PARABOLIC COMPACTIFICATION OF HOMOGENEOUS SPACES

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Abstract In this article, we study compactifications of homogeneous spaces coming from equivariant, open embeddings into a generalized flag manifold $G/P$. The key to this approach is that in each case $G/P$ is the homogeneous model for a parabolic geometry; the theory of such geometries provides a large supply of geometric tools and invariant differential operators that can be used for this study. A classical theorem of Wolf shows that any involutive automorphism of a semisimple Lie group $G$ with fixed point group $H$ gives rise to a large family of such compactifications of homogeneous spaces of $H$. Most examples of (classical) Riemannian symmetric spaces as well as many non-symmetric examples arise in this way. A specific feature of the approach is that any compactification of that type comes with the notion of ‘curved analog’ to which the tools we develop also apply. The model example of this is a general Poincaré–Einstein manifold forming the curved analog of the conformal compactification of hyperbolic space. In the first part of the article, we derive general tools for the analysis of such compactifications. In the second part, we analyze two families of examples in detail, which in particular contain compactifications of the symmetric spaces $SL(n, \mathbb{R})/SO(p, n–p)$ and $SO(n, \mathbb{C})/SO(n)$. We describe the decomposition of the compactification into orbits, show how orbit closures can be described as the zero sets of smooth solutions to certain invariant differential operators and prove a local slice theorem around each orbit in these examples.

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1. Introduction

To study non-compact manifolds it can be helpful to add a suitable boundary structure so that the resulting space is compact. Within geometry this idea is perhaps historically most

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well known in hyperbolic geometry and [25], for example, provides a striking application of this point of view. A special case that has received a lot of attention in the literature is the question of compactifying symmetric and locally symmetric spaces, see for example the monograph [3] and references therein. The prototype case of such a compactification is provided by the Poincaré ball compactification of real hyperbolic space. This compactifies hyperbolic space by adding a sphere as a boundary at infinity. This example is deceptively simple, however. In general one cannot expect that it will be sufficient to just add, to a given non-compact symmetric space, a boundary of codimension one at infinity. Rather one has to expect a complicated family of boundary components of different dimension, attached to each other in a highly complicated fashion. This has led to a host of different approaches to constructing boundaries or boundary components for (classes of) symmetric spaces that are based on Lie theory, algebraic geometry, topology, and dynamics, for example, see again [3]. The resulting compactifications are often mainly understood from the point of view of topology, and obtaining an analytical structure is a very difficult problem. Thus even getting to a setting that allows for a notion of the geometry of such a compactification is often very difficult.

Here we follow a different approach based on (equivariant) embeddings between homogeneous spaces. In particular, we focus on open embeddings into generalized flag manifolds. While this seems to be a rather special situation, it turns out that such embeddings are available for a large number of symmetric spaces and more general homogeneous spaces. This not only provides an ambient manifold containing the compactification, but it provides an ambient geometric structure that provides powerful tools for the study of compactifications. Moreover, this connects a very active area of recent research in differential geometry that we describe next.

Let us return to the Poincaré ball compactification of hyperbolic space. While completeness of the hyperbolic metric implies that it cannot be smoothly extended to the boundary, its underlying conformal structure does admit a smooth extension, thus endowing the boundary sphere with its standard conformal structure. Penrose’s concept of conformal compactness provides an analogous notion for more general (pseudo-)Riemannian manifolds, with Poincaré–Einstein manifolds forming an important special case. These ideas have been extremely fruitful with applications to topics like negatively curved Riemannian manifolds, geometric scattering, general relativity (GR), conformal geometry, and the AdS/CFT (anti-de-Sitter/conformal field theory) correspondence of physics, see [1, 13, 14, 20, 23, 24, 28, 30] but also in representation theory and harmonic analysis, see e.g. [21].

During the last years a new conceptual approach to conformal compactness and Poincaré–Einstein manifolds has been developed, see [15, 16]. Rather than viewing the metric in the interior and the conformal structure on the boundary as the basic objects, these approaches are based on the conformal structure on a manifold with boundary, together with a defining density for the boundary which automatically selects a metric from the conformal class in the interior. The advantage of this approach is that, using tools of conformal geometry, it immediately leads to a host of geometric objects that admit a smooth extension to the boundary, and indeed beyond the boundary. These then provide powerful tools for efficiently and systematically treating many of the problems
linked to the applications mentioned above [17–19]. This description also leads to an interpretation of Poincaré–Einstein metrics as a certain type of reduction of conformal holonomy. The general versions of tractor calculus (see [5]) and the theory of holonomy reductions of Cartan geometries developed in [10] then show that many of these ideas can be extended from conformal geometry to the class of Cartan geometries. The latter includes the rich class of parabolic geometries [11].

In particular, analogs of the concept of conformal compactness in the setting of projective and of c-projective differential geometry have been introduced and studied in [6–9]. In all these cases, the theory of parabolic geometries and the machinery of Bernstein–Gelfand–Gelfand (BGG) sequences introduced in [12] and [4] provide conceptual ways to obtain and identify geometric quantities that automatically admit a smooth extension to the boundary. This was a crucial input for the developments in the articles referred to above. At the same time, these procedures produce families of partial differential equations (PDEs), both in the interior and on the boundary, that are naturally associated with the compactified geometry. Most critically some of these equations also come with canonical solutions that combine with the underlying higher order Cartan geometry to define the interior and boundary geometries, as well providing a concrete link between these.

The aim of the current paper is to apply the ideas on holonomy reductions of parabolic geometries, on the level of their homogeneous models, to construct and study compactifications of certain homogeneous spaces, among which there are many symmetric spaces. As briefly mentioned above, the basic strategy is to consider a generalized flag variety \(G/P\) of a semisimple Lie group \(G\) as well as a subgroup \(H \subset G\), such that the obvious action of \(H\) on \(G/P\) has at least one open orbit. A classical result of Wolf (see Theorem 2.6) implies that this is the case (for any choice of parabolic subgroup \(P \subset G\)) if \(H \subset G\) is the fixed point group of an involutive automorphism of \(G\). This immediately leads to a large number of interesting examples. Choosing a point in an open orbit and denoting by \(K\) the stabilizer of that point in \(H\), the orbit gets identified with the homogeneous space \(H/K\). Since \(G/P\) is compact, the closure of the orbit forms a natural compactification \(\overline{H/K}\) of \(H/K\). By construction \(\overline{H/K}\) can be written as a union of \(H\)-orbits, which automatically are initial submanifolds of \(G/P\), thus providing a decomposition of the boundary.

Now \(G/P\) is the homogeneous model of parabolic geometries of type \((G, P)\). Such geometries can be restricted to open subsets, so \(H/K\) carries an \(H\)-invariant locally flat parabolic geometry of that type. Moreover, by construction this geometry admits a smooth extension across the boundary of \(H/K\) in \(G/P\), thus providing a first set of geometric objects that extend smoothly. Moreover, any \(H\)-invariant element in a representation \(V\) of \(G\) defines a parallel section in a tractor bundle naturally associated to the parabolic geometry. Such a section can be projected to a section of simpler bundle, which lies in the kernel of an overdetermined linear differential operator (a so-called ‘first BGG operator’) naturally associated to the parabolic geometry. Both sections are defined on all of \(G/P\) and thus extend smoothly to (and across) the boundary and, by \(H\)-invariance, they can be often used to distinguish between different \(H\)-orbits in the boundary. The fact that one obtains parallel sections of tractor bundles and sections in
the kernel of a first BGG operator, respectively, allows one to get good control on the
derivatives (and even on higher jets) of such BGG solutions based on information coming
from representation theory. We show that this can be used to obtain detailed descriptions
of the topology, smooth structure, and geometry of the boundary structure.

We want to point out at this stage that, although in this article we formally only
consider homogeneous spaces, the ideas introduced here automatically extend to more
general settings. On the one hand, there is the possibility to pass to certain non-compact
locally homogeneous spaces by factoring by appropriate discrete subgroups of $H$. The
model example here is again provided by hyperbolic space, for which one may factor by
convex cocompact subgroups of $SO_0(n + 1, 1)$ and still attach a boundary that locally is as
in the model [29]. On the other hand, there is the possibility to replace the homogeneous
model $G/P$ by a curved parabolic geometry of type $(G, P)$ endowed with a holonomy
reduction (as a Cartan geometry) to the subgroup $H \subset G$ in the sense of [10]. The theory
developed in that reference shows that the boundary structure in such a curved holonomy
reduction can be nicely compared to the situation on the homogeneous model via the
so-called curved orbit decomposition. From the point of view of the ambient parabolic
geometry, the existence of such a holonomy reduction of course is a very restrictive
condition, but as examples like Poincaré–Einstein manifolds, the Kähler analogues of
these, and their generalizations show, it is to be expected that there are many interesting
examples in the curved case.

Let us briefly outline how the article is organized. The general theory of
compactifications, as outlined above, is developed in § 2. We start with a general definition
of homogeneous compactifications and this already leads to first results on the boundary
structure. We then specialize to parabolic compactifications defined by open $H$-orbits in
$G/P$. The interpretation of the conformal and projective compactifications of hyperbolic
space from this point of view and some generalizations are discussed in Example 2.5. Next,
we recall Wolf’s theorem and describe several examples of parabolic compactifications
arising from it in Example 2.8. Proposition 2.9 describes the infinitesimal structure of
a neighborhood of an $H$-orbit in $G/P$ and Proposition 2.11 shows that the subgroup
$H \subset G$ can always be characterized as a stabilizer of an element in an appropriate
representation of $G$, thus leading to a parallel section of the corresponding tractor bundle.
The background on tractor bundles and the machinery of BGG sequences we need is
collected in §§ 2.6 and 2.8. Theorem 2.13 describes the basic relation between parallel
sections of tractor bundles and solutions of first BGG operators, while Proposition 2.17
shows how the BGG machinery can be used to recover information on the jets of such
solutions. The last part of § 2 introduces a generalization of defining functions and
defining densities for hypersurfaces to defining sections of vector bundles for submanifolds
of higher codimension. These smooth objects provide a geometric and analytic bridge
between the different components, and in particular they provide a very convenient way
to formulate and prove many of our results.

In the remaining two sections of the article, we apply these general tools to the
study of two families of substantial examples of parabolic compactifications. Section 3
deals with the case that $H = SO(p, q) \subset SL(p + q, \mathbb{R}) = G$, which obviously is covered
by Wolf’s theorem. We focus on the case that $P \subset G$ is a maximal parabolic, so that
$G/P$ is the Grassmannian $Gr(i, \mathbb{R}^{p+q})$ in which the $H$-orbits are determined by rank and signature, and give some indication on how to deal with more general flag varieties. The orbit structure and the infinitesimal structure around an orbit on the Grassmannian are described in Proposition 3.1. In particular, for the $i = p$, the $H$-orbit of positive subspaces is the Riemannian symmetric space $SO(p, q)/S(O(p) \times O(q))$ which we take as the model example for this section. This symmetric space clearly carries an $H$-invariant Grassmannian structure (i.e. a decomposition of its tangent bundle as a tensor product) compatible with its Riemannian metric, and the main feature of the compactification we construct is that this Grassmannian structure admits a smooth extension to the boundary. The inner product stabilized by $H$ can be directly converted into a parallel section of a tractor bundle, which is the main tool used to analyze the compactification.

As a first application of the theory of parabolic geometries, we show in Theorem 3.4, how the closures of $H$-orbits in the Grassmannian can be described as zero loci of first BGG solutions that admit a smooth extension to the boundary. The main result in this section is a slice theorem, see Theorem 3.5 and Corollary 3.6 which shows that locally, a neighborhood of an orbit can always be described in terms of a product of the orbit with a neighborhood of zero in the closure of the set of non-degenerate symmetric matrices of appropriate size and signature in the space of all symmetric matrices. The key ingredient to this is the construction of a defining section for the orbit obtained by restricting an appropriate first BGG solution to an appropriate subbundle of the tautological bundle on the Grassmannian. As a last result in this case, we show in Proposition 3.7 that for the boundary components of codimension one, one may avoid such choices and find a canonical defining density, which is a solution of a first BGG operator of order three.

In § 4 of the article, we similarly study the case that $H := SO(n, \mathbb{C}) \subset SO_0(n, n) =: G$, which again is covered by Wolf’s theorem, and in particular leads to a compactification of the Riemannian symmetric space $SO(n, \mathbb{C})/SO(n)$. The generalized flag varieties of $G$ are given by manifolds of isotropic flags, and we quickly specialize to the case of the Grassmannians of maximal isotropic subspaces, where the $H$-orbits are again described in terms of rank and signature, but ranks always drop in steps of two in this case. Moreover, it is well known that there are two such Grassmannians, given by self-dual respectively anti-self-dual maximally isotropic subspaces. We briefly indicate how to deal with more general isotropic Grassmannians and flag manifolds, for which the orbit structure becomes significantly more complicated.

While initially there is no Hermitian structure in the setup, it turns out that Hermitian matrices play a key role in the slice theorem for the $H$-orbits in this case, see Theorem 4.5. While we also obtain a description of orbit closures in terms of first BGG solutions in this case (see Theorem 4.3), we do not see a way to construct a natural defining density for the hypersurface components of the boundary.

2. Parabolic compactifications

In this section, we introduce the concept of a parabolic compactification and explain the advantages of such compactifications. Finally, we describe several sources for large families of examples.
2.1. Homogeneous compactifications

A natural idea for constructing a compactification of some non-compact space $X$ is to embed it into a compact space $Y$ and then form the closure $\overline{X}$ in $Y$. This is of course compact and contains $X$ as a dense subspace, thus defining a compactification of $X$ in the usual topological sense. In case that $X$ is a smooth manifold one may of course try to embed $X$ into a compact smooth manifold $Y$ and then form the closure in there. However, at this level of generality, it is very hard to control the ‘boundary’ $\overline{X} \setminus X$ that is added to $X$ in order to obtain the compactification. For example it may be unclear whether this boundary inherits some kind of intrinsic smooth structure, and $\overline{X}$ may be very badly behaved.

Let us specialize to the case that $X$ is a homogeneous space $X = H/K$, where $H$ is a Lie group and $K \subset H$ is a closed subgroup. Then one can try to exploit the homogeneous structure by embedding $H/K$ into a compact homogeneous space in an equivariant way. This leads to the concept of a homogeneous compactification.

**Definition 2.1.** Let $H$ be a Lie group and $K \subset H$ a closed subgroup. Then a **homogeneous compactification** of $H/K$ is defined by an embedding $i : H \to G$, of $H$ as a closed subgroup of a Lie group $G$, and a closed subgroup $P \subset G$ such that $G/P$ is compact and $P \cap H = K$.

An embedding as in this definition descends to an embedding $i : H/K \to G/P$ of the homogeneous space $H/K$ into the compact homogeneous space $G/P$, which by construction is $H$-equivariant. Hence one obtains a compactification of $H/K$ by forming the closure $\overline{H/K} \subset G/P$, and we will also refer to this closure as the homogeneous compactification of $H/K$.

In this situation, we can already get some basic information on the structure of the boundary $\overline{H/K} \setminus H/K$ that is added to $H/K$ in order to obtain the compactification. Indeed, the closed subgroup $H \subset G$ naturally acts on the homogeneous space $G/P$ by the restriction of the canonical action of $G$. Using this, we can prove the following result.

**Proposition 2.2.** Consider a homogeneous compactification of $H/K$ given by the embedding $H/K \hookrightarrow G/P$. Then the boundary $\overline{H/K} \setminus H/K$ naturally is a union of $H$-orbits in $G/P$, each of which is an initial submanifold of $G/P$.

**Proof.** As a subset of $G/P$, the space $H/K$ of course is $H$-invariant, which readily implies that the closure $\overline{H/K} \subset G/P$ is $H$-invariant, too. But given a smooth action of $H$ on a manifold, any invariant subset is a union of orbits. On the other hand, since the orbits of a smooth action can be realized as leaves of a foliation (of non-constant rank), they are automatically initial submanifolds, compare with [22, Theorem 5.14].

This statement means that the smooth structure of the individual $H$-orbits in $G/P$ is completely understood: Given one of the orbits, say $O := H \cdot gP \subset G/P$ let $L$ be the stabilizer of $gP$ in $H$. Then there is an injective immersion $j : H/L \to G/P$ whose image coincides with $O$. So we can form $j^{-1} : O \to H/L$ and for any manifold $M$, a function $f : M \to G/P$ with values in $O$ is smooth if and only if $j^{-1} \circ f : M \to H/L$ is smooth.
This result is just a small first step. At this point the question of how the different orbits are ‘pieced together’, and what additional structure each might have, remains to be resolved.

2.2. Parabolic compactifications

We now specialize homogeneous compactifications to the case that $G$ is a semisimple Lie group and $P \subset G$ is a parabolic subgroup. It is well known that this automatically implies that $G/P$ is compact. Hence we consider an inclusion $H \hookrightarrow G$, and put $K = H \cap P$. This makes the $H$-orbit of $o := eP \in G/P$ isomorphic to $H/K$ and we consider the compactification of $H/K$ given by the closure of this orbit. In fact, we specialize things a bit further, as follows.

**Definition 2.3.** Let $H$ be a Lie group and $K \subset H$ a closed subgroup. Then a parabolic compactification of the homogeneous space $H/K$ is a homogeneous compactification, for which the group $G$ is semisimple, the subgroup $P \subset G$ is parabolic and which has the property that $H/K$ is open in $G/P$.

The motivation for this specialization is the following. For a parabolic subgroup $P$ in a semisimple Lie group $G$, the homogeneous space $G/P$ carries a natural geometric structure, which is fairly well understood. Indeed, $G/P$ is the homogeneous model of parabolic geometries of type $(G, P)$. This means that immediately a large number of geometric tools are available. Since parabolic geometries can be restricted to open subsets, the homogeneous space $H/K$ inherits a locally flat parabolic geometry of type $(G, P)$. Since this is an $H$-invariant geometric structure on $H/K$, its existence is usually clear in advance. However it is just one of the $H$-invariant geometric structures available on $H/K$ and for this specific geometric structure we get the crucial additional information that it admits a smooth extension across the boundary of the compactification. The same holds for all bundles and natural operations associated to this structure. All together these provide powerful tools to study the structure of the boundary and its relation to the geometry $H/K$.

The inclusion $H \hookrightarrow G$ has a nice interpretation in the language of parabolic geometries of type $(G, P)$. Indeed, it defines a holonomy reduction of the Cartan geometry $G \to G/P$ in the sense studied in [10]. The main topic of [10] is extending properties of a reduction of the homogeneous model to cases of curved geometries. Hence any parabolic compactification automatically comes with a notion of curved analog which we describe next.

Recall that a parabolic geometry of type $(G, P)$ on a smooth manifold $M$ is given by a principal $P$-bundle $\mathcal{G} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. This generalizes the projection $G \to G/P$ and the left Maurer Cartan form. It is possible to extend the structure group of $\mathcal{G}$ to $G$ by forming $\tilde{\mathcal{G}} := \mathcal{G} \times_P G$ and, on this principal $G$-bundle, $\omega$ induces a canonical principal connection $\tilde{\omega}$. Given the subgroup $H \subset G$, one can form the associated bundle $\mathcal{G} \times_P (G/H)$. This can alternatively be realized as $\tilde{\mathcal{G}} \times_G (G/H)$ and hence inherits an Ehresmann connection. In the curved setting, a holonomy reduction corresponding to $H \subset G$ is then defined as a parallel section of that
bundle. It is a classical result that this defines a holonomy reduction of \( \tilde{G} \) compatible with \( \tilde{\omega} \), but there is additional information available based on the relative position of the reduced bundle to \( G \subset \tilde{G} \). As shown in [10], one may associate to each \( x \in M \) an \( H \)-orbit in \( G/P \) and the points in \( M \) corresponding to one such orbit are said to form a curved orbit. In particular, the \((H \text{-orbit of the})\ base point \( o = eP \), for which \( K = H \cap P \), determines a type of curved orbit. Using this, we can give the definition of curved analogs.

**Definition 2.4.** Consider a parabolic compactification defined by \( H/K \hookrightarrow G/P \), so \( K = H \cap P \). Then a curved analog of this compactification is defined as follows. Consider a parabolic geometry of type \((G, P)\) on a smooth manifold \( M \) together with a holonomy reduction to the group \( H \) such that the curved orbit of the type determined by \( eP \in G/P \) is non-empty and has compact closure in \( M \). Then this closure provides the compactification of the given curved orbit.

The conditions on compactness of the closure of the curved orbit is of course satisfied automatically if the ambient manifold \( M \) is compact. Since the developments in [10] are phrased in a geometric language, they provide the basis for the geometric study of parabolic compactifications that we are initiating in this article.

**Example 2.5.** We start with two examples of parabolic compactifications for which curved analogs are already intensively studied in the literature (and for these there is even a more general notion of curved analog than that mentioned above). Then we provide one more general family of examples.

(1) Consider the connected group \( G := SO_0(n + 2, 1) \) defined by a non-degenerate bilinear form \( \langle , \rangle \) of signature \((n + 2, 1)\) on \( \mathbb{R}^{n+3} \). Choose a vector \( v_0 \in \mathbb{R}^{n+3} \) such that \( \langle v_0, v_0 \rangle = 1 \) and let \( H \subset G \) be the stabilizer of \( v_0 \) in \( G \). Via the action on the orthocomplement \((v_0)^\perp \), the group \( H \) is identified with \( SO_0(n + 1, 1) \).

It is well known that \( G \) acts transitively on the space of future directed isotropic rays in \( \mathbb{R}^{n+3} \) which is diffeomorphic to \( S^{n+1} \) and that this provides the homogeneous model for oriented Riemannian conformal structures in dimension \( n + 1 \). Now given an isotropic ray \( \ell \), we can look at \( \langle v, v_0 \rangle \) for some \( v \in \ell \). Whether this is positive, negative or zero is independent of the choice of \( v \) and it is easy to see that this strict sign describes the full decomposition of the space of isotropic rays into \( H \)-orbits. So there are two open orbits and one closed orbit. Taking \( P \subset G \) to be the stabilizer of a ray \( \mathbb{R}_+ \cdot v \) with \( \langle v, v_0 \rangle > 0 \), we thus obtain a parabolic compactification \( H/(H \cap P) \hookrightarrow G/P \). Now \( K := H \cap P \) clearly coincides with the stabilizer in \( H \) of the ray obtained by projecting \( \ell \) into \( (v_0)^\perp \), and since \( \ell \) is isotropic, this projection must be spanned by a vector that is timelike, in that it is of negative length according to \( \langle , \rangle \). Thus \( K \) coincides with the stabilizer of that timelike vector, and \( H/K \equiv SO_0(n + 1, 1)/SO(n + 1) \), so this is the hyperbolic space of dimension \( n + 1 \).

On the other hand, the closed curved obt is just the space of isotropic rays in \( (v_0)^\perp \), so this is the sphere \( S^n \) viewed as a homogeneous space of \( SO_0(n + 1, 1) \). In fact the parabolic compactification obtained in this case is the conformal compactification of hyperbolic space with the conformal sphere as its boundary at infinity (i.e. the Poincaré...
model of hyperbolic space) [16]. Curved analogs of this parabolic contactification are defined as reductions of conformal holonomy of conformal manifolds of dimension $n + 1$. As discussed in [10, § 3.5], the results of [16] show that curved holonomy reductions of type $H \hookrightarrow G$ are equivalent to Poincaré–Einstein metrics. The general concept of conformally compact metrics, as introduced by Penrose, can then be viewed as a further weakening of the concept of this type of holonomy reduction.

(2) Consider $G := \text{SL}(n + 2, \mathbb{R})$ and $H := \text{SO}(n + 1, 1) \subset G$ defined by the choice of a Lorentzian metric $(,)$ on $\mathbb{R}^{n+2}$. Then $G$ acts on the space of rays in $\mathbb{R}^{n+2}$, thus providing the homogeneous model $S^{n+1}$ of oriented projective structures in dimension $n + 1$. As in example (1) above, the $H$-orbits on $S^{n+1}$ are determined by the signature of the restriction of $(,)$ to a ray. So again there are two open orbits corresponding to positive or negative restriction and there is one closed orbit corresponding to lines which are isotropic for $(,)$. Choosing $P$ to be the stabilizer of a timelike ray, we obtain $K := P \cap H \cong \text{SO}(n + 1)$ so we obtain another parabolic compactification of hyperbolic space $H/K \hookrightarrow G/P$. As in example (1) above, the boundary is the space of isotropic lines, so this is again the conformal sphere $S^n$. However, this time we obtain the Klein model, which is a projective compactification of hyperbolic space. As discussed in detail in [9], while the compactifications from examples (1) and (2) of course are isomorphic as topological compactifications, they are different from a geometric point of view. Curved analogs of this parabolic compactification are Klein–Einstein structures, see [9] and [10, §§3.1–3.3]. This concept can be further generalized to projective compactness of order 2, as introduced in [6].

(3) Generalizing the situation from (2), consider the Grassmannian $Gr(p, \mathbb{R}^{p+q})$, which can be realized as $G/P$, with $G = \text{SL}(p + q, \mathbb{R})$ and $P$ the stabilizer of a $p$-dimensional subspace $V$ in the standard representation $\mathbb{R}^{p+q}$ of $G$. Consider the action of the subgroup $H := \text{SO}(p,q) \subset G$ on the Grassmannian. Choosing $V$ in such a way that the bilinear form defining $H$ is positive definite on $V$, we see that $K := H \cap P \cong S(O(p) \times O(q))$, so $H/K$ is a Riemannian symmetric space. Linear algebra implies that the $H$-orbit of $V$ in $G/P$ consists of all subspaces for which the restricted bilinear form is positive definite, so this orbit is clearly open, and we have constructed a parabolic compactification of $H/K$. The space $G/P$ is the homogeneous model for (almost) Grassmannian structures of type $(p,q)$. This geometry is available on manifolds of dimension $pq$ and on a manifold $M$ such a structure is basically given by an identification of $TM$ with a tensor product of two auxiliary vector bundles of rank $p$ and $q$, respectively. The additional information completing this is a suitable identification of the top exterior powers of the auxiliary bundles. Now in the symmetric decomposition $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{m}$, it is easy to see that $\mathfrak{m}$ can be identified with the space matrices of size $p \times q$ endowed with the natural representation of $\mathfrak{g} \cong o(p) \times o(q)$. This shows that $H/K$ carries an $H$-invariant almost Grassmannian structure, which is also nicely compatible with the $H$-invariant Riemannian metric. Now this Riemannian metric is complete, so there is no hope to smoothly extend is across the boundary of $H/K$ in $G/P$. In contrast, the Grassmannian structure on $H/K$ does extend across the boundary.
2.3. Wolf’s theorem

We next describe a general scheme which can be used to generate large families of parabolic compactifications. The basis for this is the following theorem of Wolf from 1974, which is the main result of [31].

**Theorem 2.6.** Let $G$ be a real semisimple Lie group, let $\theta : G \to G$ be an involutive automorphism, and let $H \subset G$ be the fixed point group of $\theta$. Then for any parabolic subgroup $P \subset G$, the $H$-action on the generalized flag manifold $G/P$ has only finitely many orbits. In particular, there are both open and closed $H$-orbits in $G/P$.

The relevance of this result for our purposes is evident. Suppose that $H = G^\theta$ is the fixed point group of an involutive automorphism of a real semisimple Lie group. Then for any parabolic subgroup $P \subset G$, there are open $H$-orbits in $G/P$. Replacing $P$ by a conjugate subgroup if necessary, we may assume that the $H$-orbit of $eP$ is open. Then using Proposition 2.2, we get:

**Corollary 2.7.** Let $G$ be a semisimple Lie group, $\theta : G \to G$ an involutive automorphism and $H := G^\theta$ its fixed point group. Let $P \subset G$ be a parabolic subgroup such that the $H$-orbit of $eP$ is open in $G/P$. Then putting $K := H \cap P$, we obtain a parabolic compactification $H/K \hookrightarrow G/P$ whose boundary $\overline{H/K} \setminus H/K$ is the union of finitely many $H$-orbits in $G/P$, each of which is an initial submanifold.

We next discuss two sources of examples of involutive automorphisms of a classical real semisimple group $G$, for which we also get a description of $H$ as a stabilizer.

**Example 2.8.** (1) Let $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, with conjugation being defined as the identity on $\mathbb{R}$ or as the usual conjugations on $\mathbb{C}$ and $\mathbb{H}$. Suppose that $G = SL_K(V)$ for a finite dimensional $K$-vector space $V$. Then let $(\langle , \rangle)$ be a non-degenerate sesquilinear form on $V$ and for $A \in GL(V)$ let $A^*$ be the adjoint of $A$ with respect to $(\langle , \rangle)$, so $(Av, w) = \langle v, A^*w \rangle$ for all $v, w \in V$. This definition readily implies that $(AB)^* = B^*A^*$ and hence $\theta(A) := (A^{-1})^*$ defines an involutive automorphism of $G$ whose fixed point group $H$ is the special unitary group of $(\langle , \rangle)$. Depending on $K$, this group is (isomorphic to) $SO(p, q)$, $SU(p, q)$ or $Sp(p, q)$, where $(p, q)$ is the signature of $(\langle , \rangle)$.

Now the generalized flag varieties $G/P$ in this case are just the manifolds of all partial flags $V_1 \subset V_2 \cdots \subset V_k \subset V$ of $K$-subspaces of fixed dimensions. In the simplest case, $G/P$ is the Grassmannian of $i$-dimensional subspaces $V_i \subset V$. Here linear algebra shows that the $H$-orbits on $G/P$ are characterized by the rank and signature of the restriction of $(\langle , \rangle)$ to $V_1$ and the open orbits are exactly those for which the restriction is non-degenerate. Taking the signatures to be $(p, q)$ on $V$ and $(r, i-r)$ on $V_1$ (which implies that $(\langle , \rangle)$ is non-degenerate on $V_1$), we get that $V = V_1 \oplus V_1^\perp$. This decomposition is preserved by the stabilizer of $V_1$ in $H$, which, for $K = \mathbb{R}$ is thus easily seen to be isomorphic to $SO(d; i-r) \times SO(p-r, q-i+r)$ and similarly in other cases. In particular taking $i = r = p$ and $K = \mathbb{C}$, we obtain a parabolic compactification of the Hermitian symmetric space $SU(p, q)/SU(p) \times U(q)$, which is easily seen to coincide with the Bailey–Borel compactification. For $K = \mathbb{R}$ and $\mathbb{H}$, we obtain analogous compactifications for Riemannian symmetric spaces of $SO(p, q)$ and $Sp(p, q)$.
Parabolic compactifications

The first parts of this directly generalize to all flag manifolds. For a flag $V_1 \subset V_2 \subset \cdots \subset V_k \subset V$, the rank and signature of the restriction of $\langle , \rangle$ to each constituent $V_j$ of the flag is of course constant along each orbit. Moreover, linear algebra easily shows that the flags for which each of these restrictions is non-degenerate of fixed signature form a single open orbit and that these are the only open orbits. In such a case, putting $\dim(V_j) = i_j$ for each $j$ and denoting the signature of the restriction by $(r_j, s_j)$, we thus obtain, for $K = \mathbb{R}$, parabolic compactifications of the homogeneous spaces

$$SO(p,q)/S\left(\prod_{j=1}^{k+1} O(r_j - r_{j-1}, s_j - s_{j-1})\right),$$

where we put $r_0 = s_0 = 0$, $r_{k+1} = p$ and $s_{k+1} = q$. This works in the same way for the other ground fields. Notice that these are not symmetric spaces for $k \geq 2$. It turns out that, in contrast to the case of Grassmannians, the degenerate orbits are not determined by the ranks and signatures of the restrictions of $\langle , \rangle$ to the constituents of the flag any more, so things get more complicated. We will discuss this further in §3.1.

(2) The second source of involutions is even simpler and more versatile. Again we put $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and we fix a $K$-vector space $V$. For $G$ we either take the group $SL_K(V)$ or the subgroup of $SL_K(V)$ preserving a bilinear form $b$ on $V$ (which may be either symmetric or skew symmetric, and either $K$-bilinear or Hermitian). Then suppose that $J : V \to V$ is either $K$-linear or conjugate linear such that $J^2 = \epsilon \text{id}_V$, where $\epsilon = \pm 1$. Then the map $\theta(A) := \epsilon JAJ$ defines an involutive automorphism on $SL_K(V)$. If a bilinear form $b$ is involved, then depending on the properties of $b$, one has to require either that $b(Jv, Jw) = b(v, w)$ or that $b(Jv, Jw) = -b(v, w)$ to ensure that $\theta$ restricts to an involutive automorphism of the subgroup $G$. In any case, the fixed point group $H$ of $\theta$ then consists of those elements $A \in G$ which commute with $J$.

The simplest instance of this is provided by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and viewing multiplication by $i$ as a linear endomorphism $J$ of $\mathbb{R}^{2n}$ such that $J^2 = -\text{id}$. Starting from $G = SL(2n, \mathbb{R})$, we easily conclude from Wolf’s theorem that $H := SL(n, \mathbb{C}) \subset G$ acts with finitely many orbits on each partial flag manifold. In particular, we can look at the Grassmannian $Gr(n, \mathbb{R}^{2n})$ for which linear algebra implies that there is only one open $H$-orbit, namely the one consisting of those subspaces $W$ which are totally real in the sense that $W \cap J(W) = \{0\}$. Taking $P \subset G$ to be the stabilizer of $\mathbb{R}^n \subset \mathbb{C}^n$, we see that $H \cap P = SL(n, \mathbb{R}) \subset SL(n, \mathbb{C})$. Thus this example gives rise to a parabolic compactification of the symmetric space $SL(n, \mathbb{C})/SL(n, \mathbb{R})$.

Similarly, we can identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$, endowed with the complex structure provided by $i \in \mathbb{H}$, and view multiplication by $j \in \mathbb{H}$ as a conjugate linear map $J$ on $\mathbb{C}^{2n}$ such that $J^2 = -\text{id}$. Hence we conclude that the subgroup $SL(n, \mathbb{H})$ acts with finitely many orbits on each generalized flag manifold of $G := SL(2n, \mathbb{C})$.

Bringing bilinear forms into the game, we obtain further interesting examples. In the situation of $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we can consider a complex bilinear form $b$ on $\mathbb{C}^n$ and view its imaginary part as a real symmetric bilinear form on $\mathbb{R}^{2n}$. It is easy to see that this bilinear form has split signature $(n, n)$ and clearly the involution defined by $J$ can be restricted to the orthogonal group $G := SO_0(n, n)$ of this split-signature form. A complex
linear map which preserves the imaginary part of \( b \) preserves the whole form \( b \), which easily implies that as a fixed point group, we obtain \( H := SO(n, \mathbb{C}) \subset G \). We will discuss this case in more detail in §4, where we shall in particular see that it gives rise to a parabolic compactification of \( SO(n, \mathbb{C})/SO(n) \), which is a Riemannian symmetric space of non-compact type.

As a final example, let \( \langle \cdot, \cdot \rangle \) be the standard quaternionic Hermitian form on \( \mathbb{H}^n \) and consider the form \( \omega \) on \( \mathbb{H}^{2n} \) defined by \( \omega((p_1), (q_1)) := \langle p_1, q_2 \rangle - \langle p_2, q_1 \rangle \). One immediately verifies that this form is skew-Hermitian in the quaternionic sense, so the group of quaternionically linear automorphisms of \( \mathbb{H}^{2n} \) that preserve \( \omega \) form a Lie group \( G \) that is commonly denoted by \( SO^*(4n) \) and is a real form of \( SO(4n, \mathbb{C}) \). On the other hand \( J(p_1^{(p_2)}) = (p_2^{(p_1)}) \) defines an \( \mathbb{H} \)-linear automorphism of \( \mathbb{H}^{2n} \) such that \( \omega(Jp, jq) = -\omega(p, q) \) holds for all \( p, q \in \mathbb{H}^{2n} \). Now one easily verifies that the Lie algebra \( g \) of \( G \) consists of block matrices of the form \( (\begin{smallmatrix} A & B \\ -B^* & A^* \end{smallmatrix}) \), where \( A, B \), and \( C \) are quaternionic \( n \times n \)-matrices, \( B = B \) and \( C = C \). As we have seen above, the fixed point group \( H \) of the involution determined by \( J \) consists of all matrices commuting with \( J \). On the Lie algebra level, this means that for the above block form, we in addition get \( A = -A^* \) and \( C = B \), which easily implies that \( H \) is identified with \( GL(n, \mathbb{H}) \) via its action on the subspace of all elements of the form \( (p_1) \), which form an \( n \)-dimensional quaternionic subspace of \( \mathbb{H}^{2n} \).

The generalized flag varieties of \( G \) are the manifolds of flags of quaternionic subspaces in \( \mathbb{H}^{2n} \), which are isotropic for \( \omega \). Looking at the example of the Grassmannian of maximal (i.e. \( n \)-dimensional) isotropic subspaces, we see that the dimension of the span of a subspace and its image under \( J \) is constant on \( H \)-orbits. In particular, if \( V \) is a maximally isotropic subspace such that \( V \) and \( J(V) \) span all of \( \mathbb{H}^{2n} \) the \( H \)-orbit of \( V \) is open. Now consider the subspace of all vectors of the form \( (p_1) \), which clearly has that property. The Lie algebra of the stabilizer of that subspace in \( H \) are matrices of the form \( (\begin{smallmatrix} A & 0 \\ 0 & A^* \end{smallmatrix}) \) with \( A^* = -A \). This shows that the stabilizer will be a subgroup isomorphic to \( Sp(n) \subset GL(n, \mathbb{H}) \). Thus this example leads to a parabolic compactification of the Riemannian symmetric space \( GL(n, \mathbb{H})/Sp(n) \).

### 2.4. Orbits and infinitesimal transversals

Given a parabolic compactification of \( H/K \) coming from an embedding into \( G/P \), we next discuss some general tools that can be used to identify the \( H \)-orbits in \( G/P \) as well as an infinitesimal model for the local structure around such an orbit. The first thing to point out here is that there is an issue of ‘relative position’ between \( H \) and \( P \) that may look unfamiliar. Looking at \( H = SO(p, q) \subset SL(p + q, \mathbb{R}) = G \) one usually does not specify the bilinear form preserved by \( H \) explicitly, since any two choices lead to conjugate subgroups. Likewise, looking at the stabilizer \( P \subset G \) of an \( i \)-dimensional subspace of \( \mathbb{R}^{p+q} \), the subspace is not specified explicitly for the same reason. However, fixing both data at the same time, the relative position is encoded in the rank and signature of the restriction of the chosen bilinear form to the chosen subspace, which is clearly invariant under simultaneous conjugations of both subgroups.

To deal with these issues in practice, we will fix an \( H \)-orbit \( O \subset G/P \). Then we will choose a subgroup \( H_O \subset G \) conjugate to \( H \) in such a way that the orbit of the base
Let \( \mathcal{O} \) be an \( H \)-orbit of \( o = eP \in G/P \) isomorphic to \( \mathcal{O} \). Basically, this means that we keep \( P \subset G \) fixed and arrange \( H_\mathcal{O} \) in such a way that the ‘right’ relative position is achieved. Having done that, that stabilizer of \( o \) in \( H_\mathcal{O} \) coincides with \( H_\mathcal{O} \cap P \) and thus \( \mathcal{O} \cong H_\mathcal{O}/(H_\mathcal{O} \cap P) \). In particular, having determined \( H_\mathcal{O} \cap P \), we can readily read off the codimension of \( \mathcal{O} \) in \( G/P \).

Identifying \( \mathcal{O} \) with a homogeneous space of \( H_\mathcal{O} \) makes all the standard tools for analyzing the geometry of homogeneous spaces available in our situation. In particular, we can easily obtain an infinitesimal model for a neighborhood of \( \mathcal{O} \) by analyzing the geometry of homogeneous spaces available in our situation. In particular, having determined \( H_\mathcal{O} \cap P \), we can readily read off the codimension of \( \mathcal{O} \) in \( G/P \).

\[ \text{Proposition 2.9.} \quad \text{Let } \mathcal{O} \cong H_\mathcal{O}/(H_\mathcal{O} \cap P) \subset G/P \text{ be an } H \text{-orbit. Then the quotient bundle } T(G/P)|_\mathcal{O}/TO \to \mathcal{O} \text{ is the homogeneous vector bundle induced by the representation of } H_\mathcal{O} \cap P \text{ on } (g/p)/(h_\mathcal{O}/(h_\mathcal{O} \cap p)) \text{ described above.} \]

**Proof.** By assumption, \( \mathcal{O} \) is the \( H_\mathcal{O} \)-orbit of \( o = eP \in G/P \). Clearly, the derivatives of the actions of elements of \( H_\mathcal{O} \) on \( G/P \) make \( T(G/P)|_\mathcal{O}/\mathcal{O} \) into a homogeneous vector bundle. It is well known that \( T_o(G/P) = g/p \) with the natural representation of \( P \) coming from the adjoint representation. For the subgroup \( H_\mathcal{O} \cap P \subset P \), we obtain the representation described above, so the general description of homogeneous bundles shows that \( T(G/P)|_\mathcal{O} \cong H_\mathcal{O} \times_{H_\mathcal{O} \cap P} (g/p) \). Thus the result follows from the standard description of \( TO \) as a homogeneous vector bundle. \( \square \)

**Remark 2.10.** If the action of \( H_\mathcal{O} \) on \( G/P \) were proper, then this result would directly imply a description of a tubular neighborhood of \( \mathcal{O} \) in \( G/P \) and its decomposition into \( H_\mathcal{O} \)-orbits via the slice theorem for proper actions, see [27]. By properness of the action one would get a positive definite inner product on \( g/p \) which is invariant under the action of \( H_\mathcal{O} \cap P \), thus defining a \( H_\mathcal{O} \)-invariant Riemannian metric on \( G/P \). The normal bundle of the orbit then is induced by the orthocomplement of \( h_\mathcal{O}/(h_\mathcal{O} \cap p) \) in \( g/p \), and a slice is obtained from exponentiating this. However, since \( G/P \) is compact, properness of the action would imply that \( H_\mathcal{O} \) has to be compact, which is absurd in our setting. Hence the general slice theorem is never applicable in our situation.

Still we shall derive local descriptions of a neighborhood of \( \mathcal{O} \) in \( G/P \) which are similar to the one obtained from the slice theorem in several cases below. Thus in these cases, the orbit structure close to \( \mathcal{O} \) again is described in terms of orbits of the isotropy group \( H_\mathcal{O} \cap P \) on the representation \((g/p)/(h_\mathcal{O}/(h_\mathcal{O} \cap p))\).

### 2.5. Characterizing the subgroup \( H \)

As indicated above, we want to study a parabolic compactification of a homogeneous space \( H/K \) using the restriction of the natural parabolic geometry on the ambient homogeneous space \( G/P \). One of the important features of parabolic geometries is the...
existence of a special type of geometric objects, which are called tractors. These are closely related to representations of the group $G$, so $H$-invariant elements in representations of $G$ will be of particular importance for the further development.

Observe first that in all the cases discussed in Example 2.8, we can characterize $H$ as the stabilizer of an appropriate element in a relatively simple representation of $G$. Indeed, in the situation of part (1) of Example 2.8, we put $G = SL_K(V)$ and $H$ is the special unitary group of a Hermitian form $(,)$ of $G$. Hermitian bilinear forms form a subrepresentation of $S^2V^*$, so we have characterized $H$ as the stabilizer in $G$ of an element in that representation. Likewise, in the situation of part (2) of Example 2.8, we have $G \subset SL_K(V)$ and an endomorphism $J$ of $V$. This defines an element in (an appropriate subrepresentation of) the $G$-representation $V^* \otimes V$ of endomorphisms of $V$, whose stabilizer in $G$ is $H$.

We next show that a similar characterization is available in a fairly general situation.

**Proposition 2.11.** Let $G$ be a classical simple Lie group with standard representation $W$ and let $H \subset G$ be a connected closed semisimple Lie subgroup. Suppose that either $W$ is a complex representation of $G$ which is irreducible for $H$ or that the complexification of $W$ is irreducible for $H$.

Then denoting by $\mathfrak{g}$ the Lie algebra of $G$, the representation $V := \mathfrak{sl}(\mathfrak{g})$ of $G$ contains an element $v_0$, whose stabilizer in $G$ is $H$.

**Proof.** Via the restriction of the adjoint representation of $G$, the Lie algebra $\mathfrak{g}$ is a representation of $H$, and of course $\mathfrak{h} \subset \mathfrak{g}$ is an $H$-invariant subspace. Since $\mathfrak{h}$ is semisimple, there is an $\mathfrak{h}$-invariant subspace $E \subset \mathfrak{g}$ which is complementary to $\mathfrak{h}$. Since $H$ is connected, the subspace $E$ is $H$-invariant. Now let $v_0 \in L(\mathfrak{g}, \mathfrak{g})$ be the map which acts as the identity on $\mathfrak{h}$ and by multiplication by $-\frac{\dim(\mathfrak{h})}{\dim(W)}$ times the identity on $W$, where the factor is chosen to ensure that $v_0$ is trace-free.

The natural action of $H$ on $V := \mathfrak{sl}(\mathfrak{g})$ is given by $h \cdot v = \text{Ad}(h) \circ v \circ \text{Ad}(h)^{-1}$, which readily implies that $v_0$ is $H$-invariant. Since $H$ is connected, we can show that it coincides with the stabilizer of $v_0$ in $G$ by proving that if $A \in \mathfrak{g}$ satisfies $0 = A \cdot v_0$, then $A \in \mathfrak{h}$. Now of course $A \cdot v_0 = \text{ad}(A) \circ v_0 - v_0 \circ \text{ad}(A)$, so $A \cdot v_0 = 0$ means that $\text{ad}(A)$ commutes with $v_0$. In particular, $\text{ad}(A)$ must preserve the eigenspaces of $v_0$, so $\text{ad}(A)(\mathfrak{h}) \subset \mathfrak{h}$. Now the restriction of $\text{ad}(A)$ to $\mathfrak{h}$ is a derivation by the Jacobi identity, and since $\mathfrak{h}$ is semisimple, any such derivation is inner. Hence there is an element $B \in \mathfrak{h}$ such that $\text{ad}(A)|_{\mathfrak{h}}$ coincides with $\text{ad}(B)|_{\mathfrak{h}}$. Hence $A - B \in \mathfrak{h}$ has the property that $[A - B, X] = 0$ for any $X \in \mathfrak{h}$. This means that, as an endomorphism of $W$, $A - B$ commutes with each element of $\mathfrak{h}$. Passing to the complexification if necessary, irreducibility implies that $A - B$ must be a multiple of the identity. Since $\mathfrak{g}$ is simple, our assumptions imply that it consists of trace-free maps on $W$ respectively its complexification, which implies $A = B$. □

**Remark 2.12.** Evidently, some choice is involved in the construction used in the proof of Proposition 2.11. One could actually use a finer decomposition of the $H$-invariant complement $W$ to $\mathfrak{h}$ in $\mathfrak{g}$, say into $\mathfrak{h}$-isotypical components or into $\mathfrak{h}$-irreducibles, and then choose a map acting by different scalars on the individual components. Any such choice works as long as $\mathfrak{h} \subset \mathfrak{g}$ is one of the eigenspaces of $v_0$. 


The disadvantage of the general construction in Proposition 2.11 is that the representation $\mathfrak{sl}(g)$ used there is already fairly complicated. So while this characterization of the subgroup $H$ is also available in the situations discussed in Example 2.8 it will be much more efficient to work with the simpler characterizations available in these cases.

2.6. Tractor bundles and the BGG machinery

We next review the machinery for parabolic geometries we need, referring to [11, Chapter 3.2] for details. Let $G$ be a semisimple Lie group, $P \subset G$ a parabolic subgroup and let $p \subset g$ be the Lie algebras. Then we can form the reductive Levi decomposition $p = g_0 \oplus p_+$ of $p$ into a reductive Lie subalgebra $g_0 \subset p$ and a nilpotent ideal $p_+ \subset p$. Next, one defines a closed subgroup $G_0 \subset P$ as consisting of those elements whose adjoint actions preserve this decomposition. It then turns out that the map $(g_0, Z) \mapsto g_0 \exp(Z)$ defines a diffeomorphism $G_0 \times p_+ \rightarrow P$, so in particular $P_+ := \exp(p_+)$ is a nilpotent normal subgroup in $P$ such that $P/P_+ \cong G_0$.

This has consequences for the representation theory of $P$. The Lie algebra $p_+$ has to act trivially on any irreducible representation of $P$. In particular, any completely reducible representation of $P$ is obtained by trivially extending a completely reducible representation of the reductive group $G_0$. (For $G_0$, complete reducibility of a representation just means that the center acts diagonalizably.) On the other hand, a general representation $V$ of $P$ inherits a $P$-invariant filtration of the form $V = V^0 \supset V^1 \cdots \supset V^N \supset \{0\}$, which is defined recursively by $V^{i+1} = p_+ \cdot V^i \subset V^i$. Assuming that the center of $G_0$ acts diagonalizably on $V$, each of the subsequent quotients $V^i/V^{i+1}$ is a completely reducible representation of $P$. In particular, $V/V^1$ is the canonical completely reducible quotient of $V$.

Via forming associated bundles, representations of $P$ give rise to natural vector bundles on manifolds endowed with a parabolic geometry of type $(G, P)$. In the case of the homogeneous model, these are the usual homogeneous vector bundles, so $V$ corresponds to the bundle $V := G \times_p V \rightarrow G/P$. An equivariant map between two representations of $P$ induces a vector bundle map between the corresponding bundles. In particular, the $P$-invariant filtration $\{V^i\}$ of $V$ gives rise to a filtration of $V$ by smooth subbundles $V^i \subset V$ such that each of the successive quotients $V^i/V^{i+1}$ is a completely reducible bundle. Here we say that a natural bundle is completely reducible if it is induced by a completely reducible representation of $P$.

Another important class of associated bundles are tractor bundles, which correspond to representations of $P$ which are obtained as restrictions of representations of $G$. An important feature of a tractor bundle is that, on general parabolic geometries, it carries a natural linear connection called the tractor connection. On the homogeneous model, the situation is even easier, see [11, §1.5.7]. Given a representation $V$ of $G$, the corresponding bundle $V = G \times_p V$ admits a canonical trivialization and the tractor connection is the flat connection induced by this trivialization. In particular, any element $v \in V$ defines a section $s_v \in \Gamma(V)$ which is parallel for the tractor connection. In particular, we can apply this to $H$-invariant elements as discussed in §2.5.

The machinery of BGG sequences, which was developed in [12] and [4], relates parallel sections of a tractor bundle $\mathcal{V}$ to sections of the completely reducible quotient $\mathcal{H}_0 := \mathcal{V}/\mathcal{V}^1$.
which satisfy a certain system of linear PDEs. Again this works for general parabolic geometries and assumes a particularly simple form on the homogeneous model. We summarize what we need here: The projection $V \to V/V^1$ induces a natural bundle map $\Pi: V \to H_0$, which in turn induces a tensorial operator $\Gamma(V) \to \Gamma(H_0)$ on the spaces of sections that will also be denoted by $\Pi$. The key fact for the BGG machinery is that there is a natural differential operator $S: \Gamma(H_0) \to \Gamma(V)$ which splits this tensorial projection, i.e. we get $\Pi(S(\sigma)) = \sigma$ for each $\sigma \in \Gamma(H_0)$. Apart from this splitting property, there is only one more property needed to characterize the operator $S$, namely that $\nabla S(\sigma)$ always has values in a certain natural subbundle $\ker(\partial^*) \subset T^*(G/P) \otimes V$.

Now there is a natural completely reducible quotient bundle $H_1$ of $\ker(\partial^*) \subset T^*(G/P) \otimes V$ and we denote by $\pi_H$ both the corresponding bundle projection and the induced tensorial operator on sections. Putting these ingredients together, we can define a differential operator $D: \Gamma(V/V_1) \to \Gamma(H_1)$ by $D(\sigma) := \pi_H(\nabla S(\sigma))$. This is the first BGG operator determined by $V$. A crucial feature is that the representation inducing $H_1$ can be determined by purely algebraic methods from $V$. It is given as a Lie algebra homology group which can be computed using Kostant’s theorem, but the details on this are not relevant for our purposes. The main point is that knowing $V$, the bundle $H_1$ as well as the order and the principal part of the first BGG operator can be determined algorithmically. We will not go into details on how this is done, but just state the corresponding results when we need them. Let us collect the information we will need for the further development.

**Theorem 2.13.** Let $G$ be a semisimple Lie group, $P \subset G$ be a parabolic subgroup and $G/P$ the corresponding generalized flag manifold. Let $V$ be a representation of $G$, $V \to G/P$ the corresponding tractor bundle, $H_0 \to G/P$ the canonical completely reducible quotient of $V$ and $\Pi: V \to H_0$ the corresponding projection.

Then there is a computable completely reducible natural bundle $H_1$ and an invariant differential operator $D: \Gamma(H_0) \to \Gamma(H_1)$ such that for $s \in \Gamma(V)$ the following are equivalent

(a) $s$ is parallel for the tractor connection $\nabla^V$ on $V$.

(b) For $\sigma := \Pi(s)$ we have $D(\sigma) = 0$ and $s = S(\sigma)$.

(c) There is an element $v \in V$ such that $s$ corresponds to the constant function $v$ in the natural trivialization of $V$.

In the situation of a parabolic compactification coming from an embedding $H \hookrightarrow G$, we can apply this theorem to an $H$-invariant element in some representation $V$ of $G$. This element gives rise to a parallel section of the tractor bundle $V$ corresponding to $V$, which in turn projects to a solution of a first BGG operator. As we shall see below such solutions can be used to separate $H$-orbits in $G/P$ and thus understand the boundary obtained by the compactification. Moreover, we can use such sections to construct local coordinates on $G/P$ which are nicely adapted to the decomposition into $H$-orbits, thus describing the local structure of the $H$-space $G/P$. For these applications, we crucially exploit that the parallel tractor captures information about the jet of the underlying BGG solution in a nice way.
It should be remarked that Theorem 2.13 partly generalizes to curved geometries. In particular, a parallel section of a tractor bundle always projects to a solution of the first BGG operator, although in the curved case not all solutions are obtained in that way. In any case, the version of Theorem 2.13 for curved geometries is sufficient to deal with curved analogs of parabolic compactifications in the sense of Definition 2.4.

2.7. Recovering jet information

The relation between a parallel tractor and the jet of the underlying solution of a first BGG operator can be described in great detail, but quite a bit of background is needed to formulate these results. Thus we restrict to a very special case, which is sufficient to deal with the examples discussed below. In particular, we only discuss the case that the parabolic subgroup $P \subset G$ corresponds to a so-called $|1|$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, which equivalently means that the nilradical $\mathfrak{p}_+$ of its Lie algebra is abelian. This allows us to avoid the use of weighted jets and of filtrations of the tangent bundle, and to formulate a uniform result without having to distinguish cases. Here we work in the setting of curved geometries, since restricting to the homogeneous model does not provide any simplification. Thus we consider a Cartan geometry $(p : G \to M, \omega)$ of type $(G, P)$ which satisfies the usual conditions of regularity and normality (cf. [10, 11]).

Let $V$ be a representation of $G$ and let $\{V^i\}$ be the $P$-invariant filtration of $V$. Then the definitions easily imply that $\mathfrak{g}_j \cdot V^i \subset V^{i+j}$ for $j = -1, 0, 1$. Hence for $v \in V^i$ we can map $X \in \mathfrak{g}_{-1}$ to $X \cdot v \in V^{i-1}$ and the class of this modulo $V^i$ depends only on the class of $v$ modulo $V^i$. Thus for each $i$, we obtain a well-defined linear map $\partial : V^i / V^{i+1} \to \mathfrak{g}_{-1}^* \otimes V^{i-1} / V^i$, which is easily seen to be $P$-equivariant. Since $\mathfrak{g}_{-1} \cong \mathfrak{g} / \mathfrak{p}$ and $TM = \mathcal{G} \times_\mathcal{P} (\mathfrak{g} / \mathfrak{p})$, the natural bundle corresponding to $\mathfrak{g}_{-1}^*$ is the cotangent bundle. Hence denoting by $\mathcal{V} = \mathcal{G} \times_\mathcal{P} V$ the tractor bundle induced by $V$ and by $\mathcal{V}^i \subset \mathcal{V}$ the subbundle corresponding to $V^i$, we get natural bundle maps $\partial : \mathcal{V}^i / \mathcal{V}^{i+1} \to T^* M \otimes \mathcal{V}^{i-1} / \mathcal{V}^i$ for each $i$. By definition $\partial$ is the zero map for $i = 0$ and it is well known that it is injective for $i > 0$.

To formulate the result, we need some facts on Weyl structures for parabolic geometries as discussed in [11, Chapter 5]. Weyl structures generalize the choice of a metric in a conformal class (or, more generally of a Weyl connection). They form a conceptual way to describe parabolic geometries as an equivalence class of simpler additional structures. Such structures always exist and choosing one of them, one gets an induced linear connection (the Weyl connection) on any natural bundle. Moreover, one also gets an isomorphism from any natural bundle to its associated graded bundle (with respect to the natural $P$-invariant filtration of the inducing representation), called a splitting of the filtration.

Suppose now that $s$ is a section of the tractor bundle $\mathcal{V} \to M$ induced by a representation $V$ of $G$. Let us denote by $\{V^i\}$ the $P$-invariant filtration of $V$ and by $\mathcal{V}^i \subset \mathcal{V}$ the subbundle corresponding to $V^i$. Via the natural projection $\Pi : \mathcal{V} \to \mathcal{V} / \mathcal{V}^1$, the section $s$ canonically determines a section $\sigma \in \Gamma(\mathcal{V} / \mathcal{V}^1)$. Now choosing a Weyl structure for the geometry in question, we obtain a corresponding Weyl connection on $\mathcal{V}$ and each of the subquotient bundles $\mathcal{V}^i / \mathcal{V}^{i+1}$. On the other hand, the splitting of the filtration defines an isomorphism $\mathcal{V} \to \bigoplus_{i \geq 0} \mathcal{V}^i / \mathcal{V}^{i+1}$. Under this isomorphism, $s$ corresponds to a
family of sections of the subquotient bundles. This isomorphism has the property that it maps each \( \mathcal{V}^j \) to \( \bigoplus_{i \geq j} \mathcal{V}^i / \mathcal{V}^{i+1} \) and the component in \( \mathcal{V}^j / \mathcal{V}^{j+1} \) is given by the natural quotient map. In particular, the component in \( \Gamma(\mathcal{V}^1) \) of the section corresponding to \( s \) coincides with \( \sigma \). Moreover, if for a point \( x \in M \), we have \( \sigma(x) = 0 \), then \( s(x) \in \mathcal{V}^1_x \), so the value of the component in \( \Gamma(\mathcal{V}^1 / \mathcal{V}^2) \) at the point \( x \) is \( s(x) + \mathcal{V}^2_x \), and does not depend on the choice of the Weyl structure.

**Proposition 2.14.** Suppose that \( P \subset G \) corresponds to a \(|1|\)-grading of \( \mathfrak{g} \). Consider a representation \( V \) of \( G \) endowed with its natural \( P \)-invariant filtration \( \{ \mathcal{V}^i \} \). Let \( M \) be a manifold endowed with a parabolic geometry of type \((G, P)\), let \( \mathcal{V} \to M \) be the tractor bundle determined by \( V \), and let \( \mathcal{V}^j \subset \mathcal{V} \) be the smooth subbundle corresponding to \( V^j \subset \mathcal{V} \). Consider the natural bundle map \( \partial : \mathcal{V}^1 / \mathcal{V}^2 \to T^*M \otimes \mathcal{V}^0 / \mathcal{V}^1 \) as defined above.

Let \( s \in \Gamma(\mathcal{V}) \) be a section that is parallel for the canonical tractor connection, and put \( \sigma := \Pi(s) \in \Gamma(\mathcal{V}^1) \). Choose a Weyl structure with Weyl connection \( \nabla \), and let \( \mu \in \Gamma(\mathcal{V}^2 / \mathcal{V}^1) \) be the component of the image of \( s \) under the splitting of the filtration as described above. Then for any point \( x \in M \), we have \( \nabla \sigma(x) = -\partial(\mu(x)) \).

In particular, if \( \sigma(x) = 0 \) (which implies that both \( \nabla \sigma(x) \) and \( \mu(x) \) are independent of the choice of the Weyl structure), we conclude that \( \sigma \) has vanishing one jet in \( x \) if and only if \( s(x) \in \mathcal{V}^2_x \subset \mathcal{V}_x \).

**Proof.** A description of the tractor connection \( \nabla^\mathcal{V} \) under the isomorphism \( \Gamma(\mathcal{V}) \cong \bigoplus_i \Gamma(\mathcal{V}^i / \mathcal{V}^{i+1}) \) defined by the Weyl structure is given in [11, Proposition 5.1.10]. This readily shows that the component of \( \nabla^\mathcal{V} \sigma \) in \( T^*M \otimes \mathcal{V}^0 / \mathcal{V}^1 \) is given by \( \nabla \sigma + \partial(\mu) \).

If \( s \) is parallel, this vanishes identically, which implies the first claim.

We have already noted above that \( \sigma(x) = 0 \) implies that \( \mu(x) \) is independent of the Weyl structure chosen. General facts about linear connections (see §2.8) imply that \( \nabla \sigma(x) \) is independent of \( V \). Vanishing one jet of \( \sigma \) in \( x \) is of course equivalent to \( \nabla \sigma(x) = 0 \) and hence to \( \partial(\mu(x)) = 0 \). We have noted above that \( \partial \) is injective, so this is equivalent to \( \mu(x) = 0 \). Assuming \( \sigma(x) = 0 \), \( \mu(x) \) coincides with the projection of \( s(x) \in \mathcal{V}^1_x \) to \( \mathcal{V}^1_x / \mathcal{V}^2_x \), which implies the last claim.

**Remark 2.15.** One can say more about the bundle map \( \partial \) from Proposition 2.14 depending on the order of the first BGG operator determined by \( V \). This is based on the developments in [2] for \(|1|\)-graded geometries, which use a different splitting operator, but can be easily adapted to the setting of the BGG splitting operator. If the first operator has order bigger than one, then \( \partial : \mathcal{V}^1 / \mathcal{V}^2 \to T^*M \otimes \mathcal{V}^0 / \mathcal{V}^1 \) is an isomorphism of natural vector bundles. If the first BGG operator has order 1, then one can naturally decompose \( T^*M \otimes \mathcal{V}^0 / \mathcal{V}^1 \) into the direct sum of \( \text{im}(\partial) \) and \( \ker(\partial^*) \), where \( \partial^* : T^*M \otimes \mathcal{V}^0 / \mathcal{V}^1 \to \mathcal{V}^1 / \mathcal{V}^2 \) is induced in the obvious way by the action of \( \mathfrak{g}_1 \) on \( V \). The first BGG operator is then given by the \( \ker(\partial^*) \)-component of \( \nabla \sigma \) (which is the same for all Weyl connections \( \nabla \)). Hence for a section in the kernel of the first BGG operator, \( \nabla \sigma \) has values in \( \text{im}(\partial) \). In general, this component depends on the choice of the Weyl connection, but along the zero locus of \( \sigma \), it has invariant meaning.

The methods of [2] lead to more general results (still in the \(|1|\)-graded case). If the order of the first BGG operator is \( r \), then for \( k \leq r \) vanishing of the \( k \)-jet of \( \sigma = \Pi(s) \) of the BGG
solution determined by a parallel section $s$ of a tractor bundle $\mathcal{V}$ in a point $x$ is equivalent to the fact that $s(x) \in \mathcal{V}_x^{k+1} \subset \mathcal{V}_x$. Moreover, if $k < r$ then assuming vanishing $k$-jet in $x$, the $k + 1$-fold symmetrized covariant derivative in $x$ can be computed algebraically from the class of $s(x)$ in $\mathcal{V}_x^{k+1}/\mathcal{V}_x^k$. The results of [2] have been extended to general parabolic geometries in [26], but in this case weighted jets and a concept of weighted order are required, so we do not go into this.

2.8. Defining sections

We next discuss a generalization of the well-known concept of a defining function (or, more generally, a defining density) for a hypersurface to an analogous notion for the case of submanifolds of higher codimension. We will efficiently use the jet information on first BGG solutions discussed in §2.6 by using them to construct defining sections.

For a vector bundle $p : E \to M$ on a smooth manifold $M$, it is well known that two linear connections on $E$ differ by a tensor field. Explicitly, for linear connections $\nabla$ and $\tilde{\nabla}$ on $E$ there is a smooth section $A \in \Gamma(T^*M \otimes L(E, E))$ such that for any vector field $\xi \in \mathfrak{X}(M)$ and any section $\sigma \in \Gamma(E)$, we get $\tilde{\nabla}_\xi \sigma = \nabla_\xi \sigma + A(\xi)(\sigma)$. In particular, this shows that for any point $x \in M$ such that $\sigma(x) = 0$, we obtain $\tilde{\nabla}_\xi \sigma(x) = \nabla_\xi \sigma(x)$, so we obtain a well-defined map $\nabla\sigma(x) : T_xM \to E_x$. (This corresponds to the fact that the kernel of the natural projection $J^1E \to E$ from the first jet prolongation of $E$ to $E$ is naturally isomorphic to $T^*M \otimes E$.) Thus the following is well defined.

Definition 2.16. Let $N \subset M$ be a smooth submanifold of codimension $k$ in a smooth manifold $M$ and let $p : E \to M$ be a smooth vector bundle of rank $k$. Then a local section $\sigma$ of $E$ defined in a neighborhood $U$ of a point $x \in N$ is called a local defining section for $N$ if and only if for each $y \in U \cap N$ we have that $\sigma(y) = 0$ and the linear map $\nabla\sigma(y) : T_yM \to E_y$ is surjective.

Similarly to defining functions, we can use defining sections to produce local coordinates around the submanifold $N$.

Proposition 2.17. Let $N \subset M$ be a smooth submanifold of codimension $k$, $p : E \to M$ a smooth vector bundle of rank $k$ and $\sigma \in \Gamma(E)$ a defining section for $N$ defined locally around a point $x \in N$. Then for any local frame $\{\tau_1, \ldots, \tau_k\}$ for $E$ defined on a neighborhood of $x$, there is an open neighborhood $U$ of $x$ in $M$ contained in the domain of definition of the frame and a surjective submersion $\pi : U \to U \cap N$ such that the map $\psi : U \to (U \cap N) \times \mathbb{R}^k$ defined by $\psi(y) := (\pi(y), \sigma_1(y), \ldots, \sigma_k(y))$ is a diffeomorphism onto an open neighborhood of $(U \cap N) \times \{0\}$. Here the functions $\sigma_i : U \to \mathbb{R}$ are the coordinate functions of $\sigma$ with respect to the frame $\{\tau_i\}$, i.e. $\sigma(y) = \sum_i \sigma_i(y)\tau_i(y)$ for all $y \in U$.

Proof. Let us start with chart $u : U \to u(U) \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$, adapted to the submanifold $N$, which is defined on an open neighborhood $U$ of $x$ in $M$ on which the frame $\{\tau_i\}$ is defined. Projection onto the first $(n-k)$-dimensional factor in that chart defines a surjective submersion $\pi : U \to U \cap N$. The chosen frame then defines a trivialization
\[ \varphi : p^{-1}(U) \to U \times \mathbb{R}^k \] of \( E \) over \( U \). Moreover, we can define a linear connection \( \nabla \) on \( E \) by declaring the elements of the frame to be parallel. By definition, for a vector field \( \xi \), we get \( \nabla_\xi \sigma = \sum \xi \cdot \sigma_i \), so since \( \sigma \) is a defining section, we see that the function \( (\sigma_1, \ldots, \sigma_k) : U \to \mathbb{R}^k \) has surjective derivative along \( U \cap N \). Since \( \pi \) is a surjective submersion, this shows that \( \psi : U \to (U \cap N) \times \mathbb{R}^k \) has surjective derivative along \( U \cap N \subset U \), so the result follows from the inverse function theorem.

3. Parabolic compactifications related to \( SO(p, q) \subset SL(p + q, \mathbb{R}) \)

In this section we use the tools we have developed to study parabolic compactifications obtained from the realization of \( SO(p, q) \) as the fixed point group of an involutive automorphism of \( SL(p + q, \mathbb{R}) \) as discussed in part (1) of Example 2.8. In particular, this includes a parabolic compactification of the Riemannian symmetric space \( SO(p, q)/S(O(p) \times O(q)) \), which we analyze in detail.

3.1. Orbits and infinitesimal transversals

We have already noted in part (1) of Example 2.8 that \( H := SO(p, q) \) acts with finitely many orbits on each flag manifold of \( G := SL(p + q, \mathbb{R}) \). There we have also noted that on Grassmannians, the orbits are determined by the rank and signature of the restriction of the symmetric bilinear form defining \( H \) to the subspace in question and open orbits correspond to non-degenerate restrictions. On the Grassmannian \( Gr(i, \mathbb{R}^{p+q}) \) we can thus index the orbits as \( O_{(r, s)} \) where \( 0 \leq r \leq p \) and \( 0 \leq s \leq q \) and \( r + s \leq i \) and the open orbits are the ones for which \( r + s = i \). We also see immediately that the closure of \( O_{(r, s)} \) is the union of the orbits \( O_{(r', s')} \) where \( r' \leq r \) and \( s' \leq s \). We next determine the structure and the codimension of each orbit and the form of the infinitesimal transversal as discussed in §2.4.

Proposition 3.1. For \( r, s \) as above, define \( v := i - r - s, \hat{r} = p - r - v, \) and \( \hat{s} = q - s - v, \) and consider the orbit \( O := O_{(r, s)} \subset Gr(i, \mathbb{R}^{p+q}) \). Further let \( IGr(v, \mathbb{R}^{(p,q)}) \) be the Grassmannian of isotropic subspaces of dimension \( v \) in \( \mathbb{R}^{(p,q)} \). Then we have:

1. The orbit \( O \) is non-empty if and only if \( v \leq \min(p, q) \) and in that case, its codimension in \( Gr(i, \mathbb{R}^{p+q}) \) is \( v(v + 1)/2 \).

2. There is an \( H \)-equivariant surjective submersion \( O \to IGr(v, \mathbb{R}^{(p,q)}) \) whose fibers are isomorphic to the symmetric space \( SL(p + q - 2v, \mathbb{R})/S(O(r, s) \times O(\hat{r}, \hat{s})) \).

3. Replacing \( H \) by a conjugate subgroup \( H_O \) as described in §2.4, there is a natural quotient homomorphism \( H_O \cap P \to GL(v, \mathbb{R}) \times S(O(r, s) \times O(\hat{r}, \hat{s})) \). Thus the representation \( S^2\mathbb{R}^v \) of \( GL(v, \mathbb{R}) \) gives rise to a representation of \( H_O \cap P \), which is isomorphic to \( (\mathfrak{g}/\mathfrak{p})/(\mathfrak{h}_O/(\mathfrak{h}_O \cap \mathfrak{p})) \).

Proof. Let \( b \) be the bilinear form defining \( H \) and let \( W \subset \mathbb{R}^{p+q} \) be a subspace of dimension \( i \) such that \( b|_{W \times W} \) has signature \((r,s)\). Then the null space \( W \cap W^\perp \) of \( b|_{W \times W} \) has dimension \( v \) and is totally isotropic for \( b \). Since the maximal dimension of a totally isotropic subspace is \( \min(p, q) \) we conclude that \( O_{(r,s)} \) is empty for \( v > \min(p, q) \). Otherwise, mapping \( W \) to \( W \cap W^\perp \) defines an \( H \)-equivariant map from
\( \mathcal{O}_{(r,s)} \) to \( IGr(v, \mathbb{R}^{(p,q)}) \), which is easily seen to be smooth. Next, the sum \( W + W^\perp \) has orthogonal space \( W \cap W^\perp \) and thus is co-isotropic, so \( b \) induces a duality between \( W \cap W^\perp \) and \( \mathbb{R}^{p+q}/(W + W^\perp) \). On the other hand, \( b \) induces a non-degenerate bilinear form \( b \) on \( (W + W^\perp)/(W \cap W^\perp) \) which has signature \( (p - v, q - v) \). By construction \( W \) and \( W^\perp \) descend to complementary subspaces in the quotient, on which the signature of \( b \) is \( (r, s) \) and \( (\hat{r}, \hat{s}) \), respectively.

Conversely, choosing a subspace \( N \subset \mathbb{R}^{p+q} \) of dimension \( v \) which is totally isotropic for \( b \), there is an induced bilinear form \( b \) on \( N^\perp/N \) which is non-degenerate of signature \( (p - v, q - v) \). Choosing a subspace of dimension \( r + s \) in \( N^\perp/N \), on which \( b \) has signature \( (r, s) \), the pre-image in \( N^\perp \) is a subspace of dimension \( i \), which clearly lies in \( \mathcal{O}_{(r,s)} \). This shows that \( \mathcal{O}_{(r,s)} \) is non-empty if \( v \leq \min(p, q) \) and that our map \( \mathcal{O}_{(r,s)} \to IGr(v, \mathbb{R}^{(p,q)}) \) is surjective. Since these are homogeneous spaces of \( H \) and the map is \( H \)-equivariant, it must be the natural projection \( H/K_1 \to H/K_2 \) for \( K_1 \subset K_2 \subset H \), which is a submersion. The fiber of this map over \( N \) is isomorphic to the space of those linear subspaces in \( N^\perp/N \equiv \mathbb{R}^{p+q-2v} \), on which \( b \) has signature \( (r, s) \), which implies the claimed description as a symmetric space.

Returning to a fixed subspace \( W \in \mathcal{O}_{(r,s)} \), the above considerations easily imply that we can find a basis \( \{v_j\} \) for \( \mathbb{R}^{p+q} \) which is adapted to the flag \( W \cap W^\perp \subset W \subset W + W^\perp \subset \mathbb{R}^{p+q} \) such that the matrix \( (b(v_j, v_j)) \) has the block form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}
\]

Here the blocks have sizes \( v, r + s, \hat{r} + \hat{s}, \) and \( v, I \) denotes the identity matrix, and \( I_{r,s} \) is the diagonal matrix with \( r \) diagonal entries equal to +1 and \( s \) entries equal to -1. (This also shows that any two subspaces for which the restriction of \( b \) has signature \( (r, s) \) are conjugate under the action of \( H \).

Now take \( H_0 \) to be the orthogonal group corresponding to the above matrix. A simple direct computation then shows that, in a block form as above, the Lie algebra \( \mathfrak{h}_0 \) consists of all matrices of the form

\[
\begin{pmatrix}
A & K & L & M \\
E & B & -I_{r,s}D' & -I_{r,s}K' \\
F & D & C & -I_{\hat{r},\hat{s}}L' \\
G & -E' & -F' & -A'
\end{pmatrix}
\] with \( B \in \mathfrak{o}(r,s), C \in \mathfrak{o}(\hat{r}, \hat{s}) \) \( G' = -G, M' = -M \).

By definition \( \mathfrak{h}_0 \cap \mathfrak{p} \) corresponds to those matrices, for which the four blocks in the lower left corner vanish. But these are exactly the matrices which are block-upper triangular with respect to the finer block decomposition. Hence \( \mathfrak{h}_0/(\mathfrak{h}_0 \cap \mathfrak{p}) \) corresponds to the four blocks in lower left corner, while \( \mathfrak{g}/\mathfrak{p} \) is represented by arbitrary matrices of size \( (n - i) \times i \) in that part. But the only restriction on these four blocks, implied by lying in \( \mathfrak{h}_0 \), is that the \( G \)-block has to be skew symmetric. As a vector space, the infinitesimal transversal \( (\mathfrak{g}/\mathfrak{p})/(\mathfrak{h}_0/(\mathfrak{h}_0 \cap \mathfrak{p})) \) thus is isomorphic to the space of symmetric matrices of size \( v \). This shows that \( \mathcal{O}_{(r,s)} \) has codimension \( v(v + 1)/2 \) in \( \mathbb{G}/P \), which completes the proof of parts (1) and (2).

Finally, the matrices which are strictly block-upper-triangular form an ideal in \( \mathfrak{h}_0 \cap \mathfrak{p} \), with the quotient corresponding to the block diagonal part. On the group level, this corresponds to the claimed quotient homomorphism in (3). Explicitly, this sends a
linear map that preserves each of the subspaces in the chain $W \cap W^\perp \subset W \subset W + W^\perp$ to the induced maps on $W \cap W^\perp$ and the quotients $W/(W \cap W^\perp)$ and $(W + W^\perp)/W$. Since in the adjoint representation, the strictly block-upper-triangular matrices act trivially on the $G$-block, we see that the natural action of $H_G \cap P$ on the infinitesimal transversal descends to the quotient, with only the $\text{GL}(v, \mathbb{R})$-factor acting non-trivially. This completes the proof of (3). \qed

Remark 3.2. Let us briefly discuss the orbit structure for more general flag manifolds. Consider the case of two-step flags $W_1 \subset W_2 \subset \mathbb{R}^{(p,q)}$ of dimension $(i_1, i_2)$. The signatures $(r_1, s_1)$ and $(r_2, s_2)$ of the restrictions of $b$ to $W_1$ and $W_2$ satisfy $r_1 \leq r_2$ and $s_1 \leq s_2$ and these signatures are constant on $H$-orbits. Next, we have $W_{1,j}^\perp \subset W_{1,j}^\perp$ and the intersections $W_1 \cap W_{1,j}^\perp$ and $W_2 \cap W_{2,j}^\perp$ are the null spaces of the restrictions of $b$, which have dimension $v_j := i_j - r_j - s_j$ for $j = 1, 2$. But at this point it is clear that, if both $v_1$ and $v_2$ are non-zero, then an additional invariant pops up: There is the natural subspace $W_1 \cap W_2 \subset W_1 \cap W_{1,j}^\perp$ of vectors in the null space of $b|_{W_1 \times W_1}$ which remain isotropic in $W_2$. The codimension $\ell$ of this subspace of course is preserved by the action of $H$.

There are evident restrictions on $\ell$. First, since $W_1 \cap W_{1,j}^\perp \subset W_2 \cap W_{2,j}^\perp$ we must have $v_1 - \ell \leq v_2$. Second, vectors in $W_1 \cap W_{1,j}^\perp \setminus W_1 \cap W_{2,j}^\perp$ are isotropic but not orthogonal to $W_2$. Hence there is a subspace of dimension $2\ell$ in $W_2$, whose intersection with $W_1$ is complementary to $W_1 \cap W_{2,j}^\perp$ in $W_1 \cap W_{1,j}^\perp$ and on which $b$ has split signature $(\ell, \ell)$. This shows that $\ell \leq r_2 - r_1$ and $\ell \leq s_2 - s_1$. These are the only restrictions on $\ell$, however.

Indeed it is an exercise in linear algebra to show that fixing $(r_j, s_j)$ for $j = 1, 2$ and $\ell$ such that the above restrictions are satisfied, one can find a basis for $\mathbb{R}^{p+q}$ adapted to $W_1$ and $W_2$ for which the inner product has a standard form. Hence these parameters completely determine the orbits. Moreover, there is a neighborhood of the orbit corresponding to $(r_j, s_j, \ell)$ on which the corresponding parameters satisfy $r_j' \geq r_j$, $s_j' \geq s_j$ and $\ell' \leq \ell$. Using this, one concludes that the closure of the orbit determined by $(r_j, s_j, \ell)$ consists of all orbits corresponding to $(r_j', s_j', \ell')$ such that $r_j' \leq r_j$, $s_j' \leq s_j$ and $\ell' \geq \ell$.

3.2. Orbit closures via solutions of BGG operators

We continue the study of the orbits of $H = \text{SO}(p, q)$ in a Grassmannian $G/P \cong \text{Gr}(i, \mathbb{R}^n)$, where $n = p + q$. We next describe certain unions of such orbits as the zero sets of solutions of appropriate first BGG operators. In particular, this provides a description of all orbit closures for the parabolic compactification of $\text{SO}(p, q)/\text{SO}(p) \times \text{O}(q)$ discussed in §2.5 as zero sets, see Remark 3.8.

Recall the construction of the two tautological vector bundles $E$ and $F$ on the Grassmannian $\text{Gr}(i, \mathbb{R}^n)$. These fit into an exact sequence of the form $0 \to E \to \mathbb{R}^n \to F \to 0$, where $\mathbb{R}^n$ indicates a trivial bundle. Viewing a point in $\text{Gr}(i, \mathbb{R}^n)$ as a linear subspace $V \subset \mathbb{R}^n$, the fibers of $E$ and $F$ over $V$ are $V$ and $\mathbb{R}^n/V$, respectively. By definition, this is the standard tractor bundle $\mathcal{T}$ for the flat Grassmannian structure with its $P$-invariant filtration structure, in the trivialization described in §2.6. In particular, the canonical completely reducible quotient of $\mathcal{T}$ is the anti-tautological bundle $F$. For our choice of parabolic subgroup $P$, the reductive Levi component $G_0$
is isomorphic to \( S(GL(i, \mathbb{R}) \times GL(n - i, \mathbb{R})) \), and the bundles \( E \) and \( F \) correspond to the standard representations of the two factors. Thus we see that all the completely reducible natural bundles on \( Gr(i, \mathbb{R}^{p+q}) \) can be built up from \( E \), \( F \), and their duals via tensorial constructions.

At this point, we need a bit of background from representation theory. Viewing the bilinear form \( b \) we have used to define \( H \) as a linear isomorphism \( \mathbb{R}^n \to \mathbb{R}^{n*} \), we can form the induced linear isomorphism \( \Lambda^k b : \Lambda^k \mathbb{R}^n \to \Lambda^k \mathbb{R}^{n*} \). On the other hand, \( b \) induces a symmetric bilinear form \( \tilde{b} \) on \( \Lambda^k \mathbb{R}^n \) defined by

\[
\tilde{b}(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) := \det((b(v_i, w_j))_{i,j=1,\ldots,k}),
\]

which can be viewed as an element of \( S^2(\Lambda^k \mathbb{R}^{n*}) \). Unless \( k = 1 \) or \( k \geq n - 1 \), this is not an irreducible representation of \( SL(n, \mathbb{R}) \). There is however a maximal irreducible component, which we denote by \( \otimes^2(\Lambda^k \mathbb{R}^{n*}) \), whose highest weight equals twice the highest weight of the irreducible representation \( \Lambda^k \mathbb{R}^{n*} \).

**Lemma 3.3.** Identifying \( L(\Lambda^k \mathbb{R}^n, \Lambda^k \mathbb{R}^{n*}) \) with \( \otimes^2 \Lambda^k \mathbb{R}^{n*} \) the element \( \Lambda^k b \) coincides with \( \tilde{b} \in S^2(\Lambda^k \mathbb{R}^{n*}) \). Moreover, this element is automatically contained in the irreducible component \( \otimes^2(\Lambda^k \mathbb{R}^{n*}) \).

**Proof.** The map \( \mathbb{R}^n \to \mathbb{R}^{n*} \) induced by \( b \) sends \( v \) to the linear functional \( b(v, \cdot) \). Hence the induced map on the \( k \)th exterior powers sends \( v_1 \wedge \cdots \wedge v_k \) to \( b(v_1, \cdot) \wedge \cdots \wedge b(v_k, \cdot) \). Evaluating this functional on \( w_1 \wedge \cdots \wedge w_k \) we obtain \( \tilde{b}(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) \), which proves the first claim.

For the second claim, we observe that there is a homomorphism of representations of \( SL(n, \mathbb{R}) \) mapping \( S^k(L(\mathbb{R}^n, \mathbb{R}^{n*})) \) to \( L(\Lambda^k \mathbb{R}^n, \Lambda^k \mathbb{R}^{n*}) \). For \( f_1, \ldots, f_k \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^{n*}) \), the element \( f_1 \vee \cdots \vee f_k \in S^k(L(\mathbb{R}^n, \mathbb{R}^{n*})) \), defines a linear map \( \Lambda^k \mathbb{R}^n \to \Lambda^k \mathbb{R}^{n*} \) via

\[
(f_1 \vee \cdots \vee f_k)(v_1 \wedge \cdots \wedge v_k) := \frac{1}{k!} \sum_{\sigma \in S(k)} f_{\sigma_1}(v_1) \wedge \cdots \wedge f_{\sigma_k}(v_k).
\]

From this explicit formula, it follows readily that if all the \( f_i \) lie in \( S^2 \mathbb{R}^{n*} \subset \mathbb{L}(\mathbb{R}^n, \mathbb{R}^{n*}) \), then the resulting map lies in \( S^2(\Lambda^k \mathbb{R}^{n*}) \). Hence we get a homomorphism \( S^k(S^2 \mathbb{R}^{n*}) \to S^2(\Lambda^k \mathbb{R}^{n*}) \) and by construction \( \Lambda^k b \) lies in its image. In terms of Young diagrams, irreducible components of \( S^k(S^2 \mathbb{R}^{n*}) \) are obtained by arranging \( k \) copies of a horizontal pair of boxes into a Young diagram. This implies that the highest weight of \( \otimes^2(\Lambda^k \mathbb{R}^{n*}) \) occurs among these weights (as a ‘tower’ of \( k \) horizontal pairs). This Young diagram corresponds to the smallest among all the highest weights of irreducible components of \( S^k(S^2 \mathbb{R}^{n*}) \). Thus no other irreducible component of \( \otimes^2(\Lambda^k \mathbb{R}^{n*}) \) can be contained in the image of our homomorphism. Hence this image coincides with \( \otimes^2 \Lambda^k \mathbb{R}^{n*} \), which completes the proof.

As we have observed above, irreducible bundles on the Grassmannian come from representations of \( S(GL(i, \mathbb{R}) \times GL(n - i, \mathbb{R})) \) with the tautological bundle \( E \) corresponding to the standard representation of the first factor. In particular, this implies that we can form the bundles \( \otimes^2(\Lambda^k E^*) \) for \( k = 1, \ldots, i \) and for \( k = 1, i - 1, i \), this bundle coincides with \( S^2(\Lambda^k E^*) \).
Theorem 3.4. Consider the decomposition $Gr(i, \mathbb{R}^{p+q}) = \bigcup_{r,s} \mathcal{O}_{(r,s)}$ into orbits of the group $H = SO(p,q)$. Then for each $k = 1, \ldots, i$, there is a section $\sigma_k \in \Gamma(\otimes^2(\Lambda^k E^*))$, which lies in the kernel of the first BGG operator naturally defined on that bundle, whose zero set is the union of all those $H$-orbits $\mathcal{O}_{(r,s)}$ for which $r + s < k$. The relevant first BGG operator is of order one for $k < i$ and of order 3 for $k = i$.

Proof. From the filtration structure of the standard tractor bundle $\mathcal{T}$ described above, it readily follows that the dual bundle $\mathcal{T}^*$ has $E^*$ as its canonical completely reducible quotient. Hence for $S^2\mathcal{T}^*$, the completely reducible quotient is $S^2E^*$. Now $b \in S^2\mathbb{R}^{n*}$ corresponds to a parallel section $s = s_1$ of $S^2\mathcal{T}^*$, which projects to $\sigma_1 = \Pi(s_1) \in \Gamma(S^2 E^*)$.

From the above description of the trivialization of the $\mathcal{T}$ it readily follows that the value of $\sigma_1$ in a point $V \in Gr(i, \mathbb{R}^{p+q})$ is simply given by the restriction of $b$ to the fiber of $E$ over $V$, which coincides with $V$. Hence we see that $V \in \mathcal{O}_{(r,s)}$ if and only if $\sigma_1(V)$ has rank $r + s$ and signature $(r, s)$ as a bilinear form. On the other hand, $\sigma_1$ lies in the kernel of the first BGG operator by Theorem 2.13.

Now for $k = 2, \ldots, i$ we can proceed similarly starting with $\Lambda^k b$, which by Lemma 3.3 lies in $\otimes^2(\Lambda^k \mathbb{R}^{n*})$. Hence it gives rise to a parallel section $s_k \in \Gamma(\otimes^2(\Lambda^k \mathcal{T}^*))$, which in turn projects onto a section $\sigma_k := \Pi(s_k)$ of the irreducible quotient, which lies in the kernel of the corresponding first BGG operator. From the description of completely reducible quotients in terms of highest weights and since $k \leq i$, it readily follows that this quotient equals $\otimes^2(\Lambda^k E^*)$. Moreover, the projection $\Pi$ is again given by mapping a bilinear form to its restriction to the fibers of $\Lambda^k E$. Hence we conclude that $\sigma_k(V)$ is the restriction of $\Lambda^k b$ to $\Lambda^k V$, so it coincides with $\Lambda^k(\sigma_1(V))$, which by Lemma 3.3 lies in $\otimes^2(\Lambda^k V^*)$.

But now consider $\sigma_1(V)$ as a map $V \to V^*$ and let $W \subset V^*$ be its image. Then we can factorize our map as $V \to W \hookrightarrow V^*$ so by functoriality, $\Lambda^k(\sigma_1(V))$ factorizes as $\Lambda^k V \to \Lambda^k W \hookrightarrow \Lambda^k V^*$. This shows that $\Lambda^k(\sigma_1(V)) = \sigma_k(V)$ vanishes if and only if the rank of $\sigma_1(V)$ is less than $k$, i.e. iff $V$ lies in an orbit $\mathcal{O}_{(r,s)}$ such that $r + s < k$. The order of the relevant first BGG operators can be read off the highest weight of the inducing representations, compare with [2].

3.3. A slice theorem

We next derive a description of a neighborhood of one of the orbits $\mathcal{O}_{(r,s)} \subset Gr(i, \mathbb{R}^{p+q})$. In view of the description of the infinitesimal transversal in Proposition 3.1, it is visible what the best possible result would be, cf. Remark 2.10. Putting $v = i - r - s$, the infinitesimal transversal can be identified with $S^2 \mathbb{R}^{n*}$, with the action of the isotropy group coming from the natural representation of $GL(v, \mathbb{R})$ on that space. Hence the orbits of the isotropy group on the infinitesimal transversal are determined by rank and signature, and following the philosophy of slice theorems, the optimal result to expect would be a parallel description of a neighborhood of $\mathcal{O}_{(r,s)}$ in the Grassmannian. The following result is the key step to showing that such a description is available locally.

Theorem 3.5. Consider the section $\sigma_1 \in \Gamma(S^2 E^*)$ from Theorem 3.4, a pair $(r,s)$ with $0 \leq r \leq p$, $0 \leq s \leq q$ and $r + s < i$, and a point $x \in \mathcal{O}_{(r,s)}$. Then there are open neighborhoods $U$ of $x$ in $G/P$ and $\bigcup U \subset U \cap \mathcal{O}_{(r,s)}$ of $x$ in $\mathcal{O}_{(r,s)}$ and there is a smooth
subbundle $\tilde{E} \subset E|_U$ of rank $v := i - r - s$ such that

- for each $y \in U$ the null space of $\sigma_1(y)$ is contained in $\tilde{E}_y \subset E_y$
- the obvious projection of $\sigma_1$ to a section of $S^2\tilde{E}^*|_U$ is a defining section for $U$.

**Proof.** From Proposition 3.1, we know that $\mathcal{O}_{(r,s)}$ has codimension $v(v + 1)/2$ in the Grassmannian, so $v$ is the right rank for a subbundle $\tilde{E}$ to have a chance for a defining section of $S^2\tilde{E}^*$. Since $\mathcal{O}_{(r,s)}$ is an initial submanifold in $G/P$, there is a connected open neighborhood $U$ of $x$ in $\mathcal{O}_{(r,s)}$, which is a true submanifold of $G/P$. (Take an adapted chart for the initial submanifold centered at $x$ and let $U$ be the pre-image of the connected component of $x$ in the image of $\mathcal{O}_{(r,s)}$ in that chart.) Next, choose a linear subspace $W_x \subset E_x$ of dimension $r + s$ on which $\sigma_1(x)$ is non-degenerate and has signature $(r, s)$. This can be extended to a smooth subbundle $W \subset E$ on some connected open neighborhood $U$ of $x$ in $G/P$, which can be assumed to contain $U$. Shrinking $U$ and $\tilde{U}$ if necessary, we may assume that for each $y \in U$, the bilinear form $\sigma_1(y)$ is non-degenerate of signature $(r, s)$ on $W_y$. In particular, this implies that $U \subset \bigcup_{r' \geq r, s' \geq s} \mathcal{O}_{(r', s')}$.

Now for each $y \in U$, we define $\tilde{E}_y$ to be the space of all $v \in E_y$ such that $\sigma_1(y)(v, w) = 0$ for all $w \in W_y$. By non-degeneracy of $\sigma_1(y)$ on $W_y$, each of these spaces has dimension $v$ and, of course, it always contains the null space of $\sigma_1(y)$. We claim that, possibly shrinking $U$ and $\tilde{U}$ further, there is a local smooth frame for $\tilde{E}$, so this is a smooth subbundle of $E|_U$. Indeed, take smooth sections $t_1, \ldots, t_{r+s} \in \Gamma(W)$ that form a local frame for $W$ and extend them by smooth sections $t_{r+s+1}, \ldots, t_i \in \Gamma(E)$ to a local frame for $E$. Then non-degeneracy of $\sigma_1$ on $W$ implies that for each $j = r + s + 1, \ldots, i$, one can add a smooth linear combination of $t_1, \ldots, t_{r+s}$ to $t_j$ in such a way that the result has values in $\tilde{E}$. Of course, the resulting sections then have to form a smooth local frame for $\tilde{E}$.

Having constructed $\tilde{E} \subset E|_U$, there is a natural projection $q : S^2E^*|_U \to S^2\tilde{E}^*$, obtained by restricting bilinear forms to taking entries from the fibers of $\tilde{E}$. For $y \in U \subset \mathcal{O}_{(r,s)}$, we know that $\tilde{E}_y$ has dimension $v$ and contains the null space of $\sigma_1(y)$, so it has to coincide with that null space. Hence $q \circ \sigma_1$ vanishes along $U$ and we only have to verify that $\nabla(q \circ \sigma)(y) : T_y(G/P) \to S^2\tilde{E}_y$ is surjective for some connection $\nabla$ on $S^2\tilde{E}^*$ and each $y \in U \subset \mathcal{O}_{(r,s)}$.

Suppose that $\nabla$ is a linear connection on $E$. Using the decomposition $E|_U = \tilde{E} \oplus W$, we obtain an induced linear connection $\tilde{\nabla}$ on $\tilde{E}$. For $v \in \Gamma(\tilde{E})$ and $\xi \in \mathfrak{X}(U)$, we simply define $\tilde{\nabla}_\xi v$ as the $\tilde{E}$-component of $\nabla_\xi v$. Then consider the induced connections on $S^2E^*$ and $S^2\tilde{E}^*$, respectively, which we also denote by $\nabla$ and $\tilde{\nabla}$. For $v, w \in \Gamma(\tilde{E})$, we thus have

$$(\tilde{\nabla}_\xi (q \circ \sigma))(v, w) = \xi \cdot (q \circ \sigma)(v, w) - (q \circ \sigma)(\nabla_\xi v, w) - (q \circ \sigma)(v, \nabla_\xi w).$$

By definition, $(q \circ \sigma)(v, w) = \sigma(v, w)$ and, along $U$, $\sigma$ vanishes identically upon insertion of either $v$ or $w$ by construction. Hence we see that for $y \in U$ and each $\xi \in \mathfrak{X}(U)$, we get $\tilde{\nabla}_\xi (q \circ \sigma)(y) = q(\nabla_\xi \sigma(y))$.

Taking $\nabla$ to be the Weyl connection defined by some Weyl structure, we can compute $\nabla \sigma$ using Proposition 2.14. For the Grassmannian, the tangent bundle $T(G/P)$ is isomorphic to $E^* \otimes F$, so $T^*(G/P) \cong E \otimes F^*$. On the other hand, for $\nabla = S^2T^*$, we have $\nabla/\nabla^1 \cong S^2E^*$ and $\nabla^1/\nabla^2 \cong E^* \otimes F^*$. Hence in our case the bundle map $\partial$ from
Proposition 2.14 maps $F^* \otimes E^*$ to $F^* \otimes E \otimes S^2 E^*$. By $P$-equivariance, this must coincide (up to a non-zero constant factor) with tensorizing with the id$_E \in E^* \otimes E$ and then symmetrizing. Viewing $\mu \in F^* \otimes E^*$ as a linear map $F \to E^*$ and $\partial(\mu)$ as a map $F \otimes E^* \to S^2 E^*$ we conclude that $\partial(\mu)(f \otimes \lambda) = \mu(f) \vee \lambda$.

Now in a point $y \in U$, we know that $\tilde{E}_y$ is the null space of $\sigma_1(y)$. Recall that $\sigma_1 = \Pi(s_1)$ for a parallel section $s_1 \in \Gamma(S^2 T^*)$, so $\sigma_1(y) = s_1(y)|_{E_y \times E_y}$. Restricting $s_1(y)$ to $T_y \times \tilde{E}_y$, the result vanishes upon insertion of an element of $E_y$ in the first factor, so this factors to a map $F_y \times \tilde{E}_y \to \mathbb{R}$. By construction, this map has to coincide with the restriction of $\mu(y) : F_y \times E_y \to \mathbb{R}$ to $F_y \times \tilde{E}_y$. Non-degeneracy of $s_1(y)$ then implies that this induces a surjection $F_y \to \tilde{E}_y^*$. But this implies that the restriction of $q \circ \partial(\mu) : F_y \otimes E^* \to S^2 \tilde{E}_y^*$ to $F_y \otimes \tilde{E}_y^*$ is onto, which completes the proof.

This easily leads to a full description of the local structure around each of the orbits $O(r,s)$. Included in this is the result that the orbits are embedded submanifolds. This part may also be seen to follow from the fact that $H$ is an algebraic group acting algebraically on the algebraic variety $G/P$.  

**Corollary 3.6.** Each of the orbits $O(r,s)$ is an embedded submanifold of $G/P$. Moreover, for each point $x \in O(r,s)$, there is an open neighborhood $U$ of $x$ in $G/P$ and a diffeomorphism $\varphi$ from $U$ onto an open neighborhood of $(U \cap O(r,s)) \times \{0\}$ in $(U \cap O(r,s)) \times S^2 \mathbb{R}^{(n-r-s)^*}$ such that $U \subset \bigcup_{r' \geq r, s' \geq s} O(r', s')$ and a point $y \in U$ lies in $O(r', s')$ if and only if the second component of $\varphi(y)$ has signature $(r' - r, s' - s)$.

**Proof.** We take $U \subset U$ and $q \circ \sigma_1$ as in Theorem 3.5 and then apply Proposition 2.17 using a local frame of $S^2 \tilde{E}^*$ determined by a local frame $\tau_1, \ldots, \tau_v$ ($v = n - r - s$) for $\tilde{E}$. Possibly shrinking $U$ and $U$, this gives a diffeomorphism from $U$ to an open neighborhood of $U \times \{0\}$ in $U \times S^2 \mathbb{R}^{v*}$. Moreover, the coordinate functions of $q \circ \sigma_1$ with respect to that frame are simply the functions $\sigma_1(\tau_a, \tau_b)$, so the rank and signature of the resulting symmetric matrix in a point $y$ coincide with the rank and signature of $\sigma_1(y)$ on $\tilde{E}_y$. By construction, this signature is $(p', q')$ if and only if the signature of $\sigma_1(y)$ on $E_y$ is $(p' + r, q' + s)$. This shows that $U = U \cap O(r,s)$, so in particular, $O(r,s)$ is an embedded submanifold, and the characterization of $U \cap O(r', s')$ follows easily.

**3.4. A natural defining density for the largest boundary component**

Consider an orbit $O(r,s) \subset Gr(i, \mathbb{R}^n)$ with $r + s = i - 1$. Theorem 3.5 produces (locally) a section of $S^2 \tilde{E}^*$ that is a local defining section for $O(r,s)$. But in this case $\tilde{E}$ is a line bundle and therefore so is $S^2 \tilde{E}^*$. Thus we obtain an analog of a defining density as discussed on [6, p. 52]. As a final step in the discussion of the parabolic compactifications related to $SO(p,q) \subset SL(p + q, \mathbb{R})$, we show that, for these largest non-open orbits, we also get a defining density that is natural. In Theorem 3.4, we have constructed a section $\sigma_1$ of the bundle $\otimes^2 (\Lambda^i E^*)$. Now since $E$ has rank $i$, $\Lambda^i E^*$ is a line bundle, so $\otimes^2 (\Lambda^i E^*) = S^2 (\Lambda^i E^*)$ is a line bundle, too. Since we are dealing with a $|1|$-graded geometry here, any natural line bundle is a density bundle.

---

1 We thank the referee for bringing this line of argument to our attention.
Proposition 3.7. The section $\sigma_i$ of the density bundle $L := S^2(\Lambda^i E^*)$ is a defining density for each of the orbits $O_{(r,s)} \subset Gr_i(\mathbb{R}^{p+q})$ with $r + s = i - 1$.

Proof. We already know that $r + s = i - 1$ implies that $O_{(r,s)}$ is an embedded hypersurface in the Grassmannian, and by Theorem 3.4, $\sigma_i$ vanishes along $O_{(r,s)}$. Thus it remains to verify that $\nabla \sigma_i$ is non-vanishing along $O_{(r,s)}$. To see this, we have to analyze the canonical $P$-invariant filtration $\{V^i\}$ of the representation $V := \otimes^2(\Lambda^i \mathbb{R}^{p+q})$ of $G$ respectively the filtration $\{V^i\}$ of the corresponding tractor bundle $\mathcal{V}$. We already know that $\mathcal{V}/V^1 \cong L$ which was used to obtain $\sigma_i$ from the parallel section $\sigma_i \in \Gamma(\mathcal{V})$.

Now Proposition 2.14 shows that for a point $x \in G/P$ simultaneous vanishing of $\sigma_i(x)$ and $\nabla \sigma_i(x)$ are equivalent to the fact that $s_i(x) \in V^2_x$. We have also observed that the first BGG operator in our case has order 3, which by Remark 2.15 implies that $\mathcal{V}^1/V^2 \cong T^*(G/P) \otimes L$.

On the other hand, consider the natural filtration of the tractor bundle $W := \Lambda^i T^*$. This is induced by inserting elements of the subbundle $E$ into multilinear maps. In particular, $\mathcal{W}^1$ consists of those maps which vanish under insertion of $i$ elements in $E$, which explains the isomorphism $\mathcal{W}/\mathcal{W}^1 \cong \Lambda^i E^* = L$. Likewise, $\mathcal{W}^2 \subset \mathcal{W}^1$ consists of maps which vanish upon insertion of $i - 1$ elements of $E$, so $\mathcal{W}^1/\mathcal{W}^2 \cong F^* \otimes \Lambda^{i-1} E^*$ and this has rank $(n-i)i$. Hence if we look at the natural filtration of $S^2(\mathcal{W}^*)$, the iterated quotients of filtration components in the first two steps are given by $S^2 L$ and $L \otimes F^* \otimes \Lambda^{i-1} E^*$, respectively. It is easy to see that the latter bundle is isomorphic to $T^*(G/P) \otimes L$.

Comparing the statements of the last two paragraphs, we conclude that the subbundle $\mathcal{V} \subset S^2 \mathcal{W}$ has the property that $\mathcal{V}^1/\mathcal{V}^2$ surjects onto $(S^2 \mathcal{W})^1/(S^2 \mathcal{W})^2$. Otherwise put, if at some point $x$ we have $\sigma_i(x) = 0$ and $\nabla \sigma_i(x) = 0$, then, viewed as a map $\Lambda^i T_x \to \Lambda^i T^*_x$, $s_i(x)$ has the property that applying it to a wedge product of $i - 1$ elements of $E_x$ and one element of $F_x$, the result vanishes upon insertion of $i$ elements of $E_x$. But this implies that the restriction of $s_1(x)$ to $E_x$ has rank less than $i - 1$. Indeed, if this rank is at least $i - 1$, then we can choose a basis $\{e_1, \ldots, e_i\}$ for $E_x$ such that $s_1(x)(e_i, e_j)$ equals 0 if $i = 1$ or $i \neq j$ and 1 for $i = j > 1$. Non-degeneracy of $s_1(x)$ then shows that there must be an element $f \in F_x$ such that $s_1(x)(e_1, f) \neq 0$. But then by construction $s_1(x)(f \wedge e_2 \wedge \cdots \wedge e_p, e_1 \wedge \cdots \wedge e_p) \neq 0$. $\blacksquare$

Remark 3.8. Let us specialize the results of this Section to the case of the Riemannian symmetric space $H/K := SO(p, q)/SO(p) \times O(q)$, which we know can be identified with the open orbit $O_p := O_{(p,0)}$ in $Gr(p, \mathbb{R}^{p+q})$. The closure $\overline{H/K}$ can be written as $\bigcup_{j \leq p} O_j$, where we briefly write $O_j$ for $O_{(j,0)}$. From Proposition 3.1, we know that $O_j$ has codimension $(p - j)(p - j + 1)/2$, and for each $j$ we get $\overline{O_j} = \bigcup_{j \leq j} O_i$. Thus Theorem 3.4 shows that the orbit closure $\overline{O_j}$ coincides with the intersection of the zero locus of $\sigma_{p-j+1}$ with $\overline{H/K}$. In particular, the zero locus of $\sigma_p$ coincides with the closure of $O_{p-1}$, and by Proposition 3.7, $\sigma_p$ is a defining density locally around each point of that orbit.

Also, the slice theorem given in Corollary 3.6 takes a particularly nice form for this example. For each $v$, we can consider the space $\mathbb{S}^2_v \mathbb{R}^{v*}$ of positive definite symmetric $v \times v$-matrices over $\mathbb{R}$, which can be identified with the symmetric space $GL(v, \mathbb{R})/O(v)$. 

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The closure of $S^2_{≥0}\mathbb{R}^{v*}$ in the space of all symmetric matrices is the space $S^2_{≥0}\mathbb{R}^{v*}$ of positive semi-definite matrices. This can be viewed as a ‘local compactification’ of $GL(v, \mathbb{R})/O(v)$ in the sense that for any compact neighborhood $W$ of $0$ in $S^2\mathbb{R}^{v*}$, the intersection $W \cap S^2_{≥0}\mathbb{R}^{v*}$ is a compactification of $W \cap S^2_{≥0}\mathbb{R}^{v*}$. Now Corollary 3.6 says that locally around a point in $O_j$, the compactification $\overline{H}/K$ looks like the product of $O_j$ with this local compactification of $GL(p - j, \mathbb{R})/O(p - j)$.

4. Parabolic compactifications related to $SO(n, \mathbb{C}) \subset SO_0(n, n)$

As in part (2) of Example 2.8, we consider a non-degenerate complex bilinear form $b$ on $\mathbb{C}^n \cong (\mathbb{R}^{2n}, J)$. There we have seen that the imaginary part of $b$ defines a split-signature inner product $\langle , \rangle$ on $\mathbb{R}^{2n}$. This gives rise to an inclusion $SO(n, \mathbb{C}) \hookrightarrow SO_0(n, n)$ as the fixed point subgroup of the involutive automorphism $A \mapsto -JAJ$. In this section we study the resulting parabolic compactifications, with an emphasis on the case of the Riemannian symmetric space $SO(n, \mathbb{C})/SO(n)$ of the non-compact type.

4.1. Orbits and infinitesimal transversals

The generalized flag manifolds of $G := SO_0(n, n)$ can be realized as the spaces of isotropic flags in the standard representation $\mathbb{R}^{(n,n)}$. In addition, one has to take into account here that in the case of maximally isotropic subspaces (i.e. of dimension $n$), one has to distinguish between self-dual and anti-self-dual subspaces, since (anti-)self-duality is preserved by the action of $G$. As before, we will mainly discuss the case of isotropic Grassmannians. In contrast to § 3.1, not even these behave uniformly, but additional complications arise in the non-maximal case.

Given a linear subspace $V \subset \mathbb{R}^{2n}$, which is isotropic for the imaginary part of $b$, one can of course look at the restriction of the real part of $b$ to $V$, which defines a symmetric bilinear form on $V$. The rank and signature of this bilinear form are evidently preserved under the action of $H := SO(n, \mathbb{C})$, so they are basic invariants of the $H$-orbit determined by $V$. In the case that $V$ has the maximal possible dimension $n$, then these data determine the orbit and also the (anti-)self-duality properties of $V$:

**Lemma 4.1.** Let $V \subset \mathbb{C}^n = \mathbb{R}^{2n}$ be a real linear subspace of real dimension $n$, which is isotropic for the imaginary part $\langle , \rangle$ of $b$ and such that $\text{Re}(b)|_V$ has signature $(r, s)$ with $r + s \leq n$. Then $v := n - r - s$ is even, say $v = 2k$, and there is a complex basis $\{z_1, \ldots, z_n\}$ for $\mathbb{C}^n$ with respect to which $b$ has the block-matrix representation $\begin{pmatrix} 0 & 0 & I \\ 0 & b_{rs} & 0 \\ I & 0 & 0 \end{pmatrix}$ with blocks of size $k$, $r + s$, and $k$, such that $V$ is the real span of the vectors $z_j$ for $j = 1, \ldots, k + r + s$ and $iz_j$ for $j = 1, \ldots, k$.

Moreover, choosing the orientation of $\mathbb{R}^{2n}$ appropriately, $V$ is self-dual if $n - s$ is even and anti-self-dual if $n - s$ is odd.

**Proof.** Let us denote by $J$ the complex structure on $\mathbb{C}^n$, by $\perp$ the orthocomplement with respect to $\langle , \rangle$ and by $\perp_b$ the orthocomplement with respect to $b$. Then the real part of $b$ can be written as $(v, w) \mapsto \langle v, Jw \rangle$, so the null space of its restriction is $W := J(V) \cap V^\perp = J(V) \cap V$, where we have used that $V$ is maximally isotropic in the
last step. Since \( W \) evidently is a complex subspace of \( V \), its real dimension \( v \) has to be even. Putting \( v = 2k \), we choose a complex basis \( \{z_1, \ldots, z_k\} \) for \( W \). On the other hand, we conclude that \( W^\perp \) is a complex subspace of \( \mathbb{C}^n \) which has complex dimension \( n - k \) and contains \( V \).

Now \( b \) descends to a non-degenerate complex bilinear form \( b \) on the complex vector space \( W^\perp/W \), which has complex dimension \( n - 2k \). Since \( W = V \cap J(V) \), the image \( V \) of \( V \subset W^\perp \) in this quotient has to be a totally real subspace of real dimension \( n - k \). Moreover, the restriction of \( b \) to this subspace has to have signature \((r, s)\) so \( b|_V \) is non-degenerate. Choose a real orthonormal basis of \( V \) and pre-images \( z_{k+1}, \ldots, z_{k+r+s} \) of the basis elements in \( V \). Then by construction these vectors descend to a complex basis of \( W^\perp/W \), so they span a complex subspace \( \tilde{V} \subset W^\perp \) such that \( W^\perp = W \oplus \tilde{V} \).

Finally, we consider \( \tilde{V}^\perp \subset \mathbb{C}^n \). This is a complex subspace of complex dimension \( n - 2k \), on which \( b \) is non-degenerate, and which contains the subspace \( W \) which is isotropic for \( b \). Thus one can find a complex subspace \( \tilde{W} \subset \tilde{V}^\perp \) which also is isotropic for \( b \) and complementary to \( W \) in there. So \( b \) identifies \( \tilde{W} \) with dual \( W^* \) and hence we can find a complex basis \( \{z_{n-k+1}, \ldots, z_n\} \) for \( \tilde{W} \) such that \( b(z_j, z_{n-k+\ell}) = \delta_{j\ell} \) for \( j, \ell = 1, \ldots, k \). By construction \( \{z_1, \ldots, z_n\} \) is a complex basis of \( \mathbb{C}^n \) which has all required properties.

To prove the statement about (anti-)self-duality, we fix the orientation of \( \mathbb{R}^{2n} = \mathbb{C}^n \) in such a way that for any complex basis \( \{z_1, \ldots, z_n\} \) the real basis \( \{z_1, \ldots, z_n, iz_1, \ldots, iz_n\} \) has positive orientation. Taking the basis \( \{z_1, \ldots, z_n\} \) adapted to \( V \) as above, we have to compute the Hodge-* of

\[
\beta := z_1 \wedge \cdots \wedge z_{n-k} \wedge iz_1 \wedge \cdots \wedge iz_k,
\]

and we use the standard formula \( \alpha \wedge *\beta = \langle \alpha, \beta \rangle \text{vol} \). Now up to sign, \( \beta \) is one of the elements of the basis for \( \Lambda^k_2 \mathbb{R}^{2n} \) induced by the basis \( \{iz_j, j \in \mathbb{Z}\} \). There is just one other element \( \alpha \) in this basis for which \( \langle \alpha, \beta \rangle \) is non-zero, namely the wedge product of the elements \( iz_{k+1}, \ldots, iz_n \) and \( z_{n-k+1}, \ldots, z_n \). We order these elements in such a way that we get a simple expression for \( \langle \alpha, \beta \rangle \) and thus use

\[
\alpha = iz_{n-k+1} \wedge \cdots \wedge iz_n \wedge iz_{k+1} \wedge \cdots \wedge iz_{n-k} \wedge z_{n-k+1} \wedge \cdots \wedge z_n.
\]

Then \( \langle \alpha, \beta \rangle \) is the determinant of the \((n \times n)\)-matrix of mutual inner products between the factors in the wedge product. We have arranged things in such a way that this matrix is diagonal with \( s \) entries equal to \(-1\) and \( n - s \) entries equal to \( 1 \), so \( \langle \alpha, \beta \rangle = (-1)^s \). On the other hand, reordering the wedge products, we conclude that \( \alpha \wedge \beta = (-1)^n \text{vol} \) and thus \( *\beta = (-1)^{n-s} \beta \).

This implies a complete description of the set of \( H \)-orbits in the isotropic Grassmannian \( IGr^\pm(n, \mathbb{R}^{(n \times n)}) \), where the superscript indicates self-duality respectively anti-self-duality. We denote by \( \mathcal{O}_{(r,s)} \) the set of those maximal isotropic subspaces on which the restriction of the real part of \( b \) has signature \((r, s)\). Then we get \( IGr^\pm(n, \mathbb{R}^{(n \times n)}) \) is the union of the \( \mathcal{O}_{(r,s)} \) with \( r + s \leq n \), such that both \( n - r - s \) and \( n - s \) are even, while \( IGr^-\,(n, \mathbb{R}^{(n \times n)}) \) is the union of the orbits for which \( n - r - s \) is even but \( n - s \) is odd. The open orbits are exactly those in which \( r + s = n \), while \( \mathcal{O}_{(r', s')} \subset \overline{\mathcal{O}}_{(r,s)} \) if and only if \( r' \leq r \) and \( s' \leq s \) and \( s' \) has the same parity as \( s \).
In particular, there is the orbit $O_{(n,0)} \subset IGr^+(n, \mathbb{R}^{(n,n)})$ which, for the standard complex bilinear form $b$, contains the subspace $\mathbb{R}^n \subset \mathbb{C}^n$. The stabilizer of this subspace in $H$ visibly is given by the matrices in $SO(n, \mathbb{C})$ for which all entries are real, so this is just $SO(n)$. Thus, we get a parabolic compactification of $H/K := SO(n, \mathbb{C})/SO(n)$ of the form $\overline{H/K} = \bigcup_{j=0}^{[n/2]} O_{(n-2j,0)}$. Similarly, one obtains parabolic compactifications of $SO(n, \mathbb{C})/SO_0(p,q)$ for $p+q = n$.

From the proof of Lemma 4.1, one can also see that already for the isotropic Grassmannians $IGr(\ell, \mathbb{R}^{(n,n)})$ with $\ell < n$, the orbits of $H = SO(n, \mathbb{C})$ are not determined by rank and signature of the restriction of the real part of $b$ alone. Indeed, for an isotropic subspace $V \subset \mathbb{R}^{(n,n)}$ of dimension $\ell< n$, the null space of $\text{Re}(b)$ is $J(V) \cap V^\perp$. Hence the co-rank does not have to be even in general, and there is an additional distinguished subspace in the null space of $\text{Re}(b)$, namely the maximal complex subspace $J(V) \cap V$ of $V$. Hence the real dimension of this subspace, which has to be even and at most equal to the co-rank, is an additional invariant preserved by the action of $H$. So the two-step flag $V \subset V^\perp$ plays a similar role as in the discussion in §3.1. Passing to more general isotropic flag manifolds, one gets additional invariants depending on the relative position of the individual constituents of the flag and their orthocomplements with respect to $J$.

Let us next pass to the description of the individual orbits and of the infinitesimal transversal. Similar to the discussion in Proposition 3.1 each orbit admits a fibration onto a generalized flag manifold of $H$ with fiber a lower dimensional analog of $H/K$. The infinitesimal transversals are described by Hermitian matrices, which is surprising, since the setup does not seem to naturally lead to a Hermitian inner product.

**Proposition 4.2.** Fix $r$ and $s$ such that $n-r-s = 2k$. Then we have

1. The orbit $O := O_{(r,s)} \subset IGr^+(n, \mathbb{R}^{(n,n)})$ has codimension $k^2$. There is an $H$-equivariant, surjective submersion from $O_{(r,s)}$ onto the Grassmannian $IGr(k, \mathbb{C}^n)$ of $k$-dimensional complex subspaces which are isotropic for $b$. The fibers of this map are isomorphic to $SO(n-2k, \mathbb{C})/SO(r,s)$.

2. Replacing $H$ by a conjugate subgroup $H_O$, as described in §2.4, the infinitesimal transversal of $O$ is given by the natural representation of $GL(k, \mathbb{C})$ on the space of Hermitian $k \times k$-matrices via a natural surjective homomorphism $H_O \cap P \to GL(k, \mathbb{C})$.

**Proof.** (1) We continue using the notation for orthocomplements from the proof of Lemma 4.1. From there we know that the complex subspace $W := J(V) \cap V$ has complex dimension $k$ and is isotropic for $b$. Sending $V$ to $W$ defines the claimed map to the complex isotropic Grassmannian. We have also seen in that proof that $V$ descends to a totally real subspace $\overline{V} \subset W^\perp/W$ which is isotropic for the imaginary part of the induced complex bilinear form $\overline{b}$, while the restriction of the real part of $b$ to $\overline{V}$ is non-degenerate of signature $(r,s)$. This leads to the description of the fiber. Since conversely starting from a complex isotropic subspace $W \subset \mathbb{C}^n$ and an appropriate totally real subspace $\overline{V} \subset W^\perp/W$, the pre-image of $\overline{V}$ in $\mathbb{C}^n$ evidently lies in $O$, we get surjectivity. The codimension of the orbit follows from standard results on the dimensions of the spaces involved. Part (2) provides an alternative proof.
(2) Passing to \( H_\mathcal{O} \) simply means that we use matrix representations with respect to a basis adapted to \( V \in \mathcal{O} \) as in Lemma 4.1, and then use the stabilizer of \( V \) in \( G \) as our parabolic subgroup \( P \). We use a basis as obtained in Lemma 4.1 to split real linear endomorphisms of \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) into blocks in two different ways. In the notation of that lemma, we order the first basis vectors by splitting them into 6 groups of vectors. We first take \( z_1, \ldots, z_k \), then \( i z_1, \ldots, i z_k \), next \( z_{k+1}, \ldots, z_{n-k} \), then \( i z_{k+1}, \ldots, i z_{n-k} \), then \( z_{n-k+1}, \ldots, z_n \) and finally \( i z_{n-k+1}, \ldots, i z_n \). Correspondingly, we write elements of \( \mathfrak{g} \) as block matrices with \( 6 \times 6 \) blocks of sizes \( k, k, n-2k, n-2k, k, k \) respectively. We will use the notation \((M_{jk})\) for this block decomposition, with \( j, k = 1, \ldots, 6 \). From the construction, it is clear how \( J \) is described in terms of such block matrices.

On the other hand, we can use the coarser decomposition into four blocks of size \( n \times n \), by collecting the first three groups of basis vectors and the last three groups of basis vectors. Here the first \( n \) basis vectors by construction form a basis of \( V \) and the span of the last \( n \) basis vectors is also isotropic for \( \langle ., . \rangle \). Correspondingly, the matrix of \( \langle ., . \rangle \) in that block decomposition has the form \( \begin{pmatrix} 0 & J \setminus 0 \\ J \setminus 0 & 0 \end{pmatrix} \). Here \( J \) is the \( n \times n \)-matrix, which split into blocks of sizes \( k, k \) and \( n-2k \) has the form \( \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} \), with notation as in Lemma 4.1.

Correspondingly, the matrices in \( \mathfrak{g} \) are exactly those which have the coarse block form

\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & -J A_{11}^{-1} J'
\end{pmatrix}
\]

with \( n \times n \)-matrices \( A_{ij} \) such that \( A_{12} = -JA_{12}J' \) and \( A_{21} = -J A_{21} J' \).

Now expressing the condition on \( A_{21} \) in terms of the finer block decomposition immediately shows that this has to have the form

\[
\begin{pmatrix}
M_{41} & M_{42} & M_{43} \\
M_{51} & M_{52} & -M_{52}^r \\
M_{61} & -M_{61}^l & -M_{61}^r
\end{pmatrix}
\]

with \( M_{52}' = -M_{52} \), \( M_{61}' = -M_{61} \) and \( M_{43} \in \mathfrak{o}(r, s) \). From the explicit form of \( J \), it is easy to describe the subalgebra \( \mathfrak{h} \subset \mathfrak{g} \). It simply consists of those block matrices in \( \mathfrak{g} \) such that for each odd \( j \) and \( k \) we have \( M_{j+1, k+1} = M_{j, k} \) and \( M_{j, k+1} = -M_{j+1, k} \). This confirms that any element of \( \mathfrak{h} \cap \mathfrak{p} \) also stabilizes \( W \) and \( W^{\perp b} \) (which are spanned by the first \( 2k \) respectively all but the last \( 2k \) basis vectors). Moreover, it shows that the action of \( H \cap P \) on \( \mathbb{C}^n / W^{\perp b} \) defines a surjective homomorphism \( H \cap P \to GL(k, \mathbb{C}) \). On the Lie algebra level, this is represented by the block

\[
\begin{pmatrix}
M_{65} & M_{66} \\
M_{56} & -M_{55}
\end{pmatrix}
\]

(for which we know that \( M_{56} = -M_{65} \) and \( M_{66} = M_{55} \)). We also know from above that we get \( M_{11} = M_{22} = -M_{55}' \) while \( M_{21} = -M_{12} = -M_{65}' \).

For the block decomposition of \( A_{21} \) from above, we get additional restrictions, namely \( M_{51} = -M_{51}' \) and \( M_{52} = -M_{61} \). A short computation shows that one does not obtain further restrictions on \( M_{41}, M_{42}, \) and \( M_{43} \), they only have to be related to blocks in the row above. From this, we can read off a complement to \( \mathfrak{h} / (\mathfrak{h} \cap \mathfrak{p}) \) in \( \mathfrak{g} / \mathfrak{p} \). The matrices whose only non-zero blocks are \( M_{51}, M_{52}, M_{61} = M_{52} \) and \( M_{62} = -M_{51} \) such that \( M_{51}' = M_{51} \) and \( M_{52}' = -M_{52} \) visibly descend to such a complement. Since \( M_{51} \) and \( M_{61} \) both are \( k \times k \)-matrices we see that \( \mathcal{O} \) indeed has codimension \( k^2 \).

To determine the action of \( \mathfrak{h} \cap \mathfrak{p} \) on the infinitesimal transversal, we can compute the adjoint action on this complementary subspace and then project to the quotient. For \((N_{jk}) \in \mathfrak{h} \cap \mathfrak{p} \), the resulting action depends only on the blocks \( N_{jk} \) for \( j, k = 1, 2 \) (which also determine the parts for \( j, k = 5, 6 \)). Indeed, a short computation shows that the action sends \( \begin{pmatrix} M_{51} & M_{52} \\ M_{52} & -M_{51} \end{pmatrix} \) (where we still have \( M_{51}' = M_{51} \) and \( M_{52}' = -M_{52} \)) to a matrix
of the same form with first column given by
\[ \begin{pmatrix} -N_{11}' M_{51} - N_{12}' M_{61} - M_{51} N_{11} + M_{61} N_{12} \\ N_{12}' M_{51} - N_{11}' M_{61} - M_{61} N_{11} - M_{51} N_{12} \end{pmatrix}. \]
This shows that our complementary subspace is invariant under the action. Moreover, then considering the Hermitian matrix \( M_{51} + i M_{61} \) this action exactly corresponds to the standard action
\[ (N_{11}' + i N_{12}') (M_{51} + i M_{61}) - (M_{51} + i M_{61}) (N_{11} - i N_{12}) \]
of \( \mathfrak{gl}(k, \mathbb{C}) \) on Hermitian \( k \times k \)-matrices.

4.2. Orbit closures via BGG solutions

The first steps of this are closely similar to the case discussed in §3.2. We consider one of the isotropic Grassmannians \( IGr^\pm(n, \mathbb{R}^{(n,n)}) = G/P \), where \( G = SO_0(n, n) \). The tautological bundles on these Grassmannians are determined by the subbundle \( E \) in the trivial bundle with fiber \( \mathbb{R}^{(n,n)} \), whose fiber at a point \( V \in IGr^\pm(n, \mathbb{R}^{(n,n)}) \) is the subspace \( V \subset \mathbb{R}^{(n,n)} \). Via the invariant inner product \( \langle , \rangle \) on \( \mathbb{R}^{2n} \), the quotient bundle \( \mathbb{R}^{(n,n)}/E \) gets identified with \( E^* \). For our choice of parabolic subgroup \( P \), the Levi factor \( G_0 \subset P \) is \( GL(n, \mathbb{R}) \) and the bundle \( E \) is the completely reducible bundle associated to the standard representation of that group. From the description in the proof of Proposition 4.2 it is clear that \( \mathfrak{g}/\mathfrak{p} \) is isomorphic to \( \Lambda^2 \mathbb{R}^{n*} \) as a representation of \( G_0 \), while the nilradical of \( \mathfrak{p} \) acts trivially on \( \mathfrak{g}/\mathfrak{p} \). Correspondingly, the tangent bundle \( T(G/P) \) is naturally identified with \( \Lambda^2 E^* \), which defines an almost spinorial structure on \( G/P \), making it into the homogeneous model for such structures.

Completely reducible natural bundles on \( G/P \) are induced by representations of the group \( G_0 = GL(n, \mathbb{R}) \), so they can be built up from the bundles \( E \) and \( E^* \) by tensorial constructions. In particular, for \( k = 1, \ldots, n \) we can form \( \Lambda^k E^* \) and then take the subbundle \( \otimes^2(\Lambda^k E^*) \subset S^2(\Lambda^k E^*) \) as described in §3.2. By construction, this is an irreducible natural bundle over \( G/P \), and for \( k = 1, n-1, \) and \( n \) it coincides with \( S^2(\Lambda^k E^*) \). In particular, it is a density bundle for \( k = n \). The simplest tractor bundle for almost spinorial structures is the standard tractor bundle \( T \), which corresponds to the standard representation \( \mathbb{R}^{(n,n)} \) of \( G \). On the homogeneous model \( G/P \), this bundle is naturally isomorphic to the trivial bundle \( (G/P) \times \mathbb{R}^{(n,n)} \). The \( P \)-invariant filtration of \( T \) just consists of the subbundle \( E \) for which \( T/E \cong E^* \). More generally, any \( G \)-irreducible subrepresentation of a tensor power of \( \mathbb{R}^{(n,n)} \) induces a tractor bundle over \( G/P \).

Theorem 4.3. Consider the decomposition of one of the isotropic Grassmannians \( IGr^\pm(n, \mathbb{R}^{(n,n)}) \) into the orbits \( \mathcal{O}_{(r,s)} \) for the subgroup \( H \subset G \) as described in Proposition 4.2. For each \( k = 1, \ldots, n \), there is a section \( \sigma_k \in \Gamma(\otimes^2(\Lambda^k E)) \), which lies in the kernel of the first BGG operator defined on that bundle whose zero locus is the union of those \( H \)-orbits \( \mathcal{O}_{(r,s)} \), for which \( r+s < k \). The relevant first BGG operator is of first order if \( k < n-1 \), of second order for \( k = n-1 \) and of third order for \( k = n \).

Proof. The basic strategy is the same as in the proof of Theorem 3.4, but some of the representation theory is more involved. We continue viewing \( \mathbb{R}^{2n} \) as \( \mathbb{C}^n \) and \( \langle , \rangle \) as the
imaginary part of the standard symmetric complex bilinear form $b$. Then multiplication by $i$ defines an endomorphism of $\mathbb{R}^{(n,n)}$ which is symmetric for $\langle \cdot, \cdot \rangle$ and trace-free. Thus it defines an element $J \in S_0^2(\mathbb{R}^{(n,n)*})$ whose stabilizer in $G$ is the subgroup $H$. This in turn defines a parallel section $s_1$ of the tractor bundle $S_0^2(T^*)$ which can be either considered as an endomorphism of $T$ or as a symmetric bilinear form on that bundle. The irreducible quotient of $S_0^2 T^*$ is just $S^2 E^*$ and projecting $s_1$, we obtain a section $\sigma_1 \in \Gamma(S^2 E^*)$, which lies in the kernel of the first BGG operator defined on that bundle. For a point $V \in G/P$, $E_V = V$ and $\sigma_1(V)$ by construction is the restriction of $\text{Re}(b)$ to $V$. Thus $V$ lies in $\mathcal{O}(r,s)$ if and only if $\sigma_1(V)$ has signature $(r,s)$ (and thus rank $r+s$).

Parallel to the discussion in §3.2, for $k = 1, \ldots, n$, the endomorphism $\Lambda^k J$ of $\Lambda^k \mathbb{R}^{(n,n)}$ can also be viewed as a bilinear form on $\Lambda^k \mathbb{R}^{(n,n)}$. By Lemma 3.3 this sits in the $GL(2n, \mathbb{R})$-irreducible component $W_k \subset S^2(\Lambda^k \mathbb{R}^{2n*})$ of maximal highest weight. Now for most $k$, $W_k$ is not irreducible for $G$, since the inner product $\langle \cdot, \cdot \rangle$ defines non-trivial traces on it. The joint kernel of these traces is the $G$-irreducible component of highest weight in $S^2(\Lambda^k \mathbb{R}^{(n,n)*})$, which we denote by $\oplus_0^2(\Lambda^k \mathbb{R}^{(n,n)*})$. It turns out that, as a representation of $G$, $W_k$ is isomorphic to $\bigoplus_{j=0}^k \oplus_0^2(\Lambda^j \mathbb{R}^{(n,n)*})$. Here for $j = 0, 1$, we obtain a trivial summand and $S_0^2(\mathbb{R}^{(n,n)})$, respectively. Now we can split $\Lambda^k J \in W_k$ according to this decomposition. The component of $\Lambda^k J$ in $\oplus_0^2(\Lambda^k \mathbb{R}^{(n,n)*})$ defines a section $s_k \in \Gamma(\oplus_0^2(\Lambda^k T^*))$, which is parallel for the canonical tractor connection. Projecting to the irreducible quotient bundle, we obtain a section $\sigma_k \in \Gamma(\oplus^2(\Lambda^k E^*))$ which lies in the kernel of the first BGG operator defined on that bundle.

Returning to the decomposition of $W_k$ into irreducibles, we can split each of the representations $\oplus_0^2(\Lambda^j \mathbb{R}^{(n,n)*})$ into irreducibles with respect to $G_0 = GL(n, \mathbb{R})$. In particular, the $P$-irreducible quotient of $\oplus_0^2(\Lambda^j \mathbb{R}^{(n,n)*})$ is $\oplus^2(\Lambda^j \mathbb{R}^{n*})$, which easily implies that for $j < k$, there is no non-zero $G_0$-equivariant map $\oplus_0^2(\Lambda^j \mathbb{R}^{(n,n)*}) \rightarrow \oplus^2(\Lambda^k \mathbb{R}^{n*})$. But this implies that the natural projection from $W_k$ to its irreducible quotient $\oplus^2(\Lambda^k \mathbb{R}^{n*})$ which we used in the proof of Theorem 3.4 factors through $\oplus_0^2(\Lambda^k \mathbb{R}^{(n,n)*})$. But this shows that $\sigma_k$ is induced by the image of $\Lambda^k J$ under the latter projection, so from the proof of Theorem 3.4 we see that $\sigma_k = \Lambda^k \sigma_1$, which leads to the description of the zero loci, as in that proof. The orders of the BGG operators can again be read off from the weights of the inducing representations.

\[ \square \]

4.3. A slice theorem

In Proposition 4.2, we have obtained infinitesimal transversals which are formed by Hermitian matrices. This is rather surprising, since initially there does not seem to be a natural notion of conjugation around. We start by proving a lemma which directly explains how Hermitian metrics enter the picture.

**Lemma 4.4.** For even $k = 2\ell$, consider the standard complex bilinear form $b$ on $\mathbb{C}^k$ and let $\langle \cdot, \cdot \rangle$ be its imaginary part. Let $V \subset \mathbb{C}^k$ be a real subspace of dimension $k$, which is isotropic for $\langle \cdot, \cdot \rangle$. Suppose that $\mathbb{C}^k = Z_1 \oplus Z_2$ is a decomposition into a sum of two complex subspaces, both of which are isotropic for $b$, and such that $Z_2 \cap V = \{0\}$, and hence the projection $\pi$ onto the first summand restricts to a (real) linear isomorphism $V \rightarrow Z_1$. 

Then the restriction of the real part $\text{Re}(b)$ of $b$ to $V$ is Hermitian with respect to the pullback along $\pi|_V$ of the complex structure on $Z_1$.

**Proof.** Composing the projection onto the second summand with $(\pi|_V)^{-1}$, we obtain a real linear map $\varphi : Z_1 \to Z_2$ such that $V = \{z + \varphi(z) : z \in Z_1\}$. For $j = 1, 2$ take $v_j \in V$ and write it as $v_j = z_j + \varphi(z_j)$ for $z_j \in Z_1$ to obtain $b(v_1, v_2) = b(z_1, \varphi(z_2)) + b(\varphi(z_1), z_2)$. Since $V$ is isotropic for $(\cdot, \cdot)$, we conclude that $\langle z_1, \varphi(z_2) \rangle = -\langle \varphi(z_1), z_2 \rangle$. Since $z_1, z_2 \in Z_1$ may be arbitrary, $\varphi$ is skew symmetric with respect to $(\cdot, \cdot)$. On the other hand, we can compute $\text{Re}(b)(v_1, v_2)$ as the real part of the above expression, which coincides with $\langle iz_1, \varphi(z_2) \rangle + \langle \varphi(z_1), iz_2 \rangle$. Skew symmetry of $\varphi$ readily implies that this remains unchanged if we replace $z_1$ by $iz_1$ and $z_2$ by $iz_2$, which implies the claim of the lemma. 

Having this at hand, we can prove the slice theorem. Given an orbit $O_{(r,s)} \subset IGr^\pm(n, \mathbb{R}^{(n,n)})$, we know from Proposition 4.2 that $n - r - s$ is even, and we denote this by $2k$. Now we denote by $\mathcal{H}_k$ the set of real matrices of size $2k \times 2k$ which represent complex matrices of size $k \times k$ that are Hermitian. In particular, these are real symmetric matrices of even rank for which both parts of the signature are even.

**Theorem 4.5.** Each of the orbits $O_{(r,s)} \subset G/P := IGr^\pm(n, \mathbb{R}^{(n,n)})$ is an embedded submanifold. For each point $x \in O_{(r,s)}$, there is an open neighborhood $U$ of $x$ in $G/P$ and a diffeomorphism $\varphi$ from $U$ onto an open neighborhood of $(U \cap O_{(r,s)}) \times \{0\}$ in $(U \cap O_{(r,s)}) \times \mathcal{H}_k$ such that $U \subset \bigcup_{r' \geq r, s' \geq s} O_{(r', s')}$ and a point $y \in U$ lies in $O_{(r', s')}$ if and only if the second component of $\varphi(y)$ has signature $(r' - r, s' - s)$.

**Proof.** The first part of this is closely parallel to the proof of Theorem 3.5: There is a connected neighborhood $U$ of $x$ in $O_{(r,s)}$, which is an embedded submanifold of $G/P$. Possibly shrinking $U$, there is a connected open neighborhood $U$ of $x \in G/P$ and a smooth subbundle $W \subset E|_U$ of rank $r + s$ such that for each $y \in U$, the restriction of the symmetric bilinear form $\sigma_1(y)$ to $W_y \subset E_y$ is non-degenerate and has signature $(r, s)$. Defining $\tilde{E}_y \subset E_y$ to be the space of those $v \in E_y$ such that $\sigma_1(y)(v, w) = 0$ for all $w \in W_y$, we obtain a smooth subbundle $\tilde{E} \subset E|_U$ of even rank $2l$. By construction, for $y \in U$ the null space of $\sigma_1(y)$ is contained in $\tilde{E}_y$ and it coincides with $\tilde{E}_y$ if $y \in O_{(r,s)}$. In particular, we conclude that $U \subset \bigcup_{r' \geq r, s' \geq s} O_{(r', s')}$ and that a point $y \in U$ lies in $O_{(r', s')}$ if and only if the restriction of $\sigma_1(y)$ to $\tilde{E}_y$ has signature $(r' - r, s' - s)$.

At this point we need additional input to get the Hermitian aspects into the picture. For each $y \in U$, $W_y$ is a real subspace in $\mathbb{C}^n$ of real dimension $r + s$ on which $(\cdot, \cdot)$ is identically zero. But $\text{Re}(b)$ is non-degenerate on $W_y$, so $W_y \cap J(W_y) = \{0\}$ and that the restriction of $b$ to the complex subspace $W_y \oplus J(W_y) \subset \mathbb{C}^n$ is non-degenerate. Otherwise put, viewing $W$ as a smooth subbundle in the standard tractor bundle $\mathcal{T}|_U$ and viewing the parallel section $s_1$, from the proof of Theorem 4.3, as a section of $L(\mathcal{T}, \mathcal{T})$, we can form a smooth subbundle $W \oplus s_1(W) \subset \mathcal{T}$, which by construction is invariant under $s_1$. Using $s_1$, we can extend the canonical inner product $(\cdot, \cdot)$ on $\mathcal{T}$ to a field of non-degenerate $\mathbb{C}$-valued bilinear forms, which are complex bilinear with respect to $s_1$. The restriction of this form to $W \oplus s_1(W)$ is non-degenerate, so we can form the complex orthocomplement, which is an $s_1$-invariant subbundle $Z \subset \mathcal{T}$ on which the complex bundle metric is non-degenerate, too.
Now by definition, $\tilde{E}_y$ is perpendicular to $W_y$ with respect to $\text{Re}(b)$ and since $\tilde{E}_y \subset E_y$, it is also perpendicular to $W_y$ with respect to $\langle \cdot, \cdot \rangle$. This implies that $\tilde{E}_y \subset Z_y$ for all $y \in U$. On the other hand, the complex co-rank of $Z$ in $\mathcal{T}$ by construction equals the real co-rank of $W$ in $E$ and thus is even. This implies that locally we can write $Z = Z_1 \oplus Z_2$ for two smooth subbundles which are invariant under $\sigma_1$ and isotropic for the complex bundle metric. Possibly shrinking $U$ and starting in such a way that in the point $x \in \mathcal{O}_{(r,s)}$ we take $Z_1$ to be $\tilde{E}_x$ (which coincides with the null space of $\sigma_1(x)$ and thus is complex isotropic), we may assume that $Z_2 \cap \tilde{E} = \{0\}$ on all of $U$. But this implies that the projection onto the first factor restricts to an isomorphism $\tilde{E} \to Z_1$ of real vector bundles. Pulling back the complex structure from $Z_1$ then makes $\tilde{E}$ into a complex vector bundle of complex rank $k$, and applying Lemma 3.4 pointwise, we conclude that the restriction of $\sigma_1$ to $\tilde{E}$ defines a Hermitian bundle metric on the complex vector bundle $\tilde{E}$.

Having this at hand, we can continue as in the proof of Theorem 3.5 and Corollary 3.6. Restricting bilinear forms defines a map $q : S^2E^* \to S^2\tilde{E}^*$ and we know that $q \circ \sigma_1$ has values in the subbundle $\mathcal{H}^2 \tilde{E}^*$ of forms which are Hermitian with respect to the complex structure constructed above. We also know that $q \circ \sigma_1$ vanishes identically along $\mathcal{U} := U \cap \mathcal{O}_{(r,s)} \subset U$. To complete the proof, it suffices to show that, possibly shrinking $U$ and $\mathcal{U}$, $q \circ \sigma_1 \in \Gamma(\mathcal{H}^2 \tilde{E}^*)$ is a defining section for $\mathcal{U}$. Since $q \circ \sigma_1$ by construction vanishes along $\mathcal{O}_{(r,s)}$ this boils down to proving that $\nabla(q \circ \sigma)(y) : T_y(G/P) \to H^2 \tilde{E}^*$ is surjective for each $y \in \mathcal{O}_{(r,s)}$. Having shown that, we can complete the proof exactly as the one of Corollary 3.6.

The first steps in the analysis of the derivative are as in the proof of Theorem 3.5 and we use the notation from that proof. Any linear connection on $E$ induces a linear connection on $\tilde{E}$, there are induced linear connections $\nabla$ on $S^2E$ and $\tilde{\nabla}$ on $S^2\tilde{E}$ and for each vector field $\xi$, we get $\tilde{\nabla}_\xi (q \circ \sigma)(y) = q(\nabla_\xi \sigma(y))$. Starting with a Weyl connection on $E$, we can use Proposition 2.17 to compute $\nabla \sigma(y)$ as $-\partial(\mu(y))$. For the isotropic Grassmannian, the tangent bundle $T(G/P)$ is isomorphic to $\Lambda^2 E^*$ (which defines the flat almost spinorial structure) while for $\mathcal{V} = S^2_0 T^*$, we get $\mathcal{V}/\mathcal{V}^1 = S^2 E^*$ and $\mathcal{V}^1/\mathcal{V}^2 = (E^* \otimes E)_0$, where the subscript indicates the trace-free part. Hence $\partial$ maps $(E^* \otimes E)_0$ to $\Lambda^2 E \otimes S^2 E^*$ and since this comes from a $GL(n, \mathbb{R})$-equivariant map between the inducing representations, it has to be given by tensoring with the identity and then symmetrizing the $E$-components and alternating the $E^*$-components. Otherwise put, if we view $\mu$ as a linear map $E^* \to E^*$, then $\partial(\mu)$ sends $\alpha \wedge \beta$ to (a non-zero multiple of) $\mu(\alpha) \vee \beta - \mu(\beta) \vee \alpha$.

Now for $y \in \mathcal{O}_{(r,s)}$ corresponding to the isotropic subspace $E_y$, we know that $\tilde{E}_y$ is the null space of $\sigma_1(y)$ and from the proof of Lemma 4.1 we know that this coincides with the maximal complex subspace of $E_y$. Hence we get $s_1(y)(E_y) \cap E_y = \tilde{E}_y$ and $s_1(y)$ makes $\tilde{E}_y$ into a complex vector space. The fact that, viewed as an endomorphism of $\mathcal{T}$, $s_1(y)$ maps $\tilde{E}_y$ to $\tilde{E}_y \subset E_y$ says that $\sigma_1(y)|_{E_y} = 0$ and hence $s_1(y)|_{\tilde{E}_y} = \mu(y)|_{\tilde{E}_y}$. Thus the restriction of $\mu(y)$ to $\tilde{E}_y$ is simply multiplication by $i$ for the complex structure on $\tilde{E}_y$ we have constructed. By definition of the structure on the dual, $\mu(y)$ is given by multiplication by $i$ as a map $\tilde{E}_y^* \to \tilde{E}_y^*$. But this exactly means that for $\alpha, \beta \in \tilde{E}_y$, we get (up to a non-zero factor) $\partial(\mu)(\alpha \wedge \beta) = i \alpha \vee \beta - \alpha \vee i \beta$, so this lies in $\mathcal{H}^2 \tilde{E}_y^*$. On the other hand, any element of $\mathcal{H}^2 \tilde{E}_y^*$ can be written as a linear combination of
elements of the form $\alpha \lor \beta + i\alpha \lor i\beta$ and such an element is obtained (up to a factor) as $-\partial(\mu)(i\alpha \land \beta)$, which completes the proof.

\textbf{Remark 4.6}. (1) Similarly to Remark 3.8, there is a nice interpretation of Theorem 4.5 in terms of local compactifications. Let us first consider the case that

\[ H/K = SO(n, \mathbb{C})/SO(n) \cong O_{(n,0)} \]

for which we know that $H/K = \bigcup_v O_{(n-2v,0)}$ with integers $v$ such that $0 \leq v \leq \frac{n}{2}$. Then the model is a local compactification of $GL(v, \mathbb{C})/U(v)$, which can be identified with the space positive definite Hermitian $v \times v$-matrices, locally compactified by the space of positive semi-definite Hermitian matrices. Theorem 4.5 then says that for sufficiently small open subsets $W \subset O_{(n-2v,0)}$, a neighborhood of $W$ in $H/K$ is isomorphic to the product of $W$ with a neighborhood of $0$ in that local compactification of $GL(v, \mathbb{C})/U(v)$.

For other signatures, i.e. $H/K = SO(n, \mathbb{C})/SO(p, q) \cong O_{(p,q)}$ with $p + q = n$, there is a similar description in terms of local compactifications of $GL(v, \mathbb{R})/U(p', q')$ for $p' + q' = v$, $p' \leq p$ and $q' \leq q$. The relevant local compactification of Hermitian matrices of signature $(p', q')$ is given as Hermitian matrices of signatures $(r, s)$ with $r \leq p'$ and $s \leq q'$.

(2) As we have seen already in Lemma 4.1 the ranks always drop in steps of two in our example. This did not cause problems in the description of orbit closures via zero loci of BGG solutions in Theorem 4.3. However, already there the strange situation occurs that, for example, the sections $\sigma_n$ and $\sigma_{n-1}$ have the same zero locus, since rank smaller than $n$ always implies rank smaller than $n - 1$. The problem becomes serious, however, when one tries to generalize Proposition 3.7 to the current setting. While the orbits $O_{(r,s)}$ of largest non-full rank (i.e. with $r + s = n - 2$) still form embedded hypersurfaces in $G/P$ by Theorem 4.5, the section $\sigma_n$ (which is the only $\sigma_i$ that has values in a line bundle), certainly is not a defining density for these hypersurfaces. Indeed, arguments similar to the ones used in the proof of Proposition 3.7 show that $\sigma_n$ and $\nabla \sigma_n$ simultaneously vanish in points where $\sigma_1$ has rank less than $n - 1$, so this happens in all points of $O_{(r,s)}$.

Indeed, in the current situation, we do not see a way how to construct a natural defining density for the orbits which are hypersurfaces. To obtain a defining density, it would seem necessary to exploit the fact that, as shown in the proof of Theorem 4.5, the metric on a two-dimensional complement to the tangent space of the orbit is Hermitian for an appropriate complex structure. However, that complex structure seems to be canonical only along the orbit itself, where this transverse metric vanishes identically.

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