Some examples of Hayward wormholes

Peter K.F. Kuhfittig
*Department of Mathematics, Milwaukee School of Engineering,
Milwaukee, Wisconsin 53202-3109, USA

Abstract

The first part of this paper discusses a model for the theoretical construction of a simple traversable wormhole with zero density that depends on a preexisting black hole. By assuming the interconvertibility of black holes and wormholes proposed by S.A. Hayward, it is shown that a toy model suggested by the first model may yield several possible transitions from the preexisting black hole to a wormhole. A final topic is the conversion to a wormhole by assuming a specific model for the exotic matter.

PAC numbers: 04.20.Jb, 04.20.Gz

1 Introduction

Wormholes are handles or tunnels in the spacetime topology linking widely separated regions of our Universe or of different universes altogether. While just as good a prediction of Einstein's theory as black holes, they have so far eluded detection. Moreover, holding a wormhole open requires a violation if the null energy condition [1].

The first part of this paper discusses wormholes that have zero density and must therefore depend on preexisting black holes. The interconvertibility of black holes and wormholes proposed by Hayward [2, 3] then leads to a toy model suggested by the first model and which results in several possible transitions from the preexisting black hole to a wormhole. The last model discussed assumes a specific equation of state for the exotic matter and shows that the singularity of the black hole dissolves to become part of the wormhole's mass.

To study the effects of a constant density, we begin with the line element describing a static spherically symmetric wormhole is given by [1]

\[ ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \]

(We are using units in which \( c = G = 1 \).) Here \( \Phi = \Phi(r) \) is the redshift function, which must be everywhere finite to avoid an event horizon, while \( b = b(r) \) is the shape function.

*kuhfitt@msoe.edu
For the shape function, \( b(r_0) = r_0 \), where \( r = r_0 \) is the \textit{throat} of the wormhole. A key requirement is the \textit{flare-out condition}, \( b'(r_0) < 1 \), which indicates a violation of the null energy condition. A concrete manifestation of this violation is the need for exotic matter in the vicinity of the throat. In this region we also require that \( b(r) < r \).

The Einstein field equations are stated next.

\[
8\pi \rho = \frac{b'}{r^2}, \tag{2}
\]

\[
8\pi p_r = -\frac{b}{r^3} + 2 \left( 1 - \frac{b}{r} \right) \frac{\Phi'}{r}, \tag{3}
\]

\[
8\pi p_t = \left( 1 - \frac{b}{r} \right) \left( \Phi'' - \frac{b'r - b}{2r(r - b)} \Phi' + \left( \Phi' \right)^2 - \frac{b'r - b}{2r^2(r - b)} \right). \tag{4}
\]

Comparing Eqs. (1) and (2), if \( b(r) \equiv \text{constant} \), then \( \rho(r) \equiv 0 \). That zero-density wormholes may be of interest is not new. Thus Visser [4] discusses the wormhole metric

\[
ds^2 = -e^{2\phi(r)} dt^2 + \frac{dr^2}{1 - r_0/r} + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{5}
\]

It is not clear what kind of material would be able to support such a wormhole, given the zero density. For that reason we will start the next section with a black hole, whose main function is to generate the gravitational field:

\[
ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{6}
\]

This approach turns out to be somewhat similar to the modified black hole discussed in Refs. [4, 5]:

\[
ds^2 = -\left( 1 - \frac{2M}{r} + \lambda^2 \right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{7}
\]

Here the main point is that for sufficiently small \( \lambda^2 \), such a wormhole may be observationally indistinguishable from a black hole.

2 Solutions

In this section we continue the theme of theoretically constructing a wormhole based on a preexisting black hole:

\[
ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - (2M + A)/r} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad A > 0. \tag{8}
\]

(Observe that \( r_0 = 2M + A \).) In spite of our reliance on a black hole, Eq. (8) does not represent a thin shell in the sense of Visser [4, 5]. Since the shape function is \( b(r) = \)}
$2M + A$, we have $\rho(r) \equiv 0$ by Eq. (2). Since $e^{2\Phi} = 1 - 2M/r$, $\Phi = \frac{1}{2} \ln (1 - 2M/r)$ and we obtain from Eq. (3),

$$8\pi p_r = \frac{-2M + A}{r^3} + \frac{2M}{r^3} \frac{1 - \frac{2M + A}{2M}}{1 - \frac{2M}{r}}.$$

(9)

As a result, $\rho + p_r < 0$ near the throat, so that the null energy condition is indeed violated. If $A \to 0$, Eq. (9) shows that $p_r$ approaches $-1/(8\pi r_0^2)$, as one would expect.

The simple shape function also produces a simple profile curve for the embedding surface [1]. From

$$\frac{dz}{dr} = \pm \left( \frac{r}{b(r)} - 1 \right)^{-1/2},$$

we get the parabolas

$$z(r) = \pm 2(2M + A) \sqrt{\frac{r}{2M + A} - 1}.$$

(10)

For the tidal acceleration at the throat [1],

$$a_T = \left| \frac{b'r - b}{2r^2} \Phi' \Delta \xi \right|$$

(11)

in terms of some distance $\Delta \xi$. (In Ref. [1], $\Delta \xi = 2$ m, the approximate height of a person.) Eq. (11) yields

$$a_T = \left| \frac{M}{2(2M + A)^3 \left( 1 - \frac{2M}{2M + A} \right)} \Delta \xi \right|.$$

(12)

2.1 Small $A$

In line element (8), if $A$ is a small constant, we are dealing with a “black-hole mimicker,” just as in line element (7). According to Eq. (12), the size of the radial tidal force depends on the size of the region of high tension. For the right choice of $A$, $a_T$ can be made to match the tidal acceleration of the black hole itself. (This value would be the smallest value allowed in our wormhole construction.)

2.2 Large $A$

A sufficiently large $A$ will result in a wormhole with low tidal forces. For example, suppose the black hole has one solar mass, i.e., $M = 1474$ m. Its event horizon is the sphere $r = 2948$ m. If $A = 5000$ km, then

$$a_T < (10^8 \text{ m})^{-2},$$

the criterion for human traversability suggested in Ref. [1]. Similar results can be obtained for any preexisting black hole.
In this section we are going to seek another connection to the preexisting black hole by replacing $A$ by the time-dependent function $A = A(t)$.

Possible connections between black holes and wormholes have been the subject of investigations for some time. Thus Hayward [2] proposed a unified framework for black holes and traversable wormholes, a synthesis that makes them essentially interconvertible. Kardashev et al. [7] considered the possibility that compact astrophysical objects such as active galactic nuclei may be current or former entrances to wormholes. Kuhfittig [8] suggested that if the Universe had indeed crossed the phantom divide, as noted by some researchers [9], then wormholes could have formed naturally and some of these wormholes could have become black holes at a time closer to the present. Some could have become quasi-black holes, as defined by Lemos and Zaslavskii [10].

The conversion of a black hole to a wormhole is made explicit by Hayward [11]. A toy model that generalizes the CGHS two-dimensional dilaton gravity model by including a ghost scalar field made it possible to describe explicitly the evolution of the black hole to a wormhole: the ghost radiation, which acts like exotic matter, causes the event horizons of the initial black hole to become timelike and may eventually merge to form the throat of the wormhole. In that event, the trapped region would have evaporated, so that the singularity itself would have disappeared.

Sec. 2 suggests another toy model which, with Ref. [11] taken into account, results in a different kind of wormhole, more akin to the thin-shell wormhole in Ref. [6]. Replacing the constant $A$ in Eq. (8) by the time-dependent function $A = A(t)$, we have

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - [2M + A(t)]/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

$$A(t) > 0 \quad \text{for} \quad t > 0. \quad (13)$$

Since the starting point is a black hole, we need to assume that $A(0) = 0$. This model has the advantage of being easy to analyze while still managing to produce results that are both interesting and physically plausible. Whenever $A(t)$ is close to zero, we are momentarily back to Subsection 2.1. So $A(t)$ has to assume larger values, as well, in order to produce a regular wormhole instead of just a black-hole mimicker.

According to Ref. [11], the most important physical consequence is that the two event horizons have now become timelike, thereby forming two throats, one on each side of the center. It is shown in Ref. [12] that this is to be expected since our wormhole is dynamic. Moreover, returning to the initial black hole, let us recall that, according to Ref. [13], page 838, a spacelike hypersurface extending from a region in one universe to that of another is not static. The reason is that the “bridge” enlarges and contracts rapidly, so rapidly, in fact, that not even a light ray can pass through. Line element (13) could model this behavior for a proper choice of $A(t)$ (with the understanding that the event horizons are now throats), as well as the continuing evolution of the structure as ever more exotic matter is added. This structure is best described by the Penrose diagram in Fig. 1. Even though it is now easy to cross the two throats following a timelike geodesic, (line a), the singularity is still present. Judging from lines b and c, however, the only way that a
traveler can crash into the singularity is by either traveling too slowly or by delaying the start of the trip for too long in any given cycle. In other words, the rates of expansion and contraction have slowed enough to enable the traveler to avoid the singularity, or, what amounts to the same thing, $A(t)$ in Eq. (13) has a longer period. Continuing this process, the two throats could eventually merge to form the throat of a static wormhole, while the singularity is no longer part of the wormhole spacetime. So Fig. 1 represents a transitional structure between a black hole and a wormhole.

It is conceivable that Fig. 1 represents an end result, also mentioned in Ref. [3], in which case we are dealing with a new kind of wormhole, one that could be described as a black hole that periodically contracts and pinches off before expanding again, but doing so slowly enough to permit passage.

If the expansions and contractions stop entirely, then the result is a static wormhole. From the shape function $b(r) = 2M + A(t)$, with $A(t)$ fixed, we still have $\rho(r) \equiv 0$, as in Sec. [1]. But some of the exotic matter would have ended up on the throat, so that the density is best described by the delta-function form $\rho(r) = A \delta(r - r_0)$. The throat is therefore a thin shell that contains all of the exotic matter. The singularity has been excised from the wormhole spacetime, thereby avoiding a naked singularity. The resulting wormhole differs somewhat from the thin-shell wormhole in Ref. [6], which is assumed to have a throat outside the event horizon of the black hole. In our wormhole, the event horizon itself has been converted to a throat.
\[ A = A(r) \]

In this section we consider the case where \( A \) in line element (8) is a function of \( r \) alone, i.e., \( A = A(r) \), representing a static wormhole. In other words, the conversion from a black hole to a wormhole has already been completed. To obtain \( A(r) \), we are going to assume a specific model for the exotic matter.

Given that the throat of the wormhole would be a considerable distance away from the original singularity, \( p_r \) is relatively small, according to Eq. (9). So to model the exotic matter, we take the equation of state to be \( p_r = \omega \rho, \omega < -1 \). In other words, the exotic matter is similar to phantom dark energy.

Since \( b(r) = 2M + A(r) \),

\[ 8\pi p_r = 8\pi \omega \rho = \omega \frac{A'(r)}{r^2} = -\frac{2M + A(r)}{r^3} + \frac{2M}{r^3} \frac{1 - \frac{2M + A(r)}{r}}{1 - \frac{2M}{r}} \] (14)

and

\[ \omega A'(r) = -\frac{2M + A(r)}{r} + \frac{2M}{r} \frac{1 - \frac{2M + A(r)}{r}}{1 - \frac{2M}{r}}. \] (15)

This equation has the following simple solution:

\[ A(r) = c(r - 2M)^{-\frac{1}{\omega}}, \quad \omega < -1, \] (16)

where \( c > 0 \) is an arbitrary constant. It is interesting to note that \( A(2M) = 0 \) for all \( c \). Furthermore,

\[ \frac{b(r)}{r} \rightarrow 0 \text{ as } r \rightarrow \infty \]

if, and only if, \( \omega < -1 \), so that the wormhole spacetime is asymptotically flat.

The location of the throat \( r = r_0 \) depends on the constant \( c \) (in addition to \( \omega \)): since \( b(r_0) = r_0 \), we can find \( r_0 \) by solving the equation \( b(r) - r = 0 \). So from

\[ 2M + c(r - 2M)^{-\frac{1}{\omega}} - r = 0, \]

we obtain

\[ r_0 = 2M + c^{\frac{1}{1+\omega}}. \] (17)

It now follows that

\[ b'(r_0) = -\frac{1}{\omega} < 1, \quad \text{since} \quad \omega < -1. \] (18)

So we are getting a well-behaved wormhole for every \( c > 0 \).

Returning to the shape function, from Eq. (2),

\[ b(r) = b(r_0) + \int_{r_0}^{r} 8\pi \rho(r')(r')^2 dr' = 2m(r). \]

Then the mass inside a sphere of radius \( r \) is given by

\[ \frac{1}{2} b(r) = M + \frac{1}{2} c(r - 2M)^{-\frac{1}{\omega}}, \] (19)
which is the mass $M$ of the black hole plus the additional matter needed to convert the black hole to a wormhole. Since the conversion is completed, Eq. (19) suggests that the singularity has been dissolved to become part of the wormhole’s mass.

The equation of state $p_r = \omega \rho$, $\omega < -1$, suggests that we are, in fact, dealing with phantom dark energy. So if the black hole continues to draw in phantom energy from the cosmological background, the result may be a wormhole, a possibility already considered by Hayward [11] for the simple reason that this could provide a solution to the information paradox.

5 Conclusion

The first part of this paper discusses the theoretical construction of a particularly simple class of traversable wormholes having zero density. Since the density of matter is not ordinarily zero, we rely for our construction on an already existing black hole. One of the parameters in the line element can in principle be adjusted to yield a whole set of solutions ranging from black-hole mimickers to low-tidal-force wormholes. A toy model suggested by the first model then discusses the possible conversion of the black hole to a wormhole by adding exotic matter, as discussed in Ref. [11]. According to our toy model, the result is either a time-dependent wormhole with two throats that periodically expands and contracts slowly enough to permit passage or a plausible transitional structure between a black hole and the resulting wormhole. The singularity is still present in both cases. If the end result is a static wormhole, it must be viewed as a thin-shell wormhole with the original singularity excised from the wormhole spacetime, thereby avoiding a naked singularity. Assuming a specific equation of state for the exotic matter yields a solution suggesting that the singularity of the black hole dissolves to become part of the mass of the wormhole.

References

[1] M.S. Morris and K.S. Thorne, “Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity,” Amer. J. Phys. 56, 395 (1988).

[2] S.A. Hayward, “Black holes and traversible wormholes: a synthesis,” arXiv: 0203051.

[3] S.A. Hayward, “Dynamic wormholes,” arXiv: gr-qc/9805019.

[4] M. Visser, Lorentzian wormholes: from Einstein to Hawking, New York, Springer-Verlag, 1996.

[5] T. Damour and S.N. Solodukhin, “Wormholes as black hole foils,” Phys. Rev. D 76, 024016 (2007).

[6] E. Poisson and M. Visser, “Thin-shell wormholes: Linearized stability,” Phys. Rev. D 52, 7318 (1995).
[7] N.S. Kardashev, I.D. Novikov, and A.A. Shatskiy, “Astrophysics of wormholes,” Int. J. Mod. Phys. 16, 909 (2007).

[8] P.K.F. Kuhfittig, “Could some black holes have evolved from wormholes?” Schol. Res. Exch. 2008, 296158 (2008).

[9] J. Sola and H. Stefancic, “Dynamical dark energy or variable cosmological parameters?” Mod. Phys. Lett. A 21, 479 (2006).

[10] J.P.S. Lemos and O.B. Zaslavskii, “Quasi-black holes: definition and general properties,” Phys. Rev. D 76, 084030 (2007).

[11] S.A. Hayward, “Dilaton wormholes: construction, operation, maintainance and collapse to black holes,” Phys. Rev. D 65, 064003 (2002).

[12] D. Hochberg and M. Visser, “Dynamic wormholes, anti-trapped surfaces, and energy conditions,” Phys. Rev. D 58, 044021 (1998).

[13] C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation, San Francisco, Freeman, 1973.