A NOTE ON THE ASYMPTOTICS FOR INCOMPLETE BETAFUNCTIONS

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Abstract. We determine the asymptotic behaviour of certain incomplete Betafunctions.

1. Introduction and results

For integers $k, \ell \geq 1$ define

$$P_{k, \ell} = \frac{(k + \ell)!}{(k - 1)!\ell!} \int_{\frac{k}{k+\ell}}^{1} t^{k-1}(1-t)^{\ell} \, dt.$$  

By repeated partial integration we obtain the representation

$$P_{k, \ell} = \left(\frac{\ell}{\ell + k}\right)^{k+\ell} \sum_{\nu=0}^{k-1} \binom{k+\ell}{\nu} \left(\frac{k}{\ell}\right)^{\nu}.$$  

Vietoris\cite{5} showed that $P_{k, \ell} \leq \frac{1}{2}$. Alzer and Kwong\cite{1} proved that $P_{k, \ell} \geq \frac{1}{4}$ holds for all $k, \ell$. Interest in bounds for $P_{k, \ell}$ stems from application to statistics, see\cite{4}.

In this note we consider the asymptotic behaviour of $P_{k, \ell}$. We prove the following.

Theorem 1. We have

(1)  
$$P_{k, \ell} = e^{-k} \sum_{\nu=0}^{k-1} \frac{k^\nu}{\nu!} + \mathcal{O}\left(\frac{k^2}{\ell}\right),$$  

(2)  
$$P_{k, \ell} = 1 - e^{-\ell} \sum_{\nu=0}^{\ell} \frac{\ell^\nu}{\nu!} + \mathcal{O}\left(\frac{\ell^2}{k}\right),$$  

and

(3)  
$$P_{k, \ell} = \frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\min(k, \ell)}}\right).$$

Note that the first two estimates are good if one of the parameters $k, \ell$ is rather small, whereas the third one gives information in the general case.

Comparing (1) and (2) with Vietoris’ result that $P_{k, \ell} \leq \frac{1}{2}$ we see that

$$\sum_{\nu=0}^{\ell-1} \frac{\ell^\nu}{\nu!} < \frac{1}{2}e^\ell < \sum_{\nu=0}^{\ell} \frac{\ell^\nu}{\nu!}.$$  

Note that equality is impossible as $e$ is transcendental. A more precise version of this inequality has been asked by Ramanujan (Question 294) and was answered by
For a detailed discussion of this result, Uhlmann’s inequalities and Vietoris bound we refer the reader to the historical notes by Vietoris.

From Theorem 1 we deduce the following.

**Corollary 1.** The only accumulation points of the set \( \{ P_{k,\ell} : k, \ell \in \mathbb{N} \} \) are \( \frac{1}{2} \) and the real numbers \( e^{-k \sum_{\nu=1}^{k-1} \frac{\nu}{k}} \) and \( 1 - e^{-\ell \sum_{\nu=1}^{\ell} \frac{\nu}{\ell}} \).

**Proof.** Suppose that \((k_n), (\ell_n)\) are integer sequences such that the pairs \((k_n, \ell_n)\) are all distinct and that the sequence \((P_{k_n,\ell_n})\) converges. If both \(k_n, \ell_n\) tend to infinity, then by (3) we have that \( P_{k_n,\ell_n} \) tends to \( \frac{1}{2} \). If one of these sequences does not tend to infinity, then we can pass to a subsequence and assume that \(k_n\) or \(\ell_n\) is constant. Then our claim follows from (1) or (2).

Together with Vietoris bound we obtain the following.

**Corollary 2.** We have

\[
\lim_{\ell \to \infty} P_{k,\ell} \geq \frac{1}{2} - \frac{1}{\sqrt{2\pi k}}.
\]

**Proof.** We have

\[
\lim_{\ell \to \infty} (P_{k,\ell} + P_{\ell,k}) = 1 - \frac{k^k}{e^{k!}}
\]

using Stirling’s formula in the form \( k! = \left( \frac{k}{e} \right)^k \sqrt{2\pi k} e^{\theta k} \) with \( 0 \leq \theta \leq 1 \), and \( P_{\ell,k} \leq \frac{1}{2} \), we conclude

\[
\lim_{\ell \to \infty} P_{k,\ell} \geq \frac{1}{2} - \frac{k^k}{e^{k!}} \geq \frac{1}{2} - \frac{1}{\sqrt{2\pi k}}.
\]

\( \square \)

2. **Preliminary estimates**

For the proof of \([3]\) we use another representation of \( P_{k,\ell} \), which is due to Raab.

**Theorem 2.** We have \( P_{k,\ell} = U_{k,\ell} V_{k,\ell} \), where

\[
U_{k,\ell} = \exp \int_0^\infty \frac{1}{t} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \left( e^{-(k+\ell)t} - e^{-kt} - e^{-\ell t} \right) dt,
\]

\[
V_{k,\ell} = \sqrt{\ell} \sum_{\nu=1}^{\infty} \frac{c_\nu(k/\ell)}{\sqrt{\nu(\nu + \ell)}},
\]

and

\[
c_\nu(x) = \exp \int_0^\infty \frac{1}{t} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \left( e^{-\nu(1+x)t} - e^{-\nu t} - e^{-\nu t} \right) dt.
\]

We first compute the occurring integrals.

**Lemma 1.** We have for \( x \geq 1 \)

\[
\int_0^\infty \frac{1}{t} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-xt} dt = \frac{1}{12x} + \mathcal{O}(\frac{1}{x^2})
\]

and

\[
\int_0^\infty \frac{1}{t} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-xt} dt = \frac{1}{2} \log x + C + \mathcal{O}(x)
\]

for \( 0 < x \leq 1 \), where \( C \) is some constant.
Proof. The series expansion of $e^x$ yields for $t \to 0$

$$\frac{1}{t} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) = \frac{1}{t^2} \left( \frac{1}{1+t/2 + t^2/6 + O(t^3)} - 1 + \frac{t}{2} \right) = 1/12 + O(t).$$

For $t \to \infty$ this expression tends to 0, in particular, it is bounded for all positive $t$. Hence for $x \to \infty$ the integral in question becomes

$$\frac{1}{12} \int_0^\infty e^{-xt} \, dt + O \left( \int_0^\infty t e^{-xt} \, dt \right) = \frac{1}{12x} + O \left( \frac{1}{x^2} \right).$$

For $x \to 0$ we use $e^{-xt} = 1 - xt + O(x^2t^2)$ and obtain

$$\int_0^1 \frac{1}{t} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-xt} \, dt = \int_0^1 \frac{1}{t} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \, dt + O(x) = C_1 + O(x)$$

and

$$\int_1^\infty \frac{1}{t(e^t - 1)} e^{-xt} \, dt = \int_1^\infty \frac{1}{t(e^t - 1)} \, dt - x \int_1^\infty \frac{1}{e^t - 1} \, dt + O(x^2).$$

As a function of $x$, the integral $\int_1^\infty e^{-xt} \, dt$ defines a function that is differentiable from the right in 0 and has bounded second derivative in $(0, \infty)$, hence, for $x \geq 0$ this integral is $1 + O(x)$.

Finally

$$\int_1^\infty \frac{e^{-xt}}{2t} \, dt = \int_x^\infty \frac{e^{-s}}{2s} \, ds =$$

$$\int_x^1 \frac{dt}{2t} + \int_0^1 \frac{e^{-t} - 1}{2t} \, dt + \int_1^\infty \frac{e^{-t}}{2t} \, dt + \int_0^x \frac{1 - e^{-t}}{2t} \, dt$$

$$= -\frac{1}{2} \log x + C_3 + O(x).$$

Combining these estimates our claim follows.

From this we obtain

Lemma 2. We have

$$U_{k,\ell} = 1 - \frac{1}{12} \left( \frac{1}{k} + \frac{1}{\ell} - \frac{1}{k + \ell} \right) + O \left( \frac{1}{\min(k, \ell)^2} \right).$$

Proof. From Lemma 1 we obtain

$$U_{k,\ell} = \exp \left( \frac{1}{12} \left( \frac{1}{k + \ell} - \frac{1}{k} - \frac{1}{\ell} \right) + O \left( \frac{1}{\min(k, \ell)^2} \right) \right),$$

inserting the Taylor series for exp and using the fact that $k, \ell \geq 1$ our claim follows.

Next we compute $c_{\nu}(x)$.

Lemma 3. If $\nu x \geq 1$, then

$$c_{\nu}(x) = 1 + \frac{1}{12\nu(1 + x)} - \frac{1}{12\nu x} + \frac{1}{12\nu} + O \left( \frac{1}{\nu^2 \min(1, x^2)} \right).$$

If $\nu x \leq 1$, then

$$c_{\nu}(x) = K\sqrt{\nu x} + O \left( \sqrt{\frac{x}{\nu} + (\nu x)^{3/2}} \right)$$

for some constant $K$. 

Proof. If \( \nu x > 1 \), then we apply Lemma 1 to obtain:

\[
c_{\nu}(x) = \exp \left( \frac{1}{12 \nu (x + 1)} - \frac{1}{12 \nu x} + \frac{1}{12 \nu} + O \left( \frac{1}{\nu^2 \min(1, x^2)} \right) \right).
\]

If \( \nu x < 1 \), then \( \nu (x + 1) \geq 1 \), and we obtain:

\[
c_{\nu}(x) = \exp \left( \frac{1}{12 \nu (x + 1)} + \frac{1}{2} \log(\nu x) + C + \frac{1}{12 \nu} + O \left( \frac{1}{\nu^2} + \nu x \right) \right)
= K \sqrt{\nu x} + O \left( \sqrt{\nu x} + (\nu x)^{3/2} \right).
\]

\( \square \)

We now compute \( V_{k,\ell} \).

**Lemma 4.** We have

\[
V_{k,\ell} = \frac{1}{2} + O \left( \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{\ell}} \right)
\]

Proof. We have

\[
\sum_{\nu \leq \ell/k} c_{\nu}(k/\ell) \ll \sum_{\nu \leq \ell/k} \frac{\sqrt{\nu k/\ell}}{\sqrt{\nu + \ell}} \ll \frac{1}{\sqrt{k}}
\]

thus,

\[
V_{k,\ell} = \frac{\sqrt{\ell}}{2\pi} \sum_{\nu > \ell/k} \frac{1}{\sqrt{\nu + \ell}} \left( 1 + \frac{1}{12 \nu (x + 1)} - \frac{1}{12 \nu x} + \frac{1}{12 \nu} + O \left( \frac{1}{\nu^2 \min(1, k^2/\ell^2)} \right) \right) + O \left( \frac{1}{\sqrt{k}} \right)
\]

\[
= \frac{\sqrt{\ell}}{2\pi} \sum_{\nu > \ell/k} \frac{1}{\sqrt{\nu + \ell}} \left( 1 - \frac{1}{12 \nu^2 \ell} + O \left( \frac{1}{\nu^2 \min(1, k^2/\ell^2)} \right) \right) + O \left( \frac{1}{\sqrt{k}} + \sum_{\nu > \ell/k} \frac{1}{\nu^{3/2} \sqrt{\ell}} \right)
\]

\[
= \frac{\sqrt{\ell}}{2\pi} \sum_{\nu > \ell/k} \frac{1}{\sqrt{\nu + \ell}} \left( 1 - \frac{1}{12 \nu^2 \ell} + O \left( \frac{1}{\nu^2 \min(1, k^2/\ell^2)} \right) \right) + O \left( \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{\ell}} \right)
\]

If \( k > \ell \) we obtain

\[
V_{k,\ell} = \frac{\sqrt{\ell}}{2\pi} \sum_{\nu = 1}^{\infty} \frac{1}{\sqrt{\nu + \ell}} + O \left( \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{\ell}} + \sqrt{\ell} \sum_{\nu = 1}^{\infty} \frac{1}{\nu^{3/2} (\nu + \ell)} \right)
\]

\[
= \frac{1}{2\pi} \int_{0}^{\infty} dt \sqrt{\ell(t + 1)} + O \left( \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{\ell}} \right)
\]

\[
= \frac{1}{2} + O \left( \frac{1}{\sqrt{\ell}} \right).
\]

We have

\[
\frac{\sqrt{\ell}}{2\pi} \sum_{\nu > \ell} \frac{1}{\sqrt{\nu + \ell}} \left( 1 - \frac{1}{12 \nu^2 \ell} + O \left( \frac{1}{\nu^2 \min(1, k^2/\ell^2)} \right) \right) = \frac{\sqrt{\ell}}{2\pi} \sum_{\nu > \ell} \frac{1}{\sqrt{\nu + \ell}} + O \left( \frac{1}{\sqrt{k}} \right),
\]
and for $\ell/k \leq N \leq \ell$

$$\sum_{N \leq \nu \leq 2N} \frac{1}{\nu^{3/2}(\nu + \ell)} \leq \frac{1}{\sqrt{N\ell}}$$

as well as

$$\sum_{N \leq \nu \leq 2N} \frac{1}{\nu^{3/2}(\nu + \ell)} \leq \frac{1}{N^{3/2}\ell}$$

and therefore

$$\sqrt{\ell} \sum_{\nu \geq \ell/k} \frac{1}{\nu^{3/2}(\nu + \ell)} \cdot \frac{1}{\nu^{3/2}} = \frac{\ell^{3/2}}{k} \sum_{\nu \geq \ell/k} \frac{1}{\nu^{3/2}(\nu + \ell)} = \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$

and

$$\sqrt{\ell} \sum_{\nu \geq \ell/k} \frac{1}{\nu^{3/2}(\nu + \ell)} \cdot \frac{1}{\nu^{2} \min(k^2/\ell^2)} = \frac{\ell^{5/2}}{k^2} \sum_{\nu \geq \ell/k} \frac{1}{\nu^{3/2}(\nu + \ell)} = \mathcal{O}\left(\frac{1}{\sqrt{k}}\right).$$

Using these estimates we obtain

$$\frac{\sqrt{\ell}}{2\pi} \sum_{\ell/k \leq \nu \leq \ell} \frac{1}{\sqrt{\nu(\nu + \ell)}} \left(1 - \frac{1}{12\nu \ell^2} + \mathcal{O}\left(\frac{1}{\nu^2 \min(1, k^2/\ell^2)}\right)\right) = \frac{\sqrt{\ell}}{2\pi} \sum_{\ell/k \leq \nu \leq \ell} \frac{1}{\sqrt{\nu(\nu + \ell)}} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right).$$

For $k \leq \ell$ we obtain

$$V_{k,\ell} = \frac{\sqrt{\ell}}{2\pi} \sum_{\nu \geq \ell/k} \frac{1}{\sqrt{\nu(\nu + \ell)}} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$

$$= \frac{1}{2\pi} \int_{1/k}^{\infty} \frac{dt}{\sqrt{t(t+1)}} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} \frac{dt}{\sqrt{t(t+1)}} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$

$$= \frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right),$$

and the proof is complete. \qed
3. Proof of Theorem 1

We first prove (1). Note that this inequality is only interesting if the error term is $o(1)$, in particular we may assume that $k^2 < \ell$. Under this assumption we have

$$P_{k,\ell} = \left( \frac{\ell}{\ell + k} \right)^{k+\ell-1} \sum_{\nu=0}^{k-1} \binom{k+\ell}{\nu} \left( \frac{k - \nu}{\ell} \right)$$

$$= \sum_{\nu=0}^{k-1} \left( \frac{\ell}{\ell + k} \right)^{k+\ell-\nu} \frac{k^\nu}{\nu!} \exp \left( \sum_{\mu=1}^{\nu} \log \frac{k + \ell - \mu}{k + \ell} \right)$$

$$= \sum_{\nu=0}^{k-1} \left( \frac{\ell}{\ell + k} \right)^{k+\ell-\nu} \frac{k^\nu}{\nu!} \exp \left( -\sum_{\mu=1}^{\nu} \frac{\mu}{k + \ell} + O \left( \frac{\mu^2}{(k + \ell)^2} \right) \right)$$

$$= \sum_{\nu=0}^{k-1} \left( \frac{\ell}{\ell + k} \right)^{k+\ell-\nu} \frac{k^\nu}{\nu!} \exp \left( O \left( \frac{k^2}{\ell} \right) \right)$$

$$= \exp \left( (k + \ell) \log \frac{\ell}{\ell + k} \right) \sum_{\nu=0}^{k-1} \frac{k^\nu}{\nu!} \exp \left( O \left( \frac{k^2}{\ell} \right) \right)$$

$$= \exp \left( -k + O \left( \frac{k^2}{\ell} \right) \right) \sum_{\nu=0}^{k-1} \frac{k^\nu}{\nu!}$$

$$= e^{-k} \sum_{\nu=0}^{k-1} \frac{k^\nu}{\nu!} + O \left( \frac{k^2}{\ell} \right).$$

The proof of (2) is quite similar. Analogously to the previous case we may assume that $\ell^2 < k$.

$$P_{k,\ell} = \left( \frac{k + \ell}{\ell + k} \right)^{k+\ell} \left( \frac{\ell}{\ell + k} \right)^{k+\ell} \sum_{\nu=0}^{k-1} \binom{k+\ell}{\nu} \left( \frac{k}{\ell} \right)^\nu$$

$$= 1 - \left( \frac{\ell}{\ell + k} \right)^{k+\ell} \sum_{\mu=0}^{\ell} \binom{k + \ell}{\mu} \left( \frac{k}{\ell} \right)^{k+\ell-\mu}$$

$$= 1 - \left( \frac{k}{\ell + k} \right)^{k+\ell} \sum_{\mu=0}^{\ell} \binom{\ell}{\mu} \left( \frac{\ell}{k} \right)^\mu$$

$$= 1 - \exp \left( -\ell + O \left( \frac{\ell^2}{k} \right) \right) \sum_{\mu=0}^{\ell} \frac{\ell^\mu}{\mu!} \left( \frac{k + \ell}{k} \right)^\mu$$

$$= 1 - e^{-\ell} \sum_{\mu=0}^{\ell} \frac{\ell^\mu}{\mu!}$$

Finally (3) follows from Theorem 2, Lemma 2 and Lemma 4.
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