On nonexistence of solutions to some time-space fractional evolution equations with transformed space argument

Mokhtar Kirane, Ahmad Z. Fino, Bashir Ahmad

Abstract

Some results on nonexistence of nontrivial solutions to some time and space fractional differential evolution equations with transformed space argument are obtained via the nonlinear capacity method. The analysis is then used for a $2 \times 2$ system of equations with transformed space arguments.

MSC[2020]: 35A01, 26A33

Keywords: Nonlinear evolution equations, nonexistence of solutions, space transformed argument, Caputo fractional derivative, fractional Laplacian

1 Introduction

There is a sizeable number of works on nonexistence of solutions or existence of blowing-up solutions to different type of classical steady or evolution equations and recently for fractional differential equations [6, 25, 23, 20, 19]. In [3], Ahmad, Alsaedi and Kirane collected all results concerning blowing-up or growing-up solutions for delay differential equations. Differential equations with transformed arguments have been treated with respect to many points (existence, large time behavior, etc) in many works [5, 1, 4, 8, 9, 11, 14, 15, 18, 31, 32], but principally in the works of Przeworska-Rolewicz [24] and Skubachevskii [27]. Very recently, Salieva in [26] obtained results on nonexistence of solutions for nonlinear differential inequalities with transformed argument. Salieva obtained sufficient conditions for the nonexistence of solutions to some classes of differential inequalities and systems of inequalities. Here, we extend the study of Salieva [26] to time-space fractional differential equations.

We consider first the time-space fractional evolution equation with space transformed argument

$$\begin{cases}
D_{0,t}^\alpha u(t, x) + (-\Delta)^{\delta/2} u(t, x) = |u(t, g(x))|^p, & t > 0, \ x \in \mathbb{R}^d, \\
u(x, 0) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}$$

where $0 < \alpha \leq 1$, $0 < \delta \leq 2$, $p > 1$, and $d \geq 1$. $D_{0,t}^\alpha$ stands for the Caputo fractional derivative for $0 < \alpha < 1$ and for the standard partial derivative in time $\partial _t$ when $\alpha = 1$, $(-\Delta)^{\delta/2}$, $0 < \delta < 2$, is the fractional power of the laplacian that will be defined here below, and $g \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ is an invertible mapping satisfying:

$$(A1) \text{ there exists a constant } c_0 > 0 \text{ such that } |J_g^{-1}(x)| \geq c_0 > 0 \text{ for all } x \in \mathbb{R}^d (J \text{ is the jacobian matrix});$$

$$(A2) |g(x)| \geq |x| \text{ for all } x \in \mathbb{R}^d.$$
As examples of such \(g\) we mention (cf [26]):

1. The dilation mapping \(g(x) = kx\), for any \(k \in \mathbb{R}\) with \(|k| > 1\), that satisfies (A1) with \(c_0 = |k|^{-d}\) and (A2).

2. The rotation transform \(g(x) = Ax\), where \(A\) is a \(d \times d\) unitary matrix (so \(|g(x)| = |x|\) for all \(x \in \mathbb{R}^d\)) that satisfies (A1) with \(c_0 = 1\) and (A2).

In some situations (A2) can be replaced by a weaker one:

(A2′) there exists positive constants \(c_0\) and \(\rho\) such that \(|g(x)| \geq c_0|x|\) for all \(x \in \mathbb{R}^d \setminus B_\rho(0)\) (\(c_0\) may be taken \(c_0 \leq 1\)).

3. The shift transform \(g(x) = x - x_0\) for a fixed \(x_0 \in \mathbb{R}^d\) with \(c = 1\), \(c_0 = 1/2\) and \(\rho = 2|x_0|\).

Then, we consider the equation

\[
\begin{align*}
\begin{cases}
D_0^\delta u(t, x) + (-\Delta)^{\delta/2} u(t, x) + D_0^\sigma u(t, x) = |u(t, g(x))|^p, & t > 0, \ x \in \mathbb{R}^d, \\
\end{cases}
\end{align*}
\]

(2)

where \(1 < \beta \leq 2\), \(0 < \delta \leq 2\), \(0 < \alpha \leq 1\), \(p > 1\), and \(d \geq 1\). Finally, we consider the following \(2 \times 2\) system

\[
\begin{align*}
\begin{cases}
D_0^\gamma u(t, x) + (-\Delta)^{\mu/2} u(t, x) = |v(t, g(x))|^p, & t > 0, \ x \in \mathbb{R}^d, \\
D_0^\theta v(t, x) + (-\Delta)^{\vee/2} v(t, x) = |u(t, f(x))|^q, & t > 0, \ x \in \mathbb{R}^d,
\end{cases}
\end{align*}
\]

(3)

where \(0 < \gamma, \theta \leq 1\), \(0 < \sigma, \mu \leq 2\), \(p, q > 1\), and \(d \geq 1\).

Notations

- Constants \(C\) and \(C_i\) with \(i \in \mathbb{N}\) stand for suitable positive constants.
- For given nonnegative \(f\) and \(g\), we write \(f \lesssim g\) if \(f \leq Cg\), for constant \(C > 0\).

## 2 Main results

For a weight function \(w\) and \(1 \leq r < \infty\), let \(L^r_w\) denote the space of all real-valued measurable functions \(f\) such that \(f|w|^{1/r} \in L^r\), the usual Lebesgue space.

\[
X_{\delta, T} = \{\varphi \in C([0, \infty), H^\delta(\mathbb{R}^d)) \cap C^1([0, \infty), L^2(\mathbb{R}^d)), \text{such that } \text{supp} \varphi \subset Q_T\},
\]

\[
Y_{\delta, T} = \{\varphi \in C([0, \infty), H^\delta(\mathbb{R}^d)) \cap C^2([0, \infty), L^2(\mathbb{R}^d)), \text{such that } \text{supp} \varphi \subset Q_T\},
\]

where \(Q_T := [0, T] \times \mathbb{R}^d\), and the fractional Sobolev space \(H^\delta(\mathbb{R}^d), \ \delta \in (0, 2),\) is defined by

\[
H^\delta(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d); (-\Delta)^{\delta/2} u \in L^2(\mathbb{R}^d)\},
\]

endowed with the norm

\[
\|u\|_{H^\delta(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{\delta/2} u\|_{L^2(\mathbb{R}^d)}.
\]

\]
Definition 1. Let \( u_0 \in L^2(\mathbb{R}^d) \) and \( T > 0 \). A function
\[
    u \in L^1((0, T), L^2(\mathbb{R}^d)) \cap L^p((0, T), L^{2p}_\text{loc}(\mathbb{R}^d)),
\]
is said to be a weak solution of \( \text{(1)} \) on \([0, T) \times \mathbb{R}^d \) if
\[
    \int_{Q_T} |u(t, g(x))|^p \varphi(t, x) \, dt \, dx + \int_{Q_T} u_0(x) D^s_{u;T} \varphi(t, x) \, dt \, dx = \int_{Q_T} u(t, x) \varphi(t, x) \, dt \, dx + \int_{Q_T} u(t, x)(-\Delta)^{\delta/2} \varphi(t, x) \, dt \, dx,
\]
holds for all \( \varphi \in X_{\delta, T} \). We denote the lifespan for the weak solution by
\[
    T_w(u_0) := \sup\{T \in (0, \infty] : \text{there exists a unique weak solution } u \text{ of } \text{(1)}\}.
\]
Moreover, if \( T > 0 \) can be arbitrary chosen, i.e. \( T_w(u_0) = \infty \), then \( u \) is called a global weak solution of \( \text{(1)} \).

Theorem 1. Let \( u_0 \in L^1(\mathbb{R}^d) \cap L^{2}(\mathbb{R}^d) \), \( 0 < \alpha \leq 1 \), \( 0 < \delta \leq 2 \), \( p > 1 \), and \( d \geq 1 \). Assume that \( g \) satisfies conditions (A1)-(A2). If
\[
    \begin{align*}
        \left\{ \begin{array}{ll}
            p < p_* & \text{when } \alpha \in (0, 1), \\
            p \leq p_* & \text{when } \alpha = 1,
        \end{array} \right.
    \end{align*}
\]
with
\[
    p_* = 1 + \frac{\alpha \delta}{\alpha d + \delta(1 - \alpha)},
\]
then problem \( \text{(1)} \) admits no global nontrivial weak solutions.

Next, define the weak solution of the corresponding \( 2 \times 2 \) system.

Definition 2. Let \( u_0, v_0 \in L^2(\mathbb{R}^d) \), and \( T > 0 \). A couple of function \((u, v)\) such that
\[
    u, v \in L^1((0, T), L^2(\mathbb{R}^d)), \quad u \in L^q((0, T), L^{2q}_\text{loc}(\mathbb{R}^d)), \quad v \in L^p((0, T), L^{2p}_\text{loc}(\mathbb{R}^d)),
\]
is said to be a weak solution of \( \text{(3)} \) on \([0, T) \times \mathbb{R}^d \) if
\[
    \int_{Q_T} |u(t, g(x))|^{q} \varphi(t, x) \, dt \, dx + \int_{Q_T} u_0(x) D^{s}_{u;T} \varphi(t, x) \, dt \, dx = \int_{Q_T} u(t, x) \varphi(t, x) \, dt \, dx + \int_{Q_T} u(t, x)(-\Delta)^{\delta/2} \varphi(t, x) \, dt \, dx,
\]
and
\[
    \int_{Q_T} |u(t, f(x))|^{q} \psi(t, x) \, dt \, dx + \int_{Q_T} v_0(x) D^{s}_{v;T} \psi(t, x) \, dt \, dx = \int_{Q_T} u(t, x) \psi(t, x) \, dt \, dx + \int_{Q_T} u(t, x)(-\Delta)^{\delta/2} \psi(t, x) \, dt \, dx,
\]
hold for all \( \varphi \in X_{\mu, T}, \psi \in X_{\sigma, T} \). We denote the lifespan for the weak solution by
\[
    T_w(u_0, v_0) := \sup\{T \in (0, \infty] : \text{there exists a unique weak solution } (u, v) \text{ of } \text{(3)}\}.
\]
Moreover, if \( T > 0 \) can be arbitrary chosen, i.e. \( T_w(u_0, v_0) = \infty \), then \( u \) is called a global weak solution of \( \text{(3)} \).
Theorem 2. Let $u_0, v_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $0 < \gamma, \theta \leq 1$, $0 < \mu, \sigma \leq 2$, $p, q > 1$, and $d \geq 1$; assume that $	heta p + \gamma pq - pq + 1 > 0$, $\gamma q + \theta pq - pq + 1 > 0$, and that $f, g$ satisfy conditions (A1)-(A2). If

$$d < \max \{ \overline{D}; \overline{E} \},$$

or

$$\begin{cases} d = \overline{D}, \text{ when } \theta = 1, \gamma \neq 1, \text{ and } pq > 1/(1 - \gamma), \\ d = \overline{D}, \text{ when } \theta = 1, \gamma \leq 1, \text{ and } \gamma \leq \frac{\nu(q-1)}{\nu(p-1)}, \\ d = \overline{D}, \text{ when } \gamma = 1, \theta \neq 1, \text{ and } pq > 1/(1 - \theta), \\ d = \overline{E}, \text{ when } \gamma = 1, \theta \leq 1, \text{ and } \theta \leq \frac{q(p-1)}{p(q-1)}, \end{cases}$$

where

$$\overline{D} := \min \{ \max \{ D_1, D_2 \} ; \max \{ D_3, D_4 \} \} \quad \text{and} \quad \overline{E} := \min \{ \max \{ E_1, E_2 \} ; \max \{ E_3, E_4 \} \},$$

with

$$\begin{align*}
D_1 &= \frac{\theta \sigma p + \theta \mu pq - \sigma pq + \sigma}{\theta(pq - 1)}, \\
D_2 &= \frac{\mu(\theta p + \gamma pq - pq + 1)}{\gamma(pq - 1)}, \\
D_3 &= \frac{\gamma \sigma p + \gamma \mu pq - \mu pq + \mu}{\gamma(pq - 1)}, \\
D_4 &= \frac{\sigma(\theta p + \gamma pq - pq + 1)}{\theta(pq - 1)}, \\
E_1 &= \frac{\theta \mu q + \theta \sigma pq - \sigma pq + \sigma}{\theta(pq - 1)}, \\
E_2 &= \frac{\mu(\gamma q + \theta pq - pq + 1)}{\gamma(pq - 1)}, \\
E_3 &= \frac{\gamma \mu q + \gamma \sigma pq - \mu pq + \mu}{\gamma(pq - 1)}, \\
E_4 &= \frac{\sigma(\gamma q + \theta pq - pq + 1)}{\theta(pq - 1)}.
\end{align*}$$

then system (3) admits no global nontrivial weak solutions.

Remark 1.

1. If $\gamma = \theta = 1$ and $\mu = \sigma = 2$, then $\overline{D} = 2(p + 1)/(pq - 1)$ and $\overline{E} = 2(q + 1)/(pq - 1)$, which are the same exponent found in [13].

2. If $\gamma = \theta = 1$, then

$$\overline{D} := \min \{ \max \{ D_1, D_2 \} ; D_3 \} \quad \text{and} \quad \overline{E} := \min \{ E_1 ; \max \{ E_3, E_4 \} \}.$$ 

If $\mu \leq \sigma$, then

$$\overline{D} := \max \{ D_1, D_2 \} \geq D_1 = \frac{\sigma p + \mu pq - \sigma pq + \sigma}{pq - 1}, \quad \text{and} \quad \overline{E} := E_1 = \frac{\mu q + \sigma}{pq - 1}.$$ 

If $\mu \geq \sigma$, then

$$\overline{D} := D_3 = \frac{\sigma p + \mu}{pq - 1}, \quad \text{and} \quad \overline{E} := \max \{ E_3, E_4 \} \geq E_3 = \frac{\mu q + \sigma pq - \mu pq + \mu}{pq - 1}.$$ 

As a conclusion, our result is an improvement of [13] and without any additional conditions.

Definition 3. Let $u_0, u_1 \in L^2(\mathbb{R}^d)$ and $T > 0$. A function

$$u \in L^1((0, T), L^2(\mathbb{R}^d)) \cap L^p((0, T), L^2_{\mu, \gamma}((\mathbb{R}^d)),$$
is said to be a weak solution of (2) on \([0, T) \times \mathbb{R}^d\) if

\[
\int_{Q_T} |u(t, g(x))|^p \varphi(t, x) \, dt \, dx + \int_{Q_T} u_0(x) \left[ D_{t^2}^\beta \varphi(t, x) + D_{t^2}^\beta \varphi(t, x) \right] \, dt \, dx + \int_{Q_T} u_1(x) D_{t^2}^\beta \varphi(t, x) \, dt \, dx
\]

\[
= \int_{Q_T} u(t, x) D_{t^2}^\beta \varphi(t, x) \, dt \, dx + \int_{Q_T} u(t, x) D_{t^2}^\beta \varphi(t, x) \, dt \, dx + \int_{Q_T} u(t, x)(-\Delta)^{\beta/2} \varphi(t, x) \, dt \, dx,
\]

holds for all \(\varphi \in \mathcal{Y}_{\delta, T}\). We denote the lifespan for the weak solution by

\[
T_w(u_0, u_1) := \sup \{T \in [0, \infty); \text{ there exists a unique weak solution } u \text{ of (2)} \}.
\]

Moreover, if \(T > 0\) can be arbitrary chosen, i.e. \(T_w(u_0, u_1) = \infty\), then \(u\) is called a global weak solution of (2).

**Theorem 3.** Let \(u_0, u_1 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), 1 < \beta \leq 2, 0 < \delta \leq 2, 0 < \alpha \leq 1, p > 1\), and \(d \geq 1\). Assume that \(g\) satisfies conditions (A1)-(A2). If

\[
\begin{cases}
  p < p_* & \text{when } \alpha \in (0, 1),
  \\
  p \leq p_* & \text{when } \alpha = 1,
\end{cases}
\]

then problem (2) admits no global nontrivial weak solutions.

### 3 Preliminaries

This section is devoted to collect some preliminaries needed in our proofs.

**Definition 4.** (Absolutely continuous functions)

A function \(f : [a, b] \rightarrow \mathbb{R}\), with \(a, b \in \mathbb{R}\), is absolutely continuous if and only if there exists a Lebesgue summable function \(\varphi \in L^1(a, b)\) such that

\[
f(t) = f(a) + \int_a^t \varphi(s) \, ds.
\]

The space of such functions is denoted by \(AC[a, b]\). Moreover, for all \(m \geq 0\), we define

\[
AC^{m+1}[a, b] := \{ f : [a, b] \rightarrow \mathbb{R} \text{ such that } D^m f \in AC[a, b] \},
\]

where \(D^m = \frac{d^m}{dt^m}\) is the usual \(m\) times derivative.

**Definition 5.** (Riemann-Liouville fractional derivatives) \([25\text{ Chapter 1}]\)

Let \(f \in AC[0, T]\) with \(T > 0\). The Riemann-Liouville left- and right-sided fractional derivatives of order \(\alpha\) are defined by

\[
D_{0^+}^\alpha f(t) := \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - s)^{-\alpha} f(s) \, ds, \quad t > 0, \quad \alpha \in (0, 1),
\]

\[
D_{0^+}^\alpha f(t) := \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dt^2} \int_0^t (t - s)^{-(\alpha - 1)} f(s) \, ds, \quad t > 0, \quad \alpha \in (1, 2),
\]

\[
D_{0^+}^\alpha f(t) := -\frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_t^T (s - t)^{-\alpha} f(s) \, ds, \quad t < T, \quad \alpha \in (0, 1),
\]

and

\[
D_{0^+}^\alpha f(t) := \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dt^2} \int_t^T (s - t)^{-(\alpha - 1)} f(s) \, ds, \quad t < T, \quad \alpha \in (1, 2),
\]

where \(\Gamma\) is the Euler gamma function.
**Definition 6** (Caputo fractional derivatives). \([28\text{, Chapter 1}]\)

Let \(f \in AC[0,T]\) with \(T > 0\). The Caputo left- and right-sided fractional derivatives of order \(\alpha\) exists almost everywhere on \([0,T]\) and are defined by

\[
\tag{6}
cD_0^\alpha f(t) := D_0^\alpha\{f(t) - f(0)\}, \quad \forall t > 0, \quad \alpha \in (0,1),
\]

and

\[
\tag{7}
cD_0^\alpha f(t) := D_0^\alpha\{f(t) - f(0) - tf'(0)\}, \quad \forall t > 0, \quad \alpha \in (1,2).
\]

Given \(T > 0\), let \(\phi: [0, \infty) \rightarrow \mathbb{R}\) be

\[
\phi(t) = \left(1 - \frac{t}{T}\right)^\mu = \begin{cases} (1 - \frac{t}{T})^\mu & \text{if } 0 \leq t \leq T, \\ 0 & \text{if } t \geq T, \end{cases}
\]

where \(\mu \gg 1\) is big enough.

**Remark:** Here one should recognize that the choice of such a function \(\phi\) to prove a blow-up result has been for the first time used by Furati and Kirane in [12].

Later on, we need the following properties concerning the function \(\phi\).

**Lemma 1.** \([28\text{, (2.45), p. 40}]\)

Let \(T > 0\), \(0 < \alpha < 1\) and \(m \geq 0\). For all \(t \in [0,T]\), we have

\[
\tag{8}
\int_0^T (\phi(t))^{-\frac{1}{p'}} |D_t^{m+\alpha} \phi(t)|^{\frac{p}{p'}} dt = C T^{1-(m+\alpha)} \frac{\mu}{p',\alpha},
\]

and

\[
\tag{9}
\int_0^T D_t^{m+\alpha} \phi(t) dt = C T^{-(m+\alpha)}.
\]

**Proof.** Using Lemma [1] we have

\[
\int_0^T (\phi(t))^{-\frac{1}{p'}} |D_t^{m+\alpha} \phi(t)|^{\frac{p}{p'}} dt = C T^{-(m+\alpha)} \frac{\mu}{p',\alpha} \int_0^T (\phi(t))^{-\frac{1}{p'}} (\phi(t))^{\frac{\mu(m+\alpha)}{p',\alpha}} dt
\]

\[
= C T^{-(m+\alpha)} \frac{\mu}{p',\alpha} \int_0^T (1 - t/T)^{\mu-(m+\alpha)} \frac{\mu}{p',\alpha} dt
\]

\[
= C T^{1-(m+\alpha)} \frac{\mu}{p',\alpha} \int_0^1 (1 - s)^{\mu-(m+\alpha)} \frac{\mu}{p',\alpha} ds
\]

\[
= C T^{1-(m+\alpha)} \frac{\mu}{p',\alpha},
\]

where we have used \(\mu \gg 1\) to guarantee the integrability of the last integral. We obtain (9) in the same way. \(\square\)
Lemma 5. Let $u \in \mathcal{S}$ be the Schwartz space of rapidly decaying $C^\infty$ functions in $\mathbb{R}^d$ and $s \in (0, 1)$. The fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^d$ is a non-local operator given by

$$(-\Delta)^s v(x) := C_{d,s} \text{ p.v.} \int_{\mathbb{R}^d} \frac{v(x) - v(y)}{|x-y|^{d+2s}} \, dy$$

$$= \begin{cases} C_{d,s} \int_{\mathbb{R}^d} \frac{v(x) - v(y)}{|x-y|^{d+2s}} \, dy, & \text{if } 0 < s < 1/2, \\ C_{d,s} \int_{\mathbb{R}^d} \frac{v(x) - v(y) - \nabla v(x) \cdot (x-y) \chi_{|x-y|<\delta}(y)}{|x-y|^{d+2s}} \, dy, & \forall \delta > 0, \text{ if } 1/2 \leq s < 1, \end{cases}$$

where p.v. stands for Cauchy’s principal value, and $C_{d,s} := \frac{s \pi^{\frac{d}{2}} \Gamma(\frac{d}{2} + s)}{\pi^{\frac{d}{2}} \Gamma(1-s)}$.

In fact, we are rarely going to use the fractional Laplacian operator in the Schwartz space; it can be extended to less regular functions as follows. For $s \in (0, 1)$, $\varepsilon > 0$, let

$$L_{s,\varepsilon}(\Omega) := \begin{cases} L_s(\mathbb{R}^d) \cap C^{0,2s+\varepsilon}(\Omega) & \text{if } 0 < s < 1/2, \\ L_s(\mathbb{R}^d) \cap C^{1,2s+\varepsilon-1}(\Omega) & \text{if } 1/2 \leq s < 1, \end{cases}$$

where $\Omega$ be an open subset of $\mathbb{R}^d$, $C^{0,2s+\varepsilon}(\Omega)$ is the space of $2s + \varepsilon$ Hölder continuous functions on $\Omega$, $C^{1,2s+\varepsilon-1}(\Omega)$ the space of functions of $C^1(\Omega)$ whose first partial derivatives are Hölder continuous with exponent $2s + \varepsilon - 1$, and

$$L_s(\mathbb{R}^d) = \left\{ u : \mathbb{R}^d \to \mathbb{R} \text{ such that } \int_{\mathbb{R}^d} \frac{u(x)}{1 + |x|^{d+s}} \, dx < \infty \right\}.$$
Proof. Let \( \tilde{x} = x/R \); by Lemma 4 we have \((-\Delta)^s \Phi_R(x) = R^{-2s}(-\Delta)^s \Phi_R(\tilde{x}) \). Therefore, using Lemma 3 we conclude that

\[
\int_{\mathbb{R}^d} (\Phi_R(x))^{-1/(p-1)} \left| (-\Delta)^s \Phi_R(x) \right|^{p/(p-1)} \, dx \\
\leq R^{-\frac{2sp}{p-1} + d} \int_{\mathbb{R}^d} (\tilde{x})^{\frac{q_0}{p} - \frac{(d+2sp)/p}{p-1}} \, d\tilde{x} \\
\leq R^{-\frac{2sp}{p-1} + d},
\]

where we have used the fact that \( q_0 < d + 2sp \iff \frac{(d+2sp)p}{p-1} - \frac{q_0}{p} > d \).

\[ \square \]

Lemma 6. Let \( 0 < \gamma, \theta \leq 1 \), \( 0 < \mu, \sigma \leq 2 \), \( p, q > 1 \), and \( d \geq 1 \). Assume that \( \theta p + \gamma pq - pq + 1 > 0 \), then

\[ d \leq \overline{D} \iff d \leq \min_{d_0>0} \max_{1 \leq i \leq 4} h_i(d_0), \]

where \( \overline{D} \) is defined in Theorem 3 and

\[
h_1(d_0) := \frac{\mu (\theta + \gamma) - pq + 1}{d_0 (pq - 1)}, \quad h_2(d_0) := \frac{\mu (\theta + d_0 \mu) - pq + 1}{d_0 (pq - 1)},
\]

\[
h_3(d_0) := \frac{\mu (\theta + d_0 \mu + \gamma) - pq + 1}{d_0 (pq - 1)}, \quad h_4(d_0) := \frac{\mu (\theta + \mu) - 1}{d_0}.
\]

Proof. We have two cases:

**Case I:** \( \frac{\theta}{\sigma} \leq \frac{\gamma}{\mu} \). Then \( d \leq \max \min_{d_0>0} h_i(d_0) \) is equivalent to

\[
d \leq \max \left\{ \max_{d_0 \in [\frac{\sigma}{\theta}, \frac{\mu}{\gamma}]}, \max_{d_0 \in [\frac{\mu}{\gamma}, +\infty]} h_4(d_0); \max_{d_0 \in [\frac{\sigma}{\theta}, \frac{\mu}{\gamma}]} h_2(d_0); \max_{d_0 \in [\frac{\sigma}{\theta}, \frac{\mu}{\gamma}]} h_1(d_0) \right\}
\]

\[
= \max \left\{ h_4 \left( \frac{\theta}{\sigma} \right); \max_{d_0 \in [\frac{\sigma}{\theta}, \frac{\mu}{\gamma}]} h_2(d_0); h_1 \left( \frac{\gamma}{\mu} \right) \right\}.
\]

If \( p \leq \frac{1}{q-\sigma} \), as \( h_2(d_0) \) is a non-increasing function of \( d_0 \), (11) is equivalent to

\[
d \leq \max \left\{ h_4 \left( \frac{\theta}{\sigma} \right); h_2 \left( \frac{\theta}{\sigma} \right); h_1 \left( \frac{\gamma}{\mu} \right) \right\} = h_2 \left( \frac{\theta}{\sigma} \right) = \frac{\theta \sigma p + \theta \mu pq - \sigma pq + \sigma}{\theta (pq - 1)} =: D_1,
\]

because \( h_2 \left( \frac{\theta}{\sigma} \right) = h_4 \left( \frac{\theta}{\sigma} \right) \) and \( h_2 \left( \frac{\theta}{\sigma} \right) \geq h_2 \left( \frac{\gamma}{\mu} \right) = h_1 \left( \frac{\gamma}{\mu} \right) \).

If \( p \geq \frac{1}{q-\sigma} \), as \( h_2(d_0) \) is a non-decreasing function of \( d_0 \), (11) is equivalent to

\[
d \leq \max \left\{ h_4 \left( \frac{\theta}{\sigma} \right); h_2 \left( \frac{\gamma}{\mu} \right); h_1 \left( \frac{\gamma}{\mu} \right) \right\} = h_2 \left( \frac{\gamma}{\mu} \right) = \frac{\mu (\theta p + \gamma pq - pq + 1)}{\gamma (pq - 1)} =: D_2,
\]

because \( h_2 \left( \frac{\gamma}{\mu} \right) = h_1 \left( \frac{\gamma}{\mu} \right) \) and \( h_2 \left( \frac{\gamma}{\mu} \right) \geq h_2 \left( \frac{\theta}{\sigma} \right) \).

Summarizing, (11) is equivalent to

\[
d \leq \max \{ D_1, D_2 \}.
\]

**Case II:** \( \frac{\theta}{\sigma} \geq \frac{\gamma}{\mu} \). Then \( d \leq \max \min_{d_0>0} h_i(d_0) \) is equivalent to

\[
d \leq \max \left\{ \max_{d_0 \in [0, \frac{\sigma}{\theta}]} h_4(d_0); \max_{d_0 \in [\frac{\sigma}{\theta}, \frac{\mu}{\gamma}]} h_3(d_0); \max_{d_0 \in [\frac{\sigma}{\theta}, \frac{\mu}{\gamma}]} h_1(d_0) \right\}
\]

\[
= \max \left\{ h_4 \left( \frac{\gamma}{\mu} \right); \max_{d_0 \in [\frac{\sigma}{\theta}, \frac{\mu}{\gamma}]} h_3(d_0); h_1 \left( \frac{\theta}{\sigma} \right) \right\}.
\]

8
Lemma 7. Let $0 < \gamma, \theta \leq 1$, $0 < \mu, \sigma \leq 2$, $p, q > 1$, and $d \geq 1$. Assume that $\gamma q + \theta pq - pq + 1 > 0$, then

$$d \leq \bar{E} \iff d \leq \max_{d_0 > 0} \min_{1 \leq i \leq 4} H_i(d_0),$$

where $\bar{E}$ is defined in Theorem 2 and

$$H_1(d_0) := \frac{pq - pq + 1}{d_0(pq - 1)}, \quad H_2(d_0) := \frac{pq - pq + 1}{d_0(pq - 1)},$$

$$H_3(d_0) := \frac{pq - pq + 1}{d_0(pq - 1)}, \quad H_4(d_0) := \frac{pq - pq + 1}{d_0(pq - 1)} - \frac{1}{d_0}.$$ 

Proof. We have two cases:

**Case I:** $\frac{\theta}{\sigma} \leq \frac{\gamma}{\mu}$. Then $d \leq \max_{d_0 > 0} \min_{1 \leq i \leq 4} H_i(d_0)$ is equivalent to

$$d \leq \max \left\{ \max_{d_0 \in [0, \frac{\theta}{\sigma}]} H_4(d_0); \max_{d_0 \in [\frac{\theta}{\sigma}, \frac{\gamma}{\mu}]} H_3(d_0); \max_{d_0 \in [\frac{\gamma}{\mu}, +\infty]} H_1(d_0) \right\} \quad \text{(17)}$$

If $(1 - \gamma) pq \leq 1$, as $H_3(d_0)$ is a non-increasing function of $d_0$, (17) is equivalent to

$$d \leq \max \left\{ H_4 \left( \frac{\theta}{\sigma} \right); H_3 \left( \frac{\gamma}{\mu} \right) \right\} = H_4 \left( \frac{\theta}{\sigma} \right) = \theta \frac{\mu q + \theta \sigma pq - \sigma pq + \sigma}{\theta pq - 1} =: E_1,$$

because $H_3 \left( \frac{\gamma}{\mu} \right) = H_4 \left( \frac{\theta}{\sigma} \right)$ and $H_3 \left( \frac{\theta}{\sigma} \right) \geq H_3 \left( \frac{\gamma}{\mu} \right) = H_1 \left( \frac{\gamma}{\mu} \right)$. 

If $(1 - \gamma) pq \geq 1$, as $H_3(d_0)$ is a non-decreasing function of $d_0$, (17) is equivalent to

$$d \leq \max \left\{ H_4 \left( \frac{\theta}{\sigma} \right); H_3 \left( \frac{\gamma}{\mu} \right) \right\} = H_3 \left( \frac{\gamma}{\mu} \right) = \frac{\mu (\gamma q + \theta pq - pq + 1)}{\gamma (pq - 1)} =: E_2,$$ 

9
because $H_3 \left( \frac{\theta}{\mu} \right) = H_1 \left( \frac{\theta}{\mu} \right)$ and $H_3 \left( \frac{\theta}{\mu} \right) \geq H_3 \left( \frac{\theta}{\sigma} \right) = H_4 \left( \frac{\theta}{\sigma} \right)$.

Summarizing, (17) is equivalent to 
\[ d \leq \max\{E_1, E_2\}. \]

Case II: $\frac{\theta}{\sigma} \geq \frac{\gamma}{\mu}$. Then $d \leq \min_{d_0 > 0} \max_{1 \leq i \leq 4} h_i(d_0)$ is equivalent to 
\[ d \leq \max\left\{ \max_{d_0 \in [0, \frac{\theta}{\sigma}]} H_4(d_0); \max_{d_0 \in \left[\frac{\theta}{\sigma}, +\infty\right]} H_2(d_0); \max_{d_0 \in \left[\frac{\theta}{\sigma}, +\infty\right]} H_1(d_0) \right\} \]
\[ = \max\left\{ H_4 \left( \frac{\theta}{\sigma} \right); H_2(d_0); H_1 \left( \frac{\theta}{\sigma} \right) \right\}. \tag{20} \]

If $q \leq \frac{1}{p-q}$, as $H_2(d_0)$ is a non-increasing function of $d_0$, (20) is equivalent to 
\[ d \leq \max\left\{ H_4 \left( \frac{\gamma}{\mu} \right); H_2 \left( \frac{\gamma}{\mu} \right); H_1 \left( \frac{\theta}{\sigma} \right) \right\} = H_2 \left( \frac{\gamma}{\mu} \right) = \frac{\gamma \mu q + \gamma \sigma pq - \mu pq + \mu}{\gamma (pq - 1)} =: E_3, \tag{21} \]

because $H_2 \left( \frac{\gamma}{\mu} \right) = H_4 \left( \frac{\gamma}{\mu} \right)$ and $H_2 \left( \frac{\gamma}{\mu} \right) \geq H_2 \left( \frac{\theta}{\sigma} \right) = H_1 \left( \frac{\theta}{\sigma} \right)$.

If $q \geq \frac{1}{p-q}$, as $H_2(d_0)$ is a non-decreasing function of $d_0$, (20) is equivalent to 
\[ d \leq \max\left\{ H_4 \left( \frac{\gamma}{\mu} \right); H_2 \left( \frac{\theta}{\sigma} \right); H_1 \left( \frac{\theta}{\sigma} \right) \right\} = H_2 \left( \frac{\theta}{\sigma} \right) = \frac{\sigma (\gamma q + \theta pq - pq + 1)}{\theta (pq - 1)} =: E_4, \tag{22} \]

because $H_2 \left( \frac{\theta}{\sigma} \right) = H_1 \left( \frac{\theta}{\sigma} \right)$ and $H_2 \left( \frac{\theta}{\sigma} \right) \geq H_2 \left( \frac{\gamma}{\mu} \right) = H_4 \left( \frac{\gamma}{\mu} \right)$.

Consequently, (20) is equivalent to 
\[ d \leq \max\{E_3, E_4\}. \]

So, Cases I and II lead to 
\[ d \leq \min\{\max\{E_1, E_2\}; \max\{E_3, E_4\}\} = \bar{E}. \]

\[ \square \]

4 Proof of Theorems 1 and 2

Proof of Theorem 1. Let $u$ be a global nontrivial weak solution of (1). Then 
\[ \int_{Q_T} |u(t, g(x))|^p \varphi(t, x) \, dt \, dx + \int_{Q_T} u_0(x) D_{\|T}^\alpha \varphi(t, x) \, dt \, dx \]
\[ = \int_{Q_T} u(t, x) D_{\|T}^\alpha \varphi(t, x) \, dt \, dx + \int_{Q_T} u(t, x)(-\Delta)^{\delta/2} \varphi(t, x) \, dt \, dx, \tag{23} \]

for all $\varphi \in X_{\delta, T}$ and all $T > 0$. By introducing $\varphi^{\frac{1}{p'}} \varphi^{-\frac{1}{p}}$ and applying the following Young’s inequality 
\[ AB \leq \frac{c_0}{4} A^p + C(p, c_0) B^{p'}, \quad A \geq 0, \quad B \geq 0, \quad p + p' = pp', \]

where $c_0$ is introduced in (A1), we get 
\[ \int_{Q_T} u(t, x) D_{\|T}^\alpha \varphi(t, x) \, dt \, dx \leq \frac{c_0}{4} \int_{Q_T} |u(t, x)|^p \varphi(t, x) \, dt \, dx + C \int_{Q_T} \varphi^{\frac{1}{p'}}(t, x) \left| D_{\|T}^\alpha \varphi(t, x) \right|^{p'} \, dt \, dx, \tag{24} \]
and
\[
\int_{Q_T} u(t, x)(-\Delta)^{\delta/2} \varphi(t, x) \, dt \, dx \leq \frac{c_0}{4} \int_{Q_T} |u(t, x)|^p \varphi(t, x) \, dt \, dx + C \int_{Q_T} \varphi^{-\frac{1}{p^*}}(t, x) \left|(-\Delta)^{\delta/2} \varphi(t, x)\right|^p \, dt \, dx.
\]  

(25)

Let
\[
\varphi(x, t) = \Phi_R(x) \phi(t),
\]

where \(\Phi_R\) is defined in Lemma 3 for \(R > 0\), and \(\phi\) is defined in (9). Observe that, using (A1)-(A2) and the monotonicity of \(\Phi_R\), we obtain the estimate
\[
\int_{Q_T} |u(t, g(x))|^p \varphi(t, x) \, dt \, dx = \int_{Q_T} |u(t, x)|^p \Phi_R(g^{-1}(x)) |J_g^{-1}(x)| \, dt \, dx 
\geq c_0 \int_{Q_T} |u(t, x)|^p \Phi_R(x) \, dt \, dx
\]

(26)

Using the estimates (24)-(26) into (23) we get
\[
\frac{c_0}{2} \int_{Q_T} |u(t, x)|^p \varphi(t, x) \, dt \, dx + \int_{Q_T} u_0(x) D^n_{t,T} \varphi(t, x) \, dt \, dx 
\leq C \int_{Q_T} \varphi^{-\frac{1}{p^*}}(t, x) \left|D^n_{t,T} \varphi(t, x)\right|^p \, dt \, dx + C \int_{Q_T} \varphi^{-\frac{1}{p^*}}(t, x) \left|(-\Delta)^{\delta/2} \varphi(t, x)\right|^p \, dt \, dx.
\]

(27)

Whereupon, using Lemmas 2 and 5, we arrive at
\[
\int_{Q_T} |u(t, x)|^p \varphi(t, x) \, dt \, dx \lesssim \int_{\mathbb{R}^d} |u_0(x)| \, dx \int_0^T D^n_{t,T} \phi(t) \, dt 
+ \int_{\mathbb{R}^d} \Phi_R(x) \, dx \int_0^T \phi^{-\frac{1}{p^*}}(t) \left|D^n_{t,T} \phi(t, x)\right|^p \, dt 
+ \int_0^T \Phi_R(x) \, dx \int_{\mathbb{R}^d} \varphi^{-\frac{1}{p^*}}(x) \left|\frac{(-\Delta)^{\delta/2} \Phi_R(x)}{\Phi_R(x)}\right|^p \, dx 
\lesssim T^{-\alpha} \int_{\mathbb{R}^d} |u_0(x)| \, dx + R^d T^{-\alpha} \frac{1}{\Phi_R} + T^{-\frac{\delta}{p^*} + d}.
\]

(27)

At this stage, two cases can be distinguished.

**Case 1:** If \(p < p_*\), we set \(R := T^{\alpha/8}\), then (27) implies
\[
\int_{Q_T} |u(t, x)|^p \varphi(t, x) \, dt \, dx \lesssim T^{-\alpha} \int_{\mathbb{R}^d} |u_0(x)| \, dx + T^{1-\alpha} \frac{1}{\Phi_R} + T^{-\frac{\delta}{p^*} + d}.
\]

(28)

Letting \(T \to +\infty\), using the fact that \(p < p_* \iff 1 - \alpha \frac{p}{p^*} + \frac{\alpha d}{d} < 0\), the assumption \(u_0 \in L^1\) and Lebesgue’s dominated convergence theorem, we conclude that
\[
\int_0^\infty \int_{\mathbb{R}^d} |u(t, x)|^p \, dt \, dx \leq 0,
\]

(29)

which leads to a contradiction.

**Case 2:** If \(p = p_*\) and \(\alpha = 1\). Let \(\tilde{\phi}\) be a smooth nonnegative non-increasing function such that \(0 \leq \tilde{\phi} \leq 1\) and
\[
\tilde{\phi}(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq 1/2, \\
0 & \text{if } t \geq 1.
\end{cases}
\]
Using $\tilde{\phi}(t), \ell > p'$, instead of $\phi(t)$ and applying Hölder’s inequality instead of Young’s inequality into (24), we obtain
\begin{align*}
\int_{Q_T} u(t,x) \varphi(t,x) dt \, dx &\leq \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) dt \, dx \right)^{1/p} \left( \int_{Q_T} \varphi^{-\frac{p'}{p}}(t,x) |\varphi_t(t,x)|^{p'} dt \, dx \right)^{1/p'} \\
&= \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) dt \, dx \right)^{1/p} \left( \int_{\mathbb{R}^d} \Phi_R(x) dx \int_0^T \tilde{\phi}^\ell(t) \left| \frac{d}{dt} \tilde{\phi}(t) \right|^{p'} dt \right)^{1/p'} \\
&\lesssim T^{-1+\frac{p}{2}} R^{\frac{p}{2}} \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) dt \, dx \right)^{1/p},
\end{align*}
where $Q_T = [T/2, T] \times \mathbb{R}^d$. We set $R := K^{\alpha/\beta} T^{\alpha/\delta}$, where $K \geq 1$ is independent of $T$. Using the estimates (25) and (30) into (23) and taking account of $p = p_*$, we get
\begin{align*}
\int_{Q_T} |u(t,x)|^p \varphi(t,x) dt \, dx &\leq T^{-\alpha} \int_{\mathbb{R}^d} |u_0(x)| \, dx + T R^{-\frac{dp}{p-1}+d} \\
&\quad + T^{-1+\frac{p}{2}} R^{\frac{p}{2}} \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) dt \, dx \right)^{1/p} \\
&= T^{-\alpha} \int_{\mathbb{R}^d} |u_0(x)| \, dx + K^{-1} \\
&\quad + K^{\frac{p}{2}} \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) dt \, dx \right)^{1/p}. \tag{31}
\end{align*}
On the other hand, using (28) with $T \to \infty$ and taking account of $p = p_*$, we obtain
\begin{align*}
u \in L^p((0, \infty), L^p(\mathbb{R}^d));
\end{align*}
consequently,
\begin{align*}
\lim_{T \to \infty} \int_{Q_T} |u(t,x)|^p \varphi(t,x) dt \, dx = \lim_{T \to \infty} \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) dt \, dx - \int_{Q_{T/2}} |u(t,x)|^p \varphi(t,x) dt \, dx \right) = 0. \tag{32}
\end{align*}
Finally, letting $T \to +\infty$ into (31) and using (32) we arrive at
\begin{align*}
\int_0^\infty \int_{\mathbb{R}^d} |u(t,x)|^p dt \, dx \leq K^{-1},
\end{align*}
which leads to a contradiction for $K \gg 1$.

\begin{flushright}
\textbf{■}
\end{flushright}

**Proof of Theorem 2** Let $(u,v)$ be a global nontrivial weak solution of (3). Then
\begin{align*}
\int_{Q_T} |v(t,g(x))|^q \varphi(t,x) dt \, dx + \int_{Q_T} u_0(x) D_{u,T}^\gamma \varphi(t,x) dt \, dx \\
= \int_{Q_T} u(t,x) D_{u,T}^\gamma \varphi(t,x) dt \, dx + \int_{Q_T} u(t,x) (-\Delta)^{\ell/2} \varphi(t,x) dt \, dx, \tag{33}
\end{align*}
and
\begin{align*}
\int_{Q_T} |u(t,f(x))|^q \psi(t,x) dt \, dx + \int_{Q_T} v_0(x) D_{v,T}^\beta \psi(t,x) dt \, dx \\
= \int_{Q_T} v(t,x) D_{v,T}^\beta \psi(t,x) dt \, dx + \int_{Q_T} v(t,x) (-\Delta)^{\delta/2} \psi(t,x) dt \, dx, \tag{34}
\end{align*}

12
hold for all \( \varphi \in X_{\mu,T}, \psi \in X_{\sigma,T} \), and all \( T > 0 \). By applying Hölder’s inequality, we have

\[
\int_{Q_T} u(t,x) D_t^{\gamma} \varphi(t,x) \, dt \, dx = \int_{Q_T} u(t,x) \psi^{1/q} \psi^{-1/q} D_t^{\gamma} \varphi(t,x) \, dt \, dx \\
\leq \left( \int_{Q_T} |u(t,x)|^q \psi(t,x) \, dt \, dx \right)^{1/q} \left( \int_{Q_T} \psi^{-\frac{1}{q'}}(t,x) \left| D_t^{\gamma} \varphi(t,x) \right|^q \, dt \, dx \right)^{1/q'},
\]

(35)

where \( q' = q/(q - 1) \). Similarly, we obtain

\[
\int_{Q_T} v(t,x) D_t^{\theta} \psi(t,x) \, dt \, dx \\
\leq \left( \int_{Q_T} |v(t,x)|^{p'} \psi(t,x) \, dt \, dx \right)^{1/p} \left( \int_{Q_T} \psi^{-\frac{1}{p'}}(t,x) \left| D_t^{\theta} \psi(t,x) \right|^{p'} \, dt \, dx \right)^{1/p'},
\]

(36)

\[
\int_{Q_T} u(t,x) (-\Delta)^{\mu/2} \varphi(t,x) \, dt \, dx \\
\leq \left( \int_{Q_T} |u(t,x)|^q \psi(t,x) \, dt \, dx \right)^{1/q} \left( \int_{Q_T} \psi^{-\frac{1}{q'}}(t,x) \left| (-\Delta)^{\mu/2} \varphi(t,x) \right|^q \, dt \, dx \right)^{1/q'},
\]

(37)

and

\[
\int_{Q_T} v(t,x) (-\Delta)^{\sigma/2} \psi(t,x) \, dt \, dx \\
\leq \left( \int_{Q_T} |v(t,x)|^{p'} \psi(t,x) \, dt \, dx \right)^{1/p} \left( \int_{Q_T} \psi^{-\frac{1}{p'}}(t,x) \left| (-\Delta)^{\sigma/2} \psi(t,x) \right|^{p'} \, dt \, dx \right)^{1/p'},
\]

(38)

where \( p' = p/(p - 1) \). Let

\[ \varphi(t,x) = \psi(t,x) = \Phi_R(x) \phi(t), \]

where \( \Phi_R \) is defined in Lemma 3 for \( R > 0 \), and \( \phi \) is defined in (6). Using (A1)-(A2), we get

\[
\int_{Q_T} |v(t,g(x))|^p \varphi(t,x) \, dt \, dx = \int_{Q_T} |v(t,x)|^p \phi(t) \Phi_R(g^{-1}(x)) |J_g^{-1}(x)| \, dt \, dx \\
\geq c_0 \int_{Q_T} |v(t,x)|^p \phi(t) \Phi_R(x) \, dt \, dx \\
= c_0 \int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx
\]

(39)

and

\[
\int_{Q_T} |u(t,f(x))|^q \varphi(t,x) \, dt \, dx = \int_{Q_T} |u(t,x)|^q \phi(t) \Phi_R(f^{-1}(x)) |J_f^{-1}(x)| \, dt \, dx \\
\geq c_0 \int_{Q_T} |u(t,x)|^q \phi(t) \Phi_R(x) \, dt \, dx \\
= c_0 \int_{Q_T} |u(t,x)|^q \varphi(t,x) \, dt \, dx
\]

(40)

Using Lemma 2 and inserting the estimates (33)-(40) into (34), we get

\[
c_0 \int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \leq \left( \int_{Q_T} |u(t,x)|^q \varphi(t,x) \, dt \, dx \right)^{1/q} \mathcal{A} + CT^{-\gamma} \int_{R^d} |u_0(x)| \, dx,
\]

(41)
and
\[ c_0 \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \leq \left( \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p} \mathcal{B} + \mathcal{C} T^{-\theta} \int_{R^d} |v_0(x)| \, dx, \]  
(42)
where
\[ \mathcal{A} := \left( \int_{Q_T} \varphi^{-\frac{1}{2q}}(t, x) \left| D_{t,T}^\sigma \varphi(t, x) \right|^{q'} \, dt \, dx \right)^{1/q'} + \left( \int_{Q_T} \varphi^{-\frac{1}{2q}}(t, x) \left| (-\Delta)^{\sigma/2} \varphi(t, x) \right|^{q'} \, dt \, dx \right)^{1/q'} \]  
and
\[ \mathcal{B} := \left( \int_{Q_T} \varphi^{-\frac{1}{2q}}(t, x) \left| D_{t,T}^\sigma \varphi(t, x) \right|^{p'} \, dt \, dx \right)^{1/p'} + \left( \int_{Q_T} \varphi^{-\frac{1}{2q}}(t, x) \left| (-\Delta)^{\sigma/2} \varphi(t, x) \right|^{p'} \, dt \, dx \right)^{1/p'}. \]

Now, combining the terms in (41)–(42), we arrive at
\[ \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \lesssim \mathcal{A}^{\frac{p}{q}} \mathcal{B}^{\frac{p}{q}} + T^{-\frac{\theta}{p}} A \left( \int_{R^d} |v_0(x)| \, dx \right)^{1/q} + T^{-\gamma} \int_{R^d} |u_0(x)| \, dx, \]  
(43)
and
\[ \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \lesssim \mathcal{A}^{\frac{q}{p}} \mathcal{B}^{\frac{q}{p}} + T^{-\frac{\theta}{q}} B \left( \int_{R^d} |u_0(x)| \, dx \right)^{1/p} + T^{-\theta} \int_{R^d} |v_0(x)| \, dx. \]  
(44)

At this stage, we distinguish five cases:

**Case 1:** \( d < D \).
In this case, we take \( R = T^{d_0} \), \( d_0 > 0 \). Then
\[ \mathcal{A}^{\frac{p}{q}} \mathcal{B}^{\frac{p}{q}} \lesssim T^{\sigma_1} + T^{\sigma_2} + T^{\sigma_3} + T^{\sigma_4}, \]  
(45)
where
\[ \sigma_1 = d_0 d + 1 - \frac{p(d_0 + \gamma q)}{pq-1}, \quad \sigma_2 = d_0 d + 1 - \frac{d_0 p(\sigma + \mu_0)}{pq-1}, \]
\[ \sigma_3 = d_0 d + 1 - \frac{\theta + \gamma q}{pq-1}, \quad \sigma_4 = d_0 d + 1 - \frac{\theta + d_0(\mu_0)}{pq-1}. \]

In order to have all exponents of \( T \) negative, it is sufficient to require \( \sigma_i < 0 \), \( 1 \leq i \leq 4 \), which in turn is equivalent to
\[ d < \max \min_{d_0 > 0, 1 \leq i \leq 4} h_i(d_0), \]
where
\[ h_1(d_0) := \frac{p(\theta + \gamma q)}{d_0 (pq-1)}, \quad h_2(d_0) := \frac{d_0 p(\sigma + \mu_0)}{d_0 (pq-1)}, \]
\[ h_3(d_0) := \frac{\theta + \gamma q}{d_0 (pq-1)}, \quad h_4(d_0) := \frac{\theta + d_0(\mu_0)}{pq-1} - \frac{1}{d_0}, \]
and this is true due to Lemma 6. Therefore
\[ \lim_{T \to \infty} \mathcal{A}^{\frac{p}{q}} \mathcal{B}^{\frac{p}{q}} = 0. \]  
(46)

In addition, we have
\[ T^{-\frac{\theta}{q}} A \lesssim T^{-\frac{\theta}{q} + \frac{d_0 d(q-1)}{pq-1} + \frac{d_0 d(q-1)}{pq-1} + \frac{1}{d_0} - \mu d_0}. \]  
(47)

Using the fact that \( \sigma_3, \sigma_4 < 0 \), we see that
\[ -\frac{\theta}{q} + d_0 d(q-1) + q - 1 - \gamma < (\theta + \gamma q)(1 - p) < 0, \]
and
\[-\frac{\theta}{q} + \frac{d_0 d(q - 1)}{q} + \frac{q - 1}{q} - \mu d_0 < \frac{(\theta + \mu d_0 q)(1 - p)}{q(pq - 1)} < 0,\]
which implies that
\[\lim_{T \to \infty} T^{-\frac{\theta}{q}} A = 0.\]
(48)

Taking to the limit when \(T \to \infty\) in (43) and using (46)-(48) and \(u_0 \in L^1(\mathbb{R}^d)\), we get
\[\lim_{T \to \infty} \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx = 0,
\]
which implies by the monotone convergence theorem that
\[\int_0^\infty \int_{\mathbb{R}^d} |v(t, x)|^p \, dx \, dt = 0,
\]
and then \(v \equiv 0\ a.e.\). Thus, combing it with (42), we derive that
\[\int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \lesssim T^{-\frac{\theta}{q}} \int_{\mathbb{R}^d} |v_0(x)| \, dx,
\]
which yields, by the monotone convergence theorem and \(v_0 \in L^1(\mathbb{R}^d)\), that
\[\lim_{T \to \infty} \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx = 0,
\]
i.e. \(v \equiv 0\ a.e.;\) contradiction.

**Case 2:** \(d < E\).
In this case, we take \(R = T^{d_0}, d_0 > 0\). Then, we have
\[\mathcal{A}^{\frac{\theta}{q}} \mathcal{B}^{\frac{\theta}{q}} \lesssim T^{\rho_1} + T^{\rho_2} + T^{\rho_3} + T^{\rho_4},\]
(49)
where
\[\rho_1 = d_0 d + 1 - \frac{q(\gamma + \theta p)}{pq - 1}, \quad \rho_2 = d_0 d + 1 - \frac{q(\gamma + d_0 \sigma p)}{pq - 1},\]
\[\rho_3 = d_0 d + 1 - \frac{q(\mu d_0 + \theta p)}{pq - 1}, \quad \rho_4 = d_0 d + 1 - \frac{d_0 (\mu + \sigma p)}{pq - 1}.
\]
Therefore, in order to have all exponents of \(T\) negative, it is sufficient to require \(\rho_i < 0, 1 \leq i \leq 4\), which is equivalent to
\[d < \max\ \min_{d_0 > 0} H_i(d_0),\]
where
\[H_1(d_0) = \frac{q(\gamma + \theta p) - pq + 1}{d_0 (pq - 1)}, \quad H_2(d_0) = \frac{q(\gamma + d_0 \sigma p) - pq + 1}{d_0 (pq - 1)},\]
\[H_3(d_0) = \frac{q(\mu d_0 + \theta p) - pq + 1}{d_0 (pq - 1)}, \quad H_4(d_0) = \frac{q(\mu + \sigma p)}{pq - 1} - \frac{1}{d_0},\]
and this is true due to Lemma 7 Therefore
\[\lim_{T \to \infty} \mathcal{A}^{\frac{\theta}{q}} \mathcal{B}^{\frac{\theta}{q}} = 0.\]
(50)

In addition, we have
\[T^{-\frac{\theta}{q}} \mathcal{B} \lesssim T^{-\frac{\theta}{q}} + \frac{d_0 d p + 1}{p} + \frac{p - 1}{p} - \theta + T^{-\frac{\theta}{q}} + \frac{d_0 d p + 1}{p} + \frac{p - 1}{p} - \sigma d_0,
\]
(51)
Using the fact that $\rho_1, \rho_2 < 0$, we can see that
\[-\frac{\gamma}{p} + \frac{d_0(p-1)}{p} + \frac{p-1}{p} - \theta < \frac{(\gamma + \theta p)(1-q)}{p(pq-1)} < 0,\]
and
\[-\frac{\gamma}{p} + \frac{d_0(p-1)}{p} + \frac{p-1}{p} - \sigma d_0 < \frac{(\gamma + \sigma d_0 p)(1-q)}{p(pq-1)} < 0,\]
which implies that
\[
\lim_{T \to \infty} T^{-\frac{\gamma}{p}} B = 0. 
\]
Taking to the limit when $T \to \infty$ in (41) and using (50)- (52) and $\nu_0 \in L^1(\mathbb{R}^d)$, we get
\[
\lim_{T \to \infty} \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx = 0,
\]
which implies by the monotone convergence theorem that
\[
\int_0^\infty \int_{\mathbb{R}^d} |u(t, x)|^q \, dx \, dt = 0,
\]
and then $u \equiv 0$ a.e.. Thus, combing it with (41), we derive that
\[
\int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \lesssim T^{-\gamma} \int_{\mathbb{R}^d} |u_0(x)| \, dx,
\]
which yields, by the monotone convergence theorem and $u_0 \in L^1(\mathbb{R}^d)$, that
\[
\lim_{T \to \infty} \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx = 0,
\]
i.e. $v \equiv 0$ a.e.; contradiction.

**Case 3:** $d = D, \theta = 1$, and $\gamma \neq 1$.

In this case, we also suppose that $pq > 1/(1-\gamma)$. Let $\tilde{\phi}$ be a smooth nonnegative non-increasing function such that $0 \leq \tilde{\phi} \leq 1$ and
\[
\tilde{\phi}(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq 1/2, \\
0 & \text{if } t \geq 1. 
\end{cases}
\]

Using $\theta = 1$ and $\tilde{\phi}^\ell(t), \ell > p'$, instead of $\phi(t)$, we may improve (36) as follows
\[
\int_{Q_T} v(t, x) \varphi(t, x) \, dt \, dx \leq \left( \int_{\tilde{Q}_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p} \left( \int_{Q_T} \varphi^{-\frac{p}{p-1}}(t, x) |\varphi(t, x)|^{p'} \, dt \, dx \right)^{1/p'}, \quad (53)
\]
where $\tilde{Q}_T = [T/2, T] \times \mathbb{R}^d$, and so (42) becomes
\[
\begin{align*}
c_0 \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \leq & \left( \int_{\tilde{Q}_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p} \left( \int_{Q_T} \varphi^{-\frac{p}{p-1}}(t, x) |\varphi(t, x)|^{p'} \, dt \, dx \right)^{1/p'} \\
& + \left( \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p} \left( \int_{Q_T} \varphi^{-\frac{p}{p-1}}(t, x) \left|(-\Delta)^{\sigma/2} \varphi(t, x)\right|^{p'} \, dt \, dx \right)^{1/p'} \\
& + C T^{-\theta} \int_{\mathbb{R}^d} |\nu_0(x)| \, dx,
\end{align*}
(54)
Now, inserting (54) into (41), we arrive at
\[
\int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \lesssim A \left( \int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \right)^{1/qp} \left( \int_{Q_T} \varphi^{-\frac{1}{p'}}(t,x) \, |\varphi(t,x)|^{p'} \, dt \, dx \right)^{1/qp'} + \left( \int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \right)^{1/qp} \left( \int_{Q_T} \varphi^{-\frac{1}{p'}}(t,x) \, \left( -\Delta \right)^{\gamma/2} \varphi(t,x) \, |\varphi(t,x)|^{p'} \, dt \, dx \right)^{1/qp'} + T^{-\frac{q}{p}} A \left( \int_{\mathbb{R}^d} |v_0(x)| \, dx \right)^{1/q} + T^{-\gamma} \int_{\mathbb{R}^d} |u_0(x)| \, dx. \tag{55}
\]
Applying the following Young’s inequality
\[ AB \leq \frac{1}{2} A^{pq} + C(p,q)B^{\frac{pq}{pq-1}}, \quad A \geq 0, \ B \geq 0, \]
on the second term in the right-hand side of (55), we arrive at
\[
\frac{1}{2} \int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \lesssim A \left( \int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \right)^{1/qp} \left( \int_{Q_T} \varphi^{-\frac{1}{p'}}(t,x) \, |\varphi(t,x)|^{p'} \, dt \, dx \right)^{1/qp'} + A^{\frac{pq}{pq-1}} \left( \int_{Q_T} \varphi^{-\frac{1}{p'}}(t,x) \, \left( -\Delta \right)^{\gamma/2} \varphi(t,x) \, |\varphi(t,x)|^{p'} \, dt \, dx \right)^{\frac{pq}{pq-1}} + T^{-\frac{q}{p}} A \left( \int_{\mathbb{R}^d} |v_0(x)| \, dx \right)^{1/q} + T^{-\gamma} \int_{\mathbb{R}^d} |u_0(x)| \, dx. \tag{56}
\]
At this stage, we set \( R = K^d \alpha^{\frac{1}{d}} t^d_0, \ \alpha > 0, \) where \( K \geq 1 \) is independent of \( T \). Consequently,
\[
\int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \\
\lesssim \left( \int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \right)^{1/qp} \left( T^{\frac{\sigma(q-1)}{q} - \frac{\sigma d}{pq - 1}} K^{\frac{d}{pq - 1}} + T^{\frac{\sigma(q-1)}{q} - \frac{\sigma d}{pq - 1}} K^{\frac{d}{pq - 1} - \mu d_0} \right) \\
+ T^{\sigma_1 K^d d_0 - \frac{\sigma d}{pq - 1} + \frac{\sigma d}{pq - 1} - \frac{\sigma d}{pq - 1}} + T^{\sigma_2 K^d d_0 - \frac{\sigma d}{pq - 1} - \frac{\sigma d}{pq - 1} - \frac{\sigma d}{pq - 1} - \mu d_0} \\
+ \left( \int_{\mathbb{R}^d} |v_0(x)| \, dx \right)^{1/q} \left( T^{-\frac{q}{p}} + \frac{d(q-1)}{pq} + \frac{d}{pq} + \frac{d(q-1)}{pq} + \frac{d}{pq} - \mu d_0 K^{\frac{d}{pq} - \frac{d}{pq}} \right) + T^{-\gamma} \int_{\mathbb{R}^d} |u_0(x)| \, dx. \tag{57}
\]
As Lemma 6 and \( d = 7 \), imply that \( \sigma_i = 0 \) for all \( 1 \leq i \leq 4 \), we infer from (57) that
\[
\int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \lesssim K^{\frac{d(q-1)}{pq}} \left( \int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \right)^{1/qp} \\
+ K^{1 - \frac{pq(q-1)}{pq - 1}} + K^{-1} \\
+ T^{-\frac{(1+\gamma)(q-1)}{q}} K^{\frac{d(q-1)}{pq}} \left( \int_{\mathbb{R}^d} |v_0(x)| \, dx \right)^{1/q} + T^{-\gamma} \int_{\mathbb{R}^d} |u_0(x)| \, dx. \tag{58}
\]
On the other hand, using (45), (47), and (43) with \( T \to \infty \), taking account of \( \sigma_i = 0 \) for all \( 1 \leq i \leq 4 \), we get
\( v \in L^p((0, \infty), L^p(\mathbb{R}^d)); \)
consequently,
\[
\lim_{T \to \infty} \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx = \lim_{T \to \infty} \left( \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx - \int_{Q_{T/2}} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right) = 0. \tag{59}
\]
So, letting \( T \to +\infty \) into \( \text{(58)} \) and using \( \text{(59)} \), we arrive at
\[
\int_0^\infty \int_{\mathbb{R}^d} |v(t, x)|^p \, dt \, dx \lesssim K^{\frac{1-pq(1-\gamma)}{pq}} + K^{-1},
\]
which leads, by letting \( K \to +\infty \), to \( v \equiv 0 \) a.e.. Combing it with \( \text{(54)} \), we derive that
\[
\int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \lesssim T^{-1} \int_{\mathbb{R}^d} |v_0(x)| \, dx,
\]
which yields
\[
\lim_{T \to \infty} \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx = 0,
\]
i.e. \( u \equiv 0 \) a.e.; contradiction.

**Case 4: \( d = \overline{D}, \theta = 1, \text{ and } \gamma \leq 1.**

In this case, we also suppose that \( \gamma \leq \frac{p \theta (p-1)}{q(p-1)}. \) Let \( \tilde{\phi} \) be as in the Case 3. Using \( \theta = 1 \) and \( \tilde{\phi}'(t), \ell > p', \) instead of \( \phi(t) \), we may improve \( \text{(53)} \) as follows
\[
\int_{Q_T} v(t, x) \varphi(t, x) \, dt \, dx \leq \left( \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p} \left( \int_{Q_T} \varphi^{-\frac{1}{p-1}}(t, x) |\varphi(t, x)|^{p'} \, dt \, dx \right)^{1/p'}, \tag{60}
\]
and so \( \text{(12)} \) becomes
\[
c_0 \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \leq \left( \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p} \left( \int_{Q_T} \varphi^{-\frac{1}{p-1}}(t, x) |\varphi(t, x)|^{p'} \, dt \, dx \right)^{1/p'}
+ \left( \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p} \left( \int_{Q_T} \varphi^{-\frac{1}{p-1}}(t, x) \left| (-\Delta)^{\sigma/2} \varphi(t, x) \right|^{p'} \, dt \, dx \right)^{1/p'}
+ C T^{-1} \int_{\mathbb{R}^d} |v_0(x)| \, dx,
\]
Set \( R = K^{\alpha_0} T^{\alpha_0}, \, \alpha_0 > 0, \) where \( K \geq 1 \) is independent of \( T. \) Consequently,
\[
\int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \lesssim T^{-1+\frac{(\ell d_0+\frac{1}{2})d(p-1)}{p}} K^{\frac{d_0 d(p-1)}{p}} \left( \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p}
+ T^{-\sigma d_0 + \frac{(\ell d_0+\frac{1}{2})d(p-1)}{p}} \left( \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p}
+ T^{-1} \int_{\mathbb{R}^d} |v_0(x)| \, dx. \tag{61}
\]
As Lemma \( \text{(6)} \) and \( d = \overline{D} \), imply that \( \sigma_i = 0 \) for all \( 1 \leq i \leq 4, \) we infer from \( \text{(61)} \) that
\[
\int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \lesssim T^{\frac{d_0 d(p-1)}{pq-1} - \frac{d_0 d(p-1)}{p-1}} \left( \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p}
+ T^{\frac{d_0 d(p-1)}{pq-1} - \frac{d_0 d(p-1)}{p-1} - \frac{d_0 d(p-1)}{p-1}} \left( \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \right)^{1/p}
+ T^{-1} \int_{\mathbb{R}^d} |v_0(x)| \, dx.
\]
Then, by using the fact that \( \gamma \lesssim \frac{p(q-1)}{q(p-1)} \), we may conclude

\[
\int_{Q_T} u(t,x)|^{q}\varphi(t,x) \, dt \, dx \lesssim K^{d(d(p-1)/p)} \left( \int_{Q_T} |v(t,x)|^{p}\varphi(t,x) \, dt \, dx \right)^{1/p} + K^{-\frac{q}{p}} \left( \int_{Q_T} |v(t,x)|^{p}\varphi(t,x) \, dt \, dx \right)^{1/p} + T^{-1} \int_{\mathbb{R}^d} |u_0(x)| \, dx.
\]

On the other hand, using (45), (47) and (43) with \( \phi \) instead of \( \varphi \), which leads, by letting \( \gamma \)

\[
\lim_{T \to \infty} \int_{Q_T} |v(t,x)|^{p}\varphi(t,x) \, dt \, dx = \lim_{T \to \infty} \left( \int_{Q_T} |v(t,x)|^{p}\varphi(t,x) \, dt \, dx - \int_{Q_{T/2}} |v(t,x)|^{p}\varphi(t,x) \, dt \, dx \right) = 0.
\]

So, letting \( T \to \infty \) into (62) and using (63), we arrive at

\[
\int_{0}^{\infty} \int_{\mathbb{R}^d} |u(t,x)|^{q} \, dt \, dx \lesssim K^{-\frac{q}{p-1}},
\]

which leads, by letting \( K \to +\infty \), to \( u \equiv 0 \) a.e.. Combining it with (41), we derive that

\[
c_0 \int_{Q_T} |v(t,x)|^{p}\varphi(t,x) \, dt \, dx \lesssim T^{-\gamma} \int_{\mathbb{R}^d} |u_0(x)| \, dx,
\]

which yields

\[
\lim_{T \to \infty} \int_{Q_T} |v(t,x)|^{p}\varphi(t,x) \, dt \, dx = 0,
\]

i.e. \( u \equiv 0 \) a.e.; contradiction.

**Case 5:** \( d = \mathbb{R}, \gamma = 1 \) and \( \theta \neq 1 \).

In this case, we also suppose that \( pq > 1/(1-\theta) \). Let \( \tilde{\phi} \) be as in the Case 3. Using \( \gamma = 1 \) and \( \tilde{\phi}^\ell(t) \), \( \ell > p' \), instead of \( \phi(t) \), we may improve (35) as follows

\[
\int_{Q_T} u(t,x)\varphi(t,x) \, dt \, dx \leq \left( \int_{Q_T} |u(t,x)|^{q}\varphi(t,x) \, dt \, dx \right)^{1/q} \left( \int_{Q_T} \varphi^{-\frac{q}{p}}(t,x) |\varphi(t,x)|^{q'} \, dt \, dx \right)^{1/q'},
\]

and so (41) becomes

\[
c_0 \int_{Q_T} |v(t,x)|^{p}\varphi(t,x) \, dt \, dx \leq \left( \int_{Q_T} |u(t,x)|^{q}\varphi(t,x) \, dt \, dx \right)^{1/q} \left( \int_{Q_T} \varphi^{-\frac{q}{p}}(t,x) |\varphi(t,x)|^{q'} \, dt \, dx \right)^{1/q'} + \left( \int_{Q_T} |u(t,x)|^{q}\varphi(t,x) \, dt \, dx \right)^{1/q} \left( \int_{Q_T} \varphi^{-\frac{q}{p}}(t,x) (-\Delta)^{\ell/2} \varphi(t,x) \, dt \, dx \right)^{1/q'} + C T^{-1} \int_{\mathbb{R}^d} |u_0(x)| \, dx,
\]
Applying the following Young’s inequality on the second term in the right-hand side of (66), we arrive at

\[ \int_{Q_T} |u(t,x)|^q \phi(t,x) \, dt \, dx \leq \mathcal{B} \left( \int_{Q_T} |u(t,x)|^q \phi(t,x) \, dt \, dx \right)^{1/pq} \left( \int_{Q_T} \phi^{- \frac{q}{q'-1}}(t,x) \| \phi(t,x) \|^{q'} \, dt \, dx \right)^{1/pq'} \\
+ \left( \int_{Q_T} |u(t,x)| \phi(t,x) \, dt \, dx \right)^{1/pq} \mathcal{B} \left( \int_{Q_T} \phi^{- \frac{q}{q'-1}}(t,x) \| (\Delta)^{\mu/2} \phi(t,x) \|^{q'} \, dt \, dx \right)^{1/pq'} \\
+ T^{-\frac{q}{q-1}} \mathcal{B} \left( \int_{\mathbb{R}^d} |u_0(x)| \, dx \right)^{1/p} + T^{-\theta} \int_{\mathbb{R}^d} |v_0(x)| \, dx. \quad (66) \]

Applying the following Young’s inequality

\[ AB \leq \frac{1}{2} A^{pq} + C(p, q) B^{\frac{p}{pq-1}}, \quad A \geq 0, \quad B \geq 0, \]

on the second term in the right-hand side of (66), we arrive at

\[ \frac{1}{2} \int_{Q_T} |u(t,x)|^q \phi(t,x) \, dt \, dx \leq \mathcal{B} \left( \int_{Q_T} |u(t,x)|^q \phi(t,x) \, dt \, dx \right)^{1/pq} \left( \int_{Q_T} \phi^{- \frac{q}{q'-1}}(t,x) \| \phi(t,x) \|^{q'} \, dt \, dx \right)^{1/pq'} \\
+ \mathcal{B}^{\frac{p}{pq-1}} \left( \int_{Q_T} \phi^{- \frac{q}{q'-1}}(t,x) \| (\Delta)^{\mu/2} \phi(t,x) \|^{q'} \, dt \, dx \right)^{\frac{q-1}{pq-1}} \\
+ T^{-\frac{pq}{p+\theta}} \mathcal{B} \left( \int_{\mathbb{R}^d} |u_0(x)| \, dx \right)^{1/p} + T^{-\theta} \int_{\mathbb{R}^d} |v_0(x)| \, dx. \quad (67) \]

At this stage, we set \( R = K^{d_0} T^{d_0}, d_0 > 0, \) where \( K \geq 1 \) is independent of \( T. \) Consequently,

\[ \int_{Q_T} |u(t,x)|^q \phi(t,x) \, dt \, dx \leq \left( \int_{Q_T} |u(t,x)|^q \phi(t,x) \, dt \, dx \right)^{1/pq} \left( T^{\frac{d_0}{p+\theta} \left( \frac{d_0}{p+\theta} - d_0 \right)} K^{-\frac{d_0(pq-1)}{pq-1}} + T^{\frac{d_0}{p+\theta} \left( \frac{d_0}{p+\theta} - d_0 \right)} K^{-\frac{d_0(pq-1)}{pq-1} - \sigma d_0} \right) \\
+ \left( \int_{\mathbb{R}^d} |u_0(x)| \, dx \right)^{1/p} \left( T^{-\frac{1}{p+\theta} + \frac{d_0(pq-1)}{p} + \frac{p-1}{p+\theta} - \theta} K^{-\frac{d_0(pq-1)}{p}} + T^{-\frac{1}{p+\theta} + \frac{d_0(pq-1)}{p} + \frac{p-1}{p+\theta} - \sigma d_0} K^{-\frac{d_0(pq-1)}{p} - \sigma d_0} \right) \\
+ T^{-\theta} \int_{\mathbb{R}^d} |v_0(x)| \, dx. \quad (68) \]

As Lemma7 and \( d = \overline{d}, \) imply that \( \rho_i = 0 \) for all \( 1 \leq i \leq 4, \) we infer from (68) that

\[ \int_{Q_T} |u(t,x)|^q \phi(t,x) \, dt \, dx \leq K^{\frac{d_0(pq-1)}{pq-1}} \left( \int_{Q_T} |u(t,x)|^q \phi(t,x) \, dt \, dx \right)^{1/pq} \\
+ K^{-\frac{q}{q-1} \left( 1 - \frac{p}{p+\theta} \right)} + K^{-1} \\
+ T^{-\frac{(1+\theta)(1-q)}{p(pq-1)}} K^{-\frac{d_0(pq-1)}{p}} \left( \int_{\mathbb{R}^d} |u_0(x)| \, dx \right)^{1/p} \\
+ T^{-\theta} \int_{\mathbb{R}^d} |v_0(x)| \, dx. \quad (69) \]

On the other hand, using (49), (51) and (44) with \( T \to \infty, \) taking account of \( \rho_i = 0 \) for all \( 1 \leq i \leq 4, \) we get

\[ u \in L^q((0, \infty), L^q(\mathbb{R}^d)); \]
consequently,

$$\lim_{T \to \infty} \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx = \lim_{T \to \infty} \left( \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx - \int_{Q_{T/2}} |u(t, x)|^q \varphi(t, x) \, dt \, dx \right) = 0.$$  \hfill (70)

So, letting $T \to +\infty$ into (69) and using (70), we arrive at

$$\int_{0}^{\infty} \int_{\mathbb{R}^{d}} |u(t, x)|^q \, dt \, dx \lesssim K^{\frac{1}{1-\frac{p}{q}(q-1)}} + K^{-1},$$

which leads, by letting $K \to +\infty$, to $u \equiv 0$ a.e.. Combing it with (65), we derive that

$$\int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \lesssim T^{-1} \int_{\mathbb{R}^{d}} |u_0(x)| \, dx,$$

which yields

$$\lim_{T \to \infty} \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx = 0,$$

i.e. $v \equiv 0$ a.e.; contradiction.

**Case 6: $d = \overline{E}$, $\gamma = 1$, and $\theta \leq 1$.**

In this case, we also suppose that $\theta \leq \frac{q(p-1)}{p(q-1)}$. Let $\bar{\varphi}$ be as in the Case 3. Using $\gamma = 1$ and $\bar{\varphi}^\ell(t)$, $\ell > p'$, instead of $\phi(t)$, we may improve (65) as follows

$$\int_{Q_T} u(t, x) \varphi(t, x) \, dt \, dx \leq \left( \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \right)^{1/q} \left( \int_{Q_T} \varphi^{-\frac{\gamma}{q'}}(t, x) |\varphi(t, x)|^{q'} \, dt \, dx \right)^{1/q'},$$

and so (61) becomes

$$c_0 \int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \leq \left( \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \right)^{1/q} \left( \int_{Q_T} \varphi^{-\frac{\gamma}{q'}}(t, x) |\varphi(t, x)|^{q'} \, dt \, dx \right)^{1/q'} + \left( \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \right)^{1/q'} \left( \int_{Q_T} \varphi^{-\frac{\gamma}{q'}}(t, x) \left| (-\Delta)^{\nu/2} \varphi(t, x) \right|^{q'} \, dt \, dx \right)^{1/q'} + C T^{-1} \int_{\mathbb{R}^{d}} |u_0(x)| \, dx,$$

Set $R = K^{d_0}T^{d_0}$, $d_0 > 0$, where $K \geq 1$ is independent of $T$. Consequently,

$$\int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \lesssim T^{-1} \frac{\mu d_0}{q} \frac{d_0(q-1)}{q} \left( \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \right)^{1/q} + T^{-\mu d_0 - \frac{d_0(q-1)}{q}} K^{-\mu d_0 - \frac{d_0(q-1)}{q}} \left( \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \right)^{1/q} + T^{-1} \int_{\mathbb{R}^{d}} |u_0(x)| \, dx.$$  \hfill (72)

As Lemma 7 and $d = \overline{E}$, imply that $\rho_i = 0$ for all $1 \leq i \leq 4$, we infer from (72) that

$$\int_{Q_T} |v(t, x)|^p \varphi(t, x) \, dt \, dx \lesssim T^{\frac{q(q-1) - q(p-1)}{pq-1}} K^{\frac{d_0(q-1)}{q}} \left( \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \right)^{1/q} + T^{\frac{q(q-1) - q(p-1)}{pq-1}} K^{\frac{d_0(q-1) - q(q-1)}{q-1}} \left( \int_{Q_T} |u(t, x)|^q \varphi(t, x) \, dt \, dx \right)^{1/q} + T^{-1} \int_{\mathbb{R}^{d}} |u_0(x)| \, dx.$$
Then, by using the fact that $\theta \leq \frac{2(p-1)}{p(q-1)}$, we may conclude
\[
\int_{Q_T} |v(t,x)|^p \varphi(t,x) \, dt \, dx \lesssim K^{\frac{2(p-1)}{p(q-1)}} \left( \int_{Q_T} |u(t,x)|^q \varphi(t,x) \, dt \, dx \right)^{\frac{1}{q}} + K^{-\frac{q+1}{q}} \left( \int_{Q_T} |u(t,x)|^q \varphi(t,x) \, dt \, dx \right)^{\frac{1}{q}} + T^{-1} \int_{\mathbb{R}^d} |u_0(x)| \, dx. \tag{73}
\]
On the other hand, using (49), (51) and (44) with $T \to \infty$, taking account of $\rho_i = 0$ for all $1 \leq i \leq 4$, we get
\[
u \in L^q((0,\infty), L^q(\mathbb{R}^d));
\]
consequently,
\[
\lim_{T \to \infty} \int_{Q_T} |u(t,x)|^q \varphi(t,x) \, dt \, dx = \lim_{T \to \infty} \left( \int_{Q_T} |u(t,x)|^q \varphi(t,x) \, dt \, dx - \int_{Q_T/2} |u(t,x)|^q \varphi(t,x) \, dt \, dx \right) = 0. \tag{74}
\]
So, letting $T \to +\infty$ into (73) and using (74), we arrive at
\[
\int_0^\infty \int_{\mathbb{R}^d} |v(t,x)|^p \, dt \, dx \lesssim K^{-\frac{q+1}{q}},
\]
which leads, by letting $K \to +\infty$, to $v \equiv 0$ a.e.. Combing it with (42), we derive that
\[
c_0 \int_{Q_T} |u(t,x)|^q \varphi(t,x) \, dt \, dx \lesssim T^{-\theta} \int_{\mathbb{R}^d} |v_0(x)| \, dx,
\]
which yields
\[
\lim_{T \to \infty} \int_{Q_T} |u(t,x)|^q \varphi(t,x) \, dt \, dx = 0,
\]
i.e. $u \equiv 0$ a.e.; contradiction.

\[\square\]

5 Proof of Theorem 3

Proof of Theorem 3. Let $u$ be a global nontrivial weak solution of (2). Then
\[
\int_{Q_T} |u(t,g(x))|^p \varphi(t,x) \, dt \, dx + \int_{Q_T} u_0(x) \left[ D^{\alpha}_{t,T} \varphi(t,x) + D^{\beta}_{t,T} \varphi(t,x) \right] \, dt \, dx + \int_{Q_T} u_1(x) D^{\beta-1}_{t,T} \varphi(t,x) \, dt \, dx = \int_{Q_T} u(t,x) D^{\beta}_{t,T} \varphi(t,x) \, dt \, dx + \int_{Q_T} u(t,x) D^{\alpha}_{t,T} \varphi(t,x) \, dt \, dx + \int_{Q_T} u(t,x) (-\Delta)^{\delta/2} \varphi(t,x) \, dt \, dx, \tag{75}
\]
holds for all $\varphi \in Y_{\delta,T}$ and all $T > 0$. By introducing $\varphi_{1/p}^{1/p} \varphi_{1/p}$ and applying the following Young’s inequality
\[
AB \leq \frac{c_0}{6} A^p + C(p,c_0)B^{p'}, \quad A \geq 0, \; B \geq 0, \; p + p' = pp',
\]
where $c_0$ is introduced in (A1), we get
\[
\int_{Q_T} u(t,x) D^{\beta}_{t,T} \varphi(t,x) \, dt \, dx \leq \frac{c_0}{6} \int_{Q_T} |u(t,x)|^p \varphi(t,x) \, dt \, dx + C \int_{Q_T} \varphi_{1/p}^{1/(2p')} (t,x) \left| D^{\alpha}_{t,T} \varphi(t,x) \right|^{p'} \, dt \, dx, \tag{76}
\]
\[
\int_{Q_T} u(t, x) D^{\alpha}_{t,x} \varphi(t, x) \, dt \, dx \leq \frac{c_0}{6} \int_{Q_T} |u(t, x)|^p \varphi(t, x) \, dt \, dx + C \int_{Q_T} \varphi^{-\frac{\alpha}{p-1}}(t, x) \left| D^{\gamma}_{t,x} \varphi(t, x) \right|^p \, dt \, dx,
\]
and
\[
\int_{Q_T} u(t, x) (-\Delta)^{\beta/2} \varphi(t, x) \, dt \, dx \leq \frac{c_0}{6} \int_{Q_T} |u(t, x)|^p \varphi(t, x) \, dt \, dx + C \int_{Q_T} \varphi^{-\frac{\alpha}{p-1}}(t, x) \left| (-\Delta)^{\beta/2} \varphi(t, x) \right|^p \, dt \, dx.
\]

Let
\[
\Phi_R(x) = \Phi(x) \phi(t),
\]
where \(\Phi_R\) is defined in Lemma 3 for \(R > 0\), and \(\phi\) is defined in 3. Observe that, using (A1)-(A2) and the monotonicity of \(\Phi_R\), we obtain the estimate
\[
\int_{Q_T} |u(t, g(x))|^p \varphi(t, x) \, dt \, dx = \int_{Q_T} |u(t, x)|^p \phi(t) \Phi_R(g^{-1}(x)) |J^{-1}_g(x)| \, dt \, dx \\
\geq c_0 \int_{Q_T} |u(t, x)|^p \phi(t) \Phi_R(x) \, dt \, dx.
\]
Using the estimates (76)-(78) into (75) we get
\[
\frac{c_0}{2} \int_{Q_T} |u(t, x)|^p \varphi(t, x) \, dt \, dx + \int_{Q_T} u_0(x) \left[ D^{\alpha}_{t,x} \varphi(t, x) + D^{\beta}_{t,x} \varphi(t, x) \right] \, dt \, dx + \int_{Q_T} u_1(x) D^{\alpha-1}_{t,x} \varphi(t, x) \, dt \, dx \\
\lesssim \int_{Q_T} \varphi^{-\frac{\alpha}{p-1}}(t, x) \left( \left| D^{\alpha}_{t,x} \varphi(t, x) \right|^p + \left| D^{\beta}_{t,x} \varphi(t, x) \right|^p \right) \, dt \, dx + \int_{Q_T} \varphi^{-\frac{\alpha}{p-1}}(t, x) \left| (-\Delta)^{\beta/2} \varphi(t, x) \right|^p \, dt \, dx.
\]
Whereupon, using Lemmas 2 and 5 we arrive at
\[
\int_{Q_T} |u(t, x)|^p \varphi(t, x) \, dt \, dx \lesssim \int_{\mathbb{R}^d} |u_0(x)| \, dx \int_{0}^{T} \left[ D^{\alpha}_{t,x} \Phi(t) + D^{\beta}_{t,x} \Phi(t) \right] \, dt \, dx + \int_{\mathbb{R}^d} u_1(x) \, dx \int_{0}^{T} D^{\alpha-1}_{t,x} \Phi(t) \, dt \, dx + \int_{\mathbb{R}^d} \Phi_R(x) \, dx \int_{0}^{T} \varphi^{-\frac{\alpha}{p-1}}(t) \left( \left| D^{\alpha}_{t,x} \varphi(t, x) \right|^p + \left| D^{\beta}_{t,x} \varphi(t, x) \right|^p \right) \, dt \, dx + \int_{\mathbb{R}^d} \Phi_R(x) \, dx \int_{0}^{T} \varphi^{-\frac{\alpha}{p-1}}(t) \left| (-\Delta)^{\beta/2} \Phi_R(x) \right|^p \, dt \, dx \\
\lesssim T^{-\alpha} \int_{\mathbb{R}^d} |u_0(x)| \, dx + C T^{-(\beta-1)} \int_{\mathbb{R}^d} u_1(x) \, dx + R^d T^{1-\beta} + R^d T^{1-\alpha} + T R^d \frac{\alpha}{p-1} + T R^d \frac{\beta}{p-1}.
\]
At this stage, two cases can be distinguished.

Case 1: If \(p < p_*\), we set \(R := T^{\alpha/\beta}\), then (80) implies
\[
\int_{Q_T} |u(t, x)|^p \varphi(t, x) \, dt \, dx \lesssim T^{-\alpha} \int_{\mathbb{R}^d} |u_0(x)| \, dx + T^{-(\beta-1)} \int_{\mathbb{R}^d} u_1(x) \, dx + T^{1-\alpha} \frac{\alpha}{p-1} + T \frac{\beta}{p-1}.
\]
Letting \(T \to +\infty\), using the fact that \(p < p_* \iff 1 - \alpha \frac{p}{p-1} + \frac{\alpha d}{2} < 0\), the assumption \(u_0 \in L^1\) and Lebesgue’s dominated convergence theorem, we conclude that
\[
\int_{0}^{\infty} \int_{\mathbb{R}^d} |u(t, x)|^p \, dt \, dx \leq 0,
\]
which leads to a contradiction.

Case 2: If \(p = p_*\) and \(\alpha = 1\). Let \(\tilde{\phi}\) be a smooth nonnegative non-increasing function such that \(0 \leq \tilde{\phi} \leq 1\) and
\[
\tilde{\phi}(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq 1/2, \\
0 & \text{if } t \geq 1.
\end{cases}
\]
Using $\tilde{\phi}'(t)$, $\ell > p'$, instead of $\phi(t)$ and applying Hölder’s inequality instead of Young’s inequality into (77), we obtain

$$\int_{Q_T} |u(t,x)|^p \varphi(t,x) \, dt \, dx \leq \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) \, dt \, dx \right)^{1/p} \left( \int_{Q_T} \varphi^{-\frac{1}{p'}}(t,x) |\varphi_\ell(t,x)|^{p'} \, dt \, dx \right)^{1/p'}$$

$$= \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) \, dt \, dx \right)^{1/p} \left( \int_{\mathbb{R}^d} \Phi_\ell(x) \, dx \int_{0}^{T} \tilde{\phi}^{\ell-p'}(t) \left| \frac{d}{dt} \tilde{\phi}(t) \right|^{p'} \, dt \right)^{1/p'}$$

$$\lesssim T^{-1 + \frac{d}{p'}} R^{\frac{d}{p'}} \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) \, dt \, dx \right)^{1/p}$$

(83)

where $Q_T = [T/2, T] \times \mathbb{R}^d$. We set $R := K^{\alpha/\beta} T^{\alpha/\beta}$, where $K \geq 1$ is independent of $T$. Using the estimates (76), (78) and (83) into (73) and taking into account the fact that $p = p_*$, we get

$$\int_{Q_T} |u(t,x)|^p \varphi(t,x) \, dt \, dx \lesssim T^{-\alpha} \int_{\mathbb{R}^d} |u_0(x)| \, dx + T^{-(\beta-1)} \int_{\mathbb{R}^d} |u_1(x)| \, dx + R^d T^{1-\beta} \frac{d}{dt} T$$

$$+ T R^{-\frac{d}{p+1}} + T^{-1 + \frac{1}{p'}} R^{\frac{d}{p'}} \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) \, dt \, dx \right)^{1/p}$$

$$= T^{-\alpha} \int_{\mathbb{R}^d} |u_0(x)| \, dx + T^{-(\beta-1)} \int_{\mathbb{R}^d} |u_1(x)| \, dx + K^{\alpha d / \beta} T^{-(\beta-\alpha) \frac{d}{p}}$$

$$+ K^{-1} + C K^{\frac{d}{p}} \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) \, dt \, dx \right)^{1/p}.$$  (84)

On the other hand, using (81) with $T \to \infty$ and taking account of $p = p_*$, we obtain

$$u \in L^p((0, \infty), L^p(\mathbb{R}^d));$$

whereupon,

$$\lim_{T \to \infty} \int_{Q_T} |u(t,x)|^p \varphi(t,x) \, dt \, dx = \lim_{T \to \infty} \left( \int_{Q_T} |u(t,x)|^p \varphi(t,x) \, dt \, dx - \int_{Q_{T/2}} |u(t,x)|^p \varphi(t,x) \, dt \, dx \right) = 0.$$  (85)

Finally, letting $T \to +\infty$ into (81) and using (85) and the fact that $u_0, u_1 \in L^1(\mathbb{R}^d)$, we arrive at

$$\int_{0}^{\infty} \int_{\mathbb{R}^d} |u(t,x)|^p \, dt \, dx \lesssim K^{-1},$$

which leads to a contradiction for $K \gg 1$.

\[ \blacksquare \]

References

[1] A.R. Aftabizadeh, Y.K. Huang, J. Wiener, Bounded solutions for differential equations with reflection of the argument, J. Math. Anal. Appl. 135 (1988), 31-37.

[2] A. Alsaeedi, B. Ahmad, and M. Kirane, A survey of useful inequalities in fractional calculus. Fractional Calculus and Applied Analysis 20 (2017), no. 3, 574-594.

[3] A. Alsaeedi, B. Ahmad, and M. Kirane, Blow-up in delay differential equations: A survey, submitted.
[4] A.A. Andreev, Analogs of classical boundary value problems for a second-order differential equation with deviating argument, Differential Equations, 40 (2004), 1192-1194.

[5] A.B. Antonevich, Boundary value problems with strong nonlocalness for elliptic equations, Izv. Akad. Nauk SSSR Ser. Mat., 53:1 (1989) 3-24.

[6] Bei Hu, Blow-up Theories for Semilinear Parabolic Equations, Lecture Notes in Mathematics, Springer, 2018.

[7] M. Bonforte, and J.L. Vázquez, Quantitative local and global a priori estimates for fractional nonlinear diffusion equations, Adv. Math. 250 (2014), 242-284.

[8] V.M. Borok, and Ya.I. Zhitomirskii, On the Cauchy problem for linear partial differential equations with linearly transformed argument, Dokl. Akad. Nauk SSSR, 200:3 (1971), 515-518.

[9] M.Sh. Burlutskaya, and A.P. Khromov, Fourier method in an initial- boundary value problem for a first-order partial differential equation with involution, Computational Mathematics and Mathematical Physics, 51 (2011) 2102-2114

[10] A. Córdoba, and D. Córdoba, A maximum principle applied to quasi-geostrophic equations. Comm Math Phys. 2004; 249(3):511-528.

[11] G. Di Blasio, K. Kunisch, and E. Sinestrari, $L^2$-regularity for parabolic partial integrodifferential equations with delay in highest-order derivatives, J. Math. Anal. Appl., 102, No. 1 (1984), 38-57.

[12] K.M. Furati, and M. Kirane, Necessary conditions for the existence of global solutions to systems of fractional differential equations, Fract. Calc. Appl. Anal. 11 (2008), 281-298.

[13] M. Guerda, and M. Kirane, Criticality for some evolution equations, Differential Equations 37 (2001) no. 4, 540-550.

[14] C.P. Gupta, Existence and uniqueness theorems for boundary value problems involving reflection of the argument, Nonlinear Anal. 11 (1987), 1075-1083.

[15] C.P. Gupta, Two-point boundary value problems involving reflection of the argument, Internat. J. Math. Math. Sci. 10 (1987) 361-371.

[16] M. Escobedo, M.A. Herrero, Boundedness and Blow Up for a Semilinear Reaction-Diffusion System. J. Differential Equation 89 (1991) 176-202.

[17] N. Ju, The maximum principle and the global attractor for the dissipative 2-D quasi-geostrophic equations, Comm. Pure. Anal. Appl. (2005), 161-181.

[18] M. Kirane, and N. Al-Salti, Inverse problems for a nonlocal wave equation with an involution perturbation, J. Nonlinear Sci. Appl. 9 (2016) 1243-1251.

[19] M. Kirane, Y. Laskri, and N.E. Tatar, Critical exponents of Fujita type for certain evolution equations and systems with spatio-temporal fractional derivatives, Journal of Mathematical Analysis and Applications, vol. 312, no. 2, pp. 488501, 2005.

[20] M. Kirane, and M. Qafasouni, Global Nonexistence for the Cauchy Problem of Some Nonlinear Reaction Diffusion Systems, Journal of Mathematical Analysis and Applications 268, 217-243.

[21] N.S. Landkof, Foundations of Modern Potential Theory, Springer, New York, NY, USA, 1972.

[22] Y. Luchko, Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation, Fract. Calc. Appl. Anal. 15 (2012), no. 1, 141-160.
[23] E. Mitidieri, and S. N. Pohozaev, A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities, Proceedings of the Steklov Institute of Mathematics 2001; 234:1-383.

[24] D. Przeworska-Rolewicz, Equations with Transformed Argument. An Algebraic Approach, Modern Analytic and Computational Methods in Science and Mathematics, Elsevier Scientific Publishing and PWN-Polish Scientific Publishers, Amsterdam and Warsaw, 1973.

[25] P. Quittner, and P. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states. Second edition. Birkhauser Advanced Texts: Basel Textbooks, Birkhauser Verlag, 2019.

[26] O.A. Salieva, On nonexistence of solutions to some nonlinear inequalities with transformed argument, Electron. J. Qual. Theory Differ. Equ. 2017, No. 3, 1-13.

[27] A.L. Skubachevskii, Elliptic Functional Differential Equations and Applications, Birkhauser, Basel-Boston-Berlin, 1997.

[28] S.G. Samko, A.A. Kilbas, and O.I. Marichev, Integrals and Derivatives: Theory and Applications, Gordon and Breach Science, Yverdon, Switzerland, 1993.

[29] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Comm. Pure Appl. Math., 60(1) (2007), 67-112.

[30] E.M. Varfolomeev, On some properties of elliptic and parabolic functional differential operators arising in nonlinear optics, Journal of Mathematical Sciences, Vol. 153, No. 5, 2008.

[31] M.A. Vorontsov, N.G. Iroshnikov, and R. L. Abernathy, Diffractive patterns in a nonlinear optical two-dimensional feedback system with field rotation, Chaos Solitons Fractals, 4 (1994), 1701-1716.

[32] J. Wiener, and A.R. Aftabizadeh, Boundary value problems for differential equations with reflection of the argument, Int. J. Math. Math. Sci. 8 (1985) 151-163.