Exceptional and regular spectra of the generalized Rabi model

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We study the spectrum of the generalized Rabi model in which co- and counter-rotating terms have different coupling strengths. It is also equivalent to the model of a two-dimensional electron gas in a magnetic field with Rashba and Dresselhaus spin-orbit couplings. Like in case of the Rabi model, the spectrum of the generalized Rabi model consists of the regular and the exceptional parts. The latter is represented by the energy levels which cross at certain parameters’ values which we determine explicitly. The wave functions of these exceptional states are given by finite order polynomials in the Bargmann representation. The roots of these polynomials satisfy a Bethe ansatz equation of the Gaudin type. At the exceptional points the model is therefore quasi-exactly solvable. An analytical approximation is derived for the regular part of the spectrum in the weak- and strong-coupling limits. In particular, in the strong-coupling limit the spectrum consists of two quasi-degenerate equidistant ladders.

I. INTRODUCTION

The Rabi model \[^1\] is a fundamental model of light-matter interaction. It describes a single-mode photonic field interacting with a single two-level emitter,

\[
\hat{H}_R = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{\sigma}_z + g (\hat{a} + \hat{a}^\dagger) (\hat{\sigma}_+ + \hat{\sigma}_-),
\]

where the bosonic operators \(\hat{a}, \hat{a}^\dagger\) describe photons, and \(\hat{\sigma}_\mu, \mu = z, \pm\), are the Pauli matrices describing a two-level emitter. When the coupling strength \(g\) is small \(\sim 10^{-2}\) and the near-resonance condition is satisfied, \(\omega \sim 2\omega_0\), it is legitimate to make the Rotating Wave Approximation (RWA) by neglecting the counter-rotating terms \(\hat{a}\hat{\sigma}_-\) and \(\hat{a}^\dagger\hat{\sigma}_+\). In this case, known as the Jaynes-Cummings (JC) model \[^2\], the operator of the total number of excitations \(\hat{N}_e = \hat{a}^\dagger \hat{a} + \hat{\sigma}_z/2\) is a conserved quantity which ensures exact solvability of the JC model. On the other hand, in the Rabi model the only conserved quantity is the parity \(\hat{\Pi} = \hat{\sigma}_z(-1)^{\hat{a}^\dagger \hat{a}}\). The question of exact solvability of the Rabi model has been debated for a long time, and the recent renewal of interest to the subject \[^3\], \[^4\] has been motivated by the rapid experimental progress in quantum optics. Several regimes of the Rabi model \[^1\] are usually distinguished in the literature depending on the coupling strength or the detuning \(\Delta = \omega - 2\omega_0\). In terms of a dimensionless parameter \(\eta = g/\omega\) these are: (i) weak-coupling regime, when the JC model is applicable, \(\eta \sim 10^{-2}\); (ii) strong-coupling regime, \(10^{-2} < \eta < 0.1\); (iii) ultra-strong-coupling regime, \(0.1 < \eta < 1\), and (iv) deep strong-coupling regime \(\eta > 1\). If the detuning is sufficiently large so that \(|\Delta| \sim \omega + 2\omega_0\), the RWA breaks down even for a relatively weak coupling. Provided that couplings of the field and the emitter to dissipative baths (\(\gamma\) and \(\Gamma\), respectively) are included, it is often assumed that the cooperativity factor \(\xi = g^2/\gamma\Gamma\) is large enough to ensure almost coherent short time evolution. It is worth noting that the standard weak-coupling master (Lindblad) equation approach to dissipative dynamics in the strong-coupling regime should be taken with caution \[^5\]. Namely, the reduced density matrix equation should be expressed in terms of exact eigenstates of an isolated subsystem. This calls for detailed studies of the spectrum in different limits (i)-(iv). Experimentally, the weak-coupling regime is achieved in cavity QED setups \[^6\], while the regimes up to the ultra-strong coupling have been recently accessed using circuit QED systems \[^7\], \[^8\].

The analytical solution of the Rabi model in terms of transcendental functions has been found recently in Ref. \[^3\]. On the other hand, several analytical approximations are also available. Thus, uniformly approximate results for energy levels valid in the whole range of parameters were found in \[^9\]; also known are the approximation based on the polaron-like transformation, which is valid in the intermediate coupling (Bloch-Siegert) regime \[^10\], the adiabatic approximation valid in the strong-coupling regime \[^11\], and the deep-strong coupling approximation \[^12\].

A complementary information on the spectrum of the Rabi model is provided by the quasi-exact solutions (QES). Indeed, it was observed \[^13\], \[^14\], \[^15\], \[^16\], \[^17\], \[^18\] that the spectrum of the Rabi model has both regular and exceptional pieces. The exceptional parts of the spectrum are those whose wave-functions are finite-order polynomials in the Bargmann representation. The energies of the exceptional solution are integer-valued \(E = n\omega - g^2/\omega\), where for every \(n\) there is a special ( polynomial) condition on the model parameters for which this solution is valid. It was proven in \[^16\] that two neighboring energy levels cross on parallel straight lines \(E = n\omega - g^2/\omega\) in the parameter space. Moreover, for each \(n\) the number of such crossings is precisely \(n\), and there are no other crossings away from these lines. The connection of exceptional solutions with the concept of quasi-exact solvability (see \[^19\] for an extensive
review and references) has been discussed in [18]. We also note that in the quasiclassical regime the model exhibits chaotic behavior, and the exceptional solutions correspond to the isolated set of periodic orbits [20].

In this paper we study the generalized Rabi model

\[ \hat{H}_{gR} = \omega \hat{a}^{\dagger} \hat{a} + \omega_0 \hat{\sigma}_z + g_1 \left( \hat{a}^{\dagger} \hat{\sigma}_+ + \hat{a} \hat{\sigma}_- \right) + g_2 \left( \hat{a}^{\dagger} \hat{\sigma}_- + \hat{a} \hat{\sigma}_+ \right) \]

which interpolates between the JC model (for \( g_2 = 0 \)) and the original Rabi model \( g_1 = g_2 \). There are several motivations to consider this model. First, as observed in [21] it can be mapped onto the model describing a two-dimensional electron gas with Rashba \((\alpha_R \sim g_1)\) and Dresselhaus \((\alpha_D \sim g_2)\) spin-orbit couplings subject to a perpendicular magnetic field (the Zeeman splitting thereby equals \(2\omega_0\)). The Rashba spin-orbit coupling can be tuned by an applied electric field while the Zeeman term is tuned by an applied magnetic field. This allows us to explore the whole parameter space of the model. Second, the model can directly emerge in quantum optics in the context of cavity QED [22] beyond the dipole approximation.

Here we describe the exceptional solutions of the model [2]. On analogy with the Rabi model we find that the exceptional points corresponding to doubly-degenerate levels crossings in the parameter space \((g_1, g_2, \omega_0, \omega)\). These degeneracies (intersection points) form curves whose equations can be determined from a set of algebraic conditions. The level intersections occur only at integer values of the energy, and no intersections are observed elsewhere. We discuss several interesting links between the structure of the exceptional solutions and quasi-exact solvability, and the Gaudin-type Bethe ansatz solvable models.

Moreover, we analyze the weak- and strong-coupling limits of the regular spectrum. In particular, we show that in the strong-coupling limit the spectrum consists of two quasi-degenerate equidistant ladders. We supplement our analytical study by comprehensive numerical calculations.

II. EXCEPTIONAL SOLUTIONS OF THE INHOMOGENEOUS RABI HAMILTONIAN

A. Hamiltonian in the Bargmann representation

The generalized Rabi Hamiltonian is defined as

\[ \hat{H}_{gR} = \omega \hat{a}^{\dagger} \hat{a} + \omega_0 \hat{\sigma}_z + g_1 \left( \hat{a}^{\dagger} \hat{\sigma}_+ + \hat{a} \hat{\sigma}_- \right) + g_2 \left( \hat{a}^{\dagger} \hat{\sigma}_- + \hat{a} \hat{\sigma}_+ \right) \]

\[ = \omega \hat{a}^{\dagger} \hat{a} + \omega_0 \hat{\sigma}_z + \hat{a}^{\dagger} (g_1 \hat{\sigma}_+ + g_2 \hat{\sigma}_-) + \hat{a} (g_1 \hat{\sigma}_- + g_2 \hat{\sigma}_+) \]

(3)

(4)

First we apply the following transformation to the Hamiltonian

\[ \hat{P} = \begin{pmatrix} \frac{-1}{2} \sqrt{g_2} & \frac{1}{2} \\ \frac{1}{2} \sqrt{g_2} & \frac{1}{2} \end{pmatrix}, \quad \hat{P}^{-1} = \begin{pmatrix} \frac{-\sqrt{g_2}}{1} & \frac{\sqrt{g_2}}{1} \\ \frac{\sqrt{g_2}}{1} & \frac{-\sqrt{g_2}}{1} \end{pmatrix} \]

(5)

Introducing

\[ g_\pm := \sqrt{g_1 g_2}, \quad g_+ := \frac{1}{2} (g_1^2 + g_2^2), \quad g_- := \frac{1}{2} (g_1^2 - g_2^2) \]

(6)

the model reads

\[ \hat{\mathcal{H}} = \hat{P} \hat{H}_{gR} \hat{P}^{-1} = \omega \hat{a}^{\dagger} \hat{a} - \omega_0 \hat{\sigma}_z + \frac{1}{g_\pm} \hat{a}^{\dagger} \left( -g_+ \hat{\sigma}_z + g_- \hat{\sigma}_+ - g_- \hat{\sigma}_- \right) - g_\pm \hat{a} \hat{\sigma}_z \]

(7)

Writing the stationary Schrödinger equation \( \hat{\mathcal{H}} \psi = E \psi \) in the Fock-Bargmann representation

\[ \hat{a} \rightarrow \frac{d}{dz}, \quad \hat{a}^{\dagger} \rightarrow z, \]

(8)

for the two component wave function

\[ \psi(z) = \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix}, \]

(9)
we obtain a system of two first-order linear differential equations for the functions \( \psi_1(z) \) and \( \psi_2(z) \)

\[
(\omega z - g_z) \frac{d\psi_1(z)}{dz} - \left( \frac{g_+ z + E}{g_z} \right) \psi_1(z) + \left( \frac{g_- z - \Omega}{2 g_z} \right) \psi_2(z) = 0,
\]

\[
(\omega z + g_z) \frac{d\psi_2(z)}{dz} + \left( \frac{g_+ z - E}{g_z} \right) \psi_2(z) - \left( \frac{g_- z + \Omega}{2 g_z} \right) \psi_1(z) = 0.
\]

We note that the pair \((\psi_2(-z), \psi_1(-z))^T\) satisfies the same set of equations. Therefore two linearly independent spinor solutions of Eqs. (10) and (11) can be found in the form \((\psi_1(z), \psi_1(-z))^T\) and \((\psi_2(z), -\psi_2(-z))^T\). Differentiating Eq. (10) one more time and eliminating \(\psi_2(z)\) from Eq. (11) and \(\psi_2'(z)\) from Eq. (10), we get a second order differential equation for \(\psi_1(z)\). Applying the following substitution

\[\psi_1(z) = \exp\left(-\frac{g_z}{\omega} z\right) \chi(z),\]

we obtain the following differential equation for \(\chi(z)\) in the short hand form

\[A_3(z) \chi'' + B_3(z) \chi' + C_2(z) \chi = 0,\]

where the polynomials are

\[A_3(z) = \sum_{j=0}^{3} a_n z^n, \quad B_3(z) = \sum_{j=0}^{3} b_n z^n, \quad C_2(z) = \sum_{j=0}^{2} c_n z^n.\]

To write down the explicit form of the coefficients of these polynomials it is convenient to introduce dimensionless quantities

\[\delta := \frac{\omega_0}{\omega}, \quad \lambda_\pm := \frac{g_\pm}{\omega^2}, \quad \nu := \frac{g_z}{\omega}, \quad e := \frac{E}{\omega}, \quad \epsilon := \frac{E}{\omega} + \frac{g_+}{\omega^2} = e + \lambda_+, \quad l(l+1) = e(e+1) - \delta^2,\]

and to divide the whole equation (13) by \(\omega^3\). The result is

\[a_0 = -\nu^2 \delta, \quad a_1 = \nu \lambda_-, \quad a_2 = \delta, \quad a_3 = -\frac{1}{\nu} \lambda_-,\]

\[b_0 = \nu (\delta - \lambda_- + 2 \nu^2 \delta), \quad b_1 = (\delta - \lambda_- - 2 \delta \epsilon - 2 \nu^2 \lambda_-), \quad b_2 = \frac{2}{\nu} \epsilon \lambda_- - 2 \nu \delta, \quad b_3 = 2 \lambda_-;\]

\[c_0 = -\left[ \delta (\delta^2 - \epsilon^2 + \lambda_+) + \epsilon \lambda_- \right] - \nu^2 (\delta - \lambda_-) - \nu^4 \delta,\]

\[c_1 = -\left[ \frac{\lambda_-}{\nu} l(l+1) + \frac{\delta \lambda_+}{\nu} \right] - \nu (\delta - \lambda_- - 2 \delta \epsilon) + \nu^3 \lambda_-;\]

\[c_2 = -2 \lambda_- \epsilon.\]

It is convenient to rewrite the Eq. (13) as

\[
\left[ \frac{d^2}{dz^2} + \left( \sum_{s=1}^{3} \frac{\nu_s}{z - \rho_s} + \nu_0 \right) \frac{d}{dz} + \frac{D_2(z)}{\prod_{s=1}^{2} (z - \rho_s)} \right] \chi(z) = 0,
\]

where \(D_2(z) = \sum_{s=0}^{2} d_s z^s\) is a polynomial of degree 2 with the coefficients given by \(d_j = c_j/a_3, \rho_j\) are the zeros of \(A_3(z)\):

\[d_0 = \frac{\nu}{\lambda_-} \left( \delta^3 - \delta \epsilon^2 + 2 \epsilon \delta \lambda_+ - \delta \lambda_+^2 + \delta \lambda_- + \lambda_- \epsilon - \lambda_- \lambda_+ + \nu^2 \delta - \nu^2 \lambda_- - \delta \nu^4 \right),\]

\[d_1 = l(l+1) + \delta \frac{\lambda_+}{\lambda_-} + \nu^2 \delta \epsilon \frac{1}{\lambda_-} - \nu^2 - 2 \nu^2 \delta \epsilon \frac{1}{\lambda_-} - \nu^4,\]

\[d_2 = 2 \nu \epsilon,\]

\[\rho_1 = \frac{g_z}{\omega} \equiv \nu, \quad \rho_2 = -\frac{g_z}{\omega} \equiv -\nu, \quad \rho_3 = \frac{\omega_0 g_z}{g_-} \equiv \frac{\delta \nu}{\lambda_-} \equiv \kappa,\]

\[\nu_1 = -\epsilon + 1, \quad \nu_2 = -\epsilon, \quad \nu_3 = -1, \quad \nu_0 = -\frac{2 g_z}{\omega} \equiv -2 \nu.\]
The differential equation $A_3(z)\chi'' + B_3(z)\chi' + C_2(z)\chi = 0$, has a polynomial solution $\chi(z) = \prod_{i=1}^n (z - z_i)$ of degree $n$, if the coefficients $c_j$ satisfy certain relations which were explicitly found in [23]. In our case they read

\begin{equation}
\begin{aligned}
d_2 &= 2\nu n, \\
d_1 &= 2\nu Z_1 - n \left[ (n-1) + 3 \sum_{s=1}^3 \nu_s + 2\nu \sum_{s=1}^3 \rho_s \right], \\
d_0 &= 2\nu Z_2 - \left[ 2(n-1) + 3 \sum_{s=1}^3 \nu_s + 2\nu \sum_{s=1}^3 \rho_s \right] Z_1 + n(n-1) \sum_{s=1}^3 \rho_s + n \left[ 2\nu \sum_{s<p} \rho_s \rho_p + \sum_{s\neq p\neq q} \nu_s \rho_p \rho_q \right]
\end{aligned}
\end{equation}

where $Z_k = \sum_{i=1}^n z_i^k$ and $z_i$ are the roots of the Bethe ansatz equations

\begin{equation}
\sum_{j\neq i}^n \frac{2}{z_i - z_j} + \sum_{s=1}^3 \frac{\nu_s}{z_i - \rho_s} + \nu_0 = 0,
\end{equation}

or

\begin{equation}
\sum_{j\neq i}^n \frac{2}{z_i - z_j} + \epsilon - 1 + \frac{\epsilon}{z_i - \nu} + \frac{1}{z_i - \kappa} + 2\nu = 0,
\end{equation}

with $i = 1, 2, \ldots, n$. Note also that $\sum_{s=1}^3 \rho_s = \kappa$, $\sum_{s=1}^3 \nu_s = -2\epsilon$, $\sum_{s<p}^3 \rho_s \rho_p = -\nu^2$, and $\sum_{s\neq p\neq q}^3 \nu_s \rho_p \rho_q = -2\kappa - \nu + \kappa$. From the first condition, Eq. (27), $d_2 = 2\nu \epsilon = 2\nu n$, we obtain the allowed energy spectrum,

\begin{equation}
\epsilon = n, \quad \text{or} \quad E = \omega(n - \lambda_+).
\end{equation}

Substituting this into the second and the third conditions, Eqs. (28), (29), gives two quadratic equations for $\lambda_+$ and $\delta$,

\begin{equation}
2\nu Z_1 = \lambda_+^2 - \left( 2n + 1 - \frac{\kappa}{\nu} \right) \lambda_+ - \left( \delta^2 + \nu(\nu - \kappa) + \nu^4 \right),
\end{equation}

\begin{equation}
2\nu^2 Z_2 = -\lambda_+^2 + (2n + 1 - \frac{\kappa}{\nu} + \kappa^2 - \nu^2) \lambda_+ + (\delta^2 + \nu(\nu - \kappa) + \kappa^2 \nu^2 + 2\nu^2(\nu^2 + 1)).
\end{equation}

By adding these two equations we obtain a condition for $\lambda_+$,

\begin{equation}
\lambda_+ = \frac{2\nu Z_1 + 2\nu^2 Z_2 - \nu^2(\kappa^2 + 2n) + \nu^4(1 - 2n)}{\kappa^2 - \nu^2}.
\end{equation}

Using the identity $\lambda_+^2 - \lambda_-^2 = \nu^4$ and that $\lambda_- = \delta \nu / \kappa$ we can express $\lambda_+$ in terms of $\delta$ as $\lambda_+ = \sqrt{\delta^2 \frac{\nu^2}{\kappa^2} + \nu^4}$. Thus, we get a condition for $\delta$:

\begin{equation}
\sqrt{\delta^2 \frac{\nu^2}{\kappa^2} + \nu^4} = \frac{2\nu Z_1 + 2\nu^2 Z_2 - \nu^2(\kappa^2 + 2n) + \nu^4(1 - 2n)}{\kappa^2 - \nu^2}.
\end{equation}

**B. Analysis of the spectrum: exceptional case**

The procedure of determining the polynomial part of the spectrum of the generalized Rabi model is the following: by fixing the number of nodes $n$ of the wave function together with parameters $\nu$ and $\kappa$ we (in general numerically) solve the Bethe ansatz equations [31] to determine $Z_1$ and $Z_2$. Then the allowed values of $\lambda_+$ or $\delta$ are determined according to (35) or (36). It is enough to determine either $\lambda_+$ or $\delta$ since they are related by the expression $\lambda_+ = \sqrt{\delta^2 \frac{\nu^2}{\kappa^2} + \nu^4}$. Remembering that $\lambda_- = \delta \nu / \kappa$ allows us to determine the explicit values of $g_1$, $g_2$, $\omega$, and $\omega_0$ for which the polynomial form of the wave function exists. We note that the polynomial solutions obtained for the exceptional part of the spectrum can be related to the generalized Heine-Stieltjes polynomials [24].

The condition for an existence of polynomial solutions implies that the two solutions mentioned after Eq. (31) are degenerate with the eigenenergies given by (32). Away from these exceptional points these degeneracies are lifted.

It is interesting to realize that the Bethe ansatz equations (31) have the same form as those for the reduced BCS (Richardson) model having three degenerate levels of energies $\rho_{1,2,3}$ with degeneracies $\nu_{1,2,3}$ respectively. This
corresponding physical model is integrable and can be derived from the generalized Gaudin models (see, e.g., \[25\] for review). Interestingly, the energy of that reduced BCS model is proportional to $\sum_{\omega} |\alpha_\omega|^2$. In general, the Bethe equations can be analyzed using the mapping to the Riccati hierarchy \[26\]. The case of $\kappa^2 = \nu^2$ requires special attention. In this case $\delta = \pm \lambda_-$ the Bethe ansatz equations are reduced to the one corresponding to the degenerate two-step model \[27\]. Namely, when $\delta = -\lambda_-$ (so that $\kappa = -\nu$) the three roots $\rho_{1,2,3}$ degenerate into two (namely to $\pm \nu$) and moreover, the polynomial $D_2(\nu)$ is factorized as $D_2(\nu) = (\nu + \nu)(2\nu^2 + d_0/\nu)$ which simplifies the differential and the Bethe ansatz equations. The corresponding conditions are given in Ref. \[23\], Eqs. A.12-A.14 for $\sigma = 0$.

One of the central results of this paper is that these points in the parameter space precisely match the exceptional solutions found above, with the number $2n$ coming from the different solutions to the Bethe ansatz equations for $n \neq 0$. The case of $n = 0$ gives one solution to the equation \[36\]. These observations are supplemented by various numerical checks (see below). Indeed, solutions corresponding to the exceptional points are precisely those where the level crossings occur. The number of different solutions to the Bethe ansatz equations is $2n$ (this is equivalent to a number of different ways to distribute $n$ particles into 3 boxes (levels $\rho_s$) of different capacities (degeneracies $\rho_s = 1, n - 1, n$). It appears that the case $n = 0$ should be treated separately since in this case there are no Bethe roots $z_i$. Nevertheless, Eq. \[36\] has the solution $\kappa^2 = \nu^2$ giving one crossing line with $\chi(z) = \text{const}$. Therefore, the total number of exceptional solutions with $\epsilon = n$ is equal to $2n$ for $\delta > 0$ and $n > 0$ and one for $n = 0$. Since the Eq. \[36\] is quadratic in $\delta$ we also expect additional $2n$ exceptional solutions for $\delta < 0$ (for $n > 0$). Therefore there are $2 \times 2n$ exceptional points in the whole range of parameters (for $n > 0$) and 2 for $n = 0$.

Using numerical diagonalization we plot the spectrum of the generalized Rabi model in Fig. \[8\] for a range of coupling parameters. These calculations fully confirm our expectations about the number and positions of exceptional points. Below we analyze several examples of the solution in more details.

1. Several examples

Let us first consider $\epsilon = n = 0$. In this case the Bethe ansatz equations degenerate, $Z_1 = Z_2 = 0$ and the two conditions Eq. \[34\] and Eq. \[35\] are satisfied simultaneously as soon as $\kappa = \nu$, that is when $\lambda_- = \delta$ (by noticing that $\lambda_+^2 - \delta^2 - \nu^4 = 0$). In Fig. \[1\] we show the two lowest eigenenergies of $\hat{H}_{\text{R}}^\nu + \gamma_1 + \gamma_2$ as a function of $g_1$ and $g_2$. They cross precisely in the plane $E = 0$ and on the curves $\lambda_- = \delta$ (bold red line). In terms of the original parameters this curves are given by $g_1^2 - g_2^2 = 2\omega_0$.

![FIG. 1: Plot of the first and the second lowest energies of $\hat{H}_{\text{R}}^\nu$ shifted by a constant $\gamma_1^2 + \gamma_2^2$, as a function of $g_1$ and $g_2$ for $\omega = \omega_0 = 1$. The energies were calculated by numerical diagonalization. The bosonic Hilbert space $\{\langle n \rangle\}$ was truncated by $n_{\text{max}} = 200$. The yellow plane corresponds to the second energy level and the cyan plane to the first. They cross exactly in the plane of $E = 0$. The lines on which these energy levels cross is given by $g_1^2 - g_2^2 = 2\omega_0$ (red line) as predicted by the quasi-exact solutions.](image)

The second case we explicitly study is $\epsilon = n = 1$, which corresponds to two crossings of energy levels at $E = \omega$ (one crossing of the fourth and the third levels and one crossing of the second and the first levels). In this case there
is only one Bethe ansatz equation

$$\frac{1}{z_1 + \nu} + \frac{1}{z_1 - \kappa} + 2\nu = 0,$$

having the two roots

$$z_{1,\pm} = \frac{\nu(\kappa - \nu) - 1 \pm \sqrt{1 + \nu^2(\kappa + \nu)^2}}{2\nu}.\quad(38)$$

First we insert the Bethe root $z_{1,-}$ into the second condition (33) which yields

$$(3\nu - \kappa)\sqrt{\frac{\delta^2}{\kappa^2} + \nu^2} = 1 + \sqrt{1 + \nu^2(\kappa + \nu)^2} + \delta^2 \left(\frac{\nu^2}{\kappa^2} - 1\right).\quad(39)$$

For the third condition (34) we get

$$\frac{3}{2} + (\nu(\nu - \kappa) + 1)\sqrt{1 + \nu^2(\kappa + \nu)^2} = \delta^2 \left(1 - \frac{\nu^2}{\kappa^2}\right) + (3 - \frac{\kappa}{\nu} + \kappa^2 - \nu^2)\sqrt{\frac{\nu^2}{\kappa^2} + \nu^4 + 2\nu^2}.\quad(40)$$

The red curves in Fig. 2 are obtained in the following way. First we substitute back the expressions for $\kappa$, $\nu$ and $\delta$

$$\delta = \frac{\omega_0}{\omega}, \quad \nu = \frac{\sqrt{g_1g_2}}{\omega}, \quad \kappa = \frac{2\omega_0\sqrt{g_1g_2}}{g_1^2 - g_2^2},\quad(41)$$

into the second and third conditions. Then we fix the values for $\omega$, $\omega_0$ and $g_1$. We solve the second condition for $g_2$ and take only those solutions that satisfy the third condition. This procedure is repeatedly applied by scanning $g_1$ in the range of $[-3.5, 3.5]$ with the step size of 0.01. The purple curves in Fig. 2 are obtained in a similar way using the Bethe root $z_{1,+}$.

![Figure 2: Plot of the fourth (orange), third (green), second (yellow) and first (cyan) energy levels of \( H_{\text{R}} + g_1^2 + g_2^2 \) as a function of \( g_1 \) and \( g_2 \). For fixed \( \omega = 1.0, \omega_0 = 1.0 \) and \( n_{\text{max}} = 200 \).](image)

For $\epsilon = n = 2$ and $\epsilon = n = 3$ we solve the Bethe equations (31) numerically. By fixing $\kappa$ and $\nu$ (and $\omega = 1$) we obtain $Z_1$ and $Z_2$. The values for $\delta$ at which the energy levels cross are then determined according to Eq. (36). The parameters $g_1$, $g_2$ and $\omega_0$ can be expressed in terms of $\kappa$, $\nu$ and $\delta$ as

$$g_1 = \omega \sqrt{\frac{\nu}{\kappa}} \sqrt{\delta + \sqrt{\delta^2 + \kappa^2\nu^2}},\quad(42)$$

$$g_2 = \omega \sqrt{\frac{\kappa}{\nu}} \sqrt{\delta + \sqrt{\delta^2 + \kappa^2\nu^2}},\quad(43)$$

$$\omega_0 = \omega \delta.\quad(44)$$
The BCS coupling is \( g \) then it is convenient to use some known results from the analysis of the Richardson equations. The correspondence is established as follows: the BCS coupling is \( g_{\text{BCS}} = (2\nu^2)^{-1} \). The BCS pair energy levels are given by \( E_{1,2} = \pm 1 \) and 

\[
\sum_{j \neq i}^{n} \frac{2}{x_j - x_i} + \frac{\epsilon - 1}{x_i - 1} + \frac{\epsilon}{x_i + 1} + \frac{1}{x_i - \kappa/\nu} + 2\nu^2 = 0.
\]

Then it is convenient to use some known results from the analysis of the Richardson equations. The correspondence is established as follows: the BCS coupling is \( g_{\text{BCS}} = (2\nu^2)^{-1} \). The BCS pair energy levels are given by \( E_{1,2} = \pm 1 \) and
\( E_3 = \delta/\lambda_- \) with corresponding degeneracies \( 2d_1 = \epsilon - 1, 2d_2 = \epsilon \) and \( 2d_3 = 1 \). The limit \( \nu \to 0 \) corresponds to the strong-coupling limit in the sense of the BCS model. In this case the structure of the roots \( \{x_j\} \) for the Richardson ground state solution (for which all the Bethe roots diverge) reads \[28\]  
\[ x_j = \frac{1}{2\nu^2}y_j + \frac{1}{2n}\left(\frac{\kappa}{\nu} - 1\right) + O(\nu^2), \]  
where \( y_j \) are the roots of the associated Laguerre polynomials \( L_n^{(-1-2\kappa)}(y) \). Representing \( L_n^{(\alpha)}(y) = ((-1)^n/n!) \prod_{j=1}^n(y-y_j) \), one can derive the sum of the roots \( \sum_{j=1}^n y_j = -n\nu \frac{d^{(n-1)}}{dy^{(n-1)}} L_n^{(\alpha)}(y) \big|_{y=0} = nL_1^{(\alpha+n-1)}(0) = n(n+\alpha) \) as well as \( \sum_{j<k} y_jy_k = n(n-1)(n+\alpha)(n+\alpha-1)/2 \) and \( \sum_{j=1}^n y_j^2 = n(n+\alpha)(2n+\alpha-1) \). It follows then  
\[ Z_1 = \frac{-n(n+1)}{2\nu} + \frac{\nu^2}{2\nu} - \frac{\kappa}{\nu} - 1, \]  
\[ Z_2 = \frac{n(n+1)}{2\nu^2} - \frac{n+1}{2}\left(\frac{\kappa}{\nu} - 1\right) + \frac{\nu^2}{4n}\left(\frac{\kappa}{\nu} - 1\right)^2. \]  
For the other solutions one must consider the various combinations of diverging and non-diverging roots. More details are given in \[29\]. Interestingly, in the opposite limit of weak-coupling \( g_{\text{BCS}} \to 0 \) the roots can be expressed in terms of the Laguerre polynomials (see the Refs. \[29\] and \[26\]) and the analytical expressions for \( Z_{1,2} \) can be also found.

In Fig. 5 we illustrate the analytically calculated limit \( \nu \to 0 \) of \( Z_1 \) and \( Z_2 \) (red line) compared with some numerical values (blue dots) for \( n = 5 \) and \( \kappa = 0.1 \). This is consistent with the ground state solutions of the Richardson equations.

![Fig. 5: Comparison of the analytical limit (red line) and numerically calculated (blue dots) values of \( Z_1 \) and \( Z_2 \) for \( n = 5 \), \( \kappa = 0.1 \) and \( \nu \to 0 \).](image)

**C. Regular spectrum: limits**

Here we consider the two limiting cases for the regular part of the spectrum: (i) the limit of either small \( g_1 \) or small \( g_2 \), and (ii) the limit of both large \( g_1 \) and \( g_2 \). We show that in the latter case the spectrum is a superposition of two quasi-degenerate harmonic ladders.

1. **Limit of small \( g_1 \) or small \( g_2 \)**

The effective model in this limit can be obtained by modifying the canonical transformation approach to the Rabi model (see, e.g., \[20\]). We focus here on the case of small \( g_2 \). The effective model has the form of the effective Jaynes-Cummings model with intensity-dependent couplings. First, applying the unitary transformation  
\[ \hat{U}_1 = \exp \left( \Lambda(\hat{a}^\dagger \hat{\sigma}_+ - \hat{a} \hat{\sigma}_-) \right) \]  
(49)
and using the Baker-Campbell-Hausdorff formula up to the second order in \( \Lambda \), \( e^{\hat{D}} e^{-\hat{D}} \approx \hat{D} + \Lambda [\hat{D}, \hat{O}] + \frac{\Lambda^2}{2} [\hat{O}, [\hat{O}, \hat{D}]] \), we obtain

\[
\begin{align*}
\hat{U}_1 \hat{H}_0 \hat{U}_1^\dagger & \approx \hat{H}_0 - 2\lambda (\hat{a}^\dagger \hat{a}^\dagger + \hat{a} \hat{a}) - \Lambda^2 (\hat{\sigma}_z (2\hat{n}_a + 1) - 1), \\
\hat{U}_1 \hat{H}_g_1 \hat{U}_1^\dagger & \approx \hat{H}_g_1 + \Lambda \hat{g}_1 \left( (\hat{a}^\dagger)^2 + \hat{a}^2 \right) \hat{\sigma}_z - \Lambda^2 \hat{g}_1 \left[ (\hat{\sigma}_z \hat{a}^\dagger)^2 + (\hat{\sigma}_z \hat{a})^2 \right] - (\hat{\sigma}_+ \hat{a} \hat{n}_a + \hat{\sigma}_- \hat{a}^\dagger \hat{n}_a), \\
\hat{U}_1 \hat{H}_g_2 \hat{U}_1^\dagger & \approx \hat{H}_g_2 + \Lambda \hat{g}_2 \left[ \hat{\sigma}_z (2\hat{n}_a + 1) - 1 \right] - 2\Lambda^2 \hat{g}_2 \left[ \hat{\sigma}_- \hat{a} \hat{n}_a + \hat{\sigma}_+ \hat{a}^\dagger \hat{n}_a \right],
\end{align*}
\]

where \( \omega \equiv (\omega + 2\omega_0)/2 \), \( \hat{n}_a \equiv \hat{a}^\dagger \hat{a} \), \( \hat{H}_0 = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{\sigma}_z \), and \( \hat{H}_g_{1,2} \) refer to co- and counter-rotating terms. It is easy to notice that if \( \Lambda = \frac{g_2}{2\omega} \) then the second term in the transformed \( \hat{H}_0 \) part cancels the counter-rotating term in \( \hat{H}_g_2 \). The last term in transformed \( \hat{H}_0 \) and the second one in the transformed \( \hat{H}_g_1 \) terms give the Bloch-Siegert shift.

Second, one can apply the transformation \( \hat{U}_2 \equiv \exp(\xi ((\hat{a}^\dagger)^2 - \hat{a}^2 \hat{\sigma}_z)) \) with \( \xi = \Lambda g_1/2\omega \). It further eliminates the non-diagonal terms \( \hat{a}^\dagger \hat{a} \) and \( (\hat{a}^\dagger)^2 \). After that the transformed Hamiltonian up to the order \( g_2^2 \) has the form

\[
\hat{U}_2 \hat{U}_1 \hat{H}_{\text{SR}} \hat{U}_1^\dagger \hat{U}_2^\dagger \approx \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{\sigma}_z + \hat{g}(\hat{n}_a) \hat{a}^\dagger \hat{\sigma}_- + \hat{\sigma}_z \hat{a} + \hat{g}(\hat{n}_a) + \frac{g_2^2}{2\omega} \left( \hat{\sigma}_z (\hat{n}_a + \frac{1}{2}) - \frac{1}{2} \right),
\]

where \( \hat{g}(\hat{n}_a) = g_1 (1 - \Lambda^2 \hat{n}_a) \). The aim of the diagonalization procedure is achieved: the effective Hamiltonian commutes with \( \hat{N}_{\text{ex}} = \hat{a}^\dagger \hat{a} + \hat{\sigma}_z/2 \) and can be easily diagonalized. The eigenenergies can be labeled by the eigenvalues of \( \hat{\sigma}_z \) and \( \hat{\sigma}_+ + 1/2 \) so that the basis is given by \( |N, k \rangle \), \( |N, k \rangle \) \( = (N + k - 1/2) |N, k \rangle \), \( 0, 1 \). Rewriting the Bloch-Siegert term as \( \Lambda^2 \hat{\sigma}_z (\hat{N} + 1/2) - \Lambda^2 \) and noticing that \( \hat{N} |N, k \rangle = (N + k - 1/2) |N, k \rangle \) we get

\[
\begin{align*}
E_{N,k=0,1} &= \omega N - \frac{g_2^2}{2\omega} + (-1)^k \Omega_N - \frac{1}{2}, \\
\Omega_N &= \sqrt{\frac{\Delta_N^2}{4} + \frac{g_N^2}{2\omega}}, \\
\Delta_N &= (2\omega_0 - \omega) + 2N \frac{g_2^2}{2\omega}, \\
g_N &= g_1 \sqrt{N(1 - \Lambda^2 N)}.
\end{align*}
\]

We note that the expected validity range of this approximation is bounded by small values of \( g_{1,2} \sim 10^{-2} \), as discussed in the Introduction. However, we found that the agreement with the numerically calculated spectrum is accidentally good even beyond this regime. This can be seen in Fig. 6 where we show that even for \( g_2 = 10^{-1} \) we have a good agreement of Eq. (51) with the numerical results over a large range of \( g_1 \). The approximation starts to fail considerably around \( g_2 \gtrsim 0.5 \).

![Fig. 6](image.png)

**FIG. 6:** The weak-coupling approximation Eq. (51) of the spectrum of the generalized Rabi model (full lines) compared with the numerical calculation of the spectrum (dots). Note that we added the constant \( \omega \lambda_z = \frac{g_2^2 + g_2^2}{2\omega} \) to the Hamiltonian \( \hat{H}_{\text{SR}} \) such that the level crossings occur at integer values \( n = 0, 1, 2, \ldots \). Further we would like to point out that the seeming crossings at half-integer eigenvalues are in fact anti-crossings which however cannot be resolved on the scale of the figure.
D. Strong-coupling limit

In the strong-coupling regime, when both $g_1$ and $g_2$ are large one can make use of the adiabatic approximation [11]. The idea behind this approximation for the Rabi model is to rotate the basis and to consider the term $\omega \hat{a}^\dagger \hat{a} + g \sigma_x (\hat{a} + \hat{a}^\dagger)$ as a leading term which can be easily diagonalized, while the term $\omega_0 \sigma_z$ is treated as a perturbation. Generalizing this to our model we first rotate the spin basis $\sigma_x \rightarrow \sigma_y \rightarrow \sigma_z \rightarrow \sigma_x$ and write

$$\hat{H}_{GR} = \omega \hat{a}^\dagger \hat{a} + \beta (\hat{a} + \hat{a}^\dagger) \sigma_z + i \lambda (\hat{a} - \hat{a}^\dagger) \sigma_x + \omega_0 \sigma_y,$$

where $\beta = (g_1 + g_2)/2$ and $\lambda = (g_1 - g_2)/2$. In the adiabatic approximation the terms proportional to $\omega_0$ and $\lambda$ should be treated as a perturbation. Considering the basis $|\sigma\rangle \otimes |N\rangle$, where $\sigma = \pm$ and $|N\rangle = \mathcal{D}(\mp \beta/\omega)|n\rangle$ with the Fock states $|n\rangle$ ($n = 0, 1, 2, \ldots$) and the displacement operator $\mathcal{D}(\beta/\omega) = \exp ((\beta/\omega)(\hat{a} - \hat{a}^\dagger))$, we obtain the eigenvalue equation for the leading term $\langle [\hat{a}^\dagger \pm \beta/\omega](\hat{a} + \mp \beta/\omega)|\phi_{\pm}\rangle = \mathcal{D}(\mp \beta/\omega) \hat{a}^\dagger \hat{a} \mathcal{D}(\mp \beta/\omega)|\phi_{\pm}\rangle = (E/\omega + \beta^2/\omega^2)|\phi_{\pm}\rangle$. In this basis the Hamiltonian approximately has a block diagonal form with the $N$th block given by

$$\hat{H}^{(N)}_{GR} = \begin{cases} E_N & i \omega_0 \langle N_- | N_+ \rangle + i \lambda \langle N_+ | (\hat{a} - \hat{a}^\dagger)| N_- \rangle \end{cases},$$

and $E_N = \omega (N - \beta^2/\omega^2)$, provided the terms containing the overlaps $\langle N_\pm | M_\mp \rangle$ for $N \neq M$ are neglected. The overlap of the two displaced coherent states is

$$\langle M_- | N_+ \rangle = e^{-2 \beta^2/\omega^2} \left( \frac{2 \beta}{\omega} \right)^N \binom{N}{M} \sqrt{\frac{M!}{N!}} \binom{N-M}{M} \sqrt{\frac{4 \beta^2}{\omega^2}}, \quad \text{for } M < N,$$

$$\langle M_- | N_+ \rangle = e^{-2 \beta^2/\omega^2} \left( \frac{-2 \beta}{\omega} \right)^M \binom{M}{N} \sqrt{\frac{N!}{M!}} \binom{M-N}{N} \sqrt{\frac{4 \beta^2}{\omega^2}}, \quad \text{for } M \geq N,$$

while $\langle M_- | N_+ \rangle = (-1)^{N-M} \langle N_- | M_+ \rangle$ and $\langle M_+ | N_- \rangle = (-1)^{M-N} \langle M_+ | N_- \rangle$. The eigenenergies of the perturbed system are

$$E^{\pm}_N = E_N \pm \omega_0 \langle N_- | N_+ \rangle - \lambda \langle N_- | (\hat{a} - \hat{a}^\dagger)| N_+ \rangle.$$

To compute the necessary matrix elements we use the identities $\langle M_- | N_+ \rangle = \langle m | \mathcal{D}(\mp 2\alpha)|n\rangle$ and $\langle M_- | \bar{\alpha}| N_+ \rangle = \langle m | \mathcal{D}(\pm 2\alpha) \hat{a} \mathcal{D}(\mp \alpha)|n\rangle$. It follows $\langle M_- | \bar{\alpha}| N_+ \rangle = \langle m | \mathcal{D}(\mp 2\alpha) (\hat{a} - \alpha)|n\rangle = \sqrt{N} \langle M_- | (N + 1)_+ \rangle - \alpha \langle M_- | N_+ \rangle$ and $\langle M_- | \bar{\alpha}| N_+ \rangle = \sqrt{N + 1} \langle M_- | N + 1 \rangle_+ - \alpha \langle M_- | N_+ \rangle$. Thus we obtain the eigenenergies in the adiabatic approximation

$$E^{\pm}_N = E_N \pm e^{-2 \beta^2/\omega^2} \left[ \omega_0 L^N_N \left( \frac{4 \beta^2}{\omega^2} \right) + \frac{2 \beta}{\omega} \left( L^N_{N-1} - L^N_N \right) + \frac{\lambda}{\omega^2} \left( \frac{4 \beta^2}{\omega^2} \right) \right].$$

In the limiting case $g_1 = g_2 = g$ this equation agrees with the one obtained for the Rabi model in the limit of large $g$. We note that the second term introduces an exponentially small splitting when $g_1$ and $g_2$ are large and therefore the spectrum is a quasi-degenerate harmonic ladder. We illustrate a good agreement between the Eq. (62) and the numerics on Fig. 11 for $\omega_0 \ll \omega$ and $\lambda \ll 1$.

III. CONCLUSIONS AND DISCUSSIONS

The connection between the polynomial solutions, Bethe ansatz equations and quasi-exact solvability is well known and has been discussed in the literature from different perspectives, see, e.g., [11, 23] and Refs. therein. Noticing that

$$J^+ = z^a \frac{d}{dz} - nz, \quad J^- = \frac{d}{dz}, \quad J^0 = z \frac{d}{dz} - \frac{n}{2}$$

is a differential realization of the $(n + 1)$-dimensional representation of the $sl(2)$ algebra in the Bargmann space, one can construct a bilinear combination of $J^{\pm,0}$ whose eigenstates are polynomials of the order $n$ and smaller. This leads to a second-order differential operator which is called quasi-exactly solvable [19]. We illustrate this construction on a simple case of $\kappa = -\nu$ of the generalized Rabi model. The differential operator acting on the function $\chi(z)$ is

$$(z^2 - \nu^2) \frac{d^2}{dz^2} - (2\nu(z^2 - \nu^2) + 2nz - 2\nu) \frac{d}{dz} + 2\nu nz - A,$$

where $A = (n - 2\lambda_z)(n + 1) - 2\nu^2$ and $dz \equiv d/dz$. Using the operators $J^{\pm,0}$ it can be represented as $J^+ J^- - \nu^2 J^+ J^- + 2\nu J^+ + 2\nu(\nu^2 + 1) J^- - n J^0 - A = -n^2/2$ which has a quasi-exactly-solvable form. This predicts the existence of the exceptional part of the spectrum in the generalized Rabi model which has been studied in this paper. In particular, we found that:
• The exceptional part of the spectrum corresponds to the level crossings; no level crossings occur outside of the exceptional points.

• All level crossings occur at integer values of energy $\epsilon_c = n$; in the parameter space there are exactly $2 \times 2n$ continuous curves (for $n > 0$), where the intersection points reside. For $n = 0$ there are two intersection lines. The wave-functions at these points have a polynomial structure in the Bargmann space.

• The avoided level crossings occur at half-integer values of the energy, $\epsilon_{ac} = n/2$, at least for $g_1 \gg g_2$ (or $g_2 \gg g_1$).

• In the strong-coupling limit $g_1/\omega_0 \gg 1$ and $g_2/\omega_0 \gg 1$, the spectrum consist of the two quasi-degenerate harmonic ladders.

The obtained results for the generalized Rabi model can be used in several physical applications, namely for the two-dimensional electron gas in a magnetic field with Rashba and Dresselhaus spin-orbit couplings and for the cavity and circuit QED systems.

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FIG. 8: Energy spectrum of the generalized Rabi model shifted by $\frac{\hbar g_2}{2}$ as a function of the coupling $g_1$ for a range of couplings $g_2$. Horizontal dashed lines at integer energies $E = n$ indicate the energy levels intersection points where the model has an exceptional spectrum. There are $2n$ intersection points for a given $n > 0$ and one for $n = 0$. No other level crossings occur at different points. The level repulsion happens at the half-integer values of energy which however can not always be resolved on the scale of the figure.