $\sigma(i(b)) = b$ for all $b \in B$. We can now define the Busby invariant for the extension $0 \to B \overset{i}{\to} E \overset{j}{\to} A \to 0$ as the homomorphism $\tau : A \to M(B)/B$ given by $\tau(a) = \pi \circ \sigma(e)$ where $e$ is a preimage of $a$ and $\pi$ is the canonical map $M(B) \to M(B)/B$. It is easy to see that $\tau$ is well defined. An extension $\tau$ is called an essential extension if $\tau$ is injective or equivalently image of $B$ is an essential ideal of $E$. We call an extension unital if it is a unital homomorphism or equivalently $E$ is a unital $C^*$-algebra. An extension $\tau$ is called a trivial (or split) extension if there exists a homomorphism $\lambda : A \to M(B)$ such that $\tau = \pi \circ \lambda$. Two extensions $\tau_1$ and $\tau_2$ are said to be weakly unitarily equivalent if there exists a unitary $u$ in $Q(B)$ such that $u\tau_1(a)u^* = \tau_2(a)$ for all $a \in A$. They are said to be unitarily equivalent if there exists a unitary $U$ in $M(B)$ such that $\pi(U)\tau_1(a)\pi(U^*) = \tau_2(a)$ for all $a \in A$. We denote unitarily equivalence relation by $\sim_u$. Let $Ext_{\sim_u}(A, B)$ denote the set of unitary equivalence classes of extensions of $A$ by $B$. One can put a binary operation $\sim^+$ on $Ext_{\sim_u}(A, B)$ as follows. Since $M(B)$ is a stable $C^*$-algebra, we can get two isometries $\nu_1$ and $\nu_2$ in $M(B)$ such that $\nu_1\nu_1^* + \nu_2\nu_2^* = 1$. Let $\tau_1$ and $\tau_2$ be two elements in $Ext_{\sim_u}(A, B)$. Define $\tau_1 + \tau_2 : A \to Q(B)$ by

$$
(\tau_1 + \tau_2)(a) := \pi(\nu_1)\tau_1(a)\pi(\nu_1^*) + \pi(\nu_2)\tau_2(a)\pi(\nu_2^*).
$$

(2.1)

This makes $Ext_{\sim_u}(A, B)$ a commutative semigroup. Moreover, the set of trivial extensions forms a subsemigroup of $Ext_{\sim_u}(A, B)$. We denote the quotient of $Ext_{\sim_u}(A, B)$ with the set of trivial extensions by $Ext(A, B)$. For a separable nuclear $C^*$-algebra $A$, the set $Ext(A, B)$ under the operation $+$ is a group (see [1]). Two extensions
\(\tau_1\) and \(\tau_2\) represent the same element in \(\text{Ext}(A, B)\) if there exists two trivial extensions \(\phi_1\) and \(\phi_2\) such that \(\tau_1 + \phi_1 \sim_u \tau_2 + \phi_2\). One can show that for a stable C*-algebra \(B\), \(\text{Ext}(A, B) = \text{Ext}(A, B \otimes K)\). Now for an arbitrary C*-algebra \(B\), define \(\text{Ext}(A, B) := \text{Ext}(A, B \otimes K)\). We denote an equivalent class in the group \(\text{Ext}(A, B)\) of an extension \(\tau\) by \([\tau]_s\). For \(B = \mathbb{C}\), we denote the group \(\text{Ext}(A, \mathbb{C})\) by \(\text{Ext}(A)\). Note that in this case, two unital essential extensions \(\tau_1\) and \(\tau_2\) are in the same equivalence class (i.e. \([\tau_1]_s = [\tau_2]_s\)) if and only if they are unitarily equivalent.

Suppose that \(Y\) is a finite dimensional compact metric space i.e. a closed subset of \(S^n\) for some \(n \in \mathbb{N}\). Let \(M(Y)\), \(Q(Y)\) and \(Q\) be the C*-algebras \(M(C(Y) \otimes K)\), \(M(C(Y) \otimes K)/C(Y) \otimes K\) and \(L(H)/K(H)\) (Calkin algebra) respectively. It is easy to show that \(M(Y)\) is the set of all continuous functions from \(Y\) to \(L(H)\) where continuity is with respect to \(*\)-strong operator topology on \(L(H)\). We call an extension \(\tau\) of \(A\) by \(C(Y) \otimes K\) homogeneous if for all \(y \in Y\), the map \(ev_y \circ \tau : A \to Q\) is injective where \(ev_y : Q(Y) \to Q\) is the evaluation map at \(y\). Let \(\text{Ext}_{PPV}(Y, A)\) be the set of unitary equivalence classes of unital homogeneous extensions of \(A\) by \(C(Y) \otimes K\). For a nuclear C*-algebra \(A\), Pimsner, Popa and Voiculescu (\cite{PimsnerPopaVoiculescu}) showed that \(\text{Ext}_{PPV}(Y, A)\) is a group with the additive operation defined in \(2.1\). We denote the equivalence class in the group \(\text{Ext}_{PPV}(Y, A)\) of an extension \(\tau\) by \([\tau]_u\). For \(y_0 \in Y\), define the set

\[\text{Ext}_{PPV}(Y, y_0, A) = \{[\tau]_u \in \text{Ext}_{PPV}(Y, A) : ev_{y_0} \circ \tau \text{ is trivial}\}.\]

The set \(\text{Ext}_{PPV}(Y, y_0, A)\) is a subgroup of \(\text{Ext}_{PPV}(Y, A)\).

### 2.1 The groups \(\text{Ext}_{PPV}(Y, A)\) and \(\text{Ext}_{PPV}(Y, \Sigma^2 A)\)

Here we will show that for a separable nuclear C*-algebra \(A\) and a finite dimensional compact metric space \(Y\) such that \(K\)-groups of \(C(Y)\) are free groups with finite generators, the groups \(\text{Ext}_{PPV}(Y, A)\) and \(\text{Ext}_{PPV}(Y, \Sigma^2 A)\) are isomorphic. Let us recall some definitions. We say that two elements \(a\) and \(b\) in \(Q(A)\) are unitarily equivalent if there exists a unitary \(U \in M(A)\) such that \([U]a[U^*] = b\). They are weakly unitarily equivalent if there exists unitary \(u \in Q(A)\) such that \(uau^* = b\). We call an element \(a\) in a C*-algebra \(B\) norm-full if it is not contained in any proper closed ideal in \(B\). Suppose that \(A\) and \(B\) are separable C*-algebras. An extension \(\tau : A \to Q(B \otimes K)\) is said to be norm-full if for every nonzero element \(a \in A\), \(\tau(a)\) is norm full element of \(Q(B \otimes K)\).

**Definition 2.1.** Let \(B\) be a separable stable C*-algebra. Then \(B\) is said to have the corona factorization property if every norm-full projection in \(M(B)\) is Murray-von Neumann equivalent to unit element of \(M(B)\).
It is easy to see that a $C^*$-algebra $A$ with corona factorization property, any norm-
full projection in $Q(B)$ is Murray-von Neumann equivalent to 1 of $Q(B)$. Further, one
can show that for a finite dimensional compact metric space $Y$, $C(Y) \otimes K$ has corona
factorization property (see [1]).

**Proposition 2.2.** Let $A$ be a unital separable nuclear $C^*$-algebra which satisfies the
Universal Coefficient Theorem. Suppose that $Y$ is a finite dimensional compact metric
space. Then the map

$$i : \text{Ext}_{PPV}(Y, A) \longrightarrow KK^1(A, C(Y))$$

$$[\tau]_u \mapsto [\tau]$$

is an injective homomorphism.

*Proof:* Since unitarily equivalence implies stable equivalence, the map $i$ is well defined.
Any unital homogeneous extension is a purely large extension and hence a norm-full
extension (see page 19, [4]). Therefore from Theorem 2.4 in [7], it follows that $i$ is
injective. □

From now on, without loss of generality, we will assume that the Hilbert space $\mathcal{H}$ is
$L_2(\mathbb{N})$. Let $\tau$ be a unital homogeneous extension of $A$ by $C(Y) \otimes K(\mathcal{H})$. Define $\tilde{\tau} : A \to
Q(C(Y) \otimes K(\mathcal{H}) \otimes K(\mathcal{H}))$ by : $\tilde{\tau}(a) = [\tau_a \otimes p]$ where $[\tau_a] = \tau(a)$. By universal property
of quantum double suspension (see proposition 2.2, [5]), we have a homomorphism

$$\Sigma^2 \tau : \Sigma^2 A \to Q(C(Y) \otimes K(\mathcal{H}) \otimes K(\mathcal{H}))$$

such that $\Sigma^2 \tau(a \otimes p) = \tilde{\tau}(a) = [\tau_a \otimes p]$ and $\Sigma^2 \tau(1 \otimes S) = [1 \otimes 1 \otimes S]$. Clearly $\Sigma^2 \tau$
is a unital extension. Since $\tau$ is homogeneous, the map $ev_y \circ \Sigma^2 \tau$ is injective on the $C^*$-
algebra $A \otimes p$ for all $y \in Y$. Making use of the fact that $(1 \otimes p)A \otimes K(1 \otimes p) = A \otimes p$, one can prove that the map $ev_y \circ \Sigma^2 \tau$ is injective on $A \otimes K$. Since $A \otimes K$ is an essential
ideal of $\Sigma^2 A$, we conclude that the map $ev_y \circ \Sigma^2 \tau$ is injective on $\Sigma^2 A$ and hence $\Sigma^2 \tau$ is
a homogeneous extension. Moreover, if $\tau_1$ and $\tau_2$ are unitarily equivalent by a unitary
$U \in M(C(Y) \otimes K(\mathcal{H}))$ then so are $\Sigma^2 \tau_1$ and $\Sigma^2 \tau_2$ by the unitary $U \otimes 1 \in M(C(Y) \otimes
K(\mathcal{H}) \otimes K(\mathcal{H}))$. This gives a well defined map

$$\beta : \text{Ext}_{PPV}(Y, A) \longrightarrow \text{Ext}_{PPV}(Y, \Sigma^2 A)$$

$$[\tau]_u \mapsto [\Sigma^2 \tau]_u$$

**Proposition 2.3.** The map $\beta : \text{Ext}_{PPV}(Y, A) \longrightarrow \text{Ext}_{PPV}(Y, \Sigma^2 A)$ given above is in-
jective group homomorphism.
Proof: It follows from straightforward calculations.

To get surjectivity of the map $\beta$, we need to put certain assumptions on the topological space $Y$.

**Proposition 2.4.** Let $Y$ be a finite dimensional compact metric space. Assume that $K_0(C(Y))$ and $K_1(C(Y))$ are free groups with finite number of generators. Let $V \in Q(C(Y) \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H}))$ be an isometry such that $VV^*$ and $1 - VV^*$ both are norm full projections. Then $V$ is weakly unitarily equivalent to $[1 \otimes 1 \otimes S]$.

Proof: We assume that $V$ is not weakly unitarily equivalent to $[1 \otimes 1 \otimes S]$. Since $C(Y) \otimes \mathcal{K}$ has corona factorization property, it follows that $VV^*$ and $1 - VV^*$ both are Murray-von Neumann equivalent to $[1]$ of $Q(C(Y) \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H}))$. Also, one can easily verify that $[1 \otimes 1 \otimes p]$ and $[1 - 1 \otimes 1 \otimes p] = [1 \otimes 1 \otimes (1 - p)]$ are Murray-von Neumann equivalent to $[1]$ of $Q(C(Y) \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H}))$. This implies that $VV^*$ is weakly unitarily equivalent to $1 - [1 \otimes 1 \otimes p]$. So, without loss of generality, we can assume that $V$ has final projection $1 - [1 \otimes 1 \otimes p]$. Take a split unital homogeneous extension $\tau$ of $C(\mathbb{T})$ by $C(Y) \otimes \mathcal{K}(\mathcal{H})$. Let $\Sigma^2_\tau$ be a unital homogeneous extension of $\Sigma^2 C(\mathbb{T})$ by $C(Y) \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H})$ given by $\Sigma^2_\tau(a \otimes p) = [\tau_a \otimes p]$ and $\Sigma^2_\tau(1 \otimes S) = V$ where $[\tau_a] = \tau(a)$. From Corollary 3.8 (7) and the fact that $V$ is not weakly unitarily equivalent to $[1 \otimes 1 \otimes S]$, it follows that $[\Sigma^2_\tau]_u$ is not in the image of the map $\beta$. Let $n[\Sigma^2_\tau]_u = \beta([\phi]_u)$ for some $n \in \mathbb{Z} - \{0\}$ and for some unital homogeneous extension $\phi$ of $C(\mathbb{T})$ by $C(Y) \otimes \mathcal{K}(\mathcal{H})$. It is easy to see that $\phi$ must be split and in that case $n[\Sigma^2_\tau]_u$ is not in the image of the map $\beta$. By proposition 2.2 and the fact that $KK^1(\Sigma^2 C(\mathbb{T}), C(Y))$ is free group, we get that $\Sigma^2_\tau$ is a split extension. This contradicts the fact that $[\Sigma^2_\tau]_u$ is not in the image of the map $\beta$. So, for any $n \in \mathbb{Z} - \{0\}$, $n[\Sigma^2_\tau]_u$ is not in the image of the map $\beta$. This shows that image of $Ext_{ppV}(Y, \Sigma^2 C(\mathbb{T}))$ in the group $KK^1(\Sigma^2 C(\mathbb{T}), C(Y))$ has one more free generator than the group $Ext_{ppV}(Y, C(\mathbb{T}))$ in $KK^1(C(\mathbb{T}), C(Y)) \cong KK^1(\Sigma^2 C(\mathbb{T}), C(Y))$. Since for all $n \in \mathbb{N}$, $KK^1(\Sigma^{2n} C(\mathbb{T}), C(Y)) \cong K_0(C(Y)) \oplus K_1(C(Y))$ are free groups with finite generators, iterating above process will lead to a contradiction. This proves that $V$ is weakly unitarily equivalent to $[1 \otimes 1 \otimes S]$.

**Remark 2.5.** Here we should point out that above proposition may hold for any finite dimensional compact metric space $Y$. But since we could not find it in literature, we prove the proposition under certain assumptions on $Y$.

**Corollary 2.6.** Let $Y$ and $V$ be as in the above proposition. Then $V$ is unitarily equivalent to $[1 \otimes 1 \otimes S]$. 

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Proof: Consider the unital extension $\Sigma^2_\nu \tau$ constructed in proposition 2.4. From Corollary 3.8 ([7]), it follows that $\Sigma^2_\nu \tau$ is unitarily equivalent to $\Sigma^2_\nu \tau$ defined in equation 2.2 with $A = C(\mathbb{T})$. Hence $V$ is unitarily equivalent to $[1 \otimes 1 \otimes S]$. \hfill $\square$

The following lemma establishes the isomorphism between the groups $Ext_{PPV}(Y, A)$ and $Ext_{PPV}(Y, \Sigma^2 A)$ under certain assumptions on the space $Y$.

Lemma 2.7. Let $Y$ be a finite dimensional compact metric space. Assume that the groups $K_0(C(Y))$ and $K_1(C(Y))$ are free groups with finite number of generators. Then the map $\beta : Ext_{PPV}(Y, A) \rightarrow Ext_{PPV}(Y, \Sigma^2 A)$ given above is an isomorphism.

Proof: We only need to show that $\beta$ is surjective thanks to proposition 2.3. Let $\phi$ be a unital homogeneous extension of $\Sigma^2 A$ by $C(Y) \otimes K(\mathcal{H}) \otimes K(\mathcal{H})$. Let $\phi(1 \otimes S) = V$. Since $\phi$ is a unital homogeneous extension and hence a norm full extension, it follows that $VV^*$ and $1 - VV^*$ are norm full projections. Therefore by Corollary 2.6, there exists a unitary $U \in M(C(Y) \otimes K(\mathcal{H}) \otimes K(\mathcal{H}))$ such that $[U]V[U^*] = [1 \otimes 1 \otimes S]$. So without loss of generality, we can assume that $\phi$ maps $1 \otimes S$ to $[1 \otimes 1 \otimes S]$. This implies that $\phi(1 \otimes p) = [1 \otimes 1 \otimes p]$. But then $\phi(A \otimes p) \subset (1 \otimes p)\phi(A \otimes p)(1 \otimes p) \subset Q(C(Y) \otimes K(\mathcal{H})) \otimes p$ which induces a map $\tau : A \rightarrow Q(C(Y) \otimes K(\mathcal{H}))$ by omitting the projection $p$. Therefore we get a unital homogeneous extension of $A$ such that $\beta([\tau]_u) = [\phi]_u$. Hence $\beta$ is surjective. \hfill $\square$

Corollary 2.8. For $y_0 \in Y$, the map

$$\beta|_{Ext_{PPV}(Y, y_0, A)} : Ext_{PPV}(Y, y_0, A) \rightarrow Ext_{PPV}(Y, y_0, \Sigma^2 A)$$

is an isomorphism.

Proof: It is easy to check that if $ev_{y_0} \circ \tau$ is split then so is $ev_{y_0} \circ \Sigma^2 \tau$ and vice versa. Now the claim will follow by Lemma 2.7. \hfill $\square$

3 Elements of $Ext_{PPV}(\mathbb{T}, C(S_0^{2\ell+1}))$

In the present section, we will write down all elements of the groups $Ext(C(S_0^{2\ell+1}))$ and $Ext_{PPV}(\mathbb{T}, C(S_0^{2\ell+1}))$ in terms of their Busby invariants. Define the $*$-homomorphisms
\( \varphi_m \) as follows:

\[
\varphi_m : C(S_{2^\ell+1}) \to Q\left( K(\bigotimes_{\ell+1 \text{ copies}} L_2(\mathbb{N})) \right)
\]

\[
S^* \otimes 1 \otimes \cdots \otimes 1 \mapsto S^* \otimes 1 \otimes \cdots \otimes 1
\]

\[
p \otimes S^* \otimes 1 \otimes \cdots \otimes 1 \mapsto p \otimes S^* \otimes 1 \otimes \cdots \otimes 1
\]

\[
\vdots
\]

\[
p \otimes p \otimes \cdots \otimes p \otimes S^* \otimes 1 \mapsto p \otimes p \otimes \cdots \otimes S^* \otimes 1
\]

\[
p \otimes p \otimes \cdots \otimes t \mapsto p \otimes p \otimes \cdots \otimes p \otimes (S^*)^m
\]

It is easy to verify that \( \varphi_m \)'s are essential unital extensions of \( C(S_{2^\ell+1}) \) by compact operators. Hence \([\varphi_m]_s \in \text{Ext}(C(S_{2^\ell+1}^2))\). We shall show that each element in the group \( \text{Ext}(C(S_{2^\ell+1}^2)) \) is of the form \([\varphi_m]_s \) for some \( m \in \mathbb{Z} \). Let \( H_0 \) be the Hilbert space \( L_2(\mathbb{N}) \otimes \cdots \otimes L_2(\mathbb{N}) \otimes L_2(\mathbb{Z}) \). For \( m \in \mathbb{Z} \), let \( \vartheta_m \) be the representation of \( C(S_{2^\ell+1}^2) \) given by

\[
\vartheta_m : C(S_{2^\ell+1}^2) \to \mathcal{L}(H_0)
\]

\[
S^* \otimes 1 \otimes \cdots \otimes 1 \mapsto S^* \otimes 1 \otimes \cdots \otimes 1
\]

\[
p \otimes S^* \otimes 1 \otimes \cdots \otimes 1 \mapsto p \otimes S^* \otimes 1 \otimes \cdots \otimes 1
\]

\[
\vdots
\]

\[
p \otimes p \otimes \cdots \otimes p \otimes S^* \otimes 1 \mapsto p \otimes p \otimes \cdots \otimes S^* \otimes 1
\]

\[
p \otimes p \otimes \cdots \otimes t \mapsto p \otimes p \otimes \cdots \otimes p \otimes (S^*)^m
\]

Let \( P \) be the self adjoint projection in \( \mathcal{L}(H_0) \) on the subspace spanned by the basis elements \( \{ e_{n_1} \otimes \cdots \otimes e_{n_{\ell+1}} : n_i \in \mathbb{N} \text{ for all } i \in \{1, 2, \cdots, \ell + 1\} \} \). One can check that \( \mathcal{F}_m := \left( C(S_{0}^{2^{\ell+1}}), \mathcal{H}_0, 2P - 1 \right) \) with the underlying representation \( \vartheta_m \) is a Fredholm module. By Proposition 17.6.5 in ([1], page 157), the group \( \text{Ext}(C(S_{0}^{2^{\ell+1}})) \) is isomorphic to the group \( K^1(C(S_{0}^{2^{\ell+1}})) \). Under this identification, one can easily show that the equivalence class of the Fredholm module \( \mathcal{F}_m \) corresponds to the equivalence class \([\varphi_m]_s \).

**Proposition 3.1.** For \( \ell \in \mathbb{N} \), one has

\[
\text{Ext}(C(S_{0}^{2^{\ell+1}})) = \{ [\varphi_m]_s : m \in \mathbb{Z} \}.
\]

**Proof:** To prove the claim, we will use the index pairing between the groups \( K_1(C(S_{0}^{2^{\ell+1}})) \) and \( K^1(C(S_{0}^{2^{\ell+1}})) \) given by Kasparov product (see [1]). The group \( K_1(C(S_{0}^{2^{\ell+1}})) \) is
generated by the unitary $u := p \otimes \cdots \otimes p \otimes t + 1 - p \otimes \cdots \otimes p \otimes 1$. For $m \in \mathbb{Z}$, let $R_m : \mathcal{PH}_0 \to \mathcal{PH}_0$ be the operator $P \partial_m(u)P$. Hence we get

$$\langle u, \mathcal{F}_m \rangle = \text{Index}(R_m) = m.$$  

This completes the proof. □

To describe all elements of $\text{Ext}_{\mathcal{PPV}}(\mathbb{T}, C(S_0^{2\ell+1}))$, we define the $\ast$-homomorphisms $\phi_m$ as follows:

$$\phi_m : C(S_0^{2\ell+1}) \to Q\left(\mathcal{K}(L_2(\mathbb{N}) \otimes \cdots \otimes L_2(\mathbb{N})) \otimes C(\mathbb{T})\right)$$

$S^* \otimes 1 \otimes \cdots \otimes 1 \mapsto S^* \otimes 1 \otimes \cdots \otimes 1$

$p \otimes S^* \otimes 1 \otimes \cdots \otimes 1 \mapsto p \otimes S^* \otimes 1 \otimes \cdots \otimes 1$

$\cdots$

$p \otimes p \otimes \cdots \otimes p \otimes S^* \otimes 1 \mapsto p \otimes p \otimes \cdots \otimes S^* \otimes 1 \otimes 1$

$p \otimes S^* \otimes 1 \otimes \cdots \otimes 1 \otimes 1$

$p \otimes p \otimes p \otimes \cdots \otimes t \mapsto p \otimes p \otimes \cdots \otimes p \otimes (S^*)^m \otimes 1$

It is easy to verify that $\phi_m$’s are essential unital extensions. Since last component is 1, these extensions are also homogeneous. Let $A_m$ be the $C^*$-subalgebra of $C(S_0^{2\ell+3})$ generated by the operators

$$S^* \otimes 1 \otimes \cdots \otimes 1 \otimes 1,$$

$$p \otimes S^* \otimes 1 \otimes \cdots \otimes 1 \otimes 1$$

$\cdots$

$$p \otimes p \otimes \cdots \otimes S^* \otimes 1 \otimes 1,$$

$$p \otimes p \otimes \cdots \otimes p \otimes (S^*)^m \otimes 1$$

and $\mathcal{K}(L_2(\mathbb{N}) \otimes \cdots \otimes L_2(\mathbb{N})) \otimes C(\mathbb{T})$. Then for each $m \in \mathbb{Z}$, we have the following exact sequence

$$0 \to \mathcal{K}(L_2(\mathbb{N}) \otimes \cdots \otimes L_2(\mathbb{N})) \otimes C(\mathbb{T}) \to A_m \to C(S_0^{2\ell+1}) \to 0$$

with the Busby invariant $\phi_m$. By using the six term sequence, one can show that

$$K_0(A_m) = \mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}, \quad K_1(A_m) = \mathbb{Z}. \quad (3.1)$$
Lemma 3.2. For $\ell \in \mathbb{N}$ and $t_0 \in \mathbb{T}$, one has

$$Ext_{PPV}(\mathbb{T}, t_0, C(S_0^{2\ell+1})) = \{0\}, \quad Ext_{PPV}(\mathbb{T}, C(S_0^{2\ell+1})) = \mathbb{Z}.$$ 

Proof: It follows from Theorem 1.5 in [13] that

$$Ext_{PPV}(\mathbb{T}, t_0, C(\mathbb{T})) = Ext_{PPV}(\mathbb{R}^+, t_0, C_0(\mathbb{R})^+) = Ext(C_0(\mathbb{R}), C_0(\mathbb{R})) = \{0\}.$$ 

The $C^*$-algebra $C(S_0^{2\ell+1})$ can be obtained by applying quantum double suspension on $C(\mathbb{T})$ repeatedly (see [6]). Therefore from Corollary 2.8 we have

$$Ext_{PPV}(\mathbb{T}, t_0, C(S_0^{2\ell+1})) = Ext_{PPV}(\mathbb{T}, t_0, C(\mathbb{T})) = \{0\}.$$ 

Further from Theorem 1.4 in [13], we get

$$Ext_{PPV}(\mathbb{T}, C(\mathbb{T})) = Ext_{PPV}(\mathbb{T}, C_0(\mathbb{R})^+) = Ext(C_0(\mathbb{R}), C(\mathbb{T})) = \mathbb{Z}.$$ 

Hence by applying Lemma 2.7 we get the claim. □

The following lemma says that each element of the group $Ext_{PPV}(\mathbb{T}, C(S_0^{2\ell+1}))$ is of the form $[\phi_m]_u$ for some $m \in \mathbb{Z}$.

Lemma 3.3. For $\ell \in \mathbb{N}$, one has

$$Ext_{PPV}(\mathbb{T}, C(S_0^{2\ell+1})) = \{[\phi_m]_u : m \in \mathbb{Z}\}.$$ 

Proof: Fix $t_0 \in \mathbb{T}$. Define a homomorphism $\Psi$ as follows:

$$\Psi : Ext_{PPV}(\mathbb{T}, C(S_0^{2\ell+1})) \longrightarrow Ext(C(S_0^{2\ell+1}))$$

$$[\tau]_u \longmapsto [ev_{t_0} \circ \tau]_s$$

Clearly $\ker \Psi = Ext_{PPV}(\mathbb{T}, t_0, C(S_0^{2\ell+1})) = \{0\}$. Therefore $\Psi$ is an injective group homomorphism. Since for all $m \in \mathbb{Z}$, $ev_{t_0} \circ \phi_m = \varphi_m$, it follows that the homomorphism $\Psi$ is surjective. This proves the claim. □

4 Quantum quaternion sphere

In this section, we first recall the definition and representation theory of the $C^*$-algebra $C(H_q^{2n})$ of continuous functions on the quantum quaternion sphere. Then we prove our main result that the $C^*$-algebra $C(H_q^{2n})$ is isomorphic to the $C^*$-algebra $C(S_q^{4n-1})$.  

Definition 4.1. The $C^*$-algebra $C(H^2_n)$ of continuous functions on the quantum quaternion sphere is defined as the universal $C^*$-algebra generated by elements $z_1, z_2, \ldots, z_{2n}$ satisfying the following relations:

$$z_iz_j = qz_jz_i \quad \text{for } i > j, i + j \neq 2n + 1 \quad (4.1)$$

$$z_iz'_i = q^2z'_iz_i - (1 - q^2) \sum_{k > i} q^{i-k}z_kz'_k \quad \text{for } i > n \quad (4.2)$$

$$z^*_iz'_i = q^2z'_iz^*_i \quad (4.3)$$

$$z^*_iz_j = qz_jz^*_i \quad \text{for } i + j > 2n + 1, i \neq j \quad (4.4)$$

$$z^*_iz_j = qz_jz^*_i + (1 - q^2)\epsilon_i\epsilon_jq^{\rho_i+\rho_j}z'_iz^*_j \quad \text{for } i + j < 2n + 1, i \neq j \quad (4.5)$$

$$z^*_iz_i = z_iz^*_i + (1 - q^2)\sum_{k > i} z_kz^*_k \quad \text{for } i > n \quad (4.6)$$

$$z^*_iz_i = z_iz^*_i + (1 - q^2)q^{2\rho_i}z'_iz^*_i + (1 - q^2)\sum_{k > i} z_kz^*_k \quad \text{for } i \leq n \quad (4.7)$$

$$\sum_{i=1}^{2n} z_iz^*_i = 1 \quad (4.8)$$

In [14], we showed that the $C^*$-algebra $C(H^2_n)$ is isomorphic to the quotient algebra $C(SP_q(2n)/SP_q(2n-2))$ that can also be described as the $C^*$-subalgebra of $C(SP_q(2n))$ generated by $\{u^1_m, u^{2n}_m : m \in \{1, 2, \cdots 2n\}\}$ i.e. elements of first and last row of fundamental matrix of $C(SP_q(2n))$. Here we briefly describe all irreducible representations of $C(H^2_n)$. For a detailed treatment on this, we refer the reader to [14]. Let $N$ be the number operator given by $N : e_n \mapsto ne_n$ and $S$ be the shift operator given by $S : e_n \mapsto e_{n-1}$ on $L_2(\mathbb{N})$. For $i = 1, 2, \cdots, n - 1$, let $\pi_{s_i}$ denote the following representation of $C(SP_q(2n))$,

$$\pi_{s_i}(u^{k}_i) = \begin{cases} 
\sqrt{1 - q^{2N+2}}S & \text{if } (k, l) = (i, i) \text{ or } (2n - i, 2n - i), \\
S\sqrt{1 - q^{2N+2}} & \text{if } (k, l) = (i + 1, i + 1) \text{ or } (2n - i + 1, 2n - i + 1), \\
q^{N+1} & \text{if } (k, l) = (i, i + 1), \\
q^N & \text{if } (k, l) = (i + 1, i), \\
q^{N+1} & \text{if } (k, l) = (2n - i, 2n - i + 1), \\
-q^N & \text{if } (k, l) = (2n - i + 1, 2n - i), \\
\delta_{kl} & \text{otherwise}.
\end{cases}$$
For $i = n,$
\[
\pi_{si}(u_i^k) = \begin{cases} 
    \sqrt{1 - q^{2N+4}} & \text{if } (k, l) = (n, n), \\
    S^* \sqrt{1 - q^{2N+4}} & \text{if } (k, l) = (n + 1, n + 1), \\
    -q^{2N+2} & \text{if } (k, l) = (n, n + 1), \\
    q^{2N} & \text{if } (k, l) = (n + 1, n), \\
    \delta_{kl} & \text{otherwise}.
\end{cases}
\]

Each $\pi_{si}$ is an irreducible representation and is called an elementary representation of $C(SP_q(2n))$. For any two representations $\varphi$ and $\psi$ of $C(SP_q(2n))$ define, $\varphi \ast \psi := (\varphi \otimes \psi) \circ \Delta$ where $\Delta$ is the co-multiplication map of $C(SP_q(2n))$. Let $W$ be the Weyl group of $sp_{2n}$ and $\vartheta \in W$ such that $s_{i_1}s_{i_2}...s_{i_k}$ is a reduced expression for $\vartheta$. Then $\pi_{\vartheta} = \pi_{s_{i_1}} \ast \pi_{s_{i_2}} \ast \cdots \ast \pi_{s_{i_k}}$ is an irreducible representation which is independent of the reduced expression. Now for $t = (t_1, t_2, \cdots , t_n) \in \mathbb{T}^n$, define the map $\tau_t : C(SP_q(2n)) \rightarrow \mathbb{C}$ by
\[
\tau_t(u_j^i) = \begin{cases} 
    t_i \delta_{ij} & \text{if } i \leq n, \\
    t_{2n+1-i} \delta_{ij} & \text{if } i > n,
\end{cases}
\]

Then $\tau_t$ is a *-algebra homomorphism. For $t \in \mathbb{T}^n, \vartheta \in W$, let $\pi_{t, \vartheta} = \tau_t \ast \pi_{\vartheta}$. Define the representation $\eta_{t, \vartheta}$ of $C(H^2_q)$ as $\pi_{t, \vartheta}$ restricted to $C(H^2_q)$. Denote by $\omega_k$ the following word of Weyl group of $sp_{2n}$,
\[
\omega_k = \begin{cases} 
    I & \text{if } k = 1, \\
    s_1s_2 \cdots s_{k-1} & \text{if } 2 \leq k \leq n, \\
    s_1s_2 \cdots s_{n-1}s_n s_{n-1} \cdots s_{2n-k+1} & \text{if } n < k \leq 2n.
\end{cases}
\]

For $k = 1$, define $\eta_{t, I} : C(H^2_q) \rightarrow \mathbb{C}$ such that $\eta_{t, I}(z_j) = t \delta_{1j}$. The set $\{\eta_{t, I} : t \in T\}$ gives all one dimensional irreducible representations of $C(H^2_q)$.

**Theorem 4.2.** \(\{\text{[14]}\}\) The set $\{\eta_{t, \omega_k} : 1 \leq k \leq 2n, t \in \mathbb{T}\}$ gives a complete list of irreducible representations of $C(H^2_q)$.

Define $\eta_{\omega_k} : C(H^2_q) \rightarrow C(\mathbb{T}) \otimes \mathcal{F} \otimes \cdots \otimes \mathcal{F}^{k-1}$ such that $\eta_{\omega_k}(a)(t) = \eta_{t, \omega_k}(a)$ for all $a \in C(H^2_q)$. Let $C^2_1 = C(\mathbb{T})$ and for $2 \leq k \leq 2n$, $C^2_k = \eta_{\omega_k}(C(H^2_q))$.

**Corollary 4.3.** The set $\{\eta_{t, \omega_l} : 1 \leq l \leq k, t \in \mathbb{T}\}$ gives a complete list of irreducible representations of $C^2_k$.

**Corollary 4.4.** $\eta_{\omega_k}$ is a faithful representation of $C^2_k$. 

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By Corollary 4.3, one can find all primitive ideals i.e. kernels of irreducible representations of $C_{k}^{2n}$. Define $y_{l}^{k} := \tau_{l}(z_{i})$ for $1 \leq l \leq k$. Let $I_{l}^{k}$ be the ideal of $C_{k}^{2n}$ generated by $\{y_{l}^{k}, y_{l+1}^{k}, \ldots, y_{k}^{k}\}$. For $t \in \mathbb{T}$, let $C_{t}(\mathbb{T})$ be the set of all continuous functions on $\mathbb{T}$ vanishing at the point $t$. Then

$$\{C_{t}(\mathbb{T}) \otimes \mathcal{K}(L_{2}(\mathbb{N}))^{\otimes (k-1)}\}_{t \in \mathbb{T}} \subset I_{k}^{k} \subset I_{k-1}^{k} \subset \cdots \subset I_{1}^{k} = C_{k}^{2n} \quad (4.9)$$

is a complete list of primitive ideals of $C_{k}^{2n}$. In Lemma 5.1 in [14], we established the following exact sequence

$$0 \rightarrow C(\mathbb{T}) \otimes \mathcal{K}(L_{2}(\mathbb{N}))^{\otimes (k)} \rightarrow C_{k+1}^{2n} \xrightarrow{\sigma_{k+1}} C_{k}^{2n} \rightarrow 0$$

where $\sigma_{k+1}$ is the restriction of $(1^{\otimes (k)} \otimes \sigma)$ to $C_{k+1}^{2n}$, the map $\sigma : \mathcal{T} \rightarrow \mathbb{C}$ is the homomorphism such that $\sigma(S) = 1$ and $\mathcal{T}$ is the Toeplitz algebra. The following lemma says that this exact sequence is a unital homogeneous extension of $C_{k}^{2n}$ by $C(\mathbb{T}) \otimes \mathcal{K}$.

**Lemma 4.5.** For $1 \leq k \leq 2n$, the following exact sequence

$$0 \rightarrow C(\mathbb{T}) \otimes \mathcal{K}(L_{2}(\mathbb{N}))^{\otimes (k)} \rightarrow C_{k+1}^{2n} \xrightarrow{\sigma_{k+1}} C_{k}^{2n} \rightarrow 0$$

is a unital homogeneous extension of $C_{k}^{2n}$ by $C(\mathbb{T}) \otimes \mathcal{K}$.

**Proof:** Since $C_{k+1}^{2n}$ is unital, the given extension is unital. Let $\tau : C_{k}^{2n} \rightarrow Q(\mathbb{T})$ be the Busby invariant corresponding to this extension. For $t_{0} \in \mathbb{T}$, let $\tau_{t_{0}} : C_{k}^{2n} \rightarrow Q$ be the map $ev_{t_{0}} \circ \tau$ where $ev_{t_{0}} : Q(\mathbb{T}) \rightarrow Q$ is the evaluation map at $t_{0}$. Assume that $J_{t_{0}} = \ker(\tau_{t_{0}})$. To show that the given short exact sequence is a homogeneous extension, we need to prove that $J_{t_{0}} = \{0\}$ for all $t_{0} \in \mathbb{T}$.

**Case 1:** $n < k < 2n$

We have

$$\tau_{t_{0}}(y_{l}^{k}) = \tau_{t_{0}}(t \otimes q^{N} \otimes \cdots \otimes q^{N} \otimes q^{2N} \otimes q^{N} \otimes \cdots \otimes q^{N})$$

$$= t_{0}[q^{N} \otimes \cdots \otimes q^{N} \otimes q^{2N} \otimes q^{N} \otimes \cdots \otimes q^{N} \otimes \sqrt{1 - q^{2N} S^{*}}] \quad (4.10)$$

$$\neq 0.$$ 

This shows $y_{l}^{k} \notin J_{t_{0}}$. Since $J_{t_{0}}$ is intersection of all primitive ideals that contains $J_{t_{0}}$, we conclude that $J_{t_{0}}$ is equal to $C_{F}(\mathbb{T}) \otimes \mathcal{K}$ for some closed subset $F$ of $\mathbb{T}$ where $C_{F}(\mathbb{T})$ is
set of all continuous functions on $\mathbb{T}$ vanishing on $F$. From equation 4.10, we get
\[
\tau_t(\{y_k^k\}^*) = [q^{2N} \otimes \cdots \otimes q^{2N} \otimes q^{4N} \otimes q^{2N} \otimes \cdots \otimes q^{2N} \otimes (1 - q^{2N})]
\]
\[
= [q^{2N} \otimes \cdots \otimes q^{2N} \otimes q^{4N} \otimes q^{2N} \otimes \cdots \otimes q^{2N}]^\otimes 1.
\]
Therefore
\[
\tau_t(1 \otimes p \otimes \cdots \otimes p)^\otimes (k-1) = [p \otimes \cdots \otimes p]^\otimes (k-1).
\]
Hence
\[
\tau_t(t \otimes p \otimes \cdots \otimes p) = t_0[p \otimes \cdots \otimes p \otimes \sqrt{1 - q^{2N}S^*}]^\otimes (k-1).
\]
Consider the function $\chi : C(\mathbb{T}) \to Q$ such that $\chi(t) = [S^*]$. Since $[S^*]$ is unitary in $Q$ with spectrum equal to $\mathbb{T}$, it follows that the map $\chi$ is injective. This shows that for any nonzero function $f$ on $\mathbb{T}$, $\tau_t(f(t) \otimes p \otimes \cdots \otimes p)^\otimes (k-1) \neq 0$ which further implies that $F = \mathbb{T}$ and $J_{t_0} = \{0\}$.

Case 2: $1 \leq k \leq n$

For $k = n$,
\[
\tau_t(\{y^n\}) = t_0[q^N \otimes \cdots \otimes q^N \otimes \sqrt{1 - q^{4N}S^*}]^\otimes (n-1).
\]
For $1 \leq k < n$,
\[
\tau_t(\{y^k\}) = t_0[q^N \otimes \cdots \otimes q^N \otimes \sqrt{1 - q^{2N}S^*}]^\otimes (k-1).
\]
Similar calculations as done in the case 1 shows that $J_{t_0} = \{0\}$. This establishes the claim.

We now state the main result of this paper.

Theorem 4.6. For all $n \in \mathbb{N}, n \geq 2$ and $1 \leq k \leq 2n$, the $C^*$-algebra $C_k^{2n}$ is isomorphic to the $C^*$-algebra $C(S_0^{2k-1})$ of continuous functions on odd dimensional quantum sphere. In particular, $C(H_q^{2n})$ is isomorphic to $C(S_0^{2n-1})$ or equivalently to $C(S_q^{4n-1})$. 

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Proof: Fix $n$. To prove the theorem, we use induction on $k$. For $k = 1$, $C_{2n}^{2n} = \mathbb{C}(T)$. So the claim is true for $k = 1$. Assume that the claim is true for $k$ i.e. $C_{2k}^{2n}$ is isomorphic to $C(S_{0}^{2k-1})$. From Lemma 4.3, it follows that following short exact sequence

$$0 \rightarrow C(T) \otimes \mathcal{K} \rightarrow C_{k+1}^{2n} \rightarrow C_{k}^{2n} \rightarrow 0$$

(4.11)

is a unital homogeneous extension. Therefore it can be viewed as an element of the group $Ext_{PV}(\mathbb{T}, C(S_{0}^{2k-1}))$. This implies that it is unitarily equivalent to $\phi_m$ or equivalently to the following exact sequence

$$0 \rightarrow C(T) \otimes \mathcal{K} \rightarrow A_m \rightarrow C(S_{0}^{2k-1}) \rightarrow 0$$

for some $m \in \mathbb{Z}$. From Theorem 5.3 in [14] and equation (3.1), we have

$$K_0(C_{k+1}^{2n}) = \mathbb{Z}, \quad K_0(A_m) = \mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}.$$ 

Since unitary equivalence gives an isomorphism of the middle $C^*$ algebras and hence an isomorphism of the $K$-groups of middle $C^*$-algebras, it follows that the exact sequence (4.11) is unitarily equivalent to $\phi_1$ or $\phi_{-1}$. This implies that $C_{k+1}^{2n}$ is isomorphic to $A_1$ or $A_{-1}$. Since $A_1 = A_{-1} = C(S_{0}^{2k+1})$, it follows that $C_{k+1}^{2n}$ is isomorphic to $C(S_{0}^{2k+1})$. Hence by induction, it follows that $C(H_{2n}^{2n})$ is isomorphic to $C(S_{0}^{4n-1})$. From Lemma 3.2 in [10], it follows that the $C^*$-algebras $C(S_{q}^{4n-1})$ are isomorphic to $C(S_{0}^{4n-1})$ for $q \in (0, 1)$. This proves that $C(H_{2n}^{2n})$ is isomorphic to $C(S_{q}^{4n-1})$. 

Remark 4.7. In case of $q = 0$, we need to be slightly careful to get the defining relations of $C(H_{0}^{2n})$. In the relation (4.2), we first start with $i = 2n$ which gives $z_{2n}z_{1} = 0$. Then we take $i = 2n - 1$ and so on and get the relation $z_{i}z_{i'} = 0$ for $i < n$. Further in the relation (4.5), it is easy to check that for $i + j < 2n + 1$, $\rho_i + \rho_j > 0$. Now by putting $q = 0$ in the relations (4.3), (4.4) and (4.4), we get $z^*_iz_j = 0$ for $i \neq j$. Other relations are obtained by putting $q = 0$ in the remaining relations. By looking at the relations, one can see that the defining relations of $C(H_{0}^{2n})$ are exactly same as those of $C(S_{0}^{4n-1})$. These facts together with Theorem 4.6 prove that for different values of $q \in [0, 1)$, the $C^*$-algebras $C(H_{q}^{2n})$ are isomorphic.

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