

Optimal Auction Design for Flexible Consumers

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Abstract—The problem of designing Bayesian incentive compatible, individually rational, revenue maximizing auction for multiple goods and flexible customers is considered. The auctioneer has M goods and N potential customers. Customer i has a flexibility set $\phi_i$, which represents the set of goods the customer is equally interested in. Customer i can consume at most one good from its flexibility set. We first characterize the optimal auction for customers with arbitrary flexibility sets and then consider the case when the flexibility sets are nested. This allows us to group customers into classes of increasing flexibility. We show that the optimal auction can be simplified in this case and we provide a complete characterization of allocations and payments in terms of simple thresholds.

I. INTRODUCTION

The problem of allocating limited resources among strategic users with private information is often addressed through the framework of auctions/mechanism design. An auctioneer/mechanism designer would typically ask for bids from potential customers and allocate resources and charge payments as a function of the received bids. The customers with private information about their utilities can be strategic about what bids they submit. The auction design problem is to find allocation and payment functions which map the customers’ bids to allocations and payments in order to achieve some objective. Typically, the auctioneer’s objectives are either maximization of its revenue or maximization of social welfare.

In this paper, we consider the problem of designing revenue-maximizing auctions for multiple goods and flexible customers. Customer flexibility about goods can arise in different scenarios. In demand response programs in power systems, some customers may be flexible about when they receive power. In airline/hotel reservation settings, customers may be flexible about their travel dates. The seller of these goods/services should be able to take into account this flexibility to improve its profits. We consider a setting where the auctioneer has M goods and N potential customers. Customer i has a flexibility set $\phi_i$, which represents the set of goods the customer is equally interested in. Customer i can consume at most one good from its flexibility set. Our goal is to design a Bayesian incentive compatible, individually rational and revenue-maximizing auction for this setting. We first characterize such an auction for customers with arbitrary flexibility sets and then consider the case when the flexibility sets are nested. This allows us to group customers into classes of increasing flexibility. We show that the optimal auction can be simplified in this case and we provide a complete characterization of allocations and payments in terms of simple thresholds.

The problem of allocating multiple goods to several customers with special preferences over the set of offered items has long been studied in the context of combinatorial auctions [1, Chapter 11]. Numerous works have addressed social welfare maximizing or efficient auction design, the most well-known of these being the Vickrey-Clarke-Groves (VCG) mechanism [2, 3, 4]. Several works have been done to address instances of the revenue-maximizing auction design problem where certain assumptions about users’ private information and their utility functional forms are imposed to make the problem more tractable. In his seminal 1981 paper [5], Myerson derives fundamental results for the Bayesian revenue maximizing single-object auction problem. Armstrong [6] extended the analysis to the case where two objects are to be allocated. Hartline et al [7], survey various solution approaches for addressing revenue-optimal auctions in cases where users private information is unidimensional. Avery et al. [8] studied products bundling effects in multi-object auctions under linearity assumptions for utility functions.

Some recent works have considered modeling complementarities among the goods in multi-object auctions through imposing the assumption of having single-minded buyers who are interested in certain subsets of items or bundles for which they are willing to pay. Ledyard [9] characterizes a revenue maximizing dominant strategy auction for single-minded buyers. Abhishek and Hajek [10] treat the same problem of revenue optimal auction design for single-minded bidders with the users’ preferred bundle being their private information.

This paper is organized as follows: in section [II] we discuss problem formulation and the mechanism setup. In section [III] we characterize incentive compatibility and individual rationality constraints for the mechanism and show that the optimal allocations are the solution to an integer program. In section [IV] we consider the case of nested flexibility sets and simplify optimal allocation and payments for this setting. In section [V] we summarize our findings and briefly point out potential extensions to the current framework.

A. Notations

$\{0, 1\}^{N \times M}$ is the space of $N \times M$ dimensional matrices with entries that are either 0 or 1. For the set $\mathcal{A}$, $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$. $\mathbb{Z}^+$ is the set of positive integers. $x^+$ is the positive part of the real number $x$, that is, $x^+ =$
max(x, 0). Vector inequalities are component-wise; that is, for two \(1 \times n\) dimensional vectors \(u = (u_1, \ldots, u_n)\) and \(v = (v_1, \ldots, v_n)\), \(u \leq v\) implies that \(u_i \leq v_i\), for \(i = 1, \ldots, n\). \(\mathbb{I}_{\{a \leq b\}}\) is the indicator function and \(\mathbb{I}_{\{a \leq b\}}\) denotes 1 if the inequality in the subscript is true and 0 otherwise.

II. PROBLEM FORMULATION

We consider a setup where an auctioneer has \(M\) goods and \(N\) potential customers. \(\mathcal{M} = \{1, 2, \ldots, M\}\) denotes the set of goods and \(\mathcal{N} = \{1, 2, \ldots, N\}\) denotes the set of potential customers. Customer \(i, i \in \mathcal{N}\), has a flexibility set \(\phi_i \subseteq \mathcal{M}\) which represents the set of goods the customer is equally interested in. Customer \(i\) can consume at most one good from its flexibility set \(\phi_i\). We assume the flexibility sets of all consumers are common knowledge.

Customer \(i\)'s utility for receiving a good from \(\phi_i\) is \(\theta_i\). We assume that \(\theta_i\) is customer \(i\)'s private information and is unknown to other users as well as the auctioneer. We assume that \(\theta_i\)'s are independent random variables each distributed over the set \(\Theta_i = [\theta_i^{min}, \theta_i^{max}]\) according to some probability density function \(f_i(.)\) with its corresponding distribution function \(F_i(.)\). We define \(\Theta := (\Theta_1, \Theta_2, \ldots, \Theta_N)\) as the customers' valuations profile. Let \(\Theta := \prod_{i=1}^{N} \Theta_i\). We assume \(\Theta_i, f_i, F_i\) and \(\phi_i, i \in \mathcal{N}\), are common knowledge.

Let us define a \(N \times M\) dimensional matrix \(A\) with the entries \(a_{i,j} = 1\) if customer \(i\) gets good \(j\) and \(a_{i,j} = 0\) otherwise

\[
a_{i,j} = \begin{cases} 1 & \text{if customer } i \text{ gets good } j \\ 0 & \text{otherwise} \end{cases}
\]

The matrix \(A\) is called an allocation matrix. We impose the natural constraint that each of the \(M\) available goods can be allocated to at most one customer and that each customer receives at most one good. This implies that \(\sum_{i=1}^{N} A(i,j) \leq 1, \forall j\) and \(\sum_{j=1}^{M} A(i,j) \leq 1, \forall i\).

A binary matrix \(A \in \{0,1\}^{N \times M}\) that satisfies these two constraints is called a feasible allocation matrix. Let \(\mathcal{S} \subseteq \{0,1\}^{N \times M}\) denote the set of all feasible allocation matrices. That is,

\[
\mathcal{S} := \left\{ A \in \{0,1\}^{N \times M} \mid \sum_{i=1}^{N} A(i,j) \leq 1, \forall j \in \mathcal{M}, \sum_{j=1}^{M} A(i,j) \leq 1, \forall i \in \mathcal{N} \right\}
\]

Given an allocation matrix \(A\) and a payment \(t_i\) charged to customer \(i\), the net utility for this customer is

\[
u_i(\theta_i, A, t_i) = \theta_i \left( \sum_{j \in \phi_i} A(i,j) \right) - t_i.
\]

We consider a direct revelation mechanism where customer \(i, i \in \mathcal{N}\), reports a valuation from the set \(\Theta_i\) to the auctioneer. The customers can lie about their valuations. The mechanism consists of an allocation rule \(q\) and a payment rule \(t\). The allocation rule \(q\) is a function from the valuation profile space \(\Theta\) to the set of feasible allocation matrices \(\mathcal{S}\). The payment rule is a mapping from \(\Theta\) to \(\mathbb{R}^N\) with the \(n\)th component \(t_i\) being the payment for customer \(i\).

The auctioneer’s objective is to find a mechanism that maximizes its expected revenue while satisfying Bayesian Incentive Compatibility and Individual Rationality constraints. We describe these constraints below.

Consider a mechanism \((q, t)\) and suppose customers report valuations \(r := (r_1, \ldots, r_N)\). The mechanism then results in an allocation matrix \(q(r)\) and payments \(t(r)\). Define \(\psi_i(r)\) as:

\[
\psi_i(r) = \sum_{j \in \phi_i} q_{ij}(r)
\]

Customer \(i\)'s utility function can then be written in terms of its true valuation \(\theta_i\) and the reported valuations \(r\) as:

\[
u_i(\theta_i, r) = \theta_i \psi_i(r) - t_i(r).
\]

In a Bayesian incentive compatible (BIC) mechanism, truthful reporting of valuations constitutes an equilibrium of the Bayesian game induced by the mechanism. In other words, each customer would prefer to report his true valuation provided that all other customers have adopted truth-telling strategy. This implies that the mechanism must satisfy the following inequalities:

\[
\mathbb{E}_{\theta_i}[\theta_i \psi_i(\theta_i) - t_i(\theta_i)] \geq \mathbb{E}_{\theta_i}[\theta_i \psi_i(r_i, \theta_i) - t_i(r_i, \theta_i)]
\]
\[
\forall \theta_i, r_i \in \Theta_i, \forall i \in \mathcal{N}.
\]

Individual Rationality (IR) constraint implies that the customer’s expected utility at the truthful reporting equilibrium is non-negative. This can be expressed as:

\[
\mathbb{E}_{\theta_i}[\theta_i \psi_i(\theta_i) - t_i(\theta_i)] \geq 0, \forall \theta_i \in \Theta_i, \forall i \in \mathcal{N}.
\]

The auction design problem can now be written as

\[
\max_{(q,t)} \mathbb{E}_{\Theta} \left\{ \sum_{i=1}^{N} t_i(\theta_i) \right\}
\]

subject to (6) and (7).

III. CHARACTERIZATION OF BIC AND IR MECHANISMS

Suppose all customers other than \(i\) report their valuations truthfully. We can then define customer \(i\)'s expected allocation and payment under the mechanism \((q, t)\) when it reports \(r_i \in \Theta_i\) as:

\[
Q_i(r_i) := \mathbb{E}_{\theta_i}[\psi_i(r_i, \theta_i)],
\]

\[
T_i(r_i) := \mathbb{E}_{\theta_i}[t_i(r_i, \theta_i)].
\]

Using the definitions given in equations (9)-(10), we can rewrite equations (6) and (7) as:

\[
\theta_i Q_i(\theta_i) - T_i(\theta_i) \geq \theta_i Q_i(r_i) - T_i(r_i), \forall \theta_i, r_i \in \Theta_i,
\]

\[1\]* Customers may not report their valuations truthfully, so \(r_i\) may be different from \(\theta_i\).
\[ \theta_i Q_i(\theta_i) - T_i(\theta_i) \geq 0, \quad \forall \theta_i \in \Theta_i \]  

Using this characterization of customers’ expected utility in terms of \( Q_i(\cdot) \) and \( T_i(\cdot) \), we can now derive a necessary and sufficient condition for a direct-revelation mechanism to be Bayesian incentive compatible and individually rational.

**Theorem 1:** A mechanism \((q, t)\) satisfies Bayesian incentive compatibility and individual rationality constraints if and only if \( Q_i(r_i) \) is non-decreasing in \( r_i \) and
\[
T_i(r_i) = K_i + r_i Q_i(r_i) - \int_{\theta_i^{\text{min}}}^{\theta_i^{\text{max}}} Q_i(s) \, ds,
\]
where,
\[
K_i = T_i(\theta_i^{\text{min}}) - \theta_i^{\text{min}} Q_i(\theta_i^{\text{min}}) \leq 0. \tag{14}
\]

**Proof:** The proof is similar to the arguments in chapters 2-3 of [11] for characterizing BIC and IR mechanisms. \(\blacksquare\)

We can now use the result of Theorem 1 to simplify the optimization problem in equation (8). Let us rewrite auctioneer’s expected revenue as
\[
\mathbb{E}_{\theta_i} \left\{ \sum_{i=1}^{N} t_i(\theta) \right\} = \sum_{i=1}^{N} \mathbb{E}_{\theta_i} \left[ \mathbb{E}_{\theta_{-i}} \left[ t_i(\theta_i, \theta_{-i}) \right] \right] 
\]
\[
= \sum_{i=1}^{N} \mathbb{E}_{\theta_i} \left[ T_i(\theta_i) \right] \tag{15}
\]
For a BIC and IR mechanism, we can use the results in Theorem 1 to write \( \mathbb{E}_{\theta_i} \left[ T_i(\theta_i) \right] \) as:
\[
\mathbb{E}_{\theta_i} \left[ T_i(\theta_i) \right] = \mathbb{E}_{\theta_i} \left[ K_i + \theta_i Q_i(\theta_i) - \int_{\theta_i^{\text{min}}}^{\theta_i^{\text{max}}} Q_i(s) \, ds \right] 
\]
\[
= K_i + \int_{\theta_i^{\text{min}}}^{\theta_i^{\text{max}}} \left[ \theta_i Q_i(\theta_i) - \int_{\theta_i^{\text{min}}}^{\theta_i^{\text{max}}} Q_i(s) \, ds \right] f_i(\theta_i) \, d\theta_i \tag{16}
\]
We substitute for \( Q_i(\cdot) \) from its definition in equation (9) in terms of \( \psi_i(\cdot) \) and after some simplifications we get:
\[
\mathbb{E}_{\theta_i} \left[ T_i(\theta_i) \right] = K_i + \int_{\theta_i^{\text{min}}}^{\theta_i^{\text{max}}} \left[ \psi_i(\theta_i) \left( \theta_i - 1 - F_i(\theta_i) \right) \right] f(\theta) \, d\theta. \tag{17}
\]
We denote it by \( w_i(\theta_i) \).

The term \( \left( \theta_i - 1 - F_i(\theta_i) \right) \) is referred to as the customer’s virtual type/virtual valuation in economics terminology and we denote it by \( w_i(\theta_i) \).

We can now rewrite the auctioneer’s total expected revenue in equation (13) as:
\[
\sum_{i=1}^{N} \mathbb{E}_{\theta_i} \left[ T_i(\theta_i) \right] = \sum_{i=1}^{N} K_i + \int_{\theta_i^{\text{min}}}^{\theta_i^{\text{max}}} \sum_{i=1}^{N} [w_i(\theta_i) \psi_i(\theta_i)] f(\theta) \, d\theta. \tag{18}
\]
Recall from (4) that \( \psi_i(\theta) = \sum_{j \in \phi_i} q_{ij}(\theta) \). Therefore, the integral in (18) is completely determined by the choice of the allocation function \( q(\cdot) \). Also, recall that \( K_i \leq 0 \) (equation (14)). Therefore, a mechanism \((q, t)\) that maximizes the integral in (13) and ensures \( K_i = 0 \) would provide the largest expected revenue. In order to simplify the maximization, we assume that virtual types \( \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \) are increasing in \( \theta_i \).

Such a condition holds if \( f_i(\theta) \) is increasing in \( \theta \), which is to say that \( f_i \) has increasing hazard rate \( ^2 \).

The following theorem characterizes the optimal mechanism under the increasing hazard rate condition.

**Theorem 2:** Consider the allocation and tax functions \((q^*, t^*)\) defined below
\[
q^*(\theta) = \arg\max_{s \in S} \sum_{i=1}^{N} w_i(\theta_i) \sum_{j \in \phi_i} x_{ij} \tag{19}
\]
Define
\[
\psi^*_i(\theta) := \sum_{j \in \phi_i} q_{ij}^* \tag{20}
\]
and,
\[
t^*_i(\theta) := \theta_i \psi^*_i(\theta) - \int_{\theta_i^{\text{min}}}^{\theta_i^{\text{max}}} \psi^*_i(s, \theta_{-i}) \, ds \tag{21}
\]
Then, under the increasing hazard rate condition, \((q^*, t^*)\) is the revenue-maximizing Bayesian incentive compatible and individually rational mechanism, i.e., \((q^*, t^*)\) is a solution of the maximization problem in (8).

**Proof:** See Appendix II \(\blacksquare\)

The optimal allocation matrix \( q^*(\theta) \) given in (19) is the solution of an integer program and hence computationally hard to obtain. Further, each valuation profile \( \theta \in \Theta \) requires the solution of a different integer program. Similarly, the characterization of payments given by (21) is not very useful from a computational viewpoint. In the next section, we will impose a structure on the flexibility sets \( \phi_i \) of the customers and show how it can be leveraged to simplify the optimal mechanism.

**IV. OPTIMAL MECHANISM FOR NESTED FLEXIBILITY SETS**

We assume now that the flexibility sets of customers can be one of \( k \) nested sets. That is, we have \( k \) nested subsets of the set of goods:
\[
B_1 \subset B_2 \subset \cdots \subset B_k \subseteq \mathcal{M}, \tag{22}
\]
and \( \phi_i \in \{B_1, B_2, \ldots, B_k\} \) for every \( i \in \mathcal{N} \).

Based on their flexibility sets, we can divide the customers into \( k \) classes: \( C_l \) is the set of customers with flexibility set \( B_l \). Clearly, \( \mathcal{N} = \bigcup_{i=1}^{k} C_i \) and \( C_i \cap C_j = \emptyset \) \( \forall i \neq j \). We define
\[
n_l := |C_l|, \quad l = 1, \ldots, k, \quad m_l := |B_l|, \quad l = 2, \ldots, k, \quad m_1 := |B_1|, \tag{23}
\]
\(^2\text{Uniform distribution satisfies this condition.}\)
We also define the vectors \( \mathbf{n} \) and \( \mathbf{m} \) as
\[
\mathbf{n} := (n_1, n_2, \ldots, n_k), \quad \mathbf{m} := (m_1, m_2, \ldots, m_k).
\] (24)
The vector \( \mathbf{n} \) is referred to as the demand profile and the vector \( \mathbf{m} \) is referred to as the supply profile.

A. Supply Adequacy Problem

Before describing the optimal mechanism for the above setup, we will need to answer two questions:

1) Given a supply profile \( \mathbf{m} \) and a demand profile \( \mathbf{n} \), can the available goods be used to satisfy all customers? In other words, does there exist an allocation matrix \( \mathbf{A} \in \{0, 1\}^{N \times M} \) such that
\[
\sum_{j \in \phi_i} \mathbf{A}(i, j) = 1, \quad \forall i \in N
\] (25)
\[
\sum_{i=1}^{N} \mathbf{A}(i, j) \leq 1, \quad j = 1, 2, \ldots, M
\] (26)
The above conditions on \( \mathbf{A} \) ensure that each customer gets a good from its flexibility set and that a good is not allocated to multiple customers. If such an allocation matrix exists, we say that the supply profile \( \mathbf{m} \) is adequate for the demand profile \( \mathbf{n} \).

2) If the supply profile \( \mathbf{m} \) is not adequate for the demand profile \( \mathbf{n} \), what is the minimum number of customers that must be removed to achieve adequacy?

The following lemma answers the first question.

**Lemma 1:** We say that \( \mathbf{n} \prec_w \mathbf{m} \) if the following \( k \) inequalities hold:
\[
\sum_{i=1}^{l} n_i \leq \sum_{i=1}^{l} m_i, \quad l = 1, 2, \ldots, k.
\] (27)
The supply profile \( \mathbf{m} \) is adequate for the demand profile \( \mathbf{n} \) if and only if \( \mathbf{n} \prec_w \mathbf{m} \).

**Proof:** See Appendix III.

If the supply profile \( \mathbf{m} \) is not adequate, we may have to remove some customers from the demand profile. Consider a demand profile \( \tilde{\mathbf{n}} \leq \mathbf{n} \) obtained by removing some customers. This new demand profile will result in adequacy if and only if \( \tilde{\mathbf{n}} \prec_w \mathbf{m} \). The minimum number of customers to be removed to achieve adequacy can be written as the following optimization problem:
\[
\min_{\tilde{\mathbf{n}}} \sum_{i=1}^{k} (n_i - \tilde{n}_i)
\] subject to \( \tilde{\mathbf{n}} \prec_w \mathbf{m} \), \( \tilde{\mathbf{n}} \leq \mathbf{n} \)
(28)

The above integer program has a simple solution described in the following lemma.

**Lemma 2:** Define \( r_1^+ := (n_1 - m_1)^+ \). For \( 2 \leq j \leq k \), recursively define \( r_j^+ \) as the solution of the following one-dimensional integer program:
\[
r_j^+ = \arg\min_{r_j \in \mathbb{Z}^+} r_j
\] subject to
\[
\sum_{i=1}^{j-1} (n_i - r_i^+) + (n_j - r_j) \leq \sum_{i=1}^{j} m_i
\] (29)
Then, \( \sum_{i=1}^{k} r_i^+ \) is the minimum value of the integer program in (27).

**Proof:** See Appendix III.

B. Allocation Rule

We can now use the results of section IV-A to find the optimal allocation for a given valuation profile \( \theta \). Recall from Theorem 2 that the optimal allocation is given as
\[
q^*(\theta) = \max_{x \in \mathcal{S}} \sum_{n=1}^{N} w_i(\theta_i) \sum_{j \in \phi_i} x_{ij}.
\]

**Optimal Allocation:** We describe this in several steps:

1) Firstly, any customer \( l \) with \( w_i(\theta_i) \leq 0 \) is immediately removed from consideration (that is, it is not allocated any good). Since virtual valuation is an increasing function of true valuation, \( w_i(\theta_i) \leq 0 \) if and only if \( \theta_i \leq \theta_i^{res} \), where \( \theta_i^{res} \) is a threshold based on the probability distribution of \( \theta_i \). This threshold is called the reserve price for a customer.

For each class of customers, we define the subset of users with positive virtual valuations:
\[
C_i^+ := \{ l \in C_i : w_i(\theta_i) > 0 \}.
\] (29)
Let \( n_i^+ = |C_i^+| \). Define \( r_1^*, \ldots, r_k^* \) as in Lemma 2 by replacing \( n_i \) with \( n_i^+ \).

2) Let \( L_1 := C_1^+ \). From \( L_1 \), \( r_1^+ \) customers with the lowest valuations are removed from consideration. The set of remaining customers in \( L_1 \) is denoted by \( N_1 \).

3) We proceed iteratively: For \( 2 \leq i \leq k \), given the set \( N_i-1 \), define \( L_i := N_i-1 \cup C_i^+ \). Remove \( r_i^+ \) customers with lowest valuations from \( L_i \). The set of remaining customers in \( L_i \) is now defined as \( N_i \).

4) After the \( k^{th} \) iteration, all customers in \( N_k \) are allocated a good from their respective flexibility sets.

The optimality of the above allocation can be intuitively explained as follows: Firstly, it is clear that an optimal allocation should not give good to customers with non-positive virtual valuations. Among the remaining customers of class \( C_1 \), at least \( r_1^+ \) customers cannot be served. It is easy to see that the \( r_1^+ \) customers with the lowest valuations should be removed. This argument can be used iteratively. At the \( i^{th} \) iteration, at least \( r_i^+ \) customers need to be removed otherwise the \( i^{th} \) adequacy inequality would be violated. It is easy to

\[\text{Ties are resolved randomly. For continuous valuations, ties happen with zero probability and therefore allocation rule for ties does not affect expected revenue.}\]
see that an optimal allocation should remove customers with lowest valuations.

The optimal allocation can also be described using $k$ thresholds. Define

$$\theta_{i_{hr}} := (r_i^*)^{th} \text{ lowest valuation in } L_i, \ i = 1, 2, \ldots, k.$$  

(30)

Therefore, at iteration $i$, customers that have valuations less than or equal to $\theta_{i_{hr}}$ defined in (30) will be removed from the set $L_i$. The optimal allocation rule for the consumers of flexibility class $C_i$ can then be characterized in terms of the threshold valuations defined in (30). Recall that $\psi_i^*(\theta) = \sum_{j \in \phi_i} q_{ij}^*(\theta)$ is 1 if customer $l$ gets a good from its flexibility set and 0 otherwise. Under the optimal allocation, customer $l$ in class $C_i$ gets a desired good if its valuation exceeds its reserve price $\theta_{i_{rs}}$ as well as the the thresholds $\theta_{i_{hr}}, \theta_{i_{hr+1}}, \ldots, \theta_{i_{hk}}$. Thus, we have

$$\psi_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i > \max\{\theta_{i_{rs}}, \theta_{i_{hr}}, \theta_{i_{hr+1}}, \ldots, \theta_{i_{hk}}\} \\ 0 & \text{otherwise} \end{cases} \quad \forall l \in C_i, \ i = 1, 2, \ldots, k.$$  

(31)

### C. Tax Functions

We can now use the optimal allocation rule described in section 4.2 to simplify users’ tax functions. For customer $l$ in class $C_i$ define $d_{il} := \max\{\theta_{i_{rs}}, \theta_{i_{hr}}, \theta_{i_{hr+1}}, \ldots, \theta_{i_{hk}}\}$. From Equation (21) the optimal tax function for customer $l$ in flexibility class $C_i$ has the following form:

$$t_i(\theta) = \theta_i \psi_i^*(\theta) - \int_{\theta_{i_{rs}}}^{\theta_{i_{hr}}} \psi_i^*(s, \theta_{i-1}) \, ds$$  

(32)

Using the definition of $\psi_i^*(\theta)$ in equation (31), $t_i(\theta)$ can be simplified as:

1) If $\theta_i > d_{il}$,

$$t_i(\theta) = \theta_i - \int_{\theta_{i_{rs}}}^{\theta_{i_{hr}}} \psi_i^*(s, \theta_{i-1}) \, ds - \int_{d_{il}}^{\theta_i} \psi_i^*(s, \theta_{i-1}) \, ds$$  

$$= d_{il}.$$  

(33)

2) If $\theta_i \leq d_{il}$, $t_i(\theta) = 0$.

If the reserve prices are the same for all customers, then the tax function can be simplified as follows: For a customer in class $C_i$ define $d_i := \max\{\theta_{i_{rs}}, \theta_{i_{hr}}, \theta_{i_{hr+1}}, \ldots, \theta_{i_{hk}}\}$. The tax function for customer $l$ in class $C_i$ simplifies to:

$$t_i(\theta) = d_i 1_{\{\theta_i > d_i\}}.$$  

It is evident in this case that more flexible customers pay less for the good than less flexible customers.

### V. Conclusion

We studied the problem of designing expected revenue maximizing auctions for allocating multiple items to flexible customers. Assuming customers’ flexibility sets to be common knowledge, we characterized the allocation rule for an incentive compatible, individually rational and revenue-maximizing mechanism as the solution to an integer program. The corresponding payment rule was described by an integral equation. We then considered the case when the flexibility sets are nested. This allows us to group customers into classes of increasing flexibility. We showed that the optimal auction can be simplified in this case and we provide a complete characterization of allocations and payments in terms of simple thresholds.

The model we considered needs to be further extended to analyze the more general problem where customers’ flexibility sets are part of their private information. In this case, customers have two-dimensional types (that is, valuation and flexibility set) which can be strategically misreported in multiple ways. This complicates the design of an incentive compatible mechanism. Another future direction is to allow for online mechanisms where the set of customers and/or goods can change over time.

### APPENDIX I

#### PROOF OF THEOREM 2

We first establish that the mechanism $(q^*, t^*)$ is Bayesian incentive compatible and individually rational. Because of Theorem 1 it suffices to show that customer $i$’s expected payment on reporting $r_i$, $T_i(r_i)$ satisfies (13) and the expected allocation, $Q_i^*(r_i)$ is non-decreasing in $r_i$.

By taking the expectation over $\theta_{i-1}$ in (21), it is easily established that the expected payment satisfies (13) with $K_i = 0$.

In order to establish monotonicity of $Q_i^*(\cdot)$, we will argue that $\psi_i^*(\theta_i, \theta_{i-1})$ is non-decreasing in $\theta_i$ for all $\theta_{i-1} \in \Theta_{i-1}$. Consider two candidate valuations $a, b \in \Theta_i$ with $a < b$. Let the two allocation matrices $(x^a_1^T, x^a_2^T, \ldots, x^a_N^T)^T$ and $(x^b_1^T, x^b_2^T, \ldots, x^b_N^T)^T$ be the solutions to the optimization problem in (19) corresponding to the valuation profiles $(a, \theta_{i-1})$ and $(b, \theta_{i-1})$ respectively. From (20), we have

$$\psi_i^*(a, \theta_{i-1}) = \sum_{j \in \phi_i} x^a_{ij} = \sum_{j \in \phi_i} x^b_{ij}.$$  

(34)

For the valuation profile $(a, \theta_{i-1})$, the maximum value of the objective function in (19) is $\psi_i^*(a, \theta_{i-1}) w_i(a) + \sum_{j \neq i} \psi_j^*(a, \theta_{i-1}) w_j(\theta_j)$. Therefore, we must have

$$\psi_i^*(a, \theta_{i-1}) w_i(a) + \sum_{j \neq i} \psi_j^*(a, \theta_{i-1}) w_j(\theta_j) \geq \psi_i^*(b, \theta_{i-1}) w_i(a) + \sum_{j \neq i} \psi_j^*(b, \theta_{i-1}) w_j(\theta_j).$$  

(35)

Similarly, when the valuation profile is $(b, \theta_{i-1})$, the maximum value of the objective function in (19) is $\psi_i^*(b, \theta_{i-1}) w_i(b) + \sum_{j \neq i} \psi_j^*(b, \theta_{i-1}) w_j(\theta_j)$. Therefore, we must have

$$\psi_i^*(b, \theta_{i-1}) w_i(b) + \sum_{j \neq i} \psi_j^*(b, \theta_{i-1}) w_j(\theta_j) \geq \psi_i^*(a, \theta_{i-1}) w_i(b) + \sum_{j \neq i} \psi_j^*(a, \theta_{i-1}) w_j(\theta_j).$$  

(36)
Now, adding the two sides of inequalities (35)-(36) gives
\[ \psi^*_i (b, \theta_{-i}) (w_i(b) - w_i(a)) \geq \psi^*_i (a, \theta_{-i}) (w_i(b) - w_i(a)) \] (37)

Since \( w_i(\theta_i) \) is increasing in \( \theta_i \), \( w_i(b) - w_i(a) > 0 \); therefore, inequality (37) implies \( \psi^*_i (b, \theta_{-i}) \geq \psi^*_i (a, \theta_{-i}) \). This establishes that \( \psi^*_i (\theta_i, \theta_{-i}) \) is non-decreasing in \( \theta_i \) and hence \( Q^*_i(\cdot) \) is a non-decreasing function.

It is straightforward to see that users allocation rule \( q^*(\theta) \) which is defined in equation (19) as the maximizer of the weighted sum \( \sum_{i=1}^N w_i(\theta_i) \psi_i(\theta) \), will naturally maximize the integral term in equation (18) and thus, it maximizes the auctioner's total expected revenue. Hence, the mechanism \((q^*, t^*)\) is a solution of the maximization problem in (3).

**APPENDIX II**

**PROOF OF LEMMA 1**

**Necessity:**
From the adequacy condition in equation (25) we have:
\[ \sum_{g \in \phi_i} A(i, g) = 1, \quad i \in \mathcal{N} \] (38)

Taking the summation of both sides over the set of customers in the union of first \( J \) classes, we get:
\[ \sum_{i \in \bigcup_{j=1}^J C_j} \sum_{g \in \phi_i} A(i, g) = \sum_{i \in \bigcup_{j=1}^J C_j} 1 = \sum_{i=1}^J n_i \] (39)

The left hand side of (39) can be written as
\[ \sum_{i \in \bigcup_{j=1}^J C_j} \sum_{g \in \phi_i} A(i, g) \leq \sum_{i \in \bigcup_{j=1}^J C_j} \sum_{g \in \mathcal{B}_j} A(i, g) = \sum_{g \in \mathcal{B}_j} \sum_{i \in \bigcup_{j=1}^J C_j} A(i, g) \leq \sum_{g \in \mathcal{B}_j} \sum_{i \in \mathcal{N}} A(i, g) \leq \sum_{g \in \mathcal{B}_j} 1 = \sum_{i=1}^J m_i \] (40)

(39) and (40) imply that:
\[ \sum_{i=1}^J n_i \leq \sum_{i=1}^J m_i, \quad J = 1, 2, \cdots, k, \] (41)

which proves the necessity part of the lemma.

**Sufficiency:**
We can enumerate the items in the sets \( \mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_k \) as:
\[ \mathcal{B}_i = \left\{ 1, 2, \cdots, \sum_{j=1}^i m_j \right\}, \quad i = 1, 2, \cdots, k. \] (42)

We can enumerate the customers in classes \( \mathcal{C}_i, i = 1, 2, \cdots, k \) as:
\[ \mathcal{C}_1 = \left\{ 1, 2, \cdots, n_1 \right\} \]
\[ \mathcal{C}_i = \left\{ 1 + \sum_{j=1}^{i-1} n_j, \cdots, 1 + \sum_{j=1}^i n_j \right\}, \quad i = 2, 3, \cdots, k. \] (43)

Consider an allocation where the \( j \)th customer (as per the above enumeration) gets the \( j \)th good (as per the above enumeration). Thus,
\[ A(i, i) = 1, \quad \forall i \in \mathcal{N} \]
\[ A(i, j) = 0, \quad \text{for } j \neq i \] (44)

Since \( \sum_{i=1}^J n_i \leq \sum_{i=1}^J m_i, \quad \forall l = 1, 2, \cdots, k \), given the enumerations in equations (42)-(43), one can verify that customer \( j \) will always get something from her flexibility set \( \phi_j \). Therefore, given the inequalities in (25), an allocation matrix can be found that satisfies the conditions in (27), which is to say that the supply profile \( m \) is adequate for the demand profile \( n \).

**APPENDIX III**

**PROOF OF LEMMA 2**

Consider any feasible solution of the optimization problem in (27) denoted as \( (\tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_k) \). We will now show inductively that:
\[ \sum_{j=1}^i (n_j - \tilde{n}_j) \geq \sum_{j=1}^i r_j^*, \quad \forall i = 1, 2, \cdots, k \] (45)

For \( i = 1 \) we have:
\[ n_1 - \tilde{n}_1 \geq n_1 - \min\{n_1, m_1\} \implies \tilde{n}_1 \leq \min\{n_1, m_1\} \] (46)

From this we can write:
\[ n_1 - \tilde{n}_1 \geq n_1 - \min\{n_1, m_1\} = (n_1 - m_1) + \] (47)

Now suppose the inequality in (45) holds for \( i \). We now want to prove they also hold for \( i + 1 \). Let us consider two cases based on the possible values of \( r^*_i+1 \): 1) \( r^*_i+1 = 0 \) and 2) \( r^*_i+1 > 0 \). When \( r^*_i+1 = 0 \), from the induction hypothesis for \( i \) in (45) we can write:
\[ \sum_{j=1}^{i+1} (n_j - \tilde{n}_j) \geq \sum_{j=1}^{i+1} r^*_j. \] (48)

Now consider the case when \( r^*_i+1 > 0 \). In this case, from the optimization constraint in (28) it can be verified that \( r^*_i+1 = n_{i+1} + \sum_{j=1}^{i+1} (n_j - r^*_j) - \sum_{j=1}^{i+1} m_j \); hence:
\[ \sum_{j=1}^{i+1} (n_j - r^*_j) = \sum_{j=1}^{i+1} m_j \iff \sum_{j=1}^{i+1} r^*_j = \sum_{j=1}^{i+1} (n_j - m_j). \] (49)

From the optimization constraints in (27) we can write:
\[ \sum_{j=1}^{i+1} \tilde{n}_j \leq \sum_{j=1}^{i+1} m_j \] (50)

Combining (49) and (50) we get:
\[ \sum_{j=1}^{i+1} (n_j - \tilde{n}_j) \geq \sum_{j=1}^{i+1} (n_j - m_j) = \sum_{j=1}^{i+1} r^*_j \] (51)
Thus the inequality in (45) holds for $i + 1$ as well. Therefore by induction we can conclude that:

$$\sum_{j=1}^{k}(n_j - \tilde{n}_j) \geq \sum_{j=1}^{k} r^*_j. \tag{52}$$

To show that the lower bound above is achievable, consider the following procedure:

1) Let $L_1 := C_1$. From $L_1$, $r^*_1$ customers are removed. The set of remaining customers in $L_1$ is denoted by $N_1$.

2) Proceed iteratively: For $2 \leq i \leq k$, given the set $N_{i-1}$, define $L_i := N_{i-1} \cup C_i$. Remove $r^*_i$ customers from $L_i$.

The set of remaining customers in $L_i$ is now defined as $N_i$.

It can be verified that the above procedure removes exactly

$$\sum_{j=1}^{k} r^*_j$$

customers and creates a supply profile $\tilde{n}$ that meets adequacy condition.

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