Determining elastic anisotropy of textured polycrystals using resonant ultrasound spectroscopy

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ABSTRACT

Polycrystalline materials can have complex anisotropic properties depending on their crystallographic texture and crystal structure. In this study, we use resonant ultrasound spectroscopy (RUS) to nondestructively quantify the elastic anisotropy in extruded aluminum alloy 1100-O, an inherently low-anisotropy material. Further, we show that RUS can be used to indirectly provide a description of the material’s texture, which in the present case is found to be transversely isotropic. By determining the entire elastic tensor, we can identify the level and orientation of the anisotropy originated during extrusion. The relative anisotropy of the compressive ($c_{11}/c_{33}$) and shear ($c_{44}/c_{66}$) elastic constants is 1.5% ± 0.5% and 5.7% ± 0.5%, respectively, where the elastic constants (five independent elastic constants for transversely isotropic) are those associated with the extrusion axis that defines the symmetry of the texture. These results indicate that the texture is expected to have transversely isotropic symmetry. This finding is confirmed by two additional approaches. First, we confirm elastic constants and the degree of elastic anisotropy by direct sound velocity measurements using ultrasonic pulse echo. Second, neutron diffraction (ND) data confirm the symmetry of the bulk texture consistent with extrusion-induced anisotropy, and polycrystal elasticity simulations using the elastic self-consistent model with input from ND textures and aluminum single-crystal elastic constants render similar levels of polycrystal elastic anisotropy to those measured by RUS. We demonstrate the ability of RUS to detect texture-induced anisotropy in inherently low-anisotropy materials. Therefore, as many other common materials have intrinsically higher elastic anisotropy, this technique should be applicable for similar levels of texture, providing an efficient general diagnostic and characterization tool.
Introduction

Most polycrystalline materials show some degree of anisotropy. In general, anisotropy may arise in these materials from the crystal symmetry of the material compounded with a distribution of crystallographic orientations of the single-crystal grains in the aggregate (i.e., texture), and/or by an inhomogeneous distribution of secondary phases [1]. Defects generated by thermomechanical processes can also produce elastic anisotropy if they exhibit a preferential alignment or orientation [2]. Anisotropic properties have profound influence on many properties, such as a material’s response to mechanical or magnetic stimuli, and can be exploited to design materials with directionally preferential properties [3–5].

Crystallographic texture can result from a variety of common manufacturing and forming processes, such as extruding, swaging, rolling and cold working [6–12]. Texture can be characterized by several distinct methods, including X-ray diffraction (XRD), electron backscatter diffraction (EBSD) and neutron scattering [4, 13–18]. However, these methods are either limited to characterizing texture on the surface of the material, or require large user facilities with competitive access.

Alternatively, acoustic characterization techniques offer the possibility of probing the bulk of a material as mechanical waves travel through it, as is the case for the pulse-echo ultrasonic technique [19]. In particular, resonant ultrasound spectroscopy (RUS) is a fast, high-resolution, nondestructive characterization technique whereby the mechanical resonance frequencies of a specimen can be measured and used to determine the elastic properties of a material. RUS, nonlinear RUS and pulse-echo characterization constitute a powerful arsenal of nondestructive characterization techniques [20, 21]. The mechanical resonance frequencies of a material are set by the sample’s geometry, mass, internal symmetry, and elastic constants. Using these parameters, one can obtain the complete elastic tensor by solving the inverse problem [22]. This can be accomplished by using a nonlinear iterative Levenberg–Marquardt algorithm, which minimizes the deviation between the measured and predicted values of resonance frequencies based on the material’s elastic constants [23–27]. Historically, there have been attempts to use RUS to quantify the elastic anisotropy induced by texture. However, the symmetry selection process and the results obtained were not fully validated against texture measurements from other methods [28, 29]. Foster et al. [30] used RUS to determine the elastic anisotropy of rolled Cu–Zn (brass) alloys in combination with analytical expressions derived from polycrystal models based on simple averaging procedures to obtain a relatively compact representation of the sample’s texture in terms of principal orientation distribution coefficients for comparison with texture measurements by neutron diffraction (ND). A related approach, but using new RUS and ND measurements on Al polycrystalline samples and more sophisticated modeling tools, is used in this work. More recently, Lan et al. [28, 29] proposed a new technique to directly measure the texture of a material using acoustic waves and contrasted this technique against resonant ultrasound spectroscopy (RUS) and neutron diffraction. Similarly, the effect of density of defects has been studied by multiple ultrasonic and microstructural characterization methods in aluminum and other material [20, 21].

The objective of the present work is to assess the potential for RUS to identify the symmetry and orientation of the anisotropy of a polycrystalline aluminum sample, as well as to quantify the degree of anisotropy. The focus of the present work is placed on indirectly quantifying the symmetry of the crystal orientation distribution rather than providing a full texture representation. Aluminum is an inherently low-anisotropy material and was selected with the intention of testing the lower limits of detection possible using RUS. Application of the RUS method on extruded aluminum samples clearly suggests that the material texture is transversely isotropic.

Further, we validate RUS results with direct measurements using ultrasonic pulse-echo technique, as well as crystallographic texture measurements by neutron diffraction combined with elastic self-consistent (ELSC) micromechanical simulations that give the polycrystal elastic constants as a function of the known single-crystal elastic constants and the texture of the sample. The ultrasonic pulse-echo technique (PE) is another method that can be used to determine the elastic properties of a material. In PE, the sound velocity of the material is measured along different orientations [31–33]. Because the sound velocity is extracted directly from the time-of-flight, PE results are not subject to fitting or convergence. For lower-symmetry materials, however, PE cannot determine
all of the elastic constants in a single measurement. Independent measurements in different orientations with respect to the anisotropy of the sample are required [34, 35]. The consequent sample manipulation may introduce variations in delicate samples, or result in complications when characterizing hazardous samples. These ultrasonic experiments are performed in the anelastic limit, i.e., the response is elastic (linear) and non-hysteretic. The level of losses is minimal but measurable, allowing RUS and PE the capability to extract changes in sound attenuation (internal friction). As a result, the measurements are nondestructive and samples are unchanged. PE measurements are found to be in very good agreement with those measured by RUS, measured by neutron diffraction, shows very good, but nonperfect, agreement with those measured by RUS, characterization. Finally, the ELSC measurements are found to be in very good agreement with RUS characterization. Alternatively, the Hashin and Shtrikman method which is based on a variational approach can also provide bounds [38, 39]. Finally, the elastic self-consistent (ELSC) model, originally proposed by Hershey [40] for aforementioned case of polycrystalline aggregates of randomly oriented single crystals with cubic symmetry, can provide a unique estimate of the homogenized elastic response of the medium [40]. These models can be generalized to aggregates of arbitrary single-crystal symmetry and non-uniform texture, in which case the polycrystal elastic constants will depend of the single-crystal anisotropy and the orientation distribution of the grains. The recent advances obtained in mathematical methods used to extract the crystallographic texture from ultrasonic response, even for hexagonal symmetries, make determining the elastic anisotropy even more relevant [41].

A measure of intrinsic anisotropy in cubic materials is given by Zener’s elastic anisotropy factor, $A$, in Eq. 4 [42]. Almost all crystalline materials have $A \neq 1$; when those materials form polycrystalline aggregates, texture means that their physical properties are orientation-dependent. Aluminum has a cubic crystal structure with a relatively small Zener factor of $A = 1.22$ [43]. Aluminum is notably less anisotropic than most other cubic metals, such as copper ($A = 3.21$) [44] and 304 stainless steel (SS) ($A = 3.77$) [45], for example.

The effective values of $c_{11}$ and $c_{44}$ for a polycrystalline material depend on the single-crystalline counterparts. There are different approximations to calculate this relationship, the most common of which are the Voigt [36, 37] and Reuss [36, 37] averages obtained by assuming homogenous strain or stress, respectively, which, respectively, provide the upper and lower bounds for the polycrystal’s elastic constants estimates. Alternatively, the Hashin and Shtrikman method which is based on a variational approach can also provide bounds [38, 39]. Finally, the elastic self-consistent (ELSC) model, originally proposed by Hershey [40] for aforementioned case of polycrystalline aggregates of randomly oriented single crystals with cubic symmetry, can provide a unique estimate of the homogenized elastic response of the medium [40]. These models can be generalized to aggregates of arbitrary single-crystal symmetry and non-uniform texture, in which case the polycrystal elastic constants will depend of the single-crystal anisotropy and the orientation distribution of the grains. The recent advances obtained in mathematical methods used to extract the crystallographic texture from ultrasonic response, even for hexagonal symmetries, make determining the elastic anisotropy even more relevant [41].

For polycrystalline materials with no texture, each grain is randomly oriented such that the bulk elastic properties are equal in all directions. In this isotropic case, the stiffness tensor only has two independent constants, i.e., two degrees of freedom (DoF), and reduces to Eq. 3 where $c_{12} = c_{11} - 2c_{44}$. For materials with cubic symmetry, the stiffness tensor has three independent elastic constants, whereby $c_{12}$ is independent of $c_{44}$ and $c_{11}$.

$$
C_{ij} = \begin{bmatrix}
c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\
c_{11} & c_{11} & c_{12} & 0 & 0 & 0 \\
c_{11} & c_{11} & c_{11} & 0 & 0 & 0 \\
\text{sym} & c_{44} & c_{44} & 0 & 0 & c_{44}
\end{bmatrix}
$$

(3)

Elastic response of a solid, anisotropy and relationship to wave velocities

The effective elastic response of a solid, as described by Hooke’s law in Eq. 1, linearly relates the Cauchy stress, $\sigma_{ij}$, to the elastic strain, $e_{kl}$, by the elastic stiffness tensor $c_{ijkl}$. In the most general anisotropic case, the stiffness tensor has 21 independent constants due to the symmetries associated with stresses and strains and the uniqueness of strain potential energy. Alternatively, Hooke’s law can be written in matrix form using Voigt notation in Eq. 2:

$$
\begin{align*}
\sigma_{ij} &= c_{ijkl}\epsilon_{kl} \\
c_{ijkl} &= c_{jikl} = c_{ijlk} = c_{klij} \\
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{bmatrix}
&= \begin{bmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\
c_{33} & c_{34} & c_{35} & c_{36} \\
c_{44} & c_{45} & c_{46} \\
c_{55} & c_{56} \\
c_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\epsilon_{44} \\
\epsilon_{55} \\
\epsilon_{66}
\end{bmatrix}
\end{align*}
$$

(2)
For isotropic materials \( A = 1 \), the Young’s modulus \( E \) (stiffness under uniaxial tension), bulk modulus \( B \) (resistance to volumetric changes), shear modulus \( \mu \) (stiffness under shear deformation) and Poisson ratio \( v \) (ratio of transversal strain to axial strain), also referred to as the engineering constants, are given by the relations in Eqs. 5, 6, 7, and 8.

\[
E = \frac{c_{11}^2 + c_{11}c_{12} - 2c_{12}^2}{c_{11} + c_{12}} \tag{5}
\]

\[
B = \frac{c_{11} + 2c_{12}}{3} \tag{6}
\]

\[
\mu = c_{44} \tag{7}
\]

\[
v = \frac{c_{12}}{c_{11} + c_{12}} \tag{8}
\]

If a manufacturing or forming process produces texture with axial anisotropy, as is the case for extrusion, the material will have transversely isotropic symmetry [46–50]. In the transversely isotropic case, the stiffness tensor has five independent constants (i.e., the same number than in a hexagonal symmetry), as shown in Eq. 9 and the orientation has been chosen such that the “z” axis is parallel to the texture axis (parallel to the rod’s extrusion direction). As a result of the transversely isotropic texture, the values of \( E, B, \mu \) and \( v \) depend on orientation. The engineering constants for a transversely isotropic solid parallel (||) and perpendicular (\( \perp \)) to the axis of symmetry are given in Eqs. 10, 11, 12 and 13 [51–53].

\[
c_{ij} = \begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
\text{sym} & \text{sym} & \text{sym} & 1 & 0 & 0 \\
c_{44} & 0 & 0 & 0 & 0 & c_{44}
\end{bmatrix} \tag{9}
\]

\[
E_{\perp} = \frac{c_{11}^2c_{33} + 2c_{13}^2c_{12} - 2c_{12}^3c_{11} - c_{12}^2c_{33}}{c_{11}c_{33} - c_{13}^2} \tag{10}
\]

\[
E_{||} = \frac{c_{11}c_{33} + c_{12}c_{33} - 2c_{13}^2}{c_{11} + c_{12}} \tag{11}
\]

\[
B = \frac{2c_{11} + c_{33} + 2(2c_{13} + c_{12})}{9} \tag{12}
\]

The relationships between the phase velocities of ultrasonic plane waves and the elastic constants of a material can be determined as shown in detail in the Appendix. For isotropic materials (two independent elastic constants), \( c_{44} = \rho v_T^2 \) and \( c_{11} = \rho v_L^2 \), where \( \rho \) is the density, \( v_T \) is the transverse wave velocity, and \( v_L \) is the longitudinal wave velocity. For materials with lower symmetries, such as the transversely isotropic case (five independent elastic constants), determining the relationships between sound velocities and elastic constants is far more complicated, as shown in the Appendix.

**Experimental and modeling methods**

**Sample preparation**

All of the textured samples were machined from the same extruded rod of the commercial aluminum alloy (AA) 1100. AA-1100, also known as commercially pure aluminum, is composed of > 99% aluminum with silicon and iron as its primary impurities. The AA-1100 rod was provided in the annealed condition with no cold working (type O temper) in accordance with ASTM B221-14 standards. The AA-1100-O rod is assumed to have transversely isotropic texture as a result of the extrusion process during manufacturing [6–8]. An illustration of the transversely isotropic AA-1100-O rod is shown in Fig. 1b from whence rectangular parallelepiped resonators (RPRs) were machined with sides \( a, b, \) or \( c \) parallel to the rod axis. RPRs are advantageous for RUS experiments in comparison with other geometries (cylinders, cubes or spheres) because they have no degenerate resonance modes when the dimensions are correctly chosen [22, 27]. The flat parallel faces in principal directions also make RPRs suitable for pulse-echo experiments. Several RPRs were
machined from this rod with dimensions shown in Fig. 1b, where $a = 8.4 \text{ mm}$, $b = 9.2 \text{ mm}$ and $c = 13.5 \text{ mm}$ with a precision of 0.0006 mm. To simplify distinguishing the texture orientation of the samples machined, we define $\hat{z}$ as the direction parallel to the axis of extrusion. All samples were machined by electrical discharge machining (EDM) using skim cut settings in order to minimize surface roughness and damage caused by machining [54–56].

The AA-1100-O RPRs were then heat-treated at 615 K for 1 h in accordance with ASM standards in order to anneal defects present in the sample [57]. The AA-1100-O RPRs have a density of $2.6899 \pm 0.0034 \text{ g cm}^{-3}$, calculated from measuring geometrical volume and mass using a high-precision micrometer and scale. Thus, these samples are 99.26% dense, assuming a theoretical density of $2.71 \text{ g cm}^{-3}$.

Resonant ultrasound spectroscopy

The RUS system was constructed at Los Alamos National Laboratory, the general description of which is described elsewhere [24, 58]. In Fig. 1a, c, we can observe the experimental setup used. The RUS system interfaces with the computer via a field-programmable gate array-based Red Pitaya detector system, which is programmed to generate the excitation signal to one transducer as well as record the response signal from the other transducer. The RUS software was developed at Los Alamos National Laboratory and is freely available online [58].

The driving and detecting transducers are identical and are composed of an aluminum housing filled with epoxy, a coaxial SMA connector on the one end, and a 5-mm-diameter lead zirconate titanate (PZT) piezoelectric transducer glued to the flat portion of an alumina hemisphere as described elsewhere and shown in Fig. 1a, c [59–61]. The alumina hemispherical cap not only provides electrical insulation for the piezoelectrics when measuring metallic samples, but also protects the piezoelectrics from physical damage while simultaneously guaranteeing excellent point contact with the sample. This allows the sample to resonate freely without having to balance it precariously by its corners. It also allows to reproduce the position of the sample with respect to the transducers. The resonance frequencies of the specimens are independent of the positioning or orientation of the sample between the driving and detecting transducers; in practice, however, some resonances may have undetectably low amplitudes which depend on positioning (e.g., if the detecting transducer is in contact with the sample which coincides with the node of a particular resonance). We found that placing the sample vertically between the transducers (i.e., parallel to the $c$ dimension) allowed the
detection of at least 30 of the first 35 resonances. This sample arrangement was used for all the samples in order to reduce the small variations associated with measuring resonances in different positions on the sample. This arrangement, combined with the optimized sample geometry, yields easily detectable/distinguishable resonances with large contributions from shear, dilatational and compressive elastic constants. In turn, the extracted resonance frequencies result in an overdetermined problem for the extraction of all the elastic constants from the measured resonance frequencies. The quality factors (Q-values, the frequency-to-bandwidth ratio, defined by the resonance frequency divided by its full-width at half-maximum) of the resonances vary, but most resonances were measured with Q-values exceeding 20,000 as shown in Fig. 2.

The RUS inverse problem fitting procedure was executed using 12-order polynomials for all fits, although no significant differences were obtained using 14-order polynomials (deviations smaller than 0.02%). The results obtained in the RUS fits are insensitive to the initial input elastic constant values. We used more than 30 of the first 35 resonance frequencies in the fitting procedure for each sample [24]. The fits assume a particular symmetry, set by the degrees of freedom and orientation selected in the fit [22, 24]. The ability to extract the elastic constants of a polycrystalline material requires matching the symmetry of the fit with that of the sample. Extruded aluminum has transversely isotropic texture, which results in hexagonal symmetry (in the single-crystal sense). In order to distinguish physical orientations of the sample a, b and c (as described in the previous section and illustrated in Fig. 1b with the fit orientations involved in the RUS algorithm, we define the orientation of a hexagonal axis of symmetry in the fit to be \( h \). The fit should only be successful if its orientation matches the physical orientation of the sample’s anisotropy (i.e., \( h \parallel \hat{z} \) for transverse isotropy). Similarly, as shown later in the paper, the symmetry and orientation can be determined by solving the inverse problem with a lower symmetry.

**Ultrasonic pulse-echo**

The conventional pulse-echo time-of-flight technique was used to determine the speed of compressive and shear ultrasonic waves along all principal axes of the samples in order to directly obtain elastic constants for validation purposes [34, 35]. The experimental setup consisted of an arbitrary function generator and digitizing oscilloscope connected to an ultrasonic transducer, fused silica buffer rod and sample as described elsewhere [62, 63]. The electrical signal applied to the transducer was a tone burst with a Tukey envelope applied to mitigate spurious high-frequency effects [62, 64]. For improved accuracy in the determined time-of-flight, measurements were performed over a range of frequencies, typically \(*\sim 10\) MHz and \(*\sim 20\) MHz for shear and compressive modes, respectively. Measurement over a range of frequencies enables a frequency-dependent correction to be applied to account for the medium used to couple the buffer rod to the sample as previously described [62]. The accuracy of measured time-of-flight (\( \tau \)) and physical dimensions (\( d \)) and thus the sound speeds (\( v = 2d/\tau \)) were of order \( 10^{-4} - 10^{-3} \).

Because the same samples were used for the RUS and the PE experiments, and the transducer must be adhered to a flat surface in the PE experiments, PE experiments could only be performed for \( \theta = 0^\circ \) and \( \theta = 90^\circ \) relative to \( \hat{z} \) (see Appendix). With this orientation constraint and using Eq. 21, the PE experiments were used to determine \( c_{11}, c_{33}, c_{44} \) and \( c_{66} \), while \( c_{13} \) could not be measured since its only contribution to the sound velocity comes from the \( \sin^{2}\theta \) component of Eq. 20 (see the Appendix). In order to measure \( c_{13} \) in rectangular parallelepiped samples machined from this textured rod, the faces of the
sample need to be oriented, for example, at 45° with respect to \( \hat{z} \).

**Neutron diffraction**

Aluminum is an inherently low-anisotropy material \((A = 1.22)\). As such, the degree of texture required (measured in multiple of random density (MRD)) in order to result in measurable changes in its elastic properties is expected to be large. Bulk extrusion-produced texture was measured using time-of-flight neutron diffraction on the high-pressure preferred orientation time-of-flight diffractometer (HIPPO) \([65, 66]\) at the pulsed spallation neutron source at the Los Alamos Neutron Science Center (LANSCE) of Los Alamos National Laboratory \([67]\). HIPPO utilizes 1200 \(^3\)He detector tubes arranged on 45 detector panels on five rings with nominal diffraction angles of 40°, 60°, 90°, 120° and 140°, covering 22.4% of the 4\(\pi\) steradians around the sample \([68]\). The RPRs were mounted on cadmium-wrapped sample holders as described elsewhere \([65]\). The mounted samples were loaded on the HIPPO sample changer robot \([69]\), and neutron diffraction data were collected at rotation angles of 0°, 67.5° and 90° for 10 min per rotation, covering 51.7% of the 4\(\pi\) steradians \([68]\). The measured diffraction patterns were analyzed using the Materials Analysis Using Diffraction (MAUD) software \([70]\) via simultaneous Rietveld refinement \([71]\) of 132 neutron time-of-flight diffraction patterns following procedures described elsewhere \([72]\). An E-WIMV representation of 7.5° resolution was used to represent the orientation distribution function (ODF) in MAUD. No specific texture symmetry was assumed in the refinement. Lastly, in order to generate input for ELSC calculations of polycrystal elastic constants, the ODF was integrated in small, regularly spaced (every 10°) volumes of orientation space \((\phi_1, \cos \Phi, \phi_2)\), where \((\phi_1, \Phi, \phi_2)\) are the Euler angles (Bunge convention), resulting in a set of 23,328 discrete orientations centered inside each volume, determining a 72x18x18 partition of \(\phi_1, \cos \Phi, \phi_2\) in the intervals \([0, 2\pi],[0, 1]\) and \([0, \pi/2]\), respectively, with associated weights corresponding to the ODF integrals inside each volume.

**Elastic self-consistent model**

Self-consistent (SC) homogenization methods are used to estimate the mechanical response behavior of polycrystals, based on the knowledge of the properties, morphology and orientation distribution of the constituent single-crystal grains. The ELSC method was proposed independently by Hershey \([37]\) and generalized by Hill \([73]\). Here, we use the most recent ELSC numerical implementation by Tomé and Lebensohn \([74]\). In the context of the SC theory, the polycrystal is represented by a set of weighted crystal orientations. The orientations represent grains, and the weights represent volume fractions. The set of orientations and weights are chosen to reproduce the crystallographic texture of the material. In this regard, the ELSC model utilizes the whole texture information contained in the full ODF measured by ND, instead of reduced representations, e.g., orientation distribution coefficients used by Foster et al. \([30]\) for cubic materials, or texture coefficients for lower-symmetry crystals (e.g., \([75]\) for hexagonal materials).

The ELSC model is a mean-field approximation based on 1-point statistics, in which each grain is treated as an ellipsoidal elastic inhomogeneity embedded in an effective homogenized elastic medium. The ellipsoidal shape represents the average morphology of the grains. The inhomogeneity character derives from the difference in elastic properties of each individual grain and the effective medium. The inhomogeneity and the medium will generally have fully anisotropic properties, deriving from the intrinsic single-crystal anisotropy, crystallographic texture and/or non-equiaxed morphology of the grains. The effective medium represents the average environment surrounding each grain. Using the equivalent Eshelby’s inclusion method \([76]\) to solve each heterogeneity problem, the ELSC estimate for the polycrystal’s elastic stiffness (in tensorial notation) is given by the expression:

\[
\epsilon_{ijkl} = \left\langle c_{ijpq}^S A_{pqkl}^S \right\rangle
\]

where \(\epsilon_{ijkl}\) and \(c_{ijpq}^S\) are the elastic stiffness tensors of the polycrystal and each individual grain, the symbol \(< . >\) indicates weighted average over the set of grains representing the aggregate, and \(A_{pqkl}^S\) is the localization tensor associated with each grain \((g)\), which relates the strain applied to the polycrystal and the local strain in the grains, i.e., \(\epsilon_{ijkl}^g = A_{pqkl}^S \epsilon_{pqkl}\) and is defined by:
\[ A_{ijkl}^{c} = (c^2 + \tilde{c}_{ijkl}^{-1})(c + \tilde{c})_{ijkl} \]  

(15)

where \( \tilde{c}_{ijkl} \) is the interaction tensor that gives the local deviations in stress in terms of the local deviations in stresses, i.e., \( \sigma_{ij} - \sigma_{ij} = \tilde{c}_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}) \), and is given by:

\[ \tilde{c}_{ijkl} = c_{ijmn}(I - S)_{mnpq} S_{pqkl}^{-1} \]  

(16)

where \( S_{ijkl} \) is the Eshelby tensor [77], a function of \( c_{ijkl} \) and the shape of the ellipsoid representing the morphology of the grains [76]. Given these dependencies, Eq. (14) is a fix-point equation that allows obtaining \( c_{ijkl} \) iteratively.

Note that Eq. (14) involves two weighted averages of the elastic moduli of each grain to give the polycrystal’s stiffness tensor. The first set of weights are the (tensorial) factors given by the localization tensors (Eq. 15), which depend on the relative stiffness and interaction between each grain and the effective medium representing the polycrystal. The second set of weights are the normalized values of the ODF integrals inside each of 23,328 volumes in orientation space, which provide the dependence with the polycrystal’s texture.

**Results and discussion**

In order to determine the errors associated with an isotropic fit (2 DoF) of an RUS spectrum for an elastically isotropic sample, a 99.95% pure polycrystalline tungsten RPR was measured using RUS and the elastic constants were determined using both isotropic (2 DoF) and orthorhombic (9 DoF) fits.

Tungsten was chosen because its elastic properties are inherently isotropic (\( A = 1.004 \) for single-crystal tungsten) [78]. This guarantees isotropic elastic properties irrespective of the polycrystal’s texture, because the elastic contributions of each grain are independently isotropic. RPRs of tungsten were machined using skim EDM with dimensions of 8.4 ± 0.0012 mm \( \times \) 9.2 ± 0.0015 mm \( \times \) 13.5 ± 0.001 mm resulting in a density of 19.135 ± 0.006 g cm\(^{-3}\), 99.2% that of the 19.3 g cm\(^{-3}\) nominal density [56].

The elastic constants determined by these two fitting methodologies, shown in Fig. 3, are identical (within error \( \sim 0.05\% \)). These results show that a root-mean-square (RMS) error of 0.1% or lower can be achieved in RUS measurements if the sample geometry and microstructure are representative of the fit used, i.e., sharp 90° corners and flat faces for RPRs, and a microstructure that is homogeneous with a correctly modeled symmetry [24]. The slight deviations in the 9-DoF elastic constants, as well as the slightly smaller errors for the 9-DoF fit in comparison with the 2-DoF fit, are attributed to the availability of more free parameters in the 9-DoF fit minimization scheme. Given that the 9-DoF fit is in such close agreement with the isotropic fit, this demonstrates that the isotropic assumption is correct for this material.

Figure 4 displays the results of fitting the RUS spectrum obtained from an AA-1100-O RPR using an isotropic fit (2 DoF) and a hexagonal symmetry (5 DoF) for different orientations. For 2-DoF, the RMS error is 0.85%. This error is nearly ten times higher than for the isotropic fit of the tungsten RPR. The increase in the RMS for the 2-DoF fit of the AA-1100 is not due to lack of accuracy in the density or geometry of these samples, as they are very similar to those of the tungsten RPRs.
The increase in RMS error indicates that the use of the isotropic fit to analyze the RUS spectra of extruded AA-1100-O is not correct. For these AA-1100 samples, the effects of extrusion-induced texture cannot be ignored and are a significant deviation from the assumption of the random polycrystalline orientation distribution. We also find a 16% deviation from the literature value for bulk modulus $B$ [79].

Thus, we analyze the same spectrum with a 5-DoF fit (see Fig. 4) for a sample machined with $c \parallel b_z$ (see Fig. 1b). The 5-DoF fit was then performed using different symmetry orientations, i.e., with $h \parallel a$, $h \parallel b$, and $h \parallel c$. Only the latter ($h \parallel c$, i.e., $h \parallel b_z$) fit matches with the sample’s symmetry. The RMS error decreases dramatically when the correct symmetry is chosen (see Fig. 4a). It is remarkable that although the RMS error of the 2-DoF fit is 0.85%, the differences in the values obtained for $B$ are as large as 16% (see Fig. 4b). This highlights the importance of choosing the correct symmetry. When fitting with the correct symmetry, the values for $B$ are consistent with the values from the literature for polycrystalline aluminum [79]. The complete determination of the elastic constants $c_{ij}$ is shown in Table 1 for the 2- and 5-DoF fits performed in the different orientations. It could also be argued that the lower RMS obtained with 5-DoF fits is an artifact of a higher number of fitting parameters used as compared to the 2-DoF fit. However, the results for $h \parallel a$ and $h \parallel b$ indicate otherwise.

There are cases when the true symmetry of the material is unknown a priori. The elastic constants can be extracted using a fitting procedure with an even lower symmetry, such as an orthorhombic fit (9 DoF) for which $c_{11}, c_{22}, c_{33}, c_{12}, c_{13}, c_{23}, c_{44}, c_{55}$ and $c_{66}$ are independent. If the material truly has transversely isotropic texture, then the 9-DoF fit should yield identical results (within errors) to the 5-DoF fit. RMS error is expected to be slightly less for a 9-DoF fit because there are more free parameters available in the minimization scheme, analogously to the tungsten case. Figure 5 compares the results obtained using 5-DoF and 9-DoF fits for the shear constants ($c_{44}, c_{55}$ and $c_{66}$) of an AA-1100-O sample with $c \parallel b_z$. The determined value of $c_{66}$ is identical to the 5-DoF and 9-DoF fits, and the values of $c_{44}$ and $c_{55}$ are within the error. More importantly, the results shown in Fig. 5 confirm the assumption of transverse isotropy for the extruded material, without the need of determining the material’s symmetry or its orientation a priori. Note that the level of anisotropy in the shear constants is clearly outside the error of determination around 6%.

RUS is capable of determining the degree of elastic anisotropy for samples that were machined regardless of the orientations with respect to the rod. The compressive, dilatational and shear elastic constants for samples machined in different orientations are shown in Fig. 6. Henceforth, all fits were performed using a 5-DoF fit with $h \parallel b_z$. Although there are small variations in the elastic constants from sample to sample, this is likely due to handling and small differences in machining quality. The RMS errors of the fits are all low, about 0.1%. The general relationships between the elastic constants are the same for each sample: $c_{11} > c_{33}, c_{13} > c_{12}$, and $c_{44} > c_{66}$. These samples have much larger anisotropy in the shear constants than in the compressive constants. Note that $c_{12}$ is set by the values of the other constants (see Eq. 9) since hexagonal symmetry is associated with

![Figure 4](image-url)
five independent constants, but both dilatational
constants are shown in Fig. 6 for comparison
purposes.

In order to obtain more statistical confidence in the
anisotropy inferred from the results shown above,
three additional AA-1100-O samples were charac-
terized, which were machined in the same fashion
from the same rod and with identical thermal histo-
ries as described before. The RUS spectra were col-
lected for each of the RPRs and were used to perform
hexagonal (5 DoF) fits with $h||b$ for each of the
samples. The elastic constants for these samples are
displayed in Fig. 7, along with the anisotropy ratios
of the compressive and shear elastic constants. The
anisotropy in the shear elastic constants
(5.7% $\pm$ 0.5%) is consistently larger than the aniso-
tropy in dilatational elastic constants (2.4% $\pm$ 0.6%)
and larger than the anisotropy in compressive elastic
constants (1.5% $\pm$ 0.5%) as determined by RUS.

These findings are in close agreement with the PE
results performed on two samples (3 and 5) for val-
idation purposes, particularly the anisotropy ratios.
From Fig. 7a, it can be seen that the $c_{ij}$ determined by
pulse echo are slightly lower than those determined...
by the RUS technique. It is possible that this is due to
the interaction of the propagating wave with the
boundaries of the sample, which can guide and slow
the wave, resulting in lower values of \( c_{ij} \). We also see
small variations in the elastic constants from sample
to sample as expected due to different aspects such as
sample inhomogeneity, handling and small differ-
ences in machining quality. The variation is relatively
bigger for the compression constants as the aniso-
tropy is smaller. The difference in density is one
order of magnitude smaller than the variations
observed in \( c_{ij} \) values from sample to sample, thus
not the origin of sample-to-sample variation found in
\( c_{ij} \). We did observe a non-systematic change in elastic
constant obtained by RUS before and after sound
velocity measurements, of the same order of the
dispersion between samples. Nevertheless, the
degree of anisotropy remains practically unchanged.

The determination of the degree of elastic aniso-
tropy is paramount toward understanding the
mechanical properties of the material. In this partic-
ular case, with extruded AA-1100-O, anisotropy in
the shear elastic constants is nearly 6% (see Fig. 7a).
The engineering constants can be calculated from the
elastic constants by using Eqs. 10–13. For these sam-
pies, the Young’s moduli and Poisson ratios vary by
3% and 11%, respectively.

Another common measure of anisotropy was pro-
posed by Thomsen [80], given by these parameters:
\[
\begin{align*}
\epsilon &= \frac{c_{11} - c_{33}}{2c_{33}} \\
\gamma &= \frac{c_{66} - c_{44}}{2c_{44}} \\
\delta &= \frac{(c_{13} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})}
\end{align*}
\]

The first two anisotropy parameters (\( \epsilon \) and \( \gamma \)) are
the normalized difference of the compressive and
shear elastic constants. The parameter \( \delta \) reflects the
near-vertical compression wave anisotropy and does
not include \( c_{11} \) (the horizontal velocity) [80]. For weak
anisotropy, \( \delta \) can be further reduced to \( \delta_{\text{weak}} = \frac{(c_{13} - c_{33})}{2c_{33}} \), so \( \delta \) can also be seen as the off-diag-
onal or dilatational constants anisotropy factor.
Interestingly, when the \( \delta \) is plotted against \( \epsilon \) and \( \gamma \),
we find a clear correlation between them with
\( \delta = 0.036 + 2.88\epsilon \) and \( \delta = -0.01 + 2.17\gamma \) in Fig. 7b.
This suggests that the differences in \( c_{ij} \) between
samples are not due to random variation or error in
their determination but in different levels of aniso-
tropy among the samples.

To further investigate the origin of the elastic ani-
sotropy, we performed neutron measurements to
determine the texture and calculated the elastic ani-
sotropy using self-consistent model using those
neutron textures.

Figure 8 shows the resulting pole figures and
illustrations of the samples machined with the
extrusion direction parallel to the c (Fig. 8a and b)
and a (Fig. 8c and d). As shown in the pole figures,
both samples have similar MRD values, indicating
that, other than the rotation, texture is identical for
the two samples. However, the plane of isotropy for
the sample shown in Fig. 8a is clearly the a-b plane,
while the plane of isotropy for the sample shown in
Fig. 8c is the b-c plane, both of which are consistent
with RUS and PE results. It is worth noting that there
are very small deviations from transverse isotropy;
e.g., the light blue “ring” in the top 111 pole figure is
not of constant density around the ring, and the
vertical band in the bottom 200 pole figure is not
vertical.

As mentioned in “Neutron diffraction” section, the
measured ODFs of samples \( c \parallel \hat{z} \) and \( a \parallel \hat{z} \) were
integrated in small regions of orientation space to
give a set of 23,328 orientations. These sets of indi-
vidual orientations representing the textures of sam-
pies \( c \parallel \hat{z} \) and \( a \parallel \hat{z} \) were used as input of ELSC
calculations. The other adopted input parameters
were the values of Al single-crystal elastic constants
\( c_{11}^{\text{Al-SX}} = 107 \) GPa, \( c_{12}^{\text{Al-SX}} = 61 \) GPa and \( c_{44}^{\text{Al-SX}} = 28 \) GPa
obtained by averaging five different mea-
surements at room temperature reported in [78], and
a grain’s ellipsoidal shape (1:1:5) along x, y and z,
consistent with the morphology of grains extruded in
\( \hat{z} \). (Note that the reported results are fairly insensitive
to the actual ratio between the long axis and the short
axes of the ellipsoid.)

Figure 9 shows the ELSC predictions of nine
crystal’s elastic stiffness components: \( c_{11}, c_{22}, c_{33}, c_{12}, c_{13}, c_{23}
\) and \( c_{44}, c_{55}, c_{66} \) for samples \( c \parallel \hat{z} \) and \( a \parallel \hat{z} \).
Note that symmetry of \( c_{ijkl} \) was not
imposed but arose naturally from the symmetry of the
input textures. It is observed that: a) the predicted
crystal elastic constants are very close to trans-
versely isotropic symmetry (or “axial symmetry” or
“hexagonal symmetry”). For example, in the case of
\( c \parallel \hat{z} \) sample, \( c_{11} \approx c_{22}, c_{13} \approx c_{13}, c_{44} \approx c_{55}
and c_{12} \approx

Compressive Constants (GPa)

- c_{11} - RUS
- c_{11} - PE
- c_{33} - RUS
- c_{33} - PE

Shear Constants (GPa)

- c_{44} - RUS
- c_{44} - PE
- c_{66} - RUS
- c_{66} - PE

Anisotropy Ratio

- c_{11}/c_{33} - RUS
- c_{11}/c_{33} - PE
- c_{44}/c_{66} - RUS
- c_{44}/c_{66} - PE

(a) Anisotropy Ratio

Thomsen's anisotropy parameter \( \delta \)

(b) Anisotropy parameter \( \delta \)
c_{11} - 2c_{66} \text{ within a tolerance better than 0.1\%; b) the difference between the } c \parallel \hat{z} \text{ and } a \parallel \hat{z} \text{ is essentially a switch between the sample directions, } z \text{ and } x, \text{ with } x \text{ becoming the direction of the symmetry axis, and consequently } c_{11} \neq c_{22} \neq c_{33}, \hspace{1mm} c_{23} \neq c_{12} \neq c_{13}, \hspace{1mm} c_{44} \neq c_{55} \neq c_{66} \text{ and } c_{23} \neq c_{33} - 2c_{44}. \text{ These results, obtained from textures of samples cut from the same rod but independently measured, show consistency between the texture measurements and modeling methodology.}

Figure 10 shows the comparisons between the polycrystal’s elastic constants measured with RUS and...
and predicted with ELSC with input from ND textures for samples c∥z and a∥z, respectively. Note that ranges of all vertical axes are 4 GPa, for a fair visual comparison of the results. It is observed that the predicted anisotropy is qualitatively consistent with the RUS measurements, e.g., in the case of sample c∥z: $c_{33} < c_{11} \approx c_{22}$, $c_{12} < c_{13} \approx c_{23}$, and $c_{66} < c_{44} \approx c_{55}$, although quantitative, the predictions and measurement are off, in some of the worst cases, by several percent. The quantitative differences between calculation and experiments may be due to systematic errors in the texture and intrinsic to the RUS analysis, namely, the measured texture by neutrons is not perfectly transverse isotropic as assumed (see Fig. 8), the single crystal elastic constants of AA-1100-O are not the ones found in a single crystal extracted from tables, and/or model shortcomings. Elucidation of the ultimate causes of quantitative disagreement between experiments and predictions is currently the subject of investigation, including the
study of materials with higher intrinsic single-crystal anisotropy.

As mentioned before, aluminum has a relatively low Zener anisotropy factor of $A = 1.22$, meaning that its elastic properties are less sensitive to anisotropic changes due to texture than, say copper ($A = 3.21$), or 304 stainless steel ($A = 3.77$) [44] and many other common materials with larger $A$ factors, for which the degree of anisotropy due to texture will be much larger. Indeed, using the same texture measured for AA-1100, we calculated the polycrystal elastic constants that would result from Cu and SS samples with identical texture, as shown in Fig. 11, using a normalization based on the corresponding isotropic elastic constants in the case of random texture, for each material. Clearly, the same texture has larger effect on the elastic anisotropy as the $A$ increases. The changes in compression constants are smaller than those found in shear. We find an overall decrease in the average of the compression constant with higher $A$. For shear, the anisotropy is larger and there is a decrease in $c_{66}$ clearly seen for Cu and SS.

Summary and conclusions

In this study, we use resonant ultrasound spectroscopy (RUS) to nondestructively determine the entire elastic tensor in extruded AA-1100-O (commercially pure aluminum in its annealed state with no cold working), which has transversely isotropic symmetry (five independent elastic constants) due to the texture induced by the extrusion process. The relative anisotropy of the compressive ($c_{11}$ vs. $c_{33}$) and shear ($c_{44}$ vs. $c_{66}$) elastic constants is $1.5\% \pm 0.5\%$ and $5.7\% \pm 0.5\%$, respectively, where $c_{33}$ and $c_{66}$ are the elastic constants associated with the axis of symmetry of the texture. Because aluminum is an inherently low-anisotropy material whose Zener anisotropy factor is 1.22, other common alloys (steels, nickel-based superalloys, copper-based alloys, etc.) are expected to have larger anisotropic properties given the same texture.

The necessity of accounting for the anisotropy resulting from the extrusion process has been demonstrated by comparing the RUS results from AA-1100-O samples fit using the isotropic assumption (2 DoF), the transverse isotropic (5 DoF) and orthotropic (9 DoF) fits. The error associated with describing extruded aluminum with an isotropic fit is supported by the extremely low-error, high-accuracy RUS results of the isotropic fit in 99.95% pure tungsten, which has a Zener anisotropy factor of unity and is therefore isotropic regardless of texture. The ability for RUS to determine the orientation of the texture axis has also been demonstrated by analyzing the results with a lower symmetry (9 DoF) and changing the orientation of the fit with respect of extrusion orientation (5 DoF). The elastic constants and the degree of elastic anisotropy as determined

---

**Figure 11** Normalized elastic constants (vs. randomly distributed elastic constant) of AA-1100, Cu and SS using the same experimental texture.
using RUS have been confirmed by direct sound velocity measurements using the ultrasonic pulse-echo technique. The elastic constants were then used to calculate the engineering constants, whereby the Young’s moduli and Poisson ratios parallel vs. perpendicular to the extrusion direction in these samples differ by 3% and 11%, respectively. Using the experimental texture obtained by neutron diffraction, elastic self-consistent micromechanical simulations show good agreement in the level of anisotropy and small disagreement in the absolute values that we assign to material changes produced by handling the soft-annealed AA-1000 samples. Using the same texture and calculation method, we show that the texture will have much larger effect for materials with larger Zener parameter, resulting even in lowering of the elastic constants.

### Acknowledgements

Research presented in this article was supported by the Laboratory Directed Research and Development program of Los Alamos National Laboratory under project 201802500ER. Work at the National High Magnetic Field Laboratory at LANL, was also supported by the National Science Foundation through NSF/DMR-1644779 and the state of Florida (F.F.B., J.B.). Work by B.T.S. was supported by LANL Office of Experimental Science Dynamic Materials Properties program (Campaign 2). The work has benefitted from the use of the Los Alamos Neutron Science Center (LANSCE) at LANL. Los Alamos National Laboratory is operated by Triad National Security, LLC, for the National Nuclear Security Administration of the U.S. Department of Energy under contract number 89233218NCA000001.

### Compliance with ethical standards

**Conflict of interest** The authors declare no competing interest.

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### Appendix

The relationship between the phase velocities of ultrasonic plane waves and the elastic constants of the material can be determined by solving the eigenvalue problem associated with the Christoffel equation in Eq. 18, where \( \rho \) is the density, \( v \) is the elastic wave phase velocity, \( \delta_{ij} \) is the Kronecker delta, \( \Gamma_{ij} = c_{ijkl}n_in_j \) is the Christoffel tensor, and \( n_i \) are the directional cosines of the elastic wave propagation direction for a given reference axis [81–83]. For isotropic materials (two independent elastic constants, see Eq. 3 where \( c_{12} = c_{11} - 2c_{44} \), the Christoffel tensor reduces to the relations in Eq. 19:

\[
\begin{vmatrix}
\Gamma_{ij} - \rho v^2 \delta_{ij}
\end{vmatrix} = 0
\]

\[
\Gamma_{ij} = \begin{vmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\
\Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\
\Gamma_{31} & \Gamma_{32} & \Gamma_{33}
\end{vmatrix}
\]

\[
\Gamma_{11} = c_{11}n_1^2 + c_{44}n_2^2 + c_{44}n_3^2 \\
\Gamma_{22} = c_{44}n_1^2 + c_{11}n_2^2 + c_{44}n_3^2 \\
\Gamma_{33} = c_{44}n_1^2 + c_{44}n_2^2 + c_{11}n_3^2 \\
\Gamma_{12} = n_1n_2(c_{12} + c_{44}) \\
\Gamma_{23} = n_2n_3(c_{12} + c_{44}) \\
\Gamma_{13} = n_1n_3(c_{12} + c_{44})
\]

Since changing the orientation in which the elastic wave velocity measurement is performed in an isotropic material yields identical results, we arbitrarily assign the z-axis as the direction of elastic wave propagation, i.e., \( n_x = n_y = 0 \) and \( n_z = 1 \). It follows that:

\[
\begin{bmatrix}
c_{44} - \rho v^2 & 0 & 0 \\
0 & c_{44} - \rho v^2 & 0 \\
0 & 0 & c_{11} - \rho v^2
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

and
\[(c_{44} - \rho v^2)^2 (c_{11} - \rho v^2) = 0 \]  

(21)

where \(u_i\) is the polarization vector which represents the atomic displacement directions. The roots of the above relationship are the eigenvalues of the Christoffel tensor which relate the transverse \((v_s)\) and longitudinal \((v_L)\) phase velocities to the elastic constants of an isotropic solid, as shown in Eq. 22:

\[
\begin{bmatrix}
    c_{11} \sin^2 \theta + c_{44} \cos^2 \theta - \rho v^2 \\
    0 \\
    (c_{13} + c_{44}) \sin \theta \cos \theta \\
    0 \\
    c_{66} \sin^2 \theta + c_{44} \cos^2 \theta - \rho v^2 \\
    0 \\
    (c_{13} + c_{44}) \sin \theta \cos \theta \\
    0 \\
    c_{44} \sin^2 \theta + c_{33} \cos^2 \theta - \rho v^2 \\
    0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    (c_{11} + c_{44}) \sin \theta \cos \theta \\
    (c_{12} + c_{66}) \sin \theta \cos \theta \\
    (c_{12} + c_{66}) \sin \theta \cos \theta \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
\end{bmatrix}
\]

\[
\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

(25)

\[c_{44} = \rho v_T^2\]
\[c_{11} = \rho v_L^2\]

(22)

Note that because there is no orientation dependence in isotropic materials, the entire elastic tensor can be determined from a single pulse-echo experiment. For transversely isotropic materials (five independent elastic constants, see Eq. 9), however, the Christoffel tensor is described by the relations in Eq. 23. In this analysis, the plane of isotropy has been assigned to the x-y plane, and the “z” direction is parallel to the textured axis.

\[
\begin{align*}
\Gamma_{11} &= c_{11} n_1^2 + c_{66} n_2^2 + c_{44} n_3^2 \\
\Gamma_{22} &= c_{66} n_1^2 + c_{11} n_2^2 + c_{44} n_3^2 \\
\Gamma_{33} &= c_{44} n_1^2 + c_{44} n_2^2 + c_{33} n_3^2 \\
\Gamma_{12} &= n_1 n_2 (c_{12} + c_{66}) \\
\Gamma_{23} &= n_2 n_3 (c_{13} + c_{44}) \\
\Gamma_{13} &= n_1 n_3 (c_{13} + c_{44})
\end{align*}
\]

(23)

Under these symmetry conditions, the direction of wave propagation is now relevant. Following the same methodology as before, we now consider a plane wave propagating in the x-y plane in a direction \(\theta\) with respect to the z-axis. Therefore, \(n_1 = \sin \theta\), \(n_2 = \cos \theta\), \(n_3 = 0\), and the Christoffel equation becomes Eq. 24. Similarly, for a plane wave propagating in the x-z plane in a direction \(\theta\) with respect to the z-axis, \(n_1 = \sin \theta\), \(n_2 = 0\), \(n_3 = \cos \theta\), and the Christoffel equation becomes Eq. 25.

The ultrasonic phase velocities relate to the elastic constants of a transversely isotropic material by Eq. 26, where \(v_{T,q}\), \(v_T\) and \(v_L\) are the quasi-transverse, pure transverse and longitudinal velocities, and \(\theta\) is the angle between the propagation direction and the z-axis of symmetry (i.e., \(\theta = 0^\circ\) for waves propagating along the z-axis).

\[
v_L = \sqrt{\frac{c_{11} \sin^2 \theta + c_{33} \cos^2 \theta + c_{44} + \sqrt{M}}{2 \rho}}
\]
\[
v_{T,q} = \sqrt{\frac{c_{11} \sin^2 \theta + c_{33} \cos^2 \theta + c_{44} - \sqrt{M}}{2 \rho}}
\]
\[
v_T = \sqrt{\frac{c_{66} \sin^2 \theta + c_{44} \cos^2 \theta}{\rho}}
\]

\[
M = \left\{ (c_{11} - c_{44}) \sin^2 \theta + (c_{33} - c_{44}) \cos^2 \theta \right\}^2 + (c_{13} + c_{44})^2 \sin^2 2\theta
\]

The relationship between sound velocities and sample orientations for \(c_{13}\) is given in Eq. 27 for an experiment performed at 45° from the texture axis. Samples were not machined in such off-angle orientations in this work; therefore, pulse echo was unable to obtain information related to the values of \(c_{13}\) for the AA-1100 samples in this study.
\[ c_{11} = \rho \frac{\partial^2 q}{\partial \rho^2} |_{\theta=90} \]
\[ c_{33} = \rho \frac{\partial^2 q}{\partial \rho^2} |_{\theta=0} \]
\[ c_{12} = c_{11} - 2\mu \rho \frac{\partial^2 q}{\partial \rho^2} |_{\theta=45} - 2\mu \frac{\partial^2 q}{\partial \rho^2} |_{\theta=0} \]
\[ c_{13} = \sqrt{4\rho^2 \frac{\partial^2 q}{\partial \rho^2} |_{\theta=45}^2 - 2\rho^2 \frac{\partial^2 q}{\partial \rho^2} |_{\theta=0}^2 - (c_{11} + c_{33} + 2c_{44}) + (c_{11} + c_{44})(c_{33} + c_{44}) - c_{44}^2} \]
\[ v_{44} = \rho \frac{\partial^2 q}{\partial \rho^2} |_{\theta=45} \]
\[ v_{66} = \rho \frac{\partial^2 q}{\partial \rho^2} |_{\theta=90} \]

\[ \text{(27)} \]

The uncertainties associated with the pulse-echo experiments are calculated by the propagation of
errors as described by the relationship shown in Eq. 28, where \( \delta c_{ij} \) is the uncertainty of the elastic constant, \( \delta \rho \) is the uncertainty of density, and \( \delta v \) is the uncertainty of the measured sound velocity.

\[ \delta c_{ij} = c_{ij} \sqrt{ \left( \frac{\delta \rho}{\rho} \right)^2 + 2 \left( \frac{\delta v}{v} \right)^2}. \]

\[ \text{(28)} \]

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