Pitch-angle scattering in magnetostatic turbulence

I. Test-particle simulations and the validity of analytical results

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1. Introduction

Recent spacecraft observations have revealed the necessity to refine the modeling of the transport of charged energetic particles to allow for strongly pitch-angle–anisotropic phase space distribution functions, which cannot be properly accounted for by the diffusion approximation. In addition to solar energetic particles (SEPs) in general, for which this requirement has been known for decades (Roelof, 1969), other heliospheric particle populations were identified to exhibit such anisotropies. Examples are the so-called Jovian distribution functions, which cannot be properly accounted for by the diffusion approximation (Dröge, 2000), other heliospheric particle populations were identified to exhibit such anisotropies. Examples are the so-called Jovian electron jets (Ferrando et al., 1993; Dunzlafl et al., 2010) and suprathermal ion species accelerated at interplanetary traveling shocks (le Roux & Webb, 2012), as well as at the solar wind termination shock (Decker et al., 2005; Florinski et al., 2008; le Roux & Webb, 2012).

Central to such a modeling refinement is the determination of the pitch-angle diffusion coefficient $D_{\mu\phi}$ that occurs in the Fokker-Planck transport equation (Schlickeiser, 2002; Shalchi, 2009). In general, one can distinguish at least three different methods of addressing this problem.

First, the wave number $k$-dependence of the turbulent power spectrum $G(k)$ can be specified to derive analytical approximations for $D_{\mu\phi}$. The quasi-linear theory (QLT) derived by Jokipii (1966) has been the standard theory, until it was realized that QLT is not only inaccurate but, in fact, invalid for some scenarios. For the example of isotropic turbulence, it has been known from both qualitative arguments (Fisk et al., 1974; Bieber et al., 1988) and detailed calculations (Tautz et al., 2006) that QLT cannot properly describe pitch-angle scattering, because it neglects $90^\circ$ scattering and leads to infinitely large mean-free paths. This problem was remedied by the application of the second-order QLT (SOQLT, see Shalchi, 2005; Tautz et al., 2008), which considers deviations from the unperturbed spiral orbits that were assumed in QLT.

Second, to allow for more complex turbulence properties and to validate the permissibility of the analytical perturbation theories, one can resort to test particle simulations in specified turbulent magnetic fields. By tracing particle trajectories, the mean square displacements and the associated diffusion parameters can be obtained. On the one hand, there have been successful attempts to confirm quasi-linear results (Qin & Shalchi, 2009). On the other, such simulations have been employed to investigate the general behavior of pitch-angle scattering as described by the direct summation of multiple particle deflections (Lemons et al., 2009). Further examples of application of this method include the studies of the effects of structured (Lahtinen et al., 2012) and balanced turbulence (Lahtinen et al., 2012) on the particle transport, and consideration of inhomogeneous magnetic background fields (Tautz et al., 2011; Kelly et al., 2012).

Third, rather than entirely specifying the turbulent magnetic fields, one can perform direct numerical simulations to compute solutions to the magnetohydrodynamic equations, while the test-particle trajectories are still integrated as in the previous method. Such computations (see, e.g., Beresnyak et al., 2011; Spanier & Wisniewski, 2011; Wisniewski et al., 2012) do not require assumptions regarding the turbulence spectrum that is seen by the energetic particles. They are, however, limited regarding the extent of the inertial range of the turbulence spectrum, owing...
to computational constraints. In the present study we, therefore, restrict ourselves to the first and second approaches.

We undertake a systematic comparison between analytical predictions of the Fokker-Planck coefficient of pitch-angle scattering and numerical simulations that are based on a Monte-Carlo code developed by one of us (Tautz, 2010). In Sect. 2, the Monte-Carlo code PADIAN is introduced, which is used for all numerical simulations. In Sects. 3 and 4 basic properties of pitch-angle scattering and of the Fokker-Planck coefficient are presented, respectively. In Sect. 5 results from the numerical simulations are compared to estimations obtained from analytical scattering theories. Sect. 6 provides a summary and a discussion of the results.

2. PADIAN simulation code

For the numerical simulations, a Monte-Carlo code was used to compute the parallel diffusion coefficient of energetic particles for the turbulence model described above. A general description of the code and the underlying numerical techniques can be found elsewhere (Tautz, 2010; Tautz & Dosch, 2013).

Specifically, the isotropic and the slab turbulence models were employed, which are defined via the wave vector of the (Fourier transformed) turbulent magnetic field with random orientation and aligned with the direction of the background magnetic field (i.e., the \( z \) axis), respectively. The corresponding generation of turbulent magnetic fields proceeded as (Tautz & Dosch, 2013)

\[
\delta B(r, t) = \sum_{n=1}^{N_m} e'_\perp A(k_n) \cos [k_n z'_\parallel + \beta_n],
\]

where the wavenumbers \( k_n \) are distributed logarithmically in the interval \( k_{\text{min}} \leq k_n \leq k_{\text{max}} \), and \( \beta \) is a random phase angle.

The slab turbulence model is motivated by Mariner 2 measurements, indicating that the solar wind is dominated by outward propagating Alfvénic turbulence (Belcher & Davis, 1971; Tautz & Shalchi, 2013). Together with a second, two-dimensional contribution, this gave rise to the composite turbulence model (Bieber et al., 1998; Mattheus et al., 1991; Rausch & Tautz, 2013). In numerical simulations, in contrast, Alfvénic modes exhibit a scale-dependent anisotropy consistent with the Goldreich-Sridhar (1994/1995) model. Nevertheless, the classic slab model remains attractive, especially for analytical investigations, thereby allowing one to isolate specific effects.

For the amplitude and the polarization vector, one has \( A(k_n) = \sqrt{G(k_n)} \) and \( e'_\parallel \cdot e'_\parallel = 0 \), respectively, with the primed co-ordinates determined by a rotation matrix with random angles so that \( k \parallel \hat{e} \), for slab modes and random \( k \) directions for isotropic modes. From the integration of the Newton-Lorentz equation for the particle motion, various diffusion coefficients can then be calculated by averaging over an ensemble of particles and by determining the mean square displacement. For example, the scattering mean free path in the direction parallel to the background magnetic field can be obtained as \( \lambda_{||} = (3/\nu) (\langle \Delta z^2 \rangle / 2 t) \) for large times (cf. Sec. 3).

For the minimum and maximum wavenumbers included in the turbulence generator, the following considerations apply: (i) the resonance condition states that there has to be a wavenumber \( k \) so that \( R_1 k \approx 1 \), where \( R_1 \) denotes the particle’s Larmor radius so that scattering predominantly occurs when a particle can interact with a wave mode over a full gyration cycle; (ii) the scaling condition requires that \( R_1 Q_{\text{rel}} t \ll L_{\text{max}} \), where \( Q_{\text{rel}} = qB/(\gamma mc) \) is the relativistic gyrofrequency and where \( L_{\text{max}} \propto 1/k_{\text{min}} \) is the maximum extension of the system, which is given by the lowest wavenumber (for which one has \( k_{\text{min}} = 2\pi/L_{\text{max}} \), thereby proving the argument). In practice, the second condition determines the minimum wavenumber, while the first one determines the maximum wavenumber.

Here, values are chosen as \( k_{\text{min}} l_0 = 10^{-4} \) and \( k_{\text{max}} l_0 = 10^{4} \), where \( l_0 \) is the turbulence bend-over scale (see Appendix A). The sum in Eq. (1) extends over \( N_m = 512 \) wave modes, which is sufficient (Tautz & Dosch, 2013) and yet saves computation time. Furthermore, the maximum simulation time is determined as \( vt/l_0 = 10^4 \) for low particle energies and \( 10^2 \) for high particle energies.

3. Pitch-angle scattering

Perhaps the most important transport process of high-energy particles is represented by pitch-angle scattering, i.e., by stochastic variations in \( \mu = \cos \theta \). \( B_0 = v_0/\nu \) with range \([-1, 1]\), where \( B_0 = B_0 \hat{d} \parallel \) is the mean magnetic field and \( \nu \) is the particle velocity. This process is related to diffusion along the mean magnetic field, which is described by the parallel diffusion coefficient, \( k_{\perp} \), or the parallel mean free path, \( \lambda_{||} = (3/\nu)k_{\parallel} \), which are also related to the cosmic ray anisotropy (Schlickeiser, 1989; Shalchi et al., 2009).

The time evolution of the pitch angle is shown in Fig. 1 for a sample of typical single-particle trajectories (without any averaging process). It is indeed confirmed that particles with \( \mu \approx [0, \pm 1] \) almost retain their original pitch angle. However, scattering through 90° can occur, a fact that is not included in QLT. An introduction to analytical transport theories can be found, e.g., in (Schlickeiser, 2002) and (Shalchi, 2009).

3.1. General remarks

The usual definition of pitch-angle scattering can be found in the so-called Taylor-Green-Kubo (TGK) formalism (Taylor, 1922).

Fig. 1. (Color online) Pitch-angle cosine, \( \mu = \nu_0/\nu \) as a function of the normalized time, \( \tau = \Omega t \), for a relative slab turbulence strength \( \delta B/B_0 = 10^{-2} \). Four particles with initial pitch angles in the range \( 0.1 \leq \mu_0 \leq 0.8 \) are highlighted to guide the eye.
Fusion in terms of the mean-square displacement (e.g., Tautz, 2012) with each other, if it can be shown (e.g., Shalchi, 2011) that both versions agree. The combination of diffusion (Fick, 1855) and random walk (Chandrasekhar, 1943) motivated the usual definition of the diffusion in terms of the mean-square displacement (e.g., Tautz, 2012). The proportionality. This gives us an estimate for the time range, during which we can expect Eqs. (2b) and (3) to be valid. The horizontal dotted bars show the corresponding standard deviations. Additionally, the vertical dotted line shows the mean (which remains almost unchanged).

Following the derivation of the quasi-linear Fokker-Planck coefficient (see Shalchi, 2005), the pitch-angle displacement, \( \Delta \mu = \mu(t) - \mu_0 \), can be expressed as

\[
\Delta \mu(t) = \frac{2 \delta B}{B_0} \int_0^t dt' v_s(t') \delta B_s(v_s(t')) \delta B_x(v_s(t')) \delta B_y(v_s(t')) \delta B_z(v_s(t'))
\]

which is again valid if \( t \) is high enough. However, the formal limit \( t \to \infty \) is forbidden since \( |\Delta \mu| \) cannot exceed a value of 2. For high enough times, \( D_{\mu\mu} \) will always be dominated by the \( 1/t \) dependence, independent of the choice of the formula. Therefore, a meaningful, time-independent value for \( D_{\mu\mu} \) can be obtained if and only if: (i) \( t \) is long enough that the initial conditions become insignificant; (ii) \( t \) is short enough that the behavior of \( D_{\mu\mu} \) is not already dominated by the \( 1/t \) proportionality. This matter is further investigated in the second paper of this series (Tautz, 2013).

3.2. Estimation of the critical time

Based on the preceding paragraph, we now calculate the critical time, \( t_{\text{max}} \), by using QLT. This gives us an estimate for the time range, within which we can expect Eqs. (2b) and (3) to be valid.
In this paragraph, it is demonstrated that particles stay in their original pitch-angle regime for all times. This is true even if, as stated above, the Fokker-Planck coefficient(s) has to be a function of time but, at the same time, depends on the \( \mu \) values. It has to be stressed that pitch-angle scattering is unique in that, unlike for normal spatial diffusion, the coefficient depends on the variable from which the mean square displacement is obtained.

Therefore, several options are possible, two of which will be described here.

### 3.3. Pitch-angle distribution

In this paragraph, it is demonstrated that particles stay in their original pitch-angle regime for all times. This is true even if, as stated above, the Fokker-Planck coefficient of pitch-angle scattering tends to zero for times \( t > t_{\text{max}} \).

In Fig. 3 the distribution function of all particles sorted for their pitch-angle displacement, \( \Delta \mu \), is shown. As time increases, particles begin to deviate from their original pitch angles. Nevertheless (note the logarithmic scaling of the vertical axis!), the distribution remains extremely small with its width growing linearly with time as \( 10^{-5} Q t \).

In Fig. 4 the time evolution of a Kolmogorov-Smirnov (KS) test statistic is shown, which is obtained from the comparison of the pitch-angle distribution, \( f(\mu, t) \), with the initial pitch-angle distribution, \( f(\mu_0, 0) \). Because the latter is obtained from a uniform random deviate, the usual result of an isotropized distribution due to pitch-angle diffusion leads to the requirement that the pitch-angle distribution be uniform. A linear fit of the otherwise relatively volatile KS statistic shows that, on average, the \( P \) value (i.e., the probability that the given distribution agrees with the assumed one) of the KS test is well above 80%; this confirms that the pitch-angle distribution remains compatible with the initial distribution.

This result serves as a second indicator that an initially homogeneous pitch-angle distribution is retained. As a side note, it should be mentioned that the pitch-angle distribution is not precisely homogeneous, i.e., the comparison of the pitch-angle distribution with a flat distribution yields a reduced KS test statistic as opposed to the comparison of \( f(\mu, t) \) with \( f(\mu_0) \).

### 4. Fokker-Planck coefficient

In this section, some of the intricacies connected to the numerical implementation of pitch-angle scattering is discussed. For an overview of analytical calculations, the reader is referred to Appendix A, where both quasi-linear and nonlinear results are summarized.

While the calculation of the mean free path values is straightforward, this is somewhat different for the Fokker-Planck coefficient(s). Especially \( D_{\mu \mu} \) has to be a function of time but, at the same time, depends on the \( \mu \) values. It has to be stressed that pitch-angle scattering is unique in that, unlike for normal spatial diffusion, the coefficient depends on the variable from which the mean square displacement is obtained.

| \( \delta \mu / \mu_0 \) | Rigidity | Theory | \( \chi^2 \) value | \( Q \) value | \( v_{\text{max}} / \ell_0 \) |
|-----------------|---------|--------|----------------|-------------|----------------|
| 10^{-2}         | 10^{-2} | QLT    | 24.47          | 0.9747      | 1              |
| 10^{-2}         | 10^{-2} | SOQLT  | 24.33          | 0.9759      | 1              |
| 10^{-1}         | 10^{-1} | QLT    | 6.55           | 1           | 10             |
| 10^{-1}         | 10^{-1} | SOQLT  | 6.43           | 1           | 10             |
| 10^{-2}         | 1       | QLT    | 2.53           | 1           | 100            |
| 10^{-2}         | 1       | SOQLT  | 28.81          | 0.9057      | 100            |
| 10^{-1.5}       | 10^{-2} | QLT    | 16.74          | 0.9956      | 1              |
| 10^{-1.5}       | 10^{-2} | SOQLT  | 16.60          | 0.9960      | 1              |
| 10^{-1.5}       | 10^{-1} | QLT    | 10.23          | 1           | 10             |
| 10^{-1.5}       | 10^{-1} | SOQLT  | 10.37          | 1           | 10             |
| 10^{-1.5}       | 1       | QLT    | 13.55          | 1           | 100            |
| 10^{-1.5}       | 1       | SOQLT  | 27.71          | 0.9296      | 100            |
| 10^{-1.5}       | 10      | QLT    | 2.24           | 1           | 100            |

Table 1. Overall agreement between numerical results for the pitch-angle Fokker-Planck coefficient, \( D_{\mu \mu} \), and both quasi-linear (QLT) and second-order quasi-linear (SOQLT) analytical results as obtained from a chi-square test.
Fig. 6. (Color online) Running Fokker-Planck coefficient of pitch-angle scattering, \( D_{\mu\nu}(\mu, t) \) for particles with rigidity \( R = 10^{-2} \) in moderate turbulence strength, \( \delta B/B_0 = 10^{-1.5} \). The black solid lines illustrate \( D_{\mu\nu}(\mu) \) at specific times. The second approach is illustrated in Fig. 5 for various pitch-angle slots, thereby obtaining \( \delta B/B_0 = 0 \). For comparison, the analytical results from QLT and SOQLT are shown as solid blue and dashed red lines, respectively. The relative turbulence strength is chosen as \( \delta B/B_0 = 10^{-2} \).

In Fig. 6 the time evolution of the pitch-angle Fokker-Planck coefficient, \( D_{\mu\nu} \), as a function of the pitch-angle cosine, \( \mu \), for three different values for the normalized rigidity, \( R \). For comparison, the analytical results from QLT and SOQLT are shown as solid blue and dashed red lines, respectively. The relative turbulence strength is chosen as \( \delta B/B_0 = 10^{-2} \).

5. Comparison with analytical results

In this section, the numerical results for the Fokker-Planck coefficient will be compared to analytical results listed in Appendix A. Error bars are obtained from the comparison of different turbulence realizations and different initial particle positions (see Tautz, 2010). In addition, it has to be stressed again that, according to Fig. 6, the correct time point has to be chosen for the evaluation of \( D_{\mu\nu} \).

5.1. Slab turbulence

In Fig. 5 the pitch-angle Fokker-Planck coefficient is shown as a function of the normalized simulation time, \( \tau = \Omega t \). After the initial free-streaming phase, most values become (almost) constant, while the initially higher values still oscillate. At \( \tau = 10^2 \), the final, diffusive values for \( D_{\mu\nu} \) are taken that are used in the following sections. It should be noted, however, that \( D_{\mu\nu} \) is slightly decreasing (with approximately \( \tau^{-0.14} \)) so that the values for
For the ratio of the turbulent and background magnetic fields, a turbulence strength is chosen as $\delta B/B_0 = 10^{-1.5} \approx 0.0316$.

$D_{\mu\mu}$ are somewhat overestimated, in agreement with the results shown below.

In the following, the two cases of low and intermediate turbulence strengths are discussed.

### 5.1. Low turbulence strength

For the ratio of the turbulent and background magnetic fields, a value of $\delta B/B_0 = 10^{-2}$ is chosen so that the ratio of the magnetic field energies is $(\delta B/B_0)^2 = 10^{-4}$. The following results were found (see Fig. 8):

- For small and intermediate rigidities ranging from $R = 10^{-2}$ to $10^{-1}$, an excellent agreement between numerical and analytical results can be found. However, QLT and SOQLT are almost indistinguishable. For example, a chi-square test yields values of $\chi^2 = 6.551$ and $6.432$ at $R = 10^{-1}$ for the comparison to QLT and SOQLT, respectively, thereby revealing that the agreement with SOQLT is slightly better. However, the difference is marginal and might have occurred purely by serendipity.

- For high rigidities such as $R = 1$ and $R = 10$, QLT and SOQLT differ more. A chi-square test yields values of $\chi^2 = 2.531$ and $28.81$ at $R = 1$ for the comparison to QLT and SOQLT, respectively, thereby revealing that the agreement with QLT is significantly better. However, it should be noted that the approximation used for the SOQLT values becomes invalid if $R$ is too large.

### 5.1.2. Intermediate turbulence strength

Here, $\delta B/B_0 = 10^{-1.5} \approx 0.0316$ is chosen so that the ratio of the magnetic field energies is $10^{-3}$. The following results were found (see Fig. 9):

- For small and intermediate rigidities, the simulation results agree equally well with both QLT and SOQLT, to the same level of significance as was found in the previous section.

- For high rigidities, it is shown that both QLT and SOQLT severely underestimate $90^\circ$ scattering, even though SOQLT was designed explicitly to remedy this shortcoming of previous, quasi-linear results. Accordingly, the chi-square test yields slightly higher values of $\chi^2 = 13.55$ and $14.05$ at $R = 1$ for QLT and SOQLT, respectively, which again shows that QLT agrees slightly better with the numerical values.

- Additionally, it is remarkable that, for $R = 10$, the overall best agreement has been found as expressed by the low value $\chi^2 = 2.24$.

In general, it has to be noted (cf. Table 1) that the agreement between analytical and numerical results depends on the maximum simulation time. For $t \neq t_{\text{max}}$, less agreement is found.

### 5.2. Isotropic turbulence

For isotropic turbulence, the numerical Fokker-Planck coefficient is shown in Fig. 10. The comparison especially with SOQLT had to be done for higher rigidities than for slab geometry simply because the numerical evaluation of Eq. (A.5) is extremely protracted for low rigidities and/or low turbulence strengths.
While the agreement between theory and simulation is generally good at pitch-angles well off 90°, it is revealed that 90° degree scattering is not equally well described either by QLT or by SOQLT. Therefore, the agreement of the parallel mean free path with simulations (cf. Tautz et al., 2008; Tautz & Lerch, 2010) has to be attributed to the fact that, at μ = 0, SOQLT overestimates D_{μμ}, thereby compensating for values that are too low at 0.3 ≤ μ < 0.

6. Summary and conclusion

In this paper, random variations in the pitch-angle of charged particles that move in a turbulent magnetic field have been investigated. In the astrophysical theater, this situation is realized by cosmic rays and solar particle events, both of which experience continuous deflections either in the interstellar turbulence or in the solar-wind induced turbulence. In both cases, the random component superposes a mean magnetic field—e.g., the galactic magnetic field or the Parker-spiral solar magnetic field—which gives rise to a preferred direction when investigating the scattering processes. This motivates the transformation to a coordinate system in which the pitch angle is taken to be a basic variable of the particle motion.

The process can be analyzed by means of analytic calculations and numerical Monte-Carlo simulations, which are methods based on the kinetic Vlasov theory and the integration of the equation of motion for a large number of test particles, respectively. Whereas the use of a Fokker-Planck approach to determining the pitch-angle scattering and, based thereupon, the system in which the pitch angle is taken to be a basic variable of the particle motion.

A.1. Slab turbulence

For slab turbulence, the result is (e.g. Qin & Shalchi, 2009)

\[ D_{\mu \mu}^{\text{QLT}}(\mu) = \frac{2\pi^2 v}{|\mu| R_L^2} \left(\frac{\delta B}{B_0}\right)^2 G_{\text{slab}} \left(k_i = \frac{1}{|\mu| R_L}\right). \]  

For the turbulence power spectrum, G(k), a kappa-type function is used (Shalchi & Weinhold, 2009):

\[ G(k) = \frac{C}{2 \sqrt{\pi} \Gamma((s-1)/2)} \left|\ell_0 k\right|^\eta. \]  

with usually q = 0 for simplicity. The turbulence bend-over scale, \( \ell_0 \approx 0.03 \) AU, reflects the transition from the energy range \( G(k) \propto k^4 \) to the Kolmogorov-type inertial range, where \( G(k) \propto k^{-5/3} \) with \( s = 5/3 \) for large wavenumbers (Kolmogorov, 1991; Bruno & Carbone, 2005). The factor C depends on the assumed geometry and is given as \( C = 1/(2\pi) \) for slab turbulence, where \( \delta B(R) = \delta B(z) \), and \( C = 4 \) for isotropic turbulence.

In the normalized variables and, the Fokker-Planck coefficient of pitch-angle scattering can be expressed as

\[ D_{\mu \mu}^{\text{QLT}}(\mu) = \frac{2\pi^2 v}{|\mu| R} \left(1 - \mu^2\right)^2 \left(\frac{\delta B}{B_0}\right)^2 G_{\text{slab}} \left(k_i = \frac{1}{|\mu| R}\right). \]

The typical \( (1 - \mu^2) \) dependence reflects the fact that particles with pitch angles close to 0° and 180° are considerably less scattered than particles with intermediate pitch angles; in addition, the rightmost factor in Eq. (A.3) approximately gives \( |\mu|^{2/3} \), thereby suppressing 90° scattering.

A nonlinear theory developed especially to enhance pitch-angle scattering through 90° (Shalchi, 2005, 2009) yields the formula

\[ D_{\mu \mu} = \frac{1}{8s} \sqrt{\pi} \Gamma\left((s-1)/2\right) \left(1 - \mu^2\right) C(s) \left(\frac{\delta B}{B_0}\right)^2 \left(\frac{R^2}{\ell_0}\right) \times \sum_{n=\pm 1} \text{sgn}(\ell_0 B) \left(\frac{\delta B}{B_0} + n |\mu|\right) \left|\mu| + n \frac{\delta B}{B_0}\right| \]  

\[ \times \delta_{0}(w) \frac{1}{w} \int_0^{\infty} d\eta J_n^2(w) dG(k) \times \sum_{n=-\infty}^{\infty} J_n(w) \left(\frac{n^2}{w^2} J_n^2(w) + \frac{n^2}{w^2} J_n^2(w)\right), \]

where and v are the normalized rigidity and the particle speed, respectively.

A.2. Isotropic turbulence

For isotropic turbulence, the analytical theory of pitch-angle scattering is considerably more difficult to solve (Tautz et al., 2006, 2008). The general form of the Fokker-Planck coefficient for pitch-angle scattering reads as

\[ D_{\mu \mu} = 2 \left(1 - \mu^2\right)^2 \left(\frac{\delta B}{B_0}\right)^2 \left(\frac{R^2}{\ell_0}\right) \times \sum_{n=-\infty}^{\infty} J_n(w) \left(\frac{n^2}{w^2} J_n^2(w) + \frac{n^2}{w^2} J_n^2(w)\right), \]  

\[ \times \delta_{0}(w) \left(\frac{\delta B}{B_0}\right)^2 \left(\frac{R^2}{\ell_0}\right) \times \sum_{n=-\infty}^{\infty} J_n(w) \left(\frac{n^2}{w^2} J_n^2(w) + \frac{n^2}{w^2} J_n^2(w)\right), \]  

with \( J_n(w) \) the Bessel function of the first kind of order \( n \) and \( w = (kv/\Omega) \sqrt{(1 - \mu^2)(1 - \eta^2)} \). Additionally, \( \eta = \cos \angle(k, B_0) \)

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Appendix A: Analytical pitch-angle scattering

Analytically, the Fokker-Planck coefficient of pitch-angle scattering can be evaluated, e.g., using quasilinear theory (Jokipii, 1966) and nonlinear extensions (see Shalchi, 2009, for an overview). In any case, the evaluation is based on the TGK formalism by using Eq. (25).
is the wave vector polar angle. The resonance function can be expressed as
\[ R_n(k, \eta) = \frac{\pi}{2} \delta \left( kv \eta + n \Omega \right) \] (A.6a)
for QLT and
\[ R_n(k, \eta) = \sqrt{\frac{\pi}{2}} \left( \xi k \eta \right)^{-1} \exp \left[ -\frac{(kv \eta + n \Omega)^2}{k \xi^2} \right] \] (A.6b)
for SOQLT, where \( \xi = \frac{v^2}{2} \left( \delta B / B_0 \right)^2 / 3. \) Using QLT, Eq. (A.5) can be simplified further (see Tautz et al., 2006), whereas, for SOQLT, the two integrals and the infinite sum have to be evaluated numerically.

Alternatively, the formulation of Tautz & Lerche (2010) can be used for the case of SOQLT, where the infinite sum over Bessel functions was reduced to a closed form analytical expression under the assumption of a Cauchy-type resonance function.

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