Exact Solutions of Integrable 2D Contour Dynamics *

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Abstract

A class of exact solutions of the dispersionless Toda hierarchy constrained by a string equation is obtained. These solutions represent deformations of analytic curves with a finite number of nonzero harmonic moments. The corresponding $\tau$-functions are determined and the emergence of cusps is studied.

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1 Introduction

Integrable contour dynamics governed by the dispersionless Toda (dToda) hierarchy is a multifaceted subject. It underlies problems of complex analysis [1],[2], interface dynamics (Laplacian growth) [3], Quantum Hall effect [4], and associativity (WDVV) equations [5]. A common ingredient in many of its applications is the presence of random models of normal $N \times N$ matrices [1]-[4], [6],[7] with partition functions of the form

$$Z_N = \int d M \, d M^\dagger \exp \left( -\frac{1}{\hbar} \text{tr} \, W(M, M^\dagger) \right),$$

(1)

where

$$W(z, \bar{z}) = z\bar{z} + v_0 - \sum_{k \geq 1} (t_k z^k + \bar{t}_k \bar{z}^k).$$

(2)

In an appropriate large $N$ limit ($\hbar \to 0$, $s := \hbar N$ fixed), the eigenvalues of the matrices are distributed within a planar domain (support of eigenvalues) with sharp edges, which depends on the parameters $t := (s, t_1, t_2, \ldots)$.

If the support of eigenvalues is a simply-connected bounded domain with boundary given by an analytic curve $\gamma (z = z(p), |p| = 1)$, then $(s, t_1, t_2, \ldots)$ are harmonic moments of $\gamma$ and the curve evolves with $(t, \bar{t})$ according to the dToda hierarchy. Moreover, the corresponding $\tau$-function represents the quasiclassical limit of the partition function (1). A particularly interesting feature is that for almost all initial conditions the evolution of $\gamma$ leads to critical configurations in which cusp-like singularities develop. This behaviour is well-known in Laplacian growth [8] and random matrix theory [9].

In order to obtain solutions of the dToda hierarchy describing contour dynamics one must impose a string equation which leads to a particular type of Riemann-Hilbert problem [10]-[12]. In this paper we present a method for finding solutions in the form of Laurent polynomials

$$z = r p + u_0 + \cdots + \frac{u_{K-1}}{p^{K-1}},$$

(3)

which describe dynamics of curves with a finite number of nonzero harmonic moments, namely $t_k = \bar{t}_k = 0$ for $k \geq K$. We exhibit examples for arbitrary $K$ and derive their corresponding $\tau$-functions. Furthermore the emergence of cusps is analytically studied.
2 dToda contour dynamics

Let $z = z(p)$ be an invertible conformal map of the exterior of the unit circle to the exterior of a simply connected domain bounded by a simple analytic curve $\gamma$ of the form

$$\bar{z} = S(z),$$

where bar stands for complex conjugation ($z(\bar{p}) = z(p^{-1})$ on $\gamma$) and the Schwarz function $S(z)$ is analytic in some domain containing $\gamma$.

The map $z(p)$ can be represented by a Laurent series

$$z(p) = rp + \sum_{k=0}^{\infty} u_k p^k,$$

with a real coefficient $r$. The coefficients $(r, u_0, u_1, \ldots)$ are functions of the harmonic moments $t = (s = \bar{s}, t_1, t_2, \ldots)$ of the exterior of $\gamma$, which in turn can be introduced through the expansion of the Schwarz function

$$S(z) = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{s}{z} + \sum_{k=1}^{\infty} \frac{v_k}{z^{k+1}},$$

with $(v_1, v_2, \ldots)$ being functions dependent on $t$. As a consequence of (1)-(3), it follows that $z(p, t, \bar{t})$ solves the dToda hierarchy

$$\partial_t z = \{H_k, z\}, \quad \partial_{\bar{t}} z = -\{\bar{H}_k, z\},$$

where $\{f, g\} := p (\partial_p f \partial_s g - \partial_p g \partial_s f)$, the function $\bar{z}(p^{-1})$ is defined by the Laurent series

$$\bar{z}(p^{-1}) := \frac{r}{p} + \sum_{k=0}^{\infty} \bar{u}_k p^k,$$

and the symbols $(\ldots)_{\geq 1}$ $(\ldots)_{\leq -1}$ and $(\ldots)_0$ mean truncated Laurent series with only positive (negative) terms and the constant term, respectively. Furthermore, this solution satisfies the string equation

$$\{z(p), \bar{z}(p^{-1})\} = 1.$$
These properties can be proved through the twistor scheme of Takasaki-Takebe [3]. It uses Orlov-Schulman functions of the dToda hierarchy

\[ m = \sum_{k=1}^{\infty} k t_k z^k + s + \sum_{k=1}^{\infty} v_k, \]

\[ \bar{m} = \sum_{k=1}^{\infty} k \bar{t}_k z^k + s + \sum_{k=1}^{\infty} \bar{v}_k, \]

and can be summarized as follows:

**Theorem** If \((z, m, \bar{z}, \bar{m})\) are functions of \((p, t, \bar{t})\) which admit expansions of the form \((5), (8), (10)\) and satisfy the equations

\[ \bar{z} = m, \quad \bar{m} = m, \]

then \((z, \bar{z})\) is a solution of the dToda hierarchy constrained by the string equation \((9)\).

### 3 Solutions

Equations \((11)\) are meaningful only when they are interpreted as a suitable Riemann-Hilbert problem on the complex plane of the variable \(p\). Thus \((z, m)\) must be analytic functions in a neighborhood \(D = \{|p| > r\}\) of \(p = \infty\) and 
\((\bar{z}, \bar{m})\) must be analytic functions in a neighborhood \(D' = \{|p| < r'\}\) of \(p = 0\).

The statement of the Theorem holds provided \(A := D \cap D' \neq \emptyset\).

We next prove that equations \((11)\) have solutions satisfying \((5), (8)\) and \((10)\) with

\[ t_k = 0, \quad k > K, \quad t_K \neq 0. \]

In this way we assume

\[ m = \sum_{k=1}^{K} nt_k z^k + s + \sum_{k=1}^{\infty} v_k, \]

\[ \bar{m} = \sum_{k=1}^{K} k \bar{t}_k z^k + s + \sum_{k=1}^{\infty} \bar{v}_k, \]
Given two integers \( r_1 \leq r_2 \) we denote by \( V[r_1, r_2] \) the set of Laurent polynomials of the form

\[
c_{r_1} p^{r_1} + c_{r_1+1} p^{r_1+1} + \cdots + c_{r_2} p^{r_2}.\]

Let us look for solutions of (11) such that \( z \) and \( \bar{z} \) are meromorphic functions of \( p \) with possible poles at \( p = 0 \) and \( p = \infty \) only. Then, as a consequence of the assumptions (5), (8) and (12), from (11) it follows that

\[
z \in V[1 - K, 1], \quad \bar{z} \in V[-1, K - 1]. \tag{13}\]

The equation \( \bar{m} = m \) is equivalent to the system:

\[
\begin{align*}
\bar{m}_{\geq 1} &= m_{\geq 1}, \tag{14} \\
\bar{m}_0 &= m_0, \tag{15} \\
\bar{m}_{\leq -1} &= m_{\leq -1}. \tag{16}
\end{align*}
\]

If we now set

\[
m = \bar{m} = \sum_{k=1}^{K} n t_k (z^k)_{\geq 1} + \bar{m}_0 + \sum_{k=1}^{K} k \bar{t}_k (\bar{z}^k)_{\leq -1}, \tag{17}\]

with

\[
\bar{m}_0 = s + \sum_{k=1}^{K} k \bar{t}_k (\bar{z}^k)_0.
\]

it can be easily seen that \( \bar{m} \) has the required expansion of the form (12) provided \( z \) and \( \bar{z} \) satisfy (5) and (8). On the other hand, the expression (17) for \( m \) has an expansion of the form (12) if the residue of \( \frac{m}{z} \) corresponding to its Laurent expansion in powers of \( z \) verifies

\[
Res(\frac{m}{z}, z) = s. \tag{18}\]

Hence the problem reduces to finding \( z \) and \( \bar{z} \) satisfying (5), (8), (18) and

\[
z = \frac{m}{\bar{z}}. \tag{19}\]
In view of (13) we look for \( z \) and \( \bar{z} \) of the form

\[ z = rp + u_0 + \cdots + \frac{u_{K-1}}{p^{K-1}}, \tag{20} \]

\[ \bar{z} = \frac{r}{p} + \bar{u}_0 + \cdots + \bar{u}_{K-1}p^{K-1}. \]

Now, in order to prevent \( z \) from having poles different from \( p = 0 \) and \( p = \infty \) we have to impose

\[ m(p_i) = 0, \tag{21} \]

where \( p_i \) denote the \( K \) zeros of

\[ r + \bar{u}_0p + \cdots + \bar{u}_{K-1}p^K = 0. \]

In this way by using the expression (17) of \( m \), the only variables appearing in (19) are

\[ (p, t, \bar{t}, r, u_0, \ldots, u_{K-1}, w_0, \ldots, w_{K-1}), \quad w_i := \bar{u}_i. \]

Thus, by identifying coefficients of the powers \( p^i \), \( i = 1-K, \ldots, 1 \) we get \( K+1 \) equations which together with the \( K \) equations (21) determine the \( 2K+1 \) unknowns variables \( (r, u_0, \ldots, u_{K-1}, w_0, \ldots, w_{K-1}) \) as functions of \( (t, \bar{t}) \). Moreover, provided \( r \) is a real coefficient, the equations (11) are invariant under the transformation

\[ Tf(p) = f\left(\frac{1}{p}\right). \]

Hence if \( (r, u_0, \ldots, u_{K-1}, w_0, \ldots, w_{K-1}) \) solves (19) so does

\[ (r, \bar{u}_0, \ldots, \bar{u}_{K-1}, \bar{u}_0, \ldots, \bar{u}_{K-1}). \]

Therefore, if both solutions are close enough, they coincide and consequently \( w_i = \bar{u}_i \), as required.

To complete our proof we must show that (18) is satisfied too. To do that let us take two circles \( \gamma \ (|p| = r) \) and \( \gamma' \ (|p| = r') \) in the complex \( p \)-plane and denote by \( \Gamma \) and \( \Gamma' \) their images under the maps \( z = z(p) \) and \( \bar{z} = \bar{z}(1/p) \), respectively. Notice that due to (5) and (8), the curves \( \Gamma \) and \( \Gamma' \) have positive
orientation if $\gamma$ and $\gamma'$ have positive and negative orientation, respectively. Then we have

$$\text{Res}(\frac{m}{z}, z) - \text{Res}(\bar{m}/\bar{z}, \bar{z}) = \frac{1}{2i\pi} \oint_{\Gamma} \frac{m}{z} \, dz - \frac{1}{2i\pi} \oint_{\Gamma'} \frac{\bar{m}}{\bar{z}} \, d\bar{z}$$

$$= \frac{1}{2i\pi} \oint_{\gamma} \bar{z} \partial_p z \, dp - \frac{1}{2i\pi} \oint_{\gamma'} z \partial_p \bar{z} \, dp = \frac{1}{2i\pi} \oint_{\gamma} \partial_p(\bar{z}z) \, dp = 0,$$

where we have taken into account that the integrands are analytic functions of $p$ in $\mathbb{C} - \{0\}$ and that $\gamma$ and the opposite curve of $\gamma'$ are homotopic with respect to $\mathbb{C} - \{0\}$. Therefore, as we have already proved that $\bar{m}$ has an expansion of the form (12), we deduce

$$\text{Res}(\frac{m}{z}, z) = \text{Res}(\bar{m}/\bar{z}, \bar{z}) = s,$$

so that (13) follows.

Let us illustrate the method with the case $K = 2$. The polynomial $p \bar{z}$ has two zeros at the points

$$p_1 = \frac{-\bar{u}_0 + \sqrt{\bar{u}_0^2 - 4 \bar{r} \bar{u}_1}}{2 \bar{u}_1}, \quad p_2 = \frac{-\bar{u}_0 - \sqrt{\bar{u}_0^2 - 4 \bar{r} \bar{u}_1}}{2 \bar{u}_1},$$

and from (21) we get two equations which lead to

$$-2 r^2 t_2 \bar{u}_0^3 + 4 r^3 t_2 \bar{u}_0 \bar{u}_1 + r t_1 \bar{u}_0^2 \bar{u}_1 + 4 r t_2 u_0 \bar{u}_0^2 \bar{u}_1 - r^2 t_1 \bar{u}_1^2 - 4 r^2 t_2 u_0 \bar{u}_1^2$$

$$- s \bar{u}_0 \bar{u}_1^2 - \bar{t}_1 \bar{u}_0 \bar{u}_1^2 - 2 \bar{t}_2 \bar{u}_0^3 \bar{u}_1 + r \bar{t}_1 \bar{u}_1^3 = 0,$$

(22)

$$-2 r^3 t_2 \bar{u}_0^2 + 2 r^4 t_2 \bar{u}_1 + r^2 t_1 \bar{u}_0 \bar{u}_1 + 4 r^2 t_2 u_0 \bar{u}_0 \bar{u}_1 - r s \bar{u}_1^2 - r \bar{t}_1 \bar{u}_0 \bar{u}_1^2$$

$$- 2 r \bar{t}_2 \bar{u}_0^2 \bar{u}_1^2 - 2 r^2 \bar{t}_2 \bar{u}_1^3 = 0.$$

(23)
Identification of the powers of $p$ in (19) implies

\[ p : \quad -2 r^2 t_2 + r \bar{u}_1 = 0, \]

\[ p^0 : \quad 2 r^2 t_2 \bar{u}_0 - r t_1 \bar{u}_1 - 4 r t_2 u_0 \bar{u}_1 + u_0 \bar{u}_1^2 = 0, \]

\[ p^{-1} : \quad -2 r^2 t_2 \bar{u}_0^2 + 2 r^3 t_2 \bar{u}_1 + r t_1 \bar{u}_0 \bar{u}_1 + 4 r t_2 u_0 \bar{u}_0 \bar{u}_1 - s \bar{u}_1^2 - \bar{t}_1 \bar{u}_0 \bar{u}_1^2
\]

\[ -2 \bar{t}_2 \bar{u}_0 \bar{u}_1^2 - 4 r \bar{t}_2 \bar{u}_1^3 + u_1 \bar{u}_1^3 = 0. \]

Then by solving equations (22)-(24) we get the solution:

\[ z = \frac{p \sqrt{s}}{\sqrt{1 - 4 t_2 t_2}} + \frac{2 \sqrt{s} \bar{t}_2}{p \sqrt{1 - 4 t_2 t_2}} - \frac{\bar{t}_1 + 2 t_1 \bar{t}_2}{-1 + 4 t_2 t_2}, \]

(25)

which corresponds to the conformal map describing an ellipse growing from a circle \[\text{[6]}\]

**Solutions for $K \geq 3$**

Exact solutions associated to arbitrary values of $K$ can be found from the previous scheme. However in order to avoid complicated expressions, we set

\[ t_1 = t_2 = \cdots = t_{K-1} = \bar{t}_1 = \bar{t}_2 = \cdots = \bar{t}_{K-1} = 0. \]

and look for particular solutions satisfying

\[ u_1 = u_2 = \cdots = u_{K-2} = \bar{u}_1 = \bar{u}_2 = \cdots = \bar{u}_{K-2} = 0, \]

or equivalently

\[ z = r p + \frac{u_{K-1}}{p^{K-1}}, \quad \bar{z} = \frac{r}{p} + \bar{u}_{K-1} p^{K-1}. \]

(26)

Under the previous assumptions and from (17) we have that

\[ m = K t_K r^K p^K + s + K^2 \bar{t}_K r^{K-1} \bar{u}_{K-1} + \frac{K \bar{t}_K r^K}{p^K}. \]

(27)

Thus, we see that (21) leads us to a unique equation since from (27) it follows that $m$ depends on $p$ through $p^K$. Furthermore, if $p_i$ satisfies $\bar{z}(p_i) = 0$, then

\[ p_i^K = -\frac{r}{\bar{u}_{K-1}}. \]
Therefore, (21) becomes
\[
s - \frac{Kr^{K+1}t_K}{\bar{u}_{K-1}} + (K - 1)Kr^{K-1}\bar{t}_K \bar{u}_{K-1} = 0. \tag{28}
\]
On the other hand, it is easy to see that
\[
\frac{m}{z} = \frac{Kr^Kt_K}{\bar{u}_{K-1}} p + \left( K^2 r^{K-1}\bar{t}_K + \frac{s \bar{u}_{K-1} - Kr^{K+1}\bar{t}_K}{\bar{u}_{K-1}^2} \right) \frac{1}{p^{K-1}},
\]
consequently, by equating coefficients and taking (26) into account, we find that (19) leads to two equations only. More precisely
\[
p : \quad r = \frac{Kr^Kt_K}{\bar{u}_{K-1}},
\]
\[
p^{-(K-1)} u_{K-1} = K^2 r^{K-1}\bar{t}_K + \frac{s \bar{u}_{K-1} - Kr^{K+1}\bar{t}_K}{\bar{u}_{K-1}^2}.
\tag{29}
\]
Then, we get three equations for the three unknowns \(r, u_{K-1}, \bar{u}_{K-1}\), which proves that there exits a solution of the form (26). In fact, by solving (28)-(29) we find that
\[
z = rp + \frac{K \bar{t}_K r^{K-1}}{p^{K-1}},
\tag{30}
\]
with \(r\) satisfying the implicit equation
\[
K^2(K - 1)t_K\bar{t}_K r^{2(K-1)} - r^2 + s = 0. \tag{31}
\]
Figures 1 and 2 show examples of the evolution of the curve \(z(p), |p| = 1\) as \(s\) grows and \(t_K\) is kept fixed.

\(\tau\)-functions

In [1] it was proved that there is a dToda \(\tau\)-function associated to each analytic curve \(z = z(p), \ |p| = 1\), given by
\[
2 \log \tau = -\frac{1}{2} s^2 + s v_0 - \frac{1}{2} \sum_{k \geq 1} (t_k v_k + \bar{t}_k \bar{v}_k),
\tag{32}
\]
where $v_k$ are the coefficients of the expansion (29), and $v_0$ is determined by

$$\frac{\partial v_0}{\partial s} = \log r^2, \quad v_0 = \frac{\partial \log \tau}{\partial s}. \quad (33)$$

For the class of solutions (30) we have

$$2 \log \tau = -\frac{1}{2} s^2 + s v_0 - \frac{1}{2} (t_K v_K + \bar{t}_K \bar{v}_K), \quad (34)$$
and from (11) and (30) it follows that
\[ v_K = \frac{1}{2i\pi} \oint_{\Gamma} \bar{z} z^K \, dz = \frac{1}{2i\pi} \oint_{\Gamma} \bar{z}(p)z(p)^K \left( r - (K - 1)\frac{u_{K-1}}{p^K} \right) \, dp \]
\[ = \frac{(r^2 - s)(Ks - (K - 2)r^2)}{2K(K - 1)s}. \]

On the other hand by differentiating (34) with respect to \( s \) and by taking into account (33) one finds
\[ v_0 = -s + s \log r^2 + \frac{(K - 2)(K - 1)K|t_K|^2(Ks - (K - 2)r^2)r^{2K}}{2((K - 1)^2K^2|t_K|^2r^{2K} - r^4)} \]
\[ + \frac{(K - 2)(s - r^2)}{2(K - 1)K} \left( K + \frac{(K - 2)r^4}{(K - 1)^2K^2|t_K|^2r^{2K} - r^4} \right). \]

Thus, (34)-(36) and (31) characterize the \( \tau \)-function of the curves determined by (30).

**Cusps**

The pictures of the curves associated with (30)-(31) show the presence of cusps at some value of \( s \) for each fixed value of \( t_K \). Indeed by using the parametric equation \( p = e^{i\theta}, \; (0 \leq \theta \leq 2\pi) \) for the unit circle, we have that cusps on the curve \( z = z(p) \) appear at points where \( z_\theta = 0, \; z_{\theta\theta} \neq 0 \) and \( z_{\theta\theta\theta}/z_{\theta\theta} \) has a nonzero imaginary part. Therefore a necessary condition for \( p = p(\theta) \) is
\[ \frac{\partial z}{\partial p}(p) = 0, \; |p| = 1. \]

Thus from (30) we deduce
\[ p^K = K(K - 1)\bar{t}_K r^{K-2}, \]
which together with the condition \( |p| = 1 \) requires that
\[ r = (K(K - 1)|t_K|)^{\frac{1}{K - 2}}, \]
\[ \text{for } K > 2. \]
at some value \( s = s(t_K) \). But according to (31) one finds that this happens at the value \( s_0 \) given by

\[
s_0 = \frac{K - 2}{K - 1} (K(K - 1)|t_K|)^{-\frac{2}{K}},
\]

which is the point at which the profile of both positive branches of \( r \), as functions of \( s \), develop an infinite slop (see figure 3).

![Figure 3: The positive branches of \( r(s) \) for \( K = 10 \)](image)

Therefore, there are \( K \) cusps given by the roots

\[
z_j = \frac{K}{K - 1} \left( \frac{r^2 - s_0}{K t_K} \right)^{\frac{1}{K}},
\]

which emerge when \( s \) reaches the extreme value \( s_0 \) of the domain of existence of the two positive branches of \( r \) as a function of \( s \).

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