Freeness over the diagonal and global fluctuations of complex Wigner matrices

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Abstract

We characterize the limiting 2nd order distributions of certain independent complex Wigner and deterministic matrices using Voiculescu’s notions of freeness over the diagonal. If the Wigner matrices are Gaussian, Mingo and Speicher’s notion of 2nd order freeness gives a universal rule, in terms of marginal 1st and 2nd order distribution. We adapt and reformulate this notion for operator-valued random variables in a 2nd order probability space. The Wigner matrices are assumed to be permutation invariant with null pseudo variance and the deterministic matrices to satisfy a restrictive property.

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Introduction

Definition 0.1. We call a Wigner matrix a Hermitian random matrix \( X_N = \sqrt{\frac{1}{N}} (x_{ij}) \), such that the entries \((x_{ij})_{i,j}\) are centered and independent, diagonal entries are identically distributed real random variables, sub-diagonal entries are identically distributed complex random variables, the distribution of \( x_{ij} \) does not depend on \( N \) and \( x_{ij} \) has bounded moments of all orders \( E(|x_{ij}|^k) < \infty \) for all \( i, j, k \geq 0 \). We say that \( X_N \) has null pseudo-variance if \( E[x_{ij}^2] = 0 \) for \( i \neq j \).

By Voiculescu’s celebrated asymptotic freeness theorem [25], we know that free probability theory describes the global joint distribution of a large class of random matrices in large dimension. This result states in particular that if \( X_N = (X_j)_{j \in J} \) is a family of independent Gaussian Wigner matrices and \( Y_N = (Y_k)_{k \in K} \) a family of deterministic matrices which converges in *-distribution to variables \( Y = (y_k)_{k \in K} \), then the joint family \( X_N \cup Y_N \) converges to \( x \cup y \) where \( x = (x_j)_{j \in J} \) is a free semicircular system, free from \( y \). Convergence is with respect to the expectation of the normalized trace, namely for all *-polynomials \( p \in C(x,y,y^*) \)

\[
\Phi_N(p) := E \left[ \frac{1}{N} Tr \, p(X_N, Y_N, Y_N^*) \right] \rightarrow_{N \rightarrow \infty} \Phi(p). \tag{0.1}
\]

The convergence toward a semicircular system is very robust, in particular it holds for non Gaussian Wigner matrices [8] and for many variations of Wigner’s model: with bistochastic variance profile [24], for the band matrix with of intermediate growth [2], the diluted matrix [15]. Remarkably, asymptotic freeness appears for certain non independent matrices, as for independent GUE matrices along with their transpose and their degree [5, 18, 6].

In this article, we study the joint global fluctuations, that is the collection \( Z_N = (Z_N(p))_p \) of complex random variables indexed by *-polynomials,

\[
Z_N(p) := Tr \, p(X_N, Y_N, Y_N^*) - E[Tr \, p(X_N, Y_N, Y_N^*)]. \tag{0.2}
\]

The limiting fluctuations for linear statistics of a single real Wigner matrix is characterized by Khorunzhy, Khoruzhenko and Pastur in [13], with the approach of the Stieltjes transform rather than moments. The limit is Gaussian but is not universal: it depends on the fourth moment \( E[|x_{ij}|^4] \) of the Wigner matrix \( X_N = (\sqrt{\frac{1}{N}} x_{ij}) \). See [9, Theorem 2.1] for a statement for moments.

When the Wigner matrices are GUE matrices, Mingo and Speicher prove in [19] that \( Z_N \) converges also to a Gaussian process whenever the family \( Y_N \) have a limiting *-distribution. They define the notion of second-order freeness which describes more generally the limit of unitarily invariant matrices at first and second order: in particular, this notion gives a formula for
the limiting covariance

\[ \Phi_N^{(2)}(p, q^*) = \mathbb{E}[Z_N(p)Z_N(q)] \xrightarrow{N \to \infty} \Phi^{(2)}(p, q^*) \]  

(0.3)

that can be interpreted in terms of the so-called spoke diagrams [19, Definition 6.3 and Figure 11]. In general Wigner matrices are not asymptotically 2nd order free: the 2nd order limit of real orthogonal invariant random matrices is given by Redelmeier’s notion of real 2nd order freeness [23]. Covariance computations for these ensembles involve also the so-called twisted spoke diagrams.

In this article, we consider complex Wigner matrices with null pseudo-variance. The dependence in the fourth moment for the fluctuation of Wigner matrices is an obstacle for a universal presentation. Nevertheless, under the hypotheses on the Wigner and deterministic matrices stated in Section 2, we see in this article that Voiculescu’s notion of operator-valued probability space over the diagonal allows to bypass this difficulty. In a sense properly clarified in next sections, freeness with amalgamation over the diagonal rules the asymptotic fluctuations of these ensembles thanks to a modification of Mingo and Speicher theory.

A motivation for this article is the following open question in random matrix theory: how to characterize the limit of spectral linear statistics for the sum \( X_N + Y_N \) of a Wigner and an independent Hermitian matrix? Khorunzhy consider in [12] the case of a GOE and a deterministic matrix; Ji and Lee in [10] consider real Wigner and diagonal matrices; Dallaporta and Février in [7] consider complex Wigner (with arbitrary real pseudovariance) and deterministic diagonal matrices. The question remains open when \( Y_N \) is not diagonal, for which the global fluctuations depend not only on the spectrum of \( Y_N \). Heuristically, studying the asymptotic of the process \( Z_N \) of (0.2) we search for a good theoretical framework for this spectral linear statistics problem, with the perspective of finding a good analogue of the 2nd order Stieltjes transform. Our investigation suggests that freeness over the diagonal could be useful for Wigner matrices with zero pseudo-variance, but only with deterministic matrices that satisfy an assumption somehow opposed to diagonality, see assumption (H5) and Remark 2.3. Finding a unifying point of view to include the models of [10, 7] remains an open question.

We pursue here a combinatorial analysis initiated in [16]. Under mild hypotheses on the deterministic matrices and assuming the Wigner matrices permutation invariant (see Section 2), we know from [16] that \( Z_N \) is bounded. Its possible limits are Gaussian and we have a combinatorial description of the limit of the 2nd order *-distribution. Two important cases can be distinguished.

1. Let \( \Delta(A) = \text{diag}(A_{ii}) \) be the diagonal of a matrix \( A \). If all Wigner matrices have null pseudo-variance, then the possible limit of \( \Phi_N^{(2)} \) de-
depends on the limit of \( \mathbb{E}\left[ \frac{1}{N} \text{Tr}(\Delta[p(Y_N)]\Delta[q(Y_N)]) \right] \), for all polynomials \( p, q \).

2. Otherwise it also depends on the limit of \( \mathbb{E}\left[ \frac{1}{N} \text{Tr}(P(Y_N)Q(Y_N)^t) \right] \)

where \( A^t \) is the transpose of the matrix \( A \).

Moreover we know that random and deterministic matrices are asymptotically free over the diagonal under mild conditions \( \text{[3]} \). With these techniques we prove that this behavior is somehow robust at \( 2^{nd} \) order for the matrices considered.

Finally, we observe a common phenomenon in \( 2^{nd} \) order freeness and its extension, formulated briefly as follow: denoting \( \Phi : E \to \mathbb{C} \) and \( \Phi^{(2)} : E^2 \to \mathbb{C} \) the \( 1^{st} \) and \( 2^{nd} \)-order distributions of a limiting multi-matrix model \( \mathbf{a} = \sqcup_{\ell \in L} a_\ell \), there are canonical subspaces \( F_n \subset E, n \geq 2 \), orthogonal w.r.t. \( \Phi^{(2)} \), where \( \Phi^{(2)} \) is collinear to the canonical bilinear form associated to \( \Phi \), namely

\[
\Phi^{(2)}(p, q^*) = n\Phi(pq^*), \quad \forall p, q \in F_n ; \tag{0.4}
\]

the spaces \( F_n \) depends only on the class of the model (e.g. unitarily invariant matrices or Wigner with null pseudo-variances) and are associated to a notion of independence (e.g. freeness or freeness over the diagonal).

For the matrix model studied in this article, \( \Phi^{(2)} \) is not universal so it cannot be determined by \( \Phi \) only. Yet we can state a collinearity property \( \text{[0.4]} \) to describe partially \( \Phi^{(2)} \) by considering the problem on a space \( E \) that extends the space of polynomials (with the \( \Delta \)-polynomials). The description of the universal part of \( \Phi^{(2)} \) comes with an additional condition, in the form of a Leibniz Formula.

1 Preliminaries

In order to describe the global fluctuations of random matrices we propose to use the following abstract setting.

**Definition 1.1.** An operator-valued \( 2^{nd} \) order probability space is the data of a quintuple \( (\mathcal{A}, \Phi, \Phi^{(2)}, \mathcal{D}, \Delta) \) as follow.

1. \( (\mathcal{A}, \Phi) \) is a tracial \(*\)-probability space, namely \( \mathcal{A} \) is a \(*\)-algebra and \( \Phi \) is a unital and tracial linear form on \( \mathcal{A} \): \( \Phi[1] = 1, \quad \Phi[ab] = \Phi[ba] \forall a, b \in \mathcal{A} \).

2. \( (\mathcal{A}, \Phi, \Phi^{(2)}) \) is a \( 2^{nd} \) order probability space, i.e \( \Phi^{(2)} \) is a symmetric bilinear form on \( \mathcal{A} \), tracial in each variable, s.t. \( \Phi^{(2)}[1, a] = \Phi^{(2)}[a, 1] = 0 \forall a \in \mathcal{A} \).
3. \((A, D, \Delta)\) is an operator-valued probability space, namely \(D \subset A\) a unital subalgebra and \(\Delta : A \to D\) is a conditional expectation, i.e. a unital linear map s.t. \(\Delta(d_1 ad_2) = d_1 \Delta(a) d_2, \ \forall a \in A, \forall d_1, d_2 \in D\).

4. We furthermore assume that \(\Phi\) is invariant under \(\Delta\), i.e \(\Phi[\Delta(a)] = \Phi[a]\) for any \(a \in A\), and that \(\Phi^{(2)}\) is invariant under \(\Delta\) in both variables.

Let \(M^N(C)\) be the set of deterministic \(N\) by \(N\) complex matrices and \(D^N(C)\) the set of diagonal matrices. For any \(N \geq 1\), fix \(\Omega^N\) a probability space in the classical sense. Denote by \(L_{\infty}^{-}(M^N(C))\) and \(L_{\infty}^{-}(D^N(C))\) the spaces of random and random diagonal matrices on \(\Omega^N\) respectively, whose entries have finite moments of all orders. We set \(\Phi^N : A \mapsto E\left[\frac{1}{N} \sum_{i=1}^{N} A_{ii}\right]\) the expectation of the normalized trace on \(L_{\infty}^{-}(M^N(C))\), we set the covariance of traces function \(\Phi^{(2)}_N : (A, B) \mapsto E\left[(Tr A - E[Tr A])(Tr B - E[Tr B])\right]\), and we set \(\Delta : A \mapsto \text{diag}(A)\) the projection of a random matrix into its diagonal part. We then get an operator-valued 2nd order probability space \((L_{\infty}^{-}(M^N(C)), \Phi^N, \Phi^{(2)}_N, L_{\infty}^{-}(D^N(C)), \Delta)\).

Let \(X_N\) be a collection of independent Wigner matrices and \(Y_N\) be deterministic matrices. As presented previously, our purpose is to understand the limit \((x, y)\) of \((X_N, Y_N)\) as \(N\) goes to infinity in the sense of 2nd order probability spaces and compute \(\Phi^{(2)}[p, q] = \lim_\epsilon \Phi^{(2)}_{N_{\epsilon}}[p(X_N, Y_N), q(X_N, Y_N)]\) for \(p, q\) *-polynomials. The convergence is already proved in [16], but here the additional operator-valued setting provides a more natural presentation of the limit.

We do not construct a 2nd order theory of operator-valued probability, with a bilinear map \(\Delta^{(2)}\) describing fluctuation operators, e.g. \(N(\Delta(A) - E[A])\) for a random matrix \(A\). Also, contrary to the other applications of freeness over the diagonal in [24, 4] we need random diagonal matrix coefficients. The strategy in this article is to introduce only the ad-hoc material sufficient to solve our matrix problem, based on a restriction of the notion of traffics distribution [14], for which we already know all asymptotics of interest.

Let \(J\) be an index set and let \(x = (x_j)_{j \in J}\) be non commutative indeterminates. We set \(\mathbb{C}(x)\) the space of non commutative polynomials in \(x\). We recall that a \(D\)-valued monomial in the variables \(x\) is a word \(p = d_0 x_{j_1} d_1 \cdots x_{j_n} d_n\), where \(n \geq 0, d_0, \ldots, d_L \in D\) (called the coefficients of \(p\)), and \(j_1, \ldots, j_n \in J\). We set \(D(x)\) the space of \(D\)-valued polynomials, finite sums of \(D\)-valued monomials in \(x\).
Definition 1.2. Let \((A, D, \Delta)\) be an operator-valued probability space.

- The operator-valued distribution \(\Delta_a\) of a collection \(a = (a_j)_{j \in J} \in A^J\) is the map \(\Delta_a : p \mapsto \Delta(p(a)) \in D\), defined for all \(p \in D(x)\).
- A tuple \((a_1, \ldots, a_n)\) is \(D\)-alternating in the collections \(a_\ell, \ell \in L\), if \(n \geq 2\) and \(a_i = p_i(a_\ell_i)\) for indices \(\ell_1 \neq \ell_2 \neq \cdots \neq \ell_n \in L\) where \(p_i\) is a \(D\)-valued polynomial. The tuple is \(\Delta\)-centered if \(\Delta(a_i) = 0, \forall i = 1, \ldots, n\).
- The collection \(a_\ell, \ell \in L\), are free over \(D\) whenever \(\Delta(a) = 0\) for any \(a = a_1 \cdots a_n\) where \(n \geq 2\) and \((a_1, \ldots, a_n)\) is \(D\)-alternating and \(\Delta\)-centered.

We instead use the following notion of \(\Delta\)-distribution, based on a representation of operator-valued polynomials coefficients. This approach is used in \(\mathbb{R}\) in the case of matrices, we introduce it in the general case and in a simpler way.

Definition 1.3. Let \(x = (x_j)_{j \in J}\) be a collection of indeterminates.

1. A bracketed word in \(x\) is a word in the \(x_j\) and in a left and a right bracket symbols \([\) and \(]\), that belongs to the smallest monoid by concatenation containing \(x\) and stable by the bracketing operation \(\Delta : w \mapsto [w]\).
2. The set of \(\Delta\)-monomials in the variables \(x\) is the quotient monoid given by the relations: \(\Delta(1) = 1\), where \(1\) is the empty word, and \(\forall \omega_1, \omega_2, \omega_3\)
   \[\Delta(\Delta(\omega_1)\omega_2\Delta(\omega_3)) \sim \Delta(\omega_1)\Delta(\omega_2)\Delta(\omega_3)\].

We set \(\mathbb{C}(x)_\Delta\) the space of \(\Delta\)-polynomials, i.e. finite complex linear combinations of \(\Delta\)-monomials in \(x\). Product and evaluation of \(\Delta\) are extended linearly. For any \(q \in \mathbb{C}(x)_\Delta\) and for any collection \(a = (a_j)_{j \in J}\) of elements of an operator-valued probability space \((A, D, \Delta)\), we set \(q(a) \in \mathbb{A}\) by substituting indeterminates by elements of \(a\) and interpreting brackets as the evaluation of \(\Delta\).

Definition 1.4. Let \((A, \Phi, \Phi(2), D, \Delta)\) be an operator-valued 2nd order probability space and \(a = (a_j)_{j \in J} \in A^J\). For each \(N \geq 1\), let \(A_N = (A_{N,j})_{j \in J}\) be a collection of random matrices in \(L^{\infty}(M_N(\mathbb{C}))\). Let \(x = (x_j)_{j \in J}\) be indeterminates.

1. The \(\Delta\)-distribution of \(a\) is the map \(\Phi_a : p \mapsto \Phi[p(a)]\) defined for \(p \in \mathbb{C}(x, x^*)_\Delta\). We say that \(A_N\) converges to \(a\) in \(\Delta\)-distribution whenever the \(\Delta\)-distribution converges pointwise: \(\forall p \in \mathbb{C}(x, x^*)_\Delta, \Phi_{A_N}[p] \xrightarrow{\quad N \rightarrow \infty} \Phi_a[p]\).
2. The 2nd order \(\Delta\)-distribution of \(a\) is the map \(\Phi^{(2)}_a: (p,q) \mapsto \Phi^{(2)}[p(a),q(a)]\), defined for \(p,q \in \mathbb{C}(x,x^*)_\Delta\). The sequence \(A_N\) converges to \(a\) in 2nd order \(\Delta\)-distribution whenever it converges to \(a\) in \(\Delta\)-distribution and the process of random variables \(Z_N = (\text{Tr} p(A_N) - E[\text{Tr} p(A_N)])_{p \in \mathbb{C}(x,x^*)_\Delta}\) converges in law to the centered Gaussian process \(z = (z_p)_{p \in \mathbb{C}(x,x^*)_\Delta}\) with covariance function \(E[z_p z_{p'}] = \Phi^{(2)}[p, q \ast_{\Delta} p']\).

We call \(\Delta\)-algebra generated by \(a\) the set \(\mathbb{C}(a)_\Delta := \{p(a), \forall p \in \mathbb{C}(x)_\Delta\}\). It is the smallest unital subalgebra of \(A\) closed by \(\Delta\) that contains \(a\). We say that \(a \in \mathbb{C}(a)_\Delta\) is \(\Delta\)-invariant if \(\Delta(a) = a\), and denote by \(D^{(a)}\) the set of \(\Delta\)-invariant elements of \(\mathbb{C}(a)_\Delta\). The triplet \((\mathbb{C}(a)_\Delta, D^{(a)}, \Delta)\) is an operator-valued probability space, and the \(\Delta\)-distributions of \(a\) is given by the restriction of \(\Phi\) on \(\mathbb{C}(a)_\Delta\).

Note that there is no known analogue of freeness for the \(\Delta\)-distributions. In particular \(D\)-freeness does not determine entirely the \(\Delta\)-distribution, see Remark 2.6 and Lemma 3.6. This is not important in this article since we consider a particular situation: at 1st order the algebra of coefficient \(D^{(a)}\) is isomorphic to \(\mathbb{C}\), see Hypothesis (H5) next section. This implies that the \(\Delta\)-distribution follows trivially from the \(*\)-distribution.

2. Statements of the results

We now state hypotheses on matrices and noncommutative random variables that we use in the statements below. Let \(X_N = (\frac{x_{ij}}{\sqrt{N}})_{i,j}\) be a Wigner matrix.

(H1) \(X_N\) is invariant in law by the permutation group, or equivalently the entry \(x_{12}\) is distributed as its complex conjugate \(\bar{x}_{12}\).

(H2) The pseudo-variance \(E(x_{ij}^2)\) of each nondiagonal entry \(i \neq j\) equals zero.

Let \(Y_N\) be a collection of deterministic matrices.

(H3) \(\sup_N \|Y_N\| < \infty\) for any \(Y_N\) matrix of \(Y_N\), for \(\|\cdot\|\) the operator norm.

(H4) \(Y_N\) has a limiting \(\Delta\)-distribution: for any \(p \in \mathbb{C}(y,y^*)_\Delta\), the limit exists

\[
\frac{1}{N} \text{Tr} p(Y_N, Y_N^*) \xrightarrow{N \to \infty} \Phi[p],
\]

where \(\Delta\) acts on \(M_N(\mathbb{C})\) by projecting a matrix on its diagonal part.

Let \(a\) be a collection of elements of \((\mathcal{A}, \Phi, \Phi^{(2)}, D, \Delta)\) as in Definition 1.4. We will invoke this property which means that \(\Delta\) is trivial at 1st order (see Remark 2.3):
(H5) Any $\Delta$-invariant element $a \in D^{(a)}$ of the $\Delta$-algebra generated by $a$ is distributed w.r.t. $\Phi$ as the deterministic scalar $\Phi(a) 1$:
$$\Phi(\Delta(a_1) \Delta(a_2)) = \Phi(a_1) \times \Phi(a_2). \quad (2.1)$$

Condition (H1) is inherent to our method and presumably technical; (H2) and (H5) are crucial for our statements. Without loss of generality, we will assume $Y_N$ closed by adjoint: if a matrix $Y_N$ belongs to $Y_N$, then so does the adjoint $Y_N^*$. This simplifies notations, e.g. with $C\langle x, y \rangle$ instead of $C\langle x, y, y^* \rangle$.

We first study the $1^{st}$ order $\Delta$-distribution and state a slight generalisation of the asymptotic freeness of Wigner and deterministic matrices.

**Theorem 2.1.** Let $X_N$ be a collection of independent Wigner matrices satisfying (H1) and $Y_N$ be a collection of deterministic matrices satisfying (H3), (H4). Then $(X_N, Y_N)$ converges in $\Delta$-distribution to a collection of variables $(x, y)$. The convergence
$$\frac{1}{N} \operatorname{Tr} p(X_N, Y_N) \rightarrow \Phi[p(x, y)], \quad \forall p \in C\langle x, y \rangle_\Delta \quad (2.2)$$
holds in expectation, in probability and almost surely. Furthermore for any $\Delta$-monomial $m$ in the variables $(x, y)$ and for any variable $x$ in the collection $x$, we have
$$\Phi[xm] = \sum_{m=\ell x r} \Phi[x^2] \Phi[\ell] \Phi[r], \quad (2.3)$$
where the sum is over all decompositions of $m$ as a product of $\Delta$-monomials $\ell \times x \times r$. Finally, $x$ satisfies (H5) and if the limit $y$ of $Y_N$ also satisfies (H5) then so does $(x, y)$.

The theorem is proved in Section 4. The convergence (2.2) and Formula (2.3) are known for ordinary $^*$-polynomials instead of $\Delta$-polynomials [1, Lemma 5.4.7]. In this case Formula (2.3) is called the Schwinger-Dyson equation and it is a classical consequence of freeness, see [22, Theorem 14.4]. In Theorem (2.1) this equation characterizes the $\Delta$-distribution of $(x, y)$ in terms of the $\Delta$-distribution of $y$, as proved in Section 3 and illustrated in the example below.

**Example 2.2.** Assuming $\Phi(x^2) = 1$, we compute $\alpha := \Phi[x^2 \Delta(x^2 y)yx yx]$. Applying (2.3) with $m = x\Delta(x^2 y)yxyx$, the different decompositions give
$$\Phi[x^2 \Delta(x^2 y)yxyx] = \Phi[\Delta(x^2 y)yx yx] + \Phi(x \Delta(x^2 y)y) \Phi(yx) + \Phi(x \Delta(x^2 y)yxy).$$

The middle term $\Phi(x \Delta(x^2 y)y)$ vanishes by (2.3) since $q = \Delta(x^2 y)y$ has no decomposition $q = \ell x r$. Moreover, by traciality the first and third terms...
are equal. By (2.3) again we have \( \Phi[x\Delta(x^2)y] = \Phi[\Delta(x^2)y] \Phi[y] \). To compute \( \Phi[\Delta(x^2)y] \), by \( \Delta \)-invariance of \( \Phi \) and the property of conditional expectation, we write
\[
\Phi[\Delta(x^2)y] = \Phi[\Delta(x^2)y] = \Phi[\Delta(x^2)y] = \Phi[x^2y\Delta(y)].
\] (2.4)
By (2.3) again, we get \( \Phi[\Delta(x^2)y] = \Phi[\Delta(y)^2] \) and finally \( \alpha = 2\Phi[\Delta(y)^2] \Phi[y] \).

**Remark 2.3.** The last part in Theorem 2.1 says, in terms of matrices, that if
\[
\frac{1}{N} \text{Tr} [\Delta(A_N)\Delta(A_N)^*] - \frac{1}{N} \text{Tr} [A_N]^2 \longrightarrow 0 \quad (2.5)
\]
for any \( A_N = p(Y_N), p \in \mathbb{C}(y)_\Delta \), then (2.5) holds for any \( A_N = p(X_N,Y_N), p \in \mathbb{C}(x,y)_\Delta \). In particular:

- If \( A_N \) is a diagonal matrix that is not asymptotically distributed as a scalar matrix, then (2.5) does not hold.

- If \( A_N \) is a Hermitian matrix whose eigenvectors \( u \) are delocalized, i.e. satisfy a uniform estimate \( |\langle u,e_j \rangle| = O(N^{-1/2}) \) for the scalar product with elementary basis vectors \( e_j \), then (2.5) holds (by an easy computation writing \( A_N = \sum_{i=1}^{N} \lambda_i u_i u_i^* \) in eigenvectors basis).

**Example 2.4.** Assumption (H5) holds almost surely for realizations of unitarily invariant matrices satisfying (H4) [6, Theorem 1.1] and of uniform permutations matrices [14, Section 3.2.2]. Traffics algebras satisfying (H5) are those for which traffic independence implies free independence [7, Corollary 2.9].

We now consider our initial problem on 2nd order distributions. Next definition is fundamental for the study of \( \Delta \)-distribution, see Section 3.

**Definition 2.5.** In a 2nd order operator-valued prob. space \((\mathcal{A}, \Phi, \Phi^{(2)}, \mathcal{D}, \Delta)\), we fix \( \mathbf{a}_\ell, \ell \in L \) collections of elements of \( \mathcal{A} \), and let \( \mathbf{a}_r \) denote the union of all \( \mathbf{a}_\ell \)'s.

1. For any \( n \geq 2 \), a tuple \((a_1, \ldots, a_n)\) is cyclically \( \mathcal{D}^{(a)} \)-alternating if \( a_i = p_i(\mathbf{a}_\ell) \) for indices \( \ell_1 \neq \cdots \neq \ell_n \neq \ell_1 \) where \( p_i \) is a \( \mathcal{D}^{(a)} \)-valued polynomial. We set \( E_n \) the vector space generated by all products \( a_1 \cdots a_n \) where \((a_1, \ldots, a_n)\) is cyclically \( \mathcal{D}^{(a)} \)-alternating and \( \Delta \)-centered.

2. We set \( E_1 \) the smallest vector space s.t. \( \mathcal{A}g(\mathbf{a}_\ell, \Delta(E_1)) \subset E_1, \forall \ell \in L \).
Remark 2.6. If the $a^\ell_i$’s are free over $D$ (in the ordinary sense) then $\Phi$ vanishes on $E_n$ for $n \geq 2$; but there is no a priori knowledge about $\Phi$ on $E_1$.

Example 2.7. For each $i = 1, 2, \ldots$, let $\ell_i \in L$ and $a^\ell_i \in a^\ell_i$.

- $m = [a^\ell_1 \Delta(a^\ell_2 a^\ell_3) a^\ell_1 - \Delta(a^\ell_1 \Delta(a^\ell_2 a^\ell_3))](a^\ell_3 - \Delta(a^\ell_3)) \in E_2$ if $\ell_1 \neq \ell_3$.
- $m = \Delta(a^\ell_1) a^\ell_2 \Delta(a^\ell_3) \Delta(\Delta(a^\ell_2) a^\ell_4) a^\ell_2$ is in $E_1$. Indeed, $m \in \text{Alg}(a_2, \Delta(E_1))$ since we have $m = d_1 a^\ell_2 d_2 a^\ell_2$ where $d_1 \in D^{(a^\ell)}$ and $d_2 \in \Delta(\text{Alg}(a_3, D^{(a^\ell)}))$.

The analysis of Wigner matrices motivates the following definition.

Definition 2.8. Let $(A, \Phi, \Phi^{(2)}, D, \Delta)$, $a^\ell, \ell \in L$, and $E_n, n \geq 1$, be as in the previous definition, and assume that the collection of all $a^\ell_i$’s satisfy (H5). The families $a^\ell_i$ are 2nd order $\Delta$-free if they are free w.r.t. $\Phi$ and the following holds.

1. Orthogonality conditions: the vector spaces $E_n, n \geq 1$, are orthogonal w.r.t. $\Phi^{(2)}$, i.e. for any $a \in E_n, b \in E_m, n \neq m$, then $\Phi^{(2)}(a, b^*) = 0$; moreover, the $*$-algebras generated by $a^\ell_i, \ell \in L$, are orthogonal w.r.t. $\Phi^{(2)}$, i.e. for any $*$-polynomials $p$ and $q$, one has $\Phi^{(2)}(p(a_\ell), q(a_\ell)^*) = 0$ if $\ell \neq \ell'$.

2. Mingo-Speicher formula, $n \geq 2$: for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ cyclically $D^{(a)}$-alternating and $\Delta$-centered, with $a = a_1 \cdots a_n$, $b = b_1 \cdots b_n$ we have

$$\Phi^{(2)}(a, b^*) = \sum_{i=0}^{n-1} \Phi(a_1 b^*_i a_2 b^*_i a_3 b^*_i \cdots a_n b^*_{i+n}),$$  

(2.6)

where indices of $b^*_i$’s are counted modulo $n$ (e.g. $b_{n+1} = b_1$).

3. Leibniz rule, $n = 1$: for all $a, b, c \in E_1$, we have

$$\Phi^{(2)}(\Delta(a) \Delta(b), c) = \Phi^{(2)}(a, c) \Phi(b) + \Phi(a) \Phi^{(2)}(b, c).$$

(2.7)

Remark 2.9. We recall that no general notion of 1st order $\Delta$-freeness exists so naming 2nd order $\Delta$-freeness is an abuse of terminology. Moreover, we name item (ii) Mingo-Speicher formula since it is formally identical to the spoke diagram formula of ordinary 2nd order freeness, but it is stated in other spaces.

In Section 3 we prove that the rules (i), (ii) and (iii) determine uniquely the 2nd order $\Delta$ distribution of $a$ in terms of the marginal 1st and 2nd order $*$-distributions of each $a^\ell_i$. We illustrate the method below.
Example 2.10. We compute $\Phi^{(2)}(m,m^*)$ for $m = a_1 a_2 \Delta(a_1 a_2)$ where $a_i \in a_1 \ell, \ell \neq \ell$. We have $\Phi^{(2)}[m, \cdot] = \Phi^{(2)}[m_1 + m_1', \cdot]$ where $m_1 = (a_1 - \Delta(a_1))(a_2 - \Delta(a_2))\Delta(a_1 a_2)$ is in the space $E_2$ and $m_1' = -\Delta(a_1)\Delta(a_2)\Delta(a_1 a_2)$. Moreover, using the same computation as for (2.3), we have $\Phi^{(2)}(m_1', \cdot) = \Phi^{(2)}(m_1', \cdot)$ where $m_1' = \Delta(a_1)(a_2 a_1 a_2)$. We write as before $\Phi^{(2)}(m_1', \cdot) = \Phi^{(2)}(m_2 + m_3, \cdot) = \Phi^{(2)}(m_2 = \Delta(a_1)\Delta(a_2)(a_1 - \Delta(a_1))(a_2 - \Delta(a_2)) \in E_2$ and $m_3 = \Delta(a_1)\Delta(a_2)\Delta(a_1)\Delta(a_2)$ is in $E_1$. Hence, by the first orthogonality rule $\Phi^{(2)}(m, m^*) = a_1 + a_2 + a_3$ where $a_i = \Phi^{(2)}(m_1, m^*_3)$. By Mingo-Speicher formula we have $a_1 = \Phi \left\{ (a_1 - \Delta(a_1)) (a_1 - \Delta(a_1)) \right\} \times \Phi \left\{ (a_2 - \Delta(a_2))(a_2 - \Delta(a_2)) \right\} \Phi(a_1 a_2)^2$. Moreover, by freeness of $a_1$ and $a_2$ and by (H5) we find $a_1 = \forall a_1 \forall a_2 \Phi(a_1)\Phi(a_2)^2$, where $\forall a_1 = \Phi \left\{ (a - \Phi(a))(a - \Phi(a)) \right\}$. Similarly one shows $a_1 = a_2$. By the Leibnitz rule with $a = \Delta(a_1)^2$ and $b = \Delta(a_2)^2$ and the second orthogonality rule, we have $a_3 = \Phi^{(2)}(\Delta(a_1)^2, \Delta(a_1)^2)\Phi(\Delta(a_2)^2)^2 + \Phi(\Delta(a_1)^2)^2 \Phi^{(2)}(\Delta(a_2)^2, \Delta(a_2)^2)$. Using the Leibnitz rule with $a = b = a_1$, we get $a_3 = 4\Phi(a_1)^4 \Phi^{(4)}(a_1, a_1')\Phi(a_1)^4 + 4\Phi(a_1)^4 \Phi(a_2)^2 \Phi^{(2)}(a_2, a_2)$. Note that the $a_i$’s have been written in terms of 1st and 2nd order ordinary moments of $a_1$ and of $a_2$.

Theorem 2.11. Let $X_N$ and $Y_N$ be collections of independent Wigner and deterministic matrices satisfying (H1), (H2), (H3), (H4) and such that the limit $y$ of $Y_N$ satisfies (H5). Then $(X_N, Y_N)$ converges in 2nd order $\Delta$-distribution to a collection of variables $(x, y)$ such that the variables of $x$ are 2nd order $\Delta$-free, and are 2nd order $\Delta$-free from $y$.

The theorem is proved in Section 5. We now reformulate property (ii).

Proposition 2.12. Let $(A, \Phi, \Phi^{(2)}, \mathcal{D}, \Delta)$, $a_\ell, \ell \in L$, and $E_n, n \geq 2$, be as in Definition 2.3. Assume that the families $a_\ell$’s are free w.r.t. $\Phi$ and denote

$$F_n = \text{Span} \left\{ \sum_{i=0}^{n-1} a_{i+1} \cdots a_{i+n}, \quad (a_1, \ldots, a_n) \text{ $\Delta$-centered cyclically } \mathcal{D}^{(a)} \text{-alternating} \right\},$$

where indices are counted modulo $n$. Then Mingo-Speicher formula (ii) is valid for all $n \geq 2$ if and only if $\Phi^{(2)}(p, q^*) = n\Phi(pq^*)$, for all $n \geq 2$ and all $p, q \in F_n$.

Hence, back to the central limit, by plugging-in the 1st order estimators we get for a large matrix $M_N = P(X_N, Y_N)$ in $F_n$ a Gaussian approximation

$$\frac{1}{N} \text{Tr}M_N \sim \mathcal{N} \left( \mathbb{E} \left[ \frac{1}{N} \text{Tr}M_N \right], \mathbb{E} \left[ \frac{1}{N^2} \text{Tr}(M_N M_N^*) \right] \right) \quad (2.8)$$

Proof of Proposition 2.12 For any $\Delta$-centered cyclically $\mathcal{D}^{(a)}$-alternating tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, let $a = a_1 \cdots a_n$, $\hat{a} = \frac{1}{n} \sum_{i=0}^{n-1} a_{1+i} \cdots a_{n+i}$ and similarly let $b$ and $\hat{b}$. By traciality, $\Phi^{(2)}(a, b^*) = \Phi^{(2)}(\hat{a}, b^*)$, so (ii) is
valid whenever $\Phi^{(2)}(\hat{a}, \hat{b}^*) = \sum_{i=0}^{n-1} \Phi(a_i b_{i+1}^*) \cdots \Phi(a_n b_{i+n}^*)$. Moreover freeness of the $a_i$’s implies [22] Lemma 5.18 that $\Phi(a_1 b_{i+1}^*) \cdots \Phi(a_n b_{i+n}^*) = \Phi(ab_{i}^*)$, where $b_{i}^* = b_{i+1} \cdots b_{i+n}$ with indices counted modulo $n$. Hence Mingo-Speicher formula is valid if and only if $\Phi^{(2)}(\hat{a}, \hat{b}^*) = \sum_{i=0}^{n-1} \Phi(ab_{i}^*) = \frac{1}{n} \sum_{i,j=0}^{n-1} \Phi(a_i b_j^*) = n\Phi(\hat{a}\hat{b}^*)$.

For comparison, we recall the definition of ordinary 2\textsuperscript{nd} order freeness. Let $(A, \Phi, \Phi^{(2)})$ be a 2\textsuperscript{nd} order probability space and $a_\ell, \ell \in L$ be families in $A$. A tuple $(a_1, \ldots, a_n)$ of elements of $A$ is called cyclically alternating and centered whenever $a_i = p_i(a_{\ell_i})$ for indices $\ell_1 \neq \cdots \neq \ell_n \neq \ell_1$ and $\Phi(a_i) = 0$ for all $i$.

**Definition 2.13.** The families $a_\ell$ are 2\textsuperscript{nd} order free if they are free, and:

1. the vector spaces $E'_n, n \geq 2$, are orthogonal w.r.t. $\Phi^{(2)}$, the $*$-algebras generated by $a_\ell, \ell \in L$, are orthogonal w.r.t. $\Phi^{(2)}$;
2. for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ cyclically alternating and centered, with $a = a_1 \cdots a_n$, $b = b_1 \cdots b_n$

$$\Phi^{(2)}(a, b^*) = \sum_{i=0}^{n-1} \Phi(a_1 b_{i+1}^*) \Phi(a_2 b_{i+2}^*) \cdots \Phi(a_n b_{i+n}^*); \tag{2.9}$$

or equivalently $\Phi^{(2)}(p, q^*) = n\Phi(pq^*)$, for all $n \geq 2$ and all $p, q$ in

$$E'_n := \text{Span}\left\{\sum_{i=0}^{n-1} a_{i+1} \cdots a_{i+n}, \left| (a_1, \ldots, a_n) \text{ cyclically alternating centered} \right\}.$$  

### 3 Preliminaries on $\Delta$-distributions

This section reviews properties of $\Delta$-distributions and useful tools:

- **Lemma 3.2** studies the (generalized) Schwinger-Dyson (SD) equation;
- **Lemma 3.3** shows that the solution of SD equation satisfies (H5);
- **Lemma 3.6** considers characterization questions of $\Delta$-distributions.

Whereas later on we will always consider matrices, we do not assume in this section that $D$ is commutative. We use the two following notions of degrees.

**Definition 3.1.** Let $m$ be a $\Delta$-monomial, bracketing of a word $\omega = a_1 \cdots a_n$. We call $n$ the full degree of $m$. We call ground degree of $m$ the integer $n'$ such that $m = d_0 a_{j_0} d_1 \cdots d_{n'-1} a_{j_{n'-1}} d_n$, where $j_k \in [n]$ and $d_j$ is $\Delta$-invariant. Full and ground degrees in a sub-family of variables are defined in the obvious way.
Note that \( n' \leq n \) with equality when \( m \) is a monomial, and \( n' = 0 \) whenever \( m \) is \( \Delta \)-invariant. For example \( m = y\Delta(xy)y \) has full degree 4, ground degree 2; in the variable \( x \) it has full degree 1 and ground degree 0.

**Lemma 3.2.** Let \( \Phi_0 \) be a tracial \( \Delta \)-invariant linear form on \( C(y)_\Delta \). There exists a unique tracial \( \Delta \)-invariant linear form \( \Phi \) on \( C(x,y)_\Delta \) such that \( \Phi|_{C(y)_\Delta} = \Phi_0 \) and \( \Phi \) is a solution of the SD equation: \( \Phi[xm] = \sum_{m=\ell xr} \Phi[\ell] \Phi[\ell]r \) for any \( \Delta \)-monomial \( m \).

*Proof of Lemma 3.2.* Assume first that such a \( \Phi \) exists. Let \( m \in C(x,y)_\Delta \) but not in \( D^{(x,y)}(y) \), i.e. \( m \) is of positive ground degree in a variable \( x \in x \). Then \( m = m_1xm_2 \) for \( \Delta \)-monomials \( m_1 \) and \( m_2 \). By traciality, \( \Phi(m) = \Phi(xm_2m_1) \), so the SD equation implies that \( \Phi(m) \) is function of \( \Delta \)-moments of smaller ground degrees in \( x \). By induction \( \Phi(m) \) is a polynomial in \( \Phi(m') \) for \( m' \) of zero ground degree in \( x \), i.e. \( m' \in D^{(x,y)}(y) \).

We now assume \( m \in D^{(x,y)}(y) \setminus D^{(y)}(y) \); it has zero ground but positive full degree in \( x \). Then \( m = m_1dm_2 \) for \( d \in D^{(x,y)} \) with positive full degree in \( x \). We can write \( d = \Delta(p) \) where \( p \) is not \( \Delta \)-invariant and is obtained by removing the outer evaluation of \( \Delta \).

By traciality, \( \Delta \)-invariance and the property of the conditional expectation

\[
\Phi(m) = \Phi(\Delta(p)m_2m_1) = \Phi(p\Delta(m_2m_1)) = \Phi(\tilde{m})
\]

where \( \tilde{m} = p\Delta(m_2m_1) \). We call this method ground block change (see also Example 2.2). If \( p \) does not have positive ground degree in \( x \) we iterate the ground block change, we get a \( \Delta \)-monomial with positive ground degree in some variable \( x \). By the previous step, we write \( \Phi(m) \) as a function of \( \Delta \)-moments of smaller full degrees in \( x \). By induction, \( \Phi(m) \) is a function of \( \Phi(m') \) for \( m' \) of zero full degree in \( x \), i.e. \( m' \in C(y)_\Delta \).

The conclusion is that necessary \( \Phi(m) = f_m(\Phi(m_1),\ldots,\Phi(m_K)) \) for a polynomial function \( f_m \) and \( m_1,\ldots,m_K \in C(y)_\Delta \). One checks that this unique proposal indeed provides a solution of the Schwinger-Dyson equation.

**Remark 3.3.** For each \( N \) let \( \Phi_N \) be a tracial \( \Delta \)-invariant linear form on \( C(x,y)_\Delta \), such that its restriction on \( C(y)_\Delta \) converges and such that \( \Phi_N[xm] = \sum_{m=\ell xr} \Phi_N[\ell] \Phi_N[\ell]r + o(1) \) for any \( \Delta \)-monomial \( m \). Then the above proof shows that \( \Phi_N(m) = f_m(\Phi_N(m_1),\ldots,\Phi_N(m_K)) + o(1) \), for the same \( f_m \) and \( m_k \)'s, and so \( \Phi_N \) converges and is limit is the unique solution the SD equation.

**Definition 3.4.** Let \( m \) be a \( \Delta \)-monomial obtained by bracketing a word \( a_1 \cdots a_n \). We define inductively a partition \( \sigma(m) \) of \( [n] \). If \( m = a_1 \cdots a_n \), i.e. \( m \) is a monomial, then \( \sigma(m) \) has a single block \( [n] \). Otherwise, there is a subword \( \Delta(a_ia_{i+1} \cdots a_{i+j}) \) in \( m \) with no internal bracket. Then \( \{i, i +
1, \ldots, i + j \} \text{ is a block of } \sigma(m). \text{ We iterate this construction on the } \Delta\text{-monomial obtained by deleting } \Delta(a_ia_{i+1} \cdots a_{i+j}) \text{ from } m, \text{ until no bracket remains. The blocks constructed along the process define a partition } \sigma(m) \text{ of } [n].

The partition } \sigma(m) \text{ is non-crossing by the nesting characterization: a partition is non-crossing whenever removing an inner block produces a non-crossing partition, see [22, Remark 9.2]. For any } \Delta\text{-monomial } m \text{ bracketing of a word } a_1 \cdots a_n \text{ and any block } B = \{i_1, \ldots, i_k\} \text{ of } \sigma(m), \text{ we denote the monomial } m_B = a_{i_1} \cdots a_{i_k}. \text{ A simple induction reasoning in the above construction shows that a family } y \text{ satisfies (H5) whenever for any } \Delta\text{-monomial } m \in C \langle y \rangle_{\Delta}, (H5') \quad \Phi(m) = \prod_{B \in \sigma(m)} \Phi(m_B);

We use this observation, the ground block change method and the SD equation to justify the last statement in Theorem 2.1 that we reformulate in next lemma:

**Lemma 3.5.** Let } \Phi \text{ be a tracial } \Delta\text{-invariant linear form on } C \langle x, y \rangle_{\Delta} \text{ solution of the SD equation (2.3). If (H5') holds on } C \langle y \rangle_{\Delta} \text{ then (H5') also holds on } C \langle x, y \rangle_{\Delta}.

Let } m \text{ be a } \Delta\text{-monomial, bracketing of the word } a_1 \cdots a_n. \text{ Writing } m = d_0a_jd_1 \cdots d_{n-1}a_jd_n' \text{ where } j_k \in [n] \text{ and } d_j \in D(x^y), \text{ we then distinguish the block } \{j_1, j_2, \ldots, j_n\} \text{ of } \sigma(m) \text{ that we call the ground block of } m \text{ (its cardinal is the ground degree of } m). \text{ The other blocks of } \sigma(m) \text{ are called the paddle blocks.

**Proof of Lemma 3.5.** We set } \tilde{\Phi}(m) = \prod_{B \in \sigma(m)} \Phi(m_B), \text{ for any } \Delta\text{-monomial } m \text{ and show } \tilde{\Phi} = \Phi \text{ by proving it is solution of the Schwinger-Dyson equation. By assumption, the functions coincide in } C \langle y \rangle_{\Delta}, \text{ so let } m \in C \langle x, y \rangle_{\Delta} \setminus C \langle y \rangle_{\Delta}. \text{ Recall the ground block change method, illustrated in (3.1): there is a } \Delta\text{-monomial } \tilde{m}, \text{ such that } \Phi(m) = \Phi(\tilde{m}) \text{ and } \Phi(m) = \Phi(\tilde{m}), \text{ with positive ground degree in a variable } x. \text{ Hence we can always assume that } m \text{ has positive ground degree in a variable } x. \text{ By traciality, we can assume } m = xm' \text{ for } x \in x \text{ and } m' \in C \langle x, y \rangle_{\Delta}.

For any } \Delta\text{-monomial } p, \text{ we set } p_{GB} \text{ the monomial associated to the ground block of } p. \text{ Then we have } m_{GB} = xm_{GB}', \text{ and by Schwinger-Dyson equation for } \Phi

\[ \Phi(m) = \sum_{m'_{GB} = \ell_{GB}x_{GB}} \Phi(\ell_{GB})\Phi(r_{GB}) \prod_{B' \in \sigma(m), B' \neq GB(m)} \Phi(m_B), \]

where } GB(m) \text{ is the ground block of } m. \text{ But for any decomposition of } m_{GB}' \text{ into a product } \ell_{GB}x_{GB} \text{ corresponds a decomposition } m' = \ell x r \text{ where } \ell \text{ and
$r$ are $\Delta$-monomials whose ground monomials are $\ell_{GB}$ and $r_{GB}$ respectively. The set of paddle blocks of $m'$ is the union of those of $\ell$ and of $r$. Hence

$$\tilde{\Phi}(x m') = \sum_{m'=\ell x r} \tilde{\Phi}(\ell) \tilde{\Phi}(r),$$

and so $\tilde{\Phi}$ satisfies the Schwinger-Dyson, and by Lemma 3.2 it coincides with $\Phi$. \qed

Recall that given families $a_{\ell}, \ell \in L$, in Definition 2.5 we denote by $A_1$ the smallest space such that $\text{Alg}(a_{\ell}, \Delta(A_1)) \subset A_1, \forall \ell \in L$. The inductive construction of Definition 3.4 shows that $A_1$ is generated by $\Delta$-monomials $m$ evaluated in the $a_{\ell}$’s in such a way each $m_B$ is evaluated in one family $a_{\ell}$, $\forall B \in \sigma(m)$, i.e. each $m_B$ is non-mixing. Recall we also denote by $A_n, n \geq 2$, the space of cyclically $D^{(a)}$-alternating products of $\Delta$-centered elements in the $a_{\ell}$’s. The following property is crucial to consider the characterization of $\Delta$-distributions in terms of marginals.

**Lemma 3.6.** With notations as above, for any tracial $\Delta$-invariant linear map $\psi$ on $A$ and any $m \in C(a)_{\Delta}$, $\psi(m)$ is a linear combination of $\psi(q)$ for $q \in A_n, n \geq 1$. 

Hence if $a_{\ell}, \ell \in L$ are free over $D$ then the $\Delta$-distribution $\Phi$ vanishes on $A_n$ for $n \geq 2$; it only remains to determine $A_1$. On the other hand if the collection of all $a_{\ell}$ satisfies (H5) then $\Phi$ is determined on $A_1$ by its restriction on monomials on each family $a_{\ell}$; using (H5) again, if moreover the $a_{\ell}$’s are free in the ordinary sense, then $\Phi$ also vanishes on $A_n$ for $n \geq 2$. By Lemma 3.6 the joint $\Delta$-distribution of the $a_{\ell}$’s is hence determined by the marginal $\Delta$-distribution.

Assume moreover that $a_{\ell}, \ell \in L$ are 2nd order $\Delta$-free. By Lemma 3.6 and by the orthogonality conditions, the 2nd order $\Delta$-distribution is determined

- **(universal part)** on each $A_n, n \geq 2$ by the marginal 1st order distribution, via Mingo-Speicher formula and ordinary freeness;

- **(non-universal part)** the marginal 1st and 2nd order distribution via Leibnitz formula on $A_1$.

**Proof of Lemma 3.6.** Let $m$ be a $\Delta$-monomial in the variables $a_{\ell}, \ell \in L$ which is not in $A_1$: there is a block $B \in \sigma(m)$ such that $m_B$ in a $\Delta$-monomial in at least two different $a_{\ell}$’s. We set $\beta(m) \geq 2$ the sum of degrees of monomials $m_B$ in at least two different families of variables $a_{\ell}$’s, for $B \in \sigma(m)$, and we reason by induction on $\beta(m)$.

By ground block change we can assume that $B$ is the ground block of $m$. Then we write $m = r_0 s_1 r_1 \cdots s_n r_n$, where $n \geq 2$, each $s_k \in D^{(a)}(a_{\ell_k})$ is of the form $s_k = a m'_k = m''_k a'$ for $a, a' \in a_{\ell_k}$, and each $r_k$ is in $D^{(a)}$ i.e. is $\Delta$-invariant, and $\ell_k \neq \ell_{k+1}$. Traciality implies that $\psi(m) = \psi(s_1 r_1 \cdots s_n r_n r_0)$,
so we can assume \( r_0 = 1 \). If \( j_n = j_1 \) then with \( r'_1 = s_n r_{n} s_1 \) we have
\[
\psi(m) = \psi(s_1 r_1 \cdots s_{n-1} r_{n-1}).
\]
So we can assume \( j_n \neq j_1 \).

We operate to the usual reduction in the free product construction: we set \( \hat{s}_k = s_k - \Delta(s_k) \), and \( q = s_1 r_1 \cdots \hat{s}_n r_n \). Then \( m = q + q' \), where \( q \in A_n \) and \( q' \) is a linear combination of \( \Delta \)-monomials \( m' \) such that \( \beta(m') < \beta(m) \). By induction, \( \psi(m) \) is a linear combination of \( \psi(m') \) where \( m' \in \bigcup_{n \geq 1} A_n \).

\[\square\]

4 First-order convergence: proof of Theorem 2.1

Let \( X_N = (X_{ij})_{i \in L} \) and \( Y_N = (Y_{ij})_{j \in J} \) be as in Theorem 2.1 and \( x = (x_{ij})_{i \in L} \), \( y = (y_{ij})_{j \in J} \) be indeterminates. For each Wigner matrix \( X_N = \frac{1}{\sqrt{N}}(x_{ij}) \), we assume that \( \mathbb{E}[|x_{12}|] = 1 \) and we assume that \( Y_N \) is closed by adjoint. We prove the convergence of
\[
\Phi_N(q) := \mathbb{E}[\frac{1}{N} \text{Tr} q(X_N, Y_N)].
\]
Convergence almost sure and in probability follow easily from variance estimates implied by Theorem 2.1 and Borel-Cantelli lemma, see for instance the proof of [9, Theorem 1.13]. Since the trace is tracial and invariant under \( \Delta \), we can assume \( q \) is \( \Delta \)-invariant.

1. Next section in Proposition 4.3 we recall an asymptotic formula based on traffic independence. It does not imply the convergence of \( \Phi_N(q) \) since it involves quantities bounded but not necessarily convergent.

2. In Section 4.2 we prove that \( \Phi_N \) satisfies asymptotically the SD equation. This implies Theorem 2.1 thanks to the preliminary results.

4.1 Recall of asymptotic first-order formulas

We call test graph a triplet \( T = (V, E, \gamma) \), where \( (V, E) \) is a finite directed graph \( (V, E) \) and \( \gamma \) is a labelling map from \( E \) to a label set. The graph may have loops and multiple edges, so that \( V \) is a set and \( E \) is a multiset of ordered pair of elements of \( V \). The map \( \gamma \) indicates that the edge \( e \) is labeled by the variable \( x_{\gamma(e)} \). Test graphs are given with a partition \( E = E_X \cup E_Y \) of the edge set: a labelling \( \gamma(e) \) of \( e \in E_X \) is an element of \( L \) which refers to the variable \( x_{\gamma(e)} \), we say with small abuse that \( e \) is labeled by the Wigner matrix \( X_{\gamma(e)} \); a labelling \( \gamma(e) \) of \( e \in E_Y \) belongs to \( J \) and refers to \( y_{\gamma(e)} \), \( e \) is labeled by the deterministic matrix \( Y_{\gamma(e)} \).

The trace of a test graph \( T = (V, E, \gamma) \) in the matrices \( X_N, Y_N \) is the scalar
\[
\text{Tr}[T(X_N, Y_N)] = \sum_{\phi: V \to [N]} \prod_{e = (v, w) \in E_X} X_{\gamma(e)}(\phi(w), \phi(v)) \times \prod_{e = (v, w) \in E_Y} Y_{\gamma(e)}(\phi(w), \phi(v)) 1
\]

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**Definition 4.1.** A well-oriented (w.o.) simple cycle is a cycle with no repetition of vertices whose edges follow a same orientation. A w.o. cactus is a test graph such that each edge belongs to a unique simple cycle and all simple cycles are well-oriented.

Let \( m \) be a \( \Delta \)-monomial obtained by bracketing a word \( a_1 \cdots a_n \), where \( a_i \in \mathbf{x} \cup \mathbf{y} \). We associate inductively a w.o. cactus \( T_m \) in the similar way we have constructed the non crossing partition \( \sigma(m) \). Let \( T^{(0)} \) be the simple directed cycle with \( n \) edges labeled \( \cdots \xleftarrow{q_{i-1}} \xleftarrow{q_i} \xleftarrow{q_{i+1}} \cdots \). If \( m = a_1 \cdots a_n \), i.e. \( m \) is a monomial, then \( T_m = T^{(0)} \). Otherwise, there is a subword \( \Delta(a_ia_{i+1} \cdots a_{i+j}) \) in \( m \) with no internal bracket. Let \( T^{(1)} \) be the graph obtained from \( T^{(0)} \) by identifying the target of the \( i \)-th edge and the source of \( (i+j) \)-th one. Then we iterate this construction on the cycle of \( T^{(1)} \) that do not contain the edges labeled \( a_i, a_{i+1} \cdots a_{i+j} \). We thus construct a sequence \( T^{(k)}, k = 0, 1, \ldots \) of graphs until a step \( k_0 \) for which there is not bracketed expression remaining. We then set \( T^{(k_0)} = T_m \). The edges are canonically labeled by the variables of the monomial.

By construction, \( T_m \) is a w.o. cactus and its cycles are correspondence with the blocks of the non-crossing partition \( \sigma(m) \). One checks that the trace of \( m \) is the trace of the graph \( T_m \) in the matrices in the sense of (4.1), namely

\[
\text{Tr } m(\mathbf{X}_N, \mathbf{Y}_N) = \text{Tr } [T_m(\mathbf{X}_N, \mathbf{Y}_N)]. \tag{4.2}
\]

For any test graph \( T \), the injective trace of \( T \) in the matrices \( \mathbf{X}_N, \mathbf{Y}_N \) is the quantity \( \text{Tr}^0[{T}(\mathbf{X}_N, \mathbf{Y}_N)] \) defined as in (4.1) but where the sum is restricted to injective maps \( \phi : V \to [N] \). Let \( T = (V, E, \gamma) \) be a test graph. For any \( \pi \in \mathcal{P}(V) \), let \( T^\pi \) be the graph obtained by identifying vertices in a same block of \( \pi \). We say that \( T^\pi \) is a quotient of \( T \). Then we have relation [14] Section 2.1: for any test graph \( T \)

\[
\text{Tr } T(\mathbf{X}_N, \mathbf{Y}_N) = \sum_{\pi \in \mathcal{P}(V)} \text{Tr}^0[T^\pi(\mathbf{X}_N, \mathbf{Y}_N)]. \tag{4.3}
\]

In this paragraph and in Definition 4.2 we fix a \( \Delta \)-monomial \( m \) and denote \( T_m = T = (V, E, \gamma) \) its w.o. cactus. For \( \pi \in \mathcal{P}(V) \), let \( T^\pi_X = (V^\pi_X, E^\pi_X, \gamma^\pi_X) \) be the subgraph of \( T^\pi \) whose edges are labeled by Wigner matrices. Let \( T^\pi_Y = (V^\pi_Y, E^\pi_Y, \gamma^\pi_Y) \) be defined similarly with deterministic matrices and let \( C_Y(T_m^\pi) \) be the set of connected components of \( T^\pi_Y \). We define the weight associated to \( \mathbf{X}_N \)

\[
\omega_X^{(1)}(\pi) = \mathbb{E} \left[ \prod_{e = (v, u) \in E^\pi_X} \sqrt{N} X_{\gamma(e)}(\psi(u), \psi(v)) \right] \tag{4.4}
\]

for any choice of \( \psi : V^\pi_X \to [N] \) injective (permutation invariance of \( \mathbf{X}_N \) implies that this quantity does not depend on \( \psi \)), and the weight associated
to $Y_N$

$$\omega_Y(\pi) = \frac{1}{N^{\|C_Y(T^\pi)\|}} \text{Tr}^0 [T_Y^\pi(Y_N)].$$

(4.5)

By (4.2), (4.3) and the permutation invariance of $X_N$, we have [10]

$$\Phi_N(m) := \mathbb{E} \left[ \frac{1}{N} \text{Tr} m(X_N, Y_N) \right] = \sum_{\pi \in \mathcal{P}(V)} N^{q(1)(\pi)} \omega_X^{(1)}(\pi) \omega_Y(\pi),$$

(4.6)

where $q(1)(\pi) = -\frac{|E_X|}{2} - 1 + |C_Y(T)|$. Mingo Speicher’s bounds for the trace of graphs [20] then allows us to replace $N^{q(1)(\pi)}$ by an indicator function for $N$ large.

**Definition 4.2.** 1. A tree is a connected graph (directed or not) such that the removal of an arbitrary edge always disconnects the graph.

2. Fix $\pi \in \mathcal{P}(V)$. The graph of deterministic components of $T^\pi$ is the undirected graph $GDC(T^\pi) = (V, E)$, where

- the vertex set $V$ is the disjoint union of the vertex set of $T^\pi$ and of the set $C_Y(T^\pi)$ of connected components of the subgraph of $T^\pi$ of edges labeled by deterministic matrices,
- the edge set $E$ is the union of $E_X^\pi$ and of the set of pairs $\{v, S\}$ where $c \in V$ belongs to the subgraph $S \in C_Y(T^\pi)$.

3. We say that $T^\pi$ is of double-tree type and denote $T^\pi \in DT$ if the graph $\overline{GDC(T^\pi)}$ obtained from $GDC(T^\pi)$ by forgetting edge multiplicity is a tree, the edges of $\overline{GDC(T^\pi)}$ labeled in $x$ have multiplicity 2, and edges forming a group of multiplicity 2 are labeled by a same variable $x$.

**Proposition 4.3.** Under (H1) and (H3), we have the first-order asymptotic formula: for all $\Delta$-monomials $m$ and all partitions $\pi$ of the vertices of $T_m$,

$$N^{q(1)(\pi)} \omega_X^{(1)}(\pi) \omega_Y(\pi) = 1 (T^\pi_m \in DT) \prod_{C \in C_Y(T^\pi)} \frac{1}{N} \text{Tr}^0 [C(Y_N)] + o(1),$$

and these quantities are uniformly bounded when $N$ goes to infinity.

The heuristic of the proposition is presented in [16] and the result is an immediate consequence of the asymptotic traffic independence theorem of [15] with the limiting traffic distribution of [14, Section 3.1].
4.2 Proof of the asymptotic SD equation

We recall that we denote $\Phi_N : m \mapsto \mathbb{E} \left[ \frac{1}{N} \text{Tr} m(X_N, Y_N) \right]$. By Remark 3.3 to prove Theorem 2.1 we shall prove: for all $\Delta$-monomial $m$, the w.o. cactus associated to $m$ is connected. With same notations as in previous section, we denote $T = T_m$ the w.o. cactus associated to $m$ and $V = V_m$ its vertex set. Let $\pi \in \mathcal{P}(V)$ such that the quotient $T^\pi$ is of double tree type. By definition, an edge $e$ labeled $x$ in $T^\pi$ is hence grouped with another edge $e'$ of same label to form a double edge. We say in short that $e$ and $e'$ are twin edges. We claim that twin edges $e$ and $e'$ in $T^\pi$ always belong to a same simple cycle in $T$. Indeed, since $T^\pi \in \mathcal{DT}$, the graph $T_{e,e'}^\pi$ obtained by removing $e$ and $e'$ from $T^\pi$ is disconnected. Contrariwise, if $e, e'$ are not in a same cycle of $T$, the graph $T_{e,e'}$, obtained by removing $\{e, e'\}$ from $T$, is connected. But $T_{e,e'}^\pi$ is a quotient of $T_{e,e'}$, and taking a quotient cannot increase the number of connected components. The same connectivity argument shows that twin edges have opposite orientation since removing $e, e'$ must disconnect $T^\pi$: the source of $e$ is the target of $e'$ and reciprocally.

Let now prove the asymptotic SD equation. Let $m$ be a $\Delta$-monomial of the form $m = xm'$, and let $T = T_m$ be its w.o. cactus. Denote by $e$ the edge of $T$ associated to the first letter $x$ of the word of $xm'$. The edges $e' \neq e$ in the same cycle as $e$ with label $x$ are in correspondence with the decompositions $m' = \ell xr$.

We fix the edge $e'$ which forms a twin edge with $e$, and so a decomposition $m' = \ell xr$. Let $\pi \in \mathcal{P}(V)$ such that $T^\pi \in \mathcal{DT}$ and $e, e'$ are twin edges. Then $T^\pi$ is a quotient of the graph $T_{e,e'}$ obtained from $T$ by identifying the source of $e$ and the target of $e'$ and reciprocally. Note that $T_{e,e'}$ consists in the two disjoint graphs $T_\ell$ and $T_r$ (defined as $T_m$ with $\ell, r$ replacing $m$) linked by the double edge $\{e, e'\}$. Seeing $T^\pi$ as a quotient of $T_{e,e'}$, there is no identification of vertices of different subgraphs $T_\ell$ and $T_r$, by the same connectivity argument as before (removing $e, e'$ must disconnect $T^\pi$). Hence the set of $\pi \in \mathcal{P}(V)$ such that $T^\pi \in \mathcal{DT}$ and $e, e'$ are twin edges is in correspondence with the set of couples $(\pi_\ell, \pi_r) \in \mathcal{P}(V_\ell) \times \mathcal{P}(V_r)$ such that $T^\pi_\ell, T^\pi_r \in \mathcal{DT}$ ($V_\ell, V_r$ denote the vertex sets of $T_\ell, T_r$). So we have

$$\sum_{\pi \in \mathcal{P}(V) \atop e \sim e'} \sum_{(T^\pi_m \in \mathcal{DT})} \prod_{C \in \mathcal{G}_Y(T^\pi_m)} \frac{1}{N} \text{Tr}^0[C(Y_N)] = \prod_{x \in \{\ell, r\}} \sum_{\pi_x \in \mathcal{P}(V_x)} \sum_{(T^\pi_x \in \mathcal{DT})} \prod_{C \in \mathcal{G}_Y(T^\pi_x)} \frac{1}{N} \text{Tr}^0[C(Y_N)].$$
By Proposition (4.3), we get the claimed result:

$$\mathbb{E}\left[ \frac{1}{N} \text{Tr}[m(X_N, Y_N)] \right] = \sum_{m' = \ell x r} \mathbb{E}\left[ \frac{1}{N} \text{Tr}[\ell(X_N, Y_N)] \right] \mathbb{E}\left[ \frac{1}{N} \text{Tr}[r(X_N, Y_N)] \right] + o(1).$$

## 5 Second-order convergence

### 5.1 Recall of asymptotic formulas

We consider $X_N = (X_\ell)_{\ell \in L}$ and $Y_N = (Y_j)_{j \in J}$ as in Theorem 2.11. As in the previous section, we assume the Wigner matrices normalized and the family of deterministic matrices closed by adjoint. For two $\Delta$-polynomials $q_1$ and $q_2$, we denote $\Phi_N^{(2)}(q_1, q_2)$ the quantity

$$\mathbb{E}\left[ (\text{Tr} q_1(X_N, Y_N) - \mathbb{E}[\text{Tr} q_1(X_N, Y_N)]) (\text{Tr} q_2(X_N, Y_N) - \mathbb{E}[\text{Tr} q_2(X_N, Y_N)]) \right].$$

1. We recall from [10] asymptotic formulas for the $2^{nd}$ order distribution on $\Delta$-monomials, in a similar way as for first-order formulas. They involve the fourth moment of the normalized entries of Wigner matrices.

2. The novelty compared to the first-order convergence is that we are now considering the $\Phi_N^{(2)}$ on alternating products of reduced elements. We show in Section 5.2 how this discards the dependence in the fourth moment. We also prove there the orthogonality of the spaces $E_n$, $n \geq 1$.

3. Section 5.3 is devoted to the proof of Mingo-Speicher formula. In Lemma 5.9, we prove that $\Phi^{(2)}$ satisfies a variant of SD equation. Comparing them with similar equations satisfied by $\Phi$, we deduce (ii).

4. Section 5.4 is devoted to the proof of (iii), i.e. the Leibniz property.

Without loss of generality, we assume that $Y_N$ contains the identity matrix and is closed by $\Delta$-monomial $(q(Y_N) \in Y_N$ for each $q \in \mathbb{C}(\gamma)\Delta$).

We first consider two words of the form

$$m_1 = x_{\ell_1} y_{j_1} \cdots x_{\ell_p} y_{j_p},$$

$$m_2 = x_{\ell_{p+1}} y_{j_{p+1}} \cdots x_{\ell_{p+q}} y_{j_{p+q}}.$$

We denote by $T_1 = T_{m_1}$ and $T_2 = T_{m_2}$ the test graphs (c.f. Section 4.1) associated to $m_1$, $m_2$ respectively, and we set $T = T_{m_1} \sqcup T_{m_2} = (V, E, \gamma)$ which hence consists of the disjoint union of the two cycles.

Then we consider two $\Delta$-monomials $p_1$ and $p_2$ obtained by bracketing the expressions $m_1$ and $m_2$ respectively. We denote by $T_{p_1}$ and $T_{p_2}$ the w.
defined as in (4.5) with the new definition of \( T \) permutation invariance of \( X \) for any injective map \( \psi \).

In [16, Section 2.1-2.4], we give an asymptotic formula for any injective map \( \psi : V^\pi \to [N] \), and \( \omega_Y(\pi) = \frac{1}{N^{[c_Y(\pi)]}} \operatorname{Tr}[T_Y^\pi(Y_N)] \) defined as in (4.5) with the new definition of \( T = T_{m_1} \sqcup T_{m_2} \). Then, by permutation invariance of \( X_N \), we have the formula, proved in [16, Section 2.1-2.4],

\[
\Phi_N^{(2)}(m_1, m_2) = \sum_{\pi \in \mathcal{P}(V)} N^{q^{(2)}(\pi)} \omega_X^{(2)}(\pi) \omega_Y(\pi),
\]

where \( q^{(2)}(\pi) = -\frac{|E^\pi| + |E^\pi|}{2} + |C_Y(T^\pi)| \) and \( C_Y(T^\pi) \) is the set of connected components of \( T_Y^\pi \). Denote by \( \mathcal{P}^\nu(V) \subset \mathcal{P}(V) \) the set of partitions \( \pi \geq \nu \), so \( \pi \in \mathcal{P}^\nu(V) \) whenever \( T^\pi \) is a quotient of \( T_{p_1} \sqcup T_{p_2}^* \). The same proof as for (5.2) yields

\[
\Phi_N^{(2)}(p_1, p_2^*) = \sum_{\pi \in \mathcal{P}^\nu(V)} N^{q^{(2)}(\pi)} \omega_X^{(2)}(\pi) \omega_Y(\pi).
\]

In [16, Section 2.1-2.4], we give an asymptotic formula \( N^{q^{(2)}(\pi)} \omega_X^{(2)}(\pi) \omega_Y(\pi) \) for any \( \pi \in \mathcal{P}(V) \). We can hence use it to compute \( \Phi_N^{(2)}(p_1, p_2^*) \).

**Definition 5.1.** Let \( T^\pi \) be a quotient of \( T = T_{m_1} \sqcup T_{m_2} \).

1. We say that \( T^\pi \) is of double-unicyclic type and denote \( T^\pi \in \mathcal{DU} \) if
   
   - the graph \( \overline{\text{GDC}}(T^\pi) \), given from \( \text{GDC}(T^\pi) \) by forgetting edge multiplicity (multip.), has a unique simple cycle, the edges of \( \text{GDC}(T^\pi) \) labeled in \( x \) have multip. 2, and twin edges are labeled by a same variable \( x \) and have opposite orientation.
Theorem 2.11 converges to a Gaussian process.

If moreover \( \omega \) normalization of Wigner matrices implies

We first introduce notations in order to compute \( \Phi \)

5.2 Traces of alternating products of reduced elements

Proposition 5.2. [16, Section 3] Under Hypotheses (H1), (H2) and (H4)

\[
N^{q(2)(\pi)} \omega_X^{(2)}(\pi) \omega_Y(\pi)
= \mathbb{1}(T^x \in \mathcal{DU} \cup \mathcal{FT}) \omega_X^{(2)}(\pi) \prod_{C \in \mathcal{C}(T^x)} \frac{1}{N} \text{Tr}^0[C(Y_N)] + o(1). \tag{5.4}
\]

If moreover \( \Phi_N^{(2)}(p_1, p_2) \) converges for any \( p_1, p_2 \), then the process \( Z_N \) of

Theorem 2.11 converges to a Gaussian process.

If \( T^x \in \mathcal{DU} \) is of double-unicyclic type (with unicycle size \( \geq 2 \)) the normalization of Wigner matrices implies \( \omega_X(\pi) = 1 \). Contrarywise, if \( T^x \in \mathcal{FT} \) then \( \omega_X(\pi) \) depends on the fourth moment of the entry of a Wigner matrix.

5.2 Traces of alternating products of reduced elements

We first introduce notations in order to compute \( \Phi^{(2)} \) for generic \( \Delta \)-polynomials. Notations \( m_i, p_i \) and \( \nu \) are before. For the sequel, we fix \( p_1 \) and \( p_2 \) of form

\[
p_1 = s_1 r_1 \cdots s_{n_1} r_{n_1}, \quad p_2 = s_{n_1+1} r_{n_1+1} \cdots s_{n_1+n_2} r_{n_1+n_2}.
\]

where \( n_1, n_2 \geq 1 \) and we assume the following. Each \( s_k \) belongs to \( \mathcal{D}^{(x,y)}(x_{\ell_k}^\prime) \) for some variable \( x_{\ell_k}^\prime \) and is of the form \( s_k = x_{\ell_k}^\prime s_k^\prime = s_k^\prime x_{\ell_k}^\prime \). Each \( r_k \) is in \( \mathcal{D}^{(x,y)}(y) \); if moreover \( r_k \) is \( \Delta \)-invariant, then the neighboring words are labeled by different variables, with the usual cyclic convention i.e. \( \ell_k \neq \ell_{k+1} \) (if \( k \notin \{n_1, n_1+n_2\} \)) or \( \ell_{n_1} \neq \ell_1 \) (if \( k = n_1 \)) or \( \ell_{n_1+n_2} \neq \ell_{n_1+1} \) (if \( k = n_1+n_2 \)). We then define

\[
q_1 = s_1 \tilde{r}_1 \cdots s_{n_1} \tilde{r}_{n_1} \tag{5.5}
\]

\[
q_2 = \tilde{s}_{n_1+1} \tilde{r}_{n_1+1} \cdots \tilde{s}_{n_1+n_2} \tilde{r}_{n_1+n_2} \tag{5.6}
\]

as follow:
• if $n_1 \geq 2$ or $n_1 = 1$ and $r_1$ is not $\Delta$-invariant, then $\bar{s}_k = s_k - \Delta(s_k)$, $\bar{r}_k = r_k - \Delta(r_k)$ if $r_k$ not $\Delta$-invariant and $\bar{r}_k = r_k$ otherwise, $\forall k = 1, \ldots, n_1$;

• if $n_1 = 1$ and $r_1$ is $\Delta$-invariant, we simply set $q_1 = p_1$;

• the similar definition holds for $q_2$.

Note that in the first case $q_1 \in E_n$ for an $n \geq 2$: it is a cyclically $D^{(a)}$-alternating product of $\Delta$-centered elements. In the second case $q_1 \in D^{(x,y)}(x')_1$: its ground monomial is a power of $x_{q_1}$ (see the resemblance with the definition of $E_1$).

In Lemma 5.4 below, we give an expression of $\Phi^{(2)}_N(q_1, q_2^*)$ obtained by ruling out certain terms in the expression (5.3) for $(\eta_0, p_1^*)$. Since $p_1 = s_1 r_1 \cdots s_{n_1} r_{n_1}$ is not $\Delta$-invariant, its partition $\sigma(p_1)$ has a ground block. This corresponds to a cycle of $\mathcal{T}_{p_1}$ that we call the ground cycle. The variables in the ground block of $p_1$ appears as labels on the ground cycle of $\mathcal{T}_{p_1}$. We call arcs the maximal connected subsets of edges of the ground cycle that are labeled by letter of a single subword $r_k$ or $s_k$. There is hence a first arc for $s_1$ if it is not $\Delta$-invariant, there is always an arc for $r_2$ and so on.

We assume $q_1 \in E_n$ for some $n \geq 2$, or equivalently $n_1 \geq 2$ or $n_1 = 1$ and $r_1$ is not $\Delta$-invariant. Hence the ground cycle is the union of at least two arcs. We denote by $(v_k, w_k), k \in \mathcal{I}_1$, the extremities of arcs. More precisely, we set $v_k$ the target of the last (w.r.t. the direct orientation) edge of the arc or $s_k$, and $w_k$ the source of its first edge. For each $k \in [n_1]$ such that $r_k$ is not $\Delta$-invariant, we define similarly the extremities $v_{n_1+k}$ and $w_{n_1+k}$ of the arc of the ground cycle labeled by variables of $r_k$. If we also have $q_2 \in E_N$, then with $p_2^*$ replacing $p_1$, the same definition stands for vertices $v_k^*, w_k^*, k \in \mathcal{I}_2$. We have denoted by $\mathcal{I}_1 \subset [2n_1]$ and $\mathcal{I}_2 \subset [2n_2]$ the set of indices such that $(v_k, w_k)$ and $(v_k^*, w_k^*)$ respectively are defined. If $q_1$ does not belong to $E_n$ we set $\mathcal{I}_1 = \emptyset$.

**Definition 5.3.** We denote by $\mathcal{P}_{\text{red}}(\nu(V)) \subset \mathcal{P}(\nu(V))$ the set of reduced partitions, such that $v_k$ and $w_k$ do not belong to a same block for $k \in \mathcal{I}_1$, and the same property holds for $v_k^*$ and $w_k^*$ and $k \in \mathcal{I}_2$.

**Lemma 5.4.** Notations are as in Proposition 5.2 and as in (5.5) for $q_1, q_2$ and at least one belongs to $E_n$ for some $n \geq 2$. Then we have

$$
\Phi^{(2)}_N(q_1, q_2^*) = \sum_{\pi \in \mathcal{P}_{\text{red}}(\nu(V))} N^{q(2)(\pi)} \omega_{X}^{(2)}(\pi) \omega_{Y}(\pi).
$$

**Proof.** We denote $\mathcal{I} = \mathcal{I}_1 \sqcup \mathcal{I}_2$ the formal union of the index sets and write $\Phi^{(2)}_N(q_1, q_2^*) = \sum_{I \subset \mathcal{I}} (-1)^{|I|}\Phi^{(2)}_N(p_1^{(I)}, p_2^{(I)^*})$, with $p_1^{(I)}$ the $\Delta$-monomial whose test graph is obtained from $\mathcal{T}_{p_1}$ by identifying $v_k$ and $w_k$ for each $k \in I \cap \mathcal{I}_1$, and $p_2^{(I_2)}$ defined similarly. For any $\pi \in V$, let $I_\pi \subset I$ the set of indices $k$ for
which \( v_k \sim w_k \), in disjoint union with indices \( k \) such that \( v_k^* \sim w_k^* \). We get
\[
\Phi_N^{(2)}(q_1, q_2^*) = \sum_{I \subseteq \mathcal{I}} (-1)^{|I|} \sum_{\pi \in \mathcal{P}^\nu(V)} N^{(2)}(\pi) \omega_X^{(2)}(\pi) \omega_Y(\pi)
\]
and we can exchange the two sums \( \Phi_N^{(2)}(q_1, q_2^*) = \sum_{\pi \in \mathcal{P}^\nu(V)} \left( \sum_{I \subseteq \mathcal{I}} (-1)^{|I|} N^{(2)}(\pi) \omega_X^{(2)}(\pi) \omega_Y(\pi) \right) \).

The sum over \( I \subseteq \mathcal{I}_\pi \) vanishes if \( \mathcal{I}_\pi \) is not the empty set. Hence the result.

\[ \Box \]

**Corollary 5.5.** Let \( q_1, q_2 \) as in (5.5) such that at least one belongs to \( E_n \) for some \( n \geq 2 \). Then we have
\[
\Phi_N^{(2)}(q_1, q_2^*) = \sum_{\pi \in \mathcal{P}_{\text{red}}^\nu(V)} 1(\tau^\pi \in \mathcal{D} \mathcal{U}) \prod_{C \in \mathcal{C}_Y(T^\pi)} \frac{1}{N} \text{Tr} \left[ C(Y_N) \right] + o(1).
\]

**Proof.** We assume that \( q_1 \) is in \( E_n \), i.e. \( n_1 \geq 2 \) or \( r_1 \) is not \( \Delta \)-invariant. By Proposition 5.2, it suffices to prove that there is no partition \( \pi \) in \( \mathcal{P}_{\text{red}}^\nu(V) \) such that \( T^\pi \) is of 4-2 tree type. We prove that if \( \pi \in \mathcal{P}^\nu(V) \) is such that the quotient \( T_1^\pi \) of the cactus associated to \( p_1 \) is of double-tree type, then \( \pi \) is not reduced, which implies the desired assertion.

Let \( \pi \in \mathcal{P}^\nu(V) \) such that \( T_1^\pi \in \mathcal{D} \mathcal{T} \). Denote by \( C \) the ground cycle of \( T_{p_1} \). Then \( C \) induces a closed path \( C^\pi \) on \( T_1^\pi \). We call **colored component** of a test graph a maximal connected subgraph (with at least one edge) whose edges are all labeled either by a same variable \( x \) in \( x \), or by variables in \( y \). We call **graph of colored components** and denote \( \mathcal{GCC}(C^\pi) \) the undirected bipartite graph such that

- the vertex set is the disjoint union of the set \( V_1 \) of colored components of \( C^\pi \), and the set \( V_2 \) of vertices of \( C^\pi \) that belong to least two colored components,
- each \( v \in V_2 \) is connected by an edge to \( S \in V_1 \) whenever \( v \in S \).

Since the graph of deterministic components \( \mathcal{GDC}(T^\pi) \) is of double-tree type, so is \( \mathcal{GCC}(C^\pi) \), and hence the graph of colored components \( \mathcal{GDC}(C^\pi) \) is a tree. Since the ground cycle of \( p_1 \) has at least two arcs labeled by different variables, \( \mathcal{GDC}(C^\pi) \) has at least two vertices in \( V_1 \). Hence the tree \( \mathcal{GDC}(C^\pi) \) has at least two leaves. A leaf of this graph is necessarily in \( V_1 \) since a vertex of \( V_2 \) is connected to at least two elements of \( V_1 \) by definition.

Let \( e \) be an edge of the ground cycle of \( T_1 \) such that the corresponding step of \( C^\pi \) belong to a leaf of \( \mathcal{GCC}(C^\pi) \). This edge belongs to an arc associated either to some \( s_k \) or to a non \( \Delta \)-invariant \( r_k \). We assume this arc associated to \( s_k \), there is no modification of the reasoning in the other case. All edges of \( s_k \) belong to the same colored component of \( C \), so in the quotient \( C^\pi \) they also belong to the same colored component. The extremities of the arc \( s_k \) also belong to another colored component. Since we have considered a leaf, there is only one vertex of the component of \( s_k \) with this property, hence \( v_k \) and \( w_k \) are equal. As a conclusion, \( \pi \) is not reduced. \( \Box \)
In this section we assume double cycle of $T^\pi \in DU$ the maximal subgraph that covers the cycle of $GDC(T^\pi)$ when multiplicity is forgotten.

**Corollary 5.6.** Assume $q_1 \in E_n$ for some $n \geq 2$. Then for any $\pi \in P'_\text{red}(V)$ such that $T^\pi \in DU$, each arc associated to $s_k$ or to a non $\Delta$-invariant $r_k$ has at least one edge on the double cycle of $T^\pi$.

**Proof.** We consider $\pi \in P'_\text{red}(V)$ such that $T^\pi \in DU$. We repeat the same reasoning as for the previous corollary. The graph of colored component $GCC(C^\pi)$ is a unicyclic graph. If it has a leaf, then corresponds an identification and so $\pi$ is not reduced.

**Corollary 5.7.** The spaces $E_n, n \geq 0$ are asymptotically orthogonal for $\Phi^{(2)}_N$, that is: $\Phi^{(2)}_N(q,q^*) \xrightarrow{N \rightarrow \infty} 0$, for any $q \in E_n$ and $q^* \in E_{n'}$ such that $n \neq n'$.

**Proof.** Each element $q \in E_n$ for some $n \geq 2$ is linear combination of element of the form $q = q_1$ as in the beginning of the section, given by reducing terms of a cyclic alternating product of $\Delta$-monomials. As well for each $\Delta$-monomial $q \in E_1$, either its has zero full degree in $x$ and so $\Phi^{(2)}_N(q, \cdot) = 0$, either by the change block method $\Phi^{(2)}_N(q, \cdot) = \Phi^{(2)}_N(\tilde{q}, \cdot)$ where $\tilde{q}$ is of the form $\tilde{q} = p_1 = s_1 r_1 \in D(x,y)(x)$ as above (in the case where $n_1 = 1$ and $r_1$ is $\Delta$-invariant). Similarly, we can assume that $q'$ is of the form $p_2$ as above.

Hence it is sufficient to prove with these notations that $\Phi^{(2)}_N(q_1, q_2)$ tends to zero if $q_1 \in E_n$ and $q_2 \notin E_n$. But if $T^\pi$ is a double-unicyclic type graph, then the sequence of labels on the unique cycle of $GCC(T^\pi)$ and the same sequence for $GCC(T^\pi)$ with reverse order must coincide. Hence by Corollary 5.6 there is a $\pi \in P'_\text{red}(V)$ such that $T^\pi \in DU$ only if $q_2 \in E_n$.

5.3 Proof of (ii)

In this section we assume $q_1, q_2 \in E_n$ for some $n \geq 2$. In order to compare $\Phi^{(2)}$ with the first-order $\Delta$-distribution $\Phi$, we also set $m_0 = m_1 m_2$, $p_0 = p_1 p_2^*$, and $q_0 = q_1 q_2^*$. We denote by $T_0$ the w.o. simple cycle associated to $\Delta(m_0)$, by $V_0$ its vertex set, and by $v_0$ the partition of $V_0$ such that $T_{p_0} = T_0^{q_0}$ is the w.o. cactus associated to $p_0$. We consider the vertices $v_k, w_k$ and $v_k^*, w_k^*$ in the ground cycle of the graph $T_0$ with same definition as before. By [3] Lemma 4.5, we have the analogue of (5.4), namely

$$\Phi_N(q_0) = \sum_{\pi \in P'_\text{red}(V_0)} Nq^{(1)}(\pi) \omega^{(1)}(\pi) \omega_Y(\pi) \quad (5.7)$$

$$= \sum_{\pi \in P'_\text{red}(V_0)} 1(T_0^\pi \in DT) \prod_{C \in \mathcal{C}(T_0^\pi)} \frac{1}{N} \text{Tr}^0 [C(Y_N)] + o(1) \quad (5.8)$$

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Moreover, the ground cycle of $T_0$ is made of arcs from the $s_k$ and the non-$\Delta$-invariant $r_k$ of both $\Delta$-monomials $p_1$ and $p_2^*$. The arc of $s_1$ is neighbor of the arc of $s_{n+1}$ and they can be labeled by a same Wigner matrix. If $r_n$ and $r_{2n}$ are not $\Delta$-invariant, they also form two neighbouring arcs in the ground cycle of $T_0$, if they are $\Delta$-invariant then $s_n$ and $s_{2n}$ have this property. Otherwise, all the other arcs have neighbors labeled by a variable in a different family.

Assume there is a partition $\pi \in P_{\text{red}}(V_0)$ such that $T_0^\pi$ is of double-tree type. Recall from the proof of Corollary 5.5 the definition of the graph of colored components $GCC(T_0^\pi)$. This graph is a tree with at least two leaves, and a leaf cannot contain an edge of an arc whose extremities are not identified if its neighbors are labeled by different families. Hence these leaves corresponds to the two possible cases of adjacent arcs with same labels family. Necessarily $s_1$ and $s_{n+1}$ are labeled by a same variable and $v_1 \sim_n v_1^*$. Moreover, $r_n$ is $\Delta$-invariant if and only if $r_{2n}$ is $\Delta$-invariant, in which case $s_n$ and $s_{2n}$ are labeled by a same variable and $v_k \sim_n v_k^*$. If $r_n$ and $r_{2n}$ are not $\Delta$-invariant, then $v_{2n} \sim_n v_{2n}^*$.

Reasoning by induction on the subgraph obtained by removing the leaves, $\Phi_N(q_0)$ does not converges to zero only if for all $k \in [n]$, $s_k$ and $s_{n+k}$ have same label, $r_k$ is $\Delta$-invariant if and only if $r_{n+k}$ is $\Delta$-invariant. In this case we get

$$\Phi_N(q_0) = \sum_{\pi \in P_{\text{red}}(V_0)} 1(T_0^\pi \in \mathcal{DT}) \prod_{C \in \mathcal{G}_\pi(T_0^\pi)} \frac{1}{N} \text{Tr}^0[C(Y_N)].$$

(5.9)

We are now in position to prove Mingo-Speicher formula for elements $q_1, q_2 \in E_n$, $n \geq 3$ as in (5.5), that $\Phi^{(2)}(q_1, q_2) = \sum_{i=0}^{n-1} \Phi(q_1(q_2)_i^*)$ where

$$(q_2)_i = \tilde{s}_{1+i}n+1\tilde{r}_{n+1+i} \cdots \tilde{s}_{n+1+i}n+1\tilde{r}_{1+i} \cdots \tilde{s}_{1+i}n+1\tilde{r}_{n+1+i}.$$

The expression computed for $\Phi(q_1(q_2)^*)$ extends obviously fo $\Phi(q_1(q_2)_i^*)$, so we have all combinatorial expressions to demonstrate the equality. Yet the exercise of writing a bijection for this proof is delicate, especially when trying to emphasis the crucial role of assumption (H5). We propose instead to prove the formula thanks to a variation of Schwinger-Dyson equation for $\Phi$ and $\Phi^{(2)}$, following the strategy of the first order convergence.

**Lemma 5.8.** With notations as in the beginning of the section, writing $s_1 = xs_1'$, we have the following recurrence

$$\Phi(q_0) = \sum_{s_1 = x\ell} \Phi(\ell) \Phi \left( (r\cdot x - \Delta(r\cdot x)) \tilde{s}_1 \cdots \tilde{s}_n \tilde{r}_n \times q_2^* \right)$$

$$+ \sum_{s_{n+1} = x\ell} \Phi(\ell) \Phi \left( s_1' \tilde{s}_2 \cdots \tilde{s}_n \tilde{r}_n \times \tilde{r}_{2n} \tilde{s}_{2n} \cdots \tilde{r}_{n+1} \ell \right).$$

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Proof. By (5.7) and Proposition 4.3

\[ \Phi(q_0) = \lim_{N \to \infty} \sum_{\pi \in \mathcal{P}_{\text{red}}(V_0)} \mathbb{1}(T_0^\pi \in \mathcal{DT}) \prod_{C \in \mathcal{C}_Y(T_0^\pi)} \frac{1}{N} T_\nu^0[C(Y_N)] \].

Moreover, by (5.9), we can restrict to partitions such that \( v_2 \sim_x w_2 \) (or equivalently \( w_1 \sim_x v_1^* \)). Let \( e \) denote the edge of \( T_{p_0} \) associated to the first letter \( x \) of the word \( p_0 \) in \( T_0 \). For any \( \pi \) as in the above sum and such that \( T_0^\pi \in \mathcal{DT} \), let \( e' \) be the twin edge of \( e \). Then \( e' \) must belong to the same simple cycle as \( e \) in the graph obtained from \( T_0 \) by identifying \( w_1 \) and \( v_1^* \).

We writing in short \( e' \in s_k \) when \( e' \) belongs to the arc defined by \( s_k \), we then get \( e' \in s_1 \) or \( e' \in s_{n+1} \).

If \( s_1 = x s_1' \neq x \), then \( s_1' \) is not the empty word and it is of the form \( s_1' = s_1'' x \). Assume \( e' \in s_1 \). Because of the condition \( v_1 \not\sim_x w_1 \), \( e' \) cannot be the edge corresponding to the last letter \( x \) of the word \( s_1' \). The possible choices of an edge \( e' \in s_1 \) corresponds then to the choices of a decomposition \( s_1' = \ell e x \).

The graph \( T_0^\pi \) is a quotient of the graph \( T_{0,e呤e'} \) obtained by identifying source of \( e \) and target of \( e' \) and reciprocally. Removing \( e \) and \( e' \) from \( T_{0,e呤e'} \) yields two connected components \( T_\ell \) and \( T_{p'} \) where \( p' = x r x_1 s_2 r_2 \cdots r_{n} r_n x r_{n+1} s_{n+1} \). Let \( \pi_1 \) and \( \pi_2 \) be the restrictions of \( \pi \) on the vertex sets of these two subgraphs respectively. Then the quotients \( T_\ell^\pi \) and \( T_{p'}^\pi \) are also of double tree type. The partition \( \pi_2 \) is subject to the constraint that extremities of the arcs of the ground cycle of \( p' \) must not be identified, we denote in short \( \pi_2 \in \mathcal{P}_{\text{red}}(V_{p'}) \). The partition of \( \pi_1 \) is subject to no constraint. Hence, by Proposition 4.3 and the same formula as (5.7) for \( p' \),

\[ \sum_{\pi \in \mathcal{P}_{\text{red}}(V_0), \text{s.t. } e' \in s_1} \mathbb{1}(T_0^\pi \in \mathcal{DT}) \omega_Y(\pi) = \sum_{s_1' = \ell e x r x_1} \mathbb{1}(T_\ell^\pi \in \mathcal{DT}) \omega_Y(\pi_1) \sum_{\pi_2 \in \mathcal{P}_{\text{red}}(V_{p'})} \mathbb{1}(T_{p'}^\pi \in \mathcal{DT}) \omega_Y(\pi_2) \overset{\rightarrow}{\longrightarrow}_{N \to \infty} \Phi[\ell] \Phi[(r x - \Delta(r x)) r_1 s_2 \cdots r_n s_n x q_1^2]. \]

Assume now that \( e' \in s_{n+1} \), which corresponds to a decomposition \( s_{n+1} = \ell e x \). As for the case \( e' \in s_1 \), \( T_\ell^\pi \) is then a quotient of \( T_{0,e呤e'} \). Removing \( e \) and \( e' \) from this graph yields two connected components \( T_\ell \) and \( T_{p'} \) where \( p' = s_1' r r_2 s_2 \cdots r_{n} r_n s_{n+1} r_{n+1} e \). The restriction of \( \pi \) on the vertex set of \( T_\ell \) has no constraints, the restriction on the vertex set of \( T_{p'} \) must not identify extremities of all arcs of the ground cycle but the first and last arcs that correspond to \( s_1' \) and \( \ell \) (conditions \( v_1 \not\sim_x w_1 \) and \( v_{n+1} \not\sim_x w_{n+1} \) are always satisfied when \( e' \in s_{n+1} \) since \( w_1 \) and \( v_{n+1} \) are vertices of \( T_\ell \) whereas \( v_1 \) and \( w_{n+1} \) are vertices of \( T_{p'} \)). Hence the claimed formula. \( \square \)
Lemma 5.9. With above notations, we write \(s_k = (xs'_k - \Delta(xs'_k))\). Then

\[
\Phi^{(2)}(q_1, q_2) = \sum_{s_1 = x} \sum_{\ell = m_1} \sum_{s_{n+1} = \ell} \Phi(s_1) \Phi^{(2)}(r\Delta(r)) \Phi(s_{n+1} - 1) \Phi(s_{n+2} - 1) \Phi(s_{n+3} - 1) \Phi(s_{n+4} - 1) ... \Phi(s_{n+2n} - 1) \Phi(s_{n+2n+1}) \Phi(s_{n+2n+2}) \Phi(s_{n+2n+3}) ... \Phi(s_{n+2n+2n+1})
\]

\[
= \sum_{s_1 = x} \sum_{\ell = m_1} \sum_{s_{n+1} = \ell} \Phi(s_1) \Phi^{(2)}(r\Delta(r)) \Phi(s_{n+1} - 1) \Phi(s_{n+2} - 1) \Phi(s_{n+3} - 1) \Phi(s_{n+4} - 1) ... \Phi(s_{n+2n} - 1) \Phi(s_{n+2n+1}) \Phi(s_{n+2n+2}) \Phi(s_{n+2n+3}) ... \Phi(s_{n+2n+2n+1})
\]

Proof. Let \(\pi\) be a partition such that \(\pi \geq \nu\), \(T^\pi\) is of double-unicyclic type, and \(v_k \neq \pi\) for any \(k\). Let \(e\) be the edge of \(T\) associated to the first letter \(x\) of \(s_1 = x\) and let \(e'\) be the edge twined with \(e\) by \(\pi\).

Condition \(v_k \neq \pi\) implies as before that \(e'\) is not associated to the last letter of \(s_1\). Assume that \(e' \in s_1\), which hence corresponds to a decomposition \(s_1 = \ell e e' x\). The double edge is not in the double cycle of \(T^\pi\), and so the graph obtained by removing \(e\) and \(e'\) from \(T^\pi\) has two connected components: one is a double tree quotient of \(T_e\), the other is a double unicyclic graph quotient of the graph obtained from \(T\) by replacing \(x m_1\) by \(r x\) (Lemma 5.6). By replacing \(s_{n+1} = \ell e\), the graph obtained by removing \(e\) and \(e'\) from \(T^\pi\) produces a graph of double-tree type. The argument of the beginning of this section implies that identification \(v_k \sim \pi\) \(v_k^\pi\) must be satisfied. By (5.3), for \(\pi\) the usual argument on the graph of colored components implies that \(e' \notin s_k\) for \(2 \leq k \leq n\). Assume hence that \(e' \in s_k\) for \(k > n\). The choice of \(k\) corresponds to the choice of \(i \in [1, \ldots, n]\), and the choice of the edge \(e'\) in the arc of \(s_k\) corresponds to the choice of a decomposition \(s_{n+i} = \ell e\). Moreover, removing \(e\) and \(e'\) from \(T^\pi\) produces a graph of double-tree type. The argument of the beginning of this section implies that identification \(w_k \sim \pi\) \(w_k^\pi\) must be satisfied. By (5.3), for \(\pi\) the usual argument on the graph of colored components implies that \(e' \notin s_k\) for \(2 \leq k \leq n\). Assume hence that \(e' \in s_k\) for \(k > n\). The choice of \(k\) corresponds to the choice of \(i \in [1, \ldots, n]\), and the choice of the edge \(e'\) in the arc of \(s_k\) corresponds to the choice of a decomposition \(s_{n+i} = \ell e\). Moreover, removing \(e\) and \(e'\) from \(T^\pi\) produces a graph of double-tree type. The argument of the beginning of this section implies that identification \(w_k \sim \pi\) \(w_k^\pi\) must be satisfied. By (5.3), for \(\pi\) the usual argument on the graph of colored components implies that \(e' \notin s_k\) for \(2 \leq k \leq n\). Assume hence that \(e' \in s_k\) for \(k > n\). The choice of \(k\) corresponds to the choice of \(i \in [1, \ldots, n]\), and the choice of the edge \(e'\) in the arc of \(s_k\) corresponds to the choice of a decomposition \(s_{n+i} = \ell e\). Moreover, removing \(e\) and \(e'\) from \(T^\pi\) produces a graph of double-tree type.

We claim that the source and target vertices of the path corresponding to \(\ell\) must be identifying. Indeed, the labels of the edges are \(x\), so they cannot be twined with the edges of \(s_n\), whose edges must appear in the double cycle just before those of \(s_1\) by Corollary 5.6. Hence in above formula we can replace \(r\) by \(\Delta(r)\). We get the analogue for formula for \(i > 1\) by shifting the indices in \(q_2\), and hence the second contribution in the lemma. \(\Box\)
By Assumption (H5), we have
\[
\Phi \left[ x'_1 \tilde{r}_1 \tilde{s}_2 \cdots \tilde{s}_n \Delta(r) \tilde{r}_{n+i} \cdots \tilde{r}_{n+i-1} \right] = \Phi(r) \Phi \left[ x'_1 \tilde{r}_1 \tilde{s}_2 \cdots \tilde{s}_n \Delta(r) \tilde{r}_{n+i} \cdots \tilde{r}_{n+i-1} \right].
\]

Assume first that \( s_1 \) has ground degree 1 or 2. Then both in Lemmas 5.8 and 5.9 there are no decomposition \( s'_1 = \ell x rx \) and the first sum is zero. We hence get that \( \Phi^{(2)}(q_1, q_2) = \sum_{i=0}^{n-1} \Phi(q_1(q_2)_i^{(2)}) \) as claimed. Assume now that \( s_1 \) has ground larger than 2. Subtracting the formulas
\[
\Phi^{(2)}[q_1, q_2] - n\Phi[q_1, \mathcal{M}(q_2)^*] = \sum_{s_1=\ell x rx} \Phi(\ell) \left( \Phi^{(2)}[q_1', q_2] - n\Phi[q_1', \mathcal{M}(q_2)^*] \right),
\]
where \( q'_1 = rx - \Delta(rx) \). By induction of the ground degree of \( s_1 \), we hence get that the formula is always valid.

5.4 Proof of (iii)

By linearity, traciality and invariance under \( \Delta \) it is sufficient to prove that for any non-mixing \( \Delta \)-monomial \( p \),
\[
\Phi^{(2)}_N(p, \cdot) = \sum_{B \in \sigma(p)} \Phi^{(2)}_N(p_B, \cdot) \prod_{B' \neq B} \Phi_N(m_{B'}) + o(1). \tag{5.10}
\]

If \( p \in \mathbb{C}(y)_\Delta \), the equality is obvious; both sides vanish. Let \( p \in \mathbb{C}(x, y)_\Delta \) of positive full degree on a variable \( x \in x \). Assume that \( \Phi^{(2)}_N(p) \) holds for all \( \Delta \)-monomials of smaller full degree. By group block change, which preserves both side of \( \Phi^{(2)}_N \), we can assume that \( p \) is actually of the form \( xm \). We hence consider \( \Delta \)-monomials of the form \( p = p_1 = xm \) and \( p_2 = m' \) as in the previous sections. In the graph \( T_{p_1} \), let \( e \) be the edge associated to the first letter \( p = xm \).

- Denote by \( \mathcal{P}_1 \subset \mathcal{P}(V) \) the set of partitions \( \pi \) such that \( T^\pi \in DU \cup FT \), \( e \) is twinned with an edge \( e' \) of \( m \) (so \( e \) is not in the double cycle if \( T^\pi \in DU \)) and is not in the group of edges of multi. 4 if \( T^\pi \in FT \).

- We denote by \( \mathcal{P}_2 \subset \mathcal{P}(V) \) the set of partitions \( \pi \) such that \( T^\pi \in FT \) and \( e \) belongs to the group of edges of multip. 4.

- Let \( \mathcal{P}_k \subset \mathcal{P}(V) \) be the set of partitions \( \pi \) such that \( T^\pi \in DU \), \( e \) belongs to the double cycle of \( T_\pi \) and the double cycle of \( T_\pi \) is of length \( k \geq 3 \).

Setting \( \alpha_k(p_1, p_2) := \sum_{\pi \in \mathcal{P}_k} \omega^{(2)}_{\mathcal{P}}(\pi) \omega_{\mathcal{P}}(\pi) \) for any \( k \geq 1 \) (the sum is finite), by Proposition 5.2 we have \( \Phi^{(2)}(p_1, p_2) = \sum_{k \geq 1} \alpha_k(p_1, p_2) + o(1) \). The same reasoning as for SD equation yields the following relations. Firstly
\[
\alpha_1(p_1, p_2) = \sum_{p_1=\ell x rx} (\Phi_N^{(2)}(\ell, m') \Phi_N(r) + \Phi_N(\ell) \Phi_N^{(2)}(r, m')) + o(1),
\]

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since removing \(e, e'\) form \(T^{\pi}_{p_1}\) gives two connected components, one of double tree type, and the other will contribute with \(T^{\pi}_{p_2}\) to form an element of \(\mathcal{DU} \cup \mathcal{FT}\). By the Schwinger-Dyson equation, (2.1) and the induction hypothesis, denoting by \(GB\) the ground block of \(p = p_1\), we verify easily that \(\alpha_1(p, \cdot)\) is equal to

\[
\sum_{B \in \sigma(p), \ B \neq GB} \Phi_N^{(2)}(p_B, \cdot) \prod_{B' \neq B} \Phi_N(p_{B'}) + \alpha_1(p_{GB}, \cdot) \prod_{B \in \sigma(p)} \Phi_N(p_B) + o(1). \tag{5.11}
\]

Similarly, setting \(m_4(x) = E[|x_{12}|^4]\) when \(x\) is associated to \(X_N = \left(\frac{x_i}{\sqrt{n}}\right)\),

\[
\alpha_2(p_1, p_2) = \sum_{p_1 = x \ell x r} \sum_{B \in \sigma(p_2), p_2B = \ell' x m'' x r'} (m_4(x) - 1) \left( \Phi_N(\Delta(\ell)\Delta(m')) \Phi_N(\Delta(r)\Delta(r'\ell')) + \Phi_N(\Delta(\ell)\Delta(r'\ell')) \Phi_N(\Delta(r)\Delta(m')) \right) + o(1),
\]

Indeed the choice of a block \(B\) and of a decomposition \(p_{2B} = \ell' x m'' x r'\) correspond to the choice of edges \(e'', e'''\) such that \(\{e, e', e'', e''\}\) form the edge of multip. 4. The graph obtained by removing the edge of multip. 4 is then a union of double-tree type graphs. By (2.1), we can split the terms \(\Phi_N(\Delta(\ell)\Delta(m')) = \Phi_N(\ell)\Phi_N(m') + o(1)\). This allows to prove, with the same arguments as for (5.11)

\[
\alpha_2(p, \cdot) = \alpha_2(p_{GB}, \cdot) \prod_{B \in \sigma(p), B \neq GB} \Phi_N(p_B) + o(1). \tag{5.12}
\]

Finally, for any \(k \geq 3\), we consider all decompositions \(p_1 = x m_1 x m_2 x \cdots x m_k\), and \(p_{2B} = \ell x m'_1 \cdots x m'_{k-1} x r\), for any block \(B \in \sigma(p_2)\) in the formula below

\[
\alpha_k(p_1, p_2) = \sum_{p_1 = \cdots} \sum_{p_{2B} = \cdots} \Phi(\Delta(m_1)\Delta(m_{i+1})) \cdots \Phi(\Delta(m_k)\Delta(m_{i+k})) + o(1),
\]

with indices modulo \(k\) and \(m'_k := r \ell\). The decompositions of \(p_1\) and \(p_2\) represent the choice of the edges of the double cycles. The sum over \(i\) represent the \(n\) different way we can merge the \(n\)-cycles of \(T^\pi_1\) and \(T^\pi_2\) in a double cycle. As in the previous case, this allows to obtain that (5.12) is valid for \(\alpha_k\) replacing \(\alpha_1\) in both sides. Together with (5.11), these relations imply Formula (5.11).

**Remark 5.10.** The Leibniz property is stated only on \(E_1\). It is expected to hold for all \(\Delta\)-polynomials, the restriction has only purpose to simplify the proof.
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