SUPERABUNDANT CURVES AND THE ARTIN FAN

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ABSTRACT. We prove that every balanced 1-dimensional polyhedral complex arises as the tropicalization of a smooth curve over a non-Archimedean field mapping to a toric Artin fan, namely the quotient of a toric variety by its dense torus.

1. INTRODUCTION

Let $K$ be a non-Archimedean field and $T$ an algebraic torus with character lattice $M$. The Berkovich analytic space $T^{\text{an}}$ is a space of real valuations on $K[M]$. There is a natural continuous map $\text{trop}: T^{\text{an}} \to \text{Hom}(M, \mathbb{R})$, obtained by restricting valuations to the character lattice. When $C$ is a curve in $T$, the image of $C^{\text{an}}$ under $\text{trop}$ is a metric graph, embedded in $\text{Hom}(M, \mathbb{R})$ as a rational polyhedral complex $\mathcal{P}$ of dimension 1. The edges satisfy a so-called balancing condition, see [13, 20]. Define an embedded tropical curve to be a balanced, connected, rational polyhedral complex of dimension 1 in a vector space. It is natural to consider the following inverse problem:

**Question.** Given an embedded tropical curve $\mathcal{P}$ in $\text{Hom}(M, \mathbb{R})$, does there exist an algebraic curve $C$ in $T$ whose tropicalization is $\mathcal{P}$?

The answer to this question is positive when $M$ has rank 2, or when the genus of $\mathcal{P}$ is zero, see [21, 23]. However, there exist higher genus tropical curves in $\mathbb{R}^n$ that are not the tropicalizations of algebraic curves in $(K^*)^n$. For instance Speyer [25, 26] constructed an example of a genus 1 tropical curve in $\mathbb{R}^3$ that does not arise as a tropicalization, see Figure 1. This is due to the phenomenon of superabundance. That is, of tropical curves in $\mathbb{R}^n$ having spaces of deformations that are strictly larger than the expected dimension. See [21, Definition 2.2] and the discussion in [19, Section 1]. A number of authors have studied this lifting problem in a variety of contexts, see [10, 12, 19, 22, 23] and references therein.

In this paper, we consider the question of whether these superabundant tropical curves are tropicalizations in some more general sense. Recently, Ulirsch [29] has shown that the tropicalization map for a proper toric variety $X(\Delta)$ coincides with the map from $X(\Delta)^{\text{an}}$ to the analytic stack quotient $[X(\Delta)^{\text{an}}/T_0^{\text{an}}]$, where, $T_0^{\text{an}}$ is the non-Archimedean analytic compact torus, consisting of valuations that are identically 0 on the character lattice of $T$. In particular, the topological space underlying the stack $[X(\Delta)^{\text{an}}/T_0^{\text{an}}]$ is canonically identified with the Kajiwara-Payne extended tropicalization of $X(\Delta)$.

Let $\mathcal{P}$ be an embedded tropical curve. Denote by $\overline{\mathcal{P}}$ be the compactification of $\mathcal{P}$ obtained by adding a point at infinity to compactify each unbounded edge.

**Main Theorem.** There exists a smooth curve over a non-Archimedean field $K$ and a toric variety $X(\Delta)$, such that the extended polyhedral complex $\overline{\mathcal{P}}$ coincides with the topological image of a map of analytic stacks

$$C^{\text{an}} \to [X(\Delta)^{\text{an}}/T_0^{\text{an}}].$$

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Figure 1. Speyer’s example of a genus 1 tropical curve that cannot be lifted to an algebraic curve [25, Figure 5.1]. This graph fails to fulfill Speyer’s well-spacedness condition [26, Theorem 3.3].

In other words, the realizability problem for tropical curves amounts precisely to lifting this map $\text{Can} \rightarrow [X(\Delta)^{\text{an}}/T^{\text{an}}]$ to a map $\text{Can} \rightarrow X(\Delta)^{\text{an}}$.

The approach to the proof of the main theorem in similar in spirit to the techniques developed by Nishinou–Siebert [23], and extended by Cheung–Fantini–Park–Ulirsch [12] and Nishinou [22]. However, while toric degenerations of toric varieties play a central role in the cited works, we work entirely within the framework of logarithmic stable maps, as developed in the series of recent papers [2, 11, 14]. As a result, we avoid the “expansion of target” via toric degenerations, and instead rely heavily on the log structure of the stack $[X(\Delta)/T]$. The existence of algebraic moduli spaces for logarithmic stable maps also allows us to handle tropical curves with edge lengths that are not necessarily rational. We offer a simplified proof in the case where the edge lengths are rational, which requires fewer technical details from the theory of logarithmic stable maps.

We freely use the Fontaine–Illusie–Kato theory logarithmic geometry in this note, and refer the reader to the surveys [3, 4] and K. Kato’s article [18] for an introduction to the subject.

2. Analytification, Tropicalization, and Artin Fans

2.1. Analytification. Let $K$ be a field complete with respect to a rank-1 valuation $v$. Throughout we will assume that $K$ is an extension of $\mathbb{C}$ and that $v$ induces the trivial valuation on $\mathbb{C}$. The valuation ring and maximal ideal will be denoted $R$ and $m$ respectively.

Let $X$ be a finite type $K$-scheme. The Berkovich analytification $X^{\text{an}}$ is constructed as a locally ringed space in [6]. For affine $X = \text{Spec}(A)$, $X^{\text{an}}$ can be defined as a set of ring valuations on $A$,

$$X^{\text{an}} = \{ \text{val} : A \to \mathbb{R} \cup \{ \infty \} : \text{val}|_K = v \}.$$

The set $X^{\text{an}}$ is given the weak topology for the evaluation functions

$$\text{ev}_f : X^{\text{an}} \to \mathbb{R} \cup \{ \infty \},$$
$$\text{val}_x \mapsto \text{val}_x(f).$$

For arbitrary $X$, one uses the above construction on affines and glues the resulting pieces, giving rise to a functor

$$( - )^{\text{an}} : \text{Schemes}/K \to \text{An.Spaces}/K.$$

In particular, every point of $X^{\text{an}}$ can be represented by a map $\text{Spec}(L) \to X$, where $L$ is a valued field extending $K$, compatible with the valuation on $K$. 

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2.2. Generic fiber. Assume the valuation on $K$ is nontrivial. The analytic space $X^a$ is Hausdorff (resp. compact) if and only if the scheme $X$ is separated (resp. proper) \cite[Theorem 3.4.8]{6}. Since the algebraic stacks $[X(\Lambda)/T]$ are never separated, to relate them to tropicalizations we need a variation of this analytification, which is known as the Raynaud generic fiber \cite[Section 7.4]{9}.

Given a flat $R$-scheme $\mathcal{Z}$, the Raynaud generic fiber, usually denoted $\mathcal{Z}_0^\Delta$, is a Berkovich analytic space over $K$, associated to the formal completion of $\mathcal{Z}$ along the maximal ideal of $R$. We work exclusively with integral $R$-schemes, as opposed to formal schemes. In order to avoid confusion we define $\mathcal{Z}_0^\Delta := \mathcal{Z}_0$. The Raynaud generic fiber can be described as follows. Let $\mathcal{Z}$ be an affine finite-type scheme over $\text{Spec}(R)$ with generic fiber $Z$. In this case $\mathcal{Z}_0^\Delta$ is the compact analytic domain in $Z^a$ consisting of points represented by maps $\text{Spec}(L) \rightarrow Z$, that extend to maps $\text{Spec}(R_L) \rightarrow \mathcal{Z}$ from the valuation ring of $L$. This construction can be extended to all $R$-schemes by a gluing process, yielding a functor \cite[Section 0.3.3]{8}

$$(-)_0^\Delta : \text{Schemes}/R \rightarrow \text{An.Spaces}/K.$$ If $\mathcal{Z}$ is proper, the valuative criterion of properness ensures that $\mathcal{Z}_0^\Delta$ coincides with $Z^a$. If $\mathcal{Z}$ is separated, $\mathcal{Z}_0^\Delta$ will be a compact analytic domain in $Z^a$. In general this is not the case. In particular, for non-separated $\mathcal{Z}$, the analytic space $\mathcal{Z}_0^\Delta$ can be compact Hausdorff, and does not include into $Z^a$. In \cite[Section 6]{31} Yu extends the Raynaud generic fiber to a functor

$$(-)_0^\Delta : \text{Alg.Stacks}/R \rightarrow \text{An.Stacks}/K.$$ See \cite{29, 31} for details on non-Archimedean analytic stacks. Note that the two notions of analytic stack developed in the cited works coincide for the situations we consider here.

**Example 1.** Suppose that $M$ is a lattice and $\mathcal{Z} = \text{Spec}(R[M])$ is a split torus over $R$, with generic fiber $T$. The space $\mathcal{Z}_0^\Delta$ consists of those valuations $\text{val}_l \in T^a$ such that $\text{val}_l(x^u) = 0$ for all $u \in M$. That is, $\mathcal{Z}_0^\Delta$ is the non-Archimedean analytic analogue of the real torus.

As the torus $T$ over $K$ has a canonical model over $R$, we denote its Raynaud fiber by $T_0^\Delta$.

2.3. $\mathcal{Z}$–space. When $K$ carries the trivial valuation, there is an analogue of the Raynaud generic fiber, defined by Thuillier \cite{27}:

$$(-)^\Delta : \text{Schemes}/K \rightarrow \text{An.Spaces}/K.$$ Intuitively, this may be thought of as a Raynaud generic fiber in the case where the field $K$ and its valuation ring $R$ coincide. If $X = \text{Spec}(A)$ is an affine $K$-scheme, the space $X^\Delta$ is a compact analytic domain in $X^a$ consisting of points $\text{Spec}(R_L) \rightarrow X$. As before, one may glue the $\Delta$-spaces of affine patches to obtain a $K$-analytic space. If $X$ is separated, $X^\Delta$ is a subspace of $X^a$, and if $X$ is proper, $X^a = X^\Delta$. We refer the reader to \cite{27}, for details. In \cite{30}, the $\mathcal{Z}$-space construction is extended to a functor from algebraic stacks over $K$ to analytic stacks over $K$, in analogous fashion to the Raynaud generic fiber.

2.4. Tropicalization and the Artin fan. Let $T = \text{Spec}(K[M])$ be a torus with character lattice $M$ and dual lattice $N$. Given a point $\text{val}_l \in T^a$, one may restrict the valuation to the character lattice $M$ of $T$ to obtain a point $\text{trop}(t)$ of $\text{Hom}(M, \mathbb{R})$. This yields a continuous tropicalization map

$$\text{trop} : T^a \rightarrow N_{\mathbb{R}}.$$
In [24], this construction is extended, replacing the torus $T$ by an arbitrary toric variety $X(\Delta)$. This yields a continuous map

$$X(\Delta)^{an} \to N(\Delta),$$

where $N(\Delta)$ is a partial compactification of the vector space $N_{\mathbb{R}}$. The tropicalization of a subvariety $Y$ of $X(\Delta)$ is defined to be the image of $Y^{an} \to X(\Delta)^{an}$ under this map.

To simplify the discussion, we henceforth assume that $\Delta$ is a complete fan. Given a toric variety $X(\Delta)$ with dense torus $T$, the quotient stack $\mathcal{A}(\Delta) := [X(\Delta)/T]$ is referred to as the Artin fan of $X(\Delta)$. See [5, 30] for a more complete treatment of Artin fans.

**Theorem 2** ([29, Theorem 1.4]). There is a natural isomorphism of extended cone complexes $\mu_{\Delta} : [\mathcal{A}(\Delta)^{an}] \to N(\Delta)$, making the diagram

$$\begin{array}{ccc}
\mathcal{A}(\Delta)^{an} & \xrightarrow{\mu_{\Delta}} & N(\Delta) \\
\text{Stack Quotient} \downarrow & & \downarrow \text{trop} \\
X(\Delta)^{an} & & N(\Delta)
\end{array}$$

commute.

Here, $|-|$ is the functor associating to an analytic stack $Y$, its underlying topological space $|Y|$, as defined in [29, Section 5].

**Remark 3.** The fact that the map from $X(\Delta)^{an}$ to its skeleton is the quotient by the analytic group $T^0_{\mathbb{C}}$ is implicit in Berkovich’s work on local contractibility [7], and in Thuillier’s work in the trivially valued setting [27]. Ulirsch’s results allows one to enhance the topological retraction maps to analytic maps, by providing the skeleton with the structure of an analytic stack.

**Remark 4.** We bring to the reader’s attention an instructive analogy. If $P$ is a simple lattice polytope in $M$, the polarized complex toric variety $X(P)$ has the structure of a smooth symplectic manifold with an action of the compact torus $T_\mathbb{C} = \text{Hom}(M,S^1)$. The quotient of the symplectic manifold $X(P)$ by $T_\mathbb{C}$ coincides with the moment polytope $P$ of $X(P)$. In fact, the moment polytope $P$ and the Kajiwara–Payne extended tropicalization of $X(P)$, giving $\mathbb{C}$ the trivial valuation, are isomorphic as abstract polytopes. See [24, Remark 3.3].

### 3. Proof of the Main Theorem

The general approach to the proof of the main theorem is to first construct a logarithmic map from the “would-be” special fiber of a degenerating curve to $\mathcal{A}(\Delta)$, based on the expected tropicalization. We then use log smooth deformation theory to explain why this map can always be smoothed. The logarithmic tangent bundle of a toric variety is $\mathcal{O}^{\dim X}_{X(\Delta)^{an}}$. Thus, deformations of maps from rational curves are unobstructed, but maps from higher genus curves can be obstructed. On the other hand, the Artin stack $[X(\Delta)/T]$ is logarithmically étale over $K$, so logarithmic maps to it are unobstructed.
3.1. **Proof: Rational edge lengths.**

**Step I: Building the log curve.** Fix the polyhedral complex $\mathcal{P}$ as before, and let $\Delta$ be a complete fan, containing the recession fan $\text{rec}(\mathcal{P})$ as a subfan. We assume that all edge lengths of $\mathcal{P}$ are rational, and all vertices of $\mathcal{P}$ have rational coordinates. By rescaling the valuation, we may and do assume that all vertices have integer coordinates.

Suppose $v$ is a vertex of $\mathcal{P}$, incident to edges $e_1^v, \ldots, e_r^v$. Associate to $v$ a marked rational curve $C_v \cong \mathbb{P}^1$, marked at distinct points $p_1^v, \ldots, p_r^v$, in bijection with the edges emanating from $v$. Identify marked points $p_1^v$ and $p_j^v$ when the edges $e_1^v$ and $e_j^v$ coincide in $\mathcal{P}$. Denote the resulting nodal curve over $\mathbb{C}$ by $C_v'$. Suppose $q$ is a node of $C_v'$ corresponding to an edge of length $\ell$. Consider the model $\text{Spec}(\mathbb{R}[x, y]/(fg - \tau))$ such that $\nu(\tau) = \ell$. Since the special fiber has simple normal crossings, the total family carries the structure of a logarithmically smooth scheme over $\text{Spec}(\mathbb{R})$ with its divisorial structure. We give the curve $C_v'$ the logarithmic structure associated to the special fiber of this family. Each marked point inherits a canonical log structure with characteristic $\mathbb{N}$. By [16, Proposition 1.1], this yields a log smooth curve $C_0'$ over $\text{Spec}(\mathbb{P} \to \mathbb{C})$, where $\mathbb{P} \cong \mathbb{N}^k$ and $k$ is the number of nodes of $C_0'$. Choosing a uniformizer $\pi$ for $\mathbb{R}$, the factor of $\mathbb{P}$ corresponding to a node of length $\ell$ can be canonically identified with the monoid $(\pi^{\ell/\nu(\pi)})$. After identifying the monoid $\mathbb{N}$ with $(\pi)$, there is a natural map $\mathbb{P} \to \mathbb{N}$. Consequently we obtain a map

$$\text{Spec}(\mathbb{N} \to \mathbb{C}) \to \text{Spec}(\mathbb{P} \to \mathbb{C}),$$

and we may pull back $C_0'$ along this map to obtain a curve $C_0 \to \text{Spec}(\mathbb{N} \to \mathbb{C})$.

**Step II: Constructing the logarithmic map.** We now define a logarithmic map

$$\begin{array}{ccc}
C_0 & \longrightarrow & \mathbb{A}(\Delta) \\
\downarrow & & \\
\text{Spec}(\mathbb{N} \to \mathbb{C}).
\end{array}$$

If $v \in \mathcal{P}$ lies in the interior of a cone $\sigma$, map the component $C_v$ of $C_0$ to the image of the torus orbit associated to $\sigma$ in the quotient $[X(\Delta)/T]$. Let $\eta$ be a generic point of a component of $C_0$ corresponding to a vertex $v$. The log structure at $\eta$ is given by the pullback of the structure from the base, and thus to give a log map at $\eta$, it suffices to give a map of monoids $f_n^\eta : M_{\sigma} \to \mathbb{N}$, where $M_{\sigma}$ is the monoid of positive characters on the orbit of $X(\Delta)$ associated to $\sigma$. That is, we need to specify an element $\varphi \in \text{Hom}(M_{\sigma}, \mathbb{N})$. The vertex $v \in \mathbb{N}$ itself gives rise to such a homomorphism, and since $v \in \sigma$, we take $f_n^\eta = v$.

Let $q$ be a node of $C_0$. Following F. Kato’s description of log smooth curves [17, p. 222], the characteristic of $C_0$ at $q$ is given by the monoid push-out $\mathbb{N} \oplus^H \mathbb{N}^2$, where $\mathbb{N} \to \mathbb{N}^2$ is the diagonal embedding, and $\mathbb{N} \to \mathbb{N}$ is the homothety $1 \mapsto \rho_q$. Here, $\rho_q$ is the quantity $\ell_q/\nu(\pi)$, where $\ell_q$ is the length of the edge corresponding to $q$. In other words, $\rho_q$ is a logarithmic smoothing parameter for $q$.

By [14, Remark 1.2], the push-out above can be alternatively described by

$$\mathbb{N} \oplus^H \mathbb{N}^2 = \{(n_1, n_2) \in \mathbb{N}^2 : n_2 - n_1 \in \rho_q \mathbb{Z}\}.$$

Let $M_q$ be the dual lattice of the stratum to which $q$ maps. Following [14, Remark 1.2], to give a map $f_q^\eta : M_q \to \mathbb{N} \oplus^H \mathbb{N}^2$, it is sufficient to give a homomorphism $u_q : M_q \to \mathbb{Z}$.

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1Note that this is the step at which we are crucially using the fact that edge lengths of $\mathcal{P}$ are rational.
such that

\[(n_2 - n_1) \circ f^v_\Delta (m) = u_q (m) \cdot \rho_q.\]

Equivalently, we may dualize, and set \(u_q\) equal to the quantity \((v^q_1 - v^q_2) / \rho_q\), where \(v^q_1\) and \(v^q_2\) are the vertices corresponding to the components meeting at \(q\). Set \(\rho_q\) equal to the weight on the edge corresponding to an edge \(q\). Similarly, for a marked point \(p \in C_0\), one obtains an element of the dual lattice \(N\) in the direction of the associated unbounded edge in \(\mathcal{P}\). This yields a logarithmic map \(f_0 : C_0 \to \mathcal{A}(\Delta)\).

**Step III: Smoothing the map.** We now explain how to smooth this map. Let \(O\) denote the log scheme \(\text{Spec}(\mathbb{R})\) with its divisorial structure. For each \(k\), let \(O_k\) denote the quotient \(\text{Spec}(\mathbb{R}/\pi^k)\) equipped with the pullback structure for its inclusion into \(O\). We wish to lift the map \(f_0\) to a diagram

\[
\begin{array}{ccc}
C_0 & \xrightarrow{f_0} & \mathcal{A}(\Delta) \times \text{Spec}(\mathbb{R}) \\
\downarrow & & \downarrow \\
O_0 & \to & O_k \\
\end{array}
\]

As explained in [16, Proposition 8.6], the obstructions to the existence of a \(k\)th order lifting lie in \(H^1(C_0, f_0^* T^\log_{\mathcal{A}(\Delta)})\), where \(T^\log_{\mathcal{A}(\Delta)}\) is the logarithmic tangent bundle. Since the stack \(\mathcal{A}(\Delta)\) is logarithmically étale and \(T^\log_{\mathcal{A}(\Delta)} = 0\), this obstruction vanishes. Taking the direct limit over \(k\) and algebraizing [15], we obtain a family of curves \(C \to \text{Spec}(\mathbb{R})\), together with a map \(f : C \to \mathcal{A}(\Delta)\). Taking the product of the map \(f\) and the structure map \(C \to \text{Spec}(\mathbb{R})\), we have \(C \to \mathcal{A}(\Delta) \times C \text{Spec}(\mathbb{R})\). Let \(C\) denote the generic fiber of \(C\). Applying \((-)^\text{an}\) yields the proposed analytic map \(\varphi : C^\text{an} \to \mathcal{A}(\Delta)^\text{an}\).

**Step IV: Computing the tropicalization.** Since \(\mathbb{R}\) is a discrete valuation ring containing \(\mathbb{C}\), we may view \(C\) as a scheme over \(\mathbb{C}\), with its divisorial logarithmic structure coming from the special fiber. Similarly, we may view \(\mathcal{A}(\Delta) \times \text{Spec}(\mathbb{R})\) as a logarithmic stack over \(\mathbb{C}\). The formation of skeletons for \(\Sigma\)-spaces is functorial [28, Theorem 1.1], so we obtain a diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f^b} & (\mathcal{A}(\Delta) \times_C \text{Spec}(\mathbb{R}))^\Sigma \\
\downarrow & & \downarrow \\
\Sigma(\mathcal{C}) & \to & \Sigma(\mathcal{A}(\Delta) \times_C \text{Spec}(\mathbb{R})).
\end{array}
\]

The arrow on the right is a canonical isomorphism [29, Theorem 1.4], and moreover, we have \(\Sigma(\mathcal{A}(\Delta) \times_C \text{Spec}(\mathbb{R})) \cong N(\Delta) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})\). The left vertical arrow is a surjection, so to compute the image of \(\varphi\), we may compute the image \(\Sigma(\mathcal{C})\) in \(\Sigma(\mathcal{A}(\Delta) \times_C \text{Spec}(\mathbb{R}))\), and slice with the plane at height \(1\). The complex \(\Sigma(\mathcal{C})\) is canonically identified with the cone over the abstract metric graph underlying \(\mathcal{C}\). It is straightforward to check from the construction of \(f\) that the image of \(\varphi\) is precisely \(\mathcal{P}\). The result follows. □

The proof in the general case necessitates the use of the stack of pre-stable minimal logarithmic maps to a logarithmic scheme or stack.
3.2. The stack of minimal logarithmic maps to the Artin fan. Logarithmic (pre)-stable maps were introduced in the papers \cite{2, 11, 14}. In \cite{5}, an algebraic stack of minimal logarithmic pre-stable maps to $\mathcal{A}(\Delta)$ is constructed, and denoted $\mathcal{M}(\mathcal{A}(\Delta))$. The notion of minimality can be understood as follows. One wishes to work with the object $\mathcal{M}(\mathcal{A}(\Delta))$ as a moduli stack over the category of schemes, rather than over logarithmic schemes. Thus, a map from a test scheme $\mathcal{S} \to \mathcal{M}(\mathcal{A}(\Delta))$ should parametrize families logarithmic pre-stable maps over $\mathcal{S}$. However, in order to build such a family, it is necessary to give the base scheme $\mathcal{S}$ a logarithmic structure. A priori there are numerous logarithmic structures that one may place on $\mathcal{S}$. A major insight in \cite{11, 14} is that there are distinguished minimal logarithmic structures, which can be understood as the minimal requirements that a logarithmic (pre)-stable map needs to satisfy. Giving $\mathcal{S}$ this minimal logarithmic structure, $\mathcal{M}(\mathcal{A}(\Delta))$ can be understood as a moduli stack parametrizing minimal logarithmic maps. We refer the reader to loc. cit. for further details.

We now explain how to extend the proof of the main theorem to arbitrary edge lengths, by using the existence of the algebraic moduli space of pre-stable minimal logarithmic maps to $\mathcal{A}(\Delta)$.

3.3. Proof: Arbitrary edge lengths. Let $\mathcal{P}$ be a tropical curve, with no restriction on the vertices or lengths of edges. Construct the map $f_0 : \mathcal{L}_0 \to \mathcal{A}(\Delta)$, as in the previous proof.

Let $\sigma(\mathcal{P})$ be the cone of all tropical curves having the same combinatorial type as $\mathcal{P}$. That is, tropical curves whose underlying combinatorial graph is the same as $\mathcal{P}$, such that the vertex $v_i$ lies in the cone $\sigma_i$, and the edges $e_q$ are proportional to $u_q \in \mathbb{N}$. Moreover, if $v_1^q$ and $v_2^q$ are the vertices adjacent to $e_q$, we have $v_1^q - v_2^q = \ell_q u_q$ for some $\ell_q \in \mathbb{R}_{\geq 0}$. Denote by $Q^\vee$ the monoid of integral points of in $\sigma(\mathcal{P})$. Let $Q$ be the dual monoid of $Q^\vee$. By \cite[Remark 1.21]{14}, there exists a map $\text{Spec}(Q \to \mathbb{C}) \to \mathcal{M}(\mathcal{A}(\Delta))$. Pulling back the universal family, we obtain a logarithmic curve $\mathcal{C}_0$ over $\text{Spec}(Q \to \mathbb{C})$ together with a logarithmic map to $\mathcal{A}(\Delta)$. Since the logarithmic tangent bundle of $\mathcal{A}(\Delta)$ is 0, deformations of this map are unobstructed, and by \cite[Corollary 3.1.3]{5}, we obtain a family

$$
\begin{array}{ccc}
\mathcal{C}' & \longrightarrow & \mathcal{A}(\Delta) \\
\downarrow & & \\
\text{Spec}(Q \to \mathbb{C}[Q]) & & \\
\end{array}
$$

(1)

The curve $\mathcal{P}$ corresponds to a point in the cone $\sigma(\mathcal{P}) = \text{Hom}(Q, \mathbb{R}_{\geq 0})$, thus inducing a map $\text{Spec}(\mathbb{C}[\mathbb{R}_{\geq 0}]) \to \text{Spec}(\mathbb{C}[Q])$. Pulling back $\mathcal{C}'$ by this map, we obtain a family of curves $\mathcal{C} \to \text{Spec}(\mathbb{C}[\mathbb{R}_{\geq 0}])$, together with a map $f : \mathcal{C} \to \mathcal{A}(\Delta)$. Taking $K = \mathbb{C}[\mathbb{R}]$ and proceeding as in the previous proof, taking Raynaud generic fibers yields a map

$$
\varphi : \mathbb{C} \to \mathcal{A}(\Delta)_{an}.
$$

To compute the image of $\varphi$, consider (1) as a diagram over $\text{Spec}(\mathbb{C})$. The formation of skeletons for $\mathfrak{D}$-spaces is functorial \cite[Theorem 1.1]{28} so we obtain

$$
\begin{array}{ccc}
(\mathcal{C}')^\Sigma & \longrightarrow & |\mathcal{A}(\Delta) \times \text{Spec}(\mathbb{C}[Q])| \\
\downarrow & & \\
\Sigma(\mathcal{C}') & \longrightarrow & N(\Delta) \times \text{Hom}(Q, \mathbb{R}_{\geq 0}).
\end{array}
$$

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The tropical curve gives rise to a point $[\mathcal{P}] \in \text{Hom}(Q, \mathbb{R}_{\geq 0})$, and moreover there is a natural map

$$p : \Sigma(\mathcal{C}') \to \text{Hom}(Q, \mathbb{R}_{\geq 0}).$$

The image of $\varphi$ coincides with the image of the slice $p^{-1}([\mathcal{P}]) \subset \Sigma(\mathcal{C}')$ in $N(\Delta) \times \{[\mathcal{P}]\}$. It is straightforward to check that this image is precisely $\mathcal{P}$, and the result follows. $\square$

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