High-energy bounds on total cross sections in $\mathcal{N}=4$ SYM from AdS/CFT

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Using the AdS/CFT correspondence, we study the high-energy behavior of scattering amplitudes in $\mathcal{N}=4$ SYM gauge theory for dipole-dipole soft elastic scattering, described in the Wilson-loop correlator formalism. The amplitudes are evaluated in the dual picture, at large impact-parameter, by considering the exchange of supergravity fields between certain minimal surfaces in Euclidean $AdS_5$, and performing the appropriate analytic continuation to Minkowskian signature. The purely elastic behavior at large impact-parameter is then combined with the unitarity constraint in the central region, in order to derive an absolute bound on the high-energy behavior of the dipole-dipole total cross section. The possibility to obtain a stronger bound by assuming a larger domain of validity for the AdS/CFT result is also discussed.

I. INTRODUCTION

High-energy soft hadron-hadron scattering is known to be one of the hardest open problems of strong interactions. Indeed, since the momentum transfer is small (typically $\sqrt{|t|} \lesssim 1\,\text{GeV}$), this kind of processes involves the nonperturbative, strong coupling regime of the underlying microscopic theory, namely Quantum Chromodynamics. Nevertheless, a few results can be obtained using the fundamental properties of a consistent quantum field theory, established long ago in the S-matrix formalism: unitarity, coming from the conservation of probabilities, and analyticity. These properties, combined with the existence of a “mass gap” in the asymptotic particle spectrum, lead to the celebrated Froissart bound for the total cross section [1], which corresponds (up to logarithms) to an intercept not greater than 1 for the leading Regge singularity, usually called the “Pomeron”.

In recent years, a new tool to deal with strong coupling physics has appeared, namely the Gauge/Gravity duality, which has raised the hope to understand high-energy soft scattering amplitudes in gauge theories by mapping them into appropriate quantities in the dual gravity theory. The first realization of the duality is the well known AdS/CFT correspondence [2], which relates type IIB string theory on $AdS_5 \times S^5$ in the weak-coupling, supergravity limit to $\mathcal{N}=4$ SYM theory at strong coupling and large number of colours. Since this theory is a conformal field theory and thus non-confining, its behaviour is expected to differ from that of a confining theory like QCD. However, attacking the problem of soft amplitudes in this context may be a useful laboratory for further developments in QCD. Indeed, the study of soft high-energy scattering amplitudes in $\mathcal{N}=4$ SYM using the AdS/CFT correspondence, and more generally Gauge/Gravity duality, has attracted much attention in the literature [3–10].

In the conformal case there is no mass gap, and thus the Froissart bound is not expected to be valid for $\mathcal{N}=4$ SYM theory. However, unitarity and analyticity are still expected to hold, and so it is interesting to examine the question of high-energy bounds in this context. In [11] we have used unitarity, analyticity and the AdS/CFT correspondence to give a precise account of soft high-energy elastic amplitudes in the $\mathcal{N}=4$ supersymmetric gauge theory. In particular, we have combined the knowledge obtained from AdS/CFT in the region of applicability of the supergravity approximation, i.e., the large impact-parameter region where the amplitude is essentially elastic, with the constraints coming from analyticity and unitarity, in order to obtain high-energy bounds on total cross sections. More precisely, the following ingredients have been used.

1. The rôle of massive quarks and antiquarks ($Q, \bar{Q}$) in the AdS/CFT correspondence is played, as in [12], by the massive $W$ bosons arising from breaking $U(N+1) \to U(N) \times U(1)$, where one brane is considered away from the $N \to \infty$ others. In turn, the rôle of hadrons is devoted to “onia” defined as linear combinations of $Q\bar{Q}$ colorless “dipole” states [13] of average transverse size $\langle |\vec{R}| \rangle$, which sets the scale for the onium mass.

2. The dipole amplitudes are obtained using the Wilson loop formalism [14, 15] in Euclidean space, in order to avoid the complications of the Lorentzian AdS/CFT correspondence (see e.g. [16]). Performing the appropriate analytic continuation [17, 18], one obtains the physical amplitude in Minkowski space.

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After analytic continuation, one obtains the scattering amplitude in the impact-parameter representation at large impact parameter; combining this with unitarity to fix a bound in the lower impact-parameter domain, it leads to a convenient expression in terms of the normalized connected correlator

\[ A(s, t; \vec{R}_1, \vec{R}_2) = -2is \int d^2\vec{b} e^{i\vec{q} \cdot \vec{b}} C_M(\chi, \vec{b}; \vec{R}_1, \vec{R}_2) \equiv -2is \int d^2\vec{b} e^{i\vec{q} \cdot \vec{b}} \left[ \frac{\langle W_1 W_2 \rangle}{\langle W_1 \rangle \langle W_2 \rangle} - 1 \right], \]

where \( t = -q^2 \), \( q \) being the transverse transferred momentum (here and in the following we denote with \( \vec{v} \) a two-dimensional vector), and the Wilson loops follow the classical straight-line trajectories for quarks (antiquarks, in parenthesis) [20]:

\[ W_1 \rightarrow X_1^\mu = b^\mu + u_1^\mu \tau (+r_1^\mu) ; \quad W_2 \rightarrow X_2^\mu = u_2^\mu \tau (+r_2^\mu). \]

Here \( u_{1,2}^\mu \) are unit time-like vectors along the directions of the momenta defining the dipole classical trajectories, and moreover \( b^\mu = (0, 0, \vec{b}) \) and \( r_{1,2}^\mu = (0, 0, \vec{R}_{1,2}) \), with \( \vec{R}_i = |\vec{R}_i| \) the quark-antiquark transverse separation. The loop contours are then closed at positive and negative infinite proper-time \( \tau \) in order to ensure gauge-invariance. The scattering amplitude for two onium states can then be reconstructed from the dipole-dipole amplitude after folding with the appropriate wave-functions for the onia. The mass of an onium state is expected to be of the order of the inverse of its average radius, \( \langle R \rangle \). The geometrical parameters of the configuration can be related to the energy scales by the relation \( \cosh \chi \equiv s/2m^2 - 1 \equiv \varsigma - 1 \), which at high energy reads \( \chi \sim \log \varsigma \), where \( \chi \) is the hyperbolic angle (rapidity) between the trajectories of the dipoles, \( Y = \log \varsigma \) the total rapidity and \( m \) the masses of the two onia, taken to be equal for simplicity.

As already mentioned in the Introduction, it is convenient to exploit the Euclidean version of the correspondence, and then to reconstruct the relevant correlation function \( C_M \) from its Euclidean counterpart \( C_E \) by means of analytic continuation [17, 18]. The Euclidean approach has already been employed in the study of high-energy soft scattering amplitudes by means of non-perturbative techniques [3, 4, 21], including numerical lattice calculations [22]. The Euclidean normalized connected correlation function is defined as

\[ C_E(\theta, \vec{b}; \vec{R}_1, \vec{R}_2) = \left[ \frac{\langle W_1 W_2 \rangle}{\langle W_1 \rangle \langle W_2 \rangle} - 1 \right]. \]
where $W_i$ are now Euclidean Wilson loops evaluated along the straight-line paths $W_1 \rightarrow X_1^i = b^i + u_1^i \tau + r_1^i$ and $W_2 \rightarrow X_2^i = u_2^i \tau + r_2^i$, closed at infinite proper time, see Fig. 1. The variables $b$ and $r_i$ are the same defined above in the Minkowskian case (we take Euclidean time to be the first coordinate to keep the notation close to the Minkowskian case). Here $u_1$ and $u_2$ are unit vectors forming an angle $\theta$ in Euclidean space.

The physical correlation function $C_M$ in Minkowski space is obtained by means of the analytic continuation \[ C_M(\chi, \vec{b}; R_1, R_2) = C_E(-i\chi, \vec{b}; R_1, R_2), \quad \chi \in \mathbb{R}^+, \] where the analytic continuation of $C_E$ is performed starting from the interval $\theta \in (0, \pi)$ for the Euclidean angle (the restriction on the range of $\chi$ and $\theta$ does not imply any loss of information, due to the symmetries of the two theories).

III. WILSON LOOP CORRELATORS FROM ADS/CFT

Within the AdS/CFT correspondence, the correlators of Wilson loops in the gauge theory, such as those of Eq. (1), are related to a minimal surface in the bulk of AdS$_5$ having as boundaries the two Wilson loops, corresponding to the minimization of the Nambu-Goto action. When $L \equiv |\vec{b}| \lesssim R_1, R_2$ there exists a connected minimal surface with the sum of the two loops as its disjoint boundary (see e.g. [23]), although its explicit expression is difficult to obtain. However, when $L \gg R_1, R_2$ the solution simplifies, and the minimal surface has two independent, (quasi-)disconnected components: in order to calculate the correlator one then exploits, as in Ref. [24], the explicit solutions corresponding to the two loops connected by the classical supergravity interaction, i.e., by the exchange between them (see Fig. 2-1) of the lightest fields of the AdS$_5$ supergravity, namely the graviton ($G$), the anti-symmetric tensor ($B$), the dilaton ($D$), and the (tachyonic) Kaluza-Klein (KK) scalar mode ($S$). This is the case we have considered in [11], using the large-$L$ behavior of the dipole-dipole impact-parameter amplitude evaluated in [3], with a slight generalization to unequal dipole sizes.

For $L \gg R_1, R_2$, and in the weak gravitational field domain, the Euclidean normalized connected correlation function has the form

\[
C_E = \exp \left( \sum_\psi \delta_\psi \right) - 1, \quad \delta_\psi \equiv \frac{1}{4\pi^2\alpha'^2} \int d\tau_1 d\tau_2 \frac{d\theta_1 d\theta_2}{z_{1x} z_{2x}} \frac{\delta S_{NG}(\tau_1, \theta_1)}{\delta \psi}(\tau_1, \theta_1) \frac{\delta S_{NG}(\tau_2, \theta_2)}{\delta \psi}(\tau_2, \theta_2),
\] (5)

where $S_{NG}$ is the Nambu-Goto action, $\alpha' = 1/\sqrt{4\pi g_s N_c}$ and $g_s$ is the string coupling, related to the gauge theory coupling by $g_s^2 = 2\pi g_s$. Here $\delta S_{NG}/\delta \psi$ is the coupling of the world-sheet minimal surfaces attached to the two Wilson loops to the supergravity field $\psi$. Moreover, $\tau_i$ is the proper time on world-sheet $i = (1, 2)$, and $z_i$ are the fifth coordinates of points $X_i$ in AdS$_5$, namely

\[
X_1 = (u_1^i \tau_1 + x_1^i + b^i, z_1), \quad X_2 = (u_2^i \tau_2 + x_2^i, z_2),
\]

\[
u_1^i = (1, 0, \vec{0}), \quad u_2^i = (\cos \theta, -\sin \theta, \vec{0}), \quad x_1^i = \sigma_i(z_i) \frac{\tau_1^i}{R_i}, \quad \sigma_i \in [0, R_i], \quad i = 1, 2,
\] (6)

where $\sigma_i(z_i)$ is determined by inverting the solution of the minimal surface equation $z_i = z_i(\sigma_i)$. The derivatives $z_{ix} = \partial z_i / \partial \sigma_i$ are given by [12]

\[
z_{ix} = \left( \frac{z_{i\max}}{z_i} \right)^{2/3} \sqrt{1 - \left( \frac{z_i}{z_{i\max}} \right)^4}, \quad z_{i\max} = R_i \left( \frac{\Gamma(1/4)}{2\pi} \right)^{3/2}.
\] (7)

In Eq. (5), $G_\psi(X_1, X_2)$ is the Green function relevant to the exchange of field $\psi$, which depends only on invariant bitensors and scalar functions [25] of the AdS invariant

\[
u = \frac{(z_1 - z_2)^2 + \sum_{j=1}^4 (X_1^j - X_2^j)^2}{2z_1z_2}.
\] (8)

Working out the Green functions and the couplings corresponding to the exchange of the various supergravity fields, it is found that the leading dependence (the “leading” term in $\theta$ is understood as the leading term in $\chi$ after analytic continuation, see below) on $\theta$, $L$ and $R_i$ for the various terms of (5) is the following:

\[
\delta_S = \kappa_S \frac{1}{\sin \theta} \left( \frac{R_1 R_2}{L^2} \right)^2 \equiv a_S \frac{1}{\sin \theta} \left( \frac{R_1 R_2}{L^2} \right)^3 \equiv a_D \left( \frac{R_1 R_2}{L^2} \right)^3 \equiv a_D \frac{1}{\sin \theta},
\]

\[
\delta_D = \kappa_D \left( \frac{R_1 R_2}{L^2} \right)^3 \equiv a_D \left( \frac{R_1 R_2}{L^2} \right)^3 \equiv a_D \frac{1}{\sin \theta},
\]

\[
\delta_B = \kappa_B \cos \theta \left( \frac{R_1 R_2}{L^2} \right)^2 \equiv a_B \cos \theta \left( \frac{R_1 R_2}{L^2} \right)^3 \equiv a_G \left( \frac{R_1 R_2}{L^2} \right)^3 \equiv a_G \frac{(\cos \theta)^2}{\sin \theta},
\]

\[
\delta_G = \kappa_G \left( \frac{R_1 R_2}{L^2} \right)^3 \equiv a_G \left( \frac{R_1 R_2}{L^2} \right)^3 \equiv a_G \frac{(\cos \theta)^2}{\sin \theta}.
\] (9)
factorizing explicitly the angular dependence from the rest. The results of (9) correspond to the case in which \( \vec{b}, \vec{R}_1 \) and \( \vec{R}_2 \) lie along the same direction in the transverse plane. The factors \( \kappa_\psi \) for each supergravity field are numerical factors of order \( O(\lambda/N_c^2) \), with \( \lambda = g_s^2 N_c/4\pi \) the 't Hooft coupling, as one expects from the topology of the configuration. Performing now the analytic continuation \( \theta \to -i\chi \), leading to the phase shifts \( i\delta_\psi \equiv \delta_\psi(\theta \to -i\chi) \), one finally obtains for the Minkowskian correlation function

\[
C_M = \exp \left( i \sum_\psi \delta_\psi \right) - 1, \quad \delta_S = a_S \frac{1}{\sinh \chi}, \quad \delta_D = a_D \frac{1}{\sinh \chi}, \quad \delta_B = a_B \frac{\cosh \chi}{\sinh \chi}, \quad \delta_G = a_G \frac{(\cosh \chi)^2}{\sinh \chi}.
\]

We notice that under crossing, i.e., under \( \chi \to i\pi - \chi \) [18], the phases \( \delta_S, \delta_D \) and \( \delta_G \) are symmetric, while \( \delta_B \) is antisymmetric.

### IV. EIKONAL AMPLITUDE IN IMPACT-PARAMETER SPACE

#### A. The weak field constraint

The range of validity of the results above is determined by requiring [3] that the effect of the gravitational perturbation \( \delta G_{tt} \) generated by each of the string world-sheets on the other one is smaller than the background metric \( G_{tt} \), therefore ensuring that one is actually working in the weak-field limit. Considering the effect of world-sheet 2 on world-sheet 1, the strongest constraint is obtained from the evaluation of the maximal gravitational field produced at the point \( \tau_1 = 0 \), where the distance between the loops is minimal. The weak gravitational field requirement reads

\[
\frac{\delta G_{tt}}{G_{tt}} \ll 1, \quad G_{tt} \equiv \frac{1}{z_1^1},
\]

where \( G_{tt} \) is the background metric term in the Fefferman-Graham parameterization of \( AdS_5 \). In order to find the explicit expression of condition (11), we note that \( \delta_\psi \) in (5) can be also interpreted as an integral over the string world-sheet 1 of the corresponding supergravity field \( \psi(X_2) \) produced by the other, tilted world-sheet 2, namely

\[
\delta_\psi(X_1(\tau_1, z_1)) = \frac{1}{2\pi \alpha'} \int 2d\tau_2 dz_2 \frac{\delta S_{NG}}{\delta \psi}(\tau_2, z_2).
\]

Applying this generic equation to the dominant graviton contribution, one finds the constraint

\[
\frac{\delta G_{tt}}{G_{tt}} \propto \frac{z_1^1 R_1^3}{L^2} (\cos \theta)^2 \ll 1.
\]

This constraint is most restrictive when evaluated at \( z_1 = z_1^{\text{max}} \propto R_1 \), which is as far as the string world-sheet extends into the 5th dimension of \( AdS_5 \). Performing the analytic continuation \( \theta \to -i\chi \) and interchanging the rôles of the two world-sheets by switching the subscripts 1 and 2 in the result above in order to get the maximal constraint, one finally obtains (\( \zeta = s/2m^2 - 1 \))

\[
L^2 \gg L_{\text{max}}^2 = \frac{R_1 R_2 \zeta^2}{\left[ \min \left( \sqrt{\frac{R_1}{R_2}}, \sqrt{\frac{R_2}{R_1}} \right) \right]^2}.
\]

#### B. The elastic eikonal hypothesis

From expression (1) one can determine the impact-parameter partial amplitude \( a(\chi, \vec{b}) \) corresponding to the dipole-dipole elastic amplitude \( A \), i.e., suppressing the dependence on the sizes of the dipoles, \( a(\chi, \vec{b}) = -iC_M(\chi, \vec{b}) \). In the large-\( L \) region the AdS/CFT result gives

\[
a_{\text{tail}}(\chi, \vec{b}) = i \left[ 1 - \exp \left( i \sum_\psi \delta_\psi \right) \right],
\]

with the phase shifts specified by (10). This expression can be trusted as long as the solution for the minimal surface problem is disconnected and, as discussed above, as long as the weak gravitational field constraint (14) is satisfied.
Note that expression (15) corresponds to a purely elastic amplitude, in agreement with the planar limit implied by the AdS/CFT correspondence.

Although the result (14) expresses a stringent constraint on the impact-parameter range due to the weak gravitational field condition required in applying the AdS/CFT correspondence, one can try to extend the results by adopting an \( S \)-matrix point-of-view. Indeed, the exponential form of (15) is typical of a resummation of non-interacting (i.e., independent) colorless exchanges (on the gauge theory side) which can be taken into account in order to possibly enlarge its domain of validity, assuming then the validity of the eikonal approximation for a purely elastic scattering amplitude. From the AdS/CFT correspondence point-of-view, we expect that the gravitational field is strong for an impact-parameter distance below the weak-field limit, so inducing graviton self-interactions which would spoil this independent resummation; nevertheless, for completion, we will suppose that the eikonal formalism for the elastic amplitude may be extended in some larger phase-space region. We will then examine, from the empirical \( S \)-matrix point-of-view, whether and down to which value of the impact-parameter separation the formula (15) could be used beyond the constraint (14), i.e., up to what impact-parameter distance it is mainly the exchange of independent gravitons which builds the whole amplitude.

As a first step beyond our AdS/CFT correspondence result, one could infer from an \( S \)-matrix model formulation that the amplitude (15) is reliable as long as the dominant graviton-induced phase shift \( \delta_G \) is small. Following formulas (10), this means that (at large energy) \[ \frac{L^2}{R_1 R_2} \gg (\kappa_G \varsigma)^{\frac{1}{3}} \]; more precisely,

\[
L^2 > L_{\min}^2 \equiv R_1 R_2 \left( \frac{\kappa_G}{\pi} \varsigma \right)^{\frac{1}{3}}
\]

(16)
in order for the eikonal formula (15) to be physically sensible, requiring the phase shift \( \delta_G \leq \pi \). This extreme minimal bound ensures that \( \text{Im} a_{\text{tail}}(\chi, \vec{b}) \) be not oscillating with \( L \), so yielding a non-oscillating behavior for the \( L \)-dependent partial cross section. Indeed, it is reasonable to expect that more and more inelastic channels would open up when going from the peripheral to the central impact-parameter domain. We see that the request of a weak gravitational field gives a stronger constraint than the one coming from the \( S \)-matrix model point-of-view.

C. Characteristic impact-parameter scales

Let us consider a range of validity of (15) varying from its AdS/CFT determination (14) to its maximal \( S \)-matrix model extension (16). We are lead to define a characteristic distance \( L_{\text{tail}} \) such that for \( L = |\vec{b}| > L_{\text{tail}}(s) \) the impact-parameter scattering amplitude is given by Eq. (15). One can then divide the whole impact-parameter space into a tail region \( (L > L_{\text{tail}}) \), and a core region \( (L < L_{\text{tail}}) \) where inelastic channels are supposed to open up. More specifically, the following regions are identified (see Fig. 2).

1. At large distances \( L > L_{\max} \), whose exact expression is given by (14), the gravitational field in the bulk is weak enough, and the contribution of the disconnected minimal surface gives a rigorous holographic determination of the impact-parameter tail of the scattering amplitude (Fig. 2-1).

2. At moderately large distances \( L_{\min} < L < L_{\max} \), where \( L_{\min} \) has been defined in (16), the strong gravitational field is expected to generate a non zero \( \text{Im} \delta_G \) leading to inelastic contributions on the gauge theory side. The minimal surface is still disconnected but the gravitational field begins to become strong in some relevant region in the bulk. Nevertheless, for the sake of completeness, we will investigate what happens assuming the validity of the elastic eikonal expression up to \( L_{\text{tail}} \), lying somewhere in the range \( L_{\min} \leq L_{\text{tail}} \leq L_{\max} \) (Fig. 2-2).

3. For \( L_{\text{connect}} < L < L_{\min} \) the elastic eikonal expression (15) is no more reliable, even from the \( S \)-matrix point-of-view. An eikonal formula may still be valid with an imaginary contribution to the phase shifts but it cannot be obtained through the weak gravity regime of the AdS/CFT correspondence, even if the minimal surface is still made of disconnected surfaces joined by interacting fields (Fig. 2-3).

4. Finally, for even smaller distances \( L \leq L_{\text{connect}} \) the Gross-Ooguri transition [19] takes place, and the minimal surface solution becomes connected. In this region, the AdS/CFT description goes beyond the interaction mediated by supergravity fields (Fig. 2-4).

Region 1 and possibly part of region 2 constitute the impact-parameter tail region, while regions 3 and 4 constitute the central impact-parameter core region. Since we know precisely \( a(\chi, \vec{b}) \) only in the tail region, we are able to determine only part of the full scattering amplitude, i.e., the large impact-parameter contribution \( A_{\text{tail}} \),

\[
A \equiv A_{\text{core}} + A_{\text{tail}} : \quad A_{\text{tail}}(s, t; \vec{R}_1, \vec{R}_2) = 2is \int_{L \geq L_{\text{tail}}} d^2 \vec{b} \ e^{i \vec{q} \cdot \vec{b}} \left[ 1 - e^{i \sum \delta_\nu} \right] ;
\]

(17)
nevertheless, constraining $A_{\text{core}}$ with the unitarity bound for the impact-parameter amplitude $\text{Im} a(\chi, \vec{b}) \leq 2$, we will be able to set a lower and an upper bound on the large-$s$ behavior of the total cross section.

V. TOTAL CROSS SECTIONS AT HIGH ENERGY IN $\mathcal{N} = 4$ SYM

In order to evaluate the contribution $\sigma_{\text{tail}}$ to the total cross section of the large impact-parameter region, as obtained from AdS/CFT, we need only the imaginary part of the amplitude at $t = 0$, which is related to the dipole-dipole total cross section by means of the optical theorem. We can then ignore the divergence in the real part, due to the KK scalar exchange, discussed in [11]. We have

$$\sigma_{\text{tail}} \sim \frac{\text{Im} A_{\text{tail}}(s, 0; \vec{R}_1, \vec{R}_2)}{s} = 4\pi \int_{L_{\text{tail}}}^{\infty} dL L \left[ 1 - \cos \left( \sum_\psi \delta_\psi \right) \right].$$

The $\chi$-dependence at large energy induces a hierarchy between the different contributions, clearly revealed after performing the change of variables $\lambda \equiv L/(\sinh \chi)^{\frac{1}{2}} \sqrt{R_1 R_2}$, $\lambda_{\text{tail}} \equiv L_{\text{tail}}/(\sinh \chi)^{\frac{1}{2}} \sqrt{R_1 R_2}$, which yields

$$\sigma_{\text{tail}} = 4\pi (\sinh \chi)^{\frac{1}{2}} R_1 R_2 \int_{\lambda_{\text{tail}}}^{\infty} d\lambda \lambda \left[ 1 - \cos \left( \frac{\kappa_S}{\lambda^2} \frac{1}{(\sinh \chi)^2} + \frac{\kappa_D}{\lambda^6} \frac{1}{(\sinh \chi)^2} + \frac{\kappa_B}{\lambda^4} \frac{6}{(\sinh \chi)^2} + \frac{\kappa_G}{\lambda^6} (\coth \chi)^2 \right) \right]$$

(rescaling with $\sinh \chi$ instead of $\cosh \chi$ allows to keep manifest the symmetry under crossing, i.e., under $\chi \rightarrow i\pi - \chi$, of the various phase shifts). Note that the leading term, coming from graviton exchange, is crossing-symmetric, thus corresponding to "Pomeron exchange" in the $S$-matrix language, while the first subleading term, coming from antisymmetric-tensor exchange, is crossing-antisymmetric, thus corresponding to "Odderon exchange". At large energy, retaining only the dominant contribution, we have

$$\sigma_{\text{tail}} \sim 4\pi R_1 R_2 \frac{2}{3} \int_{\lambda_{\text{tail}}}^{\infty} d\lambda \lambda \left[ 1 - \cos \left( \frac{\kappa_G}{\lambda^6} \right) \right] = \frac{2\pi}{3} R_1 R_2 \left( \frac{\varsigma}{\mu_{\text{tail}}} \right)^\frac{1}{3} \int_0^1 dx x^{-\frac{4}{3}} \left[ 1 - \cos \left( \kappa_G \mu_{\text{tail}} x \right) \right],$$

where we have set $\mu_{\text{tail}} = \lambda_{\text{tail}}^{-6}$, $x = \frac{\varsigma}{\lambda_{\text{tail}}^6}$. To complete the calculation of the high-energy behavior of $\sigma_{\text{tail}}$ we need to know the limit of validity of the eikonal expression, and thus how $L_{\text{tail}}$ depends on $s$. Let us consider the parameterization

$$L_{\text{tail}} = \lambda_0 \sqrt{R_1 R_2} \varsigma^\beta \Rightarrow \lambda_{\text{tail}} = \lambda_0 \varsigma^{\frac{\beta - 6}{6}}, \quad \mu_{\text{tail}} = \lambda_0^{-6} s^{1 - 6\beta},$$

where $\lambda_0$ may have some residual dependence on $R_{1,2}$ (see e.g. (14)). According to the value of $\beta$ we have for large $s$

$$\sigma_{\text{tail}} \sim \frac{2\pi}{3} R_1 R_2 \left\{ \begin{array}{ll} \frac{3\varsigma^\frac{1}{2} \kappa_G^\frac{1}{2}}{\Gamma(1/3)} & \beta < \frac{1}{6} \\ \varsigma^\frac{1}{3} \chi^2 \int_0^1 dx x^{-\frac{4}{3}} \left[ 1 - \cos \left( \kappa_G \lambda_0^{-6} x \right) \right] & \beta = \frac{1}{6} \\ \varsigma^{2-10\beta} \frac{1}{2} \kappa_G^2 \lambda_0^{-10} & \beta > \frac{1}{6} \end{array} \right.$$
FIG. 3. Upper and lower bounds on the high-energy behavior of total cross sections. The bounds on the exponent $\gamma$ of the total cross section $\sigma_{\text{tot}} \propto \varsigma^\gamma$, i.e., on the Pomeron intercept minus one, are displayed as a function of the power law exponent $\beta$ of $L_{\text{tail}} \propto \varsigma^\beta$. Solid line: upper bound, coming from the core contribution for $\frac{1}{6} \leq \beta \leq \frac{7}{6}$. Long-dashed line: lower bound, coming from the tail contribution for $\beta \geq \frac{7}{6}$. Short-dashed line: weaker upper bound for $\beta > \frac{7}{6}$, obtained by overestimating the core contribution (see text).

We are now in the position to determine a lower and an upper bound on the high-energy behavior of the dipole-dipole total cross section. Since obviously $\sigma_{\text{tot}} > \sigma_{\text{tail}}$, Eq. (22) provides a lower bound. The overall unitarity constraint allows one to put an upper bound on the contribution from the core region $L < L_{\text{tail}}$, i.e., $\sigma_{\text{core}} \leq 4\pi L_{\text{tail}}^2 = 4\pi \lambda_0^2 R_1 R_2 \varsigma^{2\beta}$, and thus on the whole total cross section. The bounds can be written in a compact way as

$$\min \left( \frac{1}{3}, 2 - 10\beta \right) \leq \lim_{\varsigma \to \infty} \frac{\log \sigma_{\text{tot}}}{\log \varsigma} \leq \max \left( \frac{1}{3}, 2\beta \right),$$

and they are shown in Fig. 3. In particular, using the value $\beta = \frac{7}{6}$ coming from the weak field constraint (14), one obtains the rigorous bound $-6/7 \leq \lim_{\varsigma \to \infty} \log \sigma_{\text{tot}}/\log \varsigma \leq 4/7$. The following remarks are in order.

1. For $\beta < \frac{1}{6}$, at sufficiently high energy one would have $L_{\text{tail}} < L_{\text{min}}$, thus entering the unphysical region where the impact-parameter partial amplitude is infinitely oscillating; moreover, the total cross section would become purely elastic at high energy, while one expects the opening of more and more inelastic channels as the energy increases: this means that one lies beyond the applicability of the elastic eikonal approximation.

2. At $\beta = \frac{1}{6}$, corresponding to $L_{\text{tail}}/L_{\text{min}} = \text{const.}$, the tail and core contributions have the same high-energy behavior. In this case $\lambda_{\text{tail}}$ does not depend on energy. However, one has to verify the non-oscillating behavior condition $\lambda_{\text{tail}} \geq (k_G/\pi)^{\frac{3}{2}}$. In this case the bounds transform into a prediction, $\sigma_{\text{tot}} \sim \varsigma^{\frac{1}{3}}$ (see also [9]).

3. For $\frac{1}{6} < \beta \leq \frac{7}{6}$, which corresponds to $L_{\text{min}} < L_{\text{tail}} < L_{\text{max}}$ (strictly speaking, at sufficiently high energy), the core region dominates, while the tail region gives a subleading contribution as $s \to \infty$. The two bounds determine a window of possible power-law behaviors.

4. For the maximal value $\beta = \frac{7}{6}$, i.e., for $L_{\text{tail}}/L_{\text{max}} = \text{const.}$, the total cross section behavior is constrained to be such that $\sigma_{\text{tot}} \leq \text{const.} \times \varsigma^{\frac{1}{3}}$. This maximal value is determined from the requirement that the AdS/CFT correspondence can be reliably applied, expressed through the constraint (14). In fact, this is the rigorous result obtained by means of the AdS/CFT correspondence, since for smaller $\beta$ one expects inelastic contributions coming from a strong dual gravitational field. Note that it restricts the total cross section to be below the bare graviton exchange contribution, namely $\sigma_{\text{tot}} \sim \varsigma^{\frac{1}{3}}$.

5. One could also consider $\beta > \frac{7}{6}$, but in that case one would only obtain a weaker bound on the total cross section. Indeed, in doing so one would overestimate the contribution of the core, including in it the impact-parameter region $L_{\text{max}} < L < L_{\text{tail}}$, where the amplitude is reliably described by the eikonal AdS/CFT expression.

The absolute bound we obtain is linked to the precise derivation of a weak gravitational field limitation of the AdS/CFT correspondence in the supergravity formulation. Our result appears as the analogue of the Froissart bound, but in the context of the non confining $\mathcal{N} = 4$ SYM theory, since it is the combination of the unitarity bound on impact-parameter
amplitudes with the determination of a precise power-like bound on the impact-parameter radius from AdS/CFT. A more stringent bound would be obtained if one assumed the validity of the elastic eikonal approximation in a region with strong gravitational field in the bulk; however, in this region other contributions are expected to modify the gravitational sector.

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