DENSE LINEABILITY AND ALGEBRABILITY OF $\ell^\infty \setminus c_0$

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Abstract. We show that the set $\ell^\infty \setminus c_0$ is maximal dense-lineable and densely strongly $c$-algebrable answering a question posed by Nestoridis and complementing a result by García-Pacheco, Martín and Seoane-Sepúlveda.

1. Introduction

Let $\ell^\infty$ be the Banach space of real bounded sequences normed with the uniform norm, and $c_0$ be the subspace of $\ell^\infty$ of sequences converging to zero. Since $c_0$ is a closed subspace of $\ell^\infty$, the set $\ell^\infty \setminus c_0$ is a dense $G_\delta$ subset of $\ell^\infty$. Rosenthal [5] showed that $c_0$ is quasi-complemented in $\ell^\infty$ from which it immediately follows that the set $(\ell^\infty \setminus c_0) \cup \{0\}$ contains a closed and infinite dimensional subspace. In the same direction, García-Pacheco, Martín and Seoane-Sepúlveda [3] showed that if we consider $\ell^\infty$ as a Banach algebra, endowed with the coordinatewise product, then the set $(\ell^\infty \setminus c_0) \cup \{0\}$ even contains a closed infinitely generated subalgebra.

In the recent paper [4], Nestoridis proved that if $0 < p < q < \infty$, the set $(\ell^q \setminus \ell^p) \cup \{0\}$ contains a dense linear subspace and posed the following question:

Question 1.1 (Nestoridis). Does the set $(\ell^\infty \setminus c_0) \cup \{0\}$ contain a dense linear subspace?

Nestoridis also pointed out that the fact that $\ell^\infty$ is non-separable could add difficulties in trying to answer the above mentioned question. Indeed, the algebraic dimension of a dense linear subspace of $\ell^\infty$ must be $\mathfrak{c}$ (see Proposition 1.5) which makes it difficult to control that all its non-zero elements lie outside $c_0$. On the other hand, one could argue that the fact that $\ell^\infty$ is non-separable, in contrast to $c_0$ provides enough space in the set $(\ell^\infty \setminus c_0) \cup \{0\}$ to find a dense linear subspace.

We will provide a positive answer to Nestoridis’ question by proving the following more detailed result. Borrowing the terminology from Aron et al. [1] we will call a subset $M$ of a topological vector space $X$ maximal dense-lineable provided $M \cup \{0\}$ contains a dense linear subspace of $X$ of dimension $\dim(X)$.

Theorem 1.2. The set $\ell^\infty \setminus c_0$ is maximal dense-lineable.

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Motivated by the result of García-Pacheco, Martín and Seoane-Sepúlveda, we also considered the question of whether the set \( (\ell^\infty \setminus c_0) \cup \{0\} \) contains a dense subalgebra, where as before, we endow \( \ell^\infty \) with the coordinatewise product. Following Bartoszewicz and Głąb [2] we will call a subset \( M \) of a Banach algebra \( X \) \textit{densely strongly }\( c \)-\textit{algebrable} provided \( M \cup \{0\} \) contains a dense subalgebra of \( X \) generated by an algebraically independent set of cardinality \( c \).

**Theorem 1.3.** The set \( \ell^\infty \setminus c_0 \) is densely strongly \( c \)-algebrable.

Theorems 1.2 and 1.3 immediately yield the following corollary which fills a gap in the literature of lineability and algebrability.

**Corollary 1.4.** If \( 0 < p < \infty \) then the following hold.

(i) The set \( (\ell^\infty \setminus \ell^p) \cup \{0\} \) is maximal dense-lineable.

(ii) If we endow \( \ell^\infty \) with the coordinatewise product, the set \( (\ell^\infty \setminus \ell^p) \cup \{0\} \) is densely strongly \( c \)-algebrable.

The idea behind the proofs of Theorems 1.2 and 1.3 is common. For \( A \subset \mathbb{N} \) we denote by \( \chi_A : \mathbb{N} \rightarrow \mathbb{R} \) the characteristic sequence of \( A \). The set \( \{\chi_A : A \subset \mathbb{N}\} \) generates a dense linear subspace of \( \ell^\infty \). We want to conveniently perturb those characteristic sequences in such a way that the subspace or the subalgebra generated by the modified sequences intersects \( c_0 \) trivially.

Theorem 1.3 is clearly stronger than Theorem 1.2 however, we provide two separate proofs for those results since we feel that the proof of Theorem 1.2 is more intuitive than the proof of Theorem 1.3. In this way, one does not have to go through the more technical proof of Theorem 1.3 if interested in mere dense-lineability.

We also observe that due to the following simple fact, in our setting the maximality part of Theorem 1.2 and the cardinality part of Theorem 1.3 follow for free.

**Proposition 1.5.**

(i) If \( E \) is a dense subspace of \( \ell^\infty \) then \( \dim(E) = c \).

(ii) If \( A \) is a dense subalgebra of \( \ell^\infty \) (endowed with the coordinatewise product) then each set of generators of \( A \) has cardinality \( c \).

**Proof.** To prove (i), we observe that since \( \overline{E} = \ell^\infty \), \( E \) has to be infinite-dimensional. Now, considering linear combinations with rational coefficients of a Hamel basis of \( E \), and taking into account that any dense subset of \( \ell^\infty \) has cardinality \( c \), it follows that \( \dim(E) = c \).

Concerning (ii), let \( G \) be any set generating \( A \). That means that if \( \tilde{G} \) is the set of all elements of the form \( g_{i_1}^{n_1} \cdots g_{i_k}^{n_k} \), where \( k, n_1, \ldots, n_k \in \mathbb{N} \) and \( \{g_{i_1}, \ldots, g_{i_k}\} \) is a finite subset of \( G \), then,

\[
A = \text{span}(\tilde{G}).
\]
Now, $G$ has to be infinite since otherwise, $\dim(A) \leq \aleph_0$ which by (i), contradicts the fact that $A$ is dense in $\ell^\infty$. For the same reason we get that

$$c = \text{card}(\tilde{G}) \leq \text{card}(G)\aleph_0 = \text{card}(G)$$

from which it follows that $\text{card}(G) = c$.

2. Proof of Theorem 1.2

We will need two lemmas, the first is a well known fact that appears in many places in literature, but we state it and prove it here for the sake of self containment.

**Lemma 2.1.** There are $c$ infinite subsets of $\mathbb{N}$ pairwise finitely intersected.

**Proof.** Let $(q_n)_{n=1}^\infty$ be an enumeration of $\mathbb{Q} \cap [0,1]$. For each $r \in [0,1] \setminus \mathbb{Q}$ we may find an increasing sequence of natural numbers $(n_k(r))_{k=1}^\infty$ such that $q_{n_k(r)} \to r$, as $k \to \infty$. The sets $(n_k(r))_{k=1}^\infty$, for $r \in [0,1] \setminus \mathbb{Q}$ have the desired properties. $\square$

The next simple fact provides our main tool in perturbing sequences with finitely many values to sequences that lie outside $c_0$ and will also be used in the proof of Theorem 1.3.

**Lemma 2.2.** Let $(X, \|\cdot\|)$ be a (real or complex) normed space, $a = (a_n)$ a sequence in $X$ with finite range, and $b = (b_n)$ a sequence somewhere dense in $X$. Then the sequence $a + b$ is also somewhere dense in $X$.

**Proof.** Let $\{c_1, \ldots, c_k\}$, for some $k \in \mathbb{N}$ be the range of $a$ and $\{N_i : i = 1, \ldots, k\}$ a partition of $\mathbb{N}$ such that

$$a_n = c_i, \quad \text{for} \quad n \in N_i.$$

Now, since

$$\{b_n : n \in \mathbb{N}\} = \bigcup_{i=1}^k \{b_n : n \in N_i\}$$

there is $j \in \{1, \ldots, k\}$ such that

$$\{b_n : n \in N_j\}$$

is somewhere dense. Since

$$\{a_n + b_n : n \in N_j\} = c_j + \{b_n : n \in N_j\}$$

the latter set is a somewhere dense subset in $X$. $\square$

We may now proceed with the proof of Theorem 1.2.

**Proof of Theorem 1.2.** If $A \subset \mathbb{N}$, we want to associate to $\chi_A$ a sequence of suitable elements from $\ell^\infty \setminus c_0$ converging to $\chi_A$. To this end, we let by Lemma 2.1

$$\{(a_{(A,j)}(k))_{k=1}^\infty : (A,j) \in \mathcal{P}(\mathbb{N}) \times \mathbb{N}\}$$

be a family of infinite, pairwise finitely intersected subsets of $\mathbb{N}$, indexed by $\mathcal{P}(\mathbb{N}) \times \mathbb{N}$ and enumerated as increasing sequences. Note that we are using the fact that
the cardinality of the set $\mathcal{P}(\mathbb{N}) \times \mathbb{N}$ is $c$. Let also $x = (x_n)_{n=1}^{\infty} \in \ell^\infty$ be a sequence which is somewhere dense in $\mathbb{R}$ (an enumeration of the rationals of $[0,1]$ would do the job). For $A \subset \mathbb{N}$ and $j \in \mathbb{N}$, we define the following elements of $\ell^\infty$,

$$f_{(A,j)} = \chi_A + \frac{1}{j} \sum_{k=1}^{\infty} x_k \chi_{\{a_{(A,j)}(k)\}}.$$ 

Clearly,

$$f_{(A,j)} \to \chi_A \text{ as } j \to \infty$$

hence, the set

$$Y = \text{span}\{f_{(A,j)} : A \subset \mathbb{N}, j \in \mathbb{N}\}$$

is dense in $\ell^\infty$. We want to show that

$$Y \cap c_0 = \{0\}.$$ 

For $p \geq 1$, let $c_1, \ldots, c_p \in \mathbb{R}$ be such that $c_i \neq 0$, for some $i \in \{1, \ldots, p\}$ and $(A_1, j_1), \ldots, (A_p, j_p)$ be pairwise distinct elements of $\mathcal{P}(\mathbb{N}) \times \mathbb{N}$. We consider $k_0 \in \mathbb{N}$ satisfying that

$$\{a_{(A_i,j_i)}(k) : k \geq k_0\} \cap \{a_{(A_l,j_l)}(k) : k \geq k_0\} = \emptyset$$

for all $l \neq i$, with $l \in \{1, \ldots p\}$. For $k \geq k_0$, we then have that

$$c_1 f_{(A_1,j_1)}(a_{(A_i,j_i)}(k)) + \cdots + c_p f_{(A_p,j_p)}(a_{(A_i,j_i)}(k)) =$$

$$c_1 \chi_{A_1}(a_{(A_i,j_i)}(k)) + \cdots + c_p \chi_{A_p}(a_{(A_i,j_i)}(k)) + \frac{1}{j_i} c_i x_k.$$ 

This, together with Lemma 2.2 give that the sequence

$$(c_1 f_{(A_1,j_1)}(a_{(A_i,j_i)}(k)) + \cdots + c_p f_{(A_p,j_p)}(a_{(A_i,j_i)}(k)))_{k=1}^{\infty}$$

is somewhere dense in $\mathbb{R}$. In particular,

$$c_1 f_{(A_1,j_1)} + \cdots + c_p f_{(A_p,j_p)} \notin c_0$$

which shows that $\ell^\infty \setminus c_0$ contains a dense linear subspace. An application of Proposition 1.5 (i) concludes the proof. □

3. Proof of Theorem 1.3

The way we perturbed the characteristic sequences in the proof of Theorem 1.2 does not work for proving Theorem 1.3. The problem is that the product of two sequences supported on finitely intersected subsets of $\mathbb{N}$ will be a finitely supported sequence thus it will belong to $c_0$. Obviously, we cannot hope on separating totally an uncountable family of subsets of $\mathbb{N}$. Thus, we establish the following result that allows us to overcome this obstacle. We thank the anonymous referee for indicating to us this short, topological argument.
Lemma 3.1. There is a family of cardinality $c$

$$\{f_\gamma\}_{\gamma \in [0,1]^\mathbb{N}}$$

such that for each finite subset $\{\gamma_1, \ldots, \gamma_k\} \subset c$, where $k \in \mathbb{N}$, the set

$$\{(f_{\gamma_1}(n), \ldots, f_{\gamma_k}(n)) : n \in \mathbb{N}\}$$

is dense in $[0,1]^k$.

Proof. The proof is an immediate consequence of the fact that the Tychonoff cube $[0,1]^c$ endowed with the product topology, is separable (see for instance [6, Theorem 16.4 (c)]). Indeed, let

$$\{x_n : n \in \mathbb{N}\}$$

be a dense subset of $[0,1]^c$. For each $\gamma < c$, we define the element $f_\gamma \in [0,1]^\mathbb{N}$ by

$$f_\gamma(n) = x_n(\gamma), \quad n \in \mathbb{N}.$$ 

The fact that the family $(f_\gamma)_{\gamma < c}$ satisfies the claim follows by the definition of the product topology. 

Proof of Theorem 1.3. By Lemma 3.1 and the fact that the cardinality of $\mathcal{P}(\mathbb{N}) \times \mathbb{N}$ is $c$, for $A \subset \mathbb{N}$ and $j \in \mathbb{N}$, we get sequences

$$f_{(A,j)} \in \ell^\infty$$

with $\|f_{(A,j)}\| \leq \frac{1}{j}$ and satisfying that if

$$\{(A_1, j_1), \ldots, (A_k, j_k)\}$$

with $k \in \mathbb{N}$, is a finite subset of $\mathcal{P}(\mathbb{N}) \times \mathbb{N}$, then

$$\{(f_{(A_1,j_1)}(n), \ldots, f_{(A_k,j_k)}(n)) : n \in \mathbb{N}\}$$

is somewhere dense in $\mathbb{R}^k$. Now, let us set

$$h_{(A,j)} = \chi_A + f_{(A,j)}, \quad (A, j) \in \mathcal{P}(\mathbb{N}) \times \mathbb{N}.$$ 

Clearly,

$$h_{(A,j)} \to \chi_A \quad \text{as} \quad j \to \infty$$

hence, the set

$$\{h_{(A,j)} : A \subset \mathbb{N}, j \in \mathbb{N}\}$$

spans a dense subspace of $\ell^\infty$. We will show that if $B$ is the algebra generated by

$$\{h_{(A,j)} : A \subset \mathbb{N}, j \in \mathbb{N}\},$$

then

$$B \cap c_0 = \{0\}.$$ 

Let for $k \in \mathbb{N}$, $P \in \mathbb{R}[x_1, \ldots, x_k]$ be a non-zero polynomial vanishing at the origin. The zero set of $P$ is nowhere dense in $\mathbb{R}^k$. If $(A_1, j_1), \ldots, (A_k, j_k)$ are distinct elements of $\mathcal{P}(\mathbb{N}) \times \mathbb{N}$, since the range of the sequence

$$(\chi_{A_1}(n), \ldots, \chi_{A_k}(n))_{n=1}^\infty$$
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is finite, Lemma 2.2 ensures that 
\[ \{(h_{(A_1,j_1)}(n_1), \ldots, h_{(A_k,j_k)}(n_k)) : n \in \mathbb{N}\} \]
is somewhere dense in \( \mathbb{R}^k \). Hence, there is \( \delta > 0 \) and a subsequence \( \langle n_l \rangle \) of natural numbers, such that
\[ |P(h_{(A_1,j_1)}(n_l), \ldots, h_{(A_k,j_k)}(n_l))| \geq \delta. \]
In particular, since the underlying product is the coordinatewise, we get that
\[ |P(h_{(A_1,j_1)}(n_l), \ldots, h_{(A_k,j_k)}(n_l))| = |P(h_{(A_1,j_1)}(n_l), \ldots, h_{(A_k,j_k)}(n_l))| \geq \delta \]
which yields that
\[ P(h_{(A_1,j_1)}(n_l), \ldots, h_{(A_k,j_k)}(n_l)) \notin c_0 \]
hence, \( B \cap c_0 = \{0\} \) and the family \( \{h_{(A,j)} : A \subset \mathbb{N}, j \in \mathbb{N}\} \) is algebraically independent. Now, either by noticing that
\[ \text{card}\{h_{(A,j)} : A \subset \mathbb{N}, j \in \mathbb{N}\} = c \]
or by an immediate application of Proposition 1.5 (ii) the proof is complete. \( \square \)

Remark 3.2. The proofs of Theorem 1.2 and Theorem 1.3 can be easily modified to provide the same results in the complex case.

Remark 3.3. The proofs of Theorem 1.2 and Theorem 1.3 actually show that the set
\[ \ell^\infty \setminus c \]
is maximal dense-lineable and densely strongly \( c \)-algebrable, where \( c \) is the space of real convergent sequences. For the analogue of Theorem 1.3 one would need to use the fact that the image under any non-constant polynomial of several variables of an open set has non-empty interior.

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