Letter

Sudden death and rebirth of entanglement for different dimensional systems driven by a classical random external field

N Metwally\textsuperscript{1,2,5}, H Eleuch\textsuperscript{3} and A-S Obada\textsuperscript{4}

\textsuperscript{1} Mathematics Department, College of Science, Bahrain University, Bahrain
\textsuperscript{2} Mathematics Department, College of Science, Aswan University, Aswan, Egypt
\textsuperscript{3} Department of Physics, McGill University, Montreal, H3A 2T8, Canada
\textsuperscript{4} Mathematics Department, Faculty of Science, Al-Azhar University, Cairo, Egypt

E-mail: nmetwally@gmail.com

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Abstract

The entangled behavior of different dimensional systems driven by a classical external random field (CERF) is investigated. The amount of the survival entanglement between the components of each system is quantified. There are different behaviors of entanglement that come into view: decay, sudden death, sudden birth and long-lived entanglement. The maximum entangled states which can be generated from any of these suggested systems are much more fragile than the partially entangled ones. The systems of larger dimensions are more robust than those of smaller dimension systems, where the entanglement decays smoothly and gradually and may vanish for a very short time. For the class of $2 \times 3$ dimensional system, the one-parameter family is found to be more robust than the two-parameter family. Although the entanglement of a driven $2 \times 3$ dimensional system is very sensitive to the CERFs, one can use them to generate a long-lived entanglement.

Keywords: external fields, entanglement, qutrit, qubit

(Some figures may appear in colour only in the online journal)

1. Introduction

It is well known that noise represents one of the unavoidable physical phenomena in the context of quantum information and computation. The effect of different types of noises on many systems has been investigated extensively for two-dimensional systems, (see for example [1–6]). The dynamics of higher dimensional systems which travel in different noise channels have been discussed. For example, the separability of entangled qutrits in a noise channel is investigated in [7]. The time evolution of classical and quantum correlations of hybrid qubit–qutrit systems in a classical dephasing environment has been studied [8]. Also, the decoherent dynamics of quantum correlations in a qubit–qutrit system under various noise channels has been discussed in [9]. The dynamics of entanglement for a qutrit system in the presence of global, collective, multilocal and local noise channels is studied [10]. The possibility of protecting entanglement in a qubit–qutrit system from decoherence via weak measurements and reversal was considered by Xiao [11]. The phenomenon of bipartite entanglement revivals under local operations in systems subject to classical noise sources is investigated [12, 13].

Entangled systems which are driven by a classical field usually lose their correlations, and consequently their efficiency to perform quantum communication decreases. Recently this type of study has been implemented experimentally [14, 15]. Most of these studies focused on small dimensional systems (qubits). For example, [16] investigated the dynamics of the

\textsuperscript{5} Author to whom any correspondence should be addressed.
encoded information in a pulsed-driven qubit within and outside the rotating wave approximation. The dynamics of the tripartite entanglement, where two qubits are driven non-resonantly coupled to the cavity, is discussed [17]. The effect of rectangular and exponential pulses on the degree of correlations between two qubit systems was discussed in [18].

Larger dimensional systems as $\times 2^3$ and $\times 3^3$, which are driven by classical external random fields (CERFs), have not been investigated in detail. There are only a few research works in this direction that have been carried out. Recently, it has been shown that an external classical driving field for a qubit can speed-up the evolution of an open system. [19]. Therefore, we are motivated to investigate the entanglement degradation between different or similar dimensional subsystems which are driven by CERF. This study is devoted to qubit-qubit, qubit-qutrit and qutrit-qutrit systems, where it is assumed that only one particle is driven by CERF.

This paper is organized as follows: in section 2, the suggested models and their evolution are introduced. The analytical solutions are given in section 3 as well as the behaviors of entanglement for different initial state settings. We summarize our results in section 4.

2. Models

The suggested model consists of three different systems, a two-qubits system, which represents a $2 \times 2$ dimensional system, a qubit–qutrit ($2 \times 3$) system which consists of two and three level subsystems, and a two-qutrit system, where each subsystem is defined by $3 \times 3$ dimensions. Figure 1 represents the schematic diagram for the suggested models.

2.1. Dynamics of 2-qubit systems

It is assumed that one of the subsystems of each system interacts locally with its own environment which is described by a single CERF. This field has a frequency $\omega$ and a random phase $\phi$ which equals either 0 or $\pi$ with probability $P = 0.5$. The schematic description is shown in figure 1(a). In the rotating frame approximations, the Hamiltonian, $\mathcal{H}$ of the single qubit-field system is given by,

$$
\mathcal{H}_{q-f} = \frac{\omega_0 - \omega}{2} \sigma_z + i g (\sigma_+ e^{-i\phi} - \sigma_- e^{i\phi}),
$$

where $g$ is the coupling strength between the qubit and the CERF, $\sigma_z$ are the raising and lowering operators of the single qubit with frequency $\omega_0$. For the suggested 2-qubit systems, it is assumed that,

- Only one qubit is driven by the CERF.
- Only the resonances case is considered, i.e. $\omega_0 = \omega$.
- The interaction between the qubit and its environment is strong enough, where the dissipation between the vacuum and the qubit is forbidden.

By considering the above assumptions, the final state at $t > 0$ can be written as,

$$
\rho_{2 \times 2}(t) = \frac{1}{2} \sum_{j=1}^{2} \{ e^{-i\mathcal{H}_j} \rho_{2 \times 2}(0) e^{i\mathcal{H}_j} \}.
$$
where $\mathcal{H}_j = i\hbar g_j (\sigma_x e^{-i\phi_j} - \sigma_x e^{i\phi_j})$, $j = 1, 2$ with $\phi_1 = 0$ and $\phi_2 = \pi$, respectively and we set $\hbar = 1$ for simplicity. The initial state $\rho_{2\times2}(0)$ represents the state of a system consisting of $2 \times 2$ dimensions, i.e. a 2-qubits system.

2.2. Dynamics of qubit–qutrit systems

This model represents a $2 \times 3$ dimensional system. It is assumed that only one subsystem is driven by the CERF. However, if the qubit is driven by the CERF, then the dynamics of the system is governed by equation (2) (see figure 1(b)). On the other hand, if we allow the 3D subsystem, namely the qutrit, to be driven by the local CERF, then the Hamiltonian which describes this system depends on the configuration of the qutrit-system (see figure 1(c)). Let us consider that the qutrit is initially prepared in a $\Lambda$ configuration [20]. In this case, the Hamiltonian which governs the interaction between the CERF and the single qutrit is given by,

$$\mathcal{H}_{qutrit} = \sum_{i=0}^{2} \omega_i S_i + g_1 (e^{-i\phi_1} S_1 + e^{i\phi_1} S_2) + g_2 (e^{-i\phi_2} S_2 + e^{i\phi_2} S_3),$$

where $\omega_i, g_i, \ell = 1, 2$ are the frequencies of the external fields and the coupling strength between the field and the qutrit. The operators $S_i$, where $(i,j) \in \{12, 21, 20, 02\}$ obey the $SU(3)$ algebra. For this suggested system, we consider the following assumptions:

- The single qutrit system has one upper level $|2\rangle$ with frequency $\Omega_2$ and two lower levels $|0\rangle$ and $|1\rangle$ with frequencies $\Omega_0$ and $\Omega_1$, respectively.
- The transitions between the $|0\rangle \leftrightarrow |2\rangle$ and $|1\rangle \leftrightarrow |2\rangle$ are dipole allowed, while between $|0\rangle \leftrightarrow |1\rangle$ is dipole forbidden, i.e. there is no interaction.
- The non-degeneracy case is considered [21, 22], namely $\Omega_2 - \Omega_0 = \Omega_2 - \Omega_1 \gg g_1$, $\ell = 1, 2$.
- The resonance case is considered, i.e. $\Omega_2 - \Omega_0 = \omega_1, \Omega_2 - \Omega_1 = \omega_2$.

By considering the above assumptions, the dynamics of the qubit–qutrit system is given by,

$$\rho_{2\times3}(t) = \frac{1}{2} \sum_{i=0}^{2} \{e^{-i\mathcal{H}_{int} t} \rho_{2\times3}(0) e^{i\mathcal{H}_{int} t}\},$$

with

$$\mathcal{H}_{int} = g_1 (e^{-i\phi_1} S_1 + e^{i\phi_1} S_2) + g_2 (e^{-i\phi_2} S_2 + e^{i\phi_2} S_3),$$

where $\rho_{2\times3}(0)$ represents the state of a system consisting of $2 \times 2$ dimensions, i.e. a 2-qubits system.

2.3. Dynamics of a qutrit–qutrit system

In this case, the initial system consists of two three-level subsystems, namely two qutrits, which represent a generalization of a qubit. The state of the qutrit can be spanned by the three orthogonal basis $|0\rangle, |1\rangle, |2\rangle$. Physically, qutrit can be represented by three-level atoms [23–25] (see figure 1(d)). In this treatment, we consider that only one qutrit is driven by the CERF, where both qutrits are initially prepared in a $\Lambda$ configuration. Moreover, we consider the same assumptions listed above (the qubit–qutrit case). The dynamics of the qutrit–qutrit system $\rho_{3\times3}(t)$ are given by,

$$\rho_{3\times3}(t) = \frac{1}{2} \sum_{i=1}^{2} \{e^{-i\mathcal{H}_{int} t} \rho_{3\times3}(0) e^{i\mathcal{H}_{int} t}\},$$

where $\mathcal{H}_{int}$ is given from equation (4).

3. Survival entanglement

In this section, we obtain analytical solutions for all the above systems. The behavior of entanglement between each two subsystems will be discussed, where the amount of entanglement is quantified by using a measure called negativity. This measure is an acceptable measure for any bipartite system of any dimension [26]. For a state $\rho_{12}$, with dimensions $d_1 \times d_2$ where $d_1 < d_2$, the negativity, $N$ is defined as,

$$N = \frac{1}{d_1-1} \{||\rho_{12}^T|| - 1\},$$

where $\rho_{12}^T$ is the partial transpose with respect to the largest party (second-partite) and $||.||$ is the trace norm [26, 27]

3.1. Qubit–qubit systems

Let us assume that the two qubits state is given by X-state, which can be described by the computational basis ‘0’ and ‘1’ as,

$$\rho_{12}(0) = \frac{1}{2} \left[ \begin{array}{cc} (|00\rangle + |11\rangle)(|00\rangle + |11\rangle) & (|00\rangle + |11\rangle)(|01\rangle + |10\rangle) \\ (|01\rangle + |10\rangle)(|00\rangle + |11\rangle) & (|01\rangle + |10\rangle)(|01\rangle + |10\rangle) \end{array} \right]$$

$$\begin{align*}
\rho_{2\times2}(0) &= \frac{1 + c_{33}}{2} \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] + \frac{1 - c_{33}}{2} \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] + \frac{c_{11} + c_{22}}{2} \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] + \frac{c_{11} - c_{22}}{2} \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right].
\end{align*}$$
where $c_{ij} = \text{tr}(\rho_i(0)\sigma_j)$, $i,j = 1,2,3$ and $\sigma_i$ are the Pauli matrices for the first and the second qubits, respectively. From this state, one can obtain a maximum entangled class of states by setting $c_{ij} = \pm 1$, a Werner state by setting $c_{ij} = x$ and a generalized Werner state for $c_{ij} = x_1$, $c_{ij} = x_2$ and $c_{ij} = x_3$.

If only one qubit is driven by the CERF, then the final state of the system at a time $\tau > 0$ is given by,

$$\rho(t) = R_1(00)\{|00\rangle\langle 00| + R_2|01\rangle\langle 01| + R_3|10\rangle\langle 10| + R_4|11\rangle\langle 11|angle + R_5|01\rangle\langle 01| + R_6|10\rangle\langle 10| + R_7|11\rangle\langle 11|\rangle + R_8|10\rangle\langle 10| + R_9|11\rangle\langle 11|\rangle + R_{10}|10\rangle\langle 10| + R_{11}|11\rangle\langle 11|\rangle + R_{12}|11\rangle\langle 11|\rangle,$$  

(8)

where,

$$R_1 = \frac{1 + c_{33}}{2} \cos^2(\tau) + \frac{1 - c_{33}}{2} \sin^2(\tau),$$

$$R_2 = -\frac{1 + c_{33}}{2} \sin(\tau)\cos(\tau),$$

$$R_3 = \frac{c_{11} - c_{22}}{2} \cos^2(\tau) - \frac{c_{11} + c_{22}}{2} \sin^2(\tau),$$

$$R_4 = \frac{1 - c_{33}}{2} \cos^2(\tau) + \frac{1 + c_{33}}{2} \sin^2(\tau),$$

$$R_5 = \frac{c_{11} + c_{22}}{2} \cos^2(\tau) - \frac{c_{11} - c_{22}}{2} \sin^2(\tau) = R_6,$$

$$R_7 = \frac{1 - c_{33}}{2} \cos^2(\tau) - \frac{1 + c_{33}}{2} \sin^2(\tau),$$

$$R_8 = -\frac{c_{11} + c_{22}}{4} \sin(2\tau),$$

$$R_9 = R_3,$$

$$R_{10} = -\frac{c_{11} - c_{22}}{4} \sin(2\tau),$$

$$R_{11} = R_4,$$

$$R_{12} = \frac{\tau}{2} \sin(\tau)^2(\tau),$$

(9)

and $\tau = \tau$.

In figure 2, we investigate the effect of the CERF on three different classes of two-qubit systems: maximum entangled class, where we set $c_{11} = c_{22} = c_{33} = -1$, a Werner state, with $(c_{11} = c_{22} = c_{33} = x = -0.8)$ and partially entangled classes with $c_{11} = -0.8, c_{22} = -0.7, c_{33} = -0.6$. The general behavior shows that the entanglement decays as the time increases for all these initial states. The phenomena of the sudden death of entanglement is depicted for each maximum entangled state (MES) and partially entangled state (PES). However, the death time increases for initially less entangled states. For a PESs, the upper bounds of entanglement are obtained for different families. For MES, the upper bounds of entanglement are larger than the initial entanglement in some intervals of time. Moreover, the upper bounds are reached at the same time for all states.

3.2. Qubit–qutrit systems

In this section, a system consisting of two different dimensional subsystems is considered: one is a qubit (2D) and the other is a qutrit (3D) system. Analytical solutions for the final states are obtained for different families. The first state is known by a one-parameter family [27] and the second is known by a two-parameter family [29]. The time evolution is obtained when only one of their subsystems is driven by a CERF.

3.2.1. One-parameter family. This state represents a qubit–qutrit system which is defined by $2 \times 3$ dimensions. It is described by one parameter as,

$$\rho_{0p}(t) = \frac{P}{2} (|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 11|) + \frac{P}{2} (|12\rangle\langle 12| + |12\rangle\langle 12|) + \frac{1 - 2P}{2} (|02\rangle\langle 02| + |10\rangle\langle 10| + |10\rangle\langle 10| + |02\rangle\langle 02|),$$

(10)

This state has quantum correlation for $P \in [0,\frac{1}{3}]$, i.e. it is disentangled at $P = \frac{1}{3}$ only. Now we consider the following two cases:

- Only the qubit is driven by the CERF

The time evolution of the initial state (10) can be obtained by using equation (2). For $\tau > 0$, the final state $\rho_{0p}(t)$ can be written explicitly as,

$$\rho_{0p}^q(t) = \rho^q_1(00)\{|00\rangle\langle 00| + \rho^q_2(00)|01\rangle\langle 01| + \rho^q_3(00)|11\rangle\langle 11| + \rho^q_4(00)|12\rangle\langle 12| + \rho^q_5(00)|02\rangle\langle 02| + \rho^q_6(00)|10\rangle\langle 10| + \rho^q_7(00)|12\rangle\langle 12| + \rho^q_8(00)|11\rangle\langle 11|),$$

(11)

where the superscript $q_0$ refers to the qubit while the subscript $1p$ means a one-parameter family. The coefficients $\rho^q_1, \rho^q_2, \rho^q_3, \rho^q_4, \rho^q_5, \rho^q_6, \rho^q_7, \rho^q_8, \rho^q_9, \rho^q_{10}$ are given by,

$$\rho^q_1(t) = \frac{P}{2} \cos^2(\tau) + \frac{1 - 2P}{2} \sin^2(\tau),$$

$$\rho^q_2(t) = \frac{P}{2} \cos^2(\tau) + \frac{1 - 2P}{2} \sin^2(\tau),$$

$$\rho^q_3(t) = \frac{1 - 2P}{2} \cos^2(\tau) - \frac{P}{2} \sin^2(\tau),$$

$$\rho^q_4(t) = \frac{1 - 2P}{2} \cos^2(\tau) - \frac{P}{2} \sin^2(\tau),$$

$$\rho^q_5(t) = \frac{1 - 2P}{2} \cos^2(\tau) - \frac{P}{2} \sin^2(\tau),$$

$$\rho^q_6(t) = \rho^q_7(t),$$

$$\rho^q_8(t) = \rho^q_9(t),$$

$$\rho^q_{10}(t) = \rho^q_{10}(t),$$

(12)

- Only the qutrit is driven by CERF

In this case, it is assumed that only the qutrit is allowed to be driven by the CERF, where the non-degenerate state is considered [21, 22]. The final state is given by,

$$\rho_{0p}^q(t) = \rho^q_{10}(00)|00\rangle\langle 00| + \rho^q_{20}(00)|01\rangle\langle 01| + \rho^q_{30}(00)|11\rangle\langle 11| + \rho^q_{40}(00)|12\rangle\langle 12| + \rho^q_{50}(00)|02\rangle\langle 02| + \rho^q_{60}(00)|10\rangle\langle 10| + \rho^q_{70}(00)|12\rangle\langle 12| + \rho^q_{80}(00)|11\rangle\langle 11| + \rho^q_{90}(00)|10\rangle\langle 10| + \rho^q_{10}(00)|12\rangle\langle 12|,$$

(13)

where, the subscript $q_0$ refers to the qutrit. The coefficients $\rho^q_1, j = 1...25$ are given by,
where $g_1 = \tau$ and $g_2 = \tau$.

The behavior of entanglement for a system is prepared initially in a qubit–qutrit state of a one-parameter family type which is depicted in figure 3, where it is assumed that only the qubit or the qutrit are driven by the CERF. The general behavior shows that the entanglement decreases as soon as the interaction is switched on. The decay rate is much smaller than that shown in figure 3(a), where only the qubit is driven by the CERF. It is clear that the entanglement decays slowly and the phenomena of the sudden changes of entanglement appear clearly [28]. The decay time is much larger than that depicted in figure 3(a), where the two subsystems are completely separable for the first time at $\tau \approx 1.5$.

3.2.2. Two-parameter family. In a computational basis, the state which describes this family can be written as [29],

$$
\rho_{2p}(0) = \alpha [(|02\rangle \langle 02| + |12\rangle \langle 12|) + \beta (|00\rangle \langle 00| + |11\rangle \langle 11|) + \frac{\beta + \gamma}{2} (|01\rangle \langle 01| + |10\rangle \langle 10|)]
$$

$$
+ \frac{\beta - \gamma}{2} (|01\rangle \langle 10| + |10\rangle \langle 01|),
$$

(15)

where $\gamma + 2\alpha + 3\beta = 1$. The two-parameter family state (15) is entangled for $\beta = 0$, $\alpha \in [0, \frac{1}{2}]$ and $\gamma \in [\frac{1}{2}, 1]$. Moreover, if we set $\alpha = \frac{1}{2}$ and $\gamma = 1$, then the state (15) turns into a MES. Now, let us consider the following two cases:

- **Only the qubit is driven by the CERF**

In this case, the time evolution of the initial state (15) is given by the following density operator,

$$
\rho_{2p}(t) = M_0|00\rangle \langle 00| + M_2|01\rangle \langle 01| + M_6|10\rangle \langle 10|
$$

$$
+ M_4|01\rangle \langle 10| + M_6|10\rangle \langle 01| + M_6|11\rangle \langle 11| + M_2|00\rangle \langle 11| + M_6|11\rangle \langle 00| + M_6|02\rangle \langle 02|
$$

$$
+ M_6|12\rangle \langle 12|,
$$

(16)

where,

$$
M_1 = \beta \cos^2(\tau) + \frac{\beta + \gamma}{2} \sin^2(gt),
$$

$$
M_2 = \beta \sin^2(\tau) + \frac{\beta + \gamma}{2} \cos^2(\tau),
$$

$$
M_3 = M_2, \quad M_4 = M_6 = \frac{\beta - \gamma}{2} \cos^2(\tau), \quad M_6 = M_4,
$$

$$
M_7 = M_8 = \frac{\beta - \gamma}{2} \sin^2(\tau), \quad M_9 = M_{10} = \alpha.
$$

(17)

- **Only the qutrit is driven by the CERF**

In this case, the final state of the system is described by a matrix of $6 \times 6$ dimensions, where the non-zero elements are given by,
In this context, the behavior of entanglement when only the qubit is driven by the CERF is depicted in figure 4(a). If the initial state is initially prepared in a MES, then the entanglement suddenly decays to reach its minimum value for the first time at \( \tau \approx 0.8 \), then suddenly increases to be maximum \((\Lambda = 1)\). This behavior is periodically repeated as \( \tau \) increases. The phenomenon of long-lived entanglement is predicted for systems that are initially prepared in PESs, where the phenomena of sudden decay and death of entanglement are shown for systems that are initially prepared with small entanglement.

Figure 4(b) shows the behavior of entanglement when only the qubit is driven by CERF. In this case, for initial MES, the entanglement decays very fast to death completely at \( \tau' \approx 0.7 \). For this class of states, the phenomenon of the sudden birth of entanglement is depicted, where the death time is much larger than that shown for a one-parameter family. For initially less entangled states, the entanglement decreases then increases gradually. The phenomenon of entanglement death can be seen only for a short time for systems which have small initial entanglement.

From this figure, one concludes that systems which are initially prepared in MES are more sensitive to the CERF than those which are prepared in PES. If the larger dimensional systems are driven by the CERF, then the entanglement of the MES are liable to sudden death for a longer time. Systems which are initially prepared in PES are more robust than those prepared in MES for the CERF. The long-lived entanglement can be observed if one has the ability to control the parameters which describe the initial state. Consequently, one can perform quantum information tasks even in the presence of these types of noises.

### 3.3. Qutrit–qutrit system

A system of qutrit–qutrit can be defined by,

\[
\rho_{\text{q}q}(0) = \left| \psi_{\text{q}q}(0) \right\rangle \left\langle \psi_{\text{q}q}(0) \right| ,
\]

where

\[
\left| \psi_{\text{q}q}(0) \right\rangle = a_1 |00\rangle + a_2 |11\rangle + a_3 |22\rangle.
\]

with \( a_i, i = 1, 2, 3 \) are real and \( a_1^2 + a_2^2 + a_3^2 = 1 \). In this treatment, it is assumed that both qutrits are prepared in a \( \Lambda \) configuration and only the first qutrit is driven by the CERF. By using equation (5), one gets the final state of the qutrit–qutrit system at any value of \( \tau = \tau' > 0 \). This final state is defined by a \( 9 \times 9 \) matrix, where the non-zero elements are given by,

\[
\rho_{00,00} = a_1^2 c_1^2, \quad \rho_{00,11} = a_1 a_2 c_1 c_2, \quad \rho_{00,22} = a_2 a_3 c_1 c_2, \\
\rho_{00,02} = -a_1 a_3 c_1 c_2, \quad \rho_{01,00} = a_1 a_2 c_2^2, \quad \rho_{01,11} = a_1^2 c_2^2, \\
\rho_{01,12} = -a_1 a_2 c_1 c_2, \quad \rho_{11,00} = a_2 a_3 c_2^2, \quad \rho_{11,11} = a_2^2 c_2^2, \\
\rho_{11,12} = -a_2 a_3 c_1 c_2, \quad \rho_{12,00} = -a_2 a_3 c_1 c_2, \\
\rho_{22,00} = a_3 a_1 c_1 c_2, \quad \rho_{22,11} = a_3 a_2 c_1 c_2, \quad \rho_{22,22} = a_3^2 c_2^2, \\
\rho_{22,12} = -a_3 a_2 c_1 c_2, \quad \rho_{02,02} = -a_1 a_3 c_1 c_2, \\
\rho_{02,11} = -a_1 a_2 c_1 c_2, \quad \rho_{02,22} = -a_1 a_2 c_1 c_2, \\
\rho_{02,00} = -a_1 a_2 c_1 c_2, \quad \rho_{02,12} = -a_1 a_2 c_1 c_2, \\
\rho_{21,10} = a_2 a_3 c_1 c_2, \quad \rho_{21,21} = a_2 a_3 c_1 c_2, \\
\rho_{21,12} = a_2 a_3 c_1 c_2, \quad \rho_{21,02} = a_2 a_3 c_1 c_2, \\
\rho_{12,10} = a_2 a_3 c_1 c_2, \quad \rho_{12,12} = a_2 a_3 c_1 c_2, \\
\rho_{12,02} = a_2 a_3 c_1 c_2, \quad \rho_{10,10} = a_3^2 c_2^2, \quad \rho_{10,21} = a_3 a_2 c_1 c_2, \\
\rho_{10,12} = a_3 a_2 c_1 c_2, \quad \rho_{00,02} = -a_1 a_3 c_1 c_2, \\
\rho_{00,12} = -a_1 a_2 c_1 c_2, \quad \rho_{00,22} = -a_2 a_3 c_1 c_2, \\
\rho_{01,02} = -a_1 a_3 c_1 c_2, \quad \rho_{01,12} = -a_1 a_2 c_1 c_2, \\
\rho_{01,01} = -a_1 a_2 c_1 c_2, \quad \rho_{01,22} = -a_1 a_2 c_1 c_2, \\
\rho_{02,01} = -a_1 a_3 c_1 c_2, \quad \rho_{02,11} = -a_1 a_2 c_1 c_2, \\
\rho_{02,21} = -a_1 a_3 c_1 c_2, \quad \rho_{02,00} = -a_1 a_2 c_1 c_2.
\]
In figure 5, we plot the behavior of the entanglement when the first qutrit is driven by the CERF. Let us consider that the system is initially prepared in MES, i.e. we set $a_i = \frac{1}{\sqrt{3}}, i = 1, 2, 3$. In this case, the general behavior shows that the entanglement is maximum at $\tau' = 0$ (before the interaction is switched on). As soon as the interaction between the first qutrit and the CERF is switched on, the entanglement decays as the interaction time is increased. However, the entanglement vanishes for the first time at $\tau' \simeq 2.4$. For larger times, the entanglement re-increases gradually to reach its maximum value ($N = 1$) for the first time at $\tau' \simeq 3$. This behavior is repeated while increasing the interaction time between the qutrit and CERF.

The effect of CERF on the qutrit–qutrit system (initially prepared in PES) is also described in figure 5, where two classes are considered: the first one is obtained by setting $a_1 = 0.3, a_2 = 0.4$, while the second is obtained by setting $a_1 = 0, a_2 = 1/\sqrt{3}$. However, the phenomenon of sudden entanglement changes is depicted for the two cases. For less initial entangled systems, the lower bounds of entanglement are larger than those which start with a larger degree of entanglement. Moreover, the upper bounds of entanglement are reached at the same time.

From the behavior of negativity, one can conclude that as soon as the interaction is switched on, the phenomenon of the sudden decay of entanglement is depicted for MES and PES. The entanglement suddenly increases to reach their maximum bounds which are dependent on the initial entanglement. However, the upper and lower bounds of entanglement are reached at the same time for all considered states.

4. Conclusion

In this paper, the entanglement of three types of different dimensional systems are discussed. It is assumed that only one subsystem is driven by an external random field. The suggested systems are: qubit–qubit, qubit–qutrit and qutrit–qutrit systems, which are described by $2 \times 2$, $2 \times 3$ and $3 \times 3$ dimensions, respectively. The general behavior shows that the entanglement is very sensitive to the CERF. However, this sensitivity depends on the size of the driven subsystem and the initial degree of entanglement of the system.

MESs are the most sensitive states for the external field, whereas the phenomena of the sudden decay of entanglement, changes, death and sudden re-birth are displayed clearly. On the other hand, PESs are found to be more robust than the MESs to this CERF.

The dimensions of the driven subsystems by the external field play an important role on the behavior of entanglement. This result can be observed clearly for the qubit–qutrit systems, where if only the qubit is driven, then the sudden changes of entanglement (death or birth) are faster than that shown for the case in which only the qutrit is driven by this external field. For the two-parameter class, one can observe the phenomenon of long-lived entanglement if the qubit is driven by the external field. However, for a one-parameter family, PESs are more robust than the two-parameter class if the qutrit is driven by the classical external random field.

For a qutrit–qutrit system, the entanglement is more robust to the CERF than that displayed for qubit–qutrit systems, where the entanglement decays smoothly, whereas only the MES lose their entanglement for a very short period of time and the rebirth occurs again shortly. The effect of the dimensions of the driven subsystems by the CERF can be observed by comparing the behavior of survival entanglement between

Figure 4. Entanglement of the two-parameter family qubit–qutrit system (a) when only the qubit is driven by CERF and (b) only the qutrit is driven by the CERF, where $\tau_1 = \tau_2 = \tau'$. The dot curve for MES with $(\beta = 0, \gamma = 1, \alpha = 0.5)$, while the dash and the solid curves for PES with $(\beta = 0.2, \gamma = 0.7)$, and $(\beta = 0.2, \gamma = 0.6)$, respectively.

Figure 5. Entanglement of a two qutrits-system. The dash curve for MES ($a_1 = a_2 = a_3 = \frac{1}{\sqrt{3}}$) the dot for ($a_1 = 0, a_2 = \frac{1}{\sqrt{3}}, a_3 = \frac{\sqrt{2}}{\sqrt{3}}$) and solid curves for ($a_1 = 0.3, a_2 = 0.4$), respectively.
the qubit–qubit and the qutrit–qutrit systems. It is clear that the entanglement of larger dimensional systems are more robust than smaller systems.

From previous discussions one can conclude that: (i) MESs are very sensitive to the CERF; (ii) the entanglement and hence the information are lost fast for smaller dimensional systems; (iii) a one-parameter family is more powerful than a two-parameter family, where the possibility of losing its correlation is much smaller than that shown for the two-parameter family, and consequently its efficiency to perform quantum information tasks is better; (iv) the long-lived entanglement that can be depicted for systems are initially prepared in the two-parameter family’s class; (v) the two-qutrit system is more robust than the two-qubit system for the CERF. Therefore we can deduce that in the context of quantum information and communication, the 2-qutrit systems are much better than the 2-qubit system in the presence of a CERF.

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