OPTIMAL BOUNDARY GRADIENT ESTIMATES FOR LAMÉ SYSTEMS WITH PARTIALLY INFINITE COEFFICIENTS

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Abstract. In this paper, we derive the pointwise upper bounds and lower bounds on the gradients of solutions to the Lamé systems with partially infinite coefficients as the surface of discontinuity of the coefficients of the system is located very close to the boundary. When the distance tends to zero, the optimal blow-up rates of the gradients are established for inclusions with arbitrary shapes and in all dimensions.

1. Introduction and main results

It is a common phenomenon that high concentration of extreme mechanical loads occurs in high-contrast fiber-reinforced composites in the zones that include the narrow regions between two adjacent inclusions and the thin gaps between the inclusions and the exterior boundary of the background medium. Extreme loads are always amplified by such composite microstructure, which will cause failure or fracture initiation. Stimulated by the well-known work of Babuška et al [12], where computational analysis of damage and fracture in fiber composite systems is investigated, we consider the Lamé system in linear elasticity with partially infinite coefficients to characterize the high-contrast composites. This paper is a continuation of [15, 16], where the upper bound of the gradient estimate for two adjacent inclusions is established, which can be regarded as interior estimates for this problem.

Due to the interaction from the boundary data, solutions of these systems become more irregular near the boundary. In this paper, we mainly investigate the boundary gradient estimates for the Lamé system with partially infinite coefficients when the inclusion is spaced very close to the matrix exterior boundary. The novelty of these estimates is that they give not only the pointwise upper bounds but also lower bounds of the gradient, which shows that the blow-up rate of the gradient with respect to the distance between the inclusion and the matrix exterior boundary that we obtain is optimal. The role of the boundary data is embodied in these estimates. Especially, an explicit factor that determines whether the blow-up occurs or not is singled out in the lower bound estimates. We would like to emphasize that the gradient estimates obtained in this paper hold for inclusions with arbitrary convex shapes and in all dimensions.

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Let $D \subset \mathbb{R}^d (d \geq 2)$ be a bounded open set with $C^{2,\gamma}$ boundary, and $D_1$ be a strictly convex open set in $D$ with $C^{2,\gamma}$ boundary, $0 < \gamma < 1$, and spaced very close to the boundary $\partial D$. More precisely,

$$D_1 \subset D, \quad \text{the principle curvatures of } \partial D, \partial D_1 \geq \kappa_0 > 0,$$

where $\kappa_0$ is constant independent of $\varepsilon$. We also assume that the $C^{2,\gamma}$ norms of $\partial D, \partial D_1$ are bounded by some constant independent of $\varepsilon$. This implies that $D_1$ contains a ball of radius $r_0^*$ for some constant $r_0^* > 0$ independent of $\varepsilon$. See Figure 1.

Denote
$$\Omega := D \setminus D_1.$$

We assume that $\Omega$ and $D_1$ are occupied, respectively, by two different isotropic and homogeneous materials with different Lamé constants $(\lambda, \mu)$ and $(\lambda_1, \mu_1)$. Then the elasticity tensors for the background and the inclusion can be written, respectively, as $C_0$ and $C_1$, with

$$C_0^{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

and

$$C_1^{ijkl} = \lambda_1 \delta_{ij} \delta_{kl} + \mu_1 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

where $i, j, k, l = 1, 2, \cdots, d$ and $\delta_{ij}$ is the kronecker symbol: $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$.

Let $u = (u^1, u^2, \cdots, u^d)^T : D \rightarrow \mathbb{R}^d$ denote the displacement field. For a given vector valued function $\varphi = (\varphi^1, \varphi^2, \cdots, \varphi^d)^T$, we consider the following Dirichlet problem for the Lamé system:

$$\begin{align*}
\nabla \cdot (\chi_\Omega C_0 + \chi_{D_1} C_1) e(u) &= 0, \quad \text{in } D, \\
\varphi &= \text{on } \partial D,
\end{align*}$$

where $\chi_\Omega$ is the characteristic function of $\Omega \subset \mathbb{R}^d$,

$$e(u) = \frac{1}{2} (\nabla u + (\nabla u)^T)$$

is the strain tensor.
Assume that the standard ellipticity condition holds for (1.2), that is,
\[ \mu > 0, \quad d\lambda + 2\mu > 0, \quad \mu_1 > 0, \quad d\lambda_1 + 2\mu_1 > 0. \]
For \( \varphi \in H^1(D; \mathbb{R}^d) \), it is well known that there exists a unique solution \( u \in H^1(D; \mathbb{R}^d) \) to the Dirichlet problem (1.2), which is also the minimizer of the energy functional
\[
J_1[u] := \frac{1}{2} \int_\Omega \left( (\chi_0 C^0 + \chi_D C^1)e(u), e(u) \right) dx
\]
on
\[
H^1_d(D; \mathbb{R}^d) := \{ u \in H^1(D; \mathbb{R}^d) \mid u - \varphi \in H^1_0(D; \mathbb{R}^d) \}.
\]
We introduce the linear space of rigid displacement in \( \mathbb{R}^d \):
\[ \Psi := \{ \psi \in C^1(\mathbb{R}^d; \mathbb{R}^d) \mid \nabla\psi + (\nabla\psi)^T = 0 \}. \]
With \( e_1, \ldots, e_d \) denoting the standard basis of \( \mathbb{R}^d \),
\[ \{ e_i, x_j e_k - x_k e_j \mid 1 \leq i \leq d, \ 1 \leq j < k \leq d \} \]
is a basis of \( \Psi \). Denote this basis of \( \Psi \) as \( \{ \psi_\alpha \mid \alpha = 1, 2, \ldots, \frac{d(d+1)}{2} \} \).
For fixed \( \lambda \) and \( \mu \) satisfying \( \mu > 0 \) and \( d\lambda + 2\mu > 0 \), denote \( u_{\lambda, \mu} \) as the solution of (1.2). Then similarly as in the Appendix of [15], we also have
\[ u_{\lambda_1, \mu_1} \to u \quad \text{in} \ H^1(D; \mathbb{R}^d), \quad \text{as} \min\{\mu_1, d\lambda_1 + 2\mu_1\} \to \infty, \]
where \( u \) is a \( H^1(D; \mathbb{R}^d) \) solution of
\[
\begin{cases}
\mathcal{L}_{\lambda, \mu} u := \nabla \cdot (C^0 e(u)) = 0, & \text{in } \Omega, \\
u|_+ = u|_-, & \text{on } \partial D_1, \\
e(u) = 0, & \text{in } D_1, \\
\int_{\partial D_1} \frac{\partial u}{\partial \nu_0} \cdot \psi_\alpha = 0, & \alpha = 1, 2, \ldots, \frac{d(d+1)}{2}, \\
u = \varphi, & \text{on } \partial D,
\end{cases}
\]
where
\[ \frac{\partial u}{\partial \nu_0} := (C^0 e(u))\tilde{n} = \lambda (\nabla \cdot u)\tilde{n} + \mu (\nabla u + (\nabla u)^T)\tilde{n}, \]
and \( \tilde{n} \) is the unit outer normal of \( D_1 \). Here and throughout this paper the subscript ± indicates the limit from outside and inside the domain, respectively. The existence, uniqueness and regularity of weak solutions to (1.3) are proved in the Appendix of [15], where multiple inclusions case is studied. In particular, the \( H^1 \) weak solution to (1.3) is in \( C^1(\overline{\Omega}; \mathbb{R}^d) \cap C^1(\overline{D_1}; \mathbb{R}^d) \). The solution is also the unique function which has the least energy in appropriate functional spaces, characterized by
\[
I_\infty[u] = \min_{v \in A} I_\infty[v], \quad I_\infty[v] := \frac{1}{2} \int_\Omega (C^0 e(v), e(v)) dx,
\]
where
\[ A := \left\{ v \in H^1_0(D; \mathbb{R}^d) \mid e(v) = 0 \quad \text{in} \ D \right\}. \]
It is well known that for any open set \( O \) and \( u, v \in C^2(O) \),
\[ \int_O (C^0 e(u), e(v)) dx = - \int_O (\mathcal{L}_{\lambda, \mu} u) \cdot v + \int_{\partial O} \frac{\partial u}{\partial \nu_0} \cdot v. \]
A calculation gives
\[
(\mathcal{L}_{\lambda, \mu} u)_k = \mu \Delta u_k + (\lambda + \mu) \partial x_k (\nabla \cdot u), \quad k = 1, \ldots, d.
\]
We assume that for some \( \delta_0 > 0 \),
\[
\delta_0 \leq \mu, d\lambda + 2\mu \leq \frac{1}{\delta_0}.
\]  

(1.6)

It is clear that there exist two points \( P_1 \in \partial D_1 \) and \( P \in \partial D \), such that
\[
\text{dist}(P, P_1) = \text{dist}(D, \partial D) = \varepsilon.
\]
We use \( \overline{P_1 P} \) to denote the line segment connecting \( P_1 \) and \( P \). Denote
\[
\rho_d(\varepsilon) = \left\{ \begin{array}{ll}
\sqrt{\varepsilon}, & \text{if } d = 2, \\
\frac{1}{|\log \varepsilon|}, & \text{if } d = 3, \\
1, & \text{if } d \geq 4.
\end{array} \right.
\]

Remark 1.2. Actually, for \( d \geq 2 \), we have the following pointwise upper bound of \( |\nabla u| \) in \( \Omega \):
\[
|\nabla u(x)| \leq C \left[ \frac{\rho_d(\varepsilon)}{\varepsilon} + \frac{\text{dist}(x, \overline{P_1 P})}{\varepsilon + \text{dist}(x, \overline{P_1 P}) + 1} \right] \| \varphi \|_{C^2(\partial D, \mathbb{R}^d)}^2, \quad x \in \Omega,
\]  

(1.7)

and
\[
|\nabla u(x)| \leq C \| \varphi \|_{C^2(\partial D, \mathbb{R}^d)}^2, \quad x \in D_1,
\]  

(1.8)

where \( C \) depends only on \( \kappa_0, \delta_0, d \), the \( C^{2, \gamma} \) norm of \( \partial D_1 \) and \( \partial D \), but not on \( \varepsilon \).

Theorem 1.1. (Upper bound). Assume that \( \Omega, D \subset \mathbb{R}^d \), \( \varepsilon \) are defined in (1.3), \( \varphi \in C^2(\partial D; \mathbb{R}^d) \). Let \( u \in H^1(D; \mathbb{R}^d) \cap C^1(\overline{\Omega}; \mathbb{R}^d) \) be a solution to (1.3). Then for \( 0 < \varepsilon < 1/2 \), we have
\[
|\nabla u(x)| \leq \frac{C \rho_d(\varepsilon)}{\varepsilon} \| \varphi \|_{C^2(\partial D, \mathbb{R}^d)}, \quad x \in \Omega,
\]  

(1.9)

This shows that the right hand side archives its maximum at \( \overline{P_1 P} \), with value \( \frac{C \rho_d(\varepsilon)}{\varepsilon} \| \varphi \|_{C^2(\partial D, \mathbb{R}^d)}^2 \) for \( \varepsilon \) sufficiently small.

In order to show that the blow-up rate of the gradients obtained in Theorem 1.1 is optimal, we need to investigate its lower bound. Denote \( D^*_1 := \{ x \in \mathbb{R}^d \mid x + P_1 \notin D_1 \} \). Set \( \Omega^* := D \setminus \overline{D^*_1} \). Let \( u_0^* \) be the solution of the boundary value problem:
\[
\begin{cases}
\mathcal{L}_{\lambda, \mu} u_0^* = 0, & \text{in } \Omega^*, \\
u_0^* = 0, & \text{on } \partial D^*_1, \\
u_0^* = \varphi(x) - \varphi(P), & \text{on } \partial D.
\end{cases}
\]  

(1.10)

Define
\[
b_0^* := \int_{\partial D^*_1} \frac{\partial u_0^*}{\partial n_0} \cdot \psi_k, \quad \alpha = 1, 2, \ldots, \frac{d(d + 1)}{2},
\]  

which is a functional of \( \varphi \), playing an important role in the following establishment of lower bounds of \( |\nabla u| \) on the segment \( \overline{P_1 P} \).
Theorem 1.3. (Lower bound). Under the assumption as in Theorem 1.1, let \( u \in H^1(D; \mathbb{R}^d) \cap C^1(\Omega; \mathbb{R}^d) \) be a solution to (1.3). Then

(i) for \( d = 2 \), if there exists some integer \( 1 \leq k_0 \leq d \) such that \( b_{k_0}^* \neq 0 \) and \( \nabla \phi_{k_0}(P) = 0 \);

(ii) for \( d = 3 \), if there exists some integer \( 1 \leq k_0 \leq d \) such that \( b_{k_0}^* \neq 0 \);

(iii) for \( d \geq 4 \), if there exists some integer \( 1 \leq k_0 \leq d \) such that \( b_{k_0}^* \neq 0 \) and \( b_{\alpha}^* = 0 \) for all \( \alpha \neq k_0 \),

then for sufficiently small \( 0 < \varepsilon < 1/2 \),

\[
|\nabla u(x)| \geq \frac{\rho_d(\varepsilon)}{C_\varepsilon}, \quad x \in \overline{P_1P},
\]

where \( C \) depends only on \( \kappa_0, \delta_0, d \), the \( C^2 \) norm of \( \partial D_1 \) and the \( C^2 \) norm of \( \partial D \), but not on \( \varepsilon \).

Remark 1.4. In Theorem 1.3 we do not try to find the most general assumptions to guarantee blow-up occur, but instead give simple conditions (i)-(iii), which show, however, the essential role of the boundary data in this problem. Since \( u_0 \) is uniquely determined by (1.10) with given data \( \phi(x) - \phi(P) \), Theorem 1.3 shows that whether \( |\nabla u| \) blows up or not totally depends only on the boundary data \( \phi(x) - \phi(P) \). Furthermore, if the blow-up occurs, then from Theorem 1.1 and 1.3 we know that it may occur only on the segment \( \overline{P_1P} \).

Remark 1.5. Theorem 1.1 and 1.3 give not only the upper bound but also a lower bound of the blow-up rate of the strain tensor in all dimensions, which shows the optimality of our estimates. Especially for the lower bound, new difficulties need to be overcome and a number of refined estimates are used in our proof. More important, a blow-up factor, totally depending on the given boundary data, is captured.

Remark 1.6. The strict convexity assumption on \( \partial D \) and \( \partial D_1 \) in Theorem 1.1 and 1.3 can be extended to a weaker relative strict convexity assumption, see (2.6)–(2.8) below.

The organization of this paper is as follows. In Section 2 we first decompose the solution \( u \) of (1.3) as a linear combination of \( u_\alpha, \alpha = 1, 2, \cdots, d(d+1)/2 \), defined by (2.3) and (2.4) below, and then deduce the proof of Theorem 1.1 to two aspects: the estimates of \( |\nabla u_\alpha| \) and those of the coefficients \( C^\alpha \) and \( C^\alpha - \phi^\alpha(0) \). In Section 3 we establish an upper bound of the gradient of solutions to a boundary problem of Lamé system on \( \Omega \) with general Dirichlet boundary data in Theorem 2.1 of independent interest, and then obtain the estimates of \( |\nabla u_\alpha| \) as a consequence of Theorem 2.1. In Section 4 we present the estimates of the coefficients \( C^\alpha \) and \( C^\alpha - \phi^\alpha(0) \). Theorem 1.3 on the lower bound of \( \nabla u \) on the segment \( \overline{P_1P} \) is proved by studying the functional \( b^*_\alpha \) of boundary data \( \phi \) in Section 5. In the rest of the introduction we review some earlier results on interior gradient estimates for high contrast composites.

As mentioned before, Babuška, Andersson, Smith and Levin [12] computationally analyzed the damage and fracture in composite materials and observed numerically that the size of the strain tensor remains bounded when the distance \( \varepsilon \), between two inclusions, tends to zero. This was proved by Li and Nirenberg in [31]. Indeed such \( \varepsilon \)-independent gradient estimates was established there for solutions of
divergence form second order elliptic systems, including linear systems of elasticity, with piecewise Hölder continuous coefficients in all dimensions. See Bonnetier and Vogelius [19] and Li and Vogelius [32] for corresponding results on divergence form elliptic equations.

The estimates in [31] and [32] depend on the ellipticity of the coefficients. If ellipticity constants are allowed to deteriorate, the situation is very different. Consider the simplified scalar model, also called as conductivity problem,

\[
\begin{aligned}
\nabla \cdot (a_k(x) \nabla u_k) &= 0, & \text{in } \Omega, \\
u_k &= \varphi, & \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded open set of \( \mathbb{R}^d \), \( d \geq 2 \), containing two \( \varepsilon \)-apart convex inclusions \( D_1 \) and \( D_2 \), \( \varphi \in C^2(\partial \Omega) \) is given, and

\[
a_k(x) = \begin{cases} 
k \in (0, \infty), & \text{in } D_1 \cup D_2, \\
1, & \text{in } \Omega \setminus (D_1 \cup D_2).
\end{cases}
\]

When \( k = \infty \), the \( L^\infty \)-norm of \( |\nabla u_\infty| \) for the solutions \( u_\infty \) of the following perfect conductivity problem

\[
\begin{aligned}
\Delta u &= 0, & \text{in } \Omega \setminus (D_1 \cup D_2), \\
u_+ &= u_- \mid \partial D_1 \cup \partial D_2, & \text{on } \partial D_1 \cup \partial D_2, \\
\nabla u &= 0, & \text{in } D_1 \cup D_2, \\
\int_{\partial D_i} \gamma \mid \partial u \mid &= 0, & i = 1, 2, \\
u &= \varphi, & \text{on } \partial \Omega,
\end{aligned}
\]

(1.11)

generally becomes unbounded as \( \varepsilon \) tends to 0. There have been much more important progress on the interior gradient estimate of the solution of (1.11), in contrast to the elasticity vector case. The blow up rate of \( |\nabla u_\infty| \) is respectively \( \varepsilon^{-1/2} \) in dimension \( d = 2 \), \( (\varepsilon|\ln \varepsilon|)^{-1} \) in dimension \( d = 3 \), and \( \varepsilon^{-1} \) in dimension \( d \geq 4 \).

We now describe our methods of proof. By a translation and rotation of the coordinates if necessary, we may assume without loss of generality that

\[
P_1 = (0, \varepsilon) \in \partial D_1, \quad P = (0, 0) \in \partial D.
\]
In order to prove Theorem 1.1, it suffices to consider the following problem, by replacing \( u \) by \( u - \varphi(0) \),

\[
\begin{aligned}
\mathcal{L}_{\lambda, \mu} u &= 0, & \text{in } \Omega, \\
u|_+ &= u|_- , & \text{on } \partial D_1, \\
e(u) &= 0, & \text{in } D_1, \\
\int_{\partial D_1} \frac{\partial u}{\partial n} \cdot \psi_\alpha &= 0, & \alpha = 1, 2, \cdots, \frac{d(d+1)}{2}, \\
u = \varphi(x) - \varphi(0), & \text{on } \partial D.
\end{aligned}
\tag{2.1}
\]

By the third line of (2.1) and the definition of \( \Psi \), \( u \) is a linear combination of \( \{ \psi_\alpha \} \) in \( D_1 \). Since it is clear that \( \mathcal{L}_{\lambda, \mu} \xi = 0 \) in \( \Omega \) and \( \xi = 0 \) on \( \partial \Omega \) imply that \( \xi = 0 \) in \( \Omega \), we decompose the solution of (2.1), in the spire of [13], as follows:

\[
u = \sum_{\alpha=1}^{\frac{d(d+1)}{2}} C^\alpha \psi_\alpha - \varphi(0) = \sum_{\alpha=1}^{d} (C^\alpha - \varphi^\alpha(0)) \psi_\alpha + \sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha \psi_\alpha, \quad \text{in } \overline{D}_1,
\]

for some constants \( C^\alpha, \alpha = 1, 2, \cdots, \frac{d(d+1)}{2}, \) (to be determined by the forth line in (2.1)) and

\[
u = \sum_{\alpha=1}^{d} (C^\alpha - \varphi^\alpha(0)) u_\alpha + \sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha u_\alpha + u_0, \quad \text{in } \Omega, \tag{2.2}
\]

where \( u_\alpha \in C^1(\overline{\Omega}; \mathbb{R}^d) \cap C^2(\Omega; \mathbb{R}^d), \alpha = 1, 2, \cdots, \frac{d(d+1)}{2}, \) respectively, satisfy

\[
\begin{aligned}
\mathcal{L}_{\lambda, \mu} u_\alpha &= 0, & \text{in } \Omega, \\
u_\alpha &= \psi_\alpha , & \text{on } \partial D_1, \\
u_\alpha &= 0, & \text{on } \partial D,
\end{aligned}
\tag{2.3}
\]

and \( u_0 \in C^1(\overline{\Omega}; \mathbb{R}^d) \cap C^2(\Omega; \mathbb{R}^d) \) satisfies

\[
\begin{aligned}
\mathcal{L}_{\lambda, \mu} u_0 &= 0, & \text{in } \Omega, \\
u_0 &= 0, & \text{on } \partial D_1, \\
u_0 &= \varphi(x) - \varphi(0), & \text{on } \partial D.
\end{aligned}
\tag{2.4}
\]

By the decomposition (2.2), we write

\[
\nabla u = \sum_{\alpha=1}^{d} (C^\alpha - \varphi^\alpha(0)) \nabla u_\alpha + \sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha \nabla u_\alpha + \nabla u_0, \quad \text{in } \Omega. \tag{2.5}
\]

To estimate \( \| \nabla u \| \), two ingredients are in order: (i) estimates of \( \| \nabla u_\alpha \|, \alpha = 0, 1, \cdots, \frac{d(d+1)}{2} \); (ii) estimates of \( C^\alpha - \varphi^\alpha(0), \alpha = 1, 2, \cdots, d \) and \( C^\alpha, \alpha = 1, 2, \cdots, \frac{d(d+1)}{2} \).

Since the singular behavior of \( \nabla u \) may occur only in the narrow region between \( D_1 \) and \( \partial D \), we are particularly interested in such narrow region. See Figure 2.

Fix a small constant \( 0 < R < 1 \), independent of \( \varepsilon \), such that the portions of \( \partial D_1 \) near \( P_1 \) and \( \partial D \) near \( P \) can be represented, respectively, by

\[
x_d = \varepsilon + h_1(x'), \quad \text{and} \quad x_d = h(x'), \quad \text{for } |x'| < 2R.
\]

Moreover, in view of the assumptions of \( \partial D_1 \) and \( \partial D \), \( h_1 \) and \( h \) satisfy

\[
\varepsilon + h_1(x') > h(x'), \quad \text{for } |x'| < 2R, \tag{2.6}
\]
Figure 2. The narrow region between $\partial D_1$ and $\partial D$.

$$h_1(0') = h(0') = 0, \quad \nabla_{x'} h_1(0') = \nabla_{x'} h(0') = 0,$$

$$\nabla_{x'}^2 h_1(0'), \nabla_{x'}^2 h(0') \geq \kappa_0 I, \quad \nabla_{x'}^2 (h_1 - h)(0') \geq \kappa_1 I,$$

(2.7)

and

$$\|h_1\|_{C^2(\overline{B_{2R}(0')})} + \|h\|_{C^2(\overline{B_{2R}(0')})} \leq \kappa_2,$$

(2.9)

where $\kappa_0, \kappa_1$ and $\kappa_2$ are some positive constants. Throughout the paper, unless otherwise stated, we use $C$ to denote some positive constant, whose values may vary from line to line, which depend only on $\delta_0, \kappa_0, \kappa_1$ and $\kappa_2$, but not on $\varepsilon$. Also, we call a constant having such dependence a universal constant.

For $0 < r < 2R$, we denote

$$\Omega_r := \{ x = (x', x_d) \in \mathbb{R}^d \mid h(x') < x_d < \varepsilon + h_1(x'), |x'| < r \}.$$

The top and bottom boundaries of $\Omega_r$ are

$$\Gamma^+_r = \{ x \in \mathbb{R}^d \mid x_d = \varepsilon + h_1(x'), |x'| < r \}, \quad \Gamma^-_r = \{ x \in \mathbb{R}^d \mid x_d = h(x'), |x'| < r \},$$

respectively.

To estimate $|\nabla u|$, we consider the following general boundary value problems:

$$\begin{cases}
L_{\lambda, \mu} v := \nabla \cdot (C_0 e(v)) = 0, & \text{in } \Omega, \\
v = \psi(x), & \text{on } \partial D_1, \\
v = 0, & \text{on } \partial D,
\end{cases}$$

(2.10)

where $\psi(x) = (\psi^1(x), \psi^2(x), \cdots, \psi^d(x))^T \in C^2(\partial D_1; \mathbb{R}^d)$ is given vector-valued functions. Locally piontwise gradient estimates for problem (2.10) is as follows:

**Theorem 2.1.** Assume that hypotheses (2.7) - (2.9) are satisfied, and let $v \in H^1(\Omega; \mathbb{R}^d)$ be a weak solution of problem (2.10). Then for $0 < \varepsilon < 1/2$,

$$|\nabla v(x', x_d)| \leq \frac{C}{\varepsilon + |x'|^2} \left| \psi(x', \varepsilon + h_1(x')) \right| + C\|\psi\|_{C^2(\partial D_1; \mathbb{R}^d)}, \quad \forall x \in \Omega_R,$$

(2.11)

and

$$|\nabla v(x)| \leq C\|\psi\|_{C^2(\partial D_1; \mathbb{R}^d)} \quad \forall x \in \Omega \setminus \Omega_R.$$

**Remark 2.2.** Theorem 2.1 is of independent interest. We also can deal with more general case when $v = \phi(x)$ on $\partial D$, instead of the condition $v = 0$ there. The proof of Theorem 2.1 is given in Section 3.
Without loss of generality, we only need to prove Theorem \ref{thm:1.1} for \( \| \varphi \|_{C^2(\partial D; \mathbb{R}^d)} = 1 \), and for general case by considering \( u / \| \varphi \|_{C^2(\partial D; \mathbb{R}^d)} \) if \( \| \varphi \|_{C^2(\partial D; \mathbb{R}^d)} > 0 \). If \( \varphi|_{\partial D} = 0 \), then \( u \equiv 0 \). First, the estimates of \( |\nabla u_\alpha| \) are some immediate consequences of Theorem \ref{thm:2.1} only taking \( \psi = \psi_\alpha \), \( \alpha = 1, \cdots, \frac{d(d+1)}{2} \), respectively, or \( \psi = \varphi(x) - \varphi(0) \) with minor modifications.

**Corollary 2.3.** Under the hypotheses of Theorem 1.1 and with the normalization \( \| \varphi \|_{C^2(\partial D; \mathbb{R}^d)} = 1 \). Then for \( 0 < \varepsilon < 1/2 \),

\[
|\nabla u_\alpha(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad \alpha = 1, 2, \cdots, d, \quad \forall \ x \in \Omega_R; \tag{2.12}
\]

\[
|\nabla u_\alpha(x)| \leq \frac{C(\varepsilon + |x'|)}{\varepsilon + |x'|^2}, \quad \alpha = d + 1, \cdots, \frac{d(d+1)}{2}, \quad \forall \ x \in \Omega_R; \tag{2.13}
\]

\[
|\nabla u_0(x)| \leq \frac{C|\nabla \varphi(0)||x'|}{\varepsilon + |x'|^2} + C, \quad \forall \ x \in \Omega_R; \tag{2.14}
\]

and

\[
|\nabla u_\alpha(x)| \leq C, \quad \alpha = 0, 1, 2, \cdots, \frac{d(d+1)}{2}, \quad \forall \ x \in \Omega \setminus \Omega_R.
\]

On the other hand, we need the following estimates on \( C^\alpha \) and \( |C^\alpha - \varphi^\alpha(0)| \).

The proof is given in Section \ref{sec:4}.

**Proposition 2.4.** Under the hypotheses of Theorem 1.1 and with the normalization \( \| \varphi \|_{C^2(\partial D; \mathbb{R}^d)} = 1 \). Then

\[
|C^\alpha| \leq C, \quad \alpha = 1, 2, \cdots, \frac{d(d+1)}{2}, \tag{2.15}
\]

and

\[
|C^\alpha - \varphi^\alpha(0)| \leq C \rho_d(\varepsilon), \quad \alpha = 1, 2, \cdots, d. \tag{2.16}
\]

We are now in position to prove Theorem \ref{thm:1.1}.

**Proof of Theorem 1.1** Since

\[
\nabla u = \sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha \nabla \psi_\alpha = \begin{pmatrix} 0 & C^{d+1} & C^{d+2} & \cdots & C^{2d-1} \\ -C^{d+1} & 0 & C^{2d} & \cdots & C^{3d-3} \\ -C^{d+2} & -C^{2d} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & C^{d(d+1)} \\ -C^{2d-1} & -C^{3d-3} & \cdots & -C^{d(d+1)} & 0 \end{pmatrix} \text{ in } D_1,
\]

The estimate \( (1.8) \) immediately follows from \( (2.15) \).

By \( (2.16) \), Corollary 2.3 and Proposition 2.4 we have, for \( x \in \Omega_R \),

\[
|\nabla u(x)| \leq \sum_{\alpha=1}^{d} |C^\alpha - \varphi^\alpha(0)||\nabla u_\alpha| + \sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha |\nabla u_\alpha| + |\nabla u_0| \leq C \left( \frac{\rho_d(\varepsilon)}{\varepsilon + |x'|^2} + \frac{|x'|}{\varepsilon + |x'|^2} + 1 \right).
\]

Thus, \( (1.9) \) is proved, so \( (1.7) \). \hfill \square
To complete this section, we recall some properties of the tensor $C$. For the isotropic elastic material, let

\[ C := (C_{ijkl}) = (\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})), \quad \mu > 0, \quad d \lambda + 2 \mu > 0. \]

The components $C_{ijkl}$ satisfy the following symmetric condition:

\[ C_{ijkl} = C_{klij} = C_{klji}, \quad i, j, k, l = 1, 2, \cdots, d. \tag{2.17} \]

We will use the following notations:

\[ (CA)_{ij} = \sum_{k,l=1}^{d} C_{ijkl}A_{kl}, \quad \text{and} \quad (A, B) \equiv A : B = \sum_{i,j=1}^{d} A_{ij}B_{ij}, \]

for every pair of $d \times d$ matrices $A = (A_{ij})$, $B = (B_{ij})$. Clearly,

\[ (CA, B) = (A, CB). \]

If $A$ is symmetric, then, by the symmetry condition (2.17), we have that

\[ (CA, A) = C_{ijkl}A_{kl}A_{ij} = \lambda A_{ii}A_{kk} + 2\mu A_{kj}A_{kj}. \]

Thus $C$ satisfies the following ellipticity condition: For every $d \times d$ real symmetric matrix $\eta = (\eta_{ij})$,

\[ \min\{2\mu, d \lambda + 2 \mu\} \eta^2 \leq (C\eta, \eta) \leq \max\{2\mu, d \lambda + 2 \mu\} \eta^2, \tag{2.18} \]

where $|\eta|^2 = \sum_{ij} \eta_{ij}^2$. In particular,

\[ \min\{2\mu, d \lambda + 2 \mu\} |A + A^T|^2 \leq (C(A + A^T), (A + A^T)). \tag{2.19} \]

3. Proof of Theorem 2.1 and estimates of $|\nabla u_\alpha|$ 

In this section, we first prove Theorem 2.1, then give some much finer estimates on $|\nabla u_\alpha|$, which will be useful for the establishment of the low bound estimates in Section 4 and Section 5.

We decompose the solution of (2.10) as follows:

\[ v = v_1 + v_2 + \cdots + v_d, \]

where $v_l = (v_l^1, v_l^2, \cdots, v_l^d)^T$, $l = 1, 2, \cdots, d$, with $v_l^j = 0$ for $j \neq l$, and $v_l$ satisfy the following boundary value problem, respectively,

\[ \begin{cases} \mathcal{L}_{\lambda, \mu} v_l := \nabla \cdot (C^0 e(v_l)) = 0, & \text{in } \Omega, \\ v_l = (0, \cdots, 0, \psi^l, 0, \cdots, 0)^T, & \text{on } \partial D_1, \\ v_l = 0, & \text{on } \partial D. \end{cases} \tag{3.1} \]

Then

\[ \nabla v = \sum_{l=1}^{d} \nabla v_l. \tag{3.2} \]

In order to estimate $|\nabla v_l|$ one by one, we first introduce a scalar auxiliary function $\bar{v} \in C^2(\mathbb{R}^n)$ such that $\bar{v} = 1$ on $\partial D_1$, $\bar{v} = 0$ on $\partial D$ and

\[ \bar{v}(x) = \frac{x_d - h(x')}{\varepsilon + h_1(x') - h(x')}, \quad \text{in } \Omega_{2R}, \]
and

\[ \| \tilde{v} \|_{C^2(\Omega; \mathbb{R}^{n/2})} \leq C. \]  

By a direct calculation, we obtain that for \( k, j = 1, \cdots, d-1 \), and \( x \in \Omega_{2R} \),

\[ |\partial_{x_k} \tilde{v}(x)| \leq \frac{C|x'|}{\varepsilon + |x'|^2}, \quad \frac{1}{C(\varepsilon + |x'|^2)} \leq |\partial_{x_d} \tilde{v}(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \]

and

\[ |\partial_{x_kx_j} \tilde{v}(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad |\partial_{x_kx_d} \tilde{v}(x)| \leq \frac{C|x'|}{(\varepsilon + |x'|^2)^2}, \quad \partial_{x_kx_d} \tilde{v}(x) = 0. \]  

Extend \( \psi \in C^2(\partial D; \mathbb{R}^d) \) to \( \tilde{\psi} \in C^2(\overline{\Omega}, \mathbb{R}^d) \) such that \( \| \psi^l \|_{C^2(\overline{\Omega} \setminus \Omega_1)} \leq C \| \psi^l \|_{C^2(\partial D_1)} \), for \( l = 1, 2, \cdots, d \). We can find \( \rho \in C^2(\overline{\Omega}) \) such that

\[ 0 \leq \rho \leq 1, \quad |\nabla \rho| \leq C, \quad \rho = 1 \text{ on } \overline{\Omega}, \quad \rho = 0 \text{ on } \overline{\Omega} \setminus \Omega_2R. \]

Define

\[ \tilde{v}_l(x) = (0, \cdots, 0, [\rho(x)\psi^l(x', \varepsilon + h_1(x')) + (1 - \rho(x))\psi^l(x')] \tilde{v}(x), 0, \cdots, 0)^T \] in \( \Omega \).

In particular,

\[ \tilde{v}_l(x) = (0, \cdots, 0, \psi^l(x', \varepsilon + h_1(x'))\tilde{v}(x), 0, \cdots, 0)^T \] in \( \Omega_R \),

and in view of (3.3),

\[ \| \tilde{v}_l \|_{C^2(\Omega \setminus \Omega_{R/2})} \leq C\| \psi^l \|_{C^2(\partial D_1)}. \]  

Due to (3.4), and (3.5), for \( l = 1, 2, \cdots, d \), and \( k, j = 1, 2, \cdots, d-1 \), for \( x \in \Omega_R \),

\[ |\partial_{x_k} \tilde{v}_l(x)| \leq \frac{C|x'|\| \psi^l(x', \varepsilon + h_1(x')) \|}{\varepsilon + |x'|^2} + C\| \nabla \psi^l \|_{L^\infty}, \]

and

\[ \frac{|\psi^l(x', \varepsilon + h_1(x'))|}{C(\varepsilon + |x'|^2)} \leq |\partial_{x_d} \tilde{v}_l(x)| \leq \frac{C\| \psi^l(x', \varepsilon + h_1(x')) \|}{\varepsilon + |x'|^2}; \]

and

\[ |\partial_{x_kx_j} \tilde{v}_l(x)| \leq \frac{C|x'|}{\varepsilon + |x'|^2} + C \left( \frac{|x'|}{\varepsilon + |x'|^2} + 1 \right) \| \nabla \psi^l \|_{L^\infty} + C\| \nabla^2 \psi^l \|_{L^\infty}, \]

\[ |\partial_{x_kx_d} \tilde{v}_l(x)| \leq \frac{C|x'|}{(\varepsilon + |x'|^2)^2} |\psi^l(x', \varepsilon + h_1(x'))| + \frac{C}{\varepsilon + |x'|^2} \| \nabla \psi^l \|_{L^\infty}, \]

and

\[ \partial_{x_kx_d} \tilde{v}_l(x) = 0. \]

Here and throughout this section, for simplicity we use \( \| \nabla \psi \|_{L^\infty} \) and \( \| \nabla^2 \psi \|_{L^\infty} \) to denote \( \| \nabla \psi \|_{L^\infty(\partial D_1)} \) and \( \| \nabla^2 \psi \|_{L^\infty(\partial D_1)} \), respectively.

Let

\[ w_l := v_l - \tilde{v}_l, \quad l = 1, 2, \cdots, d. \]

Lemma 3.1. Let \( v_l \in H^1(\Omega; \mathbb{R}^d) \) be a weak solution of (3.1). Then

\[ \int_{\Omega} |\nabla w_l|^2 \, dx \leq C\| \psi^l \|_{C^2(\partial D_1)}, \quad l = 1, 2, \cdots, d. \]
Proof. For simplicity, we denote 
\[ w := w_l, \quad \tilde{v} := \tilde{v}_l. \]
Thus, \( w \) satisfies
\[
\begin{align*}
\mathcal{L}_{\lambda, \mu} w &= -\mathcal{L}_{\lambda, \mu} \tilde{v}, & \text{in } \Omega, \\
w &= 0, & \text{on } \partial \Omega.
\end{align*}
\] (3.17)
Multiplying the equation in (3.17) by \( w \) and applying integration by parts, we get
\[
\int_\Omega (C_0 e(w), e(w)) \, dx = \int_\Omega w (\mathcal{L}_{\lambda, \mu} \tilde{v}) \, dx.
\] (3.18)
By the Poincaré inequality,
\[
\|w\|_{L^2(\Omega \setminus \Omega_R)} \leq C \|\nabla w\|_{L^2(\Omega \setminus \Omega_R)}.
\] (3.19)
Note that the above constant \( C \) is independent of \( \epsilon \). Using the Sobolev trace embedding theorem,
\[
\int_{|x'|=R, \, h(x')<x_d<\epsilon+h_1(x')} |w| \, dx \leq C \left( \int_{\Omega \setminus \Omega_R} |\nabla w|^2 \, dx \right)^{\frac{1}{2}}.
\] (3.20)
According to (3.10), we have
\[
\int_\Omega |\nabla_x \tilde{v}|^2 \, dx
\leq C \int_{|x'|<R} (\epsilon + h_1(x') - h(x')) \left( \frac{|x'|^2 |\psi'(x', \epsilon + h_1(x'))|^2}{(\epsilon + |x'|^2)^2} + \|\nabla \psi'\|_{L^\infty}^2 \right) \, dx'
\leq C \|\psi'\|_{C^1(\partial D_1)}^2,
\] (3.21)
where \( C \) depends only on \( d \) and \( \kappa_0 \).
The first Korn’s inequality together with (2.19), (3.18), (3.9) and (3.19) implies
\[
\int_\Omega |\nabla w|^2 \, dx \leq 2 \int_\Omega |e(w)|^2 \, dx
\leq C \left( \int_{\Omega \setminus \Omega_R} w (\mathcal{L}_{\lambda, \mu} \tilde{v}) \, dx \right) + C \left( \int_{\Omega \setminus \Omega_R} w (\mathcal{L}_{\lambda, \mu} \tilde{v}) \, dx \right)
\leq C \int_{\Omega_R} w (\mathcal{L}_{\lambda, \mu} \tilde{v}) \, dx + C \|\psi'\|_{C^2(\partial D_1)} \int_{\Omega \setminus \Omega_R} |w| \, dx
\leq C \int_{\Omega_R} w (\mathcal{L}_{\lambda, \mu} \tilde{v}) \, dx + C \|\psi'\|_{C^2(\partial D_1)} \left( \int_{\Omega \setminus \Omega_R} |\nabla w|^2 \right)^{1/2}.
\]
while, due to (3.14), (3.20) and (3.21),
\[
\left| \int_{\Omega_R} w(\mathcal{L}_{\lambda,\mu} \tilde{v}) \, dx \right| \leq C \sum_{k+l \leq 3d} \int_{\Omega_R} w \partial_{x_k x_l} \tilde{v} \, dx
\]
\[
\leq C \int_{\Omega_R} |\nabla w||\nabla x' \tilde{v}| \, dx + \int_{|x'|=R, h(x')<x_1+h_1(x')} C|\nabla_x \tilde{v}| |w| \, dx
\]
\[
\leq C \left( \int_{\Omega_R} |\nabla w|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_R} |\nabla x' \tilde{v}|^2 \, dx \right)^{\frac{1}{2}} + C\|\psi^l\|_{C^1(\partial D_1)} \left( \int_{\Omega_R} |\nabla w|^2 \, dx \right)^{\frac{1}{2}}
\]
\[
\leq C\|\psi^l\|_{C^1(\partial D_1)} \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{\frac{1}{2}}.
\]
Therefore,
\[
\int_{\Omega} |\nabla w|^2 \, dx \leq C\|\psi^l\|_{C^2(\partial D_1)} \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{\frac{1}{2}}.
\]
The proof of Lemma 3.1 is completed. \(\Box\)

For convenience, we denote
\[
\delta(z) := \varepsilon + h_1(z') - h(z'), \quad \text{for } z = (z', z_d) \in \Omega_R.
\]
By (2.8), we have
\[
\frac{1}{C}(\varepsilon + |z'|^2) \leq \delta(z) \leq C(\varepsilon + |z'|^2).
\]
Set
\[
\hat{\Omega}_s(z) := \{ x \in \Omega_{2R} \mid |x' - z'| < s \}, \quad \forall \ 0 \leq s \leq R.
\]
It follows from (3.7) and (3.10)–(3.14) that for \(x \in \Omega_R, l = 1, 2, \ldots, d,\)
\[
|\mathcal{L}_{\lambda,\mu} \hat{v}_l| \leq C|\nabla^2 \hat{v}_l| \leq \left( \frac{C}{\varepsilon + |x'|^2} + \frac{C|x'|}{(\varepsilon + |x'|^2)^2} \right) |\psi^l(x', \varepsilon + h_1(x'))|
\]
\[
\leq \frac{C}{\varepsilon + |x'|^2} \|\nabla \psi^l\|_{L^\infty} + C\|\nabla^2 \psi^l\|_{L^\infty}, \tag{3.22}
\]
where \(C\) is independent of \(\varepsilon.\)

**Lemma 3.2.** For \(\delta = \delta(z) \leq R, z \in \Omega_R,\) and \(l = 1, 2, \ldots, d,\)
\[
\int_{\hat{\Omega}_s(z)} |\nabla w|^2 \, dx \leq C\delta^{d+1} \left( |\psi^l(z', \varepsilon + h_1(z'))|^2 + \delta(\|\nabla \psi^l\|^2_{C^2(\partial D_1)} + 1) \right). \tag{3.23}
\]

**Proof.** Still denote \(w := w_l,\) and \(\hat{v} := \hat{v}_l.\) For \(0 < t < s < 1,\) let \(\eta(x')\) be a smooth cutoff function satisfying \(0 \leq \eta(x') \leq 1, \eta(x') = 1\) if \(|x' - z'| < t, \eta(x') = 0\) if \(|x' - z'| > s\) and \(|\nabla \eta(x')| \leq \frac{2}{s^2}.\) Multiplying \(\eta^2 w\) on both side of the equation in (3.17) and applying integration by parts leads to
\[
\int_{\hat{\Omega}_s(z)} \left( \mathcal{C}_0 e(w), e(\eta^2 w) \right) dx = \int_{\hat{\Omega}_s(z)} (\eta^2 w) \mathcal{L}_{\lambda,\mu} \hat{v} \, dx. \tag{3.24}
\]
By the first Korn’s inequality and the standard arguments, we have
\[
\int_{\hat{\Omega}_s(z)} \left( \mathcal{C}_0 e(w), e(\eta^2 w) \right) dx \geq \frac{1}{C} \int_{\hat{\Omega}_s(z)} |\eta \nabla w|^2 \, dx - C \int_{\hat{\Omega}_s(z)} |\nabla \eta|^2 |w|^2 \, dx. \tag{3.25}
\]
This, together with (3.24) and (3.25), implies that
\[ \left| \int_{\tilde{\Omega}_{x}(z)} (\eta^2 w) \mathcal{L}_{\lambda, \mu} \tilde{v} \, dx \right| \leq \left( \int_{\tilde{\Omega}_{x}(z)} |w|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\tilde{\Omega}_{x}(z)} |\mathcal{L}_{\lambda, \mu} \tilde{v}|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{1}{(s - t)^2} \int_{\tilde{\Omega}_{x}(z)} |w|^2 \, dx + (s - t)^2 \int_{\tilde{\Omega}_{x}(z)} |\mathcal{L}_{\lambda, \mu} \tilde{v}|^2 \, dx. \]

It follows from (3.22) and the mean value theorem that
\[ \int_{\tilde{\Omega}_{x}(z)} |\nabla w|^2 \, dx \leq \frac{C}{(s - t)^2} \int_{\tilde{\Omega}_{x}(z)} |w|^2 \, dx + C(s - t)^2 \int_{\tilde{\Omega}_{x}(z)} |\mathcal{L}_{\lambda, \mu} \tilde{v}|^2 \, dx. \] (3.26)

We know that \( w = 0 \) on \( \Gamma_{R} \). By using (2.6)–(2.9) and Hölder inequality, we obtain
\[ \int_{\tilde{\Omega}_{x}(z)} |w|^2 \, dx \leq \int_{\tilde{\Omega}_{x}(z)} \int_{h(x')} (\varepsilon + h_1(x')) \int_{h(x')} |\nabla w(x', \xi)|^2 \, d\xi \, dx \leq C (\varepsilon + (|z'| + s)^2) \int_{\tilde{\Omega}_{x}(z)} |\nabla w|^2 \, dx. \] (3.27)

It follows from (3.22) and the mean value theorem that
\[ \int_{\tilde{\Omega}_{x}(z)} |\mathcal{L}_{\lambda, \mu} \tilde{v}|^2 \, dx \leq \psi'(z', \varepsilon + h_1(z'))^2 \int_{\tilde{\Omega}_{x}(z)} \left( \frac{C}{\varepsilon + |x'|^2} + \frac{C|x'|}{(\varepsilon + |x'|^2)^2} \right)^2 \, dx \] 
\[ + ||\nabla \psi'||_L^2 \int_{\tilde{\Omega}_{x}(z)} \left( \frac{C}{\varepsilon + |x'|^2} + \frac{C|x'|}{(\varepsilon + |x'|^2)^2} \right)^2 |x' - z'|^2 \, dx \]
\[ + ||\nabla \psi'||_L^2 \int_{\tilde{\Omega}_{x}(z)} \left( \frac{C}{\varepsilon + |x'|^2} \right)^2 \, dx + C\delta(z)s^{d-1}||\nabla^2 \psi'||_{L^\infty}^2 \]
\[ \leq C|\psi'(z', \varepsilon + h_1(z'))|^2 \int_{|x' - z'| < s} \frac{dx'}{(\varepsilon + |x'|^2)^2} \]
\[ + C||\nabla \psi'||_L^2 \int_{|x' - z'| < s} \left( \frac{1}{\varepsilon + |x'|^2} + \frac{s^2}{(\varepsilon + |x'|^2)^2} \right) \, dx' + C\delta(z)s^{d-1}||\nabla^2 \psi'||_{L^\infty}^2. \] (3.28)

**Case 1.** For \( 0 \leq |z'| \leq \sqrt{\varepsilon} \), (i.e. \( \varepsilon \leq \delta(z) \leq C\varepsilon \)), and \( 0 < t < s < \sqrt{\varepsilon} \), by means of (3.27) and (3.28), we have
\[ \int_{\tilde{\Omega}_{x}(z)} |w|^2 \, dx \leq C\varepsilon^2 \int_{\tilde{\Omega}_{x}(z)} |\nabla w|^2 \, dx, \] (3.29)
and
\[ \int_{\tilde{\Omega}_{x}(z)} |\mathcal{L}_{\lambda, \mu} \tilde{v}|^2 \, dx \]
\[ \leq C|\psi'(z', \varepsilon + h_1(z'))|^2 \frac{s^{d-1}}{\varepsilon^2} + C||\nabla \psi'||_L^2 \frac{s^{d-1}}{\varepsilon} + C\varepsilon s^{d-1}||\nabla^2 \psi'||_{L^\infty}^2. \] (3.30)
Denote

$$F(t) := \int_{\Omega_1(z)} |\nabla w|^2 dx.$$  

By (3.26), (3.29) and (3.30), for some universal constant $c_1 > 0$, we get for $0 < t < s < \sqrt{\varepsilon},$

$$F(t) \leq \left( \frac{c_1 \varepsilon}{s-t} \right)^2 F(s) + C(s-t)^2 s^{d-1}.$$

Using (3.31) with $F(t)$, (3.29) and (3.30), for some universal constant $c$, here we used the fact that $F(t)$ is sufficiently small. This implies that for $0 \leq |z'| \leq \sqrt{\varepsilon},$

$$\|\nabla w\|^2_{L^2(\Omega_1(z))} \leq C \varepsilon^{d+1} \left( \|\psi'(z', \varepsilon + h_1(z'))\|^2 + \varepsilon \|\nabla \psi'\|^2_{\hat{L}^\infty} + \varepsilon \|\nabla^2 \psi'\|^2_{\hat{L}^\infty} \right).$$

**Case 2.** For $\sqrt{\varepsilon} \leq |z'| < R$, (i.e. $C|z'|^2 \leq \delta(z) \leq (C+1)|z'|^2), 0 < t < s < \frac{2|z'|^3}{3},$ by using (3.27) and (3.28) again, we have

$$\int_{\Omega_2(z)} |\nabla w|^2 dx \leq C |z'|^4 \int_{\Omega_2(z)} |\nabla w|^2 dx,$$

$$\int_{\hat{\Omega}_2(z)} |L_{\lambda, u} \tilde{v}|^2 dx$$

$$\leq C |\psi'(z', \varepsilon + h_1(z'))|^2 s^{d-1} \frac{s^{d-1}}{|z'|^4} + C \|\nabla \psi'\|^2_{\hat{L}^\infty} \frac{s^{d-1}}{|z'|^2} + C |z'|^2 s^{d-1} \|\nabla^2 \psi'\|^2_{\hat{L}^\infty}.$$
Thus, for $0 < t < s < \frac{2|s'|}{3}$,

$$F(t) \leq \left( \frac{c_2 |s'|^2}{s-t} \right)^2 F(s) + C(s-t)^2 s^{d-1},$$

$$\left( \frac{|\psi_l(z', \varepsilon + h_1(z'))|^2}{|s'|^4} + \frac{\|\nabla \psi_l\|_{L^\infty}}{|s'|} + |s'|^2 \|\nabla^2 \psi_l\|_{L^2}^2 \right),$$

where $c_2$ is another universal constant. Taking the same iteration procedure as in Case 1, setting $t_i = \delta + 2c_2|s'|^2$, $i = 0, 1, \cdots$ and $k = \left[ \frac{1}{4c_2} \right] + 1$, by (3.32) with $s = t_{i+1}$ and $t = t_i$, we have, for $i = 0, 1, 2, \cdots, k$,

$$F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i+2)^{d-1}|s'|^2(d+1),$$

$$(|\psi_l(z', \varepsilon + h_1(z'))|^2 + |s'|^2(\|\nabla \psi_l\|_{L^\infty}^2 + \|\nabla^2 \psi_l\|_{L^2}^2)).$$

Similarly, after $k$ iterations, we have

$$F(t_0) \leq \left( \frac{1}{4} \right)^k F(t_k) + C \sum_{i=1}^k \left( \frac{1}{4} \right)^{i-1} (i+1)^{d-1}|s'|^2(d+1),$$

$$\left(|\psi_l(z', \varepsilon + h_1(z'))|^2 + |s'|^2(\|\nabla \psi_l\|_{L^\infty}^2 + \|\nabla^2 \psi_l\|_{L^2}^2)\right)$$

$$\leq \left( \frac{1}{4} \right)^k F(|s'|)$$

$$+ C|s'|^2(d+1) \left(|\psi_l(z', \varepsilon + h_1(z'))|^2 + |s'|^2(\|\nabla \psi_l\|_{L^\infty}^2 + \|\nabla^2 \psi_l\|_{L^2}^2)\right)$$

$$\leq C|s'|^2(d+1) \left(|\psi_l(z', \varepsilon + h_1(z'))|^2 + |s'|^2 \|\psi_l\|_{C^2(\partial D_1)}^2\right),$$

which implies that, for $\varepsilon \leq |s'| < R$,

$$\|\nabla \psi_l\|_{L^2(\Omega_D(z))} \leq C|s'|^2(d+1) \left(|\psi_l(z', \varepsilon + h_1(z'))|^2 + |s'|^2 \|\psi_l\|_{C^2(\partial D_1)}^2\right).$$

The proof of Lemma 3.2 is completed. \hfill \Box

Lemma 3.3. For $l = 1, 2, \cdots, d$,

$$|\nabla w_l(x)| \leq \frac{C|\psi_l(x', \varepsilon + h_1(x'))|}{\sqrt{\delta(x)}} + C\|\psi_l\|_{C^2(\partial D_1)}, \quad \forall \ x \in \Omega_R. \quad (3.33)$$

Consequently, by (3.10), (3.11) and (3.14), we have for sufficiently small $\varepsilon$ and $x \in \Omega_R$,

$$\frac{|\psi_l(x', \varepsilon + h_1(x'))|}{C(\varepsilon + |x|^2)} \leq |\nabla w_l(x', x_d)| \leq \frac{C|\psi_l(x', \varepsilon + h_1(x'))|}{\varepsilon + |x|^2} + C\|\psi_l\|_{C^2(\partial D_1)}.$$ \quad (3.34)

Proof. Take $w := w_l$ and $\hat{\psi} := \hat{\psi}_l$ for simplicity. Given $z = (z', z_d) \in \Omega_R$, making a change of variables

$$\begin{cases}
  x' - z' = \delta y', \\
  x_d = \delta y_d,
\end{cases}$$

where $\delta = \delta(y')$. Define

$$\hat{h}_1(y') := \frac{1}{\delta}(\varepsilon + h_1(\delta y' + z')), \quad \hat{h}(y') := \frac{1}{\delta} h(\delta y' + z').$$
Then, the region $\hat{\Omega}_3(z)$ becomes $Q_1$, where

$$Q_r = \{ y \in \mathbb{R}^d \mid \hat{h}(y') < y_d < \hat{h}_1(y'), \ |y'| < r \}, \quad 0 < r \leq 1,$$

and the top and bottom boundaries of $Q_r$ become

$$\hat{\Gamma}_r^+ := \{ y \in \mathbb{R}^d \mid y_d = \hat{h}_1(y'), \ |y'| \leq r \}$$

and

$$\hat{\Gamma}_r^- := \{ y \in \mathbb{R}^d \mid y_d = \hat{h}(y'), \ |y'| \leq r \},$$

respectively. From (2.6)-(2.9) and the definition of $\hat{h}_1$ and $\hat{h}$, it follows that

$$\hat{h}_1(0') - \hat{h}(0') = 1,$$

and for $|y'| < 1$,

$$|\nabla \hat{h}_1(y')| + |\nabla \hat{h}(y')| \leq C(\delta + |z'|), \quad |\nabla^2 \hat{h}_1(y')| + |\nabla^2 \hat{h}(y')| \leq C\delta.$$

Since $R$ is small, $||\hat{h}_1||_{C^{1,1}(\overline{B_1(0')})}$ and $||\hat{h}||_{C^{1,1}(\overline{B_1(0')})}$ are small and $Q_1$ is approximately a unit square (or a cylinder-shaped domain) as far as applications of the Sobolev embedding theorems and classical $L^p$ estimates for elliptic systems are concerned.

Let

$$\hat{v}(y', y_d) := \hat{v}(\delta y' + z', \delta y_d), \quad \hat{w}(y', y_d) := w(\delta y' + z', \delta y_d).$$

Thus, $\hat{w}(y)$ satisfies

$$\begin{cases}
\mathcal{L}_{\lambda, \mu} \hat{w} = -\mathcal{L}_{\lambda, \mu} \hat{v} & \text{in } Q_1, \\
\hat{w} = 0 & \text{on } \hat{\Gamma}_1^\pm.
\end{cases}$$

In view of $\hat{w} = 0$ on the upper and lower boundaries of $Q_1$, we have, by Poincaré inequality, that

$$\|\hat{w}\|_{H^1(Q_1)} \leq C\|\nabla \hat{w}\|_{L^2(Q_1)}.$$  

Using the Sobolev embedding theorem and classical $W^{2,p}$ estimates for elliptic systems (see e.g. [2], or theorem 2.5 in [24]), we have, for some $p > n$,

$$\|\nabla \hat{w}\|_{L^\infty(Q_{1/2})} \leq C\|\hat{w}\|_{W^{2,p}(Q_{1/2})} \leq C\left( \|\nabla \hat{w}\|_{L^2(Q_1)} + \|\mathcal{L}_{\lambda, \mu} \hat{v}\|_{L^\infty(Q_1)} \right).$$

Since

$$\|\nabla \hat{w}\|_{L^\infty(Q_{1/2})} = \delta\|\nabla w\|_{L^\infty(\tilde{\Omega}_{1/2}(z))}, \quad \|\nabla \hat{w}\|_{L^2(Q_1)} = \delta^{1-\frac{d}{2}}\|\nabla w\|_{L^2(\tilde{\Omega}_3(z))}$$

and

$$\|\mathcal{L}_{\lambda, \mu} \hat{v}\|_{L^\infty(Q_1)} = \delta^2\|\mathcal{L}_{\lambda, \mu} \hat{v}\|_{L^\infty(\tilde{\Omega}_3(z))}$$

Tracking back to $w$ through the transforms, we have

$$\|\nabla w\|_{L^\infty(\tilde{\Omega}_{1/2}(z))} \leq \frac{C}{\delta^\frac{d}{2}} \left( \delta^{1-\frac{d}{4}}\|\nabla w\|_{L^2(\tilde{\Omega}_3(z))} + \delta^2\|\mathcal{L}_{\lambda, \mu} \hat{v}\|_{L^\infty(\tilde{\Omega}_3(z))} \right). \quad (3.35)$$

By (3.22) and (3.23), we have

$$\delta^{-\frac{d}{2}}\|\nabla w\|_{L^2(\tilde{\Omega}_3(z))} \leq \frac{C}{\sqrt{\delta}}|\psi^1(z', \varepsilon + h_1(z'))| + C\|\psi^1\|_{C^2(\partial D_1)},$$

and

$$\delta\|\mathcal{L}_{\lambda, \mu} \hat{v}\|_{L^\infty(\tilde{\Omega}_3(z))} \leq \frac{C}{\sqrt{\delta}}|\psi^1(z', \varepsilon + h_1(z'))| + C(\|\nabla \psi^1\|_{L^\infty} + \|\nabla^2 \psi^1\|_{L^\infty}).$$

Plugging these estimates above into (3.35) yields (3.33). The proof of Lemma 3.3 is finished.
Proof of Theorem 2.1. By using (3.34) and the decomposition of $\nabla u$, (3.2),

$$|\nabla v(x)| \leq \sum_{l=1}^{d} |\nabla v_l| \leq \frac{C|\psi(x', \varepsilon + h_1(x'))|}{\varepsilon + |x'|^2} + C\|\psi\|_{C^2(\partial D_1)}, \quad x \in \Omega_R.$$ 

Note that for any $x \in \Omega \setminus \Omega_R$, by using the standard interior estimates and boundary estimates for elliptic systems (2.10) (see Agmon et al. [1] and [2]), we have

$$\|\nabla v\|_{L^\infty(\Omega \setminus \Omega_R)} \leq C\|\psi\|_{C^2(\partial D_1)}.$$ 

The proof of Theorem 2.1 is completed. □

The following finer estimates in $\Omega_R$ will be useful in Section 4 and Section 5. We assume that $\|\phi\|_{C^2(\partial D; \mathbb{R}^d)} = 1$ without loss of generality. For problem (2.3), taking $\psi = \psi_\alpha$, and $\tilde{u}_\alpha := \bar{v}_\psi \alpha$, $\alpha = 1, \cdots, d$
in the proof of Lemma 3.1-Lemma 3.3, respectively, we have

**Corollary 3.4.** For $\alpha = 1, 2, \cdots, d$,

$$|\nabla (u_\alpha - \tilde{u}_\alpha)(x)| \leq \frac{C}{\sqrt{\delta(x)}}, \quad \forall x \in \Omega_R. \quad (3.36)$$

Consequently, by the definition of $\tilde{u}_\alpha$ and (3.4), we have, for $\alpha = 1, \cdots, d$,

$$|\nabla x' u_\alpha(x)| \leq \frac{C}{\sqrt{\delta(x)}}, \quad \forall x \in \Omega_R, \quad (3.37)$$

and

$$\frac{1}{C\delta(x)} \leq |\partial_{x^d} u_\alpha(x)| \leq \frac{C}{\delta(x)}, \quad x \in \Omega_R. \quad (3.38)$$

**Proof.** According to the definition of $\tilde{u}_\alpha$ and (3.4), we have

$$|\nabla x' \tilde{u}_\alpha(x)| \leq \frac{C|x'|}{\varepsilon + |x'|^2}, \quad x \in \Omega_R,$$

$$\frac{1}{C(\varepsilon + |x'|^2)} \leq |\partial_{x^d} \tilde{u}_\alpha(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad x \in \Omega_R;$$

and

$$|\mathcal{L}_{\lambda, \mu} \tilde{u}_\alpha(x)| \leq C \sum_{k+l<2d} |\partial_{x^k} x^l \tilde{u}_\alpha(x)|$$

$$\leq \left(\frac{C}{\varepsilon + |x'|^2} + \frac{C|x'|}{(\varepsilon + |x'|^2)^2}\right) |\psi^l(x', \varepsilon + h_1(x'))|, \quad x \in \Omega_R.$$ 

Clearly, (3.36) follows from the proof of Lemma 3.1-Lemma 3.3 □

For problem (2.4), we decompose the solution $u_0$ as

$$u_0 = u_{01} + u_{02} + \cdots + u_{0d},$$

where $u_{0l}, l = 1, 2, \cdots, d$, satisfy, respectively,

$$\begin{cases}
\mathcal{L}_{\lambda, \mu} u_{0l} = 0, & \text{in } \Omega, \\
u_{0l} = 0, & \text{on } \partial D_1, \\
u_{0l} = (0, \cdots, 0, \varphi^l(x) - \varphi^l(0), 0, \cdots, 0)^T, & \text{on } \partial D. 
\end{cases} \quad (3.39)$$
Similar as (3.7), we define
\[ \tilde{u}_{0l}(x) := (0, \ldots, 0, [\rho(x)(\varphi(x', h(x')) - \varphi(0)) + (1 - \rho(x))(\varphi'(x) - \varphi'(0))](1 - \tilde{v})(x), 0, \ldots, 0)^T, \quad \forall x \in \Omega. \]
\[ (3.40) \]
where \( \rho \in C^2(\Omega) \) is a cutoff function satisfying (3.6) as before. In particular,
\[ \tilde{u}_{0l} = (0, \ldots, 0, (\varphi(x', h(x')) - \varphi(0))(1 - \tilde{v})(x), 0, \ldots, 0)^T, \quad \forall x \in \Omega_R. \]
Adapting the proofs of Lemma 3.1–Lemma 3.2 and using (3.44), we obtain the following corollary.

**Corollary 3.5.** For \( l = 1, 2, \ldots, d \),
\[ |\nabla u_{0l} - \tilde{u}_{0l}(x)| \leq C|\varphi'|_{C^2(\partial D)}, \quad x \in \Omega_R. \] (3.41)
Consequently,
\[ |\nabla_x u_{0l}(x)| \leq C|\varphi'|_{C^2(\partial D)}, \quad x \in \Omega_R, \] (3.42)
and
\[ \frac{|\varphi'(x', h(x')) - \varphi'(0)|}{C\delta(x)} \leq |\partial_x u_{0l}(x)| \leq \frac{C|\nabla_x \varphi'(0)||x'|}{\delta(x)} + C|\varphi'|_{C^2(\partial D)}, \quad x \in \Omega_R. \] (3.43)

**Proof.** For (3.39), it is clear from (3.40) that \( \tilde{u}_{0l} = u_{0l} = 0 \) on \( \partial D_1 \), \( \tilde{u}_{0l} = u_{0l} \) on \( \partial D \). Note that \( \tilde{u}_{0l} = 0 \) if \( k \neq l \), and for \( x \in \Omega_R \),
\[ \nabla_x \tilde{u}_{0l} = - (\varphi'(x', h(x')) - \varphi'(0)) \nabla_x \tilde{v}(x) \]
\[ + \left[ \nabla_x \varphi'(x', h(x')) + \partial_x \varphi'(x', h(x'))\nabla_x h(x') \right](1 - \tilde{v})(x), \]
\[ \partial_x \tilde{u}_{0l} = - (\varphi'(x', h(x')) - \varphi'(0)) \partial_x \tilde{v}(x). \]

By the Taylor expansion and (2.7)–(2.8),
\[ \varphi'(x', h(x')) = \varphi'(0) + \nabla_x \varphi'(0)|x'| \]
\[ + \frac{1}{2} |x'|^2 \nabla^2_x \varphi'(0) + \partial_x \nabla^2_x \varphi'(0)h(0') \] (3.44)
Hence, using (3.4), we have
\[ |\nabla x \tilde{u}_{0l}(x)| \leq \frac{C|\nabla_x \varphi'(0)||x'|^2}{\varepsilon + |x'|^2} + C|\varphi'|_{C^2(\partial D)} \leq C|\varphi'|_{C^2(\partial D)}, \quad x \in \Omega_R, \] (3.45)
and
\[ \frac{|\varphi'(x', h(x')) - \varphi'(0)|}{C(\varepsilon + |x'|^2)} \leq |\partial_x \tilde{u}_{0l}(x)| \leq \frac{C|\nabla_x \varphi'(0)||x'|}{\varepsilon + |x'|^2} + C|\varphi'|_{C^2(\partial D)}, \quad x \in \Omega_R. \] (3.46)
Adapting the proof of Lemma 3.1–Lemma 3.2 and using (3.44), we obtain
\[ |\nabla(u_{0l} - \tilde{u}_{0l})(x)| \leq \frac{C|\nabla_x \varphi'(0)||x'|}{\sqrt{\varepsilon + |x'|^2}} + C|\varphi'|_{C^2(\partial D)} \leq C|\varphi'|_{C^2(\partial D)}, \quad x \in \Omega_R, \]
which, together with (3.45) and (3.46), implies that (3.42) and (3.43). □
4. Proof of Proposition 2.4 and estimates of $C^\alpha$

In this Section, we are devoted to prove Proposition 2.4 under the normalization $\|\varphi\|_{C^2(\partial D; \mathbb{R}^d)} = 1$.

Denote
\[
a_{\alpha\beta} := -\int_{\partial D} \partial_u \alpha \cdot \psi_\beta, \quad b_\beta := \int_{\partial D} \partial_u \beta \cdot \psi_\beta, \quad \alpha, \beta = 1, 2, \ldots, \frac{d(d+1)}{2}.
\]

Multiplying the first line of (2.3) and (2.4) by $\psi_\beta$, respectively, and applying integration by parts over $\Omega$ leads to
\[
a_{\alpha\beta} = \int_{\Omega} (C^0 e(u_\alpha), e(u_\beta)) dx, \quad b_\beta = -\int_{\Omega} (C^0 e(u_0), e(u_\beta)) dx.
\]

By (2.5) and the linearity of $e(u)$,
\[
e(u) = \sum_{\alpha=1}^d (C^\alpha - \varphi^\alpha(0)) e(u_\alpha) + \sum_{\alpha=d+1}^{d(d+1)/2} C^\alpha e(u_\alpha) + e(u_0), \quad \text{in } \Omega.
\]

Then, it follows from the forth line of (2.1) that for $\beta = 1, 2, \ldots, \frac{d(d+1)}{2}$,
\[
\sum_{\alpha=1}^d (C^\alpha - \varphi^\alpha(0)) a_{\alpha\beta} + \sum_{\alpha=d+1}^{d(d+1)/2} C^\alpha a_{\alpha\beta} = b_\beta. \tag{4.1}
\]

Denote
\[
X^1 = (C^1 - \varphi^1(0), \ldots, C^d - \varphi^d(0))^T, \quad X^2 = (C^{d+1}, \ldots, C^{d(d+1)/2})^T,
\]
\[
P^1 = (b_1, \ldots, b_d)^T, \quad P^2 = (b_{d+1}, \ldots, b_{d(d+1)/2})^T,
\]

and
\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1d} \\
\vdots & \ddots & \vdots \\
a_{d1} & \cdots & a_{dd}
\end{pmatrix}, \quad B = \begin{pmatrix}
a_1 & d_1 & \cdots & a_{d1} \\
\vdots & \ddots & \ddots & \vdots \\
a_{d1} & \cdots & a_{dd}
\end{pmatrix},
\]
\[
D = \begin{pmatrix}
a_{d+1} & \cdots & a_{d(d+1)/2} \\
\vdots & \ddots & \vdots \\
a_{d(d+1)/2} & \cdots & a_{d(d+1)/2}
\end{pmatrix}.
\]

Thus, by using the symmetry property of $a_{\alpha\beta}$, (4.1) can be rewritten as
\[
\begin{pmatrix}
A & B \\
B^T & D
\end{pmatrix}
\begin{pmatrix}
X^1 \\
X^2
\end{pmatrix} =
\begin{pmatrix}
P^1 \\
P^2
\end{pmatrix}. \tag{4.2}
\]

Lemma 4.1. There exists a positive universal constant $C$, independent of $\varepsilon$, such that
\[
\sum_{\alpha, \beta=1}^{d(d+1)/2} a_{\alpha\beta} \xi_\alpha \xi_\beta \geq \frac{1}{C}, \quad \forall \xi \in \mathbb{R}^{d(d+1)/2}, \|\xi\| = 1. \tag{4.3}
\]
Proof. To emphasize the dependence on $\varepsilon$, we use $\Omega_\varepsilon := D \setminus \overline{D_1}$ and $u_{\varepsilon}^\alpha$ to denote the corresponding solution of (2.3) with $\alpha = 1, \cdots, d$. For $\xi \in \mathbb{R}^{\frac{d(d+1)}{2}}$ with $|\xi| = 1$, using (1.6), we have

$$
\int_{\Omega^*} \left| e \left( \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha u_{\varepsilon}^\alpha \right) \right|^2 \, dx \geq \frac{1}{C} \int_{\Omega} \left| e \left( \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha u_{\varepsilon}^\alpha \right) \right|^2 \, dx.
$$

We claim that there exists a constant $C > 0$, independent of $\varepsilon$, such that

$$
\int_{\Omega} \left| e \left( \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha u_{\varepsilon}^\alpha \right) \right|^2 \, dx \geq \frac{1}{C}, \quad \forall \xi \in \mathbb{R}^{\frac{d(d+1)}{2}}, \ |\xi| = 1.
$$

Indeed, if not, then there exist $\varepsilon_i \to 0^+$, $|\xi_i| = 1$, such that

$$
\int_{\Omega} \left| e \left( \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_i_\alpha u_{\varepsilon_i}^\alpha \right) \right|^2 \, dx \to 0, \quad \text{as} \ i \to \infty. \quad (4.4)
$$

Here and in the following proof, we use the notations $D_1^* := \{ x \in \mathbb{R}^d \mid x + (\vec{y}, \varepsilon) \in D_1 \}$, $\Omega^* := D \setminus \overline{D_1}$. Since $u_{\varepsilon_i}^\alpha = 0$ on $\partial D$, it follows from the second Korn’s inequality (see theorem 2.5 in [36]) that there exists a constant $C_i$, independent of $\varepsilon_i$, such that

$$
\| u_{\varepsilon_i}^\alpha \|_{H^1(\Omega_\varepsilon \setminus \overline{B_{\vec{r}}}; \mathbb{R}^d)} \leq C_i,
$$

where $\vec{r} > 0$ is fixed. Then there exists a subsequence, we still denote $\{ u_{\varepsilon_i}^\alpha \}$, such that

$$
u_{\varepsilon_i}^\alpha \rightharpoonup \bar{u}_\alpha, \quad \text{in} \ H^1(\Omega_\varepsilon \setminus \overline{B_{\vec{r}}}; \mathbb{R}^d), \quad \text{as} \ i \to \infty.
$$

By (4.4), there exists $\bar{\xi}$ such that

$$
\xi_i \to \bar{\xi}, \quad \text{as} \ i \to \infty, \quad \text{with} \ |\bar{\xi}| = 1,
$$

and

$$
\int_{\Omega^*} \left| e \left( \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \bar{\xi}_\alpha \bar{u}_\alpha \right) \right|^2 \, dx = 0.
$$

This implies that

$$
e \left( \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \bar{\xi}_\alpha \bar{u}_\alpha \right) = 0, \quad \text{in} \ \Omega^*.$$
That means that \[ \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha \bar{u}_\alpha \in \Psi \text{ in } \Omega^*. \] Hence, there exist some constants \( c_\beta, \beta = 1, 2, \cdots, \frac{d(d+1)}{2}, \) such that
\[
\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha \bar{u}_\alpha = \sum_{\beta=1}^{\frac{d(d+1)}{2}} c_\beta \psi_\beta, \quad \text{in } \Omega^*. 
\]
Since \[
\sum_{\beta=1}^{\frac{d(d+1)}{2}} c_\beta \psi_\beta = 0 \text{ on } \partial D, 
\]
it follows from lemma 6.1 in [16] that \( c_\beta = 0, \beta = 1, \cdots, \frac{d(d+1)}{2}. \) Thus,
\[
\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha \bar{u}_\alpha = 0, \quad \text{in } \Omega^*. 
\]
Restricted on \( \partial D_1^* \), it says that \[
\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha \psi_\alpha = 0 \text{ on } \partial D_1^*. \] This yields, using again lemma 6.1 in [16], \( \bar{\xi}_\alpha = 0, \alpha = 1, \cdots, d \), which contradicts with \( |\bar{\xi}| = 1. \)

Lemma 4.2. For \( d \geq 2 \), we have
\[
\frac{1}{C \rho_d(\varepsilon)} \leq a_{\alpha\alpha} \leq \frac{C}{\rho_d(\varepsilon)}, \quad \alpha = 1, \cdots, d; \tag{4.5}
\]
\[
\frac{1}{C} \leq a_{\alpha \alpha} \leq C, \quad \alpha = d+1, \cdots, \frac{d(d+1)}{2}; \tag{4.6}
\]
\[
a_{\alpha \beta} \leq C, \quad \alpha = 1, 2, \cdots, \frac{d(d+1)}{2}, \beta = d+1, \cdots, \frac{d(d+1)}{2}, \alpha \neq \beta; \tag{4.7}
\]
and if \( d = 2 \), then
\[
|a_{12}| = |a_{21}| \leq C|\log \varepsilon|; \tag{4.8}
\]
if \( d \geq 3 \), then
\[
|a_{\alpha \beta}| = |a_{\beta \alpha}| \leq C, \quad \alpha, \beta = 1, \cdots, d, \alpha \neq \beta. \tag{4.9}
\]
Consequently,
\[
\frac{1}{C(\rho_d(\varepsilon))^2} \leq \det \mathbf{A} \leq \frac{C}{(\rho_d(\varepsilon))^2}, \quad \frac{1}{C} I \leq \mathbf{D} \leq CI. \tag{4.10}
\]

Proof. STEP 1. Proof of (4.5). In view of (2.18), (3.37) and (3.38), we have, for \( \alpha = 1, \cdots, d, \)
\[
a_{\alpha \alpha} = \int_{\Omega} (C^0 e(u_\alpha), e(u_\alpha)) dx \leq C \int_{\Omega} |\nabla u_\alpha|^2 dx 
\]
\[
\leq C \int_{\Omega_R} \frac{dx}{\varepsilon + |x'|^2} + C 
\]
\[
\leq C \int_{|x'|<R} \frac{dx'}{\varepsilon + |x'|^2} + C 
\]
\[
= C \int_{0}^{R} \rho^{d-2} \frac{dp}{\varepsilon + \rho^2} + C \leq \frac{C}{\rho_d(\varepsilon)},
\]
and

\[ a_{\alpha\alpha} = \int_{\Omega} (C^0 e(u_\alpha), e(u_\alpha)) \, dx \geq \frac{1}{C} \int_{\Omega} |e(u_\alpha)|^2 \, dx \]

\[ \geq \frac{1}{C} \int_{\Omega} |e_{\alpha\alpha}(u_\alpha)|^2 \, dx \geq \frac{1}{C} \int_{\Omega_R} |\partial_x u_{\alpha\alpha}|^2 \, dx. \]

Notice that \( u_{\alpha\alpha}|_{\partial D_1} = \bar{v}|_{\partial D_1} = 1, u_{\alpha\alpha}|_{\partial D} = \bar{v}|_{\partial D} = 0 \), and recalling the definition of \( \bar{v}, \bar{v}(x', x_d) \) is linear in \( x_d \) for fixed \( x' \), so \( \bar{v}(x', \cdot) \) is harmonic, hence its energy is minimal, that is

\[ \int_{h(x')}^{h_1(x') + \varepsilon} |\partial_x u_{\alpha\alpha}|^2 \, dx_d \geq \int_{h(x')}^{h_1(x') + \varepsilon} |\partial_x \bar{v}|^2 \, dx_d = \frac{1}{\varepsilon + h_1(x') - h(x')} \]

Integrating on \( B_R(0') \) for \( x' \), we obtain

\[ \int_{\Omega_R} |\partial_x u_{\alpha\alpha}|^2 \, dx = \int_{|x'| < R} \int_{h(x')}^{h_1(x') + \varepsilon} |\partial_x u_{\alpha\alpha}|^2 \, dx_d \, dx' \]

\[ \geq \frac{1}{C} \int_{|x'| < R} \frac{(\varepsilon + |x'|)^2}{\varepsilon + |x'|^2} \, dx' \geq \frac{1}{C} \rho_d(\varepsilon). \]

Estimate (4.5) is proved.

**STEP 2.** Proof of (4.6) and (4.7). By means of (2.13), for \( \alpha, \beta = d + 1, \cdots, \frac{d(d + 1)}{2} \), we have

\[ a_{\alpha\beta} = \int_{\Omega} (C^0 e(u_\alpha), e(u_\beta)) \, dx \leq C \int_{\Omega} |\nabla u_\alpha| |\nabla u_\beta| \, dx \]

\[ \leq C \int_{\Omega_R} \frac{(\varepsilon + |x'|)^2}{\varepsilon + |x'|^2} \, dx + C \leq C. \]

On the other hand, it follows immediately from Lemma 4.1 that there exists a universal constant \( C \) such that

\[ a_{\alpha\alpha} \geq \frac{1}{C}, \quad \alpha = d + 1, \cdots, \frac{d(d + 1)}{2}. \]

We now consider the elements for \( \alpha = 1, 2, \cdots, d, \beta = d + 1, d + 2, \cdots, \frac{d(d + 1)}{2} \). We take the case that \( \alpha = 1, \beta = d + 1 \) for instance. The other cases are the same. Let \( \psi_{d+1} = (x_2, -x_1, 0, \cdots, 0)^T \). Then using (3.37) and the boundedness of \( |\nabla u_\alpha| \) on \( \partial D_1 \setminus B_R \), we have

\[ a_{1(d+1)} = -\int_{\partial D_1} \frac{\partial u_1}{\partial n_0} \cdot \psi_{d+1} \]

\[ = -\int_{\partial D_1 \cap B_R} \left( \lambda (\nabla \cdot u_1) \bar{n} + \mu (\nabla u_\alpha + (\nabla u_1)^T) \bar{n} \right) \cdot (x_2, -x_1, 0, \cdots, 0)^T \]

\[ = -\int_{\partial D_1 \cap B_R} \left( \lambda \left( \sum_{k=1}^{d} \partial_{x_k} u_1^k \right) n_1 + \mu \sum_{l=1}^{d} \left( \partial_{x_1} u_1^l + \partial_{x_1} u_1^1 \right) n_l \right) x_2 \]

\[ + \int_{\partial D_1 \cap B_R} \left( \lambda \left( \sum_{k=1}^{d} \partial_{x_k} u_1^k \right) n_2 + \mu \sum_{l=1}^{d} \left( \partial_{x_2} u_1^l + \partial_{x_2} u_1^2 \right) n_l \right) x_1 \]

is bounded for \( d \geq 2 \), so \( a_{1(d+1)} \).

Thus, estimates (4.6) and (4.7) are established.
STEP 3. Proof of (4.8) and (4.9). Firstly, we estimate \( |a_{\alpha\beta}| \) for \( \alpha, \beta = 1, \cdots, d \) with \( \alpha \neq \beta \). By the definition,

\[
a_{\alpha\beta} = a_{\beta\alpha} = -\int_{\partial D_1} \frac{\partial u_\alpha}{\partial n} + \psi_\beta
\]

Due to (2.7), for \( k \),

\[
\partial_x u_\alpha^k_n + \partial_x u_\alpha^2 n_l
\]

where

\[
\alpha \beta = \frac{1}{\sqrt{1 + |\nabla \cdot h(x')|^2}}
\]

Due to (2.7), for \( k = 1, 2, \cdots, d \), \( d - 1 \),

\[
|n_k| = \left| \frac{-\partial_x h(x')}{\sqrt{1 + |\nabla \cdot h(x')|^2}} \right| \leq C|x'|, \quad \text{and} \quad |n_d| = \frac{1}{\sqrt{1 + |\nabla \cdot h(x')|^2}} \leq 1.
\]

For \( \alpha = 1, 2, \cdots, d, \beta = 1, 2, \cdots, d - 1 \), it follows from (3.37) and (4.11) that

\[
|I_{\alpha\beta}| \leq \int_{\partial D_1 \cap B_R} \left| \sum_{k=1}^d \partial_x u_\alpha^k n_\beta \right|
\]

\[
\leq \int_{\partial D_1 \cap B_R} \frac{C|x'|}{\varepsilon + |x'|^2} \leq \begin{cases} C\log \varepsilon, & d = 2, \\ C, & d \geq 3, \end{cases}
\]

while,

\[
|\Pi_{\alpha\beta}^1| \leq \int_{\partial D_1 \cap B_R} \left| \sum_{l=1}^{d-1} \left( \partial_x u_\alpha^l + \partial_x u_\alpha^2 \right) n_l \right| \leq \int_{\partial D_1 \cap B_R} \frac{C|x'|}{\varepsilon + |x'|^2} \leq C;
\]

\[
|\Pi_{\alpha\beta}^2| \leq \int_{\partial D_1 \cap B_R} |\partial_x u_\alpha^d n_d| \leq \int_{\partial D_1 \cap B_R} \frac{C}{\varepsilon + |x'|^2} \leq \begin{cases} C\log \varepsilon, & d = 2, \\ C, & d \geq 3, \end{cases}
\]
and by the definition of $\tilde{u}_\alpha$ and (3.36),

$$|II_{\alpha\beta}| \leq \int_{\partial D_1 \cap B_R} |\partial_x u^\beta_n d| \leq \int_{\partial D_1 \cap B_R} |(\partial_x \tilde{u}^\beta) n_d| + \int_{\partial D_1 \cap B_R} |(\partial_x (u^\beta_n - \tilde{u}^\beta) n_d|$$

$$\leq \int_{\partial D_1 \cap B_R} \frac{C}{\sqrt{\varepsilon + |x'|^2}} \leq \begin{cases} C|\log \varepsilon|, & d = 2, \\ C, & d \geq 3. \end{cases}$$

Here we used the fact that $\tilde{u}^\beta_\alpha = 0$ if $\alpha \neq \beta$. Hence,

$$|II_{\alpha\beta}| \leq \begin{cases} C|\log \varepsilon|, & d = 2, \\ C, & d \geq 3. \end{cases}$$

This, together with (4.12), the boundedness of $|\nabla u_\alpha|$ on $\partial D_1 \setminus B_R$, and the symmetry of $a_{\alpha\beta} = a_{\beta\alpha}$, implies that for $\alpha, \beta = 1, \cdots, d$ with $\alpha \neq \beta$,

$$|a_{\alpha\beta}| = |a_{\beta\alpha}| \leq |\lambda||I_{\alpha\beta}| + |\mu||II_{\alpha\beta}| + C \leq \begin{cases} C|\log \varepsilon|, & d = 2, \\ C, & d \geq 3. \end{cases}$$

Therefore, (4.8) and (4.9) are proved. (4.10) is an immediate consequence of (4.5)–(4.9). The proof of the Lemma 4.2 is finished. □

Lemma 4.3.

$$|b_\beta| \leq C, \quad \beta = 1, \cdots, \frac{d(d + 1)}{2}. \quad (4.13)$$

Consequently,

$$|P^i| \leq C, \quad i = 1, 2. \quad (4.14)$$

Proof. STEP 1. To estimate $|b_\beta|$ for $\beta = 1, \cdots, d$. We take $\beta = 1$ for instance. The other cases are the same. Denote

$$b_1 = \sum_{l=1}^d \int_{\partial D_1} \frac{\partial u_{01}}{\partial v_0} |\cdot | \psi_1 =: \sum_{l=1}^d b_1.$$

where $u_{0l}, l = 1, 2, \cdots, d$, is defined by (3.39). By definition,

$$b_{11} = \int_{\partial D_1} \frac{\partial u_{01}}{\partial v_0} |\cdot | \psi_1 = \int_{\partial D_1} \left[ \lambda (\nabla \cdot u_{01}) n_1 + \mu \left( (\nabla u_{01} + (\nabla u_{01})^T)^T n_1 \right) \right]$$

$$= \int_{\partial D_1} \left[ \lambda \sum_{k=1}^d \partial_{x_k} u_{01}^k n_1 + \mu \sum_{i=1}^d (\partial_{x_i} u_{01}^i + \partial_{x_i} u_{01}^i) n_1 \right].$$

Denote

$$I := \int_{\partial D_1} \sum_{k=1}^d \partial_{x_k} u_{01}^k n_1 = \int_{\partial D_1} \sum_{k=1}^{d-1} \partial_{x_k} u_{01}^k n_1 + \int_{\partial D_1} \partial_{x_1} u_{01}^d n_1 =: I_1 + I_2,$$
and
\[
\Pi := \int_{\partial D_1} \sum_{i=1}^{d} (\partial_{x_i} u_{01}^i + \partial_{x_i} u_{01}^1)n_i
\]
\[
= \int_{\partial D_1} \sum_{i=1}^{d-1} (\partial_{x_i} u_{01}^i + \partial_{x_i} u_{01}^1)n_i + \int_{\partial D_1} \partial_{x_i} u_{01}^d n_d + \int_{\partial D_1} \partial_{x_d} u_{01}^1 n_d
\]
\[
= : \Pi_1 + \Pi_2 + \Pi_3.
\]
According to (3.42)-(3.43),
\[
|I_1| \leq \left| \int_{\partial D_1 \cap B_R} C|x'| + C \leq C; \right|
\]
\[
|I_2| \leq \left| \int_{\partial D_1} \partial_{x_d} u_{01}^d n_1 \right| \leq \int_{\partial D_1 \cap B_R} \frac{C|\nabla_x \varphi^1(0)||x'|^2}{\varepsilon + |x'|^2} + C \leq C.
\]
So that,
\[
|I| \leq |I_1| + |I_2| \leq C. \tag{4.15}
\]
By (3.42), (4.11) and the definition of \( \tilde{u}_{01} \),
\[
|I_1| \leq \int_{\partial D_1} \sum_{i=1}^{d-1} (\partial_{x_i} u_{01}^i + \partial_{x_i} u_{01}^1)n_i \leq \int_{\partial D_1 \cap B_R} C|x'| + C \leq C, \tag{4.16}
\]
and
\[
|I_2| \leq \left| \int_{\partial D_1} \partial_{x_d} u_{01}^d n_d \right|
\]
\[
\leq \int_{\partial D_1 \cap B_R} |\partial_{x_d} \tilde{u}_{01}^d n_d| + \int_{\partial D_1 \cap B_R} |\partial_{x_1} (u_{01}^d - \tilde{u}_{01}^d) n_d| + C
\]
\[
\leq C|\partial D_1 \cap B_R| + C \leq C. \tag{4.17}
\]
Now, we need only to estimate \( \Pi_3 \). Note that
\[
\Pi_3 = \int_{\partial D_1 \cap B_R} \partial_{x_d} \tilde{u}_{01}^1 n_1 d + \int_{\partial D_1 \cap B_R} \partial_{x_d} (u_{01}^1 - \tilde{u}_{01}^1) n_d =: \Pi_3^1 + \Pi_3^2.
\]
By the definitions of \( \tilde{u}_{01}^1 \) and \( \tilde{v} \),
\[
\partial_{x_d} \tilde{u}_{01}^1 = -(\varphi^1(x', h(x')) - \varphi^1(0))\partial_{x_d} \tilde{v} = -\frac{\varphi^1(x', h(x')) - \varphi^1(0)}{\varepsilon + h_1(x') - h(x')}.
\]
From the expression of \( \partial D_1 \cap B_R : x_d = \varepsilon + h_1(x') \), \( |x'| < R \), we have \( dS = \sqrt{1 + |\nabla_x h_1(x')|^2}dx' \). Then, by the Taylor expansion (3.44), we have
\[
\Pi_3^1 = \int_{\partial D_1 \cap B_R} \partial_{x_d} \tilde{u}_{01}^1 n_d
\]
\[
= \int_{|x'| < R} \frac{-(\varphi^1(x', h(x')) - \varphi^1(0))}{\varepsilon + h_1(x') - h(x')} dx'
\]
\[
= -\int_{|x'| < R} \frac{\nabla_x \varphi^1(0)x' + O(|x'|^2)}{\varepsilon + h_1(x') - h(x')} dx'.
\]
Since
\[
\frac{1}{C(\varepsilon + |x'|^2)} \leq \frac{1}{\varepsilon + h_1(x') - h(x')} \leq \frac{C}{\varepsilon + |x'|^2}, \quad |x'| \leq R,
\]
(4.18)
it follows that
\[
\left| \int_{|x'| < R} \frac{O(|x'|^2)}{\varepsilon + h_1(x') - h(x')} dx' \right| \leq C.
\]
While, according to (4.18), we have
\[
\int_{|x'| < R} \nabla x' \varphi^1(0)x' dx' = \int_{|x'| < R} \nabla x' \varphi^1(0)O_T dy' + \int_{|x'| < R} O(|x'|^{3+\gamma})(\varepsilon + \frac{1}{2}x' (\nabla^2_{x'}(h_1-h)(0'))x')(\varepsilon + h_1(x') - h(x')) dx'.
\]
For the positive matrix \((\nabla^2_{x'}(h_1-h)(0'))\), there exists orthogonal matrix \(O\), such that
\[
O^T(\nabla^2_{x'}(h_1-h)(0'))O = \text{diag}(\lambda_1, \cdots, \lambda_{d-1}),
\]
where \(\lambda_i \geq \kappa_1, \ i = 1, \cdots, d-1\). Under the orthogonal transform \(y' = O x'\), we obtain
\[
\int_{|x'| < R} \nabla x' \varphi^1(0)x' dx' = \int_{|y'| < R} \nabla x' \varphi^1(0)O^T y' dy' = 0,
\]
and
\[
\left| \int_{|x'| < R} \frac{O(|x'|^{3+\gamma})}{\varepsilon + \frac{1}{2}x' (\nabla^2_{x'}(h_1-h)(0'))x'}(\varepsilon + h_1(x') - h(x')) dx' \right| \leq C.
\]
Therefore,
\[
|\Pi^1_3| = \left| \int_{\partial D_1 \cap B_R} \partial_{x_d} \tilde{u}_{01}^1 n_d \right| \leq C.
\]
(4.19)
On the other hand, in view of (3.41),
\[
|\Pi^2_3| \leq \left| \int_{\partial D_1 \cap B_R} \partial_{x_d} (u_{01}^1 - \tilde{u}_{01}^1) n_d \right| \leq C.
\]
This, together with (4.19), implies that
\[
|\Pi_3| \leq C.
\]
(4.20)
Combining (4.15)-(4.17) and (4.20), we have
\[
|b_{11}| \leq C.
\]
Next, for $l = 2, \ldots, d$, \( \int_{\partial D_l} \frac{\partial u_{\theta_l}}{\partial \nu_0} + \psi_1 \) 

\[
\int_{\partial D_1} \left[ \lambda \nabla \cdot u_{\theta_l} n_1 + \mu \left( (\nabla u_{\theta_l})^T \vec{n} \right)_1 \right] 
\int_{\partial D_1} \left[ \lambda \sum_{k=1}^{d} \partial x_k u_{\theta_l}^k n_1 + \mu \sum_{i=1}^{d} \left( \partial x_i u_{\theta_l}^i + \partial x_i u_{\theta_l}^d \right) n_i \right] 
\]

Similarly, making use of (3.42), (3.43) and (4.11), we have

\[
\left| \int_{\partial D_1} \sum_{k=1}^{d} \partial x_k u_{\theta_l}^k n_1 \right| \leq \left| \int_{\partial D_1} \sum_{k=1}^{d-1} \partial x_k u_{\theta_l}^k n_1 \right| + \left| \int_{\partial D_1} \partial x_d u_{\theta_l}^d n_1 \right| \leq C. \hspace{1cm} (4.21)
\]

and recalling the definition of \( \tilde{u}_{\theta_l} \), \( \tilde{u}_{\theta_l}^1 = 0 \),

\[
\left| \int_{\partial D_1} \sum_{i=1}^{d} \partial x_i \tilde{u}_{\theta_l}^i n_d \right| \leq \left| \int_{\partial D_1} \sum_{i=1}^{d-1} \partial x_i \tilde{u}_{\theta_l}^i n_d \right| + \left| \int_{\partial D_1} \partial x_d (u_{\theta_l}^d - \tilde{u}_{\theta_l}^d) n_d \right| 
\]

This implies that

\[
|b_{1l}| \leq C, \quad l = 2, \ldots, d.
\]

Hence,

\[
|b_{1l}| \leq C.
\]

**STEP 2.** To estimate \( |b_{\beta}| \) for \( \beta = d + 1, \ldots, \frac{d(d+1)}{2} \). By using (2.13) and (2.14), we have

\[
|b_{1l}| = \left| \int_{\Omega} \left( C^0 e(u_0), e(u_{\beta}) \right) dx \right| 
\leq C \int_{\Omega} |\nabla u_0| |\nabla u_{\beta}| dx 
\leq C \int_{\Omega} \frac{C|\nabla x' \varphi(0)||x'| (\varepsilon + |x'|)}{(\varepsilon + |x'|)^2} dx + C 
\leq C.
\]

The proof of Lemma 4.3 is completed. \( \Box \)

**Proof of Proposition 2.4**

**Step 1.** Proof of (2.15).

Let \( u^\varepsilon \) be the solution of (2.1). By theorem 6.6 in the appendix in [15], \( u^\varepsilon \) is the minimizer of

\[
I_{\infty}[u] := \frac{1}{2} \int_{\Omega} \left( C^0 e(u), e(u) \right) dx
\]
on \( A \) defined by (1.4). It follows that

\[
\|u^\varepsilon\|_{H^1(\Omega)} \leq C\|e(u^\varepsilon)\|_{L^2(\Omega)} \leq CI_{\infty}[u^\varepsilon] \leq C.
\]
By the Sobolev trace embedding theorem,
\[ \|u^\varepsilon\|_{L^2(\partial D_1 \cap B_R)} \leq C. \]

Recalling that
\[ u^\varepsilon = \sum_{\alpha=1}^{\frac{d(d+1)}{2}} C^{\alpha} \psi_\alpha, \quad \text{on } \partial D_1. \]

If \( C := (C^1, C^2, \ldots, C^{\frac{d(d+1)}{2}}) = 0 \), there is nothing to prove. Otherwise,
\[ C \geq |C| \left\| \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \tilde{C}_\alpha \psi_\alpha \right\|_{L^2(\partial D_1 \cap B_R)}, \quad (4.22) \]

where \( \tilde{C}_\alpha = \frac{C^\alpha}{|C|} \) and \( |\tilde{C}| = 1 \). It is easy to see that
\[ \left\| \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \tilde{C}_\alpha \psi_\alpha \right\|_{L^2(\partial D_1 \cap B_R)} \geq \frac{1}{C}. \quad (4.23) \]

Indeed, if not, along a subsequence \( \varepsilon \to 0 \), \( \tilde{C}_\alpha \to C_\alpha \), and
\[ \left\| \sum_{\alpha=1}^{\frac{d(d+1)}{2}} C_\alpha \psi_\alpha \right\|_{L^2(\partial D_1 \cap B_R)} = 0, \]

where \( \partial D_1^* \) is the limit of \( \partial D_1 \) as \( \varepsilon \to 0 \) and \( |\tilde{C}| = 1 \). This implies
\[ \sum_{\alpha=1}^{\frac{d(d+1)}{2}} C_\alpha \psi_\alpha = 0 \quad \text{on } \partial D_1^* \cap B_R. \]

But \( \{\psi_\alpha|_{\partial D_1^* \cap B_R}\} \) is easily seen to be linear independent, according to lemma 6.1 in the appendix of [16], we must have \( C = 0 \). This is a contradiction. (2.15) follows from (4.22) and (4.23).

**Step 2.** Proof of (2.16). According to Lemma 4.1, the matrix
\[ \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \]

is positive definite, so invertible. Moreover,
\[ A \geq \frac{1}{C} I_{d \times d}, \quad D \geq \frac{1}{C} I_{\frac{d(d-1)}{2} \times \frac{d(d-1)}{2}}. \quad (4.24) \]

Therefore, from (4.12), we have
\[ \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}^{-1} \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}. \]

For \( d \geq 4 \), it is easy to see from Lemma 4.1 and Lemma 4.3 that
\[ |X^1| \leq C. \]
Next, we prove (2.16) for $d = 2, 3$. By lemma 6.2 in Appendix of [16] and Lemma 4.2,
\[
\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} + (\text{Errors}),
\]
where
\[
\left| (\text{Errors}) \right| \sim o(\rho_d(\varepsilon)). \tag{4.25}
\]
Then,
\[
\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} P^1 \\ P^2 \end{pmatrix} + (\text{Errors}) \begin{pmatrix} P^1 \\ P^2 \end{pmatrix} = \begin{pmatrix} A^{-1} P^1 \\ D^{-1} P^2 \end{pmatrix} + (\text{Errors}).
\]
Therefore,
\[
X^1 = A^{-1} P^1 + \text{Errors} = \frac{1}{\det A} A^* P^1 + \text{Errors}, \tag{4.26}
\]
where $A^* = \left( a^*_{\alpha \beta} \right)$ is the adjoint matrix of $A$. Following Lemma 4.2, it is clear that
\[
A^* \sim \begin{pmatrix} \hat{c}_1 & \cdots & o \left( \frac{1}{(\rho_d(\varepsilon))^{d/2}} \right) \\ \vdots & \ddots & \vdots \\ o \left( \frac{1}{(\rho_d(\varepsilon))^{d/2}} \right) & \cdots & \hat{c}_d \end{pmatrix},
\]
for some constants $\hat{c}_\alpha \neq 0, \alpha = 1, \cdots, d$, independent of $\varepsilon$. In view of (4.10) and (4.14), we obtain
\[
|X^1| \leq C\rho_d(\varepsilon).
\]
Therefore,
\[
|C^\alpha - \varphi^\alpha(0)| \leq C\rho_d(\varepsilon), \quad \alpha = 1, \cdots, d.
\]
Proposition 2.4 is established. \hfill \blacksquare

5. Proof of Theorem 1.3 (Lower bound)

In order to prove Theorem 1.3, we first prove $b_\beta \to b^*_\beta$, as $\varepsilon \to 0$, $\beta = 1, \cdots, d$.

Lemma 5.1. For $d \geq 3$, $\beta = 1, 2, \cdots, d$,
\[
|b_\beta - b^*_\beta| \leq C \left( \|\nabla_x \varphi(0)\| + \|\nabla^2 \varphi\|_{L^\infty(\partial D)} \right) \varepsilon^{\gamma_d};
\]
for $d = 2$, if $\nabla_x \varphi^\beta(0) = 0$ for $\beta = 1$ or 2, then, for $\alpha \neq \beta$,
\[
|b_\beta - b^*_\beta| \leq C \left( \|\nabla_x \varphi^\alpha(0)\| + \|\nabla^2 \varphi\|_{L^\infty(\partial D)} \right) \varepsilon^{\gamma_2},
\]
where
\[
\gamma_d = \begin{cases} \frac{d-2}{2(d-1)}, & d \geq 3, \\ \frac{1}{6}, & d = 2. \end{cases}
\]
Consequently,
\[
b_\beta \to b^*_\beta, \quad \text{as } \varepsilon \to 0, \quad \beta = 1, 2, \cdots, d.
\]
Proof. We here prove the case \( \beta = 1 \) for instance. The other cases are the same. It follows from the definitions of \( u_0 \) and \( u_1 \) and the integration by parts formula (1.5) that

\[
b_1 = \int_{\partial D_1} \frac{\partial u_0}{\partial v_0} \cdot u_1 = \int_{\Omega} (C^0 e(u_1), e(u_0)) = \int_{\partial D} \frac{\partial u_1}{\partial v_0} \cdot (\varphi(x) - \varphi(0)).
\]

Similarly,

\[
b_1^* = \int_{\partial D_1^*} \frac{\partial u_0^*}{\partial v_0} \cdot \psi_1 = \int_{\partial D} \frac{\partial u_1^*}{\partial v_0} \cdot (\varphi(x) - \varphi(0)),
\]

where \( u_1^* \) satisfies

\[
\begin{cases}
L_{\lambda, \mu} u_1^* = 0, & \text{in } \Omega^*, \\
u_1^* = \psi_1, & \text{on } \partial D_1^* \setminus \{0\}, \\
u_1^* = 0, & \text{on } \partial D.
\end{cases}
\]

Thus,

\[
b_1 - b_1^* = \int_{\partial D} \frac{\partial(u_1 - u_1^*)}{\partial v_0} \cdot (\varphi(x) - \varphi(0)).
\]

Similarly as before, in order to estimate the difference \( u_1 - u_1^* \), we introduce two auxiliary functions

\[
\tilde{u}_1 = \begin{pmatrix} \tilde{v} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{and } \tilde{u}_1^* = \begin{pmatrix} \tilde{v}^* \\ 0 \\ \vdots \\ 0 \end{pmatrix},
\]

where \( \tilde{v} \) is defined in Section 3 and \( \tilde{v}^* \) satisfies \( \tilde{v}^* = 1 \) on \( \partial D_1^* \setminus \{0\} \), \( \tilde{v} = 0 \) on \( \partial D \), and

\[
\tilde{v}^* = \frac{x - h(x)}{h(x) - h(x')}, \quad \Omega^*_R, \quad ||\tilde{v}^*||_{C^2(\Omega^*_R \setminus \Omega^*_R/2)} \leq C,
\]

where \( \Omega^*_R := \{ x \in \Omega^* \mid |x'| < r \} \), for \( r < R \). By (2.7) and (2.8), we have, for \( x \in \Omega^*_R \),

\[
|\nabla_x \tilde{u}_1 - \tilde{u}_1^*| \leq \frac{C}{|x'|},
\]

and

\[
|\partial_{x'} (\tilde{u}_1 - \tilde{u}_1^*)| = \left| \frac{1}{h(x') - h(x)} - \frac{1}{\varepsilon + h(x') - h(x')} \right| \leq \frac{C \varepsilon}{|x'|^2 (\varepsilon + |x'|^2)}. \tag{5.3}
\]

Applying Corollary 3.4 to (5.1), we obtain

\[
|\nabla (u_1^* - \tilde{u}_1^*)| \leq \frac{C}{|x'|}, \quad x \in \Omega^*_R, \tag{5.4}
\]

and

\[
|\nabla_x u_1^*(x)| \leq \frac{C}{|x'|}, \quad |\partial_{x'} u_1^*(x)| \leq \frac{C}{|x'|^2}, \quad x \in \Omega^*_R. \tag{5.5}
\]
Define a cylinder

\[ C_r := \left\{ x \in \mathbb{R}^d \mid |x'| < r, 0 \leq x_n \leq \varepsilon + 2 \max_{|x'|=r} h_1(x') \right\}, \]

for \( r < R_0 \). Next, we divide into two steps to estimate the difference \( u_1 - u_1^* \).

**STEP 1.** Notice that \( u_1 - u_1^* \) satisfies

\[
\begin{cases}
L_{\lambda, \mu}(u_1 - u_1^*) = 0, & \text{in } D \setminus (D_1 \cup D_1^*), \\
u_1 - u_1^* = \psi_1 - u_1^*, & \text{on } \partial D_1 \setminus D_1^*, \\
\partial_x(u_1 - u_1^*) = u_1 - \psi_1, & \text{on } \partial D^*_1 \setminus (D_1 \cup \{0\}), \\
u_1 - u_1^* = 0, & \text{on } \partial D.
\end{cases}
\]

We first estimate \(|u_1 - u_1^*|\) on \( \partial(D_1 \cup D_1^*) \setminus C_{\varepsilon^\gamma} \), where \( 0 < \gamma < 1/2 \) to be determined later. For \( \varepsilon \) sufficiently small, in view of the definition of \( u_1^* \),

\[ |\partial_{x_d}u_1^*(x)| \leq C, \quad x \in \Omega^* \setminus \Omega^*_{R}, \]

we have, for \( x \in \partial D_1 \setminus D_1^* \),

\[ |(u_1 - u_1^*)(x', x_d)| = |u_1^*(x', x_d - \varepsilon) - u_1^*(x', x_d)| \leq C\varepsilon. \quad (5.6) \]

For \( x \in \partial D_1^* \setminus (D_1 \cup C_{\varepsilon^\gamma}) \), by (2.12),

\[ |(u_1 - u_1^*)(x', x_d)| = |u_1(x', x_d) - u_1(x', x_d + \varepsilon)| \leq \frac{C\varepsilon}{\varepsilon + |x'|^2} \leq C\varepsilon^{1-2\gamma}. \quad (5.7) \]

By using (5.3), (3.36) and (5.4), we have, for \( x \in \Omega^*_{R} \) with \(|x'| = \varepsilon^\gamma \),

\[ |\partial_{x_d}(u_1 - u_1^*)(x', x_d)| = |\partial_{x_d}(\tilde{u}_1 - \tilde{u}_1^*) + \partial_{x_d}(u_1 - \tilde{u}_1) + \partial_{x_d}(u_1^* - \tilde{u}_1^*)(x', x_d)| \leq \frac{C\varepsilon}{\varepsilon + |x'|^2} \frac{C}{\varepsilon^\gamma} \leq \frac{C}{\varepsilon^{4\gamma - 1}} + \frac{C}{\varepsilon^\gamma}. \]

Thus, for \( x \in \Omega^*_{R} \) with \(|x'| = \varepsilon^\gamma \), recalling \( u_1 - u_1^* = 0 \) on \( \partial D \), we have

\[ |(u_1 - u_1^*)(x', x_d)| = |(u_1 - u_1^*)(x', x_d) - (u_1 - u_1^*)(x', h(x'))| \leq \sup_{h(x') \leq x_d \leq h_1(x')} |\partial_{x_d}(u_1 - u_1^*)(x', x_d)|_{|x'|=\varepsilon^\gamma} \cdot (h_1(x') - h(x')) \leq \frac{C}{\varepsilon^{4\gamma - 1}} + \frac{C}{\varepsilon^\gamma} \cdot \varepsilon^{2\gamma} \leq C(\varepsilon^{1-2\gamma} + \varepsilon^\gamma). \quad (5.8) \]

Letting \( 1 - 2\gamma = \gamma_\ast \), we take \( \gamma = 1/3 \). Combining (5.6), (5.7) and (5.8), and recalling \( u_1 - u_1^* = 0 \) on \( \partial D \), we obtain

\[ |(u_1 - u_1^*)(x)| \leq C\varepsilon^{1/3}, \quad x \in \partial(D \setminus (D_1 \cup D_1^* \cup C_{\varepsilon^\gamma})). \]

Applying the maximum principle for Lamé systems, see \[33\],

\[ |(u_1 - u_1^*)(x)| \leq C\varepsilon^{1/3}, \quad x \in D \setminus (D_1 \cup D_1^* \cup C_{\varepsilon^\gamma}). \]

Then using the standard interior and boundary estimates for Lamé system, we have, for any \( 0 < \gamma' < 1/3 \),

\[ |\nabla(u_1 - u_1^*)(x)| \leq C\varepsilon^{\gamma'}, \quad x \in D \setminus (D_1 \cup D_1^* \cup C_{\varepsilon^{\frac{1}{3} - \gamma'}}). \]
This implies that
\[ |B^\text{out}| := \left| \int_{\partial D \setminus C_{x_{\gamma}^+}} \frac{\partial(u_1 - u_1^*)}{\partial \nu_v} \right| (\varphi(x) - \varphi(0)) \leq C \varepsilon^\gamma, \tag{5.9} \]
where 0 < \gamma < 1/3 will be determined later.

**STEP 2.** In the following, we estimate
\[ B^\text{in} := \int_{\partial D \cap C_{x_{\gamma}^+}} \frac{\partial(u_1 - u_1^*)}{\partial \nu_v} \right| (\varphi(x) - \varphi(0)) \]
\[ = \int_{\partial D \cap C_{x_{\gamma}^+}} \frac{\partial(\tilde{u}_1 - \tilde{u}_1^*)}{\partial \nu_v} \right| (\varphi(x) - \varphi(0)) \]
\[ + \int_{\partial D \cap C_{x_{\gamma}^+}} \frac{\partial(w_1 - w_1^*)}{\partial \nu_v} \right| (\varphi(x) - \varphi(0)) \]
\[ = : B_a + B_w \]
where \( w_1 = u_1 - \tilde{u}_1, \ w_1^* = u_1^* - \tilde{u}_1^* \). By definition,
\[ B_a = \int_{\partial D \cap C_{x_{\gamma}^+}} \left\{ \lambda \sum_{k=1}^{d} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^*) n_k(\varphi^k(x) - \varphi^k(0)) \right. \]
\[ \left. + \mu \sum_{k=1}^{d} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^*) \left[ n_1(\varphi^1(x) - \varphi^1(0)) + n_k(\varphi^k(x) - \varphi^k(0)) \right] \right\} \]
\[ = : \lambda (B_a^1 + B_a^2) + \mu (B_a^3 + B_a^4 + B_a^5 + B_a^6), \]
where
\[ B_a^1 := \int_{\partial D \cap C_{x_{\gamma}^+}} \sum_{k=1}^{d} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^*) n_k(\varphi^k(x) - \varphi^k(0)), \]
\[ B_a^2 := \int_{\partial D \cap C_{x_{\gamma}^+}} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^*) n_d(\varphi^d(x) - \varphi^d(0)), \]
\[ B_a^3 := \int_{\partial D \cap C_{x_{\gamma}^+}} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^*) n_1(\varphi^1(x) - \varphi^1(0)), \]
\[ B_a^4 := \int_{\partial D \cap C_{x_{\gamma}^+}} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^*) n_1(\varphi^d(x) - \varphi^d(0)), \]
\[ B_a^5 := \int_{\partial D \cap C_{x_{\gamma}^+}} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^*) n_k(\varphi^1(x) - \varphi^1(0)), \]
\[ B_a^6 := \int_{\partial D \cap C_{x_{\gamma}^+}} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^*) n_d(\varphi^1(x) - \varphi^1(0)). \]
According to (5.2), (5.3) and the Taylor expansion of \( \varphi^k(x) \),
\[
|B^4_0| + |B^3_0| + |B^5_0|
\leq \int_{\partial D \cap C_{\frac{1}{4}, -\gamma}} \frac{C}{|x'|} \cdot |x'| \cdot |\nabla x' \varphi(0)||x'| + \int_{\partial D \cap C_{\frac{1}{4}, -\gamma}} C\|
abla^2 \varphi\|_{L^\infty(\partial D)} |x'|^2
\leq C|\nabla x' \varphi(0)|\varepsilon(\frac{1}{4} - \gamma)d + C\varepsilon(\frac{1}{4} - \gamma)(d+1)\|
abla^2 \varphi\|_{L^\infty(\partial D)}; \quad (5.10)
\]
\[
|B^2_0| \leq \int_{\partial D \cap C_{\frac{1}{4}, -\gamma}} \frac{C}{|x'|} \cdot |\nabla x' \varphi(0)||x'| + \int_{\partial D \cap C_{\frac{1}{4}, -\gamma}} C\|
abla^2 \varphi\|_{L^\infty(\partial D)} |x'|^2
\leq C|\nabla x' \varphi(0)|\varepsilon(\frac{1}{4} - \gamma)(d-1) + C\|
abla^2 \varphi\|_{L^\infty(\partial D)}\varepsilon(\frac{1}{4} - \gamma)d; \quad (5.11)
\]
and
\[
|B^4_0| \leq \int_{\partial D \cap C_{\frac{1}{4}, -\gamma}} \frac{C}{|x'|^2} \cdot |x'| \cdot |\nabla x' \varphi(0)||x'| + \int_{\partial D \cap C_{\frac{1}{4}, -\gamma}} C\|
abla^2 \varphi\|_{L^\infty(\partial D)} |x'|^2
\leq C|\nabla x' \varphi(0)|\varepsilon(\frac{1}{4} - \gamma)(d-1) + C\|
abla^2 \varphi\|_{L^\infty(\partial D)}\varepsilon(\frac{1}{4} - \gamma)d. \quad (5.12)
\]
For \( d = 2 \) if \( \nabla x' \varphi^1(0) = 0 \), we have
\[
|B^5_0| \leq \int_{\partial D \cap C_{\frac{1}{4}, -\gamma}} C\|
abla^2 \varphi^1\|_{L^\infty(\partial D)} \leq C\|
abla^2 \varphi^1\|_{L^\infty(\partial D)}\varepsilon(\frac{1}{4} - \gamma)(d-1). \quad (5.13)
\]
If \( d \geq 3 \), we have
\[
|B^5_0| \leq \int_{\partial D \cap C_{\frac{1}{4}, -\gamma}} \frac{C}{|x'|^2} \cdot |\nabla x' \varphi(0)||x'| + \int_{\partial D \cap C_{\frac{1}{4}, -\gamma}} C\|
abla^2 \varphi\|_{L^\infty(\partial D)}
\leq C|\nabla x' \varphi(0)|\varepsilon(\frac{1}{4} - \gamma)(d-2) + C\|
abla^2 \varphi\|_{L^\infty(\partial D)}\varepsilon(\frac{1}{4} - \gamma)(d-1). \quad (5.14)
\]
Hence, combining (5.10)–(5.14) yields that for \( \varepsilon > 0 \) sufficiently small, if \( d = 2 \) and \( \nabla x' \varphi^1(0) = 0 \),
\[
|B^4_0| \leq C \left( |\nabla x' \varphi^2(0)| + \|
abla^2 \varphi\|_{L^\infty(\partial D)} \right) \varepsilon^{\frac{1}{4} - \gamma}; \quad (5.15)
\]
and if \( d \geq 3 \),
\[
|B^5_0| \leq C \left( |\nabla x' \varphi(0)| + \|
abla^2 \varphi\|_{L^\infty(\partial D)} \right) \varepsilon(\frac{1}{4} - \gamma)(d-2). \quad (5.16)
\]
We now estimate \( B_w \). It follows from Corollary 3.4 that
\[
|\nabla w_1(x)| \leq \frac{C}{\sqrt{\delta(x)}}, \quad 0 < |x'| \leq R, \quad (5.17)
\]
and
\[
|\nabla w_1^r(x)| \leq \frac{C}{|x'|}, \quad 0 < |x'| \leq R. \quad (5.18)
\]
By definition,
\[
B_w = \int_{\partial D \cap C_{\frac{1}{4}, -\gamma}} \left\{ \lambda \sum_{k,l=1}^{d} \partial_{x_k} (w_1^k - w_1^k) n_l (\varphi^l(x) - \varphi^l(0)) + \mu \sum_{k,l=1}^{d} \partial_{x_k} (w_1^k - w_1^k) n_l (\varphi^k(x) - \varphi^k(0)) \right\}.
\]
By (5.17), (5.18) and the Taylor expansion of \( \varphi'(x) \),

\[
|\mathcal{B}_w| \leq \int_{\partial D \cap C_{\frac{1}{4}-\delta}} \frac{C}{|x'|} \cdot (|\nabla_{x'} \varphi'(0)||x'| + \|\nabla^2 \varphi'||L^\infty(\partial D)|x'|^2)
\]

\[
\leq C |\nabla_{x'} \varphi'(0)| \varepsilon^{\left(\frac{1}{4} - \gamma\right)(d-1)} + \|\nabla^2 \varphi'||L^\infty(\partial D)\varepsilon^{\left(\frac{1}{4} - \delta\right)d}
\]

\[
\leq C \left( |\nabla_{x'} \varphi(0)| + \|\nabla^2 \varphi'||L^\infty(\partial D)\varepsilon^{\left(\frac{1}{4} - \delta\right)(d-1)} \right). \quad (5.19)
\]

This, together with (5.15), implies that, for \( d = 2 \), if \( \nabla_{x'} \varphi'(0) = 0 \),

\[
|\mathcal{B}_n| \leq |\mathcal{B}_a| + |\mathcal{B}_w| \leq C \left( |\nabla_{x'} \varphi(0)| + \|\nabla^2 \varphi'||L^\infty(\partial D)\varepsilon^{\frac{1}{4} - \delta}\right).
\]

Combining with (5.9), we now simply choose \( \tilde{\gamma} = \gamma_2 = 1/6 \), such that \( \frac{1}{4} - \gamma_2 = \gamma_2 \).

Thus, we have, for \( d = 2 \),

\[
|b_1 - b_1^*| \leq |\mathcal{B}_n| + |\mathcal{B}^{out}| \leq C \left( |\nabla_{x'} \varphi'(0)| + \|\nabla^2 \varphi'||L^\infty(\partial D)\varepsilon^{1/6}\right).
\]

For \( d \geq 3 \), combining (5.19) together with (5.19) yields that

\[
|\mathcal{B}_n| \leq |\mathcal{B}_a| + |\mathcal{B}_w| \leq C \left( |\nabla_{x'} \varphi(0)| + \|\nabla^2 \varphi'||L^\infty(\partial D)\varepsilon^{(\frac{1}{4} - \delta)(d-2)}\right).
\]

Therefore, using (5.9) again and picking \( \tilde{\gamma} = \gamma_d = \frac{d-2}{3(d-1)} \) (such that \( \frac{1}{4} - \gamma_d = \gamma_d \)), we have, for \( d \geq 3 \),

\[
|b_1 - b_1^*| \leq |\mathcal{B}_n| + |\mathcal{B}^{out}| \leq C \left( |\nabla_{x'} \varphi(0)| + \|\nabla^2 \varphi'||L^\infty(\partial D)\varepsilon^{\gamma_d}\right).
\]

The proof of Lemma 5.1 is completed. \( \square \)

**Proof of Theorem 1.3** Under the assumptions of Theorem 1.3 that \( b_{k_0}^* \neq 0 \) for some integer \( 1 \leq k_0 \leq d \), it follows from Lemma 5.1 that there exists a universal constant \( C_0 > 0 \) and a sufficiently small number \( \varepsilon_0 > 0 \), such that, for \( 0 < \varepsilon < \varepsilon_0 \),

\[
|b_{k_0}| > C_0 > 0. \quad (5.20)
\]

By the definition of \( \mathcal{A}^* = (a_{\alpha \beta}^*)_{d \times d} \), where \( a_{\alpha \beta}^* \) is the cofactor of \( a_{\alpha \beta} \), and Lemma 4.2, we have

\[
a_{\alpha \alpha}^* \sim \frac{1}{(\rho_d(\varepsilon))^{d-1}}, \alpha = 1, \ldots, d; \quad a_{\alpha \beta}^* \sim \frac{1}{(\rho_d(\varepsilon))^{d-2}}, \alpha \neq \beta. \quad (5.21)
\]

According to (4.13), (4.25), (4.26), (5.20) and (5.21), for sufficiently small \( \varepsilon \),

\[
|C^{k_0} - \varphi^{k_0}(0)| = \left| \frac{1}{\det \mathcal{A}} \left[ a_{k_0k_0}^* b_{k_0} + \sum_{\beta \neq k_0} a_{k_0\beta}^* b_\beta \right] + \text{Errors} \right|
\]

\[
\geq \frac{1}{2} \frac{1}{\det \mathcal{A}} a_{k_0k_0}^* |b_{k_0}| \geq \frac{\rho_d(\varepsilon)}{C}. \quad (5.22)
\]

On the other hand, in view of Corollary 3.4, we obtain

\[
|\partial_{x_d} \tilde{u}^{k_0}_{k_0}| = |\partial_{x_d} w^{k_0}_{k_0} + \partial_{x_d} (u^{k_0}_{k_0} - \tilde{u}^{k_0}_{k_0})|
\]

\[
\geq |\partial_{x_d} \tilde{u}^{k_0}_{k_0}| - |\partial_{x_d} (u^{k_0}_{k_0} - \tilde{u}^{k_0}_{k_0})| \geq \frac{1}{C(\varepsilon + |x'|^2)}, \quad x \in \Omega_R. \quad (5.23)
\]
At the same time, since \( \tilde{u}_{k_0}^\alpha = 0 \) if \( \alpha \neq k_0 \), it is easy to see from Corollary 3.4 that
\[
|\partial_{x_d} u_{k_0}^\alpha| = |\partial_{x_d} \tilde{u}_{k_0}^\alpha + \partial_{x_d} (u_{k_0}^\alpha - \tilde{u}_{k_0}^\alpha)|
= |\partial_{x_d} (u_{k_0}^\alpha - \tilde{u}_{k_0}^\alpha)| \leq \frac{C}{\sqrt{\varepsilon + |x'|^2}}, \quad \alpha \neq k_0, \quad x \in \Omega_R.
\] (5.24)

Therefore, by a combination of the estimates (5.22), (5.23), (5.24), we get, for \((0', x_d) \in P_1 P\),
\[
\left| \sum_{\alpha=1}^{d} (C^\alpha - \varphi^\alpha(0)) \nabla u_{\alpha} \right| \geq \left| \sum_{\alpha=1}^{d} (C^\alpha - \varphi^\alpha(0)) \partial_{x_d} u_{k_0}^\alpha \right|
\geq \left| (C^k_0 - \varphi^k_0(0)) \partial_{x_d} u_{k_0}^\alpha \right| - \left| \sum_{\alpha \neq k_0}^{d} (C^\alpha - \varphi^\alpha(0)) \partial_{x_d} u_{\alpha} \right|
\geq \rho_d(\varepsilon) \frac{C}{\varepsilon}.
\] (5.25)

Here we used the assumption that \( b_{\alpha}^* = 0 \) for \( \alpha \neq k_0, \ 1 \leq \alpha \leq d \) when \( d \geq 4 \).

By means of Corollary 2.3, (2.15), Lemma 4.2 and Lemma 4.3, especially for \( x = (0', x_d) \in P_1 P\),
\[
\left| \sum_{\alpha=d+1}^{d+1} C^\alpha \nabla u_{\alpha} \right| \leq \frac{C(\varepsilon + |x'|)}{\varepsilon + |x'|^2} \leq C,
\] (5.26)
\[
|\nabla u_0| \leq \frac{C|\nabla \varphi(0)||x'|}{\varepsilon + |x'|^2} + C \leq C.
\] (5.27)

Combining (5.25), (5.26), (5.27) and (2.5) immediately yields that for \( x = (0', x_d) \in P_1 P\),
\[
|\nabla u(0', x_d)| \geq \frac{\rho_d(\varepsilon)}{C \varepsilon}, \quad 0 < x_d < \varepsilon.
\]

Theorem 1.3 is thus established. \( \square \)

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**References**

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Comm. Pure Appl. Math. 12 (1959), 623-727.

[2] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. Comm. Pure Appl. Math. 17 (1964), 35-92.

[3] H. Ammari, E. Bonnetier, F. Triki, M. Vogelius, Elliptic estimates in composite media with smooth inclusions: an integral equation approach. Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 2, 453-495.

[4] H. Ammari, G. Ciraolo, H. Kang, H. Lee and K. Yun, Spectral analysis of the Neumann-Poincaré operator and characterization of the stress concentration in anti-plane elasticity. Arch. Ration. Mech. Anal. 208 (2013), 275-304.
[5] H. Ammari, G. Dassios, H. Kang and M. Lim, Estimates for the electric field in the presence of adjacent perfectly conducting spheres, Quart. Appl. Math. 65 (2007), no. 2, 339-355.

[6] H. Ammari, P. Garapon, H. Kang; H. Lee, A method of biological tissues elasticity reconstruction using magnetic resonance elastography measurements. Quart. Appl. Math. 66 (2008), no. 1, 139-175.

[7] H. Ammari, H. Kang and M. Lim, Gradient estimates for solutions to the conductivity problem. Math. Ann. 332 (2005), no. 2, 277-286.

[8] H. Ammari, H. Kang, K. Kim, H. Lee, Strong convergence of the solutions of the linear elasticity and uniformity of asymptotic expansions in the presence of small inclusions. J. Differential Equations 254 (2013), no. 12, 4446-4464.

[9] H. Ammari; H. Kang; H. Lee; J. Lee; M. Lim, Optimal estimates for the electrical field in two dimensions. J. Math. Pures Appl. 88 (2007), 307-324.

[10] H. Ammari, H. Kang, H. Lee, M. Lim and H. Zribi, Decomposition theorems and fine estimates for electrical fields in the presence of closely located circular inclusions. J. Differential Equations. 247 (2009), no. 11, 2897-2912.

[11] I. Babuška, B. Andersson, P. Smith and K. Levin, Damage analysis of fiber composites. I. Statistical analysis on fiber scale. Comput. Methods Appl. Mech. Engrg. 172 (1999), no 1-4, 27-77.

[12] E.S. Bao, Y.Y. Li and B.Yin, Gradient estimates for the perfect conductivity problem, Arch. Ration. Mech. Anal., 193 (2009), no. 1, 195-226.

[13] E.S. Bao, Y.Y. Li and B.Yin, Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions. Comm. Partial Differential Equations 35 (2010), no. 11, 1982-2006.

[14] J.G. Bao, H.G. Li and Y.Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients. Arch. Rational Mech. Anal. 215 (2015), 307-351.

[15] J.G. Bao, H.G. Li and Y.Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients in dimensions greater than two, arXiv:1601.07879v1.

[16] E. Bonnetier and F. Triki, Asymptotics in the presence of inclusions of small volume for a conduction equation: a case with a non-smooth reference potential. Imaging microstructures, Contemp., Math., vol. 494, Amer.Math. Soc., Providence, RI, 2009, 95-111.

[17] E. Bonnetier and F. Triki, On the spectrum of the Poincaré variational problem for two close-to-touching inclusions in 2D. Arch. Ration. Mech. Anal. 209 (2013), no. 2, 541-567.

[18] E. Bonnetier and M. Vogelius, An elliptic regularity result for a composite medium with “touching” fibers of circular cross-section. SIAM J. Math. Anal. 31 (2000), no. 3, 651-677.

[19] B. Budiansky and G.F. Carrier, High shear stresses in stiff fiber composites. J. Appl. Mech. 51 (1984), 733-735.

[20] H.J. Dong, Gradient estimates for parabolic and elliptic systems from linear laminates. Arch. Ration. Mech. Anal. 205 (2012), no. 1, 119-149.

[21] H.J. Dong and J.G. Xiong, Boundary gradient estimates for parabolic and elliptic systems from linear laminates. Int. Math. Res. Not. IMRN 2015, no. 17, 7734-7756.

[22] H.J. Dong; H. Zhang, On an elliptic equation arising from composite materials. arXiv: 1505.01042.

[23] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems. Annals of Mathematics Studies, 105, Princeton University, Princeton, N.J., 1983.

[24] Y. Gorb; A. Novikov, Blow-up of solutions to a p-Laplace equation, Multiscal Model. Simul. 10 (2012), 727-743.

[25] H. Kang, M. Lim and K. Yun, Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities. J. Math. Pures Appl. (9) 99 (2013), no. 2, 234-249.

[26] H. Kang, H. Lee, K. Yun, Optimal estimates and asymptotics for the stress concentration between closely located stiff inclusions. Math. Ann. 363 (2015), no. 3-4, 1281-1306.

[27] J.B. Keller, Stresses in narrow regions. Trans. ASME J. Appl. Mech., 60 (1993), 1054-1056.

[28] J.B. Keller, Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders or nonconducting cylinders. J. Appl. Phys. 3 (1963), 991-993.

[29] H.G. Li, Y.Y. Li, E.S. Bao and B. Yin, Derivative estimates of solutions of elliptic systems in narrow regions. Quart. Appl. Math. 71 (2014), no.3, 589-596.
[31] Y.Y. Li and L. Nirenberg, Estimates for elliptic systems from composite material. Dedicated to the memory of Jürgen K. Moser, Comm. Pure Appl. Math. 56 (2003), no. 7, 892-925.
[32] Y.Y. Li and M. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. Arch. Ration. Mech. Anal. 153 (2000), no. 2, 91-151.
[33] M. Lim and K. Yun, Blow-up of electric fields between closely spaced spherical perfect conductors, Comm. Partial Differ. Equa.,34 (2009), no. 10-12, 1287-1315.
[34] X. Markenscoff, Stress amplification in vanishingly small geometries. Computational Mechanics 19 (1996), 77-83.
[35] V.G. Maz’ya, A.B. Movchan and M.J. Nieves, Uniform asymptotic formulor for Green’s tensors in elastic singularly perturbed domains, Asym. Anal. 52 (2007), 173-206.
[36] O. Oleinik, A. Shamaev and G. Yosifian, Mathematical problems in elasticity and homogenization. Studies in Mathematics and its Applications, vol. 26, North-Holland Publishing Co., Amsterdam, 1992.
[37] K. Yun, Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape. SIAM J. Appl. Math., 67 (2007), no. 3, 714-730.
[38] K. Yun, Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections. J. Math. Anal. Appl. 350 (2009), no. 1, 306-312.

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