Clifford-symmetric polynomials

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ABSTRACT

Based on the NilHecke algebra $NH_n$, the odd NilHecke algebra developed by Ellis, Khovanov and Lauda, and on Kang, Kashiwara and Tsuchioka’s quiver Hecke superalgebra, we develop the Clifford Hecke superalgebra $NH_C_n$ as another super-algebraic analogue of $NH_n$. We show that there is a notion of symmetric polynomials fitting in this picture, and we prove that these are generated by an appropriate analogue of elementary symmetric polynomials, whose properties we shall discuss in this text.

1. Introduction

For the entire paper, let $k$ be a field with char $k \neq 2$. Consider the polynomial ring $Pol_n := k[x_1, \ldots, x_n]$. The symmetric group $S_n$ acts on $Pol_n$ by interchanging indeterminates. Let $\alpha_i = x_{i+1} - x_i$; then $Pol_n$ has a decomposition $Pol_n = Pol_n^+ \oplus Pol_n^-$ into direct summands on which the simple transposition $s_i$ acts by $id$ and $-id$, respectively.

The NilHecke algebra $NH_n$ of type $A_n$ is the $k$-subalgebra of $End_k(Pol_n)$ generated by multiplication with the indeterminates $x_i$, and by the Demazure operators $\partial_i : Pol_n \to Pol_n^+, f \mapsto \frac{f - s_i(f)}{x_{i+1} - x_i}$, for $i = 1, \ldots, n - 1$. The latter assigns to a polynomial its $\partial_i$-invariant part and satisfy $\partial_i^2 = 0$. Thus, $NH_n$ is the $k$-algebra generated by $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_{n-1}$ and subject to the relations

$$x_ix_j = x_jx_i \text{ for all } 1 \leq i, j \leq n,$$

$$\partial_i x_j - x_j \partial_i = -1 \text{ for all } 1 \leq i < j \leq n,$$

$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \text{ for all } 1 \leq i < n - 1,$$

which acts faithfully on the polynomial ring $Pol_n$. The subalgebra $NC_n$ of $NH_n$ generated by the symbols $\partial_1, \ldots, \partial_{n-1}$ is called NilCoxeter algebra.

The invarient algebra $\Delta Pol_n := Pol_n^{\partial n}$, whose elements are called symmetric polynomials, is the largest subalgebra of $Pol_n$ annihilated by $NC_n$. As a $k$-algebra, $\Delta Pol_n$ is isomorphic to the polynomial ring $k[e_1^{(n)}, \ldots, e_n^{(n)}]$ in the elementary symmetric polynomials

$$e_k^{(n)} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad \text{(1.1)}$$

and to the polynomial ring $k[h_1^{(n)}, \ldots, h_n^{(n)}]$ in the complete symmetric polynomials

$$h_k^{(n)} = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}. \quad \text{(1.2)}$$
As a $\Lambda \text{Pol}_n$-algebra, $\text{Pol}_n$ is free of rank $n$!

Ellis, Khovanov and Lauda have developed a theory of odd symmetric polynomials and have constructed an odd analogue $\text{ONH}_n$ of the NilHecke algebra [7]. By odd, it is meant that odd polynomials are generated by indeterminates $\xi_1, \ldots, \xi_n$ that, instead of commutativity, satisfy the relation $\xi_i \xi_j = - \xi_j \xi_i$ whenever $i \neq j$. Our goal is to mimic the theory of symmetric and odd symmetric polynomials for a super polynomial algebra $\text{Pol}_I$ associated to a $\mathbb{Z}/2\mathbb{Z}$-graded index set $I = I_0 \sqcup I_1$. For $i \in I$, let $|i|$ be the index such that $i \in I_{|i|}$, called the parity of $i$. Then $\text{Pol}_I$ is the $\mathbb{Z}/2\mathbb{Z}$-graded algebra generated by indeterminates $\xi_i$ for $i \in I$, that are subject to the relations $\xi_i \xi_j = (-1)^{|i||j|} \xi_j \xi_i$.

In Section 2, we start with a brief review on super algebras and the construction of a quiver Hecke Clifford superalgebra $\text{NH}\mathcal{C}(C)$ from [11] in order to construct a faithful polynomial for $\text{NH}\mathcal{C}(C)$ along the lines of [12] in Section 3.

In Section 4, we develop our theory of Clifford polynomials $\text{Pol}\mathcal{C}(I)$ and Clifford symmetric polynomials $\Lambda \text{Pol}\mathcal{C}(I)$. We show that the latter are generated by a generalization of elementary symmetric polynomials and that $\text{Pol}\mathcal{C}(I)$ is a free $\Lambda \text{Pol}\mathcal{C}(I)$-module. Our main results, which are stated in Theorem 4.17, lead us to a super-analogue for the construction of cyclotomic quotients of the NilHecke algebra and for the cohomology rings of Grassmannians and flag varieties, which generalize from the purely odd constructions of the respective notions defined in [7].

2. Superalgebras, supermodules and supercategories

A $k$-super vector space $V$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $k$-vector space $V = V_0 \oplus V_1$. The $\mathbb{Z}/2\mathbb{Z}$-degree of a homogeneous element $m$ is called its parity, which is denoted by $|m|$. A homogeneous element is called even and odd according to its parity. The dimension of $V$ is written $\dim_k V = \dim_k V_0 | \dim_k V_1$, and we write $k^{n|m}$ for the canonical $m|n$-dimensional $k$-super vector space. A morphism $f : V \to W$ of super vector spaces is a morphism of vector spaces that satisfies $f(V_p) \subseteq W_p$ for all $p \in \mathbb{Z}/2\mathbb{Z}$. We denote the thus defined category of super vector spaces by $k\text{S Vect}$.

For $V, W \in k\text{S Vect}$, we denoted by $\text{hom}_k(V, W)$ the super vector space with $\text{hom}_k(V, W)_p = \{ f \in \text{Hom}_k(V, W) \mid f(V_q) \subseteq W_{p+q} \forall q \in \mathbb{Z}/2\mathbb{Z} \}$; i.e., the super vector space whose even and odd parts consist of parity-preserving and -exchanging morphisms, respectively. In particular, $\text{Hom}_{k\text{S Vect}}(-, -) = \text{hom}_k(-, -)_0$.

The tensor product $V \otimes_k W$ with components $(V \otimes_k W)_p := \bigoplus_{q \in \mathbb{Z}/2\mathbb{Z}} V_q \otimes W_{p-q}$ fits into the usual adjunction with $\text{hom}_k(-, -)$. Furthermore, with the braiding $V \otimes W \to W \otimes V$, $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$, $k\text{S Vect}$ becomes a closed symmetric monoidal category; i.e., the tensor product of morphisms satisfies

\begin{equation}
(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w) \quad (f \otimes g) \circ (f' \otimes g') = (-1)^{|f'||g'|} f' \otimes g' \quad (2.1)
\end{equation}

for all homogeneous elements $v \in V, w \in W$ and homogeneous morphisms $f \in \text{hom}_k(V', V), f' \in \text{hom}_k(V'', V'), g \in \text{hom}_k(W', W)$ and $g' \in \text{hom}_k(W'', W')$.

Remark 2.1. Given finite dimensional super vector spaces $V = k^{n|m}$ and $W = k^{p|q}$ with homogeneous bases, a homomorphism in $\text{hom}_k(V, W)$ can be written as an $(m|n) \times (p|q)$-block matrix

\[
\begin{pmatrix}
  m & n \\
  p & q \\
\end{pmatrix}
\begin{pmatrix}
  T_{0,0} & T_{0,1} \\
  T_{1,0} & T_{1,1} \\
\end{pmatrix}
\]

where $\text{hom}_k(V, W)_0$ and $\text{hom}_k(V, W)_1$ comprise the diagonal and the anti-diagonal blocks, respectively.

2.1. Superalgebras

Definition 2.2. A superalgebra $A$ over a field $k$ is a super vector space $A$ with a graded $k$-algebra structure. It is called supercommutative or simply commutative if the multiplication commutes with the braiding; i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous $a, b \in A$. In particular, this means that in a commutative
superalgebra, every odd element squares to zero. The tensor product of two superalgebras \( A \) and \( B \) (over \( k \)) carries a superalgebra structure by \((a \otimes b) \cdot (a' \otimes b') = (-1)^{|a||b|}aa' \otimes bb'\).

**Example 2.3.** For every super vector space \( V \), the space \( \text{hom}_k(V, V) \) is a superalgebra. In general, it is not commutative, unless \( V = k^{1|0} \) or \( V = k^{0|1} \). For super vector spaces \( V \) and \( W \), the space \( \text{hom}_k(V, V) \otimes \text{hom}_k(W, W) \subseteq \text{hom}_k(V \otimes W, V \otimes W) \) is a super subalgebra, according to (2.1).

**Example 2.4.** The exterior algebra on \( n \) generators is the \( k \)-algebra

\[
\Lambda[\omega_1, \ldots, \omega_n] := k\langle \omega_1, \ldots, \omega_n \rangle/(\omega_i\omega_j + \omega_j\omega_i)_{1 \leq i,j \leq n}.
\]

It is a commutative superalgebra. The polynomial superalgebra on \( m|n \) indeterminates is the commutative superalgebra \( k[y_1, \ldots, y_m|\omega_1, \ldots, \omega_n] := k[y_1, \ldots, y_m] \otimes_k \Lambda[\omega_1, \ldots, \omega_n] \).

**Example 2.5.** The Clifford algebra on \( n \) generators is the \( k \)-superalgebra

\[
\mathbb{C}_n := \langle c_1, \ldots, c_n \mid c_i^2 = 1 \text{ and } c_ic_j = -c_jc_i \text{ if } i \neq j \rangle
\]

(2.2)

all of whose generators \( c_i \) are odd. It is not supercommutative.

**Definition 2.6.** Given two \( k \)-superalgebras \( A \) and \( B \), an \( A-B \)-super bimodule \( M \) is a super vector space endowed with even superalgebra morphisms \( A \to \text{hom}_k(M, M) \) and \( B^{op} \to \text{Hom}_k(M, M) \). According to (2.1), a morphism \( f \) of bimodules satisfies \( f(amb) = (-1)^{|f||a|}af(m)b \) for \( a \in A, b \in B \). We denote the category of \( A-B \)-super bimodules by \( A[B] \).

One-sided modules are defined in the obvious way. Over a supercommutative superalgebra, a left module obtains right module structure over the opposite superalgebra by imposing \( ma = (-1)^{|a||m|}am \). Tensor products of bimodules over superalgebras are defined in the usual way.

### 2.2. Supercategories

**Definition 2.7.** A \( k \)-linear supercategory, superfunctor and supernatural transformation respectively are a category, a functor and a natural transformation enriched in the monoidal category \( k \)-sVect. Explicitly:

- A supercategory \( C \) is a category such that \( \text{Hom}_C(X, Y) \) is a \( k \)-super vector space for all \( X, Y \in C \), and composition is an even map

\[
\circ_{X,Y,Z} : \text{Hom}_C(X, Y) \otimes_k \text{Hom}_C(Y, Z) \to \text{Hom}_C(X, Z)
\]

of \( k \)-super vector spaces for all \( X, Y, Z \in k \)-sVect.

- A superfunctor \( F : C \to D \) of supercategories is a functor such that the map \( F_{XY} : \text{Hom}_C(X, Y) \to \text{Hom}_D(FX, FY) \) is an even map of super vector spaces for all \( X, Y \in C \).

- Let \( F, G : C \to D \) be two superfunctors, then the space \( \text{hom}(F, G) \) of supernatural transformations from \( F \) to \( G \) is the super vector space such that \( \text{hom}(F, G)_p \) is the \( k \)-vector space of all (ordinary) natural transformations that satisfy \( \eta_Y \circ F(f) = (-1)^{p|f|} \circ G(f) \circ \eta_X \) for all \( X, Y \in C \) and homogeneous \( f \in \text{Hom}_C(X, Y) \).

A supercategory \( C \) that is also a monoidal category is a **monoidal supercategory** if its associators and unitors are even supernatural transformations and satisfy (2.1); It is **closed and symmetric** if it is as an ordinary monoidal category; see [2] for a comprehensive treatment.

**Example 2.8.** The super vector spaces \( \text{hom}_k(\_, \_) \) turn \( k \)-sVect into a closed symmetric monoidal supercategory.
Example 2.9. The category $A$ of modules over a super $k$-algebra $A$ is a closed symmetric monoidal supercategory, where $\text{Hom}_A(M, N)$ is defined as the equalizer

$$\text{Hom}_A(M, N) := \text{eq}(\text{hom}_k(M, N) \xrightarrow{a^*} \prod_{a \in A} \text{hom}_k(M, N)).$$

The forgetful functor $A \to k$-$s\text{Vect}$ is a superfunctor.

2.3. Super-diagrams

We draw vertical lines for morphisms; these are placed next to each other to represent their tensor product. The super interchange law (2.1) is thus depicted as

$$f \otimes g = (f \otimes 1) \circ (1 \otimes g) = (-1)^{|f||g|}(1 \otimes g) \circ (f \otimes 1).$$

Such diagrams are to be understood as \textit{first vertical and then horizontal composition} of morphisms [cf. 2].

3. Super-KLR algebras

We now recall the notion of KLR-superalgebras following [11]. This is a generalization of the ordinary KLR-algebra from [12, 19].

3.1. NilHecke superalgebra

Recall the Clifford algebra $C_n$ from Example 2.5. The following definitions are motivated by the definition of the quiver Hecke Clifford superalgebra from [11, Section 3.3]; see Definition 3.11.

Definition 3.1. For any field $k$, the \textit{NilHecke Clifford superalgebra} $\mathcal{NH}_C$, of type $A_n$ is the $k$-superalgebra with even generators $y_1, \ldots, y_n$ and $\partial_1, \ldots, \partial_{n-1}$, and odd generators $c_1, \ldots, c_n$, subject to the relations

$$y_iy_j = y_jy_i \quad \forall i, j, \quad c_i^2 = 1 \quad \forall i, \quad y_1c_j = (-1)^{\delta_{ij}}c_1y_i \quad \forall i, j, \quad \partial_iy_j - y_j\partial_i = \begin{cases} -1 - c_ic_{i+1} & \text{if } j = i \\ 1 - c_ic_{i+1} & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

(3.1)

and the NilCoxeter-relations

$$\partial_i^2 = 0 \quad \partial_i\partial_j = \partial_j\partial_i \quad \text{for } |i - j| > 1 \quad \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}.$$ 

(3.2)

The \textit{NilCoxeter Clifford superalgebra} $\mathcal{NC}_C \subseteq \mathcal{NH}_C$ is the subalgebra generated by the symbols $\partial_i$ and $c_j$.

Definition 3.2. Let $A$ be a superalgebra over a field $k$. An $A$-superalgebra $B$ is an $A$-$A$-bimodule equipped with an associative multiplication map $B \otimes_A B \to B$ that is an even $A$-$A$-bimodule homomorphism.

We can consider $\mathcal{NH}_C$ as a $C_n$-superalgebra generated by the even symbols $y_i$ and $\partial_i$. We introduce the following analogues to the polynomial representation of the classical NilHecke algebra:
**Definition 3.3.** The polynomial Clifford superalgebra \( \text{Pol}\mathcal{C}_n \) is the \( \mathcal{C}_n \)-superalgebra with generators and relations
\[
\text{Pol}\mathcal{C}_n := \langle y_1, \ldots, y_n \mid y_i y_j = y_j y_i, \, y_i c_j = (-1)^{\delta_{ij}} c_j y_i, \rangle_{\mathcal{C}_n}.
\]

**Lemma 3.4.** \( \text{Pol}\mathcal{C}_n \) is a free as left and right \( \mathcal{C}_n \)-module with basis \( \{ y_1^{\alpha_1} \cdots y_n^{\alpha_n} \mid \alpha_i \in \mathbb{N} \} \). In particular, \( \text{Pol}\mathcal{C}_n \) has the same graded rank as left and right \( \mathcal{C}_n \)-module.

**Proof.** Take any monomial from \( \text{Pol}\mathcal{C}_n \). According to the defining relation \( y_i c_j = (-1)^{\delta_{ij}} c_j y_i \) of \( \text{Pol}\mathcal{C}_n \), we may move all symbols \( c_j \) to the left (or right), possibly at the cost of introducing a sign when sliding them past a symbol \( y_i \) with the same index. Since the \( y_i \)'s commute, we may sort powers of \( y_i \)'s by their index. As a left (or right) \( \mathcal{C}_n \)-module, \( \text{Pol}\mathcal{C}_n \) is thus isomorphic to \( \mathcal{C}_n \otimes_k \text{Pol}_n \) (or \( \text{Pol}_n \otimes_k \mathcal{C}_n \)). \( \square \)

**Definition 3.5.** As a Clifford-analogue for the Demazure operator \( \partial_i \) on \( \text{Pol}_n \), we define an even endomorphism \( \partial_i \) of \( \text{Pol}\mathcal{C}_n \) by
\[
\partial_i(y_i) := -1 - c_i c_{i+1} \quad \forall i,
\]
\[
\partial_i(y_{i+1}) := 1 - c_i c_{i+1} \quad \forall i,
\]
such that \( \partial_i \) is a \( s_i \)-derivation, i.e.,
\[
\partial_i(fg) := \partial_i(f)g + s_i(f)\partial_i(g) \quad \forall j \neq i, i + 1
\]
Here, the symmetric group \( S_n \) acts on \( \text{Pol}\mathcal{C}_n \) by permuting the symbols \( y_i \) and \( c_j \) independently.

**Lemma 3.6.** The Clifford Demazure operator \( \partial_i \) is a well-defined morphism of right \( \mathcal{C}_n \)-modules.

**Proof.** To show well-definedness inductively, take an \( f \in \text{Pol}\mathcal{C}_n \). Without loss of generality, we assume that \( i = 1 \). We show that \( \partial_1 \) is compatible with the relations of \( \text{Pol}\mathcal{C}_n \).

- Relations involving only symbols \( y_j \): since
\[
\partial_1(y_1y_2) = (-1 - c_1 c_2)y_2 + y_2(1 - c_1 c_2) = 0,
\]
\[
\partial_1(y_2y_1) = (1 - c_1 c_2)y_1 + y_1(-1 - c_1 c_2) = 0,
\]
we get for all \( f \) that
\[
\partial_1(y_1y_2f) = y_1y_2\partial_1(f) = y_2y_1\partial_1(f) = \partial_1(y_2y_1f).
\]
- Relations involving symbols \( y_j \) and \( c_j \): since \( \partial_1(c_1y_1) = c_2(-1 - c_1 c_2) = c_1 - c_2 = -\partial_1(y_1c_1) \), we get
\[
\partial_1(c_1y_1f) = \partial_1(c_1y_1)f + c_2y_2\partial_1(f) = -\partial_1(y_1c_1)f - y_2c_2\partial_1(f) = -\partial_1(y_1c_1f).
\]
The other relations \( \partial_1(c_2y_1f) = -\partial_1(y_1c_2f) \) and \( \partial_1(y_2c_1f) = \partial_1(c_2y_2f) \) and \( \partial_1(c_2y_2f) = -\partial_1(y_2c_2f) \) are shown similarly.

It follows by induction on the length of \( f \) (in terms of the generators) that \( \partial_i \) is well-defined. The operator \( \partial_i \) is a right \( \mathcal{C}_n \)-module homomorphism because \( \partial_i(f c_j) = \partial_i(f)c_j + s_i(f)\partial_i(c_j) = \partial_i(f)c_j \) for all \( f \in \text{Pol}\mathcal{C}_n \) and \( 1 \leq j \leq n \).

**Remark 3.7.** The kernel \( \ker \partial_i \) is a \( \mathcal{C}_n \)-super subalgebra of \( \text{Pol}\mathcal{C}_n \) because \( \partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g) = 0 \) for \( f, g \in \ker \partial_i \). Trivially, we have \( \mathcal{C}_n \subseteq \ker \partial_i \).

**Lemma 3.8.** 1. The Clifford Demazure operators \( \partial_i \) and multiplications by \( y_i \) and \( c_i \) render \( \text{Pol}\mathcal{C}_n \) a representation of the NilHecke Clifford superalgebra.

2. \( \ker \partial_i = \text{im} \partial_i \).

**Proof.** 1. We have to make sure that the Clifford Demazure operators satisfy the relations from Definition 3.1. From the definition of \( \partial_i \) one can see that the relations (3.1) are satisfied. We show that the operators \( \partial_i \) satisfy (3.2):

\[
\partial_i y_j = -y_j \partial_i + c_j c_{j+1} \quad \forall j < i,
\]
\[
\partial_i c_j = c_j \partial_i - c_i c_{i+1} \quad \forall j > i.
\]
To show $\partial_i^2(f) = 0$: Assume that the statement holds for $f, g \in \text{Pol}\mathcal{C}_n$. It follows from
\[
\partial_i^2(fg) = \partial_i(\partial_i(f)g + s_i(f)\partial_i(g)) = s_i(\partial_i(f))\partial_i(g) + \partial_i(s_i(f)\partial_i(g))
\]
that it suffices to show that $s_i\partial_i = -\partial_i s_i$. To show this inductively, assume that $s_i\partial_i(f) = -\partial_i(s_i f)$. We need to show that $s_i\partial_i(y_j f) = -\partial_i(s_i(y_j f))$, which is obvious for $j \neq i$. Applying the induction hypothesis in * below, one calculates
\[
s_i\partial_i(y_j f) = s_i\left[(-1 - c_i c_{i+1}) f + y_{i+1} \partial_i(f)\right] = (-1 + c_i c_{i+1}) s_i f - y_{i+1} \partial_i s_i f = -\partial_i(y_{i+1} s_i f),
\]
The calculation for $j = i + 1$ is similar.

To show $\partial_i\partial_j(f) = \partial_j\partial_i(f)$ for $|i - j| > 1$: Clear from the definition of $\partial_i$.

To show $\partial_i\partial_{i+1}\partial_i(f) = \partial_{i+1}\partial_i\partial_i(f)$: Assume the statement holds for $f \in \text{Pol}\mathcal{C}_n$. It is clear that the statement then also holds true for $c_j f$ and $y_j f$ for all $j \neq i, i+1, i+2$. For these cases, we assume w.l.o.g. that $i = 1$. Applying the induction hypothesis in *, one computes
\[
\partial_1\partial_2\partial_1(y_1 f) = \partial_1\partial_2((-1 - c_1 c_2) f + y_2 \partial_2 f) = \partial_1((-1 - c_1 c_3) \partial_2 f + (-1 - c_2 c_3) \partial_1 f + y_3 \partial_2 \partial_1 f) = (-1 - c_2 c_3) \partial_1 \partial_2 f + y_3 \partial_1 \partial_2 \partial_1 f = \partial_2((-1 - c_1 c_2) \partial_2 f + y_2 \partial_1 \partial_2 f) = \partial_2 \partial_1(y_1 \partial_2 f) = \partial_2 \partial_1 \partial_2(y_1 f);
\]
\[
\partial_1\partial_2\partial_1(y_2 f) = \partial_1\partial_2((-1 - c_1 c_2) f + y_1 \partial_1 f) = \partial_1((-1 - c_1 c_3) \partial_2 f + y_1 \partial_2 \partial_1 f) = (-1 - c_2 c_3) \partial_1 \partial_2 f + (-1 - c_1 c_2) \partial_2 \partial_1 f + y_2 \partial_1 \partial_2 \partial_1 f = \partial_2((-1 - c_1 c_3) \partial_1 f + y_3 \partial_1 \partial_2 f) = \partial_2 \partial_1((-1 - c_2 c_3) f + y_3 \partial_2 f) = \partial_2 \partial_1 \partial_2(y_2 f);
\]
\[
\partial_1\partial_2\partial_1(y_3 f) = \partial_1\partial_2(y_3 \partial_1 f) = \partial_1((-1 - c_2 c_3) f + y_2 \partial_2 \partial_1 f) = (1 - c_1 c_2) \partial_2 \partial_1 f + y_1 \partial_1 \partial_2 \partial_1 f = \partial_2((-1 - c_1 c_3) \partial_1 f + y_1 \partial_1 \partial_2 f) = \partial_2 \partial_1(1 - c_2 c_3) f + y_2 \partial_2 f = \partial_2 \partial_1 \partial_2(y_3 f).
\]

Therefore, $\text{Pol}\mathcal{C}_n$ is a representation of $\text{NH}\mathcal{C}_n$. We show in Lemma 4.16 that it is faithful.

2. Because $\partial_i^2 = 0$, we may regard $(\text{Pol}\mathcal{C}_n, \partial_i)$ as a chain complex of right $\mathcal{C}_n$-modules, using its polynomial grading in the $y_i$'s. We claim that this chain complex $\text{Pol}\mathcal{C}_n$ is contractible by the homotopy
\[
h_k: (\text{Pol}\mathcal{C}_n)_k \to (\text{Pol}\mathcal{C}_n)_{k+1, f} \mapsto \begin{cases} 
\frac{1}{2}(-1 - c_i c_{i+1}) y_j f & \text{if } k \text{ is even,} \\
\frac{1}{2} (+1 - c_i c_{i+1}) y_{i+1} f & \text{if } k \text{ is odd.}
\end{cases}
\]

Assume $f$ is even. We obtain
\[
h_{k-1}(\partial_i f) + \partial_i h_k(f) = \frac{1}{2}(1 - c_i c_{i+1}) y_{i+1}(\partial_i f) + \frac{1}{2} \partial_i((-1 - c_i c_{i+1}) y_j f)
\]
The computation is similar if \( f \) is odd. Therefore, \(-h\) indeed is a homotopy from the identity to the zero morphism, and the chain complex \((\Pol C_n, \partial_i)\) is acyclic. This proves the statement. \(\square\)

### 3.2. Quiver Hecke (Clifford) superalgebra

Let \( I = I_0 \sqcup I_1 \) be a finite, \( \mathbb{Z}/2\mathbb{Z} \)-graded set as in the introduction.

**Definition 3.9.** [11, Section 3] A generalized Cartan matrix on \( I \) is a matrix \( C = (d_{ij}) \in \mathbb{Z}^{I \times I} \) such that

1. \( d_{ii} = 2 \) for all \( i \in I \),
2. \( d_{ij} \leq 0 \) for distinct \( i, j \in I \),
3. \( d_{ij} = 0 \) if and only if \( d_{ji} = 0 \), and
4. \( d_{ij} \) is even if \( i \in I_1 \).

We define a \( \mathbb{Z} \)-valued bilinear form on \( N^I \) by \( (i,j) := -d_{ij} \). Equivalently, we may describe the same datum by a graph \( \Gamma_C \) with vertex set \( I \) and \( d_{ij} \) directed edges from \( i \) to \( j \) for \( i \neq j \). For distinct indices \( i, j \), we write \( i - j \) if \( d_{ij} \neq 0 \) and \( i \neq j \) if \( d_{ij} = 0 \).

For the remainder of this section, let \( C \in \mathbb{Z}^{I \times I} \) be a symmetrisable generalized Cartan matrix and fix an orientation for each edge \( i - j \) of \( \Gamma_C \). We write \( i \to j \) if the edge is oriented from \( i \) to \( j \).

The following diagrammatic calculus for the quiver Hecke super algebra from [11, Section 3] mimics the constructions from [12, 14].

**Definition 3.10.** A string diagram with \( n \) strings consists of \( n \) continuous paths \( \phi_k : [0, 1] \hookrightarrow \mathbb{R} \times [0, 1] \) for \( 1 \leq k \leq n \), called strings, such that

1. each string starts in \( N \times \{0\} \) and ends in \( N \times \{1\} \),
2. the projection of any string to the second coordinate is strictly monotonically increasing,
3. each endpoint has precisely one string attached to it, and
4. at most two strings may intersect in a single point.

A string diagram is defined up to isotopy. Strings in a diagram may be endowed with certain point-like decorations, whose positions are also specified up to isotopy. Crossings and decorations are endowed with parities. Unless decorations commute, we make sure they appear at distinct heights in one diagram. Strings are numbered from the left at their bottom endpoints.

**Definition 3.11.** \((\tilde{H}_n(C), \text{diagrammatically})\). The quiver Hecke superalgebra \( \tilde{H}_n(C) \) is the \( k \)-linear supercategory consisting of the following data.

- Its objects are sequences \( v \in I^n \).
- If \( v' \) is not a permutation of \( v \), then \( \Hom(v, v') := 0 \). Otherwise, \( \Hom(v, v') \) is generated by string diagrams on \( n \) strings which connect identical entries of \( v \) and \( v' \). We say that the \( k \)-th string is labeled by \( v_k \) and usually write its label below the string. Strings may be decorated by an arbitrary non-negative number of dots, distant from the crossings.
- The dots \( x_{k,v} := \bullet_v \) and crossings \( r_{k,v} := v_k \times v_{k+1} \) have parities \( |x_{k,v}| := |v_k| \) and \( |r_{k,v} \times v_{k+1}| := |v_k| |v_{k+1}| \).
• Composition is given by vertically stacking diagrams and subject to the following local relations:

\[
\begin{align*}
\bullet_i \bullet_j = \bullet_j \bullet_i = (-1)^{|i||j|} \bullet_i \bullet_j, \quad \forall i, j \\
\end{align*}
\]  
\[\text{(3.3)}\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j \\
\end{array}
\end{array} - (-1)^{|i||j|} \begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j
\end{array}
\end{array} &= \begin{cases}
0 & \text{if } i \neq j \\
\bullet_i \bullet_j & \text{if } i = j
\end{cases} \\
\begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j \\
\end{array}
\end{array} - (-1)^{|i||j|} \begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j
\end{array}
\end{array} &= \begin{cases}
0 & \text{if } i \neq j \\
\bullet_i \bullet_j & \text{if } i = j
\end{cases} \\
\begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j \\
\end{array}
\end{array} - (-1)^{|i||j|} \begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j
\end{array}
\end{array} &= \begin{cases}
0 & \text{if } i = j \\
\bullet_i \bullet_j & \text{if } i \neq j
\end{cases}
\end{align*}
\]  
\[\text{(3.4)}\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j \\
\end{array}
\end{array} - (-1)^{|i||j|} \begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j
\end{array}
\end{array} &= \begin{cases}
0 & \text{if } i = j \\
\bullet_i \bullet_j & \text{if } i \neq j
\end{cases} \\
\begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j \\
\end{array}
\end{array} - (-1)^{|i||j|} \begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j
\end{array}
\end{array} &= \begin{cases}
0 & \text{if } i = j \\
\bullet_i \bullet_j & \text{if } i \neq j
\end{cases}
\end{align*}
\]  
\[\text{(3.5)}\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j \\
\end{array}
\end{array} - (-1)^{|i||j|} \begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j
\end{array}
\end{array} &= \begin{cases}
0 & \text{if } i = k \text{ and } i - j \\
\bullet_i \bullet_j & \text{otherwise}
\end{cases}
\end{align*}
\]  
\[\text{(3.6)}\]

The exponents in (3.6) are meant as powers; i.e., as repetition of the symbol.

**Remark 3.12.** There is an embedding of \( \tilde{H}_n(C) \) into Brundan’s diagrammatic super Kac-Moody 2-category [3] by adding an arrow tip pointing upwards to every string.

For calculations it is advantageous to *adjoin* odd generators for all odd indices \( i \in I_1 \) instead of endowing the \( x_i \)'s with a super grading themselves. This is captured by the superalgebra defined in the following diagrammatically along the lines of [11].

**Definition 3.13 ((\( \text{HC}_n(C) \), diagrammatically)).** With the same data as above, the *quiver Hecke Clifford superalgebra* \( \text{HC}_n(C) \) is the supercategory with the following data:

- It has the same objects \( v \in I^n \) as \( \tilde{H}_n(C) \).
- It has the generating morphisms \( y_{k,v} = \bullet_{v} \) and \( \sigma_{k,v} = v_{k} \bigotimes v_{k+1} \) of even parity, and the generating morphism \( c_{k,v} = v_{k} \) of odd parity. We refer to the symbols \( y_{k,v} \) and \( c_{k,v} \) as *diamonds*, and to symbols \( \sigma_{k,v} \) as *crossings*.
- The morphisms are subject to the relations \( c_{k,v} = 0 \) if \( |v_{k}| = 0 \), \( \sigma_{i,v} = \sigma_{i,v} \) the commutativity relations \( \begin{array}{c}
\begin{array}{c}
\bullet_i \bullet_j \\
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\bullet_j \bullet_i
\end{array}
\end{array} = \begin{cases}
0 & \text{if } i \neq j, \\
\bullet_i \bullet_j & \text{if } i = j
\end{cases} \\
\end{array} \)  
\[\text{(3.7)}\]
\[
\begin{align*}
\begin{cases}
0 & \text{if } i \neq j, \\
- & \text{if } i = j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{for all } i, j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{if } i = k \text{ and } i - j, \\
0 & \text{otherwise.}
\end{cases} & \quad \text{if } i = k \text{ and } i - j,
\end{align*}
\]

(3.9) for all \( i, j \), (3.10) if \( i = k \) and \( i - j \), (3.12)

For every odd index, \( \mathbb{H}^*_n(C) \) contains a copy of \( \mathbb{NH}^*_n \).

**Lemma 3.14.** Choose numbers \( \gamma_{ij} \in k \) such that \( \gamma_{ij} = 1 \) if at least one of \( i, j \) has even parity, \( \gamma_{ii} = \frac{1}{2} \) if \( i \) has odd parity and \( \gamma_{ij}\gamma_{ji} = -\frac{1}{2} \) otherwise. There is a morphism

\[
\iota : \tilde{H}_n(C) \longrightarrow \mathbb{H}^*_n(C),
\]

\[
\begin{align*}
\tau_{k,v} & \longmapsto \gamma_{v_k,v_{k+1}}(c_{k,v} - c_{k+1,v})|v_k||v_{k+1}|\sigma_{k,v};
\end{align*}
\]

see [11, thm. 3.13].

**Proof.** We verify the well-definedness of this map. We have to check compatibility with the following relations:

- Compatibility with (3.4): If one of the indices \( i \) and \( j \) has even parity, domain and codomain locally reduce to the ordinary KLR-algebra, so nothing remains to prove. Let us thus assume that both \( i \) and \( j \) have odd parity.

\[
\frac{1}{y_j} \iota(\tau_{i,v}x_{k+1,v} + \tau_{i,v}x_{k+1,v}) = \frac{1}{y_j} \iota \left( \begin{array}{c}
\begin{cases}
0 & \text{if } i \neq j, \\
- & \text{if } i = j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{for all } i, j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{if } i = k \text{ and } i - j, \\
0 & \text{otherwise.}
\end{cases} & \quad \text{if } i = k \text{ and } i - j,
\end{array}
\right)
\]

(3.9) for all \( i, j \), (3.10) if \( i = k \) and \( i - j \), (3.12)

by (3.10). If \( i \neq j \), then (3.8) asserts that we can slide all diamonds across the crossing. According to the commutativity relations for \( \dagger \)'s and \( \ddagger \)'s, we obtain for \( i \neq j \):

\[
\begin{cases}
\begin{cases}
0 & \text{if } i \neq j, \\
- & \text{if } i = j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{for all } i, j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{if } i = k \text{ and } i - j, \\
0 & \text{otherwise.}
\end{cases} & \quad \text{if } i = k \text{ and } i - j,
\end{cases}
\]

(3.9) for all \( i, j \), (3.10) if \( i = k \) and \( i - j \), (3.12)

by (3.10). If \( i \neq j \), then (3.8) asserts that we can slide all diamonds across the crossing. According to the commutativity relations for \( \dagger \)'s and \( \ddagger \)'s, we obtain for \( i \neq j \):

\[
\begin{cases}
\frac{1}{y_j} \iota(\tau_{i,v}x_{k+1,v} + \tau_{i,v}x_{k+1,v}) = \left( \begin{array}{c}
\begin{cases}
0 & \text{if } i \neq j, \\
- & \text{if } i = j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{for all } i, j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{if } i = k \text{ and } i - j, \\
0 & \text{otherwise.}
\end{cases} & \quad \text{if } i = k \text{ and } i - j,
\end{array}
\right)
\end{cases}
\]

(3.9) for all \( i, j \), (3.10) if \( i = k \) and \( i - j \), (3.12)

by (3.10). If \( i \neq j \), then (3.8) asserts that we can slide all diamonds across the crossing. According to the commutativity relations for \( \dagger \)'s and \( \ddagger \)'s, we obtain for \( i \neq j \):

\[
\begin{cases}
\frac{1}{y_j} \iota(\tau_{i,v}x_{k+1,v} + \tau_{i,v}x_{k+1,v}) = \left( \begin{array}{c}
\begin{cases}
0 & \text{if } i \neq j, \\
- & \text{if } i = j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{for all } i, j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{if } i = k \text{ and } i - j, \\
0 & \text{otherwise.}
\end{cases} & \quad \text{if } i = k \text{ and } i - j,
\end{array}
\right)
\end{cases}
\]

(3.9) for all \( i, j \), (3.10) if \( i = k \) and \( i - j \), (3.12)

by (3.10). If \( i \neq j \), then (3.8) asserts that we can slide all diamonds across the crossing. According to the commutativity relations for \( \dagger \)'s and \( \ddagger \)'s, we obtain for \( i \neq j \):

\[
\begin{cases}
\frac{1}{y_j} \iota(\tau_{i,v}x_{k+1,v} + \tau_{i,v}x_{k+1,v}) = \left( \begin{array}{c}
\begin{cases}
0 & \text{if } i \neq j, \\
- & \text{if } i = j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{for all } i, j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{if } i = k \text{ and } i - j, \\
0 & \text{otherwise.}
\end{cases} & \quad \text{if } i = k \text{ and } i - j,
\end{array}
\right)
\end{cases}
\]

(3.9) for all \( i, j \), (3.10) if \( i = k \) and \( i - j \), (3.12)

by (3.10). If \( i \neq j \), then (3.8) asserts that we can slide all diamonds across the crossing. According to the commutativity relations for \( \dagger \)'s and \( \ddagger \)'s, we obtain for \( i \neq j \):

\[
\begin{cases}
\frac{1}{y_j} \iota(\tau_{i,v}x_{k+1,v} + \tau_{i,v}x_{k+1,v}) = \left( \begin{array}{c}
\begin{cases}
0 & \text{if } i \neq j, \\
- & \text{if } i = j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{for all } i, j,
\end{cases} & \quad \text{if } i = j, \\
\begin{cases}
0 & \text{if } i = k \text{ and } i - j, \\
0 & \text{otherwise.}
\end{cases} & \quad \text{if } i = k \text{ and } i - j,
\end{array}
\right)
\end{cases}
\]

(3.9) for all \( i, j \), (3.10) if \( i = k \) and \( i - j \), (3.12)
If \( i = j \), we obtain from (3.8):
\[
\frac{1}{y_j}t(\tau_{i,v}x_{k+1,i} + \tau_{i,v}x_{k+1,i}) = \left( \begin{array}{c}
\tau_{i,j} \\
\tau_{i,i}
\end{array} \right) + \left( \begin{array}{c}
\tau_{i,i} \\
\tau_{i,j}
\end{array} \right) + \left( \begin{array}{c}
\tau_{i,i} \\
\tau_{i,j}
\end{array} \right) + \left( \begin{array}{c}
\tau_{i,j} \\
\tau_{i,i}
\end{array} \right) = 2 \left( \begin{array}{c}
\tau_{i,i} \\
\tau_{i,j}
\end{array} \right).
\]

- Compatibility with (3.5): Similar.
- Compatibility with (3.6): If \( i = j \), this relation is obviously satisfied. For \( i \neq j \), nothing remains to be proven if one index is even; we thus assume \( |i||j| = 1 \).

\[
\frac{1}{y_j y_i} \left( \begin{array}{c}
\tau_{i,j} \\
\tau_{i,i}
\end{array} \right) = \frac{1}{y_j} \left( \begin{array}{c}
\tau_{i,j} \\
\tau_{i,i}
\end{array} \right) t(\nabla) = \left( \begin{array}{c}
\tau_{i,i} \\
\tau_{i,j}
\end{array} \right) + \left( \begin{array}{c}
\tau_{i,j} \\
\tau_{i,i}
\end{array} \right) = \left( \begin{array}{c}
-2 \\
-2
\end{array} \right)
\]

because we may drag a \( \nabla \) across crossings by (3.10). If \( i \neq j \), this equals \(-2\nabla\nabla\), because we may resolve the double crossings by (3.11). If \( i = j \), we calculate
\[
\left( \begin{array}{c}
\tau_{i,j} \\
\tau_{i,i}
\end{array} \right) = (-1)^{d_{ij}} d_{ij} + (-1)^{d_{ij}} d_{ij},
\]

- Compatibility with (3.7): Assume w.l.o.g. that \( k = 1 \). The braid relation then follows from the computation
\[
\tau_{1,2,1}(v) \tau_{2,1}(v) \tau_{1,1}(v) = (c_1 - c_2)^{[s_2]^{(v)}_{1}}[s_2]^{(v)}_{1} [s_2]^{(v)}_{2} [s_2]^{(v)}_{2} [s_2]^{(v)}_{3} [s_2]^{(v)}_{3} \sigma_{1} (c_1 - c_2)^{[s_1]^{(v)}_{1}} [s_1]^{(v)}_{1} [s_1]^{(v)}_{2} [s_1]^{(v)}_{2} \sigma_{1}
\]
\[
= (c_1 - c_2)^{[v_1]} [v_2] [c_1 - c_3]^{[v_1][v_2]} [c_2 - c_3]^{[v_1][v_2]} [c_1 - c_2]^{[v_2]} [c_1 - c_2]^{[v_2]} \sigma_{2} \sigma_{1} \sigma_{1} 
\]
\[
= (c_1 - c_2)^{[v_1]} [v_2] [c_1 - c_3]^{[v_1][v_2]} [c_2 - c_3]^{[v_1][v_2]} [c_1 - c_2]^{[v_2]} [c_1 - c_2]^{[v_2]} \sigma_{2} \sigma_{2} 
\]
\[
= (c_1 - c_2)^{[v_1]} [v_2] [c_1 - c_3]^{[v_1][v_2]} [c_2 - c_3]^{[v_1][v_2]} [c_1 - c_2]^{[v_2]} [c_1 - c_2]^{[v_2]} \sigma_{2} 
\]
\[
= \tau_{2,1}(v) \tau_{1,2}(v) \tau_{1,1}(v).
\]

The equation \( * \) is clear if at least one index is even and is verified by multiplying out the \( c_k \)'s if every index is odd.

\[\square\]

### 3.3. Faithful polynomial representation

We define a representation of the quiver Hecke Clifford superalgebra \( NH\mathbb{C}_n(C) \) as an analogue to the construction in [12, Section 2.3].

**Definition 3.15.** Let \( Pol\mathbb{C}_n(C) \) be the \( k \)-linear supercategory defined as follows.
- Its objects are the free \( k \)-super vector spaces \( Pol\mathbb{C}_v := k[y_{1,v}, \ldots, y_{n,v}, c_{1,v}, \ldots, c_{n,v}] \), indexed by sequences \( v \in \mathbb{N}^n \). The symbols \( y_{k,v} \) and \( c_{k,v} \) have the same parities and satisfy the same relations as in Definition 3.13.
- Its morphisms are super vector space homomorphisms.
Given a sequence \( v \in \mathbb{I}^n \), \( \text{Pol}\mathcal{C}_n(C) \) and \( \mathcal{H}\mathcal{C}_n(C) \) both have full subcategories \( \text{Pol}\mathcal{C}(v) \) and \( \text{NH}_n(v) \) with objects \( \{ \text{Pol}\mathcal{C}_\mu | \mu \in S_n v \} \) and string diagrams between these, respectively.

**Proposition 3.16.** \( \mathcal{H}\mathcal{C}(v) \) acts faithfully on \( \text{Pol}\mathcal{C}(v) \), where \( y_{k,v} \) and \( c_{k,v} \) act by multiplication and \( \sigma_{k,v} \) acts by

\[
\sigma_{k,v} = \begin{cases} 
  s_k & \text{if } v_k \neq v_{k+1} \text{ or } v_k \leftarrow v_{k+1}, \\
  d_k & \text{if } v_k = v_{k+1}, \\
  \left( (c_{k,v} y_{k,v})^{d_k, v_{k+1}} + (c_{k+1,v} y_{k+1,v})^{d_{k+1, v_k}} \right) s_k & \text{if } v_k \rightarrow v_{k+1}.
\end{cases} \tag{3.14}
\]

**Proof.** For the non-super case, faithfulness of the polynomial representation of the quiver Hecke algebra has been proven in [12, Section 2.3]; Our super-algebraic setup necessitates only minor alterations, which we make explicit in the following. Success

---

**Spanning set:** As a vector space, \( \mathcal{H}\mathcal{C}_n(v) \) is isomorphic to a direct sum \( \bigoplus_{\mu, \mu' \in S_n} (\mu')\mathcal{H}\mathcal{C}_\mu \), each of which contains diagrams connecting the bottom sequence \( \mu \) with the top sequence \( \mu' \). Let \( \mu S_\mu \) be the subgroup \( \mu S_\mu = \{ w \in S_n | \mu w(k) = \mu'_k \} \) of \( S_n \).

Given a diagram in \( \mu' \mathcal{H}\mathcal{C}_\mu \), assume there are two strings intersecting more than once. According to the relation (3.11), it can be replaced by a linear combination of diagrams with fewer crossings, possibly at the cost of introducing more diamonds. Diamonds can be slid across crossings by (3.8)–(3.10), possibly at the cost of introducing summands with less crossings, and past other diamonds, possibly introducing signs.

The vector space \( \mu' \mathcal{H}\mathcal{C}_\mu \) therefore is spanned by diagrams with any two strands intersecting at most once, all black diamonds below all crossings, and all white diamonds below the black ones such that there is at most one white diamond per string. The white ones may be arranged descending to the right.

A typical diagram in this spanning set looks as follows:

![Typical diagram](image)

This spanning set can be written as

\[
\mu' \mathcal{B}\mathcal{C}_\mu = \{ \sigma_{\pi, \mu, \mu' y_{\mu'}^{\alpha_1, \mu} c_{\beta_1, \mu}^{\beta_1, n_{i_1, \mu}} \cdots y_{\mu'}^{\alpha_n, \mu} c_{\beta_n, \mu}^{\beta_n, n_{i_n, \mu}} \} \tag{3.15}
\]

where \( \mu \) and \( \mu' \) are permutations of \( v \), \( \pi \) is a reduced expression for an element of \( \mu' S_\mu, \alpha \in \mathbb{N}^n \), \( \beta \in \{0, 1\}^n \) are multiindices, and \( c_{\beta, \mu} \) is an ordered monomial. Choose a complete order \( \leq \) on \( I \) such that \( i < j \) whenever there is an edge \( i \rightarrow j \). This order induces a lexicographic order on \( \mathbb{I}^n \). We show by induction on \( \mu \) w. r. t. this order on \( \mathbb{I}^n \) that \( \mu' \mathcal{B}\mathcal{C}_\mu \) is a \( k \)-basis on which \( \mathcal{H}\mathcal{C}(v) \) acts faithfully.

---

**Base of induction:** Let \( n_i \) be the number of entries \( i \) in \( v \). Let

\[
\mu = (i_1, \ldots, i_{n_1}, i_{n_1} + 1, \ldots, i_{n_1 + n_2}, \ldots)
\]

such that \( i_1 < i_2 < \cdots \). The tuple \( \mu \) is the lowest element in the orbit \( S_n v \) w. r. t. the order \( \leq \). We may write a permutation \( w \in v S_v \) as \( w = uv \) such that \( u \in S_{n_1} \times S_{n_2} \times \cdots \) permutes the \( i_1, i_2 \) etc. independently, and \( v \) is of minimal length, i.e., it does not interchange identically labeled strings and interchanges distinct labels at most once. The spanning set can thus be written as

\[
v' \mathcal{B}\mathcal{C}_v = \{ \sigma_{u, \mu} \sigma_{v, u} y_{\mu'}^{\alpha, \mu} c_{\beta, \mu} \}.
\]
We make sure that these elements act linearly independently on $\text{Pol}\mathcal{C}(\nu)$.

- The terms $y_{\mu}^{a}e_{\mu}^{b}$ take $y_{\mu}^{a'}e_{\mu}^{b'}$ to $\pm y_{\mu}^{a'+a}e_{\mu}^{b+b'}$.
- Since $\nu$ permutes strings with distinct labels, $\sigma_{\nu,\mu}$ acts by the permutation $\nu$.
- Since the permutation $u = u_1 \times \cdots \times u_m \in S_{n_1} \times \cdots \times S_{n_m}$ permutes only identically labeled strings, $\sigma_{\mu,u}$ acts by $\sigma_{u,\nu} = \nu \in N\mathcal{C}n_1 \times N\mathcal{C}n_2 \times \cdots \times N\mathcal{C}n_m$. We shall prove in Lemma 4.16 that the action of $N\mathcal{C}n$ on polynomials is faithful.

—Induction step: It suffices to show that $\mu' \mathcal{B}C_{\mu}$ is linearly independent. We have that $s_k(\mu') \mathcal{B}C_{\mu}$ is linearly independent if $\mu_k', \mu_{k+1}$ are distinct and connected by an edge; otherwise, $s_k(\mu') \mathcal{B}C_{\mu}$ maps bijectively to $\mu' \mathcal{B}C_{\mu}$ by $\sigma_{\mu_k,\mu'}$. The multiplication map $(\sigma_k \cdot) : s_k(\mu') \mathcal{B}C_{\nu} \hookrightarrow \mu' \mathcal{B}C_{\mu}$ is seen to be injective by the same argument as in [12].

Let $\mu' \mathcal{B}C_{\mu}$ be endowed with a partial order $\leq$ such that if we assign to an element $\sigma_{w,\mu}y_{\mu}^{a}e_{\mu}^{b}$ of the spanning set the tuple $(\ell(w), \alpha, \beta)$, then the order $\leq$ coincides with the lexicographic ordering $\leq$ on these tuples. Define a map $\varsigma : s_k(\mu') \mathcal{B}C_{\mu} \rightarrow \mu' \mathcal{B}C_{\mu}$

$$\begin{align*}
D \mapsto \begin{cases} 
\varsigma(D) & \text{if the } k\text{-th and } (k + 1)\text{-st strand do not intersect,} \\
D^* & \text{otherwise},
\end{cases}
\end{align*}$$

(3.16)

where $D^*$ is obtained from $D$ by removing the crossing. The map $\varsigma$ is injective, and multiplication by $\sigma_{k,\mu'}$ satisfies

$$\sigma_{k,\mu'} \cdot D \in \{ \pm \varsigma(D) \} + \sum_{D' < \varsigma(D)} D' \leq \mu' \mathcal{B}C_{\mu}.$$ 

By the induction hypothesis on $\mu' \mathcal{B}C_{\mu}$, we obtain that the multiplication map $(\sigma_{k,\mu'} \cdot)$ must be injective.

**Definition 3.17.** We endow $\text{Pol}\mathcal{C}(\nu)$ with a polynomial grading by setting $\deg(y_{\mu,\nu}) = 1$ and $\deg(c_{\mu,\nu}) = 0$. Additionally setting $\deg(\sigma_{k,\mu}) = -1$, we also endow $H\mathcal{C}(\nu)$ with a grading.

**Corollary 3.18.** With this grading, $\text{Pol}\mathcal{C}(\nu)$ is a faithful graded $H\mathcal{C}(\nu)$-module.

### 4. Clifford symmetric polynomials

Consider a $\mathbb{Z}/2\mathbb{Z}$-graded index set $I = \{1, \ldots, n\} = I_0 \sqcup I_1$ as above. From now on, we consider the quotient superalgebra $\mathcal{C}_I := \mathcal{C}_n/(c_i)_{i \in I_0}$ by the two-sided ideals $(c_i)_{i \in I_0}$. $\text{Pol}\mathcal{C}_I$ and $\partial_i$ are defined analogously to Definition 3.3.

**Definition 4.1.** By Lemma 3.8, $\Lambda\text{Pol}\mathcal{C}_I := \bigcap_{k=1}^{n-1} \ker \partial_k = \bigcap_{k=1}^{n-1} \text{im} \partial_i$ is a $\mathcal{C}_I$-subalgebra of $\text{Pol}\mathcal{C}_n(I)$. We call $\Lambda\text{Pol}\mathcal{C}_I$ the algebra of $\partial$-symmetric polynomials.$^1$

In this section, we want to investigate some of their properties. To this end, we shall introduce the notion of elementary $\partial$-symmetric polynomials and show in Theorem 4.17 that they generate $\Lambda\text{Pol}\mathcal{C}_I$ as $\mathcal{C}_I$-superalgebra.

$^1$One may be tempted to call such symmetric polynomials; this term is already a coined for another notion though [20].
4.1. Interlude: counting graded ranks

**Definition 4.2.** Given a graded ring \( R \) and a free \( \mathbb{Z} \)-graded \( R \)-module \( M = \bigoplus_{i \in \mathbb{Z}} M_i \), its **graded rank** or **Poincaré series** is the formal power series \( \text{rk}_{q,R}(M) := \sum_{i \in \mathbb{Z}} \text{rk}_R(M_i)q^i \), seen as an element of the localization \( \mathbb{Z}[[q]]((1-q)) \). We write \( \text{rk}_{q,Z} \) for the graded rank of graded free abelian groups. With respect to the field \( k \), we denote the graded dimension by \( \dim_q k \).

**Definition 4.3.** Let \( (W, S) \) be a Coxeter system with a fixed generating set \( S \), which determines a length function \( \ell \) on \( W \). Its **\( q \)-order** is \( \text{ord}_q(W) := \sum_{w \in W} q^{\ell(w)} \).

We compute some examples that we shall use later on. Note that we are working with classical, non-super polynomial rings in this section. Since \( 1 - q \) is a unit in \( \mathbb{Z}[[q]]((1-q)) \), the \( q \)-integers \( (n)_q := 1 + \cdots + q^{n-1} \) can be expressed as \((n)_q = \frac{1-q^n}{1-q}\); see [17] for a general introduction.

**Lemma 4.4.** [cf. 21, Section 1] The Coxeter group of type \( A_n, \, BC_n, \) and \( D_n \) have the following \( q \)-orders:

1. \( \text{ord}_q(A_n) = (n)_q \) with the \( q \)-factorial \((n)_q := (n)(n-1)_q \cdots (1)_q \).
2. \( \text{ord}_q(BC_n) = (2n)_q \)!! with the \( q \)-double factorial \((2n)_q \)!! := \((2n)(2n-2)_q \cdots (2)_q \).
3. \( \text{ord}_q(D_n) = (2n - 2)_q \)!! \((n)_q \).

We have included \( \text{ord}_q(BC_n) \) and \( \text{ord}_q(D_n) \) for the sake of completeness but will not need them in the following.

**Proof.** 1. The length \( \ell(w) \) of a permutation \( w \in S_n \) equals the number of inversions [1, prop. 1.5.2], i.e., the cardinality of \( \text{inv} w := \{ (i,j) \mid 1 \leq i < j \leq n, w(i) > w(j) \} \). Assume that \( \sum_{w \in S_n} p(w) = (n)_q \). Consider the permutations

\[
\pi_0: (1, \ldots, n, n+1) \mapsto (1, \ldots, n, n+1),
\pi_1: (1, \ldots, n, n+1) \mapsto (1, \ldots, n+1, n),
\vdots
\pi_n: (1, \ldots, n, n+1) \mapsto (n+1, 1, \ldots, n),
\]

contained in \( S_{n+1} \), where \( \pi_0 = e \) is the trivial permutation. Moving the rightmost entry in \( \pi_0 \) to the left will subsequently create new inversions such that \( \pi_k \) has the \( k \) inversions

\[
\text{inv} \pi_k = \left\{ (n-k+1, n-k+2), \ldots, (n-k+1, n+1) \right\},
\]

and \( \sum_k q^{\ell(\pi_k)} = (1 + q + \cdots q^n) = (n+1)_q \). Since any \( w \in S_n \) interchanges only the first \( n \) slots, the inversions of \( w \) and \( \pi_k \) do not interfere, which gives \( \text{inv}(\pi_k w) = w^{-1}(\text{inv} \pi_k) \cup \text{inv}(w) \). Therefore,

\[
\sum_k q^{\ell(\pi_k w)} = (n+1)_q q^{\ell(w)}.
\]

By the induction hypothesis, letting \( w \) traverse all of \( S_n \) proves the claim.

2. The Coxeter group \( BC_n \), called the hyperoctahedral group or signed permutation group, is the group of permutations \( \pi \) of the set \( \{ \pm 1, \ldots, \pm n \} \) such that \( \pi(-k) = -\pi(k) \). It has as generators the simple transpositions \( s_k \): \( \pm k \leftrightarrow \pm k+1 \) for \( 1 \leq k \leq n-1 \) and the additional generator \( s_0 \): \( 1 \leftrightarrow -1 \). A signed permutation \( w \) has

\[
\text{inv}(w) := \{ (i,j) \mid 1 \leq i < j \leq n, w(i) > w(j) \} \cup \{ (-i,j) \mid 1 \leq i < j \leq n, w(-i) > w(j) \}.
\]

We count the number of inversions as in part (i). Assume that \( \text{ord}_q(BC_n) = (n)_q \)!!, and consider the permutations

\[
\pi_0: (1, \ldots, n, n+1) \mapsto (1, \ldots, n, n+1),
\pi_1: (1, \ldots, n, n+1) \mapsto (n+1, 1, \ldots, n),
\pi_n: (1, \ldots, n, n+1) \mapsto (-n+1, 1, \ldots, n),
\pi_{2n+1}: (1, \ldots, n, n+1) \mapsto (1, \ldots, w(n), -n-1).
\]
The following k-algebras have the respective graded dimensions:

**Lemma 4.5.** The following k-algebras have the respective graded dimensions:

\[
\begin{align*}
\text{rk}_{q,Z} \text{Pol}_n &= \text{rk}_{q,Z} \text{Z}[y_1, \ldots, y_n] = \frac{1}{(1 - q)^n}, \\
\text{rk}_{q,Z} \wedge \text{Pol}_n &= \text{rk}_{q,Z} \text{Pol}_n^{S_n} = \frac{1}{(n)_q!(1 - q)^n}, \\
\text{rk}_{q,Z} \text{NC}_n &= \text{rk}_{q,Z} \text{Z}[\partial_1, \ldots, \partial_n] = \frac{1}{(n)_q!}, \\
\text{rk}_{q,Z} \text{NH}_n &= \text{rk}_{q,Z} \text{Pol}_n \otimes \text{Z} \text{NC}_n = \frac{1}{(n)_q!(1 - q)^n},
\end{align*}
\]

where we set \(\deg \partial_i = -1\).

**Proof.** Pol\(_n\) Every indeterminate \(y_i\) independently generates a free abelian group \((1, y_i, y_i^2, \ldots)\text{Z}\) of graded rank \(1 + q + q^2 + \cdots = \frac{1}{1-q}\). Since \(\text{Pol}_n \cong \text{Z}[y]^{\otimes n}\) as abelian groups, \(\text{Pol}_n\) has graded rank \(\text{rk}_{q,Z} \text{Pol}_n = (1 - q)^{-n}\).
\( \Lambda \text{Pol}_n \) Recall that \( \Lambda \text{Pol}_n \cong \mathbb{Z}[\varepsilon_1^{(n)}, \ldots, \varepsilon_n^{(n)}] \), where \( \varepsilon_m^{(n)} \) denotes the elementary symmetric polynomials in \( n \) indeterminates of degree \( m \). Each polynomial \( \varepsilon_m^{(n)} \) generates a subgroup of graded rank \( \frac{1}{1-q^m} \);
\( \Lambda \text{Pol}_n \) thus has graded rank \( \text{rk}_{q,Z} \Lambda \text{Pol}_n = \prod_{m=1}^n \frac{1}{1-q^m} \). Since \( (1 - q^m) = (m)q(1 - q) \) (telescoping sum), induction shows that \( \text{rk}_{q,Z} \Lambda \text{Pol}_n = \frac{1}{(n)q!(1-q^n)} \).

\( \text{NC}_n \) By the relations of the NilCoxeter-algebra \( \text{NC}_n \), there is a bijection \( ZS_n \rightarrow \text{NC}_n, s_i \mapsto \partial_i \) such that we can write \( \partial_w := \partial_{i_1} \cdots \partial_{i_n} \) for any reduced expression \( w = s_{i_1} \cdots s_{i_n} \) in \( S_n \). In particular, \( \deg \partial_w = -\ell(w) \). By Lemma 4.4, this shows that \( \text{rk}_{q,Z} \text{NC}_n = (n)q! \).

\( \text{NH}_n \) This follows from the fact that \( \text{NH}_n \cong \text{Pol}_n \otimes \mathbb{Z} \text{NC}_n \) as abelian groups.

\textbf{Remark 4.6.} Similar statements hold for the invariants under Coxeter groups of type \( BC_n \) and \( D_n \):

1. The hyperoctahedral group \( BC_n \) acts on the polynomial ring \( \text{Pol}_n \) by permuting the indeterminates and flipping their signs. Thus it has invariants \( \text{Pol}^{BC_n}_n = Z[y_1^2, \ldots, y_n^2]S_n \), which yields
   \[ \text{rk}_{q,Z} \text{Pol}^{BC_n}_n = \frac{1}{(1-q^2)(1-q^4) \cdots (1-q^{2n})} = \frac{1}{(2n)q!(1-q^n)}. \]

2. The demihypercube group \( D_n \) acts on \( \text{Pol}_n \) by permuting the indeterminates and flipping two signs at once. The invariants thus are
   \[ \text{Pol}^{D_n}_n = Z[\varepsilon_1(y_1^2, \ldots, y_n^2), \ldots, \varepsilon_{n-1}(y_1^2, \ldots, y_n^2), \varepsilon_n(y_1, \ldots, y_n)] \]
   with graded rank
   \[ \text{rk}_{q,Z}(\text{Pol}_n) = \frac{1}{(1-q^2) \cdots (1-q^{2n-2})(1-q^n)} = \frac{1}{(2n-2)q!(1-q^n)}. \]

We notice that \( \text{rk}_{q,Z} \text{Pol}_n / \text{rk}_{q,Z} \text{Pol}^W_n = \text{ord}_q W \) for \( W \in \{ S_n, BC_n, D_n \} \). In fact, graded ranks satisfy \( \text{rk}_{q,Z}(\text{Pol}_n / \text{Pol}^W_n) = \text{ord}_q W \) for general finite reflection groups [5]. Even stronger, \( \text{Pol}_n \) is a free \( \Lambda \text{Pol}_n \)-module of graded rank \( \text{ord}_q W \) for other finite reflection groups \( W \) [6, thm. 6.2]. We shall show a respective statement in the Clifford setup for \( W = S_n \) in Theorem 4.17.

\textbf{Corollary 4.7.} We endow \( \text{Pol}\mathcal{E}_I \) with a (polynomial) grading such that the odd generators \( c_i \) are of degree zero. Lemma 4.5 then shows that
   \[ \text{rk}_{q,c_i} \text{Pol}\mathcal{E}_I = \frac{1}{(1-q)^n}, \quad \text{rk}_{q,c_i} \text{NH}\mathcal{E}_I = \frac{(n)q!}{(1-q^n)} \]
both as left and right \( \mathcal{E}_I \)-modules.

\textbf{4.2. Elementary \( \partial \)-symmetric polynomials}

Our aim is to construct analogues for the elementary symmetric polynomials for \( \Lambda \text{Pol}_I \) that coincide with the ordinary elementary symmetric polynomials if all indices are even. For \( i, j \) with \( |i - j| = 1 \), we define
   \[ \gamma_{ij} := \begin{cases} c_i & \text{if } i \text{ and } j = i + 1 \text{ both have odd parity}, \\ -c_i & \text{if } i \text{ and } j = i - 1 \text{ both have odd parity}, \\ 1 & \text{otherwise}. \end{cases} \]

\textbf{Lemma 4.8.} The operator \( \partial_i \) on \( \text{Pol}\mathcal{E}_I \) has the kernel
   \[ \ker \partial_i = \langle \gamma_{i,i+1}y_i + \gamma_{i+1,i}y_{i+1}, \gamma_{i,i+1}y_{i+1}, y_j | j \neq i, i + 1 \rangle. \] (4.1)
Proof. In the purely even case, the kernel of the Demazure operator \( \partial_i \) is the subalgebra
\[
\ker \partial_i = \langle y_{i} + y_{i+1}, y_{i} y_{i+1}, y_j \mid j \neq i, i + 1 \rangle \subseteq \text{Pol}_n.
\]
In the super-setting, we have already seen that \( \ker \partial_1 \subseteq \text{Pol}_2 \) is a \( \mathcal{C}_t \)-subalgebra in Lemma 3.8. Assume w.l.o.g. that \( i = 1 \) and \( I = \{1, 2\} \). The only indeterminates are thus \( y_1, y_2 \) and \( c_1, c_2 \). Let \( \Lambda := \langle y_1 y_2, y_1 y_1 + y_2, y_2 \rangle \subseteq \text{Pol}_2 \) as \( \mathcal{C}_{1,2} \)-superalgebra. We have to show that \( \Lambda = \ker \partial_{\{1,2\}} \) as subalgebras of \( \text{Pol}_2 \).

(\( \subseteq \)) We have already shown in the proof of Lemma 3.6 that \( y_1 y_2 = y_2 y_1 \in \ker \partial_1 \). From
\[
\partial_1 (\partial_1 (y_1) y_1 - \partial_1 (y_2) y_2) = \partial_1 ((-1 - c_1 c_2) y_1 - (1 - c_1 c_2) y_2) = 0,
\]
we get that \( \ker \partial_1 \) also contains the polynomial
\[
- \frac{1}{2} (c_1 - c_2) \big|_{\mathbb{Z}/2} \big[ \partial_1 (y_1) y_1 - \partial_1 (y_2) y_2 \big] = \begin{cases} c_1 y_1 - c_2 y_2 & \text{if both } 1, 2 \text{ have odd parity,} \\ y_1 + y_2 & \text{otherwise} \end{cases} = y_1 y_2.
\]
(\( \supseteq \)) The superalgebra \( \text{Pol}_2 \) contains an element \( \alpha_1 := y_1 - y_2 \), which satisfies \( \partial_1 (\alpha_1) = -2 \). Recall that \( \text{Pol}_2 \) is a free left and right \( \mathcal{C}_{1,2} \)-module of graded rank \( \text{rk}_{\mathcal{C}_t, \mathcal{C}_{1,2}} \text{Pol}_2 = \frac{1}{1-q^2} \). Its subalgebra \( \Lambda \) is also \( \mathcal{C}_{1,2} \)-free of graded rank \( \text{rk}_{\mathcal{C}_t, \mathcal{C}_{1,2}} \Lambda = \frac{1}{1-q^2} \). The \( \mathcal{C}_{1,2} \)-submodule \( \Lambda \alpha_1 \subseteq \text{Pol}_2 \) does not intersect \( \ker \partial_1 \) because for any \( \lambda \in \Lambda \) nonzero, we have \( \partial_1 (\lambda \alpha_1) = -2s_1 (\lambda) \neq 0 \). It has graded rank \( \text{rk}_{\mathcal{C}_t, \mathcal{C}_{1,2}} \Lambda \alpha_1 = q \text{rk}_{\mathcal{C}_t, \mathcal{C}_{1,2}} \Lambda \).

we obtain that \( \text{Pol}_2 = \Lambda \oplus \Lambda \alpha_1 \) as \( \mathcal{C}_{1,2} \)-module and in particular that the inclusion \( \Lambda \subseteq \ker \partial_1 \) in fact is an equality. \( \square \)

Let \( y_{n,n+1} := 1 \) if \( n + 1 \) exceeds the number of indeterminates. Since
\[
y_{i} y_{i+1} = y_{i+1} + y_{i+1} y_{i+1} y_{i+1} + y_{i+1} + y_{i+1} + y_{i+1} y_{i+1} + y_{i+1} y_{i+1} \quad (4.2)
\]
\[
= (-1)^{|i|+|i+1|+|i+1|+2} y_{i+1} y_{i+1} y_{i+1} + y_{i+1} y_{i+1} + y_{i+1} y_{i+1} y_{i+1} \quad (4.3)
\]
we may replace the second generator by \( y_{i+1} y_{i+1} y_{i+1} y_{i+1} y_{i+1} y_{i+1} y_{i+1} y_{i+1} \) without changing the \( \mathcal{C}_t \)-span.

Definition 4.9. We denote the generators of \( \ker \partial_1 \) by
\[
\epsilon_1^{(i-1,i+1)} := y_{i+1} + y_{i+1} y_{i+1} + y_{i+1} y_{i+1} y_{i+1} = y_{i+1} y_{i+1} y_{i+1} y_{i+1} y_{i+1} y_{i+1} y_{i+1} y_{i+1}.
\]
If \( i = 1 \), we simply write \( \epsilon_1^{(2)} := \epsilon_1^{(i-1,i+1)} \). We can thus write \( \ker \partial_1 = \langle \epsilon_1^{(2)}, \epsilon_2^{(2)} \rangle \subseteq \text{Pol}_2 \) as \( \mathcal{C}_{1,2} \)-superalgebra.

---

\(^2\) It seems more natural to put all Clifford-generators on the right, because \( \partial_1 \) is \( \mathcal{C}_t \)-right linear. We shall stick to putting them on the left though in order to preserve compatibility with the notation used in [11].
Our goal is to show that there are polynomials $\epsilon_m^{(n)}$, $1 \leq m \leq n$ of polynomial degree $m$ that generate $\Lambda$ Pol$\mathcal{C}_f$ of $\mathcal{C}_f$-superalgebras. These are meant to serve as a Clifford-replacement for the ordinary elementary symmetric polynomials.

Our first task is to find a recursive formula for polynomials lying in $\Lambda$ Pol$\mathcal{C}_f$. Proving that they are indeed generators requires some more work. We shall prove this in Theorem 4.17.

Before coming to the $n$-fold intersection $\Lambda$ Pol$\mathcal{C}_f = \bigcap_{k=1}^{n-1} \ker \mathfrak{d}_k$, we consider the intersection $\ker \mathfrak{d}_1 \cap \ker \mathfrak{d}_2 \subseteq \text{Pol}_{\{1,2,3\}}$ of just the first two kernels. We can multiply the first generator of $\ker \mathfrak{d}_2$ with the unit $\gamma_{2,1}\gamma_{2,3}$ from the left without changing its span. Thus

\[
\ker \mathfrak{d}_1 \cap \ker \mathfrak{d}_2 = \left( \gamma_{1,2}\gamma_1 + \gamma_{2,1}\gamma_2 + \gamma_{2,1}\gamma_{2,3} \gamma_{3,2} \gamma_3, \frac{\gamma_{2,1}\gamma_{2,3}}{\gamma_{1,2}\gamma_{1,2}\gamma_{2,3} \gamma_{3,2} \gamma_3} \right)
\]

as a $\mathcal{C}_f$-superalgebra. This depiction serves as a template for the following:

**Definition 4.10.** The *elementary $\mathfrak{d}$-symmetric polynomials* $\epsilon_m^{(n)}$ *of degree* $m$ *in* $n$ *indeterminates* are the polynomials in Pol$\mathcal{C}_f$ defined recursively as follows:

\[
\epsilon_1^{(1)} = \gamma_{1,2}\gamma_1 \quad (4.5)
\]

\[
\vdots
\]

\[
\epsilon_1^{(n)} = \epsilon_1^{(n-1)} + (\gamma_{2,1}\gamma_{2,3} \gamma_{3,2} \gamma_{3,4} \cdots \gamma_{n-1,1,n-2} \gamma_{n-1,1,n}) \gamma_{n,n-1} \gamma_n, \quad (4.6)
\]

\[
\epsilon_2^{(n)} = \epsilon_1^{(n-1)} + \epsilon_1^{(n)} (\gamma_{2,1}\gamma_{2,3} \gamma_{3,2} \gamma_{3,4} \cdots \gamma_{n-1,1,n-2} \gamma_{n-1,1,n}) \gamma_{n,n-1} \gamma_n, \quad (4.7)
\]

\[
\vdots
\]

\[
\epsilon_m^{(n)} = \epsilon_{m-1}^{(n-1)} + \epsilon_{m-1}^{(n-1)} (\gamma_{m+1,n+1,m+1,2} \cdots \gamma_{n-1,1,n-2} \gamma_{n-1,1,n}) \gamma_{n,n-1} \gamma_n, \quad (4.8)
\]

\[
\vdots
\]

\[
\epsilon_n^{(n)} = \epsilon_{n-1}^{(n-1)} \gamma_{n+1,n+2} \gamma_n. \quad (4.9)
\]

We define the subalgebra $\Lambda_f := \langle \epsilon_1^{(n)}, \ldots, \epsilon_n^{(n)} \rangle \subseteq \text{Pol}\mathcal{C}_f$.

**Lemma 4.11.** $\Lambda_f \subseteq \Lambda$ Pol$\mathcal{C}_f$.

**Example 4.12.** Before proving the lemma let us make the definition of the $\epsilon$’s explicit for small numbers of indeterminates.

1. For $n = 1, \ldots, 4$ variables, the elementary $\mathfrak{d}$-symmetric polynomials are listed explicitly in Table 1.
2. If all indices are even, then all $\gamma_{i,i+1} = 1$. In this case, the $\mathfrak{d}$-elementary symmetric polynomials from Table 1 specialize to those from Table 2 at $\gamma_{i,i+1} = 1$, which we recognize at the ordinary elementary
symmetric polynomials $e_m^{(n)}$. In this case, the induction formula from Definition 4.10 indeed gives

$$
e^{(1)}_1 = x_1$$

$$
e^{(n)}_m = e^{(n-1)}_m + e^{(n-1)}_{m-1} x_m$$

(4.10)

by setting each of the $\gamma$’s in (4.5)–(4.9) to 1. This is a well-known recursive formula for the ordinary elementary symmetric polynomials

$$e^{(n)}_m = \sum_{1 \leq k_1 < \cdots < k_m \leq n} x_{k_1} \cdots x_{k_m}.$$  

(4.11)

3. If all indices are odd, then $\gamma_{i,i+1} = \pm \epsilon_i$. We set $x_i := \gamma_{i,i+1} y_i = \epsilon_i y_i$ (cf. Lemma 3.14). The polynomials from Table 1 then specialize at $\gamma_{i,i+1} = \pm \epsilon_i$ to the ones listed in Table 2. We refer to these as odd
elementary symmetric polynomials and denote them by \( e(n)_{m} \). The recursion formula reduces to

\[
\begin{align*}
\epsilon_1^{(1)} &= x_1 \\
\epsilon_m^{(n)} &= \epsilon_m^{(n-1)} + (-1)^{n-m} \epsilon_{m-1}^{(n-1)} x_n \\
\epsilon_n^{(n)} &= \epsilon_{m-1}^{(n-1)} x_n
\end{align*}
\]

(4.12)

Consider also the odd elementary symmetric polynomials

\[EKLE(n)_{m} := \sum_{1 \leq k_1 < \cdots < k_m \leq n} (-1)^{k_m-1} x_{k_1} \cdots (-1)^{k_1-1} x_{k_m}\]

as defined in [7, (2.21)–(2.23)], which satisfy the recursion formula

\[EKLE(n)_{m} = EKL \epsilon_{m-1}^{(m-1)} + (-1)^{n-1} EKL \epsilon_{m}^{(n-1)} x_n.\]

Our odd elementary symmetric polynomials indeed coincide with those from [7] up to an overall sign \((-1)^{\frac{m(m-1)}{2}}\):

\[
\begin{align*}
\epsilon_{m}^{(n)} &= \epsilon_{m}^{(n-1)} + (-1)^{n-m} \epsilon_{m-1}^{(n-1)} x_n \\
&= (-1)^{\frac{m(m-1)}{2}} \cdot EKL \epsilon_{m}^{(n-1)} + (-1)^{\frac{(m-1)(m-2)}{2}+(n-m)} \cdot EKL \epsilon_{m}^{(n-1)} x_n \\
&= (-1)^{\frac{m(m-1)}{2}} \cdot EKL \epsilon_{m}^{(n-1)} + (-1)^{n-1} x_n \cdot EKL \epsilon(n-1)_{m} \\
&= (-1)^{\frac{m(m-1)}{2}} \cdot EKL \epsilon_{m}^{(n)}. \quad (4.14)
\end{align*}
\]

According to our sign convention, the first monomial \(x_1 \cdots x_m\) of the odd polynomials always has +1 as coefficient. Odd symmetric functions have already been treated extensively in [7, 8, 10].

**Proof of Definition 4.10.** The statement is proven by induction on \( n \) and \( m \). It is convenient to set \( \epsilon_0^{(n)} = 1 \) for all \( n \geq 0 \). The claim then trivially holds for \( n = 1 \) and \( m = 0, 1 \).

— *Induction step on \( n \) for \( m = 1 \):* Assume that \( \epsilon_1^{(k)} \in \mathfrak{d}_1 \cap \cdots \cap \mathfrak{d}_{n-1} \) for all \( k \leq n \). We want to show that \( \text{ker} \mathfrak{d}_1 \cap \cdots \cap \mathfrak{d}_n \) contains the polynomial

\[\epsilon_1^{(n+1)} = \epsilon_1^{(n)} + (\gamma y_1 y_2, \cdots, y_{n+1,n-1} y_{n+1,n}) y_{n+1,n}.\]

The first summand \( \epsilon_1^{(n)} \) is contained in \( \text{ker} \mathfrak{d}_1 \cap \cdots \cap \text{ker} \mathfrak{d}_{n-1} \) by the induction hypothesis. The second summand contains no \( y_k \) for \( k \leq n \) and thus also lies in \( \text{ker} \mathfrak{d}_1 \cap \cdots \cap \text{ker} \mathfrak{d}_{n-1} \) (cf. the definition of \( \mathfrak{d} \) in (3.1)). In order to show that \( \epsilon_1^{(n+1)} \in \text{ker} \mathfrak{d}_n \), we expand the first summand \( \epsilon_1^{(n)} \) and obtain

\[\epsilon_1^{(n+1)} = \epsilon_1^{(n-1)} + (\gamma y_1, \cdots, y_{n+1,n-2} y_{n+1,n}) y_{n+1,n} \gamma y_{n+1,n} + (\gamma y_1, \cdots, y_{n+1,n} y_{n+1,n}) y_{n+1,n} \gamma y_{n+1,n+1}.\]

Regrouping the terms gives

\[\epsilon_1^{(n+1)} = \epsilon_1^{(n-1)} + (\gamma y_1, \cdots, y_{n+1,n-1} y_{n+1,n}) (\gamma y_{n+1,n} + \gamma_{n+1,n} y_{n+1,n}) \gamma_{n+1,n} \epsilon_1^{(n+1,n+1)}.\]

The first summand \( \epsilon_1^{(n-1)} \) is clearly contained in \( \text{ker} \mathfrak{d}_n \) because it contains no \( y_{n+1,n} \). The first expression in parentheses lies in \( \mathfrak{c}_1 \). The second summand lies in the \( \mathfrak{c}_1 \)-superalgebra \( \epsilon_1^{(n+1,n+1)} \subseteq \text{ker} \mathfrak{d}_n \) (see Lemma 4.8 and Definition 4.9). Therefore, \( \epsilon_1^{(n+1)} \in \text{ker} \mathfrak{d}_1 \cap \cdots \cap \text{ker} \mathfrak{d}_n \).

— *Induction step for \( m > 1 \):* Assume \( \epsilon_m^{(k)} \in \text{ker} \mathfrak{d}_1 \cap \cdots \cap \text{ker} \mathfrak{d}_{n-1} \) for all \( k \leq n \) and \( m \leq k \). We want to show that \( \text{ker} \mathfrak{d}_1 \cap \cdots \cap \text{ker} \mathfrak{d}_n \) contains

\[\epsilon_m^{(n+1)} = \epsilon_m^{(n)} + \epsilon_m^{(n-1)} (\gamma y_{n+1,m} y_{n+1,m+1} y_{n+1,n} + \cdots, y_{n+1,n-1} y_{n+1,n}) y_{n+1,n} \epsilon_{m+1}^{(n+1)}. \quad (4.15)\]
\[ e_{m}^{(n+1)} \in \ker \partial_k \text{ for } k \leq n - 1 \] By the induction hypothesis, we have \( e_{m}^{(n)} \in \ker \partial_k \). The second summand satisfies

\[
\partial_k \left( e_{m}^{(n-1)} \frac{Y_{m+1,m}Y_{m+1,m+2} \cdots Y_{n,n-1}Y_{n+1,n}}{Y_{n+1,n}} \right) = 5 \left( e_{m-1}^{(n)} \cdot Y_{m+1,m}Y_{m+1,m+2} \cdots Y_{n,n-1}Y_{n+1,n} \right) \partial_k (Y_{n+1,n}) = 0
\]

by the induction hypothesis on \( e_{m-1}^{(n)} \), where the expression * lies in \( \mathfrak{S}_n^* \).

\[ e_{m}^{(n+1)} \in \ker \partial_n \] For \( \partial_n \), we expand \( e_{m}^{(n)} \) and \( e_{m-1}^{(n)} \) in (4.15) by the recursion formula and obtain

\[
e_{m}^{(n+1)} = e_{m}^{(n-1)} + e_{m-1}^{(n-1)} \left( Y_{m+1,m} \cdots Y_{n,n-2}Y_{n,n-1}Y_{n+1,n} \right) + \left[ e_{m-1}^{(n-1)} + e_{m-2}^{(n-1)} \left( Y_{m+1,m} \cdots Y_{n,n-1}Y_{n+1,n} \right) \right] \left( Y_{m+1,m} \cdots Y_{n,n-1}Y_{n+1,n} \right)
\]

We regroup both terms containing \( e_{m-1}^{(n-1)} \) and get

\[
e_{m}^{(n+1)} = e_{m}^{(n-1)} + e_{m-2}^{(n-1)} \left( Y_{m+1,m} \cdots Y_{n,n-1}Y_{n,n-2}Y_{n,n-1}Y_{n+1,n} \right) + \left[ e_{m-1}^{(n-1)} \left( Y_{m+1,m} \cdots Y_{n,n-1}Y_{n+1,n} \right) \right] \left( Y_{n,n-1}Y_{n+1,n} \right) + e_{m-1}^{(n-1)} \left( Y_{m+1,m} \cdots Y_{n,n-1}Y_{n+1,n} \right)
\]

where the expression * lies in \( \mathfrak{S}_n^* \). The operator \( \partial_n \) maps the terms \( e_{m}^{(n-1)} \), \( e_{m-2}^{(n-1)} \) and \( e_{m-1}^{(n-1)} \) to zero because they do not contain \( y_n \) and \( y_{n+1} \). The second and the last summand of (4.16) are mapped to zero by \( \partial_n \) because they are contained in the \( \mathfrak{S}_n \)-bimodule \( e_{m-1}^{(n-1)} \cdot \langle e_2^{(n-1,n+1)} \rangle \subseteq \ker \partial_n \).

Therefore, \( e_{m}^{(n+1)} \in \ker \partial_n \). We have already seen that \( e_{m}^{(n+1)} \in \ker \partial_1 \cap \cdots \cap \ker \partial_{n-1} \). This proves the assertion. \( \square \)

**Remark 4.13.** The \( \mathfrak{S}_n \)-superalgebra \( \Lambda_t \) has a basis \( \{ (e_1^{(n)})^{a_1}, \ldots, (e_n^{(n)})^{a_n} \} \), \( a_i \in \mathbb{N} \), as left and as right \( \mathfrak{S}_n \)-module (cf. the proof of Lemma 3.4). With the same argumentation as in Lemma 4.5, we see that its graded \( \mathfrak{S}_n \)-rank is \( \frac{1}{(n-1)!} (1-q)^n \) both as left and right \( \mathfrak{S}_n \)-module.

### 4.3. Schubert polynomials; freeness of \( \text{Pol}\mathfrak{S}_n \)

**Definition 4.14.** The \( \partial \)-Schubert polynomial \( s_w \in \text{Pol}\mathfrak{S}_n \) associated to a reduced word \( w \in S_n \) is

\[
s_w := \partial_{w^{-1}} n_0 (\gamma_{1}^{n-1} \cdots \gamma_{n-1}) \]

(17)

This makes sense, since according to Lemma 3.8, the \( \partial \) satisfy the braid relations from (3.2). This is the definition of the ordinary Schubert polynomials [18, Section 2.3] with \( \partial \) instead of \( \partial \). Hence, in the purely even case, the \( \partial \)-Schubert polynomials coincide with the ordinary ones.

We get the following lemma, which generalizes from the statement [9, 173] about ordinary Schubert polynomials, and from the statement [7, (2.42)–(2.43)] about odd ones:
Lemma 4.15. The $\partial$-Schubert polynomials have the following properties:

1. If $wv^{-1}$ is a reduced expression, then $\partial_v s_w = s_{wv^{-1}}$.
2. For any tuple $\alpha \in \mathbb{N}^n$, we have $y^\alpha \partial_v s_w$ if $v = w^{-1}$, and $y^\alpha \partial_v s_w = 0$ if $v \neq w^{-1}$ and $\ell(v) \geq \ell(w)$. In particular, $s_e$ is non-zero.

Proof. The first statement is clear from the definition of $s_w$. For the second one, we argue as follows:

- If $\ell(v) > \ell(w)$: Every $\partial_1$ reduces the polynomial degree by one. Recall that the longest element $w_0$ of the symmetric group has $\ell(w_0) = \frac{n(n-1)}{2} = \deg(y_1^{n-1} \cdots y_n^{1-1})$. Thus, $s_w$ has polynomial degree $\ell(w)$. Since $\partial_1$ acts on constants by zero, the assertion follows.
- If $\ell(v) = \ell(w)$ and $v \neq w^{-1}$: In this case, $v(w^{-1}w_0)$ is not a reduced expression, which implies that $\partial_v w^{-1}w_0 = 0$.
- If $v = w^{-1}$: Take the reduced expression $w_0 = s_{n-1} \cdots s_2 s_{n-2} \cdots s_{3s_2 s_1}$. We have

$$\partial_w (y_1^{n-1} \cdots y_{n-1}^{1-1})$$

$$= \partial_w s_1 \partial_1 (y_1 y_2 y_1 y_3 y_2 y_1 \cdots y_{n-1} y_{n-2} \cdots y_1)$$

$$= \partial_w s_1 [\partial_1 (y_1) (y_2 y_1 y_3 y_2 y_1 \cdots) + y_2 \partial_1 y_1 y_3 y_2 y_1 \cdots].$$

The second summand vanishes because in the argument of $\partial_1$, the indeterminates $y_1, y_2$ only occur in products $y_1 y_2$, which lie in ker $\partial_1$ (see Lemma 4.8). We continue with the next operators. It turns out that in every step, we can apply $\partial_i$ to precisely one factor $y_i$, since the remaining $y_i, y_i+1$'s occur pairwise. We obtain

$$= \partial_{w_1 s_2} s_2 (\partial_1 y_1) [\partial_2 (y_1 y_3 y_2 y_1 \cdots) + \partial_2 (y_1 y_3 y_2 y_1 \cdots)]$$

$$= \partial_{w_1 s_2} s_1 s_2 (\partial_1 y_1) s_1 (\partial_2 y_1) [\partial_1 (y_1) y_3 y_2 y_1 \cdots]$$

$$= (w_1 s_1) (\partial_1 y_1) (w_0 s_1 s_2) (\partial_2 y_1) (w_0 s_1 s_2) (\partial_1 y_1) \cdots s_1 (\partial_2 y_1) \partial_1 (y_1),$$

$$\in \mathcal{C}_I^\partial.$$
Theorem 4.17.  
1. PolCₐ is a free Λ₁-module of graded rank \( (n)q! \).
2. The inclusion \( Λ₁ ⊆ ΛPolCₐ \) from Definition 4.10 is in fact an equality.
3. The faithful action of NHCᵢ on PolCᵢ gives an isomorphism \( NHCᵢ \nrightarrow ΛPolCᵢ(\text{PolCᵢ}) \).

Proof. Since Schubert polynomials behave completely analogously to the classical case (apart from \( Cᵢ \)), the classical proof applies. Nevertheless, we briefly explain the argument. For details, consider [15, Section 3.2] for the even and [7, prop. 2.15] for the odd case.

1. Define the \( Cᵢ \)-bimodule \( \mathfrak{H}_I \) := \( \langle y^{a_1}_1 · · · y^{a_{n−1}}_{n−1} \mid a_i ≤ n−i \rangle Cᵢ \). This module has a basis given by the Schubert polynomials \( s_w \); see the proof of [7, prop. 2.12]. For every index \( i \), one may chose an exponent \( 0 ≤ a_i ≤ n−i \), which yields a generator \( y^{a_i}_i \) of (polynomial) degree \( a_i \). The submodule generated by \( y^*_i \) thus has graded rank \( (n−i)q \); therefore, \( rk_{q, Cᵢ}(\mathfrak{H}_n) = (n)q! \). The same proof as in the even (or odd) case shows that the multiplication

\[ ΛPolCᵢ ⊗ Cᵢ\mathfrak{H}_I \rightarrow PolCᵢ \]

is injective. As both sides have the same graded \( Cᵢ \)-rank \( (1−q)^n \) (see Lemma 4.5 and Remark 4.13), it is an isomorphism of \( Cᵢ \)-modules. This exhibits \( PolCᵢ \) as a free \( ΛPolCᵢ \)-module of graded rank \( (n)q! \).

2. In 4.3, we have shown that \( PolCᵢ \) has a \( Λ₁ \)-basis by Schubert polynomials \( s_w \). Given a polynomial \( p ∈ PolCᵢ \), we take a linear combination \( \sum_{w∈S_n} q_w s_w \) for \( q_w ∈ Λ₁ \) with \( q_w \) non-zero for some \( w > e \). Assume the image \( \partial_i(p) = \partial_i(\sum_w \partial_i(q_w)s_w) = \sum_w s_i(q_w)s_{s_iw} \)

were zero. By assumption, there was some non-zero \( q_w \) for \( w > e \), so there is still a non-zero term \( s_i(q_w)s_{s_iw} \) in \( \partial_i(p) \). However, since \( \partial_is_v = s_{s_iv} \) if \( s_iv \) is a reduced expression by Lemma 4.15.(i), \( \partial_i \) maps no other Schubert polynomial to \( s_{s_iv} \). Therefore, \( \partial_i(p) \) cannot be zero. There are thus no \( Λ₁ \)-linear combinations of Schubert polynomials in \( PolCᵢ \) not already contained in \( ker \partial_i \). This proves the statement.

3. We have already seen in Lemma 4.16 that \( NHCᵢ \) acts faithfully on the free \( Cᵢ \)-module \( PolCᵢ \). This action is compatible with the \( ΛPolCᵢ \)-module structure from 4.3: as \( NHCᵢ = NHCᵢ ⊗ Cᵢ, PolCᵢ \), we see that \( PolCᵢ \) acts on itself, \( NHCᵢ \) acts trivially on \( ΛPolCᵢ \) by the very definition of the latter and that \( NHCᵢ \) indeed acts on \( \mathfrak{H}_n \) because it acts by decreasing exponents. A graded rank computation

\[ rk_{q, Cᵢ} End_{ΛPolCᵢ}(\text{PolCᵢ}) = rk_{q, Cᵢ}(ΛPolCᵢ) · rk_{q, ΛPolCᵢ}(\text{PolCᵢ})^2 = \frac{(n)q!}{(n)q!} = rk_{q, Cᵢ} NHCᵢ \]

shows the asserted isomorphism. □

4.4. Complete symmetric polynomials

We give another collection of generators for \( ΛPolCᵢ \), namely a Clifford-analogue for the complete homogeneous symmetric polynomials.

Definition 4.18. Let \( M_{(n)} \) be the \( n × n \)-matrix

\[
M_{(n)} :=
\begin{pmatrix}
\begin{array}{cccc}
κ_{1,2}^{(n)} & & \\
-κ_{2,1}^{(n)} & κ_{1,3}^{(n)} & & \\
& \ddots & \ddots & \\
(-1)^{n-2}κ_{n−1,1}^{(n)} & & κ_{1,n}^{(n)} \\
(-1)^{n-1}κ_{n,1}^{(n)} & & & 0
\end{array}
\end{pmatrix}
\]

(4.18)
where for \( k < l - 1 \) we set

\[
\kappa_{k,l} := \gamma_{k+1,k} \gamma_{k+1,k+2} \gamma_{k+2,k+1} \gamma_{k+2,k+3} \cdots \gamma_{l-2,l-3} \gamma_{l-2,l-1} \gamma_{l-1,l-2} \gamma_{l-1,l-1}
\]

(4.19)

\[
\kappa_{k-1,l} := \gamma_{l-1,l-1},
\]

\[
\kappa_{l,l} := \gamma_{l,l+1}.
\]

Furthermore, let

\[
\tilde{\kappa}_{k,l} := \kappa_{k,l} \kappa_{l,l} = \gamma_{k+1,k} \gamma_{k+1,k+2} \cdots \gamma_{l-1,l-1} \gamma_{l-1,l} \gamma_{l,l+1}.
\]

Note that we can “split” the \( \kappa \)'s by

\[
\kappa_{k,l} = \gamma_{k+1,k} \gamma_{k+1,k+2} \cdots \gamma_{m-1,m-2} \gamma_{m-1,m} \gamma_{m,m+1} + \gamma_{m+1,m} \gamma_{m+1,m+2} \cdots \gamma_{l-1,l-2} \gamma_{l-1,l} \gamma_{l,l+1}
\]

(4.20)

The complete symmetric polynomial \( h_m^{(n)} \) of (polynomial) degree \( m \) on \( n \) indeterminates is the top left entry of the matrix power \( M_m^{(n)} \). In particular, \( h_1^{(n)} = \epsilon_1^{(n)} \) and \( h_0^{(n)} = \epsilon_0^{(n)} = 1. \)

**Remark 4.19.** The top row of \( M_m^{(n)} \) has entries

\[
(M_m^{(n)})_{1,*} = (h_m^{(n)}, h_{m-1}^{(n)} \tilde{k}_{1,2}, h_{m-2}^{(n)} \tilde{k}_{1,2} \tilde{k}_{1,3}, \ldots, \tilde{k}_{1,2} \cdots \tilde{k}_{1,m+1}, 0, \ldots, 0).
\]

**Example 4.20.** 1. If all indices are even and thus \( \epsilon_m^{(n)} = \epsilon_m^{(n)} \) and \( \kappa_{k,l} = 1 \) for all \( k, l \), these polynomials coincide with the ordinary complete symmetric polynomials

\[
h_m^{(n)} = \sum_{1 \leq k_1 \leq \cdots \leq k_m \leq n} x_{k_1} \cdots x_{k_m}.
\]

2. If all indices are odd, the resulting complete homogeneous symmetric polynomials coincide with those of [7] up to a renormalization involving the \( \kappa \)'s.

**Proposition 4.21.** We have

\[
\sum_{l=0}^{m} (-1)^l h_{m-l}^{(n)} \tilde{k}_{1,2} \cdots \tilde{k}_{1,l} \epsilon_l^{(n)} = \sum_{l=0}^{m} (-1)^l \tilde{k}_{1,2} \cdots \tilde{k}_{1,l} \epsilon_l^{(n)} h_{n-l}^{(n)} = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** The complete homogeneous symmetric polynomial \( h_m^{(n)} \) is the top left entry of the matrix power \( M_m^{(n)} = M_{m-1}^{(n)} \cdot M_m^{(n)} \) and therefore equals

\[
h_m^{(n)} = (h_{m-1}^{(n)}, h_{m-2}^{(n)} \tilde{k}_{1,2}, \ldots, \tilde{k}_{1,m+1}, 0, \ldots, 0) \cdot \left( \epsilon_1^{(n)}, -\epsilon_2^{(n)}, \ldots, (-1)^n \epsilon_n^{(n)} \right)^T.
\]

We obtain the recursive formula

\[
h_m^{(n)} = \sum_{l=1}^{m} (-1)^{l-1} h_{m-l}^{(n)} \tilde{k}_{1,2} \cdots \tilde{k}_{1,l} \epsilon_l^{(n)}
\]

for \( m \geq 1 \). Moving \( h_m^{(n)} \) to the right side proves the statement. For the second equality, we denote the remaining entries of \( M_m^{(n)} \) by

\[
\begin{pmatrix}
    h_m^{(n)} & h_{m-1}^{(n)} \tilde{k}_{1,2} & h_{m-2}^{(n)} \tilde{k}_{1,2} \tilde{k}_{1,3} & \cdots \\
    h_{(1),m+1}^{(n)} & h_{(1),m}^{(n)} \tilde{k}_{1,2} & \cdots \\
    \vdots & \ddots & \ddots \\
    h_{(2),m+2}^{(n)} & \ddots & \ddots
\end{pmatrix} := M_m^{(n)}.
\]
such that every $h_{(k,l)}^{(n)}$ is a polynomial of degree $l$. The polynomial $h_m^{(n)}$ is the top left entry of $M_m^{(n)} = M(n) \cdot M_m^{(n)-1}$ and is thus given by

$$h_m^{(n)} = \epsilon_1 h_{m-1}^{(n)} + \tilde{\kappa}_{1,2} h_{(1,m)}^{(n)}$$

$$h_{(1,m)}^{(n)} = -\epsilon_2 h_{m-2}^{(n)} + \tilde{\kappa}_{1,3} h_{(2,m)}^{(n)}$$

$$h_{(k-1,m)}^{(n)} = (-1)^k \epsilon_k h_{m-k}^{(n)} + \tilde{\kappa}_{1,k+1} h_{(k,m)}^{(n)}.$$ 

We obtain a recursion formula

$$h_m^{(n)} = \epsilon_1 h_{m-1}^{(n)} + \tilde{\kappa}_{1,2} \left( -\epsilon_2 h_{m-2}^{(n)} + \tilde{\kappa}_{1,3} (\ldots (\pm \epsilon_{m-1} h_1^{(n)} \mp \tilde{\kappa}_{1,m} \epsilon_m^{(n)} ) \ldots ) \right)$$

$$= \sum_{l=1}^{m} (-1)^{l-1} \tilde{\kappa}_{1,2} \cdots \tilde{\kappa}_{1,1} \epsilon_l^{(n)} h_{m-l}^{(n)}.$$

Again, moving $h_m^{(n)}$ to the right side proves the claim. \qed

### 4.5. Interlude: cohomology of Grassmann and partial flag varieties

We recall some classical (non-super) theory on the cohomology rings of Grassmann and partial flag varieties. Our aim is to build super-analogues for these rings.

**Definition 4.22.** Fix natural numbers $k \leq n$. The set of all $k$-dimensional subspaces $F \subseteq k^n$ of the $n$-dimensional $k$-vector space forms an algebraic variety, called Grassmann variety and denoted by $\text{Gr}(k, n)$. This is a special case of the following construction:

**Definition 4.23.** Fix natural numbers $\ell \leq n$. A partial flag $F_\bullet$ of length $\ell$ is a chain $F_\bullet$ of $k$-vector subspaces $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_\ell = k^n$. The set of all partial flags of length $\ell$ is the (partial) flag variety $\text{Fl}(\ell; n)$. If $\ell = n$, then $\text{Fl}(n) := \text{Fl}(n; n)$ is called the full flag variety. To a flag $F_\bullet$ of length $\ell$, one associates its dimension vector $k = (\dim_k F_0, \ldots, \dim_k F_\ell)$. The connected components of $\text{Fl}(\ell; n)$ are the varieties $\text{Fl}(k)$ of partial flags with a common dimension vector $k$. In particular, $\text{Gr}(k, n) = \text{Fl}(k)$ with $k = (0, k, n)$.

From now on, we assume that $k = \mathbb{R}$ or $k = \mathbb{C}$. In this case, the flag varieties are smooth manifolds. In the following, cohomology $H^*$ always means singular cohomology with coefficients in the field $k$.

**Proposition 4.24.** 18, prop. 3.6.15, rmk. 3.6.16; 9, 161] The full flag variety $\text{Fl}(n)$ has cohomology ring $H^*(\text{Fl}(n)) \cong \text{Pol}_n / (\Lambda \text{Pol}_n)_+$. For the dimension vector $k$ with $k_0 = 0$, $k_\ell = n$, the partial flag variety $\text{Fl}(k)$ has the cohomology ring

$$H_k := H^*(\text{Fl}(k)) \cong \Lambda \text{Pol}_{k_1 - k_0} \otimes_k \cdots \otimes_k \Lambda \text{Pol}_{k_\ell - k_{\ell-1}} / (\Lambda \text{Pol}_n)_+.$$ \hspace{1cm} (4.21)

We denote the nominator by $\Lambda \text{Pol}_k$.

**Corollary 4.25.** The Grassmann variety $\text{Gr}(k, n)$ has cohomology ring

$$H_{(k,n)} := H^*(\text{Gr}(k, n)) \cong \frac{k[\epsilon_1^{(n)}, \ldots, \epsilon_m^{(n)}]}{h_{n-k+1}^{(n)}, \ldots, h_{n}^{(n)}},$$ \hspace{1cm} (4.22)

where the $\epsilon$’s and $h$’s are the elementary and the complete symmetric polynomials, respectively.

Recall the notion of graded ranks from Section 4.1.
Lemma 4.26. The cohomology ring $H(k, n) := H^*(\text{Gr}(k, n))$ of the Grassmannian has graded dimension $\dim_{q,k} H(k, n) = \binom{n}{k}_q$ with the $q$-binomial coefficient $\binom{n}{k}_q := \frac{(n)_q!}{(k)_q!(n-k)_q!}$.

Proof. Without quotienting out the ideal, the polynomial ring $\Lambda \text{Pol}_k := k[\varepsilon_1, \ldots, \varepsilon_k]$ in the symmetric polynomials has graded dimension $\dim_{q,k} \Lambda \text{Pol}_k = (1 - q)^{-1} \cdots (1 - q^k)^{-1}$. To count the graded dimension of the ideal $a = (h_{n-k+1}, \ldots, h_n)$ to be quotiented out, we note that each generator $h_m$ contributes a principal ideal with graded dimension $\dim_q \Lambda \text{Pol}_m$. We then proceed by the inclusion/exclusion principle to obtain

$$
\dim_{q,k}(\Lambda \text{Pol}_k / a) = \frac{(1 - q^{n-k+1}) \cdots (1 - q^n)}{(1 - q) \cdots (1 - q^k)} = \frac{(1 - q)^{n-(n-k)}(n)_q!}{(n-k)_q!} = \binom{n}{k}_q^*.
$$

The proof is complete.

Corollary 4.27. Let $k := (0 = k_0 \leq \cdots \leq k_r = n)$ be a dimension vector. The cohomology ring $H_k$ of the partial flag variety $\text{Fl}(k)$ has as graded dimension the $q$-multinomial coefficient

$$
\dim_{q,k} H(k) = \binom{n}{k_1, k_2 - k_1, \ldots, k_r - k_{r-1}}_q := \frac{(n)_q!}{(k_1)_q!(k_2 - k_1)_q! \cdots (k_r - k_{r-1})_q!}.
$$

Proof. The cohomology ring is the quotient

$$
\Lambda \text{Pol}_{k_1} \otimes \cdots \otimes \Lambda \text{Pol}_{k_{r-1}} / \left(1 - (1 + \varepsilon_1^{(k_1)}) \cdots + \varepsilon^{(k_1)k_1}_1 \cdots (1 + \varepsilon_1^{(k_{r-1})} + \varepsilon^{(k_{r-1})k_{r-1}}) \right)
$$

of $\Lambda \text{Pol}_k := \Lambda \text{Pol}_{k_1} \otimes \cdots \otimes \Lambda \text{Pol}_{k_{r-1}}$. Denote the ideal quotiented out by $a$ and proceed as in the proof of Lemma 4.26. We see that the nominator has graded dimension

$$
\dim_{q,k} \Lambda = \prod_{a=1}^l \frac{1}{1 - q} \cdots \frac{1}{1 - q^{k_a - k_{a-1}}} = \frac{1}{(1 - q)^n} \prod_{a=1}^l \frac{1}{(k_a - k_{a-1})_q!},
$$

and the ideal a quotiented out has graded dimension

$$
\dim_{q,k} a = (1 - q)^n(n)_q! - 1 \dim_{q,k} \Lambda.
$$

This shows the assertion.

4.6. Cyclotomic quotients

It is known that in the non-super case, the cohomology ring $H(k,n)$ of Grassmann varieties (see Corollary 4.25) is Morita equivalent to the cyclotomic quotient of the NilHecke algebra [16, Section 5]. For the odd setting, the respective Morita equivalence has been established in [7, Section 5]. We introduce analogues of both rings in the Clifford setting and prove that they are Morita equivalent.

Definition 4.28. The $m$-th cyclotomic quotient of the Hecke Clifford superalgebra $N\mathbb{C}_n$ is the quotient $N\mathbb{C}_n^m := N\mathbb{C}_n / (\kappa_1, n y_m)^m$ by the two-sided ideal $\left(\kappa_1, n y_m\right)^m$, with the coefficients $\kappa_{k,l}$ from (4.18).

We call the quotient $H \mathbb{C}_{(m,n)} := \Lambda \text{Pol} \mathbb{C}_I / (h_k^{(n)})_{k_1 \geq n-m}$ by the two-sided ideal $(h_k^{(n)})_{k_1 \geq n-m}$ the Clifford Grassmann ring.
Remark 4.29. In contrast to the standard definition of the cyclotomic quotient, we mod out some power of $y_n$ instead of $y_1$; the latter would necessitate taking the transpose of the matrix $M_{(n)}$ in Definition 4.18.

Theorem 4.30. The rings $\mathcal{N}\mathcal{H}_n^m$ and $\mathcal{H}_c(m,n)$ are Morita-equivalent.

Proof. The proof in [16, Section 5] does not rely on commutativity of the rings involved and thus applies immediately in our setting. We review the argument briefly. The reader is strongly encouraged to verify the following calculation for $n = 3$. Let us start with

$$
\mathcal{C}_n^{(n)} = \gamma_1,2 y_1 \gamma_2,3 y_2 \cdots y_{n,n+1} y_n
$$

$$
= \left( \gamma_1,2 y_1 \gamma_2,3 y_2 \cdots y_{n-1,n} y_{n-1} \right) y_{n,n+1} y_n
$$

$$
= \left( \mathcal{C}_{n-1}^{(n)} - \tilde{\mathcal{C}}_{n-1}^{(n)} \right) y_{n,n+1} y_n
$$

where we set

$$
\tilde{\mathcal{C}}_{n-1}^{(n)} = \gamma_1,2 y_1 \cdots y_{n-2,n-1} y_{n-2} y_{n-1,n} + \cdots + y_2,1 y_2 \gamma_3,2 y_3 \cdots y_{n-1,n-1} y_n
$$

$$
= \left( \gamma_1,2 y_1 \cdots y_{n-2,n-1} y_{n-2} + \cdots + y_2,1 y_2 \gamma_3,2 y_3 \cdots y_{n-1,n-2} y_{n-1} \right) y_{n-1,n} y_n
$$

$$
= \left( \mathcal{C}_{n-2}^{(n)} - \tilde{\mathcal{C}}_{n-2}^{(n)} \right) y_{n,n-1} y_n
$$

where we set

$$
\tilde{\mathcal{C}}_{n-2}^{(n)} = \gamma_1,2 y_1 \cdots y_{n-3,n-2} y_{n-3} + \cdots + y_3,2 y_3 \cdots y_{n-1,n-2} y_{n-1} y_{n,n-1} y_n
$$

$$
= \left( \mathcal{C}_{n-3}^{(n)} - \tilde{\mathcal{C}}_{n-3}^{(n)} \right) y_{n-1,n-2} y_{n-1,n} y_{n,n-1} y_n
$$

...}

where

$$
\tilde{\mathcal{C}}_1^{(n)} = \gamma_{2,1} y_{2,3} \cdots y_{n-1,n-2} y_{n-1,n} y_{n,n-1} y_n.
$$

To state the above calculation differently, we have that

$$
0 = \mathcal{C}_n^{(n)} - \left( \mathcal{C}_{n-1}^{(n)} - \left( \cdots \left( \mathcal{C}_1^{(n)} - \kappa_{1,n} y_n \right) \cdots \right) \kappa_{n-1,n} y_n \right) \kappa_{n,n} y_n
$$

$$
= \sum_{k=0}^n (-1)^k \mathcal{C}_{n-k}^{(n)} \kappa_{n-k+1,n} y_n \cdots \kappa_{n,n} y_n
$$

$$
= (-1)^n \sum_{k=0}^n (-1)^k \mathcal{C}_k^{(n)} \prod_{l=k+1}^n \kappa_{l,n} y_n.
$$

(4.23)

Set $b_l := \prod_{l=k+1}^n \kappa_{l,n} y_n$ for $1 \leq k \leq n$; in particular $b_n := 1$. Recall the $\mathcal{C}_l$-bimodule $B_l$ from the proof of Theorem 4.17.(i). We define a similar free $\mathcal{C}_l$-bimodule of graded rank $(n)_q$. For multiindices $\alpha$, let

$$
B_\alpha := \left\langle y_1^\alpha b_1, \ldots, y_n^\alpha y_n^{n-1} \cdot b_n \right\rangle \mathcal{C}_l,
$$

(4.24)

$$
\tilde{\mathcal{C}}_l := \left\langle y_1^\alpha \cdots y_{n-1}^\alpha y_n^{n-1} \mid \alpha_i < \alpha_i \right\rangle \mathcal{C}_l
$$

(4.25)

It is analogous to the proof of Theorem 4.17.(i) that multiplication gives an isomorphism $\tilde{\mathcal{C}}_l \otimes \mathcal{C}_l \xrightarrow{\cong} \mathcal{C}_l$. Recall from (4.20) how to split the $\kappa$’s. Multiplication from the left by $\kappa_{1,n} y_n$ acts
on this basis of $B_{\alpha}$ by

$$(\kappa_{1,n} y_n) \colon B_{\alpha} \to B_{\alpha},$$

1 = b_n \mapsto \kappa_{1,n} y_n = \kappa_{1,n} \kappa_n y_n = \frac{\delta_{1,n} b_n}{\kappa_{1,n} \kappa_n y_n}.$$

Thus, the entries of the first column

$$M_m \cdot M_k n + m = \frac{\delta_{1,n} b_n}{\kappa_{1,n} \kappa_n y_n}.$$ 

This shows that multiplication with $\kappa_{1,n} y_n$ from the left acts on the basis from (4.25) by the matrix

$$\begin{pmatrix}
\xi_{1}^{(n)} & \tilde{k}_{1,2} \\
-\xi_{2}^{(n)} & \tilde{k}_{1,3} \\
\vdots & \ddots \\
(-1)^{n-2} \xi_{n-1}^{(n)} & \tilde{k}_{1,n} \\
(-1)^{n-1} \xi_{n}^{(n)} & 0
\end{pmatrix} = M_{(n)}.$$

Quotienting out the two-sided ideal $(\kappa_{1,n} y_n)^m$ of $NH_{C_n}$ is the same as requiring that $M_{(n)}^m = 0$.

**Claim.** The ideal $(h_{n-m+1}, \ldots, h_n)$ of $Pol_{\mathcal{I}}$ is also generated by the first column of $M_{(n)}^{m+1}$.

By definition, $h_{k+1}$ is the top left entry of $M_{(n)} \cdot M_{(n)}^k$ for any $k$. Recall that we denoted the entries of the first column $(M_{(n)}^{k+1})_{*1}$ of $M_{(n)}^{k+1}$ by

$$(M_{(n)}^{k+1})_{*1} := (h_{k+1}, h_{(1),k+2}, h_{(2),k+3}, \ldots, h_{(n-k),n})^T.$$

We thus have

$$h_{n-m+2} = \xi_{1}^{(n)} h_{n-m+1} + \tilde{k}_{1,2} h_{(1),n-m+2}$$

$$\equiv \tilde{k}_{1,2} h_{(1),n-m+2} \mod h_{n-m+1}$$

$$h_{n-m+3} = \xi_{2}^{(n)} h_{n-m+2} + \tilde{k}_{1,2} h_{(1),n-m+3}$$

$$\equiv \tilde{k}_{1,2} \tilde{k}_{1,3} h_{(2),n-m+3} \mod h_{n-m+1}, h_{n-m+2}$$

$$h_{n} \equiv \tilde{k}_{1,2} \cdots \tilde{k}_{1,n} h_{(n-1),n} \mod h_{n-m+1}, \ldots, h_{n-1}.$$

This proves the claim, since all $\tilde{k}$’s are units. Taking the product $M_{(n)}^m = M_{(n)}^{m-1} \cdot M_{(n)}^{n-1}$ shows that the last column of $M_{(n)}^m$ has entries

$$(M_{(n)}^m)_{*n} := (h_{n-m+1}, h_{(1),n-m+2}, \ldots, h_{(m-1),n})^T \cdot \tilde{k}_{1,2} \cdots \tilde{k}_{1,n}.$$

Thus, the entries of $(M_{(n)}^m)_{*n}$ also generate the ideal $(h_{n-m+1}, \ldots, h_n)$ of $Pol_{\mathcal{I}}$.

**Claim.** All entries of $M_{(n)}^m$ are $\Lambda Pol_{\mathcal{I}}$-linear combinations of entries of the last column $(M_{(n)}^m)_{*n}$.

Since $M_{(n)}^{k+1} = M_{(n)}^k M_{(n)}^1$ for any $k$, the entries of $(M_{(n)}^{k+1})_{*1}$ are $\Lambda Pol_{\mathcal{I}}$-linear combinations of entries of $(M_{(n)}^k)_{*1}$. Since $M_{(n)}^{k+1} = M_{(n)}^k M_{(n)}^1$, we have $(M_{(n)}^{k+1})_{*2} = \tilde{k}_{1,2} (M_{(n)}^k)_{*1}$. The entries of the first column
Table 4. Elementary \( \partial \)-symmetric polynomials \( \epsilon_m^{(k,n)} \) for \( n = 3 \), as defined in Definition 4.9.

| \( m \) | \( k+1 \) | 1 | 2 | 3 |
|-------|-------|---|---|---|
| 1 | \( \gamma_1,2 \gamma_1 + \gamma_2,1 \gamma_2 + \gamma_2,1 \gamma_2,3 \gamma_3,2 \gamma_3 \) | \( \gamma_2,3 \gamma_2 + \gamma_3,2 \gamma_3 \) | \( \gamma_3,4 \gamma_3 \) |
| 2 | \( \gamma_2,1 \gamma_2,3 \epsilon_1^{(1,3)} + \gamma_2,1 \gamma_2,3 \gamma_2,3 \gamma_3 \gamma_3 + \gamma_2,1 \gamma_2,2 \gamma_3,2 \gamma_3 \) | \( \gamma_3,2 \gamma_3 \gamma_3 + \gamma_3,2 \gamma_3 \gamma_3 \) | \( \gamma_3,4 \gamma_3 \) |
| 3 | \( \gamma_2,1 \gamma_2,3 \gamma_3,4 \gamma_3 \) | \( \gamma_3,4 \gamma_3 \) | \( \gamma_3,4 \gamma_3 \) |

The leftmost column is the same as in Table 1. The braces illustrates how one can construct the polynomials recursively as stated in Corollary 4.31, starting with the rightmost polynomial \( \epsilon_3^{(2,3)} = \gamma_3,4 \gamma_3 \).

Thus are linear combinations of entries of the second column \( (M_{(n)}^{k+1})_{m,1} \); the claim follows iteratively.

Altogether, we have shown that requiring \( M_{(n)}^m = 0 \) is the same as quotienting out the ideal \( (h_{n−m+1}, \ldots, h_n) \) of \( \text{Pol} \mathcal{C}_I \). Applying this to each summand \( B_{\alpha} \) of (4.25) yields

\[
\text{NHe}_{n}^{m} \equiv \text{Mat}_{n!}((\text{Pol} \mathcal{C}_I)/(M_{(n)}^m)) \cong \text{Mat}_{n!}((\text{He}(\mathcal{C}_{(m,n)})),
\]

where \( (M_{(n)}^m) \) is the two-sided ideal generated by matrices \( M_{(n)}^m \). This establishes the asserted Morita equivalence.

**4.7. Clifford superalgebras associated to partial flag varieties**

Let \( \epsilon_m^{(k,n)} \) be the polynomial of degree \( m \) in the indeterminates \( y_k, \ldots, y_n \) that is obtained from \( \epsilon_m^{(n−k)} \) by replacing every index \( i \) by \( i + k \). Recall from Definition 4.10 the recursion formula for the elementary \( \partial \)-symmetric polynomials. We can easily derive the following corollary by regrouping the terms of (4.5)–(4.9):

**Corollary 4.31.** The elementary \( \partial \)-symmetric polynomials satisfy the recursion formula

\[
\epsilon_m^{(n)} = \gamma_1,2 \epsilon_1^{(1,n)} + \gamma_2,1 \gamma_2,3 \epsilon_1^{(1,n)}
\]

for the elementary \( \partial \)-symmetric polynomials, where \( \gamma_{i,i±1} \) is as defined in Lemma 4.8.

**Example 4.32.** The “regrouping” of terms is best made explicit by considering the first \( \partial \)-symmetric polynomials listed explicitly in Table 1. We can start with the polynomial \( \epsilon_1^{(2,3)} := \gamma_3,4 \gamma_3 \) as defined in Definition 4.9 and construct the polynomials \( \epsilon_m^{(k,3)} \) for \( 0 \leq k < 3 \) and \( m \leq n − k \) as described in the corollary. The resulting polynomials are listed in Table 4 with the grouping indicated by braces.

The recursive relations in (4.5)–(4.9) and (4.26) from Definition 4.10 and Corollary 4.31 are subsumed in the following definition:

**Definition 4.33.** For \( 1 < k < n \), let \( \lambda_{m,l}^{(0,k,n)} \in \mathbb{C}_I \) be the coefficients defined by

\[
\epsilon_m^{(n)} = \sum_{0 \leq l \leq m} \lambda_{m,l}^{(0,k,n)} \epsilon_l^{(k,n)}.
\]

(4.27)
Example 4.34. By (4.5)–(4.9) and (4.26) we know the coefficients \( \lambda_{m,l}^{(0,k,n)} \) for two special cases:

1. \( \lambda_{m,l}^{(0,n-1,n)} \) is the coefficient of the expansion \( e_m^{(n)} = \sum_l \lambda_{l}^{(n-1)} \gamma_{l,n+1}^{y_n} \). The recursion formula from Definition 4.10 gives

\[
\lambda_{m,l}^{(0,n-1,n)} = \begin{cases} 
1 & \text{if } l = m \\
\gamma_{m+1,m} \gamma_{m+1,m+2} \cdots \gamma_{n-1,n-2} \gamma_{n,n-1} & \text{if } l = m - 1 \\
0 & \text{otherwise.}
\end{cases}
\]

2. \( \lambda_{m,l}^{(0,1,n)} \) is the coefficient of the expansion \( e_m^{(n)} = \sum \gamma_{1,2} \lambda_i^{(n-1)} \). By Corollary 4.31 these are given by

\[
\lambda_{m,l}^{(0,1,n)} = \begin{cases} 
\gamma_{2,1} \gamma_{2,3} & \text{if } l = 0 \\
1 & \text{if } l = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

In general, however, it seems to be difficult to provide an explicit formula for \( \lambda_{m,l}^{(0,k,n)} \) for arbitrary \( k \).

Definition 4.35. Recall from Definition 4.10 that we defined \( \Lambda Pol I \) by (4.30). The two inclusions can be written explicitly in terms of \( \lambda_{m,l}^{(0,k,n)} \). By Corollary 4.31 these are given by

\[
\lambda_{m,l}^{(0,1,n)} = \begin{cases} 
\gamma_{2,1} \gamma_{2,3} & \text{if } l = 0 \\
1 & \text{if } l = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

In particular, the \( \lambda_{m,l}^{(0,1,n)} \) is the coefficient of the expansion \( e_m^{(n)} = \sum \gamma_{1,2} \lambda_i^{(n-1)} \). By Corollary 4.31 these are given by

\[
\lambda_{m,l}^{(0,1,n)} = \begin{cases} 
\gamma_{2,1} \gamma_{2,3} & \text{if } l = 0 \\
1 & \text{if } l = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Remark 4.36. Note that Theorem 4.17.(ii) implies that for each \( \lambda_{m,l}^{(k_j,k_{j+1})} \) the obvious inclusion in (4.28) is indeed an equality.

4.7.1. One step flag varieties

For \( 0 < j < k \), let \((k_0, \ldots, k_j, \ldots, k_l)\) denote the tuple \((k_0, \ldots, k_l)\) with the entry \( k_j \) omitted. Consider

\[
\Lambda Pol I^{(k_0, \ldots, k_j, \ldots, k_l)} = \bigcap_{0 < i < k_l} \ker \partial_i; \quad \Lambda Pol I^{(k_0, \ldots, k_j, \ldots, k_l)} = \bigcap_{0 < i < k_l} \ker \partial_i
\]

that is, in contrast to (4.28), the intersection with \( \ker \partial_{k_j} \) is not omitted. Clearly there is an inclusion

\[
\Lambda Pol I^{(k_0, \ldots, k_j, \ldots, k_l)} \subseteq \Lambda Pol I^{(k_1, \ldots, k_l)}
\]

of superalgebras.

Lemma 4.37. Let \( k = (0,k,k+1,\ldots,n) \) for some \( k \), so that there are inclusions

\[
\Lambda Pol I^{(0,\ldots,k,n)} \subseteq \Lambda Pol I^{(0,\ldots,k,k+1,\ldots,n)} \subseteq \Lambda Pol I^{(0,\ldots,k+1,\ldots,n)}
\]

by (4.30). The two inclusions can be written explicitly in terms of \( \delta \)-symmetric polynomials, namely

\[
\Lambda Pol I^{(0,k,n)} \subseteq \Lambda Pol I^{(0,k+1,n)}
\]
These inclusions turn
\[ \Lambda \text{Pol}_\bullet(0,k+1,n) \hookrightarrow \Lambda \text{Pol}_\bullet(0,k,k+1,n) \]
\[ e_m^{(0,k)} \mapsto e_m^{(0,k)} \]
\[ e_m^{(k,n)} \mapsto \gamma_{k+1,k+2} y_{k+1} e_{m-1}^{(k+1,n)} + y_{k+1,k} y_{k+1,k+2} e_m^{(k+1,n)} \]
(4.33)

Proof. A polynomial from \( \Lambda \text{Pol}_\bullet(0,k,n) \) is contained in all \( \ker \delta_i \) for \( i \in \{1, \ldots, k-1, k+1, k+2, \ldots, n\} \) by Definition 4.10. It is a fortiori contained in all \( \ker \delta_i \) for \( i \in \{1, \ldots, k-1, k+2, \ldots, n\} \); i.e., in \( \ker \delta_1 \cap \cdots \cap \ker \delta_{k-1} \cap \ker \delta_{k+2} \cap \cdots \cap \ker \delta_n \).

We know from Theorem 4.17.(ii) and Remark 4.36 that this intersection equals \( \Lambda \text{Pol}_\bullet(0,k,k+1,n) \). The coefficients for writing elementary \( \delta \)-symmetric polynomials from \( \Lambda \text{Pol}_\bullet(0,k,n) \) in terms of those polynomials from \( \Lambda \text{Pol}_\bullet(0,k,k+1,n) \) are given in Example 4.34. This proves the statement. \( \square \)

Remark 4.38. These inclusions turn \( \Lambda \text{Pol}_\bullet(0,k,k+1,n) \) into a \( \Lambda \text{Pol}_\bullet(0,k,n) \)-\( \Lambda \text{Pol}_\bullet(0,k+1,n) \)-bimodule. In the even case, this bimodule structure corresponds to the one described in [13, (5.17)]. In contrast to the notation in [13], we let \( \Lambda \text{Pol}_\bullet(0,k,n) \) act from the left and \( \Lambda \text{Pol}_\bullet(0,k+1,n) \) from the right.

4.7.2. Clifford flag rings
Recall the Clifford Grassmann ring \( HC_{(m,n)} \) from Definition 4.28 and the ordinary cohomology ring of partial flag varieties from Proposition 4.24. The following definition generalizes both:

Definition 4.39. For dimension vector \( k = (0 = k_0 \leq k_1 \leq \cdots \leq k_\ell = n) \), let the Clifford flag ring be the quotient
\[ HC_k := \Lambda \text{Pol}_\bullet(k_1,\ldots,k_\ell) / (\Lambda \text{Pol}_\bullet(0,n) + \Lambda \text{Pol}_\bullet(0,k,n)) \] by the two-sided ideal generated by non-constant \( \delta \)-symmetric polynomials.

Remark 4.40. The graded dimension computations carried out in Lemma 4.26 and Corollary 4.27 remain valid for \( HC_k \) if one replaces \( \dim_{q,k} \) by \( \text{rk}_{q,k} \).

Proposition 4.41. The quotient \( \Lambda \text{Pol}_\bullet(0,k,n) / (\Lambda \text{Pol}_\bullet(0,n) + \Lambda \text{Pol}_\bullet(0,k,n)) \) is isomorphic to the Clifford Grassmann ring \( HC_{(k,n)} \) defined in Definition 4.28.

Proof. We may expand the generators \( e_k^{(0,n)} \) as in (4.27). In the quotient, we therefore obtain relations
\[ \sum_{l=0}^m \epsilon_l^{(0,k)} \lnu_{m,l}^{(0,k,n)} \lnu_{m-l}^{(k,n)} = \delta_{m,0} \]
Comparing coefficients with those of the identity
\[ \sum_{l=0}^m (-1)^l h_{m-l}^{(k,n)} \lnu_{k,k+1} \cdots \lnu_{k,k+l} \epsilon_l^{(k,n)} = \delta_{m,0} \]
from Proposition 4.21 implies that
\[ (-1)^l h_{m-l}^{(k,n)} \lnu_{k,k+1} \cdots \lnu_{k,k+l} = \epsilon_{m-l}^{(0,k)} \lnu_{m,l}^{(0,k,n)} \] (4.35)
we thus indeed have an isomorphism \( \Lambda \text{Pol}_\bullet(0,k,n) / (\Lambda \text{Pol}_\bullet(0,n) + \Lambda \text{Pol}_\bullet(0,k,n)) \cong HC_{(k,n)} \). \( \square \)
We have arrived at a super-generalization for the cohomology rings of Flag varieties. In the ordinary set-up, it has been proven in [13] that these rings admit an action by the Kac-Moody 2-category from [4]. The latter has a super-analogue constructed in [3]. A natural question to finish this manuscript is if one can construct an action of the Kac-Moody 2-supercategory from [3].

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