Lattice distortion inducing electron-hole pair condensation in two dimensional systems

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Abstract. A stability of the electron-hole pair condensation at low temperature has been addressed by involving the coupling of exciton to vibrational degrees of freedom in two-dimensional two-band $f - c$ electron system. By mean of the unrestricted Hartree-Fock approximation, we find a formation of the insulating state typifying an excitonic condensate accompanied by a finite lattice distortion if the exciton-phonon coupling is large enough. As functions of temperature both excitonic condensation order parameter and lattice distortion behave in a same way which manifests the continuous transition in analogy to the superconductivity in the BCS theory. Inspecting to the microscopic properties in momentum space we strongly specify the BCS type of the excitonic condensation driven by the exciton-phonon interaction at low temperature.

1. Introduction

Exciton is an electrically neutral quasiparticles formed by coupling of electron and hole. It exists in insulators, semiconductors and in some liquids. The first theory of exciton was proposed by Yakov Frenkel in 1931 [1] when he was describing excitation of the atoms in the crystal lattice of insulators. Normally, quasiparticle excitons do not form the ground state because electrons and holes can reunite easily and radiate photons, or excitons can decay due to lattice defects. However, at the semimetal-semiconductor transition, the ground state of the crystals may become unstable with the spontaneous formation of excitons, provided the very small overlap or band gap between the valence and conduction bands. At low temperature, these composite quasiparticle bosons might condense and a macroscopic phase-coherent quantum state can be created. The semimetallic or semiconducting conformation in this situation would transform to an insulating state. This new state is called an insulator excitonic - EI phase, which had been theoretically proposed since the 1960s [2, 3]. The EI phase received below the critical temperature $T_{EI}$ may be similar to BCS condensate in the semimetal region or Bose Einstein condensate in the semiconductor region [4].
In Eq.(1) we have brought in Fourier-transformed quantities interaction when considering to annihilate an f electron correspondingly create a hole in the excitations and lattice displacements. It can be regarded as an effective "exciton"-phonon interaction with coupling constant $g$.\( \mu \) denotes the chemical potential. The last term in Eq.(1) expresses a local electron-phonon interaction, in the expanded Falicov-Kimball model in both one-dimensional \[9\] and two-dimensional \[10\]. Besides, the quasi-2D transition-metal dichalcogenide $1T - TiSe_2$, is the semimetal material, the temperature dependence of electron spectroscopy confirmed the existence of EI phase \[7\].

In this paper, we study the phase transition in two-dimensional two-band $f - c$ electron system. Our main purpose is considering the electron-hole pairs condensed process due to temperature and intensity electron-phonon interaction by the mean-field theory but only being interested in the effects of electron-phonon interaction and disregarding the Coulomb interaction. This model has been examined meticulously in one-dimensional by the projector-based renormalization method at the ground state temperature $T = 0$ and only noticing the effects of electron-phonon interaction, too \[8\]. Or investigating the influence of the Coulomb interaction strength, excitation energy of $f$ electrons,... while ignoring electron-phonon interaction, the temperature dependence of electron spectroscopy confirmed the existence of EI phase \[7\].

This paper is organized as follows. Section 2 presents the model. In Section 3 we use the mean-field theory to calculate directly the energies and the average particle densities. Section 4 presents our numerical results: temperature dependent of both order parameter and lattice distortion in the EI phase. The conclusion is given in Section 5.

2. Model

We start to consider the Hamiltonian of the system contains two types of spinless electrons ($c, f$) carrying momentum $k$ and dispersionless phonons ($b$)

$$
\mathcal{H} = \sum_k \varepsilon_{k}^{f} \hat{f}_{k}^{\dagger} \hat{f}_{k} + \sum_k \varepsilon_{k}^{c} \hat{c}_{k}^{\dagger} \hat{c}_{k} + \omega_0 \sum_{\mathbf{q}} \hat{b}_{\mathbf{q}}^{\dagger} \hat{b}_{\mathbf{q}} + \frac{g}{\sqrt{N}} \sum_{k \mathbf{q}} [\hat{c}_{k+\mathbf{q}}^{\dagger} \hat{f}_{k} (\hat{b}_{\mathbf{q}}^{\dagger} + \hat{b}_{\mathbf{q}}) + H.c.],
$$

(1)

where $\omega_0$ is the dispersionless phonon energy and the electronic excitation energies are given by

$$
\varepsilon_{k}^{f,c} = \varepsilon_{k}^{f,c} - tf,c \delta_{k} - \mu,
$$

(2)

with $\varepsilon_{k}^{f,c}$ are the on-site energies, $tf,c$ are the nearest-neighbor particle transfer amplitudes, $\gamma_{k} = 2(\cos k_x + \cos k_y)$ indicates a nearest-neighbor hopping in a two-dimensional lattice and $\mu$ denotes the chemical potential. The last term in Eq.(1) expresses a local electron-phonon interaction with coupling constant $g$, written in $k$ space, between local $f - c$ particle-hole excitations and lattice displacements. It can be regarded as an effective "exciton"-phonon interaction when considering to annihilate an $f$ electron correspondingly create a hole in the valence band.

In Eq.(1) we have brought in Fourier-transformed quantities $f_k^\dagger = \left( \frac{1}{\sqrt{N}} \right) \sum_i f_i^\dagger e^{i \mathbf{k} \mathbf{R}_i}$, $c_k^\dagger = \left( \frac{1}{\sqrt{N}} \right) \sum_i c_i^\dagger e^{i \mathbf{k} \mathbf{R}_i}$, and $b_q^\dagger = \left( \frac{1}{\sqrt{N}} \right) \sum_i b_i^\dagger e^{i \mathbf{q} \mathbf{R}_i}$, where $f_i^\dagger$, $c_i^\dagger$, and $b_i^\dagger$ are the local quantities. $N$ counts the number of lattice sites $i$.

In what follows, we study the half - filled band case, with total electron density

$$
\langle n \rangle = \langle n_i^f \rangle + \langle n_i^c \rangle = 1,
$$

(3)
with \( n^f_i = \left( \frac{1}{\pi} \right) \sum_k f^\dagger_k f_k \) and \( n^c_i = \left( \frac{1}{\pi} \right) \sum_k c^\dagger_k c_k \). The chemical potential \( \mu \) has to satisfy Eq.(3). In this paper, we choose \( \varepsilon^c = 1 \) as the unit of energies and calculating with \( \varepsilon^c = 0, \varepsilon^f = -1, t^f = -0.3 \). Therefore, the \( c \) electrons will be considered as "light" while the \( f \) electrons (respectively holes) are "heavy", i.e., \( |t_f| < 1 \). For \( t_c t_f < 0 \), leading to a picture of direct \( c - f \) hopping, which proposes a possible condensation of bound \( c - f \)-electron-hole pairs with finite momentum

\[
d_k = \langle c_{k+Q} f_k \rangle \neq 0,
\]

where \( Q = (\pi, \pi) \) in two-dimensional. Nonvanishing \( d_k \) leads to the spontaneous symmetry breaking, so small infinitesimal fields must be taken into Eq.(1). We write

\[
\mathcal{H} = \sum_k \varepsilon_k^c f_k^\dagger f_k + \sum_k \varepsilon_k^c c_k^\dagger c_k + \omega_0 \sum_q b_q^\dagger b_q + \Delta_0 \sum_k \left( c_{k+Q}^\dagger f_k + f_k^\dagger c_{k+Q} \right) + \sqrt{N} h_0 \left( b_{-Q}^\dagger + b_Q \right) + \frac{g}{\sqrt{N}} \sum_{kq} \left[ c_{k+Q}^\dagger f_k \left( b_{-q}^\dagger + b_q \right) + H.c. \right],
\]

where \( \Delta_0 = 0^+ \) and \( h_0 = 0^+ \) while the fields \( h_0 \) and \( \Delta_0 \) are mutually dependent. Furthermore, from \( b_Q = b_{-Q} \), we can substitute \( \sqrt{N} h_0 \left( b_{-Q}^\dagger + b_Q \right) \) for the field contribution \( \sqrt{N} h_0 \left( b_{-Q}^\dagger + b_Q \right) \).

### 3. Mean-field theory

In the mean-field limit, we can perform approximately interaction operators in Hamiltonian (5) by introducing fluctuation operators \( \delta \mathcal{A} = \mathcal{A} - \langle \mathcal{A} \rangle \) in the electron (exciton)-phonon interaction. We get

\[
\begin{align*}
\delta \left( c_{k+Q}^\dagger f_k \right) \delta \left( b_{-q}^\dagger + b_q \right) = & \quad \varepsilon_k^c \varepsilon^f_k + \sum_{kq} \varepsilon_k^c \varepsilon^f_k c_k^\dagger c_k + \omega_0 \sum_q b_q^\dagger b_q + \left[ \langle c_{k+Q}^\dagger f_k \rangle \left( b_{-q}^\dagger + b_q \right) \right. \\
+ & \left. \left. \left( b_{-q}^\dagger + b_q \right) \delta_{Q,0} \right] + \langle c_{k+Q}^\dagger f_k \rangle \left( b_{-q}^\dagger + b_q \right) \delta_{Q,0}, \right.
\end{align*}
\]

Then, the Hamiltonian \( \mathcal{H} \) is rewritten

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1
\]

with

\[
\mathcal{H}_0 = \sum_k \varepsilon_k^c f_k^\dagger f_k + \sum_k \varepsilon_k^f c_k^\dagger c_k + \omega_0 \sum_q b_q^\dagger b_q + \Delta \sum_k \left( c_{k+Q}^\dagger f_k + f_k^\dagger c_{k+Q} \right) + \sqrt{N} h_0 \left( b_{-Q}^\dagger + b_{-Q} \right),
\]

\[
\mathcal{H}_1 = \frac{g}{\sqrt{N}} \sum_{kq} \left[ \delta \left( c_{k+Q}^\dagger f_k \right) \delta \left( b_{-q}^\dagger + b_q \right) + H.c. \right].
\]

Here the fields have gotten additional shifts, which will play as order parameters in the following

\[
\begin{align*}
\Delta &= \Delta_0 + \frac{g}{\sqrt{N}} \langle b_{-Q}^\dagger + b_{-Q} \rangle, \\
h &= h_0 + \frac{g}{N} \sum_k \langle c_{k+Q}^\dagger f_k + f_k^\dagger c_{k+Q} \rangle.
\end{align*}
\]
Note that the infinitesimal $\Delta_0 = 0^+$ and $\hbar_0 = 0^+$ can be neglected.

Finally, we exclude the term $\propto (b_+^{\dagger}Q + b_-Q)$ in Eq. (8) by defining new phonon operators

$$B_q^{\dagger} = b_q^{\dagger} + \sqrt{N} (\hbar/\omega_0) \delta_{q,Q}. \quad (12)$$

Using $\delta B_{-q}^{\dagger} = \delta b_{-q}^{\dagger}$ and $\delta B_q = \delta b_q$ we get

$$\mathcal{H}_0 = \sum_k \varepsilon_k f_k^{\dagger} f_k + \sum_k \varepsilon_k c_k^{\dagger} c_k + \omega_0 \sum_q B_q^{\dagger} B_q$$

$$+ \Delta \sum_k \left( e_{k+Q}^{\dagger} f_k + f_k^{\dagger} c_{k+Q} \right) + \text{const}, \quad (13)$$

$$\mathcal{H}_1 = \frac{g}{\sqrt{N}} \sum_{kq} \left[ \delta \left( c_{k+Q}^{\dagger} f_k \right) \delta \left( B_{-q}^{\dagger} + B_q \right) + \text{H.c.} \right]. \quad (14)$$

Neglecting the fluctuation part $\mathcal{H}_1$ and annuling constant in the Eq. (13), the Hamiltonian in mean-field approximation is given

$$\mathcal{H}_{MF} = \sum_k \varepsilon_k f_k^{\dagger} f_k + \sum_k \varepsilon_k c_k^{\dagger} c_k + \omega_0 \sum_q B_q^{\dagger} B_q$$

$$+ \Delta \sum_k \left( e_{k+Q}^{\dagger} f_k + f_k^{\dagger} c_{k+Q} \right). \quad (15)$$

We have defined quasiparticle operators

$$C_{1,k}^{\dagger} = \xi_k c_{k+Q} + \eta_k f_k, \quad (16)$$

$$C_{2,k}^{\dagger} = -\eta_k c_{k+Q} + \xi_k f_k, \quad (17)$$

with

$$\xi_k^2 = \frac{1}{2} \left[ 1 + \text{sgn} \left( \varepsilon_k^f - \varepsilon_{k+Q}^c \right) \frac{\varepsilon_k^f - \varepsilon_{k+Q}^c}{W_k} \right], \quad (18)$$

$$\eta_k^2 = \frac{1}{2} \left[ 1 - \text{sgn} \left( \varepsilon_k^f - \varepsilon_{k+Q}^c \right) \frac{\varepsilon_k^f - \varepsilon_{k+Q}^c}{W_k} \right], \quad (19)$$

and

$$W_k = \sqrt{(\varepsilon_{k+Q}^c - \varepsilon_k^f)^2 + 4|\Delta|^2}. \quad (20)$$

Using the Bogoliubov transformation, the electronic part of $\mathcal{H}_{MF}$ is diagonalized and then the Hamiltonian $\mathcal{H}_{MF}$ reads

$$\mathcal{H}_{MF} = \sum_k E_k^c C_{1,k}^{\dagger} C_{1,k} + E_k^f C_{2,k}^{\dagger} C_{2,k} + \omega_0 \sum_q B_q^{\dagger} B_q, \quad (21)$$

here the electronic quasiparticle energies are given

$$E_k^c = \frac{\varepsilon_k^f + \varepsilon_k^c + \text{sgn} \left( \varepsilon_k^f - \varepsilon_{k+Q}^c \right)}{2}, \quad (22)$$

$$E_k^f = \frac{\varepsilon_k^f + \varepsilon_k^c - \text{sgn} \left( \varepsilon_k^f - \varepsilon_{k+Q}^c \right)}{2} \frac{W_k}{2}, \quad (23)$$
The quadratic form of Eq.(21) allows us to calculate all expectation values. From Eqs.(10)-(12) we easily take the following implicit equation for the order parameters ∆ = \(-\frac{2g}{\omega_0}h\)

\[1 = 4g^2 \frac{1}{\omega_0} \sum_k \text{sgn}\left(\varepsilon_k^f - \varepsilon_k^c + Q\right) \frac{f^F(E_k^c) - f^F(E_k^f)}{W_k},\]  

(24)

where \(f^F(E_{k}^{\pm f})\) are the Fermi-Dirac functions which satisfy the equation

\[f^F(\epsilon) = \frac{1}{e^{\beta \epsilon} - 1},\]  

(25)

with \(\beta = 1/T\) is the inverse of temperature.

Note that Eq. (24) stands for a BCS-like equation for ∆. A nonzero ∆ explains an exciton condensation phase as was explained above.

Now we consider forming of the EI state, here \(d_k = \langle c_{k+Q}^\dagger f_k^\dagger \rangle \neq 0 [\text{Eq.}(4)]\) plays as EI order parameter. From Eqs.(16)-(20) we get the mean particle number densities of \(c\) and \(f\) electrons for a system with \(N\) lattice sites

\[n_{k+Q}^c = \langle c_{k+Q}^\dagger c_k^c \rangle = \xi_k^2 f^F(E_k^c) + \eta_k^2 f^F(E_k^f),\]  

(26)

\[n_f^k = \langle f_k^\dagger f_k^f \rangle = \eta_k^2 f^F(E_k^c) + \xi_k^2 f^F(E_k^f),\]  

(27)

\[n_{k}^{c_f} = \langle c_{k+Q}^\dagger f_k^f \rangle = -\left[f^F(E_k^c) - f^F(E_k^f)\right] \text{sgn}\left(\varepsilon_k^f - \varepsilon_k^c + Q\right) \frac{\Delta}{W_k}.\]  

(28)

Finally, using Eq.(12), the lattice displacement in the EI state at momentum \(Q\) is given by

\[x_Q = \frac{1}{\sqrt{N}} \frac{1}{\sqrt{2\omega_0}} \langle b_{-Q}^\dagger + b_Q \rangle = -\frac{h}{\omega_0} \sqrt{\frac{2}{\omega_0}}.\]  

(29)

4. Numerical results and discussion
This section presents the numerical results which are obtained by solving self-consistently equations above to consider the picture of electron-hole pairs condensation due to exciton-phonon interaction. The calculation results are implemented in the unit of energy for \(\hbar = k_B = 1\).

4.1. The electronic quasiparticle energies
The first, let us describe momentum dependent of the quasiparticle energies of \(c\) and \(f\) electrons.
Figure 1. Quasiparticle dispersions $E^c_\mathbf{k}$ (black lines) and $E^f_\mathbf{k}$ (red lines) along the major axes of the $\mathbf{k}$.

Figure 1 shows the quasiparticle dispersions $E^c_\mathbf{k}$ of $c$ electrons (black lines) and $E^f_\mathbf{k}$ of $f$ electrons (red lines) along the major axes of the momentum $\mathbf{k}$ at the temperature $T = 0$ with the coupling constant of electron-phonon interaction $g = 1.1$ and the phonon frequency $\omega_0 = 2.5$. The excitonic condensate state is formed indicating by opening a gap at the Fermi level. This is similar to the condensation of electron-electron pairs with their opposite spin in the BCS theory. These condensate states are broken if there exists a sufficient external energy which decouples the bound states, then the gap vanishes.

The following are detailed examination of the picture of phase transition with variation of the temperature and the coupling constant of electron-phonon interaction.

Figure 2. Quasiparticle dispersions $E^c_\mathbf{k}$ (filled symbols) and $E^f_\mathbf{k}$ (open symbols) along the major axes of the $\mathbf{k}$ at various temperatures.

Figure 2 presents the renormalized quasiparticle bands $E^c_\mathbf{k}$ (filled symbols) and $E^f_\mathbf{k}$ (open symbols) along the major axes of the momentum $\mathbf{k}$ at various temperatures with the coupling constant of electron-phonon interaction $g = 1.1$. At high temperature, such as $T = 0.36$ the two quasiparticle-energy bands overlap, thus no gap exists. Decreasing the temperature, an energy
gap opens at the Fermi level indicating the formation of an excitonic insulating state. This gap expands when the temperature decreases.

Figure 3. Quasiparticle dispersions $E^c_k$ (filled symbols) and $E^f_k$ (open symbols) along the major axes of the $k$ at various coupling constants of electron-phonon interaction.

Figure 3 describes the renormalized quasiparticle energies $E^c_k$ (filled symbols) and $E^f_k$ (open symbols) along the major axes of the momentum $k$ at various coupling constants of electron-phonon interaction $g$ for $T = 0$. We see that in a case of small electron-phonon interaction such as $g = 0.4$, the two quasiparticle-energy bands of $c$ and $f$ electron overlap, the gap does not exist. Increasing the coupling constant, an energy gap opens at the Fermi surface. This shows that the EI state is formed. This gap expands when the coupling constant increases.

The above results show that the temperature and also the coupling constant of electron-phonon interaction affect to the formation of the excitonic condensate. Clearly, the EI phase is only formed at low enough temperature and large enough coupling constant.

4.2. The mean particle number densities

In order to consider more meticulously the picture of the excitonic condensation, firstly we survey temperature and coupling constant of electron-phonon interaction dependent of the mean particle number densities.
Figure 4. The mean particle number densities of $c$ and $f$ electrons along the major axes of the $k$ at various temperatures.

Figure 4 displays the mean particle number densities of $c$ and $f$ electrons as a function of the momentum $k$ when the temperature was changed at coupling constant of electron-phonon interaction $g = 1.1$. For low temperatures, when existing a small energy gap at the Fermi level, $n_{k}^{cf}$ is strongly peaked at $k$ which closes to the Fermi surface. Increasing the temperature above some critical temperature $T_{EI}$, the mean particle number density $n_{k}^{cf}$ is more weakly peaked. As a matter of course, at higher temperature, when no gap exists, $n_{k}^{cf}$ also vanishes, resulting in a state of Fermi liquid. While $n_{k}^{c}$ and $n_{k}^{f}$ monotonously increases (decreases) as $k$ varies from 0 to $\pi$ (or from $-\pi$ to 0).

Figure 5. The mean particle number densities of $c$ and $f$ electrons along the major axes of the $k$ at various coupling constant of electron-phonon interaction.

Figure 5 shows the mean particle number densities of $c$ and $f$ electrons as a function of the momentum $k$ at various coupling constant of electron-phonon interaction, for temperature...
$T = 0$. We see that for high coupling constant, a large gap opens at the Fermi surface, $n_{cf}^k$ is peaked. Decreasing the coupling constant close to critical coupling constant $g_{EI}$, $n_{cf}^k$ is strongly peaked. However, decreasing the coupling constant below some $g_{EI}$, when both quasiparticle bands $c$ and $f$ electrons overlap leads to the gap vanishes, the mean particle number density of $c$ and $f$ electrons also vanishes. While $n_c^k$ and $n_f^k$ monotonously increases (decreases) as $k$ varies from 0 to $\pi$ (or from $-\pi$ to 0).

These results showed that the mean particle number density of $c$ and $f$ electrons only is nonzero when system is existing in EI phase, i.e. at low enough temperature and large enough coupling constant.

### 4.3. The EI order parameter

This section examines in detail the influence of the electron-phonon interaction intensity and the temperature on the formation of exciton condensate by performing dependence the EI order parameter as a function of the coupling constant of electron-phonon interaction and the temperature.

Figure 6 describes the EI order parameter along the major axes of the temperature $T$ at various coupling constant $g$. We see that, when the temperature is smaller than a critical value $T_{EI}$, the order parameter is nonzero. Whereas, above that temperature, the EI order parameter completely equal zero. That temperature is so called an EI transition temperature. Below the critical temperature, the order parameter increases rapidly and then saturates at low temperature. Increasing the coupling constant the order parameter and $T_{EI}$ also increases.
Figure 7. The EI order parameter along the major axes of the coupling constant at various phonon frequency $\omega_0$, for $T = 0$.

Figure 7 shows that when the coupling constant reaches a critical value, the EI order parameter $\Delta$ is nonzero, where the EI phase transitions occurs. Critical value of the coupling constant of electron-phonon interaction increases when the phonon frequency increased. The order parameter decreases, this means that quasiparticles energy gap also decreases with increasing of the phonon frequency. Clearly, largering the phonon frequency, phonon excitation is harder, so strong enough interaction intensity needs to form the bound state of the electron-hole pairs, corresponding to exciton condensation.

4.4. The lattice displacement in the EI state
The following considering the relationship between the lattice displacement at the momentum $Q$ with the exciton condensate state. Like above, we express the dependence of the EI lattice distortion in the electron-phonon interaction intensity and the temperature.

Figure 8. The EI lattice distortion along the major axes of the temperature $T$ at various coupling constant $g$. 
Figure 9. The EI lattice distortion along the major axes of the coupling constant at various phonon frequency $\omega_0$, for $T = 0$.

Figure 8 and 9 indicate that at low temperature and large coupling corresponding to forming the EI state, a finite lattice distortion has been found. The lattice distortion enhances nearly linearly if the interaction increases. The critical value of the coupling constant of electron-phonon interaction increases if the phonon is harder.

We can affirm that the condensate state of the electron-hole pairs leads to an finite crystal lattice distortion. This only occurs in the situation that the electron-phonon interaction is large and the temperature is small enough.

4.5. The dependence on the momentum of the order parameter

Let us examine more meticulously the nature of the electron-hole pairs condensation state due to the electron-phonon interaction. This section presents momentum dependent of the order parameter at various temperature and the coupling constant of electron-phonon interaction.

Figure 10 displays the dependence on the momentum of the EI order parameter at various coupling constant of electron-phonon interaction $g$, for $T = 0$ and $\omega_0 = 2.5$. With small coupling constant $g = 0.7$, the order parameter $\Delta_k$ also is very small and the system in the uniform state. However, increasing $g$ to $g = 1.0$, $\Delta_k$ increases, particularly at the momentum closes the Fermi momentum. This asserts that the Fermi level plays a role in the formation of the excitonic condensate. This is similar to the superconducting picture which is described by BCS theory. Therefore, in this case, the excitonic condensate is called the BCS-type excitonic condensation state. Increasing strength of interaction with $g = 1.3$, we see that the order parameter also has the maximum value at momentum around the Fermi momentum. Although, there is still significant contribution of electron-hole pairs at the momenta which different from the Fermi momentum, even in the center or the bound Brillouin zone. Nevertheless, we still affirm that the exciton condensate in the BCS-type state.
Figure 10. Momentum dependent of the order parameter at various coupling constant of electron-phonon interaction, for $T = 0$.

The next, we consider the dependence on temperature of the order parameter for $g = 1.3$ and $\omega_0 = 2.5$ (see Fig. 11). When fixing the coupling constant of electron-phonon interaction, the maximum value of the order parameter which has momentum is around the Fermi momentum sharply reduces if the temperature increased. However, only the electron-hole pairs which close the Fermi level play important role to form bound state. So we still have the picture of BCS-type electron-hole condensed. If the temperature is higher than the transition temperature, the order parameter becomes very small, even at the Fermi momentum. System exists in the non-order state.

Figure 11. Momentum dependent of the order parameter at various temperature, for $g = 1.3$.

Lastly, let us describe magnitude of the EI order parameter depending on momentum and the temperature or the coupling constant of electron-phonon interaction.
Figure 12 shows the profile of the EI order parameter $\Delta_k$ dependents on the temperature $T$ for $g = 1.3$ (left-hand panel) and the electron-phonon coupling $g$ for $T = 0$ (right-hand panel) at phonon frequency $\omega_0 = 2.5$. This quantity points out the range in momentum space where $c$ electrons and $f$ holes are perceptibly involved in the electron-hole pair formation and exciton condensation process. Obviously $\Delta_k$ vanishes for all $k$ above a critical temperature $T \simeq 0.45$ and below a critical coupling strength $g \simeq 0.7$. At and just below (above) the critical temperature (coupling strength) $\Delta_k$ is solely finite at and near the Fermi momentum $k_F$, respectively, indicating a BCS-type electron-hole pairing instability. However, a further increasing electron-phonon coupling implicates more and more electron and hole states in the pairing process up to the point where Fermi-surface (nesting) effects are inefficient. Thus we expect that local, closely bound excitons will form in the strong interaction limit and, as a result, Bose-Einstein rather than BCS-like condensation occurs.

5. Conclusion
This paper presents the results of theoretical research about the effects on the electron-hole pairs condensation state (exciton) by the mean-field theory in two-dimensional two-band $f-c$ electron system that only caring about the effects of electron-phonon interaction. The results confirm the excitonic condensate state is formed only at low temperature and large electron-phonon interaction. Particularly, we have found that in the EI state, the EI order parameter is non-zero and a finite lattice distortion can be detected. And by reviewing the microscopic properties in momentum space we strongly specify the BCS type of the excitonic condensation driven by the exciton-phonon interaction at low temperature.

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