On Superstable Expansions of Free Abelian Groups

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Abstract

We prove that \((\mathbb{Z},+,0)\) has no proper superstable expansions of finite Lascar rank. Nevertheless, we exhibit a superstable proper expansion.

1 Introduction

This paper fits into the general framework of adding a new predicate to a well behaved structure and asking whether the obtained structure is still well behaved. A similar line of thought is to impose the desired properties on the expanded structure and ask for which predicates these properties are fulfilled. Even more, one might ask whether there exist proper expansions fulfilling the desired properties.

Many results that belong to the above mentioned framework have been obtained by various authors. For example Pillay and Steinhorn [PS87] proved that there no (proper) o-minimal expansions of \((\mathbb{N},\leq)\). On the other hand, Marker [Mar87] proved that there are (proper) strongly minimal expansions of \((\mathbb{N},s)\), i.e. the natural numbers with the successor function. In a more abstract context Baldwin and Benedikt proved that if \(M\) is a stable structure and \(I\) is a “small” set of indiscernibles then \((M,I)\) is still stable. Finally, Chernikov and Simon [CS12] proved the analogous result for NIP theories, i.e. NIP is preserved after naming a “small” indiscernible sequence.

In this short paper we are mainly interested in (finitely generated) free abelian groups. We are motivated by the recent addition of torsion-free hyperbolic groups to the family of stable groups (see [Sel13]). In a torsion-free hyperbolic group centralizers of (non-trivial) elements are infinite cyclic and one is interested in the induced structure on them. It seems that understanding the induced structure on these centralizers boils down to understanding whether they are superstable and if so calculate their Lascar rank.

Our main result generalizes a theorem in the thesis of the second named author proving that every Lascar rank 1 expansion of \((\mathbb{Z},+,0)\) is a pure group (see [Skl11, Theorem 8.2.3]).

**Theorem 1:** There are no (proper) superstable finite Lascar rank expansions of \((\mathbb{Z},+,0)\).

We also show that one cannot strengthen the above result any further by proving:

**Theorem 2:** The theory of \((\mathbb{Z},+,0,\Pi)\), where \(\Pi\) denotes the set of powers of 2, is superstable of Lascar rank \(\omega\).

On the other hand, if one moves to higher rank free abelian groups Theorem 1 is no longer true, and it is not hard to find proper superstable Lascar rank 1 expansions of \((\mathbb{Z}^n,+,0)\), for \(n \geq 2\). The main reason being that there exist finite index subgroups of \(\mathbb{Z}^n\) (for \(n \geq 2\) that
are not definable in \((\mathbb{Z}^n, +, 0)\). Still, we record, that a superstable finite Lascar rank expansion of \((\mathbb{Z}^n, +, 0)\) is one-based.

The essential tools for proving Theorem 1 come from geometric stability. We combine results from Hrushovski’s thesis together with Buechler’s dichotomy theorem, the characterization of one-based groups by Hrushovski-Pillay and a result on one-based types due to Wagner.

Remarks While checking our results, the second named author figured out in a talk of Bruno Poizat that Theorem 2 was already proved in [Poi14, Théorème 25]. However, since our approach seems to be different than that of B.Poizat we believe our proof is worth recording.

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2 Preliminaries

In this section we collect some well-known facts and definitions on geometric stability theory and free abelian groups.

2.1 Geometric stability

We start this section recalling the notions of internality.

**Definition 2.1:** Let \(\pi(x)\) be a partial type over \(A\) and let \(\mathcal{P}\) be a family of partial types. We say that \(\pi\) is \(\mathcal{P}\)-internal if there exists a subset \(B \supseteq A\) such that for any realization \(a \models \pi\) we can find a finite tuple \(\bar{b}\) of realizations of types in \(\mathcal{P}\) over \(B\) such that \(a \in \text{dcl}^{eq}(B, \bar{b})\).

When dealing with type-definable stable groups, Hrushovski [Hru86] showed the existence of internal factors. A proof of this fact can be found in [Wag97, Theorem 3.1.1].

**Fact 2.2:** Let \(G\) be a stable type-definable group, and assume that a generic type of \(G\) is non-orthogonal to a type \(q\). Then there exists an \(\emptyset\)-definable normal subgroup \(H\) of infinite index in \(G\) such that the quotient \(G/H\) is internal to the family of \(\emptyset\)-conjugated of \(q\).

**Definition 2.3:** Let \(\pi(x)\) be a partial type over \(A\) and let \(\mathcal{P}\) be a family of partial types. We say that \(\pi\) is \(\mathcal{P}\)-analysable if for any \(a \models \pi\) there exists a sequence \((a_i : i < \alpha)\) in \(\text{dcl}^{eq}(A, a)\) such that \(\text{tp}(a_i/A, a_j : j < i)\) is \(\mathcal{P}\)-internal and \(a \in \text{acl}^{eq}(A, a_i : i < \alpha)\).

In fact, a type-definable group \(G\) is \(\mathcal{P}\)-analysable if and only if it admits a series of relatively definable normal subgroups \(G = H_0 \supseteq H_1 \supseteq \ldots \supseteq H_\alpha = \{1\}\) such that \(H_\beta = \bigcap_{i < \beta} H_i\) for limit ordinals \(\beta\), and every quotient \(H_\iota/H_{\iota+1}\) is \(\mathcal{P}\)-internal, see [Wag97] Lemma 3.1.3].

Recall that a partial type \(\pi(x)\) over \(A\) is said to be one-based if for any tuple \(\bar{a}\) of realizations of \(\pi\) and any set of parameters \(B \supseteq A\), the type \(\text{tp}(\bar{a}/B)\) does not fork over \(\text{acl}^{eq}(\bar{a}, A) \cap \text{acl}^{eq}(B)\). One-basedness generalizes linear independence in vector spaces, and it is a tame geometric property. Buechler [Pil96, Corollary 3.3] had shown that a type of Lascar rank one is either one-based or has Morley rank one. This fact is known as the Buechler’s dichotomy theorem. Another useful property of one-basedness was proved by Wagner in [Wag04, Corollary 12].

**Theorem 2.4:** Any type analysable in a family of one-based types is one-based.
A well-known example of a one-based stable structure is an abelian group $A$ equipped with predicates for subgroups of powers of $A^n$, see [Wag97 Theorem 4.2.8] for a proof. Furthermore, Hrushovski and Pillay characterized all one-based groups in stable theories, see [Pil96 Corollary 4.8]

**Theorem 2.5:** Let $G$ be a type-definable stable one-based group. A set $X \subseteq G^n$ is definable if and only if it is a boolean combination of cosets of almost $\emptyset$-definable subgroups of $G^n$. Moreover, the group $G$ is abelian-by-finite.

### 2.2 Free abelian groups

In this subsection we collect some well known results about model theoretic properties of free abelian groups. One can see free abelian groups as $\mathbb{Z}$-modules and apply results from the model theory of modules to them. In any case, easy ad-hoc proofs can be produced for all results (mentioned without a proof) in this subsection.

**Fact 2.6:** Let $\equiv_n$ be the congruence modulo $n$ relation on the integers. Then:

- every set definable in $(\mathbb{Z},+,0,\{\equiv_n\}_{n<\omega})$ is definable in $(\mathbb{Z},+,0)$;
- $(\mathbb{Z},+,0,\{\equiv_n\}_{n<\omega})$ admits quantifier elimination;
- $(\mathbb{Z},+,0)$ does not have ordinal Morley rank;
- $(\mathbb{Z},+,0)$ is superstable of Lascar rank 1;
- $(\mathbb{Z},+,0)$ does not have the finite cover property.

The following theorem characterizes subgroups of finitely generated free abelian groups.

**Theorem 2.7:** Let $G$ be a subgroup of $\mathbb{Z}^n$. Then there is a basis $(z_1,\ldots,z_n)$ of $\mathbb{Z}^n$ and a sequence of natural numbers $d_1,\ldots,d_k$ (with $d_i$ dividing $d_{i+1}$ for $i<k$), such that $(d_1 z_1,\ldots,d_k z_k)$ forms a basis of $G$.

One can use Theorem 2.7 to prove the following lemma, which we consider as being part of the folklore.

**Lemma 2.8:** Let $G$ be a subgroup of $\mathbb{Z}^n$. Then $G$ is definable in $(\mathbb{Z},+,0)$.

We note, in contrast, that not all subgroups of $\mathbb{Z}^n$ are definable in $(\mathbb{Z}^n,+,0)$. For example, the finite index subgroup $3\mathbb{Z} \oplus 2\mathbb{Z}$ of $\mathbb{Z}^2$ is not definable in $(\mathbb{Z}^2,+)$, and of course any infinite index subgroup of $\mathbb{Z}^n$, for $n \geq 2$, is not definable in $(\mathbb{Z}^n,+,0)$.

The following proposition follows immediately from Theorem 2.5 and the above lemma.

**Proposition 2.9:** There are no proper one-based stable expansions of $(\mathbb{Z},+,0)$.

### 3 Superstable Expansions of $(\mathbb{Z}^n,+,0)$

We start this section by proving:

**Theorem 3.1:** Any finite Lascar rank expansion of $(\mathbb{Z}^n,+,0)$ is one-based.

**Proof.** Consider a finite Lascar rank expansion $\mathcal{Z} = (\mathbb{Z}^n,+0,\ldots)$ of $(\mathbb{Z}^n,+,0)$. Take an enough saturated model $\Gamma \succeq \mathcal{Z}$ and fix some stationary generic type of $\Gamma$. Since a generic type has finite Lascar rank, we know by [Pil96 Lemma 2.3.1] that it is non-orthogonal to a type $q_1$
of Lascar rank one and hence, by Fact 2.2 we can find an $\emptyset$-definable normal subgroup $H_1$ of infinite index in $\Gamma$ such that $\Gamma/H_1$ is $Q_1$-internal, where $Q_1$ is the family of all $\emptyset$-conjugates of $q_1$. As the Lascar rank of $\Gamma/H_1$ has decreased, iterating this process we obtain a finite series of $\emptyset$-definable normal subgroups $\Gamma \geq H_1 \geq H_2 \geq \ldots \geq H_{m+1} \geq \{0\}$ such that $H_{m+1}$ is finite, and each factor $H_i/H_{i+1}$ is infinite and internal to a family $Q_i$ of $\emptyset$-conjugates of some type $q_i$ of Lascar rank one.

Since free abelian groups are torsion-free they do not have any finite (non-trivial) subgroups, and so neither does $\Gamma$. This implies that $H_{m+1}$ is trivial. Furthermore, by Theorem 2.4 we obtain that no infinite quotient of $Z^n$ has ordinal Morley rank. As all subgroups $H_i$ are $\emptyset$-definable, we deduce that the quotients $H_i/H_{i+1}$ cannot have ordinal Morley rank, and neither do the types from the families $Q_i$. Whence, we conclude by Buechler’s dichotomy theorem that all of them are one-based, and so is $\Gamma$ by Theorem 2.4.

Applying Proposition 2.9 we obtain Theorem 1. On the other hand, by the observation of the previous section on the existence of non definable finite index subgroups of $(Z^n, +, 0)$, we can easily see that there are even Lascar rank one proper expansions of non cyclic free abelian groups.

Next we shall see that there are proper superstable expansions of $(Z, +, 0)$, necessarily, by our theorem, of infinite Lascar rank.

**Definition 3.2:** Let $\mathcal{L}$ be a first-order language and $P(x)$ a unary predicate. We denote by $\mathcal{L}_P$ the first-order language $\mathcal{L} \cup \{P\}$. We say that an $\mathcal{L}_P$-formula $\phi(\bar{y})$ is bounded (with respect to $P$) if it has the form $Q_1x_1 \in P \ldots Q_nx_n \in P, \phi(\bar{x}, \bar{y})$, where the $Q_i$’s are quantifiers and $\psi(\bar{x}, \bar{y})$ is an $\mathcal{L}$-formula.

The following theorem will be useful for proving Theorem 2 we refer the reader to [CZ01] for the proof.

**Theorem 3.3:** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. Consider $(\mathcal{M}, A)$ as a structure in the expanded language $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Suppose every $\mathcal{L}_P$-formula in $(\mathcal{M}, A)$ is equivalent to a bounded one. Then, for every $\lambda \geq |\mathcal{L}|$, if both $\mathcal{M}$ and $A_{ind}$ are $\lambda$-stable, we have that $(\mathcal{M}, A)$ is $\lambda$-stable.

The observation that $a \not\equiv_n b$ is equivalent to $a \equiv_n b + 1 \lor a \equiv_n b + 2 \lor \ldots \lor a \equiv_n b + (n - 1)$, leads to the following remark.

**Remark 3.4:** Let $\mathcal{L}_{mod}$ be the language of groups expanded with countably many 2-place predicates. We recall that an $\mathcal{L}_{mod}$-formula $\phi(\bar{x})$, is equivalent, in $(\mathbb{Z}, +, 0, \{\equiv_n\}_{n < \omega})$, to a finite disjunction of formulas of the form:

$$t_1(\bar{x}) = 0 \lor \ldots \lor t_k(\bar{x}) = 0$$

$$r_1(\bar{x}) \neq 0 \lor \ldots \lor r_l(\bar{x}) \neq 0$$

$$s_1(\bar{x}) \equiv_{n_1} 0 \lor \ldots \lor s_m(\bar{x}) \equiv_{n_m} 0$$

where $t_i(\bar{x}), s_i(\bar{x}), r_i(\bar{x})$ are terms in the above language.

Before moving to the next lemma we introduce for convenience the notion of “consecutive elements” of a subset of the integers. We say that two distinct elements $a_1, a_2$ of $A \subseteq \mathbb{Z}$ with $a_1 < a_2$ are consecutive in $A$, if there is no $a \in A$ such that $a_1 < a < a_2$.

Since our main focus will be on the subset of the integers consisting of powers of 2, we fix the following notation $\Pi := \{2^n \mid 1 \leq n < \omega\}$. 

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Lemma 3.5: Let $\bar{b}$ be a tuple in $\mathbb{Z}$ and $\phi(\bar{b},y,\alpha)$ be an $\mathcal{L}$-formula, where $\mathcal{L}$ is the language of groups. Suppose $\Gamma(y) := \{ \phi(\bar{b},y,\alpha) \mid \alpha \in \Pi \}$ is consistent with $\text{Th}(\mathbb{Z},+,0)$. Then there exists $c \in \mathbb{Z}$ realizing the set $\Gamma(y)$.

Proof. We may assume that $\phi(\bar{x},y,\alpha)$ is a formula as in Remark 3.4. If we fix some $\alpha_0$ in $\Pi$, then each disjunctive clause in $\phi(\bar{b},y,\alpha_0)$ asserts that $y$ is equal to each element from a finite list of elements in $\mathbb{Z}$, and $y$ is not equal to any element from a finite list of elements in $\mathbb{Z}$ and $y$ belongs to the intersection of finitely many cosets of fixed subgroups of $\mathbb{Z}$, where these fixed subgroups only depend on $\phi$ (not $\bar{b}$ or $\alpha_0$).

Our assumption that $\Gamma(y)$ is consistent implies that for each $\alpha_0$ in $\Pi$ we may choose a disjunctive clause in $\phi(\bar{b},y,\alpha_0)$ such that the set of these clauses is again consistent. Note that if one of the chosen clauses involves an equality, then the result holds trivially. So we will assume that no equality is involved in any disjunctive clause of $\phi$. On the other hand the intersection of cosets of subgroups of a group is either empty or a coset of the intersection of the subgroups, thus we will assume that a disjunctive clause that involves congruence modulo relations, it involves exactly one.

Next we prove that a finite union of sets of the form $\{k + l \cdot \alpha \mid \alpha \in \Pi \}$ cannot cover any coset of any (non-trivial) subgroup of $\mathbb{Z}$. Suppose not, and let $n\mathbb{Z} + m \subseteq \{k_1 + l_1 \cdot \alpha \mid \alpha \in \Pi \} \cup \{k_2 + l_2 \cdot \alpha \mid \alpha \in \Pi \} \cup \ldots \cup \{k_p + l_p \cdot \alpha \mid \alpha \in \Pi \}$. We may assume that there are $l_i$’s which are positive. Then we can find a natural number $r > 0$ which is bigger than any $k_i$ with $l_i < 0$, and so that any two consecutive elements bigger than $r$ in any set of the above union differ by distance more than $(p + 1) \cdot n$. Thus, we get an interval $(\gamma, \delta)$ in $\mathbb{Z}$, that contains $p + 1$ elements of $n\mathbb{Z} + m$ but not more than $p$ elements of the above union, a contradiction.

![Figure 1: The union of finitely many sets of the form $\{k + l\alpha \mid \alpha \in \Pi \}$.](image)

Now the consistency of $\Gamma(y)$ implies that $y$ belongs to the intersection of finitely many cosets of subgroups of $\mathbb{Z}$ and $y$ is not equal to any element of a union of sets of the form $\{k + l\alpha \mid \alpha \in \Pi \}$. By our previous claim, a solution can be found in $\mathbb{Z}$ and this finishes the proof.

Now we are able to prove the following technical lemma.
Lemma 3.6: Let $\mathcal{L}$ be the language of groups and $P(x)$ be a unary predicate. Let $\mathcal{Z} := (\mathbb{Z}, +, 0, \Pi)$ be an $\mathcal{L}_P$-structure.

Let $\phi(x, y, \alpha)$ be an $\mathcal{L}$-formula. Then there exists $k < \omega$ such that:

$$\mathcal{Z} \models \forall \bar{x} \left( (\forall \alpha_0 \in P \ldots \forall \alpha_k \in P \exists y \phi(\bar{x}, y, \alpha_0) \wedge \ldots \wedge \phi(\bar{x}, y, \alpha_k) \right) \rightarrow \exists y \forall \alpha \in P \phi(\bar{x}, y, \alpha).$$

Proof. Since $(\mathbb{Z}, +, 0)$ has nfcp we can assign to each formula $\phi$ a natural number $k$ such that any set of instances of the formula $\phi$ is consistent if and only if it is $k$-consistent. By Lemma 3.5 if a set of instances of a formula $\phi$, $\{\phi(b, y, \alpha) \mid \alpha \in \Pi\}$ is consistent, then a solution can be found in $\mathcal{Z}$ and this is enough to conclude. \qed

The following proposition is an easy corollary of Lemma 3.6 and the proof is left to the reader, see [CZ01] Proposition 2.1.

Proposition 3.7: Let $\mathcal{L}$ be the language of groups and $P(x)$ be a unary predicate. Let $\mathcal{Z} := (\mathbb{Z}, +, 0, \Pi)$ be an $\mathcal{L}_P$-structure. Then every $\mathcal{L}_P$-formula in $\mathcal{Z}$ is bounded.

As a consequence we deduce:

Corollary 3.8: Let $\mathcal{L}$ be the language of groups and $P(x)$ be a unary predicate, and let $(\Gamma, +', 0, \Pi') \equiv (\mathbb{Z}, +, 0, \Pi)$ be elementarily equivalent $\mathcal{L}$-structures.

Two tuples of $\Gamma$ realize the same $\mathcal{L}_P$-formulas over any set of parameters $C \subseteq \Gamma$ whenever they realize the same $\mathcal{L}$-formulas over $\Pi' \cup C$.

Proof. Let $a$ and $b$ be two tuples realizing the same $\mathcal{L}$-formulas over $\Pi', C$. It is easy to see by induction on the number of quantifiers that $a$ and $b$ realize the same formulas of the form

$$Q_1 x_1 \in P \ldots Q_n x_n \in P \psi(\bar{x}, \bar{y}),$$

where the $Q_i$'s are quantifiers and $\psi(\bar{x}, \bar{y})$ is an $\mathcal{L}(\Pi' \cup C)$-formula. Hence, we conclude by Proposition 3.7. \qed

Our last task is to prove that the induced structure on the subset of the integers that consists of powers of 2, coming from $(\mathbb{Z}, +, 0)$, is tame. Recall that if $B$ is a subset of the domain $M$, of a first order structure $\mathcal{M}$, then by the induced structure on $B$ we mean the structure with domain $B$ and predicates for every subset of $B^n$ of the form $B^n \cap \phi(M^n)$, where $\phi(x)$ is a first order formula (over the empty set). We denote this structure by $B_{\text{ind}}$.

Proposition 3.9: The structure $\Pi_{\text{ind}}$ (with respect to $(\mathbb{Z}, +, 0)$) is $\omega$-stable and has Lascar rank one.

The proof is split in a series of lemmata. We first prove some results, we believe well known, in the spirit of Diophantine analysis.

Lemma 3.10: Let $k < n$ be natural numbers such that $n$ is coprime with 2, and let $[k]_n$ denote the congruence class of $k$ modulo $n$. Then $\Pi \cap [k]_n = \{2^{m_0+\varphi(n) \cdot m} : m < \omega\}$, where $\varphi(n)$ is the Euler’s phi function and $m_0$ is the smallest natural number for which $2^{m_0} \equiv k \mod n$.

Proof. We fix $k, n, m_0$ satisfying the hypothesis of the lemma and we define $\lambda_m$ recursively as follows:

$$\lambda_0 := \frac{2^{m_0} - k}{n},$$

$$\lambda_{m+1} := \lambda_m \cdot 2^{\varphi(n)} + k \cdot \frac{2^{\varphi(n)-1}}{n},$$

for $0 \leq m < \omega$. \qed
Note that, by Euler’s theorem, all the $\lambda_m$’s are integers. Furthermore, one can easily see, by induction on $m$, that $\lambda_m \cdot n + k$ is a power of 2 of the form $2^{m_0 + c(n) \cdot m}$ and therefore $\{\lambda_m \cdot n + k \mid m < \omega\} \subseteq \Pi \cap [k]_n$.

In fact, the other inclusion also holds. To see this, let $2^l$ be an arbitrary power of 2. Then we can find some $m$ such that $l = m_0 + \varphi(n) \cdot m + s$ with $s < \varphi(n)$. As $\varphi(n)$ is the order of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$, we have that $2^s \in [1]_n$ only when $s = 0$, and therefore $2^l \in [k]_n$ if and only if $s = 0$. This concludes the proof.

Remark 3.11: Note that $\Pi \cap [k]_n$ when $n$ is some power of 2, is either empty or cofinite.

If $\bar{a} := (2^{k_1}, \ldots, 2^{k_n})$ is an element of $\Pi^n$, then we denote by $\langle \bar{a} \rangle^+$ the following set of positive powers of $\bar{a}$, $\{2^{m-k_1}, \ldots, 2^{m-k_n} \mid m < \omega\}$.

Lemma 3.12: Let $k_1 x_1 + \ldots + k_n x_n = 0$ be an equation over the integers and $S \subseteq \mathbb{Z}^n$ be its solution set. Then $\Pi \cap \Pi^n$ is either empty or the union of finitely many sets of positive powers of elements in $\Pi^n$.

Proof. It is immediate that if a tuple $(a_1, \ldots, a_n)$ of $\Pi^n$ is a solution of $k_1 x_1 + \ldots + k_n x_n = 0$, then each element of $\langle (a_1, \ldots, a_n) \rangle^+$ is a solution as well. Consider the set of all such (maximal for inclusion) sets of positive powers of solutions of the above equation. Suppose, for a contradiction, that this set is infinite. Then by an easy pigeonhole principle argument there exists a co-ordinate $i \leq n$ such that the $i$-th element of the “generating” tuple is arbitrarily larger than the rest. Thus, we can choose $a_i$ to be larger than $|k_1| \cdot |k_2| \cdot \ldots |k_{i-1}| \cdot |k_{i+1}| \cdot \ldots |k_n|$ times larger from each $a_j$, $j \leq n$, $j \neq i$. Now it is easy to see that $k_1 a_1 + \ldots + k_n a_n \neq 0$, a contradiction.

Lemma 3.13: Let $\mathcal{N} := (\mathbb{N}, s, \{Q_{k,n}\}_{n<\omega, k<n})$ be a first order structure where the function symbol $s$ is interpreted as the successor function and the predicate $Q_{k,n}$ is interpreted as the set of natural numbers which are residual $k$ modulo $n$. Then $\Pi^{\text{ind}}$ is definably interpreted in $\mathcal{N}$.

Proof. Throughout the proof the symbol $s^m$ will be used to denote $s \circ s \circ \ldots \circ s$ $m$-times. We also allow $m$ to be negative, in which case $s^m$ denotes the composition of the predecessor function $m$-times (which is clearly definable).

We first interpret $\Pi$ to be the domain of $\mathcal{N}$ which is clearly definable. Now let $P$ be a predicate of $\Pi^{\text{ind}}$. By the construction of $\Pi^{\text{ind}}$ we have that $P$ is a subset of the form $\phi(\mathbb{Z}^n) \cap \Pi^n$ for some quantifier free formula $\phi$ in $(\mathbb{Z}, +, 0, \{\equiv_n\}_{n<\omega})$. Since a quantifier free formula is a boolean combination of formulas of the form $t(x) = 0$ and $s(x) \equiv_1 0$, we only need to interpret in $\mathcal{N}$ solution sets of equations and congruence relations of the above simple form intersected with $\Pi^n$.

Suppose $\phi(x)$ is the equation $t(x) = 0$. Then, by Lemma 3.12, $\phi(\mathbb{Z}^n) \cap \Pi^n$ can be interpreted as a finite union of sets of the form $x_1 = s^{m_1}(x_2) \wedge x_1 = s^{m_2}(x_3) \wedge \ldots \wedge x_1 = s^{m_n}(x_n) \wedge x_1 \neq 1$ and $x_1 \neq 2$ and $\ldots \wedge x_1 \neq k$.

Suppose $\phi(x)$ is the congruence relation $s(x) \equiv_1 0$. If $(r_1, \ldots, r_n)$ is a tuple of integers that satisfy the congruence relation, then any tuple $(g_1, \ldots, g_n)$ for $q_i \in [r_i]$ satisfies this relation. Note that we can only have finitely many solutions up to $l$-congruence. Moreover, we may assume, by the Chinese remainder theorem, that $l$ is a power of a prime number. Thus, by Lemma 3.10 and Remark 3.11, $\phi(\mathbb{Z}^n) \cap \Pi^n$ can be interpreted as a finite union of sets of the
form $Q_{k_1,m_1}(x_1) \land \ldots \land Q_{k_n,m_n}(x_n) \land \text{"}x_1\text{ is not equal to finitely many elements" } \land \ldots \land \text{"}x_n\text{ is not equal to finitely many elements"}$. And this finishes the proof.

Lemma 3.14: The theory of $\mathcal{N} := \langle \mathbb{N}, s, \{Q_{k,n}\}_{n<\omega, k<n} \rangle$ admits quantifier elimination after adding a constant and a unary function symbol. Moreover it is $\omega$-stable and has Lascar rank one.

Proof. We add a constant to name 1 and a function symbol $s^{-1}$ to name the predecessor function; observe that both are definable in $\mathcal{N}$.

We prove elimination of quantifiers by induction on the complexity of the formula $\phi$. It is enough to consider the case where $\phi(\bar{x}, \bar{y})$ is a consistent formula of the form $\exists y \psi(\bar{x}, y)$, where $|\bar{y}| = 1$ and $\psi(\bar{x}, y)$ is a quantifier free formula. We can clearly assume that $\psi$ is in normal disjunctive form. Thus, since the negation of $Q_{k,n}$ is equivalent to the conjunction $\bigwedge_{i \neq k} Q_{i,n}$, it is enough to consider the case where $\psi(\bar{x}, y)$ is a finite conjunction of formulas of the following form:

$$Q_{k,n}(x_i) \land Q_{l,m}(y) \land x_i = c \land y = d \land x_i \neq a \land y \neq b$$

$$\land s^g(x_i) = x_j \land s^l(x_l) = y \land s^f(x_i) \neq x_j \land s^f(x_l) \neq y$$

Furthermore, we split $\psi$ to a conjunction $\psi_0(\bar{x}, \bar{y}) \land \psi_1(\bar{x})$, where $\psi_1$ is the conjunction of the atomic formulas of $\psi$ that do not contain $y$. Clearly we may assume that $\psi_0(\bar{x}, \bar{y})$ do not contain instances of the form $y = d$ or $s^g(x_i) = y$. We claim that $\exists y \psi_0(\bar{x}, y)$ is equivalent to $\bar{x} = \bar{d}$. Indeed, the projection of any formula of the form $s^g(x_i) \neq y \land \ldots \land s^g(x_k) \neq y \land y \neq d_1 \land \ldots \land y \neq d_k \land Q_{k,n}(y)$ is equivalent to $x = x$, thus the claim follows and $\psi(\bar{x}, y)$ is equivalent to $\psi_1(x)$. So, we obtain the first part of our statement.

Quantifier elimination allows us to prove by an easy counting types argument that the theory is $\omega$-stable. Fix a set of parameters $B$. We start by counting the number of complete types without parameters in one variable. Notice that any type without parameters is determined by positive formulas since, as noted before, the formula $\neg Q_{k,n}(x)$ is equivalent to a disjunction of formulas $Q_{l,n}(x)$ for $l \neq k$. In addition, as for any $n \in \mathbb{N}$ the formula $Q_{k,n}(x) \land Q_{l,n}(x)$ is inconsistent for distinct $k, l < n$, every complete type contains only one predicate of the form $Q_{k,n}(x)$ for a given $n$. Consider the family of maps $\mathcal{F} := \{f : \mathbb{N} \to \mathbb{N} \mid f(n) < n\}$ and for each $f \in \mathcal{F}$ let $\pi_f := \{Q_{f(n),n}(x) \mid n \in \mathbb{N} \} \cup \{x \neq n \mid n \in \mathbb{N}\}$. Clearly, every non algebraic complete type over the emptyset is determined by some $\pi_f$. Since $\mathcal{F}$ is countable we have that the set of complete types over the empty set is at most countable. Note in passing that not for all $f$, $\pi_f$ is consistent.

Now fix a parameter set $B$. Clearly any non-algebraic type over $B$ extends the set $\pi(x)$ given by $\{s^n(x) \neq a : a \in B, n \in \mathbb{Z}\}$. Whence, by the elimination of quantifier, we obtain that any complete non-algebraic type over $B$ (in one variable) is equivalent to $\pi(x) \cup \pi_f(x)$ for some $f \in \mathcal{F}$. Hence, $|S(B)| \leq |B| + \omega$, as desired.

Finally, again by quantifier elimination it is easy to see that the only formulas that divide are the algebraic ones. This shows that the theory has Lascar rank one; the details are left to the reader.

Now, the proof of Proposition 3.9 follows from Lemma 3.13 and 3.14. We can prove our second main theorem.

Proof of Theorem 3.3 It follows from Proposition 3.9 together with Theorem 3.3 that the expanded structure $(\mathbb{Z}, +, 0, \Pi)$ is superstable. As it is a proper expansion of $(\mathbb{Z}, +, 0)$, it has
infinite Lascar rank by Theorem 1. Whence, it remains to see that it has Lascar rank $\omega$. For this, it is enough to show that any forking extension of the principal generic has finite Lascar rank.

We shall work in an enough saturated extension of $(\mathbb{Z}, +, 0, \Pi)$, where $\Pi$ is interpreted as $\Pi'$. Let $p \in S(\emptyset)$ be the generic of the connected component, and let $q = tp(b/B)$ be an extension of $p$. Consider a realization $a$ of $p|B$, and note using Lemma 3.14 that $\Pi'$ has Lascar rank one. Now, working in the theory of $(\mathbb{Z}, +, 0)$, we obtain that $tp(b/\Pi', B)$ is the principal generic whenever $b \not\in acl(\Pi', B)$. Moreover, if a finite tuple is algebraic over $\Pi' \cup B$ and this is exemplified by some finite tuple $(c_1, \ldots, c_n)$ in $\Pi'$, then we have in $Th(\mathbb{Z}, +, 0, \Pi)$ that $U(d/B) \leq U(c/B) < \omega$ as the set $\Pi' \times \ldots \times \Pi'$ has Lascar rank $n$. Whence $a \not\in acl(\Pi', B)$ in the sense of $(\mathbb{Z}, +, 0)$ and hence its type over $\Pi' \cup B$ is the principal generic. Thus, by Corollary 3.8 we deduce that $p|B = tp(b/B)$ whenever $b$ is not algebraic in the sense of $(\mathbb{Z}, +, 0)$ over $\Pi' \cup B$. Therefore, in case that $tp(b/B)$ is a forking extension of $p$ we conclude that $b \in acl(\Pi', B)$ and so $tp(b/B)$ has finite Lascar rank, as desired.

One can see directly that the structure $(\mathbb{Z}, +, 0, \Pi)$ has infinite Lascar rank, without using Theorem 1, showing that the set $\Pi^+ \ldots + \Pi$ has Lascar rank $n$. This is left to the reader.

Finally, there is nothing special about adding a predicate for powers of 2. Our proofs work equally well if we add the powers of any prime number. Thus we get:

**Theorem 3.15:** Let $p$ be a prime and $\Delta$ be the set consisting of powers of $p$. Then $(\mathbb{Z}, +, 0, \Delta)$ is superstable of Lascar rank $\omega$.

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