Singularities of Noncompact Charged Objects

M. Sharif * and G. Abbas †
Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract

We formulate a model of noncompact spherical charged objects in the framework of noncommutative field theory. The Einstein-Maxwell field equations are solved with charged anisotropic fluid. We choose the forms of mass and charge densities which belong to two parameter family of density distribution functions instead of densities as Gaussian width length. It is found that the corresponding densities and the Ricci scalar are singular at origin whereas the metric is nonsingular indicating a spacelike singularity. The numerical solution of the horizon equation implies that there are either two or one or no horizon depending on the mass. We also evaluate the Hawking temperature which implies that a black hole with two horizons is evaporated to an extremal black hole with one horizon.

Keywords: Noncommutative geometry; Electromagnetic field; Spacetime singularity.

PACS: 04.20.Cv; 04.20.Dw

1 Introduction

Recently, there has been a growing interest to study the nature of spacetime singularity and black holes (BHs) in the context of noncommutative (NC)
field theory. This is perhaps due to the fact that some problems of the BH physics remain unanswered even after passing many years \cite{1}. For example, the satisfactory description of the final stage of BH evaporation is still an unknown. According to BH correspondence principle \cite{2}, one cannot neglect the string effects during the last stage of BH evaporation. This has inspired the researchers to adopt the string field theoretic approach in the various aspects of theoretical physics. Noncommutative field theory is one of the outcomes of the string theory in which spacetime coordinates become noncommuting operators on a $D$-brane \cite{3},\cite{4}. This motivates the researchers to reconsider the older ideas of Snyder \cite{5}.

Zade et al. \cite{6} discussed the gravitational collapse of radiating Dyon field and clarified the status of CCH. Nashed \cite{7} derived the regular charged solutions in Mull"or's tetrad theory of gravitation. Ying et al. \cite{8} formulated the double NC form of a spacetime and introduced the complex symmetric theory of gravitation. Guang \cite{9} investigated the singularities of static spherically symmetric charged scalar field solutions.

The spacetime noncommutativity can be expressed by the following relation \cite{10}

$$[x^\mu, x^\nu] = i\sigma^{\mu\nu},$$  \hspace{1cm} (1)

where $\sigma^{\mu\nu}$ is an anti-symmetric matrix that determines the spacetime cell discretization as $\hbar$ (Planck's constant) discretizes the phase space. There are several approaches to formulate the NC field theory out of which one is based on the $\star$-product and another on the coordinate coherent state formalism. Using the second approach, Smailagic and Spallucci \cite{11} explored that the problems of Lorentz invariance and unitary arising in the $\star$-product can be solved by assuming $\sigma^{\mu\nu} = \sigma diag(\epsilon_{ij}, \epsilon_{ij},....)$, where $\sigma$ is constant having dimensions of length squared. Also, the coordinate coherent state modifies the Feynman propagators. It is believed that NC would remove the singularities (divergences) appearing in general relativity (GR). In GR, the metric field is a geometrical structure and curvature measures its strength. Since noncommutativity is the fundamental property of the metric, so it affects the geometry through the field equations and hence the energy-momentum tensor. In the NC GR, we take the geometry part of the field equations unchanged, while a small change is introduced in the matter part.

It was pointed out by Doplicer et al. \cite{12} that when matter density is extremely large, a BH is formed. According to Heisenberg uncertainty principle, the measurement of a spacetime separation causes an uncertainty in
momentum, i.e., momentum preserves inverse proportionality to the extent of the separation. For small enough separation, momentum becomes large and system leads to the BH formation. The behavior of BH in the NC field theory has been studied by many people. Banerjee et al. [13] examined the behavior of the NC Schwarzschild BH while Modesto and Nicolini [14] discussed the charged rotating NC BH. Bastos et al. [15],[16] explored the non-canonical phase space, singularity problem and BH in the context of NC geometry. Also, Bartolami and Zarro [17] investigated the NC correction to pressure, particle numbers and energy density for fermion gas and radiations. The NC correction to these quantities lead to the fact that NC affects the matter dispersion relation and equation of state. Inspired by the NC correction to BH physics, Oh and Park [18] explored the gravitational collapse of shell with smeared gravitational source in the NC Schwarzschild geometry. We have extended this work for NC Reissner-Nordström background [19]. Sun et al. [20] studied gravitational collapse of spherically symmetric star in NC GR using spacetime quantization approach.

In all these papers, the matter density is taken as Gaussian distribution function. However, Castro [21] studied the singularity associated with non-compact matter source by using matter density in a particular form which belongs to the most general two parametric form of the density distribution. Here we extend this work to study the nature of singularity associated with noncompact charged matter sources extending from $r = 0$ to $r = \infty$. The plan of the paper is as follows: In the next section, we formulate the noncompact charged object model. Section 3 is devoted to discuss some properties of this model. In the last section, we conclude our results.

2 Noncompact Charged Objects Model

To formulate noncompact charged objects model, we solve the Einstein-Maxwell field equations with anisotropic fluid and electromagnetic field. In the paper [21], instead of choosing density in the form of Gaussian width length [10]

\[
\rho = \frac{M_0 e^{-\frac{r^2}{4\sigma^2}}}{(4\pi\sigma^2)^{\frac{3}{2}}},
\]
it was taken as
\[ \rho(r, \sigma) = \frac{M_0}{4\pi r^2} \frac{3\sigma^3}{2} \frac{1}{r^4[1 + \left(\frac{\sigma}{r}\right)^3]^2}, \] (3)
where \( \sigma \) is NC parameter and the constant \( M_0 \) corresponds to the total gravitational mass of the system.

The form of matter density \( \rho \), given in Eq.(3) belongs to the most general two parameters family of the density distribution function
\[ \rho(r, \sigma, k) = \frac{M_0}{4\pi r^2} \frac{k\sigma^k}{2} \frac{1}{r^{4+k}[1 + \left(\frac{\sigma}{r}\right)^3]^2}, \quad k > 2. \] (4)

We follow [21] and take the same expression for matter density as in (3) and use the charge density in the form
\[ \rho(r, \sigma) = \frac{q_0}{4\pi r^2} \frac{3\sigma^3}{2} \frac{1}{r^4[1 + \left(\frac{\sigma}{r}\right)^3]^2}, \] (5)
instead of
\[ \rho = \frac{q_0 e^{-r^2/(4\sigma^2)}}{(4\pi\sigma^2)^{3/2}}, \] (6)
as given in [22]. Here constant \( q_0 \) corresponds to the total charge of the system.

Taking matter density \( \rho \) and electric density \( \rho_{el} \) as defined in Eqs.(2) and (6), respectively, many authors [10],[13],[14],[22] have derived the neutral as well as charged and rotating NC BH solutions. All these exhibit a de-Sitter core at the origin of the underlying geometry due to the quantum fluctuations. Applying the same strategy as in above studies but using a modified definition of the matter density (as defined in Eq.(3)), Castro [21] derived a NC solution which has no de-Sitter core and exhibits the timelike naked singularity at origin. The present letter is the extension of [21] to the electromagnetic field case.

We take a spherically symmetric spacetime in the following form
\[ ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \] (7)
where \( f(r) \) is unknown function to be determined from the field equations.
The matter under consideration is charged anisotropic fluid. The components of anisotropic fluid energy-momentum tensor are

\[ T_\mu^\nu = (\rho(r), -p_r, -p_\theta, -p_\phi). \] (8)

The energy-momentum tensor of the electromagnetic field is

\[ T_\mu^\nu(\text{em}) = \frac{1}{4\pi}(-F_\nu^\lambda F_\mu^\lambda + \frac{1}{4}\delta_\mu^\nu F_\pi^\lambda F_\pi^\lambda). \] (9)

The Maxwell equations are given by

\[ F_\mu^\nu = \phi_{\nu,\mu} - \phi_{\mu,\nu}, \quad F_\mu^{\nu,\nu} = 4\pi J_\mu, \] (10)

where \( F_{\mu\nu} \) is the Maxwell field tensor, \( \phi_{\mu} \) is the four potential and \( J_{\mu} \) is the four current. For the static spherically symmetric charge distribution, the four potential and the four current are taken in the following form

\[ \phi_{\mu} = \phi^0, \quad J_{\mu} = \rho_{\text{el}} \delta_{\mu}^0, \] (11)

where \( \phi \) is electric potential and \( \rho_{\text{el}} \) is the electric charge density. Solving the Maxwell field equations, we get

\[ F_{01} = -\frac{\partial \phi}{\partial r} = \frac{q(r, \sigma)}{r^2}, \] (12)

where

\[ q(r, \sigma) = 4\pi \int_0^r r^2 \rho_{\text{el}} dr = \frac{q_0}{\sqrt{1 + (\frac{r}{r_0})^2}}. \] (13)

In order to find the relation between different components of the energy-momentum tensor, we use the conservation of energy-momentum tensor, i.e, \( \tilde{T}_{\nu,\mu} = 0 \ (\tilde{T}_\nu^\mu = T_\nu^\mu + T_{\nu}^{\text{em}}) \). For the Schwarzschild like nature of the solution, i.e, \( g_{00} = (g_{rr})^{-1} \), we restrict that \( T_0^0 = T_r^r \), which implies that \( p_r = -\rho(r) \) (negative radial pressure pointing towards the center \( r = 0 \)) and \( p_\theta = p_\phi \). With these conditions, the conservation equation provides the following relation

\[ p_\theta = -\frac{r}{2} \frac{\partial_r}{r^2} (\rho + \frac{q^2}{8\pi r^4}) - \rho - \frac{q^2}{4\pi r^4}. \] (14)
The Einstein field equations with Eqs. (3), (7)-(9), (12) and (14) lead to the following solution

\[ ds^2 = (1 - \frac{2m(r, \sigma)}{r} - \frac{Q^2(r, \sigma)}{r})dt^2 - (1 - \frac{2m(r, \sigma)}{r} - \frac{Q^2(r, \sigma)}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  

(15)

where

\[ m(r, \sigma) = 4\pi \int_0^r \rho r^2 dr = \frac{m_0}{\sqrt{1 + \left(\frac{2}{r}\right)^3}} \]  

(16)

and

\[ Q^2(r, \sigma) = \frac{q_0^2}{18\sigma} \left( \sqrt{3\pi} - 6\sqrt{3} \arctan \left( \frac{\sigma - 2r}{\sqrt{3}\sigma} \right) + \log \left( \frac{(\sigma^2 - \sigma r + r^2)^3}{(\sigma + r)^6} \right) \right) \]  

(17)

The solution given in Eq. (15) represents a noncompact charged object.

3 Properties of the Model

This section is devoted to explore the properties of charged object model. In the limit \( \sigma \to 0 \), the line element (15) reduces to the classical Reissner-Nordström solution. We observe that the metric component

\[ g_{tt} = 1 - \frac{2m(r, \sigma)}{r} - \frac{Q^2(r, \sigma)}{r} \]  

(18)

with Eqs. (16) and (17) satisfies the condition \( g_{tt}(r = 0) = g_{tt}(r = \infty) = 1 \). This implies that the solution (15) is asymptotically flat and not singular at \( r = 0 \). The corresponding Ricci scalar is

\[ R = -\frac{3}{2} \left( \frac{\sigma}{r^3} \right)^3 \left( 5\sigma^3 m + 2r^2 q_0^2 \sqrt{1 + \left( \frac{\sigma}{r^3} \right)^3 - 4mr^3} \right) \]  

(19)

which is singular at \( r = 0 \). Also, the Ricci scalar can be found directly from the field equations as \( R = -T^\mu_\mu \), where we have used \( T^\mu_\nu^{(em)} = 0 \). Further, using the values of \( T^\mu_\nu \) along with \( p_r \) and \( p_\theta \), we get

\[ R = -\left( 4\rho + \frac{r}{\partial_r} (\rho + \frac{q^2}{8\pi r^4}) + \frac{q^2}{4\pi r^4} \right) \]  

(20)
After a straightforward but laborious calculations, we get the same relation for $R$ as in (19). This verifies the fact that the solution (15) is the valid solution of the Einstein field equations with the charged anisotropic source. Despite the fact that the metric is nonsingular at $r = 0$, the Ricci scalar is singular at $r = 0$. This is due to the direct dependence of the Ricci scalar on the matter density, which is singular at $r = 0$. When $g_{00}(r) = 0$ at the horizon radius $r = r_H$, then the metric (15) becomes singular, however, this is a coordinate singularity.

The horizon equation, $g_{00}(r) = 0$, for the solution (15) indicates that it is not possible to find exact solution of the horizon equation for $r$. Thus with the arbitrary choice of parameters, we evaluate the values of $r$ for $g_{00}(r) = 0$ shown in the left graph of Figure 1. This shows that for $m_0 = 0.44$, there exists one horizon (extremal black hole green curve), when mass is smaller than this value, there exists no horizon (blue curve) and two horizons exist (inner and outer horizons pink curve) for mass larger than $m_0 = 0.44$. Thus the case $m_0 < 0.44$ corresponds to the existence of naked singularity, as there do not exist horizons to hide the singularity at $r = 0$. Thus the minimum mass for the existence of BH in this framework is $m_0 = 0.44$. Since matter density as well as the Ricci scalar diverge at $r = 0$, while the metric components are finite there so singularity is spacelike. Thus, we find that the solution represents a spacelike naked singularity, extremal BH and a BH with inner and outer horizons depending on the mass.

The Hawking temperature for the BH is

$$T_H = \left( \frac{1}{4\pi} \frac{dg_{00}}{dr} \right)_{r=r_H}. \quad (21)$$

The behavior of $T_H$ (after eliminating $m = m(r, q_0, \sigma)$ from $\frac{dg_{00}}{dr}$ and $g_{00}(r) = 0$) is shown in the right graph of Figure 1. As $r_H$ decreases, temperature increases and attains a maximum value at $r_H \simeq 2$, then suddenly drops to zero at $r_H \simeq 1.1$ corresponding to the radius of the extremal BH. Since further decrease in $r_H$ results to negative temperature which is not acceptable. Thus there does not exist naked singularity at $r_H = 0$ in this case. Consequently, singularity at $r = 0$ is covered by extremal horizons, hence the possibility of mass less than critical mass is excluded from the discussion.

The radial null geodesics from the singularity at $r = 0$ reach to an observer in the finite (naked singularity case) and infinite (one and two horizons cases)
Figure 1: The left graph is $g_{00}$ versus $r$, $q_0 = 1$, $\sigma = 1$, the intercepts on the horizontal axis give the radius of event horizons. For $m_0 = 1$ (pink curve) two horizons; $m_0 = 0.44$ (green curve) extremal BH and $m_0 = 0.3$ (blue curve) no horizon. The right graph shows that $T_H$ versus $r_H$. In this graph $T_H = 0$ for $r_H \simeq 1.1$, i.e., for the extremal BH, while $T_H \simeq 0.32$ corresponds to mass $m \simeq 0.6$.

time. The equation of the radial null geodesics is

$$ds^2 = (1 - \frac{2m(r, \sigma)}{r} - \frac{Q^2(r, \sigma)}{r}) dt^2 - (1 - \frac{2m(r, \sigma)}{r} - \frac{Q^2(r, \sigma)}{r})^{-1} dr^2 = 0, \quad (22)$$

which can be written as

$$t = \int_0^r \frac{dr}{1 - \frac{2m(r, \sigma)}{r} - \frac{Q^2(r, \sigma)}{r}}. \quad (23)$$

For one and two horizon cases, i.e., $m_0 = 0.44$ and $m_0 > 0.44$, respectively, the graph shows that the integral diverges. In both cases, we get $t = \infty$ for $r \neq \infty$. This implies that the light signal coming from the singularity along the null radial geodesics take infinite time to reach the distant observer, these never reach to the observer at $r \neq \infty$.

4 Conclusion

In this letter, we have formulated a new static, spherically symmetric charged solution of the Einstein-Maxwell field equations in the framework of NC geometry. This describes the final stage of the collapsing charged noncompact object. This work extends the work of Castro [21] to the charge case and its results can be recovered by taking the charged parameter $Q = 0$. The present solution reduces to the classical RN solution, by taking the NC parameter $\sigma \rightarrow 0$. It is found that the solution is asymptotically flat and not singular.
at the origin, while the Ricci scalar and matter density are singular, hence spacelike singularity appears in this case. The horizons analysis implies that for a suitable set of initial data and $m_0 < 0.44$, $m_0 = 0.44$ and $m_0 > 0.44$, there exist no horizon, single horizon and two horizons respectively. However, the thermodynamical results imply that the final stage of the object is extremal BH containing a singularity at $r = 0$.

In the paper [21], the horizons equation is cubic polynomial which admits three possible exact solutions that correspond to two horizons, one horizon and no horizon cases. It was concluded on the basis of Hawking temperature (with the assumption that given matter configuration has mass less than the critical mass) that there exists timelike naked singularity. In our case, we conclude that spacelike singularities associated with the noncompact charged object are covered by extremal BH horizon. There is curvature singularity at $r = 0$, while there is de-Sitter geometry around the origin [10] instead of curvature singularity.

Acknowledgment

We would like to thank the Higher Education Commission, Islamabad, Pakistan for its financial support through the Indigenous Ph.D. 5000 Fellowship Program Batch-IV.

References

[1] Padmanabhan, T.: Phys. Rep. 406(2005)49.
[2] Susskind, L.: Phys. Rev. Lett. 71(1993)2367.
[3] Witten, E.: Nucl. Phys. B460(1996)335.
[4] Siberg, N. and Witten, E.: JHEP 9909(1999)032.
[5] Snyder, S.H.: Phys. Rev. 71(1947)38.
[6] Zade, S.S., Patil, K.D and Mohod, A.D.: Chinese Phys. Lett. 25(2008)854.
[7] Nashed, G.G.L.: Chinese Phys. Lett. 24(2007)3059.
[8] Ying, S., Ya-Bo, W. and Peng, D.: Acta Physica Sinica, 53(2004)2846.

[9] Guang, C.: Chinese Phys. B10(2001)787.

[10] Nicolini, P., Smailagic, A. and Spallucci, E.: Phys. Lett. B632(2006)547.

[11] Smailagic, A. and Spallucci, E.: J. Phys. A36(2003)L 467.

[12] Doplicher, S., Fredenhagen, K. and Robert, J.E.: Commun. Math. Phys. 172(1995)187.

[13] Banerjee, R., Majhi, R.B. and Modak, S.K.: Class. Quantum Grav. 26(2009)085010.

[14] Modesto, L. and Nicolini, P.: Phys. Rev. D82(2010)104035.

[15] Bastos, C., Bertolami, O., Dias, N.C. and Prata. J.N.: Phys. Rev. D80(2009)124038.

[16] Bastos, C., Bertolami, O., Dias, N.C. and Prata. J.N.: Phys. Rev. D84(2011)024005.

[17] Bertolami, O. and Zaro, C.D.A.: Phys. Rev. D81(2010)025005.

[18] Oh, J.J. and Park, C.: JHEP 1003(2010)86.

[19] Sharif, M. and Abbas, G.: Noncommutative Correction to Thin Shell Collapse in Reissner-Nordström Geometry, submitted for publication.

[20] Sun et al.: Eur. Phys. J. C69(2010)271.

[21] Castro, C.: Phys. Lett. B665(2008)384.

[22] Ansoldi, S., Nicolini, P., Smailagic, A. and Spallucci, E.: Phys. Lett. B645(2007)261.