On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions

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Contents

1 Introduction 1

2 Preliminaries 7
   2.1 A weighted Sobolev space 7
   2.2 Weak solutions of degenerate elliptic equations 9
   2.3 Local Schauder estimates 14

3 Proof of Theorem 1.5 20

4 Local analysis near isolated blow up points 24

5 Estimates on the sphere and proofs of main theorems 41

A Appendix 49
   A.1 A Kazdan-Warner identity 49
   A.2 A proof of Lemma 4.4 50
   A.3 Two lemmas on maximum principles 51
   A.4 Complementarities 54

1 Introduction

The Nirenberg problem concerns the following: For which positive function $K$ on the standard sphere $(\mathbb{S}^n, g_{\mathbb{S}^n}), n \geq 2$, there exists a function $w$ on $\mathbb{S}^n$ such that the scalar curvature (Gauss
curvature in dimension \( n = 2 \) \( R_g \) of the conformal metric \( g = e^w g_{\mathbb{S}^n} \) is equal to \( K \) on \( \mathbb{S}^n \)? The problem is equivalent to solving

\[
-\Delta_{g^w} w + 1 = Ke^{2w}, \quad \text{on } \mathbb{S}^2,
\]

and

\[
-\Delta_{g^w} v + c(n)R_0 v = c(n)Kv^{\frac{n+2}{n-2}}, \quad \text{on } \mathbb{S}^n \text{ for } n \geq 3,
\]

where \( c(n) = (n-2)/(4(n-1)) \), \( R_0 = n(n-1) \) is the scalar curvature of \( (\mathbb{S}^n, g_{\mathbb{S}^n}) \) and \( v = e^{\frac{n+2}{n-2}w} \).

The first work on the problem is by D. Koutroufiotis [65], where the solvability on \( \mathbb{S}^2 \) is established when \( K \) is assumed to be an antipodally symmetric function which is close to 1. Moser [78] established the solvability on \( \mathbb{S}^2 \) for all antipodally symmetric functions \( K \) which is positive somewhere. Without assuming any symmetry assumption on \( K \), sufficient conditions were given in dimension \( n = 2 \) by Chang and Yang [50] and [51], and in dimension \( n = 3 \) by Bahri and Coron [6]. Compactness of all solutions in dimensions \( n = 2, 3 \) can be found in work of Chang, Gursky and Yang [29], Han [55] and Schoen and Zhang [88]. In these dimensions, a sequence of solutions can not blow up at more than one point. Compactness and existence of solutions in higher dimensions were studied by Li in [68] and [69]. The situation is very different, as far as the compactness issues are concerned: In dimension \( n \geq 4 \), a sequence of solutions can blow up at more than one point, as shown in [69]. There have been many papers on the problem and related ones, see, e.g., [1, 2, 3, 7, 9, 10, 16, 17, 25, 26, 29, 30, 31, 32, 34, 35, 37, 44, 47, 55, 56, 58, 61, 62, 67, 75, 77, 83, 84, 93, 95, 96].

In [54], Graham, Jenne, Mason and Sparling constructed a sequence of conformally covariant elliptic operators, \( \{P^\sigma_k\} \), on Riemannian manifolds for all positive integers \( k \) if \( n \) is odd, and for \( k \in \{1, \ldots, n/2\} \) if \( n \) is even. Moreover, \( P^\sigma_1 \) is the conformal Laplacian \(-L_g := -\Delta_g + c(n)R_g\) and \( P^\sigma_2 \) is the Paneitz operator. The construction in [54] is based on the ambient metric construction of [49]. Up to positive constants \( P^\sigma_k(1) \) is the scalar curvature of \( g \) and \( P^\sigma_2(1) \) is the \( Q \)-curvature. Prescribing \( Q \)-curvature problem on \( \mathbb{S}^n \) was studied extensively, see, e.g., [8, 41, 42, 43, 50, 91, 92].

Making use of a generalized Dirichlet to Neumann map, Graham and Zworski [53] introduced a meromorphic family of conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds. Recently, Chang and González [23] reconciled the way of Graham and Zworski to define conformally invariant operators \( P^\sigma_g \) of non-integer order \( \sigma \in (0,\frac{1}{2}) \) and the localization method of Caffarelli and Silvestre [22] for fractional Laplacian \((-\Delta)^\sigma\) on the Euclidean space \( \mathbb{R}^n \). These lead naturally to a fractional order curvature \( R^\sigma_g \) which will be called \( \sigma \)-curvature in this paper. A typical example is that standard conformal spheres \( (\mathbb{S}^n, [g_{\mathbb{S}^n}]) \) are the conformal infinity of Poincaré disks \( (\mathbb{B}^{n+1}, [g_{\mathbb{B}^n}]) \). In this case, \( \sigma \)-curvature can be expressed in the following explicit way. Let \( g \) be a representative in the conformal class \([g_{\mathbb{S}^n}]\) and write \( g = v^{-\frac{2}{n-2}}g_{\mathbb{S}^n} \), where \( v \) is positive and smooth on \( \mathbb{S}^n \). Then the \( \sigma \)-curvature for \( (\mathbb{S}^n, g) \) can be computed as

\[
R^\sigma_g = v^{-\frac{n+2\sigma}{n-2}}P_\sigma(v), \quad (1.1)
\]

where \( P_\sigma \) is an intertwining operator and

\[
P_\sigma = \frac{\Gamma(B + \frac{1}{2} + \sigma)}{\Gamma(B + \frac{1}{2} - \sigma)} B = \sqrt{-\Delta_{g^w} + \left(\frac{n-1}{2}\right)^2}, \quad (1.2)
\]

\( \Gamma \) is the Gamma function and \( \Delta_{g^w} \) is the Laplace-Beltrami operator on \( (\mathbb{S}^n, g_{\mathbb{S}^n}) \). The operator \( P_\sigma \) can be seen more concretely on \( \mathbb{R}^n \) using stereographic projection. The stereographic projection
from $S^n \setminus \{N\}$ to $\mathbb{R}^n$ is the inverse of

$$ F : \mathbb{R}^n \to S^n \setminus \{N\}, \quad x \mapsto \left( \frac{2x}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right), $$

where $N$ is the north pole of $S^n$. Then

$$ (P_\sigma(\phi)) \circ F = |J_F|^{-\frac{n + 2\sigma}{n}} (-\Delta)^\sigma (|J_F|^{\frac{n + 2\sigma}{n}} (\phi \circ F)), \quad \text{for } \phi \in C^\infty(S^n) \quad (1.3) $$

where

$$ |J_F| = \left( \frac{2}{1 + |x|^2} \right)^n, $$

and $(-\Delta)^\sigma$ is the fractional Laplacian operator (see, e.g., page 117 of [86]). When $\sigma \in (0, 1)$, Pavlov and Samko [81] showed that

$$ P_\sigma(v)(\xi) = P_\sigma(1)v(\xi) + c_{n,-\sigma} \int_{S^n} \frac{v(\xi) - v(\zeta)}{|\xi - \zeta|^{n+2\sigma}} \text{div}_{g^\sigma} (\zeta) \quad (1.4) $$

for $v \in C^2(S^n)$, where $c_{n,-\sigma} = \frac{n^{2\sigma} \Gamma(n+2\sigma)}{\pi^n \Gamma(1-\sigma)}$ and $\int_{S^n}$ is understood as $\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon}$.

For the $\sigma$-curvatures on general manifolds we refer to [53], [28], [52] and references therein. Corresponding to the Yamabe problem, fractional Yamabe problems for $\sigma$-curvatures are studied in [51], [52] and [82], and fractional Yamabe flows on $S^n$ are studied in [64].

From (1.1), we consider

$$ P_\sigma(v) = c(n, \sigma) K v^{\frac{n + 2\sigma}{n - 2\sigma}}, \quad \text{on } S^n, \quad (1.5) $$

where $c(n, \sigma) = P(1)$, and $K > 0$ is a continuous function on $S^n$. When $K = 1$, (1.5) is the Euler-Lagrange equation for a functional associated to the fractional Sobolev inequality on $S^n$ (see [8]), and all positive solutions must be of the form

$$ v_{\xi_0, \lambda}(\xi) = \left( \frac{2\lambda}{2 + (\lambda^2 - 1)(1 - \cos \text{dist}_{g^n}(\xi, \xi_0))} \right)^{\frac{n + 2\sigma}{2}}, \quad \xi \in S^n \quad (1.6) $$

for some $\xi_0 \in S^n$ and positive constant $\lambda$. This classification can be found in [74], [36] and [70].

In general, (1.5) may have no positive solution, since if $v$ is a positive solution of (1.5) with $K \in C^1(S^n)$ then it has to satisfy the Kazdan-Warner type condition

$$ \int_{S^n} \langle \nabla_{g^n} K, \nabla_{g^n} \xi \rangle v^{\frac{n - 2\sigma}{n - 2\sigma}} \, d\xi = 0. \quad (1.7) $$

Consequently if $K(\xi) = \xi_{n+1} + 2$, (1.5) has no solutions. The proof of (1.7) is provided in Appendix A.1.

In this and a subsequent paper [63], we study (1.5) with $\sigma \in (0, 1)$, a fractional Nirenberg problem. Throughout the paper, we assume that $\sigma \in (0, 1)$ without otherwise stated.

**Definition 1.1.** For $d > 0$, we say that $K \in C(S^n)$ has flatness order greater than $d$ at $\xi$ if, in some local coordinate system $\{y_1, \cdots, y_n\}$ centered at $\xi$, there exists a neighborhood $\mathcal{O}$ of $0$ such that $K(y) = K(0) + o(|y|^d)$ in $\mathcal{O}$. 

3
Theorem 1.1. Let \( n \geq 2 \), and \( K \in C^{1,1}(\mathbb{S}^n) \) be an antipodally symmetric function, i.e., \( K(\xi) = K(-\xi) \forall \xi \in \mathbb{S}^n \), and be positive somewhere on \( \mathbb{S}^n \). If there exists a maximum point \( \xi_0 \) of \( K \) at which \( K \) has flatness order greater than \( n - 2\sigma \), then (1.5) has at least one positive \( C^2 \) solution.

For \( 2 \leq n < 2 + 2\sigma \), \( K \in C^{1,1}(\mathbb{S}^n) \) has flatness order greater than \( n - 2\sigma \) at every maximum point. When \( \sigma = 1 \), the above theorem was proved by Escobar and Schoen [46] for \( n \geq 3 \).

Theorem 1.2. Let \( n \geq 2 \). Suppose that \( K \in C^{1,1}(\mathbb{S}^n) \) is a positive function satisfying that for any critical point \( \xi_0 \) of \( K \), in some geodesic normal coordinates \( \{y_1, \cdots, y_n\} \) centered at \( \xi_0 \), there exist some small neighborhood \( \mathcal{O} \) of \( 0 \) and positive constants \( \beta = \beta(\xi_0) \in (n - 2\sigma, n) \), \( \gamma \in (n - 2\sigma, \beta] \) such that \( K \in C^{[\gamma], \gamma - [\gamma]}(\mathcal{O}) \) (where \([\gamma]\) is the integer part of \( \gamma \)) and

\[
K(y) = K(0) + \sum_{j=1}^{n} a_j |y_j|^\beta + R(y), \quad \text{in } \mathcal{O},
\]

where \( a_j = a_j(\xi_0) \neq 0 \), \( \sum_{j=1}^{n} a_j \neq 0 \), \( R(y) \in C^{[\beta]-1,1}(\mathcal{O}) \) satisfies

\[
\sum_{s=0}^{[\beta]} |\nabla^s R(y)||y|^{-\beta + s} \to 0 \text{ as } y \to 0.
\]

If

\[
\sum_{\xi \in \mathbb{S}^n \text{ such that } \nabla_{g_{\mathbb{S}^n}} K(\xi) = 0, \sum_{j=1}^{n} a_j(\xi) < 0} (-1)^{i(\xi)} \neq (-1)^n,
\]

where

\[
i(\xi) = \# \{ a_j(\xi) : \nabla_{g_{\mathbb{S}^n}} K(\xi) = 0, a_j(\xi) < 0, 1 \leq j \leq n \},
\]

then (1.5) has at least one \( C^2 \) positive solution. Moreover, there exists a positive constant \( C \) depending only on \( n, \sigma \) and \( K \) such that for all positive \( C^2 \) solutions \( v \) of (1.5),

\[
1/C \leq v \leq C \quad \text{and} \quad \|v\|_{C^2(\mathbb{S}^n)} \leq C.
\]

For \( n = 3, \sigma = 1 \), the existence part of the above theorem was established by Bahri and Coron [6], and the compactness part were given in Chang, Gursky and Yang [29] and Schoen and Zhang [88]. For \( n \geq 4, \sigma = 1 \), the above theorem was proved by Li [68].

We now consider a class of functions \( K \) more general than that in Theorem 1.2, which is modified from [88].

Definition 1.2. For any real number \( \beta > 1 \), we say that a sequence of functions \( \{K_i\} \) satisfies condition \((*)_\beta \) for some sequence of constants \( L(\beta, i) \) in some region \( \Omega_i \), if \( \{K_i\} \in C^{[\beta], \beta - [\beta]}(\Omega_i) \) satisfies

\[
|\nabla^{[\beta]} K_i|_{C^{\beta - [\beta]}(\Omega_i)} \leq L(\beta, i),
\]

and, if \( \beta \geq 2 \), that

\[
|\nabla^s K_i(y)| \leq L(\beta, i)|\nabla K_i(y)|^{(\beta - s)/(\beta - 1)},
\]

for all \( 2 \leq s \leq [\beta], y \in \Omega_i, \nabla K_i(y) \neq 0 \).

Note that the function \( K \) in Theorem 1.2 satisfies \((*)_\beta \) condition.
**Remark 1.1.** For $1 \leq \beta_1 \leq \beta_2$, if $\{K_i\}$ satisfies $(\star)_{\beta_2}$ for some sequences of constants $\{L(\beta_2, i)\}$ in some regions $\Omega_i$, then $\{K_i\}$ satisfies $(\star)_{\beta_1}$ for $\{L(\beta_1, i)\}$, where

$$L(\beta_1, i) = \begin{cases} L(\beta_2, i) \max \left(\frac{\max_{2 \leq s \leq [\beta_1]} \|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{2s-1}{2}}}{\|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{2s-1}{2}}}, \text{diam}(\Omega_i)^{\beta_2-\beta_1}\right), & \text{if } [\beta_2] = [\beta_1] \\ L(\beta_2, i) \max \left(\frac{\max_{2 \leq s \leq [\beta_1]} \|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{2s-1}{2}}}{\|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{2s-1}{2}}}, \|\nabla K_i\|_{L^\infty(\Omega_i)}^{\frac{2s-1}{2}} \text{diam}(\Omega_i)^{1+\beta_1-\beta_2}\right), & \text{if } [\beta_2] > [\beta_1] \end{cases}$$

in the corresponding regions.

The following theorem gives a priori bounds of solutions in $L^{\frac{2n}{n-2\sigma}}$ norm.

**Theorem 1.3.** Let $n \geq 2$, and $K \in C^{1,1}(\mathbb{S}^n)$ be a positive function. If there exists some constant $d > 0$ such that $K$ satisfies $(\star)_{(n-2\sigma)}$ for some constant $L > 0$ in $\Omega_d := \{\xi \in \mathbb{S}^n : |\nabla g_0 K(\xi)| < d\}$, then for any positive solution $v \in C^2(\mathbb{S}^n)$ of (1.5),

$$\|v\|_{L^{\frac{2n}{n-2\sigma}}(\mathbb{S}^n)} \leq C,$$

where $C$ depends only on $n$, $\sigma$, $\inf_{\mathbb{S}^n} K > 0$, $\|K\|_{C^{1,1}(\mathbb{S}^n)}$, $L$, and $d$.

The above theorem was proved by Schoen and Zhang [88] for $n = 3$ and $\sigma = 1$, and by Li [68] for $n \geq 4$ and $\sigma = 1$.

Denote $H^\sigma(\mathbb{S}^n)$ by the closure of $C^\infty(\mathbb{S}^n)$ under the norm

$$\int_{\mathbb{S}^n} v P_\sigma(v) \, d\text{vol}_{g_0}.$$ 

The estimate (1.8) for the solution $v$ is equivalent to

$$\|v\|_{H^\sigma(\mathbb{S}^n)} \leq C.$$

However, the estimate (1.8) is not sufficient to imply $L^\infty$ bound for $v$ on $\mathbb{S}^n$. For instance,

$$\int_{\mathbb{S}^n} v_{\xi_0, \lambda}^{\frac{2n}{n-2\sigma}}(\xi) \, d\text{vol}_{g_0} = \int_{\mathbb{S}^n} \, d\text{vol}_{g_0},$$

but $v_{\xi_0, \lambda}(\xi_0) = \lambda^{\frac{2n}{n-2\sigma}} \to \infty$ as $\lambda \to \infty$. Furthermore, a sequence of solutions $v_i$ may blow up at more than one point, and it is the case when $\sigma = 1$ (see [69]). The following theorem shows that the latter situation does not happen when $K$ satisfies a little stronger condition.

**Theorem 1.4.** Let $n \geq 2$. Suppose that $\{K_i\} \in C^{1,1}(\mathbb{S}^n)$ is a sequence of positive functions with uniform $C^{1,1}$ norm and $1/A_1 \leq K_i \leq A_1$ on $\mathbb{S}^n$ for some $A_1 > 0$ independent of $i$. Suppose also that $\{K_i\}$ satisfying $(\star)_{\beta_2}$ condition for some constants $\beta > n - 2\sigma$, $L_0, d > 0$ in $\Omega_d$. Let $\{v_i\} \in C^2(\mathbb{S}^n)$ be a sequence of corresponding positive solutions of (1.5) and $v_i(\xi_i) = \max_{\mathbb{S}^n} v_i$ for some $\xi_i$. Then, after passing to a subsequence, $\{v_i\}$ is either bounded in $L^\infty(\mathbb{S}^n)$ or blows up at exactly one point in the strong sense: There exists a sequence of Möbius diffeomorphisms $\{\varphi_i\}$ from $\mathbb{S}^n$ to $\mathbb{S}^n$ satisfying $\varphi_i(\xi_i) = \xi_i$ and $\det d\varphi_i(\xi_i) |\frac{2n}{n-2\sigma}| = v_i^{-1}(\xi_i)$ such that

$$\|T_{\varphi_i} v_i - 1\|_{C^0(\mathbb{S}^n)} \to 0, \quad as \ i \to \infty,$$

where $T_{\varphi, v_i} := (v \circ \varphi_i) |\det d\varphi_i| |\frac{2n}{n-2\sigma}|.$
For $n = 3, \sigma = 1$, the above theorem was established by Chang, Gursky and Yang in [29] and by Schoen and Zhang in [88]. For $n \geq 4, \sigma = 1$, the above theorem was proved by Li in [68].

Möbius diffeomorphisms $\varphi$ from $S^n$ to $S^n$ are those given by $\varphi = \phi \circ F$ where $\phi$ is a Möbius transformation from $\mathbb{R}^n \cup \{ \infty \}$ to $\mathbb{R}^n \cup \{ \infty \}$ generated by translations, multiplications by nonzero constant and the inversion $x \to x/|x|^2$.

Our local analysis of solutions of (1.5) relies on a localization method introduced by Caffarelli and Silvestre in [22] for the fractional Laplacian $(-\Delta)\sigma$ on the Euclidean space $\mathbb{R}^n$, through which (1.5) is connected to a degenerate elliptic differential equation in one dimension higher (see section 2).

The proofs of Theorem 1.3 and Theorem 1.4 make use of blow up analysis of solutions of (1.5), which is an adaptation of the analysis for $\sigma = 1$ developed in [88] and [68]. Our blow up analysis requires a Liouville type theorem. For the definitions of weak solutions and the space $H_{loc}(t^{1-2\sigma}, \mathbb{R}^{n+1})$ in the following Liouville type theorem we refer to Definition 2.1 and the beginning of section 3.

**Theorem 1.5.** Let $U \in H_{loc}(t^{1-2\sigma}, \mathbb{R}^{n+1}), U(X) \geq 0$ in $\mathbb{R}^{n+1}_+$ and $U \not\equiv 0$, be a weak solution of

$$
\begin{align*}
\text{div}(t^{1-2\sigma}\nabla U(x,t)) &= 0, \quad \text{in } \mathbb{R}^{n+1}_+, \\
\lim_{t \to 0} t^{1-2\sigma}\partial_t U(x,t) &= U^{\frac{n+2\sigma}{n-2\sigma}}(x,0), \quad x \in \mathbb{R}^n.
\end{align*}
$$

Then $U(x,0)$ takes the form

$$
(N_\sigma c_{n,\sigma} 2^{2\sigma}) \frac{n-2\sigma}{2} \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2}\right)^{\frac{n-2\sigma}{2}}
$$

where $\lambda > 0, x_0 \in \mathbb{R}^n, c_{n,\sigma}$ is the constant in (1.5) and $N_\sigma$ is the constant in (2.4). Moreover,

$$
U(x,t) = \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x-y,t)U(y,0) \, dy
$$

for $(x,t) \in \mathbb{R}^{n+1}_+$, where $\mathcal{P}_\sigma(x)$ is the kernel given in (2.2).

**Remark 1.2.** If we replace $U^{\frac{n+2\sigma}{n-2\sigma}}(x,0)$ by $U^p(x,0)$ for $0 \leq p < \frac{n+2\sigma}{n-2\sigma}$ in (1.9), then the only nonnegative solution of (1.9) is $U \equiv 0$. Moreover, for $p < 0$, (1.9) has no positive solution. These can be seen from the proof of Theorem 1.3 with a standard modification (see, e.g., the proof of Theorem 1.2 in [24]). For $\sigma \in (1/2, 1)$ and $1 < p < \frac{n+2\sigma}{n-2\sigma}$, this result has been proved in [40].

**Remark 1.3.** We do not make any assumption on the behavior of $U$ near $\infty$. If we assume that $U \in H(t^{1-2\sigma}, \mathbb{R}^{n+1})$, the theorem in the case of $p = \frac{n+2\sigma}{n-2\sigma}$ follows from [59] and [70]. When $\sigma = \frac{1}{2}$, the above theorem can be found in [59], [60], [73], [80] and [72].

Given the pages needed to present the proofs of all the results, we leave the proofs of Theorem 1.1 and the existence part of Theorem 1.2 to the subsequent paper [63]. The needed ingredients for a proof of the existence part of Theorem 1.2 are all developed in this paper. With these ingredients, the existence part of Theorem 1.2 follows from a perturbation result and a degree argument which are given in [63].
The present paper is organized as the following. In section 2, we derive some properties for solutions of fractional Laplacian equations. In particular we prove that local Schauder estimates hold for positive solutions. In section 3, using the method of moving spheres, we establish Theorem 1.3. This Liouville type theorem and the local Schauder estimates are used in the blow up analysis of solutions of (1.5). In section 4, we establish accurate blow up profiles of solutions of (1.5) near isolated blow up points. In fact most of the estimates hold also for subcritical approximations to solutions of fractional Laplacian equations. In particular we prove that local Schauder estimates hold for positive solutions. In section 3, using the method of moving spheres, we establish Theorem 1.3, under appropriate hypotheses on \( \Omega \). The proofs of Theorem 1.2, 1.3 and 1.4 are given in this section. In the Appendix we provide a Kazdan-Warner identity, Lemma 4.4 that is in the same spirit of the classical Bôcher theorem, two lemmas on maximum principles and some complementarities.

2 Preliminaries

2.1 A weighted Sobolev space

Let \( \sigma \in (0, 1) \), \( X = (x, t) \in \mathbb{R}^{n+1} \) where \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R} \). Then \(|t|^{1-2\sigma}\) belongs to the Muckenhoupt \( A_2 \) class in \( \mathbb{R}^{n+1} \), namely, there exists a positive constant \( C \), such that for any ball \( B \subset \mathbb{R}^{n+1} \)

\[
\left( \frac{1}{|B|} \int_B |t|^{1-2\sigma} \, dX \right) \left( \frac{1}{|B|} \int_B |t|^{2\sigma-1} \, dX \right) \leq C.
\]

Let \( D \) be an open set in \( \mathbb{R}^{n+1} \). Denote \( L^2(|t|^{1-2\sigma}, D) \) as the Banach space of all measurable functions \( U \), defined on \( D \), for which

\[
\|U\|_{L^2(|t|^{1-2\sigma}, D)} := \left( \int_D |t|^{1-2\sigma} U^2 \, dX \right)^{\frac{1}{2}} < \infty.
\]

We say that \( U \in H(|t|^{1-2\sigma}, D) \) if \( U \in L^2(|t|^{1-2\sigma}, D) \), and its weak derivatives \( \nabla U \) exist and belong to \( L^2(|t|^{1-2\sigma}, D) \). The norm of \( U \) in \( H(|t|^{1-2\sigma}, D) \) is given by

\[
\|U\|_{H(|t|^{1-2\sigma}, D)} := \left( \int_D |t|^{1-2\sigma} U^2(X) \, dX + \int_D |t|^{1-2\sigma} |\nabla U(X)|^2 \, dX \right)^{\frac{1}{2}}.
\]

It is clear that \( H(|t|^{1-2\sigma}, D) \) is a Hilbert space with the inner product

\[
\langle U, V \rangle := \int_D |t|^{1-2\sigma} (UV + \nabla U \nabla V) \, dX.
\]

Note that the set of smooth functions \( C^\infty(D) \) is dense in \( H(|t|^{1-2\sigma}, D) \). Moreover, if \( D \) is a domain, i.e. a bounded connected open set, with Lipschitz boundary \( \partial D \), then there exists a linear, bounded extension operator from \( H(|t|^{1-2\sigma}, D) \) to \( H(|t|^{1-2\sigma}, \mathbb{R}^{n+1}) \) (see, e.g., [39]).

Let \( \Omega \) be an open set in \( \mathbb{R}^n \). Recall that \( H^\sigma(\Omega) \) is the fractional Sobolev space defined as

\[
H^\sigma(\Omega) := \left\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{2} + \sigma}} \in L^2(\Omega \times \Omega) \right\}
\]
with the norm
\[ \|u\|_{H^\sigma(\Omega)} := \left( \int_\Omega u^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2\sigma}} \, dx \, dy \right)^{1/2}. \]

The set of smooth functions \( C^\infty(\Omega) \) is dense in \( H^\sigma(\Omega) \). If \( \Omega \) is a domain with Lipschitz boundary, then there exists a linear, bounded extension operator from \( H^\sigma(\Omega) \) to \( H^\sigma(\mathbb{R}^n) \). Note that \( H^\sigma(\mathbb{R}^n) \) with the norm \( \| \cdot \|_{H^\sigma(\mathbb{R}^n)} \) is equivalent to the following space
\[ \{ u \in L^2(\mathbb{R}^n) : |\xi|^\sigma \mathcal{F}(u)(\xi) \in L^2(\mathbb{R}^n) \} \]
with the norm
\[ \| \cdot \|_{L^2(\mathbb{R}^n)} + \| |\xi|^\sigma \mathcal{F}(\cdot)(\xi)\|_{L^2(\mathbb{R}^n)} \]
where \( \mathcal{F} \) denotes the Fourier transform operator. It is known that (see, e.g., [76]) there exists \( C > 0 \) depending only on \( n \) and \( \sigma \) such that for \( U \in H^1(\mathbb{R}^n) \cap C(\mathbb{R}_+^{n+1}) \), \( \|U(\cdot,0)\|_{H^\sigma(\mathbb{R}^n)} \leq C\|U\|_{H^1(\mathbb{R}^n) \cap C(\mathbb{R}_+^{n+1})} \). Hence by a standard density argument, every \( U \in H^1(\mathbb{R}^n) \) has a well-defined trace \( u := U(\cdot,0) \in H^\sigma(\mathbb{R}^n) \).

We define \( H^\sigma(\mathbb{R}^n) \) as the closure of the set \( C^\infty_c(\mathbb{R}^n) \) of compact supported smooth functions under the norm
\[ \|u\|_{H^\sigma(\mathbb{R}^n)} = \| |\xi|^\sigma \mathcal{F}(u)(\xi)\|_{L^2(\mathbb{R}^n)}, \]
Then there exists a constant \( C \) depending only on \( n \) and \( \sigma \) such that
\[ \|u\|_{L^{\frac{2n}{n+2\sigma-2}}(\mathbb{R}^n)} \leq C\|u\|_{H^\sigma(\mathbb{R}^n)}, \quad \text{for all } u \in C^\infty_c(\mathbb{R}^n). \quad (2.1) \]

For any \( u \in H^\sigma(\mathbb{R}^n) \), set
\[ U(x,t) = \mathcal{P}_\sigma[u] := \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x - \xi, t)u(\xi) \, d\xi, \quad (x,t) \in \mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, +\infty), \quad (2.2) \]
where
\[ \mathcal{P}_\sigma(x,t) = \beta(n,\sigma) \frac{t^{2\sigma}}{(|x|^2 + t^2)^{n+2\sigma}} \]
with constant \( \beta(n,\sigma) \) such that \( \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x,1) \, dx = 1 \). Then \( U \in C^\infty(\mathbb{R}_+^{n+1}), U \in L^2(t^{1-2\sigma},\mathbb{R}_+^{n+1}), \) and \( \nabla U \in L^2(t^{1-2\sigma},\mathbb{R}_+^{n+1}) \). Moreover, \( U \) satisfies (see [22])
\[ \text{div}(t^{1-2\sigma}\nabla U) = 0 \quad \text{in } \mathbb{R}_+^{n+1}, \quad (2.3) \]
\[ \|\nabla U\|_{L^2(t^{1-2\sigma},\mathbb{R}_+^{n+1})} = N_\sigma \|u\|_{H^\sigma(\mathbb{R}^n)}, \quad (2.4) \]
and
\[ -\lim_{t \to 0} \frac{1}{t^{1-2\sigma}} \partial_t U(x,t) = N_\sigma (-\Delta)^\sigma u(x), \quad \text{in } \mathbb{R}^n \quad (2.5) \]
in distribution sense, where \( N_\sigma = 2^{1-2\sigma} \Gamma(1-\sigma)/\Gamma(\sigma) \). We refer \( U = \mathcal{P}_\sigma[u] \) in (2.2) to be the extension of \( u \) for any \( u \in H^\sigma(\mathbb{R}^n) \).

For a domain \( D \subseteq \mathbb{R}_+^{n+1} \) with boundary \( \partial D \), we denote \( \partial' D \) as the interior of \( \overline{D} \cap \partial\mathbb{R}_+^{n+1} \) in \( \mathbb{R}^n = \partial\mathbb{R}_+^{n+1} \) and \( \partial'' D = \partial D \setminus \partial' D \).
Proposition 2.1. Let $D = \Omega \times (0, R) \subset \mathbb{R}^n \times \mathbb{R}_+$, $R > 0$ and $\partial \Omega$ be Lipschitz.

(i) If $U \in \mathcal{H}(t^{1-2\sigma}, D) \cap C(D \cup \partial' D)$, then $u := U(\cdot, 0) \in \mathcal{H}^\sigma(\Omega)$, and 
\[ \|u\|_{\mathcal{H}^\sigma(\Omega)} \leq C\|U\|_{\mathcal{H}(t^{1-2\sigma}, D)} \]
where $C$ is a positive constant depending only on $n, \sigma, R$ and $\Omega$. Hence every $U \in \mathcal{H}(t^{1-2\sigma}, D)$ has a well-defined trace $U(\cdot, 0) \in \mathcal{H}^\sigma(\Omega)$ on $\partial' D$. Furthermore, there exists $C_{n,\sigma} > 0$ depending only on $n$ and $\sigma$ such that 
\[ \|U(\cdot, 0)\|_{L_{n,\sigma,R}^{2/(1-2\sigma)}(\Omega)} \leq C_{n,\sigma}\|\nabla U\|_{L^2(t^{1-2\sigma}, D)} \quad \text{for all } U \in C_c^\infty(D \cup \partial' D). \tag{2.6} \]

(ii) If $u \in \mathcal{H}^\sigma(\Omega)$, then there exists $U \in \mathcal{H}(t^{1-2\sigma}, D)$ such that the trace of $U$ on $\Omega$ equals to $u$ and 
\[ \|U\|_{\mathcal{H}(t^{1-2\sigma}, D)} \leq C\|u\|_{\mathcal{H}^\sigma(\Omega)} \]
where $C$ is a positive constant depending only on $n, \sigma, R$ and $\Omega$.

Proof. The above results are well-known and here we just sketch the proofs. For (i), by the previously mentioned result on the extension operator, there exists $\tilde{U} \in \mathcal{H}(t^{1-2\sigma}, \mathbb{R}^{n+1})$ such that $\tilde{U} = U$ in $D$ and 
\[ \|\tilde{U}\|_{\mathcal{H}(t^{1-2\sigma}, \mathbb{R}^{n+1})} \leq C\|U\|_{\mathcal{H}(t^{1-2\sigma}, D)}. \]

Hence by the previously mentioned result on the trace from $\mathcal{H}(t^{1-2\sigma}, \mathbb{R}^{n+1})$ to $\mathcal{H}^\sigma(\mathbb{R}^n)$, we have 
\[ \|u\|_{\mathcal{H}^\sigma(\Omega)} \leq \|\tilde{U}(\cdot, 0)\|_{\mathcal{H}^\sigma(\mathbb{R}^n)} \leq C\|\tilde{U}\|_{\mathcal{H}(t^{1-2\sigma}, \mathbb{R}^{n+1})} \leq C\|U\|_{\mathcal{H}(t^{1-2\sigma}, D)}. \]

For (2.6), we extend $U$ to be zero in the outside of $\overline{\Omega}$ and let $V$ be the extension of $U(\cdot, 0)$ as in (2.2). The inequality (2.6) follows from (2.1), (2.4) and 
\[ \|\nabla V\|_{L^2(t^{1-2\sigma}, \mathbb{R}^{n+1})} \leq \|\nabla U\|_{L^2(t^{1-2\sigma}, D)} \]
where Lemma A.3 is used in the above inequality.

For (ii), since $\partial \Omega$ is Lipschitz, there exists $\tilde{u} \in \mathcal{H}^\sigma(\mathbb{R}^n)$ such that $\tilde{u} = u$ in $\Omega$ and $\|\tilde{u}\|_{\mathcal{H}^\sigma(\mathbb{R}^n)} \leq C\|u\|_{\mathcal{H}^\sigma(\Omega)}$. Then $U = \mathcal{P}_\sigma[\tilde{u}]$, the extension of $\tilde{u}$, satisfies (ii). \hfill \Box

2.2 Weak solutions of degenerate elliptic equations

Let $D$ be a domain in $\mathbb{R}^{n+1}_{+}$ with $\partial' D \neq \emptyset$. Let $a \in L^{\frac{2\sigma}{2-2\sigma}}_{loc}(\partial' D)$ and $b \in L^1_{loc}(\partial' D)$. Consider 
\[ \begin{cases} 
\text{div}(t^{1-2\sigma}\nabla U(X)) = 0 & \text{in } D \\
- \lim_{t \to 0^+} t^{1-2\sigma}\partial_t U(x, t) = a(x)U(x, 0) + b(x) & \text{on } \partial' D.
\end{cases} \tag{2.7} \]

Definition 2.1. We say that $U \in \mathcal{H}(t^{1-2\sigma}, D)$ is a weak solution (resp. supersolution, subsolution) of (2.7) in $D$, if for every nonnegative $\Phi \in C_c^\infty(D \cup \partial' D)$ 
\[ \int_D t^{1-2\sigma}\nabla U \nabla \Phi = (\text{resp.} \geq, \leq) \int_{\partial' D} aU\Phi + b\Phi. \tag{2.8} \]
We denote $Q_R = B_R \times (0, R)$ where $B_R \subset \mathbb{R}^n$ is the ball with radius $R$ and centered at 0.

**Proposition 2.2.** Suppose that $a(x) \in L^{\frac{2n}{n+2}}(B_1)$ and $b(x) \in L^{\frac{2n}{n+2}}(B_1)$. Let $U \in H(t^{1-2\sigma}, Q_1)$ be a weak solution of (2.7) in $Q_1$. There exists $\delta > 0$ depending only on $n$ and $\sigma$ such that if $\|a^+\|_{L^{\frac{2n}{n+2}}(B_1)} < \delta$, then there exists a constant $C$ depending only on $n, \sigma$ and $\delta$ such that

$$\|U\|_{H(t^{1-2\sigma}, Q_{1/2})} \leq C(\|U\|_{L^2(t^{1-2\sigma}, Q_1)} + \|b\|_{L^{\frac{2n}{n+2}}(B_1)}).$$

Consequently, if $a \in L^p(B_1)$ for $p > \frac{n}{2\sigma}$, then $C$ depends only on $n, \sigma, \|a\|_{L^p(B_1)}$.

**Proof.** Let $\eta \in C^\infty_c(Q_1 \cup \partial Q_1)$ be a cut-off function which equals to 1 in $Q_{1/2}$ and supported in $Q_{3/4}$. By a density argument, we can choose $\eta^2 U$ as a test function in (2.8). Then we have, by Cauchy-Schwarz inequality,

$$\int_{Q_1} t^{1-2\sigma} \eta^2 |\nabla U|^2 \, dX \leq 4 \int_{Q_1} t^{1-2\sigma} (|\nabla \eta|^2 U^2 + 2 \int_{\partial Q_1} a^+(\eta U)^2 + b \eta^2 U \, dx.$$

By Hölder inequality and Proposition 2.1,

$$\int_{\partial Q_1} a^+(\eta U)^2 \, dx \leq \delta \|\eta U\|_{L^\infty(\partial Q_1)}^2 \leq \delta C(n, \sigma) \|\nabla (\eta U)\|^2_{L^2(t^{1-\sigma}, Q_1)}.$$

By Young’s inequality $\forall \varepsilon > 0$,

$$\int_{\partial Q_1} b \eta^2 U(\cdot, 0) \, dx \leq \varepsilon \|\eta U\|_{L^\infty(\partial Q_1)}^2 + C(\varepsilon) \|b\|_{L^\infty(\partial Q_1)}^2 \leq \varepsilon C(n, \sigma) \|\nabla (\eta U)\|^2_{L^2(t^{1-\sigma}, Q_1)} + C(\varepsilon) \|b\|^2_{L^\infty(\partial Q_1)}.$$

The first conclusion follows immediately if $\delta$ is sufficiently small.

If $a \in L^p(B_1)$, we can choose $r$ small such that $\|a\|_{L^{\frac{2n}{n+2}}(B_r(x_0))} < \delta$ for any ball $B_r(x_0) \subset B_1$. Then $\hat{U}(x, t) = r^{\frac{2\sigma}{n+2}} U(rx + x_0, rt)$ satisfies (2.7) with $\hat{a}(x) = r^{2\sigma} a(rx + x_0)$ and $\hat{b}(x, t) = r^{\frac{2\sigma}{n+2}} b(rx + x_0)$ in $Q_1$. Since $\|\hat{a}\|_{L^{\frac{2n}{n+2}}(B_1)} < \delta$, applying the above result to $\hat{U}$, we have

$$\|U\|_{H(t^{1-2\sigma}, B_{r/2} \times (0, r/2))} \leq C(\|U\|_{L^2(t^{1-2\sigma}, Q_1)} + \|b\|_{L^{\frac{2n}{n+2}}(B_1)}),$$

where $C$ depends only on $n, \sigma, \|a\|_{L^\infty(B_1)}$. This, together with the fact that (2.7) is uniformly elliptic in $B_1 \times (r/4, 1)$, finishes the proof.

**Proposition 2.3.** Suppose that $a(x) \in L^{\frac{2n}{n+2}}(B_1)$. There exists $\delta > 0$ which depends only on $n$ and $\sigma$ such that if $\|a^+\|_{L^{\frac{2n}{n+2}}(B_1)} < \delta$, then for any $b(x) \in L^{\frac{2n}{n+2}}(B_1)$, there exists a unique solution in $H(t^{1-2\sigma}, Q_1)$ to (2.7) with $U|_{\partial Q_1} = 0$.

**Proof.** We consider the bilinear form

$$B[U, V] := \int_{Q_1} t^{1-2\sigma} \nabla U \nabla V \, dX - \int_{\partial Q_1} a U V \, dx, \quad U, V \in \mathcal{A}$$

where $\mathcal{A} := \{U \in H(t^{1-2\sigma}, Q_1) \setminus U|_{\partial Q_1} = 0 \text{ in trace sense}\}$. By Proposition 2.1 it is easy to verify that $B[\cdot, \cdot]$ is bounded and coercive provided $\delta$ is sufficiently small. Therefore the proposition follows from the Riesz representation theorem.
Lemma 2.1. Suppose \( U \in H(t^{1-2\sigma}, D) \) is a weak supersolution of (2.7) in \( D \) with \( a \equiv b \equiv 0 \). If \( U(X) \geq 0 \) on \( \partial^D D \) in trace sense, then \( U \geq 0 \) in \( D \).

Proof. Use \( U^- \) as a test function to conclude that \( U^- \equiv 0 \). \( \square \)

The following result is a refined version of that in [90]. Such De Giorgi-Nash-Moser type theorems for degenerated equations with Dirichlet boundary conditions have been established in [43].

Proposition 2.4. Suppose \( a, b \in L^p(B_1) \) for some \( p > \frac{n}{2\sigma} \).

(i) Let \( U \in H(t^{1-2\sigma}, Q_1) \) be a weak subsolution of (2.7) in \( Q_1 \). Then for any \( 0 < \mu < \tau < 1, 0 < \nu \leq \frac{n+1}{n} \) we have

\[
\inf_{Q_\mu} U + \|b^+\|_{L^p(B_1)} \geq C\|U\|_{L^p(t^{1-2\sigma}, Q_\tau)}
\]

where \( C > 0 \) depends only on \( n, \sigma, p, \mu, \tau \) and \( \|a^+\|_{L^p(B_1)} \).

(ii) Let \( U \in H(t^{1-2\sigma}, Q_1) \) be a nonnegative weak supersolution of (2.7) in \( Q_1 \). Then for any

\[
\sup_{Q_{1/2}} U^+ \leq C(\|U^+\|_{L^p(t^{1-2\sigma}, Q_1)} + \|b^+\|_{L^p(B_1)})
\]

where \( C > 0 \) depends only on \( n, \sigma, p, \nu, \mu, \tau \) and \( \|a^+\|_{L^p(B_1)} \).

(iii) Let \( U \in H(t^{1-2\sigma}, Q_1) \) be a nonnegative weak solution of (2.7) in \( Q_1 \). Then we have the following Harnack inequality

\[
\sup_{Q_{1/2}} U \leq C(\inf_{Q_{1/2}} U + \|b\|_{L^p(B_1)}),
\]

where \( C > 0 \) depends only on \( n, \sigma, p, \|a\|_{L^p(B_1)} \). Consequently, there exists \( \alpha \in (0, 1) \) depending only on \( n, \sigma, p, \|a\|_{L^p(B_1)} \) such that any weak solution \( U(X) \) of (2.7) is of \( C^\alpha(Q_{1/2}) \). Moreover,

\[
\|U\|_{C^\alpha(Q_{1/2})} \leq C(\|U\|_{L^\infty(Q_1)} + \|b\|_{L^p(B_1)})
\]

where \( C > 0 \) depends only on \( n, \sigma, p, \|a\|_{L^p(B_1)} \).

Proof. The proofs are modifications of those in [90], where the method of Moser iteration is used. Here we only point out the changes. Let \( k = \|b^+\|_{L^p(B_1)} \) if \( b^+ \neq 0 \), otherwise let \( k > 0 \) be any number which is eventually sent to 0. Define \( \overline{U} = U^+ + k \) and, for \( m > 0 \), let

\[
\overline{U}_m = \begin{cases} 
\overline{U} & \text{if } U < m, \\
k + m & \text{if } U \geq m.
\end{cases}
\]

Consider the test function

\[
\phi = \eta^2(\overline{U}^\beta_{m} \overline{U} - k^{\beta+1}) \in H(t^{1-2\sigma}, Q_1),
\]

for some \( \beta \geq 0 \) and some nonnegative function \( \eta \in C^1_c(Q_1 \cup \partial^D Q_1) \). Direction calculations yield that, with setting \( W = \overline{U}^\beta_{m} \overline{U} \),

\[
\frac{1}{1 + \beta} \int_{Q_1} t^{1-2\sigma} |\nabla(\eta W)|^2 \leq 16 \int_{Q_1} t^{1-2\sigma} |\nabla \eta|^2 W^2 + 4 \int_{\partial^D Q_1} (a^+ + \frac{b^+}{k}) \eta^2 W^2.
\]

(2.10)
By Hölder’s inequality and the choice of $k$, we have
\[ \int_{\partial^c Q_1} (a^+ + \frac{b^+}{k}) \eta^2 W^2 \leq \left(\|a^+\|_{L^p(B_1)} + 1\right) \|\eta^2 W^2\|_{L^{p'}(B_1)} \]
where $p' = \frac{p}{p-1} < \frac{n}{n-2\sigma}$. Choose $0 < \theta < 1$ such that $\frac{1}{p'} = \theta + \frac{(1-\theta)(n-2\sigma)}{n}$. The interpolation inequality gives that, for any $\varepsilon > 0$,
\[ \|\eta^2 W^2\|_{L^{p'}(B_1)} \leq \varepsilon \|\eta W\|^2_{L^{2\sigma}(B_1)} + \varepsilon^{-\frac{2\sigma}{n-2\sigma}} \|\eta^2 W^2\|_{L^1(B_1)}. \]
By the trace embedding inequality in Proposition 2.1, there exists $C > 0$ depending only on $n, \sigma$ such that
\[ \|\eta W\|^2_{L^{2\sigma}(B_1)} \leq C \int_{Q_1} t^{1-2\sigma} |\nabla (\eta W)|^2. \]
By Lemma 2.3 in [90], there exist $\delta > 0$ and $C > 0$ both of which depend only on $n, \sigma$ such that
\[ \|\eta^2 W^2\|_{L^1(B_1)} \leq \varepsilon^\frac{1}{2} \int_{Q_1} t^{1-2\sigma} |\nabla (\eta W)|^2 + \varepsilon^{-\frac{2\sigma}{n}} \int_{Q_1} t^{1-2\sigma} \eta^2 W^2. \]
By choosing $\varepsilon$ small, the above inequalities give that
\[ \int_{Q_1} t^{1-2\sigma} |\nabla (\eta W)|^2 \leq C(1 + \beta)^{\delta/\theta} \int_{Q_1} t^{1-2\sigma}(\eta^2 + |\nabla \eta|^2)W^2 \]
where $C$ depends only on $n, \sigma$ and $\|a^+\|_{L^p(B_1)}$. Then the proof of Proposition 3.1 in [90] goes through without any change. This finishes the proof of (i) for $\nu = 2$. Then (i) also holds for any $\nu > 0$ which follows from standard arguments. For part (ii) we choose $k = \|b^+\|_{L^p(B_1)}$ if $b^- \not\equiv 0$, otherwise let $k > 0$ be any number which is eventually sent to 0. Then we can show that there exists some $\nu_0 > 0$ for which (ii) holds, by exactly the same proof of Proposition 3.2 in [90]. Finally use the test function $\phi = U^{-\beta} \eta^2$ with $\beta \in (0, 1)$ to repeat the proof in (i) to conclude (ii) for $0 < \nu \leq \frac{n+1}{2}$. Part (iii) follows from (i), (ii) and standard elliptic equation theory.

**Remark 2.1.** Harnack inequality (2.9), without lower order term $b$, has been obtained earlier in [23] using a different method.

The above proofs can be improved to yield the following result.

**Lemma 2.2.** Suppose $a \in L^\infty(B_1), b \in L^p(B_1)$ with $p > \frac{n}{2\sigma}$ and $U \in H(t^{1-2\sigma}, Q_1)$ is a weak subsolution of (2.7) in $Q_1$. There exists $\delta > 0$ which depends only on $n$ and $\sigma$ such that if
\[ \|a^+\|_{L^{\frac{n}{2\sigma}}(B_1)} < \delta, \]
then
\[ \|U^+(\cdot, 0)\|_{H(t^{1-2\sigma}, Q_1)} \leq C(\|U^+\|_{H(t^{1-2\sigma}, Q_1)} + \|b^+\|_{L^p(B_1)}). \]
where $C > 0$ depends only on $n, p, \sigma, \delta$, and $q = \min \left(\frac{2(n+1)}{n-2\sigma}, \frac{n(p-1)}{(n-2\sigma)p}, \frac{2n}{n-2\sigma}\right)$.

**Remark 2.2.** Analogues estimates were established for $-\Delta u = a(x)u$ in [13] (see Theorem 2.3 there) and for $-\text{div}(|\nabla u|^{p-2}\nabla u) = a(x)|u|^{p-2}u$ in [4] (see Lemma 3.1 there).
Proof of Lemma 2.2. We start from (2.10), where we choose β = \min\left(\frac{2}{p}, \frac{2(2p - n)}{2n - 2mp}\right). By Hölder inequality and Proposition 2.1,

\[
\int_{\partial'Q_1} (a^+ + \frac{b^+}{k}) \eta^2 W^2 \leq \delta \|\eta^2 W^2\|_{L^{\frac{n}{n-2\sigma}}(B_1)} + \|\eta^2 W^2\|_{L^{\sigma'}(B_1)} \leq C(n, \sigma) \delta \int_{Q_1} t^{1-2\sigma} |\nabla(\eta W)|^2 + C_{n,\sigma,p} \|\eta W\|_{H(t^{-2\sigma},Q_1)}.
\]

By Poincaré’s inequality in [48], we have

\[
\int_{Q_1} t^{1-2\sigma} |\nabla\eta|^2 W^2 \leq C_{n,\sigma,p} \|\eta W\|_{H(t^{-2\sigma},Q_1)}.
\]

If δ is sufficiently small, the above together with (2.10) imply that

\[
\int_{Q_1} t^{1-2\sigma} |\nabla(\eta W)|^2 \leq C_{n,\sigma,p} \|\eta W\|_{H(t^{-2\sigma},Q_1)}.
\]

Hence it follows from Hölder inequality and Proposition 2.1 that, by sending \(m \to \infty\),

\[
\|\eta W\|_{L^p(\partial'Q_1)} \leq C_{n,\sigma,p} \int_{Q_1} t^{1-2\sigma} |\nabla(\eta W)|^2 \leq C_{n,\sigma,p} \|\eta W\|_{H(t^{-2\sigma},Q_1)}.
\]

This finishes the proof.

Corollary 2.1. Suppose that \(K \in L^\infty(B_1), U \in H(t^{1-2\sigma},Q_1)\) and \(U \geq 0\) in \(Q_1\) satisfies, for some \(1 \leq p \leq (n+2\sigma)(n-2\sigma),\)

\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla U(X)) = 0 & \text{in } Q_1 \\
- \lim_{t \to 0^+} t^{1-2\sigma} \partial_t U(x,t) = K(x)U(x,0)^p & \text{on } \partial'Q_1.
\end{cases}
\]

Then (i) \(U \in L^\infty_{\text{loc}}(Q_1 \cup \partial'Q_1)\), and hence \(U(\cdot,0) \in L^\infty_{\text{loc}}(B_1)\).

(ii) There exist \(C > 0\) and \(\alpha \in (0,1)\) depending only on \(n, \sigma, p\), \(\|u\|_{L^\infty(B_{3/4})},\) \(\|K\|_{L^\infty(B_{3/4})}\) such that \(U \in C^\alpha(Q_{1/2})\) and

\[
\|U\|_{H(t^{1-2\sigma},Q_{1/2})} + \|U\|_{C^\alpha(Q_{1/2})} \leq C.
\]

Note that the regularity of solution of \(-\Delta u = u^{\frac{n+2\sigma}{n-2\sigma}}\) was proved by Trudinger in [89].

Proof of Corollary 2.1. By Proposition 2.1, \(U(\cdot,0) \in H^\sigma(B_1) \subset L^{\frac{n}{n-2\sigma}}(B_1)\). Thus \(U(\cdot,0)^{p-1} \in L^{\frac{n}{n-2\sigma}}(B_1)\). Then part (i) follows from Lemma 2.2 and Proposition 2.4. Part (ii) follows from Proposition 2.2 and Proposition 2.4.

\(\square\)
2.3 Local Schauder estimates

Let $\Omega$ be a domain in $\mathbb{R}^n$, $a \in L^\frac{2n}{n+2\sigma}(\Omega)$ and $b \in L^1_{loc}(\Omega)$. We say $u \in \dot{H}^\sigma(\mathbb{R}^n)$ is a weak solution of

$$(\Delta)^\sigma u = a(x)u + b(x) \quad \text{in } \Omega$$

if for any $\phi \in C^\infty(\mathbb{R}^n)$ supported in $\Omega$,

$$\int_{\mathbb{R}^n} (\Delta)^\sigma \hat{u}(\Delta)^\sigma \phi = \int_{\Omega} a(x)u\phi + b(x)\phi.$$

Then by (2.5), $u \in \dot{H}^\sigma(\mathbb{R}^n)$ is a weak solution of

$$(\Delta)^\sigma u = \frac{1}{N_\sigma}(a(x)u + b(x)) \quad \text{in } B_1$$

if and only if $U = \mathcal{P}_\sigma[u]$, the extension of $u$ defined in (2.2), is a weak solution of (2.7) in $Q_1$.

For $\alpha \in (0, 1)$, $C^\alpha(\Omega)$ denotes the standard Hölder space over domain $\Omega$. For simplicity, we use $C^\alpha(\Omega)$ to denote $C^{[\alpha],\alpha-\alpha}(\Omega)$ when $1 < \alpha \notin \mathbb{N}$ (the set of positive integers).

In this part, we shall prove the following local Schauder estimates for nonnegative solutions of fractional Laplace equation.

**Theorem 2.1.** Suppose $a(x), b(x) \in C^\alpha(B_1)$ with $0 < \alpha \notin \mathbb{N}$. Let $u \in \dot{H}^\sigma(\mathbb{R}^n)$ and $u \geq 0$ in $\mathbb{R}^n$ be a weak solution of

$$(\Delta)^\sigma u = a(x)u + b(x), \quad \text{in } B_1.$$ 

Suppose that $2\sigma + \alpha$ is not an integer. Then $u \in C^{2\sigma+\alpha}(B_{3/4})$. Moreover,

$$\|u\|_{C^{2\sigma+\alpha}(B_{3/4})} \leq C(\inf_{B_{3/4}} u + \|b\|_{C^\alpha(B_{3/4})}) \quad (2.11)$$

where $C > 0$ depends only on $n, \sigma, \alpha, \|a\|_{C^\alpha(B_{3/4})}$.

**Remark 2.3.** Replacing the assumption $u \geq 0$ in $\mathbb{R}^n$ by $u \geq 0$ in $B_1$, estimate (2.11) may fail (see (65)). Without the sign assumption of $u$, (2.11) with $\inf_{B_{3/4}} u$ substituted by $\|u\|_{L^\infty(\mathbb{R}^n)}$ holds, which is proved in (27), (20) and (19) in a much more general setting of fully nonlinear nonlocal equations.

The following proposition will be used in the proof of Theorem 2.1

**Proposition 2.5.** Let $a(x), b(x) \in C^k(B_1)$, $U(X) \in H^{1-2\sigma}(Q_1)$ be a weak solution of (2.7) in $Q_1$, where $k$ is a positive integer. Then we have

$$\sum_{i=0}^k \|\nabla_x^i U\|_{L^\infty(Q_{1/2})} \leq C(\|U\|_{L^2(1-2\sigma,Q_1)} + \|b\|_{C^k(B_1)}),$$

where $C > 0$ depends only on $n, \sigma, k, \|a\|_{C^k(B_1)}$. 

14
Proof. We know from Proposition 2.4 that \( U \) is Hölder continuous in \( Q_{8/9} \). Let \( h \in \mathbb{R}^n \) with \( |h| \) sufficiently small. Denote \( U^h(x,t) = \frac{U(x+h,t)-U(x,t)}{|h|} \). Then \( U^h \) is a weak solution of

\[
\begin{align*}
\text{div}(t^{1-2\sigma}\nabla U^h(X)) &= 0 & \text{in } Q_{8/9} \\
- \lim_{t \to 0^+} t^{1-2\sigma} \partial_t U^h(x,t) &= a(x+h)U^h + a^h U + b^h & \text{on } \partial' Q_{8/9}.
\end{align*}
\]

By Proposition 2.2 and Proposition 2.4,

\[
\|U^h\|_{H^{(1-2\sigma),Q_{2/3}}} + \|U^h\|_{C^\alpha(Q_{2/3})} \leq C(\|U^h\|_{L^2(Q_{2/3})} + \|b\|_{C^\alpha(B_1)}).
\]

for some \( \alpha \in (0, 1) \) and positive constant \( C > 0 \) depending only on \( n, \sigma, \|a\|_{C^\alpha(B_1)} \). Hence \( \nabla_x U \in H^{(1-2\sigma),Q_{2/3}} \cap C^\alpha(Q_{2/3}) \), and it is a weak solution of

\[
\begin{align*}
\text{div}(t^{1-2\sigma}\nabla(U^h)) &= 0 & \text{in } Q_{2/3} \\
- \lim_{t \to 0^+} t^{1-2\sigma} \partial_t (U^h) &= a \nabla_x U + U \nabla_x a + \nabla_x b & \text{on } \partial' Q_{2/3}.
\end{align*}
\]

Then this Proposition follows immediately from Proposition 2.2 and Proposition 2.4 for \( k = 1 \). We can continue this procedure for \( k = 2, 3, \ldots \) (by induction).

To prove Theorem 2.1 we first obtain Schauder estimates for solutions of the equation

\[
\begin{align*}
\text{div}(t^{1-2\sigma}\nabla U(X)) &= 0 & \text{in } Q_R \\
- \lim_{t \to 0^+} t^{1-2\sigma} \partial_t U(x,t) &= g(x) & \text{on } \partial' Q_R.
\end{align*}
\]

Theorem 2.2. Let \( U(X) \in H^{(1-2\sigma),Q_2} \) be a weak solution of (2.7) with \( R = 2 \) and \( g(x) \in C^\alpha(B_2) \) for some \( 0 < \alpha \notin \mathbb{N} \). If \( 2\sigma + \alpha \) is not an integer, then \( U(\cdot,0) \) is of \( C^{2\sigma+\alpha}(B_{1/2}) \). Moreover, we have

\[
\|U(\cdot,0)\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C(\|U\|_{L^\infty(Q_2)} + \|g\|_{C^\alpha(B_2)}),
\]

where \( C > 0 \) depends only on \( n, \sigma, \alpha \).

This theorem together with Proposition 2.4 implies the following

Theorem 2.3. Let \( U(X) \in H^{(1-2\sigma),Q_1} \) be a weak solution of (2.7) with \( D = Q_1 \) and \( a(x), b(x) \in C^\alpha(B_1) \) for some \( 0 < \alpha \notin \mathbb{N} \). If \( 2\sigma + \alpha \) is not an integer, then \( U(\cdot,0) \) is of \( C^{2\sigma+\alpha}(B_{1/2}) \). Moreover, we have

\[
\|U(\cdot,0)\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C(\|U\|_{L^\infty(Q_1)} + \|b\|_{C^\alpha(B_1)}),
\]

where \( C > 0 \) depends only on \( n, \sigma, \alpha, \|a\|_{C^\alpha(B_1)} \).

Proof. From Proposition 2.4 \( U \) is Hölder continuous in \( Q_{3/4} \). Theorem 2.3 follows from bootstrap arguments by applying Theorem 2.2 with \( g(x) := a(x)U(x,0) + b(x) \).
Proof of Theorem 2.2. Our arguments are in the spirit of those in [18] and [71]. Denote $C$ as various constants that depend only on $n$ and $\sigma$. Let $\rho = \frac{1}{2}$, $Q_k = Q_{\rho^k}(0)$, $\partial' Q_k = B_k, k = 0, 1, 2, \ldots$. (Note that we have abused notations a little bit. Only in this proof we refer $Q_k, B_k$ as $Q_{\rho^k}, B_{\rho^k}$.) We also denote $M = \|g\|_{C^\alpha(B_k)}$. From Proposition 2.4 we have already known that $U$ is Hölder continuous in $Q_0$. First we assume that $\alpha \in (0, 1)$.

Step 1: We consider the case of $2\sigma + \alpha < 1$. Let $W_k$ be the unique weak solution of (which is guaranteed by Proposition 2.3)

\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla W_k(X)) = 0 & \text{in } Q_k \\
- \lim_{t \to 0^+} t^{1-2\sigma} \partial_t W_k(x, t) = g(0) - g(x) & \text{on } \partial' Q_k \\
W_k(X) = 0 & \text{on } \partial'' Q_k
\end{cases}
\]  
(2.14)

Let $U_k = W_k + U$ in $Q_k$ and $h_{k+1} = U_{k+1} - U_k$ in $Q_{k+1}$, then

\[\|W_k\|_{L^\infty(Q_k)} \leq CM\rho^{(2\sigma+\alpha)k}.\]  
(2.15)

Indeed (2.15) follows by applying Lemma 2.1 to the equation of $\rho^{-2\sigma} W_k(\rho^k x) \pm (t^{2\sigma} - 3) \rho^{\alpha k}$ in $Q_0$. Hence by weak maximum principle again we have

\[\|h_{k+1}\|_{L^\infty(Q_k)} \leq CM\rho^{(2\sigma+\alpha)k}.\]

By Proposition 2.5, we have, for $i = 0, 1, 2, 3$

\[\|\nabla W_k\|_{L^\infty(Q_{k+i})} \leq CM\rho^{(2\sigma+\alpha-i)k}.\]  
(2.16)

Similarly apply Proposition 2.5 to $U$, we have

\[\|\nabla U_0\|_{L^\infty(Q_2)} \leq C(\|U_0\|_{L^\infty(Q_1)} + M) \leq C(\|U\|_{L^\infty(Q_0)} + M).\]  
(2.17)

For any given point $z$ near 0, we have

\[
|U(z, 0) - U(0, 0)| \\
\leq |U_k(0, 0) - U(0, 0)| + |U(z, 0) - U_k(z, 0)| + |U_k(z, 0) - U_k(0, 0)| \\
= I_1 + I_2 + I_3
\]

Let $k$ be such that $\rho^{k+4} \leq |z| \leq \rho^{k+3}$. By (2.15),

\[I_1 + I_2 \leq CM\rho^{(2\sigma+\alpha)k} \leq CM|z|^{2\sigma+\alpha}.
\]

For $I_3$, by (2.16) and (2.17),

\[
I_3 \leq |U_0(z, 0) - U_0(0, 0)| + \sum_{j=1}^{k} |h_j(z, 0) - h_j(0, 0)| \\
\leq C|z| \left( \|\nabla U_0\|_{L^\infty(Q_{k+1})} + \sum_{j=1}^{k} \|\nabla h_j\|_{L^\infty(Q_{k+3})} \right) \\
\leq C|z| \left( \|U\|_{L^\infty(Q_0)} + M + M \sum_{j=1}^{k} \rho^{(2\sigma+\alpha-1)j} \right) \\
\leq C|z| \left( \|U\|_{L^\infty(Q_0)} + M(1 + |z|^{2\sigma+\alpha-1}) \right).
\]
Thus, for $2\sigma + \alpha < 1$, we have

$$|U(z, 0) - U(0, 0)| \leq C(M + \|U\|_{L^\infty(Q_0)})|z|^{2\sigma + \alpha}.$$ 

which finishes the proof of Step 1.

**Step 2:** For $1 < 2\sigma + \alpha < 2$, the arguments in Step 1 imply that

$$\|\nabla_x U(\cdot, 0)\|_{L^\infty(B_1)} \leq C\left(\|U\|_{L^\infty(Q_0)} + M\right). \quad (2.18)$$

Apply (2.18) to the equation of $W_k$ we have, together with (2.15),

$$\|\nabla_x W_k(\cdot, 0)\|_{L^\infty(B_{k+1})} \leq CM\rho^{(2\sigma + \alpha - 1)k} \quad (2.19)$$

By (2.16) and (2.17),

$$|\nabla_x U_k(z, 0) - \nabla_x U_k(0, 0)|$$

$$\leq |\nabla_x U_0(z, 0) - \nabla_x U_0(0, 0)| + \sum_{j=1}^k |\nabla_x h_j(z, 0) - \nabla_x h_j(0, 0)|$$

$$\leq C|z|\left(\|\nabla^2_x U_0\|_{L^\infty(Q_{k+3})} + \sum_{j=1}^k \|\nabla^2_x h_j\|_{L^\infty(Q_{k+3})}\right)$$

$$\leq C|z|\left(\|U\|_{L^\infty(Q_0)} + M + M\sum_{j=1}^k \rho^{(2\sigma - 2 + \alpha)j}\right)$$

$$\leq C|z|\left(\|U\|_{L^\infty(Q_0)} + M(1 + |z|^{2\sigma + \alpha - 2})\right).$$

Hence

$$|\nabla_x U(z, 0) - \nabla_x U(0, 0)|$$

$$\leq |\nabla_x W_k(0, 0)| + |\nabla_x W_k(z, 0)| + |\nabla_x U_k(z, 0) - \nabla_x U_k(0, 0)|$$

$$\leq CM\rho^{(2\sigma + \alpha - 1)k} + C|z|\left(\|U\|_{L^\infty(Q_0)} + M(1 + |z|^{2\sigma + \alpha - 2})\right)$$

$$\leq C(M + \|U\|_{L^\infty(Q_0)})|z|^{2\sigma + \alpha - 1}.$$ 

which finishes the proof of Step 2.

**Step 3:** For $2\sigma + \alpha > 2$, the arguments in Step 2 imply that

$$\|\nabla^2_x U(\cdot, 0)\|_{L^\infty(B_1)} \leq C\left(\|U\|_{L^\infty(Q_0)} + M\right), \quad (2.19)$$

Apply (2.19) to the equation of $W_k$ we have, together with (2.15),

$$\|\nabla^2_x W_k(\cdot, 0)\|_{L^\infty(B_{k+1})} \leq CM\rho^{(2\sigma + \alpha - 2)k}.$$
By (2.16) and (2.17),

\[
|\nabla^2_x U_k(z,0) - \nabla^2_x U_k(0,0)| \\
\leq |\nabla^2_x U_0(z,0) - \nabla^2_x U_0(0,0)| + \sum_{j=1}^{k} |\nabla^2_x h_j(z,0) - \nabla^2_x h_j(0,0)| \\
\leq C|z|\left(\sum_{j=1}^{k} \|\nabla^2_x U_0\|_{L^\infty(Q_{k+3})} + \sum_{j=1}^{k} \|\nabla^3_x h_j\|_{L^\infty(Q_{k+3})}\right) \\
\leq C|z|\left(\|U\|_{L^\infty(Q_0)} + M + M\sum_{j=1}^{k} |\nabla^{2\alpha+3}h_j|\right) \\
\leq C|z|\left(\|U\|_{L^\infty(Q_0)} + M(1 + |z|^{2\alpha+3})\right).
\]

Hence

\[
|\nabla^2_x U(z,0) - \nabla^2_x U(0,0)| \\
\leq |\nabla^2_x W_k(0,0)| + |\nabla^2_x W_k(z,0)| + |\nabla^2_x U_k(z,0) - \nabla_x U_k(0,0)| \\
\leq CM\rho^{(2\alpha+3)k} + C|z|\left(\|U\|_{L^\infty(Q_0)} + M(1 + |z|^{2\alpha+3})\right) \\
\leq C(M + \|U\|_{L^\infty(Q_0)})|z|^{2\alpha+3}.
\]

which finishes the proof of Step 3. This finishes the proof of Theorem 2.2 for \(\alpha \in (0, 1)\).

For the case that \(\alpha > 1\), we may apply \(\nabla_x\) to (2.13) \([\alpha]\) times, as in the proof of Proposition 2.3 and repeat the above three steps. Theorem 2.2 is proved.

Proof of Theorem 2.4 Since \(u \in \dot{H}^{\alpha}(\mathbb{R}^n)\) is nonnegative, its extension \(U \geq 0\) in \(\mathbb{R}^{n+1}_+\) and \(U \in H^{(1-2\alpha)}(Q_1)\) is a weak solution of (2.7) in \(Q_1\). The theorem follows immediately from Theorem 2.3 and Proposition 2.4.

Remark 2.4. Another way to show Theorem 2.1 is the following. Let \(u \in \dot{H}^{\alpha}(\mathbb{R}^n)\) and \(u \geq 0\) in \(\mathbb{R}^n\) be a solution of

\[(\Delta)^\alpha u = g(x), \quad \text{in } B_1\]

where \(g \in C^\alpha(B_1)\). Let \(\eta\) be a nonnegative smooth cut-off function supported in \(B_1\) and equal to 1 in \(B_{\tau/8}\). Let \(v \in H^\alpha(\mathbb{R}^n)\) be the solution of

\[(\Delta)^\alpha v = \eta(x)g(x), \quad \text{in } \mathbb{R}^n\]

where \(\eta g\) is considered as a function defined in \(\mathbb{R}^n\) and supported in \(B_1\), i.e., \(v\) is a Riesz potential of \(\eta g\)

\[v(x) = \frac{\Gamma(n-2\alpha)}{2^{\alpha-\sigma} \pi^{n/2} \Gamma(\alpha)} \int_{\mathbb{R}^n} \eta(y)g(y) \frac{1}{|x-y|^{n-2\alpha}} dy.
\]

Then if \(2\alpha + \alpha\) and \(\alpha\) are not integers, we have (see, e.g., [86])

\[\|v\|_{C^{2\alpha+\alpha}(B_1/2)} \leq C\|v\|_{L^\infty(\mathbb{R}^n)} + \|\eta g\|_{C^{\alpha}(\mathbb{R}^n)} \leq C\|g\|_{C^\alpha(B_1)}.
\]
Let \( w = u - v \) which belongs to \( \dot{H}^\sigma(\mathbb{R}^n) \) and satisfies

\[
(-\Delta)^\sigma w = 0, \quad \text{in } B_{7/8}.
\]

Let \( W = \mathcal{P}_\sigma[w] \) be the extension of \( w \), and \( \tilde{W} = W + \|v\|_{L^\infty(\mathbb{R}^n)} \geq 0 \) in \( \mathbb{R}^{n+1} \). Notice that \( \tilde{W} \) is a nonnegative weak solution of (2.7) with \( a = b = 0 \) and \( D = Q_1 \). By Proposition 2.5 and Proposition 2.4, we have

\[
\|w + \|v\|_{L^\infty(\mathbb{R}^n)}\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C \|\tilde{W}\|_{L^2(t^{-2\sigma}, Q_{7/8})} \leq C \inf_{Q_{3/4}} \tilde{W} \leq C(\inf_{Q_{3/4}} u + \|v\|_{L^\infty(\mathbb{R}^n)}).
\]

Hence

\[
\|u\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq \|v\|_{C^{2\sigma+\alpha}(B_{1/2})} + \|u\|_{C^{2\sigma+\alpha}(B_{1/2})} \leq C(\inf_{B_{3/4}} u + \|g\|_{C^\alpha(B_1)}).
\]

Using bootstrap arguments as that in the proof of Theorem 2.3, we conclude Theorem 2.7.

Remark 2.5. Indeed, our proofs also lead to the following. If we only assume that \( a(x), b(x), g(x) \in L^\infty(B_1) \), and let \( U, u \) be those in Theorem 2.2 and in Theorem 2.7 respectively, then the estimates

\[
\|U(\cdot, 0)\|_{C^{2\sigma}(B_{1/2})} \leq C_1(\|U\|_{L^\infty(Q_1)} + \|g\|_{L^\infty(B_1)})
\]

\[
\|u\|_{C^{2\sigma}(B_{1/2})} \leq C_2(\inf_{B_{3/4}} u + \|b\|_{L^\infty(B_{3/4})})
\]

hold provided \( \sigma \neq 1/2 \), where \( C_1 > 0 \) depends only on \( n, \sigma, \alpha \) and \( C_2 > 0 \) depends only on \( n, \sigma, \alpha, \|a\|_{L^\infty(B_{3/4})} \). For \( \sigma = \frac{1}{2} \), we have the following log-Lipschitz property: for any \( y_1, y_2 \in B_{1/4}, y_1 \neq y_2 \),

\[
\frac{|U(y_1, 0) - U(y_2, 0)|}{|y_1 - y_2|} \leq C_1(\|U\|_{L^\infty(Q_1)} - \|g\|_{L^\infty(B_1)} \log |y_1 - y_2|),
\]

\[
\frac{|u(y_1) - u(y_2)|}{|y_1 - y_2|} \leq -C_2 \log |y_1 - y_2| \inf_{B_{3/4}} u + \|b\|_{L^\infty(B_{3/4})}
\]

where \( C_1 > 0 \) depends only on \( n, \sigma \) and \( C_2 > 0 \) depends only on \( n, \sigma, \|a\|_{L^\infty(B_{3/4})} \).

Next we have

Lemma 2.3. (Lemma 4.5 in [23]) Let \( g \in C^\alpha(B_1) \) for some \( \alpha \in (0, 1) \) and \( U \in L^\infty(Q_1) \cap H(t^{1-2\sigma}, Q_1) \) be a weak solution of (2.13). Then there exists \( \beta \in (0, 1) \) depending only on \( n, \sigma, \alpha \) such that \( t^{1-2\sigma} \partial_t U \in C^\beta(Q_{1/2}) \). Moreover, there exists a positive constant \( C > 0 \) depending only on \( n, \sigma \) and \( \beta \) such that

\[
\|t^{1-2\sigma} \partial_t U\|_{C^\beta(Q_{1/2})} \leq C(\|U\|_{L^\infty(Q_1)} + \|g\|_{C^\alpha(B_1)}).
\]
Proposition 2.6. Suppose that $K \in C^1(B_1)$, $U \in H(t^{1-2\sigma}, Q_1)$ and $U \geq 0$ in $Q_1$ is a weak solution of
\[\begin{align*}
\text{div}(t^{1-2\sigma} \nabla U) &= 0, \\
- \lim_{t \to 0} t^{1-2\sigma} \partial_t U(x, t) &= K(x)U^p(x, 0),
\end{align*}\]
(2.20)
where $1 \leq p \leq \frac{n+2\sigma}{n+1}$. Then there exist $C > 0$ and $\alpha \in (0, 1)$ both of which depend only on $n, \sigma, p, \|U\|_{L^\infty(Q_1)}$, $\|K\|_{C^1(Q_1)}$ such that
\[\nabla_x U \text{ and } t^{1-2\sigma} \partial_t U \text{ are of } C^\alpha(Q_{1/2})
\]
and
\[\|\nabla_x U\|_{C^\alpha(Q_{1/2})} + \|t^{1-2\sigma} \partial_t U\|_{C^\alpha(Q_{1/2})} \leq C.
\]
Proof. We use $C$ and $\alpha$ to denote various positive constants with dependence specified as in the proposition, which may vary from line to line. By Corollary 2.1, $U \in L^\infty_{loc}(Q_1)$ and $\|U\|_{C^\alpha(Q_{1/2})} \leq C$.

With the above, we may apply Theorem 2.5 to obtain $U(\cdot, 0) \in C^{1,\sigma}(\overline{B_{7/8}})$ and
\[\|U(\cdot, 0)\|_{C^{1,\sigma}(\overline{B_{7/8}})} \leq C.
\]
Hence we may differentiate (2.20) with respect to $x$ (which can be justified from the proof of Proposition 2.5) and apply Proposition 2.3 to $\nabla_x U$ to obtain
\[\|\nabla_x U\|_{C^\alpha(Q_{1/2})} \leq C.
\]
Finally we can apply Lemma 2.3 to obtain
\[\|t^{1-2\sigma} \partial_t U\|_{C^\alpha(Q_{1/2})} \leq C.
\]

3 Proof of Theorem 1.5

We first introduce some notations. We say that $U \in L^\infty_{loc}(\mathbb{R}^{n+1}_+)$ if $U \in L^\infty(\overline{Q_R})$ for any $R > 0$. Similarly we say $U \in H_{loc}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$ if $U \in H(t^{1-2\sigma}, \overline{Q_R})$ for any $R > 0$.

In the following $B_R(X)$ is denoted as the ball in $\mathbb{R}^{n+1}$ with radius $R$ and center $X$, and $B_R^+(X)$ as $B_R(X) \cap \mathbb{R}_+^{n+1}$. We also write $B_R(0), B_R^+(0)$ as $B_R, B_R^+$ for short respectively. We start with a Lemma, which is a version of the strong maximum principle.

Proposition 3.1. Suppose $U(X) \in H(t^{1-2\sigma}, D_\varepsilon) \cap C(B_1^+ \cup B_1 \setminus \{0\})$ and $U > 0$ in $B_1^+ \cup B_1 \setminus \{0\}$ is a weak supersolution of (2.4) with $a \equiv b \equiv 0$ and $D = D_\varepsilon := B_1^+ \setminus B_\varepsilon^+$ for any $0 < \varepsilon < 1$, then
\[\liminf_{(x,t) \to 0} U(x,t) > 0.
\]
Proof. For any $\delta > 0$, let
\[ V_\delta = U + \frac{\delta}{|x,t|^{n-2\sigma}} - \min_{\partial^\nu B_{0,8}^+} U. \]
Then $V$ is also a weak supersolution in $D_{\delta, \frac{1}{\delta^{n-2\sigma}}}$. Applying Lemma 2.1 to $V_\delta$ in $D_{\delta, \frac{1}{\delta^{n-2\sigma}}}$ for sufficiently small $\delta$, we have $V_\delta \geq 0$ in $D_{\delta, \frac{1}{\delta^{n-2\sigma}}}$. For any $(x,t) \in B_{0,8}^+ \setminus \{0\}$, we have $\lim_{\delta \to 0} V_\delta(x,t) \geq 0$, i.e., $U(x,t) \geq \min_{\partial^\nu B_{0,8}^+} U$.

The proof of Theorem 1.5 uses the method of moving spheres and is inspired by [73], [72], and [24]. For each $\lambda$, from classical elliptic equations theory, we can choose a positive constant $\lambda_0(x)$ such that for any $0 < \lambda < \lambda_0(x)$,
\[ U_{\lambda}(\xi) = \left( \frac{\lambda}{|\xi|^{n-2\sigma}} \right)^{n-2\sigma} U \left( \frac{\lambda^2(\xi - X)}{\xi - X} \right), \quad \xi \in \mathbb{R}^{n+1}_{+} \setminus \{X\}, \quad (3.1) \]
the Kelvin transformation of $U$ with respect to the ball $B_{\lambda}(X)$. We point out that if $U$ is a solution of (1.9), then $U_{\lambda, \lambda}$ is a solution of (1.9) in $\mathbb{R}^{n+1}_{+} \setminus \{0\}$, for every $x \in \partial \mathbb{R}^{n+1}_{+}$, $\lambda > 0$, and $\varepsilon > 0$.

By Corollary 2.1 any nonnegative weak solution $U$ of (1.9) belongs to $L^\infty_{\text{loc}}(\mathbb{R}^{n+1}_{+})$, and hence by Proposition 2.2 $U$ is Holder continuous and positive in $\mathbb{R}^{n+1}_{+}$. By Theorem 2.2 $U(\cdot, 0)$ is smooth in $\mathbb{R}^n$. From classical elliptic equations theory, $U$ is smooth in $\mathbb{R}^{n+1}_{+}$.

Lemma 3.1. For any $x \in \mathbb{R}^n$, there exists a positive constant $\lambda_0(x)$ such that for any $0 < \lambda < \lambda_0(x)$,
\[ U_{\lambda}(\xi) \leq U(\xi), \quad \text{in } \mathbb{R}^{n+1}_{+} \setminus B_{\lambda_x}^+(X). \quad (3.2) \]

Proof. Without loss of generality we may assume that $x = 0$ and write $U_{\lambda} = U_{0, \lambda}$.

Step 1. We show that there exist $0 < \lambda_1 < \lambda_2$ which may depend on $x$, such that
\[ U_{\lambda}(\xi) \leq U(\xi), \quad \forall \quad 0 < \lambda < \lambda_1, \quad \lambda < |\xi| < \lambda_2. \]

For every $0 < \lambda < \lambda_1 < \lambda_2$, $\xi \in \partial^\nu B_{\lambda_2}$, we have $\frac{\lambda^2}{|\xi|^2} \in B_{\lambda_2}^+$. Thus we can choose $\lambda_1 = \lambda_1(\lambda_2)$ small such that
\[ U_{\lambda}(\xi) = \left( \frac{\lambda}{|\xi|^{n-2\sigma}} \right)^{n-2\sigma} U \left( \frac{\lambda^2(\xi - X)}{\xi - X} \right) \leq \left( \frac{\lambda_1}{\lambda_2} \right)^{n-2\sigma} \sup_{B_{\lambda_2}^+} U \leq \inf_{\partial^\nu B_{\lambda_2}^+} U \leq U(\xi) \]

Hence
\[ U_{\lambda} \leq U \quad \text{on } \partial^\nu(B_{\lambda_2}^+ \setminus B_{\lambda}^+) \]
for all $\lambda_2 > 0$ and $0 < \lambda < \lambda_1(\lambda_2)$.

We will show that $U_{\lambda} \leq U$ on $(B_{\lambda_2}^+ \setminus B_{\lambda}^+)$ if $\lambda_2$ is small and $0 < \lambda < \lambda_1(\lambda_2)$. Since $U_{\lambda}$ satisfies (1.9) in $B_{\lambda_2}^+ \setminus B_{\lambda}^+$, we have
\[ \begin{cases} \text{div}(t^{1-2\sigma} \nabla(U_{\lambda} - U)) = 0, & \text{in } B_{\lambda_2}^+ \setminus B_{\lambda}^+; \\ \lim_{t \to 0} t^{1-2\sigma} \partial_t(U_{\lambda} - U) = U_{\lambda_2}^{n+2\sigma}(x, 0) - U_{\lambda}^{n+2\sigma}(x, 0), & \text{on } \partial^\nu(B_{\lambda_2}^+ \setminus B_{\lambda}^+). \end{cases} \quad (3.3) \]
Lemma 3.1 is proved. We will make use of the narrow domain technique from [11]. With the help of the mean value theorem, we have

\[
\int_{B^+_\lambda \setminus B^+_{\lambda_2}} t^{1-2\sigma} |\nabla (U_\lambda - U)^+|^2 \\
= \int_{B^+_\lambda \setminus B^+_{\lambda_2}} (U_\lambda^{n+2\sigma} (x, 0) - U_\lambda^{n+2\sigma} (x, 0)) (U_\lambda - U)^+ \\
\leq C \int_{B^+_\lambda \setminus B^+_{\lambda_2}} ((U_\lambda - U)^+)^2 U_\lambda^{n-2\sigma} \\
\leq C \left( \int_{B^+_\lambda \setminus B^+_{\lambda_2}} ((U_\lambda - U)^+) \right)^{\frac{n-2\sigma}{n}} \left( \int_{B^+_\lambda \setminus B^+_{\lambda_2}} U_\lambda^{2\sigma} \right)^{\frac{2\sigma}{n}}
\]

where Proposition 2.1 is used in the last inequality and C is a positive constant depending only on n and \( \sigma \). We fix \( \lambda_2 \) small such that

\[
C \left( \int_{B^+_{\lambda_2}} U_\lambda^{2\sigma} \right)^{\frac{2\sigma}{n}} < 1/2.
\]

Then \( \nabla (U_\lambda - U)^+ = 0 \) in \( B^+_{\lambda_2} \setminus B^+_{\lambda_1} \). Since \( (U_\lambda - U)^+ = 0 \) on \( \partial^\nu (B^+_{\lambda_2} \setminus B^+_{\lambda_1}) \), \( (U_\lambda - U)^+ = 0 \) in \( B^+_{\lambda_1} \setminus B^+_{\lambda_2} \). We conclude that \( U_\lambda \leq U \) on \( (B^+_{\lambda_1} \setminus B^+_{\lambda_2}) \) for \( 0 < \lambda < \lambda_1 := \lambda_1 (\lambda_2) \).

Step 2. We show that there exists \( \lambda_0 \in (0, \lambda_1) \) such that \( \forall 0 < \lambda < \lambda_0 \)

\[
U_\lambda (\xi) \leq U (\xi), \quad |\xi| > \lambda_2, \quad \xi \in \mathbb{R}^{n+1}.
\]

Let \( \phi (\xi) = \left( \frac{\lambda_2}{|\xi|} \right)^{n-2\sigma} \inf_{\partial^\nu B^+_{\lambda_2}} U \), which satisfies

\[
\begin{aligned}
\text{div}(t^{1-2\sigma} \nabla \phi) &= 0, \quad \text{in } \mathbb{R}^{n+1} \setminus B^+_{\lambda_2} \\
\lim_{t \to 0} t^{1-2\sigma} \partial_t \phi (x, t) &= 0, \quad \text{in } \mathbb{R}^n \setminus B^+_{\lambda_2},
\end{aligned}
\]

and \( \phi (\xi) \leq U (\xi) \) on \( \partial^\nu B^+_{\lambda_2} \). By the weak maximum principle Lemma 2.1

\[
U (\xi) \geq \left( \frac{\lambda_2}{|\xi|} \right) \inf_{\partial^\nu B^+_{\lambda_2}} U, \quad \forall |\xi| > \lambda_2, \quad \xi \in \mathbb{R}^{n+1}.
\]

Let \( \lambda_0 = \min (\lambda_1, \lambda_2 \left( \inf_{\partial^\nu B^+_{\lambda_2}} U / \sup_{\partial^\nu B^+_{\lambda_2}} U \right)^{\frac{1}{2-\sigma}} ) \). Then for any \( 0 < \lambda < \lambda_0 \), \( |\xi| \geq \lambda_2 \), we have

\[
U_\lambda (\xi) \leq \left( \frac{\lambda_0}{|\xi|} \right)^{n-2\sigma} U \left( \frac{\lambda_2 \xi}{|\xi|} \right)^{n-2\sigma} \inf_{B^+_{\lambda_2}} U \leq \left( \frac{\lambda_2}{|\xi|} \right)^{n-2\sigma} \inf_{\partial^\nu B^+_{\lambda_2}} U \leq U (\xi).
\]

Lemma 3.1 is proved.

\[
\square
\]
With Lemma 3.1, we can define for all $x \in \mathbb{R}^n$,
\[
\bar{\lambda}(x) = \sup\{\mu > 0 : U_{X,\lambda} \leq U \text{ in } \mathbb{R}^{n+1}_+ \setminus B_x^+, \forall 0 < \lambda < \mu\}.
\]
By Lemma 3.1, $\bar{\lambda}(x) \geq \lambda_0(x)$.

**Lemma 3.2.** If $\bar{\lambda}(x) < \infty$ for some $x \in \mathbb{R}^n$, then
\[
U_{X,\bar{\lambda}(x)} = U.
\]

**Proof.** Without loss of generality we assume that $x = 0$ and write $U_{\lambda} = U_{0,\lambda}$ and $\bar{\lambda} = \bar{\lambda}(0)$. By the definition of $\bar{\lambda}$,
\[
U_{\bar{\lambda}} \geq U \text{ in } B_{\bar{\lambda}}^+ \setminus \{0\},
\]
and therefore, for all $0 < \varepsilon < \bar{\lambda}$,
\[
\left\{\begin{array}{ll}
\text{div}(t^{1-2\sigma}\nabla(U_{\lambda} - U)) &= 0, \quad \text{in } B_{\lambda}^+ \setminus \overline{B_{\bar{\lambda}}^+}; \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t(U_{\lambda} - U) &\geq 0 \quad \text{on } \partial'(B_{\lambda}^+ \setminus \overline{B_{\bar{\lambda}}^+}).
\end{array}\right.
\]
(3.4)

We argue by contradiction. If $U_{\bar{\lambda}}$ is not identically equal to $U$, applying the Harnack inequality Proposition 2.4 to (3.4), we have
\[
U_{\bar{\lambda}} > U \text{ in } B_{\bar{\lambda}}^+ \setminus \{0\},
\]
and in view of Proposition 3.1
\[
\liminf_{\xi \to 0}(U_{\bar{\lambda}}(\xi) - U(\xi)) > 0.
\]

So there exist $\varepsilon_1 > 0$ and $c > 0$ such that $U_{\bar{\lambda}}(\xi) > U(0) + c$, $\forall \varepsilon_1 < |\xi| < \varepsilon_1$. Choose $\varepsilon_2$ small such that
\[
\left(\frac{\bar{\lambda}}{\lambda + \varepsilon_2}\right)^{n-2\sigma}(U(0) + c) > U(0) + \frac{c}{2}.
\]
Thus for all $0 < |\xi| < \varepsilon_1$ and $\bar{\lambda} < \lambda < \bar{\lambda} + \varepsilon_2$,
\[
U_{\lambda}(\xi) = \left(\frac{\bar{\lambda}}{\lambda}\right)^{n-2\sigma} U_{\bar{\lambda}}(\bar{\lambda}^2 \xi) \geq \left(\frac{\bar{\lambda}}{\lambda + \varepsilon_2}\right)^{n-2\sigma}(U(0) + c) \geq U(0) + c/2.
\]

Choose $\varepsilon_3$ small such that for all $0 < |\xi| < \varepsilon_3$, $U(0) > U(\xi) - c/4$. Hence for all $0 < |\xi| < \varepsilon_3$ and $\bar{\lambda} < \lambda < \bar{\lambda} + \varepsilon_2$,
\[
U_{\lambda}(\xi) > U(\xi) + c/4.
\]

For $\delta$ small, which will be fixed later, denote $K_{\delta} = \{\xi \in \mathbb{R}^{n+1}_+ : \varepsilon_3 \leq |\xi| \leq \bar{\lambda} - \delta\}$. Then there exists $c_2 = c_2(\delta)$ such that
\[
U_{\bar{\lambda}}(X) - U(X) > c_2 \text{ in } K_{\delta}.
\]

By the uniform continuous of $U$ on compact sets, there exists $\varepsilon_4 \leq \varepsilon_2$ such that for all $\bar{\lambda} < \lambda < \bar{\lambda} + \varepsilon_4$
\[
U_{\lambda} - U_{\bar{\lambda}} > c_2/2 \text{ in } K_{\delta}.
\]
Hence
\[ U_\lambda - U > c_2/2 \text{ in } K_\delta. \]
Now let us focus on the region \( \{ \xi \in \mathbb{R}^{n+1}_+ : \tilde{\lambda} - \delta \leq |\xi| \leq \lambda \} \). Using the narrow domain technique as that in Lemma 3.1 we can choose \( \delta \) small (notice that we can choose \( \varepsilon_4 \) as small as we want) such that
\[ U_\lambda \geq U \text{ in } \{ \xi \in \mathbb{R}^{n+1}_+ : \tilde{\lambda} - \delta \leq |\xi| \leq \lambda \}. \]
In conclusion there exists \( \varepsilon_4 \) such that for all \( \bar{\lambda} < \lambda < \bar{\lambda} + \varepsilon_4 \)
\[ U_\lambda \geq U \text{ in } \{ \xi \in \mathbb{R}^{n+1}_+ : 0 < |\xi| \leq \lambda \} \]
which contradicts with the definition of \( \bar{\lambda} \).

Proof of Theorem 1.5. It follows from the same arguments in [72], with the help of Lemma 3.2, that:
(i) Either \( \bar{\lambda}(x) = \infty \) for all \( x \in \mathbb{R}^n \) or \( \bar{\lambda}(x) < \infty \) for all \( x \in \mathbb{R}^n \); (Lemma 2.3 in [72])
(ii) If for all \( x \in \mathbb{R}^n \), \( \bar{\lambda}(x) = \infty \) then \( U(x,t) = U(0,t) \), \( \forall (x,t) \in \mathbb{R}^{n+1}_+ \); (Lemma 11.3 in [72])
(iii) If \( \bar{\lambda}(x) < \infty \) for all \( x \in \mathbb{R}^n \), then by Lemma 11.1 in [72]
\[ u(x) := U(x,0) = a \left( \frac{\lambda}{1 + \lambda^2|x-x_0|^2} \right)^{\frac{n-2\sigma}{2}} \] (3.5)
where \( \lambda > 0, a > 0 \) and \( x_0 \in \mathbb{R}^n \).

We claim that (ii) never happens, since this would imply, using (1.9), that
\[ U(x,t) = U(0) - U(0) \frac{t^{2\sigma}}{2\sigma} \]
which contradicts to the positivity of \( U \). Then (iii) holds.

We are only left to show that \( V := U - P_{\sigma}[u] \equiv 0 \) where \( u(x) \) is given in (3.5) and belongs to \( \dot{H}^\sigma(\mathbb{R}^n) \). Hence, \( V \) satisfies
\[
\begin{cases}
\text{div}(t^{1-2\sigma}\nabla V) = 0, & \text{in } \mathbb{R}^{n+1}_+ \\
V = 0 & \text{on } \partial \mathbb{R}^{n+1}_+.
\end{cases}
\]
By Lemma 3.2 we know that \( V_\lambda \) can be extended to a smooth function near 0. Multiplying the above equation by \( V \) and integrating by parts, it leads to \( \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla V|^2 = 0 \). Hence we have \( V \equiv 0 \).

Finally \( a = (N_\sigma c_{n,\sigma^2})^{\frac{n-2\sigma}{-2\sigma}} \) follows from (1.3) with \( \phi = 1 \) and (2.5).

4 Local analysis near isolated blow up points

The analysis in this and next section adapts the blow up analysis developed in [88] and [68] to give accurate blow up profiles for solutions of degenerate elliptic equations. For \( \sigma = \frac{1}{2} \), similar results have been proved in [57] and [45], where equations are elliptic.
Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a domain, $\tau_i \geq 0$ satisfy $\lim_{i \to \infty} \tau_i = 0$, $p_i = (n + 2\sigma)/(n - 2\sigma) - \tau_i$, and $K_i \in C^{1,1}(\Omega)$ satisfy, for some constants $A_1, A_2 > 0$, that

$$1/A_1 \leq K_i(x) \leq A_1 \quad \text{for all } x \in \Omega,$$

$$||K_i||_{C^{1,1}(\Omega)} \leq A_2. \quad (4.1)$$

Let $u_i \geq 0$ in $\mathbb{R}^n$ and $u_i \in L^\infty(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n)$ satisfying

$$(\Delta)^{\sigma} u_i = c(n, \sigma) K_i(x) u_i^{p_i}, \quad \text{in } \Omega. \quad (4.2)$$

We say that $\{u_i\}$ blows up if $||u_i||_{L^\infty(\Omega)} \to \infty$ as $i \to \infty$.

**Definition 4.1.** Suppose that $\{K_i\}$ satisfies (4.1) and $\{u_i\}$ satisfies (4.2). We say a point $\overline{y} \in \Omega$ is an isolated blow up point of $\{u_i\}$ if there exist $0 < \overline{\tau} < \text{dist}(\overline{y}, \Omega)$, $\overline{C} > 0$, and a sequence $y_i$ tending to $\overline{y}$, such that, for some $y_i$ is a local maximum of $u_i$, $u_i(y_i) \to \infty$ and

$$u_i(y) \leq \overline{C}|\overline{y} - y_i|^{-2\sigma/(p_i - 1)} \quad \text{for all } y \in B_{\overline{\tau}}(y_i).$$

Let $y_i \to \overline{y}$ be an isolated blow up of $u_i$, define

$$\overline{\pi}_i(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(y_i)} u_i, \quad r > 0, \quad (4.3)$$

and

$$\overline{\pi}_j(r) = r^{2\sigma/(p_i - 1)} \overline{\pi}_i(r), \quad r > 0.$$ \hspace{1cm} \text{Definition 4.2.}

We say $y_i \to \overline{y} \in \Omega$ is an isolated simple blow up point, if $y_i \to \overline{y}$ is an isolated blow up point, such that, for some $\rho > 0$ (independent of $i$) $\overline{\pi}_i$ has precisely one critical point in $(0, \rho)$ for large $i$.

In this section, we are mainly concerned with the profile of blow up of $\{u_i\}$. And under certain conditions, we can show that isolated blow up points have to be isolated simple blow up points.

Let $u_i \in C^2(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n)$ and $u_i \geq 0$ in $\mathbb{R}^n$ satisfy (4.2) with $K_i$ satisfying (4.1). Without loss of generality, we assume throughout this section that $\overline{B}_2 \subset \Omega$ and $y_i \to \infty$ as $i \to \infty$ is an isolated blow up point of $\{u_i\}$ in $\Omega$. Let $U_i = \mathcal{P}_\sigma [u_i]$ be the extension of $u_i$ (see (2.2)). Then we have

$$\begin{align*}
\text{div}(t^{1-2\sigma} \nabla U_i) &= 0, \quad \text{in } \mathbb{R}^{n+1}, \\
- \lim_{t \to 0} t^{1-2\sigma} \frac{\partial U_i(x,t)}{\partial t} &= c_0 K_i(x) U_i(x,0)^{p_i}, \quad \text{for all } x \in \Omega,
\end{align*}$$

where $c_0 = N_\sigma c(n, \sigma)$ with $N_\sigma = 2^{1-2\sigma} \Gamma(1 - \sigma)/\Gamma(\sigma)$.

**Lemma 4.1.** Suppose that $u_i \in C^2(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n)$ and $u_i \geq 0$ in $\mathbb{R}^n$ satisfy (4.2) with $\{K_i\}$ satisfying (4.1), and $y_i \to \infty$ is an isolated blow up point of $\{u_i\}$, i.e., for some positive constants $A_3$ and $\overline{r}$ independent of $i$,

$$|y - y_i|^{2\sigma/(p_i - 1)} u_i(y) \leq A_3, \quad \text{for all } y \in B_{\overline{r}} \subset \Omega. \quad (4.5)$$

Denote $U_i = \mathcal{P}_\sigma [u_i]$, and $Y_i = (y_i, 0)$. Then for any $0 < r < 1/\overline{\tau}$, we have the following Harnack inequality

$$\sup_{B^+_r(Y_i) \setminus B^+_r(Y_i)} U_i \leq C \inf_{B^-_r(Y_i) \setminus B^-_r(Y_i)} U_i,$$

where $C$ is a positive constant depending only on $n, \sigma, A_3, \overline{r}$ and $\sup_i ||K_i||_{L^\infty(B_{\overline{r}}(y_i))}$. 

25
Proof. For $0 < r < \frac{\bar{r}}{3}$, set

$$V_i(Y) = r^{2\sigma/(p_i-1)}U_i(Y + rY), \quad \text{in } Y \in B_3^+.$$  

It is easy to see that

$$\text{div}(s^{1-2\sigma}\nabla V_i) = 0, \quad \text{in } B_3^+,$$

and

$$-\lim_{s \to 0} s^{1-2\sigma} \partial_s V_i(y, s) = c_0 K(y_i + ry)V_i(y, 0)^{p_i}, \quad \text{on } \partial'B_3^+.$$  

Since $y_i \to 0$ is an isolated blow up point of $u_i$,

$$V_i(y, 0) \leq A_3|y|^{-2\sigma/(p_i-1)}, \quad \text{for all } y \in B_3.$$  

Lemma 4.1 follows after applying Proposition 2.4 and the standard Harnack inequality for uniform elliptic equation together to $V_i$ in the domain $Q_2 \setminus Q_{1/2}$.  

\[ \square \]

Proposition 4.1. Suppose that $u_i \in C^2(\Omega) \cap H^\sigma(\mathbb{R}^n)$ and $u_i \geq 0$ in $\mathbb{R}^n$ satisfies (4.2) with $K_i \in C^{1,1}(\Omega)$ satisfying (4.7). Suppose also that $y_i \to 0$ be an isolated blow up point of \{ $u_i$ \} with (4.5). Then for any $R_i \to \infty$, $\varepsilon_i \to 0^+$, we have, after passing to a subsequence (still denoted as \{ $u_i$ \}, \{ $y_i$ \}, etc. ...), that

$$\|m_i^{-1} u_i(m_i^{-(p_i-1)/2\sigma} \cdot + y_i) - (1 + k_i) \cdot |\cdot |^{(2\sigma-n)/2}\|_{C^2(B_{2R_i}(0))} \leq \varepsilon_i,$$

$$R_i m_i^{-(p_i-1)/2\sigma} \to 0 \quad \text{as} \quad i \to \infty,$$

where $m_i = u_i(y_i)$ and $k_i = K_i(y_i)^{1/\sigma}/4$.

Proof. Let

$$\phi_i(x) = m_i^{-1} u_i(m_i^{-(p_i-1)/2\sigma} x + y_i), \quad \text{for } x \in \mathbb{R}^n.$$  

It follows that

$$(-\Delta)^\sigma \phi_i(x) = c(n, \sigma) K_i(m_i^{-(p_i-1)/2\sigma} x + y_i) \phi_i^{p_i},$$

$$0 < \phi_i \leq A_3|x|^{-2\sigma/(p_i-1)}, \quad |x| < \bar{r} m_i^{-(p_i-1)/2\sigma},$$

and

$$\phi_i(0) = 1, \quad \nabla \phi_i(0) = 0.$$  

Let $\Phi_i = \mathcal{P}_\sigma[\phi_i]$ be the extension of $\phi_i$ (see (2.2)). Then $\Phi$ satisfies

$$\left\{ \begin{array}{l}
\text{div}(t^{1-2\sigma}\nabla \Phi_i(x, t)) = 0, \quad |x, t| < \bar{r} m_i^{\frac{p_i-1}{2\sigma}}, \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t \Phi_i(x, t) = N_\sigma c(n, \sigma) K_i(m_i^{-(p_i-1)/2\sigma} x + y_i) \Phi_i(x, 0)^{p_i}, \quad |x| < \bar{r} m_i^{\frac{p_i-1}{2\sigma}}.
\end{array} \right.$$  

By the weak maximum principle we have, for any $0 < r < 1$, $1 = \phi_i(0) = \Phi_i(0, 0) \geq \min_{\partial'B_r} \Phi_i$. It follows from Lemma 4.1 that

$$\max_{\partial B_r} \phi_i \leq \max_{\partial'^*B_r} \Phi_i \leq C \min_{\partial'^*B_r} \Phi_i \leq C.$$
Namely,
\[
\max_{B_R} \phi_i \leq C
\]
for some \(C > 0\) depending on \(n, \sigma, A_1, A_2, A_3\). This and (4.6) implies that for any \(R > 1\)
\[
\max_{B_R} \phi_i \leq C(R)
\]
for some \(C(R) > 0\) depending on \(n, \sigma, A_1, A_2, A_3\) and \(R\). Then by Corollary 2.1 there exists some \(\alpha \in (0, 1)\) such that for every \(R > 1\),
\[
\|\Phi_i\|_{H^1(Q_R) + H^{\alpha}(\Omega_R)} \leq C_1(R),
\]
where \(\alpha\) and \(C_1(R)\) are independent of \(i\). Bootstrap using Theorem 2.1 we have, for every \(0 < \beta < 2\) with \(2\sigma + \beta \notin \mathbb{N}\),
\[
\|\Phi_i\|_{C^{2, \beta}(\Omega_R)} \leq C_2(R, \beta)
\]
where \(C_2(R, \beta)\) is independent of \(i\). Thus, after passing to a subsequence, we have, for some non-negative functions \(\Phi(x) \in H_{loc}(I^{1-2\sigma}, \mathbb{R}^{n+1}) \cap C^2_{loc}(\mathbb{R}^{n+1})\) and \(\phi \in C^2(\mathbb{R}^n)\),
\[
\begin{cases}
\Phi_i \rightarrow \Phi \text{ weakly in } H_{loc}(I^{1-2\sigma}, \mathbb{R}^{n+1}), \\
\Phi_i \rightarrow \Phi \text{ in } C^2_{loc}(\mathbb{R}^{n+1}), \\
\phi_i \rightarrow \phi \text{ in } C^2_{loc}(\mathbb{R}^n).
\end{cases}
\]
It follows that
\[
\Phi(\cdot, 0) \equiv \phi, \quad \phi(0) = 1, \quad \nabla \phi(0) = 0,
\]
and \(\Phi\) satisfies
\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla \Phi) = 0, \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t \Phi(x, t) = c_0 K \Phi(x, 0)^{(n+2\sigma)/(n-2\sigma)} \quad \text{on } \partial^\sigma \mathbb{R}^{n+1},
\end{cases}
\]
with \(K = \lim_{i \to \infty} K_i(y_i)\). By Theorem 1.5 we have
\[
\phi(x) = (1 + \lim_{i \to \infty} k_i |x|^{2(\sigma-n)/2},
\]
where \(k_i = K_i(y_i)^{1/\sigma} \). Proposition 4.1 follows immediately.

Note that since passing to subsequences does not affect our proofs, we will always choose \(R_i \to \infty\) first, and then \(\varepsilon_i \to 0^+\) as small as we wish (depending on \(R_i\)) and then choose our subsequence \(\{u_i\}\) to work with.

**Proposition 4.2.** Under the hypotheses of Proposition 4.1 there exists some positive constant \(C = C(n, \sigma, A_1, A_2, A_3)\) such that,
\[
u_i(y) \geq C^{-1} m_i(1 + k_i m_i^{(p_i - 1)/\sigma}|y - y_i|^2/(2\sigma-n)^{2(\sigma-n)/2)} |y - y_i| \leq 1.
\]
In particular, for any \(\varepsilon \in \mathbb{R}^n, |\varepsilon| = 1\), we have
\[
u_i(y_i + \varepsilon) \geq C^{-1} m_i^{-1 + (n-2\sigma)/2\sigma} \tau_i,
\]
where \(\tau_i = (n + 2\sigma)/(n - 2\sigma) - p_i\).
Proof. Denote $r_i = R_i m_i^{-(p_i - 1)/2\sigma}$. It follows from Proposition 4.1 that $r_i \to 0$ and
\[ u_i(y) \geq C^{-1} m_i R_i^{2\sigma - n}, \quad \text{for all } |y - y_i| = r_i. \]
By the Harnack inequality Lemma 3.1 we have
\[ U_i(Y) \geq C^{-1} m_i R_i^{2\sigma - n}, \quad \text{for all } |Y - Y_i| = r_i, \]
where $U_i = P_\sigma[u_i]$ is the extension of $u_i$, $Y = (y, s)$ with $s \geq 0$, and $Y_i = (y_i, 0)$.
Set
\[ \Psi_i(Y) = C^{-1} R_i^{2\sigma - n} r_i^{n - 2\sigma} m_i (|Y - Y_i|^{2\sigma - n} - (3/2)^{2\sigma - n}), \quad r_i \leq |Y - Y_i| \leq 3/2. \]
Clearly, $\Psi_i$ satisfies
\[
\begin{align*}
\text{div}(s^{1 - 2\sigma} \nabla \Psi_i) &= 0 = \text{div}(s^{1 - 2\sigma} \nabla U_i), \quad r_i \leq |Y - Y_i| \leq 3/2, \\
\Psi_i(Y) &\leq U_i(Y), \quad \text{on } \partial B_{r_i} \cup \partial B_{3/2}, \\
- \lim_{s \to 0^+} s^{1 - 2\sigma} \partial_s \Psi_i(y, s) &= 0 \leq - \lim_{s \to 0^+} s^{1 - 2\sigma} \partial_s U_i(y, s), \quad r_i \leq |y - y_i| \leq 3/2.
\end{align*}
\]
By the weak maximum principle Lemma 2.1 applied to $U_i - \Psi_i$, we have
\[ U_i(Y) \geq \Psi_i(Y) \quad \text{for all } r_i \leq |Y - Y_i| \leq 3/2. \]
Therefore, Proposition 4.2 follows immediately from Proposition 4.1.

Lemma 4.2. Under the hypotheses of Proposition 4.1 and in addition that $y_i \to 0$ is also an isolated simple blow up point with the constant $\rho$, there exist $\delta_i > 0$, $\delta_i = O(R_i^{-2\sigma + o(1)})$, such that
\[ u_i(y) \leq C_i u_i(y_i)^{-\lambda_i} |y - y_i|^{2\sigma - n + \delta_i}, \quad \text{for all } r_i \leq |y - y_i| \leq 1, \]
where $\lambda_i = (n - 2\sigma - \delta_i)(p_i - 1)/2\sigma - 1$ and $C_i$ is some positive constant depending only on $n, \sigma, A_1, A_3$ and $\rho$.
Proof. From Proposition 4.1 we see that
\[ u_i(y) \leq C u_i(y_i) R_i^{2\sigma - n} \quad \text{for all } |y - y_i| = r_i. \tag{4.7} \]
Let $\pi_i(r)$ be the average of $u_i$ over the sphere of radius $r$ centered at $y_i$. It follows from the assumption of isolated simple blow up and Proposition 4.1 that
\[ r^{2\sigma/(p_i - 1)} \pi_i(r) \quad \text{is strictly decreasing for } r_i < r < \rho. \tag{4.8} \]
By Lemma 4.1 (4.8) and (4.7), we have, for all $r_i < |y - y_i| < \rho$,
\[
|y - y_i|^{2\sigma/(p_i - 1)} u_i(y) \leq C |y - y_i|^{2\sigma/(p_i - 1)} \pi_i(|y - y_i|) \leq r_i^{2\sigma/(p_i - 1)} \pi_i(r_i) \leq C R_i^{2\sigma - n + o(1)},
\]
28
where \( o(1) \) denotes some quantity tending to 0 as \( i \to \infty \). Applying Lemma 4.1 again, we obtain
\[
U_i(Y)^{p_i-1} \leq O(R_i^{-2\sigma + o(1)})|Y - Y_i|^{-2\sigma} \quad \text{for all } r_i \leq |Y - Y_i| \leq \rho. \tag{4.9}
\]

Consider operators
\[
\begin{align*}
\mathcal{L}(\Phi) &= \text{div}(s^{1-2\sigma}\nabla \Phi(Y)), \\
L_i(\Phi) &= -\lim_{s \to 0^+} s^{1-2\sigma} \partial_s \Phi(y, s) - c_0 K_i u_i^{p_i-1}(y) \Phi(y, 0),
\end{align*}
\]
for \( i \) such that for \( Y \)
\[
\text{denotes some quantity tending to } U.
\]

Due to (4.9), we can choose \( \delta_i = O(R_i^{-2\sigma + o(1)}) > 0 \) such that for \( r_i < |y - y_i| < \rho, \)
\[
\begin{align*}
L_i(|Y - Y_i|^{-\mu} - \varepsilon s^{2\sigma} |Y - Y_i|^{-(\mu+2\sigma)}) &= |Y - Y_i|^{-(\mu+2\sigma)}(2\varepsilon s^{2\sigma} - c_0 K_i u_i^{p_i-1}|Y - Y_i|^{2\sigma}). \\
&= \varepsilon s^{2\sigma} |Y - Y_i|^{-\mu} + \varepsilon(\mu + 2\sigma) (n - \mu) s^{2\sigma}|Y - Y_i|^{2\sigma}
\end{align*}
\]
and
\[
L_i(|Y - Y_i|^{-\mu} - \varepsilon s^{2\sigma} |Y - Y_i|^{-(\mu+2\sigma)}) = |Y - Y_i|^{-(\mu+2\sigma)}(2\varepsilon s^{2\sigma} - c_0 K_i u_i^{p_i-1}|Y - Y_i|^{2\sigma}).
\]

It follows from (4.9) that we can choose \( \varepsilon_i = O(R_i^{-2\sigma + o(1)}) > 0 \) and then choose \( \delta_i = O(R_i^{-2\sigma + o(1)}) > 0 \) such that for \( r_i < |y - y_i| < \rho, \)
\[
\begin{align*}
L_i(|Y - Y_i|^{-\delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-(\delta_i+2\sigma)}) &\geq 0, \\
L_i(|Y - Y_i|^{2\sigma - n + \delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-n + \delta_i}) &\geq 0
\end{align*}
\]
and for \( r_i < |Y - Y_i| < \rho, \)
\[
\begin{align*}
\mathcal{L}(|Y - Y_i|^{-\delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-(\delta_i+2\sigma)}) &\leq 0, \\
\mathcal{L}(|Y - Y_i|^{2\sigma - n + \delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-n + \delta_i}) &\leq 0.
\end{align*}
\]

Set \( M_i = 2 \max_{\partial^* \mathcal{B}_2^+} U_i, \lambda_i = (n - 2\sigma - \delta_i)(p_i - 1)/2\sigma - 1 \) and
\[
\Phi_i = M_i \rho^\delta_i |Y - Y_i|^{-\delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-(\delta_i+2\sigma)} + A u_i(y_i)^{-\lambda_i} |Y - Y_i|^{2\sigma - n + \delta_i} - \varepsilon_i s^{2\sigma} |Y - Y_i|^{-n + \delta_i},
\]
where \( A > 1 \) will be chosen later. By the choice of \( M_i \) and \( \lambda_i \), we immediately have
\[
\Phi_i(Y) \geq M_i \geq U_i(Y) \quad \text{for all } |Y - Y_i| = \rho.
\]
\[
\Phi_i \geq A U_i(Y_i) R_i^{2\sigma - n + \delta_i} \geq A U_i(Y_i) R_i^{2\sigma - n} \quad \text{for all } |Y - Y_i| = r_i.
\]

Due to (4.9), we can choose \( A \) to be sufficiently large such that
\[
\Phi_i \geq U_i \quad \text{for all } |Y - Y_i| = r_i.
\]
Therefore, applying maximum principles in section 4.3 to \( \Phi_i - U_i \) in \( \mathcal{B}_\rho \setminus \mathcal{B}_{r_i} \), it yields
\[
U_i \leq \Phi_i \quad \text{for all } r_i \leq |Y - Y_i| \leq \rho.
\]
For $r_i < \theta < \rho$, the same arguments as that in (4.5) yield
\[
\rho^{2\sigma/(p_i-1)} M_i \leq C_R^{2\sigma/(p_i-1)} t^i(\rho)
\leq C_R^{2\sigma/(p_i-1)} t^i(\theta)
\leq C_R^{2\sigma/(p_i-1)} \{ M_i \rho^{\delta_i} \theta^{-\delta_i} + A u_i(y_i) - \lambda_i \theta^{2\sigma-n+\delta_i} \}.
\]
Choose $\theta = \theta(n, \sigma, \rho, A_1, A_2, A_3)$ sufficiently small so that
\[
C_R^{2\sigma/(p_i-1)} \rho^{\delta_i} \theta^{-\delta_i} \leq \frac{1}{2} \rho^{2\sigma/(p_i-1)}.
\]
It follows that
\[
M_i \leq C u_i(y_i)^{-\lambda_i}.
\]
Then Lemma 4.2 follows from the above and the Harnack inequality. \qed

Below we are going to improve the estimate in Lemma 4.2. First, we prove a Pohozaev type identity.

**Proposition 4.3.** Suppose that $K \in C^1(B_{2R})$. Let $U \in H(t^{1-2\sigma}, B_{2R}^+)$ and $U \geq 0$ in $B_{2R}^+$. Then
\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla U) = 0, & \text{in } B_{2R}^+, \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x, t) = K(x) U^p(x, 0), & \text{on } \partial B_{2R}^+,
\end{cases}
\]
(4.10)
where $p > 0$. Then
\[
\int_{\partial B_{2R}^+} B'(X, U, \nabla U, R, \sigma) + \int_{\partial B_{2R}^+} t^{1-2\sigma} B''(X, U, \nabla U, R, \sigma) = 0,
\]
(4.11)
where
\[
B'(X, U, \nabla U, R, \sigma) = \frac{n-2\sigma}{2} K U^{p+1} + \langle X, \nabla U \rangle K U^p
\]
and
\[
B''(X, U, \nabla U, R, \sigma) = \frac{n-2\sigma}{2} U \frac{\partial U}{\partial \nu} - \frac{R}{2} |\nabla U|^2 + \frac{R}{2} |\partial U|^2.
\]

**Proof.** Let $\Omega_\varepsilon = B_{2R}^\varepsilon \cap \{ t > \varepsilon \}$ for small $\varepsilon > 0$. Multiplying (4.10) by $\langle X, \nabla U \rangle$ and integrating by parts in $\Omega_\varepsilon$, we have, with notations $\partial' \Omega_\varepsilon = \text{interior of } \Omega_\varepsilon \cap \{ t = \varepsilon \}$, $\partial'' \Omega_\varepsilon = \partial \Omega_\varepsilon \setminus \partial' \Omega_\varepsilon$ and $\nu = \text{unit outer normal of } \partial \Omega_\varepsilon$,
\[
\begin{align*}
- \int_{\partial' \Omega_\varepsilon} t^{1-2\sigma} \partial_t U \langle X, \nabla U \rangle + & \int_{\partial'' \Omega_\varepsilon} t^{1-2\sigma} R \frac{\partial U}{\partial \nu}^2 \\
= & \int_{\Omega_\varepsilon} t^{1-2\sigma} |\nabla U|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} t^{1-2\sigma} X \cdot \nabla (|\nabla U|^2) \\
= & -\frac{n-2\sigma}{2} \int_{\Omega_\varepsilon} t^{1-2\sigma} |\nabla U|^2 + \frac{1}{2} \int_{\partial' \Omega_\varepsilon} t^{1-2\sigma} R |\nabla U|^2 \\
- & \frac{1}{2} \int_{\partial'' \Omega_\varepsilon} t^{2-2\sigma} |\nabla U|^2.
\end{align*}
\]
(4.12)
Combining the above two, together with Proposition 4.3, we conclude that

$$
\int_{\Omega_{s}} t^{1-2\sigma} |\nabla U|^2 = -\int_{\partial^{0} \Omega_{s}} t^{1-2\sigma} U \partial_{\nu} U + \int_{\partial' \Omega_{s}} t^{1-2\sigma} \frac{\partial U}{\partial \nu}. \quad (4.13)
$$

By Corollary 2.1 and Proposition 2.6 there exists some $\alpha \in (0, 1)$ such that $U$, $\nabla U$, and $t^{1-2\sigma} \partial_{\nu} U$ belong to $C^\alpha(B^0_r)$ for all $r < 2R$. With this we can send $\varepsilon \to 0$ as follows. By (4.10),

$$
-t^{1-2\sigma} \partial_t U(x, t) \to K(x) U^p(x, 0) \quad \text{uniformly in } B_{3R/2} \text{ as } t \to 0.
$$

Hence (4.11) follows by sending $\varepsilon \to 0$ in (4.12) and (4.13).

**Lemma 4.3.** Under the assumptions in Lemma 4.2 we have

$$
\tau_i = O(u_i(y_i)^{-2/(n-2\sigma)+o(1)}),
$$

and thus

$$
u_i(y_i)^\tau_i = 1 + o(1).
$$

**Proof.** Since $U_i$ satisfies (4.4) and $\text{div}(y - y_i) = n$, we have, using integration by part,

$$
\frac{1}{c_0} \int_{\partial^0 B^1_{r_i}(y_i)} B'(Y, U_i, \nabla U_i, 1, \sigma)

= \frac{n - 2\sigma}{2n} \int_{\partial^0 B^1_{r_i}(y_i)} \text{div}(y - y_i) K_i U_i^{p_i+1}

+ \frac{1}{p_i+1} \int_{\partial^0 B^1_{r_i}(y_i)} \langle y - y_i, \nabla_y U_i^{p_i+1} \rangle K_i

= -\frac{n - 2\sigma}{2n} \int_{\partial B_1(y_i)} K_i U_i^{p_i+1} + \frac{1}{p_i+1} \int_{\partial B_1(y_i)} \langle y - y_i, \nabla_y U_i^{p_i+1} \rangle K_i

= \frac{\tau_i(n - 2\sigma)^2}{2n(2n - \tau_i(n - 2\sigma))} \int_{\partial^0 B^1_{r_i}(y_i)} \langle y - y_i, \nabla_y U_i^{p_i+1} \rangle K_i

- \frac{n - 2\sigma}{2n} \int_{\partial^0 B^1_{r_i}(y_i)} \langle y - y_i, \nabla_y K_i \rangle U_i^{p_i+1} + \frac{n - 2\sigma}{2n} \int_{\partial B_1(y_i)} K_i U_i^{p_i+1}

and

$$
\int_{\partial^0 B^1_{r_i}(y_i)} \langle y - y_i, \nabla_y U_i^{p_i+1} \rangle K_i

= -n \int_{\partial^0 B^1_{r_i}(y_i)} K_i U_i^{p_i+1} - \int_{\partial B_1(y_i)} \langle y - y_i, \nabla_y K_i \rangle U_i^{p_i+1} + \int_{\partial B_1(y_i)} K_i U_i^{p_i+1}.
$$

Combining the above two, together with Proposition 4.3, we conclude that

$$
\tau_i \int_{\partial^0 B^1_{r_i}(y_i)} U_i^{p_i+1} \leq C(n, \sigma, A_1, A_2) \Big\{ \int_{\partial^0 B^1_{r_i}(y_i)} |y - y_i| U_i^{p_i+1}

+ \int_{\partial B_1(y_i)} U_i^{p_i+1} + \int_{\partial^0 B^1_{r_i}(y_i)} t^{1-2\sigma} |B''(Y, U_i, \nabla U_i, 1, \sigma)| \Big\}. \quad (4.14)
$$
Lemma 4.4. Let

\[ \frac{\partial}{\partial t} B_1(Y_i) = B_1(y_i) \times \{0\} \]

it follows from Proposition 4.2 that

\[ \int_{\partial B_1(Y_i)} U_i^{p_i+1} = \int_{B_1(y_i)} u_i^{p_i+1} \]

\[ \geq C^{-1} \int_{B_1(y_i)} \left( 1 + \frac{m_i^{(p_i-1)/2\sigma}}{(y-y_i)^2} (n-2\sigma)(p_i+1)/2 \right) \]

\[ \geq C^{-1} m_i^{(n/2\sigma-1)} \int_{B_{m_i^{(p_i-1)/2\sigma}}} \frac{1}{(1+|z|^2)^{(n-2\sigma)(p_i+1)/2}} \]

\[ \geq C^{-1} m_i^{(n/2\sigma-1)}, \]

where we used change of variables \( z = m_i^{(p_i-1)/2\sigma}(y-y_i) \) in the second inequality.

By Proposition 2.6 and Lemma 4.2, it is easy to see that the last two integral terms of right-hand side of (4.14) are in \( O(m_i^{-2+o(1)}) \). By Proposition 4.1 we have

\[ \int_{\partial B_i(Y_i)} |Y-Y_i| U_i^{p_i+1} = \int_{B_i(y_i)} |y-y_i| u_i^{p_i+1} \]

\[ \leq C \int_{B_i(y_i)} \frac{|y-y_i|m_i^{p_i+1}}{(1+|m_i^{(p_i-1)/2\sigma}(y-y_i)^2|^{(n-2\sigma)(p_i+1)/2}} \]

\[ \leq C m_i^{-2/(n-2\sigma)+o(1)} \int_{B_{R_i}} \frac{|z|}{(1+|z|^2)^{n+o(1)}} \]

\[ \leq C m_i^{-2/(n-2\sigma)+o(1)}. \]

By Lemma 4.2 and that \( R_i \rightarrow \infty \), we have

\[ \int_{\partial B_1(Y_i) \setminus \partial B_i(Y_i)} |Y-Y_i| U_i^{p_i+1} = \int_{B_1(y_i) \setminus B_i(y_i)} |y-y_i| u_i^{p_i+1} \]

\[ \leq m_i^{-\lambda_i(p_i+1)} F_i^{n+1+2\sigma-n+\delta_i(p_i+1)} \]

\[ = o(m_i^{-2/(n-2\sigma)+o(1)}). \]

Combining (4.14), (4.15), (4.16), (4.17) and that \( \tau_1 = o(1) \), we complete the proof.

Proposition 4.4. Under the assumptions in Lemma 4.2 we have

\[ u_i(y) \leq C u_i^{-1}(y_i) |y-y_i|^{2\sigma-n}, \text{ for all } |y-y_i| \leq 1. \]

Our proof of this Proposition makes use of the following

Lemma 4.4. Let \( n \geq 2 \). Suppose that for all \( \varepsilon \in (0,1) \), \( U \in H(t^{1-2\sigma}, B_1^+ \setminus \overline{B_1^+}) \) and \( U > 0 \) in \( B_1^+ \setminus \overline{B_1^+} \) be a weak solution of

\[ \begin{align*}
\text{div}(t^{1-2\sigma} \nabla U) &= 0, & \text{in } B_1^+ \setminus \overline{B_1^+}, \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x,t) &= 0, & \text{in } B_1 \setminus \overline{B_1^+}. 
\end{align*} \]
Then
\[ U(X) = A|X|^{2\sigma - n} + H(X), \]
where \( A \) is a nonnegative constant and \( H(X) \in H((1-2\sigma, B_1^+) \) satisfies
\[
\begin{align*}
\text{div}(t^{1-2\sigma} \nabla H) &= 0, \\
- \lim_{t \to 0} t^{1-2\sigma} \partial_t H(x, t) &= 0, \quad \text{in } B_1.
\end{align*}
\] (4.19)

The proof of Lemma 4.4 is provided in Appendix A.2.

**Proof of Proposition 4.4** For \(|y - y_i| < r_i\), it follows from Proposition 4.1 that
\[
\begin{align*}
u_i(y) &\leq C m_i \left( \frac{1}{1 + m_i^{p_i - 1})/2\sigma} (y - y_i)^2\right)^{(n - 2\sigma)/2} \\
&\leq C m_i^{(n - 2\sigma)/2\sigma} |y - y_i|^{2\sigma - n} \\
&\leq C m_i^{-1}|y - y_i|^{2\sigma - n},
\end{align*}
\] (4.20)
where Lemma 4.3 is used the last inequality.

Suppose \(|y - y_i| \geq r_i\). Let \( e \in \mathbb{R}^{n+1}_+ \) with \(|e| = 1\), and set \( V_i(Y) = U_i(Y_i + e)^{-1} U_i(Y) \). Then \( V_i \) satisfies
\[
\begin{align*}
\text{div}(s^{1-2\sigma} \nabla V_i) &= 0, \\
- \lim_{s \to 0} s^{1-2\sigma} \partial_s V_i(y, s) &= c(n, \sigma) K U_i(Y_i + e)^{p_i - 1} V_i^{p_i},
\end{align*}
\] for \( y \in B_2^+ \).

Note that \( U_i(Y_i + e) \to 0 \) by Lemma 4.2 and for any \( r > 0 \)
\[
V_i(Y) \leq C(n, \sigma, A_1, r), \quad \text{for all } r < |y - y_i| \leq 1
\] (4.21)
which follows from Lemma 4.1. It follows that \( \{V_i\} \) converges to some positive function \( V \) in \( C_\text{loc}^\infty(B_3^+) \cap C_\text{loc}^\alpha(B_3^+ \setminus \{0\}) \) for some \( \alpha \in (0, 1) \), and \( V \) satisfies
\[
\begin{align*}
\text{div}(s^{1-2\sigma} \nabla V) &= 0, \\
- \lim_{s \to 0} s^{1-2\sigma} \partial_s V(y, s) &= 0 \quad \text{for } y \in B_1^+ \setminus \{0\}.
\end{align*}
\]
Hence \( \lim_{i \to \infty} r^{2\sigma/(p_i + 1)} u_i(r) = r^{n-2\sigma} \bar{v}(r) \), where \( v(y) = V(y, 0) \). Since \( r_i \to 0 \) and \( y_i \to 0 \) is an isolated simple blow up point of \( \{u_i\} \), it follows from Lemma 4.1 that \( r^{(n-2\sigma)/2} \bar{v}(r) \) is *almost decreasing* for all \( 0 < r < \rho \), i.e., there exists a positive constant \( C \) (which comes from Harnack inequality in Lemma 4.1) such that for any \( 0 < r_1 \leq r_2 < \rho \),
\[
r_1^{(n-2\sigma)/2} \bar{v}(r_1) \geq C r_2^{(n-2\sigma)/2} \bar{v}(r_2).
\]
Therefore, \( V \) has to have a singularity at \( Y = 0 \). Lemma 4.4 implies
\[
V(Y) = A|Y|^{2\sigma - n} + H(Y),
\] (4.22)
where $A > 0$ is a constant and $H$ is as in Lemma 4.3.

We first establish the inequality in Proposition 4.4 for $|Y - Y_i| = 1$. Namely, we prove that

$$U_i(Y_i + e) \leq CU_i^{-1}(Y_i)$$  \hspace{1cm} (4.23)

Suppose that (4.23) does not hold, then along a subsequence we have

$$\lim_{i \to \infty} U_i(Y_i + e)U_i(Y_i) = \infty.$$  \hspace{1cm} (4.24)

By integration by parts (using $\Omega_\varepsilon$ and sending $\varepsilon \to 0$, as in the proof of Proposition 4.3), we obtain

$$0 = -\int_{B_i^+} \text{div}(s^{1-2\sigma} \nabla V_i)$$

$$= \int_{\partial^* B_i^+} s^{1-2\sigma} \frac{\partial V_i}{\partial \nu} + c(n,\sigma) U_i(Y_i + e)\int_{\partial^* B_i^+} K U_i^{p_i}. \hspace{1cm} (4.25)$$

By Lemma 4.3 and similar computation in (4.16) and (4.17), we see that

$$\int_{\partial^* B_i^+} K U_i^{p_i} \leq CU_i(Y_i)^{-1}. \hspace{1cm} \text{Due to (4.24),}$$

$$\lim_{i \to \infty} U_i(Y_i + e)\int_{\partial^* B_i^+} K U_i^{p_i} = 0.$$

A direct computation yields with (4.21) (again using $\Omega_\varepsilon$ and sending $\varepsilon \to 0$)

$$\lim_{i \to \infty} \int_{\partial^* B_i^+} s^{1-2\sigma} \frac{\partial V_i}{\partial \nu} = \lim_{i \to \infty} \int_{\partial^* B_i^+} s^{1-2\sigma} \frac{\partial}{\partial \nu}(A|Y|^{2\sigma-n} + H(Y))$$

$$= A(2\sigma - n) \int_{\partial^* B_i^+} s^{1-2\sigma} < 0,$$

which contradicts to (4.25). Thus we proved (4.23). By Lemma 4.1, we have established the inequality in Proposition 4.4 for $\rho \leq |Y - Y_i| \leq 1$.

By a standard scaling argument, we can reduce the case of $r_i \leq |Y - Y_i| < \rho$ to $|Y - Y_i| = 1$. We refer to [68] (page 340) for details.

Proposition 4.2 and 4.4 give a clear picture of $u_i$ near the isolated simple blow up point. By the estimates there, it is easy to see the following result.

Lemma 4.5. We have

$$\int_{|y - y_i| \leq r_i} |y - y_i|^s u_i(y)^{p_i+1}$$

$$= \begin{cases} O(u_i(y_i)^{-2s/(n-2\sigma)}), & -n < s < n, \\ O(u_i(y_i)^{-2n/(n-2\sigma)} \log u_i(y_i)), & s = n, \\ o(u_i(y_i)^{-2n/(n-2\sigma)}), & s > n, \end{cases}$$
and
\[
\int_{r_i < |y - y_i| \leq 1} |y - y_i|^s u_i(y)^{p_i+1} = \begin{cases} 
\omega(u_i(y_i))^{-2s/(n-2\sigma)}, & -n < s < n, \\
O(u_i(y_i)^{-2n/(n-2\sigma)} \log u_i(y_i)), & s = n, \\
O(u_i(y_i)^{-2n/(n-2\sigma)}), & s > n.
\end{cases}
\]

Proof. The first estimate in the above Lemma follows from Proposition 4.4 and Lemma 4.3 and the second one follows from Proposition 4.4 and Lemma 4.3.

For later application, we replace \( K \) by \( K(x)H_i(x)^{\tau_i} \) in (4.2) and consider
\[
(-\Delta)^\sigma u_i(x) = c(n, \sigma)K_i(x)H_i(x)^{\tau_i} u_i^{p_i}(x), \quad \text{in } B_2,
\]
where \( \{H_i\} \in C^{1,1}(B_2) \) satisfies
\[
A_4^{-1} \leq H_i(y) \leq A_4, \quad \text{for all } y \in B_2, \quad \text{and } \|H_i\|_{C^{1,1}(B_2)} \leq A_5
\]
for some positive constants \( A_4 \) and \( A_5 \).

Lemma 4.6. Suppose that \( \{K_i\} \) satisfies (4.1) and (\( \ast \)) condition with \( \beta < n \) for some positive constants \( A_1, A_2, \{L(\beta, i)\} \), and that \( \{H_i\} \) satisfies (4.2) with \( A_4, A_5 \). Let \( u_i \in \dot{H}^\sigma(\mathbb{R}^n) \cap C^2(B_2) \) and \( u_i \geq 0 \) in \( \mathbb{R}^n \) be a solution of (4.26). If \( y_i \to 0 \) is an isolated simple blow up point of \( \{u_i\} \) with (4.3) for some positive constant \( A_3 \), then we have
\[
\tau_i \leq C u_i(y_i)^{-2} + C|\nabla K_i(y_i)||u_i(y_i)|^{-2/(n-2\sigma)} + C(L(\beta, i) + L(\beta, i)^{\beta-1}) u_i(y_i)^{-2\beta/(n-2\sigma)},
\]
where \( C > 0 \) depends only on \( n, \sigma, A_1, A_2, A_3, A_4, A_5, \beta \) and \( \rho \).

Proof. Using Lemma 4.3 and arguing the same as in the proof of Lemma 4.3 we have
\[
\tau_i \leq C u_i(y_i)^{-2} + C \left| \int_{B_1(y_i)} \langle y - y_i, \nabla_y (K_iH_i^{\tau_i}) \rangle u_i^{p_i+1} \right|
\]
\[
\leq C u_i(y_i)^{-2} + C \tau_i \left| \int_{B_1(y_i)} |y - y_i| u_i^{p_i+1} \right|
\]
\[
+ C \left| \int_{B_1(y_i)} \langle y - y_i, \nabla K_i \rangle H_i^{\tau_i} u_i^{p_i+1} \right|.
\]
Making use of Lemma 4.5, we have
\[
\left| \int_{B_t(y_i)} \langle y - y_i, \nabla K_t \rangle H_t^\ast u_{i}^{p_i+1} \right|
\leq C|\nabla K_t(y_i)| \int_{B_t(y_i)} |y - y_i| u_{i}^{p_i+1}
\]
\[
+ C \int_{B_t(y_i)} |y - y_i| |\nabla K_t(y) - \nabla K_t(y_i)| u_{i}^{p_i+1}
\leq C|\nabla K_t(y_i)| u_i(y_i)^{-2/(n-2\sigma)}
\]
\[
+ C \int_{B_t(y_i)} |y - y_i| |\nabla K_t(y) - \nabla K_t(y_i)| u_{i}^{p_i+1}.
\]

Recalling the definition of $(\ast)_\beta$, a directly computation yields
\[
|\nabla K_t(y) - \nabla K_t(y_i)|
\leq \left\{ \sum_{s=2}^{[\beta]} |\nabla^s K_t(y_i)||y - y_i|^{s-1} + |\nabla^s K_t|_{C^{s-\beta}(B_t(y_i))}|y - y_i|^{\beta-1} \right\}
\leq CL(\beta, i) \left\{ \sum_{s=2}^{[\beta]} |\nabla^s K_t(y_i)||y - y_i|^{(s-\beta)/(\beta-1)}|y - y_i|^{\beta-1} + |y - y_i|^{\beta-1} \right\}.
\tag{4.28}
\]

By Cauchy-Schwartz inequality, we have
\[
L(\beta, i)|\nabla K_t(y_i)|^{(s-\beta)/(\beta-1)}|y - y_i|^{s}
\leq C(|\nabla K_t(y_i)||y - y_i| + L(\beta, i) + L(\beta, i)^{\beta-1}|y - y_i|^{\beta}).
\tag{4.29}
\]

Hence, by Lemma 4.5, we obtain
\[
\int_{B_t(y_i)} |y - y_i| |\nabla K_t(y) - \nabla K_t(y_i)| u_{i}^{p_i+1}
\leq C|\nabla K_t(y_i)| u_i(y_i)^{-2/(n-2\sigma)} + C(L(\beta, i) + L(\beta, i)^{\beta-1}) u_i(y_i)^{-2/(n-2\sigma)}.
\tag{4.30}
\]

Lemma 4.6 follows immediately. \hfill \Box

Lemma 4.7. Under the hypotheses of Lemma 4.6
\[
|\nabla K_t(y_i)| \leq C u_i(y_i)^{-2} + C(L(\beta, i) + L(\beta, i)^{\beta-1}) u_i(y_i)^{-2(\beta-1)/(n-2\sigma)},
\]
where $C > 0$ depends only on $n, \sigma, A_1, A_2, A_3, A_4, A_5, \beta$ and $\rho$.

Proof. Choose a cutoff function $\eta(Y) \in C_c^\infty(B_{1/2})$ satisfying
\[
\eta(Y) = 1, \quad |Y| \leq \frac{1}{4} \text{ and } \eta(Y) = 0, \quad |Y| \geq \frac{1}{2},
\]
\[36\]
Let $U_i(y)$ be the extension of $u_i(y)$, namely,

$$
\begin{align*}
\text{div}(s^{1-2\sigma}\nabla U_i) &= 0, & \text{in } \mathbb{R}_t^{n+1} \\
-\lim_{s\to 0} s^{1-2\sigma}\partial_t U(y, s) &= c_0 K_i(y) H_{i}^{\tau_i} u_{i}^{p_i}, & y \in B_2.
\end{align*}
$$

Multiplying (4.31) by $\eta(Y - Y_i)\partial_j U_i(y, s)$, $j = 1, \ldots, n$, and integrating by parts over $B_1$, we obtain

$$
0 = \int_{B_1^+} \text{div}(s^{1-2\sigma}\nabla U_i) \eta \partial_j U_i
- \int_{B_1} s^{1-2\sigma}\nabla U_i \nabla (\eta \partial_j U_i) + c_0 \int_{\partial B_1^+} (Y_i) \eta K_i H_i^{\tau_i} \partial_j U_i U_i^{p_i}
= \frac{1}{2} \int_{B_1^+} s^{1-2\sigma}(|\nabla U_i|^2 \eta - 2\nabla U_i \nabla \eta \partial_j U_i)
- \frac{c_0}{p_i + 1} \int_{\partial B_1^+} \partial_j (K_i H_i^{\tau_i} \eta) U_i^{p_i+1}.
$$

By Proposition 4.4, we have

$$
U_i(Y) \leq CU_i(Y_i)^{-1}, \quad \text{for all } 1/2 \geq |Y| \geq 1/4
$$

and

$$
\int_{B_1^+ \setminus B_1^{1/4}} s^{1-2\sigma}|\nabla U_i| \leq CU_i(Y_i)^{-2}.
$$

Therefore by Lemma 4.5, we conclude that

$$
\left| \int_{B_1} \partial_j K_i(y) H_i^{\tau_i} u_i^{p_i+1} \right| \leq Cu_i(y_i)^{-2} + C\tau_i.
$$

Hence

$$
\left| \partial_j K_i(y_i) \int_{B_1} H_i^{\tau_i} u_i^{p_i+1} \right| - Cu_i(y_i)^{-2} - C\tau_i
\leq \int_{B_1} |\partial_j K_i(y_i) - \partial_j K_i(y)| H_i^{\tau_i} u_i^{p_i+1}
$$

Summing over $j$, then making use of (4.28), (4.29) and Lemma 4.5, we have

$$
|\nabla K_i(y_i)| \leq Cu_i(y_i)^{-2} + C\tau_i + \frac{1}{2} |\nabla K_i(y_i)|
+ C(L(\beta, i) + L(\beta, i)^{\beta-1}) u_i(y_i)^{-2(\beta-1)/(n-2\sigma)}.
$$

Then Lemma 4.7 follows from Lemma 4.6.

**Lemma 4.8.** Under the assumptions of Lemma 4.6, we have

$$
\tau_i \leq Cu_i(y_i)^{-2} + C(L(\beta, i) + L(\beta, i)^{\beta-1}) u_i(y_i)^{-2\beta/(n-2\sigma)}.
$$
Proof. It follows immediately from Lemma 4.6 and Lemma 4.7. \hfill \Box

**Corollary 4.1.** In addition to the assumptions of Lemma 4.6, we further assume that one of the following two conditions holds: (i)

\[ \beta = n - 2\sigma \text{ and } L(\beta, i) = o(1), \]

and (ii)

\[ \beta > n - 2\sigma \text{ and } L(\beta, i) = O(1). \]

Then for any \( 0 < \delta < 1 \) we have

\[ \lim_{i \to \infty} u_i(y_1)^2 \int_{B_{\delta}(y_1)} (y - y_1) \cdot \nabla (K_i H_i^{\tau_i}) u_i^{p_i + 1} = 0. \]

Proof.\[ \left| \int_{B_{\delta}(y_1)} (y - y_1) \cdot \nabla (K_i H_i^{\tau_i}) u_i^{p_i + 1} \right| \]

\[ \leq \int_{B_{\delta}(y_1)} (y - y_1) \cdot \nabla K_i H_i^{\tau_i} u_i^{p_i + 1} + \tau_i \int_{B_{\delta}(y_1)} (y - y_1) \cdot \nabla H_i^{\tau_i} - K_i u_i^{p_i + 1} \]

\[ \leq C |\nabla K_i(y_1)| \int_{B_{\delta}(y_1)} |y - y_1| u_i^{p_i + 1} + C \int_{B_{\delta}(y_1)} |y - y_1| |\nabla K_i(y) - \nabla K_i(y_1)| u_i^{p_i + 1} + \tau_i \int_{B_{\delta}(y_1)} |y - y_1| u_i^{p_i + 1}. \]

The corollary follows immediately from Lemma 4.7 and Lemma 4.8. \hfill \Box

**Proposition 4.5.** Suppose that \( \{K_i\} \) satisfies (4.1) and (\( \ast \))\( n - 2\sigma \) condition for some positive constants \( A_1, A_2, L \) independent of \( i \), and that \( \{H_i\} \) satisfies (4.27) with \( A_4, A_5 \). Let \( u_i \in \tilde{H}^{\sigma}(\mathbb{R}^n) \cap C^2(B_2) \) be a solution of (4.26). If \( y_i \to 0 \) is an isolated blow up point of \( \{u_i\} \) with (4.3) for some positive constant \( A_3 \), then \( y_i \to 0 \) is an isolated simple blow up point.

Proof. Due to Proposition 4.1, \( r^{2\sigma/(p_i - 1)\mu_i}(r) \) has precisely one critical point in the interval \( 0 < r < r_i \), where \( r_i = R_i u_i(y_i) \) as before. Suppose \( y_i \to 0 \) is not an isolated simple blow up point and let \( \mu_i \) be the second critical point of \( r^{2\sigma/(p_i - 1)\mu_i}(r) \). Then we see that

\[ \mu_i \geq r_i, \quad \lim_{i \to \infty} \mu_i = 0. \]

Without loss of generality, we assume that \( y_i = 0 \). Set

\[ \phi_i(y) = \mu_i^{2\sigma/(p_i - 1)} u_i(\mu_i y), \quad y \in \mathbb{R}^n. \]

Clearly, \( \phi_i \) satisfies

\[ (-\Delta)^\sigma \phi_i(y) = \tilde{K}_i(y) \tilde{H}_i^{\tau_i}(y) \phi_i^{p_i}(y), \]

\[ |y|^{2\sigma/(p_i - 1)} \phi_i(y) \leq A_3, \quad |y| < 1/\mu_i, \]

\[ \lim_{i \to \infty} \phi_i(0) = \infty. \]
\[ t^{2\sigma/(p_1-1)} \bar{\phi}_i(r) \] has precisely one critical point in \( 0 < r < 1 \), and
\[
\frac{d}{dr} \left\{ t^{2\sigma/(p_1-1)} \bar{\phi}_i(r) \right\} \bigg|_{r=1} = 0,
\]
where \( \bar{K}_i(y) = K_i(\mu_i y), \bar{H}_i(y) = H_i(\mu_i y) \) and \( \bar{\phi}_i(r) \) is \( |\partial B_r|^{-1} \int_{\partial B_r} \phi_i \).

Therefore, 0 is an isolated simple blow up point of \( \phi_i \). Let \( \Phi_i(Y) \) be the extension of \( \phi_i(y) \) in the upper half space. Then Lemma 4.1, Proposition 4.4, Lemma 4.4 and elliptic equation theory together imply that
\[
\Phi_i(0) \Phi_i(Y) \to G(Y) = A|Y|^{2\sigma-n} + H(Y) \quad \text{in } C^\alpha_{\text{loc}}(\mathbb{R}^{n+1}_+ \setminus \{0\}) \cap C^2_{\text{loc}}(\mathbb{R}^{n+1}_+).
\]
and
\[
\phi_i(0) \phi_i(y) \to G(y, 0) = A|y|^{2\sigma-n} + H(y, 0) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \quad \text{(4.34)}
\]
as \( i \to \infty \), where \( A > 0 \), \( H(Y) \) satisfies
\[
\begin{cases}
\text{div}(s^{1-2\sigma} \nabla H) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\
- \lim_{s \to 0} s^{1-2\sigma} \partial_s H(y, s) = 0 & \text{for } y \in \mathbb{R}^n.
\end{cases}
\]

Note that \( G(Y) \) is nonnegative, we have \( \liminf_{|Y| \to \infty} H(Y) \geq 0 \). It follows from the weak maximum principle and the Harnack inequality that \( H(y) \equiv H \geq 0 \) is a constant. Since
\[
\frac{d}{dr} \left\{ t^{2\sigma/(p_1-1)} \phi_i(0) \bar{\phi}_i(r) \right\} \bigg|_{r=1} = \phi_i(0) \frac{d}{dr} \left\{ t^{2\sigma/(p_1-1)} \bar{\phi}_i(r) \right\} \bigg|_{r=1} = 0,
\]
we have, by sending \( i \) to \( \infty \) and making use of (4.34), that
\[
A = H > 0.
\]

We are going to derive a contradiction to the Pohozaev identity Proposition 4.3 by showing that for small positive \( \delta \)
\[
\limsup_{i \to \infty} \Phi_i(0)^2 \int_{\partial^C B^+_{\delta}} B'(Y, \Phi_i, \nabla \Phi_i, \delta, \sigma) \leq 0, \quad \text{(4.35)}
\]
and
\[
\limsup_{i \to \infty} \Phi_i(0)^2 \int_{\partial^C B^+_{\delta}} s^{1-2\sigma} B''(Y, \Phi_i, \nabla \Phi_i, \delta, \sigma) < 0. \quad \text{(4.36)}
\]
And thus Proposition 4.5 will be established.

By Proposition 2.6, it is easy to verify (4.36) by that
\[
\limsup_{i \to \infty} \Phi_i(0)^2 \int_{\partial^C B^+_{\delta}} s^{1-2\sigma} B''(Y, \Phi_i, \nabla \Phi_i, \delta, \sigma) = \int_{\partial^C B^+_{\delta}} s^{1-2\sigma} B''(Y, G, \nabla G, \delta, \sigma) = -\frac{(n-2\sigma)^2}{2} A^2 \int_{\partial^C B^+_{\delta}} t^{1-2\sigma} < 0,
\]

which shows (4.36). On the other hand, via integration by parts, we have

\[
\int_{\partial B_i^+} B'(Y, \Phi_i, \nabla \Phi_i, \delta, \sigma)
= \frac{n-2\sigma}{2} \int_{B_i} \tilde{K}_i \tilde{H}_i \phi_i^{p_i+1} + \int_{B_i} \langle y, \nabla \phi_i \rangle \tilde{K}_i \tilde{H}_i \phi_i^{p_i+1}
= \frac{n-2\sigma}{2} \int_{B_i} \tilde{K}_i \tilde{H}_i \phi_i^{p_i+1} - \frac{n}{p_i+1} \int_{B_i} \tilde{K}_i \tilde{H}_i \phi_i^{p_i+1}
\]
\[
- \frac{1}{p_i+1} \int_{B_i} \langle y, \nabla (\tilde{K}_i \tilde{H}_i) \rangle \phi_i^{p_i+1} + \frac{\delta}{p_i+1} \int_{\partial B_i} \tilde{K}_i \tilde{H}_i \phi_i^{p_i+1}
\leq - \frac{1}{p_i+1} \int_{B_i} \langle y, \nabla (\tilde{K}_i \tilde{H}_i) \rangle \phi_i^{p_i+1} + C \phi_i(0)^{-(p_i+1)}.
\]

where Proposition 4.4 is used in the last inequality. It is easy to see that \(\{\tilde{K}_i\}\) satisfies (\(\ast\)) with \(L(\beta, i) = o(1)\). Therefore, (4.35) follows from Corollary 4.1.

Proposition 4.6. Suppose the assumptions in Proposition 4.5 except the (\(\ast\)) condition for \(K_i\).

Then

\[
|\nabla K_i(y_i)| \to 0, \quad \text{as } i \to \infty.
\]

Proof. Suppose that contrary that

\[
|\nabla K_i(y_i)| \to d > 0.
\]

(4.37)

Without loss of generality, we assume \(y_i = 0\). There are two cases.

Case 1. \(0\) is an isolated simple blow up point.

In this case, we argue as in the proof of Lemma 4.7 and obtain

\[
\left| \int_{B_1} \nabla K_i \tilde{H}_i u_i^{p_i+1} \right| \leq C u_i^{-2}(0) + C r_i.
\]

It follows from the mean value theorem, Lemma 4.3 and Lemma 4.5 that

\[
|\nabla K_i(0)| \leq C \int_{B_1} |\nabla K_i(y) - \nabla K_i(0)| \tilde{H}_i u_i^{p_i+1} + o(1) = o(1).
\]

Case 2. \(0\) is not an isolated simple blow up point.

In this case we argue as the proof of Proposition 4.5. The only difference is that we cannot derive (4.35) from Corollary 4.1 since (\(\ast\)) condition for \(K_i\) is not assumed. Instead, we will use the condition (4.37) to show (4.35).

Let \(\mu_i, \phi_i, \Phi_i, \tilde{K}_i\) and \(\tilde{H}_i\) be as in the proof of Proposition 4.5. The computation at the end of the proof of Proposition 4.5 gives

\[
\int_{\partial B_i^+} B'(Y, \Phi_i, \nabla \Phi_i, \delta, \sigma)
\leq \frac{1}{p_i+1} \int_{B_i} \langle y, \nabla (\tilde{K}_i \tilde{H}_i) \rangle \phi_i^{p_i+1} + C \phi_i(0)^{-(p_i+1)}.
\]

40
Now we estimate the integral term $\int_{B_i}(y, \nabla(H^i_\tau))\phi_i^{p_i+1}$. Using Lemma 4.3 and arguing the same as in the proof of Lemma 4.3, we have

$$\tau_i \leq C\phi_i(0)^{-2} + C\int_{B_i} |y| |\nabla \tilde{K}_i(y)| H^i_\tau \phi_i^{p_i+1}$$

By (4.32),

$$\int_{B_i} |\tilde{K}_i H^i_\tau \phi_i^{p_i+1}| \leq C\phi_i(0)^{-2} + C\tau_i.$$ 

It follows that

$$|\nabla \tilde{K}_i(0)| \leq C \int_{B_i} |\nabla \tilde{K}_i(y) - \nabla \tilde{K}_i(0)| \phi_i^{p_i+1} + C\phi_i(0)^{-2} + C\tau_i$$

$$\leq C\mu_i \phi_i(0)^{-2/(n-2\sigma)} + C\phi_i(0)^{-2} + C\tau_i.$$ 

Since $|\nabla \tilde{K}_i(0)| = \mu_i |\nabla K_i(0)| \geq (d/2)\mu_i$, we have

$$\mu_i \leq C\phi_i(0)^{-2} + C\tau_i.$$ 

It follows that

$$\tau_i \leq C\phi_i(0)^{-2} \quad \text{and} \quad \mu_i \leq C\phi_i(0)^{-2}.$$ 

Therefore,

$$\left| \int_{B_i} \langle y, \nabla(H^i_\tau) \rangle \phi_i^{p_i+1} \right| \leq C\phi_i(0)^{-2/(n-2\sigma)}$$

and (4.35) follows immediately.

5 Estimates on the sphere and proofs of main theorems

Consider

$$P_\sigma(x) = c(n, \sigma)Kv^n, \quad \text{on} \ S^n, \quad (5.1)$$

where $p \in (1, \frac{n+2\sigma}{n-2\sigma}]$ and $K$ satisfies

$$A_1^{-1} \leq K \leq A_1, \quad \text{on} \ S^n, \quad (5.2)$$

and

$$\|K\|_{C^{1,1}(S^n)} \leq A_2. \quad (5.3)$$

**Proposition 5.1.** Let $v \in C^2(S^n)$ be a positive solution to (5.1). For any $0 < \varepsilon < 1$ and $R > 1$, there exist large positive constants $C_1, C_2$ depending on $n, \sigma, A_1, A_2, \varepsilon$ and $R$ such that, if

$$\max_{S^n} v \geq C_1,$$
then \( \frac{n+2\sigma}{2\sigma} - p < \varepsilon \), and there exists a finite set \( \varphi(v) \subset \mathbb{S}^n \) such that

(i). If \( P \in \varphi(v) \), then it is a local maximum of \( v \) and in the stereographic projection coordinate system \( \{y_1, \ldots, y_n\} \) with \( P \) as the south pole,

\[
\|v^{-1}(P)v(u^{-(n-1)/2\sigma}(P)y) - (1 + k\|y\|^2)^{(2\sigma-n)/2}\|_{C^2(B_{2r}(P))} \leq \varepsilon, \tag{5.4}
\]

where \( k = K(P)^{1/\sigma}/4 \).

(ii). If \( P_1, P_2 \) belonging to \( \varphi(v) \) are two different points, then

\[
B_{Rv(n^{-1/2\sigma})(P_1)} \cap B_{Rv(n^{-1/2\sigma})(P_2)} = \emptyset.
\]

(iii). \( v(P) \leq C_2(d\{\text{dist}(P, \varphi(v))\})^{-2\sigma/(p-1)} \) for all \( P \in \mathbb{S}^n \).

**Proof.** Given Theorem 1.5, Remark 1.2, and the proof of Proposition 4.1, the proof of Proposition 5.1 is similar to that of Proposition 4.1 in [59] and Lemma 3.1 in [88], and is omitted here. We refer to [68] and [88] for details. \( \square \)

**Proposition 5.2.** Assume the hypotheses in Proposition 5.1. Suppose that there exists some constant \( d > 0 \) such that \( K \) satisfies \( (n)_2 \), for some \( L \in \Omega_d = \{P \in \mathbb{S}^n : |\nabla K(P)| < d\} \). Then, for \( \varepsilon > 0, R > 1 \) and any solution \( v \) of \( (5.1) \) with \( \max_{\mathbb{S}^n} v > C_1 \), we have

\[
|P_1 - P_2| \geq \delta^* > 0, \quad \text{for any } P_1, P_2 \in \varphi(v) \text{ and } P_1 \neq P_2,
\]

where \( \delta^* \) depends only on \( n, \sigma, \delta, \varepsilon, R, A_1, A_2, L_2, d \).

**Proof.** Suppose the contrary, then there exists sequences of \( \{p_i\} \) and \( \{K_i\} \) satisfying the above assumptions, and a sequence of corresponding solutions \( \{v_i\} \) such that

\[
\lim_{i \to \infty} |P_{1i} - P_{2i}| = 0, \tag{5.5}
\]

where \( P_{1i}, P_{2i} \in \varphi(v_i) \), and \( |P_{1i} - P_{2i}| = \min_{P_1, P_2 \in \varphi(v_i)} |P_1 - P_2| \).

Since \( B_{Rv(n^{-1/2\sigma})(P_{1i})} \) and \( B_{Rv(n^{-1/2\sigma})(P_{2i})} \) have to be disjoint, we have, because of \( (5.5) \), that \( v_i(P_{1i}) \to \infty \) and \( v_i(P_{2i}) \to \infty \). Therefore, we can pass to a subsequence (still denoted as \( v_i \)) with \( R_i \to \infty, \varepsilon_i \to 0 \) as in Proposition 4.1 (\( \varepsilon_i \) depends on \( R_i \) and can be chosen as small as we need in the following arguments) such that, for \( y \) being the stereographic projection coordinate with south pole at \( P_{ji} \), \( j = 1, 2 \), we have

\[
\|m_i^{-1}v_i(m_i^{-(n-1)/2\sigma}y) - (1 + k_{ji}\|y\|^2)^{(2\sigma-n)/2}\|_{C^2(B_{2R_i}(0))} \leq \varepsilon_i, \tag{5.6}
\]

where \( m_i = v_i(0), k_{ji} = K_i(q_{ji})^{1/\sigma}, j = 1, 2, i = 1, 2, \ldots \).

In the stereographic coordinates with \( P_{1i} \) being the south pole, the equation \((5.1)\) is transformed into

\[
(-\Delta)^\sigma u_i(y) = c(n, \sigma)K_i(y)H_i(y)u_i^\sigma(y), \quad y \in \mathbb{R}^n, \tag{5.7}
\]

where

\[
u_i(y) = \left(\frac{2}{1 + |y|^2}\right)^{(n-2\sigma)/2} v_i(F(y)),
\]

\[
H_i(y) = \left(\frac{2}{1 + |y|^2}\right)^{(n-2\sigma)/2},
\]

(5.8)
and $F$ is the inverse of the stereographic projection. Let us still use $P_{2i} \in \mathbb{R}^n$ to denote the stereographic coordinates of $P_{2i} \in S^n$ and set $\vartheta_i = |P_{2i}| \to 0$. For simplicity, we assume $P_{2i}$ is a local maximum point of $u_i$. Since we can always reselect a sequence of points as in the proof of Proposition 5.1 to substitute for $P_{2i}$.

From (ii) in Proposition 5.1, there exists some constant $C$ depending only on $n, \sigma$, such that

$$\vartheta_i > \frac{1}{C} \max \{ R_i u_i(0)^{-\alpha_1/2}, R_i u_i(0)^{-\alpha_2/2} \}. \quad (5.9)$$

Set

$$w_i(y) = \vartheta_i^{2\sigma/(\alpha_1-\alpha_2)} u_i(\vartheta_i y), \quad \text{in } \mathbb{R}^n.$$ 

It is easy to see that $w_i$ which is positive in $\mathbb{R}^n$, satisfies

$$(-\Delta)^\sigma w_i(y) = c(n, \sigma) K_i(y) \tilde{H}_i(y) w_i(y), \quad \text{in } \mathbb{R}^n \quad (5.10)$$

and

$$w_i(y) \in C^2(\mathbb{R}^n), \quad \lim \inf_{|y| \to \infty} w_i(y) < \infty,$$

where $\tilde{K}_i(y) = K_i(\vartheta_i y), \tilde{H}_i(y) = H_i(\vartheta_i y)$.

By Proposition 5.1, $w_i$ satisfies

$$u_i(y) \leq C_2 \vartheta_i y^{-2\sigma/(\alpha_1-\alpha_2)} \quad \text{for all } |y| \leq \vartheta_i$$

$$u_i(y) \leq C_2 |y - P_{2i}|^{-2\sigma/(\alpha_1-\alpha_2)} \quad \text{for all } |y - P_{2i}| \leq \vartheta_i/2.$$ 

In view of (5.9), we therefore have

$$\lim_{i \to \infty} w_i(0) = \infty, \quad \lim_{i \to \infty} w_i(|P_{2i}|^{-1} P_{2i}) = \infty$$

$$|y|^{2\sigma/(\alpha_1-\alpha_2)} w_i(y) \leq C_2, \quad |y| \leq 1/2,$$

$$|y - P_{2i}|^{-1} P_{2i}^{2\sigma/(\alpha_1-\alpha_2)} w_i(y) \leq C_2, \quad |y - P_{2i}|^{-1} P_{2i} \leq 1/2.$$ 

After passing a subsequence, if necessary, there exists a point $\overline{P} \in \mathbb{R}^n$ with $|\overline{P}| = 1$ such that $|P_{2i}|^{-1} P_{2i} \to \overline{P}$ as $i \to \infty$. Hence $0$ and $\overline{P}$ are both isolated blow up points of $w_i$.

If $|\nabla K_i(0)| \leq d/2$, then $0$ is an isolated simple blow up point of $w_i$ because of the $(*)_{n-2\sigma}$ condition and Proposition 4.5. If $|\nabla K_i(0)| \geq d/2$, arguing as in the proof of Proposition 4.6, we can conclude that $0$ is an isolated simple blow up point of $w_i$. Similarly, $\overline{P}$ is also an isolated simple blow up point of $w_i$.

By Proposition 4.4,

$$w_i(0) w_i(y) \leq C_\varepsilon, \quad \text{for all } \varepsilon \leq |y| \leq 1/2,$$

where $C_\varepsilon$ is independent of $i$. Let $W_i$ be the extension of $w_i$. Due to Proposition 5.1, Harnack inequality Lemma 4.1, and the choice of $P_{1i}, P_{2i}$, there exists an at most countable set $\varphi \subset \mathbb{R}^n$ such that

$$\inf \{|x - y| : x, y \in \varphi, x \neq y\} \geq 1,$$

and

$$\lim_{i \to \infty} W_i(0) W_i(Y) = G(Y), \quad \text{in } C_0^0(\overline{\mathbb{R}_{+}^{n+1} \setminus \varphi})$$

$$G(Y) > 0, \quad Y \in \mathbb{R}_{+}^{n+1} \setminus \varphi.$$
Let \( \varphi_1 \subset \varphi \) contain those points near which \( G \) is singular. Clearly, \( 0, \overline{P} \in \varphi_1 \). Since \( p_i > 1 \), it follows from (5.10) that

\[
\begin{align*}
\text{div}(s^{1-2\sigma} \nabla G) &= 0, & \text{in } \mathbb{R}^{n+1}_+, \\
\lim_{s \to 0} s^{1-2\sigma} \partial_s G(y, s) &= 0, & \text{for all } y \in \mathbb{R}^n \setminus \varphi_1.
\end{align*}
\]

By Lemma 4.4 and maximum principle, there exist positive constants \( N_1, N_2 \) and some nonnegative function \( H \) satisfying

\[
\begin{align*}
\text{div}(s^{1-2\sigma} \nabla H) &= 0, & \text{in } \mathbb{R}^{n+1}_+, \\
\lim_{s \to 0} s^{1-2\sigma} \partial_s H(y, s) &= 0, & \text{for all } y \in \mathbb{R}^n \setminus \{ \varphi_1 \setminus \{ 0, \overline{P} \} \}
\end{align*}
\]

such that

\[
G(Y) = N_1 |Y|^{2\sigma-n} + N_2 |Y - \overline{P}|^{2\sigma-n} + H(Y), \quad Y \in \mathbb{R}^{n+1}_+ \setminus \{ \varphi_1 \}.
\]

Applying Proposition 2.6 to \( H \), it is not difficult to verify (4.36) with \( \Phi^i \) replaced by \( W^i \). On the other hand, we can establish (4.35) with \( \Phi^i \) replaced by \( W^i \) if \( |\nabla K^i(0)| \leq d/2 \), because \( (\ast)_{n-2\sigma} \) condition with \( L = o(1) \) holds for \( \tilde{K}^i \) and thus Corollary 4.1 holds. If \( |\nabla K^i(0)| \geq d/2 \), we can apply the argument in the proof of Proposition 4.6 to conclude that \( \vartheta^i, \tau^i \leq w_i(0)^{-2} \), and hence (4.35) also holds for \( W^i \).

Proposition 5.2 is established.

Consider

\[
P_{\sigma}(v) = c(n, \sigma) K^i v_i^p_i \quad \text{on } \mathbb{S}^n, \\
v_i > 0, \quad \text{on } \mathbb{S}^n, \\
p_i = \frac{n + 2\sigma}{n - 2\sigma} - \tau_i, \quad \tau_i \geq 0, \quad \tau_i \to 0.
\]  

(5.11)

**Theorem 5.1.** Suppose \( K_i \) satisfies the assumption of \( K \) in Proposition 5.2. Let \( v_i \) be solutions of (5.11), we have

\[
\|v_i\|_{H^\sigma(\mathbb{S}^n)} \leq C,
\]

(5.12)

where \( C > 0 \) depends only on \( n, \sigma, A_1, A_2, L, d \). Furthermore, after passing to a subsequence, either \( \{v_i\} \) stays bounded in \( L^\infty(\mathbb{S}^n) \) or \( \{v_i\} \) has only isolated simple blow up points and the distance between any two blow up points is bounded by some positive constant depending only on \( n, \sigma, A_1, A_2, L, d \).

**Proof.** The theorem follows immediately from Proposition 5.2 Proposition 4.6 Proposition 4.5 Proposition 4.1 and Lemma 4.5.

**Proof of Theorem 1.3.** It follows immediately from Theorem 5.1.

In the next theorem, we impose a stronger condition on \( K_i \) such that \( \{u_i\} \) has at most one blow up point.
Theorem 5.2. Suppose the assumptions in Theorem 5.1. Suppose further that \( \{K_i\} \) satisfies \((*)_{n-2\sigma}\) condition for some sequences \( L(n-2\sigma,i) = o(1) \) in \( \Omega_{\delta,i} = \{ q \in \mathbb{S}^n : |\nabla_{g_0}K_i| < \delta \} \) or \( \{K_i\} \) satisfies \((*)_\beta\) condition with \( \beta > n-2\sigma \) in \( \Omega_{\delta,i} \). Then, after passing to a subsequence, either \( \{v_i\} \) stays bounded in \( L^\infty(\mathbb{S}^n) \) or \( \{v_i\} \) has precisely one isolated simple blow up point.

Proof. The strategy is the same as the proof of Proposition 5.2. We assume there are two isolated blow up points. After some transformation, we can assume that they are in the same half sphere. The condition of \( \{K_i\} \) guarantees that Corollary 4.1 holds for \( u_i \), where \( u_i \) is as in (5.3). Hence (4.35) holds for \( U_i \), which is the extension of \( u_i \). Meanwhile (4.36) for \( U_i \) is also valid, since the distance between these blow up points is uniformly lower bounded which is due to Proposition 5.2. \( \square \)

Proof of Theorem 1.4. By Theorem 5.2, we only need to show the latter case of theorem. After passing a subsequence, \( \xi_i \to \xi \) is the only isolated simple blow up point of \( v_i \). For simplicity, assume that \( \xi_i \) is identical to the south pole and \( K(\xi_i) = 1 \). Let \( F : \mathbb{R}^n \to \mathbb{S}^n \) be the inverse of stereographic projection defined at the beginning of the paper. Define, for any \( \lambda > 0 \),

\[
\psi_\lambda : x \mapsto \lambda x, \quad \forall x \in \mathbb{R}^n.
\]

Set \( \varphi_i = F \circ \psi_\lambda \circ F^{-1} \) with \( \lambda_i = v_i(\xi_i)^{-\frac{2}{n-2\sigma}} \). Then \( T_{\varphi_i}v_i \) satisfies

\[
P_\sigma(T_{\varphi_i}v_i) = c(n,\sigma)K \circ \varphi_iT_{\varphi_i}v_i^{-\frac{2}{n-2\sigma}}, \quad \text{on } \mathbb{S}^n.
\]

Let

\[
u_i(x) = \left(\frac{2}{1 + |x|^2}\right)^{\frac{n-2\sigma}{2}} v_i \circ F(x), \quad x \in \mathbb{R}^n
\]

and

\[
u_i(x) = \left(\frac{2}{1 + |x|^2}\right)^{\frac{n-2\sigma}{2}} T_{\varphi_i}v_i \circ F(x), \quad x \in \mathbb{R}^n.
\]

Note that

\[
|\det d\varphi_i(F(x))|^{-\frac{n-2\sigma}{2}} = \left(\frac{2}{1 + |\lambda_i x|^2}\right)^n \lambda_i^n \left(\frac{2}{1 + |x|^2}\right)^{-n} \left(\frac{2}{1 + |\lambda_i x|^2}\right)^{-n}.
\]

Hence, \( \ddot{u}_i(x) = \lambda_i^{-\frac{2}{n-2\sigma}} u_i(\lambda_i x) \) for any \( x \in \mathbb{R}^n \) and \( 0 < u_i \leq 2^{-\frac{2}{n-2\sigma}} \). Arguing as before, we see that

\[
\ddot{u}_i(x) \to \left(\frac{2}{1 + |x|^2}\right)^{\frac{n-2\sigma}{2}}, \quad \text{in } C^2_{loc}(\mathbb{R}^n).
\]

Therefore, \( v_i \to 1 \) in \( C^2_{loc}(\mathbb{S}^n \setminus \{N\}) \), where \( N \) is the north pole of \( \mathbb{S}^n \).

Since \( T_{\varphi_i}v_i \) is uniformly bounded near the north pole, it follows from Hölder estimates that there exists a constant \( \alpha \in (0, 1) \) such that \( T_{\varphi_i}v_i \to f \) in \( C^\alpha(B_\delta(N)) \) for small constant \( \delta > 0 \) and some function \( f \in C^\alpha(B_\delta(N)) \). It is clear that \( f = 1 \). Therefore, we complete the proof. \( \square \)

Theorem 5.3. Let \( v_i \) be positive solutions of (5.11). Suppose that \( \{K_i\} \subset C^\infty(\mathbb{S}^n) \) satisfies (5.3), and for some point \( P_0 \in \mathbb{S}^n, \epsilon_0 > 0, A_1 > 0 \) independent of \( i \) and \( 1 < \beta < n \), that

\[
\{K_i\} \text{ is bounded in } C^{[\beta],[\beta]-[\beta]}(B_{\epsilon_0}(g_0)), \quad K_i(P_0) \geq A_1
\]
and
\[ K_i(y) = K_i(0) + Q_i^{(β)}(y) + R_i(y), \quad |y| ≤ ε_0, \]
where \( y \) is the stereographic projection coordinate with \( P_0 \) as the south pole, \( Q_i^{(β)}(y) \) satisfies
\[ Q_i^{(β)}(λy) = λ^β Q_i^{(β)}(y), \quad ∀ λ > 0, \quad y ∈ \mathbb{R}^n, \quad \text{and } R_i(y) \] satisfies
\[ \sum_{s=0}^{[β]} |\nabla^s R_i(y)||y|^{-β+s} → 0 \]
uniformly for \( i \) as \( y → 0 \).

Suppose also that \( Q_i^{(β)} → Q^{(β)} \) in \( C^1(\mathbb{S}^{n-1}) \) and for some positive constant \( A_6 \)
that
\[ A_6|y|^{β-1} ≤ |∇Q^{(β)}(y)|, \quad |y| ≤ ε_0, \quad (5.13) \]
and
\[ \left( \int_{\mathbb{R}^n} ∇Q^{(β)}(y + y_0)(1 + |y|^2)^{-n} \, dy \right) ≠ 0, \quad ∀ y_0 ∈ \mathbb{R}^n. \quad (5.14) \]

If \( P_0 \) is an isolated simple blow up point of \( v_i \), then \( v_i \) has to have at least another blow up point.

**Proof.** Suppose the contrary, \( P_0 \) is the only blow up point of \( v_i \).

We make a stereographic projection with \( P_0 \) being the south pole to the equatorial plane of \( \mathbb{S}^n \), with its inverse \( π \). Then the Eq. (5.11) is transformed to
\[ (-Δ)^{σ} u_i = c(n, σ)K_i(y)u_i^{\frac{n+2σ}{n-2σ}}, \quad \text{in } \mathbb{R}^n, \quad (5.15) \]
with
\[ u_i(y) = \left( \frac{2}{1 + |y|^2} \right)^{(n-2σ)/2} v_i(π(y)). \]
Let \( y_i → 0 \) be the local maximum point of \( u_i \). It follows from Lemma 4.7 that
\[ |∇K_i(y_i)| = O(u_i(y_i)^{-2} + u_i(y_i)^{-2(β-1)/(n-2σ)}). \]

First we establish
\[ |y_i| = O(u_i(y_i)^{-2/(n-2σ)}). \quad (5.16) \]
Since we have assumed that \( v_i \) has no other blow up point other than \( P_0 \), it follows from Proposition 4.4 and Harnack inequality that for \( |y| ≥ ε > 0 \), \( u_i(y) ≤ C(ε)|y|^{2σ-n}u_i(y_i)^{-1} \).

By Proposition A.1 we have
\[ \int_{\mathbb{R}^n} ∇K_i(y)u_i^{\frac{n-2σ}{n}} = 0. \quad (5.17) \]
It follows that for \( ε > 0 \) small we have
\[ \left| \int_{B_ε} ∇K_i(y + y_i)u_i(y + y_i)^{\frac{n-2σ}{n}} \right| ≤ C(ε)u_i(y_i)^{-2n/(n-2σ)}. \]
Using our hypotheses on $\nabla Q_i^{(\beta)}$ and $R_i$, we have

$$\left| \int_{B_{\rho_i}(x)} (1 + o_{\epsilon}(1)) \nabla Q_i^{(\beta)}(y + y_i) u_i(y + y_i)^{\frac{2m_i}{n}} \right| \leq C(\epsilon) u_i(y_i)^{-2/(n-2\sigma)}.$$

Multiplying the above by $m_i^{2/(n-2\sigma)}(\beta - 1)$, where $m_i = u_i(y_i)$, we have

$$\left| \int_{B_{\rho_i}(x)} (1 + o_{\epsilon}(1)) \nabla Q_i^{(\beta)}(m_i^{2/(n-2\sigma)} y + y_i) u_i(y + y_i)^{\frac{2m_i}{n}} \right| \leq C(\epsilon) u_i(y_i)^{(2/(n-2\sigma))(\beta - 1 - n)}$$

where $\tilde{y}_i = m_i^{2/(n-2\sigma)} y_i$. Suppose (5.1.6) is false, namely, $\tilde{y}_i \to +\infty$ along a subsequence. Then it follows from Proposition 4.1 (we may choose $R_i \leq |\tilde{y}_i|/4$) that

$$\left| \int_{|y| \leq R_i} (1 + o_{\epsilon}(1)) \nabla Q_i^{(\beta)}(m_i^{2/(n-2\sigma)} y + \tilde{y}_i) u_i(y + y_i)^{\frac{2m_i}{n}} \right| \leq C \left| \int_{|z| \leq R_i} (1 + o_{\epsilon}(1)) \nabla Q_i^{(\beta)}(z + \tilde{y}_i) \left( m_i^{-1} u_i (m_i^{2/(n-2\sigma)} z + y_i) \right)^{\frac{2m_i}{n}} \right| \sim |\tilde{y}_i|^{\beta - 1}.$$

On the hand, it follows from Lemma 4.5 that

$$\left| \int_{R_i} (1 + o_{\epsilon}(1)) \nabla Q_i^{(\beta)}(m_i^{2/(n-2\sigma)} y + \tilde{y}_i) u_i(y + y_i)^{\frac{2m_i}{n}} \right| \leq C \left| \int_{R_i} (1 + o_{\epsilon}(1)) \nabla Q_i^{(\beta)}(m_i^{2/(n-2\sigma)} y + \tilde{y}_i) u_i(y + y_i)^{\frac{2m_i}{n}} \right| \leq o(1)|\tilde{y}_i|^{\beta - 1}.$$

It follows that

$$|\tilde{y}_i|^{\beta - 1} \leq C(\epsilon) m_i^{2/(n-2\sigma)(\beta - 1 - n)},$$

which implies that

$$|y_i| \leq C(\epsilon) m_i^{-2/(n-2\sigma))(n/(\beta - 1)) = o(m_i^{-2/(n-2\sigma)}).$$

This contradicts to that $\tilde{y}_i \to \infty$. Thus (5.1.6) holds.

We are going to find some $y_0$ such that (5.1.4) fails.

It follows from Kazdan-Warner condition Proposition A.1 that

$$\int_{\mathbb{R}^n} \langle y, \nabla K_i(y + y_i) \rangle u_i(y + y_i)^{2n/(n-2\sigma)} = 0. \quad (5.1.8)$$

Since $P_0$ is an isolated simple blow up point and the only blow up point of $u_i$, we have for any $\epsilon > 0$,

$$\left| \int_{B_{\rho_i}(x)} \langle y, \nabla K_i(y + y_i) \rangle u_i(y + y_i)^{2n/(n-\sigma)} \right| \leq C(\epsilon) u_i(y_i)^{-2n/(n-2\sigma)}.$$
It follows from Lemma 4.3 and expression of $K_i$ that
\[
\left| \int_{B_\varepsilon} \langle y, \nabla Q_i^{(\beta)}(y + y_i) \rangle u_i(y + y_i)^{2n/(n-2\sigma)} \right| \\
\leq C(\varepsilon) u_i(y_i)^{-2n/(n-2\sigma)} \\
+ o(1) \int_{B_\varepsilon} |y| |y + y_i|^{\beta-1} u_i(y + y_i)^{-2n/(n-2\sigma)} \\
\leq C(\varepsilon) u_i(y_i)^{-2n/(n-2\sigma)} \\
+ o(1) \int_{B_\varepsilon} (|y|^{\beta} + |y||y_i|^{\beta-1}) u_i(y + y_i)^{-2n/(n-2\sigma)} \\
\leq C(\varepsilon) u_i(y_i)^{-2n/(n-2\sigma)} + o(1) u_i(y_i)^{-2\beta/(n-2\sigma)},
\]
where we used (5.16) in the last inequality.

Multiplying the above by $u_i(y_i)^{2\beta/(n-2\sigma)}$, due to $\beta < n$ we obtain
\[
\lim_{i \to \infty} u_i(y_i)^{2\beta/(n-2\sigma)} \left| \int_{B_\varepsilon} \langle y, \nabla Q_i^{(\beta)}(y + y_i) \rangle u_i(y + y_i)^{2n/(n-2\sigma)} \right| = o(1). \tag{5.19}
\]

Let $R_i \to \infty$ as $i \to \infty$. We assume that $r_i := R_i u_i(y_i)^{-\frac{2}{n-2\sigma}} \to 0$ as we did in Proposition 4.1. By Lemma 4.3, we have
\[
u_i(y_i)^{2\beta/(n-2\sigma)} \left| \int_{r_i \leq |y| \leq \varepsilon} \langle y, \nabla Q_i^{(\beta)}(y + y_i) \rangle u_i(y + y_i)^{2n/(n-2\sigma)} \right| \\
\leq \lim_{i \to \infty} u_i(y_i)^{2\beta/(n-2\sigma)} \left| \int_{r_i \leq |y| \leq \varepsilon} (|y|^{\beta} + |y||y_i|^{\beta-1}) u_i(y + y_i)^{2n/(n-2\sigma)} \right| \to 0 \tag{5.20}
\]
as $i \to \infty$. Combining (5.19) and (5.20), we conclude that
\[
\lim_{i \to \infty} u_i(y_i)^{2\beta/(n-2\sigma)} \left| \int_{B_{r_i}} \langle y, \nabla Q_i^{(\beta)}(y + y_i) \rangle u_i(y + y_i)^{2n/(n-2\sigma)} \right| = o(1).
\]

It follows from changing variable $z = u_i(y_i)^{-\frac{2}{n-2\sigma}} y$, applying Proposition 4.1 and then letting $\varepsilon \to 0$ that
\[
\int_{\mathbb{R}^n} \langle z, \nabla Q_i^{(\beta)}(z + z_0) \rangle (1 + k|z|^{2})^{-n} = 0, \tag{5.21}
\]
where $z_0 = \lim_{i \to \infty} u_i(y_i)^{2/(n-2\sigma)} y_i$ and $k = \lim_{i \to \infty} K_i(y_i)^{1/\sigma}$.

On the other hand, from (5.17)
\[
\int_{\mathbb{R}^n} \nabla K_i(y + y_i) u_i(y + y_i)^{2n/(n-2\sigma)} = 0. \tag{5.22}
\]

Arguing as above, we will have
\[
\int_{\mathbb{R}^n} \nabla Q_i^{(\beta)}(z + z_0)(1 + k|z|^{2})^{-n} = 0. \tag{5.23}
\]
It follows from (5.21) and (5.23) that
\[
\int_{\mathbb{R}^n} Q^{(\beta)}(z + z_0)(1 + k |z|^2)^{-n} \, dz = \beta^{-1} \int_{\mathbb{R}^n} \left( z + z_0, \nabla Q^{(\beta)}(z + z_0) \right)(1 + k |z|^2)^{-n} \, dz = 0.
\]
(5.24)

Therefore, (5.13) does not hold for \( y_0 = \sqrt{k} z_0 \).

Theorem 5.4. Let \( \sigma \in (0, 1) \) and \( n \geq 3 \). Suppose that \( K \in C^{1,1}(S^n) \), for some constant \( A_1 > 0 \),
\[
1/A_1 \leq K_\xi(\xi) \leq A_1 \quad \text{for all } \xi \in S^n.
\]
Suppose also that for any critical point \( \xi_0 \) of \( K \), under the stereographic projection coordinate system \( \{ y_1, \ldots, y_n \} \) with \( \xi_0 \) as south pole, there exist some small neighborhood \( \mathcal{O} \) of 0, a positive constant \( L \), and \( \beta = \beta(\xi_0) \in (n - 2\sigma, n) \) such that
\[
\| \nabla [\beta] K \|_{C^{\beta-1}(\mathcal{O})} \leq L
\]
and
\[
K(y) = K(0) + Q^{(\beta)}_{\xi_0}(y) + R_{\xi_0}(y) \quad \text{in } \mathcal{O},
\]
where \( Q^{(\beta)}_{\xi_0}(y) \in C^{[\beta]-1,1}(S^{n-1}) \) satisfies \( Q^{(\beta)}_{\xi_0}(\lambda y) = \lambda^\beta Q^{(\beta)}_{\xi_0}(y), \forall \lambda > 0, y \in \mathbb{R}^n \), and for some positive constant \( A_6 \)
\[
A_6 |y|^{\beta-1} \leq |\nabla Q^{(\beta)}(y)|, \quad y \in \mathcal{O},
\]
and
\[
\left( \int_{\mathbb{R}^n} \nabla Q^{(\beta)}(y + y_0)(1 + |y|^2)^{-n} \, dy \int_{\mathbb{R}^n} Q^{(\beta)}(y + y_0)(1 + |y|^2)^{-n} \, dy \right) \neq 0, \quad \forall y_0 \in \mathbb{R}^n,
\]
and \( R_{\xi_0}(y) \in C^{[\beta]-1,1}(\mathcal{O}) \) satisfies \( \lim_{y \to 0} \sum_{s=0}^{[\beta]} \nabla^s R(\xi_0(y)|y|^{-\beta+s} = 0. \)
Then there exists a positive constant \( C \geq 1 \) depending on \( n, \sigma, K \) such that for any solution \( v \) of
\[
1/C \leq v \leq C, \quad \text{on } S^n.
\]

Proof. It follows directly from Theorem 5.2 and Theorem 5.3.

Proof of the compactness part of Theorem 1.2. It is easy to check that, if \( K \) satisfies the condition in Theorem 1.2 then it must satisfy the condition in the above theorem. Therefore, we have the lower and upper bounds of \( v \). The \( C^2 \) norm bound of \( v \) follows immediately.

A Appendix

A.1 A Kazdan-Warner identity
In this section we are going to show (1.7), which is a consequence of the following
Proposition A.1. Let $K > 0$ be a $C^1$ function on $S^n$, and let $v$ be a positive function in $C^2(S^n)$ satisfying

$$P_\sigma(v) = Kv^{\frac{n+2\sigma}{2n}}, \text{ on } S^n. \quad (A.1)$$

Then, for any conformal Killing vector field $X$ on $S^n$, we have

$$\int_{S^n} (\nabla_X K)v^{\frac{2n}{n+2\sigma}} \, dV_{g_{S^n}} = 0. \quad (A.2)$$

Let $\varphi_t : S^n \to S^n$ be a one parameter family of conformal diffeomorphism (in this case they are Möbius transformations), depending on $t$ smoothly, $|t| < 1$, and $\varphi_0 = \text{identity}$. Then

$$X := \left. \frac{d}{dt}(\varphi_t)^{-1} \right|_{t=0} \text{ is a conformal Killing vector field on } S^n. \quad (A.3)$$

Proof. The proof is standard (see, e.g., [12] for a Kazdan-Warner identity for prescribed scalar curvature problems) and we include it here for completeness. Since $P_\sigma$ is a self-adjoint operator, (A.1) has a variational formulation:

$$I[v] := \frac{1}{2} \int_{S^n} vP_\sigma(v) \, dV_{g_{S^n}} - \frac{n-2\sigma}{2n} \int_{S^n} Kv^{\frac{2n}{n+2\sigma}} \, dV_{g_{S^n}}. \quad (A.1)$$

Let $X$ be a conformal Killing vector field, then there exists $\{\varphi_t\}$ satisfying (A.3). Let

$$v_t := (v \circ \varphi_t)w_t,$$

where $w_t$ is given by

$$g_t := \varphi_t^*g_{S^n} = \frac{4}{x^2}g_{S^n}.$$

Then

$$I[v_t] = \frac{1}{2} \int_{S^n} vP_\sigma(v) \, dV_{g_{S^n}} - \frac{n-2\sigma}{2n} \int_{S^n} K(\varphi_t^{-1}(x))v^{\frac{2n}{n+2\sigma}} \, dV_{g_{S^n}}.$$

It follows from (A.1) that

$$0 = I'[v]\left.\left(\frac{d}{dt}\right)_{t=0}v_t\right) = \left.\frac{d}{dt}I[v_t]\right|_{t=0} = -\frac{n-2\sigma}{2n} \int_{S^n} (\nabla_X K)v^{\frac{2n}{n+2\sigma}} \, dV_{g_{S^n}}. \quad \square$$

A.2 A proof of Lemma 4.4

The classical Böcher theorem in harmonic function theory states that a positive harmonic function $u$ in a punctured ball $B_1 \setminus \{0\}$ must be of the form

$$u(x) = \begin{cases} -a \log |x| + h(x), & n = 2, \\ a|x|^{2-n} + h(x), & n \geq 3, \end{cases}$$

where $a$ is a nonnegative constant and $h$ is a harmonic function in $B_1$.

We are going to establish a similar result, Lemma 4.4 in our setting. Denote $B_R^+ = \{X : |X| < R, t > 0\}$, $\partial^R B_R^+ = \{(x, t) : |x| < R\}$ and $\partial'' B = \partial B_R^+ \setminus \partial^R B_R^+$. 

50
Proof of Lemma 4.4. We adapt the proof of the Bôcher theorem given in [5]. Define

\[ A[U](r) = \frac{\int_{\partial^+ B_r} t^{1-2\sigma} U(x,t) dS_r}{\int_{\partial^+ B_r} t^{1-2\sigma} dS_r} \]

where \( r = |(x,t)| > 0 \) and \( dS_r \) is the volume element of \( \partial^+ B_r \).

By direct computation we have

\[ \frac{d}{dr} A[U](r) = \frac{\int_{\partial^+ B_r} t^{1-2\sigma} \nabla U(x,t) \cdot \frac{x(t)}{r} dS_r}{\int_{\partial^+ B_r} t^{1-2\sigma} dS_r}. \]

Let

\[ f(r) = \int_{\partial^+ B_r} t^{1-2\sigma} \nabla U(x,t) \cdot \frac{x(t)}{r} dS_r. \]

Since \( U \) satisfies (4.18), by integration by parts we have

\[ f(r_1) = f(r_2), \quad \forall \ 0 < r_1, r_2 < 1. \]

Notice that

\[ \int_{\partial^+ B_1} t^{1-2\sigma} dS_r = r^{n+1-2\sigma} \int_{\partial^+ B_1} t^{1-2\sigma} dS_1. \]

Thus there exists a constant \( b \) such that

\[ \frac{d}{dr} A[U](r) = b r^{-n-1+2\sigma}. \]

So there exist constants \( a \) and \( b \) such that

\[ A[U](r) = a + b r^{2\sigma-n}. \]

Since we have the Harnack inequalities for \( U \) as in the proof of Lemma 4.1, the rest of the arguments are rather similar to those in [5] and are omitted here. We refer to [5] for details.

A.3 Two lemmas on maximum principles

Lemma A.1. Let \( Q_1 = B_1 \times (0,1) \subset \mathbb{R}_+^{n+1} \), then there exists \( \varepsilon = \varepsilon(n, \sigma) \) such that for all \( |a(x)| \leq \varepsilon |x|^{-2\sigma} \), if \( U \in H(\mathbb{R}^{1-2\sigma}, Q_1) \), \( U \geq 0 \) on \( \partial^0 Q_1 \), and

\[ \int_{Q_1} t^{1-2\sigma} \nabla U \nabla \varphi \geq \int_{B_1} aU(\cdot,0) \varphi \quad \text{for all } 0 \leq \varphi \in C_c^\infty(Q_1). \]

Then

\[ U \geq 0 \quad \text{in } Q_1. \]

Proof. By a density argument, we can use \( U^- \) as a test function. Hence we have

\[ \int_{Q_1} t^{1-2\sigma} |\nabla U^-|^2 \leq \int_{B_1} |a|((U^-)(\cdot,0))^2. \quad (A.4) \]
We extend $U^-$ to be zero outside of $Q_1$ and still denote it as $U^-$. Then the trace

$$U^-(\cdot, 0) \in \dot{H}^\sigma(\mathbb{R}^n).$$

Since

$$\|U^-(\cdot, 0)\|^2_{H^\sigma(\mathbb{R}^n)} = \int_{\mathbb{R}^{n+1}} t^{1-2\sigma} |\nabla \mathcal{P}_t \ast U^-(\cdot, 0)|^2 \leq \int_{\mathbb{R}^{n+1}} t^{1-2\sigma} |\nabla U^-|^2,$$

we have

$$\|U^-(\cdot, 0)\|^2_{H^\sigma(\mathbb{R}^n)} \leq \int_{B_1} |a|(U^-(\cdot, 0))^2.$$

By Hardy’s inequality (see, e.g., [94])

$$C(n, \sigma) \int_{\mathbb{R}^n} |x|^{-2\sigma}(U^-(\cdot, 0))^2 \leq \|U^-(\cdot, 0)\|^2_{H^\sigma(\mathbb{R}^n)}$$

where $C(n, \sigma) = 2^{2\sigma} \Gamma((n+2\sigma)/4) / (n-2\sigma)/4)$ is the best constant. Hence if $\varepsilon < C(n, \sigma)$, $U^-(\cdot, 0) \equiv 0$ and hence by (A.4), $U^- \equiv 0$ in $Q_1$.

**Lemma A.2.** Let $a(x) \in L^\infty(B_1)$. Let $W \in C(Q_1) \cap C^2(Q_1)$ satisfying $\nabla_x W \in C(Q_1)$, $t^{1-2\sigma} \partial_t W \in C(Q_1)$, and

$$\begin{cases}
-\text{div}(t^{1-2\sigma} \nabla W) & \geq 0 \quad \text{in } Q_1 \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t W(x, t) & \geq a(x)W(x, 0) \quad \text{on } \partial' Q_1 \tag{A.5}
\end{cases}$$

If $U \in C(Q_1) \cap C^2(Q_1)$ satisfying $\nabla_x U \in C(Q_1)$, $t^{1-2\sigma} \partial_t U \in C(Q_1)$, and

$$\begin{cases}
-\text{div}(t^{1-2\sigma} \nabla U) & \geq 0 \quad \text{in } Q_1 \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t U(x, t) & \geq a(x)U(x, 0) \quad \text{on } \partial' Q_1 \tag{A.6}
\end{cases}$$

Then $U \geq 0$ in $Q_1$.

**Proof.** Let $V = U/W$. Then

$$\begin{cases}
-\text{div}(t^{1-2\sigma} \nabla V) - 2t^{1-2\sigma} \nabla V \nabla W - \frac{\text{div}(t^{1-2\sigma} \nabla W)V}{W} & \geq 0 \quad \text{in } Q_1 \\
-\lim_{t \to 0} t^{1-2\sigma} \partial_t V + \frac{V}{W} \left( -\lim_{t \to 0} t^{1-2\sigma} \partial_t W(x, t) - a(x)W(x, 0) \right) & \geq 0 \quad \text{on } \partial' Q_1 \tag{A.7}
\end{cases}$$

We are going to show that $V \geq 0$ in $Q_1$. If not, then we choose $k$ such that $\inf_{Q_1} V < k \leq 0$. Let

$$V_k = V - k \quad \text{and } V_k^- = \max(-V_k, 0).$$

Multiplying $V_k^-$ to (A.7), we have

$$\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \leq 2 \int_{Q_1} t^{1-2\sigma} W^{-1} V_k^- \nabla V_k^- \nabla W. \tag{A.8}$$
\textit{Case 1}: Suppose $1 - 2\sigma \leq 0$. Denote $\Gamma_k = \text{Supp}(\nabla V_k^-)$. Then by the Hölder inequality and the bounds of $\nabla_x W$, $t^{1-2\sigma} \partial_t W$,
\[
2 \int_{Q_1} t^{1-2\sigma} W^{-1} V_k^- \nabla V_k^- \nabla W \leq C \left( \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \right)^{\frac{1}{2}} \left( \int_{\Gamma_k} t^{1-2\sigma} |V_k^-|^2 \right)^{\frac{1}{2}}.
\]
Hence it follows from (A.8) that
\[
\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \leq C \int_{\Gamma_k} t^{1-2\sigma} |V_k^-|^2. \tag{A.9}
\]
Since $V_k^- = 0$ on $\partial^\alpha Q_1$, by Lemma 2.1 in [90],
\[
\left( \int_{Q_1} t^{1-2\sigma} |V_k^-|^2 (n+1)/n \right)^{\frac{n}{n+1}} \leq C \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2. \tag{A.10}
\]
By (A.9), (A.10) and Hölder inequality,
\[
\int_{\Gamma_k} t^{1-2\sigma} \geq C.
\]
This yields a contradiction when $k \to \inf Q_1$, since $\nabla V = 0$ on the set of $V \equiv \inf Q_1, V$.

\textit{Case 2}: Suppose $1 - 2\sigma > 0$. Denote $\Gamma_k = \text{Supp}(V_k^-)$. Then by Hölder inequality and the bounds of $\nabla_x W$, $t^{1-2\sigma} \partial_t W$,
\[
\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \leq 2 \int_{Q_1} t^{1-2\sigma} W^{-1} V_k^- \nabla V_k^- \nabla W \leq C \int_{Q_1} V_k^- \nabla V_k^- \leq C \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \left( \int_{Q_1} t^{2\sigma-1} |V_k^-|^2 \right)^{1/2}.
\]
Hence
\[
\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \leq C \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{Q_1} t^{2\sigma-1} |V_k^-|^2.
\]
Since $V_k^- = 0$ on $\partial^\alpha Q_1$, by the proof of Lemma 2.3 in [90], for any $\beta > -1$,
\[
\int_{Q_1} t^\beta |V_k^-|^2 \leq C(\beta) \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2.
\]
In the following we choose $\beta = \sigma - 1$. Hence,
\[
\int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{Q_1} t^{\sigma-1} |V_k^-|^2 \leq C \int_{Q_1} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{Q_1} t^{2\sigma-1} |V_k^-|^2.
\]
i.e.
\[
\int_{\Gamma_k} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{\Gamma_k} t^{\sigma-1} |V_k^-|^2 \leq C \int_{\Gamma_k} t^{1-2\sigma} |\nabla V_k^-|^2 \int_{\Gamma_k} t^{2\sigma-1} |V_k^-|^2.
\]
Fixed $\varepsilon > 0$ sufficiently small which will be chosen later. By the strong maximum principle $\inf_{Q_1} V$ has to be attained only on $\partial' Q_1$, then we can choose $k$ sufficiently closed to $\inf_{Q_1} V$ such that $\Gamma_k \subset B_1 \times [0, \varepsilon]$. Then

$$\varepsilon^{-\sigma} \int_{\Gamma_k} t^{2\sigma - 1} |V_k|^2 \leq C \int_{\Gamma_k} t^{\sigma - 1} |V_k|^2.$$  

Choose $\varepsilon$ small enough such that $\varepsilon^{-\sigma} > C + 1$. It follows that

$$\int_{\Gamma_k} t^{1-2\sigma} |\nabla V|^2 \leq \int_{\Gamma_k} t^{2\sigma - 1} |V_k|^2 = 0.$$  

Hence one of them has to be zero, which reaches a contradiction immediately. \hfill \Box

### A.4 Complementarities

**Lemma A.3.** Let $u(x) \in C^\infty_c(\mathbb{R}^n)$ and $V(\cdot, t) = \mathcal{P}_\sigma(\cdot, t) * u(\cdot)$. For any $U \in C^\infty_c(\mathbb{R}^{n+1}_+ \cup \partial \mathbb{R}^{n+1}_+)$ with $U(x, 0) = u(x)$,

$$\int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla V|^2 \leq \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} |\nabla U|^2.$$  

**Proof.** Let $0 \leq \eta(x, t) \leq 1$, $\text{Supp}(\eta) \subset B_{2R}^+$, $\eta = 1$ in $B_R^+$ and $|\nabla \eta| \leq 2/R$. In the end we will let $R \to \infty$ and hence we may assume that $U$ is supported in $B_{R/2}^+$. Since $\text{div}(t^{1-2\sigma} \nabla V) = 0$, then

$$0 = \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} \nabla V \nabla (\eta(U - V))$$

$$= \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} \eta \nabla U \nabla V - \int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} \eta \nabla V |^2 - \int_{B_{2R}^+ \setminus B_R^+} t^{1-2\sigma} V \nabla \eta \nabla V$$  

where we used $\eta(U - V) = 0$ on the boundary of $B_{2R}^+$ in the first equality.

Note that for $(x, t) \in B_{2R}^+ \setminus B_R^+$

$$|V(x, t)| = \beta(n, \sigma) \left| \int_{\mathbb{R}^n} \frac{t^{2\sigma}}{(|x - \xi|^2 + t^2)^{(n+2)\sigma/2}} u(\xi) \, d\xi \right|$$

$$\leq \beta(n, \sigma) \int_{\mathbb{R}^n} \frac{(|x|^2 + t^2)^{\sigma}}{(|x|^2/4 + t^2)^{(n+2)\sigma/2}} |u(\xi)| \, d\xi$$

$$\leq C(n, \sigma) (|x|^2 + t^2)^{-\frac{\sigma}{2}} \|u\|_{L^1}$$

where in the first inequality we have used that $U$ is supported in $B_{R/2}^+$.  

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54
Direct computations yield

\[
\left| \int_{B_{2R}^+ \setminus B_R^+} t^{1-2\sigma} V \nabla \eta \nabla V \right| \\
\leq \left( \int_{B_{2R}^+ \setminus B_R^+} t^{1-2\sigma} |\nabla V|^2 \right)^{1/2} \left( \int_{B_{2R}^+ \setminus B_R^+} t^{1-2\sigma} V^2 |\nabla \eta|^2 \right)^{1/2} \\
\leq \left( \int_{B_{2R}^+ \setminus B_R^+} t^{1-2\sigma} |\nabla V|^2 \right)^{1/2} \cdot C(n, \sigma) \|u\|_{L^1(\mathbb{R}^n)} (R^{n+2-2\sigma}-2-2n)^{1/2} \to 0 \text{ as } R \to \infty
\]

where we used (2.4) that \( \int_{R^{n+1}} t^{1-2\sigma} |\nabla V|^2 < \infty \). Therefore, we have

\[
\int_{R^{n+1}} t^{1-2\sigma} |\nabla V|^2 \leq \left| \int_{R^{n+1}} t^{1-2\sigma} \nabla U \nabla V \right|.
\]

Finally, by Hölder inequality,

\[
\int_{R^{n+1}} t^{1-2\sigma} |\nabla V|^2 \leq \int_{R^{n+1}} t^{1-2\sigma} |\nabla U|^2.
\]

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