Spectra of quantum chains
without the Yang-Baxter equation

I. Peschel,\textsuperscript{a} V. Rittenberg\textsuperscript{b} and U. Schultze\textsuperscript{c}

\textsuperscript{a}Max-Planck-Institut für Physik Komplexer Systeme,
Bayreuther Str. 40, Haus 16, 01187 Dresden, Germany
Freie Universität Berlin, Fachbereich Physik,
Arnimallee 14, 14195 Berlin, Germany (permanent address)

\textsuperscript{b}Physikalisches Institut, Universität Bonn,
Nussallee 12, 53115 Bonn, Germany

\textsuperscript{c}Laboratoire de Physique Théorique des Liquides,
Université Pierre et Marie Curie, 4 Place Jussieu,
75252 Paris, Cedex 05, France

Abstract

We study one-dimensional reaction-diffusion models described by master equations and their associated two-state quantum Hamiltonians. By choosing appropriate rates, the equations of motion decouple into certain subsets. We solve the first subset which has a close relation to the problem of lattice electrons in an electric field. In this way we obtain $L(L-1)+1$ energy levels of a quantum chain with $L$ sites. The corresponding Hamiltonian depends on 7 parameters and does not look integrable using conventional methods. As an application, we compute the dynamical critical exponent of a new type of kinetic Ising model.
1 Introduction

One-dimensional quantum chain Hamiltonian show up in different contexts in equilibrium statistical mechanics and many examples of integrable ones are known. They appear as the logarithmic derivative of the monodromy matrix related to the Yang-Baxter equation [1]. However, there is also a connection to stochastic problems, because a master equation can be related to a suitable Schrödinger equation [2]. For example, the diffusion of classical particles with hard-core repulsion on a lattice leads to the ferromagnetic Heisenberg model [3], [4]. It was realised recently [5], [6], [7], [8] that in various other stochastic models the corresponding Hamiltonian is integrable and one can use this integrability in order to determine the phase diagram of the system and compute the time evolution of various average quantities. In the Schrödinger equation the wave function is the time dependent probability distribution, the Hamiltonian is in general non-hermitian and its matrix elements are constrained by the condition that the sum of rates for a given process have to add to one (conservation of probabilities).

In the present paper we adopt the opposite strategy. We ask ourselves if there are not known examples in non-equilibrium statistical mechanics where one can solve the master equation or part of it (in a sense to be defined soon) and thus get in this way the spectrum and wave functions of a Hamiltonian. We first looked at the Hamiltonian related to the master equation studied by Kimball [9] and Deker and Haake [10] where two wave functions were known for any number of sites, a few more can be found [11] but we were not able to go beyond this point. Another approach which was proven very powerful is to consider various moments of the probability distribution (the knowledge of all the moments giving back the probability distribution) and to find special rates in the master equation such that the differential equations for various moments decouple into subsets. The first example of this kind is the Glauber model [12] used to study the relaxation of the one-dimensional classical Ising model.

Going to the dual variables description [13], the Glauber model corresponds to solve the one-spin-string problem (see Sec. 2) which decouples from the many-spin-strings problem. Subsequently [14], [15] the many strings problem was also solved and implicitly the whole spectrum of the Hamiltonian derived. The Hamiltonian is however trivial: it corresponds to the XY model in a $Z$ field which can be brought through a Jordan-Wigner transform-
mation to a bilinear problem in fermionic creation and annihilation operators and can thus be trivially diagonalized.

Another case is the one considered in Refs. [16], [17], [18] and [19]. This is an example where the master equation describes chemical processes and brownian motions [20]. In this case (see Sec. 2 for the terminology) one considers first processes where two identical molecules $A$ give a molecule $A$ and a vacancy (coagulation) with a rate "tuned" to the diffusion rate and the reverse process (decogulation) with an arbitrary rate ($A + \text{vacancy} \rightarrow A + A$). This is a one-parameter process since one rate can be used to fix the time scale. In this case one considers a one-hole probability function, two-holes etc. ... as moments of the probability function. It was shown that the equations for the one-hole probability function decouple from the others. The Hamiltonian in this case is again that of the $XY$ model in a $Z$ field. Some work was done also for the two-holes problem [21], [22]. Actually, as will be shown in this paper knowing the two spin-string solution of [14] and [15] for the Glauber problem solves the two-holes problem of the coagulation-decoagulation case. The Glauber model corresponds to chemical processes in which two molecules annihilate into two vacancies: $A + A \rightarrow \text{vacancy} + \text{vacancy}$ with a rate tuned to the diffusion rate and the reverse process with an arbitrary rate. The investigation of Refs. [16] - [19] has, however, one more case, namely to the coagulation-decoagulation processes one adds the birth process (two vacancies give a molecule $A$ and a vacancy). They were able to solve the continuum limit of this problem in the case of one-hole. The spectrum can be expressed in terms of zeroes of one Airy function and the Hamiltonian is not any more a trivial one. This observation triggered the present paper which is organized as follows.

In Sec. 2 we give the master equation for a chain of $L$ sites, having on each site a stochastic variable taking two values. The rates are assumed to depend on the configuration of two neighbouring sites only. Next we present the dictionary necessary for chemistry which is also useful in order to understand our results. The one-dimensional quantum chain associated with the master equation is also given.

In Sec. 3 we first define the holes and spin-strings. Next we derive the conditions under which the differential equations for one-hole decouple from the others. These conditions assure also that the two-holes problem is decoupled from the three or more holes, etc. since there are five conditions for the twelve rates. This leaves the Hamiltonian dependent on six parameters.
(the normalisation fixes another one). The Hamiltonian one obtains is non-
hermitian. We write the differential equations for the closed ring case and
notice that they depend on one parameter less. This is not the case for
the open chain. We also show that the same differential equations describe
the one-string problem and give the relation between the parameters which
appear in the differential equations and the rates of the one-string problem
(the conditions on the rates are different in this case but their number is of
course the same).

In Sec. 4 we solve the one-hole subset of differential equations for a finite
chain on a closed ring. Their structure depends on one parameter (called $\delta$
in the text) which is non-negative in the stochastic problems. If $\delta$ is non-zero,
the eigenvalues are given by zeroes of Lommel polynomials. If $\delta$ is zero, the
eigenvalues are of trigonometric type. We also check under which conditions
the spectrum corresponds to a free fermionic one.

The continuum limit is considered in Sec. 5. If $\delta \neq 0$ one can take the
limit $L \to \infty$ and find that like in the special case of Refs. 16 - 19 the
eigenvalues are related to zeroes of one Airy function. For $\delta = 0$, they have
a simple expression. The eigenvalues are in general complex.

As a simple application of our approach, we consider in Sec. 6 the critical
dynamics of the Ising model 13, 23, 24. This process is described by
the same master equation as the one used for diffusion-reaction processes
with supplementary conditions on the rates which assure that for large times
the system reaches the equilibrium distribution of the one-dimensional Ising
model (detailed balance conditions). We have checked whether our seven
parameter Hamiltonian is compatible with the detailed balance conditions
and found two solutions: one is the old Glauber one 12, 13, another one
is new. Its physical significance is discussed.

One last comment before going to the bulk of this paper: one-dimensional
diffusion-reaction processes are apparently seen experimentally 25. This
observation might make the reader have a closer look at the subject.
2 The Master Equation and 1 - d Quantum Chains

We consider a one-dimensional chain with \( L \) sites and on each site \( k (k = 1, 2, \ldots, L) \) we take a variable \( \beta_k \) which takes two values: 0 and 1. For \( \beta_k = 0 \) the site \( k \) is empty (vacancy), for \( \beta_k = 1 \) the site \( k \) is occupied by a molecule \( A \). At \( t = 0 \) the probability to find a certain configuration of molecules and vacancies on the chain is given by the probability distribution

\[
P_0 (\beta_1, \beta_2, \ldots, \beta_L) = P_0 (\{ \beta \})
\]

(2.1)

If we assume that we have two-body processes only, the time evolution of the system is given by a master equation:

\[
\frac{\partial P (\beta; t)}{\partial t} = \sum_{k=1}^{L-1} \left[ -V (\beta_k, \beta_{k+1}) P (\beta_1, \ldots, \beta_L; t) + \sum_{\gamma_k, \gamma_{k+1}}' \Gamma_{\beta_k, \beta_{k+1}}^{\gamma_k, \gamma_{k+1}} P (\beta_1, \ldots, \beta_{k-1}, \gamma_k, \gamma_{k+1}, \beta_{k+2}, \ldots, \beta_L; t) \right]
\]

(2.2)

which gives the probability distribution at the time \( t \) for a given \( P (\{ \beta \}, t = 0) = P_0 \). The first term on the r. h. s. of (2.2) describes the losses and the second one the gains. A prime is used to indicate that in the sum the couple \( \gamma_k = \beta_k, \gamma_{k+1} = \beta_{k+1} \) is excluded. \( \Gamma_{\beta_k, \beta_{k+1}}^{\gamma_k, \gamma_{k+1}} \) represents the probability per unit time that the configuration \((\gamma_k, \gamma_{k+1})\) on neighbouring sites changes into the configuration \((\beta_k, \beta_{k+1})\). The conservation of probabilities gives:

\[
V (\gamma_k, \gamma_{k+1}) = \sum_{\beta_k, \beta_{k+1}}' \Gamma_{\beta_k, \beta_{k+1}}^{\gamma_k, \gamma_{k+1}}
\]

(2.3)

where again the prime in the sum indicates that the couple \( \gamma_k = \beta_k, \gamma_{k+1} = \beta_{k+1} \) is excluded. The master equation (2.2) is quite general and can have various physical interpretations (in Sec. 6 we will apply it to study critical dynamics), in the present section however we use the language of diffusion-reactions processes where the rates have a simple meaning. The following processes are included in the master equation (with 0 a vacancy \( \beta = 0 \) and
A a molecule \((\beta = 1)\)

\[
\begin{align*}
\text{Diffusion to the right:} & \quad A + 0 \rightarrow 0 + A \quad \text{(rate } \Gamma_{01}^{10}) \\
\text{Diffusion to the left:} & \quad 0 + A \rightarrow A + 0 \quad \text{(} \Gamma_{10}^{01} \text{)} \\
\text{Coagulation at the right:} & \quad A + A \rightarrow 0 + A \quad \text{(} \Gamma_{11}^{01} \text{)} \\
\text{Coagulation at the left:} & \quad A + A \rightarrow A + 0 \quad \text{(} \Gamma_{11}^{10} \text{)} \\
\text{Decoagulation at the right:} & \quad A + 0 \rightarrow A + A \quad \text{(} \Gamma_{11}^{01} \text{)} \\
\text{Decoagulation at the left:} & \quad 0 + A \rightarrow A + A \quad \text{(} \Gamma_{11}^{10} \text{)} \\
\text{Birth at the right:} & \quad 0 + 0 \rightarrow 0 + A \quad \text{(} \Gamma_{00}^{01} \text{)} \\
\text{Birth at the left:} & \quad 0 + 0 \rightarrow A + 0 \quad \text{(} \Gamma_{00}^{10} \text{)} \\
\text{Death at the right:} & \quad 0 + A \rightarrow 0 + 0 \quad \text{(} \Gamma_{01}^{01} \text{)} \\
\text{Death at the left:} & \quad A + 0 \rightarrow 0 + 0 \quad \text{(} \Gamma_{10}^{00} \text{)} \\
\text{Pair-annihilation:} & \quad A + A \rightarrow 0 + 0 \quad \text{(} \Gamma_{11}^{00} \text{)} \\
\text{Pair-creation:} & \quad 0 + 0 \rightarrow A + A \quad \text{(} \Gamma_{00}^{11} \text{)}
\end{align*}
\]

\((2.4)\)

In Ref. [7] (see also the references included therein), several special cases have been considered in detail. The notations for the rates used in Ref. [7] are related to those used in the present paper:

\[
\Gamma_{\alpha,\beta}^{\gamma,\delta} = w_{\gamma - \alpha,\delta - \beta}(\alpha, \beta) \quad V(\alpha, \beta) = w_{00}(\alpha, \beta) \quad (2.5)
\]

To a given master equation one can associate [2] a Schrödinger equation

\[
\frac{\partial}{\partial t} | P > = -H | P > \quad (2.6)
\]

as follows. Take a \(2^L\)-dimensional orthogonal basis:

\[
| \{\beta\} > = | \beta_1, \beta_2, \ldots, \beta_L > , \quad < \{\beta\} | \{\beta'\} > = \delta_{\{\beta\},\{\beta'\}} \quad (2.7)
\]

and denote:

\[
| P > = \sum_{\beta} P(\{\beta\}; t) | \{\beta\} > ; \quad | P_0 > = \sum_{\beta} P_0(\{\beta\}) | \{\beta\} > \quad (2.8)
\]

In order to write the Hamiltonian, we first define a basis in the space of \(2 \times 2\) matrices taking \(E_{\alpha,\beta}^{\alpha,\beta}(\alpha, \beta = 0, 1)\) with matrix elements:

\[
(E_{\alpha,\beta})_{\gamma,\delta} = \delta_{\alpha,\gamma} \delta_{\beta,\delta} \quad (2.9)
\]
On each site $k$ of the chain we take the matrices $E_k^{\alpha,\beta}(k = 1, 2, \ldots L)$. It was shown in Ref. [7] that in this basis the Hamiltonian $H$ occurring in Eq. (2.6) has the following expression:

$$H = \sum_{k=1}^{L-1} H_k$$  \hspace{1cm} (2.10)

where

$$H_k = -\sum T_{\gamma,\delta}^{\alpha,\beta} E_k^\gamma E_k^{\delta,\beta}$$  \hspace{1cm} (2.11)

with

$$T_{\gamma,\delta}^{\alpha,\beta} = \Gamma_{\gamma,\delta}^{\alpha,\beta} (\alpha = \gamma, \beta = \delta \text{ excluded})$$  \hspace{1cm} (2.12)

$$T_{\alpha,\beta}^{\alpha,\beta} = -V(\alpha, \beta)$$  \hspace{1cm} (2.13)

where we have used Eq. (2.3). The Hamiltonian $H$ is in general non-hermitian and due to the probability conservation (Eq. (2.3)), it satisfies the remarkable relation:

$$\langle 0 | H = 0$$  \hspace{1cm} (2.14)

where the bra ground-state $\langle 0 |$ is

$$\langle 0 | = \sum_{\{\beta\}} <\beta| = \left< \left( \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array} \right) \otimes \cdots \otimes \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right>$$  \hspace{1cm} (2.15)

From Eq. (2.14) it follows that the ground-state energy is zero. The Hamiltonian might have more than one eigenstate of zero energy i.e. there may be more than one stationary solution.

It is instructive to write the $4 \times 4$ matrix given by Eq. (2.11) explicitly:

$$H_k = \begin{pmatrix}
\Gamma_{00}^{00} + \Gamma_{01}^{00} + \Gamma_{11}^{00} & -\Gamma_{01}^{00} & -\Gamma_{10}^{00} & -\Gamma_{11}^{00} \\
-\Gamma_{00}^{00} & \Gamma_{00}^{00} + \Gamma_{01}^{00} + \Gamma_{11}^{00} & -\Gamma_{10}^{00} & -\Gamma_{11}^{00} \\
-\Gamma_{00}^{01} & -\Gamma_{10}^{01} & \Gamma_{00}^{10} + \Gamma_{01}^{10} + \Gamma_{11}^{10} & -\Gamma_{11}^{10} \\
-\Gamma_{00}^{11} & -\Gamma_{10}^{11} & -\Gamma_{11}^{11} & \Gamma_{00}^{11} + \Gamma_{01}^{11} + \Gamma_{11}^{11}
\end{pmatrix}$$  \hspace{1cm} (2.16)

Notice that the sum of each column in (2.16) vanishes, this is a consequence of Eq. (2.14).
Let us now consider an observable \( X(\beta_1, \ldots \beta_L) = X(\{\beta\}) \), its average quantity can be computed as follows:

\[
< X > (t) = \sum_{\{\beta\}} X(\{\beta\}) P(\{\beta\}, t) = < 0 \mid X \mid P > = < 0 \mid X e^{-Ht} \mid P_0 > \tag{2.17}
\]

The time derivative of \( < X > (t) \) is obviously:

\[
\frac{d < X > (t)}{dt} = - < 0 \mid X H e^{-Ht} \mid P_0 > = < 0 \mid [H, X] e^{-Ht} \mid P_0 > \tag{2.18}
\]

where we have used Eq. (2.14). In the next Section two types of observables will be used. The first one is a hole of length \( n \) starting at the site \( j \):

\[
X = \prod_{k=j}^{j+n-1} \delta(\beta_k) \tag{2.19}
\]

Its average is:

\[
K(j, n + j - 1) = \sum_{\beta} \delta(\beta_j) \delta(\beta_{j+1}) \ldots \delta(\beta_{j+n-1}) P(\{\beta\}, t) = < 0 \mid \prod_{k=j}^{n+j-1} E_k^{00} e^{-Ht} \mid P_0 > \tag{2.20}
\]

where we have used Eq. (2.9). The time dependent function \( K(j, n + j - 1) \) gives the probability that \( n \) consecutive sites starting with the site \( j \), are empty (a hole). This observable is sometimes used also in equilibrium problems [26]. A second observable which will be of interest is a spin-string of length \( n \) starting at the site \( j \):

\[
X = \prod_{k=j}^{j+n-1} (\delta(\beta_k) - \delta(\beta_k + 1)) \tag{2.21}
\]

Its average quantity is:

\[
S(j, n + j - 1) = < 0 \mid \prod_{k=j}^{n+j-1} \sigma_k^z e^{-Ht} \mid P_0 > \tag{2.22}
\]
where
\[
\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = E^{00} - E^{11}
\] (2.23)

Actually one can define an observable interpolating between the one-hole and the one-spin-string taking
\[
X = \prod_{k=j}^{j+n-1} \left( \frac{1}{1 + \xi} \delta(\beta_k) - \frac{\xi}{1 + \xi} \delta(\beta_k + 1) \right)
\] (2.24)

where \(\xi\) is a parameter. Its average quantity being:
\[
X(j, n + j - 1) = \langle 0 | \prod_{k=j}^{n+j-1} \tau_k e^{-Ht} | P_0 \rangle
\] (2.25)

where
\[
\tau = \frac{1}{1 + \xi} E^{00} - \frac{\xi}{1 + \xi} E^{11}
\] (2.26)

3 Decoupling of the master equation into sub-sets

As we have seen in the proceeding Section, the master equation (2.2) is equivalent to the Schrödinger equation (2.6). Thus finding the solution of the \(2^L\) linear differential equations (2.2) will give the eigenvalues and eigenfunctions of the Hamiltonian given by the Eqs. (2.10) - (2.11). The main idea that we are going to use in this paper is to find conditions under which the \(2^L\) differential equations decouple into subsets which can be solved independently. This corresponds to bringing the Hamiltonian to a block-diagonal form. There are some obvious sub-sets that one might consider:

a) Holes The definition of a hole was given by Eq. (2.20). One can ask for conditions on the rates such that the one-hole differential equations decouple from the others. The two-holes equations decouple from more than two-holes etc.... Since obviously knowing the solution of all the holes probabilities gives the probability distribution \(P(\{\beta\}, t)\), this corresponds to the diagonalisation of the Hamiltonian.
b) Spin-strings The definition of a spin-string was given by Eq. (2.22).

One can consider the hierarchy of the sub-sets of one string, two strings etc. Again knowing all the spin-strings functions is equivalent to the knowledge of \( P(\{\beta\}, t) \) (the proof is trivial).

c) Vacancies The \( N \)-vacancies function is

\[
V(i_1, i_2, \ldots, i_N) = \sum_{\beta} \delta(\beta_{i_1}) \delta(\beta_{i_2}) \ldots \delta(\beta_{i_N}) P(\{\beta\}; t)
\]

\[
= \langle 0 | E_{i_1}^{00} E_{i_2}^{00} \ldots E_{i_N}^{00} e^{-Ht} | P_0 \rangle
\]  

The hierarchy is one vacancy, two vacancies etc. ... (see also [27], [28], [29]).

d) Spins The \( N \)-spins function is

\[
S(i_1, i_2, \ldots, i_N) = \sum_{\beta} (\delta(\beta_{i_1}) - \delta(\beta_{i_2})) \ldots (\delta(\beta_{i_N}) - \delta(\beta_{i_N+1})) P(\{\beta\}; t)
\]

\[
= \langle 0 | \sigma_{i_1}^{z} \sigma_{i_2}^{z} \ldots \sigma_{i_N}^{z} e^{-Ht} | P_0 \rangle
\]  

with the hierarchy one spin, two spins etc. ...

As we are going to show soon, once we solve the holes case we can get through a similarity transformation the spin-strings solution.

It is interesting at this point to solve a small counting problem. Take first an open chain with \( L \) sites and count how many kind of holes \( N_m \) (various lengths and positions) one can have for a given total of \( m \) holes \( (m = 1, 2, \ldots, \frac{L}{2} \text{ for } L \text{ even}, m = 1, 2, \ldots, \frac{L+1}{2} \text{ for } L \text{ odd}) \). The answer is:

\[
N_m = \frac{(L+1)!}{(2m)!(L-2m+1)!} = C_L^{2m} + C_L^{2m-1}
\]  

(3.3)

The meaning of the right hand side of Eq. (3.3) will be apparent immediately. Consider now for example the following Hamiltonian:

\[
H = \sum_{i=1}^{L-1} H_i
\]  

(3.4.a)

\[
H_i = -\frac{\eta}{2} \left[ \eta \sigma_i^z \sigma_{i+1}^z + \eta^{-1} \sigma_i^y \sigma_{i+1}^y + \sigma_i^z + \sigma_{i+1}^z - \eta - 1 \right]
\]  

(3.4.b)
where \( \eta \) is a free parameter and \( \sigma^x, \sigma^y \) and \( \sigma^z \) are Pauli matrices. This Hamiltonian will correspond to a special choice of the parameters in our solution of the decoupling problem. Here we are only interested in the structure of the problem. The Hamiltonian (3.4) can be diagonalized through a Jordan-Wigner transformation [30]. The result for the open chain has the form:

\[
H = \sum_{k=1}^{L} \varepsilon_k n_k
\]

(3.5)

where \( n_k \) are fermionic occupation numbers \( (n_k = 0, 1) \) and \( \varepsilon_1 = 0 \) (there is a zero fermionic mode). If one computes now the number of states in the sectors with \( 2m \) and \( (2m-1) \) fermions one finds precisely (3.3). The fermionic vacuum does not appear in the holes calculation. The reason is simple: in a probabilistic calculation one does not need to know \( 2^L \) probabilities \( P(\beta_1, \ldots, \beta_L) \) but only \( 2^L - 1 \) since the sum of probabilities adds to one.

We now consider a closed ring with \( L \) sites. In this case the number of holes in the subset with \( m \)-holes \( (m = 1, 2, \ldots, \frac{L}{2} \) for \( L \) even, \( m = 1, 2, \ldots, \frac{L-1}{2} \) for \( L \) odd) is

\[
N_m = \frac{2L!}{(2m)!(L-2m)!}
\]

(3.6)

except for \( m = 1 \) where we have one more hole than one obtains from Eq. (3.6). This special hole corresponds to the situation where all the \( L \) sites are empty. The reason for this choice of the supplementary hole comes from a decoupling of this hole from the other functions in the 1-hole subset (see Sec. 4). Adding all states coming from all the \( m \)-holes subsets one gets \( 2^L - 1 \) as one should (one more state comes from probabilities conservation).

We now return to the Hamiltonian (3.4) with periodic boundary conditions which can be diagonalized through a Jordan-Wigner transformation [31], [32] and its spectrum can be obtained taking the even number of fermions sector of two free-fermionic Hamiltonians, one with periodic boundary conditions and the other with antiperiodic boundary conditions. Notice that both Hamiltonians have a zero ground state energy. The number of states in the \( 2m \) fermions sector of both Hamiltonians give together \( N_m \) of Eq. (3.6). We are left with two fermionic vacua. One corresponds to the total empty one-hole the other to the redundant state (the probabilities
add to one). This calculation suggests that for special choices of the parameters of our problem we will get Hamiltonians which correspond to free fermions. This will be proven to be indeed the case. We will also find an unexpected phenomenon, where the closed chain problem will be described by free fermions but not the open chain. We will also get that in general the spectrum is not given by free fermions at all, but we believe that our counting could help clarify the algebraic structure of the problem.

The vacancies problem will not be considered here but can be found in [29]. One finds a non-hermitian Hamiltonian depending on ten parameters whose spectrum is given by the XXZ Heisenberg chain in a Z field. The same similarity transformation used for the holes-spin-strings correspondence can be used to solve the spins problem once the vacancies solution is known.

We now start with the holes problem and ask for the conditions on the rates such that the time derivative of a one hole probability function

$$\frac{dK(j, j + n - 1)}{dt} = \langle 0 | \left[ H, \prod_{k=j}^{j+n-1} E_k^{00} \right] e^{-Ht} | P_0 \rangle$$

(3.7)

can be expressed in terms of one-hole probability functions only. A straightforward calculation leads to the following five conditions:

$$\Gamma_{00}^{11} = \Gamma_{01}^{01} = \Gamma_{00}^{10} = 0; \quad \Gamma_{10}^{11} = \Gamma_{01}^{01}; \quad \Gamma_{01}^{10} = \Gamma_{01}^{10} = \Gamma_{01}^{11}$$

(3.8)

The physical meaning of the conditions (3.8) is the following. All processes creating A molecules (creation, birth and decoagulation) are allowed. From the processes which ”kill” molecules only coagulation processes are allowed and they are tuned to the diffusion rates: the rate of coagulation at the right (left) is equal to the diffusion rate to the right (left). To shorten the notations, we will denote:

$$a_R = \Gamma_{01}^{10} = \Gamma_{01}^{11}, \quad a_L = \Gamma_{01}^{01} = \Gamma_{10}^{11},$$

$$d_R = \Gamma_{11}^{10}, d_L = \Gamma_{11}^{01}, \quad b_R = \Gamma_{00}^{01}, \quad b_L = \Gamma_{00}^{01}, \quad c = \Gamma_{00}^{01}$$

(3.9)

Thus the a’s describe diffusion and coagulation, the d’s decoagulation, the b’s birth and c pair-creation. If we take into account Eqs. (3.9), the $4 \times 4$ matrix $H_k$ of Eq. (2.16) becomes:

$$H_k = \begin{pmatrix}
    c + b_L + b_R & 0 & 0 & 0 \\
    -b_R & a_L + d_L & -a_R & -a_R \\
    -b_L & -a_L & a_R + d_R & -a_L \\
    -c_L & -d_L & -d_R & a_L + a_R
\end{pmatrix}$$

(3.10)
and it is the Hamiltonian (2.10) with (3.10) that we want to diagonalize. This Hamiltonian is non-hermitian and depends on seven parameters, one of which will fix the time scale. It is now useful to use the chemical picture given by Eq. (2.4) in order to look for global symmetries of the Hamiltonian. In general one finds none. All one can say is that if the rates are all positive, all the eigenvalues are non-negative.

The differential equations derived from Eq. (3.7) for the one-hole probability functions for a chain with \( L \) sites depend on the boundary conditions:

- Closed ring, \( L(L - 1) + 1 \) equations

\[
\frac{dK(j, j + n - 1)}{dt} = -[\gamma + (n - 1)\delta]K(j, j + n - 1) + a_L K(j + 1, j + n - 1) + a_R K(j, j + n - 2) + \alpha_L K(j, j + n) + \alpha_R K(j - 1, j + n - 1) \quad (1 \leq n \leq L - 1) \tag{3.11}
\]

\[
\frac{dK(j, j + L - 1)}{dt} = -L\delta K(j, j + L - 1) \quad (n = L) \tag{3.12}
\]

with the notation:

\[
K(j, j) = 1 \quad (n = 0) \tag{3.13}
\]

Obviously \( K(j, j + L - 1) \) is \( j \) independent. The open chain, \( L(L + 1)/2 \) equations. Equation (3.11) stays unchanged except when \( j = 1 \) or \( j = L - n + 1 \) when we have:

\[
\frac{dK(1, n)}{dt} = -[\gamma_L + (n - 1)\delta]K(1, n) + a_R K(1, n - 1) + \alpha_L K(1, n + 1) \tag{3.14}
\]

\[
\frac{dK(j, L)}{dt} = -[\gamma_R + (n - 1)\delta]K(j, L) + a_L K(j + 1, L) + \alpha_R K(j - 1, L) \tag{3.15}
\]

Equation (3.12) is replaced by:

\[
\frac{dK(1, L)}{dt} = -(L - 1)\delta K(1, L) \tag{3.16}
\]
and the notation (3.13) used in Eqs. (3.11), (3.14) and (3.15) stays unchanged. In the above Equations we have denoted:

\[ \alpha_R = a_R + d_R - b_R - c = \Gamma_{01}^{10} + \Gamma_{11}^{10} - \Gamma_{11}^{00} - \Gamma_{01}^{00} \]

\[ \alpha_L = a_L + d_L - b_L - c = \Gamma_{01}^{01} + \Gamma_{11}^{01} - \Gamma_{11}^{00} - \Gamma_{01}^{00} \]

\[ a_R = \Gamma_{01}^{10}, \quad a_L = \Gamma_{01}^{01} \]

\[ \gamma_R = a_L + a_R + d_R = \Gamma_{01}^{10} + \Gamma_{11}^{01} + \Gamma_{11}^{10} \]

\[ \gamma_L = a_L + a_R + d_L = \Gamma_{01}^{01} + \Gamma_{11}^{01} + \Gamma_{11}^{01} \]

\[ \gamma = \gamma_R + \gamma_L \]

\[ \delta = b_R + b_L + c = \Gamma_{01}^{00} + \Gamma_{10}^{00} + \Gamma_{11}^{00} \]

Notice that the equations for the open chain depend on seven parameters \((\alpha_R, \alpha_L, a_R, a_L, \gamma_R, \gamma_L \text{ and } \delta)\). In the case of the ring however, the differential equations depend only on six parameters since \(\gamma_R \text{ and } \gamma_L\) appear in the combination \(\gamma_R + \gamma_L\) only. One also sees that the time derivative of the probability to find a hole of length \(n\) is related to the probability functions for one-hole of length \(n-1, n \text{ and } n+1\).

We now consider the two-holes probability function:

\[ K(j, n+1; k, k+m) = \langle 0 | \prod_{r=j}^{n+j-1} E_k^{00} \prod_{s=k}^{j+m-1} E_s^{00} e^{-Ht} | P_0 \rangle \]

which describes the probability of having a hole of length \(n\) starting on the site \(j\) and a second hole of length \(m\) starting on the site \(k\). A straightforward calculation gives the following differential equations for the two-holes probability function in the case of the ring geometry:

\[ \frac{dK(j, j+n-1; k, k+m-1)}{dt} = - \left[ 2\gamma + (m+n-2)\delta \right] K(j, j+n-1; k, k+m-1) \]

\[ + a_L[K(j+1, n-1; k, k+m-1) + K(j, j+n-1; k+1, k+m-1)] \]

\[ + a_R[K(j, j+n-2; k, k+m-1) + K(j, j+n-1; k, k+m-2)] \]

\[ + \alpha_L[K(j, n; k, k+m) + K(j, j+n-1; k, k+m)] \]

\[ + \alpha_R[K(j-1, n-1; k, k+m-1) + K(j, j+n-1; k-1, k+m-1)] \]

\[ (3.19) \]
As one notices the conditions (3.8) which decouple the one-hole problem from more holes, decouples the two-holes problem from three and more holes. The two-holes problem contains one-hole probability functions. This happens when functions of the type

\[ K(j, j + n - 1; j + n, k + m - 1) = K(j, j + m + n - 1) \]  \hspace{1cm} (3.20)

occur in the right hand side of Eqs. (3.19). The same pattern occurs in the problem of three or more holes and thus the conditions (3.8) have produced a decoupling into sub-sets of the master-equation. The same happens for the open chain.

We now turn to the decoupling problem in the case of spin-strings. The one-spin string is defined by Eq. (2.22), the probability for two-spin-strings is:

\[ S(j, j + n - 1; k, k + m - 1) = \langle 0 \prod_{r=j}^{j+n-1} \sigma_r^z \prod_{s=k}^{k+m-1} \sigma_s^z e^{-Ht} | P_0 \rangle \] \hspace{1cm} (3.21)

etc. ... Let us show that the decoupling problem for the spin-strings takes us to the same differential equations as the ones for the holes with a simple replacement \( K \rightarrow S \). The expressions of the constants \( a_{R,L}, b_{R,L}, d_{R,L} \) and \( c \) respectively \( \alpha_{R,L}, \gamma_{R,L} \) and \( \delta \) in terms of rates have to be changed.

Consider two matrices \( b \) and \( b^{-1} \):

\[ b = \begin{pmatrix} 1 - 1 \\ 0 \end{pmatrix}, \quad b^{-1} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \] \hspace{1cm} (3.22)

and correspondingly the matrices:

\[ B = \prod_{k=1}^{L} b_k, \quad B^{-1} = \prod_{k=1}^{L} b_k^{-1} \] \hspace{1cm} (3.23)

we perform a similarity transformation [32] which takes us from the Hamiltonian \( H \) given by Eq. (2.10) with \( H_k \) given by Eq. (3.10) to \( H^S \) respectively \( H^S_k \):

\[ H^S = B^{-1} H B \] \hspace{1cm} (3.24)

and
Notice that as in Eq. (2.16) the sum of the matrix elements in each column are zero and consequently the matrix elements of (3.25) can be interpreted as rates provided that they are positive, we will denote them by $\tilde{\Gamma}_{\gamma,\delta}^{\alpha,\beta}$. Instead of the five conditions (3.8) on the rates $\Gamma_{\gamma,\delta}^{\alpha,\beta}$ we now find five different conditions on the rates $\tilde{\Gamma}_{\gamma,\delta}^{\alpha,\beta}$:

\[
\tilde{\Gamma}_{11}^{11} + \tilde{\Gamma}_{00}^{00} = \tilde{\Gamma}_{10}^{10} + \tilde{\Gamma}_{01}^{01}; \\
\tilde{\Gamma}_{10}^{10} + \tilde{\Gamma}_{11}^{11} = \tilde{\Gamma}_{00}^{00} + \tilde{\Gamma}_{01}^{01}; \\
\tilde{\Gamma}_{01}^{01} + \tilde{\Gamma}_{11}^{11} = \tilde{\Gamma}_{00}^{00} + \tilde{\Gamma}_{01}^{01}.
\]

The spectrum of the Hamiltonian (3.25) is again non-negative if the $\tilde{\Gamma}_{\gamma,\delta}^{\alpha,\beta}$ are positive. This implies a different domain for the parameters $a_{R,L}, b_{R,L}, d_{R,L}$ and $c$ than the one obtained from the positivity of the rates $\Gamma_{\gamma,\delta}^{\alpha,\beta}$ occuring in Eq. (3.10).

Let us now consider the one-hole function (2.19) and make use of the similarity transformation (3.24):

\[
K(j, n + j - 1) = <0 | \prod_{k=j}^{j+n-1} F_k^{00} B e^{-H^S t} B^{-1} | P_0>
\]

Let us now notice (see Eq. (2.15)) that

\[
<0 | b = <0 |
\]

(3.28)
\begin{align*}
&<0 | E_k^{00} B = <0 | (E_k^{00} - E_k^{01}) = <0 | (E_k^{00} - E_k^{11}) = <0 | \sigma_k^z \quad (3.29) \\
&\text{With (3.28) and (3.29) we get instead of Eq. (3.27)} \\
&K(j, n + j - 1) = <0 | \prod_{k=j}^{j+n-j} \sigma_k^z e^{-H^{St}t} | P_0^S > \quad (3.30)
\end{align*}

where

\begin{align*} 
&| P_0^S > = B^{-1} | P_0 > \\ 
&\text{Comparing the Eqs. (2.22) and (3.30) we learn that the one-hole probability} \\
&\text{function at time } t \text{ given by a Hamiltonian } H \text{ with an initial condition } P_0(\{\beta\}) \\
&\text{coincides with the one-spin-string function given by the Hamiltonian } H^S \text{ and} \\
&\text{initial condition } P_0^S(\{\beta\}). \text{ The rates } \Gamma_{\alpha\beta}^{\gamma\delta} \text{ and } \tilde{\Gamma}_{\alpha\beta}^{\gamma\delta} \text{ are both dependent on the} \\
&\text{seven parameters } a_{R,L}, b_{R,L}, d_{R,L} \text{ and } c \text{ but as we mentioned the domain of} \\
&\text{the parameters which makes the rates } \Gamma_{\alpha\beta}^{\gamma\delta} \text{ positive is in general different} \\
&\text{from the domain where } \tilde{\Gamma}_{\alpha\beta}^{\gamma\delta} \text{ are positive. The differential equations satisfied} \\
&\text{by the spin-string functions which depend on the parameters } a_{R,L}, \alpha_{R,L}, \gamma_{R,L} \text{ and } \delta \text{ are identical with those satisfied by the holes probability functions}, \\
&\text{the expression of these parameters in terms of the rates } \tilde{\Gamma}_{\alpha\beta}^{\gamma\delta} \text{ however, differs} \\
&\text{from (3.17) where the } \Gamma_{\alpha\beta}^{\gamma\delta} \text{ rates were used. For example:}
\end{align*}

\begin{align*} 
&\delta = 2(\tilde{\Gamma}_{00}^{00} + \tilde{\Gamma}_{01}^{00}) ; \quad \gamma = 2(\tilde{\Gamma}_{01}^{00} + \tilde{\Gamma}_{10}^{00} + \tilde{\Gamma}_{00}^{00} + \tilde{\Gamma}_{01}^{01}) \\
&(3.32)
\end{align*}

Notice that \( \gamma \) and \( \delta \) are non-negative as for the rates \( \Gamma_{\alpha\beta}^{\gamma\delta} \) (see Eq. (3.17)). Obviously the transformation (3.23) takes the two-holes probability functions into the two-spin-string functions etc.... The same similarity transformation takes vacancies (Eq. (3.1)) into spins (Eq. (3.2)). Finally let us note that the more general transformation

\begin{align*} 
&b = \begin{pmatrix} 
1 & -\xi \\
1 + \xi & 1 + \xi \\
0 & 1
\end{pmatrix} \\
&(3.33)
\end{align*}

takes the one-hole function into the more general observable (2.25). This observation is relevant for two reasons. On one side it shows that more general observables with the same structure bring nothing new from the point of view of integrability and on the other side, for each value of \( \xi \) one obtains new rates which, when positive, give a spectrum with a non-negative real part.
4 The spectrum in the one-hole subset. Closed ring

Our aim is to find the spectrum of the Hamiltonian $H$ given by Eq. (3.10) in the one-hole sector. This implies solving the equations (3.11) - (3.13) in the case of the closed ring. For the special case of the left-right symmetric coagulation-decoagulation model:

$$a_L = a_R = a; d_L = d_R = d; b_{R,L} = c = 0$$ (4.1)

or

$$\alpha_R = \alpha_L = a + d; \gamma_R = \gamma_L = 2a + d, \delta = 0$$ (4.2)

the one-hole probability equations were solved for both the ring and the open chain case [32]. The spectrum is that of free fermions. One can easily show that in this particular case one can use a similarity transformation in order to bring the Hamiltonian given by (3.10) into the form (3.4) with $\eta = (1 + d)^{1/2}$ (we choose $a = 1$). The Glauber model [12] which again is described by free fermions [13] corresponds to the choice:

$$a_L = a_R = a, b_{L,R} = d_{R,L} = c = 0$$ (4.3)

or

$$\alpha_R = \alpha_L = a, \gamma_R = \gamma_L = a, \delta = 0$$ (4.4)

As mentioned already, the Glauber model was solved in the framework of the one-spin-string problem. One other special case was solved in Refs. [33], [34], which again is a free fermionic case:

$$a_L \neq a_R, b_{R,L} = d_{R,L} = c = 0$$ (4.5)

or

$$a_L = \alpha_L, a_R = \alpha_R, \gamma_R = \gamma_L = a_L + a_R, \delta = 0$$ (4.6)

We now consider the general case. It is convenient to make a change of variables:

$$K(j, j + n - 1) = R(n; j + \frac{n - 1}{2}) = R(n; p)$$ (4.7)
which implies to consider the length of the hole and its center as variables \((p)\) is integer or half-integer). We next take the Fourier transform of \(R(n; p)\):

\[
R(n, p) = \sum_{q=0}^{L-1} e^{-\frac{2\pi i pq}{L}} Q(n; q) \quad (4.8)
\]

and get

\[
\frac{1}{D(q)} \left\{ \frac{dQ(n; q)}{dt} + [(\gamma - \delta) + n\delta]Q(n; q) \right\} = u(q)Q(n - 1; q) + u^{-1}(q)Q(n + 1; q) \quad (q = 0, 1, \ldots, L - 1) \quad (4.9)
\]

with the boundary conditions

\[
Q(0; q) = Q(L; q) = 0 \quad (q \neq 0) \\
Q(0; 0) = 1; \; Q(L; 0) = A e^{-L \delta t} \quad (4.10)
\]

where

\[
u(q) = \left[ \frac{a_R + a_L + i(a_R - a_L)\tan \frac{\pi q}{L}}{\alpha_R + \alpha_L + i(\alpha_R - \alpha_L)\tan \frac{\pi q}{L}} \right]^{1/2} \quad (4.11)
\]

\[
D(q) = \left[ (a_R + a_L)(\alpha_R + \alpha_L) \right]^{1/2} \left[ \cos^2 \frac{\pi q}{L} - \left( \frac{a_R - a_L}{a_R + a_L} \right) \left( \frac{\alpha_R - \alpha_L}{\alpha_R + \alpha_L} \right) \sin^2 \frac{\pi q}{L} + \frac{i}{2} \sin \frac{2\pi q}{L} \left( \frac{\alpha_R - \alpha_L}{\alpha_R + \alpha_L} + \frac{a_R - a_L}{a_R + a_L} \right) \right]^{1/2} \quad (4.12)
\]

and \(A\) is an arbitrary constant. The boundary conditions (4.10) for the \(q = 0\) case give an inhomogenous system of differential equations. Since in the present paper we are interested in the spectrum only, we consider only the homogenous equations and take

\[
Q^{\text{hom}}(n; q) = u^n(q)F(n, q)e^{-\lambda t} \quad (4.13)
\]

where \(\lambda\) are the eigenvalues. The \(F(n; q)\) satisfy the equation:

\[
(E - n)F(n, q) = V(F(n + 1; q) + F(n - 1; q)) \quad (4.14)
\]

where

\[
F(0; q) = F(L; q) = 0 \quad (4.15)
\]
The Equation (4.14) is already known. It appears in the energy spectrum problem of an electron in a finite one-dimensional crystal in an uniform electric field \[35\], \[36\]. The electric potential \( E_x \) on the lattice gives a contribution \( E_n \) on the lattice which appears in the left-hand side of Eq. (4.14). (We have divided by \( E \) the expression of the corresponding Hamiltonian). For real \( D(q) \) the energy levels form the so-called Wannier-Stark ladder which has been investigated in great detail. Eq. (4.14) also appears in the problem of the diagonalization of the XX Hamiltonian in a Z field whose strength is position dependent \[37\]:

\[
H = - \sum_{k=-N}^{N} \left[ \sigma^x_k \sigma^x_{k+1} + \sigma^y_k \sigma^y_{k+1} + (h + Rk)\sigma^z_k \right]
\]  

where \( \sigma^x, \sigma^y \) and \( \sigma^z \) are Pauli matrices and \( h \) and \( R \) are constants. We will shortly present the solution of Eq. (4.14) as presented in Ref. \[35\], it is given in terms of Lommel functions \[38\]:

\[
R_m,\mu(z) = \sum_{n=0}^{\lfloor m/2 \rfloor} \frac{(-1)^n(m-n)!}{n!(m-2n)!} \frac{\Gamma(\mu+m-n)}{\Gamma(\mu+n)} \left( \frac{z}{2} \right)^{-m+2n}
\]  

which satisfy the identity

\[
R_m,\mu(z) = (-1)^m R_{m,-\mu-m+1}(z)
\]  

The eigenvalues \( E \) of the equation (4.14) are given by the equation:

\[
R_{L-1,1-E} \left( 2V(q) \right) = 0
\]  

There are \( (L-1) \) solutions to the equation (4.20). To each of these solution corresponds an eigenfunction:

\[
F(n, q) = R_{n-1,1-E} \left( 2V(q) \right) F(1, q) ; (n = 2, \ldots, L-1)
\]  

Because of the identity (4.19) if \( E \) is an eigenvalue so is \( \tilde{E} \) where

\[
\tilde{E} = -E + L
\]
This implies that the eigenvalues $E$ are distributed symmetric around $\frac{L}{2}$. For $L$ even, $\frac{L}{2}$ is an eigenvalue. For various properties of the zeroes of the Lommel polynomials see Ref. [38]. To the eigenvalues obtained from Eqs. (4.16) and (4.20), one has to add the eigenvalue $\lambda = L\delta$ obtained directly from Eq. (4.10) for $q = 0$. The general character of the spectrum for real $V$ is the following. For not too large $V$, the eigenvalues in the center are equidistant, forming a simple ladder with unit spacing which results from the l. h. s. of Eq. (4.14). At the upper and lower end of the spectrum the spacing increases. In this region the $V$-term in the equation becomes important.

In the limit of large $V$ or small $\delta$ (i. e. for vanishing field in the electronic problem) one comes back to a simple tight-binding model with eigenvalues

$$\frac{E}{V} = 2 \cos \frac{\pi k}{L} \quad (k = 1, \ldots L - 1) \quad (4.23)$$

and the corresponding eigenfunction is

$$F(n, q) = \frac{\sin \left(\frac{n\pi k}{L}\right)}{\sin \frac{\pi k}{L}} F(1, q) \quad (4.24)$$

This implies that for $\delta = 0$, the eigenvalues of the Hamiltonian are

$$\lambda(q) = \gamma - 2 D(q) \cos \frac{\pi k}{L} \quad (k = 1, 2, \ldots L - 1; q = 0, 1, \ldots L - 1) \quad (4.25)$$

to which we have to add $\lambda = 0$, obtained directly from Eq. (4.10) in the case $q = 0$. From the expression (4.12) of $D(q)$ we can obtain the conditions under which the spectrum is that of free fermions. The energy levels should correspond to two-fermionic excitations (see Sec. 3). If

$$\frac{a_R}{a_L} = \frac{\alpha_R}{\alpha_L} \quad (4.26)$$

one obtains

$$\lambda(q) = \gamma + G(q + k) + G(q - k) \quad (4.27.a)$$

where

$$G(k) = - \left[ (a_R + a_L)(\alpha_R + \alpha_L) \right]^{1/2} \left( \cos \frac{\pi k}{L} + i \frac{a_R - a_L}{a_R + a_L} \sin \frac{\pi k}{L} \right) \quad (4.27.b)$$
which indeed looks like a two fermions excitation one of momentum \((q + k)\), the other of momentum \((q - k)\) with a total momentum \(q\). The known examples (4.2), (4.4) and (4.6) satisfy the condition (4.26). If \(a_R/a_L \neq \alpha_R/\alpha_L\) and if \(\delta = 0\) the spectrum is not given by free fermions, nevertheless the spectrum and wave-functions have a simple expression.

We have checked on small chains that if the condition (4.26) is fulfilled the spectrum of the Hamiltonian (not only the one-hole subset) is indeed given by free fermions. We didn’t look for the similarity transformation which would bring the Hamiltonian to the corresponding form.

The problem of the open chain, also probably solvable, was not considered yet. We have only looked at small chains for the simple case where \(b_R = b_L = c = 0\). The levels are simply or doubly degenerate (the \(E = 0\) level is always doubly degenerate). This is in contrast to the free fermionic picture where all levels should be double degenerate (see Sec. 3). This is to be expected as long as the condition (4.26) is not satisfied. The surprise is that if one takes

\[
\frac{a_R}{a_L} = \frac{\alpha_R}{\alpha_L} \neq 1 \tag{4.28}
\]

although the periodic chain is "fermionic" this is not the case for the open chain.

Finally, we didn’t touch the other sectors of the Hamiltonian (more than one hole). We will return to this problem in Sec. 7.

### 5 The spectra in the continuum limit (closed ring)

It is interesting to consider the continuum limit of the spectra. We first take the case \(\delta = 0\). Then Eq. (4.14) becomes

\[
\left(\frac{E}{V} - 2\right)F(n, q) = F(n + 1; q) + F(n - 1; q) - 2F(n; q)
\]

\[
\approx \frac{d^2F(n; q)}{dn^2} \tag{5.1}
\]

with the boundary condition

\[
F(0; q) = F(L; q) = 0 \tag{5.2}
\]
which implies
\[ \frac{E}{V} - 2 = -\frac{\pi^2 k^2}{L^2} \quad (k = 1, \ldots) \quad (5.3) \]
and using Eq. (4.16):
\[ \lambda = \gamma - 2D(q) + D(q)\frac{\pi^2 k^2}{L^2} \quad (k = 1, 2, \ldots; q = 0, \ldots) \quad (5.4) \]
where (see Eq. (4.12))
\[ D(q) = \sqrt{(a_R + a_L)(\alpha_R + \alpha_L)} \left[ 1 + \frac{i\pi q}{2L} \left( \frac{\alpha_R - \alpha_L}{\alpha_R + \alpha_L} + \frac{a_R - a_L}{a_R + a_L} \right) + 0 \left( \frac{q^2}{L^2} \right) \right] \quad (5.5) \]
To Eq. (5.4) we have to add the value \( \lambda = 0 \) for \( q = 0 \) (keep in mind
Eq. (4.10)). The spectrum given by Eq. (5.4) is massive if \( \gamma - 2D(0) > 0 \),
massless if \( \gamma = 2D(0) \) with a quadratic dispersion relation in the real part
and a linear one in the imaginary part.

We now consider the case \( \delta \neq 0 \). Using Eqs. (4.14) and (4.16), we get in
the continuum:
\[ \frac{d^2 F(n, q)}{dn^2} = (a + bn)F(n, q) \quad (5.6) \]
where
\[ a = \frac{\gamma - \delta - \lambda}{D(q)} - 2 \quad ; \quad b = \frac{\delta}{D(q)} \quad (5.7) \]
with
\[ F(0; q) = F(L; q) = 0 \quad (5.8) \]
If \( Ai(z) \) and \( Bi(z) \) are two independent solutions of the Airy equation \[39\]:
\[ \frac{d^2 w}{dz^2} = zw \quad (5.9) \]
the general solution of Eq. (5.6) reads
\[ F(n, q) = C_A Ai(a b^{-2/3} + b^{1/3} n) + C_B Bi(a b^{-2/3} + b^{1/3} n) \quad (5.10) \]
where \( C_A \) and \( C_B \) are arbitrary constants.

The eigenvalues \( \lambda \) (contained in the constant \( a \)) are determined from the
boundary conditions (5.8). These equations have to be solved numerically.
In the infinite volume limit \((L \to \infty)\) the whole picture simplifies. Since the Airy function \(Bi(z)\) diverges for large \(z\), we are left with the function \(Ai\) only. In order to satisfy the condition

\[
F(0, q) = 0 \quad (5.11)
\]

\(a b^{-2/3}\) has to be a zero of \(Ai(z)\). It is known \footnote{See Ref. [39].} that the zeroes of \(Ai(z)\) are all on the negative real axis in the complex \(z\) phase (they are tabulated in \footnote{See Ref. [39].}), we give the first two values: \(c_1 = -2.338\), \(c_2 = -4.087\). This implies that

\[
\lambda_n = \gamma - \delta - 2D(q)+ | c_n | D(q)^{1/3} \delta^{2/3} \quad (5.12)
\]

and thus the spectrum in the continuum is known. The eigenfunctions are given by Eq. (5.10) with \(C_B = 0\) and \(a\) fixed by Eq. (5.12). To the eigenvalues (5.12) one has to add for \(q = 0\) the eigenvalue \(\lambda = 0\) that one gets from (4.10) for \(L\) infinity. At least in the domain where \(a_{R,L}, b_{R,L}, d_{R,L}\) and \(c\) are positive the spectrum has to be massive. In this case one can use the stochastic interpretation of the Hamiltonian (3.10). The birth and pair-creation processes (see Eq. (2.4)) come with independent scales in the evolution operator determining a massive phase so that as long as \(\delta \neq 0\) one is in this phase. This statement is valid not only for the one-hole subset of the Hamiltonian but for all the subsets.

## 6 The critical dynamics of the one-dimensional Ising model

As an application of the calculations of spectra presented in the last section we consider the dynamics of the one-dimensional Ising model defined by the equilibrium probability distribution

\[
P_{eq} = Z^{-1} \prod_{i=1}^{L} e^{\frac{1}{2} S_i S_{i+1}} \quad (6.1)
\]

where \(Z\) is the partition function:

\[
Z = \sum_{S_i=\pm1} \prod_{i=1}^{L} e^{\frac{1}{2} S_i S_{i+1}} \quad (6.2)
\]
The two-point function has the following large distance behaviour

\[ < S_i S_{i+R} > \sim e^{-R/\zeta} \]  
(6.3)

where the correlation length \( \zeta \) and the mass \( \mu \) are

\[ \zeta^{-1} = 2\mu \quad \mu = e^{-\frac{2}{T}} \]  
(6.4)

We can rewrite (6.2) in a slightly different form:

\[ P_{eq} = Z^{-1} \prod_{i=1}^{L} e^{\frac{1}{1}(-1)^{(\alpha_i - \alpha_{i+1})}} = Z^{-1} \prod_{i=1}^{L} e^{\frac{1}{1}(-1)^{\beta_i}} \]  
(6.5)

where \( \alpha_i, \beta_i = 0, 1 \). In the last equality of Eq. (6.5) we have performed a duality transformation. It was shown in Ref. [7] that the following relations have to be satisfied by the rates appearing in the master equation (2.2) in order to have \( P_{eq} \) as a steady state solution (detailed balance):

\[ \mu \Gamma_{0,0;1,0}^{1,0:s} - \Gamma_{1,0;0,0}^{0,0:s} = \mu^2 \Gamma_{1,1;0,1}^{1,1:s} - \mu \Gamma_{0,1;1,0}^{0,1:s} = \Gamma_{0,0;1,1}^{0,0} - \mu^2 \Gamma_{0,0;1,0}^{1,1} \]  
(6.6)

\[ \Gamma_{1,0;1,0}^{0,0:a} + \mu \Gamma_{1,1;1,0}^{1,0:a} + \mu^2 \Gamma_{1,0;1,1}^{1,1:a} + \mu \Gamma_{0,0;0,0}^{0,1:a} = 2\mu \Gamma_{0,0;1,0}^{1,0:a} \]  
(6.7)

where

\[ \Gamma_{\gamma,\delta}^{\alpha,\beta:s} = \Gamma_{\gamma,\delta}^{\alpha,\beta} + \Gamma_{\delta,\gamma}^{\beta,\alpha} \]

\[ \Gamma_{\gamma,\delta}^{\alpha,\beta:a} = \Gamma_{\gamma,\delta}^{\alpha,\beta} - \Gamma_{\delta,\gamma}^{\beta,\alpha} \]

Let us assume that we give a set of rates compatible with Eqs. (6.6) - (6.7). This defines a Hamiltonian and the physical question one asks is the behaviour of the energy gap \( E_G \) in the thermodynamical limit as \( \mu \) goes to zero (the Ising model in one dimension is critical at \( T = 0 \)). The inverse of the energy gap gives the time correlation length. As shown in Ref. [7] in several examples, two scenarios are possible. In the first, as \( \mu \to 0 \), \( E_G \) stays finite. In this case we have no critical slowing down although we take local dynamics. In the second scenario, \( E_G \) vanishes like

\[ E_G \sim \mu^z \]  
(6.8)

with \( z = 2 \) (see also Refs. [13, 23, 24]). It is our aim to check if for rates \( \Gamma_{\gamma,\delta}^{\alpha,\beta} (3.10) \) given by the hole picture or spin-string rates \( \tilde{\Gamma}_{\gamma,\delta}^{\alpha,\beta} \) given by Eq.
one can obtain new solutions. Since in some cases (see Eq. (4.20) or (4.27)) the spectrum is complex it is interesting to see if the equilibrium state can be reached in an oscillatory mode.

We first consider the holes picture. The Eqs. (6.6), (6.7) together with positivity give the following solution:

\[ b_R = b_L = c = 0; \quad a_R = a_L + \eta \mu \]
\[ d_R = \mu(a_L - \eta); \quad d_L = \mu(a_L + \eta) + \eta \mu^2 \]  \hspace{1cm} (6.9)

where \( a_L > 0 \) and \( \eta \) is a parameter satisfying the condition:

\[ a_L > |\eta| \]  \hspace{1cm} (6.10)

Notice that \( \delta = 0 \) (see Eq. (3.17)) and that

\[ \frac{a_R}{a_L} \neq \frac{d_R}{d_L} \]  \hspace{1cm} (6.11)

which implies that we do not have free fermions (see Eq. (4.26)) thus the model we consider is new. In order to compute the energy gap, we use Eqs. (5.4) and (5.5) to get:

\[ E_G = \gamma - 2\sqrt{(a_R + a_L)(\alpha_R + \alpha_L)} = \frac{a_L}{2} \mu^2 - \frac{\eta \mu^3}{4} \]  \hspace{1cm} (6.12)

where we have used Eqs. (3.17) and (6.9). As we see we get no oscillatory behaviour, the energy gap vanishes for \( \mu \to 0 \) again with \( z = 2 \) (see Eq. (6.8)) in agreement with the universality hypothesis.

We now consider the rates \( \tilde{\Gamma}_{\alpha,\beta}^{\gamma,\delta} \) given by Eq. (3.25), in this case we find:

\[ c = \frac{8\mu^2}{(1 - \mu^2)} a_R, \quad a_R = a_L, \quad d_R = d_L = -b_R = -b_L = c \]  \hspace{1cm} (6.13)

which gives:

\[ \tilde{\Gamma}_{01}^{00} = \tilde{\Gamma}_{10}^{00} = \tilde{\Gamma}_{00}^{01} = \tilde{\Gamma}_{11}^{01} = \tilde{\Gamma}_{10}^{11} = \tilde{\Gamma}_{01}^{11} = \tilde{\Gamma}_{00}^{10} = 0 \]
\[ \tilde{\Gamma}_{01}^{10} = \tilde{\Gamma}_{01}^{01}; \quad \tilde{\Gamma}_{10}^{10} + \tilde{\Gamma}_{01}^{01} = \tilde{\Gamma}_{01}^{11} + \tilde{\Gamma}_{10}^{11} \]  \hspace{1cm} (6.14)

Now we have free fermions and get back the Glauber model \([12]\). Since we want \( a_R \) to be finite for \( \mu \to 0 \), we choose:

\[ c = \frac{\mu^2 \lambda}{2} \quad (\lambda > 0) \]  \hspace{1cm} (6.15)
where \( \lambda \) is a constant and we get the known result
\[
E_G = \lambda \mu^2
\quad (6.16)
\]

Our results for \( E_G \) were obtained only within a sector of the Hamiltonian and one can always argue that other subsets of differential equations can give different results. This argument holds for Eq. (6.12) but not for Eq. (6.14) since in this case the Hamiltonian can be diagonalized completely [13].

7 Conclusions

We have shown how to bring a Hamiltonian describing a one-dimensional quantum chain with two states per site and depending on seven parameters to a block-diagonal form. The structure of the blocks is reminiscent of a Hamiltonian described by free fermions. The underlying algebraic structure of the problem is still a mystery for us. We have diagonalized the first block in the case of periodic boundary conditions. It turns out that Lommel polynomials play a fundamental role in the eigenvalues and eigenfunctions of the Hamiltonian. We have also clarified the continuum limit of the spectra. The case of the open chain was not considered but probably the method used in one special case [32] can be extended to the general one. Hopefully the methods similar to the one used for the Glauber model [13] can be extended to all the blocks. If this is not possible, we will be left with an interesting example of partially integrable systems in condensed matter.

We have paid special attention to various similarity transformations which connect stochastic observables not only in order to find properties of the spectra coming from positivity but also in order to understand how the method used in this paper can be extended to models with more than two states.

In this paper we didn’t do any specific calculations related to diffusion-reaction processes which would have implied solving also the nonhomogenous case of the differential equations since our purpose was to find the eigenvalues and eigenfunctions of the Hamiltonian.

One basic question is if our Hamiltonian cannot be diagonalized using conventional Bethe-Ansatz methods. The answer is not obvious. We first have checked if one can’t use the Baxterisation programme [40]. Before starting it we have checked if the matrices \( \hat{R}_k \):
\[
\hat{R}_k = H_k + c \hat{1}
\quad (7.17)
\]
where $H_k$ is given by Eq. (3.10), $c$ is an arbitrary constant and $\hat{1}$ the unit matrix, satisfy the braid group relations:

$$\hat{R}_k \hat{R}_{k+1} \hat{R}_k = \hat{R}_{k+1} \hat{R}_k \hat{R}_{k+1}$$  (7.18)

The answer is in general negative. If the answer would have been positive we would have had to look for some conditions to find an associative algebra for which the baxterization programme can be done. We have next tried the Reshetikhin criterium according to which the following relation has to be satisfied [41]:

$$[H_k + H_{k+1}, [H_k, H_{k+1}]] = X_k - X_{k+1}$$  (7.19)

where $X_k$ is an arbitrary $4 \times 4$ matrix. The answer is again negative in general. One gets a positive answer only in what we called the free-fermionic case (the parameters satisfy the condition $\delta = 0$ and Eq. (4.26)). As it is known the Reshetikin condition is very restrictive (the Hubbard Hamiltonian does not satisfy it) but the authors’ knowledge about conventional integrability conditions stops here.

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