Universal Description of Spherical Aberration Free Lenses Composed of Positive or Negative Index Media

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We find that the function that describes the surface of spherical aberration free lenses can be used for both positive and negative refractive index media. With the inclusion of negative index, this function assumes the form of all the conic sections and expands the theory of aplanatic optical surfaces. There are two different symmetry centers with respect to the index that create an asymmetric relationship between positive and negative index lens profiles. In the thin lens limit the familiar formulas for image position and magnification hold for any index.

It was known to René Descartes as early as 1637 that aspherical surfaces are required to convert plane waves into spherical waves and vice versa. In the short wavelength limit of geometric optics, the right conic surface can bring all parallel incident light rays to a single common focus point. Most commercial lenses have spherical profiles and only perform this function approximately; they are said to have spherical aberration. It is only somewhat well known that a hyperboloid is the ideal lens shape for this purpose when using media with a relative refractive index greater than one. Less well known, is the fact that an ellipsoid is the ideal shape for a lens composed of media with relative index between zero and one.

Much of the fundamental behavior of media with negative refractive index was predicted in 1968 by Veselago. Such media are not naturally occurring, so their properties were not extensively pursued at that time. Recently, negative index media were demonstrated using engineered composites, and the same design paradigm used for those composites also enables the construction of media with index less than one and even somewhat close to zero. This makes the question of optical element design with these media pertinent. In this paper we will discuss ideal (i.e. spherical aberration free) shapes for lenses composed of media with both positive and negative index. We will derive universal formulas for lens design that apply to media of any index, and discuss the symmetry between positive and negative index lenses. We will also demonstrate that, with the inclusion of negative index, a more compete theory of aplanatic optical surfaces is obtained.

The lenses discussed here need to be clearly distinguished from the “perfect” lens of Pendry. These lenses can focus incident plane waves, i.e. waves from a source object at infinite distance. They are said to possess an aplanatic point at infinity. Plane waves are not focused by the “perfect” lens; it has aplanatic points only at distances within one lens thickness. The importance of the “perfect” lens lies in its near field focusing capability. This work describes lenses that can focus or collect energy from distant sources where the lenses are composed of materials from an extended parameter space, namely negative refractive index.

Aspherical lenses are uncommon because they are difficult to manufacture. The requirement of $\lambda/4$ to $\lambda/10$ surface roughness for good optics is quite difficult to achieve in the visible range for non spherical surfaces. However, this requirement is automatically satisfied for surfaces on the composite materials that have been used to implement negative index media. These media are composed of unit cells that must be significantly smaller than the operational wavelength in order for them to function as a homogenous media. The manufacturing process that enables construction on the required sub unit cell length scales can by default control the surface profile with one unit cell accuracy. Though the current technology may never extend to visible light, whatever the operational wavelength of these media, sub-wavelength surface accuracy for any desired shape is available.

The results discussed here were first obtained by ray tracing. The ray tracing algorithm employed only fundamental assumptions about the media dispersion, boundary matching, and the conservation of energy. For
FIG. 2: Spherical aberration free converging and diverging lenses composed of positive and negative refracting media. The unused portion of the conic section is shown dashed. Foci are shown as circles. Virtual image rays are shown dashed. All lenses have the same focal length.

For all rays, the optical path length or phase advance (retardation) is

\[ OPL = \int_C nds \]  

The equivalence of the OPL in the transition region for the two rays shown in Fig. 1 yields,

\[ \sqrt{(f - z)^2 + \rho^2} - f = -nz \]  

From this we find

\[ \frac{(z-a)^2}{a^2} + \frac{\rho^2}{b^2} = 1 \]  

which is an equation for conic sections where the \( a, b \) parameters are given in terms of the index, \( n \), and focal distance, \( f \).

\[ a = f \frac{1}{1 + n} \]  \hspace{1cm} \[ b^2 = f \frac{1 - n}{1 + n} \]  

Since \( b^2 \) can be negative, Eq. (3) can and does generate all the conic sections for different values of the index. Further, nothing in this derivation precludes \( n \) from being negative, as long as the optical path length as given by Eq. (1) is applicable. We can see this as follows. Geometric rays follow the energy direction indicated by the Poynting vector, with each ray carrying power into a surface being required to carry power out.

Here, however, we show an analytical argument using optical path length (OPL) following Hecht [1]. We wish to find a surface that refracts parallel rays to a single point (Fig. 1). Parallel rays have planar wave fronts on which the phase is constant. Rays converging to a point have spherical wave fronts on which the phase is constant. The desired lens surface must be contoured such that, the optical path length in the transition region is the same.

\[ z(\rho) = a \left( 1 - \sqrt{1 - \frac{\rho^2}{b^2}} \right) \]  

This equation is valid for all values of \( f \) and \( n \), except \( n = -1 \), where it is singular. This limiting case is handled by letting \( n = -1 + \delta \) with \( \delta \ll 1 \), and substituting into Eq. (5) with Eq. (4). We then find

\[ z(\rho) \simeq \frac{1}{4f} \rho^2 \]  

which is a paraboloid.

From Eqs. (4), (5), and (6), we find the following behavior for the ideal lens surface. These results are also

| \( n = 2 \) | \( n = -2 \) |
| --- | --- |
| \( n = 0.45 \) | \( n = -0.45 \) |
| \( n = 0 \) | \( n = -1 \) |
FIG. 3: Eccentricity (black line) and major axis radius of curvature (gray line) vs. refractive index. An eccentricity, $\varepsilon$, of one is the boundary between hyperbolic and ellipsoidal lens surfaces. A radius, $R$, of zero is the boundary between concave and convex surfaces. The concavity indicated on this boundary in the figure is for converging lenses. The inset shows the definition of $R$ and $\varepsilon$ for concave and convex hyperbolic lenses.

confirmed by ray tracing, Fig. 2

| index | conic | converge/diverge | focus |
|-------|-------|-----------------|-------|
| $1 < n$ | hyperboloid | convex/concave | far |
| $n = 1$ | no solution | - | - |
| $0 < n < 1$ | ellipsoid | concave/convex | far |
| $n = 0$ | sphere | concave/convex | far = near |
| $-1 < n < 0$ | ellipsoid | concave/convex | near |
| $n = -1$ | paraboloid | concave/convex | near |
| $n < -1$ | hyperboloid | concave/convex | near |

The conic sections characterized by the $a, b$ parameters above, can be equally well characterized by another set of parameters, the eccentricity, $\varepsilon$, and the major axis radius of curvature, $R$. (Fig. 3 inset). This pair of parameters is useful in that it separates the shape, given by $\varepsilon$, from the length scale, given by $R$. Using the usual definition, eccentricities less than one indicate ellipses and greater than one indicate hyperbolas. Applying Eq. 4 leads to the simplification

$$\varepsilon \equiv \sqrt{a^2 - b^2} / |a| = |n|$$  \hspace{1cm} (7)

Thus the eccentricity is symmetric in $n$, (Fig. 3). If this were the only parameter, we could take any positive index lens design and use it for lenses composed of negative index media.

The radius of curvature can be calculated from Eq. 5.

$$R \equiv \frac{1}{z''(0)} = \frac{b^2}{f(1 - n)}$$  \hspace{1cm} (8)

This is recognized as the familiar lens makers formula usually derived using spherical optics with the paraxial approximation. This is not surprising since any of the conic sections approximate a sphere when examined with small enough aperture; the two derivations must agree on the lens central radius of curvature. Ray tracing confirms this formula applies for both positive and negative index and correctly gives a change in the concavity at $n = 1$. Because this formula is antisymmetric about $n = 1$, (Fig. 3), we cannot use the same lens design for positive and negative index. If we do, the positive and negative index lenses will have different focal lengths. In fact, for $|n| > 1$, one lens will be converging and the other diverging. Notably, the positive index analog of the $n = -1$ parabolic lens is absent, since the $n = 1$ lens requires zero radius of curvature.

We have not yet shown how, if at all, the position of the optical focus of one of these lenses relates to the geometric foci of its conic surface. It can be easily shown algebraically or by ray tracing, that the optical focus lies exactly at the position of one of the conic section's foci. Ray tracing also confirms a simple rule that specifies which focus. For positive index the correct focus is the one furthest from the branch of the conic that is used for the lens surface. For negative index, the opposite is true. For the lenses with parabolic ($n = -1$) and spherical ($n = 0$) surfaces that possess just one geometric focus, it is also the optical focus.

As pointed out in Hecht [1], two conic surfaces can be combined to construct lenses that convert diverging spherical waves to converging spherical waves. Ray tracing confirms that in the thin lens limit, the usual formulas
apply for image distance
\[ \frac{1}{f} = \frac{1}{o} + \frac{1}{i} \] (9)

and magnification
\[ M = -\frac{i}{o} = \frac{f}{f - o} \] (10)

regardless of the sign of the index. In Fig. 4 a double concave parabolic lens is shown to function as a converging lens when composed of \( n = -1 \) material.

The extension of spherical aberration free lens design to negative index media yields a more complete theory, but the practical limitations of current negative index media, notably strong chromatic aberration (due to media dispersion) may preclude technological interest in frequency ranges where good positive index materials are available. However, in frequency ranges where this is not the case, (e.g. the millimeter wave range), and composite media are already in use, negative index lenses can be advantageous. Where larger radius of curvature is desirable, it is worth noting that an \( n = -2 \) lens has three times the radius of curvature of an \( n = 2 \) lens of the same focal length, (Fig. 2). Even more significant is the fact that an \( n = -1 \) lens is impedance matched to the surrounding media \( \Box \); none of the incident radiation is reflected. This is impossible with a positive index lens.

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