An integral model of a convective jet with a pressure force and forms of vertical fluxes in the atmospheric surface layer

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Abstract. In the article, it is considered a modification of an integral model of an unsteady turbulent jet with a pressure force. Stationary solutions of the presented model are compared with well-known analytical results of classical models. It is shown that the inclusion of the pressure forces changes the dynamic parameters of a jet by about 12\%. Analytical solutions of a steady forced buoyant jet of the atmospheric convective boundary layer and a spontaneous jet of a surface layer are presented. The simplest model of an ensemble of spontaneous jets of convective surface layer is constructed. It is shown that an ensemble of spontaneous jets forms a dependence of the turbulent moments and heat eddy diffusivity on the altitude within the convective boundary layer.

1. Introduction

Integral models of convective jets (plumes) are widely used for a description of volcanic activity and turbulent mixing within the boundary layers of the atmosphere and ocean.

Most of the well-known integral models of convective jets (e.g.: \cite{1-7}) use the approximation of the vertical boundary layer. It enables the pressure force in the equation of motion to be neglected.

The present paper discusses modifications of the integral model of a convective jet by including the pressure force. The modifications have exact solutions that can be compared against the analytical solutions of classical models. It allows us to estimate the effect of a pressure forced, that changes the vertical velocity and the buoyancy of a jet by about 15\%.

In geophysical environments, we can observe both isolated convective jets \cite{8} and ensemble of buoyant jets \cite{9}. An ensemble of convective thermals forms in the atmosphere over a heated surface of the land or ocean. A steady spontaneous jet and its analytical solution may be considered as a simplest model of an isolated element of the ensemble of thermals. It is shown that an ensemble of the dynamically identical spontaneous jets forms the vertical profile of turbulent eddy diffusivity and the profiles of the turbulent moments of velocity up to fourth order.

2. Integral model of a convective jet in a neutrally stratified atmosphere

Suppose that a convective jet forms over a point source of heat and buoyancy. Experimental measurements in a neutrally stratified environment show that the jet has a near-conical shape with the
radius $R_w$. The shape of the cone is defined by $\partial R_w / \partial z = \alpha_g$, where $\alpha_g \approx 0.1$ is a constant coefficient. The hypothesis about a conical shape of a jet is a part of the integral model [4].

The vertical velocity and buoyancy in each horizontal cross section of the jet are assumed to have stepwise $\Pi$-shaped profiles with amplitudes $\hat{w}$ and $\hat{g} \hat{\theta}$. Amplitude equations of the model [3, 4] in the vertical boundary-layer approximation with the hypothesis about a conical shape of a jet are

$$
\begin{align*}
\left[\frac{\partial}{\partial t} \hat{w} R_w^2 + \frac{\partial}{\partial z} \hat{w}^2 R_w^2\right] &= g \hat{\theta} R_w^2, \\
\left[\frac{\partial}{\partial t} \hat{\theta} R_w^2 + \frac{\partial}{\partial z} \hat{\theta} \hat{w} R_w^2\right] &= 0, \\
\frac{\partial}{\partial z} R_w &= \alpha_g
\end{align*}
$$

The boundary conditions for the point sources of heat and momentum are

$$
\lim_{z \to a} \hat{w} R_w^2 = P_s(t), \quad \lim_{z \to a} \hat{\theta} \hat{w} R_w^2 = S_s(t)
$$

Here, $P_s = P_s(t) > 0$ and $S_s = S_s(t) > 0$ are the time dependent strengths of momentum and heat fluxes, $[P_s] = m^3/s^2$, $[S_s] = m^3/s$.

A comparison between the numerical calculation by the model [3, 4] and other known integral models [1-5] was made in [5]. Those results are shown in Fig. 1. Here we use normalized vertical velocity $\hat{\tilde{w}}$, buoyancy $\hat{g} \hat{\tilde{\theta}}$ and altitude $\tilde{z}$ which description provides in models [2-5].

**Figure 1**: A comparison between integral models [1-5] and experimental data [2]. The normalized velocity $\hat{\tilde{w}}$ field. Circles and triangular dots are experimental data. Dash-and-dot line – model [1]; solid line – model [2]; thin solid line – model [3,4]; dotted line – model [5].

Let us consider an alternative way of establishing the amplitude equations of unsteady convective jet that does not use the approximation of the vertical boundary layer.

According to [10], in the structure of a turbulent jet, we can distinguish a coherent system consisting of a sequence of moving eddies that can be stylized as a chain of spherical thermals (Fig. 2).

Let us consider an equation of the isolated thermal in the chain. Suppose that $V_w = V_w(t)$ is a volume of the spherical thermal with radius $R_w = R_w(t)$; $\hat{w} = \hat{w}(t)$, $\hat{\theta} = \hat{\theta}(t)$ are an averaged mean velocity and dimensionless potential-temperature fluctuation and $\tilde{z} = \tilde{z}(t)$ is the height of the centre of mass of a thermal. The equation of the motion of a chain spherical thermal was suggested in [10]

$$
\frac{d}{dt} \hat{w} = \frac{2}{3} g \hat{\theta} - \frac{\alpha_g}{R_w} \hat{w}^2, \quad \frac{d}{dt} \hat{\tilde{w}} = \frac{2}{3} g \hat{\tilde{\theta}} - \frac{1}{\tilde{z}} \hat{\tilde{w}}^2
$$

The factor $2/3$ in (3) is related to introduction of a pressure force, that is parameterized by the added-mass, (see [11]).

The concept of a jet as a chain of thermals allows us to construct an alternative model of a convective jet. Instead of (1) we consider a modified system that include a parametric form of the pressure force

$$
\begin{align*}
\left[\frac{\partial}{\partial t} \hat{w} R_w^2 + \frac{1}{2} \frac{\partial}{\partial z} \hat{w}^2 R_w^2\right] &= \frac{2}{3} g \hat{\theta} R_w^2, \\
\left[\frac{\partial}{\partial t} \hat{\theta} R_w^2 + \frac{\partial}{\partial z} \hat{\theta} \hat{w} R_w^2\right] &= 0, \\
\frac{\partial}{\partial z} R_w &= \alpha_g
\end{align*}
$$

$$
\lim_{\tilde{z} \to \infty} \hat{w} R_w^2 = P_s(t), \quad \lim_{\tilde{z} \to \infty} \hat{\theta} \hat{w} R_w^2 = S_s(t)
$$

(4)

Suppose that $d/dt = \partial / \partial t + \hat{w} \partial / \partial \tilde{z}$ is an individual time derivative. When $\hat{w} = \hat{\tilde{w}}$ and $g \hat{\tilde{\theta}} = g \hat{\theta}$, equation of impulse (4) turns into the form (3).
Figure 2. Internal coherent structure of a turbulent jet and its geometric representation. (a) a convective turbulent ash plume from the eruption of Shiveluch volcano (Kamchatka), September 24, 2014; (b) geometric representation of a convective plume as a chain of spherical thermals.

The equation (4) includes an approximation of pressure force and therefore cannot be derived in the vertical boundary-layer approximation.

The Euler form of the modify stationary equations (4) with the boundary conditions (2) is

$$\frac{1}{2} \frac{d}{dz} \hat{w}^2 R_w = \frac{2}{3} g \hat{\theta} R_w, \quad \frac{d}{dz} \hat{\theta} R_w = 0, \quad R_w = \alpha_R z,$$

$$\lim_{z \to \infty} \hat{w} R_w^2 = P, \quad \lim_{z \to 0} g \hat{\theta} R_w^2 = g S_s.$$  (5)

The system (5) is considered on the unbounded domain $0 < z < \infty$.

3. Steady convective, buoyant and spontaneous jets with a pressure force

Consider a steady convective jet from a point source of buoyancy $S_s > 0$ and $P_v = 0$. The analytical solution to the modified model of a steady jet (4) is

$$R_w(z) = \alpha_R z, \quad \hat{w}(z) = \alpha_R z \left( g S_s \right)^{1/3} z^{-1}, \quad \hat{\theta}(z) = \alpha_R z \left( g S_s \right)^{2/3} z^{-4/3}.$$  (6)

A comparison between solution of the model (1) and the expression (6) was presented in [10]. It shows that the inclusion of the pressure forces in the model decreases the amplitude of vertical velocity and increases the amplitude of buoyancy by about 10%.

The power-law dependences of the vertical velocity and buoyancy on the altitude correspond to the similarity law for the convective jet, presented in [12].

Buoyant jets are of much interest for problems of volcanic outbursts. An analytical solution to the volcanic outburst problem (5) is found as

$$R_w(z) = \alpha_R z, \quad \hat{w}(z) = \frac{1}{\alpha_R z^{3/2}} \left[ \frac{P_v^{1/2}}{\alpha_R} + \alpha_R g S_s z^2 \right]^{2/3}, \quad \hat{\theta}(z) = \frac{g S_s}{\alpha_R z^{3/2}} \left[ \frac{P_v^{1/2}}{\alpha_R} + \alpha_R g S_s z^2 \right]^{-4/3}.$$  (7)

A solution similar to (7) also exists in the model described in [13]. When $P_v = 0$ equalities (7) turn into equalities of convective jet (6). When $g = 0$ equalities (7) turn into equalities of submerge jet [14].

The free-convection sublayer in developed turbulence is the fluid or gas layer adjacent to a heated surface. The surface sublayer has thickness $0 < z / h < 0.1$; where $h$ is the height of the convective layer. It is also described by a constancy of buoyancy flux per unit area $g S_s = const > 0$, where $[g S_s] = m^2 / s^3$.

Let $\Gamma(z) = d \hat{\theta} / dz \leq 0$ be a stratification parameter. Then,

$$g \Gamma(z) = g \frac{d \hat{\theta}}{dz} = -\lambda_0 \left( g S_s \right)^{2/3} z^{-4/3}$$  (8)

where $0.9 \leq \lambda_0 \leq 1.1$, according to the atmospheric field observations (e.g., [15, 16]).

Equality (8) follows from the classical Prandtl turbulence theory [17].

Spontaneous jets arise in unstable atmospheric layers, for more details see [18]. They gain energy from the unstable layer and so do not require any additional heat and momentum sources at the underlying surface.

Extending the modified integral model (5) to the case of unstable stratification, we obtain equations for a steady isolated spontaneous jet that include pressure. Assuming $\lambda_0 = 1$ in the form (8) we have
\[
\frac{1}{2} \frac{d}{dz} \hat{w}^2 R_w^2 = \frac{2}{3} g \hat{\theta} \hat{w} R_w^2, \quad \frac{d}{dz} g \hat{\theta} \hat{w}^2 = (g S_0)^{2/3} \frac{2}{3} \hat{w} R_w^2, \quad R_w = \alpha_k z, \quad \lim_{z \to 0} \hat{w}^2 R_w^2 = 0, \quad \lim_{z \to 0} g \hat{\theta} \hat{w}^2 = 0
\]

(9)

The analytical solution of (9) is

\[
R_w(z) = \alpha_k z, \quad \hat{w}(z) = \frac{1}{2} (g S_0)^{1/3} z^{1/3}, \quad g \hat{\theta}(z) = \frac{1}{2} (g S_0)^{2/3} z^{-1/3}
\]

(10)

Equations (10) are consistent with the similarity law for a spontaneous jet obtained in [18].

The exact solution (10) that include a pressure force can be compared with results of a classical integral model [18], that based on the vertical boundary layer approximation. The comparison results show that the inclusion of the pressure forces in the model conserves the buoyancy and increases the amplitude of vertical velocity by about 15%.

4. Ensemble of dynamically identical jets and turbulent diffusivity within a convective boundary layer

It is known that in a turbulent convective layer an ensemble of convective elements forms over a horizontally homogenous heated surface (see [9]). An idea of what an ensemble of thermals looks like can be gained from laboratory experiments at large Rayleigh numbers. The results of laboratory experiments [19], presented in Fig. 3, clearly exhibit the fine structure of the convective layer.

Figure 3. Ensemble of thermals in the form of dense salt fingers descending in a water layer, according to [19].

Let \( h \) be the height of a convective layer. It should be noted that the ensemble of convective jets existed mainly within the lower part of a convective layer \( 0 < z/h < 0.5 \). Within the upper part of a layer \( 0.5 < z/h < 1 \), the amount of jets is low.

We construct a “generalized” convective jet based on the hypothesis that within the surface layer \( 0 < z/h < 0.1 \) parameters of the “generalized” jet are defined as parameters of the spontaneous jet (10); and within the mixing layer \( 0.1 < z/h < 0.5 \) they defined as parameters of the convective jet (6).

Using the results of laboratory experiments [19] and the ensemble model [20], we assume that the convective surface layer is packed with the system of identical “generalized” convective jets.

Suppose that \( g S_0 \) is a buoyancy flux at the underlying surface. According to [21], we introduce Deardorff parameters of velocity and buoyancy

\[
w_D = h^{1/3} (g S_0)^{1/3}, \quad \theta_D = h^{-1/3} (g S_0)^{2/3}
\]

(11)

Based on the concept of statistical ensemble, we deduce a relationship for the heat eddy diffusivity. In the framework of the semi-empirical theory of turbulence by Prandtl [17], it is assumed that

\[
<K_z> = \langle w \rangle l_p, \quad \frac{l_p}{h} = k_v \left( \frac{z}{h} \right)^2 \left( 1 - \frac{z}{h} \right)
\]

(12)

Where \(<w>\) is the average velocity of updrafts; \( l_p \) is Prandl mixing length; \( k_v = 0.4 \) is von Karman constant.

Substituting the vertical velocity of the “generalized” convective jet into (12) and rewriting deduced relationships with regard to Deardorff parameters (11) yield
\begin{equation}
\frac{<K_h>}{w_D h} = \begin{cases}
\left(\frac{z}{h}\right)^{2/3} \left[1 - \left(\frac{z}{h}\right)\right] \quad \text{when} \quad 0.1 < z / h < 0.5.
\left(\frac{z}{h}\right)^{4/3} \quad \text{when} \quad 0 < z / h < 0.1
\end{cases}
\end{equation}

Well-known approximations of the real heat eddy diffusivity $K_h$ are

\begin{equation}
\frac{K_h}{w_D h} = \begin{cases}
\left(\frac{z}{h}\right)^{2/3} \left[1 - \left(\frac{z}{h}\right)\right] \quad \text{when} \quad 0.1 < z / h < 0.5.
\left(\frac{z}{h}\right)^{4/3} \quad \text{when} \quad 0 < z / h < 0.1
\end{cases}
\end{equation}

An approximation for $K_h$ in the surface layer $0 < z / h < 0.1$ was constructed in [22] within the framework of similarity theory. An approximation for $K_h$ in the mixing layer $0.1 < z / h < 0.5$ was constructed by numerical modelling in [23].

Similarity between approximations (13) and (14) means that the ensemble of “generalized” convective jets is a propulsion of the turbulent diffusivity within the convective boundary layer.

5. Ensemble of dynamically identical spontaneous jets and turbulent moments of a convective surface layer

Based on an integral model similar to (10), we deduce the forms of the second, third and fourth statistical turbulent moments of vertical velocity. Then

\begin{equation}
\begin{aligned}
\frac{<\bar{w}^2>}{w_D^2} &= \frac{\sigma \left( gS_0 \right)^{2/3}}{4} \frac{z}{h}^{2/3} = \frac{\sigma \left( gS_0 \right)^{2/3}}{4} \frac{h}{\bar{w}^2} \left( \frac{z}{h} \right) = \frac{\sigma \left( z \right)^{2/3}}{4} \frac{h}{\bar{w}^2} \\
\frac{<\bar{w}^3>}{w_D^3} &= \frac{\sigma \left( gS_0 \right)^{4/3}}{8} \frac{z}{h}^{4/3} = \frac{\sigma \left( gS_0 \right)^{4/3}}{8} \frac{h}{\bar{w}^3} \left( \frac{z}{h} \right) = \frac{\sigma \left( z \right)^{4/3}}{8} \frac{h}{\bar{w}^3} \\
\frac{<\bar{w}^4>}{w_D^4} &= \frac{\sigma \left( gS_0 \right)^{4/3}}{16} \frac{z}{h}^{4/3} = \frac{\sigma \left( gS_0 \right)^{4/3}}{16} \frac{h}{\bar{w}^4} \left( \frac{z}{h} \right) = \frac{\sigma \left( z \right)^{4/3}}{16} \frac{h}{\bar{w}^4}
\end{aligned}
\end{equation}

Where $<\bar{w}^2>$, $<\bar{w}^3>$ and $<\bar{w}^4>$ are statistical turbulent moments, related to an ensemble of spontaneous jets and downdrafts; $\sigma_{ww} / 4 \approx 1.1$, $\sigma_{ww} / 8 \approx 1.1$ and $\sigma_{ww} / 16 \approx 0.9$ are a dimensionless constant, which depends on the relative area of the updraft flows, the density of jets per area unit, and a description method of interference between jets.

We assume that $\bar{w}^2$, $\bar{w}^3$ and $\bar{w}^4$ are the real second, third and fourth order moments of vertical velocity of the atmospheric surface layer. According to results of dimension theory [15, 24] transformed by Deardorff parameters (11) we get

\begin{equation}
\frac{\bar{w}^2}{w_D^2} = \lambda_{ww} \left( \frac{z}{h} \right)^{2/3}, \quad \frac{\bar{w}^3}{w_D^3} = \lambda_{ww} \left( \frac{z}{h} \right)^{4/3}, \quad \frac{\bar{w}^4}{w_D^4} = \lambda_{ww} \left( \frac{z}{h} \right)^{4/3}
\end{equation}

Field data of the second [25], third [25] and fourth [26] moments of vertical velocity and their approximations (16) with $\lambda_{ww} = 1.8$, $\lambda_{ww} \approx 1$ and $\lambda_{ww} \approx 6.5$. are presented in Fig. 4.

**Figure 4.** Field data of the second [25], third [25] and fourth [26] moments of vertical velocity – the dots. Their approximations (16) – the solid lines.
According to (15), (16) the power-law dependences of statistical and real moments of vertical velocity on the dimensionless altitude $z/h$ are identical. Hence an ensemble of spontaneous jets forms a dependence of momentum and heat diffusivity on the altitude within the convective surface layer.

6. Conclusion
Presented results show that the classical model of convective thermals and jets can be modified by including a pressure force.

It is demonstrated that the modified integral model of convective jets retain a power-law dependence of vertical velocity and buoyancy on the altitude, that deduced in the classical models.

For convective jets, within a neutrally stratified atmosphere, an effect of a pressure force increases the amplitude of buoyancy and decreases the amplitude of vertical velocity. The total varying is about 10%. For spontaneous jets within an unstable stratified atmosphere, an effect of a pressure force retain the amplitude of buoyancy and increases the amplitude of vertical velocity by about 15%.

The relatively weak influence of the pressure force on jet’s parameters enables the approximation of vertical boundary-layer in the theory of buoyant jets to be regarded as a first approximation.

Ensembles of buoyant jets form vertical profiles of turbulent moments and eddy diffusivity within convective surface layers.

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