Target Tracking in Boost Stage with Quantized Measurements of Space-based Infrared Cameras*

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A novel state estimation method is proposed for target tracking in the boost stage using space-based infrared cameras (SBIRC) whose measurements are essentially corrupted by both Gaussian noise and quantization noise. As the quantization noise has non-Gaussian properties, conventional extended Kalman filtering (EKF) suffers from poor performance. The quantization noise of SBIRC is modelled using the mid-riser quantizer, which is usually adopted in the digital signal processing field. A novel minimum square error (MMSE) state estimation algorithm with quantized measurements, named the quantized extended Kalman filtering (QEKF), is then proposed. The time update is given based on first-order linearization of the nonlinearities, and the measurement update is derived based on the conditional mean estimate given the quantized measurements. As the multidimensional integrals in the measurement update derived doesn’t have analytical solutions, a numerical integration method is proposed by combining Genz’s transformation and quasi-Monte Carlo (QMC) method, which can avoid the curse of dimensionality. To further improve the tracking accuracy, quantized high-degree cubature Kalman filtering (QHCKF) is developed by integrating the fifth-degree cubature rule into the framework of the QEKF. Numerical simulation results illustrate the superiority of the proposed QEKF and QHCKF methods.

Key Words: Target Tracking, Quantized Measurements, Space-based Infrared Cameras, Spacecraft

Nomenclature

- $\alpha$: magnitude of thrust and drag net acceleration
- $\beta$: auxiliary variable
- $\omega_0$: Earth rotation rate
- $\omega_0$: weights of cubature points
- $\mu$: gravitational parameter
- $e_y, e_z$: length of the pixel in the direction of $y_p$ and $z_p$
- $w_y, w_z$: zero-mean Gaussian white noise in the direction of $y$ and $z$
- $x_{k|k-1}^\iota$: transformed cubature points from the state model
- $Y_{k|k-1}^\iota$: transformed cubature points from the measurement model
- $\Delta$: length of the quantization interval
- $a$: target acceleration during the boost phase
- $a_G$: acceleration of gravity
- $a_N$: net acceleration of the thrust and drag
- $A$: quantization interval
- $C^p_r$: transformation matrix from the ECEF-CS to the body coordinate system
- $e_y, e_z$: quantization noise in the direction of $y$ and $z$
- $f$: focal length of SBIRC
- $F_{k-1}^f$: Jacobian of the vector $f$
- $h$: nonlinear measurement function
- $H_k^i$: Jacobian of the measurement function $h$
- $K_k^i$: filter gain
- $m$: target mass
- $P_{k|k-1}$: state prediction covariance
- $Q$: mid-riser quantizer
- $r$: dimension of $y_k$
- $S_k$: innovation covariance
- $v$: target speed
- $[x, y, z]$: coordinate of the target in ECI-CS
- $x$: state vector
- $\hat{x}_{k|k-1}^i$: predicted state
- $y_k^i$: ideal measurement of the $i$th satellite at time $k$
- $\hat{y}_k^i$: predicted measurement
- $y_k^f$: measurement vector after quantization
- $(y_p^i, z_p^i)$: ideal coordinates of the target in the pixel coordinate system
- $(y_p^i, z_p^i)$: coordinates of the target in the pixel coordinate system
- ECEF-CS: Earth center Earth fixed coordinate system
- ECI-CS: Earth center inertial coordinate system
- EKF: extended Kalman filtering
- IR: infrared cameras
- MMSE: minimum square error
- QEKF: quantized extended Kalman filtering
- QHCKF: quantized high-degree cubature Kalman filtering
- QMC: quasi-Monte Carlo
- RMSE: root-mean-square error
- SBIRC: space-based infrared cameras

Subscripts

- $k$: time $k$
- $0$: initial
Kalman filter to locate targets, which makes them more defensible. A com-
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Quantization is a word in digital signal processing (DSP). It is generally recognized that it is caused by finite computa-
precision in microprocessors or the limit of bandwidth. The statistical theory of quantization was well
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was given and the statistical analysis of the quantization
noise was carried out. Estimation with quantized measure-
ments has been studied for decades. As shown in Asmar et
al., the authors compared several filters for vertical state
estimation with quantized barometric altitude and illustrate
the poor performance of conventional Kalman filters in the
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ality.

Among various estimation algorithms, the extended Kal-
man filtering (EKF), which is based on first-order lineariza-
tion of the non-linear state model and measurement model,
is widely used. Many studies have applied the cubature
Kalman filter to target tracking problems in which the
spherical-radial cubature rule is used to compute statistics
of a nonlinearly transformed Gaussian random variable.

Some studies applied particle filtering, but with heavy com-
putational burden.

In this paper, the characteristics of SBIRC measurements
are analyzed theoretically. Using the mid-riser quantizer in
digital signal processing to describe the discrete image plane,
the measurement noise model of a SBIRC is given. To deal
with the quantized measurements of SBIRCs, quantized ex-
tended Kalman filtering (QEKF) is proposed, in which the
measurement update is derived based on the conditional
mean estimate given the quantized measurements. To avoid
the curse of dimensionality when calculating the integrals in
measurement updates, the Genz’s transformation method is
used to transform the original integral into an integral over
a unit hyper-cube. The quasi-Monte Carlo (QMC) method is
then used to efficiently calculate the integrals. To further improve the tracking accuracy, a quantized
high-degree cubature Kalman filtering (QHCKF) is devel-
opled by integrating the fifth-degree cubature rule into the
framework of the QEKF.

The main contributions of this paper are: 1) Development
of the SBIRC measurement model and analysis of its meas-
urement noise (i.e., we first propose to use the mid-riser
quantizer to model the discrete image plane of SBIRC); 2)
state estimation methods with quantized measurements for
target tracking; and 3) a combination of the EKF/high-
degree cubature rule and the QMC integration method to
avoid the curse of dimensionality.

The paper is outlined as follows. Section 2 introduces the
SBIRC-based target tracking problem and developments the
dynamic model and measurement model of SBIRCs. Section
3 introduces the proposed QEKF and QHCKF algorithms.
The simulation conditions and results are shown in Section
4. Finally, the conclusions are drawn in Section 5.

2. Problem Statement

In this section, the SBIRC-based target tracking problem
is described. Section 2.1 gives the dynamic model of a target
in the boost phase. In Section 2.2, the derivation of SBIRC-
based measurement equations is given. Different from most
existing works, it is shown that the SBIRC-based measure-
ments are corrupted by not only Gaussian noise, but also
quantization noise. To describe the non-Gaussian measure-
ment noise, a quantizer-based SBIRC measurement model
is obtained by analyzing the working principles of SBIRCs.

2.1. Dynamic model of target in the boost phase

Target acceleration during the boost phase can be expressed as

\[ \mathbf{a} = \mathbf{a}_N + \mathbf{a}_G \]

(1)

where, \( \mathbf{a}_N \) denotes the net acceleration of the thrust \( T \) and
drag \( D \) and \( \mathbf{a}_G \) denotes the acceleration of gravity.

In this paper, the gravity-turn model is presented to de-
scribe the boost phase motion of the target, in which the
thrust and drag are assumed to be parallel to the relative ve-
locity vector \( \mathbf{v}_r \) in an Earth-center/Earth-fixed coordinate
system (ECEF-CS). Thus, we have

1: Introduction

Space-based infrared cameras (SBIRCs) are commonly
used for target tracking in the boost stage. SBIRCs com-
bine the advantages of space-based platform and optical
sensor, passively recording target thermal radiation data to
detect and track targets, which makes them more defensible.

Commonly employed strategy is to represent the statistical
properties of the noise of infrared cameras (IR) by Gaussian dis-
tributions, or mixture Gaussian distributions. However, the
image plane of SBIRCs is composed of discrete pixels,
which leads to discontinuous measurements. The Gaussian
and mixture Gaussian are continuous probability density
functions. Therefore, the Gaussian or mixture Gaussian ap-
proximation of the measurement noise of SBIRC is obvi-
ously unreasonable and may lead to poor tracking perfor-
mance. In order to improve the accuracy of trajectory
estimation of a target in the boost stage, a more appropriate
description of the SBIRC measurement noise and relevant
estimation methods are therefore necessary to be considered.

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man filtering (EKF), which is based on first-order lineariza-
tion of the non-linear state model and measurement model,
is widely used. Many studies have applied the cubature
Kalman filter to target tracking problems in which the
spherical-radial cubature rule is used to compute statistics
of a nonlinearly transformed Gaussian random variable.
where, $\alpha = (T - D)/m$ denotes the magnitude of the net acceleration of the thrust and drag, $m$ denotes the target mass, and $v_t = \|v_t\|$ is the target speed.

Let $[x, y, z]$ denote the coordinate of the target in ECI-CS. We have

$$[\dot{x}, \dot{y}, \dot{z}] = \frac{\alpha}{v_t}[\dot{x} - \omega_3 y, \dot{y} + \omega_2 x, \dot{z}]^T$$

where, $\omega_e$ is the Earth rotation rate and

$$v_t = \sqrt{r_3^2 + (\dot{\rho} + \omega_3)^2}$$

Note that $\alpha$ is unknown and needs to be estimated. To derive the dynamic model of $\alpha$, we make the following assumptions:

1. The magnitude of the non-gravitational net forces $F(t) \triangleq T(t) - D(t) = F(t_0)$ is constant; where, $t_0$ is an arbitrary initial time.

2. The target mass $m$ is linearly decreasing at a constant rate (i.e., $m = -\delta_m$) is a constant value.

We have, for $t \geq t_0$

$$\alpha(t) = \frac{F(t)}{m(t)} = \frac{F(t_0)}{m_0} - \frac{\delta_m(t - t_0)}{1 - \beta(t_0)(t - t_0)}$$

$$\beta(t) = \frac{\delta_m}{m(t)} = \frac{\delta_m}{m_0} \frac{1}{1 - \beta(t_0)(t - t_0)}$$

where, $\beta(t)$ is an auxiliary variable, and it indicates the proportion of the mass reduced from the total mass of the target.

Calculating the derivatives of Eq. (5) and Eq. (6), we get

$$\dot{\alpha} = \alpha \cdot \beta$$

$$\dot{\beta} = \beta^2$$

According to the well-known inverse-square model of gravity, the acceleration of gravity can be expressed as $a_G = -\mu/r^3 \mathbf{r}$, where, $\mu$ is the gravitational parameter, $\mathbf{r} = [x, y, z]^T$ and $r = \|\mathbf{r}\|$. Then, the gravity-turn dynamic model in the Earth-center inertial coordinate system (ECI-CS) is given as follows:

$$[\dot{x}, \dot{y}, \dot{z}] = [v_x, v_y, v_z]^T$$

$$[\dot{x}, \dot{y}, \dot{z}] = \frac{\alpha}{v_t} [\dot{x} - \omega_3 y, \dot{y} + \omega_2 x, \dot{z}]^T$$

$$- \left(\frac{\mu}{r^3}\right)[x, y, z]^T$$

The state vector $x$ is defined as

$$x = [x, y, z, v_x, v_y, v_z, \alpha, \beta]^T$$

According to Eqs. (9), (10), (11) and (12), the dynamic model is as follows:

$$\dot{x} = \left[
\begin{array}{c}
v_x \\
v_y \\
v_z \\
\frac{\alpha}{v_t} (\dot{x} - \omega_3 y) - \frac{\mu}{r^3} x \\
\frac{\alpha}{v_t} (\dot{y} + \omega_2 x) - \frac{\mu}{r^3} y \\
\frac{\alpha}{v_t} (\dot{z}) - \frac{\mu}{r^3} z \\
\alpha \cdot \beta \\
\beta^2
\end{array}
\right]$$

### 2.2. SBIRC-based measurements

SBIRCs are commonly used for target tracking in the boost stage and use computer vision and a photogrammetry technique to determine the location of the target. In real applications, at least two satellites are used simultaneously for target tracking, and each satellite is equipped with one SBIRC (see Fig. 1). To represent the statistical properties of the SBIRC noise measured, Gaussian distribution is widely used.6,7 However, the image plane of the SBIRC is composed of discrete pixels, which leads to quantized measurements; that is, the image point is not consistent on the image plane. In order to improve target tracking accuracy, the characteristics of SBIRC measurement need to be theoretically analyzed rather than assumed as Gaussian distribution, which will be given below.

#### 2.2.1. Derivation of measurement equations

To derive the measurement equations, we make the following assumptions:

![Fig. 1. Target tracking using SBIRCs.](image-url)
1) The camera coordinate system coincides with the body coordinate system of satellite $O_bX_bY_bZ_b$.

2) The SBIRCs are simple pinhole cameras, which means the target space point $(x, y, z)$, camera perspective center $O_b$, and image point $(y_q, z_p)$ lie in a straight line (see Fig. 2).

3) The image plane coordinate system $O_1Y_1Z_1$ (in mm) coincides with the pixel coordinate system $O_1Y_1Z_1$ (in pixels).

Let $y_i^t$ denote the ideal measurement of the $i$th satellite at time $k$. $y_i^t$ consists of $y_i^p$ (i.e., pixel coordinate of the image point in the $y$ direction) and $z_i^p$ (i.e., pixel coordinate of image point in the $z$ direction). According to the transformation between the body coordinate system and the pixel coordinate system, we get

$$
\begin{bmatrix}
y_i^p \\
y_i^z
\end{bmatrix} =
\begin{bmatrix}
-f y_i^b \\
-f z_i^b
\end{bmatrix}
\begin{bmatrix}
\frac{1}{e_x} x_i^b \\
\frac{1}{e_z} z_i^b
\end{bmatrix}
$$

(15)

where, $(x_i^b, y_i^b, z_i^b)$ is the position of the target in the body coordinates of the satellite $i$, $f$ is the focal length of SBIRC, $e_x$ and $e_z$ are the length of the pixel in the direction of $y_i^p$ and $z_i^p$, respectively.

Then, according to the transformation between the body coordinate system and ECEF-CS, we get

$$
\begin{bmatrix}
x_b \\
y_b \\
z_b
\end{bmatrix} =
\begin{bmatrix}
x - x_s \\
y - y_s \\
z - z_s
\end{bmatrix}
\begin{bmatrix}
C_i^b
\end{bmatrix}
$$

(16)

where, $C_i^b$ is the transformation matrix from the ECEF-CS to the body coordinate system, $(x, y, z)$ is the position of the target in the ECEF-CS, and $(x_s, y_s, z_s)$ is the position of the satellite in the ECEF-CS.

Thus, the ideal measurement $y_i^t$ is given as

$$
\begin{bmatrix}
y_i^p \\
y_i^z
\end{bmatrix} =
\begin{bmatrix}
f y_i^b + C_{i1}^b (x - x_s) + C_{i2}^b (y - y_s) + C_{i3}^b (z - z_s) \\
f z_i^b + C_{i1}^b (x - x_s) + C_{i2}^b (y - y_s) + C_{i3}^b (z - z_s)
\end{bmatrix}
$$

(17)

where, the superscript $i$ denotes the $i$th satellite, and $C_{ab}$ denotes the value at the $a$th row and the $b$th column of the transformation matrix $C_i^b$.

### 2.2.2. SBIRC noise measurement model

For the discrete image plane of SBIRC, a light beam illuminating within a pixel will yield a coordinate output corresponding to the center of that pixel (see Fig. 3). Therefore, the coordinates of the target in the pixel coordinate system $(y_p, z_p)$ can suffer from quantization errors of up to 0.5 pixels (i.e., $-0.5 \leq y_p - y_q < 0.5$ and $-0.5 \leq z_p - z_q < 0.5$; where, $(y_p, z_p)$ is the ideal coordinate of the target in the pixel coordinate system).

From the output of the discrete image plane, the only thing we can determine is the pixel in which the actual coordinate $(y_p, z_p)$ is located. It is not possible to specify the actual coordinate any more precisely than containment within the pixel. Therefore, the SBIRC measurements are “imprecise measurements” and are highly non-Gaussian. (Details of “imprecise measurements” can be seen in Karlsson and Gustafsson.)

Based on the above analysis, a mid-riser quantizer is used to model the discrete image plane of the SBIRC. The coordinate output of the SBIRC can be seen as the output of a mid-riser quantizer.

In digital signal processing, a quantizer is defined as a non-linear operator having the input-output staircase relation. (Details of the quantizer can be seen in Widrow and István.) The mathematical expression of the mid-riser quantizer is as follows

$$
y^\theta = Q(y) = \Delta \cdot \left\lfloor \frac{y}{\Delta} \right\rfloor + \frac{\Delta}{2}
$$

(18)

where, $Q$ denotes the mid-riser quantizer, $y$ denotes the input of the quantizer, the $\lfloor \cdot \rfloor$ operator rounds downward to the nearest integer, and $\Delta$ denotes the length of the quantization interval. From the output of the quantizer $y^\theta$, the only thing we can infer is that $y \in A_i$; where, $A_i = [a_i, b_i)$ is the $i$th quantization interval, which is known. The characteristics of the mid-riser quantizer can be seen in Fig. 4.

Based on the mid-riser quantizer, the relationship between the ideal coordinates and quantized coordinates of the target in the pixel coordinate system can be expressed as

$$
y^\theta = Q(y_p + w_p) = y_p + e_y
$$
$$
z^\theta = Q(z_p + w_z) = z_p + e_z
$$

(19)

where, $w_p$ and $w_z$ are zero-mean Gaussian white noise caused...
by parameters such as poor illumination and high temperature, $e_i$ and $e_z$ are the quantization noise, which depends on the size of pixels, $y_p^p$ and $z_p^p$ are measurements after quantization, which can be obtained directly from the sensor, $\bar{y}_p$ and $\bar{z}_p$ are the measurements before quantization, and $y_p$ and $z_p$ are the real measurements.

It can be seen from Eq. (19) that the SBIRC noise measurement consists of two parts: 1) Gaussian noise $w_i$ and $w_z$, and 2) quantization noise $e_i$ and $e_z$. For the case that the Gaussian noise is larger (i.e., quantization noise has less influence on measurements), traditional EKF is adoptable. However, when the quantization noise is larger, the non-Gaussian property of the measurements makes traditional EKF unadoptable, which motivates our work.

Substituting Eq. (17) into Eq. (19), we get the measurement equation for satellite $i$

$$y_k^p = \begin{bmatrix} y_p^p \\ z_p^p \end{bmatrix} = \begin{bmatrix} f(x_i^t) \\ C_1 x_i^t + w_i \\ C_2 y_i^t + w_z \end{bmatrix} + \begin{bmatrix} e_i^t \\ e_z^t \end{bmatrix}$$

where, $x_i^t = x - x_i$, $y_i^t = y - y_i$, $z_i^t = z - z_i$, $w_i$ and $w_z$ are the Gaussian part measurement noise of satellite $i$, and $e_i^t$ and $e_z^t$ are the quantization part. Therefore, the measurement noise is non-Gaussian. The focus of this paper is on the state estimation methods dealing with this kind of non-Gaussian noise for target tracking using SBIRCs.

3. Filtering with Quantized Measurements

Consider the following nonlinear discrete system model with quantized measurements

$$x_{k+1} = f(x_k) + v_k$$

$$y_k = h(x_k) + w_k$$

$$y_k^q = Q(y_k) = y_k + e_k$$

where, $x_k \in \mathbb{R}^n$ is the state vector of the dynamic system, $y_k \in \mathbb{R}^m$ is the measurement vector before quantization, which is unknown, $v_k$ and $w_k$ are mutually independent zero-mean Gaussian noise with covariance $Q_k$ and $R_k$, subscript $k$ represents discrete time $k$, $f$ and $h$ are the nonlinear state function and measurement function of the system, respectively, symbol $Q$ denotes the mid-riser quantization operator, $y_k^q$ is the measurement vector after quantization which is what can be obtained directly from the sensor, and $e_k$ is the quantization noise.

Note that, even if measurement function $h$ is linear and $w_k$ is Gaussian, $Q(y_k)$ will be nonlinear and $e_k$ will be non-Gaussian because of the quantization operator.

3.1. Review of extended Kalman filtering

It is well known that Kalman filtering is capable of achieving optimal solutions for estimation problems of linear systems with Gaussian noise. Furthermore, EKF is a generalization of KF for nonlinear systems.30) Considering the nonlinear discrete system model shown in Eq. (21) and Eq. (22), the EKF is given by the following equations.

Define

$$y_k \triangleq \{ y, i \leq k \}$$

**EKF time update**

Given the latest state $\hat{x}_{k|k-1}$, the predicted state $\hat{x}_{k|k-1}$ is

$$\hat{x}_{k|k-1} = f(\hat{x}_{k|k-1})$$

Similarly, the predicted measurement $\hat{y}_k$ is

$$\hat{y}_k = h(\hat{x}_{k|k-1})$$

The state prediction covariance $P_{k|k-1}$ is

$$P_{k|k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^T + Q_{k-1}$$

where, $F_{k-1}$ is the Jacobian of the vector $f$ evaluated at the latest estimate of the state, and $P_{k-1|k-1}$ is covariance matrix at time $k - 1$.

**EKF measurement update**

The innovation covariance $S_k$ is

$$S_k = R_k + H_k P_{k|k-1} H_k^T$$

where, $H_k$ is the Jacobian of the measurement function $h$. The filter gain $K_k$ is

$$K_k = P_{k|k-1} H_k^T S_k^{-1}$$

The update equation for state and its covariance are

$$\hat{x}_{k|k} = E(x_k | Y^k)$$

$$= \hat{x}_{k|k-1} + K_k (y_k - \hat{y}_k)$$

$$P_{k|k} = MSE(\hat{x}_{k|k} | Y^k)$$

$$= P_{k|k-1} - K_k S_k K_k^T$$

3.2. Quantized extended Kalman filtering

Consider the nonlinear discrete system model with quantized measurements shown in Eqs. (21)–(23), the EKF suffers from poor performance due to the mismatching Gauss hypothesis. In this section, an improved algorithm named...
QKEF is developed to deal with the quantized measurements. Given $\hat{x}_{i|k-1}$ and $P_{i|k-1}$, under the Gaussian assumption of a prior probability density function (pdf), the time update equations of QKEF are the same as that of EKF (i.e., Eqs. (25)–(27)) because of the Markov property of the system. The differences between the QKEF algorithm proposed and the conventional EKF algorithm lie in the measurement update, which will be introduced below.

### 3.2.1. Measurement update

Define the sequence of quantized measurements $Y_i^q \triangleq \{y_i^q, i \leq k\}$. Using the minimum square error estimation theory and property of conditional expectation, the state estimation $\hat{x}_{k|k}$ based on the quantized measurement $Y_k^q$ is calculated as follows:\(^{(15)}\)

$$
\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k[E[y_k|Y_{k-1}^q, y_k \in A_i] - \hat{y}_k]
$$

where,

$$
E[y_k|Y_{k-1}^q, y_k \in A_i] = \int_{a_i}^{b_i} y_k p(y_k|Y_{k-1}^q, y_k \in A_i) dy_k
$$

and $K_k$ can be calculated using Eq. (29). Furthermore, the covariance $P_{k|k}$ is\(^{(15)}\)

$$
P_{k|k} = MSE(x_k|Y_{k-1}^q, y_k) = MSE(x_k|Y_{k-1}^q, y_k \in A_i) = E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T|Y_{k-1}^q, y_k \in A_i]
$$

where,

$$
x_k - \hat{x}_{k|k} = x_k - E[x_k|Y_{k-1}^q, y_k] + E[x_k|Y_{k-1}^q, y_k] - \hat{x}_{k|k}
$$

and

$$
P_{k|k} = MSE(x_k|Y_{k-1}^q, y_k) = MSE(x_k|Y_{k-1}^q, y_k \in A_i) = E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T|Y_{k-1}^q, y_k \in A_i]
$$

Then, $P_{k|k}$ can be calculated using Eq. (29).

$$
P_{k|k} = MSE(x_k|Y_{k-1}^q, y_k) = MSE(x_k|Y_{k-1}^q, y_k \in A_i) = E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T|Y_{k-1}^q, y_k \in A_i]
$$

where $r$ is the dimension of $y_k$ and

$$
P(y_k \in A_i|Y_{k-1}^q) = \int_{a_i}^{b_i} p(y_k|Y_{k-1}^q) dy_k
$$

Then,

$$
P(y_k \in A_i|Y_{k-1}^q) = \frac{p(y_k|Y_{k-1}^q)}{P(y_k \in A_i|Y_{k-1}^q)} \text{ if } y_k \in A_i
$$

Inserting Eqs. (42) and (43) into Eq. (44), we get the expression of $p(y_k|Y_{k-1}^q, y_k \in A_i)$. It is well known that there is no analytical solution to the integrals of Gaussian density in Eqs. (40), (41) and (43). Therefore, a multi-dimensional numerical integration method is needed. This is presented in the next section.

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3.2.2. Multi-dimensional numerical integration using Genz’s transformation and quasi-Monte Carlo method

The multi-dimensional numerical integration methods can be divided into two categories: product rules and non-product rules. Product rules such as the product Gauss-Hermite quadrature rule successively apply the Gauss-Hermite quadrature rule in a tensor-product of one-dimensional integrals. Hence, \( m^n \) points are needed to compute an \( n \)-dimensional integral. Non-product rules such as Monte Carlo methods integrate. Before that, we adopt Genz to avoid the curse of dimensionality while calculating the computational complexity of the gridding method in which \( \mathbf{y}_1 \) are calculated using a gridding method in which \( \mathbf{y}_2 \) is computed over a unit hyper-cube.

The integral in Eq. (43) can be written as

\[
P(\mathbf{y}_k \in A_s | Y^n_{k-1} ) = \int_{\mathbf{y}_k \in A_s} p(\mathbf{y}_k | Y^n_{k-1}) d\mathbf{y}_k
\]

\[
= \frac{1}{(2\pi)^{n/2}|S_k|^{1/2}} \int_{a_0}^{b_0} \cdots \int_{a_{n-1}}^{b_{n-1}} e^{-\frac{1}{2} \mathbf{y}^T \mathbf{S}_k^{-1} \mathbf{y}} d\mathbf{y}_k
\]

Defining \( \theta_k = \mathbf{y}_k - \mathbf{\hat{y}}_{k|k-1} \), we get

\[
P(\mathbf{y}_k \in A_s | Y^n_{k-1} ) = \frac{1}{(2\pi)^{n/2}|S_k|^{1/2}} \int_{a'_0}^{b'_0} \cdots \int_{a'_{n-1}}^{b'_{n-1}} e^{-\frac{1}{2} \mathbf{\theta}_k^T \mathbf{S}_k^{-1} \mathbf{\theta}_k} d\mathbf{\theta}_k
\]

Defining \( d_i' = a_i - \mathbf{\hat{y}}_{k|k-1} \), \( b_i' = b_i - \mathbf{\hat{y}}_{k|k-1} \) for \( i = 1, 2, \ldots, r \), we get

\[
P(\mathbf{y}_k \in A_s | Y^n_{k-1} ) = \frac{1}{(2\pi)^{n/2}|S_k|^{1/2}} \int_{a'_0}^{b'_0} \cdots \int_{a'_{n-1}}^{b'_{n-1}} e^{-\frac{1}{2} \mathbf{\theta}_k^T \mathbf{S}_k^{-1} \mathbf{\theta}_k} d\mathbf{\theta}_k
\]

Define \( u_i = \Phi(z_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_i} e^{-\frac{1}{2} t^2} dt \), therefore,

\[
du_i = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z_i^2} dz_i
\]

Then,

\[
P(\mathbf{y}_k \in A_s | Y^n_{k-1} ) = \int_{d_0}^{e_0} \cdots \int_{d_{n-1}}^{e_{n-1}} du_0 \cdots du_{n-1}
\]

where, \( d_i = \Phi(a_i') \) and \( e_i = \Phi(b_i') \) for \( i = 1, 2, \ldots, r \). Defining \( u_i = d_i + w_i (e_i - d_i) \), we get

\[
P(\mathbf{y}_k \in A_s | Y^n_{k-1} ) = \int_{0}^{1} \int_{0}^{1} \cdots \int_{0}^{1} (e_1 - d_1) (e_2 - d_2) \cdots (e_r - d_r) dw
\]

Note that \( d_i \) and \( e_i \) are related to \( u_i, u_2, \ldots, u_{n-1} \), and not related to \( u_1 \). Therefore, \( d_1 \) and \( e_1 \) are not related to \( w \), and \( d_i \) and \( e_i \) are not related to \( w_i \). Therefore, Eq. (51) can be written as

\[
P(\mathbf{y}_k \in A_s | Y^n_{k-1} ) = (e_1 - d_1) \int_{0}^{1} \int_{0}^{1} \cdots \int_{0}^{1} (e_r - d_r) dw
\]

Therefore, the number of integration variables is reduced by one.

Then, the \( N \)-points quasi-Monte Carlo method is used to approximate the integral in Eq. (52).

\[
P(\mathbf{y}_k \in A_s | Y^n_{k-1} ) \approx \frac{1}{N} \sum_{i=1}^{N} f(w_i)
\]

where, \( w_i \) is produced using the Halton sequence, \( f(w_i) = (e_1 - d_1) (e_2 - d_2) \cdots (e_r - d_r) \).

For the integrals in Eqs. (40) and (41), only minor modifications to the above algorithm are needed. The general numerical integration algorithm for Eqs. (40), (41) and (43) is given in Table 1.

3.3. Quantized high-degree cubature Kalman filtering

As used in EKF, for the QEKF algorithm proposed in Section 3.2, a linearization technique is used to approximate the nonlinear model. To further improve the tracking accuracy, we propose to integrate the fifth-degree cubature rule into the framework of the QEKF to get create QHCKF.

Based on the fifth-degree cubature rule, the time update equations (Eqs. (25)-(27)) can be rewritten as

\[
\mathbf{\hat{x}}_{k|k-1} = \sum_{i=0}^{2n^2} \alpha_i \mathbf{X}_{k|k-1, i}
\]

\[
\mathbf{\hat{y}}_{k|k-1} = \sum_{i=0}^{2n^2} \alpha_i \mathbf{Y}_{k|k-1, i}
\]

\[
P_{\mathbf{\hat{x}}_{k|k-1}} = \sum_{i=0}^{2n^2} \alpha_i (\mathbf{X}_{k|k-1, i} - \mathbf{\hat{x}}_{k|k-1})(\mathbf{X}_{k|k-1, i} - \mathbf{\hat{x}}_{k|k-1})^T + \mathbf{Q}_{k-1}
\]
The parameter $\xi_k = \frac{[0 \ 0 \ \cdots \ 0]^T}{i = 0}$. 

$\beta_s^+ = \left\{ \begin{array}{ll} i = 1, 2, \ldots, n(n - 1)/2 \\ -\beta_s^+ = \frac{i = n(n - 1)/2 + 1, \ldots, n(n - 1)}{i = n(n - 1) + 1, \ldots, 3n(n - 1)/2} \\ -\beta_s^+ = \frac{i = 3n(n - 1)/2 + 1, \ldots, 2n(n - 1)}{i = 2n(n - 1) + 1, \ldots, 2n^2} \end{array} \right.$ 

The parameter $s_j^+$ and $s_j$ are given by

$$ s_j^+ = \sqrt{1/2(e_p + e_q)} \quad p < q, p, q = 1, 2, \ldots, n \quad (62) $$

$$ s_j = \sqrt{1/2(e_p - e_q)} \quad p < q, p, q = 1, 2, \ldots, n \quad (63) $$

To embed the fifth-degree cubature rule into the QEKF algorithm, measurement Eqs. (28) and (29) need to be rewritten. According to the solution for the statistical linear regression of the nonlinear measurement equation, $^{32}$ the measurement matrix $H_k$ is given by

$$ H_k = P_{xy,k|k-1} \cdot (P_{k|k-1})^{-1} \quad (64) $$

where, $P_{xy,k|k-1}$ is the cross-covariance

$$ P_{xy,k|k-1} = \sum_{i=0}^{2n^2} \omega_i (\mathbf{X}_{k|k-1,i} - \hat{\mathbf{x}}_{k|k-1})(\mathbf{Y}_{k|k-1,i} - \hat{\mathbf{y}}_{k|k-1})^T \quad (65) $$

According to Eqs. (28) and (56), it is easy to obtain that

$$ P_{xy,k|k-1} &= \mathbf{S}_k \\
&= \sum_{i=0}^{2n^2} \omega_i (\mathbf{Y}_{k|k-1,i} - \hat{\mathbf{y}}_{k|k-1})(\mathbf{Y}_{k|k-1,i} - \hat{\mathbf{y}}_{k|k-1})^T + \mathbf{R}_k \quad (66) $$

Substitute Eqs. (64) and (66) into Eq. (29), and the gain matrix $K_k$ is rewritten as follows

$$ K_k = P_{xy,k|k-1}(P_{xy,k|k-1})^{-1} \quad (67) $$

Therefore, Eqs. (66) and (67), and Eqs. (32), (37) and (38) compose the measurement update of QHCKF. Note that $E(\mathbf{Y}_{k|k-1,i}, \mathbf{y}_k \in A_i)$ and $\text{cov}(\mathbf{Y}_{k|k-1,i}, \mathbf{y}_k \in A_i)$ in Eqs. (32) and (37) can be calculated as that in QEKF.

### 4. Simulation

In this section, numerical simulation studies are performed to verify the effectiveness of the proposed QKCKF and QHCKF methods for target tracking in the boost stage using quantized measurements of SBIRCs. Two GEO satellites are used for tracking the target. The measurements of two satellites are stacked into a four-dimensional vector.

#### 4.1 Simulation scene and simulation conditions

The target trajectory in the boost phase is simulated in ECEF-CS as shown in Fig. 5, and the simulation parameters are given in Table 2.

#### 4.2 Simulation results

We compare the proposed QHCKF and QEKF methods and the original EKF (EKF-Q), the EKF with quantized measurement based on the AUN assumption (EKF-AUN), $^{14}$ the approximate MMSE filtering algorithm (NMMSE), $^{15}$ and the EKF with pure Gauss measurements (EKF-G). For each method, we ran 100 Monte Carlo simulations and obtained the time history of RMSE of the position.
and velocity estimates (see Figs. 6 and 7). The RMSE for time step $k$ is calculated by

$$RMSE_k = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (\hat{x}_k^i - x_k)^2} \quad (68)$$

where, $M$ denotes the number of Monte Carlo simulations (i.e., 100 times), $\hat{x}_k^i$ denotes the estimated state at time $k$ in the $i$th run, and $x_k$ denotes the true state at time $k$.

Figures 6 and 7 show the RMSEs from 100 Monte Carlo simulations for position and velocity at every time instance, respectively. Furthermore, the average position and velocity RMSEs during 10–60 s for five algorithms are given in Table 3. It can be seen that the EKF-Q algorithm performs the worst. EKF-AUN performs better than EKF-Q but worse than NMMSE, QEKF and QHCKF. This is because the AUN assumption does not include saturation nor correlation properties from quantization, which results in poor estimation performance. Comparing EKF-AUN, EKF-Q and NMMSE, the estimation accuracy of the proposed QHCKF and QEKF methods are closer to that of EKF-Q, which demonstrates the effectiveness of QHCKF and QEKF in target tracking using quantized measurements. The QHCKF algorithm performs even better than QEKF, which means that QHCKF is more effective in dealing with nonlinear models than QEKF. In addition, under the same Monte Carlo simulations, the curve of the RMSEs of NMMSE, QEKF and QHCKF are smoother than those of EKF-Q and EKF-AUN, which means that NMMSE, QEKF and QHCKF eliminate more uncertainty than EKF-Q and EKF-AUN.

To investigate the effects of pixel size and the focal length on the algorithms, the algorithms with different sized pixels (i.e., $e_x$, $e_z$; note that the pixels are assumed to be square, $e_x = e_z$) and different focal lengths ($f$) are compared. The mean RMSE of position and velocity with different pixel sizes and different focal lengths are shown in Figs. 8 and 9, respectively.

It can be seen from Fig. 8 that performance improved when the pixel size decreases. Furthermore, the smaller the pixels, the closer the performance between different algorithms, which is not hard to explain because the smaller the pixels, the smaller the impact of quantization noise in the system, thereby eliminating the differences between different algorithms. In Fig. 9, the RMSEs of the algorithms with different focal lengths are given. It can be seen that the proposed QHCKF algorithm performed the best and performance improved when the focal length increased.
In order to further compare the efficiency of NMMSE and the proposed QEKF algorithm, 100 Monte Carlo simulations were carried out for two algorithms under a different number of grids (i.e., NMMSE) and different integration points (i.e., QEKF). The average running time and corresponding RMSE are calculated and the relationship between them is shown in Fig. 10 and Fig. 11, respectively. It can be seen that NMMSE spends more time to obtain the same estimation accuracy as QEKF.

5. Conclusion

In this paper, the quantization noise model of SBIRCs is given using a mid-riser quantizer. Additionally, estimation algorithms with quantized measurements named QEKF and QHCKF are proposed based on the conditional mean estimate for target tracking in the boost stage. To calculate the multi-dimensional integrals in the derived measurement update, a numerical integration method is proposed by combining Genz’s transformation and the QMC method. In simulations, the measurements are estimated based on the imaging principle of vision sensors. The simulation results show that the QHCKF and QEKF methods proposed outperform other algorithms in the presence of quantization noise and have the closest performance to that of pure Gauss noise. When the pixel size decreases and focal length increases, the tracking performance improves. In addition, simulation results show that the proposed QHCKF fifth-degree cubature rule embedded algorithm performs better than QEKF. Therefore, we claim that the model proposed is applicable for modeling measurements of vision sensors and that the QEKF and QHCKF methods are effective. Future work will focus on out-of-sequence measurements caused by communication and processing latency and real data experiments.

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