A TANNAKIAN APPROACH TO PATCHING

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Abstract. We use Tannakian methods to show that patching for coherent sheaves implies patching for objects in any Noetherian algebraic stack with affine stabilizers. Among other things, this gives a straightforward way to prove patching for torsors under linear algebraic groups, as well as patching for sheaves and torsors on proper algebraic spaces.

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1. Introduction

In this paper, we use Tannakian methods to show that the following holds in all typical circumstances.

Meta-Theorem. If patching holds for coherent sheaves then it holds for objects in any Noetherian algebraic stack with affine stabilizers.

Thus, for example, patching holds for principal bundles for linear algebraic groups over proper schemes. As a consequence, we obtain new contexts to apply methods of field patching, as well as new Meyer-Vietoris type sequences for étale cohomology groups. This holds for various types of patching contexts (formal, rigid, field patching, etc.).

The technique of field patching has been exploited in various ways to obtain information about algebraic structures (particularly torsors for linear algebraic groups) over function fields of curves over complete discretely valued fields. The question of finding similar techniques to handle function fields of higher dimensional varieties is still quite open, and of much interest. The present manuscript provides a relative context in which we can now obtain results in this direction.

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The methods of this paper break up naturally into two parts. In the first part, we explain how the Tannakian formalism gives a general context in which one can extend patching results for coherent sheaves to similar results for objects in any Noetherian algebraic stack with affine stabilizers, including $G$-torsors for $G$ a linear algebraic group (Theorem 3.0.1). This can then be applied to obtain new cases of patching for torsors over rings and fields (see Corollaries 3.1.2, 3.1.5 and 3.1.8).

In the second part, we show that patching for coherent sheaves over a base scheme $X$ can often be extended to patching for coherent sheaves relative to a proper morphism $S \to X$. Combined with the above results, we then obtain patching results for $G$-torsors on proper algebraic spaces over $X$ (see Theorem 4.0.1).

2. Notation and preliminaries

2.1. Generalities on 2-equalizers. In this paper, by 2-category, we will mean what is often referred to as a strict 2-category, and will follow the definitions of, for example, [Gra74, Section I.2.1]. The concept of a (strict) 2-equalizer can both refer to a specific construction in the 2-category of categories, or an object with universal properties in a general 2-category. We will have use for both and define them below:

**Definition 2.1.1 (2-equalizers of morphisms in 2-categories).** Let $\mathcal{C}$ be a 2-category, and $f_0, f_1 : A_0 \to A_1$ a pair of morphisms in $\mathcal{C}$. A 2-equalizer of the morphisms $f_0, f_1$ is

1. an object $A$,
2. a 1-morphism $f : A \to A_0$, and
3. a 2-isomorphism $\alpha : f_1 f \Rightarrow f_0 f$

satisfying the following properties:

1. given any other object $B$, morphism $g : B \to A_0$ and 2-isomorphism $\beta : f_1 g \Rightarrow f_0 g$, there is a unique morphism $h : B \to A$ such that

\[
\begin{array}{c}
A \xrightarrow{f} A_0 \\
\downarrow h \\
B \\
\uparrow g
\end{array}
\]

commutes,

2. in the above setting, the horizontal composition $\alpha : f_1 g = f_1 fh \Rightarrow f_0 fh = f_0 g$ coincides with $\beta$,

3. given a pair of morphisms $g_1, g_2 : B \to A_0$, 2-isomorphisms $\beta_i : f_1 g_i \Rightarrow f_0 g_i$, and corresponding morphisms $h_i : B \to A$ as above, there is a bijection between natural transformations $\gamma : h_1 \to h_2$ and natural transformations $\gamma' : g_1 \to g_2$ such that we have a commutative diagram of natural transformations:

\[
\begin{array}{c}
\begin{array}{c}
f_1 g_1 \xrightarrow{\gamma} f_1 g_2 \\
\downarrow o_{g_1} \\
f_0 g_1 \xrightarrow{f_1 \gamma} f_0 g_2
\end{array}
\end{array}
\]

It is interesting to note that this is somewhat implicit in [Fal94] lines 22-23, page 358].
Note that the universal property is very strong, in that the diagrams are required to commute on the nose, not up to 2-isomorphism. It follows that the 2-equalizer, when it exists, is unique up to canonical isomorphism.

**Definition 2.1.2** (2-equalizers of functors). Let $C_0 \xrightarrow{\delta_0} C_1$ be a pair of functors between (l-)categories. We define the equalizer of $\delta_0$ and $\delta_1$, written $\text{Eq}(\delta_0, \delta_1)$, to be the category whose objects are pairs $(a, \phi)$ with $a \in \text{ob}(C_0)$ and $\phi : \delta_1 a \to \delta_0 a$ is an isomorphism, and whose morphisms $(a, \phi) \to (a', \phi')$ are morphisms $f : a \to a'$ in $C_0$ such that the diagram

$$
\begin{array}{ccc}
\delta_1 a & \xrightarrow{\phi} & \delta_0 a \\
\downarrow{\delta_1 f} & & \downarrow{\delta_0 f} \\
\delta_1 a' & \xrightarrow{\phi'} & \delta_0 a'
\end{array}
$$

commutes.

It turns out that this is a 2-equalizer in the 2-category of categories, and hence the notation is (relatively) unambiguous.

**Lemma 2.1.3.** Let $C_0 \xrightarrow{\delta_0} C_1$ be a pair of functors between (l-)categories. Then $\text{Eq}(\delta_0, \delta_1)$ is a 2-equalizer of $\delta_0, \delta_1$ in the 2-category of categories.

**Proof.** Let $C = \text{Eq}(\delta_0, \delta_1)$, and let $D$ be any category. We first need to show that functors $F : D \to C$ are in bijection with pairs $(F_0, \beta)$, where $F_0 : D \to C_0$ is a functor, and $\beta : \delta_1 F_0 \to \delta_0 F_0$ is a natural isomorphism. By definition, a morphism $F : D \to C$ gives, for every object $d$ of $D$, a pair $F d = (d_0, \phi_d)$, where $d_0 \in C_0$ and $\phi_d : \delta_1 d_0 \to \delta_0 d_0$ is an isomorphism. Set $F_0 d = d_0$, and $\beta d = \phi_d : \delta_1 F_0 d \to \delta_0 F_0 d$. For a morphism $a : d \to d'$, $F a : (F_0 d, \beta d) \to (F_0 d', \beta d')$ is a morphism in $C$, and by definition, this gives a morphism $F_0 a : F_0 d \to F_0 d'$ such that the diagram

$$
\begin{array}{ccc}
\delta_1 F_0 d & \xrightarrow{\delta_1 F_0 a} & \delta_1 F_0 d' \\
\downarrow{\beta d} & & \downarrow{\beta d'} \\
\delta_0 F_0 d & \xrightarrow{\delta_0 F_0 a} & \delta_0 F_0 d'
\end{array}
$$

commutes. But this exactly says that $\beta$ is a natural isomorphism from $\delta_1 F_0$ to $\delta_0 F_0$.

The above shows how to obtain a pair $(F_0, \beta)$ from a morphism $F : D \to C$. For the reverse, suppose we have a pair $F_0 : D \to C_0$ and a natural isomorphism $\beta : \delta_1 F_0 \to \delta_0 F_1$. Then for each $d$ in $D$, we have an isomorphism $\beta d : \delta_1 F_0 d \to \delta_0 F_0 d$, which gives an object $(F_0 d, \beta d)$ of the 2-fiber product of functors $C$. For a morphism $a : d \to d'$, we have a commutative square

$$
\begin{array}{ccc}
\delta_1 F_0 d & \xrightarrow{\delta_1 F_0 a} & \delta_1 F_0 d' \\
\downarrow{\beta d} & & \downarrow{\beta d'} \\
\delta_0 F_0 d & \xrightarrow{\delta_0 F_0 a} & \delta_0 F_0 d'
\end{array}
$$
which by definition gives a map to $C$.

Let $G : C \to C_0$ be the canonical morphism, and $\alpha : \delta_1 G \to \delta_0 G$ the canonical natural transformation defined by $\alpha(c, \phi) = \phi$.

Now consider a pair of functors $F, F' : D \to C$, and corresponding pairs $(F_0, \beta), (F'_0, \beta')$. We want to show that horizontal composition with $\alpha$ gives a bijection between natural transformations $F \to F'$ and natural transformations $\gamma : F_0 \to F'_0$ such that we have a commutative diagram

\[
\begin{array}{c}
\delta_1 F_0 \\ ^\beta \downarrow \\
\delta_0 F_0
\end{array}
\begin{array}{c}
\delta_1 F'_0 \\ \downarrow \gamma \\
\delta_0 F'_0
\end{array}
\]

But this follows from the fact that by definition of $\alpha$, $\alpha F = \beta$ and $\alpha F' = \beta'$. \qed

Further, we can use this categorical 2-equalizer to express the universal property of 2-equalizers in general as follows:

**Lemma 2.1.4.** Suppose that we have a 2-category $\mathcal{C}$, and morphisms $f_0, f_1 \in \text{Hom}_\mathcal{C}(A_0, A_1)$. Suppose $(A, f, \alpha)$ is a 2-equalizer of $f_0, f_1$. Then for every object $B$ of $\mathcal{C}$, we have an isomorphism (not just an equivalence!) of categories

\[\text{Hom}_\mathcal{C}(B, A) \to \text{Eq}(\text{Hom}_\mathcal{C}(B, A_0), \text{Hom}_\mathcal{C}(B, A_1))\]

defined on objects by taking $h : B \to A$ to $(fh, \alpha h)$, where $\alpha h$ is the horizontal composition of $\alpha : f_1 f \to f_0 f$ with $h$.

**Proof.** We note that the fact that the above extends to a fully faithful functor follows from the previous lemma.

We will illustrate the inverse isomorphism on the level of objects. Suppose we have an object in the equalizer category on the right hand side. This consists of a morphism $g : B \to A_0$ together with an isomorphism of morphisms (a 2-isomorphism) $\beta : f_0 g \to f_1 g$. By definition, this gives a unique morphism $B \to A$. \qed

**Lemma 2.1.5.** Suppose $\delta_0, \delta_1 : \mathcal{A}_0 \to \mathcal{A}_1$ are a pair of functors between abelian categories. Let $\eta : \mathcal{A}_1 \to \mathcal{A}_0$ be the 2-equalizer of these functors.

1. If $\delta_0, \delta_1$ are additive, then $\eta$ is additive and faithful.
2. If in addition $\delta_0, \delta_1$ are exact, then $\eta$ is faithfully exact (i.e. a sequence is short exact in $\mathcal{A}$ if and only if its image in $\mathcal{A}_0$ is short exact).

**Proof.** Suppose $\delta_0, \delta_1$ are additive. For objects $a = (a, \phi), a' = (a', \phi')$ of $\mathcal{A}$, by definition $\text{Hom}_{\mathcal{A}_0}(a, a')$ is the subgroup of $\text{Hom}_{\mathcal{A}_0}(a, a')$ consisting of those $f : a \to a'$ such that $(\phi')(\delta_1 f) = (\delta_0 f)(\phi)$. It follows that $\eta$ is additive and faithful.

Suppose in addition that $\delta_0, \delta_1$ are exact. Given a morphism $f : a' \to a$, if we set $k, c$ to be the kernel and cokernel of $\eta f$ in $\mathcal{A}_0$, then considering the diagram:

\[
\begin{array}{ccccc}
0 & \to & \delta_1 k & \to & \delta_1 a' & \to & \delta_1 a & \to & \delta_1 c & \to & 0 \\
& & \downarrow \phi' & & \downarrow \phi & & & & & \\
0 & \to & \delta_0 k & \to & \delta_0 a' & \to & \delta_0 a & \to & \delta_0 c & \to & 0
\end{array}
\]
We find that since the rows are exact (since \( \delta_0, \delta_1 \) are exact), and the vertical maps are isomorphisms, there are unique isomorphisms \( \psi_k : \delta_1^k \to \delta_0^k \) and \( \psi_c : \delta_1 c \to \delta_0 c \) which make the diagram commute. One can then check that \((k, \psi_k)\) and \((c, \psi_c)\) (together with the above isomorphisms) are the kernel and cokernel in \( \mathcal{A} \) of the map \( a' \to a \). It follows that formation of kernels and cokernels commutes with \( \eta \) (and hence images as well, viewed as the cokernel of a kernel). It follows that \( \eta \) is faithfully exact as claimed. \( \square \)

**Remark 2.1.6.** As a consequence of Lemma 2.1.5, if \( \delta_0, \delta_1 \) are both additive exact functors, and if \( \mathcal{B} \to \mathcal{A} \) is an additive functor to the 2-equalizer, then it is (faithfully) exact if the associated functor \( \mathcal{B} \to \mathcal{A}_0 \) is (faithfully) exact.

### 2.2. 2-equalizers of abelian monoidal categories.

**Lemma 2.2.1.** If \( \delta_0, \delta_1 : \mathcal{C}_0 \to \mathcal{C}_1 \) are monoidal functors between monoidal categories, then \( \text{Eq}(\delta_0, \delta_1) \) has a canonical monoidal structure given by \((a, \phi) \otimes (a', \phi') = (a \otimes a', \phi \otimes \phi')\), such that the map \( \delta : \text{Eq}(\delta_0, \delta_1) \to \mathcal{C}_0 \) can be naturally extended to a monoidal functor, and such that the canonical natural transformation \( \delta_1 \delta \to \delta_0 \delta \) is monoidal.

Further, if \( \mathcal{C}_0, \mathcal{C}_1 \) have a braiding (resp. symmetric braiding), and \( \delta_0, \delta_1 \) are braided monoidal functors, then \( \text{Eq}(\delta_0, \delta_1) \) has a natural braided (resp. symmetric braided) structure so that the morphism \( \delta \) is braided monoidal.

Said another way, the forgetful (2-)functor from monoidal (braided, resp. symmetric) categories to categories, preserves 2-equalizers.

In order to prove this lemma, we will begin by fixing some language and notation. Following [EGNOL5] when we say that \( \mathcal{C}_i \) is abelian monoidal \((i = 0, 1)\) we mean that have a 6-tuple \((\mathcal{C}_i, \otimes, \alpha, \ell^i, r^i)\) consisting of additive bifunctors \( \otimes : \mathcal{C}_i \times \mathcal{C}_i \to \mathcal{C}_i \), “associativity constraints,” which are isomorphisms \( \alpha^i_{a,b,c} : (a \otimes b) \otimes c \cong a \otimes (b \otimes c) \), natural in \( a, b, c \), unit objects \( 1_i \in \mathcal{C}_i \) with isomorphisms \( \ell^i_a : 1_i \otimes a \to a \), \( r^i_a : a \otimes 1_i \to a \) both natural in \( a \), such that for every \( a, b, c, d \) objects in \( \mathcal{C}_i \), we have a commutative pentagon

\[
\begin{array}{ccc}
\alpha^i_{a,b,c \otimes \text{id}_d} & \rightarrow & \alpha^i_{a \otimes b, c, d} \\
((a \otimes b) \otimes c) \otimes d & \rightarrow & (a \otimes b) \otimes (c \otimes d) \\
\text{id}_{a \otimes b} \otimes \alpha^i_{b,c,d} & \rightarrow & \alpha^i_{a,b \otimes c,d} \\
(a \otimes (b \otimes c) \otimes d) & \rightarrow & a \otimes (b \otimes (c \otimes d))
\end{array}
\]

and such that for every \( a, b \) objects in \( \mathcal{C}_i \), we have a commutative triangle

\[
\begin{array}{ccc}
\alpha^i_{a,1_i,b} & \rightarrow & a \otimes (1_i \otimes b) \\
(a \otimes 1_i) \otimes b & \rightarrow & a \otimes (1_i \otimes b) \\
\text{id}_a \otimes r^i_{b} & \rightarrow & a \otimes (1_i \otimes b)
\end{array}
\]

In this case, we note [EGNOL5 Corollary 2.2.5] that we have \( \ell^i_1 = r^i_1 \), which we denote as \( \iota_i : 1_i \otimes 1_i \to 1_i \). To say that the functors \( \delta_i : \mathcal{C}_0 \to \mathcal{C}_1 \) are monoidal is to say that we have
specified pairs \((\delta_i, J')\), where \(\delta_i\) is an additive functor, and \(J_{a,b}^\iota: \delta_i(a) \otimes \delta_i(b) \xrightarrow{\delta_i} (a \otimes b)\) are isomorphisms, natural in \(a, b\) such that for each \(a, b, c\) we have a commutative hexagon:

\[
\begin{array}{ccc}
\delta_i(a) \otimes \delta_i(b) & \xrightarrow{\alpha_{\delta_i(a), \delta_i(b), \delta_i(c)}} & \delta_i(a) \otimes (\delta_i(b) \otimes \delta_i(c)) \\
J_{a,b}^\iota \otimes \text{id}_{\delta_i(c)} & & \text{id}_{\delta_i(a)} \otimes J_{b,c}^\iota \\
\delta_i(a \otimes b) \otimes \delta_i(c) & & \delta_i(a) \otimes (\delta_i(b \otimes c)) \\
J_{a \otimes b, c}^\iota & & \delta_i((a \otimes b) \otimes c)
\end{array}
\]

and as in [EGNO15, Remark 2.4.6] we require that there is an isomorphism \(\epsilon_i : 1 \rightarrow \delta_i(1_0)\) such that for every object \(a \in \mathcal{C}_0\), we have commutative squares

\[
\begin{array}{ccc}
1 \otimes \delta_i(x) & \xrightarrow{\iota_{\delta_i(x)}} & \delta_i(x) \\
\epsilon_i \otimes \text{id}_{\delta_i(x)} & & \delta_i(\iota_{\delta_i(x)}) \\
\delta_i(1_0) \otimes \delta_i(x) & \xrightarrow{J_{\delta_i(x)}} & \delta_i(1_0 \otimes x) \\
J_{a \otimes b, x}^\iota & & \delta_i(1_0 \otimes 1_1)
\end{array}
\]

but which we will simply consider as an identification \(1_1 = \delta_i(1_0)\).

We say that \(\omega\) is a monoidal natural transformation between monoidal functors \((f, J), (f', J')\) if it is a natural transormation from \(f\) to \(f'\) which commutes with \(J, J'\) in the sense that we have a commutative diagram for every \(a, b:\)

\[
\begin{array}{ccc}
f(a) \otimes f(b) & \xrightarrow{J_{a,b}} & f(a \otimes b) \\
\phi(a) \otimes \phi(b) & & \phi(a \otimes b) \\
f'(a) \otimes f'(b) & \xrightarrow{J'_{a,b}} & f'(a \otimes b)
\end{array}
\]

**Proof of Lemma 2.2.7.** We define a monoidal structure on \(\text{Eq}(\delta_0, \delta_1)\) by setting \((a, \phi) \otimes (b, \psi) = (a \otimes b, \phi \otimes \psi)\), where \(\phi \otimes \psi\) is defined via the commutative square

\[
\begin{array}{ccc}
\delta_1(a \otimes b) & \xrightarrow{J^1_{a,b}} & \delta_1(a) \otimes \delta_1(b) \\
\phi \otimes \psi & & \phi \otimes \psi \\
\delta_0(a \otimes b) & \xrightarrow{J^0_{a,b}} & \delta_0(a) \otimes \delta_0(b).
\end{array}
\]

We claim that the morphisms

\[
\alpha_{(a, \phi), (b, \psi), (c, \theta)} : (a, \phi) \otimes ((b, \psi) \otimes (c, \theta)) \rightarrow ((a, \phi) \otimes (b, \psi)) \otimes (c, \theta)
\]

given by the associativity constraint for \(\mathcal{C}_0\), which we write as \(\alpha_{a, b, c} : a \otimes (b \otimes c) \rightarrow (a \otimes b) \otimes c\), constitutes an associativity constraint for \(\text{Eq}(\delta_0, \delta_1)\). Note that this makes sense as a definition,
because $\delta$ is a faithful functor – we need only check that this is a valid morphism in the equalizer category. By definition, we may write

$$
(a, \phi) \otimes ((b, \psi) \otimes (c, \theta)) = (a \otimes (b \otimes c), \phi \otimes (\psi \otimes \theta))
$$

and

$$
((a, \phi) \otimes (b, \psi)) \otimes (c, \theta) = ((a \otimes b) \otimes c, \phi \otimes (\psi \otimes \theta))
$$

where “$\phi \otimes (\psi \otimes \theta)$” and “$(\phi \otimes \psi) \otimes \theta$” are defined by the diagrams

and we need to check that the proposed associativity constraint induces a morphism in the equalizer category, which is to say that the following diagram commutes:
But expanding out the vertical arrows in this diagram, this is equivalent to observing that the following diagram commutes:

\[
\begin{array}{c}
\delta_1(a \otimes (b \otimes c)) \quad \delta_1((a \otimes b) \otimes c) \\
\downarrow J^1_{a,b,c} \quad \downarrow J^1_{a,b,c} \\
\delta_1(a) \otimes (\delta_1(b) \otimes \delta_1(c)) \quad \delta_1(a \otimes b) \otimes \delta_1(c) \\
\downarrow \alpha \delta_1(a) \otimes \delta_1(b) \otimes \delta_1(c) \quad \downarrow J^1_{a,b,c} \otimes \delta_1(c) \\
\delta_0(a) \otimes (\delta_0(b) \otimes \delta_0(c)) \quad (\delta_0(a) \otimes \delta_0(b)) \otimes \delta_0(c) \\
\downarrow \alpha \delta_0(a) \otimes \delta_0(b) \otimes \delta_0(c) \quad \downarrow J^0_{a,b,c} \otimes \delta_0(c) \\
\delta_0(a) \otimes \delta_0(b) \otimes c \quad \delta_0(a \otimes b) \otimes \delta_0(c) \\
\downarrow J^0_{a,b,c} \otimes \delta_0(c) \quad \downarrow J^0_{a,b,c} \\
\delta_0((a \otimes b) \otimes c) \quad \delta_0(a \otimes b) \otimes \delta_0(c)
\end{array}
\]

But this diagram commutes since the top and bottom portions are the compatibility hexagons of Diagram 3 for the monoidal functors \((\delta_1, J^1)\) and \((\delta_0, J^0)\) respectively, and the middle square commutes due to the naturality of \(\alpha\) in its three variables.

It follows that Eq\((\delta_0, \delta_1)\) has a monoidal structure (note that by convention with units, we are implicitly identifying 1 \(\in\) Eq\((\delta_0, \delta_1)\) with \((1_0, \epsilon_0^{-1} \epsilon_1^{-1})\)). The fact that the associativity constraint satisfies the pentagon condition of diagram 2 follows from the fact that \(\delta\) is faithful, and it similarly follows that the left and right unit morphisms satisfy the condition of diagram 2.

We may extend \(\delta : \text{Eq}(\delta_0, \delta_1) \to \mathcal{C}_0\) to a monoidal functor \((\delta, J)\), by defining \(J_{a,b} : \delta(a, \phi) \otimes (b, \psi) = a \otimes b \to \delta((a, \phi) \otimes (b, \psi)) = a \otimes b\) to be the identity morphism (and \(\epsilon : 1_0 \to \delta(1_0, \epsilon_0^{-1} \epsilon_1^{-1}) = 1_0\) to also be the identity morphism). The fact that the compatibility hexagon ensuring that this defines a morphism holds (similarly to diagram 3) is immediate. Finally, the fact that the canonical natural transformation \(\delta_1 \delta \to \delta_2 \delta\) is monoidal follows from the definition of the monoidal structure as in diagram 3.

If in addition, the categories \(\mathcal{C}_i\) are given a braiding defined by a natural isomorphism \(C^i_{a,b} : a \otimes b \to b \otimes a\), then one can check that \(C^0\) induces isomorphisms

\[
C_{(a, \phi), (b, \psi)} : (a, \phi) \otimes (b, \psi) \to (b, \psi) \otimes (a, \phi)
\]

provided that \(\delta_0, \delta_1\) are braided morphisms (see [EGNO15, Section 8.1] for precise definitions). Consequently, it again follows quickly from the fact that \(\delta\) is faithful that \(\delta\) is a morphism of braided monoidal categories. Again, by faithfulness, it is easy to see that if the braiding \(C^0\) on \(\mathcal{C}_0\) is symmetric, i.e. \(C^0_{b,a} C^0_{a,b} = \text{id}_{a \otimes b}\), then so is the braiding \(C\) on Eq\((\delta_0, \delta_1)\).

Suppose we are given symmetric monoidal categories \(\mathcal{A}, \mathcal{B}\). Let \(\text{Fun}_{\text{Ex}, \otimes}(\mathcal{A}, \mathcal{B})\) denote the category whose objects are right exact symmetric monoidal functors and whose morphisms are natural isomorphisms.
Lemma 2.2.2. Let \( \mathcal{A}_0, \mathcal{A}_1 \) be abelian, symmetric monoidal categories. Suppose we are given functors \( d^0, d^1 \in \text{Fun}_{r-\mathcal{Ex}}(\mathcal{A}_0, \mathcal{A}_1) \). Let \( \mathcal{A} \) be the 2-equalizer of these functors. Then the natural functor \( \mathcal{A} \to \mathcal{A}_0 \) is in \( \text{Fun}_{r-\mathcal{Ex}}(\mathcal{A}, \mathcal{A}_0) \).

Proof. This is an immediate consequence of Lemma 2.1.5. □

Let \( r\text{ExAbSMon} \) denote the 2-category of (small) abelian symmetric monoidal categories with morphisms being right exact symmetric monoidal functors and 2-morphisms being natural isomorphisms of functors. Let \( \text{Ab} \) the 2-category of (small) abelian categories. As a corollary to the above, we have the following.

Corollary 2.2.3. Suppose that \( d^0, d^1 \in \text{Fun}_{r-\mathcal{Ex}}(\mathcal{A}, \mathcal{B}) \). Then the 2-equalizer in \( \text{Ab} \) of these functors is naturally a symmetric monoidal category, and coincides with the 2-equalizer in \( r\text{ExAbSMon} \).

2.3. Patching with respect to a fibered category.

Definition 2.3.1. Let \( F \to C \) be a fibered category, and suppose that we have a diagram of morphisms \( U_* \) in \( C \):

\[
U_* = \begin{bmatrix}
U_1 & d_0 & U_0 \\
d_1 & & \end{bmatrix}
\]

Choosing pullbacks \( d^*_i \) gives us functors (well defined up to natural equivalence)

\[
F(U_0) \xrightarrow{d^*_0} F(U_1)
\]

and we define \( F(U_*) \) to be the 2-equalizer of this diagram.

Notation 2.3.2. If \( U_* \) is a diagram as above, we write \( d : U_* \to U \) for an object \( U \) to mean a morphism \( d : U_0 \to U \), such that \( dd_1 = dd_0 \). We will refer to such a diagram as a patching context.

Definition 2.3.3. Given \( d : U_* \to U \), we have a functor (well defined up to natural isomorphism)

\[
F(U) \to F(U_*)
\]

given by taking an object \( a \) of \( F(U) \) to \( (d^*a, d^*_1 d^*a \xrightarrow{\sim} d^*_0 d^*a) \), where \( \sim \) is the unique morphism of \( F \) such that we have a commutative diagram:

we say that patching holds for \( F \) with respect to the patching context \( U_* \to U \) if \( F(U) \to F(U_*) \) is an equivalence of categories.

In particular, we can regard a presheaf over \( C \) as a fibered category where each \( F(a) \) is a set for \( a \in \text{ob}(C) \) (i.e. no nontrivial morhisms). In this case, we note that patching holding for \( U_* \to U \) is the statement that \( F(U) \to F(U_0) \) is the equalizer of \( \delta^*_0, \delta^*_1 : F(U_0) \to F(U_1) \).
3. Beyond coherent sheaves

In this section, we will show that if patching holds for coherent sheaves on a system of locally excellent algebraic stacks, then patching also holds for morphisms to any sufficiently nice algebraic stack.

**Theorem 3.0.1.** Let \( \mathcal{X}, \mathcal{X}_1, \mathcal{X}_2 \) be locally excellent algebraic stacks, and suppose that we have morphisms \( \mathcal{X}_* = \left[ \mathcal{X}_1 \xrightarrow{d_0} \mathcal{X}_2 \right] \) and \( \mathcal{X}_* \to \mathcal{X} \). Suppose further that patching holds for coherent sheaves with respect to \( \mathcal{X}_* \to \mathcal{X} \). Let \( \mathcal{G} \) be a Noetherian algebraic stack with affine stabilizers. Then patching also holds for \( \text{Hom}(\_ , \mathcal{G}) \) with respect to \( \mathcal{X}_* \to \mathcal{X} \).

**Corollary 3.0.2.** Suppose we are given \( \mathcal{X}_* \to \mathcal{X} \) as in Theorem 3.0.1 for which patching holds for coherent sheaves. Then for \( G \) an affine group scheme over \( \mathcal{X} \), we have a 6-term exact sequence:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^0(\mathcal{X}, G) & \rightarrow & H^0(\mathcal{X}_0, G) & \rightarrow & H^0(\mathcal{X}_1, G) \\
\end{array}
\]

where the map \( H^0(\mathcal{X}_0, G) \rightarrow H^0(\mathcal{X}_1, G) \) is given by \( g \mapsto d_1^* g d_0^* (g^{-1}) \). In the case that \( G \) is abelian, this can be interpreted as an exact sequence of abelian groups:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^0(\mathcal{X}, A) & \rightarrow & H^0(\mathcal{X}_0, A) & \rightarrow & H^0(\mathcal{X}_1, A) \\
\end{array}
\]

**Proof:** Exactness of the top row is exactly the statement of Theorem 3.0.1 in the case \( \mathcal{G} \) is the scheme \( G \). We define the connecting map \( H^0(\mathcal{X}_1, G) \) as follows. For \( g \in H^0(\mathcal{X}_1, G) = G(\mathcal{X}_1) \), consider \( G \)-torsor on \( \mathcal{X}_* \) (i.e. the object of \( BG(\mathcal{X}_*) \)) described by \((1, g)\), where \( 1 \) denotes the trivial \( G \)-torsor on \( \mathcal{X}_0 \), and \( g \) is considered as an automorphism of the trivial \( G \)-torsor on \( \mathcal{X}_1 \). By Theorem 3.0.1 this gives a \( G \)-torsor on \( \mathcal{X} \), well defined up to isomorphism, and hence a class of \( H^1(\mathcal{X}, G) \), which by construction is trivial when restricted to \( \mathcal{X}_0 \). This is the definition of the connecting map, and why it maps into the pointed kernel. To see that it is the entire pointed kernel, if \( P \) is a \( G \)-torsor on \( \mathcal{X} \) which is trivial over \( \mathcal{X}_0 \), then Theorem 3.0.1 implies that \( BG(\mathcal{X}) \cong BG(\mathcal{X}_*) \), and we have that the image of \( P \) in \( BG(\mathcal{X}_*) \) must have the form \((1, g)\) as above.

Finally, it is fairly direct to see that the image of \( H^1(\mathcal{X}, G) \rightarrow H^1(\mathcal{X}_0, G) \) lies in the equalizer. Conversely, suppose that \([P] \in H^1(\mathcal{X}_0, G)\) is in the equalizer of the map to \( H^1(\mathcal{X}_1, G) \). In this case, we may find some isomorphism \( \phi : d_1^* P \rightarrow d_0^* P \). But in this case, it is clear that the \( G \)-torsor \( \tilde{P} \) which must exist and map to the object \((P, \phi)\), showing exactness of the bottom row. \( \Box \)

Before giving the proof of Theorem 3.0.1 let us set up a bit of notation. Let \( \mathcal{G}, \mathcal{X} \) be locally Noetherian algebraic stacks and let \( \text{Coh}(\mathcal{G}) \) and \( \text{Coh}(\mathcal{X}) \) denote their categories of coherent sheaves. Note that these are symmetric monoidal categories with respect to the tensor product.
A morphism \( f : \mathcal{X} \to G \) induces a functor

\[ f^* : \text{Coh}(G) \to \text{Coh}(\mathcal{X}) \]

Taking a map of algebraic stacks \( f : \mathcal{X} \to G \) to \( f^* \), we obtain a functor

\[ \text{Hom}(\mathcal{X}, G) \to \text{Fun}_{r-\text{Ex}, \otimes}(\text{Coh}(G), \text{Coh}(\mathcal{X})). \]

via pullback.

For a fixed algebraic stack \( G \), we define a fibered category \( G^{\otimes}_{r-\text{Ex}} \) over the category of algebraic stacks as follows. The objects of \( G^{\otimes}_{r-\text{Ex}} \) are pairs \((\mathcal{X}, F)\) where \( \mathcal{X} \) is an algebraic stack and \( F \) is an object of \( \text{Fun}_{r-\text{Ex}, \otimes}(\text{Coh}(G), \text{Coh}(\mathcal{X})) \). A morphism \((\mathcal{X}', F') \to (\mathcal{X}, F)\) consists of a morphism \( f : \mathcal{X}' \to \mathcal{X} \) together with a morphism \( F' \to f^* F \). The association \( f \mapsto f^* \) gives a morphism of fibered categories \( G \to G^{\otimes}_{r-\text{Ex}} \), where we write \( G \) for the representable fibered category it defines.

In [HR14], Hall and Rydh prove that this morphism is an equivalence under certain conditions:

**Theorem 3.0.3** (Theorem 1.1 in [HR14]). Let \( G \) be a Noetherian algebraic stack with affine stabilizers. Then, for every locally excellent algebraic stack \( X \), the natural functor

\[ G(X) = \text{Hom}(\mathcal{X}, G) \to \text{Fun}_{r-\text{Ex}, \otimes}(\text{Coh}(G), \text{Coh}(\mathcal{X})) = G^{\otimes}_{r-\text{Ex}}(\mathcal{X}) \]

is an equivalence.

**Proof of Theorem 3.0.3.** Since patching holds for coherent sheaves for \( \mathcal{X}_\bullet \to \mathcal{X} \), we have an equivalence of categories:

\[ G^{\otimes}_{r-\text{Ex}}(\mathcal{X}) = \text{Fun}_{r-\text{Ex}, \otimes}(\text{Coh}(G), \text{Coh}(\mathcal{X})) \cong \text{Fun}_{r-\text{Ex}, \otimes}(\text{Coh}(G), \text{Coh}(\mathcal{X}_\bullet))) \]

and by Lemma 2.1.4 we can identify this last category with the 2-equalizer

\[ \text{Eq} \left( \text{Fun}_{r-\text{Ex}, \otimes}(\text{Coh}(G), \text{Coh}(\mathcal{X}_0)) \longrightarrow \text{Fun}_{r-\text{Ex}, \otimes}(\text{Coh}(G), \text{Coh}(\mathcal{X}_1)) \right) \]

But this in turn is by definition the category \( G^{\otimes}_{r-\text{Ex}}(\mathcal{X}_\bullet) \), showing that patching holds. \( \square \)

3.1. **Examples of patching for coherent sheaves.** We will now record, some contexts in which patching results are known to hold for categories of coherent sheaves, which thereby gives patching for maps to Noetherian algebraic stacks with affine stabilizers. In each of the following examples, we will let Coh denote the stack of coherent sheaves.

3.1.1. **Formal patching.**

**Theorem 3.1.1.** ([FR70, FR, Prop. 4.2]) Let \( X \) be a Noetherian, 1-dimensional scheme, \( Z \subset X \) a finite subset of closed points, \( U \subset X \) its open complement. For a closed point \( \xi \in Z \), let \( X_\xi = \text{Spec}(\mathcal{O}_{X, \xi}) \), \( K_\xi \) the fraction field of \( \mathcal{O}_{X, \xi} \), \( X' = \bigsqcup_{\xi \in Z} X_\xi \) and \( U' = \bigsqcup_{\xi \in Z} \text{Spec}(K_\xi) \). Then patching holds for coherent sheaves with respect to the patching context

\[ U' \xrightarrow{d_0} X' \times U \xrightarrow{d_1} X. \]

**Corollary 3.1.2.** In the situation of Theorem 3.1.1, patching holds for morphisms to Noetherian stacks with affine stabilizers. In particular, patching holds for categories of \( G \)-torsors for any linear algebraic group \( G \).
3.1.2. Thickened formal patching.

**Notation 3.1.3.** Let $T$ be a complete discrete valuation ring with uniformizer $t$, and let $\mathcal{X}$ be a proper $T$-curve with reduced closed fiber $X$. Let $F$ be the function field of $\mathcal{X}$. Suppose that $\mathcal{P} \subset X$ a finite subset of closed points, such that $X \setminus Z$ is a disjoint union of connected affine components. Let $\mathcal{U}$ be the set of irreducible components of $X \setminus Z$.

For any connected affine open $W \subset X$, let $R_W$ denote the subring of elements of $F$ which are regular at every point of $W$ – that is to say

$$R_U = \cap_{x \in W} O_{\mathcal{X}, x}.$$ 

Let $\hat{R}_W$ be the $t$-adic completion $R_W$. By [HHK13], $\hat{R}_W$ is a domain, and we let $F_W$ denote its fraction field.

For $P \in X$, a closed point, let $\hat{R}_P$ be the complete local ring of $\mathcal{X}$ at $P$. Finally, if $\wp$ is a height one prime of $\hat{R}_P$ lying over $t \hat{R}_P$, let $\hat{R}_P[\wp]$ be the localization of $\hat{R}_P$ at $\wp$ and $\hat{R}_P[\wp]$ its $\wp$-adic completion (also coinciding with its $t$-adic completion). We let $\mathcal{B}$ denote the set of all such height one primes $\wp$. Note that these are the branches of the closed subschemes $\mathcal{U} \subset \mathcal{X}$ at the points $P \in \mathcal{P}$.

We note (see for example [HHK09, Section 3.1]), that whenever $\wp$ is a branch along $P$, there are natural inclusions of rings $\hat{R}_P \rightarrow \hat{R}_P[\wp]$, and when $P$ is in the closure of a component $W$ of the reduced closed fiber, for each branch $\wp$ along $W$ (i.e., cut out by the ideal defining the closed set $W$ in $\mathcal{X}$), there is an inclusion $\hat{R}_W \rightarrow \hat{R}_P[\wp]$. Taken together, these give natural maps

$$\prod_{P \in \mathcal{P}} \hat{R}_P \rightarrow \prod_{\wp \in \mathcal{B}} \hat{R}_P \rightarrow \prod_{U \in \mathcal{U}} \hat{R}_U.$$

**Theorem 3.1.4.** (see Pri00, Pries, Theorem 3.4]) In the language of Notation 3.1.3, patching holds for coherent sheaves with respect to the patching context

$$\text{Spec} \left( \prod_{\wp \in \mathcal{B}} \hat{R}_\wp \right) \xrightarrow{d_0 \to d_1} \text{Spec} \left( \prod_{P \in \mathcal{P}} \hat{R}_P \right) \bigsqcup \text{Spec} \left( \prod_{U \in \mathcal{U}} \hat{R}_U \right) \longrightarrow \mathcal{X}$$

**Proof.** This is precisely the statement of [Pri00] with the additional assumption that the schemes $U$ are affine, which allows us to replace the categories of modules over the formal completions with the modules over the corresponding complete rings. \qed

**Corollary 3.1.5.** In the situation of Theorem 3.1.4, patching holds for morphisms to Noetherian stacks with affine stabilizers. In particular, patching holds for categories of $G$-torsors for any linear algebraic group $G$.

3.1.3. Field patching.

**Theorem 3.1.6.** ([HHK15, Prop. 3.9]) In the language of Notation 3.1.3, suppose we are given an open affine connected subset $W \subset X$ and a finite collection of closed points $Q \subset W$. Let $\mathcal{V}$ be the set of connected components of $W \setminus Q$, and let $\mathcal{B}$ denote the collection of branches along $W$ at the points $Q \in Q$. Then patching holds for coherent sheaves with respect to the patching context

$$\text{Spec} \left( \prod_{\wp \in \mathcal{B}} F_\wp \right) \xrightarrow{\pi_1 \to \pi_2} \text{Spec} \left( \prod_{Q \in \mathcal{Q}} F_Q \right) \bigsqcup \text{Spec} \left( \prod_{V \in \mathcal{V}} F_V \right) \longrightarrow \text{Spec} F_W$$

**Theorem 3.1.7.** ([HHK15, Prop. 3.10]) In the language of Notation 3.1.3, suppose we are given a proper birational morphism $f : Y \rightarrow X$ and a closed point $P \in \mathcal{P}$. Let $V \subset Y$ be inverse image of $P$
in Y, and let \( \tilde{X} \) be the proper transform of \( X \). Suppose that \( \dim(V) = 1 \), and that \( f \) restricts to an isomorphism \( Y \setminus V \to \tilde{X} \setminus \{P\} \). Choose \( Q \) a finite collection of closed points \( \mathcal{P} \) of \( V \) including all the points of \( V \cap \tilde{X} \). Let \( \mathcal{B}' \) be the set of connected components of \( V \setminus \mathcal{P} \), and let \( \mathcal{B} \) be the set of \( X \) along the components of \( V \). Then patching holds for coherent sheaves with respect to the patching context

\[
\text{Spec} \left( \prod_{\tilde{P} \in \mathcal{P}} F_{\tilde{P}} \right) \xrightarrow{\pi_1} \text{Spec} \left( \prod_{Q \in q} F_Q \right) \coprod \text{Spec} \left( \prod_{U \in \mathcal{U}} F_U \right) \xrightarrow{\pi_2} \text{Spec} F_{\mathcal{P}}
\]

**Corollary 3.1.8.** In the situation of Theorems 3.1.6 and 3.1.7, patching holds for morphisms to Noetherian stacks with affine stabilizers. In particular, patching holds for categories of \( G \)-torsors for any linear algebraic group \( G \) defined over \( F_W \) and \( F_P \) in the respective contexts.

We note that this was known to hold for groups which were defined over \( F_w \), the function field of \( X \), but not necessarily for groups over \( F_W \) and \( F_P \).

4. Relative categories of coherent sheaves

The main result of the section is the following.

**Theorem 4.0.1.** Suppose we are given morphisms

\[
X_1 \xrightarrow{d_1} X_0 \to X
\]

of excellent Noetherian schemes such that patching holds for coherent sheaves with respect to \( X_0 \to X \). Let \( S \to X \) be a proper algebraic space, and let \( S_* = S \times_X X_0 \) be the associated diagram of spaces induced by pullback. Then patching holds for coherent sheaves with respect to \( S_* \to S \) (and hence it also holds for morphisms to Noetherian algebraic stacks with affine stabilizers by Theorem 3.0.1).

The proof of this theorem will occupy the remainder of the section.

**Lemma 4.0.2.** The map \( S_0 \to S \) is faithfully flat.

**Proof.** By Lemma 2.1.5 and the fact that pullback induces an isomorphism \( \text{Coh}(X) \xrightarrow{\sim} \text{Coh}(X_0) \), it follows that the pullback map \( \text{Coh}(X) \to \text{Coh}(X_0) \) is faithfully exact, which tells us that \( X_0 \to X \) is faithfully flat. But since \( S_0 \to S \) is obtained from this by pullback, it is also faithfully flat. \( \square \)

**Lemma 4.0.3.** The functor \( \text{Coh}(S) \to \text{Coh}(S_*) \) commutes with the formation of kernels and cokernels (i.e. is faithfully exact).

**Proof.** Since \( S_0 \to S \) is faithfully flat, we have that \( \text{Coh}(S) \to \text{Coh}(S_0) \) is faithfully exact. The conclusion follows from Remark 2.1.6 \( \square \)

**Lemma 4.0.4.** The functor \( \text{Coh}(S) \to \text{Coh}(S_*) \) is fully faithful.

**Proof.** Suppose we have coherent sheaves \( W, V \) on \( S \). Consider the (additive) group scheme \( G \) on \( X \) whose values on \( T \to X \) are \( \text{Hom}_{\text{et}}(W_T, V_T) \). Thinking of this as a stack (with trivial inertia), Theorem 3.0.1 gives a \( 2 \)-equalizer diagram (of setoids with abelian group structure)

\[
G(X) \xrightarrow{d_1} G(X_0) \xrightarrow{d_0} G(X_1)
\]
which can be identified with an isomorphism of abelian groups
\[ \text{Hom}_S(W, V) \cong \text{Hom}_{S_\ast}(W_{S_\ast}, V_{S_\ast}) \]

**Lemma 4.0.5.** The essential image of \( \text{Coh}(S) \to \text{Coh}(S_\ast) \)

- (1) is closed under \( \otimes \),
- (2) contains the image of \( \text{Coh}(X_\ast) \) under pullback,
- (3) is closed under formation of kernels and cokernels, and
- (4) is closed under extensions.

**Proof.** Part [1] follows from Remark [2.2.1] Part [2] follows from the fact that we have a commutative diagram
\[
\begin{array}{ccc}
\text{Coh}(X) & \longrightarrow & \text{Coh}(X_0) \\
\downarrow & & \downarrow \\
\text{Coh}(S) & \longrightarrow & \text{Coh}(S_0) \\
\end{array}
\]

For part [3] we will consider the case of cokernels (the case of kernels is similar). Suppose we are given a right exact sequence
\[ F \to G \to C \to 0 \]
in \( \text{Coh}(S_\ast) \), with \( F, G \) in the image of \( F', G' \) in \( \text{Coh}(S) \). Let
\[ F' \to G' \to C' \to 0 \]
be right exact in \( \text{Coh}(S) \) (i.e. \( C' \) is the cokernel in \( \text{Coh}(S) \)). Since \( \text{Coh}(S) \to \text{Coh}(S_\ast) \) is faithfully exact (by Lemma 4.0.3), it follows that we have an isomorphism between \( C \) and the image of \( C' \) showing that \( C \) is in the essential image as desired.

Using [Yon54, Thm 3.5] or [Oor64], part [4] follows so long as formation of Ext groups commutes with the functors they induce between the categories \( \text{Coh}(S_\ast) \) and \( \text{Coh}(S) \). This follows from the fact that the functor is fully faithful (and hence preserves the Hom functor) and is faithfully exact (and hence commutes with the construction of the derived functor).

**Lemma 4.0.6.** If \( S \) is a projective \( X \)-scheme then \( \text{Coh}(S) \to \text{Coh}(S_\ast) \) is essentially surjective.

**Proof.** Let \( V_\ast = (V, \phi) \) be an object of \( \text{Coh}(S_\ast) \). Let \( \pi : S \to X \) be the structure morphism. We will write \( \pi_\ast \) and \( \pi^\ast \) for the standard pushforward and pullback maps, as well as for the natural functors they induce between the categories \( \text{Coh}(S_\ast) \) and \( \text{Coh}(X_\ast) \).

By tensoring with image of the relatively ample invertible sheaf \( \mathcal{O}_S(n) \), we can define \( V_\ast(n) \) for any \( n \in \mathbb{Z} \). For any such integer \( n \), note that we have a canonical morphism \( \pi^\ast \pi_\ast V_\ast(n) \to V_\ast(n) \), and since the functor \( \text{Coh}(S_\ast) \to \text{Coh}(S) \) is faithfully exact by Lemma 4.0.3, it follows that this is surjective for \( n \gg 0 \). Let \( K_\ast \) be the kernel of this morphism. Again, for some \( m \gg 0 \), we have a surjection \( \pi^\ast \pi_\ast K_\ast(m) \to K_\ast(m) \). Taken together, this gives a right exact sequence:
\[ \pi^\ast \pi_\ast K_\ast(m) \to (\pi^\ast \pi_\ast V_\ast(n))(m) \to V_\ast(n + m) \]

By Lemma 4.0.5(2), \( \pi^\ast \pi_\ast K_\ast(m) \) and \( (\pi^\ast \pi_\ast V_\ast(n))(m) \) are in the essential image. By Lemma 4.0.4 the morphism between them is as well, and by Lemma 4.0.5(3), so is their cokernel. Therefore \( V_\ast(n + m) \) is in the essential image. But since we may tensor with \( \mathcal{O}(n - m) \) by Lemma 4.0.1 and stay in the essential image as well, this says that \( V_\ast \) is in the essential image as claimed. □
Proof of Theorem 4.0.1. We proceed by Noetherian induction (on $S$), the case of $S = \emptyset$ being clear. By Chow’s Lemma [Sta18, Tag 088U], there is a proper birational map $\theta : \tilde{S} \to S$ with $\tilde{S} \to X$ a projective morphism. There results a map of diagrams

$$\theta_* : \tilde{S}_* \to S_*$$

and associated functors

$$(\theta^*)_* : \text{Coh}(S_*) \to \text{Coh}(\tilde{S}_*)$$

and

$$(\theta_*)^* : \text{Coh}(\tilde{S}_*) \to \text{Coh}(S_*)$$

Given an object $V_* \in \text{Coh}(S_*)$, there is a canonical map

$$V_* \to (\theta^*)_*(\theta^*)_* V_*$$

that is an isomorphism over a dense open subspace of $S$. By Lemma 4.0.6, there is thus a coherent sheaf $F$ on $\tilde{S}$ and a map

$$s : V_* \to (\theta_*)*(\theta_*)_* F_*$$

that is an isomorphism over a dense open of $S$. The kernel and cokernel of $s$ are supported over proper closed subspaces of $S$.

By Lemma 4.0.5(3) and Lemma 4.0.5(4), it suffices to show that $\ker(s)$ and $\text{coker}(s)$ lie in the essential image of $\text{Coh}(S)$. But this follows from the Noetherian induction hypothesis. □

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