AN OPTIMAL CONTROL MODEL OF CARBON REDUCTION AND TRADING

HUAYING GUO AND JIN LIANG*

Department of Mathematics, Tongji University
Shanghai 200092, China

(Communicated by Qing Zhang)

ABSTRACT. In this study, a stochastic control model is established for a country to formulate a carbon abatement policy to minimize the total carbon reduction costs. Under Merton’s consumption framework, by considering carbon trading, carbon abatement and penalties in a synthetic manner, the model is converted into a two-dimensional Hamilton–Jacobi–Bellman equation. We rigorously prove the existence and uniqueness of its viscosity solution. We also present the numerical results and discuss the properties of the optimal carbon reduction policy and the minimum total costs.

1. Introduction. The greenhouse effect and climate change have been a major concern in recent years. Increases in the concentrations of greenhouse gases in the atmosphere due to human activities are considered to be largely responsible for climate change and global warming. Due to increased awareness of the severity of climate change, the Kyoto protocol was signed in 1997 and mandatory emission limits for greenhouse gases, primarily carbon dioxide (CO₂), were assigned to the signatory nations ([12]). To implement the Kyoto Protocol in European Union, the European Climate Change Program was launched by the European Commission in June 2000. Subsequently, in January 2005, the European Union Emission Trading Scheme (EU ETS) ([5]) began operating. The EU ETS was the first and is still by far the largest international system for trading greenhouse gas emission allowances, where it covers around 45% of the total greenhouse gas emissions from the 28 EU countries.

The EU ETS employs a cap and trade system, which means that the total amount of permissible greenhouse gas emissions by a country or factory are predetermined for a specific period. If the actual greenhouse gas emissions by signatory parties reach their limit, i.e., the emission cap, then they need to purchase the European Union Allowance from other countries or pay penalties for excess emissions. In addition, they can sell their spare allowances in the market to make profits ([5, 3]). Furthermore, in order to strengthen cooperation among countries, the signatory...
parties are allowed to buy international credits from emission-saving projects performed under the Kyoto Protocol’s Clean Development Mechanism and Joint Implementation instruments around the world.

Many studies have investigated different aspects of carbon reduction and trading in recent years. Daskalakis et al. ([8]) studied the three main markets for emission allowances within the EU ETS, where they developed a framework for the pricing and hedging of futures as well as options on futures. Carmona et al. ([2]) showed that the economic mechanism of carbon allowance price formation can be formulated in the framework of competitive stochastic equilibrium models and they identified the main allowance price drivers. Jan Seifert et al. ([16]) proposed a tractable stochastic equilibrium model that reflects stylized features of the EU ETS and they analyzed the resulting CO₂ spot price dynamics. Wang et al. ([18]) analyzed the emission reduction pathways of enterprises in practical processes based on China’s emission abatement target, where they developed a framework to derive the magnitude of investment required in each pathway. Zagheni and Billari ([21]) provided a stochastic differential equations model to evaluate the costs incurred for a country to reduce emissions to satisfy the cap from the perspective of options pricing ([1, 13]). Yang and Liang ([19, 20]) proposed an optimal control model, where they used a stochastic process to describe the carbon emissions of a country and they set the objective of minimizing the total costs, including the costs of reducing emissions and the penalties for exceeding the cap according to the EU ETS. They obtained the corresponding Hamilton–Jacobi–Bellman (HJB) equations, as well as proving the existence and uniqueness of the classical solutions. In our previous study ([11]), we built a stochastic optimal control model of carbon reduction by considering carbon trading and we obtained a semi-closed solution of the corresponding HJB equation in certain conditions. However, some important factors were not considered in this preliminary model, such as the penalty mechanism and the limited carbon reduction capability of a country.

In the present study, to describe the carbon emissions process using a modified environmental impact model, we build a stochastic control model for analyzing the carbon reduction policy of a country in a more general setting. In this model, we consider carbon abatement, trading, and penalties, as well as the limited capacity for carbon abatement in a synthetic manner. Our goal is to find the carbon abatement policy that minimizes the total carbon reduction costs. Under Merton’s classical consumption framework ([14, 15]), the model is converted into a semi-linear degenerate two-dimensional parabolic HJB problem. In general, the value function is not sufficiently smooth to satisfy a HJB equation in the classical sense. Therefore, it is natural to search for a weak solution such that the value function is unique although it is not smooth. A weak solution called the viscosity solution was introduced by Crandall and Lions ([6, 7]). In the present study, we rigorously prove that the value function is a unique viscosity solution of the HJB equation using the dynamic programming principle, Itô’s Lemma, and the theory of viscosity solutions. Finally, we discuss the properties of the optimal policy and the minimum total costs based on numerical calculations.

The remainder of this paper is organized as follows. In Section 2, under some assumptions, we establish an optimal control model to minimize the total costs while considering carbon emissions abatement, trading, and penalties, as well as a limited capacity for carbon abatement. According to the dynamic programming principle, we reduce this model to the corresponding HJB equation problem. In Section 3,
we present the continuity and quadratic growth properties of the value function. In Section 4, we prove the existence and uniqueness of the viscosity solution of the HJB equation problem. In Section 5, we present the numerical results and their analysis. We also discuss the properties of the optimal policy and the minimum total costs. We give our conclusions in Section 6.

2. Model formulation. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a filtered probability space that satisfies the usual conditions. To describe the environmental impact of a country, Commoner ([4]) devised the IPAT equation, which states that the environmental impact \((I)\) is the product of population \((P)\), affluence \((A)\), and technology \((T)\). In 1994, Dietz and Rosa ([9]) reformulated the IPAT equation as “Stochastic Impacts by Regression on Population, Affluence and Technology” (STIRPAT) to make the model suitable for estimating parameters and hypothesis testing. Zagheni and Bilibari ([21]) provided a stochastic representation of the IPAT equation and obtained a stochastic differential equations model. In this framework, Yang and Liang ([19]) assumed that the population \((P)\) of a country satisfies a logistic model ([17]) and they obtained a modified environmental impact model. This model is used in the present study.

Let \(I_t\) denote the carbon emissions by a country in one year. Suppose that the initial amount of emissions \(I_0 > 0\). In the modified environmental impact model ([19, 20]), the carbon emissions from an area are determined by its economy and the population. The economy is represented by the GDP and the population is described by a logistic model. The process \(I_t\) can be described as

\[
\begin{align*}
\left\{ \begin{array}{l}
dI_t &= I_t a_1 \mu_1 + a_2 f(t) - q_t) dt + I_t a_1 \sigma_1 dW^1_t, \\
I(0) &= I_0,
\end{array} \right.
\end{align*}
\]

where \(a_1\) and \(a_2\) are constant parameters, which represent the effect of the GDP and the population of a country on the carbon emissions, respectively, \(\mu_1\) is the growth rate, and \(\sigma_1\) is the volatility of the GDP. \(f(t) = \frac{\rho (P_m - P_0)}{P_m e^{\rho t} + P_0 - P_0}\) is the growth rate of the population at time \(t > 0\). \(P_m\) is the carrying capacity of the population of the country, \(\rho\) is the intrinsic population growth, and \(P_0\) is the initial population size. \(W^1_t\) is the \(\mathcal{F}_t\)-adapted standard Brownian motion. \(q_t \geq 0\) is the control policy, which is a progressively measurable (with respect to \(\{\mathcal{F}_t\}_{t \geq 0}\) process that represents the reduced growth rate of carbon emission. Furthermore, The capability of reducing carbon emissions is limited in the normal case. Thus, we assume that the control policy satisfies \(0 \leq q_t \leq \bar{q}\) and \(\bar{q}\) is the upper bound, which represents the maximum capacity for carbon abatement.

In our model, a country needs to satisfy the carbon reductions target, i.e., there is an emissions cap at the given time \(T\). If the amount of carbon emissions is below this cap, then the country can sell its spare allowance; otherwise, it needs to purchase permits for its excess emissions or accept the penalty. We assume that the price process \(C_t\) for the emissions allowance in the emissions trading market satisfies geometric Brownian motion:

\[
dC = C_t \mu_2 dt + C_t \sigma_2 dW^2_t, \tag{2}
\]

where \(\mu_2\) is the constant drift parameter and \(\sigma_2\) is the constant volatility parameter. \(W^2_t\) is \(\mathcal{F}_t\)-adapted Brownian motion and we assume that

\[
dW^1_t dW^2_t = \rho dt,
\]
where \( \rho \) is the correlation coefficient. At time \( T \), if the amount of emissions by the country exceeds the emissions limit \( \bar{I} \), i.e., \( I_T - \bar{I} > 0 \), then the country needs to buy emissions allowances from the market at price \( C_T \), or accept the penalty in the form of a fine at price \( P \). However, if \( I_T - \bar{I} < 0 \), this means that the country has spare allowances, so the allowances can also be sold at price \( C_T \) for a profit. Thus, the payoff at the terminal time should be rewritten as

\[
\hat{V}(I_T, C_T) = (I_T - \bar{I})^+ \min\{C_T, P\} - (I_T - \bar{I})^- C_T.
\]

In this model, the goal of the decision maker is to minimize the total costs spent on carbon reduction activities. Thus, we consider the corresponding value function:

\[
V(I, c, t) = \inf_{0 \leq q \leq \bar{q}} J(I, c, t; q)
\]

where \( V(I, c, t) \) comprises all of the costs for reducing carbon emissions spent by the country, which have three parts, i.e., the costs of the carbon abatement process, the costs or profit of carbon trading activities, and penalties. \( g(\cdot) \) is the price function for reducing the growth rate of carbon emissions, which means that \( g(q_t) \) represents the costs incurred by the country on the reductions project during the time interval \([t, t + dt]\). Thus, \( \int_t^T g(q_s)ds \) is the total cost of carbon abatement processes in the country from \( t \) to terminal time \( T \). \( \beta \) is the discount factor. By the dynamic programming principle ([10]), we obtain the corresponding HJB equation as

\[
\frac{\partial V}{\partial t} + I(a_1 \mu_1 + a_2 f(t)) \frac{\partial V}{\partial I} + \mu_2 c \frac{\partial V}{\partial c} + \frac{1}{2} I^2 a_1^2 \sigma_1^2 \frac{\partial^2 V}{\partial I^2} + Ic\rho a_1 \sigma_1 \sigma_2 \frac{\partial^2 V}{\partial I \partial c} + \frac{1}{2} c^2 \sigma_2^2 \frac{\partial^2 V}{\partial c^2} - \beta V + \inf_{0 \leq q \leq \bar{q}} \sqrt{[g(q) - q I \frac{\partial V}{\partial I}]} = 0, \quad I > 0, c > 0, T > t \geq 0 \tag{4}
\]

The terminal condition is

\[
V(I, c, T) = (I - \bar{I})^+ \min\{c, P\} - (I - \bar{I})^- c, \quad I > 0, c > 0. \tag{5}
\]

For the abatement cost function \( g(\cdot) \), the statistical analysis in [20] shows that the marginal abatement cost increases with the abatement amount. \( g(\cdot) \) is assumed to be an increasing quadratic function in the present study, i.e.

\[
g(q) = m_1 q^2, \tag{6}
\]

where \( m_1 \) is a positive constant. The same assumption was used in ([16]). From (6) and by taking the boundedness of the control process into consideration, the optimal control policy \( q^* \) can be written as:

\[
q^* = \begin{cases} 
0, & I \frac{\partial V}{\partial I} < 0, \\
\frac{1}{2m_1} \frac{\partial V}{\partial I}, & 0 \leq I \frac{\partial V}{\partial I} \leq 2m_1 \bar{q}, \\
\bar{q}, & I \frac{\partial V}{\partial I} > 2m_1 \bar{q}.
\end{cases}
\]

3. Properties of the value function. Now, we focus on the value function (3) and we show some of its properties in preparation for the subsequent analysis.

**Lemma 3.1.** (i) For each \((c, t) \in \mathbb{R}^+ \times [0, T]\), the value function \( V(I, c, t) \) is non-decreasing in \( I \). (ii) The value function \( V(I, c, t) \) is continuous on \( \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T] \).
\textit{Proof.} For a fixed policy $0 \leq q \leq \bar{q}$ and $(c, t) \in \mathbb{R}^+ \times [0, T]$, we denote $I_{t}^{1, I}$ as the emissions process with $I_t = I$, $s \geq t$. If we suppose that $0 < I^1 \leq I^2$ and define

$$Z_s = I_{s}^{1, I^2} - I_{s}^{1, I^1},$$

then we have $Z_s \geq 0$ for all $s \geq t$, particularly $I_{T}^{1, I^2} \geq I_{T}^{1, I^1}$. In addition, because the term in the expectation of the value function (3) increases with $I_t$, then we obtain

$$J(I_{T}^{2, c, t}; q) \geq J(I_{T}^{1, c, t}; q) \geq V(I_{T}^{1, c, t}), \quad \forall 0 \leq q \leq \bar{q}.$$ 

Since $q$ is arbitrary, then $V(I_{T}^{2, c, t}) \geq V(I_{T}^{1, c, t})$, which means that (i) holds.

For (ii), we first prove that $V(I, c, t)$ is continuous with respect to $I$, uniformly in $t$. Fix $c, t \in \mathbb{R}^+ \times [0, T], \forall I_2 > I_1 > 0$, then by the definition of the value function (3), $0 \leq q^*_1, q^*_2 \leq \bar{q}$ exist such that

$$0 < J(I_{i}, c, t; q^*_i) - \inf_{0 \leq q \leq \bar{q}} J(I_{i}, c, t; q) \leq \frac{1}{2}|I_2 - I_1|, \quad i = 1, 2.$$ 

Therefore,

$$|V(I_{2}, c, t) - V(I_{1}, c, t)| = \left| \inf_{0 \leq q \leq \bar{q}} J(I_{2}, c, t; q) - \inf_{0 \leq q \leq \bar{q}} J(I_{1}, c, t; q) \right| \leq |J(I_{2}, c, t; q^*_1) - \inf_{0 \leq q \leq \bar{q}} J(I_{1}, c, t; q_1)| + |J(I_{1}, c, t; q^*_2) - \inf_{0 \leq q \leq \bar{q}} J(I_{2}, c, t; q_2)| \leq |J(I_{2}, c, t; q^*_1) - J(I_{1}, c, t; q^*_1)| + \frac{1}{2}|I_2 - I_1| + |J(I_{2}, c, t; q^*_2) - J(I_{1}, c, t; q^*_2)| + \frac{1}{2}|I_2 - I_1|. \ (7)$$

For any $0 \leq q^* \leq \bar{q}$, a constant $M_1 > 0$ exists such that

$$|J(I_{2}, c, t; q^*) - J(I_{1}, c, t; q^*)| \leq E\left[e^{-\beta(T-t)}|((I_{2}, T) - \tilde{I})^+ - (I_{1}, T) - \tilde{I}^+)\min(C_T, P) - (I_{2}, T) - \tilde{I}) - (I_{1}, T) - \tilde{I})C_T\right]_{1, t} = I_1, I_2, t = I_2, C_t = c \leq E\left[e^{-\beta(T-t)}2C_T|I_{2}, T - I_{1}, T|\right]_{1, t} = I_1, I_2, t = I_2, C_t = c \leq e^{2c|I_2 - I_1|}\exp\left(\int_{t}^{T} (a_1 \mu_1 + a_2 f(s) - \frac{1}{2} u_1^2 \sigma_1^2 - q^* - \beta + \mu_2 - \frac{1}{2} \sigma_2^2) ds + \int_{t}^{T} a_1 \sigma_1 dW^1_s + \int_{t}^{T} \sigma_2 dW^2_s\right)|_{1, t} = I_1, I_2, t = I_2, C_t = c \leq \frac{M_1 c}{2}|I_2 - I_1|.$$ 

Thus, we have

$$|V(I_{2}, c, t) - V(I_{1}, c, t)| \leq (M_1 c + 1)|I_2 - I_1|. \ (8)$$

Similarly, using the techniques described above, we can prove that $V(I, c, t)$ is continuous with respect to $c$, uniformly in $t$. Next, we prove the continuity property of $V(I, c, t)$ in $t$. Fix $I, c \in \mathbb{R}^+ \times \mathbb{R}^+$, $\forall t_2 \geq t_1 \geq 0$, and $0 \leq q^*_1, q^*_2 \leq \bar{q}$ exist such that

$$0 < J(I_{i}, c, t; q^*_i) - \inf_{0 \leq q \leq \bar{q}} J(I_{i}, c, t; q) \leq \frac{1}{2}|t_2 - t_1|, \quad i = 1, 2.$$
Furthermore, the results given above, we obtain the continuity property of the value function which means that the value function Lemma 3.2.  

Similar to (7), it can be verified that
\[
|V(I, c, t_2) - V(I, c, t_1)| = \left| \inf_{0 \leq q_2 \leq \bar{q}} J(I, c, t_2; q_2) - \inf_{0 \leq q_1 \leq \bar{q}} J(I, c, t_1; q_1) \right| 
\leq |J(I, c, t_2; q_2^*) - J(I, c, t_1; q_1^*)| + |J(I, c, t_2; q_2^*) - J(I, c, t_1; q_2^*)| + |t_2 - t_1|. 
\]  

(9) 

For any \(0 \leq q \leq \bar{q}\), denote 
\[
S_i := (I_{i,T} - \bar{I})^+ \min\{C_{i,T}, P\} - (I_{i,T} - \bar{I})^- C_{i,T}, \quad i = 1, 2, 
\]
where \(C_{1,T}, C_{2,T}\) are the processes in (2) at \(T\) with initial values \(C_{t_1} = c, C_{t_2} = c\). \(I_{1,T}, I_{2,T}\) are the processes in (1) at \(T\) with initial values \(I_{t_1} = I, I_{t_2} = I\). Then,
\[
|J(I, c, t_2; q) - J(I, c, t_1; q)| \leq \frac{\bar{q}(\bar{q})}{\bar{\beta}}[1 - e^{-\bar{\beta}(t_2 - t_1)}] 
+ e^{-\bar{\beta}(T - t_2)}E\left[|S_2 - S_1| I_{2,t_2} = I, C_{2,t_2} = c, I_{1,t_1} = I, C_{1,t_1} = c\right] 
+ e^{-\beta(T - t_2)}E\left[|S_1| I_{2,t_2} = I, C_{2,t_2} = c, I_{1,t_1} = I, C_{1,t_1} = c\right]. 
\]  

(10) 

Furthermore,
\[
E\left[|S_2 - S_1| I_{2,t_2} = I, C_{2,t_2} = c, I_{1,t_1} = I, C_{1,t_1} = c\right] 
\leq E\left[\left|\min\{C_{2,T}, P\}(I_{2,T} - \bar{I})^+ - (I_{1,T} - \bar{I})^-\right|\right] 
+ |I_{1,T} - \bar{I}|^+(\min\{C_{2,T}, P\} - \min\{C_{1,T}, P\}) 
+ |C_{2,T}(I_{2,T} - \bar{I})^- - (I_{1,T} - \bar{I})^-| 
+ |I_{1,T} - \bar{I}|^- (C_{2,T} - C_{1,T})\left|I_{2,t_2} = I, C_{2,t_2} = c, I_{1,t_1} = I, C_{1,t_1} = c\right| 
\leq E\left[2C_{2,T}|I_{2,T} - I_{1,T}|\right] 
+ 2|C_{2,T} - C_{1,T}| |I_{1,T} - \bar{I}|\left|I_{2,t_2} = I, C_{2,t_2} = c, I_{1,t_1} = I, C_{1,t_1} = c\right| 
\leq M_2 I c \sqrt{t_2 - t_1} + M_3 c \sqrt{t_2 - t_1}, 
\]  

(11) 

where \(M_2 > 0, M_3 > 0\) are some constants, and thus it is not difficult to obtain
\[
E\left[|S_1| I_{2,t_2} = I, C_{2,t_2} = c, I_{1,t_1} = I, C_{1,t_1} = c\right] \leq M_4 I c + M_5 c, 
\]  

(12) 

where \(M_4 > 0, M_5 > 0\) are some constants. Thus, from (10), (11), and (12), the constants \(M_6 > 0, M_7 > 0\) exist such that
\[
|J(I, c, t_2; q) - J(I, c, t_1; q)| \leq \frac{M_6}{2}(I + 1) c \sqrt{t_2 - t_1} + \frac{M_7}{2} |t_2 - t_1|. 
\]  

(13) 

Substituting (13) into (9) yields
\[
|V(I, c, t_2) - V(I, c, t_1)| \leq M_6(I + 1) c \sqrt{t_2 - t_1} + (M_7 + 1) |t_2 - t_1|, 
\]  

(14) 

which means that \(V(I, c, t)\) is continuous with respect to \(t\). Hence, according to the results given above, we obtain the continuity property of the value function \(V(I, c, t)\). \(\Box\)

**Lemma 3.2.** The value function \(V(I, c, t)\) grows at most quadratically, i.e., 
\(\forall (I, c, t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T], \) \(\exists M > 0\) such that 
\[
|V(I, c, t)| \leq M(1 + |I|^2 + |c|^2). 
\]
Proof. From the definition (3) of the value function $V(I, c, t)$, we can obtain

$$|V(I, c, t)| \leq E\left[\int_0^T g(\dot{q})e^{-\beta(s-t)}ds + |(I_T - \bar{I})^+ C_T e^{-\beta(T-t)}|I_t = I, C_t = c\right].$$

Thus, the constants $M_1 > 0$, $M > 0$ exist such that

$$|V(I, c, t)| \leq M_1 E[1 + \sup_{s \geq t} |I_s|^2 + \sup_{s \geq t} |C_s|^2 | I_t = I, C_t = c]$$

$$\leq M(1 + |I|^2 + |c|^2).$$

\hfill \Box

In general, we do not know the continuous differentiability of the value function, so in the next section, we use the concept of the viscosity solution and we prove that the value function is the unique viscosity solution to the HJB equation (4).

4. Existence and uniqueness of the viscosity solution. For our problem, the viscosity solution ([10]) is defined as follows.

**Definition 4.1.** (Viscosity solution)

Denote $Q_T = \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T]$, $\bar{Q}_T = \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T]$ and

$$F(D^2u(x, y, t), Du(x, y, t), u(x, y, t), x, y, t) = x(a_1 \mu_1 + a_2 f(t)) \frac{\partial u}{\partial x} + \mu_2 y \frac{\partial u}{\partial y} + \frac{1}{2} x^2 \sigma_1^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} y^2 \sigma_2^2 \frac{\partial^2 u}{\partial y^2} + xy \rho a_1 \sigma_1 \sigma_2 \frac{\partial^2 u}{\partial x \partial y} - \beta u + \inf_{0 \leq \tilde{q} < q} |g(\tilde{q}) - q \frac{\partial u}{\partial x}|.$$

Assume that $u : \bar{Q}_T \to \mathbb{R}$ is locally bounded. Then, we define

1. The lower-semicontinuous function $u(x, y, t)$ is a viscosity supersolution of equation (4) in $Q_T$, if for all $\varphi \in C^{2,2,1}(\bar{Q}_T)$, such that $u - \varphi$ has a local minimum at $(\bar{x}, \bar{y}, \bar{t}) \in Q_T$ and $u(\bar{x}, \bar{y}, \bar{t}) = \varphi(\bar{x}, \bar{y}, \bar{t})$; then,

$$\frac{\partial \varphi}{\partial t}(\bar{x}, \bar{y}, \bar{t}) + F(D^2 \varphi(\bar{x}, \bar{y}, \bar{t}), D\varphi(\bar{x}, \bar{y}, \bar{t}), \varphi(\bar{x}, \bar{y}, \bar{t}), \bar{x}, \bar{y}, \bar{t}) \leq 0.$$

2. The upper-semicontinuous function $u(x, y, t)$ is a viscosity subsolution of equation (4) in $Q_T$, if for all $\varphi \in C^{2,2,1}(\bar{Q}_T)$, such that $u - \varphi$ has a local maximum at $(\bar{x}, \bar{y}, \bar{t}) \in Q_T$ and $u(\bar{x}, \bar{y}, \bar{t}) = \varphi(\bar{x}, \bar{y}, \bar{t})$; then,

$$\frac{\partial \varphi}{\partial t}(\bar{x}, \bar{y}, \bar{t}) + F(D^2 \varphi(\bar{x}, \bar{y}, \bar{t}), D\varphi(\bar{x}, \bar{y}, \bar{t}), \varphi(\bar{x}, \bar{y}, \bar{t}), \bar{x}, \bar{y}, \bar{t}) \geq 0.$$

3. $u(x, y, t)$ is the viscosity solution of equation (4) in $\bar{Q}_T$ if and only if it is simultaneously a supersolution and subsolution of (4). In addition, if $u|_{T} = (I - I)^+ \min\{c, P\} - (I - I)^- c$ on $\mathbb{R}^+ \times \mathbb{R}^+$, then $u(x, y, t)$ is the viscosity solution of problem (4) (5).

Next, according to the method in [10], we obtain the following result, which implies the existence of a viscosity solution to problem (4) (5).

**Theorem 4.2.** The value function $V(I, c, t)$ is a viscosity solution of equation (4) (5) in $\bar{Q}_T$. 
Proof. We have proved the continuity of the value function $V(I, c, t)$ and the terminal condition is obviously satisfied by $V(I, c, t)$. Next, we show that $V(I, c, t)$ is a viscosity subsolution of equation (4) in $\bar{Q}_T$. Suppose that a test function $\varphi \in C^{2,1}(\bar{Q}_T)$, such that $V - \varphi$ has a local maximum at $(I_0, c_0, t_0) \in Q_T$ and $V(I_0, c_0, t_0) = \varphi(I_0, c_0, t_0)$, which means that $V(I, c, t) \leq \varphi(I, c, t)$ in a neighborhood $O(I_0, c_0, t_0) \subset Q_T$. Given $h > t_0$ and an arbitrary constant control $q_t \equiv q$, then according to the dynamic programming principle, we have

$$V(I_0, c_0, t_0) \leq E\left[\int_{t_0}^h e^{-\beta(s-t_0)}g(q)ds + e^{-\beta(h-t_0)}V(I_h, C_h, h)|I_{t_0} = I_0, C_{t_0} = c_0\right]$$

$$\leq E\left[\int_{t_0}^h e^{-\beta(s-t_0)}g(q)ds + e^{-\beta(h-t_0)}\varphi(I_h, C_h, h)|I_{t_0} = I_0, C_{t_0} = c_0\right].$$

By applying Itô formula, we can obtain

$$\varphi(I_h, C_h, h) = \varphi(I_0, c_0, t_0) + \int_{t_0}^h \left[\frac{\partial \varphi}{\partial t} + I(a_1 \mu_1 + a_2 f(t) - \frac{1}{2}I \sigma_1^2 \sigma_2^2 \partial^2 \varphi}{\partial I}\right] + \mu_2 \partial \varphi}{\partial c} + \frac{1}{2}I^2 \sigma_1^2 \sigma_2^2 \partial^2 \varphi}{\partial t^2}$$

$$+ \frac{1}{2}C^2 \sigma_2^2 \partial^2 \varphi}{\partial c^2} + IC \rho a_1 \sigma_1 \sigma_2 \partial^2 \varphi}{\partial t \partial c} + \int_{t_0}^h \partial \varphi}{\partial I} a_1 \sigma_1 dW_1 + \int_{t_0}^h \partial \varphi}{\partial c} C \sigma_2 dW_2.$$

By substituting $\varphi(I_h, C_h, h)$ with the equation above and dividing both sides by $h$, and finally letting $h \to t_0$, then since $q$ is arbitrary, we obtain

$$\frac{\partial \varphi}{\partial t}(I_0, c_0, t_0) + F(D^2 \varphi(I_0, c_0, t_0), D \varphi(I_0, c_0, t_0), \varphi(I_0, c_0, t_0), I_0, c_0, t_0) \geq 0.$$

Then, we show that $V(I, c, t)$ is a viscosity supersolution of equation (4) in $\bar{Q}_T$. Similarly, we take a test function $\varphi \in C^{2,1}(\bar{Q}_T)$ such that $V - \varphi$ has a local minimum at $(I_0, c_0, t_0) \in Q_T$ and $V(I_0, c_0, t_0) = \varphi(I_0, c_0, t_0)$, which means that $V(I, c, t) \geq \varphi(I, c, t)$ in a neighborhood $O(I_0, c_0, t_0) \subset Q_T$.

By the dynamic programming principle, for every $m \in \mathbb{N}^+$, a control process $0 \leq q^m \leq \tilde{q}$ exists such that

$$V(I_0, c_0, t_0) + \frac{1}{m^2} \geq E\left[\int_{t_0}^{I_m} e^{-\beta(s-t_0)}g(q^m)ds + e^{-\beta(t_m-t_0)}V(I_{t_m}, C_{t_m}, t_m)|I_{t_0} = I_0, C_{t_0} = c_0\right]$$

$$\geq E\left[\int_{t_0}^{I_m} e^{-\beta(s-t_0)}g(q^m)ds + e^{-\beta(t_m-t_0)}\varphi(I_{t_m}, C_{t_m}, t_m)|I_{t_0} = I_0, C_{t_0} = c_0\right],$$

where $t_m = t_0 + \frac{1}{m}$ and $I_m, C_m$ are the solutions of (1) and (2) with control $q^m$. By Itô’s formula,

$$\frac{1}{m^2} \geq E\left[\int_{t_0}^{I_m} e^{-\beta(s-t_0)}[g(q^m) + \frac{\partial \varphi}{\partial t} + I^m (a_1 \mu_1 + a_2 f(s) - q^m) \frac{\partial \varphi}{\partial I} + \mu_2 \partial \varphi}{\partial c} + \frac{1}{2}I^m \sigma_1^2 \sigma_2^2 \frac{\partial^2 \varphi}{\partial t^2} + \frac{1}{2}C^2 \sigma_2^2 \frac{\partial^2 \varphi}{\partial c^2} + \frac{1}{2}C \rho a_1 \sigma_1 \sigma_2 \frac{\partial^2 \varphi}{\partial t \partial c} + I^m C \sigma_2 dW_2]ds|I^m_{t_0} = I_0, C^m_{t_0} = c_0\right].$$

(15)

Since

$$\lim_{m \to \infty} E\left[\sup_{\xi \in [t_0, t_m]} |I^m_{\xi} - I_0|\right] = 0,$$
\[
\lim_{m \to \infty} E\left[ \sup_{\xi \in [t_0, t_m]} |C^m_\xi - c_0| \right] = 0,
\]
multiply (15) by \(m\) and let \(m \to \infty\), then we can obtain

\[
\begin{align*}
0 & \geq \lim_{m \to \infty} mE\left[ \int_{t_0}^{t_m} e^{-\beta(s-t_0)} [g(q^m) + \frac{\partial \varphi}{\partial t} + I_0(a_1 \mu_1 + a_2 f(s) - q^m) \frac{\partial \varphi}{\partial I}] \\
& \quad + \mu_2 c_0 \frac{\partial \varphi}{\partial c} + \frac{1}{2} (I_0)^2 a_1^2 \sigma_1^2 \frac{\partial^2 \varphi}{\partial I^2} + \frac{1}{2} (c_0)^2 \sigma_2^2 \frac{\partial^2 \varphi}{\partial c^2} \\
& \quad + I_0 c_0 \rho a_1 \sigma_2 \frac{\partial^2 \varphi}{\partial I \partial c} \right] ds | I_0^m = I_0, C_0^m = c_0].
\end{align*}
\]

Then, we have

\[
\frac{\partial \varphi}{\partial t} (I_0, c_0, t_0) + F(D^2 \varphi (I_0, c_0, t_0), D \varphi (I_0, c_0, t_0), \varphi (I_0, c_0, t_0), I_0, c_0, t_0) \leq 0.
\]

Thus, \(V (I, c, t)\) is a viscosity supersolution of equation (4) in \(\bar{Q}_T\). This completes the proof. \(\square\)

Next, we prove the comparison principle, which enables us to verify the uniqueness of the viscosity solution. By taking \(x = \ln I, y = \ln c\), we can derive the following from equation (4) (5).

\[
\begin{align*}
\frac{\partial V}{\partial t} + (a_1 \mu_1 + a_2 f(s)) - \frac{1}{2} a_1^2 \sigma_1^2 \frac{\partial V}{\partial x} + (\mu_2 - \frac{1}{2} \sigma_2^2) \frac{\partial V}{\partial y} \\
+ \frac{1}{2} a_1^2 \sigma_1^2 \frac{\partial^2 V}{\partial x^2} + \rho a_1 \sigma_2 \frac{\partial^2 V}{\partial x \partial y} + \frac{1}{2} \sigma_2^2 \frac{\partial^2 V}{\partial y^2} - \beta V \\
+ \inf_{0 \leq q \leq q^*} [g(q) - q \frac{\partial V}{\partial x}] = 0, \quad (x, y, t) \in \Omega_T := \mathbb{R} \times \mathbb{R} \times [0, T)
\end{align*}
\]

with the terminal condition

\[
V(x, y, T) = (e^x - \bar{I})^+ \min\{e^y, P\} - (e^x - \bar{I})^- e^y, \quad x \in \mathbb{R}, y \in \mathbb{R}.
\]

**Lemma 4.3.** (Comparison Principle)

*Assume that \(U \in \Omega_T := \mathbb{R} \times \mathbb{R} \times [0, T]\) is a viscosity supersolution of equation (16), \(V \in \Omega_T\) is a viscosity subsolution of equation (16), and they satisfy

\[
|U(x, y, t)|, |V(x, y, t)| \leq M(1 + e^{2x} + e^{2y}),
\]

where \(M > 0\) is constant. If

\[
U(x, y, T) \geq V(x, y, T), x \in \mathbb{R}, y \in \mathbb{R},
\]

then

\[
U(x, y, t) \geq V(x, y, t), \forall (x, y, t) \in \overline{\Omega}_T.
\]

**Proof.** By contradiction, we assume that \((x^*, y^*, t^*) \in \Omega_T\) and \(\delta > 0\) exist such that

\[
V(x^*, y^*, t^*) \geq U(x^*, y^*, t^*) + 2\delta,
\]

where \(t^* > 0\). In fact, if \(t^* = 0\), then by the continuity of \(U\) and \(V\), we can find \((x, y, t)\) near \((x^*, y^*, t^*)\), which satisfies (19). We define the auxiliary function

\[
\Phi(x_1, y_1, x_2, y_2, t) = V(x_1, y_1, t) - U(x_2, y_2, t) - \phi(x_1, y_1, x_2, y_2, t),
\]

thus

\[
\Phi(x_1, y_1, x_2, y_2, t) = \inf_{(x, y, t) \in \bar{Q}_T} \Phi(x, y, t) = 0.
\]

This completes the proof. \(\square\)
where \((x_1, y_1, x_2, y_2, t) \in \mathbb{R}^4 \times (0, T]\) and
\[
\phi = \frac{\alpha}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2) + \frac{\epsilon}{2}e^{\lambda(T-t)}(e^{4x_1} + e^{4y_1}) + e^{4x_2} + e^{4y_2} + |x_1|^4 + |y_1|^4 + |x_2|^4 + |y_2|^4 + \frac{\sqrt{\epsilon}}{t}.
\]

Denote
\[
F_\alpha = \sup_{\mathbb{R}^4 \times (0, T]} (V - U - \phi).
\]

Since (17) and by the semicontinuity of \(U\) and \(V\), \(F_\alpha\) can reach the maximum and we denote
\[
F_\alpha = \Phi(x_{\alpha}, y_{\alpha}, x_{2\alpha}, y_{2\alpha}, t_\alpha) < \infty.
\]

For sufficiently small \(\epsilon > 0\), we have
\[
F_\alpha \geq V(x^*, y^*, t^*) - U(x^*, y^*, t^*) - \epsilon e^{\lambda(T-t^*)}(e^{4x^*} + e^{4y^*} + |x^*|^4 + |y^*|^4) - \frac{\sqrt{\epsilon}}{t^*} \geq \delta.
\]

\(\phi(\cdot) \geq 0\), so we can obtain
\[
V(x_{\alpha}, y_{\alpha}, t_\alpha) \geq U(x_{2\alpha}, y_{2\alpha}, t_\alpha) + \delta.
\]

(20)

From the inequality \(\Phi(0, 0, 0, 0, T) \leq \Phi(x_{\alpha}, y_{\alpha}, x_{2\alpha}, y_{2\alpha}, t_\alpha)\), we have
\[
\alpha \left((x_1 - x_2)^2 + (y_1 - y_2)^2\right) + \frac{\epsilon}{2}e^{\lambda(T-t_\alpha)}(e^{4x_{\alpha}} + e^{4y_{\alpha}}) + e^{4x_{2\alpha}} + e^{4y_{2\alpha}} + |x_1|^4 + |y_1|^4 + |x_2|^4 + |y_2|^4 + \frac{\sqrt{\epsilon}}{t_\alpha}
\]
\[
\leq V(x_{\alpha}, y_{\alpha}, t_\alpha) - U(x_{2\alpha}, y_{2\alpha}, t_\alpha) - V(0,0,T) + U(0,0,T) + 2\epsilon + \frac{\sqrt{\epsilon}}{T}
\]
\[
\leq M_2(1 + e^{2x_{\alpha}} + e^{2y_{\alpha}} + e^{2x_{2\alpha}} + e^{2y_{2\alpha}}),
\]

(21)

where \(M_2\) is a positive constant. (21) implies that positive constants \(M_{1\epsilon}\) and \(M_{2\epsilon}\) exist such that
\[
x_{\alpha}^4 + y_{\alpha}^4 + x_{2\alpha}^4 + y_{2\alpha}^4 \leq M_{1\epsilon}
\]

and \(t_\alpha > M_{2\epsilon} > 0\). Thus, a subsequence exists, which is still denoted by \((x_{\alpha}, y_{\alpha}, x_{2\alpha}, y_{2\alpha}, t_\alpha)\), and it converges to some point \((x_\epsilon, y_\epsilon, x_\epsilon, y_\epsilon, t_\epsilon) \in \mathbb{R}^4 \times (0, T]\). In addition, (21) implies that a constant \(M_{3\epsilon} > 0\) exists such that
\[
\frac{\alpha}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2) \leq M_{3\epsilon}.
\]

Thus, we have \(x_{\alpha} \rightarrow x_\epsilon \rightarrow x_\epsilon, y_{\alpha} \rightarrow y_\epsilon \rightarrow y_\epsilon\), when \(\alpha \rightarrow \infty\).

Therefore, we can denote \(x_{2\epsilon} = x_\epsilon, y_{2\epsilon} = y_\epsilon = y_\epsilon\).

\(F_\alpha \geq \Phi(x^*, y^*, x^*, y^*, t^*)\), so we can obtain
\[
V(x_{\alpha}, y_{\alpha}, t_\alpha) - U(x_{2\alpha}, y_{2\alpha}, t_\alpha) \geq V(x^*, t^*, t^*) - U(x^*, y^*, t^*) - \epsilon e^{\lambda(T-t^*)}(e^{4x^*} + e^{4y^*} + |x^*|^4 + |y^*|^4) - \frac{\sqrt{\epsilon}}{t^*}.
\]

If we let \(\alpha \rightarrow \infty\), then we have
\[
V(x_{\alpha}, y_{\alpha}, t_\alpha) - U(x_{2\alpha}, y_{2\alpha}, t_\alpha) \rightarrow V(x_\epsilon, y_\epsilon, t_\epsilon) - U(x_\epsilon, y_\epsilon, t_\epsilon).
\]

If \(t_\epsilon = T\), let \(\epsilon \rightarrow 0\) and because (18), then we can obtain
\[
V(x^*, y^*, t^*) \leq U(x^*, y^*, t^*),
\]
which contradicts the assumption (19); thus, \( t_e \neq T \), which means that for any
sufficiently large \( \alpha \), we have \( (x_{1\alpha}, y_{1\alpha}, t_\alpha), (x_{2\alpha}, y_{2\alpha}, t_\alpha) \in \mathbb{R} \times \mathbb{R} \times (0, T) \). By Ishii’s
Lemma (10), \( \theta, \hat{\theta}, A, B \) exist such that

\[
(\theta, p, A) \in \mathcal{P}^{2,+} V(x_{1\alpha}, y_{1\alpha}, t_\alpha),
(\hat{\theta}, \hat{p}, B) \in \mathcal{P}^{2,-} U(x_{2\alpha}, y_{2\alpha}, t_\alpha).
\]

\[
p = \begin{pmatrix}
\alpha(x_{1\alpha} - x_{2\alpha}) + \frac{e^{\lambda(T-t_\alpha)}}{2}(4e^{4x_{1\alpha}} + 4x_{1\alpha}^3) \\
\alpha(y_{1\alpha} - y_{2\alpha}) + \frac{e^{\lambda(T-t_\alpha)}}{2}(4e^{4y_{1\alpha}} + 4y_{1\alpha}^3)
\end{pmatrix},
\]

\[
\hat{p} = \begin{pmatrix}
\alpha(x_{1\alpha} - x_{2\alpha}) - \frac{e^{\lambda(T-t_\alpha)}}{2}(4e^{4x_{2\alpha}} + 4x_{2\alpha}^3) \\
\alpha(y_{1\alpha} - y_{2\alpha}) - \frac{e^{\lambda(T-t_\alpha)}}{2}(4e^{4y_{2\alpha}} + 4y_{2\alpha}^3)
\end{pmatrix},
\]

\[
\theta - \hat{\theta} = -\frac{e}{2} \lambda e^{\lambda(T-t_\alpha)} (e^{4x_1} + e^{4y_1} + e^{4x_2} + e^{4y_2})
+ |x_1|^4 + |y_1|^4 + |x_2|^4 + |y_2|^4 - \frac{\sqrt{e}}{t_\alpha^2}
\]

and the 4 \times 4 symmetric matrix \( \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \) satisfies

\[
\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq 3\alpha \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}
+ \frac{e^2 e^{2\lambda(T-t_\alpha)}}{\alpha} \begin{pmatrix} (8e^{4x_{1\alpha}} + 6x_{1\alpha}^2)^2 & 0 & 0 & 0 \\ 0 & (8e^{4y_{1\alpha}} + 6y_{1\alpha}^2)^2 & 0 & 0 \\ 0 & 0 & (8e^{4x_{2\alpha}} + 6x_{2\alpha}^2)^2 & 0 \\ 0 & 0 & 0 & (8e^{4y_{2\alpha}} + 6y_{2\alpha}^2)^2 \end{pmatrix}
+ 8\epsilon e^{\lambda(T-t_\alpha)} \begin{pmatrix} -e^{4x_{1\alpha}} & 0 & -(e^{4x_{1\alpha}} + e^{4x_{2\alpha}}) & 0 \\ 0 & -e^{4y_{1\alpha}} & 0 & -(e^{4y_{1\alpha}} + e^{4y_{2\alpha}}) \\ -(e^{4x_{1\alpha}} + e^{4x_{2\alpha}}) & 0 & -e^{4y_{1\alpha}} + e^{4y_{2\alpha}} & 0 \\ -e^{4x_{2\alpha}} & 0 & -e^{4y_{2\alpha}} & -e^{4y_{2\alpha}} \end{pmatrix}
+ 12\epsilon e^{\lambda(T-t_\alpha)} \begin{pmatrix} 3x_{1\alpha}^2 & 0 & -(x_{1\alpha}^2 + x_{2\alpha}^2) & 0 \\ 0 & 3y_{1\alpha}^2 & 0 & -(y_{1\alpha}^2 + y_{2\alpha}^2) \\ -(x_{1\alpha}^2 + x_{2\alpha}^2) & 0 & 3x_{2\alpha}^2 & 0 \\ 0 & -(y_{1\alpha}^2 + y_{2\alpha}^2) & 0 & 3y_{2\alpha}^2 \end{pmatrix}
\]

where \( \mathcal{P}^{2,+} V(x_{1\alpha}, y_{1\alpha}, t_\alpha) \) is the superjet (10) of \( V \) at point \( (x_{1\alpha}, y_{1\alpha}, t_\alpha) \) and \( \mathcal{P}^{2,-} U(x_{2\alpha}, y_{2\alpha}, t_\alpha) \) is the subjet (10) of \( U \) at point \( (x_{2\alpha}, y_{2\alpha}, t_\alpha) \). Moreover, because \( V \) and \( U \) are the viscosity supersolution and viscosity subsolution of equation (16), we can find \( q^* \in [0, \bar{q}] \), such that

\[
\left( a_1 \mu_1 + a_2 f(t_\alpha) - \frac{1}{2} a_1^2 \sigma_{11}^2 - q^* \right) \sqrt{2} \left( \frac{\alpha(x_{1\alpha} - x_{2\alpha}) + \frac{e^{\lambda(T-t_\alpha)}}{2}(4e^{4x_{1\alpha}} + 4x_{1\alpha}^3)}{\alpha(y_{1\alpha} - y_{2\alpha}) + \frac{e^{\lambda(T-t_\alpha)}}{2}(4e^{4y_{1\alpha}} + 4y_{1\alpha}^3)} \right)
+ \theta + \frac{1}{2} \left[ \begin{pmatrix} a_1^2 \sigma_{11}^2 & \rho a_1 \sigma_{11} \sigma_{21} & \sigma_{21}^2 \\ \rho a_1 \sigma_{11} \sigma_{21} & \sigma_{21}^2 & \sigma_{22}^2 \end{pmatrix} A \right] - \beta V(x_{1\alpha}, y_{1\alpha}, t_\alpha) + g(q^*) \geq 0,
\]
By choosing a sufficiently large $\mu_2$, we obtain
\[
\Delta^+ - \beta U(x_{2\alpha}, y_{2\alpha}, t_{\alpha}) + g(q^*) \leq 0.
\]

Denote
\[
\Delta_1 = -\lambda e^{\lambda(T-t_{\alpha})}[e^{4x_\alpha} + e^{4y_\alpha}] + 8e^{\lambda(T-t_{\alpha})}[a^2_1\sigma_1^2 e^{4x_\alpha} + \sigma_2^2 e^{4y_\alpha}] \\
+4e^{\lambda(T-t_{\alpha})}[(a_1\mu_1 + a_2f(t_{\alpha}) - \frac{1}{2}a_1^2\sigma_1^2 - q^*)(e^{4x_\alpha} + x_{\alpha}^2)] + (\mu_2 - \frac{1}{2}\sigma_2^2)(e^{4y_\alpha} + y_{\alpha}^2),
\]
\[
\Delta_2 = -\lambda e^{\lambda(T-t_{\alpha})}[x_\alpha^4 + y_\alpha^4] + 6e^{\lambda(T-t_{\alpha})}[a^2_1\sigma_1^2 x_\alpha^2 + \sigma_2^2 y_\alpha^2] \\
+4e^{\lambda(T-t_{\alpha})}[(a_1\mu_1 + a_2f(t_{\alpha}) - \frac{1}{2}a_1^2\sigma_1^2 - q^*)(x_\alpha^2) + (\mu_2 - \frac{1}{2}\sigma_2^2)y_{\alpha}^2].
\]

By choosing a sufficiently large $\lambda$, we have $\Delta_1 < 0$. If $|x_\alpha| \geq 1, |y_\alpha| \geq 1$, we can also obtain $\Delta_2 < 0$. If $|x_\alpha| < 1$ or $|y_\alpha| < 1$, we can choose a sufficiently small $\epsilon$ such that $\Delta_2 < \beta \delta$. Thus, from (20) and (23), we can obtain
\[
0 \leq -\beta \delta + \Delta_1 + \Delta_2 < 0,
\]
which is a contradiction. This means that
\[
U(x, y, t) \geq V(x, y, t), \forall (x, y, t) \in \bar{\Omega}_T.
\]

This completes the proof. \hfill \Box

Finally, according to the comparison principle, it is easy to show the following.

**Theorem 4.4.** The value function $V(I, c, t)$ is the unique viscosity solution of equation (4) (5) in $Q_T$. 
5. **Numerical results.** In this section, we present some numerical results for the HJB equation, which are obtained using the finite difference method. We then analyze the properties of the optimal policy and the minimum costs. The specific parameters used in calculations are as follows if they are not used as variables or indicated specifically. Some of the following data come from the study of Yang ([20]): $P_m = 1.6811$ (billion), $P_0 = 0.6574$ (billion), $\hat{\rho} = 0.0368$ /year, $\mu_1 = 0.0987$ /year, $\sigma_1 = 0.2636$, $a_2 = 2.1919$, $m_1 = 0.3$, $\bar{I} = 1.9915$, $\sigma_2 = 0.5$, $\rho = 0.2$, $\mu_2 = 0.1$, $P = 0.05$, $I_0 = 2.3$.

![Figure 1. Optimal policy, $I_0$, and $C_0$](image1)

![Figure 2. Optimal policy and $m_1$](image2)

5.1. **Optimal policy analysis.** Figure 1 shows the relationship between the optimal policy, initial amount of emissions $I_0$, and the initial price of the emissions allowance. The parameter $I_0 \in [1.6, 2.3]$. Recall that $I_0 \geq \bar{I}$ is the initial amount of emissions that exceeds the cap. First, we can see that the optimal policy curve increases with the price of the emissions allowance in the market, which means that a country should increase its energy-saving emissions reduction efforts if the initial price of the allowance is high. In fact, a higher initial price means that the country may face a higher price at the terminal time $T$, and thus, they should pay a higher cost at $T$ if the amount of emissions exceeds the limit, so the best policy is to vigorously reduce the carbon emissions before the checking time $T$. Second, under conditions where the other parameters are the same, a higher initial amount of emissions $I_0$ means that the country needs to implement more energy-saving measures and make greater efforts to reduce carbon emissions at the beginning.

Figure 2 shows the optimal policy with different $m_1$, which indicates that the optimal policy is higher if $m_1$ is lower, i.e., the country should increase its carbon reduction efforts if the cost of cutting carbon is lower.

Figure 3 shows the optimal policy with different $\bar{q}$, which represents the capability of reducing carbon emissions. A country with a higher capacity for reduction is more capable of adjusting the reduction policy when the price changes. Thus, a country with a lower reduction capability should increase its carbon reduction efforts to respond to possible increases in the price of the allowance in the future.

Figure 4 shows the optimal policy with different values of the penalty standard $P$. The penalty standard $P$ actually provides us with the highest price for purchasing the allowance at $T$. A higher value of $P$ means that a country is likely to pay more
if the amount of emissions exceeds the emission cap $\bar{I}$. Thus, it would be better to enforce a stricter policy to cut emissions.

5.2. Minimum costs analysis. Figure 5 shows the minimum total costs with different values of the initial emission amount $I_0$ and the initial price of the emissions allowance. Under the same conditions, completing more reduction tasks will cost more. Thus, we can only expect more countries to participate in carbon reduction activities if we propose a fair carbon emissions obligation.

Figure 6 shows the minimum total costs with different values of $m_1$. Similar to the conclusions given in Lemma 3.1, the results in Figure 6 imply that the cost will be higher for a country if it has a higher cost of cutting emissions under the same conditions. From a long-term perspective, developing and introducing advanced emissions reduction technologies would be beneficial for reducing the total emissions reduction costs.

Figure 7 shows the minimum total costs with different values of $\bar{q}$. A country with a higher capacity for emissions reduction will incur lower costs to satisfy the same target. In general, developing countries have a greater capacity for reducing carbon
emissions. The Kyoto Protocol stipulates that developed countries can cooperate with developing countries through the Clean Development Mechanism to facilitate reductions in carbon emissions. This method is an economical means of achieving our goals. Figure 8 shows the minimum total costs with different values of the penalty standard $P$, which indicates that a higher penalty standard increase the costs of reduction activities for a country.

6. Conclusions. In this study, we built a stochastic optimal control model to study the optimal carbon reduction and trading policy for a country. By considering carbon abatement, trading, and penalties in a synthetic manner, we established a model under Merton’s consumption framework and we obtained the corresponding HJB equation with the dynamic programming principle. We also proved the existence and uniqueness of the viscosity solution of the equation. Based on numerical calculations, we discussed the properties of the optimal carbon reduction policy and the total minimal costs.

Our results demonstrate that a policy maker should formulate carbon reduction decisions by considering the initial amount of emissions, the price of the allowance, and the penalty level, as well as the capability and efficiency of reducing carbon emissions. According to the results obtained using our model, policy makers can adjust the carbon reduction policy to minimize the total carbon reduction costs.

REFERENCES
[1] F. Black and M. Scholes, The pricing of options and corporate liabilities, Journal of Political Economy, 81 (1973), 637–654.
[2] R. Carmona, M. Fehr and J. Hinz, Optimal stochastic control and carbon price formation, SIAM Journal on Control and Optimization, 48 (2009), 2168–2190.
[3] R. Carmona, M. Fehr, J. Hinz and A. Porchet, Market design for emission trading schemes, SIAM Review, 52 (2010), 403–452.
[4] B. Commoner, The environmental cost of economic growth, In R.G. Ridker (Ed.), Population, Resources and the Environment, Washington, DC, U.S. Government Printing Office, (1972), 339–363.
[5] E. Commission, The EU emissions trading system (EU ETS), 2013, Available from: http://ec.europa.eu/clima/publications/docs/factsheet_ets_en.pdf.
[6] M. G. Crandall and P. L. Lions, Viscosity solutions of Hamilton-Jacobi equations, Transactions of the American Mathematical Society, 277 (1983), 1–42.
[7] M. G. Crandall and P. L. Lions, User’s guide to viscosity solutions of second order partial differential equations, *Bulletin of the American Mathematical Society*, 27 (1992), 1–67.

[8] G. Daskalakis, D. Pychyljios and P. N. Markellos, Modeling CO₂ emission allowance prices and derivatives: Evidence from the European trading scheme, *Journal of Banking and Finance*, 33 (2009), 1230–1241.

[9] T. Dietz and E. A. Rosa, Rethinking the environmental impacts of population, affluence and technology, *Human Ecology Review*, 1 (1994), 277–300.

[10] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, New York, 2 edition, 2006.

[11] H. Guo and J. Liang, An optimal control model for reducing and trading of carbon emissions, *Physica A: Statistical Mechanics and its Applications*, 446 (2016), 11–21, Available from: http://dx.doi.org/10.1016/j.physa.2015.10.076.

[12] C. Hepburn, Carbon trading: A review of the Kyoto mechanisms, *The Annual Review of Environment and Resources*, 32 (2007), 375–393.

[13] R. C. Merton, Theory of rational option pricing, *Bell Journal of Economics and Management Sciences*, 4 (1973), 141–183.

[14] R. C. Merton, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory*, 3 (1971), 373–413.

[15] R. C. Merton, Lifetime portfolio selection under uncertainty: The continuous-time case, *The Review of Economics and Statistics*, 51 (1969), 247–257.

[16] J. Seifert, M. Uhrig-Homburg and M. Wagner, Dynamic behavior of CO₂ spot prices, *Journal of Environmental Economics and Management*, 56 (2008), 180–194.

[17] A. Tsoularis and J. Wallace, Analysis of logistic growth models, *Mathematical Biosciences*, 179 (2002), 21–55.

[18] M. Wang, M. Wang and S. Wang, Optimal investment and uncertainty on China’s carbon emission abatement, *Energy Policy*, 41 (2012), 871–877.

[19] X. Yang and J. Liang, Minimization of the nation cost due to carbon emission, *Systems Engineering - Theory and Practice*, 34 (2014), 640–647.

[20] X. Yang, Optimal control problems associated with carbon emission abatement and leveraged credit derivatives, Ph. D Thesis, Tongji University, 2015.

[21] E. Zagheni and F. C. Billari, A cost valuation model based on a stochastic representation of the IPAT equation, *Population and Environment*, 29 (2007), 68–82.

Received October 2015; revised January 2016.

E-mail address: wosghy@163.com
E-mail address: liang_jin@tongji.edu.cn