Non-perturbative Solutions to
\( N = 2 \) Supersymmetric Yang-Mills Theories
–Progress and Perspective–

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Abstract

This note reviews the progress on the low energy dynamics of \( N = 2 \) supersymmetric Yang-Mills theories after the works of Seiberg and Witten. Specifically, the theory of prepotential for non-specialists is reviewed.

\(^1\)Note based on the talk presented in the colloquia in mathematics at RIMS, February 24 (Wed), 1999.
1 Introduction

The low energy effective theory of $N = 2$ supersymmetric Yang-Mills theory which has two kinds of supersymmetry is generated by a holomorphic function called prepotential and any information concerning the theory is available from the prepotential if it is determined. However, unfortunately, for this theory instantons are expected to contribute as a non-perturbative effect which can not be detected in the perturbation theory, and therefore the determination of the prepotential including this effect was not correctly proceeded so far.

However, in the summer in 1994, Seiberg and Witten \[1, 2\] pointed out that in the case of SU(2) gauge theory the prepotential correctly including the instanton effect, often called non-perturbative or exact solution, can be obtained, provided the periods of 1-form on a Riemann surface (elliptic curve) are given. Their proposal was immediately extended by many authors to the cases of other gauge groups with or without quarks, and a number of papers was appeared in almost three years since then. Most of them was written for miscellaneous aspects of the prepotential and related works, but the results of these studies supported that the approach taken by Seiberg and Witten produced the physically meaningful prepotential.

In this talk, I will show the main results concerning the prepotential from the works proceeded in this brief period. Specifically, I will talk on the method of Picard-Fuchs equation, the evaluation of the periods and the prepotentials via the Barnes type integral representation, the relation between instanton effect and the well-known scaling relation of prepotential of Matone, by using explicit examples. Finally, a perspective will be presented.

2 $N = 2$ supersymmetric Yang-Mills theory

2.1 $N = 2$ supersymmetry

Firstly, let us explain what is the supersymmetry [3]. As is well-known, there are particles called bosons with integer spins and fermions with half-integer spins, and these have different statistical properties. Supersymmetry is the symmetry which connects these particles. In the case that one boson (fermion) corresponds to one fermion (boson), supersymmetry is only one, so it is called $N = 1$, while the case of two fermions (bosons) it is called $N = 2$. The Yang-Mills theory to be discussed is the gauge theory enjoying two supersymmetries, namely, it is $N = 2$ supersymmetric Yang-Mills theory.

As the particles appearing in this $N = 2$ Yang-Mills theory, there are gauge fields $A_\mu$ ($\mu = 0, \cdots, 3$), Weyl fermions $(\lambda, \psi)$ and complex scalar field $\phi$ and these as a whole are referred as $N = 2$ chiral multiplet or gauge multiplet. This chiral multiplet is often arranged by

$$
\begin{pmatrix}
\lambda \\
A_\mu \\
\phi \\
\psi
\end{pmatrix}
$$

(2.1)
which means that \( A_\mu \) supersymmetrically transform to \( \lambda \) and \( \psi \), etc. Note that in this multiplet since the gauge fields are included all particles belonging to this multiplet must take the value in the adjoint representation of the gauge group \( G \) and the number of components (associated with \( G \)) is the same with the dimension of the gauge group.

As an \( N = 2 \) theory, other particles can be included, and in that case there is the multiplet called scalar or hypermultiplet consisting of Weyl fermions \((\psi_q, \psi_q^\dagger)\) and the complex bosons \((q, q^\dagger)\)

\[
\begin{align*}
\psi_q \\
q \\
\psi_q^\dagger
\end{align*}
\]

but we will consider the theory dictated by only the chiral multiplet for the moment.

For the description of a theory with supersymmetry, it is convenient to consider a field (super field) on super space which is the space with usual space-time real coordinates \( x_\mu \) and the Grassman coordinates \( \theta_\alpha \) \((\alpha = 1, 2)\). In the present case of \( N = 2 \) chiral multiplet, since it can be seen that it consists of two \( N = 1 \) multiplets \((A_\mu, \lambda)\) and \((\psi, \phi)\) in view of \( N = 1 \) supersymmetry, the Lagrangian of the \( N = 2 \) supersymmetric Yang-Mills theory is compactly written as

\[
\mathcal{L} = \frac{1}{8\pi} \text{Im} \text{ tr} \left[ \tau_0 \int d^2 \theta W^\alpha W_\alpha + 2 \int d^2 \theta d^2 \bar{\theta} \Phi \dagger e^{-2V} \Phi \right] \quad (2.3)
\]

by using the super fields

\[
W_\alpha = -i \lambda_\alpha - \frac{i}{2} (\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha F_{\mu \nu} + \cdots, \quad \Phi = \phi + \sqrt{2} \theta \psi + \cdots, \quad (2.4)
\]

where \( \sigma^\mu = (1, \sigma^i) \), \( 1 \) is an unit matrix of the size \( 2 \times 2 \), \( \sigma^i(i = 1, 2, 3) \) are the Pauli matrices, \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu] \). Ellipses are omission. \( V \) is expressed by superfields, but the detail is not necessary here. In (2.3), the complexified coupling constant

\[
\tau_0 = \frac{\theta_0}{2\pi} + i \frac{4\pi}{g^2}, \quad (2.5)
\]

where \( \theta_0 \in \mathbb{R} \) is the vacuum angle and \( g \) is the gauge coupling constant, is introduced. The introduction of the vacuum angle is an analogy of \( N = 0 \) (non-supersymmetric) QCD, where the vacuum angle is related to strong CP problem. Actually, \( \theta_0 \) in this \( N = 2 \) theory can be set to zero because of chiral rotation of fermions.

From the Grassman integral and expansion of the products of superfields, \( \mathcal{L} \) becomes

\[
\mathcal{L} = \frac{1}{g^2} \text{tr} \left[ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{g^2 \theta_0}{32\pi^2} F_{\mu \nu} \bar{F}^{\mu \nu} - \frac{1}{2} [\phi \dagger, \phi]^2 + \cdots \right], \quad (2.6)
\]

where \( \bar{F}^{\mu \nu} \) is the dual field strength defined by using the anti-symmetric tensor \( \epsilon^{\mu \nu \rho \sigma} \) \( (\epsilon^{0123} = +1) \)

\[
F^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} F_{\rho \sigma}. \quad (2.7)
\]
2.2 Vacuum

In field theories, a complex scalar field often plays a role of Higgs field, and it can be shown that the present case is also the case. From (2.6), it is easy to see that φ has the potential

$$ U = \frac{1}{2g^2} [\phi^\dagger, \phi]^2 \geq 0, \quad (2.8) $$

and the vacuum of the $N = 2$ Yang-Mills theory is characterized by the minimum of this potential. In the case at hand, the vacuum corresponds to $U = 0$, and $\phi = 0$ can be considered as the candidate of $\phi$ which realizes the vacuum configuration. However, $U = 0$ can be realized by some $\phi$ such that the commutator vanishes, so it is sufficient to parameterize it by using the generators $H_i$ of Cartan subalgebra of the gauge group $G$ \[4, 5, 6, 7\]

$$ \phi = \sum_{i=1}^{\text{rank}(G)} a_i H_i, \quad (2.9) $$

where $a_i$ are complex parameters. Since the Weyl symmetry still remains in this parameterization, it would be convenient to characterize the theory by using quantities which are invariant under this symmetry. For example, in the case of SU($N_c$) gauge group the candidates are given by

$$ u_k = \text{tr} \langle \phi^k \rangle, \quad k = 2, \cdots, N_c. \quad (2.10) $$

There exist more convenient parameters than $u_k$, defined by the symmetric polynomials

$$ s_k = (-1)^k \sum_{i_1 < \cdots < i_k = 1}^{N_c} a_{i_1} \cdots a_{i_k}, \quad k = 2, \cdots, N_c, \quad (2.11) $$

but $\{u_k\}$ and $\{s_i\}$ are related each other by the Newton’s formula

$$ ks_k + \sum_{i=1}^{k} s_{k-i} u_i = 0, \quad s_0 = 1, s_1 = u_1 = 0. \quad (2.12) $$

$s_k$ are called moduli of the theory and the space of $s_k$ is called moduli space.

3 Effective action and prepotential of SU(2) gauge theory

3.1 Effective action

When $\phi$ has a non-zero vacuum expectation value, gauge fields gain masses by Higgs mechanism. For example, in the case of SU(2) gauge group, $\phi$ can be written by $\phi = a\sigma^3$, where $a = \langle \phi \rangle \in C$, and the gauge symmetry breaks down from SU(2) to U(1) for $a \neq 0$. Then $W^\pm_\mu := (A^1_\mu \pm A^2_\mu)/2$ and their supersymmetric particles (superpartners) gain mass $\sim a$, while $A^3_\mu$ and their superpartners remain massless. Furthermore, when the vacuum expectation value of $\phi$ is zero, the original SU(2) gauge symmetry is restored, therefore the classical moduli space has a singularity at $u \equiv \text{tr} \langle \phi^2 \rangle = 2a^2 = 0$. 
On the other hand, in a quantum theory, the dynamics of particles is described by an action including quantum effects called effective action. In the case of \( a \neq 0 \), for large \( a \) the masses of \( W_\mu^\pm \) and corresponding superpartners also become large. These heavy particles do not “actively” move and only massless particles play an essential role. Then these massive particles can be dropped out in the case that the energy scale, say \( \Lambda \), is small (\( |a| \gg \Lambda \)). If these massive particles are integrated out in the path integral, only the light particles like \( A_\mu^3 \) will be left. The action obtained in this way is called low energy effective action (below, simply called effective action) and the theory is said to be in weak coupling region.

Actually, the explicit form of the effective action is expected to be complicated due to quantum fluctuation, but because of the \( N = 2 \) supersymmetry this effective action (Lagrangian) is known to be simply represented by

\[
\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left[ \int d^2 \theta d^2 \bar{\theta} \frac{\partial \mathcal{F}(A)}{\partial A} \bar{A} + \frac{1}{2} \int d^2 \theta d^2 \bar{\theta} \frac{\partial^2 \mathcal{F}(A)}{\partial A^2} W^a W_a \right],
\]

where

\[
W_a = -i\lambda^3 + \cdots, \quad A = \phi^3 + \sqrt{2}\theta \psi^3 + \cdots
\]

are the \( U(1) \) multiplets and \( \phi^3 \) etc are the third component of \( \phi \) etc. \( \mathcal{F}(A) \) is a holomorphic function called prepotential satisfying \( \partial \mathcal{F}(A)/\partial A = 0 \), and it’s second order derivative

\[
\tau = \frac{\partial^2 \mathcal{F}(A)}{\partial A^2}
\]

is called as effective coupling constant. The lowest order term of (3.3) is (2.5). Furthermore, it is convenient to introduce

\[
A_D := \frac{\partial \mathcal{F}}{\partial A},
\]

especially, for a later purpose.

Note that the effective action is generated by the prepotential and once it is determined the quantum dynamics of the particles will be clarified. However, we are interested in the vacuum configuration of the theory, so \( A \) can be replaced by it’s scalar component \( a \). Then \( \mathcal{F}(A) \) reduces to \( \mathcal{F}(a) \) and in this case (3.4) is replaced by

\[
a_D := \frac{\partial \mathcal{F}}{\partial a}.
\]

\( \mathcal{F}(a) \) is a solution to the problem how to determine the low energy dynamics and the determination of \( \mathcal{F}(a) \) is the subject of the discussions below.

### 3.2 Prepotential

In the classical theory, the prepotential is given by

\[
\mathcal{F}_{cl} = \frac{\tau_0}{2} a^2,
\]
which is available from (2.3) and (3.3). On the other hand, perturbative part of the quantum prepotential can be determined from the beta function for coupling constant. In a gauge theory, coupling constant is not simply a constant, but is a quantity receiving quantum corrections and is determined by the beta function of renormalization group. In the case at hand, the beta function at 1-loop level is given by

$$\mu \frac{dg}{d\mu} = \beta, \ \beta = -bg^3, \ b = \frac{1}{4\pi^2}, \ (3.7)$$

where \(\mu\) is the renormalization scale, but since there is the non-renormalization theorem which states that there are not corrections beyond 1-loop in perturbation theory \([8, 9]\), (3.7) exactly holds in this sense. Solving (3.7) by imposing the condition \((g(a), \mu) = a\), one finds

$$\frac{1}{g(a)^2} = \frac{1}{g^2} + 2b \ln \frac{a}{\mu}, \ (3.8)$$

where \(g = g(\mu)\) at \(\mu\). Therefore,

$$g(a)^2 = \frac{1}{\frac{1}{g(\mu)^2} + 2b \ln \frac{a}{\mu}} = \frac{1}{2b \ln \frac{a}{\Lambda}} \equiv \frac{1}{2b \ln \frac{a}{\Lambda}}, \ (3.9)$$

where \(\Lambda\) is identified by

$$\Lambda \equiv \mu e^{-1/(2bg(\mu)^2)} \ (3.10)$$

\(\Lambda\) introduced in this way is called QCD (scale) parameter. Accordingly, the 1-loop prepotential can be determined from

$$\frac{d^2F}{da^2} \sim i\frac{4\pi^2}{g(a)^2} \ (3.11)$$

as

$$F_{1\text{-loop}} = i\frac{a^2}{\pi} \ln \frac{a}{\Lambda} \ (3.12)$$

### 3.3 Instanton effect and prepotential

Here, let us consider instantons and introduce the (Euclidean) action of Yang-Mills fields

$$S = -\frac{1}{4g^2} \int d^4x \text{tr}(F_{\mu\nu}F^{\mu\nu}). \ (3.13)$$

Now, consider a configuration of gauge fields at infinity \((x \to \infty)\). Then \(F_{\mu\nu} = 0\) must be satisfied at infinity. This means that the gauge field tends to an equivalent configuration to the vacuum. The solution to classical equations of motion satisfying this condition is known as instanton solution. With the aid of this instanton solution \((3.13)\) becomes

$$S = \frac{8\pi^2}{g^2}, \ (3.14)$$

and therefore the amplitude of instanton of unit topological charge is given by \(e^{-S}\).
From (3.9) comparing the QCD parameter and the instanton amplitude, one finds that \( (\Lambda/a)^{4k} \) corresponds to the amplitude of instanton with topological charge \( k \) (\( k \)-instanton)

\[
\left( \frac{\Lambda}{a} \right)^{4k} = e^{-\frac{8\pi^2}{9(a)^2}k}.
\]

(3.15)

This factor is not proportional to powers in the coupling constant, therefore this can be considered as non-perturbative effect.

Seiberg [10] conjectured that for the actual prepotential these instantons contributed and it’s form was predicted by

\[
F = ia^2 \pi \left[ \ln \frac{a}{\Lambda} - \sum_{k=0}^{\infty} F_k \left( \frac{\Lambda}{a} \right)^{4k} \right].
\]

(3.16)

Note that (3.16) takes the form of a sum of (3.6), (3.12) and instants. \( F_1 \neq 0 \) was pointed by Seiberg [10], but it was not known whether general \( F_k \) were 0 or not. However, Seiberg and Witten [1, 2] showed that \( F_k \) could be exactly determined by using data of a Riemann surface.

### 3.4 Strong coupling region

So far we have concentrated on the region \(|a| \gg \Lambda\), but next, let us consider the case of small \( a \). Taking \( a \) to be small, one finds that \( a \) will arrive at the region \(|a| \sim \Lambda\). This corresponds to \( u \sim \pm \Lambda^2 \) in terms of moduli. In this case, the analysis of the theory is very complicated, but according to the detailed study of Seiberg and Witten [1, 2] it turned out that at \( u = +\Lambda^2 \) magnetic monopole (a particle carrying only unit magnetic charge) becomes massless and at \( u = -\Lambda^2 \) dyon (a particle carrying both electric and magnetic charges) becomes massless. This indicates that quantum mechanically the moduli space has three singularities at \( u = \pm \Lambda^2 \) and \( \infty \). Note that the classical singularity disappears in the quantum theory.

In the description of the effective action presented in the previous (sub)sections, only the massless gauge fields and their superpartners were concerned, but in the present case monopoles and dyons must be taken into account. This indicates that the theory is in strongly coupled region and in this region it is not easy to write down the effective action, but by using a notion of duality the theory in this strong coupling region can be mapped to a weakly coupled dual theory. As a result, it becomes possible to gain understandings on the original theory by studying weakly coupled dual theory without directly treating the strongly coupled original theory. Seiberg and Witten [1, 2] showed that this was in fact possible.

### 3.5 Duality

According to Seiberg and Witten’s result, the duality group \( \Gamma(2) \) which is a subgroup of \( SL(2,\mathbb{Z}) \)

\[
\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| a \equiv d \equiv 1 \mod 2, \ b \equiv c \equiv 0 \mod 2 \right\}
\]

(3.17)
is derived from the monodromy properties of $a$ and $a_D$ at each singularity and then it acts for the effective coupling constant which satisfies

$$\text{Im } \tau > 0$$

(3.18)

as

$$\tau \rightarrow \frac{a \tau + b}{c \tau + d}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2).$$

(3.19)

This fact implies that $\tau$ is a modulus of Riemann surface of genus one. From these consideration, they conjectured the existence of a torus which satisfied this condition and identified the complex $u$-plane with the quotient space $H/\Gamma(2)$ of the upper half plane $H$.

Of course, it can be observed that one of the generators of $\Gamma(2)$

$$S : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(3.20)

causes the exchange of $a$ and $a_D$

$$S : \tau = \frac{da_D}{da} \rightarrow -\frac{1}{\frac{da_D}{da}} \equiv \tau_D,$$

(3.21)

therefore the strong coupling region can be mapped to the weak coupling region of the theory having $\tau_D$ as the effective coupling constant.

## 4 The Seiberg-Witten solution

### 4.1 The SU(2) case

Searching whether a Riemann surface which satisfies this property does exist or not, Seiberg and Witten [1] found that it was given by

$$y^2 = (x^2 - \Lambda^4)(x - u)$$

(4.1)

on local complex coordinates $(x, y) \in \mathbb{C}^2$.

Furthermore, if $\tau$ is identified with the modulus of a Riemann surface of genus one (torus), it should be represented by a ratio of periods of a holomorphic 1-form $dx/y$ along $\alpha$- and $\beta$-cycles on the torus. Regarding $a$ and $a_D$ as functions in $u$

$$\tau = \frac{da_D/du}{da/du},$$

(4.2)

one may insure that this implies

$$\frac{da}{du} = \oint\alpha \frac{dx}{y}, \quad \frac{da_D}{du} = \oint\beta \frac{dx}{y}.$$
Accordingly, in this viewpoint, $a$ and $a_D$ can be interpreted as periods

$$a = \oint_\alpha \lambda_{SW}, \quad a_D = \oint_\beta \lambda_{SW},$$

(4.4)

of the meromorphic 1-form

$$\lambda_{SW} = \oint \frac{dx}{y} du = \frac{\sqrt{2}}{8\pi} \sqrt{\frac{x-u}{x^2 - \Lambda^4}} dx,$$

(4.5)

where in the second equality the normalization factor is introduced. The 1-form $\lambda_{SW}$ obtained in this way is usually referred to Seiberg-Witten differential or Seiberg-Witten 1-form, and the approach based on Riemann surface is often called Seiberg-Witten solution or Seiberg-Witten theory.

### 4.2 Other gauge group cases

In the case of classical Lie gauge groups except SU(2), it is known that the Seiberg-Witten curves are written by hyperelliptic curves. The SU(2) case is the only exception, and the Seiberg-Witten curve can have two representations, i.e., elliptic and hyperelliptic types. In the case not including quark hypermultiplets, these curves and associated Seiberg-Witten differentials are explicitly represented as follows.

**$A_n = SU(n+1)$ ($n > 0$):** [11 12 13 14]

$$y^2 = W_{A_n}^2 - \Lambda_{A_n}^{2(n+1)}, \quad \lambda_{A_n} = \frac{x \partial_x W_{A_n}}{y} dx,$$

(4.6)

where

$$W_{A_n} = x^{n+1} - \sum_{i=2}^{n+1} s_i x^{n+1-i}.$$  

(4.7)

**$B_n = SO(2n+1)$ ($n > 1$):** [11 12 13 14]

$$y^2 = W_{B_n}^2 - \Lambda_{B_n}^{4n-2} x^2, \quad \lambda_{B_n} = \frac{W_{B_n} - x \partial_x W_{B_n}}{y} dx,$$

(4.8)

where

$$W_{B_n} = x^{2n} - \sum_{i=1}^n s_{2i} x^{2(n-i)}.$$  

(4.9)

**$C_n = Sp(2n)$ ($n > 1$):** [11 12]

$$x^2 y^2 = W_{C_n}^2 - \Lambda_{C_n}^{4(n+1)}, \quad \lambda_{C_n} = -\frac{\partial_x W_{C_n}}{y} dx,$$

(4.10)

where

$$W_{C_n} = x^2 \left[ x^{2n} - \sum_{i=1}^n s_{2i} x^{2(n-i)} \right] + \Lambda_{C_n}^{2(n+1)}.$$  

(4.11)

**$D_n = SO(2n)$ ($n > 2$):** [11 14 15]

$$y^2 = W_{D_n}^2 - \Lambda_{D_n}^{4(n-1)} x^4, \quad \lambda_{D_n} = \frac{2W_{D_n} - \partial_x W_{D_n}}{y} dx,$$

(4.12)

where

$$W_{D_n} = x^{2n} - \sum_{i=1}^n s_{2i} x^{2(n-i)}.$$  

(4.13)
Note that the normalization factors of Seiberg-Witten 1-forms are ignored in each case. The list used here is that given in [16].

These hyperelliptic curves can be compactified to a Riemann surface with appropriate genus by adding an infinity (see figure 1), and taking canonical (symplectic) 1-cycles on this surface and denoting them as $\alpha_i$ and $\beta_i$, one sees that the Seiberg-Witten periods for a theory in gauge group $G$ can be written by

$$a_i = \oint_{\alpha_i} \lambda_G, \quad a_{D_i} = \oint_{\beta_i} \lambda_G.$$  \hspace{1cm} (4.14)

It is convenient to summarize these periods by

$$\Pi = \begin{pmatrix} a_{D_i} \\ a_i \end{pmatrix}.$$  \hspace{1cm} (4.15)

$\Pi$ is called period vector.

Figure 1: 1-cycles on $A_n$ type Seiberg-Witten curve identified with genus $n$ Riemann surface

Also in the case including quarks, similar hyperelliptic curves and Seiberg-Witten differentials can be constructed, but since the number of quarks which can be added is restricted from asymptotic freedom and the form of the curve differs according to the number of quarks. For a later convenience, let us write down the Seiberg-Witten curve and Seiberg-Witten differential for the case of SU($N_c$) ($N_f < N_c = n + 1$) gauge group with $N_f$ quarks of mass $m_i$ ($i = 1, \cdots, N_f$) [7, 17]

$$y^2 = W_{A_n}^2 - \Lambda_{A_n}^{2N_c-N_f} G, \quad G = \prod_{i=1}^{N_f} (x + m_i), \quad \lambda = \frac{xdx}{y} \left( \frac{W_{A_n} \partial_x G}{2G} - \partial_x W_{A_n} \right),$$  \hspace{1cm} (4.16)

where $W_{A_n}$ is the simple singularity given in (4.7). Note that $\lambda$ has poles at $x = -m_i$.

Remark: Also in the case of exceptional Lie gauge groups, Seiberg-Witten curves are known to be written in terms of hyperelliptic curves [18, 19, 20].

Below, the suffix of the QCD parameter is ignored, but this will not cause any confusion.

5 Picard-Fuchs equations

5.1 SU(2) case

As an example, let us consider the periods in the SU(2) gauge theory. These periods have been introduced by [14], but in order to evaluate them it is necessary to explicitly express 1-cycles as
appropriate integral intervals. The simplest one to specify these intervals is to use branching points of Seiberg-Witten curve as

$$
\alpha : -\Lambda^2 \rightarrow +\Lambda^2, \quad \beta : +\Lambda^2 \rightarrow u
$$

(5.1)

and to recognize these as loops running counterclockwise (but $\alpha \cap \beta = +1$). Actually, since the evaluation of periods depends on the behavior of cycles, it is complicated to treat in general. For example, considering the calculation of periods at weak coupling region, one finds that the QCD parameter approaches to zero and $u$ moves to the other branching point, so the torus collapses as a result (see figure 2). Therefore the period $a$ ultimately reduced to an integral within infinitesimal interval near the origin, so the evaluation is easy, but since the dual period $a_D$ reduces to an integral from 0 to $\infty$, if under this situation $a_D$ is calculated, the integrand also diverges for $u \rightarrow \infty$, so that $a_D$ finally involves logarithmical divergence. Since we encounter such situation even if the gauge group is any group, the calculation of dual period (in weak coupling region) is not easy in general.

![Figure 2: Torus in the weak coupling region](image)

Then, how should we do? One of the simplest way to evaluate the periods is to realize these periods as solutions to differential equation. In fact, it can be shown that the derivatives are

$$
\frac{d\Pi}{du} = -\frac{\sqrt{2}}{16\pi} \oint_\gamma \frac{dx}{y},
$$

$$
\frac{d^2\Pi}{du^2} = \frac{\sqrt{2}}{32\pi} \frac{1}{(\Lambda^4 - u^2)} \oint_\gamma \sqrt{\frac{x-u}{x^2 - \Lambda^4}} dx,
$$

(5.2)

where the 1-cycles (or their linear combinations) are summarized as $\gamma$. The second equation in (5.2) indicates the existence of the differential equation

$$
\frac{d^2\Pi}{du^2} + \frac{\Pi}{4(u^2 - \Lambda^4)} = 0
$$

(5.3)

with regular singularities and such equations are in general referred to Picard-Fuchs equation (for a history of Picard-Fuchs equation, see [21]). Note that the periods are now constructed as solutions to differential equation.

### 5.2 General case

Even if the gauge group is any group, in order to derive Picard-Fuchs equations, it is sufficient to express parameter derivatives of period by another parameter derivatives. In fact, Alishahiha wrote
the general form of Picard-Fuchs equations for all classical Lie gauge groups \[22\]. For example, in the case of SU\((n + 1)\) \[3, 22\], they are

\[
\left[(n + 1)\partial_{s_2} \partial_{s_n} - \sum_{i=2}^{n} (n + 1 - i) s_i \partial_{s_{n+1}} \partial_{s_{i+1}}\right] \Pi = 0,
\]

\[
\left[k \partial_{s_{n+2-k}} - (n + 1) \partial_{s_2} \partial_{s_{n-k}} + \sum_{i=2}^{n} (n + 1 - i) s_i \partial_{s_i} \partial_{s_{n+2-k}}\right] \Pi = 0, \quad k = 0, \ldots, n - 2
\]

\[
\left[1 + \sum_{i=2}^{n+1} i(i - 2) s_i \partial_{s_i} + \sum_{i,j=2}^{n+1} ij s_i s_j \partial_{s_i} \partial_{s_j} - (n + 1)^2 \Lambda^{2(n+1)} \partial_{s_{n+1}}^2\right] \Pi = 0,
\]

\[
\left[\partial_{s_i} \partial_{s_j} - \partial_{s_p} \partial_{s_q}\right] \Pi = 0, \quad i + j = p + q.
\]

(5.4)

Of course, since the derivation of Picard-Fuchs equations is mechanical, some algorithms suitable for computer program can be made. One day Klemm et al. \[23\] derived differential equations in Landau-Ginzburg model by using a method in singularity theory, but the algorithm of Isidro et al. \[24\] corresponds to the generalization of this construction by replacing the holomorphic 1-form by Seiberg-Witten differential on Seiberg-Witten curve.

### 5.3 Hypergeometric differential equations

Picard-Fuchs equations arising in this supersymmetric Yang-Mills theory are often identified with well-known hypergeometric differential systems. Gaussian hypergeometric function (in multiple variables) is the function which reduces to the single variable hypergeometric function \( \text{}_{2}F_{1} \) when one of the variables is non-zero and the remainings are set to zero, and Gaussian hypergeometric differential equations are the equations satisfied by such Gaussian hypergeometric functions \[25\].

Picard-Fuchs equation of SU(2) theory is given in (5.3), but for this equation performing the transformation of variable

\[
z = u^2 / \Lambda^4
\]

(5.5)

one finds that (5.3) reduces to Gauss’s hypergeometric equation \[3\]

\[
z(1 - z) \frac{d^2 \Pi}{dz^2} + \left(\frac{1}{2} - \frac{z}{2}\right) \frac{d \Pi}{dz} - \frac{\Pi}{16} = 0.
\]

(5.6)

On the other hand, in the case of the SU(3) theory \[4\], Picard-Fuchs equations are represented by a system of simultaneous partial differential equations

\[
[(27 \Lambda^6 - 4u^3 - 27v^2) \partial_u^2 - 12u^2 v \partial_u \partial_v - 3uv \partial_v - u] \Pi = 0,
\]

\[
[(27 \Lambda^6 - 4u^3 - 27v^2) \partial_v^2 - 36uv \partial_u \partial_v - 9v \partial_v - 3] \Pi = 0,
\]

(5.7)

where \( u \equiv s_2 \) and \( v \equiv s_3 \) are SU(3) moduli, but this system is known to be equivalent to the two variable hypergeometric system of Appell’s \( \text{F}_4 \) \[25, 26, 27, 28, 29\]

\[
\left[\theta_x \left(\theta_x - \frac{1}{3}\right) - x \left(\theta_x + \theta_y - \frac{1}{6}\right) \left(\theta_x + \theta_y - \frac{1}{6}\right)\right] \Pi = 0,
\]

\[
\left[\theta_y \left(\theta_y - \frac{1}{2}\right) - y \left(\theta_x + \theta_y - \frac{1}{6}\right) \left(\theta_x + \theta_y - \frac{1}{6}\right)\right] \Pi = 0
\]

(5.8)
by the transformation of variables
\[ x = 4u^3/(27\Lambda^6), \quad y = v^2/\Lambda^6. \]
(5.9)

\[ \theta_x = x \partial/\partial x \text{ and } \theta_y = y \partial/\partial y \] are Euler partial derivatives.

As for the other rank two gauge groups, there is \( B_2 = C_2 \), whose Picard-Fuchs equations can be recognized as Horn \( H_5 \) \[^3]\). Note that \( 2F_1, F_4 \) and \( H_5 \) are all Gaussian.

5.4 Solutions to the Picard-Fuchs equation

Let us consider solutions to the SU(2) Picard-Fuchs equation (5.3) or (5.6) at weak coupling region \( (u = \infty) \) \[^5]\). According to Frobenius’s method, it turns out that because of degeneracy of the solutions to indicial equation the solutions involve logarithm \( (z = u^2/\Lambda^4) \)
\[ \rho_1 = z^{1/4} \sum_{i=0}^{\infty} \frac{a_i}{z^i}, \quad \rho_2 = \rho_1 \log \frac{1}{z} + z^{1/4} \sum_{i=1}^{\infty} \frac{b_i}{z^i}, \]
(5.10)
where the first several coefficients are given by
\[ a_0 = 1, \quad a_1 = -\frac{1}{16}, \quad a_2 = -\frac{15}{1024}, \quad a_3 = -\frac{105}{16384}, \quad a_4 = -\frac{15015}{4194304}, \quad a_5 = -\frac{153153}{67108864} \]
(5.11)
and
\[ b_1 = \frac{1}{8}, \quad b_2 = \frac{13}{1024}, \quad b_3 = \frac{163}{49152}, \quad b_4 = \frac{31183}{25165824}, \quad b_5 = \frac{74791}{134217728}. \]
(5.12)

The next task is to relate these solutions to the periods, but is easily done by simply consider appropriate linear combination of them. However, the combination must be uniquely fixed by computing lower order expansion of the periods. This manipulation is equivalent to give an initial condition for the Picard-Fuchs equation. In this way, the result follows
\[ a = \frac{\Lambda}{\sqrt{2}} \rho_1, \quad a_D = i \frac{\Lambda}{\sqrt{2\pi}} (-4 + 6 \log 2) \rho_1 - i \frac{\Lambda}{\sqrt{2\pi}} \rho_2. \]
(5.13)

6 Prepotentials

6.1 The SU(2) prepotential

Let us derive the SU(2) prepotential. The prepotential is a function in \( a \) and is available from \[^3]\). However, since \( a_D \) is obtained as a function in \( u \), one must consider \( a_D = a_D(a) \) by eliminating \( u \) from this as a first step. This can be realized by inversely solving \( a = a(u) \) in (5.13)
\[ u = 2a^2 + \frac{\Lambda^4}{16a^2} + \frac{5\Lambda^8}{4096a^6} + \frac{9\Lambda^{12}}{131072a^{10}} + \cdots. \]
(6.1)

Next, after substituting this expression into (5.3), expanding it for large \( a \) and further integrating it over \( a \), one finds the prepotential \[^3]\)
\[ \mathcal{F} = i \frac{a^2}{\pi} \left[ \log \left( \frac{a}{\Lambda} \right)^2 + 4 \log 2 - 3 - \sum_{k=1}^{\infty} F_k \left( \frac{a}{\Lambda} \right)^{4k} \right]. \]
(6.2)
where $O(a)$-terms including integration constant are ignored because they do not affect to the effective coupling constant. The first two coefficients of the instanton expansion are given by

$$F_1 = \frac{1}{64}, \quad F_2 = \frac{5}{32768}. \quad (6.3)$$

Since $F_1$ is $F_1 \sim 0.016$, it is a number near 0 and also $F_2$ is a very small number $F_2 \sim 0.00015$. From these values, the contributions from instantons are small quantities which may be ignored as a matter of fact. Nevertheless, they contributes non-zero effects. The calculation of these very small but non-zero non-perturbative effects by using the methods in field theory requires much labour, but the point that gave a systematic method which can calculate such effect was one of the brilliant success of Seiberg and Witten.

### 6.2 The case including quarks

So far we have considered pure Yang-Mills theory, that is, the theory not including quarks. However, since QCD not including quarks is physically unnatural, let us consider the prepotential with quarks.

Again restricting the gauge group to SU(2), one can see that quarks can be added up to three, provided the asymptotic freedom is preserved. Seiberg and Witten conjectured that the prepotential including $N_f$ massless quarks was given by

$$\mathcal{F}_{N_f} = \frac{ia^2}{\pi} \left[ \frac{4 - N_f}{4} \ln \left( \frac{a}{\Lambda} \right)^2 + \sum_{i=0}^{\infty} F_k \left( \frac{\Lambda^2}{a^2} \right)^{(4-N_f)i} \right], \quad (6.4)$$

and the validity of this formula was later proved by Ito and Yang (for the analysis of periods, see [32]).

Actually, since the masses of the quarks are expected to be non-zero, the prepotential in the theory including quarks with masses $m_i$ is known to be modified to

$$\mathcal{F}_{N_f} = \frac{i\tilde{a}^2}{\pi} \left[ \frac{4 - N_f}{4} \ln \left( \frac{\tilde{a}}{\Lambda} \right)^2 + \mathcal{F}^{N_f}_0 - \frac{\sqrt{2} \pi}{4i \tilde{a}} \sum_{i=1}^{N_f} n_i' m_i + \frac{N_f}{2} \tilde{a} \right]$$

$$+ \frac{1}{4\tilde{a}^2} \left[ \frac{3}{2} \sum_{i=1}^{N_f} m_i^2 - \mathcal{F}_s^{N_f} \right] + \sum_{i=2}^{\infty} F_i(\Lambda^{4-N_f}, m_1, \cdots, m_{N_f}) \tilde{a}^{-2i}, \quad (6.5)$$

where $n, n' \in \mathbb{Z}$ are winding numbers of 1-cycles which enclose the poles of the Seiberg-Witten differential (c.f (1.10)) and

$$\mathcal{F}^{N_f}_0 = \sum_{i=1}^{N_f} \left( \frac{\tilde{a} - m_i}{\sqrt{2}} \right)^2 \ln \left( \frac{\tilde{a} - m_i}{\sqrt{2}} \right) + \sum_{i=1}^{N_f} \left( \frac{\tilde{a} + m_i}{\sqrt{2}} \right)^2 \ln \left( \frac{\tilde{a} + m_i}{\sqrt{2}} \right). \quad (6.6)$$

Furthermore, $\mathcal{F}_s^{N_f}$ is some constant (corresponding to classical effective coupling constant) and $\tilde{a}$ is a quantity that the residue contribution of Seiberg-Witten 1-form is subtracted from $a$. The first two expansion coefficients of $N_f = 1$ theory are given by

$$F_2 = -\frac{\Lambda^3 m_1}{64}, \quad F_3 = \frac{3\Lambda^6}{16384}. \quad (6.7)$$
It is now easy to see the dependence of the masses of the quarks. (6.3) is complicated than (6.4), but note that in (6.3) for \( m_i \to 0 \) it reduces to (6.4), so (6.3) includes the Seiberg and Witten’s formula (6.4).

### 6.3 Check

The prepotential has been obtained in this way, but does it really have field theoretic meaning? The calculation presented so far is based on the data of a Riemann surface, but if it has a physical meaning, the validity of it must be discussed by a method of field theory. One can see that the prepotential up to one-loop level certainly coincided with the result of perturbative calculus, but the instanton effects, specifically, its expansion coefficients coincide with the value expected from field theory?

As a method to check this, there is a complicated method called instanton calculus and the instanton contribution can be determined by this. However, since the actual calculation is very complicated, explicit calculation is usually proceeded up to 2-instanton level, but at least up to this level it is confirmed that the result based on Riemann surface is not contradict to the instanton calculus. In this way, Seiberg and Witten’s approach gained supports that it is correct also as a physics [16, 34, 35, 36, 37, 38, 39, 40].

Though once it was pointed out that there was a contradiction with the result of instanton calculus in the case of SU(2) with three quarks, this was resolved by admitting a linear transformation which shifts moduli by QCD parameter [38, 39, 40]. The linear transformation mentioned here is to specify where is the origin of the moduli space, so this is trivial in a sense, but note that in order to derive instanton contribution to prepotential from Seiberg-Witten curve this shift is important. This kind of discrepancy seems to admit generally for the hyperelliptic curves including quarks constructed so far, in fact, also in the case of SU(3) with four and five quarks, it is observed by Ewen and Förger that such constant shift of moduli is necessary to correctly include instanton effects [41].

Moreover, when the gauge group is exceptional Lie groups, it is known that there is a contradiction between the results by hyperelliptic curves and by instanton calculus. In these cases, it is known that if particular spectral curves, non-hyperelliptic curves often referred to square root type [42, 43, 44], are regarded as Seiberg-Witten curves then the instanton contribution to the prepotentials from these curves coincide with the instanton calculus [45, 46]. In this sense Seiberg-Witten curves should be formulated by spectral curves of integrable system rather than hyperelliptic curves not only in classical but also in exceptional gauge groups.

### 7 Analytic continuation of Period integrals

We have overviewed the theory of prepotential from the view point of Picard-Fuchs equation. Next, let us consider prepotentials from other standpoint.
7.1 Period integrals

We have introduced Picard-Fuchs equations in order to remove the labour of direct evaluation of periods because it is very intractable in general. As a matter of fact, it is sufficient to obtain solutions at regular singularities of Picard-Fuchs equations in order to study the behavior of periods at each singularities on the moduli space, but in the method intermediated by Picard-Fuchs equations, since the analysis depends on case by case such as SU(2) or SU(3), the derivation of general form of instanton correction terms is not easy. However, for about 1-instanton level, it can be slightly easily derived by using application of analytic continuation of period integrals [30, 47, 48, 49], although the derivation of higher order instanton corrections in this method will not work well because of technical problems. This approach was taken in the analysis of periods of Calabi-Yau manifolds, and the case at hand can be regarded as a version of them.

Let us consider the periods of SU($N_c$) gauge theory as an example, but details are omitted here. Since the Seiberg-Witten curve is $2N_c$-order polynomial, it has $2N_c$ zeros $e_i$ (branching points) on $x$-plane. In the weak coupling region, expanding the Seiberg-Witten differential around $\Lambda \sim 0$ and then integrating it over $\alpha_i$-cycle which enclose $e_i$, then one gets

$$a_i = e_i + \sum_{n=1}^{\infty} \frac{(1/2)^n \Lambda^{2N_c n}}{n!(2n)!} \left( \frac{\partial}{\partial e_i} \right)^{2n-1} \prod_{k,k \neq i} (e_k - e_i)^{-2n}. \quad (7.1)$$

The calculation over $\beta_i$-cycle defined in a similar way involves logarithmical divergence. Then rewriting Seiberg-Witten differential as

$$\lambda = dx \int_{-\infty}^{\infty} ds \frac{\Gamma(-s)\Gamma(s + 1/2)}{2s\Gamma(1/2)} \prod_{k=1}^{N_c} (x - e_k)^{-2s} (-\Lambda^{2N_c})^s \quad (7.2)$$

and performing residue calculus over $s = \{0\} \cup N$ and integrating it over $\beta_i$-cycle, one finds that $a_{D_i}$ is given by

$$a_{D_i} = \frac{i}{2\pi} \sum_k (a_i - a_k) \ln \left( \frac{e_i - e_k}{\Lambda} \right)^2 - \frac{i}{\pi} \sum_k (e_i - e_k) + \frac{i\ln 2}{\pi N_c} \sum_k (a_i - a_k)$$

$$-\frac{i\Lambda^{2N_c}}{8} \frac{\partial}{\partial e_i} \sum_k \frac{1}{\prod_{l \neq k}(e_k - e_l)^2}. \quad (7.3)$$

Rewriting this by $a_j$ and integrating it over the period, one can find the 1-instanton contribution to the prepotential and the result coincides with the instanton calculus.

This approach to get a general form of prepotential, in particular, the instanton correction part, from period integrals was proceeded in [30, 47, 48, 49], but the most characteristic advantage of this method was that the general form of prepotential could be derived both in weak and strong coupling region without using Picard-Fuchs equations. D’Hoker et al. [48] determined the prepotentials based on all classical Lie gauge groups and the result showed the agreement with the instanton calculus. Furthermore, D’Hoker and Phong [49] succeeded to give a formula of the prepotential in the strong coupling region for the SU($N_c$) gauge theories.
8 The scaling relation

It is well-known that the prepotential satisfies a very helpful homogeneous relation (Euler equation) called scaling relation \[50\]. In this section, the basics of the scaling relation are discussed.

8.1 Transformation rule of prepotential

Firstly, let us see how the prepotential transforms under the action of \( \Gamma(2) \) according to Matone \[50\]. We have already seen that the effective coupling constant transforms under the action of \( \Gamma(2) \). Since the effective coupling constant is a second order derivative of the prepotential, this is equivalent to

\[
\frac{\partial^2 \tilde{F}(\tilde{a})}{\partial \tilde{a}^2} = \frac{A \frac{\partial^2 F}{\partial a^2} + B}{C \frac{\partial^2 \tilde{F}}{\partial \tilde{a}^2} + D},
\]

where \( \tilde{a} = Ca_D + D \). The left hand side of \((8.1)\) is written by \( \tilde{a} \), but by rewriting this by \( a \), one finds

\[
\frac{\partial^2 F}{\partial a^2} = \left[ -\left( \frac{\partial a}{\partial \tilde{a}} \right)^3 \frac{\partial^2 \tilde{a}}{\partial a^2} \frac{\partial}{\partial \tilde{a}} + \left( \frac{\partial a}{\partial \tilde{a}} \right)^2 \frac{\partial^2}{\partial a^2} \right] \tilde{F}(\tilde{a}),
\]

so from \((8.1)\)

\[
(CF'' + D)\frac{\partial^2 \tilde{F}}{\partial \tilde{a}^2} - CF'''\frac{\partial}{\partial a} - (AF'' + B)(CF'' + D)^2 = 0
\]

\((8.3)\) can be obtained. Here, \( F'' = \frac{\partial}{\partial a} F(a) \) and \( \tilde{F} = \tilde{F}(\tilde{a}) \). \((8.3)\) can be solved over \( \tilde{F} \) and in fact it has a solution

\[
\tilde{F}(\tilde{a}) = F(a) + \frac{1}{2}(ACa_D^2 + BDa^2) + BCaa_D + c(Ca_D + D),
\]

where \( c \) is the integration constant, but can be set to zero because the linear term concerning periods does not contribute to the effective coupling constant. In this way, the transformation rule is obtained.

Next, considering the quantity

\[
\mathcal{G}(a) = F - \frac{a}{2} \frac{\partial F}{\partial a},
\]

one finds that \( \mathcal{G} \) is invariant under the action of \( \Gamma(2) \). Accordingly, \( \mathcal{G} \) is a modular invariant. \( \mathcal{G} \) is a function in \( a \), but by regarding \( a = a(u) \) \( (\prime = d/du) \), it follows that

\[
\frac{d\mathcal{G}}{du} = \frac{1}{2}(a' a_D - aa'_D).
\]

Now, look at the right hand side of \((8.6)\). Since it corresponds to the Wronskian of the Picard-Fuchs equation, it is actually a constant. To determine this constant, it is enough to substitute the periods obtained by Picard-Fuchs equation \((5.13)\) into this expression. Thus,

\[
\frac{d\mathcal{G}}{du} = \frac{-i}{2\pi}.
\]

\((8.7)\)

Therefore,

\[
F - \frac{a}{2} a_D = -\frac{i}{2\pi} u.
\]

\((8.8)\)

This is the scaling relation of prepotential.

Remark: After the discovery of this relation by Matone, it was checked that \((8.8)\) holds exactly in view of instanton calculus by Fucito and Travaglini \[51\].
8.2 Other view points

After the discovery of Matone the same scaling relation was rederived in various view points. For example, there are the degree counting of the effective coupling constant [52], Whitham theory of soliton [53], anomalous superconformal Ward identity [54] and direct derivation by Picard-Fuchs equation [55].

Here let us derive the scaling relation for SU($N_c$) as an example. Firstly, recall the Seiberg-Witten curve given in (4.6). For the variables in this curve, associating degree (mass dimension)
\[ \text{deg } (y) : \text{deg } (x) : \text{deg } (s_i) : \text{deg } (\Lambda) = n + 1 : 1 : i : 1 \]
(8.9)

one can see that this curve is homogeneous. On the other hand, since the periods are determined from Seiberg-Witten differential, that the periods found to have degree 1 in a similar way.

Accordingly, periods can be seen as a homogeneous function in moduli and QCD scale parameter, and as a result the Euler equations are derived to satisfy
\[ a_i = \sum_{j=2}^{N_c} s_j \frac{\partial a_i}{\partial s_j} + \Lambda \frac{\partial a_i}{\partial \Lambda}, \quad a_{D_i} = \sum_{j=2}^{N_c} s_j \frac{\partial a_{D_i}}{\partial s_j} + \Lambda \frac{\partial a_{D_i}}{\partial \Lambda}. \]
(8.10)

Since $a_{D_i}$ must be expressed by $a_j$ when we think of prepotential, and by interpreting $a_{D_i} = a_{D_i}(a_j(s_k, \Lambda), \Lambda)$ and using the second equation in (8.10) to the first equation, one gets
\[ a_{D_i} = \sum_{j=1}^{N_c} a_j \frac{\partial a_{D_i}}{\partial a_j} + \Lambda \frac{\partial a_{D_i}}{\partial \Lambda}, \]
(8.11)

where $\partial/\partial \Lambda$ acts only the second argument of $a_{D_i} = (a_j, \Lambda)$. Substituting (3.5) into (8.11) and further integrating it over $a_i$, one gets
\[ \sum_{i=1}^{N_c} a_i \frac{\partial \mathcal{F}}{\partial a_i} + \Lambda \frac{\partial \mathcal{F}}{\partial \Lambda} - 2\mathcal{F} = 0, \]
(8.12)

where integration constants are ignored because it can be absorbed by redefinition of prepotential. (8.12) is the one known as scaling relation of prepotential.

Actually, it is known that $\Lambda \partial \mathcal{F}/\partial \Lambda$ is proportional to the product of the coefficient of 1-loop beta function and the moduli [53], therefore in the case of SU(2) the scaling relation is given by
\[ a a_D - 2\mathcal{F} = \frac{i}{\pi} u, \]
(8.13)

which coincides with (8.8).

8.3 Scaling relation including massive quarks

Next, let us consider the scaling relation of prepotential of the case including quarks. According to the result, in the case of SU($N_c$) with $N_f$ quarks of mass $m_i$, the prepotential satisfies
\[ 2\mathcal{F} - \sum_{i=2}^{N_c} a_i \frac{\partial \mathcal{F}}{\partial a_i} = \sum_{i=2}^{N_f} m_i \frac{\partial \mathcal{F}}{\partial m_i} + \Lambda \frac{\partial \mathcal{F}}{\partial \Lambda}. \]
(8.14)
but in the case that the masses are not all zero (8.14) does not give a simple relation between moduli, quark masses and prepotential in contrast with the previous example. However, the right hand side of (8.14) can be calculated by another method, and in the case of SU(2) with $N_f = 1$ massive quark of mass $m$ the scaling relation takes the form $(\xi' = d/du)$ \[56\]

\[\begin{align*}
\frac{\partial F}{\partial a} - 2F &= \frac{m}{2\sqrt{2}}(n'a - na_D) - \frac{m^2}{16} (i + 4i\ln 2 - 2\pi nn') \\
&- \frac{1}{4} \int \left[ a'n' \int \frac{a'Z}{4m^2 - 3u} du - a' \int \frac{a'n'Z}{4m^2 - 3u} du \right] du,
\end{align*}\]

(8.15)

where

\[Z = -8m^2 + \frac{3m\Lambda^3 - 4u^2}{4m^2 - 3u}\]

(8.16)

and $n,n' \in \mathbb{Z}$ are the winding numbers of 1-cycles around the pole $x = -m$ of Seiberg-Witten differential.

### 8.4 Examples of prepotentials obtained by scaling relation

In general, the calculation of prepotential in higher rank gauge group (with massive quarks) is very complicated, but instanton coefficients can be directly determined if the scaling relation is used. The SU(2) example is presented in appendix B. As for the other examples of prepotentials calculated in this approach, there are: SU(3) with quarks \[41, 57\], $G_2$ \[45\] and $E_6$ \[46\].

### 9 WDVV equations and perspective

#### 9.1 WDVV equations

We have seen that the prepotential satisfies a relation called scaling relation, but it also satisfies another important relation. It is the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations, often appears in two-dimensional topological field theory \[58\].

In two-dimensional topological field theory, the topological free energy $F = F(t^1, \ldots, t^n)$, where $t^i$ are flat coordinates, which is a generating function of correlation functions, has following properties.

For the third order derivative

\[c_{ijk}(t) := \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}, \quad (9.1)\]

it satisfies

\[\eta_{ij} := c_{1ij}(t) = \frac{\partial^3 F}{\partial t^i \partial t^j} = \text{constant}, \quad \eta^{ij} := (\eta_{ij})^{-1}, \quad c_{ij}^k := \eta^{kl}c_{lijk}(t) \quad (9.2)\]

and from commutativity of the structure constant $c_{ijk}(t)$, the WDVV equations follow

\[F_{ijk} \eta^{kl} F_{lmn} = F_{njk} \eta^{kl} F_{lim}, \quad F_{ijk} \equiv \frac{\partial^3 F(t)}{\partial t^i \partial t^j \partial t^k}. \quad (9.3)\]

On the other hand, the ”WDVV equations” in four-dimensional $N = 2$ Yang-Mills theory of gauge group $G$ look like \[9.3\] but take the form \[59, 60, 61, 62\]

\[(\mathcal{F}_i)(\mathcal{F}_k)^{-1}(\mathcal{F}_j) = (\mathcal{F}_j)(\mathcal{F}_k)^{-1}(\mathcal{F}_i), \quad i,j,k = 1, \ldots, \text{rank}(G), \quad (9.4)\]
where the symbols mean the matrix notation

\[(F_i)_{jk} = \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k}.\] (9.5)

Marshakov et al. [60, 61] pointed out that the WDVV equations in this form are satisfied by not only the prepotentials in four-dimensional \(N = 2\) Yang-Mills theories but also those in five-dimensional supersymmetric Yang-Mills theories of \(S^1\) compactification models. This indicates that these two kinds of theories are simply one of solutions to the WDVV equations (9.4). Accordingly, the manipulation to get prepotential extensively stated in the previous sections was simply a labour to obtain one of the solutions to (9.4)! For this reason, in order to gain understandings on these supersymmetric Yang-Mills theories, it is necessary to study them in the framework of WDVV equations.

### 9.2 Perspective

In view of WDVV equations, the low energy effective theory of four-dimensional supersymmetric Yang-Mills theory seems to imply that it has a nature as a topological field theory, but since this effective theory is not necessary to be topological field theory, it is interesting that the equations (9.4) like (9.3) hold. However, it is not still turned out whether this effective theory can be really interpreted as topological field theory. Moreover, the WDVV equations in this form is known to widely hold also in the case including massive quarks, and in such case the masses of quarks are regarded as if they are one of periods [61]. In this sense, the origin of the mass of the quarks may be explained in the frame of topological field theory.

Nevertheless, since the meaning and importance of the role of (9.4) in this supersymmetric Yang-Mills theory are not still clarified, studying (9.4) in detail may provide something new aspect of \(N = 2\) supersymmetric Yang-Mills theory.

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### A. Other representation of Picard-Fuchs equations

Usually in most cases, Picard-Fuchs equations are represented by using moduli derivatives, but in the cases except \(SU(2)\) are represented by a system of simultaneous partial differential equations. However, these Picard-Fuchs equations can be expressed into a more convenient form.

According to the result [63], it is better to make differential equation by using QCD scale parameter. This is because Seiberg-Witten differentials in any (classical) gauge group always take the form

\[\frac{d^k \lambda}{d \Lambda^k} = \frac{\text{polynomial in } x}{y} dx\] (A.1)
and this quantity always representable by a linear combination of Abelian differentials \cite{23}, and therefore summing up over various $k$ produces ordinary differential equation \cite{33}. However, the equation must have periods as independent solutions, so the order of the equation is the same with the total number of periods.

Let us cite the SU(3) Picard-Fuchs ordinary differential equation \cite{33} for a reference. Denoting the equation by $(z = \Lambda^6)$

$$\frac{d^4 \Pi}{dz^4} + \sum_{i=0}^{3} c_i \frac{d^3 \Pi}{dz^i} = 0,$$

(A.2)

one finds that the coefficients $c_i$ are

\begin{align*}
c_0 &= -\frac{45 (3z - 4u^3 + 27v^2)}{2z^2 \Delta_{SU(3)}}, \\
c_1 &= \frac{45 (1053z^2 - 538zu^3 + 40u^6 + 3267zv^2 - 54u^3v^2 - 1458v^4)}{2z^2 \Delta_{SU(3)}}, \\
c_2 &= \frac{1}{4z^2 \Delta_{SU(3)}} \left[445905z^3 - 8 \left(4u^3 - 27v^2\right)^3 + z^2 \left(-217368u^3 + 734589v^2\right)
+ 36z \left(676u^6 - 135u^3v^2 - 29889v^4\right)\right], \\
c_3 &= \frac{1}{z \Delta_{SU(3)}} \left[76545z^3 - 162z^2 \left(244u^3 - 297v^2\right) - 4 \left(4u^3 - 27v^2\right)^3
+ 9z \left(656u^6 - 1080u^3v^2 - 22599v^4\right)\right],
\end{align*}

(A.3)

where $\Delta_{SU(3)}$ is the product

$$\Delta_{SU(3)} = (15z - 4u^3 + 27v^2) \Delta_{SU(3)}$$

(A.4)

of the discriminant of SU(3) Seiberg-Witten curve

$$\Delta_{SU(3)} = \left[729z^2 + \left(4u^3 - 27v^2\right)^2 - 54z \left(4u^3 + 27v^2\right)\right].$$

(A.5)

At first sight, this ordinary differential system seems to be more complicated than the original SU(3) Picard-Fuchs system \cite{28}, but it has an advantage that the solutions take a convenient form.

The moduli space of the SU(2) theory can be interpreted as a Riemann sphere with three singularities (after suitable compactification), but that of the SU(3) theory can be seen as the complex projective space $CP^2$ \cite{33}, which can be covered by the three coordinate neighborhood

\begin{align*}
P_1 : \left(\frac{4u^3}{(27\Lambda^6)^2} : \frac{v^2}{27\Lambda^3} : 1\right),
P_2 : \left(\frac{4u^3}{v^2} : 1 : \frac{27\Lambda^{12}}{v^2}\right),
P_3 : \left(1 : \frac{27v^2}{4u^3} : \frac{(27\Lambda^6)^2}{4u^3}\right).
\end{align*}

(A.6)

We can choose $P_2$ and $P_3$ as weak coupling region. We faced on a similar situation when we discussed the mirror symmetry of Calabi-Yau manifold with several complex structure moduli. There, how to choose the large radius limit was a problem, so some people might remember this. In the case at hand, there are two choices, but in view of QCD parameter both $P_2$ and $P_3$ can be thought to be in the region $\Lambda \sim 0$. Thus the solutions to (A.2) give common basis on these two coordinates.
B. Application of scaling relation

Since the scaling relation is simply an Euler equation, it is not so interesting at first sight, but let us show that this relation is very useful by taking a derivation of the SU(2) prepotential as an example \[50\].

Firstly, let us recall that the prepotential was determined by calculating the periods by Picard-Fuchs equation. In that calculation, the period \(a\) was a function in the moduli \(u\), but \(u\) was finally represented by \(a\). What happens, if this calculation is applied to Picard-Fuchs equation \((5.3)\)? \((5.3)\) is written by \(u\) derivative of \(a\), but regarding \(u\) as a function in \(a\), i.e., \(u = u(a)\), one finds \((\prime = d/da)\)

\[
\frac{da}{du} = \left(\frac{du}{da}\right)^{-1} = u^{-1}, \quad \frac{d^2 a}{du^2} = -u^{-3} u''.
\] (B.1)

Substituting this into \((5.3)\), one can arrive at the cerebrated Matone’s differential equation \[50\]

\[4(\Lambda^4 - u^2)u'' + au'^3 = 0.\] (B.2)

This equation enables to determine \(u\) as a function in \(a\), if \(u\) is solved over \(a\) \((6.1)\)

\[u = \sum_{i=0}^{\infty} f_i a^{2-4i},\] (B.3)

where the expansion coefficients are denoted by \(f_i\) with \(f_0 = 2\).

Here, let us recall the scaling relation \((8.8)\). Since up to 1-loop level prepotential can be determined from perturbation theory, this part may be thought as already known, although the general form of prepotential is always written in the form (classical)+(1-loop)+(instantons). For this, regarding only the instanton expansion coefficients are unknowns and setting

\[\mathcal{F} = i \frac{a^2}{\pi} \left[ \ln \left(\frac{a}{\Lambda}\right)^2 - \sum_{i=0}^{\infty} \mathcal{F}_i \left(\frac{\Lambda}{a}\right)^{4i} \right]\] (B.4)

and substituting this and \((B.3)\) into \((8.8)\), one finds

\[f_k = 4k \mathcal{F}_k.\] (B.5)

As was already stated, the calculation of \(a_D\) is very complicated, but this method directly determine the expansion coefficients.

Moreover, since \((8.8)\) directly relates the moduli and prepotential, the differential equation satisfied by prepotential

\[\mathcal{F}''' = \frac{\pi^2}{4} \frac{(a\mathcal{F}'' - \mathcal{F}')^3}{4 \Lambda^4 + \pi^2 (a\mathcal{F}' - 2\mathcal{F})^2}\] (B.6)

is available, provided \(u\) of \((B.2)\) is substituted into \(u\) of \((8.8)\).

**Remark:** The method of scaling relation played an important role also in the proof \[55\] of Nekrasov’s insist that states the Seiberg-Witten theory can be obtained from a circle compactification of five-dimensional supersymmetric Yang-Mills theory \[64\].

22
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