A Topological Ramsey Theorem

Paul J. Szeptycki

Department of Mathematics and Statistics
York University
Toronto Canada

and

Czech Academy of Sciences
szeptyck@yorku.ca
Joint work with Wiesław Kubiś

**Ramsey’s Theorem**

For any $r, n \in \omega$ and any $f : [\omega]^r \to n$ there is $H \subseteq \omega$ and $i < n$ such that

$$f(x) = i \text{ for all } x \in [H]^r$$

*H is homogeneous for f*

I.e., $\omega \to (\omega)^r_n$

Given $f : [\omega]^r \to K$ where $K$ is compact, in what sense can we assert that there is an $H$ homogeneous for $f$?
A generalized notion of a convergent sequence

**Definition**

Let $r \in \omega \setminus \{0\}$, $X$ a space, $S \subseteq \omega$ infinite and $f : [S]^r \to X$, $f$ converges to $p \in X$ if for every neighborhood $U$ of $p$ there is a finite set $F$ such that $f''[S \setminus F]^r \subseteq U$.

1. If $r = 1$, then $f : [S]^1 \to X$ is a sequence and this notion is the same as usual.
2. If $(x_n : n \in \omega) \to p$ and we define $f$ on $[\omega]^r$ by $f(s) = x_{\min(s)}$, then $f$ converges to $p$.
3. If $f : [S]^r \to X$ converges to $p$ and $\{s_i : i \in \omega\}$ is pairwise disjoint, then $(f(s_i) : i \in \omega) \to p$. 
Definition

Given \( r \in \omega \), a space \( X \) is said to be \( r \)-Ramsey if, for each \( f : [\omega]^r \to X \), there is \( S \subseteq \omega \) infinite such that \( f \upharpoonright [S]^r \) converges. \( X \) has the Ramsey property if it is \( r \)-Ramsey for all \( r \in \omega \).

1. 1-Ramsey \( \iff \) sequentially compact.
2. \( r + 1 \)-Ramsey \( \Rightarrow \) \( r \)-Ramsey
3. Ramsey’s Theorem can be restated as every finite space has the Ramsey property.
Compact metrizable spaces

**Theorem**

If $X$ is compact metrizable then it has the Ramsey property.

**Observations:**

1. Applying the theorem to finite $X$, we obtain Ramsey’s classical theorem as a corollary.

   $$\forall r, n \in \omega \left( \omega \rightarrow (\omega)^r_n \right)$$

2. $r = 1$: Compact metrizable spaces are sequentially compact.

3. $r = 2$: Due to M. Bojańczyk, E. Kopczyński, S. Toruńczyk. Applied to obtain idempotents in compact metrizable semigroups as limits of some particular functions on $[\omega]^2$. 

Paul J. Szeptycki  
A Topological Ramsey Theorem
Theorem

If X is compact metrizable then it is r-Ramsey for all r ∈ ω.

Proof: For each n fix a finite cover \( \mathcal{U}_n \) by \( 1/2^n \) balls and let \( f : [\omega]^r \to X \).

\( f \) and \( \mathcal{U}_n \) induce a finite coloring of \( [\omega]^r \).

Using Ramsey’s Theorem, let

\[ S_0 \supseteq S_1 \supseteq ... S_n \supseteq ... \] so that for all \( n \)

1. \( S_n \subseteq \omega \) is infinite.
2. the diameter of \( F_n = f''[S_n]^r \) is less than \( 1/2^n \)

If \( p \in \bigcap \{ F_n : n \in \omega \} \) and \( S \subseteq^* S_n \) for all \( n \)

then \( f \upharpoonright [S]^r \) converges to \( p \). ⊣
Corollary

If $X$ is compact and the closure of every countable set is first countable, then $X$ has the Ramsey property.

1. Any 1-point compactification of a discrete space is Ramsey
2. and so is any Corson compact,
3. and any compact linearly ordered space.

This can be improved a bit:

Theorem

*Sequentially compact spaces of character $< \frak{b}$ have the Ramsey property.*
Let $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ be almost disjoint, $\Psi(\mathcal{A})$ its Isbell-Mrówka space and $K(\mathcal{A})$ its one-point compactification.

**Example**

If $\mathcal{A}$ is a maximal almost disjoint family, then $K(\mathcal{A})$ is not 2-Ramsey (but is sequentially compact).

**Proof:** $K(\mathcal{A})$ is $r$-Ramsey if and only if it is $r$-Ramsey with respect to $f : [\omega]^r \rightarrow \omega$.

$f : [S]^r \rightarrow \omega$ converges to $a \in \mathcal{A}$ if and only if there is $n$ such that

$$f''[S \setminus n]^r \subseteq a$$

$f : [S]^r \rightarrow \omega$ converges to $\infty$ if and only if for every $a \in \mathcal{A}$ there is $n$ such that

$$f''[S \setminus n]^r \cap a = \emptyset$$
We may assume $A \subseteq [\omega \times \omega]^{\aleph_0}$ and $\{n\} \times \omega \in A$ for all $n$.

Define $f : [\omega]^2 \to K(A)$ by $f(\{k, n\}) = (k, n)$ ($k < n$)

Then, for any infinite $S \subseteq \omega$, and any $n$

$$f''[S \setminus n]^2 \in I^+(A)$$

**Lemma (Mathias)**

For $A$ mad, for any decreasing sequence $B_n \in I^+(A)$ there is $B \in I^+(A)$ such that $B \subseteq^* B_n$ for all $n$.

So, for any $S$, there is $A \in A$ such that for all $n$

$$f''[S \setminus n]^2 \cap A \text{ is infinite}$$

So, no $f \upharpoonright [S]^2$ can be convergent. $\dashv$
Preservation under operations

**Theorem**

The $r$-Ramsey property is preserved under

1. Closed subspaces
2. Continuous images
3. Countable products and $\Sigma$-products

**Theorem (van Douwen)**

The minimal cardinal $\kappa$ such that $2^\kappa$ is not sequentially compact is the splitting number $s$

\[ s = \min\{|F| : F \subseteq 2^\omega \text{ is splitting. I.e., for no } A \text{ is } f \upharpoonright A \text{ constant mod finite for all } f \in F. \} \]
$2^\kappa$ may be sequentially compact and not Ramsey

**Definition (Blass)**

1. $A$ is **almost homogeneous** for a family of functions $\mathcal{F} \subseteq 2^{[\omega]^r}$ if for each $f \in \mathcal{F}$ there is $n$ such that $f$ is constant on $[A \setminus n]^r$.
2. $\text{par}_r$ is the minimal cardinality of a family of functions $[\omega]^r \to 2$ with no almost homogeneous set.

**Theorem (Blass)**

For each $r \geq 2$, $\text{par}_r = \text{par}_2 = \min\{b, s\}$

Analogous to van Douwen’s characterization of $s$, we have

**Theorem**

$\text{par}_2$ is the **minimal cardinal** $\kappa$ such that $2^\kappa$ is not $r$-Ramsey.

And so, $b < s$ implies that $2^b$ is sequentially compact not 2-Ramsey.
Assuming CH ($\mathfrak{b} = \mathfrak{c}$ should suffice). For each $r$ there is an almost disjoint family $\mathcal{A}$ on $\omega$ such that $K(\mathcal{A})$ is $r$-Ramsey and not $(r + 1)$-Ramsey.

**Proof.** Build $\mathcal{A} = \{a_\alpha : \alpha \in \omega_1\}$ on $\omega^{r+1}$ starting with

$$\{a_n : n \in \omega\} = \{\{s\} \times \omega : s \in \omega^r\}$$

Not $(r + 1)$-Ramsey will be witnessed by $G$ defined by

$$G(\{k_0, k_1, \ldots, k_r\}_<) = (k_0, \ldots, k_r)$$

$(B_\alpha)_\alpha$ enumerate $[\omega]^\aleph_0$ and $(f_\alpha)_\alpha$ enumerate all $f : [\omega]^r \to \omega^{r+1}$

To make the construction work, we need to fix $S_\alpha$ convergent for $f_\alpha$ and add a new $a_\alpha$ witnessing $G \upharpoonright [B_\alpha]^{r+1}$ is not convergent.
Definition

FIN is the ideal of finite subsets of $\omega$.

$FIN^n$ is the Fubini product of FIN: defined recursively by $X \in FIN^{n+1}$ if

$$\{s \in \omega^n : \{k : s \upharpoonright k \in X\} \notin FIN\} \in FIN^n$$

1. $\{a_n : n \in \omega\} = \{\{s\} \times \omega : s \in \omega^r\} \subseteq FIN^{r+1}$,
2. and $a \in FIN^{r+1}$ whenever $a$ is a.d. from all $a_n$
3. $G''[B]^{r+1} \notin FIN^{r+1}$ for any $B$

Lemma

For every $f : [\omega]^r \to \omega^{r+1}$, there is $S \subseteq \omega$ such that

$$f''[S]^r \in FIN^{r+1}$$
(P. Simon): The productivity number for sequential compactness is $\mathfrak{h}$.

$\mathfrak{h}$ is the minimal number of mad families needed to split every infinite subset of $\omega$.

If $\{A_\alpha : \alpha < \mathfrak{h}\}$ witness, then

$$\prod_{\alpha < \mathfrak{h}} K(A_\alpha)$$

is not sequentially compact.

The productivity number for the Ramsey property is $\geq \mathfrak{h}$

Question

Are there $\mathfrak{h}$ many 2-Ramsey spaces whose product is not 2-Ramsey?