CAPITAL ALLOCATION UNDER FUNDAMENTAL REVIEW OF TRADING BOOK

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ABSTRACT. The Fundamental Review of Trading Book (FRTB) from the Basel Committee overhauls the regulatory framework for minimum capital requirements for market risk. Facing the tightened regulation, banks need to allocate their capital to each of their risk positions to evaluate the capital efficiency of their strategies. This paper proposes two computational efficient allocation methods under the FRTB framework. Simulation analysis shows that both these two methods provide more liquidity horizon weighted, more stable, and less negative allocations than the standard methods under the current regulatory framework.

Keywords: Asset allocation, Capital requirement, Risk management

1. INTRODUCTION

The Fundamental Review of Trading Book (FRTB) is a revised global risk management framework which aims to address shortcomings of the Basel II and its current amendments. The FRTB sets out revised standards for minimum capital requirements for market risk and shifts from Value-at-Risk (VaR) to an Expected Shortfall (ES) measure.

In the new Internal Model Approach (IMA), tail risk and liquidity risk are considered and the capital-reducing effects of hedging are constrained. As a result, bank’s global capital charge is facing significant changes. It therefore becomes increasingly important for banks to reposition their resources strategically to business units with high Return on Capital (ROC). To calculate the ROC, the capital charge of a bank, which is calculated based on the firm-wide portfolio, needs to be allocated to each business unit. On the other hand, calculating the ES measure under the FRTB framework is computational more demanding than calculating the VaR under the current practice. Therefore any allocation method under FRTB needs to be computationally efficient to handle the complicated portfolio structure in a bank.

We propose in this paper two allocation methods for the capital charge under the FRTB IMA. Both allocation methods consist of two stages. In the first stage, the FRTB capital charge is allocated to each bucket of different liquidity horizons and risk factors. Then, in

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\[ ^1 \text{In the industry Quantitative Impact Study (QIS), 44 banks report an average of 54\% increase of capital charge under the new IMA.} \]
the second stage, allocations in different buckets are decomposed, realigned, and aggregated again.

In the first allocation method, we examine the Euler allocation principle under the FRTB framework. The Euler allocation principle has been studied extensively. Tasche [10] proves that the Euler allocation provides signal to optimise firm’s portfolio return on risk-adjusted capital. Denault [2] provides axiomatic characterisations of the Euler allocation. When the Euler allocation principle is applied under the FRTB framework, we show that the resulting allocation to each risk factor and liquidity horizon bucket is a scaled version of the standard Euler allocation. This scaling factor depends on the stand-alone ES of this bucket and the total FRTB ES of the same risk factor category. Our second allocation method is motivated by the constrained Aumann-Shapley allocation by Li et al. [4]. Applying the Aumann-Shapley allocation to each risk factor category, we reduce the resulting allocation to another scaled version of the standard Euler allocation, where the scaling factor depends on the stand-alone ES of this bucket and its induced increment of FRTB ES. These two allocation methods are further extended, where the impact of additional risk positions on the stress period scaling factor is incorporated. Reducing the new allocation methods to the standard Euler allocation ensures computational efficiency. The same scenario extraction method can be used to compute the standard Euler allocation, without any revaluation of capital charges.

We illustrated our allocation methods via three groups of simulation analysis. Our analysis shows that risk factors with longer liquidity horizons are allocated with a larger proportion of the total FRTB capital charge. Secondly, negative allocations, resulting from hedging positions, in the standard Euler allocation are largely reduced or even reversed. Hedging between the different risk factor and liquidity horizon buckets rarely leads to negative allocations under FRTB. Meanwhile hedging positions within the same bucket could still lead to negative allocations. However, the magnitude of negative allocation to the same hedging position is much less in the FRTB than in the framework where the regular ES is evaluated on unconstrained P&L. Moreover, both allocation methods under the FRTB produce less dispersive allocations across different buckets than the Euler allocation of the regular ES. Therefore, both methods produce more stable allocations than the standard Euler allocation of the regular ES. Finally, our third simulation analysis demonstrates that allocation under the FRTB is sensitive to the choice of the reduced set of risk factors.

The rest of the paper is organised as follows: Section 2 introduces the expected shortfall measure under the FRTB and investigates its homogeneity and sub-additivity properties. Allocation methods and their extensions are introduced in Section 3, followed by the simulation analysis in Section 4.
2. FRTB expected shortfall

2.1. Risk factor and liquidity horizon bucketing. Under the FRTB IMA framework, the P&L of a risk position is attributed to risk factors of five different categories

\{RF_i : 1 \leq i \leq 5\} = \{CM, CR, EQ, FX, IR\}.

Each risk factor in the each category is assigned with a liquidity horizon with lengths

\{LH_j : 1 \leq j \leq 5\} = \{10, 20, 40, 60, 120\}.

Directly observable and frequently updated prices have shorter liquidity horizons. Risk factors associated to illiquid products and quantities which are calculated from direct observations typically have longer liquidity horizons. A table of liquidity horizons of various risk factors is presented in [8, Paragraph 181 (k)].

We call negative of the P&L of a risk position the loss of this risk position. The sign convention that positive value indicates the magnitude of loss will be employed throughout this paper. Consider a portfolio with \(N\) risk positions. For the risk position \(n, 1 \leq n \leq N\), we decompose its loss over 10 days into different risk factor and liquidity horizon buckets, and denote by \(\tilde{X}_n(i, j)\) the loss (over 10 days) attributed to RF\(_i\) and LH\(_j\). Then \(\sum_{i,j} \tilde{X}_n(i, j)\) is the loss (over 10 days) of the risk position \(n\). We record this risk factor and liquidity horizon bucketing by a \(5 \times 5\) matrix \(\tilde{X}_n = \{\tilde{X}_n(i, j)\}_{1 \leq i,j \leq 5}\).

Now define the liquidity horizon adjusted loss as

\[
X_n(i, j) = \sqrt{\frac{LH_j - LH_{j-1}}{10}} \sum_{k=j}^{5} \tilde{X}_n(i, k), \quad 1 \leq i, j \leq 5,
\]

where LH\(_0\) = 0. Considering the sum of losses attributed to all risk factors in the category RF\(_i\) with liquidity horizons at least as long as LH\(_j\), and scaling the sum by the factor \(\sqrt{\frac{LH_j - LH_{j-1}}{10}}\), we obtain \(X_n(i, j)\). We record the liquidity horizon adjusted bucketing of the risk position \(n\) by a \(5 \times 5\) matrix \(X_n = \{X_n(i, j)\}_{1 \leq i,j \leq 5}\). We call the matrix \(X_n\) as the risk profile of the position \(n\). Summing up the risk profiles of all risk positions, we get the net risk profile of the portfolio

\[
X = \sum_n X_n,
\]

where the sum is computed component-wise. We call the matrix \(X = \{X(i, j)\}_{1 \leq i,j \leq 5}\) the risk profile of the portfolio.

The FRTB ES for the portfolio loss attributed to RF\(_i\) is defined in [8, Paragraph 181 (c)] as

\[
\text{ES}(X(i)) = \sqrt{\sum_{j=1}^{5} \text{ES}(X(i, j))^2},
\]
where each $\text{ES}(X(i,j))$ is the expected shortfall of $X(i,j)$ calculated at the 97.5% quantile.

**Example 2.1.** Consider a risk position whose loss is attributed only to RF$_i$ on LH$_5 = 120$. Then $\tilde{X}(i,j) = 0$ for any $j = 1, \ldots, 4$. Assume that the 10 days loss $\tilde{X}(i,5)$ is normally distributed with zero mean and standard deviation $\sigma$. Then the loss over 120 days is normally distributed with zero mean and standard deviation $\sqrt{\frac{120}{10}} \sigma$, hence its expected shortfall is $\sqrt{\frac{120}{10}} \sigma \text{ES}(N(0, 1))$, where $\text{ES}(N(0, 1))$ is the expected shortfall at the 97.5% quantile of the standard normal distribution. On the other hand, if we calculate expected shortfall of the 120 days loss via (3), we obtain the same expression. Indeed, note that $X(i,j) = \sqrt{\frac{LH_j - LH_{j-1}}{10}} \tilde{X}(i,5)$, for $1 \leq j \leq 5$. Then

$$
\text{ES}(X(i)) = \sqrt{\sum_{j=1}^{5} \frac{LH_j - LH_{j-1}}{10} \text{ES}(\tilde{X}(i,5))^2} = \sqrt{\frac{120}{10} \text{ES}(N(0, \sigma))} = \sqrt{\frac{120}{10} \sigma \text{ES}(N(0, 1))}.
$$

**Remark 2.2.** It is not explicitly required in [8, Paragraph 181] to floor each $\text{ES}(i,j)$ at zero. This means that profit in the liquidity horizon adjusted loss attribution (i.e., negative $X(i,j)$) would lead to positive contribution in the risk measure $\text{ES}(X(i))$. To avoid this counter intuitive behavior, we suggest to floor each $\text{ES}(X(i,j))$ at zero, and introduce

$$
\text{ES}^+(X(i)) = \sqrt{\sum_{j=1}^{5} \text{ES}^+(X(i,j))^2},
$$

where $\text{ES}^+(X(i,j)) = \max\{\text{ES}(X(i,j)), 0\}$. This modification introduces better properties to the FRTB ES (see Section 2.3 later), but still retains the positive homogeneity property. Hence the allocation methods that we introduce later can be applied to both $\text{ES}(X(i))$ and $\text{ES}^+(X(i))$.

### 2.2. Stress period scaling and capital charge

After the liquidity horizon adjustment, FRTB also requires to calibrate the risk measure to a period of stress by introducing a scaling factor. For each risk factor category, calculate $\text{ES}(X(i))$ in (3) based on the current (most recent) 12-month observation period with a full set of risk factors which are relevant to the risk position, and denote this risk measure as $\text{ES}^{F,C}(X(i))$. Then identify a reduced set of risk factors, calculate its associated $\text{ES}(X(i))$ over the same period, and denote it as $\text{ES}^{R,C}(X(i))$. It is required that the reduced set of risk factors is large enough so that $\text{ES}^{R,C}(X(i))$ is at least 75% of $\text{ES}^{F,C}(X(i))$. Subsequently, identify a 12-month stress period in which the portfolio experiences the largest loss, calculate $\text{ES}(X(i))$ with the reduced set of risk factors but use the observations from the stress period, and denote this risk measure as $\text{ES}^{R,S}(X(i))$. FRTB IMA introduces the following expected shortfall capital charge (see [8]...
Paragraph 181 (d)):

\[
\text{IMCC}(X(i)) = \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))}\text{ES}^{F,C}(X(i)), \quad 1 \leq i \leq 5. \tag{5}
\]

In order to consider the unconstrained portfolio, we extend the risk profile for each risk position by adding another row:

\[
X_n(6,j) = \sum_{i=1}^{5} X_n(i,j), \quad 1 \leq j \leq 5,
\]

which records the net loss attributed to all risk factors from different categories with the liquidity horizon LH\(_j\). We call \(X_n(6,\cdot)\) as the \textit{unconstrained risk profile} of the risk position \(n\), and extend the risk profile by adding this row. We call the \(6 \times 5\) matrix \(X_n = \{X_n(i,j)\}_{1 \leq i \leq 6, 1 \leq j \leq 5}\) the \textit{extended} risk profile for the position \(n\). We also perform a similar extension to the risk profile of a portfolio, and calculate \(\text{IMCC}(X(6))\) as (5) with \(i = 6\).

Now we are ready to introduce the capital charge for modellable risk factors for under FRTB IMA (see [8, Paragraph 189]).

**Definition 2.3.** The aggregate capital charge for modellable risk factors is a weighted sum of the constrained and unconstrained expected shortfall charges:

\[
\text{IMCC}(X) = \rho \text{IMCC}(X(6)) + (1 - \rho)\sum_{i=1}^{5} \text{IMCC}(X(i)), \tag{6}
\]

where the relative weight \(\rho\) is set to be 0.5.

**2.3. Properties of IMCC.**

**Lemma 2.4.** For any constant \(a \geq 0\) and risk profiles \(X\) and \(Y\), the following statements hold:

(i) (Positive homogeneity) \(\text{IMCC}(aX) = a \text{IMCC}(X)\).

(ii) (Sub-additivity for ES) For \(i = 1, \ldots, 6\), if \(\text{ES}((X + Y)(i,j)) \geq 0\) for any \(j\), then

\[
\text{ES}((X + Y)(i)) \leq \text{ES}(X(i)) + \text{ES}(Y(i)). \tag{7}
\]

(iii) (Sub-additivity for IMCC) For any \(i = 1, \ldots, 6\), if

\[
\frac{\text{ES}^{R,S}((X + Y)(i))}{\text{ES}^{R,C}((X + Y)(i))} \leq \min \left\{ \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))}, \frac{\text{ES}^{R,S}(Y(i))}{\text{ES}^{R,C}(Y(i))} \right\}, \tag{8}
\]

and \(\text{ES}^{F,C}((X + Y)(i,j)) \geq 0\) for any \(j\), then

\[
\text{IMCC}((X + Y)(i)) \leq \text{IMCC}(X(i)) + \text{IMCC}(Y(i)). \tag{9}
\]

Items (ii) and (iii) in the previous lemma present the sub-additivity property for the ES and IMCC capital charges under conditions [7] and [8]. Without these conditions, the following examples show that the sub-additivity property may not hold.
Example 2.5. Consider two risk positions whose losses concentrate on RF\textsubscript{i} and LH\textsubscript{j}. \( X(i,j) \) has a Bernoulli distribution with \( \mathbb{P}(X(i,j) = -1) = \mathbb{P}(X(i,j) = 0) = 0.5 \), and \( Y(i,j) = -1 - X(i,j) \). Hence \( \mathbb{P}((X + Y)(i,j) = -1) = 1 \). Then \( \text{ES}(X(i)) = \text{ES}(Y(i)) = 0 \), but

\[
\text{ES}((X + Y)(i)) = |\text{ES}((X + Y)(i,j))| = |-1| = 1 > \text{ES}(X(i)) + \text{ES}(Y(i)).
\]

However, if the expected shortfall is floored at zero as in Remark 2.2, then the sub-additivity property for ES and IMCC holds without the positivity assumption \( \text{ES}((X + Y)(i,j)) \geq 0 \) for all \( j \).

Example 2.6. We consider two risk positions whose losses concentrate on RF\textsubscript{i} and LH\textsubscript{j}. Assume that \( X(i,j) \) and \( Y(i,j) \) are i.i.d. standard normal, moreover, the losses attributed to reduced sets account for 75\% and 100\%, respectively, of the standard deviations of the losses on full sets. Hence

\[
\text{ES}^{R,C}(X(i)) = 0.75 \text{ES}^{F,C}(X(i)), \quad \text{ES}^{R,C}(Y(i)) = \text{ES}^{F,C}(Y(i)).
\]

Under stress scenarios, we assume that \( X(i,j) \) and \( Y(i,j) \) have independent normal distributions, but their standard deviations are scaled up by 1.2 and 9, respectively, of their values under current period. Then

\[
\min \left\{ \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))}, \frac{\text{ES}^{R,S}(Y(i))}{\text{ES}^{R,C}(Y(i))} \right\} = \min \{1.2, 9\} = 1.2.
\]

For the aggregated portfolio, the standard deviation of \( X(i,j) + Y(i,j) \) attributed to the full set is \( \sqrt{2} \), and \( \sqrt{0.75^2 + 1} = 1.25 \) to the reduced set. Under the stress scenarios, the standard deviation of \( X(i,j) + Y(i,j) \) attributed to the reduced set becomes \( \sqrt{(0.75 \times 1.2)^2 + 9^2} \approx 9.04 \). Hence

\[
\frac{\text{ES}^{R,S}((X + Y)(i))}{\text{ES}^{R,C}((X + Y)(i))} = \frac{9.04}{1.25} = 7.23 > 1.2.
\]

Therefore, the condition (8) is violated. Now we have

\[
\text{IMCC}((X + Y)(i)) = \frac{\text{ES}^{R,S}((X + Y)(i))}{\text{ES}^{R,C}((X + Y)(i))} \text{ES}^{F,C}((X + Y)(i))
\]

\[
= 7.23 \times \sqrt{2} \text{ES}(N(0,1)).
\]

On the other hand, comparing with the sum of two IMCCs that

\[
\text{IMCC}(X(i)) + \text{IMCC}(Y(i)) = \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))} \text{ES}^{F,C}(X(i)) + \frac{\text{ES}^{R,S}(Y(i))}{\text{ES}^{R,C}(Y(i))} \text{ES}^{F,C}(Y(i))
\]

\[
= (1.2 + 9) \text{ES}(N(0,1)),
\]

we find

\[
7.23 \times \sqrt{2} = 10.22 > 10.20.
\]

Hence (9) fails.
3. Capital allocation

We introduce in this section several methods to allocate the aggregate capital charge in Definition 2.3 to different components of a portfolio. All allocation methods have two steps. Given the extended risk profile $X$ of a portfolio, which is aggregated from extended risk profiles $\{X_n\}_{1 \leq n \leq N}$ using (2), the first step allocates capital to $\{X_n(i, j)\}_{1 \leq n \leq N, 1 \leq i \leq 6, 1 \leq j \leq 5}$. Given a risk measure $\rho$ and a portfolio risk profile $X$, we denote the allocation to $X_n(i, j)$ by $\rho(X_n(i, j)|X)$.

Recall from (1) that $X_n(i, j)$ is aggregated from $\tilde{X}_n(i, k)$ with $k \geq j$. In the second step, we further allocate $\rho(X_n(i, j)|X)$ to $\tilde{X}_n(i, k)$, and denote the resulting allocations $\rho(\tilde{X}_n(i, k)|X_n(i, j))$, $k \geq j$.

Finally, to obtain the allocation for $\tilde{X}_n(i, k)$, we sum up all contributions from $X_n(i, j)$ with $j \leq k$:

$$\rho(\tilde{X}_n(i, k)|X) = \sum_{j=1}^{k} \rho(X_n(i, k)|X_n(i, j)).$$

In both methods, the second step is the same, we will focus on the first step first in what follows.

3.1. Euler allocation. The Euler allocation has been studied extensively; see [5], [10], [2], [11], and many others. We introduce in this section a computational efficient scheme for the Euler allocation of the IMCC capital charge.

For each risk factor category $RF_i$, we first allocate $ES(X(i))$ in (3) to each $X_n(i, j)$. To this end, let us introduce some notation. Let $v = (v_n)_{1 \leq n \leq N}$ be a sequence of real numbers. Given a collection of risk profiles $\{X_n\}_{1 \leq n \leq N}$, we denote

$$X_v^i(i) = \sum_n X_n^v(i),$$

where the sum is computed component-wise and

$$X_n^v(i) = (X_n(i, 1), \ldots, X_n(i, j - 1), v_n X_n(i, j), X_n(i, j + 1), \ldots, X_n(i, 5)).$$

For each $RF_i$, we define the allocation to each $X_n(i, j)$ as follows.

**Definition 3.1 (Euler allocation of FRTB ES).** For $1 \leq n \leq N, 1 \leq i \leq 6, 1 \leq j \leq 5$, let

$$ES(X_n(i, j) | X(i)) := \frac{\partial}{\partial v_n} ES(X_v^i(i)) \bigg|_{v=1^i},$$

where $ES(X_v^i(i))$ is the FRTB ES of the row $X_v^i(i)$, and $v = 1$ represents $v_n = 1$ for all $n$. We call $ES(X_n(i, j) | X(i))$ the Euler allocation of FRTB ES.

The chain rule in differentiation yields the following representation.
Lemma 3.2. For $1 \leq n \leq N, 1 \leq i \leq 6, 1 \leq j \leq 5$,

$$ES(X_n(i,j) \mid X(i)) = \frac{ES(X(i,j))}{ES(X(i))} \frac{\partial}{\partial v_n} ES(X^v(i,j)) \bigg|_{v=1},$$

(12)

where $X^v(i,j) = \sum_n v_n X_n(i,j)$.

Note that $\frac{\partial}{\partial v_n} ES(X^v(i,j)) \bigg|_{v=1}$ on the right-hand side of (12) is the standard Euler allocation of $ES(X(i,j))$. Then the Euler allocation under FRTB ES is the weighted version of the standard Euler allocation. The scaling factor $\frac{ES(X(i,j))}{ES(X(i))}$ reflects the ratio between the stand-alone ES of $X(i,j)$ and the FRTB ES of $X(i)$. This scaling factor is applied to all risk positions of the same liquidity horizon.

When the distribution of $X(i,j)$ satisfies certain regularity conditions (cf. [10, Assumption (S)]), then the standard Euler allocation can be calculated as a conditional expectation (cf. [10]):

$$\frac{\partial}{\partial v_n} ES(X^v(i,j)) \bigg|_{v=1} = \mathbb{E} \left[ X_n(i,j) \mid X(i,j) \geq \text{VaR}(X(i,j)) \right] =: \text{SE}(X_n(i,j) \mid X(i,j)), \quad (13)$$

where $\text{VaR}(X(i,j))$ is the Value-at-Risk of $X(i,j)$ calculated at the 97.5% quantile. The conditional expectation above can be calculated by the scenario-extraction method and hence is denoted by $\text{SE}(X_n(i,j) \mid X(i,j))$. Applying the scaled scenario-extraction method to (12) is also computationally efficient. Rather than calculating the element-wise derivative in (11) using a numeric differential scheme

$$\frac{\partial}{\partial v_n(i,j)} ES(X^v(i,j)) \bigg|_{v=1} = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left( ES(X(i) + \epsilon X_n(i,j)) - ES(X(i)) \right),$$

which typically requires revaluation on the bumps for each position, the scenario-extraction method calculates the conditional expectation by averaging $X_n(i,j)$ on scenarios when the portfolio loss $X(i,j)$ violates $\text{VaR}(X(i,j))$.

After applying the Euler allocation to the FRTB ES under full set of risk factors, and scaling the allocations by the stress period scaling factor, we have the following allocation to the IMCC capital charge.

Definition 3.3 (Euler allocation of IMCC). For $1 \leq n \leq N, 1 \leq i \leq 6, 1 \leq j \leq 5$, let

$$\text{IMCC}^E(X_n(i,j) \mid X(i)) := 0.5 \frac{ES^{R,S}(X(i))}{ES^{R,C}(X(i))} \frac{ES^{F,C}(X_n(i,j) \mid X(i))}{ES^{F,C}(X(i))} \times \frac{ES^{R,C}(X_n(i,j) \mid X(i))}{ES^{F,C}(X(i))}.$$

(14)

We call $\text{IMCC}^E(X_n(i,j) \mid X(i))$ the Euler allocation of IMCC. For the risk profile $X_n$, we define its Euler allocation of IMCC as

$$\text{IMCC}^E(X_n \mid X) = \sum_{i,j} \text{IMCC}^E(X_n(i,j) \mid X(i)).$$
Proposition 3.4. The Euler allocation of IMCC is a full allocation, i.e.,

$$\sum_n \text{IMCC}^E(X_n \mid X) = \sum_{n, i, j} \text{IMCC}^E(X_n(i, j) \mid X(i)) = \text{IMCC}(X).$$

Remark 3.5. If the expected shortfall for $X^v(i, j)$ is floored at zero as in Remark 2.2, (13) can be replaced by

$$\text{ES}^+(X_n(i, j) \mid X(i)) = \begin{cases} \frac{\text{ES}^+(X(i,j))}{\text{ES}^+(X(i))} \text{SE}(X_n(i, j) \mid X(i)) & \text{if } \text{ES}(X(i,j)) > 0 \\ 0 & \text{otherwise} \end{cases}.$$ 

The resulting Euler allocation of IMCC is still a full allocation, since $\text{ES}^+$ is still homogeneous of degree 1.

When a portfolio contains sub-portfolios which hedge each other, the standard Euler allocation under expected shortfall could produce negative allocations to some sub-portfolios. Because the FRTB ES discourages hedging across different risk factor classes and different liquidity horizons, negative allocations could be reduced or reversed under FRTB. The following example illustrates this point.

Example 3.6. Consider a portfolio with two risk positions whose risk profiles are denoted by $Y$ and $Z$ respectively. We assume that $Y$ concentrates on RF$_i$ and LH$_j$, and $Z$ concentrates on RF$_k$ and LH$_j$, with $1 \leq i \neq k \leq 5$. Therefore, $Y = Y(i, j)$ and $Z = Z(k, j)$. We assume that $Y = -Z$ and both of them follow standard normal distributions. Then the net loss of the portfolio $X = Y + Z = 0$, and the standard Euler allocation of regular ES would be negative for either $Y$ or $Z$, say $\text{SE}(Y \mid X) < 0$.

However, under FRTB framework, $X(i) = Y(i, j) = Y$. Then

$$\text{IMCC}^E(Y(i, j) \mid X(i)) = \frac{0.5 \text{ES}^R_S(Y)}{\text{ES}^R_C(Y)} \text{ES}^F_C(Y) \text{SE}(Y \mid X(i)) = \frac{0.5 \text{ES}^R_S(Y)}{\text{ES}^R_C(Y)} \text{ES}^F_C(Y) > 0.$$ 

Then this positive allocation could compensate the negative allocation $\text{IMCC}^E(Y(6, j) \mid X(6))$. Therefore, $\text{IMCC}^E(Y \mid X)$ could be less negative, or even positive, comparing to $\text{SE}(Y \mid X)$.

3.2. Constrained Aumann-Shapley allocation. The Shapley and Aumann-Shapley allocations were introduced in [2], where the results of [9] and [1] on coalitional games were applied to capital allocation problems. The concepts in those two allocations were combined in [4] to introduce the Constrained Aumann-Shapley allocation, where permutations of different risk positions are restricted to each business unit. In the FRTB IMA framework, the risk factor bucketing rule produces a natural constraint on risk profile organisations. Therefore, motivated by [4], we constrain permutations of different buckets within the same risk factor category.
To record liquidity horizon permutations, we introduce the following full permutation matrix:

\[ \mathcal{L} := \begin{bmatrix}
10 & 20 & 40 & 60 & 120 \\
10 & 20 & 40 & 120 & 60 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
120 & 60 & 40 & 20 & 10 \\
\end{bmatrix}_{5 \times 5} \]

Each row of \( \mathcal{L} \) records a permutation of liquidity horizons \{10, 20, 40, 60, 120\}. There are \( 5! = 120 \) permutations in total. For a given row \( r \) and a liquidity horizon \( \text{LH}_j \), we denote \( \mathcal{L}^{-1}(r,j) \) the column of \( \mathcal{L} \) in which \( \text{LH}_j \) locates. For example, \( \mathcal{L}^{-1}(2,5) = 4 \), or equivalently, \( \mathcal{L}(2,4) = \text{LH}_5 = 120 \).

Given a risk profile \( X_n \), a risk factor category \( \text{RF}_i \), a liquidity horizon \( \text{LH}_j \), and a permutation of liquidity horizons (say \( r \)-th row in \( \mathcal{L} \)). We want to first allocate \( ES(X(i)) \) to \( X_n(i,j) \). We call this allocation as the Constrained Aumann-Shapley (CAS) allocation of FRTB ES, and denote it as \( \text{CAS}(r,X_n(i,j)) \).

To introduce the value of \( \text{CAS}(r,X_n(i,j)) \), let \( v = (v_n)_{1 \leq n \leq N} \) be a sequence of real numbers, \( q \in [0,1] \), and

\[ X_n^{v,r,j}(i) = \sum_n X_n^{v,r,j}(i), \]  

where \( X_n^{v,r,j}(i) \) is a row with the entry \( X_n(i,\ell) \) at the \( \ell \)-th column if \( \mathcal{L}^{-1}(i,\ell) < \mathcal{L}^{-1}(i,j) \) (i.e., \( \text{LH}_\ell \) appears before \( \text{LH}_j \) in the permutation \( r \)); the entry \( v_n X_n(i,j) \) at the \( j \)-th column; and zero in all other columns. Taking the second row in matrix \( \mathcal{L} \) as an example, for \( j = 5 \) we have

\[ X_n^{2,5}(i) = (X_n(i,1),X_n(i,2),X_n(i,3),0,v_n X_n(i,5)). \]

Then define

\[ \text{CAS}(r,X_n(i,j)) := \int_0^1 \left. \frac{\partial}{\partial v_n} ES(X_n^{v,r,j}(i)) \right|_{v=q} dq, \]

where \( v = q \) means \( v_n = q \) for all \( n \). Intuitively, \( \partial_{v_n} ES(X_n^{v,r,j}(i))|_{v=q} \) is the marginal contribution, in the direction of \( X_n(i,j) \), of the FRTB ES for the portfolio risk profile consisting \( qX(i,j) \) and all \( X(i,\ell) \), if \( \text{LH}_\ell \) appears before \( \text{LH}_j \) in the permutation \( r \).

**Lemma 3.7.** For \( 1 \leq n \leq N \), \( 1 \leq i \leq 6 \), \( 1 \leq j \leq 5 \), and \( 1 \leq r \leq 5! \),

\[ \text{CAS}(r,X_n(i,j)) = \eta(r,i,j) \left. \frac{\partial}{\partial v_n} ES(X_n^{v}(i,j)) \right|_{v=1}, \]  

where

\[ \eta(r,i,j) = \frac{\sqrt{\sum_{1 \leq s \leq \mathcal{L}^{-1}(r,j)} ES(X(i,\mathcal{L}(r,s)))^2} - \sqrt{\sum_{1 \leq s < \mathcal{L}^{-1}(r,j)} ES(X(i,\mathcal{L}(r,s)))^2}}{ES(X(i,j))}. \]
When the distribution of \( X(i,j) \) satisfies Assumption (S), then the derivative on the right-hand side of (16) can be replaced by \( \text{SE}(X_n(i,j) \mid X(i,j)) \).

Similar to the Euler allocation under FRTB ES, the Constrained Aumann-Shapley allocation is also a weighted version of the standard Euler allocation. The scaling factor \( \eta(r,i,j) \) is the ratio between the \( X(i,j) \) induced increment of FRTB ES in the permutation \( r \) and the stand-alone expected shortfall of \( X(i,j) \).

After averaging over all permutations, we introduce the following allocation to the IMCC capital charge.

**Definition 3.8 (CAS allocation of IMCC).** For \( 1 \leq n \leq N, 1 \leq i \leq 6, 1 \leq j \leq 5 \),

\[
\text{IMCC}^C(X_n(i,j) \mid X(i)) := 0.5 \frac{E_{R,S}(X(i))}{E_{R,C}(X(i))} \frac{1}{5!} \sum_{r=1}^{5!} \text{CAS}^{F,C}(r, X_n(i,j)),
\]

where \( \text{CAS}^{F,C} \) is the Constrained Aumann-Shapley allocation of FRTB \( ES^{F,C} \). We call \( \text{IMCC}^C(X_n(i,j) \mid X(i)) \) the Constrainted Aumann-Shapley allocation of IMCC.

**Proposition 3.9.** The CAS allocation of IMCC is a full allocation.

If the expected shortfall for \( X^e(i,j) \) is floored at zero as in Remark 2.2, the CAS allocation can be adjusted similarly to Remark 3.5. The adjusted CAS allocation is still a full allocation.

**Remark 3.10.** An important concept for capital allocation is the additivity property. Consider a subportfolio \( Y \) in \( X \), where \( Y \) is aggregated from risk profiles \( \{Y_m\}_{1 \leq m \leq M} \). We want to know whether the allocation to the portfolio \( Y \) equals to the sum of allocations to all \( \{Y_m\} \), i.e. whether \( \rho(Y \mid X) = \sum_m \rho(Y_m \mid X) \) is true. The answer to this question is positive for both Euler and CAS allocations. This is due to the fact that both of them are scaled versions of the Euler allocation for the regular ES, which is additive itself.

### 3.3. The second step allocation.

After the first step of both allocation methods, capital is allocated to each liquidity horizon adjusted loss \( X_n(i,j) \). For the unconstrained part \( i = 6 \), we consider \( X_n(6,j) = \sum_{i=1}^{5} X_n(i,j) \) and use the standard Euler allocation to allocate unconstrained allocation to each \( X_n(i,j) \) and denote it by \( \text{IMCC}(X_n(i,j) \mid X(6)) \).

Now for each \( 1 \leq i \leq 6 \), since \( X_n(i,j) \) is aggregated from 10 days loss \( \hat{X}_n(i,k) \) with \( k \geq j \), it seems natural to extract capital associated to each \( \hat{X}(i,k) \) from the capital allocated to \( X(i,j) \). Recall from (11). We can consider \( X_n(i,j) \) as a portfolio of \( \sqrt{\frac{L_{H_j} - L_{H_{j-1}}}{10}} \hat{X}_n(i,k) \) with \( k \geq j \). Hence we use the Euler method to allocate capital from \( X_n(i,j) \) further down to each \( \sqrt{\frac{L_{H_j} - L_{H_{j-1}}}{10}} \hat{X}_n(i,k) \). We denote the resulting allocations by

\[
\text{IMCC} \left( \sqrt{\frac{L_{H_j} - L_{H_{j-1}}}{10}} \hat{X}_n(i,k) \right) | X_n(i,j) \right), \quad k \geq j.
\]
Now using the additivity property in Remark 3.10, we can sum all capital from $X_n(i,j)$ with $j \leq k$ to get the contribution of $\tilde{X}_n(i,k)$ as

$$IMCC\left(\tilde{X}_n(i,k)|X(i)\right) = \sum_{j \leq k} IMCC\left(\sqrt{\frac{LH_j - LH_{j-1}}{10}}\tilde{X}_n(i,k)|X_n(i,j)\right).$$  \hspace{1cm} (18)$$

Combining constrained and unconstrained allocations, the allocation for $\tilde{X}_n(i,j)$, with $1 \leq n \leq N$, $1 \leq i \leq 5$ and $1 \leq j \leq 5$, is given by

$$IMCC^{Total}\left(\tilde{X}_n(i,k)|X(i)\right) := IMCC\left(\tilde{X}_n(i,k)|X(i)\right) + IMCC\left(\tilde{X}_n(i,k)|X(6)\right).$$  \hspace{1cm} (19)$$

### 3.4. Extensions

In the previous two sections, the Euler and CAS allocations of IMCC are applied to FRTB ES for the full set under regular scenarios, and the stress scaling factor $\frac{ES^{R,S}(X(i))}{ES^{R,C}(X(i))}$ is treated as a constant for each RF$_i$. In other words, the $X_n(i,j)$ induced risk contribution is considered for $ES^{F,C}$, but not for $ES^{R,S}$ and $ES^{R,C}$. In this section, we will consider the impact of $X_n(i,j)$ on the stress scaling factors and introduce the associated modifications of Euler and CAS allocations. The second step of allocation is the same as in Section 3.3.

**Definition 3.11** (Euler allocation of IMCC with scaling adjustment). For $1 \leq n \leq N$, $1 \leq i \leq 6$, $1 \leq j \leq 5$, let

$$IMCC^{E,S}(X_n(i,j) | X(i)) := 0.5 \frac{\partial}{\partial v_n} \left[ \frac{ES^{R,S}(X_v^j(i))}{ES^{R,C}(X_v^j(i))} ES^{F,C}(X_v^j(i)) \right] \bigg|_{v=1}.$$

Taking differentiations to each expected shortfalls, we obtain

**Proposition 3.12.** For $1 \leq n \leq N$, $1 \leq i \leq 6$, $1 \leq j \leq 5$,  

$$IMCC^{E,S}(X_n(i,j) | X(i)) = 0.5 \left[ \frac{ES^{R,S}(X(i))}{ES^{R,C}(X(i))} ES^{F,C}(X_n(i,j) | X(i)) \right.$$

$$+ \frac{ES^{F,C}(X(i))}{ES^{R,C}(X(i))} ES^{R,S}(X_n(i,j) | X(i))$$

$$\left. - \frac{ES^{R,S}(X(i)) ES^{F,C}(X(i))}{ES^{R,C}(X(i))^2} ES^{R,C}(X_n(i,j) | X(i)) \right].$$  \hspace{1cm} (20)$$

The previous expression for $IMCC^{E,S}$ motivates us to define the following CAS allocation with scaling adjustment.
Definition 3.13. For \(1 \leq n \leq N, 1 \leq i \leq 6, 1 \leq j \leq 5\), let

\[
IMCC^{C,S}(X_n(i, j) \mid X(i)) \equiv \frac{0.5}{5!} \sum_{r=1}^{5!} \left[ \frac{ES^{R,S}(X(i))}{ES^{R,G}(X(i))} CAS^{F,C}(r, X_n(i, j)) + \frac{ES^{F,C}(X(i))}{ES^{R,C}(X(i))} CAS^{R,S}(r, X_n(i, j)) - \frac{ES^{R,S}(X(i)) ES^{F,C}(X(i))}{ES^{R,C}(X(i))^2} CAS^{R,C}(r, X_n(i, j)) \right].
\]

Proposition 3.14. Both Euler and CAS allocations of IMCC with scaling adjustment are full allocations and satisfy the additivity property.

4. Simulation Analysis

4.1. Positive correlations. This simulation exercise illustrates the difference of allocations among different risk factor categories and liquidity horizons. We assume that there is only one risk position, and all \(\tilde{X}(i, j)\) have identical normal distributions with zero mean and 30% annual volatility. We consider the following four scenarios of correlation structures:

(i) Independence: all \(\tilde{X}(i, j)\) are mutually independent;
(ii) Uniform positive correlation: each pair of \(\tilde{X}(i, j)\) and \(\tilde{X}(k, l)\) have correlation 0.99;
(iii) Positive correlation among RFs and zero correlation among LHs: \(\text{corr}(\tilde{X}(i, j), \tilde{X}(k, j)) = 0.99\) and \(\text{corr}(\tilde{X}(i, j), \tilde{X}(i, k)) = 0\) for any \(i \neq k\);
(iv) Positive correlation among LHs and zero correlation among RFs: \(\text{corr}(\tilde{X}(i, j), \tilde{X}(k, j)) = 0\) and \(\text{corr}(\tilde{X}(i, j), \tilde{X}(i, k)) = 0.99\) for any \(i \neq k\).

We simulated risk profiles for 250 days, risk profiles are independent across different days, and risk profiles in the same day follow the correlation scenarios above. The stress period scalings are assumed to be 1 for all risk factor categories. First, we report and compare the IMCC and the regular 97.5% ES values in the following table.

| Scenario              | IMCC    | Regular ES |
|-----------------------|---------|------------|
| (i) Independent       | 12.48   | 3.28       |
| (ii) Uniform Positive Corr | 28.57   | 16.70      |
| (iii) Zero-LH-Corr    | 18.28   | 7.81       |
| (iv) Zero-RF-Corr     | 21.00   | 7.59       |

Table 1. FRTB IMCC v.s. Regular ES

We can see from Table 1 that the IMCC values are between 1.7 and 3.8 times of the regular ES. Moreover, strong positive correlations among different liquidity horizons (scenario (iv))
increase the capital more than the scenario with strong positive correlations among different risk factor categories (scenario (iii)). This reflects the FRTB liquidity horizon bucketing rule.

Figure 1 illustrates the Euler allocation of FRTB ES, the CAS allocation of FRTB ES, and the Euler allocation of regular ES. It reports allocations to different $\tilde{X}(i,j)$, after combining the constrained and unconstrained allocations (see Equation (19)).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Euler allocation of FRTB ES (Euler FRTB ES), CAS allocation of FRTB ES (CAS FRTB ES), and Euler allocation of Regular ES (Euler Reg ES). Upper-left panel: scenario (i); Upper-right panel: scenario (ii); Bottom-left panel: scenario (iii); Bottom-right panel: scenario (iv). Each panel presents the percentage of allocation to different $\tilde{X}(i,j)$. The total capital charges are reported in Table 1.}
\end{figure}

Figure 1 shows that both FRTB allocation methods typically allocate more capital to risk factors with longer liquidity horizons. This feature is due to the facts that 1) longer liquidity horizon has bigger scalings (see Equation (1)); and 2) longer liquidity horizon has more allocation contributions from shorter liquidity horizon allocations (see Equation (18)). On the other hand, due to allocations from unconstrained part, when there is no strong positive
correlation among risk factor categories, allocations to each liquidity horizon vary within the same risk factor category.

The upper-left panel of Figure 1 shows that the Euler allocations of regular ES present large variations and negative allocations even when there are no negative correlations. These features are due to the instability of the Euler allocation for regular ES or VaR, which has been documented in [12]. The kernel smoothing technique (see [3]) can improve stability of the Euler allocation. Figure 2 presents the allocation results when the kernel smoothing technique is applied to each allocation method. Comparing Figures 1 and 2, we can see that the kernel smoothing technique significantly improves the stability for the Euler allocation for the regular ES, but it is less effective on FRTB allocations.

**Figure 2.** Kernel smoothed allocations

4.2. **Hedging.** In the second simulation exercise, we analyse three scenarios of hedging relations: hedging between 2 risk factor categories; hedging between 2 liquidity horizon classes; and hedging between two risk positions in the same bucket. To study the impact of hedging between liquidity horizon adjusted risk profiles, we view different buckets as
different risk positions. In this way, $X_n(i, j) = \sqrt{\frac{LH_j - LH_{j-1}}{10}} \tilde{X}_n(i, j)$, and the correlations between different $\tilde{X}_n(i, j)$ are the same as the correlations between different $X_n(i, j)$. This allows us to focus on the impact of FRTB rules on allocations with hedging.

We consider the following three correlation structures:

(i) Strong hedging between EQ and IR: $\text{corr}(\tilde{X}(3, j), \tilde{X}(5, j)) = -0.99$ for any $j$ and zero correlation between all other pairs;

(ii) Strong hedging between LH$_1$ and LH$_2$: $\text{corr}(\tilde{X}(i, 1), \tilde{X}(i, 2)) = -0.99$ for all $i$ and zero correlation between all other pairs;

(iii) Strong hedging between 2 risk positions within the same bucket: $\text{corr}(\tilde{X}_1(i, j), \tilde{X}_2(i, j)) = -0.99$ for all $i, j$, and zero correlation between all other pairs.

We simulated risk profiles for 250 days, risk profiles are independent across different days, and risk profiles in the same day follow the correlation scenarios above. Other settings remain the same as in the previous section. The IMCC and regular ES are reported in Table 2 below.

| Scenario          | IMCC | Regular ES |
|-------------------|------|------------|
| (i) RF Hedging    | 7.90 | 2.17       |
| (ii) LH Hedging   | 8.43 | 2.55       |
| (iii) Position Hedging | 0.84 | 0.33       |

Table 2. FRTB IMCC v.s. Regular ES

We can see from Table 2 that the IMCC is between 2.5 to 3.6 times to the regular ES. On the other hand, because FRTB restricts hedging among different buckets, the ratios between IMCC and ES in scenario (i) and (ii) are much larger than the ratio in scenario (iii), where hedging within the same bucket is not restricted by FRTB.

Figure 3 illustrates different allocations of IMCC and regular ES. The left and middle panels show that, even though there are negative correlations between different risk factor or liquidity horizon buckets, the Euler and CAS allocations of IMCC are all positive. This confirms our analysis in Example 3.6.

When one position hedge the other in the same bucket, the right panel in Figure 3 shows that there could be negative allocations for both Euler and CAS allocations of IMCC. But their magnitudes are smaller than the Euler allocations of the regular ES. In the Euler allocation of the regular ES, one scenario extraction is applied to each loss simulation of 250 days. However, in both Euler and CAS allocations of IMCC, one scenario extraction is applied to each bucket. Therefore, there are in total $30 = 6 \times 5$ scenario extractions applied to each loss simulation of 250 days. Then the final allocation of a risk position is a weighted
In order to further analyse negativity and stability of different allocations, we extend the hedging scenario (iii) from 2 risk positions to 20 risk positions, with each pair of risk positions following the hedging scenario (iii). We apply different allocation methods to allocate capital to each risk position and each bucket. Figure 4 illustrates histograms and kernel densities of these allocations for each allocation method. Even without aggregation from different risk factor and liquidity horizon classes, Figure 4 shows that the Euler and CAS allocations of IMCC still produce tighter histograms comparing to the case for the Euler allocation of the regular ES. Comparing the Euler and CAS allocations, we observe that the CAS allocation produces slightly more stable results with less extreme allocations. This is due to the fact that the CAS allocation is an averages of 5! permutations which further improve the stability of allocations.

Figure 4 shows that all allocations are symmetric around 0. This means all allocation methods produce roughly the same amount of positive and negative allocations in the hedging scenario (iii). If there are hedging between different risk factor and liquidity horizon buckets, results in the scenario (i) and (ii) show that allocations in these buckets are likely to be positive. This makes the allocation histograms skew to the positive side.

4.3. **Allocations with scaling adjustment.** In the third simulation exercise, we illustrate the impact of the choice of reduced sets on the IMCC allocations with scaling adjustment introduced in Section 3.4. Consider the situation where the reduced factor set is chosen so
that all \( X_n(i, j) \) have similar distributions in the stressed period and the current period, then \( \text{ES}^{R,S}(X(i)) \) is similar to \( \text{ES}^{R,C}(X(i)) \), and the allocations \( \text{ES}^{R,S}(X_n(i, j) \mid X(i)) \) and \( \text{ES}^{R,C}(X_n(i, j) \mid X(i)) \) are similar as well. Therefore, the second and the third terms on the right-hand side of (20) are similar, so

\[
\text{IMCC}^{E,S}(X_n(i, j) \mid X(i)) \approx 0.5 \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))} \text{ES}^{F,C}(X_n(i, j) \mid X(i)).
\]

This allocation will be significantly different from the case where risk factors have distinct distributions in the stressed period and the current period.

We follow the convention of the previous exercise where different buckets are treated as different risk positions. We consider a portfolio with two risk positions. During the current period, all \( \tilde{X}_n(i, j) \) are independent and have the same distribution. During the stress period, the correlations between any pairs of \( \tilde{X}_n(i, j) \) become 0.7. The standard deviations of \( \tilde{X}_1(3, 3) \) and \( \tilde{X}_2(1, 4) \) during the stress period become 9 times of the standard deviations during the current period. Distributions of all other \( \tilde{X}_n(i, j) \) in the stress period are assumed to be the same as in the current period.

We consider two reduced sets:

Set A: All risk factors except 60-days EQ and 120-days CM;
Set B: All risk factors except 40-days EQ and 60-days CM.
The reduced set B excludes risk factors which have distinct distributions in the stress period. Table 3 shows that both reduced sets satisfy the requirement that $\text{ES}^{R,C}(X(i)) \geq 75\% \text{ES}^{F,C}(X(i))$ for all risk factor categories.

|          | CM | CR | EQ | FX | IR | Unconstrained |
|----------|----|----|----|----|----|---------------|
| Set A    | 80%| 100%| 97%| 100%| 100%| 95%           |
| Set B    | 97%| 100%| 94%| 100%| 100%| 98%           |

Table 3. Ratios between ES using the reduced set and the full set.

Table 4 shows the difference of allocations with/without stress-scaling adjustment using different reduced sets. We can see from results associated to Set A that, when distributions of risk factors in the reduced set are different between the stress and current periods, allocations with stress-scaling adjustment increases the percentages of allocations on stress positions. However, when distributions of risk factors in the reduced set are similar between the stress and current periods, results associated to Set B indicate that allocations are the same with/without stress-scaling adjustment. Moreover the total IMCC is much lower using Set B than Set A.

|          | Set A (Adjustment) | Set A (Without adj) | Set B (Adjustment) | Set B (Without adj) |
|----------|--------------------|---------------------|--------------------|---------------------|
| CM.60 days.Position 2 | 4.00% | 2.24% | 1.43% | 1.43% |
| EQ.40 days.Position 1 | 5.04% | 3.26% | 2.11% | 2.11% |

Table 4. Percentages of allocations with and without stress-scaling adjustment using different reduced factor sets. Columns labeled adjustment report allocations using (20), columns labeled without adj report allocation using (14). The total IMCC are the same in both methods: IMCC(Set A)=11.55; IMCC(Set B)=3.14.

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Appendix A. Proofs

A.1. **Proof of Lemma 2.4.** The expected shortfall is positive homogeneous, then $\text{ES}(aX(i,j)) = a\text{ES}(X(i,j))$. All operations in (3), (5), and (6) are positive homogeneous. Hence the statement in (i) holds.
For (ii), recall that the expected shortfall is sub-additive, i.e.,

$$\text{ES}((X + Y)(i, j)) \leq \text{ES}(X(i, j)) + \text{ES}(Y(i, j)).$$

When $\text{ES}((X + Y)(i, j)) \geq 0$ for all $j$, then

$$\text{ES}((X + Y)(i)) = \sqrt{\sum_{j=1}^{5} \text{ES}((X + Y)(i,j))^2} \leq \sqrt{\sum_{j=1}^{5} \left[ \text{ES}(X(i, j)) + \text{ES}(Y(i, j)) \right]^2}$$

$$\leq \sqrt{\sum_{j=1}^{5} \text{ES}(X(i, j))^2} + \sqrt{\sum_{j=1}^{5} \text{ES}(Y(i, j))^2} = \text{ES}(X(i)) + \text{ES}(Y(i)),$$

where the second inequality follows from the Minkowski inequality.

For (iii), it follows from the sub-additivity for $\text{ES}^F_C$ that

$$\text{ES}^F_C((X + Y)(i)) \leq \text{ES}^F_C(X(i)) + \text{ES}^F_C(Y(i)).$$

Then, when (8) is satisfied, we have

$$\text{IMCC}((X + Y)(i)) = \text{ES}^F_C((X + Y)(i)) \frac{\text{ES}^R_S((X + Y)(i))}{\text{ES}^R_C((X + Y)(i))}$$

$$\leq \frac{\text{ES}^R_S((X + Y)(i))}{\text{ES}^R_C((X + Y)(i))} \left[ \text{ES}^F_C(X(i)) + \text{ES}^F_C(Y(i)) \right]$$

$$\leq \frac{\text{ES}^R_S(X(i))}{\text{ES}^R_C(X(i))} \text{ES}^F_C(X(i)) + \frac{\text{ES}^R_S(Y(i))}{\text{ES}^R_C(Y(i))} \text{ES}^F_C(Y(i))$$

$$= \text{IMCC}(X(i)) + \text{IMCC}(Y(i)).$$

\[\square\]

A.2. Proof of Proposition 3.4 Since the FRTB ES, defined in (3), is a risk measure homogeneous of degree 1. It then follows from Euler’s theorem on homogeneous functions (see [11] Theorem A.1) that the Euler allocation on FRTB ES is a full allocation, i.e.,

$$\sum_{n,j} \text{ES}^F_C(X_n(i, j) \mid X(i)) = \text{ES}^F_C(X(i)).$$

This identity, combined with (5) and (6), yields

$$\sum_{n,i,j} \text{IMCC}(X_n(i, j) \mid X(i)) = 0.5 \sum_{i=1}^{6} \frac{\text{ES}^R_S(X(i))}{\text{ES}^R_C(X(i))} \left( \sum_{n,j} \text{ES}^F_C(X_n(i, j) \mid X(i)) \right)$$

$$= 0.5 \sum_{i=1}^{6} \frac{\text{ES}^R_S(X(i))}{\text{ES}^R_C(X(i))} \text{ES}^F_C(X(i)) = \text{IMCC}(X).$$

\[\square\]
A.3. **Proof of Lemma 3.7.** When LH\textsubscript{j} is in the first column of the permutation r, i.e., \( L^{-1}(r, j) = 1 \), the row \( X^{v,r,j}(i) \) has only one nonzero entry \( \sum_n v_n X_n(i, j) \) at the j-th column. Then

\[
ES(X^{v,r,j}(i)) = |ES(\sum_n v_n X_n(i, j))|.
\]

Since the expected shortfall is homogeneous of degree 1, then

\[
\partial v_n ES(X^{v,r,j}(i)) \bigg|_{v=q} = \text{sgn}(ES(qX(i, j))) \partial v_n ES\left( \sum_n v_n X_n(i, j) \right) \bigg|_{v=q} = \text{sgn}(ES(X(i, j))) \partial v_n ES\left( \sum_n v_n X_n(i, j) \right) \bigg|_{v=1}.
\]

As a result,

\[
\text{CAS}(r, X_n(i, j)) = \int_0^1 \partial v_n ES(X^{v,r,j}(i)) \bigg|_{v=q} dq = \int_0^1 \partial v_n ES(X^v(i, j)) \bigg|_{v=1} dq = \partial v_n ES(X^v(i, j)) \bigg|_{v=1}.
\]

Note that \( \eta(r, i, j) = \text{sgn}(ES(qX(i, j))) \) in this case. Therefore the previous expression of \( \text{CAS}(r, X_n(i, j)) \) agrees with (16).

When LH\textsubscript{j} is not in the first column, i.e., \( L^{-1}(r, j) > 1 \),

\[
ES(X^{v,r,j}(i)) = \sqrt{ES\left( \sum_n v_n X_n(i, j) \right)^2 + \sum_{1 \leq s < L^{-1}(r,j)} ES(i, L(r, s))^2}.
\]

Denote

\[
ES(X^{q,r,j}(i)) = \sqrt{ES(qX(i, j))^2 + \sum_{1 \leq s < L^{-1}(r,j)} ES(i, L(r, s))^2}.
\]

It follows from the homogeneous property of the expected shortfall that

\[
\partial v_n ES(X^{v,r,j}(i)) \bigg|_{v=q} = \frac{\text{sgn}(ES(qX(i, j))) \partial v_n ES\left( \sum_n v_n X_n(i, j) \right) \bigg|_{v=q}}{ES(X^{q,r,j}(i))} = \frac{qES(X(i, j)) \partial v_n ES\left( \sum_n v_n X_n(i, j) \right) \bigg|_{v=1}}{ES(X^{q,r,j}(i))}.
\]

Integrating the derivative with respect to q, we obtain

\[
\int_0^1 \partial v_n ES(X^{v,r,j}(i)) \bigg|_{v=q} dq = \partial v_n ES(X^v(i, j)) \bigg|_{v=1} \int_0^1 \frac{qES(X(i, j))}{ES(X^{q,r,j}(i))} dq
\]

\[
= \frac{\partial v_n ES(X^{v,r,j}(i)) \bigg|_{v=1}}{ES(X(i, j))} \int_0^1 \frac{qES(X(i, j))^2}{ES(X^{q,r,j}(i))} dq = \frac{\partial v_n ES(X^{v,r,j}(i)) \bigg|_{v=1}}{2 ES(X(i, j))} \int_0^1 \frac{d(q^2ES(X(i, j))^2)}{ES(X^{q,r,j}(i))} dq
\]

\[
= \eta(r, i, j) \partial v_n ES(X^{v,r,j}(i)) \bigg|_{v=1}.
\]

\( \square \)
A.4. **Proof of Proposition 3.9.** From Lemma 3.7 and the fact that the standard Euler allocation is a full allocation, we have

\[
\sum_n \text{CAS}(r, X_n(i, j)) = \eta(r, i, j) \sum_n \partial_{v_n} \text{ES}(X^v(i, j)) \Bigr|_{v=1} = \eta(r, i, j) \text{ES}(X(i, j))
\]

\[
= \sqrt{\sum_{1 \leq s \leq L-1(r, j)} \text{ES}(X(i, L(r, s)))^2} - \sqrt{\sum_{1 \leq s < L-1(r, j)} \text{ES}(X(i, L(r, s)))^2}.
\]

Therefore

\[
\sum_{n,j} \text{CAS}(r, X_n(i, j)) = \text{ES}(X(i)).
\]

The rest proof is similar to the proof of Proposition 3.4. □

A.5. **Proof of Proposition 3.14.** Recall that

\[
\sum_{n,j} \text{ES}(X_n(i, j) \mid X(i)) = \text{ES}(X(i)).
\]

Then applying the previous identity to the Euler allocation for \(\text{ES}^{F,C}, \text{ES}^{R,S}, \) and \(\text{ES}^{R,C},\) respectively, we obtain

\[
\sum_{n,j} \text{IMCC}^{E,S}_{\text{ES}}(X_n(i, j) \mid X(i)) = 0.5 \left[ \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))} \frac{\text{ES}^{F,C}(X(i))}{\text{ES}^{R,C}(X(i))} \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))} \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))} \right]
\]

\[
= 0.5 \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))} \frac{\text{ES}^{F,C}(X(i))}{\text{ES}^{R,C}(X(i))} = \text{IMCC}(X(i)).
\]

The proof for \(\text{IMCC}^{C,S}\) is similar. □

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