ON BOUNDARIES OF $\varepsilon$-NEIGHBOURHOODS OF PLANAR SETS, PART II: GLOBAL STRUCTURE AND CURVATURE

J. S. W. LAMB, M. RASMUSSEN, AND K. TIMPERI

Abstract. We study the global topological structure and smoothness of the boundaries of $\varepsilon$-neighbourhoods $E_\varepsilon = \{ x \in \mathbb{R}^2 : \text{dist}(x, E) \leq \varepsilon \}$ of planar sets $E \subset \mathbb{R}^2$. We show that for a compact set $E$ and $\varepsilon > 0$ the boundary $\partial E_\varepsilon$ can be expressed as a disjoint union of an at most countably infinite union of Jordan curves and a possibly uncountable, totally disconnected set of singularities. We also show that curvature is defined almost everywhere on the Jordan curve subsets of the boundary.

CONTENTS

1. Introduction 1
   1.1. Main Results 2
   1.2. Context 4
   1.3. Outlook 4
2. Jordan Curves on the Boundary 4
   2.1. The Role of Sharp-sharp Singularities in Boundary Topology 5
   2.2. Boundaries of Connected Components of the Complement 5
3. Curvature 12
   3.1. Existence of Curvature via Bounded Variation 13
   3.2. A Lower Bound for Differences of Tangential Directions 16
Appendix A. Preliminaries 23
Acknowledgements 30
References 30

1. INTRODUCTION

For a given set $E \subset \mathbb{R}^2$ and radius $\varepsilon > 0$, the (closed) $\varepsilon$-neighbourhood of $E$ is the set

$$E_\varepsilon := \overline{B_\varepsilon(E)} := \bigcup_{x \in E} B_\varepsilon(x),$$

where the overline denotes closure and $B_\varepsilon(\cdot)$ is an open ball of radius $\varepsilon$ in the Euclidean metric. The sets $E_\varepsilon$ are sometimes also called tubular neighbourhoods [7], collars [13] or parallel sets [14,16].

Apart from being natural and fundamental objects in (Euclidean) geometry, $\varepsilon$-neighbourhoods also naturally arise in a number of specific settings. For instance, we are motivated by the classification and bifurcation of minimal invariant sets in random dynamical systems with bounded noise [10], but $\varepsilon$-neighbourhoods also naturally feature for instance in control theory [4].

Date: July 19, 2021

2020 Mathematics Subject Classification. 51F30; 57K20, 54C50, 51M15, 58C06.
1.1. Main Results. The main results of this article concern the global topological structure of the boundary $\partial E_\varepsilon$ of a closed $\varepsilon$-neighbourhood $E_\varepsilon$ of a compact planar set, and the existence of curvature on this boundary.

1.1.1. Global Topology of the Boundary. Building on the novel techniques developed in [9] allows us to make the topology of $\partial E_\varepsilon$ more precise in two ways. First, it is possible to give a description that applies for all $\varepsilon > 0$, without the need to ignore a countable set of $\varepsilon$-values, as was done in previous studies of open $1\varepsilon$-neighbourhoods [2, 3]. Second, the local geometry of boundaries of closed $\varepsilon$-neighbourhoods reveals that no simple arcs exist on $\partial E_\varepsilon$ (for any $\varepsilon > 0$), and that point components appear only in a very specific way.

Point components correspond to boundary points $x \in \partial E_\varepsilon$ that do not lie on the boundary of any connected component of the complement $E^c_\varepsilon := \mathbb{R}^2 \setminus E_\varepsilon$. Following [9], we call such points inaccessible singularities. Our first main result is that, apart from the set of inaccessible singularities, the boundary $\partial E_\varepsilon$ consists of a countable (possibly finite) union of Jordan curves.

---

1The boundaries $\partial E_{<\varepsilon}$ of open $\varepsilon$-neighbourhoods $E_{<\varepsilon} := \bigcup_{x \in E} B_\varepsilon(x)$ have been referred to as the $\varepsilon$-boundaries [6] or $\varepsilon$-level sets [12] of $E$. We note that $\partial E_{<\varepsilon} = \{ x \in \mathbb{R}^2 : \text{dist}(x, E) = \varepsilon \}$ and that in general $\partial E_\varepsilon$ is a closed subset of $\partial E_{<\varepsilon}$. In this article we deal exclusively with the closed $\varepsilon$-neighbourhoods (1.1).

2We call the image $\Gamma := \gamma([a, b])$ of a closed interval $[a, b]$ under a continuous map $\gamma : [a, b] \to \mathbb{R}^2$ a curve. If $\gamma$ is injective, we call $\Gamma$ a simple curve, and if injectivity is violated only at the endpoints, i.e. $\gamma(a) = \gamma(b)$, we say that $\Gamma$ is a simple closed curve or a Jordan curve.
Theorem 1 (Global structure of the boundary). For a compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the boundary $\partial E_\varepsilon$ is a disjoint union

$$\partial E_\varepsilon = \mathcal{I} \cup \mathcal{J},$$

where $\mathcal{I}$ is the set of inaccessible singularities and $\mathcal{J} = \bigcup_{i \in \mathcal{I}} J_i$ is a countable (possibly finite) union of Jordan curves $J_i$. Furthermore there is a unique representation with the property that each Jordan curve $J_i$ satisfies $J_i \subset \partial U$ for some connected component $U$ of the complement $E_\varepsilon^c$.

In general, the Jordan curves $J_i$ in Theorem 1 need not be disjoint. However, the bounded plane components defined by these Jordan curves are mutually disjoint, and the intersection of any two Jordan curves $J_i$ is either empty or contains finitely many points.

We have shown in [9] that each boundary point $x \in \partial E_\varepsilon$ is either smooth (in the sense that, in a neighbourhood of $x$, $\partial E_\varepsilon$ is a $C^1$-curve) or belongs to exactly one of eight distinct types of singularities. See Definition A.17 and Figure 11 in Appendix A for the definitions and schematic illustrations of the different types. From the topological point of view, these can further be divided into three groups:

(i) Point components of the boundary: the inaccessible singularities.

(ii) Boundary points, around which the boundary $\partial E_\varepsilon$ can be uniquely represented as a simple curve: these are the smooth points as well as the so-called wedges, sharp and sharp-chain singularities and shallow singularities.

(iii) Boundary points, around which there is more than one way to locally represent the boundary as a union of simple curves: the so-called sharp-sharp singularities.

Whenever sharp-sharp singularities exist on the boundary $\partial E_\varepsilon$, there are thus several ways of representing the boundary as a union of curves. It turns out, however, that the boundary of each individual connected component of the complement $E_\varepsilon^c$ has a unique representation as a finite union of Jordan curves. In this article we show how to construct such representations, and use them to arrive at a global representation of the boundary $\partial E_\varepsilon$ as a union of Jordan curves and point components, cf. Figure 1.

1.1.2. Curvature of the Boundary. Another fundamental property of interest is the smoothness of the boundary. It is known that for a fixed compact set $E \subset \mathbb{R}^2$, the components of the boundary $\partial E_\varepsilon$ are Lipschitz manifolds except for a zero measure set of radii $\varepsilon > 0$ [6, 7, 15]. In light of Theorem 1 and the geometric results obtained in [9], this property can be interpreted as the existence of well-defined tangents almost everywhere on the Jordan curve components of the boundary. This holds true despite the fact that singularities of wedge type may be dense on subsets of $\partial E_\varepsilon$ that have positive one-dimensional Hausdorff measure.

It turns out that it is possible to assess the curvature of $\partial E_\varepsilon$ by viewing the boundary locally as the graph of a Lipschitz function. The construction of such local boundary representations around all boundary points $x \in \partial E_\varepsilon$ is one of the key technical results of [9]. The existence of second derivatives (defined in a suitable way) of these functions then implies the existence of curvature for the corresponding boundary segments. It should be noted that in general, the first derivative of a local boundary representation only exists almost everywhere, and one cannot simply define the second derivative naïvely as a limit in the classical sense.

Theorem 2 (Existence of curvature). Let $E \subset \mathbb{R}^2$ be compact and let $\mathcal{J} = \partial E_\varepsilon \setminus \mathcal{I}$ where $\mathcal{I}$ is the set of inaccessible singularities. Then (signed) curvature $\kappa(x)$ exists for $\mathcal{H}^1$-almost all $x \in \mathcal{J}$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

\[\text{More precisely, the boundary of the unique connected component of the complement that touches the sharp-chain singularity can be uniquely represented by a simple curve in any sufficiently small neighbourhood, cf. Lemma 2.1.}\]
We obtain Theorem 2 by showing that the derivative function is of bounded variation, and hence has a well-defined derivative almost everywhere. This second derivative allows one to define curvature locally by applying the classical formula for the curvature of a function graph. The proofs of both Theorems 1 and 2 are based on the geometric properties of ε-neighbourhoods obtained in [9], executive summary of which is provided in Appendix A.

1.2. Context. The existing literature on ε-neighbourhoods can be roughly divided into two lines of inquiry. The first one concerns the topological properties of the ε-boundaries ∂E<ε and encompasses the early work of Brown [3] and its more recent generalisation to geodesic metric spaces by Blokh, Misirewicz and Oversteegen [2]. The second starts from the work of Ferry [6] and concerns conditions on the radius ε that guarantee, for a fixed compact set E, the ε-boundary ∂E<ε to be a Lipschitz manifold [7,15].

The existing literature on the topology of ε-neighbourhoods concerns the boundaries ∂E<ε of open ε-neighbourhoods E<ε := ∪x∈E Bε(x). In [3], building on previous work on triods by Moore [11] and using geometric arguments together with local connectedness of regular continua4, Brown showed that apart from a countable set of radii ε > 0, every connected component of the ε-boundary ∂E<ε of a compact set E ⊂ R2 is either a point, an arc (a simple curve), or a Jordan curve (a simple closed curve). Brown’s results were extended in [2] to the setting of 2-manifolds with a geodesic metric, possibly with boundary. It was shown in addition that for all but a countable set of ε, Jordan curves are dense on ∂E<ε and, in case the manifold has no boundary, arc components do not exist.

Ferry showed in his pioneering article [6] that ε-neighbourhoods of sets E ⊂ Rn are Lipschitz manifolds for almost all ε > 0, when n ∈ {2,3}. A decade later Fu [7] applied Federer’s theory of curvature measures on sets of positive reach [5] to further narrow down the exceptional set of radii for which the Lipschitz manifold structure is violated. Recently, Rataj and collaborators improved further on these results for ε-neighbourhoods of compact sets in two-dimensional Euclidean spaces and Riemannian manifolds [15].

1.3. Outlook. While our results for the moment concern the properties of boundaries of ε-neighbourhoods of only planar sets E ⊂ R2, our techniques appear well-suited for obtaining local and global properties of boundaries of ε-neighbourhoods also in higher dimensions. It would be of particular interest to consider the above-mentioned results of Ferry [6] from this complementary point of view.

Finally, the current paper and [9] have arisen from our interest in bifurcations of minimal invariant sets of random dynamical systems with bounded noise, which naturally appear as dynamically defined ε-neighbourhoods. In this context, the aim is to develop a theory which allows for the characterisations of topological and/or geometric changes of such sets in parametrised families. The results in this paper provide a characterisation of boundaries at fixed values of parameters (including ε), which is a first step towards more general results concerning the classifications of qualitative changes of minimal invariant sets in (generic) parametrised families of random dynamical systems with bounded noise.

2. Jordan Curves on the Boundary

In this section we show that the boundary ∂Eε is the disjoint union of a possibly uncountable set I of inaccessible singularities and an at most countably infinite collection J of Jordan curves. By definition, inaccessible singularities do not lie on the boundary of any connected component of the complement, and it was shown in [9] that these correspond to so-called chain (S6) and

4A metric space X is a continuum, if it is compact and connected. A continuum X is regular, if every a ∈ X has arbitrarily small neighbourhoods B(a) whose boundary ∂B(a) is a finite set.
chain-chain singularities (S7), see Figure 1 (b). To prove Theorem 1, one thus needs to consider the boundaries of connected components of the complement, and obtain a representation for each as a union of Jordan curves. Intuitively, these Jordan curves can be identified by connecting adjacent simple curves that represent local boundary segments. There is an essentially unique way of connecting such curve representations around all types of boundary points, except sharp-sharp singularities, where the boundary curves ‘split’, see Figure 2. However, it turns out that for each connected component \( U \) of the complement \( E_\varepsilon \), there exists a unique way of connecting the curves around sharp-sharp singularities that results in a representation of the boundary \( \partial U \) as a finite union of Jordan curves.

2.1. The Role of Sharp-sharp Singularities in Boundary Topology. Let \( U \) be a connected component of the complement \( E_\varepsilon \) and let \( Q \) denote the set of sharp-sharp singularities on \( \partial U \). According to Lemma A.21 the set \( Q \) is finite, and Proposition A.18 implies that each \( x \in Q \) lies on the boundary of at most two connected components of the complement \( E_\varepsilon \). Hence,

\[
Q = Q_1 \cup Q_2,
\]

where \( Q_1 \) contains those sharp-sharp singularities that lie on the boundary of a unique connected component \( U \) of the complement \( E_\varepsilon \), and \( Q_2 \) contains those \( x \in Q \) for which there exist two disjoint connected components \( U, V \in E_\varepsilon \) with \( x \in \partial U \cap \partial V \). The fact that both \( Q_1 \) and \( Q_2 \) may be non-empty is the reason sharp-sharp singularities play a central role in the topology of the boundary \( \partial U \).

2.2. Boundaries of Connected Components of the Complement. Let the sets \( U \) and \( Q = Q_1 \cup Q_2 \) be as above. Proposition A.13 asserts that each \( x \in \partial U \) has a neighbourhood \( B_r(x) \cap \partial U \) for any \( x \in Q_1 \) would intersect itself at \( x \), and would thus not be a Jordan curve.

The boundary \( \partial U \) of a single connected component \( U \) may thus consist of more than one Jordan curve for two reasons. First, \( U \) need not be simply connected, and can thus surround several subsets \( A \subset E_\varepsilon \). The boundary of each such \( A \) contains at least one Jordan curve. Second, a single curve containing the boundary subset \( B_r(x) \cap \partial U \) for any \( x \in Q_1 \) would intersect itself at \( x \), and would thus not be a Jordan curve.
to use these functions to represent the boundary subset \( \partial U \cap \overline{B_x} \) with either one (for \( x \in \partial U \cap Q_1 \)) or two (for \( x \in Q_1 \)) simple curves.

**Lemma 2.1 (Boundaries of connected components of the complement as unions of simple curves).** Let \( E \subset \mathbb{R}^2 \) be compact, let \( \varepsilon > 0 \) and let \( U \) be a connected component of the complement \( E^c_\varepsilon \). For each \( x \in \partial U \), let \( B_x := B_r(x) \) be the neighbourhood in \((A.6)\), corresponding to the local boundary representation \( G(x) \) given by Proposition A.13. In addition, denote by \( Q \) the set of sharp-sharp singularities on \( \partial U \), and let the subsets \( Q_1, Q_2 \subset Q \) be as in \((2.1)\). Then

1. for each \( x \in Q_1 \) there exist continuous functions \( \gamma_x^{(1)}, \gamma_x^{(2)} : [0, 1] \to \partial U \) for which the images \( \Gamma_x^{(i)} := \gamma_x^{(i)}([0, 1]) \) with \( i \in \{1, 2\} \) are simple curves, \( \partial U \cap \overline{B_x} = \Gamma_x^{(1)} \cup \Gamma_x^{(2)} \) and \( \Gamma_x^{(1)} \cap \Gamma_x^{(2)} = \{x\} \);
2. for each \( x \in \partial U \cap Q_1 \) there exists a continuous function \( \gamma_x : [0, 1] \to \partial U \), for which the image \( \Gamma_x := \gamma_x([0, 1]) \) is simple curve and \( \partial U \cap \overline{B_x} = \Gamma_x \).

**Proof.** We begin by fixing some notations. For each \( x \in \partial U \), let \( \Xi_x(E_\varepsilon) \) be the set of outward directions and \( \Pi_E(x) \) the set of contributors at \( x \), see Definitions A.1 and A.4. We denote by \( \mathcal{P}_x^{\text{ext}}(E_\varepsilon) \) the set of *extremal pairs* (see Definition A.6):

\[
\mathcal{P}_x^{\text{ext}}(E_\varepsilon) = \{ (\xi, y) \in \Xi_x(E_\varepsilon) \times \Pi_E(x) : \langle x - y, \xi \rangle = 0 \}.
\]

Intuitively, extremal pairs define coordinate axes that are adapted to the local geometry at \( x \), see Appendix A for more details. Proposition A.13 asserts that for each extremal pair \( (\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon) \) the corresponding pair of functions \( \gamma_x^{(i)} \) satisfies the boundary condition at \( x \).
\( \mathcal{G}(x) := \{ g_{\xi,y} : (\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_x) \} \) satisfies
\[
\partial E_x \cap B_r(x) = \bigcup_{(\xi,y) \in \mathcal{P}_x^{\text{ext}}(E_x)} g_{\xi,y}(A_{\xi,y})
\]
for some \( r > 0 \) and some closed sets \( A_{\xi,y} \subset [0, r] \). We call the collection \( \mathcal{G}(x) \) a local boundary representation (of radius \( r \)) at \( x \).

We divide the proof into two steps.

Step 1. In the first step, we define the local curve representations of the boundary \( \partial U \) around sharp-sharp singularities \( x \in Q \). We also show that the images \( \Gamma_x \) and \( \Gamma_x^i \) are simple (i.e. non-self-intersecting) curves, and that \( \Gamma_x^i \cap \Gamma_x^{(2)} \cap B_r(x) = \{ x \} \) for sufficiently small radii \( r > 0 \).

Consider a boundary point \( x \in \partial U \), and let \( \mathcal{G}(x) = \{ g_{\xi,y} : (\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_x) \} \) be a local boundary representation at \( x \), with the functions \( g_{\xi,y} \) defined as in (2.2). By the definition of sharp-sharp singularities (see Definition A.17), the set of extremal outward directions for each \( x \in Q \) satisfies \( \Sigma_2^{\text{ext}}(E_x) = \{ \xi, -\xi \} \) for some \( \xi \in S^1 \subset \mathbb{R}^2 \). Proposition A.18 furthermore states that there exist connected components \( V_\xi, V_{-\xi} \) of the complement \( E_x^c \), for which
\[
\partial U \cap B_r(x) \subset (\partial V_\xi \cup \partial V_{-\xi}) \cap B_r(x).
\]
Since \( Q \subset \partial U \), we have \( U \in \{ V_\xi, V_{-\xi} \} \) and we may define \( \xi \) so that \( U = V_\xi \). The correct way of defining the local curve representation near \( x \) now depends on whether \( x \in Q_1 \) or \( x \in Q_2 \), where \( Q = Q_1 \cup Q_2 \) as in (2.1). We deal with these cases separately as follows:

(i) Let \( x \in Q_1 \). This implies that \( U \) touches the point \( x \) from both directions, so that \( U = V_\xi = V_{-\xi} \). This corresponds to the situation in Figure 2 (a). Then
\[
\partial U \cap B_r(x) = \bigcup_{i \in \{1, 2\}} \left\{ g_{\xi,y_i}([0, s_{\xi,y_i}]) \cup g_{-\xi,y_i}([0, s_{-\xi,y_i}]) \right\}.
\]
In this case, we associate a curve segment with each extremal contributor \( y_1, y_2 \in \Pi^{\text{ext}}_E(x) \). Thus, we define for each \( i \in \{1, 2\} \) a continuous map \( \gamma_x^{(i)} : [0, 1] \to \partial U \cap B_r(x) \) by
\[
\gamma_x^{(i)}(s) := \begin{cases} 
g_{-\xi,y_i}(s_{\xi,y_i} - s L^{(i)}), & \text{if } s \in \left[0, \frac{s_{\xi,y_i}}{L^{(i)}}\right], 
g_{\xi,y_i}(s L^{(i)} - s_{\xi,y_i}), & \text{if } s \in \left[\frac{s_{\xi,y_i}}{L^{(i)}}, 1\right], \end{cases}
\]
where \( g_{-\xi,y_i}, g_{\xi,y_i} \in \mathcal{G}(x) \), and \( L^{(i)} := s_{\xi,y_i} + s_{-\xi,y_i} \) with \( s_{\xi,y_i} \) and \( s_{-\xi,y_i} \) as in (2.5). We thus obtain \( \partial U \cap B_r(x) = \Gamma_x^{(1)} \cup \Gamma_x^{(2)} \), where \( \Gamma_x^{(i)} := \gamma_x^{(i)}([0, 1]) \) for \( i \in \{1, 2\} \).

(ii) Let \( x \in Q_2 \). Since in this case \( U = V_\xi \neq V_{-\xi} \), the graphs \( g([0, s_{\xi,y_i}]) \) and \( g([0, s_{-\xi,y_i}]) \), which correspond to the direction \( -\xi \), intersect the local boundary subset \( \partial U \cap B_r(x) \) only at \( x \), and are thus not relevant for representing \( \partial U \). This corresponds to the situation in Figure 2 (b). Consequently we have
\[
\partial U \cap B_r(x) = g_{\xi,y_1}([0, s_{\xi,y_1}]) \cup g_{\xi,y_2}([0, s_{\xi,y_2}])
\]
for \( g_{\xi,y_1}, g_{\xi,y_2} \in \mathcal{G}(x) \). We thus define
\[
\gamma_x(s) := \begin{cases} 
g_{\xi,y_1}(s_{\xi,y_1} - s L), & \text{if } s \in \left[0, \frac{s_{\xi,y_1}}{L}\right], 
g_{\xi,y_2}(s L - s_{\xi,y_1}), & \text{if } s \in \left[\frac{s_{\xi,y_1}}{L}, 1\right], \end{cases}
\]
where \( g_{\xi,y_1}, g_{\xi,y_2} \in G(x) \), and \( L := s_{\xi,y_1} + s_{\xi,y_2} \) with \( s_{\xi,y_1} \) and \( s_{\xi,y_2} \) as in (2.7). In this case, we obtain a representation using only one curve, since \( \partial U \cap B_x = \Gamma_x := \gamma_x([0,1]) \).

For \( x \in Q_1 \), we define \( \Gamma^{(i)}_x := \gamma^{(i)}_x([0,1]) \) for \( i \in \{1,2\} \), where the functions \( \gamma^{(i)}_x \) are as in (2.6). Then \( \partial U \cap B_x = \Gamma^{(1)}_x \cup \Gamma^{(2)}_x \), and it follows directly from the definition of the functions \( g_{\xi,y} \in G(x) \) that the image \( \Gamma^{(i)}_x \) is a simple curve for each \( i \in \{1,2\} \). Note furthermore that sharp-sharp singularities by definition exhibit sharp-type geometry (as defined in Proposition A.18 (i)) in the direction of both extremal outward directions \( \xi_1, \xi_2 \in \Xi^\text{ext}(E_x) \). Proposition A.18 thus implies the existence of some \( r_1, r_2 > 0 \) for which \( g_{\xi_1,y_1}(s) \neq g_{\xi_2,y_2}(s) \) for all \( s \in (0,r_i) \) and \( i \in \{1,2\} \).

Hence, \( \Gamma^{(1)}_x \cap \Gamma^{(2)}_x \cap \partial E_x = \{x\} \) for all \( r < \min\{r_1,r_2\} \). By choosing \( r < \min\{r_1,r_2\} \) for the local boundary representation in the first place, we obtain the equivalent property \( \Gamma^{(1)}_x \cap \Gamma^{(2)}_x = \{x\} \).

For \( x \in Q_2 \), we define \( \Gamma_x := \gamma_x([0,1]) \), where the function \( \gamma_x \) is as in (2.8). The fact that \( \partial U \cap B_x = \Gamma_x \) follows from the definition of the curve \( \gamma_x \). Similarly to above, Proposition A.18 implies that \( g_{\xi_1,y_1}(s) \neq g_{\xi_2,y_2}(s) \) for sufficiently small \( s \in (0,r) \), which implies that the curve \( \Gamma_x \) is non-self-intersecting in every sufficiently small neighbourhood \( B_x \).

Step 2. In the second step, we define curve representations, analogous to those defined in Step 1, for the other types of boundary points \( x \in \partial U \setminus Q \).

We divide \( \partial U \setminus Q \) into the following two subsets:

\[
\begin{align*}
R &:= \{x \in \partial U : x \text{ is a wedge or } x \in \text{Unp}_x(x)\}, \\
P &:= \{x \in \partial U : x \text{ is a sharp or a sharp-chain singularity}\}.
\end{align*}
\]

According to Proposition A.23, chain and chain-chain singularities (types S6 and S7) do not appear on the boundaries of connected components of the complement \( E^c_x \), which implies \( \partial U = R \cup P \cup Q \). Furthermore, it follows from Theorem A.19 that the sets \( R, P \) and \( Q \) are disjoint.

Assume first that \( x \in R \), so that either \( x \in \text{Unp}_x(x) \) or \( x \) is a wedge. Then \( \mathcal{P}^\text{ext}(E_x) = \{[\xi_1,y_1],[\xi_2,y_2]\} \), where for \( y_1 = y_2 \) in case \( x \in \text{Unp}_x(x) \). It follows immediately from Lemma A.15 and Proposition A.16 that \( \overline{B_x} \cap E^c_x = \overline{B_x} \cap U \), so that

\[
\partial U \cap \overline{B_x} = g_{\xi_1,y_1}([0,s_{\xi_1,y_1}]) \cup g_{\xi_2,y_2}([0,s_{\xi_2,y_2}]),
\]

where \( g_{\xi_1,y_1}, g_{\xi_2,y_2} \in G(x) \). By normalising the arguments with \( L := s_{\xi_1,y_1} + s_{\xi_2,y_2} \), one can thus define a function \( \gamma_x : [0,1] \to \partial U \) analogously to (2.8), so that \( \partial U \cap \overline{B_x} = \Gamma_x := \gamma_x([0,1]) \). The fact that \( \Gamma_x \) defines a simple curve in some neighbourhood \( B_x(x) \) follows directly from Proposition A.16.

Let then \( x \in P \), so that \( \Xi^\text{ext}(E_x) = \{\xi,-\xi\} \) for some \( \xi \in S^1 \). In the following, we denote by \( U_r(x,v) \) an open \( x \)-centered half-ball of radius \( r \), oriented in the direction of \( v \in S^1 \), see (A.8).

For both sharp and sharp-chain singularities, there exists a unique extremal outward direction \( v \in \{\xi,-\xi\} \), towards which the singularity exhibits sharp-type geometry, as defined in Proposition A.18 (i). This means that there exists a unique connected component \( V_\xi \) of the complement \( E^c_x \), and \( s_\xi > 0 \), for which \( U_{s_\xi}(x,\xi) \cap E^c_x = U_{s_\xi}(x,\xi) \cap V_\xi \).

Moreover, in the case of sharp-chain singularities, one can use the local boundary representation (2.2) and Proposition A.18 to show that there exist no connected components \( V \) of the complement \( E^c_x \), for which \( x \in \partial(V \cap U_r(x,\xi)) \). The detailed argument can be found in the proof of [9, Corollary 4.3] (Proposition A.23 in Appendix A). For one-sided sharp singularities this follows immediately from their definition, which states that for some \( \delta > 0 \), the set \( B_\delta(x) \cap E^c_x \) is a connected set.

Hence, for both sharp and sharp-chain singularities there exists some \( r > 0 \) for which

\[
\partial U \cap B_r(x) = g_{\xi,y_1}([0,s_{\xi,y_1}]) \cup g_{\xi,y_2}([0,s_{\xi,y_2}]),
\]
where \( g_{\xi, y_1}, g_{\xi, y_2} \in G(x) \) are as in Proposition A.13. One may thus once more define the function \( \gamma_x : [0, 1] \to \partial U \) analogously to (2.8) by concatenating the images \( g_{\xi, y_1}([0, s_{\xi, y_1}]) \) in (2.12) and normalising the argument with \( L := s_{\xi, y_1} + s_{\xi, y_2} \), so that \( \partial U \cap B_x = \Gamma_x := \gamma_x([0, 1]) \).

It follows from Lemma 2.1 that the boundary \( \partial U \) of every connected component \( U \) of the complement \( E^r_x \) can be covered by the (infinite) collection
\[
\mathcal{M} := \{ \Gamma_x : x \in \partial U \setminus Q_1 \} \cup \{ \Gamma_x^{(i)} : x \in Q_1, i \in \{1, 2\}\}
\]
of simple curves. The compactness of \( E \) implies that \( \partial U \) is compact as well, and \( \mathcal{M} \) thus always has a finite subcover. We show next that \( \mathcal{M} \) moreover has a finite, order two \( ^5 \) subcover \( \mathcal{M}^* \) that contains all the curves \( \Gamma_x^{(i)} \) with \( i \in \{1, 2\} \) which correspond to sharp-sharp singularities \( x \in Q_1 \).

**Lemma 2.2 (Finite subcover of order two).** Let \( E \subset \mathbb{R}^2 \) be compact, let \( \varepsilon > 0 \) and let \( U \) be a connected component of the complement \( E^r_x \). In addition, let the functions \( \gamma_x, \gamma_x^{(1)}, \gamma_x^{(2)} : [0, 1] \to \partial U \) and the corresponding simple curves \( \Gamma_x := \gamma_x([0, 1]) \) and \( \Gamma_x^{(i)} := \gamma_x^{(i)}([0, 1]) \) with \( i \in \{1, 2\} \), be as in Lemma 2.1. Then there exists a finite subset \( X^* \subset \partial U \setminus Q_1 \) for which the collection
\[
(2.13) \quad \mathcal{M}^* := \{ \Gamma_x : x \in X^* \} \cup \{ \Gamma_x^{(i)} : x \in Q_1, i \in \{1, 2\}\}
\]
of simple curves is a finite, minimal\(^6 \), order two cover of \( \partial U \).

**Proof.** Let the sets \( R, P \subset \partial U \) be as in (2.9) and (2.10). We now use the curve representations \( \Gamma_x \) for \( x \in R \cup P \cup Q_2 \) and \( \Gamma_x^{(i)} \) for \( x \in Q_1 \), defined above, in order to construct an order two cover for the boundary \( \partial U \). To begin with, consider the open cover \( \mathcal{U}_0 := \{ B_x : x \in \partial U \} \). Since \( \partial U \) is compact, there exists a finite subcover \( \mathcal{U}_1 = \{ B_x : x \in X_0 \} \subset \mathcal{U}_0 \), corresponding to some finite subset \( X_0 \subset \partial U \). We argue that in fact there exists a further subcover \( \mathcal{U} \subset \mathcal{U}_1 \) of order two. To see this, consider some boundary point
\[
(2.14) \quad x \in \partial U \cap B_{x_1} \cap B_{x_2} \cap B_{x_3},
\]
where \( B_{x_k} \in \mathcal{U}_1 \) for \( k \in \{1, 2, 3\} \). We claim that in this situation one of the sets \( B_{x_k} \) can always be removed from the collection so that \( \mathcal{U}_1 \setminus B_{x_k} \) is still a cover for \( \partial U \). As presented above, for each \( k \in \{1, 2, 3\} \) the set \( X \cap B_{x_k} \) has a parametrisation as the curve \( \Gamma_{x_k} = \gamma_{x_k}([0, 1]) \). Hence, there exists a continuous bijection \( h : [0, 1] \to \partial U \cap \bigcup_{k \in \{1, 2, 3\}} B_{x_k} \), for which \( \partial U \cap B_{x_k} = h((a_k, b_k)) \) for some open intervals \( (a_k, b_k) \subset [0, 1] \) and \( k \in \{1, 2, 3\} \). Equation (2.14) is then equivalent to
\[
h^{-1}(x) \in \bigcap_{k \in \{1, 2, 3\}} (a_k, b_k).
\]

On the other hand, regardless of the order in which the points \( a_k, b_k \) lie on \([0, 1] \)—as long as \( a_k < b_k \) for all \( k \in \{1, 2, 3\} \)—one of the intervals \( (a_k, b_k) \) is always contained in the union of the other two, and can therefore be removed without affecting the cover. One can thus remove any redundant balls \( B_{x_k} \) from the cover \( \mathcal{U}_1 \) and obtain a minimal, order two subcover \( \mathcal{U}_2 \).

To complete the proof, we still need to ensure that none of the balls \( B_x \) corresponding to sharp-sharp singularities \( x \in Q_1 \) were removed from the initial cover \( \mathcal{U}_0 \) along the pruning process. To this end, note that according to Lemma A.21 the set \( Q_1 \) is finite. This implies that we can assume the radii of the balls \( B_x \) to have been initially chosen sufficiently small such that \( B_x \cap Q_1 = \emptyset \) for all \( x \in \partial U \setminus Q_1 \) and \( B_x \cap Q_1 = \{x\} \) for \( x \in Q_1 \). This way, the inclusion of every ball \( B_x \) with \( x \in Q_1 \) is necessary in any subcover of \( \mathcal{U}_0 \).

\(^5\) A cover \( \mathcal{U} = \{ U_\alpha : \alpha \in A \} \) of a set \( X \subset \bigcup_{\alpha \in A} U_\alpha \), indexed by the set \( A \), is said to be of order \( n \), if the set \( A(x) := \{ \alpha \in A : x \in U_\alpha \} \) contains at most \( n \) elements for all \( x \in X \), and it is said to be minimal, if no \( U \in \mathcal{U} \) may be removed so that \( \mathcal{U} \setminus \{ U \} \) is still a cover for \( X \).

\(^6\) A cover \( \mathcal{U} = \{ U_\alpha : \alpha \in A \} \) of a set \( X \subset \bigcup_{\alpha \in A} U_\alpha \), indexed by the set \( A \), is said to be of order \( n \), if the set \( A(x) := \{ \alpha \in A : x \in U_\alpha \} \) contains at most \( n \) elements for all \( x \in X \), and it is said to be minimal, if no \( U \in \mathcal{U} \) may be removed so that \( \mathcal{U} \setminus \{ U \} \) is still a cover for \( X \).
Hence, we have arrived at the desired finite, minimal subcover \( \mathcal{U}_2 \) of order two, in the form

\[
M^* := \{ \Gamma_x : x \in X^* \} \cup \{ \Gamma_x^{(i)} : x \in Q_1, i \in \{1, 2\} \},
\]

where the set \( X^* \) is defined by \( X^* = \{ x \in \partial U \setminus Q_1 : B_x \in \mathcal{U}_2 \} \).

The significance of the order two property of the cover obtained in Lemma 2.2, as well as the requirement that all the curves corresponding to \( x \in Q_1 \) are included in the cover, both stem from the need to ensure that every boundary segment in the cover has a unique successor and predecessor segment. This makes it possible to construct Jordan curves on the boundary by following the boundary along adjacent curve segments.

![Figure 4](image_url)

**Figure 4.** Tracking the boundary \( \partial U \) along a finite collection of adjacent balls and corresponding simple curves. In (a) and (b), the points \( x \) and \( x_S \) correspond to simple curves \( \Gamma_x, \Gamma_x^S \in M^* \). Here \( \Gamma_x^S \) is the successor of \( \Gamma_x \), and due to minimality of \( M^* \), no point \( z \in \Gamma_x \cap \Gamma_x^S \) belongs to any other curve \( \Gamma \in M^* \). It is not ruled out that \( x_S \in \Gamma_x \), as in (b). In (c), which is a close-up of Figure 1 (a), a longer curve is formed by concatenating such overlapping local representations.

**Proposition 2.3 (Boundaries of connected components of the complement as finite unions of Jordan curves).** Let \( E \subset \mathbb{R}^2 \) be compact, let \( \varepsilon > 0 \) and let \( U \) be a connected component of the complement \( E^c_\varepsilon \). Then \( \partial U = \bigcup_{i=1}^M J_i \), where \( M \in \mathbb{N} \) and each \( J_i \) is a Jordan curve. The representation is unique up to parametrisation, and for any \( i \neq j \), the intersection \( J_i \cap J_j \) contains at most one point.

**Proof.** We divide the proof into three steps. First, we use the finite, order two cover given by Lemma 2.2 to construct the Jordan curves on the boundary. We then argue that this representation is unique up to parametrisation, and finish by showing that the intersection of any two
Jordan curves in the representation is either a singleton or the empty set. The last two facts are essentially implied by the Jordan Curve Theorem and the connectedness of the set $U$.\footnote{The Jordan Curve Theorem states that for any set $J \subset \mathbb{R}^2$ that is homeomorphic to the unit circle $S^1 \subset \mathbb{R}^2$, the complement $\mathbb{R}^2 \setminus J$ has precisely two connected components $A_1$ and $A_2$, one of which is bounded and the other unbounded, with the common boundary $\partial A_1 = \partial A_2 = J$.}

**Step 1: Construction of Jordan Curves.** Let

$$\mathcal{M}^* := \{\Gamma_x : x \in X^*\} \cup \{\Gamma_x^{(i)} : x \in Q_1, i \in \{1, 2\}\}$$

be the finite, minimal, order two cover of $\partial U$ given by Lemma 2.2. The sets $X^*$ and $Q_1$ are as in (2.15), and $\Gamma_x$ and $\Gamma_x^{(i)}$ are simple curves. Since $\mathcal{M}^*$ is minimal and order two, each $\Gamma \in \mathcal{A} := \{\Gamma_x : x \in X^*\}$ intersects exactly two other simple curves $\Gamma_P, \Gamma_S \in \mathcal{M}^* \setminus \{\Gamma\}$, which we call the predecessor and successor, respectively. It is possible that $\Gamma_P = \Gamma_S$. The parametrisations of individual curves $\Gamma \in \mathcal{A}$ can thus be combined to form longer curves, see Figure 4. Some of these eventually loop back onto themselves, forming Jordan curves, while others connect to curves $\Gamma \in \mathcal{B} := \{\Gamma_x^{(i)} : x \in Q_1, i \in \{1, 2\}\}$. By construction, $\Gamma_x^{(1)} \cap \Gamma_x^{(2)} = \{x\}$ for each $x \in Q_1$, and in addition, each $\Gamma_x^{(i)} \in \mathcal{B}$ intersects precisely two other curves $\Gamma_P, \Gamma_S \in \mathcal{M}^* \setminus \{\Gamma_x^{(1)}, \Gamma_x^{(2)}\}$. One can thus start from any $\Gamma \in \mathcal{M}^*$ and follow the boundary along uniquely defined successor (or predecessor) curves. The order two property and the finiteness of the collection $\mathcal{M}^*$ ensure that every such extended curve ultimately returns to $\Gamma$ without self-intersections, thus forming a Jordan curve. Furthermore, there are only finitely many such curves.

**Step 2: Uniqueness of Representation.** Assume there exists another representation of $\partial U$ as a union of Jordan curves. Since there is only one way to concatenate the curves $\Gamma \in \mathcal{A}$, the only way to obtain a representation that differs from the one constructed in Step 1 is to define the Jordan curves differently at the junction points $x \in Q_1$, where four simple curves meet at the same point. For each $x \in Q_1$ and $i \in \{1, 2\}$ we write $\Gamma_x^{(i)} = I_x^+ \cup I_x^-$ where the $I_x^\pm$ are simple curves that intersect each other pairwise only at $x$. The Jordan curves constructed in Step 1 are obtained by arriving to each $x \in Q_1$ via either $I_x^+$ or $I_x^-$ and leaving along $I_x^+$ or $I_x^-$, respectively. This corresponds to following either the blue or the red curve in Figure 5 (a). Assume then that for some $x \in Q_1$, a Jordan curve $J$ arrives at $x$ via $I_x^-$ (say) and leaves via either $I_x^+$ or $I_x^-$. This corresponds to following the blue curve either in Figure 5 (b) or in (c).

---

**Figure 5.** Schematic illustration of the three ways of representing the boundary $\partial U$ near a sharp-sharp singularity $x \in Q_1$ as the union of two simple curves. Since $I_x^+(x) \subset \Gamma_x^{(i)} \subset \partial U$ for $i \in \{1, 2\}$, the point $x$ must lie at the intersection of two different Jordan curves. In (a) the red and blue curves are subsets of the curves $\Gamma_x^{(1)}$ and $\Gamma_x^{(2)}$ defined in Lemma 2.1 for $x \in Q_1$. In (b) the coloured curves bounce back at $x$ and in (c) they cross at $x$. It turns out that only case (a) admits the continuation of such local curves into a representation of $\partial U$ as Jordan curves.
Figure 5 (c), respectively. The resulting curve would then not coincide with any of the Jordan curves constructed in Step 1. We show that both possibilities lead to a contradiction.

Assume first that \( J \) bounces back at \( x \) along \( I^+_2(x) \), as in Figure 5 (b). Let \( A_1, A_2 \) be the connected components of \( \mathbb{R}^2 \setminus J \), for which \( \partial A_1 = \partial A_2 = J \). By definition, \( J \) has no self-intersections, which implies that it intersects the curves \( I^+_1(x) \) and \( I^+_2(x) \) only at \( x \). It follows that \( J \cap B_r(x) = I^+_1(x) \setminus I^+_2(x) \), where \( r > 0 \) is the radius of the local boundary representation at \( x \), see Proposition A.13. Consequently, \( J \) divides the ball neighbourhood \( B_r(x) \) into two connected components \( C_1, C_2 \subset B_r(x) \), which satisfy \( C_i \subset A_i \) for \( i \in \{1, 2\} \). Let \( \xi \in \Sigma^x_\varepsilon(E) \) be the extremal outward direction for which \( \partial U \cap U_r(x, \xi) = I^+_1(x) \cup I^+_2(x) \) for some \( r > 0 \). Choose some \( x(\xi) \in U \cap U_r(x, \xi) \) and \( x(-\xi) \in U \cap U_r(x, -\xi) \). Now, since \( U \) is connected and open, there exists a path \( \gamma : [0, 1] \to U \) connecting \( x(\xi) \) and \( x(-\xi) \). But this is impossible, because it would imply that \( A_1 \cup A_2 \) is connected.

Assume then that \( J \) crosses \( x \) along \( I^+_2(x) \), as in Figure 5 (c). Since \( J \) cannot cross itself, it follows that another Jordan curve \( J_1 \subset \partial U \) contains the curves \( I^+_1(x) \) and \( I^+_2(x) \). As before, let \( A_1, A_2 \) be the connected components of \( \mathbb{R}^2 \setminus J \), for which \( \partial A_1 = \partial A_2 = J \). Now \( J_1 \cap A_i \neq \emptyset \) for both \( i \in \{1, 2\} \). But this contradicts the fact that the connected set \( U \) must be a subset of either \( A_1 \) or \( A_2 \).

**Step 3: Pairwise Intersections.** Assume, contrary to the claim, that there exist two Jordan curves \( J_1, J_2 \subset \partial U \) and some \( x_1, x_2 \in J_1 \cap J_2 \) with \( x_1 \neq x_2 \). This implies \( x_1, x_2 \in Q_1 \). One may then construct a new Jordan curve \( J^* \subset \partial U \) by following \( J_1 \) from \( x_1 \) to \( x_2 \) and returning back to \( x_1 \) via \( J_2 \). Let \( A_1, A_2 \) be the connected components of \( \mathbb{R}^2 \setminus J^* \), for which \( \partial A_1 = \partial A_2 = J^* \). Since \( U \) is connected, either \( U \subset A_1 \) or \( U \subset A_2 \). But this contradicts the fact that \( J^* \) divides the neighbourhood \( B_r(x_1) \) (say) into two disjoint connected components \( C_1 \subset A_1 \) and \( C_2 \subset A_2 \), for which \( C_i \cup U \neq \emptyset \) for both \( i \in \{1, 2\} \).

Our first main result, Theorem 1, follows from combining Propositions 2.3 and A.23.

**Proof of Theorem 1.** Denote by \( \mathcal{U} \) the collection of connected components of the complement \( E^c_\varepsilon \). Proposition A.23 asserts that every boundary point \( x \in \partial E^c_\varepsilon \) lies on the boundary \( \partial U \) of some \( U \in \mathcal{U} \) if and only if it is not a chain (S6) or chain-chain (S7) singularity. Proposition 2.3 then implies that

\[
\partial E^c_\varepsilon = \mathcal{I} \cup \bigcup_{U \in \mathcal{U}} \left( \bigcup_{n=1}^{\bar{N}_U} J_n(U) \right),
\]

where \( \bar{N}_U, N_U \in \mathbb{N} \) for each \( U \in \mathcal{U} \) and \( J_n(U) \subset \partial U \) is a Jordan curve for all \( 1 \leq n \leq \bar{N}_U \). For each \( U \in \mathcal{U} \), the collection \( \{J_n(U) : n \in N_U\} \) is furthermore unique up to parametrisation.

Due to the compactness of \( E^c_\varepsilon \), the complement \( E^c_\varepsilon \) contains only one unbounded connected component. All other connected components of the complement \( E^c_\varepsilon \) are contained in some bounded ball \( B_M(0) \) and since they are open, each of them contains an open ball. Hence, the Lebesgue measure of each such component \( U \) satisfies \( m(U) > 0 \), although for each \( n \in \mathbb{N} \) there are only finitely many \( U \) with \( m(U) > 1/n \). It follows from this that there are at most countably many connected components of the complement, which also implies that the number of Jordan curves \( J_n(U) \) in (2.17) is countable. 

\[\square\]

3. Curvature

Having established the topological structure of the boundary, we turn to investigate the curvature of the Jordan curve subsets \( J \subset \partial U \subset \partial E^c_\varepsilon \) for connected components \( U \) of the complement \( E^c_\varepsilon \). It is not clear from the outset that curvature can be defined on these sets even almost everywhere: it was shown in [9] that so-called shallow singularities (S4, S5), which are defined as
accumulation points of increasingly shallow wedge-type singularities, may lie densely on boundary segments that have positive one-dimensional Hausdorff measure. Even though Propositions A.13 and A.14 imply that it is possible to represent the boundary locally as graphs of Lipschitz-continuous functions, defining curvature at shallow singularities via limits of first derivatives is not possible, since the derivatives do not generally exist everywhere even in small neighbourhoods.

However, since there are no cusp singularities on the boundary, the tangent function on the boundary $\partial U$ may only have jumps in one direction. In the other direction, the rate of change of the tangential direction is bounded from below by the curvature of an $\varepsilon$-radius ball, since every boundary point $x \in \partial U$ lies on the surface of one. It turns out that it is possible to use these properties to show that the one-sided tangent functions are locally of bounded variation, and therefore have a derivative at almost every point. One can then use these to obtain the (signed) curvature of the boundary.

3.1. Existence of Curvature via Bounded Variation. Let $U$ be a connected component of the complement $E_\varepsilon^c$ and let $J \subset \partial U$ be one of its Jordan curve subsets, see Theorem 1. In general, $J$ has a well-defined tangent only at almost every point. However, Proposition A.11 asserts that the directional (left and right) tangents coincide with extremal outward directions, which exist at every $x \in J$ according to Proposition A.8. Lemma 2.2 asserts that $J$ can be covered by a finite collection of simple curves that are constructed from the graphs of local boundary representations, given by Proposition A.13. Hence, in order to establish the existence of curvature on the Jordan curve $J$, it suffices to work with these local representations.

In this section, we prove our second main result, Theorem 2, by applying a criterion for bounded variation to the local boundary representations of $J$. We present here the proof of the main theorem, and postpone the more technical proof of the criterion to Section 3.2.

3.1.1. Boundary Points in a Local Coordinate System. A local boundary representation at $x \in \partial E_\varepsilon$ is a finite collection $G(x)$ of continuous functions $g_{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}^2$, one for each extremal pair $(\xi,y) \in \mathcal{P}_{ext}(E_\varepsilon)$, with the property that

$$
\partial E_\varepsilon \cap B_r(x) = \bigcup_{(\xi,y) \in \mathcal{P}_{ext}(E_\varepsilon)} g_{\xi,y}(A_{\xi,y})
$$

for some $r > 0$ and some closed sets $A_{\xi,y} \subset [0,r]$, see Proposition A.13. More precisely, for each extremal pair $(\xi,y) \in \mathcal{P}_{ext}(E_\varepsilon)$ there exists a continuous function $f^{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}$ for which

$$
g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s) \frac{x - y}{\varepsilon}.
$$

According to Proposition A.23, chain singularities do not appear on the boundary $\partial U$ for connected components $U$ of the complement $E_\varepsilon^c$. It follows then from Proposition A.13 that for each $x \in \partial U$, the sets $A_{\xi,y}$ in (3.1) are intervals $A_{\xi,y} = [s_{\xi,y}, s_{\xi,y}']$ for some $s_{\xi,y} \in [0,\varepsilon/2]$.

Consider a fixed $x \in J$ and an extremal pair $(\xi,y) \in \mathcal{P}_{ext}(E_\varepsilon)$. To facilitate the subsequent analysis, we choose local coordinates given by the orthogonal unit vectors $\xi$ and $(x - y)/\varepsilon$, and place the origin of this coordinate system at $x$. However, to simplify notation, we assume (without loss of generality) that this coordinate system coincides with the standard Euclidean coordinate system, so that $x = (0,0)$, $\xi = (1,0)$ and $(x - y)/\varepsilon = (0,1)$. For each $s \in [0, s_{\xi,y}]$, we define $x_s := g_{\xi,y}(s) = (s, f^{\xi,y}(s)) \in J$, where $f^{\xi,y} : [0, s_{\xi,y}] \to \mathbb{R}$ is as in (3.2). The boundary subset

---

7For each boundary point $x \in \partial E_\varepsilon$, the set of extremal pairs $\mathcal{P}_{ext}(E_\varepsilon)$ consists of all the pairs $(\xi,y)$ of extremal outward directions $\xi \in \Xi_{ext}(E_\varepsilon)$ and extremal contributors $y \in \Pi^E_{ext}(x)$ for which $(x - y, \xi) = 0$. 

J \cap B_r(x) can thus be expressed as the union of graphs

\[ J \cap B_r(x) = \bigcup_{(\xi, y) \in P^\text{ext}_x(E_\varepsilon)} \{(s, f^{\xi,y}(s)) : s \in [0, s_{\xi,y}]\}, \]

where each extremal pair \((\xi, y) \in P^\text{ext}_x(E_\varepsilon)\) defines its corresponding local coordinates.

According to Proposition A.12, the extremal contributors \(y(s) \in \Pi^\text{ext}_x(E_\varepsilon)\) satisfy \(y(s) \in B_{\varepsilon/2}(y)\), where \(y \in \Pi^\text{ext}_x(E_\varepsilon)\) is the extremal contributor corresponding to the extremal pair \((\xi, y) \in P^\text{ext}_x(E_\varepsilon)\), see Proposition A.13. This implies that for each \(s \in [0, s_{\xi,y}]\) the extremal outward directions \(\xi_s \in \Xi^\text{ext}_x(s)\) deviate only slightly from the \(\xi\)-axis in the local coordinate system. We call them the right and left extremal outward direction at \(x_s\), and denote them by \(\xi^+_{s}\) and \(\xi^-_{s}\). According to Proposition A.11, these coincide with the tangential directions at \(x_s \in J\). In the local coordinates, the slopes of the one-sided tangents of the graph \(\{(s, f^{\xi,y}(s)) : s \in [0, s_{\xi,y}]\}\) are given by the the left and right derivatives

\[
D^\pm f^{\xi,y}(s) := \lim_{h \to \pm 0} \frac{f^{\xi,y}(s + h) - f^{\xi,y}(s)}{h}.
\]

These correspond to the extremal outward directions \(\xi^\pm_s = (\xi^+_1(s), \xi^+_2(s)) \in \Xi^\text{ext}_x(E_\varepsilon)\) via

\[
D^\pm f^{\xi,y}(s) = \frac{\xi^\pm_2(s)}{\xi^\pm_1(s)},
\]

see Figure 6.

3.1.2. A Condition for Bounded Variation. In order to prove the existence of curvature almost everywhere on \(J\), we consider a local boundary representation \(G(x)\) around an arbitrary boundary point \(x \in J\), and show that for each extremal pair \((\xi, y) \in P^\text{ext}_x(E_\varepsilon)\) the second derivative of \(f^{\xi,y}\) exists almost everywhere on the interval \([0, s_{\xi,y}]\). We begin by stating the following general criterion for bounded variation that applies to bounded functions on a closed interval. We leave the elementary proof to the reader.

**Lemma 3.1 (A criterion for bounded variation).** Let \(q > 0\) and let \(f : [a, b] \to \mathbb{R}\) be a bounded function that satisfies

\[
f(s + h) - f(s) \geq -qh
\]

for all \(s \in [a, b]\) and \(h > 0\) with \(s + h \in [a, b]\). Then \(f\) is of bounded variation on \([a, b]\).
The main step towards proving Theorem 2 is to show that the one-sided derivatives \(D^\pm f^{\xi,y}\) satisfy condition (3.5) of Lemma 3.1 for a certain \(q > 0\) and are thus of bounded variation on the interval \([0, s_{\xi,y}]\).

**Proposition 3.2 (A lower bound for differences of right derivatives).** Let \(E \subset \mathbb{R}^2\) be a compact set and let \(x \in J \subset \partial E\) where \(J\) is a Jordan curve. Let \(\mathcal{G}(x)\) be a local boundary representation at \(x\), where each \(g_{\xi,y} \in \mathcal{G}(x)\) satisfies
\[
g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)e^{-1}(x - y)
\]
for some continuous function \(f^{\xi,y} : [0, s_{\xi,y}] \rightarrow \mathbb{R}\). Then the left and right derivatives \(D^\pm f^{\xi,y}\) satisfy the inequality
\[
(3.6) \quad D^\pm f^{\xi,y}(s + h) - D^\pm f^{\xi,y}(s) \geq -\frac{8h}{3\sqrt{3}\epsilon}
\]
for all \(s \in [0, s_{\xi,y}]\) and \(s, h > 0\) with \(0 \leq s + h \leq s_{\xi,y}\).

The proof of Proposition 3.2 is the most technical part of the proof of Theorem 2, and is presented separately in Section 3.2. We now proceed to combine Proposition 3.2 and Lemma 3.1 into a global statement about the existence of curvature on \(\partial E\). We denote by \(\mathcal{H}^1\) the one-dimensional Hausdorff measure, and by \(I\) the set of inaccessible singularities, see Proposition A.23.

**Theorem 2 (Existence of curvature).** Let \(E \subset \mathbb{R}^2\) be a compact set and \(\epsilon > 0\). Then for \(\mathcal{H}^1\)-almost all \(x \in \partial E\setminus I\) the (signed) curvature \(\kappa(x)\) exists and is given by the formula
\[
\kappa(s_x) = \frac{2^2 f^{\xi,y}(s_x)}{\left(1 + (\frac{\# f^{\xi,y}(s_x)}{h})^2\right)^{3/2}},
\]
where the coordinates \(s_x\) and \(f^{\xi,y}(s_x)\) are associated with a local boundary representation \(\mathcal{G}(z)\) at some \(z \in \partial E\setminus I\), and \(x = z + s_x\xi + f^{\xi,y}(s_x)e^{-1}(z - y)\) for some \((\xi, y) \in \mathcal{P}_z^{ext}(E_z)\).

**Proof.** According to Theorem 1, \(\partial E\setminus I\) is a countable union \(J\) of Jordan curves. Since each \(J \in J\) is compact, there exists a finite collection \(Z\) of points \(z \in J\) and corresponding boundary representations \(\mathcal{G}(z)\), for which
\[
J = \bigcup_{z \in Z} \left(\bigcup_{(\xi, y) \in \mathcal{P}_z^{ext}(E_z)} g_{\xi,y}([0, s_{\xi,y}])\right).
\]
The closed intervals \([0, s_{\xi,y}]\) are as in Proposition A.13. It suffices to show that for each \(z \in Z\), curvature exists outside a \(\mathcal{H}^1\)-negligible set on each curve \(g_{\xi,y}([0, s_{\xi,y}])\), where \((\xi, y) \in \mathcal{P}_z^{ext}(E_z)\).

Let \(z \in Z\) and let \(g_{\xi,y} \in \mathcal{G}(z)\) for some \((\xi, y) \in \mathcal{P}_z^{ext}(E_z)\). According to Proposition A.13 there exists a continuous function \(f^{\xi,y} : [0, s_{\xi,y}] \rightarrow \mathbb{R}\) for which
\[
g_{\xi,y}(s) = z + s\xi + f^{\xi,y}(s)e^{-1}(z - y).
\]
Since the one-sided derivatives \(D^\pm f^{\xi,y}(s)\) are related to the extremal outward directions \(\xi^\pm(s)\) via (3.4), it follows that \(D^+ f^{\xi,y}(s) = D^- f^{\xi,y}(s)\) whenever \(\xi^+(s) = -\xi^-(s)\). This implies that the one-sided derivatives agree on \([0, s_{\xi,y}]\) apart from at most countably infinite set, since it follows from Theorem A.20 and Proposition A.23 that the set
\[
\{x \in J : \Xi_z^{ext}(E_z) = \{\xi, -\xi\} \text{ for some } \xi \in S^1\} = J \setminus \text{Unp}_z(E),
\]
where \(\text{Unp}_z(E)\) is as in Definition A.1, is at most countably infinite.

We next confirm that the assumptions of Lemma 3.1 are satisfied for the left and right derivatives \(D^\pm f^{\xi,y}(s)\) on the interval \([0, s_{\xi,y}]\). By the definition of a local boundary representation, the
extremal contributors $y^\pm_s \in \Pi_{E}^{\text{ext}}(x_s)$ satisfy $y^\pm_s \in B_{\varepsilon/2}(y)$, where $y \in \Pi_{E}^{\text{ext}}(x)$ corresponds to the extremal pair $(\xi,y)$. This imposes an upper bound on the angle between the $\xi$-axis and the corresponding extremal outward directions $\xi^\pm(s) \in \Xi_{E}^{\text{ext}}(E_s)$, which in turn implies a bound on the left and right derivatives $D^\pm f^{\xi,y}$ on $[0,s_{\xi,y}]$ through the relationship (3.4). Due to inequality (3.6) in Proposition 3.2, one can thus apply Lemma 3.1, which implies that the one-sided derivatives $D^\pm f^{\xi,y}$ are of bounded variation on $[0,s_{\xi,y}]$.

It follows that each of the functions $D^\pm f^{\xi,y}$ has a (two-sided) derivative $\frac{d}{ds} D^\pm f^{\xi,y}$ almost everywhere on $[0,s_{\xi,y}]$. Since $D^+ f^{\xi,y} = D^- f^{\xi,y}$ outside a countable set, this furthermore implies $\frac{d}{ds} D^+ f^{\xi,y} = \frac{d}{ds} D^- f^{\xi,y}$ almost everywhere. Thus, apart from a set $W \subset [0,s_{\xi,y}]$ of zero Hausdorff measure, both the first and second (two-sided) derivatives of $f^{\xi,y}$ exist, and define curvature on the graph $\{(s,f^{\xi,y}(s)) : s \in [0,s_{\xi,y}]\}$ via (3.7).

According to Proposition A.14, the function $s \mapsto (s,f^{\xi,y}(s))$ is $2/\sqrt{3}$-Lipschitz on $[0,s_{\xi,y}]$. This implies $\mathcal{H}^1(\{f^{\xi,y}(W)\}) \leq 2\mathcal{H}^1(W)/\sqrt{3} = 0$, since the Hausdorff measure of a Lipschitz transformation is bounded from above by the corresponding Lipschitz constant, see [1, Prop. 2.49]. Curvature thus exists outside a $\mathcal{H}^1$-negligible set on each curve $g_{\xi,y}([0,s_{\xi,y}])$. \hfill \Box

3.2. A Lower Bound for Differences of Tangential Directions. In this section we prove Proposition 3.2, which represents the key step in the proof of Theorem 2. Proposition 3.2 expresses the geometric observation that it is impossible for the boundary $\partial E_s$ to curve inwards more than a certain threshold, implied by the radius $\varepsilon > 0$.

Before proceeding with the proof, we establish some notation. We consider the local boundary representation $G(x)$ around $x \in J$, where $J \subset \partial E_1$ is a Jordan curve component of the boundary. We work in the local coordinates corresponding to an extremal pair $(\xi,y) \in \mathcal{P}_{x}^{\text{ext}}(E_s)$, as defined in Section 3.1.1. For each $s \in [0,s_{\xi,y}]$, define

$$x_s := g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)\frac{x - y}{\varepsilon} = (s,f^{\xi,y}(s)) \in J,$$

where $f^{\xi,y} : [0,s_{\xi,y}] \to \mathbb{R}$ is as in (3.2). Each of the extremal contributors $y^\pm_s = (y_1^\pm(s),y_2^\pm(s)) \in \Pi_{E}^{\text{ext}}(x_s)$ lies at the center of an $\varepsilon$-radius circle $B_\varepsilon(y^\pm_s)$, whose tangent at $x_s \in \partial B_\varepsilon(y^\pm_s)$ coincides with the respective extremal outward direction $\xi^\pm_s = (\xi_1^\pm(s),\xi_2^\pm(s)) \in \Xi_{E}^{\text{ext}}(E_s)$, see Definition A.6 and Proposition A.11. Hence, for each $s \in [0,s_{\xi,y}]$,

$$x_s - y^\pm_s = (-\varepsilon\xi_2^\pm(s),\varepsilon\xi_1^\pm(s)) =: (a^\pm_s,b^\pm_s).$$

On the other hand, as indicated in (3.4), the slopes of the one-sided tangents at $x_s$ satisfy

$$D^\pm f^{\xi,y}(s) = \frac{\xi_2^\pm(s)}{\xi_1^\pm(s)} = \frac{\varepsilon\xi_2^\pm(s)}{\sqrt{\varepsilon^2 - [\varepsilon\xi_1^\pm(s)]^2}} = \frac{-a^\pm_s}{\sqrt{\varepsilon^2 - [a^\pm_s]^2}} =: p(a^\pm_s).$$

In order to prove Proposition 3.2, we establish for all $s \in [0,s_{\xi,y}]$ and $h > 0$ with $0 \leq s + h \leq s_{\xi,y}$ the double inequality

$$D^\pm f^{\xi,y}(s + h) - D^\pm f^{\xi,y}(s) \geq p(a^\pm_s + h) - p(a^\pm_s) \geq -\frac{8h}{3\sqrt{3}\varepsilon},$$

where the coordinates $a^\pm_s$ and the slope function $p : (-\varepsilon,\varepsilon) \to \mathbb{R}$ are defined in (3.8) and (3.9), respectively. We initially prove (3.10) for the right derivative $D^+ f^{\xi,y}$, and deduce from this the analogous inequality also for the left derivative $D^- f^{\xi,y}$. Since $D^+ f^{\xi,y}(s) = p(a^+_s)$, the key to

---

8Every function of bounded variation can be written as the difference of two non-decreasing functions, and consequently has a finite derivative at almost every point. For details, see for instance [8, p. 331]

9Due to the orientation of the extremal contributor $y \in \Pi_{E}^{\text{ext}}(x)$ relative to $x$, these circles lie below the graph $g_{\xi,y}([0,s_{\xi,y}])$ in the $(\xi, (x - y)/\varepsilon)$-coordinates, and are thus uniquely defined for all $s \in [0,s_{\xi,y}]$. 

---
showing inequality (3.10) is to demonstrate that $D^+ f^{\xi,y}(s+h) \geq p(a^+_s + h)$ whenever $s, h \geq 0$ and $0 \leq s + h \leq s_{\xi,y}$. We divide the proof into the following steps, of which the first three correspond to Lemmas 3.3–3.5 and step (iv) is included in the proof of Proposition 3.2 below:

(i) identify a lower bound $k(T(s,h),h)$ for $D^+ f^{\xi,y}(s+h)$ in terms of

$$T(s,h) := f^{\xi,y}(s+h) - f^{\xi,y}(s);$$

(ii) show that the bound $k_0(T) := k(T,h)$ obtained in step (i) is increasing in $T$ for a fixed $h$;

(iii) show that there exists $\hat{\mathcal{T}} := \mathcal{T}(s,h) \leq T(s,h)$ for which $k(\hat{\mathcal{T}},h) = p(a^+_s + h)$;

(iv) combine steps (i)–(iii) to obtain the inequality

$$D^+ f^{\xi,y}(s+h) \geq k(T,h) \geq k(\hat{\mathcal{T}},h) = p(a^+_s + h).$$

We begin by establishing a lower bound $k(T,h)$ for $D^+ f^{\xi,y}(s+h)$ in terms of $T := T(s,h)$ and the $T$-dependent distances

$$P(T) := \frac{\sqrt{h^2 + T^2}}{2} \text{ and } A(T) := \sqrt{\varepsilon^2 - P^2(T)} = \sqrt{\varepsilon^2 - \frac{h^2 + T^2}{4}}. \tag{3.11}$$

Lemma 3.3 (The functional form of the lower bound for the right derivative). Let $E \subset \mathbb{R}^2$ be a compact set, $\varepsilon > 0$ and let $x \in J \subset \partial E_\varepsilon$ where $J$ is a Jordan curve. Let $\mathcal{G}(x)$ be a local boundary representation at $x$, where each $g_{\xi,y} \in \mathcal{G}(x)$ satisfies

$$g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)\varepsilon^{-1}(x-y)$$

for some continuous function $f^{\xi,y} : [0,s_{\xi,y}] \to \mathbb{R}$. Then for all $s,h \geq 0$ with $0 \leq s + h \leq s_{\xi,y}$, the right derivative $D^+ f^{\xi,y}$ satisfies the inequality

$$D^+ f^{\xi,y}(s+h) \geq k(T,h) := \frac{\mathcal{T}A(T) - hP(T)}{hA(T) + TP(T)}, \tag{3.12}$$

where $T := f^{\xi,y}(s+h) - f^{\xi,y}(s)$ and the distances $P(T), A(T)$ are as in (3.11).

Proof. For each $s \in [0,s_{\xi,y}]$ we write $a^+_s := -\varepsilon \xi^+_1(s)$ and $b^+_s := \varepsilon \xi^+_2(s)$, where $\xi^+_s = (\xi^+_1(s),\xi^+_2(s))$ is the right extremal outward direction at $x_s$. Fix then some $s,h \geq 0$ with $0 \leq s + h \leq s_{\xi,y}$ and consider the corresponding boundary points $x_s, x_{s+h} \in J$ whose local coordinates are given by

$$x_s := (s,f^{\xi,y}(s)) \quad \text{and} \quad x_{s+h} := (s+h,f^{\xi,y}(s+h)). \tag{3.13}$$

It follows from (3.8) that the right extremal contributors $y^+_s \in \Pi^\text{ext}_E(x_s)$ and $y^+_{s+h} \in \Pi^\text{ext}_E(x_{s+h})$ satisfy

$$y^+_s = (s - a^+_s, f^{\xi,y}(s) - b^+_s) \quad \text{and} \quad y^+_{s+h} = (s+h - a^+_{s+h}, f^{\xi,y}(s+h) - b^+_{s+h}). \tag{3.14}$$

In local coordinates, the right derivative $D^+ f^{\xi,y}(s+h)$ is the slope of the extremal outward direction $\xi^+_{s+h}$, which is by definition perpendicular to the vector $x_{s+h} - y^+_{s+h}$. Since $y^+_{s+h}$ is a contributor and thus in $E$, it must by definition lie outside the $\varepsilon$-radius ball $B_\varepsilon(x_s)$ around the boundary point $x_s \in J$. One can therefore obtain a lower bound for the right derivative $D^+ f^{\xi,y}(s+h)$ by considering how far one could rotate the contributor $y^+_{s+h}$ clockwise around the point $x_{s+h}$ before it enters the ball $B_\varepsilon(x_s)$. This would happen at the point $y^+_{s+h}$, whose distance from both $x_s$ and $x_{s+h}$ equals $\varepsilon$, see Figure 7. These three points thus form an isosceles triangle whose base is the line segment

$$S := \{(1-\varphi)x_s + \varphi x_{s+h} : \varphi \in [0,1]\}. \tag{3.15}$$
Consequently, the orthogonal projection of the apex \( y^*_{s+h} \) of this triangle onto the line segment \( S \) lands on the mid-point \( x^*_{s+h} := (x_s + x_{s+h})/2 \) of \( S \). Together, the points \( x^*_{s+h}, y^*_{s+h} \) and \( x_{s+h} \) in turn form a right triangle \( \Omega \), whose legs \( x_{s+h} - x^*_{s+h} \) and \( x^*_{s+h} - y^*_{s+h} \) satisfy
\[
\|x_{s+h} - x^*_{s+h}\| = P(T) \quad \text{and} \quad \|x^*_{s+h} - y^*_{s+h}\| = A(T).
\]

We use the above geometric relationships to obtain an explicit lower bound for \( D^+ f^{\xi,y}(s+h) \) in terms of the lengths \( P(T) \) and \( A(T) \). As noted above, \( y^*_{s+h} \not\in B_{\varepsilon}(x_s) \), which implies the inequality
\[
0 \leq \|y^+_{s+h} - x_s\|^2 - \varepsilon^2 = (\varepsilon^2 + (f^{\xi,y}(s+h) - b^+_{s+h})^2 - \varepsilon^2 = h^2 + (f^{\xi,y}(s+h) - f^{\xi,y}(s))^2 - 2(ha^+_{s+h} + (f^{\xi,y}(s+h) - f^{\xi,y}(s))b^+_{s+h}).
\]

Writing \( T := f^{\xi,y}(s+h) - f^{\xi,y}(s) \), the above inequality can be expressed more concisely as
\[
(3.16) \quad h\varepsilon^2 = T^2 + hTb^+_{s+h} \leq (h^2 + T^2)/2.
\]

Since \( x_{s+h} - y^+_{s+h} = (a^+_{s+h}, b^+_{s+h}) \) and \( x_{s+h} - x_s = (h, T) \), (3.16) is furthermore equivalent to
\[
(3.17) \quad \frac{x^*_{s+h} - y^*_s - x_s}{\|x_{s+h} - x_s\|} \leq \frac{P(T)}{2} = P(T).
\]

Geometrically, inequality (3.17) expresses the fact that the length of the orthogonal projection of the vector \( x_{s+h} - y^+_{s+h} \) onto the line \( L := x_s + \text{span}\{x_{s+h} - x_s\} \) is at most half the distance \( \|x_{s+h} - x_s\| \). Note that when \( \xi^+_{s+h} \) is parallel to \( L \), the scalar product on the left-hand side of (3.17) vanishes, and for steeper slopes it becomes negative. However, since we seek to obtain a lower bound for \( D^+ f^{\xi,y}(s+h) = \tan \tau_{s+h} \), where \( \tau_{s+h} \) is the angle between \( \xi^+_{s+h} \) and the \( \xi \)-axis, it is sufficient to restrict the analysis to angles \( \tau_{s+h} \) that are smaller than the angle \( \theta(T) \) between the line \( L \) and the \( \xi \)-axis.\textsuperscript{10}

Consider now the vector \( x^*_{s+h} - y^*_{s+h} \), where \( x^*_{s+h} := \text{proj}_L (y^*_s + hT) \) is the orthogonal projection of \( y^+_{s+h} \) onto the line \( L \). It follows from (3.17) that \( \|x_{s+h} - x^*_{s+h}\| \leq P(T) \), and consequently
\[
\|x^*_{s+h} - y^*_{s+h}\|^2 = \varepsilon^2 - \|x_{s+h} - x^*_{s+h}\|^2 \geq \varepsilon^2 - \|x_{s+h} - x^*_{s+h}\|^2 = A^2(T).
\]

This implies that the angle \( \alpha_{s+h} \) at \( y^*_{s+h} \) between the vectors \( x_{s+h} - y^*_{s+h} \) and \( x^*_{s+h} - y^*_{s+h} \) satisfies
\[
(3.18) \quad \tan \alpha_{s+h} = \frac{\|x_{s+h} - x^*_{s+h}\|}{\|x^*_{s+h} - y^*_{s+h}\|} \leq \frac{P(T)}{A(T)} = \tan \alpha(T),
\]

where \( \alpha(T) \) is the angle at \( y^*_{s+h} \) between the vectors \( x_{s+h} - y^*_{s+h} \) and \( x^*_{s+h} - y^*_{s+h} \). Since the vectors \( x_{s+h} - y^*_{s+h} \) and \( \xi^+_{s+h} \) are perpendicular, it furthermore follows that \( \tau_{s+h} = \theta(T) - \alpha_{s+h} \). This, together with (3.18), leads to the lower bound
\[
(3.19) \quad D^+ f^{\xi,y}(s+h) = \tan \tau_{s+h} = \tan(\theta(T) - \alpha_{s+h}) \geq \tan(\theta(T) - \alpha(T))
\]
\[
= \frac{\tan \theta(T) - \tan \alpha(T)}{1 + \tan \theta(T) \tan \alpha(T)} = \frac{TA(T) - hP(T)}{hA(T) + TP(T)} = k(T, h),
\]

where the third last equality is due to the standard formula for the tangent of a sum of angles.

\textsuperscript{10}Additionally, as pointed out also in the proof of Theorem 2, the definition of a local boundary representation imposes an upper bound on the angle \( \tau_{s+h} \) between the extremal outward direction \( \xi^+_{s+h} \) and the \( \xi \)-axis, since the extremal contributor \( y^*_{s+h} \in \Pi^E_{\xi}(x_{s+h}) \) lies for all \( s, h \) within the ball \( B_{\varepsilon/2}(y) \), where \( y \in \Pi^E_{\xi}(x) \).
Figure 7. An illustration of the geometric components in the proof of Lemma 3.3. One can obtain a lower bound \( k(T, h) = \tan \tau(T) \) for the slope \( D^+ f_{\xi,y}(s+h) = \tan \tau_{s+h} = \xi_2^+(s+h)/\xi_1^+(s+h) \) by considering how far one could rotate the right extremal contributor \( y_{s+h}^+ \) clockwise around the point \( x_{s+h} \) before it enters the ball \( B_\varepsilon(x_s) \). The line segment \( S := \{(1-\varphi)x_s + \varphi x_{s+h} : \varphi \in [0,1]\} \) is marked as a dashed line. Note that the illustration here does not strictly speaking apply in the situation of Lemma 3.3 because the \( \xi \)-coordinate of \( x_{s+h} \) differs from that of \( x_s \) by more than \( \varepsilon/2 \). This artistic liberty was taken in order to improve readability through the increase of the relevant angles.

Intuitively, Lemma 3.3 describes how the \( \varepsilon \)-neighbourhood geometry imposes a lower bound \( k(T, h) \) on the tangential direction \( D^+ f_{\xi,y}(s+h) \), and how this bound depends on the increment \( h \) and the difference \( T := f_{\xi,y}(s+h) - f_{\xi,y}(s) \). We show next that if the point \( x_s \) and the increment \( h > 0 \) are fixed, \( k(T, h) \) depends monotonically on \( T \).

Lemma 3.4 (Monotonicity of the lower bound for the right derivative). For a fixed \( h \in [0, 2\varepsilon] \) and the corresponding functions

\[
P(t) := \frac{\sqrt{h^2 + t^2} - \frac{t^2}{2}}{2} \quad \text{and} \quad A(t) := \sqrt{\varepsilon^2 - P^2(t)} = \sqrt{\varepsilon^2 - \frac{h^2 + t^2}{4}},
\]

the lower bound function

\[
t \mapsto k(t, h) := \frac{tA(t) - hP(t)}{hA(t) + tP(t)}
\]

in (3.12) is increasing on the interval \( (-\sqrt{4\varepsilon^2 - h^2}, \sqrt{\varepsilon^2 - (\varepsilon - h)^2}) \).
Proof. Note first that $A(t)$ is defined when $P(t) \leq \varepsilon$, which is equivalent to $|t| \leq \sqrt{\varepsilon^2 - h^2}$. We compute the sign of the derivative $\frac{d}{dt}k(t, h)$ for a fixed $h \geq 0$. Write

$$k_1(t) := tA(t) - hP(t) \quad \text{and} \quad k_2(t) := hA(t) + tP(t),$$

so that $k(t, h) = k_1(t)/k_2(t)$. Then

$$\frac{d}{dt}k(t, h) = \frac{k_1'(t)k_2(t) - k_1(t)k_2'(t)}{k_2^2(t)},$$

$$= \frac{h(A^2(t) + P^2(t)) + (h^2 + t^2)(A'(t)P(t) - P'(t)A(t))}{(hA(t) + tP(t))^2},$$

where $P'(t) = t(2\sqrt{h^2 + t^2})^{-1}$ and $A'(t) = -t\left(4\sqrt{\varepsilon^2 - \frac{h^2 + t^2}{4}}\right)^{-1}$. Thus $\frac{d}{dt}k(t, h) > 0$ if and only if

$$0 < h(A^2(t) + P^2(t)) + (h^2 + t^2)(A'(t)P(t) - P'(t)A(t))$$

$$= h\varepsilon^2 + (h^2 + t^2) - t(h^2 + t^2) - 4t\left(4\sqrt{\varepsilon^2 - \frac{h^2 + t^2}{4}}\right)$$

$$= \varepsilon^2\left(h - t\frac{\sqrt{h^2 + t^2}}{\sqrt{\varepsilon^2 - \frac{h^2 + t^2}{4}}}\right) = \varepsilon^2\left(h - \frac{tP(t)}{\sqrt{\varepsilon^2 - P^2(t)}}\right).$$

A direct computation shows that this inequality is satisfied if and only if $t < \sqrt{\varepsilon^2 - (\varepsilon - h)^2}$, where the right-hand side is defined for $h \in [0, 2\varepsilon]$. \qed

Lemma 3.5 (An explicit lower bound for the right derivative). Let $E \subset \mathbb{R}^2$ be a compact set, $\varepsilon > 0$ and let $x \in J \subset \partial E_\varepsilon$ where $J$ is a Jordan curve. Let $G(x)$ be a local boundary representation at $x$, where each $g_{x} \in G(x)$ satisfies

$$g_{x}(s) = x + s \xi + f_{\xi,y}(s)\varepsilon^{-1}(x - y)$$

for some continuous function $f_{\xi,y} : [0, s_{\xi,y}] \rightarrow \mathbb{R}$. In addition, let $T := T(s, h) := f_{\xi,y}(s + h) - f_{\xi,y}(s)$ and let

$$k(T, h) := \frac{T(A(T) - hP(T))}{hA(T) + TP(T)}$$

be as in (3.12). Then, for all $s, h \geq 0$ with $0 \leq s + h \leq s_{\xi,y}$, there exists some $\bar{T} \leq T$ for which

$$k(\bar{T}, h) = p(a_s^+ + h) := \frac{-\varepsilon^2}{\sqrt{\varepsilon^2 - (a_s^+ + h)^2}},$$

where for each $s \in [0, s_{\xi,y}]$ the coordinate $a_s^+ := -\varepsilon\xi_2^+(s)$ corresponds to the right extremal outward direction $\xi_2^+ = (\xi_1^+(s), \xi_2^+(s))$ at the boundary point $x_s = (s, f_{\xi,y}(s))$, see (3.13).

Proof. Let $s, h \geq 0$ with $0 \leq s + h \leq s_{\xi,y}$, and consider the corresponding boundary points $x_s, x_{s+h} \in J$ whose local coordinates are given by

$$x_s := (s, f_{\xi,y}(s)), \quad x_{s+h} := (s + h, f_{\xi,y}(s + h)).$$

It follows from (3.8) that the right extremal contributors $y_{s}^+ \in \Pi_{E}^{ext}(x_s)$ and $y_{s+h}^+ \in \Pi_{E}^{ext}(x_{s+h})$ satisfy

$$y_{s}^+ = (s - a_s^+, f_{\xi,y}(s) - b_s^+) \quad \text{and} \quad y_{s+h}^+ = (s + h - a_{s+h}^+, f_{\xi,y}(s + h) - b_{s+h}^+).$$

(3.22)
According to Lemma 3.3, each $T := f^{\xi,y}(s+h) - f^{\xi,y}(s)$ is associated to a unique point $y^*_{s+h}(T)$, for which $\|y^*_{s+h}(T) - x_s\| = \|y^*_{s+h}(T) - x_{s+h}\| = \varepsilon$, and which defines geometrically the lower bound $k(T, h)$ for the right derivative $D^+ f^{\xi,y}(s + h)$. Lemma 3.4 on the other hand asserts that for fixed $h$, the function $k(T, h)$ is increasing in $T$. Since $y^*_{s} \in \Pi^{\text{ext}}_E(x_s) \subset E$ and $h \leq \varepsilon/2$, there exists a lower bound for the value $f^{\xi,y}(s + h)$, and thus for the difference $T$. We aim to find an explicit value for $k(T, h)$ at this lower bound $T \leq T$.

For the boundary point $x_{s+h} \in J \subset \partial E_{\varepsilon}$ and the extremal contributor $y^*_{s} \in \Pi^{\text{ext}}_E(x_s) \subset E$, defined in (3.21) and (3.22) respectively, the lower bound $\|x_{s+h} - y^*_{s}\| \geq \varepsilon$ implies

$$\hat{T} := -b^+_s + \sqrt{\varepsilon^2 - (a^+_s + h)^2} \leq f^{\xi,y}(s + h) - f^{\xi,y}(s) = T,$$

see Figure 8. By construction, $b^+_s$ is always positive, while $a^+_s$ may either be positive or negative depending on the slope $D^+ f^{\xi,y}(s)$. We aim to show that $k(\hat{T}, h) = p(a^+_s + h)$.

![Figure 8](image-url)  
Figure 8. Illustration of the dependence of the function $k(T, h) = \tan \tau(T)$ on the difference $T := f^{\xi,y}(s+h) - f^{\xi,y}(s)$. As $T$ decreases, the corresponding point $y^*_{s+h}(T)$ on the circumference $\partial B_{\varepsilon}(x_s)$ moves clockwise towards $y^*_s$. This results in a decrease in the slope $\tan \tau(T)$, where $\tau(T) := \theta(T) - \alpha(T)$. At the minimal value $\hat{T} \leq T$, one obtains the lower bound $k(\hat{T}, h) = \tan \tau(\hat{T}) = p(a^+_s + h)$, which geometrically corresponds to the point $y^*_{s+h}(\hat{T}) = y^*_s$.

Consider the point $\hat{x}_{s+h} := (s+h, f^{\xi,y}(s) + \hat{T}) \in \partial B_{\varepsilon}(y^*_s)$ and define $z_{s+h} := (s+h, f^{\xi,y}(s) - b^+_s)$ so that $z_{s+h}$ shares its $\xi$-coordinate with $\hat{x}_{s+h}$ and its $\varepsilon^{-1}(x - y)$-coordinate with $y^*_s$. Then
\( \tilde{x}_{s+h} - y^+_s = (\tilde{a}_{s+h}, \tilde{b}_{s+h}) \), where \( \tilde{a}_{s+h} := a^+_s + h \) and \( \tilde{b}_{s+h} := \sqrt{\varepsilon^2 - (a^+_s + h)^2} \). This implies that

\[
k(\tilde{T}, h) = \tan \left( \theta(\tilde{T}) - \alpha(\tilde{T}) \right) = \tan \tau(\tilde{T}) = \frac{-\tilde{a}_{s+h}}{\tilde{b}_{s+h}} = \frac{-a^+_s + h}{\varepsilon^2 - (a^+_s + h)^2} = p(a^+_s + h),
\]
where \( \tau(\tilde{T}) \) is the angle at \( \tilde{x}_{s+h} \) between the vectors \( y^+_s - \tilde{x}_{s+h} \) and \( z_{s+h} - \tilde{x}_{s+h} \).

Combining Lemmas 3.3–3.5, we are now ready to prove Proposition 3.2.

**Proof of Proposition 3.2.** Let \( s, h > 0 \) with \( 0 \leq s + h \leq s_{\xi,y} \) and let \( T := f^{\xi,y}(s + h) - f^{\xi,y}(s) \). In addition, define \( k(T, h) := (TA(T) - hP(T))/(hA(T) + T \bar{P}(T)) \) as in (3.12) and let \( \tilde{T} \leq T \) be the lower bound, given by Lemma 3.5, for which \( k(\tilde{T}, h) = p(a^+_s + h) \). Here, the slope function \( p : (-\varepsilon, \varepsilon) \to \mathbb{R} \) is given by \( p(s) := -s/\sqrt{\varepsilon^2 - s^2} \) as in (3.9). We want to use the monotonicity of the map \( T \mapsto k(T, h) \) in order to establish for each \( h \) the inequality \( k(T, h) \geq k(\tilde{T}, h) \).

This follows immediately by combining the Lipschitz property\(^{11}\)

\[
\left| T \right| = \left| f^{\xi,y}(s + h) - f^{\xi,y}(s) \right| \leq h/\sqrt{3},
\]

given by Proposition A.14, with the inequality \( h/\sqrt{3} \leq \sqrt{\varepsilon^2 - (\varepsilon - h)^2} < \sqrt{\varepsilon^2 - h^2} \), which is implied by \( 0 \leq h \leq s_{\xi,y} \leq \varepsilon/2 \). The results in Lemmas 3.3–3.5 thus imply that

\[
D^+ f^{\xi,y}(s + h) \geq k(T, h) \geq k(\tilde{T}, h) = p(a^+_s + h)
\]
whenever \( s, h \geq 0 \) and \( 0 \leq s + h \leq s_{\xi,y} \). According to (3.9), the slope at \( s \) satisfies \( D^+ f^{\xi,y}(s) = p(a^+_s) \), which implies

\[
D^+ f^{\xi,y}(s + h) - D^+ f^{\xi,y}(s) \geq p(a^+_s + h) - p(a^+_s).
\]

We now want to establish this inequality also for the left derivatives \( D^- f^{\xi,y}(s) \) for all \( s \in [0, s_{\xi,y}] \). According to Proposition A.11, the left derivative \( D^- f^{\xi,y}(s) \) coincides with the left extremal outward direction \( \xi^+_s \) at \( x_s \in J \). Lemma A.9 (ii) (b) furthermore implies that for any sequence \( (x_{s(n)})_{n=1}^\infty \subset J \) where \( s(n) \to s \) from below, so that \( x_{s(n)} \to x \) and

\[
\frac{x_{s(n)} - x_s}{\left\| x_{s(n)} - x_s \right\|} \to \xi^+_s,
\]

the right extremal outward directions \( \xi^+_{s(n)} \in \Xi^{ext}_{x_{s(n)}}(E_x) \) satisfy \( \xi^+_{s(n)} \to -\xi^+_s \) as \( n \to \infty \). Since these in turn correspond to the right derivatives \( D^+ f^{\xi,y}(s(n)) \) via (3.9), it follows that

\[
\lim_{n \to \infty} D^+ f^{\xi,y}(s(n)) = \lim_{n \to \infty} \frac{\xi^+_{s(n)}(s(n))}{\xi^+_{s(n)}(s(n))} = \frac{-\xi^+_s(s)}{-\xi^+_s(s)} = D^- f^{\xi,y}(s).
\]

Since the choice of the sequence \( (x_{s(n)})_{n=1}^\infty \) in (3.26) is arbitrary and the slope function \( p \) is continuous on \([0, \varepsilon/2]\), it follows from this and (3.24) that for all \( s, h \geq 0 \) and \( 0 \leq s + h \leq s_{\xi,y} \),

\[
D^- f^{\xi,y}(s + h) = \lim_{\varphi \to 0^-} D^+ f^{\xi,y}(s + h + \varphi) \geq \lim_{\varphi \to 0^-} p(a^+_s + h + \varphi) = p(a^+_s + h).
\]

\(^{11}\)Note that the function \( f^{\xi,y} \) here corresponds to the orthonormal coordinate system \( (\xi, \varepsilon^{-1}(x - y)) \), and is thus scaled by a factor of \( \varepsilon \) compared to the corresponding function \( f^{\xi,y} \) in the statement of Proposition A.14, see equations (3.2) and (A.5). Hence, the Lipschitz constant here is \( 1/\sqrt{3} \) rather than \( 1/\sqrt{3}\varepsilon \).
Due to the characterisation of the sets of extremal outward directions given in Proposition A.8, there can be no cusp singularities on the boundary $J$. Given the relationship (3.9) between extremal outward directions and the one-sided derivatives, this implies that $D^- f^\xi,y(s) \leq D^+ f^\xi,y(s) = p(a^+_s)$ for all $s \in [0, s_{\xi,y}]$. Combining this inequality with (3.24) and (3.27) yields the analogue of (3.25) for the left derivatives:

$$D^- f^\xi,y(s + h) - D^- f^\xi,y(s) \geq p(a^+_s + h) - p(a^+_s).$$

A direct calculation shows that $p'(s) = (p(s) + p^3(s))/s$ and since $p$ is an odd function, it follows that $p'$ is even and non-positive. Then

$$p'_{\min} := \min_{|s| \leq \varepsilon/2} p'(s) \geq \min_{|s| \leq \varepsilon/2} p'(s) = p'\left(\frac{\varepsilon}{2}\right) = -\frac{8}{3\sqrt{3}\varepsilon},$$

which implies

$$p(a^+_s + h) - p(a^+_s) = \int_{a^+_s}^{a^+_s+h} p'(s)ds \geq \int_{a^+_s}^{a^+_s+h} p'_{\min}ds \geq -\frac{8h}{3\sqrt{3}\varepsilon}.$$

The result now follows from (3.25) and (3.28). \hfill \Box

**APPENDIX A. Preliminaries**

We present an overview of the main concepts and techniques introduced in [9] for the analysis of local geometric properties of the boundary $\partial E_\varepsilon$. For the proofs of these results we refer the reader to [9].

In [9] we introduced the following concepts that are related to the local geometry on $\partial E_\varepsilon$:

(i) contributor, \hspace{1cm} (iv) extremal pair,

(ii) outward direction, \hspace{1cm} (v) local contribution property,

(iii) extremal contributor and outward direction, \hspace{1cm} (vi) local boundary representation.

It follows from the definition of an $\varepsilon$-neighbourhood that each boundary point $x \in \partial E_\varepsilon$ has at least one contributor: a point $y \in E$ for which $\|y - x\| = \varepsilon$.

**Definition A.1 (Contributor, [9, Definition 2.1]).** Let $E \subset \mathbb{R}^d$ be closed. For each $x \in \partial E_\varepsilon$ we define the set of contributors as the collection

$$\Pi_E(x) := \{ y \in \partial E : \|y - x\| = \varepsilon \}.$$

Boundary points $x \in \partial E_\varepsilon$ with only one contributor constitute the set

$$\text{Unp}_\varepsilon(E) := \{ x \in \partial E_\varepsilon : \Pi_E(x) = \{ y \} \text{ for some } y \in \partial E \},$$

where Unp stands for 'unique nearest point', see [5, Definition 4.1].

Each boundary point may have more than one contributor. The set of those boundary points that have a unique contributor is denoted by Unp$_\varepsilon(E)$, where 'Unp' stands for 'unique nearest point'. It turns out to be convenient to define smooth points on the boundary using the property of unique contribution.

**Definition A.2 (Smooth point, singularity, [9, Definition 2.2]).** We call a boundary point $x \in \text{Unp}_\varepsilon(E)$ smooth, if there exists a neighbourhood $B_r(x)$ for which $\partial E_\varepsilon \cap B_r(x) \subset \text{Unp}_\varepsilon(E)$. If $x$ is not smooth, we call it a singularity and write $x \in S(E_\varepsilon)$.

Proposition A.3 below confirms that the interpretation of smoothness given in Definition A.2 is indeed justified.
Proposition A.3 (Characterisation of smooth points, [9, Proposition 4.6]). Let $E \subset \mathbb{R}^2$ and $x \in \partial E$. Then $x$ is smooth in the sense of Definition A.2 if and only if there exists a $C^1$-curve $\Gamma$ for which $\Gamma = \partial E \cap B_\delta(x)$ for some $\delta > 0$.

Intuitively, outward directions represent the directions $\xi \in S^1$ which one can take in order to exit the set $E$ at a boundary point $x \in \partial E$. We define outward directions as points on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ but think of them rather as directional vectors in the ambient space $\mathbb{R}^d$, since we want to operate with them using the Euclidean scalar product $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$.

Definition A.4 (Outward direction, [9, Definition 2.4]). Let $E \subset \mathbb{R}^d$ be closed. We say that a point $\xi \in S^{d-1}$ is an outward direction from $E_\varepsilon$ at a boundary point $x \in \partial E_\varepsilon$, if there exists a sequence $(x_n)_{n=1}^\infty \subset E_\varepsilon^c$, for which $x_n \to x$ and

$$\xi_n := \frac{x_n - x}{\|x_n - x\|} \to \xi \in S^{d-1} \subset \mathbb{R}^d,$$

as $n \to \infty$. We denote by $\Xi_x(E_\varepsilon)$ the set of outward directions from $E_\varepsilon$ at $x$.

Since we are considering closed $\varepsilon$-neighbourhoods, every boundary point has at least one outward direction.

Proposition A.5 (The set of outward directions is non-empty and closed, [9, Proposition 2.7]). Let $E \subset \mathbb{R}^d$ be closed and $x \in \partial E$. Then the set $\Xi_x(E_\varepsilon)$ of outward directions is non-empty and closed.

We single out those outward directions and contributors that are perpendicular relative to each other. These encode local geometric information about the boundary and serve as the fundamental technical building block for everything that follows.

Definition A.6 (Extremal contributor, extremal outward direction). Let $E \subset \mathbb{R}^d$ be closed and $x \in \partial E$. If an outward direction $\xi \in \Xi_x(E_\varepsilon)$ and a contributor $y \in \Pi_E(x)$ satisfy

$$\langle y - x, \xi \rangle = 0,$$

we call $\xi$ an extremal outward direction and $y$ an extremal contributor at $x$. For each $x \in \partial E_\varepsilon$, we write $\Xi^\text{ext}_x(E_\varepsilon)$ and $\Pi^\text{ext}_E(x)$ for the sets of extremal outward directions and extremal contributors, respectively.

In order to describe the geometry of the sets of outward directions, we adopt the following notation for geodesic arc-segments on the unit circle $S^1$.

Definition A.7 (Geodesic arc-segment, [9, Definition 2.11]). Let $v, w \in S^1 \subset \mathbb{R}^2$ and let

$$(A.1)\quad [v, w]_{S^1} := \{u \in S^1 : u = av + bw \text{ for some } a, b \geq 0\}.$$

For $w \neq -v$, the set $[v, w]_{S^1}$ defines a geodesic arc-segment between $v$ and $w$. We also define the corresponding open geodesic arc-segment $(v, w)_{S^1} \subset S^1$ as

$$(A.2)\quad (v, w)_{S^1} := [v, w]_{S^1} \setminus \{v, w\}.$$

We use the notations $[v, w]_{S^1}$ and $(v, w)_{S^1}$ in accordance with (A.1) and (A.2) also for the cases $v = w$ and $v = -w$, even though the corresponding sets in these cases are not arc-segments.

Proposition A.8 (Structure of sets of outward directions, [9, Proposition 2.12]). Let $E \subset \mathbb{R}^2$ be compact and $x \in \partial E$. Then the set of outward directions $\Xi_x(E_\varepsilon)$ satisfies the following.

(i) If $x \in \text{Unp}_\varepsilon(E)$, then $\Xi_x(E_\varepsilon) = \{\xi \in S^1 : \langle y - x, \xi \rangle \leq 0\}$;
(ii) If $x \notin \text{Unp}_\varepsilon(E)$, then $\Xi_x(E_\varepsilon) = [\xi_1, \xi_2]_{S^1}$, where $\xi_1, \xi_2$ are the only extremal outward directions at $x$, possibly satisfying $\xi_1 = \xi_2$. 

Lemma A.9 below summarises the limiting behaviour of outward directions $\xi_n$ and contributors $y_n$ of points $x_n$ that appear in convergent sequences on the $\varepsilon$-neighbourhood boundary. In particular, Lemma A.9 (ii)(a) establishes that for each $x \in \partial E_\varepsilon$ the set of tangent vectors $T_x(E_\varepsilon)$ is a subset of the set $\Xi^\text{ext}(E_\varepsilon)$ of extremal outward directions. According to Proposition A.11 these sets in fact coincide for all $x \in \partial E_\varepsilon$.

**Lemma A.9 (Orientation in converging sequences of boundary points, [9, Lemma 2.13]).** Let $E \subset \mathbb{R}^2$ be compact and let $x \in \partial E_\varepsilon$. Furthermore, let $(x_n)_{n=1}^{\infty}$ be a sequence on $\partial E_\varepsilon$ with $x_n \to x$ and define $\xi_n := (x_n - x)/\|x_n - x\|$ for all $n \in \mathbb{N}$. Then the following statements hold true:

(i) The sequence $(\xi_n)_{n=1}^{\infty}$ can be split into two disjoint, convergent subsequences $(\xi_{i,k})_{k=1}^{\infty}$, where $i \in \{1, 2\}$ and $\xi^{(i)} := \lim_{k \to \infty} \xi_{i,k} \in \Xi^\text{ext}_E(E_\varepsilon)$.

(ii) If the limit $\xi := \lim_{n \to \infty} \xi_n \in S^1$ exists, then

(a) every sequence $(y_n)_{n=1}^{\infty}$ in $E$ with $y_n \in \Pi(x_n)$ for all $n \in \mathbb{N}$ has a convergent subsequence $(y_{k})_{k=1}^{\infty}$ for which $y := \lim_{k \to \infty} y_{n_k} \in \Pi^\text{ext}_E(x)$. Furthermore $\langle y - x, \xi \rangle = 0$ and consequently $\xi \in \Xi^\text{ext}_E(E_\varepsilon)$.

(b) every sequence $(\eta_n)_{n=1}^{\infty}$ in $S^1$ with $\eta_n \in \Xi^\text{ext}_E(E_\varepsilon)$ for all $n \in \mathbb{N}$ satisfies

$$\lim_{n \to \infty} \|\langle \eta_n, \xi \rangle\| = 1.$$

Acknowledging that classical tangents do not necessarily exist everywhere on the boundary, we adopt a set-valued definition of tangency which allows for several tangential directions to exist at each point. Our definition is a restriction of [5, Definition 4.3] to the boundary $\partial E_\varepsilon$.

**Definition A.10 (Tangent set, [9, Definition 2.3]).** Let $E \subset \mathbb{R}^d$ be closed and $x \in \partial E_\varepsilon$. We define the set $T_x(E_\varepsilon)$ of unit tangent vectors of $E_\varepsilon$ at $x$ as all those points $v \in S^{d-1}$ for which there exists a sequence $(x_n)_{n=1}^{\infty} \subset \partial E$ of boundary points satisfying $x_n \to x$ and

$$\frac{x_n - x}{\|x_n - x\|} \to v, \quad \text{as} \ n \to \infty.$$

The usefulness of the concept of extremal outward directions for the analysis of tangential properties hinges on their geometric relationship with the extremal contributors, and the following result.
Proposition A.11 (Extremal outward directions coincide with tangents, [9, Proposition 2.14]). Let $E \subset \mathbb{R}^2$ be compact and let $x \in \partial E_\varepsilon$. Then $T_x(E_\varepsilon) = E_\varepsilon^{ext}(E_\varepsilon)$.

In order to obtain a local (instead of merely point-wise) representation for the boundary geometry in terms of extremal pairs, one needs to confirm that inside sufficiently small neighbourhoods, the orientation of the boundary cannot fluctuate too dramatically. This is guaranteed by the following local contribution property.

Proposition A.12 (Local contribution, [9, Proposition 3.1]). Let $E \subset \mathbb{R}^2$ and $x \in \partial E_\varepsilon$ with $E_\varepsilon^{ext}(E_\varepsilon) = \{\xi_1, \xi_2\}$, where we allow $\xi_1 = \xi_2$. Then for all $\delta > 0$ there exists some $r > 0$ such that given $z \in B_r(x)$, we have $z \in E_\varepsilon$ if and only if either

\begin{equation}
z \notin \cup_{\xi} U_r(x, \xi_1) \cup U_r(x, \xi_2), \text{ or}
\end{equation}

\begin{equation}
z \in B_r(E \cap B_0(\Pi_\varepsilon^ext(x))).
\end{equation}

Intuitively, Proposition A.12 establishes that the geometry of the boundary $\partial E_\varepsilon$ near each boundary point $x \in \partial E_\varepsilon$ only depends on the positions of points $y \in \partial E$ near the extremal contributors $y \in \Pi_\varepsilon^ext(x)$. It follows from this that the boundary can be represented locally as a finite union of continuous graphs.

Proposition A.13 (Local boundary representation, [9, Proposition 3.5]). Let $E \subset \mathbb{R}^2$ be closed and let $x \in \partial E_\varepsilon$. For each extremal pair $(\xi, y) \in \mathcal{P}_x^ext(E_\varepsilon)$ there exists a continuous function $f^{\xi, y} : [0, \varepsilon/2] \to \mathbb{R}$ and a corresponding function $g_{\xi, y} : [0, \varepsilon/2] \to \mathbb{R}^2$, given by

\begin{equation}
g_{\xi, y}(s) := x + s\xi + f^{\xi, y}(s)(x - y),
\end{equation}

so that the collection $\mathcal{G}(x) := \{g_{\xi, y} : (\xi, y) \in \mathcal{P}_x^ext(E_\varepsilon)\}$ satisfies

\begin{equation}
\partial E_\varepsilon \cap B_r(x) = \bigcup_{(\xi, y) \in \mathcal{P}_x^ext(E_\varepsilon)} g_{\xi, y}(A_{\xi, y})
\end{equation}

for some $r > 0$ and some closed $A_{\xi, y} \subset [0, \varepsilon/2]$. We call the collection $\mathcal{G}(x)$ a local boundary representation (of radius $r$) at $x$. For each extremal pair $(\xi, y) \in \mathcal{P}_x^ext(E_\varepsilon)$ the corresponding subset $A_{\xi, y} \subset [0, \varepsilon/2]$ is either

(a) an interval $[0, s_{\xi, y}]$ for some $0 < s_{\xi, y} \leq \varepsilon/2$, or
(b) a closed set whose complement in $[0, \varepsilon/2]$ contains a sequence of disjoint open intervals with $0$ as an accumulation point.

For wedges (type S1) and $x \in \text{Unp}_\varepsilon(E)$, case (a) holds true for all $(\xi, y) \in \mathcal{P}_x^ext(E_\varepsilon)$.

Note that if a local boundary representation $\mathcal{G}(x)$ exists around $x \in \partial E_\varepsilon$ for some $r > 0$, it exists trivially also for any smaller radius $0 < \rho < r$. According to the following Proposition, the functions $f^{\xi, y} : [0, \varepsilon/2] \to \mathbb{R}$ in (A.5) are Lipschitz-continuous on $[0, r]$ for all $(\xi, y) \in \mathcal{P}_x^ext(E_\varepsilon)$ with a Lipschitz-constant $1/\sqrt{3}\varepsilon$, which implies that the corresponding functions $g_{\xi, y} : [0, \varepsilon/2] \to \mathbb{R}^2$ are $2/\sqrt{3}$-Lipschitz on $[0, r]$.

Proposition A.14 (Local boundary representation is Lipschitz, [9, Proposition 3.6]). Let $E \subset \mathbb{R}^2$, let $x \in \partial E_\varepsilon$ and let $\mathcal{G}(x)$ be a local boundary representation at $x$ with radius $r > 0$. For each extremal pair $(\xi, y) \in \mathcal{P}_x^ext(E_\varepsilon)$, the function $f^{\xi, y}$ in (A.5) is $1/\sqrt{3}\varepsilon$-Lipschitz, and the function $g_{\xi, y} \in \mathcal{G}(x)$ is $2/\sqrt{3}$-Lipschitz on the interval $[0, r]$.

In the analysis of the global topological structure of the boundary, one needs to be able to argue that near ‘most’ boundary points, the complement $E_\varepsilon^c$ has a simple topological structure. The following results state that this is the case for wedge singularities, and whenever $x \in \text{Unp}_\varepsilon(x)$. 

Lemma A.15 (Unique connected component, [9, Proposition 3.8]). Let \( E \subset \mathbb{R}^2 \) and let \( x \in \partial E \_\varepsilon \) either be a wedge (type S1) or \( x \in \text{Unp}_\varepsilon(E) \). Then there exists a unique connected component \( V \subset E_\varepsilon \), for which \( x \in \partial V \).

Proposition A.16 (Geometry of the complement, [9, Proposition 3.9]). Let \( E \subset \mathbb{R}^2 \) and let \( x \in \partial E \_\varepsilon \) either be a wedge or \( x \in \text{Unp}_\varepsilon(E) \). Then there exists some \( r > 0 \), for which

\[
E_\varepsilon \cap B_r(x) = V \cap B_r(x) = \bigcup_{0 < \rho < r} x + A(\rho),
\]

where \( V \subset E_\varepsilon \) is connected and for each \( \rho \in (0, r) \) either \( A(\rho) = \rho(\alpha, \beta, S_1) \) or \( A(\rho) = \rho(S^1 \setminus [\alpha, \beta, S_1]) \). In our classification of boundary points, case (a) corresponds to smooth points and singularities of types S4 and S5, case (b) to type S1, and case (c) to types S2, S3 and S6–S8. See also Figure 10 and Proposition A.8 regarding the structure of the set of outward directions \( \Xi_{\text{ext}}(x) \subset \{\xi_1, \xi_2\} \).

**Figure 10.** The local geometry at each boundary point \( x \in \partial E \_\varepsilon \) reflects the three basic scenarios (a)–(c) regarding the number and positions of contributors \( y \in \Pi_\text{ext}(x) \). In our classification of boundary points, case (a) corresponds to smooth points and singularities of types S4 and S5, case (b) to type S1, and case (c) to types S2, S3 and S6–S8. See also Figure 11 and Proposition A.8 regarding the structure of the set of outward directions \( \Xi_{\text{ext}}(E_\varepsilon) \).

The first main result in [9], Theorem A.19 below, provides a classification of boundary points into eight types of singularities. The classification scheme relies geometrically on the orientation of the extremal contributors \( y \in \Pi_\text{ext} E(x) \) at each boundary point \( x \in \partial E \_\varepsilon \). In the planar case, there are essentially three different ways this orientation can be realised, depicted schematically in Figure 10. The defining property \( y_1 - x = -(y_2 - x) \) for the extremal contributors \( y_1, y_2 \in \Pi_\text{ext} E(x) \) in case (c) can be equivalently expressed by \( \langle (y_1 - x)/\varepsilon, (y_2 - x)/\varepsilon \rangle = -1 \), and we will make use of both formulations in what follows.

The different types of singularities are given in Definition A.17 below. Schematic illustrations of the different types of singularities are given in Figure 11. We denote by \( U_r(x, v) \) an open \( x \)-centered half-ball of radius \( r \), oriented in the direction of \( v \in S^1 \),

\[
U_r(x, v) := \{z \in B_r(x) : \langle z - x, v \rangle > 0\}.
\]

In addition, we denote by \( S(E_\varepsilon) \) the set of singularities on the boundary \( \partial E_\varepsilon \).

**Definition A.17 (Types of singularities, [9, Definition 4.1]).** Let \( E \subset \mathbb{R}^2 \) be closed, let \( x \in S(E_\varepsilon) \) and let \( \Xi_{\text{ext}}(E_\varepsilon) = \{\xi_1, \xi_2\} \) be the set of extremal outward directions, where we allow for the possibility \( \xi_1 = \xi_2 \). We define the following eight types of singularities.
Figure 11. Schematic illustration of Theorem A.19. The grey area represents the \( \varepsilon \)-neighbourhood \( E_{\varepsilon} \), the white area the complement \( \mathbb{R}^2 \setminus E_{\varepsilon} \). Every boundary point \( x \in \partial E_{\varepsilon} \) either is a smooth point or belongs to exactly one of eight categories of singularities. At a wedge (S1) the one-sided tangents form an angle \( 0 < \theta < \pi \). A sharp singularity (S2) and a sharp-sharp singularity (S3) can be thought of as extremal cases of a wedge, with \( \theta = 0 \). A shallow singularity (S4) and a shallow-shallow singularity (S5) have a well-defined tangent, but they are accumulation points (from one or two directions, respectively) of sequences of increasingly obtuse wedges (black dots). A chain singularity (S6), a chain-chain singularity (S7) and a sharp-chain singularity (S8) share the geometric property of being accumulation points of sequences of increasingly acute wedges (black dots). See also Figure 10.

S1: \( x \) is a wedge, if \( \xi_1 \notin \{\xi_2, -\xi_2\} \), i.e. the angle \( \theta \) between the vectors \( \xi_1, \xi_2 \) satisfies \( 0 < \theta < \pi \); 

S2: \( x \) is a (one-sided) sharp singularity, if \( \xi_1 = \xi_2 \), and there exists some \( \delta > 0 \) for which the intersection \( B_\delta(x) \cap E_{\varepsilon}^c \) is a connected set; 

S3: \( x \) is a sharp-sharp singularity, if \( \xi_1 = -\xi_2 \) and for each \( i \in \{1, 2\} \) there exists some \( \delta_i > 0 \) for which the intersection \( U_{\delta_i}(x, \xi_i) \cap E_{\varepsilon}^c \) is a connected set; 

S4: \( x \) is a (one-sided) shallow singularity if \( x \in \text{Unp}_\varepsilon(E) \) and 
(i) \( U_{\delta_1}(x, \xi_1) \cap \partial E_{\varepsilon} \subset \text{Unp}_\varepsilon(E) \) for some \( \delta_1 > 0 \), and 
(ii) \( U_{\delta_2}(x, \xi_2) \cap \partial E_{\varepsilon} \not\subset \text{Unp}_\varepsilon(E) \) for all \( \delta_2 > 0 \). 

S5: \( x \) is a shallow-shallow singularity if \( x \in \text{Unp}_\varepsilon(E) \) and \( U_{\delta}(x, \xi_i) \cap \partial E_{\varepsilon} \not\subset \text{Unp}_\varepsilon(E) \) for all \( \delta > 0 \) and \( i \in \{1, 2\} \).
S6: x is a (one-sided) chain singularity, if \( \xi_1 = \xi_2 \) and there exists a sequence of singularities \((x_n)_{n=1}^{\infty} \subset S(E_c)\), for which \( x_n \to x \) and

\[
\left\langle \frac{y_n^{(1)} - x_n}{\varepsilon}, \frac{y_n^{(2)} - x_n}{\varepsilon} \right\rangle \to -1,
\]

where \( \{y_n^{(1)}, y_n^{(2)}\} = \Pi_{E_c}^x(x_n) \) is the set of extremal contributors at each \( x_n \);

S7: x is a chain-chain singularity, if \( \xi_1 = -\xi_2 \) and for each \( i \in \{1, 2\} \) there exists some \( \delta_i > 0 \) and a sequence \((x_{i,n})_{n=1}^{\infty} \subset U_{\delta_i}(x, \xi_i) \cap S(E_c)\), for which \( x_{i,n} \to x \) and

\[
\left\langle \frac{y_{i,n}^{(1)} - x_{i,n}}{\varepsilon}, \frac{y_{i,n}^{(2)} - x_{i,n}}{\varepsilon} \right\rangle \to -1,
\]

where \( \{y_{i,n}^{(1)}, y_{i,n}^{(2)}\} = \Pi_{E_c}^x(x_{i,n}) \) is the set of extremal contributors at each \( x_{i,n} \);

S8: x is a sharp-chain singularity, if \( \xi_1 = -\xi_2 \) and

(i) there exists a \( \delta_1 > 0 \) for which the intersection \( U_{\delta_1}(x, \xi_1) \cap E_c^x \) is a connected set, and

(ii) there exists some \( \delta_2 > 0 \) and a sequence \((x_n)_{n=1}^{\infty} \subset U_{\delta_2}(x, \xi_2) \cap S(E_c)\), for which \( x_n \to x \) and

\[
\left\langle \frac{y_n^{(1)} - x_n}{\varepsilon}, \frac{y_n^{(2)} - x_n}{\varepsilon} \right\rangle \to -1,
\]

where \( \{y_n^{(1)}, y_n^{(2)}\} = \Pi_{E_c}^x(x_n) \) is the set of extremal contributors at each \( x_n \).

The key to proving that the classification given in Definition A.17 defines a partition of the set of singularities on \( \partial E_c \) is Proposition A.18, which characterises the boundary geometry near boundary points of types S2–S3 and S6–S8 in terms of the local topology of the complement \( E_c^x \).

**Proposition A.18 (Difference between sharp-type and chain-type geometry, [9, Proposition 4.2]).** Let \( E \subset \mathbb{R}^2, x \in \partial E_c \) and \( \Pi_{E_c}^x(x) = \{y_1, y_2\} \) with \( y_1 - x = -(y_2 - x) \). Furthermore, let \( G(x) \) be a local boundary representation with radius \( r > 0 \) at \( x, \xi \in \Xi_x^c(E_c) \) and let \( g_{\xi,y_1}, g_{\xi,y_2} \in G(x) \) be as in (A.5). Then exactly one of the cases (i) and (ii) below holds true:

(i) (sharp-type) There exists some \( r > 0 \), for which \( g_{\xi,y_1}(s) \neq g_{\xi,y_2}(s) \) for all \( s \in (0, r) \), and

\[
E_c^x \cap U_r(x, \xi) = V_\xi \cap U_r(x, \xi) = \bigcup_{0 < s < r} x + s(\alpha(s), \beta(s))_{S^1},
\]

where \( V_\xi \) is the unique connected component of \( E_c^x \) intersecting \( U_r(x, \xi) \), \( \alpha(s), \beta(s) \in S^1 \) for all \( s \in (0, r) \) and \( \alpha(s), \beta(s) \to \xi \) as \( s \to 0 \).

(ii) (chain-type) There exists a sequence \((s_n)_{n=1}^{\infty} \subset \mathbb{R}^+ \) with the following properties:

(a) \( s_n \to 0 \) and \( g_{\xi,y_1}(s_n) = g_{\xi,y_2}(s_n) \) for all \( n \in \mathbb{N} \). We denote this common value by \( x_n \).

(b) There exists some \( r > 0 \) and a sequence \((V_n)_{n=1}^{\infty} \subset U_r(x, \xi) \) of disjoint connected components of \( E_c^x \) with \( \text{dist}_H(x, V_n) \to 0 \) as \( n \to \infty \) and \( x_n \in \partial V_n \) for all \( n \in \mathbb{N} \).

(c) \( x_n \in S(E_c) \) for each \( n \in \mathbb{N} \), with

\[
\lim_{n \to \infty} \left\langle \frac{y_n^{(1)} - x_n}{\varepsilon}, \frac{y_n^{(2)} - x_n}{\varepsilon} \right\rangle = -1,
\]

where \( \Pi_{E_c}^x(x_n) = \{y_n^{(1)}, y_n^{(2)}\} \) for all \( n \in \mathbb{N} \).

**Theorem A.19 (Classification of boundary points, [9, Theorem 1]).** Let \( E \subset \mathbb{R}^2 \) be compact, \( \varepsilon > 0 \) and let \( x \in \partial E_c \) be a boundary point of \( E_c \) that is not smooth. Then \( x \) belongs to precisely one of the eight categories of singularities given in Definition A.17.
The second main result in [9] regards the cardinality of different types of singularities.

**Theorem A.20 (Countable sets of singularities, [9, Theorem 2]).** For a compact set $E \subset \mathbb{R}^2$, the number of wedges (S1), sharp singularities (S2, S3 and S8) and one-sided shallow singularities (S4) and chain singularities (S6) on $\partial E_\varepsilon$ is at most countably infinite.

Even though there may in general be infinitely many sharp singularities on the boundary, it turns out that only finitely many of these may lie on the boundary of any given connected component of the complement.

**Lemma A.21 (Number of sharp singularities, [9, Lemma 5.3]).** Let $E \subset \mathbb{R}^2$ be compact. For any connected component $U$ of the complement $E_\varepsilon^c$, the number of sharp singularities (S2, S3 and S8) on the boundary $\partial U$ is finite.

We refer to the set of points falling into categories S6–S8 collectively as *chain singularities* and denote this set by $C(E)$. Even though the set of chain-chain singularities (S7) may in general be uncountable and can have a positive Hausdorff measure on the boundary, the third main result in [9] establishes the fact that $C(E)$ is always closed and totally disconnected.

**Theorem A.22 (The set of chain singularities is closed and totally disconnected, [9, Theorem 3]).** For any compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the set $C(E)$ of chain singularities is closed and totally disconnected.

Theorem A.22 furthermore implies that $C(E)$ is nowhere dense on $\partial E_\varepsilon$, and is hence small in the topological sense.

Of specific importance for the topological structure of $\varepsilon$-neighbourhoods is the following result, which characterises those boundary points that do not lie on the boundary of any connected component of the complement $E_\varepsilon^c$.

**Proposition A.23 (Inaccessible singularities, [9, Corollary 4.3]).** Let $E \subset \mathbb{R}^2$ and $x \in \partial E_\varepsilon$. Then $x \notin \partial V$ for all connected components $V$ of the complement $E_\varepsilon^c$ if and only if $x$ is a one-sided chain singularity (S6) or a chain-chain singularity (S7).

**Acknowledgements**

This project has received funding from the European Union’s Horizon 2020 research and innovation Programme under the Marie Sklodowska-Curie grant agreement no. 643073. The authors express their gratitude to Gabriel Fuhrman, Vadim Kulikov, Tuomas Orponen, Sebastian van Strien and Dmitry Turaev for many useful discussions and valuable comments regarding earlier versions of this article.

**References**

1. L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, 2000.
2. A. Blokh, M. Misiurewicz, and L. Oversteegen, *Sets of constant distance from a compact set in 2-manifolds with a geodesic metric*, Proc. Amer. Math. Soc. **137** (2009), no. 2, 733–743.
3. M. Brown, *Sets of constant distance from a planar set*, Michigan Math. J. **19** (1972), 321–323.
4. F. Colonius and W. Kliemann, *The Dynamics of Control*, Birkhäuser, Boston, 2000.
5. H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.
6. S. Ferry, *When $\varepsilon$-boundaries are manifolds*, Fund. Math. **90** (1976), 199–210.
7. J.H.G. Fu, *Tubular neighborhoods in Euclidian spaces*, Duke Mathematical Journal **52** (1985), no. 4, 1025–1046.
8. A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis*, Dover Publications, 1975.
9. J. S. W. Lamb, M. Rasmussen, and K. Timperi, *On boundaries of $\varepsilon$-neighbourhoods of planar sets, part I: Singularities*, Submitted to the Journal of the London Mathematical Society. An arXiv preprint is available at https://arxiv.org/abs/2012.13515.
10. J.S.W. Lamb, M. Rasmussen, and C. Rodrigues, *Topological bifurcations of minimal invariant sets for set-valued dynamical systems*, Proceedings of the American Mathematical Society 143 (2015), no. 9, 3927–3937.

11. R.L. Moore, *Concerning triods in the plane and the function points of plane continua*, PNAS 14 (1928), 85–88.

12. I. Ya. Oleksiv and N. I. Pesin, *Finiteness of the Hausdorff measure of level sets of bounded subsets in Euclidian space*, Mat. Zametki 37 (1985), 422–431, (in Russian).

13. A. Przeworski, *An upper bound on density for packings of collars about hyperplanes in \( \mathbb{H}^n \)*, Geom. Dedicata 163 (2013), 193–213.

14. J. Rataj and S. Winter, *On volume and surface area of parallel sets*, Indiana University Mathematics Journal 59 (2010), no. 5, 1661–1685.

15. J. Rataj and L. Zajíček, *Smallness of the critical values of distance functions in two-dimensional euclidean and riemannian spaces*, Mathematika 66 (2020), 297–324.

16. L.L. Stacho, *On the volume function of parallel sets*, Acta Sci. Math. 38 (1976), 365–374.

Jeroen Lamb, Department of Mathematics, Imperial College London, 180 Queen’s Gate, South Kensington, London SW7 2AZ, United Kingdom

Martin Rasmussen, Department of Mathematics, Imperial College London, 180 Queen’s Gate, South Kensington, London SW7 2AZ, United Kingdom

Kalle Timperi, Department of Mathematics, Imperial College London, 180 Queen’s Gate, South Kensington, London SW7 2AZ, United Kingdom