Abstract. Reduced forms of monomials in an algebra are obtained via substitution rules dictated by the relations of the algebra. We show that reduced forms of monomials in the subdivision algebra are generalizations of $h$-polynomials of triangulations of flow polytopes. The latter are a way of encoding the number of faces of each dimension in the triangulations. We conclude that the coefficients of specialized reduced forms are nonnegative. We then deduce a special case of a conjecture of Kirillov for the quasi-classical Yang-Baxter algebra. We also express reduced forms in terms of Ehrhart series of flow polytopes, which in turn can be seen as Kostant partition functions.

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1. Introduction

Reduced forms of monomials in an algebra are obtained via substitution rules dictated by the relations of the algebra. We show that reduced forms can be seen as generalizations of $h$-polynomials. Recall that the $h$-polynomial of a simplicial complex is a way of encoding the number of faces of each dimension. We prove that specialized reduced forms in the subdivision algebra equal shifted $h$-polynomials of regular triangulations of flow polytopes. This result opens a new avenue for understanding reduced forms of monomials in the subdivision and related algebras. We prove nonnegativity results in the subdivision algebra as a consequence of the specialized reduced form equaling the shifted $h$-polynomial. As a corollary we establish a special case of Conjecture 2 of Kirillov appearing in [4] about reduced forms in the quasi-classical Yang-Baxter algebra. We also express specialized reduced forms in terms of Ehrhart series of flow polytopes, which in turn can be seen in terms of Kostant partition functions.
Our methods in this paper are largely geometric. In [9] we study reduced forms from the point of view of the structure of reduction trees, leaving the geometry behind.

We now highlight the main results of the paper. Some theorems are not stated mathematically here, in order to avoid introducing a lot of notation. They are explained in detail in later sections, and we point back to the corresponding statement here. For the background and definitions see Section 2.

Theorem 1. The reduced form \( Q^{S(\beta)}_G (\beta - 1) \) is equal to the \( h \)-polynomial \( h(C, \beta) \), where \( C \) is an \( R_G \)-simplicial collection on \( F_G \) for an arbitrary reduction tree \( R_G \). Its coefficients are nonnegative.

Theorem 2. [4, cf. Conjecture 2]. The reduced form of \( x_{12}x_{23} \cdots x_{n-1,n} \) evaluated at \( x = (1, \ldots, 1) \) and \( \beta - 1 \) in the quasi-classical Yang-Baxter algebra \( \tilde{ACYB}_n(\beta) \) is a polynomial in \( \beta \) with nonnegative coefficients.

Theorem 3. Reduced forms in the subdivision algebra can encode unimodular, regular and flag triangulations of flow polytopes.

Theorem 4. The reduced forms in the subdivision algebra can be explicitly described in terms of shellings of flow polytopes. They can also be expressed in terms of Kostant partition functions.

The paper is organized as follows. In Section 2 we define flow polytopes. Next we explain how to subdivide flow polytopes and how we can encode the subdivisions with a reduction tree. Then we define the subdivision algebra and show that the reduced form can be read off from the leaves of the reduction tree. In Section 3 we define simplicial collections, which are hypergraphs with additional properties and into which the subdivision algebra cuts flow polytopes. The notions of \( f \)- and \( h \)-vectors extend easily to simplicial collections from simplicial complexes. We prove that the reduced form of a monomial in the subdivision algebra specialized at certain variables is equal to the shifted \( h \)-polynomial of a simplicial collection associated to the monomial. In Section 4 we show that we can use the subdivision algebra and arrive not only at a simplicial collection on a flow polytope, but also to a triangulation of the flow polytope, in the sense of a simplicial complex. To do this we use a particular reduction order \( \sigma \). The triangulation we obtain is regular and flag. In Section 5 we describe the full set of leaves of the reduction tree in order \( \sigma \), or equivalently, the monomials in the reduced form of a monomial. In Section 6 we use the regularity of the triangulation we constructed to prove and interpret the nonnegativity of the coefficients of the reduced form of a monomial in the subdivision algebra as well as a special case of Conjecture 2 of Kirillov appearing in [4]. In Section 7 we relate reduced forms to Ehrhart series of flow polytopes, and thus obtain a generalization of [4, Theorem 3.10]. We also relate reduced forms to Kostant partition functions.

2. Definitions and more of the story

2.1. Flow polytopes and their subdivisions. Given a loopless graph \( G \) on the vertex set \([n]\), let \( \text{in}(e) \) denote the smallest (initial) vertex of edge \( e \) and \( \text{fin}(e) \) the biggest (final) vertex of edge \( e \). Let \( E(G) = \{\{e_1, \ldots, e_l\}\} \) be the multiset of edges of \( G \). We correspond variables \( x_{e_i}, i \in [l] \), to the edges of \( G \), of which we think as flows. The flow polytope \( F_G \) is naturally embedded into \( \mathbb{R}^{\#E(G)} \), where \( x_{e_i}, i \in [l] \), are thought of as the coordinates. \( F_G \) is defined by

\[
x_{e_i} \geq 0, \ i \in [l],
\]

\[
1 = \sum_{e \in E(G), \text{in}(e) = 1} x_e = \sum_{e \in E(G), \text{fin}(e) = n+1} x_e,
\]

and for \( 2 \leq i \leq n \)
The flow polytope $F_{K_{n+1}}$ can be thought of as the Chan-Robbins-Yuen polytope [2], and has received a lot of attention, since its volume is equal to $\prod_{k=0}^{n-2} \text{Cat}(k)$, where $\text{Cat}(k) = \frac{1}{k+1} \binom{2k}{k}$ is the $k$th Catalan number. There is no combinatorial proof of the aforementioned result; Zeilberger [4] provided an analytical proof. For more of the story see [10].

Flow polytopes lend themselves to subdivisions via reductions, as explained below. A similar property of root polytopes was studied in [7, 8].

**Definition 1.** Given a graph $G$ on the vertex set $[n]$ containing edges $(i,j)$ and $(j,k)$, $i < j < k$, performing the reduction on these edges of $G$ yields three graphs on the vertex set $[n]$:

\[
\begin{align*}
E(G_1) &= E(G) \setminus \{(j,k)\} \cup \{(i,k)\}, \\
E(G_2) &= E(G) \setminus \{(i,j)\} \cup \{(i,k)\}, \\
E(G_3) &= E(G) \setminus \{(i,j),(j,k)\} \cup \{(i,k)\}.
\end{align*}
\]

(1)

When performing a reduction on the edges $(i,j),(j,k)$ we say that the edge $(i,j)$ is dropped if we go towards $G_2$ or $G_3$ as in Figure 1 and $(i,j)$ is kept if we go towards $G_1$. Similarly, edge $(j,k)$ is dropped if we go towards $G_1$ or $G_3$ as in Figure 1 and $(j,k)$ is kept if we go towards $G_2$.

**Definition 2.** A reduction tree $R_G$ of a graph $G$ is a tree with nodes labeled by graphs and such that all non-leaf nodes of $R_G$ have three children. The root is labeled by $G$. If there are two edges $(i,j),(j,k) \in E(G)$, $i < j < k$, on which we choose to do a reduction, then the children of the root are labeled by $G_1, G_2$ and $G_3$ as in Figure 1. Next, continue this way by constructing reduction trees for $G_1, G_2$ and $G_3$. If some graph has no edges $(i,j),(j,k)$, $i < j < k$, then it is its own reduction tree. Note that the reduction tree $R_G$ is not unique; it depends on our choice of edges to reduce. However, the number of leaves (referring to the graph labeling a leaf) of all reduction trees of $G$ with a given number of edges is the same as Lemma 5 states below. We choose a particular embedding of the reduction tree in the plane for convenience: we root it at $G$ with the tree growing downwards, and such that the left child is $G_1$, the middle child is $G_3$ and the right child is $G_2$; see Figure 1. The leaves which have the same number of edges at the root are called full dimensional.

**Definition 3.** Let the edges of $G$ be $e_1, \ldots, e_k$, where we distinguish multiple edges. If a reduction involving edges $a = (i,j)$ and $b = (j,k)$ of $G$ is performed, then the new edge $(i,k)$ appearing in all

![Figure 1](image-url)
three graphs as in (1) is formally thought of as $a + b = b + a$. The other edges stay unchanged. To get to nodes $G_1$ and $G_2$ of $R_G$, we iterate this process, thereby expressing the edges of any node as a sum of edges of the graph being the root of the reduction tree. Two edges $c$ and $d$ in the graphs $G_1$ and $G_2$, respectively, are the same, if they are the sum of exactly the same edges of $G$. The intersection of two graphs $G_1$ and $G_2$ in a reduction tree $R_G$ is $G_1 \cap G_2 = (V(G), E(G_1) \cap E(G_2))$, where if $e \in E(G_1) \cap E(G_2)$ then as explained above $e$ is the sum of the same edges of $G$ in both $G_1$ and $G_2$.

**Lemma 5.** [\[7\]] In any reduction tree of $G$, the number of leaves with $k$ edges is the same.

**Definition 4.** The augmented graph $\tilde{G}$ of $G = ([n], E)$ is $\tilde{G} = ([n] \cup \{s, t\}, \tilde{E})$, where $s$ (source) is the smallest, $t$ (target/sink) is the biggest vertex of $[n] \cup \{s, t\}$, and $\tilde{E} = E \cup \{(s, i), (i, t) | i \in [n]\}$. Denote by $\mathcal{P}(\tilde{G})$ the set of all maximal paths in $\tilde{G}$, referred to as routes. It is well known that the unit flows sent along the routes in $\mathcal{P}(\tilde{G})$ are the vertices of $\mathcal{F}(\tilde{G})$.

**Definition 5.** Consider a node $G_1$ of the reduction tree $R_G$, where each edge of $G_1$ is considered as a sum of the edges of $G$. The image of the map $m : E(G_1) \rightarrow \mathcal{P}(\tilde{G})$ which takes an edge $(v_1, v_2) = e = e_{i_1} + \cdots + e_{i_j}, e \in G_1, e_{i_j} \in E(G), j \in [l]$, to the route $(s, v_1), e_{i_1}, \ldots, e_{i_j}, (v_2, t)$ gives the vertices of $\mathcal{F}_{G_1}^\circ$ (by taking the unit flows on these routes). In case $G_1$ is not a node of the reduction tree $R_G$, but it is an intersection of nodes of $R_G$, so that each edge of $G_1$ can still be considered as a sum of the edges of $G$, we still define $\mathcal{F}_{G_1}^\circ$, as above. This definition of $\mathcal{F}_{G_1}^\circ$ is of course with respect to $G$, and this is understood from the context.

Using the above definitions the proof of the following lemma is an easy exercise.

**Lemma 6.** [\[11\]] Proposition 1], [\[13\]] Proposition 4.1], [\[15\], [\[13\]] Given a graph $G$ on the vertex set $[n]$ and $(i, j), (j, k) \in E(G)$, for some $i < j < k$, and $G_1, G_2, G_3$ as in (1) and $\mathcal{F}_{\tilde{G}_i}, i \in [3]$, as in Definition 4 we have

$$\mathcal{F}_{\tilde{G}} = \mathcal{F}_{\tilde{G}_1} \bigcup \mathcal{F}_{\tilde{G}_2} \bigcap \mathcal{F}_{\tilde{G}_3} = \mathcal{F}_{\tilde{G}_3} \bigcap \mathcal{F}_{\tilde{G}_1} \bigcap \mathcal{F}_{\tilde{G}_2} = \emptyset,$$

where $\mathcal{F}_{\tilde{G}}, \mathcal{F}_{\tilde{G}_1}, \mathcal{F}_{\tilde{G}_2}$ are of the same dimension $d - 1$, $\mathcal{F}_{\tilde{G}_3}$ is $d - 2$ dimensional, and $\mathcal{P}^\circ$ denotes the interior of $\mathcal{P}$.

### 2.2. Encoding subdivisions by relations.

Note that the reduction of graphs given in (1) can be encoded as the following relation:

$$x_{ij}x_{jk} = x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}, \text{ for } 1 \leq i < j < k \leq n.$$  \(2\)

Namely, interpreting the double indices of the variables $x_{ij}$ as edges, the monomial $x_{ij}x_{jk}$ picks out two edges $(i, j), (j, k), i < j < k$, and replaces it with three monomials, corresponding to operation on graphs (1). The variable $\beta$ is simply a placeholder, indicating that the number of edges in the third graph is one less than in the other graphs.

These relations give rise to what we call the subdivision algebra.

**Definition 6.** The associative subdivision algebra, denoted by $\mathcal{S}(\beta)$, is an associative algebra, over the ring of polynomials $\mathbb{Z}[\beta]$, generated by the set of elements $\{x_{ij} : 1 \leq i < j \leq n\}$, subject to the relations:

(a) $x_{ij}x_{kl} = x_{kl}x_{ij}$, if $i < j, k < l$,
(b) $x_{ij}x_{jk} = x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}$, if $1 \leq i < j < k \leq n$.

The algebra $\mathcal{S}(\beta)$ has been studied for root polytopes [\[7\]].

**Definition 7.** Given a monomial $M$ in $\mathcal{S}(\beta)$, its reduced form is defined as follows. Starting with $p_0 = M$, produce a sequence of polynomials $p_0, p_1, \ldots, p_m$ in the following fashion. To obtain $p_{r+1}$
from \( p_r \), choose a term of \( p_r \) which is divisible by \( x_{ij}x_{jk} \), for some \( i, j, k \), and replace the factor \( x_{ij}x_{jk} \) in this term with \( x_{ij}x_{ik} + x_{jk}x_{ik} + \beta x_{ik} \). Note that \( p_{r+1} \) has two more terms than \( p_r \). Continue this process until a polynomial \( p_m \) is obtained, in which no term is divisible by \( x_{ij}x_{jk} \), for any \( i, j, k \). Such a polynomial \( p_m \) is a reduced form of \( M \). Note that we allow the use of the commutation relations in this process.

Given a monomial \( M \) we can encode it by a graph \( G_M \), simply by letting the edges of \( G_M \) be the given by the indices of the variables in \( M \). Denote a reduced form of \( M \) in \( S(\beta) \) by \( Q_{G_M}^{S(\beta)}(x; \beta) \). If in the reduced forms we set \( x = (1, \ldots, 1) \), then in the notation we omit \( x \): \( Q_{G_M}^{S(\beta)}(\beta) \).

It is easy to see that by definition, the reduced form of a monomial in the subdivision algebra can be read off from the leaves of the reduction tree of the corresponding graph. With this and Lemma 6 in mind, it is no surprise that we can prove results about reduced forms of monomials in the algebras using flow polytopes.

Note that the reduced form of a monomial in \( S(\beta) \) is not necessarily unique, which could be a desirable property. Amazingly, there is a noncommutative algebra, denoted \( \widetilde{ACYB}_n(\beta) \), which is much like \( S(\beta) \), yet in which the reduced forms are unique. The latter statement was proved in [7]. It was A.N. Kirillov [3, 5] who introduced \( \widetilde{ACYB}_n(\beta) \) and shed the first light on its rich combinatorial structure. The paper [9] addresses more of the story of subdivision algebras as well as the story of \( \widetilde{ACYB}_n(\beta) \).

### 3. Simplicial collections, their \( h \)-vectors, and reduced forms

In this section we define simplicial collections, which are (almost) what the subdivision algebra cuts the flow polytopes into. The notions of \( f \)- and \( h \)-vectors work naturally with simplicial collections, and we show that specialized reduced forms in the subdivision algebra equal the shifted \( h \)-polynomials of the simplicial collections we obtain.

**Definition 8.** Given a convex polytope \( P \) with vertices \( \{v_1, \ldots, v_n\} \), a simplicial collection on \( P \) is a set of simplices \( \mathcal{C} \) such that:

1. If \( \sigma \in \mathcal{C} \), then the set of vertices of \( \sigma \) is a subset of \( \{v_1, \ldots, v_n\} \),
2. \( v_i \in \sigma \) for \( i \in [n] \),
3. \( \bigcup_{\sigma \in \mathcal{C}} \sigma^\circ = P \), where \( \sigma^\circ \) denotes the interior of the simplex \( \sigma \) and \( v_i^\circ = v_i, i \in [n] \).

A simplicial collection can be thought of as a geometric realization of a hypergraph with additional special properties. To illustrate the key difference between a simplicial complex and a simplicial collection, consider the crosspolytope \( C = \text{conv}\{e_i, -e_i \mid i \in [3]\} \), where \( e_i \) is the standard \( i \)-th coordinate vector. We can construct a simplicial collection on \( C \) containing the simplices \( \text{conv}(e_1, -e_1, e_2, e_3), \text{conv}(e_1, -e_1, -e_2, e_3), \text{conv}(e_1, e_2, -e_2, -e_3), \text{conv}(-e_1, e_2, -e_2, -e_3) \). On Figure 2 the edges of \( \text{conv}(e_1, -e_1, e_2, e_3) \) are in bold and the edges of \( \text{conv}(e_1, e_2, -e_2, -e_3) \) are in dashed lines. Their intersection is \( \text{conv}(e_1, e_2, 0) \), which is not a face of either.

Given a flow polytope \( F_G \), the simplices corresponding to the leaves of a reduction tree \( R_G \) satisfy \( \bigcup_{\sigma \in \mathcal{C}} \sigma^\circ = F_G^\circ \), which is almost like condition 3 from the definition of simplicial collections. We can thus construct a simplicial collection (in fact possibly more of them) on \( F_G \) which contains all the simplices corresponding to the leaves of a reduction tree \( R_G \). Different reduction trees yield distinct simplicial collections; we call simplicial collections on \( F_G \) arising from a reduction tree \( R_G \) \( R_G \)-simplicial collections.

**Definition 9.** The dimension of a simplicial collection \( \mathcal{C} \) is the highest dimension of a simplex contained in \( \mathcal{C} \).

**Definition 10.** Let \( \mathcal{C} \) be a \( d-1 \) dimensional simplicial collection. The \( f \)-vector of \( \mathcal{C} \) is

\[
f(\mathcal{C}) = (f_{-1}, f_0, \ldots, f_{d-1}),
\]
Figure 2. Two simplices in a simplicial collection do not have to intersect in a common face.

where \( f_i = f_i(C) \) be the number of \( i \)-dimensional simplices in \( C \). By convention, \( f_{-1} = 1 \) unless \( C = \emptyset \), in which case \( f_{-1} = 0 \). The \( h \)-vector of \( C \) is \( h(C) = (h_0, h_1, \ldots, h_d) \), defined by

\[
\sum_{i=0}^{d} f_{i-1} (x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.
\]

Define the \( h \)-polynomial of a simplicial collection \( C \) to be

\[
h(C, x) = \sum_{i=0}^{d} h_i x^i.
\]

Lemma 7. For a simplicial collection \( C \) we have:

\[
h(C, x) = \sum_{F \in C} x^{|F|} (1-x)^{d-|F|}.
\]

Lemma 7 is well-known for abstract simplicial complexes [12], and the usual proof works for simplicial collections, too:

Proof of Lemma 7.

\[
h(C, x) = x^d \sum_{i=0}^{d} h_i \left( \frac{1}{x} \right)^{d-i} = x^d \sum_{i=0}^{d} f_{i-1} \left( \frac{1}{x} - 1 \right)^{d-i} = \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i} = \sum_{F \in C} x^{|F|} (1-x)^{d-|F|}.
\]

The main result of this section is the following theorem. The statement of it is part of Theorem 1 in the Introduction.

Theorem 8. We have

\[
Q^{S(\beta)}_G(C, \beta) = h(C, \beta + 1),
\]

where \( C \) is an \( R_G \)-simplicial collection on \( F_C \) for an arbitrary reduction tree \( R_G \).

Note that the polynomial \( Q^{S(\beta)}_G(C, \beta) \) is well defined; according to Lemma 5, \( Q^{S(\beta)}_G(C, \beta) \) is independent of the particular reductions performed. Another proof of this fact can be obtained from the following proof. Note that as is the proof is written for flow polytopes, though it could be made more general. We outline an alternative proof of Theorem 8 in Subsection 4.3.
Proof of Theorem 8. We prove the equivalent statement

\[ Q_G^{\beta}(\beta - 1) = h(C, \beta) \]

by induction on \( \text{dep}(G) \), which is the edge-length of the longest path from \( G \) to a leaf in a reduction tree \( R_G \).

If \( \text{dep}(G) = 0 \), then \( Q_G^{\beta}(\beta - 1) = 1 \) and \( F_G \) is a simplex. If \( \dim F_G = d - 1 \), then \( h(C, \beta) = \sum_{F \in C} \beta#F (1 - \beta)^{d - \#F} = \sum_{i=0}^{d} \binom{d}{i} \beta^i (1 - \beta)^{d-i} = 1. \)

Suppose (7) is true for all graphs \( G \) with \( \text{dep}(G) < m \). Consider the graph \( G \) with \( \text{dep}(G) = m > 0 \). Since there is a pair of alternating edges in \( G \) we can perform a reduction on \( G \) obtaining the graphs \( G_1, G_2 \) and \( G_3 \), such that \( \text{dep}(G_1), \text{dep}(G_2), \text{dep}(G_3) < m \). It follows then by definition that

\[ Q_G^{\beta}(\beta) = Q_{G_1}^{\beta}(\beta) + Q_{G_2}^{\beta}(\beta) + \beta Q_{G_3}^{\beta}(\beta). \]

Since \( \text{dep}(G_1), \text{dep}(G_2), \text{dep}(G_3) < m \), it follows by inductive hypothesis that \( Q_{G_i}^{\beta}(\beta - 1) = h(C_i, \beta), i \in [3] \), where \( C_i \) is any \( R_G \)-simplicial collection of \( F_{G_i} \), \( i \in [3] \). Next we show that

\[ h(C, \beta) = h(C_1, \beta) + h(C_2, \beta) + (\beta - 1)h(C_3, \beta), \]

which will conclude the proof of (7). Recall that by Proposition 6 we have \( F_{G_1}, F_{G_2}, \text{ and } F_{G_3} \) of the same dimension, say \( d - 1 \), and \( F_{G_3} \) is one dimension less, \( d - 2 \), so by (5) we can write

\[ h(C_1, \beta) = \sum_{i=0}^{d} f_{i-1}^{G_1} \beta^i (1 - \beta)^{d-i}, \]

(11) \[ h(C_2, \beta) = \sum_{i=0}^{d} f_{i-1}^{G_2} \beta^i (1 - \beta)^{d-i}, \]

(12) \[ h(C_3, \beta) = \sum_{i=0}^{d-1} f_{i-1}^{G_3} \beta^i (1 - \beta)^{d-1-i}. \]

Thus,

\[ h(C_1, \beta) + h(C_2, \beta) + (\beta - 1)h(C_3, \beta) = \sum_{i=0}^{d} (f_{i-1}^{G_1} + f_{i-1}^{G_2}) \beta^i (1 - \beta)^{d-i} - \sum_{i=0}^{d-1} f_{i-1}^{G_3} \beta^i (1 - \beta)^{d-i} \]

(14) \[ = \sum_{i=0}^{d} (f_{i-1}^{G_1} + f_{i-1}^{G_2} - f_{i-1}^{G_3}) \beta^i (1 - \beta)^{d-i}. \]

Since the intersection of \( F_{G_1} \) and \( F_{G_2} \) is accounted for twice in \( f_{i-1}^{G_1} + f_{i-1}^{G_2} \) (note we might be counting here different faces, but their union is ultimately the same on the intersection), we conclude that

\[ \sum_{i=0}^{d} (f_{i-1}^{G_1} + f_{i-1}^{G_2} - f_{i-1}^{G_3}) \beta^i (1 - \beta)^{d-i} = h(C, \beta), \]

thereby concluding the proof. \( \Box \)

We note that one can define a vector space \( k[C] \) associated to a simplicial collection, which is the Stanley-Reisner ring when \( C \) is a simplicial complex, and use it to show that

\[ h(C, \beta) = \sum_{m \geq 0} (i(F_{G_1}, m)\beta^m) (1 - \beta)^{\dim(F_G) + 1}, \]

where \( C \) is an \( R_G \)-simplicial collection on \( F_{G_1} \) for an arbitrary reduction tree \( R_G \) and \( i(F_{G_1}, m) \) is the Ehrhart polynomial of \( F_{G_1} \). However, since \( 16 \) is a standard result in case \( C \) is a unimodular
triangulation, we instead first show that there is an $R_G$-simplicial collection on $\mathcal{F}_G$ which is a regular unimodular triangulation, and draw our conclusions regarding the Ehrhart series of $\mathcal{F}_G$ from there.

4. Triangulating flow polytopes

In this section we prove that there is an $R_G$-simplicial collection which is a simplicial complex; in other words we can use the subdivision algebra to obtain triangulations of every flow polytope $\mathcal{F}_G$. A priori this is far from clear. Moreover, the triangulation we obtain is flag and regular, and thus shellable. Thus, as we see in Section 6 not only are the coefficients of the (shifted) specialized reduced forms nonnegative, but they can also be interpreted in terms of shellings of the aforementioned triangulation. We construct our special triangulation by picking a specific reduction order $\sigma$ on our graph $G$ and utilizing the properties of $\sigma$ to prove properties about the leaves of the reduction tree $R^\sigma_G$ obtained using $\sigma$. We then note that there is a whole class of orders for which our arguments work, but for clarity we lay out the argument for $\sigma$ first.

4.1. A family of orders $\mathcal{F}(\sigma)$. Given an arbitrary graph $G$ on the vertex set $[n]$, put a total order on the set of incoming and set of outgoing edges at each vertex $1 < v < n$. Do the reductions in $G$ proceeding from the smallest vertex towards the greatest in order. Look for the smallest vertex $v$ which is nonalternating, that is that has both an edge $(a,v)$ and an edge $(v,b)$ incident to it, with $a < v < b$. Look at the incoming and outgoing edges at $v$, $(a,v)$ and $(v,b)$, which are smallest in the ordering of the incoming and outgoing edges, respectively. Do the reduction on $(a,v)$ and $(v,b)$. In the three obtained graphs the relative ordering of the edges stays the same, with the new edge $(a,b)$ either taking the place of $(a,v)$ or $(v,b)$ if these were dropped, or directly preceeding them when they are kept. Continue in this fashion on each leaf of the partial reduction tree ultimately arriving to the reduction tree $R_G$ with all leaves alternating graphs, that is all of their vertices are alternating. Let $\sigma$ be the order where the initial ordering of the incoming and outgoing edges at each vertex is such that the topmost is the smallest, then the next topmost, etc. See Figure 3 for an example of this ordering. All results of this paper generalize for any order in $\mathcal{F}(\sigma)$, but for simplicity we state and prove them for $\sigma$ only.

The main result of this section is:

**Theorem 9.** The simplices corresponding to the full dimensional leaves of $R^\sigma_G$ yield the top dimensional simplices in a regular and flag triangulation of $\mathcal{F}_G$. Moreover, lower dimensional simplices of this triangulation which are not contained in the boundary of $\mathcal{F}_G$ are obtained from the (not full dimensional) leaves of $R^\sigma_G$.

**Proof.** The proof follows from Theorem 10, Lemma 11 and Proposition 12.

**Theorem 9** is the subject of Theorem 3 in the Introduction.

Next we explain Theorem 10 and prove Proposition 12.

4.2. Coherent routes, cliques and triangulation of flow polytopes. Given a graph $G$ on the vertex set $[n]$ with edges oriented from smaller to bigger vertices, the vertices of the flow polytope $\mathcal{F}_G$ correspond to integer flows of size one on maximal directed paths from the source (1) to the sink ($n$). Following [3] we call such maximal paths **routes**. The following definitions follow [3].
Fix a framing at each inner vertex \( v \) (that is a vertex that is not a source or a sink) of \( G \), which is the linear ordering \( \prec_{in(v)} \) on the set of incoming edges \( in(v) \) to \( v \) and the linear ordering \( \prec_{out(v)} \) on the set of outgoing edges \( out(v) \) from \( v \). We call a graph with a framing at each inner vertex framed. For a framed graph \( G \) and an inner vertex \( v \) we denote by \( In(v) \) and by \( Out(v) \) the set of maximal paths ending in \( v \) and the set of maximal paths starting at \( v \), respectively. We define the order \( \prec_{in(v)} \) on the paths in \( In(v) \) as follows. If \( P, Q \in In(v) \) then let \( w \) be the largest vertex after which \( P \) and \( Q \) coincide and before which they differ. Let \( e_P \) be the edge of \( P \) entering \( w \) and \( e_Q \) be the edge of \( Q \) entering \( w \). Then \( P \prec_{in(v)} Q \) if and only if \( e_P \prec_{in(v)} e_Q \). The linear order \( \prec_{out(v)} \) on \( Out(v) \) is defined analogously.

Given a route \( P \) with an inner vertex \( v \) denote by \( Pv \) the maximal subpath of \( P \) leaving \( v \). We say that the routes \( P \) and \( Q \) are coherent at a vertex \( v \) which is an inner vertex of both \( P \) and \( Q \) if the paths \( Pv, Qv \) are ordered the same way as \( vP, vQ \); e.g., if \( Pv \prec_{in(v)} Qv \) and \( vP \prec_{out(v)} vQ \). We say that routes \( P \) and \( Q \) are coherent if they are coherent at each common inner vertex. We call a set of mutually coherent routes a clique. The following theorem is a special case of [3, Theorems 1 & 2].

**Theorem 10.** [3] Theorems 1 & 2] Given a framed graph \( G \), taking the convex hulls of the vertices corresponding to the routes in maximal cliques yield the top dimensional simplices in a regular triangulation of \( \mathcal{F}_G \). Moreover, lower dimensional simplices of this triangulation are obtained as convex hulls of the vertices corresponding to the routes in (not maximal) cliques.

**Lemma 11.** The triangulation described in Theorem 10 is flag.

**Proof.** Consider a non-face \( N \) of the triangulation that is of cardinality greater than 2. By Theorem 10 the vertices of \( N \) are routes that are not coherent; in particular there are two routes \( P \) and \( Q \) which yield vertices of \( N \) and are not coherent. Since \( P \) and \( Q \) are not coherent, they constitute a non-face. Therefore, all minimal non-faces of the triangulation described in Theorem 10 are of size 2, and therefore the triangulation is flag.

Now we are ready to prove the proposition which together with Theorem 10 and Lemma 11 implies Theorem 9. We define the framing \( \tilde{\sigma} \) on \( \tilde{G} \) as the ordering \( \sigma \) on the edges of \( G \), that is, the incoming edges are ordered top to bottom and the outgoing edges are also ordered top to bottom, and the edges of the form \((s, i)\) and \((i, t)\), for \( i \in [n] \), are always last in the orderings. See Figure 4 for an example.

**Proposition 12.** The set of vertices of the simplices corresponding to the leaves of \( R^*_G \) form a clique of mutually coherent paths in \( \tilde{G} \) with the framing \( \tilde{\sigma} \).

**Proof.** Suppose that to the contrary, there are two vertices of a simplex corresponding to a leaf of \( R^*_G \), which correspond to non-coherent routes \( P \) and \( Q \) in \( G \). Suppose that \( P \) and \( Q \) are not coherent at the common inner vertex \( v \). Suppose that the smallest vertex after which \( Pv \) and \( Qv \) agree is \( w_1 \) and the largest vertex before which \( vP \) and \( vQ \) agree is \( w_2 \). Let the edges incoming to \( w_1 \) be \( e^1_P \) and \( e^1_Q \) for \( P \) and \( Q \), respectively, and let the edges outgoing from \( w_2 \) be \( e^2_P \) and \( e^2_Q \) for
leaves of a reduction tree yield the open polytope $F$ simplex. The notation $f_{(18)}$ where $Q_{(17)}$ as read off from the reduction tree $R$ to $s$ we know that $f_{4.3.}$ triangulation of an arbitrary polytope and defining an alternative proof of Theorem 8.

Let $P$ and $Q$, respectively. Since $P$ and $Q$ are not coherent at $v$, this implies that either $e^1_P \prec_{in(w_1)} e^1_Q$ and $e^2_Q \prec_{out(w_2)} e^2_P$ or $e^1_Q \prec_{in(w_1)} e^1_P$ and $e^2_P \prec_{out(w_2)} e^2_Q$. We also have that the segments of $P$ and $Q$ between $w_1$ and $w_2$ coincide. Note that since the edges of the form $(s, i)$ and $(i, t)$, $i \in [n]$, are last in the linear orderings of the incoming and outgoing edges, it follows that at most one of the edges $e^1_P$ and $e^2_P$ and at most one of the edges $e^1_Q$ and $e^2_Q$ could be incident to $s$ or $t$. We consider several cases based on whether any of $e^1_P, e^2_P, e^1_Q, e^2_Q$ are incident to $s$ or $t$. Denote by $p$ the sum of edges between $w_1$ and $w_2$ on $P$. If none of $e^1_P, e^2_P, e^1_Q, e^2_Q$ are incident to $s$ or $t$, then after a certain number of reductions executed according to $\sigma$ we are about the perform the reduction $(*(e^1_Z + p), e^2_Z)$, where $*(e^1_Z + p)$ denotes the sum of edges left of $w_1$ that are edges in $\bar{Z}$ not incident to $s$ (including $e^1_Z$ in particular) and $p$, $\{\bar{Z}, \bar{Z}\} = \{P, Q\}$. Note, however, that after executing this reduction we have to drop either the edge $*(e^1_Z + p)$ or the edge $e^2_Z$. However, if the former were true, it would make it impossible for $*(e^1_Z + p) + e^2_Z$ to be a subsum in an edge of the leaf we are considering, which it has to be in order for $\bar{Z}$ to be a route giving a vertex of the simplex we are considering. The latter on the other hand would make it impossible for $*(e^1_Z + p) + e^2_Z$ to be a subsum in an edge of the leaf we are considering, where $*(e^1_Z + p)$ denotes the sum of edges left of $w_1$ that are edges in $\bar{Z}$ not incident to $s$. However then $\bar{Z}$ cannot be a route giving a vertex of the simplex we are considering. Thus we see that $\bar{Z}$ and $\bar{Z}$, aka, $P$ and $Q$, cannot be incoherent in this way. It follows that we need to consider the possibilities where some of $e^1_P, e^2_P, e^1_Q, e^2_Q$ are incident to $s$ or $t$. One can construct similar arguments to the above in all those cases. □

4.3. An alternative proof of Theorem 8. Note that, by definition, the reduced form $Q^{S(\beta)}_{G}(\beta)$ read off from the reduction tree $R^G$, which yields the unimodular triangulation $C^{\sigma}$, can be written as

(17) \[ Q^{S(\beta)}_{G}(\beta) = \sum_{i=1}^{d-1} f^i_1 \beta^{d-1-i}, \]

where $C^{\sigma}$ is $d - 1$ dimensional and $f^i_1$ is the number of leaves of $R^G$ yielding an $i$-dimensional simplex. The notation $f^i_{G}$ signifies that the union of the open simplices corresponding to the leaves of a reduction tree yield the open polytope $\bar{F}_{\bar{G}}$. However, using the definitions of $f$- and $h$-polynomials one can show that

(18) \[ \sum_{i=1}^{d-1} f^i_{G} \beta^{d-1-i} = h(C^{\sigma}, \beta + 1), \]

and in fact such an equation holds for any polytope not just flow polytopes, considering any triangulation of an arbitrary polytope and defining $f^i_{G}$ in a fashion as above. Since by Lemma 5 we know that $Q^{S(\beta)}_{G}(\beta)$ does not depend on the particular reduction tree, we can thus obtain an alternative proof of Theorem 8.

5. A description of the leaves of $R^G$

In this section we describe all the leaves of the reduction tree $R^G$ in terms of shellings of the triangulation obtained from $R^G$ as in Theorem 9. Since we know that these triangulations are regular, it follows that they are also shellable. Theorem 4 was referring to the following theorem in part.

**Theorem 13.** Let $\bar{F}_{\bar{F}_1}, \ldots, \bar{F}_{\bar{F}_l}$ be a shelling order of the simplicial complex arising from $R^G$. Let $P_i := \{\{Q_1, \ldots, Q^i_{j(i)}\} = \{\{F_i \cap F_j \mid 1 \leq j < i, |E(F_i \cap F_j)| = |E(F_i)| - 1\}.}$
Then

\[ \sum_{i=1}^{l} \prod_{j=1}^{f(i)} (F_i + Q_j) \]

is the formal sum of the set of the leaves of \( R^x_G \), where the product of graphs is their intersection. If \( f(i) = 0 \) we define \( \prod_{j=1}^{f(i)} (F_i + Q_j) = F_i \).

Before proving Theorem 13 we record a few properties of flow polytopes that easily follow from the above considerations and from the fact that the dimension of \( \mathcal{F}_G \) is \( |E(G)| - |V(G)| + 1 \).

In both lemmas the meanings of \( \mathcal{F}_H \) for a node \( H \) of \( R^x_G \) or intersection of such nodes is as in Definition 5, which is the key to the proofs that are left to the interested reader.

**Lemma 14.** Let \( G_1 \) and \( G_2 \) be two leaves of \( R^x_G \). Then

\[ \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} = \mathcal{F}_{G_1 \cap G_2}. \]

Moreover, \( G_1 \cap G_2 \) is a leaf of \( R^x_G \) if and only if \( \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} \) is not contained in the boundary of \( \mathcal{F}_G \).

**Lemma 15.** Let \( G_1 \) and \( G_2 \) be two leaves of \( R^x_G \). The dimension of \( \mathcal{F}_{G_1 \cap G_2} \) is \( |E(G_1 \cap G_2)| + |V(G_1 \cap G_2)| - 1 \).

**Proof of Theorem 13.** By Lemmas 14 and 15 we see that if \( \mathcal{F}_{G_1}, \ldots, \mathcal{F}_{G_l} \) is a shelling order, then the set of facets on which \( \mathcal{F}_{G_i} \) attaches to \( \mathcal{F}_{G_1}, \ldots, \mathcal{F}_{G_{i-1}} \) is \( \{ \mathcal{F}_Q \mid Q \in P_i \} \). Moreover, since the intersection of two top dimensional simplices of a triangulation of \( \mathcal{F}_G \) is not contained in the boundary of \( \mathcal{F}_G \), it follows that every element of \( P_i, i \in [l] \), appears in \( R^x_G \) by the second part of Lemma 14 (and it is a leaf since it is the intersection of two alternating graphs, so it is alternating itself). Using this same argument repeatedly and the fact that we can built up the polytope piece by piece by following the shelling, we obtain Theorem 13.

### 6. Nonnegativity results about reduced forms

This section is devoted to two nonnegativity results, which are consequences of the above considerations. These results are referred to in Theorems 1 and 2 in the Introduction.

**Theorem 16.** The polynomial \( Q_{G}^{S(\beta)}(\beta - 1) \) is a polynomial in \( \beta \) with nonnegative coefficients.

**Proof.** Recall that by Theorem 8 we have that

\[ Q_{G}^{S(\beta)}(\beta - 1) = h(C, \beta), \]

where \( C \) is an \( R_G \)-simplicial collection on \( \mathcal{F}_G \) for an arbitrary reduction tree \( R_G \). Let \( C \) be the abstract simplicial complex obtained from \( R_G \), as in Theorem 9. Since by Theorem 9 this triangulation is regular, and therefore it is shellable, we get that \( h \) is equal to the number of top dimensional simplices which attach on \( i \) facets to the union of previous simplices in a shelling order.

Using Theorem 16 we are ready to prove a special case of Kirillov’s Conjecture 2.

**Conjecture 17.** \[\text{Conjecture 2}\] Let \( k_1, \ldots, k_{n-1} \) be a sequence of nonnegative integers and let \( M = x_1^{k_1} \cdot x_2^{k_2} \cdot \ldots \cdot x_{n-1}^{k_{n-1}} \). Then the reduced form of \( M \) evaluated at \( x = (1, \ldots, 1) \) and \( \beta - 1 \) in \( \overline{\text{ACY}}B_n(\beta) \) is a polynomial in \( \beta \) with nonnegative coefficients.

**Theorem 18.** \[\text{cf. Conjecture 2}\]. The reduced form of \( x_{12}x_{23}\cdots x_{n-1,n} \) evaluated at \( x = (1, \ldots, 1) \) and \( \beta - 1 \) in \( \overline{\text{ACY}}B_n(\beta) \) is a polynomial in \( \beta \) with nonnegative coefficients.
where

\[ K \]

\[ Q \]

Corollary 22. For any graph \( G \) we have

\[ Q_G^{S(\beta)}(\beta - 1) = \sum_{m \geq 0} (K_G(m, 0, \ldots, 0, -m)^\beta m)(1 - \beta)^{\#E(G) + \#V(G)}, \]

where \( K_G \) is the Kostant partition function for \( \overline{G} \).

7. Reduced forms and Ehrhart series

In this section we connect the \( h \)-polynomials of simplicial collections to the Ehrhart series of flow polytopes, and using Theorem 8 we tie this in with reduced forms in the subdivision algebra. As a corollary to our results, we generalize \[4, \text{Theorem 3.10}\]. The results from this sections are referred to in Theorem 4 in the Introduction.

Recall that for a polytope \( P \subset \mathbb{R}^N \), the \( t \)-th dilate of \( P \) is \( tP = \{ (tx_1, \ldots, tx_N) \mid (x_1, \ldots, x_N) \in P \} \). The number of lattice points of \( tP \), where \( t \) is a nonnegative integer and \( P \) is a convex polytope, is given by the Ehrhart function \( i(P, t) \). If \( P \) has integral vertices then \( i(P, t) \) is a polynomial.

Theorem 19. Let \( C \) be an \( RG \)-simplicial collection on \( F_{\overline{G}} \) for an arbitrary reduction tree \( RG \). Then

\[ h(C, \beta) = \sum_{m \geq 0} (i(F_{\overline{G}}, m)^\beta m)(1 - \beta)^{\dim(F_{\overline{G}}) + 1}. \]

Proof. Since the triangulation of \( F_{\overline{G}} \) obtained from \( P^R \) is unimodular, \[22\] follows for it. Moreover, by Lemma 5 the polynomial \( h(C, \beta) \) is independent of the reduction tree chosen, thus we obtain \[22\] in its full generality.

Theorem 20. We have

\[ Q_G^{S(\beta)}(\beta - 1) = \sum_{m \geq 0} (i(F_{\overline{G}}, m)^\beta m)(1 - \beta)^{\dim(F_{\overline{G}}) + 1}. \]

Proof. Follows directly from Theorems 8 and 19.

Corollary 21. \[4, \text{Theorem 3.10}\]

\[ Q_{K_n}^{\beta}(\beta - 1) = \sum_{m \geq 0} (i(CRY_{n+1}, m)^\beta m)(1 - \beta)^{\frac{n+1}{2}}. \]

Proof. Follows from Theorem 20 for \( G = K_n \), since \( \dim(F_{K_n}) = \binom{n+1}{2} - 1 \) and \( i(CRY_{n+1}, m) = i(F_{K_n}, m) \), as explained in [4].

To state our final result, also a corollary of Theorem 20, relating the reduced forms to Kostant partition functions, we remind the reader of the following definition.

The Kostant partition function \( K_G \) evaluated at the vector \( v \in \mathbb{Z}^{n+1} \) is defined as

\[ K_G(v) = \# \{(b_k)_{k \in [N]} \mid \sum_{k \in [N]} b_k \alpha_k = v \text{ and } b_k \in \mathbb{Z}_{\geq 0} \}, \]

where \([N] = \{1, 2, \ldots, N\}\) and \( \{\alpha_1, \ldots, \alpha_N\}\) is the multiset of vectors corresponding to the multiset of edges of \( G \) under the correspondence which associates an edge \((i, j), i < j, \) of \( G \) with a positive type \( A_n \) root \( e_i - e_j \), where \( e_i \) is the \( i \)-th standard basis vector in \( \mathbb{R}^{n+1} \). In other words, \( K_G(v) \) is the number of ways to write the vector \( v \) as a \( N \)-linear combination of the positive type \( A_n \) roots (with possible multiplicities) corresponding to the edges of \( G \), without regard to order.

Corollary 22. For any graph \( G \) we have

\[ Q_G^{S(\beta)}(\beta - 1) = \sum_{m \geq 0} (K_G(m, 0, \ldots, 0, -m)^\beta m)(1 - \beta)^{\#E(G) + \#V(G)}, \]

where \( K_G \) is the Kostant partition function for \( \overline{G} \).
Proof. Follows from Theorem \[ \text{20} \] since \( i(F_{\tilde{G}}, m) = K_{\tilde{G}}(m, 0, \ldots, 0, -m) \) is a simple corollary of the definitions of these objects and \( \dim(F_{\tilde{G}}) = \#E(\tilde{G}) - \#V(\tilde{G}) + 1 = \#E(G) + \#V(G) - 1 \). For detailed explanations of both of these and related results see \[ \text{10} \].

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