On the solution of the Calogero model and its generalization to the case of
distinguishable particles

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Abstract
The 3-body Calogero problem is solved by separation of variables for
arbitrary exchange statistics. A numerical computation of the
4-body spectrum is also presented. The results display new
features in comparison with the standard case of bosons and
fermions, for instance the energies are not linear with the
interaction parameter $\nu$ and Bethe ansatz as well as Haldane’s
statistics are not verified.

1 Introduction
The Calogero model, that is the 1d system of $N$ bosons or $N$
fermions interacting by the 2-body potential $\nu(x - x^{-1})$, is of
physical interest: it interpolates between free bosons and free
fermions [1, 2], it exhibits the fractional statistics of Haldane
[3], and its asymptotic wavefunction is in accordance with the
Bethe hypothesis [4, 5]. Moreover, it belongs to a large class of
integrable models [6, 7].

The Calogero model is known to be completely integrable for
a long time, and this fact remains true when the particles are
in a harmonic well. Calogero has solved himself the 2-body
and 3-body problems by separation of variables. He has also shown
that the many-body problem is integrable and he has deduced
its energy spectrum [7]. More recently, the many-body eigenstates
have been constructed from an algebra of creation and
annihilation operators [8, 9, 10, 11]. This algebra is immedi-
ately generalizable to the case of Boltzmann statistics where no
exchange symmetry is imposed to the wavefunction, however the
ground states and their energies are not all known in this case.

In this paper, I discuss the asymptotic behaviour of the wave-
function when two particles coincide and I apply it to solve the
eigenvalue problem for arbitrary statistics. I analytically con-
struct the eigenvalues and eigenfunctions for $N$ bodies in the
special case $\nu = 1/2$, and next for 3 bodies in the general case.
A numerical computation of the 4-body problem is also pre-
seated. The results obviously encompass the standard solution
for bosons and fermions. However, the states of mixed statis-
tics display a non-trivial framework and their energies are not
linear in $\nu$. Moreover, the particle current does not necessarily
vanish when two particles coincide. In conclusion, I discuss the
Calogero model for distinguishable particles with constants $\nu_j$
$\mu_j$, depending on the labels of the particles [12, 13].

2 The example of the two-body problem

2.1 The nature of the wavefunction
Let me focus on the relative motion of two particles of coordi-
nates $x_1$ and $x_2$ in a harmonic well. Then the Calogero hamil-
tonian is

$$H = -\frac{1}{2} \frac{\partial^2}{\partial x_1^2} + \frac{1}{4} \frac{\nu}{x_1^2} + \frac{\nu}{x_1^2} |x_1 - x_2|,$$  \hspace{1cm} (1)

with $x = x_1 - x_2$. The hamiltonian acts on a
usual harmonic oscillator problem. The hamiltonian acts on a
domain of definition

$$D_H \subset \{ \psi, \psi \in L^2, H \psi \in L^2, H^* = H \}$$  \hspace{1cm} (2)

which warrants finite matrix elements, real eigenvalues and the
orthogonality between states of different energies. There remains
to understand the nature of the wavefunctions $\psi \in D_H$.

At ordinary points $x \neq 0$, the wavefunction and its derivative
may be assumed continuous as usual. Around the singular point
$x = 0$, one verifies that the wavefunction must have an algebraic
behaviour as

$$\psi \sim x^\nu \text{ or } \psi \sim x^{1-\nu},$$  \hspace{1cm} (3)

otherwise its image $H \psi$ is not a square integrable function. Note
that the situations $\nu < 0$ and $\nu > 1$ will not be considered in this
paper because the wavefunction is then divergent in general.

The hamiltonian is self-adjoint when the boundary term

$$\int_0^1 dx \left( \psi_1 \frac{\partial^2}{\partial x_2} \psi_2 - \psi_2 \frac{\partial^2}{\partial x_1} \psi_1 \right) = \frac{\partial}{\partial x} \psi_1 \frac{\partial}{\partial x} \psi_2$$  \hspace{1cm} (4)

is continuous at any point and vanishes at infinity for all pair of
wavefunctions $\psi_1, \psi_2$. Notice that $\frac{\partial}{\partial x} \psi \frac{\partial}{\partial x}$ is merely the particle
current. It is equivalent to impose the previous condition on the
boundary term for all pair of basic states $\psi_1, \psi_2$, or the same
condition on the particle current for all wavefunctions $\psi$. This
condition is satisfied at ordinary points and at infinity because
the wavefunction and its first derivative are continous at these

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points and vanishes at infinity. At the singular point $x = 0$, one can always set
\[ \psi = |x|^\nu f(x) + x|x|^{-\nu}g(x) \] (5)
and then one has
\[ \psi_1 \hat{\partial} \psi_2 = (1 - 2\nu)(\tilde{f}_1 g_2 - \tilde{g}_1 f_2) + x(\tilde{f}_1 \hat{\partial} g_2 + \tilde{g}_1 \hat{\partial} f_2) + [x^2(\tilde{f}_1 \hat{\partial} f_2) + x^2]|^{-2\nu}(\tilde{g}_1 \hat{\partial} g_2). \] (6)

For $0 < \nu < 1$ and $\nu \neq 1/2$, the continuity of the particle phase is ensured at $x = 0$ if $f$ and $g$ are continuous up to a given phase shift at this point and if $\partial f$ and $\partial g$ are finite at this point. The case $\nu = 1/2$ is particular because the continuity of $f$ and $g$ can then be relaxed at $x = 0$.\footnote{Note that the regular combination $|x|^{1/2}f(x) + x|x|^{-1/2}f(x) = 2\delta_{x>0}\sqrt{|x|}f(x)$ leads to a discontinuity as $2\delta_{x>0}f(x)$.
}

The choice of the overall phase $\psi$ and its derivative are continuous as it should and the fermionic eigenstates are
\[ \psi_n = |x|^\nu L_n^\nu - \nu \frac{1}{2}(\frac{1}{2} \omega x^2) e^{-\frac{1}{2} x^2}, \quad E_n = (2n + \frac{1}{2} + \nu)\omega, \] (9)
and the fermionic eigenstates are
\[ \psi_n = x|x|^{-\nu} L_n^\nu - \nu \frac{1}{2}(\frac{1}{2} \omega x^2) e^{-\frac{1}{2} x^2}, \quad E_n = (2n + \frac{3}{2} - \nu)\omega, \] (10)

2.3 Eigenvalues and eigenfunctions

The eigenstates are obtained in a standard way. Let me choose the regular gauge characterized by the boundary condition \footnote{Note that the regular combination $|x|^{1/2}f(x) + x|x|^{-1/2}f(x) = 2\delta_{x>0}\sqrt{|x|}f(x)$ leads to a discontinuity as $2\delta_{x>0}f(x)$.}

Thus, the bosonic eigenstates are
\[ \psi_n = |x|^\nu L_n^\nu - \nu \frac{1}{2}(\frac{1}{2} \omega x^2) e^{-\frac{1}{2} x^2}, \quad E_n = (2n + \frac{1}{2} + \nu)\omega, \] (9)
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\[ \psi_n = x|x|^{-\nu} L_n^\nu - \nu \frac{1}{2}(\frac{1}{2} \omega x^2) e^{-\frac{1}{2} x^2}, \quad E_n = (2n + \frac{3}{2} - \nu)\omega, \] (10)

3 Exact results for the many-body problem

3.1 The model

The $N$-body hamiltonian is
\[ H = \sum_{i=1}^{N} (-\frac{1}{2} \partial_i^2 + \frac{1}{2} \omega^2 x_i^2) + \sum_{i,j<} \frac{\nu(\nu - 1)}{x_{ij}^2} \] (11)

with $x_{ij} = x_i - x_j$. The wavefunction $\psi$ has to be sought in an irreducible representation of the permutation group $S_N$, usually bosonic or fermionic. Let me still assume $0 < \nu < 1$, the limits $\nu = 0$ and $\nu = 1$ being special for the standard bosonic or fermionic boundary conditions. The domain of definition of the hamiltonian is chosen as for the 2-body case: the wavefunction and its derivative are continuous except at $x_i = x_j$ where
\[ \psi = |x_{ij}|^\nu f + x_{ij}|x_{ij}|^{-\nu}g \] (12)

with $f$ and $g$ locally continuous, $\partial_i f$ and $\partial_j g$ finite ($\partial_i$ is a shorthand notation for $\partial_i - \partial_j$). The case $\nu = 1/2$ is special: the continuity of $f$ and $g$ is no longer necessary and the boundary condition at $x_i = x_j$ is reduced to
\[ \psi = \sqrt{|x_{ij}|} h \] (13)

with $\partial_i h$ finite.

As for the 2-body case, we can define an anyon gauge by
\[ \psi' = e^{i\nu \sum_{i,j<} \theta_{ij} \psi} \] (14)

where $\theta_{ij} = \arg(x_{ij})$. The hamiltonian is unaltered, but the boundary condition at $x_i = x_j$ is changed into
\[ \psi' = x_{ij}^\nu f + x_{ij}^{-\nu} g. \] (15)
3.3 Raising and lowering operators

This section is a brief account of some recent progress. The main idea consists to construct a covariante derivative using the operator $P_{ij}$ which exchanges the particles $i$ and $j$.

\[ D_i = \partial_i - \sum_{j \neq i} \nu P_{ij}. \]

The adjoint of $D_i$ is merely $-D_i$, for $P_{ij}$ and $x_{ij}^{-1}$ are anticommuting. The commutation rules are $[D_i, D_j] = 0$ and

\[ [D_i, x_j] = \delta_{ij} (1 + \sum_{k=1}^{N} \nu P_{ik}) - \nu P_{ij}. \]

One next defines the hamiltonian

\[ \mathcal{H} = \frac{1}{2} \sum_{i=1}^{N} (a_i^+ a_i^- + a_i^- a_i^+) \omega, \]

in terms of creation and annihilation operators,

\[ a_i^\pm = \sqrt{\frac{\omega}{2}} x_i + \frac{D_i}{\sqrt{2\omega}}, \quad [\mathcal{H}, a_i^\pm] = \pm \omega a_i^\pm. \]

These operators are mutually adjoints.

As regards its action on bosonic wavefunctions, the hamiltonian $\mathcal{H}$ is found to be identical with the Calogero one. As a result, the bosonic eigenstates of the Calogero model are easily obtained. The Bose groundstate

\[ \prod_{i,j<} \left| x_{ij} \right|^\nu e^{-\frac{1}{4}\omega \sum_i x_i^2} \]

is annihilated by the $a_i^-$‘s, its energy is $\frac{N}{2} \omega + \frac{N(N-1)}{2} \omega \nu$, and the excitements follow from the successive actions of the $a_i^+$‘s on the groundstate. However, these excitements have to be symmetrized under particle exchanges. To avoid this symmetrization, it is desirable to define a set of operators preserving the exchange symmetries of the wavefunction. Separating the center-of-mass and the relative excitements, such a set of raising and lowering operators is given by

\[ A_i^\pm = \sum_{i=1}^{N} a_i^\pm, \quad A_{k \geq 2}^\pm = \sum_{i=1}^{N} (a_i^+ - \frac{1}{N} A_1^+) k, \]

The set is complete if $k = 1, 2, \ldots, N$. For a discussion of the algebra associated with these operators, see [11]. The main commutation rules are

\[ [\mathcal{H}, A_k^\pm] = \pm k \omega A_k^\pm. \]

In this way, the bosonic excitements are directly obtained as

\[ \psi_{n_k} = \prod_{k=1}^{N} (A_1^+) n_k \prod_{i,j<} \left| x_{ij} \right|^\nu e^{-\frac{1}{4}\omega \sum_i x_i^2}, \]

and the total energies are

\[ E_{n_k} = \sum_{k=1}^{N} k n_k \omega + \frac{N}{2} \omega + \frac{N(N-1)}{2} \omega \nu. \]
where the $n_k$’s are non-negative integers. Here, the net effect of the raising operators is to multiply the groundstate by a polynomial. The particle current $\vec{I}$ vanishes at coinciding points $x_i = x_j$ because $g = 0$ and thus there is no particle exchange.

As regards its action on an arbitrary function, the hamiltonian $H$ is a non-local operator effecting among others some interchanges on the variables of the function. It is necessarily different from the Calogero hamiltonian. However, $H$ is the only local operator which coincides with $H$ acting on symmetric functions. Let me define $A_k^\pm$ as the only local operator which coincides with $A_k^\pm$ acting on symmetric functions. The explicit form of $A_k^\pm$ can be obtained by expanding $A_k^{\pm}$ in power series of $P_{ij}$, commuting the $P_{ij}$’s in first position and then replacing $P_{ij}$ by unity. One finds in this way

\[ A_k^\pm = A_1^\pm = N \sqrt{\frac{\omega}{2}} X \mp \frac{\partial X}{\sqrt{2\omega}} \]  
\[ A_k^\mp = -\frac{H_{rel}}{\omega} + \omega r^2 + r \partial r + \frac{N + 1}{2}, \]  
\[ A_k^\pm = \sum_{i=1}^N \left( \sqrt{\frac{\omega}{2}} x_i \mp \frac{\partial x_i}{\sqrt{2\omega}} - \frac{1}{N} A_k^\pm \right)^3 \]  
\[ -3 \sum_{i,j \neq i} \nu \left( \frac{\omega}{2} x_i \mp \frac{\partial x_i}{\sqrt{2\omega}} - \frac{1}{N} A_k^\pm \right), \]

One can directly check that $A_1^+$ and $A_2^+$ respectively originate the center-of-mass excitations (33) and the radial excitations (2). By definition, the operator $A_k^\pm$ does not alter the exchange symmetries of the wavefunctions. Moreover, in a Hilbert space of symmetric functions, the commutation rule (28) can be immediately translated into

\[ [H, A_k^\pm] = \pm k \omega A_k^\pm \]  

Now, this relation involves only local operators so that it is also satisfied independently of the exchange symmetries of the wavefunctions. One concludes that the $A_k^\pm$’s are Raising and lowering operators for arbitrary exchange statistics. In fact, it remains to prove that the action of these operators on an eigenfunction having the correct behaviour at coinciding points gives a new eigenfunction having also this behaviour. Since the operators involve derivatives, it suffices to assume that the successive derivatives of $f$ and $g$ in (12) are all continuous at $x_i = x_j$. Every eigenfunction discussed in this paper confirm this hypothesis.

In conclusion, the ground states of the Calogero model are annihilated by the $A_k^\pm$’s and the excitations are obtained by the action of $\prod_{k=1}^N (A_k^\pm)^{n_k}$ on these ground states. The energy spectrum is linear in the quantum numbers $n_k$. According to the permutation theory, a complete eigenstate basis will be obtained if there are $N!$ independent ground states: one in the Bose representation, another in the Fermi representation, and $d$ ground states in each irreducible representation of degree $d$ of $S_N$.

### 3.4 Eigenstates and eigenvalues for bosons and fermions

The Bose eigenstates can be deduced from the Bose ones by using the mirror symmetry.

### 3.5 Eigenstates and eigenvalues at $\nu = 1/2$

At $\nu = 1/2$, the behaviour of the wavefunction when two particles coincide is dictated by (13) where the continuity of $h$ is no longer required. It is possible to construct an eigenstate $\psi_{p,n_k}$ equal to the Bose eigenstate $\psi_{n_k}$ in the sector $x_{p_1} > x_{p_2} > \ldots > x_{p_N}$ and sets to zero in the other sectors. The states which are non-zero on different sectors are obviously linearly independent. Hence one has the eigenstate basis

\[ \psi_{p,n_k} = \delta_{x_{p_1},x_{p_2},\ldots,x_{p_N}} \prod_{k=1}^N \left( A_k^{\pm} \right)^{n_k} \prod_{i,j < N} \sqrt{|x_{ij}|} e^{-\frac{1}{2} \omega \sum x_i^2} \]  

with the energy spectrum

\[ E_{p,n_k} = \sum_{k=1}^N k n_k \omega + \frac{N(N + 1)}{4} \omega. \]

The quantum numbers are the permutations $p$ and the non-negative integers $n_k$.

Note that the energies are completely degenerated with respect to particle exchanges because the action of the permutations on any eigenstate gives $N!$ independent eigenstates. This last property implies the completeness of the basis (13) in $L^2$ provided that the subspace of the bosonic eigenstates is complete.

Physically, at $\nu = 1/2$, the particle current always vanishes when two particles coincide, for all statistics. There is no exchange of particles and no change of the ordering of the particles as in classical mechanics (2). In this situation, the coupling strength $\nu(\nu - 1)$ is at its minimal value $-1/4$. Indeed, for a smaller strength, $\nu$ is a complex number, the energies are also complex and this indicates that the particle system collapses.

### 4 General solution for three particles

#### 4.1 System of coordinates

Introducing the following Jacobi coordinates,

\[ X = \frac{1}{3} (x_1 + x_2 + x_3), \]
\[ y = \frac{1}{\sqrt{2}} (x_1 - x_2), \]
\[ x = \frac{1}{\sqrt{6}} (x_1 + x_2 - 2x_3), \]

the relative coordinates $x, y$ are expressed as functions of the standard polar coordinates $r, \phi$ (2). The radial coordinate also satisfies the alternative definition $r^2 = \frac{1}{4} \sum_{i,j} x_i^2 x_j^2$. This system of coordinates is related to a discrete Fourier transformation of the coordinates on the line,

\[ r e^{i \phi} = -\sqrt{\frac{2}{3}} (\sqrt{3} x_1 + j x_2 + x_3) \]

where $j = e^{i2\pi/3}$ is a cube root of unity, so that its exchange properties are very simple. The cyclic exchange $(x_1, x_2, x_3) \to (x_2, x_3, x_1)$ is identical with $\phi \to \phi + \frac{2}{3} \pi$ and the interchange $x_1 \leftrightarrow x_2$ is identical with $\phi \to -\phi$. Note that these two permutations generate the whole symmetric group $S_3$. 

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4
One easily obtains the interparticle coordinates
\[
x_{12} = \sqrt{2} r \sin \phi, \quad x_{23} = \sqrt{2} r \sin (\phi + \frac{2\pi}{3}),
\]
\[
x_{31} = \sqrt{2} r \sin (\phi + \frac{4\pi}{3}).
\]
From these relations, one deduces that the value of \( \phi \) corresponds to the ordering of the three particles as follows,
\[
0 < \phi < \frac{\pi}{3} \iff x_1 > x_2 > x_3,
\]
\[
\frac{\pi}{3} < \phi < \frac{2\pi}{3} \iff x_1 > x_3 > x_2,
\]
\[
\frac{2\pi}{3} < \phi < \pi \iff x_3 > x_1 > x_2,
\]
\[
\pi < \phi < \frac{4\pi}{3} \iff x_3 > x_2 > x_1,
\]
\[
\frac{4\pi}{3} < \phi < \frac{5\pi}{3} \iff x_2 > x_3 > x_1,
\]
\[
\frac{5\pi}{3} < \phi < 2\pi \iff x_2 > x_1 > x_3,
\]
At a bound \( \phi = n\pi/3 \) with \( n \) integer, two particles coincide. On the other hand, note the useful identities
\[
\sin^2 \phi \sin (\phi + \frac{2\pi}{3}) \sin (\phi + \frac{4\pi}{3}) = -\frac{1}{4} \sin 3\phi
\]
\[
\sin^2 \phi \sin^2 (\phi + \frac{2\pi}{3}) + \sin^2 (\phi + \frac{4\pi}{3}) = \frac{9}{4} \sin^2 3\phi.
\]

### 4.2 Separation of variables

In the case of \( N = 3 \) particles, the wavefunction is completely separable as \( \psi = \Xi(X)R(r)\Phi(\phi) \) according to (19) and (20). Using (24), the angular eigenvalue equation \( \lambda \Phi = \lambda \Phi \) reads
\[
-\partial^2_\phi \Phi + \frac{9\nu(\nu - 1)}{\sin^2 3\phi} \Phi = \lambda \Phi.
\]
Since the particles are identical, the angular equation can be solved in the irreducible representations of \( S_3 \),
\[
S_3 = \text{sh} + 2\text{ph} + \text{gh}.
\]
The Bose and Fermi representations are of degree 1, and the two mixed representations are equivalent and each of degree 2. Let me choose to simultaneously diagonalize the hamiltonian and the cyclic permutations. The eigenvalues of a cyclic permutation are 1, \( j, j^2 \) because the cube of a cyclic permutation is merely the identity permutation.

The bosonic (fermionic) states are invariant under the cyclic exchanges and they are even (odd) under the interchange of two particles,
\[
\text{sh} : \Phi(\phi + \frac{2\pi}{3}) = \Phi(\phi) = \Phi(-\phi),
\]
\[
\text{ph} : \Phi(\phi + \frac{2\pi}{3}) = -\Phi(\phi) = -\Phi(-\phi),
\]
The remaining states belong to the mixed representations,
\[
\text{gh} : \Phi(\phi + \frac{2\pi}{3}) = r \Phi(\phi), \quad r = j, j^2.
\]
The states with \( r = j \) and the ones with \( r = j^2 \) are in correspondence under the complex conjugation, and also under the interchange of two particles (doing \( \Phi(\phi) \to \Phi(-\phi) \) and \( \phi \to -\phi - \frac{2\pi}{3} \)) in conformity with the degree 2 of the mixed representations. Their energy spectra are identical.

### 4.3 Angular quantization

The differential equation [3] may be transformed into the hypergeometric equation by the redefinition \( \Phi = |\sin 3\phi|^{\nu} \hat{\Phi} \) and the substitution \( z = \cos^2 3\phi \). It follows that the angular eigenvalue equation only has two linearly independent solutions,
\[
\Phi_1(\phi) = |\sin 3\phi|^{\nu} F\left(\frac{\mu + \nu}{2}, \frac{\mu - \nu}{2}; \frac{1}{2}; \cos^2 3\phi\right),
\]
\[
\Phi_2(\phi) = |\sin 3\phi|^{\nu} F\left(\frac{1 + \mu + \nu}{2}, \frac{1 - \mu - \nu}{2}; \frac{3}{2}; \cos^2 3\phi\right),
\]
where \( \mu = \frac{1}{3}\sqrt{\lambda} \). According to the theory of differential equations, only the singularities of the differential equation could be singularities of its solutions. These singularities occur at \( \phi = n\pi/3 \), when two particles coincide, and they are of algebraic type, namely
\[
\Phi_1(\phi) = \gamma_{11} |\sin 3\phi|^{\nu} F\left(\frac{\mu + \nu}{2}, \frac{\mu - \nu}{2}; \frac{1 + 2\nu}{2}; \sin^2 3\phi\right)
\]
\[
+ \gamma_{12} |\sin 3\phi|^{1-\nu} \mu F\left(\frac{1 - \mu - \nu}{2}, \frac{1 + \mu + \nu}{2}; \frac{3 - 2\nu}{2}; \sin^2 3\phi\right),
\]
\[
\Phi_2(\phi) = \gamma_{21} |\sin 3\phi|^{\nu} \text{sign} (\cos 3\phi)
\]
\[
\times F\left(\frac{\mu + \nu}{2}, \frac{\mu - \nu}{2}; \frac{1 + 2\nu}{2}; \sin^2 3\phi\right)
\]
\[
+ \gamma_{22} |\sin 3\phi|^{1-\nu} \text{sign} (\cos 3\phi)
\]
\[
\times F\left(\frac{1 - \mu - \nu}{2}, \frac{1 + \mu + \nu}{2}; \frac{3 - 2\nu}{2}; \sin^2 3\phi\right),
\]
with
\[
\gamma_{11} = \frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)} , \quad \gamma_{12} = \frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{3-\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu-1}{2}\right)},
\]
\[
\gamma_{21} = \frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)} , \quad \gamma_{22} = \frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{3-\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu-1}{2}\right)},
\]
where \( \Gamma \) is the gamma function.

At an ordinary point, the eigenfunction is a combination of these two independent solutions,
\[
\Phi(\phi) = \alpha_n \Phi_1(\phi) + \beta_n \Phi_2(\phi), \quad \phi \in [\frac{\pi}{3}, (n + 1)\frac{\pi}{3}].
\]
At a singular point \( \phi = n\pi/3 \) two particles coincide so that the asymptotic behaviour of the wave function must be
\[
\Phi(\phi) = |\sin 3\phi|^{\nu} f(\phi) + |\sin 3\phi| \sin 3\phi |^{1-\nu} g(\phi)
\]
with \( f \) and \( g \) continuous, in conformity with (23), (29) and (11). The general form of the eigenfunction [32] reproduces the algebraic singularities \( |\sin 3\phi|^{\nu} \) and \( |\sin 3\phi|^{1-\nu} \), and the functions \( f \) and \( g \) in factor are continuous at \( \phi = n\pi/3 \) when one has
\[
\left( \begin{array}{c}
\gamma_{11} \quad \gamma_{21} \\
\gamma_{12} \quad \gamma_{22}
\end{array} \right)
\left( \begin{array}{c}
(-)^{n} \gamma_{21} \\
(-)^{n} \gamma_{22}
\end{array} \right)
\left( \begin{array}{c}
\alpha_{n-1} \beta_{n-1} \\
\alpha_{n} \beta_{n}
\end{array} \right) = 
\left( \begin{array}{c}
\gamma_{11} \quad \gamma_{21} \\
\gamma_{12} \quad \gamma_{22}
\end{array} \right)
\left( \begin{array}{c}
(-)^{n} \gamma_{21} \\
(-)^{n} \gamma_{22}
\end{array} \right)
\left( \begin{array}{c}
\alpha_{n-1} \beta_{n-1} \\
\alpha_{n} \beta_{n}
\end{array} \right).
\]
Note that the functions \( f \) and \( g \) are then proportional to hypergeometric series: their successive derivatives are also continuous at \( \phi = n\pi/3 \). The recurrence relation (64) may be rewritten as

\[
\left( \frac{\alpha_n}{\beta_n} \right) = T_n \left( \frac{\alpha_{n-1}}{\beta_{n-1}} \right)
\]

with the transfer matrix

\[
T_n = \frac{1}{\gamma_{11}^n \gamma_{22} - \gamma_{12} \gamma_{21}} \times \left( \begin{array}{cc} \gamma_{11}^n \gamma_{22} + \gamma_{12} \gamma_{21} & 2(-)^n \gamma_{21} \gamma_{22} \\ -2(-)^n \gamma_{11} \gamma_{12} & -\gamma_{11} \gamma_{22} - \gamma_{12} \gamma_{21} \end{array} \right).
\]

For a given \((\alpha_0, \beta_0)\), this recurrence relation determines \((\alpha_1, \beta_1), \ldots, (\alpha_5, \beta_5)\) and thus the wave function is obtained on the period \([0, 2\pi]\). However, only the wave functions with period \(2\pi\) are physically acceptable, and such functions are only for quantized values of \(\lambda\).

Let me perform the quantization taking advantage of the classification into representations of \(S_3\). The periodicity requirement is then replaced by \(\Phi(\phi + \frac{2\pi}{3}) = r\Phi(\phi)\) with \(r = 1\) for bosons and fermions, and \(r = j, j^2\) for the mixed representation. This may be expressed as

\[
T_2T_1 \left( \frac{\alpha_0}{\beta_0} \right) = r \left( \frac{\alpha_0}{\beta_0} \right)
\]

and has a solution when \(\det(T_2T_1 - r) = 0\). This last equation is responsible for the quantization of \(\lambda\), it may be expanded as

\[
\det T_2T_1 - r \operatorname{tr} T_2T_1 + r^2 = 0.
\]

Now, one has \(\det T_n = -1\) and

\[
\operatorname{tr} T_2T_1 = 4 \left( \frac{\gamma_{11}^n \gamma_{22} + \gamma_{12} \gamma_{21}}{\gamma_{11} \gamma_{22} - \gamma_{12} \gamma_{21}} \right)^2 - 2.
\]

Applying the product formula for the gamma function,

\[
\gamma_{11} \gamma_{22} + \gamma_{12} \gamma_{21} = -1 \frac{\cos \mu \pi}{1 - 2\nu \cos \nu \pi},
\]

\[
\gamma_{11} \gamma_{22} - \gamma_{12} \gamma_{21} = -1 \frac{1}{1 - 2\nu}.
\]

Finally, the angular eigenvalue \(\lambda\) is discretized according to

\[
\cos^2 \mu \pi = \frac{(1 + r)^2}{4r} \cos^2 \nu \pi, \quad \sqrt{\lambda} = 3\mu \geq 0.
\]

The solution is immediate by now. Fig.1 displays the discrete values of \(\sqrt{\lambda}\) for the \(3!\) ground states. One clearly sees the mirror symmetry and, at \(\nu = 1/2\), the crossing associated with the degenerate solution (34).

The Bose representation is characterized by \(r = 1\) and \(\mu = \ell + \nu\) where \(\ell\) is a non-negative integer. One has then \(\gamma_{12} = 0\), \(\beta_n = 0\), \(\alpha_n = 0\) if \(\ell\) is even; \(\gamma_{22} = 0\), \(\alpha_n = 0\), \(\beta_n = 0\) if \(\ell\) is odd. In each case, the remaining hypergeometric function degenerates into a Gegenbauer polynomial \(C_\ell^\nu\), hence the Bose eigenstate basis

\[
\Phi_{\ell}(\phi) = |\sin 3\phi|^\nu C_\ell^\nu(\cos 3\phi), \quad \sqrt{\lambda_{\ell}} = 3\ell + 3\nu.
\]

At \(\nu = 1\), the free states of Fermi statistics are reproduced but with some phase shifts at \(\phi = n\pi/3\).

The Fermi representation is characterized by \(r = 1\) and \(\mu = \ell + 1 - \nu\) where \(\ell\) is a non-negative integer. One has then \(\gamma_{11} = 0\), \(\beta_n = 0\), \(\alpha_n = 0\) if \(\ell\) is even; \(\gamma_{22} = 0\), \(\alpha_n = 0\), \(\beta_n = 0\) if \(\ell\) is odd. One recovers the Fermi eigenstate basis obtained by Calogero,

\[
\Phi_{\ell}(\phi) = |\sin 3\phi|^\nu |\cos 3\phi| C_\ell^\nu(\cos 3\phi), \quad \sqrt{\lambda_{\ell}} = 3\ell + 3 - 3\nu.
\]

The fermi eigenstates correspond to the Bose ones under the mirror symmetry.

At least, the mixed representation involves the remaining two possibilities \(r = j\) and \(r = j^2\). The eigenvalues are solutions of the same equation \(\cos^2 \mu \pi = \frac{1}{4} \cos^2 3\nu \pi\) in the two cases so that they are two-fold degenerate. One obtains

\[
\sqrt{\lambda} = 3\ell + \frac{3}{2} \mp \frac{3}{\pi} \arcsin \left( \frac{\cos 3\nu \pi}{2} \right),
\]

where \(\ell\) is a non-negative integer. These eigenvalues reproduce the non-linear levels in Fig.1, the sign \(-\) is responsible for the increasing levels and the sign \(+\) is for the corresponding ones under the mirror symmetry. The eigenstates directly follow from (55) and (57).

### 4.4 On the Bethe ansatz

In the limit where the harmonic attraction vanishes, the radial eigenfunction (20) becomes a Bessel function \(J_{\sqrt{\lambda}}(kr)\) and the relative energy is then \(k^2/2\) where \(k\) is a positive real. In comparison with a system of free particles, the asymptotic behaviour of the eigenfunction

\[
J_{\sqrt{\lambda}}(kr) \sim \sqrt{k r} \cos \left( kr - \frac{\pi}{2} \sqrt{\lambda} - \frac{\pi}{4} \right)
\]

sets up a phase shift proportional to the \(\nu\)-dependence of \(\sqrt{\lambda}\). In the case of Bose and Fermi statistics, this phase shift can be viewed as a sum of 2-body phase shifts in accordance with the Bethe hypothesis (8). In the case of mixed statistics, however, it is not the case and the Bethe hypothesis (20) is no longer relevant.

### 5 A numerical evaluation for four bodies

Leaving out the center-of-mass coordinate, I use the Jacobi coordinates

\[
y = \frac{1}{\sqrt{2}}(x_1 - x_2), \quad x = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3), \quad z = \frac{1}{\sqrt{12}}(x_1 + x_2 + x_3 - 3x_4).
\]

The relative hamiltonian reads

\[
\tilde{H} = H_0 - \nu \left( \frac{\partial_y}{y} + \frac{\partial_x - \sqrt{2}\partial_z}{x - \sqrt{2}z} + \text{c.p.} \right)
\]
where c.p. stands for the cyclic permutations of \(x_1, x_2, x_3\). Expressing \(x, y, z\) in terms of the spherical coordinates \(r, \theta, \phi\), the wavefunction may be separated as \(\psi = \Xi(X) R(r) Y(\theta, \phi)\). The center-of-mass eigenstates are \(\Xi = \frac{1}{\sqrt{3}}\) and the radial eigenstates \(R = r^{\nu} \tilde{R}\) are \(2\nu\).

The first angular eigenstates may be evaluated numerically by direct diagonalization of the matrix of the angular part of the hamiltonian in a basis of spherical harmonics \(Y_{\ell m}\). For simplification, the analysis is restricted to the symmetric wavefunctions under the exchanges of the particle coordinates \(x_1, x_2, x_3\), that is, \(Y_{\ell m}\) have to be symmetrized under \(\phi \to \phi + \frac{2\pi}{3}\) and \(\phi \to -\phi\). In such a situation, there are only \(4! / 3! = 4\) ground states. The angular wave function is \(\tilde{R}(\theta, \phi)\), the one distinguishes the 4 ground states surviving in \(2\nu\) and some excitations. The Bose groundstate displays a linear energy of slope 6. The other energies are not linear but their slope is always 2 at \(\nu = 0\) and \(\nu = 1\).

6 Statistical mechanics in the thermodynamic limit

In this section, the harmonic well has to be understood as a long distance regulator alternative to the box \(\mathcal{R}\). For instance, expressing the \(n\)th cluster coefficient \(b_n = Z_n - Z_{n-1} - Z_1 + \ldots + 2 Z_3^n\) in terms of the \(N\)-body partition functions \(Z_N\), \(\nu \to 0\) limit can be identified to the infinite box thermodynamic limit if the divergent factor \(n^{-1/2}(\beta \omega)^{-1}\) surviving in \(b_n\) is identified to an additive factor, namely \(V \lambda^{-1}_T\) where \(V\) is the box volume and \(\lambda_T = \sqrt{2\pi/mk_B T}\) is the thermal wavelength.

In the case of Bose and Fermi statistics, the thermodynamic limit of the thermodynamical potential has been obtained by various means from the many-body spectrum in a harmonic well \(\mathcal{R}\). The result is in accordance with the thermodynamic Bethe ansatz and thus provides a confirmation of the Bethe hypothesis. Interesting enough, the particles obey Haldane’s generalization of the Pauli principle \(3\). Note that a correlation function has recently been analysed \(15\), \(19\).

In the case of Boltzmann statistics where no exchange symmetry is imposed to the wavefunction, I can calculate the cluster coefficients \(b_1 = Z_1, b_2 = Z_2 - 2 Z_1^2, b_3 = Z_3 - Z_2 Z_1 + \frac{1}{2} Z_1^3\) from the Boltzmann partition functions \(Z_N = \frac{1}{T} \text{tr} e^{-\beta H S}\). The thermodynamic limit gives \(b_1 = V \lambda^{-1}_T, b_2 = 0\) and the new result

\[
b_3 = \frac{V}{\lambda_T^2} \frac{1}{6 \sqrt{3}} \left( \frac{3}{\pi} \frac{\sin \nu \pi}{2} - \frac{1}{4} \right).
\]

This last result is rather different from that of a system of particles obeying Haldane statistics \(3\).

7 On the multi-species Calogero model

One considers the \(N\)-particle hamiltonian

\[
H = \sum_{i=1}^{N} \left( - \frac{1}{2} \partial_i^2 + \frac{1}{2} \omega^2 x_i^2 \right) + \sum_{i,j<k} \nu_{ij} (\nu_{ij} - 1) \frac{x_{ij}}{x_{ij}^2},
\]

(70)

where the masses have been eliminated by a suitable redefinition on the coordinates and on the parameters. The discussion displayed between \(11\) and \(21\) is immediately generalizable to the multi-species model if the change \(\nu \to \nu_{ij}\) is done. However, the wavefunction is no longer in an irreducible representation of \(S_N\). Moreover, the non-unitary transformation

\[
\psi = \prod_{i,j<k} [x_{ij}]^{\nu_{ij}} \tilde{\psi}
\]

(71)

introduces three-body terms,

\[
\tilde{H} = \sum_{i=1}^{N} \left( - \frac{1}{2} \partial_i^2 + \frac{1}{2} \omega^2 x_i^2 \right) - \sum_{i,j<k} \nu_{ij} (\partial_i - \partial_j) - \frac{1}{2} \sum_{i,j,k} \nu_{ij} \nu_{ik} \frac{x_{ij} x_{ik}}{x_{jk}^3},
\]

(72)

By symmetrizing under the cyclic exchanges of \(i, j, k\) and reducing to the same denominator the term of the sum, one verifies that the three-body potential vanishes when \(\nu_{ij} = \nu\).

The possibility to generalize the algebra of Raising and lowering operators to the multi-species Calogero model by the change \(\nu \to \nu_{ij}\) in \(22\) have been studied in \(21\). It appears that this new algebra of operators does not provided physical excitements except for center-of-mass and radial excitements. Nevertheless, every excitement provided is good at first perturbative order in \(\nu_{ij}\).

Let me exemplify the multi-species Calogero model by a numerical and perturbative analysis of the 3-body problem. Only the angular part of the eigenvalue problem have to be considered again,

\[
\Lambda = -\partial^2_\phi + \frac{\nu_{12}(\nu_{12} - 1)}{\sin^2 \phi} + \frac{\nu_{23}(\nu_{23} - 1)}{\sin^2(\phi + \frac{2\pi}{3})} + \frac{\nu_{31}(\nu_{31} - 1)}{\sin^2(\phi + \frac{4\pi}{3})},
\]

(73)

The non-unitary transformation

\[
\Phi(\phi) = \left| \sin(\phi) \right|^{\nu_{12}} \left[ \sin(\phi + \frac{2\pi}{3}) \right]^{\nu_{23}} \left[ \sin(\phi + \frac{4\pi}{3}) \right]^{\nu_{31}} \tilde{\Phi}(\phi)
\]

(74)

gives an operator

\[
\tilde{\Lambda} = -\partial^2_\phi + (\nu_{12} + \nu_{23} + \nu_{31})^2 - 2 \nu_{12} \cot \phi \partial_\phi - 2 \nu_{23} \cot(\phi + \frac{2\pi}{3}) \partial_\phi
\]

\[
-2 \nu_{31} \cot(\phi + \frac{4\pi}{3}) \partial_\phi
\]

(75)

the matrix elements of which are well defined with the principal value regularization. Indeed, the angular wave function is
periodic of period $2\pi$ and thus it may be expanded in Fourier series,

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} c_l e^{il\phi},$$

(76)

and the matrix elements of $\tilde{A}$ are expressed in the Fourier space as simple Cauchy integrals. The final result reads

$$\langle \ell | \tilde{A} | \ell' \rangle = \ell^2 \delta_{\ell=\ell'} + (\nu_{12} + \nu_{23} + \nu_{31})^2 \delta_{\ell=\ell'} + 2\ell' \text{sign}(\ell - \ell) \delta_{\ell=\ell'_{\text{even}}} \times \left( \nu_{12} + \nu_{23} \right)^{l-\ell'} + \nu_{31}^{2l-2\ell'} + \text{fact}(\ell - \ell') \delta_{\ell=\ell'_{\text{even}}} \times \left( \nu_{23} \nu_{31} + \nu_{31} \nu_{12} \right)^{l-\ell'} + \nu_{12} \nu_{23}^{2l-2\ell'}$$

(77)

with $\text{sign}(0) = 0$ and $\text{fact}(l - \ell') = 0, -2, 2$ respectively if $|\ell - \ell'| = 0, 2, 4$ modulo 6. Diagonalizing a $361 \times 361$ matrix, I have obtained a numerical evaluation of the first eigenvalues $\lambda$. Note that the energy spectrum is built from the square root of $\lambda$, as it should.

In the case $\nu_{ij} = \nu$, the numerical results reproduce the exact spectrum of Fig.1 as it should.

In the case $\nu_{12} = 0$ and $\nu_{23} = \nu_{31} = \nu$, the first numerical values of $\sqrt{\lambda}$ are displayed in Fig.3. The spectrum is nearly linear in the quantum number $\ell$. However, the computation at second perturbative order around $\nu = 0$ bears out the presence of some irregularities. Since the problem is invariant under $x_1 \leftrightarrow x_2$, the non-degenerate perturbative theory may separately be used for the unperturbed bases $\{\cos(\ell \phi)\}$ and $\{\sin(\ell \phi)\}$. For instance, starting from the unperturbed eigenstate $\cos(\ell \phi)$ with $\ell = 1$ modulo 3, one finds

$$\sqrt{\lambda} = \ell - \nu + \left( -\frac{\pi}{2\sqrt{3}} - \frac{1}{\ell} \right) \nu^2 + O(\nu^3)$$

(78)

Since these energies are not linear in the quantum number, there is no algebra of Raising and lowering operators in the multi-species Calogero model.

8 Conclusion

The nature of the wavefunction of the Calogero model has been discussed at length.

As regards the case of Boltzmann statistics, we have revisited the Raising and lowering operators and we have obtained some new exact eigenfunctions. One would like to derive the complete solution of this many-body problem, if possible analytically.

In the multi-species case, we have shown the impossibility of an algebra of Raising and lowering operators. However, the question of the integrability of this model remains open.

At least, it would be desirable to have a better understanding of the commutation rules between the Raising and lowering operators of the one-species model.

References

[1] J.M. Leinaas and J. Myrheim, Int. J. Mod. Phys. A 8, 3649 (1993)
[2] A.P. Polychronakos, Nucl. Phys. B 324, 597 (1989)
[3] S.B. Isakov, Int. J. Mod. Phys. A 9, 2563 (1994).
[4] B. Sutherland, J. Math. Phys. 12, 251 (1971).
[5] C. Marchioro, J. Math. Phys. 11, 2193 (1970).
[6] F. Calogero, Lett. Nuovo Cim. 13, 411 (1975).
[7] F. Calogero, J. Math. Phys. 10, 2191 (1969); 12, 419 (1971).
[8] P.J. Gambardella, J. Math. Phys. 16, 1172 (1974)
[9] A.P. Polychronakos, Phys. Rev. Lett. 69, 703 (1992);
[10] L. Brink, T.H. Hansson and M.A. Vasiliev, Phys. Lett. B 286, 109 (1992).
[11] S.B. Isakov and J.M. Leinaas, preprint hep-th/9510184.
[12] C. Furtlehner and S. Ouvry, Phys. Lett. B 9, 503 (1995);
[13] D. Sen, preprint hep-th (1996).
[14] Z.X. Wang and D.R. Guo, Special Functions, World Scientific 1989.
[15] A. Lerda, Anyons, Springer-Verlag 1992.
[16] A. Dasmíères de Veigy and S. Ouvry, Phys. Rev. Lett. 75, 352 (1995).
[17] Huang, Statistical Mechanics, Wiley 1987.
[18] Z.N.C. Ha, Phys. Rev. Lett. 73, 1574 (1994).
[19] P.J. Forrester, Nucl. Phys. B 416, 377 (1994);
[20] C.N. Yang, Phys. Rev. Lett. 19, 1312 (1967)
[21] A. Dasmíères de Veigy and S. Ouvry, Mod. Phys. Lett. A 10, 1 (1995);
[22] C. Marchioro and E. Presutti, Meccanica 7, 23 (1972);
[23] M.V.N. Murthy and R. Shankar, Phys. Rev. Lett. 73, 3331 (1994).