Local versus nonlocal $\alpha\alpha$ interactions in 3$\alpha$ description of $^{12}\text{C}$

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Local $\alpha\alpha$ potentials fail to describe $^{12}\text{C}$ as a 3$\alpha$ system. Nonlocal $\alpha\alpha$ potentials that renormalize the energy-dependent kernel of the resonating group method allow interpreting simultaneously the ground state and $0^+_2$ resonance of $^{12}\text{C}$ as 3$\alpha$ states. A comparison with fully microscopic calculations provides a measure of the importance of three-cluster exchanges in those states.

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The microscopic 3$\alpha$ model describes $^{12}\text{C}$ as a 12-nucleon system where the nucleons are grouped into three substructures known as $\alpha$ clusters. The merit of this microscopic model is that it starts from effective two-nucleon potentials and takes account of the antisymmetry requirement exactly. It provides a satisfactory qualitative description of several states of $^{12}\text{C}$ including the $0^+$ ground state and the astrophysically important $0^+_1$ ‘Hoyle’ resonance despite that spin-dependent effects are missing in the $\alpha$ cluster assumption. The first microscopic 3$\alpha$ calculation was performed within the resonating group method (RGM) $[1,2]$ by Kamimura $[3]$. This model provides useful wave functions for various applications but remains complicated and heavy to handle.

This difficulty motivates another development of the cluster model, a ‘macroscopic’ 3$\alpha$ model, involving three structureless $\alpha$ bosons interacting through local $\alpha\alpha$ forces which reproduce the $\alpha\alpha$ phase shifts. However, in spite of about 40 years of efforts $[4,5,6,7]$, this macroscopic treatment still meets serious problems. In the 3$\alpha$ system, any local potential whether shallow or deep yields very poor results, in disagreement with experiment and microscopic models. The macroscopic 3$\alpha$ model though physically appealing fails to reproduce even qualitatively the properties of the $^{12}\text{C}$ ground state.

Initially shallow $\alpha\alpha$ potentials $[8]$ were used. Such potentials fit the phase shifts and do not support bound states since $^8\text{Be}$ is unbound, but lead to an unrealistically weakly bound $^{12}\text{C}$. The microscopically founded suggestion that potentials should be deep $[2,3,10]$ does not improve the situation. Deep potentials involve additional bound states, called forbidden or redundant states, whose role is to simulate the effect of Pauli antisymmetrization through orthogonality with physical states. They improve physical properties of two-body wave functions but introduce many unwanted levels in the spectrum of three-body systems $[11]$. Forbidden states must thus be eliminated and various techniques have been developed $[5,12,13,14]$. Unfortunately, the $^{12}\text{C}$ ground-state energy is found to strongly depend on how accurately the forbidden states of the deep potential are eliminated from the configuration space $[6,7]$. Strangely, the most accurate calculations provide much worse results than unconverged ones; they again provide a weakly bound $^{12}\text{C}$ $[7]$. The origin of this enigmatic behavior has finally been explained by a careful consideration of the role of almost-forbidden states in this 3-cluster system $[15,16]$. That the $\alpha\alpha$ potential can not be local for describing $^{12}\text{C}$ is not surprising. The interaction between composite particles is intrinsically nonlocal because of the exchange symmetry of identical constituents. Whether nonlocal $\alpha\alpha$ potentials can provide a description of $^{12}\text{C}$ is thus a fundamental issue. Such questions have recently been examined for few-nucleon systems using nonlocal nucleon-nucleon interactions of various types $[15,18,19,20]$.

A first attempt to use energy-dependent nonlocal $\alpha\alpha$ potentials $[1,2]$ has been developed in Ref. $[21]$. This interaction contains all antisymmetrization effects in the $\alpha\alpha$ system. It accurately describes the $^8\text{Be}$ ground-state and the $\alpha\alpha$ phase shifts. The only model assumptions are a simple $\left(0s\right)^4$ description of the $\alpha$ clusters within the translation-invariant harmonic-oscillator (HO) model and the use of an effective nucleon-nucleon interaction $[22]$. Employing the resulting nonlocal $\alpha\alpha$ interaction is rendered complicated by the existence of an energy dependence and of Pauli-forbidden states. Within this semi-microscopic 3$\alpha$ model, the difficulties associated with the accuracy of the forbidden-state elimination disappear $[21,23]$. Despite a reasonable success for the ground state, the problem now is that the $0^+_2$ excited state is not obtained simultaneously. Moreover the energy dependence of the $\alpha\alpha$ nonlocal interaction raises another difficulty. This energy is well defined in a two-body system but not in a three-body system. Its choice raises another ambiguity in the model $[23]$.

The principle of the elimination of the energy dependence from the RGM equation is known for a long time $[24,25,26]$. For two-body systems, its interest is mainly academic. That three-body calculations should be performed with the complicated nonlocal potential resulting from this transformation has been suggested several times $[25,26]$ but its use has never thoroughly been examined in view of the many difficulties involved. Indeed, not only must these nonlocal potentials be derived
but their use and evaluation of validity simultaneously require mastering accurate calculations of macroscopic three-body systems with nonlocal forces and of microscopic three-cluster systems for comparison. This knowledge has been developed in recent years by our groups [16, 21, 27, 28, 29, 30, 31]. The purpose of the present study is to clarify the role of nonlocality in the \(\alpha\alpha\) potential by comparing the results of its application to a three-cluster description of \(^{12}\text{C}\) with those obtained with local potentials on one hand and with the fully microscopic \(3\alpha\) model on the other hand.

As a microscopic theory for the intercluster potential, we start from the RGM [1, 2]. The RGM wave function of the two-cluster \((A+B)\) system with the relative motion function \(\chi\) is expressed as

\[
\Psi = A[\phi_A \phi_B \chi(x)] = \int \chi(r) \Phi(r) \, dr, \tag{1}
\]

with the basis function \(\Phi(r) = A[\phi_A \phi_B \delta(x - r)]\), where \(\phi_A\) and \(\phi_B\) are the cluster internal (i.e. translation invariant) wave functions and \(A\) is an antisymmetrizer over all nucleons. Here \(x\) is the relative coordinate between the clusters, while \(r\) is a parameter coordinate corresponding to \(x\). The RGM equation for \(\chi\) reads

\[
(T + V + \varepsilon K) \chi = \varepsilon \chi, \quad V = V_D + V^{EX}. \tag{2}
\]

Here \(T\) is the intercluster kinetic energy, \(V_D\) is local and called the direct potential, and \(V^{EX} = K_T + K_V\), where \(K_T\) and \(K_V\) are the exchange nonlocal kernels for the kinetic and potential (including the Coulomb term) energies, respectively. The norm kernel is defined with respect to the \(A + B\) threshold.

Equation (2) suggests that the \(A + B\) potential is

\[
V^{\varepsilon K} = V + \varepsilon K. \tag{4}
\]

When employing this kind of energy-dependent RGM kernels in three-cluster systems, the \(\varepsilon\) value has to be provided for each pair of clusters but is not a well defined quantity. A self-consistent procedure providing an average value of the energy distribution has been proposed in Ref. [21]. This procedure leads to a reasonable energy for the \(^{12}\text{C}\) ground state but fails for the \(^{9}\text{Be}\) system [22].

To resolve this problem one has to obtain a potential which has no explicit energy-dependence and still maintains basic properties such as the phase shifts. Such a potential can be constructed for a macroscopic (or renormalized) relative motion function,

\[
g = \sqrt{\mathcal{N}} \chi, \tag{5}
\]

by requiring the condition [24]

\[
(T + V^{RGM}) g = \varepsilon g. \tag{6}
\]

The nonlocal potential \(V^{RGM}\) is expressed as

\[
V^{RGM} = V + W, \tag{7}
\]

where the new nonlocal operator \(W\) is the difference between the renormalized RGM potential \(V^{RGM}\) and the bare RGM potential \(V\),

\[
W = \mathcal{N}^{-1/2}(T + V)\mathcal{N}^{-1/2} - (T + V). \tag{8}
\]

The relative motion functions \(g\) have the nice property \(\langle g|g'\rangle = \langle A[\phi_A \phi_B \chi)|A[\phi_A \phi_B \chi']\rangle\), that is, the orthonormality of microscopic wave functions \(\Psi\) and \(\Psi'\) is precisely transmitted to \(g\) and \(g'\). In addition, the asymptotics of \(g\) is the same as that of \(\chi\) because \(\mathcal{N}\) approximates unity at large distances. The phase shift determined from \(g\) using the potential \(V^{RGM}\) is exactly equal to that determined from the RGM equation (2) for \(\chi\).

A particular function \(\chi_f\) is called a Pauli-forbidden state (PFS) if it satisfies the equation

\[
\mathcal{N} \chi_f = 0 \quad \text{or} \quad K \chi_f = \chi_f, \tag{9}
\]

which, by using Eq. (2), leads to \((T + V) \chi_f = 0\), and also \(W \chi_f = 0\). We thus have the property

\[
(T + V^{RGM}) \chi_f = 0, \tag{10}
\]

justifying the denomination of forbidden states.

More generally, consider the eigenvalue problem

\[
K \psi_n = \kappa \psi_n. \tag{11}
\]

The PFS are nothing but the eigenfunctions with eigenvalue \(\kappa = 1\). For cluster wave functions described with HO functions with a common size parameter, the eigenvalue problem (11) can be solved [32]. The eigenfunctions are HO wave functions \(\psi_{n\ell m}\), and the corresponding eigenvalues are given by \(\kappa_{n\ell} = 4(1/2)^{2n+\ell} - 3\delta_{2n+\ell,0}\) for \(\alpha + \alpha\). There are thus three PFS, \(\psi_{000}, \psi_{100}\) and \(\psi_{02m}\) for \(\ell\) even, and all states are PFS for \(\ell\) odd.

The nonlocal kernel \(V^{RGM}\) can be expanded in terms of the eigenfunctions of \(K\), e.g., \(W\) is expressed as

\[
W(r, r') = \sum_{nn'\ell m} W_{nn'\ell m}(r) \psi_{n'\ell m}(r'), \tag{12}
\]

where the prime indicates that the PFS are excluded from the sum. Here \(V\) is assumed to be rotation-invariant, which makes its matrix element independent of \(m\). In practice, we include \(n\) and \(n'\) up to 100.

The potential \(V^{RGM}\) is expanded into partial waves as

\[
V^{RGM}_\ell(r, r') = 2\pi r' \int_{-1}^{1} V^{RGM}_\ell(r, r') P_\ell(t) \, dt = V_D(r) \delta(r - r') + V^{EX}_\ell(r, r') + W_\ell(r, r') \tag{13}
\]
with $t = \mathbf{r} \cdot \mathbf{r}'/rr'$ and $P_t$ is a Legendre polynomial. The potential $V^RGM$ acts on $g_t$, the macroscopic wave function in partial wave $\ell$. The terms $V_D$ and $V^EX_{\ell}$ are the direct and nonlocal RGM potentials. Using $\psi_{n\ell m}(\mathbf{r}) = R_{n\ell}(r)Y_{\ell m}(\hat{\mathbf{r}})$ enables one to calculate $W_{\ell}$ as

$$W_{\ell}(r, r') = rr'\sum'_{nn'}W_{nn'\ell}R_{n\ell}(r)R_{n'\ell}(r'). \quad (14)$$

Figure 1 displays $V^EX_{\ell}(r, r')$ and $W_{\ell}(r, r')$ for the $\alpha + \alpha$ system. The parameters of the RGM kernels ($\nu = 0.257 \text{ fm}^{-2}$, $u = 0.94687$) taken from Ref. 29 reproduce the $\alpha + \alpha$ phase shifts. The nonlocality pattern is very different between $V^EX_{\ell}$ and $W_{\ell}$. The potentials for $\ell = 0$ and 2 show a rapidly oscillating behavior, which is related to the existence of the PFS. The absolute value of $W_{\ell}$ is smaller than that of $V^EX_{\ell}$ by about one order of magnitude. The major contribution to the nonlocal potential comes from the kinetic energy; its effect is reduced by the potential energy term.

Table I compares energies and root mean square (rms) radii of the 3$\alpha$ system obtained using local potentials (Local), the nonlocal $V^{zK}$ and $V^{RGM}$ potentials (Semi-micro) as well as with fully microscopic calculations (Micro) [28, 31]. The values $\hbar^2/\mu_N = 41.47 \text{ MeV fm}^2$ and $m_\alpha = 4m_N$ are used everywhere for consistency between Micro and non-Micro calculations. Results in Refs. 7, 28 are slightly different because of the use of other parameter values. The two-nucleon potential used in Micro is the same as the one used to derive the RGM kernels, and thus the comparison between Micro and Semi-micro indicates how closely the various potentials simulate the microscopic 3$\alpha$ calculation. We stress that all models, except for $V$ of Semi-micro, reproduce the $\alpha\alpha$ phase shifts in an essentially identical way. The rms radius of the point $\alpha$-particle distribution is obtained for Micro by subtracting $r_\alpha^2$ ($r_\alpha = 1.479 \text{ fm}$) from the $\langle r^2 \rangle$ value calculated at the nucleon level. As the 0$^+_2$ state calculated with Micro and $V^{RGM}$ is above the 3$\alpha$ threshold, the value of its rms radius depends on the basis choice. Our calculations suggest, however, that it is very large around 3.5 fm.

As explained in Refs. 7, 15, 10, the local potentials, shallow (ABd) 8 and deep (BFW) 10, never reproduce the microscopic results. They lead to a weakly bound (ground or excited) state and, sometimes, to a deeply bound ground state. The results with BFW depend on the definition of the PFS 13, i.e. whether they are (i) HO states or (ii) bound states of the BFW potential. In the latter case, the result is similar to that of the ABd case. The bare RGM potential $V$ cannot reproduce the microscopic results either. Its energies resemble those obtained with local potentials. Only the Semi-micro models with $V^{zK}$ and $V^{RGM}$, which take account of the energy-dependent nonlocality, give results close to Micro. In the $V^{zK}$ model, however, both the 0$^+_1$ and 0$^+_2$ states cannot be obtained without either losing their mutual orthogonality or loosing the self-consistency of $\varepsilon$. Only the $V^{RGM}$ potential simultaneously gives satisfactory results for the two 0$^+_1$ states. We also stress that the rms radius of Micro is reproduced very well by the $V^{RGM}$ model. Another nice property of $V^{RGM}$ is that the expectation values of the 3$\alpha$ kinetic and potential energies are almost equal to the corresponding values of Micro, while the other models including $V^{zK}$ give quite different values.

Though the Semi-micro model with $V^{RGM}$ is quali-

| Model       | Potential | $L^*$ | $E$   | $\sqrt{(r^2)}$ |
|-------------|-----------|------|------|----------------|
| Local       | ABd [8]   | 0$^+_1$ | -1.52 | 2.34           |
|             | BFW(i) [10] | 0$^+_1$ | -20.62 | 1.29           |
|             |           | 0$^+_2$ | -1.25  | 2.34           |
|             | BFW(ii) [10] | 0$^+_1$ | -0.66  | 2.31           |
| Semi-micro  | $V^{zK}$ | 0$^+_1$ | -19.50 | 1.42           |
|             | $V^{RGM}$ | 0$^+_1$ | -9.60  | 1.45           |
|             | Micro     | 0$^+_1$ | -11.37 | 1.64           |
|             |           | 0$^+_2$ | 0.597  | —              |
|             |           | 0$^+_2$ | 0.597  | —              |
tatively successful, its ground state energy is about 1.9 MeV higher than Micro. The former takes account of binary exchanges between $\alpha$ clusters but not of simultaneous exchanges among the three $\alpha$ clusters. Such three-cluster exchanges produce attraction as observed here. This effect is rather significant in the compact ground state but negligible in the spatially extended $0^+_2$ state.

Similar calculations are in progress for $^9$Be treated as aon. A preliminary value $-2.16$ MeV obtained with $V_{RGM}$ for its $3/2^-$ ground state is close to the microscopic result $-2.61$ MeV [24] and to the experimental value $-1.57$ MeV. The use of $V_{RGM}$ thus resolves the overbinding of the Semi-micro energy ($-3.86$ MeV) with the energy-dependent $V_{\varepsilon K}$ [25].

To summarize, while the microscopic $3\alpha$ cluster model provides a simultaneous description of both $0^+$ states of $^{12}$C, macroscopic $3\alpha$ models with local forces that reproduce the $\alpha\alpha$ phase shifts always fail. Therefore the very existence of clusters in the $^{12}$C ground state could be doubted. We have shown that the introduction of microscopically founded nonlocal forces, which act in the Pauli-allowed space, solves this problem. The nonlocality arises from two different physical origins, the exchange of identical constituent particles already known in the nonlocal RGM kernels and the elimination of the energy-dependence of these kernels. Although the latter term is smaller than the former, its contribution is essential to simulate the microscopic features. The results of the semi-microscopic and microscopic calculations are then for the first time qualitatively close and even very similar for the $0^+_2$ state. The difference of about 2 MeV for the ground state provides a measure of three-cluster exchange effects in this state.

The cluster-model description of $^{12}$C is approximate mainly by its neglect of spin effects. Nevertheless it provides tractable wave functions which can be very useful in reaction and decay models. The semi-microscopic model is interesting as it provides similar wave functions in much shorter computing times with the possibility of easily improving them by tuning the effective force parameters to accurately reproduce the binding energies.

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