On total irregularity strength of caterpillar graphs with two leaves on each internal vertex

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Abstract. Let $G(V, E)$ be a graph. A function $f$ from $V(G) \cup E(G)$ to the set $\{1, 2, \ldots, k\}$ is said to be a totally irregular total $k$-labeling of $G$ if the weights of any two different vertices $x$ and $y$ in $V(G)$ satisfy $w_f(x) \neq w_f(y)$ where the weight $w_f(x)$ is the sum of label of $x$ and labels of all edges incident to $x$, and the weights of any two different edges $ux$ and $vy$ in $E(G)$ satisfy $w_f(ux) \neq w_f(vy)$ where the weight $w_f(ux)$ is obtained from the sum of: label of $x$, label of $u$ and label of edge $ux$. The total irregularity strength of the graph $G$, denoted by $ts(G)$, is the minimum number $k$ for which $G$ has a totally irregular total $k$-labeling. In this paper, we focus on a caterpillar graph with two leaves on each internal vertex $T_{2n+p,q}$, where $n$ is the number of leaves on each end vertex of the central path, $p$ is the number of leaves connected to internal vertices, $q$ is the number of vertices of the central path, $p \geq 4$ and $q \geq 4$. Firstly, we do some experiments for constructing a formula for totally irregular total $k$-labeling of the caterpillar graph $T_{2n+p,q}$. Secondly, we determine the minimum number $k$ which is $ts$ of the caterpillar graph $T_{2n+p,q}$. We obtain that $ts(T_{2(q-1)+p,q}) = \left\lceil \frac{2(q-1)+p+1}{2} \right\rceil$ and $ts(T_{2n+p,q}) = \left\lceil \frac{2n+p+1}{2} \right\rceil$.

1. Introduction

Let $G = (V, E)$ be a simple graph. A labeling of $G$ is a mapping from a set of graph elements into a set of integers, that are called labels. If the domain of the mapping is either a vertex set, or an edge set, or a union of vertex and edge sets, then the labeling is called vertex labeling, edge labeling, or total labeling, respectively (Wallis [19]; Marr and Wallis [10]).

Bača et al. [3] introduced a vertex irregular total $k$-labeling of $G$, that is a function $f : V \cup E \rightarrow \{1, 2, \ldots, k\}$ such that the weights $w_f(x) \neq w_f(y)$ for every two different vertices $x$ and $y$ in $V(G)$, where $w_f(x) = f(x) + \sum_{xy \in E} f(xy)$ and the minimum number $k$ is called a vertex irregularity strength of the graph $G$, denoted by tvs($G$). Meanwhile an edge irregular total $k$-labeling of $G$ is defined as a function $f : V \cup E \rightarrow \{1, 2, \ldots, k\}$ such that the weights $w_f(uv) \neq w_f(xy)$ for every two different edges $uv$ and $xy$ in $E(G)$, where $w_f(uv) = f(u) + f(v) + \sum_{uv \in E} f(uv)$ and the minimum number $k$ is called an edge irregularity strength of $G$, denoted by tes($G$). Further, the function $f$ is said to be a totally irregular total $k$-labeling of $G$ if the weights of any two different vertices $x$ and $y$ in $V(G)$ satisfy $w_f(x) \neq w_f(y)$ and the weights of any two different edges $uv$ and $xy$ in $E(G)$ satisfy $w_f(uv) \neq w_f(xy)$. The total irregularity strength of the graph $G$, denoted by $ts(G)$, is the minimum number $k$ for which $G$ has a totally irregular total $k$-labeling.
Bača et al. [3] gave the lower and upper bound of the total vertex irregularity strength for any graph $G$ with $p$ vertices, $q$ edges, the minimum degree $\delta$, and the maximum degree $\Delta$ as follows:

$$\left\lfloor \frac{p + \delta}{\Delta + 1} \right\rfloor \leq \text{tvs}(G) \leq p + \Delta - 2\delta + 1$$

Anholcer et al. [2] improved the bound given by Bača et al. as follows:

$$\text{tvs}(G) \leq 3 \left\lceil \frac{p}{\delta} \right\rceil + 1.$$

Nurdin [14] obtained the total vertex irregularity strength of a tree $T$ with $n$ leaves and no vertex of degree 2 as follows:

$$\text{tvs}(T) = \left\lfloor \frac{n + 1}{2} \right\rfloor.$$

Further, Nurdin [15] investigated the total vertex irregularity strength of a Caterpillar $T_{n,m}$ with $n$ leaves and $m$ vertices on the central path as follows: $\text{tvs}(T_{n,m}) = \left\lceil \frac{n+1}{2} \right\rceil$. Indriati et al. [5] found the total vertex irregularity strength of generalized helm graphs and prisms with outer pendant edges.

Ivanco and Jendrol et al. [9] investigated the total edge irregularity strength of a tree $T$, that is

$$\text{tes}(T) = \max \left\{ \left\lfloor \frac{|E(T)| + 2}{3} \right\rfloor, \left\lceil \frac{\Delta(T) + 1}{2} \right\rceil \right\}.$$

Hinding et al. [4] determined the total edge irregularity strength of subdivision of star. Miskuf and Jendrol [12] found a total edge irregularity strength (tes) of grid graph. Nurdin et al. [13] obtained the total edge-irregularity strengths of the corona product of paths with some graphs. Siddiqui [18] has obtained total edge irregularity strength of strong product of two paths.

In the problem of determining the exact value of total irregularity strength of a graph, Marzuki et al. [11] gave an observation on the lower bound for the total irregularity strength of a graph $G$ as follows:

$$\text{ts}(G) \geq \max \{\text{tvs}(G), \text{tes}(G)\}.$$  

In the same paper, they determined a total irregularity strength of cycles and paths. After that, many researchers have invented the total irregularity strength of graphs. Such as, Indriati et al.[6] invented the total irregularity strength of double star $S_{n,n}$ and caterpillar $S_{n,2,n}$ that is a caterpillar with inserted vertex of degree 2. Ramdani et al. [16] investigated a total irregularity strength of some cartesian product graphs. Also, Ramdani et al.[17] obtained a total irregularity strength of gear, fungus and disjoint union of star. Further, Indriati et al.[7] found the total irregularity strength of star graph, double star $S_{n,n}$ and caterpillar $S_{n,2,2,n}$. Recently, Indriati et al.[8] has done on the total irregularity strength of $S_{n,2,m}$.

In this paper, we focus on caterpillar graphs that have two leaves on each internal vertex. We denote the caterpillar graphs as $T_{2n+p,q}$ where $n$ is the number of leaves on each end vertex of the central path, $p$ is the number of leaves connected to internal vertices, $q$ is the number of vertices of the central path, $p \geq 2(q - 2)$, and $q \geq 4$. We construct a totally irregular total $k$-labeling and investigate the total irregularity strength of the caterpillar graphs $T_{2(q-1)+p,q}$ and $T_{2n+p,q}$.

2. Main Results

In this paper, we have found the total irregularity strength of some caterpillar Graphs with two leaves on each internal vertex as stated in Theorem 1 and Theorem 2.
Theorem 1 Let $T_{2(q-1)+p,q}$ be a caterpillar graph with $2(q-1)$ leaves connected to the two end vertices of the central path, $p = 2(q - 2)$ is the number of leaves connected to internal vertices, $q$ is the number of vertices of the central path, and $q \geq 4$. The total irregularity strength of $T_{2n+p,q}$ is

$$ts(T_{2(q-1)+p,q}) = \left\lceil \frac{2(q-1) + p + 1}{2} \right\rceil.$$ 

Proof. It is obvious that the caterpillar graph $T_{2(q-1)+p,q}$ has $2(q-1) + p$ leaves. Assume that the vertex set of the caterpillar graph is $V = \{a^1_i, a^2_i, b_k\}$ where $a^1_i, a^2_i$ are the leaves connected to vertices $b_k$ for $i = 1, 2, \ldots, 2(q-1) + p$, $j = 1, 2, k = 1, 2, \ldots, q$ and $b_k$ are the vertices in the central path of caterpillar. The edge set of the caterpillar is $E = \{a^1_k b_k, a^2_j b_k, b_k b_l\}$ for $k, l = 1, 2, \ldots, p$.

The total vertex irregularity strength of caterpillar $T_{2(q-1)+p,q}$ can be found by using a theorem given in [15], as follows:

$$tvs(T_{2(q-1)+p,q}) = \left\lceil \frac{2(q-1) + p + 1}{2} \right\rceil.$$ 

Based on the total edge irregularity strength of a tree given by Ivanco and Jendrol [9], we have:

$$tes(T) = \max \left\{ \left\lceil \frac{|E(T_{2n+p,q})| + 2}{3} \right\rceil, \left\lceil \frac{\triangle(T_{2(q-1)+p,q}) + 1}{2} \right\rceil \right\}$$

It follows from $|E(T_{2n+p,q})| = 2(q-1) + p + (q-1) = 3(q-1) + p$ and $\triangle(T_{2(q-1)+p,q}) = q - 1 + 1 = q$, we obtain

$$tes(T) = \max \left\{ \left\lceil \frac{2(q-1) + p + (q-1) + 2}{3} \right\rceil, \left\lceil \frac{q + 1}{2} \right\rceil \right\} = \frac{3(q-1) + p + 2}{3}.$$ 

Further, by using an observation from Marzuki et al. [11], the lower bound for the total irregularity strength of caterpillar $T_{2n+p,q}$ as follows:

$$ts(T_{2n+p,q}) \geq \max \{tvs(T_{2n+p,q}), tes(T_{2n+p,q})\} = \left\lceil \frac{2(q-1) + p + 1}{2} \right\rceil.$$ 

Further, we determine the exact value of the total irregularity strength of caterpillar $T_{2(q-1)+p,q}$ by constructing the totally irregular total $k$-labeling on vertex set and edge set: $f : V \cup E \to \{1, 2, \ldots, k\}$ where $k = \left\lceil \frac{2(q-1)+p+1}{2} \right\rceil$ as follows:

$$f(a^1_i) = \begin{cases} q, & \text{if } i = 1, 2 \\ q + (i - 2), & \text{if } 3 \leq i \leq q - 2 \end{cases}$$

The label of vertex $f(a^1_{q-1}) = f(a^1_{q-2}) = 2q - 4$.

$$f(a^1_i) = \left\lceil \frac{2(q-1) + p + 1}{2} \right\rceil, \ i = 1, 2 \ \text{and} \ 2 \leq k \leq q - 1.$$ 

$$f(a^q_p) = \begin{cases} 1, & \text{if } i = 1, 2 \\ i - 1, & \text{if } 3 \leq i \leq q - 1 \end{cases}$$

If $q$ is odd, then the labels of vertices $b_k$ as follows:
\begin{equation}
f(b_k) = \begin{cases} 
1, & \text{if } k = 1, q \\
2k - 2, & \text{if } 2 \leq k \leq q - 1 
\end{cases}
\end{equation}

If \( q \) is even, then the labels of \( b_k \) as follows:

\begin{equation}
f(b_k) = \begin{cases} 
1, & \text{if } k = 1, q \\
2k - 2, & \text{if } 2 \leq k \leq q - 2 \\
2k, & \text{if } k = q - 1 
\end{cases}
\end{equation}

We define the labels of edges as follows:

\begin{equation}
f(a_1^i b_1) = \begin{cases} 
1, & \text{if } i = 1 \\
2, & \text{if } 2 \leq i \leq (q - 1) - 1 \\
3, & \text{if } i = q - 1 
\end{cases}
\end{equation}

\begin{equation}
f(a_q^i b_q) = \begin{cases} 
1, & \text{if } i = 1 \\
2, & \text{if } 2 \leq i \leq (q - 1) 
\end{cases}
\end{equation}

\begin{equation}
f(a_k^i b_k) = i + 2k - 3, \text{ for } 2 \leq k \leq q - 1 \text{ and } i = 1, 2.
\end{equation}

Under the labeling \( f \), the weights of vertices of the caterpillar \( T_{2(q-1)+p,q} \) as follows:

\begin{equation}
w(a_k^i) = \begin{cases} 
\left\lfloor \frac{2(q-1)+p+1}{2} \right\rfloor + (i - q + 2), & \text{if } k = 1, 1 \leq i \leq (q - 1) \\
i + 1, & \text{if } k = q, 1 \leq i \leq (q - 1) \\
(i + 2k - 3) + \left\lfloor \frac{2(q-1)+p+1}{2} \right\rfloor, & \text{if } 2 \leq k \leq q - 1, i = 1, 2
\end{cases}
\end{equation}

If \( q \) is odd, the weights of vertices \( b_k \) as follows:

\begin{equation}
w(b_k) = \begin{cases} 
(2q - 1) + \left\lfloor \frac{2(q-1)+p+1}{2} \right\rfloor, & \text{if } k = 1 \\
2 \left\lfloor \frac{2(q-1)+p+1}{2} \right\rfloor, & \text{if } k = q \\
(6k - 5) + 2 \left\lfloor \frac{2(q-1)+p+1}{2} \right\rfloor, & \text{if } 2 \leq k \leq q - 1
\end{cases}
\end{equation}

If \( q \) is even, the weights of \( b_k \) as follows:

\begin{equation}
w(b_k) = \begin{cases} 
(2q - 1) + \left\lfloor \frac{2(q-1)+p+1}{2} \right\rfloor, & \text{if } k = 1 \\
2 \left\lfloor \frac{2(q-1)+p+1}{2} \right\rfloor, & \text{if } k = q \\
(6k - 5) + 2 \left\lfloor \frac{2(q-1)+p+1}{2} \right\rfloor, & \text{if } 2 \leq k \leq q - 2 \\
(6k - 3) + 2 \left\lfloor \frac{2(q-1)+p+1}{2} \right\rfloor, & \text{if } k = q - 1
\end{cases}
\end{equation}

The weights of edges \( a_k^i b_k \) and \( b_k b_{k+1} \) of the caterpillar \( T_{2(q-1)+p,q} \) under the labeling \( f \) as follows:

Case 1: For \( q \) is odd
\[ w(a^k b_k) = \begin{cases} 
\frac{1}{2} \left\lceil \frac{2(q-1)+p+1}{2} \right\rceil + 3, & \text{if } i = 1, k = 1 \\
\frac{1}{2} \left\lceil \frac{2(q-1)+p+1}{2} \right\rceil + (i + 2), & \text{if } 2 \leq i \leq (q - 1), k = 1 \\
\left\lceil \frac{2(q-1)+p+1}{2} \right\rceil + (4k + i - 5), & \text{if } 2 \leq k \leq q - 1, i = 1, 2 
\end{cases} \]

\[ w(b_k b_{k+1}) = \begin{cases} 
\left\lceil \frac{2(q-1)+p+1}{2} \right\rceil + (4k - 2), & \text{if } 2 \leq k \leq q - 2 \\
\left\lceil \frac{2(q-1)+p+1}{2} \right\rceil - 1, & \text{if } k = q - 1 
\end{cases} \]

**Case 2:** For \( q \) is even

\[ w(a^k b_k) = \begin{cases} 
\frac{1}{2} \left\lceil \frac{2(q-1)+p+1}{2} \right\rceil + 3, & \text{if } i = 1, k = 1 \\
\frac{1}{2} \left\lceil \frac{2(q-1)+p+1}{2} \right\rceil + (i + 2), & \text{if } 2 \leq i \leq (q - 1), k = 1 \\
\left\lceil \frac{2(q-1)+p+1}{2} \right\rceil + (4k + i - 5), & \text{if } 2 \leq k \leq q - 2, i = 1, 2 \\
\left\lceil \frac{2(q-1)+p+1}{2} \right\rceil + (4k + i - 3), & \text{if } k = q - 1, i = 1, 2 
\end{cases} \]

\[ w(b_k b_{k+1}) = \begin{cases} 
\left\lceil \frac{2(q-1)+p+1}{2} \right\rceil + (4k - 2), & \text{if } 2 \leq k \leq q - 2 \\
\left\lceil \frac{2(q-1)+p+1}{2} \right\rceil + 1, & \text{if } k = q - 1 
\end{cases} \]

It is proved that all labels of vertices and edges are at most \( \left\lceil \frac{2(q-1)+p+1}{2} \right\rceil \) and the edge and vertex weights under the labeling \( f \) are distinct. Thus, the total irregularity strength of the caterpillar is \( ts(T_{2(q-1)+p,q}) = \left\lceil \frac{2(q-1)+p+1}{2} \right\rceil \). □

An example of the construction of the totally irregular total 14-labeling of \( T_{2(q-1)+12,8} \) is given in Figure 1. The red colored number represent the weight of a vertex or the weight of an edge.
Figure 1. The totally irregular total 14-labeling of $T_{2,7+12,8}$.

Theorem 2 Let $T_{2n+p,q}$ be a caterpillar graph as stated in Theorem 1, where $n \geq 3q - 7$, and $q \geq 4$. The total irregularity strength of $T_{2n+p,q}$ is

$$ts(T_{2n+p,q}) = \left\lceil \frac{2n + p + 1}{2} \right\rceil.$$

Proof. It is obvious that the caterpillar $T_{2n+p,q}$ has $2n + p$ leaves. Let $V$ be the vertex set of the caterpillar $T_{2n+p,q}$ where $V = \{a_i^k, a_j^k, b_k\}$ where $a_i^k, a_j^k$ are the leaves connected to vertices $b_k$ for $i = 1, 2, \ldots, 2n, j = 1, 2, k = 1, 2, \ldots, p$ and $b_k$ are the vertices in the central path of caterpillar. Let $E$ be the edge set of the caterpillar where $E = \{a_i^kb_k, a_j^kb_k, b_kb_l\}$ for $k, l = 1, 2, \ldots, p$.

Based on observation from Marzuki et al. [11], the lower bound for the total irregularity strength of caterpillar $T_{2n+p,q}$ as follows:

$$ts(T_{2n+p,q}) \geq \max\{tvu(T_{2n+p,q}), tes(T_{2n+p,q})\} = \left\lceil \frac{2n + p + 1}{2} \right\rceil.$$

Further, we determine the total irregularity strength of caterpillar $T_{2n+p,q}$ for $n \geq 3q - 7$ by the totally irregular total $k$-labeling $f$ on the vertex set and edge set $f : V \cup E \rightarrow \{1, 2, \ldots, k\}$ where $k = \left\lceil \frac{2n + p + 1}{2} \right\rceil$ that is defined as follows:

$$f(a_i^k) = \begin{cases} \left\lceil \frac{2n + p + 1}{2} \right\rceil + (i - (q - 1)), & \text{if } 1 \leq i \leq q - 1, k = 1 \\ \left\lceil \frac{2n + p + 1}{2} \right\rceil, & \text{if } q \leq i \leq n, k = 1 \\ \left\lceil \frac{2n + p + 1}{2} \right\rceil, & \text{if } i = 1, 2, \quad 2 \leq k \leq q - 1 \end{cases}$$

$$f(a_i^q) = \begin{cases} i, & \text{if } 1 \leq i \leq n - q + 2 \\ i - 1, & \text{if } n - q + 3 \leq i \leq n, k = 1 \end{cases}$$

$$f(b_k) = \begin{cases} 2k + (n - q) - 1, & \text{if } n - q \quad \text{is odd, } 2 \leq k \leq q - 1 \\ 2k + (n - q), & \text{if } n - q \quad \text{is even, } 2 \leq k \leq q - 1 \\ 1, & \text{if } k = 1 \text{ and } k = q \end{cases}$$
If \((n - q) + 2(k - 1)\), if \(i = 1, 2\) and \(2 \leq k \leq q - 2\)

We obtain the weights of vertices of the caterpillar \(T_{2n+p,q}\) under the labeling \(f\) as follows:

\[
f(a_i^k b_k) = \begin{cases} 1, & \text{if } 1 \leq i \leq q - 1, k = 1 \\ i - q + 2, & \text{if } q \leq i \leq n, k = 1 \\ 1, & \text{if } 1 \leq i \leq n - q + 2, k = q \\ 2, & \text{if } n - q + 3 \leq i \leq n, k = q \\ (n - q) + (i + 2(k - 1)), & \text{if } i = 1, 2 \text{ and } 2 \leq k \leq q - 2 \\ (n - q) + i + p, & \text{if } i = 1, 2 \text{ and } k = q - 1 \\
\end{cases}
\]

\[
f(b_k b_{k+1}) = \left\lfloor \frac{2n + p + 1}{2} \right\rfloor \quad \text{for } 1 \leq k \leq q - 1
\]

Furthermore, the edge weights of the caterpillar under the labeling \(f\) as follows:

\[
w(a_i^k) = \begin{cases} \left\lfloor \frac{2n + p + 1}{2} \right\rfloor + (i - q + 2), & \text{if } k = 1, 1 \leq i \leq n \\ i + 1, & \text{if } k = q, 1 \leq i \leq n \\ (n - q) + i + 2(k - 1) + \left\lfloor \frac{2n + p + 1}{2} \right\rfloor, & \text{if } 2 \leq k \leq q - 2, i = 1, 2 \\ (n - q) + i + p + \left\lfloor \frac{2n + p + 1}{2} \right\rfloor, & \text{if } k = q - 1, i = 1, 2 \\
\end{cases}
\]

\[
w(b_k) = \begin{cases} 2 \left\lfloor \frac{2n + p + 1}{2} \right\rfloor, & \text{if } k = q \\ 2 \left\lfloor \frac{2n + p + 1}{2} \right\rfloor + 3(n - q) + 6k - 1, & \text{if } 2 \leq k \leq q - 2 \text{ and } n - q \text{ is even} \\ 2 \left\lfloor \frac{2n + p + 1}{2} \right\rfloor + 3(n - q) + 6k - 2, & \text{if } 2 \leq k \leq q - 2 \text{ and } n - q \text{ is odd} \\
\end{cases}
\]

\[
w(b_k) = \begin{cases} 2 \left\lfloor \frac{2n + p + 1}{2} \right\rfloor + 2k + 3(n - q) + 3 + 2p, & \text{if } k = q - 1 \text{ and } n - q \text{ is even} \\ 2 \left\lfloor \frac{2n + p + 1}{2} \right\rfloor + 2k + 3(n - q) + 2 + 2p, & \text{if } k = q - 1 \text{ and } n - q \text{ is odd} \\
\end{cases}
\]

If \((n - q)\) is even, then the weights:

\[
w(a_i^k b_k) = \begin{cases} i + 2, & \text{if } 1 \leq i \leq n, k = q \\ n + i + 2, & \text{if } 1 \leq i \leq n, k = 1 \\
\end{cases}
\]

If \((n - q)\) is odd, then we obtain the weights:

\[
w(a_i^k b_k) = \begin{cases} \left\lfloor \frac{2n + p + 1}{2} \right\rfloor + 2(n - q) + i + 4k - 2, & \text{if } i = 1, 2, 2 \leq k \leq q - 2 \\ \left\lfloor \frac{2n + p + 1}{2} \right\rfloor + 2(n - q) + i + 2k + p, & \text{if } i = 1, 2, k = q - 1 \\
\end{cases}
\]

If \((n - q)\) is odd, then we obtain the weights:

\[
w(a_i^k b_k) = \begin{cases} \left\lfloor \frac{2n + p + 1}{2} \right\rfloor + 2(n - q) + i + 4k - 3, & \text{if } i = 1, 2, 2 \leq k \leq q - 2 \\ \left\lfloor \frac{2n + p + 1}{2} \right\rfloor + 2(n - q) + i + 2k - 1 + p, & \text{if } i = 1, 2, k = q - 1 \\
\end{cases}
\]
\[
\begin{align*}
\text{w}(b_1b_2) &= \begin{cases} 
5 + \left\lceil \frac{2n+p+1}{2} \right\rceil + (n-q), & \text{if } (n-q) \text{ is even} \\
4 + \left\lceil \frac{2n+p+1}{2} \right\rceil + (n-q), & \text{if } (n-q) \text{ is odd}
\end{cases} \\
\text{w}(b_kb_{k+1}) &= \begin{cases} 
\left\lceil \frac{2n+p+1}{2} \right\rceil + 4k + 2(n-q), & \text{if } (n-q) \text{ is odd} \\
\left\lceil \frac{2n+p+1}{2} \right\rceil + 4k + 2(n-q) + 2, & \text{if } (n-q) \text{ is even}
\end{cases} \\
\text{w}(b_kb_{k+1}) &= \begin{cases} 
\left\lceil \frac{2n+p+1}{2} \right\rceil + 2k + (n-q), & \text{if } (n-q) \text{ is odd} \\
\left\lceil \frac{2n+p+1}{2} \right\rceil + 2k + (n-q) + 1, & \text{if } (n-q) \text{ is even}
\end{cases}
\end{align*}
\]

It is clear that all labels of vertices and edges are at most \( \left\lceil \frac{2n+p+1}{2} \right\rceil \) and the edge and vertex weights under the the labeling \( f \) are distinct. Thus, the total irregularity strength of the caterpillar is \( \text{ts}(T_{2n+p,q}) = \left\lceil \frac{2n+p+1}{2} \right\rceil \). \( \Box \)

An illustration of the totally irregular total 16-labeling of \( T_{2,11+8,6} \) can be seen on Figure 2. A number with the red color indicate the weight of a vertex or the weight of an edge.

**Figure 2.** The totally irregular total 16-labeling of \( T_{2,11+8,6} \).

### 3. Concluding Remarks

In this paper, we have found the exact value of total irregularity strength of caterpillar graphs that contain two leaves on each internal vertex and \( 2n \) leaves connected to the end vertices of central path in the caterpillar. We have obtained that \( \text{tvs}(T_{2(q-1)+p,q}) = \left\lceil \frac{2(q-1)+p+1}{2} \right\rceil \) and \( \text{tvs}(T_{2n+p,q}) = \left\lceil \frac{2n+p+1}{2} \right\rceil \) for \( p \geq 4 \) and \( q \geq 4 \).

An **Open problem** as the direction for further research is given as follows: The total irregularity strength of caterpillar graphs \( T_{2n+p,q} \) where \( q - 1 < n \leq 2q - 8 \) and \( n \geq 4 \) and the total irregularity strength of caterpillar graphs \( T_{2n+p,q} \) where \( p = \alpha(q-2) \) is even for \( \alpha = 2, 4, 6, \ldots \). Further, we can investigate the total irregularity strength of caterpillar graphs \( T_{n+n+p,q} \), that is a caterpillar graph that have different number of leaves on the end vertices of central path in the caterpillar graph, \( p = \alpha(q-2) \) for constant \( \alpha = 2, 4, 6, \ldots \) is even. Another,
the total irregularity strength of caterpillar graphs $T_{m+n+p,q}$ where $p = \alpha(q - 2)$ is odd for constant $\alpha = 1, 3, 5, \ldots$.

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