THE SHOENBERG EFFECT IN A RELATIVISTIC DEGENERATE ELECTRON GAS AND OBSERVATIONAL EVIDENCE IN MAGNETARS

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ABSTRACT

The electron gas inside a neutron star is highly degenerate and relativistic. Due to electron–electron magnetic interactions, the differential susceptibility can equal or exceed one, which causes the magnetic system of the neutron star to become metastable or unstable. The Fermi liquid of nucleons under the crust can be in a metastable state, while the crust is unstable to the formation of layers of alternating magnetization. The change of the magnetic stress acting on adjacent domains can result in a series of shifts or fractures in the crust. The release of magnetic free energy and elastic energy in the crust can cause the bursts observed in magnetars. Simultaneously, a series of shifts or fractures in the deep crust that is close to the Fermi liquid of nucleons can trigger a phase transition of the Fermi liquid of nucleons from a metastable state to a stable state. The magnetic free energy released in the Fermi liquid of nucleons corresponds to the giant flares observed in some magnetars.

Key words: magnetic fields – pulsars: general – stars: neutron

1. INTRODUCTION

In 1930, de Haas and van Alphen first observed that the magnetization $\mathbf{M}$ in normal metals oscillates at low temperatures, under an intense applied magnetic field. The oscillatory functions are sinusoidal series with a fundamental frequency that can be described by the extremal areas of the cross-section of the Fermi surface normal to the applied magnetic field (Lifshits & Kosevich 1956). Using the impulsive field method, Shoenberg (1962) found an unexpectedly high amplitude for the second harmonic, and proposed a magnetic interaction among the conduction electrons, the so-called Shoenberg Effect. He suggested that the magnetizing field is not the applied field $\mathbf{H}$ but the magnetic induction $\mathbf{B}$. When the differential magnetic susceptibility $\chi_m = \partial(4\pi M)/\partial B$ exceeds one (we use Gaussian units in this paper), the spatially uniform state of an electron gas is thermodynamically unstable. Then, the electron gas rapidly evolves into a stable state, and has a spatially inhomogeneous magnetic field with various Condon domains (Condon 1966).

The magnetization and the de Haas–van Alphen oscillating effect for a relativistic degenerate electron gas have been studied by many authors (Visvanathan 1962; Canuto & Chiu 1968; Chudnovsky 1981). Previous studies of the magnetic susceptibility of neutron stars mainly focused on whether or not the observed field resulted from spontaneous magnetization (Lee et al. 1969; O’Connell & Roussel 1971). This almost cannot occur because the neutron star is not sufficiently cool. Blandford & Hernquist (1982) and Wilkes & Ingraham (1989) proposed the domain structure and applied it to neutron star crusts. However, the inhomogeneous field resulting from domain formations in the crust of a normal neutron star is negligible and cannot produce the observable effects. On the other hand, for magnetars with magnetic fields stronger than normal neutron stars, the inhomogeneous field is large enough to produce super-Eddington X-ray outbursts.

Magnetars, including anomalous X-ray pulsars (AXPs) and soft gamma repeaters (SGRs), are characterized by inferred dipolar magnetic field strengths ranging from $5.9 \times 10^{13}$ G to $1.8 \times 10^{15}$ G and long spin periods ranging from 5.2 s to 11.8 s. Surprisingly, some magnetars have undergone giant flares (GFs) in which energies up to $10^{46}$ erg are released in a fraction of a second via $\gamma$-ray emission. Up until now, three GFs have been observed from SGR 0526–66 (Cline et al. 1982), SGR 1900+14 (Mazets et al. 1999), and SGR 1806–20 (Palmer et al. 2005).

The X-ray and $\gamma$-ray luminosities of these objects during the continuous and burst phases are too large to be powered by their kinetic energy. Where is the released magnetic energy stored prior to the GFs? Is it in the magnetospheres or in the neutron star? Duncan & Thompson (1992) and Thompson & Duncan (2001) proposed a model for magnetars and proposed that the released magnetic energy is stored in the neutron star. The controversy over this model relates to its triggering mechanism for the high energy radiation. Duncan & Thompson (1992) suggested that a helical distortion of the magnetic field in the core induces a large-scale fracture in the crust and a twisting deformation of the magnetic field in the crust and magnetospheres. A GF may involve a large disturbance that probably is driven by a rearrangement of the magnetic field in the deep crust and core (Thompson & Duncan 1995), while the continuous emission can be explained by the very slow transport of the field from the core to the crust by the Hall drift (Goldreich & Reisenegger 1992). Kondratyev (2002) first postulated the existence of magnetic domains in the magnetar crust because of the inhomogeneous crust structure. He argued that the burst activity of SGRs could originate from magnetic avalanches. However, because of the domain-forming mechanism resulting from the interaction among the spins of nucleons, the required magnetic field must be in the range of $10^{16} – 10^{17}$ G. It is still an open question whether or not the magnetic field in the crust of a neutron star can be so strong. Having considered the GF’s emission energy $(10^{42} – 10^{46}$ erg) and its mean waiting time (Mazets et al. 1979; Hurley et al. 1999; Terasawa et al. 2005), Stella et al. (2005) estimated that the internal field strength of a magnetar at its birth time can reach up to $B \geq 10^{15.7}$ G. Later, after numerically calculating the magnetic properties of magnetar matter, such as the magnetization and susceptibility of electrons, protons and neutrons, Suh & Mathews (2010) proposed a magnetic domain model to correlate smoothly between the statistics of star-quakes and the magnetic avalanches in the magnetar crust.
The phase transition to the domain phase in magnetars is different from the ferromagnetic phase transition. The magnetic ordering in iron comes from the exchange interaction of bound electron spins at a sufficiently low temperature and it is not a prerequisite to have an external magnetic field. On the other hand, the magnetic ordering in magnetars arises from the interactions among the orbital magnetic moments of free electrons under a high-quantizing field. Therefore, given a nearly uniform distribution of electrons, the phase transitions in magnetars are similar to those in beryllium where the magnetized matter is the conductive electrons (Shoenberg 1962). The relativistic Fermi sea of electrons is only slightly perturbed by the Coulomb forces of nuclei in the crust of the magnetar and does not efficiently screen nuclear charges. The dominant contribution to the differential susceptibility in a magnetar is electrons, while the contributions from nucleons are negligible (the magnetic moments of nucleons are three orders of magnitude lower than the electron orbital moments).

In this paper, we analytically calculate the differential susceptibility of a relativistic degenerate electron gas and find that it is an oscillating function of the magnetic field, which is often called the de Haas–van Alphen oscillation. In Section 2, we discuss the magnetic phase transition, while in Section 3 we calculate the differential susceptibility. In Section 4 we use our model to explain the observed evidence in magnetars, and finally we summarize our main results in Section 5.

2. THE MAGNETIC PHASE TRANSITION

The magnetization of the degenerate electron gas in a highly quantizing magnetic field at low temperature ($KT \ll \hbar \omega_c$, where $\omega_c$ is the electron cyclotron frequency in the magnetic field) should exhibit a nonlinear de Haas–van Alphen Effect. Without the magnetic interaction the oscillating magnetization is assumed to be periodical in the magnetic field $H$ rather than in $1/H$ (Pippard 1980; Shoenberg 1984):

$$\dot{M} = M_0 \sin(H/H_0),$$

where $H_0$ hardly varies over one cycle of oscillation. However, if we consider the feedback contribution from $4\pi M$, the magnetization depends not only on the quantizing magnetic field but also on the cooperating ordering of the magnetic moments. Thus, the magnetization should actually be a function of the magnetic induction, $M = \dot{M}(B)$. Then, the magnetic induction is given by $B = H + 4\pi M(B)$. Replacing $H$ by $B$ is called the Shoenberg or $H-B$ Effect. The oscillating sinusoidal function of the magnetization in the magnetic field of Equation (1) is given by

$$\dot{M} = M_0 \sin(B/H_0) = M_0 \sin[(H + 4\pi M)/H_0],$$

where $M_0$ and $H_0$ are constants.

When $4\pi M_0/H_0 > 1$, $M$ becomes a three-valued function of $H$, as shown in Figure 1, and the differential magnetic susceptibility $\chi_m = \partial(M/\dot{M})/\partial B$ can exceed one. Considering a thin rod oriented along the direction of the magnetization, Pippard (1963) found that the multi-valued function in Figure 1 is not physical. In the region of the curve between $L$ and $L'$ where the slope is less than one, the magnetized states are unstable and can never truly exist. For any weak perturbation $\delta H$ near $H_0$, the perturbation of the magnetic induction $\delta B$ is given by

$$\delta B = \frac{\delta H}{1 - \chi_m}.$$  (3)

Obviously, there is a singular point when $\chi_m = 1$ in Equation (3), where the magnetized state is unstable and the first-order phase transition should occur. Then, the magnetic system should be in a stable state in which the magnetization is inhomogeneous and magnetic domains form. The magnetizations in the adjacent domains have opposite directions. The stable magnetized states are represented by the dashed line between $N$ and $N'$ in Figure 1. But if there is a surface energy at the boundary of the two different magnetizations, they may become metastable, similar to superheating or supercooling in a gas–liquid transition (Reichl 1998). The solid line between $L$ and $N$ or $L'$ and $N'$ in Figure 1 represents the metastable state. It is not difficult to achieve the condition of the above magnetic phase transition in terrestrial labs. The experiments of Condon (1966) indicated that the magnetized phase transition can take place in some metals. However, what is the situation in compact objects such as neutron stars? In the following, we calculate the differential susceptibility of a relativistic degenerate electron gas and show the observable effects of the magnetic phase transition in neutron stars.

3. THE DIFFERENTIAL SUSCEPTIBILITY OF A RELATIVISTIC DEGENERATE ELECTRON GAS

The assembly of electrons in a neutron star under a strong magnetic field is degenerate and relativistic. The energy eigenvalues are (Johnson & Lippmann 1949)

$$E = \left[\epsilon^2 p_z^2 + \mu^2 + \mu \epsilon_c (2n + s + 1)\right]^{1/2},$$

where $\mu = m_e c^2$ and $\epsilon_c = \hbar eB/cm_e = \hbar \omega_c$ are the rest energy and cyclotron energy of an electron, respectively. Here, $p_z$ is the momentum component along the field direction that is taken as the $z$-direction, and $n = 0, 1, \ldots$ and $s = \pm 1$ are the Landau and spin quantum number, respectively. The density of states per unit volume for an energy level is given by $(p_z/h)(eB/hc) = g(p_z/h)$, where $g = eB/hc$ is the density of states per unit area for a Landau energy level. As a consequence of quantization, the spherical Fermi surface of a free electron is replaced by a set of circles located on a spherical surface with a common axis along the $B$ direction, as shown in Figure 2.

The susceptibility of the electron assembly can be obtained by finding its grand potential, which in turn depends on the density of states. It is a function of energy and magnetic induction. The density of states per unit volume is given by

$$Z(\epsilon, B) = \frac{2eB}{c\hbar^2} \sum_{n,s} \left[\epsilon^2 + 2\epsilon \mu - \mu \epsilon_c (2n + s + 1)\right]^{1/2},$$

where $H$ is the magnetic field, $L$ is the Landau level, $N$ is the number of electrons, and $\chi_m$ is the magnetic susceptibility.
where $\epsilon = E - \mu$ is the kinetic energy of an electron. The grand potential per unit volume of the assembly is given by

$$J = -\beta \int \ln[1 + e^{\beta(\psi - \epsilon)}]dZ(\epsilon, B)$$

$$= -\int_0^\infty Z(\epsilon, B)f(\epsilon)d\epsilon,$$

(6)

where $\beta = (kT)^{-1}$, $\psi$ is the chemical potential, and $f(\epsilon)$ is the Fermi–Dirac distribution function. After summing over the spin quantum number, Equation (5) reduces to

$$Z = \frac{2}{\sqrt{\pi}}g^{3/2}\left[\frac{\epsilon^2}{2\epsilon \mu} + 2 \sum_{n=1}^{[b]} (b - n)^{1/2}\right],$$

(7)

where $[b]$ is the integer of $b$ ($[b] \leq b$). For a neutron star, the electron system is almost completely degenerate ($\psi \gg \beta^{-1}$) and the Fermi–Dirac distribution function is almost a step function except for the region near $\epsilon = \psi$. The magnetic moment depends on the first-order derivative of the grand potential with $B$, and it mainly comes from the first-order derivative of Equation (6) at $\epsilon = \psi$. Hence, we define $b_m = b|_{\epsilon = \psi}$. If the chemical potential of the neutron star is $\sim 10$ MeV and the magnetic field is the quantum field $B_0 = 4,414 \times 10^{13}$ G (the cyclotron energy of an electron is equal to its rest energy), then $[b_m]$ is about 200. Based on Equation (7), we can see that the first-order derivative of the grand potential has a singularity when $b_m = [b_m]$. Therefore, the susceptibility of the relativistic degenerate electron gas can be equal to or even larger than one.

To reveal the oscillating effect of the grand potential, we calculate the sum of Equation (7) with the Poisson summation formula (Dingle 1952):

$$\frac{1}{2} F(0) + \sum_{n=1}^\infty F(n) = \sum_{r=-\infty}^{\infty} \int_0^{[b_m]} F(x)e^{2\pi rx}dx.$$  

(9)

Because $[b_m]$ is much larger than one, the density of states in Equation (5) is approximated by

$$Z = -\frac{4}{3\sqrt{\pi}}g^{3/2}\left[\frac{2}{3}b_m^3 + \sum_{v=0,2,4}^{\infty} \frac{(-1)^{v/2}(2v - 1)!b^{(v+2)/2}}{\sqrt{b}(4b)^v(v - 1)(v + 2)!}\right] + \frac{1}{2\sqrt{2\pi}}\sum_{r=1}^{\infty} \frac{\cos(2\pi r b - 3\pi/4)}{r^{3/2}}.$$  

(10)

where $B_m$ denotes the Bernoulli numbers. The first two terms on the right-hand side of the above equation are the non-oscillating parts while the third term is the oscillating part.

The non-oscillating grand potential per unit volume can be obtained from Equation (10). As a first-order approximation, we can keep only the first term in the summation and obtain

$$\tilde{J} = -\frac{4}{3\sqrt{\pi}}g^{3/2}\int_0^\infty \frac{(e^2 + 2\epsilon \mu)^{3/2}f(\epsilon)d\epsilon}{(e^2 + 2\epsilon \mu)^{3/2}}.$$  

(11)

In general, $\psi \gg \mu$ for a neutron star. Using the standard methods of evaluating integrals for the Fermi–Dirac distribution, we can evaluate the integrals in Equation (11). The non-oscillating susceptibility is given by

$$\tilde{\chi}_m = 4\pi \frac{\partial^2 J}{\partial B^2} = 4\sqrt{\frac{2\pi}{3}} \frac{\sqrt{\mu\epsilon}}{B^2}g^{3/2}\left[\ln \frac{2\psi}{\mu} - \frac{\pi^2}{6}\left(\frac{kT}{\psi}\right)^2\right].$$  

(12)

In our work, the chemical potential and the magnetic field in the deep crust of a normal neutron star are, respectively, about 10 MeV and $10^{12}$ G, while the non-oscillating susceptibility can be $\sim 5.6 \times 10^{-3}$.

Using the approximate formula for the Fermi–Dirac distribution

$$\int_0^\infty \eta(\epsilon)f(\epsilon)d\epsilon = \int_0^\frac{\psi - \mu}{\psi} \eta(\epsilon)d\epsilon + \frac{\pi}{6}(kT)^2\eta(\psi - \mu)d\epsilon,$$

(13)

and the Fresnel Integral

$$\int_0^x \cos \left(\frac{\pi}{2}t^2\right)dt \approx x(x \ll 1) \approx \frac{1}{2} + \frac{1}{\pi x} \sin \left(\frac{\pi}{2}x^2\right)(x \gg 1),$$

(14)

we obtain the oscillating term of the grand potential per unit volume,

$$\tilde{J} = \frac{\sqrt{2}}{\pi^{3/2}}g^{3/2}\sum_{r=1}^{\infty} \frac{1}{r^{3/2}} \left[\frac{1}{2} \sqrt{\frac{\pi}{2a_t}} - \mu \right] \cos \left(\frac{a_t \mu^2 + 3\pi}{4}\right) + \frac{1}{2} \sqrt{\frac{\pi}{2a_t}} \sin \left(\frac{a_t \mu^2 + 3\pi}{4}\right)$$

$$+ \frac{\sqrt{2}}{\pi^{3/2}}g^{3/2}\sum_{r=1}^{\infty} \frac{1}{r^{3/2}} \sin \left(\frac{\pi}{2}\right) (kT)^2(a_t \psi)^2) \times \sin \left(\frac{a_t \psi^2 - a_t \mu^2 - \frac{3\pi}{4}\right),$$

(15)

where $a_t = \pi r/(\epsilon_c \mu)$. In calculating the differential susceptibility as the second order partial derivatives of Equation (15), only the most rapidly varying factors need to be differentiated. That is, we only differentiate cosine and sin terms. Because $\psi \gg \mu$ for a neutron star, the last term in Equation (15) dominates. The oscillating susceptibility is approximately given by

$$\tilde{\chi}_m = A_0 \sum_{r=1}^{\infty} \left(\frac{r}{a_t \psi^2 - a_t \mu^2 - \frac{3\pi}{4}\right) \cos \left(\frac{a_t \psi^2 - \frac{3\pi}{4}\right).$$

(16)
where
\[ A_0 = \alpha \left( \frac{\psi}{\mu} \right)^3 \left( \frac{B}{B_0} \right)^{-3/2}, \]  \hspace{1cm} (17)
and
\[ A_1 = \frac{2\pi^3}{3} \left( \frac{\psi}{\mu} \right)^2. \]  \hspace{1cm} (18)

Figure 3. log $A_0$ vs. $B/B_0$.

Here, $\alpha$ in Equation (17) is the fine structure constant. The result of Equation (16) is similar to the result obtained in Viswanathan (1962). It indicates that the susceptibility oscillates as $1/B$. We replace $B$ by $B' = B - B_0$. When $B \to B_0$ we have $1/B \approx 1/B_0(1 - B'/B_0)$. Note that the first term is a constant, thus the oscillation is a periodic function of $B'$. The differences in the magnetic induction in an oscillating period are the periods of $B'$, which are given by
\[ \delta B = \frac{2\hbar e c^2 B^2}{\psi^2 r}(r = 1, 2, \ldots). \]  \hspace{1cm} (19)

The coefficients of the cosine functions in Equation (16) include two terms. The first term is the result of complete degeneracy, while the second term comes from thermal fluctuations and is proportional to the second power of temperature. For a typical neutron star, the chemical potential, magnetic field, and temperature are of the order of 10 MeV, $10^{15}$ G, and $10^{9}$ K, respectively. The second term can be ignored because $K T \ll \hbar \omega_c \ll \psi_0$. The oscillating susceptibility, $\bar{\chi}_m$, is mainly determined by $A_0$. Figure 3 shows the relation between log $A_0$ and $B/B_0$. Obviously, the differential susceptibility can equal or exceed one when $B < \sim 15 B_0$. In this work, we assume that magnetars are similar to normal neutron stars except for their magnetic fields: for a normal neutron star, $B = 10^{15}$ G, while for a magnetar $B = 15 B_0$. For a stronger magnetic field, the condition $\hbar \omega_c \ll \psi_0$ or $[b_m] \gg 1$ cannot be satisfied, and the approximate methods above cannot be used. We will discuss these appropriate methods in our next paper.

4. THE OBSERVED EVIDENCE IN MAGNETARS

As discussed in the last section, the susceptibility of a relativistic degenerate electron gas can be equal to or exceed one. The electron gas under a strong magnetic field in a neutron star may be in an unstable state, and the first-order phase transition should occur. Finally, the electron gas should be in a stable state with the Condon domain structure. The magnetizations in the adjacent domains have opposite directions.

Based on Equation (19), the relative variation of the magnetic induction between adjacent magnetic domains is
\[ \frac{\delta B}{B_0} \sim 10^{-3} \frac{B_0}{B_0}. \]  \hspace{1cm} (20)

For a normal neutron star ($B_0 \sim 10^{12}$ G), $\delta B/B_0 \sim 10^{-4}$ and it is very difficult to observe the oscillatory effects. However, for magnetars ($B_0 \sim 15 B_0$), $\delta B/B_0 \sim 10^{-2}$ and it is large enough to produce the bursts in SGRs or AXPs.

The adjacent magnetic domains have different electron densities. The different magnetizations in the adjacent magnetic domains mean that the phase difference is $\pi$. According to Equation (16), we have $\delta(\mu, \psi^2) = \pi$. For simplicity, we approximate the chemical potential $\psi$ with the zero-temperature free field Fermi energy $\psi_0 = \text{ch}(3n/8\pi)^{1/3}$. The difference in electron densities between adjacent magnetic domains is given by
\[ \frac{\delta n}{n} \sim 10^{-2} \left( \frac{\epsilon_e}{20} \right)^{-2} \left( \frac{B_0}{15 B_0} \right), \]  \hspace{1cm} (21)
where $\epsilon_e = \psi_0/\mu$ and we have taken $B = B_0$. For a magnetar, $\psi_0 \sim 10$ MeV, $B_0 \sim 15 B_0$, and the rest energy of electrons is $\mu \sim 0.5$ MeV. Obviously, here $\epsilon_e$ is scaled to 20 and $B_0$ is scaled to 15 $B_0$.

The electrons in a neutron star coexist with other components such as the nuclei in the crust or the Fermi liquid of protons and neutrons close to the deep crust. Due to the Coulomb interactions among electrons, nuclei, or protons, the matter of the neutron star may not be homogeneous once the domain structure appears in the crust. However, it cannot occur in the Fermi liquid. The mechanism of forming electron domains is the interactions among orbital magnetic moments that are counterbalanced by the degenerate pressure of electrons, $P_e$. Because the Coulomb interaction of the crystal lattices in the crust is the same order of magnitude as the interaction among the orbital magnetic moments, the electron domain structure can lead to the formation of the nuclei domain structure. However, in the Fermi liquid of nucleons, the neutron degenerate pressure $P_n \gg P_e = Y_e \rho_n$, where $Y_e$ is the fraction of electrons per neutron and it is only several tenths. The interaction among the orbital magnetic moments cannot counterbalance the neutron degenerate pressure $P_n$. Therefore, the magnetic domain structure cannot form in a Fermi liquid of nucleons. The magnetic system is in a metastable state, which is similar to the supercooled or superheated states in the first-order phase transition.

In the crust of a magnetar, we can roughly calculate the size of the domain structure. With increasing density, the quantum number of the Landau energy also increases. Using the gravitational potential energy of a nucleon and the Fermi energy per electron, we can estimate the height of the magnetic domains as (Suh & Mathews 2010)
\[ \delta z \sim 10^3 \left( \frac{\rho}{10^{14}} \right)^{-1} \left( \frac{Y_e}{0.36} \right) \left( \frac{\epsilon_e}{20} \right)^{-1} \left( \frac{B_0}{15 B_0} \right) \text{cm}, \]  \hspace{1cm} (22)
where \( g \) is the surface gravity and \( Y_e \) is scaled to 0.36 when neutrons drip out from nuclei. The actual size and shape of the domains is problematic. There seem to be two possibilities (Blandford & Hernquist 1982). The first is that the domains have a horizontal scale \( \delta z \) that is in general required if there is to be a balance in the electron pressures. The second possibility is that the domains form a two-dimensional lattice of vertical needles with a thickness given roughly by the geometrical mean of the cyclonic radius. This second configuration minimizes the magnetic and surface energies. If we assume that the volume change of the electron gas during the formation of the second domain configuration is only along the horizontal direction and that the changing magnitude (or the thickness of domain walls) is approximately the cyclonic radius of an electron, based on Equation (21), the width of the magnetic domain for the second possibility can be estimated as

\[
\delta l \sim 10^{-9} \left( \frac{\epsilon_e}{20} \right)^{3/2} \left( \frac{B_0}{15B_Q} \right)^{-2} \text{cm.} \tag{23}
\]

The Maxwell shearing stress between the adjacent domains can deform the crust and gives rise to a strain, \( \theta \). This stress is given by

\[
\frac{B_0 \delta B}{4\pi} \sim \theta \nu \sim \frac{\theta B_0^2}{4\pi}, \tag{24}
\]

where \( \nu \) is the shear modulus of the crust and \( B_0 = \sqrt{4\pi} v \). Close to the bottom of the crust, \( B_0 \simeq 6 \times 10^{15} \text{ G} \) (Baym & Pines 1971). For magnetars, \( B_0 \sim 15B_Q \). According to Equations (20) and (24), we recover \( \theta \sim 10^{-3} \). Ruderman (1991) found that the maximum strain of the crust, \( \theta_{\text{max}} \), is \( \sim 10^{-2} - 10^{-4} \). Our result is clearly within this range. When the domain structures appear in the crust of a magnetar, the shearing stress acting on adjacent domains can produce a relative shift or a fracture between them. This sudden shift or fracture can propagate with the Alfvén velocity along the domain layers, which results in a series of shifts or fractures in the magnetic domains. The timescale of shifts propagating along a domain layer is estimated as

\[
\tau \sim \frac{l}{V_A} \sim 0.1 \left( \frac{B_0}{15B_Q} \right)^{-1} \left( \frac{\rho}{10^{15}} \right) \left( \frac{l}{R_o} \right) \text{s}, \tag{25}
\]

where \( V_A \) and \( \rho \) are the Alfvén velocity (Duncan 2004) and the matter density at the deep crust, respectively, \( l \) is the length of the domain layer, and \( R_o \) the radius of the magnetar (\( \sim 10 \) km). The timescale agrees with the observed duration of SGRs bursts (Thompson & Duncan 1995). The available free energy equals the change in magnetic energy from a domain structure to a homogeneous structure. The density of the magnetic free energy is

\[
w = \frac{1}{16\pi} \left[ (B + \delta B)^2 + (B - \delta B)^2 - 2(B)^2 \right] = \frac{(\delta B)^2}{8\pi}. \tag{26}
\]

Then, the total free energy is approximately

\[
E_{\text{burst}} \sim \frac{(\delta B)^2}{8\pi} 4\pi (R_o)^2 \delta z. \tag{27}
\]

According to Equations (20) and (22), the total free energy is

\[
E_{\text{burst}} \sim 10^{41} \left( \frac{\delta B}{10^{13}} \right)^2 \left( \frac{\delta z}{10^3} \right) \text{ erg}. \tag{28}
\]

where we scaled \( \delta z \) to \( 10^3 \) cm and \( \delta B \) to \( 10^{13} \) G when \( B_0 \sim 15B_Q \) (see Equation (20)). This energy also agrees with the observations of bursts.

The Fermi liquid of nucleons in a magnetar may be in metastable state. A series of shifts in the deep crust close to the Fermi liquid of nucleons can trigger a phase transition of the Fermi liquid of nucleons from a metastable state to a stable state. The free magnetic energy in the Fermi liquid is released and it is far greater than that in the crust because the magnetic induction and the thickness of the Fermi liquid is larger than the crust. The actual size of the Fermi liquid of nucleons in a metastable state is also problematic because of the unknown configuration of electrons and magnetic field distributions. However, all Fermi liquid of nucleons may evolve into metastable state simultaneously when the deep crust is stable. Let the magnetic induction of the Fermi liquid be the same order of magnitude as that in the deep crust, and the thickness of the Fermi liquid be approximated by the radius of the magnetar. We then estimate that the energy released is

\[
E_{\text{flare}} \sim 10^{44} \left( \frac{\delta B}{10^{13}} \right)^2 \text{ erg.} \tag{29}
\]

This energy agrees with the energy released in the GFs of some magnetars.

5. SUMMARY

We discussed the magnetization effects of the relativistic degenerate electron gas in a neutron star. Having considered the magnetic interaction among electrons, we found that the magnetic systems may be unstable or metastable when the differential susceptibility equals or exceeds one. Under an ultra-strong magnetic field, the magnetic domain structures in a magnetar can appear in the solid crust, while the Fermi liquid of nucleons may be in metastable state. The shearing stress acting on adjacent domains can result in a series of shifts or fractures in the crust. The crust releases magnetic free energy, which corresponds to the bursts observed in magnetars. Simultaneously, a series of shifts or fractures in the deep crust close to the Fermi liquid of nucleons can trigger a phase transition of the Fermi liquid from a metastable state to a stable state. The magnetic free energy in the Fermi liquid of nucleons is released, which corresponds to the GFs observed in some magnetars.

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