Kinetic Reverse $k$-Nearest Neighbor Problem

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Abstract
This paper provides the first solution to the kinetic reverse $k$-nearest neighbor (R$k$NN) problem in $\mathbb{R}^d$, which is defined as follows: Given a set $P$ of $n$ moving points in arbitrary but fixed dimension $d$, an integer $k$, and a query point $q \notin P$ at any time $t$, report all the points $p \in P$ for which $q$ is one of the $k$-nearest neighbors of $p$.

1 Introduction

The reverse $k$-nearest neighbor (R$k$NN) problem is a popular variant of the $k$-nearest neighbor ($k$NN) problem and asks for the influence of a query point on a point set. Unlike the $k$NN problem, the exact number of reverse $k$-nearest neighbors of a query point is not known in advance. The R$k$NN problem is formally defined as follows: Given a set $P$ of $n$ points in $\mathbb{R}^d$, an integer $k$, $1 \leq k \leq n - 1$, and a query point $q \notin P$, find the set R$k$NN($q$) of all $p$ in $P$ for which $q$ is one of $k$-nearest neighbors of $p$. Thus R$k$NN($q$) = \{p \in P : |pq| \leq |pp_k|\}, where $|.|$ denotes Euclidean distance, and $p_k$ is the $k^{th}$ nearest neighbor of $p$ among the points in $P$. The kinetic R$k$NN problem is to answer R$k$NN queries on a set $P$ of moving points, where the trajectory of each point $p \in P$ is a function of time. Here, we assume the trajectories are polynomial functions of maximum degree bounded by some constant $s$.

Related work. The reverse $k$-nearest neighbor problem was first posed by Korn and Muthukrishnan [14] in the database community, and then considered extensively in this community due to its many applications, e.g., decision support systems, profile-based marketing, traffic networks, business location planning, clustering and outlier detection, and molecular biology [14, 15, 16]. The reverse $k$-nearest neighbor queries for a set of continuously moving objects has also attracted the attention of the database community; see [17] and references therein. Examples of moving objects
include players in multi-player game environments, soldiers in a battlefield, tourists in dangerous environments, and mobile devices in wireless ad-hoc networks.

To our knowledge, in computational geometry, there exist two data structures \[17, 10\] that give solutions to the R\(^k\)NN problem. Both of these solutions answer R\(^k\)NN queries for a set \(P\) of stationary points and both only work for \(k = 1\).

Maheshwari et al. (2002) \[17\] gave a data structure to solve the R1NN problem in \(\mathbb{R}^2\). Their data structure, which supports insertions and deletions of points, creates an arrangement of largest empty circles centered at the points of \(P\) and answers R1NN queries by point location in the arrangement. Their data structure uses \(O(n)\) space and \(O(n \log n)\) preprocessing time, and an R1NN query can be answered in time \(O(\log n)\). Cheong et al. (2011) \[10\] considered the R1NN problem in fixed dimension \(\mathbb{R}^d\), where \(d = O(1)\). Their method, which uses a compressed quadtree, partitions space into cells such that each cell contains a small number of candidate points. To answer an R1NN query, their solution finds a cell that contains the query point and then checks all the points in the cell. Their approach uses \(O(n)\) space and \(O(n \log n)\) preprocessing time, and can answer an R1NN query in \(O(\log n)\) time; it seems that the approach by Cheong et al. can be extended to answer R\(^k\)NN queries with preprocessing time \(O(kn \log n)\), space \(O(kn)\), and query time \(O(\log n + k)\).

For a set \(P\) of \(n\) stationary points, one can report all the 1-nearest neighbors in time \(O(n \log n)\) \[20\], and all the \(k\)-nearest neighbors, for any \(k \geq 1\), in time \(O(kn \log n)\) \[13\], where the neighbors are reported in order of increasing distance from each point; reporting the unordered set takes time \(O(n \log n + kn)\) \[6, 11, 13\].

For a set of moving points, there are two kinetic data structures \[2, 19\] to maintain all the \(k\)-nearest neighbors, but they only work for \(k = 1\).

**Our contribution.** We provide the first solution to the kinetic R\(^k\)NN problem for any \(k \geq 1\) in any fixed dimension \(d\). To answer an R\(^k\)NN query for a query point \(q \notin P\) at any time \(t\), we partition the \(d\)-dimensional space into a constant number of cones around \(q\), and then among the points of \(P\) in each cone, we examine the \(k\) points having shortest projections on the cone axis. We obtain \(O(k)\) candidate points for \(q\) such that \(q\) might be one of their \(k\)-nearest neighbors at time \(t\). To check which if any of these candidate points is a reverse \(k\)-nearest neighbor of \(q\), we maintain the \(k^{th}\) nearest neighbor \(p_k\) of each point \(p \in P\) over time. By checking whether \(|pq| \leq |pp_k|\) we can easily check whether a candidate point \(p\) is one of the reverse \(k\)-nearest neighbors of \(q\) at time \(t\).

For a set \(P\) of \(n\) continuously moving points in \(\mathbb{R}^d\), where the trajectory of each point is a polynomial function of at most constant degree \(s\), we provide a simple kinetic approach to answer R\(^k\)NN queries on the moving points. In the preprocessing step, we introduce a method for reporting all the \(k\)-nearest neighbors for all the points \(p \in P\) in order of increasing distance from \(p\). For \(k = \Omega(\log^{d-1} n)\), both our method and the method of Dickerson and Eppstein \[13\] give the same complexity, but in our view, our method is simpler in practice.
In order to answer RkNN queries, our kinetic approach maintains all the k-nearest neighbors over time. This is the first KDS for maintenance of all the k-nearest neighbors in \( \mathbb{R}^d \), for any \( k \geq 1 \). Our KDS uses \( O(n \log^d n + kn) \) space and \( O(n \log^d n + kn \log n) \) preprocessing time, and processes \( O(\phi(s, n) \cdot n^2) \) events, each in amortized time \( O(\log n) \). Here, \( \phi(s, n) \) is the complexity of the \( k \)-level of a set of \( n \) partially-defined polynomial functions, such that each pair of them intersects at most \( s \) times. The current bounds on \( \phi(s, n) \) are as follows.

\[
\phi(s, n) = \begin{cases} 
O(n^{3/2} \log n), & \text{for } s = 2 \ [8]; \\
O(n^{5/3} \text{poly log } n), & \text{for } s = 3 \ [7]; \\
O(n^{31/18} \text{poly log } n), & \text{for } s = 4 \ [7]; \\
O(n^{161/90-\delta}), & \text{for } s = 5, \text{ for some constant } \delta > 0 \ [8]; \\
O(n^{2-1/2s}), & \text{for odd } s \ [7]; \\
O(n^{2-1/2(s-1)}), & \text{for even } s \ [2]. 
\end{cases}
\]

At any time \( t \), an RkNN query can be answered in time \( O(\log^d n + k \log \log n) \). Note that if an event occurs at the same time \( t \), we first spend amortized time \( O(\log n) \) to update all the \( k \)-nearest neighbors, and then we answer the query.

**Outline.** Section 2 provides two key lemmas, and in fact introduces a new supergraph, namely the \( k \)-Semi-Yao graph, of the \( k \)-nearest neighbor graph. In Section 3 we show how to report all the \( k \)-nearest neighbors. Section 4 gives a (kinetic) data structure for answering RkNN queries on moving points, where the trajectory of each point is a bounded-degree polynomial. Also included in this section is an analysis of our kinetic data structure in terms of the kinetic data structure performance criteria. Section 5 concludes.

## 2 Key Lemmas

Partition the plane around the origin \( o \) into six wedges, \( W_0, ..., W_5 \), each of angle \( \pi/3 \) (see Figure 1(a)). Denote by \( W_l(p) \) the translation of wedge \( W_l \), \( 0 \leq l \leq 5 \), such that its apex moves from \( o \) to point \( p \) (see Figure 1(b)). Denote by \( x_l \) (resp. \( x_l(p) \)) the vector along the bisector of \( W_l \) (resp. \( W_l(p) \)) directed outward from the apex at \( o \) (resp. \( p \)) Denote the reflection of \( W_l(p) \) through \( p \) by \( W_{l'}(p) \). Note that \( l' = (l + 3) \mod 6 \); see Figure 1(b).

Consider the \( i^{th} \) nearest neighbor \( p_i \) of \( p \). Denote by \( L(P \cap W_l(p_i)) \) the list of the points in \( P \cap W_l(p_i) \), sorted by increasing order of their \( x_l \)-coordinates (projections). The following lemma provides a key insight.

**Lemma 1** Let \( p_i \) be the \( i^{th} \) nearest neighbor of \( p \) among a set \( P \) of points in \( \mathbb{R}^2 \), and let \( W_l(p_i) \) be the wedge of \( p_i \) that contains \( p \). Then point \( p \) is among the first \( i \) points in \( L(P \cap W_l(p_i)) \).
Proof. Let $P' = P \setminus \{p_1, ..., p_{i-1}\}$. Then the point $p_i$ is the closest point to $p$ among the points in $P'$; see Figure 2(a) below. We now prove by contradiction that the point $p$ has the minimum $x_l$-coordinate among the points in $P' \cap W_l(p_i)$: Assume there is a point $r \in P$ inside the wedge $W_l(p_i)$ whose $x_l$-coordinate is less than the $x_l$-coordinate of $p$; see Figure 2(b) for an example where $i = 3$. Consider the triangle $pp_i r$. Since $p_i$ is the closest point to $p$ among the points in $P'$, $|pp_i| < |pr|$ which implies that the angle $\angle pp_i r > \angle prp_i$. This is a contradiction, because $\angle pp_i r \leq \pi/3$ and $\angle prp_i > \pi/3$.

Now we add the points $p_1, ..., p_{i-2}$, and $p_{i-1}$ to the point set $P'$. Consider the worst case scenario that all these $i - 1$ points insert inside the wedge $W_l(p_i)$, and that the $x_l$-coordinates of all these points are less than the $x_l$-coordinate of $p$. Then the point $p$ is still among the first $i$ points in the sorted list $L(P \cap W_l(p_i))$. \[\square\]

The $k$-nearest neighbor graph ($k$-NNG) of a point set $P$ is constructed by connecting each point in $P$ to all its $k$-nearest neighbors. If we connect each point $p \in P$ to the first $k$ points in the sorted list $L(P \cap W_l(p))$, for $l = 0, ..., 5$, we obtain what we call the $k$-Semi-Yao graph ($k$-SYG). Lemma 1 gives a necessary condition for $p_i$ to be the $i^{th}$ nearest neighbor of $p$: the point $p$ is among the first $i$ points in $L(P \cap W_l(p_i))$, where $l$ is such that $p \in W_l(p_i)$. Therefore, the edge set of the $k$-SYG covers the edges of the $k$-NNG. In summary, we have the following.

**Lemma 2** The $k$-NNG of a set $P$ of points in $\mathbb{R}^2$ is a subgraph of the $k$-SYG of the set $P$. 4
(a) (b)

Figure 2: Point $p_3$ is the 3rd nearest neighbor of $p$. After deleting the points $p_1$ and $p_2$, point $p_3$ is the closest point to $p$; among the points in $W_0(p_3)$, $p$ has the minimum length projection on the bisector $x_0(p_3)$.

3 Reporting All $k$-Nearest Neighbors

Here we give a simple method for reporting all the $k$-nearest neighbors via a construction of the $k$-SYG.

Let $C$ be a right circular cone in $\mathbb{R}^d$ with opening angle $\theta$ with respect to some given unit vector $v$. Thus $C$ is the set of points $x \in \mathbb{R}^d$ such that the angle between $\vec{ox}$ and $\vec{v}$ is at most $\theta/2$. The angle between any two rays inside $C$ emanating from the apex $o$ is at most $\theta$. From now on, we assume $\theta = \pi/3$.

Now consider a polyhedral cone inscribed in the right circular cone $C$ where the polyhedral cone is formed by the intersection of $d$ distinct half-spaces, bounded by $f_1, \ldots, f_d$, passing through the apex of $C$. Assuming $d$ is arbitrary but fixed, the $d$-dimensional space around the origin $o$ can be tiled by a constant number of polyhedral cones $W_0, \ldots, W_{c-1}$. Denote by $C_l$ the associated right circular cone of the polyhedral cone $W_l$. Let $x_l$ be the vector in the direction of the symmetry of $C_l$. Denote by $W_l(p)$ the translation of the wedge (polyhedral cone) $W_l$ where $o$ moves to $p$.

A similar approach and analysis as that in Section 2 can be easily used to state (key) Lemmas 1 and 2 for a set of points in $\mathbb{R}^d$.

To construct the $k$-SYG efficiently, we need a data structure to perform the following operation efficiently: For each $p \in P$ and any of its wedges $W_l(p)$, $0 \leq l \leq c - 1$, find the first $k$ points in $L(P \cap W_l(p))$. Such an operation can be performed by using range tree data structures. For each wedge $W_l$ with apex at origin $o$, we construct an associated $d$-dimensional range tree $T_l$ as follows.

Consider a particular wedge $W_l$ with apex at $o$. The wedge $W_l$ is the intersection of $d$ half-spaces $f_1^+, \ldots, f_d^+$ bounded by $f_1, \ldots, f_d$ (see Figure 3). Let $\hat{u}_j$ denote the
normal to $f_j$ pointing to $f_j^+$. We define $d$ coordinate axes $u_j$, $j = 1, ..., d$, through $\hat{u}_j$, where $\hat{u}_j$ gives the respective directions of increasing $u_j$-coordinate values.

The range tree $T_l$ is a regular $d$-dimensional range tree based on the $u_j$-coordinates, $j = 1, ..., d$. The points at level $j$ are sorted at the leaves according to their $u_j$-coordinates (for more details about range trees, see Chapter 5 of [5]). From Theorem 5.8 in [5], any $d$-dimensional range tree, e.g., $T_l$, uses $O(n \log^{d-1} n)$ space and can be constructed in time $O(n \log^{d-1} n)$; for any point $r \in \mathbb{R}^d$, the points of $P$ inside the query wedge $W_l(r)$ whose sides are parallel to $f_j$, $j = 1, ..., d$, can be reported in time $O(\log^{d} n + z)$, where $z$ is the cardinality of the set $P \cap W_l(r)$. In particular, in time $O(\log^{d} n)$ one can determine a set of $O(\log^{d} n)$ internal nodes $v$ at level $d$ of $T_l$, such that $P \cap W_l(r) = \bigcup_v P(v)$, where $P(v)$ is the set of points at the leaves of subtree rooted at $v$.

Now we add a new level to $T_l$, based on the coordinate $x_l$. Let $C_l(p)$ be the set of the first $k$ points in $L(P \cap W_l(p))$. To find $C_l(p)$ in an efficient time, we use the level $d + 1$ of $T_l$, which is constructed as follows: For each internal node $v$ at level $d$ of $T_l$, we create a list $L(P(v))$ sorted by increasing order of $x_l$-coordinates of the points in $P(v)$. For the set $P$ of $n$ points in $\mathbb{R}^d$, the range tree $T_l$, which now is a $(d + 1)$-dimensional range tree, uses $O(n \log^{d} n)$ space and can be constructed in time $O(n \log^{d} n)$.

The following lemma establishes the processing time for obtaining a $C_l(p)$.

**Lemma 3** Given $T_l$, the set $C_l(p)$ can be found in time $O(\log^{d} n + k \log \log n)$.

**Proof.** The proof is by construction. Recall that the set $P \cap W_l(p)$ is the union of $O(\log^{d} n)$ sets $P(v)$, where $v$ ranges over internal nodes at level $d$ of $T_l$. Consider the associated sorted lists $L(P(v))$. 

![Figure 3: The wedge $W_0$ in $\mathbb{R}^2$ is bounded by $f_1$ and $f_2$. The coordinate axes $u_1$ and $u_2$ are orthogonal to $f_1$ and $f_2.$](image)
We construct a priority queue on the first elements of these $O(\log^d n)$ sorted lists $L(P(v))$ in time $O(\log^d n)$.

By repeating the following two steps $k$ times we can find $C_l(p)$:

- Delete the element $\hat{p}$ with highest priority from the priority queue, and
- insert the next element into the priority queue from the sorted list $L(P(v_j))$, where $v_j$ is such that $\hat{p} \in P(v_j)$.

Since $d$ is fixed and the size of the priority queue is $O(\log^d n)$, all together these $k$ iterations take $O(k \log \log n)$ time. 

By Lemma 3, we can find all the $C_l(p)$, for all the points $p \in P$. This gives the following lemma.

**Lemma 4** Using a data structure of size $O(n \log^d n)$, the edges of the $k$-SYG of a set of $n$ points in fixed dimension $d$ can be reported in time $O(n \log^d n + kn \log \log n)$.

Next, suppose we are given the $k$-SYG and we want to report all the $k$-nearest neighbors. Let $E_p$ be the set of edges incident to the point $p$ in the $k$-SYG. By sorting these edges in non-decreasing order according to their Euclidean lengths, which can be done in time $O(|E_p| \log |E_p|)$, we can find the $k$-nearest neighbors of $p$ ordered by increasing distance from $p$. Since the number of edges in the $k$-SYG is $O(kn)$ and each edge $pp'$ belongs to exactly two sets $E_p$ and $E_{p'}$, the time to find all the $k$-nearest neighbors, for all the points $p \in P$, is $\sum_p O(|E_p| \log |E_p|) = O(kn \log n)$.

From the above discussion and Lemmas 2 and 4, the following results.

**Theorem 1** For a set of $n$ points in fixed dimension $d$, our data structure can report all the $k$-nearest neighbors, in order of increasing distance from each point, in time $O(n \log^d n + kn \log n)$. The data structure uses $O(n \log^d n + kn)$ space.

## 4 R$k$NN Queries on Moving Points

We are given a set $P$ of $n$ continuously moving points, where the trajectory of each point in $P$ is a polynomial function of bounded degree $s$. To answer R$k$NN queries on the moving points, we must keep a valid range tree and track all the $k$-nearest neighbors during the motion. This section first shows how to maintain a (ranked-based) range tree, and then provides a KDS for maintenance of the $k$-SYG, which in fact gives a supergraph of the $k$-NNG over time. Using the kinetic $k$-SYG, we can easily maintain all the $k$-nearest neighbors over time. Finally we show how to answer R$k$NN queries on the moving points.
Kinetic RBRT. Let $u_j$, $1 \leq j \leq d$, be the coordinate axis orthogonal to the half-space $f_j$ of the wedge $W_l$, $0 \leq l \leq c - 1$ (see Figure 3). Abam and de Berg [1] introduced a variant of the range tree, namely the ranked-based range tree (RBRT), which has the following properties. Denote by $T_l$ the RBRT corresponding to the wedge $W_l$.

- $T_l$ can be described as a set of pairs $\Psi_l = \{(B_1, R_1), ..., (B_m, R_m)\}$ such that:
  - For any two points $p$ and $q$ in $P$ where $q \in W_l(p)$, there is a unique pair $(B_i, R_i) \in \Psi_l$ such that $p \in B_i$ and $q \in R_i$.
  - For any pair $(B_i, R_i) \in \Psi_l$, if $p \in B_i$ and $q \in R_i$, then $q \in W_l(p)$ and $p \in W_l(q)$; here $W_l(q)$ is the reflection of $W_l(p)$ through $q$.

The $\Psi_l$ is called a cone separated pair decomposition (CSPD) for $P$ with respect to $W_l$. Each pair $(B_i, R_i)$ is generated from an internal node $v$ at level $d$ of the RBRT $T_l$.

- Each point $p \in P$ is in $O(\log^d n)$ pairs of $(B_i, R_i)$, which means that the number of elements of all the pairs $(R_i, B_i)$ is $O(n \log^d n)$.
- For any point $p \in P$, all the sets $B_i$ (resp. $R_i$) where $p \in B_i$ (resp. $p \in R_i$) can be found in time $O(\log^d n)$.
- The set $P \cap W_l(p)$ is the union of $O(\log^d n)$ sets $R_i$, where $p \in B_i$.
- When the points are moving, $T_l$ remains unchanged as long as the order of the points along axes $u_j$, $1 \leq j \leq d$, remains unchanged.
- When a $u$-swap event occurs, meaning that two points exchange their $u_j$-order, the RBRT $T_l$ can be updated in worst-case time $O(\log^d n)$ without rebalancing operations.

4.1 Kinetic $k$-SYG

Here we give a KDS for the $k$-SYG, for any $k \geq 1$, extending [IS].

To maintain the $k$-SYG, we must track the set $C_l(p)$ for each point $p \in P$. So, for each $1 \leq i \leq m$, we need to maintain a sorted list $L(R_i)$ of the points in $R_i$ in ascending order according to their $x_l$-coordinates over time. Note that each set $R_i$ is some $P(v)$, the set of points at the leaves of the subtree rooted at some internal node $v$ at level $d$ of $T_l$. To maintain these sorted lists $L(R_i)$, we add a new level to the RBRT $T_l$; the points at the new level are sorted at the leaves in ascending order according to their $x_l$-coordinates. Therefore, in the modified RBRT $T_l$, in addition to the $u$-swap events, we handle new events, called $x$-swap events, when two points exchange their $x_l$-order. The modified RBRT $T_l$ behaves like a $(d + 1)$-dimensional RBRT. From the last property of an RBRT above, when a $u$-swap event or an $x$-swap event occurs, the RBRT $T_l$ can be updated in worst-case time $O(\log^{d+1} n)$. 

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Denote by $\tilde{p}_{l,k}$ the $k^{th}$ point in $L(P \cap W_l(p))$. To track the sets $C_l(p)$, for all the points $p \in P$, we need to maintain the following over time.

- A set of $d+1$ kinetic sorted lists $L_j(P)$, $j = 1, ..., d$, and the $L_l(P)$ of the point set $P$. We use these kinetic sorted lists to track the order of the points in the coordinates $u_j$ and $x_l$, respectively.

- For each $B_i$, a sorted list $L(B'_i)$ of the points in $B'_i$, where $B'_i = \{(p, \tilde{p}_{l,k}) | p \in B_i\}$. The order of the points in $L(B'_i)$ is according to a label of the second points $\tilde{p}_{l,k}$. This sorted list $L(B'_i)$ is used to answer the following query efficiently: Given a query point $q$ and a $B_i$, find all the points $p \in B_i$ such that $\tilde{p}_{l,k} = q$.

- The $k^{th}$ point $r_{i,k}$ in the sorted list $L(R_i)$. We track the values $r_{i,k}$ in order to make necessary changes to the $k$-SYG when an $x$-swap event occurs.

Handling $u$-swap events. W.l.o.g., let $q \in W_l(p)$ before the event. When a $u$-swap event between $p$ and $q$ occurs, the point $q$ moves outside the wedge $W_l(p)$; after the event, $q \notin W_l(p)$. Note that the changes that occur in the $k$-SYG are the deletions and insertions of the edges incident to $p$ inside the wedge $W_l(p)$.

Whenever two points $p$ and $q$ exchange their $u_j$-order, we do the following updates.

- We update the kinetic sorted list $L_j(P)$. Each swap event in a kinetic sorted list can be handled in time $O(\log n)$.

- We update the RBRT $T_l$ and if a point is deleted or inserted into a $B_i$, we update the sorted list $L(B'_i)$. Since each insertion/deletion to $L(B'_i)$ takes $O(\log n)$ time, and since each point is in $O(\log^d n)$ sets $B_i$, this takes $O(\log^{d+1} n)$ time.

- We update the values of $r_{i,k}$. After updating the RBRT $T_l$, point $q$ might be inserted or deleted from some $R_i$ and change the values of $r_{i,k}$. So, for all $R_i$ where $q \in R_i$, before and after the event, we do the following. We check whether the $x_l$-coordinate of $q$ is less than or equal to the $x_l$-coordinate of $r_{i,k}$; if so, we take the successor or predecessor point of $r_{i,k}$ in $L(R_i)$ as the new value for $r_{i,k}$. This takes $O(\log^{d+1} n)$ time.

- We query to find $C(p)$. By Lemma 3, this takes $O(\log^d n + k \log \log n)$ time.

- If we get a new value for $\tilde{p}_{l,k}$, we update all the sorted lists $L(B'_i)$ such that $p \in B_i$. This takes $O(\log^{d+1} n)$ time.

Considering the complexity of each step above, and assuming the trajectory of each point is a bounded degree polynomial, the following results.
Lemma 5 Our KDS for maintenance of the k-SYG handles $O(n^2)$ $u$-swap events, each in worst-case time $O((\log^{d+1} n + k \log \log n))$.

Handling $x$-swap events. When an $x$-swap event between two consecutive points $p$ and $q$ with $p$ preceding $q$ occurs, it does not change the elements of the pairs $(B_i, R_i)$ of the CSPD $\Psi_i$. Such an event changes the k-SYG if both $p$ and $q$ are in the same $W_l(w)$, for some $w \in P$, and $w_{l,k} = p$.

We apply the following updates to our KDS when two points $p$ and $q$ exchange their $x_l$-order.

1. We update the kinetic sorted list $L_l(P)$; this takes $O(\log n)$ time.
2. We update the RBRT $T_l$, which takes $O(\log^d n)$ time.
3. We find all the sets $R_i$ where both $p$ and $q$ belong to $R_i$ and such that $r_{i,k} = p$. Also, we find all the sets $R_i$ where $r_{i,k} = q$. This takes $O(\log n)$ time.
4. For each $R_i$, we extract all the pairs $(w, \tilde{w}_{l,k})$ from the sorted lists $L(B'_i)$ such that $\tilde{w}_{l,k} = p$. Note that each change to the pair $(w, \tilde{w}_{l,k})$ is a change to the k-SYG.
5. For each $w$, we update all the sorted lists $L(B'_i)$ where $(w, \tilde{w}_{l,k}) \in B'_i$: we replace the previous value of $\tilde{w}_{l,k}$, which is $p$, by the new value $q$.

Denote by $\chi_k$ the number of exact changes to the k-SYG of a set of moving points over time. For each found $R_i$, the fourth step takes $O(\log n + \xi_i)$ time, where $\xi_i$ is the number of pairs $(w, \tilde{w}_{l,k})$ such that $\tilde{w}_{l,k} = p$. For all these $O(\log^d n)$ sets $R_i$, this step takes $O(\log^{d+1} n + \sum_i \xi_i)$ time, where $\sum_i \xi_i$ is the number of exact changes to the k-SYG when an $x$-swap event occurs. Therefore, for all the $O(n^2)$ $x$-swap events, the total processing time for this step is $O(n^2 \log^{d+1} n + \chi_k)$.

The processing time for the fifth step is a function of $\chi_k$. For each change to the k-SYG, this step spends $O(\log^{d+1} n)$ time to update the sorted lists $L(B'_i)$. Therefore, the total processing time for all the $x$-swap events in this step is $O(\chi_k \log^d n)$.

From the above discussion and an upper bound for $\chi_k$ in Lemma 6 Lemma 7 results.

Lemma 6 The number of changes to the k-SYG of a set of $n$ moving points, where the trajectory of each point is a polynomial function of at most constant degree $s$, is $\chi_k = O(\phi(s, n) * n)$.

Proof. Fix a point $p \in P$ and one of its wedges $W_l(p)$. There are $O(n)$ insertions/deletions into the wedge $W_l(p)$ over time. The $x_l$-coordinates of these points create $O(n)$ partial functions.
The $k$-SYG changes if a change to $\tilde{p}_{l,k}$ occurs. The number of all changes to $\tilde{p}_{l,k}$ is equal to $\phi(s, n)$, the complexity of the $k$-level of partially-defined polynomial functions of bounded degree $s$.

Therefore, considering all the $n = |P|$ points, the number of changes to the $k$-SYG is within a linear factor of $\phi(s, n)$: $\chi_k = O(\phi(s, n) * n)$. \hfill \Box

**Lemma 7** Our KDS for maintenance of the $k$-SYG handles $O(n^2)$ $x$-swap events with a total cost of $O(\phi(s, n) * n \log^{d+1} n)$.

From Lemmas 5 and 7 the following theorem results.

**Theorem 2** For a set of $n$ moving points in $\mathbb{R}^d$, where the trajectory of each point is a polynomial of at most constant degree $s$, our $k$-SYG KDS uses $O(n \log^d n)$ space and handles $O(n^2)$ events with a total cost of $O(kn^2 \log \log n + \phi(s, n) * n \log^{d+1} n)$.

### 4.2 Kinetic All $k$-Nearest Neighbors

Given a KDS for maintenance of the $k$-SYG (from Theorem 2), a supergraph of the $k$-NNG, this section shows how to maintain all the $k$-nearest neighbors over time. For maintenance of the $k$-nearest neighbors of each point $p \in P$, we only need to track the order of the edges incident to $p$ in the $k$-SYG according to their Euclidean lengths. This can easily be done by using a kinetic sorted list. The following theorem summarizes the complexity of our kinetic approach.

**Theorem 3** For a set of $n$ moving points in $\mathbb{R}^d$, where the trajectory of each point is a polynomial of at most constant degree $s$, our KDS for maintenance of all the $k$-nearest neighbors, ordered by distance from each point, uses $O(n \log^d n + \phi(s, n) * n^2)$ space and $O(n \log^d n + kn \log n)$ preprocessing time. Our KDS handles $O(\phi(s, n) * n^2)$ events, each in $O(\log n)$ amortized time.

**Proof.** Let $E_p(t)$ be the set of edges incident to point $p \in P$ in the $k$-SYG at time $t$. Let $L(E_p(t))$ denote a kinetic sorted list that maintains the edges in $E_p(t)$ sorted by their Euclidean lengths.

Let $m_p$ be the number of insertions/deletions to the set $E_p(t)$ over time. Since the cardinality of $E_p(t)$ is $O(n)$, each insertion into a kinetic sorted list $L(E_p(t))$ can cause $O(n)$ swaps. Each change, e.g., inserting/deleting an edge $pq$, to the $k$-SYG creates two insertions/deletions in the kinetic sorted lists $L(E_p(t))$ and $L(E_q(t))$; this implies that $\sum_p m_p = O(\chi_k)$. By Lemma 3, the kinetic sorted lists handle a total of $O(n \sum_p m_p) = O(\phi(s, n) * n^2)$ events. Each event in a kinetic sorted list is handled in time $O(\log n)$. Thus from this and Theorem 2 the total processing time for swap events is $O(kn^2 \log \log n + \phi(s, n) * n \log^{d+1} n + \phi(s, n) * n^2 \log n) = O(\phi(s, n) * n^2 \log n)$. \hfill \Box
KDS performance criteria. The KDS framework [4] measures the performance of a KDS by four standard criteria, which we now apply to our KDS for maintenance of all the $k$-nearest neighbors in $\mathbb{R}^d$.

- **Efficiency:** This is the ratio of the number of events that a KDS processes to the number of exact changes to the attribute of interest over time. The exact number of changes for maintenance of all the $k$-nearest neighbors can be computed as follows. Fix a point $p \in P$. The distances of the $n - 1$ points of $P \setminus \{p\}$ to $p$ as functions of time create $2s$-intersecting curves, meaning that each pair intersects at most $2s$ times. The number of changes to the $i^{th}$ nearest neighbor $p_i$ of $p$ equals $\Phi(2s, n - 1)$, the complexity of the $i$-level of the $n - 1$ $2s$-intersecting curves. Thus the number of changes to the $k$-nearest neighbors $p_1, \ldots, p_k$ of $p$ is $O(\Phi(2s, n) * k)$. The total for all points $p \in P$ is $O(\Phi(2s, n) * kn)$. Since the number of events in our KDS is $O(\phi(s, n) * n^2)$, the efficiency of our KDS is $O(\frac{n}{k})$.

- **Responsiveness:** This is the cost of updating the KDS when an event occurs. In our KDS each event can be handled in amortized time $O(\log n)$. Thus the responsiveness of our KDS is $O(\log n)$ on average.

- **Locality:** The number of updates to a KDS when a point changes its trajectory gives the locality of the KDS. In our KDS, for each two consecutive elements in each of the kinetic sorted lists $L_j(P)$, $L_l(P)$, and $L(E_p(t))$, we have a boolean function of time, called a certificate. Each certificate has a failure time, the time when the two consecutive elements exchange their order. If a point changes its trajectory, we update a constant number of these certificates in the kinetic sorted lists $L_j(P)$ and $L_l(P)$. Since the number of edges in the $k$-SYG is $O(kn)$, if a point changes its trajectory, the number of updates to the certificates in the kinetic sorted lists $L(E_p(t))$ is $O(k)$ on average. Therefore, the locality of our KDS is $O(k)$ on average.

- **Compactness:** This is the number of certificates in the KDS. Since the number of certificates of the kinetic sorted lists $L_j(P)$ and $L_l(P)$ is $O(n)$, and the number of certificates of the kinetic sorted lists $L(E_p(t))$ is $O(kn)$, the compactness of our KDS is $O(kn)$.

Therefore, we can obtain the following.

**Lemma 8** In terms of the KDS performance criteria, the “efficiency”, “responsiveness”, “locality”, and “compactness” of our KDS are $O(nk)$, $O(\log n)$ on average, $O(k)$ on average, and $O(kn)$, respectively.
4.3 R\textsubscript{k}NN Queries

Suppose we are given a query point \( q \notin P \) at some time \( t \). To find the reverse \( k \)-nearest neighbors of \( q \), we seek the points in \( P \cap W_l(q) \) and find \( C_l(q) \), the set of the first \( k \) points in \( L(P \cap W_l(q)) \). The set \( \cup_l C_l(q) \) contains \( O(k) \) candidate points for \( q \) such that \( q \) might be one of their \( k \)-nearest neighbors. In time \( O(\log d n) \) we can find a set of \( R_i \) where \( P \cap W_l(q) = \sum_i R_i \). From Lemma 3, and since we have sorted lists \( L(R_i) \) at level \( d + 1 \) of \( T_l \), the \( O(k) \) candidate points for the query point \( q \) can be found in worst-case time \( O(\log d n + k \log \log n) \). Now we check whether these candidate points are the reverse \( k \)-nearest neighbors of the query point \( q \) at time \( t \) or not; this can be easily done by application of Theorem 3, which in fact maintain the \( k \)th nearest neighbor \( p_k \) of each \( p \in P \). Therefore, checking a candidate point can be done in \( O(1) \) time by comparing distance \( |pq| \) to distance \( |pp_k| \). This implies that checking which elements of \( C_l(q) \), for \( l = 0, ..., c - 1 \), are reverse \( k \)-nearest neighbors of the query point \( q \) takes time \( O(k) \).

If a query arrives at a time \( t \) that is simultaneous with the time when one of the \( O(\phi(s, n) \ast n^2) \) events occurs, our KDS first spends time \( O(\log n) \) in an amortized sense to handle the event, and then spends time \( O(\log d n + k \log \log n) \) to answer the query. Thus we have the following.

**Theorem 4** Consider a set \( P \) of \( n \) moving points in \( \mathbb{R}^d \), where the trajectory of each one is a bounded-degree polynomial. The number of reverse \( k \)-nearest neighbors for a query point \( q \notin P \) is \( O(k) \). Our (kinetic) data structure uses \( O(n \log d n + kn) \) space and \( O(n \log d n + kn \log n) \) preprocessing time. At any time \( t \), an R\textsubscript{k}NN query can be answered in time \( O(\log d n + k \log \log n) \). If an event occurs at time \( t \), the KDS spends \( O(\log n) \) time in an amortized sense on updating itself.

5 Discussion and Conclusion

In the kinetic setting, where the trajectories of the points are polynomials of bounded degree, to answer the R\textsubscript{k}NN queries over time we have provided a KDS for maintenance of all the \( k \)-nearest neighbors. Our KDS is the first KDS for maintenance of all the \( k \)-nearest neighbors in \( \mathbb{R}^d \), for any \( k \geq 1 \). It processes \( O(\phi(s, n) \ast n^2) \) events, each in time \( O(\log n) \) in an amortized sense. An open problem is to design a KDS that processes less than \( O(\phi(s, n) \ast n^2) \) events.

Arya et al. [3] have a kd-tree implementation to approximate the nearest neighbors of a query point that is in use by practitioners [12] who have found challenging to implement the theoretical algorithms [6, 11, 13, 20]. Since to report all the \( k \)-nearest neighbors ordered by distance from each point our method uses multi-dimensional range trees, which can be easily implemented, we believe our method may be useful in practice.
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