ETA-INVARIENTS AND VON NEUMANN ALGEBRAS

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1. Main theorem. Let $M$ be a compact oriented Riemannian manifold of dimension $4k-1$. The operator $D = *d + d*$ acting on the $2k-1$ forms on $M$ is selfadjoint, and for any Hermitian vector bundle $E \to M$ with connection $\nabla$, there is a selfadjoint operator $D \otimes \nabla$ acting on the smooth sections of $\Lambda^{2k-1}(T^*M) \otimes E$. The data $(D, E)$ defines a class $[D, E]$ in the odd analytic $K$-homology group $K^a_1(M)$. We develop in this work an equality between two methods of pairing $[D, E]$ with real-valued $K^1$-cohomology classes.

Let $\Gamma = \pi_1(M)$ and $A : \Gamma \to U_N$ be a representation, which determines a flat principal $U_N$-bundle $P_A \to M$ and an associated flat $C^N$-vector bundle $E_A \to M$. If there exists a bundle trivialization $\theta : P_A \cong M \times U_A$, then it is well known that the pair $(A, \theta)$ represents a class $\overline{A}$ in $K^1(M) \otimes \mathbb{R}$, ($\S 5$, [APS 2]). The first pairing of $[D, E]$ and $\overline{A}$ uses the relative eta-invariant to define the flat bundle index, as in ($\S 5$, ibid). Let $\nabla^0$ be the flat connection on $M \times U_N$ associated to the product structure, and let $\nabla^1$ be the flat connection associated with the push-forward under $\theta$ of the flat connection on $P_A$. Define a smooth, one-parameter family of selfadjoint operators on smooth sections of $\Lambda^{2k-1}(T^*M) \otimes E \otimes C^N$ by $D_t = t \cdot D \otimes \nabla \otimes \nabla^1 + (1 - t) \cdot D \otimes \nabla \otimes \nabla^0$. The eta-invariant $\eta(D_t)$ is smooth as a function of $t$ except for a finite number of bounded jump discontinuities, so there exists a well-defined continuous derivative function $\eta(D_t)'$ [APS 1,2]. Define

$$\langle [D, E], \overline{A} \rangle_\eta = \int_0^1 \eta(D_t)' \, dt.$$  

The second pairing uses the von Neumann algebra $W^*(\Gamma)$ associated to the universal covering $\tilde{M}$ of $M$ [A 1]. The lift to $\tilde{M}$, $D \otimes \nabla$, of the operator $D \otimes \nabla$ is essentially selfadjoint, and we introduce the projection, $P^+$, from $H = L^2(\tilde{M}, \Lambda^{2k-1}(T^*M) \otimes E \otimes C^N)$ onto the positive space $H^+$ of $D \otimes \nabla$. The operator $P^+$ is $\Gamma$-invariant, so defines an element $[P^+]$ in $K^1(W^*(\Gamma))$. The data $(A, \theta)$ defines a map $u : \tilde{M} \to U_N$ by restricting the composition $p_2 \circ \theta : P_A \to M \times U_N \to U_N$ to the leaf of $F_A$ on $P_A$ through a basepoint, where $F_A$ is the $U_N$-invariant foliation of $P_A$ associated to the flat structure. We pair $[P^+]$ to $[u] \in K^1_{(\infty)}(M)$ by constructing an associated Toeplitz operator, which is Fredholm in the sense of Breuer, and has a von Neumann index which is the continuous dimension of the
spectral flow between $P^+$ and the conjugate $u^{-1} \cdot P^+ \cdot u$. More precisely, multiplication by $u$ on the coefficients $C^N$ defines a bounded operator $M_u$ on $H$, and the compression

$$T(u) = P^+ \circ M_u : H^+ \to H^+$$

is $\Gamma$-Fredholm and $\Gamma$-invariant, so as in [A 1] has a $\Gamma$-index. Define

$$\langle [D, E], \overline{A} \rangle_\Gamma = \text{Ind}_\Gamma(T(u)).$$

The $\Gamma$-index for a fixed flat bundle $P_A$ has an alternate description in terms of the type II von Neumann algebra $W^*(F_A)$ associated to the foliation $F_A$ by Connes [C 1], with trace derived from the Haar measure on $U_N$. The operator $T(u)$ is one of a $U_N$-parametrized family of leafwise Toeplitz operators along $F_A$, giving index data which is analogous to the hull-completion formulation of the index of almost periodic operators [CDSS].

Our main result is

**Theorem.**

$$\langle [D, E], \overline{A} \rangle_\eta = \langle [D, E], \overline{A} \rangle_\Gamma.$$  

Moreover, both sides of (3) are equal to a renormalized discrete spectral flow for the lift of $D \otimes \nabla$ to a nonelliptic, pure-point-spectrum $U_N$-invariant operator on $P_A$.

The theorem is true also for any “geometric operator” on $M$ which is obtained by coupling the Spinor Dirac operator on $M$ to a coefficient vector bundle $E$, so that the theorem applies equally well, for example, to Spin manifolds of odd dimension.

The possibility of relating the eta-invariant for a family $D_t$ to a von Neumann dimension as given by (3) above was suggested in (Remark 4, p. 89, [APS 2]).

2. **Method of proof.** There are three key points to the proof.

(2.1) The invariant $\int_0^1 \eta(D_t) \, dt$ has a multiplicative property: For $E_2 \to M$ a flat vector bundle with fiber dimension $q$ and flat connection $\nabla^2$,

$$\int_0^1 \eta(D_t \otimes \nabla^2) \, dt = q \cdot \int_0^1 \eta(D_t) \, dt.$$

We take for $E_2$ the infinite-dimensional flat bundle associated to the composition of $A$ with the left regular representation of $U_N$ on $L^2(U_N)$. The Peter-Weyl Theorem decomposes this infinite bundle into an infinite direct sum of finite-dimensional bundles associated to the characters of irreducible representations of $U_N$. The equality (4) is interpreted in a renormalized sense by using the heat kernel, $H_s$, on $U_N$. We define a central eta-distribution on $U_N$ by associating to a class function the weighted sum of the eta-invariants for the operators on $M$ obtained by coupling $D \otimes \nabla$ to the finite-dimensional flat bundles associated to the characteristic subspaces of $L^2(U_N)$ in the Peter-Weyl decomposition. Then (4) becomes

$$\int_0^1 \eta(D_t; H_s) \, dt = H_s(e) \cdot \int_0^1 \eta(D_t) \, dt.$$
(2.2) The left-hand side of (5) can be written as the sum of two terms: a spectral flow summand, plus the difference of two values of the eta-distribution, one associated to \((D \otimes \nabla) \circ H_t\), and the other to \((D \otimes \nabla) \circ (u^{-1} H_t u)\) where now \(u\) is the operator on \(L^2(U_N) \otimes C^N\) given by multiplication \(u f(g) = g \cdot f(g)\) on the coefficients. Theorem 0.1 of [CG] implies that after renormalizing, the values of the eta-distribution on these two asymptotic functions agree, so that

\[
\lim_{s \to 0} H_s(e)^{-1} \int_0^1 \eta(D_t; H_s)\, dt = \text{Ind}_{\text{n}}([D, E], \overline{A})
\]

where \(\text{Ind}_{\text{n}}([D, E], \overline{A})\) is the renormalized spectral flow of the eta-distribution for the family \(D_t\) of operators obtained by lifting \(D_t \otimes \nabla\) to \(P_A\) via the leaves of \(F_A\) which cover \(M\). This renormalized spectral flow is also the index, in a suitably renormalized sense, for a Toeplitz operator associated to the p.p.s. operator \(D_0\) and the multiplier \(M_u\). We call this a renormalized transverse index, as it is based on the construction of the odd \(K_1\)-class for a transversally elliptic operator for the group \(U_N\)-action on \(P_A\), in analogy with the even transversal index theory of [A 2, S 1, 2].

(2.3) The third point of the proof is to use a Fubini principle, applied to the trace of kernels for operators on \(P_A\), to show that the renormalized index \(\text{Ind}_{\text{n}}([D, E], \overline{A})\) is equal to the index of the leafwise Toeplitz operator on \(F_A\) described in §1. The Weyl asymptotic theorem provides the final step, establishing that the renormalized trace converges to the foliation algebra trace formed from Haar measure on \(U_N\).

3. Final remarks. The main theorem is part of the authors' study of analytic invariants associated to selfadjoint operators on a manifold which are regularized by a foliation on the manifold. This includes both leafwise and transversally elliptic operators for foliations, and we conclude with a discussion of some aspects of this program.

The details of the proof of the main theorem are given in [DHK 3], where we also define for much more general classes of operators than \(D = *d + d^*\), two types of cyclic cocycles, based on viewing the operator \(D \otimes \nabla\) as either leafwise for \(F_A\), or transverse for the action of \(U_N\). A topological index theorem can be derived for the first, while renormalization of the second expresses the relative eta-invariant for \(D = *d + d^*\). A Fubini principle for transverse foliations establishes equality of the first cocycle with the renormalization of the second, and as a corollary yields the index theorem for flat vector bundles of [APS 2]. This procedure can be applied more generally to yield topological formulas for higher order regularized spectral invariants (cf. [D]).

The interpretation of the main theorem applied to \(U_N = U_1\) via quasiperiodic functions was given in [DHK 2], where the main theorem was compared to the results of [CDSS] and a proof was given via Fourier analysis on \(U_1\).

The index theorem for leafwise selfadjoint elliptic operators can be interpreted via Toeplitz extensions of the foliation \(C^*\)-algebra by the algebra
of functions on the manifold. In [DHK 4] we identify the analytic index class as an extension in the spirit of the Brown-Douglas-Fillmore theory.

The odd index theorem for coverings is discussed in [H 1], where we give an alternate proof of the main theorem in terms of the eta-invariant for coverings based on [CG]. The spectral properties of leafwise elliptic operators is studied in [H 2]; it is expected that the spectral flow interpretation of the eta will extend to all leafwise eta-invariants.

A preliminary report on this work appeared in [DHK 1], where the main theorem was announced with the additional hypothesis that the fundamental group of $M$ be amenable. The outline of our program and the result for the amenable case was discussed in a plenary address at the conference *Operator Algebras and Geometry* at the Mathematical Sciences Research Institute, Berkeley, June 1985.

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