Renormalisation Group Equations in BRST-Restored Dimensional Regularisation: Application to Two-Loop Chiral-QED

Hermès Bélusca-Maïto

1Department of Physics, University of Zagreb, Bijenička cesta 32, HR-10000 Zagreb, Croatia

Abstract

We discuss how the renormalisation group equations can be consistently formulated using the algebraic renormalisation framework, in the context of a dimensionally-renormalised chiral field theory in the BMHV scheme, where the BRST symmetry, originally broken at the quantum level, is restored via finite counterterms. We point how it differs from the more classical approach used in multiplicative renormalisation, whose application would be more cumbersome in this setting. As an illustration, the procedure is applied on the example of a chiral right-handed QED model, and beta-function and anomalous dimensions are evaluated up to two-loop orders.

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*hbelsca@phy.hr
1 Introduction

The formalism and properties of the Renormalisation Group (RG) Equations (RGEs) for dimensionally-
regularised theories have been established by ’t Hooft in [1], in the language of multiplicative/additive
renormalisation, that is most suitable for vector-like (non-chiral) theories. They are now part of stan-
dard textbook methods, and have been, amongst other cases, applied to a generic chiral gauge theory
at two-loop order by Machacek and Vaughn [2, 3, 4].

However, it is worth mentioning that the previous cited example has been studied using a naive
treatment of the Dirac $\gamma_5$ matrix, the consistency of which is under question. It is known that
when using, for the regularisation and renormalisation of chiral theories, a consistent $\gamma_5$ scheme in
dimensional regularisation [5, 6, 7], namely the “Breitenlohner–Maison–’t Hooft–Veltman” (BMHV)
scheme [8, 9, 10, 11, 12, 13, 14], Ward and BRST symmetries are inevitably broken at each order in
perturbation, by spurious anomalies. These symmetries can nonetheless be restored, order by order,
using a minimal set of local finite counterterms, see e.g. [15, 16, 17, 18].

In such a setup, the renormalisation of the parameters of chiral theories, i.e. the couplings and
wave-function renormalisations, cannot be handled easily anymore by using a naive multiplicative
renormalisation scheme [19, 20]. For this reason, another way of organising and evaluating the RGEs
can be employed, based on the framework of Algebraic renormalisation [21, 22, 23, 24]. Our goal is to
derive the corresponding RG equations for an instance of such theory, namely a chiral-QED model,
in the dimensional renormalisation scheme, taking into account all the necessary counterterms that
ensure the restoration of the BRST symmetry.

We first present in Section 2, the definition of the Chiral-QED ($\chi$QED) model, that will serve
here as the toy model for illustrating the technique, while referring to our previous results obtained
in [17, 18] wherever possible.

In Section 3, we present the theoretical framework employed for the defining and evaluating the
RGEs. We will also evoke the problems that would surface for theories where BRST symmetry has to
be restored at each order in perturbation, if the standard multiplicative-renormalisation framework
were to be used instead. The algebraic formulation for the RGE is introduced in Section 4.

Its structure is further analysed in Section 5, applied to one- and two-loop (more accurately, $\hbar^1$ and $\hbar^2$) orders, and we finally attack the calculations proper in Sections 5.1 and 5.2. Discussion of
the obtained results follows in Section 5.3, and conclusions in Section 6 ends up this paper.

2 Chiral-QED Toy-model Action and Counterterms

For illustrating the topic of this work, we employ the Chiral-QED toy-model ($\chi$QED) without scalars, previously studied in [18] at two-loop order. (For a version of this model with scalars at one-loop order, we refer the reader to [16].) In this section, we define the action of the theory, based from its 4-dimensional formulation. We thus implement support for ensuring BRST invariance in the theory, and extend it to $d$ dimensions as required by dimensional regularisation. From this “defining” tree-level action, singular counterterms, as well as finite BRST-restoring counterterms, are evaluated, using methods from [17, 18], therefore we will not reproduce the details here.

2.1 Tree-level action in 4 and $d$ dimensions

The 4-dimensional defining tree-level action of $\chi$QED is:

$$S^{(4D)}_0 = \int d^4x \bar{\psi}^i D^i_\mu \psi^i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \bar{c} \partial^2 c + \rho^\mu sA_\mu + \bar{R}^i s\psi_R i + s\bar{\psi}_R i \bar{R}^i \equiv (S_{\bar{\psi}\psi}^R + S_{\bar{\psi}A\psi}^R) + S_{\bar{A}A} + S_{\bar{R}\text{fix}} + S_{\bar{c}c} + S_{\rho c} + S_{\bar{R}c\psi} + S_{\bar{\psi}cR}.$$  (2.1a)

(All fields are function of the same space-time point $x$, therefore we will omit this dependence wherever it is evident.) Until now, all the field monomials are implicitly in 4 dimensions (their Lorentz tensor structures are 4-dimensional, and we use a “barred” notation for them). The only $U(1)$ generator is the hypercharge, which we can assume to be diagonal, $\mathcal{Y}_{Rij} \equiv (\text{diag}(\mathcal{Y}_R^1, \ldots, \mathcal{Y}_R^{N_f}))_{ij}$, where $N_f$ is the number of fermion flavours. In this formulation, the fermions are explicitly right-handed: $\psi_R i \equiv \bar{F}_R \psi_i$; their corresponding covariant derivative is:

$$D^i_\mu = \partial_\mu \delta_{ij} - ieA^\mu \mathcal{Y}_{Rij},$$  (2.1b)

and the field strength tensor is defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  (2.1c)

2.1.1 BRST invariance

The $\chi$QED tree-level action is originally gauge-invariant; however, a gauge fixing has to be chosen for implementing the quantisation procedure: we employ here a linear $R_\xi$ gauge for the photon field. After fixing its gauge, a residual symmetry remains nonetheless: this is the BRST symmetry, that this action still satisfies.

This BRST symmetry is generated by corresponding transformations of the fields, that can be obtained from their original infinitesimal gauge transformations by formally replacing the local gauge parameter by the product of a Grassmann parameter $\theta$ with an anticommuting ghost field $c(x)$, $\delta_{\text{BRST}} \phi = (\delta_{\text{gauge}} \phi)_{c(x) \to \theta c(x)} \equiv \theta s\phi$, where $s\phi$ is the so-called BRST transformation of the field $\phi$. For the $\chi$QED model fields, their non-vanishing transformations are defined by:

$$sA_\mu = \partial_\mu c,$$  (2.2a)

$$s\psi_i = s\psi_R i = i e c \mathcal{Y}_{Rij} \psi_R j,$$  (2.2b)

$$s\bar{\psi}_i = s\bar{\psi}_R i = i e \bar{\psi}_R j c \mathcal{Y}_{Rji},$$  (2.2c)

$$s\bar{c} = B \equiv -\frac{1}{\xi} \partial^\mu A_\mu.$$  (2.2d)
This ”$s$” operator has the property to be nilpotent in this formulation, $s^2 \phi = 0$ for any field of the theory.

The last equation also introduces the auxiliary non-dynamical Nakanishi-Lautrup field $B$, used in the formulation of the linear $R_\xi$ gauge-fixing. Both gauge-fixing and the ghost kinetic terms are obtained from the BRST transformation of the expression $\tau(\xi B/2 + \partial^\mu A_\mu)$, leading to the (i) ghost kinetic term: $-\bar{c} \partial^2 c$, and (ii) the $R_\xi$ gauge-fixing term: $\xi B^2/2 + B \partial^\mu A_\mu$. Finally, the auxiliary $B$ field is integrated out\textsuperscript{1} from the action (as it will be in the rest of this paper, see also Eq. (3.21a)), giving rise to the usual gauge-fixing term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ in the action, Eq. (2.1a).

The next step consists in coupling the non-linear BRST transformations with external sources, $\mathcal{J} = \rho^\mu, \bar{R}^i, R^i$, that are invariant under BRST transformations: $s \mathcal{J} = 0$, resulting in the new contributions $\rho^\mu s A_\mu + \bar{R}^i s \psi_{R_i} + s \bar{\psi}_{R_i} R^i$ in the action. This allows to define a procedure for verifying, via a functional formulation of a Slavnov-Taylor identity, the BRST invariance of the theory at higher-orders of perturbation, as we will see in Section 2.2.

### 2.1.2 Extension to $d$ dimensions

In order to implement dimensional regularisation in \( \chi QED \), the tree-level action previously defined is extended to $d$ dimensions. Several possible ways of doing the extension exist, regarding the fermion-photon interaction vertex and the fermion kinetic term, see the discussion in [15, 17, 18]. Following these references, we choose:

$$S_0 = \int d^d x \; i \bar{\psi}_{R_i} \partial_{ij} \psi_{R_j} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \bar{c} \partial^2 c + \rho^\mu s_d A_\mu + \bar{R}^i s_d \psi_{R_i} + s_d \bar{\psi}_{R_i} R^i$$

$$\equiv (\bar{S}_{\bar{\psi}\psi} + \bar{S}_{\bar{\psi}\bar{c}} + \bar{S}_{\bar{c}\bar{c}} + \bar{S}_{\bar{c}cR} + \bar{S}_{\bar{c}cR}) + S_{AA} + S_{c-fix} + S_{cc} + S_{pc} + S_{R\psi} + S_{\bar{c}cR}, \quad (2.3a)$$

where $s_d$ is the $d$-dimensional pointwise BRST transformation, defined by extending its 4-dimensional version defined above to the $d$-dimensional case (for vector fields, their BRST transformation apply to the whole $d$-dimensional fields), and the fermionic kinetic interaction term is now:

$$\int d^d x \; i \bar{\psi}_{R_i} \partial_{ij} \psi_{R_j} = \int d^d x \; i \bar{\psi}_{j} \partial_{ij} \psi_{i} + e Y_{ij} \bar{\psi}_{j} \bar{\psi}_{L} \bar{A} \bar{A} R_{\psi} \psi_{j}$$

$$= \int d^d x \; i \bar{\psi}_{j} \partial_{ij} \psi_{i} + \bar{\psi}_{i} \partial_{ij} \psi_{i} + e Y_{ij} \bar{\psi}_{j} \bar{\psi}_{L} \bar{A} \bar{A} R_{\psi} \psi_{j} \equiv \bar{S}_{\bar{\psi}\psi} + \bar{S}_{\bar{\psi}\bar{c}} + \bar{S}_{\bar{c}cR}. \quad (2.3b)$$

The vector fields are in $d$ dimensions, unless being explicitly projected. Taking the $d \to 4$ restriction of $S_0$, we see that the evanescent component of the fermion pure-kinetic term, $\bar{S}_{\bar{\psi}\psi}$, disappears, while its 4-dimensional component $\bar{S}_{\bar{\psi}\bar{c}} = \bar{S}_{\bar{\psi}\bar{c}R} + \bar{S}_{\bar{\psi}\bar{c}L}$, contains a left-handed fermion. While this degree of freedom naively decouples at tree-level in $d \to 4$ since it disappears due to all the other couplings being right-chiral-projected, its implicit presence in dimensional-regularised propagators at loop-level plays a crucial role for the renormalisation procedure in dimensional regularisation and renormalisation.

### Evanescent tree-level action

The $d$-dimensional tree-level action contains evanescent components: contributions containing tensor structures existing in $4 - d$ dimensions. We define the actual evanescent component of the tree-level

\textsuperscript{1}When doing so, the BRST transformations are not off-shell nilpotent anymore, with $s^2 \tau = 0$, but now give instead: $s^2 \tau = -(\partial^2 c)/\xi$. They become nilpotent on-shell, only by using a (anti)ghost equation of motion.
action by the difference of the $d$ and the 4-dimensional tree-level actions\(^2\),
\[
\hat{S}_0 = S_0 - \overline{S}_0 = S_{\psi\psi L} - \overline{S}_{\psi\psi} + (S_{AA} - \overline{S}_{AA}) + (S_{g-fix} - \overline{S}_{g-fix}) \\
+ (S_{ce} - \overline{S}_{cc}) + (S_{pc} - \overline{S}_{pc}) + (S_{R\psi} - \overline{S}_{R\psi}) + (S_{\psi\psi R} - \overline{S}_{\psi\psi R}).
\] (2.4a)

Few comments can be made on this equation:
- The evanescent $\overline{S}_{\psi\psi}$ is the element of the $d$-dimensional action that generates the genuine tree-level BRST breaking, see Section 2.2.
- The $S_{\psi\psi}$ interaction vertex does not appear in $\overline{S}_0$, due to the fact it is chiral-symmetric, and is always 4-dimensional.
- All the other interaction/2-point vertices are differences of BRST-symmetric quantities. Only those containing vector bosons do matter, since these contain genuine Lorentz vector quantities whose dimensionality differ (and therefore, their difference is evanescent). This is also why the last two quantities, differences between $d$ and 4-dimensional fermion/BRST external source fields, vanish, since they do not contain any explicit Lorentz vector quantity: they are both the same.

For all these reasons, and for all practical purposes, the evanescent action $\hat{S}_0$ is equivalent to:
\[
\hat{S}_0 \equiv S_{\psi\psi L} - \overline{S}_{\psi\psi} + (S_{AA} - \overline{S}_{AA}) + (S_{g-fix} - \overline{S}_{g-fix}) + (S_{ce} - \overline{S}_{cc}) + (S_{pc} - \overline{S}_{pc}).
\] (2.4b)

From the definitions:
\[
S_{ce} = \int d^d x \left( -\bar{c} \hat{\partial}^2 c \right), \\
S_{cc} = \int d^d x \left( -\bar{c} \hat{\partial}^2 c \right), \\
S_{pc} = \int d^d x \rho^\mu s_\mu A_\mu = \int d^d x \rho^\mu s_\mu \hat{A}_\mu = \int d^d x \rho^\mu (\hat{\partial}_\mu c), \\
S_{\rho c} = \int d^d x \bar{\rho}^\mu \overline{s_\mu \hat{A}_\mu} = \int d^d x \bar{\rho}^\mu (\overline{\hat{\partial}_\mu c}),
\] (2.5a) (2.5b)

we define the following evanescent operators:
\[
\hat{S}_{ce} = S_{ce} - \overline{S}_{cc} = \int d^d x \left( -\bar{c} \hat{\partial}^2 c \right), \\
\hat{S}_{pc} = S_{pc} - \overline{S}_{pc} = \int d^d x \bar{\rho}^\mu (\overline{\hat{\partial}_\mu c}).
\] (2.6a) (2.6b)

We do not explicitly write such expressions for $S_{AA} - \overline{S}_{AA}$ and $S_{g-fix} - \overline{S}_{g-fix}$, just denoting them by $\hat{S}_{AA}$ and $\hat{S}_{g-fix}$, respectively, as their expansion can be lengthy, generating more than one evanescent term. See Appendix B for the structure of $\hat{S}_{AA}$. Finally, we recall that:
\[
\overline{S}_{\psi\psi} = \int d^d x \bar{i} \psi_\dag \hat{\partial} \psi_\dag.
\] (2.6c)

Using these definitions, we can re-express the evanescent $\hat{S}_0$:
\[
\hat{S}_0 \equiv S_{\psi\psi L} - \overline{S}_{\psi\psi} + \hat{S}_{AA} + \hat{S}_{g-fix} + \hat{S}_{ce} + \hat{S}_{pc}.
\] (2.7)

\(\text{2Now, we implicitly understand that when subtracting the } d\text{-dimensional action from the 4-dimensional one, the space-time integral of the 4-dimensional action gets promoted to } d \text{ dimensions, while all the fields remain in 4 dimensions. (This is one fundamental point of this formal procedure!)}\)
2.2 Slavnov-Taylor identities and BRST invariance

The original 4-dimensional tree-level action $S_0^{(4D)}$ is invariant under the BRST transformation

$$sS_0^{(4D)} = 0,$$

and equivalently, satisfies a functional version of BRST invariance via the Slavnov-Taylor identity (STI)

$$S(S_0^{(4D)}) = 0,$$

where the Slavnov-Taylor (ST) operator is given for a general functional $F$ as

$$S(F) = \int d^4x \left( \frac{\delta F}{\delta \rho^\mu} \frac{\delta F}{\delta A_\mu} + \frac{\delta F}{\delta R^i} \frac{\delta F}{\delta \psi_i} + \frac{\delta F}{\delta \bar{\psi}} \frac{\delta F}{\delta \bar{c}} + B \frac{\delta F}{\delta \bar{c}} \right);$$

again, $B$ is treated as an abbreviation to its value given in Eq. (2.2d). This definition is then straightforwardly extended to $d$ dimensions, thus defining a $d$-dimensional ST operator $S_d(F)$ for any $d$-dimensional functional $F$, fields, and integration measure $d^d x$.

The $d$-dimensional tree-level action $S_d$, Eq. (2.3a), however, breaks the BRST invariance and generates a tree-level evanescent breaking, when acting with $S_d$,

$$S_d(S_0) = s_d S_0 = \widehat{\Delta} (\neq 0),$$

where the breaking

$$\widehat{\Delta} = \int d^d x \left\{ \bar{\psi} \left( \frac{\partial}{\partial \rho} \bar{R}_i + \frac{\partial}{\partial \bar{\rho}} L \right) \psi \right\} \equiv \int d^d x \widehat{\Delta}(x)$$

originates from the evanescent fermion kinetic term $\widehat{S}_{\psi \tilde{\psi}}$. This $d$-dimensional Slavnov-Taylor operator $S_d$ constitutes the backbone for studying BRST invariance and restoring the BRST symmetry via Slavnov-Taylor identities at higher-loop orders, that will be employed in our study.

2.3 One and two-loop counterterms and their structure

In the context of perturbative renormalisation, the theory is defined in terms of renormalised parameters: couplings and wave-function renormalisations of fields. A new set of counterterms, $S_{ct}^{(n)}$, is defined at each order of perturbation. Their primary role is to absorb any infinities arising in loop diagrams calculations (parameterised with a regulator), as well as to introduce any optional finite renormalisation.

They can also include any finite symmetry-restoring counterterms, whose aim is to restore any spurious breakings of symmetries, due to local breaking terms that were induced by the regularisation and renormalisation procedure. In dimensional renormalisation, those are needed in particular in the case of chiral theories, where the BRST symmetry is broken.

In such a setup, the renormalisation of the parameters of the theory is not of multiplicative nature anymore, at least in a naive way (more on that in Section 3.1.1), and it is useful to further distinguish five types of counterterms [17, 18]:

$$S_{ct} = S_{ct, inv} + S_{ct, noninv} + S_{ct, inv} + S_{ct, restore} + S_{ct, evan} \equiv S_{ct} + S_{fct},$$

where

$^{3}$The $B \frac{\delta F}{\delta \bar{c}}$ term can be absorbed in $(\frac{\delta F}{\delta \rho^\mu})(\frac{\delta F}{\delta A_\mu})$ in Eq. (2.9), iff $F$ satisfies both the gauge-fixing condition and the ghost equation, Eq. (3.21).
• $S_{\text{ct,inv}}$ and $S_{\text{fct,inv}}$ correspond, respectively, to the UV divergent (“singular” $1/(d-4)$ poles) and finite counterterms, that are symmetric-invariant (they respect the fundamental symmetries: BRST, etc., originally present at tree-level), and are generated by a renormalisation transformation as in Eq. (3.5) of Section 3.1.

• $S_{\text{ct,noninv}}$ corresponds to additional singular counterterms needed to cancel additional $1/(d-4)$ poles, that are however symmetric-noninvariant and can be evanescent or non-evanescent. Such counterterms exist in other dimensional schemes as well (e.g. in dimensional reduction, see [25] for a recent review). Several of their usages include, establishing scheme equivalence [26, 27], ensuring unitarity, finiteness, and consistency with infrared factorisation in higher-order computations [28, 29, 30, 31, 32, 33].

• $S_{\text{fct,restore}}$ are the finite counterterms needed to restore the symmetries: in our case the BRST symmetry. Their determination has been done, for example, at one-loop level for a generic massless chiral Yang-Mills theory in [15], supplemented with scalar fields (without VEV) in [17], and at two-loop level for a chiral QED model in [18]. In principle, one can add to them any additional finite BRST-symmetric counterterms, without changing any physical predictions.

• $S_{\text{fct,evan}}$ corresponds to additional counterterms which are both finite and evanescent. Adding or changing such counterterms can swap e.g. between different choices of fermion-photon interaction vertex, as mentioned in Section 2.1.2; these counterterms vanish in the 4-dimensional limit, but they can affect calculations at higher orders (i.e. renormalisation-scheme dependence).

The sum of the renormalised tree-level action and its counterterms defines the bare action,

$$S_B \equiv S_0 + S_{\text{ct}},$$

from which the path integral formulation of the generating functional $Z$ of the theory can be defined.

**Loop versus $\hbar$ order expansion**

Singular counterterms $S_{\text{ct}}^{(N_l)}$ are determined by the renormalisation procedure, recursively on the number of loops $N_l$ of the corresponding diagrams they sub-renormalise. The procedure is as follows: for a given $N_l$-loop divergent diagram, (i) the divergences of its sub-loops are recursively subtracted (sub-renormalised), using corresponding lower-order singular counterterms, and (ii) the overall remaining local divergence of the sub-renormalised diagram, is subtracted by an $N_l$-loop-order local singular counterterm. These singular counterterms thus encode the divergent structure of the corresponding loop diagrams, in their local $1/(d-4)$ poles.

Simultaneously, at each $\hbar$-order $m$, local finite counterterms $S_{\text{fct}}^{(m)}$ are determined so as to restore the fundamental symmetries (i.e. the BRST symmetry) of the theory. However, because of their finiteness, they behave, when inserted into loop diagrams, like the (finite) vertices from the tree-level action $S_0$, although they admit an $\hbar$ expansion. As such, they introduce a mismatch between the loop-counting and the $\hbar$-order of diagrams in which they are inserted. Consequently, a divergent $N_l$-loop diagram, constructed out of the vertices of $S_0$ and vertices $i$ from the finite counterterms $S_{\text{fct}}^{(m_i)}$, has an overall $\hbar$-order $m = N_l + \sum_i m_i$ and generates, after renormalisation, a corresponding $\hbar^m$-order singular counterterm.

Therefore, it will be useful to distinguish the $\hbar$-expansion order $m$ of the singular counterterms from their loop-order $N_l$. They will be denoted by

$$S_{\text{ct}}^{(m,N_l)},$$

with $m \geq N_l$. Singular counterterms with $m > N_l$ necessarily sub-renormalise diagrams containing at least one finite counterterm.
One and two-loop counterterms

We quote below, the explicit results of the singular and finite symmetry-restoring counterterms at both one and two-loops ($h^1$ and $h^2$ respectively) for the $\chi$QED theory, working in $d = 4 - 2\epsilon$ dimensions, obtained in [17, 18].

At $h^1$, one-loop order, the singular counterterms are:

$$S_{\text{ct}}^{(1)} = S_{\text{ct}}^{(1,1)} = -\frac{h e^2}{16\pi^2\epsilon} \left( \frac{2}{3} \text{Tr}(Y_R^2) S_{AA} + \xi \sum_j (Y_R^j)^2 \left( \frac{S^2_{\psi R}}{\psi R} + \frac{S_{\psi R A\psi R}}{\psi R A\psi R} \right) + \frac{1}{2} \text{Tr}(Y_R^2) \int d^d x \frac{1}{2} \bar{A}_\mu \partial^2 A^\mu \right),$$

where the monomials $\frac{S^2_{\psi R}}{\psi R}$, $\frac{S_{\psi R A\psi R}}{\psi R A\psi R}$ are the fully right-chiral-projected equivalents to their usual $d$-dimensional versions, (at fixed index $i$ not summed over)

$$\frac{S^2_{\psi R}}{\psi R} = \int d^d x i \bar{\psi} \Gamma_P R \psi \equiv \int d^d x \frac{i}{2} \bar{\psi} \partial \Gamma_P R \psi,$$

$$\frac{S_{\psi R A\psi R}}{\psi R A\psi R} = \int d^d x e \bar{Y}_R \bar{\psi} A \Gamma_P R \psi.$$

The finite symmetry-restoring counterterms are:

$$S_{\text{ct}}^{(1)} = \left( \frac{h e^2}{16\pi^2\epsilon} \left( \frac{2}{3} \text{Tr}(Y_R^2) S_{AA} + \frac{55}{24} \int d^d x \frac{1}{2} \bar{A}_\mu \partial^2 A^\mu \right) \right),$$

$$+ \frac{\xi + 5}{6} \sum_j (Y_R^j)^2 \left( \frac{S^2_{\psi R}}{\psi R} + \frac{S_{\psi R A\psi R}}{\psi R A\psi R} \right),$$

where our choice is not to add any additional BRST-symmetric term.

At $h^2$ order, the singular counterterm for pure two-loop (sub-renormalised) diagrams, $S_{\text{ct}}^{(2,2)}$, are, in Feynman gauge $\xi = 1$,

$$S_{\text{ct}}^{(2,2)} = \left( \frac{h}{16\pi^2} \right)^2 \frac{e^4}{3\epsilon} \text{Tr}(Y_R^4) \left( 6 S_{AA} + \frac{1}{2\epsilon} + \frac{55}{24} \int d^d x \frac{1}{2} \bar{A}_\mu \partial^2 A^\mu \right)$$

$$+ \left( \frac{h}{16\pi^2} \right)^2 \frac{e^4}{3\epsilon} \sum_j (Y_R^j)^2 \left[ \left( \frac{3}{2\epsilon} - \frac{7}{4} \right) (Y_R^j)^2 + \frac{2\epsilon}{3} \text{Tr}(Y_R^2) \left( \frac{S^2_{\psi R}}{\psi R} + \frac{S_{\psi R A\psi R}}{\psi R A\psi R} \right) \right]$$

$$+ \left( \frac{h}{16\pi^2} \right)^2 \frac{e^4}{3\epsilon} \sum_j (Y_R^j)^2 \left( \frac{1}{2} (Y_R^j)^2 + \frac{2}{3} \text{Tr}(Y_R^2) \right) \frac{S^2_{\psi R}}{\psi R}. \tag{2.15a}$$

There appear, as well, counterterms for one-loop diagrams containing one insertion (Appendix A) of a finite $S_{\text{ct}}^{(1)}$ counterterm. These counterterms, $S_{\text{ct}}^{(2,1)}$, have a genuine one-loop structure, even though they are of order $h^2$. The diagrams whose divergent parts define those counterterms are listed in Fig. 2 of Appendix C, and these corresponding counterterms are:

$$S_{\text{ct}}^{(2,1)} = -\left( S_{\text{ct}}^{(1)} \cdot \Gamma_{\text{ren}} \right) \text{div} = \left( \frac{h}{16\pi^2} \right)^2 \frac{e^4}{3\epsilon} \left\{ \xi + \frac{5}{6} \text{Tr}(Y_R^4) \left( 4 S_{AA} + 3 \int d^d x \frac{1}{2} \bar{A}_\mu \partial^2 A^\mu \right) \right.$$

$$+ \left. \frac{\xi + 5}{2} \sum_j (Y_R^j)^4 \left( \frac{S^2_{\psi R}}{\psi R} + \frac{S_{\psi R A\psi R}}{\psi R A\psi R} \right) - \xi^2 \text{Tr}(Y_R^2) \sum_j (Y_R^j)^2 \left( \frac{S^2_{\psi R}}{\psi R} + \frac{S_{\psi R A\psi R}}{\psi R A\psi R} \right) \right\}. \tag{2.15b}$$

As presented in [18], one can alternatively quote the total $h^2$-order singular counterterms, at the expense of losing their corresponding loop-counting information, that would be necessary for establishing
the RG equations, as we will see in Section 4.2. We obtain, in Feynman gauge $\xi = 1$:

$$S^{(2)}_{\text{act}} = S^{(2,1)}_{\text{act}} + S^{(2,2)}_{\text{act}}$$

$$= -\left(\frac{\hbar}{16\pi^2}\right)^2 e^4 \int d^4 x \frac{1}{2} \bar{A}_\mu \tilde{\partial}^2 \vec{A}^\mu + \frac{3e^2 S_{\bar{\psi} \psi}^{(2)}(\bar{A}^2)^2}{16} - \sum_j (\bar{Y}_R^j)^2 \left(\frac{127}{36} (\bar{Y}_R^j)^2 - \frac{1}{27} \frac{\bar{Y}_R^j}{\bar{Y}_R^j} S_{\bar{\psi} \psi}^{(2)}\right)$$

The $\hbar^2$-order BRST-restoring finite counterterms are:

$$S^{(2)}_{\text{ct}} = S^{(2)}_{\text{ct}} = \left(\frac{\hbar}{16\pi^2}\right)^2 e^4 \left\{ \frac{11}{24} \int d^4 x \frac{1}{2} \bar{A}_\mu \tilde{\partial}^2 \vec{A}^\mu + \frac{3e^2 S_{\bar{\psi} \psi}^{(2)}(\bar{A}^2)^2}{16} - \sum_j (\bar{Y}_R^j)^2 \left(\frac{127}{36} (\bar{Y}_R^j)^2 - \frac{1}{27} \frac{\bar{Y}_R^j}{\bar{Y}_R^j} S_{\bar{\psi} \psi}^{(2)}\right) \right\}$$

where our choice again is not to add any additional BRST-symmetric term.

The most crucial observation is, that new local field-operators with new tensorial structures appear in the loop-generated counterterms:

- A “new” evanescent two-point photon vertex, $\int d^4 x \frac{1}{2} \bar{A}_\mu \tilde{\partial}^2 \vec{A}^\mu$. To be accurate, this structure also appears in the expansion of the $d$-dimensional $S_{AA} = -1/4 F_{\mu\nu} F^{\mu\nu}$ kinetic term, but we observe that it acquires an extra evanescent renormalisation compared with the other components;

- Two fully 4-dimensional two-point and 4-point photon vertices, $\int \frac{1}{2} \bar{A}_\mu \tilde{\partial}^2 \vec{A}^\mu$ and $\int e^2 (\bar{A}^2)^2$. The list of local BRST-restoring finite operators is therefore:

$$\left\{ \frac{S_{\bar{\psi} \psi}}{\bar{Y}_R^j}, \int d^4 x \frac{1}{2} \bar{A}_\mu \tilde{\partial}^2 \vec{A}^\mu, \int d^4 x \frac{e^2}{4} (\bar{A}^2)^2 \right\}$$

Only the first operator was present in the original 4-dimensional component of the tree-level action.

3 The Renormalisation Group Equation in Dimensional Renormalisation

Renormalisation group equations describe how correlation (Green’s) functions relate in different renormalisation prescriptions. More accurately, one can distinguish between $(i)$ the renormalisation group (RG) equation [34], that describes the response of Green’s functions (more precisely, their invariance) under a change of renormalisation point $\mu$ (e.g. the “unit of mass” introduced in dimensional regularisation, or an arbitrary energy scale in off-shell schemes), and $(ii)$ the Callan-Symanzik (CS) equation [35, 36, 37], that describes the breaking of scale dilatations by mass-dimensioned terms, for off-shell Green’s functions, independently of the chosen renormalisation point. (See also the review [38] for a coverage of these equations.)

While they both present some similarities in their formulations as differential equations, we will focus only, for the purpose of this paper, on the RG equation, in the context of Dimensional regularisation and renormalisation. Indeed, both equations become identical in massless theories, like the $\chi$QED model we are studying.
In dimensional regularisation and renormalisation, a “unit of mass”, $\mu$, is introduced \cite{1, 39} and is associated to each loop of any loop diagram. This unit of mass serves as the arbitrary renormalisation point. The 1-particle-irreducible (1-PI) quantum effective action $\Gamma$ (generating functional of the 1-PI Green’s functions), as a functional of all the fields $\phi$ and parameters of the theory (for $\chi$QED, the fields being: the photon $A$, the right and left fermions collectively denoted by $\psi$, the ghost $c$ and the BRST external sources $\rho, R, \overline{R}$, and its parameters being: the coupling constant $e$ and gauge parameter $\xi$), depends on $\mu$ both explicitly, and implicitly via the $\mu$-dependence of the renormalised parameters and the fields renormalisations $Z_1^2$:

$$\Gamma[\{\phi(\mu)\}; e(\mu), \xi(\mu), \mu]. \quad (3.1)$$

The RG equation then represents the invariance of the 1-PI effective action $\Gamma$ under a total variation of the unit of mass $\mu$,

$$\mu \frac{d\Gamma}{d\mu} = 0 = \mu \frac{\partial \Gamma}{\partial \mu} + \left( \beta_e \frac{\partial}{\partial e} + \beta_\xi \frac{\partial}{\partial \xi} - \sum_{\phi=A,\psi, c, \rho, R, \overline{R}} \gamma_\phi N_\phi \right) \Gamma. \quad (3.2a)$$

In Eq. (3.2a), $\mu \partial / \partial \mu$ is the RGE differential operator. The $N_\phi$ are the field-numbering (“leg-counting”) differential operators for each field $\phi$ of the theory, defined for bosonic and ghosts fields by

$$N_\phi \equiv \int_x \phi(x) \frac{\delta}{\delta \phi(x)}, \quad (3.2b)$$

and, for the right and left-handed fermions respectively, by

$$N^R_\psi \equiv \int_x (\overline{P}_R \psi_i(x)) \frac{\delta}{\delta \overline{\psi}_i(x)}, \quad N^L_\psi \equiv \int_x (\overline{\psi}_i(x) P_L) \frac{\delta}{\delta \overline{\psi}_i(x)}. \quad (3.2c)$$

The coefficient functions $\beta_e$ and $\gamma_\phi$ are, respectively, the beta-function for the coupling constant $e$, and the anomalous dimensions for the fields $\phi$, defined by

$$\beta_e = \mu \frac{d e}{d \mu}, \quad \gamma_\phi = \frac{1}{2} \mu \frac{d \ln Z_\phi}{d \mu}. \quad (3.3)$$

It is sometimes convenient to factor one power of the gauge coupling(s) out of their corresponding beta-function(s) in the RG equation, and thus write the corresponding term as

$$\beta_e e \frac{\partial}{\partial e}, \quad (3.4)$$

together with a suitable modification of the definition for $\beta_e$. We will choose to do so in the following. Notice also that one should consider a contribution coming from the gauge parameter as well (for e.g. in $R_\xi$ gauges), $\beta_\xi \partial / \partial \xi$; this is not usually done when calculations are performed in a fixed gauge (like Feynman gauge $\xi = 1$), but is required when the gauge parameter is left unspecified.

### 3.1 The RG equation in Multiplicative Renormalisation

Usual multiplicative renormalisation formulation, and its usage for the RG equation, is best suited for theories where the fundamental symmetries (BRST, etc.) that are present at tree-level, remain respected at loop-level: the counterterms do not break these symmetries (they are “symmetric”), i.e. their structures are similar to those in the defining tree-level action, and where different operators
containing the same fields renormalise in a consistent way between each other. Such formulation is also mostly applied for the regularised theory (still expressed in terms of its regulator).

It is based on the following renormalisation transformations. Applied for example to standard non-chiral QED, these transformations consist of renormalisation of physical parameters\footnote{We employ additive renormalisation for the physical parameters since multiplicative renormalisation for them would not be sufficient in general.},
\[
    e \rightarrow (e + \delta e) Z_A^{-1/2} Z_\psi^{-1} = Z_{\psi A \psi}^{-1/2} Z_\psi^{-1} e = Z_e e ,
\]
obtained from the transformation
\[
    S_{\psi A \psi R} = \int e Y_{Rij} \bar{\psi}_i A \psi_i \rightarrow Z_{\psi A \psi} S_{\psi A \psi R} = Z_{\psi} Z_A^{-1/2} \int e Y_{ij} \bar{\psi}_i A \psi_i = (e + \delta e) \int Y_{ij} \bar{\psi}_i A \psi_i ,
\]
and fields, via their wave-function renormalisation,
\[
    A_\mu \rightarrow Z_1^{1/2} A_\mu ,
\]
\[
    (\psi_i, \bar{\psi}_i) \rightarrow Z_{\psi}^{1/2} (\psi_i, \bar{\psi}_i) ,
\]
\[
    c \rightarrow Z_c^{1/2} c .
\]

The remaining fields, sources and the gauge parameter renormalise in a dependent way\footnote{It is possible to use a ghost field renormalisation different from the antighost, since only the field-renormalisation of the ghost-antighost combination is constrained.}, as
\[
    \{ B, \bar{c}, \xi \} \rightarrow \left\{ Z_A^{-1/2} B, Z_A^{-1/2} \bar{c}, Z_A \xi \right\} ,
\]
\[
    \rho^\mu \rightarrow Z_A^{-1/2} \rho^\mu ,
\]
\[
    (R^i, \bar{R}^i) \rightarrow Z_{\psi}^{-1/2} (R^i, \bar{R}^i) .
\]

\section*{Beta functions and anomalous dimensions in DimReg}

Explicit formulae in dimensional regularisation for the beta-functions of couplings and anomalous dimensions of fields, that directly make contact with the counterterms, have been evaluated by ’t Hooft \cite{1}, and employed by e.g. Machacek and Vaughn \cite{2, 3, 4} for evaluating the two-loop beta-functions and anomalous dimensions in “naive” chiral QED.

Working in \( d = 4 - 2\epsilon \) dimensions, renormalised coupling constants \( x_k \) are related to their (divergent) bare values \( x_k^0 \), of mass-dimensionality \( \eta_k \), whose \( \epsilon \)-expansion is:
\[
    x_k^0 \mu^{-\eta_k \epsilon} = x_k + \sum_{n=1}^{+\infty} C_k^{(n)}(\{x_l\}) / \epsilon^n ,
\]
and (divergent) wave-function renormalisation factors \( Z_\phi \) have an \( \epsilon \)-expansion:
\[
    Z_\phi = 1 + \sum_{n=1}^{+\infty} C_\phi^{(n)}(\{x_l\}) / \epsilon^n .
\]

The beta functions for the couplings \( x_k \) are then given by
\[
    \beta_k = \frac{d x_k}{d \mu} \bigg|_{\epsilon \rightarrow 0} = -\eta_k C_k^{(1)}(\{x_l\}) + \sum_{x_l} \eta_l x_l \frac{\partial C_k^{(1)}(\{x_l\})}{\partial x_l} ,
\]
\[
    \beta_k = \frac{d x_k}{d \mu} \bigg|_{\epsilon \rightarrow 0} = -\eta_k C_k^{(1)}(\{x_l\}) + \sum_{x_l} \eta_l x_l \frac{\partial C_k^{(1)}(\{x_l\})}{\partial x_l} ,
\]
and the anomalous dimension for the field \( \phi \) is given by
\[
\gamma_\phi = \frac{1}{2} \mu \left. \frac{d \ln Z_\phi}{d \mu} \right|_{\epsilon \to 0} = -\frac{1}{2} \sum_{x_l} \eta_l x_l \frac{\partial C^{(1)}_\phi}{\partial x_l}.
\] (3.9)

Both \( \beta_k \) and \( \gamma_\phi \) thus depend on the single \( 1/\epsilon \) pole coefficient of the expansions of \( x_0^k \) and \( Z_\phi \), and their higher \( 1/\epsilon \)-order coefficients are also related to the lower-order ones via recurrence relations. In the dimensional-regularised \( \chi \)QED, the mass-dimensions of the derivative and the bare fields are:

\[
[\partial_\mu] = 1, \text{ fixed}, \quad \frac{d-2}{2} = 1 - \epsilon, \quad \frac{d-1}{2} = \frac{3}{2} - \epsilon.
\] (3.10a, b, c)

The dimensionality of the bare charge is such that the corresponding renormalised charge has the same dimensionality as in the renormalised 4-dimensional theory:
\[
[e_B] = \frac{4 - d}{2} = \epsilon, \quad [e] = 0,
\] (3.10d)

meaning that \( \eta_e = 1 \).

### 3.1.1 Shortcomings of the original RGE; extension of the tree-level action

When working in dimensional regularisation, the problem we face with chiral theories such as \( \chi \)QED is that there exist non-symmetric singular counterterms, that may (or may not) be evanescent. For example, we observe in practice that only the 4-dimensional fermion kinetic term \( S_{\bar{\psi} \psi} \) receives corrections, while the evanescent one \( \hat{S}_{\bar{\psi} \psi} \) does not. This is in contrast with non-chiral theories (e.g. standard QED, QCD, ...) where both types of fermion kinetic terms renormalise the same way: the whole \( d \)-dimensional kinetic term renormalise as a whole. Further, in \( \chi \)QED, new local, singular but evanescent, and finite operators arise, the latter ones constituting the BRST-restoring finite renormalisations.

By following an approach similar to the one exposed by Bos [19] and Schubert [20], it is still possible to employ a multiplicative renormalisation formulation, at the condition of enlarging the operator basis in the tree-level action with those non-symmetric operators. \( S_0 \) is thus supplemented by terms of the form:
\[
\int d^d x \rho_\mathcal{O} \mathcal{O}(x),
\] (3.11a)

for these evanescent and finite operators \( \mathcal{O} \). They are multiplied by new dimensionless auxiliary “coupling constants” \( \rho_\mathcal{O} \), whose role is to absorb the different renormalisations of the \( \mathcal{O} \), since they have a different renormalisation behaviour from their 4-dimensional counterparts. In this formalism the amended tree-level action becomes:
\[
S_0^* = S_0 + \int d^d x \rho_\mathcal{O} \mathcal{O}(x).
\] (3.11b)

Applying this to the example of \( \chi \)QED, we associate to the operators the following auxiliary (bare) couplings, that vanish at tree-level, but acquire non-zero values through quantum corrections:

- \( \sigma_i \) for those corresponding to the evanescent operators already present in the \( d \)-dimensional tree-level action \( S_0 \), and \( \int d^d x \frac{1}{2} A^\mu \delta^2 A^\mu \) (present within the evanescent \( \hat{S}_{AA} \)).
• \(\rho_i\) for the couplings corresponding to the 4-dimensional operators appearing in the \(\chi\)QED finite BRST-restoring counterterms, that are not already present in \(S_0\), namely: \(\int \frac{1}{2} \hat{A}_\mu \partial^2 \hat{A}^\mu\) and \(\int \frac{e^2}{4} (\hat{A}^2)^2\); and the tree-level action is modified into:

\[
S_0 \rightarrow S_0^* = S_0 + \sigma_1 S_{\psi\psi}^c + \sigma_2 \overline{S}_{AA} + \int d^d x \left( \sigma_3 \frac{1}{2} \hat{A}_\mu \partial^2 \hat{A}^\mu + \rho_1 \frac{1}{2} \hat{A}_\mu \partial^2 \hat{A}^\mu + \rho_2 \frac{e^2}{4} (\hat{A}^2)^2 \right)
\]

where \(\overline{S}_{AA} \equiv S_{AA} - \overline{S}_{AA}\). Compared to the physical coupling (the electric charge) \(e\), these auxiliary couplings have been chosen such that their dimensionality remain zero: \([\sigma_{i,B}] = [\sigma_i] = 0\) and \([\rho_{i,B}] = [\rho_i] = 0\), meaning that \(\eta_{\sigma_i} = 0\) and \(\eta_{\rho_i} = 0\).

These auxiliary couplings receive counterterm corrections. Taking for illustration the example of the evanescent fermion kinetic term \(\overline{S}_{\psi\psi}^c\), this operator renormalises as follows:

\[
(1 + \sigma_1) \overline{S}_{\psi\psi}^c \rightarrow Z_{\psi\psi} \left( 1 + \sigma_1 \right) \overline{S}_{\psi\psi}^c = (1 + Z_{\sigma_1} \sigma_1 Z_{\psi\psi}^{-1}) Z_{\psi\psi} \overline{S}_{\psi\psi}^c
\]

where the fermion wave-function renormalisation \(Z_{\psi\psi}\) takes both singular and finite (BRST-restoring) counterterm contributions into account. A similar transformation holds for the other operators. Since they usually do not renormalise multiplicatively (because they break both Lorentz and gauge invariance), it is better to consider an additive renormalisation for them:

\[
(1 + \sigma_1) \overline{S}_{\psi\psi}^c \rightarrow (1 + (\sigma_1 + \delta_{\sigma_1}) Z_{\sigma_1}) Z_{\psi\psi} \overline{S}_{\psi\psi}^c
\]

and similarly for the other ones. These operators are such that, in the renormalised limit with \(d \rightarrow 4\) and evanescent terms removed, all the renormalised tree-level “couplings” \(\sigma_i\) and \(\rho_i\) are set to zero as well; only the potentially finite or divergent counterterms \(\delta\) remain.

This tree-level action \(S_0^*\) formally generates a quantum effective action \(\Gamma_{\text{DReg}}[\phi, \rho_0]\), and if perturbative expansion in couplings holds, then \(\Gamma_{\text{DReg}}[\phi, \rho_0]\) accepts a formal perturbative expansion in the couplings \(\rho_0\), provided they are considered to be “small”. Some of these auxiliary couplings are associated to two-point operators: in the \(\chi\)QED model, the evanescent component of the fermion and the photon kinetic terms: \(\overline{S}_{\psi\psi}^c\) and \(\overline{S}_{AA}\), an extra singular evanescent contribution to the photon self-energy \(\int \hat{A}_\mu \partial^2 \hat{A}^\mu\), or the 4-dimensional operator \(\int \hat{A}_\mu \partial^2 \hat{A}^\mu\). These operators are already present in the tree-level action, within the \(d\)-dimensional \(\overline{S}_{\psi\psi}^c\) and \(\overline{S}_{AA}\) kinetic terms; therefore, they will modify in a non-trivial way the corresponding tree-level and resummed \(d\)-dimensional fermionic and photon propagators. However, these operators also receive loop-generated corrections, proportional to powers of \(\hbar\) and of the gauge coupling. Therefore, only a limited number of these operators will contribute at any given order of perturbation.

3.1.2 RGE structure in modified multiplicative renormalisation; Expectations in the renormalised limit \(d \rightarrow 4\)

The modified multiplicative renormalisation procedure we have discussed so far, formally introduces the extra couplings \(\rho_0\). As such, an RGE for the dimensional-regularised quantum effective action

\[\text{s}\]
\( \Gamma_{\text{DReg}}^* \) will depend on variations with respect to \( \rho_O \) and its corresponding \( \beta \) functions:

\[
\mu \frac{\partial \Gamma_{\text{DReg}}^*}{\partial \mu} = \left( -\tilde{\beta}_c e \frac{\partial}{\partial e} - \tilde{\beta}_\xi \frac{\partial}{\partial \xi} - \tilde{\beta}_{\sigma_i} \frac{\partial}{\partial \sigma_i} - \tilde{\beta}_{\rho_i} \frac{\partial}{\partial \rho_i} + \sum_{\phi=A,\psi,\varphi} \tilde{\gamma}_\phi N_\phi \right) \Gamma_{\text{DReg}}^*[e, \{\sigma_i\}, \{\rho_i\}] \quad (3.14)
\]

Note that here, these \( \tilde{\beta} \) and \( \tilde{\gamma}_\phi \) coefficients are not the true beta-functions and anomalous dimensions of the renormalised theory, but are instead to be understood as auxiliary intermediate quantities. The appearance of beta functions for the evanescent operators is also a sign of that fact. Therefore, the true \( \beta \) and \( \gamma \) functions of the theory are expected to depend on these other auxiliary quantities, and of the beta functions for the evanescent operators in particular.

This RG equation structure so far is suitable for the “intermediate” dimensional-regularised theory, where the extra evanescent and the non-evanescent BRST-breaking operators are still associated with the effective couplings \( \sigma_i \) and \( \rho_i \). However, these couplings are unphysical and do not appear in the renormalised theory: their renormalised values are defined to be zero, and the 4-dimensional renormalised effective action \( \Gamma_{\text{ren}} \) (that we will simply denote by \( \Gamma \) in what follows, when no ambiguities can appear) is defined by

\[
\Gamma[e] = \text{LIM}_{d \to 4} \lim_{\sigma_i \to 0, \rho_i \to 0} \Gamma_{\text{DReg}}^*[e, \{\sigma_i\}, \{\rho_i\}] \quad (3.15)
\]

that is, in the limit (denoted as \( \text{LIM}_{d \to 4} \)) where: (i) divergences are MS-subtracted from \( \Gamma \), where \( (ii) \ d \to 4 \), with (iii) remaining (finite) evanescent quantities set to zero. (The implementation of (i) is ensured by the presence of the suitable singular counterterms.)

Therefore we would need to find the RG equation for the corresponding renormalised \( \Gamma \), defined as above, whose form, since no explicit \( \sigma_i \), \( \rho_i \) dependence is present in \( \Gamma \) anymore, should be:

\[
\mu \frac{\partial \Gamma}{\partial \mu} = \left( -\beta_e e \frac{\partial}{\partial e} - \beta_\xi \frac{\partial}{\partial \xi} + \sum_{\phi} \gamma_\phi N_\phi \right) \Gamma \quad (3.16)
\]

This equation would have to be the equivalent of Eq. (3.14), but where both of its sides are taken in the limits \( \sigma_i \to 0, \rho_i \to 0 \), and \( \text{LIM}_{d \to 4} \). One should note that the new RGE coefficients, \( \beta_e \) and \( \gamma_\phi \), are not in direct one-to-one correspondence with their tilded counterparts, but have to contain the effects from the \( \sigma_i \) and \( \rho_i \) contributions as well. This implies determining what

\[
\left. \left( -\tilde{\beta}_{\sigma_i} \frac{\partial}{\partial \sigma_i} - \tilde{\beta}_{\rho_i} \frac{\partial}{\partial \rho_i} \right) \Gamma_{\text{DReg}}^*[e, \{\sigma_i\}, \{\rho_i\}] \right|_{\sigma_i \to 0, \rho_i \to 0}
\]

equals to. More precisely, one would need to determine what the action of each of these operators, \( \partial/\partial \sigma_i \) and \( \partial/\partial \rho_i \), is, in terms of the remaining operators \( \partial/\partial e \) and \( N_\phi \) for \( \phi = A, \psi, c \), and eliminate the former in favour of the latter ones. But observe, from perturbativity of \( \Gamma_{\text{DReg}} \) in the \( \sigma_i \) and \( \rho_i \) in the limit \( \to 0 \), that \( \Gamma_{\text{DReg}}^* \) can be expanded and only its first-order in those couplings be kept. Because these are linear differential operators with respect to parameters of \( \Gamma_{\text{DReg}}^* \), by the Regularised Action Principle [12, 13, 14] they are equivalent to the insertion (Appendix A) of a corresponding local field-operator at zero momentum in \( \Gamma_{\text{DReg}}^* \), that will be nothing but the operator \( \mathcal{O} \) associated with \( \rho_O \), plus a dependence from the theory’s finite BRST-restoring counterterms \( S_{\text{ft}}^{\gamma} \),

\[
\frac{\partial \Gamma_{\text{DReg}}^*}{\partial \rho_O} \bigg|_{\rho_O \to 0} = \frac{\partial (S_0^* + S_{\text{ft}}^{\gamma})}{\partial \rho_O} \cdot \Gamma_{\text{DReg}}^* \bigg|_{\rho_O \to 0} = \left( \mathcal{O} + \frac{\partial S_{\text{ft}}^{\gamma}}{\partial \rho_O} \right) \cdot \Gamma_{\text{DReg}}^* \bigg|_{\rho_O \to 0} \quad (3.18)
\]

The redundancy of the auxiliary couplings \( \rho_O \) has been motivated in a scalar-theory model in [19], and has been used separately in a fermion+scalar Yukawa theory in [20].
3.2 Algebraic properties of the RG equation; BRST invariance

One of the goals of proper perturbative renormalisation of a dimensionally-regularised theory is to ensure that BRST symmetry can be preserved at each loop (more exactly, at each $\hbar$) order, with a finite set of local finite BRST-restoring counterterms. Explicit calculations show that this is possible e.g. for the $\chi$QED model. This also signify that, at each order, this set of counterterms is sufficient to ensure the restoration of the BRST symmetry at the level of the whole 1-PI quantum effective action $\Gamma$.

However, this BRST restoration is generally possible only at the level of the 4-dimensional renormalised effective action $\Gamma$ and not for the dimensional-regularised effective action $\Gamma_{\text{DReg}}$ in its full generality. In explicit diagrammatic calculations, this appears as the fact that there exist residual non-local finite evanescent structures, and thus in $\Gamma_{\text{DReg}}$ as well, that therefore cannot be absorbed by local finite counterterms. These residual structures however vanish in the renormalised limit as defined above.

Once the BRST symmetry is restored, $\Gamma$ is now BRST invariant, i.e. in a functional form,

$$\mathcal{S} \Gamma = 0,$$

and as a consequence the RGE operator is linear-BRST invariant (BRST equation),

$$\mu \frac{\partial (\mathcal{S} \Gamma)}{\partial \mu} = 0 = \mathcal{S}_\Gamma \left( \mu \frac{\partial \Gamma}{\partial \mu} \right),$$

with the linearised BRST operator with respect to $\Gamma$,

$$\mathcal{S}_\Gamma (\mathcal{F}) = \int d^4 x \left( \frac{\delta \Gamma}{\delta \rho^\mu} \frac{\delta \mathcal{F}}{\delta A_\mu} + \frac{\delta \mathcal{F}}{\delta \rho^\mu} \frac{\delta \Gamma}{\delta A_\mu} + \frac{\delta \Gamma}{\delta R^i} \frac{\delta \mathcal{F}}{\delta \bar{\psi}_i} + \frac{\delta \mathcal{F}}{\delta \bar{R}^i} \frac{\delta \Gamma}{\delta \psi_i} + \frac{\delta \Gamma}{\delta R^c} \frac{\delta \mathcal{F}}{\delta \bar{c}} + B \frac{\delta \mathcal{F}}{\delta \bar{c}} \right).$$

More generally, the RGE possesses the same symmetries as $\Gamma$, and thus, similarly to BRST invariance, it also satisfies [22] the gauge-fixing condition,

$$\frac{\delta \Gamma}{\delta B} = \xi B + \partial^\mu A_\mu = 0 \quad \text{at all orders} \quad \rightarrow \quad \frac{\delta}{\delta B} \mu \frac{\partial \Gamma}{\partial \mu} = 0,$$

and the ghost equation,

$$\mathcal{G} \Gamma = 0 \quad \text{at all orders} \quad \rightarrow \quad \mathcal{G} \mu \frac{\partial \Gamma}{\partial \mu} = 0,$$

with the operator

$$\mathcal{G} = \frac{\delta}{\delta \bar{c}} + \partial^\mu \frac{\delta}{\delta \rho^\mu} \equiv \frac{\delta}{\delta \bar{c}},$$

where the last expression is valid for the functional dependence of $\Gamma$ on $\tilde{\rho}^\mu = \rho^\mu + \partial^\mu \bar{c}$ and $\bar{c}$. In the case of abelian models, e.g. $\chi$QED, these last equations are automatically verified since the ghost sector is very simple and does not receive any quantum corrections.

In the previous RG equations (3.14) and (3.16), the anomalous dimensions $\tilde{\gamma}_\phi$ were associated so far to the basic field-counting operators $N_\phi$ that are not BRST-invariant, and, in the multiplicative approach Eq. (3.14), the RGE also had explicit variations with respect to the auxiliary couplings. These two facts show that it was not explicit and evident, in that method, that the RGE in the renormalised limit is a BRST-invariant quantity. In this section we argued however that the RGE has to satisfy similar symmetry requirements as those of the theory (and thus the action), in the renormalised limit. In the next section, the RGE will be re-expressed in terms of explicit BRST-invariant quantities, based on the Algebraic Renormalisation framework.
Support for describing BRST invariance of the theory is implemented by introducing external BRST source fields in the tree-level action, that couple to the BRST transformations of the corresponding gauge/ghost and matter fields. These source fields should be present as well, via suitable field-counting operators, in the structure of the RGE, and so will provide separate contributions to it. This has not been done in e.g. [20]. However, these contributions with (at least one) BRST sources would decouple from the other RGE contributions without the sources, justifying a posteriori the possibility of studying an RGE without them.

4 Formulation of the RGE in Algebraic Renormalisation

In the Algebraic Renormalisation formulation, the form of the RGE is based on its linearity with respect to the 4-dimensional renormalised quantum effective action $\Gamma$, as well as the formal properties of the latter, namely, its BRST invariance, and the satisfying of the gauge-fixing condition and the ghost equation, as seen in the previous section.

As such, it is expected that $\mu \frac{\partial \Gamma}{\partial \mu}$ can be expanded in a basis of 4-dimensional operators, with ghost number $= 0$, that also satisfy these same constraints. We can easily see that $e \frac{\partial}{\partial e}$ already verifies such conditions, much like $\mu \frac{\partial}{\partial \mu}$, but it is not the case for the field-numbering operators $N_{\phi}$. Instead, one can show that the following expansion holds:

$$\mu \frac{\partial \Gamma}{\partial \mu} = \left( -\beta_{e} e \frac{\partial}{\partial e} + \sum_{\phi=A,\psi,c} \gamma_{\phi} N_{\phi} \right) \Gamma,$$

where the “curly” $N_{\phi}$ operators are BRST-invariant field-counting operators [17]. They are linear combinations of the basic field-numbering operators $N_{\phi}$ previously introduced, and are defined from their action on the tree-level action $S_{0}$, such that the functionals $N_{\phi} S_{0}$, for each $\phi$, are linear-BRST-invariants of the theory7:

$$N_{A} S_{0} = (N_{A} - N_{e} - N_{B} - N_{\rho} + 2\xi \frac{\partial}{\partial \xi}) S_{0} \equiv \overline{S_{\psi A\psi R}} + 2S_{AA} - S_{cc} - S_{pc} ,$$

$$N_{\psi} S_{0} = (N_{R}^{\psi} + N_{L}^{\psi} - N_{R} - N_{R}) S_{0} \equiv 2(\overline{S_{\psi \psi}} + S_{\psi A A R}) + \overline{S_{\psi \psi}},$$

$$N_{c} S_{0} = N_{c} S_{0} \equiv S_{cc} + S_{pc} + S_{Rc} + S_{cc R} .$$

Additionally, the action of these $N_{\phi}$ operators on the evanescent action $\hat{S}_{0}$ is:

$$N_{A} \hat{S}_{0} = 2\overline{S_{AA}} - \overline{S_{cc}} - \overline{S_{pc}} ,$$

$$N_{\psi} \hat{S}_{0} = \overline{S_{\psi \psi}} ,$$

$$N_{c} \hat{S}_{0} = \overline{S_{cc}} + \overline{S_{pc}} .$$

Their structure arise from the requirements exposed in Section 3.2. (i) BRST invariance implies that they will be differences of field-counting operators for a field $\phi$ and its associated BRST source $K_{\phi}$: $N_{\phi} - N_{K_{\phi}}$, or, in the (anti)ghost sector, as the sum: $N_{e} + N_{B}$ for the antighost and Nakanishi-Lautrup $B$ field. (ii) The requirement of satisfying the gauge and gauge-fixing equations implies, for the gauge-sector only, that $N_{A}$ not only contains the difference: $N_{A} - N_{\rho}$, but also: $-(N_{e} + N_{B})$ (with a relative sign), and a gauge-parameter dependence via $\xi \frac{\partial}{\partial \xi}$.

Notice that in $N_{A} S_{0}$ only, the gauge-fixing term dependence completely disappears:

$$\left( N_{A} + 2\xi \frac{\partial}{\partial \xi} - N_{B} \right) S_{g-fix} = 0 ,$$

7There is a sign typo in Eq.(4.7) of [17]: no global sign should be in front of the defining $N_{\phi}$ combination for $N_{\psi}$.
proving that $N_A S_0$ respects the gauge-fixing condition. However, when acting on some generic field monomials (like the counterterms), a non-trivial $\xi \frac{\partial}{\partial \xi}$ contribution will remain: this shows that knowing the gauge dependence of the involved quantities is of utmost importance, as we will see later.

### 4.1 Parameters-Variation of the RGE

We start by analysing the right-hand side of the RG equation (4.1). Since both $\epsilon \frac{\partial}{\partial \epsilon}$ and $N_\phi$ are linear differential vertex operators (DVOs), that are variations with respect to parameters (couplings) or fields naturally present in the action, their application on the renormalised effective action $\Gamma$ is, according to the Quantum Action Principle [37, 40, 41, 42, 12, 13, 14, 21, 22], equivalent to a zero-momentum insertion of a linear combination of local $d$-dimensional field-operators\(^8\) in $\Gamma$:

$$e \frac{\partial}{\partial e} \Gamma = N[e \frac{\partial}{\partial e} (S_0 + S_{\text{fct}})] \cdot \Gamma,$$

$$N_\phi \Gamma = N[N_\phi (S_0 + S_{\text{fct}})] \cdot \Gamma.$$  

(4.3a)

(4.3b)

This representation as vertex insertions is useful, as it makes contact with Feynman diagram calculations. One crucial observation is that the differential operators act on the non-singular component of the dimensional-regularised action, $S_0 + S_{\text{fct}}$, constituted by the full $d$-dimensional tree-level action $S_0$, and the finite BRST-restoring counterterms $S_{\text{fct}}$. Here, these finite counterterms should be promoted to their $d$-dimensional versions, to be used for insertion into dimensional-regularised loops. This promotion is not unique, and any choice made constitutes a choice of regularisation. Our choice, in the rest of this paper, is to keep the tensor structure of the finite counterterms to be the same as in 4 dimensions.

Since all the involved quantities have a perturbative $\hbar$ expansion: the (singular and) finite counterterms, and the renormalised action $\Gamma$, the previous equations become, truncated up to a given $\hbar^n$ order:

$$\left. e \frac{\partial}{\partial e} \Gamma \right|^{(n)} = N[e \frac{\partial}{\partial e} (S_0 + S_{\text{fct}})] \cdot \Gamma \bigg|^{(n)} \equiv N[e \frac{\partial S_0}{\partial e}] \cdot \Gamma^{(n)} + \sum_{k=1}^{n} N[e \frac{\partial S_{\text{fct}}^k}{\partial e}] \cdot \Gamma^{(n-k)},$$

$$N_\phi \Gamma^{(n)} = N[N_\phi (S_0 + S_{\text{fct}})] \cdot \Gamma^{(n)} + \sum_{k=1}^{n} N[N_\phi S_{\text{fct}}^k] \cdot \Gamma^{(n-k)}.$$  

(4.4a)

(4.4b)

The finite BRST-restoring counterterm action is expanded up to $\hbar^n$-order, $S_{\text{fct}}^{(n)} \equiv \sum_{k=1}^{n} S_{\text{fct}}^k$. Thus, up to order $\hbar^n$, the following equation holds ("RGE-Parameters-Variation"):

$$\frac{\partial \Gamma}{\mu} \bigg|^{(n)} = \left( -\beta_{\epsilon} N[\frac{\partial (S_0 + S_{\text{fct}}^{(n)})}{\partial \epsilon} \bigg] + \sum_{\phi=A,e,c} \gamma_{\phi} N[N_\phi (S_0 + S_{\text{fct}}^{(n)})] \bigg) \cdot \Gamma^{(n)}.$$  

(4.5)

We should point out that, besides the $\hbar$ expansion present in the counterterm operators and $\Gamma$, the $\beta_{\epsilon}$ and $\gamma_{\phi}$ coefficients are of $O(\hbar)$ and admit an $\hbar$-expansion, and thus, the knowledge of the finite counterterms only up to $\hbar^{n-1}$ (and not $n$) is necessary.

### 4.2 The $\mu$-dependence of $\Gamma$ in the RGE

We now turn our attention to the left-hand side of the RG equation (4.1). Contrary to the previous case where the operators $e \partial/\partial e$ and $N_\phi$ were variations with respect to parameters or fields present

---

\(^8\)The notation $N[O] \cdot \Gamma$ is the Zimmermann-like “normal product” definition (see [43, 21] and references within) of a renormalised local operator, defined as an insertion of a local operator $O$ in $\Gamma$, and followed by a minimal subtraction prescription [44] in the context of Dimensional Renormalisation.
in the tree-level action $S_0$, the “unit of mass” (renormalisation scale) $\mu$ introduced in dimensional regularisation is not a parameter of $S_0$, and therefore the Quantum Action Principle does not directly apply for the differential operator $\mu \partial / \partial \mu$.

The problem of expressing $\mu \partial \Gamma / \partial \mu$ as an insertion of normal product operators into the (renormalised) effective action $\Gamma$ was solved by Bonneau [39] (Section 4) and generalised in [15] (Eq. 78) to the case of different types of fields and external sources, evanescent contributions and finite counterterms, previously calculated from lower orders. Therefore, the only remaining divergence is sub-loop diagrams are subtracted: this is equivalent to consider them with insertions of singular 

\[ \int \mu \partial \Gamma / \partial \mu = \sum_{N_1 \geq 1} N_1 N[r.s.p.] \Gamma_{N_1 \text{loops}}^{D\text{Reg}} \cdot \Gamma. \]  

(4.6)

This equation establishes the formal connection between explicit loop calculations, where singular counterterms are extracted from sub-renormalised diagrams, with the corresponding beta-functions and anomalous dimensions in the RG equation in dimensional regularisation, as first being investigated by 't Hooft.

The notation $\Gamma_{N_1 \text{loops}}^{D\text{Reg}}$ requires some detailed explanation. This quantity designates 1-particle-irreducible (1-PI) Feynman diagrams having precisely $N_1$ loops, made from Feynman rules derived from the action $(S_0 + S_{\text{ct}})$, that are sub-renormalised (but not overall-subtracted). That is, their sub-loop diagrams are subtracted: this is equivalent to consider them with insertions of singular counterterms previously calculated from lower orders. Therefore, the only remaining divergence is an overall one that is only polynomial in momenta and masses.

In practice, the RGE (and thus, $\mu \partial \Gamma / \partial \mu$) has to be evaluated at some $\hbar^n$ order, and therefore the previous expression needs to be truncated. For this purpose, all $N_1$-loop diagrams of interest are enumerated, with ghost number $= 0$, for $N_1 = 1$ to $n$. (In addition, at $d = 4 - 2\epsilon$ these diagrams have a mass-dimension $\leq 4$, and therefore they can only have a finite upper bound on the number of external legs).

Let us sketch a $\hbar^2$-order calculation, the sum over $N_1$ running up to $N_1 = 2$ loops:

- A $N_1 = 2$-loop diagram, constructed in dimensional regularisation out of tree-level vertices from $S_0$, is of $\hbar^2$-order. This diagram is then sub-renormalised, by subtracting the divergences from each sub-loop: this is equivalent to adding its corresponding one-loop diagrams with one-loop counterterm insertions; however these diagrams are still counted as $N_1 = 2$ loop diagrams! Their remaining overall divergence is polynomial (thus, local). One then takes the residue of simple pole (r.s.p.) of their Laurent expansion in $\nu = 4 - d \equiv 2\epsilon$, i.e. the coefficient of the $1/\nu$ pole. This coefficient is therefore finite and does not depend explicitly on any powers of the dimensional regulator $\epsilon$. Since this contribution is of order $\hbar^2$, there is no need to insert it as a local operator back into $\Gamma$.

- Note however that a one-loop diagram with insertion of a “1-loop” ($\hbar^1$)-order finite counterterm $S_{\text{ct}}^{(1)}$ would have $N_1 = 1$ loops but counts as $\hbar^2$-order. The loop would again need to be subtracted, so that only an overall local divergence remains, and its r.s.p. can be extracted.

- Separately, there are genuine one-loop diagrams ($N_1 = 1$) constructed out of tree-level vertices from $S_0$ and thus counting as $\hbar^3$-order. There, their divergent part is already local and the corresponding r.s.p. can be extracted. But then this residue, interpreted as a local field operator, needs to be inserted into the overall $N[...].\Gamma$. This insertion does not need to be explicitly evaluated, however, except for insertions of evanescent operators. Sections 5.2.2 and 5.2.3 show how this is done.

It should be emphasised again, that in any of these previous points, the evaluation of r.s.p. $\Gamma_{N_1 \text{loops}}^{D\text{Reg}}$ will correspond to some singular counterterms that have been already evaluated as part of the necessary calculations for Section 2.3, since:

\[ \text{r.s.p. } \Gamma_{N_1 \text{loops}}^{D\text{Reg}} = -\text{r.s.p. } S_{\text{ct}}^{N_1 \text{loops}} \equiv -\text{r.s.p. } S_{\text{ct}}^{(m, N_1)}, \]

18
for some value \( m \) of their actual \( h \)-expansion, where \( m \geq N_i \).

### 4.3 RGE structure and overview of its resolution

The RG equation (4.1) for the renormalised effective action \( \Gamma \), acquires two equivalent forms: \( \mu \frac{\partial \Gamma}{\partial \mu} = (4.5) = (4.6) \), with both sides re-expressed as field-operator insertions in \( \Gamma \), namely

\[
N[\text{r.s.p. } \Gamma_D^\text{lopes}^{N_i}] \cdot \Gamma, \quad N[e^{\mu \frac{\partial (S_0 + S_{\text{ct}}^{(n)})}{\partial \mu}}] \cdot \Gamma, \quad N[\mathcal{N}_\phi (S_0 + S_{\text{ct}}^{(n)})] \cdot \Gamma. \tag{4.7}
\]

From there, it is possible to reformulate the RGE as a system of equations for the different \( \beta_\epsilon \) functions and anomalous dimensions \( \gamma_\phi \), if these insertions can be expanded into a common non-redundant operator basis.

The evaluation of the r.s.p. \( \Gamma_D^\text{lopes}^{N_i} \), as well as the finite part of the dimensional-regularised action,

\[
S_0 + S_{\text{ct}}^{(n)} = (\overline{S}_0 + \overline{S}_{\text{ct}}^{(n)}) + (S_0 - \overline{S}_0) + (S_{\text{ct}}^{(n)} - \overline{S}_{\text{ct}}^{(n)}) \equiv (\overline{S}_0 + \overline{S}_{\text{ct}}^{(n)}) + \delta S_0 + \delta S_{\text{ct}}^{(n)},
\]

generate 4-dimensional and evanescent structures, further inserted in \( \Gamma \). While the non-evanescent 4-dimensional insertions can be straightforwardly expanded into a basis of common 4-dimensional operators\(^9\) \( \overline{\mathcal{M}}_i \), the evanescent \( \delta S \) operator insertions are not linearly independent in full generality, and need to be further expanded into the \( \overline{\mathcal{M}}_i \) basis, by using Bonneau identities [39, 45]:

\[
N[\overline{\mathcal{M}}_j] \cdot \Gamma = \sum_i c_{ji} N[\overline{\mathcal{M}}_i] \cdot \Gamma. \tag{4.8}
\]

The coefficients \( c_{ji} \) in this expansion are at least of order \( h \).

The RGE-\( \mu \)-Dependence Eq. (4.6) (denoted by \( \mathfrak{R} \) in what follows) for \( \mu \frac{\partial \Gamma}{\partial \mu} \) can thus be expanded as:

\[
\mathfrak{R} \equiv \sum_{N_i} N_i N[\text{r.s.p. } \Gamma_D^\text{lopes}^{N_i}] \cdot \Gamma = \sum_i r_i N[\overline{\mathcal{M}}_i] \cdot \Gamma + \sum_j \delta r_j N[\delta \overline{\mathcal{M}}_j] \cdot \Gamma \equiv \sum_i r_i N[\overline{\mathcal{M}}_i] \cdot \Gamma, \tag{4.9a}
\]

with coefficients:

\[
r_i = r_i + \sum_j \delta r_j c_{ji}. \tag{4.9b}
\]

Furthermore, expanding the 4-dimensional insertions on the operator basis \( \overline{\mathcal{M}}_i \):

\[
N[e^{\mu \frac{\partial (\overline{S}_0 + \overline{S}_{\text{ct}}^{(n)})}{\partial \mu}}] \cdot \Gamma = \sum_i w_{ei} N[\overline{\mathcal{M}}_i] \cdot \Gamma, \tag{4.10a}
\]

\[
N[\mathcal{N}_\phi (\overline{S}_0 + \overline{S}_{\text{ct}}^{(n)})] \cdot \Gamma = \sum_i w_{\phi i} N[\overline{\mathcal{M}}_i] \cdot \Gamma, \tag{4.10b}
\]

and similarly for the evanescent insertions \( N[e^{\mu \frac{\partial (\delta S_0 + \delta S_{\text{ct}}^{(n)})}{\partial \mu}}] \cdot \Gamma, N[\mathcal{N}_\phi (\delta S_0 + \delta S_{\text{ct}}^{(n)})] \cdot \Gamma \) in terms of evanescent monomials \( N[\delta \overline{\mathcal{M}}_j] \cdot \Gamma \) (using coefficients \( w_{ej}, w_{\phi j} \) respectively), and using again the Bonneau identities,

\(^9\)Such basis of renormalised insertions is completely characterised by the corresponding classical basis [22]. If
\[
\{ \Delta^p \cdot \Gamma = \Delta^p_{\text{class}} + \mathcal{O}(h) \mid p = 1, 2, \ldots ; \dim(\Delta^p) \leq d \}
\]
is the set of insertions whose classical approximations form a basis for classical insertions up to dimension \( d \), then the same set is a basis for the quantum insertions bounded by \( d \). This means that a convenient choice for the set of monomials are the field operators contained in the tree-level action \( S_0 \), plus any new linearly-independent four-dimensional ones contained in the counterterms.
the RGE-Parameters-Variation Eq. (4.5) (denoted by \( \mathcal{W} \) in what follows) for \( \mu \frac{\partial \Gamma}{\partial \mu} \) can be expanded as:

\[
\mathcal{W} \equiv \sum_i \left( -\beta_e w_{ei} + \sum_{\phi=A,\psi,c} \gamma_\phi w_{\phi i} \right) N [\mathcal{M}_i] \cdot \Gamma,
\]

(4.11a)

with the coefficients:

\[
w_{ei} = w_{ei} + \sum_j \hat{w}_{ej} c_{ji}, \quad \text{and} \quad w_{\phi i} = w_{\phi i} + \sum_j \hat{w}_{\phi j} c_{ji}.
\]

(4.11b)

We observe here the influence of the evanescent operators, via the \( c_{ji} \) coefficients originating from the Bonneau Identity Eq. (4.8), on the \( r_i, w_{ei} \) and \( w_{\phi i} \) coefficients used to solve the RGE.

Finally, equating both forms of the RGE \( \mu \frac{\partial \Gamma}{\partial \mu} \), provides the overconstrained system of equations:

\[
\mathcal{R} = \mathcal{W} \Rightarrow r_i = -\beta_e w_{ei} + \sum_{\phi=A,\psi,c} \gamma_\phi w_{\phi i},
\]

(4.12)

from which the different \( \beta_e \) functions and the anomalous dimensions \( \gamma_\phi \) can be extracted at the given \( \hbar^n \) order. In the next section, we will apply this procedure to determine \( \beta_e \) and \( \gamma_\phi \) up to \( \hbar^2 \)-order.

5 The RGE Expansion up to “2-loop” (\( \hbar^2 \)) order

In this section, we analyse the \( \hbar \)-expansion of the two equivalent forms of the RG equation,

\[
\mu \frac{\partial \Gamma}{\partial \mu} \bigg|_2 = \mu \frac{\partial \Gamma}{\partial \mu} \bigg|_1 + \mu \frac{\partial \Gamma}{\partial \mu} \bigg|_2,
\]

(5.1)

then solve the resulting system of equations, from which the beta functions and anomalous dimensions are evaluated. The overconstrained nature of this system allows for ensuring consistency checks of the obtained solutions.

5.1 The “1-loop” (\( \hbar^1 \)) order RGE

Structure of the RGE-Parameters-Variation Eq. (4.5): Analysing Eq. (4.5) (terms \( \mathcal{W} \)) at \( \hbar \)-order, we observe that, since both the \( \beta_e \) and \( \gamma_\phi \) coefficients, and the finite BRST-restoring counterterms \( S_{\text{fct}} \), are at least of \( \mathcal{O}(\hbar) \), those latter ones can be neglected in the expansion of the equation. The insertions \( N[\beta(S_0+S_{\text{fct}})] \cdot \Gamma \) and \( N[\gamma_\phi(S_0+S_{\text{fct}})] \cdot \Gamma \) reduce to the tree-level values \( \frac{\partial S_0}{\partial e} \) and \( N_\phi \overline{S}_0 \), respectively, where \( \overline{S}_0 \equiv S_0^{(4D)} \) is the 4-dimensional restriction of the \( \chi \)QED tree-level action, Eq. (2.1a). The evanescent contributions in \( S_0 \) drop out, as we are studying the RGE in the renormalised \( d \rightarrow 4 \) limit: they would, from the Bonneau Identity Eq. (4.8), generate higher \( \hbar \)-order contributions. From these considerations, the contributing terms \( \mathcal{W} \) to the RGE at \( \mathcal{O}(\hbar) \) are:

\[
\mu \frac{\partial \Gamma}{\partial \mu} \bigg|_1 = -\beta_e^{(1)} e \frac{\partial S_0}{\partial e} + \sum_{\phi=A,\psi,c} \gamma_\phi^{(1)} N_\phi \overline{S}_0.
\]

(5.2)

Structure of the RGE-\( \mu \)-Dependence Eq. (4.6): Analysing Eq. (4.6) (terms \( \mathcal{R} \)) at \( \hbar \)-order, the calculation is limited to the usual \( N_l = 1 \)-loop 1-PI diagrams. These have been already calculated when evaluating the singular one-loop counterterms \( S_{\text{act}}^{(1)} \): indeed,

\[
\Gamma_{\text{DReg}}^{1\text{-loop}} = -S_{\text{act}}^{(1)} + \text{finite}.
\]
We stress here that BRST-restoring finite counterterms \( S^{(1)}_{\text{fct}} \) are excluded in the evaluation of \( \Gamma_{\text{1-loop DReg}} \): they would otherwise increase the \( \hbar \) order of such one-loop diagrams, resulting in higher \( \hbar \)-order contributions. Finally, as previously, we can ignore the evanescent contributions as we study the RGE in the \( d \rightarrow 4 \) limit. From these considerations, the contributing terms \( \mathfrak{R} \) to the RGE at \( \mathcal{O}(\hbar) \) are\(^{10}\):

\[
\left. \frac{\partial \Gamma}{\partial \mu} \right|_{\mu = 1} = \text{r.s.p.} \left( \frac{\Gamma_{\text{1-loop DReg}, \text{No } S^{(1)}_{\text{fct}}}}{\partial \mu} \right) \equiv -\text{r.s.p.} S^{(1)}_{\text{sct}}. \tag{5.3}
\]

\textbf{Resolution:} Equating both contributions \( \mathfrak{R} \) and \( \mathfrak{W} \) of the RG equation, evaluated in \( d = 4 - 2\epsilon \), we obtain:

\[
\left. \frac{\partial \Gamma}{\partial \mu} \right|_{\mu = 1} = -\beta_{e}^{(1)} e \frac{\partial S_{0}}{\partial e} + \sum_{\phi = A, \psi, c} \gamma_{\phi}^{(1)} N_{\phi} S_{0} = -\text{r.s.p.} S^{(1)}_{\text{sct}} \equiv -2\epsilon S^{(1)}_{\text{sct}}. \tag{5.4}
\]

Since the operators appear here directly (they are not insertions) and are not evanescent, it is possible to unambiguously define the beta-functions and the anomalous-dimensions. We need the singular one-loop counterterms \( S^{(1)}_{\text{sct}} \), Eq. (2.14a), from where we take their r.s.p., neglecting at this order the evanescent contributions.

The system of equations corresponding to the RGE is:

\[
\begin{align*}
S_{AA} & \rightarrow 2\gamma_{A}^{(1)} = \frac{2h}{16\pi^{2}} e^{2} \frac{2 \text{Tr}(Y_{R}^{2})}{3}, \quad (5.5a) \\
S_{\psi\psi R} & \rightarrow 2\gamma_{\psi}^{(1)} = \frac{2h}{16\pi^{2}} e^{2} \xi(Y_{R}^{2}), \quad (5.5b) \\
S_{\psi R A\psi R} & \rightarrow -\beta_{e}^{(1)} + \gamma_{A}^{(1)} + 2\gamma_{\psi}^{(1)} = \frac{2h}{16\pi^{2}} e^{2} \xi(Y_{R}^{2}), \quad (5.5c) \\
S_{c R c R}, S_{c R c} & \rightarrow -\beta_{c}^{(1)} + \gamma_{c}^{(1)} = 0, \quad (5.5d) \\
S_{Re\psi}, S_{c e R} & \rightarrow -\beta_{e}^{(1)} + \gamma_{c}^{(1)} = 0. \quad (5.5e)
\end{align*}
\]

This is an overdetermined system of equations that provides the following solutions for the \( \beta \)-functions and anomalous dimensions at one-loop level:

\[
\begin{align*}
\beta_{e}^{(1)} &= \gamma_{A}^{(1)} = \gamma_{c}^{(1)} = \frac{h}{16\pi^{2}} e^{2} \frac{2 \text{Tr}(Y_{R}^{2})}{3}, \quad (5.6a) \\
\gamma_{\psi}^{(1)} &= \frac{h}{16\pi^{2}} e^{2} \xi(Y_{R}^{2}). \quad (5.6b)
\end{align*}
\]

\textbf{5.2 The “2-loop” (\( \hbar^{2} \)) order RGE}

In this section, we discuss the \( \hbar^{2} \)-order structure of the RG equation, then evaluate each of its contributions, type by type.

\(^{10}\)Where \( S^{(1)}_{\text{sct}} \) was denoted \( S^{(1), 4D}_{\text{sct}} \) in Section 7.2 of [17].
5.2.1 Structure of the $h^2$-order RGE

Structure of the RGE-Parameters-Variation Eq. (4.5): We analyse the $h^2$ order of Eq. (4.5) (terms $\mathcal{M}$) and obtain:

$$\mu \frac{\partial \Gamma}{\partial \mu} = 2 = - \beta^{(1)}_e eN[\mu S_0^{(0)} \Gamma = 1 + \sum_{\phi = A, B, C} \gamma^{(1)}_\phi N[\phi, S_0^{(0)}] \Gamma = 1 \sim \mathcal{M}_1$$

$$- \beta^{(1)}_e eN[\mu S_0^{(0)} \Gamma = 1 + \sum_{\phi = A, B, C} \gamma^{(1)}_\phi N[\phi, S_0^{(0)}] \Gamma = 1 \sim + \mathcal{M}_2$$

$$- \beta^{(2)}_e e \frac{\partial S_0^{(1)}_{\text{fct}}}{\partial \epsilon} + \sum_{\phi = A, B, C} \gamma^{(1)}_\phi S_0^{(1)}_{\text{fct}} \sim + \mathcal{M}_3$$

$$- \beta^{(2)}_e e \frac{\partial S_0^{(0)}}{\partial \epsilon} + \sum_{\phi = A, B, C} \gamma^{(1)}_\phi N[\phi, S_0^{(0)}] \sim + \mathcal{M}_4.$$  

(5.7)

In the first two lines (terms $\mathcal{M}_1 + \mathcal{M}_2$), the $h$-order beta function $\beta^{(1)}_e$ and anomalous dimensions $\gamma^{(1)}_\phi$ from the previous section, multiply the insertion into one-loop diagrams of contributions from the tree-level action $\overline{S}_0$, and the evanescent tree-level action $\overline{S}_0$, respectively. In the third line (term $\mathcal{M}_3$), the finite BRST-restoring counterterms $S_{\text{fct}}^{(1)}$ are already of $h$ order, so they do not need to be inserted in loop diagrams, and are multiplied by the $h$-order $\beta^{(1)}_e$ and $\gamma^{(1)}_\phi$. In the last line (term $\mathcal{M}_4$), the genuine $h^2$-order beta function $\beta^{(2)}_e$ and anomalous dimensions $\gamma^{(2)}_\phi$ appear together with the 4-dimensional (non-evanescent) contributions from the tree-level action only, for the same reasons as explained in Section 5.1.

Structure of the RGE-$\mu$-Dependence Eq. (4.6): The $h^2$ order of Eq. (4.6) (terms $\mathfrak{R}$) now contains two types of terms: those of $N_1 = 1$ and $N_2 = 2$ loops 1-PI sub-renormalised diagrams. Its expansion reads:

$$\mu \frac{\partial \Gamma}{\partial \mu} = 2 = N[r.s.p. \Gamma_{\text{1-loop DReg, No S}_{\text{fct}}^{(1)}}] \Gamma = 1 + N[r.s.p. \Gamma_{\text{1-loop DReg, No S}_{\text{fct}}^{(1)}}] \Gamma = 1$$

$$+ r.s.p. S_{\text{fct}}^{(1)} \Gamma = 1 + 2(r.s.p. \Gamma_{\text{2-loops DReg, No S}_{\text{fct}}^{(1,2)}})$$

$$= \mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3 + \mathfrak{R}_4.$$  

(5.8)

We now discuss the nature of each of these terms:

- The first term $\mathfrak{R}_1$, corresponds to a standard 4-dimensional contribution from $\Gamma_{\text{1-loop DReg, No S}_{\text{fct}}^{(1)}}$, that is inserted in a larger loop to provide an overall $h^2$ contribution. As in Section 5.1, we exclude the presence of $S_{\text{fct}}^{(1)}$ vertices (“No $S_{\text{fct}}^{(1)}$” in the equation), as they would increase the $h$ order of that loop, resulting in a $h^3$-order term.

- The second term $\mathfrak{R}_2$, corresponds to the contributions from the evanescent component of $\Gamma_{\text{1-loop DReg, No S}_{\text{fct}}^{(1)}}$ that need to be inserted in a larger loop to provide a $h^2$ contribution, according to the Bonneau identities. For the same reasons as before, the $S_{\text{fct}}^{(1)}$ contributions need to be discarded at this order.

- The 1-PI diagrams $\Gamma_{\text{1-loop DReg, No S}_{\text{fct}}^{(1)}}$ without $S_{\text{fct}}^{(1)}$ are related to the one-loop singular 4-dimensional and evanescent counterterms, by:

$$\Gamma_{\text{1-loop DReg, No S}_{\text{fct}}^{(1)}} + \Gamma_{\text{1-loop DReg, No S}_{\text{fct}}^{(1)}} = \Gamma_{\text{1-loop DReg, No S}_{\text{fct}}^{(1)}} = -S_{\text{act}}^{(1)} - S_{\text{act}}^{(1)} + \text{finite} \equiv -S_{\text{act}}^{(1)} + \text{finite}.$$  

(5.9a)
• The third term $\mathcal{R}_3$, corresponds to the contribution from $\Gamma^{1\text{-loop}}_{\text{DReg}}$ that now contains one insertion of the finite BRST-restoring counterterm $S^{(1)}_{\text{fct}}$. This term is therefore of overall order $\hbar^2$, thus, does not need to be (re-)inserted in a larger loop. It constitutes part of the diagrammatic evaluation of the two-loop singular counterterms, whose corresponding diagrams are shown in Fig. 2 of Appendix C. Its divergent part, captured by the r.s.p., corresponds to the 4-dimensional projection of its corresponding counterterm $S^{(2,1)}_{\text{act}}$, Eq. (2.15b):

$$\left( S^{(1)}_{\text{fct}} \cdot \Gamma^{1\text{-loop}} \right)^{\text{div}} = -S^{(2,1)}_{\text{act}}. \quad (5.9b)$$

• Finally, the last term $\mathcal{R}_4$, corresponds to pure sub-renormalised two-loop diagrams, that size as $\hbar^2$: actual two-loop diagrams where their sub-loops are rendered finite with lower-order singular counterterms; this corresponds to considering the actual two-loop diagrams together with one-loop diagrams with insertions of the $S^{(1)}_{\text{act}}$ counterterms, Eq. (2.14a). In effect, we need here the pure two-loop counterterms, Eq. (2.15a),

$$\Gamma^{2\text{-loops, div}}_{\text{DReg}, \text{No } S^{(1)}_{\text{fct}}} = -S^{(2,2)}_{\text{act}}. \quad (5.9c)$$

Notice that on the contrary, one-loop diagrams with one insertion of finite counterterms $S^{(1)}_{\text{fct}}$ are genuine one-loop diagrams (even though they scale as $\hbar^2$) and therefore need to be excluded from this contribution.

Hence, this $\hbar^2$-order equation (5.8) can be re-expressed in terms of already-known quantities, as follows:

$$\left. \frac{\partial \Gamma}{\partial \mu} \right|_{\mu=2} = - N[\text{r.s.p. } S^{(1)}_{\text{act}}] \cdot \Gamma = - N[\text{r.s.p. } S^{(1)}_{\text{act}}] \cdot \Gamma^{1\text{-loop}} - N[\text{r.s.p. } S^{(1)}_{\text{act}}] \cdot \Gamma = - N[\text{r.s.p. } S^{(1)}_{\text{act}}] \cdot \Gamma^{1\text{-loop}} - 2(\text{r.s.p. } S^{(2,2)}) = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4. \quad (5.10)$$

In the following sub-sections, we study in pairs the following contributions: $\mathfrak{M}_1$ with $\mathcal{R}_1$, $\mathfrak{M}_2$ with $\mathcal{R}_2$, $\mathfrak{M}_3$ with $\mathcal{R}_3$, and $\mathfrak{M}_4$ with $\mathcal{R}_4$, as defined above.

### 5.2.2 4-dimensional insertions

We start by studying the contribution $\mathfrak{M}_1$ from the Eqs. (4.5) and (5.7), arising from the insertion of the 4-dimensional action $\overline{S}_0$, which is:

$$\mathfrak{M}_1 = -\beta^{(1)}_e N[\frac{\partial S_0}{\partial \phi}] \cdot \Gamma^{1\text{-loop}} = -\sum_{\phi=A,\psi,c} \gamma^{(1)}_\phi N[\bar{N}_{\phi} \overline{S}_0] \cdot \Gamma^{1\text{-loop}}$$

$$= 2\gamma^{(1)}_A N[S_{AA}] \cdot \Gamma^{1\text{-loop}} + \sum_i 2\gamma^{(1)}_{\psi_i} N[S_{\bar{\psi}\psi}] \cdot \Gamma^{1\text{-loop}} + \sum_i \left(-\beta^{(1)}_e + \gamma^{(1)}_A + 2\gamma^{(1)}_{\psi_i} \right) N[S_{\bar{\psi}A\psi_R}] \cdot \Gamma^{1\text{-loop}}$$

$$+ \left(-\gamma^{(1)}_A + \gamma^{(1)}_c \right) N[S_{Cc} + S_{\psi e}] \cdot \Gamma^{1\text{-loop}} + \left(-\beta^{(1)}_e + \gamma^{(1)}_c \right) N[S_{\psi c R} + S_{\bar{\psi} e R}] \cdot \Gamma^{1\text{-loop}}$$

$$= 2\gamma^{(1)}_A N[S_{AA}] \cdot \Gamma^{1\text{-loop}} + 2\sum_i \gamma^{(1)}_{\psi_i} N[S_{\bar{\psi}\psi} + S_{\bar{\psi}A\psi_R}] \cdot \Gamma^{1\text{-loop}}. \quad (5.11a)$$

The insertions of terms containing ghosts do not contribute, because (i) in QED they are not insertable in loop diagrams, and (ii) from the one-loop RGE solutions Eq. (5.6), $\beta^{(1)}_e = \gamma^{(1)}_A = \gamma^{(1)}_c$, so their coefficients cancel. The coefficient of $N[S_{\bar{\psi}A\psi_R}] \cdot \Gamma^{1\text{-loop}}$ simplifies also for the same reason.
On the other hand, using Eq. (2.14a), the contribution \( \mathcal{R}_1 \) from Eqs. (4.6) and (5.10), arising from the insertion of the 4-dimensional component of the one-loop singular counterterms, is:

\[
\mathcal{R}_1 = N[-\text{r.s.p.} S_{\text{act}}^{(1)}] \cdot \Gamma = \frac{\hbar}{16\pi^2} \frac{4\epsilon^2}{3} \text{Tr}(S_R^2) N[S_{\text{AA}}] \cdot \Gamma = \frac{\hbar}{16\pi^2} 2\epsilon^2 \xi \sum_j (Y_R^j)^2 N[S_{\psi\psi R}] + \frac{S_{\psi R}^2}{N[S_{\psi R}A\psi R]} \cdot \Gamma = 1.
\]

We observe that the results for \( \mathcal{W}_1 \) and \( \mathcal{R}_1 \) are equal. This analysis is valid and consistent with the \( \mathcal{O}(\hbar) \) one-loop RGEs obtained in Section 5.1, since these non-evanescent insertions constitute an operator basis.

### 5.2.3 Evanescent insertions

In this section we study the contributions \( \mathcal{W}_1 \) and \( \mathcal{R}_1 \) arising respectively, from the insertions of the evanescent component of the tree-level action: \( \hat{S}_0 \), Eq. (2.7), and from the one-loop singular counterterms: \( \hat{S}_{\text{act}}^{(1)} \), Eq. (2.14a). The calculation is done using Bonneau identities [39, 45], whose details are found in Appendix B, and the results are collected below. The following insertions\(^{11}\):

\[
N[S_{\text{cc}}] \cdot \Gamma = 0, \quad N[S_{\text{pc}}] \cdot \Gamma = 0,
\]

both vanish in \( \chi \)QED, as well as in a generic abelian \( U(1) \) theory: in those theories, the ghost field does not couple, thus neither \( \hat{S}_{\text{cc}} \) nor \( \hat{S}_{\text{pc}} \) can be inserted in diagrams in a 1-particle-irreducible manner. The only non-vanishing insertions are:

\[
N[S_{\psi\psi}] \cdot \Gamma = \frac{\hbar}{16\pi^2} \frac{e^2}{3} \left\{ 4 \text{Tr}(S_R^2) S_{\text{AA}} + \sum_j (Y_R^j)^2 \left( \frac{5\epsilon - 1}{2} S_{\psi\psi R} + (3\epsilon + 1) S_{\psi R A\psi R} \right) \right\},
\]

\[\text{(5.13a)}\]

\[
N[S_{\text{AA}}] \cdot \Gamma = \frac{\hbar}{16\pi^2} e^2 \sum_j (Y_R^j)^2 \left\{ 2 S_{\psi\psi R} + S_{\psi R A\psi R} \right\},
\]

\[\text{(5.13b)}\]

\[
N[\int d^d x \left( \frac{1}{2} \tilde{A}_\mu \partial^2 \tilde{A}^\mu \right)] \cdot \Gamma = \frac{\hbar}{16\pi^2} e^2 \sum_j (Y_R^j)^2 \left\{ \left( \frac{7\epsilon + 1}{12} + \frac{(\epsilon - 1)^2}{40} \right) S_{\psi\psi R} \right.
\]
\[
+ \left. \frac{3\epsilon - 1}{6} + \frac{(\epsilon - 1)^2}{5} \right\} S_{\psi R A\psi R} \right\},
\]

\[\text{(5.13c)}\]

One does not need to evaluate the insertion of \( \hat{S}_{\text{fix}} \) contained in \( N_A \hat{S}_0 \), since, due to the gauge-fixing condition Eq. (3.21a), no explicit gauge-fixing term appears in the RG equations.

Since the evanescent tree-level action \( \hat{S}_0 \) does not contain any term depending on the coupling constant \( e \), one has \( \frac{\partial \hat{S}_0}{\partial e} = 0 \). Collecting the above results together, and taking into account that \( Y_R \) is diagonal, the \( \mathcal{W}_2 \) contribution from Eq. (5.7) becomes:

\[
\mathcal{W}_2 = \sum_{\phi = A, \psi, c} \gamma_\phi^{(1)} N[S_{\phi\hat{S}_0}] \cdot \Gamma = 2 \gamma_A^{(1)} N[S_{\text{AA}}] \cdot \Gamma \Gamma = 1 + \sum_i \gamma_{\psi i}^{(1)} N[S_{\psi\psi}] \cdot \Gamma \Gamma = 1 + \left( -\gamma_A^{(1)} + \gamma_c^{(1)} \right) N[S_{\text{cc}} + S_{\text{pc}}] \cdot \Gamma = 1
\]

\[
= \frac{\hbar}{16\pi^2} 4\epsilon^2 \text{Tr}
\]

\[
\left( \gamma_A^{(1)} Y_{\psi i}^{(1)} \right) \frac{S_{\text{AA}}}{3} + \frac{\hbar}{16\pi^2} \frac{e^2}{3} \gamma_{\psi i}^{(1)} (Y_R^j)^2 \left\{ 2 S_{\psi R}^2 + S_{\psi R A\psi R} \right\}
\]

\[\text{(5.13a)}\]

\[
\frac{\hbar}{16\pi^2} \frac{e^2}{3} \sum_j \gamma_{\psi j}^{(1)} (Y_R^j)^2 \left( \frac{5\epsilon - 1}{2} S_{\psi R}^2 + (3\epsilon + 1) S_{\psi R A\psi R} \right).
\]

\[\text{(5.13b)}\]

\[\text{(5.13c)}\]

\(^{11}\)The only possible non-zero insertion could be the ghost one-loop vacuum bubble from \( N[S_{\text{cc}}] \cdot \Gamma = 1 \), but it would vanish nevertheless in dimensional regularisation because of its zero mass.
Using the one-loop RGE results Eq. (5.6), and in Feynman gauge $\xi = 1$, one obtains:

$$\mathfrak{m}_2 \leftarrow \left( \frac{\hbar}{16\pi^2} \right)^2 \frac{4e^4}{3} \frac{\text{Tr}(\mathcal{Y}^2_R)}{\mathcal{S}_{AA}} + \left( \frac{\hbar}{16\pi^2} \right)^2 \frac{4e^4}{9} \frac{\text{Tr}(\mathcal{Y}^2_R)}{3} \sum_j (\mathcal{Y}^j_R)^2 \left\{ 2\hat{s}_{\psi_R}^j + \hat{s}_{\psi_R A\psi_R}^j \right\}$$

$$+ \left( \frac{\hbar}{16\pi^2} \right)^2 \frac{2e^4}{3} \sum_j (\mathcal{Y}^j_R)^4 \left( \hat{s}_{\psi_R}^j + 2\hat{s}_{\psi_R A\psi_R}^j \right). \quad (5.14a)$$

Using the evanescent component of $S^{(1)}_{\text{sct}}$, Eq. (2.14a), the $\mathfrak{r}_2$ contribution from Eq. (5.10) is:

$$\mathfrak{r}_2 = N[-\text{r.s.p.} \overline{S}^{(1)}_{\text{sct}}] \cdot \Gamma = \left( \frac{\hbar}{16\pi^2} \right)^2 \frac{2e^2}{3} \frac{\text{Tr}(\mathcal{Y}^2_R)}{N} \int d^4 x \left( \frac{1}{2} \hat{A}_\mu \hat{\partial}^2 \hat{A}_\mu \right) \cdot \Gamma$$

$$\left( \frac{\hbar}{16\pi^2} \right)^2 \frac{2e^2}{9} \sum_j (\mathcal{Y}^j_R)^2 \left\{ 2\hat{s}_{\psi_R}^j + \hat{s}_{\psi_R A\psi_R}^j \right\}. \quad (5.14b)$$

We observe that there is no direct correspondence between the $\mathfrak{m}_2$ and $\mathfrak{r}_2$ contributions of the RGE with $\hat{S}_0$ and $S^{(1)}_{\text{sct}}$ insertions. This signify that these contributions have to mix together with other ones in order to give the final result satisfying the RGE.

### 5.2.4 Finite symmetry-restoring counterterm contributions

We now study the $\mathfrak{m}_3$ contribution from Eq. (5.7), arising from the action of the differential operators $e \partial / \partial e$ and $\mathcal{N}_0$ on the finite symmetry-restoring counterterms $S^{(1)}_{\text{sct}}$, Eq. (2.14d). Its gauge-dependent term, associated with the $\hat{s}_{\psi_R}^j$ operator, is of great importance, since the action of the operator $\mathcal{N}_A$ generates, via $\xi \frac{\partial}{\partial \xi},$ an extra gauge-parameter-dependent contribution. The terms of interest are:

$$\mathcal{N}_A S^{(1)}_{\text{sct}} = \mathcal{N}_A S^{(1)}_{\text{sct}} + 2\xi \frac{\partial}{\partial \xi} S^{(1)}_{\text{sct}} = \mathcal{N}_A S^{(1)}_{\text{sct}} + \xi \frac{\partial}{\partial \xi} S^{(1)}_{\text{sct}}$$

$$= \left( \frac{\hbar}{16\pi^2} \right)^2 \frac{e^2}{3} \int d^4 x \left( - \text{Tr}(\mathcal{Y}^2_R) \hat{A}_\mu \hat{\partial}^2 \hat{A}_\mu + e^2 \text{Tr}(\mathcal{Y}^2_R)(\hat{A}^2) \right) \quad (5.15a)$$

$$+ \left( \frac{\hbar}{16\pi^2} \right)^2 \frac{e^2}{3} \sum_j (\mathcal{Y}^j_R)^2 \hat{s}_{\psi_R}^j,$$

$$\mathcal{N}_\psi S^{(1)}_{\text{sct}} = (\mathcal{N}_0 - \mathcal{N}_A) S^{(1)}_{\text{sct}} = \left( \frac{\hbar}{16\pi^2} \right)^2 \frac{\xi}{6} \frac{5}{2} e^2 \sum_j (\mathcal{Y}^j_R)^2 \hat{s}_{\psi_R}^j, \quad (5.15b)$$

$$\mathcal{N}_{\text{c-}} S^{(1)}_{\text{sct}} = 0, \quad (5.15c)$$

$$e \frac{\partial S^{(1)}_{\text{sct}}}{\partial e} = \left( \frac{\hbar}{16\pi^2} \right)^2 \frac{e^2}{3} \int d^4 x \left( - \text{Tr}(\mathcal{Y}^2_R) \hat{A}_\mu \hat{\partial}^2 \hat{A}_\mu + e^2 \text{Tr}(\mathcal{Y}^2_R)(\hat{A}^2) \right)$$

$$\left( \frac{\hbar}{16\pi^2} \right)^2 \frac{e^2}{3} \sum_j (\mathcal{Y}^j_R)^2 \hat{s}_{\psi_R}^j$$

$$\equiv N_A S^{(1)}_{\text{sct}} + \mathcal{N}_\psi S^{(1)}_{\text{sct}}. \quad (5.15d)$$
Using the equality $\beta_e^{(1)} = \gamma_A^{(1)}$, the $\mathcal{M}_3$ contribution becomes:

$$
\mathcal{M}_3 = -\beta_e^{(1)} e \frac{\partial S_{\text{ct}}^{(1)}}{\partial e} + \sum_{\phi=A,\psi,c} \gamma_{\phi}^{(1)} N_{\phi} S_{\text{ct}}^{(1)}
$$

$$
= \left( -\beta_e^{(1)} + \gamma_A^{(1)} \right) N_A S_{\text{ct}}^{(1)} + \sum_i \left( -\beta_e^{(1)} + \gamma_A^{(1)} \xi \frac{\partial}{\partial \xi} + \gamma_{\psi_i}^{(1)} \right) N_{\psi_i} S_{\text{ct}}^{(1)}
$$

$$
= \sum_i \left( \gamma_A^{(1)} \left( \xi \frac{\partial}{\partial \xi} - 1 \right) + \gamma_{\psi_i}^{(1)} \right) N_{\psi_i} S_{\text{ct}}^{(1)}
$$

$$
= \left( 2 e \right)^2 \frac{\hbar}{16 \pi^2} \sum_j \left( -5 \text{Tr}(Y_R^j)^2 (Y_R^j)^2 + \frac{\xi + 5}{6} \text{Tr}(Y_R^j)^4 \right) S_{\psi\psi R}^{2}
$$

$$
\frac{\xi=1}{\hbar} \left( \hbar \right)^2 \left( 2 e \right)^4 \sum_j \left( -\text{Tr}(Y_R^j)^2 (Y_R^j)^2 + (Y_R^j)^4 \right) S_{\psi\psi R}^{2}.
$$

The “pure-photon” BRST-breaking structures disappear from the equation, due to the equality between $\beta_e^{(1)}$ and $\gamma_A^{(1)}$. The gauge-dependent term has the effect of removing the gauge dependence in the $\text{Tr}(Y_R^j)^2 \sum_j (Y_R^j)^2$ term.

The contribution $\mathcal{R}_3$ from Eq. (5.10), arises from the one-loop diagrams with one insertion of the finite counterterms $S_{\text{ct}}^{(1)}$, corresponding to the 4-dimensional component of the two-loop singular counterterms $S_{\text{ct}}^{(2,1)}$:

$$
\mathcal{R}_3 = -\text{r.s.p.} S_{\text{ct}}^{(2,1)} = \text{r.s.p.} S_{\text{ct}}^{(1)} \cdot \Gamma = 1
$$

$$
\xi=1 \left( \hbar \right)^2 \left( 2 e \right)^4 \frac{1}{3} \left\{ 4 \text{Tr}(Y_R^j)^2 S_{AA} + 3 \sum_j (Y_R^j)^4 \left[ S_{\psi\psi R}^{2} + 2 S_{\psi R A \psi R}^{2} \right] \right\}
$$

$$
- \text{Tr}(Y_R^j)^2 \sum_j (Y_R^j)^2 \left( S_{\psi\psi R}^{2} + S_{\psi R A \psi R}^{2} \right).
$$

We observe again that there is no direct correspondence between the $\mathcal{M}_3$ and $\mathcal{R}_3$ contributions of the RGE with $S_{\text{ct}}^{(1)}$ and $S_{\text{ct}}^{(2,1)}$.

### 5.2.5 Genuine $h^2$-induced contributions

We terminate our study with the $\mathcal{M}_4$ contribution from Eq. (5.7), function of the genuine $h^2$-order beta function $\beta_e^{(2)}$ and anomalous dimensions $\gamma_{\phi}^{(2)}$, and arising from the action of $e \partial / \partial e$ and $N_{\phi}$ on $S_0$. Following the same procedure as in the one-loop case, the $\mathcal{M}_4$ contribution is:

$$
\mathcal{M}_4 = -\beta_e^{(2)} e \frac{\partial S_0}{\partial e} + \sum_{\phi=A,\psi,c} \gamma_{\phi}^{(2)} N_{\phi} S_0
$$

$$
= 2 \gamma_A^{(2)} S_{AA} + \sum_i 2 \gamma_{\psi_i}^{(2)} S_{\psi\psi} + \sum_i \left( -\beta_e^{(2)} + \gamma_A^{(2)} + 2 \gamma_{\psi_i}^{(2)} \right) \frac{S_{\psi R A \psi R}}{S_{\psi R}} + \left( -\gamma_A^{(2)} + \gamma_c^{(2)} \right) \left( S_{\psi\psi c} + S_{\psi \psi e} \right) + \left( -\beta_e^{(2)} + \gamma_c^{(2)} \right) \left( S_{R c \psi} + S_{\psi \psi R} \right).
$$

The $\mathcal{R}_4$ contribution from Eq. (5.10), arises from the 4-dimensional component of the pure two-loop singular counterterms $S_{\text{ct}}^{(2,2)}$, Eq. (2.15a) (thus ignoring any evanescent contribution like $\int x \tilde{A}_{\mu} \tilde{\partial}^{2} \tilde{A}_{\mu}$),
from where we evaluate:

\[
\mathcal{R}_4 = -2 \text{r.s.p. } S^{(2,2)}_{\text{act}} = \left( \frac{\hbar}{16\pi^2} \right)^2 4e^4 \frac{3}{4} \text{Tr}(Y_R^4) S_{AA} - \left( \frac{\hbar}{16\pi^2} \right)^2 4e^4 \sum_j (Y_R^j)^2 \left[ -\frac{7}{4} (Y_R^j)^2 + \frac{2}{3} \text{Tr}(Y_R^j) \right] \left( S_{\bar{\psi}\psi_R}^L + S_{\bar{\psi}_R A\psi_R}^j \right) - \left( \frac{\hbar}{16\pi^2} \right)^2 4e^4 \sum_j (Y_R^j)^2 \left( \frac{1}{2} (Y_R^j)^2 + \frac{2}{3} \text{Tr}(Y_R^j) \right) S_{\bar{\psi}\psi_R}^j .
\]

(5.17b)

Note that the r.s.p. (residue of simple pole) operation only picks the coefficients of the $1/\epsilon$ poles, and not those of higher poles $1/\epsilon^2$.

From the analysis of all these previous results, we see that at two-loop order the ghost terms do not receive corrections as well, therefore, similarly to the one-loop case, the following equations must be satisfied:

\[
\beta^{(2)}_c = \gamma^{(2)}_A = \gamma^{(2)}_c ,
\]

leading to the conclusion that in $\mathcal{M}_4$, both terms $S_{\bar{\psi}\psi}$ and $S_{\bar{\psi}_R A\psi_R}$ must have the same coefficient: $2\gamma^{(2)}_c$. Since $\mathcal{R}_4$ does not satisfy this condition, there cannot be a direct equality between $\mathcal{M}_4$ and $\mathcal{R}_4$. This means that the contributions coming from the one-loop singular evanescent ($\mathcal{M}_2$, $\mathcal{R}_2$) and the finite counterterms ($\mathcal{M}_3$, $\mathcal{R}_3$) must combine with those $\mathcal{M}_4$, $\mathcal{R}_4$ contributions, in such a way that the overall RGE is fulfilled.

5.3 Summary of the contributions; Resolution

In the previous sections, we evaluated, in Feynman gauge ($\xi = 1$), each of the two equivalent forms of the RG equation (4.1), namely, the contributions $\mathcal{R}_i$ from Eq. (4.6), and the contributions $\mathcal{M}_i$ from Eq. (4.5). We request that the $\hbar^2$-order of the RG equation satisfies the system (4.12):

\[
\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4 = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 .
\]

(5.19a)

The contributions $\mathcal{R}_1$ and $\mathcal{M}_1$ from the 4-dimensional insertions, are already equal to each other: $\mathcal{R}_1 = \mathcal{M}_1$. We can therefore isolate the contribution containing the genuine $\hbar^2$ beta functions and anomalous dimensions: $\mathcal{M}_4$, Eq. (5.17a), that must satisfy:

\[
\mathcal{M}_4 = \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4 - \mathcal{M}_2 - \mathcal{M}_3 .
\]

(5.19b)

Each of these contributions are summarised in Table 1 by rows, exhibiting how they individually account to each of the different independent 4-dimensional operators and their related combinations of $\beta$ and $\gamma_\phi$ functions (columns). For example, in non-chiral theories like QED or QCD, only the $\mathcal{R}_4$ contribution (from the $1/(d-4)$ coefficients of the pure 2-loop $\hbar^2$-order singular counterterms) would define the $\beta$ and $\gamma_\phi$ functions, as expected, while all the other contributions would vanish.

From this, a system of equations is obtained, one for each independent 4-dimensional operator that belongs to the operator basis (equivalently found by summing separately each column of the table):

\[
\begin{align*}
S_{AA} & \rightarrow 2\gamma^{(2)}_A = \left( \frac{\hbar}{16\pi^2} \right)^2 4e^4 \text{Tr}(Y_R^4) , \\
S_{\bar{\psi}\psi_R} & \rightarrow 2\gamma^{(2)}_\psi = - \left( \frac{\hbar}{16\pi^2} \right)^2 \frac{2e^4}{3} \left( \frac{2}{3} \text{Tr}(Y_R^2)(Y_R^j)^2 + \frac{9}{2}\left(Y_R^j\right)^4 \right) ,
\end{align*}
\]

(5.20a) (5.20b)
### Contributions to operators from (normalised) $\mathfrak{M}_4$:

\[
\left( \frac{\hbar}{16\pi^2} \right)^2 e^{-4} \times \left( -\beta_e^{(2)} e^\partial_e + \sum_\phi \gamma_\phi^{(2)} N_\phi \right) \overline{S}_0
\]

| Contrib. from $(\mathfrak{R}_i - \mathfrak{M}_i)$ | $S_{AA}$ | $S_{cc} + S_{pc}$ | $S_{R\psi} + S_{\bar{c}cR}$ |
|------------------------------------------------|---------|-----------------|-----------------------------|
| $-\mathfrak{M}_2 = - \left( -\beta_e^{(1)} N[e\partial_e \bar{S}_0] \cdot \Gamma = 1 \right) + \sum_\phi \gamma_\phi^{(1)} N[N_\phi \bar{S}_0] \cdot \Gamma = 1 \right) \rightarrow$ | $-4 \text{Tr}(Y_R^2)$ | $0$ | $0$ |
| $-\mathfrak{M}_3 = - \left( -\beta_e^{(1)} e^\partial_e S_{\text{flt}}^{(1)} \right) + \sum_\phi \gamma_\phi^{(1)} N_\phi S_{\text{flt}}^{(1)} \rightarrow$ | $0$ | $0$ | $0$ |
| $\mathfrak{R}_2 = N[-\text{r.s.p.} S_{\text{flt}}^{(1)}] \cdot \Gamma = 1 \rightarrow$ | $0$ | $0$ | $0$ |
| $\mathfrak{R}_3 = -\text{r.s.p.} S_{\text{flt}}^{(2,1)} \rightarrow$ | $-8 \text{Tr}(Y_R^2)$ | $0$ | $0$ |
| $\mathfrak{R}_4 = -2 \text{r.s.p.} S_{\text{flt}}^{(2,2)} \rightarrow$ | $24 \text{Tr}(Y_R^2)$ | $0$ | $0$ |

### Contributions to operators from (normalised) $\mathfrak{M}_4$:

\[
\left( \frac{\hbar}{16\pi^2} \right)^2 e^{-4} \times \left( -\beta_e^{(2)} e^\partial_e + \sum_\phi \gamma_\phi^{(2)} N_\phi \right) \overline{S}_0
\]

| Contrib. from $(\mathfrak{R}_i - \mathfrak{M}_i)$ | $S_{\overline{\psi}_R \psi_R}$ | $S_{\overline{\psi}_R A \psi_R}$ |
|------------------------------------------------|-----------------|-----------------------------|
| $-\mathfrak{M}_2 = - \left( -\beta_e^{(1)} N[e\partial_e \bar{S}_0] \cdot \Gamma = 1 \right) + \sum_\phi \gamma_\phi^{(1)} N[N_\phi \bar{S}_0] \cdot \Gamma = 1 \right) \rightarrow$ | $-2 \left( 4 \text{Tr}(Y_R^2) \right) (Y_R) + 3(\psi_R)^2 \left( \frac{4}{3} \text{Tr}(Y_R^2) \right) (Y_R)^2 + (\psi_R)^4$ | $-4 \left( \frac{4}{3} \text{Tr}(Y_R^2) \right) (\psi_R)^2 + (\psi_R)^4$ |
| $-\mathfrak{M}_3 = - \left( -\beta_e^{(1)} e^\partial_e S_{\text{flt}}^{(1)} \right) + \sum_\phi \gamma_\phi^{(1)} N_\phi S_{\text{flt}}^{(1)} \rightarrow$ | $-2 \left( -5 \text{Tr}(Y_R^2) \right) (Y_R)^2 + (\psi_R)^4$ | $0$ |
| $\mathfrak{R}_2 = N[-\text{r.s.p.} S_{\text{flt}}^{(1)}] \cdot \Gamma = 1 \rightarrow$ | $4 \text{Tr}(Y_R^2)$ | $2 \text{Tr}(Y_R^2)$ |
| $\mathfrak{R}_3 = -\text{r.s.p.} S_{\text{flt}}^{(2,1)} \rightarrow$ | $\frac{2}{3} \left( \text{Tr}(Y_R^2) \right) (Y_R)^2 - 3(\psi_R)^4$ | $\frac{2}{3} \left( \text{Tr}(Y_R^2) \right) (Y_R)^2 - 6(\psi_R)^4$ |
| $\mathfrak{R}_4 = -2 \text{r.s.p.} S_{\text{flt}}^{(2,2)} \rightarrow$ | $\frac{2}{3} \left( -8 \text{Tr}(Y_R^2) \right) (Y_R)^2 + \frac{5}{2}(\psi_R)^4$ | $\frac{2}{3} \left( -4 \text{Tr}(Y_R^2) \right) (Y_R)^2 + \frac{7}{2}(\psi_R)^4$ |

Table 1: Contributions to the gauge (upper table) and fermionic (lower table) operators of the genuine $\hbar^2$-order term $\mathfrak{M}_4 = \left( -\beta_e^{(2)} e^\partial_e + \sum_\phi \gamma_\phi^{(2)} N_\phi \right) \overline{S}_0$ from the $(\mathfrak{R}_i - \mathfrak{M}_i)$ combination of the RG equation.

\[
\begin{align*}
\mathfrak{M}_4 &= \text{overdetermined system of equations that provides the following solutions for the } \beta \text{-functions}
\end{align*}
\]
and anomalous dimensions of $\chi QED$ at two-loop level:

\[
\begin{align*}
\beta_e^{(2)} &= \gamma_A^{(2)} = \gamma_e^{(2)} = \left( \frac{\hbar}{16\pi^2} \right)^2 2e^4 \text{Tr}(Y_R^4), \\
\gamma_{\psi_i}^{(2)} &= -\left( \frac{\hbar}{16\pi^2} \right)^2 \frac{e^4}{3} \left( \frac{2}{3} \text{Tr}(Y_R^2)(Y_R^2)^2 + \frac{9}{2}(Y_R^2)^4 \right).
\end{align*}
\]

(5.21a) (5.21b)

**Comparison with results from the literature [2]:** In this paper, the authors evaluated the beta-functions and anomalous dimensions for the gauge/fermion sector of a generic gauge theory with scalars and chiral fermions, in dimensional regularisation. However, it appears that they employed a “naive” treatment of the Dirac $\gamma_5$ matrix, as it is customarily done in most cases where no pathologies may arise. The presence of chirality-specific Weyl, or full Dirac fermions, is there parameterised with a coefficient $\kappa$. Using $\kappa = 1/2$ for chiral 2-component fermions (i.e., one single chirality as in right-$\chi QED$), we agree with their evaluation of $\gamma_A^{(2)}$ and $\beta_e^{(2)}$ (Eqs. (5.2), (5.5) and (6.1)) in the gauge/fermion sector (excluding scalars). We also agree with the second contribution $\propto [C_2(F)]^2 \equiv \sum_j (Y_R^j)^4$ in their evaluation of $\gamma_{\psi_i}^{(2)}$. However, we note that this (unphysical) anomalous dimension has a different coefficient associated to the $S_2(F)C_2(F) \equiv \text{Tr}(Y_R^2) \sum_j (Y_R^j)^2$ structure: we obtain $-2/9$, instead of $-1$ ($\kappa = 1/2$) as in [2].

6 Conclusions

When applied to “vector-like” theories, with non-chiral fermion contents, dimensional regularisation and renormalisation in the BMHV scheme (DimRen scheme) generates singular counterterms that are symmetric-invariant, preserving the fundamental symmetries such as the BRST symmetry. Their structure is simple and mirrors the one already present in the defining tree-level action. Renormalisation group equations, providing the $\beta$-function(s) and anomalous dimensions of the theory, can then be simply derived with the “multiplicative” renormalisation method, based on wave-function and vertex $Z$ renormalisation factors (Section 3.1).

In contrast, when being applied to the renormalisation of chiral theories, such as the Chiral QED ($\chi QED$) toy-model, containing right-handed fermions (Section 2 and [18]), the DimRen scheme generates non-symmetric (evanescent or not) singular counterterms. This is due to the treatment of the Dirac $\gamma_5$ matrix, which is an inherently 4-dimensional object. The BRST symmetry is spuriously broken as well, and finite symmetry-restoring counterterms need to be introduced to restore the symmetry at each loop order [15, 16, 17, 18]. In this case, the multiplicative renormalisation method is not well suited anymore. Indeed, a host of new bare operators, corresponding to all the new structures generated by both the singular and the finite symmetry-restoring counterterms, need to be introduced a posteriori back into the original tree-level action, accompanied by new bare auxiliary couplings (Section 3.1.1), making such a method cumbersome to use. They would also generate a modified RG equation with supplemental $\beta$-functions for those couplings (Section 3.1.2), not directly corresponding to the one for the renormalised 4-dimensional theory. Such a procedure has already been explored in [19, 20] in the case of (fermion+)scalar models.

Instead, the framework of Algebraic Renormalisation provides an alternative method for elegantly deriving, in the DimRen scheme, the RG equation for chiral theories such as the $\chi QED$ model. This method provides a systematic way to straightforwardly construct the structure of the actual RG equation, based on its algebraic properties (Section 3.2), inherited from the invariances of the renormalised effective action of the theory, most importantly the (restored) BRST symmetry invariance. This formulation takes also directly into account all the counterterms operators, including the singular evanescent as well as the BRST-restoring finite ones, without having to redefine the theory as in the case of multiplicative renormalisation.
Using our previous results from [18], we were able to establish the expression for the χQED RG equation, that depends only on its physical couplings (here, the electric charge $e$) and fields. The “physical” β-function and the anomalous dimensions $\gamma_\phi$ for the physical coupling and the fields of the theory, are directly obtained (Section 4) by the resolution of a simple overconstrained system of linear equations (Section 5). The overconstrained nature of this system allows for verifying the consistency of the calculations. The resulting β-function for the physical electric charge, as well as the expected equality between $\beta_e$ and the photon anomalous dimension $\gamma_A$, agree with results from the literature [2]. However, the (non-physical) anomalous dimension for the chiral right-handed fermion fields, $\gamma_{\psi_i}$, differs slightly in one of its specific structures. Since it appears that [2] instead employed a “naive” treatment of the $\gamma_5$ matrix, one may speculate about the validity of that treatment and its conclusions at higher-loop orders.

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A Operator Insertions in the Effective Action

The insertion of an operator $O(x)$ in the effective action $\Gamma$ is constructed as follows. The $d$-dimensional tree-level action $S_0$ is formally supplemented by a term $\int d^d x \ Y_O(x)O(x)$: in other words, a new action is defined,

$$S_0'[Y_O] = S_0 + \int d^d x \ Y_O(x)O(x). \tag{A.1}$$

$Y_O$ is a new external source associated to the operator. From the corresponding generating functional $Z'[Y_O]$, its connected component, and taking a Legendre transform, a quantum effective action $\Gamma'[\phi, Y_O]$ is obtained. After inclusion of singular counterterms to render the action finite (and finite symmetry-restoring counterterms as well), $\Gamma'[\phi, Y_O]$ becomes equal, in the limit where $Y_O$ is set to zero, to: $\Gamma'[\phi, Y_O = 0] = \Gamma[\phi]$, the effective action generated by $S_0$ only.

This allows describing the insertion of one $O$ operator in $\Gamma$: $O \cdot \Gamma$, as the functional derivative of $\Gamma'$ with respect to $Y_O$, when $Y_O \to 0$:

$$O(x) \cdot \Gamma \equiv \left. \frac{\delta \Gamma'}{\delta Y_O(x)} \right|_{Y_O = 0}, \quad \quad O \cdot \Gamma \equiv \int d^d x \ O(x) \cdot \Gamma. \tag{A.2}$$

Note that during renormalisation, the external sources receive counterterm corrections $\delta Y_O$, opposite of those from $O$, such as to maintain $Y_OO$ “invariant” under renormalisation (like e.g. the gauge-fixing term). When evaluating the insertions as functional derivatives with respect to $Y_O \to 0$, $Y_O$ is its renormalised value. Very often, when no ambiguities can appear, $\Gamma'$ is conflated with $\Gamma$.

B 1-loop $\hat{S}_0$ Evanescent Insertions for Section 5.2.3

In the following calculations, the overall loop $\Gamma=1$ does not contain any finite counterterm $S_{1st}^{(1)}$. See Section 5.2.1 for an extensive discussion. We use the evanescent tree-level action defined by Eq. (2.7).
\( \mathcal{N}[\hat{S}_{\psi\bar{\psi}}] \cdot \Gamma = 1 \)

We evaluate each contribution from the insertion of the evanescent \( \hat{S}_{\psi\bar{\psi}} \) by using the one-loop version of the Bonneau identities \([39, 45]\): for a singly-evanescent operator \( \hat{O} \) (i.e. having one single evanescent structure), the identity reads:

\[
\mathcal{N}[\hat{O}] \cdot \Gamma = \mathcal{N} \left[ \text{r.s.p. } \mathcal{N}[\hat{O}] \cdot \Gamma \right]^{(1)} \cdot \Gamma = \text{LIM}_{d \to 4} \mathcal{N} \left[ \text{r.s.p. } \hat{O} \cdot \Gamma \right]^{(1)},
\]

(B.1a)

here denoting \( \hat{O} \equiv \hat{S}_{\psi\bar{\psi}} \), and where this operator is replaced by \( -\hat{S}_{\psi\bar{\psi}} \), the evanescent structure \( \hat{g}_{\mu \nu} \) being replaced by a symbol \( \tilde{g}_{\mu \nu} \) whose properties \([17]\) are:

\[
\tilde{g}_{\mu \nu} g_{\nu \rho} = \tilde{g}_{\mu \nu} \hat{g}_{\nu \rho} = \tilde{g}_{\mu} \,, \quad \tilde{g}_{\mu \nu} \tilde{g}^{\nu \rho} = 0 \,, \quad \tilde{g}_{\mu} = 1.
\]

(B.1b)

\[
\mathcal{N}[\hat{O}] \cdot \Gamma = \mathcal{N} \left[ \text{r.s.p. } \mathcal{N}[\hat{O}] \cdot \Gamma \right]^{(1)} \cdot \Gamma = \text{LIM}_{d \to 4} \mathcal{N} \left[ \text{r.s.p. } \hat{O} \cdot \Gamma \right]^{(1)},
\]

(B.1a)

by imposing \( \text{Tr}(\mathcal{Y}_R^3) = 0 \) (anomaly cancellation) or permuting the external legs.

Summing all these contributions, provides the result quoted in Eq. (5.13a) of Section 5.2.3.
\[ N[\widehat{S}_{AA}] \cdot \Gamma = 1 \]

The expansion of \( \widehat{S}_{AA} \) gives (using an integration by parts, IBP):

\[
\widehat{S}_{AA} = S_{AA} - \overline{S}_{AA} = \int d^d x \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{4} \overline{F}_{\mu \nu} F^{\mu \nu} \right)
= \int d^d x \left( -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) - \cdots \right)
\overset{\text{IBP}}{=} \int d^d x \left( \frac{1}{2} A^\mu (g_{\mu \nu} \partial^2 - \partial_\mu \partial_\nu) A^\nu - \cdots \right), \tag{B.7}
\]

and the corresponding Feynman rule:

\[
i \frac{\partial^2 \widehat{S}_{AA}}{A_\nu (-p) \delta A_\mu (p)} \longrightarrow -i (g_{\mu \nu} p^2 - p_\mu p_\nu) - \cdots \right), \tag{B.8a}
\]

Now, expanding this Feynman rule, we see that:

\[
(B.8a) = -i ( (g_{\mu \nu} p^2 - \overline{g}_{\mu \nu} \overline{p}^2) - (p_\mu p_\nu - \overline{p}_\mu \overline{p}_\nu)) , \tag{B.8b}
\]

and the contributions

\[
g_{\mu \nu} p^2 - \overline{g}_{\mu \nu} \overline{p}^2 = \overline{g}_{\mu \nu} \overline{p}^2 + \overline{g}_{\mu \nu} \overline{p}^2 , \tag{B.9a}
\]
\[
p_\mu p_\nu - \overline{p}_\mu \overline{p}_\nu = \overline{p}_\mu \overline{p}_\nu + \overline{p}_\mu \overline{p}_\nu \tag{B.9b}
\]

contain a non-reducible “double-evanescent” term (the last one in the previous expansions). Such term does not have the “singly-evanescent” form for being used in the usual Bonneau identities that have been derived primarily in that case. One could derive instead new Bonneau-like identities for such multiple-evanescent insertions, but we won’t proceed to do that here. Instead, we will evaluate \( N[\widehat{S}_{AA}] \cdot \Gamma = 1 \) in the most direct way: evaluate the related (sub-renormalised) loop diagrams, with their overall divergence subtracted as well.

Here, only the following one-loop diagrams Fig. 1 need to be evaluated, in a direct way, and their overall divergence (if it exists) be removed. We also take the opportunity to tag the contributions generated by each of the tensor structures, by multiplying each individual structure appearing in the Eq. (B.9) by a coefficient:

\[
g_{\mu \nu} p^2 - \overline{g}_{\mu \nu} \overline{p}^2 \longrightarrow \alpha_1 \overline{g}_{\mu \nu} \overline{p}^2 + \alpha_2 \overline{g}_{\mu \nu} \overline{p}^2 + \alpha_3 \overline{g}_{\mu \nu} \overline{p}^2 , \tag{B.10a}
\]
\[
p_\mu p_\nu - \overline{p}_\mu \overline{p}_\nu \longrightarrow \beta_1 \overline{p}_\mu \overline{p}_\nu + \beta_2 \overline{p}_\mu \overline{p}_\nu + \beta_3 \overline{p}_\mu \overline{p}_\nu \tag{B.10b}
\]

and replacing these inside the Feynman rule Eq. (B.8b). We obtain the following results:
Diagram 1:

\[ i \left. N[S_{AA}] \cdot \Gamma = \right|_{\psi_j(\bar{\psi} R, -p)} = \frac{i \hbar}{16\pi^2} \alpha_1 \left( \frac{7\xi + 1}{12} + \frac{9(\xi - 1)^2}{40} \right) e^2 \gamma_{Rij}^2 \bar{\rho}^2 \Gamma_R \]

\[ + \frac{i \hbar}{16\pi^2} \frac{(\xi - 1)^2}{120} (27\alpha_2 + 4 \times 2(\alpha_3 - \beta_3)) e^2 \gamma_{Rij}^2 \bar{\rho}^2 \Gamma_R \]

\[ - \frac{i \hbar}{16\pi^2} \frac{\xi - 1}{120} (27\xi + 8)(\beta_1 + \beta_2) e^2 \gamma_{Rij}^2 \bar{\rho}^2 \Gamma_R \]

\[ \alpha_i = 1 = \beta_i \quad \frac{i \hbar}{16\pi^2} \frac{2 e^2 \gamma_{Rij}^2 \bar{\rho}^2 \Gamma_R}{3}. \]

Diagram 2:

\[ i \left. N[S_{AA}] \cdot \Gamma = \right|_{\psi_j A_\mu \bar{\psi} R_i} = \frac{i \hbar}{16\pi^2} \alpha_1 \left( \frac{3\xi - 1}{6} + \frac{(\xi - 1)^2}{5} \right) e^3 \gamma_{Rij}^3 \bar{\rho}^\mu \Gamma_R \]

\[ + \frac{i \hbar}{16\pi^2} \frac{(\xi - 1)^2}{40} (8\alpha_2 + 2(\alpha_3 - \beta_3)) e^3 \gamma_{Rij}^3 \bar{\rho}^\mu \Gamma_R \]

\[ - \frac{i \hbar}{16\pi^2} \frac{\xi - 1}{20} (4\xi + 1)(\beta_1 + \beta_2) e^3 \gamma_{Rij}^3 \bar{\rho}^\mu \Gamma_R \]

\[ \alpha_i = 1 = \beta_i \quad \frac{i \hbar}{16\pi^2} \frac{e^3 \gamma_{Rij}^3 \bar{\rho}^\mu \Gamma_R}{3}. \]

In this diagram, we observe that most of the gauge-independent part resides in the \( \alpha_1 \) coefficient, i.e. the \(-i \bar{\rho}^2 \gamma^2 \) contribution from the Feynman rule Eq. (B.8b), while all the others are directly related to the gauge-dependent part. When setting these coefficients to their intended values (\( \alpha_i = 1 = \beta_i \)), all gauge dependency naturally drops out, without having to fix the gauge. Also, the extra factor 2 associated with the \( \alpha_3, \beta_3 \) coefficients (here and in the next diagram) appears, because they correspond to the doubly-evanescent structures \( \tilde{g}_{\mu\nu} \bar{\rho}^2 \) and \( \bar{\rho}_\mu \bar{\rho}_\nu \). Hence we obtain:

\[ N[S_{AA}] \cdot \Gamma = \left| \psi \bar{\psi} R \right| = \frac{\hbar}{16\pi^2} \frac{2 e^2}{3} \int d^4 x \bar{\psi} \partial \bar{\rho}^\mu \Gamma_R \psi_j \]  

\[ \gamma_{R} \text{ diag.} \quad \frac{\hbar}{16\pi^2} \frac{2 e^2}{3} \sum_j (\gamma_{R})^2 S_{\psi \bar{\psi} R} \Gamma_R. \]

Diagram 2:

\[ i \left. N[S_{AA}] \cdot \Gamma = \right|_{\psi_j A_\mu \bar{\psi} R_i} = \frac{i \hbar}{16\pi^2} \alpha_1 \left( \frac{3\xi - 1}{6} + \frac{(\xi - 1)^2}{5} \right) e^3 \gamma_{Rij}^3 \bar{\rho}^\mu \Gamma_R \]

\[ + \frac{i \hbar}{16\pi^2} \frac{(\xi - 1)^2}{40} (8\alpha_2 + 2(\alpha_3 - \beta_3)) e^3 \gamma_{Rij}^3 \bar{\rho}^\mu \Gamma_R \]

\[ - \frac{i \hbar}{16\pi^2} \frac{\xi - 1}{20} (4\xi + 1)(\beta_1 + \beta_2) e^3 \gamma_{Rij}^3 \bar{\rho}^\mu \Gamma_R \]

\[ \alpha_i = 1 = \beta_i \quad \frac{i \hbar}{16\pi^2} \frac{e^3 \gamma_{Rij}^3 \bar{\rho}^\mu \Gamma_R}{3}. \]

Again, after setting the \( \alpha_i, \beta_i \) coefficients to their intended values, all gauge dependency naturally drops out. Hence we obtain:

\[ N[S_{AA}] \cdot \Gamma = \left| \psi A \bar{\psi} R \right| = \frac{\hbar}{16\pi^2} \frac{e^2}{3} \int d^4 x \gamma_{Rij} \bar{\psi} \Gamma_R \bar{L} \bar{A} \Gamma_R \psi_j \]  

\[ \gamma_{R} \text{ diag.} \quad \frac{\hbar}{16\pi^2} \frac{e^2}{3} \sum_j (\gamma_{R})^2 S_{\bar{\psi} \psi A \bar{\psi} R} \Gamma_R. \]

Summing all these contributions, provides the result quoted in Eq. (5.13b) of Section 5.2.3.

\[ N \left[ \int d^4 x \frac{1}{2} \bar{A}_\mu \hat{\partial}^2 \bar{A}^\mu \right] \cdot \Gamma = 1 \]

The Feynman rule of \( \int d^4 x \frac{1}{2} \bar{A}_\mu \hat{\partial}^2 \bar{A}^\mu \) is:

\[ i \left. \frac{\delta^2}{\delta A_\mu(-p) \delta A_\mu(p)} \int d^4 x \frac{1}{2} \bar{A}_\mu \hat{\partial}^2 \bar{A}^\mu \rightarrow -i \bar{\rho}_{\mu \nu} \hat{\partial}^2. \]

It can be obtained from the Feynman rule for \( \tilde{S}_{AA} \), Eqs. (B.8b), (B.9) and (B.10), by setting the coefficient \( \alpha_1 = 1 \) and all the other coefficients to zero. Therefore, we can directly re-use the previous calculations, and set the arbitrary \( \alpha_i, \beta_i \) coefficients to their specified values. Alternatively, one could use again the one-loop version of the Bonneau identities, Eq. (B.1), since the operator of interest is singly-evanescent; there, \( \hat{\partial}^2 \) would be replaced by \( -\hat{\partial}^2 \). In both cases one obtains:
Diagram 1:
\[
N \int d^d x \left( \frac{1}{2} \bar{A}_\mu \partial^2 A^\mu \right) \cdot \Gamma = \left. \left. \frac{\hbar}{16\pi^2} \left( \frac{7\xi + 1}{12} + \frac{9(\xi - 1)^2}{40} \right) \right|_{\psi R} \right) e^2 \sum_j (\mathcal{Y}_R^j)^2 S_{\psi R}^j. \tag{B.14}
\]

Diagram 2:
\[
N \int d^d x \left( \frac{1}{2} \bar{A}_\mu \partial^2 A^\mu \right) \cdot \Gamma = \left. \left. \frac{\hbar}{16\pi^2} \left( \frac{3\xi - 1}{6} + \frac{(\xi - 1)^2}{5} \right) \right|_{\psi A R} \right) e^2 \sum_j (\mathcal{Y}_R^j)^2 S_{\psi A R}^j. \tag{B.15}
\]

Summing all these contributions, provides the result quoted in Eq. (5.13c) of Section 5.2.3.

C 1-loop \( S_{\text{fct}}^{(1)} \) Insertions for Section 5.2.4

We evaluate the divergent parts of the diagrams Fig. 2, using the finite counterterms from Eq. (2.14d).

Diagram 1:
\[
i S_{\text{fct}}^{(1)} \cdot \Gamma = \left. \left. \frac{\hbar}{16\pi^2} \right|_{A_{\nu(-p)} A_{\nu(p)}} \right) e^2 \left( \frac{2}{3} \left( p^\mu p^\nu - p^2 g^\mu\nu \right) - \frac{1}{2} p^2 g^\mu\nu \right) + \text{finite}, \tag{C.1a}\]

hence
\[
S_{\text{fct}}^{(1)} \cdot \Gamma = \left. \left. \frac{\hbar}{16\pi^2} \right|_{A A} \right) e^2 \left( \frac{2}{3} \left( \mathcal{Y}_R^4 \right)^2 - \frac{1}{2} p^2 g^\mu\nu \right) S_{AA}, \tag{C.1b}
\]

and the divergent evanescent contribution:
\[
S_{\text{fct}}^{(1)} \cdot \Gamma = \left. \left. \frac{\hbar}{16\pi^2} \right|_{A A} \right) e^2 \left( \frac{2}{3} \left( \mathcal{Y}_R^4 \right)^2 - \frac{1}{2} p^2 g^\mu\nu \right) S_{AA}, \tag{C.1c}
\]

Diagram 2:
\[
S_{\psi R}^{(1)} \cdot \Gamma = \left. \left. \frac{\hbar}{16\pi^2} \right|_{\psi R} \right) e^2 \left( \frac{\hbar}{16\pi^2} \right) \left( \mathcal{Y}_R^j \right)^2 S_{\psi R}^j. \tag{C.2}
\]

Diagram 3:
\[
\left[ \int d^d x \left( \frac{1}{2} \bar{A}_\mu \partial^2 A^\mu \right) \cdot \Gamma = \left. \left. \frac{\hbar}{16\pi^2} \right|_{\psi R} \right) e^2 \left( \frac{\hbar}{16\pi^2} \right) \left( \mathcal{Y}_R^j \right)^2 S_{\psi R}^j \right. \tag{C.3}
\]

Total 2+3: Using the coefficients from the finite counterterms Eq. (2.14d), associated with the inserted operators in diagrams 2 and 3, we obtain:
\[
S_{\text{fct}}^{(1)} \cdot \Gamma = \left. \left. \frac{\hbar}{16\pi^2} \right|_{\psi R} \right) e^2 \left( \frac{\hbar}{16\pi^2} \right) \left( \mathcal{Y}_R^j \right)^2 S_{\psi R}^j. \tag{C.4}
\]
Diagram 4:
\[
\frac{\mathcal{S}_R^j}{\psi_R} \cdot \Gamma_{\text{ren}}^{\text{div}} \psi_{A_R} \mathcal{Y}_{\text{diag.}} = -\frac{\hbar}{16\pi^2} \frac{2e^2\xi}{\epsilon} (\mathcal{Y}_R^j)^2 \frac{\mathcal{S}_R^j}{\psi_R A_R},
\]

Diagram 5:
\[
\left[ \int d^dx \frac{1}{2} \partial_\mu \bar{A}_\mu \partial_\nu A_\nu \right] \cdot \Gamma_{\text{ren}}^{\text{div}} \psi_{A_R} \mathcal{Y}_{\text{diag.}} = -\frac{\hbar}{16\pi^2} \frac{e^2\xi^2}{\epsilon} \sum_j (\mathcal{Y}_R^j)^2 \frac{\mathcal{S}_R^j}{\psi_R A_R},
\]

Total 4+5: Using the coefficients from the finite counterterms Eq. (2.14d), associated with the inserted operators in diagrams 4 and 5, we obtain:
\[
\mathcal{S}_{(1)}^{\text{fct}} \cdot \Gamma_{\text{ren}}^{\text{div}} \left|_{\psi_{A_R}} \right. = - \left( \frac{\hbar}{16\pi^2} \right)^2 \frac{e^4\xi}{3\epsilon} \sum_j ((\xi + 5)(\mathcal{Y}_R^j)^4 - \xi \text{Tr}(\mathcal{Y}_R^j)(\mathcal{Y}_R^j)^2) \frac{\mathcal{S}_R^j}{\psi_R A_R},
\]

Summing all these contributions, provides the result quoted in Eq. (5.16) of Section 5.2.4.

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