A theoretical and numerical approach to “magic angle” of stone skipping

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We investigate the condition for the bounce of circular disks which obliquely impacts on fluid surface. An experiment [Clanet, C., Hersen, F. and Bocquet, L., Nature 427, 29 (2004)] revealed that there exists a “magic angle” of 20° between a disk’s face and water surface in which condition the required speed for bounce is minimized. We perform three-dimensional simulation of the disk-water impact by means of the Smoothed Particle Hydrodynamics (SPH). Furthermore, we analyze the impact with a model of ordinal differential equation (ODE). Our simulation is in good agreement with the experiment. The analysis with the ODE model gives us a theoretical insight for the “magic angle” of stone skipping.

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Problem of impacts and ricochets of solid bodies against water surface have been received a considerable amount of attention [1, 2, 3, 4, 5, 6]. In the early stage, the problem was of importance in naval engineering concerning the impacts of canon balls on sea-surface [7]. Investigations then revealed that there exists a maximum angle of incidence \( \theta_{\text{max}} \) for impacts of spheres, above which the rebound does not occur [8]. Besides, it was empirically found that the \( \theta_{\text{max}} \) relates to specific gravity of sphere \( \sigma \) as \( \theta_{\text{max}} = \frac{18}{\sqrt{\sigma}} \). This relation was theoretically explained using a simple model of an ordinal differential equation (ODE) [8, 9]. In military engineering today, the problem of water impacts may be not as important as that of a century ago, however, recently it attracts renewed interest under the studies of locomotion of basilisk lizards [10] and stone-skip [11].

This study is motivated by experimental study of stone-skip – bounce of a stone against water surface – by C. Clanet et al. [12]. They investigated impacts of a circular disk (stone) on water surface and found that an angle about \( \phi = 20^\circ \) between the disk’s face and water surface would be the “magic angle” which minimizes required velocity for bounce. In this paper, we study theoretically and numerically the oblique impact of disks and water surface. Our simulation successfully agrees with the experiment. Moreover, we apply an ODE model [17] to the disk-water impact and obtain an analytical form of the required velocity \( v_{\text{min}} \) and maximum angle \( \theta_{\text{max}} \) as a function of initial disk conditions.

To perform a numerical simulation of the disk-water impact, we solve the Navier-Stokes equation using the technique of Smoothed Particle Hydrodynamics (SPH) [13, 14]. Fig. 1 is the snapshots of our simulation. The SPH method is based on Lagrangian description of fluid and has an advantage to treat free surface motion. Several representation of the viscous term have been proposed for this method. In this work, we adopt an artificial viscous term [15] which is simple for computation and sufficiently examined with Couette flow [16]. In our simulation, we neglect surface tension and put the velocity of sound of the fluid, at least, 25 times larger than the incident velocity of the disk.

In the following discussion, we analyze the ODE model which was originally introduced by Birkoff et al [17]. The model is based on the following assumptions; (i) Hydrodynamic pressure \( p \) acting from water is proportional to \((v \cdot n)^2\), where \( v \) is the speed of the body and \( n \) is the unit vector to the surface of the disk. (ii) For the part of the surface facing air, there is no hydrodynamic force. (iii) During the whole process, deformation of water surface is negligible, and the boundary between immersed and non-immersed area is simply given as the cross section to a horizontal plane at water level. We notice that the first assumption is reasonable because Reynolds number would be of order \( 10^5 \) for typical cases of stone-skip [18].

FIG. 1: Snapshots of the SPH simulation of a disk-water impact with \( \phi = 20^\circ \) and incident angle \( \theta = 15 \). Specific gravity of the disk \( \sigma = 1.5 \) and number of fluid particles \( N = 12600 \).
pressions for Due to the nature of the criterion B, these analytical parameters as that of the experiment: λ, the limit of the stone-skip domain respectively. Note that, in the limit $F \to \infty$, this equation again gives $\theta_{cr} + \phi = \pi/2$.

A position of the inflection point $\xi^*$ still remains unknown. We treat $\xi^*$ as a fitting parameter, which should be determined so as to agree with experiments. However we cannot make a direct comparison between the Eq. \ref{eq:7} and the experimental data \cite{12} because $v_{\min}$ and $\theta_{\max}$ are acquired under the criterion A in the experiment. We thus fit the Eq. \ref{eq:7} with the result of our SPH simulations performed under criterion B and evaluate $\xi^*$ = 2.6. Due to the nature of the criterion B, these analytical expressions for $v_{\min}$ and $\theta_{\max}$ should give a lower and upper limit of the stone-skip domain respectively.

Then let us discuss our results. We chose the same parameters as that of the experiment: $\lambda = 9.1$ and $\sigma = 2.7$ unless particularly mentioned. Froude number $F$ typically ranges from 4.0 to 200. For the SPH simulation, $\lambda = 2.5$ and angular velocity of the disk $\omega = 65$[rounds/s]. Figure\ref{fig:3} shows the domains of stone-skip in $(\phi, v)$ and $(\theta, \phi)$ planes. For the minimum velocity $v_{\min}$, the SPH simulation successfully agrees with the experiment, and the theoretical results under criterion A and B also show the qualitative agreement.

The experiment indicates that the stone-skip domain shrinks at $\theta < 20^\circ$ in $(\theta, \phi)$ plane. The theoretical curve under criterion B does not reproduce this tendency while that of criterion A shows the similar behavior. This inconsistency is due to the assumption that the disk is fully immersed in the water when it reaches to the inflection point. However, in the case that the $\theta$ is much smaller relative to the tilt angle $\phi$ this is totally incorrect: only small part of the disk is immersed during the impact process. The SPH simulation also shows the different behavior with the experiment under $\theta < 20^\circ$. We cannot present a clear explanation for this discrepancy. As for SPH simulations, we mention that, the depth of immersion of the disk would be of the order of the fluid particle size of SPH at very small incident angle. The numerical error hence becomes larger for small $\theta$ and for the domain $\theta < 10^\circ$ simulation is not attainable.

The angle $\phi \approx 20^\circ$ is a characteristic for both $(\phi, v)$ and $(\theta, \phi)$ planes in the experiment. C. Clanet et. al. hence suggested that the angle $\phi = 20^\circ$ is the “magic angle” for stone-skip. However Eq. \ref{eq:7} implies that $\phi$ depends on $\theta$. In Fig. \ref{fig:4} we show how the “magic angle” $\phi_m$ is affected by the incident angle $\theta$. Our theory suggests $\phi_m$ decreases as incident angle increases and SPH simulation also shows a decreasing tendency. However, the change in $\phi_m$ is sufficiently small: $\phi_m$ changes only about 15% relative to the change of incident angle under $\theta = 40^\circ$. We therefore conclude that the “magic angle” still remains around $\phi = 20^\circ$ for the ordinal incident angle at stone skipping.

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![Figure 3](image1.png)

**FIG. 3:** A comparison of the stone-skip domains obtained from the experiment (12), SPH simulations and our theory. (i) The stone-skip domain in the $(v, \phi)$ plane for a fixed $\theta = 20^\circ$. (ii) The stone-skip domain in the $(\theta, \phi)$ plane for a fixed $v = 3.5$[m/s]. The boundary of the stone-skip domain under the criterion A in each graph are numerically drawn and those of B are the plot of Eqs. 7 and 8 respectively.

![Figure 4](image2.png)

**FIG. 4:** Relation of incident angle $\theta$ and the angle $\phi^*$. The SPH simulation is performed with $\sigma = 2.0$. The solid line is obtained numerically seeking the minimum of Eq. 7.
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