DARBOUX TRANSFORMATIONS FROM REDUCTIONS
OF THE KP HIERARCHY

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ABSTRACT
The use of effective Darboux transformations for general classes Lax pairs is discussed. The general construction of “binary” Darboux transformations preserving certain properties of the operator, such as self-adjointness, is given. The classes of Darboux transformations found include the multicomponent BKP and CKP reductions of the KP hierarchy.

1. Introduction
Darboux transformations define a mapping between the solutions of a linear differential equations and a similar equation containing different coefficients. Since integrable nonlinear evolution equations frequently arise as the compatibility condition for a pair for such equations, Darboux transformations may be used to construct families of exact solutions of the nonlinear equations. Typically these are multi-soliton solutions. A good introduction to this topic, including the following example, is given in the monograph by Matveev and Salle\(^1\).

As an example which motivates the work to be presented, consider the Lax pair\(^\ast\)

\[
L = i\partial_y + \partial^2 + u, \quad M = \partial_t + 4\partial^3 + 3\partial u + 3u\partial + 3i\partial^{-1}(u_y).
\]

for the variant of the Kadomtsev-Petviashvili equation known as KPI

\[
(u_u + 6uu_x + u_{xxx})_x - 3u_{yy} = 0,
\]

in which \(u\) is a real variable. This means that \([L, M] = 0\) if and only if \(u\) satisfies KPI.

For all non-zero \(\theta\) such that \(L(\theta) = M(\theta) = 0\), a Darboux transformation is defined by the operator \(G = \theta\partial\theta^{-1}\) in the sense that

\[
L(\psi) = M(\psi) = 0 \implies \hat{L}(G(\psi)) = \hat{M}(G(\psi)) = 0,
\]

\(\ast\)Here and below \(\partial = \partial/\partial x\) and \(\partial_y = \partial/\partial y\) and so on.
where \( \tilde{L} \) and \( \tilde{M} \) are the operators obtained from \( L \) and \( M \) by replacing \( u \) by \( \tilde{u} = u + 2(\log \theta)_{xx} \). This result is readily proved by observing that

\[
\tilde{L}G = GL \quad \text{and} \quad \tilde{M}G = GM,
\]

i.e.

\[
\tilde{L} = GLG^{-1} \quad \text{and} \quad \tilde{M} = GMG^{-1}.
\]

In this way the Darboux transformation manifests itself as a (differential) gauge transformation. It also follows that

\[
[L, M] = 0 \implies [\tilde{L}, \tilde{M}] = 0,
\]

i.e. \( \tilde{u} \) satisfies KPI whenever \( u \) does. Hence the Darboux transformation induces an auto-Bäcklund transformation.

There is a problem with this transformation however. For almost all \( u \), since \( \theta \) is the solution of a complex equation, \( \tilde{u} \) is not real and so we do not obtain solutions of KPI. At the root of the problem is the fact that, while \( L \) and \( M \) are self-adjoint, \( \tilde{L} \) and \( \tilde{M} \) are not. In order to overcome this problem one may use a “binary” transformation (to be defined in the next section) which does preserve the self-adjointness of \( L \) and \( M \).

This paper is concerned with the use of binary transformations to preserve the structure of two classes of operators with matrix coefficients and arbitrary order.

2. The structure of the binary transformation

For an (matrix) operator \( L \), let \( S = \{ \theta, \text{non-singular} : L(\theta) = 0 \} \) (and define \( \tilde{S}, S^\dagger \) for operators \( \tilde{L}, L^\dagger \) etc.). A (formally invertible) gauge transformation \( G_\theta \), for \( \theta \in S \), defines a mapping

\[
G_\theta: S \to \tilde{S}, \quad \text{where} \quad \tilde{L} = G_\theta LG_\theta^{-1}.
\]

Consider also the (formal) adjoint operator \( G_\theta^\dagger \). (Taking the formal adjoint is, as usual, the linear operation defined by

\[
(a\partial^i)^\dagger = (-1)^i \partial^i a^\dagger,
\]

for a matrix \( a \), where \( a^\dagger \) denotes the Hermitian conjugate of \( a \).) Since \( \tilde{L}^\dagger = G_{\theta^\dagger}^{-1} L^\dagger G_{\theta^\dagger}^\dagger \), we have

\[
G_{\theta^\dagger}^\dagger: \tilde{S}^\dagger \to S^\dagger.
\]

By determining the kernel of \( G_{\theta^\dagger} \) we obtain some nontrivial solution in \( \tilde{S}^\dagger \). Typically, we may identify this subset in terms of \( \theta \) and denote a member as \( i(\theta) \). For example, in the classical case when \( G_{\theta} = \theta \partial \theta^{-1} \) and \( G_{\theta}^\dagger = -\theta^{-1} \partial \theta^{-1} \), we find that

\[
G_{\theta}^\dagger(\rho) = 0 \iff \rho = \theta^{-1} c,
\]
where $c$ is independent of $x$.

We represent this situation in the diagram below.

To describe the general form of the binary transformation we consider operators $L$, $\tilde{L}$ and $\hat{L}$ and the corresponding sets of non-singular solutions matrices $S$, $\tilde{S}$ and $\hat{S}$. Let $\theta \in S$ and $\hat{\theta} \in \hat{S}$ be such that $G_{\theta}: S \rightarrow \tilde{S}$ and $G_{\hat{\theta}}: \hat{S} \rightarrow \tilde{S}$. Then we get the mapping

$$G_{\hat{\theta}}^{-1}G_{\theta}: S \rightarrow \hat{S}.$$  

The difficulty with this definition of a transformation is that to define it we need one of the solutions we are trying to determine, namely $\hat{\theta}$. To overcome this, we use the fact that there corresponds to $\hat{\theta} \in \hat{S}$ a solution $i(\hat{\theta}) \in \hat{S}^\dagger$ and then use the mapping $G_{\hat{\theta}}^{-1}: S \rightarrow \hat{S}^\dagger$ to obtain $\theta = i^{-1}(G_{\hat{\theta}}^{-1}(\phi))$ for any $\phi \in S^\dagger$. This is shown in the diagram below.

In this way we obtain the definition of a general binary transformation.

**Definition**  Consider an operator $L$ and gauge operator $G_\theta$, where $\theta \in S$, such that $G_\theta^\dagger(i(\theta)) = 0$. For each $\phi \in S^\dagger$, define

$$G_{\theta,\phi} = G_{\hat{\theta}}^{-1}G_\theta,$$

where $\hat{\theta} = i^{-1}(G_{\hat{\theta}}^{-1}(\phi))$. Then

$$G_{\theta,\phi}: S \rightarrow \hat{S},$$

where $\hat{L} = G_{\theta,\phi}LG_{\theta,\phi}^{-1}$, is called a binary transformation.

In the next section we will consider two concrete examples of such a binary transformation.

Now suppose that the operator $L$ has a constraint of the form

$$L^\dagger R^\dagger = RL,$$
where \( R \) is in some formally invertible (matrix differential) operator\(^\dagger\). We wish to find binary transformations that preserve this constraint. That is—using the notation of the above definition—we want \( \hat{L} \) to satisfy the constraint whenever \( L \) does.

Examples for the choice of \( R \) include

- \( R = I \). \( L \) is self-adjoint. An example of the application of this is quoted in the introduction.
- \( R = iI \). \( L \) is skew-adjoint. This corresponds to the CKP reduction of the KP hierarchy\(^2\) and the reduction of the Kuperschmidt “\( k = 0 \)” non-standard hierarchy\(^3, 4\).
- \( R = \partial \). This corresponds to the BKP\(^5\) or the Kuperschmidt “\( k = 1 \)” reduction.

The binary transformation we will discuss in the next section will preserve generalizations of these three reductions.

Let the gauge transformation \( G \), such that \( \hat{L} = GLG^{-1} \), preserve the constraint \( L^\dagger = RLR^\dagger \). Then

\[
\hat{L}^\dagger - R\hat{L}R^\dagger = 0
\]

which means that

\[
G^\dagger R^\dagger L^\dagger G^\dagger - RGLG^{-1} R^\dagger \dagger = G^\dagger RLR^\dagger G^\dagger - RGLG^{-1} R^\dagger = 0.
\]

This leads to the single condition \( RG = G^\dagger R \).

Note that the relation between \( L \) and its adjoint imposes a relationship between the solution sets \( S \) and \( S^\dagger \). In particular, for each \( \theta \in S \), \( R^\dagger(\theta) \in S^\dagger \). Hence, in the case of a binary transformation \( G = G_{\theta, \phi} \), we may make the choice \( \phi = R^\dagger(\theta) \).

3. Darboux transformations for general operators

In this section we describe two classes of Darboux transformation for general classes of matrix differential operators of arbitrary order. The first is originally due to Matveev\(^6\) and has also been considered recently by Oevel\(^7\). We will present a very simple proof of this result. The second was found by Oevel & Rogers\(^8\) in the case of scalar operators in the context of Sato theory. We will derive a more general version here.

In both cases, the results are remarkably general. There is however a serious drawback. There is, in this general case, absolutely no guarantee that the transformed operator we have the same “form” as the original and so only in special cases does one get a transformation that induces an auto-Bäcklund transformation.

\(^\dagger\)It is tempting to look for a constraint of the form \( L^\dagger S = RL \) but this in fact corresponds to two constraints since on taking adjoints \( L^\dagger R^\dagger = S^\dagger L \).
First, consider

\[ L = \partial t + \sum_{i=0}^{n} u_i \partial^i \quad \text{and} \quad \tilde{L} = \partial t + \sum_{i=0}^{n} \tilde{u}_i \partial^i, \]

where \( u_i \) and \( \tilde{u}_i \) are \( N \times N \) (not necessarily constant) matrices. Let the operator \( G \) be such that

\[ \tilde{L} = GLG^{-1} = L + [G, L]G^{-1}. \]

Hence \( G \) must satisfy

\[ [G, L]G^{-1} = \sum_{i=0}^{n} (\tilde{u}_i - u_i) \partial^i. \]

Taking \( G = \theta \partial \theta^{-1} \), where \( \theta \) is a non-singular \( N \times N \) matrix, and hence \( G^{-1} = \theta \partial^{-1} \theta^{-1} \), we get

\[ [G, L]G^{-1} = [G, L] \theta \partial^{-1} \theta^{-1} = \sum_{i=0}^{n} a_i \partial^{i-1} \theta^{-1}, \]

for some matrices \( a_i \). For \( i = 1, \ldots, n, a_i = \tilde{u}_{i-1} - u_{i-1} \) and in order that \( G \) define a Darboux transformation we must have

\[ a_0 = 0. \]

This condition gives \( [G, L](\theta) = 0 \) i.e. \( G(L(\theta)) = 0 \) since \( G(\theta) = 0 \). Hence we only need require that \( L(\theta) = \theta C \), for some \( x \)-independent matrix \( C \). Note that if \( L(\theta) = 0 \) then for \( \theta' = \theta \exp(\partial^{-1}_t(C)) \), \( L(\theta') = \theta' C \). Also, \( G_{\theta'} = G_\theta \) and so we may suppose, without loss of generality, that \( C = 0^k \). Thus we find that \( \theta \in S \).

The second case we consider is

\[ L = \partial t + \sum_{i=1}^{n} u_i \partial^i \quad \text{and} \quad \tilde{L} = \partial t + \sum_{i=1}^{n} \tilde{u}_i \partial^i, \]

where \( u_i \) and \( \tilde{u}_i \) are again \( N \times N \) matrices. Note that the multiplicative term in \( L \) and \( \tilde{L} \) is omitted.

As in the first case, a gauge operator \( G \) must satisfy

\[ [G, L]G^{-1} = \sum_{i=1}^{n} \tilde{u}_i - u_i. \]

There are now two simple choices. First, let \( G = G^{(1)}(\theta) = \theta^{-1} \), an (invertible) \( N \times N \) matrix. Then

\[ [G, L]G^{-1} = \sum_{i=0}^{n} a_i \partial^i, \]

\[ ^4 \text{Note that if } L \text{ is an ordinary differential operator, then taking } C \neq 0 \text{ is a genuine generalization. For example, this is exploited in the classical "discrete eigenvalue adding" Darboux transformation for the time-independent Schrödinger operator.} \]
and so $a_0 = [G, L](\theta) = G(L(\theta)) = 0$, i.e. $\theta \in S$.

Second, let $G = G^{(2)}_\rho = \rho_\sigma^{-1}\partial$, where $\rho_\sigma$ is an invertible $N \times N$ matrix. Now

$$[G, L]G^{-1} = [G, L]\partial^{-1}\rho_\sigma = \sum_{i=1}^{n+1} a_i\partial^{-1}\rho_\sigma,$$

and we must have $a_1 = 0$, i.e. $[G, L](\partial^{-1}(\rho_\sigma)) = G(L(\rho)) = 0$. Thus $L(\rho) = C$, an $x$-independent matrix. Again, we may take $C = 0$ without loss of generality and so $\rho \in S$.

As in the scalar case, it is the composition of the two gauge transformations which is of most interest, and we take $G_\theta = G^{(2)}_{\theta}G^{(1)}_{\theta} = (\theta^{-1})_x\partial\theta^{-1}$.

4. Binary transformations and reductions

To determine the binary transformations $G_{\theta,\phi}$ corresponding to the two Darboux transformations found above we must determine two additional things: the mapping $i: \hat{S} \to \hat{S}^\dagger$ and then the element $\hat{\theta} \in \hat{S}$ in terms of $\theta$ and $\phi$.

First consider $L = \partial_t + \sum_{i=0}^n u_i\partial^i$, $G_\theta = \theta\partial\theta^{-1}$. Here the condition $G_\theta^\dagger(i(\theta)) = -\theta^{-1}\partial(\theta^\dagger i(\theta)) = 0$ is satisfied by the choice $i(\theta) = \theta^{-1}$. Further,

$$\hat{\theta} = \left(G_\theta^\dagger(\phi)\right)^{\dagger^{-1}} = -\left(\theta^{\dagger^{-1}}\partial^{-1}(\theta^\dagger\phi)\right)^{\dagger^{-1}} = -\theta\Omega^{-1},$$

where $\Omega = \partial^{-1}(\phi^\dagger\theta)$.

It may be shown that for all operators $L = \sum_{i=0}^n u_i\partial^i$, $\Omega$ is exact in the sense that $d\Omega = \phi^\dagger\theta dx + A(u_1, \ldots, u_n, \theta, \phi)dt$.

In this case the binary transformation is

$$G_{\theta,\phi} = G_\theta^{-1}G_\theta = \theta\Omega^{-1}\partial^{-1}\Omega\partial\theta^{-1} = \theta\Omega^{-1}\partial^{-1}(\partial\Omega - \Omega_\sigma)\theta^{-1} = 1 - \theta\Omega^{-1}\partial^{-1}\phi^\dagger.$$
Theorem 1 Let the matrix operator \( L = \sum_{i=0}^{n} u_i \partial^i \) satisfy the constraint
\[
L^\dagger A = AL,
\]
where \( A \) is an Hermitian or skew-Hermitian matrix. Then the binary transformation
\[
G = 1 - \theta \Omega^{-1} \partial^{-1} \partial^\dagger A
\]
where \( \Omega = \partial^{-1}(\partial^\dagger A \theta) \), preserves the above constraint, i.e. \( \hat{L} = GLG^{-1} \) satisfies
\[
\hat{L}^\dagger A = A \hat{L}.
\]

For the second case, \( L = \sum_{i=1}^{n} u_i \partial^i \), \( G_\theta = (\theta^{-1})_x \partial^{-1} \partial \theta^{-1} \) and hence \( i(\theta) = (\theta^{\dagger -1})_x \).

To determine the binary transformation it is notationally convenient to write an element of \( S^\dagger \) as \( \phi_x \) rather than \( \phi \) as we did above. Also, it is necessary to introduce two integrals
\[
\Omega = \partial^{-1}(\phi^\dagger \theta_x) \quad \text{and} \quad \Omega' = \partial^{-1}(\phi_x^\dagger \theta),
\]
where
\[
\Omega + \Omega' = \phi^\dagger \theta.
\]

Now
\[
i(\hat{\theta}) = (\hat{\theta}^{\dagger -1})_x = G_{\theta}^{\dagger -1}(\phi_x)
\]
\[
= -(\theta^{\dagger -1})_x \partial^{-1}(\theta^\dagger \phi_x)
\]
and so
\[
(\hat{\theta}^{-1})_x = -\partial^{-1}(\phi_x^\dagger \theta)(\theta^{-1})_x = -\Omega'/(\theta^{-1})_x.
\]

Integrating by parts and taking inverses, we get
\[
\hat{\theta} = \left(-\Omega' \theta^{-1} + \partial^{-1}(\Omega' \theta^{-1})\right)^{-1}
\]
\[
= \left(\phi^\dagger - \Omega' \theta^{-1}\right)^{-1}
\]
\[
= \theta \Omega^{-1}.
\]

We may now obtain
\[
G_{\theta,\phi_x} = \hat{\theta} \partial^{-1}(\hat{\theta}^{-1})_x \theta^{-1} \partial \theta^{-1}
\]
\[
= -\theta \Omega^{-1} \partial^{-1} \Omega' \partial \theta^{-1}
\]
\[
= 1 - \theta \Omega^{-1} \partial^{-1} \phi^\dagger \partial,
\]
and in a similar way
\[
G_{\theta,\phi_x}^{\dagger -1} = 1 - \partial \phi \Omega'^{-1} \partial^{-1} \theta^\dagger.
\]
Suppose that $L$ satisfies the constraint $L^\dagger R^\dagger = R L$ where $R = A \partial$, $A$ a matrix, and choose $\phi_x = R^\dagger (\theta)_x = -(A^\dagger \theta)_x$, i.e. $\phi = -A^\dagger \theta$. The condition $R G_{\theta, \phi_x} = G_{\theta, \phi_x}^\dagger R$ is

$$A \partial - A \partial \Omega^{-1} \partial^{-1} \theta^\dagger A \partial = A \partial - \partial A^\dagger \theta \Omega \partial^{-1} \partial^{-1} \theta^\dagger A \partial \iff A^\dagger = \pm A \text{ and } A_x = 0.$$ 

With these conditions on $A$, $\Omega = \pm \Omega^\dagger$.

This establishes a second theorem.

**Theorem 2** Let the matrix operator $L = \sum_{i=1}^n u_i \partial^i$ satisfy the constraint

$$L^\dagger A \partial + A \partial L = 0,$$

where $A$ is an $x$-independent Hermitian or skew-Hermitian matrix. Then the binary transformation

$$G = 1 - \theta \Omega^{-1} \partial^{-1} \theta^\dagger A \partial$$

where $^8 \Omega = \partial^{-1}(\theta^\dagger A \theta_x)$, preserves the above constraint, i.e. $\hat{L} = G L G^{-1}$ satisfies $\hat{L}^\dagger A \partial + A \partial \hat{L} = 0$.

5. **Examples**

5.1. **Davey-Stewartson I**

This system has Lax pair

$$L = \partial_y + \alpha \partial + Q, \quad M = i \partial_t + \alpha \partial^2 + \frac{1}{2}(Q \partial + \partial Q + \alpha Q_y) + D,$$

where $Q = \begin{pmatrix} 0 & u \\ 0 & \bar{u} \end{pmatrix}$ and $D = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ is real.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon \end{pmatrix}$. Then

$$L^\dagger (iA)^\dagger = (iA)L, \quad M^\dagger A^\dagger = AL.$$

Hence we may use Theorem 1 (with $A = I$) to obtain a binary transformation. This transformation has been used to obtain a wide class of solutions including dromions.$^9$

5.2. **Sawada-Kotera equation**

The equation is a reduction of the BKP equation and so has Lax pair admitting the BKP reduction:

$$L = (\partial^2 + 3u) \partial, \quad M = \partial_t + (9 \partial^3 - 15 \partial u \partial + 30(\partial^2 u + u \partial^2) + 15u^2) \partial,$$

$^9$For a better notation we have replaced $\Omega$ with $-\Omega$ in the statement of the theorem.
where

\[ L^\dagger \partial + \partial L = M^\dagger \partial + \partial M = 0. \]

We may use Theorem 2 (with \( A = I \)) to obtain the binary transformation. Note that
this is given by \( G = \theta^{-1} \partial^{-1} \theta^2 \partial \theta^{-1} \) and coincides the with the well-known “Darboux”
transformation\(^{10, 11}\).

5.3. Modified Novikov-Veselov equation

This system\(^{12, 13, 14}\) belongs to the two component BKP hierarchy and has a Lax pair

\[ L = \partial_y + S \partial, \quad M = \partial_t + (S \partial^2 + T \partial + \partial T + U) \partial, \]

where \( S = \begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix} \), and \( T \) and \( U \) are give skew-symmetric and symmetric
real matrices respectively. Again

\[ L^\dagger \partial + \partial L = M^\dagger \partial + \partial M = 0 \]

and Theorem 2 gives the binary transformation.

6. Conclusions

We have discussed the general construction of binary transformations from Darboux transformations. In particular we have carried this out for two classes of operators. More importantly, we have shown that the well studied reductions of these classes (the multi-component BKP and CKP reductions) are among those that the binary transformations preserve.

Iteration of the binary transformation is, of course, possible and leads to closed-form expressions for solutions—of the linear problems and for the integrable systems that are their compatibility conditions—in terms of (multi-component) Grammian determinants. In the case of the reduction described in Theorem 2, one may see how these Grammians are transformed into Pfaffians by the reduction process. These features will be discussed in more details elsewhere\(^{15}\).

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