Comparison of the algebraic and the symplectic
Gromov-Witten invariants

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1 Introduction

As Witten suggested in [W1], [W2], the GW-invariants for a symplectic manifold
X are multi-linear maps
\[ \gamma_{X}^{A,g,n} : H^*(X; \mathbb{Q})^n \times H^*(\overline{\mathcal{M}_{g,n}}; \mathbb{Q}) \to \mathbb{Q}, \]
where \( A \in H_2(X, \mathbb{Z}) \) is any homology class, \( n, g \) are two non-negative integers, and \( \overline{\mathcal{M}_{g,n}} \) is the Deligne-Mumford compactification of \( \mathcal{M}_{g,n} \), the space of smooth \( n \)-pointed genus \( g \) curves. The basic idea of defining these invariants is
to enumerate holomorphic maps from Riemann surfaces to the manifolds. To
illustrate this, we let \( X \) be a smooth projective manifold and form the moduli
space \( \mathcal{M}_{g,n}(X, A) \) of all holomorphic maps \( f : \Sigma \to X \) from smooth \( n \)-pointed
Riemann surfaces \( (\Sigma; x_1, \ldots, x_n) \) to \( X \) such that \( f_*(\Sigma) = A \). \( \mathcal{M}_{g,n}(X, A) \) is
a quasi-projective scheme and its expected dimension can be calculated using
the Riemann-Roch theorem. We will further elaborate the notion of expected
dimension later, and for the moment we will denote it by \( r_{\text{exp}} \). Note that it
depends implicitly on the choice of \( X, A, g \) and \( n \). When \( r_{\text{exp}} = 0 \), then
\( \mathcal{M}_{g,n}(X, A) \) is expected to be discrete. If \( \mathcal{M}_{g,n}(X, A) \) is discrete, then the degree of \( \mathcal{M}_{g,n}(X, A) \), considered as a 0-cycle, is a GW-invariant of \( X \). We remark that we have and will ignore the issue of non-trivial automorphism groups of maps in \( \mathcal{M}_{g,n}(X, A) \) in the introduction. When \( r_{\text{exp}} > 0 \), then \( \mathcal{M}_{g,n}(X, A) \) is

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expected to have pure dimension $r_{\text{exp}}$. If it does, then we pick $n$ subvarieties of $X$, say $V_1, \ldots, V_n$, so that their total codimension is $r_{\text{exp}}$. We then form a subscheme of $\mathcal{M}_{g,n}(X,A)$ consisting of maps $f$ so that $f(x_i) \in V_i$. This subscheme is expected to be discrete. It it does, then its degree is the GW-invariant of $X$. Put them together, we can define the GW-invariants $\gamma_{A,g,n}^X$ of $X$. This is similar to construction of the Donaldson polynomial invariants for 4-manifolds.

Here are the two big ifs in carry out this program are

**Question I**: Whether the moduli scheme $\mathcal{M}_{g,n}(X,A)$ has pure dimension $r_{\text{exp}}$.

**Question II**: Whether the subschemes of $\mathcal{M}_{g,n}(X,A)$ that satisfy certain incidence relations have the expected dimensions.

Similar to Donaldson polynomial invariants, the affirmative answer to the above two questions are in general not guaranteed. One approach to overcome this difficulty, beginning with Donaldson’s invariants of 4-manifolds, is to “deform” the moduli problems and hope that the answers to the “deformed” moduli problems are affirmative. In the case of GW-invariants, one can deform the complex structure of the smooth variety $X$ to not necessary integrable almost complex structure $J$ and study the same moduli problem by replacing holomorphic maps with pseudo-holomorphic maps. This was investigated by Gromov in [Gr], Ruan [R], in which he constructed certain GW-invariants of rational type for semi-positive symplectic manifolds. The first mathematical theory of GW-invariants came from the work of Ruan and the second author, in which they found that the right set up of GW-invariants for semi-positive manifolds can be provided by using the moduli of maps satisfying non-homogeneous Cauchy-Riemann equations. In this set up, they constructed the GW-invariants of all semi-positive symplectic manifolds and proved fundamental properties of these invariants. All Fano-manifolds and Calabi-Yau manifolds are special examples of semi-positive symplectic manifolds. Also any symplectic manifold of complex dimension less than 4 is semi-positive.

Attempts to push this to cover general symplectic manifolds so far have failed. New approaches are needed in order to get a hold on the GW-invariants of general varieties (or symplectic manifolds). The first step is to convert the problem of counting mappings, which essentially is homology in nature, into the frame work of cohomology theory of the moduli problem. More precisely, we first compactify the moduli space $\mathcal{M}_{g,n}(X,A)$ to, say, $\overline{\mathcal{M}}_{g,n}(X,A)$. We require that the obvious evaluation map

$$e : \mathcal{M}_{g,n}(X,A) \rightarrow X^n$$

that sends $(f; \Sigma; x_1, \ldots, x_n)$ to $(f(x_1), \ldots, f(x_n))$ extends to

$$\bar{e} : \overline{\mathcal{M}}_{g,n}(X,A) \rightarrow X^n.$$  

We further require that if $\mathcal{M}_{g,n}(X,A)$ has pure dimension $r_{\text{exp}}$, then $\overline{\mathcal{M}}_{g,n}(X,A)$ supports a fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,A)] \in H_{2r_{\text{exp}}} (\overline{\mathcal{M}}_{g,n}(X,A); \mathbb{Q}).$$
Then the GW-invariants of $X$ are multi-linear maps

$$\gamma_{A,g,n}^X : H^*(X)^{\times n} \times H^*(\overline{M}_{g,n}) \to \mathbb{Q}$$

(1.2)

that send $(\alpha, \beta)$ to

$$\gamma_{A,g,n}^X(\alpha, \beta) = \int_{\overline{M}_{g,n}(X,A)} \bar{e}^*(\alpha) \cup \pi^*(\beta),$$

where $\pi : \overline{M}_{g,n}(X,A) \to \overline{M}_{g,n}$ is the forgetful map. Note that in such cases the GW-invariants are defined without reference to the answer to question II.

Even when the answer to question I is negative, we can still define the GW-invariants if a virtual moduli cycle $[\overline{M}_{g,n}(X,A)]^{\text{vir}} \in H_{2\text{exp}}(\overline{M}_{g,n}(X,A); \mathbb{Q})$

can be found that function as the fundamental cycle $[\overline{M}_{g,n}(X,A)]$ should the dimension of $\overline{M}_{g,n}(X,A)$ is $r_{\text{exp}}$. In this case, we simply define $\gamma_{A,g,n}^X$ as before with $[\overline{M}_{g,n}(X,A)]$ replaced by $[\overline{M}_{g,n}(X,A)]^{\text{vir}}$.

The standard compactification of $\overline{M}_{g,n}(X,A)$ is the moduli space of stable morphisms from $n$-pointed genus $g$ curves, possibly nodal, to $X$ of the prescribed fundamental class. This was first studied for pseudo-holomorphic maps by T. Parker and J. Wolfson [PW] and in algebraic geometry by Kontsevich [Ko]. Because points of the compactification $\overline{M}_{g,n}(X,A)$ are maps $f$ whose domains have $n$-marked points $x_1, \ldots, x_n$, the evaluation map $e$ extends canonically to $\bar{e}$ that sends such map $f$ to $(f(x_1), \ldots, f(x_n))$.

The virtual moduli cycles $[\overline{M}_{g,n}(X,A)]^{\text{vir}}$ for projective variety $X$ were first constructed by the authors. Their idea is to construct a virtual normal cone embedded in a vector bundle based on the obstruction theory of stable morphisms [LT1]. An alternative construction of such cones was achieved by Behrend and Fantechi [BF] and [Be]. For general symplectic manifolds, such virtual moduli cycles were constructed by the authors, and independently, by Fukaya and Ono [FO, LT2]. Shortly after them, B. Siebert [Si] and later, Y. Ruan [Ru2] gave different constructions of such virtual moduli cycles. Both Siebert and Ruan’s approach needs to construct global, finite-dimensional resolutions of so called cokernel bundles (cf. [Si] and [Ru2], Appendix).

However, one question remains to be investigated. Namely, if $X$ is a smooth projective variety then on one hand we have the algebraically constructed GW-invariants, and on the other hand, by viewing $X$ as a symplectic manifold using the Kähler form on $X$, we have the GW-invariants constructed using analytic method. These two approaches are drastically different. One may expect, although far from clear, that for smooth projective varieties the algebraic GW-invariants and their symplectic counterparts are identical.

The main goal of this paper is to prove what was expected is indeed true.

**Theorem 1.1.** Let $X$ be any smooth projective variety with a Kähler form $\omega$. Then the algebraically constructed GW-invariants of $X$ coincide with the analytically constructed GW-invariants of the symplectic manifold $(X^{\text{top}}, \omega)$.
This result was first announced in [LT2]. Its proof was outlined in [LT3]. During the preparation of the paper, we learned from B. Siebert that he was able to prove a similar result.

We now outline the proof of our Comparison Theorem. We begin with a few words on the algebraic construction of the virtual moduli cycle. Let \( w \in \overline{\mathcal{M}}_{g,n}(X, A) \) be any point associated to the stable morphism \( f : \Sigma \to X \). It follows from the deformation theory of stable morphisms that there is a complex \( C_w \), canonical up to quasi-isomorphisms, such that its first cohomology \( H^1(C_w) \) is the space of the first order deformations of the map \( w \), and its second cohomology \( H^2(C_w) \) is the obstruction space to deformations of the map \( w \).

Let \( \varphi_w \) be a Kuranishi map of the obstruction theory of \( w \). Note that \( \varphi_w \) is the germ of a holomorphic map from a neighborhood of the origin \( o \in \mathbb{C}^{m_1} \) to \( \mathbb{C}^{m_2} \), where \( m_i = \dim H^i(C_w) \). Let \( \hat{o} \) be the formal completion of \( \mathbb{C}^{m_1} \) along \( o \) and let \( \hat{w} \) be the subscheme of \( \hat{o} \) defined by the vanishing of \( \varphi_w \). Note that \( \hat{w} \) is isomorphic to the formal completion of \( \overline{\mathcal{M}}_{g,n}(X, A) \) along \( w \) (Here as before we will ignore the issue of non-trivial automorphism groups of maps in \( \overline{\mathcal{M}}_{g,n}(X, A) \)). This says that “near” \( w \), the scheme \( \overline{\mathcal{M}}_{g,n}(X, A) \) is a “subset” of \( \mathbb{C}^{m_1} \) defined by the vanishing of \( m_2 \)-equations. Henceforth, it these equations are in general position, then \( \dim \hat{w} = m_1 - m_2 \), which is the expected dimension \( r_{exp} \) we mentioned before. The case where \( \overline{\mathcal{M}}_{g,n}(X, A) \) has dimension bigger than \( r_{exp} \) is exactly when the vanishing locus of these \( m_2 \)-equations in \( \varphi_w \) do not meet properly near \( o \). Following the excess intersection theory of Fulton and MacPherson [Pl], the “correct” cycle should come from first constructing the normal cone \( C_{\hat{w}/\hat{o}} \) to \( \hat{w} \) in \( \hat{o} \), which is canonically a subcone of \( \hat{w} \times \mathbb{C}^{m_2} \), and then intersect the cone with the zero section of \( \hat{w} \times \mathbb{C}^{m_2} \to \hat{w} \). The next step is to patch these cones together to form a global cone over \( \overline{\mathcal{M}}_{g,n}(X, A) \). The main difficulty in doing so comes from the fact that the dimensions \( H^2(C_w) \) can and do vary as \( w \) vary, only \( \dim H^1(C_w) - \dim H^2(C_w) \) is a topological number. This makes the cones \( C_{\hat{w}/\hat{o}} \) to sit inside bundles of varying ranks. To overcome this difficulty, the authors came with the idea of finding a global \( \mathbb{Q} \)-vector bundle \( E_2 \) over \( \overline{\mathcal{M}}_{g,n}(X, A) \) and a subcone \( N \) of \( E_2 \) such that near fibers over \( w \), the cone \( N \) is a fattening of the cone \( C_{\hat{w}/\hat{o}} \) (See section 3 or [LT2] for more details). In the end, we let \( j \) be the zero section of \( E_2 \) and let \( j^* \) be the Gysin map

\[
A_* E_2 \to A_* \overline{\mathcal{M}}_{g,n}(X, A),
\]

where \( A_* \) denote the Chow-cohomology group (see [Pl]). Then the algebraic virtual moduli cycle is

\[
[\overline{\mathcal{M}}_{g,n}(X, A)]^{\text{vir}} = j^*([N]) \in A_{r_{exp}} \overline{\mathcal{M}}_{g,n}(X, A).
\]

Now let us recall briefly the analytic construction of GW-invariants of symplectic manifolds. Let \( (X, \omega) \) be any smooth symplectic manifold with \( J \) a tamed almost complex structure. For \( A, g \) and \( n \) as before, we can form the moduli space of \( J \)-holomorphic maps \( f : \Sigma \to X \) where \( \Sigma \) are \( n \)-pointed smooth Riemann surfaces such that \( f_*(\Sigma) = A \). We denote this space by \( \mathcal{M}_{g,n}(X, A)^J \). It is a finite dimensional topological space. As before, we compactify it to include
all $J$-holomorphic maps whose domains are possibly with nodal singularities. We denote the compactified space by $\mathfrak{M}_{g,n}(X,A)^J$. To proceed, we will embed $\mathfrak{M}_{g,n}(X,A)^J$ inside an ambient space $B$ and realize it as the vanishing locus of a section of a “vector bundle”. Without being precise, the space $B$ is the space of all smooth maps $f \in B$ from possibly nodal $n$-pointed Riemann surfaces to $X$, the fiber of the bundle over $f$ are all $(0,1)$-forms over domain($f$) with values in $f^*TX$ and the section is the one that sends $f$ to $\partial f$. We denote this bundle by $E$ and the section by $\Phi$. Clearly, $\Phi^{-1}(0)$ is homeomorphic to $\mathfrak{M}_{g,n}(X,A)^J$. Defining the GW-invariants of $(X,\omega)$ is essentially about constructing the Euler class of $[\Phi:B \to E]$. This does not make much sense since $B$ is an infinite dimensional topological space. Although at each $w \in B^{-1}(0)$ the formal differential $d\Phi(w):T_wB \to E_w$ is Fredholm, which has real index $2\text{r}_\text{exp}$, the conventional perturbation scheme does not apply directly since near maps in $B$ whose domains are singular the space $B$ is not smooth and $E$ does not admit local trivializations. To overcome this difficulty, the authors introduced the notion of weakly $Q$-Fredholm bundles, and showed in [LT2] that $[\Phi:B \to E]$ is a weakly $Q$-Fredholm bundle and that any weakly $Q$-Fredholm bundle admits an Euler class. Let

$$e[\Phi:B \to E] \in H_{2\text{r}_\text{exp}}(B;\mathbb{Q})$$

be the the Euler class of $[\Phi:B \to E]$. Since the evaluation map of $\mathfrak{M}_{g,n}(X,A)^J$ extends to an evaluation map $e:B \to \mathbb{R}^n$, the Euler class, which will also be referred to as the symplectic virtual cycle of $\mathfrak{M}_{g,n}(X,A)^J$, defines a multi-linear map $\gamma_{A,g,n}^{X,J}$ as in [LT1]. We will review the notion of weakly smooth Fredholm bundles in section 2. Here to say the least, $[\Phi:B \to E]$ is weakly Fredholm means that near each point of $\Phi^{-1}(0)$ we can find a finite rank subbundle $V$ of $E$ such that $W = \Phi^{-1}(V)$ is a smooth finite dimensional manifold, $V|_W$ is a smooth vector bundle and the lift $\phi:W \to V|_W$ of $\Phi$ is smooth. (Note that the rank of $V$ may vary but $\dim\mathbb{R}W = \text{rank}\mathbb{R}V = 2\text{r}_\text{exp}$). For such finite models $[\phi:W \to V|_W]$, which are called weakly smooth approximations, we can perturb $\phi$ slightly to obtain $\phi'$ so that $\phi'^{-1}(0)$ are smooth manifolds in $W$. To construct the Euler class, we first cover a neighborhood of $\Phi^{-1}(0)$ in $B$ by finitely many such approximations that satisfy certain compatibility condition. We then perturb each section in the approximation and obtain a collection of locally closed $Q$-submanifolds of $B$ of dimension $2\text{r}_\text{exp}$. By imposing certain compatibility condition on the perturbations, this collection of $Q$-submanifolds patch together to form a $2\text{r}_\text{exp}$-dimensional cycle in $B$, which represents a homology class in $H_{2\text{r}_\text{exp}}(B;\mathbb{Q})$. This is the Euler class of $[\Phi:B \to E]$.

Now we assume that $X$ is a smooth projective variety and $\omega$ is a Kähler form of $X$. Let $J$ be the complex structure of $X$. Then $\mathfrak{M}_{g,n}(X,A)$ is homeomorphic to $\mathfrak{M}_{g,n}(X,A)^J$. Hence the two GW-invariants $\gamma_{A,g,n}^X$ and $\gamma_{A,g,n}^{X,J}$ are identical if the homology classes $[\mathfrak{M}_{g,n}(X,A)]^\text{vir}$ and $e[\Phi:B \to E]$ will be identical. Here we view $[\mathfrak{M}_{g,n}(X,A)]^\text{vir}$ as a class in $H_*(B;\mathbb{Q})$ using

$$\mathfrak{M}_{g,n}(X,A) \cong \text{homeo} \mathfrak{M}_{g,n}(X,A)^J \subset B.$$ 

To illustrate why these two classes are equal, let us first look at the following
simple model. Let $Z$ be a compact smooth variety and let $E$ be a holomorphic vector bundle over $Z$ with a holomorphic section $s$. There are two ways to construct the Euler classes of $E$. One is to perturb $s$ to a smooth section $r$ so that the graph of $r$ is transversal to the zero section of $E$, and then define the Euler class of $E$ to be the homology class in $H_*(Z; \mathbb{Q})$ of $r^{-1}(0)$. This is the topological construction of the Euler class of $E$. The algebraic construction is as follows. Let $t$ be a large scalar and let $\Gamma_{ts}$ be the graph of $ts$ in the total space of $E$. Since $s$ is an algebraic section, it follows that the limit $\Gamma_\infty s = \lim_{t \to \infty} \Gamma_{ts}$ is a complex dimension $\dim Z$ cycle supported on union of subvarieties of $E$. We then let $r$ be a smooth section of $E$ in general position and let $\Gamma_\infty s \cap \Gamma_r$ be their intersection. Its image in $Z$ defines a homology class, which is the image of the Gysin map $j^*([C])$, where $j$ is the zero section of $E$. The reason that $e(E) = [r^{-1}(0)] = j^*([\Gamma_\infty s]) \in H_*(Z; \mathbb{Q})$ is that if we choose $r$ to be in general position, then $[r^{-1}(0)] = [\Gamma_r \cap \Gamma_0] = [\Gamma_r \cap \Gamma_s]$, and the family $\{\Gamma_{ts} \cap \Gamma_r\}_{t \in [1, \infty)}$ forms a homotopy of the cycles $\Gamma_r \cap \Gamma_r$ and $\Gamma_\infty s \cap \Gamma_r$. One important remark is that the cone $\Gamma_\infty s$ only relies on the restriction of $s$ to an “infinitesimal” neighborhood of $s^{-1}(0)$ in $Z$, and can also be reconstructed using the Kuranishi maps of the obstruction theory to deformations of points in $s^{-1}(0)$ induced by the defining equation $s = 0$. Along this line, to each finite model $[\phi : W \to V|_W]$ we can form a cone $\Gamma_{\infty \phi} = \lim_{t \to \infty} \Gamma_{t\phi}$ in $V|_{\phi^{-1}(0)}$. Hence to show that the two virtual moduli cycles coincide, it suffices to establish a relation, similar to quasi-isomorphism of complexes, between the cone $N$ constructed based the obstruction theory to deformations of points in $s^{-1}(0)$ induced by the defining equation $s = 0$. Along this line, to each finite model $[\phi : W \to V|_W]$ we can form a cone $\Gamma_{\infty \phi} = \lim_{t \to \infty} \Gamma_{t\phi}$ in $V|_{\phi^{-1}(0)}$. Hence to show that the two virtual moduli cycles coincide, it suffices to establish a relation, similar to quasi-isomorphism of complexes, between the cone $N$ constructed based the obstruction theory of $\mathcal{M}_{g,n}(X, A)$ and the collection $\{\Gamma_{\infty \phi}\}$. In the end, this is reduced to showing that the obstruction theory to deformations of maps in $\mathcal{M}_{g,n}(X, A)$ is identical to the obstruction theory to deformations of elements in $\phi^{-1}(0)$ induced by the defining equation $\phi$. This identification of two obstruction theories follows from the canonical isomorphism of the Čech cohomology and the Dolbeault cohomology of vector bundles.
The layout of the paper is as follows. In section two, we will recall the analytic construction of the GW-invariants of symplectic manifolds. We will construct the Euler class of $[\Phi : B \to E]$ in details using the weakly smooth approximations constructed in [LT2]. In section three, we will construct a collection of holomorphic weakly smooth approximations for projective manifolds. The proof of the Comparison Theorem will occupy the last section of this paper.

2 Symplectic construction of GW invariants

The goal of this section is to review the symplectic construction of the GW-invariants of algebraic varieties. We will emphasize on those parts that are relevant to our proof of the Comparison Theorem.

In this section, we will work mainly with real manifolds and will use the standard notation in real differential geometry.

We begin with the symplectic construction of GW-invariants. Let $X$ be a smooth complex projective variety, and let $A \in H_2(X, \mathbb{Z})$ and let $g, n \in \mathbb{Z}$ be fixed once and for all. We recall the notion of stable $C^\ell$-maps [LT2, Definition 2.1].

**Definition 2.1.** An $n$-pointed stable map is a collection $(f; \Sigma; x_1, \ldots, x_n)$ satisfying the following property: First, $(\Sigma; x_1, \ldots, x_n)$ is an $n$-pointed connected prestable complex curve with normal crossing singularity; Secondly, $f : \Sigma \to X$ is continuous, and the composite $f \circ \pi$ is smooth, where $\pi : \tilde{\Sigma} \to \Sigma$ is the normalization of $\Sigma$; And thirdly, if we let $S \subset \Sigma$ be the union of singular locus of $\Sigma$ with its marked points, then any rational component $R \subset \Sigma$ satisfying $(f \circ \pi)_*([R]) = 0 \in H_2(X, \mathbb{Z})$ must contains at least three points in $\pi^{-1}(S)$.

For convenience, we will abbreviate $(f; \Sigma; x_1, \ldots, x_n)$ to $(f; \Sigma; \{x_i\})$. Later, we will use $C$ to denote an arbitrary stable map and use $f_C$ and $\Sigma_C$ to denote its corresponding mapping and domain. Two stable maps $(f; \Sigma; \{x_i\})$ and $(f'; \Sigma'; \{x'_i\})$ are said to be equivalent if there is an isomorphism $\rho : \Sigma \to \Sigma'$ such that $f' \circ \rho = f$ and $x'_i = \rho(x_i)$. When $(f; \Sigma; \{x_i\}) \equiv (f'; \Sigma'; \{x'_i\})$, such a $\rho$ is called an automorphism of $(f; \Sigma; \{x_i\})$.

We let $B$ be the space of equivalence classes $[C]$ of $C^\ell$-stable maps $C$ such that the arithmetic genus of $\Sigma_C$ is $g$ and $f_C_*([\Sigma]) = A \in H_2(X, \mathbb{Z})$. Note that $B$ was denoted by $\mathcal{F}_\Delta^\ell(X, g, n)$ in [LT2]. Over $B$ there is a generalized bundle $E$ defined as follows. Let $C$ be any stable map and let $f_C : \Sigma_C \to X$ be the composite of $f_C$ with $\pi : \tilde{\Sigma}_C \to \Sigma_C$. We define $\Lambda^{0,1}_C$ to be the space of all $C^{\ell - 1}$-smooth sections of $(0, 1)$-forms of $\Sigma$ with values in $\tilde{f}^*TX$. Assume $C$ and $C'$ are two equivalent stable maps with $\rho : \Sigma_C \to \Sigma_{C'}$ the associated isomorphism, then there is a canonical isomorphism $\Lambda^{0,1}_{C'} \cong \Lambda^{0,1}_C$. We let $\Lambda^{0,1}_C$ be $\Lambda^{0,1}_C / \Aut(C)$. Then the union

$$E = \bigcup_{[C] \in B} \Lambda^{0,1}_C$$
is a fibration over $B$ whose fibers are finite quotients of infinite dimensional linear spaces. There is a natural section

$$\Phi : B \rightarrow E$$

defined as follows. For any stable map $C$, we define $\Phi(C)$ to be the image of $\partial f([C]) \in \Lambda^{0,1}_C$ in $\Lambda^{0,1}_C$. Obviously, for $C \sim C'$ we have $\Phi(C) = \Phi(C')$. Thus $\Phi$ descends to a map $B \rightarrow E$, which we still denote by $\Phi$.

From now on, we will denote by $\mathcal{M}_{g,n}(X, A)$ the moduli scheme of stable morphisms $f : C \rightarrow X$ with $n$-marked points such that $C$ is (possibly with nodal singularities) has arithmetic genus $g$ with $f_*([C]) = A$.

**Lemma 2.2.** The vanishing locus of $\Phi$ is canonically homeomorphic to the underlying topological space of $\mathcal{M}_{g,n}(X, A)$.

**Proof.** A stable $C^l$-stable map $C$ in $B$ belongs to the vanishing locus of $\Phi$ if and only if $f_0$, $f_0$, $f_0$, $f_0$, $f_0$ is holomorphic. Since $\Sigma_C$ is compact, $C$ is the underlying analytic map of a stable morphism. Hence there is a canonical map $\Phi^{-1}(0) \rightarrow \mathcal{M}_{g,n}(X, A)^{\text{top}}$, which is one-to-one and onto. This proves the lemma.

To discuss the smoothness of $\Phi$, we need the local uniformizing charts of $\Phi : B \rightarrow E$ near $\Phi^{-1}(0)$. Let $w \in B$ be any point represented by the stable map $(f_0; \Sigma_0; \{x_i\})$ with automorphism group $G_w$. We pick integers $r_1, r_2 > 0$ and smooth ample divisors $H_1, \ldots, H_{r_2}$ with $[H_i] \cdot [A] = r_1$ such that all $f_0^{-1}(H_i)$ are contained in the smooth locus of $\Sigma_0$ and that for any $x \in f_0^{-1}(H_i)$ we have

$$\text{Im}(df_0(x)) + T_{f(x)}H_i = T_{f(x)}X. \tag{2.1}$$

Now let $U \subset B$ be a sufficiently small neighborhood of $w \in B$ and let $\bar{U}$ be the collection of all $(C; z_{n+1}, \ldots, z_{n+r_1r_2})$ such that $C \in U$ and the $z_i$’s is a collection of smooth points of $\Sigma_C$ such that for each $1 \leq j \leq r_2$ the subcollection $(z_{n+(j-1)r_1+1}, \ldots, z_{n+jr_1})$ contains distinct points and is exactly $f_0^{-1}(H_j)$. Note that we do not require $(z_{n+1}, \ldots, z_{n+r_1r_2})$ to be distinct. \[\]

Let $\pi_U : U \rightarrow U$ be the projection that sends $(C; z_{n+1}, \ldots, z_{n+r_1r_2})$ to $C$. Clearly, $G_w$ acts on $\pi_U^{-1}(w)$ canonically by permuting their $(n + r_1r_2)$-marked points. Namely, for any $\sigma \in G_w$ and $C \in \pi_U^{-1}(w)$ with marked points $z_1, \ldots, z_{n+r_1r_2}$, $\sigma(C)$ is the same map with the marked points $\sigma(z_1), \ldots, \sigma(z_{n+r_1r_2})$. In particular, we can view $G_w$ as a subgroup of the permutation group $S_{n+r_1r_2}$. Hence $G_w$ acts on $\bar{U}$ by permuting the marked points of $C \in \bar{U}$ according to the inclusion $G_w \subset S_{n+r_1r_2}$. Note that if $H_i$’s are in general position then elements in $\bar{U}$ has no automorphisms and have distinct marked points. Let $G_0 = G_w$. Since fibers of $\pi_U$ are invariant under $G_0$, $\pi_U$ induces a map $\bar{U}/G_0 \rightarrow U$, which is

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obviously a covering\footnote{In this paper we call $p:A \to B$ a covering if $p$ is a covering projection $p^*$ and $\#(p^{-1}(x))$ is independent of $x \in B$. We call $p:A \to B$ a local covering if $p(A)$ is open in $B$ and $p:A \to p(A)$ is a covering.} if $U$ is sufficiently small. Further, if we let

$$E_{\bar{G}} = \bigcup_{c \in \bar{U}} \Lambda^0_{c}$$

and let $\Phi_{\bar{G}} : \bar{U} \to E_{\bar{G}}$ be the section that sends $C$ to $\bar{\partial}f_C$, then $\Phi_{\bar{G}}$ is $G_{\bar{G}}$-equivariant and $\Phi_{\bar{U}} : U \to E_{\bar{U}}$ is the descent of $\Phi_{\bar{G}}/G_{\bar{G}} : \bar{U}/G_{\bar{G}} \to E_{\bar{G}}/G_{\bar{G}}$. Note that fibers of $E_{\bar{G}}$ over $\bar{U}$ are linear spaces. Following the convention, we will call $\Lambda = (\bar{U},E_{\bar{G}},\Phi_{\bar{G}},G_{\bar{G}})$ a uniformizing chart of $(\bar{B},E,\Phi)$ over $U$. Let $V \subset U$ be an open subset and let $\bar{V} = \pi_{\bar{U}}^{-1}(V)$, let $G_{\bar{V}} = G_{\bar{G}}$, let $E_{\bar{V}} = E_{\bar{G}}|_{\bar{V}}$ and let $\Phi_{\bar{V}} = \Phi_{\bar{G}}|_{\bar{V}}$. We will call $\Lambda' = (\bar{V},E_{\bar{V}},\Phi_{\bar{V}},G_{\bar{V}})$ a uniformizing chart of $(\bar{B},E,\Phi)$ that is the restriction of the original chart to $V$, and denoted by $\Lambda|_{V}$. We can also construct uniformizing charts by pull back. Let $G_{\bar{V}}$ be a finite group acting effectively on a topological space $\bar{V}$, let $G_{\bar{V}}$ be a homomorphism and $\varphi : \bar{V} \to \bar{U}$ be a $G_{\bar{V}}$-equivariant map so that $\bar{V}/G_{\bar{V}} \to \bar{U}/G_{\bar{U}}$ is a local covering map. Then we set $E_{\bar{V}} = \varphi^*E_{\bar{G}}$ and $\Phi_{\bar{V}} = \varphi^*\Phi_{\bar{G}}$. The data $\Lambda' = (\bar{V},E_{\bar{V}},\Phi_{\bar{V}},G_{\bar{V}})$ is also a uniformizing chart. We will call $\Lambda'$ the pull back of $\Lambda$, and denoted by $\varphi^*\Lambda$. In the following, we will denote the collection of all uniformizing charts of $(\bar{B},E,\Phi)$ by $\mathcal{C}$.

The collection $\mathcal{C}$ has the following compatibility property. Let

$$\Lambda_i = (\bar{U}_i,E_{\bar{U}_i},\Phi_{\bar{U}_i},G_{\bar{U}_i}),$$

where $i = 1,\ldots,k$, be a collection of uniformizing charts in $\mathcal{C}$ over $U_i \subset B$ respectively. Let $p \in \cap_{i=1}^{k} U_i$ be any point. Then there is a uniformizing chart $\Lambda = (\bar{V},E_{\bar{V}},\Phi_{\bar{V}},G_{\bar{V}})$ over $V \subset \cap_{i} U_i$ with $p \in V$ such that there are homomorphisms $G_{\bar{V}} \to G_{\bar{U}_i}$ and equivariant local covering maps $\varphi_i : \bar{V} \to \pi_{\bar{U}_i}^{-1}(V) \subset \bar{U}_i$ compatible with $\bar{V} \to V$ and $\pi_{\bar{U}_i}^{-1}(V) \subset U_i$, such that $\varphi_i^*(E_{\bar{U}_i},\Phi_{\bar{U}_i}) \cong (E_{\bar{V}},\Phi_{\bar{V}})$. In this case, we say $\Lambda$ is finer than $\Lambda_i|_{V}$.

The main difficulty in constructing the GW invariants in this setting is that the smoothness of $(\bar{U},E_{\bar{G}},\Phi_{\bar{G}})$ is unclear when $U$ contains maps whose domains are singular. To overcome this difficulty, the authors introduced the notion of generalized Fredholm bundles in \cite{LT2}. The main result of \cite{LT2} is the following theorems, which enable them to construct the GW invariants for all symplectic manifolds.

\textbf{Theorem 2.3.} The data $[\Phi : B \to E]$ is a generalized oriented Fredholm $V$-bundle of relative index $2r_{\exp}$, where $r_{\exp} = c_1(X) \cdot A + n + (n-3)(1-g)$ is half of the virtual (real) dimension of $\Phi^{-1}(0)$.

\textbf{Theorem 2.4.} For any generalized oriented Fredholm $V$-bundle $[\Phi : B \to E]$ of relative index $r$, we can assign to it an Euler class $e([\Phi : B \to E])$ in $H_{r}(B;\mathbb{Q})$ that satisfies all the expected properties of the Euler classes.
As explained in the introduction, the pairing of the Euler class of \([\Phi : B \to E]\)
with the tautological topological class will give rise to the symplectic version of the GW invariants of \(X\). Further, the Comparison Theorem we set out to prove amounts to compare this Euler class with the image of the virtual moduli cycle \([\mathcal{M}_{g,n}(X, A)\text{vir}] \in H_* (B; \mathbb{Q})\) via the inclusion \(\mathcal{M}_{g,n}(X, A)\text{top} \subset B\). In the remainder part of this section, we will list all properties of \([\Phi : B \to E]\) that are relevant to the construction of its Euler class. This list is essentially equivalent to saying that \([\Phi : B \to E]\) is a generalized oriented Fredholm V-bundle. After that, we will construct the Euler class of \([\Phi : B \to E]\) in details.

We begin with the notion of weakly smooth structure. A local smooth approximation of \([\Phi : B \to E]\) over \(U \subset B\) is a pair \((\Lambda, V)\), where \(\Lambda = (U, E_U, \Phi_U, G_U)\) is a uniformizing chart over \(U\) and \(V\) is a finite rank \(G_U\)-vector bundle over \(U\) that is a \(G_U\)-equivariant subbundle of \(E_U\) such that \(R_V := \Phi_U^{-1}(V) \subset U\) is an equi-dimensional smooth manifold, \(V|_{R_V}\) is a smooth vector bundle and the lifting \(\phi_V : R_V \to V|_{R_V}\) of \(\Phi_U|_{R_V}\) is a smooth section.

An orientation of \((\Lambda, V)\) is a \(G_U\)-invariant orientation of the real line bundle \(\Lambda^{top}(TR_V) \otimes \Lambda^{top}(V|_{R_V})^{-1}\) over \(R_V\). We call rank \(V\) − \(\dim R_V\) the index of \((\Lambda, V)\) (We remind that all ranks and dimensions in this section are over reals). Now assume that \((\Lambda', V')\) is another weakly smooth structure of identical index over \(W \subset B\). We say that \((\Lambda', V')\) is finer than \((\Lambda, V)\) if the following holds. First, the restriction \(\Lambda'|_{W \cap U}\) is finer than \(\Lambda|_{W \cap U}\). Second, if we let \(\varphi : \pi_W^{-1}(W \cap U) \to \pi_U^{-1}(W \cap U)\) be the covering map then \(\varphi^* V \subset \varphi^* E_U \equiv E_W|_{\pi_W^{-1}(W \cap U)}\) is a subbundle of \(V|_{\pi_W^{-1}(W \cap U)}\). Thirdly, for any \(w \in W\) the homomorphism \(T_wR_V \to (V'/\varphi^* V)|_w\) induced by \(d\phi_V(w) : T_wR_V \to V'|_w\) is surjective, and the map \(\phi_V^{-1}(\varphi^* V) \to R_V\) induced by \(\varphi\) is a local diffeomorphism between smooth manifolds. Note that the last condition implies that if we identify \(T_{\phi(w)}R_V\) with \(T_w\phi_V^{-1}(\varphi^* V) \subset T_wR_V\), then the induced homomorphism
\[
T_{wR_V}/T_{\phi(w)}R_V \longrightarrow (V'/\varphi^* V)|_w
\] (2.2)
is an isomorphism. In case both \((\Lambda, V)\) and \((\Lambda', V')\) are oriented, then we require that the orientation of \((\Lambda, V)\) coincides with that of \((\Lambda', V')\) based on the isomorphism
\[
\Lambda^{top}(T_{wR_V'}) \otimes \Lambda^{top}(V'|_w)^{-1} \cong \Lambda^{top}(T_{\phi(w)}R_V) \otimes \Lambda^{top}(V|_{\varphi(w)})^{-1}\] (2.3)
induced by (2.2).

Now let \(\mathfrak{A} = \{(\Lambda_i, V_i)\}_{i \in K}\) be a collection of oriented smooth approximations of \((B, E, \Phi)\). In the following, we will denote by \(U_i\) the open subsets of \(B\) such that \(\Lambda_i\) is a smooth chart over \(U_i\). We say \(\mathfrak{A}\) covers \(\Phi^{-1}(0)\) if \(\Phi^{-1}(0)\) is contained in the union of the images of \(U_i\) in \(B\).

**Definition 2.5.** An index \(r\) oriented weakly smooth structure of \((B, E, \Phi)\) is a collection \(\mathfrak{A} = \{(\Lambda_i, V_i)\}_{i \in K}\) of index \(r\) oriented smooth approximations such that \(\mathfrak{A}\) covers \(\Phi^{-1}(0)\) and that for any \((\Lambda_i, V_i)\) and \((\Lambda_j, V_j)\) in \(\mathfrak{A}\) with \(p \in U_i \cap U_j\),
there is a \((A_k, V_k) \in \mathfrak{A}\) such that \(p \in U_k\) and \((A_k, V_k)\) is finer than \((A_i, V_i)\) and \((A_j, V_j)\).

Let \(\mathfrak{A}'\) be another index \(r\) oriented weakly smooth structure of \((B, E, \Phi)\). We say \(\mathfrak{A}'\) is finer than \(\mathfrak{A}\) if for any \((\Lambda, V) \in \mathfrak{A}\) over \(U \subset B\) and \(p \in U \cap \Phi^{-1}(0)\), there is a \((\Lambda', V') \in \mathfrak{A}'\) over \(U'\) such that \(p \in U'\) and \((\Lambda', V')\) is finer than \((\Lambda, V)\).

We say that two weakly smooth structures \(\mathfrak{A}_1\) and \(\mathfrak{A}_2\) are equivalent if there is a third weakly smooth structure that is finer than both \(\mathfrak{A}_1\) and \(\mathfrak{A}_2\).

**Proposition 2.6 ([LT2])**. The tuple \((B, E, \Phi)\) constructed at the beginning of this section admits a canonical oriented weakly smooth structure of index \(2r_{exp}\).

We remark that the construction of such a weakly smooth structure is the core of the analytic part of [LT2].

In the following, we will use the weakly smooth structure of \(\Phi: B \to E\) to construct its Euler class. The idea of the construction is as follows. Given a local smooth approximation \((\Lambda, V)\) over \(U \subset B\), we obtain a smooth manifold \(R_V\), a vector bundle \(V|_{R_V}\) and a smooth section \(\phi_V: R_V \to V|_{R_V}\). Following the topological construction of the Euler classes, we shall perturb \(\phi_V\) to a new section \(\tilde{\phi}_V: R_V \to V|_{R_V}\) so that \(\tilde{\phi}_V\) is transversal to the zero section of \(V|_{R_V}\). Here by a section transversal to the zero section, we mean that the graph of this section is transversal to the zero section in the total space of the vector bundle. Hence the Euler class will be the cycle represented by \(\tilde{\phi}_V^{-1}(0)\) near \(U\). Since the weakly smooth structure of \(\Phi: B \to E\) is given by a collection of compatible by not necessarily matching local smooth approximations, we need to work out this perturbation scheme with special care so that \(\{\tilde{\phi}_V^{-1}(0)\}\) patch together to form a well-defined cycle.

Let \(\mathfrak{A} = \{(\Lambda_\alpha, V_\alpha)\}_{\alpha \in K}\) be the weakly smooth structure provided by Proposition 2.6. For convenience, for any \(\alpha \in K\) we will denote the corresponding uniformizing chart \(\Lambda_\alpha\) by \((U_\alpha, B_\alpha, \tilde{\Phi}_\alpha, G_\alpha)\) and will denote its descent by \((U_\alpha, E_\alpha, \Phi_\alpha)\). Accordingly, we will denote the projection \(\pi_{U_\alpha}: \tilde{U}_\alpha \to U_\alpha\) by \(\pi_\alpha\), denote \(\tilde{\Phi}_\alpha^{-1}(V_\alpha)\) by \(R_\alpha\), denote \(V_\alpha|_{R_\alpha}\) by \(W_\alpha\) and denote the lifting of \(\tilde{\Phi}_\alpha|_{R_\alpha}: R_\alpha \to E_\alpha|_{R_\alpha}\) by \(\phi_\alpha: R_\alpha \to W_\alpha\). Without loss of generality, we can assume that for any approximation \((\Lambda_\alpha, V_\alpha) \in \mathfrak{A}\) over \(U_\alpha\) and any \(U' \subset U_\alpha\), the restriction \((\Lambda_\alpha, V_\alpha)|_{U'}\) is also a member in \(\mathfrak{A}\). In the following, we call \(S \subset R_\alpha\) symmetric if \(S = \pi_\alpha^{-1}(\pi_\alpha(S))\).

Next, we pick a covering data for \(\Phi^{-1}(0) \subset B\) provided by the following covering lemma.

**Lemma 2.7 ([LT2])**. There is a finite collection \(\mathcal{L} \subset K\) and a total ordering of \(\mathcal{L}\) of which the following holds. the set \(\Phi^{-1}(0)\) is contained in the union of \(\{R_\alpha\}_{\alpha \in \mathcal{L}}\) and for any \(\alpha, \beta \in \mathcal{L}\) such that \(\alpha < \beta\) then approximation \((\Lambda_\beta, V_\beta)\) is finer than the approximation \((\Lambda_\alpha, V_\alpha)\).

**Proof.** The lemma is part of Proposition 2.2 in [LT2]. It is proved there by using the stratified structures of \((B, E, \Phi)\). Here we will give a direct proof of this by using the definition of smooth approximations, when \(\Phi^{-1}(0)\) is triangulable, which is true when \(X\) is projective. Let \(k\) be the real dimension of \(\Phi^{-1}(0)\). To
prove the lemma, we will show that there are \( k + 1 \) subsets \( \mathcal{L}_k, \ldots, \mathcal{L}_0 \subset \mathcal{K} \) and that for each \( \alpha \in \bigcup_{k=0}^{k} \mathcal{L}_k \) there is an open symmetric subset \( U'_\alpha \subset U_\alpha \) such that \( R'_\alpha = R_\alpha \cap \pi^{-1}_\alpha(U'_\alpha) \subset R_\alpha \), of which the following holds: first, for each \( i \leq k \) the set \( Z_i = \Phi^{-1}(0) - \bigcup_{j=1}^{i} \cup_{\alpha \in \mathcal{L}_j} U'_\alpha \) is a triangulable space whose dimension is at most \( i - 1 \), and secondly, for any pair of distinct \( (\alpha, \beta) \in \mathcal{L}_1 \times \mathcal{L}_j \) with \( i \leq j \), the restriction \( (A_\alpha, V_\alpha)\mid_{U'_\alpha \cap U'_\beta} \) is finer than \( (A_\beta, V_\beta)\mid_{U'_\alpha \cap U'_\beta} \). We will construct \( \mathcal{L}_i \) inductively, starting from \( \mathcal{L}_k \). We first pick a finite \( \mathcal{L}_k \subset \mathcal{K} \) so that \( \cup_{\alpha \in \mathcal{L}_k} U_\alpha \supset \Phi^{-1}(0) \). This is possible since \( \Phi^{-1}(0) \) is compact. Since it is also triangulable, we can find a symmetric \( U'_\alpha \subset U_\alpha \) for each \( \alpha \in \mathcal{L}_k \) so that \( \{U'_\alpha\}_{\alpha \in \mathcal{L}_k} \) is disjoint, \( R'_\alpha \subset R_\alpha \), and \( Z_0 \) is triangulable with dimension at most \( k - 1 \). Now we assume that we have found \( \mathcal{L}_k, \ldots, \mathcal{L}_i \) as desired. Then for each \( x \in Z_i \) we can find a neighborhood \( O \) of \( x \in B \) such that for any \( \alpha \in \cup_{j \geq i} \mathcal{L}_\alpha \) either \( x \in U_\alpha \) or \( O \cap U'_\alpha = \emptyset \). Let \( \mathcal{L}_\alpha \) be those \( \alpha \in \cup_{j \geq i} \mathcal{L}_\alpha \) such that \( x \in U_\alpha \). Then by the property of \( \mathfrak{A} \) there is a \( \beta \in \mathcal{K} \) so that \( (A_\beta, V_\beta) \) is finer than \( (A_\alpha, V_\alpha) \) for all \( \alpha \in \mathcal{L}_\alpha \). Without loss of generality, we can assume that \( U_\beta \subset O \). Then \( (A_\beta, V_\beta) \) is finer than \( (A_\alpha, V_\alpha) \) for all \( \alpha \in \cup_{j \geq i} \mathcal{L}_\alpha \). Since \( Z_i \) is compact, we can cover it by finitely many such \( (A_\beta, V_\beta) \)'s, say indexed by \( \mathcal{L}_{i-1} \subset \mathcal{K} \). On the other hand, since \( Z_i \) is triangulable with dimension at most \( i - 1 \), we can find symmetric \( U'_\alpha \subset U_\alpha \) for each \( \alpha \in \mathcal{L}_{i-1} \) so that \( R'_\alpha \subset R_\alpha \) for all \( \alpha \in \mathcal{L}_{i-1} \) and \( Z_i \cup_{\alpha \in \mathcal{L}_{i-1}} U'_\alpha \) is triangulable with dimension at most \( i - 2 \). This way, we can find the set \( \mathcal{L}_i, \ldots, \mathcal{L}_0 \) as desired. In the end, we simply put \( \mathcal{L} = \bigcup_{i=0}^{k} \mathcal{L}_i \). We give it a total ordering so that whenever \( \alpha \in \mathcal{L}_i \), \( i \geq j \), then \( \alpha \leq \beta \). This proves the Lemma.

We now fix such a collection \( \mathcal{L} \) once and for all. Since \( \mathcal{L} \) is totally ordered, in the following we will replace the index by integers that range from 1 to \#(\mathcal{L}) \) and use \( k \) to denote an arbitrary member of \( \mathcal{L} \). We first build the comparison data into the collection \( \{R_k\}_{k \in \mathcal{L}} \) and \( \{W_k\}_{k \in \mathcal{L}} \). To distinguish the projection \( \pi_k: U_k \rightarrow U_k \) from the composite \( U_k \rightarrow U_k \rightarrow B \), we will denote the later by \( \pi_k \). For any pair \( k \geq l \), we set \( R_{k,l} = \pi_k^{-1}\{l_l(R_l)\} \). Then there is a canonical map and a canonical vector bundle inclusion

\[
f^l_k: R_{k,l} \rightarrow R_l(f_{l}^k)^* (W_l) \xrightarrow{i} W_k|_{R_k,l},
\]

that is part of the data making \( (A_k, V_k) \) finer than \( (A_l, V_l) \). Note that \( R_{k,l} \subset R_k \) is a locally closed submanifold, \( f_l^k(R_{k,l}) \) is open in \( R_l \) and \( f_l^k: R_{k,l} \rightarrow f_l^k(R_{k,l}) \) is a covering map. Because of the compatibility condition, for any \( k > l > m \) if \( R_{k,l} \cap R_{k,m} \neq \emptyset \) then \( f_l^m(R_{k,l} \cap R_{k,m}) \subset R_{l,m} \) and

\[
f_l^m \circ f_l^k = f_l^m : R_{k,l} \cap R_{k,m} \rightarrow R_{l,m}.
\]

Further, restricting to \( R_{k,l} \cap R_{k,m} \), the pull backs

\[
(f_l^m)^* (W_m)|_{R_{k,l} \cap R_{k,m}} = (f_l^k)^* (f_l^m)^* (W_m)|_{R_{k,l} \cap R_{k,m}} \subset W_k|_{R_{k,l} \cap R_{k,m}}.
\]

In the following, we will use \( \mathfrak{A} \) to denote the collection of data \( \{(R_{k,l}, f_l^k)\} \) and use \( \mathfrak{W} \) to denote the data \( \{(W_k, f_l^k)^*\} \). We will call the pair \( (\mathfrak{A}, \mathfrak{W}) \) a
good atlas of the weakly smooth structure $\mathfrak{A}$ of $[\Phi : \mathcal{B} \to \mathcal{E}]$. For technical reason, later we need to shrink each $R_k$ slightly. More precisely, let $\{S_k\}_{k \in \mathcal{L}}$ be a collection of symmetric open subsets $S_k \subset R_k$ such that $\{S_k\}$ still covers $\Phi^{-1}(0)$. We then let $S_{k,l} = (f_k^l)^{-1}(S_l) \cap S_k$, let $W_k' = W_k|_{S_k}$ and let $g_k^l$ and $(g_k^l)^*$ be the restriction to $S_{k,l}$ of $f_k^l$ and $(f_k^l)^*$ respectively. Then $(\mathcal{G}, \mathcal{W}')$, where $\mathcal{G} = \{(S_{k,l}, g_k^l)\}$ and $\mathcal{W}' = \{(W_k', (g_k^l)^*)\}$, is also a good atlas of $[\Phi : \mathcal{B} \to \mathcal{E}]$.

We call it a precompact sub-atlas of $(\mathfrak{A}, \mathcal{W})$, and denote it in short by $\mathcal{G} \subset \mathfrak{A}$.

To describe the collection $\{\phi_k\}$, we need to introduce the notion of regular extension. Let $M$ be a manifold and $M_0 \subset M$ be a locally closed submanifold. Let $V \to M$ be a smooth vector bundle and $V_0 \to M_0$ a subbundle of $V|_{M_0}$. We assume that both $(M, V)$ and $(M_0, V_0)$ are oriented. We say that a section $h : M \to V$ is a smooth extension of $h_0 : M_0 \to V_0$ if both $h_0$ and $h$ are smooth and if the induced section $M_0 \xrightarrow{h_0} V_0 \to V|_{M_0}$ is identical to the restriction $h|_{X_0} : X_0 \to V_0$. We say $h$ is a regular extension of $h_0$ if in addition to $h$ being a smooth extension of $h_0$ we have that for any $x \in X_0$ the homomorphism

$$dh(x) : T_x M/T_x M_0 \longrightarrow (V/V_0)|_x$$

is an isomorphism and the orientation of $(M, V)$ and $(M_0, V_0)$ are compatible over $M_0$ based on the isomorphism (2.7).

**Definition 2.8.** A collection $\{h_k\}_{k \in \mathcal{L}}$ is called a smooth section of $\mathcal{W}$ if $h_k$ is a smooth section of $W_k$ for each $k \in \mathcal{L}$ and $h_k$ is a smooth extension of $h_l$ for any pair $k \geq l$ in $\mathcal{L}$. If in addition that $h_k$ is a regular extension of $h_l$ for all $k \geq l$, then we call $\{h_k\}$ a regular section of $\mathcal{W}$.

In the following, we will use $\mathfrak{h} : \mathfrak{A} \to \mathcal{W}$ to denote a smooth section with $\mathfrak{h}$ understood to be $\{h_k\}_{k \in \mathcal{L}}$. We set $\mathfrak{h}^{-1}(0)$ to be the collection $\{h_k^{-1}(0)\}$ and set $\iota(\mathfrak{h}^{-1}(0))$ to be the union of $\iota_k(h_k^{-1}(0))$ in $\mathcal{B}$. We say $\mathfrak{h}^{-1}(0)$ is proper if $\iota(\mathfrak{h}^{-1}(0))$ is compact. Without loss of generality, we can assume that $\dim R_k > 0$ for all $k \in \mathcal{L}$. We say that $\mathfrak{h}$ is transversal to the zero section $0 : \mathfrak{A} \to \mathcal{W}$ if $\mathfrak{h}$ is a regular section and if for any $k \in \mathcal{L}$ the graph $\Gamma_{h_k}$ of $h_k$ is transversal to the $0$ section of $W_k$ in the total space of $W_k$.

**Lemma 2.9.** Let the notation be as before. Then $\mathfrak{h}^{-1}(0)$ is proper if and only if there is a symmetric open subsets $R_k' \subset R_k$ for each $k \in \mathcal{L}$ such that $\cup_{k \in \mathcal{L}} \iota_k(h_k^{-1}(0)) \subset \cup_{k \in \mathcal{L}} \iota_k(R_k')$ and such that for each $k \in \mathcal{L}$,

$$h_k^{-1}(0) \cap (R_k - R_k') \subset \left( \bigcup_{l<k} f_k^l(R_l') \right) \cup \left( \bigcup_{l>k} f_k^l(R_{k,l}) \right).$$

**Proof.** We first assume that $Z = \cup_{k \in \mathcal{L}} \iota_k(h_k^{-1}(0))$ is compact. Then since $\{R_k\}_{k \in \mathcal{L}}$ covers $Z$ and since $\dim R_k > 0$, for each $k \in \mathcal{L}$ we can find symmetric $R_k' \subset R_k$ so that $\{R_k'\}_{k \in \mathcal{L}}$ still covers $Z$. Obviously, this implies (2.8). Conversely, if we have found $R_k' \subset R_k$ as stated in the lemma, then $\{\text{cl}(\iota_k(R_k')) \cap Z\}$ will cover $Z$, where $\text{cl}(A)$ is the closure of $A$. Since $\text{cl}(\iota_k(R_k'))$ are compact and since $Z \cap \text{cl}(\iota_k(R_k'))$ is closed in $\text{cl}(\iota_k(R_k'))$, $Z$ is compact as well. This proves the lemma. $\square$
Lemma 2.10. Let \( \phi : R \rightarrow M \) be the collection \( \{\phi_k\} \) induced by \( \{\Phi_k\}_{k \in L} \). Then \( \phi \) is a regular section with proper vanishing locus.

Proof. This is equivalent to the fact that \( [\Phi : B \rightarrow E] \) is a weakly Fredholm \( V \)-bundle, which was introduced and proved in [LT2]. \( \square \)

Now let \( h : R \rightarrow M \) be a regular section such that \( h \) is transversal to the zero section and \( h^{-1}(0) \) is proper. We claim that the data \( \{h_k^{-1}(0)\} \) descends to an oriented current in \( B \) with rational coefficients supported on a stratified subset whose boundary is empty. In particular, it defines a singular homology class in \( H_*(B, Q) \).

Recall that for each \( k \in L \) the associated group \( G_k \) acts on \( R_k \) such that \( R_k/G_k \) is a covering of \( \bar{\iota}_k(R_k) \). We let \( m_k \) be the product of the order of \( G_k \) with the number of the sheets of the covering \( R_k/G_k \to \bar{\iota}_k(R_k) \). Note that then the covering \( R_{k,l} \to f_k^l(R_{k,l}) \) is an \( m_k/m_l \)-fold covering. Because \( h_k \) is a regular extension of \( (f_k^l)^*(h_l), (f_k^l)^*(h_l)^{-1}(0) \) is an open submanifold of \( h_k^{-1}(0) \) with identical orientations. Hence \( \bar{\iota}_k(h_k^{-1}(0)) \) and \( \bar{\iota}_l(h_l^{-1}(0)) \) patch together to form a stratified subset, and consequently the collection \( \{h_k^{-1}(0)\}_{k \in L} \) patch together to form a stratified subset, say \( Z \), in \( B \). Now we assign multiplicities to open strata of \( Z \). Let \( O_k = \bar{\iota}_k(h_k^{-1}(0)) \). Since \( O_k \subset Z \) is an open subset, we can assign multiplicities to \( O_k \) so that as oriented current \( [O_k] = \iota_*(\sum_{k} [h_k^{-1}(0)]) \), where \( [h_k^{-1}(0)] \) is the current of the oriented manifold \( h_k^{-1}(0) \) with multiplicity one. Here the orientation of \( h_k^{-1}(0) \) is the one induced by the orientation of \( (R_k, W_k) \). Using the fact that \( R_{k,l} \to f_k^l(R_{k,l}) \) is a covering with \( m_k/m_l \) sheets, the assignments of the multiplicities of \( O_k \) and \( O_l \) over \( \bar{\iota}_k(R_k) \cap \bar{\iota}_l(R_l) \) coincide. Therefore \( Z \) is an oriented stratified set of pure dimension with rational multiplicities. We let \( [Z] \) be the corresponding current. It remains to check that \( \partial[Z] = 0 \) as current. Clearly, \( \partial[Z \cap O_k] \subset \text{cl}(O_k) - O_k \). Since \( O_k \cap Z \) is an open covering of \( Z \), \( \partial[Z] = 0 \) if \( Z \) is compact. But this is what we have assumed in the first place. Later, we will denote the so constructed cycle by

\[ [h^{-1}(0)] \in H_*(B, Q). \]

In the remainder of this section, we will perturb the section \( \phi : R \rightarrow M \) to a new section so that it is transversal to the zero section and so that its vanishing locus is compact. The current defined by the vanishing locus of the perturbed section will define the Euler class of \( [\Phi : B \rightarrow E] \).

We begin with a collection \( S = \{S_l\}_{l \in L} \) of symmetric open \( S_l \subset R_l \) such that \( \{\iota_l(S_l)\} \) cover \( \iota(\phi^{-1}(0)) \). For technical reason, we assume that for each \( k \in L \) the boundary \( \partial S_k \), which is defined to be \( \text{cl}(S_k) - S_k \), is a smooth manifold of dimension \( \dim S_k - 1 \). By slightly altering \( S_k \), if necessary, we can and do assume that \( \partial S_k \) is transversal to \( R_{k,l} \) along \( \partial S_k \cap (f_k^l)^{-1}(\text{cl}(S_l)) \) for all \( l < k \). (We will call such \( S \) satisfying the transversality condition on its boundary.) Following the convention, we set \( S_{k,l} := (f_k^l)^{-1}(S_l) \subset S_k \). We now construct a collection of (closed) tubular neighborhoods of \( S_{k,l} \) in \( R_k \). We fix the index \( k \) and consider the closed submanifold (with boundary) \( \Sigma_l := \text{cl}(S_{k,l}) \subset R_k \). Because of the transversality condition on \( \partial S_l \) and on \( \partial S_k \), we can find a \( Dh \)-bundle.
$p_i : T_i \to \Sigma_i$, where $D^h$ is the closed unit ball in $\mathbb{R}^h$ and $h = \dim R_k - \dim R_k, l$, and a smooth embedding $\eta : T_i \to R_k$ of which the following two conditions holds. First, the restriction of $\eta$ to the zero section $\Sigma_l \subset T_i$ is the original embedding $\Sigma_l \subset R_k$, and secondly

$$\eta(p_i^{-1}(\Sigma_l \cap \partial S_k)) \subset \partial S_k \eta(p_i^{-1}(S_k, l)) \subset S_k.$$  \hspace{1cm} (2.9)

For any $0 < \epsilon < 1$, we let $T_i^\epsilon \subset T_i$ be the closed $\epsilon$-ball subbundle of $T_i$. By abuse of notation, in the following we will not distinguish $T_i^\epsilon$ from its image $\eta(T_i^\epsilon)$ in $R_k$. We will call $T_i^\epsilon$ the $\epsilon$-tubular neighborhood of $\Sigma_l$ in $R_k$. One property we will use later is that if $R_k, l \cap R_k, l' \neq \emptyset$ for $l' < l < k$, then $R_k, l \cap R_k, l'$ is an open subset of $R_k, l'$, and hence for $0 < \epsilon \ll 1$ we have $\Sigma_l \cap T_i^\epsilon \subset \Sigma_l \cap \Sigma_i$.

Now consider $\Sigma_l \subset R_k$. Since $T_i$ is a disk bundle over $\Sigma_l$, it follows that we can extend the subbundle $(f_k^*)^*(W_l)_{\Sigma_l} \subset W_k|_{\Sigma_l}$ to a smooth subbundle of $W_k|_{T_i}$, denoted by $F_l \subset W_k|_{T_i}$. We then fix an isomorphism and the inclusion

$$p_i((f_k^*)^*(W_l)_{\Sigma_i}) \cong F_i.$$  \hspace{1cm} (2.10)

In this way, we can extend any section $\zeta$ of $(f_k^*)^*(W_l)_{\Sigma_i}$ to a section of $W_k|_{T_i}$ as follows. We first let $\zeta : T_i \to F_i$ be the obvious extension using the isomorphism (2.10). We then let $\zeta_{\text{ex}} : T_i \to W_k|_{T_i}$ be the induced section using the inclusion $F_l \subset W_k|_{T_i}$. We will call $\zeta_{\text{ex}}$ the standard extension of $\zeta$ to $T_i$. We fix a Riemannian metric on $R_k$ and a metric on $W_k$. For any section $\zeta$ as before, we say $\zeta$ is sufficiently small if its $C^2$-norm is sufficiently small. We now state a simple but important observation.

**Lemma 2.11.** Let the notation be as before. Then there is an $\epsilon > 0$ such that for any section $g : T_i \to F_i \subset W_k|_{T_i}$ such that $\|g\|_{C^2} < \epsilon$, the section $h_k|_{T_i} + g$ is non-zero over $T_i^\epsilon - \Sigma_i$.

**Proof.** This follows immediately from the fact that $\Sigma_l$ is compact and that for any $x \in R_k, l$ the differential

$$dh_k : T_x R_k / T_x R_k, l \to (W_k / (f_k^*)^*(W_l)|_x$$

is an isomorphism. \hfill \Box

We now state and prove the main proposition of this section.

**Proposition 2.12.** Let $\mathfrak{h} : \mathfrak{M} \to \mathfrak{M}$ be a regular section with $\mathfrak{h}^{-1}(0)$ proper, let $\mathfrak{M} \subset \mathfrak{M}$ be a good sub-atlas and let $\mathfrak{h}'$ be the the restriction of $\mathfrak{h}$ to $\mathfrak{M}'$. We assume that the vanishing locus of $\mathfrak{h}'$ is still proper. Then there is a smooth family of regular sections $g(t) : \mathfrak{M}' \to \mathfrak{M}'$, where $\mathfrak{M}'$ be the restriction of $\mathfrak{M}$ to $\mathfrak{M}'$, parameterized by $t \in [0, 1]$ such that

$$\bigcup_{t \in [0, 1]} g(t)^{-1}(0) \times \{t\} \subset B \times [0, 1]$$  \hspace{1cm} (2.11)

is compact, that $g(0) = \mathfrak{h}'$ and that $g(1)$ is transversal to the zero section of $\mathfrak{M}'$. \hfill 15
Proof. We will construct the perturbation over $R'_1$ and then successively extends it to the remainder of $\{R'_k\}$. We first fix a collection of symmetric open subsets $\{S_k\}_{k \in \mathcal{L}}$ such that $R'_k \subset S_k \subset R_k$ and that $S_k$ satisfies the transversality condition on its boundary. Let $k$ be any positive integer no bigger than $\#(\mathcal{L}) + 1$.

The induction hypothesis $\mathcal{H}_k$ states that for each integer $l < k$ we have constructed a symmetric open $S'_l$ satisfying $R'_l \subset S'_l \subset S_l$ and a smooth family of small enough sections $e_l(t) : R_l \to W_l$ such that $e_l(0) = 0$ of which the following holds. First, let $h_l(t) = h_l + e_l(t)$, then for any $l < m < k$ the section $h_m(t)|_{S'_m}$ is a regular extension of $(f'_m)^*(h_l(t))|_{S'_m}$; Secondly, for any $l < k$, the section $h_l(1)$ is transversal to the zero section of $W_l$ over $S'_l$, and finally, for any $l < k$ and $t \in [0, 1]$,

$$h_l(t)^{-1}(0) \cap (S'_l - R'_l) \subset \left(\bigcup_{i \leq l} (f'_i)^{-1}(R'_i)\right) \bigcup \left(\bigcup_{m \geq l} f'_m(R'_{m,l})\right).$$

(2.12)

Clearly, the condition $\mathcal{H}_3$ is automatically satisfied. Now assume that we have found $\{S'_l\}_{l < k}$ and $\{e_l\}_{l < k}$ required by the condition $\mathcal{H}_k$. We will demonstrate how to find $e_k$ and a new sequence of open subsets $\{S'_l\}_{l < k}$ so that the condition $\mathcal{H}_{k+1}$ will hold for $\{e_l\}_{l < k}$ and $\{S'_l\}_{l < k}$.

We continue to use the notation developed earlier. In particular, we let $\Sigma_l$ be the closure of $S_{k,l}$, let $T_l$ be the (closed) tubular neighborhood of $\Sigma_l \subset R_k$ with the projection $\pi_l : T_l \to \Sigma_l$ and let $F_l$ be the subbundle of $W_k|_{T_l}$ with the isomorphism \([2.10]\). Let $\zeta_l(t)$ be the standard extension of $(f'_l)^*(e_l(t))|_{\Sigma_l}$ to $T_l$. Note that $h_l|_{T_l} + \zeta_l(t)$ is a regular extension of $(f'_l)^*(h_l(t))|_{\Sigma_l}$. Because $(h_l)_{l < k}$ satisfies condition $\mathcal{H}_k$, for $l < m < k$ and $x \in \Sigma_l \cap \Sigma_m$ we have $(f'_l)^*(h_l(t))(x) = (f'_m)^*(h_m(t))(x)$. Now let

$$A_l = p_l^{-1}((f'_k)^{-1}(S'_l)) - \bigcup_{l < m < k} p_m^{-1}((f'_k)^{-1}(\text{cl}(R'_m))).$$

and let

$$B_l = \text{cl}(R'_{k,l}) - \bigcup_{k > m > l} (f'_k)^{-1}(S'_m).$$

Note that $\{A_l\}_{l < k}$ covers $\text{Int}(\cup_{l < k} T_l)$, that $B_l \subset A_l$ and that $\{B_l\}_{l < k}$ is a collection of compact subsets of $R_k$. Now let $\epsilon > 0$ be sufficiently small. We choose a collection of non-negative smooth functions $\{\rho_l\}_{l < k}$ that obeys the requirement that $\text{Supp}(\rho_l) \subset \text{Int}(A_l \cap T_l)$, that $\rho_l \equiv 1$ in a neighborhood of $B_l$ and that $\sum_{l < k} \rho_l \equiv 1$ in a neighborhood of $\cup_{l < k} \text{cl}(R'_{k,l})$. This is possible because the last set is compact and is contained in $\text{Int}(\cup_{l < k} A_l)$. We set

$$\zeta_l(t) = \sum_{l < k} \rho_l \cdot \zeta_l(t).$$

Now we check that for each $l < k$ the section $h_k + \zeta_l(t)$ is a regular extension of $(f'_k)^*(h_l(t))$ in a neighborhood of $\text{cl}(R'_{k,l})$. Let $x$ be any point in $\text{cl}(R'_{k,l})$. We first consider the case where $x$ is contained in $B_m$ for some $m \geq l$. Let $y =$
Note that \( y \in S'_m \). Then restricting to a sufficiently small neighborhood of \( x \) the section \( h_k + \zeta(t) \) is equal to \( h_k + \zeta_m(t) \). Since \( h_k + \zeta(t) \) is a regular extension of \( (f^m_k)^*(h_m(t)) \) near \( x \) and since \( h_m(t) \) is a regular extension of \( (f^m_k)^*(h_i(t)) \) in a neighborhood of \( y \in S'_m \), \( h_k + \zeta(t) \) is a regular extension of \( (f^m_k)^*(h_i(t)) \) near \( x \). We next consider the case where \( x \) is not contained in any of the \( B_m \)'s. Let \( \Lambda \) be the set of all \( m > l \) such that \( x \in (f_k^m)^{-1}(S'_m) \). Then for any \( m < k \) that is not in \( \Lambda \), \( \rho_m \equiv 0 \) in a neighborhood of \( x \). Here we have used the fact that \( \Sigma_m \cap T'_i \subset \Sigma_m \cap \Sigma_i \) for \( 0 < \epsilon \ll 1 \). On the other hand, by induction hypothesis for each \( m \in \Lambda \) the section \( h_k + \zeta_m(t) \) is a regular extension of \( (f^m_k)^*(h_i(t)) \) near \( x \). Therefore since \( \sum_{m \in \Lambda} \rho_m \equiv 1 \) near \( x \), in a small neighborhood of \( x \)

\[
h_k + \zeta(t) = \sum_{m \in \Lambda} \rho_m \cdot (h_k + \zeta_m(t))
\]

is also a regular extension of \( (f^m_k)^*(h_i(t)) \).

Our last step is to extend \( \zeta(t) \) to \( R_k \). We let \( e_k(t) \) be a smooth family of sufficiently small sections of \( W_k \) such that \( e_k(0) \equiv 0 \), that the restriction of \( e_k(t) \) to a neighborhood of \( \cup_{l < k} \text{cl}((f^m_k)^{-1}(R'_l)) \) is \( \zeta(t) \) and such that the section \( h_k(1) \) is transversal to the zero section in a neighborhood of \( \text{cl}(R'_k) \). The last condition is possible because \( h_k + \zeta(1) \) is transversal to the zero section in a neighborhood of \( \cup_{l < k} \text{cl}(R'_k) \). Therefore, by possibly shrinking \( S'_l \) while still keeping \( R'_l \subset S'_l \) for \( l < k \) if necessary, we can find an \( S'_k \subset S_k \) satisfying \( R'_k \subset S'_k \) such that the induction hypothesis \( \mathcal{H}_k \) holds for \( \{e_l\}_{l \leq k} \) and \( \{S'_l\}_{l \leq k} \), except possibly the third condition.

We now show that the third condition of \( \mathcal{H}_k \) holds as well. We only need to check the inclusion \( (2.12) \) for \( l = k \). First, by Lemma \( 2.9 \) we can find an open \( S \subset S_k \) such that \( R_k \subset S \) and that

\[
\mathcal{H}_k^{-1}(0) \cap \text{cl}(S) - R'_k \subset (\bigcup_{i < k} (f^i_k)^{-1}(R'_i)) \bigcup \bigcup_{i > k} f^i_k(R'_{i,k}).
\]  

(2.13)

Now let

\[
D_1 = \mathcal{H}_k^{-1}(0) \cap (\text{cl}(S) - R'_k) \bigcap (\bigcup_{i < k} (f^i_k)^{-1}(R'_i))
\]

and let

\[
D_2 = \mathcal{H}_k^{-1}(0) \cap (\text{cl}(S) - R'_k) \bigcap (\bigcup_{i > k} f^i_k(R'_{i,k})).
\]

Since \( \mathcal{H}_k(t) \) are small perturbations of \( h_k \), we can assume that \( \mathcal{H}_k(t) \) are chosen so that for any \( t \in [0,1] \) the left hand side of \( (2.13) \) is contained in the union of neighborhood \( V_1 \) of \( D_1 \) and a neighborhood \( V_2 \) of \( D_2 \). We remark that if we choose \( \{e_l\}_{l \leq k} \) so that their \( C^2 \)-norms are sufficiently small, then we can make \( V_1 \) and \( V_2 \) arbitrary small. Then by Lemma \( 2.11 \) the vanishing locus of \( \mathcal{H}_k(t) \) inside \( V_1 \) is contained in \( \cup_{i \leq k} (f^i_k)^{-1}(S'_i) \). On the other hand, since \( \cup_{i \geq k} f^i_k(R'_{i,k}) \) is open, it contains \( V_2 \) since \( D_2 \) is compact and \( V_2 \supset D_2 \) is sufficiently small. This proves the inclusion \( (2.12) \).
Therefore, by induction we have found \( \{ S_k' \}_{k \in \mathcal{L}} \) and \( \{ t_k(\ell) \}_{k \in \mathcal{L}} \) that satisfy the condition \( H_k \) for \( k = \#(\mathcal{L}) + 1 \). Now let \( g_0(\ell) = h_0(\ell) |_{R_k'} \). Then \( g(\ell) = \{ g(\ell) \}_{\ell \in \mathcal{L}} \) satisfies the condition of the proposition. Note that the left hand side of (2.11) is compact because it is contained in the union of compact sets \( \{ t_k(\text{cl}(R_k)) \}_{k \in \mathcal{L}} \). This proves the proposition. \( \square \)

Let \( g(\ell) \) be the perturbation constructed by Proposition 2.12 with \( h = \phi \). We define the Euler class of \( [\Phi : B \to E] \) to be the homology class in \( H_*(B; \mathbb{Q}) \) represented by the current \( [g(1)^{-1}(0)] \). In the remainder of this section, we will sketch the argument that shows that this class is independent of the choice of the chart \( R \) and the perturbation \( g \).

**Proposition 2.13.** Let the notation be as before. Then the homology class \( [g(1)^{-1}(0)] \in H_*(B; \mathbb{Q}) \) so constructed is independent of the choice of perturbations.

**Proof.** First, we show that if we choose two perturbations \( g_1(\ell) \) and \( g_2(\ell) \) based on identical sub-atlas \( \mathcal{R}' \in \mathcal{R} \) as stated in Proposition 2.12, then we have \( [g_1(1)^{-1}(0)] = [g_2(1)^{-1}(0)] \). To prove this, all we need is to construct a family of perturbations \( g_s(\ell) \), where \( s \in [0, 1] \), that satisfies conditions similar to that of the perturbations constructed in Proposition 2.12. Since then we obtain a current
\[
\bigcup_{s \in [0, 1]} \iota(g_s(1)^{-1}(0)) \times \{ s \} \subset B \times [0, 1]
\]
is a homotopy between the currents \( g_0(1)^{-1}(0)_{\text{cur}} \) and \( g_1(1)^{-1}(0)_{\text{cur}} \). The construction of \( g_s(\ell) \) is parallel to the construction of \( g(\ell) \) in Proposition 2.12 by considering the data over \( \{ R_k \times [0, 1] \}_{k \in \mathcal{L}} \).

Next, we show that the cycle \( [g(1)^{-1}(0)] \) does not depend on the choice of \( \mathcal{R}_1 \in \mathcal{R} \). Let \( \mathcal{R}_1 \in \mathcal{R} \) and \( \mathcal{R}_2 \in \mathcal{R} \) be two good sub-atlas and let \( g_1(\ell) \) and \( g_2(\ell) \) are two perturbations subordinate to \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) respectively. Clearly, we can choose a sub-atlas \( \mathcal{R}_0 \in \mathcal{R} \) such that \( \mathcal{R}_1 \subset \mathcal{R}_0 \) and \( \mathcal{R}_2 \subset \mathcal{R}_0 \). Let \( g_0(\ell) \) be a perturbation given by Proposition 2.12 subordinate to \( \mathcal{R}_0 \). Then \( g_0(\ell) \) is also subordinate to \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \). Hence by the previous argument
\[
[g_1(1)^{-1}(0)] = [g_0(1)^{-1}(0)] = [g_2(1)^{-1}(0)].
\]

It remains to show that the class \( [g(1)^{-1}(0)] \) does not depend on the choice of the good atlas \( \mathcal{R} \). For this, it suffices to show that for any two good atlas \( \mathcal{R} \) and \( \mathcal{R}' \) so that \( \mathcal{R} \) is finer than \( \mathcal{R}' \), the respective perturbations \( g(\ell) \) and \( g'(\ell) \) gives rise to identical homology classes \( [g(1)^{-1}(0)] = [g'(1)^{-1}(0)] \). Let \( \mathcal{R} = \{ R_k \}_{k \in \mathcal{K}} \) and \( \mathcal{R}' = \{ R_k' \}_{k \in \mathcal{K}} \), and let \( U_k \subset B \) (resp. \( U_l \subset B \)) be the open subsets so that \( (R_k, W_k, \phi_k) \) (resp. \( (R_l, W_l, \phi_l) \)) are the smooth approximations of \( [\Phi : B \to E] \) over \( U_k \) (resp. \( U_l \)) for \( k \in \mathcal{K} \) (resp. \( l \in \mathcal{L} \)). As before, we denote \( t_k : R_k \to U_k \) be the tautological map with \( k \in \mathcal{K} \) or \( k \in \mathcal{L} \). Let \( U_{kl} = U_k \cap U_l \). We consider the good atlas \( \mathcal{R}_0 \) with charts \( R_{kl} = t_k^{-1}(U_{kl}) \), where \((k, l) \in \mathcal{K} \times \mathcal{L} \), with bundles \( W_{kl} = W_k |_{R_{kl}} \) and \( \phi_{kl} \) the restriction of \( \phi_k \), where \( R_{kl} \) is considered to be an open subset of \( R_k \). Using the extension
technique in the proof of Proposition 2.12 we can construct a perturbation \( g_0(t) \) that is a regular extension of \( g'(1)^{-1}(0) \) under the obvious \( \iota_1^{-1}(U_{kl}) \to R_{kl} \) and \( W_l|_{\iota_l^{-1}(U_{kl})} \subset W_{kl} \). Therefore, \( [g_0(1)^{-1}(0)] = [g'(1)^{-1}(0)] \). On the other hand, since \( W_{kl} = W_k|_{R_{kl}} \), \( g(t) \) induces a perturbation \( g'_0(t) \) subordinate to \( R_0 \). Hence, \( [g_0(1)^{-1}(0)] = [g_0'(1)^{-1}(0)] = [g(1)^{-1}(0)] \). This proves the proposition. □

3 Analytic charts

The goal of this section is to construct a collection of local smooth approximations \((\Lambda, V)\) so that the data \( \phi_V : R_V \to V|_{R_V} \) are analytic. Namely, \( R_V \) are complex manifolds, \( V|_{R_V} \) are holomorphic vector bundles and \( \phi_V \) are holomorphic sections. In the next section we will show that such \( \phi_V \)'s are Kuranishi maps, and hence the cones \( \lim_{t \to \infty} \Gamma_{t\phi} \) are the virtual cones constructed in [T1].

We will use the standard notation in complex geometry in this section. For instance, if \( M \) is a complex manifold, we will denote by \( T_x M \) the complex tangent space of \( M \) at \( x \) unless otherwise is mentioned. We will use complex dimension throughout this section, unless otherwise is mentioned. Accordingly the complex dimension of a set is half of its real dimension. We will use the words analytic and holomorphic interchangeably in this section as well.

We begin with the construction of such local smooth approximations. Let \( w \in B \) be any point representing a holomorphic stable map \( f : \Sigma \to X \) with \( n \)-marked points. We pick a uniformizing chart \( \Lambda = (\tilde{U}, E_\tilde{U}, \Phi_\tilde{U}, G_\tilde{U}) \) of \( w \) over \( U \subset B \) such that the elements of \( \tilde{U} \) are stable maps \( f_1 : \Sigma_1 \to X \) with \((n + k)\)-marked points \( \{x_i\} \) so that \( \{f_1(x_m)\}_{m=n+1}^{n+k} \) are the \( k \)-distinct points of \( f_1^{-1}(H) \), where \( H \) is a smooth complex hypersurface of \( X \) in general position of degree \( k = |H| \cdot |A| \) and \( A = f_1(\Sigma) \), and that the stable maps resulting from discarding the last \( k \) marked points of \( f_1 \) are in \( U \). Here as usual we assume that \( U \) is sufficiently small so that all stable maps in \( U \) intersect \( H \) transversally and positively. Note that the later correspondence is the projection \( \pi_U : U \to \tilde{U} \).

Let \( \mathcal{Y} \) be the universal (continuous) family of curves with \((n + k)\) marked sections and let \( F : \mathcal{Y} \to X \) be the universal map.

We let \( \pi : \tilde{U} \to M_{g,n+k} \) be the tautological map induced by the family \( \mathcal{Y} \) with its marked sections. Here \( M_{g,n+k} \) is the moduli space of \((n + k)\)-pointed stable curves of genus \( g \). Without loss of generality, we can assume that no fibers of \( \mathcal{Y} \) with the marked points have non-trivial automorphisms. It follows that \( M_{g,n+k} \) is smooth near \( \pi(\tilde{U}) \). As in section 1, we view \( G_\tilde{U} \) as a subgroup of \( S_{n+k} \). Then \( G_\tilde{U} \) acts on \( M_{g,n+k} \) by permuting the \((n + k)\)-marked points of the curves in \( M_{g,n+k} \), and the map \( \pi : \tilde{U} \to M_{g,n+k} \) is \( G_\tilde{U} \)-equivariant. Now let \( O \subset M_{g,n+k} \) be a smooth \( G_\tilde{U} \)-invariant open neighborhood of \( \pi(\tilde{U}) \subset M_{g,n+k} \) and let \( p : \mathcal{X} \to O \) be the universal family of stable curves over \( O \) with \((n + k)\) marked sections (In this section we will work with the analytic category unless otherwise is mentioned). It follows that the \( G_\tilde{U} \)-action on \( O \) lifts to \( \mathcal{X} \) that permutes its marked sections. For convenience, we let \( \mathcal{X} \times_O \tilde{U} \) be the topological
subspace of $\mathcal{X} \times \tilde{U}$ that is the preimage of $\Gamma_{\pi} \subset O \times \tilde{U}$ under $\mathcal{X} \times \tilde{U} \to O \times \tilde{U}$, where $\Gamma_{\pi} \subset O \times \tilde{U}$ is the graph of $\pi: \tilde{U} \to O$. Since no fibers of $\mathcal{Y}$ (with marked points) have non-trivial automorphisms, there is a canonical $G_{\tilde{U}}$-equivariant isomorphism $\mathcal{Y} \cong \mathcal{X} \times_O \tilde{U}$ as family of pointed curves. Let $\pi_X$ and $\pi_{\tilde{U}}$ be the first and the second projection of $\mathcal{X} \times_O \tilde{U}$.

Next, we let $(\mathcal{X}_n, O_n; \Sigma, p_n, \varphi_n)$ be a semi-universal family of the $n$-pointed curve $\Sigma$. Namely, $\mathcal{X}_n$ is a (holomorphic) family of pointed prestable curves over the pointed smooth complex manifold $p_n \in O_n$ whose dimension is equal to $\dim C \operatorname{Ext}^1(\Omega_{\Sigma}(D), \Omega_\Sigma)$, where $D \subset \Sigma$ is the divisor of the $n$-marked points of $\Sigma$, $\varphi_n: \Sigma \to \mathcal{X}_n|_{p_n}$ is an isomorphism of $\Sigma$ with the fiber of $\mathcal{X}_n$ over $p_n$ as $n$-pointed curve, and the Kuranishi map $T_{p_n}O_n \to \operatorname{Ext}^1(\Omega_{\Sigma}(D), \Omega_\Sigma)$ of the family $\mathcal{X}_n$ is an isomorphism. Note that $G_{\tilde{U}}$ acts canonically on $\Sigma$. For convenience, we let $\Pi_n(\mathcal{X})$ be the family of curves over $O$ that is derived from $\mathcal{X}$ by discarding its last $k$-marked sections. We now let $B = \pi_{\tilde{U}}^{-1}(w)$ and fix a $G_{\tilde{U}}$-equivariant isomorphism

$$\prod_{z \in B} \Pi_n(\mathcal{X})|_z \longrightarrow B \times \Sigma \quad (3.1)$$

over $B$. Let $\operatorname{Aut}_{p_n}(\mathcal{X}_n)$ be the group of those biholomorphisms of $\mathcal{X}_n$ that keep the fiber $\mathcal{X}_n|_{p_n}$ invariant, that send fibers of $\mathcal{X}_n$ to fibers of $\mathcal{X}_n$ and that fix the $n$-sections of $\mathcal{X}_n$. Possibly after shrinking $O_n$ if necessary, we can assume that there is a homomorphism $\rho: G_{\tilde{U}} \to \operatorname{Aut}_{p_n}(\mathcal{X}_n)$ such that for any $\sigma \in G_{\tilde{U}}$ the $\rho(\sigma)$ action on $\mathcal{X}_n|_{p_n}$ is exactly the $\sigma$ action on $\Sigma$ via the isomorphism $\varphi_n$. Finally, possibly after shrinking $U$ and $O$, we can pick a $G_{\tilde{U}}$-equivariant holomorphic map $\varphi: O \to O_n$ such that $\varphi(B) = p_n$ and that there is a $G_{\tilde{U}}$-equivariant isomorphism of $n$-pointed curves $\tilde{\varphi}: \tilde{\mathcal{X}} \to O \times O_n \mathcal{X}_n$ that extends the isomorphism (3.1). We remark that the reason for doing this is to ensure that the smooth approximation we are about to construct is $G_{\tilde{U}}$-equivariant.

Next, we let $l$ be an integer to be specified later and let $U_i \subset \Sigma$, $i = 1, \ldots, l$, be $l$ disjoint open disks away from the marked points and the nodal points of $\Sigma$. We assume that $\bigcup_{i=1}^l U_i$ is $G_{\tilde{U}}$-invariant and that for any $\sigma \in G_{\tilde{U}}$ whenever $\sigma(U_i) = U_i$ then $\sigma|_{U_i} = 1_{U_i}$. By shrinking $\tilde{U}$, $O$ and $O_n$ if necessary, we can find disjoint open subsets $U_{n,i} \subset \mathcal{X}_n$ such that $\bigcup_{i=1}^l U_{n,i}$ is $G_{\tilde{U}}$-invariant, that $\bigcup_{i=1}^l U_{n,i}$ is $G_{\tilde{U}}$-equivariantly biholomorphic to $O \times \bigcup_{i=1}^l U_i$, that $U_{n,i} \cap \Sigma = U_i$ and that the projections $U_{n,i} \to O$ induced by the projection $\mathcal{X} \to O$ is the first projection of $O \times U_i (= U_{n,i})$. For convenience, for each $i$ we will fix a biholomorphism between $U_i$ and the unit disk in $C$, and will denote by $U_i^{1/2}$ the open disk in $U_i$ of radius $1/2$. We let $U_i$ be the disjoint open subsets of $\mathcal{Y}$ defined by

$$U_i = U_{n,i} \times_{O_n} \tilde{U} \subset \mathcal{X}_n \times_{O_n} \tilde{U} \cong \mathcal{X} \times_O \tilde{U} \cong \mathcal{Y}.$$ We will call $U_i$ and $U_{i}'$ the distinguished open subsets of $\Sigma$ and $\mathcal{Y}$ respectively. Without loss of generality, we can assume that $\bigcup_{i=1}^l U_i$ is disjoint from the $(n+k)$-sections of $\mathcal{Y}$. We also assume that there are holomorphic coordinate charts $V_i \subset X$ so that $\mathcal{F}(U_i) \subset V_i$. We let $(w_i, \ldots, w_{i,m})$, where $m = \dim X$, be the coordinate variable of $V_i$ and let $v_i = \partial/\partial w_{i,1}$. For each $i$ we pick a nontrivial
(0, 1)-form $\gamma_i$ on $U_i$ with $\text{Supp} (\gamma_i) \subseteq U_i^{1/2}$. We demand further that if there is a $\sigma \in G_\mathcal{U}$ so that $\sigma (U_i) = U_j$ then $V_i = V_j$, as coordinate chart and $\sigma^* (\gamma_i) = \gamma_j$. We then let $\sigma_i$ be the $\{0, 1\}$-form over $\mathcal{U}_i$ with values in $\mathcal{F}^* (X) |_{\mathcal{U}_i}$ that is the product of the pull back of $\gamma_i$ via $\mathcal{U}_i \times_X \bar{U} \rightarrow U_i$ with $\mathcal{F}^* (V_i) |_{\mathcal{U}_i}$, and let $\bar{\sigma}_i$ be the section over $\mathcal{U}$ that is the extension of $\sigma_i$ by zero. Obviously, $\bar{\sigma}_i$ is a section of $\mathcal{E}_0$, and $(\bar{\sigma}_1, \ldots, \bar{\sigma}_l)$ is linearly independent. Hence it spans a complex subbundle of $\mathcal{E}_0$, denoted by $V$. It follows from the construction that $V$ is $G_\mathcal{U}$-equivariant.

As in the previous section, we let $R = \Phi_0^{-1} (V)$, let $W = V |_R$ and let $\phi : \mathcal{U}_i \rightarrow W$ be the lifting of $\Phi_0 |_{\mathcal{U}_i} : \mathcal{U}_i \rightarrow \mathcal{E}_0 |_{\mathcal{U}_i}$. The main task of this section is to show that we can choose $\hat{U}_i$, $\gamma_i$ and $V_i$ so that $R$ admits a canonical complex structure and that the section $\phi$ is holomorphic when $W$ is endowed with the holomorphic structure so that the basis $\hat{\sigma}_1 |_R, \ldots, \hat{\sigma}_l |_R$ is holomorphic.

To specify our choice of $U_i$, $\gamma_i$ and $V_i$, we need first to define the Dolbeault cohomology of holomorphic vector bundles over singular curves. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_\Sigma$-modules and let $E$ be the associated vector bundle, namely, $\mathcal{O}_\Sigma (E) = \mathcal{E}$. We let $\Omega_{\text{cpt}}^0 (E)$ be the sheaf of smooth sections of $E$ that are holomorphic in a neighborhood of $\text{Sing} (\Sigma)$ and let $\Omega_{\text{cpt}}^{0, 1} (E)$ be the sheaf of smooth sections of $(0, 1)$-forms with values in $E$ that vanish in a neighborhood of $\text{Sing} (\Sigma)$. Let

$$\bar{\partial} : \Gamma (\Omega_{\text{cpt}}^0 (E)) \rightarrow \Gamma (\Omega_{\text{cpt}}^{0, 1} (E))$$

be the complex that send $\varphi \in \Omega_{\text{cpt}}^0 (E))$ to $\bar{\partial} (\varphi)$. Since $\varphi$ is holomorphic near nodes of $\Sigma$, $\bar{\partial} (\varphi)$ vanishes near nodes of $\Sigma$ as well, and hence the above complex is well defined. We define the Dolbeault cohomology $H^0_\partial (E)$ and $H^1_\partial (E)$ to be the kernel and the cokernel of $\bar{\partial}$.

**Lemma 3.1.** Let $H^1 (\mathcal{E})$ be the Cech cohomology of the sheaf $\mathcal{E}$. Then there are canonical isomorphisms $H^0_\partial (E) \cong H^0 (\mathcal{E})$ and $\Psi : H^{0, 1}_\partial (E) \cong H^1 (\mathcal{E})$.

**Proof.** The proof is identical to the proof of the classical result that the Dolbeault cohomology is isomorphic to the Cech cohomology for smooth complex manifolds. Obviously, $H^0_\partial (E)$ is canonically isomorphic to $H^0 (\mathcal{E})$. We now construct $\Psi$. We first cover $\Sigma$ by open subsets $\{ W_i \}$ so that the intersection of any of its subcollection is contractible. Now let $\varphi$ be any global section in $\Omega^{0, 1}_{\text{cpt}} (E)$. Then over each $W_i$ we can find a smooth function $\eta_i \in \Gamma (W_i \Omega^0_{\text{cpt}} (E))$ such that $\bar{\partial} \eta_i = \varphi |_{W_i}$. Clearly, the class in $H^1 (\mathcal{E})$ represented by the cocycle $[ \eta_{ij} ]$, where $\eta_{ij} = \eta_i |_{W_i \cap W_j} - \eta_j |_{W_i \cap W_j}$, is independent of the choice of $\eta_i$, and thus defines a homomorphism $\Gamma (\Omega^{0, 1}_{\text{cpt}} (E)) \rightarrow H^1 (\mathcal{E})$. It is routine to check that it is surjective and its kernel is exactly $\text{Im} (\bar{\partial})$. Therefore, we have $H^{0, 1}_\partial (E) \cong H^1 (\mathcal{E})$. Also, it is direct to check that this isomorphism does not depend on the choice of the covering $\{ W_i \}$. This proves the lemma. \square

For any $z \in \bar{U}$, we denote by $\bar{\sigma}_i (z)$ the restriction of $\bar{\sigma}_i$ to the fiber of $\mathcal{U}$ over $z$. We now choose the $l$ open disks $U_i \subset \Sigma$, the $(0, 1)$-forms $\gamma_i$ on $U_i$ and the coordinate charts $V_i \subset X$ such that for any $w \in \bar{\pi}_U^{-1} (w)$ the collection
and the Riemann-Roch theorem. This is certainly possible if we choose $l$ large because the locus of $U_i$ are arbitrary as long as they are away from the nodal points of $\Sigma$ and the marked points, and the charts $V_i$ can also be chosen with a lot of choice. We fix once and for all such choices of $U_i$, $V_i$ and $\gamma_i$. We then let $U_i \subset \mathcal{Y}$, $V \to E_{\mathcal{U}}$ and $R = \Phi_{\mathcal{U}}^{-1}(V)$ be the objects constructed before according to this choice of $U_i$, $\gamma_i$ and $V_i$. Let $\mathcal{Y}_R \to R$ be the restriction to $R \subset \hat{U}$ of the family $\mathcal{Y} \to \hat{U}$ with the marked sections and let $F : \mathcal{Y}_R \to X$ be the associated map. We also fix a smooth function $\eta_i$ over $U_i$ so that $\partial \eta_i = \gamma_i$. We extend the collection $\{U_i\}_{i=1}^l$ to an open covering $\{U_i\}_{i=1}^L$ so that the intersection of any subcollection of $\{U_i\}$ is contractible, and that for any $i \leq l$ and $j \geq l + 1$ the sets $U_i^{1/2}$ and $U_j$ are disjoint. For convenience, we agree that $\eta_j = 0$ for $j > l$.

From now on, we will fix an $\tilde{w} \in R$ over $w$.

**Lemma 3.2.** There is a constant $A$ such that for any Čech 1-cocycle $[\tau_{ij}]$, where $\tau_{ij} \in \Gamma_{U_i \cap U_j}(f^*T_X)$, there are constants $a_i$ and holomorphic sections $\zeta_i \in \Gamma_{U_i}(f^*T_X)$ for $i = 1, \ldots, L$ such that

$$
(\zeta_j + a_j \eta_j)|_{U_j \cap U_i} - (\zeta_i + a_i \eta_i)|_{U_i \cap U_j} = \tau_{ji}
$$

and

$$
\sum_{i=1}^L (\| \zeta_i \|_{L_2} + |a_i|) \leq A \left( \sum_{i,j} \| \tau_{ij} \|_{L_2} \right).
$$

**Proof.** The existence of $\{a_i\}$ and $\{\zeta_i\}$ follows from the fact that the images of $\tilde{\sigma}_1(\tilde{w}), \ldots, \tilde{\sigma}_l(\tilde{w})$ spans $H^{0,1}_\mathcal{U}(f^*T_X)$ and that $H^{0,1}_\mathcal{U}(f^*T_X)$ is isomorphic to $H^1(f^*T_X)$. The elliptic estimate is routine, using the harmonic theory on the normalization of $\Sigma$. We will leave the details to the readers. \qed

We let $\Gamma(\Omega^{0,1}_{\text{cpt}}(f^*T_X))^\dagger$ be the quotient of $\Gamma(\Omega^{0,1}_{\text{cpt}}(f^*T_X))$ by the linear span of $\tilde{\sigma}_1(\tilde{w}), \ldots, \tilde{\sigma}_l(\tilde{w})$. Because $\{\tilde{\sigma}_i(\tilde{w})\}_{i=1}^L$ is invariant under the automorphism group of the stable map $f$, $\Gamma(\Omega^{0,1}_{\text{cpt}}(f^*T_X))^\dagger$ is independent of the choice of $\tilde{w} \in \pi_{\mathcal{U}}^{-1}(w)$. We let

$$
\tilde{\partial}^! : \Gamma(\Omega^{0,1}_{\text{cpt}}(f^*T_X)) \to \Gamma(\Omega^{0,1}_{\text{cpt}}(f^*T_X))^\dagger
$$

be the induced complex. We define $H^0_{\partial}(f^*)^\dagger$ and $H^{0,1}_{\partial}(f^*T_X)^\dagger$ be the kernel and the cokernel of the above complex.

**Corollary 3.3.** Let the notation be as before. Then $H^{0,1}_{\partial}(f^*T_X)^\dagger = 0$. Further, the complex dimension of $H^0_{\partial}(f^*T_X)^\dagger$ is $\deg(f^*T_X) + m(1 - g) + l$.

**Proof.** The vanishing of $H^{0,1}_{\partial}(f^*T_X)^\dagger$ follows from the surjectivity of $\tilde{\partial}^!$. The second part follows from

$$
\dim H^0_{\partial}(f^*T_X)^\dagger - \dim H^{0,1}_{\partial}(f^*T_X)^\dagger = \dim H^0_{\partial}(f^*T_X) - \dim H^{0,1}_{\partial}(f^*T_X) + l = \chi(f^*T_X) + l
$$

and the Riemann-Roch theorem. \qed
Next, we will describe the tangent space of $R$ at $\tilde{w}$. By the smoothness result of [LT2], we know that $R$ is a smooth manifold of (complex) dimension $r_{\text{exp}}$. As before, we let $D \subset \Sigma$ be the divisor of the first $n$-marked points of $\tilde{w}$. Since $f$ is holomorphic, $df^\vee$ is a homomorphism of sheaves $f^*\Omega_X \to \Omega_\Sigma$. We let

$$D^*_{\tilde{w}} = [f^*\Omega_X \xrightarrow{\eta} \Omega_\Sigma(D)]$$

be the induced complex indexed at $-1$ and $0$. We will first define the extension space $\text{Ext}^1(D^*_{\tilde{w}}, \Omega_\Sigma)^{\dagger}$ and then show that it is canonically isomorphic to $T_{\tilde{w}}R$.

We begin with some more notations. Let $E$ be a sheaf of $\mathcal{O}_\Sigma$-modules that is locally free away from the nodal points of $\Sigma$. Then there is a holomorphic vector bundle $E$ over $\Sigma^0$, where $\Sigma^0$ is the smooth locus of $\Sigma$, such that $\mathcal{O}_{\Sigma^0}(E) = \mathcal{E}|_{\Sigma^0}$.

We define $\mathcal{E}^A$ to be the sheaf so that the germs of $\mathcal{E}^A$ at nodal points $p \in \Sigma$ (resp. smooth points $p \in \Sigma^0$) are isomorphic to the germs of $\mathcal{E}$ at $p$ (resp. germs of $\mathcal{O}_{\Sigma^0}(E)$ at $p$). The set $\text{Ext}^1(D^*_{\tilde{w}}, \Omega_\Sigma)^{\dagger}$ is the set of equivalence classes of pairs $(v_1, v_2)$ as follows. The data $v_1$ is an element in $\text{Ext}^1(\Omega_\Sigma(D), \Omega_\Sigma)$, which defines an exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma \xrightarrow{v_1} \mathcal{B} \xrightarrow{v_2} \Omega_\Sigma(D) \longrightarrow 0,$$

and equivalently a family of $n$-pointed nodal curves over $T = \text{Spec } \mathbb{C}[t]/(t^2)$, say $\mathcal{C}_T$ with $n$-marked sections $\tilde{x}_i$ (See [LT1] section 1). Note that $\mathcal{B}$ is locally free over $\Sigma^0$. The data $v_2$ is a homomorphism $f^*\Omega_X \to \mathcal{B}^A$ such that, first of all, the diagram

$$\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}_\Sigma^A \\
\downarrow & & \downarrow \quad df^\vee \\
0 & \longrightarrow & \mathcal{B}^A
\end{array}$$

is commutative, where the lower sequence is induced by (3.2). Secondly, since $v_2$ is holomorphic near nodes of $\Sigma$, the differential $\partial v_2$ vanishes near nodes of $\Sigma$, and since $df^\vee$ is holomorphic, $\partial v_2$ lifts to a global section $\beta$ of $\Omega^0_{\text{cpt}}(f^*T_X)$. We require that there are constants $a_1, \ldots, a_l$ such that $\beta = \sum a_i \tilde{\sigma}_i(\tilde{w})$.

The equivalence relation of such pairs are the usual equivalence relation of the diagrams (3.3). Namely, two pairs $(v_1, v_2)$ and $(v'_1, v'_2)$ with the associated data $\{\mathcal{B}, \varphi_1\}$ and $\{\mathcal{B}', \varphi'_1\}$ are equivalent if there is an isomorphism $\eta : \mathcal{B} \to \mathcal{B}'$ so that $\eta \circ \varphi_1 = \varphi'_1$, $\varphi_2 = \varphi'_2 \circ \eta$ and $\eta \circ v_2 = v'_2$.

**Lemma 3.4.** Let the notation be as before. Then $\text{Ext}^1_{\Sigma}(D^*_{\tilde{w}}, \mathcal{O}_\Sigma)^{\dagger}$ is canonically a complex vector space of complex dimension $r_{\text{exp}}$.

**Proof.** The fact that $\text{Ext}^1_{\Sigma}(D^*_{\tilde{w}}, \mathcal{O}_\Sigma)^{\dagger}$ forms a complex vector space can be established using the usual technique in homological algebra. For instance, if $r \in \text{Ext}^1_{\Sigma}(D^*_{\tilde{w}}, \mathcal{O}_\Sigma)^{\dagger}$ is represented by $\{\mathcal{B}, \varphi_1, v_2\}$ shown in the diagram (3.3), then for any complex number $a$ the element $ar$ is represented by the same diagram with $\varphi_1$ replaced by $a \varphi_1$. We now prove that

$$\dim \text{Ext}^1_{\Sigma}(D^*_{\tilde{w}}, \mathcal{O}_\Sigma)^{\dagger} = r_{\text{exp}}. \quad (3.4)$$
Clearly, the following familiar sequence is still exact in this case:

\[
\text{Ext}^0(D_f^†, \mathcal{O}_X^\natural) \rightarrow \text{Ext}^0(\Omega_{\Sigma}(D), \mathcal{O}_\Sigma) \rightarrow H^0_\partial(f^*T_X) = 0
\]

Since \( f \) is stable, \( \text{Ext}^0(D_f^†, \mathcal{O}_X^\natural) = 0 \). Hence (3.4) follows from Corollary 3.3 and the Riemann-Roch theorem. This proves the lemma.

Recall that should \( R \) be a scheme then the Zariski tangent space of \( T_{\bar{w}}R \) would be the space of morphisms \( \text{Spec} \mathbb{C}[t]/(t^2) \rightarrow R \) that send their only closed points to \( \bar{w} \) modulo certain equivalence relation. In the following, we will imitate this construction and construct the space of pre-\( \mathbb{C} \)-tangents of \( R \) at \( \bar{w} \). We still denote by \( U_1, \ldots, U_l \) the \( l \)-distinguished open subsets of \( \Sigma \) and let \( \{U_i\}_{i=1}^L \) be an extension of \( \{U_i\}_{i=1}^l \) to an open covering of \( \Sigma \) such that the intersection of any of its subcollection are contractible. Without loss of generality, we assume \( U_j \cap U_i^{1/2} = \emptyset \) for \( j > l \) and \( i \leq l \). We also assume that there are coordinate charts \( V_i \) of \( X \) such that \( f(U_i) \subseteq V_i \). By abuse of notation, we will fix the embedding \( V_i \subset \mathbb{C}^m \) and view any map to \( V_i \) as a map to \( \mathbb{C}^m \). We let \( \iota_i : V_i \rightarrow X \) be the tautological inclusion and let

\[
g_{ij} : \iota_j^{-1}(\iota_i(V_i)) \rightarrow \iota_i^{-1}(\iota_j(V_j)) \subset V_i \subset \mathbb{C}^m
\]

be the transition functions of \( X \).

We define a pre-\( \mathbb{C} \)-tangent \( \xi \) of \( R \) at \( \bar{w} \) to be a collection of data as follows:
First, there is a flat analytic family of \( n \)-pointed pre-stable curves \( C_T \) over an open neighborhood \( T \) of \( \bar{w} \) such that the fiber of \( C_T \) over \( 0 \), denoted by \( C_0 \), is isomorphic to \( \Sigma \) as \( n \)-pointed curve; Secondly, there is an open covering \( \{U_i\}_{i=1}^L \) of \( C_T \) such that \( U_j \cap C_0 = U_i \), and that for each \( i \leq l \), there is a biholomorphism \( U_i \cong U_i \times T \) such that its restriction to \( U_i = U_i \cap C_0 = U_i \) is compatible to the identity map of \( U_j \); Thirdly, there is a collection of smooth maps \( \bar{f}_i : \bar{U}_i \rightarrow V_i \) such that for \( i > l \), all \( \bar{f}_i \) are holomorphic and that for each \( i \leq l \) we have \( \partial_0(\bar{f}_i) = 0 \) and

\[
\tilde{\partial}_i(\bar{f}_i) = \pi_{T}^* \gamma_i \cdot \pi_{U_i}^* (\gamma_i \cdot f^* (\nu_i)), \tag{3.5}
\]

where \( \pi_{U_i} \) and \( \pi_T \) are the first and the second projection of \( U_i \times T \), \( \varphi_i \) are holomorphic functions over \( T \) and \( \partial_0 \) (resp. \( \partial_1 \)) is the \( \bar{\partial} \)-differential with respect to the holomorphic variable of \( T \) (resp. \( U_i \)) using \( U_i \cong U_i \times T \) and the \( \gamma_i \) and \( \nu_i \) are the \((0,1)\)-form and the vector field chosen before; Forthly, if we let \( z_0 \) be the holomorphic variable of \( \mathbb{C} \supset T \), then we require that

\[
\tilde{f}_{ij} = \bar{f}_i - g_{ij} \circ \bar{f}_j : \bar{U}_{ij} \rightarrow \mathbb{C}^m, \tag{3.6}
\]

where \( \bar{U}_{ij} \) is a neighborhood of \( U_i \cap U_j \) in \( \bar{U}_i \cap \bar{U}_j \) over which \( \tilde{f}_{ij} \) is well-defined, is divisible by \( z_0^n \) (Namely, \( \tilde{f}_{ij} \) has the form \( \pi_T^* (z_0^n) \cdot h_{ij} \) for some smooth function \( h_{ij} : \bar{U}_{ij} \rightarrow \mathbb{C}^m \)). Intuitively, a pre-\( \mathbb{C} \)-tangent is a scheme analogue of a morphism \( \text{Spec} \mathbb{C}[t]/(t^2) \rightarrow R \) should \( R \) be a scheme. We denote the set of all pre-\( \mathbb{C} \)-tangents by \( T_{\bar{w}}^\text{pre} R \). Note that \( T_{\bar{w}}^\text{pre} R \) is merely a collection of all pre-\( \mathbb{C} \)-tangents.
We next define a canonical map

\[
T_{\widetilde{w}}^{\text{pre}} R \longrightarrow \text{Ext}^1(\mathcal{D}_{\widetilde{w}}^\bullet, \mathcal{O}_{\Sigma})^\dagger. \tag{3.7}
\]

Let \( \xi \) be any pre-C-tangent given by the data above. By the theory of deformation of \( n \)-pointed curves [LT1, section 1], the analytic family \( C_T \) defines canonically an exact sequence

\[
0 \longrightarrow \mathcal{O}_{\Sigma} \longrightarrow \mathcal{B} \longrightarrow \Omega_{\Sigma}(D) \longrightarrow 0, \tag{3.8}
\]

associated to an extension class \( v_1(\xi) \in \text{Ext}^1(\Omega_{\Sigma}(D), \mathcal{O}_{\Sigma}) \), where away from the nodes of \( \Sigma \) and the support of \( D \) the sheaf \( \mathcal{B} \) is canonically isomorphic to \( \Omega_{C_T} \otimes_{\mathcal{O}_{C_T}} \mathcal{O}_{C_S}. \) Because \( f_i : \tilde{U}_i \to V_i \) are holomorphic for \( i > l \), it follows from [LT1] that there is a canonical homomorphism of sheaves \( u_i : f^*\Omega_X|_{U_i} \to \mathcal{B}|_{U_i} \) such that

\[
f^*\Omega_X|_{U_i} \xrightarrow{u_i} f^*\Omega_X|_{U_i} \xrightarrow{d\Gamma_{|U_i}} \Omega_{U_i}(D)^{\mathcal{A}} \longrightarrow 0
\]

is commutative, where the lower sequence is induced by (3.8). Indeed, at smooth point \( p \in U_i \) away from the support of \( D \) the dual of \( u_i \otimes k(p) \) is the differential

\[
df_i(p) : T_p C_T = \mathcal{B}^\vee \otimes k(p) \longrightarrow f^*T_X|_p.
\]

Note that by our choice of \( U_i \), for \( i \leq l \) the distinguished open subsets \( U_i \) are disjoint from the support of the \( (n + k) \)-marked points of \( \tilde{w} \). Hence \( \mathcal{B}^{\mathcal{A}}|_{U_i} \) are canonically isomorphic to \( \Omega_{U_i}^\dagger(\mathcal{B} |_{U_i}) \), and the dual of \( df_i \) define canonical homomorphisms \( u_i : f^*\Omega_X|_{U_i} \to \mathcal{B}|_{U_i} \) that make the above diagrams commutative. Because of the condition (3.5), the lift of \( \partial u_i \) is a constant multiple of \( \sigma_i(\tilde{w})|_{U_i} \). Further, because of the condition that \( \tilde{f}_i \) is divisible by \( \tilde{z}^\vee_0 \), the collection \( \{ u_i \}_{i \leq l} \) patch together to form a homomorphism \( v_2(\xi) : f^*\Omega_X \to \mathcal{B}^{\mathcal{A}} \) that makes the diagram (3.3) commutative. Hence \( (v_1(\xi), v_2(\xi)) \) defines an element in \( \text{Ext}^1(\mathcal{D}_{\widetilde{w}}^\bullet, \mathcal{O}_{\Sigma})^\dagger \), which is defined to be the image of \( \xi \).

We remark that in this construction we have only used the fact that the stable map associated to \( \tilde{w} \) is holomorphic, that the domain \( \Sigma \) of \( \tilde{w} \) has \( l \) distinguished open subsets \( U_i \) with \( (0, 1) \)-forms \( \tilde{\sigma}_i(\tilde{w}) \). Because for any \( z \in R \) its domain \( \Sigma_z \) also has \( l \) distinguished open subsets, namely \( U_i \cap \Sigma_z \cong U_i \), and the forms \( \tilde{\sigma}(z) \), we can define the extension group \( \text{Ext}^1(\mathcal{D}_{\widetilde{w}}^\bullet, \mathcal{O}_{\Sigma})^\dagger \), the space of pre-C-tangents \( R \) at \( z \) and the analoguous canonical map as in (3.7) if the map \( f_z \) of \( z \) is holomorphic.

To justify our choice of \( \text{Ext}^1(\mathcal{D}_{\widetilde{w}}^\bullet, \mathcal{O}_{\Sigma})^\dagger \), we will construct, to each \( v \in \text{Ext}^1(\mathcal{D}_{\widetilde{w}}^\bullet, \mathcal{O}_{\Sigma})^\dagger \), a pre-C-tangent \( \xi^v \in T_{\widetilde{w}}^{\text{pre}} R \) so that the image of \( \xi^v \) under (3.7) is \( v \). Let \( v = (v_1, v_2) \) be any element in \( \text{Ext}^1(\mathcal{D}_{\widetilde{w}}^\bullet, \mathcal{O}_{\Sigma})^\dagger \) defined by the diagram (3.3). Let \( T \subseteq \mathbb{C} \) be a neighborhood of \( 0 \) and let \( C_T \) be an analytic family of \( n \)-pointed curves so that \( C_0 \cong \Sigma \) and the Kuranishi map \( T_0 \mathbb{C} \to \text{Ext}^1(\Omega_{\Sigma}(D), \mathcal{O}_{\Sigma}) \)
send 1 to \( v_1 \). For instance, we can take \( C_T \) be the pull back of \( X_n \) via an analytic map \((T, 0) \to (O_n, p_n)\). We let \( \{ U_i \}_{i=1}^l \) be a covering of \( \Sigma \) as before and let \( \{ U_i \}_{i=1}^l \) be a covering of \( C_T \) that are the pull back of \( U_{n,i} \). Note that for \( i \leq l \), they come with biholomorphisms \( \tilde{U}_i \cong U_i \times T \). Let \( V_i \) be open charts of \( X \) as before with \( f(U_i) \subset V_i \). For \( i > l \), since the restriction of \((8.3)\) to \( U_i \) is analytic, we can find analytic \( \tilde{f}_i : U_i \to V_i \), possibly after shrinking \( T \) if necessary, such that \( \tilde{f}_i \) are related to \( v_2|_{U_i} \) as to how \( u_i \) are related to \( v_2(\xi)|_{U_i} \) before. By analytic analogue of deformation theory (see \([LT1]\)) such \( \tilde{f}_i \) do exist. For \( i \leq l \), since \( U_i \) are smooth and \( B^A|_{U_i} \) are the sheaves \( \Omega^1_{U_i}(T^*C_T|_{U_i}) \), we simply let \( \tilde{f}_i \) be smooth so that in addition to \( \tilde{f}_i \) satisfying the condition on pre-\( \mathcal{C} \)-tangents we require that \( v_2|_{U_i} \) coincide with the dual of \( df_i|_{U_i} \). Note that \((C_T, \{ \tilde{f}_i \})\) will be a pre-\( \mathcal{C} \)-tangent if \( \tilde{f}_i \) in \((8.3)\) is divisible by \( \pi_i^*(z_0^2) \). But this is true because for any \( p \in U_i \cap U_j \), the differential \( df_i(p) \) and \( df_j(p) \) from \( T_p C_T \) to \( T_{f(p)} X \) are identical. We let the so constructed pre-\( \mathcal{C} \)-tangent be \( \xi^c \). Of course \( \xi^c \) are not unique. It is obvious from the construction that the image of \( \xi^c \) under \((8.7)\) is \( v \). We remark that it follows from the construction that for any complex number \( c \neq 0 \) the pull back of \((C_T, \{ \tilde{f}_i \})\) under \( L_c : \mathbb{C} \to \mathbb{C} \) defined by \( L_c(z_0) = cz_0 \) is a pre-\( \mathcal{C} \)-tangent, say \( \xi^{cv} \), whose image under \((8.7)\) is \( cv \).

We next construct a holomorphic coordinate chart of \( R \) at \( \tilde{w} \). Let \( r = \dim R \), which is \( r_{\exp} + l = \dim \text{Ext}^1(D_{\tilde{w}}^*, \mathcal{O}_\Sigma) \). We fix a \( \mathbb{C} \)-isomorphism \( T_0 D^r \cong \text{Ext}^1(D_{\tilde{w}}^*, \mathcal{O}_\Sigma) \). Composed with the canonical

\[
\text{Ext}^1(D_{\tilde{w}}^*, \mathcal{O}_\Sigma) \to \text{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma),
\]

we obtain

\[
T_0 D^r \to \text{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma).
\]

(3.10)

Let \( X_n \) over \( O_n \) be the semi-universal family of the \( n \)-pointed curve \( \Sigma \) given before. We let \( S \) be a neighborhood of \( 0 \in \mathbb{C}^r \) and let \( \varphi : S \to O_n \) be a holomorphic map with \( \varphi(0) = 0 \) such that

\[
d\varphi(0) : T_0 S \equiv T_0 D^r \to T_{p_0} O_n \cong \text{Ext}^1(\Omega_\Sigma(D), \mathcal{O}_\Sigma)
\]

is the homomorphism \((3.10)\). We let \( \pi_S : C_S \to S \) be the family of \( n \)-pointed curves over \( S \) that is the pull back of \( X_n \). Note that \( C_S|_0 \), denoted by \( C_0 \), is canonically isomorphic to \( \Sigma \).

We keep the open covering \( \{ U_i \}_{i=1}^l \) of \( \Sigma \) \((\cong C_0)\) chosen before. We let \( \{ W_i \}_{i=1}^l \) be an open covering of a neighborhood of \( C_0 \subset C_S \) so that \( W_i \cap C_0 = U_i \). For \( i \leq l \), we let \( W_i \) be the pull back of \( U_{n,i} \subset X_n \). For \( i > l \) and \( U_i \), smooth, we choose \( W_i \) so that there is a holomorphic map \( \pi_i : W_i \to U_i \) so that the restriction of \( \pi_i \) to \( U_i \) is the identity map. For \( i > l \) and \( U_i \) contains a nodal point, we assume that \( W_i \) is biholomorphic to the unit ball in \( \mathbb{C}^{r+1} \) so that \( U_i \subset W_i \) is defined by \( w_1 w_2 = 0 \) and \( w_i = 0 \) for \( i \geq 3 \), where \((w_i)\) are the coordinate variables of \( \mathbb{C}^{r+1} \), and the restriction of \( \pi_S \) to \( W_i \) is given by

\[
z_1 = w_1 w_2, \ z_2 = w_3, \ldots, z_r = w_{r+1},
\]
where \((z_i)\) are the coordinate variables of \(C^r\). The upshot of this is that if \(h\) is a holomorphic function on \(U_i\), then we can extend it canonically to \(W_i\) as follows. In case \(U_i\) is smooth, then the extension of \(h\) is the composite of \(W_i \to U_i\) with \(h\); in case \(U_i\) is singular, then \(\varphi\) has a unique expression

\[
a + w_1h_1(w_1) + w_2h_2(w_2),
\]

where \(a \in \mathbb{C}\) and \(h_1, h_2\) are holomorphic. We then let its extension be the holomorphic function on \(W_i\) that has the same expression.

We fix the choice of \(\{U_i\}\) and \(\{W_i\}\). Without loss of generality, we can assume that there are coordinate charts \(V_i \subset X\) so that \(f(U_i) \subset V_i\). Of course, for \(i \leq l\) the charts \(V_i\) are the charts we have chosen before. Our construction of the local holomorphic chart of \(R\) is parallel to the original construction of Kodaira-Spencer of semi-universal family of deformation of holomorphic structures without obstructions. To begin with, possibly after shrinking \(W_i\) if necessary we can assume that the maps \(f|_{U_i} : U_i \to V_i\) can be extended to a holomorphic \(F_{0,i} : W_i \to V_i\) (Recall \(f\) is holomorphic). We now let \(A(W_i, V_i)\) be the space of smooth maps from \(W_i\) to \(\mathbb{C}^m\) defined as follows. If \(i > l\), then \(A(W_i, V_i)\) consists of holomorphic maps from \(W_i\) to \(\mathbb{C}^m\); if \(i \leq l\), then using the isomorphism \(W_i \cong U_i \times S\) and holomorphic coordinate \(z = (z_i)\) of \(S\) and holomorphic coordinate \(\xi\) of \(U_i\), any smooth function \(\varphi : W_i \to \mathbb{C}^m\) can be expressed in terms of its \(m\) components \(\varphi_j(z, \xi), \, j = 1, \ldots, m\). We define \(A(W_i, V_i)\) to be the set of those smooth maps \(\varphi : W_i \to \mathbb{C}^m\) so that

\[
\begin{align*}
\partial_{z_k}\varphi_j &= 0 \quad \text{for} \quad k = 1, \ldots, r_j = 1, \ldots, m; \\
\partial_{\xi}\varphi_j &= 0 \quad \text{for} \quad j \geq 2\partial_{\xi}\varphi_1 = c\sigma'_i \quad \text{for some} \quad c \in \mathbb{C},
\end{align*}
\]

where \(\sigma'_i\) is a \((0, 1)\)-form taking values in \(\varphi^*\mathbb{C}^n\) corresponding to the form \(\sigma_i\) using the canonical embedding \(V_i \subset \mathbb{C}^n\). Note that \(A(W_i, V_i)\) are \(O_S\)-modules. In particular, if we let \(I \subset O_S\) be the ideal sheaf of \(0 \in S\), then we denote by \(I^q \circ A(W_i, V_i)\) the image of \(I^q \subset A(W_i, V_i)\) in \(A(W_i, V_i)\).

In the following, we will construct a sequence of maps \(F_{s,i} \in A(W_i, V_i)\) indexed by \(s \geq 1\) and \(1 \leq i \leq L\) of which the following holds:

1. For each \(i\), \(F_{s+1,i} - F_{s,i} \in I^s \circ A(W_i, V_i)\);

2. The restrictions \(F_{1,i}|_{U_i} : U_i \to \mathbb{C}^m\) factor through \(V_i \subset \mathbb{C}^m\) and \(\iota_i \circ (F_{1,i}|_{U_i}) : U_i \to X\) is identical to \(f|_{U_i} : U_i \to X\);

3. In a neighborhood \(W_{ij}\) of \(U_i \cap U_j\) in \(W_i \cap W_j\) over which the map

\[
F_{s,ij} = g_{ij} \circ F_{s,j} - F_{s,i} : W_{ij} \to \mathbb{C}^m
\]

is well defined, \(F_{s,ij} \in I^s \circ H(W_{ij}, \mathbb{C}^m)\), where \(H(W_{ij}, \mathbb{C}^m)\) is the \(O_S\)-module of holomorphic maps from \(W_{ij}\) to \(\mathbb{C}^m\);

4. For any vector \(\eta \in \mathbb{C}^r\), we let \(L_\eta : \mathbb{C} \to \mathbb{C}^r\) be the unique \(C\)-linear map so that \(L_\eta(1) = \eta\), and let \(\eta^{\text{pre}}\) be the pre-\(C\)-tangent associated to the pull back of \((C_S, \{F_{2,i}\})\) under \(L_\eta\). Using the standard isomorphism \(T_0S \cong T_0C^r \cong \mathbb{C}^r\), we obtain a map

\[
T_0S \to \text{Ext}^1(D^*_0, O_S)^\dagger
\]
that send $\eta \in T_0 S$ to the image of $\eta^\preceq$ under (3.7). We require that this map is the isomorphism (3.10).

For $s = 1$ we simply let $F_{1,i}$ be the standard extension of $f|_{U_i} : U_i \to V_i$ to $W_i \to \mathbb{C}^m$. We now show that we can construct $\{F_{2,i}\}$ as required. We let $\pi_1$ and $\pi_2$ be the first and the second projection of $\mathbb{C}^r \times \Sigma$, where we view $\mathbb{C}^r$ as the total space of $\text{Ext}^1(D_w^*, O_X)^!$. It follows from the definition of the extension group that there is a universal diagram

$$
\begin{array}{c}
\pi_2^* f^* \Omega_X \\
\downarrow \nu_2 \\
B^A \\
\downarrow \pi_2^*(d\nu^\vee) \\
\pi_2^* \Omega_{S(D)^A} \\
\end{array}
\xrightarrow{(3.13)}
\begin{array}{c}
\pi_2^* f^* \Omega_X \\
\end{array}
$$

such that its restriction to fibers of $\mathbb{C}^r \times \Sigma$ over $\xi \in \mathbb{C}^r$ are the diagrams (3.3) associated to $\xi \in \text{Ext}^1(D_w^*, O_X)^!$. By deformation theory of pointed curves, for any smooth point $p \in \Sigma$ the vector space $B \otimes k(p)$ is canonically isomorphic to the cotangent space $T_p^* C_S$. By applying the construction of $\xi^v \in T_w^\preceq R$ from $v \in \text{Ext}^1(D_w^*, O_X)^!$ to the family version, we can construct the family $\{F_{2,i}\}$ as required. We will leave the details to the readers.

Now we show that we can successively construct $F_{s,i}$ that satisfies the four conditions above. Assume that for some $s \geq 2$ we have constructed $\{F_{s,i}\}$ that satisfies the four conditions above. Let $W_{ij}$ be the neighborhood of $U_{ij} = U_i \cap U_j \subset W_i \cap W_j$ so that (3.10) is well-defined. Then by the condition 3 above, $F_{s,ij} \in T^s H(W_{ij}, \mathbb{C}^m)$. Let $I = (i_1, \ldots, i_r)$ be any length $s$ multiple index, namely, $i_j \geq 0$ and $\sum i_j = s$. As usual, we will denote by $\partial^I$ the symbol $\partial^{i_1} / \partial z_1^{i_1} \cdot \cdots \partial^{i_r} / \partial z_r^{i_r}$ and by $z^I$ the term $z_1^{i_1} \cdots z_r^{i_r}$. Then because of the condition 3 above, $\varphi_{I,ij} = \partial^I F_{s,ij}|_{U_{ij}}$ is a holomorphic section of $f^* T_X|_{U_{ij}}$ using the standard isomorphism

$$TX|_{V_i} \cong TV_i \cong V_i \times \mathbb{C}^m,$$

and the collection $[\varphi_{I,ij}]$ defines a Čech 1-cocycle of $f^* T_X$. We let $\{\phi_{I,i}\}$, where $\phi_{I,i} = \zeta_i + a_i \eta_i$, be the collection provided by Lemma 3.3. Using the standard isomorphism $TX|_{V_i} \cong V_i \times \mathbb{C}^m$, we can view $\phi_{I,i}$ as a map $V_i \to \mathbb{C}^m$. We let $\tilde{\phi}_{I,i} : W_i \to \mathbb{C}^m$ be the standard extension of $\phi_{I,i}$ and let $G_{I,i} = \pi_2^s(z^I) \tilde{\phi}_{I,i}$. Clearly, $\partial^I G_{I,i} = \phi_{I,i}$. Now we let

$$F_{s+1,i} = F_{s,i} + \sum_{I(I) = s} G_{I,i}.$$ 

It is direct to check that the collection $\{F_{s+1,i}\}$ satisfies the condition 1-4 before. Finally, by the estimate in Lemma 3.3, there is a neighborhood of $U_i \subset W_i$, say $W_i^0$, such that $\lim_{l} F_{s,i}$ converges over $W_i^0$. Let $F_{\infty,i}$ be its limit. Because $f(U_i) \subset V_i$, there is a neighborhood $W_i$ of $U_i \subset W_i^0$ such that $F_{\infty,i}(W_i) \subset V_i \subset \mathbb{C}^m$. It follows that we can find a neighborhood $S^0 \subset S$ of $0 \in S$ such that $\pi_S^{-1}(S^0) \subset \cup W_i$. Finally, because $F_{\infty,i}$ is analytic near $U_i$ for $i > l$ and
is analytic in $S$ direction using $W_i \cong U_i \times S$ otherwise, the condition 3 implies that the collection $F_{\infty,i}|_{W_i \cap \pi_S^{-1}(S')} \equiv U_i \times S$ defines a map

$$F_S : \pi_S^{-1}(S') \to X.$$ 

Clearly, $F_S$ is holomorphic away from the union of $W_i, \ldots, W_l$. Further, for each $i \leq l$ if we let $\xi_i$ be a holomorphic variable of $U_i$ and let $\pi_{U_i}$ and $\pi_{S'}$ be the first and the second projection of $W_i \cap \pi_S^{-1}(S') \cong U_i \times S'$, then

$$\frac{\partial}{\partial \xi_i} F_S|_{W_i \cap \pi_S^{-1}(S')} d\xi_i = \pi_{S'}^* (\varphi_1) \pi_{U_i}^* (\gamma_i) F_S^* (v_i)|_{W_i} \tag{3.14}$$

where $\varphi_1$ is a holomorphic function over $S'$. Finally, we let $Z$ be the subset of

$$\pi_S^{-1}(S') \times_S \cdots \times_S \pi_S^{-1}(S') \quad (k \text{ times})$$

consisting of $(s; x_{n+1}, \ldots, x_{n+k})$ such that $s \in S'$ and that $x_{n+1}, \ldots, x_{n+k}$ are distinct points in $\pi_S^{-1}(s)$ that lie in $F_S^{-1}(H)$. Note that if we choose $U$ to be small enough, then $F_S^{-1}(H)$ has exactly $k$ points. Let $C_Z$ be the family of $(n+k)$-pointed curves over $Z$ so that its domain is the pull back of $C_S$ via $Z \to S$, its first $n$-marked sections is the pull back of the $n$-marked sections of $C_S$ and its last $k$-sections of the fiber of $C_Z$ over $(s; x_{n+1}, \ldots, x_{n+k})$ is $x_{n+1}, \ldots, x_{n+k}$. Coupled with the pull back of $F_S$, say $F_Z : C_Z \to X$, we obtain a family of stable (continuous) maps from $(n+k)$-pointed curves to $X$. Let $\eta : Z \to U$ be the tautological map.

We claim that $\eta(Z) \subset R$. Indeed, let $z \in Z$ be any point and let $C_z$ be the domain of $z$. It follows from our construction that $C_z$ has $l$ distinguished open subsets, denoted by $U_1, \ldots, U_l$, such that $f_z = F_Z|_{C_z}$ is holomorphic away from $\bigcup_{i=1}^l U_i$ and $\bar{\partial} f_z|_{U_i}$ is a constant multiple of $\gamma_i : f_z^* (v_i)$. Hence the value of the section $\Phi_{\bar{U}_i} : U \to E_{\bar{U}_i}$ at $\eta(z)$ is contained in the subspace $V|_{\eta(z)} \subset E_{\bar{U}_i}|_{\eta(z)}$. This shows that $\eta(z) \in R$.

**Proposition 3.5.** The induced map $\eta : Z \to R$ is a local diffeomorphism near those $z \in R$ whose associated map $f_z : C_z \to X$ are holomorphic.

**Proof.** This follows immediately from the proof of the basic Lemma in [LT2]. We will omit the details here. □

By shrinking $S'$ if necessary, we can assume that $\eta : Z \to R$ is a local diffeomorphism. We can further assume that there is an open subset $Z' \subset Z$ containing $\tilde{w}$ such that $\eta' = \eta|_{Z'} : Z' \to R$ is one-to-one and the image $\eta(Z') \subset R$ is invariant under $G_{\tilde{U}}$. $\eta' : Z' \to R$ is the analytic coordinate of $\tilde{w} \in R$ we want. For convenience, we will view $Z'$ as an open subset of $R$.

**Proposition 3.6.** Let $V'$ be the restriction of $W$ to $Z'$ endowed with the holomorphic structure so that $\sigma_1|_{Z'}, \ldots, \sigma_l|_{Z'}$ is a holomorphic frame. Then $\phi' \equiv \phi|_{Z'} : \tilde{Z'} \to V'$ is holomorphic.

**Proof.** This follows immediately from (3.14). □
Let $\phi^{-1}(0)$ be any point and let $f_z : C \to X$ be the associated (analytic) stable map with $D_z$ the divisor of its first $n$-marked points. Then there is a canonical exact sequence of vector spaces

$$\text{Ext}^1(\Omega_{C_z}(D_z), \mathcal{O}_{C_z}) \to H^1(f_z^*T_X) \to \text{Ext}^2(D_z^*, \mathcal{O}_{C_z}) \to 0$$

induced by the short exact sequence of complexes

$$0 \to [0 \to \Omega_{C_z}(D_z)] \to [f_z^*\Omega_X \to \Omega_{C_z}(D_z)] \to [f_z^*\Omega_X \to 0] \to 0.$$

Similarly, the differential $d\phi_V(z) : T_zR \to W_z$ induces an exact sequence of vector spaces

$$\text{Ext}^1(D_z^*, \mathcal{O}_{C_z}) \xrightarrow{d\phi_V(z)} W|_z \to \text{Coker}(d\phi_V(z)) \to 0.$$

Note that there are canonical homomorphisms $\text{Ext}^1(D_z^*, \mathcal{O}_{C_z}) \to \text{Ext}^1(D_z^*, \mathcal{O}_{C_z})$ and $W|_z \to H^{0,1}(f_z^*T_X) \cong H^1(f_z^*T_X)$.

**Lemma 3.7.** There is a canonical isomorphism $\xi$ (as shown below) that fits into the diagram

$$\begin{array}{ccc}
\text{Ext}^1(D_z^*, \mathcal{O}_{C_z}) & \xrightarrow{d\phi_V(z)} & W|_z \\
\downarrow & & \downarrow \\
\text{Ext}^1(\Omega_{C_z}(D_z), \mathcal{O}_{C_z}) & \to & H^1(f_z^*T_X) \to \text{Ext}^2(D_z^*, \mathcal{O}_{C_z}) \to 0.
\end{array}$$

**Proof.** This is obvious and will be left to the readers. \qed

## 4 The proof of the comparison theorem

In this section, we will prove that the algebraic and the symplectic construction of GW-invariants yield identical invariants.

We will work with the category of algebraic schemes as well as the category of analytic schemes. Specifically, we will use the words schemes, morphisms and étale neighborhoods to mean the corresponding objects in algebraic category and use the word analytic maps and open subsets to mean the corresponding objects in analytic category. As before, the words analytic and holomorphic are interchangeable. Also, we will use $\mathcal{O}_S$ to mean the sheaf of algebraic sections or the sheaf of analytic sections depending on whether $S$ is an algebraic scheme or an analytic scheme. We will continue to use the complex dimension throughout this section.

We now clarify our usage of the notions of cycles and currents. Let $W$ be a scheme. We denote by $Z^k_{\text{alg}}W$ the group of formal sums of finitely many $k$-dimensional irreducible subvarieties of $W$ with rational coefficients. We call elements of $Z^k_{\text{alg}}W$ $k$-cycles of $W$. Now let $W$ be any stratified topological space with stratification $\mathcal{S}$. We say that a (complex) $k$-dimensional current $C$
is stratifiable if there is a refinement of $S$, say $S'$, such that there are finitely many $k$-dimensional strata $S_i$ and rationals $a_i \in \Q$ such that $C = \sum a_i [S_i]$ (All currents in this paper are oriented). Here we assume that each stratum of $S'$ was given an orientation a priori and $[S_i]$ is the oriented current defined by $S_i$. We identify two currents if they define identical measures in the sense of rectifiable currents. We denote the set of all stratifiable $k$-dimensional currents modulo the equivalence relation by $Z_k W$. Clearly, if $W$ is a scheme then any $k$-cycle has an associated current in $Z_k W$, which defines a map $Z_k W \to Z_k W$. In the following, we will not distinguish a cycle from its associated current. Hence $H$ defines canonically an element in $F$ and $Z$.

Before be fixed once and for all. We let $M$ for $C$ the following, we will not distinguish a cycle from its associated current. Hence $H$ defines canonically an element in $F$ and $Z$.

We begin with a quick review of the algebraic construction of GW-invariants. Let $X$ be any smooth projective variety and let $A \in H^2(X, \Z)$ and $g, n \in \Z$ as before be fixed once and for all. We let $\mathfrak{M}_{g,n}(X, A)$ be the moduli scheme of stable morphisms defined before. $\mathfrak{M}_{g,n}(X, A)$ is projective. The GW-invariants of $X$ is defined using the virtual moduli cycle

$[\mathfrak{M}_{g,n}(X, A)^{\text{vir}}] \in A, \mathfrak{M}_{g,n}(X, A)$.

To review such a construction, a few words on the obstruction theory of deformations of morphisms are in order. Let $w \in \mathfrak{M}_{g,n}(X, A)$ be any point associated to the stable morphism $X$. Let $(B, I, X_{B/I})$ be any collection where $B$ is an Artin ring, $I \subset B$ is an ideal annihilated by the maximal ideal $\mathfrak{m}_B$ of $B$ and $X_{B/I}$ is a flat family of stable morphisms over $\text{Spec} B/I$ whose restriction to the closed fiber of $X_{B/I}$ is isomorphic to $X$. An obstruction theory to deformation of $X$ consists of a $\C$-vector space $V$, called the obstruction space, and an assignment that assigns any data $(B, I, X_{B/I})$ as before to an obstruction class

$\text{Ob}(B, B/I, X_{B/I}) \in I \otimes \C V$

to extending $X_{B/I}$ to $\text{Spec} B$. Here by an obstruction class, we mean that its vanishing is the necessary and sufficient condition for $X_{B/I}$ to be extendable to a family over $\text{Spec} B$. We also require that such an assignment satisfies the obvious base change property (For reference on obstruction theory please consult [3]). In case $X$ is the map $f: C \to X$ with $D \subset C$ the divisor of its $n$ marked points, the space of the first order deformations of $X$ is parameterized by $\text{Ext}^1(D_X^*, \mathcal{O}_C)$, where $D_X^* = [f^* \Omega_X \to \Omega_C(D)]$ is the complex as before, and the standard obstruction theory to deformation of $X$ takes values in $\text{Ext}^2(D_X^*, \mathcal{O}_C)$.

An example of obstruction theories is the following. Let $R$ be the ring of formal power series in $m$ variables and let $\mathfrak{m}_R \subset R$ be its maximal ideal. Let $F$ be a vector space and let $f \in \mathfrak{m}_R \otimes \C F$. We let $(f) \subset R$ be the ideal

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generated by components of \( f \). Then there is a standard obstruction theory to deformations of \( 0 \) in \( \text{Spec} R/(f) \) taking values in \( V \), where \( V \) is the cokernel of \( df : (m_R/m_R^2)^\vee \to F \), defined as follows. Let \( I \subset B \) be an ideal of an Artin ring as before and let \( \varphi_0 : \text{Spec} B/I \to \text{Spec} R/(f) \) be any morphism. To extend \( \varphi_0 \) to \( \text{Spec} B \), we first pick a homomorphism \( \sigma : R \to B \) extending the induced \( R \to B/I \), and hence a morphism \( \varphi_{\text{pre}} : \text{Spec} B \to \text{Spec} R \). The image \( \sigma(f) \in B \otimes F \) is in \( I \otimes F \), and is the obstruction to \( \varphi_{\text{pre}} \) factor through \( \text{Spec} R/(f) \subset \text{Spec} R \). Let \( \text{Ob}(B, B/I, \varphi_0) \) be the image of \( \sigma(f) \) in \( I \otimes V \) via \( F \to V \). It is direct to check that \( \text{Ob}(B, B/I, \varphi_0) = 0 \) if and only if there is an extension \( \varphi : \text{Spec} B \to \text{Spec} R/(f) \) of \( \varphi_0 \). This assignment

\[
(B, B/I, \varphi_0) \mapsto \text{Ob}(B, B/I, \varphi_0) \in I \otimes V \tag{4.1}
\]

is the induced obstruction theory of \( \text{Spec} R/(f) \).

**Definition 4.1.** A Kuranishi family of the standard obstruction theory of \( \mathcal{X} \) consists of a vector space \( F \), a ring of formal power series \( R \) with \( m_R \) its maximal ideal, an \( f \in m_R \otimes F \), a family \( \mathcal{X}_{R/(f)} \) of stable morphisms over \( \text{Spec} R/(f) \) whose closed fiber over \( 0 \in \text{Spec} R/(f) \) is isomorphic to \( \mathcal{X} \) and an exact sequence

\[
0 \longrightarrow \text{Ext}^1(D_X^\bullet, \mathcal{O}_C) \overset{\alpha}{\longrightarrow} (m_R/m_R^2)^\vee \overset{df}{\longrightarrow} F \longrightarrow \text{Ext}^2(D_X^\bullet, \mathcal{O}_C) \longrightarrow 0 \tag{4.2}
\]

of which the following holds: First, the composite

\[
\text{Ext}^1(D_X^\bullet, \mathcal{O}_C) \overset{\alpha}{\longrightarrow} \ker(df) \equiv T_0 \text{Spec} R/(f) \longrightarrow \text{Ext}^1(D_X^\bullet, \mathcal{O}_C),
\]

where the second arrow is the Kodaira-Spencer map of the family \( \mathcal{X}_{R/(f)} \), is the identity homomorphism; Secondly, let \( I \subset B \) and \( \varphi_0 : \text{Spec} B/I \to \text{Spec} R/(f) \) be as before and let

\[
\text{Ob}(B, B/I, \varphi_0)_{\mathcal{X}_{R/(f)}} \in I \otimes \text{Ext}^2(D_X^\bullet, \mathcal{O}_C)
\]

be the obstruction to extending \( \varphi_0 \mathcal{X}_{B/I} \) to \( \text{Spec} B \). Then it is identical to \( \text{Ob}(B, B/I, \varphi_0) \) under the isomorphism

\[
\text{Coker}(df) \cong \text{Ext}^2(D_X^\bullet, \mathcal{O}_C),
\]

where \( \text{Ob}(B, B/I, \varphi_0) \) is the obstruction class in \( [\mathcal{X}] \).

We now sketch how the virtual moduli cycle \([\mathcal{M}_{g,n}(X, A)]^{\text{vir}}\) was constructed. Similar to the situation of the moduli of stable smooth maps, we need to treat \( \mathcal{M}_{g,n}(X, A) \) either as a \( \mathbb{Q} \)-scheme or as a Deligne-Mumford stack. The key ingredient here is the notion of atlas, which is a collection of charts of \( \mathcal{M}_{g,n}(X, A) \). A chart of \( \mathcal{M}_{g,n}(X, A) \) is a tuple \( (S, G, \mathcal{X}_S) \), where \( G \) is a finite group, \( S \) is a \( G \)-scheme (with effective \( G \)-action) and \( \mathcal{X}_S \) is a \( G \)-equivariant family of stable morphisms so that the tautological morphism \( \iota : S/G \to \mathcal{M}_{g,n}(X, A) \) induced by the family \( \mathcal{X}_S \) is an étale neighborhood. For details of such an notion, please consult \([\text{DM}, \text{VI}], \text{LT}\)]. We now let \( f : C \to X \) be the representative of \( \mathcal{X}_S \) with \( D \subset C \) the divisor of the \( n \)-marked sections of \( \mathcal{X}_S \). Let \( \pi : C \to S \)
be the projection. We consider the relative extension sheaves $\mathcal{E}\text{xt}_w^i(\mathcal{D}^*_X, \mathcal{O}_C)$, where $\mathcal{D}^*_X = [f^*\Omega_X \to \Omega_C]/(D)$ as before. For short, we denote the sheaves $\mathcal{E}\text{xt}_w^i(\mathcal{D}^*_X, \mathcal{O}_C)$ by $\mathcal{T}_S^i$. Because they vanish for $i = 0$ and $i > 2$, for any $w \in S$, the Zariski-tangent space $T_wS$ is $\mathcal{T}_S^1 \otimes_{\mathcal{O}_S} k(w)$ and the obstruction space to deformations of $w$ in $S$ is $V_w = \mathcal{T}_S^2 \otimes_{\mathcal{O}_S} k(w)$. Now we choose a complex of locally free sheaves of $\mathcal{O}_S$-modules $\mathcal{E}\text{o}^\bullet = [\mathcal{E}_1 \to \mathcal{E}_2]$ so that it fits into the exact sequence

$$0 \longrightarrow \mathcal{T}_S^1 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{T}_S^2 \longrightarrow 0. \quad (4.3)$$

We let $F_i(w) = \mathcal{E}_i \otimes_{\mathcal{O}_S} k(w)$. Then we have the exact sequence of vector spaces

$$0 \longrightarrow T_wS \longrightarrow F_1(w) \longrightarrow F_2(w) \longrightarrow V_w \longrightarrow 0. \quad (4.4)$$

We let $K_w \in R(w)$ be a Kuranishi map of the obstruction theory to deformations of $w$, where $R(w) = \lim_{\overset{\longrightarrow}{\mathcal{K}_w \subseteq \mathcal{K}_w}} \mathcal{K}_w$, so that $\mathcal{K}_w$ is part of the data of the Kuranishi family specified in Definition 4.1. Let $(\mathcal{K}_w) \subseteq R(w)$ be the ideal generated by the components of $\mathcal{K}_w$ and let $\text{Spec } R_w/(\mathcal{K}_w) \subseteq \text{Spec } R_w$ be the corresponding subscheme. It follows that $\text{Spec } R_w/(\mathcal{K}_w)$ is isomorphic to the formal completion of $S$ along $w$, denoted $\hat{w}$. We let $N_w$ be the normal cone to $\text{Spec } R_w/(\mathcal{K}_w)$ in $\text{Spec } R_w$. Then $N_w$ is canonically a subcone of $F_2(w) \times \hat{w}$. Here, by abuse of notation we will use $F_2(w)$ to denote the total space of the vector space $F_2(w)$. Note that $N_w$ is the infinitesimal normal cone to $S$ in its obstruction theory at $w$. To obtain a global cone over $S$, we need the following existence and uniqueness theorem, which is the main result of [LT1].

In this paper, we will call a vector bundle $E$ the associated vector bundle of a locally free sheaf $\mathcal{E}$ if $\mathcal{O}(E) \cong \mathcal{E}$. For notational simplicity, we will not distinguish a vector bundle from the total space (scheme) of this vector bundle.

**Theorem 4.2 ([LT1]).** Let $E$ be the associated vector bundle of $\mathcal{E}_2$. Then there is a cone scheme $N_S \subseteq E$ such that for each $w \in S$ there is an isomorphism

$$F_2(w) \times \hat{w} \cong E \times_{\mathcal{O}_S} \hat{w} \quad (4.5)$$

of cones over $\hat{w}$ extending the canonical isomorphism $F_2(w) \cong E \times_{\mathcal{O}_S} w$ such that under the above isomorphism $N_w$ is isomorphic to $N_S \times_{\mathcal{O}_S} \hat{w}$. In particular, the cycle defined by the scheme $N_S$ is uniquely characterized by this condition.

In the previous discussion, if we replace $F_1(w)$ and $F_2(w)$ by $T_wS$ and $V_w$ respectively, we obtain a Kuranishi map and correspondingly a cone scheme in $V_w \times \hat{w}$, denoted by $N^0_w$.

**Theorem 4.3 ([LT1]).** Let the notation be as before. Then there is a vector bundle homomorphism $r: E \times_{\mathcal{O}_S} \hat{w} \to V_w \times \hat{w}$ extending the canonical homomorphism $E|_w \to V_w$ induced by (4.4) such that

$$N^0_w \times_{V_w \times \hat{w}} E \times_{\mathcal{O}_S} \hat{w} = N_S \times_{\mathcal{O}_S} \hat{w}.$$
To construct the virtual cycle $[\mathcal{M}_{g,n}(X, A)]^{vir}$, we need to find a global complex over $\mathcal{M}_{g,n}(X, A)$ analogous to $\mathcal{E}^\bullet$. For the purpose of comparing with the analytic construction of the virtual cycles, we will use atlas of analytic charts. We let $\{(R_i, W_i, \phi_i)\}_{i \in \Lambda}$ be the good atlas of the smooth approximation of $[\Phi : \mathcal{B} \to \mathcal{E}]$ chosen in section 2. Then the collection $Z_i = \phi_i^{-1}(0)$ with the tautological family of stable analytic maps (with the last k-marked points discarded) form an atlas of the underlying analytic scheme of $\mathcal{M}_{g,n}(X, A) \cong \Phi^{-1}(0)$. Since we are only interested in constructing and working with cone cycles in $\mathbb{Q}$-bundles (known as V-bundles) over $\mathcal{M}_{g,n}(X, A)$, there is no loss of generality that we work with $\mathcal{M}_{g,n}(X, A)$ with the reduced scheme structure. Hence, for simplicity we will endow $Z_i$ be the tautological family of the n-pointed stable analytic maps over $Z_i$ that is derived by discarding the last k marked points of the restriction to $Z_i$ of the tautological family over $U_i$. We let $G_i$ be the finite group associated to the chart $(R_i, W_i, \phi_i)$, and let $\mathcal{X}_i$ be represented by $f_i : C_i \to X$ with $D_i \subset C_i$ be the divisor of the n-marked sections of $C_i$ and let $\pi_i : C_i \to Z_i$ be the projection. In Appendix, to each $i$, we have constructed a $G_i$-equivariant complex of locally free sheaves of $O_{Z_i}$-modules $\mathcal{E}^\bullet_i = [\mathcal{E}_{i,1} \to \mathcal{E}_{i,2}]$ such that $\mathcal{E}_i \cong (\mathcal{E}^\bullet_i, O_{C_i})$ is the sheaf cohomology of $\mathcal{E}^\bullet_i$. It follows from the algebraic and the analytic constructions of charts that each $(Z_i, \mathcal{X}_i)$ can be realized as an analytic open subset of an algebraic chart, say $(\mathcal{S}, \mathcal{G}, \mathcal{X}_S)$, and the complex $\mathcal{E}^\bullet_i$ is the restriction to this open subset of an algebraic complex $\mathcal{E}^\bullet$, as in (4.4). Therefore we can apply Theorem 4.2 to obtain a unique analytic cone cycle $M^\text{alg}_{i} \in Z_\ast E_i$, where $E_i$ is the associated vector bundle of $\mathcal{E}_{i,2}$, Let $t_i : Z_i/G_i \to \mathcal{M}_{g,n}(X, A)$ be the tautological map induced by the family $\mathcal{X}_i$. One property that follows from the construction of the complexes $\mathcal{E}^\bullet_i$ which we did not mention is that to each $i$, the cone bundle $E_i/G_i$ over $Z_i/G_i$ descends to a cone bundle over $t_i(Z_i/G_i)$, denoted by $E_i$, and $\{E_i\}_{i \in \Lambda}$ patch together to form a global cone bundle over $\mathcal{M}_{g,n}(X, A)$, denoted by $\tilde{E}$. Further, by the uniqueness of the cone cycles $M^\text{alg}_{i} \in Z_\ast E_i$ in Theorem 3.2 and 3.3, to each $i$ the cone cycle $M^\text{alg}_{i}/G_i$ in $E_i/G_i$ descends to a cone cycle $M^\text{alg}_{i} \in Z_\ast \tilde{E}i$, and $\{M^\text{alg}_{i}\}_{i \in \Lambda}$ patch together to form a cone cycle in $Z_\ast \tilde{E}$, denoted by $M^\text{alg}$. It follows from Appendix that $\tilde{E}$ is an algebraic cone over $\mathcal{M}_{g,n}(X, A)$ and $M^\text{alg}$ is an algebraic cone cycle in $\tilde{E}$. In the end, we let $\eta_E : \mathcal{M}_{g,n}(X, A) \to \tilde{E}$ be the zero section and let

$$\eta_E^* : \{\text{algebraic cycles in } Z_\ast \tilde{E}\} \longrightarrow H_\ast(\mathcal{M}_{g,n}(X, A); \mathbb{Q})$$

be the Gysin homomorphism. Then the virtual moduli cycle is

$$[\mathcal{M}_{g,n}(X, A)]^{vir} = \eta_E^*[M^\text{alg}] \in A_\ast \mathcal{M}_{g,n}(X, A).$$

There is an analogous way to construct the $GW$-invariants of algebraic varieties using analytic method. We continue to use the notion developed in section 1. Let $(R, W, \phi)$ be a smooth approximation of $[\Phi : \mathcal{B} \to \mathcal{E}]$ constructed in Lemma 6.3. Then we can construct a cone current in the total space of $W$ as follows. Let $\Gamma_{t\phi}$ be the graph of $t\phi$ in $W$ and let $N_{0/\phi}$ be the limit current
Lemma 4.4. Let the notation be as before and let
\[
\lim_{t \to \infty} \Gamma_{t \phi},
\]
when it exists. Clearly, if such a limit does exist, then it is contained in \( W|_{\phi^{-1}(0)} \). In general, though \( \phi \) is smooth there is no guarantee that such a limit will exist. However, if the approximation is analytic, then we will show that such limit does exist as an stratifiable current. Indeed, assume \((R, W, \phi)\) is an analytic smooth approximation. Since the existence of \( \lim \Gamma_{t \phi} \) is a local problem, we can assume that there is a holomorphic basis of \( W \), say \( e_1, \ldots, e_r \). Then \( \phi \) can be expressed in terms of \( r \) holomorphic functions \( \phi_1, \ldots, \phi_r \). Now let \( C \) be the complex line with complex variable \( t \), let \( w_i \) be the dual of \( e_i \) and let \( \Theta \subset W \times C \) be the analytic subscheme defined by the vanishing of \( tw_i - \phi_i \), \( i = 1, \ldots, r \). We let \( \Theta_0 \) be the smallest closed analytic subscheme of \( \Theta \) that contains \( \Theta \cap (W \times C^*) \), where \( C^* = C - \{0\} \). By the Weierstrass preparation theorem, such \( \Theta_0 \) does exist. Then we define \( N_{0/\phi} \) to be the associated cycle of the intersection of the scheme \( \Theta_0 \) with \( W \times \{0\} \). By \( \text{Ext} \), \( N_{0/\phi} \) is the limit of \( \Gamma_{t \phi} \).

Obviously, \( N_{0/\phi} \) is stratifiable. This shows that for any analytic approximation \((R, W, \phi)\) the limit \( \lim \Gamma_{t \phi} \) does exist.

We now state a simple lemma which implies that if \((R', W', \phi')\) is a smooth approximation that is finer than the analytic approximation \((R, W, \phi)\), then \( \lim \Gamma_{t \phi} \) exists as well. We begin with the following situation. Let \( V \) be a smooth oriented vector bundle over a smooth oriented manifold \( M \) and let \( \varphi : M 
rightarrow V \) be a smooth section. Let \( V' \subset V \) be a smooth submanifold such that for any \( x \in \varphi^{-1}(0) \) we have \( \text{Im}(d\varphi(x)) + V'_x = V_x \). Then \( M_0 = \varphi^{-1}(V') \) is a smooth submanifold of \( M \) near \( \varphi^{-1}(0) \). Let \( V_0 \) be the restriction of \( V' \) to \( M_0 \) and let \( \varphi_0 : M_0 \to V_0 \) be the induced section. We next let \( N \subset TM|_{\varphi^{-1}(0)} \) be a subbundle complement to \( TM|_{\varphi^{-1}(0)} \) in \( TM|_{\varphi^{-1}(0)} \). Then the union of \( d\varphi(x)(N_x) \) for all \( x \in \varphi^{-1}(0) \) forms a subbundle of \( V|_{\varphi^{-1}(0)} \). We denote this bundle by \( d\varphi(N) \).

Since \( V|_{\varphi^{-1}(0)} \equiv V_0|_{\varphi^{-1}(0)} \oplus d\varphi(N) \), there is a unique projection \( P : V|_{\varphi^{-1}(0)} \to V_0|_{\varphi^{-1}(0)} \) such that whose kernel is \( d\varphi(N) \) and the composite of the inclusion \( V_0|_{\varphi^{-1}(0)} \to V|_{\varphi^{-1}(0)} \) with \( P \) is the identity map.

**Lemma 4.4.** Let the notation be as before and let \( l = \dim M \) and \( l_0 = \dim M_0 \). Then \( \lim \Gamma_{t \varphi} \) exists as an \( l \)-dimensional current in \( V|_{\varphi^{-1}(0)} \) if and only if \( \lim \Gamma_{t \varphi_0} \) exists as an \( l_0 \)-dimensional oriented current in \( V_0|_{\varphi^{-1}(0)} \). Further, if they do exist then
\[
\lim \Gamma_{t \varphi} = P^*(\lim \Gamma_{t \varphi_0}).
\]
Hence \( \lim \Gamma_{t \varphi} \) is stratifiable if \( \lim \Gamma_{t \varphi_0} \) is stratifiable.

**Proof.** This is obvious and will be left to the readers. \( \square \)

Now let \( \{ (R_\alpha, W_\alpha, \phi_\alpha) \}_{\alpha \in \Xi} \) be a collection of analytic smooth approximations of \( \Phi : B \to E \) such that the images of \( Z_\alpha = \phi_\alpha^{-1}(0) \) (in \( \Phi^{-1}(0) \)) covers \( \Phi^{-1}(0) \). It follows that we can choose a good atlas \( \{ (R_i, W_i, \phi_i) \}_{k \in \Lambda} \) constructed in Lemma 4.3 so that all approximations in \( \Lambda \) are finer than approximations in \( \Xi \). Now let \( i \in \Lambda \) and let \( x \in Z_i = \phi_i^{-1}(0) \subset R_i \) be any point. Because charts in \( \Xi \) cover \( \Phi^{-1}(0) \), there is an \( \alpha \in \Xi \) such that the image of \( R_\alpha \) in \( B \) contains the image of \( x \) in \( B \). Then because \( (R_\alpha, W_\alpha, \phi_\alpha) \) is finer than \( (R_i, W_i, \phi_i) \), by definition, there is a locally closed submanifold \( R_{i,\alpha} \subset R_i \), a local diffeomorphism \( f_i^\alpha : R_{i,\alpha} \to R_\alpha \) and a vector bundle inclusion \( (f_i^\alpha)^*W_\alpha \subset W_i|_{R_{i,\alpha}} \) such
that \((f^*_i)^* (\phi_\alpha) = \phi_i\), as in (2.4). This is exactly the situation studied in Lemma 4.4. Hence \(\lim \Gamma_{t\phi_i}\) exists near fibers of \(W\) over \(x\). Because \(\{Z_\alpha\}\) covers \(\Phi^{-1}(0)\), \(\lim \Gamma_{t\phi_i}\) exists and is a pure dimensional stratifiable current of dimension \(\dim R_i\). We denote this current by \(N_{0/\phi_i}\).

Now it is clear how to construct the GW-invariants of algebraic varieties using these analytically constructed cones. By the property of good coverings, for \(j \leq i \in \Lambda\) the approximation \((R_i, W_i, \phi_i)\) is finer than \((R_j, W_j, \phi_j)\). We let \(Z_i = \phi_i^{-1}(0)\) be as before and let \(Z_{i,j} = Z_i \cap R_{i,j} \subset Z_i\), where \(R_{i,j}\) is defined before (2.4). Let \(\rho^i_j: Z_{i,j} \to Z_j\) be the restriction of \(f^*_i\) to \(Z_{i,j}\). Note that \(Z_{i,j}\) is an open subset of \(Z_i\) and \(\rho^i_j: Z_{i,j} \to \rho^i_j(Z_{i,j})\) is a covering. Let \(F_i\) be the restriction of \(W_i\) to \(Z_i\) and let \(p_i: F_i \to Z_i\) be the projection. Hence, \((\rho^i_j)^*(F_j)\) is canonically a subvector bundle of \(F_i|_{Z_{i,j}}\). By Lemma 4.4, \((\rho^i_j)^*(F_j)\) intersects transversally with \(N_{0/\phi_i} \cap p_i^{-1}(Z_{i,j})\) and as currents, \(N_{0/\phi_i} \cap (\rho^i_j)^*(F_j) = (\rho^i_j)^*(N_{0/\phi_i})\).

For convenience, in the following we will call the collection \(\{F_i\}\) with transition functions \(f^*_i\) a semi-\(\mathbb{Q}\)-bundle and denote it by \(F\), and will denote \(\{N_{0/\phi_i}\}\) by \(N^{an}\). As in section two, we will call a collection \(s = \{s_i\}_{i \in \Lambda}\) of smooth sections \(s_i: Z_i \to F_i\) a global section of \(F\) if for \(j \leq i \in \Lambda\) the restriction \(s_i|_{Z_{i,j}}: Z_{i,j} \to F_i|_{Z_{i,j}}\) coincides with the pull back section \((\rho^i_j)^* s_j: Z_{i,j} \to (\rho^i_j)^* F_j\) under the canonical inclusion \((\rho^i_j)^* F_j \subset F_i|_{Z_{i,j}}\). We say that the section \(s\) is transversal to \(N^{an}\) if for each \(i \in \Lambda\), the graph of the section \(s_i\) is transversal to \(N_{0/\phi_i}\) in \(F_i\).

Obviously, if \(s\) is a global section of \(F\) that is transversal to \(N^{an}\), then following the argument after Lemma 2.10, currents

\[
\frac{1}{m_i} \iota^* \pi_\Lambda (N_{0/\phi_i} \cap \Gamma_{s_i}), \quad i \in \Lambda,
\]

where \(\iota^*_i: Z_i \to B\) is the restriction of \(\iota_i: R_i \to B\) to \(Z_i \subset R_i\) and \(m_i\) is the number of sheets of the branched covering \(\iota^*_i: Z_i \to \iota^*_i(Z_i)\), patch together to form an oriented current in \(B\) without boundary. We denote this current by \(s^*(N^{an})\).

It has pure dimension \(r_{exp}\), since the currents \(N_{0/\phi_i}\) has dimension \(\dim R_i = \text{rank } F_i + r_{exp}\). Hence it defines a homology class \([s^*(N^{an})]\) in \(H_{2r_{exp}}(B; \mathbb{Q})\).

**Proposition 4.5.** \([s^*(N^{an})]\) is the Euler class \(e[\Phi: B \to E]\) constructed in section one.

**Proof.** Recall that the class \(e[\Phi: B \to E]\) was constructed by first selecting a collection of perturbations \(h_i(s): R_i \to W_i\) of \(\phi_i\) parameterized by \(s \in [0, 1]\) satisfying certain property and then form the current that is the patch together of the currents \(\frac{1}{m_i} \iota^* \pi_\Lambda (\Gamma_{h_i(1)} \cap \Gamma_0)\), where \(\Gamma_{h_i(1)}\) and \(\Gamma_0\) are the graph of \(h_i(1)\) and \(0: R_i \to W_i\). Alternatively, we can perturb the 0-section instead of \(\{\phi_i\}\) to obtain the same cycle. Namely, we let \(h'_i(s): R_i \to W_i\) be a collection of perturbations of the zero section \(0: R_i \to W_i\), such that it satisfies the obvious compatibility and properness property similar to that of \(h_i(s)\) in section two. Moreover, we require that the graph \(\Gamma_{h'_i(1)}\) is transversal to \(N_{0/\phi_i}\) and transversal to the graph \(\Gamma_{0\phi_i}\) for sufficiently large \(t\). Of course such perturbations do exist following the proof of Proposition 2.12. Let \(C_t\) be the current in \(B\) that is the result of
patching together the currents \( \frac{1}{m} t_i \ast p_* (\Gamma_{h_i(1)} \cap \Gamma_{t \phi_i}) \), where \( p_i \) is the projection \( W_i \to R_i \). Clearly, for \( t \gg 0 \), we have \( \partial C_t = 0 \) and \( \text{Supp}(C_t) \) is compact. Hence \( C_t \) defines a homology class in \( H_{2r_{\exp}}(\mathcal{B}; \mathbb{Q}) \), denoted by \([C_t]\). It follows from the uniqueness argument in the end of section two that for sufficiently large \( t \), the homology class \([C_t]\) in \( H_{2r_{\exp}}(\mathcal{B}; \mathbb{Q}) \) is exactly the Euler class. On the other hand, we let \( C_{t \phi_i} \) be the current in \( \mathcal{B} \) that is the patch together of the currents \( \frac{1}{m} t_i \ast p_* (\Gamma_{h_i(1)} \cap N_{t \phi_i}) \). Because \( N_{t \phi_i} \) is the limit of \( \Gamma_{t \phi_i} \), and because \( \Gamma_{h_i(1)} \) intersects transversally with \( \Gamma_{t \phi_i} \) for \( t \gg 0 \) and with \( N_{t \phi_i} \), the union
\[
\bigcup_{t \in [0, \epsilon]} \{t\} \times C_{1/t} \subset [0, \epsilon] \times \mathcal{B},
\]
where \( 1 \gg \epsilon > 0 \), is a current whose boundary is \( C_{1/\epsilon} - C_{t \phi_i} \). This implies that
\[
[C_{t \phi_i}] = [C_t] \in H_{2r_{\exp}}(\mathcal{B}; \mathbb{Q}) \quad \text{for } t \gg 0.
\]
Further, because the currents \( N_{t \phi_i} \) are contained in \( F_i = W_i | Z_i \), \( p_* (N_{t \phi_i} \cap \Gamma_{h_i(1)}) \) as current is identical to \( \pi_* (N_{t \phi_i} \cap \Gamma_{h_i(1)}) \), where \( r_i : Z_i \to F_i \) is the restriction of \( h_i(1) \) to \( Z_i \). Hence \( C_{t \phi_i} = r^* (\mathcal{N}^{an}) \) with \( r = \{r_i\} \). Finally, it is direct to check that the homology classes \([s^* (\mathcal{N}^{an})]\) do not depend on the choices of the sections \( s \) of \( \mathcal{F} = \{F_i\} \) so long as they satisfy the obvious transversality conditions. Therefore,
\[
[s^* (\mathcal{N}^{an})] = [r^* (\mathcal{N}^{an})] = [C_{1/\epsilon}] = e(\Phi : \mathcal{B} \to \mathcal{E}).
\]
This proves the Proposition.

In the end, we will compare the algebraic normal cones with the analytic normal cones to demonstrate that the algebraic and analytic construction of the GW-invariants give rise to the identical invariants.

Here is our strategy. Taking the good atlas \( \{(Z_i, \mathcal{X}_i)\}_{i \in \Lambda} \) of \( \mathfrak{g} \mathfrak{l}_{g,n}(X, A) \) as before, we have two collections of semi-\( \mathbb{Q} \)-vector bundles, namely \( \mathcal{E} = \{E_i\} \) and \( \mathcal{F} = \{F_i\} \), and two collections of cone currents \( \mathcal{M}^{alg} = \{M_i^{alg}\} \) and \( \mathcal{N}^{an} = \{N_{t \phi_i}\} \) such that \( \eta_{E_i}(\mathcal{M}^{alg}) \) and \( \eta_{F_i}(\mathcal{N}^{an}) \) are the algebraic and the symplectic virtual moduli cycles of \( \mathfrak{g} \mathfrak{l}_{g,n}(X, A) \) respectively. Here \( \eta_{E_i} \) and \( \eta_{F_i} \) are generic sections of \( \mathcal{E} \) and \( \mathcal{F} \) respectively. To compare these two classes, we will form a new semi-\( \mathbb{Q} \)-vector bundle \( \mathcal{V} = \{V_i\} \), where \( V_i = E_i \oplus F_i \), and construct a stratifiable cone current \( \mathcal{P} \) in \( \mathcal{V} \) such that the cycle \( \mathcal{P} \) intersect \( \mathcal{E} \subset \mathcal{V} \) and \( \mathcal{F} \subset \mathcal{V} \) transversally and the intersection \( \mathcal{P} \cap \mathcal{E} \) and \( \mathcal{P} \cap \mathcal{F} \) are \( \mathcal{M}^{alg} \) and \( \mathcal{N}^{an} \) respectively. Therefore, if we let \( \eta_{\mathcal{V}} \) be a generic section of \( \mathcal{V} \), then
\[
[\eta_{E_i}(\mathcal{M}^{alg})] = [\eta_{F_i}(\mathcal{P})] = [\eta^*_F(\mathcal{N}^{an})] \in H_*(\mathfrak{g} \mathfrak{l}_{g,n}(X, A); \mathbb{Q}).
\]
This will prove the Comparison Theorem.

We now provide the details of this argument. We begin with any index \( i \in \Lambda \) and an open subset \( S \subset Z_i \). Let \( f : C \to X \) be the restriction to \( S \) of the tautological family \( X_i \) of stable maps over \( Z_i \), with \( D \subset C \) the divisor of its \( n \)-marked sections and \( \pi : C \to S \) the projection. Note that \( f \) is the restriction of a
family of stable morphisms over a scheme to an analytic open subset of the base scheme. Following the construction in [LT1] section 3, after fixing a sufficiently ample line bundle over $X$, we canonically construct a locally free sheaf of $\mathcal{O}_C$-modules $K$ so that $f^*\Omega_X$ is canonically a quotient sheaf of $K$. Let $L$ be the kernel of $K \to f^*\Omega_X$. Then the restriction to $S$ of the sheaf $\mathcal{E}_{s,1}$ (resp. $\mathcal{E}_{s,2}$) mentioned before is the relative extension sheaf $\mathcal{E}xt^1(\mathcal{K} \to \Omega_{C/S}(D), \mathcal{O}_C)$ (resp. $R\pi_*(\mathcal{L}^\vee)$). We denote them by $\mathcal{E}_{S,1}$ and $\mathcal{E}_{S,2}$ respectively. As usual, we let $\mathcal{E}_{S,1}$ and $\mathcal{E}_{S,2}$ be the associated vector bundle of $\mathcal{E}_{S,1}$ and $\mathcal{E}_{S,2}$ respectively. Following the notation in [LT1], the tangent-obstruction complex $[\mathcal{T}_S^1 \to \mathcal{T}_S^2]$ of $\mathcal{X}_1|_S$ is

$$\left[ \mathcal{E}xt^1_\mathcal{K}(f^*\Omega_X \to \Omega_{C/S}(D), \mathcal{O}_C) \rightarrow \mathcal{E}xt^2_\mathcal{K}(f^*\Omega_X \to \Omega_{C/S}(D), \mathcal{O}_C) \right],$$

and that there is a canonical homomorphism $\epsilon: \mathcal{E}_{S,1} \to \mathcal{E}_{S,2}$ so that the kernel and the cokernel of $\epsilon$ are $\mathcal{T}_S^1$ and $\mathcal{T}_S^2$ respectively. The homomorphism $\epsilon$ is the middle arrow in the sequence (4.3).

We now assume that there is an analytic approximation $\alpha \in \Xi$ so that $(R_1, W_1, \phi_1)$ is finer than $\alpha$ and $\iota_1(S) \subset B$ is contained in $\iota_\alpha(Z_\alpha)$. Let $\rho_\alpha: Z_\alpha \to Z_\alpha$ be induced by $f_\alpha^*: R_\alpha \to R_\alpha$ (see (4.2)). Let $\mathcal{F}_{S,\alpha}$ be the vector bundle over $Z_\alpha$ that is the pull back of $\mathcal{F}_\alpha$. Note that $\mathcal{F}_{S,\alpha}$ is a smooth vector bundle. Let $G_{S,\alpha,2} = \mathcal{E}_{S,2} \oplus \mathcal{F}_{S,\alpha}$. In the following, we will construct a holomorphic vector bundle $G_{S,\alpha,1}$ and a possibly degenerate vector bundle homomorphism $\beta$ and non-degenerate vector bundle inclusions $\tau_{\alpha,j}$ as shown below so that

$$\begin{array}{c}
\mathcal{E}_{S,1} \xrightarrow{\epsilon} \mathcal{E}_{S,2} \\
\downarrow \tau_{\alpha,1} \quad \quad \quad \quad \downarrow \tau_{\alpha,2} \\
\mathcal{G}_{S,\alpha,1} \xrightarrow{\beta} \mathcal{G}_{S,\alpha,2}
\end{array} \tag{4.6}$$

is commutative. Let $w$ be any point in $S$. We denote by $C_w$ the fiber of $C$ over $w$ and let $f_w$ (resp. $K_w$, resp. $\mathcal{L}_w$) be the restriction of the respective objects to $C_w$. As before, for any locally free sheaf of $\mathcal{O}_{C_w}$-modules $W$ that is locally free away from the nodal points of $C_w$, we denote by $W^A_w$ the sheaf whose stalk at nodal points $z$ of $C_w$ are $W_z$ and its stalks at smooth points $z$ of $C_w$ are germs of smooth sections of the associated vector bundle of $W$ at $z$. We let $G_{S,\alpha,1}|_w$ be the vector space of the equivalence classes of commutative diagrams

$$\begin{array}{cccc}
\mathcal{K}_w & \longrightarrow & f_w^*\Omega_X \\
\downarrow h & & \downarrow df_w^\vee \\
0 & \longrightarrow & \mathcal{O}_{C_w}^A & \longrightarrow & \mathcal{B}_w^A \longrightarrow \Omega_{C_w}(D_w)^A \longrightarrow 0
\end{array} \tag{4.7}$$

such that the lower exact sequences are induced by the exact sequences of sheaves of $\mathcal{O}_{C_w}$-modules

$$\begin{array}{cccc}
0 & \longrightarrow & \mathcal{O}_{C_w} & \longrightarrow & \mathcal{B}_w \longrightarrow \Omega_{C_w}(D_w) \longrightarrow 0
\end{array}$$

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and that $h$ satisfies the following two requirements. First, let $c: \mathcal{L}_w \to \mathcal{B}^1_w$ be the composite of $\mathcal{L}_w \to \mathcal{K}_w$ with $h$. Since $\mathcal{L}_w$ is the kernel of $\mathcal{K}_w \to f^*_w \Omega_X$, $c$ automatically lifts to $h_E: \mathcal{L}_w \to \mathcal{O}^1_{C_w}$. The first requirement is that $h_E$ is holomorphic. Secondly, since both $\mathcal{K}_w$ and $\mathcal{L}_w$ are sheaves of $\mathcal{O}_{C_w}$-modules and since $h$ is analytic near nodal points of $C_w$, $\partial h$ is a $(0,1)$-form with compact support $[\square]$ taking values in the associated vector bundle of $\mathcal{K}_w^\vee \otimes \mathcal{O}_{C_w}^{-1}$. Because of the first requirement, it factors through a section $h_F$ of $\Omega^1_{\mathcal{O}_{C_w}}(f^*_w T_X)$. We require that $h_F$ is an element in $\rho^*_w W_\alpha|_w$. Using Lemma 3.2 and Corollary 3.3 and the fact that $\mathcal{K}_w^\vee$ is sufficiently ample which was the precondition on our choice of $\mathcal{K}$, it is direct to check that the collection $\{G_{S,\alpha,1}|w \mid w \in S\}$ forms a smooth vector bundle, denoted $G_{S,\alpha,1}$, and the correspondence that sends (4.7) to $h_E - h_F$ form a possibly degenerate vector bundle homomorphism $\beta: G_{S,\alpha,1} \to G_{S,\alpha,2}$.

We next define the homomorphisms $\tau_{\alpha,j}$. The homomorphism $\tau_{\alpha,2}: E_{S,2} \to G_{S,\alpha,2}$ is the obvious homomorphism based on the definition $G_{S,\alpha,2} = E_{S,2} \oplus F_{S,\alpha}$. For $\tau_{\alpha,1}$, we recall that for any $w \in S$ the vector space $E_{S,1}|w$ is the set of equivalence classes of the diagrams (4.7) of which the $h$ are holomorphic. Namely, $h$ are induced by homomorphisms $f^*_w \Omega_X \to \mathcal{B}$. Hence $E_{S,1}$ is canonically a subbundle of $G_{S,\alpha,1}$. This shows that both $\tau_{\alpha,1}$ and $\tau_{\alpha,2}$ are inclusions of vector bundles. Finally, let $\xi \in E_{S,1}|w$ be any element associated to the diagram (4.7), then $\epsilon(\xi)$ is the section of $\mathcal{L}_w^\vee$ that is the lift of $\mathcal{L}_w \to \mathcal{K}_w \to \mathcal{B}_w$ to $\mathcal{L}_w \to \mathcal{O}_{C_w}$. It follows that the square of (1.6) is commutative. We now show that $\text{Coker}(\tau_{\alpha,1}) = \text{Coker}(\tau_{\alpha,2})$. It suffices to show that the sequence

$$0 \to E_{S,1} \xrightarrow{\tau_{\alpha,1}} G_{S,\alpha,1} \xrightarrow{c} F_{S,\alpha} \to 0,$$

where $c$ is the composite of $\beta$ with $G_{S,\alpha,2} \to F_{S,\alpha}$, is an exact sequence. But this follows directly from the definition of $G_{S,\alpha,1}$ and Lemma 3.2 and Corollary 3.3. This proves that $\text{Coker}(\tau_{\alpha,1}) = \text{Coker}(\tau_{\alpha,2})$, and consequently

$$\text{Coker}(\beta|_w) = \text{Coker}(\epsilon|_w) = T^2_{S}|_w$$

for any $w \in S$.

In the following, we will construct the cone current $Q_{S,\alpha} \in Z_* G_{S,\alpha,2}$. We first pick a subbundle $H_{\alpha} \subset G_{S,\alpha,2}$ such that $H_{\alpha} \to G_{S,\alpha,2} \to \text{Coker}(\tau_{\alpha,1})$ is an isomorphism. We let $P_{\alpha}: G_{S,\alpha,2} \to E_{S,2}$ be the projection so that $\text{ker}(P_{\alpha}) = \beta(H_{\alpha})$ and $P_{\alpha} \circ \tau_{\alpha,2} = 1_{E_{S,2}}$. We then take $Q_{S,\alpha}$ to be the flat pull back current $P_{\alpha}^*(M_{i_{\alpha}})$ in $Z_* G_{S,\alpha,2}$. It follows that $Q_{S,\alpha}$ intersects the subbundle $E_{S,2} \subset G_{S,\alpha,2}$ transversally and the intersection $Q_{S,\alpha} \cap E_{S,2}$ is exactly $M_{i_{\alpha}}^{\text{alg}}$. In the following, we will demonstrate that $Q_{S,\alpha}$ intersects the subbundle $F_{S,\alpha} \subset G_{S,\alpha,2}$ transversally as well and that the intersection $Q_{S,\alpha} \cap F_{S,\alpha}$ is the current $\rho^*_\alpha(N^\alpha_m) \subset Z_* F_{S,\alpha}$. Let $w \in S \subset Z_i$ be any point. Since $T^w_{S,\alpha}$, where $w' = \rho^*_\alpha(w)$, is the vector space $\text{Ext}^1(D^*_w, \mathcal{O}^1_{C_w})^\dagger$, there is a canonical injective homomorphism $\sigma_w: T^w_{S,\alpha} \to G_{S,\alpha,1}|w$ of vector spaces that send the

$^3$By which we mean that $\partial h$ vanishes in a neighborhood of the nodal points of $C_w$. 39
diagram (3.3) to (4.7) with $K_w \to B^\alpha_w$ the composite of $K|_w \to f^*_w\Omega_X$ with the $v_2$ in (3.3). It is easy to see that the collection $\{\sigma_w\}_{w \in S}$ forms a smooth non-degenerate vector bundle homomorphism $\sigma : \rho^*_\alpha(TR_\alpha) \to G_{S,\alpha,1}$. If follows from the description of

$$\rho^*_\alpha(d\phi_\alpha) : \rho^*_\alpha(TR_\alpha) \to F_{S,\alpha}$$

that the diagram of vector bundle homomorphisms

$$\begin{array}{ccc}
G_{S,\alpha,1} & \xrightarrow{\beta} & G_{S,\alpha,2} \\
\uparrow & & \uparrow \\
\rho^*_\alpha(TR_\alpha) & \xrightarrow{\rho^*_\alpha(d\phi_\alpha)} & F_{S,\alpha}
\end{array} \tag{4.10}$$

is commutative, where the second vertical arrow is the obvious inclusion.

To compare $Q_{S,\alpha}$ with $\rho^*_\alpha(N^\alpha_{\text{an}})$, we need the following two lemmas.

**Lemma 4.6.** Let $w \in S$ be any point and let $w' = \rho_\alpha(w)$. Let $d_2 : G_{S,\alpha,2|w} \to T^2_\alpha|_w$ be the homomorphism induced by (4.9) and let $F_{S,\alpha}|_w \to T^2_\alpha|_w$ be the canonical homomorphism given in Lemma 3.7. Then the following squares are commutative:

$$\begin{array}{ccc}
F_{S,\alpha}|_w & \xrightarrow{\subset} & G_{S,\alpha,2|w} \\
d_3 & & d_1 \\
T^2_\alpha|_w & \xrightarrow{\tau_{\alpha,2}} & E_{S,2}
\end{array} \tag{4.11}$$

**Lemma 4.7.** For any point $w \in Z_\alpha$, the germ of $\phi_\alpha : R_\alpha \to W_\alpha$ at $w$ is a Kuranishi map of the standard obstruction theory of the deformation of stable morphisms associated to the exact sequence

$$0 \to T^1_\alpha|_w \to T_w R_\alpha \to F_\alpha|_w \to T^2_\alpha|_w \to 0.$$

**Proof.** We first prove Lemma 4.6. Since $G_{S,\alpha,2} = E_{S,2} \oplus F_{S,\alpha}$, $d_1$ and $d_3$ induces a homomorphism $G_{S,\alpha,2|w} \to T^2_\alpha|_w$. To prove the lemma, it suffices to show that $d_2 = d_1 \oplus d_3$. To accomplish this, we only need to show that for any $\xi \in G_{S,\alpha,1|w}$ with $\xi_E$ and $-\xi_F$ its two components of $\beta(\xi)$ according to the direct sum decomposition $G_{S,\alpha,2|w} = E_{S,2|w} \oplus F_{S,\alpha}|_w$, then $d_1(\xi_E) = d_3(\xi_F)$.

To prove this, we first pick an $h_0 : f^*_w\Omega_X \to B^\alpha_w$ such that

$$\begin{array}{ccc}
f^*_w\Omega_X & \xrightarrow{\subset} & f^*_w\Omega_X \\
\downarrow h_0 & & \downarrow df^+_w \\
B^\alpha_w & \xrightarrow{\Omega_{C_w}(D_w)^A} & \Omega_{C_w}(D_w)^A
\end{array} \tag{4.12}$$

is commutative. Let $h'_0$ be the composite of $K_w \to f^*_w\Omega_X$ with $h_0$. Then $h' - h_0$ factor through $O^A_{C_w} \to B^\alpha_w$, say $h : K_w \to O^A_{C_w}$. Clearly, $h$ composed with
\( \mathcal{L}_w \to \mathcal{K}_w \) is the section \( \xi_E \in H^0(\mathcal{L}_w^\vee) \). On the other hand, the lift of \( \bar{\partial}h \) to \( \Omega^0_{\text{cpt}}(f^*_wT_X) \) is \( \xi_E - (\bar{\partial}h_0)\text{lift} \). By the definition of the connecting homomorphism \( \delta : H^0(\mathcal{L}_w^\vee) \to H^1(f^*_w\Omega^1_X) \),

\[ \delta(\xi_E) = \text{the image of } (\xi_E - (\bar{\partial}h_0)\text{lift}) \text{ in } H^0(\mathcal{L}_w^\vee) \cong H^1(f^*_w\Omega^1_X). \]

However, the image of \( (\bar{\partial}h_0)\text{lift} \) is contained in the image of the connecting homomorphism

\[ \text{Ext}^1(\mathcal{O}_{\mathcal{C}_w}(D_w), \mathcal{O}_{\mathcal{C}_w}) \to \text{Ext}^2([f^*_w\Omega_X \to 0], \mathcal{O}_{\mathcal{C}_w}) \cong H^1(f^*_w\Omega^1_X). \]

Hence \( d_1(\xi_E) = d_3(\xi_E). \) This proves Lemma 4.6. ☐

**Proof.** We now prove Lemma 4.6. Let \( I \subset B \) be an ideal of an Artin ring annihilated by the maximal ideal \( \mathfrak{m}_B \) and let \( \varphi : \text{Spec } B/I \to R_\alpha \) be a morphism that sends the closed point of \( \text{Spec } B/I \) to \( w \) and such that \( \varphi^*(\phi_\alpha) = 0 \). By the description of the tautological family \( X_\alpha \) over \( R_\alpha \), the pull back \( \varphi^*(X_\alpha) \) forms an algebraic family of stable morphisms over \( \text{Spec } B/I \). We continue to use the open covering of the domain \( X_\alpha \) used before. Since \( R_\alpha \) is smooth, we can extend \( \varphi \) to \( \tilde{\varphi} : \text{Spec } B \to R_\alpha \). Let \( C_B \) over \( \text{Spec } B \) be the domain of the pull back of the domain of \( X_\alpha \) via \( \tilde{\varphi} \) and let \( C_B/I \) be the domain of \( C_B \) over \( \text{Spec } B/I \). We let \( \{U_i\} \) (resp. \( \{U_i\} \)) be the induced open covering of \( C_B/I \) (resp. \( C_B \)) and let \( f_i : U_i \to X \) be the restriction to \( U_i \) of the pull back of the stable maps in \( X_\alpha \). Because \( \varphi^*(\phi_\alpha) = 0 \), \( f_i \) are holomorphic. Hence they define a morphism \( f : C_B/I \to X \). Now we describe the obstruction to extending \( f \) to \( \text{Spec } B \). Let \( C_0 \) be the closed fiber of \( C_B \) and let \( f_0 : C_0 \to X \) be the restriction of \( f \). For each \( i \), we pick a holomorphic extension \( \tilde{f}_i : \tilde{U}_i \to X \) of \( f_i \) over \( U_i \). Then over \( \tilde{U}_{ij} = \tilde{U}_i \cap \tilde{U}_j \), \( \tilde{f}_j - \tilde{f}_i \) is canonically an element in \( \Gamma(f^*_i\Omega_X|_{\tilde{U}_{ij}}) \otimes I \), denoted by \( f_{ij} \). Further, the collection \( \{f_{ij}\} \) is a cocycle and hence defines an element \( f_{ij} \in H^2(\text{Ext}^1(D^*, \mathcal{O}_{\mathcal{C}_w}) \otimes I \). The obstruction to extending \( f \) to \( \text{Spec } B \) is the image of \( f_{ij} \) in \( H^2(\text{Ext}^1(D^*, \mathcal{O}_{\mathcal{C}_w}) \otimes I \). By the construction of \( \mathcal{O} \) and \( \mathcal{O}_{\mathcal{C}_w} \), the obstruction is annihilated by \( \mathfrak{m}_B \) and \( \mathfrak{m}_B \) is the section \( \xi_E \). Hence \( \delta h \) is a section of \( \Gamma(\Omega^0_{\text{cpt}}(f^*_wT_X)|_{U_i \cap C_0}) \otimes I \). Clearly they patch together to form a global section \( \gamma \) of \( \mathcal{O}^0_{\text{cpt}}(f^*_wT_X) \otimes I \). The element \( \gamma \) can be also defined as follows. Let \( \varphi^* : \mathcal{O}_{R_\alpha} \to B \) be the induced homomorphism on rings. Then since the image of \( \varphi^*_w(\phi_\alpha) \in B \otimes \mathcal{O}_{R_\alpha} \mathcal{O}_{R_\alpha}(W_\alpha) \) in \( B/I \otimes \mathcal{O}_{R_\alpha} \mathcal{O}_{R_\alpha}(W_\alpha) \) vanishes, it induces an element \( \gamma' \in I \otimes W_\alpha \). By our construction of \( R_\alpha \) and \( \phi_\alpha \), \( \gamma' \) coincides with \( \gamma \) under the inclusion \( W_\alpha|_w \subset \Gamma_{R_\alpha}(\Omega^0_{\text{cpt}}(f^*_wT_X)) \). Let \( \text{ob}^\text{an} \) be the image of \( \gamma \) in the cokernel of \( d\phi_\alpha(w) : T_w R_\alpha \to W_\alpha|_w \). By definition, \( \text{ob}^\text{an} \) is the obstruction to extending \( \varphi \) to \( \tilde{\varphi} : \text{Spec } B \to \{\phi_\alpha = 0\} \).
To finish the proof of the lemma, we need to show that $ab^{\text{alg}} = ob^{\text{an}}$ under the isomorphism

$$\text{Coker}\{\delta_\alpha(w)\} \cong \text{Ext}^1(D^*_w, O_{C_u})$$

given in Lemma 3.7. For this, it suffices to show that the Dolbeault cohomology class of $\gamma$, denoted $[\gamma] \in H^0_\partial(f_0^*T_X) \otimes I$, coincides with the Cech cohomology class $[f_{ij}] \in H^1(f_0^*T_X) \otimes I$ under the canonical isomorphism $H^0_\partial(f_0^*T_X) \cong H^1(f_0^*T_X)$. But this is obvious since $\varphi_i = \tilde{f}_i - g_i$ is in $\Gamma_{U_i \cap C_0}(\Omega^0_{\text{cpt}}(f_0^*T_X)) \otimes I$ such that $\varphi_j - \varphi_i = f_{ij}$ and $\partial \varphi_i = -\partial g_i$. Hence, $[f_{ij}] = [\gamma]$ under the given isomorphism. This proves the lemma.

Now we come back to $Q_{S,\alpha} \in Z_4G_{S,\alpha,2}$. Let $w \in S$ be any point, let $\tilde{w}$ be the formal completion of $S$ along $w$, let $V_w$ be the total space of $T^*_S|w$ and let $N^0_w \subset V_w \times \tilde{w}$ be the the cone in Theorem 4.3. We let $M^\text{alg}_S$, $N^\text{an}_S = \rho^*_i(N^\text{an}_S)$ and $Q_{S,\alpha}$ be the cone currents in $E_{S,2}$, $F_{S,\alpha}$ and $G_{S,\alpha,2}$ respectively as before. Note that they are supported on union of closed subsets each diffeomorphic to analytic variety. By Theorem 4.2, we have vector bundle homomorphisms

$$e_1 : E_{S,2} \times_S \tilde{w} \to V_w \times \tilde{w}, e_3 : F_{S,\alpha} \times_S \tilde{w} \to V_w \times \tilde{w}$$

extending $E_{S,2}|w \to T^*_S|w$ and $F_{S,\alpha}|w \to T^*_S|w$ such that $e_1^*(N^0_w)$ and $e_3^*(N^0_w)$ are the restrictions of $M^\text{alg}_S$ and $N^\text{an}_S$ to fibers over $\tilde{w}$ in $S$ respectively. Let $e_2 : G_{S,\alpha,2} \times_S \tilde{w} \to V_w \times \tilde{w}$ be induced by $P_\alpha : G_{S,\alpha,2} \to E_{S,2}$ and $e_1$. Then $e_2^*(N^0_w)$ is the restriction of $Q_{S,\alpha}$ to $G_{S,\alpha,2} \times_S \tilde{w}$. Because the squares in 4.11 are commutative,

$$e_2 : F_{S,\alpha} \times_S \tilde{w} \to G_{S,\alpha,2} \times_S \tilde{w} \xrightarrow{\oplus} V_w \times \tilde{w}$$

is surjective. Hence $F_{S,\alpha} \times \tilde{w}$ intersects $Q_{S,\alpha}$ transversally along fiber over $w$. Let $e'_3 : F_{S,\alpha} \times_S \tilde{w} \to V_w \times \tilde{w}$ be induced by $F_{S,\alpha} \to G_{S,\alpha,2}$ and $e_2$, then the intersection of $Q_{S,\alpha}$ with $F_{S,\alpha} \times_S \tilde{w}$ is $(e'_3)^*(N^0_w)$. However, by the choice of $P_\alpha$, we have $e'_3 \equiv e_3|w$, therefore the support of $Q_{S,\alpha} \cap F_{S,\alpha}|w$ is identical to the support of $N^\text{an}_{S,\alpha}|w$. Because $w \in S$ is arbitrary, the support of $Q_{S,\alpha} \cap F_{S,\alpha}$ is identical to the support of $N^\text{an}_{S,\alpha}$. Further, for the same reason, for general point $p$ in $N^\text{an}_{S,\alpha}$, the multiplicity of $N^\text{an}_{S,\alpha}$ at $p$ is identical to the multiplicity of the corresponding point in $Q_{S,\alpha} \cap F_{S,\alpha}$. This proves that the cycles (or currents) $Q_{S,\alpha}$ intersect $F_{S,\alpha} \subset G_{S,\alpha,2}$ transversally and $Q_{S,\alpha} \cap F_{S,\alpha} = N^\text{an}_{S,\alpha}$. We remark that for the same reason, the current $Q_{S,\alpha}$ is independent of the choice of the subbundles $H_\alpha \subset G_{S,\alpha,2}$.

We now let $F_S = F_1|S$ and let $G_{S,2} = E_{S,2} \oplus F_S$. Note that $G_{S,\alpha,2} \subset G_{S,2}$. Because $R_1$ is finer than $R_\alpha$, $\rho^*_\alpha TR_\alpha$ is a subbundle of $TR_1|S$. Let $K_\alpha \subset TR_1|S$ be a complement of $\rho^*_\alpha TR_\alpha \subset TR_1|S$ and let $d\delta_i(K_\alpha) \subset F_S$ be the image of this subbundle. Let $P_{S,\alpha} : F_S \to F_{S,\alpha}$ be the projection so that $\ker P_{S,\alpha} = d\delta_i(K_\alpha)$ and the composite of $F_{S,\alpha} \subset F_S$ with $P_{S,\alpha}$ is $1_{F_{S,\alpha}}$. By Lemma 4.4, $N^\text{an}_{S,\alpha} = P_{S,\alpha}^*(N^\text{an}_{S,\alpha})$. Now let $P_S$ be the projection

$$P_S = P_\alpha \circ (1_{E_{S,2}} \oplus P_{S,\alpha}) : G_{S,2} \to G_{S,\alpha,2} \to E_{S,2}$$

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and let $Q_S = P^*_S(M_{\text{tr}}^{\text{an}})$ be the pull back cone. Let $\tilde{d}_3$ be

$$
\tilde{d}_3 : F_S|_w \xrightarrow{P_{S,\alpha}|_w} F_{S,\alpha}|_w \xrightarrow{d_3} T^2_S|_w,
$$

then clearly we have a commutative diagram of vector spaces

$$
\begin{array}{c}
F_S|_w \xrightarrow{d_3} G_{S,2}|_w \leftarrow E_{S,2}|_w \\
\downarrow \quad \downarrow \quad \downarrow \\
T^2_S|_w \quad \quad \quad T^2_S|_w \quad \quad \quad T^2_S|_w
\end{array}
$$

(4.13)

Because $w$ is arbitrary, similar to the previous case, we have that $F_S$ intersects $Q_S$ transversally and $F_S \cap Q_S = N^\text{an}_{\text{tr}}|_S$, as stratifiable currents.

To enable us to patch $Q_S$, where $S \subset Z_i$, to form a current in $G_{i,2} = E_{i,2} \oplus F_i$, we need to show that $Q_S$ is independent of the choice of analytic chart $\alpha$. Namely if we let $\beta \in \Xi$ be another analytic chart so that $i_\beta(S) \subset i_\beta(Z_\beta)$, then the cone current $Q'_S \subset G_{S,2}$ constructed using $F_\beta$, etc., is identical to $Q_S$. Again, following the same argument before, it suffices to show that the homomorphism $d_1 : F_S|_w \rightarrow T^2_S|_w$ does not depend on the choice of $\alpha$. Note that $\tilde{d}_3$ also fits into the commutative diagram of exact sequences

$$
\begin{array}{c}
T_{\rho_\alpha(w)} R_\alpha \xrightarrow{d\phi_\alpha(\rho_\alpha(w))} F_\alpha|_{\rho_\alpha(w)} \xrightarrow{} T^2_\alpha|_{\rho_\alpha(w)} \xrightarrow{} 0 \\
\downarrow \quad \downarrow \quad \| \\
T_w R_i \xrightarrow{d\phi_i(w)} F_S|_w \xrightarrow{} T^2_S|_w \xrightarrow{} 0.
\end{array}
$$

(4.14)

Now assume $\beta \in \Xi$ as before. Without loss of generality, we can assume that near $w$, the vector subbundles $\rho_\alpha F_\alpha$ and $\rho_\beta F_\beta$ span a $2l$-dimensional subvector bundle of $F_i$. Now let $V_\alpha \rightarrow \tilde{U}_\alpha$ and $V_\beta \rightarrow \tilde{U}_\beta$ be the vector bundles that define $R_\alpha$ and $R_\beta$ as in section 2 and let $V_{\alpha \beta} \rightarrow \tilde{U}_i$ be the direct sum of the pull back of $V_\alpha$ and $V_\beta$ via the tautological map $\tilde{U}_i \rightarrow \tilde{U}_\alpha$ and $\tilde{U}_i \rightarrow \tilde{U}_\beta$. Then near a neighborhood of $w \in \tilde{U}_i$, the set $\tilde{\Phi}^{-1}(V_{\alpha \beta})$ will form a base of a smooth approximation containing $w$. We denote $R_{\alpha \beta} = \tilde{\Phi}_i^{-1}(V_{\alpha \beta})$ and let $\phi_{\alpha \beta} : R_{\alpha \beta} \rightarrow V_{\alpha \beta}|_{R_{\alpha \beta}}$ be the lift of $\Phi_i$. Clearly, $R_i$ is still finer than $R_{\alpha \beta}$. Hence we have commutative diagrams

$$
\begin{array}{c}
T_{\rho_\alpha(w)} R_\alpha \xrightarrow{d\phi_\alpha(\rho_\alpha(w))} V_\alpha|_w \xrightarrow{} T^2_\alpha|_{\rho_\alpha(w)} \xrightarrow{} 0 \\
\downarrow \quad \downarrow \quad \| \\
T_w R_{\alpha \beta} \xrightarrow{d\phi_{\alpha \beta}(w)} V_{\alpha \beta}|_w \xrightarrow{} T^2_i|_w \xrightarrow{} 0 \quad (4.15) \\
\downarrow \quad \downarrow \quad \| \\
T_w R_i \xrightarrow{d\phi_i(w)} F_i|_w \xrightarrow{} T^2_i|_w \xrightarrow{} 0
\end{array}
$$

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with exact rows. Note that \( V_{\alpha \beta} |_w \to T^2_i |_w \) is equal to
\[
V_{\alpha \beta} |_{\rho_i (w)} \oplus V_{\beta} |_{\rho_j (w)} \to \Gamma (\Omega^{0,1}_{\text{cpt}}(f^*_w T_X)) \to H^{0,1}_j (f^*_w T_X) \to T^2_i |_w.
\]
(Here that \( V_{\alpha \beta} |_w \to T^2_i |_w \) is defined apriori but not \( F_i |_w \to T^2_i |_w \) because elements of \( V_{\alpha \beta} |_w \) and \( V_{\beta} |_w \) are \((0,1)\)-forms with compact support.) Therefore, the homomorphism \( d_3 \) defined earlier is independent of the choice of \( \alpha \).

Now we are ready to prove the theorem. Let \( i \in \Lambda \) be any approximation and let \( \{ S_a \} \) be an open covering of \( Z_i \) so that to each \( a \) there is an \( \alpha_a \in \Xi \) so that \( \iota_\alpha(S_a) \subset \iota_\alpha(Z_{\alpha_a}) \). We let \( G_{i,2} = E_{i,2} \oplus F_i \) and let \( Q_{S_a} \) be the cone in \( G_{i,2} |_{S_a} \) constructed before using the analytic chart \( \alpha \). We know that over \( G_{i,2} |_{S_a \cap S_b} \), the currents \( Q_{S_a} \) and \( Q_{S_b} \) coincide. Hence \( \{ Q_{S_a} \} \) patches together to form a stratifiable current, denoted \( Q_i \). Assume that \( j < i \in \Lambda \) be any two indices. Let \( Z_{i,j} \subset Z_i \) be the open subset \( \iota_\alpha^{-1}(\iota_j(Z_j)) \) and let \( f^j_i : Z_{i,j} \to Z_j \) be the map induced by \( Z_i \) being finer than \( Z_j \). Then \( (f^j_i)^*(F_j) \) is canonically a subbundle of \( F_i |_{Z_{i,j}} \), and \( (f^j_i)^*(E_{j,2}) \) is canonically isomorphic to \( E_{i,2} |_{Z_{i,j}} \). Let \( (f^j_i)^*(G_{j,2}) \to G_{i,2} |_{Z_{i,j}} \) be the induced homomorphism. It follows from the previous argument that \( Q_i \) intersects \( (f^j_i)^*(G_{j,2}) \) transversally and the intersection \( Q_i \cap (f^j_i)^*(G_{j,2}) \) is \( (f^j_i)^*(Q_j) \). Finally, by our construction, \( Q_i \) intersects transversally with \( E_{i,2} \) and \( F_i \subset G_{i,2} \), and \( E_{i,2} \cap G_i = M^\text{alg}_i \) and \( F_i \cap G_i = N^\text{an}_i \). Let \( \mathcal{G} \) be the semi-\( \mathbb{Q} \)-vector bundle \( \{ G_{i,2} \} \), which is \( \mathcal{E} \oplus \mathcal{F} \), and let \( \mathcal{Q} \) be the cone \( \{ Q_i \} \). It follows from the perturbation argument in section two that for generic sections \( \eta_E, \eta_F \) and \( \eta_G \) of \( \mathcal{E} \), \( \mathcal{F} \) and \( \mathcal{G} \) respectively, we have
\[
[\mathcal{M}_{g,n}(X, A)]^{\text{vir}} = [\eta_E^* \mathcal{M}^\text{alg}_d] = [\eta_F^* \mathcal{N}^\text{an}_d] = e[\Phi : B \to E].
\]
This proves the comparison theorem.

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