Geometric Origin of Staggered Fermion: Direct Product $K$-Cycle

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Abstract

Staggered formalism of lattice fermion can be cast into a form of direct product $K$-cycle in noncommutative geometry. The correspondence between this staggered $K$-cycle and a canonically defined $K$-cycle for finitely generated abelian group where lattice appears as a special case is proved.

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I Introduction and Preliminary

Staggered formalism for lattice fermion is one of the earliest solutions to the puzzle of species doubling in lattice field theory (LFT) [1]: thorough exploration of this formalism were carried out in a series of work, especially on the problems of flavor interpretation and gauge coupling [2]: the dynamical properties of staggered fermion were considered in [3]. Recently, a nontrivial correspondence between staggered Dirac operator and noncommutative geometry (NCG) was figured out [4]; however, this correspondence was still a conjecture in general since rigid proof was

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just given for lattice whose dimension is one, two or four. In this letter, with the power of NCG being fully employed, a proof of this correspondence for any dimensional lattice is presented. This article is organized in the following way. A concise introduction of the central objects for NCG, $K$-cycles, and their direct product will be given below. In Sect. II, direct product $K$-cycle for finitely generated abelian group is introduced, with lattice being treated as a special example. In Sect. III, staggered formalism is also cast in the shape of direct product $K$-cycle, so the proof of the above-mentioned correspondence is reduced to show that the correspondence exists within each factor. Some remarks and discussions are put in Sect. IV.

Comprehensive introduction to NCG can be found in [5]. Only concepts relevant to our work are recalled below.

**Definition 1** An even $K$-cycle in Connes’ operator-algebraic approach towards NCG is presented as a quadruple $(\mathcal{A}, \mathcal{H}, D, \gamma)$, in which $\mathcal{A}$ is a pre-$C^*$ algebra being represented faithfully and unitarily on a separable Hilbert space $\mathcal{H}$ by $\pi$, Dirac operator $D$ is a selfadjoint operator on $\mathcal{H}$ with compact resolvent, and $\gamma$ is a selfadjoint involution on $\mathcal{H}$, providing $\mathcal{H}$ with a $\mathbb{Z}_2$-grading such that $[\pi(\mathcal{A}), \gamma] = 0$, $\{D, \gamma\} = 0$.

A collection of axioms is imposed on every $K$-cycle, being even or not, such that when $\mathcal{A}$ is commutative, a $K$-cycle will recover a spin-manifold [3].

**Definition 2** Direct product of two even $K$-cycle $(\mathcal{A}_i, \mathcal{H}_i, D_i, \gamma_i), i = 1, 2$ is another even $K$-cycle $(\mathcal{A}, \mathcal{H}, D, \gamma)$ where $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\gamma = \gamma_1 \otimes \gamma_2$, $D = D_1 \otimes I + \gamma_1 \otimes D_2$.

In a rigid sense, axiomatic $K$-cycle does not exist for finite sets or lattices [7], unless $\mathcal{H}$ is large enough, though being unnatural. Therefore, the term $K$-cycle in this paper is in a weak form.

II Direct Product $K$-Cycle over Finitely Generated Abelian Group

Lattice can be fitted into a more general category, the category of finitely generated abelian group (FGAG). Since the classification of FGAG is totally clear in group theory, it is possible to assign canonical $K$-cycle over this category. In fact, any FGAG can be uniquely decomposed
as a direct product group whose factors are either \( \mathbb{Z} \) or \( \mathbb{Z}_{p^k} \), where \( p \) is a prime number and \( k = 1, 2, 3, \ldots \). Hence, if a canonical \( K \)-cycle is settled for each fundamental block, \( \mathbb{Z} \) and \( \mathbb{Z}_{p^k} \), then \( K \)-cycle for any FGAG can be naturally defined by direct product of \( K \)-cycles of each factor group. Note importantly that \( \mathbb{Z}_2 \) factors will not be considered in this letter, due to their speciality; treatment of differential structure on direct product of \( \mathbb{Z}_2 \) can be found in [9].

Let \( G \) be a FGAG without \( \mathbb{Z}_2 \) factors,

\[
G = \bigotimes_{f=1}^{N} G_f, G_f \in \{ \mathbb{Z}, \mathbb{Z}_{p^k} : k \in \mathbb{N}, \text{if } p > 2; k \in \mathbb{N} \setminus \{1\}, \text{if } p = 2 \}
\]

Fixing one \( f \), a canonical \( K \)-cycle is specified as below. \( A_f \) is the algebra of complex functions over \( G_f \); if \( G_f \) is \( \mathbb{Z} \), then the functions are required to be bounded. \( \mathcal{H}_f = l^2(G_f) \otimes \mathbb{C}^2 \) gives the Hilbert space; the choice of factor \( \mathbb{C}^2 \) will be clarified below. \( A_f \) is represented on \( \mathcal{H}_f \) by multiplication \( \pi_f(u) = u \otimes \text{Id}, \forall u \in A_f \). The generator of \( G_f \) is write as \( T_f \), and it induces the regular representation of \( G_f \) on \( A_f \) in which the image of \( T_f \) is denoted as \( R_{T_f} \). Define formal partial derivatives \( \partial_{T_f} = R_{T_f} - \text{Id} \), and \( \partial_{T_f^{-1}} = R_{T_f^{-1}} - \text{Id} \), then Dirac operator is specified as

\[
\mathcal{D}_f = b^\dagger \partial_{T_f} + b \partial_{T_f^{-1}}
\]

where \( b, b^\dagger \) is a pair of standard fermionic annihilation/creation operators, being represented irreducibly on \( \mathbb{C}^2 \). Grading \( \gamma_f \) is \([b^\dagger, b]/2\). One can verify that \( \mathcal{D}_f^\dagger = \mathcal{D}_f \), \( \mathcal{D}_f^2 = \partial_{T_f} \partial_{T_f^{-1}} \), \([\pi(A_f), \gamma_f] = 0 \), and \( \{\mathcal{D}_f, \gamma_f\} = 0 \). The direct product \( K \)-cycle on \( G \) can be established straightforwardly. Namely following Definition 4, \( \mathcal{A}[G] = \bigotimes_f \mathcal{A}_f \), \( \mathcal{H} = \bigotimes_f \mathcal{H}_f \),

\[
\mathcal{D} = \mathcal{D}_1 \otimes \text{Id} \otimes \text{Id} \otimes \ldots \otimes \text{Id} + \gamma_1 \otimes \mathcal{D}_2 \otimes \text{Id} \otimes \ldots \otimes \text{Id} + \ldots + \gamma_1 \otimes \gamma_2 \otimes \ldots \otimes \gamma_{N-1} \otimes \mathcal{D}_N,
\]

\[
\gamma = \gamma_1 \otimes \gamma_2 \otimes \ldots \otimes \gamma_3.
\]

A \( d \)-dimensional lattice can be considered as \( \mathbb{Z}^d \), so that the above canonical \( K \)-cycle for FGAG can be applied to this lattice simply.
III Staggered Formalism as Direct Product \( K \)-Cycle

Staggered Dirac operator has standard expression

\[
D_S = i\eta^\mu \nabla_\mu
\]

where \( \nabla_\mu \) is symmetric difference operator defined as \( (T_\mu - T_{-\mu})/2 \), \( \eta^\mu \) is called staggered phase expressed as \( (-)^{\sum_{\nu < \mu} x_\nu} \), and an additional “i” is inserted in this definition since here a selfadjoint convention is adopted instead of an anti-selfadjoint one. The main advantage of staggered formalism is that there exists a chirality \( \gamma_S = (-)^{x_1 + x_2 + \cdots + x_d} \), such that \( \{D_S, \gamma_S\} = 0 \). Observe Eqs.(2) and Eq.(4), staggered formalism can be cast into a direct product \( K \)-cycle in the following way. Let \( \mathbb{Z}_d = \bigotimes_{\mu=1}^d \mathbb{Z}[\mu] \), and for each factor, let \( \mathcal{A}_{[\mu]} \) be algebra of bounded functions on \( \mathbb{Z}_{[\mu]} \), \( \mathcal{H}_{[\mu]} \) be the Hilbert space which is the restriction of \( \mathcal{A}_{[\mu]} \) by \( \ell^2 \)-condition; Dirac operator for \( \mathbb{Z}_{[\mu]} \) is just \( i\nabla_\mu \) and grading is taken to be chirality along \( \mu \)-direction, \( \gamma_{[\mu]} = (-)^x \). Then it is easy to check the direct product property of staggered formalism.

\[
-iD_S = \nabla_1 + \gamma[1] \nabla_2 + \gamma[1] \gamma[2] \nabla_3 + \cdots + \gamma[1] \gamma[2] \cdots \gamma[d-1] \nabla_d
\]

The canonical \( K \)-cycle \((\mathcal{A}[\mathbb{Z}], \mathcal{H}, D, \gamma)\) of d-dimensional lattice defined in the last section is equivalent to the direction product \( K \)-cycle \((\mathcal{A}[\mathbb{Z}], \mathcal{H}_S, D_S, \gamma_S)\) for staggered fermion, if and only if the equivalence holds for each factor, namely Dirac operator defined in Eq.(1) is equivalent to \( i\nabla \) for \( \mathbb{Z} \). However, this statement has been shown in [4] for one-dimensional lattice. For the integrality of this letter, this proof is rewritten below.

Staggered \( K \)-cycle of one-dimensional lattice is \((\mathcal{A}[\mathbb{Z}], \mathcal{A}[\mathbb{Z}], i\nabla, (-)^x)\). The spectral of \( \mathcal{A}[\mathbb{Z}] \) correspondences to \( \mathbb{Z} \) by Gelfand-Naimark theorem [11], namely there is bijection between pure states \( \{|x\}\} \) over \( \mathcal{A}[\mathbb{Z}] \) and \( \mathbb{Z} \). Moreover, \( \{|x\}\} \) also provides the Hilbert space with a basis. Now define fermionic operators \( c^\dagger \) by \( c^\dagger |2n\rangle = \sqrt{2} |2n+1\rangle \), \( c^\dagger |2n+1\rangle = 0 \), and \( c \) by conjugation. It is obvious that \( c^2 = 0 \), \( c^\dagger c = 2 \), \( \{c, c^\dagger\} = 2 \). Reparametrize \( \{|x\}\} \) as \( \{|n\}\} = |2n\rangle, c^\dagger |n\rangle = |2n+1\rangle \), which is the reformulation of well-known “half-spacing transformation” in [2]. Then one can
check that under this parametrization,

$$i \nabla = \frac{i}{2}(c^\dagger \partial_T^\dagger - c \partial_T)$$  \hspace{1cm} (5)

where $\partial_T |n\rangle = |n+1\rangle - |n\rangle$, and $[c, \partial_T] = 0$. So, identify that $c = ib^\dagger$ in Eqs.(1)(5), the correspondence between staggered fermion and NCG is proved for lattice of any dimension $d$.

IV Geometric Interpretation

1) The result in the last section is not very striking in the sense that a lot of clues have been shown in in [2] where the so-called spin diagonalization tech were adopted extensively. What appears as a surprise is such a simplicity of the proof in the language of direct product $K$-cycle which is almost a trivial object in NCG.

2) The proof within one factor inspires a geometric interpretation of staggered formalism. In fact, when a continuum space, say $\mathbb{R}^4$, is discretized, the tangent space at each point of the resulting lattice can be discretized also to be a linear space over character-2 field, namely for each direction, there are only two points in the tangent space, corresponding to two states $|n\rangle, c^\dagger |n\rangle$, for all $n \in \mathbb{Z}$. The chirality can be expressed in operator form $\gamma = e^{i\pi c^\dagger c}$ which is just $R$-parity in supersymmetry.

3) $\gamma$ is essentially a distinguish between even and odd numbers, hence only exists globally for prime factor groups $\mathbb{Z}$ and $\mathbb{Z}_{2k}, k = 1, 2, ....$

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