Gravitational Perfect Fluid Collapse With Cosmological Constant

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Abstract

In this paper, the effect of a positive cosmological constant on spherically symmetric collapse with perfect fluid has been investigated. The matching conditions between static exterior and non-static interior spacetimes are given in the presence of a cosmological constant. We also study the apparent horizons and their physical significance. It is concluded that the cosmological constant slows down the collapse of matter and hence limit the size of the black hole. This analysis gives the generalization of the dust case to the perfect fluid. We recover the results of the dust case for $p = 0$.

Keywords : Gravitational Collapse, Perfect Fluid, Cosmological Constant

1 Introduction

The cosmological constant is an energy associated with the vacuum, i.e., with empty space. The inclusion of the non-zero cosmological constant into the Einstein field equations has been discussed several times in the past for theoretical and observational reasons [1]. First, it has been introduced by Einstein to save the universe from expanding and rejected by him after

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expansion has been discovered by Hubble. The results of type Ia supernova [2,3] show that the universe is accelerating rather than decelerating. These results suggest that our universe can have a non-zero cosmological constant. Another analysis [4] of the peculiar motion of low-red shift galaxies give further evidence for the possibility of finite cosmological constant. These results have increased interest to study the properties of the universe with a non-zero cosmological constant. The physical applications of a cosmological constant are huge, restricting not only the growth of the universe but also the structure formation and age problems. The cosmological constant affects the properties of spacetime and matter. Since the metric of the spacetime and the stress-energy tensor of matter are related through the Einstein field equations, the effects of a cosmological constant can be analyzed by specifying the metric and the stress-energy tensor. We study gravitational collapse to see such effects.

Gravitational collapse is one of the important issues in General Relativity. This theory predicts solutions with singularities and such solutions can be produced by the gravitational collapse of non-singular, asymptotically flat initial data [5-7]. Spacetime singularities can be classified into two kinds whether they can be observed or not. A spacetime singularity is said to be naked when it is observable to local or distant observer. If such singularity can reach the neighboring or asymptotic regions of spacetime, the singularity is called locally or globally naked singularity. A spacetime singularity which can not be observed is called a black hole. Is such a singularity formed in our universe? Penrose [8] proposed so-called the cosmic censorship conjecture to resolve this problem. According to this conjecture, the singularities that appear in the gravitational collapse are always covered by an event horizon. This conjecture has provided a strong motivation for researchers in this field. The compact stellar objects such as white dwarf and neutron star are formed by a period of gravitational collapse. It is interesting to consider the appropriate geometry of interior and exterior regions and determine proper junction conditions which allow the matching of these regions.

Most of the problems related to gravitational collapse have been discussed by considering spherically symmetric system. The gravitational collapse of dust was first shown by Oppenheimer and Snyder [9]. They studied collapse by considering static Schwarzschild in the exterior and Friedman like solution in the interior. Many people [10-14] extended the above study of collapse by taking an appropriate geometry of interior and exterior regions. Markovic and Shapiro [15] generalized the work done by Oppenheimer and Snyder [9]
in the presence of positive cosmological constant. Later, Lake [16] generalized the results of Markovic and Shapiro [15] for both positive and negative cosmological constant. Cissoko et al. [17] discussed explicitly gravitational dust collapse with positive cosmological constant. Recently, the same work has been generalized by Ghosh and Deshkar [18] for higher dimensional dust collapse with cosmological constant.

In this paper, we discuss the gravitational collapse with cosmological constant for perfect fluid case. It is verified that our results reduce to the dust case as given by Cissoko et al. [17]. The paper is outlined as follows. In section 2, we give the junction conditions between a static and a non-static spherically symmetric spacetimes. Section 3 yields the spherically symmetric perfect fluid solution of the Einstein field equations with a cosmological constant. In section 4, we discuss the solution with some assumptions. Section 5 is devoted to investigate the apparent horizons and the role of the cosmological constant. Finally, we summarize the results in section 6.

2 Junction Conditions

We consider a timelike 3D hypersurface $\Sigma$, which divides $4D$ spacetime into two regions interior and exterior spacetimes, denoted by $V^+$ and $V^-$ respectively. For the interior spacetime, we consider spherically symmetric system given by

$$ds_+^2 = dt^2 - X^2 dr^2 - Y^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(1)

where $X$ and $Y$ are functions of $t$ and $r$ only. For the exterior spacetime, we take the Schwarzschild-de Sitter metric,

$$ds_+^2 = F dT^2 - \frac{1}{F} dR^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(2)

where

$$F(R) = 1 - \frac{2M}{R} - \frac{\Lambda}{3} R^2,$$

(3)

$M$ is a constant and $\Lambda$ is the cosmological constant. According to the junction conditions [19,20], it is assumed that the first and second fundamental forms from the interior and the exterior spacetimes are the same. These conditions can be expressed as
(i) The continuity of the first fundamental form over $\Sigma$ gives
\[(ds^2)_\Sigma = (ds^2)_{\Sigma} = ds^2_{\Sigma}.\] (4)

(ii) The continuity of the second fundamental form over $\Sigma$ gives
\[[K_{ab}] = K^+_ab - K^-_{ab} = 0, \quad (a,b = 0,2,3),\] (5)

where $K_{ab}$, the extrinsic curvature, is given by
\[K^\pm_{ab} = -n^\pm_\sigma \left( \frac{\partial^2 x^\sigma_\pm}{\partial \xi^a \partial \xi^b} + \Gamma^\sigma_{\mu \nu} \frac{\partial x^\mu_\pm}{\partial \xi^a} \frac{\partial x^\nu_\pm}{\partial \xi^b} \right), \quad (\sigma, \mu, \nu = 0, 1, 2, 3).\] (6)

Here the Christoffel symbols $\Gamma^\sigma_{\mu \nu}$ are calculated from the interior or exterior metrics (1) or (2), $n^\pm_\mu$ are the components of outward unit normals to $\Sigma$ in the coordinates $x^\sigma_\pm$. The equations of hypersurface $\Sigma$ in the coordinates $x^\sigma_\pm$ are written as
\[f_-(r,t) = r - r_\Sigma = 0,\] (7)
\[f_+(R,T) = R - R_\Sigma(T) = 0,\] (8)

where $r_\Sigma$ is a constant.

Using Eq.(7) in (1), the metric on $\Sigma$ takes the form
\[(ds^2)_\Sigma = dt^2 - [Y(r_\Sigma,t)]^2(d\theta^2 + \sin^2 \theta d\phi^2).\] (9)

Similarly, Eqs.(2) and (8) yield
\[(ds^2)_{\Sigma} = [F(R_\Sigma) - \frac{1}{F(R_\Sigma)} \left( \frac{dR_\Sigma}{dT} \right)^2]dT^2 - R_\Sigma^2(d\theta^2 + \sin^2 \theta d\phi^2),\] (10)

where we assume that
\[F(R_\Sigma) - \frac{1}{F(R_\Sigma)} \left( \frac{dR_\Sigma}{dT} \right)^2 > 0\] (11)

so that $T$ is a timelike coordinate. From Eqs.(4), (9) and (10), it follows that
\[R_\Sigma = Y(r_\Sigma,t),\] (12)
\[\left[ F(R_\Sigma) - \frac{1}{F(R_\Sigma)} \left( \frac{dR_\Sigma}{dT} \right)^2 \right] \frac{1}{2} dT = dt.\] (13)
Now from Eqs. (7) and (8), the outward unit normals in $V^-$ and $V^+$, respectively, are given by

\begin{align*}
  n^-_\mu &= (0, X(r_\Sigma, t), 0, 0), \\
  n^+_\mu &= (-\dot{R}_\Sigma, \dot{T}, 0, 0),
\end{align*}

where dot means differentiation with respect to $t$. The components of the extrinsic curvature $K^\pm_{ab}$ are

\begin{align*}
  K^-_{00} &= 0, \\
  K^-_{22} &= \csc^2 \theta K^-_{33} = \left(\frac{YY'}{X}\right)_\Sigma, \\
  K^+_{00} &= (\dot{R}\ddot{T} - \dot{T}\ddot{R} - \frac{F}{2} \frac{dF}{dR} t^3 + \frac{3}{2F} \frac{dF}{dR} \ddot{T}\ddot{R})_\Sigma; \\
  K^+_{22} &= \csc^2 \theta K^+_{33} = (FR\ddot{T})_\Sigma.
\end{align*}

The continuity of the extrinsic curvature gives

\begin{align*}
  K^+_{00} &= 0, \\
  K^-_{22} &= K^+_{22}.
\end{align*}

When we use Eqs. (16)-(21) along with Eq. (3), the junction conditions turn out to be

\begin{align*}
  (X\dot{Y}' - \dot{X}Y')_\Sigma &= 0, \\
  M &= \left(\frac{Y}{2} - \frac{\Lambda}{6} Y^3 + \frac{Y}{2} \dot{Y}^2 - \frac{Y}{2X^2} Y''^2\right)_\Sigma.
\end{align*}

### 3 Solution of the Field Equations

The Einstein field equations for perfect fluid with cosmological constant are given by

\[ R_{\mu\nu} = 8\pi[(\rho + p)u_\mu u_\nu + \frac{1}{2}(p - \rho)g_{\mu\nu}] - \Lambda g_{\mu\nu}, \]

where $\rho$ is the energy density, $p$ is the pressure and $u_\mu = \delta^0_\mu$ is the four-velocity in co-moving coordinates. These equations for the line element (1) take the form

\[ R_{00} = -\frac{\ddot{X}}{X} - 2\frac{\ddot{Y}}{Y} = 4\pi(\rho + 3p) - \Lambda, \]
Now we solve these equations. When we integrate Eq.(29) w.r.t. \( t \), we get

\[
X = \frac{Y'}{W},
\]

where \( W = W(r) \) is an arbitrary function of \( r \). Using this value of \( X \) in Eqs.(25)-(27), it follows that

\[
2\frac{\ddot{Y}}{Y} + \left(\frac{\dot{Y}}{Y}\right)^2 + \frac{1 - W^2}{Y^2} = \Lambda - 8\pi p.
\]

Integrating this equation w.r.t. \( t \), it turns out that

\[
\dot{Y}^2 = W^2 - 1 + 2m + (\Lambda - 8\pi p)\frac{Y^2}{3},
\]

where \( m = m(r) \) is an arbitrary function of \( r \) and is related to the mass of the collapsing system. When we use Eqs.(30) and (32) in (25), we obtain

\[
m' = 4\pi(\rho + p)Y^2Y' - \frac{1}{3}8\pi pY^3.
\]

For physical reasons, we assume that density and pressure are non-negative. Integration of Eq.(33) w.r.t \( r \) yields

\[
m(r) = 4\pi \int_0^r (\rho + p)Y^2Y' dr - \frac{8\pi}{3} \int_0^r pY^3 dr.
\]

Here we take constant of integration to be zero. The function \( m(r) \) must be positive, because \( m(r) < 0 \) implies negative mass which is not physical. Using Eqs.(30) and (32) into the junction condition (23), we obtain

\[
M = m - \frac{4\pi p}{3}Y^3.
\]
We see from Eq.(3) that the exterior spacetime becomes the Schwarzschild spacetime for $\Lambda = 0$ and $M$ as the total energy inside the surface $\Sigma$ due to its Newtonian asymptotic behaviour. The total energy $\tilde{M}(r, t)$ up to a radius $r$ at time $t$ inside the hypersurface $\Sigma$ can be evaluated by using the definition of the mass function [10,17]. For the metric (1), it takes the following form

$$\tilde{M}(r, t) = \frac{1}{2} Y^3 R_{23} = \frac{1}{2} Y[1 - \left(\frac{Y'}{X}\right)^2 + \dot{Y}^2].$$  \hspace{1cm} (36)

Using Eqs.(30) and (32) in Eq.(36), it follows that

$$\tilde{M}(r, t) = m(r) + (\Lambda - 8\pi p)\frac{Y^3}{6}.$$ \hspace{1cm} (37)

### 4 Solution With $W(r) = 1$

In this section, we consider the case $\Lambda - 8\pi p > 0$ and the condition $W(r) = 1$. \hspace{1cm} (38)

Using Eqs.(30), (32) and (38), we obtain the analytic solutions in closed form as

$$Y(r, t) = \left(\frac{6m}{\Lambda - 8\pi p}\right)^{\frac{1}{4}} \sinh^{\frac{3}{4}} \alpha(r, t),$$ \hspace{1cm} (39)

$$X(r, t) = \left(\frac{6m}{\Lambda - 8\pi p}\right)^{\frac{1}{4}} \left[\left\{\frac{m'}{3m} + \frac{16\pi mp'}{(\Lambda - 8\pi p)^2}\right\} \sinh \alpha(r, t) \right.$$

$$\left. + \left\{-8\pi p'(t_0 - t) \right\}\sqrt{3(\Lambda - 8\pi p)} + t_0' \sqrt{\frac{\Lambda - 8\pi p}{3}} \right] \cosh \alpha(r, t) \sinh^{\frac{1}{4}} \alpha(r, t),$$ \hspace{1cm} (40)

where

$$\alpha(r, t) = \frac{\sqrt{3(\Lambda - 8\pi p)}}{2} [t_0(r) - t].$$ \hspace{1cm} (41)

Here $t_0(r)$ is an arbitrary function of $r$. In the limit $p \to \frac{\Lambda}{8\pi}$, the above solution corresponds to the Tolman-Bondi solution [21]

$$\lim_{p \to \frac{\Lambda}{8\pi}} Y(r, t) = \left[\frac{9m}{2(t_0 - t)^2}\right]^{\frac{1}{4}},$$ \hspace{1cm} (42)

$$\lim_{p \to \frac{\Lambda}{8\pi}} X(r, t) = \frac{m'(t_0 - t) + 2mt'_0}{[6m^2(t_0 - t)]^{\frac{1}{4}}}. \hspace{1cm} (43)$$
5 Apparent Horizons

When the boundary of trapped two spheres is formed, we obtain the apparent horizon. Here we find this boundary of the trapped two spheres whose outward normals are null. For Eq.(1), this is given as follows

\[ g^{\mu\nu}Y_{,\mu}Y_{,\nu} = \dot{Y}^2 - \left(\frac{Y'}{X}\right)^2 = 0. \] (44)

Using Eqs.(30) and (32) in Eq.(44), we obtain

\[ (\Lambda - 8\pi p)Y^3 - 3Y + 6m = 0. \] (45)

The solutions of the above equation for \( Y \) give the apparent horizons. For \( \Lambda = 8\pi p \), it becomes the Schwarzschild horizon, i.e., \( Y = 2m \). When \( m = 0, \ p = 0 \), it yields the de-Sitter horizon \( Y = \sqrt{\frac{3}{\Lambda}} \). The case \( 3m < \frac{1}{\sqrt{\Lambda - 8\pi p}} \) leads to two horizons

\[ Y_1 = \frac{2}{\sqrt{\Lambda - 8\pi p}} \cos \frac{\psi}{3}, \] (46)
\[ Y_2 = \frac{2}{\sqrt{\Lambda - 8\pi p}} \left( \cos \frac{\psi}{3} - \sqrt{3} \sin \frac{\psi}{3} \right), \] (47)

where

\[ \cos \psi = -3m\sqrt{\Lambda - 8\pi p}. \] (48)

For \( m = 0 \), it follows from Eqs.(46) and (47) that \( Y_1 = \sqrt{\frac{3}{\Lambda - 8\pi p}} \) and \( Y_2 = 0 \). \( Y_1 \) is called the cosmological horizon and \( Y_2 \) is referred to be black hole horizon which can be generalized for \( m \neq 0 \) and \( \Lambda \neq 8\pi p \) respectively [22]. It is mentioned here that both horizons coincide for \( 3m = \frac{1}{\sqrt{\Lambda - 8\pi p}} \), i.e.,

\[ Y_1 = Y_2 = \frac{1}{\sqrt{\Lambda - 8\pi p}} = Y \] (49)

which gives a single horizon. It is obvious that the range for the cosmological horizon and the black hole horizon turns out to be

\[ 0 \leq Y_2 \leq \frac{1}{\sqrt{\Lambda - 8\pi p}} \leq Y_1 \leq \sqrt{\frac{3}{\Lambda - 8\pi p}}. \] (50)

The black hole horizon has its largest proper area \( 4\pi Y^2 = \frac{4\pi}{\Lambda} \) and the cosmological horizon has an area between \( \frac{4\pi}{\Lambda - 8\pi p} \) to \( \frac{12\pi}{\Lambda - 8\pi p} \). For \( 3m > \frac{1}{\sqrt{\Lambda - 8\pi p}} \),
there are no horizons. The formation time of the apparent horizon can be calculated with the help of Eqs. (38), (39) and (45) and is given by

\[ t_n = t_0 - \frac{2}{\sqrt{3(\Lambda - 8\pi p)}} \sinh^{-1}(\frac{Y_n}{2m} - 1)\frac{1}{2}, \quad (n = 1, 2). \] (51)

In the limit \( p \to \frac{\Lambda}{8\pi} \), we obtain the result corresponding to Tolman-Bondi [21]

\[ t_{ah} = t_0 - \frac{4}{3} m. \] (52)

From Eq. (51), it can be seen that both the black hole horizon and the cosmological horizon form earlier than the singularity \( t = t_0 \). From Eq. (51), it follows that

\[ \frac{Y_n}{2m} = \cosh^2 \alpha_n. \] (53)

Eq. (50) yields that \( Y_1 \geq Y_2 \) and also Eq. (51) gives \( t_1 \leq t_2 \), i.e., cosmological horizon forms earlier than the formation of the black hole horizon. To see the time difference between the formation of the cosmological horizon and singularity and the formation of the black hole horizon and singularity respectively, using Eqs. (46)-(48), we need to calculate the following quantities

\[ \frac{d(Y_n)}{dm} = \frac{1}{m} \left( -\frac{\sin \psi}{\sin \psi} + 3 \cos \frac{\psi}{3} \right) < 0, \] (54)

\[ \frac{d(Y_n)}{dm} = \frac{1}{m} \left( -\frac{\sin (\psi+4\pi)}{3 \sin \psi} + 3 \cos \frac{(\psi+4\pi)}{3} \right) > 0. \] (55)

We define the time difference between the formation of singularity and the apparent horizon, denoted by \( \tau \) as follows

\[ \tau_n = t_0 - t_n. \] (56)

It follows from Eq. (53) that

\[ \frac{d\tau_n}{d\left(\frac{Y_n}{2m}\right)} = \frac{1}{\sinh \alpha_n \cosh \alpha_n \sqrt{3(\Lambda - 8\pi p)}}. \] (57)

Using Eqs. (54) and (57), it turns out that

\[ \frac{d\tau_1}{dm} = \frac{d\tau_1}{d\left(\frac{Y_1}{2m}\right)} \frac{d\left(\frac{Y_1}{2m}\right)}{dm} = \frac{1}{m \sqrt{3(\Lambda - 8\pi p)} \sinh \alpha_1 \cosh \alpha_1 \left( -\frac{\sin \psi}{\sin \psi} + 3 \cos \frac{\psi}{3} \right)} < 0. \] (58)
This shows that $\tau_1$ decreases and hence the time difference between the formation of singularity and cosmological horizon decreases. From Eqs.(55) and (57), it follows that

$$\frac{d\tau_2}{dm} = \frac{1}{m\sqrt{3(\Lambda - 8\pi p)} \sinh \alpha_2 \cosh \alpha_2} \left( -\frac{\sin \left(\frac{\psi + 4\pi}{3}\right)}{\sin \psi} + \frac{3 \cos \left(\frac{\psi + 4\pi}{3}\right)}{\cos \psi} \right) > 0. \quad (59)$$

This implies that $\tau_2$ increases which means that the time difference between the formation of singularity and black hole horizon increases.

6 Conclusion

In this paper, we have studied the gravitational collapse of a perfect fluid in the presence of a cosmological constant. The effects of the cosmological constant on gravitational collapse have been discussed in the following two ways.

Firstly, the cosmological constant plays the role of repulsive force, i.e., it slows down the collapsing process. The cosmological term behaves like a Newtonian potential given by $\phi = \frac{1}{2}(1 - g_{00})$. Using Eqs.(12) and (35) for the exterior metric, the Newtonian potential takes the following form

$$\phi(R) = \frac{m}{R} + (\Lambda - 8\pi p) \frac{R^2}{6}. \quad (60)$$

The corresponding Newtonian force turns out to be

$$F = -\frac{m}{R^2} + (\Lambda - 8\pi p) \frac{R}{3} \quad (61)$$

which vanishes for $R = \sqrt[3]{\frac{1}{\Lambda - 8\pi p}}$ and $m = \frac{1}{3\sqrt[3]{\Lambda - 8\pi p}}$. Thus the force becomes repulsive/attactive for larger/smaller mass and radius respectively than these values. This means that the size of the black hole can be visualized by comparing the repulsive and attractive forces. The repulsive force generates from the cosmological constant for $\Lambda > 8\pi p$. It is worth mentioning here that for the perfect fluid $\Lambda$ can play the role of a repulsive force only for $\Lambda > 8\pi p$ while in the dust case this is true for all values of $\Lambda > 0$. From Eq.(32), the rate of collapse turns out to be

$$\ddot{Y} = -\frac{m}{Y^2} + (\Lambda - 8\pi p) \frac{Y}{3}. \quad (62)$$
For collapsing process, the force should be attractive, i.e., the acceleration should be negative which implies that $Y < \left(\frac{3p}{\Lambda - 8\pi p}\right)^{\frac{1}{3}}$. Thus Eq.(62) shows that the cosmological constant slows down the collapsing process if $\Lambda > 8\pi p$. This means that, for $p > \frac{\Lambda}{8\pi}$, the force becomes attractive and hence the cosmological constant does not slow down the collapsing process.

Secondly, there are two physical horizons instead of one due to the presence of the term $\Lambda - 8\pi p$, i.e., the black hole horizon and the cosmological horizon respectively. The cosmological constant influences the time difference between the formation of the apparent horizon and singularity. We find that the cosmological constant affects the process of collapse and hence it limits the size of the black hole. In perfect fluid case, these results are valid only for $\Lambda > 8\pi p$ while in the dust case these are valid for all $\Lambda > 0$. Thus we conclude that the pressure term creates a bound for the cosmological constant to act as a repulsive force. It is mentioned here that if we take $p = 0$, the results reduce to the dust case [17].
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