GENERAL MANIPULABILITY THEOREM FOR A
MATCHING MODEL

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ABSTRACT: In a many-to-many matching model in which agents’ preferences satisfy substitutability and the law of aggregate demand, we proof the General Manipulability Theorem. We result generalizes the presented in Sotomayor (1996 and 2012) for the many-to-one model. In addition, we show General Manipulability Theorem fail when agents’ preferences satisfy only substitutability.

KEYWORDS: Many-to-many matching model, Manipulability, Matching stable rule, Matching game, Law of aggregated demand.

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1 Introduction

Many-to-many matching models have been useful for studying assignment problems with the distinctive feature that agents can be divided into two disjoint subsets: the set of firms and the set of workers. The nature of the assignment problem consists of matching each agent with a subset of agents from the other side of the market. Thus, each firm may hire a subset of workers while each worker may work for a number of different firms.

Stability has been considered the main property to be satisfied by any matching. A matching is called stable if all agents have acceptable partners and there is no unmatched worker-firm pair who both would prefer to be
matched to each other rather than staying with their current partners. Unfortunately, the set of stable matchings may be empty. Substitutability is the weakest condition that has so far been imposed on agents’ preferences under which the existence of stable matchings is guaranteed. An agent has substitutable preferences if he wants to continue being matched to an agent on the other side of the market even if other agents become unavailable.

The college admissions problem is the name given by Gale and Shapley (1962) to a many-to-one matching model. Colleges have responsive preferences over students and students have preference over colleges; each college $c$ has a maximum number of positions to be filled (its quota $q_c$), it ranks individual students and orders subsets of students in a responsive manner (Roth 1985); namely, to add “good” students to a set leads to a better set, whereas to add “bad” students to a set leads to a worst set. In addition, for any two subsets that differ in only one student, the college prefers the subset containing the most preferred student. In this model the set of stable matchings satisfy the following additional properties: (i) there is a polarization of interests between the two sides of the market along the set of stable matchings, (ii) the set of unmatched agents is the same under every stable matching, (iii) the number of workers assigned to a firm through stable matchings is the same, and (iv) if a firm does not complete its quota under some stable matching then it is matched to the same set of workers at any stable matching.

The case in which all quotas are equal to one is called the marriage problem and is symmetric between the two sides of the market. The college admissions problem with substitutable preferences is the name given by Roth and Sotomayor (1990) to the most general many-to-one model with ordinal preferences. Firms are restricted to have substitutable preferences over sub-

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1Hatfield and Kojima (2010) in matching models with contracts introduce a weaker condition called bilateral substitutability and show that this condition is sufficient for the existence of a stable matching. Also, they consider a strengthening of the bilateral substitutability condition called unilateral substitutability. Both conditions reduce to standard substitutability in matching problems without contracts. See Definitions 3 and 5 in Hatfield and Kojima (2010) for a precise and formal definition of bilateral and unilateral substitutability, respectively.

2Kelso and Crawford (1982) were the first to use substitutability to show the existence of stable matchings in a many-to-one model with money. Roth (1984) shows that, if all agents have substitutable preferences, the set of many-to-many stable matchings is non-empty.

3Property (i) is a consequence of the decomposition lemma proved by Gale and Sotomayor (1985). Properties (ii) and (iii) were proved independently by Gale and Sotomayor (1985) and Roth (1984). Property (iv) was proved by Roth (1986).

4It is the name given to the one-to-one matching model. See Roth and Sotomayor (1990) for a precise and formal definition of such model.
sets of workers, while workers may have all possible preferences over the set of firms. Under this hypothesis Roth and Sotomayor (1990) showed that the deferred-acceptance algorithms produce either the firm-optimal stable matching or the worker-optimal stable matching, depending on whether the firms or the workers make the offers. The firm (worker)-optimal stable matching is unanimously considered by all firms (respectively, workers) to be the best among all stable matchings.

The adoption of a specific rule for some matching model induces a strategic game where the players are the agents of the model, and the strategies are the preferences they can state. The payoff function is defined by the given matching rule. Questions on incentives facing agents naturally emerge.

The first important result in this direction is the Non-Manipulability Theorem due to Dubins and Freedman (1981) in the marriage model. These authors proved that: “under the X-stable matching rule, the agents of the side of the market X of any coalition cannot get preferred mates by falsifying their preferences”\(^5\). In addition, in the college admissions problem, this result is true of the side of students (Dubins and Freedman, 1981) and Roth (1985) showed this result is false of the side of college. In a more general preference domain, for example, in the many-to-one matching model with substitutable and \(q\)-separable preferences the Non-Manipulability Theorem is true of the side of workers (Martínez et. al, 2004) and is false of the side of firms (Roth, 1985). In the many-to-one matching model with substitutable preferences the Non-Manipulability Theorem is false in both side of the market (Roth, 1985 and Martínez et. al, 2004).

The second important result is the General Manipulability Theorem: “If the matching produced by the allocation rule is not the optimal stable matching for one of the sides of the market, it is always true that some participant of this side of the market can be better off by misrepresent his/her/its preferences”. In the college admission problem this result was prove by Sotomayor (1996) of the side of students and Sotomayor (2012) of the side of college.

The third important result is a General Impossibility Theorem for the college admission model: “Under any stable matching rule for a given college admission problem, in which there is more than one stable matching, at least one agent can profitably misrepresent his/her/its preferences, assuming the others tell the truth”. This result was prove by Sotomayor (2012).

Another important result is the Impossibility Theorem: “No stable matching rule for the general matching problem exists for which truthful revelation of preferences is a dominant strategy for all agents”. In the marriage

\(^5\)Here, we considerer X the set agents of one side of the market.
model this result was prove by Roth (1982) and in the college admission was prove by Roth (1985).

The Non-Manipulability Theorem and the General Manipulability Theorem are the central results of the theory on incentives for the matching model. They gave origin to or motivated all important results of the theory of stable matching rules for those models.

Note that if General Manipulability Theorem is true, then General Impossibility Theorem and Impossibility Theorem holds.

The strict inclusion relationships between the preference domains (q-responsive implies q-separability and q-separability implies law of aggregated demand), and the absence of inclusion relationship with respect to substitutability. The domain of q-separable and substitutable preferences is much richer than the domain of responsive preferences. For instance, consider the case where larger coalitions are preferred to smaller coalitions (in terms of cardinality). Then, responsiveness imposes a substantial number of restrictions on the ranking of coalitions of the same size, whereas separability (together with substitutability) does not impose any restriction at all.

Hence, in this work, we generalize the General Manipulability Theorem to a many-to-many matching model such that agents’ preferences satisfy substitutability and law of aggregated demand. In addition, we show that General Manipulability Theorem fail when the agents’ preferences satisfy only substitutability.

The paper is organized as follow: In Section 2, we present the model. In Section 3, we present the preliminaries of matching game and state some results, already proved in the literature, which will be needed in Section 4. In Section 4, we present the manipulability property and the main result. In Section 5, we present two example show the main result fail when the model only have substitutable preferences.

2 The Model

There are two finite and disjoint sets of agents, the set of $n$ firms $F = \{f_1, ..., f_n\}$ and the set of $m$ workers $W = \{w_1, ..., w_m\}$. To simplify the notation, sometimes, we denote by $f$ (instead of $f_i$) whichever firm in $F$. Similarly, we denote by $w$ (instead of $w_j$) whichever worker in $W$. Each firm $f \in F$ has a strict linear ordering $\succ_f$ over $2^W$. And each worker $w \in W$ has a strict linear ordering over $2^F$. Preferences profiles are $(n+m)$-tuples $^6$

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$^6$The college admission model is a many-to-one matching model with $q$-responsive preferences.
of preference relations and they are represented by $\succeq = (\succeq_{f_1}, \ldots, \succeq_{f_n}, \succeq_{w_1}, \ldots, \succeq_{w_m}) = ((\succeq_f)_{f \in F}, (\succeq_w)_{w \in W})$.

We denote by $a \in F \cup W$ a generic agent of either set. Given a preference relation of an agent $\succeq_a$, the subsets of partners preferred to the empty set by $a$ are called acceptable.

To express preference relations in a concise manner, and since only acceptable sets of partners will matter, we will represent preference relations as lists of acceptable partners. For instance, $\succeq_{f_i}: \{w_1, w_3\}, \{w_2\}, \{w_1\}, \{w_3\}$ and $\succeq_{w_j}: \{f_1, f_3\}, \{f_1\}, \{f_3\}$, indicate that $\{w_1, w_3\} \succeq_{f_i} \{w_2\} \succeq_{f_i} \{w_1\} \succeq_{f_i} \{w_3\} \succeq_{f_i} \emptyset$ and $\{f_1, f_3\} \succeq_{w_j} \{f_1\} \succeq_{w_j} \{f_3\} \succeq_{w_j} \emptyset$.

The assignment problem consists of matching workers with firms keeping the bilateral nature of their relationship and allowing for the possibility that both, firms and workers, may remain unmatched. Formally,

**Definition 1** A matching $\mu$ is a mapping from the set $F \cup W$ into the set of all subsets of $F \cup W$ such that, for all $w \in W$ and $f \in F$:

1. $\mu(f) \in 2^W$.
2. $\mu(w) \in 2^F$.
3. $w \in \mu(f)$ if and only if $f \in \mu(w)$.

We say that an agent $a$ is single in a matching $\mu$ if $\mu(a) = \emptyset$. Otherwise, the agent is matched. A matching $\mu$ is said to be one-to-one if firms can hire at most one worker, and workers can work for at most one firm. The model in which all matchings are one-to-one is also known in the literature as the marriage model. A matching $\mu$ is said to be many-to-one if workers can work for at most one firm but firms may hire many workers.

Given a set of workers $S \subseteq W$, each firm $f \in F$ can determine which subset of $S$ would most prefer to hire. We will call this $f$’s choice set from $S$, and denote it by $Ch(S, \succeq_f)$. Formally:

$$Ch(S, \succeq_f) = \max_{\succeq_f} \{T : T \subseteq S\}$$

Symmetrically, given a set of firms $S \subseteq F$ for each worker $w \in W$, we define:

$$Ch(S, \succeq_w) = \max_{\succeq_w} \{T : T \subseteq S\}$$

\[\text{We will often abuse notation by omitting the brackets to denote a set with a unique element. For instance here, we write } w \in \mu(f) \text{ instead of } \{w\} \in \mu(f).\]
A matching $\mu$ is blocked by agent $a$ if $\mu(a) \neq Ch(\mu(a), \succ_a)$. We say that a matching is individually rational if it is not blocked by any individual agent.\footnote{In a many-to-one model, a matching $\mu$ is individually rational (at $\succ$) if $\mu(w) \succeq_w \emptyset$, for all $w \in W$ and $\mu(f) = Ch(\mu(f), \succ_f)$ for all $f \in F$. Note, $\mu(w) \succeq_w \emptyset$ is equivalent to $\mu(w) = Ch(\mu(w), \succ_w)$.} Denote by $IR(\succ)$ the set of individually rational matchings at $\succ$. A matching $\mu$ is blocked by a worker-firm pair $(w, f)$ if $w \notin \mu(f)$, $w \in Ch(\mu(f) \cup \{w\}, \succ_f)$, and $f \in Ch(\mu(w) \cup \{f\}, \succ_w)$. A matching $\mu$ is pairwise-stable (stable, for short) if it is not blocked by any individual agent or any worker-firm pair.

We restrict our attention to a many-to-many matching model where the agents’ preferences satisfy substitutability and Law of Aggregated Demand (LAD, for short).\footnote{Law of Aggregated Demand was introduce by Alkan (2002), and it called cardinal monotone. Hatfield and Milgrom (2005) in a model with Contracts call to this condition Law of Aggregated Demand.}

**Definition 2** A firm $f$’s preference relation $\succ_f$ satisfies substitutability if for any set $S'$ containing workers $w$ and $w'$ ($w \neq w'$), if $w \in Ch(S', \succ_f)$ then $w \in Ch(S' \setminus \{w'\}, \succ_f)$.

That is, if $f$ has substitutable preferences, then if its preferred set of employees from $S$ includes $w$, so will its preferred set of employees from any subset of $S$ that still includes $w$. A preference profile $\succ$ is substitutable if for each firm $f$, the preference relation $\succ_f$ satisfies substitutability.

In an analogous manner, is defined that each worker $w \in W$ has substitutable preferences over $2^F$.

**Definition 3** The preference $\succ_f$ of a firm $f \in F$ satisfies law of aggregated demand (LAD) if for all $Y \subseteq X \subseteq W$:

$$|Ch(Y, \succ_f)| \leq |Ch(X, \succ_f)|.$$  

A preference profile $(\succ_f)_{f \in F}$ satisfies LAD if for each firm $f \in F$, the preference relation $\succ_f$, satisfies LAD.

In an analogous manner, is defined that each worker $w \in W$ satisfies law of aggregated demand.

Let $M = (F, W, \succ, LAD)$ be a specific many-to-many matching model such that the agents satisfy substitutability and LAD. We denote by $S(M)$ the set of all stable matchings in the model $M$. 

\[8\]
For each \( a \in F \cup W \), we will use the notation \( \mu \succeq_a \mu' \) which will denote that \( \mu(a) \succ_a \mu'(a) \) or \( \mu(a) = \mu'(a) \); \( \mu \succeq_a \mu' \) if \( \mu \succeq_a \mu' \) and \( \mu(a) \neq \mu'(a) \). We denote \( \mu \succeq_F \mu' \) if for each \( f \in F \), \( \mu \succeq_f \mu' \). We denote \( \mu \succ_F \mu' \) if \( \mu \succeq_F \mu' \) and \( \mu \neq \mu' \). Similarly, we denote \( \mu \succeq_W \mu' \) and \( \mu \succ_W \mu' \). Blair (1988) defines a partial order in the follow way: for each agent \( a \in F \cup W \), we denote the Blair order by \( \mu \succ \mu' \). We denote \( \mu \succeq_B \mu' \) if \( \mu \succeq_B \mu' \) and \( \mu \neq \mu' \).

**Definition 4** For a given market \((F, W, \succ)\), the stable matching \( \mu_W \) is called the worker-optimal stable matching if \( \mu_W \succeq_W \mu \) for every stable matching \( \mu \). The firm-optimal stable matching \( \mu_F \) satisfy that \( \mu_F \succeq_F \mu \), for every stable matching \( \mu \).

**Remark 1** The firm-optimal stable matching \( \mu_F \) satisfy \( \mu_F \succeq_B \mu \), for every stable matching \( \mu \). In an analogous manner, for \( \mu_W \).

**Remark 2** Note that substitutability does not imply LAD and LAD does not imply substitutability. The preference relation \( \succ_f : w_2, w_1 w_3, w_1, w_3, \emptyset \) shows that not all substitutable preference relations satisfy LAD.

## 3 Matching Game: Preliminaries

Given the market \((F, W, \succ)\), consider that each agent \( a \in F \cup W \) may replace his/her true preference list, \( \succ_a \), by any list of preference \( \succ'_a \), and \( \succ' = (\succ'_a, \succ'_a) \) denote the profile of such lists of preference, where \( \succ'_a \) indicates the restriction of \( \succ \) to \([F \cup W] \backslash \{a\}\). Once \( \succ' \) is selected, \( W, F \), and \( \succ' \) are used as “input” for some algorithm that yields a stable matching for the market \((F, W, \succ')\), as a final “output”. This procedure is described by a function \( h \) that we call stable matching rule. For each profile of preferences \( \succ' \), \( h(\succ') \) is the stable matching for \((F, W, \succ')\), which is selected by \( h \).

Two special stable matching rules are the worker-optimal stable matching rule \( h_W \) and the firm-optimal stable matching rule \( h_F \). Under the first one (respectively second one) the participants are assigned in accordance with the worker-optimal (respectively firm-optimal) stable matching for \((F, W, \succ')\). If \( \succ' \) is a profile of preferences, we denote by \( h_W(\succ') = \mu_W(\succ') \) the worker-optimal stable matching for \((F, W, \succ')\) and by \( h_F(\succ') = \mu_F(\succ') \) the firm-optimal stable matching for \((F, W, \succ')\).

The adoption of a stable matching rule \( h \) for a given market \((F, W, \succ)\) induces a strategic game where the set of players is the set of agents \( F \cup W \);

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\(^{10}\)We denote \( w_1 w_3 = \{w_1, w_3\} \)
a strategy of agent \( a \in F \cup W \) is any list of preferences \( \succ'_a \); the outcome function is determined by \( h \) and the true preferences of the players, called true or sincere strategies, are given by \( \succ \). We refer to this game as the matching game induced by \( h \), and we denote by \( (F \cup W, \sqcup, h, \succ) \) where, \( F \cup W \) is the set of agents, \( \sqcup \) is the set of preference profile such that all \( \succ' \in \sqcup \) is substitutable, \( h \) is the matching stable rule and \( \succ \) is the true or sincere profile of strategy. We denote by \( (F \cup W, \sqcup, h, \succ, LAD) \) the matching game where for all \( \succ' \in \sqcup \), and \( \succ \) are substitutable and \( LAD \).

In a many-to-many matching model, the following propositions will be needed for the proofs of our results.

**Proposition 1**  
a) **(Alkan, 2002)** Consider the matching market with strict substitutable and LAD preferences. Each agent is matched with the same number of partners in every stable matching.

b) **(Klijn and Yazici, 2011)** Consider the matching market with strict, substitutable and \( q_X \)-separable preferences. Then, if an agent does not fill its quota at some stable matching, it is matched to the same set of agents at every stable matching.

### 4 Manipulability Property

We assume matching model \((F, W, \succ)\) is substitutable.

Given the matching model \((F, W, \succ)\), \( f \in F \), and \( T \subseteq W \) satisfying \( T = Ch(T, \succ_f) \) we define \( \succ_f \mid_T \) the preference \( \succ_f \) restricted to \( T \) as, for all \( S' \subseteq W \),

i. \( S' \nsubseteq T \) then \( \emptyset \succ_f \mid_T S' \),

ii. \( S' \subseteq T \), and \( S' \succ_f \emptyset \) if and only if \( S' \succ_f \mid_T \emptyset \), and

iii. \( S', S'' \subseteq T \) and \( S' \succ_f S'' \) if and only if \( S' \succ_f \mid_T S'' \).

In an analogous manner, is defined \( \succ_w \mid_T \) for a worker \( w \in W \) and \( T \subseteq F \).

**Definition 5** Let \((F \cup W, \sqcup, h, \succ)\) be a matching game, we say that \( a \in F \cup W \) satisfies a **manipulability property** if \( h(\succ)(a) \neq \mu_X(\succ)(a) \) (where, either \( X = F \), if \( a \in F \) or \( X = W \), if \( a \in W \)), then there exists \( \prec'_a \) such that \( h(\prec'_{a}) \succ_a h(\succ) \).

**Definition 6** Let \((F \cup W, \sqcup, h, \succ)\) be a matching game, we say that \( a \in F \cup W \) satisfies a **manipulability property with Blair order** if \( h(\succ)(a) \neq \mu_X(\succ)(a) \) (where, either \( X = F \), if \( a \in F \) or \( X = W \), if \( a \in W \)), then there exists \( \prec'_a \), such that \( h(\prec'_{a}) \succ_a h(\succ)(a) \).
The following example show in a many-to-one model with substitutable and $q$-separable preferences profile, an agent can manipulate and the result of the stable rule $h$ is a non stable matching in the true profile strategy.

**Example** Let $(F, W, \succ)$ be where $F = \{f_1, f_2, f_3\}$, $w = \{w_1, w_2, w_3, w_4\}$, such that the agents preferences satisfy substitutability and $q$-separability. The preferences profile $\succ$ is given by:

\[
\begin{array}{ccc}
\succ_{f_1}: w_2w_3, w_2w_4, w_1w_3, w_1w_2, w_1w_4, w_3w_4, \\
\succ_{f_2}: w_1, w_2, w_3, w_4, \\
\succ_{f_3}: w_4, w_1
\end{array}
\]

\[
\begin{array}{c}
\succ_{w_1}: f_1, f_3, f_2, \\
\succ_{w_2}: f_2, f_1, \\
\succ_{w_3}=\succ_{w_4}: f_1, f_3.
\end{array}
\]

We have that

\[
\begin{array}{ccc}
\mu_F(\succ) & : & f_1 \quad f_2 \quad f_3 \\
\mu_W(\succ) & : & w_2w_3 \quad w_1 \quad w_4
\end{array}
\]

Consider that worker $w_1$ reject firms $f_1$ and $f_2$ i.e., $\succ_{w_1}': f_3$, and $\succ'=(\succ_{-w_1},\succ_{w_1}')=(\succ_{w_1},\succ_{w_2},\succ_{w_3},\succ_{w_4},\succ_{f_1},\succ_{f_2},\succ_{f_3})$ then $h_F(\succ') = \mu_F'$ and we have that

\[
h_F(\succ') = \mu_F' : w_3w_4 \quad w_2 \quad w_1.
\]

We note that $\mu_F' \notin S(\succ)$ because the pair $(f_1, w_1)$ is a blocking pair.

Hence, $h_F(\succ) \neq \mu_W$ and $w_1$ manipulate with $\succ_{w_1}': f_3$, since $h_F(\succ')(w_1) \succ_{w_1}$ $h_F(\succ)(w_1)$ and $h_F(\succ') = \mu_F' \notin S(\succ)$. □

We are interesting that each agent manipulate and the result of $h$ is a stable matching in the true preference profile. Then, for each $a \in F \cup W$ we define

\[
H_a = \{ \mu \in S(\succ) : \mu(a) \succ_B h(\succ)(a) \}.
\]

**Remark 3** Note that $H_f (H_w)$ is non empty because $\mu_F \in H_f (\mu_W \in H_w$, respectively).

**Lemma 1** Let $f \in F$ and $\mu \in H_f$. If $\succ''=\succ_f|_{\mu(f)}$, then $\mu \in S(\succ'')$ where $\succ''=(\succ_{-f}, \succ_f')$.

**Proof.** First, we show $\mu \in IR(\succ'')$. For all $a \in F \cup W$, if $a \neq f$ we have $\succ''_a=\succ_a$. Thus, $Ch(\mu(a), \succ'') = Ch(\mu(a), \succ_a) = \mu(a)$.

For $f$, we have $\succ''_f=\succ_f|_{\mu(f)}$, then $Ch(\mu(f), \succ'') = \mu(f)$. In effect, if $Ch(\mu(f), \succ'') \neq \mu(f)$, hence $Ch(\mu(f), \succ'') = T$ such that $T \subseteq \mu(f)$. In particular, $T \succ''_f \mu(f)$, if and only if, by definition of $\succ''_f$, $T \succ_f \mu(f)$. Thus,
\( \mu(f) \neq Ch(\mu(f), \succ_f) \), that is, \( \mu \notin IR(\succ) \). But, this contradicts that \( \mu \in S(\succ) \).

Second, there is not exist blocking pair. If there exists \( (\overline{f}, w) \) blocks \( \mu \), i.e., \( \overline{f} \in Ch(\mu(w) \cup \overline{f}, \succ'_f) \) and \( w \in Ch(\mu(\overline{f}) \cup w, \succ'_f) \).

Note that, \( \succ''_w \succ w \) and \( \succ''_f \succ \overline{f} \) if \( \overline{f} \neq f \). Then, if \( \overline{f} \neq f \) we have that \( \mu \notin S(\succ) \).

If \( \overline{f} = f \), \( Ch(\mu(f) \cup w, \succ'_f) = \mu(f) \), and this contradicts that \( w \in Ch(\mu(f) \cup w, \succ''_f) \).

Finally, \( \mu \in S(\succ''_w) \). \( \blacksquare \)

**Proposition 2** Let \( f \in F \), \( \mu \in H_f \) and \( \succ''_f = \succ_f |_{\mu(f)} \), then \( \mu(f) = h(\succ''_w)(f) \) where \( \succ'' = (\succ'_f, \succ'_w) \).

**Proof.** By Lemma 1 \( \mu \in S(\succ''_w) \) and we have that \( h(\succ''_w) \in S(\succ''_w) \). Hence by definition of order \( \succ'_f \) we have that \( h(\succ''_w)(f) \subseteq \mu(f) \). In effect, if \( h(\succ''_w)(f) \subsetneq \mu(f) \), then by definition of order \( \succ'_f \), we have that \( \emptyset \succ'_f h(\succ''_w)(f) \). But this contradicts that \( h(\succ''_w) \in S(\succ) \). Then, by Proposition 1(a), \( |h(\succ''_w)(f)| = |\mu(f)| \). Thus, \( h(\succ''_w)(f) = \mu(f) \). \( \blacksquare \)

**Proposition 3** An agent \( a \in F \cup W \) satisfies manipulability property with Blair order, if and only if satisfies manipulability property.

**Proof.** In one direction, without lost of generality we assume that \( a = f \). If \( f \in F \) satisfies firm manipulability property with Blair order, following Definition 6 there exists \( \succ'_f \) such that \( h(\succ)(f) \neq \mu_F(\succ)(f) \) and \( h(\succ'_f, \succ'_w) \succ'_f h(\succ) \). We denote \( \succ' = (\succ'_f, \succ'_w) \), and thus,

\[
\begin{align*}
\h (\succ') (f) &= Ch(h(\succ')(f) \cup h(\succ)(f), \succ_f) \\
\text{and } h(\succ')(f) &= h(\succ)(f) \neq h(\succ)(f) \text{.}
\end{align*}
\]

We define \( T = h(\succ')(f) \cup h(\succ)(f) \), then by definition of choice set we have that \( h(\succ')(f) \succ_f S \) for all \( S \subset T \). We note that, in particular, \( h(\succ)(f) \subset T \). Thus, \( h(\succ')(f) \succ_f h(\succ)(f) \) and \( h(\succ')(f) \neq h(\succ)(f) \). Finally, following Definition 5 we have that \( f \) satisfies firm manipulability property.

In other direction, without lost of generality we assume that \( a = f \). If \( f \in F \) satisfies firm manipulability property, following Definition 5 there exists \( \succ'_f \) such that \( h(\succ)(f) \neq \mu_F(\succ)(f) \) and \( h(\succ'_f, \succ'_w) \succ'_f h(\succ) \).

Let \( \mu \in H_f \) and defined \( \succ''_f = \succ_f |_{\mu(f)} \), with \( \succ'' = (\succ'_f, \succ'_w) \) the profile strategy, then by Lemma 1 we have that \( \mu \in S(\succ''_w) \). Thus, \( \mu \in S(\succ''_w) \) and \( h(\succ''_w) \in S(\succ''_w) \), hence by Proposition 2 we have that \( h(\succ''_w)(f) = \mu(f) \).
Since $\mu \in H_f$, we have that $\mu(f) \succ_B h(\succ)(f)$ and $h(\succ''(f)) = \mu(f)$. That is, we prove that if $h(\succ)(f) \neq \mu_F(\succ)(f)$ then there exists $\succ''$ such that $h(\succ''(f)) \succ_B h(\succ)(f)$, i.e., $f$ satisfies manipulability property with Blair order.

We present the main result:

**Theorem 1 (General Manipulability Theorem)** Let $(F \cup W, \sqsubseteq, h, \succ)$ be a matching game with substitutable and LAD preference profile, then for all $a \in F \cup W$ the manipulability property with Blair order holds.

**Proof.** Without lost of generality we assume that $a = f$. Let $f$ be a firm satisfies $h(\succ)(f) \neq \mu_F(\succ)(f)$. Let $\mu \in H_f$ and defined $\succ'' = \{ \mu \} \cup (\succ_f, \succ_f)$ the profile strategy, then by Lemma 1 we have that $\mu \in S(\succ'')$. Thus, $\mu \in S(\succ'')$ and $h(\succ'')(f) \in S(\succ'')$, hence by Proposition 2 we have that $h(\succ'')(f) = \mu(f)$.

Since $\mu \in H_f$, we have that $\mu(f) \succ_B h(\succ)(f)$ and $h(\succ'')(f) = \mu(f)$. That is, we prove that there exists $\succ''$ such that $h(\succ''(f)) \succ_B h(\succ)(f)$, i.e., $f$ satisfies manipulability property with Blair order. We note that, a particular case is $\mu = \mu_F$ and the strategy will be $\mu_F$.

Following corollary is consequence of the Proposition 3 and of the previous Theorem:

**Corollary 1** Let $(F \cup W, \sqsubseteq, h, \succ)$ be a matching game with substitutable and LAD preference profile, then for all $a \in F \cup W$ the manipulability property holds.

**Remark 4** As consequence of General Manipulability Theorem in a many-to-many matching game with substitutable and LAD preference profile, the General Impossibility Theorem and the Impossibility Theorem holds.

## 5 General Manipulability Theorem with substitutable preferences

In this Section, in the many-to-one model with substitutable preferences we present two examples where show the General Manipulability Theorem is false. We presents the following two examples due to this model does not symmetric. Sotomayor (2012) proved the General Manipulability Theorem for the case many-to-one with responsive preferences. Thus, both examples
show Sotomayor (2012) results’ can not be generalized to substitutable preference profile. The Example 1, shows the firms can not manipulated, and Example 2, shows the workers can not manipulated.

Example 1 Let \( F, W, \succ \) be where \( F = \{ f_1, f_2, f_3 \} \), \( w = \{ w_1, w_2, w_3, w_4 \} \), and the preferences profile \( \succ \) is given by:

\[
\begin{align*}
\succ_{f_1} &: w_1w_2, w_1, w_2, w_3w_4, w_3, w_4, \\
\succ_{f_2} &: w_3, w_1w_4, w_4, w_1w_2, w_1, w_2, \\
\succ_{f_3} &: w_4, w_2w_3, w_1w_2, w_3, w_1, w_2,
\end{align*}
\]

\( \succ_{w_1} = \succ_{w_2} : f_2, f_3, f_1, \)

\( \succ_{w_3} : f_1, f_3, f_2, \)

\( \succ_{w_4} : f_1, f_2, f_3. \)

Let \( h_W \) be a optimal-worker stable matching rule, i.e., for all preference \( \succ' \), \( h_W(\succ') = \mu_W(\succ') \).

Note that \( \succ \) is substitutable but does not satisfy LAD. For example, consider the preference \( \succ_{f_1} \), and \( A = \{ w_2, w_3, w_4 \} \), and \( B = \{ w_3, w_4 \} \). Thus, \( B \subseteq A \), \( Ch(A, \succ_{f_1}) = \{ w_3 \} \) and \( Ch(B, \succ_{f_1}) = \{ w_3, w_4 \} \). Hence, \( |Ch(B, \succ_{f_1})| > |Ch(A, \succ_{f_1})| \), that is, the preferences \( \succ_{f_1} \) does not satisfy LAD. In the same way, the preferences \( \succ_{f_2} \) and \( \succ_{f_3} \) do not satisfy LAD.

We have that

\[
\begin{align*}
\mu_W(\succ) &= \{ w_3w_4, w_1w_2, \emptyset \}, \\
\mu_F(\succ) &= \{ w_1w_2, w_3, w_4 \}.
\end{align*}
\]

Consider that firm \( f_1 \) reject workers \( w_3 \) and \( w_4 \) i.e., \( \succ_{f_1}' = \succ_{f_1} \mid \mu_F(\succ)(f_1) : w_1w_2, w_1, w_2 \) and \( \succ' = (\succ_{f_1}', \succ_{f_1}') \) then \( h_W(\succ') = \mu_W(\succ') \) and we have that

\[
\begin{align*}
\mu_W(\succ') &= \{ w_3w_4, w_1w_2, w_2w_3 \}, \\
\mu_F(\succ') &= \{ w_3w_4, w_1w_2, w_2w_3 \}.
\end{align*}
\]

Thus, \( h_W(\succ')(f_1) = \emptyset \) and \( h_W(\succ')(f_1) = \{ w_3, w_4 \} \). Hence, \( h_W(\succ)(f_1) \succ_{f_1} h_W(\succ')(f_1) \), in particular \( h_W(\succ)(f_1) \succ_{B_1}^B h_W(\succ')(f_1) \). Therefore, firm \( f_1 \) has not incentive to misrepresent its preference.

Consider that firm \( f_2 \) reject any set that contain worker \( w_1, w_2 \) and \( w_4 \), i.e., \( \succ_{f_2}' : w_3 \) and \( \succ' = (\succ_{f_2}', \succ_{f_2}') \) then \( h_W(\succ') = \mu_W(\succ') \) we have that

\[
\begin{align*}
\mu_W(\succ') &= \{ w_3w_4, w_1w_2, \emptyset \}, \\
\mu_F(\succ') &= \{ w_3w_4, w_1w_2, \emptyset \}.
\end{align*}
\]

Thus, \( h_W(\succ')(f_2) = \emptyset \) and \( h_W(\succ')(f_2) = \{ w_1, w_2 \} \). Hence, \( h_W(\succ)(f_2) \succ_{f_2} h_W(\succ')(f_2) \), in particular \( h_W(\succ)(f_2) \succ_{B_2}^B h_W(\succ')(f_2) \). Therefore, firm \( f_2 \) has not incentive to misrepresent its preference.

\[\text{\textsuperscript{11}}\text{Remember, substitutability is a weaker condition than responsiveness.}\]
Consider that firm $f_3$ reject any set that contain worker $w_1, w_2$ and $w_3$, i.e., $\succ_{f_3}': w_4$ and $\succ' = (\succ_-f_3, \succ_{f_3}')$ then $h_W(\succ') = \mu_W(\succ')$ we have that

$$h_W(\succ') = \mu_W(\succ') : f_1 \ f_2 \ f_3 \ w_3 w_4 \ w_1 w_2 \ \emptyset.$$  

Thus, $h_W(\succ')(f_3) = \emptyset$ and $h_W(\succ)(f_3) = \emptyset$. Hence, $h_W(\succ)(f_3) = h_W(\succ')(f_3)$. Therefore, firm $f_3$ has not incentive to misrepresent its preference. \qed

**Example 2**  Let $(F, W, \succ)$ be where $F = \{f_1, f_2\}$, $W = \{w_1, w_2, w_3, w_4\}$, and the preferences profile $\succ$ is given by;

\[
\begin{align*}
\succ_{f_1}: & w_1 w_2, w_1, w_2, w_3 w_4, w_3, w_4, w_1, w_2, \succ_{w_1} = \succ_{w_2}: f_1, f_1, \\
\succ_{f_2}: & w_1 w_2, w_1, w_2, w_3 w_4, w_3, w_4, w_1 w_2, w_1, w_2, \succ_{w_3} = \succ_{w_4}: f_1, f_2.
\end{align*}
\]

Let $h_F$ be a firm-optimal stable matching rule.

Note that $\succ$ is substitutable but does not satisfy LAD. For example, consider the preference $\succ_{f_1}$, and $A = \{w_2, w_3, w_4\}$, and $B = \{w_3, w_4\}$. Thus, $B \subset A$, $Ch(A, \succ_{f_1}) = \{w_2\}$ and $Ch(B, \succ_{f_1}) = \{w_3, w_4\}$. Hence, $|Ch(B, \succ_{f_1})| > |Ch(A, \succ_{f_1})|$, that is, the preferences $\succ_{f_1}$ does not satisfy LAD. In the same way, the preferences $\succ_{f_2}$ does not satisfy LAD.

We have that

$$h_F(\succ) = \mu_F(\succ) : f_1 \ f_2 \ f_3 \ f_4 \ w_1 w_2 w_3 \ w_4.$$  

Then $h_F(\succ) \neq \mu_W$. Consider $\succ' = (\succ_{w_1}, \succ_{w_2}) = (\succ_{w_1} \succ_{w_2}, \succ_{w_3} \succ_{w_4}, \succ_{f_1}, \succ_{f_2})$ then $h_F(\succ') = \mu_F(\succ')$ we have that for any $\succ_{w_1}$, $h_F(\succ')(w_3) = \mu_F(\succ')(w_3) = f_2$. Hence, worker $w_1$, only can get firm $f_1$, then he has not incentive to misrepresent its preference. Similarly worker $w_2$, $w_3$, and $w_4$ have not incentive to misrepresent its true preference $\succ$. \qed

6 Concluding remarks

This paper contributes to literature in proves three important results which fill a gap in the theory of incentives for the many-to-many model. This results are responsible for the success that the incentives theory has had in explaining empirical economic phenomena in the many-to-many matching models.
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