CARTAN COVERS AND DOUBLING BERNSTEIN TYPE INEQUALITIES ON ANALYTIC SUBSETS OF $\mathbb{C}^2$

MICHAEL GOLDSTEIN, WILHELM SCHLAG, MIRCEA VODA

Abstract. We prove a version of the doubling Bernstein inequalities for the trace of an analytic function of two variables on an analytic subset of $\mathbb{C}^2$. The estimate applies to the whole analytic set in question including its singular points. The proof relies on a version of the Cartan estimate for maps in $\mathbb{C}^2$ which we establish in this work.

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1. Introduction

In a series of papers [FN93],[FN94],[FN96], Fefferman and Narasimhan investigated the local behavior of a polynomial $f$ of $N$ real or complex variables, restricted to a given $n$-dimensional algebraic variety $X$. Conceptually, the problem is to quantify to what extent the local behavior of the trace of $f$ on $X$ deteriorates relative to an $N$-dimensional ball. Of particular interest here is to determine the dependence of quantitative estimates on the degree of the polynomials. Fefferman and Narasimhan chose the classical Bernstein inequalities for polynomials of several variables to measure the distortion of a polynomial restricted to an algebraic variety.

The authors’ interest in this particular problem arose as part of their work on the Chulaevsky-Sinai conjecture. In their pioneering paper [CS89], Chulaevsky and Sinai analyze the spectrum of a discrete Schrödinger operator on $\mathbb{Z}$ with a quasi-periodic potential given by evaluating a generic smooth function on $\mathbb{T}^2$ along the orbit of an ergodic shift. In [GSV16b] (building on work from [GSV16a]) the authors found that some “generic versions” of these restricted Bernstein estimates play a crucial role in addressing this conjecture.

The second author was partially supported by the NSF, DMS-1500696. The first author thanks the University of Chicago for its hospitality during the months of July and August of 2016. The authors are grateful to János Kollár and Miheea Popa for helpful discussions on Bézout’s theorem.
There are two major differences between the current paper and [FN96]: (i) we obtained estimates at singular points and the estimates at regular points don’t depend on the distance to the singular points (ii) we allow analytic functions and analytic sets in place of polynomials and algebraic varieties.

As for (i), Fefferman and Narasimhan had considered compact subsets of algebraic varieties away from the singular points. For polynomials and algebraic varieties, Roytwarf and Yomdin [RY97] extended their Bernstein estimates to be independent of the distance to the singular points. However, the aforementioned spectral analysis forces us to consider analytic functions and sets, rather than algebraic ones. Our estimates for analytic functions are not as sharp as for polynomials.

Regarding (ii), we note that Coman-Poletsky [CP07] (for \( n = 1 \)) and Brudnyi [Bru08] (for all \( n \geq 1 \)) studied Bernstein estimates (amongst other local properties) for the restriction of analytic functions of \( n + 1 \) variables to the graph of an analytic function of \( n \) variables. Both these papers require a certain transversality condition of the zeros sets of the functions in question. We shall also impose a condition of this nature in our approach.

We proceed to discuss the main results of the paper. First we need to introduce some notation related to Cartan sets and to Bernstein exponents. The Cartan sets will appear in our transversality condition to allow the application of the Cartan-type estimate established in Theorem C.

**Definition 1.1.** (1) Let \( H \geq 0, K \geq 1 \). For \( \mathcal{B} \subset \mathbb{C}^2 \) we say that \( \mathcal{B} \in \text{Car}_{2,0}(H,K) \) if

\[
\mathcal{B} \subset \bigcup_{j=1}^{j_0} B(v_j, r)
\]

with \( r = e^{-H} \) and \( j_0 \leq K \).

(2) Let \( f \) be analytic on the ball \( B(\mathfrak{u}_0, R) \subset \mathbb{C}^2, S \subset \mathbb{C}^2 \), and \( \mu \in (0, 1) \). Define

\[
M_f(\mathfrak{u}_0, R) = \sup_{B(\mathfrak{u}_0, R)} \log |f|,
M_f(S, \mathfrak{u}_0, R) = \sup_{B(\mathfrak{u}_0, R) \cap S} \log |f|,
B_f(\mu; \mathfrak{u}_0, R) = M_f(\mathfrak{u}_0, R) - M_f(\mathfrak{u}_0, \mu R),
B_f(\mu; S, \mathfrak{u}_0, R) = M_f(S, \mathfrak{u}_0, R) - M_f(S, \mathfrak{u}_0, \mu R).
\]

We call \( B_f(\mu; \mathfrak{u}_0, R), B_f(\mu; S, \mathfrak{u}_0, R) \) Bernstein exponents. We make the natural convention that if the function \( f \) vanishes identically, its Bernstein exponents are zero.

(3) Let \( f \) be analytic on \( B(0, 1) \), \( \mu \in (0, 1) \). We define

\[
B_f(\mu) = \sup_{\mathfrak{u}_0 \in B(0,1/4), 0 < R \leq 1/4} B_f(\mu; \mathfrak{u}_0, R).
\]

(4) Given an analytic function \( f \) on a disk \( \mathcal{D}(z_0, R) \subset \mathbb{C} \), the quantities \( M_f(z_0, R) \) and \( B_f(\mu; z_0, R) \) are defined analogously to the above.

The classical Bernstein doubling inequality for a univariate polynomial \( f \) can be expressed using the above notation as

\[
B_f(\mu; z_0, R) \leq (\log \mu^{-1}) \times \deg f,
\]

where \( \mu \in (0, 1), z_0 \in \mathbb{C}, R > 0 \).

Throughout we will impose the following transversality condition. Suppose the functions \( f_1, f_2 \) are analytic in the ball \( B(0,1) \subset \mathbb{C}^2 \), and are normalized so that...
$M_f(0,1) \leq 0$, $i = 1,2$. We let $F = (f_1, f_2)$ and we define

$$N_F(\varepsilon) := \{w \in B(0,1) : |F(w)| < \varepsilon\}.$$ 

We require that

$$(1.1) \quad N_F(\exp(-H_0)) \cap B(0,1/2) \in \text{Car}_{2,0}(H_1, K_1), \quad \log K_1 \ll H_1,$$

for some $H_0 \gg H_1 \gg B_0 := \max_i B_{f_i}(1/4)$.

Remark 1.2. A priori it might appear that $K_1$ can be exponentially large, i.e., $\exp(cH_0)$ for some small $c > 0$. However, a simple argument, presented in Lemma 6.1, shows that we always have the polynomial bound $K_1 \leq H_0^C$, where $C$ is some absolute constant.

Let $\mathcal{Z} = \{w \in B(0,1) : f_2(w) = 0\}$. It is well known that there exists a discrete set of singular points $\text{sng} \mathcal{Z}$ (relative to $B(0,1)$) such that the set of regular points $\text{reg} \mathcal{Z} := \mathcal{Z} \setminus \text{sng} \mathcal{Z}$ is a one dimensional complex manifold (see, for example, [Chi89]).

**Theorem A.** Assume the transversality condition holds and let $\mathcal{Z}$ be as above. Let $C_0 = \log(K_1 B_{f_1}^2 H_0^2)$. Then the following statements hold.

1. For any $z_0 \in B(0,1/8) \cap \mathcal{Z}$ and $0 < R \leq 1/4$,

$$B_{f_1}(1/4; z_0, R) \preceq \max(\log R^{-1}, C_0) B_0^2 H_0.$$

2. There exists an atlas of $\text{reg} \mathcal{Z}$ with charts defined on $\mathcal{D}(0,1)$ such that for any chart $\phi$ satisfying $\phi(\mathcal{D}(0,1)) \cap B(0,1/8) \neq \emptyset$ and any $\mathcal{D}(z_0, R) \subset \mathcal{D}(0,1)$, we have

$$B_{f_1 \circ \phi}(1/4; z_0, R) \leq C(f_2) C_0 B_0^2 H_0.$$

Remark 1.3. The log $R^{-1}$ factor from part (1) of Theorem A is needed because the estimate covers singular points. See Example 7.2.

In Theorem B we obtain a sharper version of part (2) of the previous theorem for the polynomial case. Such a result is also known from [RY97]. The work of Roytwarf and Yomdin relies on a classical inequality for the Taylor coefficients of $p$-valent functions due to Biernacki [Bie36]. In turn [Bie36] relies on a deeper growth bound for $p$-valent functions obtained by Cartwright [Car35] (see [Hay94] for a more detailed account of these issues). In Theorem B we show that in the context of algebraic curves the Bernstein estimates by Roytwarf and Yomdin follow from more elementary arguments in the spirit of the argument principle, without any reference to properties of $p$-valent functions. We also require basic properties of the harmonic conjugate and of course Bezout’s theorem (which is also needed in Roytwarf and Yomdin in order to estimate the valency). It seems that this approach can be developed for a general algebraic variety.

**Theorem B.** Assume that $f_1, f_2$ are polynomials. Let $\mathcal{Z}$ be as above. Then there exists an atlas of $\text{reg} \mathcal{Z}$ with charts defined on $\mathcal{D}(0,1)$ such that for any chart $\phi$ and any $\mathcal{D}(z_0, R) \subset \mathcal{D}(0,1)$, we have

$$B_{f_1 \circ \phi}(1/4; z_0, R) \leq C(f_2) \times \deg f_1.$$

For our application in [GSV16b] we use the Cartan estimate for maps in $\mathbb{C}^2$ which is Theorem C we state below. The proof of Theorem A relies on Theorem C. The Cartan estimate for an analytic function $f(w), w \in \mathbb{C}^2$ (see Lemma 2.2), basically says that if the set $\{|f| < \varepsilon_0\}$ is “not two-dimensional” then $\{|f| < \varepsilon\}$ is “one-dimensional” for any $\varepsilon \ll \varepsilon_0$. We prove an analogue statement for mappings. Let
$F : B(0, 1) \subset \mathbb{C}^2 \to \mathbb{C}^2$ be analytic. We show that if the set \( \{ |F| < \varepsilon_0 \} \) is "zero-dimensional", then \( \{ |F| < \varepsilon \} \) is "zero-dimensional" for any \( \varepsilon \ll \varepsilon_0 \). Of course, the quantitative details of the statement here are as important as the topological ones.

**Theorem C.** Assume the transversality condition holds. Then for any \( H \gg 1 \) we have

\[
N_F(\exp(-H B_0^2 H_0)) \cap B(0, 1/4) \in \text{Car}_{2,0}(H, K), \quad K \lesssim K_1 B_0^2 H_0^2.
\]

The proof of Theorem C proceeds in four steps: (a) apply the Weierstrass preparation theorem to the given analytic functions in one of the two coordinates (b) determine the resultant of the two polynomials obtained in the previous step (c) apply Cartan’s theorem in one variable so as to guarantee that this resultant is not too small off of a union of small disks in \( \mathbb{C} \), which in turn gives that at least one of the two analytic functions is not too small outside of thin cylinders in \( \mathbb{C}^2 \) (d) repeat the previous steps with respect to the other variable. The intersection of the two families of thin cylinders gives a Car\(_{2,0}\) set.

It would be interesting to extend this method to higher dimensions, i.e., to the construction of Car\(_{d,0}\)(\( H, K \)) sets with \( d \geq 3 \) – at least for polynomials in \( d \) variables. In principle, this appears possible but it seems to require the use of multivariate resultants, which are more delicate than the univariate ones. If Theorem C extends to \( d \geq 3 \), then one would obtain a Bernstein estimate as in Theorem A. As our applications do not require this extension, we do not pursue these matters here.

We conclude this introduction by providing some details of the aforementioned spectral theory applications. Consider a trigonometric polynomial of two variables

\[
V(z, w) = \sum_{|m|, |n| \leq k} c_{m,n} e(mz + nw),
\]

\( e(\zeta) := e^{2\pi i \zeta} \). To normalize the setting we consider the unit sphere in the space of the coefficients

\[
\mathcal{C}_1 = \{ (c_{m,n}) \in \mathbb{R}^{4k+2} : \sum_{m,n} |c_{m,n}|^2 = 1 \}.
\]

We use mes for the Lebesgue measure on the sphere. Take arbitrary \( \omega \in \mathbb{T}^2 \), \( \lambda \in \mathbb{R} \). Consider the determinant

\[
f_N(\nu) = \begin{vmatrix} \nu V(\nu) & -1 & 0 & \cdots & 0 \\ -1 & \nu V(\nu + \omega) & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & \cdots & 0 & -1 & \nu V(\nu + (N-1)\omega) \\ 0 & \cdots & \cdots & \cdots & \cdots & \nu V(\nu + (N-1)\omega) \end{vmatrix}
\]

For \( \nu \in \mathbb{R}^2 \), \( f_N(\nu) \) is the characteristic determinant of the Schrödinger operator with potential \( V(\nu + n\omega), \) \( n \in \mathbb{Z} \) on the interval \([0, N - 1]\) subject to Dirichlet boundary conditions. In [GSV16b] we establish the following results: Given arbitrary \( \varepsilon > 0 \), there exists a set \( \mathcal{C} \subset \mathbb{R}^{4k+2} \) with \( \text{mes}(\mathcal{C}_1 \setminus \mathcal{C}) < \varepsilon \) and \( \lambda_0 = \lambda_0(\varepsilon) \) depending only on \( \varepsilon \) such that for any \( V \) with \( (c_{m,n}) \in \mathcal{C}_1 \) and any \( |\lambda| \geq \lambda_0 \) there exists a set \( \Omega(V) \subset \mathbb{T}^2 \) with \( \text{mes}(\Omega(V)) < \varepsilon \) such that for any \( \omega \in \mathbb{T}^2 \setminus \Omega(V) \), any \( N \), and any \( \omega_0 \in \mathbb{T}^2 \) the functions \( f_N(\omega_0 + r_0 \omega) \) and \( f_N(\omega_0 + n \omega), \omega_0 = \omega_0 + n \omega, |n| > N, \nu \in B(0, 1), r_0 = \exp(-\log N)^A, A \gg 1 \) being an absolute constant, obey all conditions of Theorem A and Theorem C with \( B_0, H_0 \leq (\log N)^c, c \ll 1 \).
The exceptional sets in this result are not artificial. In fact, the theorem fails for some \((c_{m,n}) \in \mathcal{E}\). A similar fact is true for the exceptional frequencies.

2. Cartan’s Estimate

Recall the following definition from [GS08].

**Definition 2.1.** Let \(H \geq 0, K \geq 1\). For an arbitrary set \(\mathcal{B} \subset \mathbb{C}\) we say that \(\mathcal{B} \in \text{Car}_1(H, K)\) if \(\mathcal{B} \subset \bigcup_{j=1}^{j_0} \mathbb{D}(z_j, r_j)\) with \(j_0 \leq K\), and \(\sum_j r_j < e^{-H}\).

If \(d \geq 1\) is an integer and \(\mathcal{B} \subset \mathbb{C}^d\), then we define inductively that \(\mathcal{B} \in \text{Car}_d(H, K)\) if for any \(1 \leq j \leq d\) there exists \(\mathcal{B}_j \subset \mathbb{C}\), \(\mathcal{B}_j \in \text{Car}_1(H, K)\), so that \(\mathcal{B}_j^{(1)} \subset \text{Car}_{d-1}(H, K)\) for any \(z \in \mathbb{C} \setminus \mathcal{B}_j\), here \(\mathcal{B}_j^{(1)} = \{ (z_1, \ldots, z_d) \in \mathcal{B} : z_j = z \}\).

The above definition of Cartan sets is motivated by the following statement, known as Cartan estimate on the lower bound of an analytic function of several variables.

**Lemma 2.2 ([GS08, Lem. 2.15]).** Let \(\varphi(z_1, \ldots, z_d)\) be an analytic function defined in a polydisk \(\mathcal{D} = \prod_{j=1}^d \mathbb{D}(z_{j, 0}, 1)\), \(z_{j, 0} \in \mathbb{C}\). Let \(M \geq \sup_{z \in \mathcal{D}} |\varphi(z)|\), \(m \leq \log |\varphi(z_0)|\), \(z_0 = (z_{1, 0}, \ldots, z_{d, 0})\). Given \(H \geq 1\) there exists a set \(\mathcal{B} \subset \mathcal{D}\), \(\mathcal{B} \in \text{Car}_d(H^{1/d}, K)\), \(K = C_d H(M - m)\) such that
\[
\log |\varphi(z)| > M - C_d H(M - m)
\]
for any \(z \in \prod_{j=1}^d \mathbb{D}(z_{j, 0}, 1/6) \setminus \mathcal{B}\). Furthermore, when \(d = 1\) we can take \(K = C(M - m)\) and keep only the disks of \(\mathcal{B}\) containing a zero of \(\varphi\) in them.

**Remark 2.3.** (1) The choice of the constant 1/6 in [GS08, Lem. 2.15] was so that one could invoke the one-dimensional Cartan estimate as stated in Theorem 4 of [Lev96, Lecture 11]. However, it is straightforward to adjust the result from [Lev96] and the proof from [GS08] to replace 1/6 by any \(r < 1\). Of course, the constant \(C_d\) would depend (explicitly) on the particular choice of \(r\).

(2) The definition of Cartan sets gives implicit information about their measure. For example, using Fubini and the definition of Car\(_d\), one gets by induction that the set exceptional set \(\mathcal{B}\) in the previous lemma satisfies \(\operatorname{mes}_{C_d}(\mathcal{B}) \leq C(d) \exp(-H)\).

The following notion will be needed for our discussion of Weierstrass’ preparation theorem.

**Definition 2.4.** Let \(f\) be analytic on the ball \(B(0, R_0) \subset \mathbb{C}\). Let \(\epsilon \in \mathbb{C}\) be an arbitrary unit vector. We say that \(\epsilon\) is \(m\)-regular for \(f\) at \(0\) (or just \(m\)-regular if it is clear from the context what \(0\) is) if
\[
\sup_{z \in \mathbb{D}(0, R_0/4)} \log |f(\epsilon z)| \geq m.
\]

We show that Cartan’s estimate implies that most directions are regular. We use \(\sigma\) to denote the standard spherical measure.

**Lemma 2.5.** Let \(f\) be as in Definition 2.4 and let
\[
M \geq \sup_{B(0, R_0)} \log |f|, \quad \sup_{B(0, R_0/4)} \log |f| \geq m.
\]
Take arbitrary $H \gg 1$ and set $m = M - C_2 H(M - m)$, with $C_2$ as in Lemma 2.2. Denote by $B$ the set of $z$ which are not $m$-regular. Then
\[ \sigma(B) \lesssim \exp(-H^{1/2}). \]

Proof. Apply the Cartan estimate to find a set $\hat{B}$, $\meas(\hat{B}) \lesssim R_0^3 \exp(-H^{1/2})$, such that $\log|f(z)| > m$ for any $z \in B(z_0, R_0/8) \setminus \hat{B}$. Using spherical coordinates write
\[ \meas(\hat{B}) \geq \int_B d\sigma(z) \int_0^{R_0/4} r^3 dr \gtrsim R_0^3 \sigma(B) \]
and the statement follows. \qed

3. Bernstein Exponent and Number of Zeros

In this section we provide a relation between Bernstein exponents for one variable analytic functions and the number of their zeros.

Lemma 3.1. Let $\phi$ be a non-vanishing analytic function on $D(z_0, R)$ Then for any $z, |z - z_0| = r < R$, we have
\[ -\frac{2r}{R - r} (M - \log |\phi(z_0)|) \leq \log |\phi(z)| - \log |\phi(z_0)| \leq \frac{2r}{R + r} (M - \log |\phi(z_0)|), \]
where $M = M_\phi(z_0, R)$.

Proof. The estimates follows immediately from Harnack’s inequality applied to $u(z) = M - \log |\phi(z)|$. \qed

Proposition 3.2. Let $\phi$ be an analytic function on $D(0, 1)$ such that
\[ M_\phi(0, 1) \leq 0, \quad M_\phi(0, 1/4) \geq m. \]
Let $n$ be the total number of zeros of $\phi$ in $D(0, 3/4)$. Then for any $|z_0| < 1/8, r < 1/8, \mu \in (0, 1)$, we have
\[ B_{\phi}(\mu; z_0, r) \leq C r(n - m) - n \log \mu \lesssim - (r - \log \mu) m, \]
(3.1)

Proof. Take $\zeta_0 \in D(0, 1/4)$ wit log $|f(\zeta_0)| = m$. Using Jensen’s formula applied to
\[ f \left( \frac{z + \zeta_0}{1 + \zeta_0 z} \right) \]
we get $n \lesssim -m$. So, we just have to prove the first estimate in (3.1).

Let $a_1, \ldots, a_n$, be the zeros of $\phi$ in $D(0, 7/8)$, repeated according to their multiplicities. Let $P(z) = \prod_{k=1}^n (z - a_k)$, $h = \phi/P$, and $z_1, |z_1 - z_0| = \mu r$, be such that $\log |h(z_1)| = M_h(z_0, \mu r)$. Note that $h$ is non-vanishing and analytic on $D(0, 3/4)$. Using Lemma 3.1 we have that for any $z \in D(z_0, \mu r)$
\[ \log |h(z)| \geq \log |h(z_1)| - \frac{2|z - z_1|}{1/2 - |z - z_1|} (M_h(z_0, 1/2) - \log |h(z_1)|) \]
\[ \geq M_h(z_0, \mu r) - C \mu r (M_h(0, 3/4) - M_h(z_0, \mu r)). \]
Therefore
\[ M_\phi(z_0, \mu r) \geq M_h(z_0, \mu r) - C \mu r (M_h(0, 3/4) - M_h(z_0, \mu r)) + M_P(z_0, \mu r) \]
and

\[(3.2) \quad B_\phi(\mu; z_0, r) \leq M_h(z_0, r) - M_h(z_0, \mu r) + C\mu r(M_h(0, 3/4) - M_h(z_0, \mu r)) + M_P(z_0, r) - M_P(z_0, \mu r).\]

Let \(z_2, |z_2 - z_0| = r\), such that \(\log |h(z_2)| = M_h(z_0, r)\) and \(z_3, |z_3| = 1/4\), such that \(\log |h(z_3)| = M_h(0, 1/4)\). Using Lemma 3.1 we get

\[
M_h(z_0, r) - M_h(z_0, \mu r) = \log |h(z_2)| - \log |h(z_1)|
\]
\[
\leq \frac{2|z_2 - z_1|}{1/2 + |z_2 - z_1|}(M_h(z_1, 1/2) - \log |h(z_1)|) \leq Cr(M_h(0, 3/4) - M_h(z_0, \mu r)),
\]

\[
M_h(z_0, \mu r) - M_h(0, 1/4) = \log |h(z_1)| - \log |h(z_3)|
\]
\[
\geq -\frac{2|z_3 - z_1|}{1/2 - |z_3 - z_1|}(M_h(z_3, 1/2) - \log |h(z_3)|) \geq -C(M_h(0, 3/4) - M_h(0, 1/4)).
\]

Plugging these estimates in (3.2) we get

\[(3.3) \quad B_\phi(\mu; z_0, r) \leq Cr(M_h(0, 3/4) - M_h(0, 1/4)) + B_P(\mu; z_0, r).\]

Recall that we know \(B_P(\mu; z_0, r) \leq -n \log \mu\), so to get the conclusion we just have to estimate \(M_h(0, 3/4) - M_h(0, 1/4)\). Given \(z \in \mathbb{D}(0, 3/4)\), apply the submean value property to get

\[
\log |h(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |\phi(z + e^{i\theta}/4)| \, d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log |P(z + e^{i\theta}/4)| \, d\theta \leq n \log 4
\]
and conclude \(M_h(0, 3/4) \leq n \log 4\). We used the assumption that \(M_\phi(0, 1) \leq 0\) and the fact that

\[(3.4) \quad \frac{1}{2\pi} \int_0^{2\pi} \log |z - a_k + e^{i\theta}/4| \, d\theta = \begin{cases} \log \frac{1}{4}, & |z - a_k| \leq \frac{1}{4}, \\ \log |z - a_k|, & |z - a_k| > \frac{1}{4} > \frac{1}{4}. \end{cases}
\]

Since clearly \(M_P(0, 1/4) \leq 0\), we have \(M_h(0, 1/4) \geq M_\phi(1/4)\). So,

\[
M_h(0, 3/4) - M_h(0, 1/4) \leq C(n - m)
\]
and the conclusion follows. \(\square\)

**Remark 3.3.** (1) It is not true that conclusion of Proposition 3.2 can be made just in terms of the number \(n\) of zeros of \(\phi\). Some estimate for \(M_\phi(0, 1/4)\) is really needed. Here is an elementary example \(\phi(z) = \exp(-N + Nz), z \in \mathbb{D}(0, 1)\) and \(N > 0\) is arbitrary. Clearly, \(M_\phi(0, 1) = 0, n = 0\). On the other hand \(M_\phi(0, 1/4) \simeq -N, B_\phi(1/4, 0, 1/8) \simeq N\).

(2) It is known from [RY97] that if we have control on the valency of the function \(\phi\), instead of just the number of zeros, then the estimate for \(M_\phi(0, 1/4)\) is not needed anymore.

4. **Weierstrass’ Preparation Theorem and Bernstein Exponents**

We start with a statement of the classical Weierstrass’ preparation theorem attuned to our purposes.
Lemma 4.1. Let \( f(z, w) \) be analytic function on a polydisk
\[
\mathcal{P} := \mathcal{D}(z_0, R_0) \times \mathcal{D}(w_0, R_0) \subset \mathbb{C}^2, \quad R_0 > 0.
\]
Assume that \( f(z, w) \) has no zeros on some circle \( \Gamma_{\rho_0} = \{ z : |z - z_0| = \rho_0 \} \), \( 0 < \rho_0 < R_0/2 \), for any \( w \in \mathcal{D}(w_0, r_1) \), \( 0 < r_1 < R_0 \). Then there exists a Weierstrass polynomial \( P(z, w) = z^k + a_{k-1}(w)z^{k-1} + \cdots + a_0(w) \) with \( a_j(w) \) analytic in \( \mathcal{D}(w_0, r_1) \) and an analytic function \( g(z, w), (z, w) \in \mathcal{P}' := \mathcal{D}(z_0, \rho_0) \times \mathcal{D}(w_0, r_1) \) so that the following properties hold:

(a) \( f(z, w) = P(z, w)g(z, w) \) for any \( (z, w) \in \mathcal{P}' \).

(b) \( g(z, w) \neq 0 \) for any \( (z, w) \in \mathcal{P}' \).

(c) For any \( w \in \mathcal{D}(w_0, r_1) \), \( P(z, w) \) has no zeros in \( C \setminus \mathcal{D}(z_0, \rho_0) \).

(d) We have
\[
\begin{align*}
\left( \inf_{\Gamma_{\rho_0} \times \mathcal{D}(w_0, r_1)} \log |f| \right) - k \log (2\rho_0) & \leq \inf_{\mathcal{P}'} \log |g|, \\
\sup_{\mathcal{P}'} \log |g| & \leq \left( \sup_{\mathcal{P}} \log |f| \right) + k \log \frac{2}{R_0}.
\end{align*}
\]

Proof. By the usual Weierstrass argument, one notes that
\[
b_p(w) := \sum_{j=1}^{k} \zeta_j^p(w) = \frac{1}{2\pi i} \oint_{\Gamma_{\rho_0}} z^p \frac{\partial f(z, w)}{f(z, w)} \, dz
\]
are analytic in \( \mathcal{D}(w_0, r_1) \). Here \( \zeta_j(w) \) are the zeros of \( f(z, w) \) in \( \mathcal{D}(z_0, \rho_0) \). Since the coefficients \( a_j(w) \) are linear combinations of the \( b_p(w) \), they are analytic in \( w \). Analyticity of \( g \) follows by standard arguments. We just have to prove (d). Since all the roots of \( P(z, w) \) are in \( \mathcal{D}(z_0, \rho_0) \), we have \( \sup_{\mathcal{P}'} |P| \leq (2\rho_0)^k \) and (4.1) follows using the minimum modulus principle. Note that actually the function \( g \) can be defined on \( \mathcal{P} \) as \( g = f/P \) and it is analytic there. Given \( (z, w) \in \mathcal{P}' \), apply the sub-mean value property for subharmonic functions to get
\[
\log |g(z, w)| \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(z + R_0 e^{i\theta}/2, w)| \, d\theta - \frac{1}{2\pi} \int_{0}^{2\pi} \log |P(z + R_0 e^{i\theta}/2, w)| \, d\theta
\]
\[
\leq \left( \sup_{\mathcal{P}} \log |f| \right) + k \log \frac{2}{R_0}.
\]
The estimate on the mean value of the polynomial follows by considerations analogous to (3.4).

Next we describe how Bernstein exponents rule the application of Lemma 4.1.

Lemma 4.2. Let \( f \) be analytic on \( B(0, 1) \), \( M \geq \sup_{B(0,1)} \log |f|, \, \underline{m} = M - B, \, B \gg 1, \, \underline{e_1} \) a \( \underline{m} \)-regular direction for \( f \) at 0 (recall Definition 2.4), and \( \underline{e_2} \) another non-collinear direction. With a slight abuse of notation we denote by \( f(z, w) \) the function in the new coordinates with respect to the basis \( \underline{e}_1, \underline{e}_2 \). Then there exists a circle \( \Gamma_{\rho_0} = \{ |z| = \rho_0 \} \), \( 1/8 < \rho_0 < 1/4 \), and \( r_1 = \exp(-CB) \), with \( C > 1 \) an absolute constant, such that
\[
\inf_{\Gamma_{\rho_0} \times \mathcal{D}(0, r_1)} \log |f| \geq \exp(M - CB).
\]
In particular, Lemma 4.1 applies for \( f(z, w) \) with this choice of \( \rho_0 \) and \( r_1 \), as well as with \( k \lesssim B \) and \( \delta \geq M - CB \).
Let \( g \) be any resultant of \( f \), for any \( z \) polydisk on which we have the Weierstrass factorization.

\[
\begin{align*}
\text{(4.5)} &\quad |f(z,0)| \geq \exp(M - CB) \\
\end{align*}
\]

for any \( z \in \Gamma_{\rho_0} \). Note that due to Cauchy’s estimates

\[
|f(z,w) - f(z,0)| \lesssim e^M|w|
\]

for any \( z \in \mathcal{D}(0,1/2), w \in \mathcal{D}(0,1/2) \). Taking into account (4.5), one obtains

\[
|f(z,w)| > \exp(M - CB)
\]

for any \( z \in \Gamma_{\rho_0} \), provided \( w \in \mathcal{D}(0,r_1) \), \( r_1 = \exp(-CB) \), with \( C \) large enough (of course, \( C \) is larger than in (4.5)). This proves (4.3) and allows us to apply Lemma 4.1 as stated. For the bound on the degree of the Weierstrass polynomial note that by Jensen’s formula applied to \( f(z,0), z \in \mathcal{D}(z_1,1/2) \),

\[
k \leq \# \{ z \in \mathcal{D}(0,1/4) : f(z,0) = 0 \} \leq \# \{ z \in \mathcal{D}(z_1,1/2) : f(z,0) = 0 \} \lesssim B.
\]

\[\square\]

Remark 4.3. (1) Due to Lemma 2.5, we will always apply the previous lemma with \( B \approx B_f(1/4;0,1) \). This is how the Bernstein exponent determines the size of the polydisk on which we have the Weierstrass factorization.

(2) If we are given two functions \( f_1, f_2 \) satisfying the assumptions of Lemma 4.2 with the same \( M \) and \( B \), then it is clear from the proof of the lemma that we can arrange for the conclusion to hold for both functions with the same choice of \( \rho_0 \) and \( r_1 \). Indeed, one only needs to choose \( \rho_0 \) such that \( \Gamma_{\rho_0} \cap (\mathcal{B}_1 \cup \mathcal{B}_2) = \emptyset \), where \( \mathcal{B}_i \) are the Cartan sets needed to guarantee (4.4) for \( f_i \).

5. Resultants

We briefly recall the definition of the resultant of two univariate polynomials and some of the basic properties that we’ll use. Let \( f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0 \), \( g(z) = b_mz^m + b_{m-1}z^{m-1} + \cdots + b_0 \) be polynomials, \( a_i, b_j \in \mathbb{C}, a_n \neq 0, b_m \neq 0 \). Let \( \zeta_i, 1 \leq i \leq n \) and \( \eta_j, 1 \leq j \leq m \) be the zeros of \( f(z) \) and \( g(z) \) respectively. The resultant of \( f \) and \( g \) is defined as follows:

\[
\text{(5.1)} \quad \text{Res}(f,g) = a_n^m b_m^n \prod_{i,j} (\zeta_i - \eta_j) = (-1)^{mn} b_m^n \prod_j f(\eta_j) = (-1)^{mn} a_n^m \prod_i g(\zeta_i).
\]
The resultant \( \text{Res}(f, g) \) can be expressed explicitly in terms of the coefficients (see [Lan02]):

\[
\begin{array}{ccc|ccc}
  a_n & 0 & \cdots & b_m & 0 & \cdots \\
  a_{n-1} & a_n & \cdots & b_{m-1} & b_m & \cdots \\
  a_{n-2} & a_{n-1} & \cdots & b_{m-2} & b_{m-1} & \cdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
  a_0 & a_1 & \cdots & 0 & a_0 & \cdots \\
\end{array}
\]

\[ (5.2) \quad \text{Res}(f, g) = \sum_{k=0}^{\min(m, n)} a_k b_{n-k} \]

Lemma 5.1. Let \( f, g, \zeta, \eta_j \) as above. Set

\[ t_f = \min(|a_n|, 1), \quad t_g = \min(|b_m|, 1), \quad T_f = \max_{i}(\max|a_i|, 1), \quad T_g = \max_{j}(\max|b_j|, 1), \]

\[ R_f = t_f^{-1}T_f m, \quad R_g = t_g^{-1}T_g n. \]

The following statements hold.

1. If \( \left| \text{Res}(f, g) \right| < \delta^m t_g, \quad 0 < \delta < 1, \)

then there exists \( j \) such that

\[ |f(\eta_j)| < \delta. \]

In particular, there exists \( \delta \leq R_g \) such that \( \max(|f(z)|, |g(z)|) < \delta. \)

2. If there exists \( z \) such that with \( s = \max(m, n), t = \min(t_f, t_g) \) holds

\[ \max([f(z)], |g(z)|) < t\delta^s, \quad 0 < \delta < 1, \]

then

\[ |\text{Res}(f, g)| < t^{2s} (2R)^s \delta, \]

\[ R = \max(R_f, R_g). \]

Proof. (0) follows by noting that, for example,

\[ |a_n|^n \leq (\max|a_i|) |\zeta|^n + (\max|a_i|) n \max(|\zeta|^{n-1}, 1). \]

(1) follows by contradiction from (5.1). For (2) note that there must exist \( \zeta_i, \eta_j \) such that \( |z - \zeta_i| < \delta, |z - \eta_j| < \delta \) and therefore, using (0) and (5.1),

\[ |\text{Res}(f, g)| \leq t^{2s} (2R)^s \delta \]

6. Refinement of the Assumption (1.1)

We give a simple argument showing that by making some small adjustments we actually have \( K_1 \leq H_0^C \) in (1.1).

Lemma 6.1. Using the notation and assumptions of Theorem C we have that

\[ \mathcal{N}(F, \varepsilon_0/2) \cap B(0, 1/2) \in \text{Car}_2,0(H_1/2, H_0^C) \]

where \( C \) is some large absolute constant.
Proof. Let $f_{i,N}$ be the degree $N$ Taylor polynomials (at the origin) associated with $f_i$, $i = 1, 2$ (recall that $F = (f_1, f_2)$). Since $M_{f_1}(0, 1) \leq 0$, a standard application of the Cauchy estimates yields that

$$|f_i - f_{i,N}| < \varepsilon_0/100$$

for $N = C \log \varepsilon_0^{-1} = CH_0$, $C \gg 1$. Let $F_N = (f_{1,N}, f_{2,N})$. We have

$$(6.1) \quad N(F, \varepsilon_0/2) \cap B(0, 1/2) \subset N(F_N, 3\varepsilon_0/4) \cap B(0, 1/2) \subset N(F, \varepsilon_0) \cap B(0, 1/2).$$

The set $N(F_N, 3\varepsilon_0/4) \cap B(0, 1/2)$ is semialgebraic of degree less than $CN$ and therefore it has at most $N^C$ connected components. We refer to [Bon05, Ch. 9] for a brief review of semialgebraic sets and their properties. It follows from our assumptions that $N(F_N, 3\varepsilon_0/4) \cap B(0, 1/2)$ is covered by less than $K_1$ balls of radius $\exp(-H_1)$. Therefore, each connected component of $N(F_N, 3\varepsilon_0/4) \cap B(0, 1/2)$ can be covered by just one ball of radius smaller than

$$CK_1 \exp(-H_1) \leq \exp(-H_1/2)$$

(recall that $\log K_1 \ll H_1$) and so

$$N(F_N, 3\varepsilon_0/4) \cap B(0, 1/2) \in \text{Car}_{2,0}(H_1/2, N^C).$$

The conclusion now follows from (6.1). \hfill \Box

7. Proofs of Theorems A,B,C

We start with the proof of Theorem C.

Proof of Theorem C. Take $\underline{w} = (z_0, w_0) \in B(0, 1/4)$. By our assumptions

$$B_{f_i}(1/4; \underline{w}, 1/4), \quad M_{f_i}(\underline{w}, 1/4) \leq 0.$$  

Due to Lemma 2.5, we can find unit vectors $\ell_1, \ell_2$, $|\ell_1, \ell_2| \ll 1$, that are $m$-regular at $\underline{w}_0$ for both $f_1, f_2$ restricted to $B(\underline{w}_0, 1/4)$, with $m = -CB_0$, $C \gg 1$. Then Lemma 4.2 applies to both $f_1, f_2$ and to both directions $\ell_1, \ell_2$. As in Lemma 4.2, with a slight abuse of notation we denote by $f_i(z, w)$ the functions in the coordinates with respect to the basis $\ell_1, \ell_2$ centered at $\underline{w}_0$ and with the obvious rescaling needed to apply the lemma. Applying Lemma 4.2 (see also Remark 4.3) in the direction of $\ell_1$ (and $\ell_2$ as the choice of non-collinear direction) we can write

$$f_i(z, w) = P_i(z, w)g_i(z, w),$$

$$P_i(z, w) = z^{k_1} + a_i,k_i-1(w)z^{k_i-1} + \cdots + a_0(w)$$

on $P := \mathcal{D}(0, \rho_0) \times \mathcal{D}(0, r_1)$, $1/8 < \rho_0 < 1/4$, $r_1 = \exp(-CB_0)$, where the coefficients $a_{i,j}(w)$ are analytic on $\mathcal{D}(0, r_1)$, $g_i$ are analytic and non-vanishing on $P$, the polynomials $P_i(\cdot, w)$, $w \in \mathcal{D}(0, r_1)$, have no zeroes in $\mathbb{C} \setminus \mathcal{D}(0, \rho_0)$, and $k_i \lesssim B_0$. Furthermore, using part (d) of Lemma 4.1,

$$(7.1) \quad -B_0 \lesssim \inf_{\mathcal{P}} \log |g_i| \leq \sup_{\mathcal{P}} \log |g_i| \lesssim B_0.$$  

Let

$$R(w) = \text{Res}(P_1(\cdot, w), P_2(\cdot, w)).$$

Note that by (5.2), $R$ is analytic on $\mathcal{D}(0, r_1)$. Since we chose $\underline{w}_0 \in B(0, 1/4)$, the polydisk $\mathcal{P}$ is a subset of $B(0, 1/2)$, as a set in the standard coordinates. This allows
to use the hypothesis to guarantee that there exist points \( \nu_j = (z_j, w_j) \) (expressed in the \( e_1, e_2 \) coordinates), \( 1 \leq j \leq J \leq K_1 \) such that for

\[
(z, w) \in \mathcal{P} \setminus \left( \bigcup_{j=1}^{J} B(\nu_j, C \exp(-H_1)) \right)
\]

we have

\[
\max(|f_1(z, w)|, |f_2(z, w)|) \geq \exp(-H_0) \sqrt{2}
\]

and by (7.1)

\[
(7.2) \quad \max(|P_1(z, w)|, |P_2(z, w)|) \geq \exp(-H_0 - CB_0).
\]

Note that we used the radius \( C \exp(-H_1) \) instead of \( \exp(-H_1) \) to account for the distortion under the change of coordinates. Since we are assuming that \( H_1 \gg B_0 \) and \( \log K_1 \ll H_1 \), we can find

\[
w \in \mathcal{D}(0, r_1/4) \setminus \bigcup_{j=1}^{J} \mathcal{D}(w_j, C \exp(-H_1)).
\]

For any such \( w \) (7.2) holds for any \( z \in \mathcal{D}(0, \rho_0) \) and by part (1) of Lemma 5.1

\[
\log |R(w)| \geq -B_0 H_0 - B_0^2 \geq -B_0 H_0.
\]

Note that by the definition of the resultant (5.1), we have \( \sup_{\mathcal{D}(0, r_1)} |R(w)| \leq 1 \). Take \( H \gg 1 \). Applying Cartan’s estimate, we get

\[
\log |R(w)| \geq -H B_0 H_0
\]

for any

\[
w \in \mathcal{D}(0, r_1/4) \setminus \mathcal{B}, \quad \mathcal{B} = \bigcup_{1 \leq k \leq K} \mathcal{D}(w'_k, r_1 \exp(-H)), \quad K \lesssim B_0 H_0.
\]

By part (2) of Lemma 5.1,

\[
\max(|P_1(z, w)|, |P_2(z, w)|) \geq \exp(-CHB_0^2 H_0)
\]

for any \( w \in \mathcal{D}(0, r_1/4) \setminus \mathcal{B} \) and \( z \in \mathbb{C} \). Using (7.1), we get

\[
(7.3) \quad |F(z, w)| \geq \exp(-CHB_0^2 H_0 - CB_0) \geq \exp(-H B_0^2 H_0)
\]

for any \( w \in \mathcal{D}(0, r_1/4) \setminus \mathcal{B} \) and \( z \in \mathcal{D}(0, \rho_0) \). Applying Lemma 4.2 again in the direction of \( e_2 \) (and with \( e_1 \) as the choice of non-collinear direction) and repeating the above argument we get that there exist \( 1/8 < \hat{\rho}_0 < 1/4, \hat{r}_1 = \exp(-CB_0) \), such that (7.3) also holds for any \( z \in \mathcal{D}(0, \hat{r}_1/4) \setminus \hat{\mathcal{B}} \) and

\[
w \in \mathcal{D}(0, \hat{\rho}_0), \quad \hat{\mathcal{B}} = \bigcup_{1 \leq \ell \leq L} \mathcal{D}(z'_\ell, \hat{r}_1 \exp(-H)), \quad L \lesssim B_0 H_0.
\]

In particular, (7.3) holds for any

\[
(z, w) \in \mathcal{D}(0, \hat{r}_1/4) \times \mathcal{D}(0, r_1/4) \setminus \left( \bigcup_{k, \ell} \mathcal{D}(z'_{k, \ell}, \hat{r}_1 \exp(-H)) \times \mathcal{D}(w'_k, r_1 \exp(-H)) \right).
\]
Going back to standard coordinates we obtained that there exist less than $CB_0^2H_0^2$ points $\mathbf{v}_j$ such that (7.3) holds for any 

$$(z, w) \in B(\mathbf{v}_j, \exp(-CB_0)) \setminus \left( \bigcup_j B(\mathbf{v}_j', \exp(-H)) \right).$$

Since (7.3) holds outside the initial Car\'2,0 set, we only need to apply the above argument on $K_1$ balls covering the initial set to get the conclusion. □

We will need the following lemma for the proof of Theorem A.

Lemma 7.1. Let $f$ be analytic on the ball $B(0, 1)$, $Z = \{ \mathbf{v} \in B(0, 1) : f(\mathbf{v}) = 0 \}$. Let $B \in \text{Car}_{2,0}(H_1, K_1)$, $H \gg 1$, $\log K \ll H$. If $0 \in Z$, then 

$$B(0, 1/4) \cap Z \setminus B \neq \emptyset.$$ 

Proof. We argue by contradiction. Assume $B(0, 1/4) \cap Z \subset B$. By the assumptions on $B$ we can find $1/8 < r < 1/4$ such that $B \cap B(0, r)$ is compactly contained in $B$. Therefore the zero set of $f$ restricted to $B(0, r)$ is compactly contained in $B(0, r)$ and $Z \cap B(0, r)$ is a compact analytic variety in $\mathbb{C}^2$. This cannot be, because compact analytic varieties in $\mathbb{C}^2$ are necessarily finite sets (see for example [Chi89]) and analytic functions of several variables cannot have isolated zeros (recall that $0 \in Z$).

Proof of Theorem A. (1) Take $\mathbf{v}_0 \in B(0, 1/8) \cap Z = \{ f_2 = 0 \}$, $0 < R \leq 1/4$. Let $H = C \max(\log^{-1} R^0, C_0)$ with $C$ large enough (recall that $C_0 = \log(K_1B_0^2H_0^2)$). By Theorem C we have

$$|F(\mathbf{v})| \geq \exp(-HB_0^2H_0)$$

for all $\mathbf{v} \in B(0, 1/4) \setminus B$, $B \in \text{Car}_{2,0}(H, K)$, $K \ll K_1B_0^2H_0^2$. Note that $B(\mathbf{v}_0, R) \subset B(0, 1/4)$ and our choice of $H$ is such that we can apply Lemma 7.1 to $f_2$ restricted to $B(\mathbf{v}_0, R)$ and the above $B$ (after an obvious rescaling). So, there exists $\mathbf{v}_1 \in B(\mathbf{v}_0, R/4) \cap Z \setminus B$. Note that we have

$$|f_1(\mathbf{v}_1)| = |F(\mathbf{v}_1)| \geq \exp(-HB_0^2H_0)$$

and therefore

$$M_{f_1}(Z, \mathbf{v}_0, R/4) \geq -HB_0^2H_0.$$ 

The first statement now follows by recalling that

$$M_{f_1}(Z, \mathbf{v}_0, R) \leq M_{f_1}(0, 1) \leq 0.$$ 

(2) Take $\mathbf{v}_0 \in B(0, 1/8)$. By our assumptions

$$B_{f_2}(1/4; \mathbf{v}_0, 1/4) \leq B_0, \quad M_{f_2}(\mathbf{v}_0, 1/4) \leq 0.$$ 

Due to Lemma 2.5, we can find a unit vector $\xi_1$, that is $m$-regular at $\mathbf{v}_0$, for $f_2$ restricted to $B(\mathbf{v}_0, 1/4)$, with $m = -CB_0$, $C \gg 1$. Let $\xi_2$ be another unit vector orthogonal to $\xi_1$. As in Lemma 4.2, with a slight abuse of notation we denote by $f_1(z, w)$ the functions in the coordinates with respect to the basis $\xi_1, \xi_2$ centered at $\mathbf{v}_0$ and with the obvious rescaling needed to apply the lemma. Applying Lemma 4.2 in the direction of $\xi_1$ (with $\xi_2$ as the choice of non-collinear direction) we can write

$$f_2(z, w) = P(z, w)g(z, w),$$

$$P(z, w) = z^k + a_{k-1}(w)z^{k-1} + \cdots + a_0(w)$$
on \( P := \mathcal{D}(0, \rho_0) \times \mathcal{D}(0, r_1), 1/8 < \rho_0 < 1/4, r_1 = \exp(-CB_0) \), where the coefficients 
\( a_j(w) \) are analytic on \( \mathcal{D}(0, r_1) \) and \( k \lesssim B_0 \). Since we also have that \( g \) is analytic 
and non-vanishing on \( P \),

\[
\mathcal{Z} \cap P = \mathcal{Z}_P \cap P, \quad \mathcal{Z}_P := \{(z, w) \in \mathbb{C} \times \mathcal{D}(0, r_1) : P(z, w) = 0\}.
\]

It is well known (see [Chi89]) that for any point \((z, w)\) of the variety \( \mathcal{Z}_P \), there exist \( \varepsilon > 0, \delta > 0 \) such that the following statements hold.

(i) If \((z, w)\) is a regular point, then there exists an analytic function \( \zeta : \mathcal{D}(0, \varepsilon) \to \mathcal{D}(0, \delta) \) such that

\[
\mathcal{Z}_P \cap (\mathcal{D}(z, \delta) \times \mathcal{D}(w, \varepsilon)) = \{(z + \zeta(w' - w), w') : w' \in \mathcal{D}(w, \varepsilon)\}.
\]

(ii) If \((z, w)\) is a singular point, then there exist integers \( p_i \geq 1 \) and analytic functions \( \zeta_i : \mathcal{D}(0, \varepsilon) \to \mathcal{D}(0, \delta), 1 \leq i \leq i_0(z, w) \leq k \), such that \( \sum_i p_i \leq k \) and

\[
\mathcal{Z}_P \cap (\mathcal{D}(z, \delta) \times \mathcal{D}(w, \varepsilon)) = \bigcup_i \{(z + \zeta_i((w' - w)\tilde{w}^i), w') : w' \in \mathcal{D}(w, \varepsilon)\}.
\]

By compactness we can cover \( B(0, 1/8) \cap \mathcal{Z} \) by finitely many polydisks \( \frac{1}{2} P \) (more 
precisely, by their preimages under the change of variables we assumed above) and 
in turn \( \mathcal{Z} \cap \frac{1}{2} P \) can be covered by finitely many polydisks \( \mathcal{D}(z_j, \delta_j) \times \mathcal{D}(w_j, \varepsilon_j/8) \) 
with \((z_j, w_j) \in \mathcal{Z}_P \) and \( \varepsilon_j, \delta_j \) as above. We will also use \( \zeta_j \) and \( \zeta_{i,j} \) the functions 
associated with \((z_j, w_j)\). Let \( r_0 > 0 \) be the minimum over all the \( \varepsilon_j \) needed to 
cover \( B(0, 1/8) \cap \mathcal{Z} \). Near each \((z_j, w_j)\) we will define local charts and show we 
can control the Bernstein exponent of \( f_1 \) in the local charts. The control over the 
Bernstein exponent will follow from Theorem \( C \) and Proposition 3.2. To this end 
we take \( H = C(\log r_0^{-1})C_0 \), with \( C \gg 1 \) large enough and we note that, with this 
choice of \( H \), Theorem \( C \) guarantees that

\[
|F(z, w)| \geq \exp(-HB_0^2H_0), \quad \forall (z, w) \in P \setminus (\mathbb{C} \times B)
\]

where \( B \) is a union of disks with the sum of the radii much smaller than \( r_0^k \) (recall 
that \( k \lesssim B_0 \ll H_0 \)). To define the charts we distinguish two cases.

(i) \((z_j, w_j)\) is regular. Let

\[
\psi_j(w) = (z_j + \zeta_j(w), w_j + w), \quad w \in \mathcal{D}(0, \varepsilon_j).
\]

It follows from (7.4) that

\[
M_{f_1\psi_j}(0, 1/4) \geq -HB_0^2H_0.
\]

By Proposition 3.2 (recall that \( M_{f_1}(0, 1) \leq 0 \)) it is clear that

\[
B_{f_1\psi_j}(1/4; z, r) \lesssim H \leq C(f_2)C_0B_0^2H_0
\]

when \( \mathcal{D}(z, r) \subset \mathcal{D}(0, \varepsilon_j/8) \). This shows the conclusion of part (2) holds if we define 
the local chart by rescaling \( \psi_j \mid_{\mathcal{D}(0, \varepsilon_j/8)} \).

(ii) \((z_j, w_j)\) is singular. Let

\[
\psi_{i,j}(w) = (z_j + \zeta_{i,j}(w), w_j + w^{\nu_i}), \quad w \in \mathcal{D}(0, \varepsilon_j).
\]

It follows from (7.4) that

\[
M_{f_1\psi_{i,j}}(0, 1/4) \geq -HB_0^2H_0
\]

(recall that \( p_i \leq k \)) and therefore Proposition 3.2 guarantees that

\[
B_{f_1\psi_{i,j}}(1/4; z, r) \lesssim HB_0^2H_0 \leq C(f_2)C_0B_0^2H_0
\]
when $\mathcal{D}(z, r) \subset \mathcal{D}(0, \varepsilon_j / 8)$. This shows that the conclusion holds if corresponding to each $w \in \mathcal{D}(0, \varepsilon_j / 8) \setminus \{0\}$ we define a local chart by rescaling $\psi_{i,j}|_{\mathcal{D}(w, r)}$, where $\mathcal{D}(w, r)$ is the largest disk about $w$ in $\mathcal{D}(0, \varepsilon_j / 8)$ on which $w^{p_i}$ is one-to-one.

Clearly the above charts cover $\text{reg} \, \mathcal{Z} \cap B(0, 1/8)$ and we can complete an atlas of $\text{reg} \, \mathcal{Z}$ by adding charts whose ranges don’t intersect $B(0, 1/8)$. This concludes the proof. \hfill \Box

Next we give an example showing that the $\log R^{-1}$ is actually necessary in part (1) of Theorem A.

**Example 7.2.** Let

$$f_1(z, w) = z^2 + w, \quad f_2(z, w) = zw, \quad \mathcal{Z} = \{ f_2 = 0 \}.$$  

Let $R \ll 1$, $\mathcal{U}_0 = (R/4, 0)$. Then straightforward computations show that

$$\sup_{\mathcal{B}(\mathcal{U}_0, R/4) \cap \mathcal{Z}} \log |f_1(z, w)| = \sup_{|z-R/4|<R/4} \log |z^2| = \log \left( \frac{R}{2} \right)^2,$$

and

$$\sup_{\mathcal{B}(\mathcal{U}_0, R) \cap \mathcal{Z}} \log |f_1(z, w)| = \max \left( \sup_{|z-R/4|<R} \log |z^2|, \sup_{|w|^2+(R/4)^2<R^2} \log |w| \right)$$

$$= \max \left( \log \left( \frac{5R}{4} \right)^2, \log \frac{\sqrt{15}R}{4} \right) = \log \frac{\sqrt{15}R}{4},$$

provided $R$ is small enough ($R < 1/2$ is enough). Therefore,

$$B(f_1, (1/4; \mathcal{Z}, \mathcal{U}_0, R) = C + \log R^{-1}.$$

Finally, we will prove Theorem B, but we first establish an auxilliary result.

To this end we will need the following extension of the classical Bézout theorem. Suppose we have a system of $n$ complex polynomial equations $f_i(z_1, \ldots, z_n) = 0$, $i = 1, \ldots, n$. Let $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ be the irreducible components of the variety defined by the system. Then

$$\text{deg}(\mathcal{Z}_1) + \cdots + \text{deg}(\mathcal{Z}_n) \leq \text{deg} f_1 \times \cdots \times \text{deg} f_n.$$  

The authors are grateful to János Kollár and Mihnea Popa for pointing out this version of the Bézout bound (for a more general result see [Ful98, Thm. 12.3]).

**Lemma 7.3.** Let $f(z, w)$, $g(z, w)$ be non-constant polynomials with no common factors. Let $\zeta(w)$ be an analytic function on $\mathcal{D}(0, r_0)$ such that

$$\{ \zeta(w), w : w \in \mathcal{D}(0, r_0) \} \subset \text{reg}\{ f(z, w) = 0 \}.$$  

Then there exists at most one straight line $\mathcal{L} \subset \mathcal{C}$ through the origin such that

$$\#\{ \xi \in (-r_0, r_0) : g(\zeta(\xi), \xi) \in \mathcal{L} \} > (\deg f)^2 \deg g.$$  

**Proof.** Let $\mathcal{L}$ be a line through the origin. We first argue that if (7.7) holds, then we must have $\{ g(\zeta(\xi), \xi) : \xi \in (-r_0, r_0) \} \subset \mathcal{L}$. Write

$$f(z, \xi) := P(x + iy, \xi) + iQ(x + iy, \xi) = \bar{P}(x, y, \xi) + i\bar{Q}(x, y, \xi)$$

where $P, Q$ are the real and imaginary parts of $f$, and $\bar{P}, \bar{Q}$ are real polynomials of three real variables $x, y, \xi$. Clearly $\deg \bar{P} = \deg \bar{Q} = \deg f$. Similarly write

$$g(z, \xi) := U(x + iy, \xi) + iV(x + iy, \xi) = \bar{U}(x, y, \xi) + i\bar{V}(x, y, \xi).$$
Without loss of generality we may assume that the line $\mathcal{L}$ is horizontal. If (7.7) holds, then the system

$$
\dot{P} = 0, \quad \dot{Q} = 0, \quad \dot{V} = 0
$$

has more than $\text{deg} \dot{P} \times \text{deg} \dot{Q} \times \text{deg} \dot{V}$ solutions $v_j = (x_j, y_j, \xi_j)$ with

$$
\xi_j \in (-r_0, r_0), \quad x_j + iy_j = \zeta(\xi_j), \quad \xi_{j_1} \neq \xi_{j_2}.
$$

Complexify the variables $x, y, \xi$, and let $Z_1, \ldots, Z_s$ be the irreducible components of the complex variety defined by the system (7.8). By the Bézout bound (7.5), there exists a component $Z_k$ that contains at least two of the solutions $v_j$, and therefore has dimension at least one. Let $v_0$ be one of the solutions contained in $Z_k$. We will argue that there exists an analytic mapping

$$
t \to \mathcal{Z}(t) = (x(t), y(t), \xi(t)) \in Z_k, \quad t \in \mathcal{D}(0, \delta)
$$

such that $v(0) = v_0$ and $\xi(t)$ is non-constant. By [Shi70] we know that there exists a neighborhood $N$ of $v_0$ in $Z_k$, such that for any $\mathcal{Z} \in N \setminus \{v_0\}$, there exists a one dimensional irreducible variety $V$ through both $\mathcal{Z}$ and $v_0$. Since $V$ can be parametrized by a Riemann surface (see [Chi89, Prop. 6.2]) we get the existence of a mapping of the form (7.9). If $\xi(t)$ is constant, then by the uniqueness theorem (see [Chi89, Cor. 5.3.2]), we must have $V \subset \{\xi = \xi_0\}$. If this happens for all such mappings obtained by choosing different $\mathcal{Z} \in N \setminus \{v_0\}$, then $N \subset \{\xi = \xi_0\}$, and by the uniqueness theorem, $Z_k \subset \{\xi = \xi_0\}$. This would contradict the fact that $Z_k$ contains two of the solutions $v_j$ (recall that $\xi_{j_1} \neq \xi_{j_2}$). So we proved the existence of the mapping (7.9) with the desired properties. We have

$$
f(x(t) + iy(t), \xi(t)) = 0, \quad V(x(t) + iy(t), \xi(t)) = 0, \quad t \in \mathcal{D}(0, \delta).
$$

By the assumption (7.6), we get

$$
x(t) + iy(t) = \zeta(\xi(t)),
$$

provided we choose $\delta$ small enough. Therefore

$$
V(\zeta(\xi(t)), \xi(t)) = 0, \quad t \in \mathcal{D}(0, \delta)
$$

and since $\xi(t)$ is non-constant, there exists $\varepsilon > 0$ so that

$$
V(\zeta(\xi), \xi) = 0, \quad \xi \in (\xi_0 - \varepsilon, \xi_0 + \varepsilon).
$$

So $V(\zeta(\xi), \xi) = 0$ for all $\xi \in (-r_0, r_0)$, that is $\{g(\zeta(\xi), \xi) : \xi \in (-r_0, r_0)\} \subset \mathcal{L}$.

Now we can finish the proof by arguing by contradiction. If the conclusion doesn’t hold, it follows that we have $\{g(\zeta(\xi), \xi) : \xi \in (-r_0, r_0)\} \subset \mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$ and therefore the system $f = g = 0$ has infinitely many solutions. By the classical Bézout theorem, this would contradict the assumption that $f$ and $g$ don’t have common factors.

**Proof of Theorem B.** Let $Z_1, \ldots, Z_s$ be the irreducible components of $Z$. Each of them is the zero set of an irreducible factor of $f_2$. Let $f_{2,1}, \ldots, f_{2,s}$ be such irreducible factors. Fix $k \in \{1, \ldots, s\}$ and $(z_0, w_0) \in \text{reg} Z \cap Z_k$. We can make a change of variables (as in the proof of part (2) of Theorem A) such that $(z_0, w_0)$ is mapped to the origin and we can find an analytic function $\zeta : \mathcal{D}(0, \varepsilon_0) \to \mathcal{D}(0, \delta_0)$ so that

$$
\phi(w) = (\zeta(w), w), \quad w \in \mathcal{D}(0, \varepsilon_0)
$$

is a chart for $\text{reg} Z \cap Z_k$ around the origin.
If \( f_{2,k} \) divides \( f_1 \), then \( f_1 \) vanishes identically on \( Z_k \), and its Bernstein exponent is 0 by convention (in any chart). So, we just need to treat the case when \( f_{2,k} \) and \( f_1 \) have no common factors. Let \( \psi(w) = f_1(\phi(w)) \). We claim that
\[
B_\psi(1/4; w_0, R) \leq C(f_2) \deg f_1
\]
provided \( D(w_0, R) \subset D(0, \varepsilon_0/8) \). We will check this claim by using the previous lemma and Proposition 3.2. Let \( a_1, \ldots, a_n \) be the zeros of \( \psi \). Since \( f_1 \) and \( f_{2,k} \) are co-prime, using the classical Bézout theorem, we have
\[
n \leq \deg f_{2,k} \times \deg f_1.
\]
Factorize
\[
\psi(w) = h(w)P(w), \quad P(w) = \prod_{k=1}^n (w - a_k).
\]
From the proof of Proposition 3.2 (see (3.3)) we have
\[
B_\psi(1/4; w_0, R) \leq CR(M_h(0, 3\varepsilon_0/4) - M_h(0, \varepsilon_0/4)) + B_P(1/4; w_0, R).
\]
Recall that \( B_P(1/4; w_0, R) \leq n \log 4 \leq C(f_2) \deg f_1 \). So, to check the claim (7.10) we just need to estimate \( M_h(0, 3\varepsilon_0/4) - M_h(0, \varepsilon_0/4) \). Without loss of generality we can assume \( h(0) = 1 \) and therefore \( M_h(0, \varepsilon_0/4) \geq 0 \). Since \( h \) does not vanish we have
\[
h(w) = e^{u(w) + iv(w)},
\]
where \( u + iv \) is analytic and \( u, v \) are real-valued. Then
\[
M := M_h(0, 3\varepsilon_0/4) = \sup_{w \in D(0, 3\varepsilon_0/4)} |u(w)|.
\]
Due to the Borel-Carathéodory estimate (see [Lev96, Thm. 11.1.1])
\[
N := \sup_{w \in D(0, 7\varepsilon_0/8)} |v(w)| \gtrsim M.
\]
Choose \( |\tilde{w}| = 7\varepsilon_0/8 \) such that \( |v(\tilde{w})| \geq N/2 \) and at the same time no root \( a_k \) falls on the straight line through \( \tilde{w} \) and the origin. This allows us to define the continuous functions \( \theta_k(\xi) := \arg(\xi \tilde{w} - a_k) \in [0, 2\pi], \xi \in (-\infty, +\infty) \). Set
\[
\theta(\xi) = \sum_{1 \leq k \leq n} \theta_k(\xi).
\]
Take \( \theta \in (0, 2\pi) \) arbitrary. We have
\[
\text{Im} e^{-i\theta} \psi(\xi \tilde{w}) = e^{u(\xi \tilde{w})} |P(w)| \sin(v(\xi \tilde{w}) + \theta(\xi) - \theta).
\]
It is clear form this formula that if \( N \gg n \), then for any \( \theta \),
\[
\#\{\xi \in (-7\varepsilon_0/8, 7\varepsilon_0/8) : f_1(\xi \tilde{w}), \xi \tilde{w} \in \mathcal{L}_\theta \} \geq N/4,
\]
where \( \mathcal{L}_\theta \) is the line of angle \( \theta \) through the origin. This and Lemma 7.3 imply that we must have
\[
N \lesssim (\deg f_{2,k})^2 \deg f_1.
\]
Putting the above together we have
\[
M_h(0, 3\varepsilon_0/4) - M_h(0, \varepsilon_0/4) \lesssim C(f_2) \deg f_1,
\]
which completes the proof of claim (7.10).

Finally, it is clear that the conclusion holds by choosing the charts to be rescaled versions of \( \phi|_{D(0, \varepsilon_0/8)} \), for each \((z_0, w_0) \in \mathbb{R} \). \qed
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Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 1A1
E-mail address: gold@math.toronto.edu

Department of Mathematics, The University of Chicago, 5734 S. University Ave., Chicago, IL 60637, U.S.A.
E-mail address: schlag@math.uchicago.edu

Department of Mathematics, The University of Chicago, 5734 S. University Ave., Chicago, IL 60637, U.S.A.
E-mail address: mvoda@uchicago.edu