Group-groupoid actions and liftings of crossed modules

Osman Mucuk\textsuperscript{a} and Tunçar Şahan\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Erciyes University, Kayseri, TURKEY
\textsuperscript{b}Department of Mathematics, Aksaray University, Aksaray, TURKEY

Abstract

The aim of this paper is to define the notion of lifting of a crossed module via a group morphism and give some properties of this type of the lifting. Further we obtain a criterion for a crossed module to have a lifting of crossed module. We also prove that the liftings of a certain crossed module constitute a category; and that this category is equivalent to the category of covers of that crossed module and hence to the category of group-groupoid actions of the corresponding groupoid to that crossed module.

Key Words: Crossed module, covering, lifting, action groupoid

Classification: 20L05, 57M10, 22AXX, 22A22, 18D35

1 Introduction

The theory of covering groupoids has an important role in the applications of groupoids (cf. \cite{2} and \cite{12}). In this theory it is well known that for a groupoid $G$, the category $\text{GpdAct}(G)$ of groupoid actions of $G$ on sets, these are also called operations or $G$-sets, are equivalent to the category $\text{GpdCov}/G$ of covering groupoids of $G$ (see \cite{5} Theorem 2] for the topological version of this equivalence).

On the one hand in \cite[Proposition 3.1]{6} it was proved that if $G$ is a group-groupoid, which is an internal groupoid in the category of groups and widely used in literature under the names 2-group (see for example \cite{3}), $G$-groupoid or group object \cite{7} in the category of
groupoids, then the category $\text{GpGpdCov}/G$ of group-groupoid coverings of $G$ is equivalent to the category $\text{GpGpdAct}(G)$ of group-groupoid actions of $G$ on groups. In [1, Theorem 4.2] this result has been recently generalized to the case where $G$ is an internal groupoid for an algebraic category $C$ which includes groups, rings without identity, $R$-modules, Lie algebras, Jordan algebras, and many others; acting on a group with operations in the sense of Orzech [18].

On the other hand in [7, Theorem 1] it was proved that the categories of crossed modules and group-groupoids, under the name of $G$-groupoids, are equivalent (see also [13] for an alternative equivalence in terms of an algebraic object called $\text{cat}^n$-groups). By applying this equivalence of the categories, normal and quotient objects in the category of group-groupoids have been recently obtained in [16]. In [19, Section 3] it was proved that a similar result to the one in [7, Theorem 1] can be generalized for a certain algebraic category introduced by Orzech [18], and adapted and called category of groups with operations. The study of internal category theory was continued in the works of Datuashvili [10] and [11]. Moreover, she developed cohomology theory of internal categories, equivalently, crossed modules, in categories of groups with operations [8] and [9]. The equivalences of the categories in [7, Theorem 1] and [19, Section 3] enable us to generalize some results on group-groupoids to the more general internal groupoids for a certain algebraic category $C$ (see for example [1], [14], [15] and [17]).

In this paper we use the equivalence of group-groupoids and crossed modules proved in [7, Theorem 1] to determine a notion in crossed modules, called lifting of a crossed module, corresponding to a group-groupoid action, investigate some properties of lifting of crossed modules and give a criteria for a crossed module to have a lifting. Finally we prove that for a group-groupoid $G$, the group-groupoid actions of $G$ on groups and liftings of the crossed module corresponding to $G$ are categorically equivalent.

2 Preliminaries

Let $G$ be a groupoid. We write $\text{Ob}(G)$ for the set of objects of $G$ and write $G$ for the set of morphisms. We also identify $\text{Ob}(G)$ with the set of identities of $G$ and so an element of $\text{Ob}(G)$ may be written as $x$ or $1_x$ as convenient. We write $d_0, d_1: G \to \text{Ob}(G)$ for the source and target maps, and, as usual, write $G(x, y)$ for $d_0^{-1}(x) \cap d_1^{-1}(y)$, for $x, y \in \text{Ob}(G)$. The composition $h \circ g$ of two elements of $G$ is defined if and only if $d_0(h) = d_1(g)$, and so the map $(h, g) \mapsto h \circ g$ is defined on the pullback $G_{d_0} \times_{d_1} G$ of $d_0$ and $d_1$. The inverse of $g \in G(x, y)$ is denoted by $g^{-1} \in G(y, x)$.

If $x \in \text{Ob}(G)$, we write $\text{St}_G x$ for $d_0^{-1}(x)$ and call the star of $G$ at $x$. Similarly we write
called object group at $x$, and denoted by $G(x)$. The set of all morphisms from $x$ to $x$ is a group, called object group at $x$, and denoted by $G(x)$.

A groupoid $G$ is transitive (resp. simply transitive, 1-transitive and totally intransitive) if $G(x, y) \neq \emptyset$ (resp. $G(x, y)$ has no more than one element, $G(x, y)$ has exactly one element and $G(x, y) = \emptyset$) for all $x, y \in \text{Ob}(G)$ such that $x \neq y$.

A totally intransitive groupoid is determined entirely by the family $\{G(x) \mid x \in \text{Ob}(G)\}$ of groups. This totally intransitive groupoid is sometimes called totally disconnected or bundle of groups \cite[pp.218]{2}.

Let $p: \tilde{G} \to G$ be a morphism of groupoids. Then $p$ is called a covering morphism and $\tilde{G}$ a covering groupoid of $G$ if for each $\tilde{x} \in \text{Ob}(\tilde{G})$ the restriction $\text{St}_{\tilde{G}} \tilde{x} \to \text{St}_{Gp}(\tilde{x})$ is bijective.

Assume that $p: \tilde{G} \to G$ is a covering morphism. Then we have a lifting function $S_p: G_{d_0} \times \text{Ob}(p) \to \text{Ob}(\tilde{G})$ assigning to the pair $(a, x)$ in the pullback $G_{d_0} \times \text{Ob}(p) \to \text{Ob}(\tilde{G})$ the unique element $b$ of $\text{St}_{\tilde{G}} \tilde{x}$ such that $p(b) = a$. Clearly $S_p$ is inverse to $(p, d_0): \tilde{G} \to G_{s_0} \times \text{Ob}(p) \to \text{Ob}(\tilde{G})$. So it is stated that $p: \tilde{G} \to G$ is a covering morphism if and only if $(p, s)$ is a bijection.

A covering morphism $p: \tilde{G} \to G$ is called transitive if both $\tilde{G}$ and $G$ are transitive. A transitive covering morphism $p: \tilde{G} \to G$ is called universal if $\tilde{G}$ covers every cover of $G$, i.e., for every covering morphism $q: \tilde{H} \to G$ there is a unique morphism of groupoids $\tilde{p}: \tilde{G} \to \tilde{H}$ such that $q \tilde{p} = p$ (and hence $\tilde{p}$ is also a covering morphism), this is equivalent to that for $\tilde{x}, \tilde{y} \in \text{Ob}(\tilde{G})$ the set $\tilde{G}(\tilde{x}, \tilde{y})$ has no more than one element.

Recall that an action of a groupoid $G$ on a set $S$ via a function $\omega: S \to \text{Ob}(G)$ is a function $G_{d_0} \times \omega S \to S, (g, s) \mapsto g \bullet s$ satisfying the usual rules for an action: $\omega(g \bullet s) = d_1(g), 1_{\omega(s)} \bullet s = s$ and $(h \circ g) \bullet s = h \bullet (g \bullet s)$ whenever $h \circ g$ and $g \bullet s$ are defined. A morphism $f: (S, \omega) \to (S', \omega')$ of such actions is a function $f: S \to S'$ such that $w'f = w$ and $f(g \bullet s) = g \bullet f(s)$ whenever $g \bullet s$ is defined. This gives a category $\text{GpdAct}(G)$ of actions of $G$ on sets. For such an action the action groupoid $G \ltimes S$ is defined to have object set $S$, morphisms the pairs $(g, s)$ such that $d_0(g) = \omega(s)$, source and target maps $d_0(g, s) = s, d_1(g, s) = g \bullet s$, and the composition

$$(g', s') \circ (g, s) = (g \circ g', s)$$

whenever $s' = g \bullet s$. The projection $q: G \ltimes S \to G, (g, s) \mapsto s$ is a covering morphism of groupoids and the functor assigning this covering morphism to an action gives an equivalence of the categories $\text{GpdAct}(G)$ and $\text{GpdCov}/G$. 

3
3 Group-groupoids and crossed modules

A group-groupoid is a groupoid $G$ with morphisms of groupoids $G \times G \to G$, $(g, h) \mapsto g + h$ and $G \to G$, $g \mapsto -g$ yielding a group structure internal to the category of groupoids. Since the addition map is a morphism of groupoids, we have an interchange rule that $(b \circ a) + (d \circ c) = (b + d) \circ (a + c)$ for all $a, b, c, d \in G$ such that $b \circ a$ and $d \circ c$ are defined. If the identity of $\text{Ob}(G)$ is $e$, then the identity of the group structure on the morphisms set is $1_e$.

Let $G$ be a group-groupoid. An action of the group-groupoid $G$ on a group $X$ via $\omega$ consists of a morphism $\omega: X \to \text{Ob}(G)$ from the group $X$ to the underlying group of $\text{Ob}(G)$ and an action of the groupoid $G$ on the underlying set $X$ via $\omega$ such that the following interchange law holds:

$$(g \cdot x) + (g' \cdot x') = (g + g') \cdot (x + x')$$

whenever both sides are defined. A morphism $f: (X, \omega) \to (X', \omega')$ of such actions is a morphism $f: X \to X'$ of groups and of the underlying operations of $G$. This gives a category $\text{GpGpdAct}(G)$ of actions of $G$ on groups. For an action of $G$ on the group $X$ via $\omega$, the action groupoid $G \rtimes X$ has a group structure defined by

$$(g, x) + (g', x') = (g + g', x + x')$$

and with this operation $G \rtimes X$ becomes a group-groupoid and the projection $p: G \rtimes X \to G$ is an object of the category $\text{GpGpdCov}/G$. By means of this construction the following equivalence of the categories was given in [6, Proposition 3.1].

**Proposition 3.1.** The categories $\text{GpGpdCov}/G$ and $\text{GpGpdAct}(G)$ are equivalent.

We recall that as defined by Whitehead in [20, 22] a crossed module of groups consists of two groups $A$ and $B$, an action of $B$ on $A$ denoted by $b \cdot a$ for $a \in A$ and $b \in B$; and a morphism $\alpha: A \to B$ of groups satisfying the following conditions for all $a, a_1 \in A$ and $b \in B$

$$\text{CM1} \quad \alpha(b \cdot a) = b + \alpha(a) - b,$$

$$\text{CM2} \quad \alpha(a) \cdot a_1 = a + a_1 - a.$$

We will denote such a crossed module by $(A, B, \alpha)$.

Here are some examples of well known crossed modules:

(i) The inclusion map $N \hookrightarrow G$ of a normal subgroup is a crossed module with the conjugation action of $G$ on $N$. 


(ii) If $M$ is a $G$-module, then the zero map $0: M \to G$ has the structure of crossed module.

(iii) The inner automorphism map $G \to \text{Aut}(G)$ is a crossed module.

(iv) If $X$ is a topological group, then the fundamental group $\pi X$ is a group-groupoid, the star $\text{St}_{\pi X}0$ at the identity $0 \in X$ becomes a group and the final point map $d_1: \text{St}_{\pi X}0 \to X$ becomes a crossed module.

(v) As a motivating geometric example of crossed module due to Whitehead [20, 22] if $X$ is topological space and $A \subseteq X$ with $x \in A$, then there is a natural action of $\pi_1(A, x)$ on second relative homotopy group $\pi_2(X, A, x)$ and with this action the boundary map

$$\partial: \pi_2(X, A, x) \to \pi_1(A, x)$$

becomes a crossed module. This crossed module is called fundamental crossed module and denoted by $\Pi(X, A, x)$ (see for example [4] for more details).

The following are some standard properties of crossed modules.

**Proposition 3.2.** Let $(A, B, \alpha)$ be a crossed module. Then

(i) $\alpha(A)$ is a normal subgroup of $B$.

(ii) $\ker \alpha$ is central in $A$, i.e. $\ker \alpha$ is a subset of $Z(A)$, the center of $A$.

(iii) $\alpha(A)$ acts trivially on $Z(A)$.

(iv) $Z(A)$ and $\ker \alpha$ inherit an action of $\text{Coker} \alpha$ to become $(\text{Coker} \alpha)$-modules.

Let $(A, B, \alpha)$ and $(A', B', \alpha')$ be two crossed modules. A morphism $(f_1, f_2)$ from $(A, B, \alpha)$ to $(A', B', \alpha')$ is a pair of morphisms of groups $f_1: A \to A'$ and $f_2: B \to B'$ such that $f_2\alpha = \alpha'f_1$ and $f_1(b \cdot a) = f_2(b) \cdot f_1(a)$ for $a \in A$ and $b \in B$.

Crossed modules with morphisms between them form a category denoted by $\text{XMod}$.

The following result was proved in [7] Theorem 1 and since we need some details of the proof, we give a sketch proof of the equivalence.

**Theorem 3.1.** The category $\text{XMod}$ of crossed modules and the category $\text{GrpGpd}$ of group-groupoids are equivalent.

**Proof.** Let GrpGpd be the category of group-groupoids and $\text{XMod}$ the category of crossed modules.

5
A functor $\delta: \text{GrpGpd} \to \text{XMod}$ is defined as follows. For a group-groupoid $G$ let $\delta(G)$ be the crossed module $(A, B, d_1)$ where $A = \text{Ker} \, d_0$, $B = \text{Ob}(G)$ and $d_1: A \to B$ is the restriction of the target point map. Then $A, B$ inherit group structures from that of $G$, and the target point map $d_1: A \to B$ is a morphism of groups. Further we have an action $B \times A \to A$, $(b, a) \mapsto b \cdot a$ of $B$ on the group $A$ given by

$$b \cdot a = 1_b + a - 1_b$$

for $a \in A, b \in B$ and we clearly have

$$d_1(b \cdot a) = b + d_1(a) - b$$

and

$$d_1(a) \cdot a_1 = a + a_1 - a$$

for $a, a_1 \in A, b \in B$. Thus $(A, B, d_1)$ is a crossed module.

Conversely define a functor $\eta: \text{XMod} \to \text{GrpGpd}$ in the following way. For a crossed module $(A, B, \alpha)$ let $\eta(A, B, \alpha)$ be the group-groupoid whose object set (group) is the group $B$ and whose group of the morphisms is the semi-direct group $A \rtimes B$ with the usual group structure

$$(a_1, b_1) + (a, b) = (a_1 + b_1 \cdot a, b_1 + b)$$

The source and target point maps are defined to be $d_0(a, b) = b$ and $d_1(a, b) = \alpha(a) + b$, while the groupoid composition is given by

$$(a_1, b_1) \circ (a, b) = (a_1 + a, b)$$

whenever $b_1 = \alpha(a) + b$. $\square$

**Proposition 3.3.** Let $G$ be a group-groupoid and $(A, B, \alpha)$ the crossed module corresponding to $G$. If $G$ is transitive (resp. simply transitive, 1-transitive and totally intransitive) then $\alpha$ is surjective (resp. injective, bijective; and zero morphism such that $A$ is abelian).

**Proof.** The proof can be followed by the details of the proof of Theorem 3.1. $\square$

Hence we can state the following definition.

**Definition 3.1.** Let $(A, B, \alpha)$ be a crossed module. Then $(A, B, \alpha)$ is called transitive (resp. simply transitive, 1-transitive and totally intransitive) if $\alpha$ is surjective (resp. injective, bijective; and zero morphism such that $A$ is abelian).
**Example 3.1.** If $X$ is a topological group whose underlying topology is path-connected (resp. totally disconnected), then the crossed module $(St_{\pi X}0, X, d_1)$ is transitive (resp. totally intransitive).

### 4 Lifting of crossed modules

In this section using Theorem 3.1, we determine the notion in crossed modules, corresponding to the action of a group-groupoid on a group and interpret the properties in crossed modules corresponding to group-groupoid actions.

Let $G$ be a group-groupoid acting on a group $X$ by an action $G_{d_0} \times X, (g, x) \mapsto g \cdot x$, via a group morphism $\omega: X \to \text{Ob}(G)$ and let $(A, B, \alpha)$ be the crossed module corresponding to $G$. Then we have a morphism $\omega: X \to B$ of groups and an action of $X$ on $A = \text{St}_{G0}$ defined by

$$X \times A \to A, x \cdot a = 1_{\omega(x)} + a - 1_{\omega(x)}$$  \hspace{1cm} (6)

By the group-groupoid action of $G$ on $X$ we have a group morphism

$$\varphi: A \to X, a \mapsto \varphi(a) = a \cdot 0_X$$  \hspace{1cm} (7)

such that $\omega \varphi = \alpha$, where $0_X$ is the identity element of the group $X$.

Then we prove the following theorems.

**Theorem 4.1.** By the action of $X$ on $A$ defined above, $(A, X, \varphi)$ becomes a crossed module.

**Proof.** We prove that the conditions [CM1] and [CM2] are satisfied.

[CM1] For all $a \in A$ and $x \in X$ we have the following evaluations

$$\varphi(x \cdot a) = \varphi(1_{\omega(x)} + a - 1_{\omega(x)}) \quad \text{(by Eq. 6)}$$

$$= (1_{\omega(x)} + a - 1_{\omega(x)}) \cdot 0_X \quad \text{(by Eq. 7)}$$

$$= (1_{\omega(x)} + (a + 1_{\omega(-x)})) \cdot (x + (-x))$$

$$= (1_{\omega(x)} \cdot x) + ((a + 1_{\omega(-x)}) \cdot (-x)) \quad \text{(by Eq. 1)}$$

$$= (0_A + 1_{\omega(x)}) \cdot (0_X + x) + (a + 1_{\omega(-x)}) \cdot (0_X + (-x))$$

7
and by the interchange law \((1)\)

\[
\varphi(x \cdot a) = (0_A \cdot 0_X) + (1_{\omega(x)} \cdot x) + (a \cdot 0_X) + (1_{\omega(-x)} \cdot (-x))
\]

\[
= \varphi(0_A) + x + \varphi(a) - x \tag{by Eq. 7}
\]

\[
= x + \varphi(a) - x
\]

\[\text{[CM2]}\] For all \(a, a_1 \in A\) we have the following steps

\[
\varphi(a) \cdot a_1 = (a \cdot 0_X) \cdot a_1 \tag{by Eq. 7}
\]

\[
= 1_{\omega(a \cdot 0_X)} + a_1 - 1_{\omega(a \cdot 0_X)} \tag{by Eq. 6}
\]

\[
= 1_{d_1(a)} + a_1 - 1_{d_1(a)} \tag{by \(\omega(a \cdot 0_X) = d_1(a)\)}
\]

\[
= d_1(a) \cdot a_1 \tag{by Eq. 2}
\]

\[
= a + a_1 - a \tag{by Eq. 4}
\]

\[\bigstar\]

**Theorem 4.2.** Let \((A, B, \alpha)\) be a crossed module and \(\omega: X \to B\) a group morphism. Then any group morphism \(\varphi: A \to X\) such that \(\omega \varphi = \alpha\) is a crossed module with the action defined via \(\omega\) if and only if the map \(\varphi: A \rtimes X \to X\) defined by

\[
\varphi(a, x) = \varphi(a) + x \tag{8}
\]

is a group morphism.

**Proof.** Suppose that the map \(\varphi: A \rtimes X \to X\) defined by \(\varphi(a, x) = \varphi(a) + x\) is a group morphism. Then we have the following evaluations to prove that the axioms [CM1] and [CM2] of crossed module are satisfied for the group morphism \(\varphi: A \to X\).

\[\text{[CM1]}\] For \(x \in X\) and \(a \in A\) we have that

\[
\varphi(x \cdot a) = \varphi((0, x) + (a, -x)) \tag{by Eq. 5}
\]

\[
= \varphi(0, x) + \varphi(a, -x)
\]

\[
= \varphi(0) + x + \varphi(a) - x \tag{by Eq. 8}
\]

\[
= x + \varphi(a) - x
\]

\[\text{[CM2]}\] Since the action of \(X\) on \(A\) is defined by means of the action of \(B\) on \(A\) via \(\omega\), the morphism \(\varphi\) satisfies the condition [CM2] of crossed module.
Conversely assume that $\varphi: A \to X$ is a crossed module. Then by the following evaluation the map $\overline{\varphi}: A \times X \to X$ becomes a group morphism.

\[
\overline{\varphi}((a, x) + (a_1, x_1)) = \overline{\varphi}(a + a_1, x + x_1) \\
= \varphi(a + a_1) + (x + x_1) \\
= \varphi(a) + \varphi(x \cdot a_1) + (x + x_1) \\
= \varphi(a) + x + \varphi(a_1) - x + x + x_1 \\
= \varphi(a) + x + \varphi(a_1) + x_1 \\
= \overline{\varphi}(a, x) + \overline{\varphi}(a_1, x_1).
\]

(by Eq. 5)  
(by Eq. 8)  
(by CM1)  
(by Eq. 8)

We now define the notion of lifting of a crossed module as follows:

**Definition 4.1.** Let $(A, B, \alpha)$ be a crossed module and $\omega: X \to B$ a morphism of groups. Then a crossed module $(A, X, \varphi)$ in which the action of $X$ on $A$ is defined via $\omega$, is called a **lifting of $\alpha$ over $\omega$** and denoted by $(\varphi, X, \omega)$ if the following diagram is commutative

\[
\begin{array}{c}
\varphi \downarrow \\
X \xrightarrow{\omega} B \\
A \xrightarrow{\alpha} B
\end{array}
\]

It is obvious that every crossed module $(A, B, \alpha)$ lifts to itself over the identity morphism $1_B$ on $B$.

**Remark 4.1.** In the following diagram if $(\varphi, X, \omega)$ is a lifting of $(A, B, \alpha)$, then $\text{Ker} \varphi \subseteq \text{Ker} \alpha$ and $(1_A, \omega)$ is a morphism of crossed modules

\[
\begin{array}{c}
\varphi \downarrow \text{Ker} \varphi \\
A - \xrightarrow{\varphi} X \xrightarrow{\omega} B \\
A \xrightarrow{\alpha} B
\end{array}
\]

Therefore if $(A, B, \alpha)$ is a simply transitive crossed module then $\text{Ker} \alpha$ is trivial and so also $\text{Ker} \varphi$ is. Hence the crossed module $(A, X, \varphi)$ is also simply transitive.

The following can be stated as examples of lifting crossed modules.

**Example 4.1.** A crossed module $(A, B, \alpha)$ is a lifting of the automorphism crossed module
\((A, \text{Aut}(A), \iota)\) over the action of \(B\) on \(A\), i.e., \(\theta: B \to \text{Aut}(A), \theta(b)(a) = b \cdot a\)

\[
\begin{array}{c}
A \xrightarrow{\iota} \text{Aut}(A) \\
\downarrow \theta \\
B
\end{array}
\]

**Example 4.2.** Let \(N\) be a normal subgroup of a group \(G\). Since \((N, G, \text{inc})\) is a simply transitive crossed module, any lifting crossed module \((N, X, \varphi)\) is also simply transitive. Moreover \(\varphi(N)\) is a normal subgroup of \(X\). Since \(\varphi\) is injective \(N\) also can be consider as a normal subgroup of \(X\).

**Example 4.3.** Let \(G\) be a group with trivial center. Then the automorphism crossed module \(G \to \text{Aut} G\) is simply transitive and hence every lifting of \(G \to \text{Aut} G\) is also simply transitive.

**Example 4.4.** Let \(\rho: \tilde{X} \to X\) be a covering morphism of topological groups. Hence \((\text{St}_{\pi X0}, X, d_1)\) is a crossed module. If \(\alpha\) is a path in \(X\) with initial point \(0 \in X\), the identity, then by the path lifting property there exists a unique path \(\tilde{\alpha}\) in \(\tilde{X}\) such that \(p\tilde{\alpha} = \alpha\) and \(\tilde{\alpha}(0) = \tilde{0} \in \tilde{X}\), the identity. Hence we can define a function \(\tilde{d}_1: \text{St}_{\pi X0} \to \tilde{X}\) assigning \([\alpha] \in \text{St}_{\pi X0}\) to the final point \(\tilde{\alpha}(1)\) of the lifting path of \(\alpha\) at \(\tilde{0}\). It follows that \(\tilde{d}_1\) is well defined and \((\tilde{d}_1, \tilde{X}, \rho)\) is a lifting of \((\text{St}_{\pi X0}, X, d_1)\).

We now state some results on lifting crossed modules.

**Lemma 4.1.** Let \((A, B, \alpha)\) be a crossed module and \(\varphi\) a lifting of \(\alpha\) over \(\omega: X \to B\). If there are isomorphisms \(f: B \to B'\) and \(g: X' \to X\) for some groups \(B'\) and \(G'\) then \(\varphi'\) is a lifting of \(\alpha'\) over \(\omega': X' \to B'\) where \(\varphi' = g^{-1}\varphi, \alpha' = f\alpha\) and \(\omega' = f\omega g\).

**Proof.** It follows that \(\alpha' = f\alpha\) and \(\varphi' = g^{-1}\varphi\) are crossed modules since \(f\) and \(g\) are isomorphisms; and

\[
\omega'\varphi' = (f\omega g)(g^{-1}\varphi)
\]

\[
= f\omega\varphi
\]

Since \((\varphi, X, \omega)\) is a lifting of \((A, B, \alpha)\), i.e., \(\omega\varphi = \alpha\) we have that

\[
\omega'\varphi' = f\omega\varphi
\]

\[
= f\alpha
\]

\[
= \alpha'.
\]
Proposition 4.1. Let \((A, B, \alpha)\) be a crossed module and \((\varphi, X, \omega)\) a lifting of \((A, B, \alpha)\). If \((\varphi', X', \omega')\) is a lifting of \((A, X, \varphi)\) then \((\varphi', X', \omega\omega')\) is also a lifting of \((A, B, \alpha)\).

Proof. The proof is immediate. □

Let \((\varphi, X, \omega)\) and \((\varphi', X', \omega')\) be two liftings of \((A, B, \alpha)\). A morphism \(f\) from \((\varphi, X, \omega)\) to \((\varphi', X', \omega')\) is a group homomorphism \(f : X \to X'\) such that \(f \varphi = \varphi'\) and \(\omega' f = \omega\). Hence lifting crossed modules of \((A, B, \alpha)\) and morphisms between them form a category which we denote by \(\text{LXMod}/(A, B, \alpha)\). By Proposition 4.1 it follows that if \(\varphi\) is a lifting of \((A, B, \alpha)\) over \(\omega : X \to B\) then \(\text{LXMod}/(A, X, \varphi)\) is a full subcategory of \(\text{LXMod}/(A, B, \alpha)\).

Let \((A, B, \alpha)\) be a transitive crossed module. Then a lifting \((\varphi, X, \omega)\) of \((A, B, \alpha)\) is called an \(n\)-lifting when \(|\text{Ker}\omega| = n\).

Corollary 4.1. If \((\varphi, X, \omega)\) is a 1-lifting of \((A, B, \alpha)\) then \(\omega\) is an isomorphism. Hence \((A, X, \varphi) \cong (A, B, \alpha)\).

Proof. If \((\varphi, X, \omega)\) is a 1-lifting of \((A, B, \alpha)\), then \(\omega\) becomes surjective and \(|\text{Ker}\omega| = 1\), i.e., \(\omega\) is injective. Hence \(\omega\) is an isomorphism. □

Theorem 4.3. Let \((f, g) : (\tilde{A}, \tilde{B}, \tilde{\alpha}) \to (A, B, \alpha)\) be a morphism of crossed modules where \((\tilde{A}, \tilde{B}, \tilde{\alpha})\) is transitive and let \((\varphi, X, \omega)\) be a lifting of \((A, B, \alpha)\). Then there is a unique morphism of crossed modules \((f, \tilde{g}) : (\tilde{A}, \tilde{B}, \tilde{\alpha}) \to (A, X, \varphi)\) such that \(\omega\tilde{g} = g\) if and only if \(f(\text{Ker}\tilde{\alpha}) \subseteq \text{Ker}\varphi\).

Proof. Assume that \(f(\text{Ker}\tilde{\alpha}) \subseteq \text{Ker}\varphi\). For the existence; let \(\tilde{b} \in \tilde{B}\). Since \((\tilde{A}, \tilde{B}, \tilde{\alpha})\) is transitive, \(\tilde{\alpha}\) is surjective and there exists an \(\tilde{a} \in \tilde{A}\) such that \(\tilde{\alpha}(\tilde{a}) = \tilde{b}\). Hence \(\tilde{g}(\tilde{b}) = \varphi f(\tilde{a})\).

It is easy to see that \(\tilde{g}\) is well defined since \(f(\text{Ker}\tilde{\alpha}) \subseteq \text{Ker}\varphi\). Also

\[
\omega\tilde{g}(\tilde{b}) = \omega\varphi f(\tilde{a}) = \alpha f(\tilde{a}) = g\tilde{\alpha}(\tilde{a}) = g(\tilde{b}).
\]

So \(\omega\tilde{g} = g\). By the definition of \(\tilde{g}\) it implies that \((f, \tilde{g})\) is a crossed module morphism. For any other morphism \(g' : \tilde{B} \to X\) such that \(\omega\tilde{g}' = g\), the pair \((f, g')\) is a crossed module morphism and coincides with \(\tilde{g}\).

Conversely suppose that \((f, \tilde{g}) : (\tilde{A}, \tilde{B}, \tilde{\alpha}) \to (A, X, \varphi)\) is a crossed module morphism with \(\omega\tilde{g} = g\). If \(\tilde{a} \in \text{Ker}\tilde{\alpha}\), then \(f(\tilde{a}) \in f(\text{Ker}\tilde{\alpha})\) and \(\tilde{\alpha}(\tilde{a}) = 0\). Since \((f, \tilde{g})\) is a crossed module morphism \(\tilde{g}\tilde{\alpha} = \varphi f\) and hence \(\varphi f(\tilde{a}) = 0\). So \(f(\tilde{a}) \in \text{Ker}\varphi\) and hence \(f(\text{Ker}\tilde{\alpha}) \subseteq \text{Ker}\varphi\). □
Corollary 4.2. Let \((A, B, \alpha)\) be a crossed module. Assume that \((\varphi, X, \omega)\) and \((\tilde{\varphi}, \tilde{X}, \tilde{\omega})\) are two liftings of \((A, B, \alpha)\) such that \((A, \tilde{X}, \tilde{\varphi})\) is transitive. Then \(\tilde{\varphi}\) is a lifting of \(\varphi\) if and only if \(\ker \tilde{\varphi} \subseteq \ker \varphi\).

Corollary 4.3. Let \((A, B, \alpha)\) be a crossed module. Assume that \((\varphi, X, \omega)\) and \((\tilde{\varphi}, \tilde{X}, \tilde{\omega})\) are two liftings of \((A, B, \alpha)\) such that \((A, X, \varphi)\) and \((A, \tilde{X}, \tilde{\varphi})\) are both transitive. Then \((\varphi, X, \omega) \cong (\tilde{\varphi}, \tilde{X}, \tilde{\omega})\) if and only if \(\ker \varphi = \ker \tilde{\varphi}\).

Theorem 4.4. Let \((A, B, \alpha)\) be a crossed module, \(X\) a group and let \(\omega: X \to B\) be an injective group morphism. Then any group morphism \(\varphi: A \to X\) such that \(\omega \varphi = \alpha\) becomes a lifting of \(\alpha\) over \(\omega\).

Proof. According to Theorem 4.2 we only need to show that \(\varphi: A \ltimes X \to X\) defined by \(\varphi(a, x) = \varphi(a) + x\) is a group morphism, i.e. \(\varphi(x \cdot a) = x + \varphi(a) - x\) for all \(x \in X, a \in A\).

\[
\begin{align*}
\omega(\varphi(x \cdot a)) &= \omega(\varphi(\omega(x) \cdot a)) \\
&= \alpha(\omega(x) \cdot a) \quad \text{(since } \omega \varphi = \alpha) \\
&= \omega(x) + \alpha(a) - \omega(x) \quad \text{(by CM1)}
\end{align*}
\]

and

\[
\begin{align*}
\omega(x + \varphi(a) - x) &= \omega(x) + \omega(\varphi(a)) - \omega(x) \\
&= \omega(x) + \alpha(a) - \omega(x) \quad \text{(since } \omega \varphi = \alpha.)
\end{align*}
\]

Therefore since \(\omega\) is injective and \(\omega(\varphi(x \cdot a)) = \omega(x + \varphi(a) - x)\) then \(\varphi(x \cdot a) = x + \varphi(a) - x\) for all \(x \in X, a \in A\). This completes the proof. \(\square\)

Corollary 4.4. Every crossed module \((A, B, \alpha)\) lift to the crossed module \((A, A/N, p)\) over \(\omega: A/N \to B, a + N \mapsto \alpha(a)\) where \(N = \ker \alpha\).

\[
\begin{array}{ccc}
A/N & \xrightarrow{p} & B \\
\downarrow{\omega} & & \\
A & \xrightarrow{\alpha} & B
\end{array}
\]

This lifting is called natural lifting. By Corollary 4.4 and first isomorphism theorem for groups we can say that for every crossed module \((A, B, \alpha)\), \((A, \im \alpha, \alpha)\) is a transitive crossed module with the same action.

Now we will give a criterion for the existence of the lifting crossed module.
Theorem 4.5. Let \((A, B, \alpha)\) be a crossed module and \(C\) a subgroup of \(\text{Ker} \alpha\). Then there exists a lifting \((\varphi, X, \omega)\) of \(\alpha\) such that \(\text{Ker} \varphi = C\). Moreover in this case \(\text{Ker} \omega = \text{Ker} \alpha / C\).

Proof. Since \(\text{Ker} \alpha\) is central, \(C\) is a normal subgroup of \(A\). By Corollary 4.4 \((\varphi, A/C, \omega)\) is a lifting of \(\alpha\) where \(\varphi(a) = a + C\) and \(\omega(a + C) = \alpha(a)\). Obviously \(\text{Ker} \varphi = C\). Since \(\varphi\) is surjective \((A, A/C, \varphi)\) is transitive. Further

\[
\text{Ker} \omega = \{a + C \mid \omega(a + C) = 0\} = \{a + C \mid \alpha(a) = 0\} = \{a + C \mid a \in \text{Ker} \alpha\} = \text{Ker} \alpha / C
\]

and this completes the proof. \qed

Definition 4.2. Let \((A, B, \alpha)\) be a crossed module. A lifting \((\varphi, X, \omega)\) of \((A, B, \alpha)\) is called transitive if both crossed modules \((A, B, \alpha)\) and \((A, X, \varphi)\) are transitive.

Corollary 4.5. Let \((\varphi, X, \omega)\) be a transitive lifting of \((A, B, \alpha)\). Then \((\varphi, X, \omega)\) is an \(n\)-lifting if and only if \(|\text{Ker} \alpha / \text{Ker} \varphi| = n\).

Corollary 4.6. Let \(X\) be a topological group and \(C\) a subgroup of the fundamental group \(\pi_1(X, 0)\) of \(X\) at the identity. Then there exists a lifting \((p, \text{St}_{\pi_1X0}/C, \omega)\) of \((\text{St}_{\pi_1X0}, X, d_1)\) such that \(\text{Ker} p = C\). Moreover if the underlying space of \(X\) is path-connected, then \(\text{St}_{\pi_1X0}/\pi_1(X, 0) \cong X\) as groups.

Corollary 4.7. If \(X\) is a topological group whose underlying space is totally disconnected, then \(\text{St}_{\pi_1X0} \cong \pi_1(X, 0)\) as groups.

Proposition 4.2. If \((\varphi, X, \omega)\) is a 1-transitive lifting of a crossed module \((A, B, \alpha)\), then \((\varphi, X, \omega)\) is a lifting of any lifting of \((A, B, \alpha)\).

Proof. Since \((A, X, \varphi)\) is 1-transitive, \(\text{Ker} \varphi\) is trivial and hence for any lifting \((\bar{\varphi}, \bar{X}, \bar{\omega})\) the kernel \(\text{Ker} \varphi\) is contained in \(\text{Ker} \bar{\varphi}\). Further by Corollary 4.2 \((\varphi, X, \omega)\) is a lifting of \((\bar{\varphi}, \bar{X}, \bar{\omega})\). \qed

Hence one can state the following definition.

Definition 4.3. A lifting of a crossed module \((A, B, \alpha)\) is called universal if it lifts to every lifting of \((A, B, \alpha)\).

By Proposition 4.2 a 1-transitive lifting is universal.

As a consequence of Theorem 4.5 we can give the following corollary.
Corollary 4.8. Every crossed module has a universal lifting.

Proof. Let \((A, B, \alpha)\) be a crossed module. If we choose \(C\) as trivial in Theorem 4.5 then we have that \((1_A, A, \alpha)\) is a lifting of \((A, B, \alpha)\). Moreover \((1_A, A, \alpha)\) is a lifting of every lifting of \((A, B, \alpha)\). \qed

5 Equivalences of the categories

In this section we prove that for a certain group-groupoid \(G\), there is a categorical equivalence between the group-groupoid actions of \(G\) on groups and lifting crossed modules of the crossed module corresponding to \(G\). We also prove that the liftings of a crossed module are categorically equivalent to the covering morphisms of the same crossed module.

Theorem 5.1. Let \(G\) be a group-groupoid and \((A, B, \alpha)\) the crossed module corresponding to \(G\). Then the category \(\text{GpGpdAct}(G)\) of group-groupoid actions of \(G\) on groups and the category \(\text{LXMod}/(A, B, \alpha)\) of lifting crossed modules of \((A, B, \alpha)\) are equivalent.

Proof. Define a functor \(\theta: \text{GpGpdAct}(G) \to \text{LXMod}/(A, B, \alpha)\) assigning each object \((X, \omega)\) of \(\text{GpGpdAct}(G)\) to a lifting \((\varphi, X, \omega)\) of \((A, B, \alpha)\) where

\[
\varphi: A \to X, a \mapsto a \cdot 0_X
\]

Conversely define a functor \(\psi: \text{LXMod}/(A, B, \alpha) \to \text{GpGpdAct}(G)\) assigning each lifting \((\varphi, X, \omega)\) of \((A, B, \alpha)\) to a group-groupoid action \((X, \omega)\) of \(G\) on the group \(X\) via an action map defined by

\[
G_{d_0} \times \omega X \to X, (g, x) \mapsto g \cdot x = \varphi(g - 1_{d_0}(g)) + x.
\]

Now prove that \(\theta \circ \psi\) and \(\psi \circ \theta\) are respectively naturally isomorphic to the identity functors based on the categories \(\text{LXMod}/(A, B, \alpha)\) and \(\text{GpGpdAct}(G)\). If \((\varphi, X, \omega)\) is a lifting of \((A, B, \alpha)\), then \((\theta \circ \psi)(\varphi, X, \omega) = (\varphi', X, \omega)\) where \(\varphi'\) is defined by

\[
\varphi'(a) = a \cdot 0_X = \varphi(a - 1_{d_0(a)}) + 0_X = \varphi(a)
\]

for all \(a \in A\) and hence \(\theta \circ \psi = 1\).

On the other hand if \((X, \omega)\) is an object of \(\text{GpGpdAct}(G)\) with an action of \(G\) on \(X\) given by

\[
G_{d_0} \times \omega X \to X, (g, x) \mapsto g \cdot x
\]
then the new action obtained from the functor $\psi$ is defined by

$$g \cdot' x = (g - 1_{do(g)}) \bullet (0_X + x)$$

$$= (g - 1_{do(g)}) \bullet (0_X + (1_{\omega(x)}) \bullet x)$$

$$= (g - 1_{do(g)} + 1_{\omega(x)}) \bullet (0_X + x)$$

$$(1_{do(g)} = 1_{\omega(x)})$$

$$= g \cdot x.$$  

and therefore $\psi \circ \theta = 1$ which completes the proof. \hfill \Box

Recall that as a result of [6, Proposition 4.2] a morphism $(f_1, f_2): (\tilde{A}, \tilde{B}, \tilde{\alpha}) \rightarrow (A, B, \alpha)$ of crossed modules such that $f_1: \tilde{A} \rightarrow A$ an isomorphism is called covering morphism. Hence we can give the following theorem.

**Theorem 5.2.** For a crossed module $(A, B, \alpha)$, the category $\text{LXMod}/(A, B, \alpha)$ of lifting crossed modules and the category $\text{CovXMod}/(A, B, \alpha)$ of covering crossed modules of $(A, B, \alpha)$ are equivalent.

**Proof.** Clearly if $(\varphi, X, \omega)$ is a lifting of $(A, B, \alpha)$, then $(1_A, \omega): (A, X, \varphi) \rightarrow (A, B, \alpha)$ is a crossed module morphism. Hence $(A, X, \varphi)$ is a covering crossed module of $(A, B, \alpha)$. Moreover if $(\varphi', X', \omega')$ is another lifting of $(A, B, \alpha)$ and $f$ is a morphism in $\text{LXMod}/(A, B, \alpha)$ from $(\varphi, X, \omega)$ to $(\varphi', X', \omega')$, then $(1_A, f)$ is a morphism in $\text{CovXMod}/(A, B, \alpha)$ from $(A, X, \varphi)$ to $(A, X', \varphi')$. This construction is clearly functorial.

Conversely if $(f_2, f_1): (\tilde{A}, \tilde{B}, \tilde{\alpha}) \rightarrow (A, B, \alpha)$ is a covering morphism of crossed modules, then $f_2: \tilde{A} \rightarrow A$ an isomorphism and $\varphi = \tilde{\alpha}f_2^{-1}$ is the lifting of $\alpha$ over $f_1$. It is easy to see that $f_1\varphi = \alpha$ and $(A, \tilde{B}, \tilde{\varphi})$ is a crossed module. Further if

$$(g_2, G): (A', B', \alpha') \rightarrow (A, B, \alpha)$$

is another covering of $(A, B, \alpha)$ and $(h_2, h_1): (\tilde{A}, \tilde{B}, \tilde{\alpha}) \rightarrow (A', B', \alpha')$ is a morphism in $\text{CovXMod}/(A, B, \alpha)$ then $h_1: (\varphi, \tilde{B}, f_1) \rightarrow (\varphi', B', G)$ is a morphism in $\text{LXMod}/(A, B, \alpha)$. This construction is also functorial and the other details of the equivalence is straightforward. \hfill \Box

By Example 4.1 and Theorem 5.2, we can state the following corollary.

**Corollary 5.1.** Every crossed module $(A, B, \alpha)$ is a covering of the automorphism crossed module $(A, \text{Aut}(A), \iota)$ constructed by $A$.

The category $\text{CovXMod}/(A, B, \alpha)$ of crossed module coverings of a certain crossed module $(A, B, \alpha)$ is equivalent to the category $\text{GpGpdCov}/G$ of group-groupoid coverings of group-groupoid corresponding to $(A, B, \alpha)$ (see also [11, 15] for more general case of crossed modules over groups with operations for an algebraic category C). Hence by Theorem 5.2, we can obtain the following corollary.
Corollary 5.2. Let $G$ be a group-groupoid and $(A, B, \alpha)$ the corresponding crossed module. Then the category $\text{GpGpdCov}/G$ of group-groupoid coverings of $G$ and the category $\text{LXMod}/(A, B, \alpha)$ of lifting crossed modules of $(A, B, \alpha)$ are equivalent.

By Corollary 4.8 and Corollary 5.2 we can obtain the following result.

Corollary 5.3. Every group-groupoid has a universal covering group-groupoid.

By Corollary 4.4 and Corollary 5.2 we can obtain the following result.

Corollary 5.4. Let $G$ be a group-groupoid. Then $G$ acts on $X = \text{Ker} \ d_0/G(0)$ over $\omega : \text{Ker} \ d_0/G(0) \rightarrow \text{Ob}(G), \omega(a + G(0)) = d_1(a)$ as a group-groupoid action with

$$\varphi : G_{d_0} \times \omega \ X \rightarrow X, \varphi(b, a + G(0)) = (b \circ a) + G(0)$$

In [6, Proposition 2.3] it was proved that for a topological group $X$ whose underlying space has a universal cover, the category $\text{TGrCov}/X$ of topological coverings of $X$ and the category $\text{GpGpdCov}/\pi X$ of coverings of group-groupoid $\pi X$ are equivalent; and in [11, Theorem 4.1] a similar result was given for more general topological groups with operations. Hence we can state the following corollary.

Corollary 5.5. Let $X$ be a topological group whose underlying space has a universal cover. Then the category $\text{TGrCov}/X$ of covers of $X$ in the category of topological groups and the category

$$\text{LXMod}/(\text{St}_{\pi X}0, X, d_1)$$

of liftings of $(\text{St}_{\pi X}0, X, d_1)$ are equivalent.

6 Conclusion

Using the results of the paper [10] it could be possible to develop normal and quotient notions of lifting crossed modules. Moreover using the equivalence of the categories in [19, Section 3] as stated in the introduction, parallel results of this paper could be obtained for crossed modules of group with operations and internal groupoids working in a certain algebraic category $C$. 

16
References

[1] H.F. Akız, N. Alemdar, O. Mucuk, T. Şahan, Coverings of internal groupoids and crossed modules in the category of groups with operations, Georgian Math. Journal, 20 (2) (2013), 223–238.

[2] R. Brown, Topology and Groupoids, BookSurge LLC, North Carolina, 2006.

[3] J.C. Baez, A.D. Lauda, Higher-dimensional algebra V: 2-groups Theory Appl. Categ. 12 (2004), 423–491.

[4] R. Brown, P.J. Higgins, R. Sivera, Nonabelian Algebraic Topology: filtered spaces, crossed complexes, cubical homotopy groupoids, European Mathematical Society Tracts in Mathematics 15, 2011.

[5] R. Brown, G. Danesh-Naruie, J.P.L. Hardy, Topological Groupoids: II. Covering Morphisms and G-Spaces, Math. Nachr. 74 (1976), 143–156.

[6] R. Brown, O. Mucuk, Covering groups of non-connected topological groups revisited, Math. Proc. Camb. Phil. Soc. 115 (1994), 97–110.

[7] R. Brown, C.B. Spencer, G-groupoids, crossed modules and the fundamental groupoid of a topological group, Proc. Konn. Ned. Akad. v. Wet. 79 (1976), 296–302.

[8] T. Dнатuashvili, Cohomologically trivial internal categories in categories of groups with operations, Appl. Categ. Structures 3 (1995), 221–237.

[9] T. Dнатuashvili, Cohomology of internal categories in categories of groups with operations, Categorical Topology and its Relation to Analysis, algebra and combinatorics, Ed. J. Adamek and S. Mac Lane (Prague, 1988), World Sci. Publishing, Teaneck, NJ, 1989.

[10] T. Dнатuashvili, Kan extensions of internal functors: Nonconnected case, J.Pure Appl.Algebra 167 (2002), 195–202.

[11] T. Dнатuashvili, Whitehead homotopy equivalence and internal category equivalence of crossed modules in categories of groups with operations, Proc. A. Razmadze Math.Inst. 113 (1995), 3–30.

[12] P.J. Higgins, Categories and groupoids, Van Nostrand, New York, 1971.

[13] J.-L. Loday, Spaces with finitely many non-trivial homotopy groups, J. Pure Appl. Algebra 24 (1982), 179–202.
[14] O. Mucuk, B. Kılıçarslan, T. Şahan, N. Alemdar, Group-groupoids and monodromy groupoids Topology Appl. 158 (15) (2011), 2034–2042.

[15] O. Mucuk, T. Şahan, Coverings and crossed modules of topological groups with operations, Turk. J. Math. 38 (5) (2014), 833–845.

[16] O. Mucuk, T. Şahan, N. Alemdar, Normality and Quotients in Crossed Modules and Group-groupoids, Appl. Categor. Struct. 23 (2015), 415–428.

[17] O. Mucuk, H.F. Akız, Monodromy groupoids of an internal groupoid in topological groups with operations, Filomat 29 (10) (2015), 2355–2366.

[18] G. Orzech, Obstruction theory in algebraic categories I, II. J. Pure Appl. Algebra 2 (1972), 287–314, 315–340.

[19] T. Porter, Extensions, crossed modules and internal categories in categories of groups with operations, Proc. Edinb. Math. Soc. 30 (1987), 373–381.

[20] J.H.C. Whitehead, Note on a previous paper entitled “On adding relations to homotopy group”. Ann. of Math. 47 (1946), 806–810.

[21] J.H.C. Whitehead, On operators in relative homotopy groups, Ann. of Math. 49 (1948), 610–640.

[22] J.H.C. Whitehead, Combinatorial homotopy II, Bull. Amer. Math. Soc. 55 (1949), 453–496.