MULTIPLIERS FOR VON NEUMANN-SCHATTEN BESSEL SEQUENCES IN SEPARABLE BANACH SPACES

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ABSTRACT. In this paper we introduce the concept of von Neumann-Schatten Bessel multipliers in separable Banach spaces and obtain some of their properties. Finally, special attention is devoted to the study of invertible Hilbert-Schmidt frame multipliers. These results are not only of interest in their own right, but also they pave the way for obtaining some new results for diagonalization of matrices in finite dimensional setting as well as for dual HS-frames. In particular, we show that a HS-frame is uniquely determined by the set of its dual HS-frames.

1. INTRODUCTION

Due to the fundamental works done by Feichtinger and his coauthors [14, 15], Fourier and Gabor multipliers were formally introduced and popularized from then on. Now the theory of Fourier and Gabor multipliers plays an important role in theoretics and applications; For more information about the history of this class of operators, some of their properties and their applications in scientific disciplines and in modern life the reader can consult Section 1 of the papers [5, 26] and the references (for examples) [6, 10, 11, 12, 19]. Balazs [3] extended the notion of Gabor multipliers to arbitrary Hilbert spaces. In details, he considered the operators of the form

$$M_{m,\Phi,\Psi}(f) = \sum_{i=1}^{\infty} m_i \langle f, \psi_i \rangle \phi_i \quad (f \in \mathbb{H}),$$

where $\Phi = \{\phi_i\}_{i=1}^{\infty}$ and $\Psi = \{\psi_i\}_{i=1}^{\infty}$ are ordinary Bessel sequences in Hilbert space $\mathbb{H}$, and $m = \{m_i\}_{i=1}^{\infty}$ is a bounded complex scalar sequence in $\mathbb{C}$. It is worthwhile to mention that this class of operators is not only of interest for applications in modern life, but also it is of utmost importance in different branches of linear algebra, matrix analysis and functional analysis. For example, they are used for the diagonalization of matrices [17, Definition 3.1], the diagonalization of operators [4, 11, 25] and for solving approximation problems [12, 13, 19]. We also recall that by the spectral theorem, every self-adjoint compact operator on a Hilbert space can be represented as a multiplier using an orthonormal system. In

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addition to all these, multipliers generalize the frame operators, approximately dual frames \([8, 20]\), generalized dual frames \([20, \text{Remark } 2.8(\text{ii})]\), atomic systems for subspaces \([16, 21]\) and frames for operators \([18]\). Therefore, the study of Bessel multipliers also leads us to new results concerning dual frames and local atoms, two concepts at the core of frame theory.

Various generalization of Bessel multipliers have been introduced and studied in a series of papers recently. This paper continues these investigations. In details, we investigate Bessel multipliers for von Neumann-Schatten Bessel sequences in separable Banach spaces. Here it should be noted that, just because the study of this class of operators paves the way for obtaining some new results for von Neumann-Schatten frame \([1, 24]\), this inspires us to investigate this class of operators in separable Banach spaces. Let us recall that the von Neumann-Schatten frames in a separable Banach space was first proposed by Arefijamaal and Sadeghi \([24]\) to deal with all the existing frames as a united object. In fact, the von Neumann-Schatten frame is an extension of \(g\)-frames for Hilbert spaces \([27]\) and \(p\)-frames for Banach spaces \([9]\), two important generalization of ordinary frames.

2. von Neumann-Schatten \(p\)-Bessel sequences: an overview

In this section, we give a brief overview of von Neumann-Schatten \(p\)-Bessel sequences from \([24]\). Nevertheless, we shall require some facts about the theory of von Neumann-Schatten \(p\)-class of operators. For background on this theory, we use \([23]\) as reference and adopt that book’s notation. Moreover, our notation and terminology are standard and, concerning frames in Hilbert respectively Banach spaces, they are in general those of the book \([7]\) respectively the paper \([9]\).

2.1. von Neumann-Schatten \(p\)-class of operators. Let \(\mathbb{H}\) be a separable Hilbert space and let \((B(\mathbb{H}), \| \cdot \|_{\text{op}})\) denotes the \(C^*\)-algebra of all bounded linear operators on \(\mathbb{H}\). For a compact operator \(A \in B(\mathbb{H})\), let \(s_1(A) \geq s_2(A) \geq \cdots \geq 0\) denote the singular values of \(A\), that is, the eigenvalues of the positive operator \(|A| = (A^*A)\frac{1}{2}\), arranged in a decreasing order and repeated according to multiplicity. For \(1 \leq p < \infty\), the von Neumann-Schatten \(p\)-class \(C_p(\mathbb{H})\) is defined to be the set of all compact operators \(A\) for which \(\sum_{i=1}^{\infty} s_i^p(A) < \infty\). For \(A \in C_p(\mathbb{H})\), the von Neumann-Schatten \(p\)-norm of \(A\) is defined by

\[
\|A\|_{C_p(\mathbb{H})} = \left( \sum_{i=1}^{\infty} s_i^p(A) \right)^{\frac{1}{p}} = \left( \text{tr}|A|^p \right)^{\frac{1}{p}},
\]  

(2.1)

where \(\text{tr}\) is the trace functional which defines as \(\text{tr}(A) = \sum_{e \in \mathcal{E}} \langle A(e), e \rangle\) and \(\mathcal{E}\) is any orthonormal basis of \(\mathbb{H}\). The special case \(C_1(\mathbb{H})\) is called the trace class and \(C_2(\mathbb{H})\) is called the Hilbert-Schmidt class. Recall that an operator \(A\) is in \(C_p(\mathbb{H})\) if and only if \(A^p \in C_1(\mathbb{H})\). In particular, \(\|A\|_{C_p(\mathbb{H})}^p = \|A^p\|_{C_1(\mathbb{H})}\). It is proved that \(C_p(\mathbb{H})\) is a two sided \(*\)-ideal of \(B(\mathbb{H})\) and the finite rank operators are dense.
in \((C_p(\mathbb{H}), \| \cdot \|_{C_p(\mathbb{H})})\). Moreover, for \(A \in C_p(\mathbb{H})\), one has \(\|A\|_{C_p(\mathbb{H})} = \|A^*\|_{C_p(\mathbb{H})}\), \(\|A\|_{op} \leq \|A\|_{C_p(\mathbb{H})}\) and if \(B \in B(\mathbb{H})\), then

\[
\|BA\|_{C_p(\mathbb{H})} \leq \|B\|_{op} \|A\|_{C_p(\mathbb{H})},
\]

and if \(B \in B(\mathbb{H})\), then

\[
\|AB\|_{C_p(\mathbb{H})} \leq \|B\|_{op} \|A\|_{C_p(\mathbb{H})}.
\]

In particular, \(C_p(\mathbb{H}) \subseteq C_q(\mathbb{H})\) if \(1 \leq p \leq q < \infty\). We also recall that the space \(C_2(\mathbb{H})\) with the inner product \([T, S]_{tr} := tr(S^*T)\) is a Hilbert space.

Now, for a fixed \(1 \leq p < \infty\), we define the Banach space

\[
\oplus C_p(\mathbb{H}) = \{ \mathcal{A} = \{ A_i \}_{i=1}^\infty : A_i \in C_p(\mathbb{H}) \text{ for all } i \in \mathbb{N}, \text{ and } \}
\]

\[
\| \mathcal{A} \|_p := \left( \sum_{i=1}^\infty \| A_i \|_{C_p(\mathbb{H})}^p \right)^{\frac{1}{p}} < \infty
\]

In particular, \(\oplus C_2(\mathbb{H})\) is a Hilbert space with the inner product

\[
\langle \mathcal{A}, \mathcal{A}' \rangle := \sum_{i=1}^\infty [A_i, A'_i]_{tr},
\]

and so \(\| \mathcal{A} \|_2^2 = \langle \mathcal{A}, \mathcal{A} \rangle\).

We conclude this subsection by recalling the notion of the tensor product of two arbitrary elements of \(\mathbb{H}\) which will be useful in our subsequent analysis. To this end, suppose that \(x, y \in \mathbb{H}\) and define the operator \(x \otimes y\) on \(\mathbb{H}\) by

\[
(x \otimes y)(z) = \langle z, y \rangle x \quad (z \in \mathbb{H}).
\]

It is obvious that \(\|x \otimes y\| = \|x\| \|y\|\) and the rank of \(x \otimes y\) is one if \(x\) and \(y\) are non-zero. Moreover,

\[
\|x \otimes y\|_{C_p(\mathbb{H})} = \|x\| \|y\| \quad \text{and} \quad tr(x \otimes y) = \langle x, y \rangle.
\]

Thus \(x \otimes y\) is in \(C_p(\mathbb{H})\) for all \(p \geq 1\). Furthermore, the following equalities are easily verified:

\[
(x \otimes x') (y \otimes y') = \langle y, y' \rangle (x \otimes y') ;
\]

\[
(x \otimes y)^* = y \otimes x ;
\]

\[
T (x \otimes y) = T(x) \otimes y ;
\]

\[
(x \otimes y) T = x \otimes T^*(y) ,
\]

where \(x', y' \in \mathbb{H}\) and \(T \in B(\mathbb{H})\). Clearly, the operator \(x \otimes x\) is a rank-one projection if and only if \(\langle x, x \rangle = 1\), that is, \(x\) is a unit vector. Conversely, every rank-one projection is of the form \(x \otimes x\) for some unit vector \(x\).

Having reached this state it remains to recall the definition and some properties of von Neumann-Schatten \(p\)-frames for separable Banach spaces. This is the subject matter of the next subsection.
2.2. von Neumann-Schatten $p$-frames. To simplify the later discussion, we make the following blanket assumption.

**Convention.** For the rest of this paper we assume that $\mathcal{H}$ is a Hilbert space with orthonormal basis $\mathcal{E} = \{e_i\}_{i \in I}$, $1 < p < \infty$ and $q$ is the conjugate exponent to $p$, that is, $1/p + 1/q = 1$. Moreover, the notation $\mathcal{C}_p$ [respectively, $\mathcal{C}_q$] is used to denote the space $\mathcal{C}_p(\mathcal{H})$ [respectively, $\mathcal{C}_q(\mathcal{H})$] without explicit reference to the Hilbert space $\mathcal{H}$.

Recall from [24] that a countable family $\mathcal{G} = \{G_i\}_{i=1}^{\infty}$ of bounded linear operators from separable Banach space $\mathcal{X}$ to $\mathcal{C}_p$ is a von Neumann-Schatten $p$-frame for $\mathcal{X}$ with respect to $\mathcal{H}$ if constants $0 < A_G \leq B_G < \infty$ exist such that

$$A_G \|f\|_X \leq \left( \sum_{i=1}^{\infty} \|G_i(f)\|_{\mathcal{C}_p}^p \right)^{\frac{1}{p}} \leq B_G \|f\|_X \tag{2.2}$$

for all $f \in \mathcal{X}$. It is called a von Neumann-Schatten $p$-Bessel sequence with bound $B_G$ if the second inequality holds. In [24], the authors have shown that the von Neumann-Schatten $p$-frame condition is satisfied if and only if $\sum_{i=1}^{\infty} A_i G_i$ is a well defined mapping from $\oplus \mathcal{C}_q$ onto $\mathcal{X}^*$. Motivated by this fact, they considered the following operators:

$$U_G : \mathcal{X} \to \oplus \mathcal{C}_p; \quad f \mapsto \{G_i(f)\}_{i=1}^{\infty}, \tag{2.3}$$

and

$$T_G : \oplus \mathcal{C}_q \to \mathcal{X}^*; \quad \{A_i\}_{i=1}^{\infty} \mapsto \sum_{i=1}^{\infty} A_i G_i. \tag{2.4}$$

As usual, the operator $U_G$ is called the analysis operator, and $T_G$ is the synthesis operator of $\mathcal{G}$.

Recall also from [24] that $\mathcal{G}$ is called a von Neumann-Schatten $q$-Riesz basis for $\mathcal{X}^*$ with respect to $\mathcal{H}$ if

1. $\{f \in \mathcal{X} : G_i(f) = 0 \forall i \in \mathbb{N}\} = \{0\}$,
2. there are positive constants $A_G$ and $B_G$ such that for any finite subset $I \subseteq \mathbb{N}$ and $\{A_i\}_{i=1}^{\infty} \in \oplus \mathcal{C}_q$

$$A_G \left( \sum_{i \in I} \|A_i\|_{\mathcal{C}_q}^q \right)^{\frac{1}{q}} \leq \|T_G(\{A_i\}_{i=1}^{\infty})\|_{\mathcal{X}^*} \leq B_G \left( \sum_{i \in I} \|A_i\|_{\mathcal{C}_q}^q \right)^{\frac{1}{q}}. \tag{2.5}$$

The assumption of latter definition implies that $\sum_{i=1}^{\infty} A_i G_i$ converges unconditionally for all $\mathcal{A} = \{A_i\}_{i=1}^{\infty} \in \oplus \mathcal{C}_q$ and

$$A_G \|A\|_q \leq \|T_G(\mathcal{A})\|_{\mathcal{X}^*} \leq B_G \| \mathcal{A} \|_q. \tag{2.5}$$

Thus $\mathcal{G} \subseteq B(\mathcal{X}, \mathcal{C}_p)$ is a von Neumann-Schatten $q$-Riesz basis for $\mathcal{X}^*$ with respect to $\mathcal{H}$ if and only if the operator $T_G$ defined in (2.4) is both bounded and bounded below. Specially, in this case the operators $T_G$ and $U_G$ are bijective.
The reader will remark that if \( \mathcal{H} = \mathbb{C} \), then \( B(\mathcal{H}) = \mathcal{C}_p = \mathbb{C} \) and thus \( \oplus \mathcal{C}_p = \ell^p \). Hence the above definitions is consistent with the corresponding definitions in the concept of \( p \)-frames for separable Banach spaces.

We conclude this section with the following result which can be proved with a similar argument as in the proof of [9, Corollary 2.5].

**Lemma 2.1.** Let \( \mathcal{X} \) be a reflexive Banach space and let \( \mathcal{G} = \{G_i\}_{i=1}^{\infty} \subseteq B(\mathcal{X}, \mathcal{C}_p) \) be a von Neumann-Schatten \( q \)-Riesz basis for \( \mathcal{X}^* \) with respect to \( \mathcal{H} \). If the \( q \)-Riesz basis bounds of \( \mathcal{G} \) are \( A_G \) and \( B_G \), then \( \mathcal{G} \) is a von Neumann-Schatten \( p \)-frame for \( \mathcal{X} \) with \( p \)-frame bounds \( A_G \) and \( B_G \).

3. **Von Neumann-Schatten Bessel Multipliers: Basic results**

All over in this section \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) are separable Banach spaces and the space \( \ell^r \) \((1 \leq r \leq \infty)\) has its usual meanings.

Our starting point of this section is the following lemma which play a crucial rule in this paper.

**Lemma 3.1.** Let \( \mathcal{G} \subseteq B(\mathcal{X}_1^*, \mathcal{C}_p) \) be a von Neumann-Schatten \( p \)-Bessel sequence with bound \( B_G \) and \( \mathcal{F} \subseteq B(\mathcal{X}_2, \mathcal{C}_q) \) be a von Neumann-Schatten \( q \)-Bessel sequence with bound \( B_F \). If \( \mathbf{m} = \{m_i\}_{i=1}^{\infty} \in \ell^\infty \), then the operator \( M_{\mathbf{m}, \mathcal{F}, \mathcal{G}} : \mathcal{X}_1^* \rightarrow \mathcal{X}_2^* \) defined by

\[
M_{\mathbf{m}, \mathcal{F}, \mathcal{G}}(f) = \sum_{i=1}^{\infty} m_i G_i(f) F_i \quad (f \in \mathcal{X}_1^*),
\]

is well-defined and \( \|M_{\mathbf{m}, \mathcal{F}, \mathcal{G}}\|_{\text{op}} \leq B_F B_G \|\mathbf{m}\|_\infty \).

**Proof.** It is easy to check that \( \{m_i G_i(f)\}_{i=1}^{\infty} \subseteq \oplus \mathcal{C}_p \) for all \( f \in \mathcal{X}_1^* \). On the other hand

\[
M_{\mathbf{m}, \mathcal{F}, \mathcal{G}}(f) = T_F\left(\{m_i G_i(f)\}_{i=1}^{\infty}\right) \quad (f \in \mathcal{X}_1^*).
\]

It follows that \( M_{\mathbf{m}, \mathcal{F}, \mathcal{G}} \) is well defined. Moreover, we observe that

\[
\left\| \sum_{i=1}^{n} m_i G_i(f) F_i \right\|_{\text{op}} = \sup_{h \in \mathcal{X}_2, \|h\| \leq 1} \left\{ \left\| \sum_{i=1}^{n} \text{tr}(m_i G_i(f) F_i(h)) \right\| \right\}
\]

\[
\leq \sup_{h \in \mathcal{X}_2, \|h\| \leq 1} \left\{ \sum_{i=1}^{n} \left\| \text{tr}(m_i G_i(f) F_i(h)) \right\| \right\}
\]

\[
\leq \sup_{h \in \mathcal{X}_2, \|h\| \leq 1} \left\{ \sum_{i=1}^{n} \left\| m_i G_i(f) F_i(h) \right\|_{C_1} \right\}
\]

\[
\leq \sup_{h \in \mathcal{X}_2, \|h\| \leq 1} \left\{ \sum_{i=1}^{n} \|m_i G_i(f)\|_{C_p} \|F_i(h)\|_{C_q} \right\}
\]

\[
\leq \|\mathbf{m}\|_\infty \left( \sum_{i=1}^{n} \|G_i(f)\|_{C_p}^q \right)^{\frac{1}{q}} \sup_{h \in \mathcal{X}_2, \|h\| \leq 1} \left( \sum_{i=1}^{n} \|F_i(h)\|_{C_q}^p \right)^{\frac{1}{p}}
\]

\[
\leq B_F B_G \|\mathbf{m}\|_\infty \|f\|,
\]
for all \( n \in \mathbb{N} \) and \( f \in X_1^* \). From this, we can deduced that the operator \( M_{m,F,G} \) is bounded with \( B_F B_G \| m \|_\infty \). □

Now we are in position to introduce the main object of study of this work.

**Definition 3.2.** Let \( G \subseteq B(X_1^*, C_p) \) be a von Neumann-Schatten \( p \)-Bessel sequence, and let \( F \subseteq B(X_2, C_q) \) be a von Neumann-Schatten \( q \)-Bessel sequence. Let \( m \in \ell_\infty \). The operator \( M_{m,F,G} : X_1^* \rightarrow X_2^* \) defined by

\[
M_{m,F,G}(f) = \sum_{i=1}^\infty m_i G_i(f) F_i \quad (f \in X_1^*)
\]

is called von Neumann-Schatten \((p,q)\)-Bessel multiplier and the sequence \( m \) is called its symbol.

If \( m \) is a sequence in \( \ell^r \) (\( 1 \leq r \leq \infty \)), then the mapping

\[
\mathcal{M}_{p,m} : \oplus C_p \rightarrow \oplus C_p, \{A_i\}_{i=1}^\infty \mapsto \{m_i A_i\}_{i=1}^\infty,
\]

is well-defined and bounded. Hence, the von Neumann-Schatten \((p,q)\)-Bessel multiplier \( M_{m,F,G} \) can be written as

\[
M_{m,F,G} = T_F \mathcal{M}_{p,m} U_G.
\]

This equality paves the way for the study of some operator properties of \( M_{m,F,G} \) in terms of the properties of \( \mathcal{M}_{p,m} \). To this end, we need the following remark and lemma.

**Remark 3.3.** Recall from [23] that \( \{e_n \otimes e_m : n, m \in I\} \) is an orthonormal basis of \( C_2 \). For the convenience of citation and a better exposition we denote by \( \{E_k\} \) the orthonormal basis of \( C_2 \). Hence, putting

\[
F_{i,k} = \{\delta_{i,j} E_k\}_{j=1}^\infty \quad (i, k \in \mathbb{N}),
\]

one can see that

1. in the case where \( \dim \mathcal{H} = N \), then \( \{F_{i,k}\}_{i=1}^{N^2} \) is an orthonormal basis for \( \oplus C_2 \).
2. in the case where \( \mathcal{H} \) is an infinite dimensional Hilbert space, then \( \{F_{i,k}\}_{i=1}^\infty \) is an orthonormal basis for \( \oplus C_2 \).

In what follows, the notation \( \overline{m} \) is used to denote the sequence \( \{\overline{m_i}\}_{i=1}^\infty \), where \( \overline{m_i} \) refers to the complex conjugate of \( m_i \).

**Lemma 3.4.** The following assertions hold.

1. If \( m \in \ell^\infty \), then \( \| \mathcal{M}_{p,m} \|_{\text{op}} = \| m \|_\infty \).
2. \( \mathcal{M}_{2,m}^* = \mathcal{M}_{2,\overline{m}} \).
3. If \( \dim \mathcal{H} = N \) and \( m \in \ell^p \), then \( \mathcal{M}_{2,m} \in C_p(\oplus C_2) \) and

\[
\| \mathcal{M}_{2,m} \|_{C_p(\oplus C_2)} = N^2 \| m \|_p.
\]
Proof. (1) That \( \|M_{p,m}\|_{op} \leq \|m\|_{\infty} \) is trivial. In order to prove that \( \|m\|_{\infty} \leq \|M_{p,m}\|_{op} \), suppose that \( j \in \mathbb{N} \) and \( x \in \mathcal{H} \) with \( \|x\| = 1 \). Observe that, if \( A^{(j)} = \{\delta_{i,j} \cdot x \otimes x\}_{i=1}^{\infty} \), then \( \|A^{(j)}\|_p = 1 \) and thus
\[
\|M_{p,m}\|_{op} \geq \|mA^{(j)}\|_p \geq \|m_j \cdot x \otimes x\|_{C_p} = |m_j|.
\]
It follows that \( \|m\|_{\infty} \leq \|M_{p,m}\|_{op} \).

(2) It is suffices to note that if \( A, B \in \oplus \mathcal{C}_2 \), then
\[
\langle M_{2,m}A, B \rangle = \sum_{i=1}^{\infty} [m_i A_i, B_i]_{tr}
\]
\[
= \sum_{i=1}^{\infty} [A_i, m_i B_i]_{tr}
\]
\[
= \langle A, M_{2,m}B \rangle.
\]

(3) If \( A \in \oplus \mathcal{C}_2 \), then we have
\[
M_{2,m}(A) = M_{2,m} \left( \sum_{i=1}^{\infty} \sum_{k=1}^{N^2} \langle A, F_{i,k} \rangle F_{i,k} \right)
\]
\[
= \sum_{k=1}^{N^2} \sum_{i=1}^{\infty} m_i \langle A, F_{i,k} \rangle F_{i,k}.
\]
Hence, if we set
\[
\hat{F}_j = F_{l+1,j-lN^2} \quad \left( l \in \mathbb{N} \cup \{0\}, \ lN^2 + 1 \leq j \leq (l+1)N^2 \right),
\]
and
\[
\hat{m} = \{\hat{m}_j\}_{j=1}^{\infty} = \{m_{1N^2}, \ldots, m_{1N^2}, m_{2N^2}, \ldots, m_{2N^2} \},
\]
then we observe that
\[
M_{2,m}(A) = \sum_{j=1}^{\infty} \hat{m}_j \langle A, \hat{F}_j \rangle \hat{F}_j.
\]
Therefore, \( M_{2,m} \) is in the Schatten \( p \)-class of \( \oplus \mathcal{C}_2 \); this is because of,
\[
\sum_{j=1}^{\infty} |\hat{m}_j|^p \leq N^2 \|m\|_{p}^p.
\]
Now, in order to prove that \( \|M_{2,m}\|_{C_p} = N^2 \|m\|_{p} \), first note that \( |M_{2,m}| = M_{2,|m|} \). Hence, we observe that
\[
\|M_{2,m}\|_{C_p}^p = \text{tr}(|M_{2,m}|^p) = \sum_{i=1}^{\infty} \sum_{k=1}^{N^2} \langle M_{2,|m|}^p(F_{i,k}), F_{i,k} \rangle
\]
\[
= \sum_{j=1}^{\infty} \langle M_{2,|\hat{m}|}^p(\hat{F}_j), \hat{F}_j \rangle
\]
\[
= \sum_{j=1}^{\infty} \langle M_{2,m}^p(\hat{F}_j), \hat{F}_j \rangle
\]
\[
= \sum_{j=1}^{\infty} \hat{m}_j \langle A, \hat{F}_j \rangle \hat{F}_j.
\]
\[ \begin{align*}
\sum_{j=1}^{N^2} \sum_{k=1}^{N^2} [ |m_j|^p E_k, E_k ]_{tr} & \\
= N^2 \|m\|_p^p.
\end{align*} \]

We have now completed the proof of the lemma. \(\Box\)

By applying Lemma 3.4 with \(\mathcal{H} = \mathbb{C}\), one can obtain the following improvement of [3, Lemma 5.4(3)].

**Lemma 3.5.** If \(m \in \ell^p\), then the operator
\[ M_m : \ell^2 \to \ell^2; \ {c_i}_{i=1}^\infty \mapsto \ {m_i c_i}_{i=1}^\infty, \]
is in the Schatten \(p\)-class of \(\ell^2\). In particular, \(\|M_m\|_{C_p(\ell^2)} = \|m\|_p\).

As was mentioned in section 2, in the case where \(p = 2\), the spaces \(C_2\) and \(\oplus_2 C_2\) are Hilbert. Motivated by this fact the authors of [1, 24] provided a detailed study of the duals of a von Neumann-Schatten 2-frame for Hilbert space \(K\) with respect to \(H\). Let us recall from [24] that, a sequence \(G \subseteq B(K, C_2)\) is said to be a Hilbert–Schmidt frame or simply a HS-frame for \(K\) with respect to \(H\), if there exist two positive numbers \(A_G\) and \(B_G\) such that
\[ A_G \|f\|_K^2 \leq \sum_{i=1}^\infty \|G_i(f)\|_{C_2}^2 \leq B_G \|f\|_K^2. \]

Particularly, by using the Hilbert properties of the spaces, they observed that
\[ U_G(f) = \{G_i(f)\}_{i=1}^\infty \quad \text{and} \quad T_G(\{A_i\}_{i=1}^\infty) = \sum_{i=1}^\infty G_i^* A_i, \]
where \(f \in K\) and \(\{A_i\}_{i=1}^\infty \in \oplus C_2\). Moreover, they showed that the mapping \(S_G := T_G U_G\) is a bounded, invertible, self-adjoint and positive operator, and they called the HS-frame \(\widetilde{G} := \{G_i S_G^{-1}\}_{i=1}^\infty\) the canonical dual HS-frame of \(G\). It is worthwhile to mention that the HS-frames is a more general version of \(g\)-frames, an important generalization of ordinary frames.

The following remark is a very useful tool in our study of HS-Bessel multiplier when \(H\) is a finite dimensional space.

**Remark 3.6.** Suppose that \(G\) and \(F\) are HS-Bessel sequences for \(K\) with respect to \(H\) and that \(m \in \ell^\infty\). For each \(f \in K\), we observe that
\[ M_{m,F,G}(f) = T_F M_{p,m} U_G(f) \]
\[ = \sum_{i=1}^\infty m_i F_i^* G_i(f) \]
\[ = \sum_{i=1}^\infty \sum_{n,m \in I} m_i \langle f, G_i^* (e_n \otimes e_m) \rangle F_i^* (e_n \otimes e_m). \]
In particular, if \( \dim(\mathcal{H}) = N \), then
\[
M_{m,F,G} = \sum_{n,m=1}^{N} \sum_{i=1}^{\infty} m_i \left( F_i^*(e_n \otimes e_m) \otimes G_i^*(e_n \otimes e_m) \right).
\]

Hence, if we set \( \Phi = \left\{ \{F_i^*(e_n \otimes e_m)\}_{n,m=1}^{N^2} \right\}_{i \in \mathbb{N}} \) and \( \Psi = \left\{ \{G_i^*(e_n \otimes e_m)\}_{n,m=1}^{N^2} \right\}_{i \in \mathbb{N}} \)
then \( \Phi \) and \( \Psi \) are ordinary Bessel sequences and the operator \( M_{m,F,G} \) is equal to the operator \( M_{\hat{m},\Phi,\Psi} \) in the sense of Balazs [3, Definition 5.1] for ordinary Bessel sequences. The reader will remark that in this case \( G \) and \( F \) are HS-Riesz basis if and only if \( \Phi \) and \( \Psi \) are ordinary Riesz basis, see [1, Theorem 3.3].

In the case where \( \dim(\mathcal{H}) < \infty \), Remark 3.6 paves the way for obtaining some properties of HS-Bessel multipliers from [3, 26]. In details, as an application of this remark and Lemma 3.4, by a method similar to that of [3, Theorems 6.1 and 8.1] one can easily obtain the following generalization of those theorems. The details are omitted.

Let \( F^{(l)} = \{F_i^{(l)}\}_{i=1}^{\infty} \) be a sequence in \( B(\mathcal{K}, \mathcal{C}_2) \) indexed by \( l \in \mathbb{N} \). We say that:

1. The sequence \( F^{(l)} \) converges uniformly to some sequence \( F \subseteq B(\mathcal{K}, \mathcal{C}_2) \) with respect to operator norm, if for \( i \to \infty \) we have
   \[
   \sup_l \{ \| F_i^{(l)} - F_i \|_{\text{op}} \} \to 0.
   \]

2. The sequence \( F^{(l)} \) converges to some sequence \( F \subseteq B(\mathcal{K}, \mathcal{C}_2) \) in \( \ell^2 \)-sense if
   \[
   \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \text{such that} \; \forall l \geq N \; \left( \sum_{i=1}^{\infty} \| F_i^{(l)} - F_i \|_{\mathcal{C}_p}^2 \right)^{1/2} < \varepsilon.
   \]

**Proposition 3.7.** Suppose that \( G \) and \( F \) are HS-Bessel sequences for \( \mathcal{K} \) with respect to \( \mathcal{H} \) and that \( \dim \mathcal{H} < \infty \). Then the following assertions hold.

1. If \( m \in c_0 \), then \( M_{m,F,G} \) is compact.
2. If \( m \in \ell^p \), then \( M_{m,F,G} \) is in the Schatten \( p \)-class of \( \mathcal{K} \) and
   \[
   \| M_{m,F,G} \|_{\mathcal{C}_p(\mathcal{K})} \leq \sqrt{B_G B_F} (\dim \mathcal{H})^2 \| m \|_p.
   \]
3. Let \( m^{(l)} = \{m_i^{(l)}\}_{i=1}^{\infty} \) be a sequence indexed by \( l \) converge to \( m \) in \( \ell^p \), then
   \[
   \| M_{m^{(l)},F,G} - M_{m,F,G} \|_{\mathcal{C}_p(\mathcal{K})} \to 0.
   \]
4. For the sequences \( F^{(l)} \subseteq B(\mathcal{K}, \mathcal{C}_2) \) indexed by \( l \in \mathbb{N} \), we can say that
   a. If \( m \in \ell^1 \) and the sequence \( F^{(l)} \) is a HS-Bessel sequence converging uniformly to \( F \) with respect to operator norm, then
      \[
      \| M_{m,F^{(l)},G} - M_{m,F,G} \|_{\mathcal{C}_1(\mathcal{K})} \to 0.
      \]
(b) If $m \in \ell^2$ and the sequence $F^{(i)}$ converge to $F$ in an $\ell^2$-sense, then

$$\|M_m,F^{(i)G} - M_m,F,G\|_c(\mathcal{K}) \rightarrow 0.$$  

(5) For HS-Bessel sequences $G^{(i)}$ converge to $G$, corresponding properties as in (4) apply.

(6) (a) Let $m^{(i)} \longrightarrow m$ in $\ell^1$, $F^{(i)}$ and $G^{(i)}$ be HS-Bessel sequences with bounds $B_{F^{(i)}}$ and $B_{G^{(i)}}$ such that $\sup l B_{F^{(i)}} < \infty$ and $\sup l B_{G^{(i)}} < \infty$. If the sequences $F^{(i)}$ and $G^{(i)}$ converge uniformly to $F$ respectively $G$ with respect to operator norm, then

$$\|M_m^{(i)F^{(i)}G^{(i)}} - M_m^{F,G}\|_c(\mathcal{K}) \rightarrow 0.$$  

(b) Let $m^{(i)} \longrightarrow m$ in $\ell^2$ and let $F^{(i)}$ respectively $G^{(i)}$ converge to $F$ respectively $G$ in an $\ell^2$-sense, then

$$\|M_m^{(i)F^{(i)}G^{(i)}} - M_m^{F,G}\|_c(\mathcal{K}) \rightarrow 0.$$  

By another application of Remark 3.6 with the aid of [26, Theorem 5.1] we have the following generalization of that theorem. In this proposition and in the sequel the sequence $m$ is called semi-normalized if $0 < \inf |m_i| \leq \sup |m_i| < \infty$; In this case, the notation $1/m$ is used to denote the sequence $\{1/m_i\}_{i=1}^\infty$.

**Proposition 3.8.** Suppose that $F$ is a HS-Riesz basis for $\mathcal{K}$ with respect to $\mathcal{H}$ and that $\dim \mathcal{H} < \infty$. Then the following assertions hold.

1. If $G$ is a HS-Riesz basis, then $M_m,F,G$ is invertible on $\mathcal{K}$ if and only if $m$ is semi-normalized.
2. If $m$ is semi-normalized, then $M_m,F,G$ is invertible on $\mathcal{K}$ if and only if $G$ is a HS-Riesz basis.

There does not seem to be an easy way to extend Propositions 3.7 and 3.8 to infinite dimensional case. However, we have the following result.

**Proposition 3.9.** Let $X_1$ be a reflexive Banach space and $G \subseteq B(X_1^*,C_p)$ be a von Neumann-Schatten $q$-Riesz basis for $X_1$ with bounds $A_G$ and $B_G$. Let also $F \subseteq B(X_2,C_p)$ be a von Neumann-Schatten $p$-Riesz basis for $X_2^*$ with bounds $A_F$ and $B_F$. Then for each $m \in \ell^\infty$ we have

$$A_FA_G\|m\|_\infty \leq \|M_m,F,G\|_{\text{op}} \leq B_FB_G\|m\|_\infty.$$  

In particular, the operator $M_m,F,G$ is invertible and the mapping $m \mapsto M_m,F,G$ from $\ell^\infty$ into $B(X_1^*,X_2^*)$ is injective.

**Proof.** By Lemma 3.1, it will be enough to prove that we have the lower bound. To this end, suppose that $x$ is an arbitrary element of $\mathcal{H}$ with $\|x\| = 1$ and $j \in \mathbb{N}$. Then, $A^{(i)} = \{\delta_{ij} \cdot x \otimes x\}_{i=1}^\infty \in \oplus C_p$. Thus, by the surjectivity of the operator $U_G$, there exists an element $f_j \in X_1^*$ such that $U_G(f_j) = A^{(j)}$. In particular, $\sum_{i=1}^\infty \|G_i(f_j)\|_{C_p}^p = 1$. We now invoke Lemma 2.1 to conclude that
Define $\Gamma$.

Proof. Suppose that $\mathcal{G} = \{G_j\}_{j=1}^{\infty}$ and $\mathcal{F} = \{F_j\}_{j=1}^{\infty}$ are $H.S$-frames for $\mathcal{K}$ with respect to $\mathcal{H}$, and that the symbol $\mathbf{m}$ is semi-normalized. If $\mathbf{m}$ is an invertible multiplier, then there exists a unique bounded operator $\Gamma : \mathcal{K} \to \oplus_2 \mathcal{C}_2$ such that

$$\mathbf{M}_{\mathbf{m},\mathcal{F},\mathcal{G}} = \mathbf{M}_{1/\mathbf{m},\tilde{\mathcal{G}},\tilde{\mathcal{F}}^d} + \Gamma^* U_{\mathcal{F}^d},$$

(4.1)

for all dual $H.S$-frames $\mathcal{F}^d = \{\mathcal{F}_j\}_{j=1}^{\infty}$ of $\mathcal{F}$.

Proof. Define $\Gamma : \mathcal{K} \to \oplus_2 \mathcal{C}_2$ by

$$\Gamma(f) := U_{\mathcal{F}}(\mathbf{M}_{\mathbf{m},\mathcal{F},\mathcal{G}}^{-1}(f) - \mathcal{M}_{2,1,\mathbf{m}} U_{\mathcal{G}} S_{\mathcal{G}}^{-1}(f)) \quad (f \in \mathcal{K}).$$

(4.2)

Then the operator $\Gamma$ is bounded and

$$\mathbf{M}_{\mathbf{m},\mathcal{F},\mathcal{G}}^{-1} U_{\mathcal{F}} = \Gamma^* + S_{\mathcal{G}}^{-1} T_{\mathcal{G}} \mathcal{M}_{2,1,\mathbf{m}}.$$

Using any dual $H.S$-frame $\mathcal{F}^d$ of $\mathcal{F}$ we get

$$\mathbf{M}_{\mathbf{m},\mathcal{F},\mathcal{G}}^{-1} = S_{\mathcal{G}}^{-1} T_{\mathcal{G}} \mathcal{M}_{2,1,\mathbf{m}} U_{\mathcal{F}^d} + \Gamma^* U_{\mathcal{F}^d},$$

(4.3)

It follows that $\mathbf{M}_{\mathbf{m},\mathcal{F},\mathcal{G}}^{-1} = \mathbf{M}_{1/\mathbf{m},\tilde{\mathcal{G}},\tilde{\mathcal{F}}^d} + \Gamma^* U_{\mathcal{F}^d}$ for all dual $H.S$-frames $\mathcal{F}^d$ of $\mathcal{F}$.

Having reached this state it remains to show that $\Gamma$ is uniquely determined. To this end, suppose on the contrary that Eq. (4.3) are hold for two operators $\Gamma_1$ and $\Gamma_2$. It follows that

$$(\Gamma_1 - \Gamma_2)^* U_{\mathcal{F}^d} = 0,$$

(4.4)

for all dual $H.S$-frames $\mathcal{F}^d$ of $\mathcal{F}$. In particular, for each $f \in \mathcal{K}$, we have

$$(\Gamma_1 - \Gamma_2)^* (\mathcal{F}_j S_{\mathcal{F}}^{-1}(f))_{j=1}^{\infty} = 0.$$  

(4.5)

Now, let $i$ and $k$ be arbitrary elements in $\mathbb{N}$ and $F_{i,k} = \{\delta_{i,j} \mathcal{E}_k\}_{j=1}^{\infty}$, which introduced in Remark 3.3. If for each $j \in \mathbb{N}$, we define

$$\mathcal{F}_{i,j,k} : \mathcal{K} \to \mathcal{C}_2; \quad f \mapsto (f, \mathbf{e}_1') \delta_{i,j} \mathcal{E}_k,$$
where \( \{e'_l\}_{l \in L} \) is an orthonormal basis for \( \mathcal{K} \). Then it is not hard to check that the sequence \( \mathcal{F}'_{i,k} = \{ \mathcal{F}'_{i,j,k} \}^\infty_{j=1} \) is a \( HS \)-Bessel sequence for \( \mathcal{K} \) with respect to \( \mathcal{H} \) and the \( HS \)-Bessel sequence
\[
\mathcal{F}'_{i,k} = \{ \mathcal{F}'_{j} S^{-1}_{j} + \mathcal{F}'_{i,j,k} - \mathcal{F}'_{j} S^{-1}_{j} T_{\mathcal{F}} U_{\mathcal{F}} \}^\infty_{j=1}
\]
is a dual \( HS \)-frame of \( \mathcal{F} \). Therefore, Eq. (4.4) and (4.5), implies that
\[
0 = (\Gamma_1 - \Gamma_2)^* U_{\mathcal{F}_{i,k}} (f)
\]
\[
= (\Gamma_1 - \Gamma_2)^* \left( \{ \mathcal{F}'_{j} S^{-1}_{j} (f) + \mathcal{F}'_{i,j,k} (f) - \mathcal{F}'_{j} S^{-1}_{j} T_{\mathcal{F}} U_{\mathcal{F}} (f) \}^\infty_{j=1} \right)
\]
\[
= (\Gamma_1 - \Gamma_2)^* \left( \{ \mathcal{F}'_{i,j,k} (f) \}^\infty_{j=1} \right)
\]
for all \( f \in \mathcal{K} \). Hence, we have
\[
0 = (\Gamma_1 - \Gamma_2)^* \left( \{ \mathcal{F}'_{i,j,k} (e'_l) \}^\infty_{j=1} \right) = (\Gamma_1 - \Gamma_2)^* (F_{i,k}).
\]
This says that \( (\Gamma_1 - \Gamma_2)^* (F_{i,k}) = 0 \) for all \( i, k \in \mathbb{N} \). We now invoke Remark 3.3 to conclude that \( \Gamma_1 = \Gamma_2 \) and this completes the proof of the theorem. \( \square \)

For operator \( \Gamma \) in Proposition 4.7 it is not hard to check that \( T_{\mathcal{G}} M_{2, \pi} \Gamma = 0 \). It follows that, if \( \mathcal{G} \) is a \( HS \)-Riesz basis and \( \mathfrak{m} \) is semi–normalized, then
\[
M_{\mathfrak{m}, \mathcal{F}, \mathcal{G}}^{-1} = M_{1/m, \mathcal{G}, \mathcal{F}'},
\]
for all dual \( HS \)–frames \( \mathcal{F}' = \{ \mathcal{F}'_{i} \}^\infty_{i=1} \) of \( \mathcal{F} \).

This observation together with Proposition 3.9 give the following result.

**Corollary 4.2.** Let \( \mathcal{G} \) and \( \mathcal{F} \) be \( HS \)–Riesz basis for \( \mathcal{K} \) with respect to \( \mathcal{H} \) and let \( \mathfrak{m} \) be semi–normalized. Then \( M_{\mathfrak{m}, \mathcal{F}, \mathcal{G}} \) is invertible on \( \mathcal{K} \) and
\[
M_{\mathfrak{m}, \mathcal{F}, \mathcal{G}}^{-1} = M_{1/m, \mathcal{G}, \mathcal{F}'}.
\]

The proof of Theorem 4.5 below relies on the following remark and proposition.

**Remark 4.3.** Following [20, Remark 2.8], we say that the \( HS \)-frame \( \mathcal{G}^{gd} = \{ \mathcal{G}_i^{gd} \} \) is a generalized dual \( HS \)-frame of \( \mathcal{G} \), if \( T_{\mathcal{G}} U_{\mathcal{G}^{gd}} \) is invertible. It is noteworthy that with an argument similar to the proof of [20, Theorem 2.1] and [1, Theorem 3.1] one can show that the generalized dual \( HS \)-frames of \( \mathcal{G} \) are precisely the sequences \( \mathcal{G}^{gd} \) such that
\[
\mathcal{G}_i^{gd} = \mathcal{G}_i S^{-1}_{i} Q + \pi_i \Psi, \quad (i \in \mathbb{N})
\]
where \( \Psi \) is a bounded operator in \( B(\mathcal{K}, \oplus \mathcal{C}_2) \) such that \( T_{\mathcal{G}} \Psi = 0 \), \( \pi_i : \oplus \mathcal{C}_2 \to \mathcal{C}_2 \) is the standard projection on the \( i \)-th component and \( Q \) is an invertible operator in \( B(\mathcal{K}) \). In what follows, the notation \( GD(\mathcal{G}) \) is used to denote the set of all generalized \( HS \)-duals of \( \mathcal{G} \).

In the following result and in the sequel Inv(\( \mathcal{G}, \mathfrak{m} \)) refers to the set of all \( HS \)-Bessel sequence \( \mathcal{F} \) such that the operator \( M_{\mathfrak{m}, \mathcal{F}, \mathcal{G}} \) is invertible.
Proposition 4.4. Suppose that $G$ is a HS-frame for $K$ with respect to $H$ and that $m$ is a semi-normalized sequence. Then the mapping

$$\Theta : \text{Inv}(G, m) \to GD(G); \quad \{F_i\}_{i=1}^{\infty} \mapsto \{G_i S_G^{-1} M_{m,G}^{-1} + \pi_i M 2 \mu_1 \Gamma \}_{i=1}^{\infty},$$

is bijective, where $\Gamma$ is the unique operator in $B(K, \oplus C_2)$ which satisfies the equality (4.1).

Proof. It obviously suffices to show that $\Theta$ is an onto map. To this end, suppose that $G^{op}$ is a generalized HS-dual of $G$. Then, by Remark 4.3, we would have a bounded operator in $B(K, \oplus C_2)$ and an invertible operator $Q$ in $B(K)$ such that $T_G \Psi = 0$ and

$$G^{op}_i = G_i S_G^{-1} Q + \pi_i \Psi, \quad (i \in N).$$

Letting $F_i : K \to C_2$ by

$$F_i = (1/\mu_i)G_i S_G^{-1} Q^* + \pi_i (M 2 \mu_1 \Psi).$$

Then $F$ is a HS-Bessel sequence and, in particular, we have $Q = M_{m,F,G}$. Hence, $F$ is in $\text{Inv}(G, m)$. On the other hand, if $\Gamma$ is the unique operator which defined by (4.2), then, for each $f \in K$, we have

$$\Gamma(f) = U_F (M_{m,F,G})^* (f) - M 2 \mu_1 S_G^{-1} f = M 2 \mu_1 S_G^{-1} f - M 2 \mu_1 S_G^{-1} f = M 2 \mu_1 \Psi(f).$$

It follows that $\Theta(F) = G^{op}$. \qed

Theorem 4.5. Let $G$ be a HS-frame for $K$ with respect to $H$ and let $G'$ be another HS-frame for $K$ with respect to $H$ such that $\|T_G - T_{G'}\| < \sqrt{A_G}/2$, where $A_G$ is the lower frame bound of $G$. If $m$ is semi-normalized, then there exists a one-to-one correspondence between $\text{Inv}(G, m)$ and $\text{Inv}(G', m)$.

Proof. By Proposition 4.4 above, it will be enough to prove that there exists a one-to-one correspondence between $GD(G)$ and $GD(G')$. To this end, we define the map $\Lambda$ from $GD(G)$ into $GD(G')$ by

$$G^{op} \mapsto \{G'_i S_{G'} T_{G^{op}} + \pi_i P_{\ker(T_{G'})} U_{G^{op}} \}_{i=1}^{\infty},$$

where $P_{\ker(T_{G'})}$ denotes the orthogonal projection of $\oplus C_2$ onto $\ker(T_{G'})$. Now, with an argument similar to the proof of Proposition 4.15 in [22] one can show that $\Lambda$ is bijective and this completes the proof. \qed

The next result characterizes another invertible HS-frame multipliers $M_{m,F,G}$ whose inverses can be written as $M_{1/m,G,F}$. In details, the following result is a generalization of a result proved by Balazs and Stoeva [5, Theorem 4.6] to HS-frames as well as $g$-frames. To this end, we need the following remark.

Remark 4.6. Suppose that $G$ and $F$ are HS-frames for $K$ with respect to $H$, and that the symbol $m$ is semi-normalized.
(1) If the \( HS \)-frames \( \overline{m}_i \mathcal{F} := \{ \overline{m}_i \mathcal{F}_i \}^\infty_{i=1} \) and \( \mathcal{G} \) are equivalent; that is, there exists an invertible operator \( Q \) in \( B(\mathcal{K}) \) such that \( \overline{m}_i \mathcal{F}_i = \mathcal{G}Q \) \( (i \in \mathbb{N}) \), then \( M_{m,F,G} \) is invertible and

\[
M^{-1}_{m,F,G} = M_{1/m, \tilde{G}, \tilde{F}^d},
\]

for all dual \( HS \)-frames \( \mathcal{F}^d \) of \( \mathcal{F} \). Indeed, on the one hand we have

\[
M_{m,F,G} = T_F M_{2,m} U_G = T_{\overline{m} \mathcal{F}} U_G = Q^* S_G,
\]

and on the other hand, if the letter \( FQ^{-1} \) respectively \( (1/\overline{m})G \) refer to the \( HS \)-Bessel sequence \( \{ \mathcal{F}_i Q^{-1} \}^\infty_{i=1} \) respectively \( \{(1/\overline{m})G_i \}^\infty_{i=1} \), then we observe that

\[
M_{1/m, \tilde{G}, \tilde{F}^d} = T_{G} M_{2,1/m} U_{\mathcal{F}^d}
= S_{G}^{-1} T_{(1/\overline{m})G} U_{\mathcal{F}^d}
= S_{G}^{-1} T_F Q^{-1} U_{\mathcal{F}^d}
= (Q^* S_G)^{-1}.
\]

Conversely, if \( M_{m,F,G} \) is invertible and \( M^{-1}_{m,F,G} = M_{1/m, \tilde{G}, \tilde{F}^d} \), for all dual \( HS \)-frames \( \mathcal{F}^d \) of \( \mathcal{F} \), then Theorem 4.7 implies that \( \Gamma^* U_{\mathcal{F}^d} = 0 \). Now, with an argument similar to the proof of Theorem 4.7 one can conclude that \( \Gamma = 0 \). From this, by Eq. (4.2), we deduce that

\[
\overline{m}_i \mathcal{F}_i = \mathcal{G}_i S_{G}^{-1} M_{\overline{m}, \mathcal{F}} (i \in \mathbb{N}),
\]

and thus the \( HS \)-frames \( \mathcal{G} \) and \( \overline{m} \mathcal{F} \) are equivalent. Hence, we can give the interpretation below for equivalent \( HS \)-frames:

"If \( m \) is semi-normalized, then \( M_{m,F,G} \) is invertible and \( M^{-1}_{m,F,G} = M_{1/m, \tilde{G}, \tilde{F}^d} \), for all dual \( HS \)-frames \( \mathcal{F}^d \) of \( \mathcal{F} \) if and only if the \( HS \)-frames \( \overline{m} \mathcal{F} := \{ \overline{m}_i \mathcal{F}_i \}^\infty_{i=1} \) and \( \mathcal{G} \) are equivalent."

(2) If \( \mathcal{Y} \) denotes any one of the \( HS \)-frames \( \mathcal{G} \) and \( \mathcal{F} \), then it is easy to check that

\[
\oplus \mathcal{C}_2 = \text{ran}(U_{\mathcal{Y}}) \oplus \ker(T_{\mathcal{Y}})
\]

and thus \( P_{\ker(T_{\mathcal{Y}})} + P_{\text{ran}(U_{\mathcal{Y}})} = \text{Id}_{\oplus \mathcal{C}_2} \), where \( P_X \) denotes the orthogonal projection of \( \oplus \mathcal{C}_2 \) onto \( X \). In particular, we have

\[
P_{\text{ran}(U_{\mathcal{Y}})} = U_{G} T_{G}, \quad \text{and} \quad P_{\text{ran}(U_{\mathcal{F}})} = U_{\mathcal{F}} T_{\mathcal{F}}.
\]

The following proposition is now immediate.

**Proposition 4.7.** Suppose that \( \mathcal{G} \) and \( \mathcal{F} \) are \( HS \)-frames for \( \mathcal{K} \) with respect to \( \mathcal{H} \), and that there exists a non-zero constant \( c \) such that \( m_i = c \) for all \( i \in \mathbb{N} \). Then the following statements are equivalent.

1. \( M_{m,F,G} \) is invertible and \( M^{-1}_{m,F,G} = M_{1/m, \tilde{G}, \tilde{F}^d} \).
2. \( \mathcal{F} \) and \( \mathcal{G} \) are equivalent \( HS \)-frames.
Proof. We first note that, without loss of generality, we may consider $c = 1$. The necessity of the condition “$\mathcal{F}$ and $\mathcal{G}$ are equivalent HS-frames” follows from part (1) of Remark 4.6. We prove its sufficiency. To this end, suppose that the condition (1) is satisfied. From this, we observe that

$$U_\mathcal{G}T_{\mathcal{G}} = U_\mathcal{G}M_{1/m,\tilde{\mathcal{G}}}M_{m,\mathcal{F} \mathcal{G}}T_{\mathcal{G}} = U_\mathcal{G}T_{\mathcal{G}}U_{\tilde{\mathcal{F}}}U_{\mathcal{G}}T_{\mathcal{G}}.$$ 

We now invoke part (2) of Remark 4.6 to conclude that

$$P_{\text{ran}(U_\mathcal{G})}P_{\text{ran}(U_{\tilde{\mathcal{F}}})}P_{\text{ran}(U_\mathcal{G})} = P_{\text{ran}(U_\mathcal{G})}.$$ 

This says that $\text{ran}(U_{\tilde{\mathcal{F}}}) \subseteq \text{ran}(U_\mathcal{G})$ and thus

$$\text{ran}(U_{\tilde{\mathcal{F}}}) \subseteq \text{ran}(U_\mathcal{G}).$$

Analogously, one can show that the reverse inclusion is also true. It follows that $\text{ran}(U_{\tilde{\mathcal{F}}}) = \text{ran}(U_\mathcal{G})$. Now, we follow the proof of [2, Lemma 2.1] to show that there exists an invertible operator $Q \in B(\mathcal{K})$ such that $\mathcal{G}_i = \mathcal{F}_i Q$ ($i \in \mathbb{N}$). To this end, suppose that $\mathcal{G}' = \{\mathcal{G}_i S_{\mathcal{F}}^{-\frac{1}{2}}\}$ and $\mathcal{F}' = \{\mathcal{F}_i S_{\mathcal{F}}^{-\frac{1}{2}}\}$. Observe that $\mathcal{G}'$ and $\mathcal{F}'$ are HS-frames with lower and upper frame bounds 1 and thus $S_{\mathcal{G}'} = Id_{\mathcal{K}} = S_{\mathcal{F}'}$. Moreover, it is not hard to check that $\text{ran}(U_{\mathcal{F}'}) = \text{ran}(U_{\mathcal{G}'})$ and thus

$$U_\mathcal{G}' T_{\mathcal{G}'} = P_{\text{ran}(U_{\mathcal{G}'})} = P_{\text{ran}(U_{\mathcal{F}'})} = U_{\mathcal{F}'} U_{\mathcal{G}'}.$$ 

Hence, if we set $Q = T_{\mathcal{G}'} U_{\mathcal{F}'}$, then

$$Q^* Q = T_{\mathcal{F}'} U_{\mathcal{G}'} T_{\mathcal{G}'} U_{\mathcal{F}'} = Id_{\mathcal{K}}.$$ 

This says that $Q$ is an isometry, $Q^* (f) = \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{G}_i'(f)$ and

$$f = \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{G}_i' Q(f) \quad (f \in \mathcal{K}).$$

Now, for each $f \in \mathcal{K}$, we have

$$\sum_{i=1}^{\infty} \|\mathcal{G}_i' Q(f) - \mathcal{F}_i'(f)\|_{C_2}^2 = \sum_{i=1}^{\infty} \|\mathcal{G}_i' Q(f)\|_{C_2}^2 + \sum_{i=1}^{\infty} \|\mathcal{F}_i'(f)\|_{C_2}^2 - 2 Re \left( \left[ \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{G}_i' Q, f \right]_{tr} \right) \nonumber$$

$$= \|Q(f)\|^2 - \|f\|^2 = 0$$

It follows that $\mathcal{G}_i' Q = \mathcal{F}_i' (i \in \mathbb{N})$. Having reached this state it remains to prove that $Q$ is onto or equivalently $\ker(Q^*) = \{0\}$. To this end, suppose that $g \in \ker(Q^*)$. Hence, $U_{\mathcal{G}'}(g) \in \ker(T_{\mathcal{F}'})$ and thus, since

$$\ker(T_{\mathcal{F}'}) = \text{ran}(U_{\mathcal{F}'})^\perp = \text{ran}(U_{\mathcal{G}'})^\perp = \ker(T_{\mathcal{G}'})$$

we can deduce that $U_{\mathcal{G}'}(g) \in \ker(T_{\mathcal{G}'})$. It follows that $g = T_{\mathcal{G}'} U_{\mathcal{G}'}(g) = 0$. We have now completed the proof of the proposition. \qed
We conclude this work by the following result which is of interest in its own right. In details, as an another application of Theorem 4.7, we have the following surprising new results about dual HS-frames as well as dual g-frames which shows that a HS-frame [respectively, g-frame] is uniquely determined by the set of its dual HS-frames [respectively, g-frames].

**Theorem 4.8.** Let $\mathcal{G}$ and $\mathcal{F}$ be HS-frames for $\mathcal{K}$ with respect to $\mathcal{H}$. If every dual HS-frame $\mathcal{G}^d$ of $\mathcal{G}$ is a dual HS-frame of $\mathcal{F}$, then $\mathcal{G} = \mathcal{F}$.

**Proof.** The assumption implies that $(T_F - T_G)U_{\mathcal{G}^d} = 0$ for all dual HS-frames $\mathcal{G}^d$ of $\mathcal{G}$. Hence, with an argument similar to the proof of Theorem 4.7 one can show that $T_F = T_G$. Whence $\mathcal{G} = \mathcal{F}$. □

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