POLLICOTT-RUELLE RESONANT STATES AND BETTI NUMBERS

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ABSTRACT. Given a closed hyperbolic manifold of dimension $\neq 3$ we prove that the multiplicity of the Pollicott-Ruelle resonance of the geodesic flow on perpendicular one-forms at zero agrees with the first Betti number. Additionally, we prove that this equality is stable under small perturbations of the Riemannian metric. Furthermore, we identify for hyperbolic manifolds further resonance spaces whose multiplicities are given by higher Betti numbers.

INTRODUCTION

Pollicott-Ruelle resonances have been introduced in the 1980’s in order to study mixing properties of hyperbolic flows and can nowadays be understood as a discrete spectrum of the generating vector field (see Section 1.2 for a definition and references). Very recently it has been discovered that in certain cases some particular Pollicott-Ruelle resonances have a topological meaning. Let us recall these results:

In [DZ17] Dyatlov and Zworski prove that on a closed oriented surface $\mathcal{M}$ of negative curvature the Ruelle zeta function at zero vanishes to the order $|\chi(\mathcal{M})|$, where $\chi(\mathcal{M})$ is the Euler characteristic of $\mathcal{M}$. They prove this result as follows: By previous results on the meromorphic continuation of the Ruelle zeta function [DZ16, GLP13] the order of vanishing of the Ruelle zeta function at zero can be expressed as the alternating sum $\sum_{k=0}^{2}(-1)^{k+1}m_{\mathcal{L}_X,\Lambda^n(E^s_1)}(0)$, where $m_{\mathcal{L}_X,\Lambda^n(E^s_1)}(0)$ is the multiplicity of the resonance zero of the Lie derivative $\mathcal{L}_X$ along the geodesic vector field $X \in \Gamma^{\infty}(T(S^*\mathcal{M}))$ acting on perpendicular $k$-forms. The latter are those $k$-forms on the unit co-sphere bundle $S^*\mathcal{M}$ that vanish upon contraction with $X$ (for the precise definition of the multiplicities, see Sections 1.1 and 1.2). For closed oriented surfaces it is rather easy to see that $m_{\mathcal{L}_X,\Lambda^0(E^s_1)}(0) = m_{\mathcal{L}_X,\Lambda^2(E^s_1)}(0) = b_0(\mathcal{M}) = b_2(\mathcal{M})$, thus the central task is to prove that $m_{\mathcal{L}_X,E^s_1}(0) = b_1(\mathcal{M})$.

Dyatlov and Zworski achieve this by combining microlocal analysis with Hodge theory [DZ17, Proposition 3.1(2)]. This is a remarkable result also apart from its implications on zeta function questions because it identifies a resonance whose multiplicity has a precise topological meaning.

Let us mention a second result that establishes a connection between Pollicott-Ruelle resonances and topology: Dang and Rivière [DR17c] examine a general Anosov flow $\varphi_t = e^{\lambda t}$ on a closed manifold. The Lie derivative $\mathcal{L}_X$ has a discrete spectrum (the Pollicott-Ruelle spectrum) on certain spaces of anisotropic $p$-currents and it is shown that the exterior derivative acting on generalized eigenspaces of the eigenvalue zero forms a complex which is quasi-isomorphic to the de Rham complex $\mathbb{R}$. While this result gives no precise information about the multiplicities of the resonances, it gives lower bounds for them and it holds in very great generality.

As a third result we would like to mention [GHW18a] where the relation between Pollicott-Ruelle and quantum resonances is studied for compact and convex co-compact hyperbolic surfaces. For this correspondence the resonances at negative integers turn out to be exceptional points and it is shown that their multiplicities can be expressed by the Euler characteristic of the hyperbolic surface. The proof uses a Poisson transform to establish a bijection between the resonant states and holomorphic sections of certain line bundles, and the formula for the multiplicities follows from a Riemann-Roch theorem.

In the present article we broaden the picture regarding the topological properties of Pollicott-Ruelle resonant states. To this end, we combine some of the above approaches: In a first step we use a quantum-classical correspondence to find new examples of resonances with topological multiplicities. In particular, we prove

1We would like to point out that an analogous statement also holds for Morse-Smale flows [DR16, DR17c, DR17d] and in these cases the spectral complex defined by the Pollicott-Ruelle resonances is actually isomorphic to the Morse complex. Consequently, the spectral complex of Dang and Rivière can be considered as a generalization of the Morse complex to Anosov flows.
Proposition 0.1. For any closed hyperbolic manifold $\mathcal{M}$ of dimension $n+1$ with $n \neq 2$, one has

$$m_{\mathcal{L}_X,\mathcal{E}_+}(0) = b_1(\mathcal{M}).$$

Furthermore, the resonance zero has no Jordan block and if $n \geq 3$, then zero is the unique leading resonance and there is a spectral gap.

We prove these statements using the general framework of vector-valued quantum-classical correspondence developed by the authors [KW13] as well as a Poisson transform of Gaillard [Gai86]. Without any further effort these ingredients provide additional examples of resonance multiplicities that involve not only the first but all Betti numbers (see Proposition 2.3) which should be of independent interest. For $n = 1$ the first statement in Proposition 0.1 is the special case of [DZ17] Proposition 3.1(2) restricted to hyperbolic surfaces. Interestingly $n = 2$ is an exceptional case and the multiplicity is given by $m_{\mathcal{L}_X,\mathcal{E}_+}(0) = 2b_1(\mathcal{M})$ (see Remark 2.2). For $n > 2$ the statement can be considered as a generalization of the Dyatlov-Zworski result to higher dimensions at the cost of restricting to manifolds of constant negative curvature.

In a second step we can partially overcome this restriction and prove

Theorem 0.2. Let $(\mathcal{M}, g_0)$ be a closed hyperbolic manifold of dimension $n+1$ with $n \neq 2$ and let $\Gamma^\infty(S^2(T^*\mathcal{M}))$ be the space of smooth symmetric two-tensors endowed with its Fréchet topology and $\mathcal{R}_{\mathcal{M},<0}$ the open subset of Riemannian metrics of negative sectional curvature. Then there is an open neighborhood $U \subset \mathcal{R}_{\mathcal{M},<0}$ of $g_0$ such that for all Riemannian metrics $g \in U$ one has

$$m_{\mathcal{L}_X,\mathcal{E}_+}(0) = b_1(\mathcal{M}).$$

We prove this statement by combining Proposition 0.1, which has been obtained by a quantum-classical correspondence, with the cohomology results of Dang-Rivière as well as some recent advances concerning the perturbation theory of Pollicott-Ruelle resonances for flows [DGRS18] Proposition 6.3 [Bon18]. Note that also in dimension $n+1 = 2$ our methods give the equality (0.1) only in a neighborhood of $g_0$, whereas Dyatlov and Zworski prove the equality in this dimension for all $g \in \mathcal{R}_{\mathcal{M},<0}$. It seems thus reasonable to conjecture that the equality holds for all $g \in \mathcal{R}_{\mathcal{M},<0}$ in all dimensions $n+1 \neq 3$.

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1. Pollicott-Ruelle resonances for geodesic flows

1.1. Geodesic flows on manifolds of negative curvature. Let $(\mathcal{M}, g)$ be a closed Riemannian manifold of dimension $n+1$ with negative sectional curvature. Then the geodesic flow $\varphi_t$ on the unit co-sphere bundle $S^*\mathcal{M}$ is an Anosov flow which implies that there is a $d\varphi_t$-invariant Hölder continuous splitting of the tangent bundle $T(S^*\mathcal{M})$

$$T(S^*\mathcal{M}) = E_0 \oplus E_+ \oplus E_-,$$

where $E_0 = \mathbb{R}X$ is the neutral bundle spanned by the geodesic vector field $X$ and $E_+, E_-$ are the stable and unstable bundles, respectively (see e.g. [Kni02]). Additionally, there is a smooth contact one-form $\alpha \in \Omega^1(S^*\mathcal{M})$ which is simply the restriction of the Liouville one-form on $T^*\mathcal{M}$ to $S^*\mathcal{M}$. It fulfills

$$t_X a = 1, \quad \ker(a) = E_+ \oplus E_-, \quad da \text{ is symplectic on } \ker(a), \quad \mathcal{L}_X a = 0,$$

---

2See the paragraph below (1.2) for the definition of “having no Jordan block” and the footnote in Prop. 3.1 for the other terms used here.

3It has been noted in [DGRS18] Remark 5 (without detailing the proof) that the statement of Proposition 0.1 can alternatively be obtained by a zeta factorization argument similar to [DGRS18] Proposition 7.7 based on the work of Bunke and Olbrich [BO99].
where $\mathcal{L}_X$ denotes the Lie derivative. Note that the last two properties imply that $\alpha \wedge (d\alpha^n)$ is a nowhere-vanishing flow-invariant volume form which defines the Liouville measure on $S^*\mathcal{M}$. Using the contact one-form we get a splitting of the cotangent bundle into smooth subbundles

$$T^*(S^*\mathcal{M}) = \mathbb{R}\alpha \oplus E^+_1, \quad E^+_1 := \{ \xi \in T^*(S^*\mathcal{M}) : i_X \xi = 0 \}.$$  

We will call the smooth sections of $E^+_1$ perpendicular one-forms and denote their space by $\Omega^1_\perp(S^*\mathcal{M})$. More generally, we introduce for $p = 0, \ldots, n$ the space of perpendicular $p$-forms

$$\Omega^p_\perp(S^*\mathcal{M}) := \{ \omega \in \Omega^p(S^*\mathcal{M}) : i_X \omega = 0 \} = \Gamma^\infty(\Lambda^p(E^+_1)).$$

By the Anosov splitting, the bundle $E^+_1$ can be further split into

$$E^+_1 = E^+_\ast \oplus E^-_\ast,$$

where the dual stable and unstable bundles are defined by $E^-_\ast := E^+_1 - E^+_0 = 0$. In contrast to the smoothness of $E^+_1$, the subbundles $E^\perp_\ast$ are only Hölder continuous unless $\mathcal{M}$ is a locally symmetric space of rank one.

**Remark 1.1** (Complexifications). When addressing spectral questions involving an operator on any of the bundles mentioned so far, or on any subbundle of a tensor power of $T^*(S^*\mathcal{M})$, it is often more useful to work with the complexified bundle. For simplicity of notation we shall not explicitly distinguish in the following between real vector bundles and their complexifications. It will be clear from the context whether we refer to the real or the complexified bundle.

1.2. **Pollicott-Ruelle resonances on forms.** Pollicott-Ruelle resonances were introduced by Pollicott [Pol85] and Ruelle [Rue86] in order to study mixing properties of hyperbolic flows (as mentioned before). In the last years it has been found out that these resonances can also be defined as poles of meromorphically continued resolvents [Liv04, BL07, FS11, DG16, BW17] (see also [BKL02, Bal05, BT07, Bal18, BT08, GL06, GL08, Liv05, FRS08] for important related work on hyperbolic diffeomorphisms). We follow [DG16] to introduce the notion of Pollicott-Ruelle resonances on an arbitrary smooth complex vector bundle $\mathcal{V} \to S^*\mathcal{M}$. A first order differential operator $X$ on $\mathcal{V}$ is called an **admissible lift** of the geodesic vector field if

$$X(fu) = (Xf)u + fXu, \quad f \in C^\infty(S^*\mathcal{M}), \quad u \in \Gamma^\infty(\mathcal{V}).$$

An example of such an admissible lift is the Lie derivative $\mathcal{L}_X$ with respect to the geodesic vector field on any $d\varphi_t$-invariant subbundle of $\bigotimes^p T^*(S^*\mathcal{M})$ for some $p \in \mathbb{N}_0$ (taking into account Remark 1.1). In Section 2 we will additionally consider covariant derivatives which are further examples of admissible lifts. After choosing a smooth metric on $\mathcal{V}$ one defines the space $L^2(S^*\mathcal{M}, \mathcal{V})$. Note that by the compactness of $\mathcal{M}$ only the norm on this space depends on the choice of the metric but neither does the space nor its topology. One checks that there is a constant $C_X > 0$ such that $X + \lambda : L^2(S^*\mathcal{M}, \mathcal{V}) \to L^2(S^*\mathcal{M}, \mathcal{V})$ is invertible for $\text{Re}(\lambda) > C_X$.

**Proposition 1.2** ([DG16] Theorem 1]). The resolvent $R_{X,\mathcal{V}}(\lambda) := ((X + \lambda)^{-1} : L^2(S^*\mathcal{M}, \mathcal{V}) \to L^2(S^*\mathcal{M}, \mathcal{V})$, $\text{Re}(\lambda) > 0$, has a continuation to the whole complex plane as a meromorphic family of bounded operators

$$R_{X,\mathcal{V}}(\lambda) : C^\infty(S^*\mathcal{M}, \mathcal{V}) \to D'(S^*\mathcal{M}, \mathcal{V}).$$

Moreover, for any pole $\lambda_0$ the residue operators $\Pi_{\lambda_0} = \text{res}_{\lambda=\lambda_0}(R_{X,\mathcal{V}}(\lambda))$ have finite rank.

**Definition 1.3.** The poles of $R_{X,\mathcal{V}}(\lambda)$ are called **Pollicott-Ruelle resonances of $X$**. Given a resonance $\lambda_0$, the **finite-dimensional space** $\mathcal{R}_{X,\mathcal{V}}(\lambda_0) := \text{ran}(\Pi_{\lambda_0}) \subseteq D'(S^*\mathcal{M}, \mathcal{V})$ is the space of generalized Pollicott-Ruelle resonant states and we call $m_{X,\mathcal{V}}(\lambda_0) := \text{dim}_\mathbb{C} \mathcal{R}_{X,\mathcal{V}}(\lambda_0)$ the **multiplicity** of the resonance $\lambda_0$.

For any resonance $\lambda_0$, there exists a number $J(\lambda_0) \in \mathbb{N}$ such that the generalized resonant states have the following alternative description [DG16] Theorem 2]:

$$\mathcal{R}_{X,\mathcal{V}}(\lambda_0) = \{ u \in D'(S^*\mathcal{M}, \mathcal{V}) : (X + \lambda_0)^{J(\lambda_0)}u = 0, \text{WF}(u) \subset E^+_\perp \}. $$

If $J(\lambda_0) = 1$ we say that the resonance has no Jordan block. Otherwise, the space of **Pollicott-Ruelle resonant states** $\mathcal{R}_{X,\mathcal{V}}(\lambda_0) := \ker(X + \lambda_0) \cap \mathcal{R}_{X,\mathcal{V}}(\lambda_0)$ is a proper subspace of $\mathcal{R}_{X,\mathcal{V}}(\lambda_0)$.

Note that the resolvent, the Pollicott-Ruelle resonances, and the associated resonant states and multiplicities depend on the Riemannian metric $g$. For this reason we will write $R_{X,\mathcal{V}}(\lambda, g)$, $\mathcal{R}_{X,\mathcal{V}}(\lambda, g)$, $m_{X,\mathcal{V}}(\lambda, g)$, ... in order to emphasize the dependence on $g$. In the other sections we suppress the Riemannian metric in the notation.
2. Multiplicities on constant curvature manifolds

Throughout the whole section we assume that \((\mathcal{M}, g)\) is a closed hyperbolic manifold of dimension \(n + 1\).

**Proposition 2.1.** If \(n \neq 2\), then
\[
m_{E^1_0}(0) = b_1(\mathcal{M}).
\]
Furthermore, the resonance zero has no Jordan block, and if \(n \geq 3\), then zero is the unique leading resonance and there is a spectral gap.\(^4\)

The first part of this result will be a central ingredient for Theorem~\ref{thm:main}. We will prove Proposition~\ref{prop:main} using a quantum-classical correspondence. These correspondences have recently been developed in various contexts \cite{DFG15,GHW18a,GHW18b,Had18} and we will use the general framework for vector bundles developed by the authors in \cite{KW18}. Additionally, we use a Poisson transform due to Gaillard \cite{Gai86} and combining both ingredients allow us to construct an explicit bijection between the Pollicott-Ruelle resonant states in perpendicular one forms and the kernel of the Hodge Laplacian.

**Remark 2.2.** The dimension \(n + 1 = 3\) is an exception where the multiplicity is given by \(m_{E^1_0}(0) = 2b_1(\mathcal{M})\). The deeper reason for this exception is that Gaillard’s Poisson transform is not bijective in this case. The exceptional case could also be treated with our methods by a more detailed analysis of Gaillard’s Poisson transform. This special case has however been worked out already in \cite[Proposition 7.7]{DGRS18} by factorizations of zeta functions, so we refrain from taking on the additional effort.

A crucial role in these quantum-classical correspondences is played by the so-called (generalized) first band resonant states
\[
\text{Res}^{1\text{st}}_{X,Y}(\lambda_0) := \text{Res}_{X,Y}(\lambda_0) \cap \ker U_, \quad \text{res}^{1\text{st}}_{X,Y}(\lambda_0) := \text{res}_{X,Y}(\lambda_0) \cap \ker U_.,
\]
where \(U_\cdot\) is the horocycle operator which we will introduce below in (2.9). Roughly speaking, first band resonant states are resonant states that are constant in the unstable directions. In the process of proving Proposition~\ref{prop:main} we will establish the following result.

**Proposition 2.3.** On any closed hyperbolic manifold \(\mathcal{M}\) of dimension \(n + 1\) and for any \(p = 0, \ldots, n, p \neq n/2\),
\[
\dim \mathbb{C} \text{Res}^{1\text{st}}_{X,Y,\Lambda^p(\mathcal{E}^1)}(0) = \dim \mathbb{C} \text{Res}^{1\text{st}}_{X,Y,\Lambda^p(\mathcal{E}^1)}(-2\rho) = b(p)(\mathcal{M}).
\]

We consider this result to be of independent interest because it shows that also the higher Betti numbers appear as multiplicities of Pollicott-Ruelle resonances. Again the statement is obtained by constructing an explicit isomorphism onto the kernel of the Hodge Laplacian.

\[\text{2.1. Lie theoretic structure theory.}\]
Any closed connected \(^5\) hyperbolic manifold \(\mathcal{M}\) of dimension \(n + 1\) can be written as a bi-quotient \(\mathcal{M} = \Gamma \backslash G/K\), where \(G = \text{SO}(n + 1, 1)\), \(K = \text{SO}(n + 1)\), and \(\Gamma \subset G\) a cocompact torsion-free discrete subgroup. \(\mathcal{M}\) is thus an example of a Riemannian locally symmetric space of rank one. There exists a very efficient Lie-theoretic language to describe the structure of \(\mathcal{M}\), the co-sphere bundle \(S^* \mathcal{M}\), as well as the invariant vector bundles which we introduce in this subsection. For more details we refer the reader to \cite{GHW18b,KW18} and for background information to the textbooks \cite{Kna02,Hel01}.

The Lie algebra \(\mathfrak{g}\) of any semisimple Lie group \(G\) of real rank one with finite center possesses a Cartan involution \(\theta : \mathfrak{g} \to \mathfrak{g}\) and a Cartan decomposition \(\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{p}\). There is a maximal compact subgroup \(K \subset G\) with Lie algebra \(\mathfrak{k}\) and the Cartan decomposition \(\mathfrak{k} = \mathfrak{f} \oplus \mathfrak{p}\) is \(\text{Ad}(K)\)-invariant. The tangent bundle \(T(\Gamma \backslash G/K)\) can then be identified with the associated vector bundle \(\Gamma \backslash G \times_{\text{Ad}(K)} \mathfrak{p}\). Note that via the Killing form \(\mathfrak{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\) and the Cartan involution one can define an \(\text{Ad}(K)\)-invariant inner product \((\cdot, \cdot)\) on \(\mathfrak{g}\) by \((X, Y) \mapsto -\mathfrak{B}(X, \theta Y)\). The restriction to \(\mathfrak{p}\) then allows to define a Riemannian metric on \(\Gamma \backslash G/K\). For any rank one locally symmetric space

\[\text{4} \text{I.e., a Riemannian manifold of constant sectional curvature } -1. \text{ Fixing the curvature at } -1 \text{ is a common convention. By trivial rescaling arguments all results in this paper involving the resonance } 0 \text{ remain true if the metric is multiplied by a positive constant.} \]

\[\text{5} \text{I.e., there exists } \delta > 0 \text{ such that } L_X \text{ acting on } E^+_1 \text{ has no resonances with real part in the interval } (-\delta, \infty) \text{ except the resonance zero.}\]

\[\text{6} \text{If the manifold is not connected, the description given in the following applies to each connected component.}\]

\[\text{7} \text{Here the subscript } 0 \text{ indicates the identity component.}\]
this is a metric of strictly negative sectional curvature and in the special case \( G = \text{SO}(n+1,1)_{0} \) the curvature is constant. We will normalize our Killing form such that the sectional curvature is normalized to \(-1\).

We next want to describe the structure of the co-sphere bundle \( S^{+}M \) and the vector bundles \( E_{0+/−} \). The condition real rank one for \( G \) means that there is a maximal one-dimensional abelian subalgebra \( a \subset \mathfrak{p} \). After choosing a notion of positive and negative roots for the abelian subalgebra \( a \), the unions of all positive and all negative root spaces provide two additional Lie subalgebras \( \mathfrak{n}^{±} \subset \mathfrak{g} \) and one obtains the Iwasawa decompositions on the Lie algebra level

\[
\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{a} \oplus \mathfrak{n}^{+} = \mathfrak{f} \oplus \mathfrak{a} \oplus \mathfrak{n}^{-}.
\]

Also on the group level there are two corresponding Iwasawa decompositions \( G = \text{KAN}^{+} = \text{KAN}^{-} \). Here \( N^{±} \subset G \) are the analytic subgroups with Lie algebras \( \mathfrak{n}^{±} \) and \( A \subset G \) is the analytic subgroup with Lie algebra \( \mathfrak{a} \). Furthermore, the respective exponential maps provide diffeomorphisms \( \text{Ad}(\mathfrak{a}) \approx \text{N} \approx \text{A} \) for the abelian subalgebra \( \mathfrak{a} \), the unions of all positive and all negative root spaces provide two additional Lie subalgebras \( \mathfrak{n}^{±} \subset \mathfrak{g} \) and one obtains the Iwasawa decompositions on the Lie algebra level

\[
\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{a} \oplus \mathfrak{n}^{+} = \mathfrak{f} \oplus \mathfrak{a} \oplus \mathfrak{n}^{-}.
\]

In addition, define the group

\[
\Gamma := Z_{K}(\mathfrak{a}) = \text{Z}_{K}(A),
\]

where \( K \) acts on \( a \) by the adjoint action, and let \( \mathfrak{m} \) be the Lie algebra of \( M \). The groups \( N^{±} \) are normalized by \( A \) and \( M \). On the Lie algebra level we have the Bruhat decomposition

\[
\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^{+} \oplus \mathfrak{n}^{-}
\]

which turns out to be invariant under the \( \text{Ad}(M) \) action.

Now the co-sphere bundle \( S^{+}M \) can be identified with \( \Gamma \backslash G/M \). The Lie group \( A \cong \mathbb{R} \) acts from the right on \( \Gamma \backslash G/M \) because it commutes with \( M \), and this action precisely coincides with the geodesic flow. Furthermore, the tangent bundle of \( S^{+}M \) can be identified as follows:

\[
T(S^{+}M) = \Gamma \backslash G \times_{\text{Ad}(M)} (\mathfrak{a} \oplus \mathfrak{n}^{+} \oplus \mathfrak{n}^{-}) = \mathbb{R} \mathbf{X} \oplus \Gamma \backslash G \times_{\text{Ad}(M)} \mathfrak{n}^{+} \oplus \Gamma \backslash G \times_{\text{Ad}(M)} \mathfrak{n}^{-}.
\]

There is an analogous identification of \( T^{+}(S^{+}M) \). In view of these identifications all bundles of interest in Propositions 2.1 and 2.3 are of the type \( Y_{\mathbf{r}} := \Gamma \backslash G \times_{\mathbf{r}} V \to S^{+}M \) for some finite-dimensional complex \( M \)-representation \( (r, V) \). On such bundles there always exists a canonical connection that we denote by

\[
\nabla : \Gamma^{\infty}(Y_{\mathbf{r}}) \to \Gamma^{\infty}(Y_{\mathbf{r}} \otimes T^{+}(S^{+}M)).
\]

To describe how \( \nabla \) is defined, let us regard a section \( s \in \Gamma^{\infty}(Y_{\mathbf{r}}) \) as a right-\( M \)-equivariant function \( \hat{s} \in C^{\infty}(\Gamma \backslash G, V) \). Moreover, by (2.6) we regard a vector field \( \mathbf{X} \in \Gamma^{\infty}(T(S^{+}M)) \) as a right-\( M \)-equivariant function \( \hat{\mathbf{X}} \in C^{\infty}(\Gamma \backslash G, \mathfrak{n}^{±} \oplus \mathfrak{a} \oplus \mathfrak{n}^{-}) \), that is, \( \hat{\mathbf{X}}(\Gamma \mathbf{g} \mathbf{m}) = \text{Ad}(m^{-1}) \hat{\mathbf{X}}(\Gamma \mathbf{g}) \). Then \( \nabla \) is defined by the covariant derivative

\[
\nabla_{\mathbf{X}}(s)(\Gamma \mathbf{g}) := \frac{d}{dt}|_{t=0} \hat{s}(\Gamma \mathbf{g} e^{t_{\mathbf{g}}(\Gamma \mathbf{X} \hat{\mathbf{g}})}).
\]

So far all constructions were valid for general rank one locally symmetric spaces and we have not yet used the fact that we work on hyperbolic manifolds. The fact of working with \( G = \text{SO}(n+1,1)_{0} \) has however the following implications that will simplify the analysis in the sequel:

- \( M = \text{SO}(n) \)
- \( \mathfrak{a} \cong \mathbb{R} \) by the following canonical identification: As mentioned above the geodesic flow can be identified with the right-\( A \)-action on \( \Gamma \backslash G/M \) and the geodesic vector field corresponds to a unique Lie algebra element \( H_{0} \in \mathfrak{a} \). We then identify \( \mathfrak{a} \cong \mathbb{R} \) by setting \( H_{0}=1 \).
- \( \mathfrak{n}^{±} \cong \mathbb{R}^{n} \) are abelian Lie algebras and the \( \text{Ad}(M) \)-action on \( \mathfrak{n}^{±} \) is the defining representation of \( \text{SO}(n) \) on \( \mathbb{R}^{n} \).
- With \( H_{0} \in \mathfrak{a} \) as above one has for \( Y \in \mathfrak{n}^{±} \) the relation \( [H_{0}, Y] = \pm Y \). In structure theoretic terms this means that \( \text{SO}(n+1,1)_{0} \) has only one positive restricted root. Recall that we normalized the Killing form in such a way that the sectional curvature equals \(-1\), which implies that the restricted root has length \( 1 \).
2.2. Horocycle operators. Horocycle operators have been introduced in [DFG15] as a crucial tool for establishing quantum-classical correspondences. We already mentioned them in the definition of the first band resonant states (2.1). They are defined as follows: If \( \tilde{\mathcal{P}}_E^\pm : \Gamma^\infty(\mathcal{V}_\tau \otimes T^*(S^\ast\mathcal{M})) \to \Gamma^\infty(\mathcal{V}_\tau \otimes E^\pm) \) denotes the map induced by the fiber-wise canonical projection \( \mathcal{P}_E^\pm : T^*(S^\ast\mathcal{M}) \to E^\pm \) onto the subbundle \( E^\pm \), then

\[
\mathcal{U}_\tau^- := \mathcal{P}_E^- \circ \nabla : \Gamma^\infty(\mathcal{V}_\tau) \to \Gamma^\infty(\mathcal{V}_\tau \otimes E^-).
\]

The horocycle operators are not only used to define the first band resonant states but also allow to relate any resonant state to a first band resonant state:

**Lemma 2.4.** For a closed hyperbolic manifold \( \mathcal{M} \) let \( \lambda \in \mathbb{C} \) be a Pollicott-Ruelle resonance of \( \nabla_X \) on \( \mathcal{V}_\tau \) and take \( s \in \text{Res}_{\nabla_X, \mathcal{V}_\tau}(\lambda) \setminus \{0\} \). Then there is a unique \( k \in \mathbb{N}_0 \) such that

\[
\mathcal{U}_\tau^k s \in \text{Res}_{\nabla_X, \mathcal{V}_\tau \otimes (E^-)^k}(\lambda + k) \setminus \{0\}.
\]

**Proof.** See [KW18, Lemma 4.3]. \( \square \)

2.3. First band resonant states and principal series representations. The homogeneous space \( K/M \cong S^n \) can be regarded as the boundary at infinity of the Riemannian symmetric space \( G/K = \mathbb{H}^{n+1} \) and using the Iwasawa projection we can define a left-\( G \)-action

\[
g(kM) := k^-(gk)M, \quad g \in G, \; k \in K.
\]

Given a finite-dimensional complex \( M \)-representation \( (\tau, V) \) we define the boundary vector bundle

\[
\mathcal{V}_C^\tau = (K \times \mathcal{V}_, \pi_{\mathcal{V}}), \quad \pi_{\mathcal{V}}(k, v) = kM.
\]

The total space \( K \times \mathcal{V}_\) of \( \mathcal{V}_C^\tau \) carries the \( G \)-action

\[
g[k, v] := [k^-(gk), v], \quad g \in G, \; k \in K,
\]

that lifts the \( G \)-action (2.10) on the base space \( K/M \). Consequently, we get an induced action on smooth sections:

\[
(gs)(kM) := g(s(g^-(kM))), \quad s \in \Gamma^\infty(\mathcal{V}_C^\tau), \; g \in G.
\]

If we consider a section \( s \in \Gamma^\infty(\mathcal{V}_C^\tau) \) as a right-\( M \)-equivariant smooth function \( \tilde{s} : K \to V \), the action (2.12) corresponds to assigning to \( \tilde{s} \) for any \( g \in G \) the right-\( M \)-equivariant smooth function \( \tilde{g}s : K \to V \) given by

\[
\tilde{g}s(k) = \tilde{s}(k^-g^{-1}k), \quad g \in G, \; k \in K.
\]

To describe how the principal series representation of \( G \) associated to an \( M \)-representation \( \tau \) and a parameter \( \lambda \in \mathbb{C} \) acts on smooth sections of \( \mathcal{V}_C^\tau \), let us regard a section \( s \in \Gamma^\infty(\mathcal{V}_C^\tau) \) as a right-\( M \)-equivariant function \( \tilde{s} \in \Gamma^\infty(K, V) \). We then set

\[
\pi_{\text{comp}}^{\tau, \lambda}(g)(s)(k) := e^{(\lambda+n/2)H^-g^{-1}k}\tilde{s}(k^-g^{-1}k), \quad s \in \Gamma^\infty(\mathcal{V}_C^\tau), \; kM \in K/M.
\]

This representation extends by continuity to a representation \( \pi_{\text{comp}}^{\tau, \lambda} : G \to \text{End}(D'(K/M, \mathcal{V}_C^\tau)) \). One has the following important relation between first band resonant states and the \( \Gamma \)-invariant distributional sections of the boundary vector bundle with respect to the principal series representation \( \pi_{\text{comp}}^{\tau, -\lambda-n/2} \).

**Proposition 2.5 ([KW18 Lemma 2.15]).** For each \( \lambda \in \mathbb{C} \) there is an explicit isomorphism

\[
Q_\lambda : \text{Res}^{1\text{st}}_{\nabla_X, \mathcal{V}_\tau}(\lambda) \cong \Gamma(D'(K/M, \mathcal{V}_C^\tau), \pi_{\text{comp}}^{\tau, -\lambda-n/2}).
\]

\[\footnote{We follow Olbrich’s convention as in [Olb85] between Satz 2.8 and Satz 2.9. In [Kmo86, p. 169], the definition differs from ours in such a way that \( \lambda \) is replaced by \( -\lambda \). Furthermore recall that we identified \( a \cong \mathbb{R} \) in Section 2.4.}\]
2.4. Relating resonances of the Lie- and covariant derivatives. Proposition 2.3 provides a powerful way to handle first band resonant states of the covariant derivative $\nabla_X$ along the geodesic vector field. In Proposition 2.1 and 2.3, we are however interested in resonant states of the Lie derivative. Therefore we have to relate these states:

**Lemma 2.6.** For $p \in \{0, 1, 2, \ldots\}$, suppose that $\tau$ is a subrepresentation of $\bigotimes^p(\text{Ad}(M))|_{\mathbb{R}^n}$. Then the covariant derivative and the Lie derivative along the geodesic vector field $X$, acting on smooth sections of $V_\tau$, are related by

$$L_X = \nabla_X \equiv p \text{id}_{\Gamma^p(V_\tau)}.$$

Consequently, one has for every $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$

$$\text{Res}_{L_X-V_\tau} (\lambda) = \text{Res}_{\nabla_X-V_\tau} (\lambda \mp p) \quad \text{and} \quad \text{res}_{L_X-V_\tau} (\lambda) = \text{res}_{\nabla_X-V_\tau} (\lambda \mp p).$$

**Proof.** Recall that the geodesic flow on $S^*(\Gamma \setminus G/K) = \Gamma \setminus G/M$ is given by

$$\phi_t (\Gamma g M) = \Gamma g e^{itH_0} M, \quad t \in \mathbb{R}.$$

Its derivative $d\phi_t : T(\Gamma \setminus G/M) = \Gamma \setminus G \times_{\text{Ad}(M)} (\mathfrak{n}^+ \oplus \mathfrak{a} \oplus \mathfrak{n}^-) \to \Gamma \setminus G \times_{\text{Ad}(M)} (\mathfrak{n}^+ \oplus \mathfrak{a} \oplus \mathfrak{n}^-)$ reads

$$d\phi_t (\Gamma g M)([\Gamma g M, v]) = [\Gamma g M, \text{Ad}(e^{-itH_0})v], \quad t \in \mathbb{R}, \quad [\Gamma g M, v] \in \Gamma \setminus G \times_{\text{Ad}(M)} (\mathfrak{n}^+ \oplus \mathfrak{a} \oplus \mathfrak{n}^-).$$

Any vector $v \in \mathfrak{n}^\pm$ is an eigenvector of the adjoint action:

$$\text{Ad}(e^{-itH_0})v = e^{-itH_0}v = e^{\mp t}v.$$

Let now $\omega \in \Gamma^\infty(V_\tau)$, identified with a left-$\Gamma$-, right-$M$-equivariant function $\overline{\omega} : G \to V$, where $V \subset \bigotimes^p(\mathfrak{n}^\pm)$. Considering $\overline{\omega}$ as a left-$\Gamma$-, right-$M$-equivariant map $\overline{\omega} : G \to G$, let $\overline{\phi} : G \to G$ be the left-$\Gamma$-, right-$M$-equivariant function corresponding to $\overline{\omega} : \Gamma \setminus G \to \Gamma \setminus G$. Then we get with (2.19) for $g \in G$ and $v_1, \ldots, v_p \in \mathfrak{n}^\pm$:

$$\overline{\phi^*} (\overline{\omega}(g)(v_1, \ldots, v_p)) = \overline{\omega}(g)(e^{itH_0})(v_1, \ldots, e^{itH_0}v_p) = e^{\mp it}(\overline{\omega}(g)(e^{tH_0})(v_1, \ldots, v_p)).$$

For the Lie derivative of $\omega$ we then obtain with the analogous "\("-notation and the product rule

$$L_X \omega(g)(v_1, \ldots, v_p) = \frac{d}{dt} \bigg|_{t=0} \overline{\phi^*} (\overline{\omega}(g)(v_1, \ldots, v_p)) = \frac{d}{dt} \bigg|_{t=0} \left( e^{\mp it}(\overline{\omega}(g)(e^{tH_0})(v_1, \ldots, v_p)) \right) = \frac{d}{dt} \bigg|_{t=0} \overline{\omega}(g)(e^{tH_0})(v_1, \ldots, v_p) \mp \overline{\omega}(g)(v_1, \ldots, v_p).$$

2.5. Proof of Proposition 2.3. Let us collect what we have obtained so far: By Lemma 2.6

$$\text{res}_{L_X-E_1}^1 (0) = \text{res}_{\nabla_X-E_1}^1 (-p) \quad \text{and} \quad \text{res}_{L_X-E_1}^1 (-2p) = \text{res}_{\nabla_X-E_1}^1 (-p).$$

As the adjoint action of $M$ on $\mathfrak{n}^\pm$ is given by the defining representation of $\text{SO}(n)$ on $\mathbb{R}^n$ we deduce from (2.6) that $\Lambda^p(E_\pm) = \Gamma \setminus G \times_{\tau_p} \Lambda^p(\mathbb{R}^n)$ with $\tau_p$ being the $p$-th exterior power of the standard action of $\text{SO}(n)$ on $\mathbb{R}^n$. By Proposition 2.3, we can thus identify

$$\text{res}_{L_X-E_1}^1 (0) \cong \text{res}_{L_X-E_1}^1 (-2p) \equiv \Gamma \left( \text{D'}(K/M, \nabla^B), \pi_{\tau_p-n/2}^{\tau_p-n/2} \right).$$

We now use a vector-valued Poisson transform. To this end, let $\Delta_H = \delta + \delta H$ be the Hodge Laplacian on $\Omega^p(\mathbb{H}^{p+1})$.

**Theorem 2.7.** (Poisson transform for $p$-forms). Let $K = \text{SO}(n+1)$, $M = \text{SO}(n)$ and $\tau_p$ the exterior product of the defining representation of $\text{SO}(n)$ on $\Lambda^p(\mathbb{R}^n)$. Then for any $\lambda \in \mathbb{C}$ with $\lambda \neq n - p$ and $\lambda \neq n + 1, n + 2, \ldots$, there is an isomorphism of vector spaces

$$P_{\tau_p, \lambda} : \text{D'}(K/M, \nabla^B) \to \{ \omega \in \Omega^p(\mathbb{H}^{p+1}) : \Delta_H \omega = (\lambda - p)(n - \lambda - p)\omega, \delta \omega = 0 \}$$

that intertwines the $G = \text{SO}(n+1)_0$ representation $\pi_{\tau_p-n/2}^{\tau_p-n/2}$ with the pullback action of $G$ on $\Omega^p(\mathbb{H}^{p+1})$. 

This result is due to Gaillard [Gai86] although it requires some work (see Section 2.5.1) to translate his statements into the form stated above that we can apply in our setting: For \( p \neq n/2 \) the Poisson transform \( P_{\tau_p} \) is bijective and thus

\[
\Gamma (\mathcal{D}'(K/M, \gamma_B^p), \pi^{\tau_p, p-n/2}_{\text{comp}}) \cong \{ \omega \in \Omega^p(\mathbb{H}^{p+1}), \Delta_H \omega = 0, \delta \omega = 0 \}.
\]

As on compact manifolds any harmonic form is co-closed, the right-hand side is simply the kernel of the Hodge Laplacian and Hodge theory implies that its dimension equals the \( p \)-th Betti number of \( \mathbb{H}^{p+1} = \mathcal{M} \). We thus have shown

\[
\dim \text{Res}_{\mathcal{L}_p}^{1st}(\mathcal{E}_\mathcal{H}(E_x^p))(-p) = \dim \text{Res}_{\mathcal{L}_p}^{1st}(\mathcal{E}_\mathcal{H}(E_x^p))(-2p) = b_p(\mathcal{M}).
\]

Now using once more that \( p \neq n/2 \) [KW18 Theorem 6.2] implies that the resonance at \(-p\) of \( \mathcal{N}_X \) has no Jordan block and consequently

\[
(2.20) \quad \dim \text{Res}_{\mathcal{L}_p}^{1st}(\mathcal{E}_\mathcal{H}(E_x^p)) = \dim \text{Res}_{\mathcal{L}_p}^{1st}(\mathcal{E}_\mathcal{H}(E_x^p))(-2p) = \dim \text{Res}_{\mathcal{L}_p}^{1st}(\mathcal{E}_\mathcal{H}(E_x^p))(-p) = b_p(\mathcal{M}).
\]

This finishes the proof of Proposition 2.5.

2.5.1. Gaillard’s Poisson transform. In his article [Gai86] Gaillard considers the vector-valued Poisson transform for forms on \( K/M \) to which we refer in Theorem 2.7. His notation and conventions are however quite different from ours. In the following we will translate his results into the form stated in Theorem 2.7.

Gaillard studies \( p \)-currents on \( \mathcal{D}'(K/M) \) which we will denote by \( D'_p(K/M) := (\Omega^p(K/M))' \), and we have the canonical dense embedding \( \Omega^p(K/M) \hookrightarrow \mathcal{D}'(K/M) \). As \( G \) acts by diffeomorphisms on \( K/M \) the pullback action on \( D'_p(K/M) \) provides a \( G \)-representation.

**Lemma 2.8.** The pullback action of \( G \) on the space \( D'_p(K/M) \) of \( p \)-currents is equivalent to the principal series representation \( \pi^{\tau_p, p-n/2}_{\text{comp}} \) on \( \mathcal{D}'(K/M, \gamma_B^p) \).

**Proof.** Denote by \( \mathfrak{m}^{1t} \subset \mathfrak{f} \) the orthogonal complement of \( \mathfrak{m} \) in \( \mathfrak{f} \). Then \( M \) acts via the adjoint action on \( \mathfrak{m}^{1t} \). Note that in our setting \( \mathfrak{m}^{1t} \cong \mathbb{R}^n \) and \( \text{Ad}(M)\big|_{\mathfrak{m}^{1t}} \) is simply the standard action of \( \text{SO}(n) \) on \( \mathbb{R}^n \). In the following, we shall write simply \( \text{Ad}(M) \) instead of \( \text{Ad}(M)\big|_{\mathfrak{m}^{1t}} \). Note that there is a canonical identification

\[
(2.21) \quad K \times \text{Ad}(M) \mathfrak{m}^{1t} \cong T(K/M) \quad \text{by} \quad [k, Y] \mapsto \left. \frac{d}{dt} \right|_{t=0} k e^{Y} M.
\]

Let \( g \in G \) and \( a_g : kM \mapsto k g kM \) be the diffeomorphism on \( K/M \) given by the left-\( G \)-action, then the derivative \( da_g \) acts on \( T(K/M) \). In order to prove our lemma we have to determine how \( da_g \) acts on \( K \times \text{Ad}(M) \mathfrak{m}^{1t} \) under the identification \( (2.21) \). We have for \( [k, Y] \in T(K/M) \)

\[
(2.22) \quad da_g ([k, Y]) = \left. \frac{d}{dt} \right|_{t=0} k^- (g k e^{Y} M) \cong \left[ k^- (g k), \left. \frac{d}{dt} \right|_{t=0} k^- (g k) \right]^{-1} k^- (g k \exp(tY)) = \left[ k^- (g k), \left. \frac{d}{dt} \right|_{t=0} k^- (a^- (g k) n^- (g k) \exp(tY) n^- (g k) a^- (g k)^{-1}) \right] = \left[ k^- (g k), \text{pr}_-^\mathfrak{f} \text{Ad}(a^- (g k) n^- (g k))(Y) \right] \quad \text{for} \quad \text{pr}_-^\mathfrak{f} = d k^- |_{\mathfrak{e}} : \mathfrak{g} \to \mathfrak{f} = \mathfrak{a} \oplus \mathfrak{n}^- \oplus \mathfrak{h}^- \quad \text{being the projection onto} \quad \mathfrak{f} \text{defined by the opposite Iwasawa decomposition of} \quad \mathfrak{g}.
\]

We can now proceed by studying for fixed \( g \in G, k \in K, Y \in \mathfrak{m}^{1t} \) the element

\[
(2.23) \quad \text{pr}_-^\mathfrak{f} \text{Ad}(a^- (g k) n^- (g k))(Y) \in \mathfrak{m}^{1t}.
\]

By the orthogonal Bruhat decomposition \( \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^- \) and the fact that \( \mathfrak{a} \) lies in the orthogonal complement of \( \mathfrak{f} \) in \( \mathfrak{g} \), we have \( \mathfrak{m}^{1t} \subset \mathfrak{n}^+ \oplus \mathfrak{n}^- \), so we can write \( Y = Y^+ + Y^- \) with \( Y^+ \in \mathfrak{n}^+ \) and \( \partial Y^+ = Y^- \). The space \( \mathfrak{n}^- \) is \( \text{Ad}(AN^-) \)-invariant. Consequently \( \text{Ad}(a^- (g k) n^- (g k))(Y^-) \in \mathfrak{n}^- \), so \( \text{pr}_-^\mathfrak{f} \text{Ad}(a^- (g k) n^- (g k))(Y^-) = \text{pr}_-^\mathfrak{f} \text{Ad}(a^- (g k) n^- (g k))(Y^-) = \ldots \).
0 by the opposite Iwasawa decomposition. This shows that only \( Y^+ \) contributes to (2.24). Let us write \( n^-(gk) = \exp(N) \) with \( N \in n^- \). Then we get
\[
\text{Ad}(n^-(gk))(Y^+) = e^{\text{ad}(N)}(Y^+) = Y^+ + \frac{1}{2} [N, [N, Y^+]].
\]
Here we use that \( g = g_0 \oplus n^+ \oplus n^- \) is the root-space decomposition of \( g = \phi(n + 1) \) and consequently
\[
n^+ \xrightarrow{\text{ad}(N)} g_0 \xrightarrow{\text{ad}(N)} n^- \xrightarrow{\text{ad}(N)} 0.
\]
Furthermore, the map \( \text{Ad}(a(gk)) \) acts on \( n^\pm \) by scalar multiplication with \( e^\pm h_-(gk) \) and leaves \( g_0 = m \oplus a \) invariant. The opposite Iwasawa projection \( pr^-_f \) maps \( n^- \) to 0 and the space \( g_0 \) onto \( m \). However, the Lie algebra element considered in (2.24) is by construction in \( m^\perp_1 \). We therefore arrive at
\[
pr^-_f \text{Ad}(a(gk)n^-(gk))(Y) = pr^-_f \left( e^{h_-(gk)y^+} \right).
\]
Writing
\[
Y^+ = Y^+ + \theta Y^+ - \frac{\theta Y^+}{\theta \in \mathfrak{t}} - \frac{\theta Y^+}{\theta \in n^-}
\]
reveals \( pr^-_f \text{Ad}(a(gk)n^-(gk))(Y) = e^{h_-(gk)y^+} \).

In summary, we have proved
\[
(2.25)\quad da_g([k, Y]) = [k^-(gk), e^{h_-(gk)}Y].
\]
Finally, note that \( T(K/M) \cong K \times_{\text{Ad}(M)} m^\perp_1 \) induces for each \( p \in \{1, 2, \ldots\} \) an isomorphism \( \Lambda^pT^*(K/M) \cong K \times_{\Lambda^p\text{Ad}(M)} \Lambda^p(m^\perp_1)^s \). Under that isomorphism, a \( p \)-form \( s \in \Gamma^\infty(\Lambda^pT^*(K/M)) \) corresponds to a section \( \tilde{s} \in \Gamma^\infty(K \times_{\Lambda^p\text{Ad}(M)} \Lambda^p(m^\perp_1)^s) \), and by our above computations the pullback action \( gs \equiv (g^{-1})^s s \) of an element \( g \in G \) on \( s \) corresponds to the following action on \( \tilde{s} \):
\[
(2.26)\quad (g\tilde{s})(k)(X_1, \ldots, X_p) = \tilde{s}(k^{-1}(gk))e^{h_-(g^{-1}k)}X_1, \ldots, e^{h_-(g^{-1}k)}X_p
\]
\[
= e^{h_-(g^{-1}k)}\tilde{s}(k^{-1}(g^{-1}k))(X_1, \ldots, X_p) \quad \forall X_1, \ldots, X_p \in n^\pm, k \in K.
\]
Recalling the definition (2.14) of the principal series representations, and taking into account that the pullback action of \( G \) on \( p \)-currents as well as the principal series representations of \( G \) on distributional sections of \( \Lambda^pT^*(K/M) \) are the continuous extensions of the respective actions on smooth \( p \)-forms, the proof is complete.

For the definition of his Poisson transform Gaillard generalizes his setting to currents with values in complex line bundles \( D^t \to K/M \) parametrized by a complex number \( s \in \mathbb{C} \). Let us recall their construction [Gaib86, Section 2.2]: It is based on a \( G \)-invariant function \( Q \)
\[
(2.27)\quad Q : G/K \times K/M \times G/K \to \mathbb{C} \setminus \{0\}, \quad Q(gK, kM, eK) = \|D(V^{-1}_{gK} \circ V_{eK})|_{kM}\|.
\]
where Gaillard’s “application visuelle” \( V_{gK} : S^s_{gK}(G/K) \to K/M \), \( gK \in G/K \), is defined by
\[
V_{gK} : \{g \cdot M : gK = gK\} = S^s_{gK}(G/K) \to K/M, \quad gK \mapsto k^-(g)M.
\]
A straightforward calculation similar to the proof of Lemma 2.8 shows that
\[
(2.28)\quad Q(gK, kM, eK) = e^{h_-(g^{-1}k)}, \quad \frac{\text{which gives us by the } G \text{-invariance of } Q \text{ for a general element } (gK, kM, gK) \in G/K \times K/M \times G/K:}{Q(gK, kM, gK) = Q(g(g^{-1}gK, k^-(g^{-1}k)M, eK))}
\]
\[
= Q(g^{-1}gK, k^-(g^{-1}k)M, eK)
\]
\[
= e^{h_-(g^{-1}gK^{-1}(g^{-1}k))}. \quad \text{[2.89] Here } G \text{ acts on all three factors in the domain } G/K \times K/M \times G/K \text{ by left multiplication.}
\]
With these preparations, let us now turn to Gaillard’s definition of the line bundle $D^s$ over $K/M$: Introduce an equivalence relation $\sim_s$ on $G/K \times K/M \times \mathbb{C}$ by

$$(gK, kM, z) \sim_s (\tilde{g}K, \tilde{k}M, \tilde{z}) \iff kM = \tilde{k}M,$$

and declare $D^s := G/K \times K/M \times \mathbb{C}/\sim_s$ with bundle projection $[gK, kM, z] \mapsto kM$. The bundle is a homogeneous $G$-bundle by defining the $G$ action as

$$g' [gK, kM, z] := [g'gK, g'(kM), z] = [g'gK, k^{-1}g'(k)M, z].$$

The stabilizer subgroup of $eM \in K/M$ with respect to the left-$G$-action on $K/M$ is $MAN^-$ and the action of the stabilizer group on the fiber of $D^s$ over $eM$ is

$$(manK, eM, z) = [eK, eM, e^{-s \log(a)}(n^{-1}a^{-1}m^{-1})z] = [eK, eM, e^{-s \log(a)}].$$

If we define the $MAN^-$-representation $\sigma_j$ by $\sigma_j (eM) = e^{-s \log(a) \in \mathbb{C}}$ then we can identify $D^s$ with the associated line bundle $G \times_{\sigma_j} \mathbb{C} \to G/(MAN^-) \cong K/M$. Thus the $G$-action on sections of this homogeneous bundle is equivalent to the principle series representation $\pi_{1,s}^{\text{comp}}$, where $\mathbb{1}$ denotes the trivial $M$-representation on $\mathbb{C}$. By Lemma 2.8 we know that the pullback action on $p$-currents is equivalent to $\pi_{\text{comp}}^{\tau_p-p-n/2}$, so the action of $G$ on $D^s$-valued currents is equivalent to $\pi_{\text{comp}}^{\tau_p-p-n/2} \otimes \pi_{1,s}^{\text{comp}}$ which is equivalent to $\pi_{\text{comp}}^{\tau_p-p-n/2}$. 

2.6. **Proof of Proposition 2.4** Let $\lambda \in \mathbb{C}$. By the decomposition \((\ref{1.4})\) and Lemma 2.6, we have

$$\text{Res}_{\mathbb{V}_X, E^+_1}(\lambda) \cong \text{Res}_{\mathbb{V}_X, E^+_1}(\lambda - 1) \oplus \text{Res}_{\mathbb{V}_X, E^-_1}(\lambda + 1).$$

As $\mathbb{V}_X$ is an antisymmetric operator in $L^2(E^+_1)$, there are no resonances of $\mathbb{V}_X$ with positive real part (see \cite{KW18} Remark 1.3.1.), so if $\text{Re } \lambda > -1$ one has

$$\text{Res}_{\mathbb{V}_X, E^+_1}(\lambda) \cong \text{Res}_{\mathbb{V}_X, E^-_1}(\lambda - 1).$$

By the definition of first band resonant states \((\ref{1.4})\) and the dimension formula for linear maps we conclude

$$(\ref{2.30}) \quad \dim \text{Res}_{\mathbb{V}_X, E^+_1}(\lambda) = \dim \text{Res}_{\mathbb{V}_X, E^-_1}(\lambda - 1) + \dim U^- (\text{Res}_{\mathbb{V}_X, E^-_1}(\lambda - 1)).$$

By Proposition 2.5 and Theorem 2.7, there is for $n \neq 2$ an isomorphism

$$(\ref{2.31}) \quad \text{res}_{\mathbb{V}_X, E^+_1}(\lambda - 1) \cong \{ \omega \in \Gamma^\infty (T^*M) : \Delta_H \omega = -\lambda(n + \lambda - 2) \omega, \delta \omega = 0 \},$$

where $\Delta_H$ is the Hodge Laplacian on $M$. When $n \geq 3$ and $\text{Re } \lambda > 1 - \frac{n}{2}$, the eigenvalue $-\lambda(n + \lambda - 2)$ is real and positive if $\lambda \in (1 - \frac{n}{2}, 0]$ and if this does not hold the right-hand side of \((\ref{2.31})\) is the zero space. It follows for $n \geq 3$ and $\text{Re } \lambda > 1 - \frac{n}{2}$ that $\text{Res}_{\mathbb{V}_X, E^+_1}(\lambda - 1) = \{ 0 \}$ unless $\lambda \in (1 - \frac{n}{2}, 0]$ because every Jordan block would contain at least one resonant state. Now, in view of Proposition 2.5 and \((\ref{2.20})\), it remains to prove $U^- (\text{Res}_{\mathbb{V}_X, E^-_1}(\lambda - 1)) = 0$ under the assumption that $\text{Re } \lambda > -\delta$ for some small $\delta > 0$ to establish Proposition 2.4. Recall from \((\ref{2.9})\) that $U^- (\text{Res}_{\mathbb{V}_X, E^-_1}(\lambda - 1)) \subseteq D'(M, E^+_1 \otimes E^+_1)$. Furthermore, the horocyclic operator $U^-$ obeys the commutation relation $\nabla_X U^- = U^- \nabla_X = 0$ which implies

$$U^- (\text{Res}_{\mathbb{V}_X, E^-_1}(\lambda - 1)) \subseteq \text{Res}_{\mathbb{V}_X, E^-_1 \otimes E^-_1}(\lambda).$$

If $\text{Re } \lambda > 0$, we immediately get the zero space on the right-hand side as otherwise there would be resonances of $\mathbb{V}_X$ with positive real part.

We are left with the proof of $U^- (\text{Res}_{\mathbb{V}_X, E^-_1}(\lambda - 1)) = 0$ in the case that $\text{Re } \lambda \in (-\delta, 0]$ for some small $\delta > 0$. Another application of the commutation relation and the absence of resonances of $\mathbb{V}_X$ with positive real part implies

$$\text{Res}_{\mathbb{V}_X, E^-_1 \otimes E^-_1}(\lambda) = \text{Res}_{\mathbb{V}_X, E^-_1 \otimes E^-_1}(\lambda) \quad \text{if } \text{Re } \lambda > -1.$$
Ad(M)|_{\mathfrak{n}}^{-} which induces an isomorphism $E_+^* \cong E_-$ that is compatible with the connections on the two bundles. Therefore,

$$\text{Res}_{V_X, E_+^* \otimes E_-}^{\text{1st}}(\lambda) \cong \text{Res}_{V_X, E_+^* \otimes E_-}^{\text{1st}}(\lambda).$$

Now the restriction of the Riemannian metric $g$ to $E_- \times E_-$ defines a smooth section of $E_-^* \otimes E_-. $

Lemma 2.9. If $n \neq 2$, there is a number $\delta > 0$ such that for all $\lambda \in \mathbb{C}$ with $\Re \lambda \in (-\delta, 0]$ one has

$$\text{Res}_{V_X, E_+^* \otimes E_-}^{\text{1st}}(\lambda) \cong \begin{cases} \{ c \, g |_{E_- \times E_-} : c : \mathcal{M} \to \mathbb{C} \text{ locally constant}, \\
0 \end{cases}, \quad \lambda = 0,$$

else.

Before proving this lemma let us see how it finishes the proof of Proposition 2.8. All that is left to prove is that if $U_- s = c \, g |_{E_- \times E_-}$ for some $s \in \text{Res}_{E_+}^\text{1st}(-1)$ and $c \in \mathbb{C}$, then $c = 0$. This is easy. If $U_- s = c \, g |_{E_- \times E_-}$, then

$$\langle U_- s, g |_{E_- \times E_-} \rangle_{L^2(S^* M, S^2(E_-^*))} = |c|^2 \| g |_{E_- \times E_-} \|_{L^2(S^* M, S^2(E_-^*))}^2,$$

Thus, if $U_-^*$ is the formal adjoint of $U_-$, we have

$$(2.32) \quad s(U_-^* (g |_{E_- \times E_-})) = |c|^2 \| g |_{E_- \times E_-} \|_{L^2(S^* M, S^2(E_-^*))}^2,$$

where the left-hand side is the pairing of the distributional section $s$ with the smooth section $U_-^* (g |_{E_- \times E_-})$. In [DFG15 Lemma 4.3] it is shown that $U_-^* = -T \circ U_-$, $T$ being the trace operator. The smooth section $g |_{E_- \times E_-}$ vanishes under all covariant derivatives as it corresponds to the constant function $G : \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ with value $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$. Therefore, we find $U_-^* (g |_{E_- \times E_-}) = 0$ and (2.32) implies $c = 0$.

It remains to prove Lemma 2.9.

Proof of Lemma 2.9. The tensor product $E_-^* \otimes E_-^*$ splits into a sum of three subbundles according to

$$E_-^* \otimes E_-^* = S_0^2(E_-^*) \otimes \Lambda^2(E_-^*) \oplus C g |_{E_- \times E_-},$$

where $S_0^2(E_-^*)$ denotes the trace-free symmetric tensors of rank 2. Note that $C g |_{E_- \times E_-}$ is a trivial line bundle and for $n = 1$ the other two bundles have rank zero. By naturality of the concept of generalized resonant states with respect to sums of vector bundles, we arrive at

$$(2.33) \quad \text{Res}_{V_X, E_+^* \otimes E_-}^{\text{1st}}(\lambda) \cong \text{Res}_{V_X, S_0^2(E_-^*)}^{\text{1st}}(\lambda) \oplus \text{Res}_{V_X, \Lambda^2(E_-^*)}^{\text{1st}}(\lambda) \oplus \text{Res}_{V_X, E_-^* \otimes E_-}^{\text{1st}}(\lambda).$$

Now we can consider the three summands on the right-hand side individually. According to [DFG15 Thm. 6], there is for $\Re \lambda > -1$ an isomorphism

$$(2.34) \quad \text{Res}_{V_X, S_0^2(E_-^*)}^{\text{1st}}(\lambda) \cong \{ \omega \in \Gamma^\infty(S_0^2(T^* M)) : \Delta_B \omega = -\lambda(n + \lambda + 2), \text{ div } \omega = 0 \},$$

where $\Delta_B$ is the Bochner Laplacian associated to the connection $\nabla$. The eigenvalue $-\lambda(n + \lambda + 2)$ appearing here is a real number iff $\text{Im } \lambda = 0$ or $\Re \lambda = -n$, so for $\Re \lambda > -1$ only numbers $\lambda \in (-1, \infty)$ remain as possible candidates for a non-zero resonance space (2.34). In addition, a Weitzenböck type formula (see [DFG15 Lemma 6.1]) says that the spectrum of $\Delta_B$ acting on $\Gamma^\infty(S_0^2(T^* M))$ is bounded from below by $n + 1$ which is strictly larger than $-\lambda(n + \lambda + 2)$ for $n \geq 2$ and $\lambda \in (-1, \infty)$. Consequently, for such $n$ and $\lambda$ the right-hand side of (2.34) is the zero space and it follows that $\text{Res}_{V_X, S_0^2(E_-^*)}^{\text{1st}}(0) = \{ 0 \}$ because every Jordan block would contain at least one resonant state. Turning to the second summand in (2.33), we apply once more Proposition 2.5 and Theorem 2.7 and obtain for $n \neq 2$ an isomorphism

$$(2.35) \quad \text{Res}_{V_X, \Lambda^2(E_-^*)}^{\text{1st}}(\lambda) \cong \{ \omega \in \Gamma^\infty(\Lambda^2(T^* M)) : \Delta_H \omega = -(\lambda + 2)(n + \lambda + 2), \delta \omega = 0 \}.$$

For $\Re \lambda > -1$ and $n \geq 3$, the eigenvalue appearing here is either imaginary or negative, so the right-hand side of (2.35) is the zero space (because $\Delta_H$ is positive) and $\text{Res}_{V_X, \Lambda^2(E_-^*)}^{\text{1st}}(\lambda) = \{ 0 \}$, $\text{Res}_{V_X, \Lambda^2(E_-^*)}^{\text{1st}}(\lambda) = 0$. Finally, let us turn to the third summand in (2.33). As $\nabla_X (c \, g) |_{E_- \times E_-} = (X c) \, g |_{E_- \times E_-}$ we see that the distribution $c$ has to be a scalar resonant state of a resonance $\lambda$. In the scalar case we can however applyLiverani’s result on

\footnote{We thank Colin Guillarmou for suggesting the slick argument.}
the spectral gap for contact Anosov flows [Liv04] to see that zero is the unique leading resonance (with resonant states the locally constant functions) and there is a spectral gap $\delta > 0$, so the proof is finished. \hfill \Box

3. Non-constant curvature perturbations

We now address the question how the equality $m_{\mathcal{L}_X, E^*_1}(0) = b_1(\mathcal{M})$ for constant negative curvature manifolds behaves under perturbations of the Riemannian metric. Therefore, throughout this section, let $\mathcal{M}$ be a closed manifold admitting a hyperbolic metric and $\Gamma_0(S^2(T^*\mathcal{M}))$ the space of symmetric two-tensors endowed with the Fréchet topology. Let $\mathcal{R}_{\mathcal{M},<0} \subset \Gamma_0(S^2(T^*\mathcal{M}))$ be the open subset of Riemannian metrics with negative sectional curvature and $\mathcal{H}_{\mathcal{M}} \subset \mathcal{R}_{\mathcal{M},<0}$ the nonempty subset of constant negative curvature metrics. For each $g \in \mathcal{R}_{\mathcal{M},<0}$ the geodesic vector field $X \in \Gamma_0(T(S^*\mathcal{M}))$ is an Anosov vector field.

**Theorem 3.1.** If $\dim \mathcal{M} \neq 3$ then there exists an open neighborhood $U \subset \mathcal{R}_{\mathcal{M},<0}$ of $\mathcal{H}_{\mathcal{M}}$ such that

$$m_{\mathcal{L}_X, E^*_1}(0, g) = b_1(\mathcal{M}) \quad \text{for all } g \in U.$$ 

In order to prove this result we first need a perturbation theory argument.

**Lemma 3.2.** For each $\lambda_0 \in \mathbb{C}$, $m_{\mathcal{L}_X, E^*_1}(\lambda_0, g)$ is upper semicontinuous with respect to variations of the Riemannian metric $g \in \mathcal{R}_{\mathcal{M},<0}$.

**Proof.** Dyatlov-Zworski [DZ16] have shown that for any metric $g \in \mathcal{R}_{\mathcal{M},<0}$ and $t_0 > 0$ such that $d(x, \varphi_{t_0}(x)) > \varepsilon$ for all $x \in S^*\mathcal{M}$ the flat trace

$$f(\lambda, g) := -e^{-it_0} \text{tr}^b(e^{-t_0} R_{\mathcal{L}_X, E^*_1}(\lambda, g))$$

is well defined and has a meromorphic continuation to $\mathbb{C}$ (see (4.2) in [DZ16] and discussions below). Furthermore, they prove that the poles of this function coincide with the resonances of $\mathcal{L}_X$ on $E^*_1$, and the poles $\lambda_i$ of $f(\lambda, g)$ are simple with integer residues given by $m_{\mathcal{L}_X, E^*_1}(\lambda_i, g)$. In the recent article on Fried’s conjecture [DGRS18] the main technical ingredient is the continuous dependence of $f(\lambda, g)$ on the Anosov vector field $X$. A slight reformulation is

**Proposition 3.3.** Let $g_0 \in \mathcal{R}_{\mathcal{M},<0}$ and $Z \subset \mathbb{C}$ an open set whose closure $\overline{Z}$ is compact and does not contain any resonance of $\mathcal{L}_X$ on $E^*_1$ with respect to $g_0$. Then there is an open neighborhood $U \subset \mathcal{R}_{\mathcal{M},<0}$ of $g_0$ such that for none of the metrics $g \in U$ a resonance is contained in $\overline{Z}$. Furthermore, the map

$$U \ni g \mapsto f(\lambda, g) \in \Theta(\overline{Z})$$

is continuous. Here $\Theta(\overline{Z})$ denotes the space of holomorphic functions on $\overline{Z}$.

**Proof.** The statement follows from the estimates on the resolvent kernels in [DGRS18 Proposition 6.3], the definition of the flat trace (see e.g. [DZ16 Section 2.4]) and continuous pullbacks of distributions [Hör83 Thm 8.2.4]. Note that inspecting the proof of [DGRS18 Proposition 6.3] shows that the assumptions $0 \in Z$ and $Z$ being simply connected are not necessary for the statement of [DGRS18 Proposition 6.3] but simply technical assumptions necessary for other parts of [DGRS18].

With this result the upper semicontinuity follows by a simple residue argument: Choose a metric $g_0 \in \mathcal{R}_{\mathcal{M},<0}$ and a resonance $\lambda_0$ for $g_0$ as well as an annulus $Z$ around $\lambda_0$ such that no resonances are contained in $\overline{Z}$ and $\lambda_0$ is the only resonance in the interior of the annulus. Let $\gamma$ be a closed path in $Z$ winding around $\lambda_0$, then $m_{\mathcal{L}_X, E^*_1}(\lambda_0, g) = (2\pi i)^{-1} \int_{\gamma} f(\lambda, g) d\lambda$. If $U$ is the open neighborhood of $g_0$ from Proposition 3.3 consider the continuous map $U \ni g \mapsto (2\pi i)^{-1} \int_{\gamma} f(\lambda, g) d\lambda$. As the residues of $f(\lambda, g)$ are the multiplicities of the resonances with respect to $g$ one has

$$(2\pi i)^{-1} \int_{\gamma} f(\lambda, g) d\lambda = \sum_{\lambda_j} m_{\mathcal{L}_X, E^*_1}(\lambda_j, g).$$

Note that the unit co-sphere bundles for two different Riemannian metrics are diffeomorphic. In particular, up to continuous isomorphism, all the spaces of functions and distributions considered below that involve $S^*\mathcal{M}$ do not depend on $g$. We can therefore essentially treat $S^*\mathcal{M}$ as a smooth manifold that does not depend on $g$.\footnote{Note that the unit co-sphere bundles for two different Riemannian metrics are diffeomorphic. In particular, up to continuous isomorphism, all the spaces of functions and distributions considered below that involve $S^*\mathcal{M}$ do not depend on $g$. We can therefore essentially treat $S^*\mathcal{M}$ as a smooth manifold that does not depend on $g$.}
where the sum runs over all resonances with respect to $\mathcal{L}$ in the interior of the annulus. So $(2\pi)^{-1} \int f(\lambda, g) d\lambda$ can only take on integer values and is therefore constant. This proves the upper semicontinuity of $m(\lambda_0, g)$. □

A second ingredient to Theorem 3.1 is a lower bound on $m_{\mathcal{L}_X, T^*(S^*M)}(\lambda)$:

**Lemma 3.4.** For any $g \in \mathcal{M}_{<0}$, $m_{\mathcal{L}_X, T^*(S^*M)}(\lambda) \geq b_1(S^*M) + b_0(S^*M)$.

**Proof.** Dang-Riviére [DR17] proved that

$$0 \to \text{Res}_{\mathcal{L}_X, \Lambda^0(T^*(S^*M))}(0, g) \to \text{Res}_{\mathcal{L}_X, \Lambda^1(T^*(S^*M))}(0, g) \to \cdots \to \text{Res}_{\mathcal{L}_X, \Lambda^2\dim\mathcal{M} - 1(T^*(S^*M))}(0, g) \to 0$$

forms a finite-dimensional complex whose cohomology is isomorphic to the de Rham cohomology of $S^*M$. As $\Lambda^0(T^*(S^*M))$ is simply the trivial line bundle its resonant states at zero are the locally constant functions and thus

$$d(\text{Res}_{\mathcal{L}_X, \Lambda^0(T^*(S^*M))}(0, g)) = 0,$$

which implies $\dim(\text{Res}_{\mathcal{L}_X, \Lambda^0(T^*(S^*M))}(0, g) \cap \ker(d)) = b_1(S^*M)$. Furthermore, there is the contact one-form $\alpha \in \Gamma^\infty(T^*(S^*M))$ fulfilling $L_X \alpha = 0$. By the wavefront characterization of resonant states (1.2) we know that $\alpha \in \text{Res}_{\mathcal{L}_X, \Lambda^1(T^*(S^*M))}(0, g)$. But as $\alpha$ is a contact one-form $d\alpha \neq 0$. The same holds for $c\alpha$ with $c$ being a locally constant function (thus an element in the 0-th de Rham cohomology) and the statement follows. □

We can now prove Theorem 3.1.

**Proof.** Let $g \in \mathcal{M}_{<0}$ and $\alpha$ the contact one-form. From the fact that $L_X \alpha = 1$ we can uniquely decompose any $u \in \text{Res}_{\mathcal{L}_X, T^*(S^*M)}(0, g)$ into $u = u_\perp + (i_X u)\varpi$ where $i_X u_\perp = 0$ and $u_\perp$ is a distribution section of $E^\perp$. We have for some $J \in \mathbb{N}$

$$0 = L^J_X u = L^J_X u_\perp + (X^J(i_X u))\alpha. \tag{3.1}$$

Using Cartan's magic formula for $L_X$ one checks $L^J_X u_\perp = 0$, so the wavefront characterization of resonant states (1.2) implies $u_\perp \in \text{Res}_{\mathcal{L}_X, E^\perp}(0, g)$ and $i_X u \in \text{Res}_{\mathcal{L}_X, \Lambda^0(T^*(S^*M))}(0, g)$. This implies

$$m_{\mathcal{L}_X, T^*(S^*M)}(0, g) = m_{\mathcal{L}_X, E^\perp}(0, g) + b_0(S^*M)$$

and Lemma 3.4 implies the lower bound

$$m_{\mathcal{L}_X, E^\perp}(0, g) \geq b_1(S^*M) \quad \text{for all } g \in \mathcal{M}_{<0}. \tag{3.2}$$

From Proposition 2.1 we know that for $\dim\mathcal{M} \neq 3$ and $g_0 \in \mathcal{M}$

$$m_{\mathcal{L}_X, E^\perp}(0, g_0) = b_1(M) = b_1(S^*M),$$

where the last equality follows from [CS50] (4.1). By Lemma 3.2 the multiplicity cannot increase under small perturbations of $g_0$ and by Lemma 3.4 it cannot decrease under any perturbation. This completes the proof of Theorem 3.1 □

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