Low analytic rank implies low partition rank for tensors

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Abstract

A tensor defined over a finite field $\mathbb{F}$ has low analytic rank if the distribution of its values differs significantly from the uniform distribution. An order $d$ tensor has partition rank 1 if it can be written as a product of two tensors of order less than $d$, and it has partition rank at most $k$ if it can be written as a sum of $k$ tensors of partition rank 1. In this paper, we prove that if the analytic rank of an order $d$ tensor is at most $r$, then its partition rank is at most $f(r, d, |\mathbb{F}|)$. Previously, this was known with $f$ being an Ackermann-type function in $r$ and $d$ but not depending on $\mathbb{F}$. The novelty of our result is that $f$ has only tower-type dependence on its parameters. It follows from our results that a biased polynomial has low rank; there too we obtain a tower-type dependence improving the previously known Ackermann-type bound.

1 Introduction

1.1 Bias and rank of polynomials

For a polynomial $P : \mathbb{F}^n \to \mathbb{F}$, we say that $P$ is unbiased if the distribution of the values $P(x)$ is close to the uniform distribution on $\mathbb{F}$; otherwise we say that $P$ is biased. It is an important direction of research in higher order Fourier analysis to understand the structure of biased polynomials.

Note that a generic degree $d$ polynomial should be unbiased. In fact, as we will see below, if a degree $d$ polynomial is biased, then it can be written as a function of not too many polynomials of degree at most $d - 1$. Let us now make this discussion more precise.

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Definition 1.1. Let $\mathbb{F}$ be a finite field and let $\chi$ be a nontrivial character of $\mathbb{F}$. The bias of a function $f : \mathbb{F}^n \to \mathbb{F}$ with respect to $\chi$ is defined to be $\text{bias}_\chi(f) = \mathbb{E}_{x \in \mathbb{F}^n}[\chi(f(x))]$. (Here and elsewhere in the paper $\mathbb{E}_{x \in \mathbb{F}} h(x)$ denotes $\frac{1}{|\mathbb{F}|} \sum_{x \in \mathbb{F}} h(x).$)

Remark 1.2. Most of the previous work is on the case $\mathbb{F} = \mathbb{F}_p$ with $p$ a prime, in which case the standard definition of bias is $\text{bias}(f) = \mathbb{E}_{x \in \mathbb{F}^n} \omega^{f(x)}$ where $\omega = e^{\frac{i \pi}{p}}$.

Definition 1.3. Let $P$ be a polynomial $\mathbb{F}^n \to \mathbb{F}$ of degree $d$. The rank of $P$ (denoted $\text{rank}(P)$) is defined to be the smallest integer $r$ such that there exist polynomials $Q_1, \ldots, Q_r : \mathbb{F}^n \to \mathbb{F}$ of degree at most $d - 1$ and a function $f : \mathbb{F}^r \to \mathbb{F}$ such that $P = f(Q_1, \ldots, Q_r)$.

As discussed above, it is known that if a polynomial has large bias, then it has low rank. The first result in this direction was proved by Green and Tao [4] who showed that if $\mathbb{F}$ is a field of prime order and $P : \mathbb{F}^n \to \mathbb{F}$ is a polynomial of degree $d$ with $d < |\mathbb{F}|$ and $\text{bias}(P) \geq \delta > 0$, then $\text{rank}(P) \leq c(\mathbb{F}, \delta, d)$. Kaufman and Lovett [6] proved that the condition $d < |\mathbb{F}|$ can be omitted. In both results, $c$ has Ackermann-type dependence on its parameters. Finally, Bhowmick and Lovett [1] proved that if $d < \text{char}(\mathbb{F})$ and $\text{bias}(P) \geq |\mathbb{F}|^{-s}$, then $\text{rank}(P) \leq c'(d, s)$. The novelty of this result is that $c'$ does not depend on $\mathbb{F}$. However, it still has Ackermann-type dependence on $d$ and $s$.

One of our main results is the following theorem, which improves the result of Bhowmick and Lovett unless $\mathbb{F}$ is very large. In this result, and in the rest of the paper, $\text{tower}_{8|\mathbb{F}|}(h, x)$ denotes a tower of $8|\mathbb{F}|$’s of height $h$ with an $x$ on top, that is, $\text{tower}_{8|\mathbb{F}|}(0, x) = x$ and $\text{tower}_{8|\mathbb{F}|}(h, x) = (8|\mathbb{F}|)^{\text{tower}_{8|\mathbb{F}|}(h-1, x)}$.

Theorem 1.4. Let $\mathbb{F}$ be a finite field and let $\chi$ be a nontrivial character of $\mathbb{F}$. Let $P$ be a polynomial $\mathbb{F}^n \to \mathbb{F}$ of degree $d < \text{char}(\mathbb{F})$. Suppose that $\text{bias}_\chi(P) \geq \epsilon > 0$. Then

$$\text{rank}(P) \leq 2^d \text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3}, (1/\epsilon)^{2^d}) + 1$$

Recall that if $G$ is an Abelian group and $d$ is a positive integer, then the Gowers $U^d$ norm (which is only a seminorm for $d = 1$) of $f : G \to \mathbb{C}$ is defined to be

$$\|f\|_{U^d} = \left\| \mathbb{E}_{x,y_1,\ldots,y_d \in G} \prod_{S \subset [d]} C^{|S|-|\delta|} f(x + \sum_{i \in S} y_i) \right\|^{1/2^d},$$

where $C$ is the conjugation operator. It is a major area of research to understand the structure of functions $f$ whose $U^d$ norm is large. Our next theorem is a result in this direction.
Theorem 1.5. Let $\mathbb{F}$ be a finite field and let $\chi$ be a nontrivial character of $\mathbb{F}$. Let $P$ be a polynomial $\mathbb{F}^n \rightarrow \mathbb{F}$ of degree $d < \text{char}(\mathbb{F})$. Let $f(x) = \chi(P(x))$ and assume that $\|f\|_{U^d} \geq c > 0$. Then

$$\text{rank}(P) \leq 2^d \text{tower}_{\mathbb{F}^{|P|}}((d + 3)^{d+3}, (1/c)^{2d}) + 1$$

Our result implies a similar improvement to the bounds for the quantitative inverse theorem for Gowers norms for polynomial phase functions of degree $d$.

Theorem 1.6. Let $\mathbb{F}$ be a field of prime order and let $P$ be a polynomial $\mathbb{F}^n \rightarrow \mathbb{F}$ of degree $d < \text{char}(\mathbb{F})$. Let $f(x) = \omega^{P(x)}$ where $\omega = e^{2\pi i}$ and assume that $\|f\|_{U^d} \geq c > 0$. Then there exists a polynomial $Q : \mathbb{F}^n \rightarrow \mathbb{F}$ of degree at most $d - 1$ such that

$$|\mathbb{E}_{x \in \mathbb{F}^n} \omega^{P(x)} \omega^{Q(x)}| \geq |\mathbb{F}|^{-2^d \text{tower}_{\mathbb{F}^{|P|}}((d + 3)^{d+3}, (1/c)^{2d}) - 1}$$

Theorems 1.4 and 1.6 easily follow from Theorem 1.5.

Proof of Theorem 1.4. Note that when $f(x) = \chi(P(x))$, then $\|f\|_{U^d}^2 = |\mathbb{E}_{x,y \in \mathbb{F}^n} f(x)f(x+y)| = |\mathbb{E}_{x \in \mathbb{F}^n} f(x)|^2$, so $\|f\|_{U^d} = |\mathbb{E}_{x \in \mathbb{F}^n} f(x)| = |\text{bias}_d(P)|$. However, $\|f\|_{U^d}$ is increasing in $k$ (see eg. Claim 6.2.2 in [5]), therefore $\|f\|_{U^d} \geq |\text{bias}_d(P)| \geq \epsilon$. The result is now immediate from Theorem 1.5.

Proof of Theorem 1.6. By Theorem 1.5, there exists a set of $r \leq 2^d \text{tower}_{\mathbb{F}^{|P|}}((d + 3)^{d+3}, (1/c)^{2d}) + 1$ polynomials $Q_1, \ldots, Q_r$ such that $P(x)$ is a function of $Q_1(x), \ldots, Q_r(x)$. Then $\omega^{P(x)} = g(Q_1(x), \ldots, Q_r(x))$ for some function $g : \mathbb{F}^r \rightarrow \mathbb{C}$. Let $G = \mathbb{F}^r$. Note that $|g(y)| = 1$ for all $y \in G$, therefore $|\hat{g}(\chi)| \leq 1$ for every character $\chi \in \hat{G}$. Now $\omega^{P(x)} = \sum_{\chi \in \hat{G}} \hat{g}(\chi) \chi((Q_1(x), \ldots, Q_r(x))$, so

$$1 = \mathbb{E}_{x \in \mathbb{F}^n} |\omega^{P(x)}|^2 = \sum_{\chi \in \hat{G}} \hat{g}(\chi)^2 |\mathbb{E}_{x \in \mathbb{F}^n} \omega^{P(x)} \chi((Q_1(x), \ldots, Q_r(x))|^2.$$ 

Thus, there exists some $\chi \in \hat{G}$ with $|\mathbb{E}_{x \in \mathbb{F}^n} \omega^{P(x)} \chi((Q_1(x), \ldots, Q_r(x))| \geq 1/|G| = 1/|\mathbb{F}^r|$. But $\chi$ is of the form $\chi(y_1, \ldots, y_r) = \omega^{\sum_{i \in \mathbb{F}} a_i y_i}$ for some $a_i \in \mathbb{F}$. Then $\chi(Q_1(x), \ldots, Q_r(x)) = \omega^{Q_{\alpha}(x)}$, where $Q_{\alpha}$ is the degree $d - 1$ polynomial $Q_{\alpha}(x) = \sum_{i \in \mathbb{F}} a_i Q_i(x)$. So $Q = Q_{\alpha}$ is a suitable choice.

1.2 Analytic rank and partition rank of tensors

Related to the bias and rank of polynomials are the notions of analytic rank and partition rank of tensors. Recall that if $\mathbb{F}$ is a field and $V_1, \ldots, V_d$ are finite dimensional vector spaces over $\mathbb{F}$, then an order $d$ tensor is a multilinear map $T : V_1 \times \cdots \times V_d \rightarrow \mathbb{F}$. Each $V_k$ can be identified
work of Bhowmick and Lovett \[1\] if there is some \(S\) partition rank 1. This number is denoted \(\text{prank}(T)\) but \(\text{rank}\) of \(T\) is the smallest \(r\) such that \(T\) can be written as the sum of \(r\) tensors of partition rank 1.

**Definition 1.9.** Let \(F\) be a finite field, let \(V_1, \ldots, V_d\) be finite dimensional vector spaces over \(F\) and let \(T : V_1 \times \cdots \times V_d \to F\) be an order \(d\) tensor. Then the analytic rank of \(T\) is defined to be \(\text{arank}(T) = -\log_{|F|} \text{bias}(T)\), where \(\text{bias}(T) = \mathbb{E}_{v^1, \ldots, v^d} \chi(T(v^1, \ldots, v^d))\) for any nontrivial character \(\chi\) of \(F\).

**Remark 1.8.** This is well-defined. Indeed, if \(\chi\) is a nontrivial character of \(F\), then

\[
\mathbb{E}_{v^1, \ldots, v^d} [\chi(T(v^1, \ldots, v^d))] = \mathbb{E}_{v^1, \ldots, v^{d-1}} [\mathbb{E}_{v^d} \chi(T(v^1, \ldots, v^d))] = \mathbb{E}_{v^1, \ldots, v^{d-1}} [T(v^1, \ldots, v^{d-1}, x) = 0],
\]

where \(T(v^1, \ldots, v^{d-1}, x)\) is viewed as a function in \(x\). The second equality holds because \(\mathbb{E}_{v^d} \chi(T(v^1, \ldots, v^d)) = 0\) unless \(T(v^1, \ldots, v^{d-1}, x) \equiv 0\), in which case it is 1.

Thus, \(\mathbb{E}_{v^1, \ldots, v^d} [\chi(T(v^1, \ldots, v^d))]\) does not depend on \(\chi\), and is always positive. Moreover, it is at most 1, therefore the analytic rank is always nonnegative.

A different notion of rank was defined by Naslund \[8\].

**Definition 1.9.** Let \(T : V_1 \times \cdots \times V_d \to F\) be a (non-zero) order \(d\) tensor. We say that \(T\) has partition rank 1 if there is some \(S \subset [d]\) with \(S \neq \emptyset, S \neq [d]\) such that \(T(v^1, \ldots, v^d) = T_1(v^i : i \in S)T_2(v^i : i \notin S)\) where \(T_1 : \prod_{i \in S} V_i \to F, T_2 : \prod_{i \notin S} V_i \to F\) are tensors. In general, the partition rank of \(T\) is the smallest \(r\) such that \(T\) can be written as the sum of \(r\) tensors of partition rank 1. This number is denoted \(\text{prank}(T)\).

Lovett \[7\] has proved that \(\text{arank}(T) \leq \text{prank}(T)\). In the other direction, it follows from the work of Bhowmick and Lovett \[1\] that if an order \(d\) tensor \(T\) has \(\text{arank}(T) \leq r\), then \(\text{prank}(T) \leq f(r, d)\) for some function \(f\). Note that \(f\) does not depend on \(|F|\) or the dimension of the vector spaces \(V_k\). However, \(f\) has an Ackermann-type dependence on \(d\) and \(r\). We prove a different bound under the same assumptions, which is stronger unless \(|F|\) is very large.

**Theorem 1.10.** Let \(T : V_1 \times \cdots \times V_d \to F\) be an order \(d\) tensor with \(\text{arank}(T) \leq r\). Then

\[
\text{prank}(T) \leq 2^{d-1} \text{tower}_8(|F|((d + 3)^{d+3} - 1, r)).
\]
It is not hard to see that Theorem 1.10 implies Theorem 1.5. Indeed, let \( P \) be a polynomial \( \mathbb{F}^n \to \mathbb{R} \) of degree \( d < \text{char}(\mathbb{F}) \), let \( f(x) = \chi(P(x)) \) and assume that \( \|f\|_{U^d} \geq c > 0 \). Define \( T : (\mathbb{F}^n)^d \to \mathbb{F} \) by \( T(y_1, \ldots, y_d) = \sum_{S \subset [d]} (-1)^{|S|-|I|} P(\sum_{i \in S} y_i) \). By Lemma 2.4 from [3], \( T \) is a tensor of order \( d \). Moreover, by the same lemma, we have \( T(y_1, \ldots, y_d) = \sum_{S \subset [d]} (-1)^{|S|-|I|} P(x + \sum_{i \in S} y_i) \) for any \( x \in \mathbb{F}^n \). Thus,

\[
\text{bias}(T) = \mathbb{E}_{y_1, \ldots, y_d \in \mathbb{F}^n} \chi(T(y_1, \ldots, y_d)) = \mathbb{E}_{y_1, \ldots, y_d \in \mathbb{F}^n} \prod_{S \subset [d]} C^{d-|S|} f(x + \sum_{i \in S} y_i)
\]

for any \( x \in \mathbb{F}^n \). By averaging over all \( x \in \mathbb{F}^n \), it follows that \( \text{bias}(T) = \|f\|_{U^d}^2 \geq c^{2d} \). Thus, \( \text{arank}(T) \leq 2^d \log_{\mathbb{F}}(1/c) \). Therefore, by Theorem 1.10, we get

\[
\text{prank}(T) \leq 2^{d-1}\text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3} + 1, 2^d \log_{|\mathbb{F}|}(1/c)) = 2^{d-1}\text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3} + (1/c)^{2d}).
\] (1)

Note that \( T(y_1, \ldots, y_d) = D_{j_1} \ldots D_{j_d} P(x) \) where \( D_j g(x) = g(x+y) - g(x) \). Thus, by Taylor’s approximation theorem, since \( d < \text{char}(\mathbb{F}) \), we get \( P(x) = \frac{1}{d!} D_1 \ldots D_d P(0) + W(x) = \frac{1}{d!} T(x, \ldots, x) + W(x) \) for some polynomial \( W \) of degree at most \( d - 1 \).

By equation (1), \( T \) can be written as a sum of at most \( 2^{d-1}\text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3} + (1/c)^{2d}) \) tensors of partition rank 1. Hence, \( P_1 \) can be written as a sum of at most \( 2^{d-1}\text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3} + (1/c)^{2d}) \) expressions of the form \( QR \) where \( Q, R \) are polynomials of degree at most \( d - 1 \) each. Thus, \( P - W \) has rank at most \( 2 \cdot 2^{d-1}\text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3} + (1/c)^{2d}) \), and therefore \( P \) has rank at most

\[
2 \cdot 2^{d-1}\text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3} + (1/c)^{2d}) + 1.
\]

2 The proof of Theorem 1.10

2.1 Notation

In the rest of the paper, we identify \( V_i \) with \( \mathbb{F}^n \). Thus, the set of all tensors \( V_1 \times \cdots \times V_d \to \mathbb{F} \) is the tensor product \( \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_d} \), which will be denoted by \( \mathcal{G} \) throughout this section. Also, \( \mathcal{B} \) will always stand for the multiset \( \{u_1 \otimes \cdots \otimes u_d : u_i \in \mathbb{F}^{n_i} \text{ for all } i\} \). Note that \( \mathcal{G} = \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_d} \) can be viewed as the set of \( d \)-dimensional \((n_1, \ldots, n_d)\)-arrays over \( \mathbb{F} \) which in turn can be viewed as \( \mathbb{F}^{n_1 \ldots n_d} \), equipped with the entry-wise dot product.

For \( I \subset [d] \), we write \( \mathbb{F}^I \) for \( \bigotimes_{i \in I} \mathbb{F}^{n_i} \) so that we naturally have \( \mathcal{G} = \mathbb{F}^I \otimes \mathbb{F}^I \), where \( I^c \) always denotes \([d] \setminus I\).

If \( r \in \mathbb{F}^{[d]} = \mathcal{G} \) and \( s \in \mathbb{F}^{[k]} \) (for some \( k \leq d \)), then we define \( rs \) to be the tensor in \( \mathbb{F}^{[k+1,d]} \) with
coordinates \((rs)_{i_1,\ldots,i_d} = \sum_{l_1\leq n_1,\ldots,l_d\leq n_d} r_{i_1,\ldots,i_d}s_{l_1,\ldots,l_d}\). If \(k = d\), then \(rs\) is the same as the entry-wise dot product \(r.s\). Also, note that viewing \(r\) as a \(d\)-multilinear map \(R: \mathbb{F}^{n_1} \times \cdots \times \mathbb{F}^{n_d} \to \mathbb{F}\), we have \(R(v^1, \ldots, v^d) = \sum_{l_1\leq n_1,\ldots,l_d\leq n_d} r_{i_1,\ldots,i_d}v^1_{l_1} \cdots v^d_{l_d} = r(v^1 \otimes \cdots \otimes v^d)\).

Finally, we use a non-standard notation and write \(kB\) to mean the set of elements of \(G\) which can be written as a sum of at most \(k\) elements of \(B\), where \(B\) is some fixed (multi)subset of \(G\), and similarly, we write \(kB - lB\) for the set of elements that can be obtained by adding at most \(k\) members and subtracting at most \(l\) members of \(B\).

### 2.2 The main lemma and some consequences

Theorem 1.10 will follow easily from the next lemma, which is the main technical result of this paper. See [2] for another application of this lemma.

**Lemma 2.1.** Let \(f_1(d) = 2^{d+1}\), \(f_2(d) = 2^{-d+1}\) and \(f_3(d, \delta) = \text{tower}_{\mathbb{F}[\delta]}((d + 4)^{d+4}, 1/\delta)\). If \(B' \subset B\) is a multiset such that \(|B'| \geq \delta|B|\), then there exists a multiset \(Q\) whose elements are chosen from \(f_1(d)B' - f_2(d)B'\) (but with arbitrary multiplicity) with the following property. The set of arrays \(r \in G\) with \(r,q = 0\) for at least \((1 - f_2(d))|Q|\) choices \(q \in Q\) is contained in \(\sum_{l \in [d], l \neq 0} V_l \otimes \mathbb{F}^f\) for subspaces \(V_l \subset \mathbb{F}^f\) of dimension at most \(f_3(d, \delta)\).

The proof of this lemma goes by induction on \(d\). In what follows, we shall prove results conditional on the assumption that Lemma 2.1 has been verified for all \(d' < d\). Eventually, we will use these results to prove the induction step.

**Definition 2.2.** Let \(k\) be a positive integer. We say that \(r \in G\) is \(k\)-degenerate if for every \(I \subset [d], I \neq \emptyset, I \neq [d]\), there exists a subspace \(H_I \subset \mathbb{F}^f\) of dimension at most \(k\) such that \(r \in \sum_{l \in [d-1], l \neq 0} H_l \otimes H_F\).

If \(r \in H_I \otimes \mathbb{F}^f\) with \(\dim(H_I) \leq k\), then \(r \in H_I \otimes H_F\) for some \(H_F \subset \mathbb{F}^f\) of dimension at most \(k\). (This follows by writing \(r\) as \(\sum_{j \in m} s_j \otimes t_j\) with \(\{s_j\}\) a basis for \(H_I\) and letting \(H_F\) be the span of all the \(t_j\).) Thus, \(r\) is \(k\)-degenerate if and only if \(r \in \sum_{l \in [d-1], l \neq 0} H_l \otimes \mathbb{F}^f\) for some \(H_l \subset \mathbb{F}^f\) of dimension at most \(k\), or equivalently, if and only if \(r \in \sum_{l \in [d-1], l \neq 0} \mathbb{F}^l \otimes H_F\) for some \(H_F \subset \mathbb{F}^f\) of dimension at most \(k\). Moreover, note that if \(r\) is \(k\)-degenerate, then \(\text{prank}(r) \leq 2^{d-1}k\). This is because if \(I \neq \emptyset, I \subset [d-1]\) and \(w \in H_I \otimes H_F\) for subspaces \(H_I \subset \mathbb{F}^f\) and \(H_F \subset \mathbb{F}^f\) of dimension at most \(k\), then \(w = \sum_{l \in k} s_l \otimes t_l\) for some \(s_l \in H_I, t_l \in H_F\). But clearly, \(s_l \otimes t_l\) has partition rank 1.

**Lemma 2.3.** Suppose that Lemma 2.1 has been proved for \(d' = d - 1\). Let \(r \in G\) be such that \(r(v_1 \otimes \cdots \otimes v_{d-1}) = 0 \in \mathbb{F}^{d'}\) for at least \(\delta|\mathbb{F}^{n_1-\cdots-n_{d-1}}|\) choices \(v_1 \in \mathbb{F}^{n_1}, \ldots, v_{d-1} \in \mathbb{F}^{n_{d-1}}\). Then \(r\) is \(f\)-degenerate for \(f = \text{tower}_{\mathbb{F}[\delta]}((d + 3)^{d+3}, 1/\delta)\).
Proof. Write \( r = \sum_i s_i \otimes t_i \) where \( s_i \in \mathbb{F}^{[d-1]} \) and \( \{t_i\}_i \) is a basis for \( \mathbb{F}^d \). Let \( D \) be the multiset \( \{u_1 \otimes \cdots \otimes u_{d-1} : u_1 \in \mathbb{F}^1, \ldots, u_{d-1} \in \mathbb{F}^{d-1}\} \) and \( D' = \{w \in D : rw = 0\} \). Since \( |D'| \geq \delta |D| \), by Lemma 2.1 there is a multiset \( Q \) with elements from \( \mathbb{F}^{(d-1)} \) such that the set of arrays \( r' \in \mathbb{F}^{(d-1)} \) with \( r'.q = 0 \) for all choices \( q \in Q \) is contained in some \( \sum_{I \subseteq [d-1], I \neq \emptyset} V_I \otimes \mathbb{F}^{(d-1)} \), where \( \dim(V_I) \leq \text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3}, 1/\delta) \). Note that for every \( i \) we have \( s_i.w = 0 \) for all \( w \in D' \) and so also \( s_i.q = 0 \) for all \( q \in Q \). Thus, \( r \in \sum_{I \subseteq [d-1], I \neq \emptyset} V_I \otimes \mathbb{F}^d \). \( \square \)

Now we are in a position to prove Theorem 1.10 conditional on Lemma 2.1.

Proof of Theorem 1.10. Let \( T : \mathbb{F}^{p_1} \times \cdots \times \mathbb{F}^{p_d} \to \mathbb{F} \) be an order \( d \) tensor with \( \text{rank}(T) \leq r \). By Remark 1.8, we have \( \mathbb{F}_{v_1 \in \mathbb{F}^{p_1}, \ldots, v_{d-1} \in \mathbb{F}^{p_{d-1}}} [T(v_1, \ldots, v_{d-1}, x) \equiv 0] \geq |\mathbb{F}|^{-r} \). Writing \( t \) for the element in \( G \) corresponding to \( T \), we get that \( t(v_1 \otimes \cdots \otimes v_{d-1} \otimes x) \equiv 0 \) as a function of \( x \) for at least \( \delta |\mathbb{F}|^{p_1 \cdots p_d} \) choices \( v_1 \in \mathbb{F}^{p_1}, \ldots, v_{d-1} \in \mathbb{F}^{p_{d-1}} \), where \( \delta = |\mathbb{F}|^{-r} \). But \( t(v_1 \otimes \cdots \otimes v_{d-1} \otimes x) = (t(v_1 \otimes \cdots \otimes v_{d-1})).x \), so we have \( t(v_1 \otimes \cdots \otimes v_{d-1}) = 0 \) for all these choices of \( v_i \). Thus, by Lemma 2.3, \( t \) is \( f \)-degenerate for \( f = \text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3}, |\mathbb{F}|') \). Hence, \( \text{rank}(T) \leq 2^{d-1} \text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3}, |\mathbb{F}|') = 2^{d-1} \text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3} + 1, r) \). \( \square \)

Let us continue the preparation for the induction step in Lemma 2.1.

Lemma 2.4. Suppose that Lemma 2.1 has been proved for \( d' = d-1 \). Let \( B' \subset B \) be such that \( |B'| \geq \delta |B| \) for some \( \delta > 0 \). Then there exist some \( Q \subset 2B' - 2B' \) and a subspace \( V_{[d]} \subset \mathbb{F}^{[d]} \) of dimension at most \( 5|\mathbb{F}|^{4/d^2} \) with the following property. Any array \( r \) with \( r.q = 0 \) for at least \( \frac{3}{4} |Q| \) choices \( q \in Q \) can be written as \( r = x + y \) where \( x \in V_{[d]} \) and \( y \) is \( f \)-degenerate for \( f = \text{tower}_{8|\mathbb{F}|}((d + 3)^{d+3} + 3, 1/\delta) \).

Proof. Let \( D \) be the multiset \( \{u_1 \otimes \cdots \otimes u_{d-1} : u_1 \in \mathbb{F}^1, \ldots, u_{d-1} \in \mathbb{F}^{d-1}\} \) and \( D' = \{t \in D : t \otimes u \in B' \} \) for at least \( \frac{3}{4} |\mathbb{F}|^d \) choices \( u \in \mathbb{F}^{p_d} \). Clearly, we have \( |D'| \geq \frac{3}{4} |D| \). Moreover, by Bogolyubov’s lemma (see, eg. Proposition 4.39 in [9]), for every \( t \in D' \), there exists a subspace \( U_t \subset \mathbb{F}^{p_d} \) of codimension at most \( \frac{1}{(8|\mathbb{F}|)^d} \) such that \( t \otimes U_t \subset 2B' - 2B' \) for every \( t \in D' \). After passing to suitable subspaces, we may assume that all \( U_t \) have the same codimension \( k \leq 4/\delta^2 \). Now let \( Q = \bigcup_{t \in D'} (t \otimes U_t) \).

Let \( r_1, \ldots, r_m \in G \) have the properties that

(i) for all \( i \), we have \( r_i.q = 0 \) for at least \( \frac{3}{4} |Q| \) choices \( q \in Q \) and

(ii) for all \( i \neq j \), \( r_i - r_j \) is not \( f \)-degenerate.
Note that $r_i(t \otimes s) = (r_it) . s$ for every $s \in U_i$. If $r_it \not\in U_i^+$, then $(r_it) . s = 0$ holds for only a proportion $1/|\mathbb{F}| \leq 1/2$ of all $s \in U_i$. Thus, by (i) it follows that for all $i$, we have $r_it \in U_i^+$ for at least $|D|/2$ choices $t \in \mathcal{D}'$.

Suppose that $m = 5|\mathbb{F}|^k$. By averaging, for at least $|\mathcal{D}'|$ choices $t \in \mathcal{D}'$ there are at least $m/4$ choices $i \leq m$ with $r_it \in U_i^+$. For every such $t$ there exist $i \neq j$ with $r_it = r_jt$ since $|U_i^+| \leq |\mathbb{F}|^k$ for all $j$. Thus, there exist $i \neq j$ such that $r_it = r_jt$ holds for at least $|\mathcal{D}'|/4m$ choices $t \in \mathcal{D}'$. Take $\tilde{r} = r_i - r_j$. Then $\tilde{r}t = 0$ for at least $\frac{d}{8m^2} |\mathcal{D}|$ choices $t \in \mathcal{D}$. By Lemma 2.3, $\tilde{r}$ is $g$-degenerate for $g = \text{tower}_{\mathbb{F}}((d + 3)^{d+1}, \frac{8m^2}{d})$. But $\frac{8m^2}{d} \leq \frac{200|\mathbb{F}|^d/\delta^2 \leq (8|\mathbb{F}|)^{3/\delta^2}$. Moreover, $8|\mathbb{F}|^k|\mathbb{F}|^{1/\delta} \geq 16^{16/\delta} \geq \frac{32}{\delta^2}$. Thus, $\tilde{r}$ is in fact $f$-degenerate, contradicting (ii).

Thus, if $r_1, \ldots, r_m$ is a maximal set with properties (i) and (ii), then $m < 5|\mathbb{F}|^k$. Hence we may take $V_{[d]}$ to be the span of $r_1, \ldots, r_m$ and this satisfies the conclusion of the lemma. □

Remark 2.5. In the proof above and later in the paper we are using the bound $1/\delta^2$ on the codimension of the subspace obtained in Bogolyubov’s lemma (where $\delta$ is the density of our initial set). This is not the best known bound but this choice is simple and makes no difference in the final bound in Lemma 2.1. Later (see Remark 2.12) we will highlight the most expensive step of the argument.

2.3 Construction of some auxiliary sets

The next definition describes a type of set that will be useful for us when constructing $Q$ in Lemma 2.1. Its key properties are described in this subsection.

Definition 2.6. Suppose that we have a collection of vector spaces as follows. The first one is $U \subset \mathbb{F}^n$, of codimension at most $l$. Then, for every $u_1 \in U$, there is some $U_{u_1} \subset \mathbb{F}^2$. In general, for every $2 \leq k \leq \ell$ and every $u_1 \in U, u_2 \in U_{u_1}, \ldots, u_{k-1} \in U_{u_1, \ldots, u_{k-2}}$, there is a subspace $U_{u_1, \ldots, u_{k-1}} \subset \mathbb{F}^n$. Assume, in addition, that the codimension of $U_{u_1, \ldots, u_{k-1}}$ in $\mathbb{F}^m$ is at most $l$ for every $u_1 \in U, \ldots, u_{k-1} \in U_{u_1, \ldots, u_{k-2}}$. Then the multiset $Q = \{u_1 \otimes \cdots \otimes u_d : u_1 \in U, \ldots, u_d \in U_{u_1, \ldots, u_{d-1}}\}$ is called an $l$-system.

Lemma 2.7. Let $Q$ be an $l$-system and let $Q'$ be a $l'$-system. Then $Q \cap Q'$ contains an $(l + l')$-system.

Proof. Let $Q$ have spaces as in Definition 2.6 and let $Q'$ have spaces $U'_{u_1, \ldots, u_{k-1}}$. We define an $(l+l')$-system $P$ contained in $Q \cap Q'$ as follows. Let $V = U \cap U'$. Suppose we have defined $V_{v_1, \ldots, v_{j-1}}$ for all $j \leq k$. Let $v_1 \in V, v_2 \in V_{v_1}, \ldots, v_{k-1} \in V_{v_1, \ldots, v_{k-2}}$. We let $V_{v_1, \ldots, v_{k-1}} = U_{v_1, \ldots, v_{k-1}} \cap U'_{v_1, \ldots, v_{k-1}}$. This is well-defined and has codimension at most $l + l'$ in $\mathbb{F}^m$. Let $P$ be the $(l+l')$-system with spaces $V_{v_1, \ldots, v_{k-1}}$. □
Lemma 2.8. Let $\mathcal{B}' \subset \mathcal{B}$ be a multiset such that $|\mathcal{B}'| \geq \delta |\mathcal{B}|$. Then there exists an $f_1$-system whose elements are chosen from $f_2\mathcal{B}' - f_2\mathcal{B}'$ with $f_1 = \frac{16d}{\alpha^2}$ and $f_2 = 4d$.

Proof. The proof is by induction on $d$. The case $d = 1$ is an easy application of Bogolyubov’s lemma. Suppose that the lemma has been proved for all $d' < d$ and let $\mathcal{B}' \subset \mathcal{B}$ be a multiset such that $|\mathcal{B}'| \geq \delta |\mathcal{B}|$. Let $\mathcal{D}$ be the multiset \{v_2 \otimes \cdots \otimes v_d : v_2 \in \mathbb{F}^2, \ldots, v_d \in \mathbb{F}^d\}. For each $u \in \mathbb{F}^n$, let $\mathcal{B}_u = \{s \in \mathcal{D} : u \otimes s \in \mathcal{B}'\}$ and let $T = \{u \in \mathbb{F}^n : |\mathcal{B}_u| \geq \frac{4}{3}|\mathcal{D}|\}. By averaging, we have that $|T| \geq \frac{1}{3}|\mathcal{D}|$. Now by the induction hypothesis, for every $t \in T$, there exists a $g_1$-system in $\mathbb{F}^n \otimes \cdots \otimes \mathbb{F}^d$ (whose definition is analogous to the definition of a system in $\mathbb{F}^n \otimes \cdots \otimes \mathbb{F}^d$), called $P_t$, contained in $g_2\mathcal{B}'_t - g_2\mathcal{B}'_t$ where $g_1 = \frac{16d^{l-1}}{(\alpha/2)^2}$ and $g_2 = 4d^{l-1}$. By Bogolyubov’s lemma, $2T - 2T$ contains a subspace $U \subset \mathbb{F}^n$ of codimension at most $\frac{1}{10}$. For each $u \in U$, write $u = t_1 + t_2 - t_3 - t_4$ arbitrarily with $t_i \in T$, and let $Q_u = P_{t_1} \cap P_{t_2} \cap P_{t_3} \cap P_{t_4}$, which is a $g_3$-system with $g_3 = 4g_1 = \frac{64d^{l-1}}{\alpha^2}$, by Lemma 2.7. Thus, $Q = \bigcup_{u \in U} (u \otimes Q_u)$ is indeed an $f_1$-system. Moreover, for any $u \in U$, we have $u \otimes s = t_1 \otimes s + t_2 \otimes s - t_3 \otimes s - t_4 \otimes s$ for some $t_i \in T$ and $s \in \bigcap_{i \leq 4} P_i$. Then $t_1 \otimes s \in g_2\mathcal{B}'_t - g_2\mathcal{B}'_t$, therefore $u \otimes s \in 4g_2\mathcal{B}'_t - 4g_2\mathcal{B}'_t$, so the elements of $Q$ are indeed chosen from $f_2\mathcal{B}' - f_2\mathcal{B}'$. \qed

Lemma 2.9. Let $Q$ be a $k$-system and for every non-empty $I \subset [d]$, let $L_I \subset \mathbb{F}^d$ be a subspace of codimension at most $l$. Let $T = \bigcap_I (L_I \otimes \mathbb{F}^d)$. Then $Q \cap T$ contains an $f$-system for $f = k + 2^dl$.

Proof. Let the spaces of $Q$ be $U_{u_1, \ldots, u_{j-1}}$. It suffices to prove that for every $1 \leq j \leq d$, and every $u_1 \in U_{u_1, \ldots, u_{j-1}}$, the codimension of $(u_1 \otimes \cdots \otimes u_{j-1} \otimes U_{u_1, \ldots, u_{j-1}}) \cap \bigcap_{j \in I} (L_I \otimes \mathbb{F}^d)$ in $u_1 \otimes \cdots \otimes u_{j-1} \otimes U_{u_1, \ldots, u_{j-1}}$ is at most $2^dl$. Thus, it suffices to prove that for every $I \subset [j]$ with $j \in I$, the codimension of $(u_1 \otimes \cdots \otimes u_{j-1} \otimes U_{u_1, \ldots, u_{j-1}}) \cap (L_I \otimes \mathbb{F}^d)$ in $u_1 \otimes \cdots \otimes u_{j-1} \otimes U_{u_1, \ldots, u_{j-1}}$ is at most $l$. But this is equivalent to the statement that $((\bigotimes_{i \in I(j)} u_i) \otimes U_{u_1, \ldots, u_{j-1}}) \cap L_I$ has codimension at most $l$ in $(\bigotimes_{i \in I(j)} u_i) \otimes U_{u_1, \ldots, u_{j-1}}$, which clearly holds. \qed

2.4 Finishing the proof of Lemma 2.1

Definition 2.10. Let $k$ be a positive integer and let $\epsilon > 0$. Let $Q$ be a multiset with elements chosen from $\mathcal{G}$ (with arbitrary multiplicity). We say that $Q$ is $(k, \alpha)$-forcing if the set of all arrays $r \in \mathcal{G}$ with $r.q = 0$ for at least $\alpha |Q|$ choices $q \in Q$ is contained in a set of the form \( \sum_{I \subset [d], |I| \neq 0} V_I \otimes \mathbb{F}^d \) for some $V_I \subset \mathbb{F}^d$ of dimension at most $k$.

We now turn to the main part of the proof of Lemma 2.1. For each $I \subset [d - 1]$ we will construct a corresponding $Q_I$ as defined in the next result, and (roughly) we will take $Q = \bigcup_I Q_I$.
Lemma 2.11. Suppose that Lemma 2.1 has been proved for every $d' < d$. Let $B' \subset B$ have $|B'| \geq \delta |B|$ for some $\delta > 0$. Let $k \geq \text{tower}_{8|k|}((d + 3)^{d+3}, 1/\delta)$ arbitrary, let $I \subset [d - 1], I \neq \emptyset$, and let $W_J \subset \mathbb{F}^J$ be subspaces of dimension at most $k$ for every $J \subset I, J \neq I, J \neq \emptyset$. Then there exist a multiset $Q'$, and a multiset $Q_s$ for each $s \in Q'$ with the following properties.

1. The elements of $Q'$ are chosen from $\bigcap_{J \subset I, J \neq I}(W_J^I \otimes \mathbb{F}^{|J|})$
2. $Q'$ is $(f_1, 1 - f_2)$-forcing with $f_1 = \text{tower}_{8|I|}((d + 3)^{|J|+4} + 2, k)$, $f_2 = 2^{-3^{d+2}}$
3. For each $s \in Q'$, the elements of $Q_s$ are chosen from $\mathbb{F}^{|I|}$
4. For each $s \in Q'$, $Q_s$ is $(f_3, 1 - f_4)$-forcing with $f_3 = \text{tower}_{8|I|}(4, k)$, $f_4 = 2^{-3^{d+2}}$
5. $\max_{s \in Q'} |Q_s| \leq 2 \min_{s \in Q'} |Q_s|$
6. The elements of the multiset $Q_I := \{s \otimes t : s \in Q', t \in Q_s\} = \bigcup_{s \in Q'} (s \otimes Q_s)$ are chosen from $f_3B' - f_3B'$ with $f_3 = 2^{3^{d+1}}$.

Proof. By symmetry, we may assume that $I = [a]$ for some $1 \leq a \leq d - 1$. Let $C$ be the multiset $\{u_1 \otimes \cdots \otimes u_a : u_i \in \mathbb{F}^{|I|}\}$ and let $D$ be the multiset $\{u_{a+1} \otimes \cdots \otimes u_d : u_i \in \mathbb{F}^{|I|}\}$. For each $s \in C$, let $D_s = \{t \in D : s \otimes t \in B'\}$. Also, let $C' = \{s \in C : |D_s| \geq \delta |D|\}$. Clearly, $|C'| \geq \delta |C|$. By Lemma 2.8, there exists a $g_1$-system $R$ (with respect to $\mathbb{F}^{|I|}$) with elements chosen from $g_2C' - g_2C'$ with $g_1 = \frac{4^{16d}}{\delta^2}$ and $g_2 = 4^d$. By Lemma 2.9, $R \cap \bigcap_{J \subset I, J \neq I}(W_J^I \otimes \mathbb{F}^{|J|})$ contains a $g_3$-system $T'$ for $g_3 = \frac{4^{16d}}{\delta^2} + 2^d k$. Now $|T'| \geq |\mathbb{F}^{|I|}|C|$. By Lemma 2.1 (applied to $a$ in place of $d$), it follows that there exists a multiset $Q'$ whose elements are chosen from $g_4T' - g_4T'$ and which is $(g_5, 1 - g_6)$-forcing for $g_4 = 2^{3^{a+3}} \leq 2^{3^{d+2}}$, $g_5 = \text{tower}_{8|I|}((a + 4)^{a+4}, |\mathbb{F}|^{d+3}) \leq \text{tower}_{8|I|}((d + 3)^{a+4}, |\mathbb{F}|^{d+3})$ and $g_6 = 2^{-3^{a+3}} \geq 2^{-3^{d+2}}$. But $g_3 \leq 2 \cdot 2^d k$, so $|\mathbb{F}|^{d+3} \leq \text{tower}_{8|I|}(2, k)$, therefore $Q'$ satisfies (1) and (2) in the statement of this lemma.

By Lemma 2.8, for each $s \in C'$ there exists a $g_7$-system $R_s$ (with respect to $\mathbb{F}^{|I|}$) contained in $g_8D_s - g_8D_s$, where $g_7 = \frac{4^{16d}}{\delta^2}$ and $g_8 = 4^d$. For every $s \in Q'$, choose $s_1, \ldots, s_{l'+l'} \in C'$ with $l, l' \leq 2^{3^{d+3}}$ such that $s = s_1 + \cdots + s_l - s_{l+1} - \cdots - s_{l+l'}$ (this is possible, since the elements of $Q'$ are chosen from $2g_2g_4C' - 2g_2g_4C'$ and $2g_2g_4 \leq 2^{3^{d+3}}$), and let $P_s = \bigcap_{l \leq l'} R_s$. By Lemma 2.7, $P_s$ contains a $g_9$-system with $g_9 = 2 \cdot 2^{3^{d+3}} \cdot \frac{4^{16d}}{\delta^2}$, therefore $|P_s| \geq g_{10} |D|$ for $g_{10} = |\mathbb{F}|^{-d |I|}$. By Lemma 2.1 (applied to $d - a$ in place of $d$), for every $s \in Q'$ there exists a multiset $Q_s$ with elements chosen from $g_{11}P_s - g_{11}P_s$ which is $(g_{12}, 1 - g_{13})$-forcing for $g_{11} = 2^{-3^{a+3}} \leq 2^{3^{d+2}}$, $g_{12} = \text{tower}_{8|I|}((d-a+4)^{d-a+4}, |\mathbb{F}|^{d+3}) \leq \text{tower}_{8|I|}((d+3)^{d+3}, |\mathbb{F}|^{d+3}) \leq \text{tower}_{8|I|}((d+3)^{d+3} + 4, 1/\delta) \leq \text{tower}_{8|I|}(4, k)$ and $g_{13} = 2^{-3^{a+3}} \geq 2^{-3^{d+2}}$. Notice that if we repeat every element of $Q_s$ the same number of times, then the multiset obtained is still $(g_{12}, 1 - g_{13})$-forcing, so we may assume that
\[ \max_{s \in Q'} |Q_s| \leq 2 \min_{s \in Q'} |Q_s| \]  
Define \( Q_I = \{ s \otimes t : s \in Q', t \in Q_s \} = \bigcup_{s \in Q'} (s \otimes Q_s) \). Note that as 
\( R_s \subset g_8 D_s - g_8 D_s \) for all \( s \in C' \), we have \( s \otimes R_s \subset g_8 B' - g_8 B' \) for all \( s \in C' \). But the elements of \( Q' \) are chosen from \( 2g_2 g_4 C' - 2g_2 g_4 C' \), so \( s \otimes P_s \subset 4g_2 g_4 g_8 B' - 4g_2 g_4 g_8 B' \) for all \( s \in Q' \). 
Finally, the elements of \( Q_s \) are chosen from \( g_{11} P_s - g_{11} P_s \), so the elements of \( s \otimes Q_s \) are chosen from \( 8g_2 g_4 g_8 g_{11} B' - 8g_2 g_4 g_8 g_{11} B' \) for every \( s \in Q' \). Since \( 8g_2 g_4 g_8 g_{11} \leq 8 \cdot (4^d)^2 \cdot (2^{3d+2})^2 = 2^{3+4d+2.3d+2} \leq 2^{3d+3} \), property (6) is satisfied. \( \square \)

**Remark 2.12.** Forcing condition (1) in Lemma 2.11 is the most expensive step in the proof of Lemma 2.1. That condition is the reason why \( f_i \) is so large in (2).

We have already seen in Lemma 2.4 that (for suitably chosen \( Q \)) the set of \( r \in G \) that satisfy \( r.q = 0 \) for most \( q \in Q \) are of the form \( r_1 + r_2 \) where \( r_1 \) lives in a fixed small subspace of \( G \) and \( r_2 \) is \( k \)-degenerate for some small \( k \). The next lemma allows us to turn the subspaces witnessing the \( k \)-degeneracy of \( r_2 \) into slightly larger subspaces which however do not depend on \( r \). This is done one by one, in an order determined by which \( \mathbb{P}^F \) the subspace lives in. The order in which these index sets are considered is not arbitrary: we define \( < \) to be any total order on the set of non-empty subsets of \( [d - 1] \) such that if \( J \subset I \) then \( J < I \).

**Lemma 2.13.** Let \( k \geq \text{tower}_{8g}(d + 3)^{4d+4}, 1/\delta \) arbitrary. Let \( I \subset [d - 1], I \neq \emptyset \) and let \( W_J \subset \mathbb{P}^I, W_{J'} \subset \mathbb{P}^F \) be subspaces of dimension at most \( k \) for every \( J < I \). Moreover, let \( W_{[d]} \subset \mathbb{P}^d \) have dimension at most \( k \). Suppose that \( Q', Q_s, (and Q_I) \) have the six properties described in Lemma 2.11. Then any array \( r \in W_{[d]} + \sum_{J \not\subset I} (W_J \otimes \mathbb{P}^F + \mathbb{P}^I \otimes W_{J'}) + \sum_{J \subset I} \mathbb{P}^I \otimes H_J(r) \) with \( \dim(H_J(r)) \leq k \) and the property that \( r.q = 0 \) for at least \( 1 - \frac{1}{4}(2^{-3d+2})^2 |Q_I| \) choices \( q \in Q_I \) is contained in \( W_{[d]} + \sum_{J \not\subset I} (U_J \otimes \mathbb{P}^F + \mathbb{P}^I \otimes U_{J'}) + \sum_{J \subset I} \mathbb{P}^I \otimes K_J(r) \) for some \( U_J \subset \mathbb{P}^I, U_{J'} \subset \mathbb{P}^F \) not depending on \( r \) and some \( K_J(r) \subset \mathbb{P}^F \) possibly depending on \( r \), all of dimension at most \( \text{tower}_{8g}(d + 3)^{4d+4}, 3, k \).

**Proof.** By (4) in Lemma 2.11, for every \( s \in Q' \) there exist subspaces \( V_J(s) \subset \mathbb{P}^I \) for every \( J \subset I', J \neq \emptyset \), with dimension at most \( g_1 = \text{tower}_{8g}(4, k) \) such that the set of arrays \( t \in \mathbb{P}^F \) with \( t.q = 0 \) for at least \( 1 - g_2 |Q_s| \) choices \( q \in Q_s \) is contained in \( \sum_{J \not\subset I} V_J(s) \otimes \mathbb{P}^F \setminus t \), where \( g_2 = 2^{-3d+2} \). If \( r \in G \) has \( r.q = 0 \) for at least \( 1 - \frac{1}{4}(2^{-3d+2})^2 |Q_I| \) choices \( q \in Q_I \), then by averaging and using (5) from Lemma 2.11, for at least \( 1 - g_3 |Q'| \) choices \( s \in Q' \) we have \( r.s \otimes t = 0 \) for at least \( 1 - g_2 |Q_s| \) choices \( t \in Q_s \), where \( g_3 = \frac{1}{4} 2^{-3d+2} \). Thus, (noting that \( r.s \otimes t = (r.s).t \), \( r.s \in \sum_{J \subset I', J \neq \emptyset} V_J(s) \otimes \mathbb{P}^F \setminus t \) holds for at least \( 1 - g_3 |Q'| \) choices \( s \in Q' \). Let \( Q'(r) \) be the submultiset of \( Q' \) consisting of those \( s \in Q' \) for which \( r.s \in \sum_{J \subset I', J \neq \emptyset} V_J(s) \otimes \mathbb{P}^F \setminus t \). Then we have \( |Q'(r)| \geq (1 - g_3)|Q'| \).
Note that we can write $r = r_1 + r_2 + r_3 + r_4$ where $r_1 \in \sum_{J\cup I, J\neq I} W_J \otimes \mathbb{P}^J$, $r_2 \in \sum_{J\cup I, J\neq I} (W_J \otimes \mathbb{P}^J \oplus \mathbb{P}_J \otimes W_I)$, $r_3 \in W_{[d]} + \sum_{J\cup I, J\neq I} \mathbb{P}^J \otimes W_I$ and $r_4 \in \mathbb{P}^J \otimes H_P(r)$. By (1) in Lemma 2.11, the elements of $Q'$ belong to $\bigcap_{J\cup I, J\neq I} (W_J \otimes \mathbb{P}^J)$, so we have $r_1 s = 0$ for every $s \in Q'$. Since $\text{dim}(W_J), \text{dim}(W_I), \text{dim}(H_P(r)) \leq k$, $r_2 s$ is $2^d k$-degenerate. Also, $r_3 s \in \sum_{J\cup I, J\neq I} ((\mathbb{P}^J \otimes W_I) s)$. (Here and below, for a subspace $L \subset G$ and an array $s \in \mathbb{P}^J$, we write $L s$ for the subspace $\{rs : r \in L\} \subset \mathbb{P}^J$.) It follows that for every $s \in Q'(r)$, there exists some $t(s) \in V_P(s) + \sum_{J\cup I, J\neq I} ((\mathbb{P}^J \otimes W_I) s)$ such that $r_4 s - t(s)$ is $g_4$-degenerate for $g_4 = g_1 + 2^d k$ (we have used that $\text{dim}(V_P(s)) \leq g_1$).

Now let $g_5 = \frac{g_1}{g_4}$. Let $X$ be the subset of $\mathbb{P}^J$ consisting of those arrays $x$ for which for at least $g_5 |Q'|$ choices $s \in Q'$, there exists some $t(s) \in V_P(s) + \sum_{J\cup I, J\neq I} ((\mathbb{P}^J \otimes W_I) s)$ such that $x - t(s)$ is $g_4$-degenerate. Choose a maximal subset $\{x_1, \ldots, x_m\} \subset X$ such that no $x_i - x_j$ with $i \neq j$ is $2 g_4$-degenerate. Then there do not exist $i \neq j$, $s \in Q'$ and $t \in V_P(s) + \sum_{J\cup I, J\neq I} ((\mathbb{P}^J \otimes W_I) s)$ with $x_i - t$ and $x_j - t$ both $g_4$-degenerate. It follows, by the definition of $X$, and using that the dimension of $V_P(s) + \sum_{J\cup I, J\neq I} ((\mathbb{P}^J \otimes W_I) s)$ is at most $g_1 + 2^d k$ that $m g_5 |Q'| \leq |Q'| \cdot |\mathbb{P}^{g_4+2^d k}|$, therefore $m \leq g_6 = \frac{\left(\frac{g_1}{g_4}\right)^{g_4+2^d k}}{g_5}$. Let $Z = \text{span}(x_1, \ldots, x_m)$. Then $\text{dim}(Z) \leq g_6$ and for every $x \in X$, there is some $z \in Z$ such that $x - z$ is $2 g_4$-degenerate.

By the definition of $X$, if $t \notin X$, then the number of choices $s \in Q'(r)$ for which $r_4 s = t$ is at most $g_5 |Q'|$. On the other hand, notice that $r_4 s \in H_P(r)$ for every $s \in Q'$. Since $|H_P(r)| \leq |\mathbb{P}^k|$, it follows that $r_4 s \in X$ for at least $|Q'(r)| - |\mathbb{P}^k g_5 |Q'| = |Q'(r)| - g_5 |Q'| \geq (1 - 2 g_5) |Q'| \geq (1 - 2^{-3^{d + 2}}) |Q'|$ choices $s \in Q'$.

Let $X(r) = X \cap H_P(r)$. Let $t_1, \ldots, t_\alpha$ be a maximal linearly independent subset of $X(r)$ and extend it to a basis $t_1, \ldots, t_\alpha, t'_1, \ldots, t'_\beta$ for $H_P(r)$. Now if a linear combination of $t_1, \ldots, t_\alpha, t'_1, \ldots, t'_\beta$ is in $X$, then the coefficients of $t'_1, \ldots, t'_\beta$ are all zero. Write $r_4 = \sum_{i \leq \alpha} s_i \otimes t_i + \sum_{\alpha < j \leq \beta} s'_j \otimes t'_j$ for some $s_i, s'_j \in \mathbb{P}^J$. Since $r_4 q \in X$ for at least $(1 - 2^{-3^{d+2}}) |Q'|$ choices $q \in Q'$, we have, for all $j$, that $s'_j q = 0$ for at least $(1 - 2^{-3^{d+2}}) |Q'|$ choices $q \in Q'$. Thus, by (2) in Lemma 2.11 there exist subspaces $L_j \subset \mathbb{P}^I (J \subset I, J \neq \emptyset)$ not depending on $r$, and of dimension at most $\text{tower}_{8|\mathbb{P}|}(d + 3)^{|I|+4} + 2, k)$ such that $s'_j \in \sum_{J \cup I, J \neq I} L_j \otimes \mathbb{P}^I$ for all $j$. Thus, $r_4 \in \sum_{i \leq \alpha} s_i \otimes t_i + \sum_{\emptyset \neq J} \sum_{J \cup I \neq J} L_j \otimes \mathbb{P}^I$. Moreover, for every $i \leq \alpha$, we have $t_i \in X$, so there exist $z_i \in Z$ such that $t_i - z_i$ is $2 g_4$-degenerate. It follows that $r_4 \in \mathbb{P}^I \otimes Z + \sum_{J \cup I, J \neq I} \mathbb{P}^J \otimes K'_j(r) + \sum_{J \cup I, J \neq I} L_j \otimes \mathbb{P}^I$ for some $K'_j(r) \subset \mathbb{P}^I$ of dimension at most $\alpha \cdot 2 g_4 \leq k \cdot 2 g_4$.

We claim that $\text{dim}(Z), \text{dim}(K'_j)$ and $\text{dim}(L_j)$ are all bounded by $\text{tower}_{8|\mathbb{P}|}((d + 3)^{|I|+4} + 2, k)$. If this holds, then the proof of this lemma is complete, since $k + \text{tower}_{8|\mathbb{P}|}((d + 3)^{|I|+4} + 2, k) \leq \text{tower}_{8|\mathbb{P}|}((d + 3)^{|I|+4} + 3, k)$.

Now $\text{dim}(K'_j) \leq 2 k g_4 = 2 k (g_1 + 2^d k) \leq 2 k (\text{tower}_{8|\mathbb{P}|}(4, k) + 2^d k) \leq \text{tower}_{8|\mathbb{P}|}((d + 3)^{|I|+4} + 2, k)$.
Also, \( \dim(Z) \leq g_6 = \frac{[\mathcal{R}^{d+2}]_{g_5}}{g_5} = 2^{3d+2+1} [\mathcal{R}]_{I+2d+k} \leq [\mathcal{R}]_{d+4} \leq \text{tower}_{\mathcal{R}[5]}((d + 3)^{d+2} + 2, k) \).

Finally, as we have already noted, \( \dim(L_j) \leq \text{tower}_{\mathcal{R}[5]}((d + 3)^{d+2} + 2, k) \). This completes the proof of the claim and the lemma.

\[ \square \]

**Proof of Lemma 2.1.** As stated earlier, the proof goes by induction on \( d \). For \( d = 1 \), by Bogolyubov’s lemma there is a subspace \( U \subset \mathbb{R}^n \) of codimension at most \( 1/\delta^2 \) contained in \( 2B' - 2B' \). Choose \( Q = U \). Now if \( r \cdot q = 0 \) for at least \((1 - 2^{-3})|Q|\) choices \( q \in Q \) then the same holds for all \( q \in Q \), therefore \( r \in U^\perp \), but \( \dim(U^\perp) \leq 1/\delta^2 \leq \text{tower}_{\mathcal{R}[5]}(5^3, 1/\delta) \), so the case \( d = 1 \) is proved.

Now let us assume that \( d \geq 2 \). Extend the total order \( \prec \) defined above such that it now contains \( 0 \) which has \( 0 < I \) for every non-empty \( I \subset [d-1] \). Say \( 0 = I_0 < I_1 < I_2 < \cdots < I_{2^d-1-1} \) where \( \{I_0, \ldots, I_{2^d-1-1}\} = P([d-1]) \).

**Claim.** For every \( 0 \leq i \leq 2^d-1 - 1 \) there exists a multiset \( Q_{\ell} \) with elements chosen from \( 2^{3d+3}B' - 2^{3d+3}B' \), and subspaces \( W_j(i) \subset \mathbb{R}^{d_j}, W_{(i)}(i) \subset \mathbb{R}^{d_{(i)}} \) for every \( j \leq i \) (for \( j = 0 \), we only require \( W_{[d]}(i) \) and not \( W_0(i) \)) of dimension at most

\[
g_1(i) = \text{tower}_{\mathcal{R}[5]}((d + 3)^{d+3} + 3 + \sum_{1 \leq j \leq i} ((d + 3)^{d+3} + 3, 1/\delta))
\]

with the following property. If \( r \in \mathcal{G} \) has \( r \cdot q = 0 \) for at least \((1 - \frac{1}{4}(2^{-3d-2}))|\mathcal{Q}_i|\) choices \( q \in \mathcal{Q}_i \) for all \( j \leq i \), then \( r \in W_{[d]}(i) + \sum_{1 \leq j \leq i} (W_j(i) \otimes \mathcal{R}^{d_{(j)}} + \mathcal{R}^{d_j} \otimes W_{(j)}(i)) + \sum_{j \geq i} \mathcal{R}^{d_j} \otimes H_{(j)}(i, r) \) holds for some \( \mathcal{H}_{(j)}(i, r) \) possibly depending on \( r \) and of dimension at most \( g_1(i) \).

**Proof of Claim.** This is proved by induction on \( i \). For \( i = 0 \), by Lemma 2.4, there exist \( Q_0 \subset 2B' - 2B' \) and \( V_{[d]} \subset \mathcal{R}^{[d]} \) of dimension at most \( 5[\mathcal{R}]_{d} / 2^2 \) such that if \( r \cdot q = 0 \) for at least \( \frac{1}{4} |Q_0| \) choices \( q \in Q_0 \), then \( r \) can be written as \( r = x + y \) where \( x \in V_{[d]} \) and \( y \) is \( g_2 \)-degenerate for \( g_2 = \text{tower}_{\mathcal{R}[5]}((d + 3)^{d+3} + 3, 1/\delta) \). Hence we can take \( W_{[d]}(0) = V_{[d]} \).

Once we have found suitable sets \( W_j(i-1) \) and \( W_{(i)}(i-1) \) for all \( j \leq i-1 \), we can apply Lemmas 2.11 and 2.13 with \( I = I_i \) and \( k = g_1(i-1) \) to find a suitable \( Q_i, W_j(i) \) and \( W_{(i)}(i) \) for all \( j \leq i \), and the claim is proved, since \( g_1(i) = \text{tower}_{\mathcal{R}[5]}((d + 3)^{d+3} + 3, g_1(i-1)) \).

Now, after taking several copies of each \( Q_i \), we may assume that additionally \( \max_{I} |Q_I| \leq 2 \min_{I} |Q_I| \). Let \( Q = \bigcup_{I \subset [d-1]} Q_I \) and suppose that \( r \cdot q = 0 \) for at least \((1 - 2^{-3d+2})|Q|\) choices \( q \in Q \). Since \( 2^{-3d+2} \leq \frac{1}{2} 2^{-d} \cdot \frac{1}{4} (2^{-3d+1})^2 \), it follows that for every \( I \subset [d-1] \) we have \( r \cdot q = 0 \) for at least \((1 - \frac{1}{4}(2^{-3d+2}))|Q_I|\) choices \( q \in Q_I \). By the Claim with \( i = 2^d - 1 \), we get that

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\[ r \in \sum_{I \subseteq [d]} V_I \otimes \mathbb{F}^r \] for some \( V_I \subset \mathbb{F}^d \) not depending on \( r \), and of dimension at most
\[
g_1(2^{d-1} - 1) = \text{tower}_{8\mathbb{N}}((d + 3)^{d+3} + 3 + \sum_{\emptyset \neq I \subseteq [d-1]} ((d + 3)^{|I|+4} + 3), 1/\delta).\]

But
\[
(d + 3)^{d+3} + 3 + \sum_{\emptyset \neq I \subseteq [d-1]} ((d + 3)^{|I|+4} + 3) \leq (d + 3)^{d+3} + 3 \cdot 2^{d-1} + (d + 3)^d \sum_{1 \leq k \leq d-1} \binom{d-1}{k} (d + 3)^k \\
\leq (d + 3)^{d+3} + 3 \cdot 2^{d-1} + (d + 3)^d (d + 4)^{d-1} \\
\leq (d + 3)^{d+3} + 3 \cdot 2^{d-1} + (d + 4)^{d+3} \\
\leq (d + 4)^{d+4}
\]

This completes the proof of the lemma. \(\square\)

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