The 4-loop slope of the Dirac form factor

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Abstract. The 4-loop contribution to the slope of the Dirac form factor in QED has been evaluated with 1100 digits of precision. The value is

\[ m^2 F_1^{4\text{L}}(0) = 0.886545673946443145836821730610315359390424032660064745 \ldots \left( \frac{\pi}{4} \right)^4. \]

We have also obtained a semi-analytical fit to the numerical value. The expression contains harmonic polylogarithms of argument \(e^{\pm}, e^{\pm 2}, e^{\pm 3}\), one-dimensional integrals of products of complete elliptic integrals and six finite parts of master integrals, evaluated up to 4800 digits. We show the correction to the energy levels of the hydrogen atom due to the slope.

1 The slope of the Dirac form factor

In QED the vertex can be written

\[ \bar{u}(p_1) \left( \gamma_{\mu} F_1(t) + \frac{\sigma_{\mu\nu} q}{2m} F_2(t) \right) u(p_2), \]
(1)

where \(m\) is the electron mass, \(F_1(t)\) and \(F_2(t)\) are the Dirac and Pauli form factors. At \(t = 0\), the charge conservation implies that

\[ F_1(0) = 1, \]
(2)

whereas the value of the Pauli form factor is the \(g\)-2

\[ F_2(0) = \frac{g - 2}{2}. \]
(3)

The quantity \( \frac{d}{dt} F_1(t) \big|_{t=0} = F'_1(0) \) is the slope of the Dirac form factor.

1.1 Theoretical expression

We expand perturbatively the slope in powers of \( \left( \frac{t}{2} \right) \)

\[ m^2 F_1'(0) = A_1 \left( \frac{\alpha}{\pi} \right) + A_2 \left( \frac{\alpha}{\pi} \right)^2 + A_3 \left( \frac{\alpha}{\pi} \right)^3 + A_4 \left( \frac{\alpha}{\pi} \right)^4 + \ldots \]
(4)

The coefficients known in analytical form are [2–4]

\[ A_1 = -\frac{1}{8} = \frac{1}{6\epsilon}. \]

\[ A_2 = \frac{-4819}{5184} = \frac{49}{432} \pi^2 + \frac{1}{2} \pi^2 \ln 2 - \frac{3}{4} \zeta(3). \]

\[ A_3 = \frac{17}{24} \pi^2 \zeta(3) + \frac{25}{8} \zeta(5) - \frac{217}{9} \left( \ln(1/2) \right)^2 + \frac{\ln^4 2}{24} \]
(5)

\[ - \frac{103}{1080} \pi^2 \ln 2 + \frac{3899}{25920} \pi^4 - \frac{2929}{288} \zeta(3) + \frac{41671}{2160} \pi^2 \ln 2 \]

\[ - \frac{45497}{38880} \pi^2 + \frac{7751}{186624} \]
(6)

\[ = 0.171 72 0 018 909 775 \ldots \]

In this paper we present the result of the calculation of \( A_4 \) with a precision of 1100 digits. The first digits of the result are

\[ A_4 = \ldots \]

\[ 0.8865456739464431458368217306103153 \ldots \]
(7)

\[ \text{Table 1. First 1100 digits of } A_4. \]

\begin{tabular}{|c|}
\hline
0.88654567394644314583682173061031535939042403266006474538880...
\hline
5999219084934646666274538684668639512365351275874721838799
687592197168846666274538684668639512365351275874721838799
49271921651159352559197598106495819624531372119372946716889
63429893495285130954959665386395981411437321894219002394882
7595153823476312326838464585680576168756418271979966
39182907261152235646619539927965162776824259236654373257
6136115939512571647369921622627939564524928545655821844282687
4547648877599211186537883513981096677781596743821267424747
0495222473092501281307042961996290156276892012895089916426
464494887878673727884367488374347898245541537243489089951947
164311558642591988532897498651264874758933459259931822056563
61325124118886651553762317312224846727668664712932892869865
681148838315727667873963219385926475289598223981753735282
91719426239118135272884342243699692642135486400649061242
5329153183936112566753158241744482817616593812766926161676675
6958564664939561191588888024564511634567571623969417388481
15656098338448794659918875421790666737828208535534195188378
61007518116338192280...
\hline
\end{tabular}

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1 0.13505317263464353572674275414103838371038
2 0.380292916520484458585555258298843579658371
3 -0.078944889367683610810962836694179823079
4 0.366278637658847004458542507235237020299
5 -1.0979832148713652010382073196531832520
6 0.6467871429585372084492397189890386216165
7 0.089589117040432421060936534902414320659
8 -0.33220862259016643680126657791885971097899
9 0.07633674793739345296122046789318739605
10 0.21186690108881812137876340161652003594809
11 -0.0541837571893367164657206136746826299854
12 0.0108761535582322105869453086735119912448
13 -0.01426466601968301162682021692409901716905
14 -0.005811741601042035783143542203438251011
15 -0.243906850619592123409557076293747890
16 0.206201257084112758622621863926017000956
17 0.0083566428673036656037790352019835488011
18 0.055392709530294934127688014591823326838
19 0.023605891191014021135877461122766184082
20 0.074016316220572405138179043210727390276
21 -0.053771160706495699908276338567906834199
22 0.181947427396666401697577215395176159307
23 0.235928927396666401697577215395176159307
24 -0.00212258953199048736532220609442649666
25 0.069036262075704991160330435886767759471

Table 3. Contribution to $A_4$ of the 25 gauge-invariant sets of Fig. 1.

$$A_{4\text{slope}}(nS, 4\text{-loop}) = \frac{4(Z\alpha)^4 m_c^2}{h n^2} \left( \frac{m_r}{m} \right)^3 \left( \frac{\alpha}{\pi} \right)^4 A_4$$

where $m_r$ is the reduced mass $m_r = mM/(m + M)$ and $M$ is the proton mass. Inserting the values of $m, M, c, h$ and $Z = 1$, the correction due to $A_4$ is

$$A_{4\text{slope}}(nS, 4\text{-loop}) = \frac{36.11}{n^2} \text{Hz}$$

and is comparable with the experimental error of the extremely precise measurement of $1S - 2S$ transition[7]

$$f(1S - 2S) = 2466 061 413 187 018 \pm 11 \text{ Hz}$$

Eq.(10) is the first calculated four-loop correction to the energy levels, of the kind $(\sqrt{\pi}/\pi)^4 (Z\alpha)^4$.

## 2 Gauge-invariant sets

There are 891 vertex diagrams contributing to $A_4$. These vertex diagrams can be arranged in 25 gauge-invariant sets (Fig.1). The sets are classified according to the number of photon corrections on the same side of the main electron line and the insertions of electron loops (see Ref.[8]). The numerical contributions of each set, truncated to 40 digits, are listed in the table 3. The separate contributions to the slope from diagrams without or with internal loops are listed in table 4.

### Table 2. Values of the known contributions to $F'_1(0)$ and $F_2(0)$

| loop | $F'_1(0)$ | $F_2(0)$ |
|------|-----------|-----------|
| 1    | $\infty$  | 0.5       |
| 2    | 0.469941487459 | -0.328478965579 |
| 3    | 0.171720018909 | 1.181241456587 |
| 4    | 0.886545673946 | -1.912245764926 |
| 5    |           | 6.737(159) |

positive alternating signs

The full-precision result is shown in table 1. In table 2 we have listed the known values of the slope and $g$-2; we see that $A_2, A_3, A_4$ and $A_5$ are all positive, in contrast with the alternating signs observed in the $g$-2.

### 1.2 Shift to the hydrogen levels

Let us now consider the shift to the hydrogen energy levels due to $A_4$. We express the energy shift in terms of the frequency shift $\Delta f = \Delta E/h$. For the level $nS$ the frequency shift is $[5, 6]$

\[
\Delta f(\text{S}) = \frac{4(Z\alpha)^4 m^2}{h n^2} \left( \frac{m_r}{m} \right)^3 \left( \frac{\alpha}{\pi} \right)^4 A_4
\]

where $m_r$ is the reduced mass $m_r = mM/(m + M)$ and $M$ is the proton mass. Inserting the values of $m, M, c, h$ and $Z = 1$, the correction due to $A_4$ is

\[
\Delta f(\text{S}) = \frac{36.11}{n^2} \text{Hz}
\]

and is comparable with the experimental error of the extremely precise measurement of $1S - 2S$ transition[7]

\[
f(1S - 2S) = 2466 061 413 187 018 \pm 11 \text{ Hz}
\]
$248832 \pi^2 = 1233637481 + 18215 \ln 2$

$T = 1177/36 \zeta(2) Cl_2 \left( \frac{\pi}{2} \right) + 38424/125 \zeta(2) Cl_2 \left( \frac{\pi}{2} \right)$

$= -118 \left( 4 \Re H_{0,0,1,1} (i \zeta (2)) + 4 \Im H_{0,1,1,1} (i \zeta (2)) \right)$

$- 2 \mathrm{Cl}_4 \left( \frac{\pi}{2} \right) \pi + \mathrm{Cl}_2 \left( \frac{\pi}{2} \right) \ln 2^2$.

$$V_b = \frac{14186171}{194400} \zeta_2 \left( \frac{\pi}{2} \right) + \frac{103028300}{583200} \left( \frac{\pi}{2} \right)^2$$

$= \frac{14186171}{194400} \zeta_2 \left( \frac{\pi}{2} \right) + \frac{103028300}{583200} \left( \frac{\pi}{2} \right)^2$

$= \frac{14186171}{194400} \zeta_2 \left( \frac{\pi}{2} \right) + \frac{103028300}{583200} \left( \frac{\pi}{2} \right)^2$

$V_b = \frac{212671}{2400} \zeta_2 \left( \frac{\pi}{2} \right) + \frac{1031987}{302400} \zeta_2 \left( \frac{\pi}{2} \right)$

$= \frac{212671}{2400} \zeta_2 \left( \frac{\pi}{2} \right) + \frac{1031987}{302400} \zeta_2 \left( \frac{\pi}{2} \right)$

$E_a = s \left( \frac{5581722922}{362880000} B_3 + \frac{1233637481}{7399680000} C_1 \right)$

$= \frac{5581722922}{362880000} B_3 + \frac{1233637481}{7399680000} C_1$

$= \frac{5581722922}{362880000} B_3 + \frac{1233637481}{7399680000} C_1$

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$= \frac{5581722922}{362880000} B_3 + \frac{1233637481}{7399680000} C_1$

$E_a = s \left( \frac{5581722922}{362880000} B_3 + \frac{1233637481}{7399680000} C_1 \right)$

$= \frac{5581722922}{362880000} B_3 + \frac{1233637481}{7399680000} C_1$
\begin{table}[h]
\centering
\begin{tabular}{lrrr}
\hline
$F$ & -2191.235469657511788413162922858828885509 & \\
$\sqrt{V_{\alpha}}$ & -648.74441479274053140037234999290048941 & \\
$V_{\beta}$ & -400.664495157660792576160481868291283752 & \\
$W$ & 1539.32919916681465350981276108756905937 & \\
$\sqrt{V_{\beta}}$ & -265.5409171010623811287632183079933994 & \\
$E_{b}$ & 1928.222536484421458451655379066123429 & \\
$U$ & 40.5201067276630676486149201782154366 & \\
\hline
\end{tabular}
\caption{Numerical values of the addends appearing in Eq.(12).
\label{tab:values}}
\end{table}

\subsection*{4 Method of calculation}

We sketch the method used to obtain $A_4$. It is the same used in Ref.[1].

1. Generation of 891 vertex diagrams (C program) from 104 self-mass diagrams. These are the same of the 4-loop $g$-2 calculation.

2. Extraction of the contribution to $A_4$ from the amplitude of each diagram by using projectors [18, 19] with a FORM program[20, 21].

3. Algebraic reduction to master integrals, obtained by building and solving large systems of integration-by-parts identities[9, 10] by using the program SYS[11].

4. For the sake of checks we generate a different system for each group of vertex diagrams obtained from the same self-mass diagram.

5. The smallest system contains $10^8$ identities, with size of 90GB. The system with the largest number of identities contains $5 \times 10^6$, with a size of 170GB. The largest system has $3 \times 10^8$ identities with a size of 1.2TB.

6. The ratio between number of independent identities and total number of generated identities is in the range $0.2 - 0.3$. The dependent identities become trivial zeroes when substituted into the system, and have been used to check the reliability of hardware and software. No hardware errors were detected. Instead, software errors have been detected in this way (frequency: one every 2-3 weeks), caused by a bug in the OpenMPI message passing library used with the highest level of threads support.

7. We algebraically check that the contribution from a diagram is invariant to the changes in the particular internal routing of the momentum of the external photon.

8. The renormalization is carried out by subtracting suitable counterterms, which are generated with C and FORM programs and calculated numerically with SYS.

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