The Einstein constraint equations on asymptotically hyperbolic manifolds.

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Licentiate Thesis
INTRODUCTION

This thesis consists of two papers:

I. Constant mean curvature solutions of the Einstein-scalar field constraint equations on asymptotically hyperbolic manifolds.

II. A large class of non constant mean curvature solutions of the Einstein constraint equations on an asymptotically hyperbolic manifold (joint with Romain Gicquaud).

The first part of this introduction contains some general theoretical background. We will recall how to formulate the Einstein field equations of General Relativity as an initial value problem and also review the role of the constraint equations in this formulation. Then we will discuss different examples of Cauchy hypersurfaces for this initial value problem and describe the class of asymptotically hyperbolic manifolds. Further we will try to explain some advantages of dealing with asymptotically hyperbolic Cauchy hypersurfaces. Finally, we will review the conformal reformulation of the constraint equations.

In the second part of the introduction we formulate the main results of Paper I and Paper II and compare them to the previous results regarding the constraint equations on asymptotically hyperbolic manifolds.

1. CONSTRAINT EQUATIONS AND ASYMPOTOTICALLY HYPERBOLIC MANIFOLDS

1.1. Constraint equations. Let us consider Einstein’s equations of General Relativity

\[ G = T. \]

Here

\[ G = \text{Ric}_{\gamma} - \frac{1}{2} R_{\gamma} \gamma \]

is the Einstein tensor of an \((n + 1)\)-dimensional Lorentz manifold \((M, \gamma)\), where \(\text{Ric}_{\gamma}\) and \(R_{\gamma}\) denote the Ricci tensor and the scalar curvature of \((M, \gamma)\) respectively. The symmetric \((0,2)\) tensor with divergence zero \(T\) is the stress energy tensor of the matter. Later we will deal with one particular choice of \(T\), namely

\[ T = d\Psi \otimes d\Psi - \left[ \frac{1}{2} |\nabla_{\gamma} \Psi|_{\gamma}^2 + V(\Psi) \right] \gamma. \]

This matter model is called a non-linear scalar field, where the scalar field \(\Psi\) is a smooth function on \(M\), and the potential \(V\) is a given smooth function from \(\mathbb{R}\) to itself. The above equation must be coupled with the equation

\[ \nabla_{\mu} \nabla^{\mu} \Psi = V'(\Psi) \]

which ensures \(\text{div} \ T = 0\). It is worth mentioning that, for certain choices of the potential, non-linear scalar fields can be used as an alternative to a positive cosmological constant, since the resulting solutions of (1) and (2) define cosmological models with accelerated expansion (see e.g. [Ren06]).

Assume now that \((M, \gamma, \Psi)\) is a solution of the Einstein-scalar field system (1)-(2). Let \(\Sigma\) be a spacelike hypersurface in \(M\), let \(h\) be the induced metric, let \(K\) be the second fundamental form, and finally let \(\psi\) and \(\pi\) be the restrictions of \(\Psi\) and
its derivative in the direction of the normal to $\Sigma$ respectively. Using the Gauss and Codazzi equations together with
\[ G_{\perp\perp} = T_{\perp\perp} \quad \text{and} \quad G_{\perp\alpha} = T_{\perp\alpha}, \]
we obtain necessary conditions which $(\bar{h}, \bar{K}, \bar{\psi}, \bar{\pi})$ must satisfy. Explicitly, the restrictions on the above data are
\[ (3) \quad R_{\bar{h}} - |\bar{K}^2_{\bar{h}} + (\text{tr} \bar{K})^2_{\bar{h}} = \bar{\pi}^2 + \left| \nabla^{\bar{h}} \bar{\psi} \right|^2_{\bar{h}} + 2V(\bar{\psi}), \]
and
\[ (4) \quad \text{div}^{\bar{h}} K - \nabla^{\bar{h}} (\text{tr} \bar{K}) = -\bar{\pi} \nabla^{\bar{h}} \bar{\psi}. \]
These equations are called the *Einstein-scalar field constraint equations*, or simply the *constraint equations*; (3) is the *Hamiltonian constraint*, and (4) is the *momentum constraint*.

Conversely, suppose that we are given an $n$-dimensional manifold $\Sigma$ with Riemannian metric $\bar{h}$, a symmetric covariant 2-tensor $\bar{K}$, and two functions $\bar{\psi}$ and $\bar{\pi}$ on $\Sigma$ such that $(\bar{h}, \bar{K}, \bar{\psi}, \bar{\pi})$ satisfy (3) and (4). Local well posedness theorems of the type developed in [FB52, CBG69] then guarantee that there exists an $(n+1)$-dimensional solution $(\Sigma \times \mathbb{R}, \gamma, \Psi)$ of the Einstein-scalar field system (1)-(2) which is consistent with the given Cauchy data $(\bar{h}, \bar{K}, \bar{\psi}, \bar{\pi})$ on the Cauchy hypersurface $\Sigma$. This is understood in the following sense: if we let $i$ denote the embedding $i : \Sigma \ni p \mapsto (p, 0) \in \Sigma \times \mathbb{R}$, then $\bar{h} = i^* \gamma$, $\Psi \circ i = \bar{\psi}$, and if $N$ is the future directed unit normal and $K$ is the second fundamental form of $i(\Sigma)$, then $i^* K = \bar{K}$ and $(N \Psi) \circ i = \bar{\psi}$.

To sum up, the above discussion shows the role of the constraint equations for the formulation of the Einstein-scalar field system as an initial value problem. They give necessary and sufficient conditions for initial data on a Cauchy hypersurface to be correctly chosen for constructing solutions of the Einstein’s equations.

1.2. Asymptotically flat spacetimes. Asymptotically flat spacetimes serve as models for isolated systems in General Relativity. In order to analyze the behaviour of a bounded gravitating object, and to define its mass, momentum, emitted radiation etc., it is natural to consider an idealized situation where the object is thought of as “being alone in the universe”. That is, it is assumed to be embedded in a spacetime which, in some appropriate sense, looks like Minkowski spacetime at large distances from the object.

One way to define asymptotically flat spacetimes is to use the concept of conformal compactification which was introduced by Penrose in the 60s. The intuitive idea is to attach boundary points to the “physical” spacetime representing end points at infinity of null geodesics. This produces an “unphysical” spacetime: a manifold with boundary whose interior is diffeomorphic to the original physical spacetime. More specifically, instead of the physical spacetime metric $\gamma$ one considers the conformally recaled metric $\delta = \Omega^2 \gamma$. If the conformal factor $\Omega$ tends to zero at the correct speed as one approaches infinity the spacetime becomes compact with respect to $\delta$. This simplifies the discussion of issues at infinity (such as e.g. radiation), because infinity can be viewed as the boundary of a compact manifold, and local arguments apply.

We define asymptotically flat spacetimes as follows [Ste90]:

**Definition 1.1.** A smooth spacetime $(M, \gamma)$ is called *asymptotically flat* if there exists another smooth Lorentz manifold $(\widehat{M}, \widehat{\gamma})$ such that

(i) $M$ is an open submanifold of $\widehat{M}$ with smooth boundary $\partial M$;
(ii) there exists a smooth function $\Omega$ on $\hat{M}$ such that $\hat{\gamma} = \Omega^2 \gamma$ on $M$ and such that $\Omega = 0$, $d\Omega \neq 0$ on $\partial M$;
(iii) every null geodesic in $M$ acquires two endpoint on $\partial M$;
(iv) $\text{Ric}_\gamma = 0$ near $\partial M$.

**Example 1.2.** The simplest particular case of the above definition is realized by Minkowski spacetime. In polar coordinates its metric is expressed as 

$$\gamma = -dt^2 + dr^2 + r^2 d\sigma^2,$$

where $d\sigma^2$ is the metric of the unit sphere. In the coordinates $u = t - r$ and $v = t + r$ defined in the non-compact region $\{ (u, v) : v - u \geq 0 \}$ the Minkowski metric becomes

$$\gamma = -du \, dv + \frac{1}{4} (v - u)^2 d\sigma^2.$$

We compactify this region by introducing the new coordinates $U$ and $V$ defined by

$$u = \tan U, \quad v = \tan V.$$

Both $U$ and $V$ are defined in the open interval $(-\pi/2, \pi/2)$ with the restriction $V - U \geq 0$. In these coordinates the metric takes the form

$$\gamma = \frac{1}{\Omega^2} [-4dUdV + \sin^2(V - U)d\sigma^2],$$

where $\Omega = 2 \cos U \cos V$. It is now obvious that $\gamma$ is not defined at points where $U = \pm \pi/2$ or $V = \pm \pi/2$, whereas $\delta = \Omega^2 \gamma$ is regular in these points. It is remarkable that the change of the coordinates $T = U + V$ and $R = V - U$ brings

$$\delta = -4dUdV + \sin^2(V - U)d\sigma^2$$

into the form

$$\delta = -dT^2 + dR^2 + \sin^2 R d\sigma^2,$$

which is exactly the metric of the Einstein static universe $\mathbb{R} \times S^3$. Therefore we can consider Minkowski spacetime as being conformally embedded into the Einstein cylinder $\mathbb{R} \times S^3$. This is demonstrated in Figure 1.

![Figure 1. The embedding of Minkowski space into the Einstein cylinder.](image-url)
The boundary of the conformally compactified Minkowski spacetime contains two three-dimensional null hypersurfaces $\mathcal{I}^+$ and $\mathcal{I}^-$, given by the conditions $V = \pi/2$, $|U| < \pi/2$ and $U = -\pi/2$, $|V| < \pi/2$ respectively. Further, the boundary contains two points $i^\pm$ with respective coordinates $U = V = \pm\pi/2$, and a point $i^0$ with coordinates $U = -V = -\pi/2$. Note that all null geodesics start at $\mathcal{I}^-$, which is called past null infinity, and end at $\mathcal{I}^+$, which is called future null infinity. All timelike geodesics start at $i^-$, called past timelike infinity, and end at $i^+$, called future timelike infinity. Finally, all spacelike geodesics start and end at the point $i^0$, called spacelike infinity.

It is convenient to represent Minkowski spacetime by means of a standard conformal diagram [Pen65] as shown in Figure 2. Each point in the interior of the triangle corresponds to a 2-sphere and the long side of the triangle corresponds to points $r = 0$ (i.e. $U = V$). The lines meeting at $i^0$ are lines of constant $t$, and the lines emanating from $i^-$ and converging into $i^+$ are the lines of constant $r$. Straight lines intersecting the long side at $45^\circ$ symbolize null cones. An arbitrary asymptotically flat spacetime can be represented by essentially the same diagram; note however that in general the unphysical metric extends smoothly beyond $\mathcal{I}$, but not to the points $i^\pm$ and $i^0$ [Fra04].

1.3. Asymptotically hyperbolic manifolds. In this section we will focus on a choice of Cauchy hypersurface for the Einstein-scalar field system (1)-(2) for an asymptotically flat spacetime on which the initial data satisfying the constraint equations (3)-(4) should be specified. For simplicity, we will discuss the case of Minkowski spacetime. The general case of an asymptotically flat spacetime can be treated similarly.

In Figure 3 we have displayed two types of spacelike hypersurfaces in Minkowski space. There are two asymptotically Euclidean spacelike hypersurfaces intersecting the time axis at two different points and reaching out to spacelike infinity. The other two hypersurfaces approach null infinity intersecting it in a two-dimensional spacelike hypersurface. It turns out that the induced metric of these hypersurfaces is asymptotically of constant negative curvature [Fra04]. Since this is a property of spacelike hyperboloids in Minkowski spacetime, such hypersurfaces are called asymptotically hyperbolic.
Figure 3. Spacelike hypersurfaces in the conformal picture.

Assume that a particle is moving along the time axis (the vertical dashed line on Figure 3), emitting radiation. For simplicity we assume that the particle emits electro-magnetic radiation which travels along the outgoing null cones to null infinity. In physical spacetime we can follow the signal up to an arbitrary, but finite distance, while in the conformal picture one can proceed up to $\mathcal{I}^+$ and read off the value of the radiation data. In this situation, the choice of spacelike hypersurface becomes important. In fact, asymptotically hyperbolic hypersurfaces are well adapted to deal with the radiation problem (see [Fra04] for more details).

Moreover, it turns out that by prescribing the initial data on a hyperboloidal hypersurface, one has better control of the asymptotic behaviour of the resulting solution of the Einstein equations. More specifically, it was shown in [Fri83] that given a hyperboloidal hypersurface $\Sigma$ which smoothly extends across $\mathcal{I}^+$ with certain smooth data given on it, the resulting unique solution of Einstein’s vacuum field equations (i.e. when (1) takes the form $G = 0$) admits a smooth conformal boundary at null-infinity in the future of $\Sigma$. Moreover, in [Fri86], a stability result was proved asserting that if the initial data is sufficiently close to Minkowski data induced on the same hyperboloidal hypersurface then the resulting solution of Einstein’s vacuum field equations behaves like Minkowski spacetime near future timelike infinity, in the sense that there is a conformal extension of the solution which contains a point $i^+$.

While it is intuitively natural to think of an asymptotically hyperbolic manifold as a non-compact manifold whose metric asymptotically approaches a negative constant curvature metric as one approaches infinity, it is more practical to work with a definition which is based on the concept of conformal compactification.

**Definition 1.3.** Let $(\Sigma, g)$ denote an oriented, compact $C^\infty$ Riemannian manifold of dimension $n \geq 3$, with nonempty boundary $\partial \Sigma$ and interior $\tilde{\Sigma}$. Assume that $\rho \in C^\infty(\Sigma)$ is such that $\rho > 0$ on $\tilde{\Sigma}$ while $\rho = 0$ and $|d\rho|_g = 1$ everywhere on $\partial \Sigma$. Then the manifold $(\tilde{\Sigma}, \tilde{g})$, where $\tilde{g} = \rho^{-2}g$, is called *asymptotically hyperbolic*.

A standard computation shows that if $(\tilde{\Sigma}, \tilde{g})$ is asymptotically hyperbolic in the sense of the above definition then its sectional curvature satisfies $K_{\tilde{g}} = -1 + O(\rho)$, which means that $(\tilde{\Sigma}, \tilde{g})$ indeed has negative constant curvature asymptotically.
1.4. The conformal method. Concluding the first part of the introduction, we will rewrite the constraint equations in a form which will facilitate their analysis. Below we review the conformal method which transforms (3)-(4) into a determined system of PDEs. The main idea is to split the Cauchy data \((\bar{h}, \bar{K}, \bar{\psi}, \bar{\pi})\) into the so called conformal data, which can be chosen freely, and the determined data, which can be found by solving the conformally formulated constraint equations.

In the case of the Hamiltonian constraint, the procedure is quite straightforward. Indeed, considered as an equation to be solved for \(\bar{h}\), it transforms into an elliptic PDE for a scalar function \(\phi > 0\) if one looks for the solution in the conformal class of a given metric \(h\), i.e. in the form \(\bar{h} = \phi^{\frac{2}{n-2}} h\). Then (3) becomes

\[
\Delta_h \phi - \frac{n - 2}{4(n - 1)} R_h \phi + \frac{n - 2}{4(n - 1)} \left( |\bar{K}|^2_h - \tau^2 + \bar{\pi}^2 + |\nabla^h \bar{\psi}|^2_h + 2V(\bar{\psi}) \right) \phi^{-\frac{n+2}{n-2}} = 0,
\]

where we have set \(\tau := \text{tr}_h \bar{K}\). Note that \(\bar{\pi} = \frac{\pi}{n} h\) is the mean curvature of the Cauchy hypersurface \(\Sigma\).

We now look at the momentum constraint as a vectorial equation for \(K\). First, we split \(\bar{K}\) into pure trace and traceless components: \(\bar{K} = \frac{\tau}{n} \bar{h} + \bar{\sigma}\). The equation (4) then becomes

\[
\text{div}_h \bar{\sigma} = \frac{n - 1}{n} \nabla^h \tau - \bar{\pi} \nabla^h \bar{\psi}.
\]

After setting \(\bar{\sigma} = \phi^{-2} \sigma'\) the momentum constraint transforms into

\[
\phi^{-\frac{2}{n-2}} \text{div}_h \sigma' = \frac{n - 1}{n} \nabla^h \tau - \bar{\pi} \nabla^h \bar{\psi},
\]

since \(\sigma\) is traceless. The general solution \(\sigma'\) of this nonhomogeneous linear system is obtained by adding a particular solution to the general solution of the associated linear homogeneous system. The latter is a traceless tensor \(\sigma\) satisfying \(\text{div}_h \sigma = 0\).

We can look for a particular solution in the form \(\mathcal{D}_h W\) for some vector field \(W\), where \(\mathcal{D}_h\) is the conformal Killing operator related to \(h\). In coordinate notation, \((\mathcal{D}_h W)_{ab} = \nabla_a W_b + \nabla_b W_a - \frac{2}{n} h_{ab} \nabla_m W^m\). Finally, (3) becomes

\[
\Delta_h \phi - \frac{n - 2}{4(n - 1)} R_h \phi + \frac{n - 2}{4(n - 1)} \left( 1 - \frac{n}{n} \tau^2 + |\sigma + \mathcal{D}_h W|^2 h_0 \phi^{-\frac{n}{n-2}} \right) \phi^{-\frac{n+2}{n-2}}
\]

and (4) becomes

\[
\text{div}_h \mathcal{D}_h W = \phi^{-\frac{n}{n-2}} \left( \frac{n - 1}{n} \nabla^h \tau - \bar{\pi} \nabla^h \bar{\psi} \right).
\]

Note that (3)–(4) comprise \(n + 1\) equations and that we have introduced \(n + 1\) unknowns, namely the scalar function \(\phi\) and the components of \(W\), which means that we should not introduce any further variables. After decomposing \(\psi\) and \(\bar{\pi}\) as \(\psi = \psi\) and \(\bar{\pi} = \phi^{-\frac{2}{n-2}} \pi\), we obtain the conformally formulated constraint equations:

\[
\Delta_h \phi - \frac{n - 2}{4(n - 1)} \left( R_h - |\nabla^h \psi|^2 h_0 \right) \phi
\]

\[
+ \frac{n - 2}{4(n - 1)} \left( |\sigma + \mathcal{D}_h W|^2 + \pi^2 \right) \phi^{-\frac{3n - 2}{n-2}}
\]

\[
- \frac{n - 2}{4(n - 1)} \left( \frac{n - 1}{n} \tau^2 - 2V(\psi) \right) \phi^{\frac{n+2}{n-2}} = 0,
\]

(5)
and
\[ \text{div}_h(D_h W) = \frac{n - 1}{n} \phi \frac{\pi}{n} \nabla^h \tau - \pi \nabla^h \psi. \]

If \( V, \psi, \) and \( \pi \) are identically zero, we obtain an important particular case of the system (5)-(6), namely the \textit{conformally formulated vacuum constraint equations}:
\[ \Delta_h \phi - \frac{n - 2}{4(n - 1)} R_h \phi + \frac{n - 2}{2(n - 1)} |\sigma + D_h W|^2 h^{-\frac{n - 2}{n}} \phi \frac{2}{n} - \frac{n - 2}{4n} \tau^2 \phi \frac{n + 2}{n} = 0, \]
and
\[ \text{div}_h(D_h W) = \frac{n - 1}{n} \phi \frac{\pi}{n} \nabla^h \tau. \]

To sum up, if for some choice of the conformal data \((h, \sigma, \tau, \psi, \pi)\) one can solve the conformally formulated constraint equations (5)-(6) for the determined data \((\phi, W)\), where \( \phi > 0 \), then
\[ \bar{h} = \phi \frac{\pi}{n} h, \]
\[ \bar{K} = \phi^{-2} (\sigma + D_h W) + \frac{\tau}{n} \phi \frac{\pi}{n} h, \]
\[ \bar{\psi} = \psi, \]
\[ \bar{\pi} = \phi \frac{\pi}{n} \pi \]
solve the constraint equations (3)-(4). Note also that when \( \tau = \text{const} \) (equivalently, when the Cauchy hypersurface \( \Sigma \) has constant mean curvature), the system (5)-(6) becomes decoupled in the sense that (6) is an equation for \( W \) only. This situation is usually referred to as the CMC case.

It is quite obvious that not every choice of \((h, \sigma, \tau, \psi, \pi)\) gives a corresponding solution of (5)-(6). For instance, consider the vacuum conformally formulated constraint equations (7)-(8) on the closed manifold \( \Sigma = S^3 \) with a round sphere metric \( h \) which has scalar curvature \( R_h = 8 \). Assume also that \( \tau = \sqrt{12} \) and \( \sigma = 0 \). The system (7)-(8) becomes
\[ \Delta_h \phi = \phi - \frac{1}{8} |D_h W|^2 h^{-7} + \phi^5, \]
and
\[ \text{div}_h(D_h W) = 0. \]

It follows from (10) that \( D_h W = 0 \), and (9) becomes
\[ \Delta_h \phi = \phi + \phi^5. \]

Multiplying both sides by \( \phi \) and integrating over \( S^3 \) one sees that this equation admits no positive solution \( \phi \).

It is therefore natural to ask: For which choices of the hypersurface \( \Sigma \) and the conformal data \((h, \sigma, \tau, \psi, \pi)\) do the conformally formulated constraint equations admit a solution \((\phi, W)\)\? Initially this question was studied in the context of vacuum spacetimes, and most of the results in this case are summarized in [BI04]. In particular, the simpler CMC case is completely resolved for \( \Sigma \) being closed, asymptotically Euclidean or asymptotically hyperbolic, while much less is known about non-CMC solutions of the vacuum conformally formulated constraint equations, however see [HNT08], [HNT09] and [Max09] for recent progress.

The first studies of the general scalar field case appeared in literature not so long ago. So far only the CMC case has been discussed, and even in this case there are large classes of the conformal data for which it still remains to determine whether (5)-(6) admits a solution or not, see [CBIP06], [CBIP07], and [HPP08].
2. Main results

In Paper I and Paper II of this thesis we present studies of the constraint equations on asymptotically hyperbolic manifolds. In this section we summarize the main results of these papers and contrast them to previous results pertaining to this case.

2.1. Paper I. The vacuum constraint equations on asymptotically hyperbolic manifolds with constant mean curvature were studied in [ACF92] and [AC96]. In [ACF92] the authors considered the situation when the second fundamental form is pure trace, i.e. $\bar{K} = \tau \bar{h}$, which implies that (8) becomes trivial and (7) turns into a prescribed scalar curvature equation in which the $\phi^{-\frac{2n-2}{n-2}}$-term is missing. They have shown that there is always a solution, which, in particular, means that any asymptotically hyperbolic manifold is conformally related to one with constant negative scalar curvature. These results were generalized in [AC96], where the authors constructed CMC solutions of the vacuum constraint equations whose second fundamental form is not pure trace assuming lower regularity of the conformal data. In both [ACF92] and [AC96] the regularity of the solution near the conformal boundary is studied in detail, and such issues as smooth extension of the resulting spacetime across $I^+$ are discussed.

Paper I is concerned with constant mean curvature solutions of the Einstein-scalar field constraint equations on asymptotically hyperbolic manifolds. In this case it is natural to assume that $\tau = n$, since the hyperboloid in Minkowski spacetime has mean curvature 1 and, consequently, $\tau = n$. The system (5)-(6) becomes decoupled. By Fredholm theorems of [Lee06] we know that (6) is solvable provided that the conformal data is regular enough. For this reason, the solvability of the Hamiltonian constraint (5), also known as the Lichnerowicz equation, becomes a central issue of the paper.

By the conformal covariance property of the Lichnerowicz equation [CBIP07] and the result of [ACF92], we can assume that the scalar curvature of the manifold is $-n(n-1)$. The equation (5) takes the form

$$\Delta_h \phi - R_{h,\psi} \phi + A_{h,W,\pi} \phi^{-\frac{2n-2}{n-2}} - B_{\tau,\psi} \phi^{\frac{n+2}{n-2}} = 0,$$

where

$$R_{h,\psi} = \frac{n-2}{4(n-1)} (-n(n-1) - |\nabla h|_h^2) < 0,$$

$$A_{h,W,\pi} = \frac{n-2}{4(n-1)} (|\sigma + D_h W|^2_h + \pi^2) \geq 0,$$

and

$$B_{\tau,\psi} = \frac{n-2}{4(n-1)} (n(n-1) - 2V(\psi)).$$

We look for the solutions of (11) satisfying $\phi \to 1$ as $\rho \to 0$. This boundary condition ensures that $\bar{h} = \phi^\frac{n}{n-2} h$ is also asymptotically hyperbolic (see Section 1.4).

Simply phrased, our non-existence result formulates as follows: if the potential $V$ is such that $B_{\tau,\psi} \leq 0$ then the Lichnerowicz equation admits no positive solution. This is an easy consequence of maximum principle for asymptotically hyperbolic geometries.

In the case when the potential $V$ is such that $B_{\tau,\psi} > 0$ we prove that (11) has a solution $\phi > 0$. Indeed, if we choose $B > 0$ and $C_+ > 0$ sufficiently large, and $C_- > 0$ sufficiently small, then $\phi_- = \max\{1-B\rho^2, C_-\}$ and $\phi_+ = \min\{1+B\rho^2, C_+\}$ are respectively a sub- and a supersolution of the Lichnerowicz equation. This
enables us to prove that there exists a positive solution of (11) such that $\phi \to 1$ as $\rho \to 0$.

If the potential $V$ is such that $B_{r,\phi} \geq 0$ is not strictly positive then the sub-
and supersolution method no longer applies, since the constant $C_+ > 0$ used in the
construction of a supersolution might not exist. In this case, we prove a theorem
aimed at facilitating the analysis of the Lichnerowicz equation. Namely, we show
that if the coefficients of (11) have certain regularity, then this equation has a
positive solution $\phi$ such that $\phi \to 1$ as $\rho \to 0$ if and only if the same equation
with $A_{h,W,\pi} \equiv 0$ has a solution with the same properties. Note that (11) with
$A_{h,W,\pi} \equiv 0$ can be interpreted as a problem of prescribed $R_{h,\psi}$.

We do not prove any result in the case which appears to be more complicated,
namely when the potential $V$ is such that $B_{r,\phi}$ changes sign. Note that in the papers
[CBIP07] and [HP08] devoted to the Einstein-scalar field constraint equations on
closed manifolds the situation when $B_{r,\phi}$ is of indeterminate sign and $R_{h,\psi} < 0$ is
not considered either.

2.2. Paper II. Paper II is concerned with the existence of non CMC solutions of
the vacuum Einstein constraint equations on asymptotically hyperbolic manifolds.
Earlier this issue was studied in the paper [IP97] using the so called sequence
method. The idea of the method is to consider the following iterated sequence of
PDEs:

\begin{equation}
\Delta_h \phi_k - \frac{n-2}{4(n-1)} R_h \phi_k + \frac{n-2}{4(n-1)} \left| \sigma + D_h W_k \right|^2 \phi_k^{\frac{n-2}{n-2}} - \frac{n-2}{4n} \tau^2 \phi_k^{\frac{n+2}{n-2}} = 0,
\end{equation}

and

\begin{equation}
\text{div}_h(D_h W_k) = \frac{n-1}{n} \phi_k^{\frac{n}{n-2}} \nabla h \tau,
\end{equation}

for $k \geq 1$, which is semi-decoupled, in the sense that if $\phi_{k-1}$ is known then (13) is
an equation for $W_k$ alone, and if $W_k$ is known then (12) is an equation for $\phi_k$ alone.
The authors work under a near CMC assumption, i.e. they assume that $\left| \nabla h \tau / \tau \right|_h$
is small. They show that if $\phi_0$ is chosen within certain bounds, then for every $k \geq 1$
one can find $(\phi_k, W_k)$ which satisfy (12)-(13) by means of the Fredholm theorems of
[Lee06] and the sub- and supersolution method. Moreover, as a consequence of the
contraction mapping argument, the sequence $\{ (\phi_k, W_k) \}$ converges to a solution
$(\phi_{\infty}, W_{\infty})$ of the Einstein vacuum constraint equations (7)-(8).

The approach that we use in Paper II to find non CMC solutions of (7)-(8) was
introduced in [DGH] and is inspired by the solution of the Yamabe problem (see
e.g. [SY94] or [LP87]). The idea is to consider the so called subcritical equations

\begin{equation}
\Delta_h \phi - \frac{n-2}{4(n-1)} R_h \phi + \frac{n-2}{4(n-1)} \left| \sigma + D_h W \right|^2 \phi^{\frac{n-2}{n-2}} - \frac{n-2}{4n} \tau^2 \phi^{\frac{n+2}{n-2}} = 0,
\end{equation}

and

\begin{equation}
\text{div}_h(D_h W) = \frac{n-1}{n} \phi^{\frac{n}{n-2}} \nabla h \tau,
\end{equation}

which arise from (7) and (8) after decreasing the exponent of $\phi$ in (8) by $\epsilon$, and
investigate what happens when $\epsilon$ tends to zero.

We prove that (14)-(15) is solvable using a method which is similar to that of
[IP97]. Note that, given $\phi$, the $\epsilon$-momentum constraint (15) can be solved for
$W = W_\phi$ as a consequence of the Fredholm theorems proved in [Lee06], and that,
given $W$, the Lichnerowicz equation (14) admits a solution $\Phi = \Phi_W$ by the sub-
and supersolution method. Let us denote the composition of the maps $\phi \to W_\phi$
and $W \to \Phi_W$ by $F$. Then it can be shown that a version of the Schauder fixed
point theorem applies to $F : C \to X$, where $X$ is a certain weighted Sobolev space
and $C$ is the set of functions that are between two well chosen barrier functions. This means that the subcritical equations (14)-(15) admit a solution.

The above method can also be applied in the case when $\epsilon = 0$ and $|\nabla h \tau / \tau|_h$ is small, i.e. for proving the existence of near CMC solutions of the vacuum Einstein constraint equations (7)-(8). In this case we can show that $F : C \to C$ is a contracting map, which implies that the solution is unique. Remark that although our method and the method used in [IP97] are based on the same idea, their implementations are different. In fact, since we assume lower regularity of the conformal data and clarify both the role of the decay conditions for the conformal data and the uniqueness of the solution, our results in the near CMC case can be viewed as an extension of the results obtained in [IP97].

We next study how the solutions of (14)-(15) behave when $\epsilon$ tends to zero. For each pair $(\phi, W)$ which solves the subcritical equations we define the energy $\gamma(\phi, W)$, a certain quantity which measures the growth of the solutions. We show that if for some sequence of $\{\epsilon_i\}$ such that $\epsilon_i \to 0$ as $i \to \infty$ the energies of the respective solutions $(\phi_i, W_i)$ of (14)-(15) are uniformly bounded then the sequence $\{(\phi_i, W_i)\}$ converges to a solution $(\phi, W)$ of the vacuum Einstein constraint equations (7)-(8). On the other hand, it turns out that if for some sequence of $\{\epsilon_i\}$ such that $\epsilon_i \to 0$ as $i \to \infty$ the energies of the solutions satisfy $\gamma(\phi_i, W_i) \to \infty$ then the sequence of $\tilde{W}_i = (\gamma(\phi_i, W_i))^{-1/2} W_i$ converges to a non-zero solution of the so-called limit equation:

$$ \text{div}_h (\mathcal{D}_h W) = \sqrt{\frac{n-1}{n}} |\mathcal{D}_h W|_h \frac{\nabla h \tau}{\tau}. $$

We can now formulate the main result of Paper II: If $\tau$ is such that the limit equation (16) admits no non-zero solution then there is a solution of the vacuum Einstein constraint equations (7)-(8). Indeed, in this case the energies of the solutions of (14)-(15) are uniformly bounded for all $\epsilon$ and we can find a solution of (7)-(8) by passing to limit when $\epsilon \to 0$. In other words, if we could describe all choices of $\tau$ for which there is no non-zero solution of the limit equation it would result in establishing a large class of non CMC solutions of the vacuum constraint equations on asymptotically hyperbolic manifolds. However, so far we have only been able to prove that the limit equation (16) admits no non-zero solution in a near CMC case, and it still remains to investigate if there is a non-zero solution for other choices of $\tau$.

It should also be noted that in Paper II we introduce a new class of weighted Sobolev spaces called weighted local Sobolev spaces, which stands in some sense halfway between weighted Sobolev and Hölder spaces. Apart from embedding theorems, density results, Young inequalities etc. for this class of functions, we prove the Fredholm theorem for weighted local Sobolev spaces, in parallel to the results of [Lee06] for weighted Sobolev and weighted Hölder spaces.

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CONSTANT MEAN CURVATURE SOLUTIONS
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ROMAIN GICQUAUD AND ANNA SAKOVICH

The work is in progress.