Regularized Stokes Immersed Boundary Problems in Two Dimensions: Well-Posedness, Singular Limit, and Error Estimates

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Abstract

Inspired by the numerical immersed boundary method, we introduce regularized Stokes immersed boundary problems in two dimensions to describe regularized motion of a 1-D closed elastic string in a 2-D Stokes flow, in which a regularized δ-function is used to mollify the flow field and singular forcing. We establish global well-posedness of the regularized problems and prove that as the regularization parameter diminishes, string dynamics in the regularized problems converge to that in the Stokes immersed boundary problem with no regularization. Viewing the unregularized problem as a benchmark, we derive error estimates under various norms for the string dynamics. Our rigorous analysis shows that the regularized problems achieve improved accuracy if the regularized δ-function is suitably chosen. This may imply potential improvement in the numerical method, which is worth further investigation. © 2020 Wiley Periodicals LLC

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1 Introduction

The immersed boundary problem models elastic structures moving and interacting with a surrounding fluid: the structures apply elastic force to the fluid and alter the flow field, while in turn the flow moves and deforms the structures [15][26].
Mathematically, it features hydrodynamics equations with time-varying forcing supported on possibly lower-dimensional moving objects, whose motion is governed by the flow. The numerical method of solving the immersed boundary problem, known as the immersed boundary method \[24–26\], has proven to be a powerful computational tool to study such coupled motion in physics, biology, and medical sciences \[3, 9, 14, 19, 20, 44\].

Inspired by the numerical method, in this paper, we propose a regularized version of the two-dimensional Stokes immersed boundary problem in the continuous setting \[15, 39\]. It is a PDE system describing motion of a 1-D closed elastic string immersed in Stokes flow in \(\mathbb{R}^2\), yet with forcing and string motion being mollified on a small spatial scale, which formally approximates the Stokes immersed boundary problem without regularization. Its precise formulation will be provided in Section 1.2. We shall study global well-posedness of the regularized problem and rigorously justify its convergence to the unregularized problem in the string dynamics as the regularization parameter diminishes. Viewing their difference as error, we show that the error bounds depend not only on the regularization parameter and regularity of the solution, but also on the mollifier in a crucial way. This work can be an important step towards the error estimates for the numerical immersed boundary method in a fully discrete case. Beyond that, it also suggests possible improvements of the numerical method, which is worth further investigation.

1.1 The 2-D Stokes Immersed Boundary Problem

Let us first introduce the two-dimensional Stokes immersed boundary problem without regularization, which we shall call the unregularized or the original problem in the rest of the paper. It describes a 1-D closed elastic string moving in 2-D Stokes flow \[15\]. We parametrize the moving string by \(X(s, t)\), where \(s \in \mathbb{T} \triangleq \mathbb{R} \setminus 2\pi\mathbb{Z} = [-\pi, \pi)\) is the Lagrangian coordinate and where \(t\) is the time variable. Note that \(s\) is not the arclength parameter. Then the unregularized problem is formally given by

\[
\begin{align*}
-\Delta u + \nabla p &= f(x, t), \\
\text{div} u &= 0, \quad |u|, |p| \to 0 \text{ as } |x| \to \infty, \\
f(x, t) &= \int_{\mathbb{T}} F_X(s, t) \delta(x - X(s, t)) \, ds, \\
\frac{\partial X}{\partial t}(s, t) &= u(X(s, t), t) \triangleq U_X(s, t), \quad X(s, 0) = X_0(s).
\end{align*}
\]

Equations (1.1) and (1.2) describe the Stokes flow in \(\mathbb{R}^2\) with decay condition at infinity: \(x\) is the spatial (Eulerian) coordinate, \(u\) represents the divergence-free velocity field, and \(p\) is the pressure. Here we implicitly assumed that the fluid viscosity is normalized; indeed, we can always achieve this by properly redefining \(u\), \(p\), and \(t\). In physics, the stationary Stokes equation is suitable for describing the
fluid motion that is far from being turbulent; i.e., the Reynolds number is close to 0, which is the case when spatial scale of the fluid motion is small, or when the fluid moves slowly, or when the fluid is highly viscous. $f$ denotes the elastic force exerted on the fluid, defined by (1.3). $F_X$ is the elastic force density in the Lagrangian coordinate associated with the string configuration $X$. In general, it is given by [26]

$$F_X(s, t) = \partial_s \left( T(|X'(s, t)|, s, t) \frac{X'(s, t)}{|X'(s, t)|} \right),$$

where $X'(s, t)$ denotes $\partial_s X(s, t)$. $T$ is the tension in the string. In the simple case of Hookean elasticity, for instance, $T(|X'(s, t)|, s, t) = k_0 |X'(s, t)|$, with $k_0 > 0$ being the Hooke’s constant, and thus $F_X(s, t) = k_0 X_{ss}(s, t)$. We may assume $k_0 = 1$ by properly redefining $u$, $p$, and $t$. In (1.3), we formally use the Dirac $\delta$-measure to bridge the Lagrangian and Eulerian coordinates. This implies that $f$ is a singular force only supported on the moving string. Finally, (1.4) specifies the initial string configuration and requires that the string to move with flow, where $U_X$ denotes the string velocity. The equations (1.1)–(1.4) readily admit an autonomous dynamics and there is no need to specify initial flow field, since $u$ is instantaneously determined by $f$.

The 2-D Stokes immersed boundary problem has received increasing attention recently from the analysis community. In recent works by Lin and the author [15, 39], we study its well-posedness with $F_X = X_{ss}$. By virtue of the stationary Stokes equation, $u$, $p$, and $U_X$ are completely determined by the present string configuration $X$. Thus, the system (1.1)–(1.4) can be reformulated into a contour dynamic equation,

$$\partial_t X(s, t) = U_X(s, t) = \frac{1}{4\pi} \text{p.v.} \int_T -\partial_s [G(X(s, t) - X(s', t))]X'(s', t)ds',
$$

with $X(s, 0) = X_0(s)$, where

$$G(x) = \frac{1}{4\pi} \left( -\ln |x| \text{Id} + \frac{x \otimes x}{|x|^2} \right)$$

is the fundamental solution of the velocity field for 2-D stationary Stokes equation [28]. Here $\text{Id}$ denotes the $2 \times 2$ identity matrix. Once (1.5) is solved, $u$ and $p$ can be recovered by (1.1)-(1.3).

We prove local well-posedness of (1.5) by utilizing its intrinsic dissipation.

**Proposition 1.1** ([15, 39]). Suppose $X_0(s) \in H^{5/2}(T)$, and assume there exists $\lambda > 0$ such that

$$|X_0(s_1) - X_0(s_2)| \geq \lambda |s_1 - s_2| \quad \forall s_1, s_2 \in T.$$

There exists $T_0 = T_0(\lambda, \|X_0\|_{H^{5/2}(T)}) \in (0, +\infty]$ and a unique solution $X(s, t) \in C_{[0, T_0]} H^{5/2} \cap L^2_{T_0} H^3(T)$ of (1.5) satisfying that

$$\|X\|_{C_{[0, T_0]} H^{5/2} \cap L^2_{T_0} H^3(T)} \leq 4 \|X_0\|_{H^{5/2}(T)}, \quad \|X_t\|_{L^2_{T_0} H^3(T)} \leq \|X_0\|_{H^{5/2}(T)}.$$
and that for $\forall s_1, s_2 \in T$ and $t \in [0, T_0]$.

$$|X(s_1, t) - X(s_2, t)| \geq \frac{\lambda}{2} |s_1 - s_2|.$$  

Moreover, the solution depends continuously on the initial data.

Here (1.7) is called the well-stretched condition, and

$$C_{[0, T_0]} H^{5/2}(T) = C([0, T_0]; H^{5/2}(T)), \quad L^2_{T_0} H^3(T) = L^2([0, T_0]; H^3(T)).$$

We also prove global well-posedness of (1.5) when $X_0$ is sufficiently close to an equilibrium, which is an evenly parametrized circular configuration. Moreover, such solution converges exponentially to an equilibrium. Regularity of $u$ recovered from $X(s, t)$ is studied in [39]. In a parallel work, Mori, Rodenberg, and Spirn [23] establish similar local and global well-posedness results for (1.5) in $C^{1, \alpha}$-spaces. They also show improved regularity of $X(s, t)$ for positive time and a blowup criterion. When there is no blowup, they characterize global behavior of the solution. Rodenberg [29] proves local well-posedness of (1.1)–(1.4) with elastic force $F_X$ of general form.

In the general immersed boundary problem, the stationary Stokes equation needs to be replaced by the Navier-Stokes equations. To the best of our knowledge, its well-posedness is still open.

1.2 The $\varepsilon$-Regularized 2-D Stokes Immersed Boundary Problem

Now we introduce the $\varepsilon$-regularized 2-D Stokes immersed boundary problem as follows:

(1.8) \[ - \Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon(x, t), \]

(1.9) \[ \text{div} u^\varepsilon = 0, \quad |u^\varepsilon|, |p^\varepsilon| \to 0 \text{ as } |x| \to \infty, \]

(1.10) \[ f^\varepsilon(x, t) = \int_T F_{X^\varepsilon}(s, t) \delta_\varepsilon(x - X^\varepsilon(s, t)) ds, \]

(1.11) \[ \frac{\partial X^\varepsilon}{\partial t}(s, t) = \int_{\mathbb{R}^2} u^\varepsilon(x, t) \delta_\varepsilon(X^\varepsilon(s, t) - x) dx \triangleq U_{X^\varepsilon}, \]

Apart from minor differences in notation, compared to (1.1)–(1.4), the singular $\delta$-measure is now replaced by $\delta_\varepsilon$, a regularized approximation of the Dirac $\delta$-measure defined by

(1.12) \[ \delta_\varepsilon(x) = \frac{1}{\varepsilon^2} \phi \left( \frac{x}{\varepsilon} \right). \]

Here $\phi$ is sufficiently regular and compactly supported; more assumptions on $\phi$ will be specified later. With $\delta_\varepsilon$, the elastic force along the string is mollified to become regular, and it is spread to a small neighborhood of the string. Besides that, the string velocity in (1.11) is now determined by averaging the ambient flow field in a
small neighborhood of the string using the same function $\delta_\varepsilon$, as opposed to setting the string velocity to be exactly equal to the fluid velocity at that point in (1.4). As in the unregularized case, $u^\varepsilon$, $p^\varepsilon$, and $U^\varepsilon_{X^\varepsilon}$ are fully determined by $X^\varepsilon$ at each time slice. Formally, the $\varepsilon$-regularized problem approximates the unregularized one.

The $\varepsilon$-regularized problem is motivated by the numerical immersed boundary method. The numerical method involves spatial discretization of the flow field using the Eulerian grid, and parametrization of the immersed elastic structures using Lagrangian coordinates. Flow field is solved on the Eulerian grid, while elastic force and velocity of the elastic object are evaluated only on the Lagrangian marker points. However, these two sets of coordinates do not agree in general. In order to let them communicate, before the flow field is computed in each time step, the elastic force needs to be spread from the Lagrangian marker points to adjacent Eulerian grid points in a suitable way, while after the flow field is solved on the Eulerian grid, motion of the immersed structure, or the velocity at the Lagrangian points, needs to be determined via interpolation. In practice, such spreading and interpolation are realized by a smoothed approximation of the Dirac $\delta$-function. See, e.g., [21, 26] for more details. Therefore, it is natural to introduce a similar regularization to the PDE problem, which can be heuristically viewed as the continuous system discretized and computed by the numerical method. It is would be interesting to rigorously study the regularized problem and figure out whether and how it approximates the original problem, as this can shed light on the analysis and justification of the numerical immersed boundary method in the fully discrete setting.

Regularization of singular physical quantities or singular integral kernels is also seen in other numerical methods, such as the vortex methods [12] and the method of regularized Stokeslet [6, 7].

1.3 Main Results

Unless otherwise stated (for example, in Section 3), we shall focus on the case where the string has Hookean elasticity with normalized Hooke’s constant, i.e., $F_Y = Y_{ss}$. We first prove global well-posedness of the $\varepsilon$-regularized problem (1.8)–(1.11).

**Theorem 1.2 (Global well-posedness of the $\varepsilon$-regularized problem).** Assume $\phi$, the profile of the regularized $\delta$-function in (1.12), is compactly supported in a ball $B_{c_0}(0) \subset \mathbb{R}^2$ centered at the origin with radius $c_0$, satisfying that $\phi(x) = \phi(-x)$ for all $x \in \mathbb{R}^2$. Fix $\varepsilon > 0$.

1. If $X_0 \in H^1(\mathbb{T})$ and $\phi \in W^{2,1}(\mathbb{R}^2)$, (1.8)–(1.11) admits a unique global solution such that

$$X^\varepsilon \in C^1_{[0, +\infty)} H^1(\mathbb{T}) \quad \text{and} \quad \nabla u^\varepsilon \in \text{Lip}([0, +\infty); L^2(\mathbb{R}^2)).$$
(2) For any $\beta > 1$, if $X_0 \in H^\beta(T)$ and $\phi \in C^{[\beta],1}(\mathbb{R}^2)$, $\text{(1.8)-(1.11)}$ admits a unique global solution such that $X^e \in C^{1}_{[0, +\infty), \text{loc}} H^\beta(T)$ and $\nabla u^e \in \text{Lip}([0, +\infty); L^2(\mathbb{R}^2))$.

**Remark 1.3.** Here the smoothness assumptions on $\phi$ may not be the sharpest. Yet, it is noteworthy that the four-point regularized $\delta$-function commonly used in the numerical immersed boundary method [26] admits $W^2,1$-regularity.

In [39], we prove Theorem 1.2 in special cases $\beta = 1$ and $\beta = 5/2$. In fact, we also show global well-posedness for the regularized problem with full Navier-Stokes equation. The proof of Theorem 1.2 for other $\beta$ is a straightforward generalization. We shall present the whole proof in Section 2 for completeness.

Since $\delta_\varepsilon$ approximates the Dirac $\delta$-function in distribution, it is natural to believe that as $\varepsilon \to 0$, $X^e(s, t)$ determined by (1.8)-(1.11) should converge in a certain sense to the solution $X(s, t)$ of (1.5), provided that $X^e$ and $X$ start from identical (or converging) initial data. In fact, we can show the following:

**THEOREM 1.4 (Convergence and error estimates of the $\varepsilon$-regularized problem).**

Assume $\phi \in C_0^\infty(\mathbb{R}^2)$ satisfies that

1. $\phi$ is radially symmetric;
2. $\phi$ is normalized, i.e., $\int_{\mathbb{R}^2} \phi(x) \, dx = 1$.

Define

\begin{equation}
(1.13) \quad m_1 = \int_{\mathbb{R}^2} |x| \cdot \phi \ast \phi(x) \, dx,
\end{equation}

and

\begin{equation}
(1.14) \quad m_2 = \int_{\mathbb{R}^2} |x|^2 \cdot \phi \ast \phi(x) \, dx.
\end{equation}

Fix $\theta \in \left[\frac{1}{4}, 1\right)$. Suppose $X_0 \in H^{2+\theta}(T)$ satisfies the well-stretched condition (1.7) with $\lambda > 0$. Suppose $X \in C_{[0,T]} H^{2+\theta}(T)$ is a (local) solution of the contour dynamic equation (1.5) of the original problem for some $T > 0$. Let $X^e \in C_{[0,T]} H^{2+\theta}(T)$ be the unique solution of the string motion in the $\varepsilon$-regularized problem (1.8)-(1.11). Assume that for all $\varepsilon \ll \lambda$ and $t \in [0, T]$:

(i) $\|X^e(\cdot, t)\|_{H^2+\theta(T)} \leq M$ and $\|X(\cdot, t)\|_{H^{2+\theta}(T)} \leq M$;

(ii) $X^e(\cdot, t)$ and $X(\cdot, t)$ satisfy the well-stretched condition (1.7) with constant $\lambda/2$.

Then as $\varepsilon \to 0$,

\begin{equation}
(1.15) \quad X^e \rightharpoonup X \quad \text{weak-* in } C_{[0,T]} H^{2+\theta}(T),
\end{equation}

and

\begin{equation}
X^e \to X \quad \text{in } C_{[0,T]} H^\nu(T).
\end{equation}
for all \( \gamma < 2 + \theta \). More precisely, define \( \tilde{\varepsilon} = \varepsilon / \lambda \) to be the normalized regularization parameter. Then for \( \tilde{\varepsilon} \ll 1 \), with \( C = C(\theta, \lambda^{-1} M, T) \),

\[
\| X^\varepsilon - X \|_{C_{0,T1} H^{1/2}(\mathbb{T})} \leq C (m_1 \tilde{\varepsilon}^1 + \tilde{\varepsilon}^{1+\theta} |\ln \tilde{\varepsilon}|^0) M, \tag{1.16}
\]

\[
\| X^\varepsilon - X \|_{C_{0,T1} \dot{H}^1(\mathbb{T})} \leq C |\ln \tilde{\varepsilon}|^{1/2} (m_1 \tilde{\varepsilon} + \tilde{\varepsilon}^{1+\theta} |\ln \tilde{\varepsilon}|^0) M, \tag{1.17}
\]

\[
\| X^\varepsilon - X \|_{C_{0,T1} \dot{H}^{2+\theta}(\mathbb{T})} \leq C |\ln \tilde{\varepsilon}|^{1/2} \tilde{\varepsilon}^{\theta} M, \tag{1.18}
\]

and \( \| X^\varepsilon - X \|_{C_{0,T1} \dot{H}^{2+\theta}(\mathbb{T})} \leq CM \). Estimates in intermediate \( H^\gamma \)-spaces can be derived by interpolation.

If \( m_2 = 0 \), the logarithmic factors \( |\ln \tilde{\varepsilon}|^0 \) in (1.16) and (1.17) can be removed.

**Remark 1.5.** The smoothness assumption on \( \phi \) may be weakened. The radial symmetry of \( \phi \) is not essential, but it simplifies the analysis significantly (see Section 3.2). Note that in the numerical immersed boundary method, the regularized \( \delta \)-functions are in the form of a product of one-dimensional profiles [26], which are not radially symmetric unless they are of Gaussian type.

**Remark 1.6.** The assumptions \( (i) \) and \( (ii) \) are crucial and cannot be removed. Unfortunately, in this work, we are not able to rigorously prove them or construct an example in which either of them fails. This stems from an essential difference between the regularized and the unregularized problems, which we heuristically explain as follows. In the unregularized problem, it has been shown that the string velocity can effectively damp high frequencies in the string configuration [15]. In the \( \varepsilon \)-regularized problem, however, the string velocity \( U^\varepsilon_X \) is obtained by first mollifying the flow field and then making restrictions onto the string. In this process, high-frequency information of \( X^\varepsilon \) that is encoded in the flow field gets almost eliminated, especially for those frequencies higher than \( O(\tilde{\varepsilon}^{-1}) \). As a result, although \( U^\varepsilon_X \) may approximate \( U_X \) pretty well over low frequencies with wave numbers up to \( O(\tilde{\varepsilon}^{-1}) \), it is not clear if the former one can damp higher frequencies in \( X^\varepsilon \) as the latter one does. It is then possible that high frequencies in \( X^\varepsilon \) may grow without being well-controlled, so it seems not trivial to prove a uniform-in-\( \varepsilon \) \( H^{2+\theta} \)-bound for \( X^\varepsilon \) over a uniform time interval. See more discussions in Section 5.3. In this work, instead of dealing with this subtle issue, we made assumptions \( (i) \) and \( (ii) \) to simplify our analysis.

**Remark 1.7.** Theorem 1.4 indicates that in general, over the time interval \([0, T]\), \( \| X^\varepsilon - X \|_{H^1(\mathbb{T})} \) is of order \( \tilde{\varepsilon} \) up to logarithmic factors. This roughly agrees with the well-known fact that the original numerical immersed boundary method can only achieve first-order accuracy in the vicinity of a truly lower-dimensional deformable object [2], although higher accuracy may be attained in smoother problems [10][11] or via sophisticated extension techniques [33][34]. However, our analysis also shows that when \( m_1 = 0 \), the error bounds get improved: in this case, for \( \gamma \in [1, 2 + \theta] \), \( \| X^\varepsilon - X \|_{H^\gamma(\mathbb{T})} \) is bounded by \( \tilde{\varepsilon}^{2+\theta-\gamma} \) up to logarithmic factors over the time interval \([0, T]\). The power of \( \tilde{\varepsilon} \) seems very natural given the
intuition that $X^\varepsilon$ should agree with $X$ very well over frequencies up to $O(\tilde{\varepsilon}^{-1})$ while they are both bounded in $\dot{H}^{2+\theta}$. Similar improvement is also seen in Theorem 1.8 below. Both of them arise from the key estimate in Proposition 1.12 or Proposition 3.5 which we will present later. Since $m_1$ is a constant depending only on $\phi$, it suggests a possible way of improving the accuracy of the $\varepsilon$-regularized problem by suitably choosing $\phi$. Its potential numerical implication is worth further investigation. See discussions on this in Sections 5.1, 5.2.

Our last result aims at establishing convergence and error estimates of a regularized problem as $\varepsilon \to 0$ without extra assumptions like (i) and (ii). To state the result, we introduce a further adaptation of the $\varepsilon$-regularized problem. With $F_X = X_{ss}$ and $N \in \mathbb{Z}_+$ to be chosen, we consider

$$
-\Delta u^{e,N} + \nabla p^{e,N} = f^{e,N}(x,t), 
$$

$$
\text{div } u^{e,N} = 0, \quad |u^{e,N}|, |p^{e,N}| \to 0 \text{ as } |x| \to \infty.
$$

$$
f^{e,N}(x,t) = \int_{\mathbb{T}} \mathcal{P}_N X^{e,N}_{ss}(s,t)\delta_\varepsilon(x - X^{e,N}(s,t))ds,
$$

$$
\frac{\partial X^{e,N}}{\partial t} = \mathcal{P}_N \left[ \int_{\mathbb{R}^2} u^{e,N}(x,t)\delta_\varepsilon(X^{e,N}(s,t) - x) dx \right] \triangleq U^{e,N}_{X^{e,N}}(s,t),
$$

$$
X^{e,N}(s,0) = \mathcal{P}_N X_0(s).
$$

Here $\mathcal{P}_N$ is the linear projection operator to the space of functions containing Fourier modes with wave numbers no greater than $N$. To be more precise, define Fourier transform in $L^2(\mathbb{T})$ and its inverse to be

$$
\hat{g}_k = \int_{\mathbb{T}} g(s)e^{-iks} ds, \quad g(s) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}_c} \hat{g}_k e^{iks}. \quad \text{(1.24)}
$$

Then for $g \in L^2(\mathbb{T})$, $\mathcal{P}_N$ is defined by

$$
\mathcal{P}_N g(s) \triangleq \frac{1}{2\pi} \sum_{|k| \leq N} \hat{g}_k e^{iks}, \quad s \in \mathbb{T}. \quad \text{(1.25)}
$$

We call (1.19)–(1.23) the $(\varepsilon, N)$-regularized 2-D Stokes immersed boundary problem. Compared with (1.8)–(1.11), the new system only allows the string configuration as well as the elastic force in the Lagrangian coordinate to have frequencies no higher than $N$. In fact, the projection in (1.21) may be omitted.

We remark that this adaptation mimics the numerical scenario where $N$ Lagrangian marker points are used to represent the string configuration, although projection to low frequencies is not what gets implemented in the numerical method. On the other hand, by properly choosing $N$, we can get rid of potential growth
of Fourier coefficients in high frequencies, which is the main reason the extra assumptions are needed in Theorem 1.4. Indeed, for the \((\varepsilon, N)\)-regularized problem, we can prove the following theorem.

**THEOREM 1.8** (Global well-posedness, singular limit, and error estimates of the \((\varepsilon, N)\)-regularized problem). Suppose \(\phi\) satisfies the assumptions in Theorem 1.4. Let \(\theta \in \left[\frac{1}{4}, 1\right]\). Assume \(X_0 \in H^{2+\theta}(\mathbb{T})\) satisfies the well-stretched condition (1.7) with \(\lambda > 0\) and \(\|X_0\|_{H^{2+\theta}(\mathbb{T})} = M_0\). Define \(\bar{\varepsilon} = \varepsilon / \lambda\) to be the normalized regularization parameter. Then:

1. For all \(\varepsilon, N > 0\), \((1.19)-(1.23)\) admits a unique global solution \(X^{\varepsilon, N}(s, t)\) such that \(X^{\varepsilon, N} \in C_{\text{val}, \text{loc}}^{1} H^{2+\theta}(\mathbb{T})\).

2. Suppose that \((1.5)\) admits a solution \(X(s, t) \in C_{[0, T_\ast]} H^{2+\theta}(\mathbb{T})\) for some \(T_\ast > 0\) such that for all \(t \in [0, T_\ast]\),

\[
\|X(\cdot, t)\|_{H^{2+\theta}(\mathbb{T})} \leq C_\ast M_0
\]

for some \(C_\ast \geq 1\), and

\[
X(\cdot, t) \text{ satisfies the well-stretched condition with constant } \lambda / 2.
\]

Then there exists \(c_\ast > 0\) and \(N_\ast > 0\), which depend on \(\theta, \lambda^{-1} M_0, X_0, T_\ast\), and \(C_\ast\), such that

(a) For all \(\bar{\varepsilon} \ll 1\) and \(N \in [N_\ast, c_\ast \bar{\varepsilon}^{-1}]\), we have for all \(t \in [0, T_\ast]\)

\[
\|X^{\varepsilon, N}(\cdot, t)\|_{H^{2+\theta}(\mathbb{T})} \leq 2C_\ast M_0,
\]

and

\(X^{\varepsilon, N}(\cdot, t)\) satisfies the well-stretched condition with constant \(\lambda / 4\).

(b) For any sequence \((\varepsilon, N) \to (0, +\infty)\) such that \(\bar{\varepsilon} N \leq c_\ast\),

\[
X^{\varepsilon, N} \rightharpoonup X \text{ weak-* in } C_{[0, T_\ast]} H^{2+\theta}(\mathbb{T}),
\]

and for all \(\gamma < 2 + \theta\),

\[
X^{\varepsilon, N} \to X \text{ in } C_{[0, T_\ast]} H^{\gamma}(\mathbb{T}).
\]

More precisely, for all \(t \in [0, T_\ast]\),

- For \(\beta\) satisfying

\[
0 < \beta < \min\left\{\theta, \frac{1}{2}\right\},
\]

we have

\[
\|(X^{\varepsilon, N} - X)(t)\|_{H^{1/2}(\mathbb{T})} \leq C(\varepsilon^{-N/4} N^{-1/2-\theta} + N^{-\beta/2}) M_0
\]

\[
+ C(N^{-1/2-\theta-\beta} + \bar{\varepsilon}^{1+\theta} \ln \bar{\varepsilon}^{\theta}) M_0,
\]

where

\[
\varepsilon = \bar{\varepsilon} / \lambda.
\]
and
\[ \| (X^{e,N} - X(t)) \|_{\dot{H}^1(\mathbb{T})} \leq C(e^{-\gamma N/4} N^{-1-\theta} + N^{-2}) M_0 \]
\[ + C(\ln N)^{1/2} (N^{-1/2-\theta-\beta} + M_1 \tilde{e} + \tilde{e}^{1+\theta} |\ln \tilde{e}|^\theta) M_0, \]
where \( C = C(\beta, \theta, \lambda^{-1} M_0, T_*) \). If, in addition, \( m_2 = 0 \), the logarithmic factors \( |\ln \tilde{e}|^\theta \) in (1.31) and (1.32) can be removed.

- For \( \beta \) satisfying
\[ 0 < \beta \leq \min \left\{ \theta, \frac{1}{2} \right\} \] and \( \beta \neq \frac{1}{2} \) when \( \theta = \frac{1}{2} \), we have
\[ \| (X^{e,N} - X(t)) \|_{\dot{H}^2(\mathbb{T})} \leq C(e^{-\gamma N/4} N^{-\theta} + N^{-1}) M_0 + C(\ln N)^{1/2} (N^{-1/2-\theta-\beta} + \tilde{e}^\theta) M_0 \]
where \( C = C(\beta, \theta, \lambda^{-1} M_0, T_*) \), and
\[ \| (X^{e,N} - X(t)) \|_{\dot{H}^2+\theta(\mathbb{T})} \leq C(e^{-\gamma N/4} + N^{-1+\theta} + (\tilde{e} N)^\theta) M_0, \]
where \( C = C(\theta, \lambda^{-1} M_0, T_*) \).

- Estimates in intermediate \( H^V \)-spaces can be derived by interpolation.

**Remark 1.9.** The assumption \( N \leq c_* \tilde{e}^{-1} \) seems to be natural and optimal. See discussions in Remark 1.6 and Section 5.3.

**Remark 1.10.** Compared to Theorem 1.4, extra terms involving \( N \) show up in the error estimates due to the projection. From a numerical point of view, the most natural choice of \( N \) would be \( N \sim O(\tilde{e}^{-1}) \), although \( N \) does not exactly correspond to the number of Lagrangian markers in the discrete setting. If we do take \( N = c\tilde{e}^{-1} \), the error estimates here reduce to
\[ \| X^{e,N} - X \|_{C[0,T_*)H^{1/2}(\mathbb{T})} \leq C (m_1 \tilde{e} + \tilde{e}^{1+\theta} |\ln \tilde{e}|^\theta) M_0, \]
\[ \| X^{e,N} - X \|_{C[0,T_*)\dot{H}^1(\mathbb{T})} \leq C |\ln \tilde{e}|^{1/2} (m_1 \tilde{e} + \tilde{e}^{1+\theta} |\ln \tilde{e}|^\theta) M_0, \]
\[ \| X^{e,N} - X \|_{C[0,T_*)\dot{H}^2(\mathbb{T})} \leq C |\ln \tilde{e}|^{1/2} \tilde{e}^\theta M_0, \]
\[ \| X^{e,N} - X \|_{C[0,T_*)\dot{H}^{2+\theta}(\mathbb{T})} \leq C M_0, \]
with \( C = C(\theta, \lambda^{-1} M_0, T_*) \), which coincide with those in Theorem 1.4. However, it is a bit surprising that, if we are allowed to ignore those exponentially decaying terms in (1.31), (1.32), (1.34), and (1.35), which will become negligible even when \( t \) is small, we may take \( N \) much smaller than \( O(\tilde{e}^{-1}) \) without worsening the error bounds. This suggests that in order to track the string dynamics in the regularized problem as accurately as possible, we may need much fewer Fourier modes than
Numerical implication of this result is worth further investigation.

Remark 1.11. The uniqueness of $X^{e,N}$ together with (1.28) implies that the solution $X$ assumed in Theorem 1.8 should be unique.

1.4 Scheme of the Proofs and Organization of the Paper

Let us take the $\varepsilon$-regularized problem as an example to sketch the idea of proving convergence and error estimates. Recall that in the analysis of the original problem with the Hookean elasticity [15, 23], (1.1)–(1.4) is first reduced (under some assumptions) to the contour dynamic equation (1.5), and it is then rewritten as $\partial_t X = \mathcal{L} X + g_X$. Here $\mathcal{L} X = -\frac{1}{4}(-\Delta)^{1/2} X$ captures the principal singular part in $U_X$ that is derived by linearizing the integrand of (1.5) around $s' = s$, while $g_X$ is a nonlinear non-local term collecting all the remaining terms. It is observed that $\mathcal{L}$ is a dissipative operator and $g_X$ turns out to be sufficiently regular, which enables us to prove well-posedness of (1.5).

We shall take a similar path for the $\varepsilon$-regularized problem by focusing on the string motion and using the original problem as a benchmark. By (1.4) and (1.11), we derive that $\partial_t X^\varepsilon = U_X^\varepsilon + (U_{X^\varepsilon}^\varepsilon - U_X^\varepsilon) = \mathcal{L} X^\varepsilon + g_X^\varepsilon + (U_{X^\varepsilon}^\varepsilon - U_X^\varepsilon)$ and $\partial_t (X^\varepsilon - X) = \mathcal{L} (X^\varepsilon - X) + (g_X^\varepsilon - g_X) + (U_{X^\varepsilon}^\varepsilon - U_X^\varepsilon)$.

In order to bound $X^\varepsilon - X$, thanks to the dissipative nature of the operator $\partial_t - \mathcal{L}$, it suffices to study $g_X^\varepsilon - g_X$ and $U_{X^\varepsilon}^\varepsilon - U_X^\varepsilon$. In particular, it would be ideal to show that the mapping $X \mapsto g_X$ is (locally) Lipschitz in suitable function spaces, and $U_{X^\varepsilon}^\varepsilon - U_X^\varepsilon$ can be treated as a small error. We shall implement this idea to prove Theorem 1.4 in Section 4.1. The proof of Theorem 1.8 uses a similar approach, with the estimates for $U_{X^\varepsilon}^\varepsilon - U_X^\varepsilon$ handled more carefully.

Now it is clear that a key ingredient to prove convergence and error estimates is to establish estimates for $U_{Y^\varepsilon}^\varepsilon - U_Y$ for a given string configuration $Y$. We call such estimates static estimates for the regularization error in the string velocity, as $U_{Y^\varepsilon}^\varepsilon - U_Y$ only depends on $Y$ at a single time but not on its history or future evolution. We can show the following:

Proposition 1.12 (Static estimates for $U_{Y^\varepsilon}^\varepsilon - U_Y$, Hookean case). Let $\phi$, $m_1$, and $m_2$ be defined in Theorem 1.4. Suppose $Y \in H^{2+\theta}(\Gamma)$ with $\theta \in [1/4, 1)$ satisfying the well-stretched condition (1.7) with $\lambda > 0$ and $F_Y = Y_{ss}$. Given $\varepsilon > 0$, let $U_Y$ and $U_{Y^\varepsilon}^\varepsilon$ be the string velocities corresponding to $Y$ in the original and the $\varepsilon$-regularized problems, defined in (1.5) and (1.11), respectively. Provided
that $\varepsilon \ll \lambda$,

$$
\|U_Y^\varepsilon - U_Y\|_{L^2(T)} \leq \frac{m_1 \varepsilon}{\pi \lambda} \left\| \frac{F_Y(s) \cdot Y'(s)}{|Y'(s)|^2} \right\|_{L^2(T)} + C \frac{\varepsilon^{1+\theta} \ln^\theta (\lambda/\varepsilon)}{\lambda^{1+\theta}} \|Y\|_{\dot{H}^{2+\theta}(T)}
$$

$$
+ \frac{C \varepsilon^2 \ln(\lambda/\varepsilon)}{\lambda^3} \|Y'\|_{L^\infty(T)} \|Y''\|_{L^1(T)}^2,
$$

where $C$’s are universal constants depending on $\theta$. Moreover,

$$
\|U_Y^\varepsilon - U_Y\|_{\dot{H}^1(T)} \leq \frac{C \varepsilon^{\theta}}{\lambda^\theta} \|Y\|_{\dot{H}^{2+\theta}(T)} + \frac{C \varepsilon^{2}}{\lambda^4} \|Y'\|_{L^\infty(T)} \|Y''\|_{L^1(T)}^2.
$$

If, in addition, $m_2 = 0$, the logarithmic factors in (1.36) can be removed.

Similar static estimates for the regularization error have been derived in fully discrete settings for the velocity field $u$ and pressure $p$ in the Stokes immersed boundary problem \[16, 17, 22\], and more generally, for solutions of differential equations with singular source terms (see, e.g., \[40\] and references therein). Since static error estimates are of independent interest, we shall study them in Section 3 with greater generality by considering elastic force $F$ of general form.

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1.2. Section 3 will be devoted to establishing static estimates for the regularization error in the string velocity. We will first formulate the problem with greater generality in Sections 3.1–3.2. For clarity, we collect statements of the static estimates in Section 3.3, of which Proposition 1.12 is a special case. We will prepare some preliminary estimates for them in Section 3.4 and show detailed proofs in Sections 3.5–3.6. First-time readers may skip these sections, so as not to get distracted by the technicality there. We then prove Theorem 1.4 in Section 4.1 and Theorem 1.8 in Section 4.2. We conclude the paper with discussions in Section 5. In Sections 5.1 and 5.2, we provide a physical interpretation for why $m_1 = 0$ gives rise to improved error estimates, and then discuss how this condition can be achieved by properly designing the regularized $\delta$-function. Section 5.3 is concerned with loss of damping of high frequencies in the string configuration in the $\varepsilon$-regularized problem, which is related to Remark \[16\]. We discuss future problems in Section 5.4. In Appendix A we will prove some auxiliary lemmas from Section 3. In Appendix B we show a priori estimates involving the operator $\mathcal{L} = -\frac{1}{2}(-\Delta)^{1/2}$. Finally, some estimates involving the nonlinear term $g_X$ in the contour dynamic equation are proved in Appendix C.

2 Global Well-Posedness of the $\varepsilon$-Regularized Problem

2.1 Well-Posedness for $H^1$-Initial Data

For completeness, we first prove Theorem 1.2 with $\beta = 1$ by recasting the proof in \[39\]. The idea of establishing local well-posedness is to view (1.11) as an ODE of $X$ in the Banach space $H^1(T)$—this is the case thanks to the regularization.
Then local well-posedness can be proved by applying the classic Picard theorem in Banach spaces [18, theorem 3.1]. Global well-posedness for $H^1$-initial data should follow from a continuation argument [18, theorem 3.3] combined with an energy estimate, which shows that $\|X\|_{\dot{H}^{1}(\mathbb{T})}$ is uniformly bounded for all time.

**Proof of Theorem 1.2** with $\beta = 1$. With $M > 1$ to be determined, we define

$$O_M = \{Z \in H^1(\mathbb{T}) : \|Z\|_{\dot{H}^1(\mathbb{T})} < M\}.$$  

It is nonempty and open in $H^1(\mathbb{T})$. We take $M$ suitably large such that $x_0 \in O_M$. It suffices to show that $U^e_Y \in H^1(\mathbb{T})$ for all $Y \in O_M$, and that the mapping $Y \mapsto U^e_Y$ is (locally) Lipschitz in $O_M$. Here $U^e_Y$ is defined in (1.11). With abuse of notation, we still use $f^e$ and $u^e$ to denote the quantities in (1.8)–(1.10) corresponding to $Y$. We also take arbitrary $Y_i (i = 1, 2)$ in $O_M$, and let $f^e_i$, $u^e_i$, and $U^e_Y$ be the quantities in (1.8)–(1.11) corresponding to $Y_i$.

**Step 1 (From the string configuration to the forcing).**

Since $Y \in H^1(\mathbb{T})$, $Y_{ss} \in H^{-1}(\mathbb{T})$ in (1.10). By a density argument,

$$f^e(x) = \int_\mathbb{T} Y_s(s) \cdot \nabla \delta_e(x - Y(s)) Y_s(s) ds$$

$$= \operatorname{div}_x \left[ \int_\mathbb{T} \delta_e(x - Y(s)) Y_s(s) \otimes Y_s(s) ds \right].$$

It is then easy to show

$$\|f^e\|_{L^2(\mathbb{R}^2)} \leq \|\nabla \delta_e\|_{L^2(\mathbb{R}^2)} M^2 \leq C(M, \varepsilon),$$

$$\|f^e\|_{H^{-1}(\mathbb{R}^2)} \leq \|\delta_e\|_{L^2(\mathbb{R}^2)} M^2 \leq C(M, \varepsilon),$$

and

$$\|f^e_1 - f^e_2\|_{H^{-1}(\mathbb{R}^2)}$$

$$\leq \int_\mathbb{T} \|\delta_e\|_{L^2(\mathbb{R}^2)} |Y_{1,s} - Y_{2,s}| |Y_{1,s}| ds$$

$$+ \int_\mathbb{T} \|\delta_e\|_{L^2(\mathbb{R}^2)} |Y_{2,s}| |Y_{1,s} - Y_{2,s}| ds$$

$$+ \int_\mathbb{T} |Y_1 - Y_2| \|\nabla \delta_e\|_{L^2(\mathbb{R}^2)} |Y_{2,s}|^2 ds$$

$$\leq C(M, \varepsilon) \|Y_1 - Y_2\|_{H^1(\mathbb{T})}.$$  

In the last line, we used Sobolev embedding $H^1(\mathbb{T}) \hookrightarrow C(\mathbb{T})$.

By Sobolev embedding, $Y(\mathbb{T})$ is contained in the $B_R(\bar{Y})$ with radius $R = CM$, and

$$\bar{Y} = \frac{1}{2\pi} \int_\mathbb{T} Y(s) ds.$$  

Since $\delta_e$ is supported on $B_{C_0 \varepsilon}(0)$, $f^e$ is supported in $B_{R_e}(\bar{Y})$ where $R_e = R + C_0 \varepsilon$. Moreover, $\int_{\mathbb{R}^2} f^e(x) dx = 0$ thanks to (2.1).
Step 2 (From the forcing to the velocity field). By classic estimates of the stationary Stokes equation in $\mathbb{R}^2$ and the fact that $f^\varepsilon$ has integral zero on $\mathbb{R}^2$, the mapping $f^\varepsilon \mapsto \nabla u^\varepsilon$ is well-defined and Lipschitz continuous from $H^{-1}(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$. Namely,

\begin{equation}
\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^2)} \leq C \|f^\varepsilon\|_{H^{-1}(\mathbb{R}^2)},
\end{equation}

\begin{equation}
\|\nabla u^\varepsilon_1 - \nabla u^\varepsilon_2\|_{L^2(\mathbb{R}^2)} \leq C \|f^\varepsilon_1 - f^\varepsilon_2\|_{H^{-1}(\mathbb{R}^2)},
\end{equation}

for some universal constant $C$.

We also would like to derive an $L^\infty$-estimate of $u^\varepsilon$ for the next step. Since $f^\varepsilon$ has integral zero, we may represent $u^\varepsilon$ by using a fundamental solution of the stationary Stokes equation, i.e.,

\[ u^\varepsilon(x) = \int_{\mathbb{R}^2} G(x - y) f^\varepsilon(y) dy, \]

where $G$ is the fundamental solution of the velocity field defined in (1.6). By Minkowski inequality, (2.2), and the fact that $f^\varepsilon$ is compactly supported,

\[ \|u^\varepsilon\|_{L^\infty(B_{2R_e}(\bar{Y}))} \leq C \|G\|_{L^\infty(B_{2R_e}(0))} \|f^\varepsilon\|_{L^2(B_{R_e}(\bar{Y}))} \leq C(R_e) \|f^\varepsilon\|_{L^2(\mathbb{R}^2)} \leq C(M, \varepsilon). \]

On the other hand,

\[ \|u^\varepsilon\|_{L^\infty(B_{2R_e}(\bar{Y}))} \leq C \|\nabla G\|_{L^\infty(B_{R_e}(0))} \left\| \int_{\mathbb{T}} \delta_\varepsilon(x - Y(s)) Y_5(s) \otimes Y_5(s) ds \right\|_{L^1(B_{R_e}(\bar{Y}))} \leq C(R_e)M^2 \leq C(M, \varepsilon). \]

Combining the above two estimates, we obtain

\begin{equation}
\|u^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leq C(M, \varepsilon).
\end{equation}

Step 3 (From the velocity field to the string motion). By (1.11) and (2.6),

\begin{equation}
|U^\varepsilon_Y(s)| \leq \int_{\mathbb{R}^2} |u^\varepsilon(x)| |\delta_\varepsilon(Y(s) - x)| dx \leq C \|u^\varepsilon(x)\|_{L^\infty(\mathbb{R}^2)} \leq C(M, \varepsilon),
\end{equation}

and

\begin{equation}
|\partial_s U^\varepsilon_Y(s)| = \left| Y_5(s) \int_{\mathbb{R}^2} u^\varepsilon(x) \nabla \delta_\varepsilon(Y(s) - x) dx \right| \leq C(M, \varepsilon)|Y_5(s)|.
\end{equation}

Combining (2.3), (2.5), (2.7), and (2.8), we find that

\[ \|U^\varepsilon_Y\|_{H^1(\mathbb{T})} \leq C(M, \varepsilon). \]

Now we turn to show that the map $Y \mapsto U^\varepsilon_Y$ is Lipschitz continuous in $O_M$. We shall derive an $H^1$-estimate for $U^\varepsilon_{Y_1} - U^\varepsilon_{Y_2}$. First we assume $Y_1$ and $Y_2$ both have
zero mean on $T$; this implies $Y_{t}(T) \subset B_{R}(0)$ and thus $B_{R_{e}}(0)$ covers the supports of $u_{1}^{e}(\cdot) \delta_{e}(Y_{t}(s) \cdot)$. Hence,

$$
\left| U_{Y_{1}}^{e} - U_{Y_{2}}^{e} \right| = \left| \int_{B_{R_{e}}(0)} u_{1}^{e}(x) \delta_{e}(Y_{1}(s) - x) dx - \int_{B_{R_{e}}(0)} u_{2}^{e}(x) \delta_{e}(Y_{2}(s) - x) dx \right|
$$

(2.9)

$$
\leq \int_{B_{R_{e}}(0)} |u_{1}^{e} - u_{2}^{e}| |\delta_{e}(Y_{1}(s) - x)| dx + |u_{2}^{e}| |\delta_{e}(Y_{1}(s) - x) - \delta_{e}(Y_{2}(s) - x)| dx
$$

$$
\leq C(\varepsilon)(\|u_{1}^{e} - u_{2}^{e}\|_{L^{2}(B_{R_{e}}(0))} + \|u_{2}^{e}\|_{L^{2}(B_{R_{e}}(0))} |Y_{1} - Y_{2}|).
$$

and

$$
|\partial_{s} U_{Y_{1}}^{e} - \partial_{s} U_{Y_{2}}^{e}| = \left| \partial_{s} \int_{B_{R_{e}}(0)} u_{1}^{e}(x) \delta_{e}(Y_{1}(s) - x) dx - \partial_{s} \int_{B_{R_{e}}(0)} u_{2}^{e}(x) \delta_{e}(Y_{2}(s) - x) dx \right|
$$

(2.10)

$$
\leq |Y_{1,s}(s)| \int_{B_{R_{e}}(0)} |u_{1}^{e} - u_{2}^{e}| |\nabla \delta_{e}(Y_{1}(s) - x)| dx + |Y_{1,s}(s) - Y_{2,s}(s)| \int_{B_{R_{e}}(0)} |u_{2}^{e}| |\nabla \delta_{e}(Y_{1}(s) - x)| dx
$$

$$
+ |Y_{2,s}(s)| \left| \int_{B_{R_{e}}(0)} u_{2}^{e}(x) (\nabla \delta_{e}(Y_{1}(s) - x) - \nabla \delta_{e}(Y_{2}(s) - x)) dx \right|
$$

$$
\leq C(\varepsilon)(\|Y_{1,s}\|_{L^{2}(B_{R_{e}}(0))} + \|Y_{2,s}\|_{L^{2}(B_{R_{e}}(0))} |Y_{1} - Y_{2}| + C(\varepsilon) \sum_{l=1}^{2} \|\partial_{s}^{l} \delta_{e}(Y_{t}(s) \cdot)\|_{L^{2}(B_{R_{e}}(0))} |Y_{1} - Y_{2}|).
$$

We need to bound $\|u_{1}^{e} - u_{2}^{e}\|_{L^{2}(B_{R_{e}}(0))}$. By Young’s inequality,

$$
\|u_{1}^{e} - u_{2}^{e}\|_{L^{2}(B_{R_{e}}(0))} \leq G \|f_{1}^{e} - f_{2}^{e}\|_{L^{1}(B_{R_{e}}(0))}.
$$

We argue as in (2.4) to obtain that

$$
\|f_{1}^{e} - f_{2}^{e}\|_{L^{1}(B_{R_{e}}(0))} \leq C(M, \varepsilon) \|Y_{1} - Y_{2}\|_{H^{1}(T)}.
$$

Here we need the assumption $\nabla^{2} \phi \in L^{1}(\mathbb{R}^{2})$; recall that $\phi$ is the profile of the regularized $\delta$-function in (1.12). Hence,

$$
\|u_{1}^{e} - u_{2}^{e}\|_{L^{2}(B_{R_{e}}(0))} \leq C(M, \varepsilon) \|Y_{1} - Y_{2}\|_{H^{1}(T)}.
$$

Combining this with (2.9) and (2.10), we conclude that

$$
\|U_{Y_{1}}^{e} - U_{Y_{2}}^{e}\|_{H^{1}(T)} \leq C(M, \varepsilon) \|Y_{1} - Y_{2}\|_{H^{1}(T)}.
$$
For general $Y_1$ and $Y_2$, since $U_{Y_i}^\varepsilon = U_{Y_i - Y_1}^\varepsilon$, the same estimate still holds. Therefore, we prove that the map $Y \mapsto U_{Y}^\varepsilon$ is Lipschitz continuous in $O_M$. By the Picard theorem in Banach space [18, theorem 3.1], there exists $T > 0$ and a unique local solution $X^\varepsilon \in C^1_{[0,T]}(O_M)$ describing the string dynamics, which depends continuously on the initial data. By the derivation above,

$$\nabla u^\varepsilon \in \text{Lip}([0,T]; L^2(\mathbb{R}^2)).$$

**Step 4 (Energy law and global well-posedness).** We shall derive an energy estimate to prove the global well-posedness. We take the $s$-derivative of (1.11):

$$\frac{\partial X_s^\varepsilon}{\partial t}(s, t) = \int_{\mathbb{R}^2} X_s^\varepsilon(s, t) \cdot \nabla \delta_\varepsilon(x - X^\varepsilon(s, t) - x) u^\varepsilon(x, t) dx.$$

It is valid to do so since $X^\varepsilon \in C^1_{[0,T]} L^1(H^1)$. Taking the inner product with $X^\varepsilon$ on $T$, we find by (2.1) and the energy estimate of the stationary Stokes equation that

$$\frac{1}{2} \frac{d}{dt} \| X_s^\varepsilon \|^2_{L^2(\mathbb{T})}$$

$$= - \int_{\mathbb{T}} ds \int_{\mathbb{R}^2} X_s^\varepsilon(s, t) \cdot \nabla \delta_\varepsilon(x - X^\varepsilon(s, t)) u^\varepsilon(x, t) \cdot X_s^\varepsilon(s, t) dx$$

$$= - \int_{\mathbb{R}^2} f^\varepsilon(x, t) \cdot u^\varepsilon(x, t) dx$$

$$= - \| \nabla u^\varepsilon \|^2_{L^2(\mathbb{R}^2)}.$$

In the first line, we used the assumption $\phi(x) = \phi(-x)$. This implies that for $\forall t \in [0, T]$, $\| X^\varepsilon \|_{H^1(\mathbb{T})} \leq \| X_0 \|_{H^1(\mathbb{T})} < M$. Then global well-posedness follows from a continuation argument [18, theorem 3.3].

This proves well-posedness for $H^1$-initial data in Theorem 1.2. 

### 2.2 Well-Posedness for Smoother Initial Data

**Proof of Theorem 1.2 with $\beta > 1$.** Again, we shall use the Picard theorem in Banach spaces. Let

$$O_{M, M'} = \{ Z \in H^\beta(\mathbb{T}) : \| Z \|_{H^1(\mathbb{T})} < M, \| Z \|_{H^\beta(\mathbb{T})} < M' \},$$

with $M' \geq M \geq 1$ suitably large such that $X_0 \in O_{M, M'}$. All the estimates derived in the previous part of the proof still hold for $\forall Y, Y_i \in O_{M, M'}$, with estimates only depending on $M$ and $\varepsilon$ but not on $M'$ or $\beta$.

By (1.11), thanks to $\phi$ being compactly supported and sufficiently smooth, by taking the $[\beta]^{th}$ derivative of $\delta_\varepsilon(Y(s) - x)$ with respect to $s$ and collecting all
possible terms,
\[
\|U^\varepsilon_x\|_{\dot{H}^\beta(T)}
\leq \int_{B_{R_0}} |\nu^\varepsilon(x)||\delta_e(Y(s) - x)||L^2(T) + \|\delta_e(Y(s) - x)||H^{\beta}(T) dx
\leq C(M, \varepsilon)
\tag{2.12}
\]
\[+
\int_{R_0} dx |\nu^\varepsilon(x)| \cdot C(\beta)
\cdot \sum_{1 \leq k \leq [\beta]} \|Y^{(n_1)} \otimes \cdots \otimes Y^{(n_k)} \cdot \nabla_k \delta_e(Y(s) - x)||H^{\beta-|\beta|}(T)\]

Here when $\beta$ is an integer, we abused the notation $\dot{H}^0(T)$ to represent $L^2(T)$. When $\beta - [\beta] \in (0, 1)$, since $n_1 \geq 1$,
\[
\|Y^{(n_1)} \otimes \cdots \otimes Y^{(n_k)} \cdot \nabla_k \delta_e(Y(s) - x)||H^{\beta-|\beta|}(T)
\leq C(\beta) \|Y^{(n_1)}\|_{H^1(T)} \cdots \|Y^{(n_k-1)}\|_{H^1(T)} \|Y^{(n_k)}\|_{H^{\beta-|\beta|}(T)}
\cdot \|\nabla_k \delta_e(Y(s) - x)||H^1(T)
\leq C(\beta) \|Y_s\|_{H^{n_1}(T)} \cdots \|Y_{n_k-1}(T)\|_{H^{n_k-1}(T)} \|Y_s\|_{H^{\beta-|\beta|+n_k-1}(T)}
\cdot (\|\nabla_k \delta_e(Y(s) - x)||L^2(T) + \|Y_s\|_{L^2(T)} \|\nabla_k \delta_e(Y(s) - x)||L^\infty(T))
\leq C(\varepsilon, \beta) \|Y_s\|_{L^2(T)} \|Y_s\|_{H^{\beta-1}(T)} \bigl(1 + \|Y_s\|_{L^2(T)}\)
\leq C(M, \varepsilon, \beta) M'.
\tag{2.13}
\]

In the second-to-last line, we used interpolation inequality among $H^s$-seminorms, as well as the assumption that $\phi \in C^{[\beta]+1}(\mathbb{R}^2)$. When $\beta - [\beta] = 0$, i.e., when $\beta$ is an integer, we may replace $\|\nabla_k \delta_e(Y(s) - x)||H^1(T)$ in (2.13) by $\|\nabla_k \delta_e(Y(s) - x)||L^\infty(T)$. In this way, we can derive an estimate of the same form, yet only assuming $\phi \in C^\beta(\mathbb{R}^2)$. Combining (2.13) with (2.12), we conclude that
\[
(U^\varepsilon_x)_y \leq C(M, \varepsilon, \beta) M'.
\tag{2.14}
\]

Using a similar argument as in (2.10), (2.12), and (2.13), but with more complicated derivation, we can also prove that the map $Y \mapsto U^\varepsilon_Y$ is Lipschitz continuous in $O_{M, M'}$:
\[
\|U^\varepsilon_Y - U^\varepsilon_{Y_2}\|_{H^{\beta}(T)} \leq C(M', \varepsilon, \beta) \|Y_1 - Y_2\|_{H^{\beta}(T)}, \quad \forall Y_1, Y_2 \in O_{M, M'}.
\]

Here we will need the assumption that $\phi \in C^{|\beta|, 1}(\mathbb{R}^2)$. We omit the details. Then the local well-posedness immediately follows from the Picard theorem in $O_{M, M'} \subset H^\beta(T)$.

In order to show global well-posedness, we use the energy estimate (2.11) as well as (2.14) to derive that
\[
\frac{d}{dt} \|X^\varepsilon\|_{H^\beta(T)} \leq C \left( \|X_0\|_{H^1(T)}, \varepsilon, \beta \right) \|X^\varepsilon\|_{H^\beta(T)}.
\]
This implies an a priori bound for the local solution
\[ \| X^\varepsilon \|_{C_{[0,T]} H^\beta(T)} \leq \exp \left[ C \left( \| X_0 \|_{\dot{H}^1(T)}, \varepsilon, \beta \right) T \right] \| X_0 \|_{\dot{H}^\beta(T)}. \]
Then the global well-posedness follows from a continuation argument. \( \square \)

**Remark 2.1.** In spite of the global well-posedness, \( \| X(t) \|_{\dot{H}^\beta(T)} \) only admits an exponentially growing bound. As \( \varepsilon \to 0^+ \), its growth rate deteriorates (i.e., increases) very quickly and diverges to \(+ \infty\), which is not helpful for proving any convergence of \( X^\varepsilon \).

**Remark 2.2.** As opposed to Proposition 1.1, Theorem 1.2 does not assume the well-stretched condition (1.7) for \( X_0 \). However, for given \( \varepsilon \), if we do impose that with the stretching constant \( \lambda \) for sufficiently smooth initial data, say \( X_0 \in H^\beta(T) \) with \( \beta > 3/2 \), the solution \( X^\varepsilon \) should satisfy the well-stretched condition with stretching constant \( \lambda/2 \) in short time, which may depend on \( \varepsilon \). In fact, this can be derived from the facts that \( X^\varepsilon(t) \) is (locally) continuous in \( H^\beta(T) \), and \( H^\beta(T) \hookrightarrow \operatorname{Lip}(T) \) for \( \beta > 3/2 \).

### 3 Static Error Estimates for the String Velocity

In this section, we shall establish static estimates for \( U^\varepsilon_Y - U_Y \) for a given string configuration \( Y \). Its motivation has been briefly discussed in Section 1.4. We will first set up the analysis in Sections 3.1–3.2. Main results of this section, of which Proposition 1.12 is a special case, are collected in Section 3.3. Their proofs are left to Sections 3.4–3.6; first-time readers may skip them so as not to get distracted from the bigger picture of the paper.

#### 3.1 Assumptions on the Elastic Force of General Form

Static estimates for the regularization error are of independent interest, and we shall discuss them with greater generality. In this section, we introduce assumptions on the elastic force of more general form that will be used throughout this section.

Given a string configuration \( Y \), the elastic force in the Lagrangian coordinate is generally given by
\[ F_Y(s) = \partial_s \left( T(|Y'(s)|, s) \frac{Y'(s)}{|Y'(s)|} \right). \]
Here \( T : \mathbb{R}_+ \times T \to [0, +\infty) \) is the tension in the string, which depends on the position and the way the string is locally stretched, characterized by \( p = |Y'(s)| \) with abuse of notation. In physics, \( T = T(p, s) \) is determined by the local constitutive law of elasticity of the string material: assuming the string material has a local elastic energy density \( \mathcal{E} = \mathcal{E}(p, s) \), which is allowed to be spatially inhomogeneous, then \( T(p, s) = \partial_p \mathcal{E}(p, s) \). For instance, Hookean elasticity admits \( \mathcal{E}(p) = k_0 p^2 / 2 \) and \( T(p) = k_0 p \), with \( k_0 \) being the Hooke’s constant.
Define
\[ S(p,s) = \frac{T(p,s)}{p} \]
to be the generalized stiffness coefficient. In the Hookean elasticity case, \( S(p) = k_0 \), which exactly characterizes stiffness of the string. With this notation,
\[ F_Y(s) = \partial_s[S(|Y'(s)|, s)Y'(s)]. \]

In the rest of this section, we will study elastic force in this general form, where \( S = S(p,s) \) satisfies the following assumption:
\( S = S(p,s) \in C^{1,1}_{\text{loc}}(\mathbb{R}_+ \times T) \). To be more precise, for \( \forall 0 < m < M \),
\[ \exists \mu(m, M) > 0 \text{ such that for all } (p,s) \in [m, M] \times T, \]
\[ |S(p,s)| + |\partial_p S(p,s)| + M |\partial_p S(p,s)| \leq \mu(m, M), \]
and for all \( p_1, p_2 \in [m, M] \) and \( s_1, s_2 \in T \),
\[ |\partial_s S(p_1, s_1) - \partial_s S(p_2, s_2)| + M |\partial_p S(p_1, s_1) - \partial_p S(p_2, s_2)| \leq \mu(m, M)(|s_1 - s_2| + M^{-1}|p_1 - p_2|). \]

Having homogeneous polynomials of \( p \) in mind as prototypical examples of \( S \), we purposefully add extra powers of \( M \) above in order not to break homogeneity of the estimates.

Although the assumption is not intended to be the weakest or the most comprehensive, it is general enough to include a broad family of elasticity models. For instance, any spatially homogeneous elasticity model with \( T = T(p) \) for \( T \in C^3(0, \infty) \) satisfies the assumptions. In particular, the linear elasticity model, with \( T(p) = k(p - p_0) \) and \( S(p) = k - kp_0/p \), is admissible. Here \( p_0 \geq 0 \) is the natural length of fully relaxed string material; when \( p_0 = 0 \), it characterizes the Hookean elasticity. An example that does not fulfill the assumption is the finitely extensible nonlinear elastic (FENE) model \[42\], given by, e.g.,
\[ T(p) = \frac{kp}{1 - (p/p_{\text{max}})^2}, \quad S(p) = \frac{k}{1 - (p/p_{\text{max}})^2}, \quad p \in [0, p_{\text{max}}), \]
and \( T(p) = S(p) = \infty \) otherwise. Even though the arguments in this section may also work for this case up to some adaptation, we simply avoid that technicality.

In practice, elasticity laws may vary over time. For example, for a parametrically-forced string that models active biological tissues, \( S(p,t) = a + b \sin(\omega t) \) with \( |b| < a \). For such models, it suffices to freeze the elasticity law and validate the assumption for each time slice. We may additionally require the bounds to be uniform in time, so that error estimates apply to all time. We leave the technical discussion to interested readers.

To this end, we state a useful lemma that roughly claims that \( F_Y \) behaves like \( Y_{ss} \) in regularity. This can be viewed as a generalization of the obvious fact in the Hookean elasticity case where \( F_Y = k_0 Y_{ss} \).
3.1 (Estimates for $F$). Assume $Y \in H^{2+\theta}(\mathbb{T})$ satisfies the well-stretched condition (1.7) with constant $\lambda$. Let $F_{Y}(s)$ be defined by (3.2). For $\theta \in [0, 1)$, under the Assumption (a) on $S$,

$$|F_{Y}(s)| \leq C\mu(|Y'(s)| + |Y''(s)|),$$

$$\|F_{Y}\|_{\tilde{H}^{\theta}(\mathbb{T})} \leq C\theta\|Y\|_{\tilde{H}^{2+\theta}(\mathbb{T})},$$

where

$$\mu = \mu(\lambda, c\|Y''\|_{L^{2}(\mathbb{T})})$$

is defined by (3.3) and (3.4). Here $c \geq 1$ is a universal constant chosen so that $\|Y'\|_{L^{\infty}} \leq c\|Y''\|_{L^{2}}$. With abuse of notations, we understand $\tilde{H}^{\theta}(\mathbb{T})$ as $L^{2}(\mathbb{T})$.

Its proof is a straightforward calculation. We leave it to Appendix A.1.

3.2 The Contour Dynamic Formulations

Assume $\phi$, the profile of the regularized $\delta$-function in (1.12), satisfies the conditions in Theorem 1.4. We also assume that $Y$ at least has $H^{2}$-regularity and satisfies the well-stretched condition (1.7) with constant $\lambda$.

We first recall the contour dynamic formulation of the original Stokes immersed boundary problem (1.1)–(1.4) [15]. Given a string configuration $Y$, thanks to the stationary Stokes equation, the flow field is instantaneously determined by the force exerted on the fluid. Combining (1.1)–(1.3), we formally derive that

$$u(x) = \int_{\mathbb{R}^{2}} G(x - y) f(y) dy$$

$$= \int_{\mathbb{R}^{2}} G(x - y) \int_{\mathbb{T}} F_{Y}(s) \delta(y - Y(s')) ds'$$

$$= \int_{\mathbb{T}} G(x - Y(s')) F_{Y}(s') ds',$$

where $G$ is the fundamental solution defined in (1.6). Take $x = Y(s)$. By (3.2) and integration by parts,

$$U_{Y}(s) = u(Y(s)) = p.v. \int_{\mathbb{T}} -\partial_{s'}[G(Y(s) - Y(s'))] SY'(s') ds'.$$

Here $S = S(|Y'(s)|, s)$.

It is shown in [15] that if $S \equiv 1$ and $Y \in H^{2}(\mathbb{T})$ satisfies the well-stretched condition (1.7), the above derivation can be made rigorous. We claim that, following an almost identical argument, (3.9) can be justified for general $S$ that satisfies the Assumptions (a) and for all $Y \in H^{2}(\mathbb{T})$ satisfying (1.7). See [15, sec. 2] for more details.

Remark 3.2. In (3.9), we treat the restriction of $u$ on $Y(\mathbb{T})$ as the string velocity, which is not obviously valid. In fact, it is noted in [19, prop. 1.1 and lemma 2.2] that higher space-time regularity of $Y$ is needed in order to uniquely define transport of a lower-dimensional object $Y(\mathbb{T})$ as well as the restriction of $u$ on it. Since in
this section we only focus on static error estimates, we shall avoid this subtlety, but simply assume (3.9) is a valid formula for the string velocity.

In the $\varepsilon$-regularized problem, for $Y \in H^2(\Gamma)$, it is not difficult to show that $f^\varepsilon$ defined as in (1.10) is as regular as $\delta_\varepsilon$; $u^\varepsilon$ is even more regular locally. Hence, we can rigorously perform a derivation similar to (3.8) and (3.9), and justify that (1.8)–(1.11) gives

$$U^\varepsilon_T(s) = \int_\Gamma G^\varepsilon(Y(s) - Y(s')) F_Y(s') ds'$$

$$= \int_\Gamma -\partial_{s'}[G^\varepsilon(Y(s) - Y(s'))] SY'(s') ds'.$$

(3.10)

Here

$$G^\varepsilon(x) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \delta_\varepsilon(x - y) G(y - z) \delta_\varepsilon(z) dy \, dz = [(\delta_\varepsilon \ast \delta_\varepsilon) \ast G](x)$$

is the regularized fundamental solution. Define

$$\varphi(x) = \phi * \phi(x), \quad \varphi_\varepsilon(x) \triangleq \frac{1}{\varepsilon^2} \varphi\left(\frac{x}{\varepsilon}\right).$$

(3.12)

Obviously, $\varphi$ is compactly supported, radially symmetric, and smooth; in addition, $\varphi_\varepsilon = \delta_\varepsilon \ast \delta_\varepsilon$. Hence, we may write $G^\varepsilon = \varphi_\varepsilon \ast G$, and (3.10) becomes

$$(U^\varepsilon_T)_j(s) = \int_\Gamma -\partial_{s'}[G_{jk}^\varepsilon(Y(s) - Y(s'))] \cdot SY'(s') ds'$$

$$= \int_\Gamma SY_j'(s') Y_k'(s') \partial_{s'}[\varphi_\varepsilon * G_{jk}](Y(s) - Y(s')) ds'.$$

(3.13)

Here $G_{jk}$ is the $(j,k)$-entry of $G$. Define the Fourier transform in $\mathbb{R}^d$ and its inverse as follows

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i x \cdot \xi} \, dx, \quad f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{i x \cdot \xi} \, d\xi.$$

With this definition,

$$\hat{f} * \hat{g} = \hat{f \ast g}.$$

Since $\partial_{s'} \varphi_\varepsilon \ast G$ is sufficiently regular, we rewrite it in (3.13) using its Fourier transform

$$(U^\varepsilon_T)_j(s) = \frac{1}{(2\pi)^2} \int_\Gamma ds' \, SY_j'(s') Y_k'(s')$$

$$\cdot \int_{\mathbb{R}^2} e^{i(Y(s) - Y(s')) \cdot \xi} (\partial_{s'} \varphi_\varepsilon \ast G_{jk}) \hat{f}(\xi) \, d\xi$$

$$= \frac{1}{4\pi^2} \int_\Gamma ds' \, S \int_{\mathbb{R}^2} e^{i(Y(s) - Y(s')) \cdot \xi} \cdot \frac{\Phi_\varepsilon(\xi)}{||\xi||^2} \left( Y_j'(s') - \frac{Y_j'(s') \cdot \xi}{||\xi||^2} \xi_j \right) d\xi.$$

(3.14)
In the last line, we used the fact that

$$[\partial_t \varphi_\xi \ast G](\xi) = \frac{i \xi \hat{\varphi}_\xi(\xi)}{|\xi|^2} \left( \text{Id} - \frac{\xi \otimes \xi}{|\xi|^2} \right).$$

In order to simplify (3.14), we introduce a new variable $\eta \triangleq (\eta_1, \eta_2) \in \mathbb{R}^2$ to replace $\xi$ such that

$$\xi = \frac{Y(s) - Y(s')}{|Y(s) - Y(s')|} \cdot \eta_1 + \left( \frac{Y(s) - Y(s')}{|Y(s) - Y(s')|} \right) \perp \eta_2.$$

Note that when $s' \neq s$, $|Y(s') - Y(s)| \neq 0$ due to (1.7). Here $\perp$ means rotating a vector in $\mathbb{R}^2$ counterclockwise by $\pi/2$. We also denote

(3.15) 
$$A(s, s') = Y'(s') \cdot (Y(s') - Y(s)).$$

(3.16) 
$$B(s, s') = Y'(s') \cdot (Y(s') - Y(s)) \perp.$$

(3.17) 
$$D(s, s') = |Y(s') - Y(s)|.$$

Hence,

(3.18) 
$$Y'(s') = \frac{1}{D^2} [A(Y(s') - Y(s)) + B(Y(s') - Y(s)) \perp].$$

and

$$Y'(s') : \xi = -\frac{A\eta_1 + B\eta_2}{D}.$$

Then (3.14) becomes

$$U^\varphi_Y(s) = \frac{1}{4\pi^2} \int_T ds' \int_{\mathbb{R}^2} d\eta e^{iD\eta_1} \cdot \frac{-i(A\eta_1 + B\eta_2)}{D} \cdot \hat{\varphi}_\eta(\eta) \frac{\eta_1^2 + \eta_2^2}{D^2} \cdot \left( \frac{1}{D^2} [A(Y(s') - Y(s)) + B(Y(s') - Y(s)) \perp] - \frac{A\eta_1 + B\eta_2}{(\eta_1^2 + \eta_2^2)D^2} [(Y(s') - Y(s))\eta_1 + (Y(s') - Y(s)) \perp \eta_2] \right).$$
Here we used $\hat{\Phi}_e(\xi) = \hat{\Phi}_e(\eta)$ thanks to the radial symmetry of $\hat{\Phi}_e$. We further simplify this using the fact that $\hat{\Phi}_e(\eta_1, \eta_2)$ is even in $\eta_2$.

\[
\begin{align*}
U_\xi(s) &= -\frac{1}{4\pi^2} \int_{\mathbb{T}} ds' \frac{S}{D^3}(Y(s') - Y(s)) \\
&\quad \cdot \int_{\mathbb{T}^2} d\eta e^{iD\eta_1} \cdot i(A\eta_1 + B\eta_2)(A\eta_2 - B\eta_1)\eta_2 \frac{\hat{\Phi}_e(\eta)}{(\eta_1^2 + \eta_2^2)^2} \\
&\quad + \frac{1}{4\pi^2} \int_{\mathbb{T}} ds' \frac{S}{D^3}(Y(s') - Y(s))^\perp \\
&\quad \cdot \int_{\mathbb{T}^2} d\eta e^{iD\eta_1} \cdot i(A\eta_1 + B\eta_2)(A\eta_2 - B\eta_1)\eta_1 \frac{\hat{\Phi}_e(\eta)}{(\eta_1^2 + \eta_2^2)^2} \\
&= \frac{1}{4\pi} \int_{\mathbb{T}} ds' \frac{S(A^2 - B^2)}{D^3}(Y(s') - Y(s)) \cdot \frac{1}{\pi} \int_{\mathbb{T}^2} d\eta e^{iD\eta_1} \frac{-i\eta_1 \eta_2^2}{(\eta_1^2 + \eta_2^2)^2} \hat{\Phi}_e(\eta) \\
&\quad + \frac{1}{4\pi} \int_{\mathbb{T}} ds' \frac{SAB}{D^3}(Y(s') - Y(s))^\perp \cdot \frac{1}{\pi} \int_{\mathbb{T}^2} d\eta e^{iD\eta_1} \frac{-i(\eta_1^2 - \eta_2^2)\eta_2}{(\eta_1^2 + \eta_2^2)^2} \hat{\Phi}_e(\eta).
\end{align*}
\]  

Here the inner integral is absolutely integrable so that Fubini’s theorem can be applied to omit terms that are odd in $\eta_2$. For $x \in \mathbb{R}$, define

\[
\begin{align*}
f_1(x) &= \frac{x}{\pi} \int_{\mathbb{T}^2} e^{i\eta_1} \frac{-i\eta_1 \eta_2^2}{(\eta_1^2 + \eta_2^2)^2} \hat{\Phi}(\eta) \ d\eta, \\
f_2(x) &= \frac{x}{\pi} \int_{\mathbb{T}^2} e^{i\eta_1} \frac{-i\eta_1 (\eta_1^2 - \eta_2^2)}{(\eta_1^2 + \eta_2^2)^2} \hat{\Phi}(\eta) \ d\eta.
\end{align*}
\]

Since $\hat{\Phi}_e(\eta) = \hat{\Phi}(\varepsilon \eta)$, (3.19) becomes

\[
U_\xi(s) = \frac{1}{4\pi} \int_{\mathbb{T}} ds' \frac{S(A^2 - B^2)}{D^4}(Y(s') - Y(s)) \cdot f_1 \left( \frac{D}{\varepsilon} \right) \\
+ \frac{1}{4\pi} \int_{\mathbb{T}} ds' \frac{SAB}{D^4}(Y(s') - Y(s))^\perp \cdot f_2 \left( \frac{D}{\varepsilon} \right).
\]

On the other hand, we rewrite $U_Y$ in (3.9) as

\[
U_Y(s) = \text{p.v.} \int_{\mathbb{T}} \partial_{s'}[G(Y(s) - Y(s'))] SY'(s') ds'
\]

\[
= \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{T}} S \left[ -\frac{|Y'(s')|^2}{|Y(s') - Y(s)|^2} + \frac{2[(Y(s') - Y(s)) \cdot Y'(s')]^2}{|Y(s') - Y(s)|^4} \right] \\
\cdot (Y(s') - Y(s)) ds'
\]

\[
= \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{T}} \frac{S(A^2 - B^2)}{D^4}(Y(s') - Y(s)) ds'.
\]
Combining (3.22) and (3.23), we obtain a representation of the regularization error in the string velocity

\[ U^e_Y(s) - U_Y(s) = \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{T}} \frac{S(A^2 - B^2)}{D^4} (Y(s') - Y(s)) \cdot f_3 \left( \frac{D}{\varepsilon} \right) ds' \]

\[ + \frac{1}{4\pi} \int_{\mathbb{T}} \frac{SAB}{D^4} (Y(s') - Y(s)) \cdot f_2 \left( \frac{D}{\varepsilon} \right) ds', \]

where

\[ f_3(x) \triangleq f_1(x) - 1. \]

Concerning \( f_2 \) and \( f_3 \), we can show the following:

**Lemma 3.3** (Estimates for \( f_2 \) and \( f_3 \)). Let \( f_2 \) and \( f_3 \) be defined by (3.20), (3.21), and (3.25). For \( k = 0, 1, 2 \),

\[ \left| f_2^{(k)}(x) \right| + \left| f_3^{(k)}(x) \right| \leq \frac{C}{1 + |x|^{k+2}}. \]

Here \( f_2^{(k)} \) and \( f_3^{(k)} \) denote \( k \)th derivatives of \( f_2 \) and \( f_3 \), respectively. Let \( m_2 \) be defined in (1.14). If in addition \( m_2 = 0 \), then for \( k = 0, 1, 2 \),

\[ \left| f_2^{(k)}(x) \right| + \left| f_3^{(k)}(x) \right| \leq \frac{C}{1 + x^4}. \]

**Lemma 3.4** (Integrals of \( f_2 \) and \( f_3 \)). Let \( m_1 \) be defined in (1.13). Then

\[ \int_{\mathbb{R}} f_2(x) dx = 4m_1, \quad \int_{\mathbb{R}} f_3(x) dx = -4m_1. \]

Their proofs involve repeated integration by parts; we leave them to Appendix A.2.

For future use, define

\[ f_4 \triangleq f_2 - f_3, \quad f_5 \triangleq f_2 - 2f_3. \]

Estimates for \( f_4 \) and \( f_5 \) follow from Lemma 3.3

### 3.3 Statements of the Static Error Estimates

For clarity, we present static error estimates for the string velocities as the main results of this section. Their proofs are left to Sections 3.4–3.6.

On the \( L^2 \)-estimates for the regularization error, we have:

**Proposition 3.5** (Static \( L^2 \)-error estimate). Assume \( \phi \), the profile of the regularized data set, satisfies the assumptions in Theorem 1.4. Let \( m_1 \) and \( m_2 \) be defined in (1.13) and (1.14), respectively.

Suppose \( Y \in H^{2+\theta}(\mathbb{T}) \) with \( \theta \in [1/4, 1) \), satisfying the well-stretched condition (1.7) with \( \lambda > 0 \). Let \( F_Y(s) \) be defined by (3.2), with \( S \) satisfying Assumption (a). Given \( \varepsilon > 0 \), let \( U^e_Y \) and \( U_Y \) be the string velocities corresponding to \( Y \) in
the $\varepsilon$-regularized problem and the original problem, defined by (3.22) and (3.23), respectively. Provided that $\varepsilon \ll \lambda$,

$$
\| U_Y^\varepsilon - U_Y \|_{L^2(\mathbb{T})} \leq \frac{m_1 \varepsilon}{\pi} \left\| \frac{F_Y(s) \cdot Y'(s)}{|Y'(s)|^2} \right\|_{L^2(\mathbb{T})} + \frac{C\mu \varepsilon^{1+\theta} \ln \frac{\lambda}{\varepsilon}}{\lambda^{1+\theta}} \| Y \|_{\dot{H}^{2+\theta}(\mathbb{T})} + \frac{C\mu \varepsilon^2 \ln \frac{\lambda}{\varepsilon}}{\lambda^{5}} \left\| Y' \right\|^2_{L^2(\mathbb{T})} \left\| Y'' \right\|^2_{L^4(\mathbb{T})}.
$$

(3.27)

Here $C$ is a universal constant depending on $\theta$, and $\mu$ is defined in (3.7).

If, in addition, $m_2 = 0$, the logarithmic factors above can be removed.

Note that we require $\theta \geq \frac{1}{4}$ in Proposition 3.5 because the estimate involves $\| Y'' \|_{L^4}$ and $H^{2+(1/4)}(\mathbb{T}) \hookrightarrow W^{2,4}(\mathbb{T})$.

**Corollary 3.6 (Improved static $L^2$-error estimate for the normal velocity).** Under the assumptions of Proposition 3.5 the regularization error of the string normal velocity satisfies

$$
\left\| \left( U_Y^\varepsilon(s) - U_Y(s) \right) \cdot \frac{Y'(s)}{|Y'(s)|} \right\|_{L^2(\mathbb{T})} \leq \frac{C\mu \varepsilon^{1+\theta} \ln \frac{\lambda}{\varepsilon}}{\lambda^{1+\theta}} \| Y \|_{\dot{H}^{2+\theta}(\mathbb{T})} + \frac{C\mu \varepsilon^2 \ln \frac{\lambda}{\varepsilon}}{\lambda^{5}} \left\| Y' \right\|^2_{L^2(\mathbb{T})} \left\| Y'' \right\|^2_{L^4(\mathbb{T})}.
$$

If, in addition, $m_2 = 0$, the logarithmic factors above can be removed.

We will prove Proposition 3.5 and Corollary 3.6 in Section 3.5. Proposition 3.5 implies that in general, $\| U_Y^\varepsilon - U_Y \|_{L^2}$ is of order $O(\varepsilon)$, while Corollary 3.6 shows that such $O(\varepsilon)$-error arises only from the tangential component of the string velocity. Improved error bounds may be achieved when $m_1 = 0$ or $(F_Y \cdot Y')/|Y'| \equiv 0$. We should highlight that this has a clear physical interpretation, which will be discussed in Section 5. At this point, we remark that $(F_Y \cdot Y')/|Y'| \equiv 0$ holds true if and only if $S(p, s) = kp^{-1}$, with $k \geq 0$ being a constant. In such case, the normal force in the Eulerian coordinate is proportional to the local curvature of the string, which is the situation when describing moving interfaces with surface tension between two fluid domains [31, 32, 37]. The condition $m_1 = 0$ may be achieved by suitably choosing the regularized $\delta$-function. Find more discussions on this in Section 5 as well.

Our analysis also indicates that another moment-type condition $m_2 = 0$ may only benefit the error bound by a logarithmic factor in the higher-order terms.

The next result is concerned with $H^1$-estimates of $U_Y^\varepsilon - U_Y$, whose proof will be provided in Section 3.6.
**Proposition 3.7** (Static $H^1$-error estimate). Under the assumptions of Proposition 3.5,

\[ \| U^*_Y - U_Y \|_{\dot{H}^1(T)} \leq \frac{C \mu \varepsilon}{\lambda^2} \| Y \|_{\dot{H}^2 + \eta(T)} + \frac{C \mu \varepsilon}{\lambda^4} \| Y' \|_{L^\infty(T)} \| Y'' \|_{L^4(T)}. \]

Proposition 1.12 follows from Proposition 3.5 and Proposition 3.7 by taking $F_Y = Y_{xx}$ and $\mu = 1$.

### 3.4 Preliminary Estimates

In what follows, we shall write $s' = s + \tau$. Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator on $T$. Define

\[ Q = \| Y' \|_{L^\infty(T)}, \quad R(s, \tau) = |\mathcal{M}Y''(s + \tau)| + |Y''(s + \tau)|. \]

We shall omit the arguments of $R$ whenever it is convenient. As before, $SY'(s) \triangleq S(|Y'(s)|, s)Y'(s)$.

**Lemma 3.8.** Suppose $Y \in H^2(T)$ and $\tau \in [-\pi, \pi]$. Let $A, B, D, \text{ and } S$ be defined in (3.15)–(3.17) and (3.1), respectively. Then:

1. We have
   \[ |Y(s + \tau) - Y(s) - \tau Y'(s)| \leq C \tau^2 R, \]
   \[ |Y(s + \tau) - Y(s) - \tau Y'(s + \tau)| \leq C \tau^2 R. \]
   Hence,
   \[ |D(s, s + \tau) - |\tau||Y'(s)|| \leq C \tau^2 R, \]
   \[ |D(s, s + \tau) - |\tau||Y'(s + \tau)|| \leq C \tau^2 R. \]

2. \[ |B(s, s + \tau)| \leq C \tau^2 QR. \]

3. Under Assumption \[a\] on $S$,
   \[ |SY'(s + \tau) - SY'(s)| \leq C \mu |\tau|(Q + R), \]
   with $\mu$ defined in (3.7).

**Proof.** We simply calculate that

\[ |Y(s + \tau) - Y(s) - \tau Y'(s)| \]

\[ = \left| \int_0^{\tau} Y'(s + \eta) - Y'(s) d\eta \right| \]

\[ \leq \int_0^{\tau} |Y'(s + \eta) - Y'(s + \tau)| + |Y'(s) - Y'(s + \tau)| d\eta \]

\[ \leq C \tau^2 |\mathcal{M}Y''(s + \tau)|. \]

(3.28) can be shown in a similar manner; then (3.29) and (3.30) follow immediately.
(3.31) can be proved by observing
\[ B(s, s + \tau) = (Y(s + \tau) - Y(s) - \tau Y'(s + \tau)) \cdot Y'(s + \tau). \]

Finally, by Assumption (a)
\[
|SY'(s + \tau) - SY'(s)|
\leq |S(Y'(s + \tau), s + \tau)| \cdot |Y'(s + \tau) - Y'(s)|
+ |S(|Y'(s + \tau)|, s + \tau) - S(|Y'(s)|, s)| \cdot |Y'(s)|
\leq |Y'(s + \tau) - Y'(s)| \left( \mu + \frac{\mu |Y'(s)|}{\|Y'\|_{L^\infty(T)}} \right) + \mu |\tau| |Y'(s)|
\leq C\mu |\tau| (|MY''(s + \tau)| + |Y'(s)|).
\]
This proves (3.32). \(\square\)

**Lemma 3.9.** Assume \( Y \in H^{2+\theta}(T) \) with \( \theta \in (0, 1) \) and \( \tau \in [-\pi, \pi] \). Let \( A, B, \) and \( D \) be defined in (3.15)–(3.17), respectively. Then
\[
\begin{align*}
|\partial_s A(s, s + \tau)| &\leq C |\tau| Q R, \\
|\partial_s B(s, s + \tau)| &\leq C |\tau| Q R, \\
|\partial_s D(s, s + \tau)| &\leq C |\tau| R.
\end{align*}
\]
Moreover, for \( \forall \beta \in (0, \theta] \),
\[
|\partial_s B(s, s + \tau)|
\leq C \tau^2 R^2 + C |\tau|^{1+\beta} |Y'(s + \tau)|
\cdot \left| \int_0^\tau \frac{Y''(s + \tau - \eta) - Y''(s + \tau)}{|\eta|^{1+2\beta}} d\eta \right|^{1/2}.
\]

**Proof.** We calculate that
\[
\begin{align*}
\partial_s A(s, s + \tau) &= Y''(s + \tau) \cdot (Y(s + \tau) - Y(s))
+ Y'(s + \tau) \cdot (Y'(s + \tau) - Y'(s)), \\
\partial_s B(s, s + \tau) &= Y''(s + \tau) \cdot (Y(s + \tau) - Y(s))
+ Y'(s + \tau) \cdot (Y'(s + \tau) - Y'(s)), \\
\partial_s D(s, s + \tau) &= \frac{(Y'(s + \tau) - Y'(s)) \cdot (Y(s + \tau) - Y(s))}{|Y(s + \tau) - Y(s)|}.
\end{align*}
\]
Then (3.34)–(3.36) follow immediately from
\[ |Y'(s + \tau) - Y'(s)| \leq |\tau| |MY''(s + \tau)|. \]
To prove (3.37), we rewrite
\[
\partial_s B(s, s + \tau) = Y''(s + \tau) \cdot (Y(s + \tau) - Y(s) - \tau Y'(s + \tau)) + \nabla Y'(s + \tau) \cdot (Y'(s + \tau) - \tau Y''(s + \tau)).
\]
Hence, by Lemma 3.8,
\[
|\partial_s B(s, s + \tau)| \leq C |Y''(s + \tau)| \cdot \tau^2 R + |Y'(s + \tau)| \left| \int_0^\tau Y''(s + \eta) - Y''(s + \tau) \, d\eta \right|.
\]
With \(\beta \in (0, \theta]\),
\[
|\partial_s B(s, s + \tau)| \leq C \tau^2 R^2 + |Y'(s + \tau)| \left| \int_0^\tau \frac{|\eta - \tau|^{1+2\beta}}{\eta^{1+2\beta}} \, d\eta \right|^{1/2} \left| \int_0^\tau |Y''(s + \eta) - Y''(s + \tau)|^{2} \, d\eta \right|^{1/2} \leq C \tau^2 R^2 + C |\tau|^{1+\beta} |Y'(s + \tau)| \left| \int_0^\tau \frac{|Y''(s + \tau - \eta) - Y''(s + \tau)|^{2}}{|\eta|^{1+2\beta}} \, d\eta \right|^{1/2}.
\]

3.5 Proof of the Static \(L^2\)-Error Estimate

PROOF OF PROPOSITION 3.5

Step 1 (Splitting the regularization error). We start by rewriting (3.24). By (3.18) and (3.26),
\[
(3.38) \quad 4\pi [U_\ell^\varepsilon(s) - U_\ell Y(s)]
= \text{p.v.} \int_T S \cdot \frac{B^2}{D^2} \cdot \frac{Y(s + \tau) - Y(s)}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) \, d\tau
+ \text{p.v.} \int_T S \left[ \frac{A^2}{D^4} (Y(s + \tau) - Y(s)) + \frac{AB}{D^2} (Y(s + \tau) - Y(s)) \cdot f_3 \left( \frac{D}{\varepsilon} \right) \, d\tau
+ \text{p.v.} \int_T S \left[ \frac{AB}{D^4} (Y(s + \tau) - Y(s)) \cdot f_4 \left( \frac{D}{\varepsilon} \right) \, d\tau =
\]
\[ \text{p.v.} \int_{\mathbb{T}} S \cdot \frac{B^2}{D^2} \cdot \frac{Y(s + \tau) - Y(s)}{D^2} \cdot f_5 \left( \frac{D}{\varepsilon} \right) d\tau + \text{p.v.} \int_{\mathbb{T}} A \frac{A}{D^2} \cdot SY'(s + \tau) \cdot f_3 \left( \frac{D}{\varepsilon} \right) d\tau + \text{p.v.} \int_{\mathbb{T}} \frac{SB}{D^2} \cdot Y'(s + \tau) \cdot f_4 \left( \frac{D}{\varepsilon} \right) d\tau \]

\( \triangleq E_1(s) + E_2(s) + E_3(s) \).

**Step 2 (Estimate for** \( E_1 \)). By (1.7), (3.3), Lemma 3.3 and Lemma 3.8

\[ |E_1(s)| \leq C \int_{\mathbb{T}} \mu \cdot \left( \frac{t^2 Q R}{\lambda |\tau|} \right)^2 \cdot \frac{1}{\lambda |\tau|} \cdot \frac{1}{1 + (\frac{\lambda \tau}{\varepsilon})^2} d\tau \]

(3.39)

\[ \leq \frac{C \mu Q^2}{\lambda^3} \int_{\mathbb{T}} R^2 \cdot \frac{|\tau|}{1 + (\frac{\lambda \tau}{\varepsilon})^2} d\tau. \]

By the Minkowski inequality,

\[ \| E_1 \|_{L^2(\mathbb{T})} \leq \frac{C \mu^2 Q^2}{\lambda^3} \int_{\mathbb{T}} \| R^2 \|_{L^2(\mathbb{T})} \cdot \frac{|\tau|}{1 + (\frac{\lambda \tau}{\varepsilon})^2} d\tau \]

(3.40)

\[ \leq \frac{C \mu^2 |\ln(\lambda / \varepsilon)|}{\lambda^5} \| Y'' \|_{L^2(\mathbb{T})}^2 \| Y'' \|_{L^4(\mathbb{T})}. \]

**Step 3 (Partial estimate for** \( E_2 \)). We further split \( E_2(s) \).

\[ E_2(s) = SY'(s) \cdot \text{p.v.} \int_{\mathbb{T}} A \frac{A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) d\tau + \text{p.v.} \int_{\mathbb{T}} \frac{A}{D^2} \cdot \left[ SY'(s + \tau) - SY'(s) \right] \cdot f_3 \left( \frac{D}{\varepsilon} \right) d\tau + \text{p.v.} \int_{\mathbb{T}} \frac{1}{\tau} \cdot \left[ SY'(s + \tau) - SY'(s) \right] \cdot \left[ f_3 \left( \frac{D}{\varepsilon} \right) - f_3 \left( \frac{|Y'(s)| |\tau|}{\varepsilon} \right) \right] d\tau + \text{p.v.} \int_{\mathbb{T}} \frac{SY'(s + \tau) - SY'(s)}{\tau} \cdot f_3 \left( \frac{|Y'(s)| |\tau|}{\varepsilon} \right) d\tau \]

\( \triangleq E_{2,1}(s) + E_{2,2}(s) + E_{2,3}(s) + E_{2,4}(s). \)

It would be clear later that \( E_{2,4} \) accounts for the most singular part in \( E_2 \).
We claim that $E_{2,1} = 0$. Indeed, since $\frac{d D}{d \tau} = \frac{A}{D}$,

$$E_{2,1}(s) = SY'(s) \cdot \left[ \lim_{\eta \to 0^+} \int_{\eta}^{\pi} \frac{A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) d \tau + \int_{-\pi}^{-\eta} \frac{A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) d \tau \right]$$

$$= SY'(s) \cdot \lim_{\eta \to 0^+} \left[ \int_{D(s,s+\pi)}^{D(s,s+\pi)} \frac{1}{D} \cdot f_3 \left( \frac{D}{\varepsilon} \right) d D \right]$$

$$= SY'(s) \cdot \left[ \int_{D(s,s-\pi)}^{D(s,s+\pi)} \frac{1}{D} \cdot f_3 \left( \frac{D}{\varepsilon} \right) d D \right]$$

The first term is 0 since $D(s, s + \pi) = D(s, s - \pi)$. For the second term,

$$|D(s, s + \eta) - D(s, s - \eta)|$$

$$\leq |(Y(s + \eta) - Y(s)) - (Y(s) - Y(s - \eta))|$$

$$\leq \left| \int_{0}^{\eta} Y'(s + \omega) - Y'(s + \omega - \eta) d \omega \right|$$

$$\leq C|\eta|^{3/2} \|Y'\|_{C^{1/2}(\mathbb{T})} \leq C|\eta|^{3/2} \|Y\|_{\dot{H}^2(\mathbb{T})}.$$}

Since $D(s, s + \eta) \geq \lambda |\eta|$ by (1.7) and $|f_3| \leq C$ by Lemma 3.3, we have that

$$|E_{2,1}(s)| \leq \mu |Y'(s)| \cdot \lim_{\eta \to 0^+} C|\eta|^{3/2} \|Y\|_{\dot{H}^2(\mathbb{T})} \cdot \frac{1}{\lambda |\eta|} \cdot 1 = 0.$$}

For $E_{2,2}$, by Lemma 3.3 and Lemma 3.8,

$$|E_{2,2}(s)| \leq C \int_{\mathbb{T}} \frac{|D^2 - \tau A|}{D^2} \cdot \left| SY'(s + \tau) - SY'(s) \right| \cdot \frac{1}{1 + (\frac{\lambda \tau}{\varepsilon})^2} d \tau$$

$$\leq C \int_{\mathbb{T}} \frac{|D||Y(s + \tau) - Y(s) - \tau Y'(s + \tau)|}{D^2}$$

$$\cdot \mu(Q + R) \cdot \frac{1}{1 + (\frac{\lambda \tau}{\varepsilon})^2} d \tau$$

$$\leq \frac{C\mu}{\lambda} \int_{\mathbb{T}} \frac{\left| \tau |R(Q + R)| \right|}{1 + (\frac{\lambda \tau}{\varepsilon})^2} d \tau.$$
For $E_{2,3}$, by Lemma 3.8 and the mean value theorem,

$$|E_{2,3}(s)| \leq C \int_{\mathbb{T}} \mu(Q + R) \cdot \left| f_3 \left( \frac{D(s, s + \tau)}{\varepsilon} \right) - f_3 \left( \frac{|Y'(s)| \tau}{\varepsilon} \right) \right| d\tau$$

(3.43)

where $\xi(s, \tau) \in \mathbb{R}$ is between $D(s, s + \tau)$ and $|\tau||Y'(s)|$. It is clear that $|\xi(s, \tau)| \geq \lambda|\tau|$. Again by Lemma 3.3 and Lemma 3.8

$$|E_{2,3}(s)| \leq C \mu \int_{\mathbb{T}} (Q + R) \cdot \frac{\tau^2 R}{\varepsilon} \cdot \frac{1}{1 + \left( \frac{\lambda|\tau|}{\varepsilon} \right)^3} d\tau$$

(3.44)

Here we used the fact that

$$\frac{|x|^p}{1 + |x|^{p+2}} \leq \frac{C}{1 + x^2} \quad \forall n \in \mathbb{N}. $$

In order to bound $E_{2,4}$, we recall that $\mathcal{P}_K$ is defined in (1.25). For convenience, define $\mathcal{Q}_K = \text{Id} - \mathcal{P}_K$; both $\mathcal{P}_K$ and $\mathcal{Q}_K$ commute with differentiation. With $K \in \mathbb{N}_+$ to be chosen,

$$|E_{2,4}(s) - \mathcal{P}_K[(SY')'](s)| \int_{\mathbb{T}} f_3 \left( \frac{|Y'(s)| \tau}{\varepsilon} \right) d\tau$$

(3.45)
\[ C \int_{\Omega} \left[ |\tau| |\mathcal{M}(\mathcal{P}_K [S Y']') (s) | + |\mathcal{M}(\mathcal{Q}_K [S Y']) (s) | \right] \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} \, d\tau \]

\[ \leq \frac{C \varepsilon^2 \ln(\lambda / \varepsilon)}{\lambda^2} |\mathcal{M}(\mathcal{P}_K F_Y') (s) | + \frac{C \varepsilon}{\lambda} |\mathcal{M}(\mathcal{Q}_K F_Y) (s) |. \]

Combining (3.41), (3.42), (3.44), and (3.45), we apply Minkowski and Sobolev inequalities to find

\[ \left\| E_2 - \mathcal{P}_K F_Y (s) \int_{\Omega} f_3 \left( \frac{|Y' (s) | |\tau|}{\varepsilon} \right) \, d\tau \right\|_{L^2 (\Omega)} \]

\[ \leq \frac{C \mu}{\lambda} \int_{\Omega} \| R (Q + R) \|_{L^2 (\Omega)} \cdot \frac{|\tau|}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} \, d\tau \]

\[ + \frac{C \varepsilon^2 \ln(\lambda / \varepsilon)}{\lambda^2} \| \mathcal{P}_K F_Y' \|_{L^2 (\Omega)} + \frac{C \varepsilon}{\lambda} \| \mathcal{Q}_K F_Y \|_{L^2 (\Omega)} \]

\[ \leq \frac{C \mu \varepsilon^2 \ln(\lambda / \varepsilon)}{\lambda^3} |Y''|_{L^2 (\Omega)}^2 \]

\[ + \left( \frac{C \varepsilon^2 \ln(\lambda / \varepsilon)}{\lambda^2} \cdot \frac{C \varepsilon}{\lambda} \cdot K^{1-\theta} \right) \| F_Y \|_{H^\theta (\Omega)}. \]

By Lemma 3.1 and taking \( K \sim \frac{\lambda}{\varepsilon \ln(\frac{\lambda}{\varepsilon})} \),

\[ \left\| E_2 - \mathcal{P}_K F_Y \int_{\Omega} f_3 \left( \frac{|Y' (s) | |\tau|}{\varepsilon} \right) \, d\tau \right\|_{L^2 (\Omega)} \]

\[ \leq \frac{C \mu \varepsilon^2 \ln(\lambda / \varepsilon)}{\lambda^3} |Y''|_{L^2 (\Omega)}^2 + \frac{C \mu \varepsilon^{1+\theta} \ln^\theta (\lambda / \varepsilon)}{\lambda^{1+\theta}} \| Y \|_{\dot{H}^{2+\theta} (\Omega)}. \]

The extra term on the left-hand side of (3.47) will be handled after we bound \( E_3 \).
Step 4 (Partial estimate for $E_3$). Again we further split $E_3(s)$. Thanks to (3.33),

$$E_3(s)$$

$$= \text{p.v.} \int_T \frac{SB}{D^2} \cdot (Y'(s + \tau) - Y'(s)) \cdot f_4 \left( \frac{D}{\varepsilon} \right) d\tau$$

$$+ \text{p.v.} \int_T \frac{SB}{D^2} \cdot Y'(s) \cdot f_4 \left( \frac{D}{\varepsilon} \right) - f_4 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau$$

$$+ \text{p.v.} \int_T SB \cdot \left( \frac{1}{D^2} - \frac{1}{\tau^2|Y'(s)|^2} \right) \cdot Y'(s) \cdot f_4 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau$$

$$+ \text{p.v.} \int_T S(Y'(s'), s') \cdot (Y(s') - Y(s) - \tau Y'(s')) \cdot (Y'(s') - Y'(s))$$

$$= \frac{Y'(s)}{\tau^2|Y'(s)|^2} f_4 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau$$

$$+ \text{p.v.} \int_T \left[ \left( \int_0^\tau S(Y'(s + \tau), s + \tau) Y'(s + \eta) - SY'(s + \eta) d\eta \right) \right.$$

$$\left. \cdot Y'(s) \right] \frac{Y'(s)}{\tau^2|Y'(s)|^2} f_4 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau$$

$$+ \text{p.v.} \int_T \left[ \left( \int_0^\tau SY'(s + \eta) - SY'(s + \tau) d\eta \right) \right.$$

$$\left. \cdot Y'(s) \right] \frac{Y'(s)}{\tau^2|Y'(s)|^2} f_4 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau$$

$$\triangleq \sum_{i=1}^6 E_{3,i}(s).$$

In $E_{3,5}$ and $E_{3,6}$, we used the identity that

$$S([Y'(s'), s'] \cdot [Y(s') - Y(s) - \tau Y'(s')])$$

$$= \int_0^\tau [S(Y'(s'), s') - S([Y'(s + \eta), s + \eta])Y'(s + \eta) d\eta$$

$$+ \int_0^\tau SY'(s + \eta) - SY'(s') d\eta.$$
\[ |E_{3,2}(s)| \leq C\mu \int_T \frac{\tau^2 Q R}{\lambda^2 \tau^2} \cdot \frac{\tau^2 R}{\varepsilon} \cdot \frac{1}{1 + \left(\frac{\lambda \tau}{\varepsilon}\right)^2} d\tau \]

\[ \leq \frac{C\mu}{\lambda^2} \int_T Q^2 R^2 \cdot \frac{|\tau|}{1 + \left(\frac{\lambda \tau}{\varepsilon}\right)^2} d\tau. \]

(3.49)

\[ |E_{3,3}(s)| \leq C\mu \int_T \frac{\tau^2 Q R}{\lambda^2 |\tau|^3} \cdot \frac{1}{1 + \left(\frac{\lambda \tau}{\varepsilon}\right)^2} d\tau \]

\[ \leq \frac{C\mu}{\lambda^2} \int_T Q^2 R^2 \cdot \frac{|\tau|}{1 + \left(\frac{\lambda \tau}{\varepsilon}\right)^2} d\tau. \]

(3.50)

\[ |E_{3,4}(s)| \leq C\mu \int_T |\tau|^3 R^2 \cdot \frac{1}{\tau^2 \lambda} \cdot \frac{1}{1 + \left(\frac{\lambda \tau}{\varepsilon}\right)^2} d\tau \]

\[ \leq \frac{C\mu}{\lambda} \int_T R^2 \cdot \frac{|\tau|}{1 + \left(\frac{\lambda \tau}{\varepsilon}\right)^2} d\tau. \]

(3.51)

Note that for \(E_{3,2}\), we applied the mean value theorem and proceeded as in (3.43) and (3.44).

To handle \(E_{3,5}\), we purposefully put an extra \(Y'(s)\) into the integral without changing its value, i.e.,

\[
E_{3,5}(s) = \text{p.v.} \int_T \left( \int_0^\tau \left[ S(|Y'(s + \tau)|, s + \tau) - S(|Y'(s + \eta)|, s + \eta) \right] \right) \cdot \left( Y'(s + \eta) - Y'(s) \right) d\eta \cdot Y'(s) \]

\[ \cdot \frac{Y'(s)}{\tau^2|Y'(s)|} \cdot f_4 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau. \]

Since

\[
|Y'(s + \eta) - Y'(s)| \leq |Y'(s + \eta) - Y'(s + \tau)| + |Y'(s + \tau) - Y'(s)| \leq C |\tau||M'Y''(s + \tau)|.
\]
by Lemma 3.3 and Assumption (a),

\[
|E_{3,5}(s)| 
\leq C \int_{\mathbb{T}} \left[ \int_{0}^{\tau} \left( \mu |\tau| + \frac{\mu |\tau| |\mathcal{M} Y''(s + \tau)|}{\| Y'' \|_{L^\infty(\mathbb{T})}} \right) \cdot |\tau| |\mathcal{M} Y''(s + \tau)| d\eta \right] 
\cdot \frac{1}{\tau^2} \cdot \frac{1}{1 + (\frac{\lambda \tau}{\varepsilon})^2} d\tau 
\leq \frac{C \mu}{\lambda} \int_{\mathbb{T}} (Q + R) \cdot \frac{|\tau|}{1 + (\frac{\lambda \tau}{\varepsilon})^2} d\tau.
\]

Finally, for \( E_{3,6} \), we take the same strategy as in estimating \( E_{2,4} \), choosing the same \( K \) as before. Indeed,

\[
E_{3,6}(s) = \text{p.v.} \int_{\mathbb{T}} \left[ \left( \int_{0}^{\tau} \mathcal{P}_K [SY'](s + \eta) - \mathcal{P}_K [SY'](s + \tau) d\eta \right) + \frac{\tau^2}{2} \mathcal{P}_K [(SY')'](s) \right] \cdot Y'(s) 
\cdot \frac{Y'(s)}{\tau^2|Y'(s)|^2} f_4 \left( \frac{|Y'(s)| |\tau|}{\varepsilon} \right) d\tau 
+ \frac{1}{2} \mathcal{P}_K [(SY')'](s) \cdot Y'(s) \cdot \frac{Y'(s)}{|Y'(s)|^2} \int_{\mathbb{T}} f_4 \left( \frac{|Y'(s)| |\tau|}{\varepsilon} \right) d\tau 
+ \text{p.v.} \int_{\mathbb{T}} \left[ \left( \int_{0}^{\tau} \mathcal{Q}_K [SY'](s + \eta) - \mathcal{Q}_K [SY'](s + \tau) d\eta \right) \right] \cdot Y'(s) 
\cdot \frac{Y'(s)}{\tau^2|Y'(s)|^2} f_4 \left( \frac{|Y'(s)| |\tau|}{\varepsilon} \right) d\tau.
\]

We derive that

\[
\left| \int_{0}^{\tau} \mathcal{P}_K [SY'](s + \eta) - \mathcal{P}_K [SY'](s + \tau) d\eta \right| + \frac{\tau^2}{2} \mathcal{P}_K [(SY')'](s) 
= \left| \int_{0}^{\tau} \int_{\eta}^{\tau} \mathcal{P}_K [(SY')'](s) - \mathcal{P}_K [(SY')'](s + \zeta) d\zeta d\eta \right| 
\leq C |\tau|^3 |\mathcal{M}(\mathcal{P}_K F')(s)|.
\]
Combine this with (3.53) and we find that

\[
|E_{3, s}(s) - \frac{1}{2} \mathcal{P}_K F_Y(s) \cdot Y'(s) \perp \frac{Y'(s) \perp}{|Y'(s)|^2} \int_T f_4 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau |
\leq C \int_T |\tau|^3 |\mathcal{M}(\mathcal{P}_K F_Y')(s)| \cdot \frac{1}{\tau^2} \cdot \frac{1}{1 + \left( \frac{\lambda}{\varepsilon} \right)^2} d\tau
\]

(3.54)

\[
+ C \int_T |\tau|^2 |\mathcal{M}(\mathcal{Q}_K F_Y)(s)| \cdot \frac{1}{\tau^2} \cdot \frac{1}{1 + \left( \frac{\lambda}{\varepsilon} \right)^2} d\tau
\]

\[
\leq \frac{C \varepsilon^2 \ln(\lambda/\varepsilon)}{\lambda^2} |\mathcal{M}(\mathcal{P}_K F_Y')(s)| + \frac{C \varepsilon}{\lambda} |\mathcal{M}(\mathcal{Q}_K F_Y)(s)|.
\]

Combining (3.48)–(3.52) and (3.54), we argue as in (3.46) and (3.47) to obtain that

\[
\left\| E_{3}(s) - \frac{1}{2} \mathcal{P}_K F_Y(s) \cdot Y'(s) \perp \frac{Y'(s) \perp}{|Y'(s)|^2} \int_T f_4 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau \right\|_{L^2(\mathbb{T})}
\leq \frac{C \mu \varepsilon^2 \ln(\lambda/\varepsilon)}{\lambda^5} \|Y''\|_{L^\infty(\mathbb{T})} \|Y''\|_{L^4(\mathbb{T})}
\]

\[
+ \frac{C \varepsilon^2 \ln(\lambda/\varepsilon)}{\lambda^2} K^{1-\theta} \|\mathcal{P}_K F_Y\|_{\dot{H}^{\theta}(\mathbb{T})}
\]

\[
+ \frac{C \varepsilon}{\lambda} K^{-\theta} \|\mathcal{Q}_K F_Y\|_{\dot{H}^{\theta}(\mathbb{T})}
\]

(3.55)

\[
\leq \frac{C \varepsilon^2 \ln(\lambda/\varepsilon)}{\lambda^5} \|Y''\|_{L^\infty(\mathbb{T})} \|Y''\|_{L^4(\mathbb{T})}
\]

\[
+ \frac{C \mu \varepsilon^{1+\theta} \ln^\theta(\lambda/\varepsilon)}{\lambda^{1+\theta}} \|Y\|_{\dot{H}^{2+\theta}(\mathbb{T})}.
\]

**Step 5 (Estimates of the extra terms).** Combining (3.40), (3.47), and (3.55),

\[
\left\| 4\pi [U_Y^s(s) - U_Y(s)] - \mathcal{P}_K F_Y(s) \cdot \frac{Y'(s) \perp}{|Y'(s)|^2} \int_T f_4 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau \right\|_{L^2(\mathbb{T})}
\leq \frac{C \mu \varepsilon^2 \ln(\lambda/\varepsilon)}{\lambda^5} \|Y''\|_{L^\infty(\mathbb{T})} \|Y''\|_{L^4(\mathbb{T})}
\]

\[
+ \frac{C \mu \varepsilon^{1+\theta} \ln^\theta(\lambda/\varepsilon)}{\lambda^{1+\theta}} \|Y\|_{\dot{H}^{2+\theta}(\mathbb{T})}
\]

(3.56)
To this end, we shall handle the extra terms on the left-hand side. By Lemma 3.4,

\[
\int_{T} f_3 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau
\]

(3.57)

\[
= \frac{\varepsilon}{|Y'(s)|} \int_{-|Y'(s)||\pi/\varepsilon|} f_3(\omega) d\omega - \frac{\varepsilon}{|Y'(s)|} \int_{|\omega|>|Y'(s)||\pi/\varepsilon|} f_3(\omega) d\omega
\]

\[
= -\frac{4m_1\varepsilon}{|Y'(s)|} \int_{|\omega|>|Y'(s)||\pi/\varepsilon|} f_3(\omega) d\omega,
\]

where

\[
\left| \frac{\varepsilon}{|Y'(s)|} \int_{|\omega|>|Y'(s)||\pi/\varepsilon|} f_3(\omega) d\omega \right| \leq \frac{C \varepsilon}{|Y'(s)|} \int_{|\omega|>|Y'(s)||\pi/\varepsilon|} \frac{1}{1+\omega^2} d\omega
\]

(3.58)

\[
\leq \frac{C \varepsilon^2}{\lambda^2}.
\]

Similarly,

\[
\int_{T} f_4 \left( \frac{|Y'(s)||\tau|}{\varepsilon} \right) d\tau
\]

(3.59)

\[
= \frac{8m_1\varepsilon}{|Y'(s)|} \int_{|\omega|>|Y'(s)||\pi/\varepsilon|} f_2(\omega) - f_3(\omega) d\omega.
\]

where the second term can be bounded as in (3.58). Hence, we combine (3.56)–(3.59) to obtain that

\[
\left| 4\pi \left[ U_Y^\varepsilon(s) - U_Y(s) \right] + \left( \mathcal{P}_K F_Y(s) \cdot \frac{4m_1\varepsilon}{|Y'(s)|} \right) \cdot \left( -\frac{1}{2} \mathcal{P}_K F_Y(s) \cdot Y'(s) \cdot \frac{Y'(s)}{|Y'(s)|^2} \cdot \frac{8m_1\varepsilon}{|Y'(s)|} \right) \right|_{L^2(T)}
\]

(3.60)

\[
\leq \frac{C \mu \varepsilon^2 \ln(\lambda/\varepsilon)}{\lambda^5} \| Y' \|_{L^2(T)}^2 \| Y'' \|_{L^4(T)}^2
\]

\[
+ \frac{C \mu \varepsilon^2 \ln(\lambda/\varepsilon)}{\lambda^{1+\theta}} \| Y \|_{H^{2+\theta}(T)}.
\]

Note that

\[
\mathcal{P}_K F_Y(s) \cdot \frac{4m_1\varepsilon}{|Y'(s)|} = \mathcal{P}_K F_Y(s) \cdot \frac{Y'(s)}{|Y'(s)|^2} \cdot \frac{8m_1\varepsilon}{|Y'(s)|}
\]

\[
= \frac{4m_1\varepsilon}{|Y'(s)|} \cdot \frac{\mathcal{P}_K F_Y(s) \cdot Y'(s)}{|Y'(s)|^2} \cdot \frac{Y'(s)}{|Y'(s)|^2} Y'(s)
\]

\[
= \frac{4m_1\varepsilon}{|Y'(s)|} \cdot \frac{F_Y(s) \cdot Y'(s)}{|Y'(s)|^2} \cdot \frac{Q_K F_Y(s) \cdot Y'(s)}{|Y'(s)|^2} \cdot \frac{Q_K F_Y(s) \cdot Y'(s)}{|Y'(s)|^2} Y'(s).
\]
By Lemma 3.1
\[
\frac{Q_K F_Y(s) \cdot Y'(s)}{|Y'(s)|^2} Y'(s) \|_{L^2(\mathbb{T})} \leq \frac{C \mu \epsilon^{1+\theta} \ln(\lambda/\epsilon)}{\lambda^{1+\theta}} \| Y \|_{\dot{H}^{2+\theta}(\mathbb{T})}.
\]
Therefore, (3.60) implies that
\[
\left\| \left[ U^h_Y(s) - U_Y(s) \right] + \frac{m_1 \epsilon}{\pi |Y'(s)|} \frac{F_Y(s) \cdot Y'(s)}{|Y'(s)|^2} Y'(s) \right\|_{L^2(\mathbb{T})} \leq \frac{C \mu \epsilon^{1+\theta} \ln(\lambda/\epsilon)}{\lambda^{1+\theta}} \| Y \|_{\dot{H}^{2+\theta}(\mathbb{T})},
\]
which proves the desired estimate in both Proposition 3.5 and Corollary 3.6.

Thanks to Lemma 3.3 if \( m_2 = 0, f_3, f_4, f_5 \), and their first derivatives will enjoy improved decay at \( \infty \). In this case, it is not difficult to verify that all the logarithmic factors in this proof can be removed. \( \square \)

Remark 3.10. All the principal value integrals in the proof except the one for \( E_{2,1} \) may be replaced by the usual integrals.

3.6 Proof of the Static \( H^1 \)-Error Estimate

Proof of Proposition 3.7. Let \( E_1(s), E_2(s) \) and \( E_3(s) \) be defined as in (3.38).

Step 1 (Estimate for \( E_1' \)). Recall that
\[
E_1(s) = \int_{\mathbb{T}} SB^2 \cdot \frac{Y(s + \tau) - Y(s)}{D^4} \cdot f_5 \left( \frac{D}{\epsilon} \right) d\tau.
\]
As mentioned in Remark 3.10, we can get rid of the principal value integral. Denote its integrand to be \( e_1(s, \tau) \).

We first give an estimate of \( \int_{\mathbb{T}} |\partial_\tau e_1(s, \tau)| d\tau \). For clarity, we start from some simpler estimates. By Lemma 3.3 and Lemma 3.8
\[
|SB^2| \leq C \mu |r|^2 Q R \cdot |D| Q,
\]
and
\[
\left| \frac{Y(s + \tau) - Y(s)}{D^4} \cdot f_5 \left( \frac{D}{\epsilon} \right) \right| \leq \frac{C}{|D|^3} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\epsilon} \right)^2}.
\]
Also by Lemma 3.9
\[ |\partial_s(SB^2)| \leq |\partial_s S(|Y'(s + \tau)|, s + \tau)|B^2| + \left| \partial_p S(|Y'(s + \tau)|, s + \tau) \cdot \frac{Y'' \cdot Y'(s + \tau)}{|Y'(s + \tau)|} \right| |B^2| \]
\[ + 2|S||B||\partial_s B| \]
(3.64)
\[ \leq C\mu \cdot (\tau^2 QR)^2 + C \frac{\mu R}{\|Y''\|_{L^\infty(\mathbb{T})}} \cdot \tau^2 QR \cdot |\tau|Q^2 \]
\[ + C\mu \cdot \tau^2 QR \cdot |\tau|QR \leq C\mu |\tau|^3 Q^2 R^2. \]

Lastly,
\[ \partial_s \left( \frac{Y(s + \tau) - Y(s)}{D^4} \cdot f_5 \left( \frac{D}{\varepsilon} \right) \right) \]
\[ \leq \frac{Y'(s + \tau) - Y'(s)}{D^4} \cdot f_5 \left( \frac{D}{\varepsilon} \right) \]
\[ + \frac{Y(s + \tau) - Y(s)}{D^4} \cdot \left( \frac{1}{\varepsilon} f_5' \left( \frac{D}{\varepsilon} \right) - \frac{4}{D} f_5 \left( \frac{D}{\varepsilon} \right) \right) \partial_s D. \]
(3.65)

Hence,
\[ \left| \partial_s \left( \frac{Y(s + \tau) - Y(s)}{D^4} \cdot f_5 \left( \frac{D}{\varepsilon} \right) \right) \right| \]
\[ \leq C \frac{|\tau|R}{|D|^3} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} \]
\[ + C \frac{1}{|D|^3} \left( \frac{1}{\varepsilon} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^3} + \frac{1}{|D|} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} \right) |\tau| R \]
\[ \leq \frac{CR}{\lambda |D|^3} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2}. \]
(3.66)

Combining (3.62)–(3.66), we find that
\[ \int_{\mathbb{T}} |\partial_s e_1(s, \tau)| d\tau \leq \int_{\mathbb{T}} \left| \partial_s (SB^2) \right| \left| \frac{Y(s + \tau) - Y(s)}{D^4} \cdot f_5 \left( \frac{D}{\varepsilon} \right) \right| \]
\[ + |SB^2| \left| \partial_s \left( \frac{Y(s + \tau) - Y(s)}{D^4} \cdot f_5 \left( \frac{D}{\varepsilon} \right) \right) \right| d\tau \]
\[ \leq C\mu Q^2 \int_{\mathbb{T}} \frac{R^2}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} d\tau. \]
By the Minkowski inequality,

\[ \left\| \int_T |\partial_s e_1(s, \tau)| d\tau \right\|_{L^2(T)} \leq \frac{C\mu}{4} \|Y'\|_{L^2(\mathbb{T})}^2 \|Y''\|_{L^4(\mathbb{T})}. \]

To this end, we claim that \( E_1'(s) = \int_T \partial_s e_1(s, \tau) d\tau \). Indeed, it is clear from the estimate (3.39) of \( E_1(s) \) that \( e_1 \in L^1(\mathbb{T} \times \mathbb{T}) \); so is \( \partial_s e_1 \) by (3.67). Hence, if we take an arbitrary \( \psi(s) \in C^\infty(\mathbb{T}) \), by Fubini’s theorem and integration by parts,

\[ \int_T ds \psi'(s) E_1(s) = \int_T d\tau \int_T -\psi(s) \partial_s e_1(s, \tau) ds = \int_T -\psi(s) \int_T \partial_s e_1(s, \tau) d\tau ds. \]

This proves the claim, and hence (3.67) is also a bound for \( \|E_1'(s)\|_{L^2(\mathbb{T})} \).

Step 2 (Estimate for \( E_2 \)). By the proof in Section 3.5,

\[ E_2(s) = \int_T \frac{SY'(s + \tau) - SY'(s)}{\tau} \cdot \frac{\tau A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) d\tau. \]

Denote its integrand by \( e_2(s, \tau) \). We shall bound \( \int_T |\partial_s e_2| d\tau \) first. Again, we derive some simple estimates.

Aiming at a sharper estimate of the leading term (see (3.75) below), we purposefully split the term

\[ \frac{\tau A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) = \left[ \frac{\tau A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) - f_3 \left( \frac{|Y'(s)|}{\varepsilon} \right) \right] + f_3 \left( \frac{|Y'(s)|}{\varepsilon} \right). \]

where

\[ \left| \frac{\tau A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) - f_3 \left( \frac{|Y'(s)|}{\varepsilon} \right) \right| \]

\[ \leq \left| \frac{\tau A - D^2}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) \right| + \left| f_3 \left( \frac{D}{\varepsilon} \right) - f_3 \left( \frac{|Y'(s)|}{\varepsilon} \right) \right| \]

\[ \leq C D |Y(s + \tau) - Y(s) - \tau Y'(s + \tau)| \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} \]

\[ + \frac{C |Y(s + \tau) - Y(s) - \tau Y'(s)|}{\varepsilon} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^3} \]

\[ \leq C \frac{\tau^2 R}{\lambda |\tau|} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} + C \frac{\tau^2 R}{\varepsilon} \cdot \frac{1}{1 + \left( \frac{\lambda |\tau|}{\varepsilon} \right)^3} \]

\[ \leq C \frac{|\tau| R}{\lambda} \cdot \frac{1}{1 + \left( \frac{\lambda |\tau|}{\varepsilon} \right)^2}. \]
By Lemma 3.1

\[ \left| \partial_s \left( \frac{SY'(s + \tau) - SY'(s)}{\tau} \right) \right| \leq \frac{|F(s + \tau) + |F(s)|}{\tau} \]

(3.72)

\[ \leq \frac{C\mu}{|\tau|}(Q + R + |Y''(s)|). \]

Lastly,

\[ \partial_s \left( \frac{\tau A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) \right) \]

(3.73)

\[ = \frac{\tau A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) + \frac{\tau A}{D^2} \left( \frac{1}{\varepsilon} f_3' \left( \frac{D}{\varepsilon} \right) - \frac{2}{\varepsilon} f_3 \left( \frac{D}{\varepsilon} \right) \right) \partial_s D. \]

By Lemma 3.3, Lemma 3.8, and Lemma 3.9

\[ \left| \partial_s \left( \frac{\tau A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) \right) \right| \]

(3.74)

\[ \leq C \frac{|\tau|^2 QR}{\lambda^2 \tau^2} \cdot \frac{1}{1 + \left( \frac{\lambda |\tau|}{\varepsilon} \right)^2} \]

\[ + \frac{C |D| \cdot |\tau| Q}{|D|^2} \cdot \left( \frac{1}{\varepsilon} + \frac{1}{1 + \left( \frac{\lambda |\tau|}{\varepsilon} \right)^2} + \frac{1}{\lambda |\tau|} \cdot \frac{1}{1 + \left( \frac{\lambda |\tau|}{\varepsilon} \right)^2} \right) |\tau| R \]

\[ \leq \frac{C Q R}{\lambda^2} \cdot \frac{1}{1 + \left( \frac{\lambda |\tau|}{\varepsilon} \right)^2}. \]

Combining (3.32) and (3.69)--(3.74), we find that

\[ \int_T |\partial_s e_2| d\tau \]

\[ \leq C \int_T \partial_s \left( \frac{SY'(s + \tau) - SY'(s)}{\tau} \right) \left| f_3 \left( \frac{|Y'(s)| |\tau|}{\varepsilon} \right) \right| d\tau \]

\[ + C \int_T \partial_s \left( \frac{SY'(s + \tau) - SY'(s)}{\tau} \right) \left| \frac{\tau A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) - f_3 \left( \frac{|Y'(s)| |\tau|}{\varepsilon} \right) \right| d\tau \]

(3.75)

\[ + C \int_T \left| SY'(s + \tau) - SY'(s) \right| \left| \partial_s \left( \frac{\tau A}{D^2} \cdot f_3 \left( \frac{D}{\varepsilon} \right) \right) \right| d\tau \]

\[ \leq C \int_T \frac{|F(s + \tau) - F(s)|}{|\tau|} \cdot \frac{1}{1 + \left( \frac{\lambda |\tau|}{\varepsilon} \right)^2} d\tau \]

\[ + \frac{C\mu}{\lambda^2} \int_T \left( Q + R + |Y''(s)| \right) |Q R| \cdot \frac{1}{1 + \left( \frac{\lambda |\tau|}{\varepsilon} \right)^2} d\tau. \]
By the Cauchy-Schwarz inequality,
\[
\int_{\mathbb{T}} \frac{|F(s + \tau) - F(s)|}{|\tau|} \cdot \frac{1}{1 + \left(\frac{\lambda \tau}{\epsilon}\right)^2} \, d\tau \\
\leq \left( \int_{\mathbb{T}} \frac{|F(s + \tau) - F(s)|^2}{|\tau|^{1+2\theta}} \, d\tau \right)^{1/2} \left( \int_{\mathbb{T}} \tau^{2\theta-1} \left(1 + \left(\frac{\lambda \tau}{\epsilon}\right)^2\right)^{-2} \, d\tau \right)^{1/2} \\
\leq \frac{C \epsilon^\theta}{\lambda^\theta} \left( \int_{\mathbb{T}} \frac{|F(s + \tau) - F(s)|^2}{|\tau|^{1+2\theta}} \, d\tau \right)^{1/2}.
\]

Therefore, by the Minkowski inequality, Sobolev inequality, and Lemma 3.1,

\[
(3.76) \quad \left\| \int_{\mathbb{T}} |\partial_s e_2| \, d\tau \right\|_{L^2(\mathbb{T})} \leq \frac{C \mu \epsilon}{\lambda^3} \|Y'\|_{L^\infty(\mathbb{T})} \|Y''\|_{L^1(\mathbb{T})}^2 + \frac{C \epsilon^\theta}{\lambda^\theta} \|F\|_{H^\theta(\mathbb{T})} \\
\leq \frac{C \mu \epsilon}{\lambda^3} \|Y'\|_{L^\infty(\mathbb{T})} \|Y''\|_{L^1(\mathbb{T})}^2 + \frac{C \epsilon^\theta}{\lambda^\theta} \|Y\|_{H^{2+\theta}(\mathbb{T})}.
\]

Then we argue as in (3.68) to show \( E_2'(s) = \int_{\mathbb{T}} \partial_s e_2 \, d\tau \); thus \( \|E_2'(s)\|_{L^2(\mathbb{T})} \) enjoys the bound in (3.76).

**Step 3 (Estimate for \( E_3' \)).** Recall
\[
E_3(s) = \int_{\mathbb{T}} SY'(s + \tau) \cdot \frac{B}{D^2} \cdot \theta_4 \left( \frac{D}{\epsilon} \right) \, d\tau.
\]

We denote its integrand to be \( e_3(s, \tau) \). By Lemma 3.3, Lemma 3.8, and Lemma 3.9 we calculate that
\[
\left| \frac{B}{D^2} \cdot \theta_4 \left( \frac{D}{\epsilon} \right) \right| \leq \frac{CQR}{\lambda^2} \cdot \frac{1}{1 + \left(\frac{\lambda \tau}{\epsilon}\right)^2},
\]

and
\[
\left| \frac{\partial_s}{D^2} \left( \frac{B}{D^2} \cdot \theta_4 \left( \frac{D}{\epsilon} \right) \right) \right| \\
\leq \frac{|\partial_s B|}{D^2} \cdot \left| \theta_4 \left( \frac{D}{\epsilon} \right) \right| + \frac{|B|}{D^2} \cdot \left( \frac{1}{\epsilon} \left| \theta_4' \left( \frac{D}{\epsilon} \right) \right| + \frac{2}{|D|} \left| \theta_4 \left( \frac{D}{\epsilon} \right) \right| \right| \left| \partial_s D \right| \\
\leq \frac{|\partial_s B|}{\tau^2 |Y'(s + \tau)|^2} \cdot \left| \theta_4 \left( \frac{D}{\epsilon} \right) \right| + \frac{1}{D^2} \cdot \frac{1}{\tau^2 |Y'(s + \tau)|^2} \cdot |\partial_s B| \cdot \left| \theta_4 \left( \frac{D}{\epsilon} \right) \right| \\
+ \frac{C\epsilon^2 QR}{\lambda^2 \tau^2} \left( \frac{1}{\epsilon} \cdot \frac{1}{1 + \left(\frac{\lambda \tau}{\epsilon}\right)^3} + \frac{2}{\lambda |\tau|} \cdot \frac{1}{1 + \left(\frac{\lambda \tau}{\epsilon}\right)^2} \right) |\tau| R.
We apply (3.37) to the first term to bound $\partial_s B$ while applying (3.35) to the second term:

$$\left| \partial_s \left( \frac{B}{D^2} \cdot f_4 \left( \frac{D}{\varepsilon} \right) \right) \right| \leq \frac{C \tau^2 R^2}{\tau^2 \lambda^2} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2}$$

$$+ \frac{C |\tau|^{1+\theta}|Y'(s + \tau)|}{\tau^2 |Y'(s + \tau)|} \left( \int_0^\pi \frac{|Y''(s + \tau - \eta) - Y''(s + \tau)|^2}{|\eta|^{1+2\theta}} \, d\eta \right)^{1/2}$$

$$\cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2}$$

$$+ \frac{C(D + |\tau||Y'(s + \tau)|)|Y(s + \tau) - Y(s) - \tau Y'(s + \tau)|}{D^2 \tau^2 |Y'(s + \tau)|^2}$$

$$\cdot |\tau|QR \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} + \frac{CQR^2}{\lambda^3} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2}$$

$$\leq C |Y'(s + \tau)|^{-1} \left( \int_0^\pi \frac{|Y''(s + \tau - \eta) - Y''(s + \tau)|^2}{|\eta|^{1+2\theta}} \, d\eta \right)^{1/2}$$

$$\cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} + \frac{CQR^2}{\lambda^3} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2}.$$

Hence, by Lemma 3.1,

$$\int_{\mathbb{T}} |\partial_s e_3| \, d\tau$$

$$\leq \int_{\mathbb{T}} |S Y'(s + \tau)| |\partial_s \left( \frac{B}{D^2} \cdot f_4 \left( \frac{D}{\varepsilon} \right) \right)| + |F(s + \tau)| \left| B \cdot f_4 \left( \frac{D}{\varepsilon} \right) \right| \, d\tau$$

$$\leq C\mu \int_{\mathbb{T}} \left( \int_0^\pi \frac{|Y''(s + \tau - \eta) - Y''(s + \tau)|^2}{|\eta|^{1+2\theta}} \, d\eta \right)^{1/2} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} \, d\tau$$

$$+ C\mu \int_{\mathbb{T}} |Q| \cdot Q \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} \, d\tau + C\mu \int_{\mathbb{T}} (Q + R) \cdot \frac{QR}{\lambda^2} \cdot \frac{1}{1 + \left( \frac{\lambda \tau}{\varepsilon} \right)^2} \, d\tau.$$

By the Minkowski inequality and Sobolev inequality,

$$\int_{\mathbb{T}} |\partial_s e_3| \, d\tau \leq \frac{C\mu E}{\lambda^4} \|Y'\|_{L^\infty(\mathbb{T})}^2 \|Y''\|_{L^4(\mathbb{T})}^2$$

(3.77)

$$+ C\mu \varepsilon \|Y\|_{H^{2+\theta}(\mathbb{T})}. $$

Then we argue as in (3.68) that $E_3'(s) = \int_{\mathbb{T}} \partial_s e_3 \, d\tau$, and $\|E_3'(s)\|_{L^2(\mathbb{T})}$ enjoys the bound in (3.77).
Combining (3.67), (3.76), and (3.77), we complete the proof of Proposition 3.7.

4 Singular Limit and Dynamic Error Estimates

In this section, we come back to the case of Hookean elasticity and prove Theorem 1.4 and Theorem 1.8.

4.1 Proof of Theorem 1.4

We will follow the blueprint sketched in Section 1.4 to prove Theorem 1.4. Recall that

\[ X^\varepsilon - X \]

satisfies the equation

\[ \partial_t (X^\varepsilon - X) = \mathcal{L}(X^\varepsilon - X) + (g_{X^\varepsilon} - g_X) + (U_{X^\varepsilon} - U_X^\varepsilon). \]

(4.1) \( (X^\varepsilon - X)(0) = 0, \)

where

\[ g_Y = U_Y - \mathcal{L}Y = U_Y + \frac{1}{4}(-\Delta)^{1/2}Y. \]

In order to bound \( X^\varepsilon - X \), we will use estimates for \( g_X^\varepsilon - g_X \) and \( U_X^\varepsilon - U_X^\varepsilon \).

Although many estimates for \( g_Y \) have been established in [15, sec. 3], we need improved ones in the proofs of Theorem 1.4 and Theorem 1.8.

**Lemma 4.1.** Let \( Y \in H^{2+\theta}(\Gamma) \) satisfy \( (1.7) \), with \( \theta \in \left[ \frac{1}{4}, 1 \right) \). Then

\[ \| g_Y \| \lesssim C \lambda^{-3} \| Y \|^{4} \hat{H}^{9/4}(\Gamma). \]

**Lemma 4.2.** Let \( Y_1, Y_2 \in H^{2+\theta}(\Gamma) \) both satisfy \( (1.7) \), with \( \theta \in \left[ \frac{1}{4}, 1 \right) \). For \( \beta \) satisfying \( (1.30) \),

\[ \| g_{Y_1} - g_{Y_2} \|_{L^2(\Gamma)} \leq C \lambda^{-2} \left( \| Y_1 \|_{\hat{H}^{2+\theta}} + \| Y_2 \|_{\hat{H}^{2+\theta}} \right)^2 \| \delta Y \|_{\hat{H}^{(1/2)-\beta}(\Gamma)}, \]

where \( \delta Y \triangleq Y_1 - Y_2 \) and \( C = C(\beta, \theta) \).

**Lemma 4.3.** Let \( Y_1, Y_2 \in H^{2+\theta}(\Gamma) \) both satisfy \( (1.7) \), with \( \theta \in \left[ \frac{1}{4}, 1 \right) \). For \( \beta \) satisfying \( (1.33) \),

\[ \| g_{Y_1} - g_{Y_2} \|_{\hat{H}^{1}(\Gamma)} \leq C \lambda^{-3} \left( \| Y_1 \|_{\hat{H}^{2+\theta}} + \| Y_2 \|_{\hat{H}^{2+\theta}} \right)^3 \| \delta Y \|_{\hat{H}^{(3/2)-\beta}(\Gamma)}, \]

where \( C = C(\beta, \theta) \).

We leave their lengthy proofs to Appendix C.

**Proof of Theorem 1.4.** Recall that \( \bar{\varepsilon} = \varepsilon / \lambda \), and \( X^\varepsilon \) and \( X \) satisfy assumptions [i] and [ii].

**Step 1 (Error estimate in the \( H^2 \)-seminorm).** Thanks to Proposition 1.12 and Lemma 4.3 for all \( t \in [0, T] \),

\[ \| U_{X^\varepsilon} - U_X \|_{\hat{H}^{1}(\Gamma)} \leq C \bar{\varepsilon}^\theta M + C \bar{\varepsilon} \lambda^{-3} M^4, \]

(4.4)

\[ \| g_{X^\varepsilon} - g_X \|_{\hat{H}^{1}(\Gamma)} \leq C \lambda^{-3} M^3 \| X^\varepsilon - X \|_{\hat{H}^{3/2}(\Gamma)}. \]

(4.5)
With $T_0 > 0$ to be determined, we apply the energy estimate of (4.1) to obtain that
\[
\|X^e - X\|_{C[0,T_0] \dot{H}^{3/2} \cap L^2_{T_0}} \leq C_0 T_0^{1/2} (\lambda^{-3} M^3 \|X^e - X\|_{L^\infty_{[0,T_0]} \dot{H}^{3/2}(\mathbb{T})} + \bar{\varepsilon}^\theta M + \bar{\varepsilon} \lambda^{-3} M^4).
\]
Taking $T_0$ be sufficiently small such that $C_0 T_0^{1/2} \lambda^{-3} M^3 \leq \frac{1}{2}$, we obtain that
\[
\|X^e - X\|_{C[0,T_0] \dot{H}^{3/2} \cap L^2_{T_0}} \leq C \bar{\varepsilon}^\theta M + C \bar{\varepsilon} \lambda^{-3} M^4.
\]
Here we used the assumption $\bar{\varepsilon} \ll 1$ and the fact $\lambda \leq C M$. If $T \geq T_0$, we may repeat this argument for $[T_0, 2T_0)$, $[2T_0, 3T_0)$, ..., until the time interval $[0, T]$ is fully covered. For instance, for $[T_0, 2T_0)$, the energy estimate of (4.1) is written as
\[
\|X^e - X\|_{C[0,2T_0] \dot{H}^{3/2} \cap L^2_{0,2T_0}} \leq C \|(X^e - X)(T_0)\|_{\dot{H}^{3/2}(\mathbb{T})} + C_0 T_0^{1/2} (\lambda^{-3} M^3 \|X^e - X\|_{L^\infty_{[0,2T_0]} \dot{H}^{3/2}(\mathbb{T})} + \bar{\varepsilon}^\theta M + \bar{\varepsilon} \lambda^{-3} M^4).
\]
With the same $T_0$ chosen earlier,
\[
\|X^e - X\|_{C[0,2T_0] \dot{H}^{3/2} \cap L^2_{0,2T_0}} \leq C \|X^e - X\|_{C[0,T_0] \dot{H}^{3/2}(\mathbb{T})} + C \bar{\varepsilon}^\theta M \leq C \bar{\varepsilon}^\theta M.
\]
We conclude that
\[
\|X^e - X\|_{C[0,T] \dot{H}^{3/2} \cap L^2_{T}} \leq C \bar{\varepsilon}^\theta M. \tag{4.6}
\]
Here $C = C(\theta, \lambda^{-1} M, T)$ depends exponentially on $T$.

Let $\mathcal{P}_N$ be defined as in (1.25) and $\mathcal{Q}_N = \text{Id} - \mathcal{P}_N$. With $N \gg 1$ to be determined, we bound $\mathcal{P}_N(X^e - X)$ and $\mathcal{Q}_N(X^e - X)$ separately as follows:
\[
\|\mathcal{Q}_N (X^e - X)\|_{\dot{H}^{2}(\mathbb{T})} \leq C N^{-\theta} \|X^e - X\|_{\dot{H}^{2+\theta}(\mathbb{T})} \leq C N^{-\theta} M. \tag{4.7}
\]
We apply $\mathcal{P}_N$ to both sides of (4.1). By Lemma B.1 and (4.4)–(4.6),
\[
\|\mathcal{P}_N (X^e - X)\|_{C[0,T] \dot{H}^{2}(\mathbb{T})} \leq C (\ln N)^{1/2} (\lambda^{-3} M^3 \|X^e - X\|_{C[0,T] \dot{H}^{3/2}(\mathbb{T})} + \bar{\varepsilon}^\theta M + \bar{\varepsilon} \lambda^{-3} M^4) \leq C(\theta, \lambda^{-1} M, T)(\ln N)^{1/2} \bar{\varepsilon}^\theta M. \tag{4.8}
\]
Combining (4.7) and (4.8), and taking $N \sim \bar{\varepsilon}^{-1}$, we obtain (1.18).

Step 2 (Error estimate in the $H^1$-seminorm). Error estimates in lower-order norms can be derived as in previous steps with minor adaptation. By Proposition 1.12 and Lemma 4.2 for all $t \in [0, T]$,
\[
\|U^e_t - U_t\|_{L^2(\mathbb{T})} \leq C m_1 \bar{\varepsilon} M + C \bar{\varepsilon}^{1+\theta} \|\ln \bar{\varepsilon}\|^{\theta} M + C \bar{\varepsilon}^{2} \ln \bar{\varepsilon} \lambda^{-3} M^4, \tag{4.9}
\]
\[
\|g X^e - g X\|_{L^2(\mathbb{T})} \leq C \lambda^{-2} M^2 \|X^e - X\|_{\dot{H}^{1/2}(\mathbb{T})}. \tag{4.10}
\]
If, in addition, \( m_2 = 0 \), the logarithmic factors in (4.9) can be removed. By a similar argument as in the previous step, we can show that

\[
(4.11) \quad \| X^e - X \|_{C_0, T \bigcap H^{1/2} \cap L^2, H^1(T)} \leq C(\theta, \lambda^{-1} M, T)(m_1 \tilde{\varepsilon} + \tilde{\varepsilon}^{1+\theta} | \ln \tilde{\varepsilon}|^\theta) M.
\]

To this end, we take \( N \sim \tilde{\varepsilon}^{-1} \), and derive that

\[
(4.12) \quad \| Q_N (X^e - X) \|_{H^1(T)} \leq C N^{-1-\theta} \| X^e - X \|_{H^{2+\theta}(T)} \leq C \tilde{\varepsilon}^{1+\theta} M.
\]

On the other hand, by (4.9)–(4.11) and Lemma B.1,

\[
(4.13) \quad \| P_N (X^e - X) \|_{C_0, T \bigcap H^{1/2}} \leq C (\ln N)^{1/2} \lambda^{-2} M^2 \| X^e - X \|_{C_0, T \bigcap H^{1/2}} + C(\theta, \lambda^{-1} M)(\ln N)^{1/2}(m_1 \tilde{\varepsilon} + \tilde{\varepsilon}^{1+\theta} | \ln \tilde{\varepsilon}|^\theta) M
\]

\[
\leq C(\theta, \lambda^{-1} M, T)(\ln N)^{1/2}(m_1 \tilde{\varepsilon} + \tilde{\varepsilon}^{1+\theta} | \ln \tilde{\varepsilon}|^\theta) M.
\]

Combining (4.12) and (4.13), we obtain (1.17).

**Step 3** (Error estimate in the \( L^2 \)-norm). By (4.9)–(4.11) and Lemma B.1,

\[
\| X^e - X \|_{C_0, T \bigcap L^2(T)} \leq C \lambda^{-2} M^2 \| X^e - X \|_{L^2, H^1(T)} + C T(m_1 \tilde{\varepsilon} M + \tilde{\varepsilon}^{1+\theta} | \ln \tilde{\varepsilon}|^\theta M + \tilde{\varepsilon}^2 | \ln \tilde{\varepsilon}| \lambda^{-3} M^4)
\]

\[
\leq C(\theta, \lambda^{-1} M, T)(m_1 \tilde{\varepsilon} + \tilde{\varepsilon}^{1+\theta} | \ln \tilde{\varepsilon}|^\theta) M,
\]

which is exactly (1.16).

**Step 4** (Weak-\(*\) convergence in the top regularity). In order to prove (1.15), it suffices to show that the only limiting point of the sequence \( \{ X^e - X \}_e \) in the weak-\(*\) topology of \( C_0, T \bigcap H^{2+\theta}(T) \) is 0. By (4.1) and (4.4)–(4.6), it is not difficult to show that \( \| \partial_t (X^e - X) \|_{L^2, H^1(T)} \) is uniformly bounded. By the Aubin-Lions lemma \([38]\) and the assumption (i) on page 371, \( \{ X^e - X \}_e \) is compact in \( L^2, H^2(T) \). This implies that any weak-\(*\) limiting point of \( \{ X^e - X \}_e \) in \( C_0, T \bigcap H^{2+\theta}(T) \) must be a limiting point of \( \{ X^e - X \}_e \) in the strong topology of \( L^2, H^2(T) \). However, it has been proved that the latter can only be 0. Hence, (1.15) is proved.

This completes the proof of Theorem 1.4.

### 4.2 Proof of Theorem 1.8

We first remark on a nice property of the \((\varepsilon, N)\)-regularized problem, which is of independent interest.

**Remark 4.4.** The \((\varepsilon, N)\)-regularized problem is volume preserving; i.e., the area of the domain enclosed by the string is invariant in time. Note that in the unregularized...
problem or the $\epsilon$-regularized, the volume conservation is a direct consequence of the flow field (or the regularized flow field) being divergence-free.

We derive as follows. The area of the domain enclosed by the string is given by

$$V(t) = \frac{1}{2} \int_{\mathbb{T}} X^{e,N}(s,t) \times (X^{e,N})'(s,t) \, ds.$$  

Taking the $t$-derivative and doing integration by parts,

$$V'(t) = \int_{\mathbb{T}} \partial_t X^{e,N}(s,t) \times (X^{e,N})'(s,t) \, ds.$$  

This can be justified rigorously since we will show that $X^{e,N}(s,t)$ is sufficiently smooth. By (1.22),

$$V'(t) = \int_{\mathbb{T}} \int_{\mathbb{R}^2} u^{e,N}(x,t) \delta_\epsilon(X^{e,N}(s,t) - x) \, dx \times (X^{e,N})'(s,t) \, ds.$$  

In the second equation, we used the fact that $(X^{e,N})' = \mathcal{P}_N(X^{e,N})'$; in the last equation, we applied the divergence theorem and noticed that $u^{e,N}(y,t) = \int_{\mathbb{R}^2} u^{e,N}(x,t) \delta_\epsilon(y-x) \, dx$ is divergence-free.

In the proof of Theorem 1.8, we will need estimates for $Q_N X$, the high-frequency portion of $X$, where $Q_N = \text{Id} - \mathcal{P}_N$. In fact, given the assumptions of Theorem 1.8, a naive one would be

$$\|Q_N X\|_{\dot{H}^{2+\theta}(\mathbb{T})} \leq C N^{-2-\theta} \|X\|_{\dot{H}^{2+\theta}(\mathbb{T})}$$

for all $\gamma \leq 2 + \theta$ and $N \geq 1$. Yet, we shall derive an improved one as follows.

**Lemma 4.5 (An improved estimate for $Q_N X$).** Under the assumptions on $X(s,t)$ in Theorem 1.8, for all $\gamma \leq 2 + \theta$ and $t \in [0,T_*]$,  

$$\|Q_N X(t)\|_{\dot{H}^{2+\theta}(\mathbb{T})} \leq C e^{-t N^\gamma/4} N^{\gamma - 2 - \theta} \|Q_N X_0\|_{\dot{H}^{2+\theta}(\mathbb{T})} + C N^{\gamma - 3} \lambda^{-3} \|X_0\|_{\dot{H}^{2+\theta}(\mathbb{T})}^4.$$  

Here the constants $C$ depend on $\theta$ and $\gamma$, but not on $T_*$ or $N$.

**Proof.** Consider the equation of $Q_N X$,

$$\partial_t Q_N X = L Q_N X + Q_N g_x, \quad Q_N X(0) = Q_N X_0.$$  

By Lemma 4.1 and Lemma 1.2, for all $\gamma \leq 2 + \theta$,  

$$\|Q_N X(t)\|_{\dot{H}^{2+\theta}(\mathbb{T})} \leq C e^{-t N^\gamma/4} N^{\gamma - 2 - \theta} \|Q_N X_0\|_{\dot{H}^{2+\theta}(\mathbb{T})} + C N^{\gamma - 3} \lambda^{-3} \|X_0\|_{\dot{H}^{2+\theta}(\mathbb{T})}^4.$$  

$\Box$
Now we are prepared to prove Theorem 1.8.

**Proof of Theorem 1.8.** Recall that $\bar{\varepsilon} = \varepsilon / \lambda$.

*Step 1* (Well-posedness). The proof of the global well-posedness of the $(\varepsilon, N)$-regularized problem is exactly the same as that in Section 2, as $\mathcal{P}_N$ is a bounded linear operator in all $H^\theta (T)$-spaces, and the energy estimate remains unchanged in spite of the presence of the projection. We omit the details, but only note that $X^{\varepsilon, N}$ is continuous from $[0, +\infty)$ to $H^{2+\theta} (T)$. Indeed, thanks to (2.14),

$$\| T^{\varepsilon, N} \|_{H^{2+\theta} (T)} \leq C(\| X_0 \|_{\dot{H}^1 (T)}, \varepsilon, \theta) \| X^{\varepsilon, N} \|_{\dot{H}^{2+\theta} (T)}.$$

This implies the continuity.

*Step 2* (Uniform estimates for $X^{\varepsilon, N}$). Since $\mathcal{P}_N X_0 \to X_0$ in $H^{2+\theta} (T)$ and $X_0$ satisfies the well-stretched condition with constant $\lambda$, whenever $N \gg 1$, $\mathcal{P}_N X_0$ satisfies the well-stretched condition with constant $\lambda / 2$. For given $\varepsilon$ and $N$, by the continuity of $X^{\varepsilon, N}$, there exists a maximal $T_{\varepsilon, N} > 0$ such that for all $t \in [0, T_{\varepsilon, N}]$,

$$\| X^{\varepsilon, N} (\cdot, t) \|_{\dot{H}^{2+\theta} (T)} \leq 2C_\ast M_0,$$

with $C_\ast$ and $M_0$ defined in the statement of Theorem 1.8 and

$$X^{\varepsilon, N} (\cdot, t) \text{ satisfies the well-stretched condition with constant } \lambda / 4.$$

By the maximality, we mean that for any $T' > T_{\varepsilon, N}$, there exists $t \in [0, T')$ such that at least one of (4.14) and (4.15) is false.

We shall prove that there exists $c_\ast > 0$ and $N_\ast > 0$ such that for any $\varepsilon \ll 1$ and $N_\ast \leq N \leq c_\ast \bar{\varepsilon}^{-1}$, we must have $T_{\varepsilon, N} \geq T_\ast$, with $T_\ast$ given in Theorem 1.8.

Assume otherwise. Fix $c_\ast$ small, which will be chosen latter, and we have $T_{\varepsilon, N} < T_\ast$ when $1 \ll N \leq c_\ast \bar{\varepsilon}^{-1}$. We start with $H^1$- and $H^2$-estimates for $X^{\varepsilon, N} - \mathcal{P}_N X$ by following the proof of Theorem 1.4. By (1.19)–(1.23),

$$\begin{align*}
\partial_t (X^{\varepsilon, N} - \mathcal{P}_N X) &= \mathcal{P}_N (U^{\varepsilon, N}_X - U_X) \\
&= \mathcal{P}_N (U^{\varepsilon, N}_X - U_X) + \mathcal{P}_N (U^{\varepsilon, N}_{X^N} - U_{X^N}) \\
&= \mathcal{L} (X^{\varepsilon, N} - \mathcal{P}_N X) + \mathcal{P}_N (g^{\varepsilon, N}_X - g_X) + \mathcal{P}_N (U^{\varepsilon, N}_{X^N} - U_{X^N}).
\end{align*}$$

Consider arbitrary $t \in [0, T_{\varepsilon, N}]$. By (1.26), (1.27), (4.14), (4.15), and Lemma 4.3 for any $\beta$ satisfying (1.33),

$$\begin{align*}
\| \mathcal{P}_N (g^{\varepsilon, N}_X - g_X) \|_{\dot{H}^1 (T)} &\leq C \lambda^{-3} M_3 \| X^{\varepsilon, N} - X \|_{\dot{H}^{1/2} - \beta (T)} \\
&\leq C \lambda^{-3} M_3 (\| X^{\varepsilon, N} - \mathcal{P}_N X \|_{\dot{H}^{1/2} - \beta (T)} + \| \mathcal{Q}_N X \|_{\dot{H}^{1/2} - \beta (T)}) \\
&\leq C (\| X^{\varepsilon, N} - \mathcal{P}_N X \|_{\dot{H}^{1/2} - \beta (T)} + N^{-\frac{1}{2} - \theta} \beta M_0).
\end{align*}$$
where $C = C(\beta, \theta, \lambda^{-1}M_0)$. Here we used
\[ \|Q_N X\|_{\dot{H}^{(3/2)-\beta}} \leq C N^{-\frac{1}{2} - \theta - \beta} \|X\|_{\dot{H}^{2+\theta}}. \]

Similarly, for any $\beta'$ satisfying (1.30),
\[ \|P_N (g_{Xe,N} - g_X)\|_{L^2(\mathbb{T})} \leq C(\|X^{e,N} - \dot{P}_N X\|_{\dot{H}^{1/2-\beta'}} + N^{-\frac{1}{2} - \theta - \beta'} M_0). \]

On the other hand, by Proposition 1.12
\[ (4.18) \quad \|P_N (U_{Xe,N}^e - U_{Xe,N})\|_{\dot{H}^1(\mathbb{T})} \leq C\bar{\varepsilon}^\theta M_0 + C\bar{\varepsilon}\lambda^{-3} M_0^4, \]
and
\[ \|P_N (U_{Xe,N}^e - U_{Xe,N})\|_{L^2(\mathbb{T})} \leq C m_1 \bar{\varepsilon} M_0 + C \bar{\varepsilon}^{1+\theta} \ln \bar{\varepsilon}^\theta M_0 + C \bar{\varepsilon}^2 \ln \bar{\varepsilon} |\lambda^{-3} M_0^4. \]

We argue as in the proof of Theorem 1.4 to obtain that
\[ (4.19) \quad \|X^{e,N} - \dot{P}_N X\|_{C_{[0,T]} L^1 H^1(\mathbb{T})} \leq C(\ln N)^{\frac{1}{2}} (N^{-\frac{1}{2} - \theta - \beta} + \bar{\varepsilon}^\theta) M_0, \]
\[ (4.20) \quad \|X^{e,N} - \dot{P}_N X\|_{C_{[0,T]} L^1 H^{(3/2)-\beta}(\mathbb{T})} \]
\[ \leq C(\ln N)^{\frac{1}{2}} (N^{-\frac{1}{2} - \theta - \beta'} + m_1 \bar{\varepsilon} + \bar{\varepsilon}^{1+\theta} \ln \bar{\varepsilon}^\theta) M_0. \]
\[ (4.21) \quad \|X^{e,N} - \dot{P}_N X\|_{C_{[0,T]} L^1 H^{1/2}(\mathbb{T})} \leq C N^{-\frac{1}{2} - \theta - \beta'} + m_1 \bar{\varepsilon} + \bar{\varepsilon}^{1+\theta} \ln \bar{\varepsilon}^\theta) M_0, \]
where $C = C(\beta, \theta, \lambda^{-1}M_0, T_*).$ By interpolation between (4.19) and (4.20),
\[ (4.22) \quad \|X^{e,N} - \dot{P}_N X\|_{C_{[0,T]} L^1 H^{(3/2)-\beta}(\mathbb{T})} \]
\[ \leq C(\ln N)^{\frac{1}{2}} (N^{-\frac{1}{2} - \theta - \beta} + \bar{\varepsilon}^\theta) \|X^{e,N} - \dot{P}_N X\|_{\dot{H}^{(3/2)-\beta}(\mathbb{T})}. \]

To this end, consider the equation for $E_{e,N} \triangleq \partial_t (X^{e,N} - \dot{P}_N X).$ By (4.16),
\[ \partial_t E_{e,N} = \mathcal{L} E_{e,N} + \dot{P}_N \partial_s (g_{Xe,N} - g_X) + \dot{P}_N \partial_s (U_{Xe,N}^e - U_{Xe,N}), \]
with $E_{e,N}(0) = 0.$ Let $E_{e,N} = E^{(1)}_{e,N} + E^{(2)}_{e,N},$ where $E^{(1)}_{e,N}$ and $E^{(2)}_{e,N}$ solve
\[ \partial_t E^{(1)}_{e,N} = \mathcal{L} E^{(1)}_{e,N} + \dot{P}_N \partial_s (g_{Xe,N} - g_X), \quad E^{(1)}_{e,N}(0) = 0, \]
\[ \partial_t E^{(2)}_{e,N} = \mathcal{L} E^{(2)}_{e,N} + \dot{P}_N \partial_s (U_{Xe,N}^e - U_{Xe,N}), \quad E^{(2)}_{e,N}(0) = 0, \]
respectively.

First we derive an estimate for $E^{(2)}_{e,N}.$ By Lemma B.1 and (4.18),
\[ \|E^{(2)}_{e,N}\|_{C_{[0,T]} L^2(\mathbb{T})} \leq \|\dot{P}_N \partial_s (U_{Xe,N}^e - U_{Xe,N})\|_{L^2_{[0,T]}L^2(\mathbb{T})} \]
\[ \leq C(\theta, \lambda^{-1}M_0, T_*) M_0 \cdot \bar{\varepsilon}^\theta. \]

On the other hand, by (4.18), for all $1 \leq n \leq N \leq c_* \bar{\varepsilon}^{-1},$
\[ (4.23) \quad \|\dot{P}_N \partial_s (U_{Xe,N}^e - U_{Xe,N})\|_{\dot{H}^\theta(\mathbb{T})} \leq C(\theta, \lambda^{-1}M_0) M_0 \cdot \bar{\varepsilon}^\theta T^\theta. \]
Hence, for all $n \in \mathbb{Z}_+$, by Lemma B.1,
\[
\| \mathcal{P}_n 2n E_{e,N}^{(2)} \|_{C_{[0,T_e,N]}^1 \mathring{H}^{1+\theta}(\mathbb{T})} \leq C \| \mathcal{P}_n \mathcal{P}_N \partial_s (U_{Xe,N}^\varepsilon - U_{Xe,N}) \|_{L^\infty_{T_e,N} \mathring{H}^{\theta}(\mathbb{T})} 
\leq C(\theta, \lambda^{-1} M_0) M_0 \cdot \bar{\varepsilon} \theta n^\theta.
\]
Suppose $N \in (2^{k_*}, 2^{k_*+1}]$ with some $k_* \geq 1$. Then by Parseval’s identity,
\[
\| E_{e,N}^{(2)} \|_{C_{[0,T_e,N]}^1 \mathring{H}^{1+\theta}(\mathbb{T})}^2 
\leq \| E_{e,N}^{(2)} \|_{C_{[0,T_e,N]}^1 L^2(\mathbb{T})}^2 + \sum_{k=0}^{k_*} \| \mathcal{P}_n 2^{k_*} 2^{k+1} E_{e,N}^{(2)} \|_{C_{[0,T_e,N]}^1 \mathring{H}^{1+\theta}(\mathbb{T})}^2 
\leq C(\theta, \lambda^{-1} M_0, T_*) M_0^2 \sum_{k=0}^{k_*} \bar{\varepsilon}^{2\theta} 2^{2k\theta} 
\leq C(\theta, \lambda^{-1} M_0, T_*) M_0^2 \cdot \bar{\varepsilon}^{2\theta} 2^{2k_*\theta} \sum_{k=0}^{k_*} 2^{-2\theta(k_* - k)} 
\leq C(\theta, \lambda^{-1} M_0, T_*) M_0^2 \cdot (\bar{\varepsilon} N)^{2\theta}.
\]

Next we consider $E_{e,N}^{(1)}$. Combining (4.17) and (4.22), for all $t \in [0, T_e, N]$,
\[
\| \mathcal{P}_n \partial_s (g_{Xe,N} - g_{X}) \|_{L^2(\mathbb{T})} 
\leq C \left[ (\ln N)^{\frac{1}{2}} (N^{-\frac{1}{2} - \theta - \beta} + \bar{\varepsilon} \theta)^{\frac{1}{2}} - \beta (N^{-\frac{1}{2} - \theta - \beta'} + \bar{\varepsilon})^{\frac{1}{2} + \beta} + N^{-\frac{1}{2} - \theta - \beta} \right] M_0,
\]
where $C = C(\beta, \theta, \lambda^{-1} M_0, T_*)$. Moreover, for all $n \in \mathbb{Z}_+$ satisfying $n \leq N$,
\[
\| \mathcal{P}_n \partial_s (g_{Xe,N} - g_{X}) \|_{\mathring{H}^{\theta}(\mathbb{T})} \leq C n^{\theta} \| \mathcal{P}_n \partial_s (g_{Xe,N} - g_{X}) \|_{L^2(\mathbb{T})}.
\]
Then we argue as above to derive that
\[
\| E_{e,N}^{(1)} \|_{C_{[0,T_e,N]}^1 \mathring{H}^{1+\theta}(\mathbb{T})} \leq C N^{\theta} \| \mathcal{P}_n \partial_s (g_{Xe,N} - g_{X}) \|_{L^\infty_{T_e,N} L^2(\mathbb{T})}.
\]
Combining (4.24)–(4.27),
\[
\| X_{e,N} - \mathcal{P}_N X \|_{C_{[0,T_e,N]}^1 \mathring{H}^{2+\theta}(\mathbb{T})} 
\leq C \left[ (\ln N)^{\frac{1}{2}} (N^{-\frac{1}{2} - \beta} + (\bar{\varepsilon} N)^{\theta})^{\frac{1}{2}} - \beta (N^{-\frac{1}{2} - \beta'} + \bar{\varepsilon} N)^{\frac{1}{2} + \beta} + N^{-\frac{1}{2} - \beta} + (\bar{\varepsilon} N)^{\theta} \right] M_0 
\leq C (N^{-\frac{1}{2} - \beta} + (\bar{\varepsilon} N)^{\theta}) M_0,
\]
where $C = C(\beta, \theta, \lambda^{-1} M_0, T_*)$. Here we simplify the estimate by fixing $\beta'$ to be any small number and using the assumption that $\bar{\varepsilon} \leq c_* N^{-1}$. 

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Let us highlight the importance of the projection in the argument so far. On one hand, it enables us to easily derive $\hat{H}^0$-estimates of the source terms in the $E^{(i)}_{\varepsilon,N}$-equations in terms of their $L^2$-estimates (see (4.18), (4.23), and (4.26)). More importantly, it allows us to make the estimate (4.28) arbitrarily small in the limit $N \to 0$ as long as we are willing to set a suitably small upper bound for $\varepsilon N$. We shall see shortly that the smallness of (4.28) is important for deriving a contradiction with the maximality of $T_{\varepsilon,N}$.

Finally, by Lemma 4.5,

$$
\| \mathcal{Q}_N X(t) \|_{\hat{H}^{2+\theta}(T)} \leq C(\theta, \lambda^{-1} M_0)(e^{-\varepsilon N/4} \| \mathcal{Q}_N X_0 \|_{\hat{H}^{2+\theta}(T)} + N^{-1+\theta} M_0),
$$

which implies

$$
\lim_{N \to +\infty} \| X - \mathcal{P}_N X \|_{C(0,T_\varepsilon \hat{H}^{2+\theta}(T))} = 0.
$$

Combining (1.26), (1.27), (4.28), and (4.29), we may take $N$ suitably large, which depends on $\theta, \lambda^{-1} M_0, X_0, T_\varepsilon, \text{and } C_\varepsilon$, and then assume $c_* \varepsilon^2$ to be suitably small, which also depends on $\theta, \lambda^{-1} M_0, T_\varepsilon, \text{and } C_\varepsilon$, such that $\| (X^{\varepsilon,N} - X)(t) \|_{\hat{H}^{2+\theta}}$ is sufficiently small for all $t \in [0, T_{\varepsilon,N}]$ and thus

$$
\| X^{\varepsilon,N} (\cdot, t) \|_{\hat{H}^{2+\theta}(T)} \leq \frac{3}{2} C_* M_0,
$$

and

$X^{\varepsilon,N} (\cdot, t)$ satisfies the well-stretched condition with constant $\lambda/3$.

This contradicts the maximality of $T_{\varepsilon,N}$, because by the well-posedness this implies that the solution $X^{\varepsilon,N}$ can be extended to a longer time interval without violating (4.14) and (4.15). Therefore, we prove that there exists $c_* > 0$ and $N_* > 0$ such that for all $N \in [N_*, c_* \varepsilon^{-1}]$, (4.14) and (4.15) hold for all $t \in [0, T_*]$. As a result, (4.19), (4.20), and (4.28) become estimates on $[0, T_*]$.

Step 3 (Convergence and error estimates). By Lemma 4.5, for $\gamma = \frac{1}{2}, 1, 2, 2 + \theta$, and $t \in [0, T_*],

$$
\| \mathcal{Q}_N X(t) \|_{\hat{H}^{\gamma}(T)} \leq C N^{\gamma-2-\theta}(e^{-\varepsilon N/4} + N^{-1+\theta} M_0),
$$

where $C = C(\gamma, \theta, \lambda^{-1} M_0)$. Combining this with (4.19)–(4.21) and (4.28), we can prove (1.31)–(1.35) and thus (1.29). The weak-* convergence (1.28) can be justified in the same way as in the proof of Theorem 1.4.

5 Discussion

5.1 Improved Error Estimates Revisited

Proposition 3.5 proves an improved $L^2$-static error estimate when $m_1 = 0$ or when the elastic force has zero tangential component, which leads to improved error estimates in Theorem 1.4 and Theorem 1.8. Corollary 3.6 implies that
$O(\varepsilon)$-leading term in the $L^2$-static error estimate occurs only in the tangential direction. In the following, we shall explain that these results have a clear physical interpretation.

Consider a model problem, in which the string is represented by $Y(s)$. Assume that $Y(0) = (0,0)$ and a local part of the elastic string coincides with the segment that connects $(-1,0)$ and $(1,0)$. The rest part of the string is assumed to be far away from the origin. This is a simplification of general cases. Indeed, if we zoom in to any local part of a string with sufficiently regular configuration, the local string segment is always close to being a straight line segment. Suppose that in the Eulerian coordinate, there is a constant force along that local string segment; in other words, we assume $F_Y(s) = (f_1, f_2)^T$ in that local segment, where the factor $|Y'(s)|^{-1}$ is the Jacobian between the Eulerian and Lagrangian coordinates. We consider the velocity field with no regularization around the origin. The elastic force from the other part of the string always contributes to a smooth velocity field $u_{\text{far}}$ around the origin. The local string segment, however, generates a flow field $u_{\text{loc}}$ that is not smooth. In a small neighborhood of the origin, the tangential component behaves like a shear flow, with opposite shear rates on two sides of the horizontal axis. Indeed,

$$u_{\text{loc}}(x_1, x_2) = \int_{-1}^{1} G((x_1, x_2) - (x'_1, 0)) \cdot (f_1, f_2)^T \, dx'_1$$

$$= \frac{1}{4\pi} \int_{-1}^{1} -\frac{(f_1, f_2)^T}{2} \ln((x_1 - x'_1)^2 + x_2^2) \, dx'_1$$

$$+ \frac{f_1}{4\pi} \int_{-1}^{1} \frac{((x_1 - x'_1)^2, (x_1 - x'_1)x_2)^T}{(x_1 - x'_1)^2 + x_2^2} \, dx'_1$$

$$+ \frac{f_2}{4\pi} \int_{-1}^{1} \frac{((x_1 - x'_1)x_2, x_2^2)^T}{(x_1 - x'_1)^2 + x_2^2} \, dx'_1$$

$$\sim f_1 \left( \frac{1}{2\pi} - \frac{|x_2|}{2}, 0 \right)^T + f_2 \left( 0, \frac{1}{2\pi} \right)^T + O(|f| |x|^2) \quad (5.1)$$

for $|x| = |(x_1, x_2)| \ll 1$. This has been characterized by the jump condition of the tangential component of $\nabla_u u$ across the immersed boundary when the elastic force has nonzero tangential component there [22]. In fact, [22] derives the jump condition for the immersed boundary problem with the Navier-Stokes equation, but the same argument applies to the stationary Stokes case as well.

Compare

$$U_Y(0) = u(0,0) = u_{\text{loc}}(0,0) + u_{\text{far}}(0,0)$$

with $U_Y^\varepsilon(0)$ in the regularized case. It is known that (see (3.9)–(3.12))

$$U_Y^\varepsilon(0) = [u \ast \varphi_\varepsilon](0,0) = [(u_{\text{loc}} + u_{\text{far}}) \ast \varphi_\varepsilon](0,0).$$
By (5.1), the smoothness of \( u \), and the fact that \( \varphi_\varepsilon \) is supported on a disc of radius \( C \varepsilon \),

\[
(U^\varepsilon_Y - U_Y)(0) = \left[ -\frac{|x_2|}{2} \ast \varphi_\varepsilon \right]_{(0,0)} \left( f_1, 0 \right)^T + O(|f|\varepsilon^2)
\]

(5.2)

\[
= \left[ -\frac{\varepsilon}{2} \int_{\mathbb{R}^2} |x_2|\varphi(x_1, x_2)dx_1 dx_2 \right] \left( f_1, 0 \right)^T + O(|f|\varepsilon^2)
\]

\[
= -\frac{m_{1\varepsilon}}{\pi} \left( f_1, 0 \right)^T + O(|f|\varepsilon^2).
\]

In the last equation, we used (A.13), which will be proved in the Appendix A. We should highlight that the leading term in (5.2) agrees with that in (3.27); see also (3.61).

The calculation (5.2) clearly shows where the \( O(\varepsilon) \)-error comes from. Given nonzero tangential force at a point on the string, the local tangential flow in the unregularized case has a velocity profile like an absolute value function in the transversal direction. When mollifying such a flow field and restricting that onto the string, an \( O(\varepsilon) \)-error is produced pointwise in the tangential component unless the mollifier \( \varphi = \phi \ast \phi \) is \( L^2(\mathbb{R}^2) \)-orthogonal to that absolute value function in the transversal direction, which is \(|x_2|\) in our case. By the radial symmetry of \( \varphi \), this orthogonality condition is equivalent to \( m_1 = 0 \). Since the normal velocity field is smoother, an \( O(\varepsilon) \)-error only occurs in the tangential component.

With this insight, it is natural to believe that when \( \phi \) and \( \varphi \) are not necessarily radially symmetric, the right condition for the improved accuracy should be

\[
\widehat{m}_1(v) = 0
\]

for any unit vector \( v \) in \( \mathbb{R}^2 \), where

\[
\widehat{m}_1(v) \triangleq \int_{\mathbb{R}^2} |v \times (x_1, x_2)| \cdot \phi \ast \phi(x_1, x_2)dx_1 dx_2.
\]

Here \( v \) should be understood as the tangential direction of the string. If \( \phi \) is radially symmetric, this condition reduces to \( m_1 = 0 \). However, it is not clear if this condition can be fulfilled by some \( \phi \) that is not radially symmetric.

It is noteworthy that the condition \( \widehat{m}_1(v) = 0 \) can be treated as a generalization of the one-sided first moment condition in 1-D \([2]\), which in the continuous setting requires

\[
\int_{\mathbb{R}} \max\{0, x\} \phi(x) dx = 0.
\]

Here with abuse of notation, we use \( \phi \) to denote the profile of a 1-D regularized \( \delta \)-function. Note that this is equivalent to

\[
\int_{\mathbb{R}} |x| \phi(x) dx = 0
\]

if \( \phi \) is \( L^2(\mathbb{R}) \)-orthogonal with the function \( x \) (e.g., \( \phi \) is an even function). However, in our work, we propose the orthogonality condition for \( \phi \ast \phi \) instead of \( \phi \).
5.2 Improved Regularized $\delta$-Functions

We want to show that the condition $m_1 = 0$ for the improved accuracy is indeed achievable.

Take an arbitrary $\rho \in C_0^\infty(\mathbb{R}^2)$ such that $\rho \geq 0$, $\rho$ is radially symmetric, and $\rho$ is normalized. Define $\rho_r(x) = r^{-2}\rho(x/r)$. We shall look for $\phi$ in the form of

$$\phi(x) = \frac{1}{1 - c} (\rho_r(x) - cp(x))$$

with some $r \in (0, 1)$ and $c \in (0, 1)$ to be determined. Obviously, such $\phi$ satisfies all the assumptions in Theorem 1.4.

Given the ansatz, the condition $m_1 = 0$ for $\phi$ becomes

$$\int_{\mathbb{R}^2} |x| \cdot (\rho_r - cp) \ast (\rho_r - cp) \, dx = 0. \tag{5.3}$$

We simplify the left-hand side as follows:

$$\int_{\mathbb{R}^2} |x| \cdot (\rho_r - cp) \ast (\rho_r - cp) \, dx$$

$$= \int_{\mathbb{R}^2} |x| \cdot \rho_r \ast \rho_r \, dx - 2c \int_{\mathbb{R}^2} |x| \cdot \rho_r \ast \rho \, dx + c^2 \int_{\mathbb{R}^2} |x| \cdot \rho \ast \rho \, dx$$

$$= r \int_{\mathbb{R}^2} |x| \cdot \rho \ast \rho \, dx - 2c \int_{\mathbb{R}^2} |x| \cdot \rho_r \ast \rho \, dx + c^2 \int_{\mathbb{R}^2} |x| \cdot \rho \ast \rho \, dx.$$

All three integrals in the last line are positive due to the assumption $\rho \geq 0$. Hence, (5.3) admits a root $c \in (0, 1)$ as long as $r \in (0, 1)$ and

$$\int_{\mathbb{R}^2} |x| \cdot \rho \ast \rho \, dx > 0 \tag{5.4}.$$ 

Observe that as $r \to 0$, the right-hand side converges to 0, while the left-hand side

$$\lim_{r \to 0} \left( \int_{\mathbb{R}^2} |x| \cdot \rho_r \ast \rho \, dx \right)^2 = \left( \int_{\mathbb{R}^2} |x| \rho \, dx \right)^2 > 0.$$

This implies that there exists $r \in (0, 1)$ such that (5.4) holds; in practice, such $r$ does not have to be extremely small. Then we can solve for desired $c \in (0, 1)$ so that (5.3) holds, which completes the construction of $\phi$.

In the numerical immersed boundary method, choice of the regularized $\delta$-function plays a crucial role in many aspects. Lots of efforts have been made to design a good regularized $\delta$-function, or to better understand its effect in the accuracy of the immersed boundary method as well as some other problems with singular source terms [12, 13, 14, 15, 16, 17, 26, 30, 35, 40, 43]. In the continuous case, our analysis suggests that one can indeed attain improved accuracy by suitably choosing the regularized $\delta$-function. It is then worthwhile to investigate if such an improvement is possible in the discrete case.
5.3 Loss of Damping of High Frequencies in the $\varepsilon$-Regularized Problem

It was noted in Remark 1.6 that high frequencies in the string configuration can be effectively damped in the unregularized problem (see also [15]), while such a damping mechanism might be missing in the $\varepsilon$-regularized problem because of the regularization. As a result, high frequencies in $X^\varepsilon$ may not be well-controlled and a uniform-in-$\varepsilon$ bound for $X^\varepsilon$ seems hard to derive, which is the reason we need assumptions (i) and (ii) in Theorem 1.4 and the projection $\mathcal{P}_N$ in Theorem 1.8.

In this section, we formally derive a 1-D toy model of the $\varepsilon$-regularized problem, illustrating more precisely the loss of damping over high frequencies.

Instead of a closed 1-D string, now we consider an elastic string of infinite length, lying along the $x$-axis in $\mathbb{R}^2$ and immersed in a 2-D Stokes flow over the entire plane. With abuse of notation, we parametrize it by $X^\varepsilon(s, t) = (s + \eta^\varepsilon(s, t), 0)$ with $s \in \mathbb{R}$. Here $\eta^\varepsilon$ represents horizontal displacement of material points along the string from its most comfortable position $s(0)$. $\eta^\varepsilon$ is assumed to be small and decays fast as $s \to \pm \infty$. The string dynamics follows (1.8)–(1.11) up to minor changes in notation, again with $F^\varepsilon = F_X^\varepsilon$. Nice initial data for $\eta^\varepsilon$ is provided but we omit the details. By symmetry of the flow field, the string should always lie along the $x$-axis if it initially does, and thus the form of $X^\varepsilon$ assumed above is valid. In what follows, we shall write $X^\varepsilon(s, t)$ as $X^\varepsilon(s)$ or $X^\varepsilon$ for conciseness.

Denote $\phi = \phi \ast \phi$ as before, where $\phi$ is the profile of the regularized $\delta$-function as in (1.12). Let $u_1^\varepsilon$ and $f_1^\varepsilon$ denote $x$-components of the velocity field and the force in (1.8)–(1.11), respectively. Formally, by Fourier transform,

$$\partial_t \eta^\varepsilon(s) = (u_1^\varepsilon \ast \delta_x)(s + \eta^\varepsilon(s), 0, t)$$

$$(5.5) = \frac{1}{4\pi^2} \int_{\mathbb{R}} d\xi e^{i\xi(s + \eta^\varepsilon(s))} \int_{\mathbb{R}} \hat{\phi}_\varepsilon(\xi', \xi') \hat{f}_1^\varepsilon(\xi, \xi') \cdot \frac{\xi'^2}{(\xi^2 + \xi'^2)^2} d\xi'. $$

The last factor in the integrand above is the Fourier transform of the $(1, 1)$-entry of $G$ (see (1.6)). On the other hand, by (1.10),

$$\hat{f}_1^\varepsilon(\xi, \xi')$$

$$(5.6) = \int_{\mathbb{R}^2} dx \, dy \, e^{-i(\xi x + \xi' y)} \int_{\mathbb{R}} \delta_e((x, y) - (s' + \eta(s'), 0)) \cdot \eta''(s') ds'$$

$$= \hat{\phi}_\varepsilon(\xi, \xi') \int_{\mathbb{R}} e^{-i\xi(s' + \eta(s'))} \eta''(s') ds'.$$

Combining (5.5) and (5.6),

$$\partial_t \eta^\varepsilon(s) = \frac{1}{4\pi^2} \int_{\mathbb{R}} d\xi' \int_{\mathbb{R}} ds' e^{i\xi(s + \eta^\varepsilon(s) - s' - \eta(s'))} \eta''(s') \int_{\mathbb{R}} \frac{\xi'^2 \hat{\phi}_\varepsilon(\xi, \xi')}{(\xi^2 + \xi'^2)^2} d\xi'$$

$$(5.7) = \frac{1}{8\pi} \int_{\mathbb{R}} d\xi |\xi|^{-1} \hat{\phi}_\varepsilon(\xi) \int_{\mathbb{R}} ds' e^{i\xi(s - s' + \eta(s) - \eta(s'))} \eta''(s').$$
where we define

\[ \hat{\Phi}_\varepsilon(\xi) := \frac{2}{\pi} |\xi| \int_{\mathbb{R}} \frac{\xi^2 \hat{\phi}_\varepsilon(\xi, \xi')}{(\xi^2 + \xi'^2)^{3/2}} \, d\xi'. \]

Formally, if \( \varepsilon = 0 \), we have \( \hat{\phi}_\varepsilon \equiv 1 \) and \( \hat{\Phi}_\varepsilon \equiv 1 \). When \( \varepsilon > 0 \), \( \hat{\phi}_\varepsilon \) is smooth on \( \mathbb{R} \setminus \{0\} \), and \( \hat{\Phi}_\varepsilon \approx 1 \) for \( |\xi| \lesssim o(\varepsilon^{-1}) \). Moreover, \( \hat{\phi}_\varepsilon \) decays fast at \( \pm \infty \) over an \( O(\varepsilon^{-1}) \)-length scale since \( \hat{\phi}_\varepsilon \) does likewise.

We shall linearize (5.7) around \( \eta \equiv 0 \). We approximate \( e^{i \xi (s-s' + \eta(s) - \eta(s'))} \) by \( e^{i \xi (s-s')} (1 + i \xi (s) - i \xi (s')) \) and formally derive the linearized equation

\[
\partial_t \eta(s) = \frac{1}{8\pi} \int_{\mathbb{R}} d\xi |\xi|^{-1} \hat{\Phi}_\varepsilon(\xi) \int_{\mathbb{R}} ds' \, e^{i \xi (s-s')} \eta''(s') \\
+ \frac{1}{8\pi} \int_{\mathbb{R}} d\xi |\xi|^{-1} \hat{\Phi}_\varepsilon(\xi) \int_{\mathbb{R}} ds' \, e^{i \xi (s-s')} \cdot i \xi \eta(s) - \eta(s') - i \xi (s') \eta''(s') \\
= -\frac{1}{4} \hat{\Phi}_\varepsilon * (-\Delta)^{1/2} \eta - \frac{1}{4} \left[ \eta \hat{\Phi}_\varepsilon * \mathcal{H} \eta'' - \hat{\Phi}_\varepsilon * \mathcal{H} (\eta \eta'') \right].
\] (5.8)

In the unregularized case with \( \varepsilon = 0 \), \( \Phi_\varepsilon \) turns out to be the Dirac \( \delta \)-function. The first term in the last line of (5.8) provides damping over all frequencies, which is the very fact used in [15, 23] to establish well-posedness of the original unregularized problem. The other two terms can be bounded by applying commutator estimates (see, e.g., [13]). In this way, one can prove well-posedness of (5.8) with \( \varepsilon = 0 \) and \( \eta \) being small. We omit the details.

However, when \( \varepsilon > 0 \), the first term in the last line of (5.8) only provides effective damping of low frequencies with wave number up to \( O(\varepsilon^{-1}) \). Beyond this frequency range, the damping diminishes quickly, and we have little control of growth in higher frequencies. For simplicity, let us assume \( \hat{\Phi}_\varepsilon \) is supported on a bounded interval of size \( O(\varepsilon^{-1}) \) in the frequency space, for which \( \hat{\phi}_\varepsilon \) instead of \( \phi \) needs to be compactly supported. Then because of nonlinearity in the second term of (5.8), part of the energy of \( \eta \) may leak out from the support of \( \hat{\Phi}_\varepsilon \) to higher frequencies, where no damping mechanism is known to exist. It is not immediately clear if that will cause fast growth in higher-order norms of \( \eta \) afterwards, with growth rate getting worse as \( \varepsilon \to 0 \).

In summary, the discussion above implies that in the original \( \varepsilon \)-regularized problem, it might not be trivial to establish a uniform-in-\( \varepsilon \) bound for the \( H^{2+\theta} \)-norm of \( \eta \) in a uniform time interval. Yet, a simplified model such as (5.8) may be a good starting point to carry out such analysis. On the other hand, the discussion also suggests that the projection in Fourier space in Theorem 1.8 can be a game-changer only when \( N \lesssim O(\varepsilon^{-1}) \), since it is in that frequency range that the damping mechanisms in the regularized and unregularized problems are similar. Hence, the assumption \( N \leq c_* \varepsilon^{-1} \) in Theorem 1.8 is natural and optimal.
5.4 Future Problems

In the paper, we introduce the regularized immersed boundary problem to mimic its discrete counterpart in the numerical immersed boundary method. As the convergence and error estimates have been obtained in the continuous case, it is natural to ask if it is possible to derive an error bound in the discrete and dynamic setting for the immersed boundary method, at least for problems with stationary Stokes equations. Some static error estimates for the velocity field and pressure have been obtained in the discrete case by prescribing regularity of the string and the force along it \([16, 17, 22]\). \([22]\) also studies a simplified dynamic model problem and obtains its time-dependent error bound. An a posteriori analysis of the time-dependent error is performed in a recent work \([36]\). However, a complete theory of dynamic error estimate for the numerical immersed boundary method is far from being established.

Besides the numerical issues, one question in analysis that has not been answered in this work is whether there is convergence from \(X^\varepsilon\) to \(X\) without extra assumptions (i) and (ii) in Theorem 1.4. It is suggested in Remark 1.6 and Section 5.2 that we may have to look more closely at high frequencies of the string configuration. Justification of the convergence or a counterexample would be very interesting.

In this work, we only focus on comparing the string dynamics in the regularized and the original Stokes immersed boundary problems. It is also important to formulate convergence and error estimates for other quantities such as velocity field and pressure. Such convergence should be expected in the \((\varepsilon, N)\)-regularized problem as the convergence in the string motion has been established. The subtlety lies in the choice of function spaces in which convergence is established, because the velocity field and pressure in the unregularized problem is not smooth around the immersed string even though the string configuration and the string velocity are smooth. It is also noteworthy that the condition \(n_1 = 0\) may not necessarily lead to improved accuracy in the velocity field and pressure, as it only aims at representing the velocity accurately along the string, but not in its neighborhood. Yet, if the string motion is tracked with higher accuracy, it is possible to come up with a separate scheme to find out the velocity field and pressure more accurately, which may involve a different regularization or other technicality.

Generalizing this work to the immersed boundary problem with Navier-Stokes equation might be challenging. To the best of our knowledge, the well-posedness of the unregularized problem in the Navier-Stokes case is still open, although there have been many related results on the interface dynamics in two-fluid systems \([31, 32, 37]\).
Appendix A  Proofs of Auxiliary Results

A.1 Proof of Lemma 3.1

PROOF. In what follows, we use $\partial_s S(|y'|, \cdot)$ to denote the partial derivative with respect to the second variable only, but we use $S'$ to denote the total derivative in $s$.

We first show (3.5). By the definition of $F_Y$ and Assumption (a),

$$|(SY')'(s)| \leq |SY''(s)| + |\partial_s S(|y'|, s)||y'|$$

$$+ |\partial_p S(|y'|, s)||y'| \cdot \frac{|y'|||y''(s)|}{|y'|(s)}$$

$$\leq C \mu (|y'(s)| + |y''(s)|).$$

(3.6) with $\theta = 0$ follows immediately.

To show (3.6) with $\theta \in (0, 1)$, we calculate

$$F_Y(s + \tau) - F_Y(s)$$

$$= S(|y'(s + \tau)|, s + \tau)Y''(s + \tau) - S(|y'(s)|, s)Y''(s)$$

$$+ \partial_s S(|y'(s + \tau)|, s + \tau)y'(s + \tau) - \partial_s S(|y'(s)|, s)y'(s)$$

$$+ \partial_p S(|y'(s + \tau)|, s + \tau)\frac{y'(s + \tau) \cdot y''(s + \tau)}{|y'(s + \tau)|}Y'(s + \tau)$$

$$- \partial_p S(|y'(s)|, s)\frac{y'(s) \cdot y''(s)}{|y'(s)|}Y'(s).$$

By Assumption (a) and the choice of $\mu$ in (3.7), it is not difficult to derive that

$$|F_Y(s + \tau) - F_Y(s)| \leq C \mu \left(|\tau| + \frac{|y'(s + \tau) - y'(s)|}{c\|y''\|_{L^2}} \right) |y''(s + \tau)| + C \mu |y''(s + \tau) - y''(s)|$$

$$+ C \mu \left(|\tau| + \frac{\|y'(s + \tau) - y'(\cdot)\|_{L^\infty}}{c\|y''\|_{L^2}} \right) |y'(s + \tau)| + C \mu |y'(s + \tau) - y'(s)|$$

$$\leq C \mu \left(|\tau| + \frac{\|y''(s + \tau) - y''(\cdot)\|_{L^2}}{c\|y''\|_{L^2}} \right) |y''(s + \tau)| + C \mu |y''(s + \tau) - y''(s)| + C \mu |y'(s + \tau) - y'(s)|.$$

Hence, by Sobolev embedding,

$$\|F_Y(s + \tau) - F_Y(s)\|_{L^2(\mathbb{T})} \leq C \mu |\tau| \|y''\|_{L^2} + C \mu \|y''(\cdot + \tau) - y''(\cdot)\|_{L^2}.$$

By the equivalent definitions of the $H^\theta(\mathbb{T})$-seminorm,

$$\|F_Y\|_{H^\theta(\mathbb{T})}^2 \leq C \int_{-1}^{1} \frac{\|F_Y(s + \tau) - F_Y(s)\|^2_{L^2}}{|\tau|^{1+2\theta}} d\tau \leq C \mu \|y\|_{H^{2+\theta}(\mathbb{T})}^2.$$
This proves (3.6). □

A.2 Properties of $f_2$ and $f_3$

In this section, we shall prove estimates for the auxiliary functions $f_2$ and $f_3$. For convenience, we recall their definitions in Section 3.2.

(A.1) \[ f_1(x) = \frac{x}{\pi} \int_{\mathbb{R}^2} e^{ix\eta_1} \frac{-i\eta_1 \eta_2^2}{(\eta_1^2 + \eta_2^2)^2} \hat{\phi}(\eta) d\eta, \]

(A.2) \[ f_2(x) = \frac{x}{\pi} \int_{\mathbb{R}^2} e^{ix\eta_1} \frac{-i\eta_1 (\eta_1^2 - \eta_2^2)}{(\eta_1^2 + \eta_2^2)^2} \hat{\phi}(\eta) d\eta, \]

\[ f_3(x) = f_1(x) - 1. \]

PROOF OF LEMMA 3.3. The proof involves repeated integration by parts in the formulas of $f_1$, $f_2$, and $f_3$.

Since $\varphi$ is smooth and compactly supported, $\hat{\varphi}$ is smooth and decays sufficiently fast at $\infty$. Hence, the integrals in (A.1) and (A.2) are absolutely integrable; in addition,

\[ \int_{\mathbb{R}} \frac{-i\eta_1 \eta_2^2}{(\eta_1^2 + \eta_2^2)^2} \hat{\varphi}(\eta) d\eta_2 \quad \text{and} \quad \int_{\mathbb{R}} \frac{-i\eta_1 (\eta_1^2 - \eta_2^2)}{(\eta_1^2 + \eta_2^2)^2} \hat{\varphi}(\eta) d\eta_2 \]

decays fast as $\eta_1 \to \pm \infty$. This implies that $f_1(0) = f_2(0) = 0$, and $f_1$ and $f_2$ are smooth functions in $x$. Therefore, $f_3(0) = -1$ and $f_3$ is also smooth on $\mathbb{R}$.

We start by considering

\[ g_1(x) = \frac{f_1(x)}{x} = \frac{1}{\pi} \int_{\mathbb{R}^2} e^{ix\eta_1} \frac{-i\eta_1 \eta_2^2}{(\eta_1^2 + \eta_2^2)^2} \hat{\varphi}(\eta) d\eta. \]

Since

\[ \frac{\eta_1 \eta_2^2}{(\eta_1^2 + \eta_2^2)^2} = \frac{\partial}{\partial \eta_2} \left( \frac{1}{2} \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{2(\eta_1^2 + \eta_2^2)} \right) \]

whenever $\eta_1 \neq 0$, by integration by parts first in $\eta_2$ and then in $\eta_1$, we derive that

(A.3) \[ g_1(x) = \frac{i}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} \left( \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_2} \hat{\varphi}(\eta) d\eta = \]
Here we did integration by parts separately in the regions divided by \( \{ \eta_1 = 0 \} \) because \( \arctan \frac{\eta_2}{\eta_1} \) is discontinuous across that line. Indeed,

\[
\lim_{\eta_1 \to 0^\pm} \left( \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) = \frac{\pi}{2} \text{sgn}(\pm \eta_2) \quad \forall \eta_2 \neq 0.
\]

By the fast decay of \( \partial_{\eta_2} \hat{\psi} \) at \( \infty \), the first term is 0. The second term is also 0 as the perimeter of the circle shrinks to 0 while the integrand stays bounded. Hence,

(A.4) \quad g_1(x) = -\frac{1}{2\pi} \int_{B_M} \text{sgn}(\eta_2) \cdot \partial_{\eta_2} \hat{\psi}(O, \eta_2) d\eta_2

\[
- \frac{1}{2\pi} \int_{B_M \setminus \{ \eta_1 = 0 \}} e^{ix\eta_1} \cdot \frac{\partial}{\partial \eta_1} \left[ \left( \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_2} \hat{\psi}(\eta) \right] d\eta.
\]
Here we used the fact that

\[
\int_{\mathbb{R}} \text{sgn}(\eta_2) \cdot \partial_{\eta_2} \hat{\varphi}(0, \eta_2) d\eta_2 = \int_{\mathbb{R}^+} \partial_{\eta_2} \hat{\varphi}(0, \eta_2) d\eta_2 - \int_{\mathbb{R}^-} \partial_{\eta_2} \hat{\varphi}(0, \eta_2) d\eta_2
\]

\[
= -2\hat{\varphi}(0) = -2.
\]

Observing that

\[
(A.5) \quad \frac{\eta_2^3}{(\eta_1^2 + \eta_2^2)^2} = \frac{\partial}{\partial \eta_2} \left( \frac{1}{2} \ln(\eta_1^2 + \eta_2^2) + \frac{1}{2} \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right)
\]

whenever \((\eta_1, \eta_2) \neq 0\), we perform integration by parts in \(\eta_2\) to obtain that

\[
(A.6) \quad f_3(x) = xg_1(x) - 1
\]

\[
= -\frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} \left( \text{arctan} \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1 \eta_2} \hat{\varphi}(\eta) d\eta
\]

\[
- \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} \left( \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_2 \eta_2} \hat{\varphi}(\eta) d\eta.
\]

Since \(\hat{\varphi}\) is symmetric with respect to the axis \(\{\eta_1 = 0\}\), \(\partial_{\eta_1 \eta_2} \hat{\varphi}(\eta) = 0\) when \(\eta_1 = 0\). This implies that the integrand in \((A.6)\) is continuous away from \((0, 0)\).
Hence, by integration by parts in $\eta_1$,

\[
f_3(x) = -\frac{i}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i\eta_1 x} \frac{\partial}{\partial \eta_1} \left[ \left( \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1 \eta_2} \hat{\varphi}(\eta) \right] d\eta \]

\[
- \frac{i}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i\eta_1 x} \frac{\partial}{\partial \eta_1} \left[ \left( \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1 \eta_2} \hat{\varphi}(\eta) \right] d\eta \]

\[
= -\frac{i}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i\eta_1 x} \frac{\partial}{\partial \eta_1} \left( \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1 \eta_2} \hat{\varphi}(\eta) d\eta \]

\[
- \frac{i}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i\eta_1 x} \frac{-2\eta^3_2}{(\eta_1^2 + \eta_2^2)^2} \cdot \partial_{\eta_1 \eta_2} \hat{\varphi}(\eta) d\eta \]

\[
- \frac{i}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i\eta_1 x} \left( \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1 \eta_2} \hat{\varphi}(\eta) d\eta \]

\[
- \frac{i}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i\eta_1 x} \frac{2\eta^3_1 + 4\eta_1 \eta^2_2}{(\eta_1^2 + \eta_2^2)^2} \cdot \partial_{\eta_2 \eta_1} \hat{\varphi}(\eta) d\eta. \]

Thanks to (A.5), the second term coincides with the third term. Moreover,

\[
\frac{2\eta^3_1 + 4\eta_1 \eta^2_2}{(\eta_1^2 + \eta_2^2)^2} = \frac{\partial}{\partial \eta_2} \left( 3 \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right). \]

Hence,

\[
f_3(x) = -\frac{i}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i\eta_1 x} \frac{\partial}{\partial \eta_1} \left( \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1 \eta_2} \hat{\varphi}(\eta) d\eta \]

\[
- \frac{i}{\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i\eta_1 x} \left( \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_2} \hat{\varphi}(\eta) d\eta \]

\[
+ \frac{i}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i\eta_1 x} \left( 3 \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_2 \eta_1} \hat{\varphi}(\eta) d\eta. \]

Recall that $m_2$ is defined in (1.14). Since $\hat{\varphi}$ is radially symmetric,

\[
\partial_{\eta_1 \eta_1} \hat{\varphi}(0) = \partial_{\eta_2 \eta_2} \hat{\varphi}(0) = -\int_{\mathbb{R}^2} \chi_1^2 \cdot \varphi(x_1, x_2) dx_1 dx_2 = -\frac{1}{2} m_2. \]
To this end, we perform further integration by parts to (A.8) and proceed as in (A.3) and (A.4) to find that

\[
\begin{align*}
  f_3(x) &= -\frac{1}{x^2} \partial_{\eta_1} \eta_1 \hat{\varphi}(0) + \frac{3}{x^2} \partial_{\eta_2} \eta_2 \hat{\varphi}(0) \\
  &\quad + \frac{1}{2\pi x^2} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} \left( \frac{\partial}{\partial \eta_1} \left( \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1} \eta_1 \eta_2 \hat{\varphi}(\eta) \right) d\eta \\
  &\quad + \frac{1}{\pi x^2} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} \left( \frac{\partial}{\partial \eta_1} \left( \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1} \eta_1 \eta_2 \hat{\varphi}(\eta) \right) d\eta \\
  &\quad - \frac{1}{2\pi x^2} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} \left( \frac{\partial}{\partial \eta_1} \left( 3 \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_2} \eta_2 \hat{\varphi}(\eta) \right) d\eta \\
  &\triangleq -m_2 x^{-2} + (f_{3,1}(x) + f_{3,2}(x) + f_{3,3}(x)) x^{-2}.
\end{align*}
\]

Note that the term \(-m_2 x^{-2}\) arises from the discontinuity of \(\arctan \frac{\eta_2}{\eta_1}\) across \(\{\eta_1 = 0\}\). Since the integrands in \(f_{3,i}\) are all absolutely integrable, by the Riemann-Lebesgue lemma,

\[
f_{3,1}(x) + f_{3,2}(x) + f_{3,3}(x) \to 0 \quad \text{as} \quad x \to \pm \infty.
\]

This together with the smoothness of \(f_3\) implies that there exists a universal \(C > 0\) such that

\[
|f_3(x)| \leq C \frac{1}{1 + x^2}.
\]

Next we prove estimates for \(f_3'\) and \(f_3''\). It suffices to consider \(f_{3,1}'\) and \(f_{3,1}''\). We start with \(f_{3,1}\) and derive as in (A.4)–(A.6),

\[
\begin{align*}
  f_{3,1}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} \left( \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1} \eta_1 \eta_2 \hat{\varphi}(\eta) d\eta \\
  &\quad + \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} \left( \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1} \eta_1 \eta_2 \hat{\varphi}(\eta) d\eta.
\end{align*}
\]
Hence, for $k = 1, 2,$

\[(x^2 f_{3,1})^{(k)}(x)\]

\[
= -\frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} (i\eta_1)^k \frac{d}{d\eta_1} \left( \frac{\arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1\eta_2}{\eta_1^2 + \eta_2^2}}{3\arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1\eta_2}{\eta_1^2 + \eta_2^2}} \right) \partial_{\eta_1}^3 \partial_{\eta_2}^2 \phi(\eta) \right] d\eta
\]

We apply the Riemann-Lebesgue lemma as before in order to claim that $(x^2 f_{3,1})^{(k)}$ stays bounded as $x \to \pm \infty$ for $k = 0, 1, 2$. Similarly, we can rewrite $f_{3,2}$ and $f_{3,3}$ as

\[f_{3,2}(x)\]

\[
= -\frac{12}{x^2} \partial_{\eta_1}^2 \partial_{\eta_2}^2 \phi(0)
\]

\[
- \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} (i\eta_1)^k \frac{d}{d\eta_1} \left( \frac{\arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1\eta_2}{\eta_1^2 + \eta_2^2}}{3\arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1\eta_2}{\eta_1^2 + \eta_2^2}} \right) \partial_{\eta_1}^3 \partial_{\eta_2}^2 \phi(\eta) \right] d\eta
\]

\[
+ \frac{2}{\pi x^2} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{ix\eta_1} (i\eta_1)^k \frac{d}{d\eta_1} \left( \frac{2 \ln \frac{\eta_1^2 + \eta_2^2}{\eta_1^2 + \eta_2^2}}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1}^4 \partial_{\eta_2}^3 \phi(\eta) \right] d\eta.
\]
and

\[ f_{3,3}(x) = -\frac{3}{x^2} \partial_{\eta_1}^2 \partial_{\eta_2}^2 \hat{\phi}(0) + \frac{5}{x^2} \partial_{\eta_2}^4 \hat{\phi}(0) \]

\[ + \frac{1}{2\pi x^2} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i x \eta_1} \frac{\partial}{\partial \eta_1} \left[ \left( 3 \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_2}^2 \partial_{\eta_2}^3 \hat{\phi}(\eta) \right] d\eta \]

\[ + \frac{1}{\pi x^2} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i x \eta_1} \frac{\partial}{\partial \eta_1} \left[ \left( 2 \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_2^2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1} \partial_{\eta_2}^2 \hat{\phi}(\eta) \right] d\eta \]

\[ - \frac{1}{2\pi x^2} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i x \eta_1} \frac{\partial}{\partial \eta_1} \left[ \left( 5 \arctan \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1}^2 \partial_{\eta_2}^2 \hat{\phi}(\eta) \right] d\eta. \]  

(A.10)

They give similar estimates as those for \( f_{3,1} \) as \( x \to \pm \infty \). Combining them with (A.9) and by the smoothness of \( f_3 \) on \( \mathbb{R} \), we prove that for \( k = 0, 1, 2 \),

\[ |f_3^{(k)}(x)| \leq \frac{C}{1 + |x|^{k+2}}. \]

If, in addition, \( m_2 = 0 \), \( f_3 \) enjoys the following improved estimate for \( k = 0, 1, 2 \):

\[ |f_3^{(k)}(x)| \leq \frac{C}{1 + x^4}. \]

We analyze \( f_2(x) \) using the same approach. Noticing that

\[ \frac{\eta_1 (\eta_1^2 - \eta_2^2)}{(\eta_1^2 + \eta_2^2)^2} = \frac{\partial}{\partial \eta_2} \left( \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right), \]

we calculate in a similar manner,

\[ f_2(x) = \frac{i x}{\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i x \eta_1} \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \partial_{\eta_2}^2 \hat{\phi}(\eta) d\eta \]

\[ = -\frac{1}{\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i x \eta_1} \frac{\partial}{\partial \eta_1} \left[ \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \partial_{\eta_2}^2 \hat{\phi}(\eta) \right] d\eta \]

\[ = -\frac{1}{\pi} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i x \eta_1} \frac{\partial}{\partial \eta_1} \left( \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \partial_{\eta_1} \partial_{\eta_2} \hat{\phi}(\eta) + \frac{-\eta_1^2 \eta_2 + \eta_2^3}{(\eta_1^2 + \eta_2^2)^2} \partial_{\eta_1}^2 \hat{\phi}(\eta) \right) d\eta. \]  

(A.11)

Since

\[ \frac{-\eta_1^2 \eta_2 + \eta_2^3}{(\eta_1^2 + \eta_2^2)^2} = \frac{\partial}{\partial \eta_2} \left( \frac{1}{2} \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right), \]
by integration by parts,

\[ f_2(x) = \frac{i}{\pi x} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i \eta_1} \frac{\partial}{\partial \eta_1} \left[ -\frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \partial_{\eta_1 \eta_2} \hat{\phi}(\eta) \right] d\eta \]

\[ + \left( \frac{1}{2} \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1 \eta_2} \hat{\phi}(\eta) \right] d\eta \]

\[ = -\frac{i}{\pi x} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i \eta_1} \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \partial_{\eta_1} \hat{\phi}(\eta) d\eta \]

\[ + \frac{2i}{\pi x} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i \eta_1} \left( \frac{1}{2} \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1 \eta_2} \hat{\phi}(\eta) d\eta \]

\[ + \frac{i}{\pi x} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i \eta_1} \frac{\eta_1^3 + 3 \eta_1 \eta_2^2}{(\eta_1^2 + \eta_2^2)^2} \partial_{\eta_2} \hat{\phi}(\eta) d\eta. \]

Since

\[ \frac{\eta_1^3 + 3 \eta_1 \eta_2^2}{(\eta_1^2 + \eta_2^2)^2} = \frac{\partial}{\partial \eta_2} \left( 2 \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right), \]

we proceed as in (A.8) and (A.9) to obtain that

\[ f_2(x) \]

\[ = 2m_2 x^{-2} + \frac{1}{\pi x^2} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i \eta_1} \frac{\partial}{\partial \eta_1} \left[ \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \partial_{\eta_1 \eta_2} \hat{\phi}(\eta) \right] d\eta \]

\[ - \frac{2}{\pi x^2} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i \eta_1} \frac{\partial}{\partial \eta_1} \left[ \frac{1}{2} \ln(\eta_1^2 + \eta_2^2) + \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right] \partial_{\eta_1 \eta_2} \hat{\phi}(\eta) \right] d\eta \]

\[ + \frac{1}{\pi x^2} \int_{\mathbb{R}^2 \setminus \{\eta_1 = 0\}} e^{i \eta_1} \frac{\partial}{\partial \eta_1} \left( 2 \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_2 \eta_2} \hat{\phi}(\eta) d\eta. \]

Again by the Riemann–Lebesgue lemma, the last three terms are \( o(x^{-2}) \) as \( x \to \pm \infty \). Combining this with the smoothness of \( f_2 \), we conclude that there exists a universal \( C > 0 \) such that

\[ |f_2(x)| \leq \frac{C}{1 + x^2}. \]

Bounds for the derivatives of \( f_2 \) and the improved estimates when \( m_2 = 0 \) can be justified in the same way as that for \( f_3 \). We omit the details. \( \Box \)
PROOF OF LEMMA 3.4. We start from \( f_2 \). By (A.11),
\[
\int_{\mathbb{R}} f_2(x) \, dx = \hat{f}_2(0)
\]
\[
= \lim_{\eta_1 \to 0} -2 \int_{\mathbb{R}} \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \partial_{\eta_1 \eta_2} \hat{\varphi}(\eta) \, d\eta_2
\]
\[
+ \lim_{\eta_1 \to 0} -2 \int_{|\eta_2| > \delta'} \left( \frac{-\eta_2}{\eta_1^2 + \eta_2^2} + \frac{2\eta_2^3}{(\eta_1^2 + \eta_2^2)^2} \right) \partial_{\eta_2} \hat{\varphi}(\eta) \, d\eta_2
\]
\[
+ \lim_{\eta_1 \to 0} -2 \int_{|\eta_2| \leq \delta'} \left( \frac{-\eta_2}{\eta_1^2 + \eta_2^2} + \frac{2\eta_2^3}{(\eta_1^2 + \eta_2^2)^2} \right) (\partial_{\eta_2} \hat{\varphi}(\eta) - \partial_{\eta_2} \hat{\varphi}(\eta, 0)) \, d\eta_2
\]
\[\triangleq I_1 + I_2 + I_3,
\]
where \( \delta' > 0 \) is arbitrary. Here in the last term, we used the fact that the first factor
in the integrand is odd in \( \eta_2 \). By the dominated convergence theorem,
\[
I_1 + I_2 = -2 \int_{|\eta_2| > \delta'} \frac{1}{\eta_2} \cdot \partial_{\eta_2} \hat{\varphi}(0, \eta_2) \, d\eta_2.
\]
For \( I_3 \),
\[
|I_3| \leq C \| \partial_{\eta_2} \hat{\varphi} \|_{L^{\infty}(\mathbb{R}^2)} \int_{|\eta_2| \leq \delta'} \frac{|\eta_2|}{|\eta_2|} \, d\eta_2 \leq C \delta',
\]
where \( C > 0 \) is a universal constant depending only on \( \varphi \). Since \( \delta' > 0 \) is arbitrary,
\[
\int_{\mathbb{R}} f_2(x) \, dx = -2 \cdot \text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta_2} \partial_{\eta_2} \hat{\varphi}(0, \eta_2) \, d\eta_2.
\]
In a similar fashion, thanks to (A.4), one can justify that
\[
\int_{\mathbb{R}} f_3(x) \, dx
\]
\[
\begin{align*}
= & \lim_{\eta_1 \to 0} \int_{\mathbb{R}} - \left( \arctan \frac{\eta_2}{\eta_1} - \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \right) \partial_{\eta_1 \eta_2} \hat{\varphi}(\eta)
\end{align*}
\]
\[
+ 2 \frac{\eta_2^3}{(\eta_1^2 + \eta_2^2)^2} \cdot \partial_{\eta_2} \hat{\varphi}(\eta) \, d\eta_2
\]
\[
= 2 \cdot \text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta_2} \partial_{\eta_2} \hat{\varphi}(0, \eta_2) \, d\eta_2.
\]
We then calculate that
\[
\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta_2} \partial_{\eta_2} \hat{\varphi}(0, \eta_2) \, d\eta_2
\]
\[
(A.12) \quad = \lim_{\delta' \to 0} \int_{|\eta_2| > \delta'} \frac{d\eta_2}{\eta_2} \cdot \frac{1}{\eta_2} \partial \eta_2 \int_{\mathbb{R}^2} e^{-i\eta_2 x_2} \varphi(x_1, x_2) \, dx_1 \, dx_2
\]
\[
= \lim_{\delta' \to 0} \int_{\mathbb{R}^2} \left[ \int_{|\eta_2| > \delta'} \frac{1}{\eta_2} e^{-i\eta_2 x_2} \, d\eta_2 \right] (-ix_2) \varphi(x_1, x_2) \, dx_1 \, dx_2.
\]
It is known that
\[
\int_{|\eta_2| \geq \delta'} \frac{1}{\eta_2} e^{-ix_2\eta_2} d\eta_2 = \int_{|\eta_2| \geq \delta'} \frac{-i \sin(x_2 \eta_2)}{\eta_2} d\eta_2 = -i \cdot \text{sgn}(x_2) \int_{|\eta_2| \geq |x_2|/\delta} \frac{\sin(\eta_2)}{\eta_2} d\eta_2
\]
is uniformly bounded in \(x_2\) and \(\delta'\), which converges to \(-i \pi \cdot \text{sgn}(x_2)\) as \(\delta' \to 0^+\). Applying the dominated convergence theorem to (A.12), we obtain that
\[
\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta_2} \partial_{\eta_2} \tilde{\varphi}(0, \eta_2) d\eta_2 = \int_{\mathbb{R}^2} (-i \pi \cdot \text{sgn}(x_2))(-ix_2)\varphi(x_1, x_2) d\eta_2 d\eta_2 = -\pi \int_{\mathbb{R}^2} |x_2|\varphi(x_1, x_2) d\eta_2 d\eta_2 = -2m_1.
\]
Indeed, by the radial symmetry of \(\varphi\),
\[
\int_{\mathbb{R}^2} |x_2|\varphi(x_1, x_2) d\eta_2 d\eta_2 = \int_0^\infty \int_0^{2\pi} r^2 |\sin \theta| \cdot \varphi(r) d\theta dr = 4 \int_0^\infty r^2 \varphi(r) dr = \frac{2m_1}{\pi}.
\]
This completes the proof. □

**Appendix B  Estimates for Fractional Heat Equations**

Recall that \(\mathcal{L} = -\frac{1}{4} (-\Delta)^{1/2}\).

**Lemma B.1.** Let \(\mathcal{P}_N\) be defined by (1.24) and (1.25) with \(N > 1\). Assume \(Z_0 \in H^{l+1}(\mathbb{T})\) with \(l \geq 0\), and \(f \in L_T^\infty H^l(\mathbb{T})\) for \(T > 0\). The model equation
\[
\partial_t Z(s, t) = \mathcal{L} Z(s, t) + \mathcal{P}_N f(s, t), \quad Z(s, 0) = Z_0(s), \quad s \in \mathbb{T}, \quad t \geq 0,
\]
has a unique solution \(Z \in C_{[0, T]} H^{l+1}(\mathbb{T})\) satisfying that
\[
\begin{align*}
\|Z\|_{C_{[0, T]} \dot{H}^{l+1}(\mathbb{T})} &\leq \|Z_0\|_{\dot{H}^{l+1}(\mathbb{T})} + \|\mathcal{P}_N f\|_{L_T^2 \dot{H}^{l}(\mathbb{T})}, \\
\|Z\|_{C_{[0, T]} \dot{H}^{l+1}(\mathbb{T})} &\leq \|Z_0\|_{\dot{H}^{l+1}(\mathbb{T})} + C(\ln N)^{1/2} \|\mathcal{P}_N f\|_{L_T^\infty \dot{H}^{l}(\mathbb{T})}. 
\end{align*}
\]
With abuse of notation, \(\dot{H}^{0}(\mathbb{T})\) is understood as \(L^2(\mathbb{T})\).

More generally, for arbitrary \(1 \leq n_1 < n_2 \leq N\), define \(\mathcal{P}_{n_1, n_2} = \mathcal{P}_{n_2} - \mathcal{P}_{n_1}\). Then
\[
\begin{align*}
\|\mathcal{P}_{n_1, n_2} Z\|_{C_{[0, T]} \dot{H}^{l+1}(\mathbb{T})} &\leq \|\mathcal{P}_{n_1, n_2} Z_0\|_{\dot{H}^{l+1}(\mathbb{T})} + C \left(1 + \ln \frac{n_2}{n_1}\right)^{1/2} \|\mathcal{P}_{n_1, n_2} f\|_{L_T^\infty \dot{H}^{l}(\mathbb{T})},
\end{align*}
\]
where \(C\) is a universal constant.
proof. Since \( \{e^{tL} \}_{t \geq 0} \) is a strongly continuous contraction semigroup on \( L^2(T) \) and \( H^{l+1}(T) \), while \( Z_0 \) and \( \mathcal{P}_N f(\cdot, t) \in H^{l+1}(T) \), we obtain
\[
(B.4) \quad Z(s, t) = e^{tL} Z_0(s) + \int_0^t e^{(t-t')L} \mathcal{P}_N f(s, t') \, dt'
\]
as a solution in \( C_{[0,T]} H^{l+1}(T) \). This also implies (B.1). The uniqueness follows from that in \( L^2(T) \), \( H^{l+1}(T) \); the latter can be proved by a classic energy estimate.

Now it suffices to show (B.3), since (B.2) follows from (B.1) and (B.3). Applying \( \mathcal{P}_{n_1, n_2} \) to (B.4), we first find that
\[
(B.5) \quad \| e^{tL} \mathcal{P}_{n_1, n_2} Z_0 \|_{\dot{H}^{l+1}(T)} \leq e^{-n_1 t/4} \| \mathcal{P}_{n_1, n_2} Z_0 \|_{\dot{H}^{l+1}(T)} \quad \forall t \geq 0.
\]
Then consider the integral in (B.4). By Parseval’s identity,
\[
(B.6) \quad \left\| \int_0^t e^{(t-t')L} \mathcal{P}_{n_1, n_2} f(s, t') \, dt' \right\|_{\dot{H}^{l+1}(T)}^2 = C \sum_{|k| \in (n_1, n_2]} |k|^{2l+1} \left| \int_0^t e^{-\frac{1}{4}|k|(t-t')} \hat{f}_k(t') \, dt' \right|^2.
\]
By the Cauchy-Schwarz inequality,
\[
\sum_{|k| \in (n_1, n_2]} |k|^{2l+1} \left| \int_0^t e^{-\frac{1}{4}|k|(t-t')} \hat{f}_k(t') \, dt' \right|^2 \leq \sum_{|k| \in (n_1, n_2]} |k|^{2l+1} \int_0^t e^{-\frac{1}{4}|k|(t-t')} \, dt' \int_0^t e^{-\frac{1}{4}|k|(t-t')} \left| \hat{f}_k(t') \right|^2 \, dt'
\]
\[
\leq C \sum_{|k| \in (n_1, n_2]} \int_0^t \left| k \right| e^{-\frac{1}{4}|k|(t-t')} \cdot |k|^{2|l|} \left| \hat{f}_k(t') \right|^2 \, dt'.
\]
Suppose \( t \geq n_1^{-1} \). Then
\[
\sum_{|k| \in (n_1, n_2]} \int_0^t \left| k \right| e^{-\frac{1}{4}|k|(t-t')} \cdot |k|^{2|l|} \left| \hat{f}_k(t') \right|^2 \, dt' = \sum_{|k| \in (n_1, n_2]} \int_0^{t-n_1^{-1}} + \int_{t-n_1^{-1}}^t \left| k \right| e^{-\frac{1}{4}|k|(t-t')} \cdot |k|^{2|l|} \left| \hat{f}_k(t') \right|^2 \, dt'.
\]
For \( t' \leq t - n_1^{-1} \),
\[
\left| k \right| e^{-\frac{1}{4}|k|(t-t')} \leq C n_1 e^{-\frac{1}{4}n_1(t-t')} \quad \forall |k| \geq n_1.
\]
For \( t' \in [t - n_1^{-1}, t] \),
\[
\left| k \right| e^{-\frac{1}{4}|k|(t-t')} \leq \min \left\{ \frac{C}{t - t'}, n_2 \right\} \quad \forall |k| \leq n_2.
\]
Hence, by Parseval’s identity, 

\[
\sum_{|k| \in \{n_1, n_2\}} \int_0^{t'} |k| e^{-\frac{1}{4} |k| (t-t')} \cdot |k|^{2l} |\hat{f}_k(t')|^2 dt' 
\leq C \int_0^{t-n^{-1}} n_1 e^{-\frac{1}{4} n_1 (t-t')} \sum_{|k| \in \{n_1, n_2\}} |k|^{2l} |\hat{f}_k(t')|^2 dt' 
+ \int_{t-n^{-1}}^t \min \left\{ \frac{C}{t-t'}, n_2 \right\} \sum_{|k| \in \{n_1, n_2\}} |k|^{2l} |\hat{f}_k(t')|^2 dt' 
\leq C \left( \int_0^{t-n^{-1}} n_1 e^{-\frac{1}{4} n_1 (t-t')} dt' + \int_{t-n^{-1}}^t \min \left\{ \frac{C}{t-t'}, n_2 \right\} dt' \right) 
\cdot \left\| \mathcal{P}_{n_1, n_2} f \right\|_{L_T^\infty \hat{H}^l(\mathbb{T})}^2 
\leq C \left( 1 + \ln \frac{n_2}{n_1} \right) \| \mathcal{P}_{n_1, n_2} f \|_{L_T^\infty \hat{H}^l(\mathbb{T})}^2.
\]  

(B.7) 

The case of \( t \leq n^{-1} \) can be justified in the same way.

Combining (B.4)–(B.7), we prove (B.3). □

**Lemma B.2.** Let \( Q_N = \text{Id} - \mathcal{P}_N \) with \( N > 1 \). Assume \( Z_0 \in H^{l+\gamma}(\mathbb{T}) \) with \( l \geq 0 \) and \( \gamma \in [0, 1] \), and \( f \in L_T^\infty \mathcal{H}^l(\mathbb{T}) \) for \( T > 0 \). The model equation 

\[
\partial_t Z(s, t) = \mathcal{L} Z(s, t) + Q_N f(s, t), \quad Z(s, 0) = Q_N Z_0(s), \quad s \in \mathbb{T}, \ t \geq 0,
\]

has a unique solution \( Z \in C_{[0, T]} H^{l+\gamma}(\mathbb{T}) \), satisfying that \( Z = Q_N Z \), and for all \( t \in [0, T] \),

\[
\| Z(t) \|_{\hat{H}^{l+\gamma}(\mathbb{T})} \leq e^{-tN/4} \| Q_N Z_0 \|_{\hat{H}^{l+\gamma}(\mathbb{T})} + C N^{-1} \| Q_N f \|_{L_T^\infty \hat{H}^l}.
\]

With abuse of notation, \( \hat{H}^0(\mathbb{T}) \) is understood as \( L^2(\mathbb{T}) \).

**Proof.** Once again, 

\[ Z(s, t) = e^{\mathcal{L} t} Q_N Z_0(s) + \int_0^t e^{(t-t')\mathcal{L}} Q_N f(s, t')dt', \]  

(B.8) 

gives a unique solution in \( C_{[0, T]} H^{l+\gamma}(\mathbb{T}) \). Obviously, \( Z = Q_N Z \).

It is known that 

\[
\| e^{\mathcal{L} t} Q_N Z_0 \|_{\hat{H}^{l+\gamma}(\mathbb{T})} \leq e^{-tN/4} \| Q_N Z_0 \|_{\hat{H}^{l+\gamma}(\mathbb{T})}.
\]

For the second term in (B.8), by Parseval’s identity, 

\[
\left\| \int_0^t e^{(t-t')\mathcal{L}} Q_N f(s, t')dt' \right\|_{\hat{H}^{l+\gamma}(\mathbb{T})}^2 
= C \sum_{|k| > N} |k|^{2(l+\gamma)} \left| \int_0^t e^{-\frac{1}{4} |k| (t-t')} \hat{f}_k(t') dt' \right|^2 \leq 
\]
\[
\leq C \sum_{|k|>N} |k|^{2(\theta+\gamma)} \int_0^t e^{-\frac{1}{4}|k|(|\theta-\theta'|)} d\theta' \int_0^t e^{-\frac{1}{4}|k|(|\theta-\theta'|)} |\hat{f}_k(\theta')|^2 d\theta'
\]
\[
\leq C \sum_{|k|>N} \int_0^t |k|^{2g-1} e^{-\frac{1}{4}|k|(|\theta-\theta'|)} |k|^{2g} |\hat{f}_k(\theta')|^2 d\theta'
\]
\[
\leq C \int_0^t |t-t'|^{-(2g-1)} e^{-\frac{1}{4}|N|(|\theta-\theta'|)} \sum_{|k|>N} |k|^{2g} |\hat{f}_k(\theta')|^2 d\theta'
\]
\[
\leq C N^{2g-2} \|Q_N f\|_{L_T^\infty H^1}^2.
\]
This completes the proof. \( \square \)

Appendix C Estimates for \( g_Y \)

This section aims at proving estimates concerning \( g_Y \) in Section 4, which improves the results in [15]. We first recall some previous results.

Let \( Y \in H^2(T) \). Denote \( \tau = s' - \tau' \in [-\pi, \pi) \). For \( s' \neq s \), with an abuse of notation, define

\[
\begin{align*}
L(s, s') &= \frac{Y(s') - Y(s)}{\tau}, \\
M(s, s') &= \frac{Y'(s') - Y'(s)}{\tau}, \\
N(s, s') &= \frac{L(s, s') - Y'(s)}{\tau}.
\end{align*}
\tag{C.1}
\]

and

\[
\begin{align*}
L(s, s) &= Y'(s), \\
M(s, s) &= Y''(s), \\
N(s, s) &= \frac{1}{2} Y''(s).
\end{align*}
\tag{C.2}
\]

Then \( L, M, \) and \( N \) enjoy the following estimates.

**Lemma C.1** ([15], Lemma 3.1). With the notations above,

1. For \( 1 < p \leq q \leq \infty \) and any interval \( I \subset T \) satisfying \( 0 \in I \),
   \[
   \|L(s, s')\|_{L^q_T L^p_I(s+I)} \leq C |I|^{1/q} \|Y'\|_{L^p(T)},
   \]
   \[
   \|M(s, s')\|_{L^q_T L^p_I(s+I)} \leq C |I|^{1/q} \|Y''\|_{L^p(T)},
   \]
   \[
   \|N(s, s')\|_{L^q_T L^p_I(s+I)} \leq C |I|^{1/q} \|Y''\|_{L^p(T)},
   \]
   where the constants \( C > 0 \) depend only on \( p \) and \( q \). Here
   \[
   \|f(s, s')\|_{L^q_T L^p_I(s+I)} \triangleq \|f(s, s')\|_{L^p_I(s+I)} \|L^q_T(s+I)\|.
   \]

2. For \( s, s' \in T \),
   \[
   |L(s, s')| \leq 2MY'(s), \quad |M(s, s')| \leq 2MY''(s), \quad |N(s, s')| \leq 2MY''(s).
   \]
   In particular,
   \[
   |L(s, s')| \leq C \|Y'\|_{L^\infty(T)}.
   \]
LEMMA C.2 ([15] remark 2.1, lemma 3.5, lemma 3.6, and lemma C.1). We have

\[ g_Y(s) = \frac{1}{4\pi} \text{p.v.} \int_T \left( -\frac{|Y'(s')|^2}{|L|^2} + \frac{2(L \cdot Y'(s'))^2}{|L|^4} - 1 \right) \frac{L}{\tau} \]

\[ + \left( \frac{1}{\tau} - \frac{\tau}{4\sin^2(\frac{\tau}{2})} \right) L \, ds', \]

and

\[ (C.3) \quad g'_Y(s) = \int_T \Gamma_1(s, s') \, ds', \]

where for \( s \neq s' \),

\[ 4\pi \Gamma_1(s, s') \]

\[ = \frac{(Y'(s) - L) \cdot N}{|L|^2} M - \frac{2(N \cdot L)(Y'(s) \cdot L)}{|L|^4} M \]

\[ - \left( \frac{\tau^2 - 4\sin^2(\frac{\tau}{2})}{4\tau \sin^2(\frac{\tau}{2})} \right) M \]

\[ + \frac{(M - 2N) \cdot M}{|L|^2} Y'(s) + \frac{2(N \cdot L)(L \cdot M)}{|L|^4} Y'(s) \]

\[ + \frac{2(L \cdot M)(L \cdot (M - N))(L \cdot Y'(s))}{|L|^6} L \]

\[ + \frac{2((N - M) \cdot M)(L \cdot Y'(s))}{|L|^4} L \]

\[ + \frac{2(L \cdot M)(L \cdot Y'(s'))}{|L|^6} N + \frac{2(N \cdot M)(L \cdot Y'(s'))}{|L|^4} L \]

\[ + \frac{2(L \cdot M)(N \cdot Y'(s'))}{|L|^4} L - \frac{6(L \cdot M)(L \cdot Y'(s'))(L \cdot N)}{|L|^6} L. \]

Moreover,

\[ g''_Y(s) = \int_T \partial_s \Gamma_1(s, s') \, ds'. \]

We first prove Lemma [4.1]

PROOF OF LEMMA [4.1] With \( \tau = s' - s \), we calculate

\[ \partial_s L(s, s') = N(s, s'), \]

\[ \partial_s M(s, s') = \frac{Y'(s') - Y'(s) - \tau Y''(s)}{\tau^2} = \frac{M(s, s') - Y''(s)}{\tau}, \]

\[ \partial_s N(s, s') = 2 \cdot \frac{Y(s') - Y(s) - \tau Y'(s) - \frac{\tau^2}{2} Y''(s)}{\tau^3}. \]
We claim that, by taking the $s$-derivative in (C.4),
\[
|\partial_s \Gamma_1(s, s')| \\
\leq C \lambda^{-2} \|Y'\|_{L^\infty(T)} (|\partial_s M| |M| + |\partial_s M| |N| + |\partial_s N| |M|) \\
+ C \lambda^{-3} \|Y''\|_{L^\infty(T)} |M| |N| (|M| + |N|) \\
+ C \lambda^{-2} |Y''(s)| |M| |(|M| + |N|) + C |M| + C |\tau| |\partial_s M|.
\]
(C.5)

In fact, on the right-hand side, the first term bounds all the terms in $\partial_s \Gamma_1(s, s')$ whenever the $s$-derivative falls on $M$ or $N$ in (C.4); the second term comes from the terms when the derivative falls on $L$, including those in the denominators; the third term shows up because the derivative may hit $Y_0(s)$; and the last two terms come from the $s$-derivative of the third term in (C.4).

By Lemma (C.2) it suffices to bound
\[
\left\| \int_T [\partial_s \Gamma_1(s, s + \tau) d\tau] \right\|_{L^2(T)}.
\]
It is not difficult to show that by Lemma (C.1) and (C.5),
\[
\left\| \int_T \lambda^{-3} \|Y'\|_{L^\infty(T)} |M| |(|M| + |N|) \\
+ \lambda^{-2} |Y''(s)| |M| |(|M| + |N|) + |M| + |\tau| |\partial_s M| d\tau \right\|_{L^2(T)} \\
\leq C \lambda^{-3} \|Y''\|_{L^\infty(T)} \|M\|_{L^\infty L^2(T)} \|N\|_{L^\infty L^2(T)} \|MY''\|_{L^2(T)} \\
+ C \lambda^{-2} \|Y''\|_{L^2(T)} \|M\|_{L^\infty L^2(T)} \|N\|_{L^\infty L^2(T)} + C \|N\|_{L^\infty L^2(T)} \\
+ C \|Y'\|_{L^2(T)}
\]
(C.6)

We still need to show
\[
\left\| \int_T [\partial_s M] |M| + |\partial_s M| |N| + |\partial_s N| |M| d\tau \right\|_{L^2(T)} \leq C \|Y\|^2_{H^{9/4}(T)}.
\]
(C.7)

Notice that
\[
N(s, s') = \frac{1}{\tau^2} \int_0^\tau Y'(s + \eta) - Y'(s) d\eta,
\]
\[
\partial_s M(s, s') = \frac{1}{\tau^2} \int_0^\tau Y''(s + \eta) - Y''(s) d\eta,
\]
\[
\partial_s N(s, s') = \frac{2}{\tau^3} \int_0^\tau \int_0^\eta Y''(s + \xi) - Y''(s) d\xi d\eta.
\]
(C.8)
For all $p \in [1, \infty]$, 

\[
\|N(s, s + \tau)\|_{L_p^p(\mathbb{T})} \leq \frac{1}{\tau^2} \int_0^\tau \|Y'(s + \eta) - Y'(s)\|_{L_p^p(\mathbb{T})} \, d\eta
\]

\[
\leq C \sup_{|\eta| \leq |\tau|} \|Y'(s + \eta) - Y'(s)\|_{L_p^p(\mathbb{T})},
\]

and similarly, 

\[
\|\partial_s M(s, s + \tau)\|_{L_p^p(\mathbb{T})} \leq C \sup_{|\eta| \leq |\tau|} \|Y''(s + \eta) - Y''(s)\|_{L_p^p(\mathbb{T})},
\]

\[
\|\partial_s N(s, s + \tau)\|_{L_p^p(\mathbb{T})} \leq C \sup_{|\eta| \leq |\tau|} \|Y''(s + \eta) - Y''(s)\|_{L_p^p(\mathbb{T})}.
\]

Hence,

\[
\left\| \int_\mathbb{T} |\partial_s M| |M| + \mid\partial_s M\mid |N| + \mid\partial_s N\mid |M| \, d\tau \right\|_{L_p^p(\mathbb{T})}
\]

\[
\leq \int_\mathbb{T} (\|\partial_s M\|_{L_3^3(\mathbb{T})} + \|\partial_s N\|_{L_3^3(\mathbb{T})})(\|M\|_{L_p^p(\mathbb{T})} + \|N\|_{L_p^p(\mathbb{T})}) \, d\tau
\]

\[
\leq C \int_\mathbb{T} \frac{1}{|\tau|^{7/12}} \sup_{|\eta| \leq |\tau|} \|Y''(s + \eta) - Y''(s)\|_{L_3^3(\mathbb{T})} \, d\tau
\]

\[
\cdot \left( \int_\mathbb{T} \frac{1}{|\tau|^{17/12}} \sup_{|\eta| \leq |\tau|} \|Y'(s + \eta) - Y'(s)\|_{L_p^p(\mathbb{T})}^2 \, d\tau \right)^{1/2}
\]

\[
\leq C \left( \int_\mathbb{T} \frac{1}{|\tau|^{7/6}} \sup_{|\eta| \leq |\tau|} \|Y''(s + \eta) - Y''(s)\|_{L_3^3(\mathbb{T})}^2 \, d\tau \right)^{1/2}
\]

\[
\cdot \left( \int_\mathbb{T} \frac{1}{|\tau|^{17/6}} \sup_{|\eta| \leq |\tau|} \|Y'(s + \eta) - Y'(s)\|_{L_p^p(\mathbb{T})}^2 \, d\tau \right)^{1/2}
\]

\[
\leq C \|Y''\|_{\dot{B}_{3,2}^{1/12}(\mathbb{T})} \|Y'\|_{\dot{B}_{2,2}^{11/12}(\mathbb{T})}
\]

\[
\leq C \|Y\|_{\dot{B}_{2,2}^{2+1/4}(\mathbb{T})}.
\]

In the last two inequalities, we used equivalent norms of Besov spaces and embedding theorems between them [41, §2.5.12 and §2.7.1]. This proves (C.7).

Combining (C.5)–(C.7), we complete the proof of Lemma 4.1. □

Next, we prove Lemma 4.2 and Lemma 4.3.

**Proof of Lemma 4.2.** Let $L_i$, $M_i$, and $N_i$ be defined as in (C.1) and (C.2) with $Y$ replaced by $Y_i$ ($i = 1, 2$). Recall $\delta Y = Y_1 - Y_2$, and let $\delta L$, $\delta M$, and $\delta N$ be defined in a similar manner. Define $V = \|Y'_1\|_{L^\infty(\mathbb{T})} + \|Y'_2\|_{L^\infty(\mathbb{T})}$. By

\[
\|N_i(s, s + \tau)\|_{L_p^p(\mathbb{T})} \leq \frac{1}{\tau^2} \int_0^\tau \|Y'(s + \eta) - Y'(s)\|_{L_p^p(\mathbb{T})} \, d\eta
\]

\[
\leq C \sup_{|\eta| \leq |\tau|} \|Y'(s + \eta) - Y'(s)\|_{L_p^p(\mathbb{T})}.
\]
Lemma C.2

\[ 4\pi(g_{Y_1} - g_{Y_2}) = \text{p.v.} \int_T \left( -\frac{\delta Y'(s') \cdot (Y'_1(s') + Y'_2(s'))}{|L_1|^2} + \frac{|Y'_2(s')|^2 (L_1 + L_2) \cdot \delta L}{|L_1|^2 |L_2|^2} \cdot \frac{L_1}{\tau} ds' \right) \]

\[ - \text{p.v.} \int_T \frac{2(L_1 \cdot Y'_1(s'))^2 (L_1 + L_2) \cdot \delta L}{|L_1|^4 |L_2|^2} \cdot \frac{L_1}{\tau} ds' \]

\[ + \text{p.v.} \int_T \frac{2(L_1 \cdot Y'_1(s') + L_2 \cdot Y'_2(s')) \cdot (\delta L \cdot Y'_1(s') + L_2 \cdot \delta Y'(s'))}{|L_1|^2 |L_2|^2} \cdot \frac{L_1}{\tau} ds' \]

\[ - \text{p.v.} \int_T \frac{2(L_2 \cdot Y'_2(s'))^2 (L_1 + L_2) \cdot \delta L}{|L_1|^4 |L_2|^2} \cdot \frac{L_1}{\tau} ds' \]

\[ + \text{p.v.} \int_T \left( \frac{|Y'_2(s')|^2}{|L_2|^2} + \frac{2(L_2 \cdot Y'_2(s'))^2}{|L_2|^4} - 1 \right) \frac{\delta L}{\tau} ds' \]

\[ + \text{p.v.} \int_T \left( \frac{1}{\tau} - \frac{\tau}{4 \sin^2(\frac{\tau}{2})} \right) \delta L ds'. \]

We shall do integration by parts to remove the derivative from the \( \delta Y'(s') \) terms in the integrand. Notice that

\[ \delta Y'(s') = \partial_{s'}(\delta Y(s') - \delta Y(s)) = \partial_{s'}(\tau \delta L). \]

Thanks to the regularity of \( Y_i \), it is not difficult to justify the integration by parts in spite of the singularity in the integrand. Indeed, we obtain

(C.12) \[ 4\pi(g_{Y_1} - g_{Y_2}) \]

\[ = \text{p.v.} \int_T \frac{\delta L}{\tau} \left( -\frac{(Y'_1(s') + Y'_2(s')) \otimes L_1}{|L_1|^2} + \frac{|Y'_2(s')|^2 (L_1 + L_2) \otimes L_1}{|L_1|^2 |L_2|^2} \right) ds' \]

\[ + \text{p.v.} \int_T \frac{\delta L}{\tau} \cdot \partial_{s'} \left( \frac{Y'_1(s') + Y'_2(s')}{|L_1|^2} \otimes L_1 \right) ds' \]

\[ - \text{p.v.} \int_T \frac{\delta L}{\tau} \cdot \frac{2(L_1 \cdot Y'_1(s'))^2 (L_1 + L_2) \otimes L_1}{|L_1|^4 |L_2|^2} ds' \]

\[ + \text{p.v.} \int_T \frac{\delta L}{\tau} \cdot \frac{2(L_1 \cdot Y'_1(s') + L_2 \cdot Y'_2(s'))(Y'_1(s') + L_2 \otimes L_1)}{|L_1|^2 |L_2|^2} ds' \]

\[ - \text{p.v.} \int_T \frac{\delta L}{\tau} \cdot \partial_{s'} \left( \frac{2(L_1 \cdot Y'_1(s') + L_2 \cdot Y'_2(s')) \cdot L_2 \otimes L_1}{|L_1|^2 |L_2|^2} \right) ds' - \]
\[ - \text{p.v.} \int_T \frac{\delta L}{\tau} \frac{2(L_2 \cdot Y'_2(s'))^2(L_1 + L_2) \otimes L_1}{|L_1|^2|L_2|^4} ds' \\
+ \text{p.v.} \int_T \left( -\frac{|Y'_2(s')|^2}{|L_2|^2} + \frac{2(L_2 \cdot Y'_2(s'))^2}{|L_2|^4} - 1 \right) \frac{\delta L}{\tau} ds' \\
+ \text{p.v.} \int_T \left( \frac{1}{\tau} - \frac{\tau}{4\sin^2(\frac{\tau}{2})} \right) \delta L ds' \]
\[ \leq \sum_{i=1}^8 d_i. \]

Take \( \beta' > \beta \) that also satisfies (1.30). Since \( Y_i \in H^{2+\theta}(T), |Y'_i(s') - L_i| \leq C |\tau|^{\frac{1}{2}+\beta'} \| Y_i \|_{\tilde{H}^{2+\theta}} \). In \( J_1 \), we have

\[
\left| -\frac{(Y'_1(s') + Y'_2(s')) \otimes L_1}{|L_1|^2} + \frac{|Y'_2(s')|^2(L_1 + L_2) \otimes L_1}{|L_1|^2|L_2|^2} \right|
= \left| (Y'_2(s') + L_2) \cdot (Y'_2(s') - L_2)(L_1 + L_2) \otimes L_1 \right|
+ \frac{(L_1 + L_2 - Y'_2(s') \otimes L_1)}{|L_1|^2}
\leq C \lambda^{-2} V |\tau|^{\frac{1}{2}+\beta'} (\| Y_1 \|_{\tilde{H}^{2+\theta}} + \| Y_2 \|_{\tilde{H}^{2+\theta}}).
\]

Hence, by the Cauchy-Schwarz inequality and \( \beta' > \beta \),

\[
|J_1| \leq C \lambda^{-2} V (\| Y_1 \|_{\tilde{H}^{2+\theta}} + \| Y_2 \|_{\tilde{H}^{2+\theta}}) \int_T \frac{|\delta Y(s + \tau) - \delta Y(s)|}{|\tau|^{\frac{1}{2}-\beta'}} d\tau
\leq C \lambda^{-2} V (\| Y_1 \|_{\tilde{H}^{2+\theta}} + \| Y_2 \|_{\tilde{H}^{2+\theta}})
\cdot \left( \int_T \frac{|\delta Y(s + \tau) - \delta Y(s)|^2}{|\tau|^{2-2\beta}} d\tau \right)^{1/2}.
\]

Similarly,

\[
|J_3 + J_4 + J_6| + |J_7|
\leq C \lambda^{-2} V (\| Y_1 \|_{\tilde{H}^{2+\theta}} + \| Y_2 \|_{\tilde{H}^{2+\theta}})
\cdot \left( \int_T \frac{|\delta Y(s + \tau) - \delta Y(s)|^2}{|\tau|^{2-2\beta}} d\tau \right)^{1/2}.
\]
Since $\partial_x L_i = M_i - N_i$, by Lemma C.1 and Sobolev embedding,
\begin{align*}
|J_2| + |J_5| & \leq C \lambda^{-2} V \int_T |\delta L| \cdot \left( |Y_1''(s')| + |Y_2''(s')| + |M_1| + |M_2| + |N_1| + |N_2| \right) d\tau \\
& \leq C \lambda^{-2} V \left( \int_T \frac{||\delta Y(s + \tau) - \delta Y(s)||^2}{|\tau|^{2-2\beta}} d\tau \right)^{1/2} \left\| |\tau|^{-\beta} \right\|_{L^{1/\beta'}} \\
& \quad \cdot \left( \int_T \frac{||Y_1''(s')| + |Y_2''(s')| + |M_1| + |M_2| + |N_1| + |N_2|}_{L^{1/(1-\beta')}} \right)^{1/2}.
\end{align*}
(C.15)

Finally,
\begin{align*}
|J_8| & \leq C \int_T |\delta Y(s + \tau) - \delta Y(s)| d\tau.
\end{align*}
(C.16)

Combining (C.12)–(C.16) and taking the $L^2$-norm in $s$, we prove the desired estimate (4.2).

Proof of Lemma 4.3. To prove (4.3), let $\Gamma_{1,i} (i = 1, 2)$ be defined by (C.3) and (C.4) with $Y$ replaced by $Y_i$. By Lemma C.2 it suffices to bound
\begin{align*}
\left\| \int_T \Gamma_{1,1}(s, s') - \Gamma_{1,2}(s, s') ds' \right\|_{L_2^2(T)}.
\end{align*}
(C.17)

For conciseness, we only show how to bound the part of the difference arising from the last term of (C.4). We write
\begin{align*}
\frac{(L_1 \cdot M_1)(L_1 \cdot Y_1'(s'))(L_1 \cdot N_1)}{|L_1|^6} L_1 \\
- \frac{(L_2 \cdot M_2)(L_2 \cdot Y_2'(s'))(L_2 \cdot N_2)}{|L_2|^6} L_2 \\
= \frac{(L_1 \cdot \delta M)(L_1 \cdot Y_1'(s'))(L_1 \cdot N_1)}{|L_1|^6} L_1 \\
+ \frac{(L_1 \cdot M_2)(L_1 \cdot Y_1'(s'))(L_1 \cdot \delta N)}{|L_1|^6} L_1 \\
+ \frac{(\delta L \cdot M_2)(L_1 \cdot Y_2'(s'))(L_1 \cdot N_2)}{|L_1|^6} L_1 \\
+ \frac{(L_2 \cdot M_2)(L_1 \cdot Y_1'(s'))(L_1 \cdot N_2)}{|L_1|^4} L_1 \cdot \frac{|L_2|^2 - |L_1|^2}{|L_1|^2 |L_2|^2} +
\end{align*}
(C.18)
\[ + \frac{(L_2 \cdot M_2)(\delta L \cdot Y_1'(s'))(L_1 \cdot N_2)}{|L_1|^4|L_2|^2} L_1 \]
\[ + \frac{(L_2 \cdot M_2)(L_2 \cdot \delta Y'(s'))(L_1 \cdot N_2)}{|L_1|^4|L_2|^2} L_1 \]
\[ + \frac{(L_2 \cdot M_2)(L_2 \cdot Y_2'(s'))(L_1 \cdot N_2)}{|L_1|^2|L_2|^2} \frac{|L_2|^2 - |L_1|^2}{|L_1|^2|L_2|^2} L_1 \]
\[ + \frac{(L_2 \cdot M_2)(L_2 \cdot Y_2'(s'))(L_2 \cdot N_2)}{|L_2|^4} \frac{|L_2|^2 - |L_1|^2}{|L_1|^2|L_2|^2} L_1 \]
\[ + \frac{(L_2 \cdot M_2)(L_2 \cdot Y_2'(s'))(L_2 \cdot N_2)}{|L_2|^6} L_1 \]

If \( \theta \in [\frac{1}{4}, \frac{1}{2}] \), we bound it as follows:

\[
\left| \frac{(L_1 \cdot M_1)(L_1 \cdot Y_1'(s'))(L_1 \cdot N_1)}{|L_1|^6} L_1 \right|
\[
- \frac{(L_2 \cdot M_2)(L_2 \cdot Y_2'(s'))(L_2 \cdot N_2)}{|L_2|^6} L_2 \right| \leq C \lambda^{-3} \left[ V^2 |\delta M||N_1| + V^2 |M_2||\delta N| \right]
\]
\[ + \left[ V(|\delta L| + |\delta Y'(s')|)|M_2||N_2| \right]. \]

Then we proceed as in (C.11). By Lemma (C.1) and (C.10), for \( \beta \) satisfying (1.33),

\[
\left\| \int_{\mathbb{T}} \left| \frac{(L_1 \cdot M_1)(L_1 \cdot Y_1'(s'))(L_1 \cdot N_1)}{|L_1|^6} L_1 \right| \right\|_{L^2(\mathbb{T})}
\[
- \frac{(L_2 \cdot M_2)(L_2 \cdot Y_2'(s'))(L_2 \cdot N_2)}{|L_2|^6} L_2 \right| ds' \right\|_{L^2(\mathbb{T})} \leq C \lambda^{-3} V \left\| \int_{\mathbb{T}} \left| \frac{\delta Y'(s + \tau) - \delta Y'(s)}{\tau} \right| |N_1| \right\|_{L^2(\mathbb{T})}
\[
+ \frac{|Y_2'(s + \tau) - Y_2'(s)}{\tau} ||\delta N| d \tau \right\|_{L^2(\mathbb{T})} \leq C \lambda^{-3} V \left( \|\delta L\|_{L^\infty} L_2^2 \right. + \left. \|\delta Y'\|_{L^2} \right) \|M_2\|_{L^2} L^\infty \|N_2\|_{L^\infty} L_2^2 \leq \]
\[ \begin{align*}
\leq C \lambda^{-3} V^2 \left\| \delta Y'(s + \tau) - \delta Y'(s) \right\|_{L^2_\tau} & \bigg/ \left| \tau \right|^{1-\beta} \left\| N_1 \right\|_{L^\infty_\tau} \bigg/ \left| \tau \right|^{\beta} \left\| Y_2'' \right\|_{L^2_\tau} \\
+ C \lambda^{-3} V^2 \left\| Y_2'(s + \tau) - Y_2'(s) \right\|_{L^\infty_\tau} & \bigg/ \left| \tau \right|^{1+\beta} \left\| \delta N \right\|_{L^2_\tau} \bigg/ \left| \tau \right|^{\beta} \left\| Y_2'' \right\|_{L^2_\tau} \\
+ C \lambda^{-3} V \left\| \delta Y \right\|_{\dot{H}^1(\mathbb{T})} & \left\| M Y_2'' \right\|_{L^2_\tau} \left\| Y_2'' \right\|_{L^2_\tau} \\
\leq C \lambda^{-3} V^2 & \left\| \delta Y' \right\|_{\dot{H}^{2-\beta}_{2,2}} \left\| Y_2' \right\|_{\dot{H}^{1+\beta}_{2,2}} \\
+ C \lambda^{-3} V^2 & \left\| \delta Y' \right\|_{\dot{H}^{2-\beta}_{2,2}} \left\| \delta Y' \right\|_{\dot{H}^{1+\beta}_{2,2}} \left\| Y_2'' \right\|_{\dot{H}^{\frac{1+\beta}{2}}_{2,2}} \\
+ C \lambda^{-3} & \left\| Y_2'' \right\|_{L^2} \left\| \delta Y \right\|_{\dot{H}^1} \\
\leq C \lambda^{-3} V^2 \left( \left\| \delta Y' \right\|_{\dot{H}^{2-\beta}_{2,2}} \left\| Y_2' \right\|_{\dot{H}^{1+\beta}_{2,2}} + \left\| Y_2'' \right\|_{\dot{H}^{\frac{1+\beta}{2}}_{2,2}} \right) \\
+ C \lambda^{-3} & \left\| Y_2'' \right\|_{L^2} \left\| \delta Y \right\|_{\dot{H}^1} \\
\leq C \lambda^{-3} & \left( \left\| \delta Y' \right\|_{\dot{H}^{2-\beta}(\mathbb{T})} + \left\| Y_2'' \right\|_{\dot{H}^{2}(\mathbb{T})} \right) \left\| \delta Y \right\|_{\dot{H}^{2-\beta}(\mathbb{T})}.
\end{align*} \]

Here we used equivalent norms and embedding theorems of Besov spaces again [31, §2.5.12 and §2.7.1]. The other terms in (C.17) can be handled in a similar manner. This proves (4.3) when \( \theta \in [\frac{1}{4}, \frac{1}{2}] \).

In the case of \( \theta \in [\frac{1}{2}, 1) \), we need to take special care of the first two terms in (C.18). Denote them as

\[
( L_1 \cdot \delta M)(L_1 \cdot Y_1'(s'))(L_1 \cdot N_1) L_1 + ( L_1 \cdot M_2)(L_1 \cdot Y_1'(s'))(L_1 \cdot \delta N) L_1
\]

where

\[ A_1(s, s') \delta M(s, s') + A_2(s, s') \delta N(s, s'), \]

where

\[ A_1(s, s') = \frac{( L_1(s, s') \otimes L_1(s, s')(L_1(s, s') \cdot Y_1'(s'))(L_1(s, s') \cdot N_1(s, s'))}{| L_1(s, s') |^6}, \]

\[ A_2(s, s') = \frac{( L_1(s, s') \otimes L_1(s, s')(L_1(s, s') \cdot Y_1'(s'))(L_1(s, s') \cdot M_2(s, s'))}{| L_1(s, s') |^6}. \]
In particular, by definition,
\[
A_1(s, s) = \frac{(Y_1'(s) \otimes Y_1'(s))(Y_2'(s) \cdot Y_1'(s))(Y_1'(s) \cdot \frac{1}{2} Y_1''(s))}{|Y_1'(s)|^6},
\]
\[
A_2(s, s) = \frac{(Y_1'(s) \otimes Y_1'(s))(Y_2'(s) \cdot Y_1'(s))(Y_1'(s) \cdot Y_2''(s))}{|Y_1'(s)|^6}.
\]

It is then not difficult to derive that
\[
|A_1(s, s') - A_1(s, s)| \leq C \lambda^{-2} |Y_1'(s') - Y_1'(s)| |N_1| + C \lambda^{-2} |L_1 - Y_1'(s)| |N_1|
+ C \lambda^{-1} \left| N_1 - \frac{1}{2} Y_1''(s) \right|
\]
\[
\leq C |\tau| (\lambda^{-2} |M_1| |N_1| + \lambda^{-2} |N_1|^2 + \lambda^{-1} |\partial_s N_1|).
\]

By (C.8), (C.9), and the fact that \( H^{2+\theta}(\mathbb{T}) \hookrightarrow C^{2,\theta-\frac{1}{2}}(\mathbb{T}) \),
\[
|A_1(s, s') - A_1(s, s)| \leq C |\tau| (\lambda^{-2} |Y_1''| H^\theta + \lambda^{-1} |\tau|^\theta |Y_1''| H^\theta) \leq C |\tau|^{\theta-\frac{1}{2}} \lambda^{-2} |Y_1''| H^\theta.
\]
Similarly,
\[
|A_2(s, s') - A_2(s, s)| \leq C |\tau|^{\theta-\frac{1}{2}} \lambda^{-2} |Y_1''| H^\theta |Y_2''| H^\theta.
\]

Hence,
\[
\left\| \int_{\mathbb{T}} A_1(s, s') \delta M(s, s') + A_2(s, s') \delta N(s, s') ds' \right\|_{L^2}
\]
\[
\leq \left\| \int_{\mathbb{T}} (A_1(s, s') - A_1(s, s)) \cdot \tau^{-1} (\delta Y'(s') - \delta Y'(s)) ds' \right\|_{L^2}
+ \left\| \int_{\mathbb{T}} (A_2(s, s') - A_2(s, s)) \cdot \tau^{-1} (\delta L(s, s') - \delta Y'(s)) ds' \right\|_{L^2}
+ \| A_1(s, s) \|_{L^\infty} \left\| \text{p.v.} \int_{\mathbb{T}} \frac{\delta Y'(s + \tau)}{\tau} \ d\tau \right\|_{L^2}
+ \| A_2(s, s) \|_{L^\infty} \left\| \text{p.v.} \int_{\mathbb{T}} \frac{\delta L(s, s + \tau)}{\tau} \ d\tau \right\|_{L^2} \leq \]
\[ \leq C \lambda^{-2} \| Y'' \|_{\dot{H}^0}^2 \int_{\mathbb{T}} |\tau|^{\theta - \frac{1}{2}} (\| \delta Y'(s + \tau) \|_{L^2} + \| \delta Y' \|_{L^2}) d\tau \]
\[ + C \lambda^{-2} \| Y'' \|_{\dot{H}^0} \| Y'' \|_{\dot{H}^0} \int_{\mathbb{T}} |\tau|^{\theta - \frac{1}{2}} (\| \delta L(s, s + \tau) \|_{L^2} + \| \delta Y' \|_{L^2}) d\tau \]
\[ + C \lambda^{-1} \| Y'' \|_{\dot{H}^0} \]
\[ \cdot \left\| \text{p.v.} \int_{\mathbb{T}} \frac{\delta Y'(s + \tau) + \delta Y'(s + \tau)}{2 \tan \frac{\tau}{2}} + \delta Y'(s + \tau) \left( \frac{1}{\tau} - \frac{1}{2 \tan \frac{\tau}{2}} \right) d\tau \right\|_{L^2} \]
\[ + C \lambda^{-1} \| Y'' \|_{\dot{H}^0} \]
\[ \cdot \left\| \text{p.v.} \int_{\mathbb{T}} \frac{\delta Y(s + \tau) - \delta Y(s)}{4 \sin^2 \frac{\tau}{2}} + \tau \delta L(s) \left( \frac{1}{\tau^2} - \frac{1}{4 \sin^2 \frac{\tau}{2}} \right) d\tau \right\|_{L^2} \]
\[ \leq C \lambda^{-2} (\| Y'' \|_{\dot{H}^0} + \| Y'' \|_{\dot{H}^0})^2 \| \delta Y' \|_{L^2} \]
\[ + C \lambda^{-1} \| Y'' \|_{\dot{H}^0} \left\| |\delta Y'| + \int_{\mathbb{T}} |\delta Y'(s + \tau)| d\tau \right\|_{L^2} \]
\[ + C \lambda^{-1} \| Y'' \|_{\dot{H}^0} \left\| |\delta Y'| + \int_{\mathbb{T}} |\delta L(s, s + \tau)| d\tau \right\|_{L^2} \]
\[ \leq C \lambda^{-2} (\| Y'' \|_{\dot{H}^0} + \| Y'' \|_{\dot{H}^0})^2 \| \delta Y' \|_{L^2}. \]

This bounds the \( L^2(\mathbb{T}) \)-norm of the first two terms in (C.18). For the other terms, we argue as in (C.19) and (C.20) to bound them by
\[ C \lambda^{-3} (\| Y'' \|_{\dot{H}^0} + \| Y'' \|_{\dot{H}^0})^3 \| \delta Y' \|_{L^2}. \]

The other terms in (C.17) can be handled in a similar manner. This completes the proof of (4.3) when \( \theta \in \left( \frac{1}{2}, 1 \right) \). \( \square \)

**Remark C.3.** A weaker estimate than (4.3) can be proved more easily, which reads
\[ \| y_1 - y_2 \|_{\dot{H}^1(\mathbb{T})} \leq C \lambda^{-3} (\| Y_1 \|_{\dot{H}^{2+\theta}} + \| Y_2 \|_{\dot{H}^{2+\theta}})^3 \| \delta Y \|_{\dot{H}^{3/2}(\mathbb{T})}. \]

In fact, this is sufficient for proving Theorem 1.4 and Theorem 1.8. However, we pursue a more refined estimate in order to obtain better error estimates in Theorem 1.8.

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