BOUND FOR THE BETTI NUMBERS OF SUCCESSIVE STELLAR SUBDIVISIONS OF A SIMPLEX

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Abstract. We give a bound for the Betti numbers of the Stanley-Reisner ring of a stellar subdivision of a Gorenstein* simplicial complex by applying unprojection theory. From this we derive a bound for the Betti numbers of iterated stellar subdivisions of the boundary complex of a simplex. The bound depends only on the number of subdivisions, and we construct examples which prove that it is sharp.

1. Introduction

Consider the class of simplicial complexes obtained from the boundary complex of a simplex with \( q + 1 \) vertices by any sequence of \( c - 1 \) stellar subdivisions. We give bounds for the (total) Betti numbers of the minimal resolution of the associated Stanley-Reisner rings. The bounds depend only on \( c \) and not on \( q \). Our main tool is the relation of stellar subdivisions of Gorenstein* simplicial complexes with the Kustin-Miller complex construction obtained in [1], which gives an easy way to control the Betti numbers of a stellar subdivision. By constructing a specific class of examples, we prove that for fixed \( c \) our bounds are attained for \( q \) sufficiently large.

There are bounds in the literature for various classes of simplicial complexes. If we only subdivide facets starting from a simplex the process will yield a stacked polytope. In this case, there is an explicit formula for the Betti numbers due to Terai and Hibi [15]. See also [6] for a combinatorial proof, and [8] Theorem 3.3, [1] for the construction of the resolutions. In [8] Theorem 2.1, Proposition 3.4, Herzog and Li Marzi consider bounds for a more general class than Gorenstein, leading to a less sharp bound in our setting. Migliore and Nagel discuss in [10] Proposition 9.5 a bound for fixed \( h \)-vector. The bounds of Römer [13] apply for arbitrary ideals with a fixed number of generators and linear resolution.

To state our results, for \( c \geq 1 \) and \( q \geq 2 \) denote by \( D_{q,c} \) the set of simplicial complexes \( D \) on \( q + c \) vertices which are obtained by \( c - 1 \) iterated stellar subdivisions of faces of positive dimension, starting from the boundary complex of a \( q \)-simplex. If \( k \) is any field, we denote by \( k[D] \) the Stanley-Reisner ring of \( D \). Note, that \( k[D] \) is the quotient of a polynomial ring by a codimension \( c \) Gorenstein ideal. Define inductively \( l_c = (l_{c,0}, l_{c,1}, \ldots, l_{c,c}) \in \mathbb{Z}^{c+1} \) by \( l_1 = (1, 1) \) and

\[
l_c = 2(l_{c-1}, 0) + 2(0, l_{c-1}) - (1, 1, 0, \ldots, 0) - (0, \ldots, 0, 1, 1) \in \mathbb{Z}^{c+1}
\]

for \( c \geq 2 \). For example \( l_2 = 2(1, 1, 0) + 2(0, 1, 1) - (1, 1, 0) - (0, 1, 1) = (1, 2, 1) \), \( l_3 = (1, 5, 5, 1) \), and \( l_4 = (1, 11, 20, 11, 1) \). The main result of the paper is the following theorem giving an upper bound for the Betti numbers of \( k[D] \) for \( D \in D_{q,c} \). The bound follows immediately from the stronger Theorem [8] and that it is sharp from Proposition [14].

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Theorem 1. Suppose \( c \geq 1, q \geq 2 \) and \( D \in \mathcal{D}_{q,c} \). Then for the Betti numbers of \( k[D] \) it holds that
\[
b_i(k[D]) \leq l_{c,i}
\]
for all \( 0 \leq i \leq c \). Moreover, the bound is sharp in the following sense: Given \( c \geq 1 \), there exists \( q \geq 2 \) and \( F \in \mathcal{D}_{q,c} \) with \( b_i(k[F]) = l_{c,i} \) for all \( 0 \leq i \leq c \).

In Section 3 we focus on bounding the Betti numbers of stellar subdivisions. The first result is Proposition 3 which gives a bound for the Betti numbers of the Stanley-Reisner ring of a stellar subdivision of a Gorenstein* simplicial complex \( D \) with respect to a face \( \tau \) in terms of those of \( D \) and of the link of \( \tau \). The proof of this proposition uses Proposition 2 which is a generalization of [11, Theorem 1.1], and the Kustin-Miller complex construction (see [9] and Section 4). To prove Theorem 1 by induction on the codimension \( c \), we have to enlarge the class of complexes \( \mathcal{D}_{q,c} \) by including also the links of faces. We give a bound for their Betti numbers in Proposition 7. According to the combinatorial Lemma 5 there are three types of links to consider.

Focussing on proving that the bound of Theorem 1 is sharp, we first analyze in Section 4 the Kustin-Miller complex construction in the setting of stellar subdivisions. In particular, we prove in Proposition 11 a sufficient condition for the minimality of the Kustin-Miller complex. In Section 5 we construct for any \( c \) an element \( F \in \mathcal{D}_{q,c} \) (for suitable \( q \)), and using Proposition 11 we show that the inequalities (1.1) are in fact equalities for \( F \). For an implementation of the construction see our package BETTI-BOUNDS [1] for the computer algebra system MACAULAY2 [7]. Using the minimality of the Kustin-Miller complex, we provide in the package a function which efficiently produces the graded Betti numbers of the extremal examples without the use of Gröbner bases.

2. Notation

For an ideal \( I \) of a ring \( R \) and \( u \in R \) write \( (I : u) = \{ r \in R \mid ru \in I \} \) for the ideal quotient. Denote by \( \mathbb{N} \) the set of strictly positive integer numbers. For \( n \in \mathbb{N} \) we set \( [n] = \{1, 2, \ldots, n\} \). Assume \( A \subset \mathbb{N} \) is a finite subset. We set \( 2^A \) to be the simplex with vertex set \( A \), by definition it is the set of all subsets of \( A \). A simplicial subcomplex \( D \subset 2^A \) is a subset with the property that if \( \tau \in D \) and \( \sigma \subset \tau \) then \( \sigma \in D \). The elements of \( D \) are also called faces of \( D \), and the dimension of a face \( \tau \) of \( D \) is one less than the cardinality of \( \tau \). We define the support of \( D \) to be
\[
supp D = \{ i \in A \mid \{ i \} \in D \}.
\]

We fix a field \( k \). Denote by \( R_A \) the polynomial ring \( k[x_a \mid a \in A] \) with the degrees of all variables \( x_a \) equal to 1. For a finitely generated graded \( R_A \)-module \( M \) we denote by \( b_i(M) \) the \( i \)-th Betti number of \( M \), by definition \( b_i(M) = \dim R_A/m \operatorname{Tor}_i^R(A/m, M) \), where \( m = (x_a \mid a \in A) \) is the maximal homogeneous ideal of \( R_A \). It is well-known that if we ignore shifts the minimal graded free resolution of \( M \) as \( R_A \)-module has the shape
\[
M \leftarrow R_A^{b_0(M)} \leftarrow R_A^{b_1(M)} \leftarrow R_A^{b_2(M)} \leftarrow \cdots
\]

For a simplicial subcomplex \( D \subset 2^A \) we define the Stanley-Reisner ideal \( I_{D,A} \subset R_A \) to be the ideal generated by the square free monomials \( x_{i_1}x_{i_2}\cdots x_{i_p} \) where \( \{i_1, i_2, \ldots, i_p\} \) is not a face of \( D \). In particular, \( I_{D,A} \) contains linear terms if and only if \( \text{supp} D = \emptyset \). The Stanley-Reisner ring \( k[D,A] \) is defined by \( k[D,A] = R_A/I_{D,A} \). Taking into account that \( \dim R_A = \# A \), we define the codimension of \( k[D,A] \) by \( \text{codim} k[D,A] = \# A - \dim k[D,A] \). For a nonempty face \( \sigma \) of \( D \) we set \( x_\sigma = \prod_{i \in \sigma} x_i \in k[D,A] \). We denote by \( b_i(k[D,A]) \) the \( i \)-th Betti number of \( k[D,A] \) considered as \( R_A \)-module. In the following, when the set \( A \) is clear we will sometimes simplify the notations \( I_{D,A} \) to \( I_D \) and \( k[D,A] \) to \( k[D] \). In some situations, however, it will be convenient to consider Stanley-Reisner ideals containing variables.
For a nonempty subset $A \subset \mathbb{N}$, we set $\partial A = 2^A \setminus \{ A \} \subset 2^A$ to be the boundary complex of the simplex $2^A$. For the Stanley-Reisner ring of $\partial A$ we have $k[\partial A, A] = R_A/(\prod_{a \in A} x_a)$.

Assume that, for $i = 1, 2$, $D_i \subset 2^A$ is a subcomplex and the subsets $A_1, A_2$ of $\mathbb{N}$ are disjoint. By the join $D_1 * D_2$ of $D_1$ and $D_2$ we mean the subcomplex $D_1 * D_2 \subset 2^{A_1 \cup A_2}$ defined by

$$D_1 * D_2 = \{ \alpha_1 \cup \alpha_2 \mid \alpha_1 \in D_1, \alpha_2 \in D_2 \}.$$  

By [5, p. 221, Exerc. 5.1.20] we have

$$k[D_1 * D_2, A_1 \cup A_2] = k[D_1, A_1] \otimes_k k[D_2, A_2].$$

As a consequence, using the well-known fact that the tensor product of the minimal resolutions of two modules is a minimal resolution of the tensor product of the modules we get that

$$b_i(k[D_1 * D_2]) = \sum_{\ell=0}^i b_i(k[D_1])b_{i-\ell}(k[D_2])$$

for all $i \geq 0$.

If $\sigma$ is a face of $D \subset 2^A$ define the link of $\sigma$ in $D$ to be the subcomplex

$$\text{lk}_D \sigma = \{ \alpha \in D \mid \alpha \cap \sigma = \emptyset \text{ and } \alpha \cup \sigma \in D \} \subset 2^A \setminus \sigma.$$

It is clear that the Stanley-Reisner ideal of $\text{lk}_D \sigma$ is equal to the intersection of the ideal $(I_{D,A} : x_{\sigma})$ with the subring $R_{A\setminus\sigma}$ of $R_A$. In other words, it is the ideal of $R_{A\setminus\sigma}$ generated by the minimal monomial generating set of $(I_{D,A} : x_{\sigma})$. Furthermore, define the star of $\sigma$ in $D$ to be the subcomplex

$$\text{star}_D \sigma = \{ \alpha \in D \mid \alpha \cup \sigma \in D \} \subset 2^A.$$

If $\tau$ is a nonempty face of $D \subset 2^A$ and $j \in \mathbb{N} \setminus A$, we define the stellar subdivision $D_\tau$ with new vertex $j$ to be the subcomplex

$$D_\tau = (D \setminus \text{star}_D \tau) \cup (2^{\{j\}} \ast \text{lk}_D \tau \ast \partial \tau) \subset 2^{A \cup \{j\}}.$$  

Note that $D_\tau$ consists of the following faces:

1. All faces of $D$ which do not contain $\tau$.
2. For each face $\beta \in D$ with $\tau \subset \beta$ the faces $(\beta \setminus \rho) \cup \{j\}$ for all nonempty subsets $\rho$ of $\tau$.

It is easy to see that

$$k[D_\tau, 2^{A\cup\{j\}}] = R_{A\cup\{j\}}/(I_{D,A}, x_\tau, x_j u \mid u \in (I_{D,A} : x_\tau)).$$

Following [14, p. 67], we say that a subcomplex $D \subset 2^A$ is Gorenstein* over $k$ if $A = \text{supp } D$, $k[D]$ is Gorenstein, and for every $i \in A$ there exists $\sigma \in D$ with $\sigma \cup \{i\}$ not a face of $D$. The last condition combinatorially means that $D$ is not a join of the form $2^{\{i\}} * D_1$, and algebraically that $x_i$ divides at least one element of the minimal monomial generating set of $I_{D,A}$. We say that $D \subset 2^A$ is generalized Gorenstein* over $k$ if $D \subset 2^{\text{supp } D}$ is Gorenstein* over $k$. When there is no ambiguity about the field $k$ we will just say Gorenstein* and generalized Gorenstein*. It is well-known (cf. [14, Section II.5]) that if $D \subset 2^A$ is Gorenstein* and $\sigma \in D$ is a face then $\text{lk}_D \sigma \subset 2^{A\setminus\sigma}$ is also Gorenstein*. It follows that if $D \subset 2^A$ is generalized Gorenstein* and $\sigma \in D$ then $\text{lk}_D \sigma \subset 2^{A\setminus\sigma}$ is also generalized Gorenstein*.

Recall also from [12, Definition 1.2] that if $I = (f_1, \ldots, f_s) \subset R$ is a homogeneous codimension 1 ideal of a graded Gorenstein ring $R$ such that the quotient $R/I$ is Gorenstein, then there exists $\psi \in \text{Hom}_R(I, R)$ such that $\psi$ together with the inclusion $I \hookrightarrow R$ generate $\text{Hom}_R(I, R)$ as an $R$-module. The Kustin–Miller unprojection ring of the pair $I \subset R$ is defined as the quotient of $R[T]$ by the ideal generated by the elements $T f_i - \psi(f_i)$, where $T$ is a new variable.
3. Bounds for the Betti numbers of successive stellar subdivisions

The main result of this section is Theorem 9 which gives bounds for the Betti numbers of complexes in $\mathcal{D}_{q,c}$ and links thereof.

In the following, let $D \subset 2^A$ be a generalized Gorenstein* simplicial complex, $\tau \in D$ a nonempty face and $D_\tau \subset 2^{A \backslash \{j\}}$ the corresponding stellar subdivision with new vertex $j \in \mathbb{N} \setminus A$. For simplicity set $R = R_A[z]/(I_{D,A}) = k[D,A][z]$, where $z$ is a new variable.

In [1] we showed that a stellar subdivision of a face of a Gorenstein* simplicial complex corresponds on the level of Stanley-Reisner rings to a certain Kustin–Miller unprojection. In the following proposition we generalize this statement for generalized Gorenstein* simplicial complexes.

**Proposition 2.** Assume that $\dim \tau \geq 1$. Consider the ideal $Q = (I_{D,A} : x_\tau, z) \subset R_A[z]$, and set

$$M = \text{Hom}_R(Q/(I_{D,A}), R).$$

Then $M$ is generated, as $R$-module, by the inclusion homomorphism together with the map $\psi$ that sends $(I_{D,A} : x_\tau)$ to 0 and $z$ to $x_\tau$. Denote by $S$ the Kustin–Miller unprojection ring of the pair $Q/(I_{D,A}) \subset R$ associated to the map $\psi$. We have that $z$ is $S$-regular and $S/(z) \cong k[D_\tau, A \cup \{j\}]$.

**Proof.** If $A = \text{supp} D$ then the statement is [1, Theorem 1.1(b)]. Now assume that $\text{supp} D$ is a proper subset of $A$. Consider $P = \{x_a \mid a \in A \setminus \text{supp} D\} \subset R_A$. We have

$$I_{D,A} = (I_{D,\text{supp} D}) + (P), \quad Q = (I_{D,\text{supp} D} : x_\tau, z) + (P)$$

and

$$I_{D_\tau, A \cup \{j\}} = (I_{D_\tau, \text{supp} D, A \cup \{j\}}) + (P).$$

The arguments in the proof of [1, Theorem 1.1] also prove that $M$ is generated by the inclusion together with the map $\psi$ that sends $(I_{D,A} : x_\tau)$ to 0 and $z$ to $x_\tau$. They also prove that $z$ is $S$-regular and that $S/(z) \cong k[D_\tau, A \cup \{j\}]$. \(\square\)

We will now study the Betti numbers $b_i$ of $k[D_\tau, A \cup \{j\}]$ as $R_{A \cup \{j\}}$-module in terms of the Betti numbers of $k[D,A]$ as $R_A$-module and the Betti numbers of $k[\text{lk}_D(\tau), A \setminus \tau]$ as $R_{A \setminus \tau}$-module.

**Proposition 3.** Denote by $L = \text{lk}_D(\tau) \subset 2^A \setminus \tau$ the link of the face $\tau$ of $D$. We then have

$$b_i(k[D_\tau]) \leq b_i(k[D]) + b_i(k[L]) + 1$$

and that, for $2 \leq i \leq \text{codim} k[D_\tau] - 2$,

$$b_i(k[D_\tau]) \leq b_{i-1}(k[D]) + b_i(k[D]) + b_{i-1}(k[L]) + b_i(k[L]).$$

**Proof.** If $\dim \tau = 0$, say $\tau = \{i\}$, then

$$I_{D_\tau, A \cup \{j\}} = (G, x_i),$$

where $G$ is the finite set obtained by substituting $x_j$ for $x_i$ in the minimal monomial generating set of $I_{D,A}$. Hence

$$b_i(k[D_\tau]) = b_{i-1}(k[D]) + b_i(k[D])$$

for all $i$.

Now assume that $\dim \tau \geq 1$. Using the notations of Proposition 2 we have that $S$ is the Kustin–Miller unprojection of the pair $Q/(I_{D,A}) \subset R$ and that $b_i(k[D_\tau]) = b_i(S)$ for all $i$.

We denote by $C_U$ the graded free resolution of $S$ obtained by the Kustin–Miller complex construction, cf. Section 3 and 4, Section 2, with initial data the minimal graded free resolutions of $R = R_A[z]/(I_{D,A})$ and $R_A[z]/Q$ over $R_A[z]$. Since $C_U$ is a graded free resolution of $S$ we have $b_i(S) \leq b_i(C_U)$ for all $i$, where $b_i(C_U)$ denotes the rank of the finitely generated free $R_A[z]$-module $(C_U)_i$. The variable $z$ does not appear
in the minimal monomial generating set of $I_{D,A}$, as a consequence $b_i(R) = b_i(k[D])$ for all $i$. Since $Q = (I_{D,A} : x, z)$ and the variable $z$ does not appear in the minimal generating set of $I_{D,A}$ we have for all $i$

$$b_i(R_A[z]/Q) = b_{i-1}(R_A/(I_{D,A} : x)) + b_i(R_A/(I_{D,A} : x))$$

(3.1)

$$= b_{i-1}(k[L]) + b_i(k[L]).$$

Moreover, by the Kustin–Miller complex construction (see [3] Section 2) we have

$$b_1(C_U) \leq b_1(k[D]) + b_1(R_A[z]/Q) = b_1(k[D]) + b_1(k[L]) + 1$$

and, for $2 \leq i \leq \text{codim } k[D] - 2$, that

$$b_i(C_U) \leq b_{i-1}(k[D]) + b_i(R_A[z]/Q) + b_i(k[D]).$$

Hence

$$b_i(k[D_U]) = b_i(S) \leq b_1(C_U) \leq b_{i-1}(k[D]) + b_i(R_A[z]/Q) + b_i(k[D])$$

which combined with Equality (3.1) finishes the proof. □

**Remark 4.** It may be interesting to investigate, perhaps with the use of Hochster’s formula or a generalization of the Kustin-Miller complex technique, whether the inequalities of Proposition 3 hold in a more general setting than Gorenstein.

For the proof of Proposition 7 we will need the following combinatorial lemma which relates a link of a stellar subdivision with links of the original simplicial complex. The straightforward but lengthy proof will be given in Subsection 3.1.

**Lemma 5.** If $\sigma$ is a nonempty face of $D_r$ the following hold:

1. **(Case I)** Assume $j \notin \sigma$ and $\tau \cup \sigma \in D$. Then $\tau \setminus \sigma$ is a nonempty face of $\text{lk}_D \sigma$ and

$$\text{lk}_{D_r} \sigma = (\text{lk}_D \sigma)_{\tau \setminus \sigma}$$

that is, $\text{lk}_{D_r} \sigma$ is the stellar subdivision of $\text{lk}_D \sigma$ with respect to $\tau \setminus \sigma$.

2. **(Case II)** Assume that $j \notin \sigma$ and $\tau \cup \sigma \notin D$. Then $\text{lk}_{D_r} \sigma$ is equal to $\text{lk}_D \sigma$ considered as a subcomplex of $2^{(A \cup \{j\}) \setminus \sigma}$.

3. **(Case III)** Assume $j \in \sigma$. Then $\tau \cup \sigma \setminus \{j\}$ is a face of $D$, $\tau \setminus \sigma$ is nonempty and

$$\text{lk}_{D_r} \sigma = \text{lk}_D (\tau \cup \sigma \setminus \{j\}) \ast \partial (\tau \setminus \sigma)$$

that is, $\text{lk}_{D_r} \sigma$ is equal to the join of $\text{lk}_D (\tau \cup \sigma \setminus \{j\})$ with $\partial (\tau \setminus \sigma)$.

**Remark 6.** Case II corresponds to faces $\sigma$ of $D \setminus \text{star}_D \tau$, while Cases I and III to faces of $2^{(j)} \ast \text{lk}_D \tau \ast \partial \tau$.

The next proposition gives bounds on the Betti numbers of links of a stellar subdivision in terms of links of the original complex.

**Proposition 7.** Let $\sigma$ be a face of $D_r$, and set $L = \text{lk}_{D_r} \sigma \subset 2^{(A \cup \{j\}) \setminus \sigma}$.

1. **(Case I)** If $j \notin \sigma$ and $\tau \cup \sigma$ is a face of $D$ then we have that

$$b_1(k[L]) \leq b_1(k[L_1]) + b_1(k[L_2]) + 1$$

and that for $2 \leq i \leq \text{codim } k[L] - 2$

$$b_i(k[L]) \leq b_{i-1}(k[L_1]) + b_i(k[L_2]) + b_i(k[L_2]),$$

where $L_1 = \text{lk}_D \sigma \subset 2^{(A \setminus \sigma)}$ and $L_2 = \text{lk}_D (\tau \cup \sigma) \subset 2^{(A \setminus (\tau \cup \sigma))}$.

2. **(Case II)** If $j \notin \sigma$ and $\tau \cup \sigma$ is not a face of $D$ then we have that for all $i$

$$b_i(k[L]) = b_{i-1}(k[L_1]) + b_i(k[L_1]).$$

3. **(Case III)** Assume $j \in \sigma$. Then $\tau \cup \sigma \setminus \{j\}$ is a face of $D$ and we have that for all $i$

$$b_i(k[L]) = b_{i-1}(k[L_3]) + b_i(k[L_3]),$$

where $L_3 = \text{lk}_D (\tau \cup \sigma \setminus \{j\}) \subset 2^{(A \setminus (\tau \cup \sigma))}$.
Proof. Assume first we are in Case I, that is $j \notin \sigma$ and $\tau \cup \sigma$ is a face of $D$. By part (1) of Lemma 5 we have $L = (L_1)_{\tau \setminus \sigma}$. Furthermore, a straightforward calculation shows that $\text{lk}_p(\tau \cup \sigma) = \text{lk}_{L_1}(\tau \setminus \sigma)$. The result follows from Proposition 3 applied to the stellar subdivision of the face $\tau \setminus \sigma$ of $L_1$.

Assume now we are in Case II, that is $j \notin \sigma$ and $\tau \cup \sigma$ is not a face of $D$. By part (2) of Lemma 5 we have

$$I_{L, (A \setminus \{j\}) \setminus \sigma} = (I_{L_1, A \setminus \{j\}} + (x_j)) \subset R_{(A \cup \{j\}) \setminus \sigma}$$

and the result is clear. In Case III, that is, $j \in \sigma$, we have by part (3) of Lemma 5 that $L = L_3 \ast \partial(\tau \setminus \sigma)$. Since $k[\partial(\tau \setminus \sigma)]$ is the quotient of a polynomial ring by a single equation, hence has nonzero Betti numbers only if $b_0 = 1$, the result follows by Equation (2.1).

For $c \geq 1$ and $q \geq 2$ recall that we defined $D_{q,c}$ as the set of simplicial subcomplexes $D \subset 2^{[q+c]}$ such that there exists a sequence of simplicial complexes

$$D_1, D_2, \ldots, D_{c-1}, D_c = D$$

with the property that $D_1 = \partial([q+1]) \subset 2^{[q+1]}$ is the boundary complex of the simplex on $q+1$ vertices, and, for $0 \leq i \leq c-1$, $D_{i+1} \subset 2^{[q+i+1]}$ is obtained from $D_i \subset 2^{[q+i]}$ by a stellar subdivision of a face of $D_i$ of dimension at least 1 with new vertex $q+i+1$. It is clear that $\text{supp} D_i = [q+i]$ and $\text{codim} k[D_i] = i$ for all $i$.

Assume $D \in D_{q,c}$ and consider the Stanley–Reisner ring $k[D] = R_{[q+c]}/I_D$. By Corollary 5.6.5 $D$ is Gorenstein*. As a consequence, since $\text{codim} k[D] = c$ the only nonzero Betti numbers $b_i$ of $k[D]$ are $b_0 = b_1, \ldots, b_{c-1}, b_c = 1$ and $b_i = b_{c-i}$ for all $i$.

To prove Theorem 1 we need to enlarge the class of ideals we consider by including the ideals of links. For $q \geq 2$ and $c \geq 1$ we define

$$\mathcal{I}_{q,c} = \{ I_D \mid D \in D_{q,c} \} \cup \{(I_D : x_0) \subset R : D \in D_{q,c}, \sigma \in D \text{ a nonempty face}\}$$

The following theorem is the key technical result.

**Theorem 8.** Suppose $c \geq 1$, $q \geq 2$, $R = k[x_1, \ldots, x_{q+c}]$ and $I \in \mathcal{I}_{q,c}$. Then for the the Betti numbers $b_i(R/I)$ of the minimal resolution of $R/I$ as $R$-module it holds that

$$b_i(R/I) \leq l_{c,i}$$

for all $0 \leq i \leq c$.

**Proof.** First note that from the definition of the bounding sequence $l_c$ it is clear that $l_{c,i} = l_{c-i,j}$ for all $0 \leq i \leq c$; that $l_{c,0} = l_{c,c} = 1$ for all $c \geq 1$; that $l_{c+1,1} = 2l_{c,1} + 1$ for all $c \geq 2$; and that $l_{c+1,i} = 2l_{c+1,i-1} + 2l_{c,i}$ for $2 \leq i \leq (c+1) - 2$.

Assume the claim is not true. Then there exist $c \geq 1$, $q \geq 2$ and an ideal $I \in \mathcal{I}_{q,c}$ with $I \not\in \mathcal{W}_c$, where by definition

$$\mathcal{W}_c = \{ I \in \mathcal{I}_{p,c} \mid p \geq 2 \text{ and } b_i(R/I) \leq l_{c,i} \text{ for all } i \}.$$

We fix such an ideal $I$ with $c$ the least possible, and we will get a contradiction. Since Gorenstein codimension 1 or 2 implies complete intersection, we necessarily have $c \geq 3$.

The first case is that $I = I_{D_1}$ for some $D_1 \in D_{q,c}$, so there exists $D \in D_{q,c-1}$ and a face $\tau$ of $D$ of dimension at least 1 such that $D_1 = D_\tau$. Since $c$ has been chosen to be the smallest possible, we have that $I_D \in \mathcal{W}_{c-1}$ and $(I_D : x_\tau) \in \mathcal{W}_{c-1}$. Using the properties of $l_c$ mentioned above, it follows by Proposition $\text{X}$ that $I \in \mathcal{W}_c$, which is a contradiction.

Assume now that $I = (I_{D_1} : x_\sigma)$ for some $D_1 \in D_{q,c}$ and face $\sigma$ of $D_1$. Write $D_1 = D_\tau$, for some $D \in D_{q,c-1}$ and face $\tau$ of $D$ of dimension at least 1. The new vertex $j$ of $D_\tau$ is $q + c$. In the remaining of the proof we will use the simplicial complexes $L$ and $L_1$, with $1 \leq i \leq 3$, defined in Proposition $\text{X}$. We have three cases. For all of them we will show that $I \in \mathcal{W}_c$, which is a contradiction.
Assume we are in Case I, that is \( j \notin \sigma \) and \( \tau \cup \sigma \) is a face of \( D \). Since by the minimality of \( c \) we have that both ideals \( I_L \) and \( I_{L_1} \) are in \( \mathcal{W}_{c-1} \), it follows by Case I of Proposition \[7\] that \( I \in \mathcal{W}_c \). Assume now we are in Case II, that is \( j \notin \sigma \) and \( \tau \cup \sigma \) is not a face of \( D \). Again by the minimality of \( c \) we have \( I_{L_1} \in \mathcal{W}_{c-1} \), so using Case II of Proposition \[7\] it follows that \( I \in \mathcal{W}_c \). Finally, assume we are in Case III, that is \( j \in \sigma \). By the minimality of \( c \) we have \( I_{L_2} \in \mathcal{W}_{c-1} \), so using Case III of Proposition \[7\] it follows that \( I \in \mathcal{W}_c \). This finishes the proof.

**Remark 9.** Combining Proposition \[7\] with Theorem \[5\] it is not hard to show that for fixed \( q \geq 2 \), there exists \( c_0 \geq 1 \) such that \( b_i(k[D]) < \ell_{c,i} \) for all \( c \geq c_0, D \in D_{q,c} \) and \( 1 \leq i \leq c - 1 \). So if we fix \( q \) for \( c \) sufficiently large the Betti bound in Theorem \[4\] is not sharp. We leave the details to the interested reader.

### 3.1. Proof of Lemma \[5\]

We will repeatedly use in the following two observations:

**If \( \alpha \in D \), we have \( \alpha \in D_\tau \) if and only if \( \tau \) is not a subset of \( \alpha \). Moreover, if \( \beta \in D_\tau \) and \( j \notin \beta \) then \( \beta \notin \tau \).**

We will also use the following notation. For \( \beta \in D \) with \( \tau \subseteq \beta \) and nonempty \( \rho \cap \tau \) we set

\[
\text{transf}(\beta, \rho) = (\beta \setminus \rho) \cup \{j\} \in D_\tau.
\]

Using this notation, \( D_\tau \) is the disjoint union of the set consisting of the faces \( \alpha \in D \) which do not contain \( \tau \) with the set

\[
\{\text{transf}(\beta, \rho) \mid \beta \in D \text{ with } \tau \subseteq \beta, \emptyset \neq \rho \cap \tau\}.
\]

**CASE I:** Assume \( j \notin \sigma \) and \( \sigma \cup \tau \in D \). Since \( \sigma \cap (\tau \setminus \sigma) = \emptyset \) and \( \sigma \cup \tau \in D \) we have that indeed \( \tau \setminus \sigma \) is a face of \( \text{lk}_D \sigma \). We prove that \( \text{lk}_{D_\tau} \sigma = (\text{lk}_D \sigma)_{\tau \setminus \sigma} \).

Given \( \alpha \in \text{lk}_D \sigma \), we show that \( \alpha \in (\text{lk}_D \sigma)_{\tau \setminus \sigma} \). There are two subcases:

**Subcase 1.1:** Assume \( j \notin \alpha \). Then \( \alpha \cup \sigma \in D_\tau \) and \( j \notin \alpha \cup \sigma \) implies \( \alpha \cup \sigma \in D \), hence by \( \alpha \cap \sigma = \emptyset \) we have \( \alpha \in \text{lk}_D \sigma \). If \( \tau \setminus \sigma \) is a subset of \( \alpha \) we get \( \tau \subseteq \alpha \cup \sigma \), which contradicts \( \alpha \cup \sigma \in D_\tau \). So \( \tau \setminus \sigma \) is not a subset of \( \alpha \), which implies that \( \alpha \in (\text{lk}_D \sigma)_{\tau \setminus \sigma} \).

**Subcase 1.2:** Assume \( j \in \alpha \). Since \( \alpha \cup \sigma \in D_\tau \) there exist \( \beta \in D \) with \( \tau \subseteq \beta \) and nonempty \( \rho \subseteq \tau \) such that

\[ (3.2) \quad \alpha \cup \sigma = \text{transf}(\beta, \rho) = (\beta \setminus \rho) \cup \{j\}. \]

As a consequence, using \( \alpha \cap \sigma = \emptyset \), we get \( \alpha = (\beta \setminus (\rho \cup \sigma)) \cup \{j\} \). Since \( j \notin \sigma \) Equation \[3.2\] also implies \( \sigma \subseteq \beta \). Set \( \beta' = \beta \setminus \sigma \in D \). It is enough to show that \( \rho, \beta' \in \text{lk}_D \sigma, \tau \setminus \sigma \subseteq \beta', \emptyset \neq \rho \subseteq \tau \setminus \sigma \), and

\[ (3.3) \quad \gamma = (\beta' \setminus \rho) \cup \{j\} = \text{transf}(\beta', \rho). \]

By Equation \[3.2\], we have \( \rho \cap \sigma = \emptyset \). By definition, \( \beta' \cap \sigma = \emptyset \). Moreover, \( \rho \cup \sigma \subseteq \tau \cup \sigma \in D \), and \( \beta' \cup \sigma = \beta \in D \). Since \( \tau \subseteq \beta \) we have \( \tau \setminus \sigma \subseteq \beta' \). By \( \rho \subseteq \tau \) and \( \rho \cap \sigma = \emptyset \) it follows that \( \rho \subseteq \tau \setminus \sigma \). Finally, Equation \[3.3\] follows from Equation \[3.2\] using \( \alpha \cap \sigma = \emptyset \) and \( j \notin \sigma \).

Conversely, assume \( \alpha \in (\text{lk}_D \sigma)_{\tau \setminus \sigma} \), that is, \( \alpha \) is in the stellar of the link. We will prove that \( \alpha \in \text{lk}_{D_\tau} \sigma \). We have two subcases:

**Subcase 2.1:** Assume \( j \notin \alpha \). Then \( \alpha \in \text{lk}_D \sigma \). We have that \( \tau \setminus \sigma \) is not a subset of \( \alpha \) (since \( \tau \setminus \sigma \) subset of \( \alpha \) implies \( \alpha \) not in \( (\text{lk}_D \sigma)_{\tau \setminus \sigma} \), a contradiction), as a consequence \( \tau \) is not a subset of \( \alpha \cup \sigma \). Hence \( \alpha \cup \sigma \in D_\tau \) which implies that \( \alpha \in \text{lk}_{D_\tau} \sigma \).

**Subcase 2.2:** Assume \( j \in \alpha \). Then there exist \( \beta \in \text{lk}_D \sigma \) with \( \tau \setminus \sigma \subseteq \beta \) and nonempty \( \rho \subseteq \tau \setminus \sigma \) with \( \alpha = \text{transf}(\beta, \rho) = (\beta \setminus \rho) \cup \{j\} \). To finish the proof of the subcase we will show that \( \beta' = \beta \cup \sigma \) is a face of \( D \) containing \( \tau \) and \( \alpha \cup \sigma = \text{transf}(\beta', \rho) \). Indeed,
\( \beta \in \text{lk}_D \sigma \) implies \( \beta' \in D \), and \( \tau \setminus \sigma \subset \beta \) implies \( \tau \subset \beta' \). Moreover, since \( \rho \cap \sigma = \emptyset \) we have
\[
\text{transf}(\beta', \rho) = (\beta' \setminus \rho) \cup \{j\} = \sigma \cup ((\beta \setminus \rho) \cup \{j\}) = \alpha \cup \sigma,
\]
which finishes the proof of CASE I.

**CASE II:** Assume \( j \notin \sigma \) and \( \tau \cup \sigma \notin D \). We prove that \( \text{lk}_{D_j} \sigma = \text{lk}_D \sigma \).

Given \( \alpha \in \text{lk}_{D_j} \sigma \), we show that \( \alpha \in \text{lk}_D \sigma \). There are two subcases (in fact, we will show the second cannot happen):

**Subcase 1.1:** Assume \( j \notin \alpha \). This implies \( j \notin (\alpha \cup \sigma) \) hence \( \alpha \cup \sigma \in D \). Therefore \( \alpha \in \text{lk}_D \sigma \).

**Subcase 1.2:** Assume \( j \in \alpha \). Then there exist a face \( \beta \) of \( D \) with \( \tau \subset \beta \) and nonempty \( \rho \subset \tau \) such that
\[
\alpha \cup \sigma = \text{transf}(\beta, \rho) = (\beta \setminus \rho) \cup \{j\}.
\]
Hence, \( (\alpha \cup \sigma) \setminus \{j\} = \beta \cap \rho \), which implies
\[
\tau \cup \sigma \subset \tau \cup (\alpha \cup \sigma) \setminus \{j\} \subset \tau \cup \beta = \beta \in D.
\]
From this it follows that \( \tau \cup \sigma \in D \), contradicting the assumption \( \tau \cup \sigma \notin D \). So \( j \in \alpha \) is impossible.

Conversely, assume \( \alpha \in \text{lk}_D \sigma \). To show \( \alpha \in \text{lk}_{D_j} \sigma \) it is enough to prove \( \alpha \cup \sigma \in D_\tau \), which follows from \( \tau \notin \alpha \cup \sigma \). So assume \( \tau \subset \alpha \cup \sigma \), then \( \tau \setminus \sigma \subset \alpha \in \text{lk}_D \sigma \) so \( (\tau \setminus \sigma) \cup \sigma \in D \), hence \( \tau \cup \sigma \in D \), contradicting the assumption \( \tau \cup \sigma \notin D \). So \( \tau \) is not a subset of \( \alpha \cup \sigma \). This finishes the proof of CASE II.

**CASE III:** We assume \( j \in \sigma \). We first show that \( \tau \cup \sigma \setminus \{j\} \) is a face of \( D \). Indeed, \( \sigma \in D_\tau \) and \( j \in \sigma \) imply that there exist a face \( \beta_1 \) of \( D \) with \( \tau \subset \beta_1 \) and nonempty \( \rho_1 \subset \tau \) such that
\[
\sigma = \text{transf}(\beta_1, \rho_1) = (\beta_1 \setminus \rho_1) \cup \{j\}.
\]
As a consequence \( \sigma \setminus \{j\} \subset \beta_1 \) which together with \( \tau \subset \beta_1 \) implies that \( \tau \cup \sigma \setminus \{j\} \subset \beta_1 \), hence \( \tau \cup \sigma \setminus \{j\} \) is a face of \( D \). We will show that
\[
\text{lk}_{D_j} \sigma = \text{lk}_D (\tau \cup \sigma \setminus \{j\}) \ast \partial (\tau \setminus \sigma).
\]
Assume \( \alpha \in \text{lk}_{D_j} \sigma \). Then \( \alpha \cap \sigma = \emptyset \), hence \( j \notin \alpha \). Since \( j \in \alpha \cup \sigma \) and \( \alpha \cup \sigma \in D_\tau \) there exists \( \beta \in D \) with \( \tau \subset \beta \) and nonempty \( \rho \subset \tau \) such that
\[
\alpha \cup \sigma = \text{transf}(\beta, \rho) = (\beta \setminus \rho) \cup \{j\},
\]
so in particular \( \sigma \setminus \{j\} \subset \beta \) and \( (\alpha \cup \sigma) \cap \rho = \emptyset \).

Set \( \alpha_1 = \cap (\tau \setminus \sigma) \) and \( \alpha_2 = \alpha \setminus \alpha_1 \), hence \( \alpha_2 \cap (\tau \setminus \sigma) = \emptyset \). Since \( \alpha \) is the (disjoint) union of \( \alpha_1 \) and \( \alpha_2 \) we need to show that \( \alpha_1 \in \partial (\tau \setminus \sigma) \) and \( \alpha_2 \in \text{lk}_D (\tau \cup \sigma \setminus \{j\}) \). If \( \alpha_1 = \tau \setminus \sigma \) we would have \( (\tau \setminus \sigma) \subset \alpha \), hence \( \tau \subset (\alpha \cup \sigma) \) which contradicts that \( (\alpha \cup \sigma) \cap \rho = \emptyset \). Hence \( \alpha_1 \in \partial (\tau \setminus \sigma) \).

Since \( \alpha \cap \sigma = \emptyset \) we get \( \alpha_2 \cap \sigma = \emptyset \), which together with \( \alpha_2 \cap (\tau \setminus \sigma) = \emptyset \) implies that \( \alpha_2 \cap (\tau \cup \sigma \setminus \{j\}) = \emptyset \). We will show \( \alpha_2 \cap (\tau \cup \sigma \setminus \{j\}) \in D_\tau \). Since \( \alpha \subset \beta \cup \{j\} \) and \( j \notin \alpha \), we have \( \alpha \subset \beta \), hence \( \alpha_2 \subset \beta \). By the definition of \( \beta \) we have \( \tau \subset \beta \) and as we showed above \( \sigma \setminus \{j\} \subset \beta \). As a consequence \( \alpha_2 \cap (\tau \cup \sigma \setminus \{j\}) \subset \beta \), hence \( (\alpha_2 \cup \tau \cup \sigma) \setminus \{j\} \in D \). This finishes the proof of \( \alpha \in \text{lk}_D (\tau \cup \sigma \setminus \{j\}) \ast \partial (\tau \setminus \sigma) \).

For the converse, assume \( \alpha_1 \in \partial (\tau \setminus \sigma) \) and \( \alpha_2 \in \text{lk}_D (\tau \cup \sigma \setminus \{j\}) \). We will show that \( \alpha_1 \cup \alpha_2 \in \text{lk}_{D_j} \sigma \). We have that \( (\alpha_2 \cup \tau \cup \sigma) \setminus \{j\} \in D_\tau \), that \( (\alpha_2 \cap (\tau \cup \sigma) \setminus \{j\}) = \emptyset \) (in particular \( \alpha_2 \cap \tau = \emptyset \) and \( \alpha_2 \cap \sigma = \emptyset \)) since \( j \notin \alpha_2 \), that \( \alpha_1 \cap \sigma = \emptyset \) and that \( \alpha_1 \) is a proper subset of \( \tau \setminus \sigma \). Hence there exists \( \gamma \in (\tau \setminus \sigma) \setminus \alpha_1 = \tau \setminus (\alpha_1 \cup \sigma) \). Taking into account that \( \alpha_1 \cap \tau = \emptyset \) it follows that \( \gamma \in \tau \setminus (\alpha_1 \cup \alpha_2 \cup \sigma) \).

Since \( \alpha_2 \cap \tau = \emptyset \) and \( \alpha_2 \subset \tau \), we have \( \alpha_1 \cap \alpha_2 = \emptyset \). We will now show that \( \alpha_1 \cup \alpha_2 \in \text{lk}_{D_j} \sigma \). First as we observed above both \( \alpha_1 \) and \( \alpha_2 \) have empty intersection with \( \sigma \). So it is enough to show that \( (\alpha_1 \cup \alpha_2 \cup \sigma) \in D_\tau \). Set \( \beta_2 = (\alpha_2 \cup \tau \cup \sigma) \setminus \{j\} \),
which, as observed above, is in $D$. Since $\gamma \in \tau$, it follows that $\text{transf}(\beta_2, \{\gamma\}) \in D_\tau$. Since, as observed above, $\gamma \in \tau \setminus (\alpha_1 \cup \alpha_2 \cup \sigma)$ we have

$$(\alpha_1 \cup \alpha_2 \cup \sigma) \subset (\alpha_2 \cup \tau \cup \sigma) \setminus \{\gamma\} = \text{transf}(\beta_2, \{\gamma\}),$$

hence $(\alpha_1 \cup \alpha_2 \cup \sigma) \in D_\tau$. This finishes the proof of CASE III, and hence the proof of Lemma 5.

4. THE STRUCTURE OF THE KUSTIN–MILLER COMPLEX IN THE STELLAR SUBDIVISION CASE

Kustin and Miller introduced in [9] the Kustin–Miller complex construction which produces a projective resolution of the Kustin–Miller unprojection ring in terms of projective resolutions of the initial data. In Proposition 11 we prove a criterion for the minimality of the resolution, which will be used in Section 5. For that, we analyze the additional structure of the construction in the case of stellar subdivisions.

We will use the graded version of the Kustin–Miller complex construction as described in [3] Section 2. Note, that there is an implementation of the construction available for the computer algebra system Macaulay2, see [loc. cit.].

In this section $D \subset 2^A$ will be a generalized Gorenstein* simplicial complex, $\tau \in D$ a face of positive dimension and $D_\tau \subset 2^{A \cup \{j\}}$ the corresponding stellar subdivision with new vertex $j \in \mathbb{N} \setminus A$.

Let $R = R_A[z]$ with the following grading: $\deg x_a = 1$ for $a \in A$ and $\deg z = \dim \tau$. Write $I \subset R$ for the ideal generated by $I_{D,A}$ and set $J = (I_{D,A} : x_{r,z}) \subset R$. Denote by

$$C_I : \quad R/J \leftarrow A_0 \leftarrow A_1 \leftarrow \ldots \leftarrow A_{g-1} \leftarrow A_g$$

$$C_I : \quad R/I \leftarrow B_0 \leftarrow B_1 \leftarrow \ldots \leftarrow B_{g-1}$$

the minimal graded free resolutions of $R/J$ and $R/I$ respectively.

By Proposition 2 Hom$_{(R/I)}(J/I, R/I)$ is generated as an $R/I$-module by the inclusion homomorphism together with the map $\psi$ that sends $(I_{D,A} : x_{r,z})$ to 0 and $z$ to $x_r$. By the Kustin-Miller complex construction we obtain the unprojection ideal $U \subset R[T]$ of the pair $J/I \subset R/I$ defined by $\psi$ with new variable $T$, and $a$, in general non-minimal, graded free resolution $C_U$ of $R[T]/U$ as $R[T]$-module. For more details see [loc. cit.].

Clearly, the $k$-algebra $S$ defined in Proposition 2 is isomorphic to $R[T]/U$, since it is obtained from $R[T]/U$ by substituting $T$ with $x_j$. By the same proposition $z$ is $R[T]/U$-regular and $(R[T]/U)/(z) \cong k[D_\tau]$.

We denote by $P$ the ideal $(I_{D,A} : x_{r,z})$ of $R_A$, and by

$$C_P : \quad R_A/P \leftarrow P_0 \leftarrow P_1 \leftarrow \ldots \leftarrow P_{g-1} \leftarrow 0$$

the minimal graded free resolution of $R/P$ as $R_A$-module. Moreover, we denote by

$$C_z : \quad k[z]/(z) \leftarrow k[z]$$

the minimal graded free resolution of $k[z]/(z)$ as $k[z]$-module. Since $J = (P, z)$ we have that $C_J$ is the tensor product (over $k$) of the complexes $C_P$ and $C_z$. Hence $A_0 = P_0$, $A_g = P_{g-1}^a$ and

$$A_i = P_i^a \oplus P_i^b$$

for all $1 \leq i \leq g - 1$, where $P_i^a = P_i \otimes_k k[z]$ considered as $R$-module. Moreover, using this decomposition, we have that

$$a_1 = (p_1 \ z), \quad a_g = \begin{pmatrix} -z \\ p_{g-1} \end{pmatrix}, \quad \text{and} \quad a_i = \begin{pmatrix} p_i & -z E \\ 0 & p_i^{-1} \end{pmatrix}$$

for $2 \leq i \leq g - 1$, where $E$ denotes the identity matrix of size equal to the rank of $P_i$.

Recall from [loc. cit.] that the construction of $C_U$ involves chain maps $\alpha : C_I \to C_J$, $\beta : C_J \to C_I[-1]$ and a homotopy map $h : C_I \to C_J$, given by maps $\alpha_i : B_i \to A_i$, $\beta_i : A_i \to B_i$, and $h_i : A_i \to C_I$. The homotopy $h$ is defined as:

$$h_i = \sum_{j=0}^{g-1} \alpha_j \beta_{j+1} - \sum_{j=0}^{g-1} \beta_j \alpha_{j+1}$$
$\beta_i : A_i \to B_{i-1}$ and $h_i : B_i \to B_i$ for all $i$. We will use that $\alpha_0$ is an invertible element of $R$, that $h_0 = h_g = 0$, and that the $h_i$ satisfy the defining property

\[(4.2) \quad \beta_i\alpha_i = h_{i-1}b_i + b_ih_i\]

for all $i$.

Using the decomposition \[(4.1),\] we can write, for $1 \leq i \leq g - 1$

\[\alpha_i = \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix}, \quad \beta_i = \begin{pmatrix} \beta_{i,1} & \beta_{i,2} \end{pmatrix}.\]

**Proposition 10.** We can choose $\alpha_i, \beta_i$ and $h_i$ in the following way:

1. $\alpha_i, \beta_i$ do not involve $z$ for all $i$,
2. $\alpha_{i,2} = \beta_{i,1} = 0$ for $1 \leq i \leq g - 1$, and
3. $h_i = 0$ for all $i$.

**Proof.** For the maps $\alpha_i$ the arguments are as follows. Since $\alpha_0$ is an invertible element of $R$ it does not involve $z$. Assume now that $i = 1$. Using that $\alpha$ is a chain map, we have $\alpha_0b_1 = a_1\alpha_1$, hence

\[\alpha_0b_1 = (p_1 \ldots z) \begin{pmatrix} \alpha_{1,1} \\ \alpha_{1,2} \end{pmatrix} = p_1\alpha_{1,1} + z\alpha_{1,2}.\]

Since $z$ does not appear in the product $\alpha_0b_1$ or in $p_1$ we can assume $\alpha_{1,2} = 0$ and that $z$ does not appear in $\alpha_{1,1}$. Assume now that $\alpha_{i,2} = 0$ and $\alpha_{i,1}$ does not involve the variable $z$ and we will show that we can choose $\alpha_{i+1}$ with $\alpha_{i+1,2} = 0$ and that $z$ does not appear in $\alpha_{i+1,1}$. Indeed, since $\alpha$ is a chain map, we have $\alpha_i b_{i+1} = a_{i+1}\alpha_{i+1}$, so

\[\begin{pmatrix} \alpha_{i,1} \\ 0 \end{pmatrix} b_{i+1} = \begin{pmatrix} p_{i+1} & -zE \\ 0 & p_i \end{pmatrix} \begin{pmatrix} \alpha_{i+1,1} \\ \alpha_{i+1,2} \end{pmatrix}.\]

Hence we get the equations

\[(4.3) \quad \alpha_{i,1}b_{i+1} = p_{i+1}\alpha_{i+1,1} - z\alpha_{i+1,2}, \quad 0 = p_i\alpha_{i+1,2}.\]

Write $\alpha_{i+1,1} = q_1 + zq_2$ with $z$ not appearing in $q_1$. Equation \[(4.3)] implies that $\alpha_{i,1}b_{i+1} = p_{i+1}q_1$. As a consequence, we can assume that $\alpha_{i+1,2} = 0$ and that $\alpha_{i+1,1} = q_1$, hence $\alpha_{i+1,1}$ does not involve $z$.

For the maps $\beta_i$ the argument is as follows. Since $\psi(u) = 0$ for all $u \in P$ and $\psi(z) = x_\tau$, we have by \[3, Section 2\] that $\beta_1 = (0 \ldots 0 \ x_\tau)$, hence $\beta_{1,1} = 0$ and $z$ does not appear in $\beta_{1,2}$. Assume now $\beta_{i,1} = 0$ and $z$ does not appear in $\beta_{i,2}$ and we will show that we can choose $\beta_{i+1}$ with $\beta_{i+1,1} = 0$ and $z$ not appearing in $\beta_{i+1,2}$. Indeed, since $\beta$ is a chain map, we have $b_i \beta_{i+1} = \beta_i a_{i+1}$, hence

\[b_i \begin{pmatrix} \beta_{i+1,1} & \beta_{i+1,2} \end{pmatrix} = \begin{pmatrix} 0 & \beta_{i,2} \end{pmatrix} \begin{pmatrix} p_{i+1} & -zE \\ 0 & p_i \end{pmatrix}.\]

Hence we get the equations

\[b_i \beta_{i+1,1} = 0, \quad b_i \beta_{i+1,2} = \beta_{i,2} p_i\]

so we can assume that $\beta_{i+1,1} = 0$ and that $z$ does not appear in $\beta_{i+1,2}$.

We will now prove the statement for the maps $h_i$. Since, as proved above, we can assume that $\alpha_{i,2} = \beta_{i,1} = 0$, we have

\[\beta_i\alpha_i = \begin{pmatrix} 0 & \beta_{i,2} \end{pmatrix} \begin{pmatrix} \alpha_{i,1} \\ 0 \end{pmatrix} = 0.\]

As a consequence, Equation \[(4.2)] can be satisfied by taking $h_i = 0$ for all $i$. \qed

In what follows, we will assume $\alpha_i, \beta_i$ and $h_i$ are chosen as in Proposition \[10\].
Proposition 11. Assume that the face $\tau$ of $D$ has the following property: every minimal non-face of $D$ contains at least one vertex of $\tau$ (algebraically it means that for every minimal monomial generator $v$ of $I$ there exists $p \in \tau$ such that $x_p$ divides $v$). Then $C_U$ is a minimal complex. As a consequence, we have that $C_U \otimes_R R/(z)$ is, after substituting $T$ with $x_j$, the minimal graded free resolution of $k[D_\tau, 2^{A(\tau)}]$. 

Proof. We first show the minimality of $C_U$. Since we have $h_i = 0$ for all $i$, it is enough to show that, for $1 \leq i \leq g - 1$ the chain maps $\alpha_i$ and $\beta_i$ are minimal, in the sense that no nonzero constants appear in the corresponding matrix representations. It follows by the defining properties of the chain maps $\alpha$ and $\beta$ in [3] Section 2 that $\beta_i$ is minimal if and only if $\alpha_{g - i}$ is. So it is enough to prove that the map $\alpha_i: B_i \to A_i$ is minimal for $1 \leq i \leq g - 1$. Denote by $M$ the monoid of exponent vectors on the variables of $R$. 

Since the ideals $I$ and $J$ of $R$ are monomial, there exist, for $1 \leq i \leq g - 1$, positive integers $q_{1,i}, q_{2,i}$ and multidegrees $\bar{a}_{i,j_1}, \bar{b}_{i,j_2}$ in $M$ with $1 \leq j_1 \leq q_{1,i}$ and $1 \leq j_2 \leq q_{2,i}$ such that 

$$A_i = \bigoplus_{1 \leq j_1 \leq q_{1,i}} R(-\bar{a}_{i,j_1}) \quad \text{ and } \quad B_i = \bigoplus_{1 \leq j_2 \leq q_{2,i}} R(-\bar{b}_{i,j_2}).$$

For the minimality of $\alpha_i$ it is enough to show (compare [11] Remark 8.30) that given $i$ with $1 \leq i \leq g - 1$ there are no $j_1, j_2$ with $1 \leq j_1 \leq q_{1,i}$, $1 \leq j_2 \leq q_{2,i}$ and $\bar{a}_{i,j_1} = \bar{b}_{i,j_2}$, which we will now prove. By the assumptions, given $v$ in the minimal monomial generating set of $I$ there exists $p \in \tau$ with $x_p$ dividing $v$ in the polynomial ring $R$. Hence, given $j_2$ with $1 \leq j_2 \leq q_{2,i}$ there is a nonzero coordinate of $\bar{b}_{i,j_2}$ corresponding to a variable $x_{p_i}$ with $p \in \tau$. This implies that the same is true for every $\bar{b}_{i,j_2}$ with $i \geq 1$ and $1 \leq j_2 \leq q_{2,i}$. On the other hand, no variable $x_p$ with $p \in \tau$ appears in any minimal monomial generators of $J$, hence the same is true for the coordinates of every $\bar{a}_{i,j_1}$ with $i \geq 1$ and $1 \leq j_1 \leq q_{1,i}$. So $\bar{a}_{i,j_1} = \bar{b}_{i,j_2}$ is impossible for $i \geq 1$. This finishes the proof that $C_U$ is a minimal complex.

By Proposition [3] 2 is $S$-regular and $S/(z) \cong k[D_\tau]$. Hence using [5] Proposition 1.1.5, since $C_U$ is minimal, the complex $C_U \otimes_R R/(z)$ is, after substituting $T$ with $x_j$, the minimal graded free resolution of $k[D_\tau]$. \hfill $\square$

Remark 12. We give an example where the condition for $\tau$ in the statement Proposition [11] is not satisfied but $C_U$ is still minimal. Let $D$ be the simplicial complex triangulating the 1-dimensional sphere $S^1$ having $n$ vertices with $n \geq 4$, and suppose $\tau$ is a 1-face of $D$. Since $n \geq 4$ there exist minimal non-faces of $D$ with vertex set disjoint from $\tau$. On the other hand $C_U$ is minimal, see, for example, [11] Section 5.2.

5. Champions

5.1. Construction. Assume a positive integer $c \geq 1$ is given. We will define a positive integer $q$ and construct a simplicial complex $F_c \in D_{q - 1,c}$ such that the inequalities of Theorem [11] are equalities. First note that for $c = 1$ we can take the boundary complex of any $\geq 3$, and for $c = 2$ any single stellar subdivision of that.

For $c \geq 3$ we define inductively positive integers $d_t$, for $0 \leq t \leq c - 1$, by $d_0 = 0$ and $d_{t+1} = d_t + (c - t)$, and set $q = d_{c-1}$. We also define inductively, for $1 \leq t \leq c - 1$, subsets $\sigma_t \subset [q]$ of cardinality $c$ by $\sigma_1 = \{1, \ldots, d_1 = c\}$ and 

$$\sigma_{t+1} = \{(\sigma_1)_t, (\sigma_2)_t, \ldots, (\sigma_t)_t\} \cup \{i \mid d_t + 1 \leq i \leq d_{t+1}\},$$

where $(\sigma_i)_p$ denotes the $p$-th element of $\sigma_i$ with respect to the usual ordering of $\mathbb{N}$. The main properties are that $\#(\sigma_i \cap \sigma_j) = 1$ for all $i \neq j$, every three distinct $\sigma_i$ have empty intersection, and the last element $d_t$ of $\sigma_i$ is not in $\sigma_j$ for $j \neq i$.

Example 13. For $c = 4$ we have $(d_1, d_2, d_3) = (4, 7, 9)$, $q = 9$, $\sigma_1 = \{1, 2, 3, 4\}$, $\sigma_2 = \{1, 5, 6, 7\}$ and $\sigma_3 = \{2, 5, 8, 9\}$. For $c = 5$ we have $(d_1, \ldots, d_4) = (5, 9, 12, 14)$, $q = 14$, $\sigma_1 = \{1, 2, 3, 4, 5\}$, $\sigma_2 = \{1, 6, 7, 8, 9\}$, $\sigma_3 = \{2, 6, 10, 11, 12\}$ and $\sigma_4 = \{3, 7, 10, 13, 14\}$.
We define inductively simplicial subcomplexes $F_t \subset 2^{[q+t-1]}$ for $1 \leq t \leq c$. Since $\sigma_i$ is not a subset of $\sigma_j$ for $i \neq j$ we will be able to apply the elementary observation that if $\sigma, \tau$ are two faces of a simplicial complex $D$ then $\tau$ not a subset of $\sigma$ implies that $\sigma$ is also a face of the stellar subdivision $D_\tau$. First set $F_1 = \partial([q]) \subset 2^{[q]}$ to be the boundary complex of the simplex on $q$ vertices $1, \ldots, q$. Clearly $\sigma_i$, for $1 \leq i \leq c - 1$, is a face of $F_1$. Set $F_2$ to be the stellar subdivision of $F_1$ with respect to $\sigma_1$ with new vertex $q+1$. Suppose $1 \leq t \leq c - 1$ and $F_t$ has been constructed. Since $\sigma_i$ is a face of $F_t$ for $i \geq t$, we can continue inductively and define $F_{t+1}$ to be the the stellar subdivision of $F_t$ with respect to $\sigma_t$ with new vertex $q+t$.

The Stanley-Reisner ring of $F_c$ has the maximal possible Betti numbers among all elements in $\bigcup_{p \geq 2} D_{p,c}$:

**Proposition 14.** For all $t$ with $1 \leq t \leq c$ and all $i \geq 0$ we have

$$b_i(R_{[q+t-1]}/I_{F_t}) = l_{t,i}.$$ 

We will give the proof in Subsection 5.2.

**Remark 15.** Note that boundary complexes of stacked polytopes do not, in general, reach the bounds.

**Remark 16.** In the MACAULAY2 package BETTI BOUNDS we provide an implementation of the construction of $F_t$. Using the minimality of the Kustin-Miller complex, we also provide a function which produces their graded Betti numbers. This works far beyond the range which is accessible by computing the minimal free resolution via Gröbner bases.

**Example 17.** We use the implementation to produce $F_4$:

```plaintext
i1: loadPackage "BettiBounds";
i2: F4 = champion 4;
i3: I4 = ideal F4
o3: ideal(x_1x_2x_3x_4, x_1x_5x_6x_7, x_2x_5x_8x_9, x_5x_6x_7x_8x_9x_10, x_2x_3x_4x_11,
     x_8x_9x_10x_11, x_1x_3x_4x_12, x_1x_6x_7x_12, x_6x_7x_10x_12, x_3x_4x_11x_12, x_10x_11x_12)
i4: betti res I4
```
```
0 1 2 3 4
0: 1 . . . .
1: . . . .
2: . 1 . .
3: . 9 9 1 .
4: . . 2 .
5: . 1 9 9 .
6: . . . 1 .
7: . . . .
8: . . . . 1
```

The command gradedBettiChampion 20, will produce the Betti table of the minimal free resolution of $I_{F_{20}}$ with projective dimension 20 and regularity 208 in 0.7 seconds\(^1\). For more examples, see the documentation of BETTI BOUNDS.

### 5.2. Proof of Proposition 14

The main idea of the proof is that when passing from $F_t$ to $F_{t+1}$ by subdividing $\sigma_t$, the ideals $I_{F_t}$ and $(I_{F_t} : x_{\sigma_t})$ have the same total Betti numbers (Proposition 21) and the Kustin-Miller complex construction yields a minimal free resolution (Lemma 20).

\(^1\)On a single core of an Intel i7-2640M at 3.4 GHz.
It is convenient to introduce the following notations, which will be used only in the present subsection. For nonzero monomials \( v = \prod_{i=1}^{l} x_i^{a_i} \) and \( w = \prod_{i=1}^{l} x_i^{b_i} \) in \( R[\ell] \) we set

\[
\frac{v}{w} = \prod_{i=1}^{l} x_i^{c_i}, \quad \text{with } c_i = \max(a_i - b_i, 0),
\]
and for a set \( S \) of monomials we set \( \frac{S}{w} = \{ \frac{v}{w} \mid v \in S \} \). Clearly \( \frac{S}{w} \) is the monomial generator of the ideal quotient \( (v : (w)) \).

For simplicity of notation write \( T = R[\ell+1] \). We will now study in more detail the Stanley–Reisner ideal \( I_{F_t} \subset T \) of \( F_t \). We set \( u_1 = \prod_{i=1}^{l} x_i \), \( u_2 = \prod_{i=1}^{l} x_i^{r_i} \) and inductively define finite subsets \( S_t \subset I_{F_t} \) by \( S_1 = \{ u_1 \}, S_2 = \{ u_2, x_{\sigma_1} \} \) and, for \( t \geq 2 \),

\[
S_{t+1} = S_t \cup \{ x_{\sigma_t} \}.
\]

Clearly \( S_1 \) (resp. \( S_2 \)) is the minimal monomial generating set of \( I_{F_1} \) (resp. \( I_{F_2} \)). Moreover, an easy induction on \( t \) using Equation (2.2) shows that \( S_t \) is a set of monomials generating \( I_{F_t} \) for all \( 1 \leq t \leq c \). In Proposition 21 we will show that \( S_t \) is actually the minimal monomial generating set of \( I_{F_t} \) for all \( t \).

Equation (5.1) and induction imply that given an element \( v \) of \( S_{t+1} \) there exists \( e_v \in \{ u_2, x_{\sigma_1}, \ldots, x_{\sigma_t} \} \) such that either \( v = e_v \) or \( v = \frac{w_1 e_v}{w_2} \), with \( w_1 = \prod_{j=1}^{l} x_{q+q_j} \) and \( w_2 = \prod_{j=1}^{l} x_{r_j} \) for some \( l \geq 1 \) and \( r_1 < r_2 < \cdots < r_l \). Moreover, if \( e_v = u_2 \) we have \( 2 \leq r_1 \), while if \( e_v = x_{\sigma_p} \) we have \( p + 1 \leq r_1 \). A priori \( e_v \) may not be uniquely determined and we fix one of them and call it the original source of \( v \). One can actually show that in our setting \( e_v \) is uniquely determined by \( v \) but we do not prove it and do not use it in the following.

**Example 18.** We have

\[
S_3 = \left\{ u_2, \frac{x_{q+2}u_2}{x_{\sigma_2}} \right\} \cup \left\{ x_{\sigma_1}, \frac{x_{q+2}x_{\sigma_1}}{x_{\sigma_2}} \right\} \cup \left\{ x_{\sigma_2} \right\}
\]

and

\[
S_4 = \left\{ u_2, \frac{x_{q+2}u_2}{x_{\sigma_2}}, \frac{x_{q+3}u_2}{x_{\sigma_3}}, \frac{x_{q+2}x_{q+3}u_2}{x_{\sigma_2}x_{\sigma_3}} \right\} \cup \left\{ x_{\sigma_1}, \frac{x_{q+2}x_{\sigma_1}}{x_{\sigma_2}}, \frac{x_{q+3}x_{\sigma_1}}{x_{\sigma_3}}, \frac{x_{q+2}x_{q+3}x_{\sigma_1}}{x_{\sigma_2}x_{\sigma_3}} \right\} \cup \left\{ x_{\sigma_2}, \frac{x_{q+3}x_{\sigma_2}}{x_{\sigma_3}} \right\} \cup \left\{ x_{\sigma_3} \right\}.
\]

We now fix \( t \) with \( t \leq c - 1 \). Part (1) of the following combinatorial lemma will be used in Lemma 20 for the proof of the minimality of the Kustin–Miller complex construction, while part (2) will be used in Proposition 21 for the proof of the equality of the corresponding Betti numbers of \( T/I_{F_t} \) and \( T/(I_{F_t} : x_{\sigma_t}) \).

**Lemma 19.**

(1) For every \( v \in S_t \) there exists \( a \in \sigma_t \) such that \( x_a \) divides \( v \).

(2) We can recover \( S_t \) from \( \frac{S_t}{x_{\sigma_t}} \) in the following way: \( S_t \) is the set obtained from \( \frac{S_t}{x_{\sigma_t}} \) by substituting, for \( p = 1, 2, \ldots, t - 1 \), the variable \( x_{(p)} \) by the product \( x_{(p)} x_{(p)_{t-1}} \), and substituting the variable \( x_{dt+1} \) by the product \( \prod_{r=d_{t-1}+1}^{d_t+1} x_r \).

**Proof.** Let \( v \in S_t \) and consider the original source \( e_v \in \{ u_2, x_{\sigma_1}, \ldots, x_{\sigma_{t-1}} \} \) of \( v \). Write

\[
v = \frac{w_1 e_v}{w_2},
\]

with either \( w_1 = w_2 = 1 \) or \( w_1 = \prod_{j=1}^{l} x_{q+q_j} \) and \( w_2 = \prod_{j=1}^{l} x_{r_j} \) for some \( l \geq 1 \) and \( r_1 < r_2 < \cdots < r_l \). Moreover, if \( e_v = u_2 \) we have \( 2 \leq r_1 \), while if \( e_v = x_{\sigma_p} \) we have \( p + 1 \leq r_1 \).
We first prove (1). If \( e_v = u_2 \), we set \( a = d_t \in \sigma_t \) and observe that \( x_a \) divides \( e_v \). Since \( d_t \) is not in any \( \sigma_i \) for \( i < t \) we have that \( x_a \) does not divide \( w_2 \), hence it follows by (5.2) that \( x_a \) divides \( v \). Assume now that \( e_v = \sigma_p \) for some \( p \) with \( 1 \leq p \leq t-1 \). We set \( a = (\sigma_p)_{t-1} \). By the definition of the sets \( \sigma_r \), we have that \( a \) is in the intersection of \( \sigma_p \) with \( \sigma_t \) and in no other \( \sigma_r \). Hence \( x_a \) divides \( e_v \) but not \( w_2 \), hence it follows by (5.2) that \( x_a \) divides \( v \).

We will now prove (2). We first fix \( p \in \{1, 2, \ldots, t-1\} \), set \( m = (\sigma_p)_t \), assume \( x_{m+1} \) divides \( v \), and prove that \( x_{m-1} \) also divides \( v \). The assumption that \( x_{m+1} \) divides \( v \) implies that, when \( v \neq e_v \), in the expression (5.2) we have \( r_1 \neq p \) for \( 1 \leq i \leq t \). Taking into account that \( m \) is not in \( \sigma_j \) for \( 1 \leq j \leq t-1 \) and \( j \neq p \) we get that \( e_v = u_2 \) or \( e_v = \sigma_p \). Since \( p < t \) we have \( m-1 = (\sigma_p)_{t-1} \). This, together with \( e_v \in \{u_2, \sigma_p\} \) implies that \( x_{m-1} \) divides \( e_v \). It also implies that \( m-1 \) is not in any \( \sigma_j \) for \( 1 \leq j \leq t-1 \) and \( j \neq p \). Hence, \( x_{m-1} \) does not divide \( w_2 \) and since it divides \( e_v \), if follows from (5.2) that it also divides \( v \).

We now assume \( x_{d_t+1} \) divides \( v \) and will show that \( \prod_{r=d_t+1}^{d_t+1} x_r \) also divides \( v \). Since \( d_t+1 \) is not in \( \sigma_t \) for \( 1 \leq i \leq t-1 \), we have that \( e_v = u_2 \). Fix \( r \) with \( d_t+1 \leq r \leq d_t \). Then \( r \) is not an element of \( \sigma_t \) for \( 1 \leq j \leq t-1 \). Hence \( x_r \) does not divide \( w_2 \) and since it divides \( u_2 \) if follows from (5.2) that it also divides \( v \). Taking into account that \( m \) and \( d_t+1 \) are not in \( \sigma_t \), this completes the proof of (2).

**Lemma 20.** Fix \( t \) with \( 2 \leq t \leq c-1 \). Then the Kustin–Miller complex construction related to the unprojection pair \((I_{F_1} : x_{\sigma_t}, z) \subset T[z]/(I_{F_1}) \) and using as initial data the minimal graded free resolutions of \( T[z]/(I_{F_1}) \) and \( T[z]/(I_{F_1} : x_{\sigma_t}, z) \) gives a minimal complex.

**Proof.** The minimal monomial generating set of \( I_{F_1} \) is a subset, say \( \tilde{S}_t \), of \( S_t \). By part (1) of Lemma 19 given \( v \in \tilde{S}_t \), there is an \( a \in \sigma_t \) with \( x_a \) dividing \( v \). As a consequence, the result follows from Proposition 11. \( \square \)

**Proposition 21.** Fix \( t \) with \( 2 \leq t \leq c \). Then

1. The set \( S_t \) is the minimal monomial generating set of \( I_{F_t} \).
2. The corresponding Betti numbers of \( T/I_{F_1} \) and \( T/(I_{F_1} : x_{\sigma_t}) \) are equal, that is \( b_i(T/(I_{F_1} : x_{\sigma_t})) = b_i(T/I_{F_1}) \) for all \( i \). In particular, the set \( \frac{S_t}{x_{\sigma_t}} \) has the same cardinality as \( S_t \) and is the minimal monomial generating set of \( (I_{F_1} : x_{\sigma_t}) \).

**Proof.** We use induction on \( t \). For \( t = 2 \) we have that both \( I_{F_1} \) and \( (I_{F_1} : x_{\sigma_t}) \) are codimension 2 complete intersections, so both (1) and (2) are obvious. Assume that (1) and (2) are true for a value \( t < c - 1 \) and we will show that they are true also for the value \( t + 1 \). By Lemma 20, the Kustin–Miller complex construction related to the unprojection pair \((I_{F_1} : x_{\sigma_t}, z) \subset T[z]/I_{F_1} \) and using as input data the minimal graded free resolutions of \( T[z]/I_{F_1} \) and \( T[z]/(I_{F_1} : x_{\sigma_t}, z) \) gives a minimal complex. In particular, this implies that \( S_{t+1} \) is the minimal monomial generating set of \( I_{F_{t+1}} \).

We now look more carefully the substitutions in part (2) of Lemma 19. Assume \( p \leq t \) and set \( m = (\sigma_p)_{t+1} \). Since \( p < t + 1 \) we have by the construction of \( \sigma_t \) that \( m-1 = (\sigma_p)_{t+1} \), so \( m-1 \) is an element of \( \sigma_t+1 \). Consequently \( x_{m-1} \) does not appear as variable in \( \frac{S_{t+1}}{x_{\sigma_t+1}} \). Similarly, for each \( r \) with \( d_t+1 \leq r \leq d_{t+1} \) we have \( r \in \sigma_t+1 \), so \( x_r \) does not appear as variable in \( \frac{S_{t+1}}{x_{\sigma_t+1}} \). Using these facts, the equality of Betti numbers in part (2) follows by arguing as in the proof of [2, Proposition 6.5]. Since we have shown that \( S_{t+1} \) is the minimal monomial generating set of \( I_{F_{t+1}} \), and \( \frac{S_{t+1}}{x_{\sigma_t+1}} \) contains the minimal monomial generating set of \((I_{F_1} : x_{\sigma_t}) \), the equality of Betti numbers we just showed implies, for \( i = 1 \), that \( \frac{S_{t+1}}{x_{\sigma_t+1}} \) has the same cardinality as \( S_{t+1} \) and is the minimal monomial generating set of the ideal \((I_{F_{t+1}} : x_{\sigma_{t+1}}) \). \( \square \)
We now give the proof of Proposition 14.

Proof. The proof is by induction on \( t \). For \( t = 1, 2 \) the result is clear. Assume that the result is true for some value \( 2 \leq t \leq c - 1 \) and we will show it is true for \( t + 1 \). We set for simplicity \( A_1 = T/I_{F_1} \) and \( A_2 = T/(I_{F_1} : x_\sigma) \).

By the inductive hypothesis \( b_i(A_1) = l_{t,i} \) and by part (2) of Proposition 21 \( b_i(A_2) = b_i(A_1) \), hence \( b_i(A_2) = b_i(A_1) = l_{t,i} \) for all \( i \). Since by Lemma 20 the corresponding Kustin–Miller construction is minimal, we get that

\[
b_1(R_{\sigma+t}/I_{F_{t+1}}) = b_1(A_1) + b_1(A_2) + 1 = 2l_{t,1} + 1 = l_{t+1,1}
\]

and that for \( i \) with \( 2 \leq i \leq \text{codim } R_{\sigma+t}/(I_{F_{t+1}}) - 2 \)

\[
b_i(R_{\sigma+t}/I_{F_{t+1}}) = b_{i-1}(A_1) + b_i(A_1) + b_{i-1}(A_2) + b_i(A_2)
\]

\[= 2l_{t,i-1} + 2l_{t,i} = l_{t+1,i}\]

which finishes the proof. \( \square \)

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References

[1] J. Böhm, and S. Papadakis, Stellar subdivisions and Stanley–Reisner rings of Gorenstein complexes, to appear in Australas. J. Combin.
[2] J. Böhm, and S. Papadakis, On the structure of Stanley-Reisner rings associated to cyclic polytopes, Osaka J. Math. 49 (2012), no. 1, 81-100.
[3] J. Böhm, and S. Papadakis, Implementing the Kustin–Miller complex construction, J. Softw. Algebra Geom. 4 (2012), 6-11.
[4] J. Böhm, and S. Papadakis, BettiBounds, Macaulay2 package (2012), available at http://www.mathematik.uni-kl.de/~boehm/Macaulay2/BettiBounds/html/
[5] W. Bruns, and J. Herzog, Cohen-Macaulay Rings, revised edition, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge, 1998.
[6] S. Choi and J. S. Kim, A combinatorial proof of a formula for Betti numbers of a stacked polytope, Electron. J. Combin. 17 (2010), no. 1, Research Paper 9.
[7] D. Grayson, and M. Stillman, Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/
[8] J. Herzog, and E. M. Li Marzi, Bounds for the Betti numbers of shellable simplicial complexes and polytopes in Commutative algebra and algebraic geometry (Ed. by F. Van Oystaeyen), Lecture notes in pure and applied mathematics 206, 157-167.
[9] A. Kustin and M. Miller, Constructing big Gorenstein ideals from small ones, J. Algebra 85 (1983), 303–322.
[10] J. Migliore, and U. Nagel, Reduced arithmetically Gorenstein schemes and simplicial polytopes with maximal Betti numbers, Adv. Math. 180 (2003), no. 1, 1–63.
[11] E. Miller, and B. Sturmfels, Combinatorial commutative algebra. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005.
[12] S. Papadakis and M. Reid, Kustin-Miller unprojection without complexes, J. Algebraic Geom. 13 (2004), 563–577
[13] T. Römer, Bounds for Betti numbers, J. Algebra 249 (2002), no. 1, 20–37.
[14] R. Stanley, Combinatorics and commutative algebra. Second edition. Progress in Mathematics, 41. Birkhäuser, 1996.
[15] N. Terai and T. Hibi, Computation of Betti numbers of monomial ideals associated with stacked polytopes, Manuscripta Math. 92 (1997), 447–453.

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