INFINITE DIMENSION OF SOLUTIONS
FOR DIRICHLET PROBLEM II

VLADIMIR RYZANOV

March 26, 2014

Abstract

It is proved that the space of solutions of the Dirichlet problem for the harmonic
functions in the unit disk with nontangential boundary limits 0 a.e. has the infinite
dimension.

2010 Mathematics Subject Classification: Primary 31A05, 31A20,
31A25, 31B25, 35Q15; Secondary 30E25, 31C05, 34M50, 35F45

By the well–known Lindelöf maximum principle, see e.g. Lemma 1.1 in [3], it
follows the uniqueness theorem for the Dirichlet problem in the class of bounded
harmonic functions $u$ on the unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$. In general there
is no uniqueness theorem in the Dirichlet problem for the Laplace equation even
under zero boundary data. In comparison with the last version [8], here we give
more elementary examples and constructions of solutions.

Many such nontrivial solutions $u$ for the Laplace equation can be given by
the Poisson-Stiltjes integral

$$ u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\vartheta - t) \, d\Phi(t) , \quad z = re^{i\vartheta} , \quad r < 1 , \quad (1) $$

with an arbitrary singular function $\Phi : [0,2\pi] \to \mathbb{R}$, i.e., where $\Phi$ is of
bounded variation and $\Phi' = 0$ a.e., where we use the standard notation for the
Poisson kernel

$$ P_r(\Theta) = \frac{1 - r^2}{1 - 2r \cos \Theta + r^2} , \quad r < 1 . $$
Indeed, $u$ in (1) is harmonic for every function $\Phi : [0, 2\pi] \to \mathbb{R}$ of bounded variation and by the Fatou theorem, see e.g. Theorem I.D.3.1 in [3], $u(z) \to \Phi' (\Theta)$ as $z \to e^{i\Theta}$ along any nontangential path whenever $\Phi' (\Theta)$ exists. Thus, $u(z) \to 0$ as $z \to e^{i\Theta}$ for a.e. $\Theta \in [0, 2\pi]$ along any nontangential paths for every singular function $\Phi$.

**Example 1.** The simplest example of such kind is given by nondecreasing step-like data $\Phi_{\vartheta_0}$ with values 0 and $2\pi$ and with the jump at $\vartheta_0 \in (0, 2\pi)$:

$$u(z) = P_r (\vartheta - \vartheta_0) = \frac{1 - r^2}{1 - 2r \cos (\vartheta - \vartheta_0) + r^2}, \quad z = re^{i\vartheta}, \quad r < 1. \quad (2)$$

We directly see that $u(z) \to 0$ as $z \to e^{i\Theta}$ for all $\Theta \in (0, 2\pi)$ except $\Theta = \vartheta_0$.

Note that the function $u$ is harmonic in the unit disk $\mathbb{D}$ because

$$u(z) = \text{Re} \frac{\zeta_0 + z}{\zeta_0 - z} = \frac{1 - |z|^2}{1 - 2 \text{Re} z \zeta_0 + |z|^2}, \quad \zeta_0 = e^{i\vartheta_0}, \quad z \in \mathbb{D}, \quad (3)$$

where the function $w = g(z) = g_{\zeta_0}(z) = (\zeta_0 + z)/(\zeta_0 - z)$ is analytic (conformal) in $\mathbb{D}$ and maps $\mathbb{D}$ onto half-plane $\text{Re} w > 0$, $g(0) = 1$, $g(\zeta_0) = \infty$.

**Example 2.** The second natural example is given by the formula (1) with $\Phi(t) = \varphi(t/2\pi)$ where $\varphi : [0, 1] \to [0, 1]$ is the well-known Cantor function, see e.g. [11] and further references therein.

The formula (2) gives a continual set of such examples. Furthermore, one can prove the following result.

**Theorem 1.** The space of all harmonic functions in $\mathbb{D}$ with nontangential limit 0 at every point of $\partial \mathbb{D}$ except a countable collection of points in $\partial \mathbb{D}$ has the infinite dimension.

**Proof.** Indeed, let us consider the sequence of functions of the form (3):

$$u_n(z) = \text{Re} \frac{\zeta_n + z}{\zeta_n - z} = \frac{1 - |z|^2}{1 - 2 \text{Re} z \zeta_n + |z|^2}, \quad \zeta_n = e^{i\vartheta_n}, \quad z \in \mathbb{D},$$

where

$$\vartheta_n = \pi (2^{-1} + \ldots + 2^{-n}), \quad n = 1, 2, \ldots$$

and denote by $\mathcal{H}_1$ the class of all series $u = \sum \gamma_n u_n$ whose sequences of coefficients $\gamma = \{\gamma_n\}$ belong to the space $l^1$ with the norm $\|\gamma\| = \sum_{n=1}^{\infty} |\gamma_n| < \infty$. 

Note that $\mathcal{H}_1$ consists of harmonic functions, see, e.g., Theorem I.3.1 in [5], because

$$0 < u_n(z) < \frac{1 + |z|}{1 - |z|} \quad \forall \ n = 1, 2, \ldots, \ z \in \mathbb{D}.$$

Note also that each function $u \in \mathcal{H}_1$ has nontangential limit 0 at every point $\zeta \in \partial \mathbb{D}$ except the points $\zeta_0 = -1 = e^{i\vartheta_0}, \vartheta_0 = \pi,$ and $\zeta_n, \ n = 1, 2, \ldots.$ Indeed, let $\zeta = e^{i\Theta}, \Theta \in (0, 2\pi), \zeta \neq \zeta_n, \ n = 0, 1, 2, \ldots.$ Then, applying the formula (2), we have the estimate

$$u_n(z) = \frac{1 - r^2}{4r \sin^2 \frac{\Theta - \vartheta_n}{4}} \leq C(1 - r), \quad z = re^{i\vartheta},$$

for all points $z$ belonging to a sector $|\vartheta - \Theta| < c(1 - r)$ and for all $r$ which are close enough to 1 where $C < \infty$ does not depend on $n = 1, 2, \ldots.$ Thus,

$$|u(z)| \leq C \|\gamma\| (1 - r) \to 0 \quad \text{as} \quad r \to 1, \quad z = re^{i\vartheta},$$

in any sector $|\vartheta - \Theta| < c(1 - r).$

Now, let us show that $u_n, \ n = 1, 2, \ldots,$ form a basis in the space $\mathcal{H}_1$ with the locally uniform convergence in $\mathbb{D}$ which is metrizable.

Indeed, firstly, $u = \sum \gamma_n u_n \neq 0$ if $\gamma \neq 0.$ Really, let us assume that $\gamma_n \neq 0$ for some $n = 1, 2, \ldots.$ Then $u \neq 0$ because $u(z) \to \infty$ as $z = re^{i\vartheta_n} \to e^{i\vartheta_n}.$ The latter follows since

$$u_n(re^{i\vartheta_n}) = \frac{1 + r}{1 - r} \to \infty \quad \text{as} \quad r \to 1,$$

and by the previous item

$$|\tilde{u}(re^{i\vartheta_n})| \leq C \|\gamma\| (1 - r) \to 0 \quad \text{as} \quad r \to 1,$$

where $\tilde{u} = u - \gamma_n u_n.$

Secondly, $u_m^* = \sum_{n=1}^{m} u_n \varphi_n \to u$ locally uniformly in $\mathbb{D}$ as $m \to \infty.$ Indeed, elementary calculations give the following estimate of the remainder term

$$|u(z) - u_m^*(z)| \leq \frac{1 + r}{1 - r} \cdot \sum_{n=m+1}^{\infty} |\gamma_n| \to 0 \quad \text{as} \quad m \to \infty \quad (4)$$

in every disk $\mathbb{D}(r) = \{z \in \mathbb{C} : |z| \leq r\}, \ r < 1.$ $\square$
Corollary 1. Given a measurable function $\varphi : \partial D \to \mathbb{R}$, the space of all harmonic functions $u : D \to \mathbb{R}$ with the limits $\lim_{z \to \zeta} u(z) = \varphi(\zeta)$ for a.e. $\zeta \in \partial D$ along nontangential paths has the infinite dimension.

Indeed, the existence at least one such a harmonic function $u$ follows from the known Gehring theorem in [4]. Combining this fact with Theorem 1, we obtain the conclusion of Corollary 1.

Remark 1. In view of Lemma 3.1 in [2], one can similarly prove the more refined result on harmonic functions than in Corollary 1 with respect to logarithmic capacity instead of the measure of the length on $\partial D$.

Moreover, the statements on the infinite dimension of the space of solutions can be extended to the Riemann-Hilbert problem because the latter is reduced in the papers [2] and [7] to the corresponding two Dirichlet problems.

Note also that harmonic functions $u$ found in Theorem 1 themselves cannot be represented in the form of the Poisson integral with any integrable function $\Phi : [0, 2\pi] \to \mathbb{R}$ because this integral would have nontangential limits $\Phi$ a.e., see e.g. Corollary IX.9.1 in [5]. Consequently, $u$ do not belong to the classes $h_p$ for any $p > 1$, see e.g. Theorem IX.2.3 in [5].

However, the functions $u \in \mathcal{H}_1$ in the proof of Theorem 1 have the representation as the Poisson-Stiltjes integral (1) with $\Phi = \sum \gamma_n \Phi_{\vartheta_n}$ where $\Phi_{\vartheta_n} : [0, 2\pi] \to \mathbb{R}$ are nondecreasing step-like functions with values 0 and $2\pi$ with jumps at the points $\vartheta_n$, $n = 1, 2, \ldots$. Thus, $\Phi$ is of bounded variation and hence $\mathcal{H}_1 \subset h_1$, see e.g. Theorem IX.2.2 in [5].

Problem 1. It remains the open question whether the basis of the space of all such singular solutions of the Dirichlet problem for the Laplace equation has the power of the continuum.

References

[1] Dovgoshey O., Martio O., Ryazanov V., Vuorinen M. The Cantor function, Expo. Math., 24 (2006), 1–37.

[2] Efimushkin A., Ryazanov V., On the Riemann-Hilbert problem for the Beltrami equations // arXiv: 1402.1111v2 [math.CV] 14 Feb. 2014, 1-25.
[3] Garnett J.B., Marshall D.E., *Harmonic Measure*, Cambridge Univ. Press, Cambridge, 2005.

[4] Gehring F.W., *On the Dirichlet problem*, Michigan Math. J., 3 (1955–1956), 201.

[5] Goluzin G.M. *Geometric theory of functions of a complex variable*, Transl. of Math. Monographs, Vol. 26, American Mathematical Society, Providence, R.I. 1969.

[6] Koosis P., *Introduction to $H_p$ spaces*, 2nd ed., Cambridge Tracts in Mathematics, 115, Cambridge Univ. Press, Cambridge, 1998.

[7] Ryazanov V., *On the Riemann-Hilbert problem IV* // arXiv: 1308.2486v10 [math.CV] 11 Feb. 2013, 1-10.

[8] Ryazanov V., *Infinite dimension of solutions for the Dirichlet problem* // arXiv: 1402.2130v3 [math.CV] 16 Feb. 2013, 1-5.

Vladimir Illich Ryazanov,
Institute of Applied Mathematics and Mechanics,
National Academy of Sciences of Ukraine,
74 Roze Luxemburg Str., Donetsk, 83114, Ukraine,
vl.ryazanov1@gmail.com