Some Properties of Cone Inner Product Spaces over Banach Algebra

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Authors’ contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

The aim of this paper is to introduce a concept of a cone inner product space over Banach algebras. This is done by replacing the co-domain of the classical inner product space by an ordered Banach algebra. Some properties such as Cauchy-Schwarz inequality, parallelogram identity and Pythagoras theorem are established in this setting. Similarly, the notion of cone normed algebra was introduced. Some illustrative examples are given to support our findings.

Keywords: Inner product space; cone inner product space; cone normed space, cone normed algebra, banach algebra.

1 Introduction

The concept of cone metric space which was first introduced in [1] by Rzepecki is one of the generalizations of metric space. In that paper, the author defined a metric such that \( d_E : X \times X \rightarrow \mathcal{S} \), where \( \mathcal{S} \) is a normal cone in a Banach space \( E \) with a partial order \( \leq \) and \( d_E \) is a metric on a set \( X \). Later on in [2] Huang and Zhang gives another definition of cone metric space where they replaced

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the codomain of a classical metric space (which is the space of real numbers) by a real Banach space ordered by a cone and proved some fixed point theorems of contractive mapping. Since then, many authors used this concept in generalizing the notion of cone metric space to cone normed space. For example, Samanta, Roy and Dinda [3] studied the cone convergence with respect to its cone norm and proved the completeness of a finite dimensional cone normed space. Sonmez and Cakalli [4] studied the main properties of cone normed space and proved some theorems of weighted means in cone normed space. Some theorems and examples of cone Banach spaces were studied by Abdeljawad, Turkoglu and Abuloha [5] whereas Eshaghi, Ramezani, Khodaei and Baghani [6] prove Baire’s category and Banach Steinhaus theorem in cone normed space. In [7] the authors studied the dual of cone normed space and proved the Hahn-Banach theorem of cone normed space. Recently, in [8] some topological concepts of non-symmetric cone normed space was studied by combining asymmetric norm and cone norm, while in [9] some fixed point theorems in cone Banach space are proved by the help of continuous and compatibility of mapping and in [10] an extension of the well-known fixed point theorems of Banach, Kannan and Chattarjee to fuzzy cone normed linear space is given. As we know in Functional analysis, it is always a tradition that when a certain generalization of either metric or normed spaces was given, it is expected for the generalization to be extended to inner product spaces. The aim of this paper is to introduce a concept of cone inner product space which will serves as the generalization of the classical inner product space and establish some properties such as the Cauchy-Schwarz inequality, parallelogram identity, orthogonality and Pythagoras theorem.

A non-empty subset $C$ of a Banach space $E$ is said to be a wedge if

\begin{enumerate} \item $C + C \subseteq C$ Closure under vector addition, \item $\lambda C \subseteq C \forall \lambda \geq 0$ Closure under scalar multiplication. \end{enumerate}

If in addition, $C \cap \{-C\} = \{0\}$, then $C$ is called cone.

For a given cone $C \subseteq E$, where $E$ is a Banach space, we define a partial ordering $\leq$ with respect to $C$ by $x \leq y$ if and only if $y - x \in C$ while the ordering $\ll$ with respect to $C$ by $x \ll y$ if and only if, $y - x \in \text{int} C$, where $\text{int} C$ denote the interior of $C$. Note that $y - x \in \text{int} C$ implies $y - x \in C$ but the reverse is not always the case.

2. Preliminaries

**Definition 2.1** [2] Let $X$ be non-empty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$, and $(X, d)$ is called cone metric space.

**Definition 2.2** [6] Let $X$ be a real vector space. Suppose that a mapping $\| \cdot \| : X \to E$ where $E$ is a real Banach space, satisfies:

1. $\|x\| \geq 0 \forall x \in X$ 
2. $\|x\| = 0 \iff x = 0 \forall x \in X$,
3. $\|\alpha x\| = |\alpha|\|x\| \forall x \in X \text{ and } \alpha \in \mathbb{R}$
4. $\|x + y\| \leq \|x\| + \|y\| \forall x, y \in X$.

Then $\| \cdot \|_c$ is called a cone norm on $X$ and $(X, \| \cdot \|_c)$ a cone normed space.

For some examples of cone metric and cone normed spaces, see [4, 6, 2].

**Definition 2.3** [11] An algebra over $K$, or $K$-algebra is a $K$-vector space $A$ with a bilinear map

$$A \times A \mapsto A$$

$$(x, y) \mapsto xy$$
such that for all $x, y, z \in A$ and $\alpha \in K$ the following holds:

a) $x(yz) = (xy)z$

b) $(x + y)z = xz + yz$

c) $x(y + z) = xy + xz$

d) $(\alpha x)y = \alpha(xy)$

An algebra $A$ is unital if there exist an element $1 \in A$ such that $1x = x1 = x$ for all $x \in A$, such an element is called an identity element in $A$.

**Definition 2.4** A left $A$-module is a $K$-vector space $M$ together with a bilinear map

$$ (a, m) \mapsto am $$

such that for all $m \in M$ and $x, y \in A$ we have $1m = m$ and $(xy)m = x(ym)$.

**Definition 2.5** [12] An algebra norm over algebra $A$ is a mapping

$$ \|\cdot\| : A \rightarrow \mathbb{R} $$

such that

i) $(A, \|\cdot\|)$ is a normed space.

ii) $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$

An algebra $A$ together with this norm is called normed algebra. A complete normed algebra is called Banach algebra.

**Definition 2.6** [13] A cone $C$ of a Banach algebra $A$ is called an algebra cone if $C$ satisfies the following:

i) $C \cdot C \subseteq C$

ii) $1 \in C$

**Definition 2.7** [13] A real or complex Banach algebra $A$ with unit 1 is called an ordered Banach algebra if $A$ is ordered by a relation $\leq$ such that for all $a, b, c \in A$ and $\alpha \in \mathbb{F}$ we have:

i) $a, b \geq 0 \Rightarrow a + b \geq 0$

ii) $a \geq 0, a \geq 0 \Rightarrow \alpha a \geq 0$

iii) $a, b \geq 0 \Rightarrow ab \geq 0$

iv) $1 \geq 0$

Thus, if $A$ is ordered by an algebra cone $C$, then $A$ or $(A, C)$ is an ordered Banach algebra.

**Definition 2.8** [14] An algebra cone is called normal if there exist a constant $K \geq 1$ such that for all $a, b \in A$ we have

$$ 0 \leq a \leq b \Rightarrow \|a\| \leq K\|b\| $$

*Throughout this paper, $E$ will be a real Banach space, $C$ a cone and $\|\cdot\|_C$ a cone norm, unless otherwise stated.*

### 3. Main Results

We begin the section by first introducing the concept of a cone normed algebra.

**Definition 3.1.** Let $A$ be an algebra and $C$ be a cone. An algebra cone norm $A$ is a mapping
where \((E, C)\) is an ordered Banach algebra, such that

\begin{itemize}
  \item[i)] \((A, \|\cdot\|_c)\) is a cone normed space,
  \item[ii)] \(\|xy\|_c \leq \|x\|_c\|y\|_c\) for all \(x, y \in A\).
\end{itemize}

An algebra \(A\) equipped with this cone norm is called a cone normed algebra. A complete cone normed algebra is a cone Banach algebra.

**Example 3.2** Let \(E = \mathbb{R}^n\) with norm \(\|x\|_\infty = \max_{1 \leq i \leq n} x_i\) and define multiplication component-wise in \(\mathbb{R}^n\). Let \(C = \{x = (x_i) \in \mathbb{R}^n, x_i \geq 0, \forall i\}\) be a cone in \(E\). Define a mapping

\[ \|\cdot\|_c : \mathbb{R}^n \to (E, C) \]

By \(\|x\|_c = (\lambda^i\|x_i\|_\infty)^n\) where \(\lambda > 0\) we say that \((\mathbb{R}^n, \|\cdot\|_c)\) is a cone normed algebra. To see this observes that the space \(\mathbb{R}^n\) with a component-wise multiplication is algebra and that \(E = (\mathbb{R}^n, \|\cdot\|_c)\) is Banach algebra. Thus, \((E, C)\) is an ordered Banach algebra. Now to show that \((\mathbb{R}^n, \|\cdot\|_c)\) is a cone normed algebra we need to show that \((\mathbb{R}^n, \|\cdot\|_c)\) is a cone normed space and \(\|xy\|_c \leq \|x\|_c\|y\|_c\) for all \(x, y \in \mathbb{R}^n\). Take any \(x, y \in \mathbb{R}^n, \alpha > 0\) and using the fact that \((\mathbb{R}^n, \|\cdot\|_c)\) is a normed space, we have

1. \( \|x\|_c = (\lambda^i\|x_i\|_\infty)^n \geq 0_E \) since \(\|x\|_\infty \) and \(\lambda > 0\),
2. Let \(\|x\|_c = 0_E\), then \(\|x\|_c = (\lambda^i\|x_i\|_\infty)^n = 0_E \Rightarrow \|\lambda\|_\infty = 0 \Rightarrow x = 0\),
3. \( \|\alpha x\|_c = (\lambda^i\|\alpha x_i\|_\infty)^n = (\lambda^i\|\alpha\|\|x_i\|_\infty)^n = (\lambda^i\|x_i\|_\infty)^n = \|x\|_c\) for all \(x \in \mathbb{R}^n\),
4. \( \|x + y\|_c = (\lambda^i\|x_i + y_i\|_\infty)^n \leq (\lambda^i\|x_i\|_\infty + \|y_i\|_\infty)^n = (\lambda^i\|x_i\|_\infty)^n + (\lambda^i\|y_i\|_\infty)^n = \|x\|_c + \|y\|_c\) .

Hence \((\mathbb{R}^n, \|\cdot\|_c)\) is a cone normed space. Also

\[ \|xy\|_c = (\lambda^i\|xy_i\|_\infty)^n \]

\[ \leq (\lambda^i\|x_i\|_\infty\|y_i\|_\infty)^n \] since \((\mathbb{R}^n, \|\cdot\|_\infty)\) is a normed algebra

\[ = (\lambda^i\|x_i\|_\infty)^n (\lambda^i\|y_i\|_\infty)^n = \|x\|_c\|y\|_c\]

Thus, \((\mathbb{R}^n, \|\cdot\|_c)\) is a cone normed algebra.

From now on, we refer to Banach algebra as a real Banach algebra.

**Definition 3.3** (Cone inner product space). Let \(E\) be an ordered Banach algebra and \(X\) be a \((\text{left})\) \(E\)-module, a cone inner product is a mapping \(\langle \cdot, \cdot \rangle_c : X \times X \to E\) such that \(\forall x, y, z \in X, \forall \alpha, \beta \in \mathbb{F}\) and \(a \in E\), we have:

\begin{itemize}
  \item[1)] \(\langle x, x \rangle_c \geq 0_E\),
  \item[2)] \(\langle x, x \rangle_c = 0_E \Rightarrow x = 0_X\),
  \item[3)] \(\langle x, y \rangle_c = \langle y, x \rangle_c\),
  \item[4)] \(\langle \alpha x + \beta y, z \rangle_c = \alpha\langle x, z \rangle_c + \beta\langle y, z \rangle_c\),
  \item[5)] \(\langle ax, y \rangle_c = \langle x, ay \rangle_c\).
\end{itemize}

The triplet \((X, E, \langle \cdot, \cdot \rangle_c)\) is called a cone inner product space over a Banach algebra \(E\). If condition 1) is not satisfied, then \((X, E, \langle \cdot, \cdot \rangle_c)\) is a semi-cone inner product space.

**Remark 3.4** Condition 5), requires the cone inner product to be linear in its first variable while condition 3), requires the inner product to be conjugate linear in its second variable. Therefore, using condition 3) and 5) we have \(\langle x, ay \rangle_c = \overline{a}\langle x, y \rangle_c\). Also, if \(X\) is a \((\text{right})\) \(E\)-module, then condition 5) becomes \(\langle ax, y \rangle_c = \langle x, ay \rangle_c\) and \(\langle x, ay \rangle_c = \langle x, y \rangle_c\).

**Example 3.5** The following are some of the examples of cone inner product space:
i) Every inner product space is a cone inner product space.

ii) Let \( E = \ell_2 \) and \( C = \{ \{ x_i \} \in E | x_i \geq 0, \forall i \} \) be a cone in \( E \) with a coordinate-wise ordering, let also \( M = \ell_2 \) be an \( E \)-module such that
\[
( \mathcal{M}, \langle \cdot, \cdot \rangle ) \text{ is an inner product space. Define a mapping}
\]
\[
\langle \cdot, \cdot \rangle_c : M \times M \to E \text{ by } \langle x, y \rangle_c = \{ \langle x_i, y_i \rangle \} | i = 1, 2, \ldots \}
\]
Then, \( ( \mathcal{M}, E, \langle \cdot, \cdot \rangle_c ) \) is a cone inner product space.

iii) Let \( \mathcal{R}(\mathbb{R}^n) \) be the linear space of all \( n \times n \) diagonal real matrices and \( E = \mathbb{R}^n \) with \( C = \{ x = [x_i]_{n} \in \mathbb{R}^n | x_i \geq 0, \forall i \} \) being a positive cone in \( E \). For \( A \in \mathcal{R}(\mathbb{R}^n) \), let \( [A]_D = \{ (a_{ii}) | a_{ii} \in A \text{ for } i = 1, 2, \ldots, n \} \) and define the module action on \( A \) as
\[
aA = a[A]_D \text{ where } a \in E.
\]
For \( A, B \in \mathcal{R}(\mathbb{R}^n) \) define the mapping \( \langle \cdot, \cdot \rangle_c : \mathcal{R}(\mathbb{R}^n) \times \mathcal{R}(\mathbb{R}^n) \to E \) by
\[
\langle A, B \rangle_c = [A]_D[B]_D.
\]
Then, \( ( \mathcal{R}(\mathbb{R}^n), E, \langle \cdot, \cdot \rangle_c ) \) is a cone inner product space.

**Proof.**

i) The space of real numbers which is the codomain of the classical inner product space is itself a Banach algebra together with the absolute value norm.

ii) To show that \( ( \mathcal{M}, E, \langle \cdot, \cdot \rangle_c ) \) satisfies all the properties of cone inner product space. Let \( x, y, z \in M, e \in E \) and \( a, b \in \mathbb{R} \). Then,

1) \( \langle x, x \rangle_c = \{ \langle x_i, x_i \rangle | i = 1, 2, \ldots \} = \{ |x_i|^2 | i = 1, 2, \ldots \} \geq 0_E \)

2) \( \langle x, x \rangle_c = \{ \langle x_i, x_i \rangle | i = 1, 2, \ldots \} = \{ x_i^2 | i = 1, 2, \ldots \} = 0_E \iff x_i = 0 \forall i \iff x = 0_M \)

3) \( \langle x, y \rangle_c = \{ \langle x_i, y_i \rangle | i = 1, 2, \ldots \} = \{ x_i\overline{y}_i | i = 1, 2, \ldots \}
\]
\[
\text{=} \{ x_i|y_i | i = 1, 2, \ldots \}
\]
\[
\text{=} \{ x_i|y_i | i = 1, 2, \ldots \}
\]

4) \( \langle ax + by, z \rangle_c = \{ \langle ax_i + by_i, z_i \rangle | i = 1, 2, \ldots \}
\]
\[
= \{ ax_iz_i + by_i\overline{z}_i | i = 1, 2, \ldots \} = a \langle x, z \rangle_c + b \langle y, z \rangle_c
\]

5) \( \langle ex, y \rangle_c = \{ \langle e_i(x_i\overline{y}_i) | i = 1, 2, \ldots \} = \{ e\langle x_i, y_i \rangle | i = 1, 2, \ldots \}
\]
\[
\text{=} \{ e(x_i\overline{y}_i | i = 1, 2, \ldots \} \text{ since the multiplication is done coordinatewise}
\]
\[
\text{=} e\langle x, y \rangle_c
\]

Hence \( ( \mathcal{M}, E, \langle \cdot, \cdot \rangle_c ) \) is a cone inner product space.

Now we need to show that \( ( \mathcal{R}(\mathbb{R}^n), E, \langle \cdot, \cdot \rangle_c ) \) is a cone inner product space. Let \( A, B, C \in \mathcal{R}(\mathbb{R}^n) \), \( a \in E \) and \( a, b \in \mathbb{R} \). Then

1) \( \langle A, A \rangle_c = [A]_D[A]_D = [A]_D^2 = \{ a_{ii}^2 | i = 1, 2, \ldots, n \} \geq 0 \text{ and that } A \geq 0 \text{ since } a_{ii} \in A = 0 \text{ for all } i \neq j. \)

2) Let \( \langle A, A \rangle_c = 0 \Rightarrow [A]_D^2 = \{ a_{ii}^2 | i = 1, 2, \ldots, n \} = 0 \). Since for any \( a_{ij} \in A \text{ with } i \neq j \), \( a_{ij} = 0 \text{ hence } A = 0. \) Assume that \( A = 0 \text{ then } \langle A, A \rangle_c = 0 \Rightarrow [A]_D^2 = \{ a_{ii}^2 | i = 1, 2, \ldots, n \} = 0. \)

3) \( \langle A, B \rangle_c = [A]_D[B]_D = [B]_D[A]_D = [B, A]_c. \)
4) \[ \langle aA + \beta B, C \rangle_c = ([aA + \beta B]_D)[C]_D = (a[A]_D + \beta[B]_D)[C]_D = a[aA]_D + \beta[B]_D[C]_D = a(A,C) + \beta(B,C) \]
5) \[ \langle aA,B \rangle_c = [aA]_D[B]_D = a[aA]_D[B]_D = a(a,A) \]

Hence, \( \langle \mathcal{F}(\mathbb{R}^N), E, (\cdot, \cdot)_c \rangle \) is a cone inner product space.

**Theorem 3.6** (Cauchy-Schwarz inequality). Let \( E \) be a Banach algebra over a cone \( C \) and \( X \) be a (left) \( E \)-module such that \( (X,E,(\cdot,\cdot)_c) \) is a cone inner product space. Then, for all \( x,y \in X \), we have

\[ ||(x,y)_c|| \leq \sqrt{\langle x,x \rangle_c \langle y,y \rangle_c} \]

Moreover, if the cone \( C \) is normal with normal constant \( K \), then

\[ ||(x,y)_c|| \leq K \sqrt{\langle x,x \rangle_c} \sqrt{\langle y,y \rangle_c} \]

**Proof.** Let \( x,y \in X \) if \( x = 0 \) or \( y = 0 \) the results holds. Assume that \( x \neq 0 \) and \( y \neq 0 \) and pick any \( a \in E \). Then

\[ 0 \leq \langle x - a \cdot y , x - a \cdot y \rangle_c = (x,x)_c - (x,a \cdot y)_c - (a \cdot y,x)_c + (a \cdot y,a \cdot y)_c \]

\[ = (x,x)_c - a \langle x,y \rangle_c - a \langle y,x \rangle_c + a \langle y,y \rangle_c \]

Choosing \( a = \frac{(x,y)_c}{\langle y,y \rangle_c} \), we have \( a((x,y)_c - \bar{a}(y,y)_c) = 0 \). Therefore,

\[ 0 \leq \langle x,x \rangle_c - \frac{(x,y)_c(y,y)_c}{\langle y,y \rangle_c} = (x,x)_c - \langle x,y \rangle_c(x,y)_c - \langle y,x \rangle_c(x,y)_c = (x,x)_c - ||x||^2 \]

**Lemma 3.7.** Let \( X \) be a left-\( E \) module and \( E \) be a Banach algebra ordered by a cone \( C \). Then, every cone inner product space \( (X,E,(\cdot,\cdot)_c) \) is a cone normed space \( (X,E,|| \cdot ||_c) \) with a cone norm defined by

\[ ||x||_c := \sqrt{\langle x,x \rangle_c} \]

For every \( x \in X \).

**Proof.** Let \( x,y \in X \) and \( \alpha \in \mathbb{F} \), then

1. \[ ||x||_c := \sqrt{\langle x,x \rangle_c} \geq 0 \] because \( \langle x,x \rangle_c \geq 0 \)
2. \[ ||x||_c = 0 \Leftrightarrow \sqrt{\langle x,x \rangle_c} = 0 \Leftrightarrow \langle x,x \rangle_c = 0 \Leftrightarrow x = 0 \]
3. \[ ||\alpha x||_c := \sqrt{\langle \alpha x, \alpha x \rangle_c} = \sqrt{||\alpha||^2 \langle x,x \rangle_c} = ||\alpha|| \sqrt{\langle x,x \rangle_c} = ||\alpha|| ||x||_c \]
4. \[ ||x+y||_c^2 = \langle x+y,x+y \rangle_c = \langle x,x \rangle_c + \langle x,y \rangle_c + \langle y,x \rangle_c + \langle y,y \rangle_c \]
   \[ = ||x||_c^2 + ||y||_c^2 + 2||x||_c \sqrt{\langle y,y \rangle_c} \]
   \[ \leq ||x||_c^2 + 2||x||_c \sqrt{||y||_c^2} \]
   \[ \leq ||x||_c^2 + 2||y||_c + ||y||_c^2 \]
   \[ \leq ||x||_c^2 + 2||x||_c \sqrt{||y||_c^2} + ||y||_c^2 \]

By Cauchy Schwarz inequality

\[ = ||x||_c^2 + 2||x||_c ||y||_c + ||y||_c^2 = (||x||_c + ||y||_c)^2. \]

Thus, the Cauchy Schwarz inequality for cone inner product space can be reformulated as

\[ ||x||_c \leq ||x||_c ||y||_c \]

**Corollary 3.8** (Parallelogram identity). Let \( (X,E,(\cdot,\cdot)_c) \) be a cone inner product space. For all \( x,y \in X \),

\[ ||x - y||_c^2 + ||x + y||_c^2 = 2(||x||_c^2 + ||y||_c^2) \]

**Example 3.9** Let \( E = L_1, C = \{x_i \in E | x_i \geq 0, \forall i \} \) be a cone in \( E \) with a coordinatewise ordering. Put \( X = \mathbb{R}^2 \) and define the cone norm.
\[ \|x\|_c = \left( \frac{\|x\|_1}{2^n} \right)_{n=1}^\infty \]

This cone norm cannot be obtained from a cone inner product space.

Observes that by taking the sum to infinity of \( \|x\|_c = \left( \frac{\|x\|_1}{2^n} \right)_{n=1}^\infty \) we obtain that \( \|x\|_c = \left( \frac{\|x\|_1}{2^n} \right)_{n=1}^\infty \) = \( \|x\|_1 \sum_{n=1}^{\infty} \frac{1}{2^n} = \|x\|_1 < \infty \). Thus, \( \|x\|_c = \left( \frac{\|x\|_1}{2^n} \right)_{n=1}^\infty \) is well defined.

Now to show that \((X, \|\cdot\|_c)\) satisfies all the properties of cone normed space.

Let \(x, y \in X\) and \(a \in \mathbb{F}\), then

1. \( \|x\|_c = \left( \frac{\|x\|_1}{2^n} \right)_{n=1}^\infty \geq 0 \) Because \( \|x\|_1 \geq 0 \)
2. \( \|x\|_c = \left( \frac{\|x\|_1}{2^n} \right)_{n=1}^\infty = 0 \Leftrightarrow \frac{\|x\|_1}{2^n} = 0 \Leftrightarrow \|x\|_1 = 0 \)
3. \( \|ax\|_c = \left( \frac{|a\| \|x\|_1}{2^n} \right)_{n=1}^\infty = |a| \left( \frac{\|x\|_1}{2^n} \right)_{n=1}^\infty = |a| \|x\|_c \)
4. \( \|x + y\|_c^2 = \left( \frac{\|x + y\|_1}{2^n} \right)_{n=1}^\infty \leq \left( \frac{\|x\|_1 + \|y\|_1}{2^n} \right)_{n=1}^\infty = \left( \frac{\|x\|_1}{2^n} \right)_{n=1}^\infty + \left( \frac{\|y\|_1}{2^n} \right)_{n=1}^\infty = \|x\|_c + \|y\|_c \).

Since all the properties of cone norm are satisfied. Now let’s pick \(x = (1,1) \in \mathbb{R}^2\) and \(y = (1, -1) \in \mathbb{R}^2\), then

\[ \|x\|_c = \left( \frac{\sum_{i=1}^{\infty} |x_i|}{2^n} \right)_{n=1}^\infty = \left( \frac{2}{2^n} \right)_{n=1}^\infty \]
\[ \|y\|_c = \left( \frac{\sum_{i=1}^{\infty} |y_i|}{2^n} \right)_{n=1}^\infty = \left( \frac{2}{2^n} \right)_{n=1}^\infty \]
\[ \|x + y\|_c = \left( \frac{\sum_{i=1}^{\infty} |x_i + y_i|}{2^n} \right)_{n=1}^\infty = \left( \frac{2}{2^n} \right)_{n=1}^\infty \text{ and} \]
\[ \|x - y\|_c = \left( \frac{\sum_{i=1}^{\infty} |x_i - y_i|}{2^n} \right)_{n=1}^\infty = \left( \frac{2}{2^n} \right)_{n=1}^\infty \]

Thus,
\[ \|x + y\|_c^2 + \|x - y\|_c^2 = \left( \frac{4}{2^n} + \frac{4}{2^n} \right)_{n=1}^\infty = \left( \frac{8}{2^n} \right)_{n=1}^\infty \]
\[ \text{And} \]
\[ 2(\|x\|_c^2 + \|y\|_c^2) = 2 \left( \frac{4}{2^n} + \frac{4}{2^n} \right)_{n=1}^\infty = \left( \frac{16}{2^n} \right)_{n=1}^\infty \]

Hence, \( \|x + y\|_c^2 + \|x - y\|_c^2 \neq 2(\|x\|_c^2 + \|y\|_c^2) \). Therefore the cone normed space \((X,E, \|\cdot\|_c)\) defined above cannot be obtained from the cone inner product space \((X,E, \langle \cdot, \cdot \rangle_c)\).

**Remark 3.10** We conclude that if a cone norm does not satisfy the parallelogram equality, it cannot be obtained from a cone inner product by the use of (3.1). So, not all cone normed spaces are cone inner product spaces. But, for a cone normed space \((X,E, \|\cdot\|_c)\) to be a cone inner product space\((X,E, \langle \cdot, \cdot \rangle_c)\), the following need to be satisfied:

i) The ordered Banach space \(E\) must be an algebra,
ii) The vector space \(X\) must be an \(E\)-module, and
iii) The Parallelogram identity must be satisfied.

**Definition 3.11** Let \((X,E, \langle \cdot, \cdot \rangle_c)\) be a cone inner product space. If \(X\) is complete with respect to the cone norm induced by the cone inner product \(\langle \cdot, \cdot \rangle_c\), then \((X,E, \langle \cdot, \cdot \rangle_c)\) is a cone Hilbert space.

**Definition 3.12** Two elements \(x\) and \(y\) in a cone inner product space \((X,E, \langle \cdot, \cdot \rangle_c)\) are said to be cone-orthogonal, denoted by \(x \perp_c y\), if \(\langle x, y \rangle_c = 0_E\).
If \( M \) is a subset of \( X \) such that \( (x, m)_c = 0 \forall m \in M \), then we say that \( x \) is cone-orthogonal to \( M \) and write \( x \perp_c M \). The set of all elements in \( X \) that are cone-orthogonal to \( M \) is given by the set

\[ M^{\perp_c} = \{ x \in X | (x, m)_c = 0 \forall m \in M \} . \]

**Proposition 3.13** If \((X, E, \langle \cdot, \cdot \rangle_c)\) is cone inner product space, then \( \{0\}^{\perp_c} = X \).

**Proof.** We know that \( \{0\}^{\perp_c} = \{ x \in X | (x, 0)_c = 0 \forall \in \{0\} \} \). This means that for any \( x \in X, (x, 0)_c = 0 \) which implies that \( x = 0 \). Hence, \( \{0\}^{\perp_c} = X \).

**Example 3.14** Let \( X = \mathbb{R}^3 \) and \( E = \mathbb{R}^3 \) with a cone \( C = \{ x \in E | x_i \geq 0 \} \). Define \( (x, y)_c = \{ x_i y_i | i = 1, 2, 3 \} \). Then, \((X, E, \langle \cdot, \cdot \rangle_c)\) is a cone inner product space.

If we take \( x = (x_1, 0, x_3) \) and \( y = (0, y_2, 0) \). Then \( x \) is cone-orthogonal to \( y \), since \( (x, y)_c = \{ x_i y_i | i = 1, 2, 3 \} = (x_1 \times 0, 0 \times y_2, x_3 \times 0) = (0, 0, 0) \in E \).

**Theorem 3.15** Let \((X, E, \langle \cdot, \cdot \rangle_c)\) be a cone inner product space. Let \( x, y \in X \). Then \( x \perp_c y \) if and only if \( ||x + y\perp_c 2 = x \perp_c 2 + y \perp_c 2 \).

**Proof.** " \( \Rightarrow \) " If \( x \perp_c y \), then

\[ ||x + y\perp_c 2 = (x + y, x + y)_c = (x, x)_c + 2(x, y)_c + (y, y)_c = ||x\perp_c 2 + ||y\perp_c 2 . \]

" \( \Leftarrow \) " Suppose that \( ||x + y\perp_c 2 = ||x\perp_c 2 + ||y\perp_c 2 \). Then

\[ (x, x)_c + 2(x, y)_c + (y, y)_c = (x, x)_c + (y, y)_c \]

This, is true only if \( 2(x, y)_c = 0 \), which implies that \( (x, y)_c = 0 \). Thus, \( x \perp_c y \).

**4 Conclusion**

This paper provided us with a new generalization of inner product spaces called a cone inner product space. We were able to establish some properties of the classical inner product space such as Cauchy Schwarz inequality, parallelogram identity and Pythagoras theorem in this setting.

**Further Research**

The concept of cone inner product space is introduced and the definition of cone Hilbert space is given. One may therefore look at some properties of cone Hilbert spaces. Also, the concept of cone normed algebra is introduced with example, but some of its properties have not been studied in this paper.

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**Competing Interests**

Authors have declared that no competing interests exist.
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