ALGEBRAIC DEGREES OF GENERALIZED NASH EQUILIBRIUM PROBLEMS

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Abstract. This paper studies algebraic degree of generalized Nash equilibrium problems (GNEPs) given by polynomials. Their generalized Nash equilibria (GNEs), as well as their KKT or Fritz-John points, are algebraic functions in the coefficients of defining polynomials. We study the degrees of these algebraic functions, which also counts the numbers of complex KKT or Fritz-John points. Under some genericity assumptions, we show that a GNEP has only finitely many complex Fritz-John points and every Fritz-John point is a KKT point. We also give formulae for algebraic degrees of GNEPs, which count the numbers of complex Fritz-John points for generic cases.

1. Introduction

We consider generalized Nash equilibrium problems (GNEPs). Suppose there are \( N \) players and the \( i \)th player’s strategy vector is \( x_i \in \mathbb{R}^{n_i} \) (the \( n_i \)-dimensional real Euclidean space). Denote

\[
x_i := (x_{i,1}, \ldots, x_{i,n_i}), \quad x := (x_1, \ldots, x_N).
\]

The total dimension of all strategy vectors is \( n := n_1 + \ldots + n_N \). When the \( i \)th player’s strategy is considered, we use \( x_{-i} \) to denote the subvector of all players’ strategies except the \( i \)th one, i.e.,

\[
x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N).
\]

For convenience, we also write \( x = (x_i, x_{-i}) \) when \( x_i \) is considered. The main purpose of the GNEP is to find a tuple of strategies \( u = (u_1, \ldots, u_N) \) such that each \( u_i \) is a minimizer of the \( i \)th player’s optimization

\[
F_i(u_{-i}) : \begin{cases} 
\min_{x_i \in \mathbb{R}^{n_i}} f_i(x_i, u_{-i}) \\
\text{s.t. } g_{i,j}(x_i, u_{-i}) = 0 (j \in \mathcal{E}_{i,1}), \\
g_{i,j}(x_i, u_{-i}) \geq 0 (j \in \mathcal{E}_{i,2}).
\end{cases}
\]

In the above, the \( \mathcal{E}_{i,1} \) and \( \mathcal{E}_{i,2} \) are disjoint labeling sets (possibly empty), the \( f_i \) and \( g_{i,j} \) are continuous functions in \( x \). Note that \( u_{-i} := (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N) \).

The \( i \)th player’s optimization is parameterized by other player’s strategies. A strategy tuple \( u = (u_1, \ldots, u_N) \) satisfying the above is called a generalized Nash equilibrium (GNE).

When all the defining functions \( f_i \) and \( g_{i,j} \) are polynomials in \( x \), the GNEP is called a generalized Nash equilibrium problem of polynomials. If for every player \( i \), each constraining function \( g_{i,j} \) depends only on \( x_i \), i.e., the \( i \)th player’s feasible
strategy set is independent of other players’ strategies, then the GNEP is called a Nash equilibrium problem (NEP) and the corresponding \( u \) is called a Nash equilibria (NE). In particular, when all players’ strategy sets are finite sets, the NEP is called a finite game.

GNEPs are generalizations of Nash equilibrium problems [23]. Now they have been widely used in broad areas, such as marketing, supply chain management, telecommunications, and machine learning. We refer to [21,22,35] for recent applications of GNEPs. There exists much work on GNEPs. The existence of GNEs under some continuity and convexity assumptions is given in [1]. For GNEPs of twice differentiable continuous functions, generic structural properties are studied in [5,6]. It is typically a quite difficult question to solve GNEPs. Some computational methods are given in [7,12]. In [29–32], Moment-SOS relaxation methods are given for solving NEPs and GNEPs. We refer to [8,9] for general surveys on GNEPs.

1.1. KKT and Fritz-John conditions. We consider the \( i \)th player’s optimization \( F_i(x_{-i}) \), which is parameterized by \( x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \), the strategies of the other players. At a feasible point \( x_i \), its active labeling set \( E_i \) is

\[
E_i := \{ j \in E_{i,1} \cup E_{i,2} : g_{i,j}(x_i, x_{-i}) = 0 \}.
\]

Clearly, \( E_{i,1} \subseteq E_i \subseteq E_{i,1} \cup E_{i,2} \). When the linear independence constraint qualification (LICQ) holds (i.e., the gradient set \( \{ \nabla_x g_{i,j}(x_i, x_{-i}) : j \in E_i \} \) is linearly independent), if \( x_i \) is a minimizer of the optimization \( F_i(x_{-i}) \), then there exist Lagrange multipliers \( \lambda_{i,j} (j \in E_i) \) such that the Karush-Kuhn-Tucker (KKT) conditions hold:

\[
\begin{align*}
\nabla_x f_i(x) - \sum_{j \in E_i} \lambda_{i,j} \nabla_x g_{i,j}(x) &= 0, \\
g_{i,j}(x) &= 0 (j \in E_i).
\end{align*}
\]

In the above, we do not mention the sign conditions \( \lambda_{i,j} \geq 0 (j \in E_i \cap E_{i,2}) \), because they are not relevant to algebraic degrees of GNEPs. When the LICQ fails, the KKT conditions may or may not hold. However, no matter if the LICQ holds or not, there always exists a new variable \( \lambda_{i,0} \) such that

\[
\begin{align*}
\lambda_{i,0} \nabla_x f_i(x) - \sum_{j \in E_i} \lambda_{i,j} \nabla_x g_{i,j}(x) &= 0, \\
g_{i,j}(x) &= 0 (j \in E_i), \\
\text{not all } \lambda_{i,0} \text{ and } \lambda_{i,j} \text{ are 0.}
\end{align*}
\]

A point \( x_i \) satisfying (1.2) is called a KKT point and \( x_i \) satisfying (1.3) is called a Fritz-John (FJ) point, for the optimization \( F_i(x_{-i}) \). The (1.2) is called the KKT system and (1.3) is called the Fritz-John system. It is important to note that every minimizer of \( F_i(x_{-i}) \) is a Fritz-John point. When \( \lambda_{i,0} \neq 0 \), the Fritz-John conditions imply the KKT conditions. We refer to [2] for KKT and Fritz-John conditions in optimization. A tuple \( x = (x_1, \ldots, x_N) \) is said to be a Fritz-John point for the GNEP if \( x \) satisfies (1.3) for each \( i = 1, \ldots, N \).

If the active label set is \( E_i := \{ 1, \ldots, m_i \} \), then the Fritz-John system (1.3) is equivalent to

\[
\begin{align*}
\text{rank } [\nabla_x f_i(x) \quad \nabla_x g_{i,1}(x) \quad \cdots \quad \nabla_x g_{i,m_i}(x)] &\leq m_i, \\
g_{i,1}(x) = \cdots = g_{i,m_i}(x) = 0.
\end{align*}
\]
1.2. Contributions. This paper studies the set of KKT and Fritz-John points of GNEPs of polynomials. We mainly focus on the finiteness for the set of KKT and Fritz-John points under genericity assumptions, and the number of KKT and Fritz-John points when there are finitely many of them. The major results of this paper are:

- We study the Fritz-John system for the GNEP given by polynomial functions. Under some genericity assumptions, we show that the GNEP only has finitely many complex Fritz-John points. Moreover, when defining polynomials are generic, we show that every Fritz-John point is a KKT point and every GNE is a KKT point.
- We give a formula for the algebraic degree of GNEPs when defining polynomials are generic, which counts the number of complex Fritz-John points, and provides an upper bound for the iteration loops of the computational methods in [29, 32].
- For non-generic GNEPs of polynomials, we give an upper-bound for the number of complex Fritz-John points, when there are finitely many of them.

This paper is organized as follows. In Section 2, we review the optimality conditions for GNEPs, and introduce some basics of complex Fritz-John points. In Section 3, we study the finiteness of complex Fritz-John points for GNEPs, under some genericity assumptions. We consider multi-projective varieties and their algebraic degrees in Section 4. The algebraic degree formulae are given in Section 5. Some symbolic computational results are presented in Section 6.

2. Preliminaries

Notation. The symbol \( \mathbb{N} \) (resp., \( \mathbb{R}, \mathbb{C} \)) stands for the set of nonnegative integers (resp., real numbers, complex numbers). For a positive integer \( k \), denote the set \( [k] := \{1, \ldots, k\} \). We use \( e_i \) to denote the vector such that the \( i \)th entry is 1 and all others are zeros. For a nonnegative integer vector \( a = (a_1, \ldots, a_k) \in \mathbb{N}^k \), denote that \( |a| := a_1 + \cdots + a_k \). Let \( a := (a_1, \ldots, a_k) \) and \( b := (b_1, \ldots, b_k) \) be tuples of nonnegative integers, the \( a \leq b \) means that \( a_i \leq b_i \) for each \( i = 1, \ldots, k \). Let \( \mathbb{R}[x] \) denote the ring of polynomials with real coefficients in \( x \), and \( \mathbb{R}[x]_d \) denote its subset of polynomials whose degrees are not greater than \( d \). The notation \( \mathbb{C}[x] \) and \( \mathbb{C}[x]_d \) are similarly defined. For the \( i \)th player’s strategy vector \( x_i \), the notation \( \mathbb{R}[x_i] \) and \( \mathbb{R}[x_i]_d \) are defined in the same way. For the \( i \)th player’s objective \( f_i(x) \), the \( \nabla_{x_i} f_i \) means its gradient with respect to \( x_i \).

In the following, we review some basic results in algebraic geometry. Let \( \mathbb{C} \) be the complex field and \( \mathbb{C}[x] \) be the ring of polynomials in the variable \( x = (x_{i,j}) \). An ideal \( I \) of \( \mathbb{C}[x] \) is a subset of \( \mathbb{C}[x] \) such that \( a + b \in I \) for all \( a, b \in I \) and \( q \cdot p \in I \) for all \( p \in I \) and \( q \in \mathbb{C}[x] \). A polynomial tuple \( (p_1, \ldots, p_m) \) generates the ideal

\[
\langle p_1, \ldots, p_m \rangle := p_1 \mathbb{C}[x] + \cdots + p_m \mathbb{C}[x].
\]

Every ideal is generated by a finite set of polynomials. This is the Hilbert’s basis theorem (see [3]). For an ideal \( I \), the set

\[
V(I) := \{ x \in \mathbb{C}^n : p(x) = 0 \forall p \in I \}
\]
is called the variety of $I$. Such a set is called an affine variety. For a set $V \subseteq \mathbb{C}^n$, the polynomial set

$$I(V) := \{ p \in \mathbb{C}[x] : p(x) = 0 \text{ for all } x \in V \}$$

is an ideal of $\mathbb{C}[x]$. It is called the vanishing ideal of $V$. There exists extensive work about optimization with polynomials and varieties (see [14, 16–20, 37]). Sum of squares polynomials and matrices are useful techniques for solving polynomial optimization (see [4, 36, 39]). Moreover, ideals and varieties are recently exploited in the research of games (see [27, 33, 34]). They are also useful for solving tensor optimization (see [28, 35, 34]).

The $n$-dimensional projective space $\mathbb{P}^n$ is the set of all lines passing through the origin of $\mathbb{C}^{n+1}$. A point $\tilde{z}$ in $\mathbb{P}^n$ has the coordinate $[z_0 : z_1 : \cdots : z_n]$ such that at least one of $z_j$ is nonzero. The coordinates of $\tilde{z}$ are unique up to a nonzero scaling of $\tilde{z} = (z_0, z_1, \ldots, z_n)$. Let $(p_1, \ldots, p_m) \subseteq \mathbb{C}[\tilde{z}]$ be a tuple of homogeneous polynomials in $\tilde{z}$, the ideal $I := \langle p_1, \ldots, p_m \rangle$ is called a homogeneous ideal. It determines the projective algebraic variety in $\mathbb{P}^n$:

$$U := \{ \tilde{z} \in \mathbb{P}^n : p_1(\tilde{z}) = \cdots = p_m(\tilde{z}) = 0 \}.$$ 

For positive dimensions $n_1, \ldots, n_N$, the Cartesian product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_N}$ is called a multi-projective space. For the tuple of positive integers $\nu := (n_1, \ldots, n_N)$, denote

$$\mathbb{P}^\nu := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_N}.$$ 

Given the tuple $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_N)$ of vector variables, a polynomial $p \in \mathbb{C}[\tilde{x}]$ is multi-homogeneous in $(\tilde{x}_1, \ldots, \tilde{x}_N)$ if for every $i \in [N]$, the polynomial $p(\tilde{x}_i, \tilde{x}_{-i})$ is homogeneous in $\tilde{x}_i$. Similarly, an ideal is said to be multi-homogeneous if it is generated by a set of multi-homogeneous polynomials. A set like

$$W = \{ \tilde{x} \in \mathbb{P}^\nu : p_1(\tilde{x}) = \cdots = p_m(\tilde{x}) = 0 \}$$

is called an multi-projective variety if every $p_j$ is multi-homogeneous.

For the affine (resp., projective, multi-projective) space, a hypersurface is given by a single (resp., homogeneous, multi-homogeneous) polynomial equation. A hyperplane is given by a single linear equation. In the Zariski topology, closed sets are varieties and open sets are complements of varieties. A variety $V$ is irreducible if it is not a union of two distinct proper subvarieties. Every variety is a union of finitely many irreducible subvarieties. Each of these irreducible subvarieties is called an irreducible component. Throughout the paper, a property is said to hold generically if it holds for all points in the space of input data except a set of Lebesgue measure zero.

The dimension of an irreducible projective variety $V$ is defined to be the length $k$ of the longest chain of irreducible subvarieties $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k$. For an irreducible multi-projective variety $W \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_N}$, the dimension of $W$ equals the largest number $k$ such that there exist nonnegative integers $l_1, \ldots, l_N$, with each $l_i \leq n_i$ and $l_1 + \cdots + l_N = k$, satisfying that

$$\{(\tilde{x}_1, \ldots, \tilde{x}_N) \in W : (u_{i,j})^T \tilde{x}_i = 0, \text{ for } i = 1, \ldots, N, j = 1, \ldots, l_i \}$$

is nonempty for all $u_{i,j} \in \mathbb{P}^{n_i}$. The dimension of a variety equals the biggest dimension of its irreducible subvarieties. For the variety $W$, its dimension $\dim W = 0$ if and only if its cardinality $|W| < \infty$. A variety is said to have a pure dimension if all of its irreducible components have the same dimension. A pure $k$-dimensional variety $W$ is said to be a complete intersection if its vanishing ideal is generated by
For every irreducible component $W$ codim $W$ of a multi-projective variety of points in the intersection of $V$ if $W$ properly intersect transversal intersections for varieties. side their singular loci. We refer to [10, 13, 38] for more details on smoothness and

For a pure $k$-dimensional projective variety $V$, its algebraic degree is the number of points in the intersection of $V$ and $k$ general hyperplanes. The algebraic degree of a multi-projective variety $W \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_N}$ is an array $\deg W$, labelled by a tuple $\ell = (l_1, \ldots, l_N) \in \mathbb{N}^N$ with $l_1 + \cdots + l_N = \dim W$ and $\ell \leq \nu$, such that $(\deg W)_\ell$ counts the number of points in the intersection given by (2.1) for general $u_{i,j} \in \mathbb{P}^{n_i}$. In particular, when $\dim W = 0$, the algebraic degree $\deg W$ counts the number of points in $W$.

Smoothness is useful when we consider algebraic degrees. For an irreducible multi-projective variety $W$, whose vanishing ideal is given by multi-homogeneous polynomials $p_1, \ldots, p_m$, its tangent space at a point $\tilde{u} \in W$, denoted by $\mathbb{T}_{\tilde{u}} W$, is the null space of the matrix

$$J(\tilde{u}) := \begin{bmatrix} \nabla_{\tilde{u}} p_1(\tilde{u}) & \cdots & \nabla_{\tilde{u}} p_m(\tilde{u}) \end{bmatrix}^T.$$ 

The dimension of $\mathbb{T}_{\tilde{u}} W$ is at least $\dim W$ for every $u \in W$ (see [38, Theorem 2.3]). The variety $W$ is said to be smooth at $\tilde{u}$ if

$$\dim \mathbb{T}_{\tilde{u}} W = \dim W.$$ 

In other words, the $\tilde{u}$ is a smooth point if and only if the rank of $J(\tilde{u})$ equals the codimension of $W$. When $W$ is reducible, the smoothness for $\tilde{u}$ is defined locally with the irreducible component which has the largest dimension over all irreducible components containing $\tilde{u}$. A point in $W$ is called singular or nonsmooth if it is not a smooth point of $W$, and the set of all singular points of $W$ is denoted as $W_{\text{sing}}$. Moreover, the $W$ is said to be smooth if $W_{\text{sing}} = \emptyset$. For two projective varieties $W_1$ and $W_2$, we say they intersect transversely at $\tilde{u} \in (W_1 \cap W_2) \setminus (\{W_1\}_{\text{sing}} \cup \{W_2\}_{\text{sing}})$ if

$$\dim (\mathbb{T}_{\tilde{u}} W_1 + \mathbb{T}_{\tilde{u}} W_2) = \dim \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_N} = n.$$ 

For every irreducible component $W'$ in $W_1 \cap W_2$, if the intersection

$$W' \cap (\{W_1\}_{\text{sing}} \cup \{W_2\}_{\text{sing}})$$

is a proper subset of $W'$, and $W_1$ and $W_2$ intersect transversely at every point in

$$(W_1 \cap W_2) \setminus (\{W_1\}_{\text{sing}} \cup \{W_2\}_{\text{sing}}),$$

then they are said to intersect transversely. If $W_1 \cap W_2 = \emptyset$, then they intersect transversely. In general, two varieties $W_1$ and $W_2$ intersect transversely if $\text{codim } W_1 + \text{codim } W_2 = \text{codim } (W_1 \cap W_2)$ and their intersection is smooth outside their singular loci. We refer to [10, 13, 38] for more details on smoothness and transversal intersections for varieties.

Bertini’s Theorem considers the smoothness of intersections. In the following, we refer to [38] for the definition of quasi-projective varieties and regular maps.

**Theorem 2.1** (Bertini’s Theorem [13]). If $X$ is any quasi-projective variety, $f : X \to \mathbb{P}^n$ a regular map, $H \subseteq \mathbb{P}^n$ a general hyperplane, and $Y = f^{-1}(H)$, then

$$Y_{\text{sing}} = X_{\text{sing}} \cap Y.$$
As a corollary of Bertini’s Theorem, the following result from [28] is frequently used in this paper.

**Proposition 2.2.** ([28 Theorem A.1]) Let $W$ be a $k$-dimensional multi-projective variety in $\mathbb{P}^v$. Let $\mathbb{P}^m$ be a projective space parameterizing hypersurfaces in $\mathbb{P}^v$. We denote by $Z(\beta)$ the hypersurface in $\mathbb{P}^v$ that is parameterized by $\beta \in \mathbb{P}^m$. Let $\mathcal{H}$ be the common points of these hypersurfaces, i.e., $\mathcal{H} := \cap_{\beta \in \mathbb{P}^m} Z(\beta)$. If $V \setminus \mathcal{H} \neq \emptyset$ for every irreducible component $V$ of $W$, then for a general $\beta \in \mathbb{P}^m$, we have $\dim(W \cap \mathcal{H}) = k - 1$, and

$$(W \cap Z(\beta))_{\text{sing}} \subseteq (W_{\text{sing}} \cup \mathcal{H}) \cap Z(\beta).$$

By Proposition 2.2 if we let $p_1(\tilde{x}), \ldots, p_k(\tilde{x})$ be general multi-homogeneous polynomials in $\mathbb{P}^v$, then hypersurfaces defined by each $p_i(\tilde{x}) = 0$ intersect transversely, i.e., the intersection is smooth and its dimension equals $n - k$. Moreover, let $W \subseteq \mathbb{P}^v$ be a multi-projective variety, and let $p_1, \ldots, p_k$ be multi-homogeneous polynomials such that the coefficients of each $p_i$ are parameterized by $\beta_i \in \mathbb{P}^m_i$. By implementing Proposition 2.2 repeatedly we may conclude: If for every $i = 1, \ldots, k$, the $p_i(\tilde{x}) = 0$ do not have any fixed point in $W \cap \{ \tilde{x} \in \mathbb{P}^v : p_1(\tilde{x}) = \cdots = p_{i-1}(\tilde{x}) = 0 \}$ when we vary $\beta_i$, then $W$ intersects $\{ \tilde{x} \in \mathbb{P}^v : p_1(\tilde{x}) = \cdots = p_k(\tilde{x}) = 0 \}$ transversely for a general choice of $\beta_1, \ldots, \beta_k$.

For an affine variety $X \subseteq \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_N}$, the embedding of $X$ in the multi-projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_N}$ is consisting of all points

$$F(x) := (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N)$$

with $\tilde{x}_i := (1, x_{i,1}, x_{i,2}, \ldots, x_{i,n_i})$ and $(x_1, \ldots, x_N) \in X$. The set of all $F(x)$ with $x \in X$ is denoted as $F(X)$. For convenience, we say that a multi-projective variety $Y$ contains the affine variety $X$ if it contains $F(X)$.

3. Finiteness of the Fritz-John Set

The set of points $x$ satisfying (1.4) is called the Fritz-John set. In this section, we prove that the Fritz-John set is finite when the polynomials are generic.

Suppose $E_i$ is the active labeling set for the $i$th player’s optimization problem $F_i(x_{-i})$, which is parameterized by $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. Note that $\mathcal{E}_{i,1} \subseteq E_i \subseteq \mathcal{E}_{i,1} \cup \mathcal{E}_{i,2}$. For convenience of discussion, we write that

$$E_i = \{1, \ldots, m_i\}$$

for the rest of this section. For each $i = 1, \ldots, N$, denote the affine varieties

$$U_i := \{x \in \mathbb{C}^n : g_{i,j}(x) = 0 (j \in [m_i])\},$$

$$U := U_1 \cap \cdots \cap U_N.$$

Consider the determinantal variety

$$\mathcal{V}_i := \{ x \in \mathbb{C}^n : \text{rank } [\nabla_{x_i} f_i(x) \quad \nabla_{x_i} g_{i,1}(x) \quad \ldots \quad \nabla_{x_i} g_{i,m_i}(x)] \leq m_i \}.$$

For each $i = 1, \ldots, N$, denote the intersection $W_i := U_i \cap \mathcal{V}_i$, then

$$(3.1) \quad W := W_1 \cap \cdots \cap W_N$$

is the set of all complex Fritz-John points. We are going to show that $W$ is a finite set when $f_i, g_{i,j}$ are generic polynomials.

We need to consider the homogenization of the varieties $U, V_i$ and $W$. Denote

$$\tilde{x}_i := (x_{i,0}, x_{i,1}, \ldots, x_{i,n_i}), \quad \tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_N).$$
Let \( d_{i,0,k} \) (resp., \( d_{i,j,k} \)) be the degree of \( f_i \) (resp., \( g_{i,j} \)) in \( x_k \). The degree tuples
\[
d_i,0 := (d_{i,0,1}, \ldots, d_{i,0,N}), \quad d_i := (d_{i,j,1}, \ldots, d_{i,j,N})
\]
are called the multi-degrees of \( f_i, g_{i,j} \) respectively. The multi-homogenization of the \( i \)th player’s objective \( f_i(x_i, x_{-i}) \) is
\[
\tilde{f}_i(\tilde{x}_i, \tilde{x}_{-i}) := f_i(x_1/x_{1,0}, \ldots, x_N/x_{N,0}) \cdot \prod_{k=1}^N (x_{k,0})^{d_{i,k}}.
\]
The multi-homogenization of \( g_{i,j}(x_i, x_{-i}) \) is given similarly. For convenience of notation, denote that
\[
\nu := (n_1, \ldots, n_N), \quad \nu_{-i} := (n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_N),
\]
\[
\mathbb{P}^{\nu_{-i}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{i-1}} \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_N}.
\]
Consider the multi-projective varieties
\[
\tilde{U}_i := \{ \tilde{x} \in \mathbb{P}^{\nu_{-i}} : \tilde{g}_{i,j}(\tilde{x}) = 0 \ (j \in [m_i]) \},
\]
\[
\tilde{U} := \tilde{U}_1 \cap \cdots \cap \tilde{U}_N.
\]
When \( g_{i,j} \) are generic polynomials in \( x \), the codimension of \( \tilde{U}_i \) is \( m_i \), by Proposition \ref{prop:1} and \( \tilde{U} \) has the codimension \( m_1 + \cdots + m_N \). In the following we consider the general case that
\[
m_1 + \cdots + m_N < n,
\]
because otherwise the variety \( \tilde{U} \) is empty or zero-dimensional (i.e., it is finite). When all \( f_i \) and \( g_{i,j} \) are generic, the multi-degree of \( \frac{\partial \tilde{f}_i(x)}{\partial x_k} \) (resp., \( \frac{\partial \tilde{g}_{i,j}(x)}{\partial x_k} \)) equals \( d_{i,0} - e_i \) (resp., \( d_{i,j} - e_i \)). Here, \( e_i \) denotes the vector of all zeros except the \( i \)th entry being 1. Define the multi-projective variety
\[
\tilde{V}_i := \left\{ \tilde{x} \in \mathbb{P}^{\nu} : \text{rank} \tilde{J}_i(\tilde{x}_i, \tilde{x}_{-i}) \leq m_i \right\},
\]
where \( \tilde{J}_i(\tilde{x}_i, \tilde{x}_{-i}) \) is the homogenized Jacobian matrix:
\[
\tilde{J}_i(\tilde{x}_i, \tilde{x}_{-i}) := \begin{bmatrix}
\frac{\partial \tilde{f}_i(\tilde{x})}{\partial x_{i,1}} & \frac{\partial \tilde{g}_{i,1}(\tilde{x})}{\partial x_{i,1}} & \cdots & \frac{\partial \tilde{g}_{i,m_i}(\tilde{x})}{\partial x_{i,1}} \\
\frac{\partial \tilde{f}_i(\tilde{x})}{\partial x_{i,2}} & \frac{\partial \tilde{g}_{i,1}(\tilde{x})}{\partial x_{i,2}} & \cdots & \frac{\partial \tilde{g}_{i,m_i}(\tilde{x})}{\partial x_{i,2}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \tilde{f}_i(\tilde{x})}{\partial x_{i,n_i}} & \frac{\partial \tilde{g}_{i,1}(\tilde{x})}{\partial x_{i,n_i}} & \cdots & \frac{\partial \tilde{g}_{i,m_i}(\tilde{x})}{\partial x_{i,n_i}}
\end{bmatrix}.
\]
For convenience, denote the degrees \( (j \neq 1, \ldots, m_i) \)
\[
b_{i,j} := \begin{cases} d_{i,0,j} + d_{i,1,j} + \cdots + d_{i,N,j}, & j \neq i, \\ d_{i,0,j} + d_{i,1,j} + \cdots + d_{i,N,j} - m_i - 1, & j = i. \end{cases}
\]
Then one can see that all \( (m_i + 1) \)-by-\( (m_i + 1) \) minors of \( \tilde{J}_i(\tilde{x}_i, \tilde{x}_{-i}) \) are multi-homogeneous of the multi-degree \( (b_{i,1}, \ldots, b_{i,N}) \). Let \( \tilde{J}^p_i(\tilde{x}_i, \tilde{x}_{-i}) \) denote the sub-matrix of \( \tilde{J}_i(\tilde{x}_i, \tilde{x}_{-i}) \) consisting of the right hand side \( m_i \) columns and let
\[
\tilde{W}^p_i := \{ \tilde{x} \in \mathbb{P}^{\nu} : \text{rank} \tilde{J}^p_i(\tilde{x}_i, \tilde{x}_{-i}) < m_i \},
\]
\[
\tilde{W}_i := \tilde{V}_i \cap \tilde{U}_i, \quad \tilde{W}^p_i := \tilde{W}^p_i \cap \tilde{U}_i.
\]
The \( \tilde{W}_i \) and \( \tilde{W}^p_i \) are multi-projective subvarieties of \( \tilde{U}_i \). Then
\[
\tilde{W} := \tilde{W}_1 \cap \cdots \cap \tilde{W}_N
\]
is a multi-projective variety that contains $\mathcal{W}$. When defining polynomials are generic, the set $\mathcal{W}$ is finite, as shown in the following.

**Theorem 3.1.** For every $i, j$, let $d_{i,j} := (d_{i,j,1}, \ldots, d_{i,j,N})$ be a nonzero tuple of degrees. If every $f_i$ is a generic polynomial of multi-degree $d_{i,0}$, and $g_{i,j}$ is a generic polynomial of multi-degree $d_{i,j}$, then we have:

(i) The Fritz-John system [14] has finitely many complex solutions, and hence the GNEP has finitely many Fritz-John points.

(ii) For every Fritz-John point $u = (u_1, \ldots, u_N)$, the linear independence constraint qualification condition holds for each minimizer of the optimization $F(u_{-1})$, and hence every Fritz-John point is a KKT point.

**Proof.** (i) First, we show that if $m_i \leq n_i$, then the intersection $\mathcal{W}_i := \tilde{U}_i \cap \mathcal{V}_i$ has the codimension $n_i$. Consider the projection map 

$$p_i : \mathcal{W}_i \to \mathbb{P}^{n_i-1}, \quad (\tilde{x}_i, \tilde{x}_{-i}) \mapsto \tilde{x}_{-i}.$$ 

When $g_{i,j}$ are all generic, by [13] Proposition 17.25, we have 

$$\text{codim} \tilde{U}_i = m_i, \quad \text{codim} \mathcal{V}_i \leq n_i - m_i.$$ 

So, the codimension of $\mathcal{W}_i$ is at most $n_i$. Let $\tilde{u}_{-i}$ be an arbitrary point in $\mathbb{P}^{n_i-1}$ and let 

$$Z_i = \{ \tilde{x} \in \mathbb{P}^n : \tilde{x}_{-i} = \tilde{u}_{-i} \}.$$ 

Then, by [38] Theorem 1.24], we know 

$$\tilde{U}_i \cap \mathcal{V}_i \cap Z_i \neq \emptyset.$$ 

In other words, $p_i(\mathcal{W}_i) = \mathbb{P}^{n_i-1}$. Therefore, by [38] Theorem 1.25, we have 

$$\text{dim} \mathcal{W}_i \leq \text{dim}(p_i(\mathcal{W}_i)) + \text{dim} F = n - n_i + \text{dim} F$$

for all $\tilde{u}_{-i} \in \mathbb{P}^{n_i-1}$ and for every irreducible component $F$ of the fibre $p_{i}^{-1}(\tilde{u}_{-i})$. For the given $\tilde{u}_{-i}$, the fibre is 

$$p_i^{-1}((\tilde{u}_{-i})) = \left\{ \tilde{x}_i \in \mathbb{P}^{n_i} \mid \hat{g}_{i,1}(\tilde{x}_i, \tilde{u}_{-i}) = \cdots = \hat{g}_{i,m_i}(\tilde{x}_i, \tilde{u}_{-i}) = 0, \quad \text{rank} \, J_i(\tilde{x}_i, \tilde{u}_{-i}) \leq m_i \right\}.$$ 

It is zero-dimensional for generic $\tilde{u}_{-i} \in \mathbb{P}^{n_i-1}$ and for generic polynomials $f_i, g_{i,j}$, by [28] Proposition 2.1. So dim $\mathcal{W}_i \leq n - n_i$, and we conclude

$$\text{dim} \mathcal{W}_i = n - n_i, \quad \text{codim} \mathcal{W}_i = n_i.$$ 

Second, we show that the codimension of $\mathcal{W}^o_i = \tilde{U}_i \cap \mathcal{V}^o_i$ equals $n_i + 1$. For the given $\tilde{u}_{-i} \in \mathbb{P}^{n_i-1}$ we have 

$$\tilde{U}_i \cap \mathcal{V}^o_i \cap Z_i = \left\{ \tilde{x}_i \in \mathbb{P}^{n_i} \mid \hat{g}_{i,1}(\tilde{x}_i, \tilde{u}_{-i}) = \cdots = \hat{g}_{i,m_i}(\tilde{x}_i, \tilde{u}_{-i}) = 0, \quad \text{rank} \, J_i^{o}(\tilde{x}_i, \tilde{u}_{-i}) < m_i \right\}.$$ 

By [28] Proposition 2.1, this intersection is empty when every $g_{i,j}$ is a generic polynomial and $\tilde{u}_{-i}$ is generic in $\mathbb{P}^{n_i-1}$. Therefore, 

$$\text{dim}(\mathcal{W}^o_i) < n - \text{dim} Z_i = n - n_i.$$ 

So, we know $\mathcal{W}^o_i$ is a proper closed subset of $\mathcal{W}_i$. Moreover, by [13] Proposition 17.25, $\text{codim} \mathcal{W}^o_i \leq n_i + 1$, so 

$$\text{codim} \mathcal{W}^o_i = n_i + 1.$$
Theorem 1.25], for all \( \tilde{x} \) we have codim \( \hat{C}_i \) \( \subseteq \) dim \( \mathcal{Y} \). Therefore, by \([38, Proposition 17.25]\), we have \( \text{(3.7)} \ \tilde{\lambda} \in \text{dim}(\tilde{C}) \in \mathbb{C}^{m_i+1} \) be the vector of variables. Consider the intersection of hypersurfaces in \( \mathbb{P}^r \times \mathbb{P}^{m_i} \) given by
\[ g_i, 1 = 0, \ldots, g_i, m_i = 0, \ J_i \cdot \tilde{\lambda}_i = 0. \]
Denote this multi-projective variety by \( C_i \). Then \( \tilde{x} \in \tilde{W}_i \) if and only if \( \tilde{x} \in p(C_i) \), where \( p \) is the projection from \( \mathbb{P}^r \times \mathbb{P}^{m_i} \) to \( \mathbb{P}^r \). By \([38, Theorem 1.25]\), for all \( \tilde{x} \in \tilde{W}_i \) and each irreducible component \( \mathcal{Y} \) of the fibre \( p^{-1}(\tilde{x}) \),
\[ \dim C_i \leq \dim \tilde{W}_i + \dim G, \]
and the equality above hold when \( \tilde{x} \) lies in an open subset of \( \tilde{W}_i \). Note that the \( \tilde{W}_i^o \) is a proper closed subset of \( \tilde{W}_i \). When \( \tilde{x} \in \tilde{W}_i \setminus \tilde{W}_i^o \), there exists the unique \( \tilde{\lambda}_i \) such that \( J_i \cdot \tilde{\lambda}_i = 0 \), since the columns of \( J_i^T(\tilde{x}) \) are linear independent. So, we have \( \dim C_i = \dim \tilde{W}_i = n - n_i \), which implies that codim \( C_i \) \( \subseteq \) \( p^{-1}(\tilde{x}) \). Therefore, by \([13, Proposition 17.25]\), we have
\[ \dim(\mathcal{C}_k \cap \mathcal{C}_{k-1}^i - 1 \geq \dim p(\mathcal{C}_k \cap \hat{C}_{k-1}^i) = \dim(\mathcal{C}_k \cap \hat{C}_{k-1}^i)) \]
\[ \dim(\mathcal{C}_k \cap \mathcal{C}_{k-1}^i) \geq \dim \mathcal{C}_k \cap \hat{C}_{k-1}^i = \dim(\mathcal{C}_k \cap \hat{C}_{k-1}^i)) \]
\[ \text{dim}(\bigcap_{i=1}^{k} \tilde{W}_i) = n - \sum_{i=1}^{k} n_i. \]
Therefore, by \([13, Proposition 17.25]\), we have
\[ \dim(\bigcap_{i=1}^{k} \tilde{W}_i) = n - \sum_{i=1}^{k} n_i. \]
From \([38, Proposition 17.25]\), it is clear that \( \dim \tilde{W}_i = n - n_i \). So we conclude the \( \bigcap_{i=1}^{k} \tilde{W}_i \) is proper by induction. Note that \( n = n_1 + \cdots + n_N \), so we have
\[ \dim(\bigcap_{i=1}^{N} \tilde{W}_i) = n - \sum_{i=1}^{N} n_i = 0, \]
which implies the finiteness of \( \tilde{W} \).

(ii) For \( x \in \mathbb{R}^n \), if \( \text{rank} J_i^T(\mathbb{P}(x)) = m_i \), then the LICQ for \( F_i(x-i) \) hold at \( x_i \). Therefore, it suffices to show that \( \tilde{W}_i^o \cap (\bigcap_{i \neq j \in [N]} \tilde{W}_j) = \emptyset \) is empty for every \( i \in [N] \). Without loss of generality, we fix \( i = 1 \) and show the emptiness of \( \tilde{W}_1^o \cap \tilde{W}_2 \cap \cdots \cap \tilde{W}_N \). In item (i), we showed that codim \( \tilde{W}_i^o = n_1 + 1 \). Besides that, for general polynomials \( f_i \) and \( g_{i,j} \), the
\[ C_{-1} := \{ (\tilde{x}, \lambda_2, \ldots, \lambda_N) \mid \bar{g}_{i,1}(\tilde{x}) = 0, \ldots, \bar{g}_{i,m_i}(\tilde{x}) = 0, \ J_i \cdot \tilde{\lambda}_i = 0 \ (i = 2, \ldots, N) \}. \]
is a complete intersection of hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^{m_2} \times \cdots \times \mathbb{P}^{m_N}$. Its codimension equals $n - n_1 + \sum_{i=2}^N m_i$, and we obtain the emptiness for $\mathcal{C}_z \cap W_{i,1}^\Phi$ by varying coefficients of $f_i$ and $g_{ij}$ with $i > 1$, and applying Proposition 2.2 repeatedly. Moreover, the emptiness for $\mathcal{W}_{i,1}^\Phi \cap \mathcal{W}_2 \cap \cdots \cap \mathcal{W}_N$ follows from the fact that it is the image under the projection mapping $(\tilde{x}, \lambda_2, \ldots, \lambda_N)$ to $\tilde{x}$ of $\mathcal{C}_z \cap \mathcal{W}_{i,1}^\Phi$.

4. Algebraic degrees of multi-projective varieties

For a multi-projective variety $X$ in $\mathbb{P}^n := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_N}$, the algebraic degree of $X$ is a nonnegative integer vector, labeled by a tuple

$$l := (l_1, \ldots, l_N) \in \mathbb{N}^N,$$

such that each $(\deg X)_l$ counts the number of points in the intersection

$$\bigcap_{i=1}^N \left\{ \tilde{x} \in X : (u_{i,1})^T \tilde{x}_i = \cdots = (u_{i,l_i})^T \tilde{x}_i = 0 \right\},$$

where $u_{i,1}, \ldots, u_{i,l_i}$ are general vectors in $\mathbb{P}^{n_i}$. For the case that $\dim X = 0$, the $\deg X$ is just a nonnegative integer which counts the cardinality $|X|$ of $X$.

Let $k$ be a positive integer and $\delta := (\delta_1, \ldots, \delta_N) \in \mathbb{N}^N$ such that $\delta_1 + \cdots + \delta_N = k$, and let $\mathbb{Z}_2 := \{0, 1\}$ be the binary set. Denote the set

$$\mathbb{Z}_2^{[\delta \times k]} := \left\{ A = (A_{i,j}) \in \mathbb{Z}_2^{N \times k} \bigg| \sum_{j=1}^k A_{i,j} = \delta_i (i \in [N]), \sum_{i=1}^N A_{i,j} = 1 (j \in [k]) \right\}.$$

For the matrix $A \in \mathbb{Z}_2^{[\delta \times k]}$, we denote the $j$th column by $A_{i,j}$. Let $z := (z_1, \ldots, z_k)$ be a tuple of vector variables with each $z_i = (z_{i,1}, \ldots, z_{i,N})$. Define the polynomial in $z$

$$A_\delta(z_1, \ldots, z_k) := \sum_{A \in \mathbb{Z}_2^{[\delta \times k]}} (z_1)^{A_{1,1}} (z_2)^{A_{2,2}} \cdots (z_k)^{A_{k,k}}.$$

In the above, the $z_i^{A_{i,i}} := (z_{i,1})^{A_{i,1}} (z_{i,2})^{A_{i,2}} \cdots (z_{i,N})^{A_{i,N}}$. One can check that $A_\delta(z_1, \ldots, z_k)$ is the coefficient of $t_1^{\delta_1} t_2^{\delta_2} \cdots t_N^{\delta_N}$ in the product

$$\prod_{i=1}^k (z_{i,1} t_1 + \cdots + z_{i,N} t_N).$$

We study the algebraic degree for intersections of multi-projective varieties. First, we consider complete intersections in $\mathbb{P}^n$. For a given nonnegative integer $s$, we denote

$$[s]^{(\nu)} := \{ l \in \mathbb{N}^N : |l| = s, \ l \leq \nu \}.$$

It stands for the set of all possible labels of the algebraic degree for any $s$-dimensional multi-projective variety in $\mathbb{P}^n$.

Lemma 4.1. Let $\nu := (n_1, \ldots, n_N)$ and $n = n_1 + \cdots + n_N$. Suppose $h_1, \ldots, h_k$ are general multi-homogeneous polynomials in $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_N)$, and the multi-degree for each $h_i$ is $d_i = (d_{i,1}, \ldots, d_{i,N})$. Consider the multi-projective variety

$$\mathcal{H} := \{ \tilde{x} \in \mathbb{P}^n : h_1(\tilde{x}) = \cdots = h_k(\tilde{x}) = 0 \}. $$
(i) If \( n = k \), then \( \dim \mathcal{H} = 0 \) and
\[
\deg \mathcal{H} = A_\nu(d_1, \ldots, d_k).
\]

(ii) If \( n > k \), then \( \dim \mathcal{H} = n - k \) and for each \( l \in [n - k]^{(\nu)} \),
\[
(\deg \mathcal{H})_l = A_{\nu - l}(d_1, \ldots, d_k).
\]

**Proof.** (i) When \( k = n \), the \( \mathcal{H} \) is a zero-dimensional multi-projective variety, since \( h_1, \ldots, h_k \) are general. The algebraic degree of \( \mathcal{H} \) equals the coefficient of \( t^\nu \) in the product
\[
\prod_{i=1}^{n} (t_1^{d_{i,1}} + \cdots + t_N d_{i,N}).
\]
This is shown in [38, Chapter 4, Section 2.1]. The coefficient of \( t^\nu \) in \((4.4)\) coincides with \( A_\nu(d_1, \ldots, d_k) \).

(ii) When \( k < n \) and \( h_1, \ldots, h_k \) are general, the variety \( \mathcal{H} \) is smooth and \( \dim \mathcal{H} = n - k \), by Proposition 2.2. For \( l = (l_1, \ldots, l_N) \in \mathbb{N}^N \) such that \( |l| = n - k \), the \( (\deg H)_l \) counts the number of points in the intersection
\[
\bigcap_{i=1}^{N} \{ \tilde{x} \in \mathcal{H} : u^{T}_{i,j} \tilde{x}_N = \cdots = u^{T}_{i,N} \tilde{x}_i = 0 \},
\]
for generic vectors \( u_{i,j} \in \mathbb{P}^{n_i} \). The multi-degree of the linear form \( u^{T}_{i,j} \tilde{x}_i \) is the unit vector \( e_i \). Furthermore, since these linear forms are general and \( \dim \mathcal{H} = l_1 + \cdots + l_N \), the intersection \( \mathcal{H}_l \) is zero dimensional. By (i), its algebraic degree is the coefficient of \( t^\nu \) in the product
\[
t_1^{l_1} \cdots t_N^{l_N} \prod_{i=1}^{k} (t_1^{d_{i,1}} + \cdots + t_N d_{i,N}),
\]
which equals the coefficient of \( t^{\nu-l} \) in \( \prod_{i=1}^{k} (t_1^{d_{i,1}} + \cdots + t_N d_{i,N}) \). Therefore, we get
\[
(\deg\mathcal{H})_l = A_{\nu - l}(d_1, \ldots, d_k).
\]

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two multi-projective varieties in \( \mathbb{P}^\nu \) such that \( \dim \mathcal{X} + \dim \mathcal{Y} = n \) and their intersection is transversal. Then the algebraic degree for the intersection \( \mathcal{X} \cap \mathcal{Y} \) is given by (the set \( [\dim \mathcal{X}]^{(\nu)} \) is given as in \((4.3)\))
\[
\deg \mathcal{X} \cap \mathcal{Y} = \sum_{\ell \in [\dim \mathcal{X}]^{(\nu)}} (\deg(\mathcal{X}))_\ell \cdot (\deg(\mathcal{Y}))_{\nu - \ell}.
\]
This is a generalization of the Bezout’s Theorem to the multi-projective spaces (see [40] for more details). Furthermore, we have the following useful lemma:

**Lemma 4.2.** Suppose \( \mathcal{X} \) and \( \mathcal{Y} \) are two multi-projective varieties in \( \mathbb{P}^\nu \) such that \( \text{codim} \mathcal{X} = k_1 \) and \( \text{codim} \mathcal{Y} = k_2 \), and both of them are equal-dimensional. If the intersection \( \mathcal{X} \cap \mathcal{Y} \) is transversal, then \( \dim \mathcal{X} \cap \mathcal{Y} = n - k_1 - k_2 \), and for each \( l \in [n - k_1 - k_2]^{(\nu)} \),
\[
(\deg \mathcal{X} \cap \mathcal{Y})_l = \sum_{\ell^{(1)} \in [n - k_1]^{(\nu)}} (\deg \mathcal{X})_{\ell^{(1)}} \cdot (\deg \mathcal{Y})_{\nu + l - \ell^{(1)}}.
\]
One may check that the polynomials in (4.7) equals the coefficient of \( \hat{\nu} \). By definition, for any fixed \( l(2) \in [k_2]^{(\nu)} \), we have
\[
(\deg \hat{\chi})_{l(2)} = (\deg \chi \cap \nu)_{l(2)} = (\deg \chi)_{l(2)}.
\]
Therefore, we conclude this lemma by letting \( l(1) := l(2) + l \).

Let \( \delta := (\delta_1, \ldots, \delta_N) \) be a vector in \( \mathbb{N}^N \), \( z_i = (z_{i,1}, \ldots, z_{i,N}) \) be a vector of variables, and \( s \) be an integer such that \( s \geq r := |\delta| \). For each \( i = 1, \ldots, s \), define the function
\[
B_\delta(z_1, \ldots, z_s) := \sum_{1 \leq i_1 < \cdots < i_r \leq s} A_\delta(z_{i_1}, \ldots, z_{i_r}).
\]
Note that \( B_\delta(z_1, \ldots, z_s) \) equals the coefficient of \( t^\delta = t_1^{\delta_1} \cdots t_N^{\delta_N} \) in the product
\[
\prod_{i=1}^N (1 + t_i z_{i,1} + \cdots + t_{N} z_{i,N}).
\]
For convenience, denote
\[
\mathbb{N}^{[\delta \times s]} := \{ B \in \mathbb{N}^{N \times s} \mid B_{1,1} + B_{1,2} + \cdots + B_{1,s} = \delta_i (i \in [N]) \}.
\]
The \( j \)th column of \( B \) is \( B_{.,j} \). We define
\[
S_\delta(z_1, \ldots, z_s) = \sum_{B \in \mathbb{N}^{[\delta \times s]}} \prod_{j=1}^s \left( \frac{|B_{.,j}|}{B_{1,j}} \right)^{B_{1,j}}.
\]
In the above, the \((|B_{.,j}|/B_{1,j})\) is the multi-monomial coefficient
\[
\left( \frac{|B_{1,j}|}{B_{1,j}} \right) = \frac{|B_{.,j}|}{B_{1,j}! B_{2,j}! \cdots B_{N,j}!}.
\]
One may check that \( S_\delta(z_1, \ldots, z_s) \) equals the coefficient of \( t^\delta = t_1^{\delta_1} \cdots t_N^{\delta_N} \) in
\[
\prod_{i=1}^s \frac{1}{1 - \sum_{j=1}^N z_{i,j} t_j} = \prod_{i=1}^s \left( 1 + \sum_{j=1}^N z_{i,j} t_j \right)^2 + \cdots.
\]
In the following proposition, we study the algebraic degree of multi-projective varieties defined by rank deficiency of general polynomial matrices. Our main tool is the Thom-Porteous formula [10, Theorem 14.4], which considers the degeneracy locus of homomorphisms between vector bundles. We refer to [10] for more details.

**Proposition 4.3.** Let \( d_1 \leq \cdots \leq d_s \in \mathbb{N}^N \) be vectors of positive integers, and let \( M \) be an \( r \times s \) matrix whose entries \( m_{i,j} \) are general multi-homogeneous polynomials in \( \mathbb{C}[\vec{x}_1, \ldots, \vec{x}_N] \) with multi-degrees \( d_j \). For the given \( l \in \mathbb{N}^N \) and \( \nu = (n_1, \ldots, n_N) \), the polynomials \( B_{\nu-l} \) and \( S_{\nu-l} \) are given in (4.8) and (4.9) respectively. Consider the multi-projective variety \( \chi_p := \{ \vec{x} \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_N} : \text{rank } M(\vec{x}) \leq p \} \).
(i) If \( r \leq s \), then \( \dim \mathcal{X}_{r-1} = n - s + r - 1 \) and for each \( l \in [n-s+r-1]^\nu \),
\[
(\deg \mathcal{X}_{r-1})_l = B_{\nu-\ell}(d_1, \ldots, d_s).
\]
(ii) If \( t > s \), then \( \dim \mathcal{X}_{r-1} = n - r + s - 1 \) and for each \( l \in [n-r+s-1]^\nu \),
\[
(\deg \mathcal{X}_{s-1})_l = S_{\nu-\ell}(d_1, \ldots, d_s).
\]

**Proof.** Define the direct sum of line bundles
\[
E := \bigoplus_{r \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^r}(d), \quad F := \bigoplus_{r \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^r}(d_1) \oplus \mathcal{O}_{\mathbb{P}^r}(d_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^r}(d_s).
\]

In the above, the \( \mathcal{O}_{\mathbb{P}^r}(d) \) is the line bundle of regular functions whose global sections are multi-homogeneous polynomials of multi-degree \( d \) in \( \mathbb{P}^r \), and \( \mathcal{O}_{\mathbb{P}^r} := \mathcal{O}_{\mathbb{P}^r}(\mathbf{0}) \), where \( \mathbf{0} \) is the zero vector. Then, for a given map \( \rho : \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}(d_i) \), there exists a multi-homogeneous polynomial \( p \) of multi-degree \( d_i \) such that
\[
\rho(\tau) = p \cdot \tau \quad \text{for all} \quad p \in \mathcal{O}_{\mathbb{P}^r}.
\]

So the map \( \tilde{x} \to M(\tilde{x}) \) defines a homomorphism \( \sigma : E \to F \) of vector bundles such that for all \( (\tau_1, \ldots, \tau_r) \in E \),
\[
\sigma(\tau_1, \ldots, \tau_r) = \left( \sum_{i=1}^r m_{i,1} \tau_i, \sum_{i=1}^r m_{i,2} \tau_i, \ldots, \sum_{i=1}^r m_{i,s} \tau_i \right).
\]

The \( \mathcal{X}_p \) is equivalent to the degeneracy locus \( \{ x \in \mathbb{P}^r : \text{rank}(\tilde{x}) \leq p \} \), whose dimension is \( n - (r-p)(r-p) \) from the genericity of entries of \( M \). Let
\[
c(F) = (1 + d_{1,1} + \cdots + d_{1,N}) \cdot \cdots \cdot (1 + d_{s,1N})
\]
Denote by \( c_i \) the sum of terms in the above product whose total degree in \( t \) equals \( i \). Then \( c(F) = c_0 + c_1 + \cdots + c_r \).

(i) Assume \( r \leq s \). By Thom-Porteous formula (see Theorem 14.4 and Example 14.4.1 in [1]), the \( \ell \)-degree of \( \mathcal{X}_{r-1} \) is the coefficient of \( t^{\nu} \) in the product \( t^\ell \cdot c_{s-r+1}(F) \), where \( c_{s-r+1}(F) \) is the sum of all terms in \( c(F) \) whose total degree in \( t \) equals \( s-r+1 \). Therefore, \( (\deg \mathcal{X})_l \) equals the coefficient of \( t^{\nu-\ell} \) in \( c_{s-r+1}(F) \), which coincides with \( B_{\nu-\ell}(1, \ldots, s) \).

(ii) When \( r > s \) and \( p = s - 1 \), instead of using the Thom-Porteous formula, we assume \( i \) and proceed by induction on \( k > s \). Assume there exists a positive integer \( k \geq s + 1 \) such that the formula hold for all \( r < k \). Then, let \( M_i \) be the submatrix of \( M \) consisting of the first \( r - i \) rows, and \( N_i \) be the submatrix consisting of the \( r - s \) to \( r - j \) rows. Let \( X_j \) be the variety where the matrix \( M_i \) has rank less than \( s \), and \( Y_j \) be the variety such that \( \text{rank} N_j \leq s - j \). For each \( i \), \( \dim X_i = n - r + i + s - 1 \), and \( \dim Y_i = n - i \). Since the intersection \( X_i \cap Y_{i-1} \) is transversal, \( \dim X_i \cap Y_{i-1} = n - r + s - 1 \). Besides that, for \( r_i, s_i \in \mathbb{N} \) such that \( |r_i| = n - r + i + s - 1 \) and \( |s_i| = n - i \),
\[
(\deg X_i)_{r_i} = S_{\nu-r_i}(d_1, \ldots, d_s), \quad (\deg Y_i)_{s_i} = B_{\nu-s_i}(d_1, \ldots, d_s).
\]

The first degree is obtained by induction, and the second equation follows (i). By Lemma 12, \( \deg(X_i \cap Y_{i-1})_l = \sum_{r_i + s_i = l + \nu} S_{\nu-r_i}(d_1, \ldots, d_f) \cdot B_{\nu-s_i}(d_1, \ldots, d_s) \).
Thus we have

\[(\deg \mathcal{X}_{s-1})_l = \sum_{i=1}^{s} (-1)^{i-1} \sum_{r_i+s_i=l+\nu} S_{\nu-r_i}(d_1, \ldots, d_s) \cdot B_{\nu-s_i}(d_1, \ldots, d_s).\]

Consider the identity

\[1 = \prod_{i=1}^{s} \left(1 + t_1d_{i,1} + \cdots + t_Nd_{i,N}\right)/ \prod_{i=1}^{s} \left(1 + t_1d_{i,1} + \cdots + t_Nd_{i,N}\right),\]

\[= \sum_{i=1}^{s} (1 + t_1d_{i,1} + \cdots + t_Nd_{i,N}) (1 - \sum_{j=1}^{N} d_{i,j}t_j + (\sum_{j=1}^{N} d_{i,j}t_j)^2 - \cdots),\]

\[= (1 + \sum_{\delta} B_{\delta}(d_1, \ldots, d_s)t^\delta)(1 + \sum_{\delta} (-1)^{|\delta|} S_{\delta}(d_1, \ldots, d_s)t^\delta).\]

By comparing the coefficient of \(t^{\nu-l}\), we get

\[S_{\nu-l}(d_1, \ldots, d_s) = \sum_{i=1}^{s} (-1)^{i-1} \sum_{r_i+s_i=l+\nu} S_{\nu-r_i}(d_1, \ldots, d_s) \cdot B_{\nu-s_i}(d_1, \ldots, d_s),\]

which implies that

\[\deg(\mathcal{X}_{s-1})_l = S_{\nu-l}(d_1, \ldots, d_s).\]

\[\square\]

5. **Algebraic degrees of GNEPs**

For a GNEP given by generic polynomials, its algebraic degree counts the number of complex solutions to the Fritz-John system (1.3). This section gives formulae for algebraic degrees of GNEPs.

5.1. **The case of given active constraints.** In this subsection, we consider the Fritz-John system (1.3) for the case that the active label sets \(E_i\) are

\[E_i = \{1, \ldots, m_i\}, \quad i = 1, \ldots, N.\]

Assume the polynomials \(f_i\) and \(g_{i,j}\) for the GNEP are generic for the given degrees. Recall the multi-projective variety \(\widetilde{W}\) as in (3.3). Note that \(\nu = (n_1, \ldots, n_N)\). For convenience, we denote that

\[\nu = (N \cdot n_1, \ldots, N \cdot n_N).\]

Let \(\mathcal{V} := \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_N\), where \(\mathcal{V}_i\) is the multi-projective determinantal variety given in (3.4). Note that \(\mathcal{W} = \widetilde{U} \cap \mathcal{V}\). Recall the set of labels \([s]^\nu\) is given in (4.3) for nonnegative integers \(s\).
Theorem 5.1. For each \(i, j\), let
\[
\hat{d}_{i,j} := \left( d_{i,j,1}, \ldots, d_{i,j,i-1}, \max(d_{i,j,i} - 1, 0), d_{i,j,i+1}, \ldots, d_{i,j,N} \right).
\]
Let \(\nu := (n_1, \ldots, n_N)\) and
\[
\Omega := \left\{ (l^{(0)}, \ldots, l^{(N)}) \mid |l^{(i)}| \in [n - m]^{(\nu)}, \right.
\]
\[
\left. |l^{(i)}| = [n - n_i + m_i]^{(\nu)}, \quad i \in [N] \right\}.
\]
Assume the active label sets are as in (5.1), each \(f_i\) is generic of the multi-degree \(d_{i,0}\) and each \(g_{i,j}\) is generic of the multi-degree \(d_{i,j}\). Then, the number of complex solutions to the Fritz-John system (1.3) is given by the following sum
\[
\sum_{(l^{(0)}, \ldots, l^{(N)}) \in \Omega} A_{\nu - l^{(0)}}(d_{1,1}, \ldots, d_{1,m_1}, \ldots, d_{N,m_N}) \cdot \prod_{i=1}^N S_{\nu - l^{(i)}}(\hat{d}_{i,0}, \ldots, \hat{d}_{i,m_i}).
\]
In the above, \(A_{\nu - l^{(0)}}\) is given as in (4.2) and \(S_{\nu - l^{(i)}}\) is given as in (4.9).

Proof. Note that the intersection \(\mathcal{W} = \cap_{i=1}^N \mathcal{W}_i\) is zero-dimensional. We consider that each \(m_i \leq n_i\), because otherwise the Fritz-John system (1.3) is empty when \(g_{i,j}\) are generic. Let \(\lambda_i := (\lambda_{i,0}, \ldots, \lambda_{i,m_i}) \in \mathbb{C}^{m_i+1}\) be variables. Consider the multi-projective variety \(\mathcal{C}\) given by
\[
\begin{aligned}
g_{1,1}(\tilde{x}) &= g_{1,2}(\tilde{x}) = \cdots = g_{N,m_N}(\tilde{x}) = 0, \\
\bar{J}_1(\tilde{x}) \cdot \lambda_1 &= 0, \ldots, \bar{J}_N(\tilde{x}) \cdot \lambda_N = 0.
\end{aligned}
\]
The \(\mathcal{C}\) is defined by vanishing \(n + m\) multi-homogeneous polynomials. It is clear \((\tilde{x}, \lambda_1, \ldots, \lambda_N) \in \mathcal{C}\) if and only if \(\tilde{x} \in \mathcal{W}\) and each \(\lambda_i\) solves \(\bar{J}_i(\tilde{x}) \cdot \lambda_i = 0\). When all \(f_i\) and \(g_{i,j}\) are general, if \(\tilde{x} \in \mathcal{W}\), then for each \(i\), the \(\bar{J}_i(\tilde{x}) \cdot \lambda_i = 0\) has a unique solution. This is because the columns of \(\bar{J}_i(\tilde{x})\) are linear independent if \(\tilde{x} \in \mathcal{W}\), by Theorem 5.1. Therefore, \(\dim \mathcal{C} = 0\) and \(\mathcal{C}\) is a complete intersection of hypersurfaces. We note that for the given constraining polynomials \(g_{1,1}, g_{1,2}, \ldots, g_{N,m_N}\), these hypersurfaces intersect without any fixed point when we vary coefficients of \(f_1, \ldots, f_N\). By Proposition 2.2 these hypersurfaces intersect transversely, and \(\mathcal{C}\) is smooth. Let \(\pi\) be the projection that maps \((\tilde{x}, \lambda_1, \ldots, \lambda_N)\) to \(\tilde{x}\). Since the fibers are linear, the finite set of points in \(\mathcal{C}\) is mapped bijectively to \(\mathcal{U} \cap \mathcal{V}_1 \cap \cdots \cap \mathcal{V}_N\). In particular, \(\mathcal{U}, \mathcal{V}_1, \ldots, \mathcal{V}_N\) intersect transversely and we may evaluate \(\deg \mathcal{W}\) using Lemma 4.2. Moreover, \(\mathcal{C} \cap \{ x \in \mathbb{P}^n : x_{i,0} = 0 \} = \emptyset\). To see this, we assume that \(i = 1\), without loss of generality. Then, the \(\mathcal{C} \cap \{ x \in \mathbb{P}^n : x_{1,0} = 0 \}\) is the intersection of hypersurfaces in \(\mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_N}\) that are defined by coefficients in \(x\) of
\[
f_{i,1}^{\text{hom}}(x) := \hat{f}_i(\tilde{x}), \quad g_{i,1,1}^{\text{hom}}(\tilde{x}) := \hat{g}_{i,1}(\tilde{x}), \ldots, \quad g_{i,m_i}^{\text{hom}}(x) := \hat{g}_{i,m_i}(\tilde{x}),
\]
where \(\tilde{x} := (0, x_{1,1}, \ldots, x_{1,n_1}, x_{2,0}, x_{2,1}, \ldots, x_{N,n_N})\). Then, the emptiness of \(\mathcal{C} \cap \{ x \in \mathbb{P}^n : x_{1,0} = 0 \}\) follows from Proposition 2.2 and Theorem 3.1. Therefore, the \(\mathcal{W}\), as the image of \(\mathcal{C}\) under the projection which maps \((\tilde{x}, \lambda_1, \ldots, \lambda_N)\) to \(\tilde{x}\), is disjoint with \(\{ x \in \mathbb{P}^n : x_{1,0} = 0 \}\).

By Proposition 4.3 for a vector \(l \in [n - n_i + m_i]^{(\nu)}\), we have
\[
(\deg \mathcal{V}_1)_l = S_{\nu - l}(\hat{d}_{i,0}, \ldots, \hat{d}_{i,m_i}).
\]
Consider the intersection
\[ \tilde{V} = \tilde{V}_1 \cap \tilde{V}_2 \cap \cdots \cap \tilde{V}_N = (\ldots((\tilde{V}_1 \cap \tilde{V}_2) \cap \tilde{V}_3) \cap \cdots \cap \tilde{V}_N). \]
This intersection is transversal and \( \dim \tilde{V} = m - n. \)

Now we show \((5.3)\) by induction. For each \( i \in [N], \) denote \( s_i := n - n_i + m_i. \) Suppose
\[
(5.4) \quad (\deg \tilde{V}_1 \cap \tilde{V}_2 \cap \cdots \cap \tilde{V}_k)_l = \sum_{l^{(1)} \in [s_1]\nu} \prod_{i=1}^k S_{l^{(1)} - l(i)}(\hat{d}_{i,0}, \ldots, \hat{d}_{i,m_i})
\]
hold for some \( k \leq N \) and for each \( l \in [n - \sum_{i=1}^k (n_i - m_i)]^{(\nu)}. \) Then, by Lemma \(4.2\)
\[
(\deg \tilde{V}_1 \cap \tilde{V}_2 \cap \cdots \cap \tilde{V}_k)_l = \sum_{l^{(1)} \in [s_1]\nu} S_{l^{(1)} - l(i)}(\hat{d}_{i,0}, \ldots, \hat{d}_{i,m_i}) \cdot \prod_{i=1}^k S_{l^{(1)} - l(i)}(\hat{d}_{i,0}, \ldots, \hat{d}_{i,m_i})
\]
For the right hand side, we also have
\[
\sum_{l^{(1)} \in [s_1]\nu} S_{l^{(1)} - l(i)}(\hat{d}_{i,0}, \ldots, \hat{d}_{i,m_i}) \cdot \prod_{i=1}^k S_{l^{(1)} - l(i)}(\hat{d}_{i,0}, \ldots, \hat{d}_{i,m_i})
\]
In the above, the
\[ \tilde{\Omega} := \left\{ (l^{(1)}, \ldots, l^{(k)}) \mid \begin{array}{c} l^{(i)} = [n - n_i + m_i]^{(\nu)}, i \in [k] \\ l^{(1)} + \cdots + l^{(k)} = k \cdot \nu + l \end{array} \right\}. \]
Clearly, \((5.4)\) holds when \( k = 1, \) by Proposition \(4.3.\) Therefore, we have
\[
(\deg \tilde{V})_l = \sum_{l^{(1)} \in [n - m]\nu} \prod_{i=1}^N S_{l^{(1)} - l(i)}(\hat{d}_{i,0}, \ldots, \hat{d}_{i,m_i})
\]
for all \( l = [m]^{(\nu)}, \) by induction. Note that the \( \tilde{U} \) is given by vanishing \( g_{i,j} \) for all \( i \in [N] \) and \( j \in [m_i], \) and all \( g_{i,j} \) are general. For every \( l \in [n - m]^{(\nu)}, \) we have
\[
(\deg \tilde{U})_l = A_{l^{(1)}, \ldots, l^{(N)}}, \quad \text{for } l^{(i)} = [n_i - m_i]^{(\nu)}, \quad i \in [N].
\]
Since \( \dim \tilde{V} \cap \tilde{U} = 0, \) by Lemma \(4.2\) the degree of \( \tilde{V} \cap \tilde{U} \) is
\[ \sum_{l^{(1)} \in [n - m]\nu} (\deg \tilde{U})_l = \sum_{l^{(1)} \in [n - m]\nu} (\deg \tilde{V})_l \cdot (\deg \tilde{U})_l. \]
\[
\sum_{(l(0), \ldots, l(N)) \in \Omega} A_{\nu-l(0)}(d_1, 1, \ldots, d_{1,m_1}, \ldots, d_{N,m_N}) \cdot \prod_{i=1}^N S_{\nu-l(i)}(\hat{\alpha}_{i,0}, \ldots, \hat{\alpha}_{i,m_i}).
\]

5.2. The case of ungiven active constraints. To count the number of all Fritz-John points, we need to consider all possibilities of active label sets. For the \(i\)th player’s optimization problem, let \(E_i\) be the active label set. Then, it is clear that \(E_{i,1} \subseteq E_i \subseteq E_{i,1} \cup E_{i,2}\). By Theorem 5.1 if we enumerate \(E_i\) by \([m_i]\) for each \(i\), and let \(E := (E_1, \ldots, E_N)\), then the algebraic degree for the Fritz-John system with the tuple of active labeling sets \(E\) is given by \(5.3\). Denote the set of all possible active sets

\[
\Lambda := \left\{ (E_1, \ldots, E_N) \mid \begin{array}{c}
E_{i,1} \subseteq E_i \subseteq E_{i,1} \cup E_{i,2} (i \in [N]), \\
|E_i| \leq n_i
\end{array} \right\}.
\]

Then, the \(\Lambda\) is the set of all possible active label sets for the GNEP when every \(g_{i,j}\) is generic.

**Theorem 5.2.** For each \(i \in [N]\) and \(j \in E_{i,1} \cup E_{i,2}\), let \(d_{i,j} := (d_{i,j,1}, \ldots, d_{i,j,N})\) be a tuple of degrees and let

\[
\hat{d}_{i,j} := \left( d_{i,j,1}, \ldots, d_{i,j,i-1}, \max(d_{i,j,i} - 1, 0), d_{i,j,i+1}, \ldots, d_{i,j,N} \right).
\]

Assume each \(f_i\) is generic of the multi-degree \(d_{i,0}\) and each \(g_{i,j}\) is generic of the multi-degree \(d_{i,j}\). Let \(\Lambda\) be the set of active label sets given in \(5.5\). For each \(E := (E_1, \ldots, E_N) \in \Lambda\), denote by \(D_E\) the algebraic degree given in \(5.3\) with every \(E_i\) enumerated as \([m_i]\). Then, the total number of all complex Fritz-John points is

\[
\sum_{E \in \Lambda} D_E.
\]

**Proof.** For the \(E \in \Lambda\), we denote by \(K_E\) the set of complex Fritz-John points with active constraints labels \(E\). Then, it suffices to show that for arbitrary two distinct \(E^{(1)}, E^{(2)} \in \Lambda\), the \(K_{E^{(1)}} \cap K_{E^{(2)}} = \emptyset\).

Since \(E^{(1)} \neq E^{(2)}\), there exists \(i \in [N]\) and \(j \in E^{(2)}\) such that \(j \notin E^{(1)}\). By Theorem 5.1 the \(K_{E^{(1)}}\) is a finite set. Then, the equations

\[
\tilde{g}_{i,j}(\tilde{u}) = 0 \quad (\tilde{u} \in K_{E^{(1)}})
\]

define a proper close subset of the space of coefficients for \(g_{i,j}\). This implies that

\[
K_{E^{(1)}} \cap \{ \tilde{x} \in \mathbb{P}^\nu : \tilde{g}_{i,j}(\tilde{x}) = 0 \} = \emptyset
\]

when the \(g_{i,j}\) is generic in \(\mathbb{C}[x]_{d_{i,j}}\). Note that all points in \(K_{E^{(2)}}\) vanish \(g_{i,j}\) since \(j \in E^{(2)}\). We know \(K_{E^{(1)}} \cap K_{E^{(2)}} = \emptyset\). \(\square\)

5.3. The case of NEPs. Now we consider the special case that the GNEP is an NEP. Recall that the GNEP reduces to an NEP if for every \(i \in [N]\), all \(g_{i,j}\) are independent of \(x_i\). For the NEP, the multi-degree for \(g_{i,j}(x_i)\) is \((0, \ldots, 0, d_{i,j}, 0, \ldots, 0)\), that the \(i\)th entry \(d_{i,j}\) is the degree of \(g_{i,j}(x_i)\) in the variable of \(x_i\), and all other entries are zero. Moreover, if \(g_{i,j}(x_i)\) is general in \(\mathbb{C}[x_i]_{d_i}\), then it is a general
polynomial in $\mathbb{C}[x]$ whose multi-degree is $d \cdot e_i$. For the given nonnegative integer $m$ and $i \in [N]$, let $z_0 \coloneqq (z_{0,1}, \ldots, z_{0,N})$ and $z_1, \ldots, z_m$ be variables, and let $\delta \coloneqq (\delta_1, \ldots, \delta_N)$ be degrees, we define

\begin{equation}
    \mathcal{T}_\delta^{(i)}(z_0, z_1, \ldots, z_m) := \sum_{(\eta_0, \ldots, \eta_m) \in \mathbb{N}^{m+1}} z_{0,i}^{\eta_0} \cdots z_m^{\eta_m} \cdot (\eta_0 + \sum_{i \neq j \in [N]} \delta_j)! \prod_{i \neq j \in [N]} \frac{\eta_j}{\delta_j}.
\end{equation}

**Theorem 5.3.** For each $i, j$, let $d_{i,0} \coloneqq (d_{i,0,1}, \ldots, d_{i,0,N})$ be a tuple of multi-degrees, and let $d_{i,j}$ be a nonnegative integer. Denote $\nu \coloneqq (n_1, \ldots, n_N)$. For each $i \in [N]$ and $j \in \mathcal{E}_{i,1} \cup \mathcal{E}_{i,2}$, suppose $f_i$ is a generic polynomial in $\mathbb{C}[x]$ with multi-degree $d_{i,0}$ in $x_i$, and $g_{i,j}$ is a generic polynomial in $\mathbb{C}[x_i]$ with degree $d_{i,j}$ in $x_i$.

(i) For the given tuple of active label sets $E \coloneqq (E_1, \ldots, E_N)$, if we enumerate each $E_i$ as $[m_i]$, and let

\[ \Theta \coloneqq \left\{ (l^{(1)}, \ldots, l^{(N)}) \mid l^{(i)} \in [n - m_i]^{m_i}, \ i = 1, \ldots, N, \ l^{(1)} + \cdots + l^{(N)} = (N - 1) \cdot \nu + (m_1, \ldots, m_N) \right\}, \]

then the number of complex Fritz-John points is

\begin{equation}
    \sum_{(l^{(1)}, \ldots, l^{(N)}) \in \Theta} \prod_{i=1}^{N} d_{i,1} \cdots d_{i,m_i} \cdot \mathcal{T}_\nu^{(i)}(d_{i,0} - e_i, d_{i,1} - 1, \ldots, d_{i,m_i} - 1).
\end{equation}

(ii) Under the same settings as in (i), denote by $\mathcal{D}_E$ the algebraic degree given in (5.8), and let $\Lambda$ be the set of all possible active tuples as in (5.3). Then, the total number of all complex Fritz-John points is

\[ \sum_{E \in \Lambda} \mathcal{D}_E. \]

**Proof.** (i) Since each $g_{i,j}$ is generic in $\mathbb{C}[x_i]_{d_{i,j}}$, so it is generic in the space of all polynomials in $\mathbb{C}[x]$ with multi-degree $h_{i,j} \coloneqq e_i \cdot d_{i,j}$. Therefore, we may apply Theorem 5.1 to compute the algebraic degree for the case that every player’s active set is $[m_i]$.

Let $m \coloneqq m_1 + \cdots + m_N$. For each $\delta \coloneqq (\delta_1, \ldots, \delta_N)$ such that $\delta_1 + \cdots + \delta_N = m$, the $A_\delta(h_{1,1}, \ldots, h_{1,m_1}, \ldots, h_{N,m_N})$ is the coefficient of $t^\delta$ in the product

\[ \prod_{i=1}^{N} \prod_{j=1}^{m_i} (h_{i,j,1}t_1 + \cdots + h_{i,j,N}t_N). \]

Therefore, $A_\delta(h_{1,1}, \ldots, h_{1,m_1}, \ldots, h_{N,m_N})$ is nonzero if and only if $\delta = (m_1, \ldots, m_N)$, and

\[ A_{(m_1, \ldots, m_N)}(h_{1,1}, \ldots, h_{1,m_1}, \ldots, h_{N,m_N}) = \prod_{i=1}^{N} d_{i,1} \cdots d_{i,m_i}. \]

For a given $i \in [N]$, consider

\[ S_\delta(d_{i,0}, \hat{h}_{i,1}, \ldots, \hat{h}_{i,m_i}) = \sum_{B \in \mathbb{N}^{[B \times (m_i + 1)]}} \left( \binom{B,1}{B,-1} \right) d_{i,0}^{B_{i,0}} \prod_{j=1}^{m_i} \left( \binom{B_{i,j+1}}{B_{i,j+1}} \right) \hat{h}_{i,j+1}. \]
where
\[ \hat{d}_{i,0} := d_{i,0} - e_i, \quad \hat{h}_{i,j} := h_{i,j} - e_i. \]
For the \( B \in \mathbb{N}^{[\delta \times (m+1)]} \), if there exists \((r, s)\) such that \( r \geq 2, s \neq i, \) and \( B_{r,s} \neq 0, \) then \( \prod_{j=1}^{\eta} (B_{r,s}^{j,s+1}) \hat{h}_{i,j}^{r,s+1} = 0 \) since \( \hat{h}_{i,r-1,s} = 0. \) On the other hand, if we assume \( B_{r,s} = 0 \) for all \( r \geq 2 \) and \( s \neq i, \) then \( B_{1,s} = \delta_s \) for all \( s = i. \) This is because the sum for the \( s \)th row of \( B \) equals \( \delta_s. \) If we denote the \( i \)th row of \( B \) by \((\eta_0, \ldots, \eta_m), \) then \( \eta_0 + \eta_1 + \cdots + \eta_m = \delta_i. \) By considering all possible \( B \in \mathbb{N}^{[\delta \times (m+1)]}, \) we get

\[ S_b(\hat{d}_{i,0}, \hat{h}_{i,1}, \ldots, \hat{h}_{i,m_i}) = T_b^{(i)}(d_{i,0} - e_i, d_{i,1} - 1, \ldots, d_{i,m_i} - 1). \]

We get the formula (5.8) by Theorem 5.1.

(ii) This is implied by Theorem 5.2 and the item (i).

5.4. The case of non-generic polynomials. When the polynomials \( f_i, g_{i,j} \) are not generic, the set \( \mathcal{W} \) may not be finite, or for the case that it is finite, the degree formula given by Theorem 5.1 and Theorem 5.2 may not be sharp. For such cases, we exploit the perturbation argument to get a better upper bound for the algebraic degree. For the given tuple \( E := (E_1, \ldots, E_N) \) of active constraint label sets, we enumerate \( E_i \) as \([m_i].\) Denote by \( \check{d}_{i,j} \) (resp., \( \check{d}_{i,0}^{(k)} \)) the multi-degree of the \( k \)th entry of \( \nabla_x, g_{i,j} \) (resp., \( \nabla_x, f_i \)). For every \( i = 1, \ldots, N \) and \( j = 0, \ldots, m_i, \) we let

\[ \check{d}_{i,j} := \left( \max_{k \in [n_i]} \check{d}_{i,j,1}^{(k)}, \max_{k \in [n_i]} \check{d}_{i,j,2}^{(k)}, \ldots, \max_{k \in [n_i]} \check{d}_{i,j,N}^{(k)} \right). \]

Recall the label set \( \Omega \) in (5.2).

Theorem 5.4. For the GNEP, let \( d_{i,0} \) be the multi-degrees of \( f_i, d_{i,j} \) be the multi-degrees of \( g_{i,j}, \) and for each \( i, j, \) the \( \check{d}_{i,j} \) is given by (5.9). Given the tuple of active constraint label set \( E = ([m_1], \ldots, [m_N]), \) if there are finitely many complex Fritz-John points, then their total number is bounded by the following sum

\[ \sum_{(\ell^{(0)}, \ldots, \ell^{(n)}) \in \Omega} A_{\nu-(\ell^{(0)})}(d_{1,1}, \ldots, d_{1,m_1}, \ldots, d_{N,m_N}) \cdot \prod_{i=1}^{N} S_{\nu-\ell^{(i)}}(\check{d}_{i,0}, \ldots, \check{d}_{i,m_i}). \]

Moreover, if we let \( \Lambda \) be the set of all possible tuples of active constraint labels, and denote by \( D_E \) the bound in (5.10) with the tuple of active constraints \( E, \) then the number of all Fritz-John points is bounded by \( \sum_{E \in \Lambda} D_E. \)

Proof. Given the active constraint \( E = (E_1, \ldots, E_m) \) such that every \( E_i = [m_i]. \) Let \( \nabla_x, f_i + t\phi_i \) with \( \phi_i \in \mathbb{C}[x]_{d_{i,0}} \) be a general perturbation of \( \nabla_x, f_i \) parameterized by \( t. \) Similarly, for each \( j \in [m_i], \) we let \( \nabla_x, g_{i,j} + t\gamma_{i,j} \) with \( \gamma_{i,j} \in (\mathbb{C}[x]_{\check{d}_{i,j}})^{m_i} \) be the general perturbation of \( \nabla_x, g_{i,j}. \) Denote the multi-homogenization in \( x \) of \( \nabla_x, g_{i,j} + t\gamma_{i,j} \) by \( \check{g}_{i,j}, \) and the multi-homogenization in \( x \) of \( \nabla_x, f_i + t\phi_i \) by \( \check{f}_i. \) Let

\[ \check{J}_i(x) := \left[ \check{f}_i(x) \quad \check{g}_{i,1}(x) \quad \cdots \quad \check{g}_{i,m_i}(x) \right], \]

\[ \check{V}_i := \left\{ (\check{J}_i, \check{V}_i) : t \in \mathbb{P}^\nu \times \mathbb{C} : \text{rank}(\check{J}_i)(\check{x}) \leq \eta_i \right\}. \]

Then \( \check{V}_i \subseteq \mathbb{P}^\nu \times \mathbb{C} \) is a variety with a projection \( \check{V}_i \rightarrow \mathbb{C}, \) such that each fiber \( \check{V}_{i=a} \subseteq \mathbb{P}^\nu \) is a multi-projective variety for each \( a \in \mathbb{C}. \) Furthermore, since all \( \phi_i \) and \( \gamma_{i,j} \) are general, the \( \check{W}(a) := (\check{V}_{i=a})_1 \cap \cdots \cap (\check{V}_{i=a})_N \cap \check{U} \) is finite for all but at most finitely many \( a \in \mathbb{C}, \) by Theorem 5.1. In fact, by generality
assumptions as above, \((\widetilde{V}_i)_1 \cap \cdots \cap (\widetilde{V}_i)_N \cap \widetilde{U}\) is 1-dimensional. Note that each irreducible component of \((\widetilde{V}_i)_1 \cap \cdots \cap (\widetilde{V}_i)_N \cap \widetilde{U}\) has a dimension not less than one, by Theorem 4.11 (see also [13, Proposition 17.24]). For each \(u \in (\widetilde{V}_i)_1 \cap \cdots \cap (\widetilde{V}_i)_N \cap \widetilde{U}\), it is contained in an irreducible component whose dimension is greater than or equal to one. So, there exists a curve \(u_i \subset (\widetilde{V}_i)_1 \cap \cdots \cap (\widetilde{V}_i)_N \cap \widetilde{U}\) that maps onto \(\mathbb{C}\) and such that \(u \in u_i \cap (\widetilde{V}_i)_1 \cap \cdots \cap (\widetilde{V}_i)_N \cap \widetilde{U}\). By genericity of \((\widetilde{g}_i)_{i,j}\) and elimination theory (see [3, Chapter 3]), for a general \(a \in \mathbb{C}\), the \(u(a) \in \widetilde{W}(a)\) satisfies

\[
\alpha_\xi(a)u_{i,j}(a)^\xi + \alpha_{\xi-1}(a)u_{i,j}(a)^{\xi-1} + \cdots + \alpha_1(a)u_{i,j}(a)^1 + \alpha_0(a) = 0,
\]

where

\[
\xi = \sum_{(l^{(0)}, \ldots, l^{(N)}) \in \Omega} A_{\nu-l^{(0)}}(d_{1,1}, \ldots, d_{1,m_i}, \ldots, d_{N,m_N}) \cdot \prod_{i=1}^{N} S_{\nu-l^{(i)}}(d_{i,0}, \ldots, d_{i,m_i})
\]

and \(\alpha_\xi(a)\) are rational functions of the coefficients of \(f_i\), \(\phi_i\), \(g_{i,j}\) and \(\gamma_{i,j}\). So there are at most \(\xi\) many such curves. Therefore, the intersection \(\widetilde{V}_1 \cap \cdots \cap \widetilde{V}_N \cap \widetilde{U}\) has at most \(\xi\) points, and (5.10) gives an upper bound for the algebraic degree of the GNEP.

If for each \(E \in \Lambda\), there exist finitely many Fritz-John points with active constraints \(E\), then the set of all complex Fritz-John points is the union over all \(E \in \Lambda\), and the total number for them is bounded by the sum of all \(\overline{D}_E\).

**Remark.** When the polynomials \(f_i\) and \(g_{i,j}\) are not generic, the variety \(\{\tilde{x} \in \mathbb{P}^\nu : \tilde{J}_i(\tilde{x}, \tilde{x} - i)\}\) contains, but is usually not equal to, the \(\mathbb{P}(V_i)\). When there are finitely many complex Fritz-John points, the formula (5.10) still gives an upper bound for the algebraic degree. However, this upper bound is usually greater than the one in (5.10), since \(d_{i,j} \leq \tilde{d}_{i,j}\) componentwise. We refer to Example 6.4 for such a GNEP.

In particular, suppose for every \(i \in [N]\) and for every \(j \in [m_i]\), the multi-degrees of all entries in \(\nabla_{x_i} g_{i,j}(x)\) are equal. Then the \(\mathbb{P}(V_i)\) equals the rank deficient variety for

\[
\tilde{J}_i(\tilde{x}_i, \tilde{x} - i) := \begin{bmatrix}
\frac{\partial g_{i,1}}{\partial x_{i,1}}(\tilde{x}) & \frac{\partial g_{i,1}}{\partial x_{i,2}}(\tilde{x}) & \cdots & \frac{\partial g_{i,1}}{\partial x_{i,N}}(\tilde{x}) \\
\frac{\partial g_{i,2}}{\partial x_{i,1}}(\tilde{x}) & \frac{\partial g_{i,2}}{\partial x_{i,2}}(\tilde{x}) & \cdots & \frac{\partial g_{i,2}}{\partial x_{i,N}}(\tilde{x}) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial g_{i,m_i}}{\partial x_{i,1}}(\tilde{x}) & \frac{\partial g_{i,m_i}}{\partial x_{i,2}}(\tilde{x}) & \cdots & \frac{\partial g_{i,m_i}}{\partial x_{i,N}}(\tilde{x})
\end{bmatrix}.
\]  

In the above, each \(\tilde{g}_{i,j}(\tilde{x})\) denotes the multi-homogenization for \(\frac{\partial g_{i,j}}{\partial x_{i,k}}(\tilde{x})\). Denote by \(\widetilde{Y}_i\) the multi-projective variety \(\{\tilde{x} \in \mathbb{P}^\nu : \text{rank } \tilde{J}_i(\tilde{x}_i, \tilde{x} - i) \leq m_i\}\), and let \(\tilde{J}_i^*(\tilde{x}_i, \tilde{x} - i)\) be the submatrix of \(\tilde{J}_i(\tilde{x}_i, \tilde{x} - i)\) which consists of the right \(m_i\) columns. If we further assume that the intersection \(\tilde{U}\) is transversal, and for each \(i \in [N]\), the

\[
\{\text{rank } \tilde{J}_i^*(\tilde{x}_i, \tilde{x} - i) \leq m_i - 1\} \cap \tilde{U} = \emptyset,
\]

then the intersection

\[
\widetilde{Y}_1 \cap \cdots \cap \widetilde{Y}_N \cap \tilde{U}.
\]
is transversal when all $f_i$ are generic. This is because each $\tilde{Y}_i$ is parameterized by coefficients of $f_i$, and the set of fixed points for this section is contained in the intersection

$$\{ \tilde{x} : \text{rank} \tilde{J}_i(\tilde{x}, \tilde{x}_{-i}) \leq m_i - 1 \} \cap \tilde{U},$$

if we vary all the coefficients of all $f_i$. In this case, if all $f_i$ are generic, then the number of Fritz-John points with active constraints labels $[m_1], \ldots, [m_N]$ is given by (5.10). From the above discussion and by Theorem 5.4, we have the following result.

**Proposition 5.5.** Under the assumptions in Theorem 5.4, suppose for all $i \in [N]$ and $j \in [m_i]$, the multi-degrees of all entries in $\nabla_x g_i(x)$ are equal, and hypersurfaces defined by all $\tilde{g}_{i,j}(\tilde{x}) = 0$ intersect transversely. Denote

$$\tilde{Y}_i := \{ \tilde{x} \in \mathbb{P}^\nu : \text{rank} \tilde{J}_i(\tilde{x}, \tilde{x}_{-i}) \leq m_i \}.$$ 

If the intersection $\tilde{Y}_1 \cap \cdots \cap \tilde{Y}_N \cap \tilde{U}$ is transversal, then the upper bound given by (5.10) is sharp.

### 6. Computational experiments

In this section, we present some computational results of complex Fritz-John points for GNEPs. For the GNEP, we use the symbolic computation software Macaulay2 [11] to evaluate the dimension and the number of complex solutions. The computation is implemented in a DELL PowerEdge R730 work station, with two Intel® Xeon E5-2683 v4 CPU at 3.0GHz×32 cores and 64GB of RAM, in an Ubuntu 20.4 LTS operating system. When inequality constraints exist, let $E_i$ be the active labeling set for the $i$th player’s optimization, and $E := (E_1, \ldots, E_N)$. We denote by $D_E$ the generic algebraic degree given by (5.4) for the FJ system with the tuple of active labeling set $E$. When the constraining polynomials $g_{i,j}$ are not generic, the upper bound for the complex Fritz John points is denoted by $\tilde{D}_E$, which is given in (5.10). We would like to remark that in the following examples, we use variables $x, y, z$ to represent $x_1, x_2$ and $x_3$ respectively, for cleanness of the paper.

#### 6.1. Unconstrained problems.

For unconstrained problems, the Fritz-John condition becomes $\nabla_x f_i(x) = 0$ for all $i = 1, \ldots, N$. For the generic $f_i$ with multi-degree $d_{i,0} = (d_{i,0,1}, \ldots, d_{i,0,N})$, let $\hat{d}_i = (d_{i,0,1}, \ldots, d_{i,0,i} - 1, \ldots, d_{i,0,N})$. Then, the upper bound of the algebraic degree for these equations given by Theorem 5.1 equals

$$A_\nu(\hat{d}_1, \ldots, \hat{d}_N).$$ 

Interestingly, one may check that it also equals

$$\sum_{\alpha \in [n_i]^{\nu}} \mathcal{S}_{\alpha}(\hat{d}_1) \cdots \mathcal{S}_{\alpha}(\hat{d}_N) = \sum_{\alpha \in [n_i]^{\nu}} \hat{d}_1^{\alpha_1} \cdots \hat{d}_N^{\alpha_N}.$$
Example 6.1. Consider the two-player NEP that each player solves an unconstrained optimization problem. The objective function for the first player is

\[
f_1(x, y) = 4x_1^2 y_2^2 - x_1^2 y_3 - x_1^2 y_1 + 4x_1^2 y_3^2 - 2x_1^2 y_1 - 2x_1 x_2 y_1 y_2 + 5x_1 x_2 y_1 y_3 + 3x_1 x_2 y_2^2 - 3x_1 x_2 y_3^2 - 2x_1 x_2 y_2 - 4x_1 x_2 y_3^2 - x_1 x_2 y_2^2 + 4x_1 x_3 y_3 - 2x_1 x_3 y_1 + 3x_1 x_3 y_2 + 3x_1 x_3 y_3 - x_1 y_3 - 2x_1^2 y_1 y_2 + 4x_1^2 y_1 y_3 - 3x_2 y_2^2 - 2x_2 x_3 y_2^2 + 3x_2 x_3 y_3 - 3x_2 x_2 y_2 + 3x_2 x_2 y_3 + 4x_2 y_2^2 + 5x_2 y_1 y_3 + 3x_2 y_1 + 4x_2 y_2 + 3x_2 y_2 + 2x_2 y_2^2 - 3x_2 y_1 y_2 + 2x_2 y_2^2 + 4x_2 y_2^2 - 2x_2 y_3 - 3x_2 y_3^2 + 4x_2 y_3^2 + 4x_2 y_3^2 - 3x_3 y_2 y_3 + 4x_3 y_3.
\]

For the second player, the objective function is

\[
f_2(x, y) = 3x_1^2 y_1^2 - x_1^2 y_1 y_2 + 2x_1^2 y_2 - 2x_1 x_2 y_1^2 + 2x_1 x_2 y_1 y_3 - x_1 x_2 y_2 + 5x_1 x_3^2 y_1 - x_1 x_3 y_1 + 5x_1 x_3 y_2 - 2x_1 x_3 y_2 + x_1 y_2 - x_1 - 2x_3 y_2 + 3x_2 y_2^2 - 3x_2 y_1 y_3 - x_1 x_2 y_2 + 4x_2 x_3 y_1 + 3x_2 x_3 y_2 - 2x_2 x_3 y_1 + 4x_2 y_1 y_3 + 2x_2 x_3 y_3 - x_2 y_3 - 3x_2 y_3^2 + 2x_3 y_2 y_3 - 3y_2^2 - 2y_2 - 3y_3.
\]

In this problem, \( N = 2 \) and \( \nu = (3, 3) \). The multi-degrees for both \( f_1 \) and \( f_2 \) are \((2, 2)\), and \( d_1 = (1, 2) \), \( d_2 = (2, 1) \). The upper bound for the algebraic degree given by (6.1) is

\[A_{(3,3)}(d_1, d_2, d_1, d_2, d_2, d_2) = 245.\]

Using the symbolic computation software \texttt{Macaulay2}, we find the Fritz-John system has exactly 245 complex solutions (counting multiplicities).

6.2. NEPs with ball constraints. Consider the two-player NEP such that each player’s feasible strategy vector is bounded within the unit ball:

(6.2) \[\begin{align*}
\min_{x \in \mathbb{R}^3} & f_1(x, y) \\
\text{s.t.} & \|x\|^2 \leq 1, \\
\min_{y \in \mathbb{R}^3} & f_2(x, y) \\
\text{s.t.} & \|y\|^2 \leq 1.
\end{align*}\]

In this problem, \( \nu = (3, 3) \), \( m = 2 \), and \( d_{1,1} = (2, 0) \), \( d_{2,1} = (0, 2) \), \( d_{1,1} = (1, 0) \), \( d_{2,1} = (0, 1) \).

There are totally four cases of active constraints, that:

\[E^{(1)} = (\emptyset, \emptyset), \ E^{(2)} = \{(1), \emptyset\}, \ E^{(3)} = (\emptyset, \{1\}), \ E^{(4)} = (\{1\}, \{1\}).\]

By Theorem 5.4 an upper bound of algebraic degree for this NEP is given by (6.3)

\[\tilde{D}_{E^{(1)}} + \tilde{D}_{E^{(2)}} + \tilde{D}_{E^{(3)}} + \tilde{D}_{E^{(4)}}.\]

Example 6.2. Consider the two-player ball constrained NEP (6.2). Let the objective functions be

\[
f_1(x, y) = 2x_2 - 2x_1 + 5x_3 + 2x_1 x_3 + 5x_2 x_3 - 3x_1 y_1 - 2x_2 y_1 - 3x_1 y_3 - 2x_2 y_2 - 3x_3 y_3 + 4x_3 y_3 + 4x_3^2 y_1 + 2x_1^2 y_1 - 3x_2 y_2 + x_1^2 y_1 + x_2 x_3 y_1 - x_2 x_3 y_2 + x_1 x_3 y_2 - x_1 x_3 y_2 - x_1 x_3 y_3 - x_1 x_2 y_1 - x_1 x_2 y_3 + 2x_3 x_3 y_3, \]

\[
f_2(x, y) = 2x_1 y_1 + 2x_1 y_2 + 5x_2 y_1 - 3x_2 y_2 - 4x_3 y_1 + 3x_3 y_2 - 6x_3 y_3 - y_1 y_2 - x_1 y_1^2 + 3x_1 y_3^2 + 2x_2 y_2^2 + 2x_2 y_3^2 + 4x_3 y_2^2 + 2y_2^2 + 4x_2 y_1 y_2 + 2x_1 x_2 y_3 - 3x_3 y_1 y_2 + 2x_3 y_3 - 3x_3 y_3 - x_3 y_3 y_2.
\]

The multi-degrees for objective functions are \( d_{1,0} = (2, 1) \) and \( d_{2,0} = (1, 2) \). We have

\[\tilde{D}_{E^{(1)}} = 20, \ \tilde{D}_{E^{(2)}} = \tilde{D}_{E^{(3)}} = 30, \ \tilde{D}_{E^{(4)}} = 76.\]
Therefore, the upper bound for the algebraic degree of this NEP given by (5.10) and (6.3) is 156. Using symbolic computation software Macaulay2, we find the Fritz-John system has exactly 156 complex solutions (counting multiplicities).

6.3. GNEP with generic joint-linear constraints. Consider the NEP such that the $i$th player solves the following optimization problem

\begin{align}
\min_{x \in \mathbb{R}^3} & \quad f_1(x, y) \\
\text{s.t.} & \quad g_1(x, y) \geq 0, \quad \min_{y \in \mathbb{R}^3} & \quad f_2(x, y) \\
\text{s.t.} & \quad g_2(x, y) \geq 0.
\end{align}

where $g_i(x, y)$ is a polynomial with multi-degree $(1, 1)$, i.e., $g_i(x)$ is bilinear in $(x, y)$. For this GNEP, the $\nu = (3, 3)$, $m = 2$, and

\begin{align*}
d_{1,1} &= d_{2,1} = (1, 1), \quad d_{1,2} = (0, 1), \quad d_{2,1} = (1, 0).
\end{align*}

There are totally four cases of active constraints, that:

\begin{align*}
E^{(1)} &= (\emptyset, \emptyset), \quad E^{(2)} = (\{1\}, \emptyset), \quad E^{(3)} = (\emptyset, \{1\}), \quad E^{(4)} = ((1), \{1\}).
\end{align*}

By Theorem 5.2, the algebraic degree for this GNEP equals

\begin{align}
D_{E^{(1)}} + D_{E^{(2)}} + D_{E^{(3)}} + D_{E^{(4)}}.
\end{align}

**Example 6.3.** Let $f_1(x, y)$ and $f_2(x, y)$ be given as in Example 6.2. Consider the two-player joint linear constrained GNEP (6.4), with

\begin{align*}
g_1(x, y) &= 4x_2 - 3x_1 + 3x_3 + 3y_1 + 2y_2 + 5y_3 + x_1y_1 - 3x_1y_2 + 4x_2y_1 + 2x_1y_3 - 6x_2y_2 + 2x_3y_3 + 3x_2y_3 - x_3y_2 - 4x_3y_3 - 2, \\
g_2(x, y) &= 8x_1 + 3x_2 + 5x_3 - y_1 + y_3 + 7x_1y_1 - 3x_1y_2 - 6x_2y_1 - x_1y_3 + 4x_2y_2 - 2x_3y_1 - 2x_3y_2 + 7x_3y_3 + 3.
\end{align*}

For objective functions, the multi-degrees are $d_{1,0} = (2, 1)$, $d_{2,0} = (1, 2)$. We have

\begin{align*}
D_{E^{(1)}} &= 20, \quad D_{E^{(2)}} = D_{E^{(3)}} = 34, \quad D_{E^{(4)}} = 62.
\end{align*}

Therefore, the upper bound for the algebraic degree of this GNEP given by (6.5) is 150. Using symbolic computation software Macaulay2, we find the Fritz-John system has exactly 150 complex solutions (counting multiplicities).

6.4. GNEP with inner product & joint ball constraints. Consider the two-player GNEP

\begin{align}
\min_{x \in \mathbb{R}^3} & \quad f_1(x, y) \\
\text{s.t.} & \quad x^T(y + e) = 1, \quad x^T x + y^T y \leq 2, \\
\min_{y \in \mathbb{R}^3} & \quad f_2(x, y)
\end{align}

For this GNEP, the $\nu = (2, 2)$, $m = 4$, and

\begin{align*}
d_{1,1} &= (1, 1), \quad d_{1,2} = d_{1,3} = (1, 0), \quad d_{2,1} = (2, 2).
\end{align*}

However, since the $g_{1,1}(x, y), g_{1,2}(x, y), g_{1,3}(x, y)$ (resp., $g_2(x, y)$) are not generic in $\mathbb{R}[x, y]|_{(1, 1)}$ (resp., $\mathbb{R}[x, y]|_{(2, 2)}$), the multi-degrees for their partial gradients are

\begin{align*}
\hat{d}_{1,1} &= (0, 1), \quad \hat{d}_{1,2} = \hat{d}_{1,3} = (0, 0), \quad \hat{d}_{2,1} = (0, 1).
\end{align*}

Suppose the multi-degrees for $f_1$ and $f_2$ are $d_{1,0}$ and $d_{2,0}$. Let $\hat{d}_{1,0}$ and $\hat{d}_{2,0}$ be defined as in Theorem 6.4. Note that when the objective functions $f_1$ and $f_2$ are generic, there do not exist FJ points of this GNEP such that all three constraints
for the first player are active. For such cases, there are totally six possibilities of active constraints, that:

\[ E(1) = \{(1), \emptyset\}, \quad E(2) = \{(1), \{1\}\}, \quad E(3) = \{(1, 2), \emptyset\}, \quad E(4) = \{(1, 2), \{1\}\}, \quad E(5) = \{(1, 3), \emptyset\}, \quad E(6) = \{(1, 3), \{1\}\}. \]

By Theorem 5.4, an upper bound for the algebraic degree is given by (5.10), which equals

\[ (6.7) \quad \mathcal{D}_{E(1)} + \mathcal{D}_{E(2)} + \mathcal{D}_{E(3)} + \mathcal{D}_{E(4)} + \mathcal{D}_{E(5)} + \mathcal{D}_{E(6)}. \]

Example 6.4. Consider the GNEP (6.6) where the objective functions are

\[ f_1(x, y) = 3x_1^2y_1y_2 - x_1^3y_2^2 - x_1^2y_2 - x_1^3 + 3x_1^2x_2y_1 + 2x_1^2x_2y_2 - x_1^2y_2^2 \]
\[ + 2x_1^2y_1y_2 - x_1^2y_1 + 4x_1^2 + 7x_1x_2y_1^2 + 2x_1x_2y_1 + 2x_1y_1 \]
\[ + 2x_1x_2y_2^2 - 4x_1x_2y_1y_2 + 2x_1x_2y_1 + 2x_1x_2y_2 - 4x_1y_1 \]
\[ - 4x_1y_2^2 + 3x_1y_2 + 3x_2y_1^2 + 4x_2y_1y_2 + 2x_2^2y_1 - x_2^2y_2 + 5x_2^2y_2^2 \]
\[ + 3x_2y_2 + 2x_2y_2^2 + 2x_2^2y_2 + 2x_2^2 - 2x_2y_2^2 - x_2y_2^2 - x_2^2y_2^2. \]

\[ f_2(x, y) = 3x_1^2y_1^3 - x_1^2y_1^2 - 3x_1^2y_1y_2^2 - x_1^2y_1y_2 - 2x_1y_2^2 - x_2^2y_1^3 \]
\[ - x_1x_2y_1y_2 - x_2y_2^2 - x_1x_2y_2 - x_1x_2y_1 - x_1x_2y_2 - 2x_1x_2y_2 - 2x_1y_1 \]
\[ + 4x_1x_2y_2 - x_1y_1 + 2x_1y_1^2y_2 - x_1y_1 + 3x_1y_1^2y_2 - x_1y_1^2 + 2x_1y_1^2 \]
\[ - 2x_1y_1 + 2x_1y_1^2 + 2x_1y_1^2y_2 + 2x_1y_1y_2 + 2x_2^2y_1 - x_2^2y_2 + 4x_2^2y_2^2 \]
\[ - 2x_2y_2 - 2x_2y_2^2 - x_2y_2^2 - x_2y_1y_2 + 2x_2y_2 + 3y_1 - y_1 + 3y_1^2 - y_2 - y_2^2. \]

For objective functions, the multi-degrees are \(d_{1,0} = (3, 2), d_{2,0} = (2, 3)\), and

\[ \mathcal{D}_{E(1)} = 60, \quad \mathcal{D}_{E(2)} = 74, \quad \mathcal{D}_{E(3)} = 12, \quad \mathcal{D}_{E(4)} = 16, \quad \mathcal{D}_{E(5)} = 12, \quad \mathcal{D}_{E(6)} = 16. \]

Therefore, the upper bound provided by (5.10) and (6.7) is 190. Using symbolic computation software Macaulay2, we find the Fritz-John system has exactly 190 complex solutions (counting multiplicities). We would like to remark that for this problem, the formula (6.3) and Theorem 5.2 give a weaker upper bound 230.

6.5. GNEP with generic quadratic constraints. Consider the two-player GNEP

\[ (6.8) \quad \min_{x \in \mathbb{R}^2} f_1(x, y) \quad \text{s.t.} \quad g_1(x, y) = 0, \quad \min_{y \in \mathbb{R}^2} f_2(x, y) \quad \text{s.t.} \quad g_2(x, y) = 0, \]

where \(g_1\) and \(g_2\) have multi-degrees \(d_{1,1} = (2, 2), d_{2,1} = (2, 2)\) respectively. Moreover, the \(n = (2, 2), m = 2\), and

\[ \mathcal{D}_{1,1} = (1, 2), \quad \mathcal{D}_{2,1} = (2, 1). \]

Suppose the multi-degrees for \(f_1\) and \(f_2\) are \(d_{1,0}\) and \(d_{2,0}\). Since all constraints are active, by Theorem 5.4, the upper bound for the algebraic degree is given by (6.3).

Example 6.5. Consider the GNEP (6.8) defined by

\[ f_1(x, y) = -2x_1^2y_1^2 - 2x_1^2y_1 - x_1^3y_2^2 - x_1^3 - 2x_1^2x_2y_1y_2 + 3x_1^2x_2y_1 \]
\[ - 4x_1^2x_2y_2 + 5x_1^2x_2 - x_1^2y_1^2 - 2x_1^2y_1y_2 + 3x_1^2y_1 - 4x_1^2y_2 \]
\[ + 4x_1x_2^2y_1^2 + 2x_1x_2^2y_2 - x_1x_2y_1^2 + 2x_1x_2y_1 - 2x_1x_2y_2 + 2x_1x_2 \]
\[ + 2x_1y_1^2 + 2x_2y_2 - x_1y_1 - x_2y_1^2 - 3x_2y_1y_2 + 2x_2y_2^2 + 2x_2y_2 \]
\[ - 4x_2^2y_1y_2 + 3x_2^2y_2^2 - x_2^2y_2 + 2x_2^2y_2 + 2x_2y_1y_2 + 2x_2y_2^2, \]
\[
\begin{align*}
f_2(x, y) &= 3x_1 y_1^2 - 2x_1^2 y_1 - 3x_1^2 y_2^2 + 4x_1^2 y_1 y_2 - x_1 x_2 y_1^2 - 6x_1 x_2 y_2^2 - x_1 x_2 y_1 y_2, \\
eg x_1 y_1 y_2 &- x_1 x_2 y_2 - 2x_1 x_2 y_1 + 2x_1 y_1^2 - x_1 y_1 y_2, \\
eg 2x_1 y_2 &- 3x_2 y_1^2 + 4x_2 y_1 y_2 - 2x_2 y_2^2 + 2x_2 y_1 - 2x_2 y_2^2 - 2x_1 y_1^2 - x_2 y_2^2, \\
eg x_2 y_1 &+ 4x_2 y_2^2 - x_2 y_2^2 + 4x_2 y_2 + 4x_2 y_2^2.
\end{align*}
\]

\[
\begin{align*}
g_1(x, y) &= -2x_1^2 y_1 + 2x_1^2 y_1 + 3x_1^2 y_2^2 + 4x_1^2 y_1 y_2 + 3x_1^2 + x_1 x_2 y_1 y_2, \\
+ 3x_1 y_1 y_2 &- x_1 y_1 y_2 + 2x_1 x_2 y_2 + x_1 x_2 + 6x_1 y_1 - 2x_1 y_1 y_2 + x_1 y_1 \\
+ 7x_1 y_2^2 + 8x_1 y_2 + x_2 y_2^2 - 5x_2 y_1 y_2 - 2x_2 y_1 + 3x_2 y_2^2 - 2x_2 y_2^2 + 2x_2^2 \\
+ 3x_2 y_2^2 + 2x_2 y_2 + 2x_2 y_1 + y_1 y_2 &+ 4y_1 - 2y_2^2 - 2
\end{align*}
\]

\[
\begin{align*}
g_2(x, y) &= 2x_1 x_2 y_1 + 2x_1^2 y_1 + x_1^2 y_2^2 + 4x_1^2 y_1 y_2 - x_1 x_2 y_1 y_2, \\
+ 3x_1 y_1 y_2 &- x_1 y_1 y_2 + 2x_1 x_2 y_2 + x_1 x_2 - 2x_1 y_1 + x_1 y_1 \\
- 3x_1 y_1 + 2x_1 y_2^2 + x_1 y_2 + x_1 - 3x_2 y_2^2 + 3x_2 y_1 y_2 - 3x_2 y_1 \\
+ 2x_2 y_2^2 + 3x_2 + x_2 y_2^2 + x_2 y_1 y_2 + 3x_2 y_1 - x_2 y_2^2 + 3x_2 y_2 \\
+ x_3 &- 3y_2^2 - y_1 y_2 + y_1 + 3y_2 + 5.
\end{align*}
\]

For objective functions, the multi-degrees are \(d_1, d_2, d_3 = (2, 3, 2)\). Therefore, the upper bound of the algebraic degree provided by \(\Box^3\) equals 296. Using symbolic computation software Macaulay2, we find the Fritz-John system has exactly 296 complex solutions (counting multiplicities).

6.6. The 3-player GNEP with inner product constraints. Consider the three-player GNEP

\[
\begin{align*}
\min_{x \in \mathbb{R}^2} f_1(x, y, z) &\quad \text{s.t.} \quad x^T y = 0, \\
\min_{y \in \mathbb{R}^2} f_2(x, y, z) &\quad \text{s.t.} \quad y^T z = 0, \\
\min_{z \in \mathbb{R}^2} f_3(x, y, z) &\quad \text{s.t.} \quad z^T x = 0,
\end{align*}
\]

where \(g_1, g_2, g_3\) has multi-degrees \(d_{1,1} = (1, 1, 0), d_{2,1} = (0, 1, 1)\) and \(d_{3,1} = (1, 0, 1)\) respectively. Moreover, the \(\nu = (2, 2, 2)\), \(m = 3\), and \(d_{1,1} = (0, 1, 0), d_{2,1} = (0, 0, 1), d_{3,1} = (1, 0, 0)\).

Suppose the multi-degree for \(f_1, f_2, f_3\) are \(d_{1,0}, d_{2,0}, d_{3,0}\). Let \(d_{1,0}, d_{2,0}, d_{3,0}\) be defined as in Theorem 5.4. Since all constraints are active, by Theorem 6.1.

The upper bound for the algebraic degree is given by \(6.10\).

Example 6.6. Consider the GNEP \(6.8\) where the objective functions are

\[
\begin{align*}
f_1(x, y, z) &= 6x_1 + 3x_2 + 2x_1 y_1 - 2x_2 y_1 - 2x_1 z_1 + 7x_1 z_2 + 3x_2 z_1 + 2x_2 z_2 \\
+ 2x_1^2 y_1 - x_1 y_1 + 2x_2^2 y_1 - 2x_2 y_1 z_1 + 2x_2 y_1 z_2 \\
- 5x_1 y_1 z_1 + 3x_2 y_1 z_2 + 3x_1 y_2 z_2 - x_1 y_1 z_2 - 2x_2 y_2 z_1 - 2x_2 y_2 z_2 \\
- 3x_1 y_1 z_1 - x_1 y_1 z_2 + 3x_1 x_2 y_2 + 2x_1 y_1 z_2 + 3x_1 z_1 y_1 + 3x_1 y_2 z_2 + 4x_2 y_2 z_1 + 2x_2 y_2 z_2 - x_1 y_2 z_2,
\end{align*}
\]

\[
\begin{align*}
f_2(x, y, z) &= -2y_2 + 2x_1 y_1 + 6x_2 y_1 - 2y_1 y_2 + 3y_1 z_2 + 2y_2 z_2 \\
- 3x_2 y_1^2 - x_2 y_1 y_2 + 4y_1 z_1 + y_1 z_2 + 2y_2 z_2 + 2y_1 z_2 + 3y_1 z_2 + 2y_2 z_2 \\
+ 4x_2 y_1 z_1 + 2x_1 y_1 z_2 - 2x_2 y_1 z_2 + 2x_2 y_2 z_2 \\
- x_1 y_1 y_2 - 2x_1 y_1 z_1 - x_1 y_1 z_2 + 3x_1 y_2 z_2 - 3x_2 y_2 z_1 \\
- y_1 z_1 y_2 + 2y_1 y_1 z_2 + 2x_1 y_1 z_2 + 2x_1 z_1 y_2 - x_1 y_2 z_2,
\end{align*}
\]

\[
\begin{align*}
f_3(x, y, z) &= -4y_2 + 2x_2 z_1 - x_2 z_2 - 3x_2 z_2 + 3y_2 z_2 + 2y_1 z_2 + 3y_2 z_2 \\
+ 3x_1 z_2 - 4x_1 z_1 - x_1 z_1 + x_1 z_2 + 2x_2 z_2 + 3y_2 z_2 - 2y_1 z_2 \\
- 2y_2 z_2^2 - z_2^2 + 2x_2 y_1 z_2 + 3y_2 z_2 - x_2 y_1 z_2 - x_2 y_2 z_2 - x_2 y_2 z_2 \\
+ 2x_1 y_1 z_1 + 3x_2 y_2 z_1 - 3x_2 y_2 z_2 - x_1 y_1 z_1 z_2 - 3x_1 y_2 z_2.
\end{align*}
\]
For objective functions, the multi-degrees are $d_{1,0} = (2, 1, 1)$, $d_{2,0} = (1, 2, 1)$, and $d_{3,0} = (1, 1, 2)$. The algebraic degree provided by (5.10) of this GNEP is 74. Using symbolic computation software Macaulay2, we find the Fritz-John system has exactly 74 complex solutions (counting multiplicities).

7. Conclusions

This paper studies the generalized Nash equilibrium problems defined by polynomials. We show that under some generic conditions, the Fritz-John system of the GNEP only has finitely many complex solutions, and all of them satisfy the KKT conditions for each player’s optimization. Moreover, when the GNEP is defined by general polynomials, we give a formula for the algebraic degree of the GNEP, which counts the number of complex Fritz-John points. When the GNEP is given by non-generic polynomials, the algebraic degree is also studied.

Acknowledgement Jiawang Nie is partially supported by NSF grant DMS-2110780. Xindong Tang is partially supported by the Start-up Fund P0038976/BD7L from The Hong Kong Polytechnic University.

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