On $\alpha$-points of $q$-analogs of the Fano plane

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Subset lattice

- Let $V$ be a $\nu$-element set.
- $(V^k) := \text{Set of all } \nu-k\text{-subsets of } V$.
- $\#(V^k) = (\nu)_k$.
- Subsets of $V$ form a distributive lattice (wrt. $\subseteq$).

Definition

$D \subseteq (V^k)$ is a $t-(\nu, k, \lambda)$ (block) design if

- each $T \in (V^t)$ is contained in exactly $\lambda$ blocks (elements of $D$).

Idea of $q$-analogs in combinatorics

Replace subset lattice by subspace lattice!
Subset lattice (repeated)

- Let $V$ be a $v$-element set.
- $(V \choose k)$ := Set of all $k$-subsets of $V$.
- $\#(V \choose k) = (v \choose k)$.
- Subsets of $V$ form a distributive lattice (wrt. $\subseteq$).

Subspace lattice

- Let $V$ be a $v$-dimensional $\mathbb{F}_q$ vector space.
- Grassmannian $[V \choose k]_q :=$ Set of all $k$-dim. subspaces of $V$.
- Gaussian binomial coefficient

$$\# [V \choose k]_q = [v \choose k]_q = \frac{(q^v - 1)(q^{v-1} - 1) \cdot \ldots \cdot (q^{v-k+1} - 1)}{(q - 1)(q^2 - 1) \cdot \ldots \cdot (q^k - 1)}$$

- Subspaces of $V$ form a modular lattice (wrt. $\subseteq$).
Projective geometry

- Subspace lattice of $V = \text{projective geometry } \text{PG}(v-1, q)$
  - Elements of $\binom{V}{1}_q$ are points.
  - Elements of $\binom{V}{2}_q$ are lines.
  - Elements of $\binom{V}{3}_q$ are planes.
  - Elements of $\binom{V}{4}_q$ are solids.
  - Elements of $\binom{V}{v-1}_q$ are hyperplanes.

Advantage of geometric point of view

- Access to deep results developed in decades of research on finite geometries.
- Geometry provides intuition.

Attention!

- Dimensions are off by 1:
  - Vector space of algebraic dimension $v$
  - $\leftrightarrow$ projective geometry of geometric dimension $v - 1$. 
Definition (block design, stated again)
Let $V$ be a $v$-element set. $D \subseteq \binom{V}{k}$ is a $t$-$(v, k, \lambda)$ (block) design if each $T \in \binom{V}{t}$ is contained in exactly $\lambda$ elements of $D$.

$q$-analog of a design?

Definition (subspace design)
Let $V$ be a $v$-dimensional $\mathbb{F}_q$ vector space. $D \subseteq \left[\binom{V}{k}\right]_q$ is a $t$-$(v, k, \lambda)_q$ (subspace) design if each $T \in \left[\binom{V}{t}\right]_q$ is contained in exactly $\lambda$ elements of $D$.

- If $\lambda = 1$: $D$ $q$-Steiner system
- If $\lambda = 1$, $t = 2$, $k = 3$: $D$ $q$-Steiner triple system $STS_q(v)$
- Geometrically:
  $STS_q(v)$ is a set of planes in $PG(v-1, q)$ covering each line exactly once.
Lemma
Let $D$ be a $t$-$(v, k, \lambda)_q$ design and $i, j \in \{0, \ldots, t\}$ with $i + j \leq t$. Then for all $I \in \binom{V}{i}_q$ and $J \in \binom{V}{v-j}_q$ with $I \subseteq J$

$$\lambda_{i,j} := \#\{B \in D \mid I \subseteq B \subseteq J\} = \lambda \frac{\binom{v-i-j}{k-i}_q}{\binom{v-t}{k-t}_q}.$$ 

In particular, $\#D = \lambda_{0,0}$.

**Corollary: Integrality conditions**
If a $t$-$(v, k, \lambda)_q$ design exists, then all $\lambda_{i,j} \in \mathbb{Z}$.

Sufficient to check: $\lambda_i := \lambda_{i,0} \in \mathbb{Z}$ \hspace{1cm} (Parameters admissible)

**Corollary**
$STS_q(v)$ admissible $\iff v \equiv 1, 3 \pmod{6}$. 
\( \text{STS}_q(\nu) \) for small admissible \( \nu \)

- \( \nu = 3 \)
\( \text{STS}_q(3) = \{ V \} \) exists trivially.

- \( \nu = 7 \)
\( q \)-analog of the Fano plane \( \text{STS}_q(7) \).
Existence undecided for every field order \( q \).

Most important open problem in \( q \)-analogs of designs.

- \( \nu = 9 \)
existence open for every \( q \).

- \( \nu = 13 \)
\( \text{STS}_2(13) \) exists (Braun, Etzion, Östergård, Vardy, Wassermann 2013)
Only known non-trivial \( \text{STS}_q \).
q-Pascal triangle for $\text{STS}_q(7)$ $D$

$$
\lambda_{0,0} = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1 \\
\lambda_{1,0} = q^4 + q^2 + 1 \\
\lambda_{0,1} = q^5 + q^3 + q^2 + 1 \\
\lambda_{2,0} = 1 \\
\lambda_{1,1} = q^2 + 1 \\
\lambda_{0,2} = q^2 + 1
$$

- Each point $P$ is contained in $\lambda_{1,0} = q^4 + q^2 + 1$ blocks.
- $\sim$ derived design wrt $P$ (“local point of view from $P$”)

$$
\text{Der}_P(D) = \{ B/P \mid B \in D \text{ with } P \subseteq B \} \subseteq V/P
$$

- In general: $\text{Der}_P(D)$ is $(t-1)-(v-1, k-1, \lambda)_q$ design.
- $\implies$ $\text{Der}_P(\text{STS}_q(7))$ is $1-(6, 2, 1)_q$ design.
  $= \text{set of lines in } \text{PG}(5, q) \text{ covering each point exactly once}$. 
- In other words: $\text{Der}(\text{STS}_q(7))$ is a line spread of $\text{PG}(5, q)$. 
\( \alpha \)-points

- Spread \( S \) called geometric if for all distinct \( L_1, L_2 \in S \):
  \[ \{ L \in S \mid L \subseteq L_1 + L_2 \} \] is spread of the solid \( L_1 + L_2 \).
- \( P \) is called \( \alpha \)-point of \( \text{STS}_{q}(7) \)
  if the derived design in \( P \) is a geometric spread.
- S. Thomas 1996: There exists a non-\( \alpha \)-point.
- O. Heden, P. Sissokho 2016: For \( q = 2 \):
  Each hyperplane contains non-\( \alpha \)-point.
- Goal: Investigate Heden-Sissokho result for general \( q \)!
Assume that $H$ is hyperplane containing only $\alpha$-points.

Fix a poor solid $S$ in $H$ (not containing any block).

Let $\mathcal{F} = \{ F \in H \mid S \subseteq F \}$.
We have $\#\mathcal{F} = q + 1$.

For $F \in \mathcal{F}$, let

$$\mathcal{L}_F := \{ B \cap S \mid B \in D \text{ and } B + S = F \}.$$ 

Lemma

- $\mathcal{L}_F$ is a line spread of $S$.
- The sets $\mathcal{L}_F$ with $F \in \mathcal{F}$ are pairwise disjoint.
Conclusion
\[ \mathcal{L} := \bigcup_{F \in \mathcal{F}} \mathcal{L}_F \] is a set of \((q + 1)(q^2 + 1)\) lines in \(\text{PG}(3, q)\) admitting a partition into \(q + 1\) line spreads.

Lemma
For each point \(P\) in \(S\), the \(q + 1\) lines in \(\mathcal{L}\) passing through \(P\) span only a plane \(E_P\).
(In other words, the lines form a pencil in \(E_P\) through \(P\).)

Lemma
\((\left[ ^S_1 \right]_q, \mathcal{L})\) is a projective generalized quadrangle of order \((q, q)\).
Classification
Classification of projective generalized quadrangles: (F. Buekenhout, C. Lefèvre 1974)
\[ (\begin{bmatrix} 1 \\ S \end{bmatrix}_q, \mathcal{L}) \text{ is symplectic generalized quadrangle } W(q). \]

Implication

- By property of \( \mathcal{L} \):
The lines of \( W(q) \) admit a partition into \( q + 1 \) line spreads.
- Equivalently: The points of the parabolic quadric \( Q(4, q) \) admit a partition into ovoids.
- Not possible for even \( q \).
  - Payne, Thas: Finite generalized quadrangles, 3.4.1(i)
- Not possible for prime \( q \).
  - Ball, Govaerts, Storme 2006:
    Each ovoid in \( Q(4, q) \) is an elliptic quadric.
  - Any two of them have non-trivial intersection.
Theorem
Let $q$ be prime or even and $D$ a $\text{STS}_q(7)$. Then each hyperplane contains a non-$\alpha$-point of $D$.

Research problem
- Investigate the remaining $q$ (i.e. $q$ a proper odd prime power).
- Can “each 5-subspace contains non-$\alpha$-point” be shown?

arXiv preprint
https://arxiv.org/abs/2105.00365
Thank you!

Slides can be found at
https://www.mathe2.uni-bayreuth.de/michaelk/