Relations Between Closed String Amplitudes at Higher-order Tree Level and Open String Amplitudes

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Abstract: KLT relations almost factorize closed string amplitudes on $S^2$ by two open string tree amplitudes which correspond to the left- and the right- moving sectors. In this paper, we investigate string amplitudes on $D^2$ and $RP_2$. We find that KLT factorization relations do not hold in these two cases. The relations between closed and open string amplitudes have new forms. On $D^2$ and $RP_2$, the left- and the right- moving sectors are connected into a single sector. Then an amplitude with closed strings on $D^2$ or $RP_2$ can be given by one open string tree amplitude except for a phase factor. The relations depends on the topologies of the world-sheets. In the low energy limits of these two cases, the factorization relations for graviton amplitudes do not hold. The amplitudes for gravitons must be given by the new relations instead.

Keywords: KLT relations, $D^2$, $RP_2$
1. Introduction

Superstring theories are theories without ultraviolet divergences. They contain both gravitational and gauge interactions as low energy limits [1]. Thus they offer a possible solution to the problem of unifying all of the fundamental interactions in a consistent quantum theory. In string theory, gravitons are massless states of closed strings and gauge particles are massless states of open strings. To study the relations between gravity and gauge field, we should explore the relations between closed and open strings. The duality between open and closed strings [2] also motivates us to explore the relations between closed and open strings.

The most simple relation is any excited mode of a free closed string $|N_L, N_R \rangle \otimes |p \rangle$ can be factorized by left- and right- moving open string excited modes:

$$|N_L \rangle \otimes |N_R \rangle \otimes |p \rangle .$$

(1.1)

However, when we consider the interactions among strings, there are nontrivial relations between closed and open string amplitudes. The first nontrivial relation was given by Kawai, Lewellen and Tye [3]. They express an amplitude for $N$ closed strings on sphere($S_2$) by the following equation:

$$\mathcal{A}^{(N)}_{S_2} = \epsilon_{\alpha\beta} \mathcal{A}^{(N)\alpha\beta}_{S_2} = \left( \frac{i}{2} \right)^{N-3} \kappa^{-N-2} \epsilon_{\alpha\beta} \sum_{P, P'} \mathcal{M}^{(N)\alpha}(P). \mathcal{M}^{(N)\beta}(P') e^{i\pi F(P, P')} .$$

(1.2)

\footnote{We use $\epsilon_{\alpha\beta}$ to denote all the polarization tensors for convenience. $\alpha$ correspond to the left indices and $\beta$ correspond to the right indices. If there are open strings on the boundary of $D_2$, we use $\epsilon_{\alpha\beta\gamma}$ to denote all the polarization tensors for convenience. $\gamma$ correspond to the indices of open strings.}
Here $A^{N} S_2$ is the amplitude for $N$ closed strings on $S_2$ and $A^{(N)\alpha\beta}_S$ is the closed string amplitude without polarization tensors. $A^{(N)\alpha}(P)$ and $A^{(N)\beta}(P')$ are the open string partial amplitudes on $D_2$ corresponding to the left- and right-moving sectors respectively. They are dependent on the orderings of the external legs. If we sum over the orderings $P$ and $P'$, we get the total amplitudes $\sum_{P} A^{(N)\alpha}(P)$ and $\sum_{P'} A^{(N)\beta}(P')$ for the left- and the right-moving open strings respectively. Then we can see, except for a phase factor, a closed string amplitude on $S_2$ can be factorized by two open string tree amplitudes corresponding to the left- and right-moving sectors (see fig 1. (a)). There is no interaction between left- and right-moving open strings. Any closed string polarization tensor has left and right indices, they correspond to the left- and the right-moving modes respectively. The left and right indices of polarization tensors must contract with the indices in the amplitude for left- and right-moving open strings respectively. The phase factor is entirely independent of which open and closed string theories we are considering. It only depends on $P$ and $P'$. Contour deformations can be used to reduce the number of the terms in eq. (1.2). The number of the terms can be reduced to

$$\begin{align}
(N - 3)!\left(\frac{1}{2}(N - 3)\right)!\left(\frac{1}{2}(N - 3)\right)! \times N, & \quad \text{odd,} \quad (1.3) \\
(N - 3)!\left(\frac{1}{2}(N - 2)\right)!\left(\frac{1}{2}(N - 4)\right)! \times N, & \quad \text{even.} \quad (1.4)
\end{align}$$

In the low energy limits, the massive modes decouple. Only massless states are left. Then KLT relations can be used to factorize the amplitudes for gravitons into products of two amplitudes for gauge particles. Gauge theory has a better ultraviolet behavior than gravity. Then KLT relations can be used to investigate the ultraviolet properties of gravity. Researches with KLT relations support that $N = 8$ supergravity may be finite [4]. However, a question arises: Do KLT factorization relations hold for any gravity amplitude? In string theory, to calculate the S-matrix, we should sum over all the topologies of worldsheets. $S_2$ is just the simplest topology. If we consider other topologies, we should reconsider the relations between closed and open strings. Then the question becomes: Do the factorization relations hold for any topology?

In this paper, we consider the amplitudes on disk ($D_2$) and real projective plane ($RP_2$). These two cases contribute to the higher-order tree amplitude [1] for closed strings. We find that the factorization relations (1.2) do not hold on $D_2$ and $RP_2$. The amplitudes with closed strings on $D_2$ and $RP_2$ can not be factorized by the left- and the right-moving open string amplitudes. The amplitudes satisfy new relations. Particularly, an amplitude for $N$ closed strings on $D_2$ can be given by an amplitude for $2N$ open strings:

\begin{equation}
A^{(N)} D_2 = \epsilon_{\alpha\beta} A^{(N)\alpha\beta} D_2 = \left(\frac{i}{4}\right)^{N-1} \kappa^{N-1} \epsilon_{\alpha\beta} \sum_{P} A^{(2N)\alpha\beta}(P) e^{i\pi \Theta(P)}. \quad (1.5)
\end{equation}
Figure 1: (a) A closed string amplitude on $S_2$ can be factorize by two open string tree amplitudes corresponding to the left- and right-sectors. (b) A closed string amplitude on $D_2$ can be given by connecting the open string world-sheets for the two sectors with a time reverse in the right-moving sector. (c) A closed string amplitude on $RP_2$ can be given by connecting the open string world-sheets for the two sectors with a time reverse and a twist in the right-moving sector.

In this equation, $\mathcal{M}^{(2N)\alpha\beta}(P)$ is the tree amplitude for $2N$ open strings. $N$ open strings come from the left-moving sector and the other $N$ open strings come from the right-moving sector. The left- and the right-moving sectors are not independent of each other. The two sectors are connected into a single sector. Then the left indices contract with the right indices. The reason is that the left-(right-)moving waves must be reflected at the
boundary of $D_2$ and then become right-(left-)moving waves. Then the interactions between the left-(right-)moving waves and their reflected waves become interactions between the two sectors. If there are open strings on the boundary of $D_2$, the left- and the right-moving sectors of closed strings also interact with the open strings, then an amplitude for $N$ closed strings and $M$ open strings on $D_2$ can be given by a tree amplitude for $2N + M$ open strings except for a phase factor:

$$A(D_2^{(N,M)}) = \epsilon_{\alpha\beta\gamma} A(D_2^{(N,M)\alpha\beta\gamma}) = \left(\frac{i}{4}\right)^{N-1} \kappa^{N-1} g^M \epsilon_{\alpha\beta\gamma} \sum_P \mathcal{M}^{(2N,M)\alpha\beta\gamma}(P) e^{i\pi\theta(P)}.$$  \hspace{1cm} (1.6)

The amplitudes on $RP_2$ can also be factorized by one amplitude for open strings:

$$A(RP_2^{(N)}) = \epsilon_{\alpha\beta} A(RP_2^{(N)\alpha\beta}) = \left(\frac{i}{4}\right)^{N-1} \kappa^{N-1} \epsilon_{\alpha\beta} \sum_P \mathcal{M}^{(2N)\alpha\beta}(P) e^{i\pi\theta(P)}.$$  \hspace{1cm} (1.7)

In this case, there is a crosscap but not a boundary here. However, the left-(right-) moving waves are also reflected at the crosscap and turn into the right-(left-)moving waves. Then there are also interactions between left- and right-moving sectors of closed strings. The two sectors are connected into one single sector again. It is noticed that the relations on $D_2$ are same with on $RP_2$. We will see any amplitude on $D_2$ is preserved by a twist of open strings come from one sector. After this twist, the amplitude becomes that on $RP_2$. So the amplitudes on $D_2$ and on $RP_2$ are equal.

We reduce the number of the terms in the relations by contour deformation. After appropriate contour deformations. An amplitude with $N$ closed strings and $M$ open strings on $D_2$ can be written by

$$\left(\frac{1}{2}(2N-4)\right)!\left(\frac{1}{2}(2N-2)\right)!M!$$

terms. The number of the terms in the case of $RP_2$ can be reduced to

$$\left(\frac{1}{2}(2N-4)\right)!\left(\frac{1}{2}(2N-2)\right)!.$$  \hspace{1cm} (1.8)

An important fact will be used in our paper is that the amplitudes with closed strings are invariant under conformal transformations in each single sector. This allows us to transform the form of the interactions between left- and right-moving sectors. After some appropriate transformation in one sector, the interactions between left- and the right-moving sectors have the same form with interactions between open strings in a same sector. Then we can treat the two sectors of $N$ closed strings as a single sector with $2N$ open strings.

In the low energy limit of an unoriented open string theory, the amplitudes for $N$ closed strings on $D_2$, $RP_2$ and $S_2$ contribute to the tree amplitudes for $N$ gravitons. In this case, we can not only use KLT factorization relations on $S_2$ but also use the relations on $D_2$ and $RP_2$ to calculate the tree amplitudes for gravitons. The amplitudes for $N$ closed strings and $M$ open strings on $D_2$ become tree amplitudes for $N$ gravitons and $M$ gauge particles. Then the gauge-gravity interactions can be given by pure gauge interactions.

The structure of this paper is as follows. In section 2 we will consider the amplitudes on $D_2$. We will give the relations between closed string amplitudes on $D_2$ and open string
tree amplitudes. We will also give the relations in the case of \( N \) closed strings and \( M \) open strings inserted on \( D_2 \). The relations between amplitudes on \( \mathbb{R}P^2 \) and open string amplitudes will be given in section 3. In section 4, we will show how the number of terms in the relations can be reduced. Our conclusion will be given in section 5. The volume of conformal Killing groups is useful in our calculations, they will be given in the appendix.

2. Relations between amplitudes on \( D_2 \) and open string tree amplitudes

In this section, we will explore the relations between amplitudes on \( D_2 \) and open string amplitudes. \( D_2 \) is a sphere with a boundary. It can be constructed from a sphere by identifying points under a reflection. Notice that the amplitudes are conformal invariant, we can use the conformal invariance to make a stereographic projection of the sphere onto the complex plane. Thus, we can take the conformal gauge \( ds^2 = \exp(2\omega(z, \bar{z}))d^2z \). On the complex plane, we can identify \( z \) and \( \frac{1}{z} \). The unit disk is the fundamental region for the identification. The boundary of the fundamental region is the unit circle. We can also identify \( z \) and \( \bar{z} \). In this case, the upper half-plane is the fundamental region and the boundary is the real axis. After fixing the conformal gauge, a gauge group which preserve the conformal gauge are left. This gauge group is called conformal Killing group (CKG) \([1]\).

The CKG of \( S_2 \) is \( PSL(2, \mathbb{C}) \). When we identify \( z \) and \( \frac{1}{z} \), \( z \) and \( \frac{1}{z} \) must transform in the same way under the \( PSL(2, \mathbb{C}) \). Then the CKG of \( D_2 \) is a subgroup of \( PSL(2, \mathbb{C}) \):

\[
z' = \frac{e^{i\alpha} \cosh \gamma z + e^{-i\beta} \sinh \gamma}{e^{i\beta} \sinh \gamma z + e^{-i\alpha} \cosh \gamma}
\]

(2.1)

where \( \alpha, \beta, \gamma \) are real parameters to describe this CKG. We can also identify \( z \) and \( \bar{z} \). In this case, the CKG becomes \( PSL(2, \mathbb{R}) \):

\[
z' = \frac{az + b}{cz + d}
\]

(2.2)

where \( a, b, c, d \) are real parameters. We fix \( \alpha\delta - \beta\gamma = 1 \) and identify under an overall sign reversal of \( a, b, c, d \).

To derive the amplitudes, we should calculate the correlation functions of the vertex operators on \( D_2 \), then integral over the fundamental region for closed strings. If there are open strings on the boundary, we should also integral along the boundary. Divide the world-sheet integrals by the volume of CKG, then we get the amplitudes. To obtain the correlation functions on \( D_2 \), we can use the boundary operator \([7]\) to create the \( D_2 \) vacuum, then move all the creation operators to the left of all the annihilation operators. We can also use the method of imagines \([1]\) to give the two-point Green function, then use the two-point Green function to give the correlation functions. Both unit disk and the upper half-plane can be chosen as the fundamental region. In this paper, we choose the unit disk as the fundamental region and use the boundary operator method to calculate the correlation functions.

An amplitude for \( N \) closed strings on \( D_2 \) is

\[
\mathcal{A}^N_{D_2} = \kappa^{N-1} \frac{1}{V^2_{\text{CKG}}} \left\langle \int_{|z_1|<1} d^2z_1 \mathcal{V}(z_1, \bar{z}_1) \ldots \int_{|z_N|<1} d^2z_N \mathcal{V}(z_N, \bar{z}_N) \right\rangle_{D_2},
\]

(2.3)
where $V_{CKG}^{D_2}$ is the volume of the CKG on $D_2$, $\int d^2 z_r \mathcal{V}(z_r, \bar{z}_r)(r = 1...N)$ are the conformal invariant vertices and $\langle ... \rangle_{D_2}$ is the correlation function on $D_2$ and $\kappa$ is the coupling constant for closed string vertices. We must choose vertices in appropriate pictures to make the total number of $\gamma$ ghost $-2$ [1,6]. In this paper we always choose the closed string vertices whose left-moving sectors have the same $\gamma$ ghost numbers with their right-moving sectors. For convenience, we use the bosonized vertex operator to discuss the relations. Any physical vertex operator can be given by a linear combination of the bosonized vertex operators. The general form of the bosonized vertex operator is

$$\int d^2 z \mathcal{V}(z, \bar{z}) = \int d^2 z : \exp (q \phi_0 + \bar{q} \tilde{\phi}_0) \exp (i \lambda \circ \phi + i \bar{\lambda} \circ \tilde{\phi} + i \epsilon \circ \partial \phi_i + i \bar{\epsilon} \circ \partial \tilde{\phi}_i) \exp (i k \cdot X + i \sum_{i=1}^{n} \epsilon^i \cdot \partial X + i \sum_{j=1}^{\tilde{n}} \bar{\epsilon}^j \cdot \partial \tilde{X})(z, \bar{z}):_{\text{multilinear}}. \tag{2.4}$$

Here $\phi_i(z)(i = 1...5)$ and $\tilde{\phi}_i(\bar{z})(i = 1...5)$ are bosonic fields. They are used to bosonize holomorphic and antiholomorphic fermionic fields and spinor fields. $\phi_0(z)$ and $\tilde{\phi}_0(\bar{z})$ are used to bosonize the holomorphic and antiholomorphic superconformal ghost respectively. $\epsilon$ and $\bar{\epsilon}$ correspond to the components of polarization tensors contracting with bosonic fields $\partial X$ and $\bar{\partial} X$ respectively. $\epsilon$ and $\bar{\epsilon}$ correspond to the components contracting with $\partial \phi$ and $\bar{\partial} \tilde{\phi}$ respectively. We pick up the pieces multilinear in $\epsilon$, $\bar{\epsilon}$ and $\epsilon$, $\bar{\epsilon}$, then replace these polarization vectors by the polarization tensor of the vertex operator. $\lambda'_i = i \lambda_i$ and $\tilde{\lambda}'_i = i \tilde{\lambda}_i$ (i = 1...5) are vectors in the weight lattice [6] of the left- and right-moving sectors respectively. $q$ and $\bar{q}$ are the $\gamma$ ghost number in the left- and right-moving sectors respectively. We use $\circ$ to denote the inner product in the five dimensional weight space and use $\cdot$ to denote the inner product in the space-time. The mass-shell condition for the vertex is

$$m_r^2 = \frac{2}{\alpha'}(\lambda_r \circ \tilde{\lambda}_r - q_r^2 - 2\bar{q}_r - 2 + 2\bar{n}_r + 2\tilde{m}_r) = \frac{2}{\alpha'}(\lambda_r \circ \lambda_r - q_r^2 - 2q_r - 2 + 2n_r + 2m_r), \tag{2.5}$$

where the level match condition have been used. To get the vertex operators in bosonic string theory, we just need to set $\lambda = \tilde{\lambda} = q = \bar{q} = \epsilon^i = \bar{\epsilon}^i = 0$ and use the 26-dimensional space-time instead of the 10-dimensional space-time.

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The normal ordering:: of an operator $O$ is defined as : $O := O^{(+)}O^{(-)}\tilde{O}^{(+)}\tilde{O}^{(-)}O^{(0)}$. $O$ and $\tilde{O}$ correspond to the left- and right-moving modes of the operator respectively. $(+)$, $(-)$ and $(0)$ correspond to the creation modes, annihilation modes and zero mode respectively. In $O^{(0)}$, we consider $x$ as creation operator and $p$ as annihilation operator, then in the normal ordered operator, $x$ must be on the left of $p$. Physical vertices containing higher derivatives can be transformed into the vertices with only first derivatives. In fact we can do partial integrals to reduce the order of the derivatives. After the integrals on the world-sheet, the surface terms turn to zero. Redefine the polarization tensor, the vertices then turn to those only contain first derivatives.
We use the boundary state to calculate the correct correlation functions on $D_2$. The boundary state for the scalar field is given in [7]. In the bosonized form, we have three kinds of scalar field. They are $X_\mu$, $\phi_i(i = 1...5)$ and $\phi_6$. Then the boundary state is

$$|B\rangle = |B_X\rangle \otimes |B_\phi\rangle \otimes |B_{\phi_6}\rangle$$

$$= \exp\left(\sum_{n=1}^{\infty} a_n^\dagger \cdot \tilde{a}_n^\dagger\right) |0\rangle_X \otimes \exp\left(\sum_{n=1}^{\infty} b_n^\dagger \cdot \tilde{b}_n^\dagger\right) |0\rangle_\phi \otimes \exp\left(\sum_{n=1}^{\infty} c_n^\dagger \cdot \tilde{c}_n^\dagger\right) |0\rangle_{\phi_6}$$

$$= B |0\rangle.$$

Here $a^\dagger$, $c^\dagger$ and $\tilde{a}^\dagger$ are creation operators corresponding to $\phi_i$, $\phi_6$ and the holomorphic modes of $X_\mu$. $b^\dagger$, $\tilde{c}^\dagger$ and $\tilde{a}^\dagger$ are creation operators corresponding to $\tilde{\phi}_i$, $\tilde{\phi}_6$ and the antiholomorphic modes of $X_\mu$. $B \equiv B_X \otimes B_\phi \otimes B_{\phi_6}$ and $|0\rangle \equiv |0\rangle_X \otimes |0\rangle_\phi \otimes |0\rangle_{\phi_6}$. Notice that all the boundary operators are constructed by creation operators, we move the boundary operators to the left of all the vertex operators. The operator $\mathcal{V}(z_r, \tilde{z}_r)$ in (2.4) is normal ordered. So it has the form $\mathcal{V}^+(z_r) \mathcal{V}^-(z_r) \mathcal{V}^+(\tilde{z}_r) \mathcal{V}^-(\tilde{z}_r) \mathcal{V}^+(0)$. The boundary operators commute with $\mathcal{V}^+$, $\mathcal{V}^- \mathcal{V}^+$ and $\mathcal{V}^0$ but not commute with $\mathcal{V}^- \mathcal{V}^-$ and $\mathcal{V}^0$. In fact, we have

$$\mathcal{V}^-(z)B$$

$$= \exp (q \phi_6^-) \exp (i \lambda \circ \phi^- + i \sum_{i=1}^{m} \varepsilon^i \circ \partial \phi_i^-) \exp (ik \cdot X^- + i \sum_{i=1}^{n} \varepsilon^i \cdot \partial X^-)(z)B$$

$$= B \mathcal{V}^-(z) \mathcal{V}^+(\frac{1}{z}).$$

$\mathcal{V}^+(\frac{1}{z})$ is defined as

$$\mathcal{V}^+(\frac{1}{z}) \equiv \exp (q \phi_6^+)$$

$$\times \exp (i \lambda \circ \phi^+ + i \sum_{i=1}^{m} \tilde{\varepsilon}^i \circ \partial \phi_i^+) \times \exp (ik \cdot \tilde{X}^+ + i \sum_{i=1}^{n} \tilde{\varepsilon}^i \cdot \partial \tilde{X}^+)(\frac{1}{z}),$$

where $\partial = \frac{\partial}{\partial z}$. In a similar way we have

$$\mathcal{V}^-(\frac{1}{z})B = B \mathcal{V}^-(\frac{1}{z}) \mathcal{V}^+(\frac{1}{z}).$$

$\mathcal{V}^+(\frac{1}{z})$ is defined as

$$\mathcal{V}^+(\frac{1}{z}) \equiv \exp (q \tilde{\phi}_6^+)$$

$$\times \exp (i \tilde{\lambda} \circ \phi^+ + i \sum_{i=1}^{m} \tilde{\varepsilon}^i \circ \partial \phi_i^+) \times \exp (ik \cdot \tilde{X}^+ + i \sum_{i=1}^{n} \tilde{\varepsilon}^i \cdot \partial \tilde{X}^+)(\frac{1}{z}),$$

$$\mathcal{V}^-(\frac{1}{z}) = B \mathcal{V}^-(\frac{1}{z}) \mathcal{V}^+(\frac{1}{z}).$$
where \( \partial = \frac{\partial}{\partial z} \). Then after moving B to the left of all the vertex operators, the creation operators in B annihilate the state \(|0\rangle \). The correlation function of \( \mathcal{V}(z_r, \bar{z}_r) \) then becomes

\[
\langle 0 | \mathcal{V}^{(+)}(z_N) \mathcal{V}^{(-)}(z_N) \hat{\mathcal{V}}^{(+)} \left( \frac{1}{z_N} \right) \hat{\mathcal{V}}^{(-)}(\bar{z}_N) \mathcal{V}^{(-)}(z_N) \langle 0 | \mathcal{V}^{(+)}(\bar{z}_N) \hat{\mathcal{V}}^{(+)} \left( \frac{1}{\bar{z}_N} \right) \hat{\mathcal{V}}^{(-)}(z_N) \mathcal{V}^{(-)}(z_N) \rangle (2.11)
\]

... \( \mathcal{V}^{(+)}(z_1) \mathcal{V}^{(-)}(z_1) \hat{\mathcal{V}}^{(+)} \left( \frac{1}{z_1} \right) \hat{\mathcal{V}}^{(-)}(\bar{z}_1) \mathcal{V}^{(-)}(z_1) \langle 0 | \mathcal{V}^{(+)}(\bar{z}_1) \hat{\mathcal{V}}^{(+)} \left( \frac{1}{\bar{z}_1} \right) \hat{\mathcal{V}}^{(-)}(z_1) \mathcal{V}^{(-)}(z_1) \rangle \).

The operators \( \hat{\mathcal{V}}^{(+)} \left( \frac{1}{z} \right) \) and \( \hat{\mathcal{V}}^{(+)} \left( \frac{1}{\bar{z}} \right) \) are the “images” of the left-and right-moving modes of the operator \( \mathcal{V}(z_r, \bar{z}_r) \). We move all the annihilation modes to the right of all the creation modes. Particularly, if we move \( \mathcal{V}^{(-)}(z_r) \) to the right of \( \mathcal{V}^{(+)}(z_s) \), then we get the interactions in the left-moving sector. If we move \( \hat{\mathcal{V}}^{(-)}(\bar{z}_r) \) to the right of \( \hat{\mathcal{V}}^{(+)}(\bar{z}_s) \), we get the interactions in the right-moving sector. The interactions between the two sectors can be derived by moving \( \mathcal{V}^{(-)}(z_r) \) and \( \hat{\mathcal{V}}^{(-)}(\bar{z}_r) \) to the right of \( \hat{\mathcal{V}}^{(+)} \left( \frac{1}{z_r} \right) \) and \( \mathcal{V}^{(+)} \left( \frac{1}{\bar{z}_r} \right) \) respectively. After moving all the annihilation modes to the left of all the creation modes, we use the annihilation operators to annihilate the state \(|0\rangle \) and use the creation operators to annihilate the state \(|0\rangle \). Zero modes give the constraints on momentum, fermion number and the superghost number. They also contribute to the interactions. Thus we get the correlation function. To get the amplitude, we must integral over the unit disk, then divide the integrals by the conformal Killing volume. The conformal Killing volume has different equivalent forms. We will give them in the appendix. Here we use the volume(\(A.6\)). It can be used to fix a complex coordinate. We use the conformal Killing volume to fix \( z_1 \):

\[
z_1 = z_0.
\]

At last we get the general form of the amplitude for \( N \) closed strings on \( D_2 \):

\[
\mathcal{A}^N_{D_2} = \kappa^{-1} \int_{|z| < 1} \prod_{i=1}^{N} d^2 z_i \frac{|1 - z_0 \bar{z}_0|^2}{2\pi d^2 z_0} \times \prod_{s>r} (z_s - z_r)^{\frac{N}{2}} \exp \left[ \sum_{i=1}^{n_r} \sum_{j=1}^{m_r} \left( -\alpha_r^i \right) \epsilon_r^i \cdot \tilde{\epsilon}_r^j - \sum_{i=1}^{\tilde{m}_r} \sum_{j=1}^{\tilde{m}_r} \tilde{\epsilon}_r^i \cdot \tilde{\epsilon}_r^j \right] \left( (z_r - z_s - 1) \right)^{\frac{N}{2}} \times \exp \left[ \sum_{s>r} \left( \sum_{i=1}^{n_s} \sum_{j=1}^{m_s} \left( -\alpha_s^i \right) \epsilon_s^i \cdot \epsilon_s^j - \sum_{i=1}^{\tilde{m}_s} \sum_{j=1}^{\tilde{m}_s} \tilde{\epsilon}_s^i \cdot \tilde{\epsilon}_s^j \right) \left( 1 - |z_s|^2 \right)^{-2} c.c. \right] \times \exp \left[ - \sum_{s>r} \left( \sum_{i=1}^{n_s} \sum_{j=1}^{m_s} \left( -\alpha_s^i \right) \epsilon_s^i \cdot \epsilon_s^j - \sum_{i=1}^{\tilde{m}_s} \sum_{j=1}^{\tilde{m}_s} \tilde{\epsilon}_s^i \cdot \tilde{\epsilon}_s^j \right) \left( (z_s - z_r - 1) \right)^{-2} c.c. \right] \times \exp \left[ \sum_{r \neq s} \left[ \sum_{i=1}^{n_s} \left( -\alpha_s^i \right) k_r \cdot \epsilon_s^i \cdot \epsilon_s^j - \sum_{i=1}^{\tilde{m}_s} \epsilon_s^i \cdot \tilde{\epsilon}_s^j \right) \left[ (z_r - z_s - 1) + (z_r - z_s) \right]^{-1} c.c. \right]
\]

(2.13)
\[
\times \exp \sum_{r=1}^{N} \left[ \left( - \frac{\alpha'}{2} \right) k_r \cdot \sum_{i=1}^{n_r} \epsilon_r^{(i)} - \lambda_r \cdot \sum_{i=1}^{m_r} \varepsilon_r^{(i)} \right] ((\bar{z}_r - 1 - z_r)^{-1} + c.c. \right]_{\text{multilinear},}
\]

where we have \( \sum_{r=1}^{N} \lambda_r = \sum_{r=1}^{N} \bar{\lambda}_r = 0, \sum_{r=1}^{N} k_r = 0 \) and \( \sum_{r=1}^{N} (q_r + \tilde{q}_r) = -2 \) correspond to the conservation of fermion number, the conservation of momentum and the fact that background superghost number is \(-2\).

The vertex operators \([2,3]\) are conformal invariant. They are also preserved by \( z \leftrightarrow \bar{z} \). Then we have
\[
\int d^2 z_r \mathcal{Y}(z_r, \bar{z}_r) |B\rangle = \int d^2 \left( \frac{1}{z_r} \right) \mathcal{Y}(\frac{1}{z_r}, \frac{1}{\bar{z}_r}) |B\rangle.
\]

Then the integrals over the unit disk is equal to the integrals over the region \(|z| > 1\). So we can use:
\[
\frac{1}{2} \int_{\mathbb{C}} d^2 z_r \mathcal{Y}(z_r, \bar{z}_r)
\]

instead of the integrals over \(|z| < 1\). Then the amplitude can be given by integrals over the complex plane, each integral should be multiplied by a factor \( \frac{1}{2} \). For any \( z_r = x_r + y_r \), the \( z_r \) integral can be given by \( \int_{-\infty}^{\infty} dx_r \int_{-\infty}^{\infty} dy_r \). We then follow the same steps as in \([3]\). We rotate the contour of the \( y \) integrals along the real axis to pure imaginary axis. The fixed point should be transformed simultaneously to guarantee the conformal invariance. Define the new variables:
\[
\xi_1 = \xi_o = x_o + iy_o, \eta_1 = \eta_o = x_o - iy_o,
\]
\[
\xi_r \equiv x_r + iy_r, \eta_r \equiv x_r - iy_r.
\]

Then the integrals become real integrals. The interactions in one sector can be considered as interactions between open strings. However, the interactions between left- and right-moving modes looks like those between open strings inserted at \( \xi_r \) and \( (\eta_r)^{-1} \). \( \eta_r \) is the coordinate of the right-moving open string. \( \eta_r^{-1} \) can be considered as a time reverse in the right-moving setor, then the interactions between the two sectors can be regarded as interactions between left- and right-moving open strings with a time reverse in the right moving sector (see fig \([1] (b)\)). We replace all the \( \eta_r^{-1} \) by \( \eta_r \), by using the mass-shell condition which is determined by the conformal invariance in one sector, the interactions between the two sectors as well as the interactions in one sector become those between open strings. Define
\[
\xi_{r+N} \equiv \eta_r, k_{r+N} \equiv k_r, \lambda_{r+N} \equiv \lambda_r, \tilde{\xi}_{r+N} \equiv \epsilon_r, \tilde{\xi}_{r+N} \equiv \varepsilon_r.
\]

After the simultaneous transformations, the volume of CKG becomes \( \frac{1}{2\pi} \int \frac{d\xi d\eta}{(\xi - \eta)^2} \). The fixed points become \( \xi_1 = \xi_o \) and \( \xi_{1+N} = \xi_o \). The conformal Killing volume has another form \( \int \frac{dx_a dx_b dx_c}{|x_a - x_6||x_b - x_6||x_c - x_6|} \), it can be used to fix three real variables. We reset the fixed points at:
\[
\xi_1 = x_a = 0, \xi_2 = x_b = 1, \xi_{2N} = x_c = \infty.
\]
The amplitude for $2N$ closed strings on $D_2$ then becomes

$$A_{D_2}^N = k^{-N} \left( \frac{i}{4} \right)^{N-1} \int \prod_{l=1}^{2N} d\xi_l \frac{\xi_a - \xi_b}{d\xi_a d\xi_b} \frac{\xi_c - \xi_d}{d\xi_c d\xi_d} \prod_{s>r} |\xi_s - \xi_r|^2 k_r k_s (\xi_s - \xi_r)^{\lambda_s + \lambda_s + q_s - q_s}$$

$$\times \exp \left[ \sum_{s>r} \left( \sum_{l=1}^{n_s} \sum_{l=1}^{n_r} (2\alpha') \epsilon^{(i)}_r \cdot \epsilon^{(j)}_s + \sum_{l=1}^{m_s} \sum_{l=1}^{m_r} \epsilon^{(i)}_r \circ \epsilon^{(j)}_s \right) (\xi_s - \xi_r)^{-2} \right]$$

$$\times \exp \left[ \sum_{r \neq s} \left( \sum_{l=1}^{n_s} (-2\alpha') k_r \cdot \epsilon^{(i)}_s - \sum_{l=1}^{m_s} \lambda_r \circ \epsilon^{(i)}_s \right) (x_s - x_r)^{-1} \right]_{\text{multilinear}} e^{i \pi \Theta(P)},$$

where we have absorbed a factor $\frac{1}{2}$ into each $\epsilon$. $\Theta(P)$ is defined as

$$\Theta(P) = \sum_{s>r} 2\alpha' k_r' \cdot k_s' \theta(\xi_s - \xi_r),$$

where $k_r' = \frac{1}{2} k_r$ is the momentum of the open string and

$$\theta(\xi_s - \xi_r) = \begin{cases} 0(\xi_s > \xi_r) \\ 1(\xi_s < \xi_r). \end{cases}$$

An open string tree amplitude for $M$ bosonized vertices has the form

$$M_{D_2}^N = (g)^{M-2} \int \prod_{l=1}^{2M} dx_l \frac{|x_a - x_b| |x_c - x_d|}{dx_a dx_b dx_c dx_d} \prod_{s>r} |x_s - x_r|^2 k_r k_s (x_s - x_r)^{\lambda_s + \lambda_s + q_s - q_s}$$

$$\times \exp \left[ \sum_{s>r} \left( \sum_{l=1}^{n_s} \sum_{l=1}^{n_r} (2\alpha') \epsilon^{(i)}_r \cdot \epsilon^{(j)}_s + \sum_{l=1}^{m_s} \sum_{l=1}^{m_r} \epsilon^{(i)}_r \circ \epsilon^{(j)}_s \right) (x_s - x_r)^{-2} \right]$$

$$\times \exp \left[ \sum_{r \neq s} \left( \sum_{l=1}^{n_s} (-2\alpha') k_r \cdot \epsilon^{(i)}_s - \sum_{l=1}^{m_s} \lambda_r \circ \epsilon^{(i)}_s \right) (x_s - x_r)^{-1} \right]_{\text{multilinear}},$$

where $g$ is the coupling constant for open strings. Open string coupling constant can be related with closed string coupling constant by $\kappa \sim g^2$. From (2.22) and (2.19) we can see amplitudes for $N$ closed strings on $D_2$ can be given by one open string tree amplitude for $2N$ open strings except for a phase factor. The phase factor is used to guarantee the integrals in the right branch cut. It only depend on the the orderings of the open strings. For a certain order, the phase factor decouple from the integrals. So we can break the integrals into pieces, then we get the relation between closed string amplitudes and partial amplitudes for open strings on $D_2$:

$$A_{D_2}^{(N)} = \kappa^{-N} \epsilon_{\alpha \beta} \frac{1}{A_{D_2}^{(N)}} \frac{1}{2} \int \prod_{l=1}^{2N} dx_l \frac{|x_a - x_b| |x_c - x_d|}{dx_a dx_b dx_c dx_d} \prod_{s>r} |x_s - x_r|^2 k_r k_s (x_s - x_r)^{\lambda_s + \lambda_s + q_s - q_s}$$

$$\times \exp \left[ \sum_{s>r} \left( \sum_{l=1}^{n_s} \sum_{l=1}^{n_r} (2\alpha') \epsilon^{(i)}_r \cdot \epsilon^{(j)}_s + \sum_{l=1}^{m_s} \sum_{l=1}^{m_r} \epsilon^{(i)}_r \circ \epsilon^{(j)}_s \right) (x_s - x_r)^{-2} \right]$$

$$\times \exp \left[ \sum_{r \neq s} \left( \sum_{l=1}^{n_s} (-2\alpha') k_r \cdot \epsilon^{(i)}_s - \sum_{l=1}^{m_s} \lambda_r \circ \epsilon^{(i)}_s \right) (x_s - x_r)^{-1} \right]_{\text{multilinear}},$$

(2.23)

$^3$Here $\mathcal{M}$ is the open string amplitude without the coupling constant $g$. 

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- 10 –
If there are open strings on the boundary of $D_2$, we can insert the open string vertices into the amplitude $A_{D_2}^{(N,M)}$. Because (2.19) is already an amplitude for open strings on the real axis except for a phase factor, we just increase the number of the open strings on the boundary of $D_2$ and adjust the phase factor to make the integrals in the right branch cuts. The phase factor should be adjusted because we must consider the interactions between closed and open strings. Then we have

$$A_{D_2}^{(N,M)} = \epsilon^{\alpha\beta\gamma} A_{D_2}^{(N,M)} \alpha\beta\gamma = \left( \frac{i}{4} \right)^{N-1} \kappa^{-N-1} \gamma^M \epsilon^{\alpha\beta\gamma} \sum_P \mathcal{M}^{(2N,M)} \alpha\beta\gamma (P) e^{i\pi \Theta^\prime (P)}, \quad (2.24)$$

where we have defined the coordinates of the left-moving open strings are $\xi_1, ..., \xi_N$, those of right moving open strings are $\xi_{1+N+M}, ..., \xi_{2N+M}$ and the coordinates of other open strings are $\xi_{1+N}, ..., \xi_{M+N}$.

$$\Theta^\prime (P) = \sum_{s>r} 2\alpha^l k^l_s : k^l_r \theta^l (\xi_s - \xi_r), \quad (2.25)$$

where $k^l_r$ are the momentums of the open strings. If $\xi_s > \xi_r$, $\theta^l (\xi_s - \xi_r) = 0$, else if $\xi_s < \xi_r$ but $N < s, r < N + M + 1$, $\theta^l (\xi_s - \xi_r) = 0$, otherwise $\theta^l (\xi_s - \xi_r) = 1$. This relation can also be derived by choosing the fundamental region as the upper half-plane, then repeat the similar steps in the case of $N$ closed strings on $D_2$. We can see if $M = 0$, (2.24) gives the relation for $N$ closed strings on $D_2$ (2.23) and if $N = 0$ it gives the open string tree amplitude (2.22).

It is not surprising the KLT factorization (1.2) relations do not hold on $D_2$. On $S_2$ the left- and right-moving sectors are independent of each other. Then we have the factorization relations on $S_2$. $D_2$ has a boundary. The boundary connect the left- and right-moving sectors into a single sector. In fact, the left-(right-) moving waves must be reflected at the boundary. The reflection waves of the left-(right-) moving waves turn into the right-(left-) moving waves. We can see this from the modes expansion. The left-moving sector can be expanded as a sum of $\frac{\alpha_n}{n}$. The reflection waves of the left-moving wave can be given as a sum of $\frac{\beta_{\bar{z}}}{n}$, where $\frac{1}{2}$ is the equivalent point of $z$. Then the reflection waves are in the right-moving sector. In general, two $\alpha$’s may not commute with each other, then there must be interactions between the left- and the right-moving sectors. The right-moving waves also interact with their reflection waves in the same way.

After the contour deformation, in the real coordinates, the amplitude is already an amplitude for open strings. However, the interactions between the open strings from the two different sectors looks like the interactions between left-moving open strings and the right moving open strings with a time reverse. We reverse the time in the right sector to get the ordinary form of open string amplitude. In this step, we have used the mass-shell condition. In fact, the mass-shell condition is determined by the conformal invariance in one sector, then it is the conformal invariance in one sector guarantee the relations.

We can consider the relations on $D_2$ as any closed strings can be split into two open strings. Each open string catch half of the momentum of the closed string. Move the open strings corresponding to the two sectors of closed strings onto the boundary of $D_2$. Then
an amplitude for $N$ closed strings and $2M$ open strings on $D_2$ is given by an amplitude for $N + 2M$ open strings.

In the low energy limit of an open string theory, gravitons are closed strings and gauge particles are open strings. Then in this case, the KLT factorization relations do not hold. We should use one amplitude for $2N$ gauge particles instead of the product of two amplitudes for $N$ gauge particles to give an amplitude for $N$ gravitons.

3. Relations between amplitudes on $RP_2$ and open string tree amplitudes

Real projective plane($RP_2$) is an unoriented surface. It can be considered as a sphere with a crosscap. $RP_2$ can be derived by identifying the diametrically opposite points on $S_2$ [1]. The $\mathbb{Z}_2$ equivalence becomes $z \leftrightarrow -\frac{1}{z}$. Though there is no boundary in this case, we can also choose the unit disk as the fundamental region. The CKG then becomes

$$z' = \frac{e^{\alpha}z - \tan \gamma e^{-i\beta}}{\tan \gamma e^{i\beta}z + e^{-i\alpha}}, \quad (3.1)$$

where $\alpha, \beta, \gamma$ are three real parameters.

The amplitude for $N$ closed strings on $RP_2$ is

$$\mathcal{A}_{RP_2}^N = \kappa^{N-1} \frac{1}{V_{CKG}^{RP_2}} \left( \int_{|z_N|<1} d^2z_N \mathcal{V}(z_N, \bar{z}_N) \ldots \int_{|z_1|<1} d^2z_1 \mathcal{V}(z_1, \bar{z}_1) \right)_{RP_2}, \quad (3.2)$$

where $V_{CKG}^{RP_2}$ is the volume of the conformal Killing group of $RP_2$, and $\langle \rangle_{RP_2}$ is the correlation function on $RP_2$. The background $\gamma$ ghost number is also $-2$ on $RP_2$. We must choose appropriate vertices to make the total $\gamma$ ghost number $-2$. In this case, we can also choose the vertices whose left-moving sector has same $\gamma$ ghost number with right-moving sector as in $D_2$ case. In the bosonized form, the boundary state for $RP_2$ is [7]

$$|C\rangle = |C_X\rangle \otimes |C_\phi\rangle \otimes |C_{\phi_6}\rangle$$

$$= \exp \left( \sum_{n=1}^\infty (-1)^n a_n^\dagger \cdot a_n^\dagger \right) |0\rangle_X \otimes \exp \left( \sum_{n=1}^\infty (-1)^n b_n^\dagger \circ b_n^\dagger \right) |0\rangle_\phi \otimes \exp \left( \sum_{n=1}^\infty (-1)^n c_n^\dagger \circ c_n^\dagger \right) |0\rangle_{\phi_6}$$

$$= C |0\rangle, \quad (3.3)$$

where $C \equiv \exp \left( \sum_{n=1}^\infty (-1)^n a_n^\dagger \cdot a_n^\dagger \right) \otimes \exp \left( \sum_{n=1}^\infty (-1)^n b_n^\dagger \circ b_n^\dagger \right) \otimes \exp \left( \sum_{n=1}^\infty (-1)^n c_n^\dagger \circ c_n^\dagger \right)$. Substitute this boundary state and the bosonized vertices into The amplitude on $RP_2$. As we have done in the case of $D_2$, move the boundary operator to the left of all the vertex operators, then move all the annihilation modes to the right of the creation modes. Integral over the unit disk. The conformal Killing volume(A.13) can be used to fix a complex coordinate. We can set $z_1 = z_0$. Then we have the amplitude for $N$ closed strings on $RP_2$:

$$\mathcal{A}_{RP_2}^N = \kappa^{N-1} \int_{|z|<1} \prod_{i=1}^N d^2z_i \frac{|1 + z_0 \bar{z}_0|^2}{2\pi d^2z_0}$$
\[
\prod_{s>r}(z_s - z_r)^{\frac{1}{2}}k_rk_s + \lambda_r\lambda_s - q_rq_s (\bar{z}_r - \bar{z}_s) \prod_{r<s}(1 + (z_r\bar{z}_s)^{-1})^{\frac{1}{2}}k_rk_s + \lambda_r\lambda_s - q_rq_s
\]
\[
\times \exp \left[ N \sum_{s>r} \left( \sum_{i=1}^{n_r} \sum_{j=1}^{n_s} \left( -\frac{\alpha'}{2} e_r \cdot \bar{e}_s - \sum_{i=1}^{m_r} \sum_{j=1}^{m_s} e_r \cdot \bar{e}_s \right) \right) \right] (1 + |z_r|^2)^{-2}
\]
\[
\times \exp \left[ \sum_{s>r} \left( \sum_{i=1}^{n_r} \sum_{j=1}^{n_s} \left( -\frac{\alpha'}{2} e_r \cdot \bar{e}_s - \sum_{i=1}^{m_r} \sum_{j=1}^{m_s} e_r \cdot \bar{e}_s \right) \right) \right] \left( z_s - z_r \right)^{-2} + c.c.
\]
\[
\times \exp \left[ -\sum_{s>r} \left( \sum_{i=1}^{n_r} \sum_{j=1}^{n_s} \left( -\frac{\alpha'}{2} e_r \cdot \bar{e}_s - \sum_{i=1}^{m_r} \sum_{j=1}^{m_s} e_r \cdot \bar{e}_s \right) \right) \right] \left( z_s - z_r \right)^{-2} + c.c.
\]
\[
\times \exp \left[ \sum_{r\neq s} \left( \sum_{i=1}^{n_r} \left( -\frac{\alpha'}{2} k_r \cdot e_i + \sum_{i=1}^{m_r} \lambda_r \cdot \epsilon_i \right) \right) \right] \left( \bar{z}_r - z_s \right)^{-1} + \left( -\bar{z}_r - 1 - z_s \right)^{-1} + c.c.
\]
\[
\times \exp \left[ \sum_{r=1}^{N} \left( \left( -\frac{\alpha'}{2} k_r \cdot e_i - \sum_{i=1}^{m_r} \lambda_r \cdot \epsilon_i \right) \right) \right] \left( -\bar{z}_r - 1 - z_r \right)^{-1} + z_r^{-1} + c.c. \right]_{\text{multilinear}},
\]
where we have \( \sum_{r=1}^{N} \lambda_r = \sum_{r=1}^{N} \tilde{\lambda}_r = 0 \), \( \sum_{r=1}^{N} k_r = 0 \) and \( \sum_{r=1}^{N} (q_r + \tilde{q}_r) = -2 \).

The vertex is preserved by conformal transformation and a transformation \( z \leftrightarrow \bar{z} \). Then we have
\[
\int d^2z \mathcal{V}(z, \bar{z}) \langle C \rangle = \int d^2 \left( \frac{1}{z} \right) \mathcal{V}(\frac{1}{\bar{z}}, \frac{1}{\bar{z}}) \langle C \rangle.
\]

Then the integrals on the unit disk can be replaced by integrals on the whole complex plane, each of them should be multiplied by a factor \( \frac{1}{2} \). We deform the integral contour, redefine the variables as we have done in the case of \( D_2 \). Then the amplitude is given by real integrals. The interactions in one sector are the open string interactions. The interaction between left- and right-moving sectors can be considered as interactions between open strings inserted at \( \xi_r \) and \( (1 - \eta_s)^{-1} \). \( \frac{1}{\eta_s} \) can be considered as a time reverse and a twist in the right-moving sector. Then the interactions between left- and right-moving sectors can be regarded as interactions between left- and right-moving open strings with a time reverse and a twist in the right-moving sector. We replace all the \( \eta_r \) by \( \frac{1}{\eta_r} \) and redefine the variables in the right-moving sector by Eq. (3.17). Then by using the mass-shell condition, the interactions between the two different sectors as well as in one sector become the interactions between open strings. The fixed point become two fixed points on the real axis. When we transform the complex variables into real variables, the variables in the conformal Killing volume must be transformed simultaneously to guarantee the conformal invariance. The volume of CKG becomes \( \frac{1}{2\pi} \int \frac{d\xi d\eta}{(\xi - \eta)^2} \) again. Then we can use \( \int \frac{d x_a d x_b d x_c}{|x_a - x_b||x_b - x_c||x_c - x_a|} \) instead. This volume can be used to fix three points as in the case of \( D_2 \):
\[
\xi_1 = x_a = 0, \xi_2 = x_b = 1, \xi_{2N} = x_c = \infty.
\]
The amplitude on \( RP_2 \) then becomes

\[
\mathcal{A}^N_{RP_2} = \left( \frac{i}{4} \right)^{N-1} \kappa^{N-1} \prod_{i=1}^{2N} d\xi_i \frac{|\xi_a - \xi_b| |\xi_b - \xi_c| |\xi_c - \xi_a|}{d\xi_a d\xi_b d\xi_c} \prod_{s>r} (|\xi_s - \xi_r|^2 k_r k_s (\xi_s - \xi_r) \lambda_r \lambda_s - q_r q_s
\times \exp \left[ \sum_{s>r} \left( \sum_{i=1}^{n_r} \sum_{j=1}^{n_s} (2\alpha') \epsilon^{(i)}_r \cdot \epsilon^{(j)}_s + \sum_{i=1}^{m_r} \sum_{j=1}^{m_s} \epsilon^{(i)}_r \circ \epsilon^{(j)}_s \right) (\xi_s - \xi_r)^{-2} \right]
\times \exp \left[ \sum_{s' \neq s} \left( \sum_{i=1}^{n_s} (-2\alpha') k_r \cdot \epsilon^{(i)}_s - \sum_{i=1}^{m_s} \lambda_r \circ \epsilon^{(i)}_s \right) (\xi_r - \xi_s)^{-1} \right] \bigg|_{\text{multilinear} e^{i\pi \Theta(P)}}.
\]

(3.7)

We can see this amplitude is just an amplitude for \( 2N \) open strings except for a phase factor. The phase factor only depend on the ordering of the open strings. Then we can break the integrals into pieces as in the case of \( D_2 \). Then Eq.(3.7) becomes

\[
\mathcal{A}^{(N)}_{RP_2} = \epsilon_{\alpha\beta} \mathcal{A}^{(N)\alpha\beta}_{RP_2} = \left( \frac{i}{4} \right)^{N-1} \kappa^{N-1} \epsilon_{\alpha\beta} \sum_{P} \mathbb{A}^{(2N)\alpha\beta}(P) e^{i\pi \Theta(P)}.
\]

(3.8)

As in the case of \( D_2 \), KLT factorization relations(1.2) do not hold on \( RP_2 \). Though \( RP_2 \) has no boundary, it can also be considered as a sphere with a \( \mathbb{Z}_2 \) equivalence. Then the left-(right-)moving waves are reflected at the crosscap and turn into the right-(left-)moving waves. The reflection waves of the left-(right-)moving waves in this case can be given as a sum of \( \left(\frac{a}{-2}\right)^n \left(\frac{a}{-2}\right)^n \). Then the interactions between the left-(right)moving waves and their reflection waves are just those between the two sectors.

From the relations (2.23) and (3.8) we can see, the amplitudes on \( D_2 \) and \( RP_2 \) with same external closed string states are equal. In fact, after we transform the complex variables into real ones, the image of a point \( \xi_r \) in the left-moving sector becomes \( \xi_r \) on \( D_2 \) and \( -\frac{1}{\eta_r} \) on \( RP_2 \). The minus means a twist in the right sector. Then the amplitude on \( D_2 \) can be derived from \( RP_2 \) by this twist(see Fig.1). The amplitude on \( D_2 \) is preserved by this twist in the right sector, so the amplitudes on \( D_2 \) and \( RP_2 \) with same external closed strings states are equal.

In the low energy limit of an unoriented string theory, the amplitudes for closed strings on \( RP_2 \) contribute to the amplitudes for gravitons. Then the KLT factorization relations do not hold in this case as in the case of \( D_2 \). The amplitudes for \( N \) gravitons can not be factorized by two amplitudes for \( N \) gauge particles. They can be given by an amplitude for \( 2N \) gauge particles.

4. Reduction of the terms in the relations

Though there are a lot of orderings of the open string external lines in the relations, if we consider the integrals as contour integrals, the relations can be reduced to sum over only a few terms. In this section, we give some examples to show how to reduce the terms in
the relations\(^4\). Because the relation is on \(D_2\) is same with that on \(RP_2\), we only need to consider the \(D_2\) case.

The first example is the amplitude for two closed strings on \(D_2\). There is only one integral in the amplitude:

\[
\int_{-\infty}^{\infty} d\xi_2 (\xi_2)^{\frac{\alpha'}{2} k_1 k_2} (1 - \xi_2)^{\frac{\alpha'}{2} k_2 k_3},
\]

where we have omitted the integer powers of \(\xi_i - \xi_s\) because the integer powers do not affect our discussions on the branch cuts. We can consider this integral as a contour integral along the real axis and around the infinite point, because the integral around the infinite point is zero. We then follow the similar steps as in [2]. We deform the contour to transform the integral along the real axis into integral from 1 to \(+\infty\). The integral becomes:

\[
2i \sin (\pi \frac{\alpha'}{2} k_2 k_3) \int_{1}^{\infty} d\xi_2 |\xi_2|^2 |1 - \xi_2|^2 |k_2 k_3|.
\]

The contour can also be deformed in another way. Then the integral is given as integral from \(-\infty\) to 0:

\[
2i \sin (\pi \frac{\alpha'}{2} k_2 k_1) \int_{-\infty}^{0} d\xi_2 |\xi_2|^2 |1 - \xi_2|^2 |k_2 k_3|.
\]

Then the relations on \(D_2\) can be given as

\[
A_{D_2}^2 = \kappa (-\frac{1}{2} \sin \frac{\pi \alpha'}{2} k_2 k_3) M(1, 3, 2, 4) = \kappa (-\frac{1}{2} \sin \pi \frac{\alpha'}{2} k_2 k_1) M(2, 1, 3, 4).
\]

The two expressions are equivalent.

The second example is three closed strings on \(D_2\). There are three integrals in this case. We first consider the \(\xi_2\) integral. The singularity at \(\xi_2 = 1\) should be avoided below the real axis and the singularities at \(\xi_3, \xi_4\) and \(\xi_5 = 1\) should be avoided above the axis. We can deform the \(\xi_2\) contour as in the first example to make the \(\xi_2\) integral from \(-\infty\) to 0. We also deform the \(\xi_4\) integral contour to make the \(\xi_4\) integral from 1 to \(\infty\). Then we deform the contour for \(\xi_3\) integral. \(\xi_3\) integral can be given as integral from 1 to \(\infty\). Then the relation becomes

\[
A_{D_2}^3 = i \kappa^2 \sin (\pi \frac{\alpha'}{-2} k_2 k_1) \sin (\pi \frac{\alpha'}{2} k_4 k_5)
\]

\[
\times \{ \sin (\pi \frac{\alpha'}{2} k_3 k_5) M(2, 1, 5, 3, 4, 6) + \sin (\pi \frac{\alpha'}{2} k_3 (k_4 + k_5)) M(2, 1, 5, 4, 3, 6) \}.\]

The \(\xi_3\) integral can also be given by integral from \(-\infty\) to 0, then the relations can be rewritten as

\[
A_{D_2}^3 = i \kappa^2 \sin (\pi \frac{\alpha'}{2} k_2 k_1) \sin (\pi \frac{\alpha'}{2} k_4 k_5)
\]

\[
\times \{ \sin (\pi \frac{\alpha'}{2} k_3 k_1) M(2, 3, 1, 5, 4, 6) + \sin (\pi \frac{\alpha'}{2} k_3 (k_1 + k_2)) M(3, 2, 1, 5, 4, 6) \}.\]

\(^4\)Though [5] have given some discussions and examples, the purpose of this paper is different. We focus on whether the factorization relations hold on \(D_2\) and \(RP_2\). We use a different method to reduce the terms.
These two expressions are equivalent. In general, the amplitude for closed strings on $D_2$ can be given as a sum of $(\frac{1}{2}(2N-4))!(\frac{1}{2}(2N-2))!$ terms. Each term correspond an ordering of the open strings.

The third example is the amplitude for 1 closed strings and 2 open strings. The integral is

$$\int_{-\infty}^{\infty} d\xi_2 (\xi_2)^{k_1} (1-\xi_2)^{k_3} k_2,$$

where we use $\xi_1 = 0$, $\xi_2$ to denote the two variables come from the closed string and $\xi_3 = 1, \xi_4 = \infty$ to denote the variables come from the open strings for convenience. Then the result is same with equation(4.4). In general, for $N \geq 2$, there are $(\frac{1}{2}(2N-4))!(\frac{1}{2}(2N-2))!M!$ terms in the relation.

Because the amplitude for $RP_2$ is equal to that for $D_2$ if they have same external states. Then the relation on $RP_2$ can be reduced in the same way. Then the number of terms can also be reduced to $(\frac{1}{2}(2N-4))!(\frac{1}{2}(2N-2))!$.

5. Conclusion

In this paper, we investigated the relations between closed and open strings on $D_2$ and $RP_2$. We have shown that the KLT factorization relations do not hold for these two topologies. The closed string amplitudes can not be factorized by tree amplitudes for left- and right-moving open strings. However, the two sectors are connected into a single sector. We can give the amplitudes with closed strings in these two cases by amplitudes in this single sector.

In the low energy limits of these two cases, we can not use KLT relations to factorize amplitudes for gravitons into products of two amplitudes for gauge particles. Interactions between the “left-” and the “right-” moving gauge fields connect the two amplitudes into one. Then an graviton amplitude in these two cases can be given by one amplitude for both left- and right-moving gauge particles.

The relations for other topologies have not been given. However, we expect there are also some relations between closed and open string amplitudes. If there are more boundaries and crosscaps on the world-sheet, the boundaries and the crosscaps also connect left- and the right-moving sectors, then in these cases, KLT factorization relations can not hold.

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A. The volume of CKG

Here we will give the conformal Killing volume of $D_2$ and $RP_2$. If we choose the unit disk as the fundamental region, the CKG on $D_2$ is(2.1). There are three real parameters in
this case. To calculate the volume of the CKG on $D_2$, we can fix one point by dividing an invariant volume of a subgroup of the CKG. This subgroup has two real parameters. Then the point is preserved by a one-parameter subgroup left $[1, 8]$. We then calculate the volume of the one-parameter subgroup. Then the volume of CKG is derived. The invariance volume of the two-parameter subgroup on $D_2$ is

$$\int \frac{d^2 z}{|1 - z\bar{z}|^2}. \quad (A.1)$$

We can use this volume to fix one point. After dividing this volume, a one-parameter subgroup which preserve the fixed point left satisfies

$$z_0 = \frac{e^{i\alpha} \cosh \gamma z_0 + e^{-i\beta} \sinh \gamma}{e^{i\beta} \sinh \gamma z_0 + e^{-i\alpha} \cosh \gamma}. \quad (A.2)$$

Then we get the one-parameter subgroup:

$$z' = \frac{z + it \left( z - \frac{2z_0}{1 + |z_0|^2} \right)}{1 - it \left( 1 - z \frac{2z_0}{1 + |z_0|^2} \right)}, \quad (A.3)$$

where $t = \tan \alpha$. The invariant volume element described by $z'$ is

$$\int \frac{d^2 z'}{|1 - z'\bar{z}'|^2} = \int \frac{2d\tau}{1 + \tau^2} 2i\mathbb{R} \left( \frac{(\bar{z} - \bar{z}_0)(1 - z_0\bar{z})}{(1 - |z_0|^2)(1 - |z|^2)^2} \right) dz. \quad (A.4)$$

Because $2i\mathbb{R} \left( \frac{(\bar{z} - \bar{z}_0)(1 - z_0\bar{z})}{(1 - |z_0|^2)(1 - |z|^2)^2} \right) dz$ is the conformal invariant, then the volume for the one-parameter subgroup is

$$\int_{-\infty}^{+\infty} \frac{2d\tau}{1 + \tau^2} = 2\pi. \quad (A.5)$$

Then the conformal Killing volume on $D_2$ is

$$2\pi \int \frac{d^2 z_0}{|1 - z_0\bar{z}_0|^2}. \quad (A.6)$$

If we choose the upper half-plane as the fundamental region. We can use the volume

$$2\pi \int \frac{dx_adx_b}{|x_a - x_b|^2} \quad (A.7)$$

to fix two points on the real axis, this is equal to (A.6). We can also use

$$\int \frac{dx_adx_bdx_c}{|x_a - x_b||x_b - x_c||x_c - x_a|} \quad (A.8)$$

to fix three points. In fact, this three expressions are equivalent.

The conformal Killing volume on $RP_2$ can be given is the same way. An invariant volume described by two real parameters is

$$\int \frac{d^2 z}{|1 + z\bar{z}|^2}. \quad (A.9)$$
We use this volume to fix one point in the unit disk. The a one-parameter subgroup left and satisfy:

\[ z_o = \frac{e^{i\alpha}z_o - \tan\gamma e^{-i\beta}}{\tan\gamma e^{i\beta}z_o + e^{-i\alpha}}. \quad (A.10) \]

The subgroup then can be given as

\[ z' = \frac{z + it\left(z - \frac{2z_o}{1 - |z_o|^2}\right)}{1 - it\left(1 + z\frac{2z_o}{1 - |z_o|^2}\right)}. \quad (A.11) \]

Then the volume of this one-parameter subgroup can be given as case of \( D_2 \):

\[ \int_{-\infty}^{+\infty} \frac{2d\tau}{1 + \tau^2} = 2\pi. \quad (A.12) \]

Then the volume of the CKG on \( RP_2 \) is

\[ 2\pi \int \frac{d^2z_o}{|1 + z_o \bar{z}_o|}. \quad (A.13) \]

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