Comparison of Exact and Perturbative Results for Two Metrics

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Abstract: We compare the exact and perturbative results in two metrics and show that
the spurious effects due to the perturbation method do not survive for physically relevant
quantities such as the vacuum expectation value of the stress-energy tensor.
INTRODUCTION

Most of the effort in theoretical particle physics is on having a consistent quantum mechanical theory which unifies the four known forces and accounts for all the known particles and the existing phenomenology. A vital point in this effort is incorporating the gravitational interactions to the already unified scheme of the strong, electromagnetic and weak forces. The last resort in this heroic endeavor seems to be the M theory /1, a theory in eleven dimensions which will reduce to the five consistent string theories at the appropriate limits.

A more modest attempt is using semi-classical methods to study gravitation. Such methods are very useful for extracting information about the theory in the absence of a full quantization. For instance, we can calculate the fluctuations in the energy of a particle that propagates through universes described by different metrics, which are exact solutions of Einstein’s equations. Extensive work was done in the seventies in this field stressing the phenomena of particle production in these metrics. This work is described in the books written by Birrell and Davis /2, Fulling /3 and Wald /4. Here we are essentially confronted with a problem of a particle in an external potential. The 2n-point functions reduce to the study of n two-point functions as clearly described in the work by Kuo and Ford /5.

We had applied these methods to the calculation of the vacuum expectation value, hereafter VEV, of the stress-energy tensor for impulsive spherical and shock wave solutions of Nutku and Penrose /6 and Nutku /7 respectively. We found a roundabout way /8 which results in a finite expression for the impulsive spherical wave. This method consisted of taking a detour in de Sitter space for regularizing the ultraviolet divergences and landing in the Minkowski space after an appropriate limit is taken. This method seems does not seem to be able to produce a finite result /9 for the shock wave /7, though.

We had doubts on whether the method actually gives the right answer. We may be taking a singular limit and changing the basic character of the problem in doing so. Another defect may be the use of perturbation theory which may give a different result compared to the exact one. In our work we stop at the second order and apply our regularization procedure to each order separately taking the first finite contribution. We thought it is
worthwhile to apply our method to a well-known case.

A second point was the presence of spurious infrared and ultraviolet divergences in the perturbation series and the loss of the Hadamard behaviour. Our previous experience shows that this spurious behaviour has no effect on physical quantities like the VEV of the stress-energy tensor. Here treating two metrics both exactly and perturbatively, we check if such spurious behaviour exists in the perturbative approach and if it exists, whether it is carried to the physically relevant quantities. Studying the same model exactly we know that this spurious behaviour has no validity.

Work of Deser and Gibbons prohibits the existence of vacuum fluctuations for the plane wave metrics. Here we will first study an impulsive plane wave. We solve the problem both exactly and perturbatively and compare the results. We will show that the application of our method to the perturbative case does not give us a result which is in contradiction with the exact case. We then apply the same method to sandwich waves. We again solve the problem exactly, and then carry the calculation to second order and show that there is no way of extraction a finite expression for the VEV of the stress-energy tensor, even if we take a detour in de Sitter space. We end with a few remarks.

I. Plane Impulsive Wave

Exact Calculation

Here we take the metric describing an impulsive plane wave,\[ ds^2 = 2dudv - |d\zeta + q\zeta v\Theta(v)d\zeta|^2. \] If we take \( q = g\zeta^2 \), we get a plane wave. If the power is higher than quadratic we get pp waves. The d'Alembertian operator in this metric is written as

\[ L = 2\partial_u\partial_v - \frac{2vg^2}{1-v^2g^2}\partial_u - \frac{1}{(1+vg)^2}\partial_x^2 - \frac{1}{(1-vg)^2}\partial_y^2 \]

where we switch to real coordinates, and define \( \zeta = x + iy \). We can reduce the problem to the Sturm-Liouville type

\[ L\phi = K\phi \]
and sum over the eigenvalues to obtain the Feynman propagator. We take

\[ \phi = f(v)e^{i(k_1 x + k_2 y + Ru)} \]

where

\[ f(v) = \frac{1}{(1 - v^2 g^2)^{\frac{1}{2}}} \sqrt{2|R|(2\pi)^2} e^{i\left(-\frac{k_1^2}{2gR(1 + vg)} - \frac{k_2^2}{2gR(1 - vg)} - \frac{K^2}{2R}\right)} \]

We form the Green’s function using the formula

\[ G_F = \sum_{\lambda} \frac{\phi(x)\phi^*(x')}{\lambda} \]

where we denote the eigenmodes \( k_1, k_2, R \) and \( K \) by \( \lambda \). We use the Schwinger prescription to raise the eigenvalue to the exponential \( \frac{1}{K} = -i \int_0^\infty d\alpha \epsilon^{i\alpha K - \alpha\epsilon} \) in the limit \( \epsilon \) goes to zero. All the integrals can be performed easily and we find

\[ G_F = -\frac{\Theta(v - v')}{2\pi\sigma^2} + \frac{\Theta(v' - v)}{2\pi\sigma^2} \]

where

\[ \sigma^2 = 2(v - v')(u - u') - (x - x')^2(1 + vg)(1 + v'g) - (y - y')^2(1 - vg)(1 - v'g) \]

and \( \Theta \) is the Heavyside unit step function. It is clear that we do not get a finite part for the VEV of the stress-energy tensor which is obtained from this expression by taking the appropriate derivatives after the coincidence limit is taken.

**Perturbative Calculation**

Here we will perform the same calculation perturbatively and see if there are spurious effects due to the perturbation algorithm. If we expand up to second order in the coupling constant \( g \), we get

\[ L \approx 2\partial_u \partial_v - 2vg^2\partial_u - (1 + 3(vg)^2)(\partial_x^2 + \partial_y^2) + 2vg(\partial_x^2 - \partial_y^2)) \]

The zeroth-order solution gives the free case resulting in a Green function that goes as

\[ \frac{1}{4\pi (u - u')(v - v')} - \frac{1}{2} \left[\frac{1}{[(x - x')^2 + (y - y')^2]}\right] \]
for constant $A$. We expand the solution in powers of $g$ and take the first order solution as $\phi^{(1)} = f \phi^0$. It is straightforward to solve for $f$ and we get

$$f = \frac{(k_1^2 - k_2^2)u}{2iR} \left[ v + \frac{i}{R} - \frac{Ku}{4R^2} \right]$$

For the second order solution we take $\phi^{(2)} = \phi^{(0)} h$. Here $h = v^2 h_1(x,y,u) + vh_2(x,y,u) + h_3(x,y,u)$. A straightforward calculation gives us

$$h_1 = \frac{3i}{2R^2}(k_1^2 + k_2^2)u - \frac{u^2}{4R^2}(k_2^2 - k_1^2)^2,$$

$$h_2 = \frac{u}{R^2}\left( \frac{K}{2} - 3(k_1^2 + k_2^2) \right) - \frac{3iu^2}{4R^2} ((k_2^2 - k_1^2)^2 + K(k_1^2 + k_2^2)) + \frac{K(k_1^2 - k_2^2)^2 u^3}{8R^4},$$

$$h_3 = -i \frac{u}{R^3}\left( \frac{K}{2} - 3(k_1^2 + k_2^2) \right) + \frac{u^2}{R^3}\left( \frac{1}{8} (3(k_2^2 - k_1^2)^2 + 3K(k_1^2 + k_2^2) - K^2) \right)$$

$$\quad + \frac{iu^3}{8R^5} (2K(k_1^2 - k_2^2)^2 + K^2(k_1^2 + k_2^2)) - \frac{K^2(k_1^2 - k_2^2)^2 u^4}{64R^6}. $$

Here we see that a peculiar thing happens. When we sum over the eigenmodes, we get the closest we get is ($h$ has one power of $(u - u')$ too many in the denominator. To get a finite result we need a term which has only $(u - u')^{-1} m^{-2}$ which will be regularized in the de Sitter space and upon differentiation gives us a finite result. This term will be multiplied by $\frac{\Lambda}{m^2}$ which will be finite when we take $\Lambda$ proportional to $m^2$. This is the only correct choice due to dimensional reasons. Here we do not get such a term. The closest we get is with $(u - u')^{-2} m^{-2}$ which has one power of $(u - u')$ too many in the denominator.

If we go to de Sitter space to cancel both the ultra-violet and infrared divergences, we have to multiply the expression for the Green’s function by $(1 + \frac{\Lambda u v}{6}) (1 + \frac{\Lambda u' v'}{6})^{15}$. We expand this expression in sums and differences of $u, u', v, v'$.

$$(1 + \frac{\Lambda u v}{6})(1 + \frac{\Lambda u' v'}{6}) = 1 + \frac{\Lambda}{12} ((u + u')(v + v') + (u - u')(v - v'))$$

$$\quad + \frac{\Lambda^2}{576} ((u + u')^2(v + v')^2 - (u - u')^2(v + v')^2 - (u + u')^2(v - v')^2 + (u - u')^2(v - v')^2)$$

This process reduces the ultraviolet divergence level of the expression by two orders at most /8,16. We aim to the term which is linear in $\Lambda$, however, We see that the finite part of $< T_{vv} >$ goes as

$$< T_{vv} > \propto -2 \frac{\Lambda^2}{m^2} \Theta(v)$$
which is finite only in de Sitter space. One power of the curvature cancels with the infrared parameter since we take \( \Lambda \propto m^2 \), but the remaining power takes the contribution to zero when we go back to Minkowski space. Terms with \( m^4 \) in the denominator that will cancel this term have divergences which are more severe than those regulated by the factor above.

This result which is in accord with general arguments of Deser \(^{10}\) and Gibbons \(^{11}\), is a check that our method does not contradict any known results.

One can show that this result does not change in the presence of a pp-wave background. Whether a wave is plane or pp type depends only on the form of the function \( q(\zeta) \) in the metric. The general behaviour of the expression for the vacuum expectation value of the stress-energy tensor does not depend on the form of the function \( q \). This form only changes an overall factor which can not decide whether the whole expression is finite or null. The same behaviour was already seen in the different warp functions we have used for the spherical wave.

2. SANDWICH WAVE

Exact Calculation

Here we use non-flat portion of the pure gravitational sandwich metric given by Halilsoy studied in reference 13. At this region the metric is described by the expression

\[
ds^2 = 2dudv - \cosh^2(gu)dx^2 - \cos^2(gu)dy^2.
\]

We can easily form the d’Alembertian operator

\[
L = 2\partial_u\partial_v - \text{sech}^2(gu)\partial_x^2 - \sec^2(gu)\partial_y^2 + g(tanh(gu) - tan(gu))\partial_v.
\]

We fourier analyze the solution in the variables \( x, y, v \) since there is translation invariance with respect to these variables. Since the remaining equation is only first order in \( u \), we can easily calculate the Feynman Green’s Function for this operator

\[
G_F = \frac{g}{8\pi^2} \Theta(u - u') \frac{1}{(\cosh bu\cos bu' \cosh g u\cos gu')^{\frac{1}{2}}} \frac{1}{(AB)^{\frac{1}{2}}} \left[ (v - v') - \frac{g(x - x')^2}{2A} - \frac{g(y - y')^2}{2B} \right].
\]
Here

\[ A = \tanh gu - \tanh gu', \quad B = \tang u - \tang u'. \]

In the coincidence limit both \( A \) and \( B \) can be written as a power series in \( u - u' \), beginning with the linear term in \( u - u' \). The Green’s Function goes as

\[ G_F \approx \frac{1}{2\pi} \frac{1}{2(u-u')(v-v') D_1 - (x-x')^2 D_2 - (y-y')^2 D_3}. \]

where

\[ D_1 = (1 + A_1(u-u') + B_1(u-u')^2 + ...), \]

\[ D_2 = (1 + A_2(u-u') + B_2(u-u')^2 + ...), \]

\[ D_3 = (1 + A_3(u-u') + B_3(u-u')^2 + ...). \]

Here \( A_i, B_i, i = 1 - 3 \) are functions of \( u \), but not that of \( u - u' \).

If we try to extract the VEV of the stress-energy tensor out of this expression we have to first regularize it and obtain the finite part in the coincidence limit before we differentiate it. Before the differentiation, say, with respect to \( u \), we can take the coincidence limit in all the other variables. Since a series expansion only in \( (u - u') \) and not in the other differences exist in the final expression, we can not get a finite term from equation 20 in this limit. If we go to the de Sitter space, we can get rid of the singularities and obtain a finite result. The curvature of the de Sitter space multiplies our expression for this case, though. The result goes to zero as we take the curvature to zero in the Minkowski limit. We did not encounter any infrared type singularities to cancel the de Sitter curvature term. We find that there are no vacuum fluctuations in this case, and the behaviour of the exact propagator is of the Hadamard form.

**Perturbative Calculation**

For the perturbative calculation, we take \( g \), the only free parameter in our model small we expand our operator \( L \), eigenfunctions and the eigenvalues of the associated Sturm-Liouville problem in powers of \( g \).
The operator $L$ reads

$$L \approx 2\partial_u \partial_v - \partial_x^2 - \partial_y^2 + \frac{m^2}{2} + g^2 u^2 (\partial_x^2 - \partial_y^2)$$

$$+ g^4 u^4 \left( -\frac{1}{2} (\partial_x^2 + \partial_y^2) - \frac{1}{3} (\partial_u \partial_v + \frac{2}{u} \partial_v + \frac{m^2}{4}) \right).$$

Here we have added a mass term that we will use as an infrared parameter in our calculations. The aim is to set this term equal to zero at the end with impunity.

The zeroth order solution gives the free Green’s Function, as given in equation 10. The first order solution is of the form $\phi_1 = f \phi_0$, where $\phi_0$ is the zeroth order contribution. We find

$$\phi_0 = \frac{1}{(2\pi)^2} \frac{1}{\sqrt{2|R|}} e^{\frac{iK}{2R}} e^{ik_{1x}} e^{ik_{2y}} e^{iR u}.$$  

This is found in terms of the fourier modes $k_1, k_2, K, R$ of $\phi_0$, and is given as

$$f = (k_1^2 - k_2^2) \left( \frac{u^2 v}{2iR} + \frac{u}{R^2} (v + \frac{K v^2}{4R}) + \frac{i}{2R^3} (2v + \frac{K v^2}{R} + \frac{K^2 v^3}{12R^2}) \right).$$

For the second order we make the same kind of ansatz, $\phi_2 = g \phi_0$. Actually this ansatz is dictated by the equations for $\phi_2$. We find that $g$ is given as a polynomial in the variable $u$,

$$g = u^4 g_1 + u^3 g_2 + u^2 g_3 + u g_4 + g_5$$

where $g_i, i = 1-5$ are functions of $v$ and the modes $k_1, k_2, R, K$ of $\phi_0$. As a typical term we give

$$g_1 = -v^2 \frac{(k_1^2 - k_2^2)^2}{8R^2} + \frac{v}{4Ri} \left[ -(k_1^2 + k_2^2) - \frac{K}{3} + \frac{m^2}{6} \right].$$

The other have higher powers of $v$, i.e.

$$g_2 = \frac{v^3 K}{8iR^4} (k_1^2 - k_2^2)^2 + O(v^2),$$

$$g_3 = \frac{5K^2 v^4}{96R^6} (k_1^2 - k_2^2)^2 + O(v^3),$$

$$g_4 = \frac{-K^3 v^5}{96iR^8} (k_1^2 - k_2^2)^2 + O(v^4),$$

$$g_5 = \frac{-K^4 v^6}{1152R^{10}} (k_1^2 - k_2^2)^2 + O(v^5).$$
Here $O(v^i)$ denotes that the highest power of $v$ is $i$ in the sequel.

The Green’s Function is calculated by summing over all the eigenmodes $k_1, k_2, K, R$. The calculation is standard but tedious. It is reported in reference 17. We just give sample expressions from the end result. It reads

$$G_F = \frac{1}{16\pi} \left( \frac{\Gamma_1}{m^6} + \frac{\Gamma_2}{m^4} + \frac{\Gamma_3}{m^2} + \Gamma_4 \ln(Sm) + \frac{\Gamma_5}{S^2} + \frac{\Gamma_6}{S^4} + \frac{\Gamma_6}{S^6} \right),$$

where

$$S^2 = (u - u')(v - v') - \frac{1}{2} [(x - x')^2 + (y - y')^2].$$

$\Gamma_i, i = 1 - 6$ are functions of $v, v', (v - v'), (x - x')^2 + (y - y')^2, (x - x')^2 - (y - y')^2$. $\Gamma_1$ contains terms that are as divergent as $(v - v')^{-10}$ in the coincidence (ultraviolet) limit. For the others the divergences are somewhat tamer but still existing.

In our previous work /8,9/, we had terms that go as $\frac{1}{m^2} \Theta(v - v')$ which had just $(u - u')$ in the denominator. Then it was possible to cancel this divergence by multiplying by $\Lambda(u - u')(v - v')$, which even gave us a finite expression upon differentiation with respect to $v$ and $v'$. Here the minimum singularity goes as $\frac{1}{(v - v')^3}$. There is no way to cancel the divergence by a detour in de Sitter space, with the $\Lambda$ or with the $\Lambda$ term.

We take this fact as a blessing. As we have shown in the previous subsection, we can not obtain a finite expression for $< T_{\mu\nu} >$ for this metric performing the calculation exactly. The perturbative calculation, although it gives rise to spurious infrared and ultraviolet divergences in the intermediate steps, can not be regularized and a finite expression can not be extracted. We interprete this fact as the absence of vacuum fluctuations for this case. This shows that the perturbative results do not contradict the exact result for the physical quantities.

**CONCLUSION**

If we calculate the fluctuations for a conformal metric, fluctuations should be absent /2. We first perform perturbation theory about the Minkowski space, and our perturbations
are not strong enough to overcome the restrictions imposed by conformal symmetry. If we go to de Sitter space, and perform perturbation around that metric, we do not have this obstruction. We always find finite fluctuations in that metric. This argument made it possible to extract a finite expression for the VEV of the stress-energy tensor in the spherical impulsive wave metric /8. We also note that going to de Sitter space also tamed our ultra-violet divergences.

We can investigate if it is generically true that taking a detour in de Sitter space cures all the divergence problems, or if it is a cure only for one kind of metric, the one given by Nutku and Penrose /6. This trick may not reliable, afterall. One should compare the results with the exactly solvable cases and check that no spurious results leak in through the perturbative method and the limits we used. Here we perform the calculation both perturbatively and exactly for two cases, and show that there is no contradiction as far as the value for $< T_{\mu\nu} >$ is concerned.

In the spherical impulsive wave calculation, there were no dimensional coupling constants. It turns out that if we have dimensional coupling constants, we have more severe ultra-violet divergences which are tamed only with having higher powers of the curvature scalar of de Sitter space, multiplying our expressions for the fluctuations. This happens in the two metrics, plane and sandwich waves, we have studied here. Either we do not have severe enough infrared divergences which will be cancelled by $\Lambda$ or we do not have sufficient powers of the scalar curvature term $\Lambda$ to cancel the existing infrared divergences while its companion, powers of $(v - v')$ is cancelling the ultraviolet ones resulting in a finite result.

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