SPECIAL VALUES OF PARTIAL ZETA FUNCTIONS OF REAL QUADRATIC FIELDS AT NONPOSITIVE INTEGERS AND THE EULER-MACLAURIN FORMULA

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ABSTRACT. We compute the special values at nonpositive integers of the partial zeta function of an ideal of a real quadratic field in terms of the positive continued fraction of the reduced element defining the ideal. We apply the integral expression of the partial zeta value due to Garoufalidis-Pommersheim (2001) using the Euler-Maclaurin summation formula for a lattice cone associated to the ideal. From the additive property of Todd series w.r.t. the (virtual) cone decomposition arising from the positive continued fraction of the reduced element of the ideal, we obtain a polynomial expression of the partial zeta values with variables given by the coefficient of the continued fraction. We compute the partial zeta values explicitly for \( s = 0, -1, -2 \) and compare the result with earlier works of Zagier (1977) and Garoufalidis-Pommersheim (2001). Finally, we present a way to construct Yokoi-Byeon-Kim type class number one criterion for some families of real quadratic fields.

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1. Introduction

Let $K$ be a number field of extension degree $[K : \mathbb{Q}] = r_1 + 2r_2$, where $r_1$ and $r_2$ denote respectively the number of real and complex embeddings of $K$. The Dedekind zeta function

$$\zeta_K(s) = \prod_{\mathfrak{p} \text{ prime ideal in } K} \frac{1}{1 - N\mathfrak{p}^{-s}}$$

is encoded with many interesting arithmetic properties of $K$. In particular, the residue at $s = 1$ is associated to the class number $h_K$ of $K$ by the class number formula

$$\text{Res}_{s=1} \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}R_K h_K}{\omega_K \sqrt{|D_K|}},$$

where $R_K$ is the regulator, $\omega_K$ is the number of roots of 1 in $K$ and $D_K$ is the discriminant. This has been the starting point of most studies of class numbers.

The simplest is the case of imaginary quadratic fields where the regulator appears to be trivial. In [16], Gauss listed nine imaginary quadratic fields of class number one and conjectured that the list was complete. This continued to be studied through the 20th century and is now quite well understood and was solved by works of Heegner, Stark, Baker, Goldfeld, Gross-Zagier and several others (e.g. [21], [36], [37], [1], [18], [19] and [20]).

The case of real quadratic fields is more complicated due to the presence of a nontrivial regulator. It was also conjectured by Gauss that there are infinitely many real quadratic fields of class number one. But since the regulator is far from being controlled in relation to the discriminant, there has been no essential progress towards the proof of the conjecture.

Instead of treating all real quadratic fields, most researchers considered some families of real quadratic fields whose regulators are controlled in relation to the discriminant. The most well-known family of this kind is the Richaud-Degert type: A Richaud-Degert type is defined by

$$d(n) = n^2 \pm r$$

for $r|4n$ and $-n < r \leq n$. For $r$ fixed as above, the family $\{K_n = \mathbb{Q}(\sqrt{d(n)})\}$ of real quadratic fields is called R-D type. In this case, we have a bound of the regulator $R_{K_n}$ given in terms of the discriminant:

$$R_{K_n} < 3 \log \sqrt{D_{K_n}}.$$ 

As in the imaginary quadratic case, a well-known estimate of Siegel, $L(1, \chi_D) \sim |D|^{-\epsilon}$, together with the class number formula implies that there are only finitely many R-D type fields of class number one. Assuming the generalized Riemann hypothesis, the class number one problems have been solved for many subfamilies in R-D type.

Quite recently, Biró obtained a Riemann-hypothesis-free answer to the class number one problem for the families $K_n = \mathbb{Q}(\sqrt{n^2 + 4})$ and $K_n = \mathbb{Q}(\sqrt{4n^2 + 1})$ in a series of papers ([2], [3]). He investigated the behavior of the special values of the partial Hecke $L$-functions at $s = 0$ in the family. The partial Hecke $L$-function of an ideal $\mathfrak{a}$ is defined for a ray class character $\chi$ as

$$L(s, \mathfrak{a}, \chi) := \sum_{b \sim \mathfrak{a}} \frac{\chi(b)}{N\mathfrak{p}^s}.$$
He discovered that the special values behave in a packet of linear forms whose
coefficients are easily computed for the family \( (K_n, O_{K_n}, \chi_n := \chi \circ N_{K_n/q}) \) for a
Dirichlet character \( \chi \). This property is called the linearity.

Inspired by Biró’s pioneering work, in [8], [9], [29] and [30] the linearity is ob-
served for more general families of Richaud-Degert types, and the class number one
and two problems have been answered for these.

In [23], the authors of the present article found a sufficient condition for the
linearity of the Hecke L-values at \( s = 0 \). That is, for families of integral ideals \( b_n \)
of \( K_n \) such that \( b_n^{-1} = [1, \omega(n)] := \mathbb{Z}1 + \mathbb{Z}\omega(n) \), where \( \omega(n) \) has purely periodic
positive continued fraction expansion of a fixed period \( r \):

\[
\omega(n) = \left( [a_0(n), a_1(n), \cdots, a_{r-1}(n)] \right)
\]

with \( a_i(n) \) being integer coefficient polynomials of \( n \) of degree 1. (See [5,1] for
the definition of continued fractions used here.)

In this setting we have, for \( n = qk + r \) with \( 0 \leq r < q \) and for a Dirichlet
character \( \chi \) of conductor \( q \), the \( L \)-value at \( s = 0 \):

\[
L_{K_n}(0, \chi \circ N_{K_n}, b_n) = \frac{1}{12q^2} \left( A_{\chi}(r)k + B_{\chi}(r) \right)
\]

with \( A_{\chi}(r), B_{\chi}(r) \in \mathbb{Z}[\chi] \). (Here \( \mathbb{Z}[\chi] \) denotes the extension of \( \mathbb{Z} \)
by the values of \( \chi \).)

In [24] a higher degree generalization of the linearity for ray class partial zeta
values was obtained. Under the same assumption, except that \( a_i(n) \) is allowed to
be polynomials of degree bounded by \( d \), the partial zeta value at \( s = 0 \) of the mod-q
ray class of \((C + D\omega(n))b_n\) is a quasi-polynomial in \( n \):

\[
\zeta_q(0, (C + D\omega(n))b_n) = \frac{1}{12q^2} \left( A_0(r) + A_1(r)k + \cdots + A_d(r)k^d \right)
\]

with \( A_i(r) \in \mathbb{Z} \) (for a precise definition, we refer the reader to [24]). In particular,
if we take \( d = 1 \) and sum the ray class zeta values twisted by \( \chi_n \), the linearity of
the partial Hecke L-values is recovered. For \( d > 1 \), the same process concludes the
polynomial behavior of the partial Hecke values at \( s = 0 \).

In this paper, we are interested in a similar expression of partial \( \zeta \)-values at
general nonpositive integers besides 0 and compute similar expression for these
values. Explicit formula for these special values is very important in interpolating
\( p \)-adic \( \zeta \)-or \( \zeta \)-functions. In particular, the integrality can be obtained from explicit
formulæ.

However, one might be curious at this point as to what is the arithmetic impor-
tance of these special values at negative integers. Besides the special value at \( s = 0 \),
it is expected that other values have direct arithmetic applications. In our case, we
focus on the special value at \( s = -1 \). In the Riemann-hypothesis-free solution of
the class number one problems mentioned in earlier paragraphs, Biró’s method of
using the linearity is accompanied by a certain type of class number one criteria.
The linearity is a direct consequence of the explicit formula of the special value at
\( s = 0 \). Similarly, the explicit formula of a partial \( \zeta \)-value at \( s = -1 \) allows one to
produce a class number one criterion: For a real quadratic field \( K \), the Dedekind \( \zeta \)-
and the partial \( \zeta \)-values at \( s = -1 \) are positive as they are expressed as the sum
of values of the sum-of-divisors function, thanks to a theorem of Siegel in [35] (see also
[17], p. 20). The sum-of-divisor function has a trivial lower bound, for example,
by summing up trivial divisors. In this way one can make a trivial lower bound
of the Dedekind zeta value \( \zeta_K(-1) \). If \( K \) is of class number one, \( \zeta_K(-1) \) equals \( \zeta(-1, O_K) \), where \( O_K \) is the ring of integers in \( K \). Then comparing \( \zeta(-1, O_K) \) with the lower bound of \( \zeta_K(-1) \), one obtains a condition on the discriminant as \( |D| \geq 4 \) of Example 9.2. (In Example 9.3 there is implicitly a class number one criterion, which is too strong and thus implies the nonexistence of a class number one field in the family.) Namely, it is when the trivial lower bound of \( \zeta_K(-1) \) equals the value itself. It is a general recipe for making a class number one criterion of Yokoi-Byeon-Kim type. In [6,7], Byeon and Kim obtained a class number one criterion of polynomial type by comparing \( \zeta(-1, b_n) \) and \( \zeta_{K_n}(-1) \) for the R-D type family \( \{K_n\} \), which generalizes Yokoi’s earlier work on class number one criterion (39). The original work of Yokoi was done for \( d = n^2 + 4 \), which is a very special case of R-D type. Then Byeon-Kim constructed a class number one criteria for general R-D type generalizing Yokoi’s work. In this paper our main tool is the homological property of Todd series of lattice cones. From the point of view that generalized Dedekind sums are realized as coefficients of Todd series of lattice cones and their reciprocity comes from the cocycle property of Todd series, our result generalizes and justifies the result of [6,7]. In Section 4 of this article, for two families of real quadratic fields, we suggest class number one criteria constructed by using our explicit formula. While one belongs to the R-D type, the other does not. Through the examples, one will be able to check how the method works concretely. We expect that other individual special values at nonpositive integers are also encoded with a similar sort of arithmetic properties.

Let us mention a close work on special values at \( s = 0 \) before going further. Recently, in [4] Biró and Granville obtained a compact expression of the special value at \( s = 0 \) of the partial Hecke’s L-function of an ideal of a real quadratic field w.r.t. a character, aiming its application to class number problems. They used a value at \( s = 0 \) of the Dirichlet L-function of a character. They also obtained a close work on special values at \( s = 0 \) before going further. Our main theorem of the article is as follows:

**Theorem 1.1.** Let \( b \) be an ideal of a real quadratic field \( K \) such that \( b^{-1} = [1, \omega] \) where \( \omega = [(a_0, a_1, \ldots, a_{r-1})] \). Let \( Q(x, y) \) be the quadratic form \( Q(x, y) := N(b)N(x\omega + y) \). Then we have

\[
\zeta(-k, b) = \sum_{i=0}^{\ell-1} (-1)^{i-1} L_k(\partial h_1, \partial h_2) Q(\alpha_i h_1 - \alpha_{i-1} h_2, \beta_i h_1 - \beta_{i-1} h_2)^k + \frac{B_{2k+2}}{(2k+2)!} \sum_{i=0}^{\ell-1} (-1)^{i} a_{\ell-i} R_k(\partial h_1, \partial h_2) Q(\alpha_{i-2} h_1 + \alpha_i h_2, \beta_{i-2} h_1 + \beta_i h_2)^k,
\]

where \( \alpha_i, \beta_i \) are fundamental units of \( \mathbb{Z}[\sqrt{-D}] \) with \( \alpha_i \) being the smallest even period of \( \partial h_1, \partial h_2 \) and \( \beta_i \) the corresponding odd period.
where \( L_k \) and \( R_k \) are the homogeneous polynomials of degree \( 2k \):

\[
L_k(X, Y) = \sum_{i=1}^{2k+1} \frac{B_i}{i!} B_{2k+2-i} X^{i-1} Y^{2k-i+1},
\]

\[
R_k(X, Y) = X^{2k} + X^{2k-1} Y + \ldots + Y^{2k}.
\]

We begin with the following formula due to Garoufalidis-Pommersheim \([15]\). They used the Euler-Maclaurin formula for simple lattice polytopes due to Brion and Vergne \([5]\).

**Theorem 1.2** (Garoufalidis-Pommersheim \([15]\)). For \( n \geq 0 \), we have

\[
\zeta(b, -n) = (-1)^n n! \left\{ \text{Todd}_{\sigma}^{(2n+2)}(\partial_{h_1}, \partial_{h_2}) - \delta_{n,0} \right\}
\]

\[
\circ \int_{\sigma(h_1, h_2)} e^{-Q(x_1, x_2)} dx_1 dx_2 \bigg|_{(h_1, h_2)=0},
\]

where

\[
\delta_{n,0} = \begin{cases} 
1 & \text{if } n = 0, \\
0 & \text{otherwise.}
\end{cases}
\]
\[ \bar{\sigma} = \delta_1 + \delta_2 + \cdots + \delta_m \]
\[ \delta_i = \delta_i(w_{i-1}, w_i) \]

**Figure 1.** Decomposition of \( \bar{\sigma} \) using a negative continued fraction

\[ \bar{\sigma} = \tau_1 + \tau_2 + \cdots + \tau_\ell \]
\[ \tau_i = \tau_i(v_{i-1}, v_i) \]

**Figure 2.** Decomposition of \( \bar{\sigma} \) using a positive continued fraction

\[ \sigma = \sigma_0 + \sigma_1 + \cdots + \sigma_{m-1} \]
\[ \sigma_i = \sigma_i(u_i, u_{i+1}) \]

**Figure 3.** Zagier’s decomposition of \( \sigma \)
higher Kronecker “limit” formula is obtained for positive continued fractions using the conversion formula. But the conversion process does not keep some important arithmetic data of positive continued fractions. In particular, the pattern of terms appearing in positive continued fractions becomes less intrinsic in the corresponding negative continued fractions. This was the main technical difficulty in earlier works. The use of the Euler-Maclaurin formula together with its additivity eases this difficulty and gives more direct and explicit computation to the special values. This additivity is in other words 1-cocycle property, which is closely related to the Eisenstein cocycle \cite{40}, which is exactly the modular pseudomeasure in the sense of Manin and Marcolli \cite{31}. The cocycle property of the Todd series is used in \cite{25} to associate certain exponential sums to the generalized Dedekind sums.

Using our main result with a small modification to deal with conductors, one can directly recover the polynomial expressions for partial Hecke L- or partial zeta values at \( s = 0 \) for families of real quadratic fields and ideals given in this way. Furthermore one can directly generalize the polynomial behavior of a partial zeta value at arbitrary non-positive integers.

The plan of this paper is as follows: First, we rewrite the partial zeta function of an ideal as a zeta function of a quadratic form weighted by a fundamental lattice cone of the Shintani decomposition (Section 2). We review some notions of toric geometry, in particular on lattice cones and those related to state the work of Garufalidis-Pommersheim evaluating the zeta values at nonpositive integers using the Todd series (Section 3). Then we decompose the formula of Garoufalidis-Pommersheim w.r.t. the cone decomposition arising from the positive continued fractions (Sections 4-6). The partial zeta values at \( s = 0, -1, -2 \) are explicitly computed using our method and are compared with already known results (Section 7). Section 8, which is technical and similar to the computation by Zagier (\cite{42}), is devoted to the proof of vanishing of a part skipped in the previous sections. Finally, we apply the formula to some families of real quadratic fields to compute the partial zeta values at \( s = -1 \). This will recover the Yokoi-Byeon-Kim type class number one criterion for the family (Section 9).

2. Partial zeta function of real quadratic fields

2.1. Partial zeta function. Let \( K \) be a real quadratic field and \( \mathfrak{b} \) be an ideal. Throughout this article, by partial zeta function we mean the partial zeta function of an ideal class in a narrow sense. The partial zeta function of an ideal \( \mathfrak{b} \) is defined as

\[
\zeta(s, \mathfrak{b}) := \sum_{\mathfrak{a} \sim \mathfrak{b}, \mathfrak{a} \text{ integral}} N(\mathfrak{a})^{-s},
\]

where \( \mathfrak{a} \sim \mathfrak{b} \) means \( \mathfrak{b} = \alpha \mathfrak{a} \) for a totally positive element \( \alpha \) in \( K \). This infinite series defines a holomorphic function in the region \( \operatorname{Re}(s) > 1 \) of the complex plane and has a meromorphic continuation to the entire complex plane. Since for an integral ideal \( \mathfrak{a} \) in the narrow class of \( \mathfrak{b} \) there exists totally positive element \( \alpha_0 \in \mathfrak{b}^{-1} \) such that \( \mathfrak{a} = \alpha_0 \mathfrak{b} \) and vice versa, we can again write

\[
\zeta(s, \mathfrak{b}) = \sum_{[\mathfrak{a}] \in (\mathfrak{b}^{-1})^+/E^+} N(\mathfrak{a} \mathfrak{b})^{-s},
\]

where \((\mathfrak{b}^{-1})^+\) denotes the set of totally positive elements of \( \mathfrak{b}^{-1} \) and \( E^+ = E^+_K \) denotes the group of totally positive units of \( K \). Now we are going to describe
Figure 4. A fundamental cone of $E_+^+$-action

the summation as taken inside the Minkowski space of $K$. Let $(\iota_1, \iota_2)$ be two real embeddings of $K$. Let us denote the Minkowski space of $K$ by

$$K_R = K \otimes \mathbb{Q} = K_{\iota_1} \times K_{\iota_2}.$$  

Then one can identify an ideal $\mathfrak{c}$ with a lattice of $K_R$ given by its image under the diagonal embedding of $K$ into $K_R$:

$$\iota = (\iota_1, \iota_2): K \to K_R \quad (\iota(a) \mapsto (\iota_1(a), \iota_2(a))).$$

This is a full lattice in the Minkowski space.

$E_+^+$ acts on the first quadrant of $K_R$ by coordinate-wise multiplication after the diagonal embedding. Let $\epsilon$ be the totally positive fundamental unit of $K$. We assume $\iota_1(\epsilon) > \iota_2(\epsilon)$. A fundamental domain of this action is given as a half-open cone $F_K$ of $K_R$ with basis $\{\iota(1), \iota(\epsilon)\}$:

$$F_K = \{x\iota(1) + y\iota(\epsilon) \in \mathbb{R}^2 | x \geq 0, y > 0\}.$$ 

For an ideal $a$ of $K$ or more generally a lattice $\Lambda$ of $K_R$, we denote its intersection with $F_K$ by $F_K(a)$ or $F_K(\Lambda)$, respectively.

For $[a] \in (b^{-1})^+/E^+$, there is a unique representative $a$ chosen in $F_K(b^{-1})$. Thus we have

$$\zeta(s, b) = \sum_{a \in F_K(b^{-1})} N(ab)^{-s}.$$ 

2.2. Zeta function of 2-dimensional cones. Consider the standard lattice $M := \mathbb{Z}^2$ in $\mathbb{R}^2$. Let $Q(x, y) = ax^2 + bxy + cy^2$ be a quadratic form. For two linearly independent vectors $v_1, v_2$, let $\sigma(v_1, v_2)$ be the cone in $\mathbb{R}^2$ as the convex hull of the two rays $\mathbb{R}^+ v_1, \mathbb{R}^+ v_2$:

$$\sigma(v_1, v_2) := \{x_1v_1 + x_2v_2 | x_i > 0 \text{ for } i = 1, 2\}.$$ 

For simplicity, we write $\sigma$ instead of $\sigma(v_1, v_2)$ if $v_1, v_2$ is clear from the context. In our convention, the origin is not contained in a lattice cone.

Define a weight function $wt_\sigma$ with respect to $\sigma$ as follows:

$$wt_\sigma(\ell) = \begin{cases} 1 & \ell \in \text{int}(\sigma), \\
\frac{1}{2} & \ell \in \partial(\sigma) - (0, 0), \\
0 & \text{otherwise.} \end{cases}$$
This strange weight is justified via identification of the partial zeta function with the zeta function of a lattice cone that will be defined below. The partial zeta function is a sum over the points of $F_K(b^{-1})$. Since the two edges of $F_K$ are related by the multiplication of the totally positive unit, the summands over both edges coincide. When we take $F_K$ as half-open cone, this repetition is automatically removed. Equivalently, we may apply this weight function so that the total contribution over an orbit equals 1. The choice of assigning $1/2$ to each edge will be found useful when we apply the Euler-Maclaurin formula to the cone.

Let us assume $Q(v) > 0$ for every $v \in \sigma$. Then for $\sigma$ and $Q(-)$, we define a zeta function as the following series on $\text{Re}(s) > 1$:

$$\zeta_Q(s, \sigma) := \sum_{m \in M} \frac{\text{wt}_\sigma(m)}{Q(m)^s}.$$ 

2.3. Comparison of zeta functions. One can choose $b$ as an integral ideal in such a way that $b^{-1} = [1, \omega]$ for $[1, \omega]$ being the free $\mathbb{Z}$-module generated by 1 and $\omega$. Taking $\iota(1), \iota(\omega)$ as basis of $K_R$, we have trivialization

$$K_R \simeq \mathbb{R}^2 \quad \text{and} \quad \iota(b^{-1}) \simeq M = \mathbb{Z}^2.$$ 

Here, we fix the order of the basis such that $x + y\omega$ reads $(y, x)$ in $\mathbb{R}^2$.

From the reduction theory of quadratic forms due to Gauss, we have a privileged choice of $\omega$ such that $\iota_1(\omega) > 1, -1 < \iota_2(\omega) < 0$.

Then the totally positive fundamental unit $\epsilon$ belongs to $b^{-1}$ and $\epsilon = p + q\omega$ for $p, q$ being a pair of relatively prime positive integers.

Let $\sigma$ be a lattice cone generated by $(0, 1)$ and $(q, p)$. $\sigma$ corresponds to $F_K$ in $K_R$. One should be aware that this identification depends on $b$ and the choice of $\omega$. Then we have a quadratic form given by

$$Q(m, n) := N(m\omega + n)n(b) \in \mathbb{Z}^2$$ 

for $m\omega + n \in b^{-1}$. $Q(m, n)$ is positive and integral for $a = m\omega + n$ in the fundamental cone $F_K$. Thus we have the following identification of two zeta functions previously defined:

Lemma 2.1. Let $Q(m, n) = N(m\omega + n)n(b)$ and $\sigma$ be a cone defined as above. Then we have

$$\zeta(s, b) = \zeta_Q(s, \sigma).$$ 

3. Euler-Maclaurin expression of partial zeta values

In this section, we recall some notions around the higher-dimensional generalization of the Euler-Maclaurin formula of Brion-Vergne ([5]). Theorem 1.2 of Garoufalidis and Pommersheim ([15]) is an application of the Euler-Maclaurin formula to the 2-dimensional lattice cone given by the considered ideal and the unit action. The statement of Theorem 1.2 needs notions for lattice cones and Todd series in several variables.
3.1. Todd series of 2-dimensional cone. Let \( M = \mathbb{Z}^2 \subset \mathbb{R}^2 \) be a fixed lattice. Recall that a lattice cone is the convex hull of two rays whose slopes are given by lattice vectors. We may assume the generating vectors of a cone are primitive (i.e. not a multiple of other lattice vector in the same ray). For two linearly independent primitive lattice vectors \( v_1, v_2 \), let \( \sigma(v_1, v_2) \) be the cone generated by \( v_1 \) and \( v_2 \). When \( v_1, v_2 \) are clear from the context, we will simply write \( \sigma \) instead of \( \sigma(v_1, v_2) \). When there appear several cones, they will be denoted by \( \sigma, \tau \ldots \) or \( \sigma_1, \sigma_2, \sigma_3, \ldots \). Since we will be concerned with a surface integral over a 2-dimensional cone the order of the basis vectors (i.e. the orientation of the cone) is important. So \( \sigma(v_1, v_2) \) is never equal to \( \sigma(v_2, v_1) \). Taking \( v_1, v_2 \) as column vectors in \( \mathbb{Z}^2 \), we associate a nonsingular \((2 \times 2)\)-matrix

\[
A_\sigma = (v_1, v_2)
\]
to a lattice cone \( \sigma = \sigma(v_1, v_2) \). Conversely, if a \((2 \times 2)\)-nonsingular matrix \( A \) with integer coefficient has column vectors \( v_1, v_2 \) which are primitive, we can associate a unique lattice cone. A cone is said to be nonsingular if the matrix is in \( GL_2(\mathbb{Z}) \). Equivalently, \( \sigma \) is nonsingular iff \( \det(A_\sigma) = \pm 1 \).

Remark 3.1. In literature on polytopes or toric geometry, a cone is said to be simple if it is generated by \( n \)-linearly independent rays in \( \mathbb{R}^n \). In this article, as we are considering only 2-dimensional cones, every cone is simple unless degenerate.

Let \( \mathcal{M}_\sigma \) be the sublattice of \( M \) generated by \( v_1, v_2 \) and \( \Gamma_\sigma = M/\mathcal{M}_\sigma \). An element \( g \in \Gamma_\sigma \) can be written as

\[
g = a_{\sigma,1}(g)v_1 + a_{\sigma,2}(g)v_2
\]
for rational numbers \( a_{\sigma,1}(g), a_{\sigma,2}(g) \) modulo \( \mathbb{Z} \). This is given ambiguously but yields two well-defined characters

\[
\chi_{\sigma,i}(g) = e^{2\pi i a_{\sigma,i}(g)}, \quad \text{for} \ i = 1, 2.
\]

For a root of \( 1 \lambda \), we first define a \( \lambda \)-twist of the classical Todd series:

\[
(3.1) \quad \text{Todd}^\lambda(S) = \frac{S}{1 - \lambda e^{-S}}.
\]

When \( \lambda = 1 \), this is the classical Todd series.

Now the Todd power series for a cone \( \sigma \) is defined as

\[
\text{Todd}_\sigma(x_1, x_2) := \sum_{g \in \Gamma_\sigma} \text{Todd}^{\chi_{\sigma,1}(g)}(x_1) \text{Todd}^{\chi_{\sigma,2}(g)}(x_2).
\]

We say two cones \( \sigma_1 \) and \( \sigma_2 \) are similar if

\[
AA_{\sigma_1} = A_{\sigma_2}
\]
for \( A \in GL_2(\mathbb{Z}) \). In this case, \( A \) induces an isomorphism of \( \mathcal{M}_{\sigma_1} \) in \( \mathcal{M}_{\sigma_2} \), which descends to an isomorphism of \( \Gamma_{\sigma_1} \) in \( \Gamma_{\sigma_2} \). Since this isomorphism takes the lattice generators of \( \sigma_1 \) to those of \( \sigma_2 \), the two characters are preserved. \textit{A priori} the Todd series of two similar cones coincide.

Remark 3.2. For two similar cones \( \sigma \) and \( \tau \), we have

\[
\text{Todd}_\sigma(x_1, x_2) = \text{Todd}_\tau(x_1, x_2).
\]
One should be aware that this is not the similarity of the matrices as a linear map in linear algebra. \( \sigma(v_1, v_2) \) and \( \sigma(v_2, v_1) \) are not similar in general.
3.2. Dual cone and its lattice. Let \( N := \text{Hom}(M, \mathbb{Z}) \) be the dual lattice of \( M \). \( N \) is a lattice in the vector space \( N_\mathbb{R} := N \otimes \mathbb{R} \). Using the standard inner product \( \langle x, y \rangle \) we will often identify \( N \) and \( M \). Associated to a lattice cone \( \sigma \) in \( M \), its dual cone \( \tilde{\sigma} \) is defined as

\[
\tilde{\sigma} := \{ y \in N_\mathbb{R} - 0 | \langle y, x \rangle \geq 0 \}.
\]

As the orientation is concerned, \( \tilde{\sigma} \) is endowed with the orientation given by the transpose of the matrix of \( \sigma \). Notice that \( \tilde{\sigma} \) is again a lattice cone generated by two primitive lattice vectors inward and normal to \( \sigma \) in the identification of \( N \) and \( M \). To \( \tilde{\sigma} \), there are two lattices naturally associated. \( M_{\tilde{\sigma}} \) is the sublattice of \( M = N \) generated by the primitive lattice vectors of \( \tilde{\sigma} \). Note that this coincides with the definition of \( M_\sigma \) in Section 3.1. \( N_\tilde{\sigma} := \text{Hom}(M_\sigma, \mathbb{Z}) \) is a lattice in \( N_\mathbb{R} \) generated by dual vectors \( \alpha_1, \alpha_2 \) to \( v_1, v_2 \) if \( \sigma = \sigma(v_1, v_2) \) (i.e. \( \langle \alpha_i, v_j \rangle = \delta_{ij} \)). Note that the vectors \( \alpha_1, \alpha_2 \) are lattice vectors only if \( \sigma \) is nonsingular. These are related by the following inclusion relation:

\[
M_{\tilde{\sigma}} \subseteq M = N = \mathbb{Z}^2 \subseteq N_\tilde{\sigma}.
\]

3.3. Evaluation of zeta values. For \( \sigma \) and the quadratic form \( Q(x_1, x_2) \) in relation to \( b \), Garoufalidis and Pommersheim computed the asymptotic expansion of the exponential series defining \( \zeta(b, s) \) using the Euler-Maclaurin formula:

\[
\sum_{l \in \sigma \cap M} \text{wt}_e e^{-Q(l)t} \sim \left\{ \text{Todd}_{\tilde{\sigma}}^{\text{even}} (\partial_{\nu_1}, \partial_{\nu_2}) - \frac{q}{2} \partial_{\nu_1} \partial_{\nu_2} \right\}.
\]

Applying a result of Zagier (Proposition 2 in \cite{42}), they obtained the special values of \( \zeta(b, s) \) for \( s \) a nonpositive integer as in Theorem 1.2. For \( n \geq 0 \),

\[
\zeta(b, -n) = (-1)^n n! \left\{ \text{Todd}_{\tilde{\sigma}}^{2n+2} (\partial_{\nu_1}, \partial_{\nu_2}) - \delta_{n,0} \frac{q}{2} \partial_{\nu_1} \partial_{\nu_2} \right\}
\]

where

\[
\delta_{n,0} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise}. \end{cases}
\]

4. Additivity of the Todd series and cone decomposition

In this section, we recall some techniques used in \cite{15} concerning additive decomposition of the Todd series w.r.t. cone decomposition.

The Todd series does not behave additively w.r.t. decomposition of its underlying cone. We need a kind of normalization of the Todd series. The normalized Todd power series \( S_\sigma \) for a cone \( \sigma \) is defined as follows:

\[
S_\sigma(x_1, x_2) = \frac{1}{\det(A_\sigma) x_1 x_2} \text{Todd}_\sigma(x_1, x_2).
\]

One should note that a different choice of the orientation of the same underlying cone yields the opposite sign in the normalized Todd series and interchanges the two variables. This is contrary to the original Todd series case, where the similarity class is determined by the sign.
Let \( v_i \in \mathbb{R}^2 \) for \( i = 1, 2, 3 \) be pairwise linearly independent primitive lattice vectors in a half-plane. An ordered pair \((v_i, v_j)\) for \( i \neq j \) determines a lattice cone \( \sigma_{ij} = \sigma_{ij}(v_i, v_j) \) with orientation.

In this case, we formally write \( \sigma_{ij} + \sigma_{jk} = \sigma_{ik} \).

Then we have the following:

**Theorem 4.1** (Pommersheim [33]). *For \( i = 1, \ldots, r + 1 \), let \( v_i \) be pairwise linearly independent lattice points in a half-plane of \( \mathbb{R}^2 \). We define cones

\[
\sigma_i := \sigma_i(v_i, v_{i+1}), \quad \sigma := \sigma(v_1, v_{r+1}).
\]

Thus

\[
\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_r.
\]

Then

\[
S_\sigma(x_1, x_2) = \sum_{i=1}^{r} S_{\sigma_i}(A_{\sigma_i}^{-1}A_\sigma(x_1, x_2)^t).
\]

In particular, if every \( \sigma_i \) is nonsingular (i.e. \( \det(A_{\sigma_i}) = \pm 1 \)) for \( i = 1, 2, \ldots, r \),

\[
S_\sigma(x_1, x_2) = \sum_{i=1}^{r} \det(A_{\sigma_i})F(A_{\sigma_i}^{-1}A_\sigma(x_1, x_2)^t),
\]

where \( F(x_1, x_2) = \frac{1}{1-e^{-x_1}} \frac{1}{1-e^{-x_2}} \).

**Proof.** See Theorem 2 in [33]. \( \square \)

**Remark 4.2.** Abusing notation, we may denote \( \sigma(v_2, v_1) \) by \( -\sigma(v_1, v_2) \) according to the above theorem. Actually by definition of the Todd power series of the cone above, we easily find that

\[
\text{Todd}_\sigma(x_1, x_2) = \text{Todd}_{-\sigma}(x_2, x_1).
\]

The matrix \( A_{\sigma}^{-1} \) represents the linear transformation \( v_1 \mapsto e_1, \ v_2 \mapsto e_2 \). So we have \( A_{\sigma}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ A_\sigma \). Let \( A_{\sigma}^{-1} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \) for two row vectors \( w_1, w_2 \). Then \( A_{-\sigma}^{-1} = \begin{pmatrix} w_2 \\ w_1 \end{pmatrix} \). Therefore,

\[
\text{Todd}_\sigma(A_{\sigma}^{-1}(x_1, x_2)^t) = \text{Todd}_\sigma((w_1, (x_1, x_2)), (w_2, (x_1, x_2))) \\
= \text{Todd}_\sigma((w_2, (x_1, x_2)), (w_1, (x_1, x_2))) \\
= \text{Todd}_{-\sigma}(A_{-\sigma}^{-1}(x_1, x_2)^t).
\]

Thus one sees immediately that for the additivity theorem to hold the orientation of \( \sigma \) must not make any problem.

5. **Cone decomposition and continued fraction**

In this section, we will decompose the cone \( \sigma(b^{-1}) \) into nonsingular cones. This decomposition follows directly the decomposition of the fundamental cone in the totally positive quadrant of Minkowski space under the action of the totally positive unit group. This is a fairly standard fact related to desingularization of a cusp of the Hilbert modular surface of the real quadratic field considered, or more generally to desingularization of quotient singularities in toric surfaces. It is described in terms
of the negative continued fraction expansion of the reduced basis of $b^{-1}$ so that the desingularization of the lattice cone $\sigma(b^{-1})$ in the sense of toric geometry follows (cf. [14, 17]). We are going to apply Theorem 4.1 to obtain an explicit formula of the zeta values using the terms of the positive continued fraction. In this case, contrary to the geometric case, one should deal with virtual cones, which could be avoidable if one used a negative continued fraction.

In general, there are many other decompositions possible for a singular cone. But a lattice cone arising from a totally real field and the action of the totally positive units can be decomposed according to the shape of its Klein polyhedron which is a geometric realization of a continued fraction. In dimension 2, this appears as follows: For each quadrant of $K_\mathbb{R}$, we take the convex hull of $\iota(b^{-1})$ in the 1st (resp. the 4th) quadrant of $\mathbb{R}^2$ with $B_0 = \iota(1)$, $B_{-1} = \iota(\omega)$ and $x(B_i) < x(B_{i-1})$, where $x(-)$ is taking the 1st coordinate. These $B_i$ arising as the vertices of the Klein polyhedron should not be confused with the Bernoulli numbers.
Let $\ell$ be the even period of the continued fraction expansion of $\omega$ (i.e. $\ell = r$ (resp. $2r$) for even $r$ (resp. for odd $r$)).

$B_i$ satisfies a periodic recursive relation read from the continued fraction of $\omega$ (cf. [17]):
\begin{equation}
B_{i-1} = a_i B_i + B_{i+1}.
\end{equation}

Since a successive pair $B_i, B_{i+1}$ is a basis of the lattice $\iota(b^{-1})$ in $K_\mathbb{R}$, this yields a change of basis:
\begin{equation}
(B_{i-1} \quad B_i) = (B_i \quad B_{i+1}) \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}.
\end{equation}

After successive change of basis, we have
\begin{equation}
(B_{-i-1} \quad B_{-i}) = (B_{-i} \quad B_0) \begin{pmatrix} a_{\ell-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{\ell-2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{\ell-i} & 1 \\ 1 & 0 \end{pmatrix}.
\end{equation}

Let $\alpha_i, \beta_i$ be the coordinate of $B_{-i-1}$ w.r.t. the basis $\{B_{-1}, B_0\}$:
\begin{equation}
B_{-i-1} = \alpha_i B_{-1} + \beta_i B_0.
\end{equation}

From (5.3), $(\alpha_i, \beta_i)$ is
\begin{equation}
\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} a_{\ell-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{\ell-2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{\ell-i+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{\ell-i} & 1 \\ 1 & 0 \end{pmatrix}.
\end{equation}

As $B_{-i-1}$ is primitive, so is $(\alpha_i, \beta_i)$ in $M = \mathbb{Z}^2$.

In the following lemma, the totally positive fundamental unit is identified w.r.t. the basis $\{B_{-1}, B_0\}$:

**Lemma 5.1.** Let $\epsilon$ be the totally positive fundamental unit of $K$. Then
\begin{equation}
\iota(\epsilon) = B_{-\ell} = \alpha_{\ell-1} B_{-1} + \beta_{\ell-1} B_0.
\end{equation}

**Proof.** Let $\epsilon_K > 1$ be the fundamental unit of $K$. Then for the period $r$ of the continued fraction expansion of $\omega$, we have
\begin{equation}
\iota(\epsilon_K) = B_{-r}.
\end{equation}

See p. 40 of [17] for details. Since the totally positive unit $\epsilon$ is either $\epsilon_K$ or $\epsilon_K^2$ according to the sign of $\iota_2(\epsilon_K)$, we then obtain that $\iota(\epsilon) = B_{-\ell}$. \hfill \Box

In Section 2 we associated a lattice cone $\sigma(b^{-1})$ in $\mathbb{R}^2$ to $b^{-1}$:
\begin{equation}
\sigma(b^{-1}) = \sigma((0,1), (\alpha_{\ell-1}, \beta_{\ell-1})).
\end{equation}

This corresponds to the cone bounded by $\iota(1)$ and $\iota(\epsilon)$ in $K_\mathbb{R}$. For the rest of this section, only the cone $\sigma(b^{-1})$ is needed to consider computing the zeta values. So we will abbreviate $\sigma(b^{-1})$ simply to $\sigma$.

**Lemma 5.2.** Let $\sigma = \sigma((0,1), (\alpha, \beta))$ be a lattice cone where $\alpha, \beta$ are relatively prime positive integers. Then $\sigma$, the dual cone of $\sigma$, is similar to
\begin{equation}
\tau = \tau((1,-1), (\alpha, \beta)).
\end{equation}

**Proof.** It is easy to see the dual cone $\tilde{\sigma}$ has primitive basis $((1,0), (-\beta, \alpha))$. See Figure 6. Since the rotation by $-90^\circ$ belongs to $SL_2(\mathbb{Z})$, we have the desired similarity of the cones. \hfill \Box
After Proposition 3.2 and Lemma 5.2 for $b$ as before, we have
\[ \sigma \sim \tau := \tau((0,-1), (\alpha_{\ell-1}, \beta_{\ell-1})]. \]
Thus
\[ \text{Todd}_{\sigma}(x_1, x_2) = \text{Todd}_{\tau}(x_1, x_2), \quad S_{\sigma}(x_1, x_2) = -S_{\tau}(x_1, x_2). \]
Let $v_{-1} = (0, 1), v_0 = (1, 0)$ and for $1 \leq i \leq \ell - 1,$
\[ v_i = (\alpha_i, \beta_i), \]
for $\alpha_i, \beta_i$ defined as in (5.4). Notice that $v_i$ corresponds to $B_{-i+1}$ and $v_{-1}, v_0$ are the two standard basis of $M.$ Then from the decomposition of $\sigma,$ we have the following decomposition of $\sigma:\]
\[ \sigma \sim \tau := \tau((v_{-1}, v_0) = \sigma_0' + \sigma_1 + \sigma_2 + \cdots + \sigma_{\ell-1}, \]
where $\sigma_0' := \sigma_0'(-v_{-1}, v_0)$ and $\sigma_i := \sigma_i(v_{i-1}, v_i),$ for $i \geq 0.$ Note that the coefficients $\alpha_i, \beta_i$ are concretely given using the continued fraction of $\omega.$

Applying Theorem 4.1 we have the following:

**Proposition 5.3.**

\[ S_{\tau}(x_1, x_2) = F(A_{\sigma_0'}^{-1}A_{\tau}(x_1, x_2)^t) + \sum_{i=1}^{\ell-1} (-1)^i F(A_{\sigma_i}^{-1}A_{\tau}(x_1, x_2)^t), \]
where $F(x_1, x_2) = \frac{1}{1-e^{-x_1}} \frac{1}{1-e^{-x_2}}.$

**Proof.** It is a simple consequence of the nonsingularity of cones $\sigma_i.$ The signs are computed from
\[ \det(A_{\sigma_i}) = \det \begin{pmatrix} \alpha_{i-1} & \alpha_i \\ \beta_{i-1} & \beta_i \end{pmatrix} = \beta_i \alpha_{i-1} - \alpha_i \beta_{i-1} = (-1)^{i-1}. \quad \square \]
Now we put explicit terms into the previously obtained expression. An easy computation shows that

\[ A_{\sigma_{i+1}}^{-1} A_{\tau} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} M_{i+1} \\ M_i \end{array} \right), \]

where

\[ A_{\sigma_{i+1}} = \left( \begin{array}{cc} \alpha_i & \alpha_{i+1} \\ \beta_i & \beta_{i+1} \end{array} \right), \quad A_{\tau} = \left( \begin{array}{cc} 0 & \alpha_{\ell-1} \\ -1 & \beta_{\ell-1} \end{array} \right), \]

\[ M_i := (-1)^{i+1}((\beta_i \alpha_{\ell-1} - \alpha_i \beta_{\ell-1})x_2 + \alpha_ix_1) \]

for \( i = 0, \ldots, \ell \), and we have that

\[
F(A_{\sigma_{i+1}}^{-1} A_{\tau}(x_1, x_2)^t) + F(A_{\sigma_{i+1}}^{-1} A_{\tau}(x_1, x_2)^t) = \frac{1}{1 - e^{-\alpha_{\ell-1}x_2}}.
\]

Also we have

\[
det(A_{\tau}) = det \left( \begin{array}{cc} 0 & \alpha_{\ell-1} \\ -1 & \beta_{\ell-1} \end{array} \right) = \alpha_{\ell-1}.
\]

Then we obtain the following expression:

**Proposition 5.4.** With above notation, we have

\[
\text{Todd}_{\sigma}(x_1, x_2) = \alpha_{\ell-1}x_1x_2 \left( \sum_{i=-1}^{\ell-2} (-1)^i F(M_i, M_{i+1}) + \frac{1}{1 - e^{-\alpha_{\ell-1}x_2}} \right).
\]

Let \( \text{Todd}_{\sigma}(x_1, x_2)^{(n)} \) be the degree \( n \) homogeneous part of \( \text{Todd}_{\sigma}(x_1, x_2) \).

**Proposition 5.5.** Let

\[
L_k(X, Y) = \sum_{i=1}^{2k+1} \frac{B_i}{i!} \frac{B_{2k+2-i}}{(2k + 2 - i)!} X^{i-1} Y^{2k-i+1},
\]

\[
R_k(X, Y) = X^{2k} + X^{2k-1} Y + \cdots + Y^{2k}.
\]

Then we have

\[
\text{Todd}_{\sigma}(x_1, x_2)^{(2k+2)} = \alpha_{\ell-1} \left( \sum_{i=-1}^{\ell-2} (-1)^i L_k(M_{i+1}, M_i)x_1x_2 + \sum_{i=1}^{\ell-1} (-1)^i \alpha_{\ell-1} R_k(M_{i-2}, M_i)x_1x_2 \frac{B_{2k+2}}{(2k + 2)!} \right)
\]

\[
+ \frac{B_{2k+2}}{(2k + 2)!} (-x_1 M_0^{2k+1} + x_2 M_{\ell-2}^{2k+1}) + \delta_{k,0} \alpha_{\ell-1} x_1 x_2.
\]

**Proof.** Reading the homogeneous terms of Proposition 5.4 we find

\[
\text{Todd}_{\sigma}(x_1, x_2)^{(2k+2)} = \alpha_{\ell-1} x_1 x_2 \sum_{i=-1}^{\ell-2} (-1)^i F(M_i, M_{i+1})^{(2k)} + x_1 \frac{\alpha_{\ell-1} x_2}{1 - e^{-\alpha_{\ell-1}x_2}}^{(2k+1)},
\]

\[
F(M_i, M_{i+1})^{(2k)} = \sum_{m=1}^{2k+1} \frac{B_m}{m!} \frac{B_{2k+2-m}}{(2k + 2 - m)!} M_{i-m+1}^{M_{i}^{2k-m+1}} + \frac{B_{2k+2}}{(2k + 2)!} \left( \frac{M_{i+1}^{2k+1}}{M_{i+1}} + \frac{M_{i+1}^{2k+1}}{M_{i}} \right).
\]

\[
= L_k(M_{i+1}, M_i) + \frac{B_{2k+2}}{(2k + 2)!} \left( \frac{M_{i+1}^{2k+1}}{M_{i+1}} + \frac{M_{i+1}^{2k+1}}{M_{i}} \right).
\]
and
\[
\frac{\alpha_{\ell-1} x_2}{1 - e^{-\alpha_{\ell-1} x_2}} (2k+1) = -\frac{B_{2k+1}}{(2k+1)!} \alpha_{\ell-1} x_2 2^{k+1} = \delta_{k,0} \frac{1}{2} \alpha_{\ell-1} x_2.
\]

Also we have
\[
\sum_{i=1}^{\ell-2} (-1)^i (M_{i+1}^{-1} M_i^{2k+1} + M_{i+1}^{2k+1} M_i^{-1})
\]
\[
= -M_0^{2k+1} M_1^{-1} + M_{\ell-2}^{2k+1} M_\ell^{-1} + \sum_{i=1}^{\ell-1} (-1)^i (M_{i+1}^{2k+1} - M_i^{2k+1}) M_i^{-1}.
\]

Remembering \((\alpha_{-1}, \beta_{-1}) = (0, 1), \alpha_{i+1} = a_{\ell-i-1} \alpha_i + \alpha_{i-1} \) and \(\beta_{i+1} = a_{\ell-i-1} \beta_i + \beta_{i-1}, \) one can write \(M_{-1} = \alpha_{\ell-1} x_2, M_\ell = \alpha_{\ell-q} x_1 \) and \(M_{i+1} = -a_{\ell-i-1} M_i + M_{i-1}. \) Then (5.7) is rephrased as
\[
\frac{M_0^{2k+1}}{\alpha_{\ell-1} x_2} + \frac{M_{\ell-2}^{2k+1}}{\alpha_{\ell-1} x_1} + \sum_{i=1}^{\ell-1} (-1)^i a_{\ell-i} R_k(M_{i-2}, M_i).
\]

This finishes the proof. \( \square \)

6. Special values of the zeta function: Proof of Theorem 1.1

Now we are going to evaluate the values of \(\zeta(s, b)\) at nonpositive integers using the expression of the degree \(n\) homogeneous part of the Todd series made in the previous section. We keep the notation and convention of the previous section including \(b\) and \(\omega = [a_0, \ldots, a_{r-1}].\)

After Proposition 5.3 and the formula of Garoufalidis-Pommersheim (Theorem 1.2), the partial zeta value has the following expression:

\[
\zeta(-k, b) = (-1)^k k!(\mathcal{L} + \mathcal{R}) \circ \int_{\sigma(h)} e^{-Q(x_1, x_2)} dx_1 dx_2 \bigg|_{h=0},
\]

where

\[
\mathcal{L} : = \sum_{i=-1}^{\ell-2} (-1)^i L_k(M_{i+1}, M_i)(\partial_{h_1}, \partial_{h_2}) \alpha_{\ell-1} \partial_{h_1} \partial_{h_2}
\]

\[
+ \frac{B_{2k+2}}{(2k+2)!} \sum_{i=0}^{\ell-1} (-1)^i a_{\ell-i} R_k(M_{i-2}, M_i)(\partial_{h_1}, \partial_{h_2}) \alpha_{\ell-1} \partial_{h_1} \partial_{h_2}
\]

and

\[
\mathcal{R} : = \frac{B_{2k+2}}{(2k+2)!} \left( (-a_{\ell} R_k(M_{-2}, M_0)(\partial_{h_1}, \partial_{h_2}) \alpha_{\ell-1} \partial_{h_1} \partial_{h_2} - \partial_{h_1} M_0^{2k+1}(\partial_{h_1}, \partial_{h_2})
\]

\[
+ \partial_{h_2} M_{\ell-2}^{2k+1}(\partial_{h_1}, \partial_{h_2}) \right).
\]

We will apply \(\mathcal{L}\) and \(\mathcal{R}\) to (6.1) one after the other. Later in Section 8 we will show that

\[
\mathcal{R} \circ \int_{\sigma(h)} e^{-Q(x_1, x_2)} dx_1 dx_2 \bigg|_{h=0} = 0.
\]
First, we change the coordinate in the integral: $(x_1, x_2) = (\alpha_{\ell-1} y_2, \beta_{\ell-1} y_2 + y_1)$. In the new coordinate $(y_1, y_2)$, the integral becomes

$$
\int_{\sigma(h)} e^{-Q(x_1, x_2)} dx_1 dx_2 = \alpha_{\ell-1} \int_{-\frac{h_2}{\alpha_{\ell-1}}}^{\infty} \int_{-\frac{h_1}{\alpha_{\ell-1}}}^{\infty} e^{-N(b)N(\epsilon y_2 + y_1)} dy_1 dy_2.
$$

Applied by $\alpha_{\ell-1} \partial_{h_1} \partial_{h_2}$, (6.5) becomes

$$
\alpha_{\ell-1} \partial_{h_1} \partial_{h_2} \int_{\sigma(h)} e^{-Q(x_1, x_2)} dx_1 dx_2 = e^{-N(b)N(\frac{h_2}{\alpha_{\ell-1}} \epsilon + \frac{h_1}{\alpha_{\ell-1}})}.
$$

The above simplifies (6.1) quite much assuming the vanishing of (6.4). Actually we have

$$
(6.7)
\zeta(-k, b) = (-1)^k k! \left( \sum_{i=0}^{\ell-1} B_{2k+2} (2k+2)! \sum_{i=0}^{\ell-1} (-1)^j a_{\ell-1} R_k(M_{i-2}, M_i)(\partial_{h_1}, \partial_{h_2}) \right) o e^{-N(b)N(\frac{h_2}{\alpha_{\ell-1}} \epsilon + \frac{h_1}{\alpha_{\ell-1}})} |_{h=0}.
$$

Individual terms in the above can be computed again by a linear coordinate change as follows:

**Lemma 6.1.** Let $A_i = \alpha_i \omega + \beta_i$. For $-1 \leq m, l \leq \ell - 1$, we have

$$
M_l(\partial_{h_1}, \partial_{h_2})^l M_m(\partial_{h_1}, \partial_{h_2})^j e^{-N(b)N(\frac{h_2}{\alpha_{\ell-1}} \epsilon + \frac{h_1}{\alpha_{\ell-1}})} |_{h=0} = \partial_{h_1}^i \partial_{h_2}^j e^{-N(b)N((-1)^{l+1} A_1 h_1 + (-1)^{m+1} A_m h_2)} |_{h=0},
$$

for $M_l(x_1, x_2) = (-1)^{l+1} ((\beta_i \alpha_{\ell-1} - \alpha_i \beta_{\ell-1}) x_2 + \alpha_i x_1)$.

For a binary quadratic form $Q(x, y)$ and $i, j$ with $i + j = 2k$, we have

$$
\partial_{h_1}^i \partial_{h_2}^j e^{-Q(h_1, h_2)} |_{h=0} = (-1)^k \frac{1}{k!} \partial_{h_1}^i \partial_{h_2}^j Q(h_1, h_2)^k |_{h=0}.
$$

Thus, from (6.7), (6.7) and Lemma 6.1 we finish the proof of Theorem 1.1.

$$
\zeta(-k, b) = \sum_{i=0}^{\ell-1} (-1)^{i-1} L_k(\partial_{h_1}, \partial_{h_2}) Q(\alpha_i h_1 - \alpha_i h_2, \beta_i h_1 - \beta_i h_2)^k |_{h=0} + \frac{B_{2k+2}}{(2k+2)!} \sum_{i=0}^{\ell-1} (-1)^i a_{\ell-1} R_n(\partial_{h_1}, \partial_{h_2}) Q(\alpha_{i-2} h_1 + \alpha_i h_2, \beta_{i-2} h_1 + \beta_i h_2)^k |_{h=0}.
$$

**Remark 6.2.** We obtained a polynomial expression of the zeta value in variables $\alpha_i, \beta_i$ and the coefficients of the quadratic form $Q(x, y)$. For the polynomial expression, it is important to show the vanishing (6.4). In Section 8 there appear $\alpha_0, \alpha_{\ell-1}$ in the denominator of the vanishing expression involving the $R$-operator. This is a crucial ingredient of the Kummer congruence and the corresponding $p$-adic zeta function.
7. Computation of $\zeta(-k, b)$ for $k = 0, 1$ and 2

In this section, we evaluate the zeta values $\zeta(-k, b)$ explicitly for small $n$. We express the values in terms of the continued fraction expansion $[a_0, a_1, \ldots, a_{\ell-1}]$ of the reduced basis $\omega$ of $b^{-1}$.

7.1. For $k = 0$, the zeta value is already known by C. Meyer (32) in terms of a negative continued fraction. Using the plus-to-minus conversion formula of the continued fraction (see [41], pp. 177-178)

$\delta = \omega + 1 = [[a_0, a_1, \ldots, a_{\ell-1}]] + 1$

$= ((a_0 + 2, 2, 2, \ldots, 2, a_2 + 2, 2, 2, \ldots, 2, a_4 + 2, \ldots, a_{\ell-2} + 2, 2, \ldots, 2)_{(a_1-1)}-times_{(a_3-1)}-times_{(a_2\ell-1)}-times)$

$= ((b_0, b_1, \ldots, b_m))$

one obtains the result in a positive continued fraction. Here we use the following convention of negative continued fractions:

$((b_0, b_1, \ldots, b_m)) := b_0 - \frac{1}{b_1 - \frac{1}{\ldots - \frac{1}{b_{m-1} - \frac{1}{b_0 - \ldots}}}}$

In our approach, we begin with the expression using a positive continued fraction as a special case of Theorem 1.1

$\zeta(0, b) = \frac{B_2}{2} \sum_{i=0}^{\ell-1} (-1)^i a_{i-1}$.

Since $B_2 = 1/6$ and $\ell$ is the even period of the continued fraction, this reduces to

(7.2) $\zeta(0, b) = \frac{1}{12} \sum_{i=0}^{\ell-1} (-1)^i a_i$.

Via (7.1), one recovers the result of Meyer:

$\zeta(0, b) = \frac{1}{12} \sum_{i=0}^{m-1} (b_i - 3)$,

where $b_i$ is the $i$-th term of the negative continued fraction.

Remark 7.1. Note that using the positive continued fraction, we have an alternating sum for the zeta value. Consequently, one sees directly the vanishing of $\zeta(0, b)$ when the actual period of the positive continued fraction of $\omega$ is odd (equivalently, if the fundamental unit is not totally positive).

7.2. $k = 1$ and 2. For $Q(x_1, x_2) = N_1(x_1 + x_2)$, let $L_i, M_i$ and $N_i$ be defined as in

(7.3) $Q(\alpha_i h_1 + \alpha_i h_2, \beta_i h_1 + \beta_i h_2) = L_i h_1^2 + M_i h_1 h_2 + N_i h_2^2$.

Similarly, $\tilde{L}_i, \tilde{M}_i$ and $\tilde{N}_i$ are defined as follows:

(7.4) $Q(\alpha_i h_1 + \alpha_i h_2, \beta_i h_1 + \beta_i h_2) = \tilde{L}_i h_1^2 + \tilde{M}_i h_1 h_2 + \tilde{N}_i h_2^2$. 
Then the special value at $s = -1$ is computed as follows:

$$
\zeta(-1, b) = \sum_{i=0}^{\ell-1} (-1)^{i+1} \frac{B_2}{4} M_i + \frac{B_4}{4!} \sum_{i=0}^{\ell-1} (-1)^i a_{\ell-i} (2\tilde{L}_i + \tilde{M}_i + 2\tilde{N}_i) \\
= \frac{1}{720} \sum_{i=0}^{\ell-1} (-1)^{i+1} \left( 5M_i + a_{\ell-i} (2\tilde{L}_i + \tilde{M}_i + 2\tilde{N}_i) \right).
$$

Similarly for $s = -2$,

$$
\zeta(-2, b) = \frac{1}{15120} \sum_{i=0}^{\ell-1} (-1)^i \left( 21M_i (N_i + L_i) + 2a_{\ell-i} (2\tilde{L}_i + \tilde{M}_i + 2\tilde{L}_i \tilde{N}_i + 3\tilde{M}_i \tilde{N}_i + 6\tilde{N}_i^2) \right).
$$

This should be compared with the expression obtained using a negative continued fraction in [15] and also in [42]. They considered the zeta function of the following quadratic form in view of a negative continued fraction:

$$
Q'(x_1, x_2) := N(b)N(x_1 \delta + x_2)
$$

for $\delta = \omega + 1$ (see (7.1) for a negative continued fraction). Let $A_i$ be the lattice points of the component of the Klein polyhedron of $b^{-1}$ in the 1st quadrant with normalization: $A_0 = 1$, $A_{-1} = \delta$ and the 1st coordinate of $A_i$ increasing according to $i$. Then we associate a lattice vector $(p_k, q_k)$ to $A_k$ for $A_k = -p_k A_{-1} + q_k A_0$. $p_k$ and $q_k$ are obtained from the reduced fraction of the truncation after $k$ of the negative continued fraction $\delta = ((b_0, b_1, \ldots, b_m))$:

$$
\frac{q_k}{p_k} = (b_0, \ldots, b_{k-1}).
$$

(This is the last line of p. 18 of [15], where $\frac{p_k}{q_k}$ should be corrected to $\frac{q_k}{p_k}$ as we just wrote above). Similarly, $\tilde{L}_i'$, $M_i'$, $N_i'$ and $\tilde{L}_i$, $\tilde{M}_i$, $\tilde{N}_i$ are defined as the coefficients of quadratic forms:

$$
Q'(-p_{i-1}h_1 - p_i h_2, q_{i-1}h_1 + q_i h_2) = L_i'h_1^2 + M_i'h_1h_2 + N_i'h_2^2
$$

and

$$
Q'(-p_{i-1}h_1 - p_i h_2, q_{i-1}h_1 + q_i h_2) = \tilde{L}_i'h_1^2 + \tilde{M}_i'h_1h_2 + \tilde{N}_i'h_2^2.
$$

In this setting, Garoufalidis-Pomersheim ([15]) obtained

$$
\zeta(-1, b) = \frac{1}{720} \sum_{i=0}^{m-1} \left( 5M_i' + b_i (-2\tilde{L}_i + \tilde{M}_i - 2\tilde{N}_i) \right)
$$

and

$$
\zeta(-2, b) = \frac{1}{15120} \sum_{i=0}^{m-1} \left( -21M_i' (L_i' + N_i') \right)
+ 2b_i (6\tilde{L}_i'^2 - 3\tilde{L}_i' \tilde{M}_i' + 2\tilde{L}_i' \tilde{N}_i' + \tilde{M}_i'^2 - 3\tilde{M}_i' \tilde{N}_i' + 6\tilde{N}_i'^2)).
$$

In [42], Zagier obtained

$$
\zeta(-1, b) = \frac{1}{720} \sum_{i=0}^{m-1} \left( -2N_i b_i^3 + 3M_i b_i^2 - 6L_i b_i + 5M_i \right).
$$
8. Vanishing part

Now, it remains to show the vanishing of \((8.3)\):

\[
\mathcal{R} \circ \int_{\sigma(h)} e^{-Q(x_1,x_2)} dx_1 dx_2 \bigg|_{h=0} = 0.
\]

This part is crucial in expressing the zeta values at nonpositive integers as polynomials of its argument coming from terms of continued fractions and the coefficients of quadratic forms. The vanishing has already been observed in related works by Zagier (142) and Garoufalidis-Pommersheim (15) in different settings. In 15, only the vanishing is mentioned without clear proof. In this section, we will recycle some notions and ideas from 12 for the proof of the vanishing.

It suffices to show the vanishing of the following expression which equals \((8.4)\) up to multiplication by a constant:

\[
\left( -\alpha_{\ell} R_k(M_{-2}, M_0)(\partial_{h_1}, \partial_{h_2}) \alpha_{\ell-1} \partial_{h_1} \partial_{h_2} - \partial_{h_1} M_0^{2k+1}(\partial_{h_1}, \partial_{h_2}) \right)
\]

\[
+ \partial_{h_2} M_0^{2k+1}(\partial_{h_1}, \partial_{h_2}) \bigg) \circ \int_{\sigma(h_1,h_2)} e^{-Q(x_1,x_2)} dx_1 dx_2 \bigg|_{h=0}.
\]

As \(M_0 = \beta_{\ell-1} x_2 - x_1\) and \(M_{-2} = x_2 - \alpha_{\ell-2} x_1\), we have

\[
x_1 M_0^{2k+1} + x_2 M_0^{2k+1}
\]

\[
= 2x_1^{2k+2} + \sum_{i=1}^{2k+1} (-1)^i \binom{2k+1}{i} (\beta_{\ell-1} + \alpha_{\ell-2}^i) x_1^{i} x_2^{2k+2-i}.
\]

Applying \(\partial_{h_1}^{2k+2}\) to \((8.5)\), we obtain

\[
\alpha_{\ell-1} \partial_{h_1}^{2k+2} \int_{h_2}^{\infty} \int_{h_1}^{\infty} e^{-N(b)N(e y_2 + y_1)} dy_1 dy_2 \bigg|_{h=0}
\]

\[
= 1 \alpha_{\ell-1} \int_{0}^{\infty} \partial_{h_1}^{2k+1} e^{-N(b)N(e y_2 + h_1 - h_1)} \bigg|_{h=0} dy_2.
\]

If we write \(P(x_1,x_2) = \frac{N(b)}{\alpha_{\ell-1}^2} (x_2^2 + (\epsilon + \epsilon') x_1 x_2 + x_1^2)\), by using \((8.2)-(8.3)\) one can simplify the 2nd half of \((8.1)\):

\[
\left( -\partial_{h_1} M_0(\partial_{h_1}, \partial_{h_2})^{2k+1} + \partial_{h_2} M_{-2}(\partial_{h_1}, \partial_{h_2})^{2k+1} \right) \circ \int_{\sigma(h_1,h_2)} e^{-Q(x_1,x_2)} dx_1 dx_2 \bigg|_{h=0}
\]

\[
= \frac{1}{\alpha_{\ell-1}} \sum_{i=1}^{2k+1} (-1)^i \binom{2k+1}{i} (\beta_{\ell-1} + \alpha_{\ell-2}^i) \partial_{h_2}^{i-1} \partial_{h_1}^{2k+1-i} \bigg|_{h=0}
\]

\[
= \frac{2}{\alpha_{\ell-1}} \int_{0}^{\infty} x_1^{2k+1} e^{-P(-x_1,x_2)} \bigg|_{x_1=0} dx_2.
\]

From \(M_{-2} = (a_0 \alpha_{\ell-1} + \beta_{\ell-1}) x_2 - x_1\) and \(M_0 = \beta_{\ell-1} x_2 - x_1\), we have

\[
R_k(M_{-2}, M_0) = \sum_{i=0}^{2k+1} (-1)^{i+1} \binom{2k+1}{i} \frac{(a_0 \alpha_{\ell-1} + \beta_{\ell-1})^i - \beta_{\ell-1}^i x_2^{i-1} x_1^{2k+1-i}}{a_0 \alpha_{\ell-1}}.
\]

From \((8.4)\) and \((8.5)\), we obtain the following lemma.
Lemma 8.1. Let \( P(x_1, x_2) = \frac{N(b)}{\alpha_{\ell-1}}(x_1^2 + (\epsilon + \epsilon')x_1x_2 + x_2^2) \). Then we have

\[
\left( -a_{\ell}R_k(M_{-2}, M_0)(\partial_{\alpha_{\ell-1}}\partial_{h_1}\partial_{h_2}) - \partial_{h_2}M_{\ell-2}^{2k+1}(\partial_{\alpha_{\ell-1}}\partial_{h_2}) \right) + \partial_{h_2}M_{\ell-2}^{2k+1}(\partial_{h_2}) = 0
\]

\[
= \frac{1}{\alpha_{\ell-1}} \sum_{i=0}^{2k} (-1)^{i+1} \binom{2k+1}{i+1} (a_0\alpha_{\ell-1} + \beta_{\ell-1})^{i+1} + (\alpha_{\ell-2})^{i+1} \partial_{h_2}^{2k-i} \circ e^{-P(h_1, h_2)}
\]

\[
\left. \frac{1}{\alpha_{\ell-1}} \int_0^\infty \partial_{x_1}^{2k+1} e^{-P(-x_1, x_2)} \right|_{x_1 = 0} dx_2.
\]

Lemma 8.2. For the totally positive fundamental unit \( \epsilon > 1 \), we have

\[ \epsilon + \epsilon' = a_0\alpha_{\ell-1} + \beta_{\ell-1} + \alpha_{\ell-2}. \]

Proof. We note that

\[
\delta := -\frac{1}{\omega'} = \left[ [a_{\ell-1}, a_{\ell-2}, \ldots, a_0] \right].
\]

Thus

\[
\delta = \left( \begin{array}{cccc}
  a_{\ell-1} & 1 & 0 & \\
  1 & 1 & 0 & \\
  \vdots & \vdots & \vdots & \\
  a_0 & 1 & 0 & \\
  1 & 0 & & \\
\end{array} \right) = \alpha_\ell \delta + \alpha_{\ell-1} \beta_\ell \delta + \beta_{\ell-1}.
\]

Also, we have

\[ \alpha_{\ell-1}\omega^2 - (\alpha_\ell - \beta_{\ell-1})\omega - \beta_\ell = 0. \]

Finally we have

\[ \omega + \omega' = \frac{\alpha_\ell - \beta_{\ell-1}}{\alpha_{\ell-1}}, \]

\[ \epsilon + \epsilon' = \alpha_{\ell-1}(\omega + \omega') + 2\beta_{\ell-1} = \alpha_\ell + \beta_{\ell-1}. \]

Thus

\[ \frac{\epsilon + \epsilon' - \beta_{\ell-1} - \alpha_{\ell-2}}{\alpha_{\ell-1}} = a_0. \]

From Lemma 8.2, if we let \( a_0\alpha_{\ell-1} + \beta_{\ell-1} = -a, \alpha_{\ell-2} = -b \) and \( \frac{N(b)}{\alpha_{\ell-1}} = A \), then we have \( \epsilon + \epsilon' = -(a + b) \). Hence one can rewrite Lemma 8.1 as

\[ (8.6) \]

\[ \sum_{i=0}^{2k} (-1)^{i+1} \binom{2k+1}{i+1} (a_0\alpha_{\ell-1} + \beta_{\ell-1})^{i+1} + (\alpha_{\ell-2})^{i+1} \partial_{h_2}^{2k-i} \circ e^{-P(h_1, h_2)}
\]

\[ + 2 \int_0^\infty \partial_{x_1}^{2k+1} e^{-P(-x_1, x_2)} \left|_{x_1 = 0} \right. \]

\[ = \sum_{i=0}^{2k} (-1)^{i+1} \binom{2k+1}{i+1} (a_i^{i+1} + b^{i+1}) \partial_{h_2}^{2k-i} \circ e^{-A(h_2^2 - (a + b)h_1h_2 + h_1^2)}
\]

\[ + 2 \int_0^\infty \partial_{x_1}^{2k+1} e^{-A(x_1^2 + (a + b)x_1x_2 + x_2^2)} \left|_{x_1 = 0} \right. \]

\[ dx_2. \]
Finally it remains to show the vanishing of the right hand side of (8.6):

\[
\begin{aligned}
&\sum_{i=0}^{2k} \left( \frac{2k+1}{i+1} \right) (a^{i+1} + b^{i+1}) \partial_{h_2}^{2k-i} \partial_{h_1}^{2k-i} e^{-A(h_2^{i+1} - \lambda A + b)} \right|_{h=0} \\
&+ 2 \int_0^\infty \partial_{x_1}^{2k+1} e^{-A(x_2^2 + (a+b)x_1x_2 + x_1^2)} \left|_{x_1=0} \right. dx_2.
\end{aligned}
\]

For the proof, we introduce \( f_k(\alpha, \beta, \gamma) \) and \( d_{r,k}(\alpha, \beta, \gamma) \) as follows:

\[
\begin{aligned}
&\int_0^\infty \partial_{x_1}^{2k+1} e^{-A(x_2^2 + \beta x_1x_2 + x_1^2)} \left|_{x_1=0} \right. dx_2 = -\frac{(2k+1)! f_k(\alpha, \beta, \gamma)}{2\gamma^{k+1}},
\\
&\sum_{i=0}^{2k} d_{i,2k-i}(\alpha, \beta, \gamma) x_1^i x_2^{2k-i} = (\alpha x_1^2 + \beta x_1x_2 + \gamma x_2^2)^k.
\end{aligned}
\]

These numbers are originally appeared in [42]. Note first that \( f_k(\alpha, \beta, \gamma) \) is an odd function in variable \( \beta \) (i.e. \( f_k(\alpha, \beta, \gamma) + f_k(\alpha, -\beta, \gamma) = 0 \)).

One may identify \( d_{i,2k-i}(\alpha, -\beta, \gamma) \) in a generating function:

\[
\begin{aligned}
&\partial_{x_1}^i \partial_{x_2}^{2k-i} e^{-A(x_2^2 - \beta x_1x_2 + x_1^2)} \left|_{(x_1,x_2)=(0,0)} \right. \\
&= \frac{(-1)^k}{k!} \partial_{x_1}^i \partial_{x_2}^{2k-i} (\alpha x_1^2 - \beta x_1x_2 + \gamma x_2^2)^k = \frac{(-1)^k}{k!} i!(2k-i)! d_{i,2k-i}(\alpha, -\beta, \gamma).
\end{aligned}
\]

Then the 1st line of (8.6) equals

\[
\begin{aligned}
&\sum_{i=0}^{2k} \left( \frac{2k+1}{i+1} \right) (a^{i+1} + b^{i+1}) \partial_{h_2}^{2k-i} \partial_{h_1}^{2k-i} e^{-A(h_2^{i+1} - \lambda A + b)} \right|_{h=0} \\
&= \frac{(-1)^k}{k!} (2k+1)! \sum_{i=0}^{2k} a^{i+1} + b^{i+1} i+1 d_{i,2k-i}(A, -A(a+b), A)
\end{aligned}
\]

and the 2nd line of (8.6) equals

\[
\begin{aligned}
&2 \int_0^\infty \partial_{x_1}^{2k+1} e^{-A(x_2^2 + (a+b)x_1x_2 + x_1^2)} \left|_{x_1=0} \right. dx_2 = -\frac{2k+1)! f_k(A, A(a+b), A)}{A^{k+1}}.
\end{aligned}
\]

Now, we are going to use an identity relating \( f_k(\alpha, \beta, \gamma) \) and \( d_{i,2k-i}(\alpha, -\beta, \gamma) \) due to Zagier:

**Lemma 8.3** (Zagier, Proposition 4 of [42]). For a real number \( \lambda \), we have

\[
\begin{aligned}
f_k(\alpha, \beta, \gamma) + f_k(\gamma, 2\lambda \gamma - \beta, \lambda^2 \gamma - \lambda \beta + \alpha) \\
= 2 \frac{(-1)^k}{k!} \gamma^{k+1} \sum_{i=0}^{2k} d_{i,2k-i}(\alpha, -\beta, \gamma) \frac{\lambda^{i+1}}{i+1}.
\end{aligned}
\]

If we put \( \alpha = A, \beta = A(a+b), \gamma = A \) and \( \lambda = a \) (resp. \( \lambda = b \)) into the above, we obtain

\[
\begin{aligned}
f_k(A, A(a+b), A) + f_k(A, A(a-b), A(-ab+1)) \\
= 2 \frac{(-1)^k}{k!} A^{k+1} \sum_{i=0}^{2k} d_{i,2k-i}(A, -A(a+b), A) \frac{a^{i+1}}{i+1}
\end{aligned}
\]
and
\[ f_k(A, A(a + b), A) + f_k(A, A(b - a), A(-ab + 1)) = 2 \frac{(-1)^k}{k!} A^{k+1} \sum_{i=0}^{2k} d_{i,2k-i}(A, -A(a + b), A) \frac{b^{i+1}}{i+1}. \]

As \( f_k \) is an odd function of its 2nd argument, summing the above two equations, we have
\[ \frac{f_k(A, A(a + b), A)}{A^{k+1}} = \frac{(-1)^k}{k!} \sum_{i=0}^{2k} d_{i,2k-i}(A, -A(a + b), A) \frac{a^{i+1} + b^{i+1}}{i+1}. \]
This identifies (8.9) and (8.10) up to sign, which concludes the vanishing of (8).

9. Application: Zeta values at nonpositive integers in the family and the class number problem

In this section, we apply the formula of Theorem 1.1 to obtain polynomial expressions for \( \zeta_{K_n}(-k, b_n) \) which generalize the result of [23] and [24] where the authors computed the special values at \( s = 0 \). Next we will show, through two examples \((d = 4n^2 + 2, (2n+1)^4 + 2(2n+1))\), how this result can be used in the construction of class number 1 criteria of the form as it appeared in works of Byeon and Kim (6, 7).

Recall the conditions on the family \((K_n, b_n)\) indexed by \( n \in N \) for a subset \( N \) of \( N \). \( b_n^{-1} = [1, \omega(n)] \) for a reduced element \( \omega(n) \in K_n \) and
\[ \omega(n) = [a_0(n), a_1(n), \ldots, a_{r-1}(n)] \]
for polynomials \( a_i(x) \in \mathbb{Z}[x] \) and the quadratic form
\[ Q(x, y) = N(b_n)(x\omega(n) + y)(x\omega(n)' + y) \]
associated with \( b_n \). Applying Theorem 1.1 to the family, we have

(9.1) \[ \zeta(-k, b_n) = \sum_{i=0}^{\ell-1} (-1)^{i-1} L_k(h_1, h_2) Q(a_i(n)h_1 - a_{i-1}(n)h_2, \beta_i(n)h_1 - \beta_{i-1}(n)h_2)^k \]
\[ + \frac{B_{2k+2}}{(2k+2)!} \sum_{i=0}^{\ell-1} (-1)^i a_{\ell-i}(n)R_k(h_1, h_2) Q(a_{i-2}(n)h_1 + a_i(n)h_2, \beta_{i-2}(n)h_1 + \beta_i(n)h_2)^k. \]

Remark 9.1. One should notice that the denominator of \( \zeta(-k, b_n) \) is unchanged under varying \( n \). In fact, one can check this for arbitrary ideals of real quadratic fields. It is important to control the denominator to interpolate the associated \( p \)-adic zeta function from the values at negative integers (cf. [12], [13] and [28]).

For the rest of the paper, for a number field \( K \), let us denote the ring of integers by \( O_K \).
Example 9.2. Consider the family \((K_n = \mathbb{Q}(\sqrt{4n^2 + 2}))\). Since the reduced element \(\omega_n = \frac{2n + \sqrt{4n^2 + 2}}{2} = \left[[4n, 2n]\right]\) has an even period, the fundamental unit \(\varepsilon_n\) is totally positive. Thus each ideal class contains exactly two narrow ideal classes. Let \(b_n, c_n\) be two different narrow ideals representing the wide class of \(O_K\). Then \(b_n, c_n\) are given by

\[
\begin{align*}
  b_n^{-1} &= [1, \omega_n], \\
  c_n^{-1} &= [1, \delta_n]
\end{align*}
\]

for \(\omega_n = \left[[4n, 2n]\right]\) and \(\delta_n = \left[[2n, 4n]\right]\). Thus we have

\[
\zeta_n^{\text{wide}}(-1, O_{K_n}) = \zeta_{K_n}(-1, b_n) + \zeta_{K_n}(-1, c_n).
\]

Now, we will evaluate \(\zeta_{K_n}(-1, b_n)\) and \(\zeta_{K_n}(-1, c_n)\), one by one.

1. For \(\omega_n = \frac{2n + \sqrt{4n^2 + 2}}{2} = \left[[2n, 4n]\right]\), we have

\[
\begin{align*}
  Q(b_n)(\alpha_0 x - \alpha_{-1} y, \beta_0 x - \beta_{-1} y) &= L_0 x^2 + M_0 xy + N_0 y^2 = -x^2 - 4nxy + 2y^2, \\
  Q(b_n)(\alpha_{-2} x + \alpha_0 y, \beta_{-2} x + \beta_0 y) &= \tilde{L}_0 x^2 + \tilde{M}_0 xy + \tilde{N}_0 y^2 = -x^2 - (2 + 8n^2)xy - y^2, \\
  Q(b_n)(\alpha_1 x - \alpha_{-1} y, \beta_1 x - \beta_{-1} y) &= L_1 x^2 + M_1 xy + N_1 y^2 = 2x^2 + 4nxy - y^2, \\
  Q(b_n)(\alpha_{-1} x + \alpha_1 y, \beta_{-1} x + \beta_1 y) &= \tilde{L}_1 x^2 + \tilde{M}_1 xy + \tilde{N}_1 y^2 = 2x^2 + (4 + 16n^2)xy + 2y^2.
\end{align*}
\]

Thus we have

\[
\begin{align*}
  \zeta(-1, b_n) &= \frac{1}{720} \sum_{i=0}^{1} (-1)^{i-1} \left(5M_i + a_{2-i}(n)(2L_1 + M_1 + 2\tilde{N}_1)\right) = \frac{1}{36}(4n^3 + 5n). \\
  \zeta(-1, c_n) &= \frac{1}{720} \sum_{i=0}^{1} (-1)^{i-1} \left(5M_i + a_{2-i}(n)(2\tilde{L}_1 + \tilde{M}_1 + 2\tilde{N}_1)\right) = \frac{1}{36}(4n^3 + 5n).
\end{align*}
\]

Therefore we obtain

\[
(9.2) \quad \zeta_{K_n}^{\text{wide}}(-1, O_{K_n}) = \zeta_{K_n}(-1, b_n) + \zeta_{K_n}(-1, c_n) = \frac{1}{18}(4n^3 + 5n).
\]

Since the partial zeta value at \(s = -1\) of any ideal is positive,

\[
\zeta_{K_n}^{\text{wide}}(-1, O_{K_n}) = \zeta_{K_n}(-1) \quad \text{if and only if} \quad h(K_n) = 1.
\]
\( \zeta_{K_n}(-1) \) has a well-known formula using the sum of divisors due to Siegel:

\[
(9.3) \quad \zeta_{K_n}(-1) = \frac{1}{60} \sum_{\substack{|b| < \sqrt{d} \ b \equiv d \pmod{4}}} \sigma_1 \left( \frac{d-b^2}{4} \right).
\]

Here \( \sigma_1(n) := \sum_{d \mid n} d \).

Thus, we have that

\[
(9.4) \quad \zeta_{K_n}(-1) = \sum_{|t| \leq 2n} \sigma_1(4n^2 + 2 - t^2) = 2 \sum_{t=1}^{2n} \sigma_1(4n^2 + 2 - t^2) + \sigma_1(4n^2 + 2)
\]

\[= 2 \sum_{i=1}^{n-1} \sigma_1 \left( 4n^2 + 2 - (2i)^2 \right) + \sigma_1(2)
\]

\[+ 2 \sum_{i=1}^{n} \sigma_1 \left( 4n^2 + 2 - (2i-1)^2 \right) + \sigma_1(4n^2 + 2).\]

In the above, each summand is bounded by sum of its trivial divisors (i.e. \( \sigma_1(2m) \geq 1 + 2 + m + 2m, \sigma_1(2m + 1) \geq 1 + (2m + 1) \)). Therefore

\[
(9.5) \quad \zeta_{K_n}(-1) \geq 2 \sum_{i=1}^{n-1} \left[ 1 + 2 + (2n^2 + 1 - 2i^2) + (4n^2 + 2 - 4i^2) \right] + 3
\]

\[+ 2 \sum_{i=1}^{n} \left[ 1 + (4n^2 + 2 - 2i-1)^2 \right] + \left[ 1 + 2 + (2n^2 + 1) + (4n^2 + 2) \right]
\]

\[= \frac{1}{18} (4n^3 + 5n),\]

where the equality holds if and only if

\[
(9.6) \quad \left\{ \begin{array}{ll}
2n^2 + 1 - 2i^2, & 1 \leq i \leq n - 1, \\
4n^2 + 2 - (2i-1)^2, & 1 \leq i \leq n,
\end{array} \right.
\]

are all primes.

Meanwhile, the above lower bound of \( \zeta_{K_n}(-1) \) agrees with the partial zeta value computed in (9.2).

Therefore (9.6) is the class number one criteria for \( d = 4n^2 + 2 \).

Example 9.3. Let \( K_n = \mathbb{Q} \left( \sqrt{(2n+1)^4 + 2(2n+1)} \right) \) and \( b_n = O_{K_n} \). In this case, the wide class of \( O_{K_n} \) contains two narrow classes. Let \( b_n, c_n \) be ideals representing each narrow class. We may take \( b_n, c_n \) such that

\[ b_n^{-1} = [1, \omega_n], \ c_n^{-1} = [1, \delta_n] \]

for \( \omega_n = [8n^2 + 8n + 2, 2n + 1] \) and \( \delta_n = [2n + 1, 8n^2 + 8n + 2] \). We then have

\[ \zeta^\text{wide}_{K_n}(-1, O_{K_n}) = \zeta_{K_n}(-1, b_n) + \zeta_{K_n}(-1, c_n). \]

Evaluating \( \zeta_{K_n}(-1, b_n) \) and \( \zeta_{K_n}(-1, c_n) \) in our formula, we obtain that

\[
(9.7) \quad \zeta^\text{wide}_{K_n}(-1, O_{K_n}) = \frac{64}{45} n^6 + \frac{64}{15} n^5 + \frac{244}{45} n^4 + \frac{64}{15} n^3 + \frac{112}{45} n^2 + \frac{29}{30} n + \frac{1}{6}.
\]
On the other hand, Siegel’s formula (9.3) gives
\[
\zeta_{K_n}(-1) = \sum_{|t| \leq 4n^2 + 4n + 1} \sigma_1 \left( (2n + 1)^4 + 2(2n + 1) - t^2 \right)
\]
\[= 2 \sum_{t=1}^{4n^2 + 4n + 1} \sigma_1 \left( (2n + 1)^4 + 2(2n + 1) - t^2 \right) + \sigma_1 \left( (2n + 1)((2n + 1)^3 + 2) \right).
\]

(9.8)

As in the previous example, we give a lower bound of the sum of the divisor function using trivial divisors.

For odd \( t = 2s + 1 (0 \leq s \leq 2n^2 + 2n) \), we have
\[(2n + 1)^4 + 2(2n + 1) - t^2 = 2(1 + 6n + 12n^2 + 16n^3 + 8n^4 - 2s - 2s^2).
\]
Thus,
\[\sigma_1 \left( (2n + 1)^4 + 2(2n + 1) - t^2 \right) \geq 1 + 2 + (1 + 6n + 12n^2 + 16n^3 + 8n^4 - 2s - 2s^2) + 2(1 + 6n + 12n^2 + 16n^3 + 8n^4 - 2s - 2s^2).\]

Furthermore, if \( t \) is odd and divisible by \((2n + 1)\) so that \( t = (2n + 1)(2r + 1) \) for \( 0 \leq r \leq n - 1 \), then we have three trivial divisors \(2, 2n + 1\) and \(1 + (2n + 1)(2r + 1)^3(2n^2 + 2n)\) of \((2n + 1)^4 + 2(2n + 1) - t^2\), which produces a sharper bound:
\[\sigma_1 \left( (2n + 1)^4 + 2(2n + 1) - t^2 \right) \geq 1 + (2n + 1)^4 + 2(2n + 1) - 4s^2.
\]

Similarly for even \( t = 2s \) \((1 \leq s \leq 2n^2 + 2n)\), we have the trivial bound
\[\sigma_1 \left( (2n + 1)^4 + 2(2n + 1) - t^2 \right) \geq 1 + (2n + 1)^4 + 2(2n + 1) - 4s^2.
\]
Again if \( t \) is even and divisible by \(2n + 1\), so that \( t = (2n + 1)2r \) for \( 1 \leq r \leq n - 1 \), then \((2n + 1)^4 + 2(2n + 1) - t^2\) has two trivial divisors:
\[2n + 1, (2n + 1)^3 + 2 - 4r^2(2n + 1).
\]
So we obtain a sharper bound:
\[\sigma_1 \left( (2n + 1)^4 + 2(2n + 1) - t^2 \right) \geq 1 + (2n + 1)^4 + 2(2n + 1) - 4s^2(2n + 1).
\]

Altogether, we have a lower bound for \( \zeta_{K_n}(-1) \):
\[\zeta_{K_n}(-1) \geq \frac{1}{3} + \frac{5n}{3} + \frac{34n^2}{9} + 6n^3 + \frac{64}{9}n^4 + \frac{16}{3}n^5 + \frac{16}{9}n^6.
\]
This happens to be strictly greater than the \( \zeta^{\text{wide}}(-1, O_{K_n}) \) evaluated in (9.7) for \( n \geq 1 \).
Therefore there is no field of class number one in the family considered. This answers the class number one problem for the family \( d = (2n + 1)^4 + 2(2n + 1) \).

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