GEOMETRY OF HOLOMORPHIC VECTOR BUNDLES AND SIMILARITY OF COMMUTING TUPLES OF OPERATORS

YINGLI HOU, KUI JI, SHANSAN JI* AND JING XU

Abstract. In this paper, a new criterion for the similarity of commuting tuples of operators on Hilbert spaces is introduced. As an application, we obtain a geometric similarity invariant of tuples in the Cowen-Douglas class which gives a partial answer to a question raised by R.G. Douglas in \[15, 20\] about the similarity of quasi-free Hilbert modules. Moreover, a new subclass of commuting tuples of Cowen-Douglas class is obtained.

1. Introduction

Let \(\mathcal{H}\) be a complex separable Hilbert space, and \(\mathcal{L}(\mathcal{H})\) the collection of bounded linear operators on \(\mathcal{H}\). Problems of operator theory often involve unitary and similarity equivalences of operators (operator tuples). For a positive integer \(m\), let \(T = (T_1, \cdots, T_m)\) be an \(m\)-tuple of bounded operators acting on \(\mathcal{H}\). If \(T\) satisfies \(T_iT_j = T_jT_i\) for all \(1 \leq i, j \leq m\), \(T\) will be referred to as a commuting tuple. Let \(S = (S_1, \cdots, S_m)\) be a commuting tuple and \(S_i \in \mathcal{L}(\tilde{\mathcal{H}}), 1 \leq i \leq m\). The equation \(XT = SX\) for some \(X \in \mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}})\) means \(XT_i = S_iX\) for all \(1 \leq i \leq m\). If \(X\) is a unitary operator, \(T\) is unitarily equivalent to \(S\) (denoted by \(T \sim_u S\)). If \(X\) is invertible, \(T\) is similar to \(S\) (\(T \sim_s S\)).

Let \(\Omega\) be a connected open subset of \(\mathbb{C}\) and \(n\) be a positive integer. In \[15\], M.J. Cowen and R.G. Douglas introduced a class of bounded linear operators, denoted by \(B^1_1(\Omega)\), which contains \(\Omega\) as eigenvalues of constant multiplicity \(n\). In \[17\], they also pointed out that some results of \[15\] can be directly extended to \(\Omega \subset \mathbb{C}^m, m > 1\), i.e. the class of operators \(B^1_1(\Omega)\) may be generalized to an operator tuple class \(B^m_m(\Omega)\). Each Cowen-Douglas tuple naturally determines a Hermitian holomorphic vector bundle, and such two tuples are unitarily equivalent if and only if there is an isometric and connection-preserving bundle map between the bundles \[15, 17\]. In particular, the unitary classification of tuples in \(B^m_m(\Omega)\) involves only the curvature of Hermitian holomorphic bundles.

In \[49\], G. Misra has introduced and discussed homogeneous operators in \(B^1_1(\mathbb{D})\). By using the curvature as the invariant, these homogeneous operators have been completely characterized. For homogeneous operators in \(B^1_n(\mathbb{D}), n > 1\), A. Koranyi and G. Misra analyzed their structure and proved a classification theorem (see \[44\]). In \[50\], G. Misra 2020 Mathematics Subject Classification. Primary 47B13, 32L05 · Secondary 51M15, 53C07.
Key words and phrases. Commuting tuple, Cowen-Douglas operator, Similarity.

The first author was supported by National Natural Science Foundation of China (Grant No. 12001159). The second author was supported by National Natural Science Foundation of China (Grant No. 11831006 and 11922108).

* Corresponding author.
estimated the curvatures of operators in $\mathcal{B}_1^1(\Omega)$, and further obtained a widely used curvature inequality, stating that the curvature $\mathcal{K}_{S^*}$ of the backward shift operator dominates the curvature $\mathcal{K}_T$ if $T$ is contractive. Subsequently, G. Misra and N.S.N. Sastry [52, 53] proved that the inequality holds also for curvatures of tuples in $\mathcal{B}_m^1(\Omega)$. Conversely, the fact that the curvature inequality implies that the operator has a stronger contraction than usual case has been proved by S. Biswas, D. K. Keshari and G. Misra in [9]. Other properties of curvature inequality have been discussed in [4, 22, 23, 24, 51, 66].

In [18], R.E. Curto and N. Salinas linked the above Cowen-Douglas operator theory to the generalized reproducing kernel theory. They also discussed the correspondence between the analytic functional Hilbert space with coordinate multiplication $M_z = (M_{z_1}, \cdots, M_{z_m})$ and the canonical module of Cowen-Douglas tuples, proving the following result

**Theorem 1.1.** [18] Under mild conditions, the tuple $M_z = (M_{z_1}, \cdots, M_{z_m})$ acting on two analytic functional Hilbert spaces are unitarily equivalent if and only if their normalized reproducing kernel functions are intertwined by a constant unitary matrix.

In [11], it has been shown that the form of the operator which intertwines the coordinate multiplication acting on holomorphic Hilbert space with matrix-valued reproducing kernels.

It is well known that unitary operators maintain rigidity, while general invertible operators destroy rigidity. Taking this into account, we expect that the study of operator similarity is challenging, even in one variable. The model theorem is given in Chapter 0.2 of [56], in the view of complex geometry, and shows that the eigenvector bundle induced by contraction in $\mathcal{B}_n^1(\Omega)$ has a kind of tensor structure. By using the main result of [64] and the model theorem for contractions, H. Kwon and S. Treil proved a theorem which allows one to decide whether a contractive operator $T$ is similar to the $n$ times copies of $M_z^*$ on Hardy space or not, which is

$$
\left\| \frac{\partial P(w)}{\partial w} \right\|_{HS}^2 - \frac{n}{(1 - |w|^2)^2} \leq \frac{\partial^2}{\partial w \partial w} \psi(w), \quad w \in \mathbb{D}
$$

for projection-valued function $P$ with $\text{ran} P(w) = \ker(T - w)$ and a bounded subharmonic function $\psi$. Then, the result was generalized to the case of weighted Bergman shift by R.G. Douglas, H. Kwon and S. Treil [47]. Subsequently, the quantity $-\left\| \frac{\partial P(w)}{\partial w} \right\|_{HS}^2$ has been proved to be the trace of the curvature of $T$ (cf. [29]). Currently, this result does not have a version for commuting m-tuples. Although there exist plenty of model theorems about the commuting operator tuples [1, 2, 3, 55], the techniques cannot be easily generalized for the lack of proper condition for the Corona theorem in several variables.

In infinite-dimensional separable Hilbert spaces, strongly irreducible operators can be regarded as a natural generalization of Jordan block matrix. Strong irreducibility is a similarity invariant of operators. In [10], Y. Cao, J.S. Fang and C.L. Jiang introduced the $K_0$-group into the similarity classification of operators and characterized when operators have a unique strongly irreducible decomposition up to similarity. Consequently, C.L. Jiang, X.Z. Guo and the second author gave a similarity theorem of Cowen-Douglas operators by using the ordered $K$-group of the commutant algebra as an invariant [38]. From the perspective of complex geometry, the similarity of Cowen-Douglas operators
may be described through the equivalence of two families of eigenvectors in \[37\]. Using the eigenvector bundle associated to \(T \in \mathcal{B}^1_n(\Omega)\), M. Uchiyama discussed when \(T\) is similar or quasi-similar to the unilateral shift \([65]\).

In \([31]\), W.W. Hastings provided a function-theoretic characterization of subnormal tuples quasi-similar to the Cauchy tuple. Concerning absolute equivalence, virtual unitary equivalence, and almost unitarily equivalence of tuples, readers are referred to \([14, 41, 63]\).

In 2007, R.G. Douglas raised an open question \([20]\) (Question 4), which has not been completely solved so far. The open question is the following.

\[\text{Question:}\] Can one give conditions involving the curvatures which imply that two quasi-free Hilbert modules of multiplicity one are similar?

In this note, the main result is above the geometric similarity invariant of arbitrary Cowen-Douglas tuples without the assumptions of \(n\)-hypercontraction and the help of the Corona theorem. To some extent, it gives a partial answer to the question above.

The paper is organized as follows. In section 2, we recall some notions and basic results above tuples in the Cowen-Douglas class. In section 3, we obtain an equivalence condition for the similarity of commuting operator tuples. Furthermore, a similarity classification theorem for tuples in \(\mathcal{B}^1_m(\Omega)\) is given by using the local equivalence of the holomorphic bundles associated with some Cowen-Douglas tuples of index two. In section 4, we introduce a new class of commuting tuples in the Cowen-Douglas class (notice that the unitary intertwining operator is not diagonal in this case). In section 5, some weakly homogeneous operators are investigated.

2. Preliminaries

In this section, we will recall some notations and basic results of tuples in the Cowen-Douglas class. Let \(\mathcal{L}(\mathcal{H})^m\) be the collection of all commuting \(m\)-tuples of bounded operators on \(\mathcal{H}\). For \(T = (T_1, \ldots, T_m) \in \mathcal{L}(\mathcal{H})^m\), we define \(T_x = (T_1 x, \ldots, T_m x)\), \(x \in \mathcal{H}\) and \(T - w = (T_1 - w_1, \ldots, T_m - w_m)\), then \(\ker(T - w) = \bigcap_{i=1}^m \ker(T_i - w_i)\) with \(w = (w_1, \ldots, w_m)\) in \(\Omega\). The class of Cowen-Douglas tuple of operators with rank \(n\) over \(\Omega\): \(\mathcal{B}^n_m(\Omega)\) is defined as follows \([15, 17]\):

\[\mathcal{B}^n_m(\Omega) := \{T \in \mathcal{L}(\mathcal{H})^m : \begin{align*}
(1) \ & \bigcup_{w \in \Omega} \ker(T - w) = \mathcal{H}, \\
(2) \ & \text{ran}(T - w) \text{ is closed for all } w \in \Omega, \\
(3) \ & \dim \ker(T - w) = n \text{ for all } w \in \Omega. \end{align*}\]

It follows that for each \(w \in \Omega\), \(\ker(T - w)\) is an \(n\)-dimensional vector subspace of \(\mathcal{H}\). Define \(E_T := \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker(T - w)\}\) with a projection map \(\pi : E_T \rightarrow \Omega\) such that \(\pi^{-1}(w) = \ker(T - w)\). It is a sub-bundle of \(\Omega \times \mathcal{H}\) and its Hermitian structure comes from \(\mathcal{H}\). Thus, \(E_T\) associated with \(T\) is an \(n\)-dimensional Hermitian holomorphic vector bundle.

\[\text{Theorem 2.1.} \ [15, 17] \ Let T, S \in \mathcal{B}^n_m(\Omega). \ Then T \sim_u S \ if \ and \ only \ if \ the \ Hermitian \ holomorphic \ vector \ bundles \ E_T \ and \ E_S \ are \ congruent \ (denoted \ by \ E_T \sim_u E_S) \ over \ some \ open \ subset \ \Omega_0 \ of \ \Omega \subset \mathbb{C}^m.\]

When \(m = 1\), the above theorem is proved in Theorem 2.6 of \([15]\), in the case of \(m > 1\), Theorem 2.1 is also valid \([17\), pp.16] due to M.J. Cowen and R.G. Douglas. They make a rather detailed study of certain aspects of complex geometry and introduce the following concepts.
Let $E$ be a $C^\infty$ vector bundle over $\Omega$. A connection $D$ is a differential operator, which takes sections of $E$ to sections with 1-form coefficients and satisfies the Leibnitz rule $D(fs) = (df)s + fDs$ for section $s$ and function $f$. Similarly, $D^2$ can be defined, $D^2s = \mathcal{K}sdzd\bar{z}$ for section $s$, bundle map $\mathcal{K}$ determined by $D^2$ is called as the curvature of bundle $E$.

For every Hermitian holomorphic vector bundle $E$ over $\Omega$, there is a unique canonical connection $\Theta$, which is a Chern connection metric-preserving and compatible with the holomorphic structure. Given a holomorphic frame $\gamma = \{\gamma_i\}_{i=1}^n$ of $E$, we have the metric $h(w) = \langle (\gamma_j(w), \gamma_i(w)) \rangle_{\gamma_i,\gamma_j}$ and $D\gamma = \gamma \Theta, \Theta = (\Theta_{ij})_{i,j=1}^n$ is the matrix of connection 1-form. The curvature of $E$ can be defined as:

$$\mathcal{K}(w) = - \sum_{i,j=1}^m \frac{\partial}{\partial w_j} (h^{-1}(w) \frac{\partial h(w)}{\partial w_i}) dw_i \wedge dw_j$$

for $w = (w_1, \cdots, w_m) \in \Omega$. When $E$ is a line bundle, equation (2.1) is equivalent to $\mathcal{K}(w) = - \sum_{i,j=1}^m \frac{\partial^2 \log |\gamma(w)|^2}{\partial w_i \partial w_j} dw_i \wedge dw_j$, where $\gamma$ is a non-zero section of $E$.

For any $C^\infty$ bundle map $\phi$ on $E$ and given frame $\sigma$ of $E$, we have that

1. $\phi_w(\sigma) = \frac{\partial}{\partial w} (\phi(\sigma))$;
2. $\phi_w(\sigma) = \frac{\partial}{\partial w} (\phi(\sigma)) + [h^{-1} \frac{\partial}{\partial w} h, \phi(\sigma)]$.

Since the curvature can also be regarded as a bundle map, we obtain covariant derivatives $\mathcal{K}_{w^t\bar{w}^j}$, $I, J \in \mathbb{Z}_+^m$ of the curvature by using the inductive formulae above, where $\mathbb{Z}_+^m$ is the collection of $m$-tuples of nonnegative integers. The curvature $\mathcal{K}$ and it’s covariant derivatives $\mathcal{K}_{w^t\bar{w}^j}$ are the unitarily invariants of Hermitian holomorphic vector bundle $E$ (see [13, 17]).

**Theorem 2.2.** [15, 17] Let $T, S \in \mathcal{B}_1^m(\Omega)$. Then $E_T \sim_u E_S$ if and only if there exists an isometry $V : E_T \to E_S$ and a number $m$ depending on $E_T, E_S$ such that $V\mathcal{K}_{T,w^t\bar{w}^j} = \mathcal{K}_{S,w^t\bar{w}^j}V, I, J \in \mathbb{Z}_+^m$.

### 3. On the similarity of commuting operator tuples

The classification of similarities of commuting operator tuples has always been a challenging problem. Even in the operator case, it is not yet clear how to describe the similarity of Cowen-Douglas operators in $\mathcal{B}_1^1(\Omega)$ using only geometric quantities, such as the curvature. M.J. Cowen and R.G. Douglas put forward the following conjecture in 4.35 of [15]: If $\mathbb{D}$ (the closure of unit disc $\mathbb{D}$) is a $k$-spectral set for $T, S \in \mathcal{B}_1^1(\mathbb{D})$, then $T \sim_s S$ if and only if their curvatures $\mathcal{K}_T$ and $\mathcal{K}_S$ satisfy $\lim_{w \to \partial \mathbb{D}} \frac{\mathcal{K}_T(w)}{\mathcal{K}_S(w)} = 1$. In [12, 13], two counter examples were constructed by D.N. Clark and G. Misra. Instead of the quotient of the curvatures, they considered the quotient of metrics $h_T$ and $h_S$ of $E_T$ and $E_S$ denoted by $a_w$. It was then proved in [13] that contraction $T$ is similar to $S_0$ (with weight sequence $\{(\frac{n+1}{n+2})^2\}_{n=0}^\infty$) if and only if $a_w$ is bounded and bounded below by $0$. This result can be regarded as a geometric version of the classical result for the weighted shifts given by A.L. Shields (see [62]). For recent developments concerning the similarity of Cowen-Douglas operators, the reader is referred to [21, 23, 24, 34, 47].

Although there are many studies on the similarity classification of Cowen-Douglas operators, the similarity classification of commuting tuples is not yet fully solved. In this chapter, we provide a different necessary and sufficient condition for the similarity
of commuting operator tuples. We introduce the following definition of $\sigma_{T_0,T_1}$, and the notation is adopted from the next.

**Definition 3.1.** Let $T_i \in \mathcal{L}(\mathcal{H}_i)^m$, $i = 0, 1$. Define $\sigma_{T_0,T_1} : \mathcal{L}(\mathcal{H}_1,\mathcal{H}_0) \rightarrow \mathcal{L}(\mathcal{H}_0)^m$ be the tuple

$$\sigma_{T_0,T_1}(X) = T_0 X - X T_1, \ X \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_0).$$

Let $\sigma_{T_0} : \mathcal{L}(\mathcal{H}_0) \rightarrow \mathcal{L}(\mathcal{H}_0)^m$ be the tuple $\sigma_{T_0,T_0}$.

### 3.1. On the similarity of commuting tuples

In order to describe clearly Theorem 3.2, Lemma 3.3, Corollary 3.4 and Theorem 3.6, we need to introduce the following notations. Unless otherwise specified, we always assume that

$$T_{ij} = (T_{ij}^1, \ldots, T_{ij}^m), \ \tilde{T}_{ij} = (\tilde{T}_{ij}^1, \ldots, \tilde{T}_{ij}^m), \ \bar{S}_{ij} = (\bar{S}_{ij}^1, \ldots, \bar{S}_{ij}^m), 0 \leq i \leq j \leq 1$$

and

$$T = (T_{11}, \ldots, T_{1m}), \ S = (S_{00}, \ldots, S_{mm})$$

for some positive integer $m$. The main theorem of this paper as follows.

**Theorem 3.2.** Let $T$, $S \in \mathcal{L}(\mathcal{H})^m$. Suppose that $\{S_{11} \in \mathcal{L}(\mathcal{H})^m : \ker \sigma_{S_{11},T} = \{0\}\} \neq \emptyset$. Then $T \sim_u S$ if and only if there exist two operator tuples $\tilde{T} = (T_1, \ldots, T_m), \tilde{S} = (S_1, \ldots, S_m) \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})^m$ such that

1. $T_i = \begin{pmatrix} T_{00}^i & T_{01}^i \\ 0 & T_{11}^i \end{pmatrix}$, $S_i = \begin{pmatrix} S_{00}^i & S_{01}^i \\ 0 & S_{11}^i \end{pmatrix}$, $1 \leq i \leq m$, where $T_{01} \in \text{ran} \sigma_{T_{00},S}, S_{01} \in \text{ran} \sigma_{S_{00},T}$, and $\ker \sigma_{T_{00},S}, \ker \sigma_{S_{00},T} = \{0\}$;
2. $\tilde{T} \sim_u \tilde{S}$.

In order to prove our main theorem, we first need a lemma which characterizes the unitary operator which intertwines two special commuting operator tuples.

**Lemma 3.3.** Let $\tilde{T} = (T_1, \ldots, T_m), \tilde{S} = (S_1, \ldots, S_m) \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})^m$, where $T_i = \begin{pmatrix} T_{00}^i & T_{01}^i \\ 0 & T_{11}^i \end{pmatrix}$, $S_i = \begin{pmatrix} S_{00}^i & S_{01}^i \\ 0 & S_{11}^i \end{pmatrix}$, $1 \leq i \leq m$, and $T_{01} = -\sigma_{T_{00},T}(X), S_{01} = -\sigma_{S_{00},T}(Y)$ for some $X, Y \in \mathcal{L}(\mathcal{H})$. Suppose that $\ker \sigma_{T_{00},S} = \ker \sigma_{S_{00},T} = \{0\}$, then there exists a unitary operator $U = \{(U_{i,j})_{2 \times 2}\}$ such that $U\tilde{T} = \tilde{S}U$ if and only if the following statements hold

1. $U_{0i}T_{00}^{-1}U_{10}^{-1} = S_{1i}^*, U_{0i}^{-1}T_{11}U_{10} = S_{0i}^*, 1 \leq i \leq m$;
2. $(I + XX^*)^{-1} = U_{00}U_{10}, (I + X^*X)^{-1} = U_{01}U_{10}$;
3. $Y - U_{01}X^*U_{10}^{-1} \in \ker \sigma_{S_{01},T}$.

**Proof.** Let $U = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix}$. From $U\tilde{T} = \tilde{S}U$, we have

1. $U_{0i}X^*T_{11} - U_{10}T_{00}X = S_{1i}^*U_{11} - U_{11}T_{11}^i$,
2. $U_{0i}T_{00}^i - YS_{11}^iU_{10} = S_{00}^iU_{00} - S_{00}^iYU_{10}, T_{00}U_{00}^* + (XT_{11}^i - T_{00}^iX)U_{00}^* = U_{00}^*S_{00}^i$ and
3. $U_{0i}^{-1} = S_{1i}^*U_{10}, T_{11}^iU_{00}^* = U_{01}^*S_{00}^i, 1 \leq i \leq m$.

First of all, we will prove that $U_{0i}$ and $U_{10}$ are invertible. By (3.1) and (3.3), we have $U_{10}XT_{11}^i - S_{1i}^*U_{10}X = S_{1i}^*U_{11} - U_{11}T_{11}^i$ and $(U_{0i}X + U_{10})T_{11}^i = S_{1i}^*U_{10}X + U_{11}, 1 \leq i \leq m$. 
Moreover, combining equation (3.5), we obtain
\begin{equation}
I
\end{equation}
By equations (3.5)-(3.7), we have
\begin{equation}
X
\end{equation}
Based on a routine computation, we obtain
\begin{equation}
U
\end{equation}
Also use that fact
\begin{equation}
Y
\end{equation}
The proof of Theorem 3.2:
\begin{equation}
I
\end{equation}
By using the fact
\begin{equation}
U
\end{equation}
on the right side of equation (3.8), we obtain
\begin{equation}
U
\end{equation}
From equation (3.3) and the invertibility of $U_{i0}$ and $U_{10}$, we imply $U_{10}T_{00}^{-1}S_{i1} = S_{i1}^i$ and $U_{i0}^*-T_{11}^iU_{i0} = S_{00}^i$ for all $1 \leq i \leq m$.

By equation (3.2), for any $1 \leq i \leq m$, we have
\begin{equation}
S
\end{equation}
Multiplying $U_{10}^{-1}$ on the right side of equation (3.8), we obtain $S_{00}^i Y U_{10} - S_{00}^i U_{00} = Y S_{11}^i U_{10} - U_{00} T_{00}^i = Y S_{11}^i U_{10} - U_{00} U_{10}^{-1} S_{11}^i U_{10}$.

By equations (3.5)-(3.7), we have $U_{10}^* U_{10}(I + XX^*) = I$ and $(I + XX^*) U_{i0}^* U_{01} = I$. Moreover, combining equation (3.5), we obtain $U_{01}$ and $U_{00}$ are invertible and also $(I + XX^*) U_{01}$ and $U_{10}(I + XX^*)$. Since $I + XX^*$ and $I + XX^*$ are invertible, it is easy to see that $U_{10}^* U_{10}^{-1} = (I + XX^*)^{-1}$ and $U_{i0}^* U_{01} = (I + XX^*)^{-1}$. From equation (3.3) and the invertibility of $U_{i0}$ and $U_{10}$, we imply $U_{10} T_{00}^{-1} S_{11}^i = S_{i1}^i$ and $U_{i0}^* T_{11}^i U_{i0} = S_{00}^i$ for all $1 \leq i \leq m$.

By equation (3.2), for any $1 \leq i \leq m$, we have
\begin{equation}
S
\end{equation}
Multiplying $U_{10}^{-1}$ on the right side of equation (3.8), we obtain $S_{00}^i Y U_{10} - S_{00}^i U_{00} U_{10}^{-1} = Y S_{11}^i U_{10} - U_{00} U_{10}^{-1} S_{11}^i U_{10}$. Since equation (3.4), it follows that $S_{00}^i (Y - U_{01} X^* U_{10}^{-1}) = (Y - U_{01} X^* U_{10}^{-1}) S_{11}^i$, $1 \leq i \leq m$. That is, $Y - U_{01} X^* U_{10}^{-1} \in \cap_{i=1}^m \ker \sigma_{S_{00}^i, S_{11}^i} = \ker \sigma_{S, S_{11}}$.

For the sufficient part, let $U = \begin{pmatrix} U_{01} X^* & U_{01}^* \\ U_{10}^* & -U_{10} X \end{pmatrix}$ which satisfies the conditions (1)-(3).

It implies $U$ is a unitary operator. In the following, we will check $U \tilde{T} U^* = \tilde{S}$, that is, $U T_1 U^* = S_{i1}$, $1 \leq i \leq m$. Note that $U T_1 U^*$ has the following form
\begin{equation}
\begin{pmatrix}
(U_{01} X^* T_{00} U_{10} + (U_{01} X^* (XT_{11}^i - T_{00}^i) + U_{01} T_{11}^i)) U_{01}^* X^* T_{00} U_{10} + (U_{01} X^* (XT_{11}^i - T_{00}^i) + U_{01} T_{11}^i) (-X^* U_{10}) \\
U_{10} T_{00} U_{01}^* + U_{10} (XT_{11}^i - T_{00}^i) U_{01}^* - U_{10} X U_{10} + U_{10} (XT_{11}^i - T_{00}^i) (-X^* U_{10}) + U_{10} X U_{10}^* U_{10}^* \end{pmatrix}.
\end{equation}

Since $Y - U_{01} X^* U_{10}^{-1} \in \ker \sigma_{S, S_{11}}$, $Y S_{11}^i - S_{00}^i Y = (U_{01} X^* U_{10}^{-1}) S_{11}^i - S_{00}^i (U_{01} X^* U_{10}^{-1})$, $1 \leq i \leq m$. By statements (1) and (2), we have
\begin{align*}
U_{01} X^* T_{00} U_{10}^* + (U_{01} X^* (XT_{11}^i - T_{00}^i) X U_{10}^*) & = U_{01} X^* T_{00} (I + XX^*) U_{10}^* - U_{01} (I + XX^*) T_{11}^i X^* U_{10}^* \\
U_{01} X^* U_{10}^{-1} S_{11}^i - S_{00}^i U_{10}^{-1} X^* U_{10}^* & = Y S_{11}^i - S_{00}^i Y, \ 1 \leq i \leq m.
\end{align*}

Based on a routine computation, we obtain $U \tilde{T} U^* = \tilde{S}$. The proof of these equations also use that fact $X^* U_{10} U_{10} = X^* (I + XX^*)^{-1} = (I + XX^*)^{-1} X^* = U_{01} U_{01} X^*$. These equalities finish the proof of sufficient part.

The proof of Theorem 3.2.
Proof. Sufficiency: Firstly, there is a tuple $S_{11}$ such that $\ker \sigma_{S_{11}} T = \{0\}$. Suppose that two commuting tuples of operators $\tilde{T} = (T_1, \cdots, T_m)$ and $\tilde{S} = (S_1, \cdots, S_m) \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})^m$ are unitarily equivalent, that is, there exists a unitary operator $U = \left( \begin{smallmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{smallmatrix} \right)$ such that $U^{*} \tilde{T} U = \tilde{S}$ if condition (1) in the theorem is satisfied by $\tilde{T}, \tilde{S}$, by Lemma \[\text{3.3}\] we know that $U_{01}^{*}$ is invertible and $T_{11} U_{01}^{*} = U_{01} S_{00}$ for all $1 \leq i \leq m$. It follows that $T U_{01}^{*} = U_{01}^{*} S$. Thus, $T$ is similar to $S$.

Necessity: Since $T$ is similar to $S$, there exists an invertible operator $X_1$ such that

$$(3.9) \quad T = X_1 S X_1^{-1},$$

that is, $T_{11} = X_1 S_{00}^{1} X_1^{-1}$ for all $1 \leq i \leq m$. Without loss of generality, we assume that $(X_1^{-1})^{*} X_1^{-1} = I \geq 0$. Otherwise, let $\alpha = \inf \{x|x \in \sigma((X_1^{-1})^{*} X_1^{-1})\}$, we have $(X_1^{-1})^{*} X_1^{-1} = I \geq 0$, since $X_1$ is invertible. Then notice that $\alpha > 0$, upon replacing $X_1^{-1}$ by $\frac{X_1^{-1}}{\alpha}$, we obtain that $(X_1^{-1})^{*} X_1^{-1} = I \geq 0$. Therefore, we find a bounded linear operator $X$, such that

$$(3.10) \quad I + X X^{*} = (X_1^{-1})^{*} X_1^{-1}.$$ 

Obviously, $I + XX^{*}$ is also invertible and positive. In the same way as constructing $X$, we know that there exists $X_2$ satisfies

$$(3.11) \quad (I + XX^{*})^{-1} = X_2^{*} X_2,$$

and $X_2$ is an invertible operator.

Choosing a non-zero commuting tuple of operators $S_{11} \in \mathcal{L}(\mathcal{H})^m$ such that $\ker \sigma_{S_{11}} T = \{0\}$, that is, $\bigcap_{i=1}^{m} \ker \sigma_{S_{11}} T_{11} = \{0\}$. Next, we will construct another tuple $T_{00} \in \mathcal{L}(\mathcal{H})^m$. Let

$$(3.12) \quad T_{00}^{i} := X_{2}^{-1} S_{11}^{i} X_2,$$

for $1 \leq i \leq m$. Then $T_{00} = X_2^{-1} S_{11} X_2$.

We claim that $\ker T_{00} S = \{0\}$. If $Z \in \ker T_{00} S$, then for all $1 \leq i \leq m$, we have $T_{00} Z = Z S_{00},$ equivalently, $S_{11}^{i} X_2 Z X_1^{-1} = X_2 Z X_1^{-1} T_{11}^{i}$, since equations (3.9) and (3.12) hold. By $\bigcap_{i=1}^{m} \ker \sigma_{S_{11}} T_{11} = \{0\}$, we obtain $X_2 Z X_1^{-1} = 0$. Note that $X_1, X_2$ are both invertible. It follows that $Z = 0$ and $\bigcap_{i=1}^{m} \ker \sigma T_{00} S_{00} = \ker T_{00} S = \{0\}$.

For any bounded linear operator $W \in \bigcap_{i=1}^{m} \ker \sigma S_{00} S_{11}^{i}$, let $Y := W + X_1 X_1 X_2^{-1}$. This implies that

$$(3.13) \quad Y - X_1 X_1 X_2^{-1} \in \bigcap_{i=1}^{m} \ker \sigma S_{00} S_{11}^{i} = \ker \sigma S S_{11}.$$ 

Based on the above discussion, we may assume that operator tuples $\tilde{T} = (T_1, \cdots, T_m)$, $\tilde{S} = (S_1, \cdots, S_m)$ with $T_i = \left( \begin{smallmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{smallmatrix} \right), S_i = \left( \begin{smallmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{smallmatrix} \right), 1 \leq i \leq m$ and $T_{00} = -\sigma_{T_{00}} T(X), S_{01} = -\sigma_{T_{00}} T(Y)$. Then a simple calculation shows that $\tilde{T}$ and $\tilde{S}$ are commuting tuples and satisfy the condition (1).

Set $U := \left( \begin{smallmatrix} X_1 X_1 & 0 \\ X_2 & -X_2 X_1 \end{smallmatrix} \right)$. From Lemma \[\text{3.3}\] and equations (3.9)-(3.13), we obtain $U$ is unital and $U T = \tilde{S} U$. Hence, $\tilde{T} \sim \tilde{S}$. \hfill \Box

Given an $m$-tuple of operators $T = (T_1, \cdots, T_m) \in \mathcal{B}_n^m(\Omega)$, by subsection 2.2 in \[\text{[12]}\], we know that $T$ is unitarily equivalent to the adjoint of an $m$-tuple of multiplication
operators $M_z = (M_{z_1}, \cdots, M_{z_m})$ by coordinate functions on some Hilbert space $\mathcal{H}_K$ of holomorphic functions on $\Omega^* = \{ \bar{w} : w \in \Omega \}$ possessing a reproducing property $\langle f, K(\cdot, \bar{w})\xi \rangle_{\mathcal{H}_K} = \langle f(\bar{w}), \xi \rangle_{\mathbb{C}^n}$. It is expressed equivalently as $T \sim_u (\mathbf{M}_z^*, \mathcal{H}_K)$. Define $e_w$ be the evaluation function of $\mathcal{H}_K$ at $w$. Given a vector $\xi \in \mathbb{C}^n$, the function $e_w^* \xi \in \mathcal{H}_K$ and is denoted by $K(\cdot, \bar{w})\xi$, which has the reproducing property $\langle f, K(\cdot, \bar{w})\xi \rangle_{\mathcal{H}_K} = \langle f(\bar{w}), \xi \rangle_{\mathbb{C}^n}$. In addition, we have $\ker(\mathbf{M}_z - w) = \{ K(\cdot, \bar{w})\xi, \xi \in \mathbb{C}^n \}$.

In order to find the minimal order $m$ of covariant partial derivatives in Theorem 2.2, M.J. Cowen and R.G. Douglas introduced the concept of coalescing set $\mathcal{H}$. The algebra $\mathcal{H}(w)$ is generated by the curvatures and their covariant derivatives at $w$. The coalescing set of $\mathcal{H}(x)$ is the set where the dimension of $\mathcal{H}(x)$ (as a function of $x$) is not locally constant. It is trivially closed and nowhere dense. Furthermore, they proved that two bundles are locally equivalent on at least one dense open set, i.e. the complement of the coalescing set for the curvature corresponding to one of bundles, which means that the two bundles are equivalent. In Corollary 3.4 and Theorem 3.6, using this geometric quantity, we characterize the similarity classification of tuples in the Cowen-Douglas class. In other words, we give a partial answer to R.G. Douglas’s question about the geometric similarity of Cowen-Douglas class for multivariable case. Our results allow one to use geometric quantities of Cowen-Douglas tuples with index one to assess whether they are similar or not.

**Corollary 3.4.** Let $T, S \in \mathcal{B}_m^m(\Omega)$ and $T \sim_u (\mathbf{M}_z, \mathcal{H}_K), S \sim_u (\mathbf{M}_z, \mathcal{H}_K)$. Suppose that $\{S_{i1} \in \mathcal{L}(\mathcal{H}) : \ker(\sigma_{S_{i1}}) = \{0\} \neq \emptyset$. Then $T \sim_s S$ if and only if there exist two tuples $\tilde{T} = (T_1, \cdots, T_m), \tilde{S} = (S_1, \cdots, S_m) \in \mathcal{B}_m^m(\Omega)$ such that

1. $T_i = \begin{pmatrix} T_{i0}^{00} & T_{i0}^{01} \\ T_{i1}^{00} & T_{i1}^{01} \end{pmatrix}, S_i = \begin{pmatrix} S_{i0}^{00} & S_{i0}^{01} \\ S_{i1}^{00} & S_{i1}^{01} \end{pmatrix}, 1 \leq i \leq m$, where $T_{01} \in \text{ran} \sigma_{T_{00}, T}, S_{01} \in \text{ran} \sigma_{S_{10}, S}$ and $\ker(\sigma_{T_{00}, T}) = \{0\}$;

2. the bundles $E_{\tilde{T}}$ and $E_{\tilde{S}}$ of $T$ and $\tilde{S}$ are locally equivalent on an open dense subset of $\Omega$, the complement of the coalescing set for the curvature of $E_{\tilde{T}}$.

**Proof.** Let $\Omega_0$ be the complement of the coalescing set for the curvature of $E_{\tilde{T}}$. Clearly, $\Omega_0 \subset \Omega$. If the bundles $E_{\tilde{T}}$ and $E_{\tilde{S}}$ are locally equivalent on $\Omega_0$, by using the main theorem of [16] due to M.J. Cowen and R.G. Douglas, we obtain that metric-preserving connections $D_{\tilde{T}}$ and $D_{\tilde{S}}$ of $E_{\tilde{T}}$ and $E_{\tilde{S}}$ are equivalent to order $2n$ on $\Omega$. From the proof of the sufficiency in Theorem 3.2, we see that $T$ is similar to $S$.

From the proof of Theorem 3.2 and the main theorem of [16], we only need to prove $\tilde{T}, \tilde{S} \in \mathcal{B}_m^m(\Omega)$.

Suppose that there exist $X, Y$ such that $T_{01} = \sigma_{T_{00}, T}(-X), S_{01} = \sigma_{S_{10}, S}(-Y)$, that is, $T_{01}^{ij} = XT_{11}^{ij} - T_{00}^{ij}X, S_{01}^{ij} = YS_{11}^{ij} - S_{00}^{ij}Y$ for all $1 \leq j \leq m$. Without losing generality, we assume that $T = (\mathbf{M}_z, \mathcal{H}_K)$, and then $\ker(T - w) = \{ K_T(\cdot, \bar{w})\xi, \xi \in \mathbb{C}^n \}, w \in \Omega$. For fixed but arbitrary $w \in \Omega$ and $\xi \in \mathbb{C}^n$, we have

$$T_{01}^{ij}K_T(\cdot, \bar{w})\xi = (XT_{11}^{ij} - T_{00}^{ij}X)K_T(\cdot, \bar{w})\xi = w_j X K_T(\cdot, \bar{w})\xi - T_{00}^{ij}X K_T(\cdot, \bar{w})\xi = (T_{00}^{ij} - w_j)(-X K_T(\cdot, \bar{w})\xi), 1 \leq j \leq m.$$
It follows that $T_{0j}^j(\ker(T-w)) \subset \text{ran}(T_{00}-w), 1 \leq j \leq m$ and $T_{0j}(\ker(T-w)) \subset \text{ran}(T_{00}-w)$. Thus, $\tilde{T} \in \mathcal{B}_{2n}^m(\Omega)$. Similarly, we also have $\tilde{S} \in \mathcal{B}_{2n}^m(\Omega)$. This completes the proof. 

C.L. Jiang, D.K. Keshari, G. Misra and the second author in [35] showed that for $T, \tilde{T} \in \mathcal{B}_{1}^1(\Omega)$, if $XT = \tilde{T}X$, then either $X = 0$ or $X$ has a dense range. In fact, we see that this result is also true when $T, \tilde{T}$ are tuples in the Cowen-Douglas class with index one. The next lemma shows that the conditions in Lemma 3.3 can be satisfied in many cases.

Lemma 3.5. Let $T, S \in \mathcal{B}_{1}^m(\Omega)$ and $T \sim_u (M^*_z, \mathcal{H}_{K_0}), S \sim_u (M^*_z, \mathcal{H}_{K_1})$. If

$$\lim_{\text{dist}(w, \partial \Omega) \to 0} \frac{K_0(w, w)}{K_1(w, w)} = 0,$$

then there exists no non-zero bounded intertwining operator $X$ such that $XT = SX$, i.e. $\ker \sigma_{S,T} = \{0\}$.

Proof. Without loss of generality, we set $T = (M^*_z, \mathcal{H}_{K_0}), S = (M^*_z, \mathcal{H}_{K_1})$, where $\mathcal{H}_{K_i}$ are vector-valued analytic functional Hilbert spaces with reproducing kernels $K_i, i = 0, 1$, respectively. Suppose that $XT = SX$ for a bounded operator $X$. This means that $X(\ker(T-w)) \subset \ker(S-w), w \in \Omega$. Since $K_0(\cdot, \bar{w}) \in \ker(T-w), K_1(\cdot, \bar{w}) \in \ker(S-w)$, there exists a holomorphic function $\phi$ on $\Omega$ such that $X(K_0(\cdot, \bar{w})) = \phi(w)K_1(\cdot, \bar{w}), w \in \Omega$ (see details in Proposition 2.4 [58]). Note that $\|X(K_0(\cdot, \bar{w}))\| = \|\phi(w)\| \leq \|X\|$. We obtain that $|\phi|$ will go to zero when $\text{dist}(w, \partial \Omega)$ goes to zero. By the maximum modulus principle of holomorphic functions, we have $\phi(w)$ is equal to zero for all $w \in \Omega$ so does $X(K_0(\cdot, \bar{w}))$. According to the spanning property $\ker(T-w) = \bigvee\{K_0(\cdot, \bar{w})\}$, we infer $X = 0$. That means $\ker \sigma_{S,T} = \{0\}$. 

Theorem 3.6. Let $T, S \in \mathcal{B}_{1}^m(\Omega)$, and $T \sim_u (M^*_z, \mathcal{H}_{K_T}), S \sim_u (M^*_z, \mathcal{H}_{K_S})$. Then $T \sim_s S$ if and only if there exist two operator tuples $\tilde{T} = (T_1, \ldots, T_m), \tilde{S} = (S_1, \ldots, S_m) \in \mathcal{B}_{1}^m(\Omega)$ such that

1. $T_i = \begin{pmatrix} T_{0i} & T_{1i}^* \\ 0 & T_{11} \end{pmatrix}, S_i = \begin{pmatrix} S_{0i} & S_{1i}^* \\ 0 & S_{11} \end{pmatrix}, 1 \leq i \leq m$, where $T_{0i} \in \text{ran} \sigma_{T_{00}, T_i}, S_{0i} \in \text{ran} \sigma_{S_{00}, S_i}$, and $\ker \sigma_{T_{00}, S} = \ker \sigma_{S_{11}, T} = \{0\}$;

2. the bundles $E_{\tilde{T}}$ and $E_{\tilde{S}}$ of $\tilde{T}$ and $\tilde{S}$ are locally equivalent on an open dense subset of $\Omega$, the complement of the coalescing set for the curvature of $E_{\tilde{T}}$.

Proof. From the proof of Theorem 3.2 and Corollary 3.3, we only need to prove that there exists an $m$-tuple $S_{1i} \in \mathcal{B}_{1}^m(\Omega)$ satisfying the condition $\ker \sigma_{S_{11}, T} = \{0\}$.

We first choose a generalized Bergman kernel $K_\Omega$ on $\Omega \times \Omega$ (this concept was introduced by R.E. Curto and N. Salinas in [18], which satisfies $\lim_{\text{dist}(\Omega, \partial \Omega) \to 0} K_\Omega(w, w) = \infty$. Set $K_{S_{1i}} := K_\Omega \cdot K_T$. By [18] and Theorem 2.6 in [60], we know that $K_{S_{1i}}(w, w)$ is also a generalized Bergman kernel and there exists $S_{1i} \in \mathcal{B}_{1}^m(\Omega)$ such that $S_{1i} \sim_u (M^*_z, \mathcal{H}_{K_{S_{1i}}})$. Furthermore, we have that

$$\lim_{\text{dist}(w, \partial \Omega) \to 0} \frac{K_T(w, w)}{K_{S_{1i}}(w, w)} = \lim_{\text{dist}(w, \partial \Omega) \to 0} \frac{1}{K_\Omega(w, w)} = 0.$$
By Lemma 3.5 we know $\ker \sigma_{S_{11}, T} = \cap_{i=1}^m \ker \sigma_{S_{11}, T_i} = \{0\}$. This completes the proof.

\[ \Box \]

3.2. Application. Let $T$ be a bounded operator on some Hilbert $\mathcal{M}$, and $\mathcal{M}$ be a subspace of Hilbert space $\mathcal{N}$. A bounded operator $S$ on $\mathcal{N}$ is a dilation of $T$ if $P_M S|_\mathcal{M} = T$. We know that the adjoint of multiplication operator $M_\gamma$ which contains $\gamma$ with spectral radius $\gamma$, is a Cowen-Douglas operator with index one over $\mathcal{K}$ of $\mathcal{M}_\gamma$, we have that there exist plenty of operators such that their dilation is $\mathcal{M}_\gamma$ and they are all similar to $\mathcal{M}_\gamma$. Thus the following question is natural:

**Question:** For any Cowen-Douglas operator $S$, is there a Cowen-Douglas operator $T$ such that $S$ is a dilation of $T$ but not similar to $T$?

By using the main theorem of this paper, we give lots of positive examples for this question.

In [25], J.S. Fang, C.L. Jiang and the second author introduced an operator in the form of $T_x = \left( \begin{array}{cc} T & x \\ 0 & M_x' \end{array} \right)$, where $M_x'$ is the adjoint of multiplication operator on Hardy space $\mathcal{K}$, $M_x' = 0$ and $T \in \mathcal{L}(\mathcal{K})$ with spectral radius $r(T) < 1$, $x \in \mathcal{K}$. They proved that $T_x$ is a Cowen-Douglas operator with index one over $\Sigma$, a connected component of $\mathbb{D} \setminus \sigma(T)$ which contains $\{w \in \mathbb{D} : r(T) < |w| < 1\}$. Here $T_x \in \mathcal{L}(\mathcal{K} + \mathcal{K})$ is a dilation of $M_x'$, since $P_{0\oplus \mathcal{K}} T_x |_{0 \oplus \mathcal{K}} = M_x'$. In the following lemma, we replace the adjoint of the multiplication operator on Hardy space with a general Cowen-Douglas operator, and find the result is still valid.

**Lemma 3.7.** [25] Let $S \in \mathcal{B}_n(1) \cap \mathcal{L}(\mathcal{K})$ and $S \sim_u (M_x', \mathcal{H}_x)$. Suppose that $T \in \mathcal{L}(\mathcal{K})$ with spectral radius $r(T) < 1$ and $T_{S,x} = \left( \begin{array}{cc} T & x \\ 0 & M_x' \end{array} \right)$, where $e_0 = K(\cdot, 0) \xi_0$ for some $\xi_0 \in \mathbb{C}^n$, $x \in \mathcal{K}$. Let $\Sigma$ be the connected component of $\mathbb{D} \setminus \sigma(T)$ which contains $\{w \in \mathbb{D} : r(T) < |w| < 1\}$. Then we have the following:

1. For $w \in \mathbb{D} \setminus \sigma(T)$, $\dim \ker(T_{S,x} - w) = n$ and $\text{ran}(T_{S,x} - w) = \mathcal{H} \oplus \mathcal{H}$;
2. $T_{S,x} \in \mathcal{B}_n(\Sigma)$ if and only if $x$ is a cyclic vector of $T$, i.e., $\bigvee_{n \geq 0} \{T_n x \} = \mathcal{K}$.

**Proof.** In the following, we will describe $\ker(T_{S,x} - w)$ for $w \in \mathbb{D} \setminus \sigma(T)$.

Suppose that $\left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) \in \ker(T_{S,x} - w)$, $w \in \mathbb{D} \setminus \sigma(T)$. This is equivalent to $(T - w)y_1 + (x \otimes e_0)y_2 = 0$ and $(S - w)y_2 = 0$, $w \in \mathbb{D} \setminus \sigma(T)$. Without loss of generality, we assume that $S = (M_x', \mathcal{H}_x)$. Then $\ker(S - w) = \{K(\cdot, w)\xi, \xi \in \mathbb{C}^n\}$. That means $y_2 = K(\cdot, \bar{w})\xi$ for some $\xi \in \mathbb{C}^n$. Note that $T - w$ is invertible when $w \in \mathbb{D} \setminus \sigma(T)$. We have

$$y_1 = -(T - w)^{-1}(x \otimes e_0)y_2 = -(T - w)^{-1}(K(0, \bar{w})\xi, \xi_0).$$

Thus, $-(T - w)^{-1}(K(0, \bar{w})\xi, \xi_0)x \oplus K(\cdot, \bar{w})\xi$ is an eigenvector of $T_{S,x}$ with eigenvalue $w \in \mathbb{D} \setminus \sigma(T)$ for $\xi \in \mathbb{C}^n$. Since $\dim \ker(S - w) = n$, we infer $\dim \ker(T_{S,x} - w) = n$.

For any $w \in \mathbb{D} \setminus \sigma(T)$, $T_{S,x} - w$ is surjective if and for every $\left( \begin{array}{c} y_1' \\ y_2' \end{array} \right) \in \mathcal{H} \oplus \mathcal{H}$, there exists $(y_2') \in \mathcal{H} \oplus \mathcal{H}$ such that $\left( \begin{array}{c} T - w & x \otimes e_0 \\ 0 & S - w \end{array} \right) \left( \begin{array}{c} y_1' \\ y_2' \end{array} \right) = \left( \begin{array}{c} (T - w)y_1 + (x \otimes e_0)y_2 \\ (S - w)y_2 \end{array} \right) = \left( \begin{array}{c} y_1' \\ y_2' \end{array} \right)$.

The existence of $y_2$ is clear, since $S$ is a Cowen-Douglas operator. If we take $y_1 = (T - w)^{-1}(y_1' - (x \otimes e_0)y_2)$, then the last equation holds. This proves the statement (1).

In order to get statement (2), by (1), we only need to prove that

$$\bigvee_{w \in \Sigma} \{-(T - w)^{-1}(K(0, \bar{w})\xi, \xi_0)x \oplus K(\cdot, \bar{w})\xi, \xi \in \mathbb{C}^n\} = \mathcal{H} \oplus \mathcal{H}.$$
and \( \bigvee_{n \geq 0} \{ T^n x \} = \mathcal{H} \) are equivalent. Suppose that there exists an \( x_1 \oplus x_2 \in \mathcal{H} \oplus \mathcal{H} \) such that \( \langle - (T - w)^{-1} \langle K(0, \bar{w}) \xi, \xi_0 \rangle x + K(\cdot, \bar{w}) \xi, x_1 \oplus x_2 \rangle = 0 \). Then
\[
\langle (w - T)^{-1} \langle K(0, \bar{w}) \xi, \xi_0 \rangle x, x_1 \rangle = - \langle K(\cdot, \bar{w}) \xi, x_2 \rangle.
\]

Note that \( K(\cdot, \bar{w}) \xi \) is analytic on \( \mathbb{D} \), \( (w - T)^{-1} = \frac{1}{w} \sum_{n=0}^{\infty} \left( \frac{w}{T} \right)^n \) for \( |w| > r(T) \) and \( (w - T)^{-1} \langle K(0, \bar{w}) \xi, \xi_0 \rangle x, x_1 \rangle \) is analytic when \( |w| > r(T) \). Then \( (w - T)^{-1} \langle K(0, \bar{w}) \xi, \xi_0 \rangle x, x_1 \rangle = - (K(\cdot, \bar{w}) \xi, x_2), r(T) < |w| < 1 \). Thus, by the analytic continuation theorem, we know that \( \langle (w - T)^{-1} \langle K(0, \bar{w}) \xi, \xi_0 \rangle x, x_1 \rangle \) is analytic on \( \mathbb{C} \). Since \( \lim_{|w| \to \infty} \sum_{n=0}^{\infty} \left( \frac{w}{T^n} \right), x_1 \) = 0, \( (w - T)^{-1} \langle K(0, \bar{w}) \xi, \xi_0 \rangle x, x_1 \rangle \) is a bounded entire function on \( \mathbb{C} \). Then \( \sum_{n=0}^{\infty} (T^n x, x_1) \frac{1}{(w^n + 1)} = 0 \). Therefore, \( \langle T^n x, x_1 \rangle = 0, n \geq 0 \). Suppose \( x \) is a cyclic vector of \( T \). This implies \( x_1 = 0 \). From \( \bigvee_{w \in \Sigma} \ker(T_{S,x} - w) = \mathcal{H} \oplus \mathcal{H} \). Suppose \( x \) is not a cyclic vector of \( T \). Let \( 0 \neq x_1 \perp \{ T^n x : n \geq 0 \} \). Then \( (x_1 \oplus 0) \perp \bigvee_{w \in \Sigma} \ker(T_{S,x} - w) \) and therefore \( \bigvee_{w \in \Sigma} \ker(T_{S,x} - w) \neq \mathcal{H} \oplus \mathcal{H} \). This completes the proof.

**Proposition 3.8.** Let \( T_{S,x} \) be the operator in Lemma 3.7 and \( T \in \mathcal{L}(\mathcal{H}) \) with spectral radius \( r(T) < 1 \). Suppose that \( \lim_{|w| \to r(T)} \| (T - w)^{-1} x \| = \infty \) and \( x \) is a cyclic vector of \( T \), then \( T_{S,x} \) is not similar to \( S \).

**Proof.** Suppose that \( T_{S,x} \) is similar to \( S \). By Theorem 3.2 without losing generality, there exists a bounded linear operator \( X \) such that \( Y = (I + X^* X)^{1/2} \) and \( T_{S,x} Y = Y S \). By Lemma 3.7 and \( x \) is a cyclic vector of \( T \), we have that \( T_{S,x} \in \mathcal{B}_1(\Sigma) \), where \( \Sigma \) is the connected component of \( \mathbb{D} \setminus \sigma(T) \) which contains \( \{ w \in \mathbb{D} : r(T) < |w| < 1 \} \). Note that \( -(T - w)^{-1} \langle K(\cdot, \bar{w}) \xi, \xi_0 \rangle x + K(\cdot, \bar{w}) \xi \in \ker(T_{S,x} - w) \) for \( \xi \in \mathbb{C} \) and \( w \in \Sigma \). By Proposition 2.4 in \[55\] and \( T_{S,x} Y = Y S \), we find \( \xi_w \in \mathbb{C} \) such that
\[
(I + X^* X)^{1/2} K(\cdot, \bar{w}) \xi_w = -(T - w)^{-1} \langle K(0, \bar{w}) \xi_0, \xi_0 \rangle x + K(\cdot, \bar{w}) \xi_w.
\]

This implies that \( \| (I + X^* X)^{1/2} K(\cdot, \bar{w}) \xi_w \| \leq (1 + \| X \|^2) (K(\bar{w}, \bar{w}) \xi_0, \xi_w) \) and
\[
\| (I + X^* X)^{1/2} K(\cdot, \bar{w}) \xi_w \|^2 = \| (T - w)^{-1} \langle K(0, \bar{w}) \xi_0, \xi_0 \rangle x \|^2 + \langle K(\bar{w}, \bar{w}) \xi_0, \xi_w \rangle
\]
\[
= \| \langle K(0, \bar{w}) \xi_0, \xi_0 \rangle \| (T - w)^{-1} x \|^2 + \langle K(\bar{w}, \bar{w}) \xi_0, \xi_w \rangle.
\]

Then we have \( 0 \leq \| \langle K(0, \bar{w}) \xi_0, \xi_0 \rangle \| (T - w)^{-1} x \|^2 \leq \| X \|^2 \| K(\bar{w}, \bar{w}) \xi_0, \xi_w \rangle \). Note that \( \langle K(\bar{w}, \bar{w}) \xi_0, \xi_0 \rangle, w \in \mathbb{D} \) and \( X \) are bounded, then \( \| (K(0, \bar{w}) \xi_0, \xi_0 \| (T - w)^{-1} x \|^2 \) is bounded. Since \( \lim_{|w| \to r(T)} \| (T - w)^{-1} x \| = \infty \), \( \| (K(0, \bar{w}) \xi_0, \xi_0 \| \to 0 \) when \( |w| \to r(T) \). We know that \( \langle K(0, \bar{w}) \xi_0, \xi_0 \rangle \) is holomorphic, by the maximum modulus principle of holomorphic function, then \( \langle K(0, \bar{w}) \xi_0, \xi_0 \rangle = 0 \) for all \( w \in \Sigma \). This is a contradiction. So \( T_{S,x} \) is not similar to \( S \).

Upon using our main theorem (Theorem 3.2), a new proof of the sufficiency of A.L. Shields’ similarity theorem in \[02\] is given.

**Example 3.9.** Let \( T = (T^1, \cdots, T^n) \), \( S = (S^1, \cdots, S^n) \in \mathcal{B}_m^2(\Omega) \) and \( T \sim_u (M^*_z, \mathcal{H}_K), S \sim_u (M^*_u, \mathcal{H}_K) \), where \( K(z, w) = \sum_{\alpha \in \mathbb{Z}^m, \rho_0(\alpha) > 0} \rho_1(\alpha) z^\alpha w^\alpha \) and \( \rho_0(\alpha) > 0 \) for \( i = 0, 1, \alpha \in \mathbb{Z}^m \). If \( \rho_0(\alpha) \) is bounded for all \( \alpha \in \mathbb{Z}^m \), then \( T \sim_s S \).

**Proof.** By section 2.1 of \[38\] due to Q. Lin and Theorem 3.6 in this paper, we find that there exist tuples \( T_0, S_1 \) such that \( \ker(T_0, S_1) = \ker(T_0, S_1) = \{ 0 \} \). Since \( \rho_0(\alpha) \) is bounded for all \( \alpha \in \mathbb{Z}^m, \) it follows that there exist \( l > 0 \) such that \( l^2 \rho_1(\alpha) - \rho_0(\alpha) > 0 \) and
Let $X : \mathcal{H}_{K_0} \to \mathcal{H}_{K_1}$ as $Xw^\alpha = b_\alpha w^\alpha$ for all $\alpha \in \mathbb{Z}_+^m$. Thus, $l^2 K_1(\bar{w}, \bar{w}) = K_0(\bar{w}, \bar{w}) + \|XK_0(\cdot, \bar{w})\|^2 = \|(1 + XX^*)\bar{K}_0(\cdot, \bar{w})\|^2$. Note that $I + XX^*$ and $I + XX^*$ are positive and invertible, there exist invertible operators $U_{10}, U_{10}$ such that $(1 + XX^*)^{-1} = U_{10}^* U_{10}, (1 + XX^*)^{-1} = U_{10}^* U_{10}$. Choosing $\bar{Y} \in \ker \sigma_{S, S_i}$. Set $Y = \bar{Y} + U_{10} X^* U_{10}^{-1}, \bar{T} = (T_1, \cdots, T_m), S = (S_1, \cdots, S_m)$ and $U = (U_{10} X^* U_{10}^{-1}, T_i = (T_{i0}, 0 \ T_{i1}), S_i = (s_{i0}, s_{i1}), 1 \leq i \leq m,$ and $T_{01} = -\sigma_{s_{10}}, T(X), S_{01} = -\sigma_{s_{11}}(Y)$. By Theorem 3.2, we know that $U$ is unitary and $U\bar{T} = \bar{S}$. Thus we have $T \sim S$.

Example 3.10. Let $T_i \in B_1^1(\mathbb{D}), T_i \sim (M_{z_i}^*, \mathcal{H}_{K_i}), i = 0, 1$, where $K_0(z, w)$ is the reproducing kernel in Example 3.9. If $K_1(z, w) - K_0(z, w) = P(z, w)$ is a polynomial and positive over $\mathbb{D} \times \mathbb{D},$ then $T_0 \sim T_1$.

Proof. Since the symmetry of $K_i, i = 0, 1, P$ is also symmetric. Without losing generality, we assume that $P(z, w) = \sum_{p=q=0}^m a_{pq} z^p w^q, z, w \in \mathbb{D}$ for some positive $m$. This implies that matrix $A := (a_{pq})_{p,q=0}^m$ is positive. By diagonalization of $A$, there exist $\{\phi_i\}_{i=0}^m \subset H^\infty(\mathbb{D}),$ a set of bounded holomorphic function on $\mathbb{D}$, such that $P(z, w) = \sum_{j=0}^m \phi_j(z) \phi_j(w), z, w \in \mathbb{D}$. Let $K_0(z, w) = \sum_{n=0}^\infty c_n(z) e_n^*(w)$ for the orthogonal normal basis $\{e_n\}_{n=0}^\infty$ of $\mathcal{H}_{K_0}$. Set $\phi_j(z) = \sum_{i=0}^m b_{ji} e_i(z), b_{ji} \in \mathbb{C}, 0 \leq j \leq m$. A linear operator $X$ is defined as the following:

$$X(e_i) := \begin{cases} \sum_{j=0}^m b_{ji} e_j, & 0 \leq i \leq m, \\ 0, & i > m. \end{cases}$$

Then

$$\|X\| = \sup_{\|y\|=1} \|Xy\| = \sup_{\|y\|=1} \|X \sum_{n=0}^\infty b_n e_n\|$$

$$= \sup_{\|y\|=1} \left\| \sum_{n=0}^m b_n \left( \sum_{j=0}^m b_{jn} e_j \right) \right\|$$

$$\leq \left( \sum_{n=0}^m b_n^2 \right)^{\frac{1}{2}} \sum_{j=0}^m \left( \left\| \sum_{n=0}^m b_{jn}^2 \right\|^{\frac{1}{2}} \right) \leq (m + 1) M,$$

where $y = \sum_{n=0}^\infty b_n e_n \in \mathcal{H}, M = \max_{0 \leq i \leq m} \{\|\phi_i\|\}$. Thus, $X$ is bounded. Note that

$$XK_0(z, w) = \sum_{i=0}^\infty X e_i(w) e_i(z) = \sum_{i=0}^m \sum_{j=0}^m b_{ji} e_j(w) e_i(z) = \sum_{j=0}^m \sum_{i=0}^m b_{ji} e_i(z) e_j(w) = \sum_{j=0}^m \phi_j(z) e_j(w),$$

we then have $\|XK_0(z, w)\|^2 = \sum_{i=0}^\infty |\phi_i(z)|^2$ and $\|XK_0(\cdot, \bar{w})\|^2 = P(w, \bar{w})$. Similarly to the proof of Example 3.9, we deduce that $T_0 \sim T_1$.

Example 3.11. Let $T_i = (T_{i1}, \cdots, T_{in}) \in B_1^m(\Omega) \cap \mathcal{L}(\mathcal{H}_i)$ and $T_i \sim (M_{z_i}^*, \mathcal{H}_{K_i}), i = 0, 1$. If there exists a uniformly bounded positive sequence $\{\lambda_\alpha\}_{\alpha \in \mathbb{Z}_+^m}$ such that $K_1(z, w) - K_0(z, w) = \sum_{\alpha \in \mathbb{Z}_+^m} \lambda_\alpha e_\alpha(z) e_\alpha^*(w)$ for some orthonormal basis $\{e_\alpha\}_{\alpha \in \mathbb{Z}_+^m}$ of $\mathcal{H}_0$, then $T_0 \sim T_1$. 


Proof. Without loss of generality, we assume that \( T_i = (M^\ast_{\varphi_i}, \mathcal{H}_{K_i}), i = 0, 1 \). Since \( T_i \in \mathcal{B}^m_1(\Omega) \), we know that \( K_i(\cdot, \bar{w}) \in \ker(T_i - w) \) for \( w \in \Omega \) and \( i = 0, 1 \). Then

\[
K_1(w, w) - K_0(w, w) = \sum_{\alpha \in \mathbb{Z^+}_m} \lambda_\alpha |e_\alpha(w)|^2
= \sum_{\alpha \in \mathbb{Z^+}_m} \lambda_\alpha \langle K_0(\cdot, \bar{w}), e_\alpha \rangle^2
= \sum_{\alpha \in \mathbb{Z^+}_m} \lambda_\alpha \langle K_0(\cdot, \bar{w}), e_\alpha \rangle \langle e_\alpha, K_0(\cdot, \bar{w}) \rangle
= \left\langle \left\{ \sum_{\alpha \in \mathbb{Z^+}_m} \lambda_\alpha \langle K_0(\cdot, \bar{w}), e_\alpha \rangle e_\alpha, K_0(\cdot, \bar{w}) \right\} \right\rangle.
\]

(3.14)

Let \( X_1 := \sum_{\alpha \in \mathbb{Z^+}_m} \lambda_\alpha e_\alpha \otimes e_\alpha \). Further, equation (3.14) can be written as \( K_1(w, w) - K_0(w, w) = \langle X_1 K_0(\cdot, \bar{w}), K_0(\cdot, \bar{w}) \rangle \). It is easy to see that \( X_1 \) is positive, since \( \lambda_\alpha > 0 \) for all \( \alpha \in \mathbb{Z^+}_m \). Thus, there exists an operator \( X_2 \) such that \( X_1 = X_2 X_2 \) and \( K_1(w, w) - K_0(w, w) = \|X_2 K_0(\cdot, \bar{w})\|^2 \). Similarly to the proof of Example 3.9, we deduce that \( T_0 \sim_s T_1 \). \( \square \)

Example 3.12. Let \( T_i = (T_{i1}, \cdots, T_{in}) \in \mathcal{B}^m_1(\Omega) \cap \mathcal{L}(\mathcal{H}_i) \) and \( T_i \sim_u (M^\ast_{\varphi_i}, \mathcal{H}_{K_i}), i = 0, 1 \). Suppose that \( L^2(X, \mu) \) is separable for some \( \sigma \)-finite measure space \( (X, \mu) \). If there exists \( \phi \in L^\infty(X, \mu) \) (need not to be holomorphic) such that \( K_1(w, w) = (1 + |\phi(w)|^2)K_0(w, w) \), then \( T_0 \sim_s T_1 \).

Proof. From \( \phi \in L^\infty(X, \mu) \) for some \( \sigma \)-finite measure space \( (X, \mu) \), then there is a multiplication operator \( M_\phi : L^2(X, \mu) \to L^2(X, \mu) \) is defined by \( M_\phi f(x) = \phi(x) f(x) \), and it satisfies \( \|M_\phi f\|_2 = (\int_X |\phi f|^2 d\mu)^{\frac{1}{2}} \leq (\int_X (|\phi| \|f\|)^2 d\mu)^{\frac{1}{2}} \leq \|\phi\|_\infty \|f\|, f \in L^2(X, \mu) \), thus \( M_\phi \) is bounded. For \( f, g \in L^2(X, \mu) \), we have \( \langle M_\phi f, g \rangle = \int_X (\phi f) g d\mu = \int_X f(\phi) g d\mu = \langle f, M_\phi g \rangle \), which implies \( M_\phi^* = M_\phi \).

From \( K_1(w, w) = (1 + |\phi(w)|^2)K_0(w, w) \), we have \( \|K_1(\cdot, \bar{w})\|^2 = (1 + |\phi(w)|^2)\|K_0(\cdot, \bar{w})\|^2 \). Since \( L^2(X, \mu) \) is separable, there is a unitary operator \( U : \mathcal{H}_{K_0} \to L^2(X, \mu) \) such that

\[
\|K_1(\cdot, \bar{w})\|^2 = \|U K_0(\cdot, \bar{w})\|^2
= \langle (1 + |\phi(w)|^2) U K_0(\cdot, \bar{w}), U K_0(\cdot, \bar{w}) \rangle
= \langle (1 + M_\phi^* M_\phi) U K_0(\cdot, \bar{w}), U K_0(\cdot, \bar{w}) \rangle
= \| (1 + U^* M_\phi^* M_\phi U) K_0(\cdot, \bar{w}) \|^2.
\]

Without loss of generality, we assume that \( T_i = (M^\ast_{\varphi_i}, \mathcal{H}_{K_i}) \), then \( K_i(\cdot, \bar{w}) \in \ker(T_i - w) \) for \( w \in \Omega \) and \( i = 0, 1 \). Similarly to the proof of Example 3.9, we deduce that \( T_0 \sim_s T_1 \). \( \square \)

4. A Subclass \( N^m \mathcal{PB}^m_{n, a, n_1}(\Omega) \) of Cowen-Douglas Tuples

Let \( \text{M"ob} \) denote the group of all biholomorphic automorphisms of \( \mathbb{D} \). Recall that a bounded operator \( T \) is said to be homogeneous if the spectrum \( \sigma(T) \) of \( T \) is contained in \( \mathbb{D} \) and for every \( \phi \in \text{M"ob} \), \( \phi(T) \) is unitarily equivalent to \( T \). The concept of homogeneous operator can be extended to the commuting operator tuple. When \( \mathcal{D} \) is a bounded symmetric domain, an \( m \)-tuple \( T = (T_1, \cdots, T_m) \) of commuting bounded operators is said to be homogeneous with respect to \( G \) if their joint Taylor spectrum is contained in \( \mathbb{D} \) and for every holomorphic automorphism \( \phi \in G \), \( \phi(T) \) is unitarily equivalent to \( T \) (see [24]). The topic of homogeneous operators and tuples received much attention.
with three parameters due to A. Koranyi. Next, another bi-lateral homogeneous 2-shift with a model, but also helps us to study the properties and similarity of other operators. Let $\{1\}$ be homogeneous. Let $\{1\}$ be homogeneous in [43]. This is the first example of irreducible bi-lateral homogeneous 2-shifts introduced by S. Hazra in [32].

For $\alpha \in \mathbb{C}$ and homogeneous operators $T_0, T_1$ acting on $\mathcal{H}$, $\left( T_0 \frac{\alpha(T_0 - T_1)}{T_1} \right)$ is homogeneous if $T_0$ and $T_1$ have the same associated unitary representation given by A. Koranyi in [43]. Let $t_n = t_n(a, b) = \sqrt{\frac{n+a}{n+b}}$, $n \in \mathbb{Z}$ for $a, b \in (0, 1), a \neq b$. Define the operator $T = T(a, b)$ as $Te_n = t_ne_{n+1}$ for the natural basis $\{e_n\}_{n \in \mathbb{Z}}$ of $l^2(\mathbb{Z})$. It is shown to be homogeneous. Let $\alpha > 0$. Defining $\tilde{T} = \tilde{T}(a, b, \alpha) = \left( \begin{array}{c} T(a, b) \alpha(T(a, b) - T(b, a)) \\ 0 \end{array} \right)$ on $\mathcal{H} \oplus \mathcal{H}$ and rearranging the bases, we obtain a block matrix so that it is $\tilde{T}_n = \tilde{T}_n(a, b, \alpha)$ at $(n+1, n)$-position and the rest are 0. It is shown that $\tilde{T}$ is homogeneous and irreducible in [43]. This is the first example of irreducible bi-lateral homogeneous 2-shifts with three parameters due to A. Koranyi. Next, another bi-lateral homogeneous 2-shift introduced by S. Hazra in [32]. Let $B(s)$ and $B$ be bi-lateral shifts, the weight sequence of $B(s)$ be $w_n = \frac{n+a+s}{n+b+s}(s \neq 0)$, and $B$ be unweighted. Then operators $B(s)$ and $B$ are homogeneous in Theorem 5.2 of [6]. For $\alpha > 0$, define $B(\lambda, s, \alpha) = \left( \begin{array}{c} B(s) \alpha(B(s) - B) \\ 0 \end{array} \right).$ It is also homogeneous by Lemma 2.1 of [43]. It is proved in [32] that $B(\lambda, s, \alpha)$ is irreducible and the homogeneous operators defined by A. Koranyi and S. Hazra, respectively, are mutually unitarily inequivalent.

Inspired by the above results, we here define a new class of tuples of commuting bounded operators. With the help of this class, we discuss the similarity of commuting tuples. In addition, we know that the Cowen-Douglas class is a very rich operator and tuple class, including many homogeneous operators, normal operators and so on. The structure of the elements in $B_n(\Omega)$ is very complicated, so that we still cannot clearly describe their similarity. Therefore, it is necessary to investigate a subclass of $B_n(\Omega)$.

4.1. Definitions. In what follows, we assume that $n_0, n_1$ are positive integers.

**Definition 4.1.** Let $T_{ii} = (T_{ii}^1, \cdots, T_{ii}^m) \in B_n(\Omega), i = 0, 1$ and $T_{01} = (T_{01}^1, \cdots, T_{01}^m)$ be a commuting $m$-tuple of bounded operators. Suppose that the $m$-tuple $T = (T_1, \cdots, T_m)$ satisfies $T_j = \left( \begin{array}{c} T_{i_0}^j \ T_{i_1}^j \\ 0 \end{array} \right)$ for $1 \leq j \leq m$. We call $T \in NFB_{n_0,n_1}(\Omega)$, if $T_{01} \in \text{ran} \sigma_{T_{00},T_{11}}$.

By Corollary 3.4 we see that tuples in $NFB_{n_0,n_1}(\Omega)$ are Cowen-Douglas tuples with index $n_0 + n_1$ over $\Omega$. If $n_0 = n_1 = n$, the class $FB_{n,n}(\Omega)$ can be expressed as $NFB_{2n}(\Omega)$.

**Remark 4.2.** Suppose that tuple $T$ satisfies the conditions of Definition 4.1 and there exists an operator $X$ such that $T_{01} = \sigma T_{00},T_{11}(-X)$. Then $T_{ii}, i = 0, 1$ are commuting
m-tuples, which means T is a commuting m-tuple, since

\[
T_pT_q = \begin{pmatrix}
T_{p0}T_{q0} & -T_{00}T_{11}X + XT_{11}T_{p0} & 0 \\
0 & T_{11}T_{11} & 0 \\
T_{p0}T_{q0} & -T_{00}T_{11}X + XT_{11}T_{p0} & T_{11}T_{11}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
T_{00} & 0 & 0 \\
0 & T_{11} & 0 \\
0 & 0 & T_{11}
\end{pmatrix}
\]

for all \(1 \leq p, q \leq m\).

Based on Theorem 1.49 in [40], C. Jiang, D.K. Keshari, G. Misra and the second author introduced an operator class, denoted by \(\mathcal{FB}_n^1(\Omega)\) in [34, 35], which is norm dense in \(\mathcal{B}_n^1(\Omega)\). They also showed that the complete unitary invariants of operators in \(\mathcal{FB}_n^1(\Omega)\) include the curvatures and the second fundamental forms of the diagonal operators. We will give the commuting tuple version of this kind of operator.

**Definition 4.3.** Let \(T_{ii} = (T_{11}, \ldots, T_{mm}) \in \mathcal{B}_1^m(\Omega) \cap \mathcal{L}(\mathcal{H})^m, i = 0, 1\). Suppose that there exists \(T_{01} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)\) such that \(T = (T_1, \ldots, T_m)\) is a commuting m-tuple with \(T_j = \begin{pmatrix} T_{00} & 0 \\ 0 & T_{11} \end{pmatrix}, 1 \leq j \leq m\). We call \(T \in \mathcal{FB}_2^m(\Omega)\), if \(T_{01} \in \ker \sigma_{T_{00}, T_{11}}\).

In order to show that the tuples in \(\mathcal{NFB}_2^m(\Omega)\) may not belong to \(\mathcal{FB}_2^m(\Omega)\), we need to introduce the following concept which is first defined in [39].

**Definition 4.4.** [39] Property (H) Let \(T_i \in \mathcal{L}(\mathcal{H}_i), i = 0, 1\) and \(T = \begin{pmatrix} T_{00} & XT_{11} - T_{00}X \\ T_{11} \end{pmatrix} \in \mathcal{B}_2^1(\Omega)\). We call T satisfies the Property (H) if and only if the following statements hold: If \(Y \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)\) satisfies

(i) \(T_{00}Y = YT_{11}\),

(ii) \(Y = T_{00}Z - ZT_{11}\) for some \(Z\).

Then \(Y = 0\). That is equivalent to \(\ker \sigma_{T_{00}, T_{11}} \cap \text{ran } \sigma_{T_{00}, T_{11}} = \{0\}\).

By Definition 4.4, we see if \(T = \begin{pmatrix} T_{00} & XT_{11} - T_{00}X \\ T_{11} \end{pmatrix}\) satisfies the Property (H), and \(XT_{11} \neq T_{00}X\) then \(T\) does not belong to \(\mathcal{FB}_2^1(\Omega)\). Otherwise, \(XT_{11} - T_{00}X \in \ker \sigma_{T_{00}, T_{11}} \cap \text{ran } \sigma_{T_{00}, T_{11}}\). That means \(XT_{11} = T_{00}X\). It is a contradiction. In the following, we will give two results to show when \(T\) would satisfy the Property (H).

**Proposition 4.5.** [39] Let \(T_0, T_1 \in \mathcal{L}(\mathcal{H})\) and \(S_1\) be the right inverse of \(T_1\). If \(\lim_{n \to \infty} \frac{\|T^n_0\| + \|S^n_1\|}{n} = 0\), then the Property (H) holds, i.e. if there exists \(X \in \mathcal{L}(\mathcal{H})\) such that \(T_0X = XT_1\) and \(X = T_0Y - YT_1\), then \(X = 0\) (i.e. \(\ker \sigma_{T_0, T_1} \cap \text{ran } \sigma_{T_0, T_1} = \{0\}\)).

**Example 4.6.** [39] Let \(A, B \in \mathcal{B}_1^1(\mathbb{D})\) be backward shift operators with weighted sequences \(\{a_i\}_{i=1}^\infty\) and \(\{b_i\}_{i=1}^\infty\). If \(\lim_{n \to \infty} \frac{\prod_{k=1}^b_i a_k}{\prod_{k=1}^b_i a_k} = \infty\), then \(\ker \sigma_{A, B} \cap \text{ran } \sigma_{A, B} = \{0\}\).

In [35], it is proved that the unitary operator of intertwining two operators \(T\) and \(\tilde{T}\) in \(\mathcal{FB}_n^1(\Omega)\) should be a diagonal matrix. From the proof of Lemma 3.3, it can be seen that the unitary operator intertwines the two tuples in the class \(\mathcal{NFB}_2^m(\Omega)\) could be non-diagonal. This is another reason why we study this new class. Although the structures of tuples in the classes \(\mathcal{NFB}_2^m(\Omega)\) and \(\mathcal{FB}_2^m(\Omega)\) are quite different, the following proposition shows that they are also closely related. The unitary equivalence of the tuples in \(\mathcal{NFB}_2^m(\Omega)\) can always be related to the similarity of the tuples in \(\mathcal{FB}_2^m(\Omega)\).
Proposition 4.7. For \( i = 0, 1 \), let \( T_{0i}, S_{0i} \in \mathcal{B}_I^{m}(\Omega), T_{00}, S_{00} \in \mathcal{L}(\mathcal{H})^m \). Let \( \widetilde{T} = (T_1, \cdots, T_m), \) \( \widetilde{S} = (S_1, \cdots, S_m) \in N \mathcal{FB}_2^{m}(\Omega) \) with \( T_j = \left( \begin{array}{c} T_{10} \ 0 \end{array} \right), S_j = \left( \begin{array}{c} S_{10} \ 0 \end{array} \right), 1 \leq j \leq m \). Suppose that \( \ker \sigma_{T_{00}, S_{00}} = \ker \sigma_{S_{11}, T_{11}} = \{0\} \). If \( \mathbf{T} \sim_u \mathbf{\tilde{S}} \), then there exist operators \( S_0, S_1 \) and tuples \( \mathbf{\tilde{T}} = (\tilde{T}_1, \cdots, \tilde{T}_m), \mathbf{\tilde{S}} = (\tilde{S}_1, \cdots, \tilde{S}_m) \) with \( \tilde{T}_i = \left( \begin{array}{c} S_{0i} \ 0 \end{array} \right), \tilde{S}_i = \left( \begin{array}{c} T_{1i} \ 0 \end{array} \right), 1 \leq i \leq m \), such that \( \mathbf{\tilde{T}}, \mathbf{\tilde{S}} \in \mathcal{FB}_2^{m}(\Omega) \) and \( \mathbf{T} \sim_s \mathbf{\tilde{S}} \).

Proof. Suppose that there exist \( X, Y \) such that \( T_{01} = \sigma_{T_{00}, T_{11}}(-X), S_{01} = \sigma_{S_{00}, S_{11}}(-Y) \). Since \( \ker \sigma_{T_{00}, S_{00}} = \ker \sigma_{S_{11}, T_{11}} = \{0\} \) and \( \mathbf{T} \sim_u \mathbf{\tilde{S}} \), by Lemma 3.3 we will find a unitary operator \( U = ((U_{ij}))_{2 \times 2} \) such that

\[
\begin{align*}
T_{00}^i &= U_{10}^{-1}S_{11}^iU_{10}, T_{11}^i = U_{01}^*S_{00}^iU_{01}^{-1},
S_{00}^i(Y - U_{01}X^*U_{10}^{-1}) &= (Y - U_{01}X^*U_{10}^{-1})S_{11}^i, 1 \leq i \leq m.
\end{align*}
\]

Multiplying \( U_{10} \) on the right side of the equation above, by equation (4.1), we have

\[
S_{00}^i(YU_{10} - U_{01}X^*) = (YU_{10} - U_{01}X^*)T_{00}^i, 1 \leq i \leq m.
\]

Then multiplying \( U_{01}^* \) on the left side of the last equation above, by equation (4.1) again and \( XU_{01}^*U_{01} = U_{10}^*U_{10} \) due to Lemma 3.3 we obtain

\[
T_{11}^i(U_{01}^*Y - X^*U_{10}) = (U_{01}^*Y - X^*U_{10})S_{11}^i, 1 \leq i \leq m.
\]

Setting \( S_0 = YU_{10} - U_{01}X^* \) and \( S_1 = U_{01}Y - X^*U_{10} \). By equations (4.2) and (4.3), we see that \( S_0 \in \ker \sigma_{S_{00}, T_{00}} \) and \( S_1 \in \ker \sigma_{T_{11}, S_{11}} \). Let \( \tilde{T}_i = \left( \begin{array}{c} S_{0i} \ 0 \end{array} \right), \tilde{S}_i = \left( \begin{array}{c} T_{1i} \ 0 \end{array} \right) \), \( 1 \leq i \leq m \) and \( \mathbf{\tilde{T}} = (\tilde{T}_1, \cdots, \tilde{T}_m), \mathbf{\tilde{S}} = (\tilde{S}_1, \cdots, \tilde{S}_m) \). That means \( \mathbf{\tilde{T}}, \mathbf{\tilde{S}} \in \mathcal{FB}_2^{m}(\Omega) \) from Definition 1.3. Set \( Z := U_{01}^* \oplus U_{10} \). Then \( Z \) is invertible. Using the equations \( XU_{01}^*U_{01} = U_{10}^*U_{10} \) and (4.1) again, we imply that \( Z\tilde{T}_iZ^{-1} = \tilde{S}_i \) for all \( 1 \leq i \leq m \). Hence, \( Z\mathbf{\tilde{T}} = \mathbf{\tilde{S}}Z \) and \( \mathbf{\tilde{T}} \sim_s \mathbf{\tilde{S}} \).

Let \( \{\mathbf{T}'\} = \{X|X \mathbf{T} = \mathbf{T}X\}, \{\mathbf{T}, \mathbf{T}'\} = \{X|X \mathbf{T} = \mathbf{T}X, \mathbf{T}' = \mathbf{T}'X\} \). The commuting tuple \( \mathbf{T} \) is said to be irreducible, if there is no nontrivial orthogonal idempotents in \( \{\mathbf{T}'\} \). The following lemma is given by J. Fang, C. Jiang and P. Wu in Lemma 3.3 of [26], which shows that the double commutant \( \{\mathbf{T}, \mathbf{T}'\} \) of irreducible operator \( \mathbf{T} \) is only scalar operators. We will prove that this result also holds for irreducible operator tuples.

Lemma 4.8. [26] If \( \mathbf{T} \in \mathcal{L}(\mathcal{H})^m \) is irreducible and there is \( X \in \mathcal{L}(\mathcal{H}) \) such that \( X \in \{\mathbf{T}, \mathbf{T}'\} \), then \( X \) is a scalar multiple of identity.

Proof. Since \( X \mathbf{T} = \mathbf{T}X, X \mathbf{T}' = \mathbf{T}'X \), we have \( X^*X \mathbf{T} = \mathbf{T}X^*X \). Then, for any spectral projection \( P \) of \( X^*X \), this implies \( P \mathbf{T} = \mathbf{T}P \). From the irreducibility of \( \mathbf{T} \), it follows that \( P = 0 \) or \( I \). Furthermore, \( \sigma(X^*X) = \{\alpha\} \) and \( X^*X = \alpha I \). Note that \( X \mathbf{T}(\ker X) = \mathbf{T}X(\ker X) = 0 \) and \( X \mathbf{T}'(\ker X) = \mathbf{T}'X(\ker X) = 0 \). We know that \( \ker X \) is a reducing subspace for \( \mathbf{T} \), then \( \ker X = \{0\} \) or \( \mathcal{H} \), since \( \mathbf{T} \) is irreducible. So, either \( X \) is injective or \( X = 0 \). Suppose that \( X \) is a injective in the dense range. By the polar decomposition of \( X \), we have \( X = U(X^*X)^{1/2} \), \( U \) is a unitary operator. We assume that \( \alpha \neq 0 \), then \( \mathbf{U} \mathbf{T} = \mathbf{T} \mathbf{U}, \mathbf{U} \mathbf{T}' = \mathbf{T}' \mathbf{U} \). Repeating the above assumption, we have \( \mathbf{U} = \beta \mathbf{I} \). Thus \( X = \sqrt{\alpha} \beta \mathbf{I} \) is a scalar multiple of identity. \( \square \)
Let $T_1, T_2$ be two bounded operators acting on $\mathcal{H}$ and $\alpha \in \mathbb{C}$. For $\tilde{T} = \begin{pmatrix} T_1 & \alpha(T_1 - T_2) \\ 0 & T_2 \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$, in Lemma 2.1 of [43], A. Koranyi proved that the operator $\tilde{T}$ is unitarily equivalent to $\begin{pmatrix} T_2 & \alpha(T_2 - T_1) \\ 0 & T_1 \end{pmatrix}$ through intertwining unitary operator $\begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}$. In the following two propositions, we discuss the conditions that the tuples in $N\mathcal{FP}^m_{n_0, n_1}$ make this conclusion hold, which is similar to the above.

**Proposition 4.9.** For $i = 0, 1$, let $T_{i \in} \in \mathcal{B}^m_n(\Omega) \cap \mathcal{L}(\mathcal{H})^m$ and $T_{01} = \sigma_{T_{00}, T_{11}}(-X)$, $S_{01} = \sigma_{T_{11}, T_{00}}(Y)$ for $X, Y \in \mathcal{L}(\mathcal{H})$. Let $\tilde{T} = (T_1, \cdots, T_m), \tilde{S} = (S_1, \cdots, S_m) \in N\mathcal{FP}^m_{2n}(\Omega)$ with $T_{i} = \begin{pmatrix} T_{01} & 0 \\ 0 & T_{11} \end{pmatrix}$, $S_{i} = \begin{pmatrix} S_{01} & 0 \\ 0 & S_{11} \end{pmatrix}$, $1 \leq i \leq m$. Suppose that $\ker \sigma_{T_{00}, T_{11}} = \{0\}$, $T_{00}, T_{11}$ are irreducible and $XX^* \in \{T_{00}\}', X^*X \in \{T_{11}\}'$. Then $\tilde{T} \sim_u \tilde{S}$ if and only if there exists $\theta \in \mathbb{R}$, such that $S_{01} = e^{i\theta} \sigma_{T_{11}, T_{00}}(-X^*)$.

**Proof.** Let $U = ([U_{i,j}])_{2 \times 2}$ be a unitary operator which satisfies $U\tilde{T} = \tilde{S}U$. By Lemma 3.3, we have $U_{01}, U_{10}$ are invertible and $(I + X^*X)^{-1} = U_{01}U_{10}, (I + XX^*)^{-1} = U_{10}U_{01}$. Since $X$ is a bounded linear operator, then $I + XX^*$ and $I + X^*X$ are positive and invertible. Furthermore, we have that $U_1 := (I + X^*X)\frac{1}{2}U_{01}, U_2 := U_{10}(I + XX^*)\frac{1}{2}$ are unitary. By using the statement (1) of Lemma 3.3, we also have $U_{10}T_{01} = T_{00}U_{01}, T_{11}U_{10} = U_{01}T_{11}, 1 \leq i \leq m$. It follows that

\begin{equation}
U_2(I + XX^*)^{-\frac{1}{2}}T_{01} = T_{00}U_2(I + XX^*)^{-\frac{1}{2}} \tag{4.4}
\end{equation}

and

\begin{equation}
T_{11}^{-1}(I + X^*X)^{-\frac{1}{2}}U_1 = (I + X^*X)^{-\frac{1}{2}}U_1T_{11}, 1 \leq i \leq m. \tag{4.5}
\end{equation}

From the conditions $XX^* \in \{T_{00}\}', X^*X \in \{T_{11}\}'$, we obtain that $(I + XX^*)T_{01} = T_{01}(I + XX^*)$ and $(I + X^*X)T_{11} = T_{11}(I + X^*X), 1 \leq i \leq m$. By functional calculus of positive operators, we have

\begin{equation}
(I + XX^*)^{-\frac{1}{2}}T_{01} = T_{01}(I + XX^*)^{-\frac{1}{2}}, (I + X^*X)^{-\frac{1}{2}}T_{11} = T_{11}(I + X^*X)^{-\frac{1}{2}}, 1 \leq i \leq m. \tag{4.6}
\end{equation}

Combining with equations (4.4) and (4.5), we imply that $U_2T_{01} = T_{01}U_{01}, U_1T_{11} = T_{11}U_1, 1 \leq i \leq m$. Thus, $U_{2} \in \{T_{00}, T_{01}\}', U_{1} \in \{T_{11}, T_{10}\}'$, since $U_{1}, U_{2}$ are unitary. From $T_{00}, T_{11}$ are irreducible and Lemma 3.3, we obtain $U_1 = e^{i\theta}I, U_2 = e^{i\theta}I$ for some $\theta_1, \theta_2 \in \mathbb{R}$. By the statement (3) of Lemma 3.3, we have $Y - U_{01}X^*U_{10}^{-1} \in \ker \sigma_{T_{11}, T_{00}}$. It follows that $T_{11}(Y - U_{01}X^*U_{10}^{-1}) = T_{11}(Y - e^{-i(\theta_1 + \theta_2)}X^*) = (Y - e^{-i(\theta_1 + \theta_2)}X^*)T_{00}$ and $YT_{00} - T_{11}Y = e^{-i(\theta_1 + \theta_2)}(X^*T_{00} - T_{11}X^*), 1 \leq i \leq m$. This finishes the proof of necessary part.

For the proof of the sufficient part, choose any $\theta_1, \theta_2 \in \mathbb{R}$ such that $\theta_1 + \theta_2 = -\theta$. Define the operator $U$ as follows

\begin{equation}
U = \begin{pmatrix} e^{-i\theta_1}(I + X^*X)^{-\frac{1}{2}}X^* & e^{-i\theta_1}(I + X^*X)^{-\frac{1}{2}} \\
e^{i\theta_2}(I + XX^*)^{-\frac{1}{2}} & -e^{i\theta_2}(I + XX^*)^{-\frac{1}{2}} \end{pmatrix}.
\end{equation}

Using the fact of $X(I + X^*X)^{-1} = (I + XX^*)^{-1}X$, we obtain $U$ is a unitary operator. By a simple calculation, we imply $UT_i = S_iU$ for $1 \leq i \leq m$, then $U\tilde{T} = \tilde{S}U$. \hfill \box

**Proposition 4.10.** For $i = 0, 1$, let $T_{i \in} \in \mathcal{B}^m_n(\Omega)$ and $T_{01} = \sigma_{T_{00}, T_{11}}(-X)$ for some self-adjoint operator $X$. Let $\tilde{T} = (T_1, \cdots, T_m) \in N\mathcal{FP}^m_{2n}(\Omega)$ with $T_i = \begin{pmatrix} T_{01} & 0 \\ 0 & T_{11} \end{pmatrix}, 1 \leq i \leq m$. \hfill \box
Suppose that \( X \in \{ T_{00} \}' \cap \{ T_{11} \}'. \) Then the operator \( \tilde{T} \) gotten by interchanging the roles of \( T_{00} \) and \( T_{11} \) is unitarily equivalent to \( T \).

**Proof.** Let \( \tilde{X} = \begin{pmatrix} x(I + X^2)^{-\frac{1}{2}} & (I + X^2)^{-\frac{1}{2}} \\ (I + X^2)^{-\frac{1}{2}} & -x(I + X^2)^{-\frac{1}{2}} \end{pmatrix} \). By functional calculus of positive operators, we have \( X(I + X^2)^{-\frac{1}{2}} = (I + X^2)^{-\frac{1}{2}}X \), thus \( \tilde{X} \) is self-adjoint. Note that

\[
\tilde{X} X^* = X^* \tilde{X} = \begin{pmatrix} x(I + X^2)^{-1} + (I + X^2)^{-1} x(I + X^2)^{-1} - (I + X^2)^{-1} x(I + X^2)^{-1} - (I + X^2)^{-1} x(I + X^2)^{-1} - x(I + X^2)^{-1} - x(I + X^2)^{-1} - x(I + X^2)^{-1} - x(I + X^2)^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},
\]

we see that \( \tilde{X} \) is unitary. From \( X \in \{ T_{00} \}' \cap \{ T_{11} \}' \), then for any \( 1 \leq i \leq m \), we have \( XT_{jj} = T_{jj} X, j = 0, 1 \). By functional calculus of \( I + X^2 \), we also have

\[
(I + X^2)^{-\frac{1}{2}} T_{jj} = T_{jj} (I + X^2)^{-\frac{1}{2}} \quad \text{and} \quad (I + X^2)^{\frac{1}{2}} T_{jj} = T_{jj} (I + X^2)^{\frac{1}{2}}, \quad 1 \leq i \leq m, j = 0, 1.
\]

Based on a simple calculation, \( \tilde{X} \tilde{T} = \tilde{T} \tilde{X} \) can be obtained. Hence, \( \tilde{T} \) is unitarily equivalent to \( \tilde{T} \). \( \square \)

4.2. **Some properties of tuples in** \( NFB_{n_0,n_1}^m(\Omega) \). The commuting operator tuple \( T \) is said to be strongly irreducible if there is no nontrivial idempotent in \( \{ T \}' \). Otherwise, it is strongly reducible. A strongly irreducible operator can be regarded as a natural generalization of a Jordan block matrix on the infinite dimensional case. In \( \text{[60]} \), C. Jiang proved that for any strongly irreducible Cowen-Douglas operator \( T \), \( \{ T \}' / \text{rad} \{ T \}' \) is commutative, where \( \text{rad} \{ T \}' \) denotes the Jacobson radical of \( \{ T \}' \). Based on this, C. Jiang gave a similarity classification of strongly irreducible Cowen-Douglas operators by using the \( K_0 \)-group of their commutant algebra as an invariant (see more details in \( \text{[61]} \)). These results are also generalized to the case of direct integrals of strongly irreducible operators by R. Shi (cf. \( \text{[61]} \)). The following proposition shows that the strong reducibility of tuples in \( NFB_{n_0,n_1}^m(\Omega) \), that is, every tuple in \( NFB_{n_0,n_1}^m(\Omega) \) can be written as the direct sum of two tuples in \( B_2^m(\Omega) \), \( i = 0, 1 \) up to similarity. For \( m \)-tuples \( T_{00} \) and \( T_{11} \), \( T_{00} \oplus T_{11} = (T_{00}^1 \oplus T_{11}^1, \ldots, T_{00}^m \oplus T_{11}^m) \).

**Proposition 4.11.** For \( i = 0, 1 \), let \( T_{ii} \in B_{n_i}^m(\Omega) \cap \mathcal{L}(\mathcal{K}_i^m) \) and \( T_{01} = \sigma_{T_{00},T_{11}}(-X) \) for \( X \in \mathcal{L}(\mathcal{K}_1,\mathcal{K}_0) \). Let \( \tilde{T} = (T_1, \ldots, T_m) \in NFB_{n_0,n_1}^m(\Omega) \) with \( T_i = \begin{pmatrix} T_{00}^i & T_{01}^i \\ T_{10}^i & T_{11}^i \end{pmatrix} \), \( 1 \leq i \leq m \).

Then \( \tilde{T} \) is strongly reducible. What is more, \( \tilde{T} \) is similar to \( T_{00} \oplus T_{11} \).

**Proof.** Let \( W = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \). We have that

\[
WT_j = \begin{pmatrix} T_{00}^j & T_{01}^j X \\ 0 & T_{11}^j \end{pmatrix} \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} = \begin{pmatrix} T_{00}^j & T_{01}^j \\ T_{10}^j & T_{11}^j \end{pmatrix} W, 1 \leq j \leq m
\]

and \( W\tilde{T} = (T_0 \oplus T_1) W \). Note that \( W \) is invertible and \( W^{-1} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \). Then we finish the proof. \( \square \)

The characterization of irreducibility of tuples in \( NFB_2^m(\Omega) \) is as follows.

**Proposition 4.12.** Let \( T_{ii} \in B_1^m(\Omega) \), \( T_{ii} \sim_u (M_z^*,\mathcal{K}_i) \), \( i = 0, 1 \) and \( T_{01} = \sigma_{T_{00},T_{11}}(-X) \) for some \( X \). Suppose that \( \tilde{T} = (T_1, \ldots, T_m) \in NFB_2^m(\Omega) \) with \( T_i = \begin{pmatrix} T_{00}^i & T_{01}^i \\ T_{10}^i & T_{11}^i \end{pmatrix} \), \( 1 \leq i \leq m \). If \( \lim_{\text{dist}(w,\mathcal{K}_i) \to 0} \frac{K_{0i}(w,w)}{K_{1i}(w,w)} = 0 \), then \( \tilde{T} \) is irreducible.
Proof. Suppose that \( \widetilde{\mathbf{T}} \) is reducible, then there exists a nontrivial orthogonal projection \( P = \left( P_{00} P_{01} \right) \in \{ \widetilde{\mathbf{T}} \}' \), such that

\[
\begin{pmatrix}
P_{00} T_{00} P_{00} (X T_{11} - T_{00} X) + P_{01} T_{11} \\
P_{10} T_{00} P_{00} (X T_{11} - T_{00} X) + P_{11} T_{11}
\end{pmatrix} = \begin{pmatrix}
T_{00} P_{00} (X T_{11} - T_{00} X) P_{00} & + (X T_{11} - T_{00} X) P_{01} \\
T_{10} P_{00} (X T_{11} - T_{00} X) + P_{11} T_{11}
\end{pmatrix}
\]

for all \( 1 \leq i \leq m \). It follows that \( P_{10} \in \bigcap_{i=1}^{m} \ker \sigma T_{11}, T_{00} = \ker \sigma T_{11}, T_{00} \).

By Lemma 3.5 if we have \( \lim_{\text{dist}(w, \Omega) \to 0} \frac{K_{n}(w, w)}{K_{1}(w, w)} = 0 \), then \( \ker \sigma T_{11}, T_{00} = \{ 0 \} \) and \( P_{10} = 0 \).

Note that \( P \) is a self-adjoint idempotent, we obtain \( P_{01} = 0 \) and \( P_{ii} = P_{ii}^2 \), \( i = 0, 1 \). From equation (4.6), we infer \( P_{00} T_{00} = P_{00} T_{00}, P_{11} T_{11} = T_{11} P_{11} \), \( 1 \leq i \leq m \). Then \( P_{00} T_{00} = T_{00} P_{00}, P_{11} T_{11} = T_{11} P_{11} \). Since tuples in \( \mathcal{B}^{m}_{n}(\Omega) \) are irreducible, we have \( P_{ii} = 0 \) or \( I \). According to \( P_{00} (X T_{11} - T_{00} X) = (X T_{11} - T_{00} X) P_{11} \), we have \( P_{00} = P_{11} = 0 \) or \( I \), that is, \( P \) is trivial. This is a contradiction. Hence, \( \widetilde{\mathbf{T}} \) is irreducible.

By the following proposition, the Hermitian holomorphic vector bundles corresponding to the tuples in \( \mathcal{N} \mathcal{F} \mathcal{B}^{n_{0}, n_{1}}(\Omega) \) is given.

**Proposition 4.13.** Let \( T_{ii} \in \mathcal{B}^{n}_{n}(\Omega) \), \( T_{ii} \sim_{u} (M_{ii}, \mathcal{H}_{K_{ii}}), i = 0, 1 \) and \( T_{01} = \sigma T_{00}, T_{11} (-X) \) for some \( X \). Suppose that \( \widetilde{\mathbf{T}} = (T_{1}, \ldots, T_{m}) \in \mathcal{N} \mathcal{F} \mathcal{B}^{m}_{n_{0} + n_{1}}(\Omega) \) with \( T_{i} = \left( \begin{smallmatrix} T_{i0} & T_{i1} \\ 0 & T_{i1} \end{smallmatrix} \right) \), \( 1 \leq i \leq m \). Then for all \( w \in \Omega \),

\[
E_{\mathbf{T}}(w) = \overline{\text{span}} \{ K_{0}(\cdot, \bar{w}) \xi_{0}, X(K_{1}(\cdot, \bar{w}) \xi_{1}) + K_{1}(\cdot, \bar{w}) \xi_{1}, \xi_{0} \in \mathcal{C}^{n_{0}}, \xi_{1} \in \mathcal{C}^{n_{1}} \}.
\]

**Proof.** Since \( E_{T_{ii}}(w) = \overline{\text{span}} \{ K_{i}(\cdot, \bar{w}) \xi_{i}, \xi_{i} \in \mathcal{C}^{n_{i}} \} \) and the dimension of \( E_{T_{ii}}(w) \) is \( n_{i}, i = 0, 1 \), it is easy to see that

\[
K_{0}(\cdot, \bar{w}) \xi_{0}, X(K_{1}(\cdot, \bar{w}) \xi_{1}) + K_{1}(\cdot, \bar{w}) \xi_{1} \in \ker(\widetilde{\mathbf{T}} - w), w \in \Omega, \xi_{0} \in \mathcal{C}^{n_{0}}, \xi_{1} \in \mathcal{C}^{n_{1}}.
\]

Note that \( \dim \ker(\widetilde{\mathbf{T}} - w) = n_{0} + n_{1}, w \in \Omega \), then we only need to prove that for each \( \xi_{0} \in \mathcal{C}^{n_{0}}, \xi_{1} \in \mathcal{C}^{n_{1}}, K_{0}(\cdot, \bar{w}) \xi_{0} \) and \( X(K_{1}(\cdot, \bar{w}) \xi_{1}) + K_{1}(\cdot, \bar{w}) \xi_{1} \) are linearly independent. For fixed but arbitrary \( \xi_{0} \in \mathcal{C}^{n_{0}}, \xi_{1} \in \mathcal{C}^{n_{1}} \), suppose that there exist \( x_{0}, x_{1} \in \mathcal{C} \) such that

\[
x_{0} K_{0}(\cdot, \bar{w}) \xi_{0} + x_{1} (X(K_{1}(\cdot, \bar{w}) \xi_{1}) + K_{1}(\cdot, \bar{w}) \xi_{1}) = 0.
\]

By taking the inner product with \( K_{1}(\cdot, \bar{w}) \xi', \xi' \in \mathcal{C}^{n_{1}} \) on both sides, we have that \( \langle x_{1} K_{1}(\cdot, \bar{w}) \xi_{1}, K_{1}(\cdot, \bar{w}) \xi' \rangle = 0 \). With the spanning property of \( \{ K_{1}(\cdot, \bar{w}) \xi', \xi' \in \mathcal{C}^{n_{1}} \} \), we infer \( x_{1} K_{1}(\cdot, \bar{w}) \xi_{1} = 0 \), then \( x_{1} = 0 \), since \( K_{1}(\cdot, \bar{w}) \xi_{1} \) is non-zero. Thus, we obtain \( x_{0} K_{0}(\cdot, \bar{w}) \xi_{0} = 0 \). Similarly, we have \( x_{0} = 0 \). This completes the proof.

**Example 4.14.** For \( i = 0, 1 \), let \( T_{ii}, S_{ii} \in \mathcal{B}^{n}_{n}(\Omega) \), \( T_{ii} = (M_{ii}, \mathcal{H}_{K_{ii}}), S_{ii} = (M_{ii}, \mathcal{H}_{K_{ii}}) \) and \( T_{01} = \sigma T_{00}, T_{11} (-X), S_{01} = \sigma S_{00}, S_{11} (-Y) \) for some \( X, Y \). Let \( \mathbf{T} = (T_{1}, \ldots, T_{m}), \mathbf{S} = (S_{1}, \ldots, S_{m}) \in \mathcal{N} \mathcal{F} \mathcal{B}^{m}_{2}(\Omega) \) with \( T_{i} = \left( \begin{smallmatrix} T_{i0} & T_{i1} \\ 0 & T_{i1} \end{smallmatrix} \right), S_{i} = \left( \begin{smallmatrix} S_{i0} & S_{i1} \\ 0 & S_{i1} \end{smallmatrix} \right) \), \( 1 \leq i \leq m \). By Lemma 4.3 we have \( \{ K_{0}(\cdot, \bar{w}), X(K_{1}(\cdot, \bar{w}) \xi_{1}) + K_{1}(\cdot, \bar{w}) \xi_{1} \} \) is a frame of \( E_{\mathbf{T}}(w) \). Similarly, a frame of \( E_{\mathbf{S}}(w) \) is obtained.

Define \( K_{\gamma}, \tilde{K}_{\gamma} \) to be the function on \( \Omega^{*} \times \Omega^{*} \) taking values in the \( 2 \times 2 \) matrices \( \mathcal{M}_{2}(\mathbb{C}) \):

\[
K_{\gamma}(z, w) = \begin{pmatrix}
K_{0}(z, w) & \langle X(K_{1}(\cdot, \bar{w}) \xi_{1}), K_{0}(\cdot, z) \rangle \\
\langle K_{0}(\cdot, w), X(K_{1}(\cdot, \bar{w}) \xi_{1}) \rangle & \langle X(K_{1}(\cdot, \bar{w}) \xi_{1}), X(K_{1}(\cdot, \bar{w}) \xi_{1}) + K_{1}(\cdot, \bar{w}) \xi_{1} \rangle
\end{pmatrix},
\]

\[
\tilde{K}_{\gamma}(z, w) = \begin{pmatrix}
\tilde{K}_{0}(z, w) & \langle Y(\tilde{K}_{1}(\cdot, \bar{w}) \xi_{1}), \tilde{K}_{0}(\cdot, z) \rangle \\
\langle \tilde{K}_{0}(\cdot, w), Y(\tilde{K}_{1}(\cdot, \bar{w}) \xi_{1}) \rangle & \langle Y(\tilde{K}_{1}(\cdot, \bar{w}) \xi_{1}), Y(\tilde{K}_{1}(\cdot, \bar{w}) \xi_{1}) + \tilde{K}_{1}(\cdot, \bar{w}) \xi_{1} \rangle
\end{pmatrix}.
\]
By subsection 2.2 in [12], we know that $\tilde{T}$ and $\tilde{S}$ are unitarily equivalent to the adjoint of multiplication operator tuple $M_z$ on some analytic functional spaces $\mathcal{H}_{K_z}$ and $\mathcal{H}_{K_{K_z}}$ with reproducing kernel $K_z(z, w)$ and $K_{K_z}(z, w)$, respectively. That means $\tilde{T} \sim_u (M_z^*, \mathcal{H}_{K_{K_z}}), \tilde{S} \sim_u (M_z^*, \mathcal{H}_{K_z})$. R.E. Curto and N. Salinas gave a necessary and sufficient condition for the unitary equivalence of commuting operator tuples acting on reproducing kernel Hilbert spaces (see Remark 3.8, [13]), that is, $M_z$ acting on $\mathcal{H}_{K_z}$ and $\mathcal{H}_{K_{K_z}}$ are unitarily equivalent if and only if $\Phi(z)K_{K_z}(z, w)\overline{\Phi(w)} = K_z(z, w)$ for some holomorphic and invertible function $\Phi$.

Now if there exist holomorphic functions $\phi$ and $\psi$ such that $\Phi(w) = \left( \begin{array}{c} 0 \\ \psi(w) \\ \phi(w) \end{array} \right)$ which satisfies $\Phi(z)K_{K_z}(z, w)\overline{\Phi(w)} = K_z(z, w)$, then $T$ is unitarily equivalent to $\tilde{T}$, that is,

$$\left( \begin{array}{c} 0 \\ \psi(z) \\ \phi(z) \end{array} \right) = \left( \begin{array}{c} K_0(z, w) \\ K_0(z, w)X(\tilde{K}_1(z, \cdot), z) + K_{\tilde{K}_1}(z, w) \\ \overline{X(\tilde{K}_1(z, \cdot), z)} \end{array} \right) \left( \begin{array}{c} 0 \\ \psi(w) \\ \phi(w) \end{array} \right)$$

Choosing $z = w$, we have that

$$K_0(z, w) = \|X(\phi(w)K_1(\cdot, w))\|^2 + |\phi(w)|^2K_1(z, w) = \|(I + X^*X)\frac{1}{2}(\phi(w)K_1(\cdot, w))\|^2,$$

$$|\psi(w)|^2K_0(z, w) = \|Y(\tilde{K}_1(\cdot, w))\|^2 + \tilde{K}_1(z, w) = \|(I + Y^*Y)^{\frac{1}{2}}\tilde{K}_1(\cdot, w)\|^2, w \in \Omega.$$

By Theorem 3.2 and the proof of Example 3.9 we have that $T_{00} \sim S_{11}, S_{00} \sim T_{11}$.

Thus, $T_{00} \oplus S_{00} \sim T_{11} \oplus S_{11}$.

Let $T, S \in \mathcal{B}_1^1(\Omega)$ and $T \sim_u (M_z^*, \mathcal{H}_{K_z}), S \sim_u (M_z^*, \mathcal{H}_{K_z})$. By Lemma 3.5, we know that if $T_{00} \sim T_{11}$, then $\frac{K_0(z, w)}{K_1(z, w)}$ is bounded and bounded below from zero. In the following proposition, we will prove that there is a similar result in the operator class $\mathcal{N}_T^1(\Omega)$. For the case of index two, $\frac{K_0(z, w)}{K_1(z, w)}$ is replaced by the ratio of the determinants of the metrics corresponding to the two bundles.

**Proposition 4.15.** Let $T_{ii}, S_{ii} \in \mathcal{B}_1^1(\Omega)$ and $T_{ii} \sim_u (M_z^*, \mathcal{H}_{K_z}), S_{ii} \sim_u (M_z^*, \mathcal{H}_{K_z}), i = 0, 1$. Suppose that $T = \left( \begin{array}{cc} T_{00} & T_{01} \\ T_{10} & T_{11} \end{array} \right), S = \left( \begin{array}{cc} S_{00} & S_{01} \\ S_{10} & S_{11} \end{array} \right) \in \mathcal{N}_T^1(\Omega)$ and there exist $X, Y$ such that $T_{01} = \sigma_{T_{10}, T_{11}}(\bar{X}), S_{01} = \sigma_{S_{00}, S_{01}}(\bar{Y})$. If $T \sim S$, then there exist metrics $h_T, h_S$ corresponding to $E_T, E_S$ such that $m \leq \frac{det h_T(\omega)}{det h_S(\omega)} \leq M, \omega \in \Omega$, for positive numbers $m$ and $M$.

**Proof.** Without loss of generality, we assume that $T_{ii} = (M_z^*, \mathcal{H}_{K_z}), S_{ii} = (M_z^*, \mathcal{H}_{K_z}), i = 0, 1$. Then $K_z(\cdot, \bar{w}), \tilde{K}_z(\cdot, \bar{w})$ are the sections of $E_{T_{ii}}$ and $E_{S_{ii}}, i = 0, 1$, respectively. By Lemma 4.13, we know that $\{K_0(\cdot, \bar{w}), XK_1(\cdot, \bar{w}) + K_1(\cdot, \bar{w})\}, \{\tilde{K}_0(\cdot, \bar{w}), Y\tilde{K}_1(\cdot, \bar{w}) + \tilde{K}_1(\cdot, \bar{w})\}$ are frames of $E_T(w), E_S(w)$, respectively. It follows that

$$h_T(\omega) = \left( \begin{array}{c} K_0(\bar{w}, \bar{w}) \\ \overline{X(\tilde{K}_1(\cdot, \bar{w}))} \end{array} \right), det h_T(\omega) = K_0(\bar{w}, \bar{w})(K_1(\bar{w}, \bar{w}) + \|X(\tilde{K}_1(\cdot, \bar{w}))\|^2 - |\overline{X(\tilde{K}_1(\cdot, \bar{w}))}|^2).$$
Similarly, we have
\[
\det h_S(\omega) = \tilde{K}_0(\bar{\omega}, \bar{w})(\tilde{K}_1(\bar{\omega}, \bar{w}) + \|Y(\tilde{K}_1(\cdot, \bar{w}))\|^2) - |\langle \tilde{K}_0(\cdot, \bar{w}), Y(\tilde{K}_1(\cdot, \bar{w})) \rangle|^2.
\]

By Proposition 4.11, we know that operators in \(NF^B_{2}(\Omega)\) are strongly reducible and \(T \sim_s T_{00} \oplus T_{11}, S \sim_s S_{00} \oplus S_{11}\). If \(T \sim_s S\), then \(T_{00} \oplus T_{11} \sim_s S_{00} \oplus S_{11}\). By the main theorem of [33], we know that every Cowen-Douglas operator has a uniquely strongly irreducible decomposition up to similarity. Thus, the equivalence relation is either \(S_{11}\) or \(S_{00}\). In either case, according to Lemma 3.5, there exist positive numbers \(m_1\) and \(M_1\) such that \(m_1 \leq \frac{\det h_{T_{00} \oplus T_{11}}}{\det h_{S_{00} \oplus S_{11}}} = \frac{K_0(\bar{\omega}, \bar{w})K_1(\bar{\omega}, \bar{w})}{K_0(\bar{\omega}, \bar{w})K_1(\bar{\omega}, \bar{w})} \leq M_1\). By Cauchy-Schwarz inequality, we have \(K_0(\bar{\omega}, \bar{w})\|X(\tilde{K}_1(\cdot, \bar{w}))\|^2 - |\langle K_0(\cdot, \bar{w}), X(\tilde{K}_1(\cdot, \bar{w})) \rangle|^2 \geq 0\). Thus,
\[
\frac{\det h_T(\omega)}{\det h_{T_{00} \oplus T_{11}}(\omega)} \leq \frac{K_0(\bar{\omega}, \bar{w})(K_1(\bar{\omega}, \bar{w}) + \|X(\tilde{K}_1(\cdot, \bar{w}))\|^2)}{K_0(\bar{\omega}, \bar{w})K_1(\bar{\omega}, \bar{w})} \leq 1 + \|X\|^2.
\]

On the other hand, since \(X\) is a bounded linear operator, we have
\[
\frac{\det h_T(\omega)}{\det h_{S_{00} \oplus S_{11}}(\omega)} = \frac{\det h_T}{\det h_{S_{00} \oplus S_{11}}} \leq 1 + \|Y\|^2.
\]

Similarly, we obtain \(1 \leq \frac{\det h_T}{\det h_{S_{00} \oplus S_{11}}} \leq 1 + \|Y\|^2\). Note that
\[
\frac{\det h_T}{\det h_S} = \frac{\det h_T}{\det h_{T_{00} \oplus T_{11}}} \cdot \frac{\det h_{T_{00} \oplus T_{11}}}{\det h_{S_{00} \oplus S_{11}}} \cdot \frac{\det h_{S_{00} \oplus S_{11}}}{\det h_S}.
\]

Let \(m := \frac{m_1}{1+\|Y\|^2}, M := M_1(1+\|X\|^2)\). We have \(m \leq \frac{\det h_T(\omega)}{\det h_S(\omega)} \leq M\). This completes the proof. \(\square\)

5. Weakly homogeneous operator tuples

An operator \(T \in L(\mathcal{H})\) is said to be weakly homogeneous if \(\sigma(T) \subset \overline{D}\) and \(\phi(T)\) is similar to \(T\) for each \(\phi\) in \(M\). When \(\mathcal{D}\) is a bounded symmetric domain, a commuting \(m\)-tuple \(T = (T_1, \ldots, T_m)\) of bounded operators is said to be weakly homogeneous with respect to \(G\) if their joint Taylor spectrum is contained in \(\overline{D}\) and \(\phi(T)\) is similar to \(T\) for every holomorphic automorphism \(\phi \in G\). Given a Hilbert space \(\mathcal{H}\) with sharp reproducing kernel \(K\) on \(\mathbb{D} \times \mathbb{D}\), S. Ghara in [27] obtain an equivalent condition that the multiplication operator \(M_x\) on \((\mathcal{H}, K)\) is weakly homogeneous. Next, we consider the weakly homogeneity of class \(NF^B_{m_0+n_1}(\mathbb{D}^m)\).

**Proposition 5.1.** Let \(T_{ij} \in B_{m_1}(\mathbb{D}^m), i = 0, 1\) and \(T_{01} \in ran \sigma_{T_{00} \oplus T_{11}}\). Suppose that \(\tilde{T} = (T_1, \ldots, T_m) \in NF^B_{m_0+n_1}(\mathbb{D}^m)\) with \(T_i = \begin{pmatrix} T_{i0} & \frac{T_{i1}}{T_{i0}} \\ 0 & \frac{T_{i1}}{T_{i0}} \end{pmatrix}\), \(1 \leq j \leq m\). If \(T_{00}, T_{11}\) are both weakly homogeneous with respect to \(M\), then \(T\) is also \(M\)-weakly homogeneous.

**Proof.** Suppose that there exists \(X\) such that \(T_{01} = \sigma_{T_{00} \oplus T_{11}}(-X)\). By Proposition 4.11 we know that \(\tilde{T}\) is similar to \(T_{00} \oplus T_{11}\) and \(\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} T_{00} & T_{01} \\ 0 & T_{11} \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} T_{00} & 0 \\ 0 & T_{11} \end{pmatrix}\) for \(1 \leq j \leq m\). Further, we have \(\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} T_{00} & T_{01} \\ 0 & T_{11} \end{pmatrix}^n \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} T_{00} & 0 \\ 0 & T_{11} \end{pmatrix}^n\) for any positive
integer $n$ and
\[
(I - X) \phi_{\alpha_j} \left( \begin{pmatrix} T_{y_0} & T_{y_1} \\ 0 & T_{y_2} \end{pmatrix} \right) \left( \begin{pmatrix} 0 & X \\ T & I \end{pmatrix} \right) = \left( \begin{pmatrix} \phi_{\alpha_j}(T_{y_0}) & 0 \\ 0 & \phi_{\alpha_j}(T_{y_1}) \end{pmatrix} \right), \phi_{\alpha_j} \in \text{Möb}, 1 \leq j \leq m.
\]

Let $\phi_\alpha = (\phi_{\alpha_1}, \phi_{\alpha_2}, \ldots, \phi_{\alpha_m})$. Then $\phi_\alpha \in \text{Möb}^m$ and $(I - X) \phi_\alpha(\tilde{T}) \left( \begin{pmatrix} 0 & X \\ 0 & I \end{pmatrix} \right) = \phi_\alpha(T_{y_0} \oplus T_{y_1})$. Since $T_{y_0}, T_{y_1}$ are both weakly homogeneous, it follows that there exists invertible operator $Y_\alpha$ depending on $\alpha$, such that $Y_\alpha^{-1} \left( \begin{pmatrix} 0 & X \\ 0 & I \end{pmatrix} \right) \phi_\alpha(\tilde{T}) \left( \begin{pmatrix} 0 & X \\ 0 & I \end{pmatrix} \right) Y_\alpha = T_{y_0} \oplus T_{y_1}$. By using Proposition 5.1, we obtain that $(I - X) Y_\alpha^{-1} \left( \begin{pmatrix} 0 & X \\ 0 & I \end{pmatrix} \right) \phi_\alpha(\tilde{T}) \left( \begin{pmatrix} 0 & X \\ 0 & I \end{pmatrix} \right) = \tilde{T}$.

Hence, $\tilde{T}$ is weakly homogeneous.

Proposition 5.2. Let $T_{ii} \in \mathcal{B}_1^m(D^m), i = 0, 1$ and $T_{01} \in \text{ran} \sigma_{T_{00}, T_{11}}$. Suppose that $\tilde{T} = (T_1, \ldots, T_m) \in N\mathcal{F}_1^m_{\text{rot}+n_1}(D^m)$ with $T_{i} = \left( \begin{pmatrix} T_{y_0} & T_{y_1} \\ 0 & T_{y_2} \end{pmatrix} \right), 1 \leq j \leq m$. If $\tilde{T}$ is weakly homogeneous with respect to $\text{Möb}^m$, then $T_{00} \oplus T_{11}$ is also $\text{Möb}^m$-weakly homogeneous.

Proof. Suppose that there exists $X$ such that $T_{01} = \sigma_{T_{00}, T_{11}}(-X)$. If $\tilde{T}$ is weakly homogeneous with respect to $\text{Möb}^m$, then there exists invertible operator $Y_\alpha$ depending on $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in D^m$ and $\phi_\alpha = (\phi_{\alpha_1}, \phi_{\alpha_2}, \ldots, \phi_{\alpha_m}) \in \text{Möb}^m$, such that $Y_\alpha^{-1} \phi_\alpha(\tilde{T}) Y_\alpha = T$. By using the strong reducibility of $T$ in Proposition 5.1, we have
\[
(I - X) Y_\alpha^{-1} \left( \begin{pmatrix} 0 & X \\ 0 & I \end{pmatrix} \right) \phi_\alpha(T_{y_0} \oplus T_{y_1}) \left( \begin{pmatrix} 0 & X \\ 0 & I \end{pmatrix} \right) Y_\alpha \left( \begin{pmatrix} 0 & X \\ 0 & I \end{pmatrix} \right) = T_{00} \oplus T_{11}.
\]

Thus, $T_{00} \oplus T_{11}$ is weakly homogeneous with respect to $\text{Möb}^m$.

Proposition 5.3. If $(T_{y_0} T_{y_1}) \in N\mathcal{F}_1^1(D)$ is weakly homogeneous and $\ker \sigma_{\phi_\alpha(T_{00}), T_{11}} = \{0\}$ for any $\phi_\alpha \in \text{Möb}$. Then $T_{00}, T_{11}$ are both weakly homogeneous.

Proof. Suppose that there exists $X$ such that $T_{01} = \sigma_{T_{00}, T_{11}}(-X)$. If $(T_{y_0} T_{y_1})$ is weakly homogeneous, by Proposition 5.2, we see that $T_{00} \oplus T_{11}$ is weakly homogeneous, that is, $\phi_\alpha(T_{00}) \oplus \phi_\alpha(T_{11})$ is similar to $T_{00} \oplus T_{11}$ for any $\phi_\alpha \in \text{Möb}$. Note that $\phi_\alpha(T_{01}) \in \mathcal{B}_1^1(D), i = 0, 1$, by the main theorem of [33], we know that every Cowen-Douglas operator has a unique strongly irreducible decomposition up to similarity, then either $\phi_\alpha(T_{ii}) \sim_{s} T_{ii}, i = 0, 1$ or $\phi_\alpha(T_{00}) \sim_{s} T_{11}, \phi_\alpha(T_{11}) \sim_{s} T_{00}$ for $\phi_\alpha \in \text{Möb}$. Since $\ker \sigma_{\phi_\alpha(T_{00}), T_{11}} = \{0\}$, we have $\phi_\alpha(T_{ii}) \sim_{s} T_{ii}, i = 0, 1$ for any $\phi_\alpha \in \text{Möb}$. Hence, $T_{00}, T_{11}$ are both weakly homogeneous.

Proposition 5.4. Let $T \in \mathcal{B}_1^1(D)$. If $T$ is weakly homogeneous, then there exists $\Psi(\cdot, \cdot) : D \times D \rightarrow \mathbb{R}^+$ such that $K_\Psi(\alpha, \phi_\alpha(w)) |\phi_\alpha(w)|^2 + K_\Psi(\alpha, w) = 0$, where $K_\Psi(\alpha, w) = -\frac{\partial^2}{\partial w \partial \overline{w}} \log \Psi(\alpha, w)$ and $\phi_\alpha \in \text{Möb}$. In particular, $\Psi(\alpha, w)$ also satisfies $K_\Psi(w, w) = -\frac{\partial^2}{\partial w \partial \overline{w}} \log |w|^2$ and $K_\Psi(0, -w) = -K_\Psi(0, w)$.

Proof. Without losing generality, we assume that $\phi_\alpha(w) = \frac{\alpha - w}{1 - \alpha w}, \alpha, w \in D$. If $T$ is weakly homogeneous, then for any $\alpha \in D$, $T$ is similar to $\phi_\alpha(T)$. Let $e$ be a non-zero section of $E_T$ associated with $T$. Note that $\phi_\alpha(T) \in \mathcal{B}_1^1(D)$ and $e(\phi_\alpha(w)) \in \ker(\phi_\alpha(T) - w)$. By Theorem 3.2, we find that a bounded operator $X_\alpha$ and $\psi_\alpha \in H^\infty(D)$ depending on $\alpha$, such that $\|e(\phi_\alpha(w))\|^2 = |\psi_\alpha(w)|^2(\|e(w)\|^2 + \|X_\alpha(e(w))\|^2), w \in D$. Further, we have $\frac{\partial^2}{\partial w \partial \overline{w}} \log \|e(\phi_\alpha(w))\|^2 = \frac{\partial^2}{\partial w \partial \overline{w}} \log \|e(w)\|^2 + \frac{\partial^2}{\partial w \partial \overline{w}} \log \left(1 + \frac{|X_\alpha(e(w))|^2}{\|e(w)\|^2}\right), w \in D$. Define
Then equation (5.1) is equivalent to

\[ \mathcal{K}_{\phi_\alpha(T)}(w) = \mathcal{K}_T(w) + \mathcal{K}_\Psi(\alpha, w). \]

From the arbitrariness of \( w \in \mathbb{D} \) in equation (5.1), we replace \( w \) with \( \phi_\alpha(w) \). It follows that

\[ \mathcal{K}_{\phi_\alpha(T)}(\phi_\alpha(w)) = \mathcal{K}_T(\phi_\alpha(w)) + \mathcal{K}_\Psi(\alpha, \phi_\alpha(w)). \]

It can be obtained by a simple application of the chain rule that \( \mathcal{K}_T(w) = \mathcal{K}_{\phi_\alpha(T)}(\phi_\alpha(w))|_{\phi_\alpha(w)} \) where

\[ \mathcal{K}_T(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log \Psi(\alpha, w). \]

Then equation (5.2) can be transformed into

\[ \mathcal{K}_T(w)|_{\phi_\alpha(w)} = \mathcal{K}_T(\phi_\alpha(w)) + \mathcal{K}_\Psi(\alpha, \phi_\alpha(w)). \]

Using the chain rule again, we have

\[ \mathcal{K}_{\phi_\alpha(T)}(\phi_\alpha(w)) = \mathcal{K}_T(\phi_\alpha(w)) + \mathcal{K}_\Psi(\alpha, \phi_\alpha(w)). \]

Then equation (5.1) is equivalent to

\[ \mathcal{K}_T(\phi_\alpha(w)) = \mathcal{K}_T(w)|_{\phi_\alpha(w)} - \mathcal{K}_\Psi(\alpha, w)|_{\phi_\alpha(w)} - \mathcal{K}_\Psi(\alpha, w)|_{\phi_\alpha(w)}. \]

Combining equations (5.3) and (5.4), we obtain that

\[ \mathcal{K}_\Psi(\alpha, \phi_\alpha(w))|_{\phi_\alpha(w)} + \mathcal{K}_\Psi(\alpha, w) = 0. \]

Note that \( \phi_\alpha(w) = \frac{\alpha - w}{1 - \alpha w} \) and \( \phi_\alpha(0) = \alpha \). It follows that \( \mathcal{K}_\Psi(\alpha, \alpha)(1 - \alpha)^2 + \mathcal{K}_\Psi(\alpha, 0) = 0 \) for all \( \alpha \in \mathbb{D} \), that is, \( \mathcal{K}_\Psi(w, w) = \frac{\mathcal{K}_\Psi(w, 0)}{(1 - |w|^2)^2} \) for all \( w \in \mathbb{D} \). Since \( \phi_0(w) = -w \), we imply \( \mathcal{K}(0, -w) + \mathcal{K}(0, w) = 0 \). \( \square \)

**Proposition 5.5.** Let \( T \in \mathcal{B}_1^1(\mathbb{D}) \) and \( e \) be a non-zero section of \( E \) determined by \( T \). For any \( \phi_\alpha \in \mathcal{M}_1, E_\alpha \) is the Hermitian holomorphic vector bundle associated with \( \phi_\alpha(T) \). If \( T \) is weakly homogeneous, then there exists vector bundle \( F_\alpha \) with \( F_\alpha(w) = \mathcal{V}\{ X_\alpha \} e(w) \}, \) such that \( E \otimes E_\alpha \sim_{\alpha} F_\alpha \otimes \alpha^\alpha \), where \( X_\alpha \) is a bounded operator depending on \( \alpha \) and \( F_\alpha(w) = \mathcal{V}\{ \chi_\alpha \} e(\phi_\alpha(w)) \}, w \in \mathbb{D} \).

**Proof.** Without losing generality, we assume that \( \phi_\alpha(w) = \frac{\alpha - w}{1 - \alpha w} \). If \( T \) is weakly homogeneous, then \( T \sim_{\alpha} \phi_\alpha(T), \alpha \in \mathbb{D} \). Then it is easy to see that \( \phi_\alpha(T) \in \mathcal{B}_1^1(\mathbb{D}) \) and \( e(\phi_\alpha(w)) \in \ker(\phi_\alpha(T) - w) \). By Theorem 3.2, we know that there exists a bounded operator \( X_\alpha \) and \( \psi_\alpha \in \mathcal{H}(\mathbb{D}) \) depending on \( \alpha \), such that

\[ \| e(\phi_\alpha(w)) \|^2 = |\psi_\alpha(w)|^2(\|e(w)\|^2 + \|X_\alpha e(w)\|^2), w \in \mathbb{D}. \]

This is equivalent to

\[ \| e(\phi_\alpha(w)) \|^2 \|e(w)\|^2 = |\psi_\alpha(w)|^2(1 + \|X_\alpha e(w)\|^2) \]

and hence

\[ \| e(\phi_\alpha(w)) \|^2 \|e(w)\|^2 = |\psi_\alpha(w)|^2 \|e(\phi_\alpha(w))\|^2 \|e(w)\|^2. \]

Further, we have

\[ \frac{\partial^2}{\partial w \partial \bar{w}} \log \| e(\phi_\alpha(w)) \| \|e(w)\|^2 \|e(\phi_\alpha(w))\|^2 = 0, w \in \mathbb{D}. \]

Let \( F_\alpha(w) = \mathcal{V}\{ \chi_\alpha \} e(w) \} \) and \( F_\alpha(w) = \mathcal{V}\{ \chi_\alpha \} e(\phi_\alpha(w)) \} \) for \( \alpha, w \in \mathbb{D} \). By Theorem 2.2 and equation (5.6), we infer \( E \otimes E_\alpha \sim_{\alpha} F_\alpha \otimes F_\alpha \). \( \square \)
Remark 5.6. Let $T$ be a Cowen-Douglas operator with index one. By Proposition 3.2 in [30], the first and second authors joint with L. Zhao proved that for any $\phi_\alpha \in \mathcal{M}$, if the holomorphic Hermitian vector bundle $E_\alpha$ associated with $\phi_\alpha(T)$ is congruent to $E_T \otimes \mathcal{L}_\alpha$ for some line bundle $\mathcal{L}_\alpha$, then $T$ is homogeneous. In this case, when the index of $T$ is one, the $\mathcal{K}_\Psi$ in Proposition 5.4 is zero, since $\|e(\phi_\alpha(w))\|^2 = |\phi_\alpha(w)|^2\|e(w)\|^2$ for some holomorphic function $\varphi$ and $\alpha, w \in \mathbb{D}$. But if $\|e(\phi_\alpha(w))\|^2 = (1 + |\phi_\alpha(w)|^2)\|e(w)\|^2$, then $T$ is not a homogeneous operator, since $1 + |\phi_\alpha(w)|^2$ is not the square of the Modulus of some holomorphic function. Although it can be regarded as $X_\alpha e(w) = \phi_\alpha(w)e(w)$, the equation $\mathcal{K}_\Psi(\alpha, \phi_\alpha(w))|\phi_\alpha(w)|^2 + \mathcal{K}_\Psi(\alpha, w) = 0$, a necessary condition of $T$ to be a weakly homogeneous operator, does not hold. We have $\mathcal{K}_\Psi(\alpha, w) = -\frac{|\phi_\alpha(w)|^2}{(1 + |\phi_\alpha(w)|^2)^2}$ and $\mathcal{K}_\Psi(\alpha, \phi_\alpha(w)) = -\frac{|\phi_\alpha(\phi_\alpha(w))|^2}{(1 + |\phi_\alpha(\phi_\alpha(w))|^2)^2}$. The reason for this phenomenon may be $(I + X_\alpha X_\alpha)^\frac{1}{2}$ is not the intertwining of $T$ and $\phi_\alpha(T)$.

In what follows, we assume that Hilbert space $\mathcal{H}_i, i \geq 0$ is analytical function space with reproducing kernel $K^i(z, w)$, where $K^0(z, w) = \frac{1}{(1 - zw)^n}$, $K^n(z, w) = \frac{1}{(1 - zw)^n}$, $n \geq 1, z, w \in \mathbb{D}$. Let $T$ be a Cowen-Douglas operator and $T \sim_u (M^*_z, \mathcal{H}_K)$. When $T \in \mathcal{B}_n^1(\mathbb{D})$ and is contracive, M. Uchiyama in [65] provide a necessary and sufficient condition for $T$ is similar to the $n$ times copies of $M^*_z$ on Hardy space, which is that there exist positive constants $m, M$ such that $m \sum_{i=1}^n |x_i|^2 \leq (1 - |w|^2)\langle K(\bar{w}, w)\xi, \xi \rangle \leq M \sum_{i=1}^n |x_i|^2$ for any $w \in \mathbb{D}$ and $\xi = \sum_{i=1}^n x_i \xi_i, x_i \in \mathbb{C}, \xi_i = (0, \ldots, 0, 1, 0, \ldots) \bar{w}^i$ with 1 on the $i$th position. When $T \in \mathcal{B}_n^1(\mathbb{D})$ and is $n$-hypercontractive, the second and third authors in [33] show that $T$ is similar to $M^*_z$ on $(\mathcal{H}_i, K^n)$ if and only if $\frac{K(w, w)}{K^n(w, w)}$ is bounded and bounded below from zero. For each $n \geq 1$, we know that the multiplication operator on $(\mathcal{H}_n, K^n)$ is homogeneous in [49] given by G. Misra. It is well known that an operator similar to a homogeneous is a weakly homogeneous. The $n$–hypercontraction $T, n \geq 1$ determined by the similarity above is weakly homogeneous. For some positive definite kernels $K$, it is shown that in Theorem 5.3 of [27] the multiplication operator $M_z$ on $(\mathcal{H}, KK^n)$, $n > 0$, is a weakly homogeneous operator due to S. Gharah.

In the following, we provide some methods to obtain weakly homogeneous tuples. We let $(M_z, \mathcal{H}_K)$ denote the tuple of multiplication operators acting on Hilbert space $\mathcal{H}_K$ determined by the unique non-negative definite kernel $K$.

Proposition 5.7. Let $\mathcal{H}_{K_i}$ be the analytic function space with reproducing kernel $K_i$ over $\Omega \subset \mathbb{C}^m, i = 0, 1$. Suppose that the tuple of multiplication operators $M_z = (M_{z_1}, \ldots, M_{z_m})$ acting on $\mathcal{H}_{K_0}$ and $\mathcal{H}_{K_1}$ are bounded. If the identity mapping $\text{id} : \mathcal{H}_{K_0} \rightarrow \mathcal{H}_{K_1}$ is bounded, then $(M_z, \mathcal{H}_{K_0+K_1})$ is similar to $(M_z, \mathcal{H}_{K_1})$.

Proof. For any analytic function $f$ and $\xi \in \mathbb{C}^n$, we have

$$
(f, (\text{id})^*K_1(\cdot, w)\xi)_{\mathcal{H}_{K_0}} = (f, K_1(\cdot, w)\xi)_{\mathcal{H}_{K_1}} = (f(w), \xi)_{\mathcal{H}_{K_0}} = (f(w), \xi)_{\mathbb{C}^n}, w \in \Omega.
$$

By the arbitrariness of $f$, it follows that $(\text{id})^*K_1(\cdot, w)\xi = K_0(\cdot, w)\xi$ for all $w \in \Omega$ and $\xi \in \mathbb{C}^n$. Let $X := (\text{id})^* : \mathcal{H}_{K_1} \rightarrow \mathcal{H}_{K_0}$. If $\text{id} : \mathcal{H}_{K_0} \rightarrow \mathcal{H}_{K_1}$ is bounded, then so is $X$. Since $\mathcal{H}_{K_i}$ is the analytic function space with reproducing kernel $K_i, i = 0, 1$, then so is $\mathcal{H}_{K_0+K_1}$ with $K_0 + K_1$ in [18, 60] due to R. E. Curto and N. Salinas. We know that the
tule of multiplication operators $M_z = (M_{z_1}, \ldots, M_{z_m})$ satisfies
\[
\bigvee_{w \in \Omega^*} \ker(M_z^* - w) = \bigvee_{w \in \Omega^*} \{K_i(\cdot, \bar{w}) \xi, \xi \in \mathbb{C}^n\} = H_{K_i}, \ i = 0, 1,
\]
then the reproducing kernel of $M_z$ on $H_{K_{00} + K_1}$ is
\[
\begin{align*}
&= \left\langle (K_0(\cdot, w), K_1(\cdot, w))e_j, (K_0(\cdot, w), K_1(\cdot, w))_0, e_i \right\rangle_{\mathcal{K}_{K_{00} + K_1}}^n_{i,j=1} \\
&= \left\langle (K_0(\cdot, w), K_1(\cdot, w))e_j, (K_0(\cdot, w), K_1(\cdot, w))_0, e_i \right\rangle_{\mathcal{K}_{K_{00} + K_1}}^n_{i,j=1} + \left\langle (K_0(\cdot, w), K_1(\cdot, w))e_j, (K_0(\cdot, w), K_1(\cdot, w))_0, e_i \right\rangle_{\mathcal{K}_{K_{00} + K_1}}^n_{i,j=1},
\end{align*}
\]
for any orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{C}^n$. Note that
\[
\begin{align*}
&\left\langle (I_{\mathcal{K}_{K_1}} + X^*X)^{\frac{1}{2}}K_1(\cdot, w)e_j, (I_{\mathcal{K}_{K_1}} + X^*X)^{\frac{1}{2}}K_1(\cdot, w)e_i \right\rangle_{\mathcal{K}_{K_1}}^n_{i,j=1} \\
&= \left\langle (I_{\mathcal{K}_{K_1}} + X^*X)K_1(\cdot, w)e_j, K_1(\cdot, w)e_i \right\rangle_{\mathcal{K}_{K_1}}^n_{i,j=1} \\
&= \left\langle (K_1(\cdot, w)e_j, K_1(\cdot, w)e_i)_{\mathcal{K}_{K_1}}^n_{i,j=1} + \left\langle (XK_1(\cdot, w)e_j, XK_1(\cdot, w)e_i)_{\mathcal{K}_{K_1}}^n_{i,j=1} \\
&= \left\langle (K_1(\cdot, w)e_j, K_1(\cdot, w)e_i)_{\mathcal{K}_{K_1}}^n_{i,j=1} + \left\langle (K_0(\cdot, w)e_j, K_0(\cdot, w)e_i)_{\mathcal{K}_{K_0}}^n_{i,j=1},
\end{align*}
\]
then means that the reproducing kernels of $(I_{\mathcal{K}_{K_1}} + X^*X)^{\frac{1}{2}}M_z(I_{\mathcal{K}_{K_1}} + X^*X)^{-\frac{1}{2}}$ on $H_{K_1}$ and $M_z$ on $H_{K_{00} + K_1}$ are the same. By Remark 3.8 of \[44\], we have $(I_{\mathcal{K}_{K_1}} + X^*X)^{\frac{1}{2}}M_z$ on $H_{K_1}$ is unitarily equivalent to $M_z$ on $H_{K_{00} + K_1}$. Since $(I_{\mathcal{K}_{K_1}} + X^*X)^{\frac{1}{2}}$ is invertible, then $(M_z, H_{K_{00} + K_1})$ is similar to $(M_z, H_{K_1})$.

Let $K_0, K_1 : \Omega \times \Omega \to M_n(\mathbb{C})$. We write $K_0 \geq 0$ means that $K_0$ is a non-negative definite kernel. We write $K_0 \preceq K_1$ or $K_1 \succeq K_0$, if $K_0$ and $K_1$ are two non-negative kernels satisfying $K_1 - K_0 \preceq 0$ on $\Omega \times \Omega$.

**Proposition 5.8.** Let $H_{K_i}$ be Hilbert space determined by reproducing kernel $K_i$ over $\Omega \subset \mathbb{C}^m$, $i = 0, 1$. Suppose that the multiplication operator $M_z = (M_{z_1}, \ldots, M_{z_m})$ is bounded on $(H_{K_i}, K_i)$ for $i = 0, 1$ and $K_1 \geq K_0$ on $\Omega \times \Omega$. Then $(M_z, H_{K_{00} + K_1})$ is similar to $(M_z, H_{K_1})$.

**Proof.** Recall that if $K_1 \geq K_0$, then $(H_{K_0}, K_0) \subset (H_{K_1}, K_1)$ and $\|h\|_{\mathcal{K}_{K_1}} \leq \|h\|_{\mathcal{K}_{K_0}}$ for $h \in (H_{K_0}, K_0)$ in Theorem 6.25 of \[57\]. Let $id : H_{K_0} \to H_{K_1}$ be the identity mapping. For any $h \in (H_{K_0}, K_0)$, we have $\frac{\|id(h)\|_{\mathcal{K}_{K_1}}}{\|h\|_{\mathcal{K}_{K_0}}} \leq 1$. It follows that $id : H_{K_0} \to H_{K_1}$ is bounded. By Proposition 5.7, we obtain that $(M_z, H_{K_{00} + K_1})$ is similar to $(M_z, H_{K_1})$.

In \[44\], A. Korányi and G. Misra explicitly construct all homogeneous holomorphic Hermitian vector bundles on $\mathbb{D}$ and give the forms of all operators similar to homogeneous operators.

**Lemma 5.9.** \[44\] Any homogeneous operator in the Cowen-Douglas class is similar to the direct sum (with multiplicity) of weighted block shifts, which is the direct sum of an ordinary (unweighted) block shift and a Hilbert-Schmidt operator.

**Remark 5.10.** If we choose $K_1$ as an arbitrary homogeneous kernel (the tuple $M_z$ of multiplications acting on $(H_{K_1}, K_1)$ is homogeneous), $K_0$ is an arbitrary nonnegative definite kernel and satisfies $K_1 \succeq K_0$, then $(M_z, H_{K_{00} + K_1})$ is weakly homogeneous by Proposition 5.8. In the following, we simply lists some examples, but does not give detailed proof.
For any nonnegative integer $k$ and $\{i_0, \cdots, i_l\} \subset \{1, \cdots, k\}$, $(M_z, \mathcal{K}_{\sum_{j=0}^l K^{i_j}})$ is similar to $(M_z, \mathcal{K}_{\sum_{j=0}^l K^{i_j}})$, where $r = \max\{i_0, \cdots, i_l\}$. Thus, $(M_z, \mathcal{K}_{\sum_{j=0}^l K^{i_j}})$ is weakly homogeneous. From Lemma 4.3 it is also similar to a weighted block shift. Let $K^{(\lambda)}(z, w) = \prod_{i=0}^{m-1} \left(\frac{1}{1 - z w_i}\right)^{\lambda_i}$ and $K^{(\alpha)}(z, w) = \prod_{i=1}^{m} \left(\frac{1}{1 - z w_i}\right)^{\alpha_i}$ for $z = (z_1, \cdots, z_m), w = (w_1, \cdots, w_m) \in \mathbb{D}^m$, positive integer $\lambda$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{Z}^m_+$. Then the tuples of multiplications acting on $(\mathcal{K}_{\sum_{i} K^{(\lambda,\mu)}})$ are weakly homogeneous for $0 < \lambda \leq \mu$ and $\alpha \leq \beta \leq \mathbb{Z}^m_+$.

Next, we consider the operator class $\mathcal{FB}^1_m(\Omega)$, which has a flag structure and introduced in [33, 34]. Let $T \in \mathcal{FB}^1_n(\Omega)$ with a form in Definition 3.1 of [35] and $T_i \sim_u (M_z, \mathcal{K}_{K^{(\lambda)}})$, $0 \leq i \leq n - 1$. If $S_i T_i+1 K^{(\lambda)}(\cdot, w) = K_i(\cdot, w)$ and it is not zero on $w \in \Omega_0 \subset \Omega$, then $(M_z, \mathcal{K}_{\sum_{i} K^{(\lambda)}})$ is similar to $(M_z, \mathcal{K}_{K^{(\lambda)}})$ on $\Omega_0$. Naturally, this result can be extended to some tuples in the class $\mathcal{FB}^1_n(\Omega)$.

Let $\mathcal{K}^{(\lambda,\mu)}$ be the reproducing kernel Hilbert space with the kernel $K^{(\lambda,\mu)}(z, w) = \left(\frac{1}{1 - z w}\right)^{\lambda} \left(\frac{1}{1 - z w}\right)^{\mu}$ on $\mathbb{D} \times \mathbb{D}$, $\lambda, \mu > 0$. It is shown in [46, 67] that every irreducible homogeneous operators in $\mathcal{B}^1_2(\mathbb{D})$ must be unitarily equivalent to the adjoint of multiplication operator acting on $\mathcal{K}^{(\lambda,\mu)}$ for some $\lambda, \mu > 0$. Let $\lambda_i, \mu_i > 0, i = 0, 1$. By a direct computation, it can be seen $K^{(\lambda_2,\mu_2)} \succeq K^{(\lambda_1,\mu_1)}$ when $\lambda_1 \geq \lambda_0$ and $\mu_1 \geq \mu_0$. Thus, from Proposition 5.8 we obtain that $(M_z, \mathcal{K}_{\sum_{i} K^{(\lambda,\mu)}})$ is similar to $(M_z, \mathcal{K}_{K^{(\lambda,\mu)}})$ and $(M_z, \mathcal{K}_{K^{(\lambda,\mu)}})$ is weakly homogeneous.

Let $\tilde{\mathcal{K}}^{(\lambda,\mu)} = \mathcal{K}_{\lambda_1} \otimes \cdots \otimes \mathcal{K}_{\lambda_m} \otimes \mathcal{K}_{\lambda_0, \mu_0} \otimes \mathcal{K}_{\lambda_0, \mu_0}$, $\lambda_i, \mu > 0, 1 \leq i \leq m$. Then $\tilde{\mathcal{K}}^{(\lambda,\mu)}$ is the reproducing kernel Hilbert space determined by the kernel

$$
\tilde{K}^{(\lambda,\mu)}(z, w) = \left(\prod_{i=1}^{m-1} K^{(\lambda_i)}(z_i, w_i)\right) K^{(\lambda,\mu)}(z_m, w_m)
$$
on $\mathbb{D}^m \times \mathbb{D}^m$ for a tuple of positive real numbers $\lambda = (\lambda_1, \cdots, \lambda_m)$ and $\mu > 0$. Based on the above discussion of kernels of $\tilde{K}^{(\alpha)}$ and $K^{(\lambda,\mu)}$, for $\lambda, \lambda \in \mathbb{Z}^m_+$ and $\mu, \tilde{\mu} > 0$, if $\lambda \geq \hat{\lambda}$ and $\mu > \frac{1}{\lambda_m} - \frac{1}{\lambda_m} + \tilde{\mu}$, we obtain that $\tilde{K}^{(\lambda,\mu)} \succeq \tilde{K}^{(\hat{\lambda},\tilde{\mu})}$ on $\mathbb{D}^m \times \mathbb{D}^m$. It follows from Proposition 5.8 that $(M_z, \mathcal{K}_{\tilde{K}^{(\lambda,\mu)} + \tilde{K}^{(\lambda,\mu)}})$ is similar to $(M_z, \mathcal{K}_{\tilde{K}^{(\lambda,\mu)}})$. By Theorem 6.7 of [19], we know that each homogeneous tuple of operators in $\mathcal{B}^1_2(\mathbb{D})$ with respect to $\tilde{\mathcal{K}}^{(\lambda,\mu)}$ is unitarily equivalent to the adjoint of the tuple of multiplication operators on the reproducing kernel Hilbert space $\tilde{\mathcal{K}}^{(\lambda,\mu)}$ with the kernel $\tilde{K}^{(\lambda,\mu)}$, where $\lambda = (\lambda_1, \cdots, \lambda_m)$ is a tuple of positive real numbers and $\mu > 0$. Thus, $(M_z, \mathcal{K}_{\tilde{K}^{(\lambda,\mu)} + \tilde{K}^{(\lambda,\mu)}})$ is weakly homogeneous.
A question raised by K. Zhu in [68] is that for the multiplication operator $M_z$ on the Bergman space and its two invariant subspaces $I$ and $J$, when are the two restriction operators $M_z|_I$ and $M_z|_J$ are similar? For this problem, we replace the multiplication operator on Bergman space with the adjoint of the direct sum of multiplication operators on two analytic functions Hilbert spaces and give a sufficient condition.

Remark 5.11. If $\mathcal{H}_{K_0}$ and $\mathcal{H}_{K_1}$ are analytic functions Hilbert spaces with reproducing kernels $K_0$ and $K_1$, respectively, then so is $\mathcal{H}_{K_0+K_1}$ with reproducing kernel $K_0 + K_1$ due to [18, 60]. In order not to cause misunderstanding, we note that the multiplication operator on $\mathcal{H}$ is similar to $(M_z, \mathcal{H}_{K_0+K_1})$. An implicit conclusion in [60] is $(M_z, \mathcal{H}_{K_0+K_1})$ is unitarily equivalent to $P_{\mathcal{M}^+} (M_{z,0} \oplus M_{z,1})|_{\mathcal{M}^+}$. For any $(g, -g) \in \mathcal{M}$, we have $(M_{z,0} \oplus M_{z,1})(g, -g) = z(g, -g) \in \mathcal{M}$ and then $\mathcal{M}$ is an invariant subspace of $M_{z,0} \oplus M_{z,1}$. A simple calculation leads to $\mathcal{M}^+$ is an invariant subspace of $M_{z,0}^* \oplus M_{z,1}^*$. It follows that $(M_{z,0}^* \oplus M_{z,1}^*)|_{\mathcal{H}_{K_0+K_1}}$ is unitarily equivalent to $(M_{z,0}^* \oplus M_{z,1}^*)|_{\mathcal{H}_{K_1}}$. A simple calculation leads to $(M_{z,0}^* \oplus M_{z,1}^*)|_{\mathcal{H}_{K_1}}$ is similar to $(M_{z,1}^* \oplus M_{z,0}^*)|_{\mathcal{H}_{K_1}}$. Hence, $(M_{z,0}^* \oplus M_{z,1}^*)|_{\mathcal{H}_{K_1}}$ is similar to $(M_{z,0}^* \oplus M_{z,1}^*)|_{\mathcal{H}_{K_1}}$.

References

[1] J. Arazy and M. Englis, Analytic models for commuting operator tuples on bounded symmetric domains, Trans. Amer. Math. Soc., 355 (2003), no. 2, 837-864.
[2] A. Athavale, Holomorphic kernels and commuting operators, Transactions of the American Mathematical Society, 304 (1987), 101-110.
[3] A. Athavale, Model theory on the unit ball in $\mathbb{C}^n$, Journal of Operator Theory, 27 (1992), 347-358.
[4] B. Bagchi and G. Misra, Contractive homomorphisms and tensor product norms, J. Int. Eqns. Operator Th., 21 (1995), 255-269.
[5] B. Bagchi and G. Misra, Homogeneous operators and projective representations of the Mobius group: a survey, Proc. Ind. Acad. Sci.(Math. Sci.), 111 (2001), 415-437.
[6] B. Bagchi and G. Misra, The homogeneous shifts, J. Func. Anal., 204 (2003), 293-319.
[7] B. Bagchi and G. Misra, Homogeneous tuples of multiplication operators on twisted Bergman space, J. Func. Anal., 136 (1996), 171-213.
[8] I. Biswas and G. Misra, $SL(2,\mathbb{R})$-homogeneous vector bundles, International Journal of Mathematics, 19 (2014), 1-19.
[9] S. Biswas, D.K. Keshari and G. Misra, Infinitely divisible metrics and curvature inequalities for operators in the Cowen-Douglas class, J. Lond. Math. Soc., 88 (2013), 941-956.
[10] Y. Cao, J. S. Fang, C. L. Jiang, K-groups of Banach algebras and strongly irreducible decompositions of operators, Journal of Operator Theory, 48 (2002), 235-253.
[11] L. Chen, On intertwining operators via reproducing kernels, Linear Algebra and its Applications, 438 (2013), 3661-3666.
[12] D. N. Clark and G. Misra, On curvature and similarity, Michigan Math. J. 30 (1983), no. 3, 361-367.
[13] D. N. Clark and G. Misra, On weighted shifts, curvature and similarity, J. London Math. Soc. 31 (1985), no. 2, 357-368.
[14] J. B. Conway, and J. Gleason. Absolute Equivalence and Dirac Operators of Commuting Tuples of Operators, Integral Equations and Operator Theory, 51 (2005), 57-71.
[15] M. J. Cowen and R. G. Douglas, Complex geometry and operator theory, Acta Math. 141 (1978), 187-261.
[16] M. J. Cowen, R. G. Douglas, Equivalence of connections, Advances in Mathematics, 56 (1985), 39-91.
[17] M. J. Cowen, R. G. Douglas, Operators possessing an open set of eigenvalues, Functions, series, operators, Vol. I, II (Budapest, 1980), Colloq. Math. Soc. János Bolyai, vol. 35, North-Holland, Amsterdam, 1983, pp. 323-341.
[18] R. E. Curto, N. Salinas, *Generalized Bergman kernels and the Cowen-Douglas theory*, American Journal of Mathematics, **106** (1984), 447-488.

[19] P. Deb and S. Hazra, *Homogeneous Hermitian holomorphic vector bundles and operators in the Cowen-Douglas class over the poly-disc*, J. Math. Anal. Appl., **510** (2022), 1-32.

[20] R. G. Douglas, *Operator Theory and Complex Geometry*, Extracta Mathematicae, **24**(2007), no. 2, 135-165.

[21] R. G. Douglas, H. Kwon, and S. Treil, *Similarity of n-hypercontractions to backward Bergman shifts*, J. Lond. Math. soc., **88**(2013), no. 3, 637-648.

[22] R.G. Douglas, G. Misra and J. Sarkar, *Contractive Hilbert modules and their dilations*, Israel J. Math., **187**(2011), 141-165.

[23] R. G. Douglas, Y. Kim, H. Kwon and J. Sarkar, *Curvature invariant and generalized canonical operator models-I*, Oper. Theory Adv. Appl., **221**(2012), 293-304.

[24] R. G. Douglas, Y. Kim, H. Kwon and J. Sarkar, *Curvature invariant and generalized canonical operator models-II*, J. Funct. Anal., **266**(2014), 2486-2502.

[25] J. Fang, C. Jiang and K. Ji, *Cowen-Douglas operators and the third of Halmos’ ten problems*, arXiv:1904.10401.

[26] J. S. Fang, C. L. Jiang, P. Y. Wu, *Direct sums of irreducible operators*, Studia Mathematica, **155**(2003), 37-49.

[27] S. Ghara, *The orbit of a bounded operator under the Möbius group modulo similarity equivalence*, Israel J. Math., **238**(2020), 167-207.

[28] S. Ghara, S. Kumar and P. Pramanick, *K-homogeneous tuple of operators on bounded symmetric domains*, arXiv:2002.01298.

[29] Y. L. Hou, K. Ji and H. K. Kwon, *The trace of the curvature determines similarity*, Stud. Math., **236**(2017), no. 2, 193-200.

[30] Y. L. Hou, K. Ji and L. L. Zhao, *Factorization of generalized holomorphic curve and homogeneity of operators*, Banach J. Math. Anal., **15**(2021), 1-23.

[31] W. W. Hastings, *Commuting subnormal operators simultaneously quasisimilar to unilateral shifts*, Illinois Journal of Mathematics, **22**(1978).

[32] S. Hazra, *Homogeneous 2-shifts*, Complex Anal. Oper. Theory, **4**(2019) 1729-1763.

[33] K. Ji and S. Ji, *The metrics of Hermitian holomorphic vector bundles and the similarity of Cowen-Douglas operators*, Indian Journal of Pure and Applied Mathematics, to appear.

[34] K. Ji, C. Jiang, D. K. Keshari and G. Misra, *Flag structure for operators in the Cowen-Douglas class*, Comptes rendus - Mathématique, **352**(2014), 511–514.

[35] K. Ji, C. Jiang, D. K. Keshari and G. Misra, *Rigidity of the flag structure for a class of Cowen-Douglas operators*, J. Func. Anal., **272**(2017), 2899-2932.

[36] C. Jiang, *Similarity classification of Cowen-Douglas operators*. Canad. J. Math. **56**(2004), no. 4, 742-775.

[37] C. L. Jiang, *Similarity, reducibility and approximation of the Cowen-Douglas operators*, J. Operator Theory, **32**(1994), 77-89.

[38] C. Jiang, X. Guo and K. Ji, *K-group and similarity classification of operators*, J. Funct. Anal., **225**(2005), 167-192.

[39] C. Jiang, K. Ji and D. K. Keshari, *Geometric similarity invariants of Cowen-Douglas operators*, In Press.

[40] C. Jiang and Z. Wang, *Strongly irreducible operators on Hilbert space*. Pitman Research Notes in Mathematics Series, 389. Longman, Harlow, 1998. x+243 pp. ISBN: 0-582-30594-2.

[41] A. Jibril, *On almost unitarily equivalent operators*, Arab Gulf J. Sci. Res., **11**(1993), 295-303.

[42] D. K. Keshari, *Trace formulae for curvature of jet bundles over planar domains*, Complex Anal. Oper. Theory, **8**(2014), 1723-1740.

[43] A. Korányi, *Homogeneous bilateral block shifts*, Proc. Indian Acad. Sci. Math. Sci., **124**(2014), 225-233.

[44] A. Korányi and G. Misra, *A classification of homogeneous operators in the Cowen-Douglas class*, Adv. Math., **226**(2010), 5338-5360.

[45] A. Korányi and G. Misra, *Homogeneous operators on Hilbert spaces of holomorphic functions*, J. Func. Anal., **254**(2008), 2419-2436.
[46] A. Korányi and G. Misra, *Multiplicity-free homogeneous operators in the Cowen-Douglas class*, Stat. Sci. Interdiscip. Res., 8 (2009), 83-101.

[47] H. Kwon and S. Treil, *Similarity of operators and geometry of eigenvector bundles*, Publ. Mat., 53 (2009), 417-438.

[48] Q. Lin, *Operator theoretical realization of some geometric notions*, Trans. Amer. Math. Soc., 305 (1988), 353-367.

[49] G. Misra, *Curvature and the backward shift operator*, Proc. Amer. Math. Soc., 91 (1984), 105-107.

[50] G. Misra, *Curvature inequalities and extremal properties of bundle shifts*, J. Operator Theory, 11 (1984), 305-317.

[51] G. Misra, A. Pal, *Contractivity, complete contractivity and curvature inequalities*, arXiv:1410.7493.

[52] G. Misra and N.S.N. Sastry, *Contractive modules, extremal problems and curvature inequalities*, J. Funct. Anal., 88 (1990), 118-134.

[53] G. Misra and N.S.N. Sastry, *Completely contractive modules and associated extremal problems*, J. Funct. Anal., 91 (1990), 213-220.

[54] V. Müller and F. H. Vasilescu, *Standard models for some commuting multioperators*. Proc. Am. Math. Soc., 117 (1993), no. 4, 979-989.

[55] N. K. Nikolskii, *Treatise on the Shift Operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer Verlag, Berlin, 1986, Spectral function theory, With an appendix by S. V. Hruščev [S. V. Khrushčëv] and V. V. Peller, Translated from the Russian by Jaak Peetre.

[56] V. I. Paulsen and M. Raghupathi, *An introduction to the theory of reproducing kernel Hilbert spaces*, Cambridge Studies in Advanced Mathematics, vol. 152, Cambridge University Press, Cambridge, 2016.

[57] S. Richter, *Invariant subspaces in Banach spaces of analytic functions*, Trans. Amer. Math. Soc., 304 (1987), 585-616.

[58] S.S. Roy, *Homogeneous operators, jet construction and similarity*, Complex Anal. Oper. Theory, 5 (2011), 261-281.

[59] N. Salinas, *Products of kernel functions and module tensor products*, Topics in operator theory, Oper. Theory Adv. Appl., vol. 32, Birkhäuser, Basel, 1988, pp. 219-241.

[60] R. Shi, On a generalization of the Jordan canonical form theorem on separable Hilbert spaces. Proc. Amer. Math. Soc., 140 (2012), no. 5, 1593-1604.

[61] A.L. Shields, *Weighted shift operators and analytic function theory*, Math. Surveys, (1974), no. 13, 49-128.

[62] E.I. Timko, *A classification of m-tuples of commuting shifts of finite multiplicity*, Integral Equations and Operator Theory, 90 (2017), 1-22.

[63] S. Treil and B.D. Wick, *Analytic projections, corona problem and geometry of holomorphic vector bundles*, J. Amer. Math. Soc., 22 (2009), no. 1, 55-76.

[64] M. Uchiyama, *Curvatures and similarity of operators with holomorphic eigenvectors*, Trans. Amer. Math. Soc., 319 (1990), 405-415.

[65] K. Wang and G. Zhang, *Curvature inequalities for operators of the Cowen-Douglas class*, Israel Journal of Mathematics, 222 (2017), 279-296.

[66] D.R. Wilkins, *Homogeneous vector bundles and Cowen-Douglas operators*, Intern. J. Math., 4 (1993), 503-520.

[67] K. Zhu, *Restriction of the Bergman shift to an invariant subspace*, Quart. J. Math. Oxford Ser.(2), 48(1997), no. 192, 519-532.

Current address, Y. Hou, K. Ji, S. Ji and J. Xu: School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, Hebei 050016, China

Email address, Y. Hou: houyingli0912@sina.com

Email address, K. Ji: jikui@hebtu.edu.cn, jikuikui@163.com

Email address, S. Ji: jishanshan15@outlook.com

Email address, J. Xu: xujingmath@outlook.com