Nonlocal gradients in bounded domains motivated by Continuum Mechanics: Fundamental Theorem of Calculus and embeddings

J. C. Bellido
E.T.S.I. Industriales, Department of Mathematics, Universidad de Castilla-La Mancha, 13071-Ciudad Real, Spain. Email JoseCarlos.Bellido@uclm.es.

J. Cueto
E.T.S.I. Industriales, Department of Mathematics, Universidad de Castilla-La Mancha, 13071-Ciudad Real, Spain. Email Javier.Cueto@uclm.es.

C. Mora-Corral
Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain. Email Carlos.Mora@uam.es

Abstract

In this paper we develop a new set of results based on a nonlocal gradient jointly inspired by the Riesz s-fractional gradient and Peridynamics, in the sense that its integration domain depends on a ball of radius $\delta > 0$ (horizon of interaction among particles, in the terminology of Peridynamics), while keeping at the same time the singularity of the Riesz potential in its integration kernel. Accordingly, we define a functional space suitable for nonlocal models in Calculus of Variations and partial differential equations. Our motivation is to develop the proper functional analysis framework in order to tackle nonlocal models in Continuum Mechanics, which requires working with bounded domains, while retaining the good mathematical properties of Riesz s-fractional gradients. This functional space is defined consistently with Sobolev and Bessel fractional ones: we consider the closure of smooth functions under the natural norm obtained as the sum of the $L^p$ norms of the function and its nonlocal gradient. Among the results showed in this investigation we highlight a nonlocal version of the Fundamental Theorem of Calculus (namely, a representation formula where a function can be recovered from its nonlocal gradient), which allows us to prove inequalities in the spirit of Poincaré, Morrey, Trudinger and Hardy as well as the corresponding compact embeddings. These results are enough to show the existence of minimizers of general energy functionals under the assumption of convexity. Equilibrium conditions in this nonlocal situation are also established, and those can be viewed as a new class of nonlocal partial differential equations in bounded domains.

Keywords: Riesz fractional gradient, Nonlocal gradient, Nonlocal fundamental theorem of Calculus, Nonlocal Poincaré inequality, Nonlocal embeddings, Nonlocal Calculus of Variations, Peridynamics

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1 Introduction

In the last decades, models based on differential equations are increasingly sharing their prominence with those based on integral or integro-differential equations. This is due to the fact that they can catch some information that the local ones cannot, such as long range interactions or multiscale behaviour; they usually require less regularity of the functions, allowing for more general admissible solutions. In fact, they can overcome some drawbacks of the local models since a typical feature is to be able to provide an effective modelling for discontinuities or singularities.

One such example is Peridynamics, a nonlocal alternative model in Solid Mechanics proposed by Silling [61]; see also [40,62,63]. One of its goals was to unify elastic and singularity phenomena, such as fracture or cavitation. The development of this theory in the last years has been impressive. As general expositions, we can mention the review paper [37], the two books [32,41] and the collaborative handbooks [12,67]. Several aspects of these models have been studied such as localization [30,45,17], existence and regularity [31], computational issues [18,19], function spaces involved [43,46], or linear theories [23,25,27,44,66,69].

Accordingly, nonlocal models are gaining attention in the modelling of various phenomena in physics, biology, geometry and more. As a consequence, it is required a more thorough mathematical analysis of the new objects and operators involved. Some of those objects are of diffusion type, where the fractional Laplacian stands out: this is an operator that generalizes the standard Laplacian to a degree of differentiability beyond derivatives of integer order (see, e.g., [1,14,15,22,49,53,55,56] among hundreds of possible references). Others, on the other hand, are of gradient type, which may provide a better scope for nonlocal vector calculus [7,16,20,24,47,59,60]. One of the features of both kind of operators is that they require less regularity of the functions than their classical counterparts. In this paper we focus on the latter type, in particular, those called one-point

\[ D^{n,s} u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy, \]

where \( c_{n,s} \) is a suitable normalizing constant. It follows the same formula as in (1) where the integration domain is considered to be \( \mathbb{R}^n \) and the kernel is \( \rho(x) = c_{n,s} \frac{1}{|x|^{n+1+s}}. \) This object has attracted a great interest in the last years, as will be reported below, mainly due to its nice properties from a functional analysis perspective. However, the fact of being defined over the whole space is a remarkable drawback for applications in realistic physical models; furthermore, this feature obviously leads to some extra difficulties from a computational or numerical point of view.

Thus, given the interest of working with bounded domains, and inspired by (1) and (2) we propose the operator

\[ D^{n,s}_\delta u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x-y|^{n+1+s}} \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|} dy, \]

as a particular case of (1), (with \( \rho(x) = c_{n,s} \frac{1}{|x|^{n+1+s}} w_\delta(x) \) and \( w_\delta \in C^\infty_c(B(0,\delta)) \) a cut-off function), except for the fact that in this case, \( u \) is assumed to be defined, at least, in \( \Omega_\delta = \Omega + B(0,\delta) \) so that
the integral in (3) is completely defined in the whole ball. This entails dealing with the ‘collar’ \( \Omega_\delta \setminus \Omega \) as a nonlocal boundary. The nonlocal gradient (3) is defined so that it keeps the fractional index \( s \) of differentiability, while at the same time the interaction of particles is restricted to a distance no larger than \( \delta \) (the horizon parameter in Peridynamics).

To put it into context, the fractional gradient (2), as recently addressed by several authors [7,8,16,38,54,57], seems to be the suitable notion, from a merely mathematical perspective, for such a differential object. In particular, it has been proved in [68] that formula (2) determines up to a multiplicative constant the unique object fulfilling some minimal consistency requirements from the physical and mathematical point of view, such as invariance under rotations and translations, \( s \)-homogeneity under dilations and some weak continuity properties. Moreover, the classical gradient can be recovered when \( s \) goes to 1 in (2). This operator is closely related to the Riesz potential, \( I_{1-s}(x) = \frac{c_{n,s}}{n-1+s} |x|^{-(n-1+s)} \), and particularly in the case of smooth functions \( u \in C^\infty_c(\mathbb{R}^n) \), it can be written as a convolution of this kernel with the classical gradient: \( D^s u = I_{1-s} \ast \nabla u \). This implies that its Fourier transform can be computed as \( \hat{D}^s u(\xi) = \frac{2\pi c_{n,s}}{|2\pi \xi|^s} \hat{u}(\xi) \), which gives another insight to the unfamiliar reader.

In the case of the nonlocal gradient \( D^s_\Omega u \), among all the properties mentioned earlier and systematized in [68] characterising the fractional gradient (2), the one that is not fulfilled by (3) is the \( s \)-homogeneity under dilations, in favour of considering bounded domains (equivalently, a compactly supported kernel). Similar operators have been studied in works like [45,47], where (3) could fit after normalizing its kernel. In particular, it was shown in [47] that these operators converge to the potential, but with compact support, making (3) integrable. Thus, the nonlocal gradient can be written as a convolution with the classical one: \( D^s_\Omega u = Q^s_\delta \ast \nabla u \) for sufficiently smooth functions. As it will be seen later, this shows that the nonlocal gradient (defined or extended to every point in \( \mathbb{R}^n \) if necessary) of test functions such as \( C^\infty_c(\mathbb{R}^n) \) or \( S \) (the Schwartz space) remains in such spaces, as opposed to the fractional gradient, making it more suitable for defining the nonlocal gradient of a distribution.

Regarding functional spaces, essential for Calculus of Variations, the one associated to Riesz fractional gradients is the Bessel space \( H^{s,p}(\mathbb{R}^n) \). Among the several equivalent definitions, the most intuitive in this context is that based on the completion of \( C^\infty_c(\mathbb{R}^n) \) functions under the norm

\[
\| u \|_{H^{s,p}(\mathbb{R}^n)} = \left( \| u \|_{L^p(\mathbb{R}^n)}^p + \| D^s u \|_{L^p(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}.
\]

There are, of course, many spaces between \( L^p(\mathbb{R}^n) \) and the Sobolev space \( W^{1,p}(\mathbb{R}^n) \) that possess a degree of differentiability of order \( s \). The most familiar one is possibly the Gagliardo space \( W^{s,p}(\mathbb{R}^n) \), which is equipped with the seminorm

\[
[u]_{W^{s,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}
\]

and the norm

\[
\| u \|_{W^{s,p}(\mathbb{R}^n)} = \left( \| u \|_{L^p(\mathbb{R}^n)}^p + [u]_{W^{s,p}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}.
\]

A great difference between \( H^{s,p}(\mathbb{R}^n) \) and \( W^{s,p}(\mathbb{R}^n) \) is that in the latter there is no suitable concept of fractional gradient, even though it possibly is the natural space to define the fractional Laplacian. Moreover, despite the analogy of the seminorms in those spaces,

\[
\| D^s u \|_{L^p(\mathbb{R}^n)} = c_{n,s} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+s}} \, dx \, dy \right)^{\frac{1}{p}} \quad \text{and} \quad [u]_{W^{s,p}(\mathbb{R}^n)}
\]
the fact that in $\|D^s u\|_{L^p(\mathbb{R}^n)}$ the absolute value affects the inner integral, while in $[u]_{W^{s,p} (\mathbb{R}^n)}$ the absolute value affects the integrand, reveals that the inclusions between these spaces are not obvious. We mention, in passing, that in [2,59] it is shown the embeddings $H^{s_2,p}(\mathbb{R}^n) \subset W^{s,p}(\mathbb{R}^n) \subset H^{s_1,p}(\mathbb{R}^n)$ for $0 < s_1 < s < s_2 < 1$, as well as the equality $H^{s,2}(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$. This feature of the absolute value affecting the inner integral in $\|D^s u\|_{L^p(\mathbb{R}^n)}$ has several consequences in the proofs of properties of $H^{s,p}(\mathbb{R}^n)$, since, in general, it hampers a direct application of the elementary inequality that the absolute value of the integral is less than the integral of the absolute value, since that inequality cannot be reversed. For example, one cannot apply directly the techniques of [50,51], which are suitable for seminorms in the style of $W^{s,p}$, but with general kernels. Although the definitions of the Riesz gradient and the Bessel spaces are rather old, it was the study [59] that initiated the attention in the community of nonlocal problems in partial differential equations and Calculus of Variations. In fact, in [59,60] it was shown the relationship between Riesz gradients and Bessel spaces, as well as a series of inequalities and embeddings mimicking those of Sobolev spaces, which constitute the basis for an analysis of the equations and minimization problems naturally related to the fractional gradient.

While [60] treated the existence of minimizers for convex scalar problems using the direct method of the Calculus of Variations, the following two papers make extensions and applications for vectorial problems: in [7] we showed that the concept of polyconvexity is also suitable in these problems, while in [38] it was shown the analogue for the concept of quasiconvexity. These notions, polyconvexity and quasiconvexity, are classical in the Calculus of Variations (see, e.g., [17]). In these three works, the functional to minimize is of the form

$$\int_{\mathbb{R}^n} W(x, u(x), D^s u(x)) \, dx,$$

with the integrand $W$ satisfying similar assumptions as in local problems.

This article can be seen as a follow-up to other works in the search of a nonlocal model suitable for hyperelasticity but also introducing a theory applicable to other phenomena. Following previous works [9,11] by some of the authors of this paper, we showed in [6] that (bond-based) Peridynamics models based on energy functionals of the form

$$\int_\Omega \int_{\Omega \cap B(x,\delta)} w(x - y, u(x) - u(y)) \, dy \, dx,$$

although defined for bounded domains, do not fit in nonlinear Solid Mechanics, since very few local nonlinear models are limit of nonlocal ones when $\delta \to 0$. As mentioned earlier, $\delta$ is the horizon: the interaction distance between the particles. On the other hand, going back to the analysis in $H^{s,p}(\mathbb{R}^n)$ we showed in [8] that the limit when $s \to 1$ of integral (4) based on the Riesz gradient is the local model

$$\int_{\mathbb{R}^n} W(x, u(x), \nabla u(x)) \, dx.$$

In this paper we propose a model that combines the good properties of the Riesz gradient and the space $H^{s,p}$ with the requirement that the energy is defined in a bounded domain of $\mathbb{R}^n$, since it is in this case where its interpretation of an elastic energy is physically meaningful, and so we think it could fit in state-based Peridynamics.

In order to reproduce similar existence results for energy functionals like

$$\int_\Omega W(x, u(x), D^s u(x)) \, dx,$$
it is necessary the study of the functional space associated to the nonlocal gradient $\delta$. To be precise, we define the space $H^{s,p,\delta}(\Omega)$ in concordance with Bessel and Sobolev spaces, as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm
\[
\|u\|_{H^{s,p,\delta}(\Omega)} = \left(\|u\|_{L^p(\Omega_\delta)}^p + \|D^s_\delta u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}},
\]
where $\Omega_\delta$ is the union of $\Omega$ with a tubular neighbourhood of the boundary of radius $\delta$. A subspace $H^{s,p,\delta}_0(\Omega)$ representing roughly $H^{s,p,\delta}(\Omega)$ functions with zero ‘boundary’ conditions (in truth, with zero values in another tubular neighbourhood of the boundary) is also studied.

We highlight as one of the main contributions of this work a nonlocal version of the Fundamental Theorem of Calculus, which is obtained despite the lack of homogeneity and semi-group properties of the kernels involved. As in other frameworks, given that it allows us to recover a function from its nonlocal gradient, we see it as a versatile tool that may prove itself useful in other situations. In particular, it helps in overcoming the aforementioned problem of an absolute value affecting the inner integral in the seminorms we are considering, and therefore, it is a key ingredient in the process of obtaining several inequalities and embeddings.

Thus, this article can be regarded as a first step to explore properties in $H^{s,p,\delta}$ that are known in $W^{1,p}$ and $H^{s,p}$, such as integration by parts, Fundamental Theorem of Calculus, Poincaré inequalities and compact embeddings. In fact, it is illustrative to compare those definitions and properties in the three contexts: classical, fractional and nonlocal. In what follows, classical will typically refer to properties for Sobolev $W^{1,p}$ or even smooth functions involving the (classical or distributional) gradient $\nabla$, fractional to properties in $H^{s,p}$ involving Riesz’ $s$-fractional gradient $D^s$, and nonlocal to properties in $H^{s,p,\delta}$ involving the nonlocal gradient $D^s_\delta$. A partial list of this comparison is as follows:

- **Gradient.** The classical gradient is just the pointwise or distributional gradient. The fractional gradient is $D^s$, while the nonlocal gradient is $D^s_\delta$.

- **Divergence.** The classical divergence is the pointwise divergence. The fractional divergence is
\[
\text{div}^s \phi(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{\phi(x) - \phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} dy,
\]
while the nonlocal divergence is
\[
\text{div}^s_\delta \phi(x) = c_{n,s} \int_{B(x,\delta)} \frac{\phi(x) - \phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} w_\delta(x-y) \cdot \frac{1}{|x-y|^{n+1+s}} dy.
\]

- **Integration by parts.** For $u \in C^1_c(\Omega)$ and $\phi \in C^1_c(\Omega, \mathbb{R}^n)$,
\[
\text{Classical: } \int_\Omega \nabla u \cdot \phi = - \int_\Omega u \text{div} \phi.
\]
\[
\text{Fractional: } \int_{\mathbb{R}^n} D^s u \cdot \phi = - \int_{\mathbb{R}^n} u \text{div}^s \phi.
\]
\[
\text{Nonlocal: } \int_{\Omega} D^s_\delta u \cdot \phi = - \int_{\Omega} u \text{div}^s_\delta \phi.
\]

- **Fundamental Theorem of Calculus.**
\[
\text{Classical: } u(x) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla u(y) \cdot \frac{x-y}{|x-y|^n} dy.
\]
\[
\text{Fractional: } u(x) = c_{n,-s} \int_{\mathbb{R}^n} D^s u(y) \cdot \frac{x-y}{|x-y|^{n-s+1}} dy.
\]
\[
\text{Nonlocal: } u(x) = \int_{\mathbb{R}^n} D^s_\delta u(y) \cdot V^s_\delta(x-y) dy.
\]
At this point we mention the attempt to unify fractional and nonlocal theories recently explored in [21], in the context of a general vector calculus (following the earlier works [24,35]) and in [20], which focuses on nonlocal gradients. We also point out the work [28], where a different approach to the fractional fundamental theorem of calculus in dimension one is addressed, as well as a study of the function spaces involved.

The role of the Fourier transform in this analysis deserves a special mention. It was pointed out in [59] that the fractional gradient behaves nicely under Fourier transform: 
\[ \hat{D^s u}(\xi) = \frac{2\pi i |\xi|^{s-1}}{2\pi |\xi|} \hat{u}(\xi). \]
This fact was used in [8] to obtain some properties that would otherwise require a much longer argument. In this paper we also use Fourier transform, which is of no surprise having in mind the fundamental theorems of Calculus above expressing \( u \) as a convolution, and, in fact, the constant presence of convolutions in this work. Again in [59] the properties of the Riesz potential and its Fourier transform were used in connection with the Riesz gradient. In this paper we also use a potential playing the role of Riesz'. In our case, this potential will no longer have the semigroup property, but yet we will succeed in capturing its main features to prove the nonlocal fundamental theorem of Calculus.

The similarities with the fractional case show that a parallel theory could potentially be developed. Actually, it turns out that when restricted to \( \Omega_\delta \), functions from Bessel spaces belong to \( H^{s,p,\delta}(\Omega) \) (Proposition 3.5). This leads to wonder if functions in Bessel spaces \( H^{s,p}(\mathbb{R}^n) \) are always an extension to \( \mathbb{R}^n \) of functions in \( H^{s,p,\delta}(\Omega) \). This inclusion also implies that functions exhibiting fracture or cavitation phenomena are admitted in \( H^{s,p,\delta}(\Omega) \) [7, Sect. 2.1].

The outline of the paper is the following. Section 2 fixes some notation used throughout the article. In Section 3 the new versions of nonlocal gradient, divergence and integration by parts are established. We also define the associated function space \( H^{s,p,\delta}(\Omega) \) and state its basic properties. Section 4 proves the nonlocal version of the fundamental theorem of Calculus. Its proof, nevertheless, depends on the existence of the kernel \( V^s_\delta \), which is addressed in Section 5. Then, in Section 6 we first define the space \( H^{s,p,\delta}_0(\Omega) \) and then use the nonlocal Fundamental Theorem of Calculus to prove versions in this context of the inequalities by Poincaré, Morrey, Trudinger and Hardy. In Section 7 we establish the compact embeddings from \( H^{s,p,\delta}_0(\Omega) \) to \( L^q(\Omega) \). Section 8 shows the existence of minimizers of scalar convex variational problems involving \( D^s_\delta \), as well as the corresponding Euler–Lagrange equation. The article finishes with two appendices: in Appendix A we point out the necessary changes needed in Section 5 for the case \( n = 1 \), while in Appendix B we state some Fourier analysis facts used throughout the paper for which we have not found a reference.

2 Notation

2.1 General notation

In all this work, we fix the dimension \( n \in \mathbb{N} \) of the space \( (n \geq 1) \), an open bounded set \( \Omega \) of \( \mathbb{R}^n \) representing the body, a number \( 0 < s < 1 \) quantifying the degree of differentiability, a \( \delta > 0 \) indicating the horizon (the interaction distance between the particles of the body), and an exponent \( 1 \leq p < \infty \) of integrability. Sometimes we will additionally require \( p > 1 \). The Hölder conjugate exponent of \( p \) is \( p' = \frac{p}{p-1} \).

The notation for Sobolev \( W^{1,p} \) and Lebesgue \( L^p \) spaces is standard. So is the notation for functions that are continuous \( C \), and of class \( C^k \) for \( k \) an integer or infinity. Their version of compact support are \( C^k_c \). The set of continuous functions vanishing at infinity is \( C_0 \). We will indicate the domain of the functions, as in \( C^1(\Omega) \); the target is indicated only if it is not \( \mathbb{R} \). When using the norm in those spaces, the target is omitted, as in \( \|\cdot\|_{L^p(\Omega)} \).
We write $B(x, r)$ for the open ball centred at $x \in \mathbb{R}^n$ of radius $r > 0$. The complement of a subset $A \subset \mathbb{R}^n$ is denoted by $A^c$, its closure by $\bar{A}$ and its boundary by $\partial A$.

We denote by $\sigma_{n-1}$ the area of the unit sphere, while the surface area in integrals is indicated by $H_{n-1}$.

We will use the multiindex notation: for $\alpha \in \mathbb{N}^n$, we give the standard meaning to the partial derivative $\partial^\alpha$, the size $|\alpha|$, the monomial $x^\alpha$ for $x \in \mathbb{R}^n$, the ordering $\beta \leq \alpha$ and the combinatorial number $\binom{\alpha}{\beta}$; see, e.g., [34, Sect. 2.2].

The vectors of the canonical basis of $\mathbb{R}^n$ are $e_j$, $j = 1, \ldots, n$.

The operation of convolution is denoted by $\ast$. We indicate the duality product between tempered distributions and Schwartz functions as $\langle \cdot, \cdot \rangle$.

2.2 Fourier transform

The convention for the Fourier transform of a function $f$ is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$$

for $f \in L^1(\mathbb{R}^n)$. This definition is extended by continuity and duality to other function and distribution spaces, notably, as isomorphisms in the Schwartz space $S$ and in the space of tempered distributions $S'$. Sometimes we will also use the alternative notation $F(f)$ for $\hat{f}$. The variable in the Fourier space is generically designed by $\xi$. The reflection of the function $f : \mathbb{R}^n \to \mathbb{R}$ is $\tilde{f}(x) = f(-x)$, and one has $\tilde{f} = F(\hat{f})$, in principle, for functions $f \in L^1(\mathbb{R}^n)$ for which $\hat{f} \in L^1(\mathbb{R}^n)$, but then by continuity and duality this property is extended to a larger class of functions and distributions. Classical texts in Fourier analysis are [26,34].

2.3 Radial functions

We recall the following definitions regarding radial functions.

**Definition 2.1.** We will say that

a) a function $f : \mathbb{R}^n \to \mathbb{R}$ is radial if there exists $\tilde{f} : [0, \infty) \to \mathbb{R}$ such that $f(x) = \tilde{f}(|x|)$ for every $x \in \mathbb{R}^n$. In such a case, $\tilde{f}$ is the radial representation of $f$.

b) a radial function $f : \mathbb{R}^n \to \mathbb{R}$ is radially decreasing if its radial representation $\tilde{f} : [0, \infty) \to \mathbb{R}$ is a decreasing function.

c) a function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is vector radial if there exists a radial function $\tilde{\phi} : [0, \infty) \to \mathbb{R}$ such that $\phi(x) = \tilde{\phi}(|x|)x$ for every $x \in \mathbb{R}^n$.

It is known (see, e.g., [34, App. B.5]) that the Fourier transform of a radial (respectively, vector radial) function is radial (respectively, vector radial).

3 Function space: nonlocal gradient, divergence and integration by parts

In this section we define the nonlocal gradient and divergence, and state their basic properties, notably, the integration by parts. We also set the natural functional space associated to the nonlocal gradient. The framework is the following. As typical in nonlocal models [3, 5, 24, 35, 36].
conditions are usually of volumetric type. In our case, we fix a distance $\delta > 0$ and consider a bounded, open domain $\Omega \subset \mathbb{R}^n$. The set $\Omega$ itself is regarded as a nonlocal interior domain, while $\Omega_\delta := \Omega + B(0, \delta)$ is considered as its nonlocal closure. Accordingly, the set $\Omega_{B, \delta} := \Omega_\delta \setminus \Omega$ plays the role of nonlocal boundary; see Figure 1. The set $\Omega_{-\delta} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \}$ will also be relevant along this work. Thus, we consider $\delta$ small enough so that $\Omega_{-\delta}$ is not empty.

Let $w_\delta : \mathbb{R}^n \to [0, +\infty)$ be a cut-off function, and $\rho_\delta : \mathbb{R}^n \to [0, +\infty)$ defined as

$$\rho_\delta(x) = \frac{1}{\gamma(1-s)|x|^{n-1+s}w_\delta(x)},$$

where the constant $\gamma(s)$ is given by

$$\gamma(s) = \frac{\pi^{\frac{n}{2}}2^{s}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)}$$

and $\Gamma$ is Euler’s gamma function. We assume the following conditions over $w_\delta$:

a) $w_\delta$ is radial and nonnegative; $\bar{w}_\delta$ is its radial representation.

b) $w_\delta \in C_\infty(B(0, \delta))$.

c) There are constants $a_0 > 0$ and $0 < b_0 < 1$ such that $0 \leq w_\delta \leq a_0$, with $w_\delta = a_0$ in $B(0, b_0\delta)$.

d) $\bar{w}_\delta$ is decreasing.

In fact, it will be apparent in the proof of Lemma 5.3 that condition d) can be considerably weakened. Note, in addition, that $\rho_\delta \in L^1(\mathbb{R}^n)$.

Given a function $f : \Omega \to \mathbb{R}$ and $x \in \Omega$ such that $f \in L^1(\Omega \setminus B(x, r))$ for every $r > 0$, the principal value centred at $x$ of $\int_\Omega f$, denoted by

$$\text{pv}_x \int_\Omega f$$

is defined as

$$\lim_{r \to 0} \int_{\Omega \setminus B(x, r)} f,$$

whenever this limit exists.

The definitions of the nonlocal gradient and divergence for smooth functions are the following.

**Definition 3.1.** Set

$$c_{n,s} := \frac{n - 1 + s}{\gamma(1-s)}.$$
a) Let \( u \in C_c^\infty(\mathbb{R}^n) \). The nonlocal gradient \( D_\delta^u \) is defined as

\[
D_\delta^u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|^{n-1+s}} \, dy, \quad x \in \mathbb{R}^n.
\]

b) Let \( \phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \). The nonlocal divergence is defined as

\[
\text{div}_\delta^\phi(x) = -\text{pv}_x \int_{B(x,\delta)} \frac{\phi(x) + \phi(y)}{|x-y|} \frac{x-y}{|x-y|^{n-1+s}} \, dy, \quad x \in \mathbb{R}^n.
\]

Notice that the integral in (6) is absolutely convergent because \( u \) is Lipschitz and \( \rho_\delta \in L^1(\mathbb{R}^n) \). It is also immediate from the definition that \( \text{supp} \, D_\delta^u \subset \text{supp} \, u + B(0, \delta) \). On the other hand, by odd symmetry,

\[
-\text{pv}_x \int_{B(x,\delta)} \frac{\phi(x) + \phi(y)}{|x-y|} \frac{x-y}{|x-y|^{n-1+s}} \, dy = \int_{B(x,\delta)} \frac{\phi(x) - \phi(y)}{|x-y|} \frac{x-y}{|x-y|^{n-1+s}} \, dy
\]

and this last integral is absolutely convergent.

Note also that, for each \( x \in \Omega \),

\[
\int_{B(x,\delta)} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|^{n-1+s}} \, dy = \int_{\Omega_\delta} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|^{n-1+s}} \, dy,
\]

and similarly for the integral in (7), since \( B(x,\delta) \subset \Omega_\delta \) and \( \text{supp} \, w_\delta \subset B(0, \delta) \).

The operators of Definition 3.1 are dual operators in the sense of integration by parts. Actually, several versions of integration by parts formulas for related fractional or nonlocal operators have already appeared in the literature \([16, 24, 47, 68]\). For the purposes of this work, we will use a particular case of \([47, \text{Th. 1.4}]\), which, for convenience, we restate here in our context.

**Proposition 3.1.** Assume \( u \in C_c^\infty(\mathbb{R}^n) \) and \( \phi \in C_c^1(\Omega, \mathbb{R}^n) \). Then

\[
\int_{\Omega} \int_{\Omega} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \cdot \phi(x) \rho_\delta(x-y) \, dy \, dx = \int_{\Omega} \frac{u(x)}{\text{pv}_x} \int_{\Omega} \frac{\phi(x) + \phi(y)}{|x-y|} \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy \, dx.
\]

The integration by parts formula of interest in this investigation is the following. Notice the presence of a boundary term, which is due to the fact that \( u \) is not assumed to have compact support in \( \Omega \). Note that the minus sign in the boundary term makes sense since the vector \( x-y \) points inwards.

**Theorem 3.2.** Suppose that \( u \in C_c^\infty(\mathbb{R}^n) \) and \( \phi \in C_c^1(\Omega, \mathbb{R}^n) \). Then \( D_\delta^u \in L^\infty(\mathbb{R}^n, \mathbb{R}^n) \) and \( \text{div}_\delta^\phi \in L^\infty(\mathbb{R}^n) \). Moreover,

\[
\int_{\Omega} D_\delta^u(x) \cdot \phi(x) \, dx = -\int_{\Omega} u(x) \text{div}_\delta^\phi(x) \, dx - (n-1+s) \int_{\Omega} \int_{\Omega_\delta} \frac{u(y)\phi(x)}{|x-y|} \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy \, dx
\]

and these three integrals are absolutely convergent.

**Proof.** Denoting by \( L > 0 \) the Lipschitz constant of \( u \), we have, for each \( x \in \mathbb{R}^n \),

\[
|D_\delta^u(x)| \leq c_{n,s} L \int_{B(x,\delta)} \frac{w_\delta(x-y)}{|x-y|^{n-1+s}} \, dy = (n-1+s)L.
\]
so $D^s_\delta u \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Analogously, the integral of the right-hand side of (7) is absolutely convergent and $\text{div}^s_\delta \phi \in L^\infty(\mathbb{R}^n)$.

We have

$$\int_{\Omega} D^s_\delta u(x) \cdot \phi(x) \, dx = (n-1+s) \int_{\Omega} \int_{\Omega_\delta} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \phi(x) \rho_\delta(x-y) \, dy \, dx$$

with

$$\int_{\Omega} \int_{\Omega_\delta} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho_\delta(x-y) \cdot \phi(x) \, dy \, dx = \int_{\Omega} \int_{\Omega} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \cdot \phi(x) \rho_\delta(x-y) \, dy \, dx$$

$$+ \int_{\Omega} \int_{\Omega_\delta} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \cdot \phi(x) \rho_\delta(x-y) \, dy \, dx.$$ 

By Proposition 3.1

$$\int_{\Omega} \int_{\Omega} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \cdot \phi(x) \rho_\delta(x-y) \, dy \, dx = \int_{\Omega} u(x) \text{pv}_x \int_{B(x,\delta)} \frac{\phi(x) + \phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy \, dx.$$

On the other hand,

$$- \int_{\Omega} u(x) \text{div}^s_\delta \phi(x) \, dx = (n-1+s) \int_{\Omega} u(x) \text{pv}_x \int_{B(x,\delta)} \frac{\phi(x) + \phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy \, dx.$$

Now, for each $x \in \Omega$,

$$\text{pv}_x \int_{B(x,\delta)} \frac{\phi(x) + \phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy = \text{pv}_x \int_{\Omega} \frac{\phi(x) + \phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy$$

$$+ \text{pv}_x \int_{\Omega_{B,\delta}} \frac{\phi(x) + \phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy$$

and, since $\phi$ vanishes in $\Omega_{B,\delta}$,

$$\text{pv}_x \int_{\Omega_{B,\delta}} \frac{\phi(x) + \phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy = \int_{\Omega_{B,\delta}} \frac{\phi(x) - \phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy$$

and this last integral is absolutely convergent, as explained in (7).

Putting together the formulas above, we have obtained that

$$\int_{\Omega} D^s_\delta u(x) \cdot \phi(x) \, dx + \int_{\Omega} u(x) \text{div}^s_\delta \phi(x) \, dx$$

$$= (n-1+s) \int_{\Omega} \int_{\Omega_{B,\delta}} \left[ \frac{u(x) - u(y)}{|x-y|} \phi(x) - u(x) \frac{\phi(x) - \phi(y)}{|x-y|} \right] \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy \, dx,$$

being the three integrals absolutely convergent. Finally, for each $x \in \Omega$ and $y \in \Omega_{B,\delta}$,

$$\frac{u(x) - u(y)}{|x-y|} \frac{\phi(x) - \phi(y)}{|x-y|} = \frac{-u(y)}{|x-y|} \frac{\phi(x) + u(x)}{|x-y|} \frac{\phi(y)}{|x-y|} = \frac{-u(y)}{|x-y|} \phi(x)$$

since $\phi \in C^1_c(\Omega, \mathbb{R}^n)$. This concludes the proof.

Although we have proved $L^\infty$ regularity for $D^s_\delta u$, much more is true, since Proposition 4.3 will show that $D^s_\delta u \in C^\infty_c(\mathbb{R}^n)$.

We now extend Definition 3.1 to a broader class of functions.
Definition 3.2. a) Let \( u \in L^1(\Omega) \) be such that there exists a sequence of \( \{u_j\}_{j \in \mathbb{N}} \subset C^\infty_c(\mathbb{R}^n) \) converging to \( u \) in \( L^1(\Omega) \) and for which \( D^s_\delta u_j \) converges to some \( U \) in \( L^1(\Omega, \mathbb{R}^n) \). We define \( D^s_\delta u \) as \( U \).

b) Let \( \phi \in L^1(\Omega, \mathbb{R}^n) \) be such that there exists a sequence of \( \{\phi_j\}_{j \in \mathbb{N}} \subset C^\infty_c(\mathbb{R}^n, \mathbb{R}^n) \) converging to \( \phi \) in \( L^1(\Omega, \mathbb{R}^n) \) and for which \( \{\text{div}^s_\delta \phi_j\}_{j \in \mathbb{N}} \) converges to some \( \Phi \) in \( L^1(\Omega) \). We define \( \text{div}^s_\delta \phi \) as \( \Phi \).

The following result shows that the above definitions are independent of the sequence chosen.

Lemma 3.3. a) Let \( u \in L^1(\Omega) \) be such that there exist sequences \( \{u_j\}_{j \in \mathbb{N}} \) and \( \{v_j\}_{j \in \mathbb{N}} \) in \( C^\infty_c(\mathbb{R}^n) \) such that \( u_j \to u \) and \( v_j \to u \) in \( L^1(\Omega) \), and for which \( D^s_\delta u_j \to U \) and \( D^s_\delta v_j \to V \) in \( L^1(\Omega, \mathbb{R}^n) \) as \( j \to \infty \). Then \( U = V \) a.e. in \( \Omega \).

b) Let \( \phi \in L^1(\Omega, \mathbb{R}^n) \) be such that there exist sequences \( \{\phi_j\}_{j \in \mathbb{N}} \) and \( \{\theta_j\}_{j \in \mathbb{N}} \) in \( C^\infty_c(\mathbb{R}^n, \mathbb{R}^n) \) such that \( \phi_j \to \phi \) and \( \theta_j \to \phi \) in \( L^1(\Omega, \mathbb{R}^n) \), and for which \( \text{div}^s_\delta \phi_j \to \Phi \) and \( \text{div}^s_\delta \theta_j \to \Theta \) in \( L^1(\Omega) \) as \( j \to \infty \). Then \( \Phi = \Theta \) a.e. in \( \Omega \).

Proof. We prove \( \text{b) \ the \ proof \ of \ (a) \ is \ analogous. \ Let \( \phi \in C^1_c(\Omega, \mathbb{R}^n) \). By Theorem 3.2

\[
\int_\Omega U \cdot \phi = \lim_{j \to \infty} \int_\Omega D^s_\delta u_j \cdot \phi
\]

\[
= -\lim_{j \to \infty} \left( \int_\Omega u_j \text{div}^s_\delta \phi + (n - 1 + s) \int_\Omega \int_{\Omega_B, \delta} \frac{u_j(y) \phi(x)}{|x - y|} \cdot \frac{x - y}{|x - y|} \rho_\delta(x - y) \, dy \, dx \right)
\]

\[
= -\left( \int_\Omega u \text{div}^s_\delta \phi + (n - 1 + s) \int_\Omega \int_{\Omega_B, \delta} \frac{u(y) \phi(x)}{|x - y|} \cdot \frac{x - y}{|x - y|} \rho_\delta(x - y) \, dy \, dx \right).
\]

Among the previous limits, only the equality

\[
\lim_{j \to \infty} \int_\Omega \int_{\Omega_B, \delta} \frac{u_j(y) \phi(x)}{|x - y|} \cdot \frac{x - y}{|x - y|} \rho_\delta(x - y) \, dy \, dx = \int_\Omega \int_{\Omega_B, \delta} \frac{u(y) \phi(x)}{|x - y|} \cdot \frac{x - y}{|x - y|} \rho_\delta(x - y) \, dy \, dx
\]

(9)

requires some explanation, but, assuming its validity, since the same reasoning can be done for \( V \), we conclude that

\[
\int_\Omega U \cdot \phi = \int_\Omega V \cdot \phi
\]

for all \( \phi \in C^1_c(\Omega, \mathbb{R}^n) \), whence \( U = V \) a.e. in \( \Omega \).

It remains to justify limit (9). For this, it is enough to show that the function

\[
F(y) := \int_\Omega \frac{\phi(x)}{|x - y|} \cdot \frac{x - y}{|x - y|} \rho_\delta(x - y) \, dx,
\]

\( y \in \Omega_B, \delta \)

is in \( L^\infty(\Omega_B, \delta) \), and this is the case, since, denoting by \( L \) the Lipschitz constant of \( \phi \) and using that \( \phi \) vanishes in \( \Omega_B, \delta \), we have

\[
|F(y)| \leq \int_\Omega \frac{|\phi(x) - \phi(y)|}{|x - y|} \rho_\delta(x - y) \, dx \leq L.
\]

This concludes the proof. \( \square \)

It is quite natural to consider the space of \( L^p \) functions whose nonlocal gradient is also an \( L^p \) function. Taking into account the previous definitions, it is also natural to define such a space as the closure of smooth, compactly supported functions. Notice that this is analogous to the definition of the Bessel space \( H^{s,p}(\mathbb{R}^n) \). [8][16].
Definition 3.3. We define the space $H^{s,p,\delta}(\Omega)$ as

$$H^{s,p,\delta}(\Omega) := C_0^\infty(\mathbb{R}^n) \|_{H^{s,p,\delta}(\Omega)}$$

equipped with the norm

$$\|u\|_{H^{s,p,\delta}(\Omega)} = \left( \|u\|_{L^p(\Omega)}^p + \|D_\delta^s u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}.$$

Similarly to Sobolev spaces, this space satisfies reflexivity and separability properties.

Proposition 3.4. Let $1 \leq p < \infty$. Then the space $H^{s,p,\delta}(\Omega)$ is a separable Banach space. Moreover, when $p > 1$, it is reflexive.

Proof. That $H^{s,p,\delta}(\Omega)$ is a Banach space is immediate since it has been defined as a closure.

For the rest of the proof, we apply a standard argument; see, for example, [45, Th. 2.1] for the nonlocal case and [13, Prop. 8.1] for the local case.

We have that the space $F_p = L^p(\Omega_\delta) \times L^p(\Omega, \mathbb{R}^n)$ is separable and, if $p > 1$, it is reflexive. Now we define the map $T : H^{s,p,\delta}(\Omega) \to F_p$ by $T(u) = (u, D_\delta^s u)$. Then $T$ is an isometry since

$$\|T(u)\|_{F_p}^p = \|u\|_{L^p(\Omega)}^p + \|D_\delta^s u\|_{L^p(\Omega)}^p = \|u\|_{H^{s,p,\delta}(\Omega)}^p.$$

By Definitions 3.2 and 3.3 it is clear that $T(H^{s,p,\delta}(\Omega))$ is a closed subspace of $F_p$. Since every closed subspace of a reflexive space is reflexive (see, e.g., [13, Prop. 3.20]) and every subset of a separable space is separable (e.g., [13, Prop. 3.25]), it follows that $T(H^{s,p,\delta}(\Omega))$ is separable and, if $p > 1$, it is reflexive. The conclusion follows since $T$ is an isometry. 

In the next result we compare the spaces $H^{s,p,\delta}(\Omega)$ for different exponents $p$, as well as with the better-known Bessel space $H^{s,p}(\mathbb{R}^n)$.

Proposition 3.5. Let $1 \leq q \leq p < \infty$. Then:

a) $H^{s,p,\delta}(\Omega) \subset H^{s,q,\delta}(\Omega)$.

b) $H^{s,p}(\mathbb{R}^n) \subset H^{s,p,\delta}(\Omega)$, with continuous embedding.

Proof. The proof of [4] is obtained in a straightforward manner applying the known inclusions $L^p(\Omega_\delta) \subset L^q(\Omega_\delta)$ and $L^p(\Omega) \subset L^q(\Omega)$ to the norms of $u$ and $D_\delta^s u$.

Regarding [3], we first prove the corresponding inequality for smooth functions. Thus, let $u \in C_0^\infty(\mathbb{R}^n)$. We have that, for $x \in \Omega$,

$$D_\delta^s u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{w_\delta(x - y)}{|x - y|^{n+1+s}} \, dy$$

$$= c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{1}{|x - y|^{n+1+s}} \, dy - c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{a_0 - w_\delta(x - y)}{|x - y|^{n+1+s}} \, dy$$

$$= a_0 D^s u(x) - c_{n,s} \int_{B(x,\delta_0)^c} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{1}{|x - y|^{n+1+s}} \, dy$$

$$= a_0 D^s u(x) + c_{n,s} \int_{B(x,\delta_0)^c} \frac{u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{a_0 - w_\delta(x - y)}{|x - y|^{n+1+s}} \, dy.$$
for some constant $c_1 > 0$. Consequently,
\[
\|D^s_\delta u\|_{L^p(\Omega)} \leq a_0 \|D^s u\|_{L^p(\Omega)} + c_1 |\Omega|^{\frac{s}{p}} \|u\|_{L^p(B(x_0,\delta)^c)} \leq c_2 \|u\|_{H^{s,p}(\mathbb{R}^n)}
\]
for some constant $c_2 > 0$. Since we also have that $\|u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\mathbb{R}^n)}$, we obtain that there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbb{R}^n)$,
\[
\|u\|_{H^{s,p}(\mathbb{R}^n)} \leq C \|u\|_{H^{s,p}(\mathbb{R}^n)}.
\]
Being the spaces $H^{s,p,\delta}(\Omega)$ and $H^{s,p}(\mathbb{R}^n)$ defined as the closure of $C_c^\infty(\mathbb{R}^n)$ with their respective norms, the result follows.

The inclusion of Proposition 3.5 is one of the main motivations for the definition of our space, since it roughly suggests that $H^{s,p,\delta}(\Omega)$ consists of Bessel $H^{s,p}$ functions defined only on $\Omega$ and without any integrability requirement at infinity. As a matter of fact, the examples of [7, Sect. 2.1] in the context of Solid Mechanics, together with the inclusion $H^{s,p}(\mathbb{R}^n) \subset H^{s,p,\delta}(\Omega)$ show that $H^{s,p,\delta}(\Omega, \mathbb{R}^n)$ contains deformations exhibiting fracture or cavitation, for some range of $s$ and $p$. One of the advantages of the space $H^{s,p,\delta}(\Omega)$ is that, contrary to $H^{s,p}(\mathbb{R}^n)$, it contains linear functions, such as the identity, which would be relevant in a future linearization process.

4 Nonlocal Fundamental Theorem of Calculus

We start by recalling the following classical representation theorem, which can be seen in [33, Lemma 7.14] or [52, Prop. 4.14].

**Proposition 4.1.** For every $\varphi \in C^\infty_c(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$, we have
\[
\varphi(x) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla \varphi(y) \cdot \frac{x - y}{|x - y|^n} \, dy.
\]

This result may be understood as a fundamental theorem of Calculus, in the sense that we recover a function from its gradient by integration; more precisely, by convolution. A fractional version of it, involving the Riesz fractional gradient is also known [52,59]. This section is devoted to a novel nonlocal version of Proposition 4.1, where a function can be recovered from its nonlocal gradient $D^s_\delta$ through a convolution with a suitable kernel $V^s_\delta$.

Our approach is inspired by the proofs of the fractional fundamental theorem of Calculus previously referred in [52,59]. However, those partly rely on the semigroup properties of Riesz potentials, which our kernels do not enjoy. Therefore, our procedure is much more involved.

To begin with, we show that the kernel in the definition of $D^s_\delta u$ (see formula (6)) can be seen, in a certain sense, as the gradient of a certain function. In order to do so, we introduce the following kernels.

**Definition 4.1.** Define
\[
\tilde{q}_\delta : [0, \infty) \to \mathbb{R}, \quad q_\delta : \mathbb{R}^n \to \mathbb{R} \quad \text{and} \quad Q_\delta^s : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}
\]
as
\[
\tilde{q}_\delta(t) = (n - 1 + s)t^{n-1+s} \int_0^t \frac{\tilde{w}_\delta(r)}{r^{n+s}} \, dr, \quad q_\delta(x) = \tilde{q}_\delta(|x|) \quad \text{and} \quad Q_\delta^s(x) = \frac{1}{\gamma(1-s)|x|^{n-1+s}} q_\delta(x).
\]

In the following result, $a_0$ and $b_0$ are the constants from the definition of $w_\delta$. 

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Lemma 4.2. a) \( \bar{q}_\delta \) is \( C^\infty((0, \infty)) \), \( C^{n-1}([0, \infty)) \), \( \text{supp} \bar{q}_\delta \subset [0, \delta) \) and there exists a constant \( z_0 \in \mathbb{R} \) such that for every \( t \in [0, b_0\delta] \),
\[
\bar{q}_\delta(t) = a_0 + z_0 t^{n-1+s}.
\]
b) \( Q^s_\delta \) is radially decreasing, \( Q^s_\delta \in L^1(\mathbb{R}^n) \), \( \text{supp} Q^s_\delta \subset B(0, \delta) \) and
\[
\frac{-1}{n-1+s} \nabla Q^s_\delta(x) = \frac{\rho_\delta(x)}{|x|} \frac{x}{|x|}.
\]

Proof. We start with \([\text{4}]\). The function \( \bar{q}_\delta \) is clearly \( C^\infty \) in \((0, \infty)\) as a product of \( C^\infty \) functions in \((0, \infty)\). We have that
\[
\left( \frac{\bar{q}_\delta(t)}{t^{n-1+s}} \right)' = -(n-1+s) \frac{\bar{w}_\delta(t)}{t^{n+s}}, \quad t > 0.
\]
Since \( \bar{q}_\delta(\delta) = 0 \) and \( \text{supp} \bar{w}_\delta \subset [0, \delta) \), we obtain that \( \text{supp} \bar{q}_\delta \subset [0, \delta) \). Now, for \( 0 < t < \delta b_0 \) we have that
\[
\left( \frac{\bar{q}_\delta(t)}{t^{n-1+s}} \right)' = -(n-1+s) \frac{a_0}{t^{n+s}} = \left( \frac{a_0}{t^{n-1+s}} \right)'
\]
so the existence of \( z_0 \) in the statement follows. In particular, \( \bar{q}_\delta \) is \( C^{n-1}([0, \infty)) \).

We now show \([\text{4}]\). We get immediately from \([\text{11}]\) that \( Q^s_\delta \) is radially decreasing and
\[
\nabla Q^s_\delta(x) = -\frac{n-1+s}{\gamma(1-s)} \frac{\bar{w}_\delta(|x|)}{|x|^{n+s}} \frac{x}{|x|}, \quad x \in \mathbb{R}^n \setminus \{0\},
\]
so \([\text{10}]\) holds. As \( \text{supp} \bar{q}_\delta \subset [0, \delta) \) we obtain that \( \text{supp} Q^s_\delta \subset B(0, \delta) \). Consequently, \( Q^s_\delta \in L^1(\mathbb{R}^n) \) because of the boundedness of \( \bar{q}_\delta \).

In the following proposition we write the nonlocal gradient as a convolution of the classical one with the kernel \( Q^s_\delta \). Its fractional version can be found in \([\text{52}, \text{Lemma 15.9}]\) and \([\text{59}, \text{Th. 1.2}]\). In fact, our proof is inspired by that of \([\text{52}]\): its idea is based on an integration by parts starting with Definition \([3.1]\) and \([\text{10}]\).

**Proposition 4.3.** For every \( u \in C^\infty_c(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \) we have
\[
D^s_\delta u(x) = \int_{\mathbb{R}^n} \nabla u(y) Q^s_\delta(x-y)dy
\]
and \( D^s_\delta u \in C^\infty(\mathbb{R}^n) \).

**Proof.** Let \( K \) be a ball containing \( \text{supp} u \) and let \( K_\delta = K + B(0, \delta) \). If \( x \in K_\delta \) then both terms of \([\text{12}]\) are zero since \( \text{supp} D^s_\delta u \subset \text{supp} u + B(0, \delta) \subset K_\delta \) and \( \text{supp} Q^s_\delta \subset B(0, \delta) \). Thus, we consider \( x \in K_\delta \), \( e \in \mathbb{R}^n \) with \( |e| = 1 \) and the vector field
\[
\beta : K_\delta \setminus \{x\} \to \mathbb{R}^n
\]
defined by
\[
\beta(y) = (u(x) - u(y)) Q^s_\delta(x-y) e.
\]
Let \( \varepsilon > 0 \) be such that \( \bar{B}(x, \varepsilon) \subset K_\delta \). From Lemma \([\text{4.2}]\) we have that
\[
\text{div} \beta(y) = (n-1+s) \frac{u(x) - u(y)}{|x-y|} \rho_\delta(x-y) \frac{x-y}{|x-y|} \cdot e - Q^s_\delta(x-y) \nabla u(y) \cdot e, \quad y \in K_\delta \setminus B(x, \varepsilon)
\]

and notice that \( \text{div} \beta \) is integrable in \( K_\delta \setminus B(x, \varepsilon) \). Applying the divergence theorem we obtain

\[
\int_{K_\delta \setminus B(x, \varepsilon)} \text{div} \beta(y) \, dy = \int_{\partial K_\delta} \beta(y) \cdot \nu_y \, d\mathcal{H}^{n-1}(y) + \int_{\partial B(x, \varepsilon)} \beta(y) \cdot \frac{x - y}{|x - y|} \, d\mathcal{H}^{n-1}(y),
\]

where \( \nu_y \) is the outer normal vector to \( K_\delta \). Now we show that \( \beta(y) = 0 \) for all \( y \in \partial K_\delta \). Indeed, if \( x \in K_\delta \setminus K \) then \( u(x) = u(y) = 0 \) for all \( y \in \partial K_\delta \), whereas if \( x \in K \), then \( Q^s_\beta(x - y) = 0 \) for every \( y \in \partial K_\delta \). Thus,

\[
\int_{K_\delta \setminus B(x, \varepsilon)} \text{div} \beta(y) \, dy = \int_{\partial B(x, \varepsilon)} \beta(y) \cdot \frac{x - y}{|x - y|} \, d\mathcal{H}^{n-1}(y).
\]

We estimate the integrand in the right-hand side. As \( u \) is Lipschitz, using the mean value theorem and the definition of \( Q^s_\beta \) (see Definition 4.1 and Lemma 4.2) we find that, for all \( y \in \partial B(x, \varepsilon) \),

\[
\left| \beta(y) \cdot \frac{x - y}{|x - y|} \right| \leq |\beta(y)| \leq \|\nabla u\|_\infty |x - y| |Q^s_\beta(x - y)| \leq \|\nabla u\|_\infty \frac{c}{|x - y|^{n+s-2}} = \|\nabla u\|_\infty \frac{c}{\varepsilon^{n+s-2}}
\]

for some constant \( c > 0 \), so

\[
\left| \int_{\partial B(x, \varepsilon)} \beta(y) \cdot \frac{x - y}{|x - y|} \, d\mathcal{H}^{n-1}(y) \right| \leq \|\nabla u\|_\infty c \sigma_{n-1} \varepsilon^{1-s},
\]

which goes to 0 when \( \varepsilon \) goes to 0. Therefore,

\[
\lim_{\varepsilon \to 0} \int_{K_\delta \setminus B(x, \varepsilon)} \text{div} \beta(y) \, dy = 0.
\]

As a result, using (13) we obtain that

\[
\lim_{\varepsilon \to 0} \int_{K_\delta \setminus B(x, \varepsilon)} (n - 1 + s) \frac{u(x) - u(y)}{|x - y|} \rho_\beta(x - y) \frac{x - y}{|x - y|} \cdot e \, dy = \lim_{\varepsilon \to 0} \int_{K_\delta \setminus B(x, \varepsilon)} Q^s_\beta(x - y) \nabla u(y) \cdot e \, dy,
\]

provided that both limits exists, which is actually true as both integrals are absolutely convergent in \( K_\delta \); see the comment after Definition 3.1 for the left integral and notice that \( Q^s_\beta \in L^1(\mathbb{R}^n) \) (see Lemma 4.2) for the right integral. Thus,

\[
\int_{K_\delta} (n - 1 + s) \frac{u(x) - u(y)}{|x - y|} \rho_\beta(x - y) \frac{x - y}{|x - y|} \cdot e \, dy = \int_{K_\delta} Q^s_\beta(x - y) \nabla u(y) \cdot e \, dy.
\]

As this is true for every \( e \in \mathbb{R}^n \) with \( |e| = 1 \), we conclude that

\[
\int_{K_\delta} (n - 1 + s) \frac{u(x) - u(y)}{|x - y|} \rho_\beta(x - y) \frac{x - y}{|x - y|} \, dy = \int_{K_\delta} \nabla u(y) Q^s_\beta(x - y) \, dy,
\]

and formula (12) is proved.

We have thus shown that \( D^s_\beta u = \nabla u * Q^s_\beta \). As \( Q^s_\beta \in L^1(\mathbb{R}^n) \), \( \nabla u \in C^\infty(\mathbb{R}^n) \) and both have compact support, we conclude that \( D^s_\beta u \in C^\infty_c(\mathbb{R}^n) \). \( \square \)

Proposition 4.3 shows that for \( u \in C^\infty_c(\mathbb{R}^n) \), its nonlocal gradient \( D^s_\beta u \) is also in \( u \in C^\infty_c(\mathbb{R}^n) \), unlike its fractional gradient \( D^s u \), which is \( C^\infty \) (see [68 Prop. 5.2] or [42]) but not of compact support (an easy counterexample can be found in [38 Sect. 2.2]). The reason for this difference is that, while Proposition 4.3 shows that \( D^s_\beta u = \nabla u * Q^s_\beta \) with \( Q^s_\beta \) of compact support, the fractional analogue states that \( D^s u = \nabla u * I_{1-s} \), with \( I_{1-s} \) the Riesz potential (see (17) below for the definition), which is not of compact support.

We continue by restating the main result of Section 5 (Theorem 5.9).
Proposition 4.4. There exists a function $V_\delta^s \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ such that
\[ \int_{\mathbb{R}^n} V_\delta^s(z) Q_\delta^s(y - z) \, dz = \frac{1}{\sigma_{n-1} |y|^n}, \quad y \in \mathbb{R}^n \setminus \{0\}. \] (14)

Moreover, $V_\delta^s \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, and for every $R > 0$ there exists $M > 0$ such that
\[ |V_\delta^s(x)| \leq \frac{M}{|x|^{n-\sigma}}, \quad x \in B(0, R) \setminus \{0\}. \]

We will study further properties of $V_\delta^s$ in Theorem 5.9. The proof of Proposition 4.4 is long and comprises the whole of Section 5. With this, the main result of this section (a nonlocal version of the fundamental theorem of Calculus) reads as follows. Its proof follows the lines from [52, Prop. 15.8], whereas the main differences are gathered in Proposition 4.4.

Theorem 4.5. Let $V_\delta^s$ be the function of Proposition 4.4. Then, for every $u \in C^\infty_c(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,
\[ u(x) = \int_{\mathbb{R}^n} D_\delta^s u(y) \cdot V_\delta^s(x - y) \, dy. \] (15)

Proof. Let $F(x)$ denote the right hand side of (15). This integral is absolutely convergent since $V_\delta^s \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ (Proposition 4.4) and $D_\delta^s u$ is bounded with compact support (Proposition 4.3). In fact, Proposition 4.3 allows us to write the equality
\[ F(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla u(z) Q_\delta^s(y - z) \cdot V_\delta^s(x - y) \, dz \, dy. \]

Next we make the changes of variables $\eta = x - y$ and $\xi = x - z$ to obtain
\[ F(x) = \int_{\mathbb{R}^n} \nabla u(x - \xi) \cdot \int_{\mathbb{R}^n} V_\delta^s(\eta) Q_\delta^s(\xi - \eta) \, d\eta \, d\xi. \]

By Proposition 4.4
\[ \int_{\mathbb{R}^n} V_\delta^s(\eta) Q_\delta^s(\xi - \eta) \, d\eta = \frac{1}{\sigma_{n-1} |\xi|^n}. \]

Thus, thanks to Proposition 4.1
\[ F(x) = \int_{\mathbb{R}^n} \nabla u(x - \xi) \cdot \frac{\xi}{\sigma_{n-1} |\xi|^n} \, d\xi = u(x) \]

and the proof is complete. \qed

5 Existence of $V_\delta^s$

This section is devoted to the proof of Proposition 4.4 as well as to the derivation of further properties of $V_\delta^s$. The idea of the proof is to convert equation (14) into
\[ \hat{V}_\delta^s(\xi) \hat{Q}_\delta^s(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi \xi|} \]
through Fourier transform (see Lemma B.1 for the Fourier transform of the right-hand side of (14)). Thus, the candidate for $V_\delta^s$ should satisfy
\[ \hat{V}_\delta^s(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi \xi|} \frac{1}{\hat{Q}_\delta^s(\xi)}. \]
In the first half of the section, we show that \( \hat{Q}_\delta^s \) is positive and, consequently, the formula above is well defined. Taking inverse Fourier transform, we then conclude that \( V_\delta^s \) is at least a tempered distribution. In the second half, we see that \( V_\delta^s \) is actually a function.

As seen in the introduction, it is illustrative to compare formula (15) with the fractional fundamental theorem of Calculus in \( H^{s,p} \) (see [16, Th. 3.11], [32, Prop. 15.8] or [59, Th. 1.12]):

\[
\hat{u}(x) = c_{n-s} \int_{\mathbb{R}^n} D^s u(y) \cdot \frac{x - y}{|x - y|^{n+s+1}} \, dy.
\]

(16)

Here \( D^s u \) is Riesz’ s-fractional gradient. Thus, \( D^2_\delta u \) and \( V_\delta^s \) in our context play the role of \( D^s u \) and \( \frac{x}{|x|^{n+s+1}} \) in \( H^{s,p} \), respectively. In fact, in the analysis in \( H^{s,p} \) an essential part is performed by the Riesz potential. We recall (see [59,64]) that given \( 0 < s < n \), the Riesz potential \( I_s : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) and its Fourier transform are

\[
I_s(x) = \frac{1}{\gamma(s)} \frac{1}{|x|^{n-s}} \quad \text{and} \quad \hat{I}_s(\xi) = |2\pi \xi|^{-s},
\]

(17)

where \( \gamma(s) \) is defined in [5]. A key study in a great part of this and the following section will be the comparison, first, of \( \hat{I}_s \) with \( \hat{V}_\delta^s \), and, then, of \( I_s \) with \( V_\delta^s \). In fact, in the comment after Proposition 4.3 it was hinted that here \( Q_\delta^s \) plays the role of \( I_{1-s} \) in the fractional case, so it becomes natural that in this section we also compare \( \hat{Q}_\delta^s \) with \( \hat{I}_{1-s} \).

5.1 Positivity of \( \hat{Q}_\delta^s \) and existence of \( V_\delta^s \) as a distribution

We start with an analysis of the Fourier transform of the function \( q_\delta \) of Definition 4.1

**Lemma 5.1.** The function \( \hat{q}_\delta \) is analytic, \( C_0(\mathbb{R}^n) \) and \( L^1(\mathbb{R}^n) \).

**Proof.** Given that \( \hat{q}_\delta \in C_c^{m-1}(0, \infty) \) (see Lemma 4.2), we have that \( q_\delta \in L^1(\mathbb{R}^n) \) and has compact support. Therefore, by known facts in Fourier analysis, \( \hat{q}_\delta \) belongs to \( C_0(\mathbb{R}^n) \) and is analytic.

It remains to show that \( \hat{q}_\delta \in L^1(\mathbb{R}^n) \), and for this we will previously check that \( q_\delta \in W^{2n-1,1}(\mathbb{R}^n) \). Indeed, as a consequence of Lemma 4.2 for \( 1 \leq j \leq 2n - 1 \) there exists \( z_j \in \mathbb{R} \) such that

\[
\bar{q}_\delta(t) = z_j t^{n-1+s-j}, \quad t \in (0, b_0 \delta),
\]

where the superindex \( j \) indicates the \( j \)-th derivative. On the other hand, \( \bar{q}_\delta(t) \) is bounded in \([b_0 \delta, \delta]\) and vanishes in \([0, \infty)\). This implies that the a.e. and weak derivative of order \( j \) of \( q_\delta \) coincide and they satisfy, for some constant \( C_j > 0 \),

\[
|D^j q_\delta(x)| \leq C_j |x|^{n-1+s-j}, \quad x \in B(0, b_0 \delta) \setminus \{0\},
\]

while \( |D^j q_\delta| \) is bounded in \( B(0, \delta) \setminus B(0, b_0 \delta) \) and vanishes in \( B(0, \delta) \). This implies that \( q_\delta \in W^{2n-1,1}(\mathbb{R}^n) \). In particular, the Fourier transform of any partial derivative of order \( 2n - 1 \) of \( q_\delta \) is bounded, so there exists \( C > 0 \) such that for any multiindex \( \alpha \) of order \( 2n - 1 \) we have

\[
|(2\pi i \xi)^\alpha \hat{q}_\delta(\xi)| = |\hat{\partial^\alpha q}_\delta(\xi)| \leq C,
\]

and, hence,

\[
|\hat{q}_\delta(\xi)| \leq \frac{C}{2\pi |\xi|^{2n-1}}.
\]

This decay at infinity of \( \hat{q}_\delta \), together with the fact that \( \hat{q}_\delta \) is continuous, implies that \( \hat{q}_\delta \in L^1(\mathbb{R}^n) \) for \( n \geq 2 \).
In the rest of the proof, we assume that \( n = 1 \). In this case, \( q_δ \) is the even extension of \( q_δ \). As shown before, there exists \( z_1 \in \mathbb{R} \) such that \( q_δ(x) = \frac{|x|^{1-\gamma}}{\gamma(s)} \) for \( x \in B(0,b_0) \). If \( z_1 = 0 \) then \( q_δ \) is \( C_\mathcal{C}^\infty(\mathbb{R}^n) \), so \( q_δ \) is in \( \mathcal{S} \) and, in particular, in \( L^1(\mathbb{R}^n) \). We assume from now on that \( z_1 \neq 0 \).

Consider a \( \varphi \in C_\mathcal{C}^\infty(\mathbb{R}) \) with \( \varphi|_{B(0,\frac{1}{4})} = 1 \) and \( \varphi|_{B(0,\frac{1}{2})} = 0 \). Then,

\[
|2\pi\xi|^{-s} - \frac{1}{z_1\gamma(s)}\hat{q}_δ(\xi) = \mathcal{F}\left(\frac{1}{\gamma(s)|x|^{1-s}} - \frac{1}{z_1\gamma(s)}q_δ(x)\right) = \mathcal{F}\left(\frac{\varphi}{\gamma(s)|x|^{1-s}} - \frac{1}{z_1\gamma(s)}q_δ(x)\right) + \mathcal{F}\left(\frac{1 - \varphi}{\gamma(s)|x|^{1-s}}\right).
\]

Looking at the expression of \( q_δ \), we notice that the functions \( \frac{\varphi}{\gamma(s)|x|^{1-s}} \) and \( \frac{1}{z_1\gamma(s)}q_δ(x) \) coincide in \( B(0, \min\{b_0\delta, \frac{1}{4}\}) \), and both have compact support. Therefore, its difference is a smooth function of compact support. In particular, it is in the Schwartz space, as well as its Fourier transform:

\[
\mathcal{F}\left(\frac{\varphi}{\gamma(s)|x|^{1-s}} - \frac{1}{z_1\gamma(s)}q_δ(x)\right) \in \mathcal{S}.
\]

On the other hand, the function \( \mathcal{F}\left(\frac{1 - \varphi}{\gamma(s)|x|^{1-s}}\right) \) is treated in [34, Ex. 2.4.9], and it is concluded that its decay at infinity is faster than any negative power of \( |\xi| \). Consequently, the decay at infinity of

\[
|2\pi\xi|^{-s} - \frac{1}{z_1\gamma(s)}\hat{q}_δ(\xi)
\]

is also faster than any negative power of \( |\xi| \). In particular, there exists \( C_1' > 0 \) such that

\[
\frac{2\pi\xi}{z_1\gamma(s)}\hat{q}_δ(\xi) \leq \frac{1}{z_1\gamma(s)}\hat{q}_δ(\xi) \leq C_1' \frac{1}{|2\pi\xi|^{1-s}},
\]

which allows us to conclude that \( \hat{q}_δ \in L^1(\mathbb{R}) \). \( \Box \)

In the following result we obtain relevant properties about \( Q_δ^s \). Recall that \( a_0 \) is the constant from the definition of \( w_δ \).

**Proposition 5.2.**

a) \( \hat{Q}_δ^s \) is analytic, bounded, radial, and \( \hat{Q}_δ^s(0) = \|Q_δ^s\|_{L^1(\mathbb{R}^n)} \).

b) \( \partial^\alpha\hat{Q}_δ^s \) is bounded for every multiindex \( \alpha \).

c) \( \lim_{|\xi|\to\infty} \frac{\hat{Q}_δ^s(\xi)}{|2\pi\xi|^{-(1-s)}} = a_0 \).

**Proof.** The proof of part a) comes directly from known facts in Fourier analysis. Indeed, as \( Q_δ^s \in L^1(\mathbb{R}^n) \) we have \( \hat{Q}_δ^s \in L^\infty(\mathbb{R}^n) \). As \( Q_δ^s \) has compact support, \( \hat{Q}_δ^s \) is analytic. Since \( Q_δ^s \) is radial, so is \( \hat{Q}_δ^s \). Finally, the equality \( \hat{Q}_δ^s(0) = \|Q_δ^s\|_{L^1(\mathbb{R}^n)} \) is a straightforward consequence of the formula of the Fourier transform.

Regarding b), we have that \( \partial^\alpha\hat{Q}_δ^s = \mathcal{F}((-2\pi i)^\alpha Q_δ^s) \). Thus, \( \partial^\alpha\hat{Q}_δ^s \) is the Fourier transform of an \( L^1(\mathbb{R}^n) \) function (since \( Q_δ^s \in L^1(\mathbb{R}^n) \) has compact support). Therefore, \( \partial^\alpha\hat{Q}_δ^s \) is bounded.

In order to show c), we apply the Fourier transform to the expression \( Q_δ^s = I_{1-s} q_δ \) (see Definition 4.1). Since the Riesz potential \( I_{1-s} \) is not an \( L^1(\mathbb{R}^n) \) function and \( q_δ \) is not Schwartz, the Fourier transform is, in principle, in the sense of tempered distributions. To wit, as \( I_{1-s} \in L^1(B(0,1)) + \)
\( L^\infty(B(0, 1)^c) \), both factors \( I_{1-s} \) and \( q_\delta \) can be seen as distributions; in addition, \( q_\delta \) has compact support, so we can use Lemma [B.2] and obtain that

\[
\hat{Q}_\delta^s(\xi) = |2\pi \xi|^{-(1-s)} \ast \hat{q}_\delta(\xi) 
\]

in the sense of distributions. Actually, by Young’s inequality for the convolution we have that

\[
\| \hat{I}_{1-s} \ast \hat{q}_\delta \|_{L^\infty(\mathbb{R}^n)} \leq \| \hat{I}_{1-s} \|_{L^1(B(0,1))} \| \hat{q}_\delta \|_{L^\infty(\mathbb{R}^n)} + \| \hat{I}_{1-s} \|_{L^\infty(B(0,1)^c)} \| \hat{q}_\delta \|_{L^1(\mathbb{R}^n)}.
\]

Therefore, the integral defining \((\hat{I}_{1-s} \ast \hat{q}_\delta)(\xi)\) is absolutely convergent for a.e. \( \xi \in \mathbb{R}^n \). Consequently, equality (18) holds a.e.

Then, we consider \( \xi = \lambda \xi_0 \) with \( \xi_0 \in B(0, 1)^c \) fixed and \( \lambda > 0 \). Using the change of variables \( x = \lambda x' \) we have

\[
\hat{Q}_\delta^s(\lambda \xi_0) = \int_{\mathbb{R}^n} |2\pi(x - \lambda \xi_0)|^{-(1-s)} \hat{q}_\delta(x) \, dx = \int_{\mathbb{R}^n} |2\pi(\lambda x - \lambda \xi_0)|^{-(1-s)} \hat{q}_\delta(\lambda x) \lambda^n \, dx
\]

\[
= \lambda^{-(1-s)} \int_{\mathbb{R}^n} |2\pi(\xi_0 - x)|^{-(1-s)} \hat{q}_\delta(\lambda x) \lambda^n \, dx.
\]

As the function \( \xi \mapsto \frac{\hat{Q}_\delta^s(\xi)}{|2\pi \xi|^{-(1-s)}} \) is radial, in order for (c) to hold, it is enough that

\[
\lim_{\lambda \to \infty} \frac{\hat{Q}_\delta^s(\lambda \xi_0)}{|2\pi \lambda \xi_0|^{-(1-s)}} = a_0,
\]

equivalently,

\[
\lim_{\lambda \to \infty} \int_{\mathbb{R}^n} |2\pi(\xi_0 - x)|^{-(1-s)} \hat{q}_\delta(\lambda x) \lambda^n \, dx = a_0.
\]

Define now \( g_\lambda(x) = \frac{1}{a_0} \hat{q}_\delta(\lambda x) \lambda^n \) and \( f(\xi) = |2\pi \xi|^{-(1-s)} \). The limit above is equivalent to

\[
\lim_{\lambda \to \infty} \int_{\mathbb{R}^n} f(\xi_0 - x) g_\lambda(x) \, dx = 1,
\]

and, in turn, equivalent to

\[
\lim_{\lambda \to \infty} \int_{\mathbb{R}^n} f(\xi_0 - x) g_\lambda(x) \, dx = f(\xi_0),
\]

in other words,

\[
\lim_{\lambda \to \infty} f \ast g_\lambda(\xi_0) = f(\xi_0).
\]

We recall from Lemma [B.2] that \( a_0 = \hat{q}_\delta(0) = q_\delta(0) = \int_{\mathbb{R}^n} \hat{q}_\delta \); note that \( \hat{q}_\delta \in L^1(\mathbb{R}^n) \) thanks to Lemma [5.1]. Thus, \( \int_{\mathbb{R}^n} g_\lambda = 1 \) for each \( \lambda > 0 \). Then, by construction, \( g_\lambda \) is a mollifier family tending to the Dirac delta at 0, when \( \lambda \to \infty \) in the sense of distributions. Thus,

\[
|f \ast g_\lambda(\xi_0) - f(\xi_0)| = \int_{\mathbb{R}^n} |f(\xi_0 - x) - f(\xi_0)| g_\lambda(x) \, dx \leq \int_{\mathbb{R}^n} |f(\xi_0 - x) - f(\xi_0)| |g_\lambda(x)| \, dx.
\]

Let \( \varepsilon > 0 \). Since \( f \) is uniformly continuous in \( B(0, 1/2)^c \), there exists \( 0 < r < \frac{1}{2} \) such that

\[
|f(\xi_0 - x) - f(\xi_0)| < \varepsilon, \quad \text{for all } \xi_0 \in B(0, 1)^c \text{ and } x \in B(0, r).
\]
Therefore, as \( f \in L^\infty(B(0,1/2)) \),
\[
|f * g_\lambda(\xi_0) - f(\xi_0)| \leq \int_{B(0,r)} |f(\xi_0 - x) - f(\xi_0)| \, |g_\lambda(x)| \, dx + \int_{B(0,r)^c} |f(\xi_0 - x) - f(\xi_0)| \, |g_\lambda(x)| \, dx
\]
\[
\leq \varepsilon \int_{B(0,r)} |g_\lambda(x)| \, dx + 2\|f\|_{L^\infty(B(0,1/2)^c)} \int_{B(0,r)^c} |g_\lambda(x)| \, dx.
\]
Finally, we use that \( \lim_{\lambda \to \infty} \int_{B(0,r)^c} |g_\lambda(x)| \, dx = 0 \). As a result, there exists \( \lambda_0 > 0 \) such that for every \( \lambda > \lambda_0 \), the inequality \( \int_{B(0,r)^c} |g_\lambda(x)| \, dx < \varepsilon \) holds. Consequently,
\[
|f * g_\lambda(\xi_0) - f(\xi_0)| \leq (\|g_\lambda\|_{L^1(\mathbb{R}^n)} + 2\|f\|_{L^\infty(B(0,1/2)^c)}) \varepsilon.
\]
As \( \|g_\lambda\|_{L^1(\mathbb{R}^n)} = \|g_1\|_{L^1(\mathbb{R}^n)} \), this proves convergence (19), and, hence, statement (3).

The following calculation is useful for showing the positivity of \( Q_\delta^* \).

**Lemma 5.3.** For all \( j \in \{1, \ldots, n\} \) and \( r > 0 \),
\[
\int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi rx_j) \, dx > 0.
\]

**Proof.** The integral is absolutely convergent since
\[
\left| \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi rx_j) \right| \leq \frac{1}{|x|^{n-1+s}} w_\delta(x) 2\pi r
\]
and \( w_\delta \) as compact support. By a change of variables we have
\[
\int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi rx_j) \, dx = r^s \int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta \left( \frac{x}{r} \right) \sin(2\pi x_j) \, dx.
\]
Recall that \( \tilde{w}_\delta \) is the radial representation of \( w_\delta \). By symmetry, the co-area formula and Fubini’s theorem we have
\[
\int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta \left( \frac{x}{r} \right) \sin(2\pi x_j) \, dx = 2 \int_{|x_j| > 0} \frac{x_j}{|x|^{n+s+1}} w_\delta \left( \frac{x}{r} \right) \sin(2\pi x_j) \, dx
\]
\[
= 2 \int_0^\infty \tilde{w}_\delta \left( \frac{t}{r} \right) t^{n-1} \int_{S_j^+} t^j \sin(2\pi t z_j) \, d\mathcal{H}^{n-1}(z) \, dt
\]
\[
= 2 \int_j^\infty \tilde{w}_\delta \left( \frac{z}{t^{s+1}} \right) \sin(2\pi t z_j) \, d\mathcal{H}^{n-1}(z),
\]
where \( S_j^+ = \{ z \in \mathbb{R}^n : |z| = 1, z_j > 0 \} \). Finally, let us show that
\[
\int_0^\infty \tilde{w}_\delta \left( \frac{t}{t^{s+1}} \right) \sin(2\pi t z_j) \, dt > 0
\]
for each \( z \in S_j^+ \). For this, consider the function \( f(t) = \frac{\tilde{w}_\delta \left( \frac{z}{t^{s+1}} \right)}{t^{s+1}} \) and express
\[
\int_0^\infty \frac{\tilde{w}_\delta \left( \frac{z}{t^{s+1}} \right)}{t^{s+1}} \sin(2\pi t z_j) \, dt = \sum_{k=0}^{\infty} \int_{\frac{k+1}{z_j}}^{\frac{k+2}{z_j}} f(t) \sin(2\pi t z_j) \, dt.
\]
We have that each term in the sum is positive; indeed, by splitting the integral in two through point \( \frac{k+1}{2z_j} \) and making the change of variables \( t = t' + \frac{1}{2z_j} \) in one of them, it is easy to obtain
\[
\int_{\frac{k+1}{2z_j}}^{\frac{k+1}{z_j}} f(t) \sin(2\pi tz_j) \, dt = \int_{\frac{k+1}{2z_j}}^{\frac{k+1}{z_j}} \left[ f(t) - f(t + \frac{1}{2z_j}) \right] \sin(2\pi tz_j) \, dt \geq 0,
\]

since \( \sin(2\pi tz_j) > 0 \) and \( f \) is decreasing (as so is \( \bar{w}_\delta \)). In fact,
\[
\int_{0}^{\frac{1}{2z_j}} \left[ f(t) - f(t + \frac{1}{2z_j}) \right] \sin(2\pi tz_j) \, dt > 0,
\]
as \( f \) is strictly decreasing in \([0, r b_0 \delta]\), so (21) holds, which concludes the proof.

As can be seen from the proof above, the assumption that \( \bar{w}_\delta \) is decreasing can be weakened to the following: the function \( f \) of the proof is decreasing in \( t \) for all \( r > 0 \). This is equivalent to \( f'(t) \leq 0 \) for all \( t > 0 \) and \( r > 0 \), which in turn, is equivalent to the differential inequality
\[
\bar{w}_\delta'(t) \leq (s + 1) \frac{\bar{w}_\delta(t)}{t}, \quad t \geq b_0 \delta.
\]
Therefore, assumption \([d]\) on \( \bar{w}_\delta \) (see Section 3) could have been replaced with the above inequality, but, for ease of reading, we prefered to state that \( \bar{w}_\delta \) is decreasing.

The following result shows the convergence of the truncations of \( \nabla Q_\delta^s \).

Lemma 5.4. The function \( \nabla Q_\delta^s \) can be identified with the tempered distribution defined component-wise as
\[
\langle \partial_j Q_\delta^s, \varphi \rangle = -c_{n,s} \int_{\{x_j > 0\}} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) (\varphi(x) - \varphi(-x)) \, dx, \quad j \in \{1, \ldots, n\}
\]
and we have the convergence
\[
\nabla Q_\delta^s \chi_{B(0,\varepsilon)^c} \to \nabla Q_\delta^s \quad \text{in } S' \quad \text{as } \varepsilon \to 0.
\]

Proof. We recall from Lemma 4.2 that
\[
\nabla Q_\delta^s(x) = -(n - 1 + s) \partial_\delta(x) \frac{x}{|x|} = -c_{n,s} \frac{x}{|x|^{n+s}} w_\delta(x).
\]
Thus, \( \nabla Q_\delta^s \chi_{B(0,\varepsilon)^c} \) is in \( L^1(\mathbb{R}^n) \) for each \( \varepsilon > 0 \), so it can be identified with a tempered distribution. Let \( j \in \{1, \ldots, n\} \); we shall prove the desired convergence for the \( j \)-th component of \( \nabla Q_\delta^s \).

Using the notation \( B_j^\varepsilon(0,\varepsilon)^c = \{x \in B(0,\varepsilon)^c : \pm x_j > 0\} \), we have
\[
\int_{B(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \varphi(x) \, dx = \int_{B_j^\varepsilon(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \varphi(x) \, dx + \int_{B_j^\varepsilon(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \varphi(x) \, dx
\]
\[
= \int_{B_j^\varepsilon(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) (\varphi(x) - \varphi(-x)) \, dx.
\]

By the mean value theorem,
\[
\left| \frac{x_j}{|x|^{n+s+1}} w_\delta(x) (\varphi(x) - \varphi(-x)) \chi_{B_j^\varepsilon(0,\varepsilon)^c}(x) \right| \leq 2 \frac{\|\nabla \varphi\|_\infty \|w_\delta\|_\infty}{|x|^{n-1+s}} \chi_{B(0,\delta)}(x).
\]
This shows that formula (22) defines a tempered distribution; moreover, by dominated convergence we obtain that
\[ \int_{B_j^+(0,e)}^* \frac{x_j}{|x|^{n+s+1}} w_\delta(x)(\varphi(x) - \varphi(-x)) \, dx \to \int_{\{x_j > 0\}} \frac{x_j}{|x|^{n+s+1}} w_\delta(x)(\varphi(x) - \varphi(-x)) \, dx \]
as \( \varepsilon \to 0 \). This proves the desired convergence. \( \square \)

We now show the positivity of \( \hat{Q}_\delta^s \).

**Proposition 5.5.** \( \hat{Q}_\delta^s(\xi) > 0 \) for all \( \xi \in \mathbb{R}^n \).

**Proof.** Since \( \hat{Q}_\delta^s(0) = \|Q_\delta^s\|_{L^1(\mathbb{R}^n)} > 0 \), we have to show that \( \hat{Q}_\delta^s(\xi) > 0 \) for every \( \xi \in \mathbb{R}^n \setminus \{0\} \). For this, we fix any \( j \in \{1, \ldots, n\} \) and claim that, despite \( \frac{\partial Q_\delta^s}{\partial x_j} \notin L^1(\mathbb{R}^n) \),
\[ \frac{\partial Q_\delta^s}{\partial x_j}(\xi) = \frac{(n - 1 + s)}{\gamma(1 - s)} i \int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi \xi \cdot x) \, dx. \] (23)
This is shown at the end of the proof. Assuming the validity of (23), by Lemma 5.3 we obtain
\[ \frac{1}{i} \frac{\partial Q_\delta^s}{\partial x_j}(\xi_je_j) > 0, \quad \xi_j > 0. \]

Now, the formula
\[ 2\pi i \xi_j \hat{Q}_\delta^s(\xi_je_j) = \frac{\partial Q_\delta^s}{\partial x_j}(\xi_je_j) \]
holds in the sense of tempered distributions. Since both terms are actually functions, the equality holds as functions for almost every point. Moreover, since both functions are continuous, the equality holds everywhere. We then conclude that
\[ \xi_j \hat{Q}_\delta^s(\xi_je_j) > 0, \quad \xi_j > 0. \]

Consequently, since \( \hat{Q}_\delta^s \) is radial, \( \hat{Q}_\delta^s(\xi) > 0 \) for all \( \xi \in \mathbb{R}^n \).

It remains to prove (23). By Lemma 4.2
\[ \frac{\partial Q_\delta^s}{\partial x_j}(x) = -(n - 1 + s) \frac{\rho_\delta(x)}{|x|} x_j. \]
We have \( \frac{\partial Q_\delta^s}{\partial x_j} \chi_{B(0,e)} \in L^1(\mathbb{R}^n) \) for all \( \varepsilon > 0 \), and by Lemma 5.4
\[ \frac{\partial Q_\delta^s}{\partial x_j} \chi_{B(0,e)} \to \frac{\partial Q_\delta^s}{\partial x_j} \text{ in } S' \text{ as } \varepsilon \to 0, \]
so
\[ \mathcal{F} \left( \frac{\partial Q_\delta^s}{\partial x_j} \chi_{B(0,e)} \right) \to \mathcal{F} \left( \frac{\partial Q_\delta^s}{\partial x_j} \right) \text{ in } S' \text{ as } \varepsilon \to 0. \]

We now compute
\[ \mathcal{F} \left( \frac{\partial Q_\delta^s}{\partial x_j} \chi_{B(0,e)} \right)(\xi) = \frac{(n - 1 + s)}{\gamma(1 - s)} i \int_{B(0,e)} \frac{x_j}{|x|^{n+s+1}} w_\delta(x)e^{-2\pi i \xi \cdot x} \, dx \]
\[ = \frac{(n - 1 + s)}{\gamma(1 - s)} i \int_{B(0,e)} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi \xi \cdot x) \, dx, \]
where we have used the odd symmetry. Now, by dominated convergence,
\[
\int_{B(0,\epsilon)} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi \xi \cdot x) \, dx \to \int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi \xi \cdot x) \, dx
\]
because of the same argument as in (20). This proves (23).

With this, we can conclude the existence of \( V_\delta^s \) as a distribution.

**Proposition 5.6.** There exists a tempered distribution \( V_\delta^s \) whose Fourier transform is given by
\[
\hat{V}_\delta^s(\xi) = -\frac{i\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}_\delta^s(\xi)}.
\] (24)

**Proof.** Denote by \( W_\delta^s \) the right-hand side of (24), which is well defined for \( \xi \in \mathbb{R}^n \setminus \{0\} \) since \( \hat{Q}_\delta^s \) is positive (Proposition 5.5). Next, we subtract from \( W_\delta^s \) a multiple of the function (and distribution) \(-\frac{i\xi}{|\xi| \cdot 2\pi} \) of Lemma B.1 (c):
\[
W_\delta^s(\xi) - -\frac{i\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}_\delta^s(\xi)} = -\frac{i\xi}{2\pi|\xi|^2} \left( \frac{1}{\hat{Q}_\delta^s(\xi)} - \frac{1}{\hat{Q}_\delta^s(0)} \right).
\] (25)

This function is in \( L^\infty(B(0,1)^c) \), as a difference of functions in \( L^\infty(B(0,1)^c) \) (see Proposition 5.2). Let us see that it is also in \( L^\infty(B(0,1)) \). By the mean value theorem, there exists \( c > 0 \) such that for all \( \xi \in B(0,1) \),
\[
\left| \frac{1}{\hat{Q}_\delta^s(\xi)} - \frac{1}{\hat{Q}_\delta^s(0)} \right| \leq c|\xi|.
\]
As a result,
\[
\left| W_\delta^s(\xi) - -\frac{i\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}_\delta^s(0)} \right| \leq \frac{c}{2\pi},
\] (26)
so the function in (25) is in \( L^\infty(B(0,1)) \), and, hence, in \( L^\infty(\mathbb{R}^n) \). In particular, this function is a tempered distribution, and, by Lemma B.1 (c), so is \( W_\delta^s \). As the Fourier transform is an isomorphism from \( S' \) into itself, there exists \( V_\delta^s \in S' \) such that (24) holds.

**5.2 Existence of \( V_\delta^s \) as a function**

In this subsection we prove that the distribution \( V_\delta^s \) of Proposition 5.6 is actually a function. First we notice that \( \hat{V}_\delta^s \) does not belong to any space where we can conclude directly that its Fourier transform is a function. The main drawback comes from the fact that the tail of \( \hat{V}_\delta^s \) is not integrable enough. So as to tackle this, we exploit the fact that, at infinity, \( \hat{V}_\delta^s \) behaves like a homogeneous function with a known Fourier transform (namely, like \( \hat{I}_\delta \)). Thus, we adapt the proof of [34, Prop. 2.4.8] (homogeneous function) to the non-homogeneous function \( \hat{V}_\delta^s \).

We first need the following decay estimate for the derivatives of \( \hat{V}_\delta^s \).

**Lemma 5.7.** For every \( \alpha \in \mathbb{N}^n \) there exists \( C_\alpha > 0 \) such that for any \( |\xi| \geq 1 \),
\[
|\partial^\alpha \hat{V}_\delta^s(\xi)| \leq \frac{C_\alpha}{|\xi|^{s(|\alpha|+1)}}.
\]

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Proof. In this proof we use the letter $C$ with some subindices to denote a generic positive constant independent of $\xi$; the relevant dependence is included in the subindices. The value of the constant may vary from line to line.

Express $\hat{V}_s^\delta = -\frac{i}{2\pi} g f$ with

$$g(\xi) = \frac{\xi}{|\xi|}, \quad f = f_1 \circ g_1, \quad f_1(t) = t^{-1}, \quad g_1(\xi) = |\xi| \hat{Q}_s^\delta(\xi).$$

By Leibniz’ formula,

$$\partial^\alpha (gf) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta g \partial^{\alpha - \beta} f.$$

Let $\beta \in \mathbb{N}^n$. By induction, it is easy to see that $\partial^\beta g(\xi)$ can be expressed as

$$P(\xi) \frac{|\xi|^{2|\beta|+1}}{|\xi|^{2|\beta|+1}},$$

for some $\mathbb{R}^n$-valued polynomial $P$, all of which components are of degree $|\beta| + 1$. Therefore,

$$|\partial^\beta g(\xi)| \leq C_\beta |\xi|^{2|\beta|+1}, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \quad (27)$$

We apply Faà di Bruno’s formula for the higher-order derivatives of a composition, and obtain that

$$\partial^\gamma f = \sum_{k=1}^{\infty} f_k^k \circ g_1 G_k$$

where $G_k$ is a linear combination of products of $k$ partial derivatives of $g_1$, the order of which adds up $|\gamma|$. 

We estimate the partial derivatives of $g_1$. We express $g_1 = h \hat{Q}_s^\delta$ with $h(\xi) = |\xi|$. Since $\nabla h = g$, we have, by (27), that

$$|\partial^\beta h(\xi)| \leq \frac{C_\beta}{|\xi|^{2|\beta|+1}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \quad (28)$$

Now we show that

$$|\partial^\beta \hat{Q}_s^\delta(\xi)| \leq \frac{C_\beta}{|\xi|^{2|\beta|}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \quad (29)$$

From Definition 4.1 and Lemma 4.2, we have that $Q_s^\delta$ is an $L^1$ function of compact support, smooth outside the origin, and that in a ball $B$ centred at the origin, one has

$$Q_s^\delta(x) = \lambda_0 + \frac{\lambda_1}{|x|^{n-1+s}}, \quad x \in B \setminus \{0\}$$

for some $\lambda_0, \lambda_1 \in \mathbb{R}$. With this expression it is easy to see that

$$|\partial^\beta (x^\beta Q_s^\delta(x))| = \left| \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^n (x^\beta) \partial^{\beta - \gamma} Q_s^\delta(x) \right| \leq \frac{C_\beta}{|x|^{n-1+s}}, \quad x \in B \setminus \{0\}$$

for some $C_\beta > 0$. Moreover, since $\partial^\beta (x^\beta Q_s^\delta)$ is smooth outside the origin and has compact support, we conclude that it is in $L^1(\mathbb{R}^n)$. Consequently, $\mathcal{F}(\partial^\beta (x^\beta Q_s^\delta))$ is bounded. But

$$\mathcal{F}(\partial^\beta ((-2\pi i)^x Q_s^\delta)) = (2\pi i)^\beta \mathcal{F}((-2\pi i)^x Q_s^\delta) = (2\pi i)^\beta \partial^\beta \hat{Q}_s^\delta(\xi),$$
which shows (29).

Now, by Leibniz’ formula, (28) and (29),

\[ |\partial^\alpha g_1(\xi)| \leq C_\alpha \sum_{\beta \leq \alpha} |\partial^\beta h(\xi)| \left| \partial^{\alpha-\beta} \hat{Q}_\delta^s(\xi) \right| \leq \frac{C_\alpha}{|\xi|^{\alpha-1}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}, \]

for some constant $C_\alpha > 0$. Hence, if we multiply $k$ partial derivatives of $g_1$, the order of which adds up $|\gamma|$, we obtain that

\[ |G_k(\xi)| \leq \frac{C_{\gamma,k}}{|\xi|^{\gamma-k}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}, \]

for some constants $C_{\gamma,k} > 0$. On the other hand, by induction,

\[ |f_1^k(t)| = \frac{C_k}{k^{k+1}}, \quad k \in \mathbb{N}, \ t > 0, \]

for some constants $C_k > 0$, and, hence,

\[ \left| f_1^k \circ g_1(\xi) \right| \leq \frac{C_k}{\left( |\xi| \hat{Q}_\delta^s(\xi) \right)^{k+1}}, \quad k \in \mathbb{N}, \ \xi \in \mathbb{R}^n \setminus \{0\}. \]

From Proposition 5.2 we know that, for $|\xi| \geq 1,$

\[ \left| \frac{1}{\hat{Q}_\delta^s(\xi)} \right| \leq C |\xi|^{1-s}, \]

so

\[ \frac{1}{\left( |\xi| \hat{Q}_\delta^s(\xi) \right)^{k+1}} \leq \frac{C}{|\xi|^{s(k+1)}}, \]

Thus,

\[ |\partial^\gamma f(\xi)| \leq \sum_{k=1}^{|\gamma|} \left| f_1^k \circ g_1(\xi) \right| |G_k(\xi)| \leq C_{\gamma} \sum_{k=1}^{|\gamma|} \frac{1}{|\xi|^{s(k+1) + |\gamma|-k}} \leq \frac{C_{\gamma}}{|\xi|^{s(|\gamma|+1)}}. \]

We conclude that, for $|\xi| \geq 1,$

\[ \left| \partial^\alpha \hat{V}_\delta^s(\xi) \right| \leq C_\alpha \sum_{\beta \leq \alpha} |\partial^\beta g(\xi)| \left| \partial^{\alpha-\beta} f(\xi) \right| \leq C_\alpha \sum_{\beta \leq \alpha} \frac{1}{|\xi|^{s(|\alpha|+1) + |\beta|(1-s)}} \leq \frac{C_\alpha}{|\xi|^{s(|\alpha|+1)}}, \]

as desired.

The decay estimate of Lemma 5.7 is not optimal. In fact, a more refined argument can possibly improve decay (29) and show that the bound

\[ |\partial^\beta \hat{Q}_\delta^s(\xi)| \leq \frac{C_\alpha}{|\xi|^{\beta+1-s}}, \quad \xi \in \mathbb{R}^n \setminus \{0\} \]

holds. With that estimate, an adaptation of the proof of Lemma 5.7 would yield

\[ \left| \partial^\alpha \hat{V}_\delta^s(\xi) \right| \leq \frac{C_\alpha}{|\xi|^{\alpha+s}}, \quad |\xi| \geq 1. \]

Nevertheless, the bound of Lemma 5.7 is enough for our purposes in Theorem 5.9. Before that, we need the following inverse Lipschitz estimate of the function $x_{\xi_{\alpha}^{\beta}}^{\alpha^{\beta}}$. 25
Lemma 5.8. For every $R_1, R_2 > 0$ there exists $m > 0$ such that for all $x \in B(0, R_1) \setminus \{0\}$ and $h \in B(0, R_2) \setminus \{x\}$,

$$m|h| \leq \left| \frac{x}{|x|^{n+1-s}} - \frac{x - h}{|x - h|^{n+1-s}} \right|. \quad (30)$$

Proof. We divide the proof into four cases, according to the position of the points $x$ and $h$. Let us define $G(x) = \frac{x}{|x|^{n+1-s}}$.

Case 1: $2|x| \leq |x - h|$. Taking

$$m \leq \frac{1 - \frac{1}{2n-s}}{R_1^{n-s} R_2}$$

we have

$$|G(x) - G(x - h)| \geq \frac{1}{|x|^{n-s}} - \frac{1}{|x - h|^{n-s}} \geq \left( 1 - \frac{1}{2n-s} \right) \frac{1}{|x|^{n-s}} \geq \frac{1}{R_1^{n-s}} \geq mR_2 \geq m|h|.$$  

Case 2: $G(x) \cdot G(x - h) \leq 0$. Taking

$$m \leq \frac{1}{R_1^{n-s} R_2}$$

we have

$$|G(x) - G(x - h)| = \left( |G(x)|^2 + |G(x - h)|^2 - 2G(x) \cdot G(x - h) \right) \frac{1}{2} \geq |G(x)|$$

$$= \frac{1}{|x|^{n-s}} \geq \frac{1}{R_1^{n-s}} \geq mR_2 \geq m|h|. $$

Case 3: $|x - h| \leq 2|x|$ and

$$\min \left\{ |G(x)|^2, |G(x - h)|^2 \right\} \leq G(x) \cdot G(x - h). \quad (31)$$

We observe that the inverse of $G$ is $G^{-1}(y) = \frac{y}{|y|^{n-s}}$, with derivative

$$DG^{-1}(y) = |y|^{-\frac{n+1}{n-s}} I - \frac{n + 1 - s}{n - s} |y|^{-\frac{n+1}{n-s}} y \otimes y,$$

where $\otimes$ denotes the tensor product, so

$$|DG^{-1}(y)| \leq d_{n,s} |y|^{-\frac{n+1}{n-s}} \quad (32)$$

for some constant $d_{n,s} > 0$. By the mean value theorem,

$$|h| = |G^{-1}(G(x)) - G^{-1}(G(x - h))| \leq \|DG^{-1}\|_{L^\infty([G(x),G(x-h)])} |G(x) - G(x - h)|. \quad (33)$$

Now, using (32),

$$\|DG^{-1}\|_{L^\infty([G(x),G(x-h)])} \leq d_{n,s} \max_{y \in [G(x), G(x-h)]]} |y|^{-\frac{n+1}{n-s}} = d_{n,s} \left( \min_{y \in [G(x), G(x-h)]} |y| \right)^{-\frac{n+1}{n-s}}.$$

Elementary geometry (see Figure 2) shows that

$$\min_{y \in [G(x), G(x-h)]} |y| = \begin{cases} |G(x - h)| & \text{if } (G(x - h) - G(x)) \cdot G(x - h) \leq 0, \\ |G(x)| & \text{if } (G(x - h) - G(x)) \cdot G(x) \geq 0. \end{cases} \quad (34)$$
Assumption \([31]\) asserts that one of the two options of \([34]\) occurs, so

\[
\min_{y \in [G(x), G(x-h)]} |y| \geq \min \{|G(x-h)|, |G(x)|\}
\]

and, hence,

\[
\left( \min_{y \in [G(x), G(x-h)]} |y| \right)^{\frac{n-s+1}{n-s}} \leq \left( \min \{|G(x-h)|, |G(x)|\} \right)^{\frac{n-s+1}{n-s}} = \max \{|x|^{n-s+1}, |x-h|^{n-s+1}\}.
\]

Finally, since \( |x-h| \leq 2|x| \),

\[
\max \{|x|^{n-s+1}, |x-h|^{n-s+1}\} \leq 2^{n-s+1}|x|^{n-s+1} \leq 2^{n-s+1}P_1^{n-s+1}.
\]

Going back to \([33]\), we find that \(|h| \leq 2^{n-s+1}d_{n,s}P_1^{n-s+1}|G(x) - G(x-h)| \), so inequality \([30]\) holds for

\[
m \leq \frac{1}{2^{n-s+1}d_{n,s}P_1^{n-s+1}}.
\]

Case 4: \(|x-h| \leq 2|x|\) and

\[
0 < G(x) \cdot G(x-h) < \min \{|G(x)|^2, |G(x-h)|^2\}.
\]

Note first that inequality \([36]\) cannot occur in dimension \(n = 1\).

Let \(\gamma : [0, 1] \to \mathbb{R}^n\) be any piecewise \(C^1\) curve such that \(\gamma(0) = G(x)\) and \(\gamma(1) = G(x-h)\). By the fundamental theorem of Calculus,

\[
|h| = \left|G^{-1}(\gamma(0)) - G^{-1}(\gamma(1))\right| = \left|\int_0^1 (G^{-1} \circ \gamma)'(t) \, dt\right| \leq \max_{\gamma([0,1])} |DG^{-1}| \ell(\gamma),
\]

where \(\ell\) denotes the length of the curve.

Assumption \([36]\) implies that none of the cases of \([34]\) occurs (hence none of the situations depicted in Figure 2), but the distance from the origin to the segment \([G(x), G(x-h)]\) is attained at a point \(P\) in the interior of the segment. Assume that \(|G(x) - P| \leq |G(x - h) - P|\), although the construction is totally analogous in the symmetric case \(|G(x) - P| \geq |G(x - h) - P|\). Let \(Q\) be the point in the segment \([G(x), G(x-h)]\) such that \(P\) is the middle point between \(G(x)\) and \(Q\). We define the curve \(\gamma\) as follows. The curve \(\gamma\) starts at \(G(x)\) and describes the arc of circumference of center the origin \(O\) and radius \(|G(x)|\) joining \(G(x)\) with \(Q\); among the two possible arcs, we choose that which subtends an angle of less than \(\pi\) radians. Then, \(\gamma\) continues joining \(Q\) and \(G(x-h)\) with a straight line. See Figure 3.

For this particular \(\gamma\) we estimate the right hand-side of \([37]\). First, using \([32]\),

\[
\max_{\gamma([0,1])} |DG^{-1}| \leq d_{n,s} \max_{y \in \gamma([0,1])} |y|^{\frac{n-s+1}{n-s}} = d_{n,s} |G(x)|^{-\frac{n-s+1}{n-s}},
\]

Figure 2: Position of the points \(G(x), G(x-h)\) and origin \(O\) when \((G(x-h) - G(x)) \cdot (G(x-h) - G(x)) \leq 0\) (left) and when \((G(x-h) - G(x)) \cdot (G(x) - G(x)) \geq 0\) (right).
since, by construction of $\gamma$, the shortest distance of $\gamma([0,1])$ to the origin is $|G(x)|$. In order to estimate $\ell(\gamma)$, let $\theta$ be the angle $G(x) \overrightarrow{OP}$ if it is positive, or else the opposite angle $\overrightarrow{POG}(x)$, so that

$$\sin \theta = \frac{\ell([G(x), P])}{|G(x)|}$$

and $\theta \in [0, \frac{\pi}{2}]$ because $0 \leq G(x) \cdot G(x - h)$. Then

$$\ell(\gamma) = 2\theta |G(x)| + \ell([Q, G(x - h)]).$$

Now we use the elementary inequality

$$t \leq \frac{\pi}{2} \sin t, \quad t \in [0, \frac{\pi}{2}]$$

to obtain that

$$2\theta |G(x)| \leq \pi \sin \theta |G(x)| = \pi \ell([G(x), P]) = \frac{\pi}{2} \ell([G(x), Q]),$$

so

$$\ell(\gamma) \leq \frac{\pi}{2} \ell([G(x), Q]) + \ell([Q, G(x - h)]) \leq \frac{\pi}{2} \ell([G(x), G(x - h)]).$$

(39)

Using (38) and (39), inequality (37) becomes

$$|h| \leq \frac{\pi}{2} d_{n,s} |G(x)|^{-\frac{n-s+1}{n-s}} \ell([G(x), G(x - h)]).$$

If we had assumed $|G(x) - P| \geq |G(x - h) - P|$ instead of $|G(x) - P| \leq |G(x - h) - P|$ we would have obtained

$$|h| \leq \frac{\pi}{2} d_{n,s} |G(x - h)|^{-\frac{n-s+1}{n-s}} \ell([G(x), G(x - h)]),$$

so, in either case,

$$|h| \leq \frac{\pi}{2} d_{n,s} \max \left\{ |G(x)|^{-\frac{n-s+1}{n-s}}, |G(x - h)|^{-\frac{n-s+1}{n-s}} \right\} \ell([G(x), G(x - h)]))$$

$$= \frac{\pi}{2} d_{n,s} \max \left\{ |x|^{-n-s+1}, |x - h|^{-n-s+1} \right\} |G(x) - G(x - h)| .$$

Now we use (35) and find that inequality (30) holds for

$$m \leq \frac{1}{2^{n-s} \pi d_{n,s} R_1^{n-s+1}}.$$

Figure 3: The curve $\gamma$ (in thick line), the points $G(x), P, Q, G(x - h)$ (aligned, in dotted line), the origin $O$ and the angle $\theta$. 28
Finally, we present the main result of this section: its statement includes that of Proposition 4.4, shows that $V_\delta^s$ is actually a function and exhibits its main properties.

**Theorem 5.9.** There exists a vector radial function $V_\delta^s \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ such that

$$
\hat{V}_\delta^s(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi \xi| Q_\delta^s(\xi)}.
$$

Furthermore, the following properties hold:

- **a)** $V_\delta^s$ is the only $L^1_{\text{loc}}$ function that satisfies
  $$
  \int_{\mathbb{R}^n} V_\delta^s(z) Q_\delta^s(y - z) \, dz = \frac{1}{\sigma_{n-1} |y|^n}, \quad y \in \mathbb{R}^n \setminus \{0\}.
  $$

- **b)** There exists a Lipschitz bounded $W : \mathbb{R}^n \to \mathbb{R}^n$ (actually, $W \in C_0(\mathbb{R}^n, \mathbb{R}^n)$ when $n \geq 2$) such that
  $$
  V_\delta^s(x) = W(x) + \frac{c_{n-s}}{a_0} \frac{x}{|x|^{n+1-s}}.
  $$

- **c)** For any $R > 0$ there exists $M > 0$ such that for all $x \in B(0, R) \setminus \{0\}$,
  $$
  |V_\delta^s(x)| \leq \frac{M}{|x|^{n-s}}.
  $$

- **d)** For any $R_1, R_2 > 0$ there exists $M > 0$ such that for all $x \in B(0, R_1) \setminus \{0\}$ and $h \in B(0, R_2) \setminus \{x\}$,
  $$
  |V_\delta^s(x) - V_\delta^s(x - h)| \leq M \left| \frac{x}{|x|^{n+1-s}} - \frac{x - h}{|x - h|^{n+1-s}} \right|.
  $$

**Proof.** We first prove that there exists $V_\delta^s \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ such that (40) holds.

We start as in the proof of [34, Prop. 2.4.8]. In order to see that $V_\delta^s$ is $C^\infty$ away from the origin we note that $\mathcal{F}(\hat{V}_\delta^s) = \hat{V}_\delta^s$ and shall see that $\mathcal{F}(\hat{V}_\delta^s)$ is $C^M$ in $\mathbb{R}^n \setminus \{0\}$ for all $M$. Thus, fix $M \in \mathbb{N}$ and let $\alpha \in \mathbb{N}^n$ be any multiindex such that

$$
\left( |\alpha| + 1 \right)^n - n \geq M.
$$

We take $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ in $B(0, 2)^c$ and $\varphi = 0$ in $B(0, 1)$. Write

$$
u = \hat{V}_\delta^s, \quad u_0 = (1 - \varphi)u \quad \text{and} \quad u_\infty = \varphi u.
$$

On the one hand, $\partial^\alpha u = \partial^\alpha u_0 + \partial^\alpha u_\infty$ in the sense of distributions and also in $\mathbb{R}^n \setminus \{0\}$. On the other hand, as $u$ is smooth outside the origin, we have that $\partial^\alpha u_\infty$ is smooth and can calculate

$$
\partial^\alpha u_\infty = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha - \beta} \varphi \partial^\beta u.
$$

Write

$$
u = \partial^\alpha u_0 + \sum_{\beta \leq \alpha, \beta \neq \alpha} \binom{\alpha}{\beta} \partial^{\alpha - \beta} \varphi \partial^\beta u.\]
Then \( v \) is a distribution with support in \( B(0, 2) \), so \( \hat{v} \) is \( C^\infty \). Moreover, \( \partial^\alpha u = v + \varphi \partial^\alpha u \). Thus, in order to see that \( \hat{\partial^\alpha u} \) is \( C^M \), it remains to show that \( \varphi \partial^\alpha u \) is \( C^M \). The function \( \varphi \partial^\alpha u \) is \( C^\infty \) and, by Lemma 5.7,

\[
|\varphi(\xi)\partial^\alpha u(\xi)| \leq \frac{C_\alpha}{1 + |\xi|^{|\alpha|+1}}, \quad \xi \in \mathbb{R}^n.
\]

Having in mind (43), a classical result relating the decay of a function at infinity with the regularity of its Fourier transform (see, e.g., Exercise 2.4.1) shows that \( \varphi \partial^\alpha u \) is \( C^M \).

Once we have shown that \( \hat{\partial^\alpha u} \) is \( C^M \), we note that \( \hat{\partial^\alpha u}(\xi) = (2\pi i \xi)^\alpha \hat{u}(\xi) \). Let \( \xi \in \mathbb{R}^n \setminus \{0\} \); then \( \xi_j \neq 0 \) for some \( j \in \{1, \ldots, M\} \). Let \( V \) be a neighbourhood of \( \xi \) such that every \( \eta \in V \) satisfies \( \eta_j \neq 0 \). Let \( m \in \mathbb{N} \) be such that \( s(m+1) - n \geq M \) and let \( \alpha \) be the multiindex \( (0, \ldots, 0, m, 0, \ldots, 0) \), with the component \( m \) in position \( j \). Then \( \alpha \) satisfies (43). Moreover, for any \( \eta \in V \),

\[
\hat{u}(\eta) = \frac{\partial^\alpha u(\eta)}{(2\pi i \eta)^m},
\]

so \( \hat{u} \) is of class \( C^M \) in \( \mathbb{R}^n \setminus \{0\} \) for every \( M \in \mathbb{N} \), and therefore, so is \( V^s_\delta \).

Once we have that \( V^s_\delta \) is a function, since \( \hat{V}^s_\delta \) is radial and imaginary-valued, standard properties of the Fourier transform show that \( \hat{V}^s_\delta \) must be radial and real-valued.

Next, we show that the function

\[
Z(\xi) := \hat{V}^s_\delta(\xi) - \frac{-i\xi}{a_0|\xi|^2} \rho \frac{1}{|\xi|^s}
\]

decays to zero at infinity faster than any negative power of \( |\xi| \). For this we observe that

\[
Z(\xi) = -i \frac{\xi}{|\xi|^2} \rho \frac{1}{|\xi|^s} - i \frac{\xi}{a_0|\xi|^s} \rho = -i \frac{\xi}{|\xi|^s} \rho \frac{a_0|\xi|^{-1+s} - \hat{Q}^s_\delta(\xi)}{a_0|\xi|^s \hat{Q}^s_\delta(\xi)}.
\]

The terms \( |2\pi|\rho | \) and \( \hat{Q}^s_\delta(\xi) \) in the denominator above only contribute as a power of \( |\xi| \) in the growth at infinity (see Proposition 5.2). Therefore, it remains to show that the numerator above

\[
a_0|2\pi|^{1+s} - \hat{Q}^s_\delta(\xi)
\]

decays faster at infinity than any negative power of \( |\xi| \). Consider a \( \varphi \in C^\infty_c(\mathbb{R}^n) \) with \( \varphi_B(0, \frac{1}{2}) = 1 \) and \( \varphi_B(0, \frac{1}{2}) = 0 \). Then, recalling (17),

\[
a_0|2\pi|^{1+s} - \hat{Q}^s_\delta(\xi) = \mathcal{F} \left( \frac{a_0}{\gamma(1-s)|x|^{n-1+s}} - Q^s_\delta(x) \right)
\]

\[
= \mathcal{F} \left( \frac{a_0 \varphi}{\gamma(1-s)|x|^{n-1+s}} - Q^s_\delta(x) \right) + \mathcal{F} \left( \frac{a_0(1-\varphi)}{\gamma(1-s)|x|^{n-1+s}} \right).
\]

Looking at the expression of \( Q^s_\delta \) (Definition 4.1 and Lemma 4.2), we notice that the difference between \( \frac{a_0 \varphi}{\gamma(1-s)|x|^{n-1+s}} \) and \( Q^s_\delta(x) \) coincide with the constant \( \frac{a_0 \varphi}{\gamma(1-s)} \) in \( B(0, \min\{b_0 \delta, \frac{1}{2}\}) \), and both have compact support. Therefore, its difference is a smooth function of compact support. In particular, it is in the Schwartz space, as well as its Fourier transform:

\[
\mathcal{F} \left( \frac{a_0 \varphi}{\gamma(1-s)|x|^{n-1+s}} - Q^s_\delta(x) \right) \in \mathcal{S}.
\]

On the other hand, the function \( \mathcal{F} \left( \frac{1-\varphi}{\gamma(1-s)|x|^{n-1+s}} \right) \) is treated in Example 2.4.9, and it is shown that its decay at infinity is faster than any negative power of \( |\xi| \).
From expression (44) we can see that \( Z \) is in \( L^1_{\text{loc}} \) when \( n \geq 2 \). Because of its decay at infinity, \( Z \) is in \( L^1 \) when \( n \geq 2 \), so it has a Fourier transform \( Z \), which is \( C_0 \). When \( n = 1 \), in Lemma A.1 it will be shown that \( Z \) is a tempered distribution, so it has a Fourier transform \( \hat{Z} \), which, in principle, is a tempered distribution. But since both \( \hat{V}_s^\delta(\xi) \) and \( \frac{1}{|\xi|^{2|\pi\xi|^2}} \) are Fourier transforms of functions (see Lemma B.1(b) for the latter), we conclude that \( \hat{Z} \) is a function.

We continue by proving that \( Z \) is Lipschitz. We have \(-\nabla \hat{Z} = F(2\pi i \xi Z(\xi))\). The function \( 2\pi i \xi Z(\xi) \) is in \( L^1_{\text{loc}} \), as can be seen from expression (44). Due to the decay of \( Z \) at infinity, \( 2\pi i \xi Z(\xi) \) is in \( L^1(\mathbb{R}^n) \), so \( F(2\pi i \xi Z(\xi)) \) is bounded, and, hence, \( \hat{Z} \) is Lipschitz.

We define \( W \) as \( W(x) = \hat{Z}(-x) \). Taking inverse Fourier transforms to the expression

\[
\hat{V}_s^\delta(\xi) = Z(\xi) + \frac{-i\xi}{a_0|\xi| |2\pi\xi|^s}
\]

we obtain equality (42) (see Lemma B.1 for the inverse Fourier transform of the last term). That expression, together with the fact that \( W \) is continuous, shows that \( \hat{V}_s^\delta \) is in \( L^1_{\text{loc}} \).

In order to show (a), we note that equality (41) is equivalent to the equality of its Fourier transforms. More precisely, the functions \( V_s^\delta \) and \( Q_s^\delta \) can also be seen as tempered distributions, and, in particular, \( Q_s^\delta \) with compact support. Hence, by Lemmas B.2 and B.1(c) we have that equality (41) is equivalent to

\[
\hat{V}_s^\delta(\xi) \hat{Q}_s^\delta(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi\xi|^s},
\]

which holds due to (40). The uniqueness of \( \hat{V}_s^\delta \) also follows from this argument, since \( \hat{V}_s^\delta \) is uniquely determined by equality (45). Thus, (a) is proved.

Fact (b) has been proved when \( n \geq 2 \), while for the case \( n = 1 \) it only remains to show that \( W \) is bounded: this is tackled in Appendix A. With this, we have that given \( R > 0 \), for all \( x \in B(0, R) \setminus \{0\} \),

\[
|V_s^\delta(x)| \leq \|W\|_{L^\infty(\mathbb{R}^n)} + \frac{|c_{n-s}|}{a_0} \frac{1}{|x|^{n-s}} \leq \left( \|W\|_{L^\infty(\mathbb{R}^n)} R^{n-s} + \frac{|c_{n-s}|}{a_0} \right) \frac{1}{|x|^{n-s}},
\]

which shows (d).

As for inequality (d), since \( W \) is Lipschitz, we estimate

\[
|V_s^\delta(x) - V_s^\delta(x - h)| \leq \|DW\|_{L^\infty(\mathbb{R}^n)} |h| + \frac{|c_{n-s}|}{a_0} \left| \frac{x}{|x|^{n+1-s}} - \frac{x - h}{|x - h|^{n+1-s}} \right| 
\leq M \left| \frac{x}{|x|^{n+1-s}} - \frac{x - h}{|x - h|^{n+1-s}} \right|,
\]

for a suitable constant \( M > 0 \) coming from Lemma 5.8. The proof is complete.

Part (a) of Theorem 5.9 shows that \( V_s^\delta \) behaves like \( \frac{x}{|x|^{n+1-s}} \) around 0. It can also be seen that it behaves like \( \frac{x}{|x|^{n+1-s}} \) at infinity. Comparing these facts with the classical and fractional fundamental theorem of Calculus (see the Introduction), we have the following picture of \( V_s^\delta \): at 0, it behaves like the kernel of the fractional fundamental theorem of Calculus, while, at infinity, like that of the classical fundamental theorem of Calculus.

### 6 Poincaré, Morrey, Trudinger and Hardy inequalities

In this section we will use the nonlocal fundamental theorem of Calculus (Theorem 4.5) to prove inequalities in the spirit of Poincaré–Sobolev, Morrey, Trudinger and Hardy. In those inequalities we
will need a boundary condition implying that the function vanishes in a tubular neighbourhood of \( \partial \Omega \).

In order to describe more precisely that boundary condition, we recall the set \( \Omega_{-\delta} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \} \) and define the subspace \( H_{0}^{s,p,\delta}(\Omega_{-\delta}) \) as the closure of \( C_{c}^{\infty}(\Omega_{-\delta}) \) in \( H^{s,p,\delta}(\Omega) \):

\[
H_{0}^{s,p,\delta}(\Omega_{-\delta}) = \overline{C_{c}^{\infty}(\Omega_{-\delta})}^{H^{s,p,\delta}(\Omega)}.
\]

It is immediate to check that any \( u \in H_{0}^{s,p,\delta}(\Omega_{-\delta}) \) satisfies \( u = 0 \) a.e. in \( \Omega_{-\delta} \setminus \Omega_{-\delta} \) and \( D_{\delta}^{s}u = 0 \) a.e. in \( \Omega_{\partial \delta} \). We leave for a future work the issue of whether \( H_{0}^{s,p,\delta}(\Omega_{-\delta}) \) actually coincides with the set of \( u \in H^{s,p,\delta}(\Omega_{-\delta}) \) such that \( u = 0 \) a.e. in \( \Omega_{-\delta} \setminus \Omega_{-\delta} \), so that \( H_{0}^{s,p,\delta}(\Omega_{-\delta}) \) can be regarded as a volumetric-type condition. Finally, given \( g \in H^{s,p,\delta}(\Omega) \) we define the affine subspace \( H_{g}^{s,p,\delta}(\Omega_{-\delta}) \) as \( g + H_{0}^{s,p,\delta}(\Omega_{-\delta}) \).

As in the space \( H^{s,p}(\mathbb{R}^{n}) \) (and \( W^{s,p} \), too), the Sobolev conjugate exponent of a \( p \in [1, \frac{n}{s}] \) is

\[
p_{s}^{*} = \frac{np}{n - sp}.
\]

The Poincaré–Sobolev inequality in \( H_{0}^{s,p,\delta}(\Omega_{-\delta}) \) is as follows. Its analogue in the fractional case can be found in [59, Th. 1.8]. As theirs, our proof is based on (our version of) the nonlocal fundamental theorem of Calculus and the Hardy–Littlewood–Sobolev inequality, but we also take advantage of the comparison between the kernel \( V_{\delta}^{s} \) and the Riesz potential given by Theorem 5.9.

**Theorem 6.1.** Let \( 1 < p < \infty \) be with \( sp < n \). Then, there exists \( C > 0 \) such that for all \( u \in H_{0}^{s,p,\delta}(\Omega_{-\delta}) \),

\[
\| u \|_{L^{q}(\Omega)} \leq C \| D_{\delta}^{s}u \|_{L^{p}(\Omega)}
\]

for every \( q \in [1, p_{s}^{*}] \).

**Proof.** By density, it is enough to prove the inequality for \( u \in C_{c}^{\infty}(\Omega_{-\delta}) \).

Fix \( x \in \Omega \) and let \( C > 0 \) denote a constant whose value may vary through this process. Notice that \( \text{supp} D_{\delta}^{s}u \subset \Omega \) and, by Proposition 4.3, \( D_{\delta}^{s}u \in C_{c}^{\infty}(\mathbb{R}^{n}) \). By Theorem 4.5 and Proposition 4.4,

\[
|u(x)| \leq \int_{\Omega} |D_{\delta}^{s}u(y)||V_{\delta}^{s}(x - y)|dy \leq C \int_{\Omega} \frac{|D_{\delta}^{s}u(y)|}{|x - y|^{n-s}} dy = C \left( |x|^{s} * |D_{\delta}^{s}u| \right)(x),
\]

where \( I_{s} \) is the Riesz potential. On the other hand, by the Hardy–Littlewood–Sobolev inequality (e.g., [48, Ch. 4, Th. 2.1]) we have that

\[
\| I_{s} * |D_{\delta}^{s}u| \|_{L^{p_{s}^{*}}(\mathbb{R}^{n})} \leq C \| D_{\delta}^{s}u \|_{L^{p}(\mathbb{R}^{n})}.
\]

Therefore, for every \( q \in [1, p_{s}^{*}] \),

\[
\| u \|_{L^{q}(\Omega)} \leq C \| u \|_{L^{p_{s}^{*}}(\Omega)} \leq C |I_{s} * |D_{\delta}^{s}u| \|_{L^{p_{s}^{*}}(\mathbb{R}^{n})} \leq C \| D_{\delta}^{s}u \|_{L^{p}(\mathbb{R}^{n})} = C \| D_{\delta}^{s}u \|_{L^{p}(\Omega)}.
\]

A nonlocal Poincaré inequality is obtained as a corollary.

**Theorem 6.2.** Let \( 1 < p < \infty \). Then there exists \( C > 0 \) such that for all \( u \in H_{0}^{s,p,\delta}(\Omega_{-\delta}) \),

\[
\| u \|_{L^{p}(\Omega)} \leq C \| D_{\delta}^{s}u \|_{L^{p}(\Omega)}.
\]
Proof. If $sp < n$, the result is a particular case of Theorem 6.1. If $sp \geq n$, we take any $q$ satisfying
\[ 1 < q \leq p, \quad sq < n \quad \text{and} \quad p \leq q^*_s, \tag{48} \]
which is easily seen to exist. Indeed, if $n \geq 2$ we can take $q = \frac{np}{n+sp}$, while if $n = 1$ we choose any $q$
such that
\[ 1 < q < \frac{1}{s}, \quad \frac{p}{1+sp} \leq q, \]
which exists because $1 < \frac{1}{s}$ and $\frac{p}{1+sp} < \frac{1}{s}$.

Once $q$ is chosen, by Theorem 6.1 and (48), we have for some $c_1, c_2, c_3 > 0,$
\[ \|u\|_{L^p(\Omega)} \leq c_1 \|u\|_{L^{q^*_s}(\Omega)} \leq c_2 \|D^s_\delta u\|_{L^q(\Omega)} \leq c_3 \|D^s_\delta u\|_{L^p(\Omega)}. \]

Next we introduce a nonlocal analogue of Morrey’s inequality, whose fractional version was shown
in [59, Th. 1.11]. Unlike their proof, which uses Morrey-type estimates of the Riesz transform, ours
is based on the nonlocal fundamental theorem of Calculus in this context (Theorem 4.5) together
with the estimates of the kernel $V^s_\delta$.

**Theorem 6.3.** Let $1 < p < \infty$ be such that $sp > n$. Then there exists $C > 0$ such that for all
$u \in H^{s,p,\delta}_0(\Omega_{-\delta}),$
\[ |u(x) - u(y)| \leq C |x - y|^{s - \frac{n}{p}} \|D^s_\delta u\|_{L^p(\Omega)}, \quad \text{a.e. } x, y \in \Omega \tag{49} \]
and
\[ \|u\|_{L^\infty(\Omega)} \leq C \|D^s_\delta u\|_{L^p(\Omega)}. \tag{50} \]

In addition, any $u \in H^{s,p,\delta}_0(\Omega_{-\delta})$ has a representative which is Hölder continuous of exponent $s - \frac{n}{p}$,
and the continuous inclusion $H^{s,p,\delta}_0(\Omega_{-\delta}) \subset C^{0,s - \frac{n}{p}}(\Omega)$ holds.

**Proof.** The core of the proof consists in showing that
\[ |u(x) - u(y)| \leq C |x - y|^{s - \frac{n}{p}} \|D^s_\delta u\|_{L^p(\Omega)}, \quad x, y \in \Omega \tag{51} \]
for all $u \in C^\infty_c(\Omega_{-\delta})$. Fix $x, y \in \Omega$ and $u \in C^\infty_c(\Omega_{-\delta})$. By Theorem 4.5 and Theorem 5.9(d) there
exists $C > 0$ such that
\[ |u(x) - u(y)| = \left| \int_{\mathbb{R}^n} D^s_\delta u(z) \cdot [V^s_\delta(x - z) - V^s_\delta(y - z)] \, dz \right| \]
\[ \leq \int_{\Omega} |V^s_\delta(x - z) - V^s_\delta(y - z)| |D^s_\delta u(z)| \, dz \]
\[ \leq C \int_{\Omega} \left| \frac{x - z}{|x - z|^{n+1-s}} - \frac{y - z}{|y - z|^{n+1-s}} \right| |D^s_\delta u(z)| \, dz. \tag{52} \]

Now define $r := |x - y|$. Continuing with (52), we have
\[ |u(x) - u(y)| \leq C \int_{B(x,2r)} |x - z|^{s-n} |D^s_\delta u(z)| \, dz + C \int_{B(x,2r)} |y - z|^{s-n} |D^s_\delta u(z)| \, dz \]
\[ + C \int_{B(x,2r)^c} \left| \frac{x - z}{|x - z|^{n+1-s}} - \frac{y - z}{|y - z|^{n+1-s}} \right| |D^s_\delta u(z)| \, dz. \tag{53} \]
For the first term we have that by Hölder’s inequality,

$$
\int_{B(x,2r)} |x - z|^{s-n} |D_\delta^s u(z)| \, dz \leq \left( \int_{B(x,2r)} |x - z|^{(s-n)p'} \, dz \right)^{\frac{1}{p'}} \left( \int_{B(x,2r)} |D_\delta^s u(z)|^p \, dz \right)^{\frac{1}{p}}
$$

$$\leq (2r)^{s-n} \left( \frac{\sigma_{n-1}(p-1)}{sp - n} \right)^{\frac{1}{p}} \|D_\delta^s u\|_{L^p(\mathbb{R}^n)}, \quad (54)
$$

since $n + (s-n)p' = \frac{2p-n}{p-1} > 0$. With respect to the second term, we use the inclusion $B(x,2r) \subset B(y,3r)$ and an analogous calculation as in (54) allows us to obtain

$$
\int_{B(x,2r)} |y - z|^{s-n} |D_\delta^s u(z)| \, dz \leq \int_{B(y,3r)} |y - z|^{s-n} |D_\delta^s u(z)| \, dz
$$

$$\leq (3r)^{s-n} \left( \frac{\sigma_{n-1}(p-1)}{sp - n} \right)^{\frac{1}{p}} \|D_\delta^s u\|_{L^p(\mathbb{R}^n)}. \quad (55)
$$

Finally, so as to tackle the last term, by the fundamental theorem of Calculus,

$$
\left| \frac{x - z}{|x - z|^{n+1-s}} - \frac{y - z}{|y - z|^{n+1-s}} \right| = \int_0^1 \frac{t x + (1-t)y - z}{|t x + (1-t)y - z|^{n+1-s}} dt
$$

$$= \int_0^1 \frac{r}{|t x + (1-t)y - z|^{n+1-s}} + (n+1-s) \frac{r}{|t x + (1-t)y - z|^{n+3-s}} dt
$$

$$= (n+2-s) \int_0^1 \frac{1}{|t x + (1-t)y - z|^{n+1-s}} dt,
$$

so

$$
\int_{B(x,2r)^c} \left| \frac{x - z}{|x - z|^{n+1-s}} - \frac{y - z}{|y - z|^{n+1-s}} \right| |D_\delta^s u(z)| \, dz
$$

$$\leq (n+2-s) \int_0^1 \int_{B(x,2r)^c} |t x + (1-t)y - z|^{s-n-1} |D_\delta^s u(z)| \, dz dt.
$$

By Hölder’s inequality,

$$
\int_{B(x,2r)^c} |t x + (1-t)y - z|^{s-n-1} |D_\delta^s u(z)| \, dz
$$

$$\leq \left( \int_{B(x,2r)^c} |t x + (1-t)y - z|^{(s-n-1)p'} \, dz \right)^{\frac{1}{p'}} \|D_\delta^s u\|_{L^p(\mathbb{R}^n)},
$$

Since $B(t x + (1-t)y, r) \subset B(x,2r)$ for all $t \in [0,1]$, we have

$$
\int_{B(x,2r)^c} |t x + (1-t)y - z|^{(s-n-1)p'} \, dz \leq \int_{B(t x + (1-t)y, r)^c} |t x + (1-t)y - z|^{(s-n-1)p'} \, dz
$$

$$= \frac{\sigma_{n-1}}{(n+1-s)p' - n^{n+(s-n-1)p'}}.
$$
since \( n + (s - n - 1)p' = \frac{(1-s)p+n}{p-1} < 0 \). Putting together the last three inequalities, we can see that there exists \( \tilde{C} = \tilde{C}(s, n, p) > 0 \) such that

\[
\int_{B(\Omega,2r)^c} \left| \frac{x - z}{|x-z|^{n+1-s}} - \frac{y - z}{|y-z|^{n+1-s}} \right| |D^s_{\delta}u(z)| \, dz \leq \tilde{C} r^{[n+(s-n-1)p']\frac{1}{p} + 1} \|D^s_{\delta}u\|_{L^p(\mathbb{R}^n)}
\]

(56)

Then, inequality (51) follows combining (53), (54), (55) and (56), as well as the inclusion \( \text{supp} \, D^s_{\delta} u \subset \Omega \), which implies \( \|D^s_{\delta} u\|_{L_p(\mathbb{R}^n)} = \|D^s_{\delta} u\|_{L_p(\Omega)} \). Once (51) is established, inequality (49) follows from a standard density argument.

In order to show inequality (50) and the continuous inclusion \( H^s_{0,\delta}(\Omega_{-\delta}) \subset C^{0,s-\frac{p}{n}}(\Omega) \), by a density argument, it is enough to prove (50) for \( u \in C^\infty_c(\Omega_{-\delta}) \). Let \( x \in \Omega \) and \( x_0 \in \Omega \setminus \Omega_{-\delta} \). By (51),

\[
|u(x)| = |u(x) - u(x_0)| \leq C |x - x_0|^{s-\frac{p}{n}} \|D^s_{\delta} u\|_{L^p(\Omega)} \leq C (\text{diam} \, \Omega)^{s-\frac{p}{n}} \|D^s_{\delta} u\|_{L^p(\Omega)},
\]

where diam stands for the diameter of a set. The proof is concluded.

The limiting case \( sp = n \) is covered by the following version of Trudinger’s inequality. Its proof is a straightforward adaptation of the classical one (see, e.g., [33, Th. 7.15]) but using inequality (47). Its fractional version can be found in [59, Th. 1.10]. We denote by \( |\Omega| \) the measure of \( \Omega \).

**Theorem 6.4.** Let \( 1 < p < \infty \) be such that \( sp = n \). Then there exist \( c_1, c_2 > 0 \) such that for all \( u \in H^s_{0,\delta}(\Omega_{-\delta}) \),

\[
\int_{\Omega} \exp \left( \frac{|u(x)|}{c_1 \|D^s_{\delta} u\|_{L^p(\Omega)}} \right)^{p'} \, dx \leq c_2 |\Omega|.
\]

**Proof.** By a standard density argument, it is enough to prove the inequality for \( C^\infty_c(\Omega_{-\delta}) \) functions, so let \( u \in C_c^\infty(\Omega_{-\delta}) \) and set

\[
g(x) = \int_{\Omega} \frac{|D^s_{\delta} u(y)|}{|x - y|^{n-s}} \, dy.
\]

By (47),

\[
|u| \leq C g.
\]

while by [33, Lemma 7.13], for some constants \( c'_1, c_2 > 0 \),

\[
\int_{\Omega} \exp \left( \frac{g(x)}{c'_1 \|D^s_{\delta} u\|_{L^p(\Omega)}} \right)^{p'} \, dx \leq c_2 |\Omega|.
\]

(57)

Putting together these two inequalities, we obtain the conclusion.

We mention that the constants \( c'_1, c_2 \) of (57) do not depend on \( \Omega \), but the constant \( C \) of (47) does. That is why the constant \( c_1 \) of Theorem 6.4 depends on \( \Omega \), but not the constant \( c_2 \).

We end this section with the analogue of Hardy’s inequality. Its fractional version can be found in [59, Th. 1.9], whose proof (as well as the classical one [65]) is easily adapted to our context.

**Theorem 6.5.** Let \( 1 < p < \infty \) be with \( sp < n \). Then, there exists \( C > 0 \) such that for all \( u \in H^s_{0,\delta}(\Omega_{-\delta}) \),

\[
\left( \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx \right)^{\frac{1}{p}} \leq C \|D^s_{\delta} u\|_{L^p(\Omega)}.
\]

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Proof. As before, it is enough to establish the inequality for \( u \in C_c^\infty(\Omega_{-\delta}) \). The proof is just a combination of inequality (47) together with the classical Hardy inequality for Riesz potentials due to Stein and Weiss [65] (see also [59, Lemma 2.8]):

\[
\int_\Omega \frac{|u(x)|^p}{|x|^{sp}} \, dx \leq c_1 \int_\Omega \frac{(I_s * |D^\delta_x u|)(x)^p}{|x|^{sp}} \, dx \leq c_1 \int_\mathbb{R}^n \frac{(I_s * |D^\delta_x u|)(x)^p}{|x|^{sp}} \, dx \leq c_2 \|D^\delta_x u\|_{L^p(\mathbb{R}^n)}^p,
\]

for some constants \( c_1, c_2 > 0 \).

7 Compact embeddings

In this section we will use the nonlocal fundamental theorem of Calculus (Theorem 4.5) to prove compact embeddings of the spaces \( H^{s,p,\delta}_g(\Omega_{-\delta}) \) into \( L^q(\Omega) \) spaces.

We start with the following Hölder estimate of the function \( |w|^{n+1-s} \). In fact, this result is included in the proof of [16 Prop. 3.14], but we provide a full proof for the comfort of the reader.

Lemma 7.1. There exists a constant \( C > 0 \), such that for every \( s \in (0,1) \) and \( h \in \mathbb{R}^n \),

\[
\int_{\mathbb{R}^n} \frac{z}{|z|^{n+1-s}} - \frac{z - h}{|z - h|^{n+1-s}} \, dz \leq \frac{C|h|^s}{s(1 - s)}.
\]

Proof. We first show that there exists a constant \( C > 0 \), such that for every \( s \in (0,1) \) we have

\[
\int_{\mathbb{R}^n} \frac{|w|}{|w|^{n+1-s}} - \frac{|w - e_1|}{|w - e_1|^{n+1-s}} \, dw \leq \frac{C}{s(1 - s)}.
\]  \[ (58) \]

On the one hand, we have that

\[
\int_{B(0,2)} \left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| \, dw \leq C \int_{B(0,2)} \frac{1}{|w|^{n-s}} \, dw \leq C \frac{2^s}{s} \leq C.
\]

On the other hand, for a fixed \( w \in B(0,2)^c \),

\[
\left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| = \left| \int_0^1 \frac{d}{dt} \frac{w - te_1}{|w - te_1|^{n+1-s}} \, dt \right| = \left| \int_0^1 \frac{w - te_1}{|w - te_1|^{n+1-s}} - \frac{e_1}{|w - te_1|^{n+1-s}} \, dt \right| \leq C \int_0^1 \frac{1}{|w - te_1|^{n+1-s}} \, dt.
\]

Now, for \( w \in B(0,2)^c \) and \( t \in [0,1] \) we have

\[
|w - te_1| \geq |w| - t \geq |w| - 1 \geq \frac{1}{2}|w|,
\]

so

\[
\int_0^1 \frac{1}{|w - te_1|^{n+1-s}} \, dt \leq 2^{n+1-s} \int_0^1 \frac{1}{|w|^{n+1-s}} \, dt \leq 2^{n+1} \frac{1}{|w|^{n+1-s}}.
\]

By integration, we obtain that

\[
\int_{B(0,2)^c} \left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| \, dw \leq C \int_{B(0,2)^c} \frac{1}{|w|^{n+1-s}} \, dw \leq C \frac{2^{1+s}}{1 - s} \leq \frac{C}{1 - s}.
\]
This yields \((58)\).

In order to complete the proof, we take a rotation \(R\) such that \(R^T h = |h|e_1\). Then, making the change of variables \(z = |h| R w\) and using \((58)\), we arrive at

\[
\int_{\mathbb{R}^n} \frac{z}{|z|^{n+1-s}} - \frac{z - h}{|z - h|^{n+1-s}}\, dz = |h|^s \int_{\mathbb{R}^n} \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}}\, dw \leq |h|^s \frac{C}{s(1 - s)}.
\]

\(\square\)

A key ingredient of the desired compactness result is the application of the Fréchet–Kolmogorov criterion, for which the following estimate on the translations is crucial. The analogous result in the Sobolev case is classical \([13] \text{ Prop. 9.3}\). The next result is inspired by \([16] \text{ Prop. 3.14}\), where they proved a fractional version when \(p = 1\).

**Proposition 7.2.** a) Let \(1 < p < \infty\). Then there exists \(C > 0\) such that for all \(u \in H_0^{s,p,\delta}(\Omega_{-\delta})\) and \(h \in \mathbb{R}^n\),

\[
\left( \int_{\Omega} |u(x + h) - u(x)|^p\, dx \right)^{\frac{1}{p}} \leq C |h|^s \|D_\delta^s u\|_{L^p(\Omega)},
\]

\((59)\).

b) Let \(p = 1\). Then for all \(M > 0\) there exists \(C > 0\) such that for all \(u \in H_0^{s,p,\delta}(\Omega_{-\delta})\) and \(h \in B(0, M)\), inequality \((59)\) holds.

**Proof.** By a standard density argument, it is enough to prove the result for \(u \in C^\infty_c(\Omega_{-\delta})\).

We start with case a). Let us fix \(M > 0\) such that \(x + h \notin \Omega_{-\delta}\) for all \(x \in \Omega\) and \(h \in B(0, M)^c\). Then, by Theorem 6.2

\[
\left( \int_{\Omega} |u(x + h) - u(x)|^p\, dx \right)^{\frac{1}{p}} = \|u\|_{L^p(\Omega)} \leq C \|D_\delta^s u\|_{L^p(\Omega)} \leq C \frac{1}{M^s} |h|^s \|D_\delta^s u\|_{L^p(\Omega)},
\]

and the proof is concluded in this case.

In the rest of the proof we consider \(h \in B(0, M)\). As \(\text{supp} \, D_\delta^s u \subset \Omega\), there exists \(R > 0\) such that \(D_\delta^s u(x - z) = 0\) for all \(x \in \Omega\) and \(z \in B(0, R)^c\). Let \(x \in \Omega\). By Theorem 4.5

\[
|u(x + h) - u(x)| = \left| \int_{\mathbb{R}^n} (V_\delta^s(z) - V_\delta^s(z + h)) \cdot D_\delta^s u(x - z)\, dz \right| \leq \int_{B(0,R)} |V_\delta^s(z) - V_\delta^s(z + h)| \|D_\delta^s u(x - z)\|\, dz.
\]

(60)

By Theorem 5.9 d), there exists \(C > 0\) such that

\[
|V_\delta^s(z) - V_\delta^s(z + h)| \leq C \left| \frac{z}{|z|^{n+1-s}} - \frac{z + h}{|z + h|^{n+1-s}} \right|,
\]

for all \(z \in B(0, R)\). Thus, applying Hölder’s inequality to the right-hand side of (60),

\[
|u(x + h) - u(x)| \leq C \left( \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z + h}{|z + h|^{n+1-s}} \right| \|D_\delta^s u(x - z)\|^p\, dz \right)^{\frac{1}{p}} \left( \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z + h}{|z + h|^{n+1-s}} \right|\, dz \right)^{\frac{1}{p}}
\]

\[
\leq \left( \frac{C |h|^s}{s(1 - s)} \right) \left( \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z + h}{|z + h|^{n+1-s}} \right|\, dz \right)^{\frac{1}{p}},
\]

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where we have used Lemma 7.1. Next, we integrate and apply Fubini's theorem to obtain
\[
\int_\Omega |u(x + h) - u(x)|^p \, dx \leq \left( \frac{C|h|^s}{s(1-s)} \right)^{p/p'} \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z + h}{|z + h|^{n+1-s}} \right| |D^s_\delta u(x - z)|^p \, dx \, dz
\]
\[
\leq \left( \frac{C|h|^s}{s(1-s)} \right)^{p/p'+1} \|D^s_\delta u\|_{L^p(\mathbb{R}^n)}^p = \left( \frac{C|h|^s}{s(1-s)} \right)^p \|D^s_\delta u\|_{L^p(\Omega)}^p,
\]
where we have applied Lemma 7.1 again. This completes the proof of (a).

Now we prove (b) by following the same lines as in case (a). As \( \text{supp } D^s_\delta u \subset \Omega \), there exists \( R > 0 \) such that \( D^s_\delta u(x - z) = 0 \) for all \( x \in \Omega \) and \( z \in B(0,R)^c \). Fix \( M > 0 \) and consider \( h \in B(0,M) \). Let \( x \in \Omega \). As in (60)–(61),
\[
|u(x + h) - u(x)| \leq C \, \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z + h}{|z + h|^{n+1-s}} \right| |D^s_\delta u(x - z)| \, dz.
\]
Using Lemma 7.1 we find that, for some \( C_1 > 0 \)
\[
\int_\Omega |u(x + h) - u(x)| \, dx \leq C \, \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z + h}{|z + h|^{n+1-s}} \right| |D^s_\delta u(x - z)| \, dx \, dz
\]
\[
\leq \frac{C_1|h|^s}{s(1-s)} \|D^s_\delta u\|_{L^1(\Omega)},
\]
which concludes the proof. \( \square \)

The main result of this section is the following compact embedding, which is an analogue of the Rellich–Kondrachov theorem. See [60, Th. 2.2] for the fractional case; their proof uses Ascoli–Arzelà’s theorem to mollifiers of the sequence, while we prefer to invoke directly the Fréchet–Kolmogorov criterion. Recall the notation \( p^*_s \) from (46).

**Theorem 7.3.** Let \( g \in H^{s,p,\delta}(\Omega) \). Then, for any sequence \( \{u_j\}_{j \in \mathbb{N}} \subset H^{s,p,\delta}_g(\Omega_\delta) \) such that
\[
u_j \to u \quad \in H^{s,p,\delta}(\Omega),
\]
for some \( u \in H^{s,p,\delta}(\Omega) \), one has \( u \in H^{s,p,\delta}_g(\Omega_\delta) \) and:

a) if \( p > 1 \),
\[
u_j \to u \quad \in L^q(\Omega),
\]
for every \( q \) satisfying
\[
\begin{cases}
q \in [1, p^*_s) & \text{if } sp < n, \\
q \in [1, \infty) & \text{if } sp = n, \\
q \in [1, \infty) & \text{if } sp > n.
\end{cases}
\]

b) if \( p = 1 \),
\[
u_j \to u \quad \in L^1(\Omega).
\]

**Proof.** Clearly, \( u \in H^{s,p,\delta}_g(\Omega_\delta) \), since \( H^{s,p,\delta}_g(\Omega_\delta) \) is a closed affine subspace of \( H^{s,p,\delta}(\Omega_\delta) \). By linearity, we can assume \( g = 0 \).

The case \( sp > n \) implies \( p > 1 \) and follows from Theorem 6.3 and the Ascoli–Arzelà theorem. The case \( sp = n \) reduces to the case \( sp < n \) thanks to Proposition 3.5. Thus, we focus on the case...
sp < n. Moreover, the case q < p reduces to the case q ≥ p thanks to the inclusions \( L^{p_2}(\Omega) \subset L^p(\Omega) \) for all 1 ≤ p_1 ≤ p_2, so we can assume that q ∈ \([p, p^*_s]\).

Let \( M > 0 \) be such that \( \|u_j\|_{H^{s,p_j}(\Omega)} \leq M \) for each \( j \in \mathbb{N} \).

We start with \( a) \), so we assume \( p > 1 \). By Proposition 7.2, we have that for \( j \in \mathbb{N} \) and \( h \in \mathbb{R}^n \),

\[
\|\tau_h u_j - u_j\|_{L^p(\Omega)} \leq C \|h^s\| \|D^s_{\alpha} u_j\|_{L^p(\Omega)},
\]

with \( \tau_h u_j = u_j(\cdot - h) \) and some \( C > 0 \). Next, as \( p \leq q < p^*_s \), we can write

\[
\frac{1}{q} = \frac{\alpha}{p} + \frac{1 - \alpha}{p^*_s} \quad \text{for some } \alpha \in (0, 1].
\]

Using the interpolation inequality, \([62]\), the triangular inequality and Theorem 6.1,

\[
\|\tau_h u_j - u_j\|_{L^q(\Omega)} \leq \|\tau_h u_j - u_j\|_{L^p(\Omega)} \|\tau_h u_j - u_j\|_{L^{p^*_s}(\Omega)}^{1-\alpha} \\
\leq (C|h|^s)^\alpha \|D^s_{\alpha} u_j\|_{L^p(\Omega)}^{\alpha} \left(2 \|u_j\|_{L^{p^*_s}(\Omega)}\right)^{1-\alpha} \\
\leq (2C_1)^{1-\alpha} (C|h|^s)^\alpha \|D^s_{\alpha} u_j\|_{L^p(\Omega)} \leq (2C_1)^{1-\alpha} M (C|h|^s)^\alpha,
\]

for some \( C_1 > 0 \). Thus,

\[
\lim_{h \to 0} \sup_{j \in \mathbb{N}} \|\tau_h u_j - u_j\|_{L^q(\Omega)} = 0.
\]

As a result, the Fréchet–Kolmogorov criterion leads to the compactness of \( \{u_j\}_{j \in \mathbb{N}} \) in \( L^q(\Omega) \).

Now we show \( b) \), so we assume \( p = 1 \). Fix \( M_1 > 0 \). By Proposition 7.2, we have that for \( j \in \mathbb{N} \) and \( h \in B(0, M_1) \)

\[
\|\tau_h u_j - u_j\|_{L^1(\Omega)} \leq C |h|^s \|D^s_{\alpha} u_j\|_{L^1(\Omega)} \leq C M |h|^s,
\]

for some \( C > 0 \). Again the Fréchet–Kolmogorov criterion concludes the compactness of \( \{u_j\}_{j \in \mathbb{N}} \) in \( L^1(\Omega) \).

Of course, Theorem 6.3 yields additionally the compact inclusion of \( H^{s,p}_0(\Omega-\delta) \) into \( C^{0,\beta}(\Omega) \) for any \( 0 < \beta < s - \frac{n}{p} \) under the range \( sp > n \).

8 Existence of minimizers and the Euler–Lagrange equation

In this final section, we prove the existence of minimizers of functionals of the form

\[
I(u) = \int_{\Omega} W(x, u(x), D^s_{\alpha} u(x)) \, dx
\]

under coercivity and convexity conditions. We also show the corresponding (nonlocal) Euler–Lagrange equations satisfied by the minimizers.

From now on, \( \mathcal{L}^n \) denotes the Lebesgue sigma-algebra in \( \mathbb{R}^n \), whereas \( \mathcal{B} \) and \( \mathcal{B}^n \) denote the Borel sigma-algebras in \( \mathbb{R} \) and \( \mathbb{R}^n \), respectively. The result on the existence of minimizers, which is a standard application of the direct method of the Calculus of Variations, is as follows.

**Theorem 8.1.** Let \( 1 < p < \infty \). Let \( u_0 \in H^{s,p,\delta}(\Omega) \). Let \( W : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \) satisfy the following conditions:

\( a) \) \( W \) is \( \mathcal{L}^n \times \mathcal{B} \times \mathcal{B}^n \)-measurable.

\( b) \) \( W(x, \cdot, \cdot) \) is lower semicontinuous for a.e. \( x \in \Omega \).
c) For a.e. $x \in \Omega$ and every $y \in \mathbb{R}$, the function $W(x, y, \cdot)$ is convex.

d) There exist $c > 0$ and $a \in L^1(\Omega)$ such that

$$W(x, y, z) \geq a(x) + c |z|^p$$

for a.e. $x \in \Omega$, all $y \in \mathbb{R}$ and all $z \in \mathbb{R}^n$.

Define $I$ as in (63), and assume that $I$ is not identically infinity in $H^{s, p, \delta}_{w_0} (\Omega_\delta)$. Then there exists a minimizer of $I$ in $H^{s, p, \delta}_{w_0} (\Omega_\delta)$.

Proof. Assumption [d] shows that the functional $I$ is bounded below by $\int_{\mathbb{R}} a$. As $I$ is not identically infinity in $H^{s, p, \delta}_{w_0} (\Omega_\delta)$, there exists a minimizing sequence $\{u_j\}_{j \in \mathbb{N}}$ of $I$ in $H^{s, p, \delta}_{w_0} (\Omega_\delta)$. Then, assumption [d] implies that $\{D^s \phi u_j\}_{j \in \mathbb{N}}$ is bounded in $L^p(\Omega, \mathbb{R}^n)$. By Theorem 6.2 applied to $u_j - u_0$, we obtain that $\{u_j - u_0\}_{j \in \mathbb{N}}$ and, hence, $\{u_j\}_{j \in \mathbb{N}}$ are bounded in $L^p(\Omega)$. Therefore, $\{u_j\}_{j \in \mathbb{N}}$ is bounded in $H^{s, p, \delta}(\Omega)$. As $H^{s, p, \delta}(\Omega)$ is reflexive (Proposition 3.4), we can extract a weakly convergent subsequence. Using Theorem 7.3, we obtain that there exists $u \in H^{s, p, \delta}(\Omega)$ such that for a subsequence (not relabelled),

$$u_j \rightharpoonup u \text{ in } H^{s, p, \delta}(\Omega) \quad \text{and} \quad u_j \to u \text{ in } L^p(\Omega).$$

Moreover, $u \in H^{s, p, \delta}_{w_0}(\Omega_\delta)$.

A standard lower semicontinuity result for convex functionals (see, e.g., [29, Th. 7.5]) shows that

$$I(u) \leq \liminf_{j \to \infty} I(u_j).$$

Therefore, $u$ is a minimizer of $I$ in $H^{s, p, \delta}_{w_0}(\Omega_\delta)$ and the proof is concluded. 

We finally show the Euler–Lagrange equation satisfied by any minimizer. The notation for partial derivatives is as follows: $D_y W(x, \cdot, z)$ is the derivative of $W(x, \cdot, z)$, and $D_z W(x, y, \cdot)$ is the derivative of $W(x, y, \cdot)$.

**Theorem 8.2.** Let $1 < p < \infty$. Let $u_0 \in H^{s, p, \delta}(\Omega)$. Let $W : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following conditions:

a) $W(\cdot, y, z)$ is $\mathcal{L}^n$-measurable for each $y \in \mathbb{R}$ and $z \in \mathbb{R}^n$.

b) $W(x, \cdot, \cdot)$ is of class $C^1$ for a.e. $x \in \Omega$.

c) There exist $c > 0$ and $a \in L^1(\Omega)$ such that

$$|W(x, y, z)| + |D_y W(x, y, z)| + |D_z W(x, y, z)| \leq a(x) + c \left( |y|^p + |z|^p \right),$$

for a.e. $x \in \Omega$, all $y \in \mathbb{R}$ and all $z \in \mathbb{R}^n$.

Define $I$ as in (63). Let $u$ be a minimizer of $I$ in $H^{s, p, \delta}_{w_0}(\Omega_\delta)$. Then, for every $\varphi \in C^\infty_c(\Omega_\delta)$,

$$\int_\Omega [D_y W(x, u(x), D^s_\delta u(x)) \varphi(x) + D_z W(x, u(x), D^s_\delta u(x)) \cdot D^s_\delta \varphi(x)] \, dx = 0. \quad (64)$$

If, in addition, $D_z W(\cdot, u(\cdot), D^s_\delta u(\cdot)) \in C^1(\Omega_\delta, \mathbb{R}^n)$ then

$$D_y W(x, u(x), D^s_\delta u(x)) = \text{div}^s_\delta D_z W(x, u(x), D^s_\delta u(x)) \quad (65)$$

for a.e. $x \in \Omega_\delta$. 

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Proof. Using a standard argument, in order to show (64) it is enough to check that one can differenti-
te under the integral sign in the function \( t \mapsto I(u + t\varphi) \), since \( u + t\varphi \in H^{s,p,\delta}(\Omega - \delta) \). Assumption (c) shows that this is the case (see, e.g., [39, Ch. 13, §2, Lemma 2.2]), so (64) is proved.

Now we make the assumption \( D_zW(\cdot,u(\cdot),Ds\delta u(\cdot)) \in C^1(\Omega - \delta, \mathbb{R}^n) \). Then there exists a \( C^1_c(\Omega, \mathbb{R}^n) \) extension of this function; we denote by \( W_z \) any such extension. In order to derive (65) from (64), we use Theorem 3.2 to obtain

\[
\int_{\Omega} W_z(x) \cdot D^s_{\delta} \varphi(x) \, dx = - \int_{\Omega} \varphi(x) \text{div}^s_{\delta} W_z(x) \, dx - (n-1+s) \int_{\Omega} \int_{\Omega_{B,\delta}} \frac{\varphi(x) W_z(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_{\delta}(x-y) \, dy \, dx.
\]

This last integral is zero; indeed, \( \varphi(x) = 0 \) for \( x \in \Omega \setminus \Omega_{-\delta} \), while for \( x \in \Omega_{-\delta} \) and \( y \in \Omega_{B,\delta} \) we have \( \rho_{\delta}(x-y) = 0 \). Therefore,

\[
\int_{\Omega} W_z(x) \cdot D^s_{\delta} \varphi(x) \, dx = - \int_{\Omega} \varphi(x) \text{div}^s_{\delta} W_z(x) \, dx.
\]

We combine this equality with (64) and apply the fundamental lemma of the Calculus of Variations to obtain that equality (65) holds for a.e. \( x \in \Omega_{-\delta} \).

Note that (65) imposes an a.e. equality in \( \Omega_{-\delta} \) and not in \( \Omega \), which is natural since in \( \Omega_{\delta} \setminus \Omega_{-\delta} \) we already have the condition \( u = u_0 \). Even though (65) prescribes a pointwise condition, it is nonlocal because of the presence of \( D^s_{\delta} u \).

Appendices

A \( \mathbb{V}^s_\delta \) when \( n = 1 \)

In this appendix we explain the necessary changes in the proof of Theorem 5.9 when \( n = 1 \). We first need a result regarding the function \( Z \) appearing therein.

Lemma A.1. Let \( n = 1 \). Then:

a) The function \( Z \) of (44) can be identified with the tempered distribution

\[
\langle Z, \varphi \rangle = \int_0^\infty Z(\xi)(\varphi(\xi) - \varphi(-\xi)) \, d\xi, \quad \varphi \in \mathcal{S}
\]

and we have the convergence

\[
Z\chi_{B(0,\varepsilon)} \rightarrow Z \quad \text{in} \, \mathcal{S}' \quad \text{as} \, \varepsilon \rightarrow 0.
\]

b) The function

\[
Y(\xi) = -\frac{i\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}_{\delta}(0)} \chi_{B(0,1)}(\xi)
\]

can be identified with the tempered distribution

\[
\langle Y, \varphi \rangle = \int_0^\infty Y(\xi)(\varphi(\xi) - \varphi(-\xi)) \, d\xi, \quad \varphi \in \mathcal{S},
\]
we have the convergence
\[ Y_{\chi_{B(0,\varepsilon)^c}} \to Y \quad \text{in } S' \quad \text{as } \varepsilon \to 0 \] (69)
and
\[ \hat{Y}(x) = -\frac{1}{\pi Q_{s}^*(0)} \int_{0}^{1} \frac{1}{\xi} \sin(2\pi \xi x) \, d\xi. \] (70)

Proof. We start with \([a]\). Let us see that formula (66) defines a tempered distribution. By Propositions 5.2 and 5.5, there exists \( C > 0 \) such that
\[ |Z(\xi)| \leq \frac{1}{2\pi} \frac{1}{|\xi|} + \frac{1}{a_0} \frac{1}{|2\pi \xi|} \leq \frac{C}{|\xi|}, \quad |\xi| \leq 1. \]
Thus, by the mean value theorem
\[ \left| \int_{0}^{1} Z(\xi)(\varphi(\xi) - \varphi(-\xi)) \, d\xi \right| \leq 2C\|\varphi'\|_{\infty}. \] (71)
On the other hand, in the proof of Theorem 5.9\([b]\), we saw that \( Z \) decays to 0 at infinity faster than any negative power of \(|\xi|\). In particular, there exists \( C > 0 \) such that
\[ |Z(\xi)| \leq \frac{C}{\xi^2}, \quad |\xi| \geq 1. \]
Consequently,
\[ \left| \int_{1}^{\infty} Z(\xi)(\varphi(\xi) - \varphi(-\xi)) \, d\xi \right| \leq 2C \int_{1}^{\infty} \frac{1}{\xi^2} \, d\xi \|\varphi\|_{\infty}. \] (72)
Estimates (71) and (72) show that \( Z \) defined by (66) is in \( S' \).

As before, the fact that \( Z \) decays to 0 at infinity faster than any negative power of \(|\xi|\) implies that \( Z_{\chi_{B(0,\varepsilon)^c}} \in L^1(\mathbb{R}) \) for all \( \varepsilon > 0 \). In particular, \( Z_{\chi_{B(0,\varepsilon)^c}} \) considered as a distribution acts as follows: for each \( \varphi \in S \),
\[ \langle Z_{\chi_{B(0,\varepsilon)^c}}, \varphi \rangle = \int_{B(0,\varepsilon)^c} Z(\xi) \varphi(\xi) \, d\xi = \int_{\varepsilon}^{\infty} Z(\xi)(\varphi(\xi) - \varphi(-\xi)) \, d\xi. \]
As we saw in (71)–(72), the function \( Z(\xi)(\varphi(\xi) - \varphi(-\xi)) \) is in \( L^1((0,\infty)) \), so, by dominated convergence, we have that
\[ \int_{\varepsilon}^{\infty} Z(\xi)(\varphi(\xi) - \varphi(-\xi)) \, d\xi \to \int_{0}^{\infty} Z(\xi)(\varphi(\xi) - \varphi(-\xi)) \, d\xi \quad \text{as } \varepsilon \to 0, \]
which justifies the identification of the function \( Z \) with the distribution (66) and shows the convergence (67).

Thus, \([a]\) is proved. The proof of \([b]\) is analogous and we only write the details for the expression of \( \hat{Y} \). The function \( Y_{\chi_{B(0,\varepsilon)^c}} \) is in \( L^1(\mathbb{R}^n) \) for \( 0 < \varepsilon < 1 \) and
\[ \mathcal{F}(Y_{\chi_{B(0,\varepsilon)^c}}) = \int_{B(0,\varepsilon)^c} Y(\xi)e^{-2\pi i x \xi} \, d\xi = \frac{-1}{\pi Q_{s}^*(0)} \int_{\varepsilon}^{1} \frac{1}{\xi} \sin(2\pi \xi x) \, d\xi, \]
because of odd symmetry. As the function \( \xi \mapsto \frac{1}{\xi} \sin(2\pi \xi x) \) is in \( L^1(0,1) \), and convergence (69) holds, we obtain expression (70). \( \square \)
We are in a position to prove Theorem 5.9b.

Proof of Theorem 5.9b when \( n = 1 \). It remains to show that \( W \), or, equivalently, \( \hat{Z} \) is bounded. To that end, it is useful to introduce the function \( Y \) of (68) and express

\[
\hat{Z} = \mathcal{F}(Z_{\chi B(0,1)} - Y) + \hat{Y} + \mathcal{F}(Z_{\chi B(0,1)^c}).
\]

Since \( Z_{\chi B(0,1)^c} \in L^1(\mathbb{R}) \), by the Riemann–Lebesgue Lemma \( \mathcal{F}(Z_{\chi B(0,1)^c}) \in C_0(\mathbb{R}) \). Now we study the function

\[
Z_{\chi B(0,1)}(\xi) - Y(\xi) = \left[ \frac{-i\xi}{2\pi |\xi|^2} \left( \frac{1}{Q_\phi^s(\xi)} - \frac{1}{Q_\phi^s(0)} \right) - \frac{-i\xi}{a_0 |\xi| |2\pi \xi|^s} \right] \chi_{B(0,1)}(\xi).
\]

Now, as we saw in (25)–(26),

\[
\sup_{\xi \in B(0,1)} \left| \frac{-i\xi}{2\pi |\xi|^2} \left( \frac{1}{Q_\phi^s(\xi)} - \frac{1}{Q_\phi^s(0)} \right) \right| < \infty,
\]

and, on the other hand,

\[
\left| \frac{-i\xi}{a_0 |\xi| |2\pi \xi|^s} \right| \leq \frac{1}{a_0} \frac{1}{|2\pi \xi|^s},
\]

which is integrable in \( B(0,1) \). Therefore, \( Z_{\chi B(0,1)} - Y \in L^1(\mathbb{R}) \), so \( \mathcal{F}(Z_{\chi B(0,1)} - Y) \in C_0(\mathbb{R}) \).

It remains to show that \( \hat{Y} \) is bounded. By Lemma A.1b,

\[
\hat{Y}(x) = \frac{-1}{\pi Q_\phi^s(0)} \int_0^1 \frac{1}{\xi} \sin(2\pi \xi x) d\xi = \frac{-1}{\pi Q_\phi^s(0)} \int_0^x \frac{1}{\xi} \sin(2\pi \xi) d\xi.
\]

This latter function is known to be bounded. Therefore, \( \hat{Z} \) is bounded and the proof is concluded. \( \square \)

### B Fourier analysis results

In this appendix we collect several Fourier analysis results needed throughout the paper. They are possibly known to experts, but we have not found a precise reference.

First, we compute the Fourier transform of the vectorial version of the Riesz potential.

**Lemma B.1.**

a) If \( n \geq 2 \) and \( 0 < \alpha < n - 1 \), then

\[
\mathcal{F} \left( \frac{n - \alpha - 1}{\gamma(1 + \alpha)} \frac{x}{|x|^{n-\alpha+1}} \right)(\xi) = -i \frac{\xi}{|\xi|} |2\pi \xi|^{-\alpha} = -i \frac{\xi}{|\xi|} \hat{I}_\alpha.
\]

b) If \( n = 1 \) and \( 0 < s < 1 \), then

\[
\mathcal{F} \left( \frac{x}{|x|^{2-s}} \right) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi \xi|^s}.
\]

c) \( \mathcal{F} \left( \frac{x}{\sigma_{n-1} |x|^n} \right)(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi \xi|^n} \).
Proof. Fix $j \in \{1, \ldots, n\}$. On the one hand, we have that
\[
\frac{1}{\gamma(1 + \alpha)} \frac{\partial}{\partial x_j} \frac{1}{|x|^{n-(\alpha+1)}} = -\frac{n - \alpha - 1}{\gamma(1 + \alpha)} \frac{x_j}{|x|^{n-\alpha+1}}.
\]
Thus,
\[
\frac{1}{\gamma(1 + \alpha)} \mathcal{F}\left(\frac{\partial}{\partial x_j} \frac{1}{|x|^{n-(\alpha+1)}}\right)(\xi) = -\frac{n - \alpha - 1}{\gamma(1 + \alpha)} \mathcal{F}\left(\frac{x_j}{|x|^{n-\alpha+1}}\right)(\xi),
\] 
(73)

On the other hand, by standard properties of the Fourier transform, and, in particular, by (17),
\[
\frac{1}{\gamma(1 + \alpha)} \mathcal{F}\left(\frac{\partial}{\partial x_j} \frac{1}{|x|^{n-(\alpha+1)}}\right)(\xi) = 2\pi i \xi_j \hat{1}_{1+\alpha} = 2\pi i \xi_j |2\pi \xi|^{-(1+\alpha)} = \frac{i \xi_j}{|\xi|} |2\pi \xi|^{-\alpha}.
\] 
(74)

Putting together (73) and (74) we obtain the conclusion of (a).

Now we present the proof of (b). We recall formula (16) of the fractional version of the fundamental theorem of Calculus: for every $u \in C_c^\infty(\mathbb{R})$,
\[
u(x) = c_{1-s} \int_\mathbb{R} D^s u(y) \frac{x - y}{|x - y|^{2-s}} dy.
\]
Next, we take Fourier transform and use the formula for the convolution of a distribution with a Schwartz function:
\[
\hat{\nu}(\xi) = \mathcal{F}\left(D^s u\right)(\xi) \mathcal{F}\left(c_{1-s} \frac{x}{|x|^{2-s}}\right) = \frac{2\pi i \xi}{|2\pi \xi|} |2\pi \xi|^s \hat{\nu}(\xi) \mathcal{F}\left(c_{1-s} \frac{x}{|x|^{2-s}}\right),
\]
where we have used the explicit expression for Fourier transform of $D^s u$ (see [59, Th. 1.4] or [8, Lemma 3.1]). Now, we multiply both terms by $-i2\pi \xi$ and obtain that
\[
-i2\pi \xi \hat{\nu}(\xi) = |2\pi \xi|^{1+s} \hat{\nu}(\xi) \mathcal{F}\left(c_{1-s} \frac{x}{|x|^{2-s}}\right).
\]
Since that equality holds for every $u \in C_c^\infty(\mathbb{R})$, statement (b) follows.

Finally, we prove (c). Since $\frac{x}{|x|^n} \in L^1(B(0,1)) + L^\infty(B(0,1)^c)$ we have that $\frac{x}{|x|^n}$ belongs to $\mathcal{S}'$, and so does its Fourier transform. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. We apply the Fourier transform to the representation formula of Proposition 4.1, obtaining that
\[
\hat{\varphi}(\xi) = \mathcal{F}\left(\hat{\varphi}\right)(\xi) = 2\pi i \xi \hat{\varphi}(\xi) \mathcal{F}\left(c_{\sigma_{n-1}} \frac{x}{|x|^{n-2-s}}\right) (\xi).
\]
Since this is true for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ we infer that
\[
1 = 2\pi i \xi \mathcal{F}\left(c_{\sigma_{n-1}} \frac{x}{|x|^{n-2-s}}\right) (\xi).
\]
Therefore, there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $\xi \cdot g(\xi) = 0$ and
\[
\mathcal{F}\left(c_{\sigma_{n-1}} \frac{x}{|x|^{n-2-s}}\right) (\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi \xi|} + g(\xi).
\]
On the other hand, $\mathcal{F}\left(c_{\sigma_{n-1}} \frac{x}{|x|^{n-2-s}}\right)$ must be a vector radial function, as the Fourier transform of a vector radial function. Consequently (recall Definition 2.1), there exists $\tilde{g} : [0, \infty) \to \mathbb{R}$ such that $g(\xi) = \xi \tilde{g}(|\xi|)$. Thus, $|\xi|^2 \tilde{g}(|\xi|) = 0$, so $\tilde{g} = 0$ and, hence, $g = 0$ a.e. The proof is concluded. 

Now, we recall the following definitions and properties about the convolution and Fourier transform of tempered distributions. Recall from Section 2.2 the notation $\hat{f}$ for the reflection of $f$.

**Remark B.1.** Let $u, v \in S'$.

a) $\hat{v} \in S'$ is defined as
$$\langle \hat{v}, \varphi \rangle = \langle v, \hat{\varphi} \rangle \quad \forall \varphi \in S.$$

b) Assume that $\hat{v} \ast \varphi \in S$ for every $\varphi \in S$. Then the tempered distribution $v \ast u$ is defined as
$$\langle v \ast u, \varphi \rangle = \langle u, \hat{v} \ast \varphi \rangle \quad \forall \varphi \in S.$$

c) We have that $\mathcal{F}(\hat{v}) = \hat{v}$.

Finally, we show that the product property of the Fourier transform of a convolution also holds for tempered distributions.

**Lemma B.2.** Let $V, Q \in S'$ be such that $Q$ is a distribution with compact support. Then
$$\hat{V} \ast \hat{Q} = \hat{V} \hat{Q}.$$  

**Proof.** Firstly, we recall that the convolution $V \ast Q$ is well defined since $\hat{Q} \ast \varphi \in S$ for every $\varphi \in S$ (see [34, Th. 2.3.20]), and its action is defined as in Remark B.1b:
$$\langle V \ast Q, \varphi \rangle = \langle V, \varphi \ast \hat{Q} \rangle \quad \forall \varphi \in S.$$

Now, by definition of the Fourier transform in the sense of distributions, for every $\varphi \in S$,
$$\langle \hat{V} \ast \hat{Q}, \varphi \rangle = \langle V \ast Q, \hat{\varphi} \rangle = \langle V, \hat{\varphi} \ast \hat{Q} \rangle. \quad (75)$$

Next, by the Fourier transform of a convolution (of a distribution with a Schwartz function),
$$\hat{\varphi} \ast \hat{Q} = \mathcal{F}(\varphi \mathcal{F}^{-1}(\hat{Q})) = \mathcal{F}(\varphi \hat{Q}),$$

since $\mathcal{F}^{-1}(\hat{Q}) = \hat{Q}$ (see Remark B.1c). This also tells us that $\varphi \hat{Q}$ belongs to $S$, because so does $\hat{\varphi} \ast \hat{Q}$ (by the bijection of the Fourier transform in $S$). Actually, it is known that $\hat{Q}$ is a smooth function (see [34, Th. 2.3.21]). Therefore, continuing with (75) and using again the duality of the Fourier transform,
$$\langle \hat{V} \ast \hat{Q}, \varphi \rangle = \langle V, \varphi \ast \hat{Q} \rangle = \langle V, \mathcal{F}(\varphi \hat{Q}) \rangle = \langle \hat{V}, \varphi \hat{Q} \rangle.$$

As $\varphi \hat{Q} \in S$, the product $\hat{V} \hat{Q}$ is well defined in a distributional sense and
$$\langle \hat{V} \hat{Q}, \varphi \rangle = \langle \hat{V}, \varphi \hat{Q} \rangle = \langle \hat{V} \ast \hat{Q}, \varphi \rangle.$$

Consequently, the desired formula holds.

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References

[1] N. ABATANGELO and E. VALDINOCI, Getting acquainted with the fractional Laplacian, in Contemporary research in elliptic PDEs and related topics, vol. 33 of Springer INdAM Ser., Springer, Cham, 2019, pp. 1–105.

[2] R. A. ADAMS, Sobolev spaces, vol. 65 of Pure and Applied Mathematics, Academic Press, New York-London, 1975.

[3] B. AKSOYLU and T. MENGESHA, Results on nonlocal boundary value problems, Numer. Funct. Anal. Optim., 31 (2010), pp. 1301–1317.

[4] B. AKSOYLU and M. L. PARKS, Variational theory and domain decomposition for nonlocal problems, Appl. Math. Comput., 217 (2011), pp. 6498–6515.

[5] F. ANDREU, J. M. MAZÓN, J. D. ROSSI, and J. TOLEDO, A nonlocal p-Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions, SIAM J. Math. Anal., 40 (2009), pp. 1815–1851.

[6] J. C. BELLIDO, J. CUETO, and C. MORA-CORRAL, Bond-based peridynamics does not converge to hyperelasticity as the horizon goes to zero, J. Elasticity, 141 (2020), pp. 273–289.

[7] ———, Fractional Piola identity and polyconvexity in fractional spaces, Ann. I. H. Poincaré – AN, 37 (2020), pp. 955–981.

[8] ———, Γ-convergence of polyconvex functionals involving s-fractional gradients to their local counterparts, Calc. Var. Partial Differential Equations, 60 (2021), p. 7.

[9] J. C. BELLIDO and C. MORA-CORRAL, Existence for nonlocal variational problems in peridynamics, SIAM J. Math. Anal., 46 (2014), pp. 890–916.

[10] ———, Lower semicontinuity and relaxation via Young measures for nonlocal variational problems and applications to peridynamics, SIAM J. Math. Anal., 50 (2018), pp. 779–809.

[11] J. C. BELLIDO, C. MORA-CORRAL, and P. PEDREGAL, Hyperelasticity as a Γ-limit of peridynamics when the horizon goes to zero, Calc. Var. Partial Differential Equations, 54 (2015), pp. 1643–1670.

[12] F. BOBARU, J. T. FOSTER, P. H. GEUBEILLE, and S. A. SILLING, eds., Handbook of peridynamic modeling, Advances in Applied Mathematics, CRC Press, Boca Raton, FL, 2017.

[13] H. BREZIS, Functional analysis, Sobolev spaces and partial differential equations, Universitext, Springer, New York, 2011.

[14] L. CAFFARELLI and L. SILVESTRE, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations, 32 (2007), pp. 1245–1260.

[15] W. CHEN, Y. LI, and P. MA, The fractional Laplacian, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020.

[16] G. E. COMI and G. STEFANI, A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up, J. Funct. Anal., 277 (2019), pp. 3373–3435.
[17] B. Dacorogna, Direct methods in the calculus of variations, vol. 78 of Applied Mathematical Sciences, Springer, New York, second ed., 2008.

[18] M. D’Elia, Q. Du, C. Glusa, M. Gunzburger, X. Tian, and Z. Zhou, Numerical methods for nonlocal and fractional models, Acta Numer., 29 (2020), pp. 1–124.

[19] M. D’Elia, Q. Du, and M. D. Gunzburger, Recent progress in mathematical and computational aspects of peridynamics, in Handbook of Nonlocal Continuum Mechanics for Materials and Structures, G. Z. Voyiadjis, ed., Springer, 2019.

[20] M. D’Elia, M. Gulian, T. Mengesha, and J. M. Scott, Connections between nonlocal operators: from vector calculus identities to a fractional Helmholtz decomposition. ArXiv preprint 2112.05317, 2021.

[21] M. D’Elia, M. Gulian, H. Olson, and G. E. Karniadakis, Towards a unified theory of fractional and nonlocal vector calculus, Fract. Calc. Appl. Anal., 24 (2021), pp. 1301–1355.

[22] M. D’Elia and M. Gunzburger, The fractional Laplacian operator on bounded domains as a special case of the nonlocal diffusion operator, Comput. Math. Appl., 66 (2013), pp. 1245–1260.

[23] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou, Analysis of the volume-constrained peridynamic Navier equation of linear elasticity, J. Elasticity, 113 (2013), pp. 193–217.

[24] ——, A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws, Math. Models Methods Appl. Sci., 23 (2013), pp. 493–540.

[25] Q. Du and X. Tian, Stability of nonlocal Dirichlet integrals and implications for peridynamic correspondence material modeling, SIAM J. Appl. Math., 78 (2018), pp. 1536–1552.

[26] J. Duoandikoetxea, Fourier analysis, vol. 29 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001.

[27] E. Emmrich and O. Weckner, On the well-posedness of the linear peridynamic model and its convergence towards the Navier equation of linear elasticity, Commun. Math. Sci., 5 (2007), pp. 851–864.

[28] X. Feng and M. Sutton, A new theory of fractional differential calculus, Analysis and Applications, 19 (2021), pp. 715–750.

[29] I. Fonseca and G. Leoni, Modern methods in the calculus of variations: $L^p$ spaces, Springer, New York, 2007.

[30] M. Foss, P. Radu, and Y. Yu, Convergence analysis and numerical studies for linearly elastic peridynamics with Dirichlet-type boundary conditions, J. Peridyn. Nonlocal Model., (2022).

[31] M. D. Foss, P. Radu, and C. Wright, Existence and regularity of minimizers for nonlocal energy functionals, Differential Integral Equations, 31 (2018), pp. 807–832.

[32] W. H. Gerstle, Introduction to practical peridynamics, vol. 1 of Frontier Research in Computation and Mechanics of Materials and Biology, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016.

[33] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[34] L. Grafakos, Classical Fourier analysis, vol. 249 of Graduate Texts in Mathematics, Springer, New York, second ed., 2008.

[35] M. Gunzburger and R. B. Lehoucq, A nonlocal vector calculus with application to nonlocal boundary value problems, Multiscale Model. Simul., 8 (2010), pp. 1581–1598.

[36] B. Hinds and P. Radu, Dirichlet’s principle and wellposedness of solutions for a nonlocal p-Laplacian system, Appl. Math. Comput., 219 (2012), pp. 1411–1419.

[37] A. Javili, R. Morasata, E. Oterkus, and S. Oterkus, Peridynamics review, Math. Mech. Solids, 24 (2019), pp. 3714–3739.

[38] C. Kreisbeck and H. Schönberger, Quasiconvexity in the fractional calculus of variations: characterization of lower semicontinuity and relaxation, Nonlinear Anal., 215 (2022), pp. 112625, 26.

[39] S. Lang, Real analysis, Addison-Wesley, Reading, MA, second ed., 1983.

[40] R. B. Lehoucq and S. A. Silling, Force flux and the peridynamic stress tensor, J. Mech. Phys. Solids, 56 (2008), pp. 1566–1577.

[41] E. Madenci and E. Oterkus, Peridynamic Theory and Its Applications, Springer, New York, NY, 2014.

[42] C. Martínez, M. Sanz, and F. Periago, Distributional fractional powers of the Laplacean. Riesz potentials, Studia Math., 135 (1999), pp. 253–271.

[43] T. Mengesha, Nonlocal Korn-type characterization of Sobolev vector fields, Commun. Contemp. Math., 14 (2012), pp. 1250028, 28.

[44] T. Mengesha and Q. Du, Nonlocal constrained value problems for a linear peridynamic Navier equation, J. Elasticity, 116 (2014), pp. 27–51.

[45] ———, On the variational limit of a class of nonlocal functionals related to peridynamics, Nonlinearity, 28 (2015), pp. 3999–4035.

[46] ———, Characterization of function spaces of vector fields and an application in nonlinear peridynamics, Nonlinear Anal., 140 (2016), pp. 82–111.

[47] T. Mengesha and D. Spector, Localization of nonlocal gradients in various topologies, Calc. Var. Partial Differential Equations, 52 (2015), pp. 253–279.

[48] Y. Mizuta, Potential theory in Euclidean spaces, vol. 6 of GAKUTO International Series. Mathematical Sciences and Applications, Gakkotosho Co., Ltd., Tokyo, 1996.

[49] G. Molica Bisci, V. D. Radulescu, and R. Servadei, Variational methods for nonlocal fractional problems, vol. 162 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2016. With a foreword by Jean Mawhin.

[50] A. C. Ponce, An estimate in the spirit of Poincaré’s inequality, J. Eur. Math. Soc., 6 (2004), pp. 1–15.

[51] ———, A new approach to Sobolev spaces and connections to Γ-convergence, Calc. Var. Partial Differential Equations, 19 (2004), pp. 229–255.
[52] ———, *Elliptic PDEs, measures and capacities*, vol. 23 of EMS Tracts in Mathematics, European Mathematical Society (EMS), Zürich, 2016. From the Poisson equations to nonlinear Thomas-Fermi problems.

[53] C. Pozrikidis, *The fractional Laplacian*, CRC Press, Boca Raton, FL, 2016.

[54] J. F. Rodrigues and L. Santos, *On nonlocal variational and quasi-variational inequalities with fractional gradient*, Appl. Math. Optim., 80 (2019), pp. 835–852.

[55] X. Ros-Oton, *Nonlocal elliptic equations in bounded domains: a survey*, Publ. Mat., 60 (2016), pp. 3–26.

[56] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. (9), 101 (2014), pp. 275–302.

[57] A. Schikorra, D. Spector, and J. Van Schaftingen, *An L1-type estimate for Riesz potentials*, Rev. Mat. Iberoam., 33 (2017), pp. 291–303.

[58] J. Scott and T. Mengesha, *A fractional Korn-type inequality*, Discrete Contin. Dyn. Syst., 39 (2019), pp. 3315–3343.

[59] T.-T. Shieh and D. E. Spector, *On a new class of fractional partial differential equations*, Adv. Calc. Var., 8 (2015), pp. 321–336.

[60] ———, *On a new class of fractional partial differential equations II*, Adv. Calc. Var., 11 (2018), pp. 289–307.

[61] S. A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, J. Mech. Phys. Solids, 48 (2000), pp. 175–209.

[62] S. A. Silling, M. Epton, O. Weckner, J. Xu, and E. Askari, *Peridynamic states and constitutive modeling*, J. Elasticity, 88 (2007), pp. 151–184.

[63] S. A. Silling and R. B. Lehoucq, *Peridynamic theory of solid mechanics*, in Advances in Applied Mechanics, H. Aref and E. van der Giessen, eds., vol. 44 of Advances in Applied Mechanics, Elsevier, 2010, pp. 73–168.

[64] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[65] E. M. Stein and G. Weiss, *Fractional integrals on n-dimensional Euclidean space*, J. Math. Mech., 7 (1958), pp. 503–514.

[66] X. Tian and Q. Du, *Analysis and comparison of different approximations to nonlocal diffusion and linear peridynamic equations*, SIAM J. Numer. Anal., 51 (2013), pp. 3458–3482.

[67] G. Z. Voyiadjis, ed., *Handbook of Nonlocal Continuum Mechanics for Materials and Structures*, Springer, Cham, 2019.

[68] M. Šilhavý, *Fractional vector analysis based on invariance requirements (critique of coordinate approaches)*, Contin. Mech. Thermodyn., 32 (2020), pp. 207–228.

[69] K. Zhou and Q. Du, *Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions*, SIAM J. Numer. Anal., 48 (2010), pp. 1759–1780.