EXACT CONVERSES TO A REVERSE AM—GM INEQUALITY, WITH APPLICATIONS TO SUMS OF INDEPENDENT RANDOM VARIABLES AND (SUPER)MARTINGALES

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Abstract. For every given real value of the ratio \( \mu := AX/GX > 1 \) of the arithmetic and geometric means of a positive random variable \( X \) and every real \( v > 0 \), exact upper bounds on the right- and left-tail probabilities \( P(X/GX \geq v) \) and \( P(X/GX \leq v) \) are obtained, in terms of \( \mu \) and \( v \). In particular, these bounds imply that \( X/GX \to 1 \) in probability as \( AX/GX \downarrow 1 \). Such a result may be viewed as a converse to a reverse Jensen inequality for the strictly concave function \( f = \ln \), whereas the well-known Cantelli and Chebyshev inequalities may be viewed as converses to a reverse Jensen inequality for the strictly concave quadratic function \( f(x) \equiv -x^2 \). As applications of the mentioned new results, improvements of the Markov, Bernstein–Chernoff, sub-Gaussian, and Bennett–Hoeffding probability inequalities are given.

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