q-DIFFERENCE EQUATION FOR GENERALIZED TRIVARIATE q-HAHN POLYNOMIALS

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Abstract. In this paper, we introduce a family of trivariate q-Hahn polynomials \(\Psi_n^{(a)}(x,y,z;q)\) as a general form of Hahn polynomials \(\psi_n^{(a)}(x;q)\), \(\psi_n^{(a)}(x,y;q)\) and \(F_n(x,y,z;q)\). We represent \(\Psi_n^{(a)}(x,y,z;q)\) by the homogeneous q-difference operator \(\mathcal{L}(a,b,c)\) introduced by Srivastava et al [H. M. Srivastava, S. Arjika and A. Sherif Kelil, Some homogeneous q-difference operators and the associated generalized Hahn polynomials, Appl. Set-Valued Anal. Optim. 1 (2019), pp. 187–201.] to derive: extended generating, Rogers formula, extended Rogers formula and Srivastava-Agarwal type generating functions involving \(\Psi_n^{(a)}(x,y,z;q)\) by the q-difference equation.

1. Introduction

In this paper, we adopt the common conventions and notations on q-series. For the convenience of the reader, we provide a summary of the mathematical notations, basics properties and definitions to be used in the sequel. We refer to the general references (see [9]) for the definitions and notations. Throughout this paper, we assume that \(|q| < 1\).

For complex numbers \(a\), the q-shifted factorials are defined by:

\[
(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1} (1-aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty} (1-aq^k) \tag{1.1}
\]

and \((a_1, a_2, \ldots, a_r; q)_m = (a_1; q)_m(a_2; q)_m \cdots (a_r; q)_m, \ m \in \{0, 1, 2 \cdots\}.

The q-binomial coefficient is defined as [5]

\[
\begin{align*}
\binom{n}{k}_q &= \frac{(q;q)_n}{(q)_k(q;q)_{n-k}} = \frac{(q^{-1};q)^k}{(q)_k} \frac{(-1)^k q^{nk}}{(q)_k} , \quad \text{for } 0 \leq k \leq n.
\end{align*}
\]

The basic or q-hypergeometric function in the variable \(z\) (see Slater [12, Chap. 3], Srivastava and Karlsson [13, p. 347, Eq. (272)] for details) is defined as:

\[
\Phi_r \left[ \begin{array}{c} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_s \end{array} \right]_q \left( z \right) = \sum_{n=0}^{\infty} \left( -1 \right)^n q^{n \binom{s}{2}} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \frac{z^n}{(q; q)_n},
\]

when \(r > s + 1\). Note that, for \(r = s + 1\), we have:

\[
\Phi_{r+1} \left[ \begin{array}{c} a_1, a_2, \ldots, a_{r+1}; \\ b_1, b_2, \ldots, b_r \end{array} \right]_q \left( z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} \frac{z^n}{(q; q)_n}.
\]

We will be mainly concerned with the Cauchy polynomials as given below [4]

\[
p_n(x,y) := (x-y)(x-qy)\cdots(x-q^{n-1}y) = (y/x)^n (x^n) \tag{1.2}
\]

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with the Srivastava-Agarwal type generating function
\[
\sum_{n=0}^{\infty} p_n(x,y) \frac{(\lambda;q)_n t^n}{(q;q)_n} = 2\Phi_1 \left[ \begin{array}{c} \lambda, y/x; \\ q; xt \end{array} \middle| 0 \right].
\] (1.3)
For \( \lambda = 0 \), we get the generating function [4]
\[
\sum_{n=0}^{\infty} p_n(x,y) \frac{t^n}{(q;q)_n} = (yt;q)_\infty.
\] (1.4)

The generating function (1.4) is also the homogeneous version of the Cauchy identity or the \( q \)-binomial theorem given by [5]
\[
\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \Phi_0 \left[ \begin{array}{c} a; \\ q, q^{-1}z \end{array} \middle| (a; q)_\infty, |z| < 1. \right.
\] (1.5)
Putting \( a = 0 \), the relation (1.5) becomes Euler’s identity [5]
\[
\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z; q)_\infty}, \quad |z| < 1.
\] (1.6)
and its inverse relation [5]
\[
\sum_{k=0}^{\infty} (-1)^k q^{k^2} \frac{z^k}{(q;q)_k} = (z; q)_\infty.
\] (1.7)

Saad and Sukhi [11] defined the \( q \)-difference operator \( \theta_{xy} \)
\[
\theta_{xy} [f(x,y)] := \frac{f(q^{-1}x,y) - f(x, qy)}{q^{-1}x - y},
\] (1.8)
which turns out to be suitable for dealing with the Cauchy polynomials. Their corresponding \( q \)-exponential operator is
\[
\mathbb{E}(z\theta_{xy}) = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q;q)_k} (z\theta_{xy})^k.
\] (1.9)

Recently, Srivastava, Arjika and Kelil [14] have introduced the \( q \)-difference operator \( \tilde{L}(a,b; \theta_{xy}) \)
\[
\tilde{L}(a,b; \theta_{xy}) = \sum_{k=0}^{\infty} \frac{q^{k^2} (a;q)_k}{(q;q)_k} \left( b\theta_{xy} \right)^k,
\] (1.10)
to study \( q \)-polynomials and related generating functions.

In this paper, our goal is to generalize the results of Srivastava, Arjika and Kelil [14], and Mohameed [1]. We first construct the following generalized trivariate \( q \)-Hahn polynomials as
\[
\Psi_n^{(a)}(x,y,z|q) = (-1)^n q^{-\frac{1}{2}n(n+1)} \sum_{k=0}^{n} \frac{n^k}{k} nx^k p_{n-k}(y,x)z^k.
\] (1.11)

\textbf{Remark 1.} For \( a = 0 \), the generalized trivariate \( q \)-Hahn polynomials \( \Psi_n^{(a)}(x,y,z|q) \) are the well known trivariate \( q \)-polynomials \( F_n(x,y,z; q) \) investigated by Mohameed (see [1] for more details), i.e.,
\[
\Psi_n^{(0)}(x,y,z|q) = F_n(x,y,z; q).
\] (1.12)
If we let \( a = 0 \), \( y = ax \) and \( z = y \), the generalized trivariate \( q \)-Hahn polynomials \( \Psi_n^{(a)}(x,y,z|q) \) reduce to the second \( H \)ahn polynomials \( \psi_n^{(a)}(x,y|q) \) [3], i.e.,
\[
\Psi_n^{(0)}(x,ax,y|q) = \psi_n^{(a)}(x,y|q).
\] (1.13)
Also, \( a = 0, y = ax \) and \( z = 1 \), the generalized trivariate \( q \)-Hahn \( \Psi_n^{(a)}(x,y,z)|q) \) reduce to Hahn polynomials \( \psi_n^{(a)}(x|q) \) [2], i.e.,

\[
\psi_n^{(a)}(x,ax,1|q) = \psi_n^{(a)}(x|q).
\]  \hspace{1cm} (1.14)

The polynomials (1.11) can be represented by the homogeneous \( q \)-difference operator (1.10) as follows.

**Proposition 2.**

\[
\Psi_n^{(a)}(x,y,z)|q) = \overline{L}(a,z;\theta_{xy})\left\{(-1)^n q^{-\frac{n}{2}}p_n(y,x)\right\}.
\]  \hspace{1cm} (1.15)

**Proof.** By identity (1.10) and taking into account \( \theta_{xy}p_n(y,x) = -(1-q^n)p_{n-1}(y,x) \), we get the result. \( \Box \)

In light of \( \theta_{xy}[(x;\theta_{xy})/\Phi \right) \neq 0 \], we have the following identity

\[
\overline{L}(a,z;\theta_{xy})\left\{(x;\theta_{xy})\Phi \right) = (x;\theta_{xy})\Phi \left[\Phi_{a; q; z; t} \right].
\]  \hspace{1cm} (1.16)

The main object of this paper is to use the \( q \)-difference equation to derive some identities such as: extended generating function, Rogers formula, extended Rogers formula and Srivastava-Agarwal type generating functions.

The paper is organized as follows. In Section 2, we state two theorems and give the proofs. We derive an extended generating function for these \( q \)-polynomials. In Section 3, we state the Rogers formula and extended Rogers formula and give the proofs by the \( q \)-difference equation. In Section 4, we obtain Srivastava-Agarwal type generating functions involving the generalized trivariate \( q \)-Hahn polynomials by the method of \( q \)-difference equation.

2. Main results and proofs

In this section, we introduce another extension of \( q \)-Hahn polynomials. Then, we represent it by the homogeneous \( q \)-difference operator and derive an extended generating function.

**Theorem 3.** Let \( f(a,b,x,y,z) \) be an 5-variable analytic function at \( (a,b,x,y,z) = (0,0,0,0,0) \in \mathbb{C}^5 \). If \( f(a,b,x,y,z) \) satisfies the \( q \)-difference equation

\[
(q^{-1} x - y)[f(a,b,x,y,z) - f(a,b,x,y,qz)] = z[f(a,b,q^{-1} x,y,qz) - f(a,b,x,qy,qz)] + az[f(a,b,x,qy,qz^2) - f(a,b,q^{-1} x,y,qz^2)],
\]  \hspace{1cm} (2.1)

then we have:

\[
f(a,b,x,y,z) = \overline{L}(a,z;\theta_{xy})\{f(a,b,x,y,0)\}.
\]  \hspace{1cm} (2.2)

**Corollary 4.** Let \( f(b,x,y,z) \) be an 4-variable analytic function at \( (b,x,y,z) = (0,0,0,0) \in \mathbb{C}^4 \). If \( f(b,x,y,z) \) satisfies the \( q \)-difference equation

\[
(q^{-1} x - y)[f(b,x,y,z) - f(b,x,y,qz)] = z[f(b,q^{-1} x,y,qz) - f(b,x,qy,qz)],
\]  \hspace{1cm} (2.3)

then we have:

\[
f(b,x,y,z) = \overline{B}(z\theta_{xy})\{f(b,x,y,0)\}.
\]  \hspace{1cm} (2.4)

**Proof.** From the theory of several complex variables [10], we begin to solve the \( q \)-difference equation (2.3). First we may assume that

\[
f(a,b,x,y,z) = \sum_{n=0}^{\infty} A_n(a,b,x,y)z^n.
\]  \hspace{1cm} (2.5)
Substituting (2.5) into (2.3), we get:

\[(q^{-1}x - y)\sum_{n=0}^{\infty} (1 - q^n)A_n(a, b, x, y)z^n = \sum_{n=0}^{\infty} q^n(1 - aq^n)[A_n(a, b, q^{-1}x, y) - A_n(a, b, x, y)]z^{n+1}.
\]

Comparing coefficients of \(z^n, n \geq 1\), we find that

\[(q^{-1}x - y)(1 - q^n)A_n(a, b, x, y) = q^{n-1}(1 - aq^{n-1})[A_{n-1}(a, b, q^{-1}x, y) - A_{n-1}(a, b, x, y)].
\]

After simplification, we get:

\[A_n(a, b, x, y) = q^{n-1}\frac{1 - aq^{n-1}}{1 - q^n}\theta_{xy}\{A_{n-1}(a, b, x, y)\}.
\]

By iteration, we gain

\[A_n(a, b, x, y) = q^{(\frac{1}{2})}(a; q)_n q^n\theta_{xy}\{A_0(a, b, x, y)\}.
\]

(2.6)

Just taking \(z = 0\) in (2.5), we immediately obtain \(A_0(a, b, x, y) = f(a, b, x, y, 0)\). Substituting (2.6) back into (2.5), we achieve (2.3).

**Theorem 5** (Extended generating function for \(\Psi_n^{(a)}(x, y, z|q)\)). For \(|yt| < 1\), we have:

\[\sum_{n=0}^{\infty} \Psi_n^{(a)}(x, y, z|q)\frac{(-1)^{n+k}q^{n+k}t^n}{(q; q)_n} = t^{-k}(xt; q)_\infty \sum_{n=0}^{k} \frac{(q^{-k}, y; q)_n q^n}{(x; q)_\infty} \Phi_1\left(\frac{a; q; ztq^n}{0}; a; q; q\right).
\]

(2.7)

**Corollary 6.** For \(|yt| < 1\), we have:

\[\sum_{n=0}^{\infty} F_{n+k}(x, y, z|q)\frac{(-1)^{n+k}q^{n+k}t^n}{(q; q)_n} = t^{-k}(xt, zt; q)_\infty \Phi_2\left(\frac{q^{-k}, y; q; q}{0}; a; q; q; \Phi_1\left(\frac{a; q; q}{0}; a; q; q\right)\right).
\]

(2.8)

**Remark 7.** For \(a = 0\), (2.7) reduces (2.8). For \(a = 0\) and \(k = 0\), (2.7) and (2.8) reduce to the generating function for \(\Psi_n^{(a)}(x, y, z|q)\)

\[\sum_{n=0}^{\infty} \Psi_n^{(a)}(x, y, z|q)\frac{(-1)^{n+k}q^{n+k}t^n}{(q; q)_n} = \frac{(xt; q)_\infty}{(y; q)_\infty} \Phi_1\left(\frac{a; q; ztq^n}{0}; a; q; q\right), \quad |yt| < 1
\]

(2.9)

and [1, Theorem 2.6].

To prove the Theorem 5, the following Lemma is necessary.

**Lemma 8.** \(q\)-Chu-Vandermonde formula [5, Eq. (II.6)]

\[2\Phi_1\left(\frac{q^{-n}a; q}{c}; \frac{c/a; q_n}{c}; q\right) = (c/a; q)_n a^n.
\]

(2.10)

**Proof of Theorem 5.** Denoting the right-hand side of equation (2.7) by \(F(a, t, x, y, z)\), we have:

\[F(a, t, x, y, z) = t^{-k}\sum_{n=0}^{k} \frac{(q^{-k}; q)_n q^n (xtq^n; q)_\infty}{(q; q)_n (ytq^n; q)_\infty} \Phi_1\left(\frac{a; q; ztq^n}{0}; a; q; q\right).
\]

(2.11)

Because equation (2.11) satisfies (2.3), we have:

\[F(a, t, x, y, z) = \frac{\bar{L}(a, z; \theta_{xy})}{\frac{\bar{L}(a, z; \theta_{xy})}{\{\bar{L}(a, z; \theta_{xy})\}}} \frac{\{\bar{L}(a, z; \theta_{xy})\}}{\{\bar{L}(a, z; \theta_{xy})\}}
\]

(2.11)
Corollary 10. We have:

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Psi^{(a)}_{n+m}(x, y, z|q)(-1)^{n+m} q^{\frac{m(n+1)}{2}} \frac{t^n}{(q; q)_n (q; q)_m} = \left(\frac{xs; q}{q}; q\right)_\infty \sum_{k=0}^{\infty} \frac{(ys; q)_k q^k}{(sq/t, xs, q; q)_k} \Phi_1 \left[ \begin{array}{c} a; \\ q; zsq^k \end{array} ; 0; \right],
\]

(3.1)

where \(\max\{|t/s|, |ys|\} < 1\).

Corollary 10. We have:

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{n+m}(x, y, z|q)(-1)^{n+m} q^{\frac{m(n+1)}{2}} \frac{t^n}{(q; q)_n (q; q)_m} = \left(\frac{xs, zs; q}{q}; q\right)_\infty \sum_{k=0}^{\infty} \frac{(ys; q)_k q^k}{(sq/t, xs, q; q)_k} \Phi_3 \left[ \begin{array}{c} yt, 0, 0; \\ q; q; \end{array} ; sq/t, xs, zs; \right],
\]

(3.2)

where \(\max\{|t/s|, |ys|\} < 1\).

Proof. Denoting the right-hand side of equation (3.1) by \(G(a, s, x, y, z)\), we have:

\[
G(a, s, x, y, z) = \frac{1}{(t/s; q)_\infty} \sum_{k=0}^{\infty} \frac{q^k}{(sq/t, q; q)_k} \left(\frac{xsq^k; q}{q}; q\right)_\infty \Phi_1 \left[ \begin{array}{c} a; \\ q; zsq^k \end{array} ; 0; \right].
\]

(3.3)

Because equation (3.3) satisfies (2.3), by (2.4), we have:

\[
G(a, s, x, y, z) = \bar{L}(a, z; \theta_{xy}) G(a, s, x, y, 0)
\]

\[
= \bar{L}(a, z; \theta_{xy}) \left\{ \frac{1}{(t/s; q)_\infty} \sum_{k=0}^{\infty} \frac{q^k}{(sq/t, q; q)_k} \left(\frac{xsq^k; q}{q}; q\right)_\infty \right\}
\]

\[
= \bar{L}(a, z; \theta_{xy}) \left\{ \frac{xs; q}{(ys; q)_\infty} \sum_{k=0}^{\infty} \frac{(ys; q)_k q^k}{(xs, q; q)_k (t/s; q)_\infty (sq/t; q)_k} \right\}
\]

which is the left-hand side of (2.7). □
Remark 12. Because equation (3.4) satisfies (3.5) by $H(a, \omega, x, y, z)$, we have:

\[
 H(a, \omega, x, y, z) = \frac{1}{(s/t, t/\omega, \omega)} \sum_{k=0}^{\infty} q^k \sum_{l=0}^{\infty} (q^{l}q^{(n+1)} \omega^k) (q^{l}q^{(n+1)} \omega^k) \Phi_1 \left[ \begin{array}{c} a; \\ q, z \omega q \end{array} \right] \left( \begin{array}{c} 0; \\ \omega \end{array} \right), \tag{3.5}
\]

which is the left-hand side of (3.1).

\[ \square \]

**Theorem 11** (Extended Roger’s-type formula for $\Psi_n(a)(x, y, z|q)$). We have:

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \Psi_n^{(a)}(x, y, z|q)(-1)^{n+m+k} q^{(n+m+k)} s^{k} \omega^k = \frac{(x \omega; q)_\infty}{(s/t, t/\omega, \omega)_\infty} \sum_{j=0}^{\infty} \frac{(y \omega; q)_j q^j}{(x \omega q^{l}; q)_j} \Phi_1 \left[ \begin{array}{c} a; \\ q; \omega q \end{array} \right] \left( \begin{array}{c} 0; \\ \omega \end{array} \right), \tag{3.4}
\]

where $\max(|s/t|, |t/\omega|, |\omega|) < 1$.

**Remark 12.** For $s = 0$, (3.4) reduces to (3.1).

**Proof of Theorem 11.** Denoting the right-hand side of equation (3.4) by $H(a, \omega, x, y, z)$, we have:

\[
 H(a, \omega, x, y, z) = \frac{1}{(s/t, t/\omega; q)_\infty} \sum_{k=0}^{\infty} q^k \sum_{l=0}^{\infty} (q^{l}q^{(n+1)} \omega^k) (q^{l}q^{(n+1)} \omega^k) \Phi_1 \left[ \begin{array}{c} a; \\ q, z \omega q \end{array} \right] \left( \begin{array}{c} 0; \\ \omega \end{array} \right). \tag{3.5}
\]

Because equation (3.5) satisfies (2.3), by (2.4), we have:

\[
 H(a, \omega, x, y, z) = \frac{1}{(s/t; q)_\infty} \sum_{k=0}^{\infty} q^k \sum_{l=0}^{\infty} (q^{l}q^{(n+1)} \omega^k) (q^{l}q^{(n+1)} \omega^k) \Phi_1 \left[ \begin{array}{c} a; \\ q, z \omega q \end{array} \right] \left( \begin{array}{c} 0; \\ \omega \end{array} \right), \tag{3.5}
\]

where $\max(|s/t|, |t/\omega|, |\omega|) < 1$.

**Proof of Theorem 11.** Denoting the right-hand side of equation (3.4) by $H(a, \omega, x, y, z)$, we have:

\[
 H(a, \omega, x, y, z) = \frac{1}{(s/t, t/\omega; q)_\infty} \sum_{k=0}^{\infty} q^k \sum_{l=0}^{\infty} (q^{l}q^{(n+1)} \omega^k) (q^{l}q^{(n+1)} \omega^k) \Phi_1 \left[ \begin{array}{c} a; \\ q, z \omega q \end{array} \right] \left( \begin{array}{c} 0; \\ \omega \end{array} \right). \tag{3.5}
\]

Because equation (3.5) satisfies (2.3), by (2.4), we have:

\[
 H(a, \omega, x, y, z) = \frac{1}{(s/t, t/\omega; q)_\infty} \sum_{k=0}^{\infty} q^k \sum_{l=0}^{\infty} (q^{l}q^{(n+1)} \omega^k) (q^{l}q^{(n+1)} \omega^k) \Phi_1 \left[ \begin{array}{c} a; \\ q, z \omega q \end{array} \right] \left( \begin{array}{c} 0; \\ \omega \end{array} \right). \tag{3.5}
\]
Theorem 14. For $|\nu t| < 1$, we have:

$$
\sum_{n=0}^{\infty} \psi^{(a)}_{n}(x, y, z|q) p_n(\nu, \mu) \frac{(-1)^n q^{\nu n}(q^2)^n}{(q; q)_n} = \frac{\nu/\lambda, y\nu t; q_\infty}{(y\nu t; q)_\infty} 1 \Phi_1 \left[ \begin{array}{c}
\nu/\lambda, y
\end{array} ; q; y\nu t^2q^\nu \right].
\tag{4.3}
$$

Corollary 15. For $|\nu t| < 1$, we have:

$$
\sum_{n=0}^{\infty} F_n(x, y, z; q) p_n(\nu, \mu) \frac{(-1)^n q^{\nu n}(q^2)^n}{(q; q)_n} = \frac{\nu/\lambda, y\nu t, z\nu t; q_\infty}{(y\nu t, z\nu t; q)_\infty} 3 \Phi_2 \left[ \begin{array}{c}
\nu/\lambda, \frac{\mu}{\nu}, y
\end{array} ; q, q; y\nu t, z\nu t \right].
\tag{4.4}
$$

Corollary 16. For $|t| < 1$, we have:

$$
\sum_{n=0}^{\infty} \psi^{(a)}_{n}(x, y, z|q) \Phi_n(x, \lambda, q) \frac{(-1)^n q^{\nu n}(q^2)^n}{(q; q)_n} = \frac{\nu/\lambda, x\nu t, z\nu t; q_\infty}{(x\nu t, z\nu t; q)_\infty} 1 \Phi_1 \left[ \begin{array}{c}
\nu/\lambda, \frac{\mu}{\nu}, x, z
\end{array} ; q, q; x\nu t, z\nu t \right].
\tag{4.5}
$$

Remark 17. For $\nu = 1$, (4.4) reduces to (4.5).
Corollary 18. For $|axt| < 1$, we have:

$$
\sum_{n=0}^{\infty} \phi_n^{(a)}(x,y|q)(\lambda; q)_n \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} = (\lambda, xt, yt| q)^\infty \frac{3 \Phi_2}{(axt; q)_\infty} \left[ \begin{array}{c} axt, 0, 0; \\ xt, yt; \end{array} \right] \tag{4.6}
$$

and

$$
\sum_{n=0}^{\infty} \phi_n^{(a)}(x|q)(\lambda; q)_n \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} = (\lambda, xt, t| q)^\infty \frac{3 \Phi_2}{(axt; q)_\infty} \left[ \begin{array}{c} axt, 0, 0; \\ xt, t; \end{array} \right]. \tag{4.7}
$$

Remark 19. Setting $\nu = 1$, $a = 0$, $y = ax$ and $z = y$ in (4.4), we get (4.6). For $\nu = 1$, $a = 0$, $y = ax$ and $z = 1$, (4.4) reduces to (4.7). For $y = 1$, (4.6) reduces to (4.7).

Before we prove the Theorem 15, the following Lemma is necessary.

Lemma 20. [6, Eq. (III.1)] For $|c, |x|| < 1$, we have:

$$
\Phi_1 \left[ \begin{array}{c} a, b; \\ q; z; \\ c; \end{array} \right] = \Phi_1 \left[ \begin{array}{c} \frac{b, az; q)^\infty}{(c, z; q)^\infty} \\ q; b; \\ a; \end{array} \right]. \tag{4.8}
$$

Proof of Theorems 15. Denoting the right-hand side of equation (4.4) by $H'(a, t, x, y, z)$, we have:

$$
H'(a, t, x, y, z) = (\mu/\nu q)^\infty \sum_{n=0}^{\infty} \frac{(xvtq^n; q)^\infty (\mu/\nu)^n}{(yvtq^n; q)^\infty (q; q)_n} \Phi_1 \left[ \begin{array}{c} a; \\ q; z; \\ \nu; \end{array} \right]. \tag{4.9}
$$

Because equation (4.9) satisfies (2.3), by (2.4), we have:

$$
H'(a, t, x, y, z) = \tilde{L}(a, z; \theta_{xy}) \{H'(a, t, x, y, 0)\}
$$

$$
= \tilde{L}(a, z; \theta_{xy}) \left\{ (\mu/\nu q)^\infty \sum_{n=0}^{\infty} \frac{(xvtq^n; q)^\infty (\mu/\nu)^n}{(yvtq^n; q)^\infty (q; q)_n} \right\}
$$

$$
= \tilde{L}(a, z; \theta_{xy}) \left\{ (\mu/\nu, xvtq; q)^\infty \frac{yvt, 0; q; \mu}{(yvt; q)^\infty} \left[ \begin{array}{c} a; \\ q; z; \\ \nu; \end{array} \right] \right\}.
$$

By using (4.8) and (1.3), the last relation becomes

$$
H'(a, t, x, y, z) = \tilde{L}(a, z; \theta_{xy}) \left\{ \sum_{n=0}^{\infty} p_n(y, x) \frac{p_n(y, \mu) t^n}{(q; q)_n} \right\}
$$

$$
= \sum_{n=0}^{\infty} \tilde{L}(a, z; \theta_{xy}) \left\{ (-1)^n q^{-\binom{n}{2}} p_n(y, x) \frac{p_n(y, \mu) t^n}{(q; q)_n} \right\} \text{ by (1.15)}
$$

$$
= \sum_{n=0}^{\infty} \phi_n^{(a)}(x, y, z|q)p_n(v, \mu) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n},
$$

which is the left-hand side of (4.4). This achieves the proof.

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q-DIFFERENCE EQUATION FOR GENERALIZED TRIVARIATE q-HAHN POLYNOMIALS

REFERENCES

[1] M. A. Abdllhusein, Two operator representations for the trivariate q-polynomials and Hahn polynomials, Ramanujan J. 40 (2016), 491–509.
[2] A. Al-Salam and L. Carlitz, Some orthogonal q-polynomials, Math. Nachr. 30 (1965), 47–61.
[3] J. Cao, On Carlitz’s trilinear generating functions, Appl. Math. Comput. 218 (2012), 9839–9847.
[4] W. Y. C. Chen, A. M. Fu and B. Zhang, The homogeneous q-difference operator, Adv. App. Math. 31 (2003), 659–668.
[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, 2nd edn. Cambridge University Press, Cambridge, 2004.
[6] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.
[7] W. Hahn, Uber Orthogonalpolynome, die q-Differenzengleichungen, Math. Nachr. 2 (1949), 434.
[8] W. Hahn, Beitrage zur Theorie der Heineschen Reihen; Die 24 Integrale der hypergeometrischen q-Differenzengleichung; Das q-Analogon der Laplace-Transformation, Math. Nachr. 2 (1949), 340–379.
[9] R. Koekock and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue report, Delft University of Technology, 1998.
[10] R. M. Range, Complex analysis: A brief tour into higher dimensions, Amer. Math. Mon. 110 (2003), 89–108.
[11] H. L. Saad and A. A. Sukhi, Another homogeneous q-difference operator, Appl. Math. Comput. 215 (2010), 4332–4339.
[12] L. J. Slater, Generalized Hypergeometric Functions, Cambridge Univ. Press, Cambridge/London/New York, 1966.
[13] H. M. Srivastava and P. W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted (Ellis Horwood, Chichester); Wiley, New York, 1985.
[14] H. M. Srivastava, S. Arjika and A. Sherif Kelil, Some homogeneous q-difference operators and the associated generalized Hahn polynomials, Appl. Set-Valued Anal. Optim. 1(2019), 187–201.
[15] H. M. Srivastava and A.K. Agarwal, Generating functions for a class of q-polynomials, Ann. Mat. Pure Appl. (Ser. 4) 154 (1989), 99–109.