Sub-threshold resonances in few–neutron systems

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Abstract

Three– and four–neutron systems are studied within the framework of the hyperspherical approach with a local $S$–wave $nn$–potential. Possible bound and resonant states of these systems are sought as zeros of three– and four–body Jost functions in the complex momentum plane. It is found that zeros closest to the origin correspond to sub–threshold $(nn)n^1$ and $(nnnn)$ $0^+$ resonant states. The positions of these zeros turned out to be sensitive to the choice of the $nn$–potential. For the Malfliet–Tjon potential they are $E_0(3n) = -4.9 - 6.9i$ (MeV) and $E_0(4n) = -2.6 - 9.0i$ (MeV). Movement of the zeros with an artificial increase of the potential strength also shows an extreme sensitivity to the choice of potential. Thus, to generate $3n$ and $4n$ bound states, the Yukawa potential needs to be multiplied by 2.67 and 2.32 respectively, while for the Malfliet–Tjon potential the required multiplicative factors are 4.04 and 3.59.

I. INTRODUCTION

The question of the existence of pure neutron nuclei is a longstanding problem, almost as old as nuclear physics itself. Many experiments have been performed in the past in order to find di–, tri–, or tetra–neutron bound states. On the theoretical front much effort has also been devoted to understand the physics of few–neutron systems.

Early experimental and theoretical investigations were controversial and their findings varied from non-existence to ‘discovery’ and ‘prediction’ of such states. Nowadays it is generally accepted that no few–neutron nuclei exist. However, the interest in pure neutron systems did not wane. Since $(np)$, $(nnp)$, $(npp)$, and $(nnpp)$ systems are bound, one could expect that due to isotopic invariance a small increase in the neutron–neutron attraction could generate bound states in their isotopic partners $(nn)$, $(nnn)$, and $(nnnn)$. Indeed, this is the case for the two–neutron system which requires only an additional 67 keV attraction for the formation of a bound di-neutron. For three and more neutrons, however, the situation is more complicated due to the Pauli principle. This becomes clear when one

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considers a multi–neutron system in even the simplest model, namely, that of neutrons moving independently in a harmonic oscillator well. Since only two neutrons can occupy the ground $S$–wave state of this well, the third, fourth, and other neutrons must be in excited states and such a multi–neutron system can form quasi–stationary states instead of bound ones. This encouraged research by both experimentalists and theorists to locate resonances in few–neutron systems.

In spite of the abundance of experimental works that exploit different reactions and methods, the situation concerning few–neutron resonances remains unclear. In many cases certain irregularities in cross–sections are observed, which can be attributed to such resonances, but their positions and widths have not been firmly established. As for theoretical studies of few–body resonances, they are complicated and cumbersome to the extend that these states have not been properly investigated. Since bound states can be located more easily, it is customary for theorists to increase the $nn$–potential until a bound multi–neutron system is supported. It was found that the interaction needs to be increased by a factor of between 2 and 4, depending on the number of neutrons, the type of potential, and the method used.

There are many different definitions of quantum resonances as well as many methods of locating them. Though all methods are physically justified and able to produce reasonable results, only few of them can be considered as rigorous. These are the methods based on the correspondence between resonances and singularities (poles) of the $S$–matrix in the unphysical sheet of the complex energy plane. Apart from being mathematically rigorous, such approaches enable one to obtain the position and width of a resonance simultaneously with the same accuracy, simply as coordinates of a pole on the energy plane. And again, there are several different methods for locating the $S$–matrix poles (a survey of them can be found in Refs. [4,5]). However due to the existence of redundant poles in the $S$–matrix it is preferable to search for zeros of the Jost function instead.

For two–body systems the Jost function is well–defined and an exact method of calculating it has been developed. However, for many–body systems, no such method exists and in fact no rigorous definition of the Jost function has been given in a general form yet. Only for a special class of many–body systems has a definition and method of calculating the Jost function been proposed, namely, for systems which cannot form clusters of any type and thus only one allowed channel exists. Some authors call them ‘democratic systems’. Wave functions which describe such systems behave asymptotically as diverging hyper–radial spherical waves. Clearly a multineutron system is one such system and it may be investigated via the hyperspherical approach.

In this work we consider three– and four–neutron systems in the minimal approximation, $L = L_{\text{min}}$, of the hyperspherical harmonic approach with local $S$–wave $nn$–potentials. Possible bound and resonant states are sought as zeros of three– and four–body Jost functions in the complex plane of the momentum conjugate to the hyperradius. These zeros are related to the total energy in a straightforward way. The many–body Jost function for the hyperradial equation is defined similarly to the two–body case and is calculated by a method.
which combines the variable constant and complex rotation methods \cite{2}. The zeros found represent the following sub-threshold resonant states: $^3n(2^-)$, $^4n(0^+)$, $^4n(1^+)$, and $^4n(2^+)$. 

Our paper is organized as follows: In Sec. II we discuss the many-body Jost function and in Sec. III we give the required matrix elements of the potential. In Sec. IV the complex coordinate rotation method is used to achieve an analytic continuation of the Jost function into the lower half of the momentum plane, while in Sec. V we describe the physical input used in the calculations and present our results together with some discussions.

**II. MANY-BODY JOST FUNCTION**

Let $\{r\}$ and $\{p\}$ be the complete sets of Jacobi vectors defining a multi-neutron configuration in coordinate and momentum space respectively. The wave function $\Psi^a(r,p)$ of an $N$-neutron system can be expanded in terms of hyperspherical harmonics $Y^a_{[L]}(\Omega)$ with unknown coefficient-functions $u^a_{[L]}(r,p)$

$$\Psi^a(r,p) = r^{2-3N/2} \sum_{[L]=[L_{min}]}^{\infty} Y^a_{[L]}(\Omega) u^a_{[L]}(r,p), \quad (1)$$

where $\{a\}$ is the set of all conserved quantum numbers (total angular momentum, isospin, and parity) and the multi-index $[L]$ represents the ‘grand orbital’ $L$ and other nonconserved quantum numbers; $r$ and $\Omega$, $\{r, \Omega\} \equiv \{r\}$, are the hyperradius and hyperangles. For two neutrons, $N=2$, the expansion is the usual partial-wave decomposition where $[L] \equiv [\ell, m]$ is the angular momentum and its third component, and $\ell_{min} = 0$. When $N>2$, $L$ starts from a generally nonzero value $L_{min}$ defined by the symmetry properties of the state under consideration \cite{11}.

Substituting the expansion (1) into the Schrödinger equation, one arrives at the following infinite system of coupled hyperradial equations

$$\left[\partial_r^2 + p^2 - \lambda (\lambda + 1)/r^2\right] u^a_{[L]}(r,p) = \sum_{[L']} W^a_{[L][L']} (r) u^a_{[L']} (r,p), \quad (2)$$

Here $[L] = [L_{min}],[L_{min}+1],\cdots$ and $p$ may be called ‘hypermomentum’ since it is related to the total energy $E$ and the neutron mass $m$ in the same manner as in the two-body case, $p^2 = 2mE$ (we use units such that $\hbar = 1$); $\lambda$ is an analog of the orbital angular momentum,

$$\lambda \equiv L + \frac{3}{2}(N-2), \quad (3)$$

which assumes half-integer values for uneven particle number $N$. The potential matrix $W$ has elements

$$W^a_{[L][L']} (r) \equiv 2m \langle Y^a_{[L]} | \sum_{i<j} V_{ij}(r) | Y^a_{[L']} \rangle, \quad (4)$$

where the integrals are over the hyperangles and $V_{ij}$ are the two-body potentials.
The required boundary conditions for the differential equations (2) are, firstly, the solutions must be regular at the origin,
\[ u^a_{[L]}(r, p) \rightarrow 0 , \quad r \rightarrow 0 , \]  
and, secondly, at infinity must be of some physically motivated form
\[ u^a_{[L]}(r, p) \rightarrow U^a_{[L]}(r, p) , \quad r \rightarrow \infty . \]  
In the general case the boundary function \( U^a_{[L]}(r, p) \) consists of terms describing each open channel. Since a multi–neutron system is a ‘democratic’ one, we shall construct \( U^a_{[L]} \) as an one–channel boundary function.

Instead of matching \( u^a_{[L]}(r, p) \) to \( U^a_{[L]}(r, p) \), we can secure the one–channel boundary condition automatically, and in an exact form, by the following procedure. Let us consider a general regular solution of (2), defined only by the condition (3) and without any restrictions at large \( r \). Certainly many such solutions exist, each having different behaviour at large distances. By choosing only those which are linearly independent, we have a regular basis. Any other solution which is regular and has a specific behaviour at large \( r \) must be a linear combination of this basis set. Thus, we can find first the regular basis and then construct the required physical solution.

Like any other basis, the regular basis may be chosen in an infinite number of ways. Exploiting this freedom, we can choose the basis regular solutions in a way which is most suitable for the subsequent construction of a physical solution. Considering the system (2) (which is always truncated at some \([L_{\text{max}}] \)) as a matrix equation, we see that it has as many independent regular column–solutions as the column dimension (number of equations in the system). We may combine all these linear independent columns in a square matrix \( \| \phi^a_{[L][L']} \| \) of the form (for simplicity we drop the superscript \( a \))

\[ \phi^a_{[L][L']}(r, p) = \frac{1}{2} \left\{ h^{(+)}_{\lambda}(pr)F^{(+)}_{[L][L']}(r, p) + h^{(-)}_{\lambda}(pr)F^{(-)}_{[L][L']}(r, p) \right\} , \]  
where \( h^{(\pm)}_{\lambda} \) are the Riccati–Hankel functions \([12]\), and \( F^{(\pm)}_{[L][L']}(r, p) \) are new unknown functions. Since instead of one unknown matrix \( \phi^a_{[L][L']} \), we have introduced two matrices \( F^{(\pm)}_{[L][L']} \), we require an additional constraint. The most convenient is the Lagrange condition
\[ h^{(+)}_{\lambda}(pr)\partial_r F^{(+)}_{[L][L']}(r, p) + h^{(-)}_{\lambda}(pr)\partial_r F^{(-)}_{[L][L']}(r, p) = 0 , \]  
which is standard in the variable–constant method for solving differential equations \([13]\).

Substituting the ansatz (6) into Eq. (2) and using the condition (8), we derive the following coupled differential matrix equations of first order:

\[ \partial_r F^{(+)}_{[L][L']}(r, p) = -\frac{h^{(-)}_{\lambda}(pr)}{2ip} \sum_{[L'']} W_{[L][L'']} (r) \left\{ h^{(+)}_{\lambda''}(pr)F^{(+)}_{[L'][L']}(r, p) + h^{(-)}_{\lambda''}(pr)F^{(-)}_{[L'][L']}(r, p) \right\} , \]  
\[ \partial_r F^{(-)}_{[L][L']}(r, p) = \frac{h^{(+)}_{\lambda}(pr)}{2ip} \sum_{[L'']} W_{[L][L'']} (r) \left\{ h^{(+)}_{\lambda''}(pr)F^{(+)}_{[L'][L']}(r, p) + h^{(-)}_{\lambda''}(pr)F^{(-)}_{[L'][L']}(r, p) \right\} . \]
These equations must be supplemented with appropriate boundary conditions at \( r = 0 \). In Ref. [14] it was shown that for an arbitrary \( N \)--body system the fundamental system of regular solutions of Eq. (3) vanishes near \( r = 0 \) in such a way that

\[
\lim_{r \to 0} \frac{\phi_{\lambda}[L\,|\,L'](r, p)}{r^{\lambda' + 1}} = \delta_{[L][L']}. \tag{10}
\]

Thus, we can define the regular basis by the following boundary condition

\[
\lim_{r \to 0} \frac{\phi_{\lambda}[L\,|\,L'](r, p)}{j_{\lambda}(pr)} = \delta_{[L][L']}, \tag{11}
\]

where \( j_{\lambda} \) is the Riccati–Bessel function [12]. This is in accordance with (10) and is a natural generalization of the corresponding boundary condition of the two--body problem.

Since, by definition, \( \phi_{\lambda}[L\,|\,L'] \) is regular at \( r = 0 \), the behaviour of the functions \( F_{\lambda}[L\,|\,L'](r, p) \) and \( F_{\lambda}[L\,|\,L'](r, p) \) of Eq. (3) near the origin is such that the singularities of \( h_{\lambda}^{(+)}(pr) \) and \( h_{\lambda}^{(-)}(pr) \) compensate each other. This can be achieved if \( F^{(+)} \) and \( F^{(-)} \) are identical as \( r \to 0 \), i.e.,

\[
F_{\lambda}[L\,|\,L'](r, p) \sim r \to 0 F_{\lambda}[L\,|\,L'](r, p), \tag{12}
\]

for then

\[
\phi_{\lambda}[L\,|\,L'](r, p) \sim j_{\lambda}(pr) F_{\lambda}[L\,|\,L'](r, p),
\]

so that the boundary conditions, Eq. (11), become

\[
\lim_{r \to 0} \left[ \frac{j_{\lambda}(pr) F_{\lambda}[L\,|\,L'](r, p)}{F_{\lambda}[L\,|\,L'](r, p)} \right] = \delta_{[L][L']}. \tag{13}
\]

However, since \( r = 0 \) is a singular point, for practical calculations one needs to solve the system (3) analytically on a small interval \((0, \delta]\) and then impose the boundary conditions at \( r = \delta \). Such an analytical solution can be easily found by choosing \( \delta \) to be small enough so that for \( r \in (0, \delta] \) we may write

\[\partial_{r} F_{\lambda}[L\,|\,L'](r, p) = \pm \frac{1}{i p} h_{\lambda}^{(\pm)}(pr) W_{[L][L']}(r) j_{\lambda'}(pr).\]

For small \( r \) the Riccati-Neumann function \( n_{\lambda} \) is dominant in \( h_{\lambda}^{(\pm)} \equiv j_{\lambda} \pm i n_{\lambda} \) and thus

\[\partial_{r} F_{\lambda}[L\,|\,L'](r, p) \approx -\frac{1}{p} n_{\lambda}(pr) W_{[L][L']}(r) j_{\lambda'}(pr).\]

Upon integrating this (approximate) equation we find

\[F_{\lambda}[L\,|\,L'](r, p) \approx -\frac{1}{p} \int n_{\lambda}(pr) W_{[L][L']}(r) j_{\lambda'}(pr) \, dr + \text{const},\]
and if the arbitrary constant of integration is taken to be $\delta_{[L][L']}$, we obtain for the short range behavior,

$$F^{(\pm)}_{[L][L']}(r,p) \approx \delta_{[L][L']} - \frac{1}{p} \int n_\lambda(pr) W_{[L][L']}(r) j_\lambda'(pr) dr ,$$

which obeys the condition (13). In practical calculations the last indefinite integral can be found analytically by using the leading terms of series expansions of $n_\lambda$, $W_{[L][L']}$ and $j_\lambda'$. In this way one finds that the second term of Eq. (14) is regular at $r = 0$ for $\lambda \leq \lambda'$ (right upper corner and the diagonal of the matrix) and may be singular for $\lambda > \lambda'$ (left lower corner of the matrix). However, this singularity is always compensated by the presence of $j_\lambda$ in Eq. (12). Thus the coupled equations, Eqs. (9), along with the boundary conditions, Eqs. (13) and (14), form a well-defined differential problem.

The regular basis obtained in the form of Eq. (7) is ideally suited for constructing physical solutions for one–channel problems. Indeed, since the right hand sides of Eqs. (9) vanish together with the potential, so do the derivatives $\partial_r F^{(\pm)}$, which in turn implies that beyond some $r_{\text{max}}$ both function $F^{(\pm)}_{[L][L']}(r,p)$ become practically constant and the asymptotic behavior of $\phi$ is totally determined by the Riccati–Hankel functions. On the other hand, in one–channel problems we can have only three types of physical solutions describing bound, Siegert, and scattering states. For all of them the boundary function $U$ of Eq. (8) must be constructed from the Riccati–Hankel functions which depend on the hyperradius and at large $r$ behave as

$$h^{(\pm)}_\lambda(pr) \rightarrow \mp i \exp[\pm i(pr - \frac{\lambda \pi}{2})] .$$

Thus the physical solution is a linear combination of the form

$$u_{[L]}(r,p) = \sum_{[L']} \phi_{[L][L']}(r,p) A_{[L']}(p) ,$$

which at $r = r_{\text{max}}$ smoothly matches, and for $r > r_{\text{max}}$ automatically coincides with, the boundary function $U$,

$$\frac{1}{2} \left\{ h^{(+)}_\lambda(pr) \sum_{[L']} F^{(+)}_{[L][L']}(r_{\text{max}},p) A_{[L']}(p) + h^{(-)}_\lambda(pr) \sum_{[L']} F^{(-)}_{[L][L']}(r_{\text{max}},p) A_{[L']}(p) \right\} = U_{[L]}(r,p) ,$$

provided the correct coefficients $A_{[L']}(p)$ are found.

In the present work we are concerned with bound and Siegert states. For both of them each element of the column $u_{[L]}$ at large $r$ must be proportional to $h^{(\pm)}_\lambda(pr)$ which exponentially decays when $p$ is on the positive imaginary axis (bound state), or represents pure outgoing waves when $p$ is in the fourth quadrant of the complex $p$–plane (resonant Siegert state). This can be achieved provided that

$$\sum_{[L']} F^{(-)}_{[L][L']}(r_{\text{max}},p) A_{[L']}(p) = 0 .$$
Here the argument of $A$ is $p$ because the wave function of a bound or Siegert state does not depend on the orientation of the incident momenta $\{p\}$.

The homogeneous system of equations (16) has a nontrivial solution if and only if

$$\det \| F_{[L][L']}^{(-)}(r_{\text{max}}, p) \| = 0 .$$  \hfill (17)

The discrete points $p = p_{01}, p_{02}, \ldots$ at which Eq. (17) is fulfilled, are the spectral points corresponding to bound and resonant states.

When the number of particles $N = 2$, the functions $F^{(±)}(r, p)$ are closely related to the Jost solutions and the limit of $F^{(-)}$,

$$f(p) = \lim_{r \to \infty} F^{(-)}(r, p) ,$$  \hfill (18)

is the Jost function. By analogy, we can call $\| F_{[L][L']}^{(-)}(\infty, p) \|$ the Jost matrix for a one–channel $N$–body problem. In practical calculations instead of $r = \infty$ we can always use $r = r_{\text{max}}$.

III. MATRIX ELEMENTS OF THE POTENTIAL

From the simplest shell–model it follows that the lowest configurations for three and four neutrons are $(0s)^2(0p)$ and $(0s)^2(0p)^2$ and so we expect to find one and two neutrons with $\ell = 1$ respectively. More elaborate few–body analyses [15,19] corroborate this simplistic argument and concluded that if the nuclei $^3n$ and $^4n$ were to exist, the most favorable quantum numbers $(J^\pi)$ would be $\frac{1}{2}^-$ for $^3n$ and $0^+, 1^+$, or $2^+$ for $^4n$. This means that even if $^3n$ and $^4n$ do not exist, the resonant poles corresponding to these states must be the closest to the origin of the complex energy plane. In the present work we shall search only for these states.

Since in the $nnn$ and $nnnn$ systems one and two particles respectively have $\ell = 1$, the minimal value of the grand orbital number for them is $L_{\text{min}} = 1$ for $nnn$ and $L_{\text{min}} = 2$ for $nnnn$. The general rule defining $L_{\text{min}}$ can be found in Ref. [11]. It has been pointed out in many papers (see, for example, Refs. [15,19]) that the minimal approximation $L = L_{\text{min}}$ where only the first equation of the system (2) is retained, provides an adequate description of the properties of $^3$He and $^4$He nuclei and that the corresponding minimal components, $u_{[L_{\text{min}}]}(r, p)$, of the wave functions contribute $\sim 95\%$ to the total normalization integral. Therefore we employ this approximation to investigate the analytical properties of the multi–neutron Jost function and thus to shed some light on the question of existence of resonances in many–neutron systems.

The general form of the matrix elements $W_{[L_{\text{min}}][L_{\text{min}}]}$ for three and four neutron systems were given in Refs. [16] and [15]. They are

$$W_{[L_{\text{min}}][L_{\text{min}}]}^{(\text{nnn}, 1/2^-)}(r) = \frac{48}{\pi} \int_0^{\pi/2} d\theta \sin^4 \theta \cos^2 \theta V_{nn}(\sqrt{2}r \cos \theta) ,$$  \hfill (19)
\[ W^{(nnn,0^+)}_{[L_{\text{min}}][L_{\text{min}}]}(r) = \frac{105 \cdot 33}{8 \cdot 16} \int_0^{\pi/2} d\theta \sin^5 \theta \cos^2 \theta \left( 4 \cos^4 \theta - 4 \cos^2 \theta \sin^2 \theta \right) \]
\[ + \frac{13}{4} \sin^4 \theta V_{nn}(\sqrt{2}r \cos \theta) , \] (20)

\[ W^{(nnn,1^+)}_{[L_{\text{min}}][L_{\text{min}}]}(r) = \frac{105 \cdot 33}{64} \int_0^{\pi/2} d\theta \sin^9 \theta \cos^2 \theta V_{nn}(\sqrt{2}r \cos \theta) , \] (21)

\[ W^{(nnn,2^+)}_{[L_{\text{min}}][L_{\text{min}}]}(r) = \frac{105 \cdot 33}{80} \int_0^{\pi/2} d\theta \sin^5 \theta \cos^2 \theta \left( \cos^4 \theta + \frac{25}{16} \sin^4 \theta \right) V_{nn}(\sqrt{2}r \cos \theta) . \] (22)

These potentials are employed in our search for sub–threshold resonances.

IV. COMPLEX ROTATION

Within the minimal approximation, only the first of Eqs. (2) remains and consequently we have only one pair of Eqs. (3). Therefore the problem is similar to the two–body one with potentials (19–22) and angular momenta obtained by Eq. (3), that is \( \lambda = 5/2 \) for \( (nnn) \) and \( \lambda = 5 \) for \( (nnnn) \). The corresponding Jost function becomes an effective two–body one.

It is known that starting with a two–body radial Schrödinger equation in its ordinary form, the Jost function for a long–range potential can be defined for \( \text{Im} \{ p \} \geq 0 \) only. The potentials (19–22) are clearly of the long–range type. This is a result of a general rule in which two–body potentials when sandwiched between hyperspherical harmonics, acquire slowly decaying tails (see Ref. [11]). For example, even if we take \( V_{nn} \) in Eq. (19) in the form of a square well, the resulting function \( W_{[L_{\text{min}}][L_{\text{min}}]}(r) \) at large \( r \) behaves as \( r^{-3} \) [16]. Thus, the limit (18) does not exist in the fourth quadrant of the complex \( p \)–plane where we shall search for possible resonances.

To overcome this difficulty we employ the complex rotation method in the form developed in Refs. [2,8]. In Eqs. (3) we replace the real hyperradius with a complex one, viz.

\[ r = x \exp(i\theta) , \quad x \geq 0 , \quad 0 \leq \theta < \frac{\pi}{2} , \]

\[ \partial_x F^+(x, \theta, p) = \frac{e^{i\theta}}{2ip} h_\lambda^{(-)}(pxe^{i\theta}) W(xe^{i\theta}) \]

\[ \times \left[ h_\lambda^{(+)}(pxe^{i\theta}) F^+(x, \theta, p) + h_\lambda^{(-)}(pxe^{i\theta}) F^-(x, \theta, p) \right] , \]

\[ \partial_x F^-(x, \theta, p) = -\frac{e^{i\theta}}{2ip} h_\lambda^{(+)}(pxe^{i\theta}) W(xe^{i\theta}) \]

\[ \times \left[ h_\lambda^{(+)}(pxe^{i\theta}) F^+(x, \theta, p) + h_\lambda^{(-)}(pxe^{i\theta}) F^-(x, \theta, p) \right] . \] (23)

Such a rotation does not change the Jost function which is \( r \)–independent. Meanwhile, it changes the functions \( F^{(\pm)} \) to the effect that \( F^-(r, p) \) can be defined above the line \( (-\infty e^{-i\theta}, +\infty e^{-i\theta}) \) in the complex \( p \)–plane. Moreover, in Ref. [2,8] it was shown that at all
points above this line the limit ([8]) exists and gives the correct Jost function. Therefore, using a rotation with large enough $\theta$, we can calculate the Jost function at the points of interest in the fourth quadrant of the $p$–plane and thus the multineutron resonances, if any, can be located.

V. RESULTS AND DISCUSSION

A multitude of $nn$–interactions can be found in the literature. We may divide them into two classes, namely, those with and those without a repulsion at small distances. In this work we performed calculations using both types of potentials.

As a purely attractive potential we choose the Yukawa type

$$V_{nn}(r) = -V_0 \frac{e^{-\alpha_0 r}}{r},$$

with $V_0 = 54.7477$ MeV fm and $\alpha_0 = 0.84034$ fm$^{-1}$. From the second class we employ the Malfliet–Tjon singlet potential ([21])

$$V_{nn}(r) = V_1 \frac{e^{-\alpha_1 r}}{r} - V_2 \frac{e^{-\alpha_2 r}}{r},$$

with $V_1 = 1438.72$ MeV fm, $V_2 = 513.968$ MeV fm, $\alpha_1 = 3.11$ fm$^{-1}$, and $\alpha_2 = 1.55$ fm$^{-1}$. Both these potentials describe a nucleon–nucleon interaction in the $^1S_0$ channel and support a virtual di-neutron state. We note that we have slightly adjusted the depth of the Yukawa potential so that the virtual bound state it generates is the same as for the Malfliet–Tjon potential. This virtual bound state energy is given in Table 1.

Solving Eqs. (23) numerically with the potential matrices (19-22), we found that, beyond $x_{max} = 40$ fm, the function $F^{(-)}(x, \theta, p)$ becomes practically constant (varying only in the 8-th digit). Further, the value of $F^{(-)}(x_{max}, \theta, p)$ did not depend on the choice of the rotation angle $\theta$ (indicative of the accuracy of the method) provided the point $p$ is above the line $(-\infty e^{-i\theta}, +\infty e^{-i\theta})$.

Using Newton’s method, we have located one zero of $F^{(-)}(x_{max}, \theta, p)$ for each of the states $^3n(1/2^-)$, $^4n(0^+)$, $^4n(1^+)$, and $^4n(2^+)$ in the fourth quadrant of the complex $p$–plane, for the Yukawa as well as Malfliet–Tjon $nn$–potential. The coordinates $p_0$ of these zeros are given in Table 1. All zeros found lie below the diagonal of the quadrant, which represents the threshold energy ($\text{Re } E = 0$); that is the corresponding energies, $E_0 = p_0^2/2m$, have negative real parts. This implies that these zeros are sub–threshold resonances.

It is evident from the results of Table 1 that the position of zeros are sensitive to the choice of the $nn$–potential. The movement of these zeros with an artificial increase of the potential strength also exhibits an extreme sensitivity to the choice of the potential. Thus, to generate $^3n$ and $^4n$ bound states, the Yukawa potential needs to be multiplied by 2.67 and 2.32 respectively, while for the Malfliet–Tjon potential by 4.04 and 3.59.
In Fig. 1 we show the movement of the three–neutron Jost function zeros on the momentum plane, when the $nn$–potential is multiplied by a factor $\alpha$, $V_{nn} \rightarrow \alpha V_{nn}$. Open and filled circles correspond to the Yukawa and the Malfliet–Tjon potentials respectively. The zeros furthest from the origin correspond to the physical potentials ($\alpha = 1$) and represent the zeros given in the Table 1. The upward sequences of points are shown for the uniform increase of $\alpha$, $\alpha = 1, 1.1, 1.2, \cdots$. In Fig. 2 the corresponding trajectory in the energy plane is depicted. In Figs. 3 and 4 the results for the four–neutron system are plotted the notation being the same as for Figs. 1 and 2. It is worth mentioning that a trajectory similar to these figures, was obtained in Ref. [22] for a sub–threshold resonance of the hypernuclear system $\Lambda nn$.

There have been several previous theoretical attempts to locate three and four neutron resonances: using the hyperspherical approach [9,16,23–25], by an analytical continuation of the Faddeev kernel [17], and by using the complex scaling method [26]. Invariably the authors of these papers searched for multineutron resonances above the threshold and close to the real energy axis. Only one of these attempts turned out to be successful, the complex–scaling calculation by Csoto, Oberhummer, and Pichler [26] where they found a three–neutron resonance at $E = (14 - i6.5)$ MeV for the $J^m = \frac{3}{2}^+$ state and that all other states up to $J = \frac{5}{2}$ are nonresonant.

In short, all the above authors agree that there are no resonances in the four neutron system, nor in the $\frac{1}{2}^-$ state of three–neutrons. On the other hand, it has been shown many times that by an artificial increase of the neutron–neutron attraction, one can always obtain bound states for the $3n(\frac{1}{2}^-)$ and $4n(0^+)$ systems. Since the corresponding poles cannot vanish or undergo a discontinuous relocation, they must be smoothly move somewhere after decreasing the potential strength to its physical value. Thus, to say that the states $3n(\frac{1}{2}^-)$ and $4n(0^+)$ are nonresonant is unsatisfactory. One would like to know in which direction and how far the poles move. Our present work sheds some light on this problem.

Using the minimal approximation $L = L_{minn}$ means that the actual position of resonances in the complex momentum plane may be different from those shown in the figures. In this connection, however, we emphasize that in bound states calculations the minimal approximation always underbinds the system and that the omitted higher harmonics amount to an additional attraction in the effective potential [4]. Since in this work we found that an increase in the attraction moves the resonances, in the energy plane, up (decreases their width) and to the right (closer to the threshold), the values given in Table 1 can be considered as lower bounds for the energies and upper bound for the widths of the resonances.

Since the physical potentials generate negative real parts of $E_0$, the corresponding resonances can be excited at negative energies only, that is when the $3n$ or the $4n$ system is placed in a sufficiently strong attractive external field. Such a situation can be realized, for example, inside a nucleus. Besides the negative energy, the interior of a nucleus provides a high frequency of multineutron collisions due to the high density of nucleons. Therefore, three or four neutrons inside a nucleus could form an unstable cluster corresponding to a sub–threshold resonance.
In conclusion, our approach which is based on the Jost function calculation, is a power method which enables us to investigate the analytical properties of the $S$-matrix in the complex momentum plane. This opens up new possibilities in locating sub-threshold resonances which is a difficult task for many other methods. Moreover the formalism given can be extended to complex values of the angular momentum $\lambda$ and therefore Regge trajectories can also be located.

ACKNOWLEDGMENTS

Financial support from the University of South Africa and the Joint Institute for Nuclear Research, Dubna, is greatly appreciated.
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### TABLE I. Jost function zeros in momentum and energy planes.

| potential | Yukawa | Malfliet–Tjon |
|-----------|-------|---------------|
|           | $p_0$ (fm$^{-1}$) | $E_0$ (MeV) | $p_0$ (fm$^{-1}$) | $E_0$ (MeV) |
| nn (0$^+$) | -i0.0401 | -0.0667 | -i0.0401 | -0.0667 |
| nnn (1/2$^-$) | 0.232 – i0.560 | -5.39 – i5.39 | 0.291 – i0.569 | -4.95 – i6.87 |
| nnnn (0$^+$) | 0.337 – i0.594 | -4.96 – i8.30 | 0.402 – i0.537 | -2.64 – i8.95 |
| nnnn (1$^+$) | 0.262 – i0.790 | -11.5 – i8.59 | 0.435 – i0.783 | -8.77 – i14.1 |
| nnnn (2$^+$) | 0.467 – i0.744 | -6.97 – i14.4 | 0.535 – i0.699 | -4.19 – i15.5 |
FIG. 1. Movement of the 3n Jost function zeros when the \( nn \)-potential is multiplied by the enhancing factor \( V_{nn} \rightarrow \alpha V_{nn} \). Open and filled circles correspond to Yukawa and Malflie-Tjon potentials respectively. The sequence of points are shown for a uniform increase of \( \alpha \) by 0.1, \( \alpha = 1.0 \) (lowestpoint), 1.1, 1.2, ....
FIG. 2. Movement of the three–neutron Jost function zeros on the energy plane. The notation is the same as in Fig. 1.
FIG. 3. Movement of the four-neutron Jost function zeros on the momentum plane. The notation is the same as in Fig. 1.
FIG. 4. Movement of the four-neutron Jost function zeros on the energy plane. The notation is the same as in Fig. 1.