THE NUMBER OF EIGENVALUES FOR AN HAMILTONIAN IN FOCK SPACE

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ABSTRACT. A model operator $H$ corresponding to the energy operator of a system with non-conserved number $n \leq 3$ of particles is considered. The precise location and structure of the essential spectrum of $H$ is described. The existence of infinitely many eigenvalues below the bottom of the essential spectrum of $H$ is proved if the generalized Friedrichs model has a virtual level at the bottom of the essential spectrum and for the number $N(z)$ of eigenvalues below $z < 0$ an asymptotics established. The finiteness of eigenvalues of $H$ below the bottom of the essential spectrum is proved if the generalized Friedrichs model has a zero eigenvalue at the bottom of its essential spectrum.

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1. INTRODUCTION

One of the remarkable results in the spectral analysis for continuous three-particle Schrödinger operators is the Efimov effect: if none of the three two-particle Schrödinger operators (corresponding to the two particle subsystem) has negative eigenvalues, but at least two of them have a zero-energy resonance, then this three-particle Schrödinger operator has an infinite number of discrete eigenvalues, accumulating at zero.

Since its discovery by Efimov in 1970 \cite{Efimov}, many works are devoted to this subject. See, for example, \cite{Sobolev, Yafaev, Efimov, Faddeev, Newton}. In particular, Yafaev gave a mathematically rigorous proof for the existence of such a phenomenon \cite{Yafaev}.

The main result obtained by Sobolev \cite{Sobolev} (see also \cite{Yafaev}) is the asymptotics of the form $U_0|\log|\lambda||$ for the number of eigenvalues on the left of $\lambda, \lambda < 0$, where the coefficient $U_0$ does not depend on the potentials $v_\alpha$ and is a positive function of the ratios $m_1/m_2, m_2/m_3$ of the masses of the three-particles.

Recently the existence of the Efimov effect for $N$-body quantum systems with $N \geq 4$ has been proved by X.P.Wang in \cite{Wang}.

In fact in \cite{Wang} a lower bound on the number of eigenvalues of the form

$$C_0|\log(E_0 - \lambda)|$$

is given, when $\lambda$ tends to $E_0$, where $C_0$ is a positive constant and $E_0$ is the bottom of the essential spectrum.

The presence of Efimov’s effect for the lattice three-particle Schrödinger operators has been proved (see, e.g., \cite{Lakaev, Rasulov, Efimov, Faddeev, Newton, Wang} for relevant discussions and \cite{Sobolev, Yafaev, Efimov, Faddeev, Newton, Wang, Wang} for the general study of the low-lying excitation spectrum for quantum systems on lattices).

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The systems considered in above mentioned works have a fixed number of quasi-particles. In the theory of solid-state physics [24] and the theory of quantum fields [12] interesting problems arise where the number of quasi-particles is bounded, but not fixed. At the same time the study of systems with a non conserved bounded number of particles is reduced to the study of the spectral properties of self-adjoint operators acting in "the cut" subspace \( \mathcal{H}_n \) of Fock's space, consisting of \( r \leq n \) particles [12, 23, 24, 35].

The perturbation problem of an operator, with point and continuous spectrum (which acts in \( \mathcal{H}(2) \)) has played a considerable role in the discussions about spectral problems connected with of the quantum theory of fields (see [12]).

The main goal of the present paper is to prove the existence of infinitely many eigenvalues below the essential spectrum for a model operator \( H \) corresponding to the energy operators of systems of three non conserved particles acting in the subspace \( \mathcal{H}_3 \).

More precisely, under some technical smoothness assumptions upon the family of the generalized Friedrichs model \( h(p), p \in \mathbb{T}^3 = (-\pi, \pi]^3 \) (see [12, 15]) we obtain the following results:

(i) We describe precisely the location and structure of the essential spectrum of \( H \) by the spectrum of the generalized Friedrichs model.

(ii) We prove that the operator \( H \) has infinitely many eigenvalues below the bottom of the essential spectrum, if the operator \( h(0) \) has a zero-energy resonance at the bottom of its essential spectrum.

(iii) In the case (ii) we establish the following asymptotic formula for the number \( N(z) \) of eigenvalues of \( H \) lying below \( z < 0 \)

\[
\lim_{z \to -0} \frac{N(z)}{|\log |z||} = \mathcal{U}_0 (0 < \mathcal{U}_0 < \infty).
\]

(iv) We prove the finiteness of eigenvalues of \( H \), if \( h(0) \) has a zero eigenvalue at the bottom of its essential spectrum.

We remark that the presence of a zero-energy resonance for the Schrödinger operators is due to the two-particle interaction operators \( V \), in particular, the coupling constant (if \( V \) has in front of it acoupling constant) (see, e.g., [1, 16, 26, 33]).

In our case it is remarkable that the presence of a zero-energy resonance at the bottom of the essential spectrum of generalized Friedrichs model (consequently the existence of the infinitely many eigenvalues of \( H \)) is due to the annihilation and creation operators acting in the symmetric Fock space.

We also notice that the assertion (iv) is also surprising and similar assertions have not been proved for the three-particle Schrödinger operators on \( \mathbb{R}^3 \) or \( \mathbb{Z}^3 \).

We remark that the operator \( H \) has been considered before, but the existence of infinitely many eigenvalues below the bottom of the essential spectrum of \( H \) has only been announced in [19] and in the location of the essential spectrum of \( H \) has been established in [20].

The organization of present paper is as follows. Section 1 is an introduction to the whole work. In section 2 the model operator is described as a bounded and self-adjoint operator \( H \) in the subspace \( \mathcal{H}_3 \) and the main results of the present paper are formulated. In Section 3, we study some spectral properties of \( h(p), p \in \mathbb{T}^3 \). In section 4 we describe the essential spectrum of \( H \). In Section 5, we reduce the eigenvalue problem to the Birman-Schwinger principle. In section 6 we prove the part (i) of Theorem 2.9. In section 7, we give an asymptotic formula for the number of eigenvalues. Some technical material is collected in Appendix A.
Throughout the present paper we adopt the following conventions: For each $\delta > 0$ the notation $U_\delta(0) = \{p \in \mathbb{T}^3 : |p| < \delta\}$ stands for a $\delta$-neighborhood of the origin.

Denote by $\mathbb{T}^3$ the three-dimensional torus, the cube $(-\pi, \pi)^3$ with appropriately identified sides. Throughout the paper the torus $\mathbb{T}^3$ will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on $\mathbb{R}^3$ modulo $(2\pi \mathbb{Z})^3$.

2. Three particle model operator and statement of results

Let us introduce some notations used in this work. Let $C^1$ be one-dimensional complex space and let $L_2(\mathbb{T}^3)$ be the Hilbert space of square-integrable functions defined on $\mathbb{T}^3$ and $L_2^s((\mathbb{T}^3)^2)$ be the Hilbert space of square-integrable symmetric functions on $(\mathbb{T}^3)^2$.

Denote by $\mathcal{K}$ a direct sum of spaces $\mathcal{K}_0 = C^1$, $\mathcal{K}_1 = L_2(\mathbb{T}^3)$, $\mathcal{K}_2 = L_2^s((\mathbb{T}^3)^2)$, that is, $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$.

Let $H$ be the operator in $\mathcal{K}$ with the entries $H_{ij} : \mathcal{K}_j \rightarrow \mathcal{K}_i$, $i,j = 0, 1, 2$:

$$(H_{00}f_0)_0 = u_0f_0, \quad (H_{01}f_1)_0 = \int_{\mathbb{T}^3} v(q')f_1(q')dq', \quad H_{02} = 0,$$

$$H_{10} = H_{01}, \quad (H_{11}f_1)_1(p) = u(p)f_1(p), \quad (H_{12}f_2)_1(p) = \int_{\mathbb{T}^3} v(q')f_2(p, q')dq',$$

$$H_{20} = 0, \quad H_{21} = H_{12}, \quad (H_{22}f_2)_2(p, q) = w(p, q)f_2(p, q),$$

where $H^*_j : \mathcal{K}_i \rightarrow \mathcal{K}_j$, $j = i + 1$, $i = 0, 1$ is the adjoint operator of $H_{ij}$.

Here $f_i \in \mathcal{K}_i$, $i = 0, 1, 2$, $u_0$ is a real number, $v(p)$ and $u(p)$ are real-analytic functions on $\mathbb{T}^3$ and $w(p, q)$ is a real-analytic symmetric function defined on $(\mathbb{T}^3)^2$.

Under these assumptions the operator $H$ is bounded and self-adjoint in $\mathcal{K}$.

We remark that, $H_{10}$ and $H_{21}$ resp. $H_{01}$ and $H_{12}$ in the Fock space are called creation resp. annihilation operators.

Throughout this paper we assume the following additional technical assumptions.

**Assumption 2.1.** The real-analytic function $w(p, q)$ which is symmetric on $(\mathbb{T}^3)^2$, is even with respect to $(p, q)$, has a unique non-degenerate zero minimum at the point $(0, 0) \in (\mathbb{T}^3)^2$ and there exist positive definite matrix $W$ and real numbers $l_1, l_2$ ($l_1 > 0, l_2 \neq 0$) such that

$$\left(\frac{\partial^2 w(0, 0)}{\partial p_i \partial p_j}\right)_{i,j=1}^3 = l_1W, \quad \left(\frac{\partial^2 w(0, 0)}{\partial p_i \partial q_j}\right)_{i,j=1}^3 = l_2W.$$  

**Assumption 2.2.** The real-analytic functions $u(p)$ and $v(p)$ on $\mathbb{T}^3$ are even and the function $u(p)$ has a unique non-degenerate minimum at $0 \in \mathbb{T}^3$.

By Assumptions 2.1 and 2.2 for any $p \in \mathbb{T}^3$ the integral

$$\int_{\mathbb{T}^3} \frac{v^2(t)dt}{w(p, t)}$$

is finite and hence we can define continuous function on $\mathbb{T}^3$, which will be denotes $\Lambda(p)$.

Since the function $w(p, q)$ has a unique non degenerate minimum at the point $(0, 0) \in (\mathbb{T}^3)^2$ and $w(0, 0) = 0$ the function $\Lambda(p)$ is positive. In particular, if $v(0) = 0$ then $\Lambda(p)$ is a twice continuously differentiable function at the point $p = 0$ (see proof of Lemma A.1).

**Assumption 2.3.** (i) For any nonzero $p \in \mathbb{T}^3$ the inequality $\Lambda(p) < \Lambda(0)$ holds.

(ii) If $v(0) = 0$, then $\Lambda(p)$ has a non-degenerate maximum at $p = 0$.  

Remark 2.4. Let
\[ u(p) = \varepsilon(p) + c, \quad v(p) = \varepsilon(p), \quad w(p, q) = \varepsilon(p) + \varepsilon(p + q) + \varepsilon(q), \]
where \( c > 0 \) is a real number and
\[ \varepsilon(p) = 3 - \cos p_1 - \cos p_2 - \cos p_3, \quad p = (p_1, p_2, p_3) \in \mathbb{T}^3. \tag{2.1} \]
Then Assumptions 2.1, 2.2 and 2.3 are fulfilled (see Lemma A.1 below).

To formulate the main results we introduce a family of the generalized Friedrichs model \( h(p), p \in \mathbb{T}^3 \), which acts in \( C^1 \oplus L_2(\mathbb{T}^3) \) with the entries
\[ (h_{00}(p)f_0)_0 = u(p)f_0, \quad h_{01} = \frac{1}{\sqrt{2}} H_{01}, \tag{2.2} \]
where \( u_p(q) = w(p, q) \).

Let \( C \) be the field of complex numbers.

The proof of the following variant of the Birman-Schwinger principle for the family \( h(p), p \in \mathbb{T}^3 \) is customary.

Proposition 2.5. For any \( p \in \mathbb{T}^3 \) the number \( z \in C \setminus [m_w(p), M_w(p)] \) is an eigenvalue of \( h(p), p \in \mathbb{T}^3 \) if and only if the number 1 is an eigenvalue of the integral operator given by
\[ (G(p, z)\psi)(q) = \frac{v(q)}{2(u(p) - z)} \int_{\mathbb{T}^3} \frac{v(t)\psi(t)dt}{w_p(t) - z}, \quad \psi \in L_2(\mathbb{T}^3), \]
where the numbers \( m_w(p) \) and \( M_w(p) \) are defined by
\[ m_w(p) = \min_{q \in \mathbb{T}^3} w(p, q) \quad \text{and} \quad M_w(p) = \max_{q \in \mathbb{T}^3} w(p, q). \]

Let \( C(\mathbb{T}^3) \) be the Banach space of continuous functions on \( \mathbb{T}^3 \).

Assumption 2.6. For any \( p \in \mathbb{T}^3 \) and \( \psi \in C(\mathbb{T}^3) \) the integral
\[ \int_{\mathbb{T}^3} \frac{v(t)\psi(t)dt}{w_p(t) - m_w(p)} \]
is finite.

Remark 2.7. Let \( v \) be an arbitrary analytic function on \( \mathbb{T}^3 \) and
\[ w_p(q) = l_1 \varepsilon(p) + l_2 \varepsilon(p + q) + l_1 \varepsilon(q), \]
where the function \( \varepsilon(p) \) is defined by 2.1 and \( l_1, l_2 > 0, \ l_1 \neq l_2 \).

Then Assumption 2.6 is fulfilled (see Lemma A.2 below).

It should be noted that under Assumption 2.6 for any \( p \in \mathbb{T}^3 \) the operator \( G(p, z) \) can be defined as a bounded operator on \( C(\mathbb{T}^3) \) even for \( z = m_w(p) \).

Definition 2.8. Let Assumption 2.6 be fulfilled. The operator \( h(p), p \in \mathbb{T}^3 \), is said to have a resonance at the point \( z = m_w(p) \) if the number 1 is an eigenvalue of the integral operator given by
\[ (G(p, m_w(p))\psi)(q) = \frac{v(q)}{2(u(p) - m_w(p))} \int_{\mathbb{T}^3} \frac{v(t)\psi(t)dt}{w_p(t) - m_w(p)}, \quad \psi \in C(\mathbb{T}^3), \]
and \( v(p) \neq 0 \) (provided \( u(p) - m_w(p) \neq 0 \)).
Let us denote by $\tau_{ess}(H)$ the bottom of the essential spectrum of the operator $H$ and by $N(z)$ the number of eigenvalues of $H$ lying below $z \leq \tau_{ess}(H)$.

The main result of this work is the following.

**Theorem 2.9.** (i) Assume Assumptions 2.1, 2.2 and 2.3 are fulfilled and let the operator $h(0)$ have a zero eigenvalue. Then the operator $H$ has a finite number of negative eigenvalues.

(ii) Assume Assumptions 2.1, 2.2 and part (i) of Assumption 2.3 are fulfilled and the operator $h(0)$ has a zero-energy resonance. Then the operator $H$ has infinitely many negative eigenvalues accumulating at $\tau_{ess}(H) = 0$ and the function $N(z)$ obeys the relation

$$\lim_{z \to -0} \frac{N(z)}{|\log |z||} = U_0 (0 < U_0 < \infty). \quad (2.3)$$

**Remark 2.10.** The constant $U_0$ does not depend on the function $v$ and is given as a positive function depending only on the ratio $l_1/l_2$ (with $l_1, l_2$ as in Assumption 2.1).

**Remark 2.11.** A zero-energy resonance (resp. a zero eigenvalue) of $h(0)$, if it does exist, is simple (see Lemma 3.4 below).

**Remark 2.12.** We remark that the assumptions for the functions $u, v$ and $w$ are far from the precise, but we will not develop this point here.

### 3. Spectral properties of the operators $h(p), p \in \mathbb{T}^3$

In this section we study some spectral properties of the family of generalized Friedrichs model $h(p), p \in \mathbb{T}^3$ given by (2.2), which plays a crucial role in the study of the spectral properties of $H$. We notice that the spectrum and resonances of the generalized Friedrichs model have been studied in detail in [4, 15].

Let the operator $h_0(p)$ act in $C^1 \oplus L_2(\mathbb{T}^3)$ as

$$h_0(p) \begin{pmatrix} f_0 \\ f_1(q) \end{pmatrix} = \begin{pmatrix} u(p)f_0 \\ w_p(q)f_1(q) \end{pmatrix}.$$  

The perturbation $h(p) - h_0(p)$ of the operator $h_0(p)$ is a two-dimensional bounded self-adjoint operator. Therefore in accordance with the invariance of the absolutely continuous spectrum under the trace class perturbations the absolutely continuous spectrum of $h(p)$ fills the following interval on the real axis:

$$\sigma_{ac}(h(p)) = [m_w(p), M_w(p)].$$

For any $p \in \mathbb{T}^3$ and $z \in \mathbb{C} \setminus \sigma_{ac}(h(p))$ we define the function

$$\Delta(p, z) = u(p) - z - \frac{1}{2} \int_{\mathbb{T}^3} \frac{v^2(t)dt}{w_p(t) - z}.$$  

Note that $\Delta(p, z)$ is real-analytic in $\mathbb{T}^3 \times (\mathbb{C} \setminus \sigma_{ac}(h(p)))$.

**Lemma 3.1.** For all $p \in \mathbb{T}^3$ the operator $h(p)$ has an eigenvalue $z \in \mathbb{C} \setminus \sigma_{ac}(h(p))$ outside of the absolutely continuous spectrum if and only if $\Delta(p, z) = 0$. 
Proof. The number $z \in \mathbb{C} \setminus \sigma_{ac}(h(p))$ is an eigenvalue of $h(p)$ if and only if (by the Proposition 2.5) $\lambda = 1$ is an eigenvalue of the operator $G(p, z)$. According to Fredholm’s theorem the number $\lambda = 1$ is an eigenvalue for the operator $G(p, z)$ if and only if
\begin{equation*}
u(p) - z - \frac{1}{2} \int_{T^3} \frac{v^2(t)dt}{w_p(t) - z} = 0, \quad \text{that is,} \quad \Delta(p, z) = 0.
\end{equation*}

Since the function $\Delta(0, \cdot)$ is decreasing on $(-\infty, 0)$ and the function $w_0(q)$ has a unique non-degenerate minimum at $q = 0$ (see Lemma A.3) by dominated convergence the finite limit
\begin{equation*}
\Delta(0, 0) = \lim_{z \to -0} \Delta(0, z)
\end{equation*}
exists.

Lemma 3.2. Let Assumption 2.1 be fulfilled. The operator $h(0)$ has a zero energy resonance if and only if $\Delta(0, 0) = 0$ and $v(0) \neq 0$.

Proof. ”Only If Part”. Let the operator $h(0)$ have a zero energy resonance. Then by Definition 2.8 the equation
\begin{equation*}
\psi(q) = v(q) \int_{T^3} \frac{v(t)\psi(t)dt}{w_0(t)}, \quad \psi \in C(T^3),
\end{equation*}
has a simple solution $\psi \in C(T^3)$.

One can check that this solution is equal to the function $v(q)$ (up to a constant factor). Therefore we see that
\begin{equation*}
v(q) = \frac{v(q)}{2u(0)} \int_{T^3} \frac{v^2(t)dt}{w_0(t)}
\end{equation*}
and hence
\begin{equation*}
\Delta(0, 0) = u(0) - \frac{1}{2} \int_{T^3} \frac{v^2(t)dt}{w_0(t)} = 0.
\end{equation*}

”If Part”. Let the equality $\Delta(0, 0) = 0$ hold and $v(0) \neq 0$. Then only the function $v(q) \in C(T^3)$ is a solution of the equation
\begin{equation*}
\psi(q) = \frac{v(q)}{2u(0)} \int_{T^3} \frac{v(t)\psi(t)dt}{w_0(t)},
\end{equation*}
that is, the operator $h(0)$ has a zero energy resonance. \hfill \Box

Lemma 3.3. Let Assumption 2.1 be fulfilled. The operator $h(0)$ has a zero eigenvalue if and only if $\Delta(0, 0) = 0$ and $v(0) = 0$.

Proof. ”Only If Part”. Suppose $f = (f_0, f_1)$, is an eigenvector of the operator $h(0)$ associated with the zero eigenvalue. Then $f_0, f_1(q)$ satisfy the system of equations
\begin{equation}
\begin{cases}
u(0)f_0 + \frac{1}{\sqrt{2}} \int_{T^3} v(q')f_1(q')dq' = 0 \\
\frac{1}{\sqrt{2}}v(q)f_0 + w_0(q)f_1(q) = 0.
\end{cases}
\end{equation}

From (3.1) we find that $f_0$ and $f_1(q)$, except for an arbitrary factor, are given by
\begin{equation}
f_0 = 1, \quad f_1(q) = -\frac{v(q)}{\sqrt{2}w_0(q)},
\end{equation}

(3.2)
and from the first equation of (3.1) we derive the equality
\[ \Delta(0, 0) = 0. \]

Since the functions \( w_0(q) \) and \( v(q) \) are analytic on \( T^3 \) and the function \( w_0(q) \) has a unique non-degenerate minimum at the origin we can conclude that \( f_1 \in L_2(T^3) \) if and only if \( v(0) = 0 \).

"If Part". Let \( v(0) = 0 \) and \( \Delta(0, 0) = 0 \) then the vector \( f = (f_0, f_1) \), where
\[ f_0 = const \neq 0, \quad f_1(q) = -\frac{v(q)}{2u(0)w_0(q)} f_0, \]
obyes the equation
\[ h(0)f = 0 \]
and
\[ f_1 \in L_2(T^3). \]
\[ \square \]

Lemma 3.4. Let Assumption 2.1 be fulfilled and the operator \( h(0) \) have a zero-energy resonance (resp. zero eigenvalue). Then the vector \( f = (f_0, f_1) \), where \( f_0 \) and \( f_1 \) are given by (3.2), is the unique solution (up to a constant factor) of the equation \( h(0)f = 0 \) and \( f_1 \in L_1(T^3) \setminus L_2(T^3) \) (resp. \( f_1 \in L_2(T^3) \)).

Proof. Let the operator \( h(0) \) have a zero-energy resonance (resp. a zero eigenvalue). Then by Lemma 3.2 (resp. Lemma 3.3) we have \( v(0) \neq 0 \) (resp. \( v(0) = 0 \)) and \( \Delta(0, 0) = 0 \).

One can check that \( f = (f_0, f_1) \) obeys the equation \( h(0)f = 0 \) or the system of equations (3.1).

Since the functions \( w_0(q) \) and \( v(q) \) are analytic on \( T^3 \) and the function \( w_0(q) \) has a unique non-degenerate minimum at the origin we can conclude that \( f_1 \in L_1(T^3) \setminus L_2(T^3) \) (resp. \( f_1 \in L_2(T^3) \)) if and only if \( v(0) \neq 0 \) (resp. \( v(0) = 0 \)).

From the representation (3.2) of \( f_0 \) and \( f_1 \) it follows that the subspace generated by the vector \( f = (f_0, f_1) \) is one dimensional. \( \square \)

Lemma 3.5. Let Assumption 2.1 be fulfilled.

(i) Let \( \max_{p \in T^3} \Delta(p, 0) < 0 \). Then for any \( p \in T^3 \) the operator \( h(p) \) has a unique negative eigenvalue.

(ii) Let \( \min_{p \in T^3} \Delta(p, 0) < 0 \) and \( \max_{p \in T^3} \Delta(p, 0) \geq 0 \). Then there exists a non void open set \( D \subset T^3 \) such that for any \( p \in D \) the operator \( h(p) \) has a unique negative eigenvalue and for \( p \in T^3 \setminus D \) the operator \( h(p) \) has no negative eigenvalues.

(iii) Let \( \min_{p \in T^3} \Delta(p, 0) \geq 0 \). Then for any \( p \in T^3 \) the operator \( h(p) \) has no negative eigenvalues.

Proof. (i) Let \( \max_{p \in T^3} \Delta(p, 0) < 0 \). Since \( T^3 \) is a compact set and the function \( \Delta(p, 0) \) is continuous on \( T^3 \) for all \( p \in T^3 \) we have the inequality
\[ \Delta(p, 0) < 0. \]

For any \( p \in T^3 \) the function \( \Delta(p, \cdot) \) is continuous and decreasing on \( (-\infty, 0] \) and
\[ \lim_{z \to -\infty} \Delta(p, z) = +\infty. \]
Then there exist a unique point \( z(p) \in (−\infty, 0) \) such that \( \Delta(p, z(p)) = 0 \). Hence by Lemma 3.1 for any \( p \in T^3 \) the point \( z(p) \) is the unique negative eigenvalue of \( h(p) \), \( p \in T^3 \).

(ii) Let \( \min_{p \in T^3} \Delta(p, 0) < 0 \) and \( \max_{p \in T^3} \Delta(p, 0) \geq 0 \).

Let us introduce the notation
\[
D = \{ p \in T^3 : \Delta(p, 0) < 0 \}.
\]

Since the function \( \Delta(p, 0) \) is continuous on \( T^3 \) and \( T^3 \) is compact there exist points \( p_0, p_1 \in T^3 \) such that
\[
\min_{p \in T^3} \Delta(p, 0) = \Delta(p_0, 0) < 0, \quad \max_{p \in T^3} \Delta(p, 0) = \Delta(p_1, 0) \geq 0
\]
and we have that \( D \neq T^3 \) is a non void open set.

If \( p \in D \), then \( \Delta(p, 0) < 0 \) and it is proved as above that for any \( p \in D \), the operator \( h(p) \) has a unique negative eigenvalue.

Since the function \( \Delta(p, \cdot) \) is decreasing on \( (−\infty, 0] \) for all \( p \in T^3 \setminus D \) and \( z < 0 \), we have
\[
\Delta(p, z) > \Delta(p, 0) \geq 0.
\]

Then by Lemma 3.1 for all \( p \in T^3 \setminus D \) the operator \( h(p) \) has no negative eigenvalues.

(iii) Let \( \min_{p \in T^3} \Delta(p, 0) \geq 0 \). Since \( T^3 \) is a compact set and the function \( \Delta(p, 0) \) is continuous on \( T^3 \) we have
\[
\Delta(p, 0) \geq 0 \quad \text{for all} \quad p \in T^3
\]
and it is proved as above that for all \( p \in T^3 \), the operator \( h(p) \) has no negative eigenvalues.

The following decomposition plays a crucial role in the proof of the asymptotics (2.3).

**Lemma 3.6.** Assume Assumptions 2.1 and 2.2 are fulfilled. Then for any \( p \in U_\delta(0) \), \( \delta > 0 \) sufficiently small, and \( z \leq 0 \) the following decomposition holds:
\[
\Delta(p, z) = \Delta(0, 0) + 2\pi^2v^2(0)\frac{\sqrt{m^2 - l^2}^2}{l^2} (\det W)^{-\frac{1}{2}} \sqrt{m_w(p) - z} + (3.3)
+ \Delta^{(02)}(m_w(p) - z) + \Delta^{(20)}(p, z),
\]
where \( \Delta^{(02)}(m_w(p) - z) \) (resp. \( \Delta^{(20)}(p, z) \)) is a function behaving like \( O(m_w(p) - z) \) (resp. \( O(|p|^2) \)) as \( |m_w(p) - z| \to 0 \) (resp. \( p \to 0 \)).

**Proof.** Let \( W(p, q) \) the function defined on \( U_\delta(0) \times T^3 \) as
\[
W(p, q) = w_p(q + q_0(p)) - m_w(p),
\]
where \( q_0(p) \in T^3 \) is an analytic function in \( p \in U_\delta(0) \) (see Lemma A.3) and is the non-degenerate minimum point of the function \( w_p(q) \) for any \( p \in U_\delta(0) \).

We define the function \( \hat{\Delta}(p, \zeta) \) on \( U_\delta(0) \times \mathbb{C}_+ \) by
\[
\hat{\Delta}(p, \zeta) = \Delta(p, m_w(p) - \zeta^2),
\]
where \( \mathbb{C}_+ = \{ z \in \mathbb{C} : Rez > 0 \} \). Using (3.4) the function \( \hat{\Delta}(p, \zeta) \) is represented as
\[
\hat{\Delta}(p, \zeta) = u(p) - m_w(p) + \zeta^2 - \frac{1}{2} \int_{T^3} \frac{v^2(q + q_0(p))}{W(p, q) + \zeta^2} dq.
\]
Let $V_\delta(0)$ be a complex $\delta$-neighborhood of the point $\zeta = 0 \in \mathbb{C}$. Denote by $\Delta^*(p, \zeta)$ the analytic continuation of the function $\tilde{\Delta}(p, \zeta)$ to the region $U_\delta(0) \times (\mathbb{C}_+ \cup V_\delta(0))$.

Since the functions $v(q), u(q), m_w(q)$ and $W(p, q)$ are even we have that $\Delta^*(p, \zeta)$ is even in $p \in U_\delta(0)$. Then by the asymptotics $u(p) = u(0) + O(|p|^2)$ as $p \to 0$ we have

$$\Delta^*(p, \zeta) = \Delta^*(0, \zeta) + \tilde{\Delta}^{(20)}(p, \zeta),$$

(3.5)

where $\tilde{\Delta}^{(20)}(p, \zeta) = O(|p|^2)$ uniformly in $\zeta \in \mathbb{C}_+$ as $p \to 0$ (see also [16]). A Taylor series expansion gives

$$\Delta^*(0, \zeta) = \Delta^*(0, 0) + \tilde{\Delta}^{(01)}(0, 0)\zeta + \tilde{\Delta}^{(02)}(0, \zeta)\zeta^2,$$

where $\tilde{\Delta}^{(02)}(0, \zeta) = O(1)$ as $\zeta \to 0$.

A simple computation shows that

$$\frac{\partial \Delta^*(0, 0)}{\partial w} = \tilde{\Delta}^{(01)}(0, 0) = 2\pi^2 v^2(0) \sqrt{\frac{l_1^2 - l_2^2}{l_1^2}} (\text{det} W)^{-\frac{1}{2}}.$$

(3.7)

The representations (3.5), (3.6) and the equality (3.7) give (3.3).

□

**Corollary 3.7.** Assume Assumptions 2.1 and 2.2 are fulfilled and let the operator $h(0)$ have a zero energy resonance. Then for any $p \in U_\delta(0), \delta > 0$ sufficiently small, and $z \leq 0$ the following decomposition holds:

$$\Delta(p, z) = 2\pi^2 v^2(0) \sqrt{\frac{l_1^2 - l_2^2}{l_1^2}} (\text{det} W)^{-\frac{1}{2}} \sqrt{m_w(p) - z} + \Delta^{(02)}(m_w(p) - z) + \Delta^{(20)}(p, z),$$

where the functions $\Delta^{(02)}(m_w(p) - z)$ and $\Delta^{(20)}(p, z)$ are the same as in Lemma 3.3.

**Proof.** The proof of Corollary 3.7 immediately follows from decomposition (3.3) and Lemma 3.2. □

**Corollary 3.8.** Assume Assumptions 2.1 and 2.2 are fulfilled and let the operator $h(0)$ have a zero energy resonance. Then there exists $\delta > 0$ such that for any $p \in U_\delta(0), p \neq 0$

$$\Delta(p, 0) > 0,$$

that is, $\Lambda(p) < \Lambda(0)$.

(3.8)

**Proof.** By Corollary 3.7 and the asymptotics (see part (ii) of Lemma 3.3)

$$m_w(p) = \frac{l_1^2 - l_2^2}{2l_1} (Wp, p) + O(p^4) \quad \text{as} \quad p \to 0$$

(3.9)

we get

$$2\pi^2 v^2(0) \sqrt{\frac{l_1^2 - l_2^2}{l_1^2}} (\text{det} W)^{-\frac{1}{2}} \sqrt{m_w(p)} > |\Delta^{(20)}(p, 0)|$$

for $p \in U_\delta(0), p \neq 0, \delta > 0$ sufficiently small. This inequality gives (3.8). □

**Lemma 3.9.** Assume Assumptions 2.1, 2.2 and 2.4 are fulfilled and let the operator $h(0)$ have a zero-energy resonance. Then there exist positive numbers $c, C$ and $\delta$ such that

$$c|p| \leq \Delta(p, 0) \leq C|p| \quad \text{for any} \quad p \in U_\delta(0)$$

(3.10)

and

$$\Delta(p, 0) \geq c \quad \text{for any} \quad p \in \mathbb{T}^3 \setminus U_\delta(0).$$

(3.11)
Proof. From (3.3) and (3.9) we get (3.10) for some positive numbers $c, C$.

By Assumptions 2.2 and 2.3 we get $\Delta(p,0) > 0$, $p \neq 0$. Since $\Delta(p,0)$ is continuous on $T^3$ and $\Delta(0,0) = 0$ we have (3.11) for some $c > 0$.

\[ \text{Lemma 3.10. Assume Assumptions 2.1, 2.2 and 2.3 are fulfilled and let the operator } h(0) \text{ have a zero eigenvalue, then there exist numbers } \delta > 0 \text{ and } c > 0 \text{ so that} \]

\[ |\Delta(p,0)| \geq cp^2 \quad \text{for all } p \in U_\delta(0), \]

\[ |\Delta(p,0)| \geq c \quad \text{for all } p \in T^3 \setminus U_\delta(0). \]

Proof. Let the operator $h(0)$ have a zero eigenvalue. By Lemma 3.3 we have $\Delta(0,0) = 0$ and $v(0) = 0$. By Assumptions 2.2 and 2.3 the function $\Delta(p,0) = u(p) - \frac{4}{3} \Lambda(p)$ has a unique non-degenerate minimum at $p = 0$. Then there exist positive numbers $\delta$ and $c$ such that the statement of the lemma is fulfilled.

4. THE ESSENTIAL SPECTRUM OF THE OPERATOR $H$

We consider the operator $\hat{H}$ acting in $\hat{\mathcal{H}} = L_2(T^3) \oplus L_2((T^3)^2)$ as

\[ \hat{H} \left( \begin{array}{c} f_1(p) \\ f_2(p,q) \end{array} \right) = \left( \begin{array}{c} u(p)f_1(p) + \frac{1}{\sqrt{2}} \int_{T^3} v(q')f_2(p,q')dq' \\ \frac{1}{\sqrt{2}}v(q)f_1(p) + w_p(q)f_2(p,q) \end{array} \right). \]

The operator $\hat{H}$ commutes with any multiplication operator $U_\gamma$ by the function $\gamma(p)$ acting in $\hat{\mathcal{H}}$ as

\[ U_\gamma \left( \begin{array}{c} f_1(p) \\ f_2(p,q) \end{array} \right) = \left( \begin{array}{c} \gamma(p)f_1(p) \\ \gamma(p)f_2(p,q) \end{array} \right), \gamma \in L_2(T^3). \]

Therefore the decomposition of the space $\hat{\mathcal{H}}$ into the direct integral

\[ \hat{\mathcal{H}} = \int_{T^3} \oplus \hat{\mathcal{H}}(p)dp, \]

where $\hat{\mathcal{H}}(p) = C^1 \oplus L_2(T^3)$, yields for the operator $\hat{H}$ the decomposition into the direct integral

\[ \hat{H} = \int_{T^3} \oplus h(p)dp, \quad (4.1) \]

where we recall that the fiber operators $h(p), p \in T^3$, are defined by (2.2).

4.1. The spectrum of the operator $\hat{H}$.

**Lemma 4.1.** For the spectrum $\sigma(\hat{H})$ of $\hat{H}$ the equality

\[ \sigma(\hat{H}) = \cup_{p \in T^3} \sigma_d(h(p)) \cup [0, M] \]

holds, where $\sigma_d(h(p))$ is the discrete spectrum of $h(p), p \in T^3$.

Proof. The assertion of the lemma follows from the representation (4.1) of the operator $\hat{H}$ and the theorem on decomposable operators (see [28]).

Set

\[ \sigma_{two}(\hat{H}) = \cup_{p \in T^3} \sigma_d(h(p)), \quad (4.2) \]

\[ a \equiv \inf \sigma_{two}(\hat{H}), \; b \equiv \sup \sigma_{two}(\hat{H}). \]
So by Lemma 3.3 the operator $h(p), p \in \mathbb{T}^3$, has in the interval $(M, +\infty)$ either one or zero number of eigenvalues. Hence the location and structure of the essential spectrum of $\hat{H}$ can be precisely described as well as in the following

**Lemma 4.2.** Assume Assumption 2.1 is fulfilled and let $\Delta(p, M) \leq 0$ for any $p \in \mathbb{T}^3$.

(i) Let $\max_{p \in \mathbb{T}^3} \Delta(p, 0) < 0$, then

$$\sigma(\hat{H}) = [a, b] \cup [0, M] \text{ and } b < 0.$$  

(ii) Let $\min_{p \in \mathbb{T}^3} \Delta(p, 0) < 0$ and $\max_{p \in \mathbb{T}^3} \Delta(p, 0) \geq 0$, then

$$\sigma(\hat{H}) = [a, M] \text{ and } a < 0.$$  

(iii) Let $\min_{p \in \mathbb{T}^3} \Delta(p, 0) \geq 0$, then

$$\sigma(\hat{H}) = [0, M].$$

**Proof.** (i). Let $\max_{p \in \mathbb{T}^3} \Delta(p, 0) < 0$. Then by Lemma 3.5 for all $p \in \mathbb{T}^3$ the operator $h(p)$ has a unique negative eigenvalue $z(p) < m_w(p)$.

By Assumptions 2.1 and 2.2 and Lemma 3.1 $z : p \in \mathbb{T}^3 \to z(p)$ is a real analytic function on $\mathbb{T}^3$.

Therefore $\text{Im} z$ is a connected closed subset of $(-\infty, 0)$, that is, $\text{Im} z = [a, b]$ and $b < 0$ and hence $\sigma_{\text{two}}(H) = [a, b]$.

(ii). Let $\min_{p \in \mathbb{T}^3} \Delta(p, 0) < 0$ and $\max_{p \in \mathbb{T}^3} \Delta(p, 0) \geq 0$. Then by assertion (ii) of Lemma 3.5 there exists a non void open set $D$ such that for any $p \in D$ the operator $h(p)$ has a unique negative eigenvalue $z(p)$.

Since for any $p \in \mathbb{T}^3$ the operator $h(p)$ is bounded and $\mathbb{T}^3$ is compact set, there exist a positive number $C$ such that $\sup_{p \in \mathbb{T}^3} ||h(p)|| \leq C$ and for any $p \in \mathbb{T}^3$ we have

$$\sigma(h(p)) \subset [-C, C].$$  

(4.3)

For any $q \in \partial D = \{ p \in \mathbb{T}^3 : \Delta(p, 0) = 0 \}$ there exist $\{p_n\} \subset D$ such that $p_n \to q$ as $n \to \infty$. Set $z_n = z(p_n)$. Then by Lemma 3.5 for any $p_n \in D$ the number $z_n$ is negative and from (4.3) we get $\{z_n\} \subset [-C, 0]$. Without loss of a generality we assume that $\{z_n\} \to z_0$ as $n \to \infty$.

The function $\Delta(p, z)$ is continuous in $\mathbb{T}^3 \times (-\infty, 0]$ and hence

$$0 = \lim_{n \to \infty} \Delta(p_n, z_n) = \Delta(q, z_0).$$

Since for any $p \in \mathbb{T}^3$ the function $\Delta(p, \cdot)$ is decreasing in $(-\infty, 0]$ and $p \in \partial D$ we can see that $\Delta(p, z_0) = 0$ if and only if $z_0 = 0$.

For any $q \in \partial D$ we define

$$z(q) = \lim_{p \to q, p \in D} z(p) = 0.$$  

Since the function $z(p)$ is continuous on the compact set $D \cup \partial D$ and $z(p) = 0$, $p \in \partial D$ and we conclude that

$$\text{Im} z = [a, 0], \quad a < 0.$$  

Hence the set

$$\{ z \in \sigma_{\text{two}}(\hat{H}) : z \leq 0 \} = \cup_{p \in \mathbb{T}^3} \sigma_d(h(p)) \cap (-\infty, 0]$$  

coincides with the set $\text{Im} z = [a, 0]$. Then Lemma 4.1 and 4.2 complete the proof of (ii).
(iii). Let \( \min_{p \in \mathbb{T}^3} \Delta(p, 0) \geq 0 \). Then by Lemma 4.3, for all \( p \in \mathbb{T}^3 \) the operator \( h(p) \) has no negative eigenvalues.

Hence we have

\[
\sigma(\hat{H}) = [0, M].
\]

\[\square\]

**Lemma 4.3.** The essential spectrum \( \sigma_{ess}(H) \) of the operator \( H \) coincides with the spectrum of \( H \), that is,

\[
\sigma_{ess}(H) = \sigma(\hat{H}).
\] (4.4)

**Proof.** In \[20\] it has been proved that the essential spectrum \( \sigma_{ess}(H) \) of the operator \( H \) coincides with \( \sigma_{ess}(\hat{H}) \cup [0, M] \). By Lemma 4.1 we have \((5.1)\). \[\square\]

From Lemmas 4.2 and 4.3 we have the following theorem.

**Theorem 4.4.** Assume Assumption \[4.7\] is fulfilled and let \( \Delta(p, M) \leq 0 \) for any \( p \in \mathbb{T}^3 \).

(i) Let \( \max_{p \in \mathbb{T}^3} \Delta(p, 0) < 0 \), then

\[
\sigma_{ess}(H) = [a, b] \cup [0, M] \quad \text{and} \quad b < 0.
\]

(ii) Let \( \min_{p \in \mathbb{T}^3} \Delta(p, 0) < 0 \) and \( \max_{p \in \mathbb{T}^3} \Delta(p, 0) \geq 0 \), then

\[
\sigma_{ess}(H) = [a, M] \quad \text{and} \quad a < 0.
\]

(iii) Let \( \min_{p \in \mathbb{T}^3} \Delta(p, 0) \geq 0 \), then

\[
\sigma_{ess}(H) = [0, M].
\]

5. THE BIRMAN-SCHWINGER PRINCIPLE

In this section we prove an analogue of the Birman-Schwinger principle.

Let \( M(z), z \in \tau_{ess}(H) \), \( z \neq u_0 \) the operator in \( \mathcal{H} \) with entries

\[
M_{00}(z) = M_{11}(z) = M_{22}(z) = M_{02}(z) = M_{20}(z) = 0,
\]

otherwise

\[
M_{\alpha\beta}(z) = -R^\frac{1}{2}_{\alpha}(z)H_{\alpha\beta}R^\frac{1}{2}_{\beta}(z),
\]

where \( R_\alpha(z) = (H_{\alpha\alpha} - z)^{-1} \), \( \alpha = 0, 1, 2 \).

**Proposition 5.1.** The number \( \lambda > 1 \) is an eigenvalue of the operator \( M(z), z < \tau_{ess}(H) \), \( z \neq u_0 \) if and only if the number \( \lambda^2 \) is an eigenvalue of the operator in \( L^2(\mathbb{T}^3) \) given by

\[
V(z) = R^\frac{1}{2}_{1}(z)H_{10}R_0(z)H_{10}R^\frac{1}{2}_{2}(z) + R^\frac{1}{2}_{1}(z)H_{12}R_2(z)H_{21}R^\frac{1}{2}_{1}(z).
\]

Moreover the eigenvalues \( \lambda \) and \( \lambda^2 \) have the same multiplicities.

**Proof.** Let \( \lambda > 1 \) be an eigenvalue of \( M(z) \), that is, the equation \( M(z)f = \lambda f \) or a system of equations

\[
\begin{cases}
-M_{01}(z)f_1 = \lambda f_0 \\
-M_{10}(z)f_0 - M_{12}(z)f_2 = \lambda f_1 \\
-M_{21}(z)f_1 = \lambda f_2 
\end{cases}
\] (5.1)

has a nontrivial solutions.

From the first and third equations of the system \((5.1)\) for \( f_\alpha, \alpha = 0, 2 \) we get

\[
f_\alpha = -\frac{1}{2}M_{\alpha1}(z)f_1, \quad \alpha = 0, 2.
\]
Substituting the latter expression for $f_{\alpha}$, $\alpha = 0, 2$ into the second equation of the system (5.1) we have the following

$$V(z)f_1 = \lambda^2 f_1, \quad f_1 \in L_2(\mathbb{T}^3)$$  \hspace{1cm} (5.2)

and this equation has a nontrivial solution if and only if the system of equations (5.2) has a nontrivial solution and the linear subspaces of solutions of (5.1) and (5.2) have the same dimensions.

In our analysis of the spectrum of $H$ the crucial role played by the compact integral operator $T(z)$, $z < \tau_{ess}(H)$, $z \neq u_0$ in the space $L_2(\mathbb{T}^3)$ with the kernel

$$\frac{v(p)v(q)}{\sqrt{\Delta(p, z)}\sqrt{\Delta(q, z)}} \left[ \frac{1}{2(w(p, q) - z)} + \frac{1}{u_0 - z} \right].$$

For a bounded self-adjoint operator $A$, we define $n(\lambda, A)$ as

$$n(\lambda, A) = \sup\{dim F : (Au, u) > \lambda, \quad u \in F, \quad ||u|| = 1\}.$$

$n(\lambda, A)$ is equal the infinity if $\lambda$ is in essential spectrum of $A$ and if $n(\lambda, A)$ is finite, it is equal to the number of the eigenvalues of $A$ bigger than $\lambda$. By the definition of $N(z)$ we have

$$N(z) = n(-z, -H), \quad -z > -\tau_{ess}(H).$$

The following lemma is a realization of well known Birman-Schwinger principle for the operator $H$ (see [3, 29, 31]).

**Lemma 5.2.** The operator $T(z)$, $z \neq u_0$ is compact and continuous in $z < \tau_{ess}(H)$ and

$$N(z) = n(1, T(z)).$$  \hspace{1cm} (5.3)

**Proof.** Since

$$H = \begin{pmatrix} H_{00} & 0 & 0 \\ 0 & H_{11} & 0 \\ 0 & 0 & H_{22} \end{pmatrix} + \begin{pmatrix} 0 & H_{01} & 0 \\ H_{10} & 0 & H_{12} \\ 0 & H_{21} & 0 \end{pmatrix},$$

and $H_{ii} - z, i = 0, 1, 2$ is positive and invertible for $z < \tau_{ess}(H)$, $z \neq u_0$ (for simplicity we assume $z > u_0$, if then $z < u_0$ the operator $-(H_{00} - z)$ is positive) one has $f \in \mathcal{H}$ and $((H - zI)f, f) < 0$ if and only if $((M(z) - I)v, v) > 0$ and $v_i = (H_{ii} - z)^2 f_i, i = 0, 1, 2$, where $I$ is the identity operator on $\mathcal{H}$.

It follows that

$$N(z) = n(1, M(z)).$$  \hspace{1cm} (5.4)

Using Proposition 5.1 we get

$$n(1, M(z)) = n(1, V(z)).$$  \hspace{1cm} (5.5)

Now we represent the operator $H_{21}$ as a sum of two operators $H_{21}^{(1)}$ and $H_{21}^{(2)}$ acting from $L_2(\mathbb{T}^3)$ to $L_2((\mathbb{T}^3)^2)$ as

$$(H_{21}^{(1)} f_1)(p, q) = \frac{1}{2} v(p)f_1(q), \quad (H_{21}^{(2)} f_1)(p, q) = \frac{1}{2} v(q)f_1(p).$$

Then $\varphi \in L_2(\mathbb{T}^3)$ and $((V(z) - I)\varphi, \varphi) > 0$ iff $\psi = R_1^2(z)\varphi$ and $\psi \in L_2(\mathbb{T}^3)$ and $((H_{11} - z - H_{12}R_2(z)H_{21}^{(2)})\varphi, \varphi) < 0$ if $H_{10}R_0(z)H_{10}\psi + (H_{12}R_2(z)H_{21}^{(1)}\psi, \psi)$, where $I$ is the identity operator on $L_2(\mathbb{T}^3)$. This fact means that

$$n(1, V(z)) = n(-z, H_{11} - H_{12}R_2(z)H_{21}^{(2)} - H_{10}R_0(z)H_{01} - H_{12}R_2(z)H_{21}^{(1)}).$$  \hspace{1cm} (5.6)
Since $z < \tau_{ess}(H)$, for any $p \in \mathbb{T}^3$ the function $\Delta(p, z)$ is positive and hence the operator $H_{11} - z - H_{12}R_2(z)H_{21}^{(2)}$ is positive and invertible and

$$(H_{11} - z - H_{12}R_2(z)H_{21}^{(2)})^{-\frac{1}{2}} = R_{11}^{\frac{1}{2}}(z) > 0.$$\hspace{1cm} \text{(5.3)}$$

A direct calculation shows that

$$n(-z, H_{11} - H_{12}R_2(z)H_{21}^{(2)} - H_{10}R_0(z)H_{01} - H_{12}R_2(z)H_{21}^{(1)}) = n(1, T(z)). \hspace{1cm} \text{(5.7)}$$

The equalities (5.4), (5.5), (5.6) and (5.7) gives (5.3).

Finally it we note that the operator $T(z), z \neq u_0$ is compact and continuous in $z < \tau_{ess}(H)$. \hfill \Box

6. The finiteness of the number of eigenvalues of the operator $H$. \hspace{1cm}

\textbf{Lemma 6.1.} Assume Assumptions 2.1, 2.2 and 2.3 are fulfilled and let the operator $h(0)$ have a zero eigenvalue. Then the operator $T(z), z \neq u_0, (u_0 \neq 0)$ belongs to the Hilbert-Schmidt class and is continuous from the left up to $z = 0$.

\textbf{Proof.} Since the function $v(p)$ is analytic, even and $v(0) = 0$ we have $|v(p)| \leq C|p|^2$ for some $C > 0$. By virtue of Lemmas 5.9, 5.10 and 5.11 the kernel of the operator $T(z), z \leq 0, z \neq u_0, (u_0 \neq 0)$ is estimated by

$$C \left( \frac{\chi_{\delta}(p)}{|p|} + 1 \right) \left( \frac{|p|^2|q|^2\chi_{\delta}(p)\chi_{\delta}(q)}{p^2 + q^2} + 1 \right) \left( \frac{\chi_{\delta}(q)}{|q|} + 1 \right),$$

where $\chi_{\delta}(p)$ is the characteristic function of $U_{\delta}(0)$.

Since the latter function is square integrable on $(\mathbb{T}^3)^2$ we have that operator $T(z), z \neq u_0$ is a Hilbert-Schmidt operator.

The kernel function of $T(z), z \neq u_0$ is continuous in $p, q \in \mathbb{T}^3, z < 0$ and square integrable on $(\mathbb{T}^3)^2$ as $z \leq 0$. Then by the dominated convergence theorem the operator $T(z)$ is continuous from the left up to $z = 0$. \hfill \Box

We are now ready for the

\textbf{Proof of (i) of Theorem 2.9.} Let the conditions part (i) of Theorem 2.9 be fulfilled.

\textbf{Case} $u_0 \neq 0$. By Lemma 5.12 we have

$$N(z) = n(1, T(z)), z \neq u_0 \text{ as } z < 0$$

and by Lemma 6.1 for any $\gamma \in [0, 1)$ the number $n(1 - \gamma, T(0))$ is finite. Then we have

$$n(1, T(z)) \leq n(1 - \gamma, T(0)) + n(\gamma, T(z) - T(0))$$

for all $z < 0, z \neq u_0$ and $\gamma \in (0, 1)$. This relation can be easily obtained by use of the Weyl inequality

$$n(\lambda_1 + \lambda_2, A_1 + A_2) \leq n(\lambda_1, A_1) + n(\lambda_2, A_2)$$

for sum of compact operators $A_1$ and $A_2$ and for any positive numbers $\lambda_1$ and $\lambda_2$.

Since $T(z), z \neq u_0$ is continuous from the left up to $z = 0$, we obtain

$$\lim_{z \to 0} N(z) = N(0) \leq n(1 - \gamma, T(0)) \text{ for all } \gamma \in (0, 1).$$

Thus

$$N(0) \leq n(1 - \gamma, T(0)) < \infty.$$\hspace{1cm}

The latter inequality proves the assertion (i) of Theorem 2.9 in case $u_0 \neq 0$. \hspace{1cm}
By the (3.9) and Corollary 3.7 for any sufficiently small negative

exists and is continuous in

Lemma 7.1.

rem 2.9 will be deduced by a perturbation argument based on the following lemma.

for all

for any

In this section we shall closely follow A. Sobolev’s method [29] to derive the asymptotics for the number of eigenvalues of

where

By Assumption 2.1 we get

For the proof of Lemma 7.1, see Lemma 4.9 of [29].

We remark that the range of the operator

is compact and continuous

where

The latter inequality proves the assertion (i) of Theorem 2.9 in case

Case \( u_0 = 0 \). First we represent the operator \( T(z), z < 0 \) as a sum of two bounded operators

acting on \( L_2(T^3) \) as

\[
(T^{(1)}(z)f)(p) = \frac{v(p)}{2\sqrt{\Delta(p, z)}} \int_{\mathbb{T}^3} \frac{v(q)f(q)dq}{\sqrt{\Delta(q, z)(v(p) - z)}},
\]

\[
(T^{(2)}(z)f)(p) = -\frac{v(p)}{z\sqrt{\Delta(p, z)}} \int_{\mathbb{T}^3} \frac{v(q)f(q)dq}{\sqrt{\Delta(q, z)}}.
\]

We remark that the range of the operator \( T^{(2)}(z) \) is one and hence \( n(1, T^{(2)}(z)) \leq 1 \)

for any \( z < 0 \). Then according to Lemma 5.2 and the Weyl inequality, for all \( z < 0 \) and \( \gamma \in (0, 1) \) we obtain

\[
N(z) = n(1, T(z)) \leq n(1 - \gamma, T^{(1)}(z)) + 1.
\]

Since \( T^{(1)}(z) \) is continuous from the left up to \( z = 0 \), we obtain

\[ N(0) \leq n(1 - \gamma, T(0)) \]

for all \( \gamma \in (0, 1) \). The latter inequality proves the assertion (i) of Theorem 2.9 in case \( u_0 = 0 \). \( \square \)

7. Asymptotics for the number of eigenvalues of the operator \( H \).

In this section we shall closely follow A. Sobolev’s method [29] to derive the asymptotics for the number of eigenvalues of \( H \).

We shall first establish the asymptotics of \( n(1, T(z)), z \neq u_0 \) as \( z \to -0 \). Then Theorem 2.9 will be deduced by a perturbation argument based on the following lemma.

**Lemma 7.1.** Let \( A(z) = A_0(z) + A_1(z) \), where \( A_0(z) (A_1(z)) \) is compact and continuous in \( z < 0 \) \((z \leq 0)\). Assume that for some function \( f(\cdot), f(z) \to 0, z \to -0 \) the limit

\[
\lim_{z \to -0} f(z)n(\lambda, A_0(z)) = \ell(\lambda),
\]

exists and is continuous in \( \lambda > 0 \). Then the same limit exists for \( A(z) \) and

\[
\lim_{z \to -0} f(z)n(\lambda, A(z)) = \ell(\lambda).
\]

For the proof of Lemma 7.1 see Lemma 4.9 of [29].

By Assumption 2.1, we get

\[
w(p, q) = \frac{1}{2}(l_1(W, p) + 2l_2(W, q) + l_1(W, q)) + O(|p|^4 + |q|^4) \text{ as } p, q \to 0. \tag{7.1}
\]

By the (5.9) and Corollary 5.7, for any sufficiently small negative \( z \) we get

\[
\Delta(p, z) = \frac{4\pi^2l_1(0)}{l_1^{3/2}\text{det}(W)^{1/2}} |n(W, p) - 2z| + O(|p|^2 + |z|) \text{ as } p, z \to 0, \tag{7.2}
\]

where

\[
n = (l_1^2 - l_2^2)/l_1.
\]

Let \( T(\delta; |z|), z \neq u_0 \neq 0 \) be an operator on \( L_2(T^3) \) defined by

\[
(T(\delta; |z|)f)(p) = d_0 \int_{\mathbb{T}^3} \frac{\chi_\delta(p)\chi_\delta(q)(n(W, p) + 2|z|)^{-1/4}(n(W, q) + 2|z|)^{-1/4}}{l_1(W, p) + 2l_2(W, q) + l_1(W, q) + 2|z|} f(q)dq,
\]

where \( \chi_\delta(\cdot) \) is the characteristic function of \( \tilde{U}_\delta(0) = \{ p \in T^3 : |W^{1/2} p| < \delta \} \) and

\[
d_0 = \frac{\text{det} W^{1/2}}{2\pi^2 l_1^3}.
\]
The main technical point to apply Lemma 7.1 is the following

**Lemma 7.2.** Let Assumptions 2.1, 2.2 and part (i) of Assumption 2.3 be fulfilled and \( u_0 \neq 0 \). Then the operator \( T(z) - T(\delta; |z|) \), \( z \neq u_0 \) belongs to the Hilbert-Schmidt class and is continuous in \( z \leq 0 \).

**Proof.** Applying the asymptotics \((7.1), (7.2)\) and Lemmas 3.9 and A.4 one can estimate the kernel of the operator \( T(z) - T(\delta; |z|) \), \( z \neq u_0 \) \((u_0 \neq 0)\) by

\[
C[(p^2 + q^2)^{-1} + |p|^{-\frac{1}{2}}(p^2 + q^2)^{-1} + |q|^{-\frac{1}{2}}(p^2 + q^2)^{-1} + 1]
\]

and hence the operator \( T(z) - T(\delta; |z|), z \neq u_0 \) belongs to the Hilbert-Schmidt class for all \( z \leq 0 \). In combination with the continuity of the kernel of the operator in \( z < 0 \) this gives the continuity of \( T(z) - T(\delta; |z|) \), \( z \neq u_0 \) in \( z \leq 0 \). The details are omitted. \( \square \)

Let us now recall some results from \([29]\), which are important in our work.

Let \( S_r : L_2((0, r) \times \sigma) \to L_2((0, r) \times \sigma) \) be the integral operator with the kernel

\[
S(y; t) = (2\pi)^{-2} \frac{l}{\cos y + st}
\]

and

\[
r = 1/2|\log |z||, y = x - x', t = <\xi, \eta >, \xi, \eta \in \mathbb{S}^2, l = \left(\frac{l_2^2}{l_1^2 - l_2^2}\right)^{\frac{1}{2}}, s = \frac{l_2}{l_1},
\]

\( \sigma = L_2(\mathbb{S}^2) \), \( \mathbb{S}^2 \) being the unit sphere in \( \mathbb{R}^3 \).

The coefficient in the asymptotics of \( N(z) \) will be expressed by means of the self-adjoint integral operator \( \hat{S}(\lambda), \lambda \in \mathbb{R} \), in the space \( L_2(\mathbb{S}^2) \) whose kernel depends on the scalar product \( t = <\xi, \eta > \) of the arguments \( \xi, \eta \in \mathbb{S}^2 \) and has the form

\[
\hat{S}(\lambda) = (2\pi)^{-1} \frac{\sinh[\lambda(\arccos st)]}{(1 - s^2 t^2)^{\frac{1}{2}} \sinh(\pi \lambda)}.
\]

For \( \mu > 0 \), define

\[
U(\mu) = (4\pi)^{-1} \int_{-\infty}^{+\infty} n(\mu, \hat{S}(y))dy
\]

and set \( \Pi_0 \equiv U(1) \).

The following lemma can be proved in the same way as Theorem 4.5 in \([29]\).

**Lemma 7.3.** The following equality

\[
\lim_{r \to +\infty} \frac{1}{2} n(\mu, S_r) = U(\mu)
\]

holds.

The following theorem is basic for the proof of the asymptotics \([2.3]\).

**Theorem 7.4.** Let \( u_0 \neq 0 \) \((u_0 < 0)\). The equality

\[
\lim_{|z| \to 0} \frac{n(1, T(z))}{|\log |z||} = \lim_{r \to +\infty} \frac{1}{2} n(1, S_r)
\]

holds.

**Remark 7.5.** Since \( \Pi(\cdot) \) is continuous in \( \mu \), according to Lemma 7.1 any perturbations of the operator \( A_0(z) \) defined in Lemma 7.1 which is compact and continuous up to \( z = 0 \) do not contribute to the asymptotics \([2.3]\). During the proof of Theorem 7.4 we use this fact without further comments.
Proof of Theorem 7.4. Let \( u_0 \neq 0 \). As in Lemma 6.4, it can be shown that \( T(z) - T(\delta; |z|), z \neq u_0 \) defines a compact operator continuous in \( z \leq 0 \) and it does not contribute to the asymptotics \( \delta \).

The space of functions having support in \( \hat{U}_0(0) \) is an invariant subspace for the operator \( T(\delta; |z|), z \neq u_0 \).

Let \( T_0(\delta; |z|), z \neq u_0 \) be the restriction of the operator \( T(\delta; |z|), z \neq u_0 \) to the subspace \( L_2(\hat{U}_0(0)) \). One verifies that the operator \( T_0(\delta; |z|), z \neq u_0 \) is unitary equivalent to the following operator \( T_0(\delta; |z|), z \neq u_0 \) acting in \( L_2(\hat{U}_0(0)) \) as

\[
(T_0(\delta; |z|)f)(p) = d_1 \int_{U_0(0)} \frac{(np^2 + 2|z|)^{-1/4}(nq^2 + 2|z|)^{-1/4}}{l_1p^2 + 2l_2(p, q) + l_1q^2 + 2|z|} f(q)dq,
\]

where

\[
d_1 = (2\pi^2)^{-1/4}.
\]

Here the equivalence is performed by the unitary dilation

\[
Y : L_2(U_0(0)) \to L_2(\hat{U}_0(0)), \quad (Yf)(p) = f(U^{-\frac{1}{2}}p).
\]

The operator \( T_0(\delta; |z|), z \neq u_0 \) is unitary equivalent to the integral operator \( T_1(\delta; |z|) : L_2(U_r(0)) \to L_2(U_r(0)) \) with the kernel

\[
d_1 \frac{(np^2 + 2|z|)^{-1/4}(nq^2 + 2|z|)^{-1/4}}{l_1p^2 + 2l_2(p, q) + l_1q^2 + 2},
\]

where \( U_r(0) = \{ p \in \mathbb{R}^3 : |p| < r \}, r = \frac{|z|}{2}, z \neq u_0 \).

The equivalence is performed by the unitary dilation

\[
B_r : L_2(U_0(0)) \to L_2(U_r(0)), \quad (B_rf)(p) = \left( \frac{r}{\delta} \right)^{-3/2} f\left( \frac{\delta}{r}p \right).
\]

Further, we may replace

\[
(np^2 + 2)^{-1/4}, (nq^2 + 2)^{-1/4} \quad \text{and} \quad l_1p^2 + 2l_2(p, q) + l_1q^2 + 2
\]

by

\[
\left( np^2 \right)^{-1/4}(1 - \chi_1(p)), \left( nq^2 \right)^{-1/4}(1 - \chi_1(q)) \quad \text{and} \quad l_1p^2 + 2l_2(p, q) + l_1q^2,
\]

respectively, since the error will be a Hilbert-Schmidt operator continuous up to \( z = 0 \). Then we get the integral operator \( T_2(r) \) on \( L_2(U_r(0) \setminus U_1(0)) \) with the kernel

\[
(n)^{-\frac{3}{2}}d_1 \frac{|p|^{-1/2}|q|^{-1/2}}{l_1p^2 + 2l_2(p, q) + l_1q^2}.
\]

By the dilation

\[
M : L_2(U_r(0) \setminus U_1(0)) \to L_2((0, \pi) \times \sigma),
\]

where \( (Mf)(x, w) = e^{3x/2}f(e^xw), x \in (0, \pi), w \in S^2 \), one sees that the operator \( T_2(r) \) is unitary equivalent with the integral operator \( S_r \).

Proof of (ii) of Theorem 2.9. Let the conditions part (ii) of Theorem 2.9 be fulfilled.

Case \( u_0 \neq 0 \). Similarly to \( \text{[29]} \), we can show that

\[
\mathcal{U}_0 = U(1) \geq \frac{1}{4\pi} \int_{-\infty}^{+\infty} n(1, \hat{S}^{(0)}(y))dy \geq \frac{1}{4\pi} \text{mes}\{y : \hat{S}^{(0)}(y) > 1\},
\]
where \( \hat{S}^{(0)}(y) \) is the multiplication operator by number
\[
\hat{S}^{(0)}(y) = i \frac{\sinh(y \text{arcsinh})}{sy \cosh \frac{y}{2}}
\]
in the subspace of the harmonics of degree zero.

The positivity of \( l_{0} \) follows from the facts that \( l > 1, \hat{S}^{(0)}(0) > 1 \) and continuity of \( \hat{S}^{(0)}(y) \). Taking into account the inequality (7.3) and Lemmas 5.2, 7.4, 7.3, we complete the proof of (ii) of Theorem 2.9 in case \( u_{0} \neq 0 \).

Case \( u_{0} = 0 \). Since \( n(1, T^{(2)}(z)) \leq 1 \) for any \( z < 0 \) according to the Weyl inequality for all \( z < 0 \) and \( \gamma \in (0, 1) \) we obtain
\[
n(1 + \gamma, T^{(1)}(z)) \leq n(1, T(z)) \leq n(1 - \gamma, T^{(1)}(z)) + 1.
\]

From the latter inequality we have
\[
U(1 + \gamma) \leq \lim_{|z| \to 0} \frac{n(1, T(z))}{|\log|z||} \leq U(1 - \gamma).
\]

After this remark from the continuity of the function \( U(\cdot) \) it follows
\[
\lim_{|z| \to 0} \frac{n(1, T(z))}{|\log|z||} = U(1) = l_{0}.
\]

The latter equality and Lemma 5.2 completes the proof of (ii) of Theorem 2.9 in case \( u_{0} = 0 \).

APPENDIX A.

Lemma A.1. Let
\[
u(p) = \varepsilon(p) + c, \quad v(p) = \varepsilon(p), \quad w(p, q) = \varepsilon(p) + \varepsilon(p + q) + \varepsilon(q),
\]
where \( c > 0 \) is a real number and the function \( \varepsilon(p) \) is defined by (2.2). Then Assumptions 2.1, 2.2, and 2.3 are fulfilled.

Proof. It is easy to see that Assumptions 2.1 and 2.2 are fulfilled.

We prove that Assumption 2.3 is fulfilled. Since \( w(p, q) \) and \( v(p) \) are even the function \( A(p) \) is also even.

Then we get
\[
A(p) - A(0) = \frac{1}{4} \int_{T^{3}} \frac{2u_{0}(t) - (w_{p}(t) + w_{-p}(t))}{w_{p}(t)w_{-p}(t)u_{0}(t)} [w_{p}(t) + w_{-p}(t)]v^{2}(t)dt - (A.1)
\]
\[
- \frac{1}{4} \int_{T^{3}} \frac{|w_{p}(t) - w_{-p}(t)|^{2}}{w_{p}(t)w_{-p}(t)u_{0}(t)} v^{2}(t)dt.
\]

From the equality
\[
w_{0}(t) - \frac{w_{p}(t) + w_{-p}(t)}{2} = \sum_{j=1}^{3} (\cos p_{j} - 1)(1 + \cos t_{j})
\]
and (A.1) we get for all nonzero \( p \in T^{3} \) the inequality
\[
A(p) - A(0) = \int_{T^{3}} \sum_{j=1}^{3} (\cos p_{j} - 1)(1 + \cos t_{j})v^{2}(t)dt - \frac{1}{4} \int_{T^{3}} \frac{|w_{p}(t) - w_{-p}(t)|^{2}}{w_{p}(t)w_{-p}(t)u_{0}(t)} v^{2}(t)dt < 0,
\]
that is, part (i) of Assumption 2.3 holds.
Since for any \( p, q \in \mathbb{T}^3, p \neq 0 \) the inequality \( w_p(q) > 0 \) holds for any nonzero \( p \in \mathbb{T}^3 \) the integrals

\[
\int_{\mathbb{T}^3} \frac{\partial^2}{\partial p_i \partial p_j} w_p(t)v^2(t)dt \quad (w_p(t))^3
\]

and

\[
2 \int_{\mathbb{T}^3} \frac{\partial}{\partial p_i} w_p(t) \frac{\partial}{\partial p_j} w_p(t)v^2(t)dt \quad (w_p(t))^3, \quad i, j = 1, 2, 3
\]

are finite and the finiteness of these integrals at the point \( p = 0 \) follows the fact that \( v(0) = 0 \). After this remark we can define bounded continuous functions on \( \mathbb{T}^3 \), which will be denotes by \( \lambda^{(1)}_{ij}(p) \) and \( \lambda^{(2)}_{ij}(p) \), respectively.

Then the function \( \Lambda(p) \) is a twice continuously differentiable function on \( \mathbb{T}^3 \) and

\[
\frac{\partial^2 \Lambda(p)}{\partial p_i \partial p_j} = -\lambda^{(1)}_{ij}(p) + \lambda^{(2)}_{ij}(p), \quad i, j = 1, 2, 3.
\]

Since

\[
\frac{\partial}{\partial p_i} w_0(t) = \sin t_i,
\]

\[
\frac{\partial^2}{\partial p_i \partial p_i} w_0(t) = 1 + \cos t_i,
\]

\[
\frac{\partial^2}{\partial p_i \partial p_j} w_0(t) = 0, \quad i \neq j, i, j = 1, 2, 3
\]

we get

\[
\frac{\partial^2 \Lambda(0)}{\partial p_i \partial p_i} = -2 \int_{\mathbb{T}^3} \sum_{s=1,s\neq i}^{3} (1 - \cos t_s)(1 + \cos t_i)v^2(t)dt \quad (w_0(t))^3,
\]

\[
\frac{\partial^2 \Lambda(0)}{\partial p_i \partial p_j} = 2 \int_{\mathbb{T}^3} \sin t_i \sin t_j v^2(t)dt \quad (w_0(t))^3, \quad i \neq j, i, j = 1, 2, 3.
\]

Since the function \( v \) is even on \( \mathbb{T}^3 \) from the latter two equalities we get

\[
\frac{\partial^2 \Lambda(0)}{\partial p_i \partial p_i} < 0, \quad \frac{\partial^2 \Lambda(0)}{\partial p_i \partial p_j} = 0, \quad i \neq j, i, j = 1, 2, 3.
\]

Using these facts, one may verify that the matrix of the second order partial derivatives of the function \( \Lambda(p) \) at the point \( p = 0 \) is negative definite. Thus the function \( \Lambda(p) \) has a non-degenerated maximum at the point \( p = 0 \).

\( \square \)

**Lemma A.2.** Let \( v \) be an arbitrary analytic function on \( \mathbb{T}^3 \) and

\[
w_p(q) = l_1 \varepsilon(p) + l_2 \varepsilon(p + q) + l_3 \varepsilon(q),
\]

where \( l_j > 0, j = 1, 2, l_1 \neq l_2 \) and the function \( \varepsilon(p) \) is defined by \( \text{(2.1)} \). Then for any \( p \in \mathbb{T}^3 \) and \( \psi \in C(\mathbb{T}^3) \) the integral

\[
\int_{\mathbb{T}^3} \frac{v(t)\psi(t)dt}{w_p(t) - m_w(p)}
\]

is finite.
Proof. The function \( w_p(q) \) can be rewritten in the form

\[
w_p(q) = \varepsilon_1(p) + 3(l_1 + l_2) - \sum_{i=1}^{3} (a(p_j) \cos q_j + c(p_j) \sin p_j), \tag{A.2}
\]

where the coefficients \( a(p_j) \) and \( c(p_j) \) are given by

\[
a(p_j) = l_2 \cos p_j + l_1, \quad c(p_j) = -l_2 \sin p_j.
\]

The equality (A.2) implies the following representation for \( w_p(q) \)

\[
w_p(q) = \varepsilon_1(p) + 3(l_1 + l_2) - \sum_{i=1}^{3} r(p_i)(\cos q_i - q_0(p_i)). \tag{A.3}
\]

where

\[
r(p_i) = \sqrt{a^2(p_i) + c^2(p_i)}, \quad q_0(p_i) = \arcsin \frac{c(p_i)}{r(p_i)}, \quad p_i \in (-\pi, \pi].
\]

Therefore

\[
m_w(p) = \min_{q \in T^3} w_p(q) = \varepsilon_1(p) + 3(l_1 + l_2) - \sum_{i=1}^{3} r(p_i).
\tag{A.4}
\]

From (A.3) and (A.4) we have

\[
w_p(q) - m_w(p) = \sum_{i=1}^{3} r(p_i)(1 - \cos(q_i - q_0(p_i))).
\]

Since \( l_j > 0, j = 1, 2, 1 \neq l_2 \) for any \( p \in T^3 \) the function \( w_p(q) - m_w(p) \) has a unique non-degenerate minimum at the point \( q = q_0(p) = (q_0(p_1), q_0(p_2), q_0(p_3)) \), therefore for any \( p \in T^3 \) the integral

\[
\int_{T^3} \frac{v(t)\psi(t)dt}{w_p(t) - m_w(p)}
\]

is finite. \( \square \)

**Lemma A.3.** Let Assumption 2.1 be fulfilled. Then there exists a \( \delta \)-neighborhood \( U_\delta(0) \subset T^3 \) of the point \( p = 0 \) and an analytic function \( q_0(p) \) defined on \( U_\delta(0) \) such that:

(i) for any \( p \in U_\delta(0) \) the point \( q_0(p) \) is a unique non-degenerate minimum of the function \( w_p(q) \) and

\[
q_0(p) = -\frac{l_2}{l_1} p + O(|p|^3) \text{ as } p \to 0.
\]

(ii) The function \( m_w(p) = w_p(q_0(p)) \) is analytic in \( U_\delta(0) \) and has the asymptotic form

\[
m_w(p) = \frac{l_2^2}{2l_2} (Wp, p) + O(|p|^4) \text{ as } p \to 0. \tag{A.5}
\]

**Proof.** (i) By Assumption 2.1 we obtain \( w_0(q) > w_0(0), \quad q \neq 0 \) and

\[
\left( \frac{\partial^2 w_0(0)}{\partial q_i \partial q_j} \right)_{i,j=1}^3 = l_1 W.
\]

Since \( W \) is a positive matrix the function \( w_0(q) \) has a unique non-degenerate minimum at \( q = 0 \), the gradient \( \nabla w_0(q) \) is equal to zero at the point \( q = 0 \).
Now we apply the implicit function theorem to the equation
\[ \nabla w_p(q) = 0, \quad p, q \in \mathbb{T}^3. \]
Then there exists a \( \delta \)-neighborhood \( U_\delta(0) \) of the point \( p = 0 \) and a vector function \( q_0(p) \) defined and analytic in \( U_\delta(0) \) and for all \( p \in U_\delta(0) \) the identity \( \nabla w_p(q_0(p)) \equiv 0 \) holds.

Denote by \( B(p) \) the matrix of the second order partial derivatives of the function \( w_p(q) \) at the point \( q_0(p) \). The matrix \( B(0) = \mathbb{I}_2 \) is positive definite and \( B(p) \) is continuous in \( U_\delta(0) \) and hence for any \( p \in U_\delta(0) \) the matrix \( B(p) \) is positive definite. Thus \( q_0(p) \), \( p \in U_\delta(0) \) is the unique non-degenerate minimum point of \( w_p(q) \).

The non-degenerate minimum point \( q_0(p) \) is an odd function in \( p \in U_\delta(0) \).

Indeed, since \( w(p, q) \) is even with respect \( (p, q) \) we get \( w_p(-q) = w_p(q) \), and we obtain
\[ w_p(-q_0(p)) = m_w(p) = m_w(-p) = w_p(q_0(-p)). \]
Since for all \( p \in U_\delta(0) \) the point \( q_0(p) \) is the unique non-degenerate minimum of \( w_p(q) \) we have
\[ q_0(-p) = -q_0(p). \]
By Assumption \( \Box \) and the Taylor expansion we get
\[ w_p(q - \frac{l_2}{l_1} p) = \frac{l_1}{2} (Wq, q) + \frac{l_1^2 - l_2^2}{2l_1} (Wp, p) + O(|q|^4 + |p|^4) \text{ as } q, p \to 0. \quad (A.6) \]
Since \( w_p(q_0(p)) = \min_{q \in \mathbb{T}^3} w_p(q) \leq w_p(-\frac{l_2}{l_1} p) \) and \( q_0(p) \) is odd we have
\[ \frac{l_1}{2} (Wq_0(p) + \frac{l_2}{l_1} p)^2 + \frac{l_1^2 - l_2^2}{2l_1} (Wp, p) + O(|p|^4) \leq \frac{l_1^2 - l_2^2}{2l_1} (Wp, p) + O(|p|^4) \text{ as } p \to 0, \]
that is,
\[ l_1 (Wq_0(p) + \frac{l_2}{l_1} p))^2 \leq O(|p|^4) \text{ as } p \in U_\delta(0). \]
This inequality is not valid if \( q_0(p) \) has the asymptotics \( q_0(p) + \frac{l_2}{l_1} p = O(|p|^3) \) as \( p \to 0. \)

(ii) Since the functions \( w(p, q), p, q \in \mathbb{T}^3 \) and \( q_0(p), p \in U_\delta(0) \) are analytic we have that the function \( m_w(p) = w_p(q_0(p)) \) is also analytic on \( p \in U_\delta(0) \).

By \( q_0(p) = -\frac{l_2}{l_1} p + O(|p|^3) \) \( p \to 0 \) and \( \Box \) we get the asymptotics \( \Box \).

**Lemma A.4.** There exist numbers \( C_1, C_2, C_3 > 0 \) and \( \delta > 0 \) such that the following inequalities

(i) \[ C_1(|p|^2 + |q|^2) \leq w(p, q) \leq C_2(|p|^2 + |q|^2) \text{ for all } p, q \in U_\delta(0), \]

(ii) \[ w(p, q) \geq C_3 \text{ for all } (p, q) \notin U_\delta(0) \times U_\delta(0). \]

**Proof.** By Assumption \( \Box \) the point \( (0, 0) \in (\mathbb{T}^3)^2 \) is unique non-degenerated minimum of \( w(p, q) \). Then by \( \Box \) there exist positive numbers \( C_1, C_2, C_3 \) and a \( \delta \)-neighborhood of \( p = 0 \) \( \in \mathbb{T}^3 \) so that (i) and (ii) hold true. \( \Box \)

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