Chemical potential as a source of stability for gravitating
Skyrmions

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Abstract

A discussion of the stability of self gravitating Skyrmions, with a large winding number \( N \), in a Schwarzschild type of metric, is presented for the case where an isospin chemical potential is introduced. It turns out that the chemical potential stabilizes the behavior of the Skyrmion discussed previously in the literature. This analysis is carried on in the framework of a variational approach using different ansaetze for the radial profile of the Skyrmion. We found a divergent behavior for the size of the Skyrmion, associated to a certain critical value \( \mu_c \) of the chemical potential. At this point, the mass of the Skyrmion vanishes. \( \mu_c \) is essentially independent of gravitating effects. The stability of a large \( N \) skyrmion against decays into single particles is also discussed.

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The stability of a Skyrme topological soliton coupled to gravity has been analyzed many times in the literature [1, 2, 3]. The purpose of this letter is to extend this discussion to the scenario where Isospin chemical potential is taken into account.

Gravitating Skyrmions turn out to have a very rich structure, with a nontrivial spectrum for the asymptotic states [2], and provide a counter example to the “non hair theorem” in the construction of black hole solutions [3]. It is therefore interesting and natural to consider such objects now in the presence of chemical potentials associated to conserved charges.

Recently, we have discussed Skyrmions as a model for baryons in the presence of isospin chemical potential ($\mu$), showing the existence of a critical value $\mu_c$ where the mass of the Skyrmion vanishes, signaling the occurrence of phase transitions. We found that the radial profile of the Skyrmion becomes broader as $\mu$ approaches the critical value. The nucleon spectra was also considered in this context [4].

Here we present, following the article by Glendenning et al [1], a variational analysis for the stability of a self gravitating SU(2) Skyrmion for large values of the winding number $N$, showing that $\mu$ plays an stabilizing role. In particular, the parameter associated to the radius of the Skyrmion profile starts to grow, whereas the mass diminishes as function of $\mu$. For a certain value $\mu_c$, the radius diverges and the mass vanishes in the same way as it happens in the non gravitating case [4]. We found strong numerical evidence that the value of $\mu_c$ is independent of the parameters associated to gravity.

It was found in [1], that the Skyrmion mass behaves like $O(N^2)$. Therefore gravity becomes important in the large N limit. Actually, the scale where gravity becomes relevant is given by $Nf_\pi \sim M_{\text{Planck}}$. The Skyrmion mass depends on the quotient $R_S/R \approx G(Nf_\pi)^2$, where $R$ is a measure of the Skyrmion size in the absence of gravity, whereas $R_S$ is essentially the Schwarzschild radius, i.e. the natural length scale associated to gravity effects. In the variational approach, there is no solution for values of $R_S/R$ bigger than a certain limit. If $\mu \neq 0$, this limit becomes bigger, signalizing the stabilizing role of the chemical potential. If $\mu \rightarrow \mu_c$, we have no bound for $N$.

The authors of [1] found also that their solution actually does not satisfy the condition $M(N) \leq NM(1)$, indicating that the solution with high winding number should decay into many Skyrmion states with lower $N$. In our case, the chemical potential also provides a source of stability near $\mu_c$.

We would like to remark that in our previous analysis [4], the variational approach was
not appropriate due to the lack of stability for large values of $\mu$. Therefore, a full numerical treatment was unavoidable. However, in the case of the self gravitating Skyrmion, a variational approach is possible for any value of $\mu < \mu_c$. The idea is to consider a certain given radial profile, which depends on a free parameter, minimizing then the mass of the Skyrmion.

Let start our analysis with the nonlinear sigma-type Skyrme lagrangian

\[ L_m = \frac{f^2}{4} Tr \left[ D_\mu U D^\mu U^\dagger \right] + \frac{1}{32\pi^2} Tr \left[ (D_\mu U)U^\dagger, (D_\nu U)U^\dagger \right]^2, \]
\[ \equiv L_2 + L_4, \]  

(1)

where

\[ D_\mu = \partial_\mu - i \frac{\mu}{2} [\sigma, U] \delta_{\mu,0}, \]

(2)
is the covariant derivative that introduces the chemical potential \[ \mu \], and

\[ U = \sigma_0 \cos F(r) + i \vec{\sigma} \cdot \hat{n} \sin F(r), \]

(3)
is the standard Hedgehog ansatz, $\vec{\sigma}$ are the Pauli matrices, $\sigma_0$ is the identity and $F(r)$ is the radial profile. The gravitating version (Einstein-Skyrme lagrangian) can be obtained through the action

\[ S = \int (L_g + L_m) \sqrt{-g} \, d^4x, \]

(4)

with $L_m$ given by (1) and

\[ L_g = -\frac{R}{16\pi G}, \]

(5)

where $R$ is the Ricci scalar curvature and $G$ the Newton constant.

The explicit form of $L_2$ and $L_4$, without $\mu$ is

\[ L_{2,0} = \frac{1}{2} f^2 \left( g^{11} F'(r)^2 + g^{22} \sin^2 F(r) + g^{33} \sin^2 F(r) \sin^2 \theta \right). \]

(6)
\[ L_{4,0} = -\frac{1}{2e^2} \left( g^{11} g^{22} F'(r)^2 \sin^2 F(r) 
+ g^{11} g^{33} F'(r)^2 \sin^2 F(r) \sin^2 \theta 
+ g^{22} g^{33} \sin^4 F(r) \sin^2 \theta \right). \] (7)

When we turn on the chemical potential, \( \mu \)-dependent terms appear, so that, in an obvious notation,

\[ L_m = L_{2,0} + L_{4,0} + L_{2,\mu} + L_{4,\mu}. \] (8)

with

\[ L_{2,\mu} = \frac{1}{2} f^2 \pi \mu^2 \sin^2 F(r) \sin^2 \theta, \] (9)

and

\[ L_{4,\mu} = -\frac{1}{2e^2} \mu^2 \sin^2 F(r) \sin^2 \theta \left( g^{11} F''(r)^2 
+ g^{22} \sin^2 F(r) \right). \] (10)

The Schwarzschild metric depends on two arbitrary functions \( \nu(r) \) and \( \lambda(r) \), in such a way that

\[ g^{00} = e^{-2\nu(r)}, \quad g^{11} = -e^{-2\lambda(r)}, \]
\[ g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2 \sin^2 \theta}. \] (11)

Our starting point is the Einstein equation

\[ G^{\mu\nu} = -8\pi G T^{\mu\nu}. \] (12)

The matter energy-momentum tensor is given by

\[ T^{\mu\nu} = -g^{\mu\nu} L_m + 2 \frac{\partial L_m}{\partial g_{\mu\nu}}. \] (13)
For our variational approach, where different ansaetze for the radial profile \( F(r) \) will be used, only the \( T^{00} \) component is needed

\[
T^{00} = \frac{f_{\pi}^2}{2} \left( \frac{2 \sin^2 F(r)}{r^2} + \frac{F''(r)}{e^{2\lambda(r)}} \right) \\
+ \frac{1}{2e^2} \left( \frac{\sin^4 F(r)}{r^4} + \frac{\sin^2 F(r) F''(r)}{e^{2\lambda(r)} r^2} \right) \\
- \frac{f_{\pi}^2 \mu^2}{2} \sin^2 F(r) \sin^2 \theta \\
- \frac{\mu^2}{2e^2} \left( \frac{\sin^2 F(r) \sin^2 \theta F''(r)}{e^{2\lambda(r)}} \right) \\
+ \frac{\sin^4 F(r) \sin^2 \theta}{r^2}.
\] (14)

For our purposes, we will only consider the angular average over the unit sphere. Since we are following a variational approach, the reasons that justify the self gravitating skyrmion stability will be not affected by taking this angular average. In this way, we replace \( \sin^2(\theta) \to 2/3 \), obtaining an averaged energy-momentum tensor component \( \langle T^{00} \rangle \)

The first component of (12) reads

\[
r^2 G^0_0 \equiv e^{-2\lambda} (1 - 2r \lambda') - 1 = 8\pi G r^2 \langle T^{00}_0 \rangle,
\] (15)

where the right hand term, written in terms of the dimensionless variable \( x = e f_{\pi} r \), becomes

\[
8\pi G r^2 \langle T^{00}_0 \rangle = 4\pi G f_{\pi}^2 \left[ \sin^2 F(x) + \frac{\sin^4 F(x)}{2x^2} \\
- \frac{\mu^2}{2e^2 f_{\pi}^2} \sin^4 F(x) \frac{2}{3} \\
- \frac{\mu^2}{2e^2 f_{\pi}^2} x^2 \sin^2 F(x) \frac{2}{3} \\
+ \left( 1 - \frac{2\nu(x)}{x} \right) \left( \frac{x^2}{2} + \sin^2 F(x) \\
- \frac{\mu^2}{2e^2 f_{\pi}^2} x^2 \sin^2 F(x) \frac{2}{3} \right) F''(r) \right].
\] (16)

Following [1], equation (15) can be written as

\[
- \frac{d}{dr} \left[ r (1 - e^{-2\lambda(r)}) \right] = -8\pi G r^2 \langle T^{00}_0 \rangle.
\] (17)
A formal integration of the previous equation leads us to the identification

$$r(1 - e^{-2\lambda(r)}) = 8\pi G \int_0^r \langle T_0^0 \rangle r'^2 \, dr'. \quad (18)$$

It is natural to define the mass inside the radius $r$ as

$$M(r) = 4\pi \int_0^r \langle T_0^0 \rangle r'^2 \, dr'. \quad (19)$$

Then, we can identify the left term in (15) as $r(1 - e^{-2\lambda(r)}) \Rightarrow 2GM(r)$.

In order to compare with [1], we introduce $R \equiv \sqrt{3/ef_\pi}$ as a relevant length scale parameter.

In terms of $R$, we define the dimensionless quantities

$$\frac{R_S}{R} \equiv \frac{8\pi^3}{3} G(Nf_\pi)^2, \quad \tilde{\mu} \equiv \mu R = \frac{\sqrt{3\mu}}{ef_\pi}. \quad (20)$$

In what follows we will use the dimensionless variable $\tilde{M}(x) \equiv ef_\pi GM(x)$. In this way, we find

$$\frac{d\tilde{M}(x)}{dx} = \left[1 - \frac{2\tilde{M}(x)}{x}\right] p(x) + q(x), \quad (21)$$

where we have introduced the functions $p(x)$ and $q(x)$ according to

$$p(x) = \frac{1}{4} \frac{R_S}{R} \left(\frac{F'(x)}{\pi N}\right)^2 \left(6\sin^2 F(x) + 3x^2 - \frac{2\tilde{\mu}^2 x^2 \sin^2 F(x)}{3}\right), \quad (22)$$

and

$$q(x) = \left(\frac{R_S}{R}\right) \frac{\sin^2 F(x)}{4\pi^2 N^2} \left(6x^2 + 3\sin^2 F(x) - \frac{2\tilde{\mu}^2 x^4}{3} - \frac{2\tilde{\mu}^2 x^2 \sin^2 F(x)}{3}\right) \quad (23)$$

Different ansaetze may be used for the radial profile, in order to proceed with the variational approach, fulfilling the boundary conditions $F(0) = N\pi$ and $F(x \to \infty) = 0$. Nevertheless, the results seem to be more or less independent of the details of the parametrization.
employed. A simple and natural profile emerges from the discussion of instanton holonomies.

\[ F(x) = N\pi \left[ 1 - \frac{x}{\sqrt{x^2 + X^2}} \right]. \quad (24) \]

This profile is also valid for curved spaces. Actually the validity of this ansatz has been proved in \[9\]. Other approaches, as for example, exponential decaying profiles \( F(x) = N\pi \exp(-x/X) \) have been used. The variational procedure leads us to the optimal value of the free parameter \( X \) in each case.

For our estimates, we will employ the first ansatz. The main reason for this, is a better numerical convergence of the integrals which give us the mass of the Skyrmion. We would like to remark that this is possible when gravity effects are taken into account. In our previous works on the stability of hadrons described as Skyrmions and their excitations, in the presence of \( \mu \), this parametrization was only valid for small values of \( \mu \). In the present case, we do not have such instability problems when going to higher values of \( \mu \). As we claimed in the introduction, gravity compensates the instability of the non gravitating Skyrmion.

In the large N limit, due to the rapid oscillating factor, we may approximate \( \sin^2 F(x) \approx 1/2 \). On the other side, since \( q(x) \propto O(1/N^2) \) and \( p(x) \propto O(1) \), we can neglect \( q(x) \), getting

\[ \frac{d\tilde{M}(x)}{dx} = \left[ 1 - \frac{2\tilde{M}(x)}{x} \right] \tilde{p}, \quad (25) \]

\[ \tilde{p} = \frac{3}{4} \frac{R_S}{R} \left( \frac{\dot{F}(x)}{\pi N} \right)^2 \left( 1 + x^2 - \frac{\mu^2}{3} x^2 \right). \quad (26) \]

Gledenning et al \[1\] found stable solutions for the Skyrmion only for values of \( R_S/R \lesssim 0.26 \). This means that for bigger values of \( R_S/R \), the mass of the Skyrmion does not have a minimum with respect to \( X \).

In our case, for each value of \( R_S/R \), we are able to find a stabilizing value of \( \mu \) (\( \mu_S \)), such that the mass of the Skyrmion has a minimum as function of \( X \). The behavior of \( \mu_S \) as function of \( R_S/R \) is shown in figure \[1\]. Notice that \( \mu_S \) asymptotically goes to a certain critical value \( \mu_c \approx 0.3/R \) (80 MeV), where the size of the Skyrmion diverges. This is shown in figure \[2\], where the divergent behavior of the parameter \( X \) is plotted as function of
\( \mu \), signalizing also a divergent behavior for the radial extension of the profile, since bigger values of \( X \) imply a broader radial profile.

In the same way, simultaneously with these effects, it turns out that \( \tilde{M} = e f_\pi M \) diminishes as function of \( \mu \), vanishing at the critical value \( \mu_c \). This is shown in figure (3). It is interesting to remark that the same kind of behavior for the mass of the Skyrmion was found in the non gravitating scenario discussed in our previous work. This means that the phase transition induced by the chemical potential is triggered by the strong dynamics.

Finally, we would like to address the problem of stability of the large N Skyrmion into particle emission. The condition for that is \( M(N) < N M(1) \). This means that

\[
\frac{d \tilde{M}(N)}{d (R_S/R)} < \frac{A}{N}.
\]

(27)

From figure (4) we see that the slope of \( \tilde{M} \) as function of \( R_S/R \) diminishes when \( \mu \) increases. For \( \tilde{\mu} \rightarrow \tilde{\mu}_c \), the slope vanishes. This means that for any value of \( R_S/R \) we can always find a value for \( \mu \), such that the condition (27) is fulfilled, implying the stability against decay.

In this letter we have discussed stability conditions for self gravitating large N \( SU(2) \) skyrmions in the presence of isospin chemical potential. Our main results show that this chemical potential plays an important role stabilizing the solution, extending the range of validity for the solutions as function of \( R_S/R \). We showed that the skyrmion mass vanishes at the same critical chemical potential where the size of the Skyrmion diverges. Finally, this scenario allows the existence of stable large N Skyrmions against decays into single Skyrmions.
FIG. 1: The behavior of $\tilde{\mu}_S = \mu_S R$ is shown as function of $R_s/R$. It goes asymptotically to the critical value $\tilde{\mu}_s \to \tilde{\mu}_c$.

FIG. 2: This figure shows the evolution of the variational parameter $X$ that minimizes the mass as function of the chemical potential. It diverges when $\tilde{\mu} \to \tilde{\mu}_c$. 
FIG. 3: Here we show the Skyrmion mass evolution as function of $\tilde{\mu}$. The mass vanishes when $\tilde{\mu} \to \tilde{\mu}_c$.

FIG. 4: Here we show the Skyrmion mass dependence as function of $R_S/R$ for three different values of $\tilde{\mu}$: continuous line $\tilde{\mu} = 2.5$; dashed line $\tilde{\mu} = 2.7$; dash-dotted line $\tilde{\mu} = 2.85$. 
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[1] N. K. Glendenning, T. Kodama, F. R. Klinkhamer, Phys. Rev. D38 (1988) 3226; T. Ioannidou, B. Kleihaus, J. Kunz, Phys. Lett. B 635 (2006) 161.
[2] P. Bizon, T. Chmaj, Phys. Lett. B297 (1992) 55.
[3] S. Droz, M. Heusler, N. Straumann, Phys. Lett. B 268 (1991) 371; M. Heusler, S. Droz, N. Straumann, Phys. Lett. B285 (1992) 21.
[4] M. Loewe, S. Mendizabal, J.C. Rojas, Phys. Lett. B
[5] H. A. Weldon, Phys.Rev.D26 (1982) 1394.
[6] A. Actor, Phys.Lett.B157 (1985) 53. 632 (2006) 512; ibid, Phys. Lett. B 638 (2006) 464.
[7] M.F. Atiyah and N.S. Manton, Phys.Lett. B 222 (1989) 438.
[8] Michael Atiyah, Paul Sutcliffe, Phys. Lett. B 605 (2005) 106.
[9] M. Eto, M. Nitta, K. Ohashi, D. Tong, Phys. Rev. Lett.95 (2005) 252003.