HEEGAARD FLOER INVARIANTS AND TIGHT CONTACT THREE–MANIFOLDS

PAOLO LISCA AND ANDRÁS I. STIPSICZ

Abstract. Let \( Y_r \) be the closed, oriented three–manifold obtained by performing rational \( r \)–surgery on the right–handed trefoil knot in the three–sphere. Using contact surgery and the Heegaard Floer contact invariants we construct positive, tight contact structures on \( Y_r \) for every \( r \neq 1 \). This implies, in particular, that the oriented boundaries of the positive \( E_6 \) and \( E_7 \) plumbings carry positive, tight contact structures, solving a well–known open problem.

1. Introduction

One of the central problems in contact topology is the construction of positive, tight contact structures on closed, oriented three–manifolds. Using contact surgery \([2, 3]\) one can easily find contact structures on various three–manifolds, but until recently it has been a hard problem to check that such structures are tight. The purpose of this paper is to show that the newly introduced Heegaard Floer contact invariants \([14]\) provide an appropriate tool to attack this problem. In fact, under favourable circumstances, a partial determination of the Heegaard Floer contact invariants suffices to prove that the structure under examination is tight. Here we present a prototype of the results obtainable along these lines. A more thorough investigation will appear later \([9]\).

Theorem 1. Let \( r \in \mathbb{Q} \cup \{\infty\} \), and denote by \( Y_r \) the closed, oriented three–manifold obtained by performing \( r \)–surgery on the right–handed trefoil knot \( K \subset S^3 \). If \( r \neq 1 \), then \( Y_r \) carries a positive, tight contact structure.

Remarks. (a) Theorem 1 solves, in particular, the well–known open problem asking whether the boundaries of the positive definite \( E_6 \) and \( E_7 \) plumbings (respectively \( Y_3 \) and \( Y_2 \) in the above notation) carry positive, tight contact structures \([6]\). Note that the contact structures on \( Y_2 \) and \( Y_3 \) given by Theorem 1 are not symplectically semi–fillable because the three–manifolds \( Y_2 \) and \( Y_3 \) do not carry symplectically semi–fillable contact structures \([5]\).

(b) The three–manifold \( Y_1 \) is the oriented boundary of the positive definite \( E_8 \) plumbing. It is known that \( Y_1 \) does not carry positive, tight contact structures \([4]\).

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In proving Theorem 1 we shall use contact surgery to define contact structures on \( Y_r \) \((r \neq 1)\) and then show that the Heegaard Floer contact invariants of those structures do not vanish, implying tightness. During the course of the proof we show that the contact invariants are nontrivial for infinitely many tight, not semi-fillable contact three-manifolds.

2. The proof of Theorem 1

Consider the contact structures defined by Figure 1(a). The meaning of such a diagram is the following: perform \((-1)\)- and \(r'\)-surgeries on the Legendrian knots of the diagram with respect to the contact framings. Extend the standard contact structure of \( S^3 \) to the new solid tori by structures which are tight on these tori. Such an extension exists if and only if \( r' \neq 0 \) and is unique if and only if \( r' = \frac{1}{k} \) for some integer \( k \in \mathbb{Z} \). In general, the number of different extensions depends on a continued fraction expansion of \( r' \) \([2, 3] \). As illustrated in Figure 1(b), this procedure defines contact structures on \( Y_r \) for \( r = \frac{1}{1-r'} \) with \( r' \neq 0 \), i.e., for all \( r \neq 1 \).

According to [3], Proposition 3, a contact \( r' \)-surgery with \( r' < 0 \) or \( r' = \infty \) can be replaced by a sequence of Legendrian (i.e., contact \((-1)\)-) surgeries, providing a Stein fillable and hence tight contact three-manifold. If \( r' > 0 \), \( r' \neq \infty \), then by [3], Proposition 7, the contact structures of Figure 1(a) can be equivalently given by the diagram of Figure 2 for any positive integer \( k \) and \( r'' = \frac{r'}{1-kr'} \).

In a series of papers [10, 11, 12, 14] Ozsváth and Szabó introduced a family of invariants associating a homology group \( \widehat{HF}(M,s) \) to a Spin\(^c \) three-manifold \((M,s)\), a homomorphism

\[
F_{W,t} : \widehat{HF}(M_1,s_1) \rightarrow \widehat{HF}(M_2,s_2)
\]
to a Spin\(^c\) cobordism \((W, t)\) and an element
\[
c(M, \xi) \in \widehat{HF}(-M, s_\xi)/\langle \pm 1 \rangle
\]
to a contact three–manifold \((M, \xi)\), where \(s_\xi\) denotes the Spin\(^c\) structure induced by the contact structure \(\xi\). In this note we shall always use this homology theory with \(\mathbb{Z}/2\mathbb{Z}\) coefficients, so that the above sign ambiguity for \(c(M, \xi)\) does not occur.

It is proved in \cite{14} that if \((M, \xi)\) is overtwisted then \(c(M, \xi) = 0\), and if \((M, \xi)\) is Stein fillable then \(c(M, \xi) \neq 0\). The following lemma is implicitly contained in Theorem 4.2 of \cite{14}.

**Lemma 2.** Suppose that \((M_2, \xi_2)\) is obtained from \((M_1, \xi_1)\) by a contact \((+1)\)–surgery. Then we have
\[
F_{-X}(c(M_1, \xi_1)) = c(M_2, \xi_2),
\]
where \(-X\) is the cobordism induced by the surgery with reversed orientation and \(F_{-X}\) is the sum of \(F_{-X,s}\) over all Spin\(^c\) structures \(s\) extending the Spin\(^c\) structures on \(-M_i\) by \(\xi_i\), \(i = 1, 2\). In particular, if \(c(M_2, \xi_2) \neq 0\) then \((M_1, \xi_1)\) is tight.

**Proof.** Let us assume that we are performing contact \((+1)\)–surgery along the Legendrian knot \(\gamma \subset (M_1, \xi_1)\). It follows from \cite{14} and \cite{3} that there is an open book decomposition \((F, \phi)\) on \(M_1\) compatible with \(\xi_1\) in the sense of Giroux and such that \(\gamma\) lies on a page and is not homotopic to the boundary. Then, an open book for \((M_2, \xi_2)\) is given by \((F, \phi')\), where \(\phi' = \phi \circ R_\gamma^{-1}\) and \(R_\gamma\) is the right–handed Dehn twist along \(\gamma\). The first part of the statement now follows applying Theorem 4.2 of \cite{14}. The second part of the statement follows immediately from the fact that the invariant of an overtwisted contact structure vanishes. \(\square\)

We will now describe an extremely effective computational tool for Heegaard Floer homology, the **surgery exact triangle**. Let \(M\) be a closed, oriented three–manifold and let \(K \subset M\) be a framed knot. For \(n \in \mathbb{Z}\), let \(M_n\) denote the three–manifold
given by $n$–surgery along $K \subset M$ with respect to the given framing, and call the resulting cobordism $X_n$. For a given Spin$^c$ structure $s$ on $M \setminus K$ let
\[
\widehat{HF}(M, [s]) = \bigoplus_{t \in \mathcal{S}} \widehat{HF}(M_n, t),
\]
where $\mathcal{S} = \{t \in \text{Spin}^c(M_n) \mid t|_{M_n \setminus K} = s\}$, and define
\[
\widehat{HF}(M_n) = \bigoplus_{t \in \text{Spin}^c(M_n)} \widehat{HF}(M_n, t).
\]
The cobordism $X_n$ induces a homomorphism
\[
F_{X_n} : \widehat{HF}(M, [s]) \to \widehat{HF}(M_n, [s])
\]
obtained by summing over Spin$^c$ structures.

**Theorem 3** ([11], Theorem 9.16). The homomorphism $F_{X_n}$ fits into an exact triangle:

\[
\begin{array}{ccc}
\widehat{HF}(M, [s]) & \xrightarrow{F_{X_n}} & \widehat{HF}(M_n, [s]) \\
& & \downarrow \\
& & \widehat{HF}(M_{n+1}, [s])
\end{array}
\]

**Remark.** Since the Spin$^c$ structures on $M_n$ are partitioned according to their restrictions to $M_n \setminus K$, by taking the direct sum of the exact triangles given by Theorem 3 one obtains an analogous exact triangle involving the groups $\widehat{HF}(M)$, $\widehat{HF}(M_n)$ and $\widehat{HF}(M_{n+1})$.

**Lemma 4.** Let $K \subset (S^3, \xi_{st})$ be the Legendrian unknot with Thurston–Bennequin invariant equal to $-1$ and vanishing rotation number. Then, the contact three–manifold $(S^1 \times S^2, \eta)$ obtained by $(+1)$–surgery along $K$ has nonvanishing Heegaard Floer invariant.

**Proof.** In view of Lemma 2, Theorem 3 and the remark following the latter, we have an exact triangle:

\[
\begin{array}{ccc}
\widehat{HF}(S^3) & \xrightarrow{F_{-X}} & \widehat{HF}(S^1 \times S^2) \\
& & \downarrow \\
& & \widehat{HF}(S^3)
\end{array}
\]

where $-X$ is the cobordism from $S^3$ to $S^1 \times S^2$ obtained by attaching a two–handle to $S^3$ along a zero–framed unknot, and

\[
F_{-X}(c(S^3, \xi_{st})) = c(S^2 \times S^1, \eta).
\]

By [11], $\widehat{HF}(S^1 \times S^2)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, while $\widehat{HF}(S^3)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Exactness of the triangle immediately implies that $F_{-X}$ is injective. Since $(S^3, \xi_{st})$ is Stein fillable we have $c(S^3, \xi_{st}) \neq 0$, therefore $c(S^2 \times S^1, \eta) \neq 0$. \qed
Let \((V_k, \xi_k)\) denote the contact three–manifold obtained by choosing \(r' = \frac{1}{k}\) in Figure 1(a), so that \(V_k \cong Y_{\frac{1}{k}}\). Notice that for \(r' = k = 1\) the three–manifold \(V_1 \cong Y_{\infty}\) is diffeomorphic to the three–sphere \(S^3\). According to [3] the contact three–manifold \((V_k, \xi_k)\) can be alternatively defined by the diagram of Figure 3, which contains \(k\) contact \((+1)\)–framed Legendrian unknots.

**Figure 3. Equivalent surgery diagram for \((V_k, \xi_k)\)**

**Lemma 5.** Let \(k \geq 1\) be an integer. Then, \(\dim_{\mathbb{Z}/2\mathbb{Z}} \hat{HF}(-V_k) = k\).

**Proof.** \((V_{k+1}, \xi_{k+1})\) is clearly obtained from \((V_k, \xi_k)\) by performing a contact \((+1)\)–surgery. Therefore, as in the proof of Lemma 4 the cobordism \(X_k\) corresponding to the surgery induces a homomorphism \(F_{-X_k}\) which fits into an exact triangle:

\[
\begin{array}{ccc}
\hat{HF}(-V_k) & \overset{F_{-X_k}}{\longrightarrow} & \hat{HF}(-V_{k+1}) \\
\downarrow & & \downarrow \\
\hat{HF}(-Y_{k+1}) & & \\
\end{array}
\]

A simple computation shows that \(Y_{k+1}\) is the Poincaré sphere \(\Sigma(2, 3, 5)\), and it follows from the calculations of [13], Section 3.2, that \(\hat{HF}(-Y_{k+1}) = \mathbb{Z}/2\mathbb{Z}\). Therefore, setting \(d(k) = \dim_{\mathbb{Z}/2\mathbb{Z}} \hat{HF}(-V_k)\), Triangle \((*)\) implies

\[
d(k + 1) = d(k) \pm 1
\]

for every \(k \geq 1\). Now observe that \(-V_k\) can be presented by the surgery diagram of Figure 4. Consider the three–manifold \(M\) obtained by surgery on the framed link of Figure 4 with the 2–framed knot \(K\) deleted. Applying Theorem 8 to the pair \((M, K)\) for \(n = 1\) yields the exact triangle:

\[
\begin{array}{ccc}
\hat{HF}(L(7k - 9, 7)) & \longrightarrow & \hat{HF}(L(8k - 9, 8)) \\
\downarrow & & \downarrow \\
\hat{HF}(-V_k) & & \\
\end{array}
\]
Figure 4. A surgery diagram for $-V_k$

Since by [11], Proposition 3.1, $\dim_{\mathbb{Z}/2\mathbb{Z}} \widehat{HF}(L(p,q)) = p$ for every $p$ and $q$, exactness of the triangle implies
\begin{equation}
(2) \quad d(k) \geq k
\end{equation}
for every $k \geq 1$, and since $V_1$ is diffeomorphic to $S^3$, we have $d(1) = 1$. Therefore, by (1) and (2) we have $d(k) = k$ for every $k \geq 1$. \hfill \Box

**Lemma 6.** Let $k \geq 1$ be an integer. Then, $c(V_k, \xi_k) \neq 0$.

**Proof.** Consider Figure 3 for $k = 1$, which represents $(V_1, \xi_1)$. By [2], performing a contact (+1)–surgery on a Legendrian pushoff of the Legendrian trefoil is equivalent to erasing the trefoil from the picture, thus resulting in $(S^1 \times S^2, \eta)$. It follows from Lemmas 2 and 4 that $c(V_1, \xi_1) \neq 0$. Since by [13] we know that $\widehat{HF}(-Y_{-1}) = \mathbb{Z}/2\mathbb{Z}$, it follows from Lemma 5 and Triangle ($\ast$) that the homomorphism $F_{-X_k}$ is injective for every $k \geq 1$. Thus,
\begin{align*}
c(V_k, \xi_k) &= F_{-X_{k-1}}(c(V_{k-1}, \xi_{k-1}))
\end{align*}
for every $k \geq 1$. \hfill \Box

**Remark.** As we have already remarked, $(V_2, \xi_2)$ is a tight but not semi–fillable contact three–manifold. Since contact (+1)–surgery on a nonfillable structure produces a nonfillable structure [2, 5], by Lemma 6 the contact three–manifold $(V_k, \xi_k)$ is tight, not symplectically semi–fillable for any $k \geq 2$.

**Proof of Theorem 1.** If $r' < 0$ or $r' = \infty$, any contact surgery given by Figure 1(a) can be realized by a Legendrian surgery, therefore the resulting contact structure is Stein fillable and hence tight. If $r' \neq \infty$ and $r' > 0$, choose an integer $k$ so large that $r'' = \frac{r'}{1-kr'} < 0$. Since contact (+1)–surgery on a Legendrian pushoff cancels Legendrian surgery, Figure 2 shows that one can perform a sequence of contact (+1)–surgeries on any contact structure given by the diagram of Figure 1(a), obtaining $(V_k, \xi_k)$. It follows from Lemma 2 and repeated applications of Lemma 2 that all the contact structures defined by Figure 1(a) are tight. \hfill \Box
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Dipartimento di Matematica, Università di Pisa, I–56127 Pisa, ITALY

E-mail address: lisca@dm.unipi.it

Rényi Institute of Mathematics, Hungarian Academy of Sciences, H-1053 Budapest, Réáltánoda utca 13–15, Hungary

E-mail address: stipsicz@math-inst.hu