Transport in a One-Dimensional Superfluid: Quantum Nucleation of Phase Slips

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We present an analytical derivation for the quantum decay rate of the superflow through a weak link in a one-dimensional (1D) Bose-Einstein-condensate. The effective action for the phase difference across the link reduces to that of a massive particle with damping subject to a periodic potential. We find an algebraic flow–pressure relation, characteristic for quantum nucleation of phase slips in the link and show how short-wave-length fluctuations renormalize the interaction between Bosons remove the quantum phase transition expected in this class of systems.

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Quantum fluids exhibit fascinating phenomena as they push quantum mechanical effects to the macroscopic level. Classic examples are the electronic condensate in superconductors and the uncharged bosonic \(^4\)He and fermionic \(^3\)He superfluids \[3\]. Furthermore, the new laser- and evaporative cooling techniques have opened up new opportunities in the fabrication and manipulation of Bose-Einstein condensates (BEC) in various alkali-metal vapors \[2\]. A spectacular phenomenon is the friction free transport, often studied in terms of the Josephson effect across a constriction or orifice separating two reservoirs. Here, we concentrate on a specific geometry, a narrow channel, and study the decay of the superflow due to quantum generation of phase slips. Such decay processes have been studied in superconducting wires, both experimentally \[3\] and theoretically \[4\]. In this letter, we investigate the quantum decay of driven uncharged condensates — the Galilean invariance and the Schrödinger dynamics in these systems pose new challenges in the theoretical description and produce drastic changes in the physical results as compared to the charged situation.

In a (thick) one-dimensional superconducting wire, topological fluctuations of the phase (phase slips) are weak, resulting in a true superconducting response \[4\] with an algebraic current-voltage (I-V) characteristic \[V \propto I^{2g-2}\] and a dimensionless coupling constant \[g = 50r_0/\lambda_L\] \((r_0\) denotes the radius of the wire and \(\lambda_L\) the London penetration length). Reducing \(g\), e.g., via decreasing the diameter of the wire, the system undergoes a \((T = 0)\) quantum phase transition at \(g = 2\) to a metallic phase \[\[4\]\ with an ohmic characteristic. In the metallic wire the dynamics of the phase derives from integration over fermionic modes and is determined through the coupling to the electromagnetic field. Here, we study an uncharged BEC which is accurately described by the Gross-Pitaevskii Lagrangian for the condensate wave function \(\psi(x,t)\). The Galilean invariance of the homogeneous system prohibits the nucleation of phase slips and the homogeneous system is always in the superfluid state. Including a weak link breaks Galilean invariance and phase slips can nucleate at the defect \[\[4\]\, however, we find that the link always remains in the superfluid state and the entire system exhibits no quantum phase transition whatsoever.

We start with the Gross-Pitaevskii Lagrangian giving the dynamics of the condensate wave function \[\[4\]\]. With \(m\) the mass of the bosons, \(\rho_0\) the condensate density, and \(U\) the strength of the repulsion, the Lagrangian reads:

\[\mathcal{L}_{GP} = i\hbar \bar{\psi} \partial \psi - \frac{\hbar^2}{2m} (\partial_x \bar{\psi})(\partial_x \psi) - \frac{U}{2} \left( \bar{\psi} \psi - \rho_0 \right)^2. \quad (1)\]

The system admits states \(\psi = \sqrt{\rho_1} e^{i m v x / h}\) carrying a flow \(I = \rho_1 v\) with the renormalized density \(\rho_1 = \rho_0 - m v^2 / 2U\) and the velocity \(v\), driven by the phase difference \(\Delta \Phi = m v L / h\) across the reservoirs. The suppression of the modulus produces the critical flow \(I_c = (2/3)^{3/2} \rho_0 \sqrt{U \rho_0 / m}\). A crucial feature of the Lagrangian \[\[4\]\] is its Galilean invariance: with \(\psi(x,t)\) a state, further solutions are generated by the boost \(\exp(i m [-v x + i v^2 t / 2] / h) \psi(x-v t, t)\), with \(v\) the relative velocity of the two coordinate systems. A flow carrying state is mapped via such a boost to a stable state with \(I = 0\), hence homogeneously flowing states are stable. An inhomogeneity in the tube breaks the symmetry and quantum nucleation of phase slips bound to the perturbation appear. Here, we introduce a weak link (see Fig.\[4\]).
modeled by a variation in the stiffness,
\[ \mathcal{L}_w = \left( \hbar^2 c(x)/2m \right) \left( \partial_x \overline{\psi} \left( \partial_x \psi \right) \right), \tag{2} \]
where \( c(x) \) vanishes for \( |x| > a/2 \) (\( a \) denotes the width of the weak link). The action of the channel then reads
\[ S = S_R + S_L + \int dt \int \frac{d \tau}{\hbar} \left[ \mathcal{L}_{GP} + \mathcal{L}_w \right], \tag{3} \]
with \( S_{R(L)} \) the contribution of the right (left) lead. Note, that the above perturbation is time independent with the weak link at rest with respect to the reservoirs. This contrasts with Ref. \[ \text{[3]} \] where the flow around a moving object has been considered. Applying the mapping \( \psi(x,t) \rightarrow \exp(-imv/\hbar)\psi(x-vt,t) \) and \( \rho_0 \rightarrow \rho_1 \) allows for a comparison of our system with the setup in \[ \text{[3]}. \]

In order to study the quantum behavior of the superfluid we go over to the Euclidean action \[ \text{[3]} \]
\[ S_E = \int \frac{d \tau}{\hbar} \int d x \left[ \hbar \overline{\psi} \partial_x \psi + H \left[ \overline{\psi}, \psi \right] \right], \tag{4} \]
with \( \tau \) the imaginary time, \( \beta = 1/T \) the inverse temperature, and \( H \left[ \overline{\psi}, \psi \right] \) the Hamiltonian density,
\[ H \left[ \overline{\psi}, \psi \right] = \frac{\hbar^2}{2m} \left[ 1 - c(x) \right] \left( \partial_x \overline{\psi} \right) \left( \partial_x \psi \right) + \frac{U}{2} \left[ \overline{\psi} \psi - \rho_0 \right]^2. \]

A saddle point solution extremizing the above action involves \( \psi \) and \( \overline{\psi} \) as independent variables; this freedom allows for an analytic continuation of the integration contour in the path integral. Performing functional derivatives produces the equations

\[ -\hbar \partial_x \psi = -\left( \hbar^2/2m \right) \partial_x^2 \psi + U \left[ \overline{\psi} \psi - \rho_0 \right] \psi, \]
\[ -\hbar \partial_x \overline{\psi} = -\left( \hbar^2/2m \right) \partial_x^2 \overline{\psi} + U \left[ \overline{\psi} \psi - \rho_0 \right] \overline{\psi}. \tag{5} \]

Note that with \( \psi(\tau), \overline{\psi}(\tau) \) satisfying Eq. \[ \text{[3]} \] the configuration \( \psi^*(\tau), \psi^*(-\tau) \) is also a solution, with \( \psi^* \) the complex conjugate of \( \psi \). Furthermore, the periodicity of \( \psi \) and \( \overline{\psi} \), i.e., \( \psi(x, \tau + \hbar \beta) = \psi(x, \tau) \), and the boundary condition \( \psi^*(x, \hbar \beta/2) = \overline{\psi}(x, \hbar \beta/2) \) imply that
\[ \overline{\psi}(\tau) = \psi^*(\tau) \tag{6} \]
for the saddle point solution.

The 1D quantum phase slips, saddle point solutions describing the quantum tunneling of the \( \psi \) field, take the topological form of vortex-antivortex pairs in the \( (x, \tau) \)-plane \[ \text{[3]}. \] Within the vortex core region the modulus of the wave function drops to zero, while in the region far from the vortex center its variation is small and the action is determined by the behavior of the phase. In the uncharged superfluid, Galilean invariance prohibits formation of a vortex-antivortex pair in the homogeneous region and thus the vortices are bound to the weak link.

In the following we derive an effective action for the phase difference across the link.

We first integrate out the weak link. For a small width \( a < \xi \) the dominant contribution in the action is the kinetic term \[ \text{[4]} \] and the equation for the wave function \( \psi \) inside the weak link simplifies to
\[ \partial_x \{ 1 - c(x) \} \partial_x \psi = 0. \tag{7} \]
The boundary conditions are \( \psi(a/2) = \psi_R(a/2) \) and \( \psi(-a/2, \tau) = \psi_L(-a/2) \) with \( \psi_R \) (\( \psi_L \)) the wave function in the right (left) lead. Inserting the solution of \[ \text{[4]} \] into the action we obtain the contribution
\[ S_w = \int dt \frac{I_w}{2 \rho_0} \left[ \psi_L \psi_L + \psi_R \psi_R - \overline{\psi}_R \psi_L - \overline{\psi}_L \psi_R \right]. \tag{8} \]

Note, that in the above expression \( \psi_R \) and \( \psi_L \) are taken at \( x = a/2 \) and \( x = -a/2 \). The critical flow of the weak link \( I_w^{-1} = (m/\hbar \rho_0) \int_a^{-a/2} dx \left[ 1 - c(x) \right]^{-1} \) is assumed to be small compared to the critical flow \( I_c \) of the homogeneous system \( I_w/I_c \ll 1 \).

Next, we focus on the contribution of the leads. The flow in the wire is limited by the small critical current \( I_w \) of the link, justifying the ansatz
\[ \psi = \sqrt{\rho_0} \left( 1 + h \right) e^{i \phi} \tag{9} \]
with \( h \), \( \partial_x \phi \), and \( \partial_\tau \phi \) small, of order \( I_w/I_c \). For the following calculations, it is convenient to define the length \( \xi = \hbar/\sqrt{m \rho_0 U} \), the sound velocity \( v_s = \rho_0 U/m \), and the time \( \tau_0 = \xi/v_s \). Inserting the ansatz \[ \text{[4]} \] into \[ \text{[3]} \], the modulus and phase decouple to lowest order in \( I_w/I_c \), and using \[ \text{[3]} \] the equation for the phase \( \phi \) takes the form
\[ -\tau_0 \partial_\tau \phi(x, \tau) + \left( \xi^2/2 \right) \partial^2_\tau \phi(x, \tau) = \sin \left[ \phi^*(x, \tau) \right], \tag{10} \]
with \( \phi^*(x, \tau) = [\phi(x, \tau) - \phi(x, -\tau)]/2 \). The left side defines a diffusion equation, while the right side describes a nonlinear and nonlocal coupling of the wave function. The boundary conditions demand that the asymmetry \( \phi^* \) vanishes in the limit \( x \rightarrow \infty \) and \( \tau \rightarrow \pm \hbar \beta/2 \), see Eq. \[ \text{[3]} \]. The solution of the linearized system then is a promising ansatz for describing the solution far from the vortex core region. Below we find that the asymmetry is of order \( I_w/I_c \) outside the core region and the relevant behavior of the phase slip is determined by the region where the linearization of Eq. \[ \text{[3]} \] is justified. Applying a Fourier transformation leads to a local and linear system of equations for \( \phi(\omega) = u(\omega) + iv(\omega) \)
\[ -\xi^2 k^2 \left( \begin{array}{c} u \\ v \end{array} \right) = A \left( \begin{array}{c} u \\ v \end{array} \right), \quad A = \left[ \begin{array}{cc} 0 & -2 \omega \tau_0 \\ 2 \omega \tau_0 & -4 \end{array} \right], \tag{11} \]
defining modes with a dispersion relation \(-\omega^2/v_s^2 = k^2 (1 + \xi^2 k^2/4) \). Next, we solve \[ \text{[11]} \] for the phase in the right lead \( x > a/2 \) with the boundary conditions \( \phi_R(x \rightarrow \infty, \tau) = 0 \) and \( \phi_R(a/2, \omega) = p(\omega) + iq(\omega) \), where \( \psi_R(x, \tau) = \exp(i \phi_R(x, \tau)) \).
\[ u(x, \omega) = \frac{1}{\lambda_- - \lambda_+} \left\{ [\lambda_- - 2\omega \tau_0 q] e^{-s_i(2x-a)/2\zeta} - [\lambda_+ + 2\omega \tau_0 q] e^{-s_i(2x-a)/2\zeta} \right\}, \]
\[ v(x, \omega) = \frac{-\lambda_- + \lambda_+}{\lambda_- - \lambda_+} \left\{ [\lambda_- - 2\omega \tau_0 \lambda_+] e^{-s_i(2x-a)/2\zeta} - [\lambda_+ + 2\omega \tau_0 \lambda_-] e^{-s_i(2x-a)/2\zeta} \right\}, \]

with \( \lambda_\pm = 2(1 \pm \sqrt{1 - \omega^2 q^2}) \) and \( s_i = \sqrt{\lambda_\pm} \). Inserting the solution back into the effective action \( S_E \), Eq. (\ref{SE}), we concentrate on the first term (the second term involves the Hamiltonian \( \mathcal{H} = \int dx \mathcal{H} \) which is conserved under time evolution and therefore does not contribute). Linearizing in the small quantities \( \hbar, \partial_x \phi, \) and \( \partial_t \phi \), the variation \( h \) of the modulus drops out and the action is determined by the phase field alone,

\[ S_E = -\hbar \rho_0 \int_{-\infty}^{\infty} \int_{a/2}^{\infty} dx \omega \, u(x, \omega) \, v(x, \omega). \]  

A straightforward calculation leads to the effective action for the phase \( \phi_R(a/2, \tau) \) on the right side of the weak link

\[ S_E = \frac{\hbar K}{\pi} \int_0^{\infty} \frac{\omega}{4\pi} \left[ (p-q)^2 + \omega \tau_0 \right] \]  

with \( K = \pi \rho_0 \xi \) the number of particles per coherence length. An expansion in \( \omega \tau_0 \) is equivalent to an expansion in \( I_w/I_c \) and we obtain to lowest order

\[ S_E = \frac{\hbar K}{\pi} \int_{-\infty}^{\infty} d\omega \left[ |\omega| p^2 - 2\omega pq \right]. \]  

The first term is a Caldeira-Leggett type damping \(^{[1]}\) for the symmetric part \( p \) of the phase \( \phi_R(a/2, \tau) \) and describes the flow of energy from the weak link to the reservoirs via sound wave excitations. This term is expected to be captured by a dynamics that couples the sound waves and is also present in the charged superconductor. The second term is a coupling between symmetric and asymmetric parts and is only present for Schrödinger dynamics.

An analogous calculation with \( \phi_L(-a/2, \tau) \) adds a similar effective action for the phase on the left side of the link. Adding up terms from the leads and the link we obtain the total effective action for the phase difference \( \phi(\tau) = \phi_R(a/2, \tau) - \phi_L(-a/2, \tau) \) across the weak link (minimization with respect to the ‘center of mass’ variable \( \bar{\theta}(\tau) = \phi_R(a/2, \tau) + \phi_L(-a/2, \tau) \) gives \( \bar{\theta} = 0 \)). Introducing the notation \( \varphi^\pm = \frac{\varphi(\tau) \pm \varphi(-\tau)}{2} \) for the symmetric and asymmetric parts we obtain

\[ \frac{S_{\text{eff}}}{\hbar} = \frac{K}{2\pi} \int d\tau \left[ \int d\tau' \frac{[\varphi^+(\tau) - \varphi^-(\tau')]^2}{4\pi (\tau - \tau')^2} - \varphi^-(\tau) \partial_t \varphi^+(\tau) \right] \]  

\[ + \int d\tau \{ I_w \left[ 1 - \cos \varphi^+(\tau) \right] - I \partial^\tau \varphi^+(\tau) \}. \]  

Integrating out \( \varphi^- \) in \( S_{\text{eff}} \) produces the mass term \( \hbar \int d\tau m_w (\partial_t \varphi^+)^2/2 \) with \( m_w = K^2/4I_w \pi^2 \) via the relation \( \varphi^- = -(K/2\pi I_w) \partial_t \varphi^+ \). Collecting terms, the effective action for the symmetric part \( \varphi^+ \) of the phase difference across the link is equal to the action for a particle with mass \( m_w \) and damping \( \eta = K/2m \) in a periodic potential and driven by the force \( I \). The general form of a phase slip is \( \varphi^+ (\tau) = g^+(\tau) + g^-(\tau) + \arcsin(I/I_w) \) where the shift described by last term is due to the driving force \( I \), while \( g^+ (g^-) \) denotes a kink (anti-kink) associated with the presence of a vortex (anti-vortex) in the link. Applying the trial function \( g^\pm = \pm 2 \arctan [(2\tau - \tau)/\tau_c] \) with the kink width \( \tau_c = K/\pi I_w \) \(^{[1]}\), the action for the phase slip takes the form

\[ S_{\text{eff}}/\hbar = 2K \ln (\tau/\tau_c) - 2\pi I \tau - \ln y, \]  

where \( \ln y \) is found by collecting terms independent of \( \tau \). The kinks interact logarithmically with strength \( 2K \), while the driving force \( I \) produces a linear repulsion within a pair. Using the above trial function \( g^\pm \), we can justify the linearization in \( \Theta(10) \): the region of large asymmetry \( \phi^- (x, \tau) \) is limited by \( \xi \) along \( x \) \(^{[1]}\) and by the width \( \tau_c \) of the kink along the \( \tau_c \) direction. The contribution from the core then is independent of \( \tau \) and only enters through the fugacity \( y \).

**Thermodynamics (I = 0):** In the following we consider a gas of quantum phase slips. The associated action describes charged particles with logarithmic interaction in 1D and a chemical potential \( \ln y \). The one-dimensionality of the gas follows from the confinement of the vortices to the weak link. This contrasts with the superconducting wire where phase slips appear throughout the homogeneous channel and the action of the quantum phase slip gas is mapped to that of 2D charged particles with logarithmic interaction of strength \( 2y \) \(^{[1]}\).

In the latter system, a quantum phase transition (at \( g = 2 \)) has been found \(^{[1]}\) as quantum fluctuations drive the system ohmic at small values of \( g \leq 2 \) (including a weak link and ignoring phase slips in the wire shifts this transition to \( g = 1 \) \(^{[1]}\)). We then may expect a similar transition to occur in the superfluid at \( K = 1 \). Indeed, a careful analysis shows that a massive particle with damping and subject to a periodic potential undergoes a quantum phase transition for a critical damping \( K = 1 \) \(^{[1]}\). However, in the above derivation of the effective action \(^{[1]}\) we have neglected high frequency fluctuations of the order parameter which affect the long wave length behavior of the homogeneous superfluid via a renormalization of the sound velocity \( v_s = \sqrt{\chi(y) \rho_0 h/m} \), where \( \gamma = m U / h^2 \rho_0 \). The universal function \( f(\gamma) \) is obtained from the exact (Bethe Ansatz) solution of interacting 1D Bosons \(^{[1]}\),

\[ f(\gamma) = \begin{cases} \gamma - \gamma^{3/2}/2\pi, & \gamma \ll 1, \\ \pi^2 (1 - 8/\gamma), & \gamma \to \infty. \end{cases} \]  

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As the original relation $\pi \rho_0 h/m = K v_s$ between the sound velocity and the dimensionless parameter $K$ is not renormalized, a consequence of the Galilean invariance \[\text{[4]},\] the renormalization of $K$ can be found from that of the sound velocity. For $\gamma \ll 1$, the result $K = \pi \rho_0 \xi \gg 1$ is recovered, while in the limit $\gamma \to \infty$ we obtain hard-core bosons with $K = 1$ \[\text{[17]}\]. For intermediate values of $\gamma$, $K$ varies smoothly between these two limits. Therefore, no quantum phase transition is present in the uncharged superfluid.

Superflow ($I \neq 0$): The shape of the phase slip is determined by the driving current $I$: minimizing the $(T = 0)$ action \[\text{[17]}\] with respect to $\gamma$ we find the kink–anti-kink separation $\gamma = K/\pi I$ (see Ref. \[\text{[1]}\] for a detailed discussion addressing the motion of a particle with damping in a periodic potential). The kink width $\tau_c$ determines the asymmetry of the phase slip via the relation $\phi^- \sim -\tau_c \phi^+$, see Fig. 2. This form of the phase slip agrees well with numerical simulations \[\text{[3]}\].

![Phase slip solution](image)

\text{FIG. 2.} Phase slip solution: the dotted line is the symmetric part $\phi^+$, while the dashed line represents the asymmetric part $\phi^-$. The width of the asymmetry is determined by $\tau_c$ and the distance between the kink–anti-kink pair is $\gamma$.

The quantum decay rate of the superflow is determined by $\Gamma = B \exp(-S_{\text{ps}}/h)$ \[\text{[3]}\] with $S_{\text{ps}}$ the action of the phase slip and $B$ arising from fluctuations around the saddle point. At $T = 0$, the action is dominated by the logarithmic interaction between the kink and the anti-kink leading to a factor $I^{2K}$. Gaussian fluctuations in the distance between the kink and anti-kink with frequency $\propto I$ lead to a preexponential factor $B \propto 1/I$ and therefore $\Gamma \propto I^{2K-1}$ (in the superconducting wire the relative positions of the kink–anti-kink pair fluctuate both along $x$ and $\tau$ and therefore $B \propto 1/I^2$, leading to $\Gamma \propto I^{2K-2}$). Note, that the prefactor produces a sizeable correction of the flow rate as compared to the saddle point result \[\text{[3]}\]. At finite temperature we have to determine the transition rates $\Gamma^\pm$ both to lower and higher energy states. These transitions then produce a time dependent phase difference between the reservoirs which in turn determine the drop $\Delta \mu$ in the chemical potential across the channel,

\[
\frac{\Delta \mu}{h} = 2\pi \left[ \Gamma^+ - \Gamma^- \right] \propto \begin{cases} 
I_w \left( \frac{I}{I_w} \right)^{2K-1}, & I \gg \frac{T}{h}, \\
I \left( \frac{T}{h I_w} \right)^{2K-2}, & I \ll \frac{T}{h}.
\end{cases}
\]

With the pressure $p \propto \Delta \mu$ we arrive at an algebraic flow-pressure characteristic for $T = 0$, while at $T \neq 0$ the channel always exhibits a linear behavior for a small superflow $I$.

In the end, we have found a remarkable similarity in the response of charged and uncharged condensates: the quantum decay of the superflow due to phase slips leads to algebraic current-voltage and flow-pressure relations, respectively. On the other hand, we find distinct differences as well: first, the quantum nucleation of phase slips in the present uncharged situation is bound to a defect, a consequence of Galilean invariance, while the same phenomenon in the charged case can take place throughout the wire. Second, the presence of high energy modes renormalizing the sound velocity removes the quantum phase transition for the uncharged superfluid, preserving the superflow.

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