Field-induced boson insulating states in a 2D superconducting electron gas with strong spin–orbit scatterings

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Abstract

The phenomenon of field-induced superconductor-to-insulator transitions observed experimentally in electron-doped SrTiO$_3$/LaAlO$_3$ interfaces, analyzed recently by means of 2D superconducting fluctuations theory, is revisited with new insights associating it with the appearance at low temperatures of field-induced boson insulating states. Within the framework of the time-dependent Ginzburg–Landau functional approach, we pinpoint the origin of these states in field-induced extreme softening of fluctuation modes over a large region in momentum space, upon diminishing temperature, which drives Cooper-pair fluctuations to condense into mesoscopic puddles in real space. Dynamical quantum tunneling of Cooper-pair fluctuations out of these puddles, introduced within a phenomenological approach, which break into mobile single-electron states, contains the high-field resistance onset predicted by the exclusive boson theory.

Keywords: interface superconductivity, superconductor-to-insulator transitions, superconducting fluctuations, spin–orbit scatterings

(Some figures may appear in colour only in the online journal)

1. Introduction

In a recent paper [1] we have shown that Cooper-pair fluctuations in a 2D electron gas with strong spin–orbit scatterings can lead at low temperatures to pronounced magnetoresistance (MR) peaks above a crossover field to superconductivity. The model was applied to the high mobility electron systems formed in the electron-doped interfaces between two insulating perovskite oxides—SrTiO$_3$ and LaAlO$_3$ [2, 3], showing good quantitative agreement with a large body of experimental sheet-resistance data obtained under varying gate voltage [4].

The model employed was based on the opposing effects generated by fluctuations in the superconducting (SC) order parameter: the nearly singular enhancement of conductivity (paraconductivity) due to fluctuating Cooper pairs below the nominal (mean-field) critical magnetic field, on one hand, and the suppression of conductivity, associated with the loss of unpaired electrons due to Cooper pairs formation, on the other hand. The self-consistent treatment of the interaction between fluctuations [5, 6], employed in these calculations, avoids the critical divergence of both the Aslamazov–Larkin (AL) paraconductivity [7] and the DOS conductivity [8], allowing to extend the theory to regions well below the nominal critical SC transition. The absence of long range phase coherence implied by this approach is consistent with the lack of the ultimate zero-resistance state in the entire data analyzed there.

The most intriguing question arising from the Cooper-pair fluctuations scenario of the superconductor–insulator transition (SIT) presented in [1] is how Cooper-pairs liquid, whose condensation (in momentum space) is customarily associated with superconductivity, could metamorphose into an insulator just by lowering its temperature under sufficiently high magnetic field? For answering this intriguing question we
note our use of the time-dependent Ginzburg–Landau (TDGL) functional approach in consistently evaluating the AL and the DOS conductivities. Within this exclusive boson approach we have found in [1] that at low temperatures the (negative) DOS conductivity prevails over the AL paraconductivity at fields that roughly indicate the presence of the observed enhanced MR. Dynamical quantum tunneling of Cooper-pair fluctuations out of mesoscopic puddles has been introduced into the theory within a complementary phenomenological approach, including the contribution of unpaired normal electron states, to account for the observed experimental data.

In the present paper we reveal the underlying origin of these low-temperature field-induced boson insulating states by exploiting a detailed analytical scheme within the framework of the TDGL functional approach. It is found that strong field-induced suppression of the fluctuation stiffness parameter at low temperatures resulting in extreme softening of fluctuating modes over a large region in momentum space, dramatically enhances the Cooper-pair fluctuations density in mesoscopic puddles of real space. The resulting large enhancement of the (negative) DOS conductivity versus the diminishing AL paraconductivity, associated with the fluctuation mass enhancement, trigger the appearance of insulating states at high field. Our detailed analysis has also illuminated the mechanism in which the exclusive fluctuation boson picture is modified within a unified phenomenological approach. It allows field-induced pair-breaking processes to develop during dynamical quantum tunneling of Cooper-pair fluctuations out of mesoscopic puddles, which result in free exchange between the systems of charge-bosons and unpaired free electrons.

2. The TDGL functional approach

The TDGL functional $\mathcal{S}(\Delta, \mathbf{A})$ of the order parameter $\Delta(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$ determines the Cooper-pairs current density [9]:

$$ j(\mathbf{r}, t) = \nabla^2 \mathcal{S}(\Delta(\mathbf{r}, t), \mathbf{A}(\mathbf{r}, t)) $$

responsible for the AL paraconductivity. In this approach the entire underlying information about the thin film of pairing electrons system (which includes in-plane spin–orbit scatterings, Zeeman spin splitting as well as out-of-plane diamagnetic energy [1]) is incorporated in the inverse fluctuation propagator (in wavevector-frequency representation) $D^{-1}(\mathbf{q} + 2e\mathbf{A}/\hbar, \omega)$, mediating between the order parameter and the GL functional. In the Gaussian approximation the relation is quadratic, i.e.:

$$ \mathcal{S}(\Delta, \mathbf{A}) = \frac{1}{2\pi} d^2 q \left( \frac{1}{2\pi} \right)^2 \times \int d\Omega |\Delta(q, \Omega)|^2 D^{-1}(\mathbf{q} + 2e\mathbf{A}/\hbar, \Omega) $$

so that the coupling to the external electromagnetic field takes place directly through the vertex of the Cooper-pair current, defined in equation (1).

The corresponding AL time-ordered current-current correlator is given by:

$$ Q_{\text{AL}}(i\Omega_n) = (4eN_{2D})^2 d^{-1} \left( \frac{1}{2\pi} \right)^2 \int d^2 q q^2 k_B T $$

$$ \sum_{\mu = -\infty}^{\infty} C(q, \Omega_\mu + i\Omega_n) C(q, \Omega_\mu) D(q, \Omega_\mu) $$

where $\Omega_\mu = 2\mu k_B T/h, \Omega_n = 2\nu k_B T/h, \mu = 0, \pm 1, \pm 2, \ldots, \nu = 0, 1, 2, \ldots$, are bosonic Matsubara frequencies, $d$ is the thickness of the detected film, and $N_{2D} = \mu^* / 2\hbar^2$ is the single-electron DOS, with an effective mass $\mu^*$. Here the electrical current is generated along the $\hat{x}$ axis, $q_x, q_y$, are the fluctuation (in-plane) wave-vector components along the magnetic and electric field directions, respectively, and $q^2 = q_x^2 + q_y^2$.

Explicitly for the model of spin–orbit scatterings employed, the fluctuation propagator $D(q, \Omega_\mu)$ and its corresponding effective current vertex $C(q, \Omega_\mu)$ are given by [1]:

$$ D(q, \Omega_\mu) = \frac{1}{N_{2D}} \Phi(\mathbf{x} + |\mu|; \varepsilon_H) $$

$$ C(q, \Omega_\mu) = \frac{1}{4\pi k_B T} \Phi'(\mathbf{x} + |\mu|; \varepsilon_H) $$

where

$$ \Phi(\mathbf{x} + |\mu|; \varepsilon_H) = \varepsilon_H + a_+ [\psi(1/2 + f_- + x + |\mu|) - \psi(1/2 + f_-) + a_- \psi[(1/2 + f_+) + x + |\mu|) - \psi(1/2 + f_+)] $$

and:

$$ \varepsilon_H \equiv \ln \left( \frac{T}{T_\text{c0}} \right) + a_+ \psi \left( \frac{1}{2} + f_- \right) + a_- \psi \left( \frac{1}{2} + f_+ \right) $$

Here $T_{\text{c0}}$ is the mean-field SC transition temperature at zero magnetic field, $\psi$ is the digamma function, $x = hDq^2 / 4\pi k_B T$, where $D \equiv \tau_{SO}E_F / m^*$ is the electron diffusion coefficient, $E_F$ the Fermi energy, and $\varepsilon_{SO} = h / \tau_{SO}$ is the spin–orbit energy. The system parameters: $f_\pm = \delta H^2 + \beta \pm \sqrt{\beta^2 - \mu^2 H^2}$, $a_\pm = \left( 1 \pm \beta \sqrt{\beta^2 - \mu^2 H^2} \right) / 2$ are dimensionless functions of the magnetic field $H$, with the basic parameters: $\beta \equiv \varepsilon_{SO} / 4\pi k_B T$, $\delta \equiv D (de^2 / 2\pi k_B T)h$, $\mu$ and $\mu_B$ the Bohr magneton.

The DOS conductivity is obtained within this TDGL functional approach by exploiting the Drude formula $\sigma_{\text{DOS}} = -2ne^2\varepsilon_{SO}/m^*$, through the Cooper-pair fluctuations density $n_x$ [8]:

$$ n_x = \frac{1}{d} \left( \frac{2\pi}{\hbar} \right)^2 \int \langle |\phi(q)|^2 \rangle d^2 q $$

with the Cooper-pair momentum distribution function $\langle |\phi(q)|^2 \rangle$ derived by exploiting the frequency-dependent GL functional, equation (2). This is done by rewriting equation (2) in terms of the frequency and wavenumber representations GL.
wave functions $\phi(q, \Omega)$, after analytic continuation to real frequencies $i\Omega_\mu \to \Omega$, i.e.:

$$\mathcal{L}(\Delta) = \int \frac{d^2 q}{(2\pi)^2} \int \frac{d\Omega}{2\pi} \left[ \langle \Delta(q, \Omega) \rangle^2 D(q, \Omega) \right]^{-1}$$

$$= \int \frac{d^2 q}{(2\pi)^2} \int \frac{d\Omega}{2\pi} \langle \phi(q, \Omega) \rangle^2 L(q, \Omega)^{-1} = \mathcal{L}(\phi) \quad (8)$$

where the transformed inverse propagator $L(q, \Omega)^{-1}$ is obtained from the equation:

$$N_{2D}D(q, \Omega) = \mathcal{A}qT L(q, \Omega) \quad (9)$$

and: $\mathcal{A} = 4\pi^2 k_B T / \zeta(3) E_F$, with: $\zeta(3) \approx 1.202$.

For the sake of clarity of the analysis that follows we expand $L^{-1}(q, \Omega)$ to leading orders in small $q$ and $\Omega$, i.e.:

$$L^{-1}(q, \Omega) \approx \mathcal{A}qT [\varepsilon_H + \tilde{\eta}(H) \xi^2 q^2] - i(\hbar \Omega) \gamma_{GL} \quad (10)$$

where:

$$\eta(H) = a_+ \psi' \left( \frac{1}{2} + f_+ \right) + a_- \psi' \left( \frac{1}{2} + f_- \right); \quad \tilde{\eta}(H) \equiv \frac{\eta(H)}{\eta(0)} = \frac{2\eta(H)}{\pi^2}$$

$$\xi = \sqrt{\pi \hbar D/8k_B T} \quad \text{is the dirty-limit coherence length and} \quad \gamma_{GL} = \tilde{\eta}(H) \pi \mathcal{A}/8 \quad \text{is the dimensionless GL Cooper-pair life time. Thus, using equation (10) the momentum distribution function is related to the fluctuation propagator through [8]$:}$

$$\left\langle |\phi^2| \right\rangle = 2k_B T \gamma_{GL} \int \frac{d(\Omega)}{2\pi} \left| L(q, \Omega) \right|^2$$

which readily yields:

$$\left\langle |\phi^2| \right\rangle \approx \left( \frac{7\zeta(3) E_F}{4\pi^2 k_B T} \right) \varepsilon_H + \eta(H) \left( \frac{\hbar D}{4\pi k_B T} \right)^2 q^2 \quad (12)$$

3. **Conductance fluctuations at very low temperatures**

In order to reveal the origin of the puzzling insulating state that emerges in our approach we will consider in this section the fluctuations contributions to the sheet conductivity in the magnetic fields region where they are rigorously derivable from the microscopic Gor’kov’s Ginzburg–Landau theory, i.e. above the nominal (mean-field) critical field, determined from the vanishing of the Gaussian critical shift-parameter $\varepsilon_H$ (equation (6)). There are no restrictions on the temperature $T$ as we are mainly interested in the low temperatures region well below $T_{c0}$ down to the limit of $T \to 0$.

### 3.1. **DOS conductivity**

Using equation (12) in equation (7) and the Drude formula, the DOS conductivity is written in the form:

$$\sigma_{DOS} \equiv -\frac{3.5\zeta(3)}{\pi} \int_0^\infty \frac{dx}{\varepsilon_H + \eta(H)x} \equiv -\frac{4.2}{\varepsilon_H} \left( \frac{e^2}{\pi^2}\right)^2 \int \frac{d(\Omega)}{\varepsilon} x^2 \left( \frac{\eta(H)}{\varepsilon_H} \right)^2 \quad (13)$$

where $G_0 = e^2/\pi \hbar$ is the conductance quantum, $x_0 = hDq_0^2/4\pi k_B T$, with $q_0$ the cutoff wave number, and $3.5\zeta(3) \approx 4.207$. It is interesting to compare this result with the result of the fully microscopic (diagrammatic) approach presented in [8] for a multilayer of 2D electron systems in the zero field limit. Using the notation employed in [8] (according to which $h = k_B = 1$ and the distance between layers is $s$) the corresponding DOS conductivity is given by:

$$\sigma_{DOS}^{s} = -\kappa(T) \pi e^2 \int \frac{d(\Omega)}{2\pi^2} \left( \frac{\eta(\Omega)}{\varepsilon_H} \right)^2 \quad (14)$$

where $\varepsilon \equiv \ln(T/T_{c0})$, $\eta(\Omega) = \pi D/8T$, and the dirty limit: $\kappa(T)_{T \to 0} \approx 8 \times 7\zeta(3)/\pi^4$.

Equation (14) is in full agreement with the zero-field limit of equation (13) derived within our TDGL functional approach. This agreement is quite remarkable since the coefficient $\kappa(T)_{T \to 0} \approx 8 \times 7\zeta(3)/\pi^4$ was obtained by summing the contributions of four diagrams following a lengthy calculation involving external impurity-scattering renormalization of pair vertices (i.e. connected to the current vertices by electron lines).

### 3.2. **Paracconductivity**

The AL contribution to the sheet conductance is calculated by analytically continuing the time-ordered current-current correlator derived by using equation (3), i.e.:

$$Q_{AL}(i\nu) = k_B T \left( \frac{2\nu}{\hbar} \right)^2 \left( \frac{1}{2\pi d} \right)^{\frac{\xi}{\nu d}} \int_0^\infty dx \Phi'(x + |\mu + \nu|, \varepsilon_H) \Phi'(x + |\mu|, \varepsilon_H) \Phi(x + |\mu|, \varepsilon_H) \Phi(x + |\mu + \nu|, \varepsilon_H) \quad (15)$$

from the imaginary Matsubara frequency $i\nu$, to the real frequency $\Omega$ in the static limit, i.e.: $Q_{AL} (i\nu) \to Q_{AL}^0 (\Omega)$; $\sigma_{AL} = \lim_{\nu \to 0} (i/\Omega) \left[ Q_{AL}^0 (\Omega) - Q_{AL}^0 (0) \right]$. It is interesting to note that under direct analytic continuation of the discrete summation in equation (15) about zero frequency, i.e. $\nu \to \hbar \Omega / 2\pi k_B T \to 0$, all nonzero Matsubara-frequency terms are cancelled out and the remaining $\mu = 0$ term can be written in the form:

$$\sigma_{AL} = \frac{1}{4} \left( \frac{G_0}{\pi} \right) \int_0^\infty \left( \frac{\Phi'(x, \varepsilon_H)}{\Phi(x, \varepsilon_H)} \right)^2 dx \quad (16)$$

Exploiting the linear approximation of equation (5), i.e.: $\Phi(x; \varepsilon_H) \approx \varepsilon_H + \eta(x) \varepsilon_H$, and performing the integration over $x$ analytically we find:
\[
\sigma_{\text{AL}d} \simeq \left( \frac{e^2}{4\pi^2\hbar} \right) \frac{\eta(H)}{\varepsilon_H (1 + \frac{\mu_H}{\eta(H)\kappa})}. \tag{17}
\]

Note, that in the zero field limit, where \(\eta(H \to 0) = \pi^2/2\), this result is by a factor of 2 larger than the well-known result obtained, e.g. in [8] by using a fully microscopic (diagrammatic) approach. The discrepancy is not related to the different calculational approaches employed but is due to the different schemes of analytic continuation, used in both approaches, in evaluating the retarded response function from the time-ordered correlator. The smaller prefactor is obtained by using the common contour-integration scheme consisting of three sub-contours (see [10]). This ambiguity seems to indicate that the electrical response in the low frequency range is more intricate than commonly thought and well established in the classic literature, however the whole issue calls for further investigation.

In any numerical computation performed in this paper we will adopt the smaller prefactor consistently with the common microscopic theory pair vertices in the AL diagram are renormalized from outside by impurity scattering ladders between single electron lines.

The exclusive boson TDGL functional approach employed here treats consistently the DOS and the AL terms as functionals of the fluctuations propagator whose field dependence exclusively determines the field dependence of the conductivity.

### 3.3. Divergent boson mass at low temperatures

Combining equation (13) with equation (17), the resulting expression for the total fluctuations contributions to the sheet conductance, \(\sigma_{\text{fluct}} - \sigma_{\text{AL}d} + \sigma_{\text{DOS}d}\), highlights the complementary roles played by the stiffness parameter \(\eta(H)\) in the AL and DOS conductivities. The importance of \(\eta(H)\) in controlling the development of an insulating bosonic state at low temperatures and high magnetic field is clearly revealed by considering the extreme situation of its zero temperature limit.

To effectively investigate this limiting situation it will be helpful to rewrite \(\eta(H)\) (see equation (11)) as a sum over fermionic Matsubara frequency, that is:

\[
\eta(H) = \sum_{n=0}^{\infty} \frac{\chi_n^2 - \pi^2 h^2}{\chi_n (\chi_n - 2\beta) + \pi^2 h^2} \tag{18}
\]

where:

\[
\chi_n = n + 1/2 + 2\beta + \delta h^2
\]

and: \(h \equiv \hbar H/\kappa_{\parallel}^*\), \(t \equiv T/T^*\), \(\beta = \mu_0/t, \mu_0 \equiv \mu_B H_{\parallel}^*/2\pi k_B T^*_c\), \(\delta = \delta_0/t, \delta_0 \equiv D (deH_{\parallel}^*/\kappa^2) /2\pi k_B T^*_c h\), with \(H_{\parallel}^*/\kappa\) and \(T^*_c\) being characteristic scales of the critical parallel magnetic field and critical temperature, respectively.

At very low temperatures, \(t \ll 1\), and finite magnetic field, \(h > 0\), the discrete summation in equation (18) transforms into integration and:

\[
\eta(h) \to t \int_0^{\infty} dv' \frac{x_{\nu}^2 - v'^2 h^2}{[x_{\nu} (x_{\nu} - 2\beta_0) + \mu^2_0 h^2]^2} = t \left( \frac{\eta_0(h)}{h^2} \right) \tag{20}
\]

where \(x_{\nu} = \nu + 2\beta_0 + \delta h^2\),

\[
\eta_0(h) \equiv \frac{\delta_0 h^2 + 2\beta_0}{(\delta_0 h^2 + 2\beta_0) \delta_0 + \mu_0^2}
\]

and \(\beta_0 \equiv \varepsilon_{SO}/4\pi k_B T^*_c\). Note that at zero magnetic field: \(\eta(h = 0) = \sum_{n=0}^{\infty} (n + 1/2)^{-2} = \psi'(1/2) = \pi^2/2\), independent of temperature. Thus, the low temperature limit of the sheet conductance at fields above the nominal critical field \(H_{\parallel}^*/\kappa\) can be written in the form:

\[
(\sigma_{\text{fluct}})_{h>1, t \ll 1} d \to \left( \frac{G_0}{\pi} \right) t \left( \frac{\eta_0(h)}{8h^2} \right) \frac{1}{\varepsilon_h} \frac{1}{1 + \left( \frac{h}{\eta_0(h)} \right)^2} - \frac{1}{7} \left( \frac{3.5 \zeta(3) h^2}{\eta_0(h)} \right) \ln \left( 1 + \frac{\eta_0(h)x_0}{h^2 \varepsilon_h} \right) \tag{22}
\]

where \(x_0 \equiv hDq^2/4\pi k_B T^*_c\) is the temperature-independent dimensionless cutoff parameter. Note the factor of 8 in the denominator of the AL term which follows the common scheme of analytic continuation, as discussed below equation (17). It should be stressed at this point that the temperature-independent argument of the logarithmic factor in equation (22) (see [11]) is consistent with the temperature-dependent cutoff parameter \(x_c = x_0/t\). It should also be noted here that, despite the divergence of \(x_c\) in the \(t \to 0\) limit, the linear approximation \(\Phi(x; \varepsilon_H) \simeq \varepsilon_H + \eta(H)x\) used in deriving equation (22) is valid in the entire range of integration below the cutoff \(x_c\) (see appendix A).

Thus, we conclude that in the \(t \to 0\) limit the AL paraconductivity follows the vanishing stiffness parameter \(\eta(h) \propto t\), equation (20), whereas the DOS conductivity diverges with \(1/\eta(h) \propto 1/t\). Both effects have the same origin: The divergent effective mass of the fluctuations, which leads directly to the former effect and indirectly to the latter effect through extreme softening of the fluctuation modes over a large region in momentum space, which results in large accumulation of Cooper-pairs within fluctuation puddles, whose characteristic spatial size (localization length):

\[
\bar{\xi} (t \to 0) = \frac{1}{\hbar} \left( \frac{\eta_0(h)}{\varepsilon_H} \frac{hD}{4\pi k_B T^*_c} \right)^{1/2}
\]

remains finite in this extreme limiting situation. The decreasing asymptotic field dependence \(\eta(h) \propto 1/h^2\) of the stiffness parameter (see equation (20)) further enhances the sheet resistance at high fields by diminishing the localization length \(\bar{\xi} (t \to 0) \propto 1/\hbar \sqrt{\varepsilon_H}\).
Finally, based on typical values of our fitting parameters (including \(x_0 \equiv hDq^2/4\pi k_b T^*_c = 0.015\)), we use equation (23) for determining the value of the cutoff wavenumber \(q_c\) on the scale of the inverse temperature-independent coherence length \(\xi^{-1} (t \to 0)\). Thus, at field just above the ‘nominal’ critical field \(H^*_{c0} = 4.5T\) (\(\varepsilon_B > 1 = 0.05\)) we estimate: \((\eta_0(h)x_0/h^2\varepsilon_B)_{\varepsilon_B > 1} \approx 1.3\), so that we find the expected relation:

\[
q_c = \left(\frac{\eta_0(h)x_0}{\varepsilon_B h^2}\right)^{1/2} = \tilde{\xi}^{-1} (t \to 0) \approx \tilde{\xi}^{-1} (t \to 0).
\]

(24)

4. Quantum tunneling and pair breaking in the boson-insulating state

It is evident that the ultimately divergent negative conductance implied by equation (22) is an unphysical result, which clearly indicates the breakdown of the thermal fluctuations approach at finite field and very low temperatures. In particular, the unlimited rising Cooper-pairs density within mesoscopic puddles, predicted by equation (22) in the zero temperature limit, can be stopped only by pair breaking into unpaired mobile electron states. Within the fully microscopic (diagrammatic) theory of fluctuations in superconductors [8], quantum fluctuations associated with renormalization of the pairing vertices by impurity-scattering (see, e.g. [12, 13]) can lead to such pair-breaking processes. However, their apparent dynamical nature have not been treated consistently in the current literature (see, e.g. the calculation of the DOS contribution in [8]).

Furthermore, the state of the art of the microscopic theory of fluctuations in superconductors is not sufficiently developed to include dynamical quantum tunneling of Cooper-pair fluctuations [1], a phenomenon which should intensify concurrently with the field-induced pair breaking processes, due to the strongly enhanced Cooper-pair fluctuations density in mesoscopic puddles.

Thus, in the absence of a complete microscopic quantum theory of fluctuations the bosonic TDGL functional approach employed here is complemented by a phenomenological scheme, which introduces quantum tunneling of Cooper-pair fluctuations jointly and consistently with dynamical pair-breaking corrections.

Within this phenomenological approach, we identify in both equations (12) and (15), ‘external’ and ‘internal’ links for quantum tunneling corrections to be inserted into both the DOS and the AL conductivities, respectively. For the DOS conductivity the ‘external’ link in the momentum distribution function, equation (12), is the inverse thermal-prefactor: \(1/k_b T\), which is interpreted as a characteristic thermal activation time \(\tau_T = h/k_b T\), whereas the ‘internal’ link is in the fluctuation energy function \(\Phi(x;\varepsilon_B) \approx \varepsilon_U + \eta(H)x\). The correction in the ‘external’ link amounts to modifying \(\tau_T\) by including the effect of quantum tunneling through the rate equation:

\[
\frac{1}{\tau_U} = \frac{1}{\tau_T} + \frac{1}{\tau_Q} = k_b (T + T_Q)/h
\]

(25)

where \(\tau_Q = h/k_b T_Q\) is the quantum tunneling time.

The corresponding correction in the ‘internal’ link reflects the dynamics of the quantum tunneling by shifting the fermionic Matsubara frequency \(\omega_0 = (2n + 1)\pi k_b T/h\) with the ‘excitation’ frequency \(\pi k_b T_Q/h\), under summation defining the digamma functions in equations (5) and (6).

For the AL conductivity the ‘external’ link in the current correlator equation (15) is the thermal-rate prefactor for the charge transfer: \(k_b T \times 1/\tau_T\), which is corrected by adding the quantum tunneling attempt rate \(1/\tau_Q \times k_b T_Q\) according to the rate equation (25), whereas the ‘internal’ links in the fluctuation energy functions and their derivatives are corrected in a way identical to that employed for the DOS conductivity (see also appendix B for more details).

The over all ‘external’ modifications result in multiplying the AL conductivity (equation (17)) and dividing the DOS conductivity (equation (13)) by the same factor \((1 + T_Q/T)\). The corresponding ‘internal’ modifications, result in shifting the arguments of the digamma functions and their derivatives in \(\varepsilon_B\), and \(\eta(h)\), respectively, with the normalized ‘excitation’ frequency term \(T_Q/2T\), which reflect the dynamical nature of the quantum tunneling introduced to the ‘external’ links.

This pattern of quantum corrections is consistent with the introduction of the unified quantum-thermal (QT) fluctuations partition function:

\[
Z_{qtuc} = \prod \int D\Delta(q) D\Delta^*(q) \times \exp \left\{ -\frac{\tau_U}{h} \left[ \varepsilon_U + \frac{\eta(h)}{4\pi k_b T} Dq^2 \right] \right\}
\]

(26)

where \(\tau_U\), defined in equation (25), is interpreted as the combined QT characteristic time for both activation over and tunneling through the GL energy barriers separating SC and normal state regimes. The significance of the unified QT electron pairing functions \(\varepsilon_U, \eta(h)\), following the ‘internal’ modifications mentioned above, will be further elaborated below. The partition function, equation (26) yields the QT fluctuations propagator:

\[
D_U(q;\varepsilon_B) = \frac{k_b (T + T_Q)}{N_{2D} \left(\varepsilon_B + \frac{Dq^2\eta(h)}{4\pi k_b T}\right)}
\]

(27)

in which the ‘dressed’ critical shift parameter, \(\varepsilon_B\), due to interaction between Gaussian fluctuations, is determined from the self-consistent field (SCF) equation [1]:

\[
\varepsilon_B^U = \varepsilon_B^U + \alpha F_U(h) (1 + T_Q/T) \ln \left(1 + \frac{\eta(h)x_0}{\varepsilon_B^U}\right).
\]

(28)

Here:

\[
F_U(h) = \frac{1}{\eta(h)} \sum_{n=0}^\infty \frac{x_n^U \left[ (x_n^U)^2 + \pi^2 h^2 \right]}{\left[ x_n^U (x_n^U - 2\beta + \pi^2 h^2)^2 \right]^T},
\]

(29)

with:

\[
x_n^U = n + 1/2 + T_Q/2T + 2\beta + 2\pi h^2
\]

(30)
is the four-electron correlator controlling the interaction between fluctuations, and:

\[
\alpha \equiv \frac{1}{l_{\text{pi}}^3 D_{\text{2D}}} = \frac{2}{\pi^2} \left( \frac{\varepsilon_{SO}}{E_F} \right)
\]

(31)

is the interaction strength parameter. Note the ‘external’ quantum tunneling correction factor \((1 + T_Q/T)\) in equation (28), which originates in the unified QT rate factor of the fluctuation propagator, as written in equation (27). The ‘external’ corrections to the AL and the DOS conductivities in equations (13) and (17) respectively are equivalent to replacing the stiffness parameter appearing in their prefactors with the hybrid expression:

\[
\eta(h) \rightarrow \left(1 + \frac{T_Q}{T}\right) \eta_U(h)
\]

(32)

where:

\[
\eta_U(h) = \sum_{n=0}^{\infty} \left[ \psi_n^\prime \left( \frac{x_n}{h} \right) - \frac{2\beta f_n}{\pi^2 h^2} \right]^2.
\]

(33)

The ‘excitation’ frequency-shift term \(T_Q/2T\) appearing in equation (33) (through equation (30)), represents pair-breaking effect associated with the tunneling process. This is seen more directly under the transformation \(\varepsilon_h \rightarrow \varepsilon_h^U\) of the critical-shift parameter:

\[
\varepsilon_h \rightarrow \varepsilon_h^U \equiv \ln \left( \frac{T}{T_Q} \right) + a + \psi \left( \frac{1}{2} + \frac{T_Q/2T}{T} - f_+ \right) + a - \psi \left( \frac{1}{2} + \frac{T_Q/2T + f_-}{T} \right) - \psi(1/2).
\]

(34)

In the absence of quantum tunneling \(\varepsilon_h\) (equation (6)) is subjected to the usual magnetic field induced pair-breaking effect [14] through the Zeeman spin-splitting energy \(\mu_B H\) and the diamagnetic energy \((D|dE/dH|^2)/h\) terms. In the zero temperature limit, the effect is dramatically reduced in the removal of the (Cooper) singularity of the logarithmic term in equation (6), due to exact cancellation by the asymptotic values of the digamma functions for \(f_n \gg 1\) (see appendix C). In the presence of quantum tunneling, the excitation frequency shift \(\pi k_{\text{B}} T_Q/h\) introduced to define \(\varepsilon_h^U\), equation (34), causes in this limit an additional, pair-breaking effect, not driven directly by magnetic field, and suppresses the Cooper-pair fluctuations density and rein-

recovery of the stiffness parameter at finite fields (see figure 1), and so suppress the Cooper-pair fluctuations density and reinforce pair-breaking into unpaired electron states.

To summarize, the frequency shift that transforms \(\varepsilon(h)\) to \(\varepsilon_U(h)\) and represents pair breaking effect, is intimately connected to the quantum tunneling process discussed above. This is clearly seen by considering the zero temperature limit of \(\eta_U(h)\) in equation (33):

\[
(\eta_U(h))_{T=0} = \left( \frac{T}{T_Q} \right) \eta_0(h)
\]

(35)

where:

\[
\eta_0(h) \equiv \int_0^\infty \frac{du}{(x^2_0 - \mu_0^2 h^2)} \left[ \frac{(x^2_0 - 2\beta_0 h^2) + \mu_0^2 h^2}{(1/2 + 2\beta_0 + \delta_0 h^2)(1/2 + \delta_0 h^2 + 2\beta_0) + \mu_0^2 h^2} \right]
\]

(36)

and \(x_0^2 = \nu + 1/2 + 2\beta_0 + \delta_0 h^2\), with \(\beta_0 = \beta_0/T_Q, \mu_0 = \mu_0/T_Q, \delta_0 = \delta_0/T_Q, T_Q = \Delta T_Q\).

The limiting function \(\eta_0(h)\) in equation (36) is a continuous smooth function of the field \(h\), including at \(h=0\). Therefore, equation (35) implies that the discontinuous plunge of \(\eta(h)\) at \(h=0\) in the zero temperature limit (see figure 1) is removed by the frequency shift term, as can be directly checked in equation (33). The magnitude of \(\eta_U(h)\) diminishes uniformly to zero with \(T/T_Q\) in this limit. However, by multiplying with the divergent quantum tunneling factor \((1 + T_Q/T)\) the resulting hybrid product in equation (32), which represents the combined effect of quantum tunneling and pair breaking, is a smooth finite function of the field \(\eta_0(h)\) (see appendix C).

A similar hybrid form and limiting behavior at zero temperature characterize the interaction term in the SCF equation (28). The four-electron correlator:

\[
(F_U(h))_{T=0} = \left( \frac{T}{T_Q} \right) F_Q(h)
\]

(37)
where:

$$F_Q(h) = \frac{1}{\eta_Q(h)} \int_0^{\infty} du \frac{\varepsilon_1^0 (\varepsilon_0^0)^2 + \mu_0^2 h^2}{\varepsilon_1^0 (\varepsilon_0^0 - 2\beta_0^0) + \mu_0^2 h^2}$$  \hspace{1cm} (38)$$

shows similar singular behavior to that of $\eta_H(h)$ (see equations (35) and (36)). The overall interaction term has the finite regular limiting form (including the logarithm):

$$\alpha F_{U}(h) (1 + T_Q/T) \ln \left( \frac{1 + \eta_{U}(h) x_0^{-1}}{\varepsilon_h^{0}} \right) \rightarrow \alpha F_{Q}(h) \ln \left( \frac{1 + \eta_{U}(h) x_0}{\varepsilon_h^{0} t_0} \right).$$  \hspace{1cm} (39)$$

This SCF approach avoids the critical divergence of both the AL paraconductivity and the DOS conductivity, and allows to extend the expression for the conductance fluctuations $\sigma_{\text{fluct}} d = \sigma_{\text{AL}} d + \delta \sigma_{\text{DOS}} d$, given in terms of equations (13) and (17), to regions well below the nominal critical SC transition. It also offers an extended proper measure of the pair-breaking effect. In contrast to $\varepsilon_h$, $\varepsilon_{\Delta h}$ is positive definite in the entire fields range, including that below the critical field where $\varepsilon_h < 0$ (see figure 2). The uniform enhancement of $\varepsilon_{\Delta h}$ with respect to $\varepsilon_h$, seen in figure 2, resulting from the introduction of the frequency shift to the SCF equation (28), is a genuine measure of the pair-breaking effect associated with the quantum tunneling. Its monotonically increasing field dependence seen in figure 2 properly reflects the field-induced pair-breaking effect in the entire fields range.

5. Discussion

In this paper we have discovered, while searching for the deep origin of the high-field insulating states appearing at diminishing temperature, that due to extreme softening of the fluctuation modes and their redistribution over a large region in momentum space, there is a propensity of Cooper-pair fluctuations to condense in real-space puddles of decreasing spatial size, $\xi(t \to 0)$ (see equation (23)). This picture is of course ideal, but basically reflects real tendency toward boson insulating states. Charge transfers between the exclusive bosons system and the normal-electron states, underlying the microscopic Gorkov GL approach employed, are introduced within a unified phenomenological approach, by allowing field-induced pair-breaking processes to develop during dynamical quantum tunneling of Cooper-pair fluctuations out of the mesoscopic puddles.

Other quantum fluctuations effects arising from coherent Andreev-like scatterings [8, 15, 16] associated with the Maki-Thompson contribution to the paracconductivity [17, 18], are expected to be suppressed by the strong spin-orbit scatterings [8], which characterize the SrTiO$_3$/LaAlO$_3$ interfaces under consideration here [4, 19].

Exploiting the complete agreement between the results of our approach and those of the fully microscopic theory at zero magnetic field, it will be meaningful at this point to compare the influence of the quantum fluctuations employed in each approach on the conductivity at finite field. Thus, on one hand, the DOS conductivity derived in our approach in the quantum limit (see equation C.4), and the renormalized single-particle conductivity derived within the fully microscopic approach in the quantum fluctuations regime [13], are both finite, with negative sign, and have the same field dependence. On the other hand, in the fully microscopic approach the vanishing rate of the AL paraconductivity is further accelerated in the quantum fluctuations regime [13], whereas in our approach the vanishing AL conductivity (see equation (22)) is recovered by the effect of quantum tunneling of Cooper-pair fluctuations (see equation (C.6)), which is fundamentally missing in the fully microscopic approach. The physical reasoning behind this recovery is elaborated in section 4 and in appendix B.

An important feature of the localization process predicted in our approach is its dynamical nature, namely that it occurs in response to the driving electric force [1], and not spontaneously in a thermodynamical process toward equilibrium state. This feature seems to distinguish it from the various approaches to the phenomenon of SIT discussed in the literature [20–23], in which disorder-induced spatial inhomogeneity in the form of SC islands is involved in generating the insulating state. However, in a similar manner the formation of fluctuation puddles in our approach is controlled by disorder, which strongly affect the Cooper-pairs amplitude correlation function in real space. This can be seen by comparing the pair correlation function derived in the dirty limit [1, 24] to that obtained in the pure limit [25].

Another important parameter in our approach of relevance to the insulating behavior that seems to have a parallel in the literature [21], is the self-consistent critical shift parameter $\varepsilon_H$, which also plays the role of an energy gap in the Cooper-pair fluctuations spectrum [1]. Thus, it is interesting to note that the two-particle gap, which characterizes the insulating state in [21], vanishes at the SIT. Analogously, in our approach the
(two-particle) Cooper-pair fluctuation gap $\tilde{\varepsilon}_H$ gradually diminishes to very small (nonvanishing) values upon decreasing field below the sheet-resistance peak (see figure 2), in accord with the lack of a critical point.

The combined effect of this nonvanishing two-particle gap $\tilde{\varepsilon}_H$ and the diminishing stiffness parameter $\eta \ (H)$ upon increasing field at very low temperatures, is responsible for the loss of long-range phase coherence and for the puddles formation. The resulting boson insulating state is reminiscent of the field-induced paired insulating phase discussed in [26], which is also closely related to the picture of the suppressed Bose insulator deliberated in [27]. The introduction of quantum tunneling of Cooper-pair fluctuations within our complementary phenomenological approach, which leads to the broadening of the sharp MR peaks at low temperatures, is clearly consistent with the conducting Josephson tunneling effect among SC islands.

Finally, a few comments about the robustness of the quantitative comparison with the experimental data [4] are in order. As the direct analytic continuation scheme of the AL correlator employed in [1] (see the remarks below equation (17)) doubles the prefactor of the corresponding paraconductance as compared to the well-known result, the implication for the fitting process of using the latter prefactor is expected to further amplify the relative magnitude of the negative DOS conductivity and so to further reinforce the appearance of the field-induced boson insulating states at low temperatures. This has been confirmed quantitatively in appendix D by repeating the fitting process described in detail in [1] with the 1/2 prefactor of the AL contribution. The results show that good agreement with the experimental data can be achieved also with the reduced AL prefactor by changing the spin–orbit scattering parameter moderately within its range of uncertainty.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Range of validity of the linear approximation

Extending the first order expansion to next order, i.e. writing: $\Phi (x; \varepsilon_h) = \varepsilon_h + \eta \ (H) \ x + \zeta \ (H) \ x^2 + \ldots$, where $\zeta \ (H) = [a_+ \ \psi^{'''} (1/2 + f_+ \ldots + a_- \ \psi^{'''} (1/2 + f_+ \ldots)] / 2$, we find at low temperature ($t \ll 1$) and finite magnetic field, $h > 0$, that in addition to equations (20) and (21), $\zeta \ (H) \to \zeta_0 \ (h) \ t^2 / h^4$, where:

$$\frac{1}{h^2} \zeta_0 \ (h) = \frac{1}{2} \left[ \int_0^\infty \frac{d\nu}{\nu + \beta_0} \frac{(1 + \beta_0 / \sqrt{\beta_0^2 - \mu_0^2 h^2})}{(\nu + \delta_0 h^2 + \beta_0 - \sqrt{\beta_0^2 - \mu_0^2 h^2})^3} \right.$$  
$$+ \int_0^\infty \frac{d\nu}{\nu + \delta_0 h^2 + \beta_0 + \sqrt{\beta_0^2 - \mu_0^2 h^2}}$$.

Evaluation of the integral leads to:

$$\zeta_0 \ (h) = \frac{1}{2} \frac{(2 \beta_0 + \delta_0 h^2)^2 - \mu_0^2 h^2}{[(2 \beta_0 + \delta_0 h^2) \delta_0 + \mu_0^2]^2}$$

so that the relevant expansion at high fields ($h \sim 1$), is:

$$\Phi (x; \varepsilon_h) \to \varepsilon_H + \frac{1}{h^2} \eta_0 \ (h) \ (tx) + \frac{1}{h^2} \zeta_0 \ (h) \ (tx)^2 + \ldots \quad \text{(A.1)}$$

with the corresponding temperature independent expansion parameter:

$$\chi t = \frac{h \ Deq^2}{4 \pi k_B T_c} \leq \frac{h \ Deq^2}{4 \pi k_B T_c} = x_0.$$

For the experimental situation encountered in [1] the diamagnetic energy term $\delta_0 h^2$ is much smaller than both the spin–orbit energy $\beta_0$ and the Zeeman splitting $\mu_0 h$, implying that the coefficients: $\eta_0 \ (h) \sim 2 \beta_0 / \mu_0^2$, $\zeta_0 \ (h) \sim (4 \beta_0^2 - \mu_0^2 h^2) / 2 \mu_0^2$, are constant, or nearly constant (since typically $(2 \beta_0)^2 \gg (\mu_0 h)^2$).

We therefore conclude that the condition for uniform convergence of the expansion is $x_0 \ll 1$. In our fitting process we have used the value $x_0 = 0.015$, well within the domain of convergence.

Appendix B. The quantum fluctuations corrections to conductivity

In this appendix we outline the physical reasoning behind our phenomenological quantum fluctuations correction to the two ingredients of the conductance fluctuations. Starting with the DOS conductivity we consider the Cooper-pair density, $n_c$, given in equation (7), with $\langle |\phi (q)|^2 \rangle$ in equation (12). Approximating $7 \zeta (3) \approx 8.4$ we rewrite:

$$\langle |\phi (q)|^2 \rangle \approx 4 \frac{1}{\Phi (x; \varepsilon_H) \lambda_T^2}$$

where $\Phi (x; \varepsilon_H) \approx \varepsilon_H + \eta \ (H) \ x$ and $n_{2D} = k_F^2 / 2 \pi$ is the density of the 2D electron gas and $\lambda_T = \sqrt{k^2 / 2 \pi m^* k_B T}$ is the thermal wavelength.

The momentum distribution function $\langle |\phi (q)|^2 \rangle$ measures the number of bosons per wave vector $q$ in the Cooper-pairs liquid, engaged in equilibrium with a 2D gas of unpaired mobile electrons with a nominal density $n_{2D}$. The prefactor
the effective frequency-shift term \( \pi \) pairs. The resulting reduction in the number of Cooper-pairs, replacing the temperature \( T \), for both thermal activation and quantum tunneling of Cooper pairs. Thus, increasing the temperature and/or shortening the time \( T \) for quantum tunneling (which also enhance pair breaking by increasing \( \Phi_U (x; \varepsilon_H^U) \)), result in larger rate of thermal and/or quantum leakage from puddles of Cooper pairs. The resulting reduction in the number of Cooper-pairs, which occurs versus a corresponding increase in the number of unpaired mobile electrons, would suppress the DOS contribution to the resistance.

The corresponding unified (quantum-thermal (QT)) density (per unit area) of the Cooper-pairs liquid is now evaluated: \( n^U = \frac{1}{2} \frac{1}{(2\pi^2)} \int \frac{d^2 q}{q^2} \frac{1}{q^2} (\frac{1}{\pi^2 (q^2 + \pi^2 \nu)^2}) \), so that the unified DOS conductivity, \( \sigma_{DOS}^U \), is given by:

\[
\sigma_{DOS}^U \equiv -2 (n^U e^2/m^*) \tau_S O, \text{is given by:}
\]

\[
\sigma_{DOS}^d = -4.2 \frac{G_0}{\pi} \int_{0}^{\tau_S O} (1 + T/Q \Phi_U (x; \varepsilon_H^U)) dx. \quad (B.2)
\]

For the AL thermal fluctuations conductivity we start with the retarded current-current correlator \( Q_{Al}^R (\Omega) \), equation (15), which was obtained from the Matsubara correlator \( Q_{Al} (i \Omega_n) \) following the analytic continuation \( i \Omega \rightarrow \Omega \). The corresponding electrical response function is seen to be proportional to the thermal energy \( k_B T \). The effects of quantum tunneling and pair breaking are introduced by adding to the thermal attempt rate \( 1/\tau_T \propto k_B T \) the quantum tunneling attempt rate \( 1/\tau_Q \propto k_B T \), and by appropriately inserting the effective frequency-shift term \( T_Q/2T \) into the function \( \Phi (x + |\mu + \nu|; \varepsilon_H) \), as explained in the main text, i.e.:

\[
Q_{AL}^R (i \Omega_n) = k_B (T + T_Q) \left( \frac{2e}{h} \right)^2 \left( \frac{1}{2 \pi d} \right) \int x dx \times \sum_{\mu = 0, \pm 1, \pm 2, \ldots} \Phi_U^R (x + |\mu + \nu|; \varepsilon_H^R) \Phi_U^R (x + |\mu + \nu|; \varepsilon_H^R)
\]

\[
\Phi_U (x + |\mu + \nu|; \varepsilon_H^R) \Phi_U (x + |\mu + \nu|; \varepsilon_H^R)
\]

Now, by repeating the procedure employed in deriving equation (16), in which the above discrete summation is directly continued analytically, \( \nu \rightarrow \pi \Omega / 2 \pi i k_B T \), and expanded about zero frequency, we arrive at the following expression for the unified QT AL conductivity:

\[
\sigma_{AL}^d = \frac{1}{4} \left( \frac{G_0}{\pi} \right) \left( 1 + \frac{T_Q}{T} \right) \int_0^{\tau_S O} \Phi_U^R (x; \varepsilon_H^R)^2 \frac{\Phi_U^R (x; \varepsilon_H^R)^2}{\Phi_U (x; \varepsilon_H)} dx. \quad (B.3)
\]

As indicated in section 3.2 below equation (17), the common scheme of analytic continuation utilizing the contour integration method for performing the Matsubara summation yields the same result as equation (B.3) but with the prefactor 1/4 replaced with 1/8.

Appendix C. The quantum limit

In this appendix we examine the zero-temperature (quantum) limit of the conductance fluctuation analyzed in appendix B. We begin by studying the field-induced pair-breaking and quantum tunneling of Cooper pairs in this limiting situation. Consider the unified (quantum-thermal) expression, equation (34), for the critical shift parameter \( \varepsilon_H^U \). Using the asymptotic expansion of \( \psi \left( \frac{1}{2} + \frac{T_Q}{2T} + f_{\pm} \right) \) for \( T_Q/T, f_{\pm} \gg 1 \), i.e. \( \psi \left( \frac{1}{2} + \frac{T_Q}{2T} + f_{\pm} \right) \rightarrow \ln \left( \frac{T/Q}{2T} + f_{\pm} \right) = \ln \left( 1 \right)/2T \), we have:

\[
\varepsilon_H^U \rightarrow \varepsilon_H^O = \ln \left( \frac{T}{T_0} \right) - \ln T + a \ln \left( \frac{T_Q + T_-}{T_Q + T_+} \right) + a \ln \left( \frac{T_Q + T_+}{T_Q + T_-} \right) - \ln 2 - \psi \left( \frac{1}{2} \right)
\]

where:

\[
T_\pm \equiv \frac{D (de)^2 H^2}{2 \pi k_B} + \frac{\varepsilon_{SO}}{2 \pi k_B} \pm \sqrt{\frac{\varepsilon_{SO}^2}{2 \pi k_B} + \left( \frac{\mu_B H}{2 \pi k_B} \right)^2}
\]

In the above expression for \( \varepsilon_H^O \) (equation (C.1)), the Cooper singular term, \( \ln \left( \frac{T}{T_0} \right) \), is exactly cancelled by the logarithmic term arising from the asymptotic expansion of the digamma functions, so that the remaining regular terms are rearranged to yield the following temperature independent expression for \( \varepsilon_H^O \):

\[
\varepsilon_H^O \rightarrow a_+ \ln \left( \frac{T_Q + T_-}{T_Q + T_0} \right) + a_- \ln \left( \frac{T_Q + T_+}{T_Q + T_0} \right) + \ln 2 + \gamma
\]

where \( \gamma = 0.5772 \ldots \) is the Euler–Mascheroni constant, and:

\[
a_\pm = \frac{1}{2} \left( 1 \pm \frac{1}{\sqrt{1 - (\mu_0/\beta_0)^2} h^2} \right)
\]
We now turn to the quantum limit of the DOS conductivity, equation (2.4):

\[
\sigma_{\text{DOS}}^Q = -4.2 \frac{G_0}{\pi} \frac{t_\eta \eta_0}{2 + h^2} \ln \left[ 1 + \frac{\eta_0}{\eta_h \left( t_\eta \eta_0 / 2 + h^2 \right)} \right],
\]

where \( \eta_h \) is determined by the SCF equation, equation (28), in the quantum limit, i.e.:

\[
\eta_h = \frac{\epsilon_h^Q}{\eta_h} + \alpha F_\eta(h) \ln \left[ 1 + \frac{\eta_0 \eta_0}{\eta_h \left( t_\eta \eta_0 / 2 + h^2 \right)} \right].
\]

In the absence of the self-consistent interaction between fluctuations the critical shift parameter reduces to \( \epsilon_h^Q \), which vanishes at the quantum critical field \( h_c \), so that near \( h_c \):

\[
\epsilon_h^Q \propto h - h_c \to 0.
\]

It is instructive to note that equation (C.4), is equivalent to the fluctuation conductivity derived in [13] in the region of quantum fluctuations within a fully microscopic (diagrammatic) approach.

Considering the unified AL conductivity, equation (B.3) in the linear approximation:

\[
\sigma_{\text{AL}}^Q d \approx \frac{1}{4} \frac{G_0}{\pi} \left( 1 + \frac{T_0}{T} \right) \int_0^{\eta_0 \eta_0} \frac{\eta_0 (h)}{\eta_h (h) \left( 2 + h^2 \right)} \frac{t_\eta \eta_0 \eta_0}{\eta_h \left( t_\eta \eta_0 / 2 + h^2 \right)} \frac{1}{\eta_h \left( 2 + h^2 \right)} \frac{1}{\eta_h \left( 2 + h^2 \right)}
\]

so that in the limit, \( T_0 / T \to \infty \):

\[
\sigma_{\text{AL}}^Q d \doteq \frac{1}{4} \frac{G_0}{\pi} \frac{t_\eta \eta_0 \eta_0}{2 + h^2} \frac{1}{\eta_h \left( t_\eta \eta_0 / 2 + h^2 \right)} \frac{1}{\eta_h \left( 2 + h^2 \right)} \frac{1}{\eta_h \left( 2 + h^2 \right)}.
\]

Note that in deriving equations (C.4)–(C.6), with the field independent parameter \( \eta_0 \), the small diamagnetic energy term was neglected.

### Appendix D. Robustness of the fitting process

In this appendix we present results (see figure C1) of a fitting process similar to that presented in [1], in which the prefactor of the AL conductivity term is 1/2 of that used in [1]. The level of agreement between these calculations and the experimental data is preserved if the values of the dimensionless spin–orbit scattering parameter used in [1], i.e. \( \beta_0 = 14 \) (\( R_N = 7.5 \) kΩ) and \( \beta_0 = 11 \) (\( R_N = 20.5 \) kΩ), are changed in the new fitting to \( \beta_0 = 16 \) and \( \beta_0 = 12 \) respectively. The phenomenological parameters determining the quantum tunneling attempt rates and the normal state conductivity should also slightly modified. All the other parameters of microscopic origins can remain fixed.

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