Mean-Field Limits: From Particle Descriptions to Macroscopic Equations

JOSÉ A. CARRILLO & YOUNG-PIL CHOI

Communicated by J. Bedrossian

Abstract

We rigorously derive pressureless Euler-type equations with nonlocal dissipative terms in velocity and aggregation equations with nonlocal velocity fields from Newton-type particle descriptions of swarming models with alignment interactions. Crucially, we make use of a discrete version of a modulated kinetic energy together with the bounded Lipschitz distance for measures in order to control terms in its time derivative due to the nonlocal interactions.

1. Introduction

In this work, we analyse the evolution of an indistinguishable $N$-point particle system given by

$$\begin{align*}
\dot{x}_i &= v_i, \quad i = 1, \ldots, N, \quad t > 0, \\
\varepsilon_N \dot{v}_i &= -\gamma v_i - \nabla_x V(x_i) - \frac{1}{N} \sum_{j=1}^N \nabla_x W(x_i - x_j) + \frac{1}{N} \sum_{j=1}^N \psi(x_i - x_j)(v_j - v_i)
\end{align*}$$

subject to the initial data

$$(x_i, v_i)(0) =: (x_i(0), v_i(0)), \quad i = 1, \ldots, N. \quad (1.2)$$

Here $x_i = x_i(t) \in \mathbb{R}^d$ and $v_i = v_i(t) \in \mathbb{R}^d$ denote the position and velocity of $i$-particle at time $t$, respectively. The coefficient $\gamma \geq 0$ represents the strength of linear damping in velocity, $\varepsilon_N > 0$ the strength of inertia, $V : \mathbb{R}^d \to \mathbb{R}_+$ and $W : \mathbb{R}^d \to \mathbb{R}$ stand for the confinement and interaction potentials, respectively. $\psi : \mathbb{R}^d \to \mathbb{R}_+$ is a communication weight function. Throughout this paper, we assume that $W$ and $\psi$ satisfy $W(x) = W(-x)$ and $\psi(x) = \psi(-x)$ for $x \in \mathbb{R}^d$. They include basic
particle models for collective behaviors, see [12,20,25,34,36,46,47,63] and the references therein.

Our main goal is to derive the macroscopic collective models rigorously governing the evolution of the particle system (1.1) as the number of particles goes to infinity. On one hand, we will derive hydrodynamic Euler-alignment models given by

\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \nabla_x \cdot (\rho u) &= 0, \\
\frac{\partial}{\partial t} (\rho u) + \nabla_x \cdot (\rho u \otimes u) &= -\gamma \rho u - \rho \nabla_x V - \rho \nabla_x W \ast \rho \\
&\quad + \rho \int_{\mathbb{R}^d} \psi(x-y)(u(y) - u(x)) \rho(y) \, dy
\end{align*}
\]  

(1.3)

in the mean-field limit: when initial particles are close to a monokinetic distribution \( \rho_0(x) \delta_{u_0(x)}(v) \) in certain sense and \( \varepsilon N = O(1) \) as \( N \to \infty \). On the other hand, we will show that the particle system can be described by aggregation equations of the form

\[
\frac{\partial}{\partial t} \bar{\rho} + \nabla_x \cdot (\bar{\rho} \bar{u}) = 0,
\]

(1.4)

where

\[
\gamma \bar{\rho} \bar{u} = -\bar{\rho} \nabla_x V - \bar{\rho} \nabla_x W \ast \bar{\rho} + \bar{\rho} \int_{\mathbb{R}^d} \psi(x-y)(\bar{u}(y) - \bar{u}(x)) \bar{\rho}(y) \, dy
\]

(1.5)

in the combined mean-field/small inertia limit when initial particles are close to a monokinetic distribution \( \rho_0(x) \delta_{u_0(x)}(v) \), \( \gamma > 0 \) and \( \varepsilon N \to 0 \) as \( N \to \infty \). For simplicity of notations when dealing with the mean-field limit, we will take \( \varepsilon N = 1 \) in the sequel.

1.1. Mean-field limits: from particles to continuum

As the number of particles \( N \) tends to infinity, microscopic descriptions given by the particle system (1.1) become more and more computationally unbearable. Reducing the complexity of the system is of paramount importance in any practical application. The classical multiscale strategy in kinetic modelling is to introduce the number density function \( f = f(x,v,t) \) in phase space \((x,v) \in \mathbb{R}^d \times \mathbb{R}^d\) at time \( t \in \mathbb{R}_+ \) and study the time evolution of that density function. Then at the formal level, we can derive the following Vlasov-type equation from the particle system (1.1) as \( N \to \infty \):

\[
\begin{align*}
\frac{\partial}{\partial t} f + v \cdot \nabla_x f - \nabla_v \cdot \left( (\gamma v + \nabla_x V + \nabla_x W \ast \rho_f) f \right) + \nabla_v \cdot (F_a(f) f) &= 0,
\end{align*}
\]

(1.6)

where \( \rho_f = \rho_f(x,t) \) is the local particle density and \( F_a(f) = F_a(f)(x,v,t) \) represents a nonlocal velocity alignment force given by

\[
\rho_f(x,t) := \int_{\mathbb{R}^d} f(x,v,t) \, dv
\]
and

\[ F_a(f)(x, v, t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x - y)(w - v) f(y, w, t) \, dy \, dw, \]

respectively. Let us briefly recall the reader the basic formalism leading to the kinetic equation (1.6) as the limiting system of (1.1). We first define the empirical measure \( \mu^N \) associated to a solution to the particle system (1.1), that is,

\[ \mu^N_t(x, v) := \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i(t), v_i(t))}. \]

As long as there exists a solution to (1.1), the empirical measure \( \mu^N \) satisfies (1.6) in the sense of distributions. To be more specific, for any \( \varphi \in C^1_0(\mathbb{R}^d \times \mathbb{R}^d) \), we get

\[ \frac{d}{dt} \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x, v) \mu^N_t(\text{d}x \, \text{d}v) \right] = \frac{d}{dt} \frac{1}{N} \sum_{i=1}^{N} \varphi(x_i(t), v_i(t)) \]

(1.7)

Notice that the particle velocity can also be rewritten in terms of the empirical measure \( \mu^N \) as

\[ \dot{v}_i(t) = -\gamma v_i - \nabla_x V(x_i) - \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x W(x_i - y) \mu^N_t(\text{d}y \, \text{d}w) \]

\[ + \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x_i - y)(w - v_i) \mu^N_t(\text{d}y \, \text{d}w). \]

This implies that the right-hand side of (1.7) can also be written in terms of the empirical measure \( \mu^N \) as

\[ \frac{d}{dt} \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x, v) \mu^N_t(\text{d}x \, \text{d}v) \right] = \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x \varphi(x, v) \mu^N_t(\text{d}x \, \text{d}v) \]

\[ - \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \varphi(x, v) \cdot \left( \gamma v + \nabla_x V(x) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x W(x - y) \mu^N_t(\text{d}y \, \text{d}w) \right) \mu^N_t(\text{d}x \, \text{d}v) \]

\[ + \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \varphi(x, v) \cdot \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x - y)(w - v) \mu^N_t(\text{d}y \, \text{d}w) \right) \mu^N_t(\text{d}x \, \text{d}v). \]

This concludes that \( \mu^N \) is a solution to (1.6) in the sense of distributions as long as particle paths are well defined. In fact, if the interaction potential \( W \) and the communication weight function \( \psi \) in the classical Cucker–Smale alignment model are regular enough, for instance, bounded Lipschitz regularity, then the global-in-time existence of measure-valued solutions can be obtained by establishing a weak-weak stability estimate for the empirical measure, see [46, Section 5] for more details. The mean-field limit has attracted lots of attention in the
last years in different settings depending on the regularity of the involved potentials \( V, W \) and communication function \( \psi \). Different approaches to the derivation of the Vlasov-like kinetic equations with alignments/interaction terms or the aggregation equations have been taken leading to a very lively interaction between different communities of researchers in analysis and probability. We refer to [3,4,10,20,30,31,35,44,47,50,54–56,64,67] for the classical references and non-Lipschitz regularity velocity fields in kinetic cases, to [48,49] for very related incompressible fluid problems, and to [7,9,16,17,37,43,45,51,52,61,63,65,66] for results with more emphasis on the singular interaction kernels both at the kinetic and the aggregation-diffusion equation cases.

1.2. Local balanced laws, the mono-kinetic ansatz, and the small inertia limit

The classical procedure in kinetic theory of deriving equations for the first 3 moments of the distribution function \( f \) leads to the standard problem of how to close the moment system since the equation for the second moment will depend on higher order moments. Suitable closure assumptions are not known so far even in cases where noise/diffusion is added to the system. However, at the formal level, we can take into account the mono-kinetic ansatz for \( f \), as done in [18,21], leading to

\[
f(x, v, t) \simeq \rho(x, t) \delta_u(x, t)(v),
\]

where \( \rho \) and \( u \) are the macroscopic density and the mean velocity of particles, that is, the first two moments of \( f \) in velocity variable

\[
\rho := \int_{\mathbb{R}^d} f \, dv \quad \text{and} \quad \rho u := \int_{\mathbb{R}^d} v f \, dv.
\]

It is standard to check that the strain tensor and heat flux become zero and the moment system closes becoming the pressureless Euler equations with nonlocal interaction forces (1.3):

\[
\begin{align*}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+ , \\
\partial_t u + u \cdot \nabla_x u &= -\gamma u - \nabla_x V - \nabla_x W \ast \rho + \int_{\mathbb{R}^d} \psi(x - y)(u(y) - u(x)) \rho(y) \, dy,
\end{align*}
\]

and

\[
\begin{align*}
\partial_t \frac{|u|^2}{2} + u \cdot \nabla_x \frac{|u|^2}{2} &= -\gamma |u|^2 - u \cdot \nabla_x V - u \cdot \nabla_x W \ast \rho \\
&+ \int_{\mathbb{R}^d} \psi(x - y) \left( u(x) \cdot u(y) - |u(x)|^2 \right) \rho(y) \, dy.
\end{align*}
\]

on the support of \( \rho \). The last equation coming from the closed equation on the evolution of the second moment is redundant but it gives a nice information about the total energy of the system. Although the monokinetic assumption is not fully rigorously justified and it does not have a direct physical motivation, it is observed by particle simulations that the derived hydrodynamic system shares some qualitative behavior with the particle system, see [12,18,20–22,33]. Note that (1.3) conserves only the
total mass in time in this generality. However, the total free energy is dissipated due
to the linear damping and the velocity alignment force as pointed out in [19] for
weak solutions of this system. The hydrodynamic system (1.9) has a rich variety of
phenomena compared to the plain pressureless Euler system. This fact is due to the
competition between attraction/repulsion and alignment leading to sharp thresholds
for the global existence of strong solutions versus finite time blow-up and decay to
equilibrium, see [13–15,26,63,68]. We emphasize that the additional alignment,
linear damping and attraction/repulsion terms can promote the existence of global
solutions depending on the initial data. We will show that these hydrodynamical
solutions can be obtained directly from particle descriptions as long as they exist,
so their physical relevance is dictated by the time of existence of these solutions.

It is worth noticing as in [18] that the mono-kinetic ansatz for $f$ is a measure-
valued solution of the kinetic equation (1.6). More precisely, one can show that
$\rho(x,t)\delta_{u(x,t)}(v)$ is a solution to the kinetic equation (1.6) in the sense of distri-
butions as long as $(\rho,u)(x,t)$ is a strong solution to the hydrodynamic equations
(1.3). Indeed, for any $\varphi \in C^1_0(\mathbb{R}^d \times \mathbb{R}^d)$, we obtain

$$
\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x,v) \rho(x,t) \delta_{u(x,t)}(dv) \, dx
= \frac{d}{dt} \int_{\mathbb{R}^d} \varphi(x,u(x,t)) \rho(x,t) \, dx
= \int_{\mathbb{R}^d} \varphi(x,u(x,t)) \partial_t \rho \, dx
+ \int_{\mathbb{R}^d} (\nabla_v \varphi)(x,u(x,t)) \cdot (\partial_t u) \rho \, dx
=: I_1 + I_2.
$$

Using the continuity equation in (1.3), $I_1$ can be easily rewritten as

$$
I_1 = \int_{\mathbb{R}^d} \nabla_x (\varphi(x,u(x,t))) \cdot (\rho u) \, dx
= \int_{\mathbb{R}^d \times \mathbb{R}^d} (\nabla_v \varphi)(x,v) \cdot (\rho v) \delta_{u(x,t)}(dv) \, dx
+ \int_{\mathbb{R}^d} (\nabla_v \varphi)(x,u(x,t)) \cdot \rho (u \cdot \nabla_x) u \, dx.
$$

By multiplying the velocity equation in (1.3) by $\rho$ and using $(\nabla_v \varphi)(x,u(x,t))$ as
a test function to the resulting equation yields

$$
I_2 = -\int_{\mathbb{R}^d} (\nabla_v \varphi)(x,u(x,t)) \cdot (\partial_t u) \rho \, dx
$$

$$
-\int_{\mathbb{R}^d} (\nabla_v \varphi)(x,u(x,t)) \cdot (\gamma u + \nabla_x V + \nabla_x W \ast \rho) \rho \, dx
$$

$$
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} (\nabla_v \varphi)(x,u(x,t)) \cdot (u(y) - u(x)) \psi(x-y) \rho(x) \rho(y) \, dx \, dy.
$$
Then similarly as before, we can rewrite the second and third terms on the right hand side of the equality by using the mono-kinetic ansatz (1.8). This implies

\[
I_2 = -\int_{\mathbb{R}^d} (\nabla v \varphi)(x, u(x, t)) \cdot (\partial_t u) \rho \, dx \\
- \int_{\mathbb{R}^d \times \mathbb{R}^d} (\nabla v \varphi)(x, v) \cdot (\gamma v + \nabla x V + \nabla x W \star \rho) \, \rho \delta_{u(x,t)}(dv) \, dx \\
+ \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} (\nabla v \varphi)(x, v) \cdot (w - v) \psi(x - y) \rho(x) \delta_{u(x,y)}(dv) \rho(y) \delta_{u(y,t)}(dw) \, dx \, dy.
\]

Combining all of the above estimates yields

\[
\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x, v) \rho(x, t) \delta_{u(x,t)}(dv) \, dx = \int_{\mathbb{R}^d \times \mathbb{R}^d} ((\nabla x \varphi)(x, v) \cdot v) \rho \delta_{u(x,t)}(dv) \, dx \\
- \int_{\mathbb{R}^d \times \mathbb{R}^d} (\nabla v \varphi)(x, v) \cdot (\gamma v + \nabla x V + \nabla x W \star \rho) \, \rho \delta_{u(x,t)}(dv) \, dx \\
+ \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} (\nabla v \varphi)(x, v) \cdot (w - v) \psi(x - y) \rho(x) \delta_{u(x,y)}(dv) \rho(y) \delta_{u(y,t)}(dw) \, dx \, dy.
\]

This shows that \( \rho(x, t) \delta_{u(x,t)}(v) \) satisfies the kinetic equation (1.6) in the sense of distributions.

Finally, we will be also dealing with the small inertia limit for both the kinetic equation (1.6) and the hydrodynamic system (1.3) combined with the mean field limit. In the small inertia asymptotic limit, we want to describe the behavior of the scaled kinetic equation

\[
\varepsilon(\partial_t f + v \cdot \nabla_x f) - \nabla v \cdot ((\gamma v + \nabla x V + \nabla x W \star \rho f) f) + \nabla v \cdot (F_a(f) f) = 0,
\]

(1.10)

and the scaled hydrodynamic system

\[
\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
\varepsilon(\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u)) = -\gamma \rho u - \rho \nabla x V - \rho \nabla x W \star \rho + \rho \int_{\mathbb{R}^d} \psi(x - y) (u(y) - u(x)) \rho(y) \, dy,
\]

(1.11)

in the limit of small inertia \( \varepsilon \to 0 \). At the formal level, the equations (1.11) will be replaced by (1.4)–(1.5) as \( \varepsilon \to 0 \). The limiting nonlinearly coupled aggregation equations (1.4)–(1.5) have been recently studied in [39,40]. Several authors have studied particular choices of interactions \( V, W \) and communication functions \( \psi \) for some of the connecting asymptotic limits from the kinetic description (1.10) with/without noise to the hydrodynamic system (1.11) in [8,11,42,57], from the hydrodynamic system (1.11) to the aggregation equation (1.4)–(1.5) in [23,59,60], and for the direct limit from the kinetic equation to the aggregation equation (1.4)–(1.5) in [8,53].
1.3. Purpose, mathematical tools and main novelties

Summarizing the main facts of the mean-field limit and the monokinetic ansatz in Sections 1.1 and 1.2, both the empirical measure $\mu^N(t)$ associated to the particle system (1.1) and the monokinetic solutions $\rho(x, t)\delta_{u(x,t)}(v)$, with $(\rho, u)(x, t)$ satisfying the hydrodynamic equations (1.3) in the strong sense, are distributional solutions of the same kinetic equation (1.6). In order to analyse the convergence of the empirical measure $\mu^N$ to $\rho(x, t)\delta_{u(x,t)}(v)$, the goal is to establish a weak-strong stability estimate where the strong role is played by the distributional solution $\rho(x, t)\delta_{u(x,t)}(v)$ associated to the strong solution of the hydrodynamic system (1.3). Our main goal is then to quantify the following convergence

$$\mu^N_t(x,v) \to \rho(x,t)\delta_{u(x,t)}(v) \quad \text{as} \quad N \to \infty$$

in the sense of distributions for both the mean-field and the combined mean-field/small inertia limit for well prepared initial data. Our main mathematical tools are the use of a modulated kinetic energy combined with the bounded Lipschitz distance in order to control terms between the discrete particle system and the hydrodynamic quantities. Let us first introduce the modulated kinetic energy as

$$\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} f \left| v - u \right|^2 \, dx \, dv, \quad (1.12)$$

where $f$ is a solution of kinetic equation (1.6) and $u$ is the velocity field as part of the solution of the pressureless Euler equations (1.3). Modulated kinetic energies were used in conjunction with relative potential energy terms for quasineutral limits of Vlasov like equations [5,6,62] for instance. We would like to emphasize that the quantity (1.12) gives a sharper estimate compared to the classical modulated macroscopic energy. Indeed, the macro energy of the system (1.3) is given by

$$E(U) := \frac{|m|^2}{2\rho} \quad \text{with} \quad U := \left(\frac{\rho}{m}\right), \quad m = \rho u.$$

Thus its modulated energy, also often refereed to as relative energy, can be defined as

$$E(U_f|U) := E(U_f) - E(U) - DE(U)(U_f - U) \quad \text{with} \quad U_f := \left(\frac{\rho_f}{m_f}\right), \quad m_f = \rho_f u_f.$$

A straightforward computation gives

$$\int_{\mathbb{R}^d} E(U_f|U) \, dx = \frac{1}{2} \int_{\mathbb{R}^d} \rho_f \left| u_f - u \right|^2 \, dx. \quad (1.13)$$

On the other hand, by Hölder inequality, we easily find that

$$\rho_f \left| u_f \right|^2 \leq \int_{\mathbb{R}^d} |v|^2 f \, dv.$$
This yields
\[
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} f |v - u|^2 \, dx \, dv = \int \int_{\mathbb{R}^d} \rho_f |u_f - u|^2 \, dx \\geq 0.
\]
In fact, we can easily show that
\[
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} f |v - u|^2 \, dx \, dv = \int \mathbb{R}^d \rho_f |u_f|^2 \, dx + \int \int_{\mathbb{R}^d \times \mathbb{R}^d} f |v - u_f|^2 \, dx \, dv.
\]
(1.14)
This shows that the convergence of the modulated kinetic energy (1.12) implies the convergence of the modulated macro energy (1.13). We notice that if \( f \) is a monokinetic distribution, \( f(x, v, t) = \rho_f(x, t) \delta_{u_f(x, t)}(v) \), then the second term on the right hand side of (1.14) becomes zero, and the two modulated energies (1.12) and (1.13) coincide. For notational simplicity, we denote by \( Z^N(t) = \{ (x_i(t), v_i(t)) \}_{i=1}^N \) the set of trajectories associated to the particle system (1.1). Then let us define the first important quantity that will allow us to quantify the distance between particles (1.1) and hydrodynamics (1.3), it is just the discrete version of the modulated kinetic energy (1.12) defined as
\[
E^N(Z^N(t) | U(t)) := \frac{1}{2} \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |u - v|^2 \, \mu^N_t (dx \, dv)
\]
\[
= \frac{1}{2N} \sum_{i=1}^N |u(x_i(t), t) - v_i(t)|^2.
\]
(1.15)
The second quantity that will allow us our quantification goal combined with the discrete modulated energy (1.15) is a classical distance between probability measures, the bounded Lipschitz distance, used already by the pioneers in kinetic theory [4,64,67] in the early works for the mean-field limit. Notice that the pressureless Euler system (1.3) includes the nonlocal position and velocity interaction and alignment forces. Furthermore, its relative energy/entropy has no strict convexity in terms of density variable due to the lack of pressure term. In order to overcome these difficulties, ideas of combining the modulated macro energy and the first or second order Wasserstein distance have been recently proposed in [8,11,32] quantifying the hydrodynamic limit from kinetic equation to the pressureless Euler type system. More recently, in [24], a general theory providing some relation between a modulated macro energy-type function and \( p \)-Wasserstein distance is also developed. In particular, in [24, Proposition 3.1], it is discussed that the \( p \)-Wasserstein distance with \( p \in [1,2] \) can be controlled by the modulated macro energy functional.
In the present work, we will employ the bounded Lipschitz distance to provide stability estimates between the empirical particle density \( \rho^N \) defined as
\[
\rho^N_t(x) := \int_{\mathbb{R}^d} \mu^N_t (dv) = \frac{1}{N} \sum_{j=1}^N \delta_{x_j(t)}(x)
\]
with $\mu^N$ be the empirical measure associated to the particle system (1.1), and the hydrodynamic particle density $\rho$ solution to (1.3). More precisely, let $\mathcal{M}(\mathbb{R}^d)$ be the space of signed Radon measures on $\mathbb{R}^d$, which can be considered as nonnegative bounded linear functionals on $C_0(\mathbb{R}^d)$. Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ be two Radon measures. Then the bounded Lipschitz distance, which is denoted by $d_{BL} : \mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d) \rightarrow \mathbb{R}_+$, between $\mu$ and $\nu$ is defined by

$$d_{BL}(\mu, \nu) := \sup_{\phi \in \Omega} \left| \int_{\mathbb{R}^d} \phi(x)(\mu(dx) - \nu(dx)) \right|,$$

where the admissible set $\Omega$ of test functions are given by

$$\Omega := \left\{ \phi : \mathbb{R}^d \rightarrow \mathbb{R} : \|\phi\|_{L^\infty} \leq 1, \ Lip(\phi) := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|} \leq 1 \right\}.$$

We also denote by $Lip(\mathbb{R}^d)$ the set of Lipschitz functions on $\mathbb{R}^d$. In Proposition 2.2 below, we provide a relation between the bounded Lipschitz distance and the discrete version of the modulated kinetic energy (1.15). This key observation allows us to overcome the difficulties mentioned above.

### 1.4. Main results and Plan of the paper

We will first assume that the particle system (1.1), the pressureless Euler-type equations (1.3), and the aggregation equations (1.4)–(1.5) have existence of smooth enough solutions up to a fixed time $T > 0$. We postpone further discussion at the end of this subsection, although we make precise now the assumptions needed on these solutions for our main results.

Our first main result shows the rigorous passage from Newton’s equation (1.1) to pressureless Euler equations (1.3) via the mean-field limit as $N \rightarrow \infty$.

**Theorem 1.1.** Let $T > 0$, $Z^N(t) = \{(x_i(t), v_i(t))\}_{i=1}^N$ be a solution to the particle system (1.1), and let $(\rho, u)$ be the unique classical solution of the pressureless Euler system with nonlocal interaction forces (1.3) satisfying $\rho > 0$ on $\mathbb{R}^d \times [0, T)$, $\rho \in C([0, T]; \mathcal{P}(\mathbb{R}^d))$ and $u \in L^\infty(0, T; \mathcal{W}^{1, \infty}(\mathbb{R}^d))$ up to time $T > 0$ with initial data $(\rho_0, u_0)$. Suppose that the interaction potential $W$ and the communication weight function $\psi$ satisfy $\nabla_x W \in \mathcal{W}^{1, \infty}(\mathbb{R}^d)$ and $\psi \in \mathcal{W}^{1, \infty}(\mathbb{R}^d)$, respectively. If the initial data for (1.1) and (1.3) are chosen such that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v - u_0(x)|^2 \mu_0^N(dx dv) + d_{BL}^2(\rho_0^N, \rho_0) \rightarrow 0 \text{ as } N \rightarrow \infty,$$

then we have

$$\int_{\mathbb{R}^d} v \mu^N(dv) = \frac{1}{N} \sum_{i=1}^N v_i \delta_{x_i} \rightarrow \rho u \text{ weakly in } L^\infty(0, T; \mathcal{M}(\mathbb{R}^d)),$$

$$\int_{\mathbb{R}^d} (v \otimes v) \mu^N(dv) = \frac{1}{N} \sum_{i=1}^N (v_i \otimes v_i) \delta_{x_i} \rightarrow \rho u \otimes u \text{ weakly in } L^\infty(0, T; \mathcal{M}(\mathbb{R}^d)),$$

and

$$\mu^N \rightarrow \rho \delta_u \text{ weakly in } L^\infty(0, T; \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)),$$

where $\delta_u$ is the Dirac measure centered at $u.$
as \( N \to \infty \). In fact, we have the following quantitative bound estimate:

\[
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - u(x, t)|^2 \mu_t^N (dx dv) + d_{BL}^2 (\rho_t^N (\cdot), \rho (\cdot, t)) \leq C \left( \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - u_0(x)|^2 \mu_0^N (dx dv) + d_{BL}^2 (\rho_0^N , \rho_0) \right),
\]

where \( C > 0 \) only depends on \( \|u\|_{L^\infty (0, T; \mathcal{W}^{1, \infty})} , \|\psi\|_{\mathcal{W}^{1, \infty}}, \|\nabla W\|_{\mathcal{W}^{1, \infty}}, \) and \( T \).

The main novelty of this first result resides in how to control the alignment terms via the modulated energy combined with the bounded Lipschitz distance.

**Remark 1.1.** (Singular repulsive interaction) The previous result also applies to singular repulsive interaction potentials. In particular, it holds for the Coulomb interaction potential on \( \mathbb{R}^d \) given by

\[
N(x) = \begin{cases} 
-\frac{|x|}{2} & \text{for } d = 1, \\
-\frac{1}{2\pi} \log |x| & \text{for } d = 2, \\
\frac{1}{d(d-2)\alpha_d} \frac{1}{|x|^{d-2}} & \text{for } d \geq 3,
\end{cases}
\]

and for Riesz potentials in a sense to be specified in Section 2.3. Here \( \alpha_d \) denotes the volume of the unit ball in \( \mathbb{R}^d \). In order to deal with the singularity on the interaction potential, the diagonal term should be eliminated in the modulated energy functional. This has been recently solved in the recent breakthrough result in [66] by introducing a different relative potential energy avoiding the diagonal terms. The details for singular interaction potentials cases are postponed to Section 2.3, see Theorem 2.1.

Section 2 is devoted to the proof of Theorem 1.1 and the generalization to singular repulsive potentials using [66] in its last subsection.

Our second main result is devoted to the asymptotic analysis for the particle system (1.1) under the small inertia regime: \( \varepsilon_N \to 0 \) as \( N \to \infty \). By Theorem 1.1, we expect that for sufficiently large \( N \gg 1 \), the system (1.1) in the mean-field/small inertia limit can be well approximated by

\[
\partial_t \bar{\rho} + \nabla_x \cdot (\bar{\rho} \bar{u}) = 0, \\
\varepsilon_N \partial_t (\bar{u} \bar{u}) + \varepsilon_N \nabla_x \cdot (\bar{\rho} \bar{u} \otimes \bar{u}) \\
= -\gamma \bar{\rho} \bar{u} - \bar{\rho} \nabla_x V - \bar{\rho} \nabla_x W \star \bar{\rho} + \bar{\rho} \int_{\mathbb{R}^d} \psi(x - y)(\bar{u}(y) - \bar{u}(x)) \bar{\rho}(y) dy.
\]

At the formal level, since \( \varepsilon_N \to 0 \) as \( N \to \infty \), it follows from the momentum equations in the above system that the hydrodynamic system (1.3) should be replaced by (1.4)–(1.5) as \( N \to \infty \). In order to apply our strategy above, we rewrite
the equations (1.4)–(1.5) as

\[
\begin{align*}
\partial_t \tilde{\rho} + \nabla_x \cdot (\tilde{\rho} \tilde{u}) &= 0, \\
\varepsilon_N \partial_t (\tilde{\rho} \tilde{u}) + \varepsilon_N \nabla_x \cdot (\tilde{\rho} \tilde{u} \otimes \tilde{u}) &= -\gamma \tilde{\rho} \tilde{u} - \tilde{\rho} \nabla_x W - \tilde{\rho} \nabla_x W \star \tilde{\rho} \\
&+ \tilde{\rho} \int_{\mathbb{R}^d} \psi(x-y)(\tilde{u}(y) - \tilde{u}(x)) \tilde{\rho}(y) \, dy + \varepsilon_N \tilde{\rho} \tilde{\epsilon},
\end{align*}
\]

(1.16)

where \(\tilde{\epsilon} := \partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u}\).

We can now state our second main result related to a weak-strong stability estimate in the combined mean-field/small inertia limit.

**Theorem 1.2.** Let \(T > 0\) and \(d \geq 1\). Let \(Z^N(t) = \{(x_i(t), v_i(t))\}_{i=1}^N\) be a solution to the particle system (1.1), and let \((\tilde{\rho}, \tilde{u})\) be the unique classical solution of the aggregation-type equation (1.4)–(1.5) satisfying \(\tilde{\rho} \in C([0, T]; \mathcal{P}(\mathbb{R}^d))\) and \(\tilde{\rho} > 0\) on \(\mathbb{R}^d \times [0, T]\), \(\tilde{u} \in L^\infty(0, T; \mathcal{W}^{1, \infty}(\mathbb{R}^d))\) and \(\partial_t \tilde{u} \in L^\infty(\mathbb{R}^d \times [0, T])\) up to time \(T > 0\) with the initial data \(\tilde{\rho}_0\). Suppose that the interaction potential \(W\) and the communication weight function \(\psi\) satisfy \(\nabla_x W \in \mathcal{W}^{1, \infty}(\mathbb{R}^d)\) and \(\psi \in \mathcal{W}^{1, \infty}(\mathbb{R}^d)\), respectively, and the strength of damping \(\gamma > 0\) is large enough. If the initial data for (1.1) and (1.4) are chosen such that

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v - \bar{u}_0(x)|^2 \mu^N_0(x, dv) + dB_L(\rho^N_0, \bar{\rho}_0) \to 0 \quad \text{as} \quad N \to \infty,
\]

then we have

\[
\int_{\mathbb{R}^d} \mu^N dv = \frac{1}{N} \sum_{i=1}^N v_i \delta_{x_i} \to \bar{\rho} \bar{u} \quad \text{weakly in} \quad L^1(0, T; \mathcal{M}(\mathbb{R}^d))
\]

(1.17)

and

\[
\mu^N \rightharpoonup \bar{\rho} \bar{u} \quad \text{weakly in} \quad L^1(0, T; \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))
\]

(1.18)

as \(N \to \infty\) (and thus \(\varepsilon_N \to 0\)). In fact, we have the following quantitative bound estimate:

\[
d^2_{BL}(\rho^N_i(\cdot), \tilde{\rho}(\cdot, t)) + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{u}(x, s)|^2 \mu^N_s(x, dv) \, ds \\
\leq C \varepsilon_N \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{u}_0(x)|^2 \mu^N_0(x, dv) + C d^2_{BL}(\rho^N_0, \bar{\rho}_0) + C \varepsilon_N
\]

and

\[
\frac{1}{\varepsilon_N} d^2_{BL}(\rho^N, \tilde{\rho}(\cdot, t)) \leq C(1 + \varepsilon_N) \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{u}(x, t)|^2 \mu^N_t(x, dv) \\
\leq C(1 + \varepsilon_N) \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{u}_0(x)|^2 \mu^N_0(x, dv) + C \varepsilon_N
\]

for all \(t \in [0, T]\), where \(C > 0\) is independent of both \(\varepsilon_N\) and \(N\) but depending on \(\|\bar{u}\|_{\mathcal{W}^{1, \infty}(\mathbb{R}^d \times \mathbb{R}^d)}\), \(\|\partial_t \bar{u}\|_{L^\infty}\), \(\|\nabla_x W\|_{\mathcal{W}^{1, \infty}(\mathbb{R}^d)}\), \(\|\psi\|_{\mathcal{W}^{1, \infty}(\mathbb{R}^d)}\), and \(\gamma\).
Remark 1.2. Theorem 1.2 implies that if the initial data satisfies
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{u}_0(x)|^2 \mu_0^N (dx dv) + d_{BL}(\rho_0^N, \bar{\rho}_0) \leq C_0 \varepsilon_N \]
for some $C_0 > 0$ which is independent of both $\varepsilon_N$ and $N$, then we have
\[ d_{BL}^2(\rho_t^N(\cdot), \bar{\rho}(\cdot,t)) + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{u}(x, s)|^2 \mu_s^N (dx dv) \, ds \leq C \varepsilon_N^2 \]
and
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{u}(x, t)|^2 \mu_t^N (dx dv) \leq C \varepsilon_N \]
for all $t \in [0, T]$, where $C > 0$ is independent of both $\varepsilon_N$ and $N$. This further yields that the convergences (1.17) and (1.18) hold in weakly in $L^\infty(0, T; M(\mathbb{R}^d))$ and $L^\infty(0, T; M(\mathbb{R}^d \times \mathbb{R}^d))$, respectively.

Remark 1.3. If $V \equiv 0$ and $\gamma > 0$ is sufficiently large, then we can check that $\|\bar{u}\|_{L^\infty(0, T; W^{1, \infty})}$ and $\|\partial_t \bar{u}\|_{L^\infty}$ can be bounded from above by some constant, which depends only on $\|\nabla_x W\|_{W^{1, \infty}}$, $\|\psi\|_{W^{1, \infty}}$, $\|\bar{\rho}\|_{L^\infty(0, T; L^1)}$, and $\gamma$. We refer to [24] for details. For general confinement potentials, we can also deal with general strong solutions for compactly supported initial data since their support remains compact for all times. We refer to [1,15] for particular instances of these results.

Remark 1.4. One may follow a similar argument as in [40, Theorem 2.4] to have the existence and uniqueness of classical solutions $(\bar{\rho}, \bar{u})$ to the equations (1.4)–(1.5) satisfying the regularity properties and assumptions of Theorem 1.2. For the Coulomb or Riesz interaction, an idea of proof proposed in [28] would be employed to establish the local-in-time existence and uniqueness of classical solutions to the equations (1.4)–(1.5) without the confinement potential.

Section 3 is devoted to the proof of Theorem 1.2 and the generalizations to singular repulsive potentials. Finally, we complement these results by showing the existence of solutions to the particle system (1.1) in Appendix A, and the existence and uniqueness of classical solutions stated in Theorem 1.1 for the hydrodynamic system (1.3) in Section 4.

2. Mean-Field Limit: From Newton to Pressureless Euler

In this section, we provide the details of the proof for Theorem 1.1. As mentioned before, one of our main mathematical tools is the discrete version of the modulated kinetic energy $E^N(Z^N(t)|U(t))$ defined in (1.15).
2.1. Modulated kinetic energy estimate

In this part, our main purpose is to give the quantitative bound estimate of the discrete modulated kinetic energy $\mathcal{E}^N(\mathcal{Z}^N(t)|U(t))$.

**Proposition 2.1.** Let $T > 0$, $\mathcal{Z}^N(t) = \{(x_i(t), v_i(t))\}_{i=1}^N$ be a solution to the particle system (1.1), and let $(\rho, u)$ be the unique classical solution of the pressureless Euler system with nonlocal interaction forces (1.3) under the assumptions of Theorem 1.1 up to time $T > 0$. Suppose that the interaction potential $W$ and the communication weight function $\psi$ satisfy $\nabla_x W \in \mathcal{W}_1^1(\mathbb{R}^d)$ and $\psi \in \mathcal{W}_1^1(\mathbb{R}^d)$, respectively. Then we have

$$\frac{d}{dt} \mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) + 2\gamma \mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) + \frac{1}{N^2} \sum_{i,j=1}^N \psi(x_i - x_j) |v_i - u(x_i)|^2 \leq C \mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) + C d_{BL}^2(\rho_i^N(\cdot), \rho(\cdot, t)),$$

(2.1)

where $C > 0$ is independent of $N$ and $\gamma$.

**Proof.** By the notion of our classical solution, we obtain from the momentum equation in (1.3) that

$$\partial_t (u(x_i(t), t)) = v_i(t) \cdot \nabla_x u(x_i(t), t) + (\partial_t u)(x_i(t), t)$$

$$= (v_i(t) - u(x_i(t), t)) \cdot \nabla_x u(x_i(t), t) - \gamma u(x_i(t))$$

$$- \nabla_x V(x_i(t)) - (\nabla_x W \star \rho)(x_i)$$

$$+ \int_{\mathbb{R}^d} \psi(x_i(t) - y) (u(y, t) - u(x_i(t), t)) \rho(y, t) dy.$$

Then using this and (1.1), we estimate the discrete modulated kinetic energy functional as

$$\frac{d}{dt} \mathcal{E}^N(\mathcal{Z}^N(t)|U(t)) = \frac{1}{N} \sum_{i=1}^N (u(x_i(t), t) - v_i(t))$$

$$\cdot (\partial_t u(x_i(t), t) + v_i(t) \cdot \nabla_x u(x_i(t), t) - \dot{v}_i(t))$$

$$= \frac{1}{N} \sum_{i=1}^N (u(x_i(t), t) - v_i(t)) \cdot ((v_i(t) - u(x_i(t), t)) \cdot \nabla_x u(x_i(t), t)$$

$$- \gamma \sum_{i=1}^N |u(x_i(t), t) - v_i(t)|^2$$

$$- \frac{1}{N} \sum_{i=1}^N (u(x_i(t), t) - v_i(t)) \cdot \left((\nabla_x W \star \rho)(x_i) - (\nabla_x W \star \rho^N)(x_i)\right)$$

$$+ \int_{\mathbb{R}^d} \psi(x_i(t) - y) (u(y, t) - u(x_i(t), t)) \rho(y, t) dy.$$
\[
+ \frac{1}{N} \sum_{i=1}^{N} (u(x_i(t), t) - v_i(t)) \cdot F(x_i(t), v_i(t))
\]
\[
=: \sum_{i=1}^{4} I_i, \tag{2.2}
\]
where
\[
F(x_i(t), v_i(t)) := \int_{\mathbb{R}^d} \psi(x_i(t) - y)(u(y, t) - u(x_i(t), t))\rho(y, t) \, dy
\]
\[- \frac{1}{N} \sum_{j=1}^{N} \psi(x_i(t) - x_j(t))(v_j(t) - v_i(t)).
\]

Here \(I_1\) can be easily estimated as
\[
I_1 = \frac{1}{N} \sum_{i=1}^{N} \nabla_x u(x_i(t), t) : (u(x_i(t), t) - v_i(t)) \otimes (v_i(t) - u(x_i(t), t))
\]
\[\leq \|\nabla_x u(\cdot, t)\|_{L^\infty} \frac{1}{N} \sum_{i=1}^{N} |u(x_i(t), t) - v_i(t)|^2
\]
\[= 2\|\nabla_x u(\cdot, t)\|_{L^\infty} \mathcal{E}^N(Z^N(t)|U(t)).
\]

By definition, we obtain \(I_2 = -2\gamma \mathcal{E}^N(Z^N(t)|U(t))\). We next estimate \(I_3\) as
\[
I_3 = -\frac{1}{N} \sum_{i=1}^{N} (u(x_i(t), t) - v_i(t)) \cdot \left( (\nabla_x W \ast \rho)(x_i(t), t) - (\nabla_x W \ast \rho^N)(x_i(t), t) \right)
\]
\[= \frac{1}{N} \sum_{i=1}^{N} (v_i(t) - u(x_i(t), t)) \cdot (\nabla_x W \ast (\rho - \rho^N))(x_i(t), t).
\]

On the other hand, the fact \(\nabla_x W \in \mathcal{W}^{1, \infty}\) gives
\[\|\nabla_x W \ast (\rho - \rho^N)(\cdot, t)\|_{L^\infty} \leq \|\nabla_x W\|_{\mathcal{W}^{1, \infty}dBL(\rho^N, \rho)},
\]
and subsequently this asserts
\[
I_3 \leq \|\nabla_x W\|_{\mathcal{W}^{1, \infty}dBL(\rho^N, \rho)} \left( \frac{1}{N} \sum_{i=1}^{N} |v_i(t) - u(x_i(t), t)| \right)
\]
\[\leq \|\nabla_x W\|_{\mathcal{W}^{1, \infty}dBL(\rho^N, \rho)} \left( \frac{1}{N} \sum_{i=1}^{N} |v_i(t) - u(x_i(t), t)|^2 \right)^{1/2}
\]
\[= \|\nabla_x W\|_{\mathcal{W}^{1, \infty}dBL(\rho^N, \rho)} \sqrt{\mathcal{E}^N(Z^N(t)|U(t))}.
\]
For the estimate of $I_4$, we note that
\[
\frac{1}{N} \sum_{j=1}^{N} \psi(x_j(t) - x_j(t))(v_j(t) - v_i(t))
= \frac{1}{N} \sum_{j=1}^{N} \psi(x_i(t) - x_j(t))(v_j(t) - u(x_j(t), t))
+ \frac{1}{N} \sum_{j=1}^{N} \psi(x_i(t) - x_j(t))(u(x_j(t), t) - v_i(t))
=: J_1 + J_2.
\]

Then we rewrite $J_2$ as
\[
J_2 = \int_{\mathbb{R}^d} \psi(x_i(t) - y)(u(y, t) - v_i(t))\rho^N(y, t) \, dy.
\]

This yields
\[
I_4 = \frac{1}{N} \sum_{i=1}^{N} (u(x_i) - v_i) \cdot \frac{1}{N} \sum_{j=1}^{N} \psi(x_i - x_j) (u(x_j) - v_j)
+ \frac{1}{N} \sum_{i=1}^{N} (u(x_i) - v_i)
\cdot \left( \int_{\mathbb{R}^d} \psi(x_i - y)(u(y) - u(x_i))\rho(y) \, dy
- \int_{\mathbb{R}^d} \psi(x_i - y)(u(y) - v_i)\rho^N(y) \, dy \right)
=: I_4^1 + I_4^2.
\]

Here we can easily estimate $I_4^1$ as
\[
I_4^1 \leq \|\psi\|_{L^\infty} \left( \frac{1}{N} \sum_{i=1}^{N} (u(x_i) - v_i) \right)^2 \leq \|\psi\|_{L^\infty} \frac{1}{N} \sum_{i=1}^{N} |u(x_i) - v_i|^2
= 2\|\psi\|_{L^\infty} E^N(\mathcal{Z}^N(t)\big|U(t)).
\]

Note that
\[
\frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \psi(x_i - y)(v_i - u(x_i))(\rho^N(y) - \rho(y)) \cdot (u(y) - u(x_i)) \, dy
= \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \psi(x_i - y)(v_i - u(x_i))\rho^N(y) \cdot (u(y) - u(x_i)) \, dy
+ I_4^2 - \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \psi(x_i - y)(v_i - u(x_i))\rho^N(y) \cdot (u(y) - v_i) \, dy
= I_4^2 + \frac{1}{N^2} \sum_{i,j=1}^{N} \psi(x_i - x_j)|v_i - u(x_j)|^2.
\]
that is,

\[ I_4^2 = \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \psi(x_i - y)(v_i - u(x_i))(\rho^N(y) - \rho(y)) \cdot (u(y) - u(x_i)) \, dy \]

\[ - \frac{1}{N^2} \sum_{i,j=1}^{N} \psi(x_i - x_j)|v_i - u(x_i)|^2. \]

On the other hand, we can estimate

\[ \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \psi(x_i - y)(v_i - u(x_i))(\rho^N(y) - \rho(y)) \cdot (u(y) - u(x_i)) \, dy \]

\[ = \frac{1}{N} \sum_{i=1}^{N} (v_i - u(x_i)) \cdot \int_{\mathbb{R}^d} \psi(x_i - y)u(y)(\rho^N(y) - \rho(y)) \, dy \]

\[ - \frac{1}{N} \sum_{i=1}^{N} (v_i - u(x_i)) \cdot u(x_i) \int_{\mathbb{R}^d} \psi(x_i - y)(\rho^N(y) - \rho(y)) \, dy \]

\[ =: K_1 + K_2, \]

where

\[ |K_1| \leq \frac{1}{N} \sum_{i=1}^{N} |v_i - u(x_i)| \left| \int_{\mathbb{R}^d} \psi(x_i - y)u(y)(\rho^N(y) - \rho(y)) \, dy \right| \]

\[ \leq \|\psi u\|_{Y_1,\infty} \frac{1}{N} \sum_{i=1}^{N} |v_i - u(x_i)| \, d_{BL}(\rho^N, \rho) \]

\[ \leq \|\psi u\|_{Y_1,\infty} \left( \frac{1}{N} \sum_{i=1}^{N} |v_i - u(x_i)|^2 \right)^{1/2} \, d_{BL}(\rho^N, \rho) \]

\[ \leq \|\psi u\|_{Y_1,\infty} \sqrt{2N \mathcal{E}(Z^N(t)|U(t))} \, d_{BL}(\rho^N, \rho). \]

Similarly, we also find that

\[ |K_2| \leq \frac{1}{N} \sum_{i=1}^{N} |v_i - u(x_i)||u(x_i)| \left| \int_{\mathbb{R}^d} \psi(x_i - y)(\rho^N(y) - \rho(y)) \, dy \right| \]

\[ \leq \|u\|_{L}\|\psi\|_{Y_1,\infty} \sqrt{2N \mathcal{E}(Z^N(t)|U(t))} \, d_{BL}(\rho^N, \rho). \]
Combining all of the above estimates, we have
\[
\frac{d}{dt} \mathcal{E}^N(Z^N(t)|U(t)) + 2\gamma \mathcal{E}^N(Z^N(t)|U(t)) + \frac{1}{N^2} \sum_{i,j=1}^N \psi(x_i - x_j)|v_i - u(x_i)|^2 \\
\leq 2 (\|\nabla_x u(\cdot, t)\|_{L^\infty} + \|\psi\|_{L^\infty}) \mathcal{E}^N(Z^N(t)|U(t)) \\
+ \sqrt{2} (\|\psi u\|_{W^{1,\infty}} + \|u(\cdot, t)\|_{L^\infty} \|\psi\|_{W^{1,\infty}} + \|\nabla_x W\|_{W^{1,\infty}}) \\
\sqrt{\mathcal{E}^N(Z^N(t)|U(t))} d_{BL}(\rho^N(\cdot), \rho(\cdot, t)).
\]

This completes the proof. \(\square\)

**Remark 2.1.** We assumed that the communication weight \(\psi\) is nonnegative, which takes into account the velocity alignment forces, however a similar bound estimate for the discrete kinetic energy \(\mathcal{E}^N\) to that in Proposition 2.1 can be obtained. Indeed, if \(\psi\) can be negative, but bounded, then the third term on the left hand side of (2.1) can be estimated as
\[
\left| \frac{1}{N^2} \sum_{i,j=1}^N \psi(x_i - x_j)|v_i - u(x_i)|^2 \right| \leq 2 \|\psi\|_{L^\infty} \mathcal{E}^N(Z^N|U).
\]

This yields
\[
\frac{d}{dt} \mathcal{E}^N(Z^N(t)|U(t)) + 2\gamma \mathcal{E}^N(Z^N(t)|U(t)) \leq C \mathcal{E}^N(Z^N(t)|U(t)) + C d_{BL}^2(\rho^N(\cdot), \rho(\cdot, t)),
\]
where \(C > 0\) is independent of \(N\) and \(\gamma\).

In order to close the estimate in Proposition 2.1, we need to estimate the bounded Lipschitz distance between \(\rho^N\) and \(\rho\). In the proposition below, we provide the relation between the bounded Lipschitz distance and the discrete modulated kinetic energy.

**Proposition 2.2.** Let \(\rho^N\) and \(\rho\) be defined as above. Then we have
\[
d_{BL}^2(\rho^N(\cdot, t), \rho(\cdot, t)) \leq C d_{BL}^2(\rho^N_0, \rho_0) + C \int_0^t \mathcal{E}^N(Z^N(s)|U(s)) \, ds,
\]
where \(C > 0\) depends only on \(\|u\|_{L^\infty(0,T;\text{Lip})}\) and \(T\).

**Proof.** Consider a forward characteristics \(\eta = \eta(x, t)\) for the system (1.3) satisfying the following ODEs:
\[
\frac{d\eta(x, t)}{dt} = u(\eta(x, t), t) \\
\text{(2.3)}
\]
subject to the initial data: \(\eta(x, 0) = x \in \mathbb{R}^d\). The characteristic \(\eta\) is well-defined because of the Lipschitz continuous regularity of \(u\). Note that along the characteristic, the solution \(\rho\) can be written in the mild form
\[
\rho(\eta(x, t), t) = \rho_0(x) \exp \left( - \int_0^t (\nabla_x \cdot u)(\eta(x, s), s) \, ds \right),
\]
and thus we get
\[
\rho_0(x) = \rho(\eta(x,t), t) \exp \left( \int_0^t (\nabla_x \cdot u)(\eta(x,s), s) \, ds \right) = \rho(\eta(x,t), t) \det ((\nabla_x \eta)(x,t)).
\]

This together with using the change of variables yields
\[
\int_{\mathbb{R}^d} \phi(\eta(x,t)) \rho_0(x) \, dx = \int_{\mathbb{R}^d} \phi(\eta(x,t)) \rho(\eta(x,t), t) \det ((\nabla_x \eta)(x,t)) \, dx = \int_{\mathbb{R}^d} \phi(x) \rho(x,t) \, dx \tag{2.4}
\]
for \( \phi \in \mathcal{W}^{1,\infty}(\mathbb{R}^d) \). Moreover, we find from (2.3) that
\[
|\eta(x,t) - \eta(y,t)| = \left| x - y + \int_0^t (u(\eta(x,s), s) - u(\eta(y,s), s)) \, ds \right| \leq |x - y| + \|u\|_{Lip} \int_0^t |\eta(x,s) - \eta(y,s)| \, ds,
\]
and applying Grönwall’s lemma to the above gives
\[
|\eta(x,t) - \eta(y,t)| \leq C|x - y|,
\]
where \( C > 0 \) depends only on \( \|u\|_{L^\infty(0,T;Lip)} \) and \( T \), that is, \( \eta \) is Lipschitz continuous in \( \mathbb{R}^d \). We also get
\[
|x_i(t) - \eta(x,t)| \leq |x_i(0) - x| + \int_0^t |v_i(s) - u(\eta(x,s), s)| \, ds.
\]

Here the second term on the right hand side of the above inequality can be estimated as
\[
\begin{align*}
\int_0^t |v_i(s) - u(\eta(x,s), s)| \, ds & \leq \int_0^t |v_i(s) - u(x_i(s), s)| \, ds + \int_0^t |u(x_i(s), s) - u(\eta(x,s), s)| \, ds \\
& \leq \int_0^t |v_i(s) - u(x_i(s), s)| \, ds + \|u\|_{Lip} \int_0^t |x_i(s) - \eta(x,s)| \, ds.
\end{align*}
\]

Thus we get
\[
|x_i(t) - \eta(x,t)| \leq |x_i(0) - x| + \int_0^t |v_i(s) - u(x_i(s), s)| \, ds + \|u\|_{Lip} \int_0^t |x_i(s) - \eta(x,s)| \, ds,
\]
and applying Grönwall’s lemma to the above deduces
\[
|x_i(t) - \eta(x,t)| \leq C|x_i(0) - x| + C \int_0^t |v_i(s) - u(x_i(s), s)| \, ds,
\]
where \( C \) depends only on \( \|u\|_{L^\infty(0,T; Lip)} \) and \( T \). In particular, by taking \( x = x_i(0) \), we get

\[
|x_i(t) - \eta(x_i(0), t)| \leq C \int_0^t |v_i(s) - u(x_i(s), s)| \, ds.
\] (2.6)

Then for any \( \phi \in \mathcal{W}^{1,\infty}(\mathbb{R}^d) \) we use (2.4) to estimate

\[
\begin{align*}
\left| \int_{\mathbb{R}^d} \phi(x)(\rho^N - \rho) \, dx \right| &= \left| \frac{1}{N} \sum_{i=1}^N \phi(x_i(t)) - \int_{\mathbb{R}^d} \phi(\eta(x, t))\rho_0 \, dx \right| \\
&= \left| \frac{1}{N} \sum_{i=1}^N (\phi(x_i(t)) - \phi(\eta(x_i(0), t))) + \frac{1}{N} \sum_{i=1}^N \phi(\eta(x_i(0), t)) - \int_{\mathbb{R}^d} \phi(\eta(x, t))\rho_0 \, dx \right| \\
&\leq \frac{1}{N} \sum_{i=1}^N |\phi(x_i(t)) - \phi(\eta(x_i(0), t))| + \frac{1}{N} \sum_{i=1}^N \phi(\eta(x_i(0), t)) - \int_{\mathbb{R}^d} \phi(\eta(x, t))\rho_0 \, dx \\
&=: L_1 + L_2.
\end{align*}
\] (2.7)

For \( L_1 \), we use the Lipschitz continuity together with (2.6) to obtain

\[
L_1 \leq \left\| \phi \right\|_{Lip} \frac{1}{N} \sum_{i=1}^N |x_i(t) - \eta(x_i(0), t)| \leq \left\| \phi \right\|_{Lip} \frac{1}{N} \int_0^t \sum_{i=1}^N |v_i(s) - u(x_i(s), s)| \, ds
\]

\[
\leq \left\| \phi \right\|_{Lip} \sqrt{T} \left( \int_0^t \frac{1}{N} \sum_{i=1}^N |v_i(s) - u(x_i(s), s)|^2 \, ds \right)^{1/2}
\]

\[
\leq \left\| \phi \right\|_{Lip} \sqrt{T} \left( \int_0^t \mathcal{E}^N(\mathcal{Z}^N(s)|U(s)) \, ds \right)^{1/2}.
\] (2.8)

For the estimate of \( L_2 \), we notice that

\[
\frac{1}{N} \sum_{i=1}^N \phi(\eta(x_i(0), t)) = \int_{\mathbb{R}^d} \phi(\eta(x, t))\rho_0^N \, dx.
\]

Using this identity, the Lipschitz estimate for \( \eta \) in (2.5), and the fact \( \phi \in \mathcal{W}^{1,\infty}(\mathbb{R}^d) \), we find

\[
L_2 = \left| \int_{\mathbb{R}^d} \phi(\eta(x, t))(\rho_0^N - \rho_0) \, dx \right| \leq \left( \left\| \phi \right\|_{L^\infty} + \left\| \phi \right\|_{Lip} \left\| \eta \right\|_{Lip} \right) d_{BL}(\rho_0^N, \rho_0).
\] (2.9)

Putting (2.8) and (2.9) into (2.7) yields

\[
d_{BL}(\rho_t^N(\cdot), \rho(\cdot, t)) \leq C d_{BL}(\rho_0^N, \rho_0) + C \left( \int_0^t \mathcal{E}^N(\mathcal{Z}^N(s)|U(s)) \, ds \right)^{1/2}
\]

for \( 0 \leq t \leq T \), where \( C > 0 \) depends only on \( \|u\|_{L^\infty(0,T; Lip)} \) and \( T \). \( \square \)
2.2. Proof of Theorem 1.1

2.2.1. Quantitative bound estimates Applying Grönwall’s lemma and Young’s inequality to the differential inequality in Proposition 2.1 yields

\[ \mathcal{E}^N(Z^N(t)|U(t)) \leq C \mathcal{E}^N(Z^N_0|U_0) + C \int_0^t d_{BL}(\rho^N_s(\cdot), \rho(\cdot, s)) \, ds, \]

where \( C > 0 \) is independent of \( N \). We then use Proposition 2.2 to have

\[ \mathcal{E}^N(Z^N(t)|U(t)) + d_{BL}^2(\rho^N_1(\cdot), \rho(\cdot, t)) \leq C \mathcal{E}^N(Z^N_0|U_0) + C d_{BL}^2(\rho^N_0, \rho_0) \]

\[ + C \int_0^t d_{BL}^2(\rho^N_s(\cdot), \rho(\cdot, s)) \, ds + C \int_0^t \mathcal{E}^N(Z^N(s)|U(s)) \, ds. \]

We finally apply Grönwall’s to the above to conclude the desired result.

2.2.2. Convergence estimates For the convergence estimates, it suffices to prove the following lemma:

**Lemma 2.1.** (i) Convergence of local moment:

\[ d_{BL} \left( \int_{\mathbb{R}^d} v \mu^N dv, \rho u \right) \leq \left( \int_{\mathbb{R}^d} |v - u(x)|^2 \mu^N(dx) dv \right)^{1/2} + C d_{BL}(\rho^N, \rho). \]

(ii) Convergence of local energy:

\[ d_{BL} \left( \int_{\mathbb{R}^d} (v \otimes v) \mu^N dv, \rho u \otimes u \right) \]

\[ \leq \int_{\mathbb{R}^d} |v - u(x)|^2 \mu^N(dx) dv + C \left( \int_{\mathbb{R}^d} |v - u(x)|^2 \mu^N(dx) dv \right)^{1/2} \]

\[ + C d_{BL}(\rho^N, \rho). \]

(iii) Convergence of empirical measure:

\[ d_{BL}^2(\mu^N, \rho^N) \leq C \int_{\mathbb{R}^d} |v - u(x)|^2 \mu^N(dx) dv + C d_{BL}^2(\rho^N, \rho). \]

Here \( C > 0 \) is independent of \( N \).

**Proof.** (i) For any \( \phi \in W^{1, \infty}(\mathbb{R}^d) \), we get

\[ \left| \int_{\mathbb{R}^d} \phi(x) \left( \int_{\mathbb{R}^d} v \mu^N(x, dv) - (\rho u)(x) \right) \, dx \right| \]

\[ = \left| \int_{\mathbb{R}^d} \phi(x) (v - u(x)) \mu^N(x, dv) + \int_{\mathbb{R}^d} \phi(x) u(x)(\rho^N(x) - \rho(x)) \, dx \right| \]

\[ \leq \| \phi \|_{L^\infty} \left( \int_{\mathbb{R}^d} |v - u(x)| \mu^N(dx) dv \right) + \| \phi u \|_{W^{1, \infty}} d_{BL}(\rho^N, \rho) \]

\[ \leq \| \phi \|_{L^\infty} \left( \int_{\mathbb{R}^d} |v - u(x)|^2 \mu^N(dx) dv \right)^{1/2} \]

\[ + \left( \| \phi \|_{L^\infty} \| u \|_{L^\infty} + \| \phi \|_{L^\infty} \| u \|_{Lip} + \| u \|_{L^\infty} \| \phi \|_{Lip} \right) d_{BL}(\rho^N, \rho). \]
(ii) Adding and subtracting, we notice that

\[
\int_{\mathbb{R}^d} (v \otimes v) \mu^N (dv) - \rho u \otimes u
\]

\[
= \int_{\mathbb{R}^d} (v - u) \otimes (v - u) \mu^N (dv) + u \otimes \left( \int_{\mathbb{R}^d} v \mu^N (dv) - \rho u \right)
\]

\[
+ \left( \int_{\mathbb{R}^d} v \mu^N (dv) - \rho u \right) \otimes u + (\rho - \rho^N) u \otimes u.
\]

This yields for \( \phi \in W^{1, \infty}(\mathbb{R}^d) \)

\[
\left| \int_{\mathbb{R}^d} \phi(x) \left( \int_{\mathbb{R}^d} (v \otimes v) \mu^N (dv) - (\rho u)(x) \otimes u(x) \right) dx \right|
\]

\[
\leq \| \phi \|_{L^\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - u|^2 \mu^N (dx dv) + 2 \| \phi u \|_{L^\infty \cap L^p} d_{BL} \left( \int_{\mathbb{R}^d} v \mu^N (dv), \rho u \right) \]

\[
+ \| \phi \|_{W^{1, \infty}}^2 d_{BL}(\rho^N, \rho).
\]

(iii) For any \( \varphi \in W^{1, \infty}(\mathbb{R}^d \times \mathbb{R}^d) \), we find that

\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x, v) \left( \mu^N (dx dv) - \rho(x) dx \otimes \delta_{u(x)} (dv) \right) \right|
\]

\[
= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x, v) \mu^N (dx dv) - \int_{\mathbb{R}^d} \varphi(x, u(x)) \rho(x) dx \right|
\]

\[
= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \varphi(x, v) - \varphi(x, u(x)) \right) \mu^N (dx dv) + \int_{\mathbb{R}^d} \varphi(x, u(x)) (\rho^N - \rho)(x) dx \right|
\]

\[
\leq \| \varphi \|_{L^p} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - u(x)|^2 \mu^N (dx dv) + (\| \varphi \|_{L^\infty} + \| \varphi \|_{L^p}) d_{BL}(\rho^N, \rho)
\]

\[
\leq C \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - u(x)|^2 \mu^N (dx dv) \right)^{1/2} + C d_{BL}(\rho^N, \rho).
\]

\( \square \)

2.3. Singular interaction potential cases: Coulomb and Riesz potentials

In this part, we discuss the singular interaction potentials. Let \( d \geq 1 \) and consider a potential \( \tilde{W} \) has the form

\[
\tilde{W}(x) = |x|^{-\alpha} \max\{d - 2, 0\} \leq \alpha < d \quad \forall d \geq 1
\]

or

\[
\tilde{W}(x) = - \log |x| \quad \text{for} \quad d = 1 \text{ or } 2.
\]

Note that the case \( \alpha = d - 2 \) with \( d \geq 3 \) or (2.11) with \( d = 2 \) corresponds to the Coulomb potential, and the other cases are called Riesz potentials. With these types of singular potentials, in a recent work [66], the quantitative mean-field limit from the particle system (1.1) to the pressureless Euler-type system when \( \gamma = 0 \),
$V \equiv 0$ and $\psi \equiv 0$. More precisely, in [66], the following modulated free energy is employed to measure the error between particle and continuum systems:

$$F_N(Z^N(t)|U(t)) := \frac{1}{2} \int \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \tilde{W}(x - y)(\rho_N - \rho)(x)(\rho_N - \rho)(y) \, dx \, dy,$$

where $\Delta$ denotes the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$.

**Theorem 2.1.** Let $T > 0$ and $Z^N(t) = \{(x_i(t), v_i(t))\}_{i=1}^N$ be a solution to the particle system (1.1), and let $(\rho, u)$ be the unique classical solution of the pressureless Euler system (1.3) with nonlocal interaction forces $\tilde{W}$, which is appeared in (2.10) or (2.11), instead of $W$ up to time $T > 0$ with initial data $(\rho_0, u_0)$. Assume that the communication weight function $\psi$ satisfies $\psi \in \mathcal{V}^{1,\infty}(\mathbb{R}^d)$. Assume that the classical solution $(\rho, u)$ satisfies $\rho \in L^\infty(0, T; \mathbb{P} \cap L^\infty)(\mathbb{R}^d)$ and $u \in L^\infty(0, T; \mathcal{V}^{1,\infty}(\mathbb{R}^d))$. In the case $\alpha \geq d - 1$, we further assume that $\rho \in L^\infty(0, T; C^{\sigma}(\mathbb{R}^d))$ for some $\sigma > \alpha - d + 1$. Then there exists $\beta < 2$ such that

$$\int \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - u(x, t)|^2 \mu^N_t (dx \, dv) + d^2_{BL}(\rho^N_t (\cdot), \rho(\cdot, t)) + \int \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \tilde{W}(x - y)(\rho_N - \rho)(x)(\rho_N - \rho)(y) \, dx \, dy \leq C \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - u_0(x)|^2 \mu^0_0 (dx \, dv) + Cd^2_{BL}(\rho^0_0, \rho_0) + CN^{\beta - 2},$$

where $C > 0$ is independent of $N$.

**Remark 2.2.** If the interaction potential $W$ is singular at the origin, then the term related to $W$ in (1.1) should be replaced by $\frac{1}{N} \sum_{j: j \neq i}^N \nabla_x W(x_i - x_j)$ since $W(0)$ can not be well defined. This is why the diagonal $\Delta$ is excluded in the integration in the modulated potential energy.

**Remark 2.3.** If the right hand side of (2.12) converges to zero as $N \to \infty$, then we also have the same convergence estimates in Theorem 1.1.

**Remark 2.4.** Our quantified mean-field limit estimate from (1.1) to (1.3) also apply with a simple combination of Theorems 1.1 and 2.1 for interaction potentials of the form $\tilde{W} := W + \tilde{W}$ with $W$ satisfying $\nabla W \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$ and $\tilde{W}$ appeared in (2.10) or (2.11).

**Proof of Theorem 2.1.** For the proof, we only need to reestimate $I_3$ term in the proof of Proposition 2.1. Although this proof is almost the same with that of [66], we provide the details here for the completeness of our work. Let us denote by

$$I := -\frac{1}{N} \sum_{i=1}^N \int_{\mathbb{R}^d} (u(x_i(t), t) - v_i(t)) \cdot \nabla_x \tilde{W}(x_i(t) - y) \rho(y, t) \, dy + \frac{1}{N^2} \sum_{i \neq j} (u(x_i(t), t) - v_i(t)) \cdot \nabla_x \tilde{W}(x_i(t) - x_j(t)).$$
On the other hand, we find that
\[
\frac{d}{dt} \mathcal{Z}^N(\mathcal{Z}^N(t)|U(t)) = \frac{1}{2} \frac{d}{dt} \left( \frac{1}{N^2} \sum_{i \neq j} \tilde{W}(x_i - x_j) \right)
\]
\[
- \frac{d}{dt} \left( \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \tilde{W}(x_i - y) \rho(y) \, dy \right)
\]
\[
+ \frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{W}(x - y) \rho(x) \rho(y) \, dx \, dy \right)
\]
\[
= \frac{1}{N^2} \sum_{i \neq j} \nabla_x \tilde{W}(x_i - x_j) \cdot v_i - \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \nabla_x \tilde{W}(x_i - y) \cdot v_i \rho(y) \, dy
\]
\[
- \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \nabla_x \tilde{W}(x_i - y) \cdot (\rho u)(y) \, dy
\]
\[
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x \tilde{W}(x - y)(\rho u)(x) \rho(y) \, dx \, dy.
\]
Here we used
\[
\nabla_x \tilde{W}(-x) = -\nabla_x \tilde{W}(x) \quad \text{for} \quad x \in \mathbb{R}^d \setminus \{0\}. \quad (2.13)
\]
This implies
\[
I := -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \tilde{W}(x - y) (\rho^N - \rho)(x) (\rho^N - \rho)(y) \, dx \, dy
\]
\[
+ \frac{1}{N^2} \sum_{i \neq j} u(x_i) \cdot \nabla_x \tilde{W}(x_i - x_j) - \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \nabla_x \tilde{W}(x_i - y) \cdot (u(x_i) - u(y)) \rho(y) \, dy
\]
\[
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x \tilde{W}(x - y)(\rho u)(x) \rho(y) \, dx \, dy.
\]
We next use (2.13) to get
\[
\frac{1}{N^2} \sum_{i \neq j} u(x_i) \cdot \nabla_x \tilde{W}(x_i - x_j) = \frac{1}{2} \frac{1}{N^2} \sum_{i \neq j} (u(x_i) - u(x_j)) \cdot \nabla_x \tilde{W}(x_i - x_j)
\]
and
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x \tilde{W}(x - y)(\rho u)(x) \rho(y) \, dx \, dy
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x \tilde{W}(x - y) (u(x) - u(y)) \rho(x) \rho(y) \, dx \, dy.
\]
Thus we obtain
\[
I := -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \tilde{W}(x - y) (\rho^N - \rho)(x) (\rho^N - \rho)(y) \, dx \, dy
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} (u(x) - u(y)) \cdot \nabla_x \tilde{W}(x - y)(\rho^N - \rho)(x) (\rho^N - \rho)(y) \, dx \, dy.
\]
This together with the estimates in Proposition 2.1 yields
\[
\frac{d}{dt} \left( E^N(Z^N(t)|U(t)) + F^N(Z^N(t)|U(t)) \right) \\
+ 2\gamma E^N(Z^N(t)|U(t)) + \frac{1}{N^2} \sum_{i,j=1}^N \psi(x_i - x_j)|v_i - u(x_i)|^2 \\
\leq C E^N(Z^N(t)|U(t)) + Cd_{BL}(\rho^N, \rho) \\
+ \frac{1}{2} \int \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} (u(x) - u(y)) \cdot \nabla_x \tilde{W}(x - y)(\rho_N - \rho)(x)(\rho_N - \rho)(y) \, dx \, dy.
\]

We then apply [66, Proposition 1.1] to have that the last term on the right hand side of the above inequality can be bounded from above by
\[
C F^N(Z^N(t)|U(t)) + C N^{\beta - 2}
\]
for some $\beta < 2$, where $C > 0$ is independent of $N$. Applying the Grönwall’s lemma to the resulting inequality concludes the desired quantitative bound estimate. The convergence result can be directly obtained by using Lemma 2.1. This completes the proof.

\[\square\]

3. Combined Small Inertia & Mean Field Limits: From Newton to Aggregation

3.1. Proof of Theorem 1.2

We first start with the case of smooth interaction potentials as in previous section and apply a similar strategy to the proof of Proposition 2.1 to the system (1.16). Then we get
\[
\frac{d}{dt} E^N(Z^N(t)|\bar{U}(t)) := \frac{1}{\varepsilon_N} \left( \sum_{i=1}^4 \bar{I}_i \right) + \bar{I}_5,
\]
where $\bar{I}_i$, $i = 1, 2, 3, 4$ are the terms $I_i$, $i = 1, 2, 3, 4$ in (2.2) with replacing $(\rho, u)$ by $(\bar{\rho}, \bar{u})$, and $\bar{I}_5$ is given by
\[
\bar{I}_5 := \frac{1}{N} \sum_{i=1}^N (\bar{u}(x_i) - v_i) \cdot \bar{e},
\]
where $\bar{e} = \partial_t \bar{u} + \bar{u} \cdot \nabla_x \bar{u}$. This can be simply estimated as
\[
|\bar{I}_5| \leq \|\bar{e}\|_{L^\infty} \frac{1}{N} \sum_{i=1}^N |\bar{u}(x_i) - v_i| \\
\leq C \frac{1}{\varepsilon_N} \sum_{i=1}^N |\bar{u}(x_i) - v_i|^2 + C \varepsilon_N \leq C \frac{1}{\varepsilon_N} E^N(Z^N(t)|\bar{U}(t)) + C \varepsilon_N.
\]
where $C > 0$ depends only $\|\tilde{e}\|_{L^\infty}$, independent of $N$ and $\varepsilon_N$. For the rest, we employ almost the same arguments as before to have

$$
\frac{1}{\varepsilon_N} \left( \sum_{i=1}^{N} \tilde{I}_i \right) \leq -2\gamma \frac{\varepsilon_N}{\varepsilon_N} \mathcal{E}^N(\bar{Z}^N(t)|\bar{U}(t)) - \frac{1}{\varepsilon_N N^2} \sum_{i,j=1}^{N} \psi(x_i - x_j)|v_i - \bar{u}(x_i)|^2 + \frac{C}{\varepsilon_N} \mathcal{E}^N(\bar{Z}^N(t)|\bar{U}(t)) + C d_{BL}^2(\rho_t^N(\cdot), \rho(\cdot, t)),
$$

where $C > 0$ is independent of $N$, $\varepsilon_N$, and $\gamma > 0$. This yields

$$
\frac{d}{dt} \mathcal{E}^N(\bar{Z}^N(t)|\bar{U}(t)) + \frac{2\gamma - C}{\varepsilon_N} \mathcal{E}^N(\bar{Z}^N(t)|\bar{U}(t)) \leq \frac{C}{\varepsilon_N} d_{BL}^2(\rho_t^N(\cdot), \rho(\cdot, t)) + C \varepsilon_N,
$$

(3.1)

where $C > 0$ is independent of $N$, $\varepsilon_N$, and $\gamma > 0$. On the other hand, by Proposition 2.2, we can bound the first term on the right hand side of the above inequality from above by

$$
\frac{C}{\varepsilon_N} d_{BL}^2(\rho_0^N(\cdot), \bar{\rho}_0) + \frac{C}{\varepsilon_N} \int_{0}^{t} \mathcal{E}^N(\bar{Z}^{N}(s)|\bar{U}(s)) \, ds,
$$

where $C > 0$ is independent of $N$, $\varepsilon_N$, and $\gamma > 0$. This together with integrating (3.1) in time implies

$$
\mathcal{E}^N(\bar{Z}^N(t)|\bar{U}(t)) + \frac{2\gamma - C}{\varepsilon_N} \int_{0}^{t} \mathcal{E}^N(\bar{Z}^{N}(s)|\bar{U}(s)) \, ds + \frac{1}{\varepsilon_N N^2} \sum_{i,j=1}^{N} \int_{0}^{t} \psi(x_i(s) - x_j(s))|v_i(s) - \bar{u}(x_i(s), s)|^2 \, ds \leq \mathcal{E}^N(\bar{Z}_0^N|\bar{U}_0) + \frac{C}{\varepsilon_N} d_{BL}^2(\rho_0^N, \bar{\rho}_0) + C \varepsilon_N.
$$

We finally apply Grönwall’s lemma to conclude the desired result in Theorem 1.2.

### 3.2. Singular interaction potential cases

Similarly as before, Theorem 1.2 can be also easily extended to the case with Coulomb or Riesz potentials $\tilde{W}$ defined in (2.10) or (2.11). More specifically, we have the following theorem.

**Theorem 3.1.** Let $T > 0$ and $\bar{Z}^N(t) = \{(x_i(t), v_i(t))\}_{i=1}^{N}$ be a solution to the particle system (1.1), and let $(\bar{\rho}, \bar{u})$ be the unique classical solution of the aggregation-type equation (1.4)–(1.5) with $\tilde{W}$, which is appeared in (2.10) or (2.11), instead of $W$, under the assumptions of Theorem 1.2 up to time $T > 0$ with the initial data $\bar{\rho}_0$. Suppose that the strength of damping $\gamma > 0$ is large enough and $(\bar{\rho}, \bar{u})$ satisfies
\( \bar{\rho} \in L^\infty(\mathbb{R}^d \times (0, T)) \). We further assume that \( \bar{\rho} \in L^\infty(0, T; C^\sigma(\mathbb{R}^d)) \) for some \( \sigma > \alpha - d + 1 \) in the case \( s \geq d - 1 \). Then there exists \( \beta < 2 \) such that

\[
d^2_{BL}(\rho_t^N(\cdot), \bar{\rho}(\cdot, t)) + \int \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \tilde{W}(x - y)(\rho^N - \bar{\rho})(x)(\rho^N - \bar{\rho})(y) \, dx \, dy \\
+ \int_0^t \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \tilde{u}(x, s)|^2 \mu^N_s(\, dx \, dv) \, ds \\
\leq C d^2_{BL}(\rho^N_0, \bar{\rho}_0) + C \int \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \tilde{W}(x - y)(\rho^N_0 - \bar{\rho}_0)(x)(\rho^N_0 - \bar{\rho}_0)(y) \, dx \, dy \\
+ C \varepsilon_N \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \tilde{u}_0(x)|^2 \mu^N_0(\, dx \, dv) + C \varepsilon_N^2 + C N^{\beta - 2}
\]

and

\[
\frac{1}{\varepsilon_N} d^2_{BL}(\rho^N_t(\cdot), \bar{\rho}(\cdot, t)) + \frac{1}{\varepsilon_N} \int \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \tilde{W}(x - y)(\rho^N - \bar{\rho})(x)(\rho^N - \bar{\rho})(y) \, dx \, dy \\
+ \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \tilde{u}(x, t)|^2 \mu^N_t(\, dx \, dv) \\
\leq \frac{C}{\varepsilon_N} d^2_{BL}(\rho^N_0, \bar{\rho}_0) + \frac{C}{\varepsilon_N} \int \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \tilde{W}(x - y)(\rho^N_0 - \bar{\rho}_0)(x)(\rho^N_0 - \bar{\rho}_0)(y) \, dx \, dy \\
+ C(1 + \varepsilon_N) \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \tilde{u}_0(x)|^2 \mu^N_0(\, dx \, dv) + C \varepsilon_N + C \frac{N^{\beta - 2}}{\varepsilon_N}
\]

for all \( t \in [0, T] \), where \( C > 0 \) is independent of \( \varepsilon_N \) and \( N \). In particular if

\[
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \tilde{u}_0(x)|^2 \mu^N_0(\, dx \, dv) \leq C \varepsilon_N
\]

and

\[
d^2_{BL}(\rho^N_0, \bar{\rho}_0) + \int \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \tilde{W}(x - y)(\rho^N_0 - \bar{\rho}_0)(x)(\rho^N_0 - \bar{\rho}_0)(y) \, dx \, dy \leq C \varepsilon_N^2
\]

for some \( C > 0 \) which is independent of \( \varepsilon_N \), then we have

\[
d^2_{BL}(\rho^N_t(\cdot), \bar{\rho}(\cdot, t)) + \int \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \tilde{W}(x - y)(\rho^N - \bar{\rho})(x)(\rho^N - \bar{\rho})(y) \, dx \, dy \\
\leq C \varepsilon_N^2 + C N^{\beta - 2}
\]

and

\[
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \tilde{u}(x, t)|^2 \mu^N_t(\, dx \, dv) \leq C \varepsilon_N + C \frac{N^{\beta - 2}}{\varepsilon_N}
\]

for all \( t \in [0, T] \), where \( C > 0 \) is independent of \( \varepsilon_N \) and \( N \).
4. Local Cauchy Problem for Pressureless Euler Equations with Nonlocal Forces

In order to make the analysis for the mean-field limit from the particle system (1.1) to the pressureless Euler-type equations (1.3) fully rigorous, we need to have the existence of solutions for both systems. As mentioned in Introduction, we postpone the existence theory for the particle system (1.1) in Appendix A, and here we provide local-in-time existence and uniqueness of classical solutions for the system (1.3). For the reader’s convenience, let us recall our limiting system:

\[
\begin{align*}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) &= -\rho u - \rho \nabla_x V - \rho \nabla_x W \ast \rho + \rho \int_{\mathbb{R}^d} \psi(x-y)(u(y) - u(x)) \rho(y) \, dy,
\end{align*}
\]

(4.1)

with the initial data

\[ (\rho(x, t), u(x, t))|_{t=0} = (\rho_0(x), u_0(x)), \quad x \in \mathbb{R}^d. \]

Here we set the coefficient of linear damping \( \gamma = 1. \)

For the one dimensional problem, the well-posedness and singularity formation for the system (4.1) without the linear damping, the confinement and interaction potentials, called Euler-alignment system, are discussed in [13]. To be more precise, the sharp critical threshold which distinguishes the global-in-time regularity of classical solutions and finite-time breakdown of smoothness is analyzed. The sharp critical threshold estimate is also obtained in [15] for the pressureless damped Euler–Poisson system with quadratic confinement potential in one dimension, that is the system (4.1) with replacing \( W \) by \( \mathcal{N}, \) \( V = \frac{|x|^2}{2}, \) and \( \psi \equiv 0. \) For the pressureless Euler–Poisson system, the critical threshold is also discussed in [2,38], see also [69] for the case with pressure. More recently, in [27], the local-in-time existence of classical solutions and finite-time singularity formation are taken into account.

We introduce the exact notion of strong solution to the system (4.1) that we will deal with.

**Definition 4.1.** Let \( s > d/2 + 1. \) For given \( T \in (0, \infty), \) the pair \((\rho, u)\) is a strong solution of (4.1) on the time interval \([0, T]\) if and only if the following conditions are satisfied:

(i) \( \rho \in C([0, T]; H^s(\mathbb{R}^d)), \) \( u \in C([0, T]; Lip(\mathbb{R}^d) \cap L^2_{loc}(\mathbb{R}^d)), \) and \( \nabla_x^2 u \in C([0, T]; H^{s-1}(\mathbb{R}^d)), \)

(ii) \((\rho, u)\) satisfy the system (4.1) in the sense of distributions.

Notice that due to the choice of \( s \) in the previous definition, these strong solutions are also classical solutions to (4.1). Our main result of this section is the following local Cauchy problem for the system (4.1).
Theorem 4.1. Let $s > d/2 + 1$ and $R > 0$. Suppose that the confinement potential $V$ is given by $V = |x|^2/2$, the interaction potential $\nabla_x W \in (W^{1,1} \cap W^{1,\infty})(\mathbb{R}^d)$, and the communication weight function $\psi$ satisfies

$$\psi \in C^1_c(\mathbb{R}^d) \quad \text{and} \quad \text{supp}(\psi) \subseteq B(0, R),$$

where $B(0, R) \subset \mathbb{R}^d$ denotes a ball of radius $R$ centered at the origin. For any $N < M$, there is a positive constant $T^*$ depending only on $R$, $N$, and $M$ such that if $\rho_0 > 0$ on $\mathbb{R}^d$ and

$$\|\rho_0\|_{H^s} + \|u_0\|_{L^2(B(0, R))} + \|\nabla_x u_0\|_{L^\infty} + \|\nabla^2_x u_0\|_{H^{s-1}} < N,$$

then the Cauchy problem (4.1) has a unique strong solution $(\rho, u)$, in the sense of Definition 4.1, satisfying

$$\sup_{0 \leq t \leq T^*} \left(\|\rho(\cdot, t)\|_{H^s} + \|u(\cdot, t)\|_{L^2(B(0, R))} + \|\nabla_x u(\cdot, t)\|_{L^\infty} + \|\nabla^2_x u(\cdot, t)\|_{H^{s-1}}\right) \leq M.$$

Remark 4.1. The assumption on the communication weight function (4.2) implies $\psi \in W^{1,p}(\mathbb{R}^d)$ for any $p \in [1, \infty]$.

Remark 4.2. By the standard Sobolev embedding theorem, the solution $(\rho, u)$ constructed as in Theorem 4.1 is a classical solution, that is $(\rho, u) \in C^1(\mathbb{R}^d \times (0, T^*))$.

Remark 4.3. The $L^2$-norm of $u$ on the ball is introduced due to the confinement potential $V$. In fact, if we ignore the confinement potential $V$ in the momentum equation in (4.1), then under the following assumption on the initial data

$$\|\rho_0\|_{H^s} + \|u_0\|_{H^{s+1}} < N,$$

we have the unique strong solution $(\rho, u)$ to the system (4.1) satisfying

$$\sup_{0 \leq t \leq T^*} \left(\|\rho(\cdot, t)\|_{H^s} + \|u(\cdot, t)\|_{H^{s+1}}\right) \leq M.$$

Remark 4.4. In case of a singular interaction potential beyond the Coulomb case, we refer to [27] for the well-posedness theory for the Euler–Riesz system. More precisely, in [27], the local-in-time existence and uniqueness of classical solutions to the system (4.1) with $\tilde{W}$ defined in (2.10) instead of the regular $W$, $\gamma = 0$, $V \equiv 0$, and $\psi \equiv 0$ are discussed. One may extend the arguments used in [27] to study the well-posedness for the system (4.1) with $\tilde{W}$. 
4.1. Linearized system

In this part, we construct approximate solutions \((\rho^n, u^n)\) for the system (4.1) and provide some uniform bound estimates of it.

Let us first take the initial data as the zeroth approximation:

\[
(\rho^0(x, t), u^0(x, t)) = (\rho_0(x), u_0(x)), \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+.
\]

We next suppose that the \(n\)th approximation \((\rho^n, u^n)\) with \(n \geq 1\) is given. Then we define the \((n + 1)\)th approximation \((\rho^{n+1}, u^{n+1})\) as a solution to the following linear system.

\[
\begin{align*}
\partial_t \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} + \rho^{n+1} \nabla \cdot u^n &= 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \\
\rho^{n+1} \partial_t u^{n+1} + \rho^{n+1} u^n \cdot \nabla u^{n+1} &= -\rho^{n+1} u^{n+1} - \rho^{n+1} (\nabla_x V + \nabla_x W \ast \rho^{n+1}) \\
&+ \rho^{n+1} \int_{\mathbb{R}^d} \psi(x - y)(u^n(y) - u^n(x)) \rho^{n+1}(y) \, dy,
\end{align*}
\]

with the initial data

\[
(\rho^n(x, 0), u^n(x, 0)) = (\rho_0(x), u_0(x)) \quad \text{for all} \quad n \geq 1, \quad x \in \mathbb{R}^d.
\]

Let us introduce a solution space \(\mathcal{Y}_{s, R}(T)\) with \(s > d/2 + 1\) as

\[
\mathcal{Y}_{s, R}(T) := \left\{ (\rho, u) : \rho \in C([0, T]; H^s(\mathbb{R}^d)), u \in C([0, T]; L^2(B(0, R))) \cap C([0, T]; W^{1, \infty}(\mathbb{R}^d)), \nabla_x^2 u \in C([0, T]; H^{s-1}(\mathbb{R}^d)) \right\}.
\]

Then by the standard linear solvability theory [58], for any \(T > 0\) we have that the approximation \(\{(\rho^n, u^n)\}_{n=0}^\infty \subset \mathcal{Y}_{s, R}(T)\) is well-defined.

For notational simplicity, in the rest of this section, we drop \(x\)-dependence of the differential operator \(\nabla_x\).

**Proposition 4.1.** Suppose that the initial data \((\rho_0, u_0)\) satisfies \(\rho_0 > 0\) on \(\mathbb{R}^d\) and

\[
\|\rho_0\|_{H^s} + \|u_0\|_{L^2(B(0, R))} + \|\nabla u_0\|_{L^\infty} + \|\nabla^2 u_0\|_{H^{s-1}} < N,
\]

and let \((\rho^n, u^n)\) be a sequence of the approximate solutions of (4.3) with the initial data \((\rho_0, u_0)\). Then for any \(N < M\), there exists \(T^* > 0\) such that

\[
\sup_{n \geq 0} \sup_{0 \leq t \leq T^*} \left( \|\rho^n(\cdot, t)\|_{H^s} + \|u^n(\cdot, t)\|_{L^2(B(0, R))} \right. \\
+ \|\nabla u^n(\cdot, t)\|_{L^\infty} + \|\nabla^2 u^n(\cdot, t)\|_{H^{s-1}} \left. \right) \leq M.
\]
Proof. For the proof, we use the inductive argument. Since we take the initial data for the first iteration step, it is clear to find
\[ \sup_{0 \leq t \leq T} \left( \| \rho^0(\cdot, t) \|_{H^s} + \| u^0(\cdot, t) \|_{L^2(B(0, R))} + \| \nabla u^0(\cdot, t) \|_{L^\infty} + \| \nabla^2 u^0(\cdot, t) \|_{H^{s-1}} \right) \]
\[ = \| \rho_0 \|_{H^s} + \| u_0 \|_{L^2(B(0, R))} + \| \nabla u_0 \|_{L^\infty} + \| \nabla^2 u_0 \|_{H^{s-1}} < N < M. \]
We now suppose that
\[ \sup_{0 \leq t \leq T_0} \left( \| \rho^n(\cdot, t) \|_{H^s} + \| u^n(\cdot, t) \|_{L^2(B(0, R))} + \| \nabla u^n(\cdot, t) \|_{L^\infty} + \| \nabla^2 u^n(\cdot, t) \|_{H^{s-1}} \right) \]
for some \( T_0 > 0 \). In the rest of the proof, upon mollifying if necessary we may assume that the communication weight function \( \psi \) is smooth. Since this proof is a rather lengthy, we divide it into four steps:

- **In Step A**, we provide the positivity and \( H^s(\mathbb{R}^d) \)-estimate of \( \rho^{n+1} \):
  \[ \rho^{n+1}(x, t) > 0 \quad \forall (x, t) \in \mathbb{R}^d \times [0, T] \quad \text{and} \quad \| \rho^{n+1}(\cdot, t) \|_{H^s} \leq \| \rho_0 \|_{H^s} e^{CMt} \]
  for \( t \leq T_0 \), where \( C > 0 \) is independent of \( n \).
- **In Step B**, we show \( \nabla \| u^{n+1}(\cdot, t) \|_{L^\infty} + \| u^{n+1}(\cdot, t) \|_{L^2(B(0, R))} \)
  \[ \leq \| \nabla u_0 \|_{L^\infty} e^{(CM-1)t} + \| u_0 \|_{L^2(B(0, R))} + E(t) \]
  for \( t \leq T_0 \), where \( C > 0 \) is independent of \( n \), and \( E : [0, T_0] \rightarrow [0, \infty) \) is continuous on \([0, T_0]\) satisfying \( E(t) \rightarrow 0 \) as \( t \rightarrow 0^+ \).
- **In Step C**, we estimate the higher order derivative of \( u^{n+1} \):
  \[ \| \nabla^2 u^{n+1} \|_{H^{s-1}} \leq \| \nabla^2 u_0 \|_{H^{s-1}} e^{CMt} + E(t) \]
  for \( t \leq T_0 \), where \( C > 0 \) is independent of \( n \), and \( E \) satisfies the same property as in **Step B**.
- **In Step D**, we finally combine all of the estimates in **Steps A, B, & C** to conclude our desired result.

**Step A.**- We first show the positivity of \( \rho^{n+1} \). Consider the following characteristic flow \( \eta^{n+1} \) associated to the fluid velocity \( u^n \) by
\[ \partial_t \eta^{n+1}(x, t) = u^n(\eta^{n+1}(x, t), t) \quad \text{for} \quad t > 0 \tag{4.4} \]
with the initial data \( \eta^{n+1}(x, 0) = x \in \mathbb{R}^d \). Since \( u^n \) is globally Lipschitz, the characteristic equations (4.4) are well-defined. Then by using the method of characteristics, we obtain
\[ \partial_t \rho^{n+1}(\eta^{n+1}(x, t), t) = -\rho^{n+1}(\eta^{n+1}(x, t), t)(\nabla \cdot u^n)(\eta^{n+1}(x, t), t), \]
and applying Grönwall’s lemma yields
\[ \rho^{n+1}(\eta^{n+1}(x, t), t) = \rho_0(x) \exp \left( - \int_0^t (\nabla \cdot u^n)(\eta^{n+1}(x, \tau), \tau) \, d\tau \right) \geq \rho_0(x) e^{-M \tau_0} > 0. \]

We next estimate \( H^s \)-norm of \( \rho^{n+1} \). We first easily find

\[
\begin{align*}
\frac{d}{dt} \| \rho^{n+1} \|_{L^2}^2 & \leq C \| \nabla u^n \|_{L^\infty} \| \rho^{n+1} \|_{L^2}^2 \leq C M \| \rho^{n+1} \|_{L^2}^2, \\
\frac{d}{dt} \| \nabla \rho^{n+1} \|_{L^2}^2 & \leq C \| \nabla u^n \|_{L^\infty} \| \nabla \rho^{n+1} \|_{L^2}^2 \\
& \quad + C \| \nabla^2 u^n \|_{L^2} \| \rho^{n+1} \|_{L^\infty} \| \nabla \rho^{n+1} \|_{L^2} \leq C M \| \rho^{n+1} \|_{H^s} \| \nabla \rho^{n+1} \|_{L^2},
\end{align*}
\]

and

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla^k \rho^{n+1}|^2 \, dx
\]

\[=: \sum_{i=1}^4 I_i \]

for \( 2 \leq k \leq s \). Here we use Moser-type inequality to estimate \( I_i, i = 1, \cdots, 4 \) as

\[
\begin{align*}
I_1 & \leq \| \nabla u^n \|_{L^\infty} \| \nabla^k \rho^{n+1} \|_{L^2}^2 \leq C M \| \nabla^k \rho^{n+1} \|_{L^2}^2, \\
I_2 & \leq \| \nabla^k (\nabla \rho^{n+1} \cdot u^n) - u^n \cdot \nabla^k \rho^{n+1} \|_{L^2} \| \nabla^k \rho^{n+1} \|_{L^2} \\
& \quad \leq C \left( \| \nabla^k u^n \|_{L^2} \| \nabla \rho^{n+1} \|_{L^\infty} + \| \nabla u^n \|_{L^\infty} \| \nabla^k \rho^{n+1} \|_{L^2} \right) \| \nabla^k \rho^{n+1} \|_{L^2} \\
& \quad \leq C M \| \nabla \rho^{n+1} \|_{H^{s-1}} \| \nabla^k \rho^{n+1} \|_{L^2}, \\
I_3 & \leq \| \rho^{n+1} \|_{L^\infty} \| \nabla^k \rho^{n+1} \|_{L^2} \| \nabla^k u^n \|_{L^2} \leq C M \| \rho^{n+1} \|_{H^s} \| \nabla^k \rho^{n+1} \|_{L^2}, \\
I_4 & \leq \| \nabla^k (\rho^{n+1} \nabla \cdot u^n) - \rho^{n+1} \nabla^k (\nabla \cdot u^n) \|_{L^2} \| \nabla^k \rho^{n+1} \|_{L^2} \\
& \quad \leq C \left( \| \nabla^k \rho^{n+1} \|_{L^2} \| \nabla u^n \|_{L^{\infty}} + \| \nabla \rho^{n+1} \|_{L^\infty} \| \nabla^k u^n \|_{L^2} \right) \| \nabla^k \rho^{n+1} \|_{L^2} \\
& \quad \leq C M \| \nabla \rho^{n+1} \|_{H^{s-1}} \| \nabla^k \rho^{n+1} \|_{L^2}.
\end{align*}
\]

Combining all of the above estimates entails

\[
\frac{d}{dt} \| \rho^{n+1} \|_{H^s} \leq C M \| \rho^{n+1} \|_{H^s}, \quad \text{that is} \quad \| \rho^{n+1} (\cdot, t) \|_{H^s} \leq \| \rho_0 \|_{H^s} e^{C M t},
\]

(4.5)
for $t \leq T_0$, where $C > 0$ is independent of $n$.

**Step B.** Due to the positivity of $\rho^{n+1}$, it follows from the momentum equation in (4.3) that $u^{n+1}$ satisfies

$$
\partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} = -u^{n+1} - \nabla V - \nabla W \star \rho^{n+1} + \int_{\mathbb{R}^d} \psi(x - y)(u^n(y) - u^n(x))\rho^{n+1}(y) \ dy.
$$

(4.6)

Taking the differential operator $\nabla$ to (4.6) gives

$$
\partial_t \nabla u^{n+1} + u^n \cdot \nabla^2 u^{n+1} = -\nabla u^n \nabla u^{n+1} - \nabla u^{n+1} - \|I_d - \nabla W \star \nabla \rho^{n+1} + \int_{\mathbb{R}^d} (u^n(y) - u^n(x)) \otimes \nabla_x \psi(x - y)\rho^{n+1}(y) \ dy
$$

$$
- \nabla u^n \int_{\mathbb{R}^d} \psi(x - y)\rho^{n+1}(y) \ dy.
$$

(4.7)

where we used $\nabla V = x$ and $I_d$ denotes the $n \times n$ identity matrix. Note that

$$
|\nabla u^n \nabla u^{n+1}| \leq M \|\nabla u^{n+1}(\cdot, t)\|_{L^\infty}
$$

and

$$
\|\nabla W \star \nabla \rho^{n+1}\|_{L^\infty} \leq \|\nabla W\|_{L^2} \|\nabla \rho^{n+1}\|_{L^2}.
$$

We also estimate the last terms on the right hand side of (4.7) as

$$
\left| \int_{\mathbb{R}^d} (u^n(y) - u^n(x)) \otimes \nabla_x \psi(x - y)\rho^{n+1}(y) \ dy \right|
$$

$$
\leq \int_{|x-y| \leq R} |u^n(y) - u^n(x)||\nabla_x \psi(x - y)|\rho^{n+1}(y) \ dy
$$

$$
\leq \|\nabla u^n\|_{L^\infty} \int_{|x-y| \leq R} |y - x||\nabla_x \psi(x - y)|\rho^{n+1}(y) \ dy
$$

$$
\leq \|\nabla u^n\|_{L^\infty} R \|\nabla \psi\|_{L^2} \|\rho^{n+1}\|_{L^2}
$$

$$
\leq CM \|\nabla \psi\|_{L^2} \|\rho^{n+1}\|_{L^2}
$$

and

$$
\left| \nabla u^n \int_{\mathbb{R}^d} \psi(x - y)\rho^{n+1}(y) \ dy \right|
$$

$$
\leq \|\nabla u^n\|_{L^\infty} \|\psi\|_{L^2} \|\rho^{n+1}\|_{L^2} \leq CM \|\psi\|_{L^2} \|\rho^{n+1}\|_{L^2}.
$$

These estimates together with integrating (4.7) along the characteristic flow $\eta^{n+1}$ implies

$$
e^t \|\nabla u^{n+1}(\cdot, t)\|_{L^\infty} \leq \|\nabla u_0\|_{L^\infty} + CM \int_0^t e^\tau \|\nabla u^{n+1}(\cdot, \tau)\|_{L^\infty} \ d\tau
$$

$$
+ C(1 + M) \int_0^t e^\tau \|\rho^{n+1}(\cdot, \tau)\|_{H^s} \ d\tau.
$$
By using Grönwall’s lemma, we obtain
\[
e^{\tau} \| \nabla u^{n+1}(\cdot, t) \|_{L^\infty} \leq \| \nabla u_0 \|_{L^\infty} e^{CM\tau} + C(1 + M) \int_0^\tau e^{\tau} \| \rho^{n+1}(\cdot, \tau) \|_{H^s} \, d\tau
\]
\[+ CM(1 + M)e^{CM\tau} \int_0^\tau e^{-CM\xi} \| \rho^{n+1}(\cdot, \tau) \|_{H^s} \, d\tau \, d\xi.
\]
This together with (4.5) asserts
\[
\| \nabla u^{n+1}(\cdot, t) \|_{L^\infty} \leq \| \nabla u_0 \|_{L^\infty} e^{(CM-1)t} + E_1(t),
\]
where \(E_1 : [0, T_0] \to [0, \infty)\) is continuous on \([0, T_0]\) satisfying \(E_1(t) \to 0\) as \(t \to 0^+\).

For the \(L^2\)-estimate of \(u^{n+1}\) on \((0, R)\), we multiply (4.6) by \(u^{n+1}\) and integrate it over \((0, R)\) to yield
\[
\frac{1}{2} \frac{d}{dt} \int_{B(0, R)} |u^{n+1}|^2 \, dx
\]
\[= \int_{B(0, R)} u^{n+1} \cdot \left( -u^n \cdot \nabla u^{n+1} - u^{n+1} - \nabla V - \nabla W \ast \rho^{n+1} \right) \, dx
\]
\[+ \int_{B(0, R)} u^{n+1} \cdot \left( \int_{\mathbb{R}^d} \psi(x-y)(u^n(y) - u^n(x))\rho^{n+1}(y) \, dy \right) \, dx
\]
\[\leq \| \nabla u^{n+1} \|_{L^\infty} \| u^n \|_{L^2(B(0, R))} \| u^{n+1} \|_{L^2(B(0, R))} - \| u^{n+1} \|_{L^2(B(0, R))}^2
\]
\[+ R \| u^{n+1} \|_{L^1(B(0, R))} + C(\| \rho^{n+1} \|_{L^2} + \| \rho^{n+1} \|_{L^\infty}) \| u^{n+1} \|_{L^1(B(0, R))}
\]
\[+ \| \nabla u^n \|_{L^\infty} R \| \psi \|_{L^2} \| \rho^{n+1} \|_{L^2} \| u^{n+1} \|_{L^1(B(0, R))}.
\]
Here we used
\[
\left| \int_{\mathbb{R}^d} \psi(x-y)(u^n(y) - u^n(x))\rho^{n+1}(y) \, dy \right|
\]
\[\leq \int_{|x-y| \leq R} \psi(x-y)|u^n(y) - u^n(x)|\rho^{n+1}(y) \, dy
\]
\[\leq \| \nabla u^n \|_{L^\infty} \int_{|x-y| \leq R} \psi(x-y)|x-y|\rho^{n+1}(y) \, dy
\]
\[\leq \| \nabla u^n \|_{L^\infty} R \| \psi \|_{L^2} \| \rho^{n+1} \|_{L^2}.
\]
Thus we obtain
\[
\frac{d}{dt} \| u^{n+1} \|_{L^2(B(0, R))} \leq CM \| \nabla u^{n+1} \|_{L^\infty} + C(1 + (1 + M)\| \rho^{n+1} \|_{H^s}),
\]
where \(C > 0\) depends only on \(R\) and \(\| \psi \|_{L^2}\). Integrating this over \([0, t]\) with \(t \leq T_0\) and using the estimates (4.5) and (4.8) imply
\[
\| u^{n+1} \|_{L^2(B(0, R))} \leq \| u_0 \|_{L^2(B(0, R))} + E_2(t),
\]
(4.9)
where \( E_2 : [0, T_0] \to [0, \infty) \) is continuous on \([0, T_0]\) satisfying \( E_2(t) \to 0 \) as \( t \to 0^+ \).

**Step C.** For \( 2 \leq k \leq s + 1 \), we find

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla^k u^{n+1}|^2 \, dx
= - \int_{\mathbb{R}^d} \nabla^k u^{n+1} \cdot (u^n \cdot \nabla^{k+1} u^{n+1}) \, dx
- \int_{\mathbb{R}^d} \nabla^k u \cdot (\nabla^k (u^n \cdot \nabla u^{n+1}) - u^n \cdot \nabla^{k+1} u^{n+1}) \, dx
- \int_{\mathbb{R}^d} |\nabla^k u^{n+1}|^2 \, dx - \int_{\mathbb{R}^d} \nabla^k u^{n+1} \cdot \nabla^k (\nabla W \ast \rho^{n+1}) \, dx
+ \int_{\mathbb{R}^d} \nabla^k u^{n+1} \cdot \nabla^k \int_{\mathbb{R}^d} \psi(x - y)(u^n(x) - u^n(y)) \rho^{n+1}(y) \, dy \, dx
= : \sum_{k=1}^{5} J_k,
\]

where \( J_1 \) and \( J_2 \) can be estimated as

\[
J_1 \leq \| \nabla u^n \|_{L^\infty} \| \nabla^k u^{n+1} \|_{L^2}^2 \leq M \| \nabla^k u^{n+1} \|_{L^2}^2
\]

and

\[
J_2 \leq C \left( \| \nabla^k u^n \|_{L^2} \| \nabla u^{n+1} \|_{L^\infty} + \| \nabla u^n \|_{L^\infty} \| \nabla^k u^{n+1} \|_{L^2} \right) \| \nabla^k u^{n+1} \|_{L^2}
\leq CM(\| \nabla u^{n+1} \|_{L^\infty} + \| \nabla^k u^{n+1} \|_{L^2}) \| \nabla^k u^{n+1} \|_{L^2}.
\]

For the estimate of \( J_4 \), we use the fact that \( W \) is the Coulombian potential to deduce

\[
|J_4| = \left| \int_{\mathbb{R}^d} |\nabla^k u^{n+1}| \| \nabla^2 W \ast \nabla^{k-1} \rho^{n+1} | \, dx \right| \leq \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^2 W \|_{L^1} \| \nabla^{k-1} \rho^{n+1} \|_{L^2}.
\]

We next divide \( J_5 \) into two terms:

\[
J_5 = \sum_{0 \leq \ell \leq k} \binom{k}{\ell} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla^\ell \psi(x - y) \nabla^{k-\ell}(u^n(y) - u^n(x)) \rho^{n+1}(y) \, dy \, dx
= \sum_{0 \leq \ell \leq k-1} \binom{k}{\ell} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla^\ell \psi(x - y) \nabla^{k-\ell} u^n(x) \rho^{n+1}(y) \, dy \, dx
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla^\ell \psi(x - y)(u^n(y) - u^n(x)) \rho^{n+1}(y) \, dy \, dx
= : J_1^5 + J_2^5.
\]
Note that
\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla^\ell y (x-y) \nabla^{k-\ell} u^n(x) \rho^{n+1}(y) \, dy \, dx \right|
\]
\[
= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla^\ell y (x-y) \nabla^{k-\ell} u^n(x) \rho^{n+1}(y) \, dy \, dx \right|
\]
\[
= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x-y) \nabla^k u^{n+1}(x) \nabla^\ell y \nabla^{k-\ell} u^n(x) \rho^{n+1}(y) \, dy \, dx \right|.
\]
Thus for \( \ell = k - 1 \) we get
\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x-y) \nabla^k u^{n+1}(x) \nabla u^n(x) \nabla^{k-1} \rho^{n+1}(y) \, dy \, dx \right|
\]
\[
\leq \| \nabla u^n \|_{L^\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x-y) |\nabla^k u^{n+1}(x)||\nabla^{k-1} \rho^{n+1}(y)| \, dy \, dx
\]
\[
\leq \| \nabla u^n \|_{L^\infty} \| \psi \|_{L^1} \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^{k-1} \rho^{n+1} \|_{L^2}
\]
\[
\leq CM \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^{k-1} \rho^{n+1} \|_{L^2},
\]
and for \( 0 \leq \ell \leq k - 2 \) we obtain
\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x-y) \nabla^k u^{n+1}(x) \nabla^{k-\ell} u^n(x) \nabla^\ell \rho^{n+1}(y) \, dy \, dx \right|
\]
\[
\leq \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^{k-\ell} u^n \|_{L^2} \| \psi \|_{L^2} \| \nabla^\ell \rho^{n+1} \|_{L^2}
\]
\[
\leq CM \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^\ell \rho^{n+1} \|_{L^2}.
\]
This asserts
\[
J_3 \leq CM \| \nabla^k u^{n+1} \|_{L^2} \sum_{0 \leq \ell \leq k-2} \binom{k}{\ell} \| \nabla^\ell \rho^{n+1} \|_{L^2} + CM \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^{k-1} \rho^{n+1} \|_{L^2}
\]
\[
\leq CM \| \nabla^k u^{n+1} \|_{L^2} \| \rho^{n+1} \|_{H^{k-1}}.
\]
Similarly, by integration by parts, we notice that
\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla_x \psi(x-y) (u^n(y) - u^n(x)) \rho^{n+1}(y) \, dy \, dx \right|
\]
\[
= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla_y \psi(x-y) (u^n(y) - u^n(x)) \rho^{n+1}(y) \, dy \, dx \right|
\]
\[
= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla_x \psi(x-y) \nabla_y u^{n+1}(x) \rho^{n+1}(y) \, dy \, dx \right|
\]
\[
- \left| \sum_{0 \leq \ell \leq k-1} \binom{k-1}{\ell} \left\langle \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla_x \psi(x-y) \nabla_y^{k-\ell} u^n(y) - u^n(x) \nabla^\ell \rho^{n+1}(y) \, dy \, dx \right\rangle.
\]
On the other hand, we find that
\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla_x \psi(x-y)(u^n(y) - u^n(x)) \nabla_y^{k-1} \rho^{n+1}(y) \, dy \right|
\]
\[
\leq \| \nabla u^n \|_{L^\infty} \int_{|x-y| \leq R} |\nabla^k u^{n+1}(x)||\nabla_x \psi(x-y)||x-y||\nabla_y^{k-1} \rho^{n+1}(y)| \, dy
\]
\[
\leq R \| \nabla u^n \|_{L^\infty} \| \psi \|_{L^1} \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^{k-1} \rho^{n+1} \|_{L^2}
\]
\[
\leq C M \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^{k-1} \rho^{n+1} \|_{L^2}
\]

and
\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla_x \psi(x-y) \nabla_y u^n(y) \nabla_y^{k-2} \rho^{n+1}(y) \, dy \right|
\]
\[
\leq \| \nabla u^n \|_{L^\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla^k u^{n+1}(x)||\nabla_x \psi(x-y)||\nabla_y^{k-2} \rho^{n+1}(y)| \, dy
\]
\[
\leq \| \nabla u^n \|_{L^\infty} \| \nabla \psi \|_{L^1} \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^{k-2} \rho^{n+1} \|_{L^2}
\]
\[
\leq C M \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^{k-2} \rho^{n+1} \|_{L^2}.
\]

Moreover, for \(0 \leq \ell \leq k - 3\) we obtain
\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^k u^{n+1}(x) \nabla_x \psi(x-y) \nabla_y^{k-1-\ell} u^n(y) \nabla_y^\ell \rho^{n+1}(y) \, dy \right|
\]
\[
\leq \| \nabla^k u^{n+1} \|_{L^2} \| \nabla \psi \|_{L^2} \| \nabla^{k-1-\ell} u^n \|_{L^2} \| \nabla^\ell \rho^{n+1} \|_{L^2}
\]
\[
\leq C M \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^\ell \rho^{n+1} \|_{L^2}.
\]

Thus we have
\[
J_5^2 \leq C M \| \nabla^k u^{n+1} \|_{L^2} \sum_{0 \leq \ell \leq k-3} \binom{k-1}{\ell} \| \nabla^\ell \rho^{n+1} \|_{L^2} + C M \| \nabla^k u^{n+1} \|_{L^2} \| \nabla^{k-2} \rho^{n+1} \|_{H^1}
\]
\[
\leq C M \| \nabla^k u^{n+1} \|_{L^2} \| \rho^{n+1} \|_{H^{k-1}},
\]

and subsequently we get
\[
J_5 \leq C M \| \nabla^k u^{n+1} \|_{L^2} \| \rho^{n+1} \|_{H^{k-1}}.
\]

We finally combine all of the above estimate to have
\[
\frac{d}{dt} \| \nabla^2 u^{n+1} \|_{H^{k-1}} + \| \nabla^2 u^{n+1} \|_{H^{k-1}} \leq C M \| \nabla^2 u^{n+1} \|_{H^{k-1}}
\]
\[
+ C M \| \nabla u^{n+1} \|_{L^\infty} + C M \| \rho^{n+1} \|_{H^s},
\]

and applying Grönwall’s lemma gives
\[
\| \nabla^2 u^{n+1} \|_{H^{k-1}} \leq \| \nabla^2 u_0 \|_{H^{k-1}} e^{C M t} + E_3(t),
\]

where we used the estimates in Steps B & C and \(E_3 : [0, T_0] \to [0, \infty)\) is continuous on \([0, T_0]\) satisfying \(E_3(t) \to 0\) as \(t \to 0^+\).
Step D.- We now combine (4.5), (4.8), (4.9), and (4.10) to have

\[
\|\rho^{n+1}(\cdot, t)\|_{H^1} + \|\nabla u^{n+1}(\cdot, t)\|_{L^\infty} + \|u^{n+1}(\cdot, t)\|_{L^2(B(0, R))} + \|\nabla^2 u^{n+1}\|_{H^{r-1}} \\
\leq \|\rho_0\|_{H^s} e^{CMt} + \|\nabla u_0\|_{L^\infty} e^{(CM-1)t} + \|u_0\|_{L^2(B(0, R))} + \|\nabla^2 u_0\|_{H^{r-1}} e^{CMt} + E(t)
\]

(4.11)

for \(t \leq T_0\), where \(C > 0\) is independent of \(n\), and \(E : [0, T_0] \to [0, \infty)\) is continuous on \([0, T_0]\) satisfying \(E(t) \to 0\) as \(t \to 0^+\). On the other hand, the right hand side of (4.11) converges to \(\|\rho_0\|_{H^s} + \|u_0\|_{L^2(B(0, R))} + \|\nabla u_0\|_{L^\infty} + \|\nabla^2 u_0\|_{H^{r-1}}\) as \(t \to 0^+\) and that is strictly less than \(N\). This asserts that there exists \(T_\ast \leq T_0\) such that

\[
\sup_{0 \leq t \lesssim T_\ast} \|\rho^{n+1}(\cdot, t)\|_{H^s} + \|\nabla u^{n+1}(\cdot, t)\|_{L^\infty} + \|u^{n+1}(\cdot, t)\|_{L^2(B(0, R))} + \|\nabla^2 u^{n+1}\|_{H^{r-1}} \leq M.
\]

This completes the proof. \(\Box\)

4.2. Proof of Theorem 4.1

We first show the existence of a solution \((\rho, u) \in \mathcal{Y}_{s, R}(T_\ast)\). Note that \(\rho^{n+1} - \rho^n\) and \(u^{n+1} - u^n\) satisfy

\[
\partial_t (\rho^{n+1} - \rho^n) + (u^n - u^{n-1}) \cdot \nabla \rho^{n+1} + \rho^{n+1} \cdot \nabla (\rho^{n+1} - \rho^n) \\
+ (\rho^{n+1} - \rho^n) \nabla \cdot u^n + \rho^n \nabla \cdot (u^n - u^{n-1}) = 0
\]

(4.12)

and

\[
\partial_t (u^{n+1} - u^n) + (u^n - u^{n-1}) \cdot \nabla u^{n+1} + \rho^{n+1} \cdot \nabla (u^{n+1} - u^n) \\
= -(u^{n+1} - u^n) - \nabla W \ast (\rho^{n+1} - \rho^n) \\
+ \int_{\mathbb{R}^d} \psi(x - y)(u^n(y) - u^{n-1}(y)) \rho^{n+1}(y) \, dy \\
- (u^n(x) - u^{n-1}(x)) \int_{\mathbb{R}^d} \psi(x - y) \rho^{n+1}(y) \, dy \\
+ \int_{\mathbb{R}^d} \psi(x - y)(u^{n-1}(y) - u^{n-1}(x))(\rho^{n+1} - \rho^n)(y) \, dy,
\]

respectively. Then multiplying (4.12) by \(\rho^{n+1} - \rho^n\) and integrating it over \(\mathbb{R}^d\) gives

\[
\|\rho^{n+1} - \rho^n\|_{L^2}^2 \leq C \int_0^t \left( \|\rho^{n+1} - \rho^n\|_2^2 + \|u^n - u^{n-1}\|_H^1 \right) \, dt,
\]

(4.13)
where $C > 0$ is independent of $n$. On the other hand, for $k = 0, 1$, we find that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla^k (u^{n-1} - u^n)|^2 \, dx$$

$$= - \int_{\mathbb{R}^d} \nabla^k (u^{n-1} - u^n) \nabla^k \left( (u^n - u^{n-1}) \cdot \nabla u^{n+1} \right) \, dx$$

$$- \int_{\mathbb{R}^d} \nabla^k (u^{n-1} - u^n) \nabla^k \left( u^{n-1} \cdot \nabla (u^{n+1} - u^n) \right) \, dx$$

$$- \int_{\mathbb{R}^d} |\nabla^k (u^{n+1} - u^n)|^2 \, dx - \int_{\mathbb{R}^d} \nabla^k (u^{n+1} - u^n) \nabla^k (\nabla W \ast (\rho^{n+1} - \rho^n)(x)) \, dx$$

$$+ \int_{\mathbb{R}^d} \nabla^k (u^{n+1} - u^n) \nabla^k \left( \int_{\mathbb{R}^d} \psi(x - y)(u^n(y) - u^{n-1}(y)) \rho^{n+1}(y) \, dy \right) \, dx$$

$$- \int_{\mathbb{R}^d} \nabla^k (u^{n+1} - u^n) \nabla^k \left( (u^n(x) - u^{n-1}(x)) \int_{\mathbb{R}^d} \psi(x - y) \rho^{n+1}(y) \, dy \right) \, dx$$

$$+ \int_{\mathbb{R}^d} \nabla^k (u^{n+1} - u^n) \nabla^k \left( \int_{\mathbb{R}^d} \psi(x - y)(u^{n-1}(y) - u^{n-1}(x)) (\rho^{n+1} - \rho^n)(y) \, dy \right) \, dx$$

$$\, dx =: \sum_{i=1}^{7} K_i,$$

where we easily estimate

$$\sum_{i=1}^{3} K_i \leq C \|u^{n+1} - u^n\|^2_{H^1} + C \|u^n - u^{n-1}\|^2_{H^1}.$$

Here $C > 0$ is independent of $n$. We next use the following estimates

$$\left| \int_{\mathbb{R}^d} (u^{n+1} - u^n)(x) \cdot (\nabla W \ast (\rho^{n+1} - \rho^n)(x)) \, dx \right|$$

$$\leq C \|u^{n+1} - u^n\|_{L^2} \|\nabla W\|_{L^1} \|\rho^{n+1} - \rho^n\|_{L^2}$$

and

$$\left| \int_{\mathbb{R}^d} \nabla (u^{n+1} - u^n)(x) : (\nabla^2 W \ast (\rho^{n+1} - \rho^n)(x)) \, dx \right|$$

$$\leq \|\nabla^2 W\|_{L^1} \|\nabla (u^{n+1} - u^n)\|_{L^2} \|\rho^{n+1} - \rho^n\|_{L^2}$$

to have $K_4 \leq C \|u^{n+1} - u^n\|^2_{H^1} + C \|\rho^{n+1} - \rho^n\|^2_{L^2}$. For the rest, if $k = 0$, then

$$K_5 \leq \|u^{n+1} - u^n\|_{L^2} \|\nabla \psi\|_{L^2} \|\nabla (u^{n-1} - u^n)\|_{L^2} \|\rho^{n+1}\|_{L^2}$$

$$\leq C \|u^{n+1} - u^n\|^2_{L^2} + C \|u^n - u^{n-1}\|^2_{L^2},$$

$$K_6 \leq \|u^{n+1} - u^n\|_{L^2} \|u^n - u^{n-1}\|_{L^2} \|\nabla \psi\|_{L^2} \|\rho^{n+1}\|_{L^2}$$

$$\leq C \|u^{n+1} - u^n\|^2_{L^2} + C \|u^n - u^{n-1}\|^2_{L^2},$$

$$K_7 \leq R \|\nabla u^{n-1}\|_{L^\infty} \|\nabla \psi\|_{L^2} \|u^{n+1} - u^n\|_{L^2} \|\rho^{n+1} - \rho^n\|_{L^2}$$

$$\leq C \|u^{n+1} - u^n\|^2_{L^2} + C \|\rho^{n+1} - \rho^n\|^2_{L^2}. $$
On the other hand, if \( k = 1 \), we obtain

\[
K_5 \leq \| \nabla (u^{n+1} - u^n) \|_{L^2} \| \nabla \psi \|_{L^2} \| u^n - u^{n-1} \|_{L^2} \| \rho^{n+1} \|_{L^2} \\
\leq C \| \nabla (u^{n+1} - u^n) \|_{L^2} + C \| u^n - u^{n-1} \|_{L^2}^2,
\]

\[
K_6 \leq \| \nabla (u^{n+1} - u^n) \|_{L^2} \left( \| \nabla (u^n - u^{n-1}) \|_{L^2} \| \nabla \psi \|_{L^2} + \| u^n - u^{n-1} \|_{L^2} \| \nabla \psi \|_{L^2} \right) \| \rho^{n+1} \|_{L^2} \\
\leq C \| \nabla (u^{n+1} - u^n) \|_{L^2} + C \| u^n - u^{n-1} \|_{H^1}^2,
\]

\[
K_7 \leq \| \nabla (u^{n+1} - u^n) \|_{L^2} \left( R \| \nabla u^{n-1} \|_{L^\infty} \| \nabla \psi \|_{L^1} + \| \nabla \psi \|_{L^1} \| \nabla u^{n-1} \|_{L^\infty} \| \rho^{n+1} - \rho^n \|_{L^2} \\
\leq C \| \nabla (u^{n+1} - u^n) \|_{L^2} + C \| \rho^{n+1} - \rho^n \|_{L^2}^2,
\]

We now combine all of the above estimates to have

\[
\frac{d}{dt} \| u^{n+1} - u^n \|_{H^1}^2 \leq C \| u^{n+1} - u^n \|_{H^1}^2 + C \| u^n - u^{n-1} \|_{H^1}^2 + C \| \rho^{n+1} - \rho^n \|_{L^2}^2,
\]

and subsequently this yields

\[
\| (u^{n+1} - u^n)(\cdot, t) \|_{H^1}^2 \leq C \int_0^t \left( \| (\rho^{n+1} - \rho^n)(\cdot, \tau) \|_{L^2}^2 + \| (u^n - u^{n-1})(\cdot, \tau) \|_{L^2}^2 \right) \, d\tau,
\]

where \( C > 0 \) is independent of \( n \). This together with (4.13) asserts that \((\rho^n, u^n)\) is a Cauchy sequence in \( C([0, T]; L^2(\mathbb{R}^d)) \times C([0, T]; H^1(\mathbb{R}^d)) \). Interpolating this strong convergences with the above uniform-in-\( n \) bound estimates gives

\[
\rho^n \to \rho \quad \text{in } C([0, T_*]; H^{s-1}(\mathbb{R}^d)),
\]

\[
u^n \to u \quad \text{in } C([0, T_*]; H^1(B(0, R))) \quad \text{as } n \to \infty,
\]

\[
\nabla u^n \to \nabla u \quad \text{in } C(\mathbb{R}^d \times [0, T_*]), \quad \text{and}
\]

\[
\nabla^2 u^n \to \nabla^2 u \quad \text{in } C([0, T_*]; H^{s-2}(\mathbb{R}^d)) \quad \text{as } n \to \infty,
\]

due to \( s > d/2 + 1 \). We then use a standard functional analytic arguments, see for instances [29, Section 2.1], to have that the limiting functions \( \rho \) and \( u \) satisfy the regularity in Theorem 4.1. We easily show that the limiting functions \( \rho \) and \( u \) are solutions to (4.1) with regularity properties and assumptions of Theorem 1.2.

We finally provide the uniqueness of strong solutions. Let \((\rho, u)\) and \((\tilde{\rho}, \tilde{u})\) be the strong solutions obtained above with the same initial data \((\rho_0, u_0)\). Set \( \Delta(t) \) a difference between two strong solutions:

\[
\Delta(t) := \| \rho(\cdot, t) - \tilde{\rho}(\cdot, t) \|_{L^2} + \| u(\cdot, t) - \tilde{u}(\cdot, t) \|_{H^1}.
\]

Then by using almost the same argument as above, we have

\[
\Delta(t) \leq C \int_0^t \Delta(s) \, ds \quad \text{with} \quad \Delta(0) = 0.
\]

This concludes that \( \Delta(t) \equiv 0 \) on \([0, T_*]\) and completes the proof.
Acknowledgements. JAC was partially supported by EPSRC Grant Numbers EP/P031587/1 and EP/V051121/1, and the Advanced Grant Nonlocal-CPD (Nonlocal PDEs for Complex Particle Dynamics: Phase Transitions, Patterns and Synchronization) of the European Research Council Executive Agency (ERC) under the European Union’s Horizon 2020 research and innovation programme (Grant Agreement No. 883363). YPC was supported by NRF Grant (No. 2017R1C1B2012918), POSCO Science Fellowship of POSCO TJ Park Foundation, and Yonsei University Research Fund of 2019-22-0212.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix A. Well-Posedness of the Particle System

In this appendix, we study the global existence and uniqueness of classical solutions to the particle system (1.1)–(1.2).

Let us first consider the case with singular interaction potentials with \( d \geq 2 \). In this case, we can use the repulsive effect from the interaction forces, and this also enables us to have the uniqueness of solutions.

**Theorem A.1.** Let \( d \geq 2 \). Suppose that \( \tilde{W} \) is of the form (2.10) or (2.11) and the confinement potential \( V \) satisfies either \( V \to +\infty \) as \( |x| \to \infty \) or \( \nabla_x V \) has linear growth as \( |x| \to \infty \). If the initial data \( x_0 \) satisfy

\[
\min_{1 \leq i \neq j \leq N} |x_{i0} - x_{j0}| > 0.
\]

Then there exists a unique global smooth solution to the system (1.1)–(1.2) with \( \tilde{W} \) instead of \( W \) satisfying

\[
C \geq \max_{1 \leq i \neq j \leq N} |x_i(t) - x_j(t)| \geq \min_{1 \leq i \neq j \leq N} |x_i(t) - x_j(t)| > 0
\]

for \( t \geq 0 \), where \( C > 0 \) is independent of \( t \).

**Proof.** For the proof, we first introduce the maximal life-span \( T_0 = T(x_0) \) of the initial data data \( x_0 \) as

\[
T_0 := \sup \{ s > 0 : \text{solution } (x(t), v(t)) \text{ for the system (1.1) exists up to the time } s \}.
\]

Then by the assumption and continuity of solutions, we get \( T_0 > 0 \). We now claim that \( T_0 = \infty \) and for this it suffices to show that there is no collision between particles for all \( t \geq 0 \) and that particles cannot escape to infinity in finite time.

A straightforward computation yields
for \( t \in [0, T_0) \). Note that

\[
\frac{1}{2} \frac{d}{dt} \sum_{i=1}^{N} |v_i|^2 = -\gamma \sum_{i=1}^{N} |v_i|^2 - \sum_{i=1}^{N} v_i \cdot \nabla_x V(x_i)
\]

\[
- \frac{1}{N} \sum_{i \neq j} v_i \cdot \nabla_x \tilde{W}(x_i - x_j) + \frac{1}{N} \sum_{i,j=1}^{N} \psi(x_i - x_j)(v_j - v_i) \cdot v_i
\]

for \( t \in [0, T_0) \). Note that

\[
\frac{d}{dt} \sum_{i=1}^{N} V(x_i) = \sum_{i=1}^{N} v_i \cdot \nabla_x V(x_i)
\]

and

\[
\frac{1}{2N} \frac{d}{dt} \sum_{i \neq j} \tilde{W}(x_i - x_j) = \frac{1}{2N} \sum_{i \neq j} \nabla_x \tilde{W}(x_i - x_j) \cdot (v_i - v_j) = \frac{1}{N} \sum_{i \neq j} \nabla_x \tilde{W}(x_i - x_j) \cdot v_i,
\]

where we used \( \nabla \tilde{W}(-x) = -\nabla \tilde{W}(x) \). Similarly, we also find

\[
\frac{1}{N} \sum_{i,j=1}^{N} \psi(x_i - x_j)(v_j - v_i) \cdot v_i = \frac{1}{2N} \sum_{i,j=1}^{N} \psi(x_i - x_j)|v_j - v_i|^2.
\]

Combining all of the above estimates, we obtain

\[
\frac{d}{dt} \mathcal{F}^N(x, v) + \gamma \sum_{i=1}^{N} |v_i|^2 + \frac{1}{2N} \sum_{i,j=1}^{N} \psi(x_i - x_j)|v_j - v_i|^2 = 0
\]

for \( t \in [0, T_0) \), where \( \mathcal{F}^N(x, v) \) denotes the discrete free energy given by

\[
\mathcal{F}^N(x, v) := \frac{1}{2} \sum_{i=1}^{N} |v_i|^2 + \sum_{i=1}^{N} V(x_i) + \frac{1}{2N} \sum_{i \neq j} \tilde{W}(x_i - x_j).
\]

If \( d = 2 \), then we have either

\[
\frac{1}{2N} \sum_{i \neq j} \frac{1}{|x_i - x_j|^\alpha} \leq \mathcal{F}^N(x_0, v_0) \quad \text{or} \quad -\frac{1}{2N} \sum_{i \neq j} \log |x_i(t) - x_j(t)| \leq \mathcal{F}^N(x_0, v_0),
\]

where \( \alpha \in (0, 2) \). On the other hand, if \( d \geq 3 \), we obtain

\[
\frac{1}{2N} \sum_{i \neq j} \frac{1}{|x_i(t) - x_j(t)|^\alpha} \leq \mathcal{F}^N(x_0, v_0)
\]

for all \( t \in [0, T_0) \), where \( \alpha \in (d - 2, d) \). Since the right hand side of the above inequality is uniformly bounded in \( t \), we conclude \( T_0 = \infty \) for the case \( d \geq 2 \). An upper bound estimate of the distance between particles is a simple consequence of the uniform-in-time bound estimate of the free energy \( \mathcal{F}^N \) due to the confinement potential whenever is present. If \( V = 0 \), one can obtain that particles cannot escape to infinity in finite time as soon as \( \nabla_x V \) has linear growth as \( |x| \to \infty \). \( \square \)
Let \( d \) be equal to 1. Proposition A.1.

Let us finally comment on the one dimensional case. If \( d = 1 \) and the interaction potential \( W \) is given by \((2.11)\), then we apply Theorem A.1 to get the global unique classical solution and uniform-in-time bound estimate. If \( W \) is given by the Coulomb potential, that is

\[
W'(x) = \frac{1}{2} \text{sgn}(x), \quad \text{where } \text{sgn}(x) := \begin{cases} x/|x| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}. \tag{A.1}
\]

Thus the interaction force \(-W'\) is discontinuous, but bounded. In this sense, it is not so singular compared to the other cases. Since the velocity alignment force is regular, we can use a similar argument as in [50, Proposition 1.2], see also [10,41], to have the following singular compared to the other cases. Since the velocity alignment force is regular, we can use a similar argument as in [50, Proposition 1.2], see also [10,41], to have the following proposition.

Proposition A.1. Let \( d = 1 \). For any initial configuration \( Z^N(0) \), there exists at least one global-in-time solution to the system of \((1.1)\) with \((A.1)\) in the sense that \((x_i(t), v_i(t))\) satisfies the integral system:

\[
x_i(t) = x_i(0) + \int_0^t v_i(s) \, ds, \quad i = 1, \ldots, N, \quad t > 0,
\]

\[
v_i(t) = v_i(0) - \gamma \int_0^t v_i(s) \, ds - \int_0^t V'(x_i(s)) \, ds - \frac{1}{N} \sum_{j \neq i} \int_0^t W'(x_i(s) - x_j(s)) \, ds
\]

\[
+ \frac{1}{N} \sum_{j=1}^N \int_0^t \psi(x_i(s) - x_j(s))(v_j(s) - v_i(s)) \, ds.
\]

Even though Proposition A.1 does not provide the uniqueness of solutions, it is not necessary for the analysis of mean-field limit or mean-field/small inertia limit from the particle system \((1.1)\) to the pressureless Euler system \((1.3)\) or the aggregation equation \((1.4)\).

References

1. Balagué, D., Carrillo, J.A., Laurent, T., Raoul, G.: Nonlocal interactions by repulsive-attractive potentials: radial ins/stability. Physica D 260, 5–25, 2013
2. Bhatnagar, M., Liu, H.: Critical thresholds in one-dimensional damped Euler–Poisson systems. Math. Models Methods Appl. Sci. 30, 891–916, 2020
3. Bolley, F., Cañizo, J.A., Carrillo, J.A.: Stochastic mean-field limit: non-Lipschitz forces and swarming. Math. Models Methods Appl. Sci. 21, 2179–2210, 2011
4. Braun, W., Hepp, K.: The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles. Comm. Math. Phys. 56, 101–113, 1977
5. Brenier, Y.: Convergence of the Vlasov–Poisson system to the incompressible Euler equations. Commun. Partial Differ. Equ. 25, 737–754, 2000
6. Brenier, Y., Mauser, N., Norbert, Puel, M.: Incompressible Euler and e-MHD as scaling limits of the Vlasov–Maxwell system. Commun. Math. Sci. 1, 437–447, 2003
7. Bresch, D., Jabin, P.-E., Wang, Z.: On mean-field limits and quantitative estimates with a large class of singular kernels: application to the Patlak–Keller–Segel model. C. R. Math. Acad. Sci. Paris. 357, 708–720, 2019
8. Carrillo, J.A., Choi, Y.-P.: Quantitative error estimates for the large friction limit of Vlasov equation with nonlocal forces. Ann. Inst. H. Poincaré Anal. Non Linéaire 37, 925–954, 2020
9. Carrillo, J.A., Choi, Y.-P., Hauray, M.: The derivation of swarming models: Mean-field limit and Wasserstein distances. In: Collective Dynamics from Bacteria to Crowds, CISM Courses and Lect., vol. 553. Springer, pp. 1–46, 2014

10. Carrillo, J.A., Choi, Y.-P., Hauray, M., Salem, S.: Mean-field limit for collective behavior models with sharp sensitivity regions. *J. Eur. Math. Soc.* 21, 121–161, 2019

11. Carrillo, J.A., Choi, Y.-P., Jung, J.: Quantifying the hydrodynamic limit of Vlasov-type equations with alignment and nonlocal forces. *Math. Models Methods Appl. Sci.* 31, 327–408, 2021

12. Carrillo, J.A., Choi, Y.-P., Pérez, S.: A review on attractive-repulsive hydrodynamics for consensus in collective behavior, Active particles. Vol. 1. Advances in theory, models, and applications. Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, pp. 259–298, 2017

13. Carrillo, J.A., Choi, Y.-P., Tadmor, E., Tan, C.: Critical thresholds in 1D Euler equations with nonlocal forces. *Math. Models Methods Appl. Sci.* 26, 85–206, 2016

14. Carrillo, J.A., Choi, Y.-P., Tse, O.: Convergence to equilibrium in Wasserstein distance for damped Euler equations with interaction forces. *Commun. Math. Phys.* 365, 329–361, 2019

15. Carrillo, J.A., Choi, Y.-P., Zatorska, E.: On the pressureless damped Euler–Poisson equations with quadratic confinement: critical thresholds and large-time behavior. *Math. Models Methods Appl. Sci.* 26, 2311–2340, 2016

16. Carrillo, J.A., Delgadino, M.G., Pavliotis, G.A.: A proof of the mean-field limit for $\lambda$-convex potentials by $\Gamma$-convergence. *J. Funct. Anal.* 279, 108734, 2020

17. Carrillo, J.A., DiFrancesco, M., Figalli, A., Laurent, T., Slepcev, D.: Global-in-time weak measure solutions and finite-time aggregation for nonlocal interaction equations. *Duke Math. J.* 156, 229–271, 2011

18. Carrillo, J.A., D’Orsogna, M.R., Panferov, V.: Double milling in self-propelled swarms from kinetic theory. *Kinet. Relat. Models* 2, 363–378, 2009

19. Carrillo, J.A., Feireisl, E., Gwiazda, P., Świerczewska-Gwiazda, A.: Weak solutions for Euler systems with non-local interactions. *J. Lond. Math. Soc.* 95, 705–724, 2017

20. Carrillo, J.A., Fornasier, M., Toscani, G., Vecil, F.: Particle, kinetic, and hydrodynamic models of swarming. Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, Series: Modelling and Simulation in Science and Technology, Birkhäuser, pp. 297–336, 2010

21. Carrillo, J.A., Klar, A., Martin, S., Tiwari, S.: Self-propelled interacting particle systems with roosting force. *Math. Models Methods Appl. Sci.* 20, 1533–1552, 2010

22. Carrillo, J.A., Klar, A., Roth, A.: Single to double mill small noise transition via semi-Lagrangian finite volume methods. *Commun. Math. Sci.* 14, 1111–1136, 2016

23. Carrillo, J.A., Peng, Y., Wróblewska-Kamińska, A.: Relative entropy method for the relaxation limit of hydrodynamic models. *Netw. Heterog. Media* 15, 369–387, 2020

24. Choi, Y.-P.: Large friction limit of pressureless Euler equations with nonlocal forces, preprint

25. Choi, Y.-P., Ha, S.-Y., Li, Z.: Emergent dynamics of the Cucker–Smale flocking model and its variants, Active particles. Vol. 1. Advances in Theory, Models, and Applications. Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, pp. 299–331, 2017

26. Choi, Y.-P., Haskovec, J.: Hydrodynamic Cucker–Smale model with normalized communication weights and time delay. *SIAM J. Math. Anal.* 51, 2660–2685, 2019

27. Choi, Y.-P., Jeong, I.-J.: On well-posedness and singularity formation for the Euler–Riesz system, preprint

28. Choi, Y.-P., Jeong, I.-J.: Classical solutions to the fractional porous medium flow. *Nonlinear Anal.* 210, 112393, 2021

29. Choi, Y.-P., Kwon, B.: The Cauchy problem for the pressureless Euler/isentropic Navier–Stokes equations. *J. Differ. Equ.* 261, 654–711, 2016
30. Choi, Y.-P., Salem, S.: Propagation of chaos for aggregation equations with no-flux boundary conditions and sharp sensing zones. *Math. Models Methods Appl. Sci.* **28**, 223–258, 2018
31. Choi, Y.-P., Salem, S.: Collective behavior models with vision geometrical constraints: truncated noises and propagation of chaos. *J. Differential Equations* **266**, 6109–6148, 2019
32. Choi, Y.-P., Yun, S.-B.: Existence and hydrodynamic limit for a Paveri–Fontana type kinetic traffic model. *SIAM J. Math. Anal.* **53**, 2631–2659, 2021
33. Chuang, Y.-L., D'Orsogna, M.R., Marthaler, D., Bertozzi, A.L., Chayes, L.: State transitions and the continuum limit for a 2D interacting, self-propelled particle system. *Physica D* **232**, 33–47, 2007
34. Cucker, F., Smale, S.: Emergent behavior in flocks. *IEEE Trans. Autom. Control.* **52**, 852–862, 2007
35. Dobrushin, R.: Vlasov equations. *Funct. Anal. Appl.* **13**, 115–123, 1979
36. D'Orsogna, M.R., Chuang, Y.-L., Bertozzi, A.L., Chayes, L.: Self-propelled particles with soft-core interactions: patterns, stability, and collapse. *Phys. Rev. Lett.* **9696**, 104302-1/4, 2006
37. Duerinckx, M.: Mean-field limits for some Riesz interaction gradient flows. *SIAM J. Math. Anal.* **48**, 2269–2300, 2016
38. Engelberg, S., Liu, H., Tadmor, E.: Critical thresholds in Euler–Poisson equations. *Indiana Univ. Math. J.* **50**, 109–157, 2001
39. Fournier, N., Hauray, M., Mischler, S.: Propagation of chaos for the 2d viscous vortex model. *J. Eur. Math. Soc.* **16**, 1423–1466, 2014
40. Golse, F.: The mean-field limit for the dynamics of large particle systems. *Journées équations aux dérivées partielles* **9**, 1–47, 2003
41. Golse, F.: On the Dynamics of Large Particle Systems in the Mean Field Limit, in Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity. Lecturer Notes Applied Mathematics and Mechanics, vol. 3. Springer, Cham, pp. 1–144, 2016
42. Ha, S.-Y., Liu, J.-G.: A simple proof of the Cucker–Smale flocking dynamics and mean-field limit. *Commun. Math. Sci.* **7**, 297–325, 2009
43. Ha, S.-Y., Tadmor, E.: From particle to kinetic and hydrodynamic descriptions of flocking. *Kinet. Relat. Models* **1**, 415–435, 2008
44. Han-Kwan, D., Iacobelli, M.: From Newton’s second law to Euler’s equations of perfect fluids. *Proc. Am. Math. Soc.*, to appear
45. Hauray, M.: Wasserstein distances for vortices approximation of Euler-type equations. *Math. Models Methods Appl. Sci.* **19**, 1357–1384, 2009
46. Hauray, M.: Mean field limit for the one dimensional Vlasov–Poisson equation. In: Sém. Laurent Schwartz 2012–2013, exp. 21, 16 pp. ,2014
47. Hauray, M., Jabin, P.-E.: N-particles approximation of the Vlasov equations with singular potential. *Arch. Ration. Mech. Anal.* **183**, 489–524, 2007
48. Hauray, M., Jabin, P.-E.: Particle approximations of Vlasov equations with singular forces: propagation of chaos. *Ann. Sci. École Norm. Sup.* **48**, 891–940, 2015
49. Jabin, P.-E.: Macroscopic limit of Vlasov type equations with friction. *Ann. Inst. H. Poincaré Anal. Non Linéaire.* **17**, 651–672, 2000
50. Jabin, P.-E., Wang, Z.: Mean field limit and propagation of chaos for Vlasov systems with bounded forces. *J. Funct. Anal.* **271**, 3588–3627, 2016
55. JABIN, P.-E., WANG, Z.: Mean Field Limit for Stochastic Particle Systems, Active Particles. Vol. 1. Advances in Theory, Models, and Applications. Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, pp. 379–402, 2017
56. JABIN, P.-E., WANG, Z.: Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels. Invent. Math. 214, 523–591, 2018
57. KARPER, T.K., MELLET, A., TRIVISA, K.: Hydrodynamic limit of the kinetic Cucker–Smale flocking model. Math. Models Methods Appl. Sci. 25, 131–163, 2015
58. KATO, T.: Linear evolution equations of “hyperbolic” type II. J. Math. Soc. Jpn. 25, 648–666, 1973
59. LATTANZIO, C., TZAVARAS, A.E.: Relative entropy in diffusive relaxation. SIAM J. Math. Anal. 45, 1563–1584, 2013
60. LATTANZIO, C., TZAVARAS, A.E.: From gas dynamics with large friction to gradient flows describing diffusion theories. Commun. Partial Differ. Equ. 42, 261–290, 2017
61. LAZAROVICI, D., PICKL, P.: A mean field limit for the Vlasov–Poisson system. Arch. Ration. Mech. Anal. 225, 1201–1231, 2017
62. MASMOUDI, N.: From Vlasov–Poisson system to the incompressible Euler system. Commun. Partial Differ. Equ. 26, 1913–1928, 2001
63. MINAKOWSKI, P., MUCHA, P. B., PESZEK, J., ZATORSKA, E.: Singular Cucker–Smale Dynamics, Active Particles. Vol. 2. Advances in Theory, Models, and Applications. Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, pp. 201–243, 2019
64. NEUNZERT, H.: An introduction to the nonlinear Boltzmann–Vlasov equation, In Kinetic theories and the Boltzmann equation (Montecatini Terme, 1981), Lecture Notes in Mathematics, vol. 1048. Springer, Berlin, 1984
65. PETRACHE, M., SERFATY, S.: Next order asymptotics and renormalized energy for Riesz interactions. J. Inst. Math. Jussieu. 16, 501–569, 2017
66. SERFATY, S.: Mean field limit for coulomb-type flows. Duke Math. J. 169, 2887–2935, 2020
67. SPOHN, H.: Large Scale Dynamics of Interacting Particles. Texts and Monographs in Physics. Springer, Berlin 1991
68. TADMOR, E., TAN, C.: Critical thresholds in flocking hydrodynamics with nonlocal alignment. Philos. Trans. A Math. Phys. Eng. Sci. 372, 20130401, 2014
69. TADMOR, E., WEI, D.: On the global regularity of subcritical Euler–Poisson equations with pressure. J. Eur. Math. Soc. 10, 757–769, 2008

José A. Carrillo
Department of Mathematics Mathematical Institute,
University of Oxford,
Oxford
OX2 6GG UK.
e-mail: carrillo@maths.ox.ac.uk

and

Young-Pil Choi
Department of Mathematics,
Yonsei University,
50 Yonsei-Ro, Seodaemun-Gu,
Seoul
03722 Republic of Korea.
e-mail: ypchoi@yonsei.ac.kr

(Received July 31, 2020 / Accepted May 14, 2021)
Published online June 1, 2021
© The Author(s) (2021)