Bergman Kernel from Path Integral

Michael R. Douglas\textsuperscript{1,2,3} and Semyon Klevtsov\textsuperscript{1,2,4}

\textsuperscript{1} Simons Center for Geometry and Physics, Stony Brook University, Stony Brook,NY 11794–3840, USA
\textsuperscript{2} NHETC and Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855–0849, USA
\textsuperscript{3} I.H.E.S., Le Bois-Marie, Bures-sur-Yvette, 91440, France
\textsuperscript{4} ITEP, Moscow, 117259, Russia

mrd@physics.rutgers.edu, klevtsov@physics.rutgers.edu

ABSTRACT: We rederive the expansion of the Bergman kernel on Kähler manifolds developed by Tian, Yau, Zelditch, Lu and Catlin, using path integral and perturbation theory, and generalize it to supersymmetric quantum mechanics.

One physics interpretation of this result is as an expansion of the projector of wave functions on the lowest Landau level, in the special case that the magnetic field is proportional to the Kähler form. This is relevant for the quantum Hall effect in curved space, and for its higher dimensional generalizations. Other applications include the theory of coherent states, the study of balanced metrics, noncommutative field theory, and a conjecture on metrics in black hole backgrounds discussed in [24]. We give a short overview of these various topics.

From a conceptual point of view, this expansion is noteworthy as it is a geometric expansion, somewhat similar to the DeWitt-Seeley-Gilkey et al short time expansion for the heat kernel, but in this case describing the long time limit, without depending on supersymmetry.
1. Introduction

A prototypical topic at the interface of geometry and theoretical physics is the study of quantum mechanics in curved space, i.e. on a Riemannian manifold $M$. Many results in this area are of great interest both to physicists and to mathematicians, with some examples being the DeWitt-Seeley-Gilkey short time expansion of the heat kernel, and the relation between supersymmetric quantum mechanics and the Atiyah-Singer index theorem.
A more recent result which, although not well known by physicists, we feel also belongs in this category, is the expansion for the Bergman kernel on a Kähler manifold developed by Tian, Yau, Zelditch, Lu and Catlin [8, 9, 10, 11]. It applies to Kähler quantization and gives an asymptotic expansion around the semiclassical limit. This has many uses in mathematics [8, 12, 13, 14]; see the recent book [15].

Here we will provide a physics derivation of the asymptotic expansion of the Bergman kernel using path integrals, and explain various possible applications of this result.

In physics terms, perhaps the simplest way to define the Bergman kernel is in the context of quantum mechanics of a particle in a magnetic field, in which it is the projector on the lowest Landau level. It is not hard to see that the limit of large magnetic field is semiclassical, so that one can get an expansion in the inverse magnetic field strength using standard perturbative methods.

Our basic result is to rederive the Tian-Yau-Zelditch et al expansion as the large time limit of the perturbative expansion for the quantum mechanical path integral. We also generalize it to $N = 1$ and $N = 2$ supersymmetric quantum mechanics.

Let us state the basic result for (nonsupersymmetric) quantum mechanics. We consider a compact Kähler manifold $M$, and a particle in magnetic field, with the field strength proportional to the Kähler form on the manifold

$$F_{ij} \sim \omega_{ij}.$$  

(1.1)

One can show (see below) that, just as for a constant magnetic field in flat space, in this situation the spectrum is highly degenerate, splitting into “Landau levels.” Let the lowest Landau level (LLL or ground state) be $N$-fold degenerate with a basis of orthonormal wave functions $\psi_I(x)$, then we define the projector on the LLL as

$$\rho(x,x') = \sum_{I=1}^{N} \psi^*_I(x') \psi_I(x).$$  

(1.2)

We could also regard this as a density matrix describing a mixed state in which each ground state appears with equal weight, describing the zero temperature state of maximum entropy.

We then consider scaling up the magnetic field by a parameter $k$, as $F \rightarrow kF$. Note that on a compact manifold, $F$ must satisfy a Dirac quantization condition; thus we take $k = 1, 2, 3, \ldots$ In the large $k$ limit, the diagonal term then satisfies

$$\rho(x,x) \sim k^n \left( 1 + \frac{\hbar}{2k} R + \frac{\hbar^2}{k^2} \left( \frac{1}{3} \Delta R + \frac{1}{24} |\text{Riem}|^2 - \frac{1}{6} |\text{Ric}|^2 + \frac{1}{8} R^2 \right) + \mathcal{O}(\hbar/k)^3 \right)$$  

(1.3)

as an asymptotic expansion [15] (see Appendix A for the precise definitions of different terms here).

In some ways this expansion is similar to the well known short time expansion of the heat kernel, but note that it is a long time expansion, because it projects on the ground states. Unlike other analogous results for ground states, it does not require supersymmetry, either for its definition or computation. Of course, similar results can be obtained for
supersymmetric theories, our point is that that they do not depend on supersymmetry (whether they depend ultimately on holomorphy is an interesting question discussed below).

There are various other physics interpretations of this result. One familiar variation is to regard $M$ as a phase space, and try to quantize it, following Berezin [16]. As a phase space, $M$ must have a structure which can be used to define Poisson brackets; it is familiar [17] that this is a symplectic structure, i.e. a nondegenerate closed two-form $\omega$. The definitions we just gave are then the standard recipe of geometric quantization [18]. They lead to a finite dimensional Hilbert space, whose dimension is roughly the phase space volume of $M$ in units of $(2\pi\hbar)^n$. In this interpretation, the parameter $k$ plays the role of $1/\hbar$, and thus the large $k$ limit is semiclassical.

From this point of view, it is intuitive that one should be able to localize a wave function in a region of volume $(2\pi\hbar)^n \sim 1/k^n$, and thus in the large $k$ limit the density matrix $\rho(x, x)$ should be computable in terms of the local geometry and magnetic field near $x$.

To do this, given a point $z \in M$, one might seek a wave function $\psi_z(z')$ which is peaked around $z$. Given an orthonormal basis for $\mathcal{H}$, a natural candidate is

$$\psi_z(z') = \sum_I \psi_I^*(\bar{z})\psi_I(z')$$

This is a coherent state (in the sense of [18]). It can be used to define the symbol of an operator, the star product [19], and related constructions. In this context the Bergman kernel is the “reproducing kernel” studied in [20], see [17] for a review. For recent work on applications of the Bergman kernel to quantization of Kähler manifolds see [21, 22]. Another recent paper discussing the topic is [23].

Our original interest in this type of result came from the study of balanced metrics in [14], and a conjecture about their relevance for black holes in string theory stated in [24]. However, after realizing that these results and techniques do not seem to have direct analogs in the physics literature and could have other applications, we decided to provide a more general introduction as well.

2. Background

Let us give a few mathematical and physical origins and applications of this type of result.

2.1 Particle in a magnetic field

We consider a particle of mass $m$ (which later we set to one) and charge $k$, moving on a $2n$-dimensional manifold $M$ which carries a general metric $g_{ij}$, and a magnetic field $F_{ij}$. It is described by a wave function $\psi(x; t)$ which satisfies the Schrödinger equation,

$$H\psi \equiv \frac{\hbar^2}{2m}\sqrt{g} D_i \sqrt{g} g^{ij} D_j \psi = E\psi, \quad (2.1)$$

where $D_i = i\partial_i + kA_i$ is the covariant derivative appropriate for a scalar wavefunction with charge $k$, and $E = i\hbar\partial/\partial t$. If $M$ is topologically nontrivial, as usual we need to work in
coordinate patches related by gauge and coordinate transformations, and this expression applies in each coordinate patch. We can of course also consider the time-independent Schrödinger equation with $E$ fixed, and seek the energy eigenstates $H\psi_i(x) = E_i\psi_i$.

Let us now consider the limit of large magnetic field or equivalently large $k$. The case of two-dimensional Euclidean space $g_{ij} = \delta_{ij}$ with a constant magnetic field $F_{ij} = B_{ei}\delta_{ij}$ is very familiar. The energy eigenstates break up into Landau levels, such that all states in the $l$'th level have energy $E_l = \hbar k B (l + \frac{1}{2})/m$. Within a Landau level, one can roughly localize a state to a region of volume $\hbar/kB$.

These results can be easily generalized to $d = 2n$ dimensions. Choose coordinates such that the magnetic field lies in the 12, 34 planes and so forth, and $B_{12} > 0$, $B_{34} > 0$ etc. Then, considering the lowest Landau level (LLL) we have

$$E = \frac{\hbar}{2m} (B_{12} + \ldots + B_{2n-1,2n}),$$

(2.2)

with states localized as before within each two-plane.

In a general metric and magnetic field, while one might not at first expect this high degree of degeneracy, it is still possible. When the magnetic field is much larger than the curvature of the metric, the intuition that wave functions localize should still be valid. Then, we might estimate the energy of a wave function in the lowest Landau level localized around a point $x$ as Eq. (2.2), where the components $B_{12}, B_{34}$ and so on are evaluated in a local orthonormal frame. If the energy $E$ in Eq. (2.2) is constant, then all states in the LLL will be degenerate, at least in the limit of large $k$.

The proper generalization of the splitting of the components of $B$ Eq. (2.2) for nonconstant magnetic fields seems to be, that the field strength should be nonzero only for mixed components $F_{a\bar{a}}$, with $F_{ab} = F_{\bar{a}\bar{b}} = 0$ in the complex coordinates $z^a, \bar{z}^{\bar{a}} (a, \bar{a} = 1, \ldots, n)$ on the manifold. Mathematically it means, that the underlying line bundle is holomorphic. In this case, the argument can be sharpened by using the identity

$$[D_i, D_j] = F_{ij},$$

to rewrite the Hamiltonian as

$$H = g^{a\bar{a}} F_{a\bar{a}} + g^{a\bar{a}} D_a \bar{D}_{\bar{a}}.$$

This makes it clear that if the following combination is constant

$$g^{a\bar{a}} F_{a\bar{a}} = \text{const}$$

(2.3)

every wave function satisfying

$$0 = \bar{D}_{\bar{a}} \psi$$

(2.4)

will be degenerate and lie in the LLL. This argument can hold away from the strict $k \to \infty$ limit.

The condition (2.3) is known as hermitean Yang-Mills equation, and is essentially equivalent to Maxwell equation in the case $F^{0,2} = 0$. Recalling, that the metric coefficients
on the Kähler manifold are related to the Kähler form ω as \( g_{\overline{a}a} = -i\omega_{\overline{a}a}, \ g_{\overline{a}a} = i\omega_{\overline{a}a}, \) one can see that our choice (1.1) of the magnetic field strength
\[
F_{\overline{a}a} = kg_{\overline{a}a}
\]
does satisfy the condition (2.3).

In fact the previous argument relies only upon the Maxwell equations and the condition \( F^{2,0} = 0. \) This suggests that there exist more general magnetic field configurations, than (2.5), for which the LLL is still highly degenerate and the expansion in large magnetic fields, analogous to (1.3), is possible. For example this includes the case when \( b^{1,1}(M) > 1. \) We will elaborate this question in the future publication.

The previous physical condition for the field strength is equivalent to the mathematical condition that \( M \) be a complex manifold with complex structure \( J = B. \) For a tensor \( J^i \) to be a complex structure, it must satisfy the conditions \( J^2 = -1 \) and \( 0 = \nabla_{[i}J^j_{k]} \). The first is manifest, and given the expression for \( J^i \) in terms of the vector potential \( J^i_{jk} = g^ik\partial_k A_j, \) so is the second.

Now, a standard trick to simplify the equations Eq. (2.4), is to do a “gauge transformation” with a complex parameter \( \theta(x). \) While at first this might seem to violate physical requirements such as the unitarity of the Hamiltonian, in fact it is perfectly sensible as long as we generalize another ingredient in the standard definitions, namely the inner product on wave functions. Explicitly, we define the wave function in terms of another function \( s(x), \)
\[
\psi(x) = e^{ik\theta(x)}s(x), \quad D_a\psi(x) = e^{ik\theta(x)}(i\partial_a + kA_a - k\partial_a\theta)s(x).
\]
This would be a standard \( U(1) \) gauge transformation if \( \theta(x) \) were real. By allowing complex \( \theta(x), \) and assuming
\[
0 = \{\overline{D_{\bar{a}}}D_{\bar{b}}\} \equiv F_{\bar{a}\bar{b}}
\]
(\textit{i.e.} \( F^{0,2} = 0), \) we can find a transformation which trivializes all the antiholomorphic derivatives,
\[
\overline{D_{\bar{a}}} \rightarrow \overline{\partial_{\bar{a}}}. \quad (2.7)
\]
In this “gauge,” wave functions in the LLL can be expressed locally in terms of holomorphic functions. The only price we pay is that the usual inner product,
\[
\langle \psi | \psi' \rangle = \int_M \sqrt{g} \psi^*(x)\psi'(x),
\]
turns into an inner product which depends on an auxiliary real function,
\[
h(x) \equiv e^{-2Im\theta(x)}, \quad (2.8)
\]
as
\[
(h(x) \overline{s}(x)) s'(x).
\]
Taking into account the gauge transformations between coordinate patches, the \( s(x) \) are holomorphic sections of a holomorphic line bundle \( L^k \).

In mathematics, one would say that \( s(x) \) is a section of \( L^k \) evaluated in a specific frame, while the quantity \( h^k(x) \) defines a hermitian metric on the line bundle \( L^k \).

### 2.2 The lowest Landau level

Since we have made a one-to-one correspondence between LLL wave functions and holomorphic sections of the line bundle, we can now find the total number of LLL states, which we denote \( N \). The number of holomorphic sections \( \dim H^0(L^k) \) can be determined for large \( k \) by the index formula \[ N = \dim H^0(L^k) = \int_M e^F \wedge \text{Td}(M) = a_0k^n + a_1k^{n-1} + \ldots \] where the coefficients \( a_i \) are given by certain integrals involving the curvature of the metric.

Now, given that there is a large degeneracy of ground states and thus a nontrivial LLL, it becomes interesting to study the projector on the LLL, or in other words the LLL density matrix

\[
P = \sum_{i; E_i = E_0} |i\rangle\langle i|.
\]

If we shift \( H \) to set the ground state energy \( E_0 = 0 \), it can also be defined as the large time limit of propagation in Euclidean time. Regarded as a function on two variables, the projector \( P \) becomes the Bergman kernel

\[
P(x, x') = \lim_{T \to \infty} \langle x|e^{-TH}|x'\rangle.
\]

Thus it can be defined as a path integral by the standard Feynman-Kac formula.

The standard example in which the projector on LLL appears in physics is the Quantum Hall Effect, see for a review \[23\]. In the simplest case, one studies the dynamics of electrons on a two-dimensional plane with a constant orthogonal magnetic field. At low temperatures and high values of the field only the lowest energy level is important. It is also interesting to consider a partly filled ground state, with number of fermionic particles \( K < N \). In this case one has to introduce a potential \( V \), then particles form an incompressible droplet, whose edge dynamics is of particular interest.

In recent years this problem has been much generalized; to Riemann surfaces in \[24\] and references therein, while higher dimensional examples include the case of \( S^4 \), \( \mathbb{R}^4 \) \[28\] and \( \mathbb{CP}^n \) \[29\]; see also \[30\] for a review.

The case of \( \mathbb{CP}^n \) is the first nontrivial case in which we can make contact with the results of this paper. The choice made in \[29\] for the \( U(1) \) background field is

\[
F_{a\bar{a}} \sim R_{a\bar{a}},
\]

proportional to the Ricci tensor. Since for \( \mathbb{CP}^n \) the Ricci tensor is equal to the Kähler metric, one immediately recognizes Eq. \ref{2.10} as the physical condition on the magnetic
field Eq. (2.5). Using the local complex coordinates $z_1, \ldots, z_n$, the LLL wave functions can be constructed explicitly as

$$\psi_\alpha \sim \frac{z_1^{\alpha_1}z_2^{\alpha_2} \cdots z_n^{\alpha_n}}{(1 + |z|^2)^{k/2}}, \quad \alpha = 1, \ldots, N$$  \tag{2.11}$$

up to a normalization constant [30]. As in Eq. (2.6) it has the form of the holomorphic function, weighted by a metric of the line bundle Eq. (2.8), or, equivalently, the magnetic potential.

The dynamics of the droplet is characterized in the following way. One starts with diagonal density matrix $\rho_0$ with $K$ states occupied, then the fluctuations, preserving the number of states are given by unitary transformation $\rho_0 \rightarrow \rho = U\rho_0U^\dagger$, and the equation of motion is the quantum Liouville equation

$$i\frac{\partial \rho}{\partial t} = [V, \rho].$$

The form of the droplet is determined by the form of the minima of the confining potential. In [29] the case of spherically symmetric potential $V = V(r = z\bar{z})$ was studied. In the limit of large number of states $N$ (i.e. large magnetic field) and large number of fermions $K < N$ the density matrix has the form

$$\rho(r^2) = \Theta(r^2 - R_d^2),$$

where $R_d$ is the radius of the droplet and $\Theta$ is the step function. In other words the density matrix is equal to constant in the region, occupied by the droplet. This is due to the fact, that the LLL is only partly filled, otherwise it would have been constant everywhere in space. The condition of the constant density matrix (Bergman kernel) turns out to have interesting consequences.

The edge dynamics of the droplet is described by Chern-Simons type action in higher dimensions [29].

One can generalize the above construction to nonabelian background gauge fields. Since $\mathbb{C}P^n = SU(n + 1)/U(n)$ and Lie algebra of $U(n) = U(1) \times SU(n)$, then in addition to $U(1)$ gauge field one can also turn on $SU(n)$ gauge field. In [29] the case of constant $SU(n)$ gauge field was considered, so that the field strength is proportional to the $SU(n)$ component of the Riemann curvature. The wave functions (2.11) as well as the density matrix now carry additional indices, corresponding to $SU(n)$ representation.

The similar generalization of the Bergman kernel was considered recently in [31]. In addition to the line bundle $L$ one can consider more general hermitian vector bundle $E$ with corresponding connection with the curvature $R^E$. Then the corresponding Bergman kernel is given by the large time limit of the exponential of Dirac operator $D$ squared, for which the expansion analogous to (1.3) exists

$$\rho(x) = \lim_{T \to \infty} e^{-TD^2(x, x)} = k^n + k^{n-1}\left(\frac{1}{2}R \cdot 1_E + iR^E\right) + \ldots.$$

The second term was computed in [31]. It would be interesting to make further contact between these results and the higher dimensional Quantum Hall Effect.
2.3 Applications in Kähler geometry

The original mathematical motivation for this development, usually attributed to Tian and to Yau, seems to have been to use Bergman metrics to study the problem of approximation of Kähler-Einstein metrics, which by definition have Ricci tensor proportional to the metric itself, on complex manifolds.

In [8] Tian considered an algebraic manifold \( M \) of complex dimension \( n \), embedded in some projective space \( \mathbb{CP}^N \), \( N > n \). Turning on the magnetic field is equivalent to considering a bundle \( L \), or its \( k \)-th power \( L^k \) for magnetic flux proportional to \( k \), whose choice corresponds to a choice of “polarization” on \( M \). In local complex coordinates \( z^a, \bar{z}^{\bar{a}}, \ a, \bar{a} = 1, \ldots n \) the Kähler form \( \omega_g \) of the metric \( g \) is defined as

\[
\omega_g = i g_{a\bar{a}} dz^a \wedge d\bar{z}^{\bar{a}}.
\]

The Kähler metric, polarized with respect to \( L \), has the associated Kähler form \( \omega_g \) in the same class as the Chern class \( C_1(L) \) of \( L \). A particularly useful choice of \( \omega_g \) is to take it to be equal to the curvature of the line bundle (magnetic field strength). If the hermitian metric of \( L \) is \( h(z, \bar{z}) \) then for \( L^k \) the metric is \( h^k \) and it’s curvature is

\[
g_{a\bar{a}} F_{a\bar{a}} = \partial_a \bar{\partial}_{\bar{a}} \log h^k,
\]

exactly as in Eq. (2.5).

Consider next some orthonormal basis \( s_0(z), \ldots, s_{N_k}(z) \) on the space \( H^0(M, L^k) \) of all global holomorphic sections of \( L^k \)

\[
(s_\alpha, s_\beta) = \int_M \sqrt{g} h^k s_\alpha \bar{s}_\beta = \delta_{\alpha\beta}.
\]

One can think of \( s_\alpha \) as of projective coordinates on \( \mathbb{CP}^{N_k} \). Therefore a particular choice of the basis of sections defines a particular embedding of the manifold \( M \) into \( \mathbb{CP}^{N_k} \) (different choices of the basis are related by \( PGL(N_k+1) \) transformation). The standard metric on the projective space is the Fubini-Study metric \( g_{FS} = \partial\bar{\partial} \log \sum_{\alpha} |s_\alpha|^2 \). One immediately realizes that \( \frac{1}{k} \)-multiple of it’s pullback \( \frac{1}{k} g_{FS}|M \) to \( M \) is in the same Kähler class as the original metric \( g \) (2.3), since

\[
\frac{1}{k} g_{FS}|M = g + \frac{1}{k} \partial\bar{\partial} \log \left( h^k \sum_{\alpha=0}^{N_k} s_\alpha \bar{s}_\alpha \right)
\]

and the expression inside the logarithm is a globally defined function. This metric is called the Bergman metric. In [8] Tian proved, that as \( k \to \infty \), the Bergman metric converges to \( g \) (at least in \( C^2 \) topology). This result opened a possibility of approximating the Kähler-Einstein metrics by the Bergman metrics.

It is interesting to look at the structure of the “density of states” function

\[
\rho_k(z) = h^k \sum_{\alpha=0}^{N_k} s_\alpha(z) \bar{s}_\alpha(\bar{z}).
\]

Zelditch [9] and Catlin [10] showed that there is an asymptotic expansion of the density function in \( 1/k \) in terms of local invariants of the metric \( g \), such as the Riemann tensor and

\[
\rho_k(z) = h^k \sum_{\alpha=0}^{N_k} s_\alpha(z) \bar{s}_\alpha(\bar{z}).
\]
its contractions. These invariants were computed by Lu \[11\] up to third nontrivial order in $1/k$ with the following result up to the second order in $1/k$

$$\rho_k(z) = k^n + \frac{1}{2}k^{n-1}R + k^{n-2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\text{Riem}|^2 - \frac{1}{6}|\text{Ric}|^2 + \frac{1}{8}R^2\right) + \mathcal{O}(k^{n-3}) \quad (2.14)$$

The computation is based on the global peak section method, developed by Tian \[8\], which is a technique to approximate sections of line bundle for large values of $k$. Another method to derive this results are the heat kernel approach of \[31\] reproducing kernel approach of \[32\]. Let us also mention their importance for the proof of holomorphic Morse inequalities \[33, 31\].

In this paper, we reproduce the expansion (2.14) by taking the large time limit of the quantum mechanical path integral for a particle in magnetic field. The function $\rho_k$ is nothing but the diagonal of the density matrix on the lowest Landau level.

Based on the results of \[8, 9, 10, 11\] Donaldson suggested to study the metrics with constant density function

$$\rho_k(z) = \text{const} = \frac{\dim H^0(M, L^k)}{\text{Vol} M}.$$ 

Solving the previous equation for $h^k$ and plugging back to the orthonormality condition Eq. (2.13) we get the equation

$$\frac{\dim H^0(M, L^k)}{\text{Vol} M} \int_M \sqrt{g} s_\alpha s_\beta \overline{s}_\gamma \overline{s}_\delta = \delta_{\alpha\beta}.$$ 

on the sections of the line bundle. This is the orthonormality condition for the basis in $H^0(M, L^k)$. The embedding $M \to \mathbb{CP}^N_k$, satisfying this condition, is called “balanced” \[34\] and the corresponding Kähler metric $g_{\alpha\bar{\alpha}}$ is the “balanced metric” (see \[33\] for the first appearance of this concept). Using the expansion from Eq. (2.14) Donaldson was able to show \[12, 13, 14\] that under assumption of existence of constant scalar curvature metric, the metric, satisfying previous equation, approaches the metric of constant scalar curvature as $k \to \infty$. In \[14\] an iterative procedure was proposed to construct these metrics numerically. One starts with an arbitrary choice of basis, parameterized by a hermitian matrix $G_{\alpha\beta}$, and defines the following integral operator

$$T(G)_{\alpha\beta} = \frac{\dim H^0(M, L^k)}{\text{Vol} M} \int_M \sqrt{g} s_\alpha \overline{s}_\beta \overline{s}_\gamma \overline{s}_\delta (G^{-1})^\gamma_{\delta\gamma}.$$ 

The fixed point of this operator $T(G) = G$ corresponds to the balanced embedding. It was shown in \[12, 14\] that for any initial choice of the matrix $G$, the iterative procedure for $T$ converges to the balanced embedding. This construction was recently used for approximating the Ricci flat metrics on Calabi-Yau hypersurfaces in projective spaces \[36\] and finding numerical solutions to the hermitian Yang-Mills equation on holomorphic vector bundles \[37\].
3. Non-supersymmetric Bergman Kernel

3.1 Density matrix

The euclidean path integral for a particle on a 2n-dimensional Kähler manifold $M$ with the magnetic field is

$$
\rho(x_i, x_f) = N \int_{x(t_i) = x_i}^{x(t_f) = x_f} \prod_{t_i < t < t_f} \det g_{ab}(x(t)) \mathcal{D}x^a \mathcal{D}\bar{x}^\bar{b} e^{-\frac{1}{\hbar} \int_{t_i}^{t_f} dt [g_{ab} \dot{x}^a \dot{\bar{x}}^{\bar{b}} + A_a \dot{x}^a + \bar{A}_{\bar{a}} \dot{\bar{x}}^{\bar{a}}]} \tag{3.1}
$$

Here we assume that $F_{ab} = F_{\bar{a}\bar{b}} = 0$ and work in the anti-holomorphic gauge $A_a = 0$ for the gauge connection\(^1\). We also set the magnetic field strength to be aligned with the metric as in Eq. (2.5), with $K = -\log h$ being the Kähler potential for the metric. The physical reason for this choice of the magnetic field strength was outlined in the introduction (namely, to get a highly degenerate ground state).

Our goal is to compute the value of the density matrix (3.1) on the diagonal $x_i = x_f = x$ and in the lowest Landau level, i.e. in the large time $T = t_f - t_i \to \infty$ limit. However, it turns out that one cannot take this limit before doing the computation. If one does this, then the kinetic term in the action is suppressed, and one obtains the result 1 for the functional integral, as is easy to check in first order in $\hbar$. Thus, we must keep $T$ finite in the process of calculation and take the $T \to \infty$ limit after doing the Feynman integrals. It turns out that this limit is free of IR divergent terms for the choice of magnetic field Eq. (3.2) if the path integral is properly regularized. Whether this limit is well-defined for more general field strength, than Eq. (3.2), is an interesting question we will address elsewhere.

The result is an asymptotic expansion in $\hbar \sim 1/k$, whose coefficients at each order can be computed using perturbation theory, and depend on local invariants of the metric, such as the Riemann and Ricci tensors, curvature scalar and their derivatives.

Let us begin. To keep track of the $T$ dependence we rescale the time parameter, defining

$$
t = t_f + (t_f - t_i)\tau = t_f + T\tau
$$

with $\tau \in [-1, 0]$. The classical solution for the trajectory with boundary conditions $x_i = x_f$ is just a constant function. Introduce normal coordinates $z^a, \bar{z}^{\bar{a}}$ in the vicinity of the classical trajectory

$$
x^a = x_f^a + z^a(\tau)
$$

$$
\bar{x}^{\bar{a}} = \bar{x}_f^{\bar{a}} + \bar{z}^{\bar{a}}(\tau)
$$

\(^1\)Although the gauge, which trivializes anti-holomorphic derivatives is rather $\bar{A}_{\bar{a}} = 0$ (2.7), the difference between gauge choices is inessential, since the density matrix is a gauge invariant object. We find it convenient to work in the anti-holomorphic gauge.
The normalization factor $\mathcal{N}$ can be fixed by considering the standard normalization of the heat kernel in the case of non-coincident initial and final points, as e.g. in [39], and is equal to
\[ \mathcal{N} = k^n, \]
where $n$ is the complex dimension of the manifold.

### 3.2 Weyl-ordering counterterm

The path integral representation (3.1) of the heat kernel has been studied since the pioneering work by DeWitt [38]. In the hamiltonian framework the path integral corresponds to transition amplitude
\[ K(x_i, x_f; T) = \langle x_f | e^{-(T/\hbar)\hat{H}} | x_i \rangle. \]

Since the kinetic term in $\hat{H}$ depends on the coordinate variable through the metric, the well known subtlety arises in this case with the operator ordering of momentum and coordinate variables – different choices of ordering lead to different lagrangians. This issue was studied in great detail in [40, 41, 4]. There it was shown that there is a convenient choice of the ordering in the hamiltonian which preserves general coordinate invariance
\[
\hat{H} = \frac{1}{2} G^{-1/4} (\hat{p}_i - A_i) G^{ij} G^{1/2} (\hat{p}_j - A_j) G^{-1/4}
\]
\[ = \frac{1}{2} g^{-1/2} \hat{\rho}_a g^{ab} g (\hat{\rho}_b - \hat{A}_b) g^{-1/2} + \frac{1}{2} g^{-1/2} (\hat{\rho}_b - \hat{A}_b) g^{ab} g \hat{\rho}_a g^{-1/2}, \quad (3.3) \]
where we specified our hamiltonian to the Kähler case. Here $G = \det g_{ij} = g^{2} = (\det g_{a\bar{a}})^2$.

To transform the hamiltonian framework to lagrangian we rewrite this expression in a Weyl-ordered form (see Appendix B) and then perform the Legendre transform with the generalized momenta
\[ p_a = g_{ab} \dot{z}^b, \quad \hat{p}_b = g_{ab} \dot{z}^a + \hat{A}_b. \]

The following action, written in euclidean time, appears then in the exponent of path integral
\[ S = \int_{t_i}^{t_f} dt \left( g_{ab} \dot{z}^a \dot{z}^b + \hat{A}_b \dot{z}^b - \frac{\hbar^2}{4} R \right). \]

The Weyl ordering corresponds to a “mid-point rule” prescription for path integral representation, which will be introduced in the next subsection. The last term in this “quantum corrected” action is necessary e.g. to obtain correct path integral representation for the small-$T$ heat kernel [3]. In the next section we will see that it is also necessary for obtaining the correct infinite-$T$ expansion.

### 3.3 Normal coordinates, free action and propagators

In Kähler normal coordinate frame (see Appendix A for conventions) all pure (anti-) holomorphic derivatives of the metric at a chosen point are set to zero. Setting $x = x_i = x_f$, we use Kähler normal coordinates, centered at $x$ and obtain the following expansions for the Kähler potential, metric and gauge connection, up to the sixth order in derivatives
\[ K(x^a + z^a, \bar{x}^\bar{a} + \bar{z}^\bar{a}) = g_{ab}(x) z^a \bar{z}^b + \frac{1}{4} K_{ab\bar{c}}(x) z^a \bar{z}^b \bar{z}^c + \frac{1}{36} K_{abc\bar{d}}(x) z^a \bar{z}^b \bar{z}^c \bar{z}^d + \ldots, \]
\[ \tilde{A}_b(x^a + z^a, \bar{x}^a + \bar{z}^a) = k \partial_b K(x^a + z^a, \bar{x}^a + \bar{z}^a) \]
\[ = k \left( g_{ab}(x)z^a + \frac{1}{2} K_{ab\bar{c}}(x)z^a z^b \bar{z}^\bar{a} + \frac{1}{12} K_{abc\bar{e}}(x)z^a z^b z^c \bar{z}^\bar{a} \bar{z}^\bar{c} + \ldots \right), \]

where \( g_{ab}(x^a + z^a, \bar{x}^a + \bar{z}^a) = \partial_a \partial_b K(x^a + z^a, \bar{x}^a + \bar{z}^a) = g_{ab}(x) + K_{ab\bar{c}}(x)z^a \bar{z}^\bar{a} + \frac{1}{4} K_{abc\bar{e}}(x)z^a z^b z^c \bar{z}^\bar{a} \bar{z}^\bar{e} + \ldots \)

in self-explanatory notations. Note that we omitted terms which turn out not to be relevant up to the second order in \( \hbar \). For example, the term with five derivatives of mixed type \( K_{abc\bar{e}}(x)z^a z^b z^c \bar{z}^\bar{a} \bar{z}^\bar{e} \) is non-zero in our coordinate frame, but it contributes to the density matrix only starting from \( \hbar^3 \), as one can check by power counting.

Using auxiliary ghost fields \( b^a \) and \( c^\bar{b} \) to raise the determinant from the measure \([3.1]\) to the exponent, we can rewrite the diagonal of the density matrix \([3.1]\) as
\[ \rho(x) = \mathcal{N} \int_{z(-1)^{=0}}^{z(0)^{=0}} \mathcal{D}z^a(\tau) \mathcal{D}z^\bar{b}(\tau) e^{-\frac{i}{\hbar} S_0 - \frac{i}{\hbar} S_{int}} \]

where we split the action into free part
\[ S_0 = \int_{-1}^{0} d\tau \left[ \frac{1}{T} g_{ab}(x)z^a \dot{z}^b + k g_{ab}(x)z^a \bar{z}^\bar{b} + g_{ab}(x)b^a c^\bar{b} \right], \]
and interaction part, which up to the sixth order in derivatives of the Kähler potential, looks like
\[ S_{int} = \int_{-1}^{0} d\tau \left[ \frac{1}{T} \left( K_{ab\bar{c}}(x)z^b \bar{z}^\bar{a} + \frac{1}{4} K_{abc\bar{e}}(x)z^b z^c \bar{z}^\bar{a} \bar{z}^\bar{e} \right) z^a \dot{z}^b + k \left( \frac{1}{2} K_{ab\bar{c}}(x)z^a z^b \bar{z}^\bar{a} + \frac{1}{12} K_{abc\bar{e}}(x)z^a z^b z^c \bar{z}^\bar{a} \bar{z}^\bar{e} \right) \dot{z}^b + \left( K_{ab\bar{c}}(x)z^b \bar{z}^\bar{a} + \frac{1}{4} K_{abc\bar{e}}(x)z^b z^c \bar{z}^\bar{a} \bar{z}^\bar{e} \right) b^a c^\bar{b} \right. \]
\[ - \frac{\hbar^2}{4} T \left( R(x) + \partial_a \partial_{\bar{c}} R(x) z^c \bar{z}^\bar{c} \right) \],

and dots denote \( \tau \) derivatives. The propagator for free theory \([3.4]\)
\[ \langle \dot{z}^{\bar{b}}(\tau) z^a(\sigma) \rangle = \hbar g^{\bar{b}a} \Delta(\tau, \sigma), \]
satisfies equation
\[ \left[ \frac{1}{T} \frac{d^2}{d\tau^2} + k \frac{d}{d\tau} \right] \Delta(\tau, \sigma) = \delta(\tau - \sigma), \]
and the path integral boundary conditions translate into boundary conditions for \( \Delta \)
\[ \Delta(-1, \sigma) = \Delta(0, \sigma) = \Delta(\tau, -1) = \Delta(\tau, 0) = 0 \]
The unique solution is
\[ \Delta(\tau, \sigma) = \frac{1}{k(e^{kT} - 1)} \left\{ \theta(\tau - \sigma) e^{kT} (1 - e^{-kT})(1 - e^{-kT(\sigma + 1)}) + \theta(\sigma - \tau) (1 - e^{-kT})(1 - e^{kT(\tau + 1)}) \right\}, \]
where the step-function is defined using the "mid-point rule"

\[
\theta(\tau - \sigma) = \begin{cases} 
1, & \tau > \sigma \\
\frac{1}{2}, & \tau = \sigma \\
0, & \tau \leq \sigma 
\end{cases} \tag{3.7}
\]

This value at zero follows from the choice of symmetric ordering in path integral and is crucial for obtaining correct results for heat kernel expansion \[4\]. Ghost propagator can be regulated with the help of \( \Delta(\tau, \sigma) \) in the following way

\[
\langle b^a(z) c^b(\bar{z}) \rangle = -\hbar g^{\bar{a}b} \delta(\tau - \sigma) = \hbar g^{\bar{a}b} \left( \frac{1}{T} \Delta(\tau, \sigma) - k \Delta(\tau, \sigma) \right),
\]

where \( \Delta(\tau, \sigma) = d\Delta(\tau, \sigma)/d\tau \), etc.

### 3.4 Perturbation theory. First Order

Now we are ready to study the perturbation theory in \( \hbar \) for the diagonal of the density matrix \((3.4)\)

\[
\rho(x) = \mathcal{N}(1 + \hbar \rho_1(x) + \hbar^2 \rho_2(x) + \ldots).
\]

From \((3.5)\) the dimension of variable \( z \) is \( \hbar^{1/2} \), therefore by power counting \( \rho_n(x) \) should contain terms with \( 2n \) covariant derivatives of the metric. For example, at first order in \( \hbar \) the only metric invariant is Ricci scalar.

At the first order in \( \hbar \) we have

\[
\hbar \rho_1 = -\frac{1}{\hbar} K_{ab\bar{a}\bar{b}} \int d\tau \left( \frac{1}{T} \langle z^a \bar{z}^\bar{a} z^b \bar{z}^\bar{b} \rangle_{\tau} + \frac{k}{2} \langle z^a \bar{z}^\bar{a} z^b \bar{z}^\bar{b} \rangle_{\tau} + \langle z^a \bar{z}^\bar{a} b^a c^b \rangle_{\tau} \right) + \frac{T}{4} R
\]

\[
= \hbar R \frac{1}{T} \int d\tau \left( \Delta(\Delta^* + \Delta^* \Delta^* \Delta) \right)_{\tau} + \hbar \frac{T}{4} R = \hbar R I_1(T, k) + \hbar \frac{T}{4} R. \tag{3.8}
\]

and from here on integration always runs from \(-1\) to \(0\). Here we apply usual Wick rule to calculate the correlators and in then we use the fact that \( R_{a\bar{a}b\bar{b}}(x) = K_{a\bar{a}b\bar{b}}(x) \) in normal frame centered at \( x \). This calculation elucidates the role of the ghost fields \( b, c \). Namely, their contribution cancels the \( \delta(0) \) terms, which appear in second derivatives of the bosonic propagators at coinciding points.

The values integrals used in the main text and their large time asymptotics are collected in Appendix C. In \( T \to \infty \) limit of the expression above becomes

\[
\hbar \rho_1 = \left(-\frac{T}{4} + \frac{1}{2k}\right) \hbar R + \frac{\hbar T}{4} R = \frac{\hbar}{2k} R. \tag{3.9}
\]

Note, that the Weyl-ordering counterterm \((3.6)\) is necessary to cancel the large-\( T \) divergence. This calculation provides an independent check of the coefficient in front of this term\(^2\) (see \[4\] for detailed consideration of this question).

\(^2\) Factor \( 1/4 \) here compared to \( 1/8 \) in \[4\] is due to our definition of scalar curvature \((A.1)\) in Kähler case.
3.5 Perturbation theory. Second Order

At the $\hbar^2$ order the following metric invariants may appear in the expansion: $\Delta R = g^{a\bar{a}}\partial_a\partial_{\bar{a}}R$, $|\text{Ric}|^2 = R_{a\bar{a}}R^{a\bar{a}}$, $|\text{Riem}|^2 = R_{a\bar{a}b\bar{b}}R^{a\bar{a}b\bar{b}}$ and $R^2$. Therefore the second order correction splits into four components, corresponding to the listed invariants. The full second-order contribution reads

$$
\hbar^2 \rho_2 = -\frac{1}{\hbar} K_{abc\bar{a}b} \int d\sigma \left( \frac{1}{4T} \langle z^{\bar{b}} z^{\bar{a}} z^{\bar{a}} z^{\bar{b}} \rangle \right) + \frac{1}{2\hbar^2} K_{a'b'c'b'} \int d\tau d\sigma \left( \frac{1}{T^2} \langle z^{\bar{b}} z^{\bar{a}} z^{\bar{a}} z^{\bar{b}} \rangle \right) 
\quad + \frac{k}{T} \langle z^{\bar{b}} z^{\bar{a}} z^{\bar{a}} z^{\bar{b}} \rangle + \frac{k^2}{4} \langle z^{\bar{b}} z^{\bar{a}} z^{\bar{a}} z^{\bar{b}} \rangle + \langle z^{\bar{b}} z^{\bar{a}} z^{\bar{a}} z^{\bar{b}} \rangle + \langle z^{\bar{b}} z^{\bar{a}} z^{\bar{a}} z^{\bar{b}} \rangle + \langle z^{\bar{b}} z^{\bar{a}} z^{\bar{a}} z^{\bar{b}} \rangle 
\quad + \frac{\hbar T}{4} \partial_\sigma \partial_\tau R \int d\sigma \langle z^c z^c \rangle \right) + \frac{\hbar T}{4} R \cdot hR \mathcal{I}_4(T, k) + \frac{1}{2} \left( \frac{\hbar T}{4} R \right)^2 \tag{3.10}
$$

We start computation from the first line in this expression. Taking into account the identity (A.2), the first line reads

$$
-\hbar^2 (-\Delta R + 2|Ric|^2 + |\text{Riem}|^2) \int d\tau \left( \frac{1}{T^2} \langle \Delta^2 \Delta^* \Delta + \Delta^2 (\Delta^* \Delta + \Delta^* \Delta) / 2 \rangle \right) \tau 
\quad = -\hbar^2 (-\Delta R + 2|Ric|^2 + |\text{Riem}|^2) \mathcal{I}_4(T, k) 
\quad \approx -\hbar^2 (-\Delta R + 2|Ric|^2 + |\text{Riem}|^2) \left( \frac{5}{6k^2} - \frac{T}{4k} \right), \text{ as } T \to \infty \tag{3.11}
$$

Consider now the integral in the second to fifth lines in (3.10). There are several nonequivalent ways to contract $z$ variables, leading to different invariants. Contraction of each of the primed indices $a'$, $b'$, $\bar{a}'$, $\bar{b}'$ with a non-primed index, leads to the $|\text{Riem}|^2$ structure. Such terms are given by the following expression

$$
\frac{\hbar^2}{2} |\text{Riem}|^2 \int d\tau d\sigma \left( \frac{1}{T^2} \langle \Delta(\sigma, \tau) \Delta(\tau, \sigma) \Delta^*(\sigma, \tau) \Delta^*(\tau, \sigma) \right) 
\quad + \Delta(\sigma, \tau) \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau) \Delta^*(\tau, \sigma) + \Delta(\sigma, \tau) \Delta(\tau, \sigma) \Delta^*(\sigma, \tau) \Delta^*(\tau, \sigma) 
\quad + \Delta^*(\sigma, \tau) \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau) \Delta^*(\tau, \sigma) 
\quad + \Delta(\sigma, \tau) \Delta(\tau, \sigma) \Delta(\sigma, \tau) \Delta(\tau, \sigma) + \frac{k}{T} \langle \Delta(\sigma, \tau) \Delta(\tau, \sigma) \Delta(\sigma, \tau) \Delta(\tau, \sigma) \rangle 
\quad + \frac{k^2}{T} \langle \Delta(\sigma, \tau) \Delta(\tau, \sigma) \Delta(\sigma, \tau) \Delta(\tau, \sigma) \rangle 
\quad - \Delta(\sigma, \tau) \Delta(\tau, \sigma) \left( \frac{1}{T} \Delta(\sigma, \tau) - k \Delta(\sigma, \tau) \right) \left( \frac{1}{T} \Delta(\sigma, \tau) - k \Delta(\sigma, \tau) \right) \right) 
\quad = \frac{\hbar^2}{2} |\text{Riem}|^2 \cdot \mathcal{I}_4(T, k) \approx \frac{\hbar^2}{2} |\text{Riem}|^2 \left( \frac{7}{4k^2} - \frac{T}{2k} \right), \text{ as } T \to \infty \tag{3.12}
$$

If we contract only two prime and two nonprime indices between each other we get the structure $|\text{Ric}|^2$

$$
\frac{\hbar^2}{2} |\text{Ric}|^2 \int d\tau d\sigma \left( \frac{1}{T^2} \langle \Delta(\tau) \Delta(\tau) \Delta(\tau, \sigma) \Delta(\tau, \sigma) \rangle + \Delta(\tau) \Delta^*(\sigma) \Delta(\tau, \sigma) \Delta^*(\tau, \sigma) \right)
$$

\[\text{--- 14 ---}\]
only disconnected diagrams. The structure of this term is just
coefficient one-half
other words we contract separately $z$’s and $\bar{z}$’s at point $\tau$ and $z$’s and $\bar{z}$’s at $\sigma$, we get only disconnected diagrams. The structure of this term is just $(\hbar R I_1)^2$. Adding up this term and last two terms from (3.10) we obtain the first order term \((3.13)\) squared with the coefficient one-half

$$\frac{1}{2} (\hbar \rho_1)^2 = \frac{1}{2} \left( \hbar R I_1(T, k) + \frac{hT}{4} R \right)^2 \approx \frac{\hbar^2}{8k^2} R^2, \text{ as } T \to \infty. \quad (3.14)$$

This term appears since we compute partition function, not the free energy, and therefore do not subtract disconnected diagrams.

Finally the first term in the last line in (3.10) reads

$$\frac{\hbar^2 T}{4} \Delta R \int d\tau \Delta(\tau, \tau) = \frac{\hbar^2 T}{4} \Delta R I_3(T, k) \approx \hbar^2 \Delta R \left( -\frac{1}{2k^2} + \frac{T}{4k} \right), \text{ as } T \to \infty. \quad (3.15)$$
Let us now collect all the terms (3.11, 3.12, 3.13, 3.15) that contribute to $\rho_2$ and compute its $T \to \infty$ limit

$$
\rho_2 = \left( I_2(T,k) + T I_3(T,k)/4 \right) \Delta R + \left( -2 I_2(T,k) + I_5(T,k)/2 \right) |\text{Ric}|^2 \\
+ (-I_2(T,k) + I_4(T,k)/2) |\text{Riem}|^2 + \frac{1}{2} (I_1(T,k) + T/4)^2 R^2 \\
\approx \frac{1}{k^2} \left( \frac{1}{3} \Delta R + \frac{1}{24} |\text{Riem}|^2 - \frac{1}{6} |\text{Ric}|^2 + \frac{1}{8} R^2 \right), \text{ as } T \to \infty \quad (3.16)
$$

Now we are ready write down the full expansion of the density matrix (3.4) up to second order in $\hbar$

$$
\rho = k^n \left( 1 + \frac{\hbar}{2k} R + \frac{\hbar^2}{k^2} \left( \frac{1}{3} \Delta R + \frac{1}{24} |\text{Riem}|^2 - \frac{1}{6} |\text{Ric}|^2 + \frac{1}{8} R^2 \right) + O((\hbar/k)^3) \right). \quad (3.17)
$$

Note that this expansion is in perfect agreement with the expansion of Bergman kernel, obtained in [11].

4. $\mathcal{N} = (1, 1)$ Supersymmetry

4.1 Action, symmetries and propagators

One can obtain expansions similar to (3.17) in other quantum mechanical theories. Here we consider $(1, 1)$-supersymmetric particle on Kähler manifold with the magnetic field turned on. The action is

$$
S = \int_{t_i}^{t_f} dt \left( g_{ab} \dot{z}^a \dot{z}^b + \bar{\psi}^\alpha (g_a^a \dot{\psi}^a + x^b \partial_b g_a^a \psi^a) + \bar{A}_a^\dot{\alpha} \bar{\psi}^\alpha \psi^a \right) \quad (4.1)
$$

This action is invariant under the following $(1, 1)$ supersymmetry transformations

$$
\delta x^a = -\bar{\epsilon} \dot{\psi}^a \\
\delta \bar{z}^{\dot{\alpha}} = -\epsilon \bar{\psi}^{\dot{\alpha}} \\
\delta \psi^a = \dot{x}^a \epsilon \\
\delta \bar{\psi}^{\dot{\alpha}} = \dot{\bar{z}}^{\dot{\alpha}} \bar{\epsilon}
$$

if the metric is Kähler and if $A_a$, $\bar{A}_{\dot{a}}$ is a connection of holomorphic vector bundle

$$
F_{ab} = F_{\dot{a}\dot{b}} = 0.
$$

Set the field strength to be proportional to the metric, exactly as before in Eq. (2.5). Consider now the path integral representation of this theory. If the boundary conditions for $x$ and $\psi$ fields are the same, no ghosts are needed in the action, because bosonic and fermionic determinants cancel in the measure. Moreover, no Weyl-ordering counterterm is needed in this theory, due to fermions. Bosonic propagator is the same a before, and fermionic propagator

$$
\langle \bar{\psi}^b(\tau) \psi^a(\sigma) \rangle = \hbar g^{ab} \Gamma(\tau, \sigma)
$$
satisfies
\[ \left[ \frac{d}{d\sigma} + T \kappa \right] \Gamma(\tau, \sigma) = -\delta(\tau - \sigma) \]

We would like to compute the “index density”, i.e. the supertrace of the density matrix, without performing the x-integral
\[ \rho(x) = \lim_{T \to \infty} Str(-1)^F e^{-TH}. \]

The right hand side here depends only on the bosonic “zero-mode” x, and all fermionic dependence is integrated out. Fermion number insertion \((-1)^F\) corresponds to periodic boundary conditions for fermions, in which case the propagator has the form
\[ \Gamma(\tau, \sigma) = \frac{1}{1 - e^{kT}} \left( e^{kT(\tau - \sigma)} \theta(\tau - \sigma) + e^{kT(\tau - \sigma + 1)} \theta(\sigma - \tau) \right). \]

### 4.2 Perturbation theory

The idea of the calculation is the same as in nonsupersymmetric case. We use Kähler normal coordinates and expand the metric around the constant configuration x.

Free part of the action is given by
\[ S_0 = \int_{-1}^{0} d\tau \left[ \frac{1}{T} g_{ab}(x) z^a \dot{z}^b + k g_{ab}(x) z^a \dot{z}^b + g_{ab}(x) \bar{\psi} \dot{\psi}^a + T k g_{ab} \bar{\psi} \dot{\psi}^a \right], \quad (4.3) \]

and the interaction part, up to the sixth order in derivatives of the Kähler potential, reads
\[ S_{int} = \int_{-1}^{0} d\tau \left[ \frac{1}{T} \left( K_{ab\hat{a}}(x) z^b \dot{z}^\hat{a} + \frac{1}{4} K_{abc\hat{a}\hat{b}}(x) z^b \dot{z}^\hat{a} z^c \dot{z}^\hat{b} \right) \right] \cdot \cdot \cdot \]

At the first order in \( h \) we get
\[ \rho_1(N = 1) = R \int d\tau \left( \frac{1}{T} (\bullet \Delta^\bullet + \bullet \Delta^\bullet) + k^* \Delta \Delta - \Delta \delta(0) + \Delta \Gamma \right) |_{\tau = 0}, \]

so \( h^0 \) term is exactly zero, even for finite T.

Computation at the second order in \( h \) proceeds in a similar fashion as in previous section. Let us only mention one shortcut. Note, that each contraction of \( \bar{\psi}(\sigma) \) and \((Tk + \partial_\tau)\psi(\tau)\) is proportional to delta-function \( \delta(\tau, \sigma) \), exactly as contraction of ghosts b and c. Therefore the first three lines in the interaction lagrangian (4.4) generate the same terms as bosonic interaction lagrangian (3.5) and only the last line in (4.4) is a new one.
With this observation the calculation simplifies significantly. We only give the final answer here

\[
\rho_2(\mathcal{N} = 1) = -(I_2(T, k) + I_6(T, k))(-\Delta R + 2|Ric|^2 + |Riem|^2) \\
+ (I_5(T, k)/2 + I_7(T, k) + I_8/2)|Ric|^2 \\
+ (I_4(T, k)/2 + I_6(T, k) - I_9(T, k))|Riem|^2
\]

(4.5)

and refer to Appendix C for the values of the integrals here. The coefficients in front of $|Ric|^2$ and $|Riem|^2$ turn out to be $T$-independent, as a consequence of supersymmetry and the index theorem, and the answer for the density matrix up to the second order in $h$ is

\[
\rho(x)(\mathcal{N} = 1) = k^n \left(1 + \frac{h^2}{24k^2} (2\Delta R - |Ric|^2 + |Riem|^2) + \mathcal{O}(h^3)\right)
\]

This is consistent with the index theorem. According to the latter the $x$-integral of $\rho(x)(\mathcal{N} = 1)$ is equal to the index of Dirac operator on the Kähler manifold $M$ for which the exact answer is

\[
\int_M dx \rho(x)(\mathcal{N} = 1) = \text{ind}D_A = \int_M \text{ch}F \wedge \hat{A}(M).
\]

If we plug $F = kg_{ab}dz^a \wedge dz^b$ and expand the A-roof genus $\hat{A}$ in powers of curvature tensors then the first two terms in this expression coincide with first two terms in $\int \rho(x)(\mathcal{N} = 1)$, up to maybe an overall constant.

5. $\mathcal{N} = (2, 2)$ Supersymmetry

5.1 Action, symmetries and propagators

The action is

\[
S = \int_{t_i}^{t_f} dt \left( g_{ab} \dot{x}^a \dot{x}^b + \bar{\psi}^a_+ (g_{aa} \dot{\psi}^a_+ + \dot{x}^b \partial_b g_{aa} \psi^a_+) \right. \\
\left. + \bar{\psi}^-_+ (g_{aa} \dot{\psi}^-_+ + \dot{x}^b \partial_b g_{aa} \psi^-_+) + \bar{A}_{\dot{a}} \dot{\psi}^\dot{a}_+ + F_{\dot{a}a} (\bar{\psi}^\dot{a}_+ \psi_+ + \bar{\psi}^\dot{a}_- \psi^-_+) \right).
\]

(5.1)

The $\mathcal{N} = (2, 2)$ supersymmetry transformations

\[
\delta x^a = -\epsilon_+ \psi^a_+ - \bar{\epsilon}_- \psi^a_- \\
\delta \bar{x}^\dot{a} = -\epsilon_+ \psi^\dot{a}_+ - \bar{\epsilon}_- \psi^\dot{a}_- \\
\delta \psi^a_+ = \dot{x}^a \epsilon_+ + \bar{\epsilon}_- \Gamma^a_{bc} \psi^b_+ \psi^c_+ \\
\delta \bar{\psi}^\dot{a}_+ = \dot{x}^\dot{a} \epsilon_+ + \bar{\epsilon}_- \Gamma^a_{bc} \bar{\psi}^\dot{b}_+ \bar{\psi}^\dot{c}_+ \\
\delta \psi^-_+ = \dot{x}^a \epsilon_- + \bar{\epsilon}_+ \Gamma^a_{bc} \psi^-_+ \psi^-_+ \\
\delta \bar{\psi}^\dot{a}_- = \dot{x}^\dot{a} \epsilon_- + \bar{\epsilon}_+ \Gamma^a_{bc} \bar{\psi}^\dot{b}_- \bar{\psi}^\dot{c}_-
\]

(5.2)

leave the action invariant if connection $A$ is holomorphic and the hermitian Yang-Mills equation is obeyed

\[
g^{ab} D_a F_{bb} = 0.
\]
Recall again, that our choice of field strength $F_{ab} = k g_{ab}$ [2.5] satisfies this equation.

The object that we would like to compute is the Dolbeault index density, which corresponds to taking the supertrace over one species of fermions, and setting the zero modes of the second species of fermions to zero. To achieve this, we choose the following propagators for the fermions

$$\langle \psi^k_+(\tau) \psi^a_+(\sigma) \rangle = \hbar g^{ab} \Gamma_+(\tau, \sigma)$$
$$\langle \psi^k_-(\tau) \psi^a_-(\sigma) \rangle = \hbar g^{ab} \Gamma_-(\tau, \sigma),$$

where $\Gamma_+$ satisfies periodic boundary conditions: $\Gamma_+(-1, \sigma) = \Gamma_+(0, \sigma)$, $\Gamma_+(\tau, -1) = \Gamma_+(\tau, 0)$, and $\Gamma_-$ satisfies Dirichlet b.c. $\Gamma_-(-1, \sigma) = \Gamma_-(\tau, 0) = 0$. These propagators are given by

$$\Gamma_+(\tau, \sigma) = \frac{1}{1 - e^{kT}} \left( e^{kT(\tau - \sigma)} \theta(\tau - \sigma) + e^{kT(\tau - \sigma - 1)} \theta(\sigma - \tau) \right)$$
$$\Gamma_-(\tau, \sigma) = e^{kT(\tau - \sigma)} \theta(\tau - \sigma),$$

One also has to add a pair of bosonic ghost fields $a^a, \bar{a}^a$, coming from the path integral measure

$$S_{gh} = \int d\tau g_{ab} a^a \bar{a}^a.$$

### 5.2 Perturbation theory

The calculation proceeds along the same lines as in the previous two sections. Here we present the final answer for the index density

$$\rho(x) = k^n \left( 1 + \hbar I_{13}(T, k) R + \hbar^2 (I_{14}(T, k) \Delta R + (I_5(T, k)/2 + I_{11}(T, k) + I_{12}(T, k)) |\text{Riem}|^2 
+ (-I_2(T, k) + I_4(T, k)/2 + I_{10}(T, k)) |\text{Riem}|^2 + I_{13}^2(T, k) R^2 / 2 + \mathcal{O}(h^3) \right)$$

$$= k^n \left( 1 + \frac{h}{2k} R + \frac{k^2}{h^2} (I_{14}(T, k) \Delta R - \frac{1}{6k^2} |\text{Riem}|^2 + \frac{1}{24k^2} |\text{Riem}|^2 + \frac{1}{8k^2} R^2) + \mathcal{O}(h^3) \right)$$

$$\approx k^n \left( 1 + \frac{h}{2k} R + \frac{k^2}{h^2} \left( \frac{3}{2} \Delta R + \frac{1}{24} |\text{Riem}|^2 - \frac{1}{6} |\text{Riem}|^2 + \frac{1}{8} R^2 \right) + \mathcal{O}(h^3) \right), \text{ as } T \to \infty.$$

Notice, that the only $T$-dependent term here is a total derivative, as is expected from the index theorem

$$\int_M dx \rho(x)(N = 2) = \text{ind} \, \delta_A = \int_M \text{ch} F \wedge \text{Td}(M).$$

Recall, that this index formula computes $\dim \sum_q (-1)^q H^{0,q}(M, L^k)$, which is equal to $\dim H^0(M, L^k)$ for large enough $k$. This explains why $N = 2$ and nonsupersymmetric Bergman kernel’s expansions coincide.

### 6. Conclusions

In this paper we derived the Tian-Yau-Zelditch et al expansion of the Bergman kernel from quantum mechanical path integral. Our results are in complete agreement with the calculation of Ref. [11], using Tian’s peak section method.
In quantum mechanics, the Bergman kernel corresponds to density matrix of a particle in strong magnetic field on Kähler manifold, projected to the lowest Landau level. The expansion in the inverse magnetic flux number can be extracted by taking the infinite time $T \to \infty$ limit of the (non-supersymmetric) path integral and using the normal coordinate expansion of the metric and the magnetic field. In this paper we considered a the configuration of magnetic field, being proportional to the Kähler form. Expansions of the same type can also be obtained using supersymmetric quantum mechanics, where they correspond to index densities. It would be interesting to extend this result to the case of particle coupled to non-abelian external fields, considered in mathematical literature [31].

The physical argument, presented in sec. 2.1, suggests that the analogous expansion for the Bergman kernel may be obtained for a more general magnetic field strength, associated with holomorphic line bundle. We plan to check this in the future publication.

One of the most interesting consequences of this result, as discussed in section 2, is that there exists a specific magnetic field and metric for which the density matrix (1.3) is constant everywhere on the manifold. This is the “maximally entropic” metric for a quantum mechanical observer, in a sense discussed in [24].

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A. Curvatures

We follow conventions of [11]

$$
R_{\bar{a}b\bar{a}} = \partial_{\bar{b}}\partial_{\bar{b}}g_{a\bar{a}} - g^{\bar{c}e}\partial_{\bar{c}}g_{ae}\partial_{\bar{c}}g_{\bar{a}},
$$

$$
R_{a\bar{a}} = -g^{\bar{b}b}R_{\bar{a}b\bar{a}b},
$$

$$
R = g^{a\bar{a}}R_{a\bar{a}},
$$

$$
\Delta R = g^{\bar{a}a}\partial_{\bar{a}}\partial_{a}R,
$$

$$
|\text{Riem}|^2 = R_{\bar{a}b\bar{a}}R_{a\bar{a}b\bar{a}},
$$

$$
|\text{Ric}|^2 = R_{a\bar{a}}R^{a\bar{a}}.
$$

Let $K$ be the Kähler potential for the metric

$$
g_{a\bar{a}} = \partial_{a}\partial_{\bar{a}}K.
$$

In Kähler normal coordinate frame the Cristoffel symbols and all pure holomorphic derivatives of the metric vanish at the origin

$$
K_{a\bar{a}1...\bar{a}n}(x) = 0,
$$
for any positive integers $m, n$. The following terms in the Taylor expansions of the Kähler potential, the metric, Riemann tensor and Ricci scalar are relevant for the present paper

$$K(x^a + z^\alpha, \bar{x}^\alpha + \bar{z}^\alpha) = K(x) + \frac{1}{2} g_{ab}(x) z^a z^b + \frac{1}{4} g_{abc}(x) z^a z^b z^c + \frac{1}{36} g_{abcde}(x) z^a z^b z^c z^d z^e + \ldots,$$

$$R_{a\bar{b}c\bar{d}}(x^a + z^\alpha, \bar{x}^\alpha + \bar{z}^\alpha) = K_{a\bar{b}}(x) + \frac{1}{2} g_{abc}(x) z^c + \frac{1}{4} g_{abcd}(x) z^c z^d + \ldots$$

Using (B.3, B.5) we get the expression for Weyl ordered Hamiltonian (B.1)

$$\hat{H} = \frac{1}{2} \left( \hat{p}_a g^{ab} \hat{p}_b + \hat{p}_a g^{ab} \hat{p}_b \right) W = \frac{1}{4} \left( \hat{p}_a g^{ab} \hat{p}_b + \hat{p}_a g^{ab} \hat{p}_b + \hat{p}_a g^{ab} \hat{p}_b + g^{ab} \hat{p}_a \hat{p}_b \right) \quad (B.1)$$

B. Hamiltonian

Here we rewrite the Hamiltonian (B.3) in a Weyl-symmetric way. First we simplify the expression without the gauge potential

$$\hat{H} = \frac{1}{2} g^{-1/2} \hat{p}_a g^{ab} g^{\hat{p}b} g^{-1/2} + \frac{1}{2} g^{-1/2} \hat{p}_b g^{ab} g \hat{p}_a g^{-1/2} = \frac{1}{2} (\hat{p}_a g^{ab} \hat{p}_b + \hat{p}_b g^{ab} \hat{p}_a) \quad (B.1)$$

where we use $\hat{p}_a = -i \hbar \partial_a$, $\hat{p}_a = -i \hbar \partial_a$. The Weyl ordered form of the first term in the previous expression is

$$(\hat{p}_a g^{ab} \hat{p}_b) W = \frac{1}{4} \left( \hat{p}_a g^{ab} \hat{p}_b + \hat{p}_a g^{ab} \hat{p}_b + \hat{p}_a g^{ab} \hat{p}_b + g^{ab} \hat{p}_a \hat{p}_b \right)$$

Therefore

$$\frac{1}{2} (\hat{p}_a g^{ab} \hat{p}_b + \hat{p}_b g^{ab} \hat{p}_a) = (\hat{p}_a g^{ab} \hat{p}_b) W + \frac{1}{8} ([\hat{p}_a, [g^{ab}, \hat{p}_b]] + [[\hat{p}_a, [g^{ab}, \hat{p}_b]]]) \quad (B.3)$$

$$= (\hat{p}_a g^{ab} \hat{p}_b) W + \frac{\hbar^2}{4} R + \frac{\hbar^2}{4} g^{ab} \Gamma_{ab}^{bc} \quad (B.4)$$

The last three terms in (B.1) can be written as

$$\hat{p}_b (g^{ab} \partial_a \ln g) = \partial_a (g^{ab} \hat{p}_b \ln g) = -R - g^{ab} \Gamma_{ab}^{bc} \quad (B.5)$$

$$g^{ab} \hat{p}_b \ln g \partial_a \ln g = g^{ab} \Gamma_{ab}^{bc} \quad (B.5)$$

Using (B.3, B.4) we get the expression for Weyl ordered Hamiltonian (B.1)

$$\hat{H} = (\hat{p}_a g^{ab} \hat{p}_b) W - \frac{\hbar^2}{4} R \quad (B.6)$$

Now it is straightforward to see that the similar expression holds in the presence of gauge connection (B.3), one just has to shift $\hat{p}_b \rightarrow \hat{p}_b - A_b$ in the previous equation.
C. Integrals

Here we collect exact expressions for the integrals that appear in the main text. The following short hand notations are used

\[ \Delta_1(T, k) = \int d\tau \frac{1}{T} (\Delta_1^*(\tau)\Delta_1(\tau) + \Delta_1(\tau)\Delta_1^*(\tau)) \]

\[ \Delta_2(T, k) = \int d\sigma \left( \frac{1}{T} \Delta_2(\tau, \sigma) \right) \]

\[ \Delta_3(T, k) = \int d\tau \Delta_3(\tau) \]

\[ \Delta_4(T, k) = \int d\tau d\sigma \left( \frac{1}{T^2} (\Delta_4(\tau)\Delta_4^*(\tau) + \Delta_4(\tau)\Delta_4^*(\tau)) \right) \]

\[ \Delta_5(T, k) = \int d\tau d\sigma \left( \frac{1}{T^2} (\Delta_5(\tau)\Delta_5^*(\tau) + \Delta_5(\tau)\Delta_5^*(\tau)) \right) \]

and so on.
\[ I_6(T, k) = \int d\tau \Gamma \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau) = \frac{1 + e^{kt}}{4k^2(-1 + e^{kt})^3}(3 + kT + 4kT e^{kt} + e^{2kt}(-3 + kT)) \quad \text{(C.6)} \]

\[ I_7(T, k) = \int d\tau d\sigma \left( \Gamma(\sigma, \sigma) \left( \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau) + \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau) \right) \right) + \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau) + \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau)) / T \]
\[ + k(\Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau) + \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau)) \Delta^*(\sigma, \tau) = \frac{1 + e^{kt}}{4k^2(-1 + e^{kt})^4}(-5 - 2kT + e^{kt}(5 - 8kT - 2k^2T^2) + e^{2kt}(5 + 8kT - 2k^2T^2) + e^{3kt}(-5 + 2kT)) \quad \text{(C.7)} \]

\[ I_8(T, k) = \int d\tau d\sigma \left( \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau) \Gamma(\tau, \sigma) - \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau) \Gamma(\tau, \sigma) \right) \]
\[ = \frac{1}{4k^2(-1 + e^{kt})^3}(1 + e^{kt}(5 + 4kT) + e^{2kt}(-5 + 4kT) - e^{3kt}) \quad \text{(C.8)} \]
\[ I_9(T, k) = \int d\tau d\sigma \ \Delta^*(\tau, \sigma) \Delta^*(\sigma, \tau) \Gamma(\tau, \sigma) \Gamma(\sigma, \tau) \]
\[ = \frac{e^{kT}}{k^2(-1 + e^{kT})^4} (-(-1 + e^{kT})^2 + k^2 T^2 e^{kT}) \tag{C.9} \]

\[ I_{10}(T, k) = \int d\tau d\sigma \ \left( -\frac{1}{2} \Delta^*(t, s) \Delta^*(s, t) (\Gamma_+(t, s) \Gamma_+(s, t) + \Gamma_-(t, s) \Gamma_-(s, t)) \right. \]
\[ + \frac{T^2}{2} \Gamma_+(t, s) \Gamma_+(s, t) \Gamma_-(t, s) \Gamma_-(s, t) \right) \]
\[ = \frac{e^{kT}}{2(1 + e^{kT})^4 k^2} ((-1 + e^{kT})^2 - e^{kT} k^2 T^2) \tag{C.10} \]

\[ I_{11}(T, k) = \int d\tau d\sigma \ \left( \Delta^*(t, s) \Delta^*(s, t)(\Gamma_+(t) + \Gamma_-(t))(\Gamma_+(s) + \Gamma_-(s)) \right. \]
\[ - \Delta^*(t) \Delta^*(s)(\Gamma_+(t, s) \Gamma_+(s, t) + \Gamma_-(t, s) \Gamma_-(s, t)) \]
\[ - T^2(\Gamma_+(t) \Gamma_+(s) \Gamma_-(t, s) + \Gamma_-(t) \Gamma_-(s) \Gamma_+(t, s) \Gamma_+(s, t)) \]
\[ + \frac{2}{T}(\Gamma_+(s) + \Gamma_-(s))(\Delta^*(t, s) \Delta^*(s, t) \Delta(t) + (\Delta^*(t) - T\delta(t - s)) \Delta^*(t, s) \Delta(s, t)) \]
\[ + (\Delta^*(t, s) - T\delta(t - s)) \Delta^*(s, t) \Delta(t) + (\Delta^*(t, s) - T\delta(t - s)) \Delta(s, t) \Delta^*(t)) \]
\[ + 2 k(\Gamma_+(s) + \Gamma_-(s))(\Delta^*(t, s) - T\delta(t - s)) \Delta(s, t) \Delta(t) + \Delta^*(t, s) \Delta(s, t) \Delta^*(t)) \]
\[ + 2 T \Delta^*(t)(\Gamma_+(t, s) \Gamma_+(s, t) \Gamma_-(s) + \Gamma_-(t, s) \Gamma_-(s, t) \Gamma_+(s)) \]
\[ = -\frac{1}{(-1 + e^{kT})^4 k^2} (3(-1 + e^{kT})^2(1 + e^{kT}) + kT(1 + e^{3kT}(-3 + kT)) \]
\[ + e^{kT}(5 + kT) + e^{2kT}(-3 + 2kT)) \tag{C.11} \]

\[ I_{12}(T, k) = \int d\tau \ \left( \Delta(\Delta^* + kT \Delta)(\Gamma_- + \Gamma_+) - \frac{1}{T}(\Delta^* - T\delta(0)) \Delta^2 + 2 \Delta \Delta^* \Delta - k^* \Delta \Delta^2 \right) \mid_{\tau} \]
\[ = \frac{1}{6(-1 + e^{kT})^3 k^2} (1 + 9 e^{kT}(2 + kT) + 9 e^{2kT}(-1 + 2kT) \]
\[ + e^{3kT}(-10 + 3kT)) \tag{C.12} \]

\[ I_{13}(T, k) = \int d\tau \ \left( \frac{1}{T}(\Delta^* - T\delta(0)) \Delta + \Delta \Delta^* \right) + k^* \Delta \Delta - T \Gamma_+ \Gamma_- + \Delta^*(\Gamma_+ + \Gamma_-) \mid_{\tau} \]
\[ = \frac{1}{2k} \tag{C.13} \]

\[ I_{14}(T, k) = \int d\tau \ \left( \frac{1}{2T}(\Delta^* - T\delta(0)) \Delta^2 + 2 \Delta \Delta^* \Delta^* \right) + \frac{k}{2} \Delta \Delta^2 - T \Delta \Gamma_+ \Gamma_- + \Delta \Delta^*(\Gamma_+ + \Gamma_-) \right) \]
\[ = \frac{1}{6(-1 + e^{kT})^3 k^2} (1 - 6 e^{kT} - 3 e^{2kT}(-1 + 2kT) + 2 e^{3kT}) \tag{C.14} \]
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