GALOIS CODESCENT FOR MOTIVIC TAME KERNELS

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Abstract. Let $L/F$ be a finite Galois extension of number fields with an arbitrary Galois group $G$. We give an explicit description of the kernel of the natural map on motivic tame kernels $H^2_M(o_L,\mathbb{Z}(i))^G \to H^2_M(o_F,\mathbb{Z}(i))^G$. Using the link between motivic cohomology and $K$-theory, we deduce genus formulae for all even $K$-groups $K_{2i-2}(o_F)$ of the ring of integers. As a by-product, we also obtain lower bounds for the order of the kernel and cokernel of the functorial map $H^2_M(F,\mathbb{Z}(i)) \to H^2_M(L,\mathbb{Z}(i))^G$.

K-theory, Motivic cohomology.

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Introduction. The motivic Bloch-Kato Conjecture, now a theorem of Voevodsky, led to a more precise description of the $K$-theory of the ring of integers of algebraic number fields. It implies in particular that the Quillen-Lichtenbaum conjecture holds for odd primes. The exact deviation in the 2-torsion between $K$-theory and étale cohomology has been determined [Ka97, RW00] using ideas in the proof of Milnor’s conjecture by Voevodsky [Vo03]. This leads to a better understanding of the relationship between $K$-theory and number theory (cf e.g. Kolster’s survey [Ko04] and Weibel’s survey [We05]). However, many problems remain unsolved. Let $L/F$ be a Galois extension of algebraic number fields with Galois group $G$. Then, for any integer $i \geq 2$, we are interested in the natural maps

$$K_{2i-2}F \to (K_{2i-2}L)^G \text{ and } (K_{2i-2}L)_G \to K_{2i-2}F.$$ By the Bloch-Kato conjecture, one can replace $K$-groups by motivic cohomology groups and then study the kernels and cokernels of the corresponding maps. The transfer map

$$\text{tr}_i : H^2_M(L,\mathbb{Z}(i))^G \to H^2_M(F,\mathbb{Z}(i))$$
is an isomorphism, up to known 2-torsion, so that the induced map on the rings of integers

$$\text{tr}_i : H^2_M(o_L,\mathbb{Z}(i))^G \to H^2_M(o_F,\mathbb{Z}(i))$$
is also surjective (up to known 2-torsion). We work out the kernel of this map explicitly for any Galois extension $L/F$ and obtain a genus formula comparing the order of the two groups $H^2_M(o_L,\mathbb{Z}(i))^G$ and $H^2_M(o_F,\mathbb{Z}(i))$ (Theorem 2.4). The link between $K$-theory and motivic cohomology yields a genus formula for even $K$-groups (Theorem 3.3). Its difficult part is a norm index as in Chevalley’s genus formula for class groups, where the units are replaced by some odd $K$-group. However, we manage to deal with this
norm index in some special cases. For instance, when \( L/F \) is cyclic of order \( p \), the genus formulae for even \( K \)-groups involve only the ramification in the extension \( L/F \) (Corollaries 4.12, 4.13). The proof of the genus formulae uses a localisation sequence in motivic cohomology due to Geisser, which relates (in our situation) the motivic cohomology groups of rings of integers of \( F \) to the motivic cohomology of \( F \) combined with ideas used in the étale setting [Ge04, Theorem 1.1] combined with ideas used in the étale setting [KM00, Ko02, Ko03, KM03, AM04, AM12].

Let \( \mu_p \) denotes the \( p \)-th roots of unity and \( E = F(\mu_p) \). There exists a subgroup \( D^{(i)}_F \) of \( E^* \) (the étale Tate kernel [Ko91]) such that

\[
H^1_{\text{ét}}(F, \mathbb{Z}_p(i))/p \simeq D^{(i)}_F/E^p.
\]

The elementary extension \( E(\sqrt[p]{D^{(i)}_F})/E \) is unramified outside \( p \) and we have the following arithmetic criterion of Galois co-descent for \( H^2_{\text{M}}(o_L, \mathbb{Z}(i)) \) (which is a special case of Theorem 4.2):

**Theorem 0.1.** Let \( L/F \) be a finite Galois extension of number fields with Galois group \( G \). Assume that \( L/F \) is unramified at infinity. Then, for \( i \geq 2 \) even, the natural map \( \text{tr}_i : H^2_{\text{M}}(o_L, \mathbb{Z}(i))_G \to H^2_{\text{M}}(o_F, \mathbb{Z}(i)) \) is surjective and the following two conditions are equivalent

(i) the map \( \text{tr}_i : H^2_{\text{M}}(o_L, \mathbb{Z}(i))_G \to H^2_{\text{M}}(o_F, \mathbb{Z}(i)) \) is an isomorphism

(ii) for every prime \( p \) dividing the ramification index \( e_v \) in \( L/F \) for some finite prime \( v \), the Frobenius automorphisms \( \sigma_v(E(\sqrt[p]{D^{(i)}_F})/E), v \in T_p \setminus S_p \), are linearly independent in the \( \mathbb{F}_p \)-vector space \( \text{Gal}(E(\sqrt[p]{D^{(i)}_F})/E) \).

Here \( T_p \setminus S_p \) stands for the set of finite non-\( p \)-adic primes \( v \) of \( F \) such that \( p \mid e_v \).

We then give the necessary and sufficient arithmetic conditions for the vanishing of étale cohomology groups \( H^1_{\text{ét}}(o_F[1/p], \mathbb{Z}_p(i)) \) to go up along a \( p \)-extension, for a given prime \( p \). For \( p = 2 \), we are led to study the vanishing of the positive étale cohomology groups \( H^2_+(o_F[1/2], \mathbb{Z}_2(i)) \). When \( i \) is odd, we have an exact sequence

\[
0 \to (\mathbb{Z}/2)^{\delta(F)} \to H^2_+(o_F[1/2], \mathbb{Z}_2(i)) \to H^2(o_F[1/2], \mathbb{Z}_2(i)) \to 0
\]

where \( (\mathbb{Z}/2)^{\delta(F)} \) is the cokernel of the signature map (see §1.2).

\[
\text{sgn}_F : H^1(F, \mathbb{Z}_2(i))/2 \to \bigoplus_{v \text{ real}} \mathbb{Z}/2.
\]

In doing so, we also answer a question raised by B. Kahn in [Ka97, page 2] concerning the image of the above signature map (Theorem 4.15).

**Theorem 0.2.** Let \( n \geq 1 \) be an integer. Then there exists a totally real number field \( F \) such that the image of the signature map \( \text{sgn}_F \) has \( 2 \)-rank \( p_i = n \), for all \( i \geq 2 \) odd.
As we will see in Section 1, the signature map $\text{sgn}_F$ is trivial for $i \geq 2$ even.

As far as the kernel and the cokernel of the natural restriction map

$$f_i : H^2_M(F, \mathbb{Z}(i)) \to H^2_M(L, \mathbb{Z}(i))^G$$

are concerned, they are described by the $G$-cohomology of $H^1_M(L, \mathbb{Z}(i))$ [Ko04, Theorem 2.13], which is in general difficult to compute. Using Borel’s results on the abelian group structure of the odd $K$-groups $K_{2i-1}F$, it is not difficult to give an upper bound for the generator rank and the order of $\text{coker } f_i$. Providing lower bounds turn out to be complicated, as pointed out in the classical case of $K_2$ in [Ka93]. Using a cup-product argument, we construct a quotient of the Galois cohomology group $H^2(G, H^1_M(L, \mathbb{Z}(i)))$ (which is essentially the cokernel of the map $f_i$) depending only on the ramification in $L/F$ (Theorem 5.1) leading to a lower bound for $\text{coker } f_i$. When $G$ is a cyclic group, the knowledge of the Herbrand quotient also allows us to give a lower bound for the order of $\ker f_i$ (Corollary 5.2).

In the final section, given any integer $i$ and any odd prime number $p$, we provide a complete list of $p$-extensions $L$ of $\mathbb{Q}$ for which the $p$-primary part of $H^2_M(o_L, \mathbb{Z}(i))$ (resp. $K_{2i-2o_L}$) vanishes. For the prime $p = 2$, we have the following (see §6)

**Theorem 0.3.** Let $L$ be a finite cyclic 2-extension of $\mathbb{Q}$ and $i \geq 2$. Then the 2-primary part of $K_{2i-2o_L}$ vanishes exactly in the following cases

1. $L$ is totally complex and unramified outside a set of primes $\{2, \ell, \infty\}$ with $\ell \equiv \pm 3 \pmod{8}$.
2. $L$ is totally real and
   a. either $2i - 2 \equiv 0 \pmod{8}$ and $L$ is unramified outside a set of primes $\{2, \ell_1, \ell_2\}$ with $\ell_1 \neq 1 \pmod{8}$, $\ell_2 \neq 1 \pmod{8}$ and $\ell_1 \neq \ell_2 \pmod{8}$.
   b. or $2i - 2 \equiv 4$ or $6 \pmod{8}$ and $L$ is unramified outside a set of primes $\{2, \ell\}$ with $\ell \equiv \pm 3 \pmod{8}$.

When $2i - 2 \not\equiv 0 \pmod{8}$ or if $L$ is totally imaginary, then cyclic extensions can be replaced by arbitrary Galois extensions in the above theorem.

As far as the integers $i$ for which $2i - 2 \equiv 2 \pmod{8}$ are concerned, we always have a surjective map $K_{2i-2o_L} \to (\mathbb{Z}/2)^{t_1}$ for any number field $L$. When $L/\mathbb{Q}$ is a Galois 2-extension, then the 2-primary part of $K_{2i-2o_L}$ is isomorphic to $(\mathbb{Z}/2)^{t_1}$ precisely when $L$ is unramified outside a set of primes $\{2, \ell, \infty\}$ with $\ell \equiv \pm 3 \pmod{8}$.

1. Preliminaries

1.1. **Notation and setting.** Notation frequently used throughout the paper will be described in this subsection. We fix an algebraic number field $F$ and let
\( F^* \) the multiplicative group of non-zero elements of \( F \);  
\( \mathfrak{o}_F \) the ring of integers of \( F \);  
\[ r_1 := r_1(F) \text{ (resp. } r_2 := r_2(F)) \] the number of real (resp. complex) places of \( F \).

We shall work with a Galois extension \( L/F \) with Galois group \( G \). For a place \( v \) of \( F \), we fix a prime \( w \) of \( L \) above \( v \). The residue fields at \( v \) and \( w \) will be denoted respectively by \( k_v \) and \( k_w \). We denote by \( e_v \) the ramification index of \( v \) in the extension \( L/F \) and let \( e := \text{lcm}\{e_v \mid \text{finite} \} \). We denote by \( q_v \) the order of \( k_v \) and for any integer \( i \in \mathbb{Z} \) let \( e_v^{(i)} := \gcd(e_v, q_v^i - 1) \).

If \( \ell \) is the residue characteristic of the completion \( F_v \) of \( F \) at \( v \), we denote by \( e'_v \) the order of the non-\( \ell \)-primary part of the inertia group \( I_v \) of \( v \) in the extension \( L/F \). Namely \( e'_v \) is the order of the factor group \( I_v/I_{v,1} \) where \( I_{v,1} \) is the first ramification group of \( v \) in the local extension \( L_w/F_v \). In particular, when the Galois group \( G_v := \text{Gal}(L_w/F_v) \) is abelian, then \( e_v^{(i)} = e'_v \) and therefore does not depend on the integer \( i \).

The notation \( S \) stands for a set of primes containing the infinite primes \( S_\infty \). We say that a prime \( v \in S_\infty \) is ramified in \( L/F \) if \( v \) is a real place of \( F \) which becomes complex in \( L \). For a prime number \( p \), we adopt the following usual notations:

\[
\begin{align*}
S_p & \quad \text{the set of the primes above } p \text{ and the infinite primes;} \\
\mathfrak{o}'_F & = \mathfrak{o}_F [1/p], \quad \mathfrak{o}'_F^\mathbb{S} \quad \text{the ring of } S_p \text{ (resp. } S)\text{-integers of } F; \\
U'_F & \quad \text{the group of units of the ring } \mathfrak{o}'_F; \\
A'_F \text{ (resp. } A_F^+) & \quad \text{the } p\text{-primary part of the (resp. narrow) class group of } \mathfrak{o}'_F; \\
F_\infty & \quad \text{the cyclotomic } \mathbb{Z}_p\text{-extension of } F; \\
E & = F(\mu_p) \quad \text{the field obtained by adding to our base field } F \text{ the } p\text{-th roots of unity;} \\
F_{S_p} & \quad \text{the maximal extension of } F \text{ unramified outside } p \text{ and infinity;} \\
G_{S_p}(F) & \quad \text{the Galois group } \text{Gal}(F_{S_p}/F); \\
\Delta & \quad \text{the Galois group } \text{Gal}(E/F) \simeq \text{Gal}(E_\infty/F_\infty).
\end{align*}
\]

The cardinality of any finite set \( T \) will be denoted by \( | T | \). If \( n \) is a non-negative integer and \( A \) is an abelian group, we denote by \( {}^nA \) the kernel of multiplication by \( n \), and by \( A/n \) the cokernel.

1.2. Signature. For any real place \( v \) of the number field \( F \), let \( i_v : F \to \mathbb{R} \) denote the corresponding real embedding. The natural signature maps \( \text{sgn}_v : F^* \to \mathbb{Z}/2 \) (where \( \text{sgn}_v(x) = 0 \) or \( 1 \)) according to whether \( i_v(x) > 0 \) or not) give rise to the following surjective signature map

\[
F^*/F^{*2} \twoheadrightarrow \oplus_{v \text{ real}} \mathbb{Z}/2 \quad \quad x \mapsto (\text{sgn}_v(x))_{v \text{ real}}
\]

The commutative diagram

\[
\begin{array}{ccc}
F^* & \xrightarrow{2} & F^* \\
\downarrow \oplus i_v & & \downarrow \oplus i_v \\
\oplus_{v \text{ real}} \mathbb{R}^* & \xrightarrow{2} & \oplus_{v \text{ real}} \mathbb{R}^* \\
& & \oplus_{v \text{ real}} \mathbb{R}^*/\mathbb{R}^{*2} \twoheadrightarrow 0
\end{array}
\]
Galois co-descent for motivic tame kernels together with Kummer theory, shows that the above signature map is the same as the localisation map

\[ H^1(F, \mu_2) \to \oplus_{v \text{ real}} H^1(F_v, \mu_2), \]

where \( H^r(K, \quad) \) denotes the (continuous) Galois cohomology of the absolute Galois group of any field \( K \).

We will also be considering restrictions of the signature map such as the following composite map

\[ H^1(F, \mathbb{Z}_2(i))/2 \hookrightarrow H^1(F, \mathbb{Z}/2(i)) \simeq F^*/F^{*2} \to \oplus_{v \text{ real}} \mathbb{Z}/2 \]

and denote by \( \delta_i := \delta_i(F) \) the rank of its cokernel:

(1) \[ H^1(F, \mathbb{Z}_2(i))/2 \xrightarrow{\text{sgn}_{F}} (\mathbb{Z}/2)^{r_1} \to (\mathbb{Z}/2)^{\delta_i} \to 0. \]

For a finite extension \( L/F \) of number fields, let \( R(L/F) \) be the set of infinite primes of \( F \) which ramify in \( L \) and \( r := r(L/F) := | R(L/F)|. \) Then we have a (partial) surjective signature map

\[ F^*/F^{*2} \to \oplus_{v \in R(L/F)} \mathbb{Z}/2 \]

\[ x \mapsto (\text{sgn}_v(x))_{v \in R(L/F)} \]

and we denote by \( s_i := s_i(L/F) \) the rank of the cokernel of its restriction \( \text{sgn}_{L/F} \) to \( H^1(F, \mathbb{Z}_2(i))/2 \)

(2) \[ H^1(F, \mathbb{Z}_2(i))/2 \xrightarrow{\text{sgn}_{L/F}} (\mathbb{Z}/2)^{r(L/F)} \to (\mathbb{Z}/2)^{s_i(L/F)} \to 0. \]

In general, we have \( \delta_i \geq s_i \) as it can readily be seen by the following commutative diagram

\[ H^1(F, \mathbb{Z}_2(i))/2 \xrightarrow{\text{sgn}_F} (\mathbb{Z}/2)^{r_1} \to (\mathbb{Z}/2)^{\delta_i} \to 0 \]

\[ H^1(F, \mathbb{Z}_2(i))/2 \xrightarrow{\text{sgn}_{L/F}} (\mathbb{Z}/2)^{r} \to (\mathbb{Z}/2)^{s_i} \to 0 \]

where the right vertical map, induced by the two others, is surjective. More precisely, we have \( r_1 - r \geq \delta_i - s_i \geq 0. \)

Suppose now that \( i \) is even and consider the following commutative diagram

(3) \[ 0 \to H^1(F, \mathbb{Z}_2(i))/2 \to H^1(F, \mathbb{Z}/2(i)) \to 2H^2(F, \mathbb{Z}_2(i)) \to 0 \]

\[ 0 \to \oplus_{v \text{ real}} H^1(F_v, \mathbb{Z}/2(i)) \xrightarrow{\cong} \oplus_{v \text{ real}} 2H^2(F_v, \mathbb{Z}_2(i)) \]

where the vertical maps are the localisation maps and the isomorphism on the bottom row comes from the following two facts for any real prime \( v \) and any even \( i \) \[ \text{Ko03, page 231} \]:

(i) the vanishing of \( H^1(F_v, \mathbb{Z}_2(i)) \);

(ii) the cohomology groups \( H^1(F_v, \mathbb{Z}/2(i)) \) and \( 2H^2(F_v, \mathbb{Z}_2(i)) = H^2(F_v, \mathbb{Z}_2(i)) \) are both cyclic of order 2.

As noticed before, the middle vertical map is the signature map so that
Proposition 1.1. (compare to [Ko03, Lemma 2.5]) For any number field $F$ and any even integer $i$, the signature map

$$\text{sgn}_F : H^1(F, \mathbb{Z}_2(i))/2 \to (\mathbb{Z}/2)^{r_1}$$

is trivial. □

The above proposition shows in particular that $s_i = r$ in the exact sequence (2) whenever $i$ is even. For $i \equiv 3 \pmod{4}$, a question [Ka97, page 2] raised by B. Kahn concerns the 2-rank $\rho_i(F)$ of the image of $\text{sgn}_F$. More precisely, it is asked whether $\rho_i(F) = r_1(F) - \delta_i = 1$ when $r_1(F) \geq 1$. In other words, is the image of $H^1(F, \mathbb{Z}_2(i))$ in $F^*/F^{*2}$ contained in $\{-1, +1\} \times \{\text{totally positive elements}\}$?

As we will see in Section 4, for any integer $n \geq 1$ and any odd integer $i \geq 3$, there exists a real number field $F$ such that $\rho_i(F) = n$.

When $i$ is odd we will often make the following hypothesis relative to the extension $L/F$:

$$(\mathcal{H}_i) \quad H^*(G, (\mathbb{Z}/2)^{\delta_i(L)}) = 0 \quad \text{for } n = -2 \text{ and } -1.$$

This hypothesis obviously holds in the following cases:

(i) the number field $L$ is totally imaginary or more generally the signature map $\text{sgn}_L : H^1(L, \mathbb{Z}_2(i))/2 \to (\mathbb{Z}/2)^{r_1(L)}$ is trivial,

(ii) the signature map $\text{sgn}_L$ is surjective.

We will also need to consider the restriction of the signature map to $S$-units $U_S$ for any set $S$ of primes of $F$ containing the infinite primes

$$U_S/U^2_S \to (\mathbb{Z}/2)^{r_1}.$$

Let $A_S(F)$ (resp. $A_S(F)^+$) be the 2-primary part of the (resp. narrow) $S$-class group of $F$. We then have the following well-known exact sequence

$$(4) \quad 0 \to U^+_S/U^2_S \to U_S/U^2_S \to (\mathbb{Z}/2)^{r_1} \to A_S(F)^+ \to A_S(F) \to 0$$

where $U^+_S$ is the subset of totally positive $S$-units. In particular, $A_S(F) \cong A_S(F)^+$ precisely when the restriction of the signature map to $S$-units is surjective.

1.3. Motivic cohomology and $K$-theory. Algebraic $K$-theory and motivic cohomology can be thought of as global cohomology theories for étale cohomology. For a survey on motivic cohomology, algebraic $K$-Theory and their connection with number theory we refer the reader to [Ko04, We05]. Let $i$ be a fixed integer. The motivic cohomology groups $H^k_M(X, \mathbb{Z}(i))$ for a smooth scheme $X$ over a base $B$ can be defined as the hypercohomology of Bloch’s cycles $\mathbb{Z}(i)$ for the Zariski topology. We will be dealing with the case where $X = B$ is the spectrum of a field $K$ or of the ring of $S$-integers $\mathcal{O}_F^S$ of a number field $F$. We denote the corresponding motivic cohomology groups respectively by $H^k_M(K, \mathbb{Z}(i))$ and $H^k_M(\mathcal{O}_F^S, \mathbb{Z}(i))$. For an integer $n$, the motivic cohomology group with coefficients $\mathbb{Z}/n$ is the cohomology of the complex $\mathbb{Z}/n(i) = \mathbb{Z}(i) \otimes \mathbb{Z}/n$. 

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For a number field $F$ with ring of integers $\mathcal{O}_F$ and a fixed prime $p$, we let $\mathcal{O}_F^{1/p}$ and simply denote by $H^k(\mathcal{O}_F^{1/p}, \mathbb{Z}_p(i))$ the étale cohomology groups $H^k_{\text{ét}}(\text{spec}(\mathcal{O}_F^{1/p}), \mathbb{Z}_p(i))$. Soulé, Dwyer and Friedlander [So79, DF85] constructed the étale Chern characters $ch_{i,k}^{(p)}$ which give the relation to the algebraic $K$-theory (tensored with $\mathbb{Z}_p$) of $\mathcal{O}_F$ and étale cohomology. They proved that for all $i \geq 2$ and $k = 1, 2$,

$$ch_{i,k}^{(p)} : K_{2i-k}\mathcal{O}_F \otimes \mathbb{Z}_p \to H^k_{\text{ét}}(\mathcal{O}_F^{1/p}, \mathbb{Z}_p(i))$$

is surjective with finite kernel provided that $p > 2$. The Quillen-Lichtenbaum conjecture asserts in fact that $ch_{i,k}^{(p)}$ is an isomorphism for $p > 2$. The odd $p$-primary part of the conjecture follows from the motivic Bloch-Kato Conjecture (cf e.g. [Ko04, Theorem 2.7]) which is now a theorem thanks to the work of Rost and Voevodsky [Vo11]. For $p = 2$, the exact information about the kernel and the cokernel of the Chern Character $ch_{i,k}^{(2)}$ [Ka97, RW00] has been determined using ideas in Voevodsky’s proof of the Milnor $K$-Conjecture [Vo03]. In particular, $ch_{i,k}^{(2)}$ is an isomorphism when $2i-k \equiv 0, 1, 2, 7 \pmod{8}$ or $F$ is totally imaginary. The link between $K$-theory and motivic cohomology is given by motivic Chern characters [Pu04]

$$ch_{i,k}^M : K_{2i-k}(F) \to H^k_M(F, \mathbb{Z}(i))$$

which, once tensored with $\mathbb{Z}_p$, induce $p$-adic Chern characters [L98, Chapter III]. The Bloch-Kato conjecture implies that this map is an isomorphism -up to 2-torsion- for all $i \geq 2$ and $k = 1, 2$. As a consequence of the Bloch-Kato conjecture, we also have the following comparison between motivic cohomology and étale cohomology. If $S$ is a set of primes of $F$ and $\mathcal{O}_F^S$ is the ring of $S$-integers of $F$, there are isomorphisms

$$H^k_M(\mathcal{O}_F^S, \mathbb{Z}(i)) \otimes \mathbb{Z}_p \cong H^k_{\text{ét}}(\mathcal{O}_F^{1/p} [1/p], \mathbb{Z}_p(i))$$

for all integers $i \geq 2$, $k = 1, 2$ and all prime numbers $p$. For the second cohomology groups, we have

$$H^2_M(\mathcal{O}_F^S, \mathbb{Z}(i)) \cong \prod_p H^2_{\text{ét}}(\mathcal{O}_F^{1/p} [1/p], \mathbb{Z}_p(i))$$

(cf e.g. [Ko04]). In fact one can use this property to give a global model for the étale cohomology groups $H^2_{\text{ét}}(\mathcal{O}_F^{1/p} [1/p], \mathbb{Z}_p(i))$. The construction of a global model in the same way for the groups $H^1_{\text{ét}}(\mathcal{O}_F^{1/p} [1/p], \mathbb{Z}_p(i))$ is more complicated and performed in [CKPS98]. Motivic cohomology groups with finite coefficients $\mathbb{Z}/n(i)$ fit into a long exact sequence

$$\cdots \to H^k_M(X, \mathbb{Z}(i)) \overset{\alpha}{\to} H^k_M(X, \mathbb{Z}(i)) \to H^k_{\text{ét}}(X, \mathbb{Z}(i)) \to H^k_{\text{ét}}^1(X, \mathbb{Z}(i)) \to \cdots$$

For a field $K$, we note that $H^k_M(K, \mathbb{Z}/n(i))$ coincides with the Galois cohomology group $H^k(K, \mu_n^{\otimes i})$. One of the main ingredients in the sequel is the following localization sequence in motivic cohomology [Get14, Theorem 1.1], which relates motivic cohomology groups of the ring $\mathcal{O}_F^S$ of $S$-integers of $F$ to the motivic cohomology of $F$, and is similar to Soulé’s localization sequence in étale cohomology [So79],

$$0 \to H^2_M(\mathcal{O}_F^S, \mathbb{Z}(i)) \to H^2_M(F, \mathbb{Z}(i)) \to \bigoplus_{v \in \mathbb{S}} H^1_M(k_v, \mathbb{Z}(i-1)) \to 0$$
and isomorphisms $H^1_M(o^S_F, \mathbb{Z}(i)) \cong H^1_M(F, \mathbb{Z}(i))$. Here $k_v$ stands for the residue field at the place $v$. Recall also that for all $i \geq 2$

$$K_{2i-2}k_v \cong H^2_M(k_v, \mathbb{Z}(i)) = 0$$

and

$$K_{2i-1}k_v \cong H^1_M(k_v, \mathbb{Z}(i)) \cong H^0(k_v, Q/Z(i)) \cong \mathbb{Z}/(q^i_v - 1)$$

where $q_v$ is the order of the residue field $k_v$. Writing the above exact sequence for both $o_F$ and $o^S_F$ leads to the exact sequence

$$0 \to H^2_M(o_F, \mathbb{Z}(i)) \to H^2_M(o^S_F, \mathbb{Z}(i)) \to \bigoplus_{v \in S \setminus S_{\infty}} H^1_M(k_v, \mathbb{Z}(i-1)) \to 0.$$  

Using Quillen’s localization sequence

$$0 \to K_{2i-2}o^S_F \to K_{2i-2}F \to \bigoplus_{v \in S} K_{2i-3}k_v \to 0$$

we obtain the isomorphisms

$$K_{2i-k}o^S_F \cong H^k_M(o^S_F, \mathbb{Z}(i))$$

for $i \geq 2$, $k = 1, 2$, up to known 2-torsion [Ka97, RW00]. Hence in view of [Ta76, So79], the sequence (9) is the analogue of the exact sequence

$$0 \to K_2o_F \to K_2o^S_F \to \bigoplus_{v \in S \setminus S_{\infty}} k_v^* \to 0$$

for $K_2$.

From the above discussions and the finiteness of $K_{2i-2}o_F$ (Borel), it is also clear that the motivic cohomology group $H^2_M(o_F, \mathbb{Z}(i))$ is finite for $i \geq 2$. We call it the (higher) motivic tame kernel. Its order $h^M_i$ is related to the special value of the Dedekind $\zeta$-function at the negative integer $1 - i$ ([BG03, KNF96 · · · ]).

Finally fix a finite prime $v$ in $F$. Denote by $\ell$ the prime integer such that $v \mid \ell$. Local class-field theory and Tate duality show that the $p$-adic Galois cohomology groups $H^k(F_v, \mathbb{Z}_p(i))$, $k = 0, 1, 2$, are finite for any prime integer $p \neq \ell$ and trivial for almost all $p$. Define

$$H^k(F_v, \mathbb{Z}(i))' := \prod_{p \neq \ell} H^k(F_v, \mathbb{Z}_p(i)).$$

The groups $H^k(F_v, Q/Z(i))'$ and $H^k(F_v, \mathbb{Z}/n(i))'$ are defined in the same way. By local duality,

$$H^2(F_v, \mathbb{Z}_p(i)) \cong H^0(F_v, Q_p/Z_p(i-1)) \cong H^0(k_v, Q_p/Z_p(i-1))$$

for any prime integer $p \neq \ell$. Therefore

$$H^2(F_v, \mathbb{Z}(i))' \cong H^1(k_v, \mathbb{Z}(i-1)) \text{ once } v \nmid p.$$  

Hence, taking the product over all the primes $p$ such that $v \nmid p$:

$$H^2(F_v, \mathbb{Z}(i))' \cong H^1_M(k_v, \mathbb{Z}(i-1)).$$
Furthermore, the local cohomology groups $H^1(F_v, \mathbb{Z}_p(i))$ being finite for any prime $p \neq \ell$, we have

$$H^1(F_v, \mathbb{Z}_p(i)) = H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(i)) \cong H^0(k_v, \mathbb{Q}_p/\mathbb{Z}_p(i))$$

so that

(12) \quad $H^1(F_v, \mathbb{Z}_p(i)) \cong H^1(k_v, \mathbb{Z}_p(i))$ once $v \nmid p$

and

(13) \quad $H^1(F_v, \mathbb{Z}(i))' \cong H^1_M(k_v, \mathbb{Z}(i))$.

We shall often use the following result which describes Galois descent and codescent for motivic cohomology groups.

**Theorem 1.2.** [Ko01] Theorem 2.13] Let $L/F$ be a finite Galois extension of number fields with group $G = \text{Gal}(L/F)$. Let $S$ be a finite set of places of $F$ containing the set $\mathcal{S}_\infty$ of infinite places and all those which ramify in $L/F$. For $i \geq 2$, we have:

(i) $H^1_M(F, \mathbb{Z}(i)) \cong H^1_M(L, \mathbb{Z}(i))^G$ and there are exact sequences

$$0 \to H^2_M(o^S_{L}, \mathbb{Z}(i))_G \to H^2_M(o^S_L, \mathbb{Z}(i)) \to (\mathbb{Z}/2)^r \to 0 \quad \text{for } i \text{ even}$$

$$0 \to (\mathbb{Z}/2)^n \to H^2_M(o^S_{L}, \mathbb{Z}(i))_G \to H^2_M(o^S_L, \mathbb{Z}(i)) \to 0 \quad \text{for } i \text{ odd}$$

(ii) For $i$ even, we have an exact sequence

$$0 \to H^1(G, H^1_M(L, \mathbb{Z}(i))) \to H^2_M(o^S_L, \mathbb{Z}(i)) \to H^2_M(o^S_L, \mathbb{Z}(i))^G \to H^2(G, H^1_M(L, \mathbb{Z}(i))) \to 0.$$  

(iii) For $i$ odd, there is an exact sequence

$$0 \to H^1(G, H^1_M(L, \mathbb{Z}(i))) \to H^2_M(o^S_L, \mathbb{Z}(i)) \to H^2_M(o^S_L, \mathbb{Z}(i))^G \to H^2(G, H^1_M(L, \mathbb{Z}(i))) \to (\mathbb{Z}/2)^{-s_i} \to 0.$$

The above theorem is proven $p$-part by $p$-part. For $p$ odd, the proof uses in an essential way the fact that the étale cohomology groups $H^k(o^S_F[1/p], \mathbb{Z}_p(i))$ vanish for $k \neq 1, 2$. When $p = 2$, one replaces étale cohomology by positive étale cohomology [CKPS98 §1] inspired by the ideas from [Ka93] and [M 86]. Recall the main properties of the positive étale cohomology groups $H^1_+(o^S_F[1/2], \mathbb{Z}_2(i))$. First notice that the positive étale cohomology groups vanish for $k \neq 1, 2$. In particular, we have isomorphisms

(14) \quad $H^1_+(o^S_F[1/2], \mathbb{Z}_2(i)) \cong H^1_+(o^S_L[1/2], \mathbb{Z}_2(i))^G$  

and

(15) \quad $H^2_+(o^S_L[1/2], \mathbb{Z}_2(i)) \cong H^2_+(o^S_F[1/2], \mathbb{Z}_2(i))$  

as well as the following dimension shifting result:

(16) \quad $\hat{H}^q(G, H^2_+(o^S_L[1/2], \mathbb{Z}_2(i))) \cong \hat{H}^{q+2}(G, H^1_+(o^S_F[1/2], \mathbb{Z}_2(i)))$

for all $q \in \mathbb{Z}$, given by cup-product [CKPS98 Proposition 3.1]. We shall use this result to give the deviation between $H^q(G, H^2(o^S_L[1/2], \mathbb{Z}_2(i)))$ and $H^{q+2}(G, H^1(o^S_F[1/2], \mathbb{Z}_2(i)))$:
For $n \in \mathbb{Z}$ and an infinite prime $v$, it can readily be seen that

$$
\hat{H}^n(G, \oplus_{w|v} \mathbb{Z}_2(i)) = \begin{cases} 
\mathbb{Z}/2 & \text{if } i + n \text{ is even, } v \text{ is a real prime becoming complex in } L, \\
0 & \text{otherwise}.
\end{cases}
$$

a) Suppose first that $i$ is even. We have the following short exact sequences [Ko04, pages 231-232]

(17) $0 \rightarrow \oplus_{w|v} \mathbb{Z}_2(i) \rightarrow H^1_+(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow H^1(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow 0$

and

(18) $0 \rightarrow H^2_+(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow H^2(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow \oplus_{v|w, v|w\text{ real}} \mathbb{Z}/2 \rightarrow 0.$

Since every direct sum $\oplus_{w|v} \mathbb{Z}/2(i)$ in the exact sequence (18) is $G$-cohomologically trivial (Shapiro), we have

$$\hat{H}^n(G, H^2_+(o_L^S[1/2], \mathbb{Z}_2(i))) \cong \hat{H}^n(G, H^2(o_L^S[1/2], \mathbb{Z}_2(i)))$$

for $n \in \mathbb{Z}$. Now, write the cohomology exact sequence corresponding to (17)

$$0 \rightarrow \hat{H}^{2q+1}(G, H^1_+(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow \hat{H}^{2q+1}(G, H^1(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow (\mathbb{Z}/2)^r$$

$$\rightarrow \hat{H}^{2q+2}(G, H^1_+(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow \hat{H}^{2q+2}(G, H^1(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow 0.$$

Therefore by the dimension shifting (10) and the above isomorphism, we obtain the exact sequence

$$0 \rightarrow \hat{H}^{2q-1}(G, H^2(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow \hat{H}^{2q+1}(G, H^1(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow (\mathbb{Z}/2)^r$$

(19) $\rightarrow \hat{H}^{2q}(G, H^2(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow \hat{H}^{2q+2}(G, H^1(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow 0$

for all $q \in \mathbb{Z}$.

b) Assume now that $i$ is odd. Then we have a six-term exact sequence [Ko04, page 231]

$$0 \rightarrow \oplus_{w, \text{complex}} \mathbb{Z}_2(i) \rightarrow H^1_+(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow H^1(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow \oplus_{v, \text{real}} \mathbb{Z}/2$$

(20) $\rightarrow H^2_+(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow H^2(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow 0$

In this exact sequence the map

$$H^1(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow \oplus_{v, \text{real}} \mathbb{Z}/2 = (\mathbb{Z}/2)^r_1(L)$$

factors through $H^1(o_L^S[1/2], \mathbb{Z}_2(i))/2$ and induces the signature map $\text{sgn}_L$. Split the above exact sequence into four short ones:

$$0 \rightarrow \oplus_{w, \text{complex}} \mathbb{Z}_2(i) \rightarrow H^1_+(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow C \rightarrow 0$$

$$0 \rightarrow C \rightarrow H^1(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow (\mathbb{Z}/2)^{r_1(L)-d(L)} \rightarrow 0$$

$$0 \rightarrow (\mathbb{Z}/2)^{r_1(L)-d(L)} \rightarrow (\mathbb{Z}/2)^{r_1(L)} \rightarrow \hat{H}^{2q}(\mathbb{Z}/2, \mathbb{Z}_2(i)) \rightarrow 0$$

(21) $0 \rightarrow (\mathbb{Z}/2)^{d(L)} \rightarrow H^2_+(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow H^2(o_L^S[1/2], \mathbb{Z}_2(i)) \rightarrow 0$.
where obviously \(C\) contains \(2H^1(o_L^S[1/2], \mathbb{Z}_2(i))\) and the factor group \(C/2H^1(o_L^S[1/2], \mathbb{Z}_2(i))\) is the kernel of the signature map

\[
\text{sgn}_L : H^1(L, \mathbb{Z}_2(i))/2 \rightarrow (\mathbb{Z}/2)^{\delta(L)}
\]

introduced in section \([1.2]\).

Assume

\[\hat{H}^n(G, (\mathbb{Z}/2)^{\delta(L)}) = 0 \text{ for all } n \in \mathbb{Z}.\]

Then \(H^2_+\) and \(H^2_-(o_L^S[1/2], \mathbb{Z}_2(i))\) have the same \(G\)-cohomology. Also, by Shapiro’s Lemma \(\hat{H}^n(G, (\mathbb{Z}/2)^{r(L)}) = 0\) for all \(n \in \mathbb{Z}\). Therefore \(\hat{H}^n(G, (\mathbb{Z}/2)^{r(L)-\delta(L)}) = 0\) for all \(n \in \mathbb{Z}\) and the two \(G\)-modules \(C\) and \(H^1(o_L^S[1/2], \mathbb{Z}_2(i))\) also have the same \(G\)-cohomology. Taking the cohomology of the first short exact sequence above and using the dimension shifting \([10]\) yields the following exact sequence

\[
0 \rightarrow \hat{H}^{2q-2}(G, H^2(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow \hat{H}^{2q}(G, H^1(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow (\mathbb{Z}/2)^r
\]

(22)

for all \(q \in \mathbb{Z}\) under the hypothesis \(\hat{H}^n(G, (\mathbb{Z}/2)^{\hat{H}(L)}) = 0\) for all \(n \in \mathbb{Z}\).

When \(q = 0\) (and \(i\) odd), we have the above exact sequence \([1.3]\) as soon as hypothesis \((H_i)\) is fulfilled. Furthermore, we have a precise description of the image of the map

\[
\hat{H}^0(G, H^1(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow (\mathbb{Z}/2)^r.
\]

Indeed, the signature map gives rise to the following commutative diagram

\[
\begin{array}{ccc}
H^1(o_L^S[1/2], \mathbb{Z}_2(i)) & \rightarrow & (\mathbb{Z}/2)^r \\
\downarrow & & \downarrow \\
\hat{H}^0(G, H^1(o_L^S[1/2], \mathbb{Z}_2(i))) & \rightarrow & (\mathbb{Z}/2)^r
\end{array}
\]

where the left hand vertical map is surjective (Theorem \([1.2]\) (i)). An easy diagram chasing yields

\[
\text{coker} (\hat{H}^0(G, H^1(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow (\mathbb{Z}/2)^r) \cong (\mathbb{Z}/2)^{s_i}
\]

so that the beginning of the exact sequence \([1.3]\) for \(q = 0\) (and \(i\) odd) is written

(23)

\[
0 \rightarrow H_1(G, H^2(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow \hat{H}^0(G, H^1(o_L^S[1/2], \mathbb{Z}_2(i))) \rightarrow (\mathbb{Z}/2)^r \rightarrow (\mathbb{Z}/2)^{s_i} \rightarrow 0
\]

under hypothesis \((H_i)\).

**Remark 1.3.** If the extension \(L/F\) is unramified at infinite prime, Theorem \([1.2]\) shows that \(H^2_M(L, \mathbb{Z}(i))\) satisfy Galois co-descent. It follows that the kernel and cokernel of the functorial map

\[
f_i : H^2_M(o_L^S, \mathbb{Z}(i)) \rightarrow H^2_M(o_L^S, \mathbb{Z}(i))^G
\]

are given by \(\ker(f_i) \cong \hat{H}^{-1}(G, H^2_M(o_L^S, \mathbb{Z}(i)) \cong H^1(G, H^1_M(o_L^S, \mathbb{Z}(i)))\) and \(\text{coker} (f_i) \cong \hat{H}^0(G, H^1_M(o_L^S, \mathbb{Z}(i)) \cong H^2(G, H^2_M(o_L^S, \mathbb{Z}(i)))\). If, Moreover, \(G\) is a cyclic group, we then have the dimension shifting :

(24)

\[
\hat{H}^n(G, H^2_M(o_L^S[1/2], \mathbb{Z}(i))) \cong \hat{H}^{n+2}(G, H^1_M(o_L^S[1/2], \mathbb{Z}(i))
\]
for all \(i \geq 2\) and all \(q \in \mathbb{Z}\), given by cup-product.

Notice that similar results hold in the local case. We mention the following isomorphisms which will be used in the next sections:

For any local extension \(L_w/F_v\) with Galois group \(G_v\), denote by \(k_v\) (resp. \(k_w\)) the residue field of \(F_v\) (resp.\(L_w\)). The group \(G_v\) acts on \(H^1_M(k_w, \mathbb{Z}(i))\) via the natural composite map \(G_v \to G_v/I_v \hookrightarrow \text{Gal}(k_w/k_v)\). It follows from the isomorphisms (11) and (13) that

\[
H^1_M(k_v, \mathbb{Z}(i)) \cong H^1_M(k_w, \mathbb{Z}(i))^{G_v},
\]

\[
H^1_M(k_w, \mathbb{Z}(i))_{G_v} \cong H^1_M(k_v, \mathbb{Z}(i))
\]

and

\[
\hat{H}^q(G_v, H^1_M(k_w, \mathbb{Z}(i-1))) \cong \hat{H}^{q+2}(G_v, H^1_M(k_w, \mathbb{Z}(i)),
\]

for all \(q \in \mathbb{Z}\) and \(i \geq 2\).

2. Genus formula for motivic tame kernels

Let \(L/F\) be a finite Galois extension of algebraic number fields with Galois group \(G\). From here on, unless specified otherwise, the set \(S\) consists of infinite primes and those which ramify in \(L/F\). The short exact sequence (9) yields the following commutative diagram:

\[
\begin{array}{c}
H^2_M(o_L, \mathbb{Z}(i))_G \rightarrow H^2_M(o_L, S(i))_G \rightarrow (\oplus_{v \in S \setminus S_{\infty}} (\oplus_{w \mid v} H^1_M(k_w, \mathbb{Z}(i-1))))_G \rightarrow 0 \\
0 \rightarrow H^2_M(o_F, \mathbb{Z}(i)) \rightarrow H^2_M(o_F, S(i)) \rightarrow \oplus_{v \in S \setminus S_{\infty}} H^1_M(k_v, \mathbb{Z}(i-1)) \rightarrow 0
\end{array}
\]

where the right hand vertical map is an isomorphism by (20). Let

\[
\alpha : H^1(G, H^2_M(o_L, \mathbb{Z}(i))) \rightarrow H^1(G, \oplus_{v \in S \setminus S_{\infty}, w \mid v} H^1_M(k_w, \mathbb{Z}(i)))
\]

be the map given by the homology of the exact sequence (9). Using Theorem 1.2 (i), we then derive the following

**Proposition 2.1.** If \(i \geq 2\) is even, we have an exact sequence

\[
0 \rightarrow \text{ker} (\alpha) \rightarrow H^2_M(o_L, \mathbb{Z}(i))_G \rightarrow H^2_M(o_F, \mathbb{Z}(i)) \rightarrow (\mathbb{Z}/2)^r \rightarrow 0.
\]

If \(i \geq 3\) is odd, the map \(\text{tr}_i : H^2_M(o_L, \mathbb{Z}(i))_G \rightarrow H^2_M(o_F, \mathbb{Z}(i))\) is surjective and its kernel fits into an exact sequence

\[
0 \rightarrow \text{coker} (\alpha) \rightarrow \text{ker} (\text{tr}_i) \rightarrow (\mathbb{Z}/2)^{s_i} \rightarrow 0.
\]

In particular

\[
\frac{|H^2_M(o_L, \mathbb{Z}(i))_G|}{|H^2_M(o_F, \mathbb{Z}(i))|} = 2^{n_i} |\text{coker}(\alpha)|
\]

where \(n_i = -r\) if \(i\) is even and \(n_i = s_i\) if \(i\) is odd.

\(\square\)
For each prime $v \in S \setminus S_\infty$, we fix a prime $w$ of $L$ above $v$ and denote by $G_v$ the local Galois group $\text{Gal}(L_w/F_v)$. We first express the homology groups $H_1(G_v, H^1_M(k_w, \mathbb{Z}(i - 1)))$ in terms of the ramification in $L/F$. Denote by $q_v$ the order of the residue field $k_v$ and by $e_v$ the ramification index of $v$ in the extension $L/F$. If $\ell$ is the residue characteristic of $F_v$, we denote by $e'_v$ the order of the non-$\ell$-primary part of the inertia group of $v$ in the extension $L/F$.

**Lemma 2.2.** For each finite prime $v$, we have

$$H_1(G_v, H^1_M(k_w, \mathbb{Z}(i - 1))) \simeq \mathbb{Z}/e_v^{(i)}$$

where $e_v^{(i)} := \gcd(e_v, q_v - 1)$. In particular, when $G_v$ is abelian, then $e_v^{(i)} = e'_v$ and therefore does not depend on the integer $i$.

**Proof** By the dimension shifting (27), we have

$$H_1(G_v, H^1_M(k_w, \mathbb{Z}(i - 1))) \cong \hat{H}^0(G_v, H^1_M(k_w, \mathbb{Z}(i))).$$

Let $E_w = \bF_v^{nr} \cap L_w$ be the maximal unramified extension of $F_v$ contained in $L_w$ and $I_v := \text{Gal}(L_w/E_w)$ be the inertia group.

Since $I_v$ acts trivially on $H^1_M(k_w, \mathbb{Z}(i))$, we have

$$\hat{H}^0(G_v, H^1_M(k_w, \mathbb{Z}(i))) \cong H^1_M(k_v, \mathbb{Z}(i))/N_{G_v/I_v}(N_{l_v}H^1_M(k_w, \mathbb{Z}(i))) \cong H^1_M(k_v, \mathbb{Z}(i))/e_vN_{G_v/I_v}(H^1_M(k_w, \mathbb{Z}(i)))$$

where $N_{G_v/I_v}$ (resp. $N_{I_v}$) is the norm map corresponding to the extension $E_w/F_v$ (resp. $L_w/E_w$). By [Qu72, section 12, page 185 remark], the norm map

$$N_{G_v/I_v} : H^1_M(k_w, \mathbb{Z}(i)) \to H^1_M(k_v, \mathbb{Z}(i))$$

is surjective. Hence

$$\hat{H}^0(G_v, H^1_M(k_w, \mathbb{Z}(i))) \cong H^1_M(k_v, \mathbb{Z}(i))/e_v \cong \mathbb{Z}/e_v^{(i)}$$

as required. The last assertion comes from the fact that $I_v/I_v,1$ is isomorphic to a quotient of $k_v^*$, where $I_v,1$ is the first ramification group. \qed

For each prime number $p$, and each finite non-$p$-adic prime $v$ of $F$, we consider the composite map

$$H^1_M(F, \mathbb{Z}(i)) \otimes \mathbb{Z}_p \xrightarrow{\sim} H^1(F, \mathbb{Z}_p(i)) \quad \text{(by the isomorphism (5))}$$

$$\xrightarrow{\text{res}} H^1(F_v, \mathbb{Z}_p(i)) \xrightarrow{\sim} H^1(k_v, \mathbb{Z}_p(i)) \quad \text{(by the isomorphism (12))}$$

$$\xrightarrow{\sim} H^1_M(k_v, \mathbb{Z}(i)) \otimes \mathbb{Z}_p \quad \text{(by the isomorphism (8)).}$$

Let $p^m$ be the exact power of $p$ dividing the order $q_v^{i} - 1$ of $H^1_M(k_v, \mathbb{Z}(i))$. We then have a canonical map

$$H^1_M(F, \mathbb{Z}(i))/p^m \to H^1_M(k_v, \mathbb{Z}(i))/p^m.$$
Taking the direct sum over all primes \( p \mid q_v - 1 \) yields a map
\[
H^1_M(F, \mathbb{Z}(i))/(q_v - 1) \to H^1_M(k_v, \mathbb{Z}(i)).
\]
Finally, composing with the canonical surjection
\[
H^1_M(F, \mathbb{Z}(i)) \to H^1_M(F, \mathbb{Z}(i))/(q_v - 1)
\]
we have a map
\[
(28) \quad \varphi_v : H^1_M(F, \mathbb{Z}(i)) \to H^1_M(k_v, \mathbb{Z}(i)).
\]
Now, introduce the map
\[
(29) \quad \beta_S : \hat{H}^0(G, H^1_M(L, \mathbb{Z}(i))) \to \hat{H}^0(G, \oplus_{w|v \in S \setminus S_{\infty}} H^1_M(k_w, \mathbb{Z}(i)))
\]
given by taking the cohomology on the direct sum
\[
\oplus_{w|v \in S \setminus S_{\infty}} \varphi_w : H^1_M(L, \mathbb{Z}(i)) \to \oplus_{w|v \in S \setminus S_{\infty}} H^1_M(k_w, \mathbb{Z}(i)).
\]

The exact sequences \((13)\) (i even), \((23)\) (i odd, under hypothesis \((H_i)\)) and the shifting \((27)\) give rise to the commutative diagram
\[
\begin{align*}
H_1(G, H^2_M(L, \mathbb{Z}(i))) &\xrightarrow{\alpha} H_1(G, \oplus_{w|v \in S \setminus S_{\infty}} H^1_M(k_w, \mathbb{Z}(i - 1))) \\
\hat{H}^0(G, H^1_M(L, \mathbb{Z}(i))) &\xrightarrow{\beta_S} \hat{H}^0(G, \oplus_{w|v \in S \setminus S_{\infty}} H^1_M(k_w, \mathbb{Z}(i)));
\end{align*}
\]
Hence:

(i) for \( i \) even, the exact sequence \((13)\) implies that the left vertical map is surjective so that
\[
\ker(\alpha) \cong \ker(\beta_S).
\]
(ii) for \( i \) odd, using \((23)\), we have the following exact sequence
\[
0 \to \ker(\alpha) \to \ker(\beta_S) \to (\mathbb{Z}/2)^{r-v_i} \to \ker(\alpha) \to \ker(\beta_S) \to 0
\]
under hypothesis \((H_i)\). Since \( H^1_M(L, \mathbb{Z}(i)) \) verifies Galois descent (Theorem \((12)\)), the above map \( \beta_S \) can be written, by Shapiro’s lemma, as
\[
H^1_M(F, \mathbb{Z}(i))/N_G(H^1_M(L, \mathbb{Z}(i))) \to \oplus_{v \in S \setminus S_{\infty}} H^1_M(k_v, \mathbb{Z}(i))/N_G(H^1_M(k_w, \mathbb{Z}(i))).
\]

**Definition 2.3.** We define the normic subgroup \( H^1_M(F, \mathbb{Z}(i)) \) to be the set of elements \( a \in H^1_M(F, \mathbb{Z}(i)) \) whose image by \( \varphi_v \) belongs to \( N_{L_w/F}(H^1_M(k_w, \mathbb{Z}(i))) \), for all \( w \mid v \in S \setminus S_{\infty} \). In other words, \( H^1_M(F, \mathbb{Z}(i)) \) is the kernel of the map
\[
H^1_M(F, \mathbb{Z}(i)) \to \oplus_{v \in S \setminus S_{\infty}} H^1_M(k_v, \mathbb{Z}(i))/N_G(H^1_M(k_w, \mathbb{Z}(i)));
\]
induced by the maps \( \varphi_v \).

Hence
\[
\text{Im}(\beta_S) \cong H^1_M(F, \mathbb{Z}(i))/H^1_M(F, \mathbb{Z}(i)).
\]
Now by Lemma 2.2, it follows that:
\[
|\ker(\beta_S)| = \prod_{v \in S \setminus S_{\infty}} e_v^{(i)} \frac{|H^1_M(F, \mathbb{Z}(i))/H^1_M(F, \mathbb{Z}(i))|}{|H^1_M(F, \mathbb{Z}(i))/H^1_M(F, \mathbb{Z}(i))|}. 
\]
Taking into account the above discussion, Proposition 2.1 yields the following genus formula:

**Theorem 2.4.** Let $L/F$ be a finite Galois extension of number fields with Galois group $G$ and $i \geq 2$. When $i$ is odd, we assume hypothesis $(H_i)$ holds. Denote by $S$ the set of infinite primes and those which ramify in $L/F$. Then

$$\frac{|H_2^i(o_L, \mathbb{Z}(i))|}{|H_2^i(o_F, \mathbb{Z}(i))|} = 2^{\nu_i} \prod_{v \in S \setminus S_{\infty}} e_v^{(i)}$$

where for each finite prime $v$, $e_v^{(i)} := \gcd(e_v, q_v^i - 1)$, $e_v$ being the ramification index of $v$ in the extension $L/F$, $\nu_i = -r$ for $i$ even and $s_i \leq \nu_i \leq r$ for $i$ odd. $\square$

**Remark 2.5.** When $L/F$ is unramified at infinite primes, the power $2^{\nu_i}$ disappears. In the general case, the quantity $\nu_i$, $i$ odd, may take any of the two bounds as the following examples show.

1. Suppose that $L/F$ is unramified at all finite primes. Then $\text{coker}(\alpha) \cong \text{coker}(\beta_S) = 0$ hence $\nu_i = s_i$.
2. Let $L/F$ be a CM-extension such that the natural map

   $$H^1_M(F, \mathbb{Z}(i))/2 \longrightarrow \bigoplus_{v \in S \setminus S_{\infty}} H^1_M(k_v, \mathbb{Z}(i))/2$$

   is an isomorphism. Then

   $$\beta_S : \hat{H}^0(G, H^1_M(L, \mathbb{Z}(i))) \rightarrow \hat{H}^0(G, \bigoplus_{v \in S \setminus S_{\infty}, v \mid \nu} H^1_M(k_v, \mathbb{Z}(i))$$

is surjective and is in fact an isomorphism taking into account the orders. An easy diagram chasing in (20) shows that, in this situation:

$$\text{coker}(\alpha) \cong (\mathbb{Z}/2)^{r-s_i}$$

so that $\nu_i = r > 0$ (Proposition 2.1). As an explicit example consider the case $F = \mathbb{Q}$. The nullity of the étale cohomology group $(i$ is odd$)$ $H^1_{et}(o_F, \mathbb{Z}(2))$ and Kummer theory show that

$$H^1_{et}(F, \mathbb{Z}(2))/2 \cong H^1_{et}(o_F, \mathbb{Z}(2))/2 \cong U'_{et}/U''_{et} = < -1, 2 >$$

where $o_F = \mathbb{Z}[1/2]$ and $U'_{et}$ denotes the group of units in $o_F$. In this case, the map

$$U'_{et}/U''_{et} \rightarrow k_3^* / k_3^{\omega} \oplus k_5^* / k_5^{\omega}$$

is an isomorphism and all we have to do is to take $L = \mathbb{Q}(\sqrt{-15})$. Here, the signature map

$$H^1_{et}(F, \mathbb{Z}(2))/2 \rightarrow \mathbb{Z}/2$$

is surjective and $s_i = 0$. Hence $\nu_i = r \neq s_i$.

To remove the ambiguity on the 2-power in the above theorem when $i$ is odd, we use positive cohomology. Start with the short exact sequence

$$0 \rightarrow H^i_+(o_F[1/2], \mathbb{Z}(2)) \rightarrow H^i_+(o_F[1/2], \mathbb{Z}(2)) \rightarrow \oplus_v H^2(F_v, \mathbb{Z}(2)) \rightarrow 0$$
where \( v \) runs over finite non-dyadic primes in \( S \) [Ko13 page 336]. According to the isomorphism (11) and the vanishing of \( H^1(k_v, \mathbb{Z}_2(i-1)) \) for a dyadic prime \( v \) (the isomorphism (8) tensored with \( \mathbb{Z}_2 \)), we obtain the exact sequence

\[
0 \to H^2_+ (o_F[1/2], \mathbb{Z}_2(i)) \to H^2_+ (o_F[1/2], \mathbb{Z}_2(i)) \to \oplus_{v \in S \setminus S_\infty} H^1(k_v, \mathbb{Z}_2(i-1)) \to 0.
\]

Hence the following commutative diagram:

\[
\begin{array}{ccc}
H^2_+ (o_F[1/2], \mathbb{Z}_2(i)) & \rightarrow & H^2_+ (o_F[1/2], \mathbb{Z}_2(i)) \\
\downarrow \text{tr}^+_i & & \downarrow \text{tr}^+_i \\
H^2_+ (o_F[1/2], \mathbb{Z}_2(i)) & \rightarrow & H^2_+ (o_F[1/2], \mathbb{Z}_2(i)) \\
\end{array}
\]

\[
(\oplus_{v \in S \setminus S_\infty, w | v} H^1(k_w, \mathbb{Z}_2(i-1)))_G \rightarrow 0
\]

\[
(\oplus_{v \in S \setminus S_\infty, w | v} H^1(k_w, \mathbb{Z}_2(i-1)))_G \rightarrow 0
\]

which shows that the left vertical map \( \text{tr}^+_i \) is surjective and that

\[
\ker(\text{tr}^+_i) \cong \text{coker}(\alpha^+: H^1_+ (G, o_L^{S}(1/2), \mathbb{Z}_2(i))) \to H^1_+ (G, \oplus_{w | v \in S \setminus S_\infty} H^1(k_w, \mathbb{Z}_2(i-1))).
\]

Now the dimension shifting (16) shows that

\[
\text{coker}(\alpha^+) \cong \text{coker}(\beta^+_S : \hat{H}^0(G, H^1_+(L, \mathbb{Z}_2(i))) \to \hat{H}^0(G, \oplus_{w | v \in S \setminus S_\infty} H^1(k_w, \mathbb{Z}_2(i)))
\]

where \( \beta^+_S \) is defined in the same way as \( \beta_S \) with positive cohomology instead of motivic cohomology. More precisely, for each finite prime \( v \), let

\[
\varphi^+_v : H^1_+ (F, \mathbb{Z}_2(i)) \to H^1(k_v, \mathbb{Z}_2(i))
\]

be the map obtained by composing the map

\[
H^1_+ (F, \mathbb{Z}_2(i)) \to H^1(F, \mathbb{Z}_2(i))
\]

(in the exact sequences (17) and (18)) with the map \( \varphi_v \) introduced in (28) tensored with \( \mathbb{Z}_2 \). The map \( \beta^+_S \) is defined by considering the direct sum

\[
\oplus_{w | v \in S \setminus S_\infty} \varphi^+_w : H^1_+(L, \mathbb{Z}_2(i))) \to \oplus_{w | v \in S \setminus S_\infty} H^1(k_w, \mathbb{Z}_2(i))
\]

and passing to the \( G \)-cohomology. In this way, we get the following +-analog of definition 2.3:

**Definition 2.6.** We define the (plus) normic subgroup \( H^1_+^{N}(F, \mathbb{Z}_2(i)) \) to be the kernel of the map

\[
H^1_+(F, \mathbb{Z}_2(i)) \to \oplus_{v \in S \setminus S_\infty} H^1(k_v, \mathbb{Z}_2(i))/N_{G_v}(H^1(k_w, \mathbb{Z}_2(i)))
\]

induced by the maps \( \varphi^+_v \).

Since \( H^1_+(o_L^{S}[1/2], \mathbb{Z}_2(i)) \) verifies Galois descent by (14), it follows exactly as in the proof of Theorem 2.3 that

\[
| \text{coker}(\beta^+_S) | = \frac{\prod_{v \in S \setminus S_\infty} 2^{n_v}}{[H^1_+(F, \mathbb{Z}_2(i)) : H^1_+^{N}(F, \mathbb{Z}_2(i))]} \]

where for \( v \in S \setminus S_\infty \), \( 2^{n_v} \) is the exact power of 2 dividing \( e_v^{(i)} \). We summarize
**Theorem 2.7.** Let $L/F$ be a finite Galois extension of number fields with Galois group $G$ and $i \geq 2$. Denote by $S$ the set of infinite primes and those which ramify in $L/F$. Then the natural map

$$\text{tr}_i^* : H^2_+(o_L[1/2], \mathbb{Z}_2(i)) \to H^2_+(o_F[1/2], \mathbb{Z}_2(i))$$

is surjective and we have the following genus formula

$$\frac{|H^2_+(o_L[1/2], \mathbb{Z}_2(i))_G|}{|H^2_+(o_F[1/2], \mathbb{Z}_2(i))_G|} = \prod_{v \in S \setminus S_\infty} 2^{n_v^{(i)}} \frac{[H^1_+(F, \mathbb{Z}_2(i)) : H^1_{\text{cr}}(F, \mathbb{Z}_2(i))]}{[H^1_+(L, \mathbb{Z}_2(i)) : H^1_{\text{cr}}(L, \mathbb{Z}_2(i))]}$$

where for $v \in S \setminus S_\infty$, $2^{n_v^{(i)}}$ is the exact power of $2$ dividing $e_v^{(i)}$. \hfill \Box

We also need the following comparison result:

**Proposition 2.8.** Let $L/F$ be a finite Galois extension of number fields with Galois group $G$ and $i \geq 2$. When $i$ is odd, we assume hypothesis $(H_i)$ holds. Then

$$\frac{|H^2_+(o_L[1/2], \mathbb{Z}_2(i))_G|}{|H^2_+(o_F[1/2], \mathbb{Z}_2(i))_G|} = 2^{u_i} \frac{|H^2_+(o_L[1/2], \mathbb{Z}_2(i))_G|}{|H^2_+(o_F[1/2], \mathbb{Z}_2(i))_G|}$$

where $u_i = r$ is the number of the real primes of $F$ ramified in $L$ for $i$ even and $u_i = -s_i = -s_i(L/F)$ for $i$ odd.

**Proof** Denote by $S$ the set of infinite primes and those which ramify in $L/F$. Assume first that $i$ is odd. In this case we have a commutative diagram (cf the exact sequences (1.3) and (23))

$$\begin{array}{cccccc}
\left(\mathbb{Z}/2\right)^{\delta(L)}_G & \to & H^2_+(o_L[1/2], \mathbb{Z}_2(i))_G & \to & H^2_+(o_L[1/2], \mathbb{Z}_2(i))_G & \to 0 \\
\downarrow & & \text{tr}_i^* \downarrow & & \text{tr}_i \downarrow & \\
0 & \to & H^2_+(o_F[1/2], \mathbb{Z}_2(i)) & \to & H^2_+(o_F[1/2], \mathbb{Z}_2(i)) & \to 0 \\
\downarrow & & \downarrow & & & \\
\left(\mathbb{Z}/2\right)^{\kappa(L/F)} & \to 0
\end{array}$$

with exact lines and columns. The left hand vertical map is injective since its kernel $\tilde{H}^{-1}(G, \left(\mathbb{Z}/2\right)^{\delta(L)})$ vanishes by hypothesis $(H_i)$. Since the maps tr$_i^*$, tr$_i$ are surjective according to Theorem 2.7 and Proposition 2.1 respectively, we have a short exact sequence

$$0 \to \ker(\text{tr}_i^*) \to \ker(\text{tr}_i) \to \left(\mathbb{Z}/2\right)^{\kappa(L/F)} \to 0.$$ (31)

In particular

$$|\ker(\text{tr}_i)| = 2^{n_i} |\ker(\text{tr}_i^*)|.$$
If instead $i$ is even, the exact sequence (1.8) yields the following commutative diagram:

$$
\begin{array}{ccc}
0 & \to & H^2_+(o_L[1/2], \mathbb{Z}_2(i))_G \\
\text{tr}_i^+ & \downarrow & \text{tr}_i \\
0 & \to & H^2_+(o_F[1/2], \mathbb{Z}_2(i)) \to \oplus_{v \text{ real}} \mathbb{Z}/2 \\
\end{array}
$$

By Shapiro’s lemma, $H_1(G, \oplus_{v \text{ real}} \mathbb{Z}/2) = 0$, hence the exactness on the left hand side of the top sequence. The right vertical map being injective, we have

$$\ker(\text{tr}_i) \cong \ker(\text{tr}_i^+)$$

and

$$\text{coker}(\text{tr}_i) \cong (\mathbb{Z}/2)^{r(L/F)}. \quad \Box$$

Combining the above proposition with Theorem 2.7 leads to

**Corollary 2.9.** Let $L/F$ be a finite Galois extension of number fields with Galois group $G$ and $i \geq 2$ odd. Assume hypothesis $(\mathcal{H}_i)$ holds. Denote by $S$ the set of infinite primes and those which ramify in $L/F$. Then

$$\frac{|H^2(o_L[1/2], \mathbb{Z}_2(i))_G|}{|H^2(o_F[1/2], \mathbb{Z}_2(i))|} = 2^{s_i} \frac{\prod_{v \in S \setminus S_\infty} 2^{n_v^{(i)}}}{[H^1_L(F, \mathbb{Z}_2(i)) : H^1_{1\text{tr}}(F, \mathbb{Z}_2(i))]}$$

where, for $v \in S \setminus S_\infty$, $2^{n_v^{(i)}}$ is the exact power of 2 dividing $e_v^{(i)}$.

As mentioned before, if the number field is totally imaginary hypothesis $(\mathcal{H}_i)$ always holds. However $(\mathcal{H}_i)$ does not necessarily hold in some extension $L/F$. For instance, take $F = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{3})$. Then, $L$ has only one dyadic prime and its class group is trivial. As explained in the general case in the proof of Proposition 4.8, for such a number field we have

$$H^1(o_L, \mathbb{Z}_2(i))/2 \cong H^1(o_L, \mathbb{Z}/2(i)) \cong U'_L/\mathbb{Z}_2.$$  

Besides, the group $U'_L$ of 2-units is generated by $-1, 2 - \sqrt{3}$ and $\sqrt{3} - 1$. Therefore the cokernel of the signature map $\text{sgn}_L : H^1(L, \mathbb{Z}_2(i))/2 \to (\mathbb{Z}/2)^{r_i(L)}$ is $\mathbb{Z}/2$ whose $G$-cohomology groups $H^n(G, \mathbb{Z}/2)$ are non trivial. When the extension $L/F$ is cyclic, we have the following genus formula independent of hypothesis $(\mathcal{H}_i)$.

**Theorem 2.10.** Let $L/F$ be a cyclic extension of number fields with Galois group $G$ and $i \geq 2$ odd. Assume that $L/F$ is unramified at infinite primes. Then

$$\frac{|H^2_M(o_L, \mathbb{Z}_2(i))_G|}{|H^2_M(o_F, \mathbb{Z}_2(i))|} = \frac{\prod_{v \in S \setminus S_\infty} e_v^{(i)}}{[H^1_M(F, \mathbb{Z}_2(i)) : H^1_{1\text{tr}}(F, \mathbb{Z}_2(i))]}$$

where for each finite prime $v$, $e_v^{(i)} := \gcd(e_v, q_v^i - 1)$, $e_v$ being the ramification index of $v$ in the extension $L/F$.  


Proof. As in the proof of Theorem 2.4 we have a commutative diagram
\[
\begin{array}{ccc}
H_1(G, H_M^1(o_L^S, \mathbb{Z}(i))) & \xrightarrow{\alpha} & H_1(G, \oplus_{v \in S \setminus S_\infty} H_M^1(k_v, \mathbb{Z}(i - 1))) \\
\downarrow & & \downarrow \\
\hat{H}^0(G, H_M(L, \mathbb{Z}(i))) & \xrightarrow{\beta} & \hat{H}^0(G, \oplus_{w | v \in S \setminus S_\infty} H_M^1(k_w, \mathbb{Z}(i)))
\end{array}
\]
by Remark 1.3. The rest of the proof goes along the same lines as Theorem 2.4. □

For relative quadratic extensions, explicit genus formulae are given in [Ko03] using totally positive elements in the Tate kernel (See also theorem 2.7, §4 and §6). In [AM04, AM12], we explicitly determined the above norm index in the particular cases where the Galois group $G$ is a cyclic $p$-group for a prime $p$. In section 4, we give the necessary and sufficient condition for the above norm index to be exactly $\prod_{v \in S \setminus S_\infty} e_v^{(i)}$.

This allows us to study the descent and co-descent for the motivic tame kernels in an arbitrary Galois extension $L/F$.

3. General genus formulae for even $K$-groups

In this section, we are going to apply the results and methods of the previous sections to give genus formulae for even $K$-groups of rings of integers of number fields. We keep the notation of the preceding sections: $L/F$ is a finite Galois extension of number fields with Galois group $G$ and $S$ is the set of infinite primes and those which ramify in $L/F$. For each finite prime $v$, $e_v^{(i)} := \gcd(e_v, q_v - 1)$, $e_v$ being the ramification index of $v$ in the extension $L/F$. Finally $r$ is the number of infinite primes of $F$ which ramify in the extension $L/F$.

The relation between $K$-theory and motivic cohomology is provided by motivic Chern characters [Pu04]
\[
ch_{i,k}^M : K_{2i-k}(F) \rightarrow H_M^k(F, \mathbb{Z}(i))
\]
which, once tensored with $\mathbb{Z}_p$, induce $p$-adic Chern characters [L08 Chapter III]. The odd part of the following theorem is a consequence of the motivic Bloch-Kato conjecture proved by Voevodsky [Vo11]. The 2-primary information has been obtained by Kahn (partly) and by Rognes-Weibel [Ka97, RW00] based on Voevodsky’s proof of the Milnor conjecture.

Theorem 3.1. Let $S$ be a (finite) set of places of $F$ containing the places at infinity and $o_F^S$ the ring of $S$-integers of $F$. Let $r_1$ be the number of real places. Then for all $i \geq 2$ and $k = 1, 2$, the Chern characters
\[
ch_{i,k}^M : K_{2i-k}(o_F^S) \rightarrow H_M^k(o_F^S, \mathbb{Z}(i))
\]
are

(i) isomorphisms for $2i - k \equiv 0, 1, 2, 7 \pmod{8}$

(ii) surjective with kernel isomorphic to $(\mathbb{Z}/2)^{r_1}$ for $2i - k \equiv 3 \pmod{8}$

(iii) injective with cokernel isomorphic to $(\mathbb{Z}/2)^{r_1}$ for $2i - k \equiv 6 \pmod{8}$.
If \( i \equiv 3 \mod 4 \), there is an exact sequence
\[
0 \to K_{2i-1}(o_F^S) \to H^1_{\mathcal{M}}(o_F^S, \mathbb{Z}(i)) \to (\mathbb{Z}/2)^{\tau_1} \to K_{2i-2}(o_F^S) \to H^2_{\mathcal{M}}(o_F^S, \mathbb{Z}(i)) \to 0
\]
where the middle map
\[
H^1_{\mathcal{M}}(o_F^S, \mathbb{Z}(i)) \to (\mathbb{Z}/2)^{\tau_1}
\]
factors through \( H^1_{\mathcal{M}}(o_F^S, \mathbb{Z}(i))/2 \) and induces the signature map \( \tau_1 \).

For each finite prime \( v \), we introduce a new map \( \varphi'_v \) making the following square commutative
\[
\begin{array}{ccc}
K_{2i-1}F & \xrightarrow{\varphi'_v} & K_{2i-1}k_v \\
\downarrow \text{ch}^M_{2i-1} & & \downarrow \ell \\
H^1_{\mathcal{M}}(F, \mathbb{Z}(i)) & \xrightarrow{\varphi'_v} & H^1_{\mathcal{M}}(k_v, \mathbb{Z}(i))
\end{array}
\]
where \( \varphi_v \) has been introduced in section 2. The map \( \varphi'_v \) will play a similar role to \( \varphi_v \) in the context of \( K \)-groups.

Consider the commutative diagram
\[
\begin{array}{ccc}
K_{2i-1}L & \xrightarrow{\oplus_{w|v}\varphi'_w} & \oplus_{w|v}K_{2i-1}k_w \\
\downarrow N_G & & \downarrow N_G \\
K_{2i-1}F & \xrightarrow{\varphi'_v} & K_{2i-1}k_v
\end{array}
\]
Summing the cokernels of the vertical maps over all \( v \in S \setminus S_\infty \), we have a canonical map
\[
\beta'_S : K_{2i-1}F/N_G(K_{2i-1}L) \to \oplus_{v \in S \setminus S_\infty} K_{2i-1}k_v/N_G(\mathbb{Z}(E_v)(K_{2i-1}k_v))
\]
playing a similar role to \( \beta_S \) defined in section 2. Here, for each \( v \) we fix a \( w \) in \( L \) above \( v \).

**Definition 3.2.** We define the normic subgroup \( K_{2i-1}^N F \) to consist of elements \( a \in K_{2i-1}F \) whose image by \( \varphi'_v \) belongs to \( N_G(\mathbb{Z}(E_v)(K_{2i-1}k_v)) \) for all \( v \in S \setminus S_\infty \) and \( w \mid v \). In other words, \( K_{2i-1}^N F \) is the kernel of the map
\[
K_{2i-1}F \to \oplus_{v \in S \setminus S_\infty} K_{2i-1}k_v/N_G(\mathbb{Z}(E_v)(K_{2i-1}k_v))
\]
induced by the maps \( \varphi'_v \).

The transfer map
\[
\text{Tr}_i : K_{2i-2}(o_L^S)_G \to K_{2i-2}(o_F^S)
\]
realises isomorphisms on the odd \( p \)-primary parts by Theorems 1(i) and 3.1 provided that \( L/F \) is unramified outside \( S \). We are interested in the kernel and cokernel of the transfer map at the level of the ring of integers:
\[
\text{Tr}_i : K_{2i-2}(o_L)_G \to K_{2i-2}(o_F)
\]
In this section we prove genus formulae for even \( K \)-groups combining the above Theorem 3.1 with Theorem 2.3 and Corollary 2.3 from the previous section.
Theorem 3.3. Let \( L/F \) be a finite Galois extension of number fields with Galois group \( G \) and \( i \geq 2 \) an integer. When \( i \) is odd, we assume hypothesis \((H_i)\) holds. Then

\[
\frac{|K_{2i-2}(o_L)_G|}{|K_{2i-2}(o_F)|} = 2^{\alpha_i} \prod_{v \mid \infty} e_v^{(i)} \frac{|K_{2i-1}F : K_{2i-1}^N F|}{|K_{2i-1}L : K_{2i-1}^N L|}
\]

where

(i) \( s_i \leq \alpha_i \leq r \), if \( 2i - 2 \equiv 0 \) (mod 8);
(ii) \( \alpha_i = -r \) if \( 2i - 2 \equiv 2 \) (mod 8);
(iii) \( \alpha_i = 0 \) if \( 2i - 2 \equiv 4 \) or 6 (mod 8).

Proof. When \( 2i - 2 \equiv 0 \) (mod 8), the motivic Chern characters \( ch^M_{i,k} : K_{2i-k}(o_F^S) \to H^2_M(o_F^S, \mathbb{Z}(i)) \) from Theorem 3.1 are isomorphisms for \( k = 1, 2 \) and the result follows from Theorem 2.4.

When \( 2i - 2 \equiv 2 \) (mod 8), Theorem 3.1 shows that \( K_{2i-2}o_F \sim H^2_M(o_F, \mathbb{Z}(i)) \) and that we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & (\mathbb{Z}/2)^{r_1(L)} & \to & K_{2i-1}L & \to & H^1_M(L, \mathbb{Z}(i)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\mathbb{Z}/2)^{r_1(F)} & \to & K_{2i-1}F & \to & H^1_M(F, \mathbb{Z}(i)) & \to & 0
\end{array}
\]

where all the vertical maps are induced by the norm \( N_G \). In particular, we obtain an exact sequence

\[
(\mathbb{Z}/2)^{r} \to K_{2i-1}F/N_G(K_{2i-1}L) \to H^1_M(F, \mathbb{Z}(i))/N_G(H^1_M(L, \mathbb{Z}(i))) \to 0.
\]

Hence a commutative diagram

\[
\begin{array}{cccc}
(K_{2i-1}F)/N_G(K_{2i-1}L) & \to & H^1_M(F, \mathbb{Z}(i))/N_GH^1_M(L, \mathbb{Z}(i)) \\
\beta'_S \downarrow & & \beta_S \downarrow & \\
\oplus_{v \in S \setminus S_{\infty}} (K_{2i-1}k_v)/N_G(k_{2i-1}k_w) & \cong & \oplus_{v \in S \setminus S_{\infty}} H^1_M(k_v, \mathbb{Z}(i))/N_GH^1_M(k_w, \mathbb{Z}(i))
\end{array}
\]

showing that

\[
\text{Im} (\beta'_S) \cong \text{Im} (\beta_S).
\]

In particular, these two images have the same order:

\[
[K_{2i-1}F : K_{2i-1}^N F] = [H^1_M(F, \mathbb{Z}(i)) : H^1_M(F, \mathbb{Z}(i))].
\]

Again, the genus formula in this case results from Theorem 2.4.
Assume now that $2i - 2 \equiv 4 \pmod{8}$. This is the case where neither $ch_{i,1}^M$ nor $ch_{i,2}^M$ is an isomorphism in general. By Theorem 3.1, we have a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \left(\mathbb{Z}/2\right)^{\delta_i(L)}_G \\
\downarrow & & \downarrow \\
\left(\mathbb{Z}/2\right)^{\delta_i(F)}_G & \rightarrow & H^2_{\mathcal{M}}(o_L, \mathbb{Z}(i))_G \\
\downarrow & & \downarrow \\
0 & \rightarrow & K_{2i-2}o_F \\
\downarrow & & \downarrow \\
\left(\mathbb{Z}/2\right)^{\kappa_i(L/F)} & \rightarrow & H^2_{\mathcal{M}}(o_L, \mathbb{Z}(i)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

where the right hand vertical map is surjective by Proposition 2.1. The left hand column is exact since $\hat{H}^{-1}(G, (\mathbb{Z}/2)^{\delta_i(L)}) = 0$ by hypothesis ($\mathcal{H}_i$). Applying Snake lemma, we get

\[
\frac{|K_{2i-2}(o_L)_G|}{|K_{2i-2}(o_F)|} = 2^{-s_i(L/F)} \frac{|H^2_{\mathcal{M}}(o_L, \mathbb{Z}(i))_G|}{|H^2_{\mathcal{M}}(o_F, \mathbb{Z}(i))|}
\]

for $2i - 2 \equiv 4 \pmod{8}$ under ($\mathcal{H}_i$).

We focus on the $2$-primary part since for the $p$-primary parts with $p$-odd the étale Chern characters realise isomorphisms between $K_{2i-k}o_F \otimes \mathbb{Z}_p$ and $H^2_{\mathcal{M}}(o_F, \mathbb{Z}(i)) \otimes \mathbb{Z}_p$ for $k = 1, 2$. The $2$-primary part of $\frac{|H^2_{\mathcal{M}}(o_L, \mathbb{Z}(i))_G|}{|H^2_{\mathcal{M}}(o_F, \mathbb{Z}(i))|}$ has been computed in Corollary 2.9. We shall prove that the two norm indices $[K_{2i-1}F : K_{2i-1}^N F]$ and $[H^1_+(F, \mathbb{Z}(i)) : H^1_+(F, \mathbb{Z}(i))]$ have the same $2$-primary parts. Theorem 3.1 and the exact sequence (1.3) yield an exact sequence

\[
0 \rightarrow \oplus_{\nu \mid \text{v complex}} \mathbb{Z}_2(i) \rightarrow H^1_+(L, \mathbb{Z}(i)) \rightarrow K_{2i-1}L \otimes \mathbb{Z}_2 \rightarrow 0.
\]

We then have a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \oplus_{\nu \mid \text{v complex}} \mathbb{Z}_2(i) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^1_+(F, \mathbb{Z}(i)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & K_{2i-1}F \otimes \mathbb{Z}_2 \\
\end{array}
\]

where the vertical maps are the norm maps $N_G$. It follows that

\[
H^1_+(F, \mathbb{Z}(i))/N_GH^1_+(L, \mathbb{Z}(i)) \cong (K_{2i-1}F \otimes \mathbb{Z}_2)/N_G(K_{2i-1}L \otimes \mathbb{Z}_2).
\]

The commutative diagram

\[
\begin{array}{ccc}
H^1_+(F, \mathbb{Z}(i))/N_GH^1_+(L, \mathbb{Z}(i)) & \cong & (K_{2i-1}F \otimes \mathbb{Z}_2)/N_G(K_{2i-1}L \otimes \mathbb{Z}_2) \\
\oplus_{\nu \in S \setminus S_{\infty}} H^1_+(k_{\nu}, \mathbb{Z}(i))/N_{G_{\nu}}H^1_+(k_{\nu}, \mathbb{Z}(i)) & \cong & \oplus_{\nu \in S \setminus S_{\infty}} (K_{2i-1}k_{\nu} \otimes \mathbb{Z}_2)/N_{G_{\nu}}(K_{2i-1}k_{\nu} \otimes \mathbb{Z}_2)
\end{array}
\]

gives the desired equality between norm indices.
Assume finally that $2i - 2 \equiv 6 \pmod{8}$. By Theorem 3.1(iii), we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & K_{2i-2}(o_L)_G & \rightarrow & H^2_M(o_L, \mathbb{Z}(i))_G & \rightarrow & (\oplus_{\text{w real}} \mathbb{Z}/2)_G & \rightarrow & 0 \\
0 & \rightarrow & K_{2i-2}(o_F) & \rightarrow & H^2_M(o_F, \mathbb{Z}(i)) & \rightarrow & \oplus_{\text{w real}} \mathbb{Z}/2 & \rightarrow & 0 \\
& & & & & & (\mathbb{Z}/2)^r & = & (\mathbb{Z}/2)^r \\
& & & & & & 0 & \rightarrow & 0 \\
\end{array}
$$

The exactness of the middle vertical sequence comes from Proposition 2.1. Hence the transfer map $\text{Tr}_i$ is surjective and has the same kernel as the map $\text{tr}_i$. Accordingly

$$
(33) \quad \frac{|K_{2i-2}(o_L)_G|}{|K_{2i-2}(o_F)|} = 2^r \frac{|H^2_M(o_L, \mathbb{Z}(i))_G|}{|H^2_M(o_F, \mathbb{Z}(i))|}
$$

for $2i - 2 \equiv 6 \pmod{8}$.

To conclude, we use again the genus formula in Theorem 2.4 and the fact that $K_{2i-1}(F) \cong H^1_M(F, \mathbb{Z}(i))$. □

4. Galois co-descent

In this section, we will carry out an arithmetic interpretation concerning Galois co-descent for motivic tame kernels $H^2_M(o_F, \mathbb{Z}(i))$.

We first give a necessary and sufficient condition under which the map

$$
\beta_S : \hat{H}^0(G, H^1_M(L, \mathbb{Z}(i))) \rightarrow \oplus_{v \in S \setminus S_{\infty}} \hat{H}^0(G_v, H^1_M(k_v, \mathbb{Z}(i)))
$$

introduced in section 2 is surjective. For each prime $p$, let

$$
T_p = \{v \in S \setminus S_{\infty} / p \mid e_v\}
$$

and

$$
\text{loc}_{T_p} : H^1_{\text{et}}(F, \mathbb{Z}_p(i)) \rightarrow \oplus_{v \in T_p} H^1_{\text{et}}(k_v, \mathbb{Z}_p(i))
$$

be the direct sum $\oplus_{v \in T_p} \varphi_v$ tensored with $\mathbb{Z}_p$. For the definition of the map $\varphi_v$ see the §2.

**Lemma 4.1.** The map $\beta_S$ is surjective precisely when all the localization maps $\text{loc}_{T_p}$ are surjective for all primes $p$ (dividing the ramification index $e_v$ for some $v \in S \setminus S_{\infty}$).
Proof. For each prime $p$ and each finite prime $v$, we have a commutative diagram

\[
\begin{array}{ccc}
H^1_{\text{d}}(L, \mathbb{Z}_p(i)) & \longrightarrow & \bigoplus_{w \mid v} \hat{H}^1_{\text{d}}(k_w, \mathbb{Z}_p(i)) \\
\downarrow N_G & & \downarrow \bigoplus N_{L_w/k_v} \\
H^1_{\text{d}}(F, \mathbb{Z}_p(i)) & \xrightarrow{\varphi_v \otimes \mathbb{Z}_p} & H^1_{\text{d}}(k_w, \mathbb{Z}_p(i)) \\
\downarrow & & \downarrow \\
\hat{H}^0(G, H^1_{\text{d}}(L, \mathbb{Z}_p(i))) & \longrightarrow & \hat{H}^0(G, \bigoplus_{w \mid v} H^1_{\text{d}}(k_w, \mathbb{Z}_p(i))) \cong \hat{H}^0(G_v, H^1_{\text{d}}(k_w, \mathbb{Z}_p(i)))
\end{array}
\]

where the vertical maps are exact since $H^1_{\text{d}}(L, \mathbb{Z}_p(i))$ and $H^1_{\text{d}}(k_w, \mathbb{Z}_p(i))$ satisfy Galois descent (Theorem 1.2 (i) and the isomorphism (25)).

Suppose first that all the maps $\text{loc}_{T_p}$ are surjective. Then, by summing the above diagram over all $v \in T_p$, we see that

\[
\beta_{T_p} : \hat{H}^0(G, H^1_{\text{d}}(L, \mathbb{Z}_p(i))) \rightarrow \bigoplus_{v \in T_p} \hat{H}^0(G_v, H^1_{\text{d}}(k_w, \mathbb{Z}_p(i)))
\]

is also surjective. Hence the surjectivity of $\beta_S$ which is, by definition, the direct sum $\bigoplus_p \beta_{S,p}$.

Conversely, suppose that $\beta_S$ is surjective. For each prime $v$ in $S \setminus S_\infty$, fix a prime $w$ of $L$ above $v$. As in the proof of Lemma 2.2 we have

\[
\hat{H}^0(G_v, H^1_M(k_w, \mathbb{Z}(i))) \cong H^1_M(k_v, \mathbb{Z}(i))/e_v.
\]

Therefore the following map

\[
H^1_M(F, \mathbb{Z}(i)) \rightarrow \bigoplus_{v \in S \setminus S_\infty} H^1_M(k_v, \mathbb{Z}(i))/e_v,
\]

obtained naturally from the $\varphi_v, v \in S \setminus S_\infty$, is surjective. Dividing the above map by $p$, we get the surjectivity of the map

\[
H^1_M(F, \mathbb{Z}(i))/p \rightarrow \bigoplus_{v \in T_p} H^1_M(k_v, \mathbb{Z}(i))/p
\]

which in turn shows that of $\text{loc}_{T_p}$.

Suppose first that $i \geq 2$ is even. In this case $\text{coker}(\alpha) \cong \text{coker}(\beta_S)$ (see the discussion before Theorem 2.4). It follows by Proposition 2.1 that the sequence

\[
0 \rightarrow H^2_M(o_L, \mathbb{Z}(i))_G \rightarrow H^2_M(o_F, \mathbb{Z}(i)) \rightarrow (\mathbb{Z}/2)^r \rightarrow 0
\]

is exactly precisely when the equivalent conditions of the above lemma hold.

If instead $i \geq 3$ is odd, we assume that hypothesis ($H_\iota$) holds (or $L/F$ cyclic) and that our extension $L/F$ is unramified at infinity. Hence, once again $\text{coker}(\alpha) \cong \text{coker}(\beta_S)$ (see the discussion before Theorem 2.4) and by Proposition 2.1

\[
H^2_M(o_L, \mathbb{Z}(i))_G \cong H^2_M(o_F, \mathbb{Z}(i))
\]

precisely when the equivalent conditions of the above lemma hold.

For a prime $p$, let $E = F(\mu_p)$ and $\Delta = \text{Gal}(E/F)$. Let $v$ be a finite non-$p$-adic prime of $F$ which splits in $E$. Let $v'$ be any prime of $E$ above $v$. If $M/E$ is a Galois extension unramified at $v'$, then the Frobenius automorphism in $M/E$ does not depend on the choice of $v'$ above $v$ and will simply be denoted by $\sigma_v(M/E)$.
We now recall the definition of the étale Tate kernel \( D_F^{(i)} \) introduced in [Ke01]. Let \( E := F(\mu_p) \) and \( \Delta := \text{Gal}(E/F) \). For \( i \geq 2 \), the exact sequence

\[
0 \to \mathbb{Z}_p(i) \to \mathbb{Z}_p(i) \to \mathbb{Z}/p\mathbb{Z}(i) \to 0
\]

induces, by cohomology, the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & H^1(F, \mathbb{Z}_p(i))/p & \to & H^1(F, \mathbb{Z}/p(i)) & \to & pH^2(F, \mathbb{Z}_p(i)) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (H^1(E, \mathbb{Z}_p(i))/p)^\Delta & \to & E^*/E^{*p}(i-1)^\Delta & \to & (pH^2(E, \mathbb{Z}_p(i)))^\Delta & \to & 0.
\end{array}
\]

This shows the existence of a subgroup \( D_F^{(i)} \) of \( E^* \) containing \( E^{*p} \) - the analogue of the Tate kernel in the case where \( i = 2, F \supseteq \mu_p \), such that

\[
H^1_{\text{ét}}(F, \mathbb{Z}_p(i))/p \simeq D_F^{(i)}/E^{*p}(i-1).
\]

Let \( G_{S_p}(F) \) be the Galois group over \( F \) of the maximal extension of \( F \) unramified outside \( p \) and infinity. Since

\[
H^1_{\text{ét}}(F, \mathbb{Z}_p(i)) \simeq H^1_{\text{ét}}(d'_F, \mathbb{Z}_p(i)) \simeq H^1(G_{S_p}(F), \mathbb{Z}_p(i))
\]

we have

\[
D_F^{(i)}/E^{*p}(i-1) \hookrightarrow H^1(G_{S_p}(F), \mathbb{Z}/p(i)) \cong H^1(G_{S_p}(E), \mathbb{Z}/p(i))^\Delta.
\]

Therefore the elementary extension \( E(\sqrt[p]{D_F^{(i)}})/E \) is unramified outside \( p \) and infinity.

This being said, since \( H^1_{\text{ét}}(k_v, \mathbb{Z}_p(i)) = 0 \) for any \( p \)-adic prime \( v \), the localization map \( \text{loc}_{T_p} \) can be written

\[
\text{loc}_{T_p} : H^1_{\text{ét}}(F, \mathbb{Z}_p(i)) \to \bigoplus_{v \in T_p \setminus S_p} H^1_{\text{ét}}(k_v, \mathbb{Z}_p(i)).
\]

Its surjectivity is equivalent to that of

\[
H^1_{\text{ét}}(F, \mathbb{Z}_p(i))/p \to \bigoplus_{v \in T_p \setminus S_p} H^1_{\text{ét}}(k_v, \mathbb{Z}_p(i))/p.
\]

For each \( v \in T_p \setminus S_p \), we have \( \mu_p \subset F_v \) and

\[
H^1_{\text{ét}}(k_v, \mathbb{Z}_p(i))/p \cong H^1(k_v, \mathbb{Z}/p(i)) \cong H^1(F_{v,\infty}/F_v, \mathbb{Z}/p(i))
\]

where \( F_{v,\infty} := F_v(\mu_{p^\infty}) \) is the cyclotomic \( \mathbb{Z}_p \)-extension of the local field \( F_v \) which coincides with the maximal unramified pro-\( p \)-extension of \( F_v \). Therefore, the last map is the dual of the map

\[
\varphi'_{T_p} : \prod_{v \in T_p \setminus S_p} \text{Gal}(F_{v,\infty}/F_v)/p \to \text{Gal}(E(\sqrt[p]{D_F^{(i)}})/E)
\]

sending each term \( \text{Gal}(F_{v,\infty}/F_v)/p \) onto the decomposition group of the prime \( v \) in the extension \( E(\sqrt[p]{D_F^{(i)}})/E \).

Hence, the surjectivity of \( \text{loc}_{T_p} \) is equivalent to the injectivity of \( \varphi'_{T_p} \). Summarizing we obtain the following Galois co-descent criterion for motivic tame kernels:
Theorem 4.2. Let $L/F$ be a finite Galois extension of number fields with Galois group $G$ and $i \geq 2$. Assume that $L/F$ is unramified at infinity. If $i$ is odd, we assume either hypothesis ($H_i$) holds or $L/F$ is cyclic. Then, the natural map $\text{tr}_i : H^2_M(o_L, Z(i))_G \to H^2_M(o_F, Z(i))$ is surjective and the following two conditions are equivalent:

(i) the map $\text{tr}_i : H^2_M(o_L, Z(i))_G \to H^2_M(o_F, Z(i))$ is an isomorphism

(ii) for every prime $p$ dividing the ramification index $e_v$ for some finite prime $v$, the Frobenius automorphisms $\sigma_v(E(\sqrt[p]{D_F^{(i)}})/E)$, $v \in T_p \setminus S_p$, are linearly independent in the $F_p$-vector space $\text{Gal}(E(\sqrt[p]{D_F^{(i)}})/E)$.

Here $T_p$ stands for the set of finite primes $v$ of $F$ such that $p \mid e_v$. \hfill \square

Remark 4.3. The hypothesis "$L/F$ is unramified at infinity" in the above theorem means that $r = 0$. When $i$ is even, it is a necessary condition for the map $\text{tr}_i$ to be an isomorphism (see Proposition [2, 7]). Hence, for $i$ even, $\text{tr}_i$ is an isomorphism precisely when $r = 0$ and Condition (ii) of the above Theorem holds.

Now, let $E_1$ be the compositum of the first layers of all $Z_p$-extensions of $E$. Denote by $A_E$ its Kummer radical: $E_1 = E(\sqrt{A_E})$. The Galois group $\Delta$ acts on $\text{Gal}(E_1/E)$ by conjugation. For $i \in Z$, let $E_1^{(i)}$ be the subfield of $E_1$ corresponding to $\text{Gal}(E_1/E)(-i)_\Delta$ and $A_E^{[i-1]} := A_E(i-1)^\Delta$ its Kummer dual:

$$E_1^{(i)} = E(\sqrt{A_E^{[1-i]}}).$$

Let $T$ be a set of primes of $F$ containing the set $S_p$ of primes of $F$ above $p$ and infinite primes. We assume that the absolute norm $Nv \equiv 1 \pmod{p}$ for all non-$p$-adic primes of $T$ since such a condition is necessary for $e_v$ to be divisible by $p$.

Let $i \in Z$. The set $T$ is called $i$-primitive for $(F, p)$ if the Frobenius automorphisms $\sigma_v(E_1^{(i)}/E)$, $v \in T \setminus S_p$, generate an $F_p$-subspace of $\text{Gal}(E_1^{(i)}/E)$ of dimension the cardinality of $T \setminus S_p$. The notion of $i$-primitive sets introduced in [As95, AM12] is the twisted version of the notion of primitive sets [GJ89, Mo88, Mo90, MN90]. When $i \equiv 0 \pmod{\Delta}$, $i$-primitive sets are exactly primitive sets. Here, we will use the following more suggestive definition of primitivity (see also [Hu05]):

Definition 4.4. Let $T$ be a set of primes of $F$ containing $S_p$ and $D$ a subgroup of $E^\times$ such that $E(\sqrt[\Delta]{D})/E$ is unramified at all $v \in T \setminus S_p$. Then $T$ is called $D$-primitive for $(F, p)$ if the Frobenius automorphisms $\sigma_v(E(\sqrt[\Delta]{D})/E)$, $v \in T \setminus S_p$, are linearly independent in the $F_p$-vector space $\text{Gal}(E(\sqrt[\Delta]{D})/E)$.

With this definition, the second condition of Theorem [4,2] means that for each prime $p$ the set $T_p$ is $D_F^{(i)}$-primitive for $(F, p)$. By Čebotarev density theorem, there exists an infinite number of $D$-primitive sets as soon as $E(\sqrt[\Delta]{D}) \neq E$. 

Number fields such that the maximal pro-$p$-quotient of $G_{\mathbb{Q}}(F)$ is a free pro-$p$-group are called $p$-rational. They were introduced in [Mo88, Mo90, MN90] to construct infinitely many examples of non-abelian extensions of $\mathbb{Q}$ satisfying Leopoldt’s conjecture at the prime $p$. More recently they have been used in order to construct continuous representations $\rho : G_1(\mathbb{Q}/\mathbb{Q}) \to GL_n(\mathbb{Z}_p)$ of the absolute Galois group of $\mathbb{Q}$ with an open image for some $n \geq 2$ and $p$ [Gr16]. When $p$ is odd, the number field $F$ is $p$-rational precisely when $H^2_\text{ét}(o'_F, \mathbb{Z}/p) = 0$. Number fields $F$ such that $H^2_\text{ét}(o'_F, \mathbb{Z}/p(i)) = 0$ are called $(p, i)$-regular [As95]. For $p$ odd, the $(p, i)$-regularity is equivalent to the nullity of $H^2_\text{ét}(o'_F, \mathbb{Z}_p(i))$. These number fields can be considered as a generalization of the cyclotomic fields $\mathbb{Q}(\mu_p)$, $p$ regular. In the same way, $(p, i)$-regular number fields are introduced to construct number fields satisfying the “twisted” Leopoldt conjecture

$$H^2_\text{ét}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0, \ i \neq 1.$$  

As remarked in [Ka92], the nullity of the étale cohomology groups $H^2_\text{ét}(\mathbb{Z}[1/p], \mathbb{Z}_p(i))$ for all odd $i \geq 3$ is equivalent to Vandiver’s conjecture for the prime $p$ (asserting the non divisibility by $p$ of the class group of the maximal real subfield of $\mathbb{Q}(\mu_p)$). For any given odd $i$, Soulé proved that $H^2_\text{ét}(\mathbb{Z}[1/p], \mathbb{Z}_p(i)) = 0$ for $p$ greater than an effective bound depending on $i$ [So99]. The vanishing of $K_4\mathbb{Z} = 0$ is already known ([R00], see also [So00]).

In [As95] a going-up property for $(p, i)$-regularity is given for a $p$-extension $L/F$ under the Leopoldt conjecture in the cyclotomic tower $L(\mu_p^\infty)$. As an immediate consequence of Theorem 4.2, we have an alternative proof of the same property without any assumption on Leopoldt’s conjecture. Recall that for $(p, i)$-regular number fields, $i$-primivity coincides with $D_F^{(i)}$-primivity [AM12, Prop. 1.2 and 1.4].

**Corollary 4.5.** Let $p$ be an odd prime number and let $L/F$ be a finite Galois $p$-extension of number fields with Galois group $G$. Then for $i \in \mathbb{Z}$, the following conditions are equivalent

(i) $H^2_\text{ét}(o'_L, \mathbb{Z}_p(i)) = 0$;
(ii) $H^2_\text{ét}(o'_F, \mathbb{Z}_p(i)) = 0$ and the set of primes of $F$ which are in $S_p$ or ramify in $L$ is $D_F^{(i)}$-primitive for $(F, p)$.

**Proof** It is clear that when $i \equiv j \mod |\Delta|$, the properties (i) and (ii) are exactly the same for $i$ and $j$. So without loss of generality we can assume that $i \geq 2$ and apply Theorem 4.2. □

We also need a co-descent criterion for the positive cohomology groups $H^2_+\text{(o}_L/\mathbb{Z}_p(i))$. Following Kolster [Ko03, Definition 2.4], we introduce the subgroup $D_F^{(i)}(\text{positive étale Tate kernel})$ to be the kernel of the signature map $\text{sgn}$ (see Subsection 1.2) restricted to $D_F^{(i)}$ so that we have an exact sequence

$$0 \to D_F^{(i)} / F^* \to D_F^{(i)} / F^{\bullet} \to (\mathbb{Z} / 2)^{r_1(F)} \to (\mathbb{Z} / 2)^{d_1(F)} \to 0.$$
Recall that, when \( i \) is even, Proposition 4.1 amounts to \( D^+_F(i) = D^+_F(i) \).

The same arguments as in the proof of Theorem 4.2 above show that

\[
H^2_+(o_L[1/2], \mathbb{Z}_2(i))_G \cong H^2_+(o_F[1/2], \mathbb{Z}_2(i))
\]

precisely when the localization composite map

\[
H^1_+(F, \mathbb{Z}_2(i)) \to H^1(F, \mathbb{Z}_2(i)) \xrightarrow{\text{loc}_{T_2}} \oplus_{v \in T_2} H^1_{\text{ét}}(k_v, \mathbb{Z}_2(i))
\]

is surjective, or equivalently the induced composite map

\[
H^1_+(F, \mathbb{Z}_2(i))/2 \to H^1(F, \mathbb{Z}_2(i))/2 \cong D^+_F/F^{*2}(i-1) \to \oplus_{v \in T_2} H^1_+(k_v, \mathbb{Z}_2(i))/2
\]

is surjective. The image of the above left map being the subgroup \( D^+_F(i)/F^{*2} \) of \( D^+_F(i)/F^{*2} \), we have the following theorem which is proved in the same way as Theorem 4.2.

**Theorem 4.6.** Let \( L/F \) be a Galois 2-extension of number fields with Galois group \( G \). Then, the surjective map \( \text{tr}_i^+: H^2_+(o'_L, \mathbb{Z}_2(i))_G \to H^2_+(o'_F, \mathbb{Z}_2(i)) \) is an isomorphism precisely when \( S_2 \cup T_2 \) is \( D^+_F(i) \)-primitive for \((F, 2)\). \( \square \)

**Corollary 4.7.** Let \( L/F \) be an imaginary 2-extension of number fields with Galois group \( G \) and \( i \geq 2 \) odd. Then the surjective map \( \text{tr}_i^+: H^2_+(o'_L, \mathbb{Z}_2(i))_G \to H^2_+(o'_F, \mathbb{Z}_2(i)) \) is an isomorphism precisely when the signature map \( \text{sgn}_F: H^1(F, \mathbb{Z}_2(i))/2 \to (\mathbb{Z}/2)^{r_1(F)} \) is surjective and \( S_2 \cup T_2 \) is \( D^+_F(i) \)-primitive for \((F, 2)\).

**Proof.** Since \( L/F \) is totally imaginary, hypothesis \((\mathcal{H}_i)\) is satisfied and \( s_i(L/F) = \delta_i(F) \). The exact sequence (31) shows that \( \text{tr}_i \) is an isomorphism exactly when \( s_i(L/F) = \delta_i(F) = 0 \) and \( \text{tr}_i^+ \) is an isomorphism. \( \square \)

Let us mention the following arithmetic criterion for the nullity of the positive étale cohomology groups \( H^2_+(o'_F, \mathbb{Z}_2(i)) \) whose proof follows the same ideas as those in the proof of [Ko03, Prop.2.6] where the case of real number fields is dealt with. See also [RO00, Prop.2.2].

**Proposition 4.8.** Let \( F \) be a number field and \( i \geq 2 \). Then \( H^2_+(o'_F, \mathbb{Z}_2(i)) \) vanishes precisely when \( F \) has only one dyadic prime and the narrow class group \( A'_F \) is 0. In particular, the vanishing of \( H^2_+(o'_F, \mathbb{Z}_2(i)) \) is independent of the integer \( i \geq 2 \).

**Proof.** the 2-rank of \( H^2(o'_F, \mathbb{Z}_2(i)) \) is given by [RW00, Prop.6.13] or [Ko03, Lemma 2.2]:

\begin{enumerate}
  \item \( \text{rk}_2 H^2(o'_F, \mathbb{Z}_2(i)) = |S_2 \setminus S_{\infty}| + \text{rk}_2(A'_F) - 1 \) for \( i \) odd;
  \item \( \text{rk}_2 H^2(o'_F, \mathbb{Z}_2(i)) = |S_2 \setminus S_{\infty}| + \text{rk}_2(A'_F) - 1 + r_1 \) for \( i \) even;
\end{enumerate}

where \( |S_2 \setminus S_{\infty}| \) represents the number of dyadic primes in \( F \) and \( r_1 \) is the number of real places in \( F \).

Also, the exact sequence

\[
0 \to H^1(o'_F, \mathbb{Z}_2(i))/2 \to H^1(o'_F, \mathbb{Z}/2(i)) \to 2 H^2(o'_F, \mathbb{Z}_2(i)) \to 0
\]
shows that $H^2(o'_F, Z_2(i)) = 0$ precisely when
\[ H^1(o'_F, Z_2(i))/2 \cong H^1(o'_F, Z/2)(i). \]

Besides, by Kummer theory we have an exact sequence
\[ 0 \to U'_F/U''_F \to H^1(o'_F, \mu_2) \to 2A'_F \to 0. \]
Therefore, $A'_F$ vanishes precisely when
\[ H^1(o'_F, \mu_2) \cong U'_F/U''_F. \]
First assume $i$ is odd. The exact sequence \([1.3]\) shows that $H^2(o'_F, Z_2(i)) = 0$ precisely when $H^2(o'_F, Z_2(i)) = 0$ and the signature map \([1]\) is surjective. By the above formula, $H^2(o'_F, Z_2(i)) = 0$ if and only if $F$ contains only one dyadic prime and $A'_F = 0$, which by the above discussion is equivalent to $|S_2 \setminus S_\infty| = 1$ and
\[ H^1(o'_F, Z_2(i))/2 \cong H^1(o'_F, Z/2)(i) \cong U'_F/U''_F. \]
In particular, the signature map \([1]\) becomes
\[ \text{sgn}_F : U'_F/U''_F \to (\mathbb{Z}/2\mathbb{Z})^r \]
whose surjectivity is equivalent to the equality $A'_F^+ = A'_F$ (cf. §1.2).

If instead $i$ is even, consider the following commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(o'_F, Z_2(i))/2 & \longrightarrow & H^1(o'_F, Z/2(i)) & \longrightarrow & 2H^2(o'_F, Z_2(i)) & \longrightarrow & 0 \\
& & \downarrow & & \text{sgn} \downarrow & & \sigma \downarrow & & \\
0 & & (\mathbb{Z}/2)^r & & \cong & & (\mathbb{Z}/2)^r & & \\
\end{array}
\]
obtained in the same manner as diagram \([3]\) where the left vertical map is induced by the signature map.

The short exact sequence \([18]\), readily yields an exact sequence
\[ 0 \to 2H^2(o'_F, Z_2(i)) \to 2H^2(o'_F, Z_2(i)) \xrightarrow{\sigma} (\mathbb{Z}/2)^r \to H^2(o'_F, Z_2(i))/2. \]
Hence, we successively have
\[ H^2(o'_F, Z_2(i)) = 0 \iff \sigma \text{ is surjective and } rk_2H^2(o'_F, Z_2(i)) = r_1 \]
\[ \iff |S_2 \setminus S_\infty| = 1, A'_F = 0 \text{ and } H^1(o'_F, Z/2(i)) \xrightarrow{\text{sgn}} (\mathbb{Z}/2)^r \text{ is surjective} \]
\[ \iff |S_2 \setminus S_\infty| = 1, A'_F = 0 \text{ and } U'_F/U''_F \xrightarrow{\text{sgn}} (\mathbb{Z}/2)^r \text{ is surjective} \]
\[ \iff |S_2 \setminus S_\infty| = 1 \text{ and } A'_F^+ = 0. \]

A number field $F$ which has only one dyadic prime and whose narrow class group $A'_F^+ = 0$ is called 2-regular \([GJ80, N90, RO00]\). According to the above proposition a number field $F$ is 2-regular precisely when the positive cohomology $H^2(o'_F, Z_2(i))$ vanishes for some (hence all) integer $i$.

Theorem \([4.6]\) leads to the following going-up property which completes Corollary \([4.5]\) for $p = 2$: 

\[ \text{□} \]
Corollary 4.9. Let \( L/F \) be a finite Galois 2-extension of number fields with Galois group \( G \). Then \( H^2(\sigma'_L, \mathbb{Z}_2(i)) = 0 \) precisely when the set of primes of \( F \) which are in \( S_2 \) or ramify in \( L \) is \( D_F^{+(i)} \)-primitive for \((F, 2)\).

Recall that \( D_F^{+(i)} = D_F^{(i)} \) for \( i \) even according to Proposition 1.1.

We now focus on the case of cyclic extensions of degree \( p \), \( p \) prime, and give general genus formulae involving only the number of some specified tamely ramified primes. As before let \( E = F(\mu_p) \) and let \( D \) be a subgroup of \( E^* \) such that \( E(\sqrt[p]{D})/E \) is unramified outside \( p \)-adic primes. We have a perfect pairing

\[
\text{Gal}(E(\sqrt[p]{D})/E) \times D/E^{p^r} \longrightarrow \mu_p \quad \sigma \mapsto \sigma(\sqrt[p]{a})/\sqrt[p]{a}.
\]

Let \( T \) be a maximal \( D \)-primitive set contained in \( S \). For each \( v \in S \), denote by \( \sigma_v \) the Frobenius at the prime \( v \) in \( E(\sqrt[p]{D})/E \). Finally, let \( H := \langle \sigma_v, v \in S - S_p \rangle \leq \sigma_v, v \in T - S_p \rangle \) and let \( H^\perp \) be the orthogonal complement of \( H \) under the above pairing (35).

We have the following lemma which will be crucial in the proof of Proposition 4.11 below.

**Lemma 4.10.** Under the pairing (35), the orthogonal complement \( H^\perp \) of \( H \) is equal to the subgroup \( D \cap \bigcap_{v \in S - S_p} N_{L_w/F_v}(L_w^*)/E^{p^r} \) of \( D/E^{p^r} \).

**Proof.** The proof is identical to that of [AM12, Theorem 2.4] (see also [AM04, Proposition 3.6]) where \( D_F^{(i,n)} \) is replaced by \( D \).

Since

\[
[D : D \cap \bigcap_{v \in S - S_p} N_{L_w/F_v}(L_w^*)] = \dim_{F_p}(\text{Gal}(E(\sqrt[p]{D})/E)) - \dim_{F_p}(H^\perp)
\]

and \( \dim_{F_p}(H^\perp) = \dim_{F_p}(\text{Gal}(E(\sqrt[p]{D})/E)) - \dim_{F_p}(H) \), we have the following

**Proposition 4.11.** Let \( L/F \) be a cyclic extension of degree \( p \). Then, with the above notation, we have the following formula for the norm index

\[
[D : D \cap \bigcap_{v \in S - S_p} N_{L_w/F_v}(L_w^*)] = p^{t_D},
\]

where \( t_D := \dim_{F_p} < \sigma_v(E(\sqrt[p]{D})/E)/v \in S - S_p > \) denotes the maximal number of tamely ramified primes in \( L/F \) belonging to a \( D \)-primitive set for \((F, p)\).
When $L/F$ is a Galois extension of degree a prime number $p$, the norm index occurring in the genus formula in Theorem 2.4 can be written as

$$[D_F^{(i)} : D_F^{(i)} \cap (\bigcap_{v \in S \setminus S_p} N_{L_v/F_v}(L_w^*))].$$

Indeed, we have by (12), an isomorphism

$$H^1(F_v, \mathbb{Z}_p(i)) \cong H^1(k_v, \mathbb{Z}_p(i))$$

for any prime $v \in S \setminus S_p$. Since $v \in S \setminus S_p$ is allowed to ramify in a $p$-extension, $F_v$ contains the group $\mu_p$ of $p$-th roots of the unity. As in the global case, there exists a subgroup $D_v^{(i)}$ such that

$$H^1(F_v, \mathbb{Z}_p(i))/p \cong D_v^{(i)}/F_v^{*p}.$$  

Since $L/F$ is cyclic of degree $p$, the norm index in Theorem 2.4 is the same as the order of the image of the map

$$H^1(F, \mathbb{Z}_p(i)) \rightarrow \bigoplus_{v \in S \setminus S_p} H^1(F_v, \mathbb{Z}_p(i))/N_{L_v/F_v}(H^1(L_w, \mathbb{Z}_p(i)) \cong D_v^{(i)}/F_v^{*p}N_{L_v/F_v}D_v^{(i)}.$$ 

By [AM12, Lemma 2.3] (whose proof goes along the same line for $p = 2$),

$$D_v^{(i)}/F_v^{*p}N_{L_v/F_v}D_v^{(i)} \cong D_v^{(i)}/D_v^{(i)} \cap N_{L_v/F_v}(L_w^*)$$

for any $v \in S \setminus S_p$.

**Corollary 4.12.** Let $L/F$ be a cyclic extension of prime degree $p$. Then,

1. If $p$ is odd,
   $$\left|\frac{(K_{2i-2}\mathcal{O}_L)_G}{K_{2i-2}\mathcal{O}_F}\right| = \frac{|H^2_M(\mathcal{O}_L, \mathbb{Z}(i))_G|}{|H^2_M(\mathcal{O}_F, \mathbb{Z}(i))|} = p^{S \setminus S_p - t_i}.$$  
2. If $p = 2$ and $2i - 2 \equiv 2 \pmod{8}$,
   $$\left|\frac{(K_{2i-2}\mathcal{O}_L)_G}{K_{2i-2}\mathcal{O}_F}\right| = \frac{|H^2_M(\mathcal{O}_L, \mathbb{Z}(i))_G|}{|H^2_M(\mathcal{O}_F, \mathbb{Z}(i))|} = 2^{S \setminus S_p - t_i - r}.$$  
3. If $p = 2$ and $2i - 2 \equiv 6 \pmod{8}$,
   $$\left|\frac{(K_{2i-2}\mathcal{O}_L)_G}{K_{2i-2}\mathcal{O}_F}\right| = 2^r \frac{|H^2_M(\mathcal{O}_L, \mathbb{Z}(i))_G|}{|H^2_M(\mathcal{O}_F, \mathbb{Z}(i))|} = 2^{S \setminus S_p - t_i}.$$ 

Here $t_i := \dim_p < \sigma_v(E(\sqrt[2^i]{D^{(i)}_F})/E)/v \in S - S_p >$ denotes the maximal number of tamely ramified primes in $L/F$ belonging to a $D_F^{(i)}$-primitive set for $(F, p)$ and $r$ is the number of infinite places of $F$ which ramify in $L$.

**Proof.** By Proposition 4.11 in all the considered cases, we have

$$[D_F^{(i)} : D_F^{(i)} \cap (\bigcap_{v \in S \setminus S_p} N_{L_v/F_v}(L_w^*))] = p^t.$$
The formulae for the motivic cohomology groups follow from Theorem 2.4. When $p$ is odd or $p = 2$ and $2i - 2 \equiv 2 \pmod{8}$, the formulae for the $K$-groups is then a consequence of Theorem 3.1. In the remaining case, we use the formula (33). □

When $p = 2$ and $i \geq 2$ is odd, we use positive cohomology. The norm index in Theorem 2.7 can be written as

$$[D_F^{+(i)} : D_F^{+(i)} \cap ( \bigcap_{v \in S \setminus S_P} N_{L_w/F_v}(L_w^*))]$$

and recall that we have an exact sequence

$$0 \to D_F^{+(i)}/F^* \to D_F^{+(i)}/F^* \to (\mathbb{Z}/2)^{r_1} \to (\mathbb{Z}/2)^{\delta_i} \to 0$$

where the second map is the restriction of the partial signature map $\text{sgn}_{R_{t}/F}$ to $D_F^{+(i)}/F^*$.

**Corollary 4.13.** Let $L/F$ be a quadratic extension of number fields and $i \geq 2$ odd. Let $t_i^+$ denote the maximal number of tamely ramified primes in $L/F$ belonging to a $D_F^{+(i)}$-primitive set for $(F, 2)$. Then

$$|H^2_F(o_L, \mathbb{Z}_2(i))_G| = 2^{s \setminus S_2 + s_i - t_i^+}.$$

Moreover if hypothesis $(H_t)$ holds, then

$$|H^2(o_L, \mathbb{Z}_2(i))_G| = 2^{s \setminus S_2 + s_i - t_i^+}$$

and

1. If $2i - 2 \equiv 0 \pmod{8}$,

$$\frac{|(K_{2i-2}o_L)_G|}{|K_{2i-2}o_F|} = \frac{|H^2_M(o_L, \mathbb{Z}_2(i))_G|}{|H^2_M(o_F, \mathbb{Z}_2(i))|} = 2^{s \setminus S_2 + s_i - t_i^+}.$$

2. If $2i - 2 \equiv 4 \pmod{8}$,

$$\frac{|(K_{2i-2}o_L)_G|}{|K_{2i-2}o_F|} = 2^r \frac{|H^2_M(o_L, \mathbb{Z}_2(i))_G|}{|H^2_M(o_F, \mathbb{Z}_2(i))|} = 2^{s \setminus S_2 - t_i^+}.$$

**Proof.** By Proposition 4.11 we have

$$[D_F^{+(i)} : D_F^{+(i)} \cap ( \bigcap_{v \in S \setminus S_P} N_{L_w/F_v}(L_w^*))] = p^{t_i^+}.$$

By Theorem 2.7 it follows that

$$\frac{|H^2_F(o_L, \mathbb{Z}_2(i))_G|}{|H^2_F(o_F, \mathbb{Z}_2(i))_G|} = 2^{s \setminus S_2 + s_i - t_i^+}.$$

The formulae for the motivic cohomology groups then follow from Proposition 2.8. Now, if $2i - 2 \equiv 0 \pmod{8}$, then the formula follows by Theorem 3.1 and if $2i - 2 \equiv 4 \pmod{8}$, then we use formula (32). □
When $2i - 2 \equiv 0 \pmod{8}$ and $L/F$ is unramified at infinite primes, we use Theorem 2.10 to obtain an explicit formula without any assumption on the cohomology of $(\mathbb{Z}/2)^{\delta_i(L)}$:

**Proposition 4.14.** Let $L/F$ be a quadratic extension of number fields and $i \geq 2$ odd. Let $t_i$ denotes the maximal number of tamely ramified primes in $L/F$ belonging to a $D_F^{(i)}$-primitive set for $(F, 2)$. Assume that $L/F$ is unramified at infinity. Then

$$\frac{|H^2_M(o_L, \mathbb{Z}(i))|}{|H^2_M(o_F, \mathbb{Z}(i))|} = 2^{|S \setminus S_2| - t_i}.$$  

In particular if $2i - 2 \equiv 0 \pmod{8}$,

$$\frac{|(K_{2i-2}o_L)|}{|K_{2i-2}o_F|} = 2^{|S \setminus S_2| - t_i}. \tag*{□}$$

We now have the ingredients to answer a question (of independent interest), raised by B. Kahn in [Ka97, page 2]: Let $\rho_i := \rho_i(F)$ denote the 2-rank of the image of the signature map

$$\text{sgn}_F : H^1(F, \mathbb{Z}_2(i))/2 \longrightarrow (\mathbb{Z}/2)^{r_1}.$$  

Is it true that $\rho_i = 1$, for any number field $F$ such that $r_1 \geq 1$? The answer turns out to be negative in general. For instance, $\mathbb{Q}(\sqrt{5})$ is 2-regular by Proposition 4.8 and we saw in the proof that the signature map for such a field is surjective. In fact, as the following theorem shows, the integer $\rho_i$ can be arbitrary large for real number fields.

**Theorem 4.15.** Let $n \geq 1$ be an integer. Then there exists a totally real number field $F$ such that the image of the signature map $\text{sgn}_F$ has 2-rank $\rho_i = n$, for all $i \geq 2$ odd.

**Proof.** Let $m$ be the unique integer such that

$$2^m \leq n < 2^{m+1}.$$  

Since $\mathbb{Q}$ is 2-regular, there exist 2-regular number fields $F$ such that $[F : \mathbb{Q}] = 2^m$ by Theorem 4.6. For instance, one may take for $F$ the $m$-th layer of the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$.

Then, as noticed in the proof of Proposition 4.8, the signature map $\text{sgn}_F$ is surjective and the exact sequence (34) shows that

$$\dim_{P_2} D_F^{+(i)} / F^{*2} = \dim_{P_2} D_F^{(i)} / F^{*2} - r_1(F).$$

Choose a maximal $D_F^{+(i)}$-primitive set $T$ for $(F, 2)$ (Definition 4.4). The set $T$ then contains the dyadic prime and exactly one non dyadic prime but we do not need this fact.

Let $S$ be a $D_F^{(i)}$-primitive set for $(F, 2)$ containing $T$ and such that $|S \setminus T| = 2^{m+1} - n$. Such a primitive set $S$ exists since $2^{m+1} - n \leq 2^{m+1} - 2^m = 2^m = r_1(F)$. Then there exists a real quadratic extension $L$ of $F$ such that the non dyadic primes which are
ramified in \( L/F \) consist precisely of \( S \setminus S_2 \) (see lemma 4.16 below). Proposition 4.14 shows that \( H^2(o'_L, \mathbb{Z}_2(i)) = 0 \). By the exact sequence (21), we then have
\[
H^2(o'_L, \mathbb{Z}_2(i)) \cong (\mathbb{Z}/2)\delta_i(L).
\]
On the other hand, Corollary 4.13 shows that
\[
|H^2(o'_L, \mathbb{Z}_2(i))_G| = 2^{|S \setminus T|}
\]
leading to \( \dim_F H^2(o'_L, \mathbb{Z}_2(i)) = |S \setminus T| \). Hence
\[
\rho_i(L) = r_1(L) - \delta_i(L) = 2^{m+1} - 2^{m+1} + n = n.
\]
It remains to prove:

**Lemma 4.16.** Let \( F \) be a number field with a trivial 2-primary narrow class group and let \( \{p_1, \ldots, p_t\} \) be a set of non dyadic primes. Then there exists a (totally real) quadratic extension \( L \) of \( F \) unramified at infinity and in which the tamely ramified primes are precisely \( p_1, \ldots, p_t \).

**Proof.** Since the 2-primary part of the narrow class group of \( F \) is trivial, there exists an odd integer \( m \) such that \( p_i^m = (d_i) \) is a principal ideal generated by a totally positive element for all \( i, 1 \leq i \leq t \). Obviously \( d_i \notin F^2 \) and the field \( L = F(\sqrt{d_1 \cdots d_t}) \) fulfils the conditions of the lemma.

### 5. Galois descent

We keep the notations of the preceding sections: \( L/F \) is a finite Galois extension of number fields with Galois group \( G \). Recall that \( S \) consists of finite primes which ramify in \( L/F \) as well as the infinite primes. Suppose that \( S \) contains a set \( T \) such that for all prime \( p \) (dividing \( e \)) the set \( T_{L/F,p} := \{ v \in T \mid p \mid e_v \} \cup S_p \) is \( D_{i}^{(i)} \)-primitive for \((F,p)\).

As before, the natural map
\[
\beta_T : \hat{H}^0(G, H^1_M(L, \mathbb{Z}(i))) \to \hat{H}^0(G, \oplus_{v \in T, w | v} H^1_M(k_w, \mathbb{Z}(i)))
\]
is surjective.

Recall that, by Theorem 1.2, the kernel and cokernel of the functorial map
\[
f_i : H^2_M(F, \mathbb{Z}(i)) \to H^2_M(L, \mathbb{Z}(i))^G
\]
are described by the \( G \)-cohomology of \( H^1_M(L, \mathbb{Z}(i)) \).

We shall use the surjectivity of \( \beta_T \) to prove that of the natural map
\[
\theta_T : H^2(G, H^1_M(L, \mathbb{Z}(i))) \to H^2(G, \oplus_{v \in T, w | v} H^1_M(k_w, \mathbb{Z}(i)))
\]
by taking the cup-product with \( H^2(G, \mathbb{Z}) \). This provides lower bounds of respectively the order and the generator rank of \( \text{coker} f_i \).
Since \( \ell \) (the residue characteristic of \( F_v \)) does not divide the order of \( H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i)) \), we have
\[
\tilde{H}^q(I_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i))) \cong \tilde{H}^q(I'_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i)))
\]
for all \( q \in \mathbb{Z} \). Here \( I'_v = I_v/I_{v,1} \) is the inertia group of \( v \) in \( L/F \) factored by the first ramification group.

Since \( I'_v \) is a cyclic group, cup product gives rise to the following commutative diagram
\[
\begin{array}{ccc}
\tilde{H}^0(I_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i))) & \otimes & H^2(I_v, \mathbb{Z}) \\
& \downarrow \text{cor} & \downarrow \text{res} \\
\tilde{H}^0(G_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i))) & \otimes & H^2(G, \mathbb{Z})
\end{array}
\]
which shows the surjectivity of the bottom map. In fact this map is an isomorphism since \( H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i)) \) is of order prime to \( \ell \) and the restriction map \( H^2(I_v, \mathbb{Z}) \to H^2(I'_v, \mathbb{Z}) \) is an isomorphism on the non-\( \ell \)-part.

Now consider the commutative diagram
\[
\begin{array}{ccc}
\tilde{H}^0(I_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i))) & \otimes & H^2(I_v, \mathbb{Z}) \\
& \downarrow \text{cor} & \downarrow \text{res} \\
\tilde{H}^0(G_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i))) & \otimes & H^2(G, \mathbb{Z})
\end{array}
\]
By (37) the cokernel of the map
\[
(37) \quad \text{cor} : H^2(I_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i))) \to H^2(G_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i)))
\]
is isomorphic to the cokernel of the corestriction
\[
(38) \quad \tilde{H}^0(I_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i-1))) \to \tilde{H}^0(G_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i-1)))
\]
induced by the norm. The map (38) is surjective since, as in the proof of Lemma 2.2, we have \( \tilde{H}^0(I_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i-1))) \cong H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i-1))/e_v \) and \( \tilde{H}^0(G_v, H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i-1))) \cong H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i-1))/e_v \). Hence the map (37) is also surjective. It follows from the above diagram that the cup-product induces a surjective homomorphism
\[
\gamma_v : \tilde{H}^0(G, \oplus_{w|v} H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i))) \otimes H^2(G, \mathbb{Z}) \to H^2(G, \oplus_{w|v} H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i))).
\]
Consider now the following commutative diagram
\[
\begin{array}{ccc}
\tilde{H}^0(G, H^1_{\mathcal{M}}(L, \mathbb{Z}(i))) & \otimes & H^2(G, \mathbb{Z}) \\
& \downarrow \theta_T & \\
\tilde{H}^0(G, \oplus_{v\in T, w|v} H^1_{\mathcal{M}}(k_w, \mathbb{Z}(i))) & \otimes & H^2(G, \mathbb{Z})
\end{array}
\]
where the horizontal maps are induced by cup-product. The left vertical map is induced by \( \beta_T \) (see (36)) and so is surjective. The surjectivity of the right vertical map \( \theta_T \) then
follows from the surjectivity of the maps $\gamma_v$. To compute the cohomology group $H^2(G, \oplus_{v \in T, w} H^1_M(k_w, Z(i)))$, we have for $v \in T$,

$$H^2(G_v, H^1_M(k_w, Z(i))) \cong \mathcal{H}^0(G_v, H^1_M(k_w, Z(i - 1))) \quad \text{(by (27))}$$

$\cong \mathbb{Z}/e_v^{(i-1)}$ (see the proof of Lemma 2.2)

where $e_v^{(i-1)} = \gcd(e_v, q_i^v - 1)$.

The above discussion together with Theorem 1.2 yield the following upper bounds for the capitulation cokernel:

**Theorem 5.1.** Let $L/F$ be a Galois extension of number fields with Galois group $G$. Assume that the set of ramified primes in the extension $L/F$ contains a set $T$ such that for all prime $p$, the set $\{v \in T/ p \mid e_v \} \cup S_p$ is $D_F^{(i)}$-primitive for $(F, p)$. Then we have a surjective homomorphism

$$H^2(G, H^1_M(L, Z(i))) \longrightarrow \oplus_{v \in T} \mathbb{Z}/e_v^{(i-1)}$$

leading to the following upper bounds:

1. $|\text{coker } f_i| \geq \prod_{v \in T} e_v^{(i-1)}$ for $i$ even and
2. $|\text{coker } f_i| \geq 2^{s_i-r} \prod_{v \in T} e_v^{(i-1)}$ for $i$ odd

where for each $v$, $e_v^{(i-1)} := \gcd(e_v, q_i^v - 1)$.

When $G$ is a cyclic group, we see from Theorem 1.2 that the Herbrand quotient $h(G, H^1_M(L, Z(i)))$ equals $2^{-r}$ for $i$ even and $2^{r}$ for $i$ odd. Hence the above theorem together with Lemma 2.2 lead to the following upper bounds for the capitulation kernel:

**Corollary 5.2.** Let $L/F$ be a cyclic extension of number fields of degree $n$ with Galois group $G$. Assume that the set of ramified primes in the extension $L/F$ contains a set $T$ such that for all prime $p$, the set $\{v \in T/ p \mid e_v \} \cup S_p$ is $D_F^{(i)}$-primitive for $(F, p)$. Then

$$|\text{ker } f_i| \geq 2^r \prod_{v \in T} e'_v \quad \text{for } i \text{ even and } |\text{ker } f_i| \geq 2^{-r} \prod_{v \in T} e'_v \quad \text{for } i \text{ odd.}$$

We finish the section by giving examples of Galois extensions $L/F$ for which the kernel and the cokernel of $f_i$ are explicitly known.

Since $H^2_M(o_F, Z(i))$ is finite there exist infinitely many integers $n$ such that $H^2_M(o_F, Z(i))/n$ vanishes. We take such an integer $n$ and a cyclic extension $L/F$ of degree $n$, with group $G$ and unramified at archimedean places. As before, $S$ consists of ramified primes as well as archimedean ones. Assume that for all prime numbers $p$, the set $\{v \in S/ p \mid e_v \} \cup S_p$
is $D_F^{(i)}$-primitive for $(F, p)$. Consider the commutative diagram obtained from the exact sequence $[\mathfrak{C}]$:

$$
\begin{array}{cccc}
0 & \rightarrow & H^2_M(o_L, \mathbb{Z}(i))^G & \rightarrow & H^2_M(o_L, \mathbb{Z}(i))^G \\
f_i' & \uparrow & f_i & \uparrow & f_i \\
0 & \rightarrow & H^2_M(o_F, \mathbb{Z}(i)) & \rightarrow & H^2_M(o_F, \mathbb{Z}(i)) \\
& & f_i & \uparrow & f_i \\
& & H^1_M(k_w, \mathbb{Z}(i-1)) & \rightarrow & 0
\end{array}
$$

By Theorem 4.2 ker $f_i$ is annihilated by $n$. Hence the vanishing of $H^2(o_F, \mathbb{Z}(i))/n$ shows that $f_i'$ is injective. Moreover, by Theorem 4.2 $H^2_M(o_L, \mathbb{Z}(i))^G$ has the same order as $H^2_M(o_F, \mathbb{Z}(i))$. Therefore $f_i'$ is an isomorphism.

Now,

$$\text{coker } f_{v,i} \simeq H^0(G_v, H^1_M(k_w, \mathbb{Z}(i-1))) \simeq \mathbb{Z}/e_v'$$

and similarly

$$\ker f_{v,i} \simeq H^{-1}(G_v, H^1_M(k_w, \mathbb{Z}(i-1))) \simeq \mathbb{Z}/e_v'.$$

This leads to the exact structure of the kernel and the cokernel of $f_i$:

$$\ker f_i \simeq \text{coker } f_i \simeq \oplus_{v \in S \setminus S_{\infty}} \mathbb{Z}/e_v'.$$

### 6. Examples

In this section, we consider the special case where $F = \mathbb{Q}$. Given a prime $p$ we are going to caracterize $p$-extensions $L$ of $\mathbb{Q}$ for which $H^2_M(o_L, \mathbb{Z}(i)) \otimes \mathbb{Z}_p \simeq H^2_M(o_L, \mathbb{Z}(i)) = 0$.

Recall that the motivic cohomology groups $H^2_M(o_L, \mathbb{Z}(i))$ and the K-groups $K_{2i-2}o_L$ are isomorphic up to known 2-torsion [Ka97, RW00]. Hence for $p$ odd, we actually determine $p$-extensions $L$ of $\mathbb{Q}$ for which $K_{2i-2}o_L \otimes \mathbb{Z}_p$ vanishes. The list of $p$-extensions $L$ of $\mathbb{Q}$ for which the $p$-part of the classical wild kernel $W(K_{2i} \otimes \mathbb{Z}_p$) vanishes can be found in [KM00] example 2.13, 2

6.1. We begin with the case where $p$ is an odd prime. In this case $H^2_M(o_L', \mathbb{Z}_p(i))$ is isomorphic to $K_{2i-2}o_L \otimes \mathbb{Z}_p$.

First, consider the case where $i = 2k$ is even. The order of $H^2_M(o_L, \mathbb{Z}(i))$ can be computed using the values of the Riemann zeta function at odd negative integers since the Lichtenbaum conjecture is true in this case thanks to the main theorem in Iwasawa theory (Theorem of Mazur-Wiles) [MW84]. Namely, let $B_k$ be the $k$-th Bernoulli number and $c_k$ be the numerator of $B_k/4k$. Then $|H^2_M(o_L, \mathbb{Z}(i))| = 2c_k$. For instance, for $p$ odd and $i = 2, 4, 6, 8, 10$, we have $H^2_M(\mathbb{Z}[1/p], \mathbb{Z}_p(i)) = 0$ whereas for $p = 691$ and $i = 12$, the group $H^2_M(\mathbb{Z}[1/p], \mathbb{Z}_p(i))$ is cyclic of order $p$.

If $i \not\equiv 0 \pmod{p-1}$, Corollary 4.5 shows that $H^2_M(o_L', \mathbb{Z}_p(i)) = 0$ for a $p$-extension $L$ of $\mathbb{Q}$, precisely when $H^2_M(\mathbb{Z}[1/p], \mathbb{Z}_p(i)) = 0$ and $L$ is contained in the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.
If \( i \equiv 0 \pmod{p-1} \), the triviality of \( H^2_{\text{ét}}(o'_L, Z_p(i)) \) is equivalent to the \( p \)-rationality of the field \( L \) and it is known that a \( p \)-extension \( L \) of \( \mathbb{Q} \) is \( p \)-rational exactly when at most one non-\( p \)-adic prime \( \ell \) is ramified in \( L \) where \( \ell \) is such that \( \ell \equiv 1 \pmod{p} \) and \( \ell \neq 1 \pmod{p^2} \). The last assertion also follows readily from Corollary 4.5.

Now, consider the case where \( i = 2k + 1 \) is odd.

Assuming the triviality of \( H^2_{\text{ét}}(\mathbb{Z}[1/p], Z_p(i)) \) (which is the case under Vandiver’s conjecture), we shall construct all \( p \)-extensions \( L \) of \( \mathbb{Q} \) such that \( H^2_{\text{ét}}(o'_L, Z_p(i)) = 0 \).

Let \( E = \mathbb{Q}(\mu_p) \) and \( \Delta = \text{Gal}(E/\mathbb{Q}) \). We have \( \dim_{\mathbb{F}_p} H^1_{\text{ét}}(\mathbb{Z}[1/p], Z_p(i))/p = 1 \) and

\[
H^1_{\text{ét}}(\mathbb{Z}[1/p], Z_p(i))/p \cong U'_E/p(i-1)^\Delta
\]

where \( U'_E \) denotes the group of units of \( o'_E \).

If \( i \equiv 1 \pmod{p-1} \), \( H^1_{\text{ét}}(\mathbb{Z}[1/p], Z_p(i))/p \cong U'_Q/p(i-1) \) and therefore \( D^{(i)}_Q/E^{\ast_p} \) is generated by the class of \( p \). Hence \( D^{(i)}_Q \)-primitive sets for \( (\mathbb{Q}, p) \) are of the form \( S_p \) or \( T = S_p \cup \{\ell\} \) with \( \ell \) inert in \( E(\sqrt[p]{\ell})/E \). Therefore \( H^2_{\text{ét}}(o'_L, Z_p(i)) = 0 \) when \( L/\mathbb{Q} \) is unramified outside \( T = S_p \cup \{\ell\} \) with \( \ell \equiv 1 \pmod{p} \) and \( p \not\in (\mathbb{Z}/\ell)^{\ast_p} \).

If \( i \neq 1 \pmod{p-1} \), the hypothesis \( H^2_{\text{ét}}(\mathbb{Z}[1/p], Z_p(i)) = 0 \) shows that \( c\ell_E/p(i-1)^\Delta \) is trivial where \( c\ell_E \) is the class group of \( E \). Hence

\[
H^1_{\text{ét}}(\mathbb{Z}[1/p], Z_p(i))/p \cong U'_E/p(i-1)^\Delta = U_E/p(i-1)^\Delta \cong C_E/p(i-1)^\Delta,
\]

where \( C_E \) is the group of cyclotomic units (for the last isomorphism see e.g. [Wa97, chapter 15, Theorem 15.7]). Let \( \omega \) be the Teichmüller character. Then, for the even integer \( j := 1-i \), the class of the cyclotomic element

\[
\xi_j := \prod_{\delta \in \Delta} (\zeta^j - 1)^{\omega^{-j}(\delta)}
\]

generates \( D^{(i)}_Q/E^{\ast_p} \). Hence \( D^{(i)}_Q \)-primitive sets for \( (\mathbb{Q}, p) \) are of the form \( S_p \) or \( T = S_p \cup \{\ell\} \) with \( \ell \) inert in \( E(\sqrt[p]{\ell})/E \). Therefore \( H^2_{\text{ét}}(o'_L, Z_p(i)) = 0 \) when \( L/\mathbb{Q} \) is unramified outside \( T = S_p \cup \{\ell\} \) with \( \ell \equiv 1 \pmod{p} \) and \( \xi_j \not\in Q^\ast_p \).

6.2. Assume now \( p = 2 \) and consider a 2-extension \( L \) of \( \mathbb{Q} \). Let, as before, \( S \) be the set consisting of infinite primes as well as those which ramify in \( L \). The triviality of the class group of \( \mathbb{Q} \) together with Kummer theory show that

\[
H^1_{\text{ét}}(\mathbb{Z}[1/2], \mathbb{Z}/2(i)) \cong U'_Q/U^2_Q = \langle -T, T \rangle
\]

where, as before, \( U'_Q \) denotes the group of units in \( \mathbb{Z}[1/2] \).

(i) If \( i \) is even, Proposition 2.1 shows that \( H^2_{\text{ét}}(o'_L, Z_2(i)) = 0 \) precisely when \( L \) is totally imaginary and \( \beta_S \) is surjective. By Lemma 4.1 \( \beta_S \) is surjective when \( S \cup S_2 \) is \( D^{(i)}_Q \)-primitive for \( (\mathbb{Q}, 2) \). Recall that \( S_2 = \{2, \infty\} \). In this case the signature map is trivial (see [Ko03, Lemma 2.5] or Proposition 4.1 above) so that \( D^{(i)}_Q/Q^\ast_2 \) is generated by the
class of 2. It follows that $D_{Q}^{(i)}$-primitive sets for $(Q, 2)$ are precisely those contained in
\{2, \infty, \ell \} with \ell inert in $Q(\sqrt{2})/Q$, or equivalently $\ell \equiv \pm 3 \pmod{8}$. Therefore

**Proposition 6.1.** Let $L$ be a finite Galois 2-extension of $Q$ and $i \geq 2$ even. Then
the étale cohomology $H^{2}_{\text{et}}(\mathcal{O}_{L}[1/2], Z_{2}(i))$ vanishes precisely when $L$ is totally imaginary
and there exists an odd prime $\ell \equiv \pm 3 \pmod{8}$ such that $L/Q$ is unramified outside
\{2, \infty, \ell \} with $\ell \equiv \pm 3 \pmod{8}$. □

(ii) If $i$ is odd, we discuss according to whether the number field $L$ is complex or real.
(a) If $L$ is totally complex, the vanishing of $H^{2}_{\text{et}}(\mathcal{O}_{L}, Z_{2}(i))$ is independent of the parity
of $i$ ([Ko03 Lemma 2.2]). Hence by the above case (i), just studied, we have

**Proposition 6.2.** Let $L$ be a totally imaginary 2-extension of $Q$ and $i \geq 2$. Then
the étale cohomology $H^{2}_{\text{et}}(\mathcal{O}_{L}[1/2], Z_{2}(i))$ vanishes precisely when $L/Q$ is unramified outside
a set of primes \{2, \infty, \ell \} with $\ell \equiv \pm 3 \pmod{8}$. □

(b) If $L$ is totally real, we assume that $L/Q$ is a cyclic 2-extension. Since $H^{2}_{\text{et}}(\mathcal{O}_{L}[1/2], Z_{2}(i))$
is trivial

\[ H^{1}_{\text{et}}(Q, Z_{2}(i))/2 \cong H^{1}_{\text{et}}(\mathcal{O}_{L}[1/2], Z_{2}(i))/2. \]

Thus

\[ D_{Q}^{(i)}/Q^{\ast^{2}} = \langle -T, \mathfrak{p} \rangle \]

by \([99]\). Hence $S \cup S_{2}$ is $D_{Q}^{(i)}$-primitive for $(Q, 2)$ precisely when $S$ is contained in
\{2, \infty, \ell_{1}, \ell_{2} \} with the Frobenius automorphisms $\sigma_{\ell_{1}}(Q(\zeta_{8})/Q)$ and $\sigma_{\ell_{2}}(Q(\zeta_{8})/Q)$ generating
$\text{Gal}(Q(\zeta_{8})/Q)$, or equivalently $\ell_{1} \not\equiv 1 \pmod{8}$, $\ell_{2} \not\equiv 1 \pmod{8}$ and $\ell_{1} \not= \ell_{2}$
(mod 8). Therefore, Theorem [472] leads to the following

**Proposition 6.3.** Let $L$ be a totally real cyclic 2-extension of $Q$ and $i \geq 2$ odd. Then
the étale cohomology $H^{2}_{\text{et}}(\mathcal{O}_{L}[1/2], Z_{2}(i))$ vanishes exactly when $L/Q$ is unramified outside
a set of primes \{2, \ell_{1}, \ell_{2} \} with $\ell_{1} \not\equiv 1 \pmod{8}$, $\ell_{2} \not\equiv 1 \pmod{8}$ and $\ell_{1} \not= \ell_{2}$
(mod 8). □

Under hypothesis ($H_{L}$), the above proposition holds for general real Galois 2-extensions
$L$ of $Q$.

6.3. In this subsection, we will apply our results to find Galois 2-extensions $L$ of $Q$
with minimal 2-parts $K_{2i-2}\mathcal{O}_{L} \otimes Z_{2}$.

(i) $2i - 2 \equiv 2 \pmod{8}$. By Theorem [3.1] and the exact sequence \([18]\) we have a
surjective map $K_{2i-2}\mathcal{O}_{L} \otimes Z_{2} \to (Z/2)^{r_{i}(L)}$ which becomes an isomorphism precisely
when $H^{2}_{+}(\mathcal{O}_{L}[1/2], Z_{2}(i)) = 0$.

(ii) $2i - 2 \equiv 4 \pmod{8}$. Theorem [3.1] and the exact sequence \([13]\) show that the two
groups $K_{2i-2}\mathcal{O}_{L} \otimes Z_{2}$ and $H^{2}_{+}(\mathcal{O}_{L}[1/2], Z_{2}(i))$ are simultaneously trivial or non trivial.

(iii) $2i - 2 \equiv 6 \pmod{8}$. By Theorem [3.1] and the exact sequence \([18]\) we have an
isomorphism $K_{2i-2}\mathcal{O}_{L} \otimes Z_{2} \cong H^{2}_{+}(\mathcal{O}_{L}[1/2], Z_{2}(i))$. 

In the above three cases we are led to study the vanishing of the positive cohomology groups $H^2_+(\mathcal{O}_L[1/2], \mathbb{Z}_2(i))$. Since $H^2_+(\mathbb{Z}[1/2], \mathbb{Z}_2) = 0$ for all integer $i \geq 2$ (cf e.g. [Ko03, Proposition 2.6]), Corollary 4.9 leads immediately to the following

**Proposition 6.4.** Let $L$ be a finite Galois 2-extension of $\mathbb{Q}$ and $i \geq 2$. Then the positive cohomology group $H^2_+(\mathcal{O}_L[1/2], \mathbb{Z}_2(i))$ vanishes exactly when $L/\mathbb{Q}$ is unramified outside a set of primes $\{2, \infty, \ell\}$ with $\ell \equiv \pm 3 \pmod{8}$. □

**Corollary 6.5.** Let $L$ be a totally real finite Galois 2-extension of $\mathbb{Q}$ and $2i - 2 \equiv 0 \pmod{8}$. Then
(i) either $2i - 2 \equiv 2 \pmod{8}$ and the surjective map $K_{2i-2}\mathcal{O}_L \otimes \mathbb{Z}_2 \to (\mathbb{Z}/2)^{r_i(L)}$ is an isomorphism;
(ii) or $2i - 2 \equiv 4 \pmod{8}$ and $K_{2i-2}\mathcal{O}_L \otimes \mathbb{Z}_2 = 0$;
(iii) or $2i - 2 \equiv 6 \pmod{8}$ and $K_{2i-2}\mathcal{O}_L \otimes \mathbb{Z}_2 = 0$;
precisely when $L$ is unramified outside a set of primes $\{2, \infty, \ell\}$ with $\ell \equiv \pm 3 \pmod{8}$. □

Finally, if $2i - 2 \equiv 0 \pmod{8}$, we need to discuss according to whether the Galois 2-extension $L$ is real or imaginary. In this case, the Chern characters

$$ch^M_{i,k} : K_{2i-k}(\mathcal{O}_F) \to H^k_M(\mathcal{O}_F, \mathbb{Z}(i))$$

are isomorphisms (Theorem 3.1).

When $L$ is complex, the vanishing of $K_{2i-2}\mathcal{O}_L \otimes \mathbb{Z}_2$ turns out to be independent of $i \geq 2$:

**Corollary 6.6.** Let $L$ be a totally imaginary finite Galois 2-extension of $\mathbb{Q}$. Then, for $i \geq 2$, the 2-primary part of $K_{2i-2}\mathcal{O}_L$ vanishes precisely when $L$ is unramified outside a set of primes $\{2, \infty, \ell\}$ with $\ell \equiv \pm 3 \pmod{8}$. □

When $L$ is real, we use Proposition 6.3 for the vanishing of $K_{2i-2}\mathcal{O}_L \otimes \mathbb{Z}_2$.

**Proposition 6.7.** Let $L$ be a totally real finite cyclic 2-extension of $\mathbb{Q}$. Assume that $2i - 2 \equiv 0 \pmod{8}$. Then the 2-primary part of $K_{2i-2}\mathcal{O}_L$ vanishes precisely when $L$ is unramified outside a set of primes $\{2, \ell_1, \ell_2\}$ with $\ell_1 \not\equiv 1 \pmod{8}$, $\ell_2 \not\equiv 1 \pmod{8}$ and $\ell_1 \not\equiv \ell_2 \pmod{8}$. □

Note again that under hypothesis $(\mathcal{H}_i)$, the above proposition holds for general real Galois 2-extensions $L$ of $\mathbb{Q}$.

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