Transverse velocities, intermittency and asymmetry in fully developed turbulence

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Using experimental transverse velocities data for very high Reynolds number turbulence, we suggest a model describing both formation of intermittency and asymmetry of turbulence. The model, called "bump-model" is a modification of ramp-model suggested earlier [1]. The connection between asymmetry and intermittency makes it possible to study the latter with relatively low moments.

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Self-similar properties of turbulence, suggested by Kolmogorov [2], have been intensively studied for a long time. The theory predicted simple scaling for the longitudinal velocity increments \( u_r = u(x+r) - u(x) \), namely, \( \langle |u_r|^p \rangle \sim r^{p/3} \). It became clear, however, that there are corrections to these scalings, usually attributed to intermittency. The only exception is the so-called 4/5th-Kolmogorov law [3] which is exact in inertial range. According to the law, the third moment, that is the structure function, \( \langle u_3 \rangle = B_{uuu}(r) = -4/5 \varepsilon r \) in inertial range, and this scaling has no intermittency corrections [4]. Nevertheless, more detailed study of this third order structure function proved to be useful in understanding the intermittency. It is indeed important to understand what contribution into the third moment give the tails. It is natural to assume that the main events from the core of \( P(u_r) \), the PDF, mainly contribute. This is certainly the case for even order structure functions, or for moments like \( \langle |u_r|^p \rangle \). As to the odd order moments, we note that \( \langle u_2^2 \rangle \neq 0 \), while \( \langle u_r \rangle = 0 \). This means that the PDF is asymmetric. However, the core of the PDF may be more or less symmetric, in which case the contribution of the tails would be substantial. And indeed, this asymmetry, described by a ramp-model, also suggest intermittency (in addition to the asymmetry) [4], [5]. Further studies showed that indeed the tails of \( P(u_r) \), responsible for the intermittency, give a substantial contribution to \( B_{uuu} \). Another way to check this connection between asymmetry and intermittency is to compare directly the PDF for positive and negative parts of \( u_r \), and we can see that the asymmetry of the PDF stretches far into the tails [6]. The transverse velocities give additional information about both asymmetry and intermittency, and this paper is devoted to their study.

The transverse (vertical) component of the velocity increments \( v_r = v(x+r) - v(x) \) is also supposed to possess asymmetry, although \( \langle v_r \rangle = 0 \). Assuming isotropic turbulence, the only non-vanishing correlation is (see [7], [8])

\[
B_{vvv} = \langle u_r v_r^2 \rangle = \frac{1}{6} \frac{d(B_{uuu}(r))}{dr}. \tag{1}
\]

We used data acquired at Brookhaven National Lab for longitudinal and transverse components of the velocity (40 million samples of each, courtesy of Sreenivasan). The estimated Taylor Reynolds number is 10680. As seen from Fig. 1, experimental \( B_{vvv} \) is close to that obtained from [1], especially at small distances between the points, in agreement with earlier observations [3] (see their Fig. 2). Indeed at small scales the statistical properties are more isotropic, in accordance with Kolmogorov ideas about local isotropy.

![Fig. 1](image-url)

**FIG. 1**: Different third order structure functions: experimental \( \langle u_r v_r^2 \rangle \), and its cumulative moments. The distance is given in terms of Kolmogorov microscale \( \eta \).

Denote \( u_r' = u_r/\sigma_u \), \( v_r' = v_r/\sigma_v \), and \( \sigma_u = \langle u_2 \rangle^{1/2} \), \( \sigma_v = \langle v_2 \rangle^{1/2} \). We will consider cumulative moments,

\[
\langle u_r' v_r'^2 \rangle_{u_r' \geq t} = \left( \int_{-\infty}^t + \int_t^{\infty} \right) du_r' \int_{-\infty}^\infty dv_r' u_r' v_r'^2 P(u_r', v_r'),
\]

\[
\langle u_r' v_r'^2 \rangle_{v_r' \geq t} = \int_{-\infty}^\infty du_r' \left( \int_{-\infty}^t + \int_t^{\infty} \right) dv_r' u_r' v_r'^2 P(u_r', v_r'),
\]

where \( P(u_r', v_r') \) is the distribution function, and \( t \) is a number. If \( t \ll 1 \), then essentially the whole distribution
works, and the cumulative moments are expected almost to coincide with \( \langle u'_t v''_t \rangle = k(r) \), where \( k = B_{uuw}/(\sigma_u \sigma_v^3) \), - analog to skewness. For not small \( t \), we are dealing with the tails of the distribution, and it is important to know what contribution they give to the distribution. Figure 1 shows these moments for \( t = 3 \) or \( 4 \). It can be seen that the moments thus constructed do not deviate much from the experimental \( B_{uuw}(r) \).

In order to have some comparison with a “regular” behavior when the tails are absent, we constructed a series of PDF’s \( I_r(u') \) for different distances \( r \), as a sum of two Gaussian functions, and satisfying \( \langle u^0 \rangle_t = \int I_r du' = 1 \), \( \langle u_1' \rangle_t = \int I_r u'du' = 0 \), \( \langle u_2' \rangle_t = 1 \), and \( \langle u_3' \rangle_t = k(r) \). Then, obviously, \( \langle u_r' \rangle_t = \langle u' \rangle_t \sigma_u = 0 \), \( \langle v_r' \rangle_t = \langle u' \rangle_t \sigma_v = \langle u_r' \rangle_t B_{uu}, \langle u_2' \rangle_t = B_{uv}, \) and \( \langle u_r v''_r \rangle_t = B_{uw} \). We now reconstruct the third moment for cumulative average \( \langle u^3 \rangle_t \), \( t = 3 \) or \( 4 \). Corresponding moments are depicted in Fig. 1. We note that even for \( t = 3 \), the cumulative moment constructed from \( I_r \) is essentially lower than \( B_{uuw} \); only for large distances it mixes with the experimental cumulative moments. As to the case \( t = 4 \), it can be seen that the difference between the \( I_r \)-cumulative moments and experimental moments is dramatic. We conclude that the contribution of the tails for experimental data is substantial, which can be seen when comparing the experimental cumulative moments with real moments, - on one hand, and, on the other hand, comparing them with those constructed from \( I_r \) - that does not contain any tails by definition.

As a non-vanishing \( B_{uuw} \) is a result of asymmetry of the PDF for \( u_r \), the correlation \( B_{uuw} \), obeying \( \#1 \), is therefore related to the asymmetry. Indeed, in order that \( B_{uuw} < 0 \), there should be an anti-correlation between \( u_r \) and \( v''_r \), - decreasing \( u_r \) is accompanied by increasing \( v''_r \), and \( v''_r \) versa. Roughly speaking, the conditional average \( B_{uuw}(u_r < 0) > B_{uuw}(u_r > 0) \). If the asymmetry is indeed related to intermittency, this conditional inequality should be satisfied for \( u_r > t \) versus \( u_r < -t \), where \( t \) is not a small number. To check this we consider, first, distributions for smallest \( r \)’s corresponding to the distance between two neighbor samples. Second, we consider cumulative moments, \( \langle u_r v''_r \rangle_{u_r \leq -t} \) and \( \langle u_r v''_r \rangle_{u_r \geq t} \), for different \( t \). Figure 2(d) presents the experimental moments. It shows, first, quite substantial tails: even when \( t = 30 \) (!), or greater (in units of \( \sigma_u \)), the contribution to the cumulative moments is substantial. Second, we see a remarkable feature: the negative contributions exceed the positive not only at small \( t \), corresponding to the core of the distribution, but also far in tails. For comparison, we constructed analogous moments based on \( I_r \) (without tails). We see that these moments are decaying fast, already for \( t > 3 \), or so, as we would indeed expect from (pseudo)-Gaussian distributions.

In \( \#1 \), \( \#3 \) a model was suggested explaining how the asymmetry appears. Figure 2(a) shows a ramp-structure. Obviously, \( \langle \partial_x u(x) \rangle = 0 \), while \( \langle \partial_x u(x)^3 \rangle < 0 \). In addition, the negative part of \( \partial_x u(x) \) is certainly intermittent, see Fig. 2(b), and that is how the idea of intermittency being connected to the asymmetry came into life.

**FIG. 2:** (a) Ramp-structure. (b) Derivative of the ramp. (c) Bump-structure. (d) Cumulative moments for different cut-off numbers \( t \). The inset corresponds to same cumulative moments, calculated for the model \( \#4 \).

This model is only heuristic, however. It was shown \( \#10 \) that Burgers vortex, embedded into a converging motion, acquires negative skewness, this picture containing both asymmetry and intermittency. The ramp-model does not exactly correspond to it. More realistic modification of this model is a bump-model \( \#3 \), see Fig. 2(c), where \( u(x) \) is a sum of the solid and dashed lines. Here again, \( \langle \partial_x u(x) \rangle = 0 \) while \( \langle \partial_x u(x)^3 \rangle < 0 \). This model simulates a converging motion (solid line in the vicinity of the dashed peak), naturally generating a vortex (dashed line). Supposedly, this structure in the longitudinal velocity appears in the vicinity of a Burgers vortex. Both ramp-model and bump-model are 1D, and therefore they, of course, do not reflect any structures appearing in the transverse field. Therefore, we need further modification of the model to make it 2D, or 3D type. Besides, a real vortex in a converging motion depicted by a solid line in Fig. 2(c) would be described by a shear of \( v_r \)-component of velocity (rather than generating a shear in \( v_x \)). If that is the case, then the \( u_r v''_r \) anti-correlation will appear. Indeed, when \( u_r < 0 \) (converging motion), the vortex is generated increasing \( v''_r \), that is, \( v''_r \), while for \( u_r > 0 \) (diverging motion), the vortex is not generated (and \( v''_r \) is smaller than averaged).

Consider, therefore, a 2D-model: \( v_x = f_1(x') \), corresponding to the solid line in Fig. 2(c), and \( v_y = f_2(x') \), depicted by a dashed line. Let \( \alpha = \alpha_1(x' + r_x) - f_1(x') \), and \( \omega = f_2(x' + r_x) - f_2(x') \). Then,

\[
\begin{align*}
u_r &= [\alpha \cos \phi + \omega \sin \phi] \cos \phi, \\
v_r &= [-\alpha \sin \phi + \omega \cos \phi] \cos \phi,
\end{align*}
\]
where \( r_x = r \cos \phi, \ r_y = r \sin \phi \). We thus have two averages: over \( x' \), and over \( \phi \). As a result, \( \langle u_r \rangle = \langle v_r \rangle = 0 \), while \( \langle u_r^3 \rangle = (\alpha^3) \phi_{60} + 3 \alpha(0) \langle \omega^2 \rangle \phi_{40}, \langle u_r v_r^2 \rangle = (\alpha^3) \langle \phi_{40} + \alpha(0) \omega^2 \rangle (\phi_{60} - 2 \phi_{40}) \), where \( \phi_{60} = \langle \cos^6 \phi \rangle = 5/16, \phi_{42} = \langle \cos^4 \phi \sin^2 \phi \rangle = 1/16 \) (and \( \langle v_r^3 \rangle = 0 \)). Here we considered small \( r \)'s, so that \( \omega \) is strongly peaked at \( x' = 0 \), and therefore, when combined with \( \omega \), the value of \( \alpha \) contributes only at \( x' = 0 \). As \( \alpha(0) < 0 \) (converging motion), and \( |\omega| \gg |\alpha| \), both \( \langle u_r^3 \rangle \) and \( \langle u_r v_r^2 \rangle \) are negative.

So far, the model is consistent with the experimental data. The model contains several free parameters. It is interesting to note that by choosing them just in a “reasonable” way, we immediately reproduce the real experimental values for \( \langle u_r^3 \rangle \), and \( \langle u_r v_r^2 \rangle \) with a decent accuracy. To do it even better, we used computer routines to optimize these parameters so that they fit the experimental values in the best way. We now are ready to calculate the cumulative moments \( \langle u_r v_r^2 \rangle_{u_r \leq -t} \), and \( \langle u_r v_r^2 \rangle_{u_r \geq t} \), for different \( t \), corresponding to this model. They are shown in the inset of Fig. 2(d). Qualitatively, we see the same features as in experimental moments. Namely, there are substantial tails, obviously related to the presence of the vortex, and the negative part always exceeds the positive one.

In conclusion we note that the cumulative moments are useful in studying the tails of the distributions: we thus consider the contribution of the tails, as if the core of the distribution does not contribute at all. We saw that the Kolmogorov law, and related third order \( u_r - v_x^2 \) correlation can be satisfactorily reproduced by the tails only. In contrast, the third order moments corresponding to some pseudo-Gaussian distributions are poorly reproduced by the contribution of the tails. As these third order moments do not vanish because of the asymmetry of the distributions, we assume that the intermittency (i.e., substantial contribution of the tails of the distributions) is related to the asymmetry. This conjecture can be checked directly, comparing positive and negative contributions. We see that predicted difference between the positive and negative parts is present not only at the core of the distribution, but also stretches far into the tails.

The intermittency related to the asymmetry comes out naturally from the ramp-model, and its 2D modification – the bump-model. It simply presents a vortex embedded into a converging motion. Some analytical representation of this model shows quantitatively the same behavior as the experimental data. We conclude that it is consistent with the above interpretation of intermittency related to the asymmetry. As the third moment is a relatively low moment, this conjecture suggests an useful tool in studying the intermittency of turbulence.

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