A Probabilistic Characterization on Navier-Stokes Equations *

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Abstract

By proposing and solving future distribution dependent SDEs, the well-posedness and regularity are derived for (generalized) incompressible Navier-Stokes equations on $\mathbb{R}^d$ or $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ ($d \geq 1$) for given initial datum and pressure.

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1 Introduction and main results

Let $d \in \mathbb{N}$. Consider the following incompressible Navier-Stokes equation on $E := \mathbb{R}^d$ or $\mathbb{R}^d/\mathbb{Z}^d$:

\begin{equation}
\partial_t u_t = \kappa \Delta u_t - (u_t \cdot \nabla) u_t - \nabla \varphi_t, \quad t \in [0, T],
\end{equation}

where $T > 0$ is a fixed time,

\[ u := (u^1, \ldots, u^d) : [0, T] \times E \to \mathbb{R}^d, \quad \varphi : [0, T] \times E \to \mathbb{R}, \]

and $u_t \cdot \nabla := \sum_{i=1}^d u_t^i \partial_i$. This equation describes viscous incompressible fluids, where $u$ is the velocity field of a fluid flow, $\varphi$ is the pressure, and $\kappa > 0$ is the viscosity constant. For the incompressible fluid we assume

\[ \nabla \cdot u_t := \sum_{i=1}^d \partial_i u_t^i = 0. \]

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The real-world model in physics is for $d = 3$. In this case, Leray [2] proved the weak existence for $u_0 \in L^3(\mathbb{R}^3)$ and studied the blow-up property. See [13, 3, 5] and references within for the blow-up in $L^p, p \geq 3$, and see [4] and references therein for the study using probabilistic approaches.

In this paper, we propose a new type stochastic differential equation (SDE) depending on distributions in the future, such that the solution of (1.1) is explicitly given by the initial datum $u_0$ and the pressure $\varphi$. By proving the well-posedness of the SDE, we derive the well-posedness of (1.1) in $\mathcal{C}_b^n(n \geq 2)$, see Theorem 1.1 below.

For any $n \in \mathbb{N}$, let $\mathcal{C}_b^n$ be the class of real functions $f$ on $E$ having derivatives up to order $n$ such that

$$\|f\|_{\mathcal{C}_b^n} := \sum_{i=0}^n \|\nabla^i f\|_{\infty} < \infty,$$

where $\nabla^0 f := f$. Moreover, for $\alpha \in (0, 1)$, we denote $f \in \mathcal{C}_b^{n+\alpha}$ if $f \in \mathcal{C}_b^n$ such that

$$\|f\|_{\mathcal{C}_b^{n+\alpha}} := \|f\|_{\mathcal{C}_b^n} + \sup_{x \neq y} \|\nabla^n f(x) - \nabla^n f(y)\| |x - y|^\alpha < \infty.$$

Let $(W_s)_{s \in [0,T]}$ be the $d$-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathcal{F}_s\}_{s \in [0,T]}, \mathbb{P})$. Consider the following future distribution dependent SDE on $\mathbb{R}^d$:

$$dX^x_{t,s} = \left[ \mathbb{E} \int_s^T \nabla \varphi_{T-r}(X^x_{y,r})dr - \mathbb{E}u_0(X^x_{s,T}) \right]_{y=X^x_{t,s}} ds + \sqrt{2\kappa} dW_s, \quad X^x_{t,t} = x, s \in [t, T].$$

See Definition 1.1 below for the definition of solution. When $E = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, we extend $u_0$ and $\varphi_t$ to $\mathbb{R}^d$ periodically, i.e. for a function $f$ on $\mathbb{T}^d$, it is extended to $\mathbb{R}^d$ by letting

$$f(x + k) = f(x), \quad x \in [0, 1)^d, k \in \mathbb{Z}^d.$$

With this extension, we also have the SDE (1.2) for the case $E = \mathbb{T}^d$.

Our first result is the following.

**Theorem 1.1.** If there exists $n \geq 2$ such that $u_0 \in \mathcal{C}_b^n$ and $\varphi_t \in \mathcal{C}_b^n$ for a.e. $t \in [0,T]$ with

$$\int_0^T (\|\nabla \varphi_t\|_{\infty}^2 + \|\varphi_t\|_{\mathcal{C}_b^n}) dt < \infty.$$

Then (1.2) is well-posed and (1.1) has a unique solution satisfying

$$\sup_{t \in [0,T]} \|u_t\|_{\mathcal{C}_b^n} < \infty,$$

and the solution is given by

$$u_t(x) = \mathbb{E}u_0(X^x_{T-t,T}) - \mathbb{E} \int_{T-t}^T \nabla \varphi_{T-s}(X^x_{T-t,s}) ds.$$
It is a challenging problem to construct regular incompressible solutions to (1.1). To this end, we consider the operators
\[ P_{t,s}^\rho f(x) := \mathbb{E}[f(X_{t,s}^x)], \quad 0 \leq t \leq s \leq T, \quad x \in \mathbb{R}^d, \quad f \in \mathcal{B}_b(\mathbb{R}^d), \]
where
\[ \varphi \in \mathcal{C}^\alpha_b(T) := \left\{ \varphi : [0, T] \to E : \int_0^T \left\{ \|\nabla \varphi_t\|_\infty^2 + \|\varphi_t\|_{\ell_b^\alpha} \right\} dt < \infty \right\}, \]
and \( X^x := (X_{t,s}^x)_{0 \leq t \leq s \leq T, x \in \mathbb{R}^d} \) is the unique solution of (1.2). Let
\[ Q_j^t \varphi := P_{T-t,T}^\rho u^0_t - \int_{T-t}^T P_{T-t,s}^\rho \partial_j \varphi_{T-s} ds, \quad t \in [0, T], \ 1 \leq j \leq d. \]
We have the following result.

**Theorem 1.2.** Let \( u_0 \in \mathcal{C}^\alpha_b \) and \( \varphi \in \mathcal{C}^\alpha_b(T) \) for some \( n \geq 2 \) with \( \nabla \cdot u_0 = 0 \). The unique solution of (1.1) in Theorem 1.1 satisfies \( \nabla \cdot u_t = 0 \) if and only if
\[
\Delta \varphi_t + \sum_{i,j=1}^d \left( \partial_i Q_i^t \varphi \right) \left( \partial_j Q_j^t \varphi \right) = 0, \quad t \in [0, T].
\]

Therefore, to construct regular incompressible solutions to (1.1), one needs to find \( \varphi \) satisfying (1.6). We do not discuss this in the present paper.

Indeed, we will prove a more general result for the following Navier-Stokes type equation on \( E := \mathbb{R}^d \) or \( \mathbb{R}^d/\mathbb{Z}^d \):
\[
\partial_t u_t = L_t u_t - (u_t \cdot \nabla) u_t + V_t, \quad t \in [0, T],
\]
where
\[ L_t := \text{tr}\{a_t \nabla^2\} + b_t \cdot \nabla \]
and
\[ V, b : [0, T] \times E \to \mathbb{R}^d, \quad a : [0, T] \times E \to \mathbb{R}^{d \otimes d} \]
are measurable, and \( a_t(x) \) is positive definite for \( (t, x) \in [0, T] \times E \).

To characterize (1.7), we consider the following SDE on \( \mathbb{R}^d \) where differentials are in \( s \in [t, T] \):
\[
dX_{t,s}^x = \sqrt{2a_{T-s}(X_{t,s}^x)}dW_s
\]
\[
+ \left\{ b_{T-s}(X_{t,s}^x) - \left[ \mathbb{E}u_0(X_{s,T}^y) + \mathbb{E} \int_s^T V_{T-r}(X_{s,r}^y) dr \right]_{y=X_{t,s}^x} \right\} ds,
\]
\[ t \in [0, T], \quad s \in [t, T], \quad X_{t,t}^x = x \in \mathbb{R}^d. \]
When \( E = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \), we extend \( a_t, b_t, V_t \) to \( \mathbb{R}^d \) as in (1.3), we also have the SDE (1.8) for the case \( E = \mathbb{T}^d \).

Regarding \( s \) as the present time, the SDE (1.8) depends on the distribution of \( (X_{s,r})_{r \in [s, T]} \) coming from the future. So, this is a future distribution dependent equation. We will use \( X := (X_{t,s}^x)_{0 \leq t \leq s \leq T, x \in E} \) to formulate the solution to (1.7).

Let \( D_T := \{(t, s) : 0 \leq t \leq s \leq T\} \). We define the solution \( X \) of (1.8) as follows.
We now make the following assumption on the operator $L^{p,q}$.

**Definition 1.1.** A family $X := (X^x_{t,s})_{(t,s,x) \in D_T \times \mathbb{R}^d}$ of random variables on $\mathbb{R}^d$ is called a solution of $(LS)$, if $X^x_{t,s}$ is $\mathcal{F}_s$-measurable, $\mathbb{P}$-a.s. continuous in $(t,s,x)$,

$$
\mathbb{E} \int_t^T \left\{ \left\| a_{T-s}(X^x_{t,s}) \right\| + \left| b_{T-s}(X^x_{t,s}) \right| - \left[ \mathbb{E}u_0(X^y_{s,T}) + \mathbb{E} \int_s^T V_{T-r}(X^y_{s,T})dr \right]_{y=X^x_{t,s}} \right\} ds < \infty
$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$, and $\mathbb{P}$-a.s.

$$
X^x_{t,s} = x + \int_t^s \sqrt{2a_{T-r}(X^x_{t,r})} dW_r + \int_t^s \left\{ b_{T-r}(X^x_{t,r}) - \left[ \mathbb{E}u_0(X^y_{r,T}) + \mathbb{E} \int_r^T V_{T-s}(X^y_{r,T})d\theta \right]_{y=X^x_{t,r}} \right\} dr, \quad (t,s,x) \in D_T \times \mathbb{R}^d.
$$

We will allow the operator $L_t$ to be singular, where the drift contains a locally integrable term introduced in $[7]$ for singular SDEs. For any $p,q > 1$ and $0 \leq t < s$, we write $f \in \tilde{L}^p_q(t,s)$ if $f$ is a measurable function on $[t,s] \times \mathbb{R}^d$ such that

$$
\| f \|_{\tilde{L}^p_q(t,s)} := \sup_{z \in \mathbb{R}^d} \left( \int_t^s \| f_r 1_{B(z,1)} \|_{L^p}^q dr \right)^{\frac{1}{q}} < \infty,
$$

where $B(z,1)$ is the unit ball at $z$, and $\| \cdot \|_{L^p}$ is the $L^p$-norm for the Lebesgue measure. We denote $f \in \tilde{H}^2_p(t,s)$ if $|f| + |\nabla f| + \| \nabla^2 f \| \in \tilde{L}^p_q(t,s)$. When $(t,s) = (0,T)$ we simply denote

$$
\tilde{L}^p_q = \tilde{L}^p_q(0,T), \quad \tilde{H}^2_p = \tilde{H}^2_p(0,T).
$$

We will take $(p,q)$ from the following class:

$$
\mathcal{K} := \left\{ (p,q) : p,q > 2, \frac{d}{p} + \frac{2}{q} < 1 \right\}.
$$

We now make the following assumption on the operator $L_t$.

**($H$)** Let $b_t = b_t^{(0)} + b_t^{(1)}$, and when $E = \mathbb{T}^d$ we extend $a_t, b_t^{(0)}$ and $b_t^{(1)}$ to $\mathbb{R}^d$ as in $(1.3)$.

1. $a$ is positive definite with

$$
\| a \|_{\infty} + \| a^{-1} \|_{\infty} := \sup_{(t,x) \in [0,T] \times E} \| a_t(x) \| + \sup_{(t,x) \in [0,T] \times E} \| a_t(x)^{-1} \| < \infty,
$$

$$
\lim_{\varepsilon \to 0} \sup_{|x-y| \leq \varepsilon, t \in [0,T]} \| a_t(x) - a_t(y) \| = 0.
$$

2. There exist $l \in \mathbb{N}$, $\{(p_i,q_i)\}_{0 \leq i \leq l} \subset \mathcal{K}$ and $0 \leq f_i \in \tilde{L}^{p_{i}}_{q_{i}}, 0 \leq i \leq l$, such that

$$
|b^{(0)}| \leq f_0, \quad \| \nabla a \| \leq \sum_{i=1}^{l} f_i.
$$
Under the metric $\| \cdot \|$, where the distribution of a random variable $\xi$ be the space of continuous maps from $\mathbb{R}$.

Under this assumption, we will prove the well-posedness of (1.8) and solve (1.7) in the class $\mathcal{U}_b$.

Assume Theorem 1.3.

Theorem 1.3. Assume (H). Let $\|u_0\|_\infty + \int_0^T \|V_t\|_\infty^2 dt < \infty$. Then the following assertions hold.

1. The SDE (1.8) has a unique solution $X := (X^x_{t,s,x})_{(t,s,x) \in DT \times \mathbb{R}^d}$.

2. If $u$ solves (1.7) and $u \in \mathcal{U}_b(p_0, q_0)$, then

   \[ u_t(x) = \mathbb{E} \left[ u_0(X_{t-t,T}^x) + \int_{T-t}^T V_{t-s}(X_{t,s}^x) ds \right], \quad (t, x) \in [0, T] \times E. \tag{1.10} \]

   Moreover, there exists a constant $c > 0$ such that for any $i \in \{1, 2\}$ and $j, j' \in \{0, 1\}$,

   \[ \| \nabla^i u_t \|_\infty \leq c t^{-\frac{i}{2}} \| \nabla^j u_0 \|_\infty + c \int_{T-t}^T (s + t - T)^{-\frac{j-j'}{2}} \| \nabla^{j'} V_{t-s} \|_\infty ds, \quad t \in (0, T]. \tag{1.11} \]

3. If $b^{(1)} = 0$ and $u_0, V_t \in \mathcal{C}_b$ with $\int_0^T \|V_t\|_\mathcal{C}_b^2 dt < \infty$, then $u$ given by (1.10) solves (1.7), and $u$ is in the class $\mathcal{U}_b(p_0, q_0)$.

In the next two sections, we prove assertions (1) and (2)-(3) of Theorem 1.3 respectively, where in Section 2 the well-posedness is proved for a more general equation than (1.8). Finally, Theorems 1.1 and 1.2 are proved in Section 4 and Section 5 respectively.

2 Proof of Theorem 1.3(1)

Let $\mathcal{P}$ be the space of probability measures on $\mathbb{R}^d$ equipped with the weak topology, let $\mathcal{L}_\xi$ be the distribution of a random variable $\xi$ on $\mathbb{R}^d$. Let

$\Gamma := C(D_T \times \mathbb{R}^d; \mathcal{P})$ be the space of continuous maps from $D_T \times \mathbb{R}^d$ to $\mathcal{P}$. For any $\lambda > 0$, $\Gamma$ is a complete space under the metric

\[ \rho_\lambda(\gamma^1, \gamma^2) := \sup_{(t,s,x) \in D_T \times \mathbb{R}^d} e^{-\lambda(T-t)} \| \gamma^1_{t,s,x} - \gamma^2_{t,s,x} \|_{\var} \quad \gamma^1, \gamma^2 \in \Gamma, \]

where $\| \cdot \|_{\var}$ is the total variation norm defined by

\[ \| \mu - \nu \|_{\var} := \sup_{|f| \leq 1} | \mu(f) - \nu(f) |, \quad \mu, \nu \in \mathcal{P} \]
for \( \mu(f) := \int f \, d\mu \). Note that the convergence in \( \| \cdot \|_{\text{var}} \) is stronger than the weak convergence.

We consider the following more general equation than \((1.8.2)\):

\[
dX^x_{t,s} = \left\{ b^{(1)}_{T-s}(X^x_{t,s}) + Z_s(X^x_{t,s}, \mathcal{L}_X) \right\} \, ds + \sqrt{2a_{T-s}(X^x_{t,s})} \, dW_s,
\]

\( t \in [0, T], s \in [t, T], X^x_{t,t} = x \in \mathbb{R}^d, \)

where \( \mathcal{L}_X \in \Gamma \) is defined by \( \{ \mathcal{L}_X \}^s_{t,s,x} := \mathcal{L}^s_{X^x_{t,s}}, \) and

\[
Z : [0, T] \times \mathbb{R}^d \times \Gamma \to \mathbb{R}^d
\]
is measurable.

It is easy to see that \((2.1)\) covers \((1.8)\) for

\[
Z_t(x, \gamma) := b^{(0)}_{T-t}(x) - \int_{\mathbb{R}^d} u_0(y) \gamma_{t,T,x}(dy) - \int_t^T ds \int_{\mathbb{R}^d} V_{T-s}(y) \gamma_{t,s,x}(dy),
\]

\( (t, x, \gamma) \in [0, T] \times \mathbb{R}^d \times \Gamma. \)

The solution of \((2.1)\) is defined as in Definition 1.1 using \( b^{(1)}_{T-s}(X^x_{t,s}) + Z_s(X^x_{t,s}, \mathcal{L}_X) \) replacing

\[
b_{T-s}(X^x_{t,s}) - \left[ \mathbb{E} u_0(X^y_{s,T}) + \mathbb{E} \int_s^T V_{T-r}(X^y_{s,r})dr \right]_{y=X^x_{t,s}}.
\]

We make the following assumption.

(A) \( b^{(1)} \) and \( a \) satisfy \((H)\), and there exists \((p_0, q_0) \in \mathcal{K} \) and \( f_0 \in \tilde{L}^{p_0}_{q_0} \) such that

\[
|Z_t(x, \gamma)| \leq f_0(t, x), \quad (t, x, \gamma) \in [0, T] \times \mathbb{R}^d \times \Gamma.
\]

Moreover, there exists \( g \in L^2([0, T]) \) such that

\[
\sup_{t \in \mathbb{R}^d} |Z_t(x, \gamma^1) - Z_t(x, \gamma^2)| \leq g_t \sup_{(s, x) \in [t, T] \times \mathbb{R}^d} \| \gamma^1_{t,s,x} - \gamma^2_{t,s,x} \|_{\text{var}}, \quad t \in [0, T], \gamma^1, \gamma^2 \in \Gamma.
\]

When \( \| u_0 \|_\infty + \int_0^T \| V_i \|_\infty \, dt < \infty \), \((H)\) implies \((A)\) for \( Z \) given by \((2.2)\). So, \( \text{Theorem 1.3 (1)} \) follows from the following result, which also includes regularity estimates on the solution.

**Theorem 2.1.** Assume \((A)\). Then the following assertions hold.

1. \((2.1)\) has a unique solution, and the solution has the flow property

\[
X^x_{t,r} = X^{X^x_{t,s}}_{s,r}, \quad 0 \leq t \leq s \leq r \leq T, \ x \in \mathbb{R}^d.
\]

2. For any \( j \geq 1 \),

\[
\nabla_v X^x_{t,s} := \lim_{\varepsilon \downarrow 0} \frac{X^x_{t,s} + \varepsilon v - X^x_{t,s}}{\varepsilon}, \quad s \in [t, T]
\]

exists in \( L^j(\Omega \to C([t, T]; \mathbb{R}^d), \mathbb{P}) \), and there exists a constant \( c(j) > 0 \) independent of such that

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E} \left[ \sup_{s \in [t, T]} |\nabla_v X^x_{t,s}|^j \right] \leq c(j) |v|^j, \quad v \in \mathbb{R}^d.
\]
(3) For any $0 \leq t < s \leq T$, $v \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$
\nabla_v \{ \mathbb{E} f(X_{t,s}) \}(x) = \frac{1}{s-t} \mathbb{E} \left[ f(X_{t,s}^x) \int_t^s \left( (\sqrt{2aT-r})^{-1}(X_{t,r}^x)\nabla_v X_{t,r}^x, dW_r \right) \right].
$$

Proof. (a) We first explain the idea of proof using fixed point theorem on $\Gamma$. For any $\gamma \in \Gamma$, we consider the following classical SDE

$$
dX_{t,s}^\gamma = \left\{ b(1)_{T-s}(X_{t,s}^\gamma) + Z_s(X_{t,s}^\gamma, \gamma) \right\} ds + \sqrt{2aT-r}(X_{t,s}^\gamma)dW_s,
$$

$t \in [0, T], s \in [t, T], X_{t,t}^\gamma = x \in \mathbb{R}^d$.

By [6, Theorem 2.1] for $[t, T]$ replacing $[0, T]$, see also [7] for $b(1) = 0$, this SDE is well-posed, such that for any $j \geq 1$ and $v \in \mathbb{R}^d$, the directional derivative

$$
\nabla_v X_{t,s}^\gamma := \lim_{\varepsilon \downarrow 0} \frac{X_{t,s}^{\gamma + \varepsilon v} - X_{t,s}^{\gamma}}{\varepsilon}, s \in [t, T]
$$

exists in $L^j(\Omega \rightarrow C([t,T]; \mathbb{R}^d), \mathbb{P})$, and there exists a constant $c(j) > 0$ such that

$$
\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \sup_{s \in [t,T]} |\nabla_v X_{t,s}^{\gamma}|^j \right] \leq c(j)|v|^j, \quad v \in \mathbb{R}^d,
$$

and for any $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$
\nabla_v \{ \mathbb{E} f(X_{t,s}^\gamma) \}(x) = \frac{1}{s-t} \mathbb{E} \left[ f(X_{t,s}^\gamma) \int_t^s \left( (\sqrt{2aT-r})^{-1}(X_{t,r}^\gamma)\nabla_v X_{t,r}^\gamma, dW_r \right) \right].
$$

By the pathwise uniqueness of (2.6), the solution satisfies the flow property

$$
X_{t,r}^\gamma = X_{s,r}^{\gamma, X_{t,s}^\gamma}, \quad 0 \leq t \leq s \leq r \leq T, \quad x \in \mathbb{R}^d.
$$

Moreover,

$$
\Phi(\gamma)_{t,s,x} := \mathcal{L}X_{t,s}^{\gamma, x}, \quad (t, s, x) \in DT \times \mathbb{R}^d
$$

defines a map $\Phi : \Gamma \rightarrow \Gamma$. If $\Phi$ has a unique fixed point $\bar{\gamma} \in \Gamma$, then (2.6) with $\gamma = \bar{\gamma}$ reduces to (2.1), the well-posedness of (2.6) implies that of (2.1), and the unique solution is given by

$$
X_{t,s}^x = X_{t,s}^{\bar{\gamma}, x}.
$$

Then (2.3), (2.4) and (2.5) follow from (2.9), (2.7) and (2.8) for $\gamma = \bar{\gamma}$ respectively. Therefore, it remains to prove that $\Phi$ has a unique fixed point.

(b) By the fixed point theorem, we only need to find constants $\lambda > 0$ and $\delta \in (0, 1)$ such that

$$
\rho_\lambda(\Phi(\gamma_1), \Phi(\gamma_2)) \leq \delta \rho_\lambda(\gamma_1, \gamma_2), \quad \gamma_1, \gamma_2 \in \Gamma.
$$

Below, we prove this estimate using Girsanov’s theorem.
For $i = 1, 2$, consider the SDE
\[
\begin{align*}
\mathrm{d}X^{i,x}_{t,s} &= \left\{ b^{(1)}(X^{i,x}_{t,s}) + Z_s(X^{i,x}_{t,s}, \gamma^i) \right\} \mathrm{d}s + \sqrt{2a_{T-s}(X^{i,x}_{t,s})} \mathrm{d}W_s, \\
t &\in [0, T], s \in [t, T], X^{i,x}_{t,t} = x \in \mathbb{R}^d.
\end{align*}
\]
By the definition of $\Phi$, we have
\[
(2.11) \quad \Phi(\gamma^i)_{t,s,x} = \mathcal{L}_{X^{i,x}_{t,s}}, \quad i = 1, 2, \ (t, s, x) \in D_T \times \mathbb{R}^d.
\]
Let
\[
\xi_s := \left( \sqrt{2a_{T-s}(X^{1,x}_{t,s})} \right)^{-1} \left\{ Z_s(X^{1,x}_{t,s}, \gamma^1) - Z_s(X^{1,x}_{t,s}, \gamma^2) \right\}, \quad s \in [t, T].
\]
By (A), there exists a constant $K > 0$ such that
\[
(2.12) \quad \|\xi_s\| \leq Kg_s \sup_{(r, x) \in [s, T] \times \mathbb{R}^d} \|\gamma^1_{s, r, x} - \gamma^2_{s, r, x}\|_{\text{var}}.
\]
By Girsanov theorem,
\[
\bar{W}_s := W_s - \int_t^s \xi_r \mathrm{d}r, \quad s \in [t, T]
\]
is a Brownian motion under the weighted probability $Q_t := R_t \mathbb{P}$, where
\[
R_t := e^{\int_t^T \langle \xi_s \mathrm{d}W_s \rangle - \frac{1}{2} \int_t^T |\xi_s|^2 \mathrm{d}s}.
\]
With this new Brownian motion, the SDE for $X^1$ becomes
\[
\begin{align*}
\mathrm{d}X^{1,x}_{t,s} &= \left\{ b^{(1)}(X^{1,x}_{t,s}) + Z_s(X^{1,x}_{t,s}, \gamma^2) \right\} \mathrm{d}s + \sqrt{2a_{T-s}(X^{1,x}_{t,s})} \mathrm{d}\bar{W}_s, \quad s \in [t, T].
\end{align*}
\]
By the (weak) uniqueness for the SDE with $i = 2$, we derive
\[
\mathcal{L}_{X^{1,x}_{t,s}\mid Q_t} = \mathcal{L}_{X^{2,x}_{t,s}} = \Phi(\gamma^2)_{t,s,x},
\]
where $\mathcal{L}_{X^{1,x}_{t,s}\mid Q_t}$ is the distribution of $X^{1,x}_{t,s}$ under $Q_t$. Combining this with (2.11), we get
\[
(2.13) \quad \|\Phi(\gamma^1)_{t,s,x} - \Phi(\gamma^2)_{t,s,x}\|_{\text{var}} = \sup_{|f| \leq 1} \left| \mathbb{E}[f(X^{1,x}_{t,s}) - f(X^{1,x}_{t,s}) R_t] \right| \leq \mathbb{E}[R_t - 1].
\]
By Pinsker’s inequality and the definition of $R_t$, we obtain
\[
(2.14) \quad (\mathbb{E}[R_t - 1])^2 \leq 2\mathbb{E}[R_t \log R_t] = 2\mathbb{E}_{Q_t}[\log R_t] = 2\mathbb{E}_{Q_t} \int_t^T |\xi_s|^2 \mathrm{d}s,
\]
where $\mathbb{E}_{Q_t}$ is the expectation under the probability $Q_t$. Combining (2.13) and (2.14) with (2.12), and using the definition of $\rho_\lambda$, we arrive at
\[
\|\Phi(\gamma^1)_{t,s,x} - \Phi(\gamma^2)_{t,s,x}\|_{\text{var}} \leq \left( 2K^2 \int_t^T g_s^2 \sup_{(r,y) \in [s, T] \times \mathbb{R}^d} \|\gamma^1_{s, r, y} - \gamma^2_{s, r, y}\|^2_{\text{var}} \mathrm{d}s \right)^{\frac{1}{2}}.
\]
\[ \leq \rho(\gamma_1, \gamma_2) \left( 2K^2 \int_t^T g(s) e^{2\lambda(T-s)} ds \right)^\frac{1}{2}, \quad (t, x) \in [0, T] \times \mathbb{R}^d. \]

Therefore
\[ \rho(\Phi(\gamma_1), \Phi(\gamma_2)) \leq \varepsilon \rho(\gamma_1, \gamma_2), \]
where
\[ \varepsilon := \sup_{t \in [0, T]} \left( 2K^2 \int_t^T g(s) e^{-2\lambda(s-t)} ds \right)^\frac{1}{2} \downarrow 0 \text{ as } \lambda \uparrow \infty. \]

By taking large enough \( \lambda > 0 \), we prove (2.10) for some \( \delta < 1 \).

For later use we present the following consequence of Theorem 2.1.

**Corollary 2.2.** Assume (A) and let
\[ P_{t,s} f(x) := \mathbb{E}[f(X_{t,s}^x)], \quad (t, s, x) \in D_T \times \mathbb{R}^d. \]

Then there exists a constant \( c > 0 \) such that for any function \( f \),
\[ \| \nabla P_{t,s} f \|_\infty \leq c \min \left\{ (s-t)^{-\frac{1}{2}} \| f \|_\infty, \| \nabla f \|_\infty \right\}, \]
\[ \| \nabla^2 P_{t,s} f \|_\infty \leq c (s-t)^{-\frac{1}{2}} \| \nabla f \|_\infty, \quad 0 \leq t < t \leq T. \]

**Proof.** By (2.5) we have
\[ \| \nabla P_{t,s} f \|_\infty \leq c(t-s)^{-\frac{1}{2}} \| f \|_\infty \]
for some constant \( c > 0 \). Next, by chain rule and (2.4),
\[ |\nabla P_{t,s} f(x)| = |\mathbb{E}[\langle \nabla f(X_{t,s}^x), \nabla X_{t,s}^x \rangle]| \leq c \| \nabla f \|_\infty, \quad (t, s, x) \in D_T \times \mathbb{R}^d \]
holds for some constant \( c > 0 \). Moreover,
\[ \nabla P_{t,s} f(x) = \mathbb{E}[\langle \nabla f(X_{t,s}^x), \nabla X_{t,s}^x \rangle] = \mathbb{E}[g(X_{t,s}^x)], \]
where \( g(X_{t,s}^x) := \langle \nabla f(X_{t,s}^x), \nabla X_{t,s}^x \rangle. \) Combining this with (2.5) and (2.4), we find a constant \( c > 0 \) such that
\[ \| \nabla^2 P_{t,s} f(x) \| \leq |\mathbb{E}[g(X_{t,s}^x)]| \]
\[ \leq \frac{1}{s-t} \mathbb{E}\left[ g(X_{t,s}^x) \cdot \int_s^t \left( (\sqrt{2(aT-r)})^{-1} (X_{t,r}^x)^n \right) \nabla X_{t,r}^x, dW_r \right] \]
\[ \leq \frac{1}{t-s} (\mathbb{E}[g(X_{t,s}^x)]^2)^\frac{1}{2} \left( \mathbb{E} \int_t^s \| a^{-1} \|_\infty \| \nabla X_{t,r}^x \|_\infty dr \right)^\frac{1}{2} \leq c \| \nabla f \|_\infty. \]

Then the proof is finished.
3 Proofs of Theorem 1.3(2)-(3)

We will need the following lemma implied by Theorem 2.1, Theorem 3.1, Lemma 3.3, see also and references within for the case \( b^{(1)} = 0 \).

**Lemma 3.1.** Assume (A)(1), (A)(3) and \( b^{(0)}\|_{L^{p_0}} < \infty \) for some \( (p_0, q_0) \in \mathcal{K} \). Let \( \sigma_t = \sqrt{2u_t} \). Then the following assertions hold.

1. For any \( p, q > 1, \lambda \geq 0, 0 \leq t_0 < t_1 \leq T \) and \( f \in \tilde{L}^p(t_0, t_1) \), the PDE
   \[
   \begin{align*}
   (\partial_t + L_t)u_t &= \lambda u_t + f_t, \quad t \in [t_0, t_1], u_{t_1} = 0, \\
   \end{align*}
   \]
   has a unique solution in \( \tilde{H}^{2,p}(t_0, t_1) \). If \( (2p, 2q) \in \mathcal{K} \), then there exist a constant \( c > 0 \) such that for any \( 0 \leq t_0 < t_1 \leq T \) and \( f \in \tilde{L}^p(t_0, t_1) \), the solution satisfies
   \[
   \|u\|_\infty + \|\nabla u\|_\infty + \|(\partial_t + \nabla b^{(1)})u\|_{\tilde{L}^p(t_0, t_1)} + \|\nabla^2 u\|_{\tilde{L}^p(t_0, t_1)} \leq c\|f\|_{\tilde{L}^p(t_0, t_1)}.
   \]

2. Let \( (X_t)_{t \in [0, T]} \) be a continuous adapted process on \( \mathbb{R}^d \) satisfying
   \[
   \begin{align*}
   (3.2) \quad X_t &= X_0 + \int_0^t b_s(X_s)ds + \int_0^t \sigma_s(X_s)dW_s, \quad t \in [0, T].
   \end{align*}
   \]
   For any \( p, q > 1 \) with \( (2p, 2q) \in \mathcal{K} \), there exists a constant \( c > 0 \) such that for any \( X_t \) satisfying \( (3.2) \),
   \[
   \mathbb{E}\left( \int_t^s |f_r(X_r)|dr \bigg| \mathcal{F}_t \right) \leq c\|f\|_{\tilde{L}^p(t, s)}, \quad (t, s) \in D_T, f \in \tilde{L}^p(t, s).
   \]

3. Let \( p, q > 1 \) with \( \frac{d}{p} + \frac{2}{q} < 1 \). For any \( u \in \tilde{H}^{2,p} \) with \( \|(\partial_t + b^{(1)})u\|_{\tilde{L}^{p_0}} < \infty \), \( \{u_t(X_t)\}_{t \in [0, T]} \) is a semimartingale satisfying
   \[
   du_t(X_t) = L_tu_t(X_t)dt + (\nabla u_t(X_t), \sigma_t(X_t)dW_t), \quad t \in [0, T].
   \]

In the following we consider \( E = \mathbb{R}^d \) and \( T^d \) respectively.

3.1 \( E = \mathbb{R}^d \)

**Proof of Theorem 1.3(2).** Let \( u \in \mathcal{U}(p_0, q_0) \) solve (1.7). Then
\[
\begin{align*}
(3.3) \quad u \in \tilde{H}^{2,p_0}, \quad \|(\partial_t + b^{(1)} \cdot \nabla)u\|_{\tilde{L}^{p_0}} < \infty
\end{align*}
\]
as required by Lemma 3.1(3). It remains to prove (1.10), which together with Corollary 2.2 implies (1.11).

Let
\[
\begin{align*}
\mathcal{L} := \text{tr}\{a_{T-t}\nabla^2\} + \tilde{b}_t \cdot \nabla, \\
\tilde{b}_t(x) := b_{T-t}(x) - \mathbb{E}u_0(X^x_{T,T}) - \mathbb{E}\int_t^T V_{T-s}(X^x_{t,s})ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d.
\end{align*}
\]
Since \( \| u_0 \|_\infty + \int_0^T \| V_t \|_\infty dt < \infty \), \( \| b^{(0)} \|_{L_0^p} < \infty \) implies \( \tilde{b}_t(x) := b^{(1)}_{T-t}(x) + \tilde{b}^{(0)}(x) \) with \( \| \tilde{b}^{(0)} \|_{L_0^p} < \infty \). Then (A) holds for \( \tilde{b} \) replacing \( b \), so that by (3.3) and Lemma 3.1(3), the following Itô’s formula holds for \( X^x_{t,s} \) solving (1.8):

\[
\frac{d u_{T-s}(X^x_{t,s})}{d s} = \left( \partial_s + \mathcal{L}_s \right) u_{T-s}(X^x_{t,s}) + \{ \nabla u_{T-s}(X^x_{t,s}) \} \sqrt{2a_{T-s}(X^x_{t,s})} dW_s, \quad s \in [t, T],
\]

where \( (\nabla u)_{i,j} := (\partial_j u^i)_{1 \leq i,j \leq d} \). By (1.7) and (3.4), we obtain

\[
(\partial_s + \mathcal{L}_s) u_{T-s}(X^x_{t,s}) + V_{T-s}(X^x_{t,s}) = \left\{ u_{T-s}(y) - \mathbb{E} u_0(X^y_{s,T}) - \mathbb{E} \int_s^T V_{T-r}(X^y_{s,r}) dr \right\} \cdot \nabla u_{T-s}(X^x_{t,s}).
\]

Combining this with the follow property (2.3) and (3.5), we derive

\[
\mathbb{E} u_0(X^x_{t,T}) - u_{T-t}(x) = \mathbb{E} \left[ u_{T-T}(X^x_{t,l}) - u_{T-t}(X^x_{t,t}) \right] = \mathbb{E} \int_t^T \left\{ (u_{T-s}(y) - \mathbb{E} u_0(X^y_{s,T}) - \mathbb{E} \int_s^T V_{T-r}(X^y_{s,r}) dr) \cdot \nabla \right\} u_{T-s}(X^x_{t,s}) ds
\]

\[- \mathbb{E} \int_t^T V_{T-s}(X^x_{t,s}) ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d.
\]

Letting

\[
h_t := \sup_{x \in \mathbb{R}^d} \left| u_{T-t}(x) - \mathbb{E} u_0(X^x_{t,T}) - \mathbb{E} \int_t^T V_{T-s}(X^x_{t,s}) ds \right|, \quad t \in [0, T],
\]

we arrive at

\[
h_t \leq \int_t^T h_s \| \nabla u \|_\infty ds, \quad t \in [0, T].
\]

By Grownwall’s inequality we prove \( h_t = 0 \) for \( t \in [0, T] \), hence (1.10) holds.

**Proof of Theorem 1.3(3).** (a) Let \( P_{t,s} f = \mathbb{E} f(X^x_{t,s}) \) for \( f \in \mathcal{B}_b(\mathbb{R}^d) \), where \( X^x_{t,s} \) solves (1.8). For \( u \) given by (1.10) we have

\[
(3.6) \quad u_t = P_{T-t,T} u_0 + \int_{T-t}^T P_{T-t,s} V_{T-s} ds, \quad t \in [0, T].
\]

By \( \| u_0 \|_\infty + \int_0^T \| V_t \|_\infty dt < \infty \) and (1.11), we find a constant \( c > 0 \) such that

\[
(3.7) \quad \| u \|_\infty + \| \nabla u \|_\infty \leq c, \quad \| \nabla^2 u_t \|_\infty \leq c t^{-\frac{1}{2}}, \quad t \in (0, T].
\]

Moreover, the SDE (1.8) becomes

\[
(3.8) \quad dX^x_{t,s} = \sqrt{2a_{T-s}(X^x_{t,s})} dW_s + \{ b_{T-s} - u_{T-s} \}(X^x_{t,s}) ds, \quad t \in [0, T], s \in [t, T], X^x_{t,t} = x \in \mathbb{R}^d.
\]
and the generator in (3.4) reduces to
\[ \mathcal{L}_s := \text{tr}\{a_{T-s} \nabla^2 \} + \{b_{T-s} - u_{T-s}\} \cdot \nabla, \quad s \in [0, T]. \]

(b) We prove the Kolmogorov backward equation
\begin{equation}
\partial_t P_{t,s}f = -\mathcal{L}_t P_{t,s}f, \quad f \in \mathcal{C}_b^2, \ t \in [0, s], s \in (0, T].
\end{equation}
For any \( f \in \mathcal{C}_b^2 \), by Itô’s formula we have
\begin{equation}
P_{t,s}f(x) = f(x) + \int_t^s P_{t,r}(\mathcal{L}_r f)(x)dr, \quad (t, s) \in D_T,
\end{equation}
where \( \int_t^s P_{t,r}(\mathcal{L}_r f)(x)dr = \mathbb{E} \int_t^s \mathcal{L}_r f(X_{t,r}^x)dr \) exists, since Krylov’s estimate in Lemma 3.1(2) holds under (A) and \( \|u\|_{\infty} < \infty \).

By (3.10), we obtain the Kolmogorov forward equation
\begin{equation}
\partial_s P_{t,s}f = P_{t,s}(\mathcal{L}_s f), \quad s \in [t, T].
\end{equation}
On the other hand, \( b^{(1)} = 0 \) and (A) imply
\begin{equation}
\|\mathcal{L}_f\|_{\dot{L}_p^{00}} \leq c_0\|f\|_{\mathcal{C}_b^2}
\end{equation}
for some constant \( c_0 > 0 \). By Lemma 3.1(1), for any \( s \in (0, T] \), the PDE
\begin{equation}
(\partial_t + \mathcal{L}_t)\tilde{u}_t = -\mathcal{L}_t f, \quad t \in [0, s], \tilde{u}_s = 0
\end{equation}
has a unique solution \( \tilde{u} \in \mathcal{C}(p_0, q_0) \), such that
\begin{equation}
\|\nabla^2 \tilde{u}\|_{\dot{L}_p^{00}(0, s)} \leq c_1\|\mathcal{L}_f\|_{\dot{L}_p^{00}(0, s)}
\end{equation}
holds for some constant \( c_1 > 0 \) independent of \( s \). By Itô’s formula in Lemma 3.1(3),
\[ d\tilde{u}_t(X_{0,t}^x) = -\mathcal{L}_t f(X_{0,t}^x) + \langle \nabla f(X_{0,t}^x), \sqrt{2a_{T-t}}(X_{0,t}^x)dW_t \rangle, \ t \in [0, s]. \]
This and (3.11) imply
\[ 0 = \tilde{u}_s(x) = \tilde{u}_t(x) - \int_t^s (P_{t,r} \mathcal{L}_r f)(x)dr, \]
\[ = \tilde{u}_t(x) - \int_t^s \frac{d}{dr}(P_{t,r}f)(dr) = \tilde{u}_t(x) - P_{t,s}f(x) + f(x), \ t \in [0, s]. \]
Thus,
\begin{equation}
\tilde{u}_t = P_{t,s}f - f, \quad t \in [0, s].
\end{equation}
Combining this with (3.13) we derive (3.9).

(c) By (3.7) and (3.9), we see that \( u \) solves (1.10) with \( u \in \mathcal{C}(p_0, q_0) \) provided
\begin{equation}
\|\nabla^2 u\|_{\dot{L}_p^{00}} < \infty.
\end{equation}
By (3.12), (3.14) and (3.15), we find a constant \( c_2 > 0 \) such that
\[ \sup_{t \in [0, s]} \|\nabla^2 P_{t,s}f\|_{\dot{L}_p^{00}(0, s)} \leq c_2\|f\|_{\mathcal{C}_b^2}, \quad s \in (0, T], f \in \mathcal{C}_b^2. \]
Combining this with (3.6), \( b^{(1)} = 0 \) and \( \|u_0\|_{\mathcal{C}_b^2} + \int_0^T \|V_t\|_{\mathcal{C}_b^2}dt < \infty \), we prove (3.10).

\[ \square \]
3.2 \( E = \mathbb{T}^d \)

In this case, all functions on \( E \) are extended to \( \mathbb{R}^d \) as in (1.3), so that the proof for \( E = \mathbb{R}^d \) works also for the present setting if we could verify the following periodic property for the solution of (1.8):

\[
X_{t,s}^{x+k} = X_{t,s}^x + k, \quad (t, s) \in D_T, \quad x \in \mathbb{R}^d, \quad k \in \mathbb{Z}^d.
\]

Let \( \tilde{X}_{s,t}^x := X_{t,s}^x + k. \) Since the coefficients of (1.8) satisfies (1.3), \( \tilde{X}_{t,s}^x \) solves (1.8) with \( \tilde{X}_{t,t}^x = x + k. \) By the uniqueness of (1.8) ensured by Theorem 1.3(1), we derive (3.17).

4 Proof of Theorem 1.1

We only prove for \( E = \mathbb{R}^d \) as the case for \( E = \mathbb{T}^d \) follows by extending functions from \( \mathbb{T}^d \) to \( \mathbb{R}^d \) as in (1.3).

Let \( I_d \) be the \( d \times d \) identity matrix. By Theorem 1.3 with \( b = 0, a = \kappa I_d \) and \( V = -\nabla \phi \), for any \((p_0, q_0) \in \mathcal{X}, (1.1) \) has a unique solution in the class \( \mathcal{U}(p_0, q_0) \), and by (1.5),

\[
u_t(x) := \mathbb{E} u_0(X_{T-t,T}^x) - \mathbb{E} \int_{T-t}^T \nabla \phi_{T-s}(X_{T-t,s}^x) ds,
\]

(4.1) \[
P_t = \mathbb{E} X_{T-t,T}^x u_0(x) - \int_{T-t}^T P_{T-t,s} \nabla \phi_{T-s}(x) ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d.
\]

By (3.8) for the present \( a \) and \( b \), \( X_{t,s}^x \) solves the SDE

\[
dX_{t,s}^x = \sqrt{2\kappa} W_s - \nu_{T-s}(X_{t,s}^x) ds, \quad X_{t,t}^x = x, \quad t \in [0, T], \quad s \in [t, T],
\]

and the generator is

\[
\mathcal{L}_s := \kappa \Delta - \nu_{T-s} \cdot \nabla, \quad s \in [0, T].
\]

It remains to prove (1.4). To this end, we present the following lemma.

Lemma 4.1. Let \( P_{t,s} f := \mathbb{E}[f(X_{t,s}^x)] \) for the SDE (1.2). Let \( m \geq 1 \) such that

(4.3) \[
\sup_{t \in [0, T]} \|u_t\|_{\mathcal{E}_{b}^m} + \|f\|_{\mathcal{E}_{b}^{m+1}} < \infty,
\]

then \( \sup_{(t,s) \in D_T} \|P_{t,s} f\|_{\mathcal{E}_{b}^{m+1}} < \infty. \)

Proof. By (4.2) and \( \sup_{t \in [0, T]} \|u_t\|_{\mathcal{E}_{b}^m} < \infty \), we have

\[
\sup_{(t,s,x) \in D_T \times \mathbb{R}^d} \mathbb{E}[\|\nabla^i X_{t,s}^x\|] < \infty, \quad 1 \leq i \leq m.
\]

By chain rule, this implies that for some constant \( c_0 > 0, \)

(4.4) \[
\sup_{(t,s) \in D_T} \|P_{t,s} g\|_{\mathcal{E}_{b}^m} \leq c_0 \|g\|_{\mathcal{E}_{b}^m}, \quad g \in \mathcal{E}_{b}^m.
\]
Let $P^0_t = e^{\kappa \Delta t}$. By $\partial_r P^0_{r-t} = P^0_{r-t}(\kappa \Delta)$ and (3.9), we have
$$
\partial_r P^0_{r-t} P_{s,t} f = P^0_{r-t} \langle \nabla P_{s,t} f, u_{T-r} \rangle, \quad r \in [t,s].
$$
So,
$$
P_{s,t} f = P^0_{s-t} f - \int_t^s P^0_{r-t} \langle \nabla P_{r,s} f, u_{T-r} \rangle \, dr.
$$
It is well known that for any $\alpha, \beta \geq 0$ there exists a constant $c_{\alpha, \beta} > 0$ such that
$$
\| P^0_t g \|_{\mathcal{C}_{\alpha, \beta}^m} \leq c_{\alpha, \beta} t^{-\frac{\beta}{2}} \| g \|_{\mathcal{C}_{\beta}^m}, \quad t > 0, g \in \mathcal{C}_{\beta}^m.
$$
This together with (4.5) implies that for some constants $c_1, c_2 > 0$,
$$
\| P_{s,t} f \|_{\mathcal{C}_{\beta}^{m+\frac{1}{2}}} \leq c_1 \| f \|_{\mathcal{C}_{\beta}^{m+\frac{1}{2}}} + c_1 \int_t^s (t + r - s)^{-\frac{3}{4}} \| \langle \nabla P_{r,s} f, u_{T-r} \rangle \|_{\mathcal{C}_{\beta}^{m-1}} \, dr.
$$
Combining this with (4.1) and $\| f \|_{\mathcal{C}_{\beta}^m} + \sup_{t \in [0, T]} \| u_t \|_{\mathcal{C}_{\beta}^m} < \infty$, we obtain
$$
\sup_{(t,s) \in D_T} \| P_{s,t} f \|_{\mathcal{C}_{\beta}^{m+\frac{1}{2}}} < \infty.
$$
By this together with (4.5) and (4.3), we find a constant $c_2 > 0$ such that
$$
\sup_{(t,s) \in D_T} \| P_{s,t} f \|_{\mathcal{C}_{\beta}^{m+1}} \leq c_2 \| f \|_{\mathcal{C}_{\beta}^{m+1}}
$$
$$
+ c_2 \int_t^s (t + r - s)^{-\frac{3}{4}} \| \langle \nabla P_{r,s} f, u_{T-r} \rangle \|_{\mathcal{C}_{\beta}^{m-\frac{3}{4}}} \, dr < \infty.
$$
We now prove (1.4) as follows. By $u \in \mathcal{U}(p_0, q_0)$, we have
$$
\| u \|_{\mathcal{C}^\infty} + \| \nabla u \|_{\mathcal{C}^\infty} < \infty.
$$
Combining this with (4.1) and Lemma 4.1, we prove (1.4) by inducing in $m$ up to $m = n$.

## 5 Proof of Theorem 1.2

Let $u_t$ be given in Theorem 1.1 with $\nabla \cdot u_0 = 0$. Let $h_t = \nabla \cot u_t$. By (1.1) we have
$$
\partial_t h_t = (\kappa \Delta - u_t \cdot \nabla) h_t - \Delta \phi_t - \sum_{i,j=1}^d (\partial_i u^*_t)(\partial_j u^*_t).
$$
Combining this with $\nabla \cdot u_0 = 0$, (1.5), the definition of $Q^i_t \phi$ and (1.6), we obtain
$$
\partial_t h_t = (\kappa \Delta - u_t \cdot \nabla + \psi_t) h_t, \quad t \in [0, T], h_0 = 0.
$$
Therefore, $\nabla \cdot u_t = h_t = 0$ holds for all $t \in [0, T]$.

On the other hand, if $\nabla \cdot u_t = h_t = 0$, then (5.1), (5.5) and the definition of $Q^i_t \phi$ imply (1.6).
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