Filtration, automorphisms and classification of the infinite dimensional odd Contact superalgebras

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Abstract: The principal filtration of the infinite-dimensional odd Contact Lie superalgebra over a field of characteristic $p > 2$ is proved to be invariant under the automorphism group by investigating ad-nilpotent elements and determining certain invariants such as subalgebras generated by some ad-nilpotent elements. Then, it is proved that two automorphisms coincide if and only if they coincide on the $-1$ component with respect to the principal grading. Finally, all the odd Contact superalgebras are classified up to isomorphisms.

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0. Introduction

As is well known, filtration techniques are of great importance in the structure and classification theories of Lie (super)algebras. A descending filtration of a Lie superalgebra $L$ is a sequence of $\mathbb{Z}_2$-graded spaces $L = L_0 \supset L_1 \supset \cdots$ for which $[L_i, L_j] \subset L_{i+j}$ holds for all $i, j$. In the situation that the Killing form on a simple Lie (super)algebra is degenerate, the filtration structure plays a particular role. A filtration $L = L_0 \supset L_1 \supset \cdots$ of a Lie superalgebra $L$ is said to be invariant provided that $\varphi(L_i) \subset L_i$ for all $i$ and all automorphisms $\varphi$ of $L$. We know that the simple Lie (super)algebras of Cartan type possess various natural filtration structures, for which the invariance may be used to make an insight for the intrinsic properties and the automorphism groups of those Lie (super)algebras. The filtration structures have been studied for the finite dimensional Lie superalgebras of Cartan type, for example, in [4, 7, 9] the invariance of the natural filtrations was determined for the generalized Witt superalgebras, the special superalgebras, the Hamiltonian superalgebras and the odd Hamiltonian superalgebras.

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The reader is also referred to [5, 8] for the similar work on certain infinite dimensional modular Lie superalgebras of Cartan type. Let us state the main results of this paper.

Write $KO(n, n + 1)$ for the infinite dimensional odd Contact Lie superalgebras over a field of prime characteristic (see Sec.1 for a definition and more details). $KO(n, n + 1)$ has a canonical filtration structure known as principal. By means of characterizing ad-nilpotent elements of $KO(n, n + 1)$ we first obtain in this paper that:

- (Theorem 4.3) The principal filtration of the odd Contact Lie superalgebra is invariant under the automorphism group of the Lie superalgebra.

As a consequence the automorphisms can be characterized as follows:

- (Theorem 4.4) Two automorphisms of the odd Contact Lie superalgebra coincide if and only if they coincide on the $−1$ component with respect to the principal grading.

Finally we classify all the infinite dimensional odd Contact Lie superalgebras up to isomorphisms:

- (Theorem 4.5) $KO(n, n + 1) ≅ KO(m, m + 1)$ if and only if $n = m$.

1. Preliminaries

Throughout $\mathbb{F}$ is a field of characteristic $p > 2$; $\mathbb{Z}_2 := \{0, 1\}$ is the additive group of two elements; $\mathbb{N}$ and $\mathbb{N}_0$ are the sets of positive integers and nonnegative integers, respectively. Let $\mathcal{O}(n)$ be the divided power algebra with $\mathbb{F}$-basis $\{x^\alpha \mid \alpha \in \mathbb{N}_0^n\}$. Note that $x^{(0)} := 1 \in \mathcal{O}(n)$, where $0 = (0, \ldots, 0) \in \mathbb{N}_0^n$. For $\bar{v} := (\delta_{i1}, \delta_{i2}, \ldots, \delta_{in}) \in \mathbb{N}_0^n$, write $x_i$ for $x^{(i)}$, where $i = 1, \ldots, n$. Let $\Lambda(m)$ be the exterior superalgebra over $\mathbb{F}$ in $m$ variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$. Set

$$\mathbb{B}(m) := \{ (i_1, i_2, \ldots, i_k) \mid n + 1 \leq i_1 < i_2 < \cdots < i_k \leq n + m, k \in [0, m] \}.$$ 

For $u := (i_1, i_2, \ldots, i_k) \in \mathbb{B}(m)$, write $|u| := k$ and $x^u := x_{i_1}x_{i_2} \cdots x_{i_k}$. Notice that we also denote the index set $\{i_1, i_2, \ldots, i_k\}$ by $u$ itself.

Let $\partial_r$, be the superderivations of $\mathcal{O}(n, m)$ defined by $\partial_r(x^\alpha) = x^{(\alpha - r)}$ for $r \in \mathbb{T}_n$ and $\partial_r(x_i) = \delta_{ir}$ for $r, s \in \mathbb{T}_{n+m}$. Here $\mathbb{T}_n$ is the set of integers $1, 2, \ldots, n$. The generalised Witt superalgebra $W(n, m)$ is $\mathbb{F}$-spanned by all $f_r\partial_r$, where $f_r \in \mathcal{O}(n, m)$, $r \in \mathbb{T}_{n+m}$. Note that $W(n, m)$ is a free $\mathcal{O}(n, m)$-module with basis $\{\partial_r \mid r \in \mathbb{T}_{n+m}\}$.

In particular, $W(n, m)$ has a standard $\mathbb{F}$-basis $\{x^\alpha x^u \partial_r \mid (\alpha, u, r) \in \mathbb{N}_0^n \times \mathbb{B}(m) \times \mathbb{T}_{n+m}\}$.

For an $n$-tuple $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, put $|\alpha| := \sum_{i=1}^n \alpha_i$. When $m = n + 1$, we usually write $\mathcal{O} := \mathcal{O}(n, n + 1)$, $W := W(n, n + 1)$, $\mathbb{A} := \mathbb{N}_0^n$ and $\mathbb{B} := \mathbb{B}(n + 1)$. If $u := (i_1, \ldots, i_r) \in \mathbb{B}$, put

$$\| u \| := \begin{cases} |u| + 1, & \text{if } 2n + 1 \in u \\ |u|, & \text{if } 2n + 1 \notin u. \end{cases}$$

Recall the standard $\mathbb{Z}$-grading, $\mathcal{O} = \bigoplus_{0 \geq i} \mathcal{O}_{\alpha, |i|}$, where

$$\mathcal{O}_{\alpha, |i|} := \mathrm{span}_{\mathbb{F}} \{ x^\alpha x^u | |\alpha| + |u| = i, \alpha \in \mathbb{A}, u \in \mathbb{B} \}.$$
Infinite-dimensional odd Contact superalgebras

It induces naturally the standard grading \( W = \oplus_{i \geq -1} W_{s,i} \), where \( W_{s,i} := \text{span}_F \{ f \partial_j \mid f \in O_{s,i+1}, j \in \{1, 2n+1\} \} \). The standard gradings of \( O \) and \( W \) are of type \((1, \ldots, 1 \mid 1, \ldots, 1)\). Let \( W_{s,i} = \sum_{j \geq i} W_{s,j} \). Then \( (W_{s,i})_{i \geq -1} \) is called the standard filtration of \( W \).

We shall also use the principal grading \( O = \oplus_{i \geq 0} O_{p,i} \), where

\[
O_{p,i} := \text{span}_F \{ x^{(a)} x^n \mid |a| + \|u\| = i \}, \quad a \in A, \ u \in B,
\]

and the principal grading \( W = \oplus_{i \geq -2} W_{p,i} \), where

\[
W_{p,i} := \text{span}_F \{ f \partial_j \mid f \in O_{p,i+1} + f_{s,j_1,2n+1}, j \in \{1, 2n+1\} \}
\]
(c.f. \([1]\)). The principal gradings of \( O \) and \( W \) are of type \((1, \ldots, 1 \mid 1, \ldots, 1, 2)\).

For a vector superspace \( V = V_0 \oplus V_1 \), we write \( p(x) := \theta \) for the parity of a \( \mathbb{Z}_2 \)-homogeneous element \( x \in V_\theta \), \( \theta \in \mathbb{Z}_2 \). Once the symbol \( p(x) \) appears, it will imply that \( x \) is a \( \mathbb{Z}_2 \)-homogeneous element.

The odd Contact superalgebra, which is a subalgebra of \( W \), is defined as follows (see \([1]\) for more details):

\[
KO(n, n + 1) := \{ D_{KO}(a) \mid a \in O \},
\]

where

\[
D_{KO}(a) := T_H(a) + (-1)^{|p(a)|} \partial_{2n+1}(a) E + (E(a) - 2a) \partial_{2n+1},
\]

\[
E := \sum_{i = 1}^{2n} x_i \partial_i, \quad T_H(a) := \sum_{i = 1}^{2n} (-1)^{\mu(i')} p(a) \partial_{i'}(a) \partial_i,
\]

\[
i' := \begin{cases} 
  i + n, & \text{if } i \in \mathbb{N}, \\
  i - n, & \text{if } i \in \mathbb{N} + 1, 2\mathbb{N}, \\
  0, & \text{if } i \in 1, n, \\
  1, & \text{if } i \in n + 1, 2n + 1.
\end{cases}
\]

For the operator \( T_H \) and further information, the reader is referred to \([2]\). Note that for \( a, b \in O \) (see \([1]\)),

\[
[D_{KO}(a), D_{KO}(b)] = D_{KO}(D_{KO}(a)(b) - (-1)^{|p(a)|} 2\partial_{2n+1}(a)b).
\] (1.1)

For simplicity, we usually write \( KO \) for \( KO(n, n + 1) \). Note that \( KO \) has a so-called principal \( \mathbb{Z} \)-grading structure denoted by

\[
KO = \oplus_{i \geq -2} KO_{p,i}, \quad \text{where } KO_{p,i} := KO \cap W_{p,i}.
\]

In particular,

\[
KO_{p,-1} = \mathbb{F} \cdot D_{KO}(1),
\]

\[
KO_{p,1} = \text{span}_F \{ D_{KO}(x_i) \mid i \in \mathbb{N}, 2\mathbb{N} \},
\]

\[
KO_{p,0} = \text{span}_F \{ D_{KO}(x_{2n+1}), D_{KO}(x_i x_j) \mid 1 \leq i \leq j \leq 2n \}.
\] (1.2)

Let

\[
X_{p,i} := \sum_{j \geq i} X_{p,j}, \quad \text{where } X = W \text{ or } KO.
\]

Then \( (X_{p,i})_{i \geq -2} \) is called the principal filtration of \( X \). Recall that the infinite-dimensional generalized Witt superalgebra \( W(n, n) \) contains the following Lie superalgebra as a subalgebra (see \([1, 2]\)):

\[
SHO'(n, n) := \{ T_H(a) \mid a \in O(n, n), \Delta(a) = 0 \}, \quad \text{where } \Delta := \sum_{i=1}^n \partial_i \partial_{i'}.
\]

**Convention:** In the sequel we shall write \( KO_{i} \) and \( KO_{i} \) for \( KO_{p,i} \) and \( KO_{p,i} \), respectively.
2. Ad-nilpotent elements

Let $L$ be a Lie superalgebra. An element $y \in L$ is ad-nilpotent if as a transformation $\text{ad}y$ is nilpotent on $L$. Let $R$ be a subalgebra of $L$. Put

$$\text{nil}(R) := \text{the set of the elements in } R \text{ which are ad-nilpotent on } L,$$

$$\text{span}_s\text{nil}(R) := \text{the subspace spanned by } \text{nil}(R),$$

$$\text{Nil}(R) := \text{the subalgebra of } L \text{ generated by } \text{nil}(R).$$

**Lemma 2.1.** Suppose $a \in \mathcal{O}$ is of $\mathbb{Z}$-degree 2 with respect to the standard grading. If $\partial_{2n+1}(a) = 0$, that is, if $a \in \mathcal{O}(n,n)$, then

$$[D\text{KO}(a), D\text{KO}(b)] = D\text{KO}(T_H(a)(b)) \text{ for all } b \in \mathcal{O}.$$

**Proof.** It follows from [11]. \qed

A nonempty subset $S$ of a Lie superalgebra $L$ is called a Lie-super subset if $S$ is closed under the multiplication of $L$ and it spans a sub-superspace (and then is a Lie superalgebra). A slight modification of [6, Theorem 1.3.1] yields the following lemma.

**Lemma 2.2.** Suppose $V$ is a vector superspace over $\mathbb{F}$ and $S$ a Lie-super subset of Lie superalgebra $\mathfrak{gl}(V)$. If $S$ consists of nilpotent linear transformations $\text{span}_S S$ is of finite dimension, then $\text{span}_S S$ is strictly triangularizable on $V$, that is, there is a finite sequence $(V_i)_{0 \leq i \leq m}$ of sub-superspaces such that

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = V; \quad x(V_i) \subset V_{i-1} \text{ for all } x \in \text{span}_S S.$$

**Lemma 2.3.** $W_{p,1} \subseteq \text{nil}(W)$.

**Proof.** There exists a sub-superspace $V \subseteq W_{s,1}$ such that

$$W_{p,1} = \text{span}_S\{x_{2n+1}\partial_1 | i \in \underline{2n}\} + V.$$

For $E \in W_{p,1}$, write $E = E_0 + E_1$, where $E_0 \in \text{span}_S\{x_{2n+1}\partial_1 | i \in \underline{2n}\}$, $E_1 \in V$. Put $E_2 := [E_0, E_1]$. Then there exists an $n$-tuple $t$ of positive integers such that $E_1, E_2 \in W_{s,1}(n, n+1; \underline{2}) := W(n, n+1; \underline{2}) \cap W_{s,1}$ [see [8] for a definition of $W(n, n+1; \underline{2})$]. Note that

$$S := \text{span}_S\{\text{ad}(x_{2n+1}\partial_1) | i \in \underline{2n}\} \cup \text{ad}W_{s,1}(n, n + 1; \underline{2})$$

is a Lie-super subset of the general linear Lie superalgebra $\mathfrak{gl}(W)$. A direct computation shows that $\text{span}_S\{x_{2n+1}\partial_1 | i \in \underline{2n}\} \subseteq \text{nil}(W)$. By virtue of [8, Theorem 2.5], we have $W_{s,1}(n, n+1; \underline{2}) \subseteq \text{nil}(W)$. Then Lemma 2.3 ensures that $E \in \text{nil}(W)$. \qed

**Lemma 2.4.** (1) $K\mathcal{O}_1 \subseteq \text{nil}(K\mathcal{O})$.

(2) Suppose $y = y_{[i]} + y_{i+1} \in \text{nil}(K\mathcal{O}_1)$, where $y_{[i]} \in K\mathcal{O}_{[i]}$ and $y_{i+1} \in K\mathcal{O}_{i+1}$. Then $y_{[i]} \in \text{nil}(K\mathcal{O}_{[i]})$.

(3) Suppose $y = y_{[-1]} + y_0 \in \text{nil}(K\mathcal{O}_0)$, where $y_{[-1]} \in K\mathcal{O}_{[-1]} \cap K\mathcal{O}_0$ and $y_0 \in K\mathcal{O}_0 \cap K\mathcal{O}_0$. Then $y_{[-1]} = 0$.

(4) $\text{Nil}(K\mathcal{O}_0) = \text{Nil}(K\mathcal{O}_{[0]} \cap K\mathcal{O}_0) + K\mathcal{O}_1 \cap K\mathcal{O}_0$. 

Infinite-dimensional odd Contact superalgebras
Proof. (1) Since $KO_1 \subseteq W_{2,1}$, by virtue of Lemma 2.4, we get $KO_1 \subseteq \text{nil}(KO)$.
(2) Similar to [8, Lemma 2.7], one may prove (2).
(3) Suppose $y_{[-1]} = \sum_{i=1}^{n} a_i D_{KO}(x_i^j)$, where $a_i \in F$. Note that $D_{KO}(x_i^{(k+1)\epsilon_i}) \in KO$ for $k \in \mathbb{N}$. If $a_i \neq 0$ for some $j \in \mathbb{N}$, then $y_{[-1]}$ is not ad-nilpotent, since $(\text{ad}y_{[-1]})^k(D_{KO}(x_i^{(k+1)\epsilon_i})) = a_i^k D_{KO}(x_i) \neq 0$. This contradicts (2). Therefore, $a_j = 0$ for all $j \in \mathbb{N}$, that is, $y_{[-1]} = 0$.
(4) It follows from (1) and (3).

Lemma 2.5. If $i \neq j' \in \mathbb{N}$, then $(T_H(x_i x_j))^2p = 0$.

Proof. Note that
$$T_H(x_i x_j) = (-1)^{\mu(i) + \mu(j)} x_i \partial_{x_j} + (-1)^{\mu(i)} x_i \partial_{x_j},$$
(2.1)
$(x_i \partial_{x_j})^p = (x_i \partial_{x_j})^p = 0$ and $[x_i \partial_{x_j}, x_j \partial_{x_i}] = 0$ for all $i \neq j' \in \mathbb{N}$. In combination with (2.1), we have $(T_H(x_i x_j))^2p = 0$.

Lemma 2.6. For $i, j \in \mathbb{N}$, the following statements hold.
(1) If $f \in \mathcal{O}$, $D_{KO}(f) \in KO_{[0]}$ and $\partial_{2n+1}(f) \neq 0$, then $D_{KO}(f) \notin \text{nil}(KO_{[0]})$.
(2) Suppose $i \neq j'$ and $a_i \in F$ for all $i \in \mathbb{N}$. Then
$$D_{KO}(x_i x_j) \in \text{nil}(KO_{[0]}), \quad \sum_{i=1}^{n} a_i D_{KO}(x_i x_j) \notin \text{nil}(KO_{[0]}) \text{ or is 0,}$$
$$D_{KO}(x_i x_j) \notin \text{Nil}(KO_{[0]}), \quad D_{KO}(x_{2n+1}) \notin \text{Nil}(KO_{[0]}),$$
$$D_{KO}(x_i x_{j'} - x_{j'} x_i) \in \text{Nil}(KO_{[0]}).$$

Proof. (1) Suppose $f \in \mathcal{O}$, $D_{KO}(f) \in KO_{[0]}$ and $\partial_{2n+1}(f) \neq 0$. Then there exist $0 \neq a \in F$ and $f_0 \in \mathcal{O}(n, n)$ such that $f = ax_{2n+1} + f_0$. Since
$$[D_{KO}(ax_{2n+1} + f_0), D_{KO}(1)] = 2a D_{KO}(1),$$
we have $D_{KO}(f) \notin \text{nil}(KO_{[0]})$.
(2) Applying Lemma 2.4, we obtain by induction on $k$ that
$$(\text{ad} D_{KO}(x_i x_j))^k(D_{KO}(f)) = D_{KO}(T_H(x_i x_j))^k(f)$$
for all $k \in \mathbb{N}$, where $f \in \mathcal{O}$. Since $\ker(D_{KO}) = 0$, one sees that $D_{KO}(x_i x_j)$ is ad-nilpotent if and only if $T_H(x_i x_j)$ is a nilpotent transformation of $\mathcal{O}$. Then by Lemma 2.5 we know that
$$D_{KO}(x_i x_j) \in \text{nil}(KO_{[0]}) \quad \text{for all } i \neq j' \in \mathbb{N}.$$ Note that
$$\left[ \sum_{i=1}^{n} a_i D_{KO}(x_i x_{j'}), D_{KO}(x_j x_{2n+1}) \right] = -a_j D_{KO}(x_j x_{2n+1}), \quad j \in \mathbb{N}.$$ If $\sum_{i=1}^{n} a_i D_{KO}(x_i x_{j'}) \neq 0$ then
$$\sum_{i=1}^{n} a_i D_{KO}(x_i x_{j'}) \notin \text{nil}(KO_{[0]}).$$
Infinite-dimensional odd Contact superalgebras

It follows from (1) that nil$(KO_{[0]}) \subseteq SHO'(n, n)$. Therefore, Nil$(KO_{[0]}) \subseteq SHO'(n, n)$. But $D_{KO}(x_i x_{i'})$, $D_{KO}(x_{2n+1}) \notin SHO'(n, n)$, thus

$$D_{KO}(x_i x_{i'}) \notin \text{Nil}(KO_{[0]}).$$

By virtue of the fact that

$$[D_{KO}(x_i x_j), D_{KO}(x_{i'} x_{j'})] = -D_{KO}(x_i x_{i'} - x_j x_{j'}),$$

we have $D_{KO}(x_i x_{i'} - x_j x_{j'}) \in \text{Nil}(KO_{[0]}).$

3. Invariant subalgebras

Let

$$\mathcal{T} := \text{Nor}_{KO_0}(\text{Nil}(KO_0)),$$

$$\Omega := \{y \in KO_1 \mid [y, KO_1] \subseteq T\},$$

$$\mathcal{M} := \{y \in KO_1 \mid [y, \Omega] \subseteq \text{Nil}(KO_0)\}.$$ 

It is easy to see that $\mathcal{T}$ is an invariant subspace under the automorphisms of $KO$ and so are $\Omega$ and $\mathcal{M}$.

**Proposition 3.1.** $\mathcal{T} = KO_0 \cap KO_0$. In particular, $KO_0 \cap KO_0$ is an invariant subalgebra of $KO$.

**Proof.** Let

$$y = \sum_{i=1}^n a_i D_{KO}(x_i') + y'' \in \mathcal{T},$$

where $a_i \in F$ for all $i \in \overline{1,n}$, $y'' \in KO_0 \cap KO_0$. Assume that $a_j \neq 0$ for some $j \in \overline{1,n}$. Take $j \neq k \in \overline{1,n}$. By Lemma 2.4(4), we have $D_{KO}(x_j x_k x_{k'}) \in KO_1 \cap KO_0 \subseteq \text{Nil}(KO_0)$. Then

$$-a_j D_{KO}(x_k x_{k'}) - a_k D_{KO}(x_j x_{k'}) + h$$

$$= \left[ \sum_{i=1}^n a_i D_{KO}(x_i') + y'', D_{KO}(x_j x_k x_{k'}) \right] \in \text{Nil}(KO_0),$$

where $h \in KO_1 \cap KO_0$. This contradicts Lemma 2.3(2) and then $\mathcal{T} \subseteq KO_0 \cap KO_0$. On the other hand, by (1.2) and Lemma 2.6(2), we have

$$KO_0 \cap KO_0 = \text{span}_F \text{nil}(KO_0) + Y,$$

where $Y := \text{span}_F \{D_{KO}(x_i x_{i'}) \mid i \in \overline{1,n} \}$. By (1.1), we have

$$[Y, \text{Nil}(KO_0)] \subseteq \text{Nil}(KO_0).$$

Then

$$[KO_0 \cap KO_0, \text{Nil}(KO_0)] = \text{span}_F \text{nil}(KO_0) + Y, \text{Nil}(KO_0)] \subseteq \text{Nil}(KO_0).$$

The proof is complete. $\square$
Lemma 3.2. $\mathcal{Q} \subseteq \text{span}_F \{D_{KO}(x_i) \mid 1 \leq i \leq j \leq n\} + KO_1 \cap KO_1$.

Proof. For $y \in \mathcal{Q}$, we may write

$$y = D_{KO}(a) + y',$$

where $a \in F$, $y' \in KO_{-1} \cap KO_1$. Note that $D_{KO}(x_{i',x_{2n+1}}) \in KO_1$. Then

$$-2aD_{KO}(x_{i'}) + h = [D_{KO}(a) + y', D_{KO}(x_{i',x_{2n+1}})] \in KO_0 \cap KO_0,$$

where $h \in KO_0 \cap KO_0$. Then $a = 0$. Thus we may write

$$y = \sum_{i=1}^{n} a_i D_{KO}(x_i) + y'',$$

where $a_i \in F$ for all $i \in \{1, \ldots, n\}$. If $a_j \neq 0$ for some $j \in \{1, \ldots, n\}$, take $j \neq k \in \{1, \ldots, n\}$. Note that $D_{KO}(x_{j'}x_{k'}) \in KO_1$. We have

$$D_{KO}(a_jx_{k'} - a_kx_{j'}) + h = \left[ \sum_{i=1}^{n} a_i D_{KO}(x_i) + y'', D_{KO}(x_{j'}x_{k'}) \right] \in KO_0 \cap KO_0,$$

where $h \in KO_0 \cap KO_0$, contradicting that $a_j \neq 0$. Thus we may write

$$y = \sum_{1 \leq i \leq j \leq n} a_{ij} D_{KO}(x_{i,j}) + \sum_{1 \leq i < j \leq n} b_{ij} D_{KO}(x_{i',x_{j'}}) + y'',$$

where $a_{ij}, b_{ij} \in F$ for all $1 \leq i \leq j \leq n$, $y'' \in KO_1 \cap KO_1$. Assume that $b_{kl} \neq 0$ for some $1 \leq k < l \leq n$. Since $D_{KO}(x_k) \in KO_1$, we have

$$\sum_{i=1}^{k-1} -b_{ik} D_{KO}(x_{i'}) + \sum_{i=k+1}^{n} b_{ki} D_{KO}(x_{i'}) + h = [y, D_{KO}(x_k)] \in KO_0 \cap KO_0,$$

where $h \in KO_0 \cap KO_0$, contradicting the assumption that $b_{kl} \neq 0$. Then

$$\mathcal{Q} \subseteq \text{span}_F \{D_{KO}(x_{i,j}) \mid 1 \leq i \leq j \leq n\} + KO_1 \cap KO_1$$

and this completes the proof. \(\square\)

Remark 3.3. For $i \neq j \in \{1, \ldots, n\}$, we have $D_{KO}(x_{i,x_{i',x_{2n+1}}}) \in \mathcal{Q}$, $D_{KO}(x_{i',x_{2n+1}}) \in \mathcal{Q}$.

Proposition 3.4. $\mathcal{M} = KO_0 \cap KO_1$. In particular, $KO_0 \cap KO_1$ is an invariant subalgebra of $KO$.

Proof. By \(12\), Lemmas \(2.4\), \(2.6\) and \(3.2\), we have

$$[KO_0 \cap KO_1, \mathcal{Q}] \subseteq [KO_0 \cap KO_1, \text{span}_F \{D_{KO}(x_{i,j}) \mid 1 \leq i \leq j \leq n\} + KO_1 \cap KO_1] \subseteq \text{Nil}(KO_0).$$

Hence $KO_0 \cap KO_1 \subseteq \mathcal{M}$. Conversely, for $y \in \mathcal{M}$, we may write

$$y = D_{KO}(a) + y',$$
Infinite-dimensional odd Contact superalgebras

where \( a \in F, y' \in KO_{-1} \cap KO_1 \). By Remark 3.3 we have
\[
2aD_{KO}(x_ix_i') + h = [D_{KO}(a) + y', D_{KO}(x_ix_i'x_{2n+1})] \in \text{Nil}(KO_0),
\]
where \( h \in KO_1 \cap KO_0 \). By Lemma 2.6(2), we have \( a = 0 \). Thus we may write
\[
y = \sum_{i=1}^{n} a_i D_{KO}(x_i) + y'',
\]
where \( a_i \in F \) for all \( i \in \mathbb{1}, n \). By Remark 3.3 we have
\[
a_jD_{KO}(x_kx_k') - a_kD_{KO}(x_jx_k') + h
\]
\[
= \left[ \sum_{i=1}^{n} a_i D_{KO}(x_i) + y'', D_{KO}(x_jx_kx_k') \right] \in \text{Nil}(KO_0),
\]
where \( h \in KO_1 \cap KO_0 \). By Lemma 2.6(2), we have \( a_j \neq 0 \). This contradicts the assumption that \( a_j \neq 0 \). Thus \( \mathfrak{M} \subseteq KO_0 \cap KO_1 \) and the proof is complete.

The key in this paper is the following proposition.

**Proposition 3.5.** \( KO_0 \) is an invariant subalgebra of \( KO \).

**Proof.** It follows from Propositions 3.1 and 3.4.

\[ \square \]

**4. Filtration, automorphisms and classification**

One of the main results is as follows, which is a direct consequence of Proposition 3.5 and the following Lemmas 4.2 and 4.3.

**Theorem 4.1.** The principal filtration of \( KO \) is invariant under the automorphisms of \( KO \), that is, \( \varphi(KO_i) = KO_i \) for all \( i \geq -2 \) and all automorphisms \( \varphi \) of \( KO \).

**Lemma 4.2.** \( KO_{-1}/KO_0 \) is the unique irreducible \( KO_0 \)-submodule of \( KO/KO_0 \). In particular, \( KO_{-1} \) is an invariant subalgebra of \( KO \).

**Proof.** Clearly, \( KO_{-1}/KO_0 \) is an irreducible \( KO_0 \)-submodule. To show the uniqueness, suppose \( M/KO_0 \) is a nonzero \( KO_0 \)-submodule of \( KO/KO_0 \), where \( M \supset KO_0 \) is a \( KO_0 \)-submodule of \( KO \). For \( 0 \neq y \in M \), one may write \( y = D_{KO}(1) + y' \), where \( y' \in KO_{-1} \). Then
\[
[D_{KO}(x_i) D_{KO}(x_{2n+1}), D_{KO}(1) + y'] \in M \quad \text{for all} \quad i \in \mathbb{1}, 2n.
\]
Note that \( [D_{KO}(x_i), D_{KO}(1)] \) \( \in M \). We have
\[
2D_{KO}(x_i) = [D_{KO}(x_i x_{2n+1}), D_{KO}(1)] \in M
\]
for all \( i \in \mathbb{1}, 2n \). Therefore, \( KO_{-1}/KO_0 \subseteq M/KO_0 \). The proof is complete.

**Lemma 4.3.** \( KO_i = \{ y \in KO_{i-1} \mid [y, KO_{-1}] \subseteq KO_{i-1} \} \) for all \( i \geq 1 \).
Proof. Put
\[ h_i := \{ y \in KO_{i-1} \mid [y, KO_{i-1}] \subseteq KO_{i-1} \}. \]
Clearly, KO \(_i \) \( \subseteq h_i \). For all \( y \in h_i \), we may write
\[ y := \sum_{j \geq i} y_j, \]
where \( y_j \in KO_{[j]} \). By the definition of \( h_i \), \([y_{i-1}, KO_{i-1}] = 0\). Let \( y_{i-1} := \sum_{\alpha, u} b_{\alpha, u} D_{KO}(x^{(\alpha)} x^u) \), where \( \alpha \in \mathbb{K}, u \in \mathbb{B}, D_{KO}(x^{(\alpha)} x^u) \in KO_{[i-1]} \subseteq KO_0 \), \( b_{\alpha, u} \in \mathbb{F} \). For any fixed \( \beta \neq 0 \) with \( \beta_k \geq 1 \) for some \( k \in \mathbb{N} \), we have
\[
0 = \left[ \sum_{\alpha, u} b_{\alpha, u} D_{KO}(x^{(\alpha)} x^u), D_{KO}(x^v) \right]
= \sum_{\alpha, u} b_{\alpha, u} D_{KO}(\partial_k (x^{(\alpha)} x^u) - (-1)^{p(x^u)} \partial_{2n+1} (x^{(\alpha)} x^u) x^v).
\]
Consequently, \( b_{\alpha, u} = 0 \) whenever \( \alpha \neq 0 \). It remains to consider the case \( \alpha = 0 \). Fix any \( v \neq 0 \). If there is \( l \in \mathbb{N} \), \( 2n \) such that \( l \in v \). Then
\[
0 = \left[ \sum_{\alpha, u} b_{\alpha, u} D_{KO}(x^u), D_{KO}(x^v) \right]
= \sum_{\alpha, u} b_{\alpha, u} D_{KO}((-1)^{p(x^u)} \partial_l (x^u) - (-1)^{p(x^u)} \partial_{2n+1} (x^u) x^v).
\]
It follows that \( b_{\alpha, u} = 0 \), where \( (2n + 1) \neq v \in \mathbb{B} \). Taking \( i \in \mathbb{N} \), since
\[
0 = [b_{\alpha, 2n+1} D_{KO}(x_{2n+1}), D_{KO}(x_i)] = b_{\alpha, 2n+1} D_{KO}(x_i),
\]
we have \( b_{\alpha, 2n+1} = 0 \). Therefore, \( y_{i-1} = 0 \) and then \( h_i \subset KO_i \).

As a corollary we give a characterization of the automorphisms of KO:

Theorem 4.4. Two automorphisms of KO coincide if and only if they coincide on the \(-1\) component \( KO_{[-1]} \).

Proof. Let \( \phi \) and \( \psi \) be the automorphisms of KO. It suffices to prove that \( \phi |_{KO_{[-1]}} = \psi |_{KO_{[-1]}} \) implies \( \phi = \psi \). As in \([\mathbb{F}] \) Corollary 18, using Theorem 1.1 one can prove that \( \phi |_{KO_{[-2]} = \psi |_{KO_{[-2]}}, \) Since
\[
\phi([D_{KO}(1)]) = \phi([D_{KO}(x_1), D_{KO}(x_1)]) = \psi([D_{KO}(x_1), D_{KO}(x_1)]) = \psi([D_{KO}(1)]),
\]
we have \( \phi |_{KO_{[-2]} = \psi |_{KO_{[-2]}}, \) Then \( \phi = \psi \) and the proof is complete.

We are now in position to state the final main result in this paper, which says that the parameter \( n \) defining the odd Contact superalgebra KO\((n, n + 1)\) is intrinsic and then all the infinite-dimensional odd Contact superalgebras are classified up to isomorphisms.

Theorem 4.5. KO\((n, n + 1) \cong KO(m, m + 1)\) if and only if \( n = m \).

Proof of Theorem 4.5. One direction is obvious. Assume that \( \sigma : KO(n, n + 1) \rightarrow KO(m, m + 1) \) is an isomorphism of Lie superalgebras. Clearly,
\[
\sigma(\text{Nor}_{KO(n, n + 1)_{0}}(\text{Nil}(KO(n, n + 1)_{0}))) = \text{Nor}_{KO(m, m + 1)_{0}}(\text{Nil}(KO(m, m + 1)_{0})).
\]
In view of the proof of Proposition 3.1 we have
\[
KO(n, n + 1)_{0} \cap KO(n, n + 1)_{0} = \text{Nor}_{KO(n, n + 1)_{0}}(\text{Nil}(KO(n, n + 1)_{0})).
\]
Therefore,
\[
\sigma(KO(n, n + 1)_0 \cap KO(n, n + 1)_0) = KO(m, m + 1)_0 \cap KO(m, m + 1)_0. \tag{4.1}
\]
Recall that
\[
\Omega = \{ y \in KO(n, n + 1)_1 | [y, KO(n, n + 1)_1] \subseteq KO(n, n + 1)_0 \cap KO(n, n + 1)_0 \}
\]
and
\[
\mathfrak{M} = \{ y \in KO(n, n + 1)_1 | [y, \Omega] \subseteq \text{Nil}(KO(n, n + 1)_0) \}.
\]
By Lemma 3.2 and Proposition 3.4, we have
\[
\sigma(KO(n, n + 1)_0 \cap KO(n, n + 1)_1) = KO(m, m + 1)_0 \cap KO(m, m + 1)_1. \tag{4.2}
\]
By (4.1) and (4.2), we have
\[
\sigma(KO(n, n + 1)_0) = KO(m, m + 1)_0.
\]
Therefore \( \sigma \) induces an isomorphism of \( \mathbb{Z}_2 \)-graded vector spaces
\[
\sigma' : KO(n, n + 1)/KO(n, n + 1)_0 \rightarrow KO(m, m + 1)/KO(m, m + 1)_0.
\]
Since \( KO(n, n + 1)/KO(n, n + 1)_0 \cong KO(n, n + 1)_{[-2]} \oplus KO(n, n + 1)_{[-1]} \) as \( \mathbb{Z}_2 \)-graded vector spaces, we have
\[
dim(KO(n, n + 1)_{[-2]} \oplus KO(n, n + 1)_{[-1]}) = dim(KO(m, m + 1)_{[-2]} \oplus KO(m, m + 1)_{[-1]}).
\]
This implies that \( n + 1 = m + 1 \), that is, \( n = m \). The proof is complete.

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