Energy conditions allow eternal inflation

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Received August 24, 2020
Revised January 21, 2021
Accepted January 27, 2021
Published March 29, 2021

Abstract. Eternal inflation requires upward fluctuations of the energy in a Hubble volume, which appear to violate the energy conditions. In particular, a scalar field in an inflating spacetime should obey the averaged null energy condition, which seems to rule out eternal inflation. Here we show how eternal inflation is possible when energy conditions (even the null energy condition) are obeyed. The critical point is that energy conditions restrict the evolution of any single quantum state, while the process of eternal inflation involves repeatedly selecting a subsector of the previous state, so there is no single state where the conditions are violated.

Keywords: inflation, quantum field theory on curved space, gravity, quantum cosmology

ArXiv ePrint: 2008.01878
1 Introduction

1.1 Eternal inflation and the null energy condition

Inflation is a period of quasi-exponential expansion driven by the potential energy of some inflaton scalar field $\phi$ [1]. Classically, the value of $\phi$ rolls slowly down a mildly sloped potential, giving time for many $e$-foldings of expansion of the universe. However, in most scenarios [2], quantum fluctuations sometimes drive $\phi$ up the potential in a particular Hubble volume, causing an increase in the expansion rate there. See figure 1. While such fluctuations are rare, the regions in which they occur then expand faster than others. Each such volume becomes many volumes, each of which may experience another upward fluctuation, causing inflation to persist eternally.

Eternal inflation, however, seems to require violation of energy conditions, which are restrictions on the stress-energy tensor $T_{\mu\nu}$ of quantum fields. (See [3] for a recent review).
Figure 1. An example of a slow-roll inflaton potential. The field classically rolls down to the true vacuum region. However, quantum fluctuations can drive the field up the slope, increasing the rate of expansion.

We will be particularly concerned with the null energy condition (NEC), which requires that

$$ T_{\mu\nu} \ell^\mu \ell^\nu \geq 0 $$

(1.1)

for all null vectors $\ell$. If $T_{\mu\nu}$ has the form of a perfect fluid with energy density $\rho$ and isotropic pressure $P$ in its rest frame, NEC takes the form

$$ \rho + P \geq 0. $$

(1.2)

Borde and Vilenkin [4], Winitzki [5], and Vachaspati [6] have argued that eternal inflation requires violations of the NEC. As we will do below, these works consider only fluctuations on superhorizon scales, so regions of horizon-size or a few horizon sizes are approximately given by the Friedman-Lemaître-Robertson-Walker (FLRW) metric,

$$ ds^2 = dt^2 - a(t)^2 d\vec{x}^2, $$

(1.3)

and the equation governing the Hubble expansion, $H = \dot{a}/a$, is

$$ \dot{H} = -4\pi G(\rho + P) + \frac{k}{a^2}. $$

(1.4)

In an inflating spacetime, we can ignore the curvature term, so set $k = 0$. Eternal inflation requires an increase in the expansion rate, meaning $\dot{H} > 0$, so $\rho + P < 0$, and the NEC is violated.

1.2 The averaged null energy condition

The need for NEC violation is not in itself a big problem, since quantum fields can easily violate NEC or any pointwise energy condition [7]. The problem becomes much worse when we consider the averaged null energy condition (ANEC),

$$ \int d\gamma T_{\mu\nu} \ell^\mu \ell^\nu \geq 0, $$

(1.5)
where the integral is taken over a complete geodesic $\gamma$. ANEC is much harder to violate than NEC. We proved [8] that achronal ANEC is obeyed by a quantum scalar field on a curved background generated by a classical field. Here we can consider the background inflation to be given by a classical inflaton field and fluctuations in that field to be quantum-mechanical. Our proof was for a massless scalar, but in inflation we have a scalar field with a potential. However the potential is nearly flat and level, so the corrections to the previous analysis are small. In fact, in the majority of this paper, as we discuss in section 1.5 below, we will model the inflationary process as taking place in a fixed background de Sitter space. There are some other technical restrictions in ref. [8], but they are all obeyed in the present case.

We should note here that null geodesics are not past-complete in an inflating spacetime [9], so ANEC is not directly applicable. But when we consider a de Sitter background, we can complete our spacetime by adding the other half of de Sitter space beyond the surface $t = -\infty$ (in the FLRW parameterization). We will not allow any dynamics there, but rather turn on the possibility of fluctuations (gradually) somewhere after the $t = -\infty$ surface. Then we will have complete achronal [10] geodesics on which to integrate, but no contribution from the extended part.

As ref. [4] showed, the ANEC integral during eternal inflation can be written

$$\int_\gamma d\lambda T_{\mu\nu}\ell^\mu\ell^\nu = -2 \int dt \frac{\dot{H}}{a}. \quad (1.6)$$

This is an integral over $\dot{H}$ with weight factor $1/a$, which falls exponentially with time. Thus an increase in expansion rate could obey ANEC if it was preceded by a decrease. The decrease could even be much smaller if it took place a while earlier. So there are some specific scenarios which have eternal inflation without ANEC violation.

Nevertheless, this idea disagrees with the usual understanding of eternal inflation, where each Hubble time the inflaton may fluctuate in either direction by an amount of order $H/(2\pi)$ [2]. This comes from the usual model of inflationary fluctuations [11, 12]. So if ANEC applies to eternal inflation in the simple form of eq. (1.6), this understanding of inflationary fluctuations would be wrong. It would have to be replaced by some different model, a dangerous endeavor because the usual model does such a good job of explaining density perturbations [13]. The new model might or might not lead to eternal inflation at all. But, fortunately, no such changes are necessary. We will see below that a careful analysis of quantum states shows that the usual fluctuation model and the eternal inflation it produces are compatible with ANEC.

There is a simple geometrical argument that shows the relationship between ANEC and eternal inflation. ANEC says, essentially, that gravity does not allow defocusing of null geodesics, so there is no “anti-lensing”. Now suppose that $H$ starts out constant but then increases. Imagine a converging spherical shell of incoming geodesics infinitesimally inside the Hubble distance $1/H$. When $H$ increases, the Hubble distance will be smaller, so the sphere is outside the Hubble distance, and the expansion of the universe carries the geodesics away. This means that the geodesics are now diverging and that requires ANEC violation.

1.3 Quantum-mechanical expectation values

One might argue that energy conditions restrict only the quantum mechanical expectation value of $T_{\mu\nu}$ and this is unproblematic. Perhaps $\langle T_{\mu\nu} \rangle$ in inflation indeed obeys NEC and ANEC, and the violations necessary for eternal inflation take place only in some sector of this
state, with other sectors outweighing the problematic sector in $\langle T_{\mu\nu} \rangle$. But this is not correct, because when a field obeys an energy condition, it obeys it in every quantum state. This was shown in [14] where quantum inequalities are reformulated from restrictions on expectation values to statements about the positivity of certain operators. Thus if the inflating state can be written as a sum of sectors, $|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle$, every sector $|\psi_i\rangle$ must obey ANEC. There cannot be any sector where ANEC is violated, allowing $H$ to increase.

1.4 A solution to the apparent paradox: selection of sectors

The right solution to this problem is slightly different. It is not that the eternally inflating state is a substate of the quantum state during inflation. Rather, eternal inflation requires (repeated) selection of a substate that has a faster expansion rate. Thus rather than being a single quantum state to which energy conditions and their consequences would apply, eternal inflation has one state at early times and a different state later. The later state is selected from the much broader earlier state to be one that happens to have faster expansion in a certain volume. Since the faster expansion arose from selection rather than just evolution of the earlier stage, we cannot prove something about the final state using energy conditions.

1.5 A simple model for eternal inflation

In this paper, we will primarily use a slightly simplified model of inflation that nevertheless supports eternality and exhibits the problem that we are trying to solve. Application to realistic scenarios is discussed in section 6. We will consider a massive scalar field $\phi$ evolving as if in a background de Sitter space with (fixed) Hubble constant $H$. We will allow $\phi$ to change the metric, but we will not allow the changed metric to affect the evolution of $\phi$. We thus ignore the rolling of the inflaton that leads to inflation being only quasi-de Sitter, and we also ignore the gravitationally-mediated effect of fluctuations of $\phi$ on its equation of motion.

This model is sufficient to lead to eternal inflation, because a superhorizon fluctuation of $\phi$ leads to a slowly rolling energy density $\rho = (1/2)m^2\phi^2$ that increases the expansion rate. This is possible in our model because we do allow the energy of the changed field to affect the metric, even though we do not allow the resulting metric changes to affect $\phi$.

We can thus take $\phi$ to be given by quantum field theory in a fixed de Sitter background. When we discuss the expansion rate, we will take the shape of spacetime to passively follow the instantaneous value of the field $\phi$. Thus we will not consider propagating modes of gravity. We take the gravitational field to have no quantum mechanics of its own but merely to follow the quantum mechanics of $\phi$. One might call this approach “passive quantum cosmology” in analogy to the passive quantum gravity of Ford and Wu [15].

This approach is very different from semiclassical gravity, where one takes the quantum expectation value $\langle T_{\mu\nu} \rangle$ as the source for gravitation. In semiclassical gravity, it is not possible to consider quantum mechanical alternatives for the shape of spacetime, as we will do here. Rather, the shape of spacetime is uniquely determined by a single quantum mechanical average.

1.6 Plan of paper

In the rest of this paper we will carefully explain the idea above. We start in section 2 with some simple models showing how selection can find values of a quantity that would otherwise be prohibited. In section 3 we derive the classical field equations for our model, and in section 4 we quantize the field assuming that it is in a general Gaussian state and
examine two particular states, the Bunch-Davies vacuum and a late time rapid expansion state which is the one corresponding to the eternally inflating part of spacetime. In section 5 we evolve this (selected) late state backward in time and show that this specific sector obeys NEC and thus ANEC. In fact this rapidly-expanding sector had even more rapid expansion earlier. In section 6 we show how these ideas extend to more usual inflationary models, and we conclude in section 7.

2 Some simple analogies

2.1 Momentum in a linear potential

We begin with a simple analogy with “uphill” fluctuations. Consider a particle in a linear potential $V(x) = -fx$, where $f$ is a constant. Classically, the particle’s momentum increases steadily, $dp/dt = f$. In quantum mechanics, Ehrenfest’s theorem tells us that $d\langle p \rangle/dt = f$.

By analogy with energy conditions, one might call this a “momentum condition” saying that $\langle p \rangle$ increases in every state.

We can write the momentum-space wave function $\psi(p)$. The Schrödinger equation is $\partial \psi/\partial t = ip^2/(2m)\psi - f\partial \psi/\partial p$ and the general solution is $\psi(p; t) = \psi_0(p - ft) \exp((ip^3 - i(p - ft)^3)/(6fm))$, where $\psi_0(p)$ is the wave function at time $t = 0$. Now let $\psi_0$ be a Gaussian centered around 0. Suppose we measure the momentum at some time $t_1$. The most likely result will be $ft_1$, but there is a possibility that we will find some $p_1 < 0$. In such a case the momentum is now less than the (average) momentum at $t = 0$, even though $\langle p \rangle$ increases in every state. This is not an error in measurement, nor is it due to uncertainty about the original state. The original state was pure, though not a state of definite momentum.

We have assumed here an ideal measurement that measures the momentum without the measurement process itself injecting momentum into the system. It is not clear whether such a measurement can really be made. However, if we consider a free particle ($f = 0$), then we can measure its momentum without affecting the system (other than the projection required for measurement), because momentum eigenstates are also eigenstates of the Hamiltonian. In that case $\langle p \rangle$ is constant in every state but can decrease by an ideal measurement with no momentum flowing from the measurement apparatus.

2.2 Scalar field in flat space

Let us now consider a field theory example: a free massless scalar in flat space. We will work in a periodic cubical box of side $L$ and volume $V = L^3$. We decompose the field in Fourier modes:

$$\phi(x, t) = \frac{1}{L^{3/2}} \sum_k e^{ik \cdot x} \phi_k(t).$$

(2.1)

Since $\phi$ is real, we must have

$$\phi_{-k} = \phi_k^*.$$  

(2.2)

We can decompose it,

$$\phi_k = \chi_k + i\xi_k,$$

(2.3)

where $\chi_k$ and $\xi_k$ are real fields. We will concentrate here on the $\chi_k$; the $\xi_k$ are entirely analogous.

Each mode is just a harmonic oscillator with Hamiltonian

$$H_k = \frac{1}{2} \left[ p_k^2 + k^2 \chi_k^2 \right].$$

(2.4)
where the momentum associated with $\chi_k$ is $p_k = \dot{\chi}_k$. The classical equation of motion is

$$\chi_k'' + k^2 \chi_k = 0. \quad (2.5)$$

We quantize the field as usual. The ground state wave function is just a product of wave functions for the individual modes, each of which is a Gaussian

$$\psi(\chi_k) = \left(\frac{k}{\pi}\right)^{1/4} e^{-k\chi_k^2/2}, \quad (2.6)$$

which has ground state energy $k/2$. (Usually we would renormalize by subtracting the ground state energies, but it does not matter here.)

Now consider some specific mode $k$ and suppose we measure $\chi_k$. We might find $\chi_k \gg k^{-1/2}$ and thus energy much larger than $k/2$. So we see that measurement sometimes raises the energy of the system.

However, the system started in the ground state, which is an eigenstate of energy, so the energy cannot fluctuate. The increasing energy in this case must be added by the measurement apparatus. Without such a process, the energy would be fixed. Thus this system is a poor analogy to inflationary cosmology. In that case, decoherence may select out specific sectors of a wave function, but there is no measurement apparatus that could add energy to the system.

2.3 Small changes to $H$ allow large changes to $E$

The reason that, as we will see, decoherence can select modes with different energies, is that the Hamiltonian for each mode changes over time due to the expansion of the universe. We will discuss this in detail below, but first we will give an example showing how a small change to the Hamiltonian can lead to large (but rare) changes to an otherwise conserved energy.

Consider a spin in a magnetic field pointing along the $z$ axis, so that the Hamiltonian is $H_1 = E(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)$. At time $t = 0$, let us rotate the magnetic field by a tiny angle $\epsilon$, so the Hamiltonian is $H_2 = E(|\uparrow'\rangle\langle\uparrow'| - |\downarrow'\rangle\langle\downarrow'|)$. Prepare the initial state $|\downarrow\rangle$, with energy $-E$. At $t = 0$, the state does not change, but we now write it $\cos(\epsilon/2)|\downarrow\rangle + \sin(\epsilon/2)|\uparrow\rangle$. We can now measure the spin along the new magnetic field direction without adding energy to the system. With probability $\sin^2 \epsilon$ we will find energy $E$ even though our system was prepared with energy $-E$.

We would not expect energy conservation when the Hamiltonian changes. But it is worth noting that the change in energy can be large even when the change in the Hamiltonian is tiny. The eigenvalues of $H_2 - H_1$ are $\pm 2E \sin(\epsilon/2) \approx \epsilon E$ if $\epsilon \ll 1$. A tiny change in the Hamiltonian can allow a large but rare change in the energy of the system.

The analogy here is that the upward fluctuations needed for eternal inflation are very rare, but the state when they happen is quite different from the original inflating state. The Hamiltonian for the modes of the inflaton is time-dependent because of the expansion of the universe. For superhorizon modes, the time dependence is rapid compared to the oscillation frequency of the mode. Thus even though each mode starts in its ground state, it is not in the ground state later. Decoherence will then select a state in which each mode has its value distributed in a narrow range. If those ranges happen to be displaced up the potential, the result will be an increase in the expansion rate, even though ANEC is never violated in any quantum state.

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1 It’s important that the change is rapid compared to the timescale set by the eigenvalue difference $2E$. Otherwise the adiabatic theorem would tell us that the system would remain in an eigenstate of energy.
3 Classical field equations

We now return to the simple eternally inflating model of section 1.5 to understand how upward fluctuations can be compatible with energy conditions. Inflaton fluctuations on a classical FLRW background have been discussed by various authors, see [11] for one of the first descriptions of both classical and quantum perturbations and [12] for a pedagogical introduction. We begin by analyzing the classical evolution of the massive scalar field $\phi$ in the de Sitter background.

As in section 2.2, we work in a box of comoving side length $L$ and volume $V = L^3$. The field $\phi(x, t)$ can then be decomposed in Fourier components, as in eqs. (2.1), (2.2). The Lagrangian density for a massive scalar is

$$L = \frac{1}{2} \left[ \dot{\phi}^2 - (\nabla \phi)^2 - m^2 \phi^2 \right], \quad (3.1)$$

and we can write the total Lagrangian $\sum L_k$ where

$$L_k = \frac{a^3}{2} \left[ |\dot{\phi}_k|^2 - \left( k^2 + m^2 a^2 \right) |\phi_k|^2 \right], \quad (3.2)$$

where $a(t)$ is the scale factor. The canonical momentum is therefore $p_k = a^3 \dot{\phi}_k^*$ and the Hamiltonian for the mode is

$$H_k = \frac{1}{2} \left[ \frac{|p_k|^2}{a^3} + a^3 \left( \frac{k^2}{a^2} + m^2 \right) |\phi_k|^2 \right]. \quad (3.3)$$

The equation of motion is

$$\ddot{\phi}_k + 3H \dot{\phi}_k + \left[ \frac{k^2}{a^2} + m^2 \right] \phi_k = 0, \quad (3.4)$$

representing a damped oscillator with varying frequency.

In this simple system where changes to the metric due to $\phi$ are prohibited from affecting the equation of motion, and with a quadratic potential only, the modes evolve independently, which allows us to study evolution of each $k$ in isolation. In a realistic system, there would be a more complicated potential and also gravitational coupling between the modes. In fact we will rely later on the existence of some such couplings to provide decoherence, but we will assume that they are small enough that we can treat the evolution of the modes separately.

To solve eq. (3.4), we first go to conformal time $\tau$, defined by $a d\tau = dt$, and write conformal time derivatives $\phi' = d\phi/d\tau = a \dot{\phi}$. The action $S = \int L dt$ should be unchanged, so the Lagrangian should be multiplied by $dt/d\tau = a$,

$$L_k = \frac{a^2}{2} \left[ |\phi'_k|^2 - \left( k^2 + a^2 m^2 \right) |\phi_k|^2 \right]. \quad (3.5)$$

Then we define a new field $\sigma_k = a \phi_k$, in terms of which

$$L_k = \frac{1}{2} \left[ |\sigma'_k|^2 - \frac{a'}{a} (\sigma'_k \sigma_k' + \sigma_k \sigma'_k) - \left( k^2 + a^2 m^2 - \frac{a'^2}{a^2} \right) |\sigma_k|^2 \right]. \quad (3.6)$$
We are free to add the total derivative \((d/d\tau)(\sigma_k^2 a'/a)\) giving
\[
L_k = \frac{1}{2} \left[ \sigma_k^2 - \left( k^2 + a^2 m^2 - \frac{a''}{a} \right) \sigma_k^2 \right] \tag{3.7}
\]
representing a varying-frequency but undamped oscillator. The equation of motion is
\[
\sigma_k'' + K(\tau) \sigma_k = 0, \tag{3.8}
\]
with
\[
K(\tau) = k^2 + a^2 m^2 - \frac{a''}{a}. \tag{3.9}
\]
We will work in de Sitter space, \(a(t) = a_0 e^{Ht}\), where \(H\) is the unchanging physical Hubble constant, so \(\tau = -(a_0/H)e^{-Ht}\), and runs from \(-\infty\) up to 0. We find \(a(\tau) = -1/(H\tau)\), \(a''/a = 2/\tau^2\) and
\[
K(\tau) = k^2 + \frac{1}{\tau^2} \left( \frac{m^2}{H^2} - 2 \right). \tag{3.10}
\]
The general solution to eqs. (3.8), (3.10) is
\[
\sigma_k = \sqrt{-\tau} \left[ |c_J| J_\nu(-k\tau) + |c_Y| Y_\nu(-k\tau) \right] \tag{3.11}
\]
with
\[
\nu^2 = \frac{9}{4} - \frac{m^2}{H^2}. \tag{3.12}
\]
We are interested in \(m \ll H\), so we can write
\[
\nu = \frac{3}{2} - \epsilon, \tag{3.13}
\]
with
\[
\epsilon = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \approx \frac{m^2}{3H^2}. \tag{3.14}
\]
The coefficients in eqs. (3.11) are complex, but since \(\sigma_k = \sigma^*_k\), we require \(c_J(-k) = c_J(k)^*\) and likewise for \(c_Y\). Thus there are 4 real degrees of freedom for the system including wave vectors \(k\) and \(-k\).

Including exponential factors and both \(\sigma_k\) and \(\sigma_{-k}\), the contribution to \(\phi\) from eq. (3.11) is
\[
e^{i k \cdot x} \phi_k(t) + e^{-i k \cdot x} \phi_{-k}(t) = \frac{2\sqrt{-\tau}}{a} \left[ |c_J| \cos(kx + \delta_J)J_\nu(-k\tau) + |c_Y| \cos(kx + \delta_Y)Y_\nu(-k\tau) \right] \tag{3.15}
\]
with \(\delta_J = \text{arg } c_J\) and similarly for \(Y\).

At early times \((a \to 0; \tau \to -\infty)\), the solutions to eq. (3.8) are simply oscillations. Using the large-argument approximation to \(J_\nu\) and \(Y_\nu\) in eq. (3.15) gives
\[
\frac{1}{a} \sqrt{\frac{8}{\pi k}} \left[ |c_J| \cos(kx + \delta_J) \cos(k\tau + \delta_J) - |c_Y| \cos(kx + \delta_Y) \sin(k\tau + \delta_J) \right] \tag{3.16}
\]
where \(\delta_J = (\pi/2)(\nu + 1/2)\). This is a superposition of two damped standing waves, and we can write it also as a superposition of damped traveling waves going in opposite directions.
At late times \((a \to \infty; \tau \to 0^-)\), we use the small-argument approximation in eq. (3.15) to get

\[
\frac{(-k\tau)^{\nu+1/2}}{2^\nu \Gamma(\nu + 1)a \sqrt{k}} |c_J| \cos(kx + \delta_J) + \frac{2^\nu \Gamma(\nu)(-k\tau)^{1/2-\nu}}{a \sqrt{k}} |c_Y| \cos(kx + \delta_Y). \tag{3.17}
\]

In the first term,

\[
\sigma \sim (-\tau)^{\nu+1/2}
\]

so

\[
\phi \sim (-\tau)^{\nu+3/2} \sim (-\tau)^{3+\epsilon}
\]

which declines rapidly. But in the second term

\[
\sigma \sim (-\tau)^{1/2-\nu}
\]

so

\[
\phi \sim (-\tau)^{3/2-\nu} = (-\tau)^{\epsilon} \sim e^{-\epsilon Ht} \sim e^{-m^2t/(3H)},
\]

which is a slow-roll mode. In the massless case it would be frozen. This is just what you would get by starting from eq. (3.4), ignoring the second-derivative term to find the slow-roll solution, and going to late times where \(a \gg k/m\).

The conventional slow-roll parameters are

\[
\varepsilon = \frac{1}{16\pi G} \left( \frac{U'}{U} \right)^2, \quad \eta = \frac{1}{8\pi G} \left( \frac{U''}{U} \right), \tag{3.22}
\]

where \(U\) is the scalar field potential. In our case, \(U' = m^2\phi\), but for \(U\) we should use instead the effective potential of the cosmological constant that is driving the underlying expansion, \(U = 3H^2/(8\pi G)\), giving

\[
\varepsilon = \frac{4\pi \phi^2}{m_{\text{Planck}}^2} \epsilon^2, \quad \eta = \epsilon \tag{3.23}
\]

for \(\epsilon \ll 1\). Thus our solution qualifies as slow roll for sub-Planckian field values.

Note that at late times there are no right-going and left-going modes. Propagating modes appear in eq. (3.16) through particular choices of the \(J\) and \(Y\) parameters. But the \(J\) sector decreases rapidly once the mode crosses the horizon, leaving only a slowly decreasing sinusoidal in the \(Y\) sector.

If we discover the field with some value \(\phi_1\) at some time \(\tau_1\), the simplest possibility would be that it is in the slow-roll mode with a pure \(Y_\nu\) solution,

\[
\phi_1 = \frac{C_Y \sqrt{-7} Y_\nu(-k\tau)}{a} = \frac{C_Y H \Gamma(\nu)}{\pi} \left( \frac{2}{k_1} \right)^{3/2} \left( \frac{2}{k_1} \right)^{\epsilon}. \tag{3.24}
\]

But we cannot rule out some admixture of the rapidly decaying mode, unless we measure \(\dot{\phi}\) very carefully.

Extending eq. (3.24) to early times, we find the solution

\[
\phi = \frac{C_Y}{a} \sqrt{\frac{7}{\pi k}} \cos(-k\tau + \delta) = -\sqrt{\frac{\pi}{k_1}} \frac{C_Y \phi_1}{2H \Gamma(\nu) a} \left( \frac{k_1}{2} \right)^{\epsilon} \cos(-k\tau + \delta). \tag{3.25}
\]

Thus we know that such a state at early times has an oscillation at least as large as that, and there could have been a much larger oscillation (with a different phase) that has decayed faster.
4 Quantum mechanics

We quantize the field $\phi$ in the Schrödinger picture. Formally we will write a wave functional $\Psi[\phi]$ obeying $i\partial \Psi/\partial t = H \Psi$. We will consider cases where $\Psi$ is just a product of terms for the various $k$, $\Psi[\phi] = \prod \Psi_k(\phi_k)$. This holds in the Bunch-Davies vacuum, and ref. [16] showed that the wavefunction after a decoherence process such as that discussed there also has this product form. We will furthermore restrict our consideration to cases where $\Psi_k$ is a product of wavefunctions operating on the real and imaginary parts of $\phi_k$. We decompose

$$\phi_k = \chi_k + i \xi_k, \quad (4.1)$$

as in section 2.2, and require $\Psi_k(\phi_k) = \psi_k(\chi_k)\zeta_k(\xi_k)$. We discuss below why this is sufficient for our purposes.

We let the momentum operator corresponding to $\chi_k$ be $p_k = -i\partial/\partial \chi_k$. The time-dependent Schrödinger equation for $\psi_k$ is then (see eq. (3.3)),

$$i \frac{\partial \psi_k}{\partial t} = H_k \psi_k = \frac{1}{2} \left[ - \frac{1}{a^3} \frac{\partial^2}{\partial \chi_k^2} + a^3 \left( \frac{k^2}{a^2} + m^2 \right) \chi_k^2 \right] \psi_k. \quad (4.2)$$

We will look for solutions where $\psi_k(\chi_k)$ has a Gaussian form. Ehrenfest’s theorem tells us that the expectation values of the position and momentum satisfy the classical equations of motion. So let us choose some classical solution $\Phi_k(t)$ and $P_k(t) = a^3 \dot{\Phi}_k(t)$, and look for $\psi_k(\chi_k)$ in the form

$$\psi_k(\chi_k, t) = N_k(t) \exp \left[ - \frac{A_k(t)}{2} (\chi_k - \Phi_k(t))^2 + i P_k(t) \chi_k + i \theta_k(t) \right], \quad (4.3)$$

where $N_k(t)$, $A_k(t)$, and $\theta_k(t)$ are functions to be determined. Here $N_k(t)$ and $\theta_k(t)$ are real, but $A_k(t)$ can be complex. Then from (4.2) we have

$$- \frac{i}{2} \dot{A}_k(t) (\chi_k - \Phi_k(t))^2 + i A_k(t) (\chi_k - \Phi_k(t)) \dot{\Phi}_k(t) - \dot{P}_k(t) \chi_k - \dot{\theta}_k(t) + i \dot{N}_k(t)/N_k(t)$$

$$= \frac{1}{2a^3} \left[ A_k(t) - (\chi_k - \Phi_k(t))^2 \right] + \frac{1}{2} a^3 K(t) \chi_k^2. \quad (4.4)$$

From eq. (3.3), $\dot{P}_k(t) = -a^3 K(t) \Phi_k$, so we can rearrange,

$$\frac{1}{2} \left[ -i \dot{A}_k(t) + \frac{A_k^2(t)}{a^3} - a^3 K(t) \right] (\chi_k - \Phi_k)^2$$

$$+ \frac{1}{2} K(t) a^3 \Phi_k^2 + i \dot{N}_k(t)/N_k(t) - \dot{\theta}_k(t) - \frac{1}{2a^3} A_k(t) - \frac{a^3}{2} \dot{\Phi}_k^2 = 0. \quad (4.5)$$

For this to hold we should require

$$- i \dot{A}_k(t) + \frac{A_k^2(t)}{a^3} - a^3 K(t) = 0, \quad (4.6)$$

and then the normalization $N_k(t)$ and the irrelevant phase $\theta(t)$ can be determined by integration (see also ref. [17], which derives the same equation using a slightly different method).

Equation (4.6) is a Riccati equation. We can rewrite $A_k$ in terms of an unknown function $f_k$,

$$i A_k = a^3 \frac{\dot{f}_k}{f_k}, \quad (4.7)$$
so that eq. (4.6) becomes
\[ \ddot{f}_k + 3H \dot{f}_k + K(t)f_k = 0 \] (4.8)
which is the just the classical equation of motion, eq. (3.4), with \( f_k(t) \) analogous to \( \phi_k(t) \) there. Then let us define \( g_k(\tau) = a f_k(\tau) \) analogous to \( \sigma_k(\tau) \) in eq. (3.8), with the general solution given by eq. (3.11),
\[ g_k = \sqrt{-\tau} \left[ g_{jJ}Y_\nu(-k\tau) + g_Y Y_\nu(-k\tau) \right] . \] (4.9)

Then we have
\[ A_k = -ia^2 \left[ \frac{g_k^*}{g_k} - H \right] = ia^2 k \left[ \frac{g_{jJ}Y_{\nu-1}(-k\tau) + g_Y Y_{\nu-1}(-k\tau)}{g_{jJ}(\nu-k\tau) + g_Y Y_{\nu}(-k\tau)} + cHa \right] . \] (4.10)

The precision (inverse variance) of the Gaussian wave function is given by
\[ \text{Re } A_k = a^2 \text{Im } \frac{g_k^*}{g_k} = a^2 \frac{g_k^* g_k - g_k^* g_k}{2|g_k|^2} . \] (4.11)

The numerator of (4.11) is the Wronskian of the solutions \( g_k \) and \( g_k^* \). It does not depend on time and its real part vanishes. To have a sensible wavefunction, we need \( \text{Re } A_k > 0 \), so \( \text{Im}(g_k^* g_k - g_k^* g_k) > 0 \). We have the freedom to multiply \( g_k \) by a constant without changing \( A_k \), and we can use this freedom to require
\[ g_k^* g_k - g_k^* g_k = i , \] (4.12)
or if \( g_k \) has the form of eq. (4.9),
\[ \text{Im}(g_j g_k^*) = \pi/4 . \] (4.13)

When the parameters of eq. (4.9) obey eq. (4.13), we have the simple form
\[ \text{Re } A_k = \frac{a^2}{2|g_k|^2} = -\frac{a^2}{2\tau |g_{jJ}Y_{\nu}(-k\tau) + g_Y Y_{\nu}(-k\tau)|^2} . \] (4.14)

For \( \text{Im } A_k \), we take the imaginary part of eq. (4.6),
\[ -\text{Re } \dot{A}_k + \frac{2(\text{Re } A_k)(\text{Im } A_k)}{a^3} = 0 \] (4.15)
so
\[ \text{Im } A_k = \frac{a^3}{2} \frac{d}{dt} \ln(\text{Re } A_k) . \] (4.16)

The imaginary part of \( A_k \) tells us how the width of the Gaussian is changing.

At late times, \( J_{\nu} \) decreases while \( Y_{\nu} \) increases. Thus most properties depend on \( g_Y \).

We can use the freedom to multiply \( g_j \) and \( g_Y \) by a common phase without affecting \( A \) or eq. (4.13) to make \( g_Y \) purely imaginary and write the general \( g_k \) satisfying eq. (4.13) in the form
\[ g_k(\tau) = (1/2) \sqrt{-\pi\tau/c} \left( (1 + i b)J_{\nu}(-k\tau) - i c Y_{\nu}(-k\tau) \right) , \] (4.17)
where \( c \) is a real constant giving the degree of squeezing.

From eq. (4.17), we find
\[ A_k = ia^2 k \left[ (1 + i b)J_{\nu-1}(-k\tau) - i c Y_{\nu-1}(-k\tau) \right] \left( (1 + i b)J_{\nu}(-k\tau) - i c Y_{\nu}(-k\tau) \right) + \frac{cHa}{2k} \] (4.18)
and
\[ \text{Re } A_k = \frac{2a^3 H c}{\pi \left( J_{\nu}(-k\tau)^2 + (c Y_{\nu}(-k\tau) - b J_{\nu}(-k\tau))^2 \right)} . \] (4.19)
4.1 Late and early time approximations

At late times, we can neglect $J_\nu$ and then use the small-argument approximation for $Y_\nu$, 

\[ \text{Re} A_k = \frac{2a^3H}{\pi c Y_\nu(-k\tau)^2} = \frac{\pi k^3}{4c\Gamma(\nu)^2H^2} \left( -\frac{2}{k\tau} \right)^{2\epsilon}. \]  

(4.20)

Note that $b$ does not enter, because it is the coefficient of a decaying mode. The width (standard deviation) of the Gaussian wavefunction is

\[ s = \frac{1}{\sqrt{\text{Re} A_k(\tau)}} = 2\sqrt{\frac{\tau \Gamma(\nu)H}{\pi}} \left( -\frac{2}{k\tau} \right)^{-\epsilon}. \]  

(4.21)

At late times, the imaginary part of $A_k$ becomes

\[ \text{Im} A_k(\tau) = a^2k \left[ \frac{Y_{\nu-1}(-k\tau)}{Y_\nu(-k\tau)} + \frac{\epsilon H a}{2k} \right] = \frac{k^2a}{2H(\nu - 1)} + \epsilon a^3H. \]  

(4.22)

We see that $\text{Im} A_k$ grows rapidly with time, so the phase of $\psi_k(\phi_k)$ oscillates rapidly.

At early times we can take

\[ \sqrt{-\tau} J_\nu(-k\tau) = \sqrt{\frac{2}{\pi k}} \cos(k\tau + \delta_\nu) \]  

(4.23)

\[ \sqrt{-\tau} Y_\nu(-k\tau) = -\sqrt{\frac{2}{\pi k}} \sin(k\tau + \delta_\nu). \]  

(4.24)

Combining these with any complex coefficients $g_J$ and $g_Y$ gives an ellipse in the complex plane. Equation (4.13) tells us that the area of this ellipse is $\pi/(2\epsilon)$. We can always write it

\[ g = \frac{e^{i\theta}}{\sqrt{2\epsilon c k}} [\cos(-k\tau + \delta) - i\bar{c}\sin(-k\tau + \delta)] \]  

(4.25)

with $\bar{c}$, $\theta$, $\delta$ real constants that depend on $c$ and $b$. Without loss of generality we can take $\bar{c} \leq 1$. Equation (4.25) gives

\[ \text{Re} A = \frac{a^2k\bar{c}}{\cos^2(-k\tau + \delta) + \bar{c}^2\sin^2(-k\tau + \delta)}, \]  

(4.26)

which is an oscillation between $\bar{c}$ and $1/\bar{c}$, with an overall factor $a^2k$, which is the $A$ of the ground state.

If $b = 0$, eq. (4.25) is just the early-time version of eq. (4.17) with $c = \bar{c}$. Otherwise, $\bar{c} < c$, i.e., some of the early oscillation in Re $A$ is damped rather than frozen in.

All of the considerations above apply also to the wave function $\zeta_k(\xi_k)$, which could in principle have its own $A_k$, $\Phi_k$ and $P_k$. An even more general Gaussian would have a general quadratic form in $\chi_k$ and $\xi_k$ in the exponent, i.e., there would be an additional term proportional to $\chi_k\xi_k$. However, in the cases we will use, the $A_k$ will be the same for both $\chi_k$ and $\xi_k$, and there will be no cross term. This means that considerations involving the width of the Gaussian will not distinguish particular locations $x$.

The values of $\Phi_k$ and $P_k$ may be different for $\chi_k$ and $\xi_k$, but this means merely that we can build our wavefunction around any classical solution such as eq. (3.15).
4.2 Bunch-Davies vacuum

Now we will consider the effect of fluctuations of $\phi$ in the background de Sitter spacetime. After a long period of inflation, we expect the quantum state to be the Bunch-Davies vacuum [18]. Consider a short-wavelength mode, so $k \gg H \gg m$, and the mode is at each moment in the flat-space ground state, which is a Gaussian around $\phi = 0$ (so $\Phi_k = 0 = P_k$) with $A_k = a^2 k$. From eq. (4.14), this requires $|g_k| = 1/\sqrt{2k}$. Thus at early times we have eq. (4.25) with $\bar{c} = 1$ (and we can take $\theta = 0$), $g_k(\tau) = (1/\sqrt{2k})e^{ik\tau}$. This becomes in general eq. (4.17) with $c = 1$ and $b = 0$, i.e., $g_k(\tau) = (\sqrt{-\pi/2})H_\nu^{(2)}(-k\tau)$ meaning $g_J = \sqrt{\pi/2}$, $g_Y = -i\sqrt{\pi/2}$ and so

$$\text{Re} A_k = \frac{2a^3 H}{\pi (J_\nu(-k\tau)^2 + Y_\nu(-k\tau)^2)}.$$ (4.27)

At late times,

$$\text{Re} A_k = \frac{\pi k^3}{4\Gamma(\nu)^2 H^2} \left( -\frac{2}{k\tau} \right)^{2\epsilon},$$ (4.28)

and the width of the wavefunction is

$$s_0 = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\nu)H}{k^{3/2}} \left( -\frac{k\tau}{2} \right)^{\epsilon}.$$ (4.29)

In the massless case, $\epsilon = 0$, the width is fixed. The width of the probability distribution $|\psi|^2 \sim e^{-\text{Re} A_k^2}$ is $s_0/\sqrt{2}$. Using $\Gamma(3/2) = \sqrt{\pi}/2$, we recover the usual perturbation spectrum from inflation, $\delta_k = H/\sqrt{2k^3}$. With a mass the Gaussian narrows with time.

4.3 State with rapid expansion

When the Bunch-Davies state above is strongly subhorizon, it is essentially in its ground state. The rate the frequency changes is small compared to the frequency itself, so conditions change adiabatically and the mode remains in the ground state. But as it crosses the horizon, this is no longer true. The mode remains a Gaussian whose width is given by eq. (4.29), but it is no longer an energy eigenstate. At late times when the mode is superhorizon, it is frozen in and its energy depends almost entirely on $\phi$. We can think of this mode as having a spread of possible energies corresponding to different $\phi$.

We now imagine that we have extended our toy model to include some small interactions between different modes, such as nonlinear gravitational coupling [16], which will lead to decoherence. Each (sufficiently superhorizon) mode wave function will decohere into some particular alternative narrowly peaked around some $\Phi_k$, with the chance of the various $\Phi_k$ given by

$$|\psi_k(\Phi_k)|^2 \sim e^{-\text{Re} A_k \Phi_k^2}$$ (4.30)

with $\text{Re} A_k$ given by eq. (4.28). More specifically, decoherence at late times will pick out a specific classical solution of the slowly rolling type given by eq. (3.21) above.

The details of the decoherence will not concern us here, but it is important that the different modes decohere independently [16] and that the “pointer basis” of decoherence is the space of field values. This latter fact is required by the unobservability of the decaying mode and the rapid phase oscillations of $\psi$ [16, 19–21].

It is important to note that the probability distribution of eq. (4.30) extends to values of $\Phi_k$ that will have much more energy than the average in the Bunch-Davies state. Indeed, such values are the ones that will lead to eternal inflation. This increase in the (average)
energy is possible even though the decoherence process (weak coupling to other modes [16], for example) does not transfer any significant energy into the mode. Since the Bunch-Davies state of a superhorizon mode is not an eigenstate of the Hamiltonian, decoherence without energy transfer can select a state with energy different from the average energy of the state before decoherence. The decoherence process is analogous to the final measurement of the system of section 2.3.

After decoherence, the mode of interest will have only some small spread of $\phi_k$. Let us take it to be a Gaussian with narrow width $s < s_0$. We will take $s$ to be the same for the real and imaginary components ($\chi_k$ and $\xi_k$) of $\phi_k$, as we discuss below. We would now like to know the history of that specific state, to see explicitly whether it obeys energy conditions.

The general Gaussian state with width $s$ is given by eqs. (4.17), (4.21). The ratio $s/s_0 = \sqrt{c}$ is a constant less than 1, which gives the degree of decoherence, i.e., it tells us how narrow the decoherent wave function is as compared to the wave function before decoherence. We cannot determine $b$, because it does not enter at late times. The expressions for $g$ and $A$ are given in eq. (4.17)–(4.22).

## 5 Energy conditions

The stress-energy tensor for the minimally coupled scalar field is

$$T_{\mu \nu} = (\partial_\mu \phi \partial_\nu \phi) - \frac{1}{2} \eta_{\mu \nu} ((\nabla \phi)^2 - m^2 \phi^2).$$  

(5.1)

We will be interested in the expectation value of $T_{\mu \nu}$ in quantum states given by products of wavefunctions in the form of eq. (4.3). Such a state has a classical part

$$\Phi = \frac{1}{L^{3/2}} \sum_k e^{ik \cdot x} \Phi_k,$$

(5.2)

and we can define the purely quantum part, $\delta \phi = \phi - \Phi$, with the usual Fourier decomposition

$$\delta \phi = \frac{1}{L^{3/2}} \sum_k e^{i k \cdot x} \delta \phi_k$$

(5.3)

and write $\delta \phi_k = \delta \chi_k + i \delta \xi_k$. Then our wavefunction $\Psi$ is the product of terms

$$\psi_k(\delta \chi_k) \sim \exp \left[ -\frac{A_k(t)}{2} \delta \chi_k^2 + i P_k(t)(\delta \chi_k + \Phi_k) \right],$$

(5.4)

and similarly for $\delta \xi_k$.

Using $\phi = \delta \phi + \Phi$ in eq. (5.1), we see that $T_{\mu \nu}$ is made up of quadratic combinations of the operators $\chi_k$, $\xi_k$, and their corresponding momenta. Applying such operators to the wavefunction $\Psi$, we get expectation values of terms up to second order in the $\delta \chi_k$ and $\delta \xi_k$. Terms with exactly one of these vanish since $\langle \delta \chi_k \rangle = \langle \delta \xi_k \rangle = 0$ by symmetry. Terms with no $\delta \chi_k$ or $\delta \xi_k$ are just classical quantities, while those with two are quantum mechanical, but they vanish unless the $k$’s are the same and they are both $\delta \chi_k$ or both $\delta \xi_k$. Thus we can write $T_{\mu \nu}$ as the sum of a classical and a quantum part,

$$T_{\mu \nu} = T_{\mu \nu}^C + T_{\mu \nu}^Q$$

(5.5)
with
\[ T^Q_{\mu\nu} = (\partial_\mu \delta \phi \partial_\nu \delta \phi) - \frac{1}{2} g_{\mu\nu} ((\nabla \delta \phi)^2 - m^2 \delta \phi^2), \tag{5.7} \]
and
\[ T^Q_{\mu\nu} = (\partial_\mu \Phi \partial_\nu \Phi) - \frac{1}{2} g_{\mu\nu} ((\nabla \Phi)^2 - m^2 \Phi^2), \tag{5.6} \]
The classical part obeys the energy conditions, while the quantum mechanical part may violate them.

Modes which are subhorizon (or those which are superhorizon but not sufficiently so to have decohered \cite{16}) are still in the Bunch-Davies state. They have no classical part and their Re A is given by eq. (4.27). Decoherent modes have classical parts around which their wave functions are strongly peaked, so their Re A is much larger than that of Bunch-Davies.

Since the overall stress-energy tensor is just the sum of quantum and classical parts, we will first calculate the energy density and the NEC in the case without any classical contribution, so \( T_{\mu\nu} \) is just \( T^Q_{\mu\nu} \) and \( \delta \phi \) is just \( \phi \). Then we can add in a possible classical contribution later. We will assume that the decoherent \( A_k \) does not depend on the particular \( \Phi_k \) that decoherence gives us, i.e., the degree of decoherence is the same in the different possible decoherent states. This is consistent with ref. \cite{16} where the decoherence functional depends only on the difference between its two arguments. In other words, \( \psi_k(\phi_k) \) is independent of \( \Phi_k \) and \( P_k \). Thus \( \psi_k(\phi) \) depends only on the magnitude of \( k \) and not on its direction, and the real and imaginary parts of \( \phi \) enter in the same way. Then \( T_{\mu\nu} \) has the perfect fluid form, \( \text{diag}(\rho, P, P, P) \).

The energy density is
\[ \rho = T_{00} = \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} \left( \frac{1}{a^2} (\partial_i \phi)^2 + m^2 \phi^2 \right), \tag{5.9} \]
and the null projection
\[ N = T_{00} + \frac{1}{3} T^i_i = (\partial_0 \phi)^2 + \frac{1}{3a^2} (\partial_i \phi)^2. \tag{5.10} \]
When we decompose the field into modes, we get
\[ \rho = \frac{1}{2V} \sum_k \left[ |\dot{\phi}_k|^2 + \frac{k^2}{a^2} + m^2 \right] |\phi_k|^2 = \frac{1}{2a^2V} \sum_k \left[ |\phi'_k|^2 + \left( k^2 + a^2 m^2 \right) |\phi_k|^2 \right], \tag{5.11} \]
and
\[ N = \frac{1}{V} \sum_k \left[ |\dot{\phi}_k|^2 + \frac{k^2}{3a^2} |\phi_k|^2 \right] = \frac{1}{a^2V} \sum_k \left[ |\phi'_k|^2 + \frac{k^2}{3} |\phi_k|^2 \right]. \tag{5.12} \]
When we quantize the field, \( \rho \) and \( N \) become operators, which we can write in terms of the operators \( \chi_k \) and \( \xi_k \), being the real and imaginary parts of \( \phi_k \), and their conjugate momenta,
\[ \rho = \frac{1}{2a^2V} \sum_k \left[ \frac{p^2_k}{a^4} + \left( k^2 + a^2 m^2 \right) \chi^2_k \right] + \text{terms involving } \xi_k, \tag{5.13} \]
\[ N = \frac{1}{a^2V} \sum_k \left( \frac{p^2_k}{a^4} + \frac{k^2}{3} \chi^2_k \right) + \text{terms involving } \xi_k. \tag{5.14} \]
Next we want to find the expectation value of the quantized energy density in a squeezed coherent state of the form of eq. (4.3). We first calculate
\[
\langle \psi_k | \chi_k^2 | \psi_k \rangle = \int_{-\infty}^{\infty} d\chi_k \chi_k^2 |\psi_k(\phi, \tau)|^2 = \frac{1}{2 \text{Re}(A_k(\tau))},
\]
(5.15)
and
\[
\langle \psi_k | p_k^2 | \psi_k \rangle = \int_{-\infty}^{\infty} d\chi_k \left| \frac{\partial}{\partial \phi} \psi_k(\phi, \tau) \right|^2 = \frac{|A_k(\tau)|^2}{2 \text{Re}(A_k(\tau))}.
\]
(5.16)

The contribution for a single mode of \(\chi_k\) is then
\[
\langle \psi_k | \rho_k | \psi_k \rangle = \frac{1}{2a^2 V} \left[ \frac{1}{a^4} \left( \frac{|A_k(\tau)|^2}{2 \text{Re}(A_k(\tau))} \right) + \left( k^2 + a^2 m^2 \right) \left( \frac{1}{2 \text{Re}(A_k(\tau))} \right) \right],
\]
(5.17)
and
\[
\langle \psi_k | N_k | \psi_k \rangle = \frac{1}{2a^2 V} \left[ \frac{1}{a^4} \left( \frac{|A_k(\tau)|^2}{2 \text{Re}(A_k(\tau))} \right) + \frac{k^2}{3} \left( \frac{1}{2 \text{Re}(A_k(\tau))} \right) \right].
\]
(5.18)

To better understand these formulas, let us define a dimensionless quantity
\[
B = \frac{1}{ka^2} A_k(\tau).
\]
(5.19)

From eq. (4.18), we find that \(B\) depends on \(k\) and \(\tau\) only through the combination \(k \tau\) and through the possible dependence (absent in the Bunch-Davies state) of the parameters \(c\) and \(b\) on \(k\).

If we consider the massless case \(m = 0\) for simplicity, we have
\[
\langle \psi_k | \rho_k | \psi_k \rangle = \frac{H^4}{V k^3} \left( \frac{(k \tau)^4}{4} \left( \frac{|B|^2}{\text{Re} B} + \frac{1}{\text{Re} B} \right) \right),
\]
(5.20)
and
\[
\langle \psi_k | N | \psi_k \rangle = \frac{H^4}{V k^3} \frac{1}{2} \left( \frac{|B|^2}{\text{Re} B} + \frac{1}{3 \text{Re} B} \right).
\]
(5.21)

We note that \(H^4 (k \tau)^4 / (V k^3) = k_{\text{phys}} / V_{\text{phys}}\) where \(k_{\text{phys}} = k/a\) and \(V_{\text{phys}} = V a^3\) are the physical wavenumber and volume respectively. So eqs. (5.20), (5.21) are physical quantities as they should be.

### 5.1 Renormalization

While expectation values in single modes, such as eqs. (5.20), (5.21), are finite, sums over \(k\) lead to the usual divergence of the expectation value of the stress-energy tensor in quantum field theory. To get finite physical values we must renormalize. There is a well-known prescription following a set of axioms described by Wald [22]. We first write a “point-split” version of \(T_{\mu \nu}\) as a differential operator acting on the two-point function. We renormalize the two-point function by subtracting the Hadamard parametrix, yielding a smooth function. We then apply the differential operator and bring the points together. One can use this procedure [18, 23, 24] to compute \(T_{\mu \nu}\) in the Bunch-Davies vacuum and in a general Gaussian state.

However, in our case there is an additional complexity. We are interested in the energy density and null-projection of a single mode. Thus we would need to renormalize not just...
the overall $\rho$ and $N$, but the contribution from each mode separately. This requires a mode expansion of the Hadamard parametrix, but that is not a clearly defined operation. (For an analysis of the corresponding problem for the Casimir effect in flat space, see ref. [25].)

For this reason, we will not try to give the contribution of a single mode to the renormalized $T_{\mu\nu}$. Instead we will give the difference $T_{\mu\nu}^{\text{diff}}$ between the contribution of a mode in a particular state and that same mode in the Bunch-Davies vacuum. This is unambiguous.\(^2\)

The Bunch-Davies vacuum obeys the symmetries of de Sitter space, which means that its renormalized $T_{\mu\nu}$ (including all modes) must be proportional to $g_{\mu\nu}$ [18]. Thus its NEC contribution is zero, and it marginally obeys NEC.

Thus we write the total renormalized energy density in any state as the positive Bunch-Davies (BD) value plus the sum of $\rho_k^{\text{diff}} = \rho_k - \rho_k^{\text{BD}}$. The total contribution to $N$ is just the sum of $N_k^{\text{diff}} = N_k - N_k^{\text{BD}}$, since the total $N = 0$ in the Bunch-Davies state.

Following that idea we subtract the Bunch-Davies vacuum which corresponds to the $c = 1$ and $b = 0$ case

$$\langle \psi_k | \rho^{\text{diff}} | \psi_k \rangle = \langle \psi_k | \rho | \psi_k \rangle - \langle \psi_k | \rho^{\text{BD}} | \psi_k \rangle .$$

(5.22)

For early times we can use eq. (4.25) which for $c = 1$ gives $B = 1$. Then

$$\langle \psi_k | \rho^{\text{diff}} | \psi_k \rangle = \frac{H^4}{V k^3} \frac{(k_4)^4}{4} \left( |B|^2 + \frac{1}{2} \frac{1}{\text{Re} B} - 1 \right) = \frac{H^4}{V k^3} \frac{(k_4)^4}{4} \text{Re} B \left( \text{Im} B^2 + (\text{Re} B - 1)^2 \right) .$$

(5.23)

Since $\text{Re} B \geq 0$ the change to the energy density is always positive: a mode in a narrower Gaussian leads to higher energy density than the Bunch-Davies state. However, at late times we can have decreases.

For the case of $N$, we start with eq. (5.21) and subtract the Bunch-Davies vacuum. At early times we have

$$\langle \psi_k | N^{\text{diff}} | \psi_k \rangle = \frac{H^4}{V k^3} \frac{(k_4)^4}{6} \left( \frac{3 |B|^2}{\text{Re} B} + \frac{1}{\text{Re} B} - 4 \right) .$$

(5.24)

but this does not have a uniform sign, so the NEC contribution of this mode may be less than in Bunch-Davies. Since NEC is obeyed only marginally in Bunch-Davies, this means that some decoherent states will have NEC violation.

However, in addition to the quantum mechanical contribution, there will be a classical contribution. The typical form for the classical $\Phi$ is one whose amplitude (not including any decaying mode) matches the standard deviation of the probability distribution $|\psi_k|^2$ in Bunch-Davies state. This has the general form (we have chosen the case where $\Phi_k > 0$ at late times),

$$\Phi_k(\tau) = -\frac{\sqrt{\pi}}{2 a} \sqrt{-\tau} Y_\nu(-k \tau)$$

(5.25)
or in the massless case,

$$\Phi_k(\tau) = \frac{H}{\sqrt{2 k^3}} \cos (k \tau) + k \tau \sin (k \tau) .$$

(5.26)

The classical contribution to $\rho$ and $N$ is always positive and for typical values outweighs any negative quantum contribution. To have eternal inflation, we need an especially large classical part, so this will be even more true in that case.

\(^2\)Alternatively one could perhaps use here the adiabatic regularization process [26] to renormalize mode-by-mode. This is a point that deserves further investigation.
Figure 2. The changes to energy density and (directionally averaged) NEC contribution, $\rho^{\text{diff}}$ and $N^{\text{diff}}$, from a single mode with $c = 0.05$, $b = 0$, and the classical part in the massless case. The vertical scale is in units of $H^4/(Vk^3)$. The energy density and $N$ diverge at early times.

Figure 3. A zoomed in part of figure 2 where the negative values of $\rho^{\text{diff}}$ and $N^{\text{diff}}$ are visible.

5.2 Numerical evolution

Now we explore these effects numerically. We vary $c$, $b$ and $\nu$ and examine the effect that each parameter has on the change to the expectation values of $\rho^{\text{diff}}$ and $N^{\text{diff}}$. We will plot these quantities in units of $H^4/(Vk^3)$.

First we look at $m = 0$ and small values of $c$, which correspond to highly decoherent states. In figure 2 we see that at early times $\rho^{\text{diff}}$, $N^{\text{diff}}$, and of course the classical part, are positive. Thus the selection of a decoherent state does not lead to negative energies or NEC violation. All effects are larger at earlier times. However, at late times (figure 3) there are small negative values of both $\rho^{\text{diff}}$ and $N^{\text{diff}}$. Still, a typical classical part is expected to be larger. A numerical analysis shows no decrease in the values of the NEC or the energy density when both the classical and quantum parts are taken into account. So a typical decoherent state does not produce negative energies or NEC violation even at late times.
Figure 4. As in figures 2 and 3, but for $c = 0.9$. Here $\rho^{\text{diff}}$ is positive at early times while $N^{\text{diff}}$ oscillates (left). But both are much smaller than their classical counterparts at late times (right).

Figure 5. Plot of the regions where $N^{\text{diff}} < 0$. For small $c$ this occurs only at late times, while for larger $c$ it occurs repeatedly, with longer duration for $c$ closer to 1.

We can ask what happens for different values of $c$. For large values of $c$ (close to 1) both $N^{\text{diff}}$ and $\rho^{\text{diff}}$ have recurring negative values. At early times, $\rho^{\text{diff}}$ becomes strictly positive as expected, but that does not happen with $N^{\text{diff}}$. However, the classical values are much larger at early times in both cases (figure 4). In the regional plot of figure 5 we see that these recurring regions of negative $N^{\text{diff}}$ become longer for $c$ closer to 1.

The effect of the $b$ parameter, the coefficient of the decaying mode, is shown in figure 6. For small values of $c$, increasing the $b$ parameter reduces the range of over which $N^{\text{diff}} < 0$ and the maximum negative magnitude that it reaches.

Finally we can examine the effect of mass on the NEC. For a larger mass, according to eq. (3.12) the $\nu$ parameter of the Bessel function is something smaller than $3/2$. As we see in figure 7 larger mass leads to less negative values of the $N^{\text{diff}}$ at late times for small values of $c$.

In every case, a decoherent state with a typical or larger classical part has a larger energy density and a more positive contribution to NEC at all times than the Bunch-Davies state. Thus the sequence of decoherent states that leads to eternal inflation always obeys NEC. The process that allows eternal inflation to take place is successive selection of subsectors of the wavefunction, each of which individually always obeys NEC.
Figure 6. The effect of parameter $b$ on $N_{\text{diff}}$. It becomes more positive with increasing $b$ (left) while at late times it is less negative and negative for a shorter time (right).

Figure 7. The effect of the mass on $N_{\text{diff}}$. For decreasing values of the $\nu$ parameter (increasing mass) the negative values of $N_{\text{diff}}$ at late times decrease in duration and maximum magnitude.

6 Beyond the simple model

In the sections above we discussed a simple model that exhibits the paradox of eternal inflation and energy conditions and showed how this paradox is resolved. It seems clear that the same resolution applies in realistic models, but it would still be good to go beyond the simple model and discuss the situation of the actual eternal inflation. However, this is very complicated. It is straightforward to consider a quasi-de Sitter background, and we do that in the next subsection. Then we discuss how one might go beyond that.

6.1 Quasi-de Sitter background

Above we considered a massive scalar field in a de Sitter background. Here we will consider instead the fluctuations of an inflaton field rolling in a linear potential, without any quadratic (mass) term. The classical slow-roll motion gives rise to a quasi-de Sitter space, and we consider the fluctuations to evolve only in this background spacetime, not in the modified
spacetime that they themselves produce. The scale factor for quasi-de Sitter is [12]

\[ a \sim \frac{1}{\tau^{1+\varepsilon}} \]  

(6.1)

where \( \varepsilon \ll 1 \) is the slow roll parameter of eq. (3.23). The motion in this background is given by eq. (3.8), but now

\[ K(\tau) = k^2 - \frac{a''}{a} = k^2 - \frac{1}{\tau^2} (2 + 3\varepsilon) \]  

(6.2)

giving solutions that are qualitatively the same as before with

\[ \nu^2 = \frac{9}{4} - 3\varepsilon, \]  

(6.3)

instead of eq. (3.12).

6.2 Going further

The next step would be to include modifications to the spacetime as a result of a fluctuating field. This is routinely done in the study of inflationary perturbations. In that case, one takes the perturbations to be small and considers only their first order effects. For example, \( T_{\mu\nu} \) is second order in the field, so one would consider only cross terms with products of the fluctuating field and the background. In this case one can also consider gauge (coordinate) choices that remove the fluctuation in the field and move it into other sectors. For example one can choose surfaces on which the field has its background value.

In our case, however, the fluctuations are not small. On the contrary, the effect of the fluctuations that lead to eternal inflation is larger than the effect of the slowly rolling background. So the first order expansion does not make sense. Furthermore, one cannot absorb field fluctuations into gauge transformations. No (monotonic) change of time coordinate, for example, can take an increasing field and reparameterize it as a decreasing field.

The large fluctuations lead to non-linear effects. The square of the field enters into the metric, which then appears in the field equation. The nonlinear equation means that the field states are no longer given by harmonic oscillator states such as the Gaussians that we have used above. Furthermore, nonlinearities couple the different Fourier modes, again invalidating the above analysis.

So we conclude that the case of the real eternal inflation is approximately given by the massless case in the analysis of sections 3–5 above, but we cannot consistently go beyond that level. The idea behind the compatibility of eternal inflation with energy conditions is the same as in our simple model, but we cannot do a detailed analysis in the realistic case.

7 Conclusion

Eternal inflation is driven by a series of quantum fluctuations that increase the rate of expansion \( H \) of the universe in a certain region. But the null energy condition (NEC) requires that \( \dot{H} \leq 0 \), so \( H \) can never increase. Nevertheless, eternal inflation can proceed without any NEC violation. The eternally inflating spacetime is a succession of more and more specific quantum states, selected by decoherence. The energy conditions apply to each of these quantum states individually, so \( H \) cannot increase in any of them. But when decoherence selects a particular sector of the quantum state, this sector can have a larger \( H \) than the
overall state. A succession of such decoherence events can lead to a repeated increase in $H$, even though NEC is obeyed and prohibits an increase $H$ in any state.

We showed this process explicitly in a simplified model that exhibits the same paradox. We considered a scalar field in de Sitter space where the field can affect the spacetime, but the resulting changes do not affect the field evolution. We start in the Bunch-Davies vacuum and allow decoherence to select narrow states in field space for a particular mode of the field. When these states have particularly large values, they lead to more rapid expansion, but the selected states themselves never violate NEC.

In cosmology there is no true decoherence, because there is no environment outside the universe with which things may interact. If we start our model in the Bunch-Davies vacuum, it remains in that state. At that level of analysis there is no increase in the expansion rate and no issue of NEC violation. Concepts of eternal inflation (and, in the real universe, thermalization and the development of structure) apply to sectors of the wavefunction that are effectively decoherent in the sense that their quantum correlations are distributed widely through the universe. It is in such an analysis that one might find eternal inflation and be concerned about compliance with NEC. We have shown here that NEC does not restrict eternal inflation in that sense, because eternal inflation does not take place in a single quantum state, but rather a succession of more finely selected states.

Would it be right to say that in the selected sector the universe has always been expanding rapidly, and its expansion rate never increases? Perhaps not. In the Bunch-Davies vacuum, when the mode of interest is strongly subhorizon it is essentially in an energy eigenstate, so there is little uncertainty about its contribution to the energy density and thus to the expansion rate. An energy eigenstate can of course be expressed as a superposition of non-eigenstates. If this state is coupled to the expansion rate, the state with known expansion rate can be expressed in terms of states with various different expansion rates. Nevertheless the superposition itself has a fixed expansion rate. Changes in the Hamiltonian as the mode becomes superhorizon allow decoherence to pick out one of these states from the superposition. That is the sense in which the universe has always been expanding rapidly in the selected state.

Acknowledgments

We would like to thank Larry Ford, Ben Freivogel, Hanno Gottschalk, Alan Guth, Tanmay Vachaspati, Alex Vilenkin, and an anonymous referee for helpful conversations. EAK’s contribution to this work is part of a project that has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 744037 “QuEST”.

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