SH-wave in a multilayered orthotropic crust under initial stress: A finite difference approach

Shishir Gupta1, Smita1* and Snehamoy Pramanik1

Abstract: The present work examines the finite difference modeling of SH-wave in a multilayered orthotropic crust to perceive the stability criterion and dispersion equation. The dispersion equation, for the case when \((n-1)\) layers underlying the earth or along its surface, is obtained in a standard form. This paper actually assigns to apply the finite difference method in seismic wave propagation. The phase and group velocity are retrieved utilizing the techniques of finite difference method. For the occurrence of SH-wave propagation, stability conditions are constituted applying the technique of finite difference method in time and space. The graphs are produced to comprehend the concepts of dispersion equation, group velocity, and phase velocity.

Keywords: finite difference method; SH-wave; orthotropic; group velocity; phase velocity

AMS subject classifications: 65M10; 78A48

ABOUT THE AUTHOR
Shishir Gupta is a professor (Head of Department, Department of Applied Mathematics) in Indian Institute of Technology (Indian School of Mines), Dhanbad. The author got PhD degree in 1987 from ISM Dhanbad. The author's area of interest is theoretical seismology, numerical methods, and special functions. He has published more than 130 research articles in reputed international journal of mathematical science, geophysics, and engineering science. He is a reviewer of several international journals. He guided many postgraduate and PhD students. Citations of his research contribution can be found in many books, monographs, PhD thesis, and journal articles.

PUBLIC INTEREST STATEMENT
In this paper, we introduced a multilayered orthotropic crust which will assist us to understand and interpret the seismic characteristics during an earthquake. We remodeled a finite difference algorithm for SH-wave propagation in an orthotropic medium under initial stress, considering primarily the effect of source terms, thereby also enabling to calculate phase and group velocities. From the present study, it can be expressed that the initial stress makes extremely important significance on the elastic waves generated by an earthquake explosion or impact. In respect of this appearance, analysis of wave propagation under initially stressed medium is not only relevant but significant also. This paper is not only of interest to mathematicians but also of special interest to civil engineers, seismologist, acoustical engineers, and others who are interested in wave propagation through different layered media. The recommendation of this abstraction is of practical importance and can be advantageously used in interpretation and simulation of data for geophysical studies.
1. Introduction

Seismology is crucially established on the application of principal of mechanics in a continuous medium. Seismologist usually employs the assumption and convention of physics, mathematics, geology, and engineering to analyze the physical characteristic of the earth. The measurement of gravity and magnetic fields, seismic waves, temperature, and natural electric current are being done by them. Geophysicists study this topic from the viewpoint of physics of solid bodies, gases, and fluids. The research area of seismologists is the earth’s interior and its vibration applying mechanics of physics. With these viewpoints, this attempt will assist us to interpret the earth’s interior. The current attempt correlates ongoing effort to figure a mathematical model which is appropriate for explanation of the crust during an earthquake. The concept of this paper represents the current state of art in geological or earthquake research. This analysis can be employed favorably in several crustal deformations during an earthquake. During an earthquake, surface wave transmits the greatest amount of energy and this leads to the reason of devastation. Surface waves in crust of the earth are substantial to seismologists because of possible applications in geographical exploration and in identifying the reason and estimation of destruction due to earthquakes. Initial stress cannot be ignored in propagation of SH-wave in a multilayered orthotropic crust. It is a stress which exists in a structure or mass not subjected to the action of external forces except gravity. The mechanical and thermal properties of an orthotropic material are unique and independent in mutually perpendicular directions. Orthotropy is a property of a point within an object rather than the object as a whole, unless the object is homogeneous. Seismic waves which propagate in interior of the earth are very useful in exploration of minerals, crystal, and metals buried inside. Liu and Sen (2009) successfully applied finite difference approach in seismology. They reviewed the conventional arbitrary order explicit finite difference method and their recent developments including arbitrary even order implicit finite difference methods for standard grids. Ewing and Jardetzky (1957) expressed a lot of discussion about seismic wave. Sevostianov and Kachanov (2008) enlightened on approximate symmetries of the elastic properties and elliptic orthotropy. They discussed a special type of orthotropic typical for a variety of heterogeneous materials elliptic orthotropic in terms of a certain symmetric second rank tensor. Son and Kang (2011) discussed the propagation behavior of SH-waves in layered piezoelectric plates. They have shown that the thickness ratio and the properties of two layers have a significant effect on the propagation of SH-waves. Pal and Mandal (2013) studied the generation of SH-type waves in a sandy layer lying over isotropic inhomogeneous elastic half spaces because of discontinuity of shear stress. Maakela and Ostlund (2003) discussed an orthotropic elastic-plastic model for paper materials. Slawinski and Krebes (2002) analyzed the modeling of SH-wave propagation in nowelded contact media. This poses a difficulty for finite difference modeling of seismic wave propagation in fractured media. Mi, Xia, and Xu (2015) discussed the finite difference modeling of SH-wave conversion in shallow shear refraction surveying. They also implemented the numerical simulation for conversion of SH-wave to P-wave in 3D heterogeneous medium with the finite difference method. Starzewski (2000) studied the random fiber networks and special elastic orthotropy of paper. Kalyani, Pallavika, Chakraborty, Mahanti, and Sinha (2008), successfully applied the finite difference method to model the propagation of SH-waves in different types of layered media expressively as monoclinic media and anisotropic porous media. The result evinces that SH-wave is dispersive in both types of media. Chattopadhyay, Gupta, Chattopadhyay, and Singh (2010) enlightened on the propagation of SH-wave in multilayered magnetoelastic self-reinforced media using finite difference technique. They applied finite difference algorithm to compute stability condition and dispersion equation. Burschil, Bellecke, and Krawezyk (2015) studied the finite difference modeling to evaluate seismic P-wave and shear wave field data.

In the present work, Biot’s theory of orthotropic media has been used to formulate the problem. Haskell’s theory of matrices has been applied to solve this particular problem. The method of finite difference is applied to calculate group and phase velocities of SH-wave propagation in a multilayered medium under initial stress. The stability criteria are established for the finite difference approximation in time and space for existence of SH-wave propagation. The attractive feature of this problem is to replace the differential equations and boundary conditions by simple finite difference
approximations in such a way that an explicit, recursive set of equations is formed, which is helpful in modeling of SH-wave propagation.

2. Formulation of the problem
A multilayered half space consisting of \( n \) orthotropic layers is considered. The stratum of layers is expressed as \( h_i \) for \( i = 1, 2, 3, \ldots, n \). For the plane wave propagation, \( x_1 \)-axis is assumed along the propagation of wave while \( x_3 \)-axis is taken vertically downwards as exhibited in Figure 1. The representation of layers is taken in top-down trend. The uppermost layer is enumerated as 1 and undermost half space is enumerated as \( n \). Initial stress \( T(l) \) along the \( x \) axis is considered in the respective layers. \( R(l)^{1}, R(l)^{2}, \) and \( R(l)^{3} \) symbolize for rigidity, density, and shear modulus of \( l \)th layer, respectively.

3. Solution of the problem
The fundamental equation of motion under initial stress in the absence of body force is (Biot, 1956).

\[
\begin{align*}
\frac{\partial \kappa_{11}}{\partial x_1} + \frac{\partial \kappa_{12}}{\partial x_2} + \frac{\partial \kappa_{13}}{\partial x_3} - T \left( \frac{\partial \Omega_{x_1}}{\partial x_2} - \frac{\partial \Omega_{x_2}}{\partial x_3} \right) &= \rho \frac{\partial^2 u_1}{\partial t^2} \\
\frac{\partial \kappa_{21}}{\partial x_1} + \frac{\partial \kappa_{22}}{\partial x_2} + \frac{\partial \kappa_{23}}{\partial x_3} - T \left( \frac{\partial \Omega_{x_2}}{\partial x_1} - \frac{\partial \Omega_{x_1}}{\partial x_3} \right) &= \rho \frac{\partial^2 u_2}{\partial t^2} \\
\frac{\partial \kappa_{31}}{\partial x_1} + \frac{\partial \kappa_{32}}{\partial x_2} + \frac{\partial \kappa_{33}}{\partial x_3} - T \left( \frac{\partial \Omega_{x_3}}{\partial x_1} - \frac{\partial \Omega_{x_1}}{\partial x_3} \right) &= \rho \frac{\partial^2 u_3}{\partial t^2}
\end{align*}
\]

where \( u_1, u_2, \) and \( u_3 \) are the displacement components. \( \Omega_{x_1}, \Omega_{x_2}, \) and \( \Omega_{x_3} \) are being used to denote the rotational components along \( x_1, x_2, \) and \( x_3 \) direction, respectively. Here \( \kappa_{ij} \) and \( \rho \) denote the incremental stress and density, respectively. Stress and strain components are related in such a manner that
\( \kappa_{11} = C_{11} f_{11} + C_{22} f_{22} + C_{13} f_{33} \)
\( \kappa_{12} = 2R_{2} f_{12} \)
\( \kappa_{22} = C_{21} f_{11} + C_{22} f_{22} + C_{23} f_{33} \)
\( \kappa_{23} = 2R_{3} f_{23} \)
\( \kappa_{33} = C_{31} f_{11} + C_{32} f_{22} + C_{33} f_{33} \)
\( \kappa_{31} = 2R_{3} f_{31} \)  

where \( C_{ij} \) and \( R_{i} \) are used to denote the incremental normal elastic coefficients and shear modulus, respectively. The component of strain \( f_{ij} \) is expressed as

\[
 f_{ij} = \frac{1}{2} \left( \frac{\partial u_{i}}{\partial x_{j}} + \frac{\partial u_{j}}{\partial x_{i}} \right) 
\]  

where \( i, j = 1, 2, 3 \). Applying conventional condition of SH-wave, \( u_{1} = 0, u_{3} = 0, \) and \( u_{2} = u_{2}(x_{1}, x_{3}, t) \) in Equation (1), we obtain

\[
 \frac{\partial}{\partial x_{1}} \left( R_{3} \frac{\partial u_{2}}{\partial x_{1}} \right) + \frac{\partial}{\partial x_{3}} \left( R_{1} \frac{\partial u_{2}}{\partial x_{3}} \right) - \frac{T}{2} \frac{\partial}{\partial x_{1}} \left( \frac{1}{2} \frac{\partial u_{2}}{\partial x_{1}} \right) = \frac{\rho}{2} \frac{\partial^{2} u_{2}}{\partial t^{2}} 
\]  

Equation (4) gives that

\[
 \left( R_{3} - \frac{T}{2} \right) \frac{\partial^{2} u_{2}}{\partial x_{1}^{2}} + R_{1} \frac{\partial^{2} u_{2}}{\partial x_{3}^{2}} = \frac{\rho}{2} \frac{\partial^{2} u_{2}}{\partial t^{2}} 
\]  

The stress components \( \kappa_{12} = 2R_{2} f_{12} \), \( \kappa_{23} = 2R_{3} f_{23} \) and other stress components will be zero. From Equation (5), the following analytical solution can be proposed for the wave propagating along \( x_{1} \)-direction,

\[
 u_{2} = U(z) e^{i(k_{1} x_{1} - \omega t)} 
\]  

where \( k \) and \( c \) appear for the wave number and phase velocity, respectively. Employing Equation (6) in Equation (5), we get

\[
 \frac{d^{2} U}{dz^{2}} + \alpha^{2} U = 0 
\]

where

\[
 \alpha^{2} = \frac{k^{2}}{R_{1}} \left( c^{2} \rho - \left( R_{3} - \frac{T}{2} \right) \right) 
\]

From, Equation (7) the solution for an initially stressed orthotropic medium will be of the form

\[
 u_{2} = \left( M_{1} e^{iux_{3}} + M_{2} e^{-iux_{3}} \right) e^{-\frac{i(\omega t - k_{1} x_{1})}{2}} 
\]

where \( M_{1} \) and \( M_{2} \) are arbitrary constants.

The equation of motion for propagation of \( l \)th layer in multilayered orthotropic media following the Equation (5) is expressed as

\[
 \left( R_{3} - \frac{T}{2} \right) \frac{\partial^{2} u_{2}^{(l)}}{\partial x_{1}^{2}} + R_{1} \frac{\partial^{2} u_{2}^{(l)}}{\partial x_{3}^{2}} = \frac{\rho}{2} \frac{\partial^{2} u_{2}^{(l)}}{\partial t^{2}} 
\]  

Explicitly the solution of Equation (9) using Equation (7) is assumed as
\[ u_2^{(l)} = (M_1^{(l)} e^{i\alpha^{(l)} x} + M_2^{(l)} e^{-i\alpha^{(l)} x}) e^{-i(\omega t - kx)} \]  \hspace{1cm} (10) 

where 

\[ \alpha^{(l)} = \sqrt{\frac{k^2}{R_i^{(l)}} \left( c^2 \rho^{(l)} - \left( R_i^{(l)} - \frac{P^{(l)}}{\mu} \right)^2 \right) } \]

The shear stress at the boundary of \( l \)th layer from Equation (10) is observed that 

\[ q^{(l)} = \mu^{(l)} i \alpha^{(l)} \left( M_1^{(l)} e^{i\alpha^{(l)} z} - M_2^{(l)} e^{-i\alpha^{(l)} z} \right) e^{-i(\omega t - kx)} \]  \hspace{1cm} (11) 

The coordinate system is translated from the free surface layer to \( (l-1) \)th interface along \( x_3 \)-axis, we get 

\[ \frac{1}{c} \frac{\partial u_2^{(l-1)}}{\partial t} = -i\omega \frac{1}{c} (M_1^{(l)} e^{i\alpha^{(l)} x} + M_2^{(l)} e^{-i\alpha^{(l)} x}) e^{-i(\omega t - kx)} \]  \hspace{1cm} (12) 

At \( x_3 = 0 \), Equation (12) becomes 

\[ \frac{1}{c} \frac{\partial u_2^{(l-1)}}{\partial t} = -i\omega \frac{1}{c} (M_1^{(l)} + M_2^{(l)}) e^{-i(\omega t - kx)} \]  \hspace{1cm} (13) 

or 

\[ \left( \frac{u_2}{c} \right)^{(l-1)} = -ik (M_1^{(l)} + M_2^{(l)}) e^{-i(\omega t - kx)} \]  \hspace{1cm} (14) 

\[ q^{(l-1)} = \mu^{(l)} i \alpha^{(l)} \left( M_1^{(l)} - M_2^{(l)} \right) e^{-i(\omega t - kx)} \]  \hspace{1cm} (15) 

At the \( l \)th interface, i.e. at \( x_3 = h_i \), the dimensionless phase velocity and shear stress are expressed as 

\[ \left( \frac{u_2}{c} \right)^{(l)} = -ik \left( M_1^{(l)} e^{i\alpha^{(l)} x} + M_2^{(l)} e^{-i\alpha^{(l)} x} \right) e^{-i(\omega t - kx)} \]  \hspace{1cm} (16) 

\[ \left( \frac{u_2}{c} \right)^{(l)} = -ik \left( \left( M_1^{(l)} + M_2^{(l)} \right) \cos (\alpha^{(l)} h_i) + i \sin \left( \alpha^{(l)} h_i \right) \left( M_1^{(l)} - M_2^{(l)} \right) \right) e^{-i(\omega t - kx)} \]  \hspace{1cm} (17) 

\[ \Rightarrow \left( \frac{u_2}{c} \right)^{(l)} = k \left( -i \left( M_1^{(l)} + M_2^{(l)} \right) \cos (\alpha^{(l)} h_i) + \sin \left( \alpha^{(l)} h_i \right) \left( M_1^{(l)} - M_2^{(l)} \right) \right) e^{-i(\omega t - kx)} \]  \hspace{1cm} (18) 

\[ q^{(l)} = \mu^{(l)} i \alpha^{(l)} \left( i \left( M_1^{(l)} + M_2^{(l)} \right) \sin (\alpha^{(l)} h_i) + \cos \left( \alpha^{(l)} h_i \right) \left( M_1^{(l)} - M_2^{(l)} \right) \right) e^{-i(\omega t - kx)} \]  \hspace{1cm} (19) 

Equations (18) and (19) accord with the dimensionless phase velocity and shear stress at the \( l \)th layer for proposed work, respectively. The matrix is formed from (16), (17), (18), and (19) in the following way 

\[ \left( \frac{u_2}{c} \right)^{(l)} = -\sin \left( \alpha^{(l)} h_i \right) q^{(l-1)} i k^{(l)} \alpha^{(l)}^{-1} + \cos \left( \alpha^{(l)} h_i \right) \left( \frac{u_2}{c} \right)^{(l-1)} \]  \hspace{1cm} (20)
\[ q^{(l)} = \left( -ik^{-1}(\mu^{(l)}a^{(l)}) \left( \frac{\dot{u}_2}{c} \right)^{(l-1)} \sin a^{(l)}h_l + \cos a^{(l)}h_l \right) q^{(l-1)} e^{-i(\omega t - \alpha k)} \]  

\[ \text{(21)} \]

Using Equations (20) and (21), the matrix is formed as follow:

\[ \begin{bmatrix} \frac{\dot{u}_2}{c} \\ q^{(l)} \end{bmatrix} = \begin{bmatrix} \cos a^{(l)}h_l & -ik^{-1}(\mu^{(l)}a^{(l)})^{-1} \sin a^{(l)}h_l \\ -ik^{-1}(\mu^{(l)}a^{(l)}) \sin a^{(l)}h_l & \cos a^{(l)}h_l \end{bmatrix} \begin{bmatrix} \frac{\dot{u}_2}{c} \\ q^{(l-1)} \end{bmatrix} \]

\[ \text{(22)} \]

Let \( g_l \) be the matrix such that

\[ g_l = \begin{bmatrix} \cos a^{(l)}h_l & -ik^{-1}(\mu^{(l)}a^{(l)})^{-1} \sin a^{(l)}h_l \\ -ik^{-1}(\mu^{(l)}a^{(l)}) \sin a^{(l)}h_l & \cos a^{(l)}h_l \end{bmatrix} \]

\[ \text{(23)} \]

Taking Equations (22) and (23) in a way that we get

\[ \begin{bmatrix} \frac{\dot{u}_2}{c} \\ q^{(l)} \end{bmatrix} = g_l \begin{bmatrix} \frac{\dot{u}_2}{c} \\ q^{(l-1)} \end{bmatrix} \]

\[ \text{(24)} \]

Similarly, for \((l-1)\)th layer, it can be recognized that

\[ \begin{bmatrix} \frac{\dot{u}_2}{c} \\ q^{(l-1)} \end{bmatrix} = g_{l-1} \begin{bmatrix} \frac{\dot{u}_2}{c} \\ q^{(l-2)} \end{bmatrix} \]

\[ \text{(25)} \]

which is obtained by replacing \( l \) to \( l-1 \). Now from Equations (24) and (25), we find

\[ \begin{bmatrix} \frac{\dot{u}_2}{c} \\ q^{(l-1)} \end{bmatrix} = g_l g_{l-1} \begin{bmatrix} \frac{\dot{u}_2}{c} \\ q^{(l-2)} \end{bmatrix} \]

\[ \text{(26)} \]

Repetition of this procedure will give

\[ \begin{bmatrix} \frac{\dot{u}_2}{c} \\ q^{(l-1)} \end{bmatrix} = g_l g_{l-1} \cdots g_1 \begin{bmatrix} \frac{\dot{u}_2}{c} \\ q^{(0)} \end{bmatrix} \]

\[ \text{(27)} \]

It is assumed that the matrix \( G = g_l g_{l-1} \cdots g_1 \). Let us suppose that

\[ G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \]

\[ \text{(28)} \]

Applying the matrix (27) and (28), it can be rewritten as follow

\[ \begin{bmatrix} \frac{\dot{u}_2}{c} \end{bmatrix}^{(l-1)} = G_{11} \begin{bmatrix} \frac{\dot{u}_2}{c} \end{bmatrix}^{(0)} + G_{12} q^{(0)} \]

\[ \text{(29)} \]

\[ q^{(l-1)} = G_{21} \begin{bmatrix} \frac{\dot{u}_2}{c} \end{bmatrix}^{(0)} + G_{22} q^{(0)} \]

\[ \text{(30)} \]

Using Equations (14), (15), (29), and (30), we get
\[ q^{(l-1)} = G_{21} \left( \frac{\mu_{2}}{c} \right) \]  

Let \( q^{0} = 0 \) and \( M^{(l)}_{3} = 0 \) in Equation (31), we get

\[ G_{21} = k^{-1} \mu^{(l)} a^{(l)} G_{s1} \]  

Equation (32) is the dispersion equation for SH-wave. When \( l = 2 \), we have,

\[ \tan \left( a^{(1)} h_{1} \right) = \frac{a^{(2)} \mu^{(2)}}{a^{(1)} \mu^{(1)}} \]  

where

\[ (a^{(1)})^2 = \frac{k^2}{R_{1}^{2}} \left( -\left( R_{3}^{(1)} - \frac{T^{(1)}}{2} \right) + c^2 \rho^{(1)} \right) \]

and

\[ (a^{(2)})^2 = \frac{k^2}{R_{1}^{2}} \left( -c^2 \rho^{(2)} + \left( R_{3}^{(2)} - \frac{T^{(2)}}{2} \right) \right) \]

which exhibits the dispersion equation of SH-wave. Dispersion equation as obtained from Equation (33) agrees with the dispersion equation as described by Chattopadhyay et al. (2010). When \( l = 3 \), we get

\[ \mu^{(2)} a^{(2)} \tan \left( a^{(2)} h_{2} \right) + \mu^{(1)} a^{(1)} \tan \left( a^{(1)} h_{1} \right) = \mu^{(3)} a^{(3)} \left( 1 - \left( \mu^{(2)} a^{(2)} \right)^{-1} \mu^{(1)} a^{(1)} \tan \left( a^{(2)} h_{2} \right) \tan \left( a^{(1)} h_{1} \right) \right) \]  

Equation (34) represents the dispersion equation for two layers lying over a half space.

4. Special cases

Case 1: When \( T^{(l)} = 0 \), that is initial stress is absent in \( l \)th layer, then we have

\[ \tan \left( \sqrt{\frac{c^2 \rho - R_{3}^{(1)}}{R_{1}^{1}} kH} \right) = \frac{\mu^{(2)}}{\mu^{(1)}} \sqrt{\frac{R_{2}^{(2)} - c^2 \rho^{(2)}}{R_{1}^{1}}} \]  

This represents (36) the dispersion equation for a multilayered orthotropic media in the absence of initial stress.

Case 2: When \( T^{(l)} = 0 \) and \( R_{1}^{(l)} \rightarrow R_{3}^{(l)} \rightarrow \mu^{(l)} \) for \( l = 1, 2 \) then we have

\[ \tan \left( \sqrt{\frac{c^2}{\beta_{1}^{2} - 1}} \right) kH = \frac{\mu^{(2)}}{\mu^{(1)}} \left( 1 - \frac{c^2}{\beta_{2}^{2}} \right) \]  

where \( \beta_{1}^{2} = \frac{R_{1}^{(1)}}{\rho} \) and \( \beta_{2}^{2} = \frac{R_{2}^{(2)}}{\rho \beta}. \)

The above equation represents a classical Love type wave when an orthotropic layer of finite thickness lying over an orthotropic half space. This equation agrees with the equation described by Kundu, Gupta and Manna (2014).
5. Stability criterion, group and phase velocity

A finite rectangular region of an orthotropic medium is considered. The \( x_1 x_3 \)-plane is discretized by representing a grid with uniform increments of \( \Delta x_1 \) and \( \Delta x_3 \) along the \( x_1 \) and \( x_3 \) axis, respectively. The time axis is introduced by the step length \( \Delta t \). Thus space–time grids are taken as

\[
\begin{align*}
x_{1n} &= l \Delta x_1 \\
x_{3m} &= m \Delta x_3 \\
t_w &= w \Delta t
\end{align*}
\]

Using the finite difference method in Equation (5), it is approximated as

\[
\left( R_3 - \frac{T}{2} \right) \frac{u_{w}^{(n)} - 2u_{w}^{(n-1)} + u_{w}^{(n-2)}}{(\Delta x_1)^2} R_1 \frac{u_{w}^{(n)} - 2u_{w}^{(n-1)} + u_{w}^{(n-2)}}{(\Delta x_3)^2} = \frac{\rho}{(\Delta t)^2} \frac{u_{w}^{(n+1)} - 2u_{w}^{(n)} + u_{w}^{(n-1)}}{l} (37)
\]

To find the stability condition of this present work, we consider initial errors in \( u_2 \) at \( t = 0 \). Assuming a disturbance errors at \( (l, m, w) \) of the form as given by Equation (37), we get

\[
E(u_{w}^{(n)}) = A \exp \left( -i\omega (\omega \Delta t) + ik(l \Delta x_1) + ik(m \Delta x_3) \right)
\]

(38)

For fixed \( k \), we shall find the dispersion relation \( \omega(k) \) which will describe the time dependence of the error. By technique of finite difference method we know that the error equation satisfies the solution, therefore substituting this error Equation (38) in solution (37) we get

\[
\sin^2(\omega \Delta t) = \frac{(\Delta t)^2}{(\Delta x_1)^2} \left( R_1 - \frac{T}{2} \right) \sin^2 k \Delta x_1 + \frac{(\Delta t)^2}{(\Delta x_3)^2} R_1 \sin^2 k \Delta x_3
\]

(39)

\[
\sin^2(\omega \Delta t) = \left( \frac{\Delta t}{\Delta x_1} \right)^2 \left( R_1 - \frac{T}{2} \right) \sin^2 k \Delta x_1 + \left( \frac{\Delta t}{\Delta x_3} \right)^2 R_1 \sin^2 k \Delta x_3
\]

(40)

Let us suppose that \( d = \frac{\Delta x_1}{\Delta x_3} \), then it gives

\[
\sin^2(\omega \Delta t) = \left( \frac{\Delta t}{\Delta x_1} \right)^2 \left( R_1 - \frac{T}{2} \right) \sin^2 k \Delta x_1 + \left( \frac{\Delta t}{\Delta x_1} \right)^2 R_1 \sin^2 k d \Delta x_1
\]

(41)

If we approximate \( \sin \delta = \delta \) and \( \cos \delta = 1 - \frac{\delta^2}{2} \) up to second order, then from Equation (41), we have

\[
c_0 = \frac{\omega}{k} = \left( \frac{R_1 + R_3 - T}{2} \right) \frac{1}{\rho}
\]

(42)

Let us assume that \( \lambda = \frac{2\pi}{k} \) and \( \rho = c_0 \frac{\Delta t}{\Delta x_1} \), where \( \rho \) is known as a Courant number, then we have

\[
c = \frac{1}{2\rho \pi} \frac{\lambda}{\Delta x_1} \sin^{-1} \left( \frac{\rho}{R_1 + R_3 - T} \left[ \left( \frac{R_1 - T}{\rho} \right) \sin^2 \left( \frac{2\pi \Delta x_1}{\lambda} \right) + \frac{1}{\rho} \frac{R_1}{\rho} \sin^2 \left( \frac{2\pi \Delta x_1}{\lambda} \right) \right] \right)
\]

(43)

where \( \frac{c}{c_0} \) denotes the phase velocity. Now we determine the group velocity as follow
\[
\frac{\omega}{c_0} = \frac{\rho}{2p} \sqrt{\frac{2}{3\pi} \sin \left[ 2\pi \frac{\Delta x}{\lambda} \right] \cos \left[ 2\pi \frac{\Delta x}{\lambda} \right] + \frac{2\pi}{\Delta \dot{\lambda}} \sin \left[ \frac{2\pi \Delta x}{\lambda} \right] \cos \left[ \frac{2\pi \Delta x}{\lambda} \right]} \]

Equation (44) represents the required phase velocity.

Now we consider the case when \( \Delta x_1 = \Delta x_3 \) for a small value. For \( \Delta x_1 = \Delta x_3 \), Equation (39) becomes

\[
\sin (\omega \Delta t) = \frac{\Delta t}{\Delta x_1} \sin \left( k \Delta x_1 \right) \left( \frac{R_1 + R_3 - \frac{1}{2}}{\rho} \right) \]

For the stability of finite difference \( |\sin (\omega \Delta t)| \) must be surely at most 1. As a consequence, \( \omega \) will be real and error will not grow with the time factor. Therefore for stable condition, we must have

\[
\frac{\Delta t}{\Delta x_1} \sin \left( k \Delta x_1 \right) \left( \frac{R_1 + R_3 - \frac{1}{2}}{\rho} \right) \leq 1
\]

The condition of stability provides relation between frequency and wave number which is satisfied by the solution. The phase and group velocities solved by Equations (43) and (44) approach the true local values if \( \Delta t, \Delta x_1, \) and \( \Delta x_2 \) are very small. For small values of \( \Delta t, \Delta x_1, \) and \( \Delta x_2 \), one can approximate \( \sin \delta = \delta \). Applying this approximation, we have the local phase velocity as follow

\[
\frac{\omega}{\delta} = \left( \frac{R_1 + R_3 - \frac{1}{2}}{\rho} \right)
\]

The stability condition (46) requires that \( \rho \leq 1 \)

\[
\rho = c_0 \frac{\Delta t}{\Delta x_1}
\]

where \( \rho \) is known as the Courant number.

Let \( c_0 \) be the local phase velocity, then

\[
c_0 = \frac{\omega}{\delta} = \left( \frac{R_1 + R_3 - \frac{1}{2}}{\rho} \right)
\]

Let us suppose that \( \lambda = \frac{\Delta x}{\rho} \) and \( \omega = kc_1 \), the phase velocity is expressed as

\[
\frac{c}{c_0} = \frac{\lambda}{2\Delta x_1 \rho \pi} \sin^{-1} \left( \frac{2\Delta x_1 \pi}{\lambda} \right)
\]

This is the required phase velocity.

Now we find the group velocity, which is determined as follow

\[
\frac{\omega}{\delta} = \frac{\cos 2\pi \Delta x_1 \rho}{\left( 1 - \left( \rho \sin \left( \frac{2\pi \Delta x_1 \rho}{\lambda} \right) \right)^2 \right)}
\]
Thus Equations (50) and (51) give the required phase velocity and group velocity, respectively, for the case when \( \Delta x_1 = \Delta x_3 \). It is important to understand how small \( \Delta t, \Delta x_1, \) and \( \Delta x_3 \) likely to be so that the above relation is approximately correct. The limit depends on wave length \( \lambda = \frac{2\pi}{k} \). The Equations (50) and (51) agree somehow with the stability condition as described by Chattopadhyay et al. (2010).

6. Graphical observation
For the graphical observation in a multilayered orthotropic media, we take data from Gubbins (1990), \( R_1 = 5.82 \times 10^{10} \text{N/m}^2, R_3 = 3.99 \times 10^{10} \text{N/m}^2, \rho = 4,500 \text{ Kg/m}^3 \). Figures 2 and 3 are plotted to show the effect of initial stress in dispersion equation.

Figure 2 is sketched in respect of dimensionless wave number against phase velocity for different values of initial stress in the layer. Phase velocity decreases with an increment in \( \frac{T(1)}{2R_1} \). Also, it is observed that phase velocity decreases with an increment in the dimensionless wave number. Finally,

\[ \frac{T(1)}{2R_1} \]

\[ \frac{T(3)}{2R_3} \]

\[ \frac{T(1)}{2R_1} \]

\[ \frac{T(3)}{2R_3} \]

Figure 3 is sketched in respect of dimensionless wave number against phase velocity for different values of initial stress in the layer.
it can be concluded that curves are monotonically decreasing with respect to phase velocity and wave number and monotonically decreasing with respect to phase velocity and initial stress.

Figure 3 renders for the effect of initial stress on phase velocity. The initial stress in half space roles just opposite to the initial stress of layer, that is phase velocity increases with an increment in the initial stress. The phase velocity decreases when wave number increases. So, we can finish that the curves in Figure 3 are monotonically increasing with respect to phase velocity and initial stress but monotonically decreasing in respect of phase velocity and wave number.

Figures 4-6 are sketched to see the consequences of various parameters with respect of group velocity and dispersion parameter. Figure 4 demonstrate the effect of initial stress on group velocity
with respect to dispersion parameter. When the initial stress increases group velocity also increases. After reaching the maximum value, group velocity decreases with an increment in dispersion parameter.

Figure 5 signifies the importance of Courant number. The group velocity significantly depends on Courant number. Group velocity increases with an increment in Courant number. Group velocity first increases with an increment in dispersion parameter then decreases after reaching the maximum value of group velocity.

Figure 6 indicates the effect of discretization region $\Delta x_3$ on group velocity. It is noticed that group velocity increases with an increment in the $\Delta x_3$. Also, it examines that, group velocity increases when dispersion parameter increases and after reaching a maximum, group velocity decreases uncertainly.

Figure 6. Variation of $\frac{\Delta x_3}{\Delta x_1}$ with respect to group velocity and dispersion parameter.

Figure 7. Variation of $\frac{\Delta x_3}{\Delta x_1}$ with respect to phase velocity and dispersion parameter.
Figures 7–9 are plotted for phase velocity and dispersion parameter. Figure 7 is designed for Variation in discretization region. The phase velocity decreases when $\frac{\Delta x}{\Delta x_1}$ increases. It is examined that phase velocity decreases with an increment in the dispersion parameter.

Figure 8 demonstrates the effect of Courant number on phase velocity. It is observed that the increment of Courant number increases phase velocity and increment of dispersion parameter decreases the phase velocity.

Figure 9 is designed to show the effect of initial stress on phase velocity. Phase velocity increases when initial stress increases. The increment in dispersion parameter decreases the phase velocity. Finally, the following outcomes can be described throughout this analysis:
(i) Dispersion parameter effects phase velocity as well as group velocity substantially. More precisely, phase velocity decreases with an increment in the wave number and the curves of group velocity are oscillatory in nature. For some particular value, first it increases and then decreases with an increment in wave number.

(ii) $\frac{\Delta x}{\Delta t}$, initial stress and Courant number have a considerable effect on phase velocity as well as group velocity.

(iii) Following validity condition is in well agreement to one study of propagation of SH-wave in given geometry $\beta_1 < \beta_2 < \beta_3 < \ldots < \beta_n-1$ or $\beta_n-1 < \beta_n-2 < \beta_n-3 < \ldots < \beta_1$.

(iv) The accuracy of finite difference approximation increases when the value of Courant number is higher and consequently the phase velocity will increase.

7. Conclusion
A finite difference approximation for dispersion equation of SH-wave in an initially stressed orthotropic medium has been developed mathematically. The compact form of group velocity, phase velocity, and dispersion equation has been constituted for $n$ orthotropic layer lying beneath the earth using finite difference technique. This method facilitates the quick modeling of SH-wave. For graphical observation, “MATHEMATICA” software has been used to generate numerous consequent results. The study has examined that the phase and group velocity are prestiged by initial stress, Courant number, and region of discretization. The study of some aspects of propagation of SH-wave in a multilayered orthotropic media under initial stress with a special reference to seismology has been incorporated into this paper. Hence it furnishes precious information for assortment of some particular structural material that exists in construction assignment which is not only related to seismologist but also to geophysicist, mathematicians including civil engineers.

Funding
The authors received no direct funding for this research.

Author details
Shishir Gupta1
E-mail: shishir_ism@yahoo.com
Smita1
E-mails: smitaism77@gmail.com, smita7906@gmail.com
Snehamoy Pramanik1
E-mail: snehamoy.pramanik@gmail.com
1 Department of Applied Mathematics, Indian Institute of Technology (Indian School of Mines), Dhanbad 826004, India.

Citation information
Cite this article as: SH-wave in a multilayered orthotropic crust under initial stress: A finite difference approach, Shishir Gupta, Smita & Snehamoy Pramanik, Cogent Mathematics (2017), 4: 1284294.

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