MSF and Connectivity in Limited Variants of the Congested Clique *

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Abstract

The congested clique is a synchronous, message-passing model of distributed computing in which each computational unit (node) in each round can send message of $O(\log n)$ bits to each other node of the network, where $n$ is the number of nodes. This model has been considered under two extreme scenarios: unicast or broadcast. In the unicast model, a node can send (possibly) different message to each other node of the network. In contrast, in the broadcast model each node sends a single (the same) message to all other nodes. Following [1], we study the congested clique model parametrized by the range $r$, the maximum number of different messages a node can send in one round.

Following recent progress in design of algorithms for graph connectivity and minimum spanning forest (MSF) in the unicast congested clique, we study these problems in limited variants of the congested clique. We present the first sub-logarithmic algorithm for connected components in the broadcast congested clique. Then, we show that efficient unicast deterministic algorithm for MSF [11] and randomized algorithm for connected components [5] can be efficiently implemented in the rcast model with range $r = 2$, the weakest model of the congested clique above the broadcast variant ($r = 1$) in the hierarchy with respect to range. More importantly, our algorithms give the first solutions with optimal capacity of communication edges, while preserving small round complexity.

1 Introduction

Recently, the congested clique model of distributed computation attracted much attention in algorithmic community. In this model, each pair of $n$ nodes of a network is connected by a separate communication link. That is, the network forms an $n$-node clique. Communication is synchronous, each node in each round can send message of $O(\log n)$ bits to each other node of the network. The main purpose of such a model is to understand the role of congestion in distributed computation.

The congested clique model has been mainly considered in two variants: unicast or broadcast. In the unicast model, a node can send (possibly) different message to each other node of the network. In contrast, in the broadcast model each node can only send a single (the same) message to all other nodes in a round.

Following [1], we study the congested clique model parametrized by the range $r$, the maximum number of different messages a node can send in one round. We call the model with such a restriction the rcast congested clique. (Note that $r = 1$ corresponds to the broadcast congested clique and $r = n$ to the unicast congested clique.)

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As the broadcast and unicast models differ significantly in the amount of information which can be exchanged in a round, it is natural to introduce an intermediate model which uses a quantitative measure of usage of the possibility of sending different messages in each outgoing link. The study of the rcast model is aimed at exploring this research direction.

1.1 The model

We considered the congested clique model with the following parameters: \( r \) – the maximum number of different messages a node can send over its outgoing links in a round; \( b \) – the maximum size of a message (bandwidth); \( n \) – the number of nodes in the network/graph. The model with the above parameters will be denoted rcast\((n, r, b)\). Usually, we consider the model with \( b = \log n \) and therefore the model rcast\((n, r, \log n)\) is also denoted rcast\((n, r)\).

We consider randomized algorithms in which a computational unit in each node of the input network can use private random bits in its computation. We say that some event holds with high probability (whp) for an algorithm \( A \) running on an input of size \( n \) if this event holds with probability \( 1 - \frac{1}{n^c} \) for a given constant \( c \).

Graph problems in the congested clique model

Graph problems in the congested clique model are considered in the following framework. The joint input to the \( n \) nodes of the network is an undirected \( n \)-node weighted graph \( G(V, E, w) \), where each node corresponds to a node of the communication network and weights of edges are integers of polynomial size (i.e., each weight is a bit sequence of length \( O(\log n) \)). Each node \( u \) initially knows the network size \( n \), its unique ID in \([n]\), the list of IDs of its neighbors in the input graph and the weights of its incident edges. All graph problems are considered in this paper in accordance with this definition.

In the paper, we consider connected components problem (CC) and minimum spanning forest problem (MSF). Our goal is to compute CC or MSF of input graph, i.e., each node should know the set of edges inducing CC/MSF at the end of an execution of an algorithm.

Complexity measures

The key complexity measure considered in context of the congested clique models is round complexity (called also time) which is equal to the number of rounds in which an algorithm works for instances of problems of a given size.

For the rcast model, the range \( r \) is also a parameter determining complexity of an algorithm. Below, we give an observation justifying the statement that the key increase in communication power is between \( r = 1 \) (broadcast congested clique) and \( r = 2 \) (the weakest variant of the congested clique above the broadcast model wrt the range).

**Fact 1.** One round of an algorithm \( A \) from rcast\((n, n)\) may be simulated in rcast\((n, r)\) in \( O(\log, n) \) rounds.

**Corollary 1.** Given an algorithm \( A \) solving a problem \( P \) in rcast\((n, n)\) in \( R(A) \) rounds, one can build an algorithm \( A' \) solving \( P \) in rcast\((n, r)\) in \( O(\frac{R(A)}{\log r}) \).

The above observations show that each algorithm designed for the unicast congested clique model might be simulated in the range cast model with \( O(\frac{\log n}{\log r}) \) overhead. Thus, for problems of large complexity in the unicast congested clique, the models unicast and rcast2 seem to be very close to each other. However, for problems with sublogarithmic round complexity in the unicast congested clique, the question about efficient rcast algorithms remains interesting and relevant.

In order to provide accurate measure of the amount of information transmitted over communication links of a network, we consider the edge capacity measure. The edge capacity \( \beta_A(i, n) \) is the
(maximal) length (in bits) of messages which can be transmitted in the \(i\)th round of executions of the algorithm \(A\) on graphs of size \(n\). The (total) edge capacity \(B(A, n)\) is the sum of edge capacities of all rounds, \(B_A(n) = \sum_i \beta_A(i)\). As \(n\) is usually known from the context, we use shorthands \(\beta_A(i)\) and \(B_A\) for \(\beta_A(i, n)\) and \(B_A(n)\), respectively.

For further references we make the following observation concerning edge capacity of algorithms solving CC and MSF.

**Fact 2.** Total edge capacity of each algorithm solving the connected components problem or the minimum spanning forest problem is \(\Omega(\log n)\).

**Proof.** At the end of an execution of an algorithm solving CC each node knows a partition of the set \(V\) of nodes into connected components. Information available to a node at the beginning of an algorithm has \(O(n)\) bits (a characteristic vector of the set of its neighbors). As there are \(2^{\Omega(n \log n)}\) partitions of a set of size \(n\), descriptions of some partitions require \(\Omega(n \log n)\) bits. On the other hand, the number of bits received by a node during an execution of \(A\) is \(O(nB(A))\). Thus, in order to collect information of size \(\Omega(n \log n)\) in each node, the capacity \(B(A)\) has to satisfy \(B(A) = \Omega(\log n)\). 

1.2 Related work

The rcast model of the congested clique was introduced in [1,2]. The authors presented examples showing the substantial difference between the case \(r = 1\) (broadcast model) and \(r = 2\). Moreover, it was shown that an exponential increase of the range \(r\) causes \(\omega(1)\) drop in round complexity for some problems. The impact of a single message size \(b\) transmitted in a round through a communication link is also studied in [1,2].

The broadcast and unicast models of congested clique were studied in several papers, e.g., [11, 6, 5, 4, 3, 10, 13]. The recent Lenzen’s [10] constant time routing and sorting algorithm in the unicast congested clique shows the power of the unicast model. (The routing problem according to the definition from [10] trivially requires \(\Omega(n)\) rounds in the broadcast congested clique.) Lotker et al. [11] designed a \(O(\log \log n)\) round deterministic algorithm for MSF (minimum spanning forest) in the unicast model. Recently, an alternative algorithm of the same complexity has been presented [3]. The best known randomized solution for MSF in the unicast model works in \(O(\log^* n)\) rounds [5], improving the recent \(O(\log \log \log n)\) bound [6]. Reduction of the number of transmitted messages in the MST algorithms was studied in [14]. If messages can have \(\sqrt{n} \log n\) bits (bandwidths \(b = \sqrt{n} \log n\)), one can compute MSF in constant number of rounds, even in the broadcast congested clique [13]. In [4] the authors proved that it is possible to simulate powerful classes of bounded-depth circuits in the unicast congested clique, which points out to the power of this model and explains difficulty in obtaining lower bounds for this model. In [3], randomized variants of the broadcast congested clique are considered.

Apart from purely theoretical and algorithmic interest in the congested clique, the model also relates to other models of processing of large-scale graphs, e.g., the \(k\)-machine model [8], MapReduce [7] and the concept of overlay networks.

1.3 Our results

We present the first sub-logarithmic algorithm for connected components in the broadcast congested clique. Our algorithm works in \(O(\log n/ \log \log n)\) rounds and scales to models with varying sizes.
of messages. Then, we show that efficient unicast deterministic algorithm for MST \[ \text{III} \] and randomized algorithm for connected components \[ \text{V} \] can be efficiently adjusted to the rcast(2) model, the weakest variant above the broadcast congested clique in the hierarchy of rcast(r) models for \( r > 1 \). More importantly, our result imply solutions with efficient (optimal, in some case) capacity of communication edges, while preserving small round complexity. An interesting direction arising from these results is to determine a relationship between adaptiveness (the number of rounds) and communication complexity (sum of sizes of transmitted messages).

2 Graph terminology and tools for capacity/range reduction

Given a natural number \( p, [p] \) denotes the set \{1, 2, \ldots, p\}.

For a graph \( G(V, E) \) and \( E' \subseteq E \), \( C_1, C_2, \ldots, C_k \subseteq V \) is a partition of \( G \) into components with respect to \( E' \subseteq E \) if \( C_i \)'s are pairwise disjoint, \( \bigcup_{i \in [k]} C_i = V \), each \( C_i \) is connected with respect to the edges from \( E' \) and there are no edges \((u, v) \in E' \) such that \( u \in C_i \) and \( v \in C_j \) for \( i \neq j \). That is, \( C_1, \ldots, C_k \) are connected components of \( G(V, E') \).

Fragment of a graph \( G(V, E, w) \) is a tree \( F \) which is a subgraph of a minimum spanning forest of \( G \). A family \( \mathcal{F} \) of fragments of \( G(V, E, w) \) is a partition of \( G \) into fragments with respect to \( E' \subseteq E \) if \( F_1 \) and \( F_2 \) have disjoint sets of nodes for each \( F_1 \neq F_2 \) from \( \mathcal{F} \), each \( v \in V \) belongs to some \( F \in \mathcal{F} \) and each edge of each tree \( F \in \mathcal{F} \) belongs to \( E' \).

Given a partition \( \mathcal{C} \) (\( F \), resp.) of a graph \( G(V, E) \) into components (fragments, resp.) and \( v \in V \), \( C^v \) (\( F^v \)) denotes the component (the fragment, resp.) containing \( v \).

We will usually consider components with respect to a set of edges which are known to all nodes in congested clique.

We say that a fragment (component, resp.) is growable if there is an edge connecting it with some other fragment/component in the considered graph. An edge \((u, v)\) is incident to a fragment \( F \) (component \( C \), resp.) wrt to some partition of a graph in fragments/components if it connects \( F \) with another fragment (component, resp.), i.e., \( F^u \neq F^v = F \) or \( F^v \neq F^u = F \) (\( C^u \neq C^v = C \) or \( C^v \neq C^u = C \), resp.).

Tools for capacity and range reduction

As tools to reduce edge capacity and range of congested clique algorithms, we introduce the local broadcast problem and the global broadcast problem. In the local broadcast problem, the following parameters are known to each node of a network

- a set \( T \subset V \),
- a set \( R \subset V \),
- a natural number \( b \).

Moreover, each node \( v \in T \) has its own message \( M^v \) of length \( b \). As a result of local broadcast, each node \( v \in R \) receives the message \( M^v \) from each \( u \in T \).

**Proposition 1.** Algorithm \[ \text{II} \] (LocalBroadcast) solves the local broadcast problem in \( O(1) \) rounds with range \( r = 2 \) and capacity \( 1 \), provided \( |T| b = O(n) \). It is possible to execute LocalBroadcast (Algorithm \[ \text{II} \]) simultaneously for \( k \) triplets \((T_i, R_i, b_i)_{i \in [k]} \), as long as \( T_i \)'s are pairwise disjoint, \( R_i \)'s are pairwise disjoint and \( |T_i| b_i \in O(n) \) for each \( i \in [k] \).

**Proof.** First, we show that it is possible to send messages from \( T \) to \( R \) in two rounds using one bit per communication link per round, provided \( |T| b \leq n \). Let us split nodes \( V \) into \( t = |T| \) segments \( S_1, \ldots, S_t \) of size \( b \). In Round 1, each \( v \in T \) sends the \( j \)th bit \((j \in [b])\) of its message \( M^v \) to the \( j \)
node of its segment. In Round 2, each node \( v \) sends the bit received in Round 1 to all nodes from \( R \). Algorithm 1 solves the problem for one pair \((T, R)\) of transmitters and receivers. However, it is possible to solve it simultaneously for multiple pairs \((T_i, R_i)_{i \in [k]}\) with messages of size \( b_i \), as long as \( T_i \) are pairwise disjoint, \( R_i \) are pairwise disjoint and \( |T_i|b_i \in O(n) \). Observe that all transmitters in Round 1 belong to \( T_i \). As \( T_i \)'s are pairwise disjoint, Round 1 can be done simultaneously for each \( i \in [k] \). On the other hand, all receivers in Round 2 belong to \( R_i \). As \( R_i \)'s are pairwise disjoint, Round 2 can be done simultaneously for each \( i \in [k] \). Finally, if \( |T_i|b_i \leq c \cdot n \) for a constant \( c > 1 \), we can solve the local broadcast problem in \( 2c \) rounds by repeating Algorithm 1 \( c \) times.

As transmission at each edge in Round 1 and Round 2 contains 1 bit, we obtain a solution with range 2 and edge capacity 1 in time \( O(1) \).

**Remark.** Design of algorithms in the unicast congested clique has been recently fostered by the Lenzen’s routing lemma [10]. For a reader familiar with Lenzen’s paper, Proposition 1 might seem to be a corollary from his result. We remark here that it is not the case, because the overall size of all copies of a message \( M^v \) for \( v \in T_i \) is \( b_i|R_i| \) which might be \( \omega(n) \).

Next, we define the **global broadcast problem**. Assume that each node from a set \( S \subseteq V \) of nodes knows (the same) message \( M \) of length \( b \). The **global broadcast problem** is to deliver \( M \) to each node \( v \in V \) of the network.

**Proposition 2.** The global broadcast problem can be solved in one round with range \( r = 1 \) and edge capacity \( \lfloor \frac{b}{|S|} \rfloor \).

**Proof.** The problem can be solved by splitting the common message into \( \lfloor \frac{b}{|S|} \rfloor \) parts assigning the \( i \)th part to the \( i \)th element of \( X \) for \( i \in \left[ \lfloor \frac{b}{|S|} \rfloor \right] \). Then, each \( v \in X \) broadcasts its part to the whole network. \( \square \)

### 3 Connected components in the broadcast congested clique

This section is devoted to the broadcast congested clique, the weakest variant of the congested clique model. First, we recall a distributed implementation of the well known Boruvka’s algorithm for MST. Then, we design a new algorithm for connectivity which (unexpectedly?) shows that the \( \log n \) bound on round complexity can be broken in the broadcast congested clique.

#### 3.1 Minimum spanning forest in broadcast congested clique

Minimum spanning forest can be computed using a distributed version of the classical Boruvka’s algorithm. The algorithm works in **phases.** At the beginning of phase \( i \) a partition \( F \) into fragments of size \( \geq 2^i \) is given. During phase \( i \) new fragments of size \( \geq 2^{i+1} \) are determined, based on the lightest edges incident to all fragments.

In the distributed implementation of the Boruvka’s algorithm each node knows the set of fragments at the beginning of a phase. During the phase each node \( v \) announces (broadcasts) the
lightest edge connecting \( v \) with a node \( u \not\in F^v \). Using those edges, each node can individually (locally) perform the next phase of the Boruvka’s algorithm and determine new (larger) fragments.

**Theorem 1.** *Boruvka’s algorithm can be implemented in \( O(\log n) \) rounds in \( \text{rcast}(n, 1) \).*

### 3.2 Connected components algorithm

To calculate connected components we could use the standard Boruvka’s algorithm as well. However, we are not forced to select the lightest edge incident to each component. Our general idea is to prefer edges which connect nodes to components of large degree. And the intended result of a phase should be that each component either has small degree or it is connected to some “host” of large degree (directly or by a path of length larger than one). As the number of such “hosts” will be relatively small, we obtain significant reduction of the number of components of large degree in each phase. Moreover, we separately deal with components of small degree by allowing them to broadcast all their neighbours at the final stage of the algorithm.

More precisely, we define \( \deg(v) \) for a vertex \( v \) wrt a partition \( C \) as the number of components connected with \( v \), i.e., \( \deg(v) = |N(v)| \), where

\[
N(v) = \{ C \in \mathcal{C} \mid \exists u \in C \text{ such that } (v, u) \in E \text{ and } C \neq C'' \}.
\]

For a component \( C \in \mathcal{C} \), \( \deg(C) = \max_{v \in C} \{ \deg(v) \} \). Note that, according to this definition, the degree of a component \( C \) might be smaller than the actual number of components containing nodes connected by an edge with nodes from \( C \). Our definition of degree is adjusted to make it possible that degrees of components can be determined in \( O(1) \) rounds.

The algorithm is parametrized by a natural number \( s \) which (intuitively) sets the threshold between components of small degree (smaller than \( s \)) and large degree (at least \( s \)). Given a partition \( \mathcal{C} \) of the graph into components (wrt edges known to all nodes), we define the linear ordering \( > \) of components, where \( C > C' \) iff \( \deg(C) > \deg(C') \) or \( \deg(C) = \deg(C') \) and \( ID(C) > ID(C') \). A component \( C \) is a *local maximum* if all its neighbors are smaller with respect to the \( > \) ordering.

Our algorithm consists of the main part and the *playoff*. The main part is split into *phases*. At the beginning of phase 1 each node is *active* and it forms a separate component. During an execution of the algorithm, nodes from non growable components and components of small degree (smaller than \( s \)) are *deactivated*. At the beginning of a phase, a partition of the graph of active nodes is known to the whole network. First, each node \( v \) determines \( N(v) \) and announces its degree \( \deg(v) \) wrt the current partition of the set of active nodes into components (Round 1). With this information, each node \( v \) knows the ordering of components of the graph of active nodes according to \( > \). Then, each *active* node \( v \) (except of members of local maxima) broadcasts its incident edge to the largest active component from \( N(v) \) according to \( > \) relation (Round 2). Next, each node \( v \) of each local maximum \( C \) checks whether edges connecting \( C \) to all components containing neighbors of \( v \) (i.e., to components from \( N(v) \)) have been already broadcasted. If it is not the case, an edge connecting \( v \) to a new component \( C' \) (i.e., to such \( C' \) that no edge connecting \( C \) and \( C' \) was known before) is broadcasted by \( v \) (Round 3). Based on broadcasted edges, new components are determined and their degrees are computed (Round 4). Each new component with degree smaller than \( s \) is *deactivated* at the end of a phase.

The playoff lasts \( s \) rounds in which each node \( v \) of each deactivated component broadcasts edges going to all components connected to \( v \) (there are at most \( s \) such components for each deactivated node). More precise description of this strategy is presented as Algorithm 2. The key property for an analysis of complexity of our algorithm is that each active component \( C \) of large degree is either connected during a phase to all its neighbors or to a component which is larger than \( C \) according to \( > \).
Algorithm 2: BroadcastCC$(v, s)$

\begin{algorithmic}
\State \textbf{while} there are active components \textbf{do}
\Comment execution at a node $v$ \hfill ($s$ is the threshold between small/large degree)
\State \textbf{Round 1:} $v$ broadcasts $\text{deg}(v)$ \hfill \Comment execution at a node $v$
\State \textbf{if} $\text{deg}(v) > 0$ \textbf{then}
\State \hspace{1em} $C_{\text{max}}(v) \leftarrow$ the largest element of $N(v)$ wrt the ordering $\succ$
\State \textbf{Round 2:}
\State \hspace{1em} \textbf{if} $C'$ is not a local maximum \textbf{then} $v$ broadcast an edge $(u, v)$ such that $u \in C_{\text{max}}$
\State \textbf{Round 3:}
\State \hspace{1em} \textbf{if} $C'$ is a local maximum \textbf{then}
\State \hspace{2em} $N_{\text{lost}}(v) \leftarrow \{ C | C \in N(v) \text{ and no edge connecting } C \text{ and } C' \text{ was broadcasted}\}$
\State \hspace{1em} \textbf{if} $N_{\text{lost}}(v) \neq \emptyset$ \textbf{then}
\State \hspace{2em} $u \leftarrow$ a neighbor of $v$ such that $u \in C$ for some $C \in N_{\text{lost}}(v)$
\State \hspace{2em} $v$ broadcasts an edge $(u, v)$
\State $v$ computes the new partition into components, using broadcasted edges
\State \textbf{Round 4:} $v$ broadcasts $\text{deg}(v)$ \hfill \Comment degrees wrt the new components!
\State \textbf{if} $\text{deg}(C') < s$ \textbf{then} deactivate $v$
\State \textbf{Playoff ($s$ rounds):} deactivated nodes broadcast edges to neighboring components.
\end{algorithmic}

\textbf{Theorem 2.} Algorithm 2 solves the spanning forest problem in $O(s + \log_s n)$ rounds for an $n$-node graph.

\textbf{Proof.} First, consider round complexity of the algorithm. It is clear that Playoff has $s$ rounds. To show the claimed complexity we show that the number of active components is decreased at least \( s \) times in each phase. An intuition is that all components join with (some) local maxima and thus each local maximum of large degree “combines” at least $s$ components in a new, larger component. However, the situation is not that simple, as there might be many local maxima.

In order to formalize the intuition, consider a directed graph $G_{\text{phase}}$ of components active at the beginning of a phase, where $(C_1, C_2)$ is an edge in $G_{\text{phase}}$ \iff a node from $C_1$ broadcasts an edge connecting it with $C_2$ in step \[6\] of the phase (edge of type 1) \textbf{or} $C_1$ is a local maximum, a node from $C_1$ broadcasts an edge connecting it with some $C'$ in step \[12\] while a node from $C'$ broadcasts an edge connecting it with $C_2$ in step \[6\] (we call it edge of type 2).

The algorithm guarantees that
\begin{enumerate}[(a)]
\item $G_{\text{phase}}$ is acyclic.
\begin{itemize}
\item Indeed, each edge $(C_1, C_2)$ resulted from broadcasts in step \[6\] satisfies $C_1 \prec C_2$. Moreover, an edge is broadcasted from $C'_1$ to $C''_1$ in step \[12\] iff all nodes from $C''_1$ broadcasted connections to components larger than $C'_1$ wrt $\succ$ ordering.
\end{itemize}
\item Each connected component $C$ (i.e., each node of $G_{\text{phase}}$) is either a sink of $G_{\text{phase}}$ connected with (at least) $\text{deg}(C)$ nodes in $G_{\text{phase}}$ or has out-degree at least one.
\begin{itemize}
\item This property follows from the fact that only nodes of local maxima are candidates for sinks, as only they do not broadcast in step \[6\]. Moreover, assume that $C$ is a local maximum and there is a neighbor $C'$ of $C$ whose nodes have not broadcasted connections with $C$ in step \[4\]. Then a node(s) from $C'$ broadcast in step \[12\] which implies that out-degree of $C$ is at least one.
\end{itemize}
\item Each connected component of a partition obtained at the end of a phase contains at least one sink of $G_{\text{phase}}$.
\begin{itemize}
\item If one ignores that edges of $G_{\text{phase}}$ are directed then certainly new components at the end of the phase correspond to connected components of $G_{\text{phase}}$. This follows from the fact that edges of $G_{\text{phase}}$ correspond to connections between components (by an edge or a path of two edges in the original graph) broadcasted during the phase. As $G_{\text{phase}}$ is acyclic, each
connected component contains a sink.

Let \( C \) be a partition into components at the beginning of a phase and \( C' \) be the partition into components at the end of that phase, before deactivating components of small degree.\(^1\) The above observations imply that each component of \( C' \) either contains only components of \( C \) of small degree (smaller than \( s \)) or it contains at least \( s + 1 \) components from \( C \). Contrary, assume that a component \( C' \) of \( C' \) contains a component \( C \in C \) of degree \( \geq s \), while \( C' \) contains altogether at most \( s \) components of \( C \). Then, there is a directed path from \( C \) to a sink \( C_{\text{sink}} \) of degree at least \( \deg(C) \geq s \). Property (b) implies that at least \( s \) components of \( C \) have edges towards \( C_{\text{sink}} \) in \( G_{\text{phase}} \). This contradicts the contrary assumption that \( C' \) contains altogether less than \( s \) components of \( C \).

Summarizing, assume that we have \( p \) active components at the beginning of a phase. Then, at the end of the phase, there are at most \( p/s \) new components which contain at least one component whose degree at the beginning of the phase was \( \geq s \). It remains to consider the final components of the phase which are composed only from components whose degree was \( < s \) at the beginning of the stage. However, as the degree of a node cannot increase during the algorithm, the degrees of these new components are \( < s \) and they are deactivated at the end of the phase. Thus, each phase decreases the number of active components at least \( s \) times – there are at most \( \log_s n \) phases.

Correctness of the algorithm follows from the fact that each node of each deactivated component can broadcast its connections with all other components during Playoff. Moreover, active components are connected subgraphs of \( G \) at each stage.

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The minimum of \( s + \log_s n \) is obtained for \( s = \frac{\log n}{\log \log n} \). Then, Algorithm 2 works in \( O(\log n / \log \log n) \) rounds.

**Corollary 2.** It is possible to solve the connected components problem in the broadcast congested clique in \( O \left( \frac{\log n}{\log \log n} \right) \) rounds.

Now, consider the model in which the maximum size of a message (bandwidth) is larger than \( \log n \). If \( s = d \) in Algorithm 2 we get \( \log_d n \) phases, each requiring \( O(\log n) \) bits per node. Edges from deactivated nodes are broadcasted during Playoff in one round, using \( O(d \log n) \) bits. This gives \( O(\log_d n) \) round algorithm using \( O(\log n (d + \frac{\log n}{\log d})) \) bits per node during the whole execution.

**Corollary 3.** It is possible to solve connectivity problem in the broadcast congested clique with bandwidth \( d \log n \) using \( \log_d n \) rounds and \( O(\log n (d + \frac{\log n}{\log d})) \) bits transmitted by each node.

The above corollary gives an improvement over a result from [12], where the total number of bits per node is \( O(d \frac{\log^2 n}{\log d}) \) in \( O(\log_d n) \) rounds. Moreover, our algorithm is simpler than that in [12], since it does not require number theoretic techniques as \( d \)-pruning and deterministic sparse linear sketches.

## 4 Deterministic rcast algorithm for minimum spanning forest

In this section we provide a deterministic algorithm for minimum spanning forest (MSF) in the rcast model. First, we describe a generic algorithm for minimum spanning tree from [11]. Then we provide a new efficient rcast\((n, 2)\) version of this general algorithm. Finally, an algorithm optimizing the range \( r \) and achieving asymptotically optimal edge capacity is presented.

\(^1\)Note that deactivation of components of degree \( < s \) at the end of a phase does not guarantee that degrees of all components are \( \geq s \) at the beginning of the next phase. This is caused by the fact that deactivation of some components might decrease degrees of components which remain active (degrees are calculated only among active nodes).
4.1 Generic MSF Algorithm

First, we introduce terminology useful in describing (distributed) algorithms for MSF.

For a graph $G(V, E, w)$ and its partition into fragments, we say that an edge $e = (v, u)$ is relevant for a set $A \subseteq V$ if $F^v \neq F^u$ and $e$ is the lightest edge connecting a node from $A$ and a node from fragment $F^w$. Let $E_{A, \mu}$ denote the set of $\mu$ lightest relevant edges incident to the set $A \subseteq V$. Moreover, $N_{F, \mu}$ for a fragment $F$ denotes the set of fragments connected with $F$ by edges from $E_{F, \mu}$.

Below, we give a lemma which is crucial for the first efficient unicast congested clique algorithm for MSF [11].

**Lemma 1.** [11] Let $F$ be a partition of a graph $G(V, E)$ into fragments, let $E_F$ be the set of edges in the trees of the partition $F$. Then, for each $\mu > 0$, the minimum spanning forest $F'$ of $G(V, E_F \cup \bigcup_{F \in F'} E_{F, \mu})$ is a partition of $G(V, E)$ into fragments, such that the size of each growable tree of $F'$ is at least $(\mu + 1) \min_{F \in F} |F|$.

In other words, the above lemma says that, in order to increase the size of fragments $\mu + 1$ times, it is sufficient to consider $\mu$ lightest relevant edges for each fragment.

Using Lemma 1 one can build MSF in phases using the idea described in Algorithm 3. First, let us fix a sequence $\mu_1, \mu_2, \ldots$ of natural numbers. Phase $i$ starts from a partition of the input graph into fragments and ends with a new partition into larger fragments. Before the first phase, each node is considered as a separate fragment. At the beginning of the $i$th phase, the set $E_{F, \mu_i}$ of $\mu_i$ lightest relevant edges (or all relevant edges, if there are at most $\mu_i$) is determined for each fragment $F$ of the current partition. Then, this information is broadcasted to all nodes of the network. Using Lemma 1 each node can compute a new partition into fragments such that the size of the smallest growable fragment is increased at least $\mu_i + 1$ times.

Lotker et al. designed a congested clique implementation of Algorithm 3 which guarantees that each phase works in $O(1)$ rounds for the sequence $\mu_1 = 1$ and $\mu_i = \mu_{i-1}(\mu_{i-1} + 1)$ for $i > 1$. As $\mu_k \geq n$ for $k = O(\log \log n)$, their algorithm works in $O(\log \log n)$ rounds.

**Algorithm 3 Minimum Spanning Forest**

1: $i \leftarrow 1$
2: $F = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$
3: while $E \neq \emptyset$ do
4: SelectEdges($\mu_i, F$)
5: announce edges from $E_{F, \mu_i}$
6: locally merge fragments, modify $F$ appropriately
7: $E \leftarrow E \setminus \{(u, v) \mid F^v = F^u\}$
8: $i \leftarrow i + 1$

In order to illustrate problems with design of algorithms with limited range and edge capacity, we first shortly describe the $O(\log \log n)$ solution for MSF from [11].

The selection of $\mu_i$ lightest edges incident to each fragment in a phase (step 4 of Alg. 3) is done after one round of communication as follows. For each node $v$ and each fragment $F \neq F_v$, $v$ sends the lightest edge from the set $\{(v, u) \mid u \in F\}$ to all nodes from $F$. Thus, the edge capacity $\Theta(\log n)$ is needed in each phase. The upper bound on the range is equal to the number of components which might be $\Omega(n/\mu_i)$ in phase $i$.

After the above described round, each node $v$ knows $E_{F_v, \mu_i}$, the set of all relevant edges incident to its fragment. Thus, the set $E_{F_{F_v, \mu_i}}$ is computed individually (and locally) by each node of $F$ (for each fragment $F$). The choice of the sequence $\mu_i$ guarantees that each growable fragment has at
Moreover, EF \( \mu \) might be broadcasted to the whole network (step 5 of Alg. 3) in one round such that each node of \( F \) broadcasts one element of \( EF, \mu \). The range \( r \) is equal to 1 in this round, while the edge capacity is \( \Theta(\log n) \).

Below, we summarize properties satisfied by MSF algorithm from [11].

**Corollary 4.** There exists a deterministic congested clique MSF algorithm which works in \( O(\log \log n) \) rounds with range \( r = O(n) \) and edge capacity \( O(\log n \log \log n) \).

### 4.2 Minimum spanning forest algorithm in rcast(\( n, 2 \))

In this section we will show an implementation of Algorithm 3 in \( O(\log \log n) \) rounds, which is also efficient with respect to the range and edge capacity. As we discussed above, the only part of the Lotker et al. [11] implementation of Algorithm 3 with large range is the selection of the set of the lightest relevant edges for the current fragments. Therefore, in order to reduce the range without increasing round complexity, it is sufficient to design a new version of this part of Algorithm 3 for the sequence \( \mu_1 = 1 \) and \( \mu_i = \mu_{i-1}(\mu_{i-1} + 1) \) for \( i > 1 \). We give such a solution in this section.

First, observe that the set of \( \mu \) lightest relevant edges incident to a fragment \( F \) (i.e., \( EF, \mu \)) is included in the union of \( \mu \) lightest relevant edges incident to each node from \( F \), i.e., \( EF, \mu \subseteq \bigcup_{v \in F} E_{v, \mu} \). Thus, in order to determine \( EF, \mu \), it is sufficient to distribute/broadcast information about \( E_{v, \mu} \) for each \( v \in F \) among nodes of \( F \). This task corresponds to the local broadcast problem (see Section 2). More precisely, given a partition \( F = \{F_1, \ldots, F_k\} \) in phase \( i \), each \( v \in F_j \) is supposed to broadcast the message \( M' \) of size \( b_i = O(\mu_i \log n) \) (i.e., description of \( \mu \) lightest relevant edges incident to \( v \)) to all nodes of \( F_j \). Using Proposition 1 we can solve this task in \( O(1) \) rounds with range \( r = 2 \) and edge capacity 1, provided

\[
|F_i|/\mu_i \log n \leq n. \tag{1}
\]

However, for large fragments and/or large \( \mu_i \), this inequality is not satisfied. Therefore, we need a more general observation saying that \( \mu \) lightest relevant edges incident to a set \( A \) (not necessarily a fragment) might be chosen from the sets of \( \mu \) lightest edges incident to subsets \( A_j \) forming a partition of \( A \).

**Fact 3.** Let \( \mathcal{F} \) be a partition of a graph in fragments and let \( A_1, \ldots, A_k \) be a partition of the set of nodes of a fragment \( F \in \mathcal{F} \). Then, for each \( \mu \in \mathbb{N} \), \( EF, \mu \subseteq \bigcup_{j \in [k]} E_{A_j, \mu} \).

Using Fact 3 we compute \( EF, \mu \) for a large fragment in the following way. The set \( F \) is split into small groups and \( \mu \) lightest relevant edges are selected for each group and knowledge about them is distributed among nodes of the group. Then, the leader of each group is chosen and the task is reduced to choosing \( \mu \) lightest relevant edges among the sets of \( \mu \) edges known to the leaders. This reduces our problem to its another instance with smaller size of nodes. Another issue to deal with is to set the value of \( \mu_i \) not too large for each \( i \), in order to satisfy (1). The choice of parameters in Algorithm 4 guarantees that the task of selecting \( \min\{\mu_i, n^{1/3}\} \) lightest relevant edges incident to each fragment is possible in \( O(1) \) rounds with edge capacity 1.

**Proposition 3.** Algorithm 4 determines the set \( EF, \mu' \) lightest relevant edges incident to each fragment \( F \in \mathcal{F} \) in \( O(1) \) rounds with edge capacity 1 and range \( r = 2 \), where \( \mu' = \min\{n^{1/3}, \mu\} \). Moreover, \( EF, \mu' \) is known to each \( v \in F \) for each \( F \in \mathcal{F} \) at the end of an execution.

Proof. Assume that \( n \) is large enough to satisfy \( n^{1/3} > \log n \). First observe that the inequality \( |A|/\mu'^{1/3} \log n \leq n \) is satisfied when the last step of the algorithm is executed. If \( |F|/\mu' \log n \leq n \) then
Using these facts, one can implement step 5 of Algorithm 3 in two rounds. In round 1, that the

Proof. After an execution of SelectEdges, a designated node
edges which should be broadcasted to all nodes in step 5. The definition of the sequence
For some $i$, $\mu_i$ edges which should be broadcasted to all nodes in step 5. The definition of the sequence

Lemma 1 guarantee that the smallest size of a fragment at the beginning of phase $i$ is at least $\mu'_i$. Using these facts, one can implement step 5 of Algorithm 3 in two rounds. In round 1, that the node $v \in F$ which knows $E_{F,\mu'}$ sends the $j$th edge from $E_{F,\mu'}$ to the $j$th element of $F$. In round 2, each node broadcasts an edge received in round 1 to the whole network. Thus, each iteration of the while-loop works (i.e., each phase) works in $O(1)$ rounds with range $r = 2$. It remains to determine the number of iterations of the while-loop (i.e., the number of phases). For some $i = O(\log \log n)$ we get $\mu_i \geq n^{1/3}$. The smallest size of a (growable) component is larger than $n^{1/3}$ after $i = O(\log \log n)$ phases. For $j > i$, the smallest size of a growable component is

\begin{algorithm}[h]
\begin{algorithmic}[1]
\State $\mu' \leftarrow \min \{ n^{1/3}, \mu \}$
\State $n_{\text{max}} \leftarrow n^{1/3}$
\For {each $F \in \mathcal{F}$ and each $v \in F$ simultaneously}
\State $A \leftarrow$ the nodes of $F$
\State $M^v \leftarrow \mu'$ lightest relevant edges incident to $v$
\EndIf \hfill \Comment{the algorithm for node $v$}
\EndFor
\State $n' \leftarrow \frac{|A|}{\mu' \log n}$
\State $k \leftarrow \lfloor |A| / n' \rfloor$
\State split $A$ into disjoint subsets $A_1, \ldots, A_k$ such that $|A_i| = n'$ for $i < k$ and $|A_k| \leq n'$
\For {each $A_i$ simultaneously}
\State \text{LocalBroadcast}($A_i, \mu' \log n$)
\EndFor
\State let $A_j$ denote the set which contains $v$
\If {ID($v$) = min\{ID($u$) $| u \in A_j$\}}
\State $M^v \leftarrow \mu'$ lightest edges incident to $A_j$
\Else
\State $v$ is removed from $A$
\EndIf
\State \text{LocalBroadcast}($A, F, \mu' \log n$)
\State $v$ determines $E_{F,\mu'}$ on the basis of received messages \Comment{see Fact 3}
\end{algorithmic}
\caption{SelectEdges($\mu, \mathcal{F}$)}
\end{algorithm}

the claimed inequality holds, since $|A| = |F|$ in this case. Otherwise, the size of $A$ is reduced to $\mu' \log n \leq n^{1/3} \log n < \frac{n^{2/3}}{\log n}$. The choice of $n'$ guarantees also that $|A_j| \mu' \log n \leq |F| \leq n$ for each $j \in [k]$. Also, all fragments are pairwise disjoint, and all sets $A_j$ are pairwise disjoint (as a disjoint subsets of fragments). Thus, all execution of LocalBroadcast last $O(1)$ rounds with edge capacity 1, by Proposition 1.

By Fact 3, the algorithm determines $\mu'$ lightest relevant edges for elements of partitions of $F$ and eventually determines $\mu'$ lightest relevant edges for each $F \in \mathcal{F}$, i.e., $E_{F,\mu'}$. For each $F \in \mathcal{F}$, the set $E_{F,\mu'}$ is known to all element of $F$ at the end of the execution of the algorithm, thanks to LocalBroadcast executed in the last step of the algorithm.

Using Algorithm 4 in the template described by Algorithm 3 we obtain the following result.

Lemma 2. Assume that $\mu_1 = 1$ and $\mu_i = \min\{n^{1/3}, \mu_{i-1}(\mu_{i-1} + 1)\}$ for $i > 1$. Then, an implementation of Algorithm 3 using the procedure SelectEdges from Algorithm 4 solves the MSF problem in $O(\log \log n)$ rounds with range $r = 2$.

Proof. After an execution of SelectEdges, a designated node $v \in F$ for each fragment $F$ knows $\mu_i$ edges which should be broadcasted to all nodes in step 5. The definition of the sequence $\mu'_i$ and Lemma 1 guarantee that the smallest size of a fragment at the beginning of phase $i$ is at least $\mu'_i$. Using these facts, one can implement step 5 of Algorithm 3 in two rounds. In round 1, that the node $v \in F$ which knows $E_{F,\mu'}$ sends the $j$th edge from $E_{F,\mu'}$ to the $j$th element of $F$. In round 2, each node broadcasts an edge received in round 1 to the whole network. Thus, each iteration of the while-loop works (i.e., each phase) works in $O(1)$ rounds with range $r = 2$.
increased (at least) \( n^{1/3} \) times in the \( j \)th round. This results in size \( n \) for the phase \( i + 2 \) and shows that the algorithm works in \( O(\log \log n) \) rounds.

\[ \square \]

**Reduction of total edge capacity**

Our solution for the MSF from Lemma \( 2 \) reduces the range \( r \) to 2, but each phase requires sending \( \Theta(\log n) \) bits by some nodes, because weights of some edges are transmitted by nodes in step \( 5 \) of Alg. \( 3 \). In order to reduce (total) edge capacity, we modify the sequence \( \{\mu_i\} \) again to make it possible that step \( 5 \) of Alg. \( 3 \) requires \( O(1) \) edge capacity for large fragments and edge capacities summarize to \( O(\log n) \) for small fragments. More precisely, let

\[
\mu_i = \begin{cases} 
1 & \text{for } i \leq 2\log \log n \text{ (Stage 1)} \\
\min\{\mu_{i-1}^2/\log n, n^{1/3}\} & \text{for } i > 2\log \log n \text{ (Stage 2)} 
\end{cases}
\]

Then, we implement Alg. \( 3 \) as described in Lemma \( 2 \) for the new sequence \( \{\mu_i\}_i \). One can verify that executions of SelectEdges can still be implemented in \( O(1) \) rounds with capacity 1. However, to reduce also total edge capacity of the whole algorithm we change implementation of the part, where the edges from \( E_{F,\mu} \) are announced for each \( F \) to the whole network (step \( 5 \) of Alg. \( 3 \)). Using Lemma\( 11 \) one can observe that the size of the smallest growable fragment is

- at least \( 2^{i-1} \) at the beginning of phase \( i \leq 2\log \log n \);
- at least \( \mu_i \) at the beginning of phase \( i > 2\log \log n \).

In a phase of \( i \leq 2\log \log n \) phases each fragment \( F \) has to broadcast a message \( M^F \) of \( \Theta(\log n) \) bits describing the lightest relevant edge incident to \( F \). We split this message into \(|F|\) fragments, each of length \( O(\log n/|F|) \).

For \( i > 2\log \log n \) and a fragment \( F \) we want to broadcast a description of \( \frac{|F|}{\log n} \) edges, which consists of \( O(\frac{|F|}{\log n} \log n) = O(|F|) \) bits. In order to do that it is enough that each node announces \( O(1) \) bits to the whole network, cf. Proposition \( 2 \).

By analyzing this algorithm, we will prove the following result.

**Theorem 3.** It is possible to calculate the minimum spanning forest in \( O(\log \log n) \) rounds and with total capacity of communication edges \( O(\log n) \) and range \( r = 2 \).

**Proof.** Number of rounds. The first stage consists of \( 2\log \log n \) rounds by definition. The second stage also consists of \( O(\log \log n) \) rounds, however, we need a slightly more detailed analysis to show this fact.

At the beginning of the second stage, the size of all growable fragments is at least \( \log^2 n \). Assume that the size of each growable fragment at the beginning of phase \( i \) is at least \( \mu_i \). Then, \( \frac{\mu_i}{\log n} \) lightest relevant edges announced by each fragment satisfies \( \frac{\mu_i}{\log n} \geq \mu_{i/2}^{1/2} \). Therefore, by Lemma \( 11 \) the size of the smallest growable fragment increases \( \mu_{i/2}^{1/2} + 1 \) times in a phase. Thus, the size of the smallest growable fragment in the \( i \)th phase during the second stage is limited from below by \( f_i \) defined as follows: \( f_{1+2\log \log n} = \log^2 n, f_i = f_{i-1}^{3/2} \) for \( i > 1+2\log \log n \). For some \( i \in \Theta(\log \log n) \), the size of the smallest fragment will be at least \( n^{1/3} \). Then, as shown in the previous section (Lemma \( 2 \)), we obtain MSF after \( O(1) \) additional phases.

**Total capacity of communication edges.** In the first stage we have \( O(\log \log n) \) phases, the size of the smallest growable fragment in the \( i \)th phase is at least \( 2^{i-1} \). Thus total capacity of communication edges of the first stage is \( O(\sum_i \frac{\log n}{2^n}) = O(\log n) \). In the second stage we have
$O(\log \log n)$ phases, each is implemented in $O(1)$ rounds with edge capacity 1, thus total capacity of communication edges of those stages is $O(\log \log n)$. Therefore total capacity of communication edges of presented algorithm is $O(\log n + \log \log n) = O(\log n)$.

5 Randomized rcast algorithm for connected components

The fastest known randomized algorithm calculating Connected Components in the unicast model works in $O(\log^* n)$ communication rounds [5]. The algorithm works in phases. At the beginning of each phase, a partition of an input graph into components is known to all nodes. In a phase of the algorithm, the number of growable components drops from $\frac{n}{\log^* x}$ to $\frac{n}{2}$, by simulating $\Theta(\log x)$ steps of the standard Boruvka’s algorithm. Each phase is implemented in $O(1)$ rounds. The key tool to make it possible is a special kind of linear sketches.

In the following, we first describe the linear sketches of Ghaffari and Parter [5]. Then, we shortly describe the $O(\log^* n)$ algorithm for connected components in the unicongested clique [5]. In the next part, we present an implementation of the algorithm in the $rcast(n, 2)$ model. Finally, we provide version of the algorithm with optimal total edge capacity and range 2.

5.1 Linear sketches

In order to build Parter-Ghaffari’s sketches for a graph with $n$ nodes, a preprocessing is necessary. During the preprocessing, each (prospective) edge $(u, v)$ is assigned an ID of size $O(\log n)$, based on a random seed of size $O(\log n)$. In order to build sketches for a given graph $G(V, E)$ with $n$ nodes and a parameter $x \leq n$, the sets $E_1, E_2, \ldots, E_{10\log x}$ included in $E$ are chosen such that each edge $e \in E$ belongs $E_j$ with probability $1/2^j$ and all random choices are independent. For $v \in V$ and $A \subset V$, let $E_j(v)$ be the set of elements of $E_j$ incident to $v$ and let $E_j(A)$ be the set of elements of $E_j$ incident to $A$, i.e., $E_j(A) = \{\{u, v\} \in E_j | u \in A, v \notin A\}$. Then, sketch($X$) ($X$ may be a set or a single node) is a table consisting of $10 \log x$ rows, each row contains a bit string of length $O(\log n)$. The $j$th row of sketch($X$) is the xor of IDs of all elements of $E_j(X)$. The sequence of $\log x$ sketches for a set or a node will be called its multi-sketch. Thus, a multi-sketch might be seen as a table consisting of $10 \log^2 x$ rows. By sketch$_r(A)$ and multi-sketch$_r(A)$ we denote the $r$th row of a sketch and a multi-sketch of $A$, respectively.

Below, we give the key properties of sketches for design of distributed algorithms for graph connectivity.

Proposition 4. [5] 1. It is possible to determine an edge $\{u, v\}$ such that $u \in A$ and $v \notin A$ from a sketch of $A \subset V$ with probability $\Omega(1)$, provided the number of edges $\{u, v\}$ such that $u \in A$, $v \notin A$ is at most $x^5$.

2. The sketch of a set $A = A_1 \cup A_2 \subset V$ for disjoint sets $A_1, A_2$ is equal to sketch($A_1$) xor sketch($A_2$). That is, the $i$th row of sketch($A$) is equal to the xor of the $i$th row of sketch($A_1$) and the $i$th row of sketch($A_2$).

Note that, by Proposition 4.2, the sketch of a set $A$ is equal to the xor of sketches of all elements of $A$.

5.2 Ghaffari-Parter $O(\log^* n)$ connected components algorithm

Ghaffari-Parter algorithm for connected components in the unicast congested clique works in $O(\log^* n)$ phases, each phase consists of $O(1)$ rounds. At the beginning of a phase, a partition
\(C\) of an input graph into \(O(n/\log^2 x)\) (growable) components is known to each node. As a result of the phase, the number of components is reduced to \(O(n/x)\), whp. During the phase (see Algorithm 5 for the pseudocode):

(i) multi-sketches are computed for each component and sent to the leader node \(u^*\);
(ii) the leader \(u^*\) locally simulates \(\log x\) steps of the Boruvka’s algorithm, using obtained multi-sketches of components;
(iii) the leader distributes (with help of other nodes) information about new partition into components;
(iv) each node \(v\) broadcast a random edge \(\{u, v\}\) such that \(C^u \neq C^v\) and a partition is updated using the broadcasted edges;
(v) non-growable components are deactivated.

**Algorithm 5 CCLogstar** \(\triangleright\) the algorithm for a node \(v \in C_k\)

1: \(C = \{\{v_1\}, \ldots, \{v_n\}\}\)

2: \textbf{while} \(C \neq \emptyset\) \(\triangleright\) i.e., while there are active components
3: \(x \leftarrow \min \{y \mid |C| < \frac{n}{10 \log^2 x}\}\) \(\triangleright\) \(C\) is the number of growable components
4: Compute multi-sketches of all components from \(C\)
5: Distribute the multi-sketches in the network
6: Update \(C\) by simulating \(\Theta(\log^2 n)\) rounds of Boruvka’s algorithm, using sketches.
7: Determine edges in the input graph which connect old components in the new ones
8: Broadcast a random edge incident to each component, update \(C\) based on these edges.
9: Deactivate (remove from \(C\)) non-growable components.

Step 7 of Alg. 5 does not require any work in [5], since sketches are computed for actual edges of the input graph (this step will become important in our new algorithm). Below, we shortly describe some other aspects of an implementation of the above steps in the unicast congested clique in [5]:

(a) For each edge \(\{u, v\} \in E\), the node with larger ID (say \(u\)) makes random choices determining to which of the sets \(E_1, E_2, \ldots, E_{10 \log^2 x}\) the edge \(\{u, v\}\) belongs.

(b) Each node \(v\) computes individually its \(\log x\) sketches. In order to make it possible, \(u\) sends (an encoding of) \(\log^2 x\) bits to \(v\) determining in which rows of multi-sketches the edge \(\{u, v\}\) is added, for each edge \(\{u, v\} \in E\) such that \(\text{ID}(u) > \text{ID}(v)\).

(c) Each component \(C_i\) for \(i \in [1, n/\log^2 x]\) has assigned a representative set of \(\log^2 x\) nodes \(V_i = \{v_{1+(i-1)y}, v_{2+(i-1)y}, \ldots, v_{iy}\}\), where \(y = \log^2 x\). Each node \(v \in C_i\) sends the \(j^{th}\) row of multi-sketch of \(v\) to the \(j^{th}\) node of \(V_i\). By xoring all obtained messages, the \(j^{th}\) node of \(V_i\) knows the \(j^{th}\) row of the multi-sketch of \(C_i\) and sends it to \(u^*\).

(d) After computing a new partition of nodes into components, \(u^*\) sends to each \(v \in V\) its new component ID (the “name” of a component might be, e.g., the smallest ID of a node inside the component). Then, each node broadcasts its new component to all nodes. In this way each node knows a new partition into components.

The above distributed implementation requires the range \(r = 2^{\log^2 x}\) in part (b), since \(\log^2 x\) bits are transmitted over each edge. As the unicast model allows for messages of length \(\log n\) only, there is a problem if \(\log^2 x > \log n\) (which appears in the last phase). In order to overcome this problem, the authors of [5] argue that random distribution of transmitted string guarantees that they can be encoded in \(\log n\), whp. The range \(r\) of part (c) in the above implementation is \(r = \log^2 x\) and part (d) requires the range equal to the number of new components which might be \(n/x\).

---

2 Random edges are necessary in order to deal with components with degree \(> x^3\), because sketches do not help much to find their neighbors.
The following result from [5] implies that Algorithm 5 determines connected components in $O(\log^* n)$ iterations of the while-loop, with high probability.

**Lemma 3.** An iteration of the while-loop Algorithm 5 reduces the number of non-growable components from $n/\log^2 x$ to at most $n/x$, whp.

### 5.3 Range efficient algorithm for connected components

In order to implement a phase of Algorithm 5 in the rcast model with the range $r = 2$ and in $O(1)$ rounds, we need a new method of computing and distributing sketches.

Assume that a partition $C$ into components is known to all nodes at the beginning of a phase. Consider a meta-graph, whose nodes correspond to the current components, where $C_i, C_j$ are connected by a meta-edge iff there is an edge $\{u, v\}$ such that $u \in C_i$ and $v \in C_j$. From the “point of view” of nodes it means that $u$ and $v$ are connected by an edge iff $C^u$ and $C^v$ are neighbors in the current meta-graph.

In our algorithm, the sketches are computed for the meta-graph and delivered to all nodes. On the basis of the sketches, each node can simulate $\log x$ steps of the Boruvka’s algorithm on the meta-graph, merging components into larger ones. After determining new larger components, information about the real edges connecting merged input components (into new larger ones) are determined and broadcasted to all nodes. Below, we describe this strategy in more detail.

**Computing sketches in a meta-graph** For computing (and broadcasting) multi-sketches in a meta-graph, each component $C_i$ is associated with a representative set $V_i$ of size $\log^2 x$. In the first round, each node $v$ sends the bit 1 to each element of $V_i$ for $i \in [\log^2 x]$ iff $(v, u) \in E$ for some $u \in C_i$. Otherwise, $v$ sends 0 to each node of $V_i$. After such a round each node of $V_i$ knows all neighbors of $C_i$ in the meta-graph. In order to compute and distribute a multi-sketch of $C_i$ in $O(1)$ rounds, we make the $j$th element of $V_i$ (say, $v_{i,j}$) responsible for the $j$th row of the multi-sketch of $C_i$. For each edge $(C_i, C_{i'})$ such that $i > i'$, $v_{i,j}$ chooses with appropriate probability (i.e., $1/2^{1+ (j-1) \mod 10 \log x}$) whether this edge is included in the $j$th row of the multi-sketch. In the second communication round $v_{i,j}$ sends 1 to $v_{i',j}$ when the edge is included and 0 otherwise. Using own random choices and messages received in both rounds, $v_{i,j}$ computes the $j$th row of the multi-sketch of $C_i$ and broadcasts it to the whole network. More precise description of the above strategy is presented in Algorithm 6. Below, we summarize efficiency of this algorithm.

**Proposition 5.** Assume that Algorithm 6 is executed for a partition of an input graph in at most $n/\log^2 x$ components. Then, the algorithm determines multi-sketches of all nodes in the meta-graph and broadcasts them to the whole network in $O(1)$ rounds, with range $r = 2$ and edge capacity $O(\log n)$. 

15
Algorithm 6 LinearSketches \(\triangleright\) the algorithm for a node \(v \in C_k\)

1: \(y \leftarrow 10 \log_2 x\)
2: Let \(C_k\) be the component containing \(v\)
3: \(V_i \leftarrow \{v_{i,1}, \ldots, v_{i,y}\}\) for \(i \in [n/y]\), where \(v_{i,j} = v_{(i-1)y+j}\)
4: Let \(v = v_{p,r}\)
5: for each \(j \in [n/y]\) do
6: \(\quad\) if \(\{\{v, u\} | u \in C_j\} \neq \emptyset\) then
7: \(\quad\) \(b_j \leftarrow 1\)
8: \(\quad\) else
9: \(\quad\) \(b_j \leftarrow 0\)
10: \textbf{Round 1}: \(v\) sends \(b_j\) to each node of \(V_j\) for each \(j \in [n/y]\)
11: \textbf{E}(\(C_p\)) \(\leftarrow\) \{\((C_p, C_l) | 1\) received from some \(u \in C_l\)\}
12: for each \(e = (C_p, C_l) \in \textbf{E}(C_p)\) such that \(l > p\): set \(b_l \leftarrow 1\) with probability \(\frac{1}{2^{1+(r-1) \mod y}}\), \(b_l \leftarrow 0\) otherwise
13: \textbf{Round 2}: \(v = v_{p,r}\) sends \(b_l\) to \(v_{l,r}\) for each \(l \in [n/y]\)
14: for each \(l < p\): set \(b_l\) to the bit received in Round 2 from \(v_{l,r}\)
15: multi-sketch\(_r(C_p)\) \(\leftarrow\) xor\(_k \in [n/y]\) \(b_k \cdot \text{ID}((C_p, C_k))\)
16: \textbf{Round 3}: \(v_{p,r}\) sends multi-sketch\(_r(C_p)\) to all nodes

Determining real edges connecting merged components An offline simulation of the Boruvka’s algorithm based on meta-edges derived from sketches gives a new partition into components \(C'\). Each component \(C'\) of this new partition is a connected subograph of the meta-graph, with meta-edges connecting elements of \(C'\) known to all nodes (determined by sketches). In order to determine real edges connecting elements of \(C'\), a rooted spanning tree for \(C'\) is chosen arbitrarily but in the same way by each node \(v \in C'\). For each node \(v\), if \(v\) is adjacent to an (“real”) edge \((u, v)\) such that \(C^u\) is the parent of \(C^v\) then \(v\) chooses such edge arbitrarily. Then, in one round, \(v\) broadcasts such chosen edge to the whole network.

Let CCLogstarR be a variant of Alg. 5 where the steps 4 and 5 of the algorithm is implemented through Algorithm 6. In order to decode the real edges in the input graph between joined components, we use the method described above which requires one round with capacity \(O(\log n)\) and range 2 in order to make step 7 of our implementation of Alg. 5.

Lemma 4. Algorithm CCLogstarR identifies the connected components of the input graph in \(O(\log^* n)\) communication rounds in the rcast\((n, 2)\) model, with high probability.

\textbf{Proof.} Prop. 5 implies that, in each phase, all nodes will know sketches of all components and each node can perform locally \(\Theta(\log x)\) steps of the Boruvka’s algorithm on the meta-graph in the way described in 5. Moreover, the real edges showing connectivity of components are decoded as described above in one round in each iteration of the while-loop. Thus, the algorithm determines connected components in \(O(\log^* n)\) rounds with range 2, by Lemma 3.\(\square\)

5.4 Reduction of total edge capacity

In this section we will show that it is possible to achieve the optimal edge capacity \(O(\log n)\) without increasing the range or round complexity of CCLogstarR. More precisely, we show the following theorem.

3One doubt which might appear here is that the authors of 5 deal with real edges in all phases, while our implementation just considers meta-edges between components. However, the reduction of the number of components in a phase holds whp for arbitrary graph, thus also for a meta-graph of components.
Theorem 4. There is a randomized algorithm in the $\text{rcast}(n,2)$ congested clique that identifies the connected components of the input graph with total edge capacity $O(\log n)$ in $O(\log^* n)$ communication rounds, with high probability.

Proof. There are three steps of our $\text{rcast}(n,2)$ implementation of CCLogStar (see Lemma 3), which require $\Omega(\log n)$ bits in each phase. The first is announcing multi-sketches (step 5 of Alg. 4), the second is determining and announcing real edges connecting components (step 7 of Alg. 5), required because of the fact that sketches are computed with respect to the meta-graph – see the description of CCLogstarR) and the third is announcing a random edge (step 8 of Alg. 5).

In order to announce multi-sketches with smaller edge capacity, we slightly change the whole algorithm. In each phase we will select $n$ require $\Omega(\log\log n)$ bits in each phase. The first is announcing multi-sketches (step 5 of Alg. 4), the second is determining and announcing real edges connecting components (step 7 of Alg. 5), required because of the fact that sketches are computed with respect to the meta-graph – see the description of CCLogstarR) and the third is announcing a random edge (step 8 of Alg. 5).

In order to announce multi-sketches with smaller edge capacity, we slightly change the whole algorithm. In each phase we will select $x$ equal to $\min\{y \mid |C| < \frac{n}{10\log x}\}$ instead of $\min\{y \mid |C| < \frac{n}{10\log x}\}$. Therefore representative sets in Alg. 6 can now have size $10\log^2 x$. In order to compute sketches in Alg. 6, we will use only the $10\log^2 x$ nodes from each representative set $V_i$ as presented before. The only part requiring a change is announcing the meta-sketch to the whole network (Round 3 of Alg. 5). Consider a representative set $V_i$ of size $10\log^2 x$ and its subset $V'_i \subset V_i$ of size $10\log^2 x$ such that the $j$th node of $V'_i$ has computed the $j$th row of the multi-sketch of $C_i$ of $O(\log n)$ bits. Then, using the local broadcast primitive (Proposition 1) for $T = V'_i$, $R = V_i$ and $b = O(\log n)$, the whole multi-sketch($C_i$) can be distributed to all nodes of $V_i$ in $O(1)$ rounds with edge capacity $1$, since $\log n \log^2 x \in O(n)$. Next, multi-sketch($C_i$) can be announced to the whole network by the global broadcast procedure with $S = V_i$ and $b = O(\log^2 x \log n)$. By Proposition 2 this task can be done in one round with range 1 and edge capacity $O(\frac{\log^2 x \log n}{\log x}) = O(\frac{\log x}{\log^2 x})$.

In order to determine and announce real edges connecting “old” components into “new” ones (step 7 of Alg. 5), we execute the following procedure. Let $C$ denote the “old” partition into components before step 6 on the meta-graph and let $C'$ denote the “new” partition after that step. After determining $C'$ and decoding meta-edges from sketches locally (the simulation of Boruvka’s algorithm in step 6 of Alg. 5), all nodes build locally a forest $F$ of rooted trees (using disclosed meta-edges), connecting old components from $C$ in the new ones from $C'$. Then, in a separate round, each node $v$ sends the bit $B = 1$ iff $v$ is incident to an edge connecting its old component $C^v$ to the parent of $C^v$ in the appropriate tree; $v$ sends $B = 0$ otherwise. Consider $C$ which is not the root of a tree in $F$. Based on transmitted bits, the node $v_C$ is chosen as the one with the smallest ID among elements of $C$ which sent $B = 1$ (i.e., among nodes incident to edges with an endpoint in the parent of $C$ in $F$). Then, $v_C$ announces the real edge connecting $C$ and the parent of $C$ to all nodes in the representative set of $C$ by local broadcast (Proposition 1 with $T = \{v_C\}$, $R = V_C$ and $b = \log n$, where $V_C$ is the representative set of the component $C$. Then, the global broadcast procedure is applied with $S = V_C$ and $b = \log n$ (see Proposition 2). In this way a real edge connecting $C$ with the parent of $C$ is announced to the whole network. Thus we implement step 7 of Alg. 5 in $O(1)$ rounds with edge capacity $O(\frac{\log^2 x \log n}{\log x}) = O(\frac{\log x}{\log^2 x})$.

In order to select and announce a random meta-edge incident to each component (step 8 of Alg. 5), we use the strategy from the previous paragraph. Namely, a random edge incident to the component $C$ is chosen by the node with the smallest ID in $C$. Then, this edge is broadcasted to the whole network using the local broadcast and the global broadcast primitives, with help of the representative set of $C$. The only issue here is that a node $v$ knows only edges incident to $v$, not the edges incident to the whole component $C^v$. However, each node can learn neighborhood of its component using Round 1 from Alg. 6. Thus, the choice of a random edge is preceded by such a round (described by steps 1–10 of Alg. 6). Thus, we chose a random edge incident to each component in the new meta-graph; we can decode the real edges corresponding to the chosen meta-
edges as described in the previous paragraph. In this way, we implement step 8 of Alg. 5 in $O(1)$ rounds with edge capacity 1 and range 2.

Summarizing, we obtain an algorithm which determines connected components in $O(T(n))$ rounds whp, where $T(n)$ is equal to the smallest $i$ such that $f_i \geq n$ for the sequence $f_1 = c$ and $f_i = 2^{f_{i-1}/3}$ for a constant $c \geq 1$.\footnote{One can make $O(\log c)$ steps of Boruvka’s algorithm at the beginning, in order to start from the components of size $\geq c$.} One can easily verify that $f_i \geq n$ for $i = O(\log^* n)$. As we showed above, the range of our algorithm is $r = 2$. The edge capacity of phase $i$ is $O\left(\frac{\log n}{\log x}\right)$, where $x \geq f_i$ whp. As $f_i > 2^i$ for each $i$ if the constant $c = f_1$ is large enough, the total edge capacity is $O\left(\sum_i \frac{\log n}{2^i}\right) = O(\log n)$.

\section{Conclusions}

We have shown the first sub-logarithmic algorithm for connected components in the broadcast congested clique. Moreover, we provided efficient rcast($n, 2$) implementations of the deterministic MSF algorithm\cite{11} and randomized algorithm for connected components\cite{5}. Both implementations are not only time efficient but also optimal with respect to maximal edge capacity of communication edges. An interesting research problem arising from these results is to determine a relationship between adaptiveness (the number of rounds) and total capacity of communication edges. Moreover, it is still not known whether MSF can be computed in $o(\log n)$ rounds or connected components can be computed in $o(\log n / \log \log n)$ rounds in the broadcast congested clique.

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7 Appendix

7.1 Proof of Fact [1]

The local computation part of the algorithm A stays the same. As for communication part, we can split each message of original protocol into blocks of size \( \lceil \log_r \rceil \) and sent them in separate rounds. Therefore, if in one round protocol sent message of size \( \log n \), after \( \frac{\log n}{\log r} = O(\log, n) \) rounds whole message would be sent to receiver, with no more than \( 2^{\lceil \log_r \rceil} \leq r \) messages per round.