TRANSITION MATRICES BINDING CARTER DIAGRAMS

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ABSTRACT. In 1972, R. Carter introduced the so-called admissible diagrams representing semi-Coxeter elements, which, together with Coxeter elements, give a complete list of conjugacy classes in the Weyl group. Carter diagrams differ from the admissible ones in that they take into account the sign of the Tits bilinear form on the pair of roots; for this, the language of solid and dotted edges is used. These diagrams are characterized by the presence of subdiagrams, which are 4-cycles. The Carter diagrams of the same ADE type and the same size are called homogeneous. Let  and  be homogeneous Carter diagrams, where  is a Dynkin diagram. The transition matrix  between  and  in composition with the Weyl group binds arbitrary root subsets associated with  and . The homogeneous Carter diagrams form a chain of diagrams in which root subsets for each successive diagram is obtained (using  from the root subsets for previous one in some canonical way, affecting exactly one root, so that this root is mapped to the minimal element in some root subsystem. It is shown that all root subsets associated with the given Carter diagram are conjugate under Weyl group. It is observed a numerical relation between enhanced Dynkin diagrams (introduced by Dynkin-Minchenko in [DM10]) and Carter diagrams. In this regard, a number of facts and conjectures are discussed in the last sections.

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1. **Introduction**

The Carter diagrams originated as a tool for studying conjugacy classes in Weyl groups. With that, these diagrams characterize root subsets of linearly independent roots that contain 4-cycles. The Carter diagrams of the same ADE type and the same size constitute a homogeneous class of Carter diagrams, see Section 1.5. In this paper, we show that each homogeneous class of Carter diagrams essentially depict the same subset of the root system given in different bases. There are several chains of diagrams containing homogeneous classes of Carter diagrams. The transition between successive bases in any chain can be done in some canonical way affecting exactly one root, which is mapped to the minimal element in some root subsystem. The transition matrix binding two successive diagrams is involution and has only one column, which is different from the vector of the form \((0, \ldots, 0, 1, 0, \ldots, 0)\) with 1 on the diagonal.

1.1. **All involved diagrams.** The Dynkin diagram is a certain convenient way to depict simple roots of the corresponding complex semisimple Lie algebra. On the other hand, this is a very restricted method to describe the root system. Root system is a much richer object. The root system associated with Dynkin diagram can have quite a few interesting subsets that cannot be described only by Dynkin diagrams and their subdiagrams. Two options for enlargement of Dynkin diagrams are considered: the first is the diagrams introduced by R.Carter (with a certain change) and the second one is the enhanced Dynkin diagrams introduced by Dynkin-Minchenko in \([DM10]\).

In \([Ca72]\), R.Carter suggested the uniform approach to the description of conjugacy classes in the Weyl groups. Basically, the conjugacy classes in \(W(\Phi)\) are represented by Coxeter elements of subsystems of the root system \(\Phi\); however, not all conjugacy classes arise in this way. The missing conjugacy classes arise from so-called *admissible diagrams*, associated with
linearly independent subsets of roots (not necessarily simple). In these diagrams, the connection between two nodes is carried out exactly as in Dynkin diagrams: two nodes are connected if and only if the corresponding nodes are not orthogonal in the sense of the Tits bilinear form.

*Carter diagrams* (introduced in [St17]) differ from admissible diagrams in that they take into account the sign of the bilinear form on pair of roots; for this, the language of *solid and dotted edges* is used, see Section 2.2. The Carter diagrams with cycles are of interest for us. In the theorem on exclusion of long cycles [St17], it was shown that any Carter diagram with cycles of arbitrary even length can be reduced to diagrams with cycles of length 4 only. In the proof of this theorem, for each specific case, it is checked that the semi-Coxeter element associated with Carter diagram with long cycles is conjugate to the semi-Coxeter element associated with another Carter diagram with cycles of length 4. The Carter diagrams with conjugate semi-Coxeter elements are said to be *equivalent*. Up to equivalence, the simply-laced Carter diagrams (with cycles\(^1\)) representing non-Coxeter conjugacy classes are given in Fig. 3.3.

### 1.2. Dynkin-Minchenko diagrams: Procedure of completion.

The Dynkin diagrams are not large enough to contain the diagrams of all subsystems of the root system \(\Phi\). In [DM10], Dynkin and Minchenko introduced *enhanced Dynkin diagrams* arising for description of regular subalgebras of the complex semisimple Lie algebras. Let \(\Gamma\) be a Dynkin diagram of the complex semisimple Lie algebra \(g\). Subdiagrams of \(\Gamma\) are Dynkin diagrams of regular subalgebras of \(g\).

However, not all regular subalgebras can be obtained in this way. Besides, non-conjugate regular subalgebras can have identical Dynkin diagrams. Both problems are efficiently solved by using enhanced Dynkin diagrams. In [DM10], Dynkin-Minchenko introduce a canonical enlargement of the basis called the *enhanced basis* and enhanced Dynkin diagrams rep- resenting an *enhanced basis*. They constructed an enhancement of \(\Gamma\) by a recursive procedure which they call the *completion*: At each step of the procedure, find a \(D_4\)-subset in the already introduced nodes, add the maximal root of this subset, and connect it by edges to the corresponding part of the already introduced nodes. For the enhanced Dynkin diagrams \(\Delta(E_6)\) and \(\Delta(E_7)\), see Section A.

In [VM21], Vavilov and Migrin combined both types of considered diagrams: *Carter diagrams* and *enhanced Dynkin diagrams*, they applied the language of solid and dotted edges to enhanced Dynkin diagrams, the obtained diagrams are called *signed enhanced Dynkin diagrams*. They showed that any Carter diagram of the homogeneous class containing Dynkin diagrams \(\Gamma = E_6, E_7, E_8\) can be embedded into the signed enhanced Dynkin diagram \(\Delta(\Gamma)\) associated with \(\Gamma\) such that the “solid and dotted” correspondence is preserved.

---

\(^1\)There exist also Carter diagrams without cycles: Dynkin diagrams or equivalent (by similarity), see Section 2.3.

\(^2\)Adding a minimal root (instead of a maximal one) leads to a topologically isomorphic enhanced Dynkin diagram, but distinguished by solid and dotted edges.
1.3. Values 2,4,8 and diagrams $E_6$, $E_7$, $E_8$. To the Vavilov-Margin observation mentioned above, I would like to add the following easily verifiable fact:

Observation 1.1. The number of extra nodes obtained by the Dynkin-Minchenko completion procedure for a simply-laced Dynkin diagram coincides with the number of Carter diagrams (with cycles) of the same type, see Fig. 3.4.

Table 1.1. Cardinality of extra nodes

| $D_{2m}$ | $D_{2m+1}$ | $E_6$ | $E_7$ | $E_8$ |
|----------|------------|------|------|------|
| $m-1$    | $m-1$     | 2    | 4    | 8    |

For a further discussion of the relationship between enhanced Dynkin diagrams and Carter diagrams, see Section C.1.

In [Rin16], in the context of Auslander-Reiten quivers, Ringel observed a completely different relation between values 2,4,8 and diagrams $E_6$, $E_7$, $E_8$, see Section C.3.

In [Ba01], Baez (in relation to Rosenfeld’s idea in [Ro97]) points to another connection between values 2,4,8 and diagrams $E_6$, $E_7$, $E_8$, see Section C.4.

1.4. McKee-Smyth diagrams: Eigenvalues in $(-2,2)$. Much to my surprise, I found a complete list of 8-vertex Carter diagrams with circles in the paper of McKee and Smyth [MS07, Figs. 12-14]. The $\{0,1\}$-matrices with zeros on the diagonal can be regarded as adjacency matrices of graphs. Assume that the off-diagonal elements of such a matrix to be chosen from the set $\{-1,0,1\}$. Then, we get so-called a signed graph, a non-zero $(\alpha,\beta)$th entry denotes a sign of $-1$ or $1$ on the edge connecting vertices $\alpha$ and $\beta$. The signed graphs exactly correspond to our diagrams with solid and dotted edges. The matrix with zeros on the diagonal is called an uncharged matrix. By [MS07, Theorem 4], the signed graphs maximal with respect to having all their eigenvalues in $(-2,2)$ are exactly Carter diagrams $E_8(a_i)$, $1 \leq i \leq 8$ and $D_l(a_i)$, $i < l/2 - 1$, see Fig. 3.4. If the diagonal matrix $2I$ is added to such an uncharged matrix, then exactly partial Cartan matrix will be obtained, see Section C.2. Then, the eigenvalues of the partial Cartan matrices should lie in the interval $(0,4)$. Using eigenvalues one can get an invariant description of Carter diagrams, see [St08, Section 4.4]. For some details on the relationship between Carter diagrams and eigenvalues of partial Cartan matrices, see Section C.2.

Similar results to [MS07] were also obtained by Mulas and Stanic in [MuS22].

1.5. Homogeneous classes of Carter diagrams. The Dynkin diagram $A_l$, where $l \geq 1$ (resp. $D_l$, where $l \geq 4$; resp. $E_l$, where $l = 6,7,8$) is said to be the Dynkin diagram of $A$-type (resp. $D$-type, resp. $E$-type). The Carter diagram $A_l$, where $l \geq 1$ (resp. $D_l$, $D_l(a_k)$, where $l \geq 4$, $1 \leq k \leq \left[\frac{l-2}{2}\right]$; resp. $E_l$, $E_l(a_k)$, where $l = 6,7,8$, $k = 1,2,3,4$) is said to be the Carter diagram of $A$-type (resp. $D$-type, resp. $E$-type).
Carter diagrams of the same type and the same index constitute a homogeneous class of Carter diagrams, which has one of the following types: \( \{E, i\}\)-type (with \( i = 6, 7, 8 \)) or by \( \{D, l\}\)-type (one for each \( l > 4 \)), see (1.1) and Fig. 3.3.

\[
\{
\{E_6, E_6(a_1), E_6(a_2)\}, \\
\{E_7, E_7(a_1), E_7(a_2), E_7(a_3), E_7(a_4)\}, \\
\{E_8, E_8(a_1), E_8(a_2), E_8(a_3), E_7(a_4), E_8(a_5), E_8(a_6), E_8(a_7), E_7(a_8)\}, \\
\{D_l, D_l(a_1), D_l(a_2), \ldots, D_l(a_l)\}, \quad \text{where } l \geq 4.
\]

(1.1)

Carter diagrams from the same homogeneous class are called homogeneous. A pair of Carter diagrams from the same homogeneous class is called a homogeneous pair. Let \( \tilde{S} \) (resp. \( S \)) be a \( \Gamma \)-set (resp. \( \Gamma \)-set). In Tables 4.2 – 4.3, the transition matrix \( M_I : \tilde{S} \rightarrow S \) is constructed for the following homogeneous pairs \( \{\Gamma, \Gamma\} \):

\[
\begin{align*}
(1) & \quad \{D_4(a_1), D_4\} & (9) & \quad \{E_6(a_1), E_6\} \\
(2) & \quad \{D_l(a_k), D_l\} & (10) & \quad \{E_8(a_2), E_8\} \\
(3) & \quad \{E_6(a_1), E_6\} & (11) & \quad \{E_8(a_3), E_8(a_2)\} \\
(4) & \quad \{E_6(a_2), E_6(a_1)\} & (12) & \quad \{E_8(a_4), E_8(a_1)\} \\
(5) & \quad \{E_7(a_1), E_7\} & (13) & \quad \{E_8(a_5), E_8(a_4)\} \\
(6) & \quad \{E_7(a_2), E_7\} & (14) & \quad \{E_8(a_6), E_8(a_4)\} \\
(7) & \quad \{E_7(a_3), E_7(a_1)\} & (15) & \quad \{E_8(a_7), E_8(a_5)\} \\
(8) & \quad \{E_7(a_4), E_7(a_3)\} & (16) & \quad \{E_8(a_8), E_8(a_7)\}
\end{align*}
\]

(1.2)

The diagrams obtained as images of the mapping \( M_I \) are considered up to equivalence of Carter diagrams, see [S17, Section 1.3]. The entire list (1.2) is called an adjacency list. The adjacency list is not a complete list, but a minimal list that binds all Carter diagrams belonging to the same homogeneous class using the transition matrices constructed in Theorem 1.1. One example of alternative transition matrices is presented in Section 4.2 along with the pair \( \{E_8(a_6), E_8(a_4)\} \) two other pairs are considered: \( \{E_8(a_5), E_8(a_3)\} \) and \( \{E_8(a_6), E_8(a_5)\} \).

1.6. Theorems on transition matrix. The main result of this paper can be formulated as follows: Let \( \{\tilde{\Gamma}, \Gamma\} \) be a pair out of the adjacency list (1.2), and let \( \tilde{S} \) (resp. \( S \)) be a \( \tilde{\Gamma} \)-set (resp. \( \Gamma \)-set). We construct the matrix \( M_I \) having the following properties (Theorem 1.1, Theorem 1.3):

(a) The matrix \( M_I \) is the transition matrix transforming the roots of \( \tilde{S} \) to the roots of \( S \).

(b) The matrix \( M_I \) transforms the only element \( \tilde{\alpha} \in \tilde{S} \) into the minimal element of some Dynkin subset \( S(\tilde{\alpha}) \subseteq \tilde{S} \), the remaining elements of \( \tilde{S} \) are left fixed.

(c) The matrix \( M_I \) acts as involution on \( \tilde{S} \):

\[
M_I \tilde{\alpha} = \alpha = -\tilde{\alpha} + \sum_{\tau_i \in \tilde{S}} \tau_i, \quad M_I \tau_i = \tau_i \quad \text{for } \tau_i \neq \tilde{\alpha},
\]

(1.3)
and $M_I$ acts also as involution on $S$:

$$M_I\alpha = \tilde{\alpha} = -\alpha + \sum_{\tau_i \in S} \tau_i, \quad M_I\tau_i = \tau_i \text{ for } \tau_i \neq \alpha, \quad (1.4)$$

(d) In most cases of Tables 4.2-4.5, the mapping $M_I$ given in (1.3) eliminates one circle (or one endpoint), the mapping $M_I$ given in (1.4) builds one circle (or one endpoint). In case (16) of Table 4.5, $M_I$ eliminates 3 cycles.

Only $ADE$ root systems are considered.

2. Diagrams containing cycles

2.1. Admissible diagrams: Conjugacy classes of $W$. Let $\Phi$ be the root system corresponding to the Weyl group $W$. Each element $w \in W$ can be expressed in the form

$$w = s_{\alpha_1}s_{\alpha_2} \ldots s_{\alpha_k}, \text{ where } \alpha_i \in \Phi \text{ for all } i. \quad (2.1)$$

Carter proved that $k$ in the decomposition (2.1) is the smallest if and only if the subset of roots $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is linearly independent; such a decomposition is said to be reduced. The admissible diagram corresponding to the given element $w$ is not unique, since the reduced decomposition of the element $w$ is not unique.

Denote by $l_C(w)$ the smallest value $k$ corresponding to any reduced decomposition (2.1). The corresponding set of roots $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ consists of linearly independent and not necessarily simple roots, see [Ca72, Lemma 3]. If $l(w)$ is the smallest value $k$ in any expression like (2.1) such that all roots $\alpha_i$ are simple, then $l_C(w) \leq l(w)$.

**Lemma 2.1.** [Ca72] Lemma 3 Let $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Phi$. Then, $s_{\alpha_1}s_{\alpha_2} \ldots s_{\alpha_k}$ is reduced if and only if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are linearly independent. \(\square\)

The Cartan matrix (resp. quadratic form) associated with $\Phi$ is denoted by $B$ (resp. $B$). The inner product induced by $B$ is denoted by $(\cdot, \cdot)$.

Let us take the subset of linearly independent, but not necessarily simple roots $S \subset \Phi$. To the subset $S$ we associate some diagram $\Gamma$ that provides one-to-one correspondence between roots of $S$ and nodes of $\Gamma$. The diagram $\Gamma$ is said to be *admissible* if the following two conditions hold:

(a) The nodes of $\Gamma$ correspond to a set of linearly independent roots in $\Phi$.
(b) If a subdiagram of $\Gamma$ is a cycle, then it contains an even number of nodes.

(2.2)

Note that the admissible diagram may contain cycles, since the roots of $S$ are non necessarily simple. [SL14] Section 1.2.1]. Let us fix some basis of roots corresponding to the given admissible diagram $\Gamma$:

$$S = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_h\}$$

The admissible diagram is bicolored, i.e., the set of nodes can be partitioned
into two disjoint subsets $S_\alpha = \{\alpha_1, \ldots, \alpha_k\}$ and $S_\beta = \{\beta_1, \ldots, \beta_h\}$, where roots of $S_\alpha$ (resp. $S_\beta$) are mutually orthogonal. The element

$$c = w_\alpha w_\beta,$$

where $w_\alpha = \prod_{i=1}^{k} s_{\alpha_i}$, $w_\beta = \prod_{j=1}^{h} s_{\beta_j}$

is said to be semi-Coxeter element; it represents the conjugacy class associated with the admissible diagram $\Gamma$ and root subset $S$ (not necessarily root system).

2.2. Carter diagrams: Language of “solid and dotted” edges. In [St17], it was observed that the cycles in the admissible diagrams with necessity contains at least one pair of roots $\{\alpha_1, \beta_1\}$ with $(\alpha_1, \beta_1) > 0$ and at least one pair of roots $\{\alpha_2, \beta_2\}$ with $(\alpha_2, \beta_2) < 0$, where $(\cdot, \cdot)$ is the Tits bilinear form associated with the root system $\Phi$. This observation motivated me to distinguish such pairs of roots: Let us draw the dotted (resp. solid) edge $\{\alpha, \beta\}$ if $(\alpha, \beta) > 0$ (resp. $(\alpha, \beta) < 0$). The admissible diagram with dotted and solid edges is said to be the Carter diagram. Up to dotted edges, the classification of Carter diagrams coincides with the classification of admissible diagrams.

In the theorem on exclusion of long cycles [St17], it was shown that any Carter diagram with cycles of arbitrary even length can be reduced to diagrams with cycles of length 4 only. This explains why the admissible diagrams $D_l(b_{l-1}), E_7(b_2), E_8(b_3), E_8(b_5)$ listed in [Ca72, Table 2] do not appear in the lists of conjugacy classes. The Carter diagrams with conjugate semi-Coxeter elements are said to be equivalent. The Carter diagrams (with cycles) representing non-Coxeter conjugacy classes are given in Fig. 3.1. For another view of these diagrams, see [St17, Table 1]).

2.3. Carter diagrams: Eliminating the cycle. The semi-Coxeter elements generated by reflections $\{s_{\alpha_1}, s_{\alpha_2}, s_{\beta_1}, s_{\beta_2}\}$ constitute exactly two conjugacy classes with representatives $w_t$ and $w_o$, see Fig. 2.1. Semi-Coxeter elements $w_t$ and $w_o$ are distinguished by orders of reflections in the decomposition of $w_t$ and $w_o$. Here, $t$ is the bicolored order $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$, and $o$ is the cyclic order $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$, see [St17, Section 1.2].

![Figure 2.1](image_url)

Figure 2.1. Diagram $\Gamma_1$ (resp. $\Gamma_2$) of type $D_4(a_1)$ (resp. equivalent to $D_4$)
The element \( w_o \) is conjugate to the \( \tilde{w}_o = s_{\alpha_1}s_{\alpha_2}s_{-(\alpha_1+\beta_1+\beta_2)}s_{\beta_2} \) since

\[
w_o = s_{\alpha_1}s_{\beta_1}s_{\alpha_2}s_{\beta_2} = s_{\alpha_1+\beta_1}s_{\alpha_2}s_{\beta_2} = s_{\alpha_1}s_{\alpha_2}s_{\beta_2}s_{\alpha_1+\beta_1} = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1+\beta_1}s_{\beta_2} = s_{\alpha_1}s_{\alpha_2}s_{-(\alpha_1+\beta_1+\beta_2)}s_{\beta_2} = \tilde{w}_o.
\]

The elements \( w_o \) and \( \tilde{w}_o \) are conjugate, the corresponding sets of roots are as follows:

\[ S_1 = \{ \alpha_1, \beta_1, \alpha_2, \beta_2 \} \] and \[ S_2 = \{ \alpha_1, \alpha_2, -(\alpha_1 + \beta_1 + \beta_2), \beta_2 \}. \]

There is a map \( M : S_1 \rightarrow S_2 \) acting as follows:

\[ M\alpha_1 = \alpha_1, \quad M\alpha_2 = \alpha_2, \quad M\beta_2 = \beta_2, \quad \beta_1 = M\beta_1 = -(\alpha_1 + \beta_1 + \beta_2). \]

Note that \( M \) is involution, \( M : S_1 \rightarrow S_2 \) and \( M : S_2 \rightarrow S_1 \), since

\[
M^2\beta_1 = -(\alpha_1 + M\beta_1 + \beta_2) = -(\alpha_1 + \beta_2) + (\alpha_1 + \beta_1 + \beta_2) = \beta_1,
\]

\[
M\beta_1 = \beta_1 \quad \text{and} \quad M\beta_1 = \beta_1.
\]

Thus, \( M \) transforms the root \( \beta_1 \) into the minimal element in the root subsystem \( \{ \beta_2, \alpha_1, \beta_1 \} \). In this paper, we will encounter a number of involution mappings \( M \) that map a certain element to the minimal element of some root subsystem of \( \Phi \). So, we observe that there are two different orders of reflections:

- The cyclic order of reflections \( o \). Then, we get a 4-cycle leading to the Coxeter class \( D_4 \) of \( W(D_4) \), see Fig. 2.2.
- The bicolored order of reflections \( t \). Then, we get an indestructible 4-cycle leading to the semi-Coxeter class \( D_4(a_1) \).

**Figure 2.2.** Conjugate elements \( \{ w_o, \tilde{w}_o \} \) corresponding to the Coxeter class \( D_4 \)

### 2.4. Connection diagrams

In \[SU17\], in addition to Carter diagrams, the so-called connection diagrams were introduced. Let \( S \) be a set of linearly independent and not necessarily simple roots, \( o \) be the order of reflections in the decomposition \[2.1\] of element \( w \) associated with the set of roots \( S \). The connection diagram is a pair \( (\Gamma, o) \), where \( \Gamma \) corresponds to the set \( S \). In the connection diagram \( \Gamma \), edges are also solid and dotted as in Carter diagrams. The connection diagrams serve to transform one Carter diagram \( \Gamma_1 \) into another \( \Gamma_2 \), since in the process of transformation we can get non-Carter diagrams – the evenness of the cycles may be violated, see \[SU17, Section 1.2.2\].
In [St17], three equivalence transformations operating on the connection diagrams and Carter diagrams were introduced: similarities, conjugations and s-permutations. The Carter diagrams are studied there up to equivalence. In what follows, we only need similarity. Let $\alpha$ be a root in the $\Gamma$-set $S$. The similarity transformation $L_{\alpha}$ reflects the root $\alpha$:

$$L_{\alpha} : \alpha \mapsto -\alpha. \quad (2.3)$$

Two connection diagrams obtained from each other by a sequence of reflections (2.3), are called similar connection diagrams, see Fig. 2.3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{similar_4_cycles}
\caption{Eight similar 4-cycles equivalent to $D_4(a_1)$}
\end{figure}

2.5. Bicolored partition. Let $\Gamma$ be a Carter diagram and

$$S = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_h\} \quad (2.4)$$

be any $\Gamma$-set (of not necessarily simple roots), where roots of the set $S_\alpha := \{\alpha_i \mid i = 1, \ldots, k\}$ (resp. $S_\beta := \{\beta_j \mid j = 1, \ldots, h\}$) are mutually orthogonal. According to (2.2(a)), there exists a certain set (2.4) of linearly independent roots. Thanks to (2.2(b)), there exists a partition $S = S_\alpha \sqcup S_\beta$ which is said to be the bicolored partition.

Let $w = w_1w_2$ be the decomposition of $w$ into the product of two involutions. By [Ca72, Lemma 5] each of $w_1$ and $w_2$ can be expressed as a product of reflections as follows:

$$w = w_1w_2, \quad \text{where} \quad w_1 = s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_k}, \quad w_2 = s_{\beta_1}s_{\beta_2}\ldots s_{\beta_h}, \quad (2.5)$$

where subset $S_\alpha = \{\alpha_1, \ldots, \alpha_k\}$ (resp. $S_\beta = \{\beta_1, \ldots, \beta_h\}$) consists of mutually orthogonal roots. Let

$$\Pi_w = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_h\} \quad (2.6)$$

be the linearly independent root subset. Then, the decomposition (2.5) is reduced, see Lemma 2.1 and $k + h = l_C(w)$. The decomposition (2.5) is said to be a bicolored decomposition.
Figure 3.4. Carter diagrams of $D$ and $E$ types
3. The Cartan matrix

3.1. The generalized Cartan matrix. the \( n \times n \) matrix \( K = (k_{ij}) \), where \( 1 \leq i, j \leq n \), such that

\[(C1) \quad k_{ii} = 2 \text{ for } i = 1, \ldots, n,\]
\[(C2) \quad -k_{ij} \in \mathbb{Z} = \{0, 1, 2, \ldots\} \text{ for } i \neq j,\]
\[(C3) \quad k_{ij} = 0 \text{ implies } k_{ji} = 0 \text{ for } i, j = 1, \ldots, n\]

is called a generalized Cartan matrix, [Kac80], [St08, Section 2.1]. For the Carter diagram \( \Gamma \), which is not a Dynkin diagram, the condition (C2) fails: The elements \( k_{ij} \) associated with dotted edges are positive.

If the Carter diagram does not contain any cycle, then the Carter diagram is the Dynkin diagram, the corresponding conjugacy class is the conjugacy class of the Coxeter element, and the partial Cartan matrix is the classical Cartan matrix, which is the particular case of a generalized Cartan matrix.

3.2. The partial Cartan matrix. Similarly to the Cartan matrix associated with Dynkin diagrams, we determine the Cartan matrix for each pair \( \{\Gamma, S\} \) consisting of the connection or Carter diagram \( \Gamma \) and \( \Gamma \)-set \( S \):

\[
B_{\Gamma} := \begin{pmatrix}
(\tau_1, \tau_1) & \ldots & (\tau_1, \tau_n) \\
\ldots & \ldots & \ldots \\
(\tau_n, \tau_1) & \ldots & (\tau_n, \tau_n)
\end{pmatrix},
\]

(3.1)

where \( S = \{\tau_1, \ldots, \tau_n\} \). We call the matrix \( B_{\Gamma} \) a partial Cartan matrix corresponding to the diagram \( \Gamma \). The partial Cartan matrix \( B_{\Gamma} \) is well-defined since products \((\tau_i, \tau_j)\) in (3.1) do not depend on the choice of the \( \Gamma \)-set \( S \). The elements of the partial Cartan matrix are uniquely determined by the diagram \( \Gamma \) as follows:

\[
(\tau_i, \tau_j) = \begin{cases}
2, & \text{if } \tau_i = \tau_j, \\
0, & \text{if } \tau_i \text{ and } \tau_j \text{ are not connected}, \\
-1, & \text{if edge } \{\tau_i, \tau_j\} \text{ is solid}, \\
1, & \text{if edge } \{\tau_i, \tau_j\} \text{ is dotted}.
\end{cases}
\]

(3.2)

Let \( L \) be the subspace spanned by the vectors \( \{\tau_1, \ldots, \tau_n\} \). We write this fact as follows:

\[
L = [\tau_1, \ldots, \tau_n],
\]

(3.3)

The subspace \( L \) is said to be the \( S \)-associated subspace. Let \( B \) be the Cartan matrix corresponding to the primary root system \( \Phi \).

Proposition 3.1. (i) The restriction of the bilinear form associated with the Cartan matrix \( B \) on the subspace \( L \) coincides with the bilinear form associated with the partial Cartan matrix \( B_{\Gamma} \), i.e., for any pair of vectors \( v, u \in L \), we have

\[
(v, u)_{\Gamma} = (v, u), \text{ and } \mathcal{B}_{\Gamma}(v) = \mathcal{B}(v).
\]

(3.4)

(ii) For every Carter diagram, the matrix \( B_{\Gamma} \) is positive definite.
Proof. (i) From (3.1) we deduce:
\[(v, u) = (\sum t_i\tau_i, \sum q_j\tau_j) = \sum t_iq_j(\tau_i, \tau_j) = (v, u).\]

(ii) This follows from (i). \(\square\)

If \(\Gamma\) is a Dynkin diagram, the partial Cartan matrix \(B_{\Gamma}\) is the Cartan matrix associated with \(\Gamma\). By (3.4) the matrix \(B_{\Gamma}\) is positive definite. The symmetric bilinear form associated with \(B_{\Gamma}\) is denoted by \((\cdot, \cdot)_{\Gamma}\) and the corresponding quadratic form is denoted by \(\mathcal{B}_{\Gamma}\).

Remark 3.2. (i) D. Leites noticed that there are a number of other cases, where some off-diagonal elements of the Cartan matrices are positive. For example, this is so in the case of Lorentzian algebras, see [GN02, CCLL10]. However, in these cases the Cartan matrices are of hyperbolic type, whereas the partial Cartan matrices are positive definite.

(ii) I will quote from a paper by S. Brenner: “...it is amusing to note that there is a surprisingly large intersection between the finite type quivers with commutativity conditions and the diagrams by Carter in describing conjugacy classes of the classical Weyl groups...”, [Br75, p.43]. On various other cases arising in the representation theory of quivers, algebras and posets with Cartan matrices containing positive off-diagonal elements, see [Bon83, GR92, 10.7], [Sim10].

4. Transition

4.1. First transition theorem. Let \(\{\tilde{\Gamma}, \Gamma\}\) be the homogeneous pair of Carter diagrams, let \(\tilde{S}\) be a \(\tilde{\Gamma}\)-set and \(S\) be a \(\Gamma\)-set. In this section, we construct a mapping connecting \(\tilde{S}\) and \(S\). This mapping represents the transition matrix connecting \(\tilde{S}\) and \(S\) as bases in the linear spaces. The transition matrix has some good properties that are presented in Theorems 4.1 and 4.3. Let \(\Gamma'\) be the subdiagram of \(\Gamma\), and subset \(S' \subset S\) be a \(\Gamma'\)-set. If \(\Gamma'\) is the Dynkin diagram, we call \(S'\) the Dynkin subset.

Theorem 4.1. For each pair of diagrams \(\{\tilde{\Gamma}, \Gamma\}\) out of the list (12), there exists the linear mapping matrix \(M_{\tilde{\Gamma}}\) transforming each \(\tilde{\Gamma}\)-set \(\tilde{S}\) to some \(\Gamma\)-set \(S\) being the image of \(M_{\tilde{\Gamma}}\) (i.e., \(S = M_{\tilde{\Gamma}}\tilde{S}\)), see Tables 4.2, 4.3. The matrix \(M_{\tilde{\Gamma}}\) is the transition matrix binding \(\tilde{S}\) and \(S\) as bases in the linear spaces.

(i) The matrix \(M_{\tilde{\Gamma}}\) acts only on one element \(\tilde{\alpha} \in \tilde{S}\) and does not change remaining elements in \(\tilde{S}\); \(M_{\tilde{\Gamma}}\) transforms \(\tilde{\alpha}\) into the minimal element \(\alpha\) of some Dynkin subset \(S(\tilde{\alpha})\) in \(\tilde{S}\):

\[
\begin{cases}
\tilde{\alpha} \in S(\tilde{\alpha}) \subset \tilde{S}, \\
M_{\tilde{\Gamma}}\tau_i = \tau_i, \text{ for all } \tau_i \in \tilde{S}, \tau_i \neq \tilde{\alpha}, \\
M_{\tilde{\Gamma}}\tilde{\alpha} = \alpha = -\tilde{\alpha} + \sum t_i\tau_i, \text{ where the sum is taken over } \tau_i \in \tilde{S}, \tau_i \neq \tilde{\alpha}, t_i \in \mathbb{Z}, \\
\alpha - \text{ minimal element in } S(\tilde{\alpha})
\end{cases}
\]
The image $S = M_f\tilde{S}$ is the set $\{\tilde{S}\setminus\tilde{\alpha}\} \cup \{\alpha\}$ that satisfies to the orthogonality relations of the Carter diagram $\Gamma$.

(ii) The mapping $M_f : \tilde{S} \mapsto S$ is an involution on the set $\tilde{S} \cup \{\alpha\}$. We have

$$M_f\tilde{\alpha} = \alpha \quad \text{and} \quad M_f\alpha = \tilde{\alpha}.$$ 

For each pair of diagrams $\{\tilde{\Gamma}, \Gamma\}$ out of the list (1.2), the matrix $M_f$ is defined in Tables 4.2–4.5. The matrix $M_f$ is the transition matrix transforming each basis $\tilde{S}$ into some basis $S$.

The proof of Theorem 4.1 is given in Section 4.5. It is carried out separately for each pair $\{\tilde{\Gamma}, \Gamma\}$ from Tables 4.2–4.5.

4.2. The chain of homogeneous pairs. Let $\tilde{\Gamma}$ be a Carter diagram. Denote by $C(\tilde{\Gamma})$ the homogeneous class containing $\tilde{\Gamma}$. For any Carter diagram $\tilde{\Gamma}$ and the Dynkin diagram $\Gamma$ from $C(\tilde{\Gamma})$ there exists the chain of homogeneous pairs connecting $\tilde{\Gamma}$ and $\Gamma$ as follows:

$$\{\{\tilde{\Gamma}, \Gamma_1\}, \{\Gamma_1, \Gamma_2\}, \ldots, \{\Gamma_{k-1}, \Gamma_k\}, \{\Gamma_k, \Gamma\}\}. \quad (4.1)$$

This fact follows easily from consideration of the adjacency list (1.2).

4.2.1. Example: from $E_8(a_8)$ to $E_8$. There are 16 cases in Tables 4.2–4.5. Denote by $M_f(n)$ the transition matrix of the $n$-th case. The similarity transformation $L_n$ from (2.3) is the diagonal matrix of the form

$$\text{diag}(1, 1, \ldots, 1, -1, 1, \ldots, 1)$$

with $-1$ in the $\{i, i\}$th entry. The homogeneous pairs are bound by matrices $M_f$ and similarity matrices $L_n$. Consider, for example, the chain diagrams $E_8(a_8), E_8(a_7), E_8(a_5), E_8(a_4), E_8(a_1), E_8$, see eq. (4.2) and Section 1.5.

$$E_8(a_8) \xrightarrow{M_f(16)} E_8(a_7) \xrightarrow{L_{\beta_2}} E_8(a_7) \xrightarrow{M_f(15)} E_8(a_5) \xrightarrow{L_{\alpha_3}} E_8(a_5) \xrightarrow{M_f(12)} E_8(a_4) \xrightarrow{M_f(9)} E_8(a_1) \xrightarrow{L_{\alpha_4} L_{\beta_4}} E_8, \quad (4.2)$$

In eq. (4.2), we mean that instead of each diagram $\Gamma$ there is some $\Gamma$-set. The matrices $M_f(n)$ are given in Appendix 13. Consider the product of matrices of (1.2):

$$F = L_{\alpha_4} L_{\beta_4} M_f(9) M_f(12) M_f(13) L_{\alpha_3} M_f(15) L_{\beta_2} M_f(16)$$

The matrix $F$ maps the $E_8(a_8)$-basis $\tilde{S}$ to a certain $E_8$-basis $S = F\tilde{S}$:

$$F : \tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\} \mapsto S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\}$$

The chain (4.2) is parallel to the ascending chain of maximal eigenvalues of the corresponding partial Cartan matrices, see Section C.2.
4.2.2. Alternative transitions. The transition matrices from the adjacency list (1.2) do not constitute a complete set of possible transitions. For example, to the transition \( \{E_8(a_6), E_8(a_4)\} \), one can add more pairs containing \( E_8(a_6) \): \( \{E_8(a_6), E_8(a_5)\} \) and \( \{E_8(a_6), E_8(a_1)\} \), see Fig. 4.5. The marked vertex corresponds to the root, which is converted with the transition matrix \( M_I \). To keep the solid and dotted edges corresponding to them in cases (9) and (13) in Tables 4.4 - 4.5, each such transition is followed by actions of similarities which result in the desired solid and dotted edges (naming of vertices for \( E_8(a_1) \) and \( E_8(a_5) \) differ from those indicated in Tables 4.4 - 4.5):

\[
\{E_8(a_6), E_8(a_4)\} : M_I \tilde{\beta}_4 = \beta_4 = -(\tilde{\beta}_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3).
\]

\[
\{E_8(a_6), E_8(a_5)\} : M_I \beta_2 = \tilde{\beta}_2 = -(\tilde{\beta}_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3),
\]

followed by actions of \( L_\tau \), with \( \tau = \alpha_1, \alpha_2, \beta_1, \tilde{\beta}_2 \).

\[
\{E_8(a_6), E_8(a_1)\} : M_I \beta_3 = \tilde{\beta}_3 = -(\tilde{\beta}_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3),
\]

followed by actions of \( L_\tau \), with \( \tau = \tilde{\alpha}_4, \tilde{\alpha}_3, \tilde{\beta}_4 \).

\[(4.3)\]

![Figure 4.5. Alternative transitions \{E_8(a_6), E_8(a_4)\} and \{E_8(a_6), E_8(a_5)\}](image-url)

4.2.3. The product of transition matrices. As in Section 4.2.1, for any chain (4.1), one can construct the matrices \( F \) and \( F^{-1} \), where \( F \) is the product of corresponding transition matrices \( M_I(n) \) and similarity matrices \( L_\tau \). The matrix \( F \) is invertible since all \( M_I(n) \) and \( L_\tau \) are invertible.

\[
F : \tilde{\Gamma} \rightarrow \Gamma, \quad F^{-1} : \Gamma \rightarrow \tilde{\Gamma}.
\]  

\[(4.4)\]
This means that for any \( \tilde{\Sigma} \)-set \( \tilde{\Sigma} \) there exists \( \Sigma \)-set \( \Sigma \) such that
\[
F \tilde{\Sigma} = \Sigma, \quad F^{-1} \tilde{\Sigma} = \tilde{\Sigma},
\]
(4.5)
The matrix \( F \) does not depend on \( \tilde{\Sigma} \) and \( \Sigma \). If \( \Sigma = \{\tau_1, \ldots, \tau_n\} \), \( F \) transforms \( \tau_i \) so that
\[
F \tau_i = \sum_{j=1}^{n} f_{ji} \tau_i,
\]
(4.6)
where \( f_{ji} \) are some coefficients that depend only on diagrams \( \tilde{\Gamma} \) and \( \Gamma \).

4.3. Action of \( W \) on Carter diagram \( \tilde{\Gamma} \).

**Lemma 4.2.** Let \( F, F^{-1} \) be the matrices described in (4.4), (4.5). Then,

(i) The matrix \( F \) commutes with the Weyl group \( W \) on any \( \Gamma \)-set \( \Sigma \) as follows:
\[
wF = Fw \quad \text{for any } w \in W.
\]
(4.7)
(ii) Let \( \Gamma \)-sets \( \Sigma \) and \( \Sigma' \) be conjugate by some \( w \in W \): \( w\Sigma = \Sigma' \). Then, \( FS \) and \( FS' \) are conjugate by the same element \( w \in W \), i.e.,
\[
w\Sigma = \Sigma' \quad \text{implies} \quad wFS = FS'.
\]

**Proof.** (i) It suffices to prove eq. (4.7) for each element \( \tau_i \in \Sigma \). Each element \( w \in W \) transforms basis \( \Sigma \) to another \( \Sigma' = w\Sigma \), where \( w\tau_i = \tau_i' \), and \( \Sigma' = \{\tau_1', \ldots, \tau_n'\} \). In our case,
\[
Fw\tau_i = F\tau_i' = \sum_{j=1}^{n} f_{ji} \tau_i' \quad \text{and}
\]
\[
wF\tau_i = w \sum_{j=1}^{n} f_{ji} \tau_i = \sum_{j=1}^{n} f_{ji} w\tau_i = \sum_{j=1}^{n} f_{ji} \tau_i'.
\]
Therefore, \( Fw\tau_i = wF\tau_i \) for any \( \tau_i \in \Sigma \).

(ii) If \( w\Sigma = \Sigma' \) then by (i), we have \( wFS = Fw\Sigma = FS' \), i.e., \( FS \) and \( FS' \) conjugate by the same element \( w \in W \). \( \Box \)

4.4. Second transition theorem.

**Theorem 4.3.** (i) For any Carter diagram \( \tilde{\Gamma} \), all \( \tilde{\Gamma} \)-sets are conjugate under the Weyl group \( W \).

(ii) Let \( \{\tilde{\Gamma}, \Gamma\} \) be any homogeneous pair of Carter diagrams, where \( \Gamma \) is the Dynkin diagram, and let \( \tilde{\Sigma} \) be any \( \tilde{\Gamma} \)-set, \( S \) be any \( \Gamma \)-set. Then, there exists \( F \), the product of transition matrices \( M_I(n) \) and some matrices of similarity maps like \( L_{\tau_i} \) as in Theorem 4.1 and Section 4.2.1 and \( w \in W \) such that \( S = wF\tilde{\Sigma} \).

**Proof.** (i) The Carter diagram \( \tilde{\Gamma} \) belongs to some homogeneous class \( C(\tilde{\Gamma}) \). Every homogeneous class contains a Dynkin diagram \( \Gamma \). As in Section 4.2.1 there exists the mapping \( F \) from any \( \tilde{\Gamma} \)-set \( \tilde{\Sigma} \) to some \( \Gamma \)-set \( S \). By Theorem 4.1, the mapping \( F \) is the product of transition matrices \( M_I(n) \) and some matrices of similarity maps like \( L_{\tau_i} \).
Let $\tilde{S}'$ and $\tilde{S}''$ be any $\tilde{\Gamma}$-sets. We will prove that $\tilde{S}'$ and $\tilde{S}''$ are conjugate under the Weyl group $W$, i.e.,

$$w\tilde{S}' = \tilde{S}'', \quad \text{for some } w \in W. \quad (4.8)$$

There exist $\Gamma$-sets $S'$ and $S''$ such that

$$F\tilde{S}' = S', \quad F\tilde{S}'' = S'', \quad (4.9)$$

see Fig. 4.6. Let $\Phi$ be the root system associated with $\Gamma$. All bases in $\Phi$ are conjugate, see [Hum90, Theorem 1.4]. Then, there exists $w \in W$, such that $wS' = S''$. By (4.9),

$$wF\tilde{S}' = wS' = S'' = F\tilde{S}''.$$

By Lemma 4.2, transformations $w$ and $F$ in (4.10) commute, so

$$Fw\tilde{S}' = F\tilde{S}'', \quad \text{and } w\tilde{S}' = \tilde{S}''. \quad (4.11)$$

(ii) First, by Theorem 4.1 we transform $\tilde{S}$ to some $\Gamma$-set $S'$ by the mapping $F$ as in (i), see Fig. 4.6. Further, as in (i), there exists $w \in W$ such that $wS' = S$. Thus,

$$wF\tilde{S} = wS' = S.$$

□
Table 4.2. \{D_4(a_1), D_4\}, \{D_l(a_k), D_l\}, \{E_6(a_1), E_6\}, \{E_6(a_2), E_6(a_1)\}

| \(\tilde{\Gamma}\)-basis \(\tilde{S}\) and \(\Gamma\)-basis \(S\) | Mapping \(M_I\) |
|---|---|
| \(D_4\) | \(S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\) |
| \(\{\tilde{\Gamma}, \Gamma\} = \{D_4(a_1), D_4\}\) | \(\tilde{S} = \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \alpha_4\}\), \(M_I \tau = \tau\) for \(\tau \in \{\alpha_1, \alpha_2, \alpha_4\}\), \(M_I \tilde{\alpha}_3 = \alpha_3 = -(\alpha_1 + \alpha_2 + \tilde{\alpha}_3)\) |
| \(D_1\) | \(S = \{\tau_1, \tau_2, \ldots, \tau_{n-1}, \beta_{k+1}\}\) |
| \(\{\tilde{\Gamma}, \Gamma\} = \{D_l(a_k), D_l\}\) | \(\tilde{S} = \{\tau_1, \ldots, \tau_k, \tau_{k+1}, \tau_{k+2}, \ldots, \tau_{n-1}\}\), \(M_I \tau = \tau\) for \(\tau \in \{\tau_1, \tau_2, \ldots, \tau_{n-1}\}\), \(M_I \tau_{k+1} = \begin{cases} \beta_{k+1} = -(\tau_1 + 2 \sum_{i=2}^{k} \tau_i + \tau_{k+1} + \tau_{k+1}), & \text{if } k \geq 2, \\ \beta_2 = -(\tau_1 + \tau_2 + \tau_3). & \end{cases}\) |
| \(E_6\) | \(S = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}\) |
| \(\{\tilde{\Gamma}, \Gamma\} = \{E_6(a_1), E_6\}\) | \(\tilde{S} = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \tilde{\beta}_3\}\), \(M_I \tau = \tau\) for \(\tau \in \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\}\), \(M_I \tilde{\beta}_3 = \beta_3 = -(\alpha_1 + \beta_1 + \alpha_3 + \tilde{\beta}_3)\) |
| \(E_6(a_1)\) | \(\{\tilde{\Gamma}, \Gamma\} = \{E_6(a_2), E_6(a_1)\}\) | \(\tilde{S} = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \tilde{\beta}_2, \tilde{\beta}_3\}\), \(M_I \tau = \tau\) for \(\tau \in \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \tilde{\beta}_3\}\), \(M_I \tilde{\beta}_2 = \beta_2 = -(\alpha_3 + \beta_1 + \alpha_2 + \tilde{\beta}_2)\) |
Table 4.3. \( \{E_7(a_1), E_7\} \), \( \{E_7(a_2), E_7\} \), \( \{E_7(a_3), E_7(a_1)\} \), \( \{E_7(a_4), E_7(a_3)\} \)

| \( \tilde{\Gamma} \)-basis \( \tilde{S} \) and \( \Gamma \)-basis \( S \) | Mapping \( M_I \) |
|---|---|
| \( E_7 \) | \( S = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4\} \) |
| \( \{\tilde{\Gamma}, \Gamma\} = \{E_7(a_1), E_7\} \) | \( \tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3, \beta_1, \beta_2, \beta_3, \beta_4\} \), \( M_I\tau = \tau \) for \( \tau \in \{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4\} \), \( M_I\tilde{\alpha}_3 = \alpha_3 = -(\beta_2 + \alpha_2 + \beta_3 + \tilde{\alpha}_3) \) |
| \( \{\tilde{\Gamma}, \Gamma\} = \{E_7(a_2), E_7\} \) | \( \tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3, \beta_1, \beta_2, \beta_3, \beta_4\} \), \( M_I\tau = \tau \) for \( \tau \in \{\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4\} \), \( M_I\tilde{\alpha}_1 = \alpha_1 = -(\beta_1 + \alpha_2 + \beta_3 + \tilde{\alpha}_1) \) |
| \( \{\tilde{\Gamma}, \Gamma\} = \{E_7(a_3), E_7(a_1)\} \) | \( \tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3, \alpha_4, \beta_1, \beta_2, \beta_3\} \), \( M_I\tau = \tau \) for \( \tau \in \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \alpha_4, \beta_1, \beta_2, \beta_3\} \), \( M_I\alpha_4 = \beta_4 = -(\alpha_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1) \) |
| \( \{\tilde{\Gamma}, \Gamma\} = \{E_7(a_4), E_7(a_3)\} \) | \( \tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3, \alpha_4, \beta_1, \beta_2, \beta_3\} \), \( M_I\tau = \tau \) for \( \tau \in \{\alpha_2, \tilde{\alpha}_3, \alpha_4, \beta_1, \beta_2, \beta_3\} \), \( M_I\tilde{\alpha}_1 = \alpha_1 = -(2\beta_1 + \tilde{\alpha}_1 + \alpha_2 + \tilde{\alpha}_3) \) |
Table 4.4. \( \{E_8(a_1), E_8\}, \{E_8(a_2), E_8\}, \{E_8(a_3), E_8(a_2)\}, \{E_8(a_4), E_8(a_1)\} \)

| \( E_8 \) | \( \tilde{\Gamma} \)-basis \( \tilde{S} \) and \( \Gamma \)-basis \( S \) | Mapping \( M_I \) |
|---|---|---|
| \( E_8 \) | \( \{\tilde{\Gamma}, \Gamma\} = \{E_8(a_1), E_8\} \) | \( S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\} \) |
| \( E_8(a_1) \) | \( \{\tilde{\Gamma}, \Gamma\} = \{E_8(a_2), E_8\} \) | \( \tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\} \); \( M_{I\tau} = \tau \) for \( \tau \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\} \); \( M_{I\tilde{\alpha}_3} = \alpha_3 = -(\tilde{\alpha}_3 + \beta_3 + \alpha_2 + \beta_2) \) |
| \( E_8(a_2) \) | \( \{\tilde{\Gamma}, \Gamma\} = \{E_8(a_3), E_8(a_2)\} \) | \( \tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\} \); \( M_{I\tau} = \tau \) for \( \tau \in \{\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\} \); \( M_{I\tilde{\alpha}_1} = \alpha_1 = -(\tilde{\alpha}_1 + \beta_3 + \alpha_2 + \beta_1) \) |
| \( E_8(a_3) \) | \( \{\tilde{\Gamma}, \Gamma\} = \{E_8(a_4), E_8(a_2)\} \) | \( \tilde{S} = \{\tilde{\alpha}_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\} \); \( M_{I\tau} = \tau \) for \( \tau \in \{\alpha_4, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4\} \); \( M_{I\tilde{\alpha}_4} = \alpha_4 = -(3\tilde{\alpha}_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + \alpha_3 + \tilde{\alpha}_4) \) |
| \( E_8(a_4) \) | \( \{\tilde{\Gamma}, \Gamma\} = \{E_8(a_4), E_8(a_1)\} \) | \( \tilde{S} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4\} \); \( M_{I\tau} = \tau \) for \( \tau \in \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4\} \); \( M_{I\tilde{\alpha}_4} = \alpha_4 = -(\tilde{\alpha}_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1 + \beta_4) \) |
Table 4.5. \{E_8(a_5), E_8(a_4)\}, \{E_8(a_6), E_8(a_4)\}, \{E_8(a_7), E_8(a_5)\}, \{E_8(a_8), E_8(a_7)\}

| \begin{align*}
\tilde{\Gamma} &- \Gamma \text{-basis } \tilde{S} \text{ and } \Gamma \text{-basis } S \\
\{\tilde{\Gamma}, \Gamma\} &- \{E_8(a_5), E_8(a_4)\} \\
M_I &- \text{Mapping } M_I
\end{align*} |
|---|
| (13) \(E_8(a_5)\) | \(\tilde{S} = \{\alpha_1,\alpha_2,\tilde{\alpha}_3,\tilde{\alpha}_4,\beta_1,\beta_2,\beta_3,\tilde{\beta}_4\}\) |
| | \(M_I\tau = \tau \text{ for } \tau \in \{\alpha_1,\alpha_2,\tilde{\alpha}_3,\tilde{\alpha}_4,\beta_1,\beta_2,\beta_3\}\) |
| | \(M_I\tilde{\beta}_4 = \beta_4 = -(\tilde{\beta}_4 + \tilde{\alpha}_4 + \alpha_2 + \beta_1 + \alpha_1)\) |
| (14) \(E_8(a_6)\) | \(\tilde{S} = \{\alpha_1,\alpha_2,\tilde{\alpha}_3,\tilde{\alpha}_4,\beta_1,\beta_2,\beta_3,\tilde{\beta}_4\}\) |
| | \(M_I\tau = \tau \text{ for } \tau \in \{\alpha_1,\alpha_2,\tilde{\alpha}_3,\tilde{\alpha}_4,\beta_1,\beta_2,\beta_3\}\) |
| | \(M_I\tilde{\beta}_4 = \beta_4 = -(\tilde{\beta}_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3)\) |
| (15) \(E_8(a_7)\) | \(\tilde{S} = \{\tilde{\alpha}_1,\alpha_2,\tilde{\alpha}_3,\tilde{\alpha}_4,\beta_1,\beta_2,\beta_3,\tilde{\beta}_4\}\) |
| | \(M_I\tau = \tau \text{ for } \tau \in \{\tilde{\alpha}_1,\alpha_2,\tilde{\alpha}_3,\tilde{\alpha}_4,\beta_1,\beta_2,\beta_3,\tilde{\beta}_4\}\) |
| | \(M_I\tilde{\alpha}_1 = \alpha_1 = -(2\beta_1 + \tilde{\alpha}_1 + \alpha_2 + \tilde{\alpha}_3)\) |
| (16) \(E_8(a_8)\) | \(\tilde{S} = \{\tilde{\alpha}_1,\alpha_2,\tilde{\alpha}_3,\tilde{\alpha}_4,\beta_1,\beta_2,\beta_3,\beta_4\}\) |
| | \(M_I\tau = \tau \text{ for } \tau \in \{\tilde{\alpha}_1,\alpha_2,\tilde{\alpha}_3,\tilde{\alpha}_4,\beta_1,\beta_2,\beta_3\}\) |
| | \(M_I\beta_4 = \tilde{\beta}_4 = -(2\tilde{\alpha}_4 + \beta_4 + \beta_2 + \beta_3)\) |

4.5. Proof of Theorem 4.1

---

**Table 4.5.** The table lists the \(\tilde{\Gamma} \text{-basis } \tilde{S} \) and the \(\Gamma \text{-basis } S\) for each case. The mapping \(M_I\) is defined for each case, along with its corresponding equations. The proof of Theorem 4.1 is not provided here but is typically included in the text following the table.
(i) Let us construct matrix $M_I$ for every pair $\{\tilde{\Gamma}, \Gamma\}$ out of the adjacency list \[1.2\].

\begin{align*}
(1) \text{Pair } \{D_4(a_1), D_4\} & \quad M_I \text{ maps } D_4(a_1)\text{-set to } D_4\text{-set} \\
\text{Mapping:} & \quad M_I \tilde{\alpha}_3 = \alpha_3 = -(\alpha_1 + \alpha_2 + \tilde{\alpha}_3) \\
\text{Root system:} & \quad S = \{\alpha_2, \alpha_1, \tilde{\alpha}_3\} \\
\Phi - \text{root system of type } A_3 & \quad \Phi = \{\alpha_2, \alpha_1, \tilde{\alpha}_3\} \\
\text{Minimal root:} & \quad \alpha_3 - \text{minimal root in } \Phi \\
\text{Eliminated edges:} & \quad \{\tilde{\alpha}_3, \alpha_1\}, \quad \{\tilde{\alpha}_3, \alpha_4\} \\
\text{Emerging edge:} & \quad \{\alpha_3, \alpha_2\} \\
\text{Checking relations:} & \quad \begin{cases} 
\alpha_3 \perp \alpha_1 & (\alpha_3 \text{ is the minimal root}) \\
\alpha_3 \perp \alpha_4 & (\alpha_2 + \tilde{\alpha}_3 \perp \alpha_4) \\
(\alpha_3, \alpha_2) = -1 & (\alpha_3 \text{ is the minimal root}) 
\end{cases}
\end{align*}
(2) Pair \( \{D_l(a_k), D_l\} \)

\( M_I \) maps \( D_l(a_k) \)-set to \( D_l \)-set

\[
M_I \tau_{k+1} = \begin{cases} 
\beta_{k+1} = 
- (\tau_1 + 2 \sum_{i=2}^{k} \tau_i + \tau_{k+1} + \overline{\tau}_{k+1}) 
& \text{for } k \geq 2, \\
\beta_2 = -(\tau_1 + \tau_{2} + \overline{\tau}_{2}) & \text{for } k = 1
\end{cases}
\]

Root systems:

- \( S_1 = \{\tau_1, \ldots, \tau_k, \tau_{k+1}, \overline{\tau}_{k+1}\} \) for \( k \geq 2 \)
- \( S_2 = \{\tau_1, \tau_2, \overline{\tau}_{2}\} \) for \( k = 2 \)
- \( \Phi(S_1) \) — root system of type \( D_{k+2} \)
- \( \Phi(S_2) \) — root system of type \( A_3 \)

Minimal roots:

- \( \beta_{k+1} \) - minimal root in \( \Phi(S_1) \)
- \( \beta_2 \) - minimal root in \( \Phi(S_2) \)

Eliminated edges:

\( \{\overline{\tau}_{k+1}, \tau_k\}, \{\overline{\tau}_{k+1}, \tau_{k+2}\} \)

Emerging edge:

\( \{\beta_{k+1}, \tau_2\} \)

Checking relations for \( k \geq 2 \):

\[
\begin{cases} 
\beta_{k+1} \perp \tau_i (1 \leq i \leq k+1, i \neq 2) \\
\beta_{k+1} \perp \tau_{k+2} \\
\beta_{k+1} \perp \tau_i (i > k + 2) \\
(\beta_{k+1} \text{ is the minimal root})
\end{cases}
\]

Checking relations for \( k = 1 \):

\[
\begin{cases} 
\beta_2 \perp \tau_1, (\beta_2, \tau_2) = -1 \\
\beta_2 \perp \tau_i (i > 3) \\
(\beta_2 \text{ is the minimal root})
\end{cases}
\]

Hereinafter, the reason of the relation is indicated in parentheses.

1. Let \( \alpha \) be the sum of several roots \( \alpha_i \): \( \alpha = \sum \alpha_i \). Hereinafter, the line "\( \alpha \perp \beta \) (disconnected)" means the case, where each summand \( \alpha_i \) in \( \alpha \) is orthogonal to \( \beta \).
(3) Pair \( \{ E_6(a_1), E_6 \} \). \( M_I \) maps \( E_6(a_1) \)-set to \( E_6 \)-set

Mapping:
\[
M_I \tilde{\beta}_3 = \beta_3 = -(\alpha_1 + \beta_1 + \alpha_3 + \tilde{\beta}_3)
\]

Root system:
\[
S = \{ \alpha_1, \beta_1, \alpha_3, \tilde{\beta}_3 \}
\]

Minimal root:
\[
\beta_3 - \text{minimal root in } \Phi
\]

Eliminated edges:
\[
\{ \tilde{\beta}_3, \alpha_3 \}, \{ \tilde{\beta}_3, \alpha_2 \}
\]

Emerging edge:
\[
\{ \beta_3, \alpha_1 \}
\]

Checking relations:
\[
\begin{align*}
\beta_3 & \perp \alpha_3, \beta_1, \\
\beta_3 & \perp \alpha_2, \\
\beta_3 & \perp \beta_2, \\
(\beta_3, \alpha_1) & = -1
\end{align*}
\]

(4) Pair \( \{ E_6(a_2), E_6(a_1) \} \).

Mapping:
\[
M_I \tilde{\beta}_2 = \beta_2 = -(\alpha_3 + \beta_1 + \alpha_2 + \tilde{\beta}_2)
\]

Root systems:
\[
S = \{ \alpha_3, \beta_1, \alpha_2, \tilde{\beta}_2 \},
\]

Minimal roots:
\[
\beta_2 - \text{minimal root in } \Phi(S)
\]

Eliminated edges:
\[
\{ \tilde{\beta}_2, \alpha_3 \}, \{ \tilde{\beta}_2, \alpha_1 \}
\]

Emerging edges:
\[
\{ \beta_2, \alpha_2 \}
\]

Checking relations:
\[
\begin{align*}
\beta_2 & \perp \alpha_3, \beta_1, \\
\beta_2 & \perp \alpha_1, \tilde{\beta}_3, \\
(\beta_2, \alpha_2) & = -1
\end{align*}
\]

Figure 4.7. Case (3). Mapping \( M_I : E_6(a_1) \mapsto E_6 \).
Figure 4.8. Case (4). Mapping $M_I: E_6(a_2) \mapsto E_6(a_1)$.

(5) Pair $\{E_7(a_1), E_7\}$

Mapping: $M_I \tilde{\alpha}_3 = \alpha_3 = -(\beta_2 + \alpha_2 + \beta_3 + \tilde{\alpha}_3)$

Root system: $S = \{\beta_2, \alpha_2, \beta_3, \tilde{\alpha}_3\}$

Minimal root: $\alpha_3$ - minimal root in $\Phi$

Eliminated edges: $\{\tilde{\alpha}_3, \beta_1\}, \{\tilde{\alpha}_3, \beta_3\}$

Emerging edge: $\{\beta_2, \alpha_3\}$

Checking relations:

\[
\begin{align*}
\alpha_3 & \perp \alpha_2, \beta_3 & (\alpha_3 \text{ is the minimal root}) \\
\alpha_3 & \perp \beta_1 & (\tilde{\alpha}_3 + \alpha_2 \perp \beta_1) \\
\alpha_3 & \perp \alpha_1, \beta_4 & (\text{disconnected}) \\
(\alpha_3, \beta_2) & = -1 & (\alpha_3 \text{ is the minimal root})
\end{align*}
\]

Figure 4.9. Case (5). Mapping $M_I: E_7(a_1) \mapsto E_7$. 
(6) Pair \( \{ E_7(a_2), E_7 \} \) \( \)\( M_I \) maps \( E_7(a_2) \)-set to \( E_7 \)-set

Mapping: \( M_I \alpha_1 = \alpha_1 = -(\beta_1 + \alpha_2 + \beta_3 + \tilde{\alpha}_1) \)

Root system: \( S = \{ \beta_1, \alpha_2, \beta_3, \tilde{\alpha}_1 \} \), \( \Phi - \) root system of type \( A_4 \)

Minimal root: \( \alpha_1 \) - minimal root in \( \Phi \)

Eliminated edges: \{\( \tilde{\alpha}_1, \beta_3 \), \{\( \tilde{\alpha}_1, \beta_2 \), \{\( \tilde{\alpha}_1, \tilde{\beta}_4 \)\}

Emerging edges: \( \{ \alpha_1, \beta_1 \), \{ \alpha_1, \beta_4 \)\]

\[
\begin{align*}
&\frac{\beta_4}{\beta_1} \quad \frac{\tilde{\alpha}_1}{\alpha_2} \quad \frac{\beta_3}{\alpha_3} \\
\end{align*}
\]

\( \Phi(S) \) - root system of type \( A_5 \)

Minimal root: \( \beta_4 \) - minimal root in \( \Phi(S) \)

Eliminated edges: \{\( \alpha_4, \beta_3 \), \{\( \alpha_4, \beta_2 \)\}

Emerging edges: \( \{ \beta_4, \alpha_1 \)\]

Checking relations:
\[
\begin{align*}
&\frac{\beta_4}{\beta_1} \perp \frac{\beta_3}{\alpha_2} \quad \frac{\beta_3}{\alpha_3} \quad (\beta_4 \text{ is the minimal root}) \\
&\frac{\beta_4}{\beta_1} \perp \frac{\tilde{\alpha}_3}{\beta_2} \quad (\beta_1 + \beta_3 \perp \tilde{\alpha}_3 \text{ and } \alpha_2 + \alpha_4 \perp \beta_2) \\
&\frac{(\beta_4, \alpha_1)}{= -1} \quad (\beta_4 \text{ is the minimal root})
\end{align*}
\]

Figure 4.10. Case (6). Mapping \( M_I : E_7(a_2) \mapsto E_7 \).

(7) Pair \( \{ E_7(a_3), E_7(a_1) \} \) \( M_I \) maps \( E_7(a_3) \)-set to \( E_7(a_1) \)-set

Mapping: \( M_I \alpha_4 = \beta_4 = -(\alpha_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1) \)

Root systems: \( S = \{ \alpha_4, \beta_3, \alpha_2, \beta_1, \alpha_1 \} \), \( \Phi(S) - \) root system of type \( A_5 \)

Minimal roots: \( \beta_4 \) - minimal root in \( \Phi(S) \)

Eliminated edges: \{\( \alpha_4, \beta_3 \), \{\( \alpha_4, \beta_2 \)\}

Emerging edges: \( \{ \beta_4, \alpha_1 \)\]

Checking relations:
\[
\begin{align*}
&\frac{\beta_4}{\beta_1} \perp \frac{\beta_3}{\alpha_2} \quad \frac{\beta_3}{\alpha_3} \quad (\beta_4 \text{ is the minimal root}) \\
&\frac{\beta_4}{\beta_1} \perp \frac{\tilde{\alpha}_3}{\beta_2} \quad (\beta_1 + \beta_3 \perp \tilde{\alpha}_3 \text{ and } \alpha_2 + \alpha_4 \perp \beta_2) \\
&\frac{(\beta_4, \alpha_1)}{= -1} \quad (\beta_4 \text{ is the minimal root})
\end{align*}
\]
(8) Pair \( \{E_7(a_4), E_7(a_3)\} \) 

\( M_I \) maps \( E_7(a_4) \)-set to \( E_7(a_3) \)-set

Mapping:

\[ M_I \bar{\alpha}_1 = \alpha_1 = -(2\beta_1 + \bar{\alpha}_1 + \alpha_2 + \bar{\alpha}_3) \]

Root system:

\[ S = \{\beta_1, \bar{\alpha}_1, \alpha_2, \alpha_3\}, \Phi(S) \text{ - root system of type } D_4 \]

Minimal root:

\( \alpha_1 \) - minimal root in \( \Phi(S) \)

Eliminated edges:

\( \{\bar{\alpha}_1, \beta_1\}, \{\bar{\alpha}_1, \beta_2\} \)

Emerging edges:

\( \{\alpha_1, \beta_1\} \)

Checking relations:

\[
\begin{align*}
\alpha_1 & \perp \alpha_2, \bar{\alpha}_3 & (\beta_1 \text{ is the minimal root}) \\
\alpha_1 & \perp \beta_3, \beta_2 & (\alpha_2 + \bar{\alpha}_3 \perp \beta_3 \text{ and } \bar{\alpha}_1 + \alpha_2 \perp \beta_2) \\
\alpha_1 & \perp \alpha_4 & (\alpha_4 \perp \bar{\alpha}_1, \bar{\alpha}_3, \alpha_2, \beta_1) \\
(\alpha_1, \beta_1) & = -1 & (\beta_1 \text{ is the minimal root})
\end{align*}
\]

Figure 4.11. Case (7). Mapping \( M_I : E_7(a_3) \mapsto E_7(a_1) \).

Figure 4.12. Case (8). Mapping \( M_I : E_7(a_4) \mapsto E_7(a_3) \) (diagram (c)). Diagrams (c) and (d) are equivalent (by similarity \( \bar{\alpha}_3 \mapsto -\bar{\alpha}_3 \))
(9) Pair \( \{E_8(a_1), E_8\} \) \( M_I \) maps \( E_8(a_1) \)-set to \( E_8 \)-set

Mapping:
\[ M_I \tilde{\alpha}_3 = \alpha_3 = -(\tilde{\alpha}_3 + \beta_3 + \alpha_2 + \beta_2) \]

Root system:
\[ S = \{\tilde{\alpha}_3, \beta_3, \alpha_2, \beta_2\} \]
\( \Phi \) - root system of type \( A_4 \)

Minimal root:
\( \alpha_3 \) - minimal root in \( \Phi \)

Eliminated edges:
\( \{\tilde{\alpha}_3, \beta_3\}, \{\tilde{\alpha}_3, \tilde{\beta}_3\} \)

Emerging edge:
\( \{\beta_2, \alpha_3\} \)

Checking relations:
\[ \begin{align*}
\alpha_3 \perp \alpha_2, \beta_3 & \quad (\alpha_3 \text{ is the minimal root}) \\
\alpha_3 \perp \beta_1 & \quad (\tilde{\alpha}_3 + \alpha_2 \perp \beta_1) \\
\alpha_3 \perp \alpha_1, \alpha_4, \beta_1 & \quad (\text{disconnected}) \\
(\alpha_3, \beta_2) &= -1 \quad (\alpha_3 \text{ is the minimal root})
\end{align*} \]

(10) Pair \( \{E_8(a_2), E_8\} \).

\( M_I \) maps \( E_8(a_2) \)-set to \( E_8 \)-set

Mapping:
\[ M_I \tilde{\alpha}_1 = \alpha_1 = -(\tilde{\alpha}_1 + \beta_3 + \alpha_2 + \beta_1) \]

Root system:
\[ S = \{\tilde{\alpha}_1, \beta_3, \alpha_2, \beta_1\} \]
\( \Phi \) - root system of type \( A_4 \)

Minimal root:
\( \alpha_1 \) - minimal root in \( \Phi \)

Eliminated edges:
\( \{\beta_3, \tilde{\alpha}_1\}, \{\tilde{\alpha}_1, \beta_4\}, \{\tilde{\alpha}_1, \beta_2\} \)

Emerging edges:
\( \{\alpha_1, \beta_1\}, \{\alpha_1, \beta_4\} \)

Checking relations:
\[ \begin{align*}
\alpha_1 \perp \alpha_2, \beta_3 & \quad (\alpha_1 \text{ is the minimal root}) \\
\alpha_1 \perp \beta_2 & \quad (\tilde{\alpha}_1 + \alpha_2 \perp \beta_2) \\
\alpha_1 \perp \alpha_4 & \quad (\text{disconnected}) \\
(\alpha_1, \beta_1) &= -1 \quad (\alpha_1 \text{ is the minimal root}) \\
(\beta_4, \alpha_1) &= +1 \quad ((\beta_4, \alpha_1) = -(\beta_4, \tilde{\alpha}_1))
\end{align*} \]
Case (10). Mapping $M$:

$E_8(a_3)$ maps to $E_8(a_2)$-set

Mapping:

$M\tilde{\alpha}_4 = \alpha_4 = -(3\tilde{\alpha}_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + \alpha_3 + \tilde{\alpha}_4)$

Root system:

$S = \{\tilde{\alpha}_1, \beta_2, \beta_3, \alpha_3, \tilde{\alpha}_4\}$

$\Phi$ - root system of type $E_6$

Minimal root:

$\alpha_4$ - minimal root in $\Phi$

Eliminated edges:

$\{\tilde{\alpha}_4, \beta_3\}$

Emerging edge:

$\{\beta_4, \alpha_4\}$

Checking relations:

$\begin{align*}
\alpha_4 &\perp \tilde{\alpha}_1, \alpha_3, \beta_2, \beta_3 \\
\alpha_4 &\perp \alpha_2 \\
\alpha_4 &\perp \beta_1 \\
(\alpha_4, \beta_4) & = -1
\end{align*}$

The minimal root $M\tilde{\alpha}_4 = \alpha_4$ is connected only with $\beta_4$. 

Figure 4.14. Mapping $M_I : E_8(a_2) \mapsto E_8$ (diagram (b)). Diagrams (b) and (c) are equivalent (by similarity $\alpha_4 \mapsto -\alpha_4$, $\beta_4 \mapsto -\beta_4$)
Figure 4.15. Case (11). Mapping $M_I : E_8(a_3) \rightarrow E_8(a_2)$ (diagram (b)). Diagrams (b) and (c) are equivalent (by similarity $\beta_2 \mapsto -\beta_2$, $\alpha_3 \mapsto -\alpha_3$)

(12) $\text{Pair } \{E_8(a_4), E_8(a_1)\}$  \hspace{1cm} $M_I$ maps $E_8(a_4)$-set to $E_8(a_1)$-set

Mapping:  
$M_I \tilde{\alpha}_4 = \alpha_4 = - (\tilde{\alpha}_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1 + \beta_4)$

Root systems:  
$S = \{\tilde{\alpha}_4, \beta_3, \alpha_2, \beta_1, \alpha_1, \beta_4\}$  
$\Phi(S)$ - root system of type $A_4$

Minimal roots:  
$\alpha_4$ - minimal root in $\Phi(S)$

Eliminated edges:  
$\{\tilde{\alpha}_4, \beta_2\}, \{\tilde{\alpha}_4, \beta_3\}$

Emerging edges:  
$\{\alpha_4, \beta_4\}$

Checking relations:  
$\begin{cases} 
\alpha_4 \perp \beta_3, \alpha_2, \beta_1, \alpha_1 & (\alpha_4 - \text{minimal root}) \\
\alpha_4 \perp \tilde{\alpha}_3, \beta_2 & (\beta_3 + \beta_1 \perp \tilde{\alpha}_3, \text{ and }) \\
\tilde{\alpha}_4 + \alpha_2 \perp \beta_2 & \\
(\alpha_4, \beta_4) = -1 & (\alpha_4 - \text{minimal root})
\end{cases}$

Figure 4.16. Case (12). Mapping $M_I : E_8(a_4) \rightarrow E_8(a_1)$. 
(13) Pair \( \{E_8(a_5), E_8(a_4)\} \)

\( M_I \) maps \( E_8(a_5) \)-set to \( E_8(a_4) \)-set

Mapping:
\[
M_I \tilde{\beta}_4 = \beta_4 = -(\tilde{\beta}_4 + \tilde{\alpha}_4 + \beta_3 + \alpha_2 + \beta_1 + \alpha_1)
\]

Root system:
\( S = \{\tilde{\beta}_4, \tilde{\alpha}_4, \beta_3, \alpha_2, \beta_1, \alpha_1\} \)
\( \Phi(S) \) - root system of type \( A_6 \)

Minimal roots:
\( \beta_4 \) - minimal root in \( \Phi(S) \)

Eliminated edges:
\( \{\tilde{\alpha}_4, \tilde{\beta}_4\} \), \( \{\tilde{\beta}_4, \alpha_1\} \)

Emerging edge:
\( \{\beta_4, \alpha_1\} \)

Checking relations:
\[
\begin{align*}
\beta_4 &\perp \beta_1, \alpha_2, \beta_2, \beta_3, \tilde{\beta}_4 \quad (\beta_4 \text{ - minimal root}) \\
\beta_4 &\perp \tilde{\alpha}_3, \beta_2 \quad (\beta_4 + \beta_1 \perp \tilde{\alpha}_3 \text{ and } \alpha_2 + \tilde{\alpha}_4 \perp \beta_2) \\
(\beta_4, \alpha_1) &=-1 \quad (\beta_4 \text{ - minimal root})
\end{align*}
\]

(14) Pair \( \{E_8(a_6), E_8(a_4)\} \)

\( M_I \) maps \( E_8(a_6) \)-set to \( E_8(a_4) \)-set

Mapping:
\[
M_I \tilde{\beta}_4 = \beta_4 = -(\tilde{\beta}_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3)
\]

Root system:
\( S = \{\tilde{\beta}_4, \alpha_1, \beta_1, \alpha_2, \beta_2, \beta_3\} \)
\( \Phi \) - root system of type \( D_6 \)

Minimal roots:
\( \beta_4 \) - minimal root in \( \Phi \)

Eliminated edges:
\( \{\tilde{\beta}_4, \tilde{\alpha}_3\} \), \( \{\tilde{\beta}_4, \alpha_1\} \)

Emerging edge:
\( \{\beta_4, \alpha_1\} \)

Checking relations:
\[
\begin{align*}
\beta_4 &\perp \beta_1, \alpha_2, \beta_2, \beta_3, \tilde{\beta}_4 \quad (\beta_4 \text{ - minimal root}) \\
\beta_4 &\perp \tilde{\alpha}_3 \quad (\beta_4, \tilde{\alpha}_3) = \\
(\beta_4 + \beta_3 + 2\beta_1, \tilde{\alpha}_3) & = -1 - 1 + 2 = 0 \\
\beta_4 &\perp \tilde{\alpha}_4 \quad (\beta_3 + \beta_2 \perp \alpha_4) \\
(\beta_4, \alpha_1) &=-1 \quad (\beta_4 \text{ - minimal root})
\end{align*}
\]

Figure 4.17. Case (13). Mapping \( M_I : E_8(a_5) \mapsto E_8(a_4) \).
Figure 4.18. Case (14). Mapping $M_I : E_8(a_6) \mapsto E_8(a_4)$.

(15) Pair $\{E_8(a_7), E_8(a_5)\}$

Mapping:

$M_I \tilde{\alpha}_1 = \alpha_1 = - (2\beta_1 + \tilde{\alpha}_1 + \alpha_2 + \tilde{\alpha}_3)$

Root systems:

$S = \{\beta_1, \tilde{\alpha}_1, \alpha_2, \tilde{\alpha}_3\}$

$\Phi(S)$ - root system of type $D_4$

Minimal roots:

$\alpha_1$ - minimal root in $\Phi(S)$

Eliminated edges:

$\{\tilde{\alpha}_1, \tilde{\beta}_1\}, \{\tilde{\alpha}_1, \beta_2\}$

Emerging edge:

$\{\beta_1, \alpha_1\}$

Checking relations:

\[
\begin{align*}
\alpha_1 & \perp \alpha_2, \tilde{\alpha}_3 \quad (\alpha_1 - \text{minimal root}) \\
\alpha_1 & \perp \tilde{\beta}_4 \quad (\text{disconnected}) \\
(\alpha_1, \beta_1) & = -1 \quad (\alpha_1 - \text{minimal root}) \\
\alpha_1 & \perp \beta_2, \beta_3 \quad (\alpha_2 + \tilde{\alpha}_1 \perp \beta_2, \text{ and } \\
& \quad \alpha_2 + \tilde{\alpha}_3 \perp \beta_3) \\
\end{align*}
\]

(16) Pair $\{E_8(a_8), E_8(a_7)\}$

Mapping:

$M_I \tilde{\beta}_4 = \beta_4 = - (2\tilde{\alpha}_4 + \beta_4 + \beta_2 + \beta_3)$

Root systems:

$S = \{\tilde{\alpha}_4, \beta_4, \beta_2, \beta_3\}$

$\Phi(S)$ - root system of type $D_4$

Minimal roots:

$\beta_4$ - minimal root in $\Phi(S)$

Eliminated edges:

$\{\tilde{\beta}_4, \tilde{\alpha}_4\}, \{\beta_4, \tilde{\alpha}_1\}, \{\beta_4, \tilde{\alpha}_3\}$

Emerging edges:

$\{\tilde{\beta}_4, \alpha_4\}$

Checking relations:

\[
\begin{align*}
\tilde{\beta}_4 & \perp \beta_2, \beta_3 \quad (\beta_4 - \text{minimal root}) \\
\tilde{\beta}_4 & \perp \tilde{\alpha}_1, \tilde{\alpha}_3 \quad (\tilde{\alpha}_1 \perp \beta_2 + \beta_4 \text{ and } \\
& \quad \beta_3 + \beta_4 \perp \tilde{\alpha}_3) \\
(\tilde{\beta}_4, \tilde{\alpha}_1) & = -1 \quad (\beta_4 - \text{minimal root})
\end{align*}
\]
(ii) Let us prove that $M_I$ is involution. There exists a certain root $\tilde{\alpha} \in \tilde{S}$ such that
\[
\begin{cases}
M_I \tau_i = \tau_i \text{ for all } \tau_i \in \tilde{S}, \tau_i \neq \tilde{\alpha}, \\
M_I \tilde{\alpha} = \alpha = -\tilde{\alpha} + \sum t_i \tau_i, \text{ where sum is taken over all } \tau_i \in \tilde{S}, \tau_i \neq \tilde{\alpha}.
\end{cases}
\]

The image \(M_I \tilde{\alpha} = \alpha\) is the root in \(S\). Then,
\[
M_I^2 \tilde{\alpha} = -M_I \tilde{\alpha} + \sum t_i \tau_i = \tilde{\alpha} - \sum t_i \tau_i + \sum t_i \tau_i = \tilde{\alpha}, \text{ and }
M_I^2 \tilde{\alpha} = M_I \alpha = \tilde{\alpha}.
\]

In other words,
\[
M_I : \tilde{\alpha} \mapsto \alpha, \text{ and } M_I : \alpha \mapsto \tilde{\alpha}.
\]

\[\square\]

4.6. Relation of partial Cartan matrices. Consider pairs \(\{\Gamma, \tilde{\Gamma}\}\) out of the adjacency list (1.2). The transition matrix \(M_I\) maps the \(\tilde{\Gamma}\)-basis \(\tilde{S}\) to the \(\Gamma\)-basis \(S\):
\[
M_I \tilde{\tau}_i = \tau_i, \text{ where } \tilde{\tau}_i \in \tilde{S}, \tau_i \in S.
\]

If the matrix \(M\) does not change a certain root \(\tilde{\tau}_i\), the designation of this root and the corresponding node is the same for \(\tilde{\Gamma}\)-basis and \(\Gamma\)-basis, namely:
\[
M_I \tau_i = \tau_i.
\]

The transition matrix \(M\) relates the partial Cartan matrices \(B_{\Gamma}\) and \(B_{\tilde{\Gamma}}\) as follows:
\[
t^t M \cdot B_{\tilde{\Gamma}} \cdot M = B_{\Gamma}.
\]

(4.12)

Eq. (4.12) is the relation of partial Cartan matrices \(B_{\Gamma}\) given in different bases.

4.6.1. Example: \(\{D_4(a_1), D_4\}\). For the pair \(\{D_4(a_1), D_4\}\) (case (1) in Table B.6),
\[
M = \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad B_{D_4(a_1)} = \begin{bmatrix}
2 & 0 & -1 & -1 \\
0 & 2 & 1 & -1 \\
-1 & 1 & 2 & 0 \\
-1 & -1 & 0 & 2
\end{bmatrix}.
\]

Then,
\[
t^t M B_{D_4(a_1)} M = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & -1 \\
0 & -1 & 2 & 0 \\
0 & -1 & 0 & 2
\end{bmatrix}.
\]

The matrix \(t^t M B_{D_4(a_1)} M\) is the Cartan matrix \(B_{D_4}\) for Dynkin diagram \(D_4\).
Example: \{E_6(a_1), E_6\}. For the pair \{E_6(a_1), E_6\} (case (3) in Table B.7),

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}, \quad B_{E_6(a_1)} = \begin{bmatrix}
2 & 0 & 0 & -1 & 0 & 0 \\
0 & 2 & 0 & -1 & -1 & 1 \\
0 & 0 & 2 & -1 & 0 & -1 \\
-1 & -1 & -1 & 2 & 0 & 0 \\
0 & -1 & 0 & 0 & 2 & 0 \\
0 & 1 & -1 & 0 & 0 & 2 \\
\end{bmatrix}.
\]

Then \( tMB_{E_6(a_1)}M \) is the Cartan matrix \( B_{E_6} \) for Dynkin diagram \( E_6 \).

Appendix A. Dynkin-Mincheno completion procedure

A.1. Enhanced Dynkin diagram \( \Delta(E_6) \). Extra nodes \( m_1 \) and \( m_2 \) for the enhanced Dynkin diagram \( \Delta(E_6) \) are as follows:

\[
m_1 = 2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5,
\]

\( m_1 \) is the maximal element in \( \{\alpha_4, \alpha_2, \alpha_3, \alpha_5\} \), \( \text{(A.1)} \)

see Fig. A.21

\[
m_2 = 2m_1 - \alpha_4 + \alpha_1 + \alpha_6,
\]

\( m_2 \) is the maximal element in \( \{m_1, -\alpha_4, \alpha_1, \alpha_6\} \) \( \text{(A.2)} \)

From (A.1) and (A.2) we get

\[
m_2 = 3\alpha_4 + 2\alpha_2 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6,
\]

Then, \( m_2 \) is also the maximal element in the \( E_6 \)-set \( \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \).

A.2. Enhanced Dynkin diagram \( \Delta(E_7) \). As in the case \( E_6 \), the extra nodes \( m_1 \) and \( m_2 \) are as follows:

\[
m_1 = 2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5,
\]

\[
m_2 = 3\alpha_4 + 2\alpha_2 + 2\alpha_3 + 2\alpha_5 + \alpha_1 + \alpha_6,
\]

Further, the extra node \( m_3 \) is as follows:

\[
m_3 = 2\alpha_6 + \alpha_5 + \alpha_7 + m_1 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7
\]
Figure A.21. Enhanced Dynkin diagrams $\Delta(E_6)$

Here, $m_3$ is the maximal element in the $D_6$-set $\{\alpha_2, \alpha_3, \alpha_4, \alpha_4\alpha_6\}$.

The extra node $m_4$:

$$m_4 = 2\alpha_1 + \alpha_3 + m_1 + m_3 = 4\alpha_4 + 3\alpha_5 + 3\alpha_3 + 2\alpha_2 + 2\alpha_6 + 2\alpha_1 + \alpha_7$$

The node $m_4$ is the maximal element in the $E_7$-set

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$,

see Fig. A.22.

Figure A.22. Enhanced Dynkin diagrams $\Delta(E_7)$

Appendix B. Transition matrices $M_I$, cases (1) - (16)
Table B.6. Transition matrices, Cases (1)-(4)

| (1) $\{D_4(a_1), D_4\}$ | (2) $\{D_1(a_k), D_1\}$ |
|-----------------------------|-----------------------------|
| $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & \tilde{\alpha}_3 \\ 0 & 0 & 0 & 1 & \alpha_4 \end{bmatrix}$ | $\begin{bmatrix} \tau_1 & \tau_2 & \ldots & \tau_{k-1} & \beta_{k+1} \\ 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & -1 & -2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & -1 & \tau_{k+1} \\ 0 & 0 & \ldots & 0 & -1 & \tau_{k+1} \end{bmatrix}$ |

| (3) $\{E_6(a_1), E_6\}$ | (4) $\{E_6(a_2), E_6(a_1)\}$ |
|-----------------------------|-----------------------------|
| $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3 \\ 1 & 0 & 0 & 0 & 0 & -1 & \alpha_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & 1 & 0 & 0 & -1 & \alpha_3 \\ 0 & 0 & 0 & 1 & 0 & -1 & \beta_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & -1 & \beta_3 \end{bmatrix}$ | $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3 & \tilde{\beta}_3 \\ 1 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\ 0 & 1 & 0 & 0 & -1 & 0 & \alpha_2 \\ 0 & 0 & 1 & 0 & -1 & 0 & \alpha_3 \\ 0 & 0 & 0 & 1 & -1 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 & -1 & 0 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & \beta_3 \end{bmatrix}$ |

Table B.7. Transition matrices, Cases (5)-(8)

| (5) $\{E_7(a_1), E_7\}$ | (6) $\{E_7(a_2), E_7\}$ |
|-----------------------------|-----------------------------|
| $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 1 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\ 0 & 1 & -1 & 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & -1 & 0 & 0 & 0 & \tilde{\alpha}_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & \beta_1 \\ 0 & 0 & -1 & 0 & 1 & 0 & \beta_2 \\ 0 & 0 & -1 & 0 & 0 & 1 & \beta_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & \beta_4 \end{bmatrix}$ | $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ -1 & 0 & 0 & 0 & 0 & 0 & \tilde{\alpha}_1 \\ -1 & 1 & 0 & 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & \alpha_3 \\ -1 & 0 & 0 & 1 & 0 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_2 \\ -1 & 0 & 0 & 0 & 0 & 1 & \beta_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_4 \end{bmatrix}$ |

| (7) $\{E_7(a_3), E_7(a_1)\}$ | (8) $\{E_7(a_4), E_7(a_3)\}$ |
|-----------------------------|-----------------------------|
| $\begin{bmatrix} \alpha_1 & \alpha_2 & \tilde{\alpha}_3 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 1 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & \tilde{\alpha}_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & \beta_3 \end{bmatrix}$ | $\begin{bmatrix} \alpha_1 & \alpha_2 & \tilde{\alpha}_3 & \alpha_4 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ -1 & 0 & 0 & 0 & 0 & 0 & \tilde{\alpha}_1 \\ -1 & 1 & 0 & 0 & 0 & 0 & \alpha_2 \\ -1 & 0 & 1 & 0 & 0 & 0 & \tilde{\alpha}_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & \alpha_4 \\ -2 & 0 & 0 & 0 & 1 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & \beta_3 \end{bmatrix}$ |
### Table B.8. Transition matrices, Cases (9)-(12)

|       | \( \{E_s(a_1), E_s\} \) | \( \{E_s(a_2), E_s\} \) |
|-------|--------------------------|--------------------------|
|       | \( \alpha_1 \alpha_2 \alpha_3 \alpha_4 \beta_1 \beta_2 \beta_3 \beta_4 \) | \( \alpha_1 \alpha_2 \alpha_3 \alpha_4 \beta_1 \beta_2 \beta_3 \beta_4 \) |
| (9)   | 1 0 0 0 0 0 0 0  \( \alpha_1 \) | -1 0 0 0 0 0 0 0 \( \tilde{\alpha}_1 \) |
|       | 0 1 -1 0 0 0 0 0 \( \alpha_2 \) | -1 1 0 0 0 0 0 0 \( \alpha_2 \) |
|       | 0 0 -1 0 0 0 0 0 \( \tilde{\alpha}_3 \) | 0 0 1 0 0 0 0 \( \alpha_3 \) |
|       | 0 0 0 1 0 0 0 \( \alpha_4 \) | 0 0 0 1 0 0 0 \( \alpha_4 \) |
|       | 0 0 0 0 1 0 0 \( \beta_1 \) | -1 0 0 0 1 0 0 \( \beta_1 \) |
|       | 0 0 -1 0 0 1 0 \( \beta_2 \) | 0 0 0 0 0 1 0 \( \beta_2 \) |
|       | 0 0 -1 0 0 0 1 \( \beta_3 \) | -1 0 0 0 0 0 \( \beta_3 \) |
|       | 0 0 0 0 0 0 1 \( \beta_4 \) | 0 0 0 0 0 0 1 \( \beta_4 \) |

|       | \( \{E_s(a_3), E_s(a_2)\} \) | \( \{E_s(a_4), E_s(a_1)\} \) |
|-------|--------------------------|--------------------------|
| (11)  | \( \tilde{\alpha}_1 \alpha_2 \alpha_3 \alpha_4 \beta_1 \beta_2 \beta_3 \beta_4 \) | \( \alpha_1 \alpha_2 \tilde{\alpha}_3 \alpha_4 \beta_1 \beta_2 \beta_3 \beta_4 \) |
|       | 1 0 0 -3 0 0 0 0 \( \tilde{\alpha}_1 \) | 1 0 0 -1 0 0 0 0 \( \alpha_1 \) |
|       | 0 1 0 0 0 0 0 0 \( \alpha_2 \) | 0 1 0 -1 0 0 0 0 \( \alpha_2 \) |
|       | 0 0 1 -1 0 0 0 0 \( \alpha_3 \) | 0 0 1 0 0 0 0 \( \alpha_3 \) |
|       | 0 0 0 -1 0 0 0 \( \tilde{\alpha}_4 \) | 0 0 0 -1 0 0 0 \( \tilde{\alpha}_4 \) |
|       | 0 0 0 0 1 0 0 \( \beta_1 \) | 0 0 0 -1 1 0 0 \( \beta_1 \) |
|       | 0 0 0 -1 0 1 0 \( \beta_2 \) | 0 0 0 0 0 1 \( \beta_2 \) |
|       | 0 0 0 -1 0 0 1 \( \beta_3 \) | 0 0 0 0 1 0 \( \beta_3 \) |
|       | 0 0 0 -1 0 0 0 \( \beta_4 \) | 0 0 0 1 0 \( \beta_4 \) |

### Table B.9. Transition matrices, Cases (13)-(16)

|       | \( \{E_s(a_5), E_s(a_4)\} \) | \( \{E_s(a_6), E_s(a_1)\} \) |
|-------|--------------------------|--------------------------|
| (13)  | \( \alpha_1 \alpha_2 \tilde{\alpha}_3 \tilde{\alpha}_4 \beta_1 \beta_2 \beta_3 \beta_4 \) | \( \alpha_1 \alpha_2 \tilde{\alpha}_3 \tilde{\alpha}_4 \beta_1 \beta_2 \beta_3 \beta_4 \) |
|       | 1 0 0 0 0 0 0 0 \( \tilde{\alpha}_1 \) | 1 0 0 0 0 0 0 0 \( \tilde{\alpha}_1 \) |
|       | 0 1 0 0 0 0 0 -1 \( \tilde{\alpha}_2 \) | 0 -1 0 0 0 0 \( \tilde{\alpha}_2 \) |
|       | 0 0 1 0 0 0 0 \( \tilde{\alpha}_3 \) | 0 0 1 0 0 0 \( \tilde{\alpha}_3 \) |
|       | 0 0 0 1 0 0 0 \( \tilde{\alpha}_4 \) | 0 0 0 1 0 \( \tilde{\alpha}_4 \) |
|       | 0 0 0 0 1 0 0 \( \beta_1 \) | 0 0 0 0 1 \( \beta_1 \) |
|       | 0 0 0 0 0 1 0 \( \beta_2 \) | 0 0 0 0 0 \( \beta_2 \) |
|       | 0 0 0 0 0 0 1 \( \beta_3 \) | 0 0 0 0 0 \( \beta_3 \) |
|       | 0 0 0 0 0 0 0 \( \beta_4 \) | 0 0 0 0 0 \( \beta_4 \) |

|       | \( \{E_s(a_7), E_s(a_5)\} \) | \( \{E_s(a_8), E_s(a_7)\} \) |
|-------|--------------------------|--------------------------|
| (15)  | \( \alpha_1 \alpha_2 \tilde{\alpha}_3 \tilde{\alpha}_4 \beta_1 \beta_2 \beta_3 \beta_4 \) | \( \alpha_1 \alpha_2 \tilde{\alpha}_3 \tilde{\alpha}_4 \beta_1 \beta_2 \beta_3 \beta_4 \) |
|       | -1 0 0 0 0 0 0 0 \( \tilde{\alpha}_1 \) | 1 0 0 0 0 0 0 0 \( \tilde{\alpha}_1 \) |
|       | -1 1 0 0 0 0 0 \( \tilde{\alpha}_2 \) | 0 1 0 0 0 0 \( \tilde{\alpha}_2 \) |
|       | -1 0 1 0 0 0 0 \( \tilde{\alpha}_3 \) | 0 0 1 0 0 0 \( \tilde{\alpha}_3 \) |
|       | 0 0 0 1 0 0 0 \( \tilde{\alpha}_4 \) | 0 0 0 1 0 \( \tilde{\alpha}_4 \) |
|       | -2 0 0 0 1 0 0 \( \beta_1 \) | 0 0 0 0 1 \( \beta_1 \) |
|       | 0 0 0 0 0 1 0 \( \beta_2 \) | 0 0 0 0 1 \( \beta_2 \) |
|       | 0 0 0 0 0 1 0 \( \beta_3 \) | 0 0 0 0 0 \( \beta_3 \) |
|       | 0 0 0 0 0 0 1 \( \beta_4 \) | 0 0 0 0 0 \( \beta_4 \) |
C. Conjectures regarding the Dynkin-Minchenko completion procedure. The numerical relation in Observation 1.1 motivates to the following assumption:

**Conjecture C.1.** There is a correspondence between Carter diagrams (with cycles) and extra nodes in the Dynkin-Minchenko completion procedure.

One more conjecture on relationship between enhanced Dynkin diagrams and Carter diagrams is as follows:

**Conjecture C.2.** If \{\tilde{\Gamma}, \Gamma\} is a homogeneous pair of Carter diagrams, then the signed enhanced Dynkin diagrams associated with \tilde{\Gamma} and \Gamma coincide up to similarities.

\[ \Delta(\tilde{\Gamma}) = \Delta(\Gamma). \]

**Remark C.3.** Conjecture C.2 is easily verified for Carter diagrams \( E_6(a_1) \) and \( E_6(a_2) \):

**Case** \( E_6(a_1) \). Extra nodes \( m_1 \) and \( m_2 \) are as follows:

\[ m_1 = 2\beta_1 + \alpha_1 + \alpha_2 + \alpha_3, \quad m_2 = 2\alpha_2 + \beta_1 + \beta_2 - \beta_3. \]

**Figure C.23.** Enhanced Dynkin diagram \( \Delta(E_6(a_1)) \)

Orthogonality relations are as follows:

\[ m_1 \perp \alpha_1, \alpha_2, \alpha_3, \tilde{\beta}_3, \quad (m_1, \beta_2) = (\alpha_2, \beta_2) = -1, \]

\[ (m_1, m_2) = (m_1, \beta_1 + \beta_2) = 1 - 1 = 0. \]

**Case** \( E_6(a_2) \). Here, extra nodes \( m_1 \) and \( m_2 \) are as follows:

\[ m_1 = 2\alpha_3 + \tilde{\beta}_3 + \tilde{\beta}_2 + \beta_1, \quad m_2 = 2\beta_1 + \alpha_1 + \alpha_2 + \alpha_3. \]

**Figure C.24.** Enhanced Dynkin diagram \( \Delta(E_6(a_2)) \)

Orthogonality relations:

\[ m_1 \perp \tilde{\beta}_3, \tilde{\beta}_2, \beta_1, \alpha_1, \alpha_2, \quad (m_1, m_2) = (m_1, \alpha_3) = 1. \]

\[ \square \]
Further, according to Vavilov-Mingin, see [VM21, Theorem 1],
\[ E_7(a_i) \subset \Delta(E_7) \text{ and } E_8(a_i) \subset \Delta(E_8), \quad 1 \leq i \leq 8, \]
therefore
\[ \Delta(E_7(a_i)) \subset \Delta(E_7) \text{ and } \Delta(E_8(a_i)) \subset \Delta(E_8). \]
Thus, to prove Conjecture C.2, it suffices to check only the reverse inclusions:
\[ E_7 \subset \Delta(E_7(a_i)) \text{ and } E_8 \subset \Delta(E_8(a_i)). \]

**C.2. Adjacency, complexity and eigenvalues.** Let us define the complexity of the Carter diagram as \( N_c + Ke \), where \( N \) is the number of cycles and \( K \) is the number of endpoints. Assume, that one cycle contributes to complexity as two endpoints. One can select another proportion. In Table C.10, Carter diagrams located side by side are the pairs from the adjacency list (1.2). The Carter diagrams from the adjacency list can be transformed to each other using the transition matrix \( M_I \) constructed in Theorem 4.1, see Section 4.2.1. Denote by \( \text{Max-E} \) the maximal eigenvalue of a partial Cartan matrix.

**Table C.10. Two chains arranged in ascending order of \( \text{Max-E} \).**

|        | \( E_8(a_8) \) | \( E_8(a_7) \) | \( E_8(a_2) \) | \( E_8(a_4) \) | \( E_8(a_1) \) | \( E_8 \) | \( E_8(a_2) \) | \( E_8(a_3) \) |
|--------|----------------|----------------|----------------|----------------|----------------|---------|----------------|----------------|
| \( N_c + Ke \) | 6c             | 3c + 1e        | 2c + 2e        | 2c + 1e        | 1c + 2e        | 3e      | 1c + 3e        | 1c + 4e        |
| \( 2N + K \) | 12             | 7              | 6              | 5              | 4              | 3       | 5              | 6              |
| \( \text{Max-E} \) | 3.73           | 3.93           | 3.956          | 3.969          | 3.982          | 3.989   | 3.975          | 3.93           |

In Table C.10 there are two chains which are arranged in ascending order of the maximal eigenvalues of the partial Cartan matrices and in descending order of the complexity parameter, see (C.1).

\[ E_8(a_8) \rightarrow E_8(a_7) \rightarrow E_8(a_5) \rightarrow E_8(a_4) \rightarrow E_8(a_1) \rightarrow E_8, \]
\[ E_8(a_3) \rightarrow E_8(a_2) \rightarrow E_8. \]  

(C.1)

The arrows in (C.1) point in the direction of increasing of maximal eigenvalue. It is not so clear the place of \( E_8(a_6) \) in the homogeneous class \( \{ E, 8 \} \):

- The complexity parameter for the diagram \( E_8(a_6) \) is equal to 3c.
- By alternative transition matrices from Section 4.2.2 the possible adjacent diagram for \( E_8(a_6) \) can be \( E_8(a_5) \), \( E_8(a_4) \), \( E_8(a_1) \).
- The maximum eigenvalue of the partial Cartan matrix for \( E_8(a_6) \) is 3.902.

Based on these three factors, \( E_8(a_6) \) can be placed, for example, between \( E_8(a_5) \) and \( E_8(a_4) \), or in the separated pair \( \{ E_8(a_5), E_8(a_6) \} \).
C.3. C. M. Ringel: **Invariants with value** 2, 4, 8. In the survey article [Rin16], C. M. Ringel provided several notes regarding representations of Dynkin quivers. As Ringel writes: “they shed some new light on properties of Dynkin and Euclidean quivers”. The following is Ringel’s 2−4−8 assertion regarding Dynkin quivers $E_n$, $n = 6, 7, 8$.

Let $\Gamma$ be the extended Dynkin diagram (=Euclidean quiver) for the Dynkin diagram $\Gamma$. If $\Gamma$ is constructed from $\Gamma$ by adding the new edge to the vertex $y$ of $\Gamma$, then $y$ is said to be the exceptional vertex. For the Dynkin quivers $E_n$, $n = 6, 7, 8$, let $x$ be the neighbor of $y$ and $\Gamma' = \Gamma \setminus \{y\}$, see Table C.11.

**Table C.11.** Ringel 2−4−8 assertion

| $\Gamma$ | $\Gamma'$ |
|---------|-----------|
| $E_6$   | $A_5$     |
| $E_7$   | $D_6$     |
| $E_8$   | $E_7$     |

Let $P'(x)$ (resp. $I'(x)$) be the indecomposable projective (resp. injective) representation of $\Gamma'$ corresponding to vertex $x$. Let $\tau'$ be the Auslander-Reiten translation in the category of finite-dimensional representations reps$(\Gamma)$. Then $P'(x)$ and $I'(x)$ belong to the same $\tau'$-orbit and there is the following 2−4−8 assertion: 

$$P'(x) = (\tau')^n I'(x),$$

where $n$ is the **Ringel invariant** given in the Table C.11 see [Rin16], Part 3.

C.4. **B. Rosenfeld: Isometry groups of the projective planes.** J. Baez in [Ba01] points out another connection between the invariants 2, 4, 8 and the diagrams $E_6$, $E_7$ and $E_8$. This connection was discovered by Rosenfeld, see [Ro97]: “... Boris Rosenfeld had the remarkable idea... that the exceptional Lie groups $E_6$, $E_7$ and $E_8$ may be considered as the isometry groups of the projective planes over the following 3 algebras, respectively:”

- the bioctonions $\mathbb{C} \otimes \mathbb{O}$,
- the quateroctonions $\mathbb{H} \otimes \mathbb{O}$,
- the octoctonions $\mathbb{O} \otimes \mathbb{O}$.

Any real finite-dimensional division algebra over the reals must be only one of these: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$\(^\dagger\). The real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$ are division algebras of dimensions, respectively: 1, 2, 4, or 8.

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