OBSTRUCTIONS TO DEFORMATION OF CURVES TO OTHER HYPERSURFACES

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Abstract. We are interested in obstructions to the FIRST order deformation of a pair of a smooth hypersurface \( f_0 \) and a smooth curve \( C_0 \) contained in \( f_0 \). In the first half of the paper, we give necessary conditions for the pair to deform in the first order. In particular, for a rational curve \( C_0 \), this necessary condition is

\[ H^1(N_{C_0,f_0}(1)) = 0. \]

In the second half, we apply the necessary conditions from the first half of the paper to study the geometry of smooth curves in hypersurfaces (theorem 4.1, theorem 5.1). The main application is for the case where \( C_0 \) is a rational curve.

1. Introduction. Let \( \mathbb{P}^n \) be the projective space of dimension \( n \geq 3 \) over complex numbers. Let \( f_0 \subset \mathbb{P}^n \) be a smooth hypersurface of degree \( h \). Let \( C_0 \subset f_0 \) be a smooth curve. We investigate the existence of a family of pairs \( C_t \subset f_t \), the curves \( C_t \) of degree \( d \) and the hypersurfaces \( f_t \) of degree \( h \) in the projective space \( \mathbb{P}^n \), where \( t \) is in a variety. Furthermore the family \( f_t \) is not a constant hypersurface. We simply call this \( C_t \) a deformation of \( C_0 \) to other hypersurfaces, or a “FULL” deformation of the pair to other hypersurface. The similar question was investigated by L. Chiantini and Z. Ran in [5]. In general there is Kodaira’s deformation theory ([6], theorem 1) about the submanifold \( C_0 \subset f_0 \) in a complex manifold \( f_0 \), that says a sufficient condition for the \( C_0 \) to deform to ALL the other submanifolds is

\[ H^1(N_{C_0,f_0}) = 0. \]

The space \( H^1(N_{C_0,f_0}) \) is often called the obstruction space to the deformation of the pair \( C_0 \subset f_0 \). But in general, it is not clear that this condition is also a necessary condition, i.e, if

\[ H^1(N_{C_0,f_0}) \neq 0, \]

\( C_0 \) may still be able to deform to all the other hypersurfaces (We don’t have a proof of that yet). So in the first half of the paper, we prove theorem 1.1 below that gives necessary conditions (i.e. obstructions) for \( C_0 \) to deform to “other hypersurfaces” in the first order. The conditions, the formula (1.6) or (1.10), are expressed in terms of the dimensions of cohomology groups of various twisted bundles over \( C_0 \). But our assumption in theorem 1.1 is weaker than the existence of the deformation \( C_t \subset f_t \). We only use the first order deformation of \( C_0 \subset f_0 \).

The second half of the paper concentrates on the applications. They can be categorized into two different kinds:

(1) Rational curves in a smooth quintic 3-fold. In this case, we see the converse of the Kodaira’s theorem ([6], theorem 1) for a rational curve \( C_0 \) in a generic quintic 3-fold \( f_0 \) is exactly the Clemens’ conjecture: the \( H^1 \) of the normal bundle of the rational curve \( C_0 \subset f_0 \) is equal to 0,

\[ H^1(N_{C_0,f_0}) = 0. \]

Thus even though our theorem 1.1 did not completely solve the Clemens’ conjecture, it still gives a new approach and a new result towards the Clemens’ conjecture. See theorem 4.1.
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(2) Numerical bounds for curves in hypersurfaces. In general there is the interest in knowing the lower bound of the geometric genus of a subvariety in a general hypersurface or a complete intersection, and numerical bounds of other invariants. There are many works on this by Clemens, Chiantini, Ein, Pacienza, Ran, Voisin and Xu, etc. Among them some bounds are sharp. Even though this paper did not produce better bounds, but our results are new in the sense that we assumed a weaker condition: our $f_0$ is not generic and the deformation of the pair of $C_0, f_0$ is in the first order only, i.e. we did not assume the pair $C_0 \subset f_0$ can actually deform to generic hypersurfaces. Please see the details in theorem 5.1, theorem 5.2. The difference between our bounds and their sharp bounds may be caused by our weaker assumption for the deformation of the pair (in the first order only).

The formal setting.

To state the theorem in a precise way, we need to give a formal description about the first order assumption. Let $H^0(\mathcal{O}_{\mathbb{P}^n}(h))$ denote the vector space of homogeneous polynomials of degree $h$ in $n + 1$ variables. We use the same letter $f_0 \in H^0(\mathcal{O}_{\mathbb{P}^n}(h))$ to denote the hypersurface $\text{div}(f_0) \subset \mathbb{P}^n$, homogeneous polynomial $f_0$, and its projectivization in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(h)))$. Let $S \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(h)))$ be a subvariety containing $f_0$ which is a smooth point of $S$. Also assume that $f_0$ is a smooth hypersurface. Let

\begin{equation}
X_S \subset S \times \mathbb{P}^n,
\end{equation}

\begin{equation}
X_S = \{(f, x) : f \in S, f(x) = 0\}.
\end{equation}

be the universal hypersurface.

Let $C$ be a smooth projective curve of genus $g$, and

$$c_0 : C \to f_0 \subset \mathbb{P}^n$$

a smooth imbedding of $C$ to $f_0$. Then

$$\tilde{c}_0 : C \to \{f_0\} \times f_0 \subset X_S$$

is the induced imbedding. The projection

$$P_S : X_S \to S$$

induces a map on the sections of bundles over $C$,

\begin{equation}
P_S^* : H^0(\tilde{c}_0^*(TX_S)) \to T_{f_0}S,
\end{equation}

where $T_{f_0}S \simeq H^0(T_{f_0}S \otimes \mathcal{O}_C)$ is the space of global sections of the trivial bundle whose each fibre is $T_{f_0}S$. In this paper we consider two specific parameter spaces for $S$:

**Assumption (1)** The first subvariety $S$ under consideration is the collection of hypersurfaces in the following form:

\begin{equation}
f_0 + \sum_{i=0}^{h} a_iL_0 \cdots \hat{L_i} \cdots L_h, \quad (\hat{L_i} \text{ is omitted})
\end{equation}

where $L_i \in H^0(\mathcal{O}_{\mathbb{P}^n}(1)), i = 0, \cdots, h$ are fixed sections whose zeros are distinct, i.e.

$$\text{div}(L_i) \neq \text{div}(L_j), i \neq j.$$
Let
\[ A' = C^{h+1} = \{(a_0, \cdots, a_h)\} \]
be the parameter space of the family. Let \( A \subset A' \) that parametrizes smooth hypersurfaces. So \( S = A \) in this case.

**Assumption (2)** Secondly \( S \) is the entire space \( \mathbf{P}(H^0(\mathcal{O}_\mathbf{P}^n(h))) \). We will denote \( \mathbf{P}(H^0(\mathcal{O}_\mathbf{P}^n(h))) \) by \( E \). So \( S = E \) in this case.

Let \( N_{c_0}V \) denote the pullback of a bundle \( V \subset TP^n|_{C_0} \) over \( C_0 \). Let \( h^i(E) \) denote the dimension of \( H^i(E) \) for any sheaf \( E \). Let \( L = c_0^*(\mathcal{O}_\mathbf{P}^n(1)) \). Then \( d = \deg(L) \), \( g = \text{genus}(C) \).

**Theorem 1.1.** Assume \( P^h_\mathbf{A} \) is surjective (see assumption (1) and formula (1.3)).

1. If \( C \) is a rational curve and \( \{L_i = 0\} \cap \{L_i = 0\} \cap C_0 = \emptyset, i \neq j, \) then
\[ (1.6) \quad H^1(N_{c_0}f_0(1)) = H^1(c_0^*(Tf_0(1))) = 0, \]
where \( N_{c_0}f_0(1) \) is the pullback of the twisted normal bundle \( N_{C_0}f_0(1) \).

2. Let
\[ (1.7) \quad \sigma(c_0, f_0) = (h + 1)h^0(L) - h - h^0(L^{h+1}) + h^1(c_0^*(Tf_0(1))) - h^1(c_0^*(TP^n(1))) + h^1(L^{h+1}). \]

If
\[ (h + 1)h^0(L) - h \geq h^0(c_0^*(TP^n(1))), \]
and
\[ (1.9) \quad \{L_i = 0\} \cap \{L_i = 0\} \cap C_0 = \emptyset, i \neq j, \]
then
\[ (1.10) \quad \sigma(c_0, f_0) = 0. \]

**Remark** This theorem proves that if \( c_0, f_0 \) satisfy conditions
\[ (h + 1)h^0(L) - h \geq h^0(c_0^*(TP^n(1))), \]
then \( C_0 \) can’t deform to all the hypersurfaces in \( A \) in the first order. Thus \( H^1(N_{C_0}f_0(1)) \neq 0 \) in the case \( g = 0 \), and \( \sigma(c_0, f_0) \neq 0 \) in general, give us obstructions to the deformation of \( C_0 \) to other hypersurfaces. Note \( \sigma(c_0, f_0) = H^1(N_{C_0}f_0(1)) \) in the case \( g = 0 \).

The rest of the paper is organized as follows. In section 2, we study the deformation of a smooth hypersurface. We mainly describe and prove a theorem by H.
Clemens, then some sequences of bundles associated with it. This is the main technique for the paper. In section 3, we study the first order deformation of smooth curves together with hypersurfaces containing it. This is a proof of theorem 1.1. In section 4, we apply theorem 1.1 to rational curves in quintic 3-folds. The main result is to determine the upper-bound of the degrees of the summand in the normal bundle of a rational curve in a smooth quintic 3-fold. In section 5, we apply theorem 1.1 to any smooth hypersurfaces. We obtain bounds for a couple of numerical invariants. These bounds are not better than the bounds in Clemens’ paper [4], however they are obtained by using our weaker first order assumption. Therefore they are new theorems (theorem 5.1, theorem 5.2).

2. Deformation of hypersurfaces. The main idea of the proof is to transform the problems of $TP^n$ to similar types of problems of some isomorphic bundle $\frac{TX_A(1)}{O(1)}$. Then the existence of the first order deformation of the pair $C_0, f_0$ allows us to work with $\frac{TX_A(1)}{O(1)}$, which is more accessible now than $TP^n$. Thus the isomorphism between $TP^n$ and $\frac{TX_A(1)}{O(1)}$ serves as an important bridge between two different realms. In this section, we introduce the construction of the vector bundle $\frac{TX_A(1)}{O(1)}$ and the associated morphisms that are used in our proof.

Let

$$F(a_1, \cdots, a_h, x) = f_0(x) + \sum_{i=0}^{h} a_i L_0(x) \cdots \hat{L}_i(x) \cdots L_h(x), \quad (\text{omit } L_i)$$

be the universal polynomial. Thus

$$\{F = 0\} = X_A \subset A \times P^n.$$

is also the universal hypersurface, which is smooth. Let $W \subset P^n$ denote the complement of the proper subvariety

$$\cup_{h \geq j > i \geq 0} \{L_i = L_j = 0\}.$$

Let

$$X_W = X_A \cap (A \times W)$$

$$f_0^W = f_0 \cap W.$$

Let

$$u_i = L_0 \frac{\partial}{\partial a_0} - L_i \frac{\partial}{\partial a_i}, i = 1, \cdots, h$$

be sections of $TA \otimes O_W(1)$. Since $u_i$ annihilate $F$, they are tangent to $X_W$. So let

$$G(1) \subset TX_W(1)$$

be the vector bundle of rank $h$ over $X_W$ that is generated by the sections $u_i$.

For any smooth varieties $V_1, V_2$, let

$$T_{V_1/V_2}$$
denote the relative tangent bundle of $V_1$ over $V_2$, i.e. it is the bundle $TV_1 \oplus \{0\}$ over the variety $V_1 \times V_2$.

The following theorem 2.1 is communicated to us by H. Clemens ([3]).

**Theorem 2.1.** (H. Clemens)

\[(2.6) \quad \frac{TX_W(1)}{G(1)} \simeq T_{W/A}(1),\]

where $T_{W/A}(1)$ is restricted to $X_W$.

**Proof.** Consider the exact sequence

\[(2.7) \quad 0 \to \frac{TX_W(1)}{G(1)} \to \frac{T(A \times W)(1)}{G(1)} \to D \to 0,\]

of bundles over $X_W$, where $D$ is some quotient bundle over $X_W$. Easy to see

\[(2.8) \quad c_1(D) = c_1(O_{P^h}(h + 1))|_{X_W}.\]

Let $s$ be a generic section of $O_{P^h}(1)$ that does not have common zeros with $L_i$, $i = 0, \cdots, h$. Let $\sigma$ be the reduction of $s\frac{\partial}{\partial a_0}$ in $\frac{T(A \times W)(1)}{G(1)}$. Notice the zeros of $\sigma$ is exactly

\[(2.9) \quad div(\sigma) = div(sL_1 \cdots L_h).\]

Since $sL_1 \cdots L_h \in H^0(O_{P^h}(h + 1))$, $\sigma$ splits the sequence (2.7). If $L_s \subset \frac{T(A \times W)(1)}{G(1)}$ is the line bundle generated by $\sigma$,

\[(2.10) \quad L_s \oplus \frac{TX_W(1)}{G(1)} = \frac{T(A \times W)(1)}{G(1)},\]

as bundles over $X_W$. Secondly, we have another exact sequence

\[(2.11) \quad 0 \to T_{W/A}(1) \to \frac{T(A \times W)(1)}{G(1)} \to D' \to 0,\]

of bundles over $X_W$, where $D'$ is some quotient bundle over $X_W$. By the direct calculation (note $G(1)$ is a trivial bundle):

\[c_1(D') = c_1(c_0^*(T_{A/W}(1))) = (h + 1)(c_1(O_{P^h}(1)))|_{X_W}\]

As above, $\sigma$ splits this sequence (2.11). Hence

\[(2.12) \quad L_s \oplus T_{W/A}(1) = \frac{T(A \times W)(1)}{G(1)} ,\]

Comparing (2.10), (2.12), we obtain

\[(2.13) \quad \frac{TX_W(1)}{G(1)} \simeq T_{W/A}(1),\]

over $X_W$. \(\square\)

Let

\[(2.14) \quad I : \frac{TX_W(1)}{G(1)} \to T_{W/A}(1) \]

\[(\frac{\partial}{\partial a_0}, v) \to -v\]
be this isomorphism in the formula (2.13) or (2.6), where \( \frac{\partial}{\partial a_0}, v \) is the decomposition in the formula (2.12) and \( v \in T_v W \).

Consider the composition map \( \mu_1 \):

\[
T_{A/W} \xrightarrow{\alpha} \frac{T(A \times W)}{T_W} \rightarrow N_{\bar{f}_0^W} W \simeq O_{\mathbb{P}^n}(h)|_{\bar{f}_0^W}
\]

where \( \bar{f}_0^W = \{ f_0 \} \times f_0^W \). The last map is the restriction map. Tensoring it with \( O_{\mathbb{P}^n}(1) \), we have the composition \( \mu_2 \):

\[
TX_W(1) \rightarrow T_{A/W}(1) \rightarrow O_{\mathbb{P}^n}(h+1)|_{\bar{f}_0^W}
\]

where the first map is the differential of the projection \( X_W \rightarrow A \). Since \( \mu_2 \) vanishes on \( G(1) \), we obtain the bundle morphism \( \mu_3 \):

\[
\frac{TX_W(1)}{G(1)}|_{\bar{f}_0^W} \xrightarrow{\mu_3} O_{\mathbb{P}^n}(h+1)|_{\bar{f}_0^W}.
\]

**Lemma 2.2.** There is a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & T_{\bar{f}_0^W/A}(1) \\
\downarrow I & & \downarrow I \\
0 & \rightarrow & T_{f_0 W}(1) \\
\end{array}
\]

\[
\begin{array}{ccc}
T_{X_W(1)}|_{\bar{f}_0^W} & \xrightarrow{\mu_3} & O_{\mathbb{P}^n}(h+1)|_{\bar{f}_0^W} \\
\downarrow I & & \downarrow I \\
TX_W(1)|_{f_0 W} & \xrightarrow{\nu} & O_{\mathbb{P}^n}(h+1)|_{f_0 W} \\
\end{array}
\]

where \( I \) is the isomorphism in the formula (2.14) and \( \nu \) is the differential map

\[
\nu : \beta \rightarrow \frac{\partial f_0}{\partial \beta}|_{f_0 W}.
\]

**Proof.** It is obvious that the two horizontal sequences are identical if \( I \) is an isomorphism. But the lemma says with the maps \( \nu, \mu_3 \) defined independently as above, they are still isomorphic. Thus it suffices to prove the commutativity for

\[
\begin{array}{ccc}
T_{X_W(1)}|_{\bar{f}_0^W} & \xrightarrow{\mu_3} & O_{\mathbb{P}^n}(h+1)|_{\bar{f}_0^W} \\
\downarrow I & & \downarrow I \\
TW(1)|_{f_0 W} & \xrightarrow{\nu} & O_{\mathbb{P}^n}(h+1)|_{f_0 W} \\
\end{array}
\]

Next we just verify it at each point. This is a tedious but straightforward verification. Because \( f_0 \) is smooth, \( X_W \rightarrow A \) is a smooth map around the point \( z_0 = (f_0, x) \in X_W \). Thus \( T_{(f_0,x)}X_W \rightarrow T_{f_0}A \) is surjective. So we can let

\[
\frac{\partial}{\partial a_i} - \beta_i
\]

be an inverse of \( \frac{\partial}{\partial a_i} \) at the point \( z_0 = (f_0, x) \). Let

\[
\sigma' = \sum_{i=0}^{h} x_i (\frac{\partial}{\partial a_i} - \beta_i) + y
\]

be a representative of an element in

\[
\frac{TX(1)}{G(1)}|_{z_0}
\]
where the number \( x_i \) is the coefficient of \( \frac{\partial}{\partial a_i} - \beta_i \), and \( y \in (T_{f_0/A}(1))|_{z_0} \).

By the definition of \( \mu_3 \),

\[ \mu_3(\sigma') = \sum_{i=1}^{h} x_i (L_1 \cdots \hat{L}_i \cdots L_h)|_{z_0}. \]

If \( x \) is not a zero of all \( L_i, i \neq 0 \), now at this point \( z_0 \), \( \sigma' \) can be written as

\[ \sigma' = \frac{\sum_{i=0}^{h} x_i L_0 \cdots \hat{L}_i \cdots L_h}{L_1 \cdots L_h}|_x \cdot \left( \frac{\partial}{\partial a_0} - \beta_0 \right) \]

\[ + \frac{1}{L_1 \cdots L_h} \sum_{i=1}^{h} x_i L_1 \cdots \hat{L}_i \cdots L_h (L_0 \beta_0 - L_i \beta_i)|_x \]

\[ + y + \sum_{i=1}^{h} x_i (\beta_i - \frac{L_0 \partial}{\partial a_i} - L_i \frac{\partial}{\partial a_0})|_x \]

Modulo \( G(1) \), it is just

\[ \sigma' \]

\[ \frac{G(1)}{G(1)} \]

\[ \sum_{i=0}^{h} x_i L_0 \cdots \hat{L}_i \cdots L_h \]

\[ L_1 \cdots L_h \]

\[ |_x \cdot \beta_0 \]

\[ + \frac{1}{L_1 \cdots L_h} \sum_{i=1}^{h} x_i L_1 \cdots \hat{L}_i \cdots L_h (L_0 \beta_0 - L_i \beta_i)|_x + y|x. \]

Using the definition of the isomorphism \( I \) (the formula (2.14)),

\[ I(\sigma') = \frac{\sum_{i=0}^{h} x_i L_0 \cdots \hat{L}_i \cdots L_h}{L_1 \cdots L_h}|_x \cdot \beta_0 \]

\[ + \frac{1}{L_1 \cdots L_h} \sum_{i=1}^{h} x_i L_1 \cdots \hat{L}_i \cdots L_h (L_0 \beta_0 - L_i \beta_i)|_x + y|x. \]

Since the last two terms lie in \( T_{f_0/A} \), we obtain that

\[ \nu(I(\sigma')) = \frac{\sum_{i=1}^{h} x_i L_1 \cdots \hat{L}_i \cdots L_h}{L_1 \cdots L_h}|_x \cdot \frac{\partial f_0(x)}{\partial \beta_0} \]

\[ \text{(because } \frac{\partial F}{\partial a_0} - \frac{\partial F}{\partial \beta_0} = 0) \]

\[ = \sum_{i=1}^{h} (x_i L_1 \cdots \hat{L}_i \cdots L_h)|_x. \]

Thus

\[ \nu(I(\sigma')) = \mu_3(\frac{\sigma'}{G(1)}). \]

This proves the lemma at this point \( z_0 \).

If \( x \) is a zero of \( L_i, i \neq 0 \), say \( L_1(x) = 0 \). Then

\[ I(\frac{\sigma'}{G(1)}) = \sum_{i=0}^{h} x_i \beta_i + y. \]
Again
\[\nu(I(\sigma' G(1))) = \mu_3(\sigma' G(1)).\]

This proves the lemma.

3. Deformation of curves to other hypersurfaces. In this section, we try to use the surjectivity of $P_s A$ to obtain some results on $T X A(1) G(1)$. We’ll denote the pull-back normal bundle over $C$ by $N_{c_0 V}$ for any smooth $V \subset P^n$. Also denote the image of a map $\mu$ by $\text{Im}(\mu)$.

Proposition 3.1. Let $L$ be the hyperplane section bundle $c_0^*(O_{P^n}(1))$ over $C$. Assume $P_s A$ is surjective.

(a) If $C$ is a rational curve and
\[\{L_i = 0\} \cap \{L_j = 0\} \cap C_0 = \emptyset, i \neq j,\]
the map
\[\nu^* : H^0(c_0^*(T P^n(1))) \rightarrow H^0(c_0^*(O_{P^n}(h + 1))),\]
is surjective.

(b) If
\[\{L_i = 0\} \cap \{L_j = 0\} \cap C_0 = \emptyset, i \neq j,\]
\[\dim(\text{Im}(\nu^*)) = (h + 1)h^0(L) - h.\]

Proof. Because the formula (3.1), the image of $c_0$ completely lies in $W$. In general we denote the induced morphism on $H^0$ groups by $\phi^*$ if the morphism on the bundles is $\phi$. In lemma (2.2), pulling back the diagram in the formula (2.11) to $C$, we obtain the commutative diagram
\[
\begin{array}{ccc}
H^0(c_0^*(TX(1))) & \rightarrow & H^0(c_0^*(O_{P^n}(h + 1))) \\
\downarrow I^* & & \downarrow \mu_3^* \\
H^0(c_0^*(T P^n(1))) & \rightarrow & H^0(c_0^*(O_{P^n}(h + 1))).
\end{array}
\]
where $I^*$ is the isomorphism induced from $I$. Thus
\[\dim(\text{Im}(\nu^*)) = \dim(\text{Im}(\mu_3^*)).\]
Adding one more space in the diagram, we obtain

\[
\begin{array}{cccc}
H^0(c_0^*(TX(1))) & \xrightarrow{\phi} & H^0(c_0^*(\mathcal{O}_{P^n}(h + 1))) & \\
\downarrow I^* & & \uparrow \mu_3^* & \\
H^0(c_0^*(TP^n(1))) & \xrightarrow{\nu_3^*} & H^0(c_0^*(\mathcal{O}_{P^n}(h + 1))). & \\
\end{array}
\]  

(3.7)

The sequence in the first row is not exact. Using the assumption for the part (a), \(C\) is a rational curve, \(H^1(c_0^*(G(1))) = 0\) (because \(c_0^*(G(1))\) is a trivial bundle over \(P^1\)). Thus \(\phi\) is surjective.

Using the assumption for the part (b), we would like to show \(\phi\) is also surjective for non-zero genus curve. First by the surjectivity of \((P_A^*)\),

\[c_0^*(TX) \simeq \oplus_{i=0}^{h+1} O_C \oplus c_0^*(Tf_0),\]

where \(O_C\) is the trivial bundle generated by the sections \((P_A^*)^{-1}(\frac{\partial}{\partial a_i}), i = 0, \ldots, h\). Thus

\[H^0(c_0^*(TX(1))) \simeq \oplus_{i=0}^{h+1} H^0(L) \oplus H^0(c_0^*(Tf_0)).\]

By the definition of \(\mu_3\), it is easy to see the image of \(\mu_3^2 = \mu_3^* \circ \phi\) is just the subspace,

\[
\{ \sum_{i=0}^{h} x_i c_0^*(L_0 \cdots \hat{L}_i \cdots L_h) \} \subset H^0(c_0^*(\mathcal{O}_{P^n}(h + 1)))
\]

where \(x_i \in H^0(c_0^*(\mathcal{O}_{P^n}(1)))\) run through all sections. Notice that because

\[c_0^*(L_i), c_0^*(L_j), \text{ for } i \neq j\]

do not have common zeros,

\[
\sum_{i=0}^{h} x_i c_0^*(L_0 \cdots \hat{L}_i \cdots L_h) = 0
\]

if and only if

\[x_i = \epsilon_i L_i, \text{ and } \sum_{i=0}^{h} \epsilon_i = 0,\]

for some complex numbers \(\epsilon_i\). Thus

\[\dim(\text{Im}(\mu_3^2)) = (h + 1)h^0(L) - h.\]

By the assumption for the part (b),

\[\dim(\text{Im}(\phi)) \geq \dim(\text{Im}(\mu_3^2)) = (h + 1)h^0(L) - h \geq h^0(c_0^*(TP^n(1))).\]

Because \(I^*\) is an isomorphism, this means that \(\phi\) is surjective.

Since \(\phi\) is surjective,

\[\dim(\text{Im}(\nu^*)) = (h + 1)h^0(L) - h.\]
This proves the part (b). If $C$ is a rational curve, it is automatic that

$$(h + 1)h^0(L) - h - h^0(O_{P^n}(h + 1)) = 0.$$  

Thus $\nu^*$ is surjective. So we proved the part (a).

**Proof.** of the theorem 1.1: Consider the exact sequence

$$0 \rightarrow c_0^*(T_{f_0}(1)) \rightarrow c_0^*(P^n(1)) \rightarrow c_0^*(O_{P^n}(h + 1)) \rightarrow 0.$$  

Then we have the long exact sequence

$$H^0(c_0^*(P^n(1))) \xrightarrow{\nu^*} H^0(c_0^*(O_{P^n}(h + 1))) \rightarrow H^1(c_0^*(T_{f_0}(1))) \rightarrow H^1(c_0^*(P^n(1))) \rightarrow H^1(c_0^*(O_{P^n}(h + 1))) \rightarrow 0.$$  

Thus the codim$(\text{Im}(\nu^*))$ is

$$(3.9) \quad h^1(c_0^*(T_{f_0}(1))) - h^1(c_0^*(TP^n(1))) + h^1(L^{h+1}).$$

Combining the result from proposition 3.1, we proved the part (b).

For the part (a), by the proposition, $\nu^*$ is surjective. Then

$$(3.10) \quad h^1(c_0^*(T_{f_0}(1))) - h^1(c_0^*(TP^n(1))) + h^1(L^{h+1}) = 0.$$  

Notice $h^1(c_0^*(TP^n(1))) = h^1(L^{h+1}) = 0$ for a rational curve, then

$$h^1(c_0^*(T_{f_0}(1))) = 0.$$  

We complete the proof.

4. **Rational curves in a smooth quintic threefold.** In this section, we apply above theorem 1.1 to rational curves in a smooth quintic 3-fold (which is not generic).

**Example 4.1** Let $n = 4, h = 5$ and $d = 1$ in theorem 1.1. Now $C$ is a line, thus a rational curve. Consider the Fermat quintic $f_0$, and the parameter space $A$ satisfying the condition in formula (1.5). In this case,

$$H^1(N_{c_0f_0}(1)) = H^0(O_{P^n}(0) \oplus O_{P^n}(-4)) \neq 0.$$  

Our theorem 1.1 says this is an obstruction to the deformations of $C_0$ to other quintics in $A$. Indeed no lines in the Fermat quintic $f_0$ can deform to a generic quintic by Albano and Katz’s result ([1], Prop. 2.1). In [1], one can find the detailed description of deformations of the pair, line $\subset$ Fermat quintic. Our result here is stronger than Albano and Katz’s because the quintics in $A$ is not generic.

This example shows that our obstruction in the theorem 1.1 is meaningful and non-trivial.

**Example 4.2** Let $n = 4, h = 5$ and $g = 0$ in theorem 1.1. Let

$$f_0 = lg_1 + qg_2$$
where \( l \) is linear and \( q \) is quadratic. Assume all \( l, q, g \) are generic. Let \( C_0 \) be a smooth rational curve of degree \( d \), lying on the quadratic surface \( \{ l = q = 0 \} \). Now \( f_0 \) is not smooth, but there are only 24 singular points. We may assume \( f_0 \) is smooth along \( C_0 \). Assume the parameter space \( A \) satisfies formula (1.5). Then the theorem 1.1 should still be valid for such pair \( C_0 \subset f_0 \). Apply it the pair \( (C_0, f_0) \).

\[ H^1(N_{C_0}f_0(1)) \simeq H^0(\mathcal{O}_{\mathbb{P}^1}(-3d) \oplus \mathcal{O}_{\mathbb{P}^1}(d - 2)), \]

which is non-zero if \( d \neq 1 \). Thus if \( C_0 \) is not a line, then the pair \( C_0 \subset f_0 \) can’t deform to all hypersurfaces in \( A \) in the first order. In particular, they are obstructed to deform to all hypersurfaces in the first order.

**Theorem 4.1.** (Upper-bound of the degrees of summands). Let \( f_0 \subset \mathbb{P}^4 \) be a smooth quintic threefold, and \( C_0 \subset f_0 \) a smooth rational curve. Then

1. \( N_{C_0}f_0 = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-2 - k) \)

where \( k \geq -1 \) is an integer.

2. If \( \{ L_i = 0 \} \cap \{ L_i = 0 \} \cap C_0 = \emptyset, i \neq j, \)

and \( P_A^* \) is surjective,

\[ k < d. \]

**Proof.** (1). By the adjunction formula \( \deg(N_{C_0}f_0) = -2 \). Since all vector bundles over \( \mathbb{P}^1 \) is decomposable. Then the part (1) is proved.

(2). The part (1) says

\[ N_{C_0}f_0(1) \simeq \mathcal{O}_{\mathbb{P}^1}(k + d) \oplus \mathcal{O}_{\mathbb{P}^1}(-2 - k + d) \]

where \( k \geq -1 \). Then \( H^1(N_{C_0}f_0(1)) \simeq H^0(\mathcal{O}_{\mathbb{P}^1}(-k - d - 2) \oplus \mathcal{O}_{\mathbb{P}^1}(k - d)) = 0 \). By theorem 1.1, \( H^1(N_{C_0}f_0(1)) \) is both \( k - d < 0 \), or \( k < d \).

In this case, we also have the Clemens’ conjecture [2] that is equivalent to the assertion that for a generic quintic \( f_0 \), Kodaira’s condition \( H^1(N_{C_0}f_0) = 0 \) (without a twist) is also a necessary condition for \( C_0 \) to deform to all quintics. Our theorem 1.1 did not prove the Clemens’ conjecture because of the twist on the normal bundle \( N_{C_0}f_0 \). Instead, we only obtain an upper bound of \( k \) above.

**5. Smooth curves in a hypersurface of a higher dimension.** In this section, we apply theorem 1.1 to hypersurfaces of dimension \( n \geq 3 \).

**Theorem 5.1.** If \( C_0 \) is a rational curve and \( P_E^* \) is surjective at a smooth hypersurface \( f_0 \subset \mathbb{P}^n \), then

\[ h \leq 2n - 2, \]

where \( h = \deg(f_0) \).
Remark This theorem has an importance in the deformation theory of a pair of varieties. Even though the bound for $h$ in the theorem is the same as Clemens’ bound in [4], but we did not assume $f_0$ is generic. Therefore our theorem 5.1 is beyond the Clemens’ result in [4]. In order to obtain a better bound (better than $2n - 2$), the condition of higher order deformations of the pair must be used. This is indeed the case in [8], in which Voisin used the integrability of a “vertical ” distribution on the versal subvariety to improve the Clemens’ bound $2n - 2$ to $2n - 3$. So this may explain, why if the “Full” deformation (or all higher orders) of the pair $C_0, f_0$ exists, the sharp bound is $1$ less than $2n - 2$, which is proved by Voisin ([7], [8]). We suspect $h \leq 2n - 2$ is the sharp bound under our assumption that $P^E_F$ is surjective. This speculation is equivalent to the assertion:

There exists a smooth hypersurface $f_0$ (not generic) of degree $h = 2n - 2$

such that it contains an irreducible rational curve $C_0$ with the surjective $P^E_F$.

Therefore the significance of Clemens’ bound $2n - 2$ might lie in the existence of the first order deformation of the pair $C_0 \subset f_0$, while the significance of Voisin’s bound $2n - 3$ lies in the existence of the “Full” deformation $C_t \subset f_t$.

Proof. Because $P^E_F$ is surjective, we choose generic sections $L_i, i = 0, \cdots, h$ in $H^0(O_{P^n}(1))$ for $A$. Then all conditions in theorem 1.1 are satisfied. By theorem 1.1, we obtain that $h^1(c_0^*(Tf_0(1))) = 0$. By Riemann-Roch,

\[(5.1) \quad h^1(c_0^*(Tf_0(1))) = h^0(c_0^*(Tf_0(1))) - \left( Ch(c_0^*(f_0(1))) \cdot Tod(TC) \right) \]

\[= h^0(c_0^*(Tf_0(1))) - \left( c_1(c_0^*(Tf_0(1))) + \frac{n-1}{2} \cdot TC \right) \]

\[= h^0(c_0^*(Tf_0(1))) - \left( c_1(c_0^*(TP^n(1))) - (h+1)d + \frac{n-1}{2}c_1(TC) \right) \]

\[= h^0(c_0^*(Tf_0(1))) + (h-2n)d + (n-1)(g-1) \]

\[= h^0(c_0^*(Tf_0(1))) + (h-2n)d - (n-1) = 0 \]

Since

\[(5.2) \quad h^0(c_0^*(Tf_0(1))) = h^0(N_{c_0}f_0) + h^0(TC(1)) \]

\[= h^0(N_{c_0}f_0) + d + 3, \]

Formula (5.1) becomes

\[(5.3) \quad (h-2n+1)d + h^0(N_{c_0}f_0)) - (n-4) = 0. \]

To show $h \leq 2n - 2$, it suffices to prove that

\[h^0(N_{c_0}f_0)) - (n-4) > 0. \]

Now we use the assumption $P^E_F$ is surjective. Let $q \in C_0$ be a point. Let $v \in Tf_0|_q$ but not in $TC_0|_q$. There is a $GL(n+1)$ action on

\[P(H^0(O_{P^n}(h))) \times P^n, \]
that preserves the universal hypersurface
\[ X_E \subset P(H^0(\mathcal{O}_{P^n}(h))) \times P^n. \]

Then we have a submanifold
\[ O = \{(g^{-1}f_0, g(C_0)) : g \in GL(n + 1)\} \subset X_E. \]

The tangent space \( T_{(f_0, q)}O \) of it at \( (f_0, q), q \in C_0 \) lies in
\[ T_{(f_0, q)}X_E \subset T_{f_0}P(H^0(\mathcal{O}_{P^n}(h))) \times T_{q}P^n. \]

It is clear that \( T_{(f_0, q)}O \) contains a vector
\[ (\bar{\alpha}_\sigma, \sigma) \in T_{f_0}P(H^0(\mathcal{O}_{P^n}(h))) \times T_{q}P^n, \]
such that \( \sigma|_q = v \). We should note
\[ \alpha_\sigma \in P(H^0(\mathcal{O}_{P^n}(h))) \]
is a hypersurface obtained by apply some action on \( f_0 \) and \( \bar{\alpha}_\sigma \) represents the directional vector of the line through \( f_0 \) and \( \alpha_\sigma \). Since \( \sigma|_q = v \in T_{f_0|q} \), the hypersurface \( \alpha_\sigma \) viewed as polynomial of \( \mathbb{C}^{n+1} \) lies in the maximal ideal of the point \( q \in \mathbb{C}^{n+1} \) (view \( q \) as a point in \( \mathbb{C}^{n+1} \)). Thus
\[ \alpha_\sigma = \sum x_i Q_i, \]
where \( x_i \) is in \( H^0(\mathcal{O}_{P^n}(1)) \) vanishing at \( q \), \( Q_i \) is a monomial in \( H^0(\mathcal{O}_{P^n}(h - 1)) \) and \( x_i Q_i \) denotes the vector in
\[ T_{f_0}P(H^0(\mathcal{O}_{P^n}(h))) \]
that represents the direction of the line in \( P(H^0(\mathcal{O}_{P^n}(h))) \) through two points, \( f_0 \) and \( x_i Q_i \). Let \( y \in H^0(\mathcal{O}_{P^n}(1)) \) such that \( y|_q = 1 \).
\[ \sum_i x_i (yQ_i) - y(x_i Q_i) \]
is in \( H^0(TX_E(1)|_{C_0}) \). Hence
\[ g_v = \sum_i x_i P_1^n \circ (P_E^n)^{-1}(yQ_i) - y\sigma \]
must be in
\[ H^0(Tf_0(1)|_{C_0}), \]
and \( (g_v)|_q = v \), where \( P_1^n \) is the projection map
\[ H^0(TX_E|_{C_0}) \to H^0(TP^n|_{C_0}). \]
This shows the dimension of
\[ H^0(Tf_0(1)|_{C_0}), \]
is at least $n - 2$, because at the point $q$, $\{g_v\}$ span

$$H^0(N_{c_0f_0(1)})|_q$$

which has dimension $n - 2$. Hence

$$h^0(N_{c_0f_0}) - (n - 4) \geq n - 2 - (n - 4) = 2.$$

We complete the proof.

\[ \square \]

**Theorem 5.2.** Assume all conditions in theorem 1.1., in particular $P_A^*$ is surjective. Then either

$$g(h - n + 1) \geq (h - 2n)d - n + 1$$

or

$$g \geq \frac{d}{2} + 1.$$

**Remark.** If $C_0$ can actually deform to all hypersurfaces ($C_t \subset f_t$ exist for generic hypersurfaces $f_t$) and $h \geq 2n - 1$, Clemens has a better bound for $g$ in ([4]),

$$g \geq \frac{1}{2}(h - 2n + 1)d + 1.$$ 

But our result is different from Clemens’ in many respects.

**Proof.** Suppose otherwise, i.e.

$$g(h - n + 1) < (h - 2n)d - n + 1$$

and

$$g < \frac{d}{2} + 1.$$ 

By Riemann-Roch and Serre-duality, the inequality

$$(h + 1)h^0(\mathcal{L}) - h \geq h^0(c_0^*(T\mathbb{P}^n(1))),$$

is reduced to $g(h - n + 1) \leq (h - 2n)d - n + 1$ which is satisfied. Thus if $P_A^*$ is surjective, theorem 1.1 says

$$\sigma(c_0, f_0) = h^1(c_0^*(Tf_0(1))) - gh = 0.$$ 

Now we calculate $h^1(c_0^*(T\mathbb{P}^n(1))) - gh$. Using the condition

$$h^1(\mathcal{L}) = h^0(\mathcal{L}^* \otimes K) = 0, h^1(\mathcal{L}^2) = h^0((\mathcal{L}^*)^2 \otimes K) = 0$$

(because $g < \frac{d}{2} + 1$), by the Riemann-Roch,
An obstruction to the deformations of curves to all hypersurfaces

(5.4) \[ h^1(c_0^*(Tf_0(1))) - gh = h^0(Tf_0(1)) - (Ch(f_0(1)) \cdot Tod(TC_0)) - hg > - (Ch(Tf_0(1)) \cdot Tod(TC_0)) - gh = -(c_1(c_0^*(Tf_0(1)))) + \frac{n-1}{2}(TC_0) - gh = -(c_1(c_0^*(TP^n(1)))) - (h+1)d + \frac{n-1}{2}c_1(TC_0)) - gh = (h - 2n)d + (n - 1)(g - 1) - gh > 0 \]

(5.5)

This is a contradiction. We complete the proof. □

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