A graph polynomial from chromatic symmetric functions

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Abstract
Many graph polynomials may be derived from the coefficients of the chromatic symmetric function $X_G$ of a graph $G$ when expressed in different bases. For instance, the chromatic polynomial is obtained by mapping $p_n \rightarrow x$ for each $n$ in this function, while a polynomial whose coefficients enumerate acyclic orientations is obtained by mapping $e_n \rightarrow x$ for each $n$.

In this paper, we study a new polynomial we call the tree polynomial arising by mapping $X_{P_n} \rightarrow x$, where $X_{P_n}$ is the chromatic symmetric function of a path with $n$ vertices. In particular, we show that this polynomial has a deletion-contraction relation and has properties closely related to the chromatic polynomial while having coefficients that enumerate certain spanning trees and edge cutsets.

KEYWORDS
algebraic combinatorics, algebraic graph theory, chromatic polynomial, chromatic symmetric function, graph polynomials

1 | INTRODUCTION

The chromatic symmetric function $X_G$ of a graph $G$ was introduced by Stanley in the 1990s [13] as a generalization of the chromatic polynomial $\chi_G(x)$, defined as

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\[ X_G(x_1, x_2, \ldots) := \sum_{\chi(G) \to \mathbb{Z}^+} x_{\chi(v_i)} \cdots x_{\chi(v_{\chi(G)})} \text{ for } x \text{ proper colouring} \]

In particular, while \( \chi_G(k) \) counts the number of proper \( k \)-colourings of \( G \) for each \( k \), if \( \lambda = (\lambda_1, ..., \lambda_k) \) is an integer partition, then when \( X_G \) is expanded in the monomial symmetric function basis, the coefficient \( [m_\lambda]X_G \) gives (up to a constant factor) the number of proper \( k \)-colourings of \( G \) such that colour \( i \) is used \( \lambda_i \) times. The study of \( X_G \) is currently an active area of research in part due to its deep connections with other areas of mathematics, including algebraic geometry \([3, 12]\), knot theory \([4, 10]\), and representation theory \([9, 11]\).

Stanley showed that when the chromatic symmetric function of a graph \( G \) is expanded over the elementary symmetric function basis \( \{e_i\} \), mapping each \( e_i \to x \) (and extending algebraically so that \( e_i \to x^{(i)} \)) yields a polynomial \( p(x) \) such that \( [x^k]p(x) \) is the number of acyclic orientations of \( G \) with \( k \) sinks \([13, \text{ theorem } 3.3]\); this polynomial and its generalizations have recently been further studied by Hwang, Jung, Lee, Oh, and Yu \([8]\). Similarly, expanding \( X_G \) into the power-sum symmetric function basis \( \{p_i\} \) and setting each \( p_i \to x^{(i)} \) yields \( \chi_G(x) \).

In recent work, Sprikil and the second author extended \( X_G \) to graphs \( (G, \omega) \) equipped with an integer vertex weighting \( \omega : V(G) \to \mathbb{Z}^+ \) to give the function a deletion-contraction relation \([7]\), and Aliniaeifard, Wang, and van Willigenburg showed that if \( \{G_i\}_{i=1}^k \) is a family of connected, vertex-weighted graphs such that \( G_i \) has weight \( i \), then their chromatic symmetric functions are algebraically independent, and so may be used to form a basis \( \{X_{G_i}\} \) by letting \( G_i \) be the disjoint union of \( G_{\lambda_1}, \ldots, G_{\lambda_k} \) \([2]\); for instance, the aforementioned \( p \) and \( e \) bases arise in this way where \( G_i \) is a single vertex of weight \( i \) and an unweighted clique of size \( i \) respectively.

Given the interpretations described above for the polynomials formed by mapping \( p_i \) and \( e_i \) to the variable \( x \), it is natural to wonder if there is meaning to polynomials formed by mapping \( X_{G_i} \to x \) for other interesting choices of the \( G_i \) as given above. One natural choice is implied by other results from the aforementioned work of Aliniaeifard, Wang, and van Willigenburg \([2]\): they note that the basis formed from chromatic symmetric functions of unweighted paths \( P_n \) have surprising algebraic properties in relation to well-known symmetric function bases. They show that the map \( \Lambda \to \Lambda \) defined by mapping \( p_n \to X_{P_n} \) and extending algebraically is an involution. Furthermore, extending to integer compositions \( \alpha \) by defining \( P_{\alpha} \) to be the path with \( l(\alpha) \) vertices of weights \( \alpha_1, \ldots, \alpha_{l(\alpha)} \) along the path, they show that when \( \beta \) is another integer composition, \( X_{P_{\alpha}} = X_{P_{\beta}} \) if and only if the ribbon Schur functions \( r_{\alpha} \) and \( r_{\beta} \) are equal.

In this work, we consider the properties of the polynomial formed by applying the above mapping when \( G_i = P_i \) for all \( i \). In fact, we show that the same polynomial arises from taking any choice of \( \{T_i\}_{i=1}^\infty \) a family of unweighted trees such that \( T_i \) has \( i \) vertices, and mapping \( X_{T_i} \to x \); we therefore call this polynomial the tree polynomial \( \tau_G(x) \).

We show that it has a surprisingly close relationship with the chromatic polynomial; both admit a natural edge deletion-contraction formula and clique-intersection formula, and in fact for unweighted graphs with \( n \) vertices they are related by the identity (Theorem 3.5)

\[ \tau_G(x) = (x - 1)^n \chi_G \left( \frac{x}{x - 1} \right). \]

Furthermore, there is a natural sort of duality between the polynomials when considered on weighted graphs. The natural vertex-weighted extension of the chromatic polynomial \( \chi_{(G, \omega)}(x) \) determines \( |V(G)| \), but not \( \omega(G) = \sum_{v \in V(G)} \omega(v) \). Conversely \( \tau_{(G, \omega)}(x) \) cannot generally determine \( |V(G)| \) (in fact, \( \tau_{(G, \omega)}(x) = x \) for every unweighted tree regardless of its
number of vertices); however, for any weighted graph \((G, \omega)\), we can derive from \(\tau_{(G, \omega)}(x)\) the ‘excess weight’ \(\omega(G) - |V(G)|\).

In Section 2, we provide the necessary background in symmetric functions and colouring. In Section 3, we introduce the tree polynomials described above, showing how they may be derived from chromatic symmetric functions in a manner analogous to the chromatic polynomial. We also use the aforementioned chromatic symmetric function involution of Aliniaeifard, Wang, and van Willigenburg [2] to show explicit relationships between the coefficients and outputs of these two polynomials. In Section 4, we give direct interpretations for the coefficients and output of the tree polynomial of a graph in terms of its internal spanning forests. Finally, in Section 5, we investigate further directions for research.

2 | BACKGROUND

We take the natural numbers to contain 0. We define \([a] = \{1, \ldots, a\}\) and \([a, b] = \{a, a+1, \ldots, b\}\) where \(a, b \in \mathbb{N}\).

2.1 | Graphs and the chromatic polynomial

We take all graphs \(G\) to be simple. In particular, we take the edge-contraction graph \(G/e\) with \(e \in E(G)\) to not have any loops or multiedges. A weighted graph (as defined in [7]) will be denoted as \((G, \omega)\) where \(\omega : V(G) \to \mathbb{Z}^+\) is a weight function. We define the contraction of a weighted graph \((G, \omega)\) by an edge \(ev \in E(G)\) to be the graph \(G_e/\omega(v)\), where \(\omega(ev) = \omega(v_1) + \omega(v_2)\) if \(v\) is the vertex obtained from contracting \(v_1v_2\) and \((\omega/e)(v) = \omega(v)\) otherwise. We say the total weight of \((G, \omega)\) is \(\omega(G) = \sum_{v \in V(G)} \omega(v)\), and the excess weight of \((G, \omega)\) is \(\omega(G) - |V(G)|\), that is, the total weight minus the number of vertices.

We let \(CG = \{G_1, \ldots, G_k\}\) denote the set of connected components of \(G\). We write \(H \leq G\) to mean \(H\) is a subgraph of \(G\). A spanning subgraph \(H \leq G\) is called maximally connected if \(|C(H)| = |C(G)|\), and in this case we also call \(H\) a spanning tree of \(G\) (we use this term even if \(G\) is disconnected for simplicity and consistency with the literature). It is straightforward to see that an equivalent condition is that there is no \(e \in E(G)\) such that \(e\) is a bridge in \(H \cup \{e\}\). If \(F\) is a spanning subgraph of a spanning tree of \(G\), we call it a spanning forest of \(G\).

Graphs will be equipped with a total ordering on their edges in the form of a bijective mapping \(\eta : E(G) \to \{1, 2, \ldots, |E(G)|\}\). Note that the precise choice of mapping is irrelevant, as all proofs and results using \(\eta\) will hold for every possible choice of \(\eta\). Whenever \(\eta\) is not necessary, it will not be mentioned.

If \(F\) is a spanning forest of \(G\) we say an edge \(e \in E(G)\) is externally active if \(F \cup e\) is not a forest and \(e\) is the \(\eta\)-minimum edge in the unique cycle of \(F \cup e\) (i.e., if \(C\) is the unique cycle of \(F \cup e\), then \(\eta(e) < \eta(e')\) for all \(e' \in E(C)\)). We say that the spanning forest \(F\) is internal if no edges \(e \in E(G)\) are externally active. For \(S \subseteq E(G)\), we let \(G(S) = (V(G), S)\), the spanning subgraph of \(G\) induced by \(S\).

Definition 2.1. Let \(G\) be a graph. An edge set \(S \subseteq E(G)\) is a broken circuit if \(S = E(C)\), where \(C\) is a cycle in \(G\) and \(e \in E(C)\) is its minimum edge in the sense that \(\eta(e) < \eta(e')\) for all \(e' \in E(C)\).
**Definition 2.2.** The broken circuit complex \( B_G \) is the set of all \( S \subseteq E(G) \) which do not contain (all the edges of) a broken circuit.

\( B_G \) has an induced partial order arising from the partial order on subsets of \( E(G) \) with respect to inclusion. From the definition of \( B_G \), we note that if \( S \in B_G \) then \( \mathcal{G}(S) \) is acyclic and hence a forest, as the existence of a cycle means the existence of the broken circuit contained in that cycle. Moreover, we note that for each \( S \in B_G \), it follows that \( \mathcal{G}(S) \) is internal, as an externally active edge would imply the existence of a broken circuit. Conversely, it is clear that the edge set of every internal forest is an element of \( B_G \), so \( B_G \) consists precisely of the edge sets of internal forests of \( G \).

**Lemma 2.1.** \( S \in B_G \) is (inclusion-wise) maximal if and only if \( \mathcal{G}(S) \) is maximally connected.

**Proof.** If \( \mathcal{G}(S) \) is maximally connected then \( E(G) \setminus S \) contains no bridges of \( \mathcal{G}(S) \). Thus, if \( E(G) \supseteq S' \supseteq S \) then \( \mathcal{G}(S') \) contains a cycle, so \( S' \notin B_G \), so \( S \) must be maximal in \( B_G \).

Conversely, suppose \( \mathcal{G}(S) \) is not maximally connected with \( S \in B_G \). Then there exists at least one bridge for \( \mathcal{G}(S) \) in \( E(G) \setminus S \); let \( f_i \) be the smallest of these, in the sense that \( \eta(f_i) < \eta(f) \) for all other such bridges \( f \in E(G) \setminus S \). Consider \( S' := S \cup \{f_i\} \). Suppose \( S' \) contains the broken circuit \( W \), which is the cycle \( C \) with the smallest edge removed. Since \( S \in B_G \), we must have \( f_i \in W \). Suppose the smallest edge in \( C \) is \( \tilde{f} \). As \( \tilde{f} \notin W \) and \( f_i \in W \), it follows that \( \tilde{f} \neq f_i \). We note that \( S \cup \{\tilde{f}\} = (S' \cup \{\tilde{f}\}) \setminus \{f_i\} \) contains no cycle as \( S' \cup \{\tilde{f}\} \) contains exactly one cycle and \( f_i \) is in that cycle. So \( \tilde{f} \) is a bridge for \( \mathcal{G}(S) \) which contradicts the minimality of \( f_i \). Thus, \( S' \in B_G \) with \( S \not\subsetneq S' \), so \( S \) is not maximal in \( B_G \). \( \square \)

With this lemma as motivation, we say (in a slight abuse of terminology arising from identifying \( S \) with \( \mathcal{G}(S) \)) that \( S \) is an internal spanning tree if \( S \) is a maximal element of \( B_G \).

Let \( W \) be a vector space and let \( A \subseteq W \) be linearly independent. For \( f \in \text{span}(A) \) and \( a \in A \), we write \([a]f\) to mean the coefficient of \( a \) when writing \( f \) as a (unique) linear combination of elements in \( A \).

A proper \( k \)-colouring of \( G \) is a function \( f : V(G) \to \{1, \ldots, k\} \) such that \( uv \in E(G) \Rightarrow f(u) \neq f(v) \). A proper colouring is a function \( g : V(G) \to \mathbb{Z}^+ \) which is a proper \( k \)-colouring for some \( k \in \mathbb{Z}^+ \). Let \( \chi_G(x) \) be the chromatic polynomial of \( G \), defined such that for each \( k \in \mathbb{N} \setminus \{0\} \), \( \chi_G(k) \) is the number of proper \( k \)-colourings of \( G \).

**Proposition 2.1.** \( \sum_i [x^i] \chi_G = 1 \) if \( E(G) = \emptyset \) and otherwise \( \sum_i [x^i] \chi_G = 0 \).

**Proof.** This follows since \( \chi_G(1) \) is the number of proper 1-colourings of \( G \). \( \square \)

We will have the opportunity to make use of Whitney’s Broken Circuit Theorem:

**Theorem 2.1** (Whitney [16]).

\[
\chi_G(x) = \sum_{S \in B_G} (-1)^{|S|} x^{|V(G)| - |S|} = \sum_{S \in B_G} (-1)^{|S|} x^{c(S)},
\]

where \( c(S) \) is the number of connected components of \( \mathcal{G}(S) \).
2.2 Chromatic symmetric functions of graphs

A partition $\lambda$ of $n \in \mathbb{Z}^+$ is a weakly decreasing tuple $(\lambda_1, ..., \lambda_k)$ of positive integers such that $\sum_{i=1}^{k} \lambda_i = n$. We write $\ell(\lambda) = k$ to be the length of $\lambda$ and $|\lambda| = n$ to be the size of $\lambda$. We write $\lambda \vdash n$ to mean $\lambda$ is a partition of $n$. When we say a partition (without specifying its size) we mean a partition of some $n \in \mathbb{Z}^+$.

For $S \in B_G$ with $C(G(S)) = \{H_1, ..., H_k\}$, we write $\lambda(S)$ to be the integer partition with parts $|V(H_1)|, ..., |V(H_k)|$.

A symmetric function is a power series $f \in \mathbb{Q}[x_1, x_2, ...]$ of finite degree such that for each $\sigma \in S_{\mathbb{Z}^+}$ we have $f(x_1, x_2, ...) = f(x_{\sigma(1)}, x_{\sigma(2)}, ...)$. Let $\Lambda$ be the algebra of symmetric functions. It is well-known that bases of $\Lambda$ are typically indexed by integer partitions $\lambda$ [14].

Let $b_\lambda \vdash n \geq 0$ be a basis for $\Lambda$. We say $b_\lambda \vdash n \geq 0$ is a multiplicative basis if $b_\lambda = b_{\lambda_1} \cdots b_{\lambda_\ell(\lambda)}$. We define the power-sum basis $p_n = \sum_{i=1}^{\infty} x_i^n$. For more information regarding symmetric functions, we defer to [14].

For a family of weighted graphs $\{(G_i, \omega_i)\}_{i=1}^{\infty}$ and a partition $\lambda$, we define the weighted graph $G_{\lambda}(\omega)$ to be the disjoint union of weighted graphs

$$G_{\lambda}(\omega) := (G_{\lambda_1}(\omega_{\lambda_1}) \cup \cdots \cup (G_{\lambda_\ell(\lambda)}(\omega_{\lambda_\ell(\lambda)}).$$

Similarly, if $\{G_i\}_{i=1}^{\infty}$ is a family of graphs, we define $G_{\lambda}$ to be the disjoint union of graphs

$$G_{\lambda} = G_{\lambda_1} \cup \cdots \cup G_{\lambda_\ell(\lambda)}.$$

In [13], the chromatic symmetric function of $G$ with $V(G) = \{v_1, ..., v_n\}$ is defined as

$$X_G := \sum_{\kappa: V(G) \rightarrow \mathbb{Z}^+} \prod_{v \in V(G)} x_{\kappa(v)}$$

which satisfies the deletion contraction relation [7, lemma 2] for $e \in E(G)$,

$$X_G = X_{G \setminus e} - X_{G/e, \omega/e}$$ 

recalling that the weight of the vertex formed by the contraction of $e$ is the sum of the weights of the endpoints of $e$. The chromatic symmetric function also satisfies the following analogue of
Whitney’s Broken Circuit Theorem, proved for unweighted graphs by Stanley [13] and for weighted graphs by Spirkl and the second author [7]:

**Theorem 2.2.**

\[ X_{(G, \omega)} = \sum_{S \in B_G} (-1)^{|S|} p_{\lambda(S)} \]

where \( \lambda(S) \) is the integer partition whose multiset of parts is the multiset of total weights of connected components of \( G(S) \).

Aliniaeifard, Wang, and van Willigenburg showed that one can form bases of \( \Lambda \) which arise from weighted chromatic symmetric functions, generalizing the work of [5] for unweighted graphs:

**Theorem 2.3** (Aliniaeifard, Wang, and Willigenburg [2, theorem 5.4]). Let \( \{(G_n, \omega_n)\}_{n=1}^{\infty} \) be a family of connected weighted graphs such that \( \omega_n(G_n) = n \). Then \( \{X_{(G_n, \omega_n)} : n \geq 0\} \) is a set of algebraically independent symmetric functions, and \( \{X_{(G_n, \omega_n)} : \lambda \vdash n \geq 0\} \) is a multiplicative basis for \( \Lambda \).

Such bases for \( \Lambda \) are called chromatic bases. For instance, among the five classical symmetric function bases [14], the \( e- \) and \( p- \) bases are chromatic, while the \( m- \), *\( h- \), and \( s- \) bases are not† [6, 7].

**Corollary 2.1.** Let \( \{T_n\}_{n=1}^{\infty} \) be a family of trees with each \( T_n \) having \( n \) vertices. Then \( \{X_{T_n} : n \geq 0\} \) is a set of algebraically independent symmetric functions, and

\[ \{X_{T_n} : \lambda \vdash n \geq 0\} \]

is a multiplicative basis for \( \Lambda \).

In particular, \( \{X_{P_n} : \lambda \vdash n \geq 0\} \) and \( \{X_{S_n} : \lambda \vdash n \geq 0\} \) where \( P_n, S_n \) are the paths and stars on \( n \) vertices, respectively, are multiplicative bases for \( \Lambda \). Aliniaeifard, Wang, and van Willigenburg found a peculiar relation between certain chromatic bases and the power-sum basis.

**Theorem 2.4** (Chromatic Reciprocity [2, theorem 6.3]). The maps \( p_{\lambda} \rightarrow X_{P_{\lambda}} \) and \( p_{\lambda} \rightarrow X_{S_{\lambda}} \) are automorphisms of \( \Lambda \) that are also involutions.

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*It is true that \( \bar{m}_\lambda = (\prod_i \eta(\lambda_i)) m_{\lambda_i} \), where \( \eta(\lambda) \) is the number of occurrences of \( i \) as a part of \( \lambda \), is the chromatic symmetric function of a complete graph with vertices of weights \( \lambda_1, \cdots, \lambda_\ell \), but this basis is not multiplicative. The \( \bar{m}_\lambda \), however, are multiplicative with respect to a different multiplication operation that agrees with the complete join of graphs [15].

†However, all five of these classical bases, as well as many bases of the space of quasisymmetric functions, are generalized chromatic functions of edge-coloured graphs in the sense of [1].
3 | THE TREE POLYNOMIAL AND THE CHROMATIC POLYNOMIAL

We begin by formalizing the transformations we are using.

3.1 | Deriving graph polynomials from the chromatic symmetric function

Using Corollary 2.1, we can define the following.

Definition 3.1. Given a symmetric function $f = \sum_\lambda a_\lambda X_\lambda$, define $\tau_f(x)$ to be the polynomial

$$\tau_f(x) = \sum_\lambda a_\lambda x^{(\lambda)} = \sum_{i=0}^\infty \left( \sum_\lambda a_\lambda \right) x^i.$$

Definition 3.2. Given a symmetric function $f = \sum_\lambda c_\lambda p_\lambda$, define $\chi_f(x)$ to be the polynomial

$$\chi_f(x) = \sum_\lambda c_\lambda x^{(\lambda)} = \sum_{i=0}^\infty \left( \sum_\lambda c_\lambda \right) x^i.$$

Equivalently, we may consider maps $\tau : \Lambda \rightarrow \mathbb{Q}[x]$ and $\chi : \Lambda \rightarrow \mathbb{Q}[x]$ as the linear transformations which extend algebraically the mappings $X_\lambda \rightarrow x$ and $p_\lambda \rightarrow x$, respectively, and define that for a symmetric function $f$, $\tau(f) = \tau_f(x)$ and $\chi(f) = \chi_f(x)$. Note that a symmetric function $f$ is of bounded degree, so sums are finite and thus, these polynomials are well-defined.

We can extend this to any symmetric function basis.

Definition 3.3. Let $B = \{b_\lambda : \lambda \vdash n \geq 0\}$ be a basis for $\Lambda$. The $B$-polynomial of $f = \sum_\lambda d_\lambda b_\lambda \in \Lambda$ is the polynomial $p(x)$ defined by

$$p(x) = \sum_\lambda d_\lambda x^{(\lambda)} = \sum_{i=0}^\infty \left( \sum_\lambda d_\lambda \right) x^i.$$

These maps are, in fact, multiplicative.

Lemma 3.1. Let $f_1, f_2$ be symmetric functions. Let $B$ be a multiplicative basis for $\Lambda$ and let $\theta_{f_i}$ denote the $B$-polynomial of $f_i$. Then $\theta_{f_1 f_2} = \theta_{f_1} \theta_{f_2}$ (so the general map $\theta : \Lambda \rightarrow \mathbb{Q}[x]$ defined by extending algebraically $b_n \rightarrow x$ is a morphism of algebras).
Proof. For $k \in \mathbb{N}$,

\[
[x^k] \theta_{f, f_2} = \sum_{\ell(\lambda) = k} [b_\lambda] f_1 f_2 
= \sum_{t=0}^{k} \sum_{\ell(\alpha) = t} \sum_{\ell(\beta) = k-t} [b_\alpha] f_1 [b_\beta] f_2 
= \sum_{t=0}^{k} \left[ \sum_{\ell(\alpha) = t} [b_\alpha] f_1 \right] \left[ \sum_{\ell(\beta) = k-t} [b_\beta] f_2 \right] 
= \sum_{t=0}^{k} [x^t] \theta_\lambda [x^{k-t}] \theta_{f_2} 
= [x^k] \theta_\lambda \theta_{f_2}
\]

and the result follows. □

Now, we shall justify our choice of notation and maps. First, we verify that our map $\chi$ agrees naturally with the chromatic polynomial $\chi_G(x)$.

**Proposition 3.1.** Let $(G, \omega)$ be a weighted graph. Then $\chi_G(x) = \chi_{X(G, \omega)}(x)$ (i.e., the chromatic polynomial of $G$ is equal to the image of the chromatic symmetric function $X(G, \omega)$ for any choice of $\omega$ under our mapping $\chi$).

**Proof.** This follows immediately from the Broken Circuit Theorems 2.1 and 2.2. □

Thus, our notion of $\chi_{X(G, \omega)}$ generalizes the chromatic polynomial in a direct way by ignoring the vertex weights. We now want to consider what precisely is output by the map $\tau$. First, we require the following straightforward auxiliary lemma.

**Lemma 3.2.** Let $n \in \mathbb{N}$. Then the set of polynomials $x^k(x - 1)^{n-k}$ for $k \in [0, n]$ are linearly independent in $\mathbb{Q}[x]$.

**Proof.** Suppose that $\sum_{k=0}^{n} r_k x^k(x - 1)^{n-k} = 0$. Note that the constant term of this expression is $(-1)^n r_0$, so it follows that $r_0 = 0$. Inductively, suppose that $r_0 = \cdots = r_t = 0$ for some $t \in [0, n-1]$. Then the coefficient of $x^{t+1}$ in the expression is $(-1)^{n-t-1} r_{t+1}$, so it follows that $r_{t+1} = 0$. By induction, $r_k = 0$ for each $k \in [0, n]$.

Now, we define polynomials as $\tau_G := \tau_{X_G}$ and $\tau_{(G, \omega)} := \tau_{X(G, \omega)}$.

**Proposition 3.2.** Let $F$ be a forest with $n$ vertices and $m$ components. Then $\tau_F = x^m$.

**Proof.** Write $X_F = \sum_{a \in \mathcal{A}} a \chi_{P_a}$. Note that $\chi_{P_a} = x^{\ell(a)}(x - 1)^{n - \ell(a)}$. Applying $\chi$ to both sides,
Because the \( x^k(x - 1)^{n-k} \) are linearly independent, the result follows by comparing coefficients. \( \square \)

**Corollary 3.1.** For a tree \( T \), \( \tau_T = x \).

Thus, all trees \( T \) have the same polynomial \( \tau_T \), independent even of the number of vertices of \( T \). Moreover, although this fact will not be required elsewhere in this paper, it is interesting to note that \( \tau \) may likewise be derived using any family of trees, not just paths.

**Theorem 3.1.** Let \( \{T_n\}_{n=1}^\infty \) be a family of trees with each \( T_n \) having \( n \) vertices and \( \{X_{T_\lambda} : \lambda \vdash n \geq 0\} \) its corresponding basis. For any weighted graph \( (G, \omega) \) and \( k \in \mathbb{N} \),

\[
\sum_{\lambda} [X_{T_\lambda}]X_{(G,\omega)} = \sum_{\lambda} [X_{P_\lambda}]X_{(G,\omega)}.
\]

**Proof.** Write \( X_{(G,\omega)} = \sum_a a_a X_{P_a} = \sum_a b_a X_{T_a} \). Letting \( N \) be the total weight of \( (G, \omega) \), applying \( \chi \) we obtain

\[
X_{(G,\omega)} = \sum_a a_a X_{P_a} = \sum a_a x^{\ell(\lambda)}(x - 1)^{N-\ell(\lambda)}
\]

and likewise also

\[
X_{(G,\omega)} = \sum_a b_a X_{T_a} = \sum a_a x^{\ell(\lambda)}(x - 1)^{N-\ell(\lambda)}
\]

so the result follows by using the linear independence of the \( x^{\ell(\lambda)}(x - 1)^{N-\ell(\lambda)} \) and comparing coefficients. \( \square \)

With Theorem 3.1 as motivation, we thus call \( \tau_f \) the tree polynomial of \( f \). For a graph \( G \) we say \( \tau_G \) is the tree polynomial of \( G \) and \( \tau_{(G,\omega)} \) the tree polynomial of \( (G, \omega) \).

### 3.2 Connections between the tree and chromatic polynomials

Using the chromatic reciprocity of \( p_\lambda \) and \( X_{P_\lambda} \) discovered by Aliniaeifard, Wang, and Willigenburg [2] (described in the Background as Theorem 2.4), we obtain

**Theorem 3.2.** For a graph \( G \) with \( n \) vertices, and any \( k \in \mathbb{N} \),

\[
[x^k] \tau_G = (-1)^{n-k} \sum_{m=1}^{k} \binom{n-m}{k-m} [x^m]X_G.
\]

**Proof.** Writing \( X_G = \sum_{\lambda \vdash n} a_\lambda P_\lambda \) we calculate
From this, we can derive many basic properties of the tree polynomial and fundamental connections to the chromatic polynomial.

Define the function \( \phi : \mathbb{Q}[x] \to \mathbb{Q}[x] \) by

\[
\phi(p(x)) = \sum_{k=1}^{n} (-1)^{n-k} k \left( \sum_{m=1}^{k} \binom{n-m}{k-m} [t^m] p(t) \right) x^k
\]

where \( n = \deg(p(x)) \). In particular, \( \phi(\chi_G) = \tau_G \).

**Proposition 3.3.** For graphs \( G, H \), \( \phi(\chi_G \chi_H) = \phi(\chi_G) \phi(\chi_H) \)

**Proof.** This follows from Lemma 3.1 as

\[
\phi(\chi_G \chi_H) = \phi(\chi_{G \cup H}) = \tau_{G \cup H} = \tau_G \tau_H = \phi(\chi_G) \phi(\chi_H).
\]

Notably, \( \phi \) is not a homomorphism on all of \( \mathbb{Q}[x] \). However, because it multiplies over chromatic polynomials, we may carry over some of their results to tree polynomials. For example, the well-known clique-gluing formula for the chromatic polynomial [17] also applies to the tree polynomial.

**Corollary 3.2.** Let \( G_1, G_2 \) be graphs containing \( k \)-vertex cliques \( K_1 \) and \( K_2 \), respectively. Write \( V(K_1) = \{u_1, \ldots, u_k\} \) and \( V(K_2) = \{v_1, \ldots, v_k\} \). Define the graph \( H \) to have vertex set

\[
V(H) = V(G_1) \cup (V(G_2) \setminus V(K_2))
\]

and edge set

\[
E(H) = \{xy : xy \in E(G_1)\} \cup \{vw : vw \in E(G_2)\} \cup \{u_iw : v_iw \in E(G_2)\}.
\]
That is, \( H \) is the graph obtained by gluing \( G_1 \) and \( G_2 \) along their \( k \)-vertex cliques, resulting in a graph with gluing clique \( K \). Then

\[
\tau_H = \frac{\tau_{G_1} \tau_{G_2}}{\tau_K}.
\]

**Corollary 3.3.** Let \( G \) be a graph and let the graph \( H \) be formed by gluing a vertex of some tree to a vertex of \( G \). Then \( \tau_G = \tau_H \).

We also show that while \( \tau_{(G,\omega)} \) cannot determine \( |V(G)| \), it instead determines \( \omega(G) - |V(G)| \). To show this we first prove the following auxiliary lemma.

**Lemma 3.3.** Let \( G \) be a graph with \( n \) vertices. Then

\[
\sum_{\lambda \vdash n} [X_{p_G}] X_G = 1.
\]

**Proof.** Let \( X_G = \sum_{\lambda \vdash n} \alpha_{\lambda} p_\lambda \). It is easy to check (e.g., using Theorem 2.2) that \( a_v^{(p')} = 1 \). Furthermore, by Proposition 2.1, for any graph \( H \), \( \sum_{\lambda \vdash m} [p_\lambda] X_H = \sum_{\lambda} [x'] \chi_H = 1 \) if \( H \) has no edges and 0 otherwise. Thus, applying Chromatic Reciprocity,

\[
\sum_{\lambda \vdash n} [X_{p_G}] X_G = \sum_{\lambda \vdash n} \alpha_{\lambda} \sum_{\lambda \vdash n} [X_{p_G}] p_\lambda
\]

\[
= a_v^{(p')} + \sum_{\lambda \vdash n} \alpha_{\lambda} \sum_{\lambda \vdash n} [p_\lambda] X_{p_G}
\]

\[
= a_v^{(p')} + \sum_{\lambda \vdash n} \alpha_{\lambda} (0)
\]

\[
= a_v^{(p')}
\]

\[
= 1.
\]

**Theorem 3.3.** Let \( (G, \omega) \) be a weighted graph with total weight \( N \) and \( n \) vertices. Let \( k \) be the largest integer such that \( (x - 1)^k |\tau_{(G,\omega)}| \). Then \( k = N - n \), and moreover, \( \tau_{(G,\omega)} = (x - 1)^k \tau_G \).

**Proof.** We proceed by induction on \( N - n \). If \( N - n = 0 \) then applying Lemma 3.3, \( \tau_G(1) = 1 \neq 0 \) and the result holds.

Now, suppose the result holds for all weighted graphs with excess weight \( N - n = m \), and suppose \( N - n = m + 1 \). Pick a vertex \( v_0 \) with a weight greater than 1. Consider the graph \( G' \) with vertex set \( V(G) \cup \{w\} \) and edge set \( E(G) \cup \{wv_0\} \). Let \( \omega' \) be defined by \( \omega'(w) = 1 \), \( \omega'(v) = \omega(v) \) for \( v_0 \neq v \in V(G) \) and \( \omega'(v_0) = \omega(v_0) - 1 \). Note that \( (G'/v_0w, \omega'/v_0w) = (G, \omega) \). So by vertex-weighted deletion-contraction (1), using the diagrams as stand-ins for their chromatic symmetric functions,

\[
\text{w} \quad \bullet (G, \omega') = \bullet (G, \omega') - (G, \omega)
\]
so taking tree polynomials and rearranging,
\[
\tau_{(G, \omega)} = x\tau_{e(G, \omega')} - \tau_{(G', \omega')}. 
\]

By the inductive hypothesis, \(\tau_{(G', \omega')} = (x - 1)^m \tau_{G'} = (x - 1)^m \tau_{G} = \tau_{(G, \omega')}\) where the second last equality follows from Corollary 4.2 (as our construction of \(G'\) is the same as gluing a path of length 2 to \(v_0\)). So,
\[
\tau_{(G, \omega)} = x\tau_{(G, \omega')} - \tau_{(G, \omega')} = (x - 1)\tau_{(G, \omega')} = (x - 1)(x - 1)^m \tau_{G} = (x - 1)^{m+1} \tau_{G} 
\]
and as \(x - 1\) does not divide \(\tau_{G}\), the result follows.

We also derive a deletion-contraction relation for the tree polynomial.

**Theorem 3.4 (Unweighted Deletion-Contraction).** Let \(G\) be a graph and \(e\) an edge of \(G\). Then
\[
\tau_{G} = \tau_{G \setminus e} - (x - 1)\tau_{G/e}. 
\]

**Proof.** The result follows immediately from vertex-weighted deletion-contraction of chromatic symmetric functions (1) and Theorem 3.3.

We may further use chromatic reciprocity to directly write the chromatic polynomial as an evaluation of the tree polynomial and vice versa without explicitly computing coefficients. We first need the following proposition, the tree polynomial analogue of the fact that the chromatic polynomial of a forest depends only on its size and number of components.

**Proposition 3.4.** \(\tau_{P_1} = x^\ell(\lambda)(x - 1)^{\lambda - \ell(\lambda)}\)

**Proof.** We have
\[
[x^k]\tau_{P_1} = \sum_{\Delta} [X_{P_1}]_{P_1} = \sum_{\ell(\Delta) = k} [P_\Delta]_{P_1} = [x^k]_{X_{P_1}} 
\]
so
\[
\tau_{P_1} = X_{P_1} = x^\ell(\lambda)(x - 1)^{\lambda - \ell(\lambda)}. 
\]

**Theorem 3.5.** Let \((G, \omega)\) be a weighted graph with \(n\) vertices and total weight \(N\). Then,
\[
(1) \quad \chi_{G} = (x - 1)^N \tau_{(G, \omega)} \left(\frac{x}{x - 1}\right) \\
(2) \quad \tau_{(G, \omega)} = (x - 1)^N \chi_{G} \left(\frac{x}{x - 1}\right) 
\]

**Proof.** Write \(X_{(G, \omega)} = \sum_{\lambda \vdash N} a_\lambda X_{P_\lambda}\). Then taking \(\chi\) of both sides,
\[
X_{(G, \omega)} = \sum_{\lambda \vdash N} a_{\lambda} X_{P_{\lambda}} \\
= \sum_{\lambda \vdash N} a_{\lambda} x^\ell(\lambda)(x - 1)^{\lambda_1 - \ell(\lambda)} \\
= \sum_{k=0}^{N} \left( \sum_{\lambda \vdash N \atop \ell(\lambda) = k} a_{\lambda} \right) x^k (x - 1)^{N-k} \\
= \sum_{k=0}^{N} \left( \sum_{\lambda \vdash N \atop \ell(\lambda) = k} a_{\lambda} \right) (t^k \tau_{(G, \omega)}(t)) x^k (x - 1)^{N-k} \\
= (x - 1)^N \sum_{k=0}^{N} \left( \sum_{\lambda \vdash N \atop \ell(\lambda) = k} a_{\lambda} \right) \left( \frac{x}{x - 1} \right)^k \\
= (x - 1)^N \tau_{(G, \omega)} \left( \frac{x}{x - 1} \right)
\]

which shows (1). Write \( X_{(G, \omega)} = \sum_{\lambda \vdash N} b_{\lambda} P_{\lambda} \). Then,

\[
\tau_{(G, \omega)} = \sum_{\lambda \vdash N} b_{\lambda} \tau_{P_{\lambda}} \\
(\text{proposition 3.4}) = \sum_{\lambda \vdash N} b_{\lambda} x^\ell(\lambda)(x - 1)^{\lambda_1 - \ell(\lambda)} \\
= \sum_{k=0}^{N} \left( \sum_{\lambda \vdash N \atop \ell(\lambda) = k} b_{\lambda} \right) x^k (x - 1)^{N-k} \\
= \sum_{k=0}^{N} \left( \sum_{\lambda \vdash N \atop \ell(\lambda) = k} b_{\lambda} \right) (t^k \chi_G(t)) x^k (x - 1)^{N-k} \\
= (x - 1)^N \sum_{k=0}^{N} \left( \sum_{\lambda \vdash N \atop \ell(\lambda) = k} b_{\lambda} \right) \left( \frac{x}{x - 1} \right)^k \\
= (x - 1)^N \chi_G \left( \frac{x}{x - 1} \right)
\]

Note that the last line follows because \( [x^k] \chi_G(x) = 0 \) for \( k > n \). \( \square \)

Thus, given \( \tau_{(G, \omega)} \) and the number of vertices of \( G \) or the total weight of \( (G, \omega) \) we can recover \( \chi_G \) and vice-versa.

4 | COEFFICIENTS AND EVALUATIONS OF TREE POLYNOMIALS

In the previous section, we derived the tree polynomial, and demonstrated its remarkable relationship with the chromatic polynomial. In this section, we focus directly on the tree polynomial, giving combinatorial interpretations for the coefficients and evaluations of the tree polynomial in terms of cutsets and colourings of the internal forests of a graph.
First, it is natural to ask if evaluating the tree polynomial at positive integers has a natural combinatorial meaning, as is true for the chromatic polynomial. From the computation of Theorem 3.5, for an unweighted graph \( G \) on \( n \) vertices,

\[
\tau_G(x) = \sum_{k=0}^{n} ([t^k] \chi_G(t)) x^k (x - 1)^{n-k}.
\]

For \( j \) a positive integer, we can thus interpret \( \tau_G(j) \) combinatorially. Using Whitney’s Broken Circuit Theorem 2.1, \([t^k]\chi_G(t)\) is \((-1)^{n-k}\) times the number of \( S \in B_G \) with \( \ell(S) = k \). Recalling that \( G(S) \) denotes the spanning subgraph of \( G \) induced by the edges of \( S \), since \( G(S) \) is a forest with \( k \) components, \( j^k (j - 1)^{n-k} \) is the number of proper \( j \)-colourings of \( G(S) \). Letting \( \ell(S) = |C(G(S))| \), we may immediately interpret this sum:

**Lemma 4.1.** Let \( G \) be an unweighted graph with \( n \) vertices. For \( j \) a positive integer,

\[
\tau_G(j) = (-1)^n \sum_{S \in B_G} (-1)^{\ell(S)} \chi_{G(S)}(j) = (-1)^n \sum_{(S, \kappa)} (-1)^{\ell(S)}
\]

where the last sum runs over ordered pairs \((S, \kappa)\) where \( S \) is an internal forest of \( G \) and \( \kappa \) is a proper \( j \)-colouring of \( G(S) \).

To interpret the coefficients of \( \tau_G(x) \), we introduce some further definitions in preparation.

**Definition 4.1.** Let \( G \) be a graph and \( H \leq G \). Let \( C(H) = \{H_1, ..., H_m\} \). Define \( V_G(H) := \bigcup_{i=1}^{m} E_G(H_i) \), where

\[
E_G(H_i) = \{uv \in E(G) : u \in V(H_i), v \in V(G) \setminus V(H_i)\}.
\]

In other words, \( E_G(H_i) \) is the edge cutset induced by \( H_i \) calculated in \( G \), and \( V_G(H) \) is the set of all edges in \( G \) that are incident with \( H \), but not contained in a block of \( H \). Equivalently, \( V_G(H) \) contains exactly those edges in \( G \setminus H \) that do not become loops in \( G/H \).

In particular, \( V_G(H) \) is larger than the edge cutset of \( G \) induced by \( V(H) \) because it may also include edges between different components of the spanning subgraph of \( G \) induced by \( H \).

**Definition 4.2.** Let \( G \) be a graph. For \( S \subseteq E(G) \), we define \( C(S) := C(G(S)) \) and as above \( \ell(S) := |C(S)| \).

**Definition 4.3.** For a graph \( G = (V, E) \) and \( S \subseteq E \) we write \( V_G(S) = V_G(G(S)) \).

**Definition 4.4.** Let \( G \) be a graph, and \( S \subseteq E(G) \) be such that \( G(S) \) is a subforest of \( G \).

If \( e \in E(G) \setminus V_G(S) \), then \( S \cup \{e\} \) contains a unique cycle called the fundamental cycle of \( e \).

If \( e \in V_G(S) \), then \( e \) has endpoints in distinct components \( H_i \) and \( H_j \) of \( G(S) \). The set of all edges in \( V_G(S) \) with endpoints in those same two components \( H_i \) and \( H_j \) is called the fundamental cocycle of \( e \).
We will give a combinatorial interpretation of the tree polynomial’s coefficients in terms of spanning trees and edge cutsets of $G$. We first prove two auxiliary lemmas, noting that the second of these lemmas contains the true heart of the final theorem.

**Lemma 4.2.** Let $G$ be a graph with edge ordering $\eta$, and let $S \in B_G$. If $\hat{e}$ is the minimal edge in $\nabla_G(S)$ with respect to $\eta$, then $S \cup \{\hat{e}\} \in B_G$.

**Proof.** Let $e \in E(G) \setminus (S \cup \{\hat{e}\})$. If $e$ is externally active in $S \cup \{\hat{e}\}$, its fundamental cycle cannot contain $\hat{e}$, since $\eta(\hat{e}) < \eta(e)$. But if the fundamental cycle of $e$ does not contain $\hat{e}$, then $e$ is also externally active in $S \cup \{\hat{e}\}$, contradicting $S \in B_G$, and the result follows. $\square$

**Lemma 4.3.** Let $G$ be a graph with $n$ vertices and edge ordering $\eta$, and let $S \in B_G$. Then, 
\[
\sum_{U \subseteq B_G} (-1)^{|U|} U \text{ is } (-1)^{|C(G)|} \text{ times the number of spanning trees } T \in B_G \text{ containing } S \text{ such that } e \in \nabla_G(S) \cap T, \text{ then } e \text{ is contained in the fundamental cycle with respect to } T \text{ of some smaller edge in } \nabla_G(S) \setminus T.
\]

**Proof.** Let $m = |\nabla_G(S)|$ and label the edges of $\nabla_G(S)$ as $\{e_1, \ldots, e_m\}$ so that $i < j$ if and only if $\eta(e_i) < \eta(e_j)$. We will use the following definition: given $U \supseteq S$ with $U \in B_G$, we say that $e \in E(G) \setminus U$ is dead to $U$ if $U \cup \{e\}$ contains a cycle, or equivalently if $e \not\in \nabla_G(U)$ (so in particular, this edge cannot be added to $U$ while keeping the set in $B_G$). Note that when $U \supseteq S$ with $U \in B_G$, we have $\nabla_G(U) \subseteq \nabla_G(S)$, so the set of edges of $G$ that are not dead to $U$ is always a subset of $\{e_1, \ldots, e_m\}$.

We will construct sets $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_m$ recursively, each containing a family of forests, by letting $S_0 := \{U \in B_G : U \supseteq S\}$ and for $k \in [0, m-1]$, $S_{k+1} = S_k \setminus \{U, U \cup \{e_{k+1}\} : U \in S_k, e_{k+1} \not\in U \text{ and } e_{k+1} \text{ not dead to } U\}$.

We claim that $S_m$ is precisely the family of spanning trees $T$ described in the lemma’s statement. To show this claim, we will inductively prove properties of the $S_i$.

Let $S$ be a family of forests in $B_G$. Given $k \in [1, m]$, we say that $S$ is $k$–correct if for all $U \in S$, the following property holds: For all $i \in [1, k]$, either $e_i$ is dead to $U$, or $e_i \in U$ and there exists $j < i$ such that $e_j \not\in U$, $U \cup \{e_j\}$ contains a cycle, and $e_i$ is in that cycle. (*)

We prove inductively the following for each $k \in [1, m]$:

(a) For $U \in S_{k-1}$ such that $e_k \not\in U$ and $e_k$ is not dead to $U$, we have $U \cup \{e_k\} \in S_{k-1}$, so the recursive operation eliminates sets in pairs.
(b) $S_k$ is the inclusion-wise maximal subset of $S_0$ that is $k$-correct.

Note that (a) holds trivially for $k = 1$, and (b) holds for $k = 1$ by Lemma 4.2.

We prove that for $k \in [2, m]$, if (b) holds for $k − 1$ then (a) holds for $k$. Suppose that $S_{k-1}$ is the maximal subset of $S_0$ that is $(k − 1)$-correct. Then given $U \in S_{k-1}$ with $e_k \not\in U$ and $e_k$ not dead to $U$, we must verify that $U \cup \{e_k\} \in S_0$, and that for each $i \in [1, \ldots, k − 1]$, $e_i$ satisfies (*) in $U \cup \{e_k\}$. It is straightforward to verify the latter property: if $e_i$ satisfies (*) for some $i \leq k − 1$, then we may check that adding the edge $e_k$ cannot now make it violate (*).

To show the former, suppose otherwise, that $U \cup \{e_k\} \notin S_0$. Certainly $U \cup \{e_k\} \supseteq S$ if $U \supseteq S$, so it must be the case that $U \cup \{e_k\} \not\in B_G$. Since $U \in B_G$, but $e_k$ is not dead to $U$, it must be the case that the addition of $e_k$ creates a broken circuit. Thus, there is a unique
There exists an edge $e_i \notin U$ for some $i < k$ such that $e_i$ is in the fundamental cycle $C$ of $e_i$ in $U \cup \{e_k\}$. But since $U \in S_{k-1}$ and $e_i \notin U$, by (*) it follows that $e_i$ is dead to $U$, so $U \cup \{e_i\}$ contains a cycle $D$. But this is impossible since then $e_i$ has distinct fundamental cycles in $U$ and $U \cup \{e_k\}$ (since one contains $e_k$ and one does not).

Thus, if $U \in S_{k-1}$ is given such that $e_k \notin U$ and $e_k$ is not dead to $U$, we have $U \cup \{e_k\} \in S_{k-1}$.

We now prove that for $k > 1$, if (b) holds for $k - 1$ and (a) holds for $1, ..., k - 1$, then (b) holds for $k$. We first show that $S_k$ is $k$-correct. If $e_k \notin U$, then $e_k$ must be dead in $U$, as otherwise, $U$ would have been removed at the latest when moving from $S_{k-1}$ to $S_k$. Now, suppose that $e_k \in U$. For $k > 1$, if $e_k \in U$, it must be the case that $U \setminus \{e_k\} \notin S_{k-1}$, as otherwise $U$ would have been eliminated since $e_k \notin U \setminus \{e_k\}$ and necessarily $e_k$ is not dead in $U \setminus \{e_k\}$, as otherwise $U \notin S_0$.

Since $U \setminus \{e_k\} \notin S_{k-1}$, applying the inductive hypothesis, there is some $i \in [1, k-1]$ such that $e_i$ violates (*) in $U \setminus \{e_k\}$. If $e_i \in U \setminus \{e_k\}$ and violates (*), then it is straightforward to check that $e_i$ must also violate (*) in $U$, contradicting that $U \in S_{k-1}$. Thus, $e_i \notin U \setminus \{e_k\}$, so $e_i$ is not dead in $U \setminus \{e_k\}$ to violate (*). However, since $e_i$ does not violate (*) in $U$, $e_i$ is dead in $U$. This can only occur if the addition of $e_k$ causes $e_i$ to be dead, showing that $e_k$ satisfies (*) with respect to $i < k$. Thus, $S_k$ is $k$-correct.

To show maximality, it suffices to show that every $U \in S_0$ that satisfies the $k$-correctness property is in $S_k$. We show this by induction. The case $k = 1$ is obvious, so suppose inductively that $S_{k-1}$ is the maximal $k$-correct subset of $S_0$, and consider $U \in S_0$ which satisfies $k$-correctness. By the inductive hypothesis, $U \in S_{k-1}$. If $e_k \notin U$, then by $k$-correctness $e_k$ is dead to $U$, and so $U$ is not removed when proceeding from $S_{k-1}$ to $S_k$. If $e_k \in U$, then by $k$-correctness there is some $i < k$ such that $e_i \notin U$ and $e_k$ is in the unique cycle of $U \cup \{e_i\}$. Then $e_i$ is not dead in $U \setminus \{e_k\}$, so this set is eliminated when moving from $S_{k-1}$ to $S_i$, and in particular $U \setminus \{e_k\} \notin S_{k-1}$, so $U$ is not removed when passing from $S_{k-1}$ to $S_k$, completing the proof of maximality.

Thus, $S_m$ is the maximal $m$-correct subset of $S_0$. For any $U \in S_m$, we note that every edge of $E(G) \setminus \{e_i\}$ in $U$ is dead to $U$: if $f \notin E(G) \setminus \{e_i\}$ is not dead to $U$, then $f \notin V_G(U) \subseteq V_G(S)$, so $f = e_i$ for some $i$, but then since $U$ is $m$-correct we have $f \in U$, a contradiction. It follows that $U$ is maximally connected, so is a spanning tree of $G$. These are exactly the trees that satisfy the conditions of the lemma: every edge $e \in V_G(S) \cup U$ is one of the $e_i$, and by $m$-correctness each such edge is in the fundamental cycle of a smaller $e_i$ outside of $U$. Conversely, it is easy to check that every spanning tree with this property is $m$-correct. Furthermore, by the maximality of $S_m$, every spanning tree satisfying the lemma’s criteria is in $S_m$, so $S_m$ is precisely the set of these spanning trees.

Finally, by (a) above, each removal indeed removes two sets satisfying $\ell(U) \equiv 1 + \ell(U \cup \{e_{k+1}\}) \pmod{2}$. Therefore, for each $k \in [0, m - 1]$, $\sum_{U \in S_{k+1}} (-1)^{\ell(U)} = \sum_{U \in S_k} (-1)^{\ell(U)}$.

Thus, the following chain of equalities holds:

$$\sum_{U \in B_G} (-1)^{\ell(U)} = \sum_{U \in S_0} (-1)^{\ell(U)} = \sum_{U \in S_1} (-1)^{\ell(U)} = \cdots = \sum_{U \in S_m} (-1)^{\ell(U)} = (-1)^{C(G)} |S_m|$$

as desired.
Using the above lemmas, we compute the coefficients of the tree polynomial. A \( k \)-cutset of a graph \( G \) is a set \( S \subseteq E(G) \) such that \( (V(G), E(G) \setminus S) \) has \( k \) components. A \( k \)-cutset \( S \) is minimal if whenever \( T \subset S \), \( T \) is not a \( k \)-cutset of \( G \).

**Theorem 4.1.** Let \( G \) be a graph with \( n \) vertices and edge ordering \( \eta \). The coefficient \( [x^k] \tau_G \) is \( (-1)^{C(G) + k} \) times the number of pairs \((E, T)\), where \( E \) is a minimal \( k \)-cutset of \( G \) and \( T \in B_G \) is a spanning tree such that for every \( e \in E \cap T \) there exists \( f \in E \) with \( \eta(f) < \eta(e) \) such that \( f \notin T \) and \( e \) is in the fundamental cycle of \( f \).

**Proof.** By Whitney’s Broken Circuit Theorem 2.1, \( [x^m] \chi_G(x) \) is equal to \( (-1)^{n-m} \) times the number of elements of \( B_G \) that induce \( m \) components of \( G \) (and therefore contain \( n-m \) edges). Then \( (-1)^{n-m} [x^m] \chi_G \) is \( (-1)^{n-m} \) times the number of pairs \((A, B)\) where \( B \in B_G \) with \( \ell(B) = m \) and \( A \subseteq B \) with \( \ell(A) = k \). Then applying Theorem 3.2,

\[
[x^k] \tau_G = (-1)^{n-k} \sum_{m=1}^{k} \binom{n-m}{k-m} [x^m] \chi_G
\]

\[
= (-1)^{n-k} \sum_{m=1}^{k} \binom{n-m}{n-k} [x^m] \chi_G
\]

\[
= (-1)^{n-k} \sum_{S \in B_G} \sum_{U \in B_G} \sum_{\ell(S) = k} (-1)^{\ell(U) + n}
\]

\[
= (-1)^k \sum_{S \in B_G} \sum_{U \in B_G} \sum_{\ell(S) = k} (-1)^{\ell(U)}.
\]

By Lemma 4.3, it now follows that \( [x^k] \tau_G \) is \( (-1)^{k + |C(G)|} \) times the number of pairs \((A, B)\) where \( A \in B_G \) with \( \ell(A) = k \), and \( B \in B_G \) is a spanning tree with \( B \supseteq A \) such that when writing \( \nabla_G(A) = \{e_1, ..., e_m\} \) with \( i < j \) if and only if \( \eta(e_i) < \eta(e_j) \), we have that if \( e_i \in B \) then there exists \( j < i \) such that \( e_i \) is in the fundamental cycle of \( e_j \).

These pairs \((A, B)\) are in bijection with the pairs \((E, T)\) described in the theorem statement via the pair of mutually inverse bijective functions

\[
(A, B) \mapsto (\nabla_G(A), B),
\]

\[
(E, T) \mapsto (T \setminus E, T).
\]

That these are mutually inverse is apparent by verifying that \( B \setminus A = B \cap \nabla_G(A) \). \( \square \)

## 5 Further Directions

There are further avenues of exploration for interpreting the tree polynomial. For a graph \( G = (V, E) \) and a set partition \( \pi = \{B_1, ..., B_k\} \) of \( V \), we say \( \pi \) is a connected partition if the subgraph of \( G \) induced by \( B_i \) is connected for each \( i \). The lattice of contractions of \( G \) (denoted \( L_G \)) is the poset of connected partitions of \( V \) ordered by refinement (i.e., \( \{D_1, ..., D_k\} \leq \{B_1, ..., B_k\} \) if each \( D_i \) is contained in some \( B_j \)). Note that the minimal element \( \hat{0} \) of \( L_G \) is the partition in which each vertex has its own block. We write \( \ell(\pi) \) to denote the number of blocks in the set partition \( \pi \).
Theorem 5.1 (Whitney [16]). The chromatic polynomial of a graph $G$ can be written as

$$
\chi_G(x) = \sum_{\pi \in L_G} \mu(\hat{0}, \pi)x^{\ell(\pi)}.
$$

where $\mu$ is the Möbius function of $L_G$.

From this, following a similar argument as in the end of the last section, we see that for a graph $G$ on $n$ vertices,

$$
\tau_G(x) = \sum_{k=0}^{n} ([k^\ell]\chi_G(t))x^k(x - 1)^{n-k}
$$

$$
= \sum_{\pi \in L_G} \mu(\hat{0}, \pi)x^{\ell(\pi)}(x - 1)^{n-\ell(\pi)}
$$

$$
= \sum_{\pi \geq \hat{0}, \pi \in L_G} \mu(\hat{0}, \pi)x^{\ell(\pi)}(x - 1)^{n-\ell(\pi)}.
$$

This motivates the following definition:

**Definition 5.1.** Let $G$ be a graph and let $\pi \in L_G$. Define the augmented tree polynomial as

$$
\tau_G(\pi, x) := \sum_{\sigma \geq \pi, \sigma \in L_G} \mu(\pi, \sigma)x^{\ell(\sigma)}(x - 1)^{n-\ell(\sigma)}.
$$

We note that $\tau_G(\hat{0}, x) = \tau_G(x)$. By Möbius inversion,

$$
\tau_G(x) = \sum_{\sigma \geq \pi} \tau_G(\sigma, x)
$$

so we can give an interpretation of $\tau_G$ in terms of $L_G$ if we can ‘reverse engineer’ $\tau_G(\pi, x)$.

In total, the tree polynomial appears to be a natural graph polynomial worth further study, particularly given that even the tree polynomial’s basic properties imply further research possibilities for the chromatic polynomial. While proper colourings of a graph and spanning forests of a graph have each received substantial study, the tree polynomial suggests that a unification of these two approaches, considering proper colourings of spanning forests, provides combinatorial meaning as well. It may be interesting to apply the sieving argument of Lemma 4.3 to the positive integer evaluation of $\tau_G(x)$ given in Lemma 4.1 to determine if some of these coloured internal forests cancel naturally to reduce the sum, perhaps even to coloured internal spanning trees.

The tree polynomial’s reciprocal relationship with the chromatic polynomial also suggests a combinatorial meaning for certain evaluations of the chromatic polynomial at rational numbers, such as $\chi_G\left(\frac{5}{4}\right)$, which to the best of the authors’ knowledge is unknown. It may be interesting to consider if other families of rational evaluations of the chromatic polynomial have similar meaning.
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CONFLICTS OF INTEREST STATEMENT
The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT
Data sharing is not applicable to this article as no data sets were generated or analysed during the current study.

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