On the expected diameter of planar Brownian motion

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Abstract

Known results show that the diameter $d_1$ of the trace of planar Brownian motion run for unit time satisfies $1.595 \leq E d_1 \leq 2.507$. This note improves these bounds to $1.601 \leq E d_1 \leq 2.355$. Simulations suggest that $E d_1 \approx 1.99$.

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Let $(b_t, t \in [0, 1])$ be standard planar Brownian motion, and consider the set $b[0, 1] = \{b_t : t \in [0, 1]\}$. The Brownian convex hull $H_1 := \text{hull } b[0, 1]$ has been well-studied from Lévy \cite{Levy} §52.6, pp. 254–256 onwards; the expectations of the perimeter length $\ell_1$ and area $a_1$ of $H_1$ are given by the exact formulae $E \ell_1 = \sqrt{8\pi}$ (due to Letac and Táňacs \cite{Tanacs}) and $E a_1 = \pi/2$ (due to El Bachir \cite{Elbachir}).

Another characteristic is the diameter $d_1 := \text{diam } H_1 = \text{diam } b[0, 1] = \sup_{x,y \in [0,1]} \|x - y\|$, for which, in contrast, no explicit formula is known. The exact formulae for $E \ell_1$ and $E a_1$ rest on geometric integral formulae of Cauchy; since no such formula is available for $d_1$, it may not be possible to obtain an explicit formula for $E d_1$. However, one may get bounds.

By convexity, we have the almost-sure inequalities $2 \leq \ell_1/d_1 \leq \pi$, the extrema being the line segment and shapes of constant width (such as the disc). In other words,

$$\frac{\ell_1}{\pi} \leq d_1 \leq \frac{\ell_1}{2}.$$

The formula of Letac and Takács \cite{Tanacs} says that $E \ell_1 = \sqrt{8\pi}$, so we get:

**Proposition 1.** $\sqrt{8/\pi} \leq E d_1 \leq \sqrt{2\pi}$.

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Note that \( \sqrt{8/\pi} \approx 1.5958 \) and \( \sqrt{2\pi} \approx 2.5066 \). In this note we improve both of these bounds.

For the lower bound, we note that \( b[0,1] \) is compact and thus, as a corollary of Lemma 3 below, we have the formula

\[
d_1 = \sup_{0 \leq \theta \leq \pi} r(\theta),
\]

where \( r \) is the parametrized range function given by

\[
r(\theta) = \sup_{0 \leq s \leq 1} (b_s \cdot e_\theta) - \inf_{0 \leq s \leq 1} (b_s \cdot e_\theta),
\]

with \( e_\theta \) being the unit vector \((\cos \theta, \sin \theta)\). Feller [2] established that

\[
E r(\theta) = \sqrt{8/\pi} \quad \text{and} \quad E(r(\theta)^2) = 4\log 2,
\]

and the density of \( r(\theta) \) is given explicitly as

\[
f(r) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \exp\{-k^2 r^2/2\}, \quad (r \geq 0).
\]

Combining (1) with (2) gives immediately \( E d_1 \geq E r(0) = \sqrt{8/\pi} \), which is just the lower bound in Proposition 1. For a better result, a consequence of (1) is that \( d_1 \geq \max\{r(0), r(\pi/2)\} \). Observing that \( r(0) \) and \( r(\pi/2) \) are independent, we get:

**Lemma 2.** \( E d_1 \geq E \max\{X_1, X_2\} \), where \( X_1 \) and \( X_2 \) are independent copies of \( X := r(0) \).

It seems hard to explicitly compute \( E \max\{X_1, X_2\} \) in Lemma 2, because although the density given at (3) is known explicitly, it is not very tractable. Instead we obtain a lower bound. Since

\[
\max\{x, y\} = \frac{1}{2} (x + y + |x - y|)
\]

we get

\[
E \max\{X_1, X_2\} = EX + \frac{1}{2} E|X_1 - X_2|.
\]

Thus with Lemma 2, the lower bound in Proposition 1 is improved given any non-trivial lower bound for \( E|X_1 - X_2| \). Using the fact that for any \( c \in \mathbb{R} \), if \( m \) is a median of \( X \), \( E|X - c| \geq E|X - m| \), we see that

\[
E|X_1 - X_2| \geq E|X - m|.
\]

Again, the intractability of the density at (3) makes it hard to exploit this. Instead, we provide the following as a crude lower bound on \( E|X_1 - X_2| \).

**Lemma 3.** For any \( a, h > 0 \),

\[
E|X_1 - X_2| \geq 2h \mathbb{P}(X \leq a) \mathbb{P}(X \geq a + h).
\]

**Proof.** We have

\[
E|X_1 - X_2| \geq E [|X_1 - X_2| \mathbb{1}\{X_1 \leq a, X_2 \geq a + h\}]
\]

\[
+ E [|X_1 - X_2| \mathbb{1}\{X_2 \leq a, X_1 \geq a + h\}]
\]

\[
\geq h \mathbb{P}(X_1 \leq a) \mathbb{P}(X_2 \geq a + h) + h \mathbb{P}(X_2 \leq a) \mathbb{P}(X_1 \geq a + h)
\]

\[
= 2h \mathbb{P}(X \leq a) \mathbb{P}(X \geq a + h),
\]

which proves the statement. \( \square \)
This lower bound yields the following result.

**Proposition 4.** For $a, h > 0$ define

$$g(a, h) := h\left(\frac{4}{\pi} \exp\left\{ -\frac{\pi^2}{2a^2} \right\} - \frac{4}{3\pi} \exp\left\{ -\frac{9\pi^2}{2a^2} \right\} \right) \left(1 - \frac{4}{\pi} \exp\left\{ -\frac{\pi^2}{8(a + h)^2} \right\} \right).$$

Then $E d_1 \geq \sqrt{8/\pi} + g(1.492, 0.337) \approx 1.6014$.

**Proof.** Consider

$$Z := \sup_{0 \leq s \leq 1} |b_s \cdot e_0|.$$

Then it is known (see [3]) that for $x > 0$,

$$\frac{4}{\pi} \exp\left\{ -\frac{\pi^2}{8x^2} \right\} - \frac{4}{3\pi} \exp\left\{ -\frac{9\pi^2}{8x^2} \right\} \leq \mathbb{P}(Z < x) \leq \frac{4}{\pi} \exp\left\{ -\frac{\pi^2}{8x^2} \right\}. \quad (5)$$

Moreover, we have

$$Z \leq X \leq 2Z.$$

Since $X \leq 2Z$, we have

$$\mathbb{P}(X \leq a) \geq \mathbb{P}(Z \leq a/2) \geq \frac{4}{\pi} \exp\left\{ -\frac{\pi^2}{2a^2} \right\} - \frac{4}{3\pi} \exp\left\{ -\frac{9\pi^2}{2a^2} \right\},$$

by the lower bound in (5). On the other hand,

$$\mathbb{P}(X \geq a + h) \geq \mathbb{P}(Z \geq a + h) \geq 1 - \frac{4}{\pi} \exp\left\{ -\frac{\pi^2}{8(a + h)^2} \right\},$$

by the upper bound in (5). Combining these two bounds and applying Lemma 3 we get $E|X_1 - X_2| \geq 2g(a, h)$. So from (1) and the fact that $EX = \sqrt{8/\pi}$ by (2) we get $Ed_1 \geq \sqrt{8/\pi} + g(a, h)$. Numerical evaluation using MAPLE suggests that $(a, h) = (1.492, 0.337)$ is close to optimal, and this choice gives the statement in the proposition.

We also improve the upper bound in Proposition 1.

**Proposition 5.** $Ed_1 \leq \sqrt{8 \log 2} \approx 2.3548$.

**Proof.** First, we claim that

$$d_1^2 \leq r(0)^2 + r(\pi/2)^2. \quad (6)$$

It follows from (4) and (2) that

$$E(d_1^2) \leq E(X_1^2 + X_2^2) = 2E(X^2) = 8 \log 2.$$

The result now follows by Jensen’s inequality.

It remains to prove the claim (5). Note that the diameter is an increasing function, that is, if $A \subseteq B$ then $\text{diam} A \leq \text{diam} B$. Note also, that by the definition of $r(\theta)$, $b[0, 1] \subseteq z + [0, r(0)] \times [0, r(\pi/2)] =: R_z$ for some $z \in \mathbb{R}^2$. Since the diameter of the set $R_z$ is attained at the diagonal,

$$\text{diam } R_z = \sqrt{r(0)^2 + r(\pi/2)^2},$$

for all $z \in \mathbb{R}^2$, and we have $\text{diam } b[0, 1] \leq \text{diam } R_z$, the result follows. \qed
We make one further remark about second moments. In the proof of Proposition 5, we saw that \( \mathbb{E}(d_1^2) \leq 8 \log 2 \approx 5.5452 \). A bound in the other direction can be obtained from the fact that \( d_1^2 \geq \ell_1^2/\pi^2 \), and we have (see \cite{7} §4.1) that

\[
\mathbb{E}(\ell_1^2) = 4\pi \int_{-\pi/2}^{\pi/2} d\theta \int_0^\infty \cos \theta \frac{\cosh(u\theta)}{\sinh(u\pi/2)} \tanh \left( \frac{(2\theta + \pi)u}{4} \right) \approx 26.1677,
\]

which gives \( \mathbb{E}(d_1^2) \geq 2.651 \).

Finally, for completeness, we state and prove the lemma which was used to obtain equation (1).

**Lemma 6.** Let \( A \subset \mathbb{R}^d \) be a nonempty compact set, and let \( r_A(\theta) = \sup_{x \in A} (x \cdot e_\theta) - \inf_{x \in A} (x \cdot e_\theta) \). Then

\[
\text{diam } A = \sup_{0 \leq \theta \leq \pi} r_A(\theta).
\]

**Proof.** Since \( A \) is compact, for each \( \theta \) there exist \( x, y \in A \) such that

\[
r_A(\theta) = x \cdot e_\theta - y \cdot e_\theta = (x - y) \cdot e_\theta \leq \|x - y\|.
\]

So \( \sup_{0 \leq \theta \leq \pi} r_A(\theta) \leq \sup_{x,y \in A} \|x - y\| = \text{diam } A \).

It remains to show that \( \sup_{0 \leq \theta \leq \pi} r_A(\theta) \geq \text{diam } A \). This is clearly true if \( A \) consists of a single point, so suppose that \( A \) contains at least two points. Suppose that the diameter of \( A \) is achieved by \( x, y \in A \) and let \( z = y - x \) be such that \( \hat{z} := z/\|z\| = e_{\theta_0} \) for \( \theta_0 \in [0, \pi] \). Then

\[
\sup_{0 \leq \theta \leq \pi} r_A(\theta) \geq r_A(\theta_0) \geq y \cdot e_{\theta_0} - x \cdot e_{\theta_0} = z \cdot \hat{z} = \|z\| = \text{diam } A,
\]

as required. \( \square \)

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