SEMICONJUGACIES TO ANGLE-DOUBLING

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Abstract. A simple consequence of a theorem of Franks says that whenever a continuous map, \( g \), is homotopic to angle-doubling on the circle, it is semi-conjugate to it. We show that when this semiconjugacy has one disconnected point inverse, then the typical point in the circle has a point inverse with uncountably many connected components. Further, in this case the topological entropy of \( g \) is strictly larger than that of angle-doubling, and the semiconjugacy has unbounded variation. An analogous theorem holds for degree-\( D \) circle maps with \( D > 2 \).

1. Introduction

The angle-doubling map, \( d \), on the circle, \( S^1 := \mathbb{R}/\mathbb{Z} \), is an often cited example of a chaotic dynamical system. If we define the itinerary of \( \theta \in S^1 \) as the sequence \( s \) defined by \( s_i = 0 \) if \( 0 < d^i(\theta) \leq 1/2 \) and \( s_i = 1 \) if \( 1/2 < d^i(\theta) \leq 1 \), then for any sequence of 0’s and 1’s we can find a \( \theta \) which has that sequence as its itinerary. Thus the system embeds the randomness of a sequence of coin tosses within its dynamics.

This dynamical complication of angle-doubling is actually topological in character in the sense that it cannot be removed by continuously deforming the system. A theorem of Franks (\cite{Franks}) shows that any circle map that is homotopic to \( d \) has dynamics at least as complicated as those of \( d \) in the precise sense given in the next theorem. (Angle-doubling on a circle is a simplest case of a much more general theorem.)

Theorem 1.1 (Franks). If \( g \) is a continuous, circle map that is homotopic to the angle-doubling map \( d \), then there exists a continuous, onto map \( \alpha : S^1 \to S^1 \) with \( \alpha \circ g = d \circ \alpha \).

An \( \alpha \) as in the theorem is called a semiconjugacy of \( g \) to \( d \). The theorem can be informally understood by noting that whenever \( g \) is homotopic to \( d \), the map \( g^n \) must of necessity wrap the circle \( 2^n \) times around itself, and so iterates of \( g \) have an unavoidable topological complication.

A useful interpretation of the theorem considers the point inverses, \( \alpha^{-1}(\theta) \), as “fibers” over the points \( \theta \). The dynamics of \( g \) can be then thought of as a twisted product with the base point \( \theta \) moved according to \( d \) while the fiber over \( \theta \) is mapped by \( g \) to the fiber over \( d(\theta) \). Thus all the information about how the dynamics of \( g \)
differ from those of \( d \) is contained in the nature of the point inverses of \( \alpha \) and in the way in which these point inverses are transformed into each other by \( g \).

If \( \alpha \) is homeomorphism, each \( \alpha^{-1}(\theta) \) is a single point, and so \( g \) and \( d \) have the same dynamics. The next simplest case is when each \( \alpha^{-1}(\theta) \) is a connected set, and thus is a point or an interval. In this case the essential difference between the dynamics of \( g \) and \( d \) is contained in the dynamics on intervals, a much studied subject. The case of interest here is when \( \alpha \) has at least one disconnected point inverse. In this case the dynamics of \( g \) are much more complicated than those of \( d \) in the sense that the typical fiber, \( \alpha^{-1}(\theta) \), has uncountably many connected components.

**Theorem 1.2.** If \( g \) is a continuous circle map that is homotopic to the angle-doubling map \( d \) and \( \alpha \) is its semiconjugacy to \( d \), then the following are equivalent:

(a) There exists a point \( \theta \in S^1 \) with \( \alpha^{-1}(\theta) \) disconnected.

(b) There exists a full measure, dense, \( G_\delta \)-set \( \Lambda \subset S^1 \) so that \( \theta \in \Lambda \) implies that \( \alpha^{-1}(\theta) \) has uncountably many connected components.

(c) The map \( \alpha \) is not of bounded variation.

Further, in this case the topological entropy of \( g \) is strictly larger than that of \( d \), 
\[ h_{\text{top}}(g) > h_{\text{top}}(d) = \log(2). \]

Note that the existence of the semiconjugacy yields that 
\[ h_{\text{top}}(g) \geq h_{\text{top}}(d), \]
so the content of the last statement of the theorem is the strict inequality. From the point of view developed before the theorem, this conclusion indicates that the action of \( g \) in permuting the fibers \( \alpha^{-1}(\theta) \) has positive entropy.

We briefly remark on related work. The case not included in the theorem, namely, when the semiconjugacy has connected point inverses, includes the situation where \( g \) is a covering map (see the first paragraph of the proof of Theorem 4.1). The semiconjugacies of degree-two covering maps have been widely studied from an analytic point of view (see, for example, section II.2 in [5], and the references therein). Also, there is a theorem in symbolic dynamics concerning a semiconjugacy between two transitive subshifts of finite type which bears a resemblance to Theorems 1.2 and 4.1 (see Remark 5.3). Finally, there are theorems analogous to Theorems 1.2 and 4.1 which hold for homeomorphisms of the two-torus which are isotopic to Anosov diffeomorphisms. These will be the subject of a subsequent paper.

While we state and prove our results for degree-two maps, it will be clear that virtually identical proofs yield the analogous theorems for degree-\( D \) circle maps with \( D > 2 \).

## 2. Preliminaries

The circle \( S^1 \) has universal cover \( \mathbb{R} \), and the phrases *lift* and *projection* always mean lifts to and projections from this cover. A circle map is said to have degree \( D \in \mathbb{Z} \) if it is homotopic to \( \theta \mapsto D\theta \). In the special case of degree two, we write the angle-doubling map as \( d(\theta) = 2\theta \) and for simplicity we choose a preferred lift, \( \tilde{d}(x) = 2x \). Note that a map \( g : S^1 \to S^1 \) has degree two if and only if any lift can be written as

\[
\tilde{g} = \tilde{d} + \varphi
\]

with \( \varphi(x + 1) = \varphi(x) \). Whenever we consider a degree-two map, we fix a lift once and for all.
Given a degree-two circle map $g$ with lift $\tilde{g}$, for each $D \in \mathbb{Z}$, let $B_D$ be the complete metric space of all lifts of continuous degree-$D$ circle maps with the sup topology, and define $F_D : B_D \to B_D$ by $F_D(f) = (\tilde{f} \circ \tilde{g})/2$. It is easy to see that $F_D$ is a contraction mapping whose fixed point $\tilde{\alpha}_D$ satisfies $\tilde{\alpha}_D \circ \tilde{g} = \tilde{d} \circ \tilde{\alpha}_D$, and so projecting to the circle for any $D \neq 0$, we obtain Theorem 1.2. This proof shows that for each $D$ the semiconjugacy $\alpha_D$ is the unique continuous, degree-$D$ map which satisfies $\alpha_D \circ g = d \circ \alpha_D$. In this paper we will only consider the case $D = 1$, and given a degree-two $g$ by its semiconjugacy we always mean $\alpha_1$, which will henceforth be denoted $\alpha$. If we begin the iteration of $F_1$ with the identity map, id, we obtain

\begin{equation}
\frac{\tilde{g}^n}{2^n} = F^n_1(id) \to \tilde{\alpha}
\end{equation}

uniformly.

It is also useful to consider an operator that acts on the periodic parts of the maps. If the given degree-two map is as in (2.1) and $C$ is the Banach space of 1-periodic functions with the sup norm, then $G : C \to C$ defined by $G(\sigma) = (\varphi + \sigma \circ \tilde{g})/2$ is also a contraction mapping, and if its fixed point is $\gamma$, then the lift of the semiconjugacy is $\tilde{\alpha} = id + \gamma$. If we begin the iteration of $G$ with the zero map $0$, we obtain that

\begin{equation}
G^n(0) = \sum_{i=0}^{n-1} \frac{\varphi \circ \tilde{g}^i}{2^{i+1}} \to \gamma
\end{equation}

uniformly and so

\[ \tilde{\alpha} = id + \sum_{i=0}^{\infty} \frac{\varphi \circ \tilde{g}^i}{2^{i+1}}, \]

as could have been confirmed directly.

The semiconjugacy gives a uniform bound on the distance between the $\tilde{g}$-orbit of $x$ and the $\tilde{d}$ orbit of $\tilde{\alpha}(x)$. Using the semiconjugacy and $\tilde{\alpha} = id + \gamma$

\begin{equation}
|\tilde{g}^n(x) - 2^n \tilde{\alpha}(x)| = |\tilde{g}^n(x) - \tilde{\alpha}(\tilde{g}^n(x))| \leq \|\gamma\| \leq \|\varphi\|,
\end{equation}

for all $n$, where for the last inequality we used (2.3). In the language of [7], this says that the orbits $o(x, g)$ and $o(\alpha(x), d)$ globally shadow, where for a given map $f$, the orbit of a point $x$ is $o(x, f) := \{f^n x : n = 0, 1, \ldots\}$. It is worth noting that Theorem 1.1 can also be proved by a slight alteration of the global shadowing proof of the semiconjugacies to pseudo-Anosov maps given in [7].

Recall that a map is called light if every point preimage is totally disconnected and monotone if every point preimage is connected. A theorem of Eilenberg and Whyburn (independently) says that for any continuous map $f : X \to Y$ with $X$ and $Y$ compact metric spaces, there exist a compact metric space $Z$, a continuous light map $\ell : Z \to Y$ and a continuous monotone map $m : X \to Z$, so that $f = \ell m$. The decomposition is particularly simple in the case at hand, $X = Y = S^1$, for since connected components of point inverses are always closed intervals, $Z = S^1$, and the monotone map $m$ simply collapses certain intervals to points.

To study semiconjugacies $\alpha$ with disconnected point preimages, it is useful at first to ignore the monotone part of $\alpha$ and assume that $\alpha$ is light. We shall see in the proof of Theorem 1.2 that by collapsing collections of invariant intervals, any degree-two $g$ can be projected to a degree-two map whose semiconjugacy is light.
The next proposition gives various dynamical characterizations of those $g$ whose semiconjugacies are light maps.

Recall that a map $f$ on a space $X$ is \textit{locally eventually onto} (leo) if for any open set $U$ there is an $n > 0$ so that $f^n(U) = X$. A map is \textit{transitive} if it has a dense orbit. A well-known characterization of transitivity on compact metric spaces is that for all open $U$ and $V$ there exists an $n > 0$ so that $f^n(U) \cap V \neq \emptyset$, and so clearly leo implies transitivity. For a one-dimensional system an interval $J$ is \textit{periodic} if there exists an $n > 0$ so that $f^n(J) \subset J$, and $J$ is \textit{wandering} if for all $i \neq j$, $i, j \geq 0$, $f^i(J) \cap f^j(J) = \emptyset$. Here and throughout this paper the terminology \textit{interval} always means a compact, nontrivial interval.

\textbf{Proposition 2.1.} If $g$ is a continuous degree-two circle map the following are equivalent:

(a) The semiconjugacy $\alpha$ of $g$ to $d$ is light.

(b) $g$ is locally eventually onto.

(c) $g$ is transitive.

(d) $g$ is light and has no periodic or wandering intervals.

\textbf{Proof.} If $J$ is a nontrivial interval and $\alpha$ is light, then there must exist $x_1, x_2 \in \bar{J}$ with $\tilde{\alpha}(x_2) > \tilde{\alpha}(x_1)$ and $\tilde{J}$ a lift of $J$. Thus we may find an $n > 0$ with $2^n\tilde{\alpha}(x_2) - 2^n\tilde{\alpha}(x_1) > 1 + 2\|\varphi\|$, where $\varphi$ is as in \cite{2.1}. Thus by (\ref{g}), $\tilde{g}^n(x_2) - \tilde{g}^n(x_1) > 1$, and so $g^n(J) = S^1$. Therefore, (a) implies (b), and as noted above the theorem, (b) implies (c). Now assume that $o(x, g)$ is dense. If $\alpha$ was not light, then for some nontrivial interval $J$, $\alpha(J) = \theta_0$, a point. Since $o(x, g)$ is dense, there is $i \neq j$ with $g^i(x) \in J$ and $g^j(x) \in J$. Thus $\alpha(g^i(x)) = \alpha(g^j(x)) = \theta_0$ for $i \neq j$, and so by the semiconjugacy, $d^\alpha(o(x)) = d^\theta(o(x))$, and so $o(o(x), d)$ is eventually periodic. On the other hand, the continuity of the semiconjugacy implies that $o(o(x), d)$ is dense since $o(x, g)$ is, a contradiction, and so (c) implies (a).

Now (b) clearly implies (d). We finish by proving the contrapositive of (d) implies (a), so assume $\alpha$ is not light, and thus there is some nontrivial interval $J$ with $\alpha(J) = \theta_0$. Now if $g^n(J)$ is a point for some $n > 0$ or if $J$ wanders, we are done. So we are left with the case when there is an $i > j$ with $g^i(J) \cap g^j(J) \neq \emptyset$. The semiconjugacy then yields that $d^\alpha(\theta_0) = \alpha(g^i(J)) = \alpha(g^j(J)) = d^\theta(\theta_0)$. Thus if $J$ is the connected component of $\alpha^{-1}(d^\theta(\theta_0))$ which contains $g^i(J)$, we must have $g^{i-j}(J) \subset \tilde{J}$, and so $g$ has a periodic interval. \hfill $\Box$

We shall make frequent use of standard results and techniques of one-dimensional dynamics without mention, but for the reader’s convenience we state the following fundamental lemma. Recall that $I$ covers $J$ means that $J \subset I$. For more information on one-dimensional dynamics see \cite{2}, \cite{3}, or \cite{5}. The version of the lemma we give essentially comes from \cite{11}.

\textbf{Lemma 2.2.} Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

(a) If $f(J)$ covers $I$, then there is an interval $J' \subset J$ so that $f(J') = I$ and no interior point of $J'$ maps to the boundary of $I$ under $f$.

(b) If $\{J_i\}$ is a finite collection of intervals such that $f(J_i)$ covers $J_{i+1}$ for all $i$, then there exists an interval $J' \subset J_0$ with $f^i(J') \subset J_i$ for all $i$. If $\{J_i\}$ is an countable collection, then there is a $y \in J_0$ with $f^i(y) \in J_i$ for all $i$. 

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3. The Main Lemmas

The first main lemma locates a copy of the dynamics of $d$ inside the dynamics of $g$. It makes no assumptions about the lightness or injectivity of the semiconjugacy.

**Lemma 3.1.** Given a degree-two circle map $g$ with semiconjugacy $\alpha$, for each $r \in \mathbb{R}$ let $p_r = \min\{\hat{\alpha}^{-1}(r)\}$.

(a) If $x < p_r$, then $\hat{\alpha}(x) < r$.
(b) The map $r \mapsto p_r$ is order preserving.
(c) Each $p_r$ satisfies $\hat{g}(p_r) = p_{2r}$.
(d) If $x < p_r$, then $\hat{g}(x) \leq \hat{g}(p_r)$.
(e) If $s \not\rightarrow r$, then $p_s \not\rightarrow p_r$.

**Proof.** If $x < p_r$, then $\hat{\alpha}(x) \neq r$ by definition. But if $\hat{\alpha}(x) > r$, then since $\alpha$ is degree one, there is a $y < x < p_r$ with $\hat{\alpha}(y) = r$, contradicting the definition of $p_r$, and so we have (a); then (b) follows immediately. Now to prove (c), since $\hat{\alpha}(p_r) = 2\hat{\alpha}(p_r) = 2r$, again by the definition of $p_r$, we have $\hat{g}(p_r) \geq p_{2r}$. Now if $x \leq p_r$ and $\hat{g}(x) > p_{2r}$, there would be a $y < x \leq p_r$ with $\hat{g}(y) = p_{2r}$. But then $2\hat{\alpha}(y) = \hat{\alpha}(\hat{g}(y)) = \hat{\alpha}(p_{2r}) = 2r$, and so $\alpha(y) = r$, contradicting the definition of $p_r$. Thus $x \leq p_r$ implies $\hat{g}(x) \leq p_{2r}$, so we have (c), and then immediately (d). Finally, if $s \not\rightarrow r$, by (b), $\{p_s\}$ is increasing in $s$ and is bounded above by $p_r$. If there was a $z < p_r$ with $p_s \not\rightarrow z$, then by the continuity of $\hat{\alpha}$, $\hat{\alpha}(z) = r$, again contradicting the definition of $p_r$. □

For $r = k/2^n$ with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ we adapt the special notation of $p_{k,n} = p_r$. Conjugation of $\hat{g}$ by a rigid translation will yield a map $\hat{g}'$ for which $p_{0,0} = 0$. Now this $\hat{g}'$ will be the lift of a degree-two $g'$ which is a conjugate of $g$ by a rigid rotation. Since such a conjugation does not change the dynamics of $g$ nor the relevant properties of $\alpha$, we may assume without loss of generality that $p_{0,0} = 0$. Since $\alpha$ is degree one, this implies that $p_{k,0} = k$ for all $k$, and so using Lemma 3.1(c), $\hat{g}^n(p_{k,n}) = k$ for all $k,n$. The next lemma gives an explicit consequence of a non-injective semiconjugacy in the form of a “fold” in the dynamics of $g$.

**Lemma 3.2.** If $g : S^1 \rightarrow S^1$ is a continuous, degree-two circle map which has been conjugated so that $p_{0,0} = 0$ and is such that its semiconjugacy $\alpha$ is light but not injective, then there exists $\bar{N}, K \in \mathbb{N}$ with $0 \leq K < 2^\bar{N}$ and $\hat{x} \in \mathbb{R}$ with $p_{K,N} < \hat{x} < p_{K+1,N}$ so that $\hat{g}^\bar{N}(\hat{x}) = K - 1$.

**Proof.** First note that there exists some $p_r$, and an $x' \in \mathbb{R}$ with $x' > p_r$, and $\hat{\alpha}(x') < r'$, for otherwise by Lemma 3.1(a), (b), $\hat{\alpha}$ would be injective. If we fix this $x'$, then the set $\{r : x' > p_r \text{ and } \hat{\alpha}(x') < r\}$ is nonempty. Let $r_0$ be its supremum and note that by Lemma 3.1(e), $x' > p_{r_0}$ and $\hat{\alpha}(x') < r' \leq r_0$. Next we prove that $s > r_0$ implies $x' < p_s$, by assuming to the contrary that $s > r_0$ and $x' \geq p_s$. Now if $x' = p_s$, then $\hat{\alpha}(x') = s > r_0$, and if $x' > p_s$, by the definition of $r_0$ we have $\hat{\alpha}(x') \geq s > r_0$. Thus in either case we have a contradiction to $\hat{\alpha}(x') < r_0$.

Letting $s_0 = \hat{\alpha}(x')$, since $s_0 < r_0$, elementary number theory yields integers $K$ and $N$ with

$$2^N s_0 + 1 + 2\|\varphi\| < K \leq 2^N r_0 < K + 1,$$

with $\varphi$ as in (2.1). Then since $2^N s_0 = 2^N \hat{\alpha}(x') = \hat{\alpha}(\hat{g}^\bar{N}(x'))$, (2.3) says that $\|\hat{g}^\bar{N}(x') - 2^N s_0\| < \|\varphi\|$ and so $\hat{g}^\bar{N}(x') < K - 1$. Now since $K/2^N \leq r_0 < (K + 1)/2^N$, using the first paragraph of the proof and Lemma 3.1(b), we have $p_{K,N} \leq p_{r_0} < x' < \bar{N}$.
4. The main theorem

The main theorem gives a number of conditions which are equivalent to \( g \) having a light semiconjugacy that is not injective. It will easily imply Theorem 1.2 of the Introduction.

**Theorem 4.1.** If \( g \) is a continuous, degree-two circle map with a light semiconjugacy \( \alpha \), then the following are equivalent:

(a) The map \( \tilde{g} \) is not injective.
(b) The map \( \alpha \) is not injective.
(c) There exists a full measure, dense, \( G_\delta \)-set \( \Lambda \subset S^1 \) so that \( \theta \in \Lambda \) implies that \( \alpha^{-1}(\theta) \) is uncountable, and thus contains a Cantor set.
(d) The topological entropy of \( g \) satisfies \( h_{\text{top}}(g) > \log(2) \).
(e) For all nontrivial intervals \( J \subset S^1 \), the map \( \alpha|_J \) is not of bounded variation.

**Proof.** If \( \alpha \) is injective, then so is \( \tilde{g} = \tilde{\alpha} \tilde{\alpha}^{-1} \), and thus (a) implies (b). Since conjugate maps have the same entropy, (d) implies (b). Now if \( g \) is injective, then by (2.2), \( \tilde{\alpha} \) is nondecreasing, but by hypothesis \( \alpha \) is light, and so \( \tilde{\alpha} \) is strictly increasing and thus is injective, therefore (b) implies (a). The fact that each of (c) and (e) imply (b) is obvious, so we henceforth assume that \( \tilde{\alpha} \) is not injective and show that this implies (c), (d), and (e).

Let \( K \) and \( N \) be as in Lemma 3.2 and continue to assume that \( g \) has been conjugated so that \( p_{0,0} = 0 \). By Lemma 3.2 and Lemma 2.2(a), we may find intervals \( I_a, I_b, \) and \( I_c \) in \([p_{K,N},p_{K+1,N}]\) with disjoint interiors and \( I_a \leq I_b \leq I_c \) so that \( \tilde{g}^{-N}(I_a) = \tilde{g}^{-N}(I_b) = [K-1,K] \) and \( \tilde{g}^{-N}(I_c) = [K,K+1] \). For each \( k \neq 0 \) with \( 0 \leq k < 2^N \), define intervals \( I_k = [p_{k,N},p_{k+1,N}] \). Define a set of “addresses” as \( A = \{0,1,2,\ldots,K-1,K+1,\ldots,2^N-1,a,b,c\} \), and for \( \eta \in A \), let \( \phi(\eta) \) be given by \( \phi(a) = \phi(b) = K-1, \phi(c) = K \), and for \( 0 \leq k < 2^N \), \( \phi(k) = k \). By Lemma 3.1(c) we now have that for all \( \eta \in A, \tilde{g}^{-N}(I_\eta) \) covers \([0,1]+\phi(\eta)\). Projecting the collection \( \{I_\eta\} \) to the circle we see that \( g^{-N} \) has a \((2^N+2)\)-fold horseshoe and so (see Theorem 4.3.2 in [2]) \( h_{\text{top}}(g^{-N}) \geq \log(2^N+2) \) and therefore \( h_{\text{top}}(g) \geq \log(2^N+2)/N > \log(2) \), yielding (d).

Returning to the covering space \( \mathbb{R} \), since \( g \) is a degree-two map, for any integer \( m, \tilde{g}^{-N}(I_0+m) \) covers \([0,1]+\phi(\eta)+2^N m \). Thus by Lemma 2.2(b) for any sequence \( \underline{s} \in A^{\mathbb{N}} \) we may find a \( y \in [0,1] \) with

\[
\tilde{g}^{-Nj}(y) \in I_{s_j} + \sum_{i=0}^{j-1} 2^{N(j-i-1)} \phi(s_i)
\]

for all \( j \in \mathbb{N} \). Now a given \( y \) can represent two or more sequences, but that can only happen if for some \( i, \tilde{g}^{-Ni}(y) \) is contained in two intervals and so must be in the boundary of some \( I_\eta \). However, then by construction of the \( I_\eta \), \( \tilde{g}^{-N(i+1)}(y) \in \mathbb{Z} \), and since \( p_{0,0} = 0 \) as noted above in Lemma 3.2 we have that for all \( j > i, \tilde{g}^{-Nj}(y) \in \mathbb{Z} \). If we assume initially that \( K \neq 0, 2^N - 1 \), then for any integer \( m, (I_{2^{N-i}} + m - 1) \cap (I_0 + m) = \{m\} \). Thus a point \( y \) can represent two sequences \( \underline{s} \) and \( \underline{s} \) only if \( s_j \) and \( s'_j \) are contained in \([2^N-1,0]\) for all sufficiently large \( j \).
Therefore, if we say a sequence has a *nontrivial tail* if there exist arbitrarily large \(j\) with \(s_j \notin \{2^N - 1, 0\}\), we see that when \(a\) has a nontrivial tail, \(a \neq a'\) implies that the corresponding \(y's\) are distinct. To make this true when \(K = 0\) the definition of nontrivial tail must be altered to require arbitrarily large \(j\) with \(s_j \notin \{2^N - 1, a\}\), and when \(K = 2^N - 1\) to require arbitrarily large \(j\) with \(s_j \notin \{c, 0\}\).

Now note that (2.2) implies that a \(y\) which satisfies (4.1) will have

\[
\hat{\alpha}(y) = \lim_{j \to \infty} \frac{1}{2^N} \sum_{i=0}^{j-1} 2^N(j-i-1) \phi(s_i) = \sum_{i=0}^\infty \phi(s_i) 2^{-N(i+1)}.
\]

Since \(\phi(a) = \phi(b) = \phi(K - 1) = K - 1\), whenever \(\phi(s_i) = K - 1\) in this sum, there are three possible choices of \(s_i\) which give the same value of \(\hat{\alpha}(y)\). Thus if \(\eta \in \{1, 2, \ldots, 2^N - 1\}^N\) is a sequence with \(t_i = K - 1\) for infinitely many \(i\), the sum

(4.2)

\[
r = \sum_{i=0}^\infty \frac{t_i}{2^N(i+1)}
\]

is equal to the sum in (4.1) for uncountably many sequences \(s\). If uncountably many of these sequences \(s\) have a nontrivial tail, then for such an \(r\) the set \(\hat{\alpha}^{-1}(r)\) is uncountable. We will prove that the collection of all such \(r\) is as in (c).

It is well known that when a map is ergodic with respect to a smooth measure on a compact manifold, the collection of \(x\) whose orbits are dense is a dense, \(G_{\delta}\), full measure set, and that the angle-doubling map \(d\) is ergodic with respect to Lebesgue measure. Thus (c) is proven after we show that whenever \(\theta\) has a dense orbit, its lift to an \(r \in [0, 1)\) is as described at the end of the previous paragraph.

The proof of this proceeds by repeating the construction that gave rise to (4.1) in the easier case of \(\hat{d}\). For \(0 \leq k < 2^N\), let \(I_k = [k/2^N, (k + 1)/2^N]\), and so for any integer \(m\), \(\hat{d}^N(I_k + m)\) covers \([0, 1] + k + 2^N m\). Thus for any sequence \(\zeta \in \{1, 2, \ldots, 2^N - 1\}^N\), we may find an \(r \in [0, 1)\) with

(4.3)

\[
\hat{d}^N(r) \in \hat{I}_{t_j} + \sum_{i=0}^{j-1} 2^N(j-i-1)t_i
\]

for all \(j \in \mathbb{N}\). This implies that \(r\) is given by (4.2). Conversely, because \(\hat{d}\) is expanding and for all \(k, m\), \(\hat{d}^N(I_k + m) = [0, 1] + k + 2^N m\), it follows that any \(r \in [0, 1)\) with \(\hat{d}^N(r) \notin \mathbb{Z}\) for all \(j \in \mathbb{N}\) satisfying (4.3) for a sequence \(\eta\) with \(t_i \notin \{0, 2^N - 1\}\) for arbitrarily large \(i\). In particular, if \(\theta \in S^1\) has a dense orbit under \(d\), then its orbit lands infinitely often in the projection to the circle of every interval \(I_k\), and thus its lift \(r \in [0, 1)\) yields a sequence \(\eta\) for which \(t_i = K - 1\) infinitely often, and any \(s\) with \(\phi(t_i) = s_i\) for all \(i\) must have a nontrivial tail. Thus for such \(r\), \(\hat{\alpha}^{-1}(r)\) is uncountable and thus \(\alpha^{-1}(\theta)\) is uncountable also, proving (c).

Now to prove (e), say that an interval \(J \subset \mathbb{R}\) unit covers if for some integer \(M, [M, M + 1] \subset J\). By construction for each \(\eta \in A, \hat{g}^N(I_\eta)\) unit covers. Since there are \(2^N + 2\) such intervals \(I_\eta\), using \(\hat{\alpha} = (\hat{\alpha} \circ \hat{g}^N)/2^N\) we obtain that the variation of \(\hat{\alpha}\) on the interval \([0, 1]\) satisfies \(\var{\alpha}([0, 1]) \geq (2^N + 2)/2^N\). Now since each \(\hat{g}^N(I_\eta)\) unit covers and each unit interval \([M, M + 1]\) contains \(I_\eta + M\) for all \(\eta\), using Lemma (2.2) there are \(2^N + 2\) intervals \(I_\eta\) in each \(I_\eta\) so that each \(\hat{g}^{2N}(I_{\eta,j})\) unit covers, so \(\var{\alpha}([0, 1]) \geq (2^N + 2)^2/2^{2N}\). An obvious induction then yields
that for all $j$,

\[(4.4) \quad \text{var}(\alpha, [0, 1]) \geq \frac{(2^N + 2)^j}{2^{Nj}},\]

which goes to infinity as $j \to \infty$, and so $\tilde{\alpha}$ has unbounded variation on $[0, 1]$.

Now as noted at the beginning of the proof of Proposition 2.1, for any interval $J \subset \mathbb{R}$ there is a $w \in \mathbb{N}$ so that $g^w(J)$ unit covers. Then using Lemma 2.2 and the intervals of the previous paragraph we get that

\[\text{var}(\tilde{\alpha}, J) \geq \frac{(2^N + 2)^j}{2^{Nj+w}} \to \infty,\]

proving (e). $\square$

**Proof of Theorem 1.2.** Assume that the semiconjugacy of $g$ to $d$ has monotone-light decomposition, $\alpha = \ell m$. Now if $J$ is an interval such that $m(J)$ is a point, then certainly $d \ell m(J) = \ell mg(J)$ is also a point, and since $\ell$ is light, this says that $mg(J)$ must also be a point. Thus the formula $\tilde{g} = mgm^{-1}$ unambiguously defines a continuous degree-two map with light conjugacy $\ell$. Now if $\alpha$ has a disconnected preimage, then $\ell$ must also, and so by Theorem 4.1(c), (a) implies (b), while the converse is trivial. The graph of $\ell$ differs from that of $\alpha$ only by the insertion of perhaps a countable number of horizontal intervals, and so assuming (a), by Theorem 4.1(c), (c) follows, and the converse is also clear. Finally, since $g$ is semiconjugate to $\tilde{g}$, $h_{\text{top}}(g) \geq h_{\text{top}}(\tilde{g})$. Assuming (a), Theorem 4.1(d) gives $h_{\text{top}}(\tilde{g}) > \log(2)$, finishing the proof. $\square$

5. REMARKS AND QUESTIONS

**Remark 5.1.** The primary distinction between the general case of Theorem 1.2 and the light semiconjugacy case of Theorem 4.1 is that a general $g$ can have an arbitrary amount of dynamical complications and thus entropy in, say, a periodic interval. In this case, one can have $h_{\text{top}}(g) > \log(2)$, which clearly implies that $\alpha$ is not injective, but it does not necessarily imply that $\alpha$ is not monotone.

**Remark 5.2.** If $g$ is piecewise monotone with a finite number of turning points, then it follows from a theorem of Misiurewicz and Szlenk [10] that the variation estimate on $\tilde{g}^Nj$ that gives rise to (4.4) is equivalent to the entropy result. See Corollary 4.1.8 in [8] as well as Section 6 in [1].

The proof of Theorem 4.1 given here has much of the flavor of this symbolic dynamics result, basically showing the existence of a diamond in the semiconjugacy. In fact, parts of the result could have been reduced to the symbolic dynamics theorem, but doing so would have resulted in a longer, less self-contained proof.

**Remark 5.4.** Parts of Theorem 4.1 can also be obtained by more topological methods. From Proposition 2.1 it follows that any $g$ with a light conjugacy is locally eventually onto, and from this is follows fairly easily that if $\alpha$ is not injective in one
open set, then it is not injective in any open set. Such an \( \alpha \) is called nowhere locally injective. Block, Oversteegen and Tymchatyn have shown that any light, nowhere locally injective map between manifolds has the property that the topologically generic point has a Cantor set as its point inverse (\([4]\)).

**Remark 5.5.** This paper has dealt primarily with combinatorial/topological aspects of degree-two circle maps. It would also be of interest to study quantitative/analytic aspects. For example, for a \( g \) with a light semiconjugacy, give an explicit relationship between properties of its semiconjugacy \( \alpha \), say the fractal dimensions of the graph of \( \alpha \), and the difference in entropy, \( h_{\text{top}}(g) - \log(2) \). In this regard we note then when \( g \) has a finite number of turning points, its semiconjugacy can be treated in the context of fractal functions. In particular, if \( g \) is piecewise linear with expanding pieces, then \( \alpha \) is an affine fractal function, and its graph is the attractor of a planar iterated function system (see \([9]\)). Also, in analogy to the degree-one case, it would also be interesting to study the transition to a nonmonotone semiconjugacy in parameterized families, for example in the standard degree-two family \( f_{b,\omega}(x) = 2x + \omega + b \sin(2\pi x) \).

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