Antipodal Hadwiger numbers of finite-dimensional Banach spaces

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Abstract

Let \( X \) be a finite-dimensional Banach space; we introduce and investigate a natural generalization of the concepts of Hadwiger number \( H(X) \) and strict Hadwiger number \( H'(X) \). More precisely, we define the antipodal Hadwiger number \( H_\alpha(X) \) as the largest cardinality of a subset \( S \subseteq S_X \), such that \( \forall x \neq y \in S \exists f \in B_{X^*} \) with

\[
1 \leq f(x) - f(y) \quad \text{and} \quad f(y) \leq f(z) \leq f(x) \quad \forall z \in S.
\]

The strict antipodal Hadwiger number \( H'_\alpha(X) \) is defined analogously. We prove that \( H'_\alpha(X) = 4 \) for every Minkowski plane and estimate (or in some cases compute) the numbers \( H_\alpha(X) \) and \( H'_\alpha(X) \), where \( X = \ell^p_n \), \( 1 \leq p < +\infty \) and \( n \geq 2 \). We also show that the number \( H'_\alpha(X) \) is bounded below by an unbounded function of \( \dim X \).

Introduction

If \( X \) is any (real) Banach space, then \( B_X \) and \( S_X \) denote its closed unit ball and unit sphere respectively. A subset \( S \) of a normed space \( X \) is said to be \( \delta \)-separated, if \( \|x - y\| \geq \delta \) for \( x \neq y \in S \). Specifically \( S \) is called equilateral, if there is a \( \lambda > 0 \) such that for \( x \neq y \in S \) we have \( \|x - y\| = \lambda \); we also call \( S \) a \( \lambda \)-equilateral set. Any equilateral set in an \( n \)-dimensional space is of cardinality at most \( 2^n \) and the maximum is attained only when \( X = \ell^\infty_n \) (see [13]).

Let \( X \) be a finite-dimensional Banach space. The \textit{Hadwiger number} \( H(X) \) of \( X \) is the largest cardinality of a set \( S \subseteq S_X \) such that \( \|x - y\| \geq 1 \), for \( x \neq y \in S \). Also the \textit{strict Hadwiger number} \( H'(X) \) of \( X \) is the largest cardinality of a set \( S \subseteq S_X \) such that \( \|x - y\| > 1 \), for \( x \neq y \in S \). It is clear that \( H'(X) \leq H(X) \). There exists an extensive bibliography on the above concepts (see [13]).

A subset \( S \) of a normed space \( X \) is said to be \textit{antipodal} if for every \( x, y \in S \) with \( x \neq y \) there exists \( f \in X^* \) such that \( f(x) < f(y) \) and \( f(x) \leq f(z) \leq f(y) \forall z \in S \). That is, for every \( x, y \in S \) with \( x \neq y \) there exist closed distinct parallel support hyperplanes \( P = \{ z \in X : f(z) = f(x) \} \) and \( Q = \{ z \in X : f(z) = f(y) \} \) with \( x \in P \) and \( y \in Q \). Every antipodal subset of an \( n \)-dimensional real vector space has cardinality at most \( 2^n \).

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by a result of Danzer and Grünbaum (see [2], also [13]) and this is attained only when
the points of the antipodal set are the vertices of an n-dimensional parallelopetope.

A bounded and separated antipodal subset of a normed space $X$ is a subset $S \subseteq B_X$, for which there is $\delta > 0$ such that $\forall x \neq y \in S$ there is $f \in B_{X^*}$ with $\delta \leq f(x) - f(y)$ and $f(y) \leq f(z) \leq f(x)$ for $z \in S$. If $X$ is a finite dimensional real vector space then this concept of antipodality coincides with the classical one (see [2], [12]). The generalization of antipodality stated above was defined in [11] where the following Theorem (Th. 3 of [11]) was proved

**Theorem.** Let $(X, || \cdot ||)$ be a Banach space and $S \subseteq X$ be a bounded and separated antipodal set with constant $\delta$. Then we have:

1. There is an equivalent norm $||| \cdot |||$ on $X$, such that $S$ is an equilateral set in $(X, ||| \cdot |||)$.

2. The Banach-Mazur distance between $(X, || \cdot ||)$ and $(X, ||| \cdot |||)$ satisfies the inequality $d((X, || \cdot ||), (X, ||| \cdot |||)) \leq \frac{\delta}{2}$.

The Banach-Mazur distance between two isomorphic Banach spaces $X$ and $Y$ is $d(X,Y) = \inf \{ \|T\| \cdot \|T^{-1}\| : T : X \to Y$ is an isomorphism}. It is easy to see that the dual spaces also have the same Banach-Mazur distance $d(X,Y) = d(X^*,Y^*)$. An equivalent (geometric) definition of the Banach-Mazur distance is (in finite dimensions), for $K, L \subseteq \mathbb{R}^n$ symmetric convex bodies $d(K,L) = \inf\{ r > 0 : L \subseteq T(K) \subseteq r \cdot L, T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation$\}$.

Let $X$ be an n-dimensional Banach space. An Auerbach basis of $X$ is a biorthogonal system $\{(e_i, e_i^*) : i = 1, 2, \ldots, n\}$ in $X \times X^*$ (i.e. $e_i^*(e_j) = \delta_{ij}$, $i,j = 1, 2, \ldots, n$) such that $\{e_i : i = 1, 2, \ldots, n\}$ is a basis of $X$ and $\|e_i\| = \|e_i^*\| = 1$ for $i = 1, 2, \ldots, n$. It is well known that any finite-dimensional Banach space admits an Auerbach basis (see [3]).

In the present paper we introduce and study some interesting analogues of the Hadwiger and the strict Hadwiger number for a finite dimensional Banach space, which we call antipodal Hadwiger ($H_\alpha(X)$) and strict antipodal Hadwiger number ($H'_\alpha(X)$). The main results are the following:

1. We prove that $H'_\alpha(X) = 4$ for every Minkowski plane (Prop.4, Th.1).

2. We estimate and in some cases find exact values of the numbers $H_\alpha(X)$ and $H'_\alpha(X)$, when $X = \ell_p^n, 1 < p \leq +\infty$ and $n \geq 2$ (Th.2). We also show that for an n-dimensional Banach space $X$, the number $H'_\alpha(X)$ is bounded below by an unbounded function $\varphi(n)$ (Th.4).

3. We compute the numbers $H_\alpha(X)$ and $H'_\alpha(X)$ when $X = \ell_1^n$ or $X$ is the Petty space, i.e. $X = (\mathbb{R}^3, || \cdot ||)$, where $||(x, y, z)|| = \sqrt{x^2 + y^2 + z^2}$ (Prop.9).
Antipodal Hadwiger number of a finite-dimensional Banach space

We will always assume that $X$ is a finite-dimensional Banach space. Given the definition of Hadwiger number and its variants (see [15]) it is natural to give the following definitions:

**Definition 1.**
1. The antipodal Hadwiger number $H_\alpha(X)$ is the largest cardinality of a set $S \subseteq S_X$ such that $\forall x \neq y \in S$ there is $f \in B_{X^*}$ with

\[ 1 \leq f(x) - f(y) \text{ and } f(y) \leq f(z) \leq f(x) \text{ for } z \in S. \]

Clearly $S$ is a bounded and separated antipodal set of unit vectors with $\delta = 1$. In particular $S \subseteq S_X$ and $\|x - y\| \geq 1$ for $x \neq y \in S$, hence $H_\alpha(X) \leq H(X)$.

2. The strict antipodal Hadwiger number $H'_\alpha(X)$ is the largest cardinality of a set $S \subseteq S_X$ such that $\forall x \neq y \in S$ there is $f \in B_{X^*}$ with

\[ 1 < f(x) - f(y) \text{ and } f(y) \leq f(z) \leq f(x) \text{ for } z \in S. \]

As above, $S \subseteq S_X$ and $\|x - y\| > 1$ for $x \neq y \in S$, hence $H'_\alpha(X) \leq H'(X)$.

**Remarks 1**

1. Since every antipodal subset of an $n$-dimensional real vector space has cardinality at most $2^n$, we get that $H'_\alpha(X) \leq H_\alpha(X) \leq 2^n$.
2. If $Y$ is a subspace of $X$, then obviously $H_\alpha(Y) \leq H_\alpha(X)$ and $H'_\alpha(Y) \leq H'_\alpha(X)$.
3. Let $S$ be an antipodal subset of $S_X$ such that $\|x - y\| \geq 1$ for $x \neq y \in S$. Since the space is finite-dimensional, $S$ is a bounded and separated antipodal set, but the constant $\delta$ may be smaller than 1.

A simple example is a set of three consecutive vertices of a regular hexagon inscribed in the unit circle of $\ell_p^n$. These form an isosceles and obtuse triangle (with an angle of $120^\circ$) with 2 equal sides of length 1. Any functional $f \in B_{\ell_p^n}$ separating vertices $x, y$ which are at distance 1 gives an evaluation $|f(x) - f(y)| < 1$.

4. Any $\lambda$-equilateral set $S \subseteq B_X$ is a bounded and separated antipodal set with $\delta = \lambda$ (see [11], Proposition 2). Since the usual basis $S = \{e_1, e_2, \ldots, e_n\}$ of $\ell_p^n, 1 < p < \infty$ is a $2^{\frac{1}{p}}$-equilateral set and $2^{\frac{1}{2}} > 1$, it follows in particular that $H'_\alpha(\ell_p^n) \geq n$. If $p = 1$, then the set $\{\pm e_k : k = 1, 2, \ldots, n\}$ is $2$-equilateral, hence $H'_\alpha(\ell_1^n) \geq 2n$.

In the sequel we will obtain lower estimates for the antipodal and the strict antipodal Hadwiger numbers of a finite-dimensional Banach space.

**Proposition 1.** Let $X$ be an $n$-dimensional Banach space. Also let $\{(e_i, e^*_i) : i = 1, 2, \ldots, n\}$ be an Auerbach basis of $X$. Then the set $A = \{\pm e_i, i = 1, 2, \ldots, n\}$ is a bounded and separated antipodal subset of $X$ with constant $\delta = 1$ and hence $H_\alpha(X) \geq 2n$.

**Proof.** We check that $\forall x \neq y \in A$ there is $f \in B_{X^*}$ with $1 \leq f(x) - f(y)$ and $f(y) \leq f(z) \leq f(x)$ for $z \in A$. We have the following cases:
(1) Let $x = e_i$ and $y = e_j$, $i \neq j$. Set $f = \frac{e_i - e_j}{2}$, then $\|f\| \leq 1$ and
$$-\frac{1}{2} = f(e_j) \leq f(\pm e_k) \leq f(e_i) = \frac{1}{2} \text{ for } k = 1, 2, \ldots, n$$

(2) Let $x = e_i$ and $y = -e_i$. Set $f = e_i^*$, then $\|f\| = 1$ and
$$-1 = f(-e_i) \leq f(\pm e_k) \leq f(e_i) = 1 \text{ for } k = 1, 2, \ldots, n$$

(3) Let $x = e_i$ and $y = -e_j$, $i \neq j$. Set $f = \frac{e_i^* + e_j^*}{2}$, then $\|f\| \leq 1$ and
$$-\frac{1}{2} = f(-e_j) \leq f(\pm e_k) \leq f(e_i) = \frac{1}{2} \text{ for } k = 1, 2, \ldots, n$$

Any other case is similar and the proof is complete. \qed

In case when the space is smooth we have the following stronger result, which was proved in [5] (Prop. 3.9); see also [6] (Prop. 1.6):

**Proposition 2.** Let $X$ be an $n$-dimensional smooth Banach space and let $\{(e_i, e_i^*) : i = 1, 2, \ldots, n\}$ be an Auerbach basis of $X$. Then the set $A = \{\pm e_i, i = 1, 2, \ldots, n\}$ is a bounded and separated antipodal subset of $B_X$ with constant $\delta = 1 + \varepsilon$ for some $\varepsilon > 0$. So when $X$ is smooth we have $H'_\alpha(X) \geq 2n$.

Strengthening our assumption about the basis (supposing it is 1-suppression unconditional) we can prove the following:

**Proposition 3.** Let $X$ be an $n$-dimensional Banach space and let $\{e_i, 1 \leq i \leq n\}$ be a 1-suppression unconditional normalized basis of $X$. If the norm of $X$ is strictly convex (or smooth), then the set $A = \{\pm e_i, i = 1, 2, \ldots, n\}$ is a bounded and separated antipodal subset of $B_X$ with constant $\delta = 1 + \varepsilon$, hence $H'_\alpha(X) \geq 2n$.

**Proof.** Recall that the basis $\{e_i, 1 \leq i \leq n\}$ of $X$ is 1-suppression unconditional, if for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ and $F \subseteq \{1, 2, \ldots, n\}$ we have $\|\sum_{k \in F} \alpha_k e_k\| \leq \|\sum_{k=1}^n \alpha_k e_k\|$. Since the basis is normalized, we get that the biorthogonal functionals $\{e_i^*, 1 \leq i \leq n\}$ are also normalized. In particular $\{(e_i, e_i^*) : 1 \leq i \leq n\}$ is an Auerbach basis of $X$.

In both cases we use the fact that $\|e_i \pm e_j\| > 1$ for $1 \leq i < j \leq n$. In any case we have $\|e_i \pm e_j\| \geq 1$, since $\{(e_i, e_i^*) : 1 \leq i \leq n\}$ is an Auerbach basis of $X$.

Let $X$ be strictly convex and assume that $\|e_i + e_j\| = 1$ for some $1 \leq i < j \leq n$. Then $e_i^*(e_i + e_j) = e_i^*(e_i) = 1$, hence the normalized functional $e_i^*$ attains its maximum at two distinct points of the unit ball, a contradiction (see [10] §3.2). So $\|e_i + e_j\| > 1$ and similarly $\|e_i - e_j\| > 1$.

Let $\delta = \min\{\|e_i \pm e_j\| : 1 \leq i < j \leq n\}$. Then $\delta = 1 + \varepsilon$ for some $\varepsilon > 0$. We distinguish the following cases:

(I) Given $1 \leq i < j \leq n$ there are $\lambda, \mu \in \mathbb{R}$ such that the functional $f = \lambda e_i^* + \mu e_j^*$ satisfies $\|f\| \leq 1$, $f(e_i - e_j) = \|e_i - e_j\| = \lambda - \mu \geq 1 + \varepsilon$ and $\lambda = f(e_i) \geq f(\pm e_k) \geq f(e_j) = \mu$, for $1 \leq k \leq n$. 

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To prove this, note that the set \( \{ e_i^* : 1 \leq i \leq n \} \) is also 1-suppression unconditional normalized basis of \( X^* \) and thus \( \|e_i - e_j\| = \sup \{(xe_i^* + ye_j^*) : \|xe_i^* + ye_j^*\| \leq 1\} \).

So there are \( \lambda, \mu \in \mathbb{R} \) such that the functional \( f = \lambda e_i^* + \mu e_j^* \) gives

\[
 f(e_i - e_j) = \|e_i - e_j\| = \lambda - \mu > 1. 
\]

It follows that

(a) \(-1 \leq \lambda, \mu \leq 1 \) (since \( f(e_i) = \lambda, f(e_j) = \mu, \|f\| \leq 1 \) and \( \|e_i\| = \|e_j\| = 1 \)).

(b) \(-1 \leq \mu < 0 < \lambda \leq 1 \) (since \( \lambda - \mu \geq 1 + \varepsilon \)).

Consequently \( \lambda = f(e_i) \geq f(\pm e_k) \geq f(e_j) = \mu \), for \( 1 \leq k \leq n \).

(II) Given \( 1 \leq i < j \leq n \) there are \( \lambda, \mu \in \mathbb{R} \) such that the functional \( g = \lambda e_i^* + \mu e_j^* \) satisfies \( \|g\| \leq 1 \), \( g(e_i + e_j) = \|e_i + e_j\| = \lambda + \mu \geq 1 + \varepsilon \) and \( -\lambda = g(-e_i) \leq g(\pm e_k) \leq g(e_j) = \mu \), for \( 1 \leq k \leq n \).

The proof of this is similar.

(III) For \( 1 \leq i \leq n \) set \( f = e_i^* \). Then we have \(-1 \leq f(-e_i^*) \leq f(\pm e_k) \leq f(e_i^*) = 1 \), for \( 1 \leq k \leq n \), which completes the proof that \( A \) is a bounded and separated antipodal subset of \( X \).

Let now \( X \) be a smooth space. Assume that \( \|e_i + e_j\| = 1 \) for some \( 1 \leq i < j \leq n \). Then \( e^*_i(e_i + e_j) = e^*_j(e_i + e_j) = 1 \), hence the normalized support functional of the vector \( e_i + e_j \) is not unique, which contradicts the smoothness of \( X \). So \( \|e_i + e_j\| > 1 \) and similarly \( \|e_i - e_j\| > 1 \). The rest of the proof proceeds as in the strictly convex case.

\[ \square \]

**Remarks 2**

(1) It is clear that the assumption of strict convexity or smoothness in Prop.3 can be replaced by \( \|e_i \pm e_j\| > 1 \) for \( 1 \leq i < j \leq n \).

(2) An obvious example realizing Prop.3 is the set \( \{ \pm e_i : 1 \leq i \leq n \} \) in \( \ell_p^n \), \( 1 < p < \infty, n \geq 2 \).

(3) If \( X \) is a Minkowski plane and \( x, y \in S_X \) with \( \|x - y\| = \|x + y\| = 1 \) then the points \( \pm(x + y) \) and \( \pm(x - y) \) lie on \( S_X \) and are the vertices of a parallelogram inscribed in the unit circle. Since \( \|\pm x\| = \|\pm y\| = 1 \), by Lemma 5, p.8 of [10] we get that all the segments joining neighbouring vertices must lie on the unit circle and hence the unit circle itself coincides with the parallelogram with vertices \( \pm(x + y), \pm(x - y) \). It follows that \( X \) is isometric to \( \ell_\infty^2 \). So if \( X \) is not isometric to \( \ell_\infty^2 \) and we have \( x, y \in S_X \) with \( \|x - y\| = 1 \), then necessarily \( \|x + y\| > 1 \). Similarly when \( x, y \in S_X \) with \( \|x - y\| = \|x + y\| = 2 \), it follows that \( X \) is isometric to \( \ell_\infty^2 \).

**Corollary 1.** Let \( X \) be a 2-dimensional Banach space and \( \{(e_i, e_i^*) : i = 1, 2\} \) be an Auerbach basis of \( X \). Then we have:

1. \( H_\alpha(X) = 4 \)

2. If \( \|e_1 \pm e_2\| > 1 \) (for instance if \( X \) is either smooth or strictly convex), then \( H'_\alpha(X) = 4 \).
Proof. Claim (1) is obvious by Prop.1. Regarding claim (2), note that any Auerbach basis of a 2-dimensional Banach space is 1-suppression unconditional, hence Prop.3 and Remark 2(1) imply the claim.

For the rest of this chapter we confine ourselves to the study of the strict antipodal Hadwiger number of a Minkowski plane, i.e. of a 2-dimensional Banach space. We will show that this number always equals 4.

**Proposition 4.** Let $(X, \| \cdot \|)$ be a Minkowski plane. Then $H'_\alpha(X) = 4$.

Proof. Let $\{(e_i, e_i^*) : i = 1, 2\}$ be an Auerbach basis of $X$. Since the vectors $e_1$ and $e_2$ are mutually orthogonal, the unit ball $B_X$ is supported at $e_1$ by a line $L_1 = \{ z \in X : e_1^*(z) = 1 \}$ parallel to $e_2$ and also supported at $e_2$ by a line $L_2 = \{ z \in X : e_2^*(z) = 1 \}$ parallel to $e_1$. The lines $L_1, L_2$ are (non-parallel) sides of the parallelogram with vertices $\{ \pm (e_1 - e_2), \pm (e_1 + e_2) \}$ (and $\pm e_1, \pm e_2$ are the midpoints of these sides). Since $\{ e_1, e_2 \}$ is an Auerbach basis, we get that $1 \leq \| e_1 \pm e_2 \|$ (and of course $\| e_1 \pm e_2 \| \leq 2$). If all the vertices of this parallelogram are of norm 2, then $X$ is isometric to $\ell^2_\infty$ (see Remark 2(3)) and the result follows.

Otherwise there is a pair of opposite vertices of norm less than 2 (say $\| \pm (e_1 + e_2) \| < 2$). Now draw the diagonals of this parallelogram and take the points of intersection of these diagonals with the unit sphere $S_X$ (see figure 1). We get 4 points (the points $\pm \frac{e_1 - e_2}{\| e_1 - e_2 \|}, \pm \frac{e_1 + e_2}{\| e_1 + e_2 \|}$) which constitute a new parallelogram $ABCD$. This is a bounded and separated antipodal set of 4 points lying on the unit sphere. Consider the two norm one functionals $f$ and $g$ whose kernels are lines parallel to $AB$ and $BC$.

Then since the homothetic copy $2ABCD$ does not touch the unit sphere, $f$ and $g$ give an evaluation $> \frac{1}{2}$ and $< -\frac{1}{2}$ on opposite sides of $ABCD$. Indeed, let $x \in S_X$ such that $f(x) = 1$, then $x \notin Ker f$, so the half-line $L^+ = \{ \lambda x : \lambda \geq 0 \}$ either intersects the line $2AB$ or the line $2DC$. Let for instance $L^+$ intersect $2AB$ at $y = \lambda z$ and $AB$ at $z$. Then clearly $\lambda > 1$ and $z = \frac{1}{2} x$, thus $f(z) = \frac{1}{2} f(x) = \frac{1}{2} > \frac{1}{2}$. In a similar manner we
get that also $g$ has the desired property. This way we obtain the constant $\delta > 1$ in the definition of bounded and separated antipodal subset of $X$. 

The following result shows that any Minkowski plane has an Auerbach basis satisfying the hypothesis of Cor.1(2). Therefore we get still another proof of Prop.4. This result is probably already known, but since we haven’t seen it stated anywhere, we include a full proof.

**Theorem 1.** Let $(X, \| \cdot \|)$ be any Minkowski plane. Then there is an Auerbach basis $\{(e_i, e_i^*) : i = 1, 2\}$ of $X$ such that $\|e_1 - e_2\| > 1$ and $\|e_1 + e_2\| > 1$. Hence Corollary 1 implies that the set $A = \{\pm e_1, \pm e_2\}$ is bounded and separated antipodal with $\delta > 1$ (so $H'_\alpha(X) = 4$).

**Proof.** If $\{(e_1, e_2)\}$ is the usual basis of $\ell^2_\infty$, then $\{(u_i, u_i^*) : i = 1, 2\}$, where $u_1 = e_1 + e_2, u_2 = e_1 - e_2, u_1^* = \frac{e_1 + e_2}{2}, u_2^* = \frac{e_1 - e_2}{2}$ is an Auerbach basis of this space satisfying the desired property. So we may assume that $X$ is not isometric to $\ell^2_\infty$.

Let $\{(e_i, e_i^*) : i = 1, 2\}$ be an Auerbach basis of $X$. Since the vectors $e_1$ and $e_2$ are mutually orthogonal, the unit ball $B_X$ is supported at $e_1$ by a line parallel to $e_2$ and also supported at $e_2$ by a line parallel to $e_1$. These lines are (non-parallel) sides of the parallelogram with vertices $\{\pm(e_1 - e_2), \pm(e_1 + e_2)\}$ and $\pm e_1, \pm e_2$ are the midpoints of these sides (see figure 2).

![figure 2](attachment://image.png)

We clearly have $\|e_1 \pm e_2\| \geq 1$ and since $X$ is not isometric to $\ell^2_\infty$, we have that $\|e_1 - e_2\| > 1$ or $\|e_1 + e_2\| > 1$ (see Remark 2(3)).

If both of these inequalities are valid, then we are done. Otherwise, suppose without loss of generality that $\|e_1 - e_2\| > 1$ and $\|e_1 + e_2\| = 1$. Then the vertices $\pm(e_1 + e_2)$ of the circumscribed parallelogram are antidiamicetric points of the unit circle $S_X$. Now since the midpoints of the sides of the parallelogram lie on the unit circle, it follows that the closed segments $[e_2, e_1 + e_2], [e_1, e_1 + e_2], [-e_2, -e_1 - e_2]$ and $[-e_1, -e_1 - e_2]$ also lie on $S_X$, by Lemma 5, p.8 of [10].
Let $Y$ be the subspace of $X$ produced by $e_1 + e_2$. Consider a point $u \in S_X$ such that $d(u, Y) = 1$ (equivalently take a support line $L$ of $B_X$ passing through $u \in S_X$ and parallel to the vector $e_1 + e_2$). Then $u$ must lie on one of the arcs of $S_X$ with endpoints $e_2, -e_1$ (second quadrant of $S_X$) or $-e_2, e_1$ (fourth quadrant of $S_X$).

Assume without loss of generality that $u$ lies on the second quadrant. Then there are two cases:

1. $u$ lies on the interior of the corresponding arc, or
2. $u$ is an endpoint of the corresponding arc.

We start by examining the first case. It is clear that $u$ is normal to the vector $e_1 + e_2$ ($1 = \|u\| \leq \|u - y\|$, $y \in Y$). Let $J$ be the subspace of $X$ produced by $u$, then we will prove that also $e_1 + e_2$ is normal to $u$ (i.e., that $1 = \|e_1 + e_2\| \leq \|e_1 + e_2 - z\|$, $z \in J$). If this is true, then the diameters of $S_X$ defined by the lines $L$ and $J$ are conjugate and therefore $\{u, e_1 + e_2\}$ is an Auerbach basis of $X$, for which we will prove that the conclusion of the Proposition holds. Note that $e_1 + e_2$ is an extreme point of $B_X$ (a vertex of the circumscribed parallelogram), so we can draw a line $L'$ through $e_1 + e_2$ parallel to the vector $u$, which supports $B_X$ at $e_1 + e_2$, hence $e_1 + e_2 \perp u$ (see figure 2). We conclude that $\{u, e_1 + e_2\}$ is an Auerbach basis of $X$.

We claim that $\|u - (e_1 + e_2)\| > 1$ and $\|u + (e_1 + e_2)\| > 1$. To prove the first inequality, draw through $O$ a parallel to the line connecting points $u$ and $e_1 + e_2$. Note that the length of the vector $\overline{OM}$ equals the length of the segment $[N, e_1 + e_2]$ (both are of length 1) which is obviously smaller than the length of the segment $[u, e_1 + e_2]$ , i.e. from $\|u - (e_1 + e_2)\|$. Similarly, drawing through $O$ a parallel to the line connecting $u$ and $-e_1 - e_2$, we conclude that the length of $\overline{OP}$ equals the length of the segment $[\Sigma, -(e_1 + e_2)]$ which is smaller than $\|u - (-e_1 - e_2)\| = \|u + e_1 + e_2\|$. In particular the points $u + e_1 + e_2$ and $u - (e_1 + e_2)$ lie outside of the circumscribed parallelogram and thus $\|u - (e_1 + e_2)\| > 1$ and $\|u + (e_1 + e_2)\| > 1$.

Assume now that $u$ coincides with one of the endpoints of the corresponding arc, e.g. with $e_2$ (see figure 3). Then the segments $[-e_1, e_2]$ and $[-e_2, e_1]$ lie completely on $S_X$ and in this second case the ball is the affine regular hexagon with vertices $e_1 + e_2, e_2, -e_1, -e_1 - e_2, -e_2$ and $e_1$. This hexagon is regular with respect to the norm of $X$, with side length equal to 1. Now pick any point $\alpha$ in the interior of the segment $[-e_1, e_2]$, then one can easily check that the pair $\{\alpha, e_1 + e_2\}$ is an Auerbach basis of $X$ satisfying $\|\alpha - (e_1 + e_2)\| > 1$ and $\|\alpha + (e_1 + e_2)\| > 1$. 

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Remark 3
If \( X \) is an \( n \)-dimensional space with \( n \geq 3 \) then \( H'_\alpha(X) \geq 4 \), since any Minkowski plane admits a bounded and separated antipodal set establishing this fact (see Prop.4). Moreover one can obtain a 3-dimensional set of 4 points yielding the same result. By a result of V.V. Makeev, in any 3-dimensional space there is an equilateral tetrahedron such that its vertices are equidistant from the barycenter (see [9]). Assuming that the distance of a vertex from the barycenter is 1 (and that the barycenter coincides with the origin) we obtain a (3-dimensional) \( \lambda \)-equilateral set of 4 points lying on the unit sphere with \( \lambda > 1 \). Taking into account Remarks 1(4) we have the result.

Antipodal Hadwiger number of \( \ell_p \) spaces, \( 1 < p \leq \infty \)

In this chapter we evaluate the antipodal and strict antipodal Hadwiger numbers of many (finite dimensional) \( \ell_p \) spaces. Let \( \alpha_n = \frac{\log n}{\log 2} = \log_2 n \), \( n \geq 2 \). This sequence is strictly increasing: \( 1 = \frac{\log 2}{\log 2} < \frac{\log 3}{\log 2} < \frac{\log 4}{\log 2} = \alpha_4 = 2 < \cdots < \alpha_n < \ldots \).

Proposition 5. Let \( n \geq 2 \) and \( 1 < p \leq \infty \), then the following hold:

1. When \( p > \alpha_n \), then \( H'_\alpha(\ell^n_p) = 2^n \) (hence also \( H_\alpha(\ell^n_p) = 2^n \)).

2. When \( p \geq \alpha_n \), then \( H_\alpha(\ell^n_p) = 2^n \).

Proof. Consider the set \( S = \{-1,1\}^n \) and note that it is an antipodal subset of \( \mathbb{R}^n \).

Assume first that \( 1 < p < \infty \). Then \( \|x\|_p = n^{1/p} \), for every \( x \in S \). Since \( \lim_{p \to \infty} n^{1/p} = 1 \) and \( 1 < n^{1/p}, n \geq 2 \) and \( p > 1 \), we get that there is \( p_0(n) \) such that

\[ p > p_0(n) \Rightarrow 1 < n^{1/p} < 2 \quad (1). \]
We will show that the least number \( p_0(n) \) such that inequality (1) holds is \( \alpha_n \). Indeed, for \( n \geq 2 \) and \( p > 1 \) we have

\[
n^{1/p} < 2 \iff \log n^{1/p} < \log 2 \iff \frac{1}{p} \log n < \log 2 \iff p > \alpha_n = \frac{\log n}{\log 2}.
\]

Set \( S' = \frac{1}{n^{1/p}} \cdot S \), for \( p \geq \alpha_n \) and \( n \geq 2 \) and observe that \( S' \) is a bounded and separated antipodal subset of \( S_p \) with constant \( \delta = \frac{2}{n^{1/p}} \geq 1 \). This is true because, given \( x \neq y \in S' \), \( x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \), there is \( 1 \leq k \leq n \) with \( x_k \neq y_k \) such that the numbers \( x_k, y_k \) have different signs and \( |x_k| = |y_k| = \frac{1}{n^{1/p}} \).

Without loss of generality, let \( x_k > 0 \) and \( y_k < 0 \). Then

\[
e_k(x) = -\frac{1}{n^{1/p}} \leq e_k(z) \leq e_k(x) = \frac{1}{n^{1/p}}, \quad \text{for} \quad z \in S'.
\]

It is clear from these inequalities that \( H'_a(\ell^p_1) = 2^n \) for \( p \in (\alpha_n, \infty) \) (since then \( \delta = \frac{2}{n^{1/p}} > 1 \)) and \( H'_a(\ell^p_2) = 2^n \) for \( p = \alpha_n \) (since then \( n^{1/p} = 2 \) and thus \( \delta = 1 \)).

In case when \( p = \infty \), it is easily verified that the set \( S \) itself is bounded and separated antipodal subset of \( S_{\ell^\infty} \) with constant \( \delta = 2 \) (actually \( S \) is a 2-equilateral set, cf. Remark 1(4)), hence \( H'_a(\ell^\infty_2) = 2^n \).

The proof of the Proposition is now complete.

\[ \Box \]

**Remarks 4**

(1) For \( n = 2 \) we already have the stronger result \( H'_a(X) = 4 \) for any Minkowski plane, by Proposition 4.

(2) For \( n = 3 \) we get that, when \( p > \alpha_3 = \frac{\log 3}{\log 2} \simeq 1.58 \), then \( H'_a(\ell^3_p) = 2^3 = 8 \). In particular \( H'_a(\ell^3_2) = 2^3 = 8 \).

The following Proposition settles the situation in case of a 3-dimensional space for the remaining \( 1 < p \leq \alpha_3 \simeq 1.58 \):

**Proposition 6.** When \( 1 < p < \infty \), then \( H'_a(\ell^3_p) = 2^3 = 8 \).

**Proof.** Let \( \beta > 2 \) be close enough to 2 such that \( \frac{2}{\beta^{1/p}} > 1 \). Take \( \alpha > 0 \) (and \( \alpha < 2 \)) such that \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \iff \alpha = \frac{\beta}{\beta - 1} \). We consider the points

\[
\begin{align*}
x_1 &= \left( \frac{1}{\alpha^{1/p}}, 0, \frac{1}{\beta^{1/p}} \right) \\
x_2 &= \left( -\frac{1}{\alpha^{1/p}}, 0, \frac{1}{\beta^{1/p}} \right) \\
x_3 &= \left( 0, \frac{1}{\alpha^{1/p}}, \frac{1}{\beta^{1/p}} \right) \\
x_4 &= \left( 0, -\frac{1}{\alpha^{1/p}}, \frac{1}{\beta^{1/p}} \right).
\end{align*}
\]

Also set \( x_5 = -x_1, x_6 = -x_2, x_7 = -x_3 \) and \( x_8 = -x_4 \). Since \( x_4 - x_1 = \left( -\frac{1}{\alpha^{1/p}}, -\frac{1}{\alpha^{1/p}}, 0 \right) = x_2 - x_3 \), the points \( x_1, x_2, x_3, x_4 \) are vertices of a parallelogram (actually an orthogonal parallelogram). Thus, the points \( \pm x_1, \pm x_2, \pm x_3, \pm x_4 \) are vertices of an orthogonal parallelepiped. Observe that \( \|x_k\|_p = 1 \), \( k = 1, 2, 3, 4 \), so \( \pm x_k \in S_{\ell^p_2}, k = 1, 2, 3, 4 \) and also \( \|x_k - x_l\|_p > 1, 1 \leq k < l \leq 8 \).
Obviously $S = \{ x_k : k = 1, 2, \ldots, 8 \}$ is an antipodal subset of $S_3^2$. We will show that it is bounded and separated antipodal with constant $\delta > 1$. Consider the functionals

\[ f_1 = e_3^*, \quad f_2 = \frac{e_1^* + e_2^*}{2^{1/q}} \quad \text{and} \quad f_3 = \frac{e_1^* - e_2^*}{2^{1/q}}, \]

where $q$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$ and $\{e_1, e_2, e_3\}$ is the usual basis of $\ell_f^3$, $1 < p < \infty$. Note that $f_1, f_2, f_3 \in S_q^2$ (the unit sphere of the dual space $\ell_q^f$). Geometrically, the kernels of $f_1, f_2, f_3$ are the planes $z = 0, y = -x$ and $y = x$ of $\mathbb{R}^3$ respectively.

The following are easy to check:

\[ f_1(x_1) = f_1(x_2) = f_1(x_3) = f_1(x_4) = \frac{1}{\beta^{1/p}} \quad \text{and} \]
\[ f_1(x_5) = f_1(x_6) = f_1(x_7) = f_1(x_8) = -\frac{1}{\beta^{1/p}}. \]

Therefore the points $\{x_1, x_2, x_3, x_4\}$ and $\{x_5, x_6, x_7, x_8\}$ are separated by the plane $z = 0$ and the difference is $\frac{1}{\beta^{1/p}} - \left(-\frac{1}{\beta^{1/p}}\right) = \frac{2}{\beta^{1/p}} > 1$. Also

\[ f_2(x_1) = f_2(x_2) = f_2(x_3) = f_2(x_4) = \frac{1}{2^{1/q}} \cdot \frac{1}{\alpha^{1/p}} > \frac{1}{2^{1/q}} \cdot \frac{1}{2^{1/p}} = \frac{1}{2} \quad \text{and} \]
\[ f_2(x_5) = f_2(x_6) = f_2(x_7) = f_2(x_8) = -\frac{1}{2^{1/q}} \cdot \frac{1}{\alpha^{1/p}} < -\frac{1}{2}. \]

So the separation now is achieved by the plane $y = -z$ and the difference is $\frac{1}{2^{1/q}} \cdot \frac{1}{\alpha^{1/p}} - \left(-\frac{1}{2^{1/q}} \cdot \frac{1}{\alpha^{1/p}}\right) = \frac{2}{2^{1/q} \cdot \alpha^{1/p}} > 1$. We also have

\[ f_3(x_1) = f_3(x_4) = f_3(x_5) = f_3(x_7) = \frac{1}{2^{1/q}} \cdot \frac{1}{\alpha^{1/p}} > \frac{1}{2} \quad \text{and} \]
\[ f_3(x_2) = f_3(x_3) = f_3(x_6) = f_3(x_8) = -\frac{1}{2^{1/q}} \cdot \frac{1}{\alpha^{1/p}} < -\frac{1}{2}. \]

The separation of the points is now achieved by the plane $y = z$ and the corresponding difference is $\frac{1}{2^{1/q}} \cdot \frac{1}{\alpha^{1/p}} - \left(-\frac{1}{2^{1/q}} \cdot \frac{1}{\alpha^{1/p}}\right) = \frac{2}{2^{1/q} \cdot \alpha^{1/p}} > 1$.

From the above calculations, we conclude that $\delta = \min\{\frac{2}{\beta^{1/p}}, \frac{2}{2^{1/q} \cdot \alpha^{1/p}}\} > 1$ and thus $H'_\alpha(\ell_f^p) = 2^4 = 8$, $\forall p > 1$.

Using the same method of proof, one can prove the following generalization:

**Proposition 7.** When $n \geq 3$ and $1 < p < \infty$, then $H'_\alpha(\ell_f^p) \geq 4n - 4$.

**Proof.** Assigning to the last coordinate the value $\pm \frac{1}{\beta^{1/p}}$ and placing $\pm \frac{1}{\alpha^{1/p}}$ each time in one of the first $n - 1$ coordinates with 0 in every other place, one obtains the $4n - 4$ required vectors. \qed
Remark 5

It is clear from Remark 4(1) that the above result also holds true for \( n = 2 \). Since the Banach space \( \ell^p_n, 1 < p < \infty, n \geq 2 \) is smooth, we get from Prop. 2 that \( H'_\alpha(\ell^p_n) \geq 2n \). Proposition 7 though provides us with a better lower bound, \( H'_\alpha(\ell^p_n) \geq 4n - 4 \).

Proposition 8. Let \( n \geq 4 (\alpha_n \geq 2) \), then the following hold:

1. When \( 2 \leq p \leq \alpha_n \), then \( 4n - 4 \leq H'_\alpha(\ell^p_n) < 2^n \).

2. When \( 2 \leq p < \alpha_n \), then \( 4n - 4 \leq H'_\alpha(\ell^p_n) \leq H_\alpha(\ell^p_n) < 2^n \).

3. When \( p = \alpha_n \), then \( 4n - 4 \leq H'_\alpha(\ell^p_n) < H_\alpha(\ell^p_n) < 2^n \).

Proof. To prove (1), given \( p, n \) satisfying \( 2 \leq p \leq \alpha_n \) assume that the contrary holds, i.e. \( H'_\alpha(\ell^p_n) = 2^n \). Then \( B_{\ell^p_n} \) contains a bounded and separated antipodal subset \( S \) with constant \( \delta > 1 \) and cardinality \( |S| = 2^n \). By Theorem 3 of [11] stated in the Introduction, there is an equivalent norm \( ||| \cdot ||| \) on \( \mathbb{R}^n \) which admits an equilateral set of cardinality \( 2^n \), hence \( (\mathbb{R}^n, ||| \cdot |||) \) is isometric to \( \ell^\infty_n \). Moreover the same Theorem yields for the Banach-Mazur distance of the norms \( \| \cdot \|_p \) and \( ||| \cdot ||| \) that

\[
d(\ell^p_n, \ell^\infty_n) \leq \frac{2}{\delta} < 2, \quad \text{since } \delta > 1.
\]

But for \( p \geq 2 \) we know that \( d(\ell^p_n, \ell^\infty_n) = n^{1/p} \), see [4]. Hence

\[
n^{1/p} = d(\ell^p_n, \ell^\infty_n) \leq \frac{2}{\delta} < 2 \Rightarrow p > \frac{\log n}{\log 2}
\]

which contradicts our assumption. Taking into account Proposition 7, the proof of (1) is complete.

The proof of (2) is similar. Concerning (3), the inequalities follow from (1) and the equality follows from Proposition 5(2). \( \square \)

The following Theorem summarizes all the previous results about \( \ell^p_n, 1 < p \leq +\infty \) spaces:

Theorem 2. Let \( 1 < p \leq +\infty \), then the following hold:

1. When \( n = 2 \) or \( 3 \), then \( H'_\alpha(\ell^p_n) = 4n - 4 = 2^n = H_\alpha(\ell^p_n) \).

2. When \( n \geq 4 \), then we have:

   (a) \( H'_\alpha(\ell^p_n) \geq 4n - 4 \)

   (b) when \( 2 \leq p < \alpha_n \), then \( 4n - 4 \leq H'_\alpha(\ell^p_n) \leq H_\alpha(\ell^p_n) < 2^n \)

   (c) when \( p = \alpha_n \), then \( 4n - 4 \leq H'_\alpha(\ell^p_n) \leq 2^n = H_\alpha(\ell^p_n) \), in particular for \( n = 4 \) we get that \( p = \alpha_4 = 2 \) and \( \frac{12}{\log 2} < 16 = H_\alpha(\ell^2_4) \) and

   (d) when \( p > \alpha_n \), then \( H'_\alpha(\ell^p_n) = 2^n = H_\alpha(\ell^p_n) \).
The following result says that the number $H'_\alpha(X)$ is bounded below by an unbounded function of the dimension of $X$. We shall briefly describe the proof of this result.

Recall that if $X$ is a finite dimensional Banach space, then $e(X)$ (=the equilateral number of $X$) denotes the largest size of an equilateral subset of $X$. We define analogously the number $e_\lambda(X)$ to be the largest size of a $\lambda$-equilateral subset of $S_X$, where $1 < \lambda \leq 2$. It is clear that

$$e_\lambda(X) \leq e(X)$$

and also by Remark 1(4) that

$$H'_\alpha(X) \geq e_\lambda(X).$$

Now, one can prove by similar arguments (we omit the details) the following variant of a significant Theorem of Brass and Dekster (see Theorem 8 of [13]):

**Theorem 3.** Let $X$ be an $n$-dimensional Banach space ($n \geq 2$) with Banach-Mazur distance $d(X, \ell_p^n) \leq 1 + \frac{1}{3(n+1)}$. Then any $\lambda$-equilateral set in $S_X$, where $\lambda \in \left(1, 1 + \frac{1}{3(n+1)}\right)$, of at most $n - 1$ points can be extended to a $\lambda$-equilateral set in $S_X$ of $n$ points.

We can prove using Dvoretzky’s Theorem and Theorem 3, in the same way as Theorem 7 of [13] is proved, that if $\text{dim} X = n$ then $e_\lambda(X) \geq c(\log n)^{\frac{1}{3}}$ for some constant $c > 0$ and $n$ sufficiently large.

So from the above observations we get the following:

**Theorem 4.** Let $X$ be an $n$-dimensional Banach space. Then

$$H'_\alpha(X) \geq e_\lambda(X) \geq c(\log n)^{\frac{1}{3}}.$$

**Note.** Using the techniques of Swanepoel and Villa in [14] (see also Th.4.3 of [7]) one can show that $H'_\alpha(X) \geq e_\lambda(X) \geq e^{c_1 \sqrt{\log n}}$, for some constant $c_1 > 0$. We also note that for $n = 2$ or 3 the inequality $e_\lambda(X) \geq n + 1$ holds; for the case $n = 2$ we refer the reader to [8], Prop. 1.2 and for $n = 3$ to Remark 3.

We also investigated the spaces $\ell_1^3$ and the Petty space on $\mathbb{R}^3$ (see [13], [15]) with respect to their (strict) antipodal Hadwiger numbers. In both spaces, the strict antipodal Hadwiger numbers are as big as possible:

**Proposition 9.**

1. $H_\alpha(\ell_1^3) = H'_\alpha(\ell_1^3) = 2^3 = 8.$

2. If $(X, \| \cdot \|)$ is the Petty space on $\mathbb{R}^3$, where $\|(x, y, z)\| = \sqrt{x^2 + y^2 + |z|}$, then $H_\alpha(X) = H'_\alpha(X) = 2^3 = 8.$

**Proof.** To prove (1), set $x_1 = (1, 1, -\frac{1}{3})$, $x_2 = (1, -\frac{1}{3}, 1)$, $x_3 = (-\frac{1}{3}, 1, 1)$ and $O = \text{conv}\{\pm x_1, \pm x_2, \pm x_3\}$. The Minkowski functional of $O$ defines a norm and the corresponding space is isometric to $\ell_1^3$ through the linear isometry designated by $T(e_i) = x_i$, $i = 1, 2, 3$ ($\{e_1, e_2, e_3\}$ is the usual basis of $\ell_1^3$). Let also $C_3 = B_{c_3} = [-1, 1]^3$. In [16] Fei Xue observed that the octahedron $O$ and the cube $C_3$ satisfy

$$\frac{5}{9} C_3 \subseteq O \subseteq C_3.$$
and obtained for the Banach-Mazur distance of the spaces $\ell_1^d$ and $\ell_\infty^d$ the upper bound $d(\ell_1^d, \ell_\infty^d) \leq \frac{9}{5} < 2$.

The set $S = \frac{5}{9}[-1, 1]^3$ is the set of vertices of a parallelepiped and all of its points belong to the boundary of the octahedron $O$ (hence are vectors of $\ell_1$-norm 1). One can readily check that

$$5 \left( \frac{1}{9}, \frac{1}{9}, \frac{1}{9} \right) = \frac{5}{3} \left( x_1 + x_2 + x_3 \right), \quad 5 \left( \frac{1}{9}, \frac{1}{9}, -\frac{1}{9} \right) = \frac{5}{3} x_1 + \frac{1}{6} (-x_2) + \frac{1}{6} (-x_3)$$

$$5 \left( -\frac{1}{9}, -\frac{1}{9}, \frac{1}{9} \right) = \frac{5}{3} x_3 + \frac{1}{6} (-x_1) + \frac{1}{6} (-x_2), \quad 5 \left( -\frac{1}{9}, -\frac{1}{9}, -\frac{1}{9} \right) = \frac{2}{3} (-x_2) + \frac{1}{6} x_3 + \frac{1}{6} x_1$$

and the other points are the symmetric of these 4. We will show that $S$ is a bounded and separated antipodal subset of $S_{\ell_1^d}$ with constant $\delta = \frac{10}{9} > 1$.

The functionals separating opposite faces of the parallelepiped are the $e_i^*$, $i = 1, 2, 3$ (since $O \subseteq C_3$, we have that $|e_i^*(x, y, z)| \leq 1$ for any $(x, y, z) \in O$, hence $e_i^* \in B(S_{\ell_1^d})$). For instance

$$e_1^* \left( \frac{5}{9}, \frac{5}{9}, \frac{5}{9} \right) = e_1^* \left( \frac{5}{9}, \frac{5}{9}, -\frac{5}{9} \right) = e_1^* \left( \frac{5}{9}, -\frac{5}{9}, -\frac{5}{9} \right) = \frac{5}{9}$$

$$e_1^* \left( -\frac{5}{9}, \frac{5}{9}, \frac{5}{9} \right) = e_1^* \left( -\frac{5}{9}, -\frac{5}{9}, -\frac{5}{9} \right) = e_1^* \left( -\frac{5}{9}, -\frac{5}{9}, \frac{5}{9} \right) = -\frac{5}{9}$$

and the evaluations for the other faces are similar (see also the proof of Proposition 6), which implies that $\delta = \frac{5}{9} - (-\frac{5}{9}) = \frac{10}{9} > 1$.

Now for the proof of (2) one may consider the points $A(-0.18, 0, 0.82)$, $B(0.82, 0, -0.18)$, $C(0.32, 0.6, 0.32)$ and $D(0.32, -0.6, 0.32)$. These points lie on the unit sphere of the Petty space and form a parallelogram, as $\overline{AC} = \overline{DB} = (0.5, 0.6, -0.5)$. So these points together with the symmetric points $A', B', C', D'$ with respect to the origin form a parallelepiped with all vertices on the unit sphere. To find the three functionals separating opposite faces, we first calculate the equations of three planes, each one defined by three vertices of the parallelepiped. We have:

$$ADB : x + z = 0.64$$

$$AD'C' : 0.6x + y - 0.6z = -0.6 \quad \text{and} \quad C'D'B : 0.6x - y - 0.6z = -0.6.$$
\[ f_1(A') = f_1(D') = f_1(B') = f_1(C') = -0.64 \]

with \( f_1(A) - f_1(A') = 1.28 > 1 \).

\[ f_2(A) = f_2(D) = f_2(B') = f_2(C') = -\frac{0.6}{\sqrt{1.36}} \]

\[ f_2(A') = f_2(D') = f_2(B) = f_2(C) = \frac{0.6}{\sqrt{1.36}} \]

also

\[ f_3(A') = f_3(D) = f_3(B) = f_3(C') = \frac{0.6}{\sqrt{1.36}} \]

\[ f_3(A) = f_3(D') = f_3(B') = f_3(C) = -\frac{0.6}{\sqrt{1.36}} \]

with \( f_2(A') - f_2(A) = f_3(A') - f_3(A) = \frac{1.2}{\sqrt{1.36}} \simeq 1.02899 > 1 \). Hence \( \delta \simeq 1.02899 > 1 \) and the conclusion follows.

\[ \square \]

**Remarks 6**

(1) Using the Theorem stated in the Introduction (Th. 3 of [11]) we obtain an upper bound for the Banach-Mazur distance of the Petty space \( X \) from \( \ell_3^\infty \). We apply the Theorem as in the proof of Proposition 8 and conclude that \( d(X, \ell_3^\infty) < 2 \). One can find a better upper bound by direct calculation. Since \( d(X, \ell_3^\infty) = d(X^*, \ell_1^3) \), it suffices to evaluate the distance between the dual spaces, which is easier. The ball of \( X^* \) is the right cylinder \( B_{X^*} = \{(x, y, z) : \sqrt{x^2 + y^2} \leq 1 \text{ and } |z| \leq 1 \} \). Consider the points \( A(0, -1, 1), B(0.8, 0.6, 1) \) and \( C(-0.8, 0.6, 1) \) of \( S_{X^*} \) along with the symmetric points \( A', B', C' \) with respect to the origin. Let \( O = \text{conv}\{\pm OA, \pm OB, \pm OC\} \) (an octahedron producing a space linearly isometric to \( \ell_1^3 \)). We calculate the largest \( \alpha > \frac{1}{2} \) such that

\[ \alpha \cdot B_{X^*} \subseteq O \subseteq B_{X^*}. \]

The right-hand inclusion is obvious. To find the optimal value of \( \alpha \) one has to check which homothetic copy of \( B_{X^*} \) touches some of the faces of the octahedron in a single point (it suffices to check the upper 4 faces due to symmetry). The upper 4 faces define the planes

\[ ABC : z = 1 \]

\[ CAB' : 10x + 5y - 3z + 8 = 0 \]

\[ BAC' : 10x - 5y + 3z - 8 = 0 \]

and

\[ AB'C' : 5y + z = -4. \]

Note that if the upper half of the octahedron touches a homothetic copy \( \alpha \cdot B_{X^*} \) of the cylinder, then the common point must lie on the circle \( x^2 + y^2 = \alpha^2, z = \alpha \). Solving the systems of equations of each of the above 4 planes together with \( x^2 + y^2 = \alpha^2 \) and \( z = \alpha \) we find that for \( \alpha \simeq 0.56416 \) there is at most one common point of a face with
the corresponding homothetic copy of $B_X^*$, hence $0.56416 \cdot B_X^* \subseteq O \subseteq B_X^*$ and thus
\[ d(X, t_{\infty}^0) = d(X^*, t_1^0) \leq \frac{1}{0.56416} \approx 1.77254. \]

(2) Concerning the original Hadwiger number of the Petty space $X$, we obtain a lower
bound of 14 by using the 1-separated set of 14 points on the unit sphere of $X$:
\[
\alpha_1 = e_1 = (1, 0, 0), \quad \alpha_2 = \left( \frac{2}{3}, \frac{2}{3}, \frac{1}{6} \right), \quad \alpha_3 = \left( \frac{0}{3}, \frac{2}{3}, \frac{1}{3} \right), \quad \alpha_4 = \left( -\frac{1}{2}, \frac{2}{3}, \frac{1}{6} \right), \quad \alpha_5 = -e_1, \\
\alpha_6 = \left( -\frac{1}{2}, -\frac{2}{3}, \frac{1}{6} \right), \quad \alpha_7 = \left( 0, -\frac{2}{3}, -\frac{1}{3} \right), \quad \alpha_8 = \left( \frac{1}{2}, -\frac{2}{3}, \frac{1}{6} \right), \quad b_1 = e_3 = (0, 0, 1) \\
b_2 = \left( \frac{1}{2}, 0, \frac{1}{2} \right), \quad b_3 = \left( -\frac{1}{2}, 0, \frac{1}{2} \right), \quad b_4 = -e_3, \quad b_5 = \left( \frac{1}{2}, 0, -\frac{1}{2} \right), \quad b_6 = \left( -\frac{1}{2}, 0, -\frac{1}{2} \right).
\]
Observe that the points $\alpha_1, \alpha_2$ are the symmetric of $\alpha_5, \alpha_4$ with respect to
the YZ-plane, the points $\alpha_8, \alpha_7, \alpha_6$ are the symmetric of $\alpha_2, \alpha_3, \alpha_4$ with respect to the XZ-plane and
the points $b_1, b_2, b_3$ are the symmetric of $b_4, b_5, b_6$ with respect to the XY-plane.

Also the set of points
\[
c_1 = \left( \frac{2}{3}, 0, -\frac{1}{3} \right), \quad c_2 = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right), \quad c_3 = \left( 0, \frac{2}{3}, \frac{1}{3} \right), \quad c_4 = \left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right), \\
c_5 = \left( -\frac{2}{3}, 0, -\frac{1}{3} \right), \quad c_6 = \left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right), \quad c_7 = \left( 0, -\frac{2}{3}, \frac{1}{3} \right), \quad c_8 = \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right), \\
b_1, b_2, b_3, b_4, b_5 = \left( 0, \frac{1}{2}, -\frac{1}{2} \right), \quad b_6 = \left( 0, -\frac{1}{2}, -\frac{1}{2} \right),
\]
shows that $H(X) \geq 14$. Both of the above pointsets are maximal 1-separated subsets
of $S_X$. Furthermore the subset \{c_1, c_2, \ldots, c_8\} \cup \{b_1, b_4\}$ of the second pointset yields
the lower bound of 10 for the strict Hadwiger number of $X$, i.e. $H'(X) \geq 10$. It seems
likely that $H(X) = 14$, but the details of such a proof are not yet clear.

(3) About the Hadwiger number of the euclidean spaces the exact values are known in
case when $n = 2, 3, 4, 8$ and 24 (see [1], also [15]). When $2 \leq n \leq 6$, the best-known
lower bounds are larger than $2^n$ and for $n = 7$ we have $126 \leq H(\ell_2^0) \leq 134$. Unlike that,
when $8 \leq n \leq 24$, the best-known upper bounds are smaller than $2^n$.

We conclude with some open problems and questions:

1. Find better upper and lower bounds for the numbers:

   (a) $H'_n(\ell_p^n)$ and $H_n(\ell_p^n)$, for $n \geq 4$ and $1 \leq p < 2$,
   (b) $H'(\ell_{\alpha_0}^n)$, for $n \geq 4$ and
   (c) $H'_n(\ell_2^n)$ and $H_n(\ell_2^n)$, for $5 \leq n \leq 24$

Of particular interest is the case $n = 4$ (then $\alpha_4 = 2$) and $n = 8, 24$, where
the exact values $H(\ell_2^n)$ are known.
2. Is there a 3-dimensional Banach space $X$, with $H_\alpha(X) < 8$? Such a space would be a candidate to have Banach-Mazur distance $d(X, \ell_3^\infty) \geq 2$ (see the proof of Prop.8).

3. Let $X$ be an $n$-dimensional Banach space with $n \geq 4$. Does the inequality $H'_\alpha(X) \geq n + 1$ hold? (See Theorem 4 and the Note following it).

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