VERIFICATION OF THE ORDINARY CHARACTER TABLE OF THE BABY MONSTER

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ABSTRACT. We prove the correctness of the character table of the sporadic simple Baby Monster group that is shown in the ATLAS of Finite Groups.

Dedicated by the three remaining authors to the memory of our friend and colleague Kay Magaard, who sadly passed away during the preparation of this paper.

1. Introduction

Jean-Pierre Serre has raised the question of verification of the ordinary character tables that are shown in the ATLAS of Finite Groups [7]. This question was partially answered in the paper [6], the remaining open cases being the largest two sporadic simple groups, the Baby Monster group $B$ and the Monster Group $M$, and the double cover $2.B$ of $B$.

The current paper describes a verification of the character table of $B$. The computations shown in [4] then imply that also the ATLAS character table of $2.B$ is correct. As in [6], one of our aims is to provide the necessary data in a way that makes it easy to reproduce our computations.

2. Strategy

We begin with two preliminary sections, whose aim is to prove that certain specified matrices do indeed generate copies of the Baby Monster. These matrices can then be used in the main computation.

The $Y_{555}$ presentation of the BiMonster implies a $Y_{433}$ presentation for $B$ (see [11]), provided that the Schur multiplier $H^2(M, C^*)$ of the Monster has odd order. We did not find a suitable reference for the latter result in the literature, and we therefore give a proof in Section 3 using only information from [10]. For completeness, we extend this to a proof that the Schur multiplier of the Monster is trivial. The latter proof, however, uses the structures of all the $p$-element centralizers, which are not so well supported by the published literature.

In Section 4 we use the $Y_{433}$ presentation to prove that three pairs of matrices, of dimension 4370 over the field with two elements and of dimension 4371 over the fields with three and five elements, respectively, generate the group $B$, and, moreover, that mapping one pair of these generators to any other such pair defines a group isomorphism.

In Section 5 we compute a first approximation to the list of conjugacy class names, by establishing invariants, in terms of the above three matrix representations of $B$, that in fact distinguish almost all conjugacy classes of cyclic subgroups of $B$. 
These invariants are then used to determine the power maps between the specified union of conjugacy classes.

In order to compute the conjugacy classes of $B$ and the corresponding centralizer orders, we apply the following general statement. For a group $G$ and an element $g \in G$, if $x, y \in G$ power to $g$ then $x \sim g \iff x \sim N g y$ where $N := N_G(\langle g \rangle)$; moreover $C_G(x) = C_N(x)$. This implies that it suffices to find the normalizers (or overgroups thereof) of prime order subgroups, and their character tables. Section 6 deals with the first problem, Section 7 with the second. At this point, we know the conjugacy classes of $B$, and their lengths. Some further calculations then match these up with the names listed in Section 5 by using the given invariants and some small extra arguments.

Finally, the irreducible characters of $B$ are computed in Section 8, using character theoretic methods such as induction from several subgroups of $B$.

3. The Schur multiplier of the Monster is trivial

**Theorem 3.1.** The Schur multiplier of the Monster sporadic simple group is trivial.

We start the proof of Theorem 3.1 by proving that if the Schur multiplier of the Monster is non-trivial, then it has odd order. In particular, this result is required at a stage where the Monster is known to exist, but the character tables and conjugacy classes of the Monster and the Baby Monster have not been established. The background knowledge we are permitted to use for this exercise is that given in Griess’s construction of the Monster [10, Table 10.1].

**Lemma 3.2.** The Schur multiplier of the Monster is either trivial or has odd order.

**Proof.** Let $\overline{M}$ be the Monster simple group and $M$ be a covering group of $\overline{M}$ with $M/Z(M) = \overline{M}$ and $M' = M$. Suppose that $Z(M)$ has order divisible by 2. Then we may as well suppose that $Z(M)$ has order 2. Let $S \in \text{Syl}_2(M)$ and $P$ and $R$ be over-groups of $S$ such that $P$ is the centralizer of a central involution in $S$ and $R$ is the normalizer of an elementary abelian subgroup of order 4. Then $P \cong 2^{1+24}.C_{O_2}$ with $O_2(P)$ extraspecial and $R \cong 2^{2+11+(11^* \otimes 2)}.(S_3 \times M_{24})$.

These two subgroups appear in [10] Table 10.1 and so are available without knowledge of the character table or conjugacy classes of the Monster. Furthermore, the data in [10] Table 10.1 describes the intersection of $P$ and $R$ as well as several other subgroups we shall encounter.

We initially study $Q = O_2(P)$. Let $Z$ be the preimage in $M$ of $Z(\overline{S})$ containing $Z(M)$. Then $Z$ has order 4 and, as $P$ is perfect, $Z$ is centralized by $P$ and so $Z = Z(S)$. If $Z = Q'$, then, as $P$ acts irreducibly on $Q/Z$ and centralizes $Z$, we have $Q$ is special. Now [13] Lemma 2.73 (i) implies that $Q/Z$ is not an absolutely irreducible module for $P/Q$, which contradicts [10] Lemma 2.41. So $Q'$ has order 2. Since $Q' \not\subseteq Z(M)$, we have $Z$ is elementary abelian. Set $U = Q'$. Then $Q/U$ is abelian. If $Q/U$ is not elementary abelian, then $P/Q$ acts non-trivially on $\Omega_1(Q/U)/\Phi(Q/U)$ which has order $2^{23}$, a contradiction as $P$ contains a non-central chief factor of order $2^{24}$. Thus $\Phi(Q) = U$ and $Q/U$ is a module for $P/Q$. Since $H^1(P/Q, Q/Z) = 0$ by [10] Lemma 2.11, we have $Q/U$ is isomorphic to the direct
sum of the Leech lattice module for and the trivial module for \( P/Q \). Let \( Q_1 < Q \) be the normal subgroup of \( P \) of order \( 2^{25} \). Then \( Q_1 Z(M) = Q \). Now \( P/Q_1 \) is an extension of \( P/Q \) by a subgroup of order \( Z(M)Q_1/Q_1 \) of order 2. This extension cannot split. Indeed, if it does split, then \( P \) contains a subgroup \( P_1 \) of index 2 which complements \( Z(M) \) and so \( S_1 = S \cap P_1 \) complements \( Z(M) \) and we see that \( M \) splits over \( Z(M) \) by the theorem of Gaschütz [1] (10.4). Now we have that \( Z(M)Q_1/Q_1 = Z(P/Q_1) \) and that \( P/Q_1 \cong 2 \cdot CO_1 \) is non-split.

Let \( g \in R \setminus P \) and set \( E = Q \cap Q^g \). Since \( Z \neq Z^g, U \neq U^g \) and we obtain

\[
\Phi(E) \leq \Phi(Q) \cap \Phi(Q^g) = U \cap U^g = 1.
\]

Hence \( E \) is elementary abelian and therefore \( |E| \leq 2^{14} \) as \( E \leq Z(M)Q_1 \cong 2 \times 2^{1+24} \).

Since \( P \cap P^g \) centralizes \( ZZ^g \) we have

\[
P \cap P^g \geq (P \cap R)′ = R^∞ \sim 2^{1+2+11+11'}..M_{24}.
\]

Therefore, \( (Q \cap O_2(R))Q^g/Q^g \) has order at most \( 2^{11} \). Further, using \( QO_2(R)/O_2(R) \) is a normal 2-subgroup in \( P \cap R \) which has index 3 in \( R \), we observe,

\[
2^{26} = |Q| = |QO_2(R)/O_2(R)| \cdot |(Q \cap O_2(R))/E| \cdot |E| \leq 22^{11}.2^{14} = 2^{26},
\]

and so we have equality in the third position. In particular \( E \) is a maximal abelian subgroup of \( Q \) and also \( U \leq E \). Similarly \( U^g \in E \). Thus \( E \) is normalized by \( R^∞ \)

and \( \langle Q, Q^g \rangle \) and so by \( R^∞ \langle Q, Q^g \rangle = R \). Now we have \( R/E \cong (2^{1+2} \otimes 2).S_3 \times M_{24} \)

with \( O_2(R/E) \) the tensor product module for \( S_3 \times M_{24} \). Let \( E_1 = Q_1 \cap E = Q_1 \cap Q^g \). Notice \( E_1 \) is a maximal abelian subgroup of \( Q_1 \) and so \( |E : E_1| = 2 \).

Because \( (Q^g \cap O_2(R))/Q^g \) is a normal subgroup of \( R^∞Q/Q \), we have \( (Q^g \cap O_2(R))/Q^g \) is elementary abelian of order \( 2^{11} \). Since \( O_2(R^∞Q/Q) \) has order \( 2^{11} \) and is an irreducible \( R^∞Q/Q \)-module, we see that \( O_2(R^∞Q/Q) \) is elementary abelian of order \( 2^{12} \). Therefore

\[
U^g \leq (Q^g \cap O_2(R))^g \leq Q_1
\]

and so

\[
[E_1, \langle Q, Q^g \rangle] \leq \langle [E_1, Q], [E_1, Q^g] \rangle = \langle U, U^g \rangle \leq E_1.
\]

Thus \( E_1 \) is normalized by \( R^∞ \leq P \cap P^g \) and by \( \langle Q, Q^g \rangle \). Since \( (Q, Q^g)O_2(R)/O_2(R) \cong S_3, E_1 \) is normalized by \( R \). Consider the group \( O_2(R/E_1) \). We know that \( O_2(R/E) \) has order \( 2^{22} \) and is a minimal normal subgroup of \( R/E \) and so either \( O_2(R)/E_1 \) is elementary abelian or extraspecial. In the latter case, the module \( O_2(R)/E \) supports a quadratic form coming from the square map to \( E/E_1 \). But the module \( O_2(R)/E \) is not self-dual as an \( R/O_2(R) \)-module. Hence \( O_2(R)/E_1 \) is elementary abelian. Let \( x \in R \) be an element of order 3 such that \( \langle x \rangle O_2(R) \) is normal in \( R \). Then \( O_2(R)/E_1, \langle x \rangle \) is normal in \( R/E_1 \) and has index \( 2 \) in \( O_2(R)/E_1 \). Let \( W \) be the preimage of \( O_2(R)/E_1, \langle x \rangle \). Then \( O_2(R) = WZ(M) \). Since the Schur multiplier of \( M_{24} \) is trivial, \( R^∞W/W \cong M_{24} \) and \( |R : R^∞W| = 12 \). Now \( Z(M)Q_1R^∞W/R^∞W \) is elementary abelian of order \( 2^2 \). Thus \( S \cap Q_1WR^∞ \) is complemented by \( Z(M) \) in \( S \) and so Gaschütz’s Theorem provides a contradiction. This proves the result.

\[ \square \]

Before we proceed to the odd primes we present a lemma which captures some of the arguments presented in the first paragraphs of the proof of Lemma 2.2.
Lemma 3.3. Suppose that $p$ is an odd prime, $X$ is a finite group and $|Z(X)| = p$. Set $\overline{X} = X/Z(X)$. Assume that $P \leq X$, $Z(X) \leq P$, $Q = O_p(P)$ and the following conditions hold:

(i) $\overline{Q}$ is extraspecial, $P \leq C_X(Q')$ and $C_{\overline{Q}}(Q) = \overline{Q}$;

(ii) $O_{p,2}(P) > Q$; and

(iii) the $\text{GF}(p)$ representation of $\overline{T}$ on $\overline{Q'/Q}$ is absolutely irreducible.

Then $Q$ splits over $Z(X)$ with complement $[Q, O_{p,2}(P)]$ normal in $P$. Furthermore, if $P$ contains a Sylow $p$-subgroup of $X$, $\overline{T}$ is perfect and the Schur multiplier of $P/Q$ has trivial $p$-part, then $X$ splits over $Z(X)$.

Proof. Assume that $|Q| = p^{1+2n}$ for some $n \geq 1$. Let $Z$ be the preimage of $\overline{Q}'$ in $X$ containing $Z(X)$. Then $Z$ has order $p^2$. It follows that $Q/C_Q(Z)$ has index at most $p$ in $Q$ and, as $C_Q(Z)$ is normal in $P$, (iii) implies $Q = C_Q(Z)$. Hence $Z = Z(Q)$. If $Q' = Z$, then [13] Lemma 2.73 yields $\text{End}_{\text{GF}(p)}(Q/Q')$ has order at least $p^2$, contrary to (iii). Hence $Q'$ has order $p$ and $Z = Z(X)Q'$ is elementary abelian. As $\overline{T}$ acts irreducibly on $\overline{Q'/Z(Q)}$, setting $Q^*$ to be the preimage of $\Omega_1(Q/Q')$, we have $Q = Q'Z(X) = Q^*$. Hence $\Phi(Q) = Q'$ and $Q/Q^*$ is elementary abelian of order $p^{2n+1}$.

Set $T = O_{p,2}(P)$. As $T > Q$ and $T$ centralizes $Z(X)$, Maschke’s Theorem gives

$$Q/Q^* = [Q/Q', T] \times C_{Q/Q'}(T) = [Q/Q', T] \times Z/Q'$$

using (iii). In particular, $[Q, T]$ has index $p$ in $Q$, $[Q, T]$ is normal in $P$ and $Q = Z(X)[Q, T]$. This proves that $Q$ splits over $Z(X)$ with complement $Q_0 = [Q, T]$ normal in $P$. This is the first claim of the lemma.

We now assume that $P$ contains a Sylow $p$-subgroup of $X$, $P/Q$ is perfect and that the Schur multiplier of $P/Q$ has trivial $p$-part. Then, as $Z(X)Q_0/Q_0 \leq Z(P/Q_0)$, there is a normal subgroup $P^*$ of $P$ of index $p$ such that $P/Q_0 = Z(X)P^*/Q_0$. In particular, there exists a Sylow $p$-subgroup $S$ of $P$ and hence of $X$ such that $S = (S \cap P^*)Z(X)$. That is $S$ splits over $Z(X)$ and so therefore does $M$ by the theorem of Gaschütz, a contradiction. $\square$

For the remainder of the proof of Theorem 3.1 we suppose that the structure of the centralizers in the Monster of $p$-elements are known for odd primes $p$. We take the structure of these subgroups from Wilson’s 1985 paper [18].

Proof of Theorem 3.1. As before let $\overline{M}$ be the Monster simple group and $M$ be a covering group of $\overline{M}$ with $M/Z(M) = \overline{M}$ and $M' = M$. Since we intend to prove that $Z(M)$ is trivial, Lemma 3.3.2 shows we may assume that it has prime order $p$ for some odd prime $p$. By [11] 33.14, we have the Sylow $p$-subgroups of $\overline{M}$ are not cyclic. This means that $p \in \{3, 5, 7, 11, 13\}$ just by looking at $|M|$.

Assume that $p \in \{3, 5, 7\}$ and let $\overline{r}$ be a $p$-central element of $\overline{M}$ and $P$ be such that $\overline{T} = C_{\overline{M}}(\overline{r})$. Then $P$ contains a Sylow $p$-subgroup $S$ of $M$ and

$$\overline{T} \sim \begin{cases} 3^{1+12}.2.Suz & p = 3 \\ 5^{1+6}.2.J_2 & p = 5 \\ 7^{1+4}.2.A_7 & p = 7. \end{cases}$$
(see [18] Theorems 3, 5 and 7). Application of Lemma 3.3 shows that \( p = 3 \) and \( P \) has a normal subgroup \( Q_1 \) such that \( P/Q_1 \cong \text{Suz} \). The following proof is similar to the proof in Lemma 3.2. Let \( P \) be a 3-local subgroup of \( M \) such that \( \mathcal{P} = 3_{+}^{1+12}.2\text{Suz} \). Let \( L \) be a subgroup of \( M \) containing \( S \) with

\[
\mathcal{P} \sim 3^{2+5+(5\otimes 2)}.(\text{GL}_2(3) \times M_{11}).
\]

Set \( Q = O_3(P), R = O_3(L) \). We have \( Q \cong 3 \times 3^{1+12} \) by Lemma 3.3. \( L = 3^{1+2+5+(5+5)}M_{11} \) and \( L^{-1}Q/Q \cong 3^{3}M_{11} \). Let \( g \in L \setminus P \). Then \( E = Q \cap Q^g \) has index at most \( 3^6 \) in \( Q \) and \( \Phi(E) \leq \Phi(Q) \cap \Phi(Q^g) = Q' \cap Q^g = 1 \). Thus \( E \) is a maximal elementary abelian subgroup of \( Q \). Since \( E \) is normalized by \( L^{-1}Q \), \( Q, Q^g \) and \( O_{3,2}(P) \), we have \( E \) is normal in \( L \). Set \( Q_1 = [Q, O_{3,2}(P)] \) and \( E_1 = Q_1 \cap E \). Then \( [E : E_1] = 3 \). Since \( (Q^g \cap R)Q_1/Q_1 \) is elementary abelian, we have \( E_1 \) is normal in \( L \). Now \( R/E_1 \) is a minimal normal subgroup of order \( 3^{10} \) in \( L/E_1 \). If \( R/E_1 \) is extraspecial, then \( L/R \) must be isomorphic to a subgroup of \( S_{10}(3) \) by [23]. This is impossible as elements of order 11 do not commute with elements of order 3 in \( S_{10}(3) \). Hence \( R/E_1 \) is elementary abelian. Now \( R = Z(M)[R, O_{3,2}(L)] \) with \( |R : [R, O_{3,2}(L)]| = 3 \) and \( O_{3,2}(L)/R \cong \text{Qs} \). Since the Schur multiplier of \( M_{11} \) is trivial, we now have \( Q_1Z(M)L^{-1}R/L^{-1}R \) is elementary abelian of order 9 and \( S \leq Q_1Z(M)L^{-1}R \). It follows that \( S \cap Q_1L^{-1} \) complements \( Z(M) \) in \( S \) and so \( M \) splits over \( Z(M) \) by Gaschütz, a contradiction. Thus we have shown \( p \in \{11, 13\} \).

Suppose then \( Z(M) \) has order 11. Let \( S \in \text{Syl}_{11}(M) \). Then \( \overline{S} \) is elementary abelian of order \( 11^2 \) and \( N_{M}(\overline{S}) \sim 11^{2} : (5 \times \text{SL}_2(5)) \). In particular, either \( S \) is elementary abelian, in which case Gaschütz’s theorem, provides a contradiction or \( S \) is extraspecial. Hence \( S \) is extraspecial. Therefore \( C_{\text{Aut}(S)}(Z(S))/\text{Im}(\text{S}) \cong \text{SL}_2(11) \) by [23]. However \( N_{M}(S)/S \) is not isomorphic to a subgroup of \( \text{SL}_2(11) \), a contradiction.

Finally, consider the case that \( Z(M) \) has order 13. Then \( \overline{S} \) contains a subgroup \( \mathcal{P} \sim 13^{2} : \text{SL}_2(13).4 \). Let \( S \in \text{Syl}_{13}(L) \) and \( Q = O_{13}(L) \). Since \( L/Q \) is not isomorphic to a subgroup of \( \text{SL}_2(13) \), \( Q \) is not extraspecial and so we have that \( Q \) is elementary abelian. Thus \( Q = [Q, L]Z(M) \) with \( |[Q, L]| = 13^{2} \). Now the fact that \( L/[Q, L] \) has shape \( 13.\text{SL}_2(13).4 \) and we find a subgroup \( S_0 < S \) so that \( S = Z(M)S_0 \) and we have a contradiction via Gaschütz’s theorem. The elimination of this final case, completes the proof of the theorem.

4. Verifying a Presentation for the Baby Monster

In this section we give words in the ‘standard generators’ for the Baby Monster, that represent the 11 transpositions in the \( Y_{433} \) presentation. This provides a relatively straightforward test to prove that a given black-box group is in fact isomorphic to the Baby Monster.

4.1. The presentation. A presentation for the Baby Monster sporadic simple group \( \mathbb{B} \) was conjectured in the ATLAS [7], and proved by Ivanov [11], subject to the Monster not having a proper double cover. This hypothesis has been proved in Lemma 3.2

The presentation is on 11 generators \( t_i \) (\( 1 \leq i \leq 11 \)), satisfying the Coxeter relations \( t_i^2 = 1 \) for all \( i \), \( (t_it_j)^3 = 1 \) for \( (i, j) = (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (5, 9), (9, 10), (10, 11) \), and \( (t_it_j)^2 = 1 \) for \( i < j \) otherwise. Adjoining one extra relation, \( (t_5t_4t_3t_2t_1t_0)^{10} = 1 \), nicknamed the ‘spider relation’, gives a presentation for \( 2 \times 2.\mathbb{B} \). To obtain a presentation for \( \mathbb{B} \) itself, we need two
extra relations, \((t_5 t_4 t_3 t_4 t_5 t_8 t_9)^9 = 1\) and \((t_5 t_4 t_3 t_6 t_9 t_{10} t_{11})^9 = 1\). Since the Coxeter diagram has three ‘arms’, of lengths 4, 3, 3, this presentation is known as the \(Y_{433}\) presentation.

Matrices generating a copy of the Baby Monster were first produced in the early 1990s [19]. These act on a vector space of dimension 4370 over the field of order 2. In order to prove, without relying on the character table, that these matrices do indeed generate the Baby Monster, a method was given for producing elements of this group that satisfy the \(Y_{433}\) presentation for the Baby Monster. However, actual words for these elements were not given in [19].

In this section we rectify this deficiency in [19], and hence enable the reader to check relatively easily that the matrices given in [22], that are claimed to generate the Baby Monster in various different representations, do in fact generate the Baby Monster. In addition to the representation over the field of order 2, already mentioned, we checked the representations over the fields of order 3 and 5 constructed in [15]. All three of these representations will be required later on, for determining certain class fusions and power maps.

We begin with the ‘standard generators’ in the sense of [20, 22], that is an element \(a \in 2C\) and an element \(b \in 3A\) such that \(ab\) has order 55 and \((ab)^4bab(ab^2)^2\) has order 23. The cited references explain how to find such generators in a group which is in fact isomorphic to the Baby Monster. All calculations described in this Section were performed using the C Meataxé written by Michael Ringe [16], based on the original Meataxé of Richard Parker [14].

4.2. Finding the generators for the presentation. The calculations in this section were performed using the standard generators for (a group that is claimed to be) the Baby Monster in its 4370-dimensional representation over the field of order 2, taken from [22]. Following the 10-step method described in [19], we proceed as follows. Steps 1–4 are devoted to finding generators \(t_1, \ldots, t_8, t_{11}\) for a particular subgroup \(2 \times S_9\). In Steps 5–8 we centralize successively the elements \(t_1, t_3 t_4, t_6 t_7\) and \(t_8\) to produce a small number of candidates for \(t_9\) and \(t_{10}\). These candidates are tested in Step 9, at which point all the required generators have been found.

Step 10 tests the relations.

**Step 1. Take an arbitrary 2A-element, and call it \(t_{11}\).** Find \(C(t_{11}) \cong 2^2 E_6(2):2\).

The element \(d = (ab)^{15}b\) has order 38, and powers to the involution \(t_{11} = d^{19}\). The centralizer of \(t_{11}\) is generated by \(d\) and \(c = (at_{11})^3\).

**Step 2. Find a subgroup \(F \cong F_{122}:2\) inside \(C(t_{11})\).**

We restrict the representation of \(B\) to \(H := \langle c, d \rangle \cong 2^2 E_6(2):2\), find the composition factors using the Meat-axe program chop, and extract a 78-dimensional irreducible representation of the quotient \(\overline{H} := 2 E_6(2):2\), in which the computational searches for steps 2–4 are performed. (This use of a small representation reduces the computation time by a factor of around \(10^5\).) The invariant \(q(x) = \text{rank}(1 + x)\) is useful for identifying conjugacy classes. In particular, \(cd\) is an element of order 26 in the outer half of \(\overline{H}\), so powers to an involution \(x \in 2D\), in the \(\text{ATLAS}\) notation for conjugacy classes in \(2 E_6(2):2\). We calculate \(q(x) = 26\). As \(\overline{\sigma}\) is an involution in the outer half and \(q(\overline{\sigma}) = 36\), we deduce that \(\overline{\sigma} \in 2E\).

Similarly, \(\overline{cd}\) has order 36 and therefore powers to a 3C-element \(x\), with \(q(x) = 54\), while \(\overline{cd^2}\) has order 48 and therefore powers to a 3A-element \(y\), with \(q(y) = 42\). We then find that \(\overline{cd^3}\) has order 30 and powers to an element \(z\) of order 3 with
$q(z) = 48$. Hence $z$ is in class $3B$. Looking at a few groups generated by conjugates of $\tau$ and $(\alpha\delta^3)^{10}$ we quickly find that if $\tau = [(\alpha\delta^3)^{10}]^d$ then $F := \langle \tau, \tau \rangle \cong F_{22} : 2$.

**Step 3. Find a subgroup $S \cong S_{10}$ inside $\mathcal{F}$.**

The elements $e$ and $e e c$ then generate the subgroup $\mathcal{F}_{22}$ of index 2 in $\mathcal{F}$, in which we find $((e e c)(e e)^3)(e e)^2 e e c$ is an element of order 10 powering to an element of $\mathcal{F}_{22}$-class $2A$. The element $(e e)^3 c(e e)^3$ has order 9 and is most likely to be in class $9C$ in $\mathcal{F}_{22}$. Since there is no simple test for this, we proceed and hope for the best. Looking at conjugates of these elements we soon find a pair $f, g$ generating $S := \langle f, g \rangle \cong S_{10}$, as follows.

\[
\begin{align*}
\overline{f} &= \frac{((e e)^3 c(e e)^3)(e e)^2 e e c)}{(e e)^3 c(e e)^3)}
\end{align*}
\]

**Step 4. In $(\overline{t_{11}}) \times S$ find transpositions $\overline{t_1}, \ldots, \overline{t_6}$ generating $T \cong S_9$ with the required Coxeter relations.**

These transpositions can be taken as $\overline{t_n} = \overline{f}$ and $\overline{t_{n+2}} = \overline{f}^n \overline{g}$ for $0 \leq n \leq 6$.

**Step 5. Find $C = C(t_1) \cong 2 \times E_6(2) : 2$.**

This step has to be carried out in the 4370 dimensional representation. Standard dihedral group methods give the element $p = ((ab)^3 t_{11} (ab)^{-3} t_1 (ab)^5)^{-1}$ that conjugates $t_{11}$ to $t_1$, so that $h = e^p$ and $i = d^p$ generate the group $\mathcal{C} := \langle h, i \rangle = C(t_1)$.

**Step 6. Find $D = C_{\mathcal{C}}(\langle t_3 t_4 \rangle) \cong 6 \times U_6(2)$.**

Again we retract the representation of $\mathcal{R}$ and extract a copy of the 78-dimensional representation of $\langle h, i \rangle$, in which to carry out steps 6–8. Following the instructions in [19] we found two elements of the centralizer of $\langle t_3 t_4 \rangle$ to be $\overline{f} = [(t_5)^{3i}, t_3 t_4]$ and $\overline{k} = [(t_5)^{3j}, t_3 t_4]$. Together with $\overline{t_{10} t_7}$ and $\overline{t_8}$, these are enough to generate $\mathcal{D} \cong 3 \times U_6(2)$.

**Step 7. Find $E = C_{\mathcal{D}}(\langle t_6 t_7 \rangle) \cong 3 \times U_4(2)$.**

Similarly we found the following elements centralizing $\overline{t_{10} t_7}$:

\[
\begin{align*}
\overline{t}_3 &= \overline{[t_6^k, t_6 t_7]}[t_6^k, t_6 t_7] \\
\overline{t}_4 &= \overline{[t_8^{jk}, t_6 t_7]} \\
\overline{t}_5 &= \overline{[t_8^{jk}, t_6 t_7]}
\end{align*}
\]

These are sufficient to generate $E \cong 3 \times U_4(2)$.

**Step 8. Find the twelve [sic] transpositions in $E$ which commute with $t_8$.**

There is a slight error in [19] at this point. There are in fact 13 transpositions in $U_4(2)$ that centralize $t_8$, rather than 12 as stated there. They are the 13 transpositions in a copy of 2 ($A_4 \times A_4$).2. One is the central involution and the other 12 are in the outer half. Together they generate a subgroup $2^{1+4} : S_3$ of index 3. Presumably the central involution was omitted from the original calculation. However, it commutes with neither $t_5$ nor $t_{11}$, so is not a candidate for $t_9$ or $t_{10}$.

First we looked for conjugates of $\overline{t_4}$ that commute with $\overline{t_8}$, using the elements $\overline{t_5} = [(t_3 t_4)^{3i}]$ and $\overline{t_5} = [(t_3 t_4)^{3j}, t_3 t_4]$, both of order 9, for the conjugation. Then the four conjugates $\overline{t_4}$, $\overline{t_4}$, $\overline{t_4}$, $\overline{t_4}$ and $\overline{t_4}$ are sufficient to generate $2^{1+4} : S_3$. The central involution is $\langle m_1 m_2 m_3 m_4 \rangle^3$. Then $\overline{t_7}$ and its conjugates by $\overline{m_2 m_3}$, $\overline{m_2 m_4}$, and $\overline{m_3 m_4}$ are four transpositions mapping to the
Table 1. Words to express $t_1, \ldots, t_{11}$ in terms of $a, b$

| $d$  | $(ab)^5 b$ |
|------|------------|
| $t_{11}$ | $d^{19}$ |
| $c$  | $(a_{11})^3$ |
| $e$  | $(cd)^{30} d$ |
| $t_1 = f$ | $(((cc)^6 c(ec)^3)^2 ece^2 c)^5 (ee)^4$ |
| $g$  | $((ee)^8 c(ec)^3)(ec^x e)^2$ |
| $t_{n+2}$ | $f^g s^n$ for $0 \leq n \leq 6$ |
| $p$  | $((ab)^5 t_{11} (ab)^{-5} t_1 (ab)^5)^{-1}$ |
| $h$  | $e^g$ |
| $i$  | $d^p$ |
| $j$  | $[(t_5)^2, t_3 t_4]$ |
| $k$  | $[(t_5)^4, t_3 t_4]$ |
| $l$  | $[(t_8)^{jk}, t_6 t_7][t_8)^{kj}, t_6 t_7]$ |
| $l_3$ | $[(t_8)^{(jk)^4}, t_6 t_7]$ |
| $l_4$ | $((t_8)^{jk})^{kj}$ |
| $l_5$ | $((t_8)^3)^2 l_3 l_4$ |
| $m_2$ | $l_3^{t_5}$ |
| $m_3$ | $(m_2)^{l_5}$ |
| $t_{10}$ | $m_3 m_2 l_4 m_2 m_3$ |
| $t_9$ | $l_4 m_2 l_{10} m_2 l_4$ |

same involution in the quotient $S_3$, so give the full set of 12 after conjugating by $m_1 m_2$ and $m_2 m_1$.

Step 9. Check these twelve [sic] transpositions for candidates for $t_9$ and $t_{10}$. There is only one possibility up to an obvious inner automorphism.

This step was again carried out in the 4370 dimensional representation. The calculations that do not involve $t_2$ could have been done in 78 dimensions, but the time saved would be of the order of one minute, so insignificant. Of the 13 transpositions, the only one which commutes with $t_2$ and $t_{11}$ but not $t_5$ is $m_1 m_2 m_3 m_2 m_1$. Hence this is the only possibility for $t_9$. There are two that commute with $t_5$ but not $t_{11}$, namely $m_1 m_2 m_3$ and $m_1 m_2 m_4 m_1 m_2$. But $t_{11}$ conjugates one to the other, so without loss of generality we may take $t_{10} = m_1 m_2 m_3$.

4.3. Verifying the presentation. We now have a straight line program for producing the elements $t_1, \ldots, t_{11}$ from the elements $a, b$. This program is given in Table I for convenience. It must be applied in every claimed representation of the Baby Monster, and then the relations of the presentation must be checked.

Step 10. Prove that $t_1, \ldots, t_{11}$ satisfy all the required relations.

We check the 66 Coxeter relations by finding the order of the elements $t_i t_j$ for all $j \geq i$. (The relations $t_i^2 = 1$ are implicit in the calculation, but were explicitly checked again.) Similarly the spider relation is checked by confirming that the element $t_5 t_4 t_5 t_6 t_7 t_5 t_9 t_{10}$ has order 10. Finally, we check that $t_5 t_4 t_3 t_7 t_8 t_9$ and $t_5 t_4 t_3 t_6 t_9 t_{10} t_{11}$ have order 9. (In the particular representations we checked, this
last check can be omitted, since it is straightforward to show in each case that the centre of the group is trivial. Indeed, Schur’s Lemma implies the centre consists of scalars, while the generators have determinant 1 and the only scalar of determinant 1 is 1.)

We verified the relations in the three representations from [22], that is in dimension 4370 over the field of order 2, and in dimension 4371 over the fields of orders 3 and 5. The total computation time was under 12 hours.

4.4. Reversing the process. To complete the proof that the matrices given in [22] generate the Baby Monster, we have to reverse the process, and find the Atlas standard generators in terms of the Y_{433} generators. First we make two elements 
\[ r = t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 \] of order 9 and 
\[ s = t_5 t_9 t_{10} t_{11} \] of order 5. These elements in fact generate the whole group, but it is not necessary to prove this at this stage. Now \( t_1 t_2 \) is an element in class 3A. A random search produces the element \((r^7 s)^{15}\) in class 2C. (Again, it is not necessary to prove that this involution is in class 2C. However, we used the conjugacy class invariants in [21] to guide us, and found that this element \( x \) has \( q(x) = 2158 \), which identifies the conjugacy class as 2C.)

Another random search gives a candidate pair of standard generators
\[
\begin{align*}
  a' &= (r^7 s)^{15} \\
  b' &= (t_1 t_2)^{(sr)^{10}}
\end{align*}
\]

Finally, we use the ‘chop’ program of the Meataxe to conjugate the matrices to a standard basis with respect to, first, the generators \( a, b \), and then, the generators \( a', b' \). This calculation must, of course, be carried out in each representation that we wish to check. We found that in all three of the representations in [22], the resulting pairs of matrices are identical, proving that all claimed generating sets do indeed generate the same group. (This does not mean that \( a', b' \) are the same elements as \( a, b \), merely that the pair \( (a', b') \) is equivalent to \( (a, b) \) under an automorphism of the group, and therefore under conjugation.) The total computation time was under 3 hours.

5. Conjugacy class invariants and power maps in the Baby Monster

5.1. Introduction. In this section, we produce a list of easily computed conjugacy class invariants for a specified list of elements of the Baby Monster, which are in fact good enough to distinguish all conjugacy classes of cyclic subgroups except 16D and 16F. As a result, we have a splitting of the elements into small unions of conjugacy classes, and power maps between these unions of classes. The final splitting into conjugacy classes, and refinements of power maps, is done later.

In [21] (and also in [22]) there is a list of words in the Atlas standard generators of the Baby Monster, suitable powers of which are in fact representatives for the 184 conjugacy classes. However, the proof given there depends on the accuracy of the ATLAS character table of the Baby Monster, and in particular on the accuracy of the power map information. It is therefore necessary to provide a new proof, which does not depend on the character table. We can of course use the words, as long as we do not quote from [21] any of the properties of the corresponding elements of the Baby Monster. We assume that the three representations of the Baby Monster given in [22] do indeed represent the Baby Monster. This was proved in Section 4.
5.2. The words and their names. In [21] there is a list of 76 words for elements of specified orders, that in fact lie in the 76 classes of maximal cyclic subgroups. There are in fact 175 classes of cyclic subgroups altogether, including the trivial group. We can therefore take suitable powers of the 76 words as a further set of 99 words defining elements of the group.

First we label the 76 words with the names given in [21]. These names will later, of course, be identified with unions of conjugacy classes, but at this stage they are simply names. We calculate the orders of the elements, and hence verify that the numerical part of the name is indeed the order of the element. We define our other 99 words and their labels as the obvious powers from the first line to the second line of each row of Table 2.

At this stage, we have a list of 175 words which give elements of the specified orders in the Baby Monster. Our job now is to find invariants that distinguish the alphabetical part of the name.

5.3. Invariants. We compute only the invariants that [21] tells us are useful. The invariants we use for an element $x$ are of the following types:

- black box type: the order of $x$, computed in the mod 2 representation;
- mod 2 type: the trace $t_2(y)$ and the rank $r(y)$ of selected polynomials $y = p(x)$ in $x$, in the mod 2 representation;
- mod 3 type: the trace $t_3(x^k)$ of selected powers of $x$, in the mod 3 representation;
- mod 5 type: the trace $t_5(x)$ of $x$, in the mod 5 representation.

### Table 2. The names of our words

| 70A | 66A | 60A | 60C | 56AB | 52A | 48B | 46AB | 44A | 66A |
|-----|-----|-----|-----|------|-----|-----|------|-----|-----|
| 35A | 33A | 30D | 30F | 28B  | 26A | 24G | 23AB | 22B | 22A |
| 42C | 40E | 40D | 60B | 60A  | 40C | 38A | 36C  | 36B | 34A |
| 21A | 20G | 20F | 20E | 20A  | 20D | 19A | 18E  | 18C | 17A |
| 32CD | 32AB | 48B | 48A | 30F  | 30A | 28E | 28A  | 42A | 42B |
| 16D | 16C | 16B | 16A | 15B  | 15A | 14E | 14D  | 14A | 14B |
| 42C | 26A | 24N | 24M | 24L  | 24K | 24J | 24H  | 24G | 24D |
| 14C | 13A | 12R | 12O | 12Q  | 12M | 12J | 12F  | 12G | 12D |
| 36C | 36B | 36A | 60A | 60B  | 22B | 20J | 20I  | 20H | 20F |
| 12N | 12K | 12B | 12C | 12E  | 11A | 10F | 10D  | 10C | 10B |
| 30A | 30E | 18F | 18E | 16H  | 16G | 16F | 16E  | 24J | 24M |
| 10A | 10E | 9B  | 9A  | 8M   | 8K  | 8H  | 8D   | 8J  | 8I  |
| 24I | 24K | 24C | 24B | 24A  | 24E | 24N | 40D  | 14D | 12T |
| 8G  | 8F  | 8E  | 8C  | 8B   | 8A  | 8N  | 8L   | 7A  | 6K  |
| 12S | 12R | 12P | 12O | 12I  | 18A | 30B | 30A  | 30E | 30C |
| 6J  | 6I  | 6H  | 6G  | 6C   | 6D  | 6A  | 6B   | 6E  | 6F  |
| 12C | 10F | 10B | 8N  | 8M   | 8L  | 8J  | 8I   | 8H  | 8E  |
| 4A  | 5B  | 5A  | 4J  | 4H   | 4G  | 4E  | 4F   | 4C  | 4B  |
| 12E | 12T | 6K  | 6A  | 4J   | 4I  | 4A  | 6A   | 2B  |     |
| 4D  | 4I  | 3B  | 3A  | 2D   | 2C  | 2B  | 2A   | 1A  |     |
The first two are cheap, and are used in all cases. The last two are expensive, and are only used when we know they will in fact be useful.

5.3.1. *Odd-order elements.* We find 24 cyclic subgroups of odd-order elements. For orders 1, 7, 11, 13, 17, 19, 21, 23, 25, 27, 31, 33, 35, 39, 47 and 55, the only invariant we shall need is the order. For the other orders, 3, 5, 9 and 15, the trace in the 4370 dimensional representation mod 2 distinguishes two names in each case:

\[
\begin{array}{cccccccc}
3 & A & 3 & B & 5 & A & 5 & B \\
3 & A & 9 & B & 15 & A & 15 & B \\
\end{array}
\]

5.3.2. *Elements of twice odd order.* For elements of order 38, 46, 66 or 70, no further invariant is required. In the remaining cases we compute the rank (or nullity) of \(1 + x\) on the 4370 dimensional representation over the field of order 2. This turns out to be a sufficient invariant to distinguish all cases except the elements of order 30 and 42. The rank of \(1 + x\) is tabulated below: note that in the case 26A the rank is given incorrectly in [21] as 4196 instead of 4198.

\[
\begin{array}{cccccccccccc}
2A & 2B & 2C & 2D & 6A & 6B & 6C & 6D & 6E & 6F \\
1860 & 2048 & 2158 & 2168 & 3486 & 3510 & 3566 & 3534 & 3606 & 3604 \\
10F & 14A & 14B & 14C & 14D & 14E & 18A & 18B & 18C & 18D \\
3932 & 3996 & 4008 & 4048 & 4034 & 4052 & 4088 & 4090 & 4110 & 4124 \\
18E & 18F & 22A & 22B & 26A & 26B & 30A/B & 30C & 30D & 30E \\
4128 & 4122 & 4140 & 4158 & 4198 & 4176 & 4190 & 4212 & 4206 & 4214 \\
30F & 30GH & 34A & 34BC & 42A/B & 42C \\
4224 & 4216 & 4238 & 4220 & 4242 & 4258 \\
\end{array}
\]

The cases 30A and 30B can be distinguished by the rank of \(1 + x^5\), which is 3510 and 3486 respectively. The cases 42A and 42B can be distinguished by the rank of \(1 + x^3\), which is 3996 and 4008 respectively.

5.3.3. *Elements of order at least 28.* For elements of order 44, 52, and 56, no further invariant is required. For elements of order 36 and 60, the rank of \(1 + x\) is sufficient:

\[
\begin{array}{cccccccc}
36A & 36B & 36C & 60A & 60B & 60C \\
4226 & 4238 & 4248 & 4280 & 4286 & 4296 \\
\end{array}
\]

For the remaining element orders, 28, 32, 40, and 48, we have the following values of the rank of \(1 + x\):

\[
\begin{array}{cccccccc}
28A/C & 28B/D & 28E & 32AB/CD & 40A/B/C & 40D & 40E & 48A/B \\
4188 & 4200 & 4210 & 4222 & 4242 & 4250 & 4258 & 4266 \\
\end{array}
\]

In particular, this invariant is of no help for elements of order 32 or 48. All necessary cases can be separated by the trace mod 3 of \(x\) or \(x^2\) or \(x^7\):

\[
\begin{array}{cccccccc}
28A & 28C & 28B & 28D & 32AB & 32CD \\
2 & 0 \\
0 & 1 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
40A & 40B & 40C & 48A & 48B \\
0 & 1 & 2 & 0 & 1 \\
\end{array}
\]
5.3.4. Elements of order 4 and 8. The rank of $1 + x$ distinguishes 7 cases of elements of order 4. Three of these split into two, according to the trace on the 4371-dimensional representation mod 3. An alternative invariant to distinguish $4H$ from $4J$ is the rank of $(1 + x)^3$ in the mod 2 representation.

| $r(1 + x)$ | $4A$ | $4B$ | $4C$ | $4D$ | $4E$ | $4F$ | $4G$ | $4H$ | $4I$ | $4J$ |
|------------|------|------|------|------|------|------|------|------|------|------|
| $t_3(x)$   |      |      |      |      |      |      |      |      |      |      |
| $r(1 + x)^3$ |      |      |      |      |      |      |      |      |      |      |

Similarly, for elements of order 8, the rank of $1 + x$ distinguishes 8 cases, one of which is split by the rank of $(1 + x)^2$, while the rank of $(1 + x)^3$ splits two more:

| $8A$ | $8B/C/E$ | $8D$ | $8F$ | $8H$ | $8G$ | $8I$ | $8J$ | $8K/M$ | $8N$ |
|------|----------|------|------|------|------|------|------|---------|------|
| 3774 | 3738     | 3778 | 3780 | 3810 | 3786 | 3812 | 3818     | 3818 |
|       |          |      |      |      |      |      |      | 3202    | 3204  |
|       |          |      |      |      |      |      |      | 2619    | 2620  |
|       |          |      |      |      |      |      |      | 2714    | 2717  |

The trace modulo 3 distinguishes the remaining cases

| $t_3(x)$ | $8B$ | $8C$ | $8E$ | $8K$ | $8M$ |
|----------|------|------|------|------|------|
|          | 1    | 0    | 2    | 1    | 2    |

5.3.5. Elements of order 12 and 24. The rank of $1 + x$ distinguishes 14 cases of elements of order 12:

| $12A/C/D$ | $12B$ | $12E$ | $12F$ | $12G/H$ | $12I$ | $12J$ |
|-----------|-------|-------|-------|----------|-------|-------|
| $r(1 + x)$ | 3936  | 3942  | 3958  | 3996     | 3962  | 3964  | 3986  |

| $12K/M$ | $12L$ | $12N$ | $12O$ | $12P$ | $12Q/R/T$ | $12S$ |
|---------|-------|-------|-------|-------|-----------|-------|
| $r(1 + x)$ | 3978  | 3986  | 3964  | 3982  | 3988      | 4002  | 4004  |

All except $12A/D$ can be split using the trace mod 3. This last case would seem to require the trace mod 5.

| $12A$ | $12C$ | $12D$ | $12G$ | $12H$ | $12K$ | $12M$ | $12Q$ | $12R$ | $12T$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $t_3(x)$ | 0    | 1    | 0    | 1    | 0    | 1    | 0    | 1    | 2    |
| $t_5(x)$ | 3    | 4    | 1    | 4    | 1    | 1    | 0    | 2    | 1    |

Similarly for elements of order 24, the rank of $1 + x$ distinguishes 8 cases.

| $24A/B/C/D$ | $24E/G$ | $24F$ | $24H$ | $24I/M$ | $24J$ | $24K$ | $24L/N$ |
|-------------|----------|-------|-------|----------|-------|-------|---------|
| 4152        | 4164     | 4170  | 4182  | 4176     | 4178  | 4174  | 4186    |

Of these, we can distinguish $24I/M$ with the rank of $1 + x^2$, which is 3986 and 3982 respectively, and $24E/G$ with the rank of $1 + x^3$, which is 3774 and 3778 respectively. All the rest are distinguished by the trace mod 3, apart from the case $24C/D$, which seems to require the trace mod 5.

| $24A$ | $24B$ | $24C$ | $24D$ | $24L$ | $24N$ |
|-------|-------|-------|-------|-------|-------|
| $t_3(x)$ | 1    | 0    | 2    | 2    | 1    | 2    |
| $t_5(x)$ | 3    | 0    | 2    | 1    |      |      |
5.3.6. **Elements of orders** 16, 20. In these cases, the rank of $1 + x$ distinguishes the following:

| $r(1 + x)$ | 16A/B | 16C/D/E/F | 16G/H |
|------------|--------|------------|--------|
| 4072       | 4074   | 4094       |

| $r(1 + x)$ | 20A/B/C/D | 20E | 20F | 20G | 20H | 20I | 20J |
|------------|------------|-----|-----|-----|-----|-----|-----|
| 4114       | 4128       | 4132| 4148| 4144| 4138| 4150|

For the elements of order 20, the rank of $1 + x^2$ distinguishes 20B from 20A/C/D, which are distinguished from each other by the trace mod 3:

| $r(1 + x^2)$ | 20A | 20B | 20C | 20D |
|--------------|-----|-----|-----|-----|
| t, 3(x)     | 3896| 3908| 3896| 3896|
| t, 3(x^2)   | 2   | 1   | 0   | 1   |

For the elements of order 16, the trace mod 3 distinguishes 16A/B, and separates 16C/E from 16D/F. Then 16C/E can be separated with the rank of $1 + x^2$, and 16G/H with the trace of $x^2$ mod 3.

| $r(1 + x^2)$ | 16A | 16B | 16C | 16E | 16D/F | 16G | 16H |
|--------------|-----|-----|-----|-----|--------|-----|-----|
| t, 3(x)     | 0   | 1   | 2   | 2   | 0      | 1   | 2   |
| t, 3(x^2)   | 3780| 3778|

There would appear to be no easily computed invariant which distinguishes 16D from 16F.

5.4. **Checking the approximate power maps.** We power up each of the given words, to every relevant power (that is, every power dividing the element order), and compute the necessary invariants of the resulting elements. We therefore know the power maps approximately. In every case the power maps agree with the character table in the ATLAS [7]. Indeed, some of the power maps form part of the definition of our set of class representatives, so the calculations in these cases can in fact be omitted. This includes the class 16D, which is defined to be the square of the classes 32CD. Hence it is not necessary to find an invariant to distinguish 16D from 16F, in order to verify the power maps. All that remains in order to verify that the power maps are actually correct, is, firstly, to prove that class list is correct, and secondly, to deal with any issues concerning algebraically conjugate classes. Details of the computations are given in [5].

6. **Centralizers of prime order elements in the Baby Monster**

In this section we determine the classes of prime order elements, and the orders of their centralizers, in the Baby Monster. Much of this information comes from Stroth’s 1976 paper [17]. In cases where [17] does not give full information, our strategy is first to use a certified copy of the Baby Monster from [22] to give lower bounds on both the number of conjugacy classes and the orders of the respective centralizers, and then to use local arguments, together with information about the permutation representation on the $\{3,4\}$-transpositions, to show these are also upper bounds. For technical reasons, we deal with the primes in the order 2, 3, 7, 17, 11, 13, 19, 23, 5, 47, 31.
6.1. Fusion of involutions. From [17] we see there are exactly four classes of involutions in $\mathbb{B}$, with representatives labelled $d, x_{36}(1), d\beta$ and $dx_{33}(1)x_{32}(1)$ respectively. In the ATLAS [7], these are labelled $2A, 2B, 2C, 2D$ respectively. The centralizers of $2A$ and $2B$ and $2C$ are given in detail in [17]. The centralizer order of $2D$ is given together with a rough description of the structure.

For the purposes of computation, it is necessary to match these classes to the names given in Section 5. Note that any element of order 38 powers into class 2A, and any element of order 22 powers into class 2B. An element of order 34 powers into either 2A or 2C. We can now compute the rank of $1 + x$ for involutions $x$ in the certified copy of the Baby Monster in dimension 4370 mod 2, obtaining the values 1860 for class 2A and 2048 for class 2B. For suitable $x$ obtained as the 17th power of an element of order 34, we obtain the value 2158, and for another involution we obtain 2168, so these are in class 2C of an element of order 34, we obtain the value 2158, and for another involution we can now easily determine the class of any explicitly given involution.

ATLAS involutions in Section 5 are indeed the same as the GAP involutions in the Baby Monster. Indeed, computation using explicit matrices, and certain of its subgroups. For simplicity we use the GAP characters of $H$ we now compute the permutation characters of the action of $H$ on the first three of these suborbits. We use standard operations in GAP, using only the character tables of $H$ and certain of its subgroups. For simplicity we use the GAP labels for characters of $H$. 

6.2. The permutation representation on $\{3,4\}$-transpositions. According to [17] the non-trivial suborbit lengths of $H$ acting on the 13571955000 cosets of $H$ are as follows:

- $3^3.5.7.13.17.19 = 3968055$, with point stabilizer $2.2^{1+20}.U_6(2).2$;
- $2^{12}.3^3.11.19 = 23113728$, with point stabilizer $2^2 \times F_4(2)$;
- $2^{20}.7.17.19 = 2370830336$, with point stabilizer $F_{23}.2$ and
- $2^8.3^2.5.7.11.13.17.19 = 11174042880$, with point stabilizer $2^{1+20}.U_4(3).2^2$.

We now compute the permutation characters of the action of $H$ on the first three of these suborbits.
In the first case, the action on the suborbit is the permutation action of \(2E_6(2).2\) on the cosets of the \(U_6(2)\) maximal parabolic, and is known to have rank 5. Using GAP, we found the only way to get the character degrees adding to the correct number is for the degrees to be \(1 + 1938 + 48620 + 1828332 + 2089164\). The trivial character is a constituent, because it is a permutation character, leaving 16 possibilities for the signs on the other four constituents. It turns out that only one of these characters has non-negative values. This character is the sum of the irreducibles labelled 1, 3, 5, 13, 15 in GAP.

In the second case, GAP computes possible class fusions from \(F_4(2)\) into \(H\), and we induce up the trivial character in each case. The answers are all the same. The permutation character is a subcharacter of this induced character, and it is easy to determine the character degrees, and then check all possibilities as above. The answer is the sum of irreducibles numbered 1, 5, 17, 24.

In the third case, similarly, we compute possible class fusions from \(\text{Fi}_{22}:2\) into \(H\). There are then two possibilities for the induced trivial character, and they differ by multiplying the outer elements of \(\text{Fi}_{22}:2\) by the central involution of \(H\). But we know that in the point stabilizer \(\text{Fi}_{22}:2\) the 2D involutions fuse to 2A in \(B\) (if necessary we can verify this computationally using the subgroup \(S_3 \times \text{Fi}_{22}:2\) in our certified copy of the Baby Monster), which distinguishes the two cases. The answer is the sum of characters numbered 1, 3, 5, 13, 17, 28, 49, 76, 190, 192, 196, 202, 210, 217.

It is not necessary to compute the full permutation character of \(B\) on the cosets of \(H\), which would involve computing the fourth suborbit case as well. Later on we will however need to compute the values on a few selected classes.

6.3. Fusion of 3-elements. Computationally, using a certified copy of the Baby Monster, and words provided in [22], we find two subgroups \(S_3 \times \text{Fi}_{22}:2\) and \(3^{1+8}2^{1+6}\cdot U_4(2).2\), which normalize cyclic subgroups of order 3. The corresponding elements of order 3 can be distinguished by the trace in the 4370 dimensional representation module 2, so do not fuse in \(B\). We use the ATLAS labels 3A and 3B for these two conjugacy classes.

Conversely, note that \(\text{Fi}_{23}\) contains a Sylow 3 subgroup of \(B\) so every 3-element in \(B\) is conjugate to an element of \(\text{Fi}_{23}\). Moreover, we know the fusion from 2.\(\text{Fi}_{22}\) to \(\text{Fi}_{23}\), and in particular, every 3-class in \(\text{Fi}_{23}\) is represented in 2.\(\text{Fi}_{22}\) and therefore in \(H\). Using the fact that \(H\)-classes \(-2A\) and \(+2D\) are in 2A, and computing structure constants in \(H\), we get that \(H\) classes 3A and 3B fuse in \(B\). Hence there are exactly two classes of elements of order 3 in \(B\).

We now show that a 3A-element \(x\) has centralizer \(C(x) \cong 3 \times \text{Fi}_{22}:2\) in \(B\). We know its centralizer is at least that (either computationally, as above, or see [17]). On the other hand, the number of \(B\)-conjugates of \(x\) is at least one-third of the product of the length of the whole orbit with the length of the relevant suborbit. This number \(13571955000 \times 2370830336/3\) is equal to the index of \(3 \times \text{Fi}_{22}:2\) in \(B\), and the claim is proved.

Next we show that the subgroup \(3^{1+8}2^{1+6}\cdot U_4(2)\) computed above is the full centralizer of a 3B-element. To do this we need to know the value on 3B of the full permutation character of \(B/H\). Equivalently, the value of the permutation character of the last orbit above on \(H\)-class 3C. Recall that the point stabilizer in the last orbit is \(H_5 = 2^{1+20}\cdot U_4(3).2^2\). We use the GAP function \text{PossibleClassFusions} to get the 3-fusion from \(H_5\) to \(H\). The result is that ATLAS class 3A in \(U_4(3)\) fuses to 3C in \(H\), while all other classes of elements of order 3 fuse to 3A or 3B in \(H\).
Hence the value on $H$-class $3C$ of the permutation character of $H$ on this orbit is $|C_H(3C)|/|C_H(3A)| = 2^{11}.3^9/2^8.3^6 = 2^3.3^3 = 216$. Therefore the value of whole permutation character of $B/H$ on class $3C$ is 1620.

Hence we know $|C_B(3B)|/|C_H(3C)| = 1620 = 2^2.3^4.5$, and the $3C$ centralizer in $H$ has order $2^{11}.3^9$, so we deduce the order of the $3B$-centralizer in $B$ is $2^{13}.3^{13}.5$ and the claim follows.

6.4. Elements of orders 7, 11, 13, 17, 19 and 23. From [17] (Lemma 6.11) we get $|N(7^2)| = 2^6.3^2.7^2$ and $|C(7^2)| = 2^4.7^2$. The only subgroup of $GL_2(7)$ of order $2^4.3^2$ is $3 \times 2S_4$, which is transitive on non-zero elements of $7^2$. Hence there is a single class of elements of order 7. (This can also be verified computationally in a certified copy of the Baby Monster.)

In $H$ we have a 7A-centralizer $7 \times 2L_3(4).2$. To show that the centralizer in $B$ is no bigger, we follow the same strategy as for $3B$ elements above, although it is slightly more complicated since both classes 7A and 7B fuse to 7A in $B$.

The 7-elements in $H_7$ fuse to class 7B in $H$. Hence the value of the permutation character of $B/H$ on $H$-class 7A is 121. (As a check, $|C_H(7B)| = 4704 = 2^5.3.7^2$ and $|C_{H_7}| = 2^4.7$, so the character value of the last orbit on $H$-class 7B is 2.3.7 = 42. This implies the value of the permutation character of $B/H$ on $H$-class 7B is also 121, as it must be.) Therefore $|C_B(7A)|/|C_H(7A)| + |C_B(7A)|/|C_H(7B)| = 121$, so that $|C_B(7A)| = 2^4.3^2.5.7$.

For the remaining primes in the list, 11, 13, 17, 19 and 23, most of the information we need is already in [17]. Lemma 6.13 of [17] says the centralizer order of an element of order 17 is $2^2.17$, and the normalizer has order $2^6.17$, so there is a single class of elements of order 17. Lemma 6.8 of [17] says that the order of the Sylow 11-normalizer is $2^4.3.5^2.11$, and the centralizer of an element of order 11 is $S_5 \times 11$. Hence the normalizer is $S_5 \times 11:10$. In Lemma 6.12 of [17] there are two possibilities for the normalizer of an element of order 13. But the normalizer of such an element in $F_4(2)$ is just 13:12, so from the proof of Lemma 6.12 we get that the 13 centralizer in $B$ is $13 \times S_4$, and the normalizer is 13:12 $\times S_4$.

The Sylow 19-subgroup is self-centralizing in $H/(d)$, so the Sylow 2-subgroup of $C(19)$ is 2, containing a 2A-element. Since $|F_{22}|$ is not divisible by 19, that forces $N(19)$ to lie in $H$. Lemma 6.20 of [17] says that the 23-normalizer contains $2 \times 23:11$, and that the Sylow 2-subgroup of the 23-normalizer has order 2; we know (from Lemmas 7.13, 7.14, 7.15 and 7.17 of [17]) that all Sylow subgroups of the normalizer are cyclic. From the discussion earlier in this section, we know that the normalizer does not contain elements of order 7, 13, 17, or 19. The normalizing 11 rules out 47 and 31, by the Frattini argument. This leaves 3, 5. We know the 3-centralizers, so 3 is ruled out. Finally 5 is ruled out because $|B|/(2.5.11.23) \neq 1$ (mod 23).

6.5. The elements of order 5. The subgroup $HN$ (constructed explicitly in our certified copy of $B$) contains a full Sylow 5-subgroup. Every element of order 5 in $HN$ centralizes an involution, which we know fuses to $2B$ or $2D$ in $B$. Moreover, every element of order 5 in $C(2B)$ or $C(2D)$ centralizes an element of order 3. But $C(3A)$ and $C(3B)$ contain just one class of elements of order 5 each, so there are at most two classes of elements of order 5 in $B$. On the other hand, we find two classes of 5-elements with different traces. Hence there are exactly two classes.
The usual argument gives the order of \( C(5A) \). We have \(|C_H(5A)| = 2\cdot3\cdot2\cdot5\cdot7\) and \(|C_{H_0}(5A)| = 2\cdot5\cdot7\) so the value of the permutation character of the last orbit on this class is \(2\cdot3\cdot2\cdot5\cdot7 = 630\). Hence the full permutation character has value 630 + 470 = 1100 on \( B \) class 5A. Therefore \(|C_{B}(5A)| = 1100|C_H(5A)| = 2^{10}\cdot3^2\cdot5^4\cdot7^1\cdot11\), which is the order of \( 5 \times HS:2 \). Hence the 5A normalizer is \( 5:4 \times HS:2 \).

Computationally, using the matrices and words provided in [22], we find a subgroup \( 5^{1+4}\cdot2^{1+4}\cdotA_5\cdot4 \), normalizing a cyclic group of order 5, which must therefore be of \( 5B \) type. We shall show that the normalizer is no bigger than this. We know that there is no \( 5B \) element in the centralizer of any element of order 7, 11, 13, 17, 19, 23, or of a \( 3A \). Also 47 and 31 are \( 3 \mod 4 \) and do not centralize an involution, so do not centralize a \( 5B \) by the Frattini argument. Hence the centralizer of a \( 5B \) is a \( \{2, 3, 5\}\)-group, and contains the full Sylow 5-subgroup of \( B \), so only the Sylow 2- or 3-subgroup could grow.

Now the centralizer of a 5-element in \( C(3B) = 3^{1+8}\cdot2^{1+6}\cdotU_4(2) \) is just a cyclic group of order 30, so the Sylow 3-subgroup of the \( 5B \)-centralizer has order 3. Since \( C(2A) \) and \( C(2C) \) contain no \( 5B \), we look in \( C(2B) \) and \( C(2D) \). In \( C(2B) \) only the 5A class of \( O_2 \) fuses to \( B \) class 5B, and we see the centralizer \( (2^{1+2} \times 5^{1+2})\cdot2A_4 \) of order \( 2^6\cdot3^3\cdot5^1 \). In \( C(2D) \) we see centralizer order \( 2^7\cdot3\cdot5^2 \). In neither case does the Sylow 2-subgroup grow. Thus we know the orders of all the Sylow subgroups of \( C(5B) \), and therefore the order of \( C(5B) \).

6.6. Primes 47 and 31. The order of the 47 normalizer now divides \( 47\cdot2\cdot23\cdot31 \) and Sylow’s theorem implies it is \( 47\cdot23 \cdot31 \cdot30 \), so is 31.15 by Sylow’s Theorem.

7. Obtaining the class list

Our strategy for obtaining the list of conjugacy classes in the Baby Monster is first to determine the classes of even order elements, by computing the character tables of subgroups containing the four distinct involution centralizers, and noting down the conjugacy classes of elements in each subgroup that power to the relevant involution class. (The centralizers of involutions in classes \( 2A, 2B, 2C \) are maximal, so the subgroup is the involution centralizer itself in these cases.) At the same time, we note down the length of each such class. A similar computation for odd-order elements in the centralizers of elements of odd prime order is trivial in comparison.

In fact, there is a great deal of redundancy in the information that we have computed, and classes of elements whose order is divisible by two primes can be computed in two different ways. This provides a robust check on these results, in particular for the large number of classes of elements of order divisible both by 2 and an odd prime.

7.1. Involution centralizers in the Baby Monster. The character table of the \( 2A \)-centralizer is known by [6] and the computations shown in [4]. The \( 2C \)-centralizer has the structure \( (2^2 \times F_4(2)):2 < D_8 \times F_4(2):2 \), and its character table is determined by those of the subgroups \( 2^2 \) and \( F_4(2) \) and the factor groups \( D_8 \) and \( F_4(2):2 \), hence it is known.

The \( 2D \)-centralizer is not itself a maximal subgroup, but is contained in maximal subgroups of the structure \( 2^{(8+1)+16}\cdotS_8(2) \) in \( B \). The character table of this maximal subgroup can be verified by restricting the 2-modular degree 4370 representation of
to the subgroup, using the straight line program from [22], finding a faithful 180-
dimensional subquotient of this module, and computing the character table from
this matrix representation using the MAGMA computer algebra system [3]. (This
had been done by E. O’Brien in 2007, but we repeat the computations in order to
make sure that only safe data are used.)

The 2B-centralizer has the structure 2^{1+22}.Co_2. The character table has been
computed in [12] but the arguments assume the character table of B. In the re-
mainder of this section, we describe briefly how we verify this character table. Full
details can be found in [5].

First we restrict the certified 3- and 5-modular representations of degree 4371 of
B to the 2B centralizer, using the straight line program from [22]; the composit-
of the module have the dimensions 23, 2300, and 2048. Next, we find an
orbit of length 4600 in the 2300-dimensional module. The action on this orbit yields
a faithful permutation representation of the factor group 2^{22}.Co_2. We compute class
representatives for this factor group, and let MAGMA compute its character table.

The 2048-dimensional module is faithful. We compute the class fusion under
the epimorphism from 2^{1+22}.Co_2 to 2^{22}.Co_2, and the Brauer characters of our 3-
and 5-modular representations for this module. These Brauer characters lift to an
ordinary irreducible character \chi of 2^{1+22}.Co_2. All faithful irreducible characters of
2^{1+22}.Co_2 arise as tensor products of \chi with the irreducibles of the factor group
Co_2.

Once the character tables of (overgroups of) all the involution centralizers are
available, we can read off from these tables all the conjugacy classes of elements
that power to each of the involutions, together with the centralizer orders. This
includes all conjugacy classes of even-order elements.

7.2. Elements of odd order. For primes p \geq 11 it is now almost a triviality
to write down the classes of elements of odd order divisible by p. For p = 7, we
have that N(7A) is contained in H, so the relevant classes can be read off from
the classes of elements of H that power into class 7A. (Note that the 7A-centralizer in
H has the shape 2.L_3(4):2_2, and the 2_2 automorphism swaps the L_3(4)-classes 5A
with 5B.)

For elements powering into 5A, we read off the classes and their centralizer orders
from the ATLAS character table of HS:2. Similarly, for elements powering into 3A,
use the table for Fi_{22}:2, but note that there are some classes missing in the ATLAS
character table for Fi_{22}:2: these only affect the calculations for elements of order
30, which have already been dealt with in the 5A-centralizer.

In the cases 3B and 5B again, GAP contains character tables of the respective
normalizers. However, it is not recorded exactly what information was used to
calculate these tables. Therefore we re-calculate them (see [4]). In conclusion, we
find that the list of odd-order elements and their centralizer orders agrees with
the ATLAS.

8. Computing the irreducible characters of the Baby Monster

From the previous sections, we know that B contains subgroups of the structures
2^2.E_6(2).2, Fi_{23}, and HN.2. The ordinary character tables of these groups have
been verified (see [6]) and thus may be used in our computations. The class fusions
from these subgroups to B can be computed with the methods available in GAP [9].
Moreover, in Section 7.1 we have computed the character table of the 2B centralizer
in $\mathbb{B}$. The class fusion from $2^{1+22} \cdot \text{Co}_2$ to $\mathbb{B}$ is determined by evaluating the three representations of $\mathbb{B}$ at the class representatives of $2^{1+22} \cdot \text{Co}_2$, and applying the invariants from Section 5.

Thus we can induce the irreducible characters from these subgroups to $\mathbb{B}$. Using the power maps of $\mathbb{B}$, we induce also the irreducibles of all cyclic subgroups of $\mathbb{B}$. Now we proceed in two steps.

In the first step, we assume that $\mathbb{B}$ has an ordinary irreducible representation $\chi$ of degree 4371 such that the reductions modulo 3 and 5 are (irreducible and) equivalent to the representations we have used in the previous sections, and such that the reduction modulo 2 has one trivial composition factor and one that is equivalent to the representation we have used above. Then the Brauer character values of our representations yield the values of $\chi$, except on the classes of element orders divisible by 30, and the missing values are uniquely determined by the obvious bounds. If we add $\chi$ and the trivial character of $\mathbb{B}$ to the list of induced characters then applying standard character-theoretic techniques such as LLL reduction yields a complete list of irreducible characters for $\mathbb{B}$, which coincides with the characters in the ATLAS table of $\mathbb{B}$.

In the second step, we do not want to assume the existence of the ordinary character $\chi$, and try to apply the character-theoretic criteria to the safe list of induced characters. This way, we do not get any irreducible character. However, we can show that 30 vectors from the list of irreducibles computed in the first step lie in the lattice spanned by the induced characters. Thus these vectors are verified as irreducible characters of $\mathbb{B}$. Now we form symmetrizations and tensor products of the known irreducible characters, and the lattice spanned by the known characters of $\mathbb{B}$ contains all the missing irreducibles computed in the first step. Thus we are done.

Again, the details of these constructions can be found in [4].

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