FREE QUANTUM FIELDS ON THE POINCARÉ GROUP.

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Summary. A class of free quantum fields defined on the Poincaré group, is described by means of their two-point vacuum expectation values. They are not equivalent to fields defined on the Minkowski spacetime and they are “elementary” in the sense that they describe particles that transform according to irreducible unitary representations of the symmetry group, given by the product of the Poincaré group and of the group $SL(2, \mathbb{C})$ considered as an internal symmetry group. Some of these fields describe particles with positive mass and arbitrary spin and particles with zero mass and arbitrary helicity or with an infinite helicity spectrum. In each case the allowed $SL(2, \mathbb{C})$ internal quantum numbers are specified. The properties of local commutativity and the limit in which one recovers the usual field theories in Minkowski spacetime are discussed. By means of a superposition of elementary fields, one obtains an example of a field that present a broken symmetry with respect to the group $Sp(4, \mathbb{R})$, that survives in the short-distance limit. Finally, the interaction with an accelerated external source is studied and it is shown that, in some theories, the average number of particles emitted per unit of proper time diverges when the acceleration exceeds a finite critical value.

PACS. 11.10.Kk – Field theories in higher dimensions.
PACS. 02.20.+b – Group theory.
I. Introduction.

Quantum field theories defined on the Poincaré group manifold $\mathcal{P}$ instead of the Minkowski spacetime have been introduced by Lurçat\textsuperscript{1} in 1964. A motivation of these investigations was a symmetric treatment of translations, rotations and Lorentz boosts, namely of all the restricted Poincaré transformations. Later\textsuperscript{2,3} it has been recognized that this point of view, in order to be really consistent, requires a symmetric treatment of velocity, angular velocity and acceleration; since in relativistic theories there is an upper bound to the velocity of material objects, one has to introduce similar limitations to angular velocity and acceleration. The existence of an upper bound to the proper acceleration has also been suggested in ref. 4. Brandt\textsuperscript{5,6} has shown that a maximal acceleration of the order of $c^2 l_P^{-1}$, where $l_P$ is the Planck length, is expected as a quantum gravitational effect.

The theories studied in ref. 1 are symmetric with respect to both the left and the right translations of the group $\mathcal{P}$. We suggest that the physical symmetry group is smaller, namely it contains all the left translations, but only the right translations generated by the homogeneous Lorentz group (or its universal covering $SL(2,\mathbb{C})$). The aim of the present paper is to analyze the free fields on $\mathcal{P}$ (or on its universal covering $\tilde{\mathcal{P}}$), that satisfy this weaker symmetry requirement, besides the natural positivity and spectral conditions. We begin by considering “elementary” fields, namely fields that describe particles that transform according to irreducible unitary representations (i. u. r.’s) of the symmetry group. Other fields can be obtained by superposition of elementary fields, which could provide the building blocks for the construction of theories implementing the physical ideas indicated above.

We do not claim to have found all the fields with the assumed properties. A formal proof of this statement would require a more precise formulation of the problem, for instance a specification of the distribution space to which the fields have to belong. We think that in a first approach it is more useful to find as many examples as possible and to introduce technical assumptions only when they are found to be necessary.

In order to clarify the physical meaning of our assumptions, it is convenient to consider the theories on an arbitrary group manifold as special cases of theories of a much larger class\textsuperscript{7–10}, based on a $n$-dimensional differentiable manifold $\mathcal{S}$ endowed with a geometric structure defined by $n$ vector fields $A_{\alpha}$ ($\alpha = 0, \ldots, n-1$) linearly independent at every point of $\mathcal{S}$. If $\mathcal{S} = \mathcal{G}$ is a group manifold, the vector fields $A_{\alpha}$ are the generators of the right translations (invariant under left translations)\textsuperscript{11} and form a basis of the Lie algebra $\mathfrak{l}$ of $\mathcal{G}$.

In another interesting case, $\mathcal{S}$ is the 10-dimensional principal bundle\textsuperscript{12} of the pseudo-orthonormal frames (tetrads) of a (3+1)-dimensional pseudo-Riemannian spacetime $\mathcal{M}$. The fields

\begin{equation}
A_4 = A_{23}, \quad A_5 = A_{31}, \quad A_6 = A_{12}, \quad A_7 = A_{10}, \quad A_8 = A_{20}, \quad A_9 = A_{30}
\end{equation}

are the generators of the structural group of the bundle, namely of the Lorentz group acting on the tetrads. They define a basis of the “vertical” subspaces of
the tangent spaces of $S$. The fields $A_0, \ldots, A_3$ describe the infinitesimal parallel displacements along the tetrad vectors, namely they generate the “horizontal” subspaces which define a connection. If $M$ is the flat Minkowski spacetime, one can identify the bundle of frames $S$ with the Poincaré group. In a similar way the de Sitter (or anti-de Sitter) group can be identified with the bundle of frames $S$ of a pseudo-Riemannian spacetime $M$ with constant positive (or negative) curvature.

Classical field theories including gravitation based on these ideas have been developed in refs. 8-10, 13, 14. If one identifies $S$ with a principal bundle with a larger structural group, one can also treat Maxwell and Yang-Mills fields\textsuperscript{8,15}. Any classical or quantum field defined on the pseudo-Riemannian space-time $M$ can easily be translated into a field defined on the bundle of frames $S$. We think, however, that the new formalism should be used to formulate new physical ideas.

The fields $A_\alpha$ define a “teleparallelism” in $S$, namely a set of isomorphisms between all the tangent spaces of $S$ and a fixed vector space $\mathcal{T} = \mathbb{R}^n$. A closed wedge $\mathcal{T}^+ \subset \mathcal{T}$ defines a field of wedges in the tangent spaces of $S$ which describes the causal properties of the theory and in particular the upper bounds to velocity, angular velocity and acceleration\textsuperscript{2,3}.

It is useful to introduce the structure coefficients $F^\gamma_{\alpha\beta}$ defined by

$$[A_\alpha, A_\beta] = F^\gamma_{\alpha\beta} A_\gamma,$$

where $[A,B]$ is the Lie bracket of two vector fields. If $S$ is a group manifold, they are the structure constants of the corresponding Lie algebra. If $S$ is a bundle of frames, some of the structure coefficients are the components of the curvature and torsion tensors and in a theory of gravitation they have a dynamical character, namely they depend on the distribution of matter.

In accord with the ideas indicated above, it is natural to consider theories in which all the structure coefficients $F^\gamma_{\alpha\beta}$ and all the vector fields $A_\alpha$ have a dynamical character. In a theory of this kind, the field equations determine both the fields $A_\alpha$ that describe the geometry and the fields $\psi_\rho$ that describe matter. We assume that this theory is invariant under all the diffeomorphisms of $S$ and under a symmetry group $F$ acting linearly in the following way

$$\psi_\rho \rightarrow S_\rho^\sigma(k)\psi_\sigma,$$

$$A_\alpha \rightarrow C_\alpha^\beta(k)A_\beta, \quad k \in F,$$

where $S$ and $C$ are linear representations of $F$. The representation $C$ is real and one can consider it as acting on the vector space $\mathcal{T}$. It is natural to require that the transformations $C(k)$ leave the wedge $\mathcal{T}^+$ invariant. It follows from the physical interpretation of the fields $A_\alpha$ that the element $k$ cannot depend on the point of $S$, namely $F$ is a global symmetry group.

In the physically most interesting example, $S$ is a ten-dimensional manifold, the group $F$ contains a subgroup isomorphic to the restricted Lorentz group.
SO$^\dagger(3, 1)$ and the representation $C$ restricted to this subgroup is the direct sum of the vector and the antisymmetric tensor representations. Actually, in order to treat matter fields with half-integral spin, it is convenient to assume that $\mathcal{F}$ contains the universal covering of $SO^\dagger(3, 1)$, namely $SL(2, \mathbb{C})$. Under these conditions, if we assume that $\mathcal{T}^+$ is a cone (namely that $\mathcal{T}^+ \cap -\mathcal{T}^+ = \{0\}$) with interior points, this cone is determined up to a change of the units of time and length$^{2,3}$. It has a large symmetry group given by $L(4, \mathbb{R})$ acting on $\mathcal{T}$ by means of its symmetric tensor representation$^2$. In this framework, it is natural to assume that the symmetry group $\mathcal{F}$ of the theory is $L(4, \mathbb{R})$ or one of its subgroups that contain $SL(2, \mathbb{C})$. Possible choices are $SL(4, \mathbb{R}), Sp(4, \mathbb{R})$ or $SL(2, \mathbb{C})$.

In the present paper we consider the fields $A_\alpha$ as fixed classical fields and we concentrate our attention on the quantum fields that describe matter. The symmetry group of this partial theory contains only the elements of the symmetry group of the complete theory that do not affect the geometric fields $A_\alpha$. For instance, the elements of $\mathcal{F}$ that do not preserve the values of the structure coefficients $F^\gamma_{\alpha\beta}$ represent broken symmetries. If we consider the values of the structure coefficients as expectation values of some fields in a vacuum state of the complete theory, this is a spontaneous symmetry breaking.

In Sec. II we discuss some general properties of quantized matter fields on an arbitrary connected Lie group. In Sec. III we begin the treatment of free quantum fields on the universal covering $\tilde{\mathcal{P}}$ of the restricted Poincaré group (namely the inhomogeneous $SL(2, \mathbb{C})$ group). The subgroup of $\mathcal{F}$ that survives the symmetry breaking must preserve both the cone $\mathcal{T}^+$ and the structure constants of the Poincaré Lie algebra. It follows that it must coincide with $SL(2, \mathbb{C})$, as we have anticipated above. The unbroken symmetry group of the theory is the product of $\tilde{\mathcal{P}}$ and $SL(2, \mathbb{C})$, considered as an internal symmetry group. The free quantum fields are completely described by the two-point Wightman distributions (or vacuum expectation values, v. e. v.’s) and we give a general representation of the distributions that satisfy the appropriate symmetry, spectral and positivity conditions. In Sec. IV we treat the commutation or anticommutation properties of the free fields and we discuss the connection between spin and statistics, which is not the usual one. For instance, a “scalar” field, namely a field with only one component, has to be quantized with commutators, but it can describe particles with any spin. In Sec. V we treat the positive-mass case with arbitrary spin and we write explicitly a wide class of v. e. v.’s in terms of the matrix elements of the i. u. r.’s of $SL(2, \mathbb{C})$. In Sec. VI we consider the more delicate zero-mass case and we find theories that describe particles with an infinite helicity spectrum (not observed in nature) and particles with an arbitrary given helicity. Scalar fields that describe particles with zero mass and given nonvanishing helicity have pathological features.

Since the non trivial i. u. r.’s of $SL(2, \mathbb{C})$ are infinite-dimensional, the mass spectrum of these theories is infinitely degenerate. In order to avoid evident contradictions with the known physical phenomena, we have to require that, when the mass is within the range of presently available energies, only a finite number
of internal states of the particles can be excited by the field with appreciable probability. This happens when the parameters which label the i. u. r.’s of SL(2, C) approach the limit $M = j = 0$, $c \to 1$. Actually, in this limit the v. e. v.’s tend to the ones that define the usual scalar free field in Minkowski spacetime. This problem is less relevant when the mass is of the order of the Planck mass. The theories with a broken higher symmetry satisfy these requirements automatically.

In Sec. VII we find the differential equations satisfied by the quantum fields defined in the preceding Sections and we compare them with some field equations in a flat ten-dimensional space. In Sec. VII we introduce the concept of “spin-mass-shell” and we discuss the relation between the v. e. v.’s on $\tilde{P}$ and the corresponding distributions defined on $\mathbb{R}^{10}$. We show that not all the free fields on the flat space have a corresponding field on $\tilde{P}$.

In the remaining Sections we give some examples, in order to illustrate the general formalism. A more complete treatment will be given elsewhere. In Sec. IX we consider a theory on the flat space symmetric with respect to the group $Sp(4, \mathbb{R})$ and we build the corresponding theory on $\tilde{P}$. In this theory the higher symmetry is broken, but the v. e. v.’s maintain the higher symmetry in the short-distance limit. This is an explicit example of a new kind of broken symmetry in quantum field theory.

In Sec. X we consider an external source, represented by an accelerated disk, interacting with one of the fields defined in the present paper. We consider with more detail the field introduced in Sec. IX and we show that the number of particles emitted per unit of proper time diverges when the acceleration exceeds a finite critical value. This result shows that the formalism really contains, in some sense, the ideas that provided its motivation. Brandt has suggested that, when the acceleration of a particle approaches a critical value, the energy radiated in the form of quantum black holes diverges, preventing a larger acceleration. The formalism presented here could provide a simplified model for this process, if the particles described by our fields are interpreted as quantum black holes. This interpretation, however, raises several difficult problems.

II. Quantum Fields on a Group Manifold.

As we have anticipated in the Introduction, we consider a $n$-dimensional manifold $\mathcal{S}$ with $n$ vector fields $A_{\alpha}$ ($\alpha = 0, \ldots, n-1$) that, being linearly independent, define a basis in each tangent space of $\mathcal{S}$. As a consequence, we can identify all the tangent spaces with a single vector space $\mathcal{T}$. These vector fields describe the gravitational field and possibly other gauge fields, while matter is described by a set of fields $\psi_{\rho}$, on the manifold $\mathcal{S}^{7-10}$. We assume that the complete field equations, including gravitation, are invariant under all the diffeomorphisms of $\mathcal{S}$ and under a group $\mathcal{F}$ which acts on the matter fields and on the geometric fields according to the linear formulas (1.3), (1.4).

We consider the fields $A_{\alpha}$ as classical external fields and we restrict our considerations to symmetry transformations that leave them invariant. The transformations which have this property are implemented by unitary or anti-unitary operators acting on the Hilbert space $\mathcal{H}$ which describes the states of matter. Since
we consider only connected symmetry groups, we deal only with unitary symmetry operators. Moreover, we assume that they form a continuous representation of the symmetry group. The fields $\psi_\rho$ are operator-valued distributions on $S$ that act in a dense linear subspace $D$ of $H$. We assume that both the smeared field operators and the symmetry operators transform $D$ into itself.

Field theories on a $n$-dimensional manifold $S$ which has a symmetry group have been treated in ref. 7. Here we consider the case in which the vector fields $A_\alpha$ generate a connected $n$-dimensional Lie group $G$ of diffeomorphisms of $S$. It is convenient to assume that $G$ acts on the right in the space $S$, namely the action of the element $g \in G$ on the element $s \in S$ is written as $(g,s) \rightarrow sg$. We assume also that $G$ acts freely and transitively; it follows that if we choose an origin $s_0 \in S$, the mapping $g \rightarrow s = s_0 g$ is a diffeomorphism of $G$ onto $S$. The action of $G$ on $S$ takes the form

$$s = s_0 g \rightarrow s' = s_0 gh = sh, \quad g, h \in G,$$

namely it corresponds to a right translation of $G$. The vector fields $A_\alpha$ can also be considered as fields on $G$, that generate the right translations. They form a basis of the Lie algebra $l$ of $G$, that, as a vector space, can be identified with $T$.

The vector fields $A_\alpha$ are invariant under the left translations of $G$; it follows that a transformation of the kind

$$s = s_0 g \rightarrow s' = s_0 hg, \quad g, h \in G$$

is a symmetry transformation that leaves the geometry of $S$ invariant. It can be interpreted as a change of the origin $s_0$.

The diffeomorphisms of the kind (2.2) provide a first class of symmetry transformations. They act on the matter fields in the following way:

$$\psi_\rho(s_0 h^{-1} g) = U(h)^{-1} \psi_\rho(s_0 g) U(h), \quad g, h \in G.$$

Note that the indices of the fields are not involved: every component behaves as a scalar field. We indicate by $U(A_\alpha)$ the generators of the continuous unitary representation $U$ corresponding to the elements $A_\alpha$ of $l$. The operators $iU(A_\alpha)$ are self-adjoint and are interpreted as the energy, the momentum, the relativistic angular momentum and possibly (if $n > 10$) the charges that generate some gauge transformations. Note that, as it is expected, these operators, as well as the unitary operators $U(h)$, depend on the choice of the frame $s_0$.

The diffeomorphisms of the kind (2.1) affect the geometric fields according to the formula

$$A_\alpha \rightarrow A_\beta B^{\beta}_{\alpha}(h),$$

where $B^{\beta}_{\alpha}(h)$ is the adjoint representation of $G$. They give rise to symmetry transformations only if eq. (2.4) can be compensated by a transformation of the kind (1.4), namely if

$$B^{\beta}_{\alpha}(h) = C^{\alpha}_{\beta}(\hat{h}^{-1}), \quad h \in G, \quad \hat{h} \in F.$$
This condition defines a subgroup \( G_2 \subset G \) and we assume that \( h \to \hat{h} \) is a continuous homomorphism of \( G_2 \) onto a subgroup \( F_2 \subset F \). In the following we write \( S(h) \) instead of \( S(\hat{h}) \). Then we have

\[
S_{\rho}^\sigma(h)\psi_\sigma(sh) = V(h)^{-1}\psi_\rho(s)V(h), \quad h \in G_2.
\]

The operators \( V(h) \) commute with the operators \( U(g) \) and do not depend on the choice of \( s_0 \). They describe a kind of internal symmetry.

The internal automorphisms of \( G \), given by \( g \to hgh^{-1} \), are the product of a right and a left translation and do not require a separate treatment. The external automorphisms of \( G \) (for instance the spacetime dilatations in the Poincaré group) give rise to a new kind of symmetry if their action on the fields \( A_\alpha \) can be compensated by a transformation of the kind (1.4). This compensation is not possible in the cases we shall treat in the following Sections.

Several general features of the quantum field theories on Minkowski spacetime \(^{19,20}\) can be extended to the theories on a group manifold. We assume that there is a vacuum state \( \Omega \in \mathcal{D} \) invariant with respect to both the representations \( U \) and \( V \), and we define the v. e. v.

\[
(\Omega, \psi_\rho(s_1)\psi_\sigma(s_2)\Omega) = \mathcal{W}_{\rho\sigma}(s_1, s_2).
\]

It follows from eq. (2.3) that this quantity can be considered as a distribution on \( G \). In fact, we have

\[
\mathcal{W}_{\rho\sigma}(s_1, s_2) = \mathcal{W}_{\rho\sigma}(s_0g_1, s_0g_2) = \mathcal{W}_{\rho\sigma}(g_1^{-1}g_2).
\]

In the following we understand a fixed choice of \( s_0 \) and we write \( \psi_\rho(g) \) instead of \( \psi_\rho(s_0g) \). The v. e. v. of \( m + 1 \) fields is defined in the following way as a distribution on \( G^m \):

\[
(\Omega, \psi_{\rho_1}(g_1)\cdots\psi_{\rho_{m+1}}(g_{m+1})\Omega) = W_{\rho_1\ldots\rho_{m+1}}^{(m)}(g_1^{-1}g_2, \ldots, g_{m+1}^{-1}g_{m+1}).
\]

If the fields \( \psi_\rho(g) \) are not Hermitean, we have to consider the Hermitean conjugate as a different field and we use the notation

\[
\psi^\dagger_\rho(g) = \psi^{\dagger_\rho}(g), \quad S_{\rho_\sigma}(a) = S_{\rho_\sigma}(a).
\]

In all the formulas any index can be replaced by a barred index and vice versa, unless it is stated otherwise. If the field is Hermitean, we have to put \( \overline{\rho} = \rho \) and the representation \( S \) must be real. Then from the definition we get

\[
W_{\rho_1\ldots\rho_{m+1}}^{(m)}(g_1, \ldots, g_m) = W_{\overline{\rho}_1\ldots\overline{\rho}_{m+1}}^{(m)}(\overline{g}_1^{-1}, \ldots, \overline{g}_m^{-1})
\]

and from eq. (2.6) we get the symmetry property

\[
W_{\rho_1\ldots\rho_{m+1}}^{(m)}(g_1, \ldots, g_m) = W_{\overline{\rho}_1\ldots\overline{\rho}_{m+1}}^{(m)}(\overline{g}_1^{-1}, \ldots, \overline{g}_m^{-1})
\]

\( h \in G_2 \).
If we deal with free fields, all the v. e. v.'s can be obtained from the two-point distributions by means of the Wick theorem. In this case, the vectors of the kind

\[(2.13) \Phi(f) = \int f^\rho(g) \psi^\dagger_\rho(g) \, dg \Omega,\]

where \(f\) is a test function and \(dg\) is an invariant measure on \(G\), are dense in the Hilbert space \(H^{(1)}\) of the “one-particle” states. The square of the norm of a the vector (2.13) is given by

\[(2.14) (\Phi(f), \Phi(f)) = \int \overline{f^\rho(g_1)} f^\sigma(g_2) W_{\rho\sigma}(g_1^{-1} g_2) \, dg_1 \, dg_2 \geq 0.\]

This is the positivity condition. For interacting field theories, we have more complicated positivity conditions that involve all the other v. e. v.'s.

The symmetry operators defined in eqs. (2.3) and (2.6) act on the one-particle states in the following way:

\[(2.15) U^{(1)}(h) \Phi(f) = \Phi(f'), \quad f'^\rho(g) = f^\rho(h^{-1} g),\]

\[(2.16) V^{(1)}(h) \Phi(f) = \Phi(f'), \quad f'^\rho(g) = f^\sigma(gh) \overline{S}_{\sigma\rho}(h^{-1}).\]

Eqs. (2.14)-(2.16) permit, in the usual way, the reconstruction of the one-particle Hilbert space (as the completion of a quotient) and of the operators \(U^{(1)}(h)\) and \(V^{(1)}(h)\). If the unitary representation \(U^{(1)} \times V^{(1)}\) of \(G \times G_2\) is irreducible, we say that the free field is “elementary”.

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We consider the fields \(A_\alpha\) as classical external fields and we restrict our considerations to symmetry transformations that leave them invariant. The transformations which have this property are implemented by unitary or anti-unitary operators acting on the Hilbert space \(H\) which describes the states of matter. Since we consider only connected symmetry groups, we deal only with unitary symmetry operators\(^{17}\). Moreover, we assume that they form a continuous representation of the symmetry group\(^{18}\). The fields \(\psi_\rho\) are operator-valued distributions on \(S\) that act in a dense linear subspace \(D\) of \(H\). We assume that both the smeared field operators and the symmetry operators transform \(D\) into itself.
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namely it corresponds to a right translation of $\mathcal{G}$. The vector fields $A_\alpha$ can also be considered as fields on $\mathcal{G}$, that generate the right translations. They form a basis of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$, that, as a vector space, can be identified with $\mathcal{T}$.

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where $B^{\beta}_\alpha(h)$ is the adjoint representation of $\mathcal{G}$. They give rise to symmetry transformations only if eq. (2.4) can be compensated by a transformation of the kind (1.4), namely if

$$B^{\beta}_\alpha(h) = C_\alpha^{\beta}(\hat{h}^{-1}), \quad h \in \mathcal{G}, \quad \hat{h} \in \mathcal{F}.$$ 

This condition defines a subgroup $\mathcal{G}_2 \subset \mathcal{G}$ and we assume that $h \rightarrow \hat{h}$ is a continuous homomorphism of $\mathcal{G}_2$ onto a subgroup $\mathcal{F}_2 \subset \mathcal{F}$. In the following we write $S(h)$ instead of $S(\hat{h})$. Then we have

$$S_\rho^\sigma(h)\psi_\alpha(sh) = V(h)^{-1}\psi_\rho(s)V(h), \quad h \in \mathcal{G}_2.$$
The operators \( V(h) \) commute with the operators \( U(g) \) and do not depend on the choice of \( s_0 \). They describe a kind of internal symmetry.

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It follows from eq. (2.3) that this quantity can be considered as a distribution on \( G \). In fact, we have

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\[
(\Omega, \psi_\rho_1(g_1) \cdots \psi_\rho_{m+1}(g_{m+1})\Omega) = \mathcal{W}^{(m)}_{\rho_1 \cdots \rho_{m+1}}(g_1^{-1}g_2, \ldots, g_m^{-1}g_{m+1}).
\]

If the fields \( \psi_\rho(g) \) are not Hermitean, we have to consider the Hermitean conjugate as a different field and we use the notation

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\psi_\rho^\dagger(g) = \psi_{\overline{\rho}}(g), \quad S_{\rho}^\sigma(a) = S^{\overline{\rho}}_{\sigma}(a).
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In all the formulas any index can be replaced by a barred index and vice versa, unless it is stated otherwise. If the field is Hermitean, we have to put \( \overline{\rho} = \rho \) and the representation \( S \) must be real. Then from the definition we get

\[
\mathcal{W}^{(m)}_{\rho_1 \cdots \rho_{m+1}}(g_1, \ldots, g_m) = \mathcal{W}^{(m)}_{\overline{\rho}_1 \cdots \overline{\rho}_{m+1}}(g_1^{-1}, \ldots, g_m^{-1})
\]

and from eq. (2.6) we get the symmetry property

\[
S_{\rho_1}^{\overline{\sigma}_1}(h) \cdots S_{\rho_{m+1}}^{\overline{\sigma}_{m+1}}(h) \mathcal{W}^{(m)}_{\sigma_1 \cdots \sigma_{m+1}}(h^{-1}g_1h, \ldots, h^{-1}g_mh) = \mathcal{W}^{(m)}_{\rho_1 \cdots \rho_{m+1}}(g_1, \ldots, g_m), \quad h \in \mathcal{G}_2.
\]

If we deal with free fields, all the v. e. v. ’s can be obtained from the two-point distributions by means of the Wick theorem. In this case, the vectors of the kind

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\Phi(f) = \int f^\rho(g)\psi_\rho^\dagger(g) \, dg \, \Omega,
\]

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where \( f \) is a test function and \( dg \) is an invariant measure on \( \mathcal{G} \), are dense in the Hilbert space \( \mathcal{H}^{(1)} \) of the “one-particle” states. The square of the norm of a the vector (2.13) is given by

\[
(\Phi(f), \Phi(f)) = \int \overline{f}(g_1) f(g_2) W_{\rho \sigma} (g_1^{-1} g_2) \, dg_1 \, dg_2 \geq 0.
\]

This is the positivity condition. For interacting field theories, we have more complicated positivity conditions that involve all the other v. e. v.’s.

The symmetry operators defined in eqs. (2.3) and (2.6) act on the one-particle states in the following way:

\[
U^{(1)}(h) \Phi(f) = \Phi(f'), \quad f'_{\rho}(g) = f_{\rho}(h^{-1}g),
\]

\[
V^{(1)}(h) \Phi(f) = \Phi(f'), \quad f'_{\rho}(g) = f_{\sigma}(gh) S_{\rho \sigma}(h^{-1}).
\]

Eqs. (2.14)-(2.16) permit, in the usual way, the reconstruction of the one-particle Hilbert space (as the completion of a quotient) and of the operators \( U^{(1)}(h) \) and \( V^{(1)}(h) \). If the unitary representation \( U^{(1)} \times V^{(1)} \) of \( \mathcal{G} \times \mathcal{G}_2 \) is irreducible, we say that the free field is “elementary”.

### III. Free Fields on the Poincaré Group.

Now we apply the general results of the preceding Section to the Poincaré group. We consider the ten-dimensional manifold \( \mathcal{S} \) of all the Lorentz reference frames in the Minkowski space \( \mathcal{M} \) which are left-handed and future-directed. If we fix a reference frame \( s_0 \), for each reference frame \( s \) there is one and only one element of the proper orthochronous Poincaré group \( \mathcal{P} \) that transforms \( s_0 \) into \( s \). Then we can identify the space \( \mathcal{S} \) with \( \mathcal{P} \) and consider quantum fields defined on it. Actually, in order to treat fields with half-integral spin, it is convenient to use the universal covering \( \tilde{\mathcal{P}} \) of \( \mathcal{P} \), namely the semidirect product of the four-dimensional translation group \( \mathbb{R}^4 \) and \( SL(2, \mathbb{C}) \). For the elements of this group and for their multiplication rule, we use the standard notation

\[
(x, a)(y, b) = (x + \Lambda(a)y, ab), \quad x, y \in \mathbb{R}^4, \quad a, b \in SL(2, \mathbb{C}),
\]

where \( \Lambda(a) \) is the \( 4 \times 4 \) Lorentz matrix corresponding to the element \( a \in SL(2, \mathbb{C}) \). For the scalar product of two four-vectors we use the notation

\[
g_{ik} x^i y^k = x \cdot y = -x^0 y^0 + x \cdot y, \quad x \cdot y = x^1 y^1 + x^2 y^2 + x^3 y^3.
\]

The one-parameter subgroups of \( SL(2, \mathbb{C}) \) that correspond to rotations around the axes and to pure Lorentz transformations along the axes are written as

\[
u_k(\theta) = \exp(-\frac{1}{2} i \theta \sigma_k), \quad a_k(\zeta) = \exp(\frac{1}{2} \zeta \sigma_k),
\]
where \( \sigma_k \) are the Pauli matrices.

We assume that the group \( \mathcal{F} \) contains \( SL(2, \mathbb{C}) \) and that it preserves the cone \( T^+ \) defined in refs. 2, 3. It is easy to see that the translations (acting on \( T \) by means of the adjoint representation of \( \mathcal{P} \)) and the spacetime dilatations (which are external automorphisms of \( \mathcal{P} \) and of the corresponding Lie algebra) do not preserve the cone \( T^+ \). It follows that we have to put \( \mathcal{G}_2 = SL(2, \mathbb{C}) \). Then the general equations (2.3) and (2.6) take the form

\[
\psi_\rho((y,b)^{-1}(x,a)) = \psi_\rho(\Lambda(b)^{-1}(x-y),b^{-1}a) = U(y,b)^{-1}\psi_\rho(x,a)U(y,b),
\]

\[
S_\rho^\sigma(b)\psi_\sigma(x,ab) = V(b)^{-1}\psi_\rho(x,a)V(b).
\]

If \( V(b) = 1 \), from eq. (3.5) we have

\[
\psi_\rho(x,a) = S_\rho^\sigma(a^{-1})\psi_\sigma(x,1)
\]

and we are dealing with a field \( \psi_\sigma(x) = \psi_\sigma(x,1) \) defined on the Minkowski space-time. From eq. (3.4), we get the usual covariance property

\[
S_\rho^\sigma(b)\psi_\sigma(\Lambda(b)^{-1}(x-y)) = U(y,b)^{-1}\psi_\rho(x)U(y,b).
\]

In the following we consider the case in which \( V(b) \) is not trivial.

A free field theory is completely described by the two-point v. e. v.’s

\[
(\Omega, \psi_\rho(x,a)\psi_\sigma(y,b)\Omega) = W_{\rho\sigma}((x,a)^{-1}(y,b)) = W_{\rho\sigma}(\Lambda(a^{-1})(y-x),a^{-1}b).
\]

From the covariance property (3.5) we get the formula

\[
W_{\rho\sigma}(\Lambda(b)x,bab^{-1}) = S_\rho^\mu(b)S_\sigma^\nu(b)W_{\mu\nu}(x,a)
\]

(remember our conventions about the introduction of barred indices). If the field is not Hermitean, we assume

\[
W_{\rho\sigma}(x,a) = W_{\rho\sigma}(x,a) = 0
\]

(here the bars over the indices cannot be modified) and the distributions \( W_{\rho\sigma}(x,a) \) and \( W_{\sigma\rho}(x,a) \) can be treated independently unless we introduce some requirement of local commutativity (see the next Section).

The states of the form

\[
\Phi(f) = \int f^\rho(x,a)\psi_\rho^\dagger(x,a)\,d^4x\,d^6a\,\Omega,
\]

where \( f \) is an arbitrary test function with compact support and \( d^6a \) is an invariant measure on \( SL(2, \mathbb{C}) \), form a dense set in the Hilbert subspace \( \mathcal{H}^{(1)} \) of the one-particle states. Their norm is given by the formula

\[
(\Phi(f), \Phi(f)) = \int \overline{f^\rho(x,a)}f^\sigma(y,b)W_{\rho\sigma}(\Lambda(a^{-1})(y-x),a^{-1}b)\,d^4x\,d^6a\,d^4y\,d^6b \geq 0,
\]

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that also gives the positivity property of the two-point distribution.

We want to find the solutions of the conditions (3.9) and (3.12) that describe elementary free fields. More general solutions can be obtained by means of sums or integrals of these solutions. If we introduce the Fourier transformations

\[ W_{\rho\sigma}(x, a) = \int \exp(-ik \cdot x) \tilde{W}_{\rho\sigma}(k, a) d^4k, \]

\[ \tilde{f}^\rho(k, a) = \int \exp(-ik \cdot x) f^\rho(x, a) d^4x, \]

eq. (3.12) takes the form

\[ (\Phi(f), \Phi(f)) = \int \tilde{f}^\rho(k, a) \tilde{f}^\sigma(k, b) \tilde{W}_{\rho\sigma}(\Lambda(a^{-1})k, a^{-1}b) d^6a d^6b d^4k \geq 0 \]

and eq. (3.9) gives

\[ \tilde{W}_{\rho\sigma}(\Lambda(b)k, bab^{-1}) = S_\rho^\mu(b) \tilde{S}_\sigma^\nu(b) \tilde{W}_{\mu\nu}(k, a). \]

We assume that the distribution \( \tilde{W}_{\rho\sigma}(k, a) \) vanishes if \( k \) does not belong to the future cone (spectral condition). Since the elementary fields have a definite mass \( \mu \), this distribution has support on the orbit defined by

\[ k \cdot k = -\mu^2, \quad k^0 > 0 \]

and has the form

\[ \tilde{W}_{\rho\sigma}(k, a) = (2\pi)^{-3} w_{\rho\sigma}(k, a) \theta(k^0) \delta(k \cdot k + \mu^2), \]

where \( \theta \) is the step function.

Following a procedure introduced by Wigner\(^{21}\), we choose a representative element on each orbit

\[ \hat{k} = (\mu, 0, 0, 0), \quad \mu > 0, \]

\[ \hat{k} = (1, 0, 0, 1), \quad \mu = 0 \]

and for each value of the four-momentum \( k \) on the orbit, we choose an element \( a_k \in SL(2, \mathbb{C}) \) with the property

\[ k = \Lambda(a_k)\hat{k}. \]

Then we see from eq. (3.16) that we can put

\[ w_{\rho\sigma}(k, a) = S_\rho^\mu(a_k) \tilde{S}_\sigma^\nu(a^{-1}a_k) w_{\mu\nu}(a_k^{-1}a_k), \]

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where

\[ (3.23) \quad w_{\rho \sigma}(a) = S_{\sigma} \nu(a) w_{\rho \sigma}(\hat{k}, a). \]

In this way we obtain the integral representation

\[ (3.24) \quad W_{\rho \sigma}(x, a) = \]

\[ = (2\pi)^{-3} \int \exp(-ik \cdot x) S_{\rho}^\mu(a_k) S_{\sigma} \nu(a^{-1} a_k) w_{\rho \sigma}(a_k^{-1} \cdot a_k \theta(k^0) \delta(k \cdot k + \mu^2) d^4 k. \]

The positivity condition (3.15) takes the form

\[ (3.25) \quad (\Phi(f), \Phi(f)) = (2\pi)^{-3} \int \tilde{f}^\rho(k, a_k b) \times \]

\[ \times S_{\rho}^\mu(a^{-1}) S_{\sigma} \nu(b^{-1}) w_{\rho \sigma}(ba^{-1}) \theta(k^0) \delta(k \cdot k + \mu^2) d^6 a d^6 b d^4 k \geq 0, \]

which is equivalent to the simpler condition

\[ (3.26) \quad \int \tilde{f}^\rho(a) f^\sigma(b) w_{\rho \sigma}(ba^{-1}) d^6 a d^6 b \geq 0. \]

The little group \( K \) corresponding to the representative element \( \hat{k} \) is defined by the condition

\[ (3.27) \quad \Lambda(u) \hat{k} = \hat{k}, \quad u \in K \subset SL(2, C). \]

For \( \mu > 0 \) we have \( K = SU(2) \), the universal covering of the rotation group \( SO(3) \). For \( \mu = 0 \), we have \( K = \tilde{E}(2) \), a double covering of the Euclidean group \( E(2) \). From eq. (3.16) we get the condition

\[ (3.28) \quad w_{\rho \sigma}(uau^{-1}) = S_{\rho}^\mu(u) S_{\sigma} \nu(u) w_{\rho \sigma}(a), \quad u \in K. \]

From eq. (2.11) we also get the property

\[ (3.29) \quad \overline{W}_{\rho \sigma}(x, a) = W_{\sigma \rho}((x, a)^{-1}). \]

An equivalent condition is

\[ (3.30) \quad \overline{w}_{\rho \sigma}(a) = w_{\sigma \rho}(a^{-1}), \]

which is a consequence of eq. (3.26).

Our problem is to find solutions of the conditions (3.26) and (3.28). Then we have to verify that the product of distributions that appears in eq. (3.24) is meaningful.

If the theory can be interpreted as a theory in Minkowski spacetime, namely if \( V(b) = 1 \), the v. e. v.’s satisfy the additional symmetry property

\[ (3.31) \quad W_{\rho \sigma}(x, ab^{-1}) = S_{\sigma} \nu(b) W_{\rho \sigma}(x, a). \]

An equivalent condition is to require that the function \( w_{\rho \sigma}(a) \) defined by eq. (3.23) does not depend on \( a \).
IV. Local Commutativity.

The study of the distribution $W_{\rho\sigma}(x,a)$ can be simplified if we find an element $b$ in such a way that

\begin{equation}
(4.1) \quad a = bu_3(\alpha)a_3(\beta)b^{-1}, \quad -2\pi < \alpha \leq 2\pi, \quad \beta \geq 0.
\end{equation}

This is possible outside the four-dimensional submanifold of $SL(2,\mathbb{C})$ where the eigenvalues of $a$ are equal. Since these eigenvalues are given by $\exp(\pm \frac{1}{2}(\beta - i\alpha))$, we have to avoid the “singular” points $\alpha = 0, 2\pi, \beta = 0$, namely we have to impose the condition

\begin{equation}
(4.2) \quad \cosh \beta - \cos \alpha = 2\sinh(\frac{1}{2}(\beta - i\alpha)) > 0.
\end{equation}

We also introduce the quantities

\begin{equation}
(4.3) \quad \rho^2 = (\cosh \beta - \cos \alpha)^{-1}(\cosh \beta x \cdot x - x \cdot \Lambda(a)x),
\end{equation}

\begin{equation}
(4.4) \quad \sigma^2 = (\cosh \beta - \cos \alpha)^{-1}(\cos \alpha x \cdot x - x \cdot \Lambda(a)x),
\end{equation}

which, when $a = u_3(\alpha)a_3(\beta)$, take the form

\begin{equation}
(4.5) \quad \rho^2 = (x^1)^2 + (x^2)^2, \quad \sigma^2 = (x^0)^2 - (x^3)^2.
\end{equation}

In order to simplify the formalism, we assume that $W_{\rho\sigma}(x,a)$ is a tempered distribution in $x$ that depends continuously on $a$. This is true for the massive free fields described in the next Section. From eqs. (3.9) and (4.1), we see that in the open set defined by eq. (4.2) we have

\begin{equation}
(4.6) \quad W_{\rho\sigma}(x,a) = S_{\rho}^{\mu}(b)\overline{S}_{\sigma}^{\nu}(b)W_{\mu\sigma}(\Lambda(b^{-1})x, u_3(\alpha)a_3(\beta)).
\end{equation}

The spectral condition implies, as in the Minkowskian field theory, that the distribution $W_{\rho\sigma}(x,u_3(\alpha)a_3(\beta))$ for fixed values of $\alpha$ and $\beta$ is the boundary value of an analytic function of $x$ defined in the tube $\text{Im} \, x \in V_+$, where $V_+$ is the open future cone. From eq. (3.9) we also obtain

\begin{equation}
(4.7) \quad W_{\rho\sigma}(\Lambda(u_3(\psi)a_3(\xi))x, u_3(\alpha)a_3(\beta)) =
\end{equation}

\begin{equation}
\quad = S_{\rho}^{\mu}(u_3(\psi)a_3(\xi))S_{\sigma}^{\nu}(u_3(\psi)a_3(\xi))W_{\mu\sigma}(x, u_3(\alpha)a_3(\beta)).
\end{equation}

If we fix the variables $\alpha, \beta$, it is a simple application of the Bargmann Hall Wightman theorem\cite{19,20} to find by means of eq. (4.7) an analytic continuation of the v. e. v. (Wightman function) which is covariant with respect to the complex two-dimensional Lorentz group acting on the coordinates $x^0, x^3$ and to the real rotations acting on the coordinates $x^1, x^2$. The real points which satisfy the condition $\sigma^2 < 0$ belong to the analyticity domain.
The universal covering of the proper complex Lorentz group is $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. We indicate its elements by the notation $(a, b)$; the real Lorentz transformations correspond to the elements of the kind $(a, \overline{a})$. If $S(a)$ is the irreducible spinor representation $S^{(s, s')}(a) = S^{(s', 0)}(a) \otimes S^{(s', 0)}(\overline{a})$, its analytic continuation is given by $S^{(s, s')}(a, b) = S^{(s, 0)}(a) \otimes S^{(s', 0)}(b)$. The universal covering of the proper complex Lorentz group is a direct sum of irreducible spinor representations all with the same value of $|s|$. From eq. (4.7) we see that if $(\sigma, x)$ is a distribution given by $\exp(-\frac{1}{2}i\pi\sigma x)$, then its analytic continuation is equivalent to $S^{(s, s')}(a, b)$. We consider the following product of a complex Lorentz transformation acting on $x^0, x^3$ and a real rotation acting on $x^1, x^2$:

\begin{equation}
\text{(4.8)} \quad \exp(-\frac{1}{2}i\pi\sigma x), \exp(-\frac{1}{2}i\pi\sigma x)) \exp(-\frac{1}{2}i\pi\sigma x), \exp(\frac{1}{2}i\pi\sigma x)) = (-1, 1),
\end{equation}

\begin{equation}
\text{(4.9)} \quad \Lambda(-1, 1) = -1, \quad S^{(s, s')}(1, 1) = (-1)^s.
\end{equation}

Then, from the covariance property of the Wightman function we obtain

\begin{equation}
\text{(4.10)} \quad W_{\rho\overline{\sigma}}(x, u_3(\alpha)a_3(\beta)) = (-1)^{s+s'}W_{\rho\overline{\sigma}}(x, u_3(\alpha)a_3(\beta))
\end{equation}

and the same equality holds for real $x$ if $\sigma^2 < 0$. The same result is valid if $S$ is a direct sum of irreducible spinor representations all with the same value of $(-1)^{s+s'}$. If we use eq. (4.6) and the general expression (4.4) for $\sigma^2$, we see that

\begin{equation}
\text{(4.11)} \quad W_{\rho\overline{\sigma}}(x, a) = (-1)^{s+s'}W_{\rho\overline{\sigma}}(x, a)
\end{equation}

in the open set $\mathcal{C} \subset \tilde{\mathcal{P}}$ defined by the condition

\begin{equation}
\text{(4.12)} \quad \cos \alpha \cdot x \cdot x \cdot \Lambda(a)x < 0.
\end{equation}

One can show that this condition implies eq. (4.2).

If we consider a scalar field and we introduce the variables

\begin{equation}
\text{(4.13)} \quad x = (\epsilon\sigma \cosh \xi', \rho \cos \psi', \rho \sin \psi', \epsilon\sigma \sinh \xi'), \quad \sigma > 0,
\end{equation}

\begin{equation}
\text{(4.14)} \quad x = (\epsilon|\sigma| \sinh \xi', \rho \cos \psi', \rho \sin \psi', \epsilon|\sigma| \cosh \xi'), \quad \sigma^2 < 0, \quad \epsilon = \pm 1,
\end{equation}

eq (4.7) shows that $W(x, u_3(\alpha)a_3(\beta))$ does not depend on the variables $\psi'$ and $\xi'$. From eq. (4.10) we see that if $\sigma^2 < 0$ it does not even depend on $\epsilon$. In conclusion, for $\sigma^2 < 0$, $W$ can be considered as a distribution in the variables $\alpha, \beta, x^1, x^2, |\sigma|$ invariant with respect to rotations acting on $x^1, x^2$. In particular, since $a$ and $a^{-1}$ have the same eigenvalues, from eqs. (4.3) and (4.4) we obtain

\begin{equation}
\text{(4.15)} \quad W(x, a) = W(-x, a) = W(-x, a^{-1}), \quad (x, a) \in \mathcal{C}.
\end{equation}

The commutator or the anticommutator of a free field is a numerical distribution given by

\begin{equation}
\text{(4.16)} \quad [\psi_\rho(x, a), \psi_{\sigma}^\dagger(y, b)]_{\pm} = W_{\rho\overline{\sigma}}((x, a)^{-1}(y, b)) \pm W_{\overline{\rho}\sigma}((y, b)^{-1}(x, a)) = W_{\rho\overline{\sigma}}((x, a)^{-1}(y, b)) \pm \overline{W_{\overline{\rho}\sigma}}((x, a)^{-1}(y, b)).
\end{equation}
If we consider a Hermitean scalar field, and we use eq. (3.9), we have the simpler relation

\[(4.16) \quad [\psi(x,a), \psi(y,b)]_\pm = W(x - y, ba^{-1}) - W(y - x, ab^{-1}) =
\]

\[= 2i \text{Im} W((x - y, ba^{-1})
\]

and from eq. (4.14) we get

\[(4.17) \quad [\psi(x,a), \psi(y,b)]_\pm = 0 \quad \text{for} \quad (x,a)^{-1}(y,b) \in C.
\]

We see that in the formulation of local commutativity, eq. (4.12) is the analog of the inequality \(x \cdot x > 0\) in Minkowski field theory.

In the general case, if we impose a local (anti)commutativity condition of the kind

\[(4.18) \quad [\psi_\rho(x,a), \psi_\sigma^+(y,b)]_\pm = 0 \quad \text{for} \quad (x,a)^{-1}(y,b) \in C,
\]

from eqs. (4.11) and (4.15) we get

\[(4.19) \quad W_{\rho\sigma}(x,a) = \mp (1)^{s+s'} W_{\rho\sigma}(x,a), \quad (x,a) \in C.
\]

This relation can be continued analytically in the tube Im \(x \in V_+\), and, as an equality of distributions, it holds for any real \(x\). It is compatible with the spectral and covariance conditions, but it satisfies the positivity condition only if \(\mp (1)^{s+s'} = 1\). In conclusion, the local (anti)commutativity relation (4.18) is satisfied if we put

\[(4.20) \quad W_{\rho\sigma}(x,a) = W_{\rho\sigma}(x,a),
\]

and the statistics is determined by \((-1)^{s+s'}\). We shall see that \(s+s'\) is not directly related to the spin of the particles described by the field and the usual relation between spin and statistics is not necessarily valid. From eq. (3.24) we see that eq. (4.20) is equivalent to the condition

\[(4.21) \quad w_{\rho\sigma}(a) = \overline{w}_{\rho\sigma}(a).
\]

For a Hermitean field this condition requires that \(w_{\rho\sigma}(a)\) is real.

It is interesting to study the integral (3.24) in the scalar case with more detail. If \(w(a) = 1\), eq. (3.24) gives the usual v. e. v. for the Minkowskian free scalar field, namely

\[(4.22) \quad W(x) = i(4\pi)^{-1}\epsilon(x^0)\delta(s^2) + (2\pi)^{-2}\theta(-s^2)\mu|s|^{-1}K_1(\mu|s|) +
\]

\[+ (8\pi)^{-1}\theta(s^2)\mu s^{-1} \left(Y_1(\mu s) - i\epsilon(x^0)J_1(\mu s)\right), \quad s^2 = -x \cdot x = \sigma^2 - \rho^2,
\]

where \(\epsilon(x^0)\) is the sign of \(x^0\). The corresponding commutator vanishes for \(\sigma^2 < \rho^2\).
If the function \( w(a_k^{-1}aa_k) \) decreases for large \( k \) the distribution \( W(x,a) \) is less singular. In order to simplify the integral (3.24), it is convenient to write the four-vector \( k \) in the following way

\[
(4.23) \quad k = (q \cosh \xi, p \cos \psi, p \sin \psi, q \sinh \xi), \quad q = (p^2 + \mu^2)^{\frac{1}{2}},
\]

\[
0 \leq p < \infty, \quad -\infty < \xi < \infty, \quad 0 \leq \psi < 2\pi,
\]

\[
(2k^0)^{-1} d^3k = \frac{1}{2} p dp d\xi d\psi
\]

and to choose

\[
(4.24) \quad a_k = u_3(\psi)a_3(\xi)a_\eta.
\]

For \( \mu > 0 \) we take

\[
(4.25) \quad a_\eta = a_1(\eta), \quad p = \mu \sinh \eta, \quad q = \mu \cosh \eta, \quad 0 \leq \eta < \infty
\]

and for \( \mu = 0 \)

\[
(4.26) \quad a_\eta = a_1(\eta)u_2(\frac{1}{2}\pi), \quad p = q = \exp \eta, \quad -\infty < \eta < \infty.
\]

From eq. (4.13) we get

\[
(4.27) \quad k \cdot x = -\epsilon q \sigma \cosh(\xi - \xi') + pp \cos(\psi - \psi'), \quad \sigma^2 > 0,
\]

\[
(4.28) \quad k \cdot x = \epsilon q |\sigma| \sinh(\xi - \xi') + pp \cos(\psi - \psi'), \quad \sigma^2 < 0.
\]

The integrations over the variables \( \xi \) and \( \psi \) in eq. (3.24) can be performed in terms of Bessel functions and we get the following formula

\[
(4.28) \quad W(x,a) = (2\pi)^{-1} \int_0^\infty \Delta(q\sigma, \epsilon(x^0))J_0(pp)w(\alpha, \beta, p)p dp,
\]

where

\[
(4.29) \quad \Delta(q\sigma, \epsilon(x^0)) = (2\pi)^{-1}\theta(-\sigma^2)K_0(q|\sigma|) - \frac{1}{8}\theta(\sigma^2) \left( Y_0(q\sigma) - \epsilon(x^0)J_0(q\sigma) \right)
\]

and

\[
(4.30) \quad w(\alpha, \beta, p) = w(a_\eta^{-1}u_3(\alpha)a_3(\beta)a_\eta).
\]

The commutator of a scalar Hermitean field follows from eqs. (4.16) and (4.28) and has the form

\[
(4.31) \quad 2i \text{Im} W(x,a) = i(4\pi)^{-1}\theta(\sigma^2)\epsilon(x^0) \int_0^\infty J_0(q\sigma)J_0(pp)w(\alpha, \beta, p)p dp.
\]
We see that it vanishes for $\sigma^2 < 0$. Eq. (4.31) can be considered as a Hankel transformation. By considering the corresponding inverse transformation, one can easily see that a necessary condition to have a commutator vanishing for large $\rho$ is that $w(\alpha, \beta, p)$ is an entire analytic function of $p^2$. This does not happen for the elementary fields considered in the following Sections.

When $w(\alpha, \beta, p)$ is an even function of $p$, it is useful to rewrite eq. (4.31) as an integral in the complex $p$ plane

\begin{equation}
2i \text{Im} W(x, a) = i2^{-3}\pi^{-1}\theta(\sigma^2)\epsilon(x^0) \int_C J_0(q\sigma)H_0^{(1)}(pp)w(\alpha, \beta, p)p\,dp,
\end{equation}

where the integration path $C$ lies just above the real axis. For $|p| \to \infty$, $\text{Im} p > 0$, we have

\begin{equation}
pJ_0(q\sigma)H_0^{(1)}(pp) \simeq \pi^{-1}(\rho\sigma)^{-\frac{1}{2}}\exp(ip(\rho - \sigma)).
\end{equation}

If $w = 1$ and $\rho > \sigma$, we can close the integration path at infinity in the upper half plane and the integral vanishes in accord with eq. (4.22). Actually, the integral (4.32) is meaningful only in the sense of distribution theory; in order to deal with a convergent integral, one can multiply the integrand by $(p + i)^{-\nu}$ and take the limit $\nu \to 0$ at the end. If $w$ has singularities in the upper half plane, one has to take into account their contributions.

V. Positive-Mass Free Fields.

In the positive-mass case, we can find the function $w_{\rho\sigma}(a)$ by exploiting the positivity properties of the matrix elements $D_{j mj' m'}^{Mc}(a)$ of the i. u. r.’s of $SL(2, \mathbb{C})$ described in refs. 22-26. The parameter $c$ is the same as in ref. 22 and is called $\lambda$ in ref. 24. The parameter $M$ is the same as in ref. 24 and corresponds to the parameter $-\frac{1}{2}m = \pm k_0$ of ref. 22. For the i. u. r.’s of the principal series, $c$ lies on the imaginary axis and $M$ is an integral or half-integral number. For the i. u. r.’s of the supplementary series, $M = 0$ and $-1 < c < 1$. The representations $D_{j mj' m'}^{Mc}$ and $D_{-M, -c}$ are unitarily equivalent. The indices $j, j'$ take the values $|M|, |M| + 1, \ldots$ and, if we indicate by $R_{mn'}^{j}(u)$ the $(2j + 1)$-dimensional representation of $SU(2)$, we have

\begin{equation}
D_{j mj' m'}^{Mc}(u) = \delta_{jj'}R_{mn'}^{j}(u), \quad u \in SU(2).
\end{equation}

Then we put

\begin{equation}
w_{\rho\sigma}(a) = (2s + 1)(2J + 1)^{-1}\sum_{mm'} C_{js}(J, m; n', \rho)C_{js}(J, m; n, \sigma)D_{j mj' m'}^{Mc}(a),
\end{equation}

where $C$ indicates the Clebsch-Gordan coefficients. The parameters $M, c, j, J, s$ characterize the theory; they are fixed and no sum over them is understood. After some calculation we have

\begin{equation}
w_{\rho\sigma}(uau^{-1}) = R_{\rho \mu}^{s}(u)R_{\sigma \nu}^{s}(u)w_{\mu\nu}(a)
\end{equation}
and we see that eq. (3.28) is satisfied if

\begin{equation}
S_\rho^\sigma(u) = R_{\rho\sigma}^*(u), \quad u \in SU(2).
\end{equation}

If we indicate by \( S^{(s,s')} \) an irreducible spinor representation, we can put, with an appropriate choice of the basis,

\begin{equation}
S(a) = S^{(s,0)}(a) \quad \text{or} \quad S(a) = S^{(0,s)}(a).
\end{equation}

In order to prove the positivity property, we substitute eq. (5.2) into eq. (3.25) and we get the positive expression

\begin{equation}
(\Phi(f), \Phi(f)) = \int \sum_{m,j,m'} |f_{m,j'm'}(k)|^2 \theta(k^0) \delta(k \cdot k + \mu^2) d^4k,
\end{equation}

where

\begin{equation}
f_{m,j'm'}(k) = (2\pi)^{-\frac{3}{2}} (2s + 1)^{\frac{1}{2}} (2J + 1)^{-\frac{1}{2}} \times \\
\times C_{j\rho}(J, m; n, \mu) \int \tilde{f}^\rho(k, a_k) \overline{S}_\rho^\mu(a^{-1}) D^{M\epsilon}_{j\rho m'j'm'}(a) d^6a.
\end{equation}

It is clear that the quantity (5.7) is the wave function in momentum space of the one-particle state \( \Phi(f) \) and that its indices represent the spin and the internal quantum numbers.

Eq. (2.15) takes the form

\begin{equation}
U(1)(y, b)\Phi(f) = \Phi(f'), \quad f'^\rho(x, a) = f^\rho(\Lambda(b^{-1})(x - y), b^{-1}a).
\end{equation}

The Fourier transformation (3.14) gives

\begin{equation}
\tilde{f}^\rho(k, a) = \exp(-ik \cdot y) \tilde{f}^\rho(k', b^{-1}a), \quad k' = \Lambda(b^{-1})k
\end{equation}

and from eq. (5.7) we obtain

\begin{equation}
f'_{m,j'm'}(k) = \exp(-ik \cdot y) R^J_{m'\rho}(a_k^{-1}b_k) f_{m', j'm'}(k'),
\end{equation}

\begin{equation}
a_k^{-1}b_k \in SU(2).
\end{equation}

We see that the wave function transforms according to the i. u. r. with mass \( \mu > 0 \) and spin \( J \) defined by Wigner. Eq. (5.7) shows that spin has a double origin, namely the dependence of \( f^\rho(x, a) \) on the index \( \rho \) and on the group element \( a \). From eqs. (3.24) and (5.2) we have

\begin{equation}
W_{\rho\sigma}(x, -a) = (-1)^{2J} W_{\rho\sigma}(x, a)
\end{equation}
and a similar formula holds for the field $\psi_{\rho}(x,a)$, which is a one or two-valued function on $P$ if $2J$ is, respectively, even or odd.

Eq. (2.16) takes the form

$$V^{(1)}(b)\Phi(f) = \Phi(f'), \quad f'^\rho(x,a) = f^\sigma(x,ab)\overline{S}_\sigma^\rho(b^{-1}),$$

and from eqs. (3.14) and (5.7) we get

$$f_{mj}'m'(k) = f_{mj}''m''(k)D_{j'm'j''m''}(b^{-1}) = \overline{D}_{j''m''j'm'}^M(b)f_{mj}''m''(k).$$

We see that the wave function transforms according to the i. u. r. $D_{j'm'm''}^M$ of the internal symmetry group $SL(2,\mathbb{C})$. This i. u. r. is equivalent to $D_{j'm'm''}^M$.

The general formalism described above becomes simpler in some special cases. If $s = 0$, $S(a) = 1$, we have a scalar field. It is $J = j$ and eq. (5.2) takes the form

$$w(a) = w^{Mcj}(a) = (2j + 1)^{-1}\sum_mD_{jmjm}(a).$$

If $j = M = 0$, we have $J = s$ and

$$w_{\rho\sigma}(a) = \delta_{\rho\sigma}D_{0000}^0(a) = \delta_{\rho\sigma}w^{0c0}(a).$$

In order to give more explicit expressions for the functions $w^{Mcj}(a)$, we write the i. u. r.’s of $SU(2)$ and of $SL(2,\mathbb{C})$ in the form

$$R_{mm'}(u_3(\phi)u_2(\theta)u_3(\psi)) = \exp(-im\phi)r_{mm'}^{j}(\theta)\exp(-im'\psi),$$

$$D_{j'mjm'}^M(ua_3(\zeta)u') = \sum_{m''}R_{mm''}^{j}(u)d_{m''jm'}^{Mc}(\zeta)R_{m'm'}^{j'}(u').$$

Then the function defined by eq. (5.14) can be written as

$$w^{Mcj}(ua_3(\zeta)u') = w^{Mcj}(a_3(\zeta)u_3(\phi + \psi)u_2(\theta)) =$$

$$= (2j + 1)^{-1}\sum_mD_{mjm}(\zeta)r_{mm}(\theta)\exp(-im(\phi + \psi)), \quad u'u = u_3(\phi)u_2(\theta)u_3(\psi).$$

The quantities $d_{mjm}(\zeta)$ are given in refs. 23, 24 in terms of elementary functions. It follows that the expression (5.18) too is a combination of elementary functions. For instance, we have

$$w^{0c0}(ua_3(\zeta)u') = d_{000}^{0c}(\zeta) = (c\sinh(\zeta))^{-1}\sinh(c\zeta),$$

but the expressions become more and more complicated when the parameters $M$ and $j$ increase. An useful integral representation of $w^{Mcj}(a)$ is given in the
Appendix A, where one derives also a simple approximate expression valid for $a \to 1$.

Note that we have

\begin{equation}
\lim_{c \to 1} w^{0c0}(a) = \lim_{c \to 1} D^{0c}_{0000}(a) = 1.
\end{equation}

In this limit the distribution $w_{\rho \sigma}(a)$ given by eq. (5.15) becomes independent of $a$ and we have a theory which can be defined in Minkowski spacetime. From the unitarity condition we also have

\begin{equation}
\lim_{c \to 1} D^{0c}_{00jm}(a) = \lim_{c \to 1} D^{0c}_{jm00}(a) = 0, \quad j > 0,
\end{equation}

and if $j = M = 0$ the components of the wave function (5.7) tend to zero for $j' > 0$. We conclude that the theories with $M = j = 0$ can possibly describe small deviations from the known physical theories. For other values of the parameters, we get theories which can only apply to the unknown physics of masses and energies beyond the Planck mass. From the mathematical point of view, eqs. (5.20) and (5.21) show that in the limit $c \to 1$, the representation $D^{0c}$ becomes reducible and we have

\begin{equation}
\lim_{c \to 1} D^{0c} = 1 \oplus D^{1,0}.
\end{equation}

\[\]

We see from eq. (3.18) that $\tilde{W}_{\rho \sigma}(k, a)$ can be considered as a tempered distribution in the variables $k$ that depends continuously on the group element $a$ and a similar statement holds for its Fourier transform $W(x, a)$. If $a = 1$, eq. (5.2) gives

\begin{equation}
w_{\rho \sigma}(1) = \delta_{\rho \sigma}
\end{equation}

and from eq. (3.24) we see that $W_{\rho \sigma}(x, 1)$ is the v. e. v. of a field in Minkowski space, given, in the scalar case, by eq. (4.22). If $a \neq 1$, the function $w_{\rho \sigma}(a^{-1}aa_k)$ decreases for large $k$ and the distribution $W_{\rho \sigma}(x, a)$ is less singular.

For a scalar field we can use eq. (4.28) and from eq. (4.30) we obtain

\begin{equation}
w(\alpha, \beta, p) = w(a_3(\zeta)u_3(\phi)u_2(\theta)),
\end{equation}

where

\begin{equation}
cosh \zeta = (\cosh \eta)^2 \cosh \beta - (\sinh \eta)^2 \cos \alpha = \\
= \cosh \beta + p^2 \mu^{-2}(\cosh \beta - \cos \alpha),
\end{equation}

\begin{equation}
\cos \frac{\theta}{2} \exp(\frac{i\phi}{2}) = \left( \cosh \frac{\zeta}{2} \right)^{-1} \cos \frac{\alpha}{2} \cosh \frac{\beta}{2} + i \left( \sinh \frac{\zeta}{2} \right)^{-1} \sin \frac{\alpha}{2} \sinh \frac{\beta}{2}.
\end{equation}

For $j = M = 0$, we see that the expression (5.19) is an analytic function of $\cosh \zeta$ with a branch point at $\cosh \zeta = -1$. From eq. (5.25) we see that it is not an entire analytic function of $p^2$ and the commutator function (4.31) cannot vanish for large $\rho$. 

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VI. Zero-Mass Free Fields.

In order to use the method described in the preceding Section in the zero-mass case, we have to write the i. u. r.’s of $SL(2, \mathbb{C})$ in a basis that evidentiates the decomposition of the representation space into spaces where i. u. r.’s of $\tilde{E}(2)$ operate. In this case, we have a direct integral decomposition and we have to introduce a “continuous” basis.

We start from the realization of the operator $D^{Mc}(a)$ in a space of functions of a complex variable $z$:

\begin{equation}
[D^{Mc}(a)f](z) = (a_{21}z + a_{11})^{-c+M-1}(a_{21}z + a_{11})^{-c-M-1}f \left((a_{22}z + a_{12})(a_{21}z + a_{11})^{-1}\right),
\end{equation}

where

\begin{equation}
a = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} \in SL(2, \mathbb{C}).
\end{equation}

This representation is equivalent to the one defined in refs. 22, 23, that in our notation is given by $D^{-M,-c}(u_1(\pi)au_1(-\pi))$.

The elements of the little group $\tilde{E}(2)$ have the form

\begin{equation}
h = \begin{pmatrix}
\exp(-\frac{1}{2}i\phi) & \xi \exp(-\frac{1}{2}i\phi) \\
0 & \exp(\frac{1}{2}i\phi)
\end{pmatrix} \in \tilde{E}(2),
\end{equation}

with $\phi$ real and $\xi$ complex. We have

\begin{equation}
[D^{Mc}(h)f](z) = \exp(-iM\phi)f(\exp(i\phi)z + \xi).
\end{equation}

We see that $\tilde{E}(2)$ acts on the complex $z$ plane by means of Euclidean transformations. If we introduce the (non normalizable) basis vectors

\begin{equation}
f_{\kappa m}(z) = (z|z|^{-1})^{-m}J_{-m}(\kappa|z|), \quad \kappa > 0, \quad m = 0, \pm 1, \pm 2, \ldots,
\end{equation}

the matrix that represents an Euclidean transformation is diagonal in the index $\kappa$, namely we have

\begin{equation}
f_{\kappa m}(\exp(i\phi)z + \xi) = f_{\kappa m'}(z)R_{m'm}^{\kappa}(h),
\end{equation}

where the matrix

\begin{equation}
R_{m'm}^{\kappa}(h) = \exp(-im'\phi)(\xi|\xi|^{-1})^{m'-m}J_{m'-m}(\kappa|\xi|)
\end{equation}

is unitary. From eq. (6.4), we get

\begin{equation}
[D^{Mc}(h)f_{\kappa m}](z) = \exp(-iM\phi)R_{m'm}^{\kappa}(h)f_{\kappa m'}(z).
\end{equation}
We see from eq. (6.7) that the factor \( \exp(-iM\phi) \) can be eliminated by means of a “translation” of the indices \( m', m \), that is irrelevant if \( M \) is an integer. If \( M \) is half-odd, we get a representation of \( \tilde{E}(2) \) that is double-valued on \( E(2) \).

In analogy with eq. (5.14) we put

\[
(6.9) \quad w(a) = \sum_m (f_{\kappa m}, D^{Mc}(a)f_{\kappa m}).
\]

Also in this case, the conditions (3.26) and (3.28) are satisfied. In the same way as in the preceding Section, we can show that the fields constructed by means of the distributions found above represent zero-mass particles with an infinite helicity spectrum and with internal quantum numbers described by the i. u. r. \( D^{Mc} \) of \( SL(2, \mathbb{C}) \). Since particles of this kind are not observed in nature, we shall not discuss these fields with more detail.

The zero-mass particles present in nature have only one value \( m \) of the helicity (two if parity is taken into account) and they are described by one-dimensional i. u. r.’s of the little group, of the kind

\[
(6.10) \quad R^m(h) = \exp(-im\phi), \quad h \in \tilde{E}(2), \quad m = 0, \pm \frac{1}{2}, \pm 1, \ldots
\]

where \( h \) is given by eq. (6.3). These i.u.r.’s are not contained in the i. u. r.’s of \( SL(2, \mathbb{C}) \) and we have to use a different method in order to find the corresponding \( w \) functions. We propose the following solutions, without explaining how they have been obtained:

\[
(6.11) \quad w^0_{0c}(a) = |a_{21}|^{2c-2}, \quad 0 < c < 1,
\]

\[
(6.12) \quad w^0_{Mc}(a) = \delta^2(a_{21})a_{22}^{c-M}\bar{\sigma}_{22}^{c+M}, \quad \Re c = 0.
\]

It is easy to show that the symmetry condition (3.28) is satisfied. The positivity condition (3.26) follows from the formulae

\[
(6.13) \quad w^0_{0c}(ba^{-1}) =
\]

\[
= \int |a_{22}|^{2c}\delta^2(a_{21} - z'a_{22})|z' - z|^{2c-2}|b_{22}|^{2c}\delta^2(b_{21} - zb_{22}) d^2z d^2z',
\]

\[
(6.14) \quad w^0_{Mc}(ba^{-1}) =
\]

\[
= \int \bar{\sigma}_{22}^{-c-M}a_{22}^{-c+M}\delta^2(a_{21} - za_{22})b_{22}^{c-M}\bar{\sigma}_{22}^{c+M}\delta^2(b_{21} - zb_{22}) d^2z.
\]

Note that \(|z' - z|^{2c-2}\) is a positive integral kernel, the one that defines the scalar product in the space of an i. u. r. of the supplementary series.
The norm (3.25) in the case (6.11) can be written as

\[(\Phi(f), \Phi(f)) = \int \mathcal{F}(k, z')|z' - z|^{2c - 2} f(k, z)\theta(k^0)\delta(k \cdot k) d^2 z d^2 z' d^4 k,\]

and in the case (6.12) we get

\[(\Phi(f), \Phi(f)) = \int |f(k, z)|^2 \theta(k^0)\delta(k \cdot k) d^2 z d^4 k,\]

where in both cases

\[(6.17) f(k, z) = (2\pi)^{-\frac{3}{2}} \int \tilde{f}(k, a_k a_{22} e^{-M} a_{22} e^{+M} \delta^2(a_{21} - za_{22}) d^6 a).\]

Of course, in the first case we have to put \(M = 0\).

If we consider a Poincaré transformation of the kind (5.8), (5.9), the wave function (6.17) transforms in the following way

\[(6.18) f'(k, z) = \exp(-ik \cdot y) R^M (a_k^{-1} b_{a_k'}) f(k', z),\]

where \(k'\) is given by eq. (5.9). This is the transformation property\(^{21}\) of the wave function of a particle of zero mass and helicity \(M\).

Under the “internal” transformation (5.12) the wave function transforms as

\[(6.19) f'(k, z) = (b_{22} + zb_{12})^{-c+M-1} (b_{22} + zb_{12})^{-c-M-1} f(k, z'),\]

\[z' = (b_{21} + zb_{11}) (b_{22} + zb_{12})^{-1},\]

namely according to the i. u. r. \(D^{-M,-c}\) defined in refs. 22, 23, which is equivalent to the i. u. r. \(D^{Mc}\). Note that, if we fix up to equivalence the i. u. r. \(D^{Mc}\), we have two possible theories with helicity \(\pm M\).

In order to compute the integral (3.24), we use the variables introduced in Sec. IV. If we put

\[(6.20) \tilde{a} = a_\eta^{-1} u_3(\alpha) a_3(\beta) a_\eta,\]

from eq. (4.26) we get

\[(6.21) \tilde{a}_{21} = -p \sinh(\frac{1}{2}(\beta - i\alpha)), \quad \tilde{a}_{22} = \cosh(\frac{1}{2}(\beta - i\alpha)).\]

We see that the quantity \(\delta^2(\tilde{a}_{21})\) that appears in eq. (6.12) is rather badly defined when considered as a distribution in the variables \(k^1, k^2\) for fixed values of \(\alpha, \beta\) satisfying eq. (4.2). Other difficulties arise when one tries to perform the integrals (3.24) or (4.28). We conclude that, even if the expression (6.12) satisfies the required positivity and symmetry conditions, it does not give rise to a well-behaved field theory. All the calculations based on this expression have a purely formal character.
When we substitute eqs. (6.11) and (6.21) into eqs. (4.28) and (4.31), the integrals can be performed in terms of Legendre functions\textsuperscript{27,29} and we get
\begin{equation}
W^{0c}_{0}(x, a) = 2^{c-3} \pi^{-2}(\Gamma(c))^{2}(\cosh \beta - \cos \alpha)^{c-1} \times \\
\times (\rho^{2} - \sigma^{2})^{-c} P_{c-1}((\sigma^{2} + \rho^{2})(\sigma^{2} - \rho^{2})^{-1}), \quad \sigma^{2} < 0.
\end{equation}

\begin{equation}
2i\text{Im } W^{0c}_{0}(x, a) = i2^{c-2} \pi^{-1} \epsilon(x^{0})\theta(\sigma^{2})\Gamma(e)(\Gamma(1-c))^{-1} \times \\
\times (\cosh \beta - \cos \alpha)^{c-1}|\sigma^{2} - \rho^{2}|^{-c} P_{c-1}((\sigma^{2} + \rho^{2})|\sigma^{2} - \rho^{2}|^{-1}).
\end{equation}

The real part for \(\sigma^{2} > 0\) can be obtained by analytic continuation of eq. (6.22). In the special case \(\rho = 0\), we have
\begin{equation}
W^{0c}_{0}(x, a) = 2^{c-3} \pi^{-2}(\Gamma(c))^{2}(\cosh \beta - \cos \alpha)^{c-1} \times \\
\times |\sigma^{2}|^{-c} (\theta(-\sigma^{2}) + \theta(\sigma^{2}) \exp(i\pi\epsilon(x^{0}))).
\end{equation}

Since we are not able to build fields starting from eq. (6.12), in order to describe particles with a non-vanishing helicity we have to use non-scalar fields. From eq. (6.3) we have
\begin{equation} h \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \exp(-\frac{1}{2}i\phi) \left( \begin{array}{c} 1 \\ 0 \end{array} \right). 
\end{equation}

We remark that the spinor representations \(S^{(s,0)}(h)\) and \(S^{(0,s)}(h)\) are equivalent to symmetrized tensor products of \(s\) matrices equal, respectively, to \(h\) or \(\overline{h}\) and we adopt conventions in agreement with eqs. (5.4) and (5.5). Then, if we put
\begin{equation}
S(a) = S^{(m,0)}(a), \quad m \geq 0, \\
S(a) = S^{(0,|m|)}(a), \quad m \leq 0,
\end{equation}
we have
\begin{equation}
S_{\rho}^{m}(h) = \exp(-im\phi)\delta_{\rho}^{m}, \quad h \in \tilde{E}(2).
\end{equation}

As a consequence, the expression
\begin{equation}
w_{\rho\sigma}(a) = \delta_{\rho m}\delta_{\sigma m}|a_{21}|^{2c-2}
\end{equation}
satisfies the symmetry condition (3.28).

Eq. (6.15) is still valid if we modify eq. (6.17) in the following way
\begin{equation}
f(k, z) = (2\pi)^{-\frac{3}{2}} \int \tilde{f}^{\rho}(k, a_{k}a)S_{\rho}^{m}(a^{-1})|a_{22}|^{2c} \delta^{2}(a_{21} - za_{22}) d^{6}a.
\end{equation}

Eq. (6.18) holds with \(M\) replaced by \(m\) and eq. (6.19) holds with \(M = 0\); this means that the theory describes zero mass particles with helicity \(m\) that transform according to the representation \(D^{0c}\) of the internal symmetry group. In the limit \(c \rightarrow 1\), the expression (6.28) becomes independent of \(a\) and we get a Minkowskian theory of the kind usually adopted to describe neutrinos, photons or gravitons.
VII. Field Equations.

Now we find the wave equations satisfied by the scalar fields defined in the preceding Sections; non-scalar fields will be treated elsewhere. We indicate by

\[ L_{ik} = -L_{ki} \]

the generators of the right translations on \( SL(2, \mathbb{C}) \), considered as differential operators acting on smooth functions defined on the group. They satisfy the commutation relations

\[ [L_{ik}, L_{rs}] = g_{ir} L_{ks} - g_{kr} L_{is} - g_{is} L_{kr} + g_{ks} L_{ir} \]

of \( sl(2, \mathbb{C}) \), commute with the left translations and have the following commutation property with the finite right translation represented by the operator \( T(b) \):

\[ T(b^{-1}) L_{ik} T(b) = \Lambda_i^r(b) \Lambda_k^s(b) L_{rs} \]

We also need the generators \( L'_ik \) of the left translations, which satisfy commutation relations with the opposite sign. We have

\[ L'_ik = \Lambda_i^r(a) \Lambda_k^s(a) L_{rs} \]

and their commutation relation with a finite left translation \( T'(b) \) is

\[ T'(b) L'_ik T'(b^{-1}) = \Lambda_i^r(b) \Lambda_k^s(b) L'_{rs} \]

We also consider the generators \( A_k \) and \( A_{ik} \) of the right translations on the group \( \tilde{\mathcal{P}} \), that satisfy the Poincaré Lie algebra. They act on functions of the kind \( f(x, a) \) in the following way:

\[ A_i = \Lambda_i^k(a) \frac{\partial}{\partial x^k}, \quad A_{ik} = L_{ik} \]

For the generators of the left translations of \( \tilde{\mathcal{P}} \) we have

\[ A'_i = \frac{\partial}{\partial x^k}, \quad A'_{ik} = L'_ik + x_i \frac{\partial}{\partial x^k} - x_k \frac{\partial}{\partial x^i} \]

From eqs. (3.24), (7.5) and (7.6) we obtain immediately the differential equation

\[ g^{ik} A_i A_k W(x, a) = g^{ik} A'_i A'_k W(x, a) = \mu^2 W(x, a) \]

which is essentially the Klein Gordon equation.

If the function \( w(a) \) is a linear combination of matrix elements of the representation \( D^{Mc}(a) \), as in eqs. (5.2) and (6.9), it satisfies the differential equations

\[ \frac{1}{2} g^{ir} g^{ks} L_{ik} L_{rs} w(a) = \frac{1}{2} g^{ir} g^{ks} L'_{ik} L'_{rs} w(a) = (1 - c^2 - M^2) w(a) \]

\[ \frac{1}{8} e^{ikrs} L_{ik} L_{rs} w(a) = \frac{1}{8} e^{ikrs} L'_ik L'_rs w(a) = -iMc w(a) \]
The functions (6.11) and (6.12) are not defined in terms of matrix elements, but it is easy to show directly that they satisfy the differential equations

\[(L'_{10} - L'_{31})w(a) = 0, \quad (L'_{20} + L'_{23})w(a) = 0,\]

\[(L'_{12}w(a) = -iMw(a), \quad L'_{30}w(a) = (1 - c)w(a).\]

By means of the formulas

\[(7.12) \quad \frac{1}{2}g^{ir}g^{ks}L'_{ik}L'_{rs} = \]
\[= -(L'_{10} + L'_{31})(L'_{10} - L'_{31}) - (L'_{20} - L'_{23})(L'_{20} + L'_{23}) + (L'_{12})^2 - (L'_{30})^2 + 2L'_{30},\]

\[(7.13) \quad \frac{1}{8}e^{ikrs}L'_{ik}L'_{rs} = \]
\[= \frac{1}{2}(L'_{20} - L'_{23})(L'_{10} - L'_{31}) - \frac{1}{2}(L'_{10} + L'_{31})(L'_{20} + L'_{23}) - L'_{30}L'_{12} + L'_{12}.\]

we can derive also in this case the eqs. (7.8) and (7.9).

The Casimir operators that appear in eqs. (7.8) and (7.9) commute with the left and the right translations on \(SL(2, \mathbb{C})\) and from eqs. (3.24) and (7.5) we get, for a scalar field,

\[(7.14) \quad \frac{1}{2}g^{ir}g^{ks}A_{ik}A_{rs}W(x, a) = (1 - c^2 - M^2)W(x, a),\]

\[(7.15) \quad \frac{1}{8}e^{ikrs}A_{ik}A_{rs}W(x, a) = -iMcW(x, a).\]

Eqs. (7.7), (7.14) and (7.15) identify the mass and the internal quantum numbers and hold for all the scalar fields. There are other equations that identify spin or helicity. In the positive-mass case the generators of the little group \(K = SU(2)\) are \(L_{12}, L_{23}, L_{31}\), and from eqs. (5.1) and (5.2), taking into account the properties of the representations of \(SU(2)\), we get

\[(7.16) \quad ((L'_{12})^2 + (L'_{23})^2 + (L'_{31})^2)w(a) = -j(j + 1)w(a).\]

If \(j = 0\), we have the stronger result

\[(7.17) \quad L'_{12}w(a) = L'_{23}w(a) = L'_{31}w(a) = 0.\]

From eqs. (3.19) and (7.16) we get

\[(7.18) \quad \frac{1}{4}e^{ijrs}k_iL'_{rs}e_j^{kpq}k_kL'_{pq}w(a) = -\mu^2j(j + 1)w(a)\]

and from eqs. (3.21), (3.22) and (7.4)

\[(7.19) \quad \frac{1}{8}e^{ikrs}k_iL'_{rs}e_j^{kpq}k_kL'_{pq}w(k, a) = -\mu^2j(j + 1)w(k, a).\]
By substitution into eq. (3.24) we obtain the differential equation

\begin{equation}
(7.20) \quad g_{ik} \Sigma^{i} \Sigma^{k} W(x, a) = \mu^2 j(j + 1) W(x, a),
\end{equation}

where

\begin{equation}
(7.21) \quad \Sigma^{i} = \frac{1}{2} e^{ijrs} A_j A^r_s = \frac{1}{2} e^{ijrs} \frac{\partial}{\partial x^j} L'_r s.
\end{equation}

If we also introduce the differential operators

\begin{equation}
(7.22) \quad \Sigma^{i} = \frac{1}{2} e^{ijrs} A_j A^r_s = \Lambda_k^i(a) \Sigma^{k},
\end{equation}

we have

\begin{equation}
(7.23) \quad g_{ik} \Sigma^{i} \Sigma^{k} W(x, a) = \mu^2 j(j + 1) W(x, a).
\end{equation}

If \( j = 0 \) we have the stronger result

\begin{equation}
(7.24) \quad \Sigma^{i} W(x, a) = \Sigma^{i} W(x, a) = 0.
\end{equation}

In the zero-mass case, the generators of the little group \( \mathcal{K} = \tilde{E}(2) \) are \( L_{12}, (L_{10} - L_{31}), (L_{20} + L_{23}) \). From eqs. (6.8) and (6.9) we obtain the differential equation

\begin{equation}
(7.25) \quad \left( (L'_1 - L'_3)^2 + (L'_2 + L'_3)^2 \right) w(a) = -\kappa^2 w(a),
\end{equation}

which can be written in the form

\begin{equation}
(7.26) \quad \frac{1}{4} e^{ijrs} \hat{k}_i L'_r s e_j^{kpa} \hat{k}_k L'_p a w(a) = -\kappa^2 w(a).
\end{equation}

By means of the procedure used above, for a field with zero mass and infinite helicity spectrum we find the equation

\begin{equation}
(7.27) \quad g_{ik} \Sigma^{i} \Sigma^{k} W(x, a) = g_{ik} \Sigma^{i} \Sigma^{k} W(x, a) = \kappa^2 W(x, a).
\end{equation}

If we consider a field with zero mass and given helicity \( M \) based on the functions (6.11) and (6.12), we see that eqs. (7.10) and (7.11) can be written in the form

\begin{equation}
(7.28) \quad \frac{1}{2} e^{ijrs} \hat{k}_i L'_r s w(a) = -iM \hat{k}^i w(a),
\end{equation}

\begin{equation}
(7.29) \quad \hat{k}^i L'_i w(a) = (c - 1) \hat{k} w(a).
\end{equation}

Proceeding as in the other cases, we obtain

\begin{equation}
(7.30) \quad \Sigma^{i} W(x, a) = -iM A^i W(x, a), \quad \Sigma_i W(x, a) = -iM A_i W(x, a),
\end{equation}
(7.31) \[ A^i A_{ik} W(x, a) = (c - 1) A_k W(x, a). \]

From eqs. (3.8), (3.29) and (4.20) we get

(7.32) \[ (\Omega, \psi(x, a) \psi^\dagger(y, b) \Omega) = \overline{W}((y, b)^{-1}(x, a)), \]

(7.33) \[ (\Omega, \psi^\dagger(y, b) \psi(x, a) \Omega) = \overline{W}((y, b)^{-1}(-x, a)). \]

If the function \( \overline{W}(x, a) \) satisfies a differential equation invariant under left translations and under the reflection \( x \to -x \), from the Wightman reconstruction theorem \(^{19, 20}\) we have that \( \psi(x, a) \) satisfies the same equation. In this way we get the field equations

(7.34) \[ g^{ik} A_i A_k \psi(x, a) = \mu^2 \psi(x, a), \]

(7.35) \[ \frac{1}{2} g^{ir} g^{ks} A_{ik} A_{rs} \psi(x, a) = (1 - c^2 - M^2) \psi(x, a), \]

(7.36) \[ \frac{1}{8} e^{ikrs} A_{ik} A_{rs} \psi(x, a) = -i Mc \psi(x, a). \]

For \( \mu > 0, \ j > 0 \) we have

(7.37) \[ g_{ik} \Sigma^i \Sigma^k \psi(x, a) = \mu^2 j(j + 1) \psi(x, a), \]

for \( \mu > 0, \ j = 0 \) we have

(7.38) \[ \Sigma^i \psi(x, a) = 0, \]

for \( \mu = 0 \) and infinite helicity spectrum we have

(7.39) \[ g_{ik} \Sigma^i \Sigma^k \psi(x, a) = \kappa^2 \psi(x, a), \]

for \( \mu = 0 \) and helicity \( M \) we have

(7.40) \[ \Sigma^i \psi(x, a) = i M A_i \psi(x, a), \]

(7.41) \[ A^i A_{ik} \psi(x, a) = (\overline{c} - 1) A_k \psi(x, a). \]

We have seen in Sec. VI that there is some difficulty in the definition of fields starting from the function (6.12). For these fields, the calculations given above have a purely formal character.
VIII. Flat-Space Theories and Spin-Mass-Shells.

We approximate a small region of the group manifold $\tilde{P}$ by means of a tangent space with coordinates $x^i$ and $x^{ik} = -x^{ki}$, which can be identified with the vector space $T$. The operators $A_i$ and $A_{ik}$ which appear in the differential equations satisfied by the fields and by their v. e. v.’s can be replaced by the partial derivatives with respect to these coordinates. We get in this way a field theory in the flat ten-dimensional space and the Fourier transforms of the fields and of the v. e. v.’s have their support in a Lorentz invariant manifold defined by some polynomial equations in the ten-dimensional “spin-momentum space” $T^*$ with coordinates $k_i$ and $k_{ik} = -k_{ki}$. These equations are obtained from the field equations by means of the substitutions $A_i \rightarrow -i\delta_{ik}$, $A_{ik} \rightarrow -i\delta_{ik}$. To indicate the positive-energy part ($k^0 = -k_0 > 0$) of this manifold, we use the term “spin-mass-shell”. The two-point v. e. v. of the flat-space theory is the Fourier transform of a positive Lorentz invariant measure on the spin-mass-shell. As we shall see, in some cases the replacement of non-commuting operators by commuting quantities may lead to inconsistencies or ambiguities. Nevertheless, the correspondence between theories in $\tilde{P}$ and in the flat space is an unavoidable heuristic instrument. Note that there is no ambiguity in the higher degree terms of the equations defining the spin-mass-shell. If we drop all the other terms, we get a set of homogeneous equations that define an unambiguous dilatation invariant “asymptotic” manifold.

From eqs. (7.7), (7.14) and (7.15) we get the following equations valid on the spin-mass-shell of all the elementary field theories.

\[(8.1) \quad g^{ik}k_i k_k = -\mu^2, \quad \frac{1}{2}g^{ir}g^{ks}k_{ik}k_{rs} = c^2 + M^2 - 1, \quad \frac{1}{8}e^{ikrs}k_{ik}k_{rs} = iMc.\]

In a similar way from eqs. (7.23), (7.24), (7.27), (7.30) and (7.31) we obtain other equations valid for the various specific cases.

Since the spin-mass-shell is Lorentz-invariant, it is determined by its intersection with the hyperplane $k_i = \hat{k}_i$. This intersection can be considered as a manifold in the six-dimensional “spin space” with coordinates $k_{ik}$ and we call it the “spin-shell”. Its dimension is given by the dimension of the spin-mass-shell minus three. It is convenient to introduce the three-dimensional vectors

\[(8.2) \quad k = (k^1, k^2, k^3), \quad k' = (k'^{23}, k'^{31}, k'^{12}), \quad k'' = (k''^{10}, k''^{20}, k''^{30})\]

and similar notations for the coordinates $x^i, x^{rs}$. Then the last two conditions of eq. (8.1) take the form

\[(8.3) \quad (k')^2 - (k'')^2 = c^2 + M^2 - 1, \quad k' \cdot k'' = iMc.\]

The spin-shell in the case $\mu > 0$ is described by the equations

\[(8.4) \quad (k')^2 = j(j + 1), \quad (k'')^2 = j(j + 1) + 1 - c^2 - M^2, \quad k' \cdot k'' = iMc.\]

Note that the right-hand sides are real and satisfy the inequality

\[(8.5) \quad j(j + 1) (j(j + 1) + 1 - c^2 - M^2) + M^2 c^2 = (k')^2 (k'')^2 - (k' \cdot k'')^2 \geq 0.\]
If $j > 0$ this manifold has dimension three, but if $j = M = 0$ eq. (8.4) takes the simpler form

\[(8.6) \quad k' = 0, \quad (k'')^2 = 1 - c^2\]

and describes a two-dimensional manifold. In both cases the spin-shell is compact and the rotation group acts transitively on it, namely the spin-shell is an orbit of the rotation group in the spin space. It follows that the spin-mass-shell is an orbit of the Lorentz group in the spin-momentum space.

The orbits of the rotation or of the Lorentz group which correspond to a field on the group $\hat{P}$ are called “allowed orbits”. It is useful to consider all the orbits, not necessarily allowed, which can be classified by means of the invariants $(k')^2$, $(k'')^2$ and $k' \cdot k''$ satisfying the condition (8.5). One can also use eq. (8.4) to introduce the parameters $j$, $c$ and $M$ even when they do not label any i. u. r. Given an orbit with $k' \cdot k'' \neq 0$, eq. (8.4) determines the real quantities $j$, $M$ and the imaginary quantity $c$, up to a common change of sign of $M$ and $c$. They satisfy the conditions

\[(8.7) \quad j > 0, \quad c^2 < 0, \quad 0 < M^2 \leq j(j + 1) \left(1 + (j(j + 1) + |c^2|^{-1})\right).\]

Note that only some discrete values of the parameters $j$ and $M$ are allowed and that they are rather uniformly distributed in the set defined by eq. (8.7). These orbits are three-dimensional in the general case, but have dimension two when the equality sign holds in eq. (8.5) or in the last eq. (8.7), namely when the vectors $k'$ and $k''$ are parallel. The two-dimensional orbits correspond to values of $M$ which are not allowed, but for large $j$ are relatively near to the allowed values $M = \pm j$.

If $k' \cdot k'' = 0$, we have two possible choices of $M$ and $c$, namely

\[(8.8) \quad M = 0, \quad 1 - c^2 = (k'')^2 - (k')^2, \quad j \geq 0,\]

or

\[(8.9) \quad c = 0, \quad M^2 - 1 = (k')^2 - (k'')^2, \quad j(j + 1) = (k')^2 \geq M^2 - 1.\]

There is some ambiguity in the parametrization of the orbits, which disappears if we consider only allowed orbits and allowed values of the parameters. These orbits are three-dimensional in the general case and two-dimensional if $k' = 0$ or $k'' = 0$. If both these vectors vanish, we have a zero-dimensional orbit, corresponding to $j = M = 0$, $c^2 = 1$, namely to a Minkowskian theory.

In the case $\mu = 0$, infinite helicity spectrum, the spin-shell is described by the equations (8.3) and

\[(8.10) \quad (k_{10} - k_{31})^2 + (k_{20} + k_{23})^2 = \kappa^2 > 0.\]

It has dimension three and it is an unbounded orbit of the little group $\hat{E}(2)$. 
In the case \( \mu = 0 \), helicity \( M \), from eq. (7.30) we get the equations

\[
(8.11) \quad k_{10} = k_{31}, \quad k_{20} = -k_{23}, \quad k_{12} = M.
\]

Eq. (7.31) contains products of non-commuting operators and gives ambiguous results. A direct substitution gives

\[
(8.12) \quad k_{30} = i(1 - c),
\]

but this result is not compatible with eq. (8.3). We consider the equation

\[
(8.13) \quad k_{30} = N(c),
\]

without specifying the function \( N(c) \), apart from the conditions \( N(1) = 0 \) and \( N(c) \approx -ic \) for large imaginary \( c \). If \( N(c) \) is real, eqs. (8.11) and (8.13) define a two-dimensional unbounded orbit of \( \tilde{E}(2) \), unless \( M = N = 0 \).

In order to complete the list of the orbits of \( \tilde{E}(2) \) in the spin space, we have to consider the zero-dimensional trivial orbit, corresponding to a Minkowskian scalar field, and a set of one-dimensional bounded orbits defined by the conditions

\[
(8.14) \quad k_{10} = k_{31}, \quad k_{20} = -k_{23}, \quad k_{12} = k_{30} = 0,
\]

\[
(k_{31})^2 + (k_{23})^2 = \nu^2 > 0.
\]

The corresponding orbits of the Lorentz group are four-dimensional and have a large symmetry group locally isomorphic to \( L(4, \mathbb{R}) \). It has been shown in ref. 7 that they are not allowed according to our definition.

Now we study the v. e. v.’s of the flat-space theories with \( \mu > 0 \). Given a positive-mass orbit of the Lorentz group in the spin-momentum space and positive Lorentz-invariant measure \( m \) on it, the v. e. v. can be written in the form

\[
(8.15) \quad V(x, x', x'') = (2\pi)^{-3} \int \exp(-ik \cdot x + ik' \cdot x' - ik'' \cdot x'') \, dm =
\]

\[
= (2\pi)^{-3} \int \exp(-ik \cdot x + ik' \cdot x' - ik'' \cdot x'') \tilde{v}(\hat{k}', \hat{k}'') \theta(k^0) \delta(k \cdot k + \mu^2) \, d^4k \, d^3k' \, d^3k'',
\]

where

\[
(8.16) \quad \hat{k}_{ik} = \Lambda^r_i(a_k) \Lambda^s_k(a_k) k_{rs},
\]

\( \tilde{v}(k', k'') \, d^3k' \, d^3k'' \) represents a positive rotation-invariant measure concentrated on the spin-shell and \( a_k \) is defined by eq. (3.21).

The integral (8.15) can be written in the form

\[
(8.17) \quad V(x, x', x'') = (2\pi)^{-3} \int \exp(-ik \cdot x) v(\hat{x}', \hat{x}'') \theta(k^0) \delta(k \cdot k + \mu^2) \, d^4k,
\]
where

\[(8.18) \quad \hat{x}^{ik} = \Lambda^i_r(a_k)\Lambda^k_s(a_k)x^{rs}\]

and

\[(8.19) \quad v(x', x'') = \int \exp(ik' \cdot x' - ik'' \cdot x'') \tilde{v}(k', k'') \, d^3k' \, d^3k''.\]

If the vectors \(k', k''\) represent an arbitrary point of the spin-shell, we can put

\[(8.20) \quad v(x', x'') = I(k', k'', x', x'') = \int_{SO(3)} \exp(iRk' \cdot x' - iRk'' \cdot x'') \, d^3R,\]

where \(R\) is a three-dimensional rotation matrix and \(d^3R\) is the normalized invariant measure on the rotation group. In the general case, this integral cannot be expressed in terms of elementary functions, but if the vectors \(k', k''\) are parallel or antiparallel we have

\[(8.21) \quad v(x', x'') = t^{-1} \sin t,\]

where

\[(8.22) \quad t^2 = (||k'||x' \mp ||k''||x'')^2.\]

If we consider a theory with \(j = M = 0\), from eq. (8.6) we get

\[(8.23) \quad t^2 = (1 - c^2)(x'')^2.\]

Eq. (8.17) is similar to eq. (3.24) and it is interesting to compare the functions \(v(x', x'')\) and \(w(a)\), where \(a\) is given by the exponential

\[(8.24) \quad a = \exp(\frac{1}{2}(x'' - ix') \cdot \sigma).\]

Eq. (A.27) gives the power expansion of the integral (8.20) up to quadratic terms in the variables \(x', x''\). A comparison with eq. (A.26) shows that, disregarding higher order terms, we have

\[(8.25) \quad w(a) \simeq v(x', x'').\]

It follows that if \(f(x)\) is a test function and \(a\) is given by eq. (8.24), we have

\[(8.26) \quad \int f(x)W(x, a) \, d^4x \simeq \int f(x)V(x, x', x'') \, d^4x\]

up to terms of the second order.
In order to simplify the integral (8.17), we proceed as in Sec. IV, namely we put
\begin{equation}
\mathbf{x}' = (0, 0, \alpha), \quad \mathbf{x}'' = (0, 0, \beta)
\end{equation}
and we use the definitions (4.23), (4.24) and (4.25). From eq. (8.18) we obtain
\begin{equation}
\mathbf{\dot{x}}' = (0, -\beta \sinh \eta, \alpha \cosh \eta) = \mu^{-1}(0, -\beta p, \alpha q), \\
\mathbf{\dot{x}}'' = (0, \alpha \sinh \eta, \beta \cosh \eta) = \mu^{-1}(0, \alpha p, \beta q).
\end{equation}
The analogue of eq. (4.28) is
\begin{equation}
V(x, \alpha, \beta) = (2\pi)^{-1} \int_{0}^{\infty} \Delta(q\sigma, \epsilon(x'^{0})) J_{0}(p\rho) v(\mathbf{x}', \mathbf{x}'') p \, dp
\end{equation}
and in a similar way one writes the analogues of eqs. (4.31) and (4.32) which determine the commutator.

If in eq. (8.20) we replace the integrand by the maximum of its modulus, we get the inequality
\begin{equation}
|I(\mathbf{k}', \mathbf{k}'', \mathbf{x}' + i\mathbf{y}', \mathbf{x}'' + i\mathbf{y}'')| \leq \\
\leq \exp \left( (\mathbf{k}')^2(y')^2 + (\mathbf{k}'')^2(y'')^2 + 2(\mathbf{k}' \cdot \mathbf{k}'')(\mathbf{y}' \cdot \mathbf{y}'') + 2||\mathbf{k}' \times \mathbf{k}''||||\mathbf{y}' \times \mathbf{y}''|| \right)^{1/2}.
\end{equation}
For large \(|p|\) we have \(\text{Im} \, q \simeq \text{Im} \, p\) and, with this approximation, from eq. (8.28) we obtain
\begin{equation}
|v(\mathbf{x}', \mathbf{x}'')| \leq \exp \left( \mu^{-1} |\text{Im} \, p| \left( \alpha^2 + \beta^2 \right)^{1/2} \left( (\mathbf{k}')^2 + (\mathbf{k}'')^2 + 2||\mathbf{k}' \times \mathbf{k}''|| \right)^{1/2} \right).
\end{equation}
If we take into account eq. (4.33) we see that in the analogue of eq. (4.32) we can close the integration path at infinity in the upper half plane if
\begin{equation}
\rho > \sigma + \mu^{-1}(\alpha^2 + \beta^2)^{1/2} \left( (\mathbf{k}')^2 + (\mathbf{k}'')^2 + 2||\mathbf{k}' \times \mathbf{k}''|| \right)^{1/2}
\end{equation}
and under this condition the commutator of the flat-space theory vanishes. We remark that the flat-space theory has stronger local commutation properties than the corresponding theory on \(\tilde{\mathcal{P}}\).

For \(j = M = 0\), we can use eqs. (8.21) and (8.23) and from eq. (8.28) we have
\begin{equation}
l^2 = (1 - c^2)(\alpha^2(\sinh \eta)^2 + \beta^2(\cosh \eta)^2) = (1 - c^2)(\beta^2 + (\alpha^2 + \beta^2)\mu^{-2}p^2).
\end{equation}
If we remark that, for small values of \(\alpha\) and \(\beta\), eq. (5.25) gives
\begin{equation}
\zeta^2 \simeq \alpha^2(\sinh \eta)^2 + \beta^2(\cosh \eta)^2,
\end{equation}
we can easily verify eq. (8.26) in this special case.
IX. A Theory with Broken $Sp(4, \mathbb{R})$ Symmetry.

Now we consider a scalar flat-space theory invariant with respect to a group $\mathcal{F}$ larger than $SL(2, \mathbb{C})$. It is defined by a spin-mass-shell which is an orbit of $\mathcal{F}$ and can be decomposed into orbits of the Lorentz group. As a consequence, its two-point v. e. v. is a superposition (an integral) of the Lorentz invariant v. e. v.’s $V(x, x', x'')$ described in the preceding Section. If all the Lorentz orbits which appear in the decomposition are allowed (apart from a set of vanishing measure), we can consider the analogous superposition of the v. e. v.’s $W(x, a)$ and we find the v. e. v. of a non-elementary theory on $\tilde{P}$ which corresponds, in some sense, to the $\mathcal{F}$-invariant theory on the flat space. In general the theory on $\tilde{P}$ has lost the symmetry under the large group $\mathcal{F}$, but some consequence of this higher symmetry remains in the short-distance limit.

In refs. 7, 30 we have described several flat-space theories symmetric with respect to $SL(4, \mathbb{R})$ or to one of its subgroups isomorphic to $Sp(4, \mathbb{R})$, which form a one-parameter family$^2$. The positivity of the energy requires that the spin-mass-shell is contained in a closed invariant cone $T^{*+}$, the dual of the cone $T^+$ that describes the causal properties of the theory. Therefore, only the allowed Lorentz orbits contained in $T^{*+}$ are interesting for the purpose we are discussing. The corresponding spin-shells must be bounded and this requirement excludes all the zero-mass allowed theories considered in the preceding Section, but the Minkowskian zero-mass theory corresponding to the spin-shell reduced to the origin. In addition to this theory, only positive-mass elementary theories can be used in the construction of a theory on $\tilde{P}$ with broken higher symmetry. If we look at the definition of the cone $T^{*+}$, we see that for $\mu > 0$ the spin-shell must be contained in the set defined by

$$ (9.1) \quad (k')^2 + (k'')^2 + 2k' \times k'' \| \leq \mu^2 $$

(see the Appendix B). This inequality gives rise to a complicated constraint on the parameters $j, M,$ and $c$, in particular we get

$$ (9.2) \quad j(j + 1) + 1 - c^2 \leq \mu^2. $$

The mass $\mu$ is measured in natural units of the order of the Planck mass and for the observable particles it is very small. It follows that we must have $j = M = 0$ and $1 - c^2 \ll 1$, namely the theory must be very near to the Minkowskian limit. We also see that the particles with small $\mu$ and non-vanishing spin cannot be described by scalar fields. We have already remarked in Section 5 that spin has two different origins: in this case the spin is generated by the field indices, as in eqs. (5.15) and (6.28). Combined with eq. (8.32), eq. (9.1) ensures that the commutator of the flat-space theory vanishes for

$$ (9.3) \quad \rho > \sigma + (\alpha^2 + \beta^2)^{\frac{1}{2}}. $$

We consider the spin-mass-shells described in refs. 7, 30 and we exclude the four-dimensional one, composed of a single zero-mass not allowed Lorentz orbit.
The decomposition of these spin-mass-shells, (disregarding a set of vanishing measure) contains only positive-mass Lorentz orbits. In general not all these orbits are allowed, but one can try to replace the integral over the continuous parameters \( j \) and \( M \) by a sum over the discrete allowed values. In this way, we may also obtain a discrete mass spectrum. This procedure will be examined elsewhere; in the following we consider a particular choice of the group \( \mathcal{F} \) isomorphic to \( Sp(4, \mathbb{R}) \) and a particular class of orbits, in such a way that all the Lorentz orbits that appear in the decomposition are allowed.

For the description of the spin-mass-shells invariant with respect to \( Sp(4, \mathbb{R}) \), locally isomorphic to the anti-de Sitter group \( SO^\uparrow(2, 3) \), it is convenient to introduce the notation

\[
(9.4) \quad k^{5i} = -k^{i5} = k^i, \quad i = 0, 1, 2, 3
\]

and to consider the quantities \( k^{uvw} \) as the components of an antisymmetric tensor in a five-dimensional space with metric \( g_{55} = g_{00} = -1, \quad g_{11} = g_{22} = g_{33} = 1 \). In the following the indices \( u, v, w, x, y, z \) take the values 5, 0, 1, 2, 3.

We consider the orbit which contains the point defined by \( k^0 = s \geq 0, \quad k = k' = k'' = 0 \). Then it also contains the Lorentz orbit that corresponds to a Minkowskian theory with mass \( \mu = s \). It is six-dimensional and in ref. 30 it has been called \( \mathcal{O}_{4, \frac{1}{2}, \frac{1}{2}} s \) for \( s > 0 \), or \( \mathcal{O}_{2, 0} \) for \( s = 0 \). The following \( O(2, 3) \)-invariant set of conditions is satisfied on the orbit for all the values of \( s \):

\[
(9.5) \quad e^{uvwxy}k_{uv}k_{xy} = 0.
\]

In the four-dimensional formalism these conditions take the form

\[
(9.6) \quad e^{ikrs}k_{ik}k_{rs} = 0, \quad e^{ikrs}k_{kr}k_{rs} = 0
\]

and in the three-dimensional formalism we can write

\[
(9.7) \quad k \cdot k' = 0, \quad k' \cdot k'' = 0,
\]

\[
(9.8) \quad k' = (k^0)^{-1}k'' \times k.
\]

It is clear that eq. (9.8) implies eq. (9.7) and therefore the whole set of conditions (9.5). We see that these conditions are not independent and define a seven-dimensional manifold.

The \( O(2, 3) \)-invariant manifold defined by eq. (9.5) can be parametrized by means of the coordinates \( k, k'', k^0 \) and it is easy to control that the measure defined by

\[
(9.9) \quad (k^0)^{-2}d^3k d^3k'' dk^0
\]

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is invariant under $O(2, 3)$. In order to get an orbit of $SO^\uparrow(2, 3)$, we have to introduce the further invariant condition

\begin{equation}
\frac{1}{2} g^{uv} g^{vy} k_{uv} k_{xy} = (k^0)^2 - (k)^2 + (k')^2 - (k'')^2 = s^2,
\end{equation}

that, together with eq. (9.8) gives

\begin{equation}
k^0 = 
\pm \left( \frac{1}{2} ( (k)^2 + (k'')^2 + s^2) \pm \frac{1}{2} \left( (k)^2 + (k'')^2 + s^2 - 4 \|k'' \times k\|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.
\end{equation}

This formula describes the orbit we are considering if we choose the sign $+$ twice.

From eqs. (9.8) and (9.10) we see that in the decomposition of this orbit into orbits of the Lorentz group, we find, besides the above mentioned orbit with mass $\mu = s$, the allowed orbits labelled, in accord with eq. (8.4), by the parameters

\begin{equation}
M = j = 0, \quad \mu^2 = 1 - c^2 + s^2 > s^2.
\end{equation}

In order to find an invariant measure on this orbit, we have to multiply eq. (9.9) by the appropriate invariant $\delta$-function and integrate over $dk^0$. The result is

\begin{equation}
dm_s = \delta \left( (k^0)^2 - (k)^2 + (k')^2 - (k'')^2 - s^2 \right) (k^0)^{-2} d^3k d^3k' dk^0 = 
\frac{1}{2} (k^0)^{-1} \left( (k^0)^2 - (k')^2 \right)^{-1} d^3k d^3k' d^3k'',
\end{equation}

where $k'$ and $k^0$ are given by eqs. (9.8) and (9.11).

The Fourier transform of this measure can be performed in two steps:

\begin{equation}
V_s(x, x', x'') = (2\pi)^{-3} \int \exp(-ik \cdot x) A(k^0, k, x', x'') d^4k,
\end{equation}

\begin{equation}
A(k^0, k, x', x'') = (k^0)^{-2} \int \exp(ik' \cdot x' - ik'' \cdot x'') \times
\delta \left( k^0 - 2\frac{1}{2} ((k)^2 + (k'')^2 + s^2) \frac{1}{2} \right) \delta \left( (k^0)^2 - (k)^2 + (k')^2 - (k'')^2 - s^2 \right) d^3k''.
\end{equation}

If $k = 0$, we also have $k' = 0$ and therefore,

\begin{equation}
A(k^0, 0, x', x'') =
\theta(k^0 - s)(k^0)^{-2} \int \exp(-ik'' \cdot x'') \delta((k'')^2 - (k^0)^2 + s^2) d^3k'' =
2\pi(k^0)^{-2}((k^0)^2 - s^2) \frac{1}{2} \theta(k^0 - s)v^{000}(x''),
\end{equation}

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where \( v^{0c0}(x'') \) is given by eqs. (8.21) and (8.23) with \( 1 - c^2 = \mu^2 - s^2 \). In general, by means of the Lorentz transformation \( a_k \) we obtain

\[
A(k^0, k, x', x'') = 2\pi\mu^{-2}(\mu^2 - s^2)^{1/2}\theta(\mu^2 - s^2)\theta(k^0)v^{0c0}(x''),
\]

where \( x'' \) is given by eq. (8.18) and \( \mu^2 = (k^0)^2 - (k)^2 \). In conclusion, we have

\[
V_s(x, x', x'') = (\pi^{-2} J_0(p\rho) t^{-1} \sin tpdpdq,
\]

\[
W_s(x, u_3(\alpha)a_3(\beta)) = 2 \int_0^\infty \int_0^\infty (q^2 - p^2)^{-1}(q^2 - p^2 - s^2)^{1/2} \times
\]

\[
\times \theta(q^2 - p^2 - s^2) \Delta(q\sigma, \epsilon(x^0))J_0(pp)(c\sinh \zeta)^{-1} \sinh(c\zeta) pdp dq,
\]

where

\[
t^2 = (q^2 - p^2)^{-1}(q^2 - p^2 - s^2)(p^2\alpha^2 + q^2\beta^2)
\]

and

\[
W_s(x, u_3(\alpha)a_3(\beta)) = 2 \int_0^\infty \int_0^\infty (q^2 - p^2)^{-1}(q^2 - p^2 - s^2)^{1/2} \times
\]

\[
\times \theta(q^2 - p^2 - s^2) \Delta(q\sigma, \epsilon(x^0))J_0(pp)(c\sinh \zeta)^{-1} \sinh(c\zeta) pdp dq,
\]

where

\[
e^2 = s^2 - q^2 + p^2 + 1, \quad \cosh \zeta = (q^2 - p^2)^{-1}(q^2\cosh \beta - p^2\cos \alpha).
\]

The field defined by \( W_s \) satisfies the equations

\[
e^{uvwx} A_{uv} A_{wx} \psi = 0,
\]
written in the five-dimensional formalism. In fact, it is a direct integral of fields that satisfy eqs. (7.34)-(7.36) and (7.38) with the parameters constrained by eq. (9.12). This field describes “particles” with vanishing spin and a continuous mass spectrum lying on the half-line \( \mu \geq s \).

The function \( V_s \) has been computed in ref. 30 and, with the conventions adopted here, is given by

\[
V_s(x,x',x'') = \lambda_1^{-1} \lambda_2^{-1} (\lambda_1 + \lambda_2)^{-1} \exp(-\frac{1}{2} s (\lambda_1 + \lambda_2)),
\]

where \( \lambda_1, \lambda_2 \) are given by

\[
\lambda_{1,2}^2 = A \pm (A^2 - B)^{1/2},
\]

\[
A = -\sigma^2 + \rho^2 - \alpha^2 + \beta^2;
B = (\sigma^2 - \rho^2 - \alpha^2 - \beta^2)^2 - 4\rho^2 (\alpha^2 + \beta^2)
\]

(see the Appendix B for more details). The signs of \( \lambda_1, \lambda_2 \) are determined in such a way that their real parts are positive or, if one of them vanishes, it becomes positive after the addition of a small positive imaginary part to \( x^0 \).

Starting from eqs. (9.20)-(9.23), and introducing the new integration variables \( p' = \epsilon p, \ q' = \epsilon q \), it is easy to prove that

\[
\lim_{\epsilon \to 0} (\epsilon^3 W_{s}(\epsilon x, u_3(\epsilon \alpha) a_3(\epsilon \beta))) = V_0(x, \alpha, \beta),
\]

\[
\lim_{\epsilon \to 0} (\epsilon^3 V_{s}(\epsilon x, \epsilon \alpha, \epsilon \beta)) = V_0(x, \alpha, \beta).
\]

This means that in the short-distance limit, namely near to the unit of the group, the v. e. v. \( W_s \) coincides with the distribution \( V_0 \), symmetric with respect to \( Sp(4, \mathbb{R}) \). In other words, the symmetry broken by the structure of the Poincaré group survives in the short distance limit. We also see that in the short-distance limit the dependence on the parameter \( s \) disappears.

X. Radiation from an Accelerated Source.

The simplest exercise with a free quantum field is its interaction with an external source. We consider a scalar Hermitean field and we write the time integral of the interaction Hamiltonian in the form

\[
F = \int H'(t) \, dt = \int f(x, a) \psi(x, a) \, d^4 x \, d^6 a.
\]
The scattering operator is given by

\[ S = \exp(-iF) = \exp\left(-\frac{1}{2}\|F\Omega\|^2\right) : \exp(-iF) : \]

(time ordering is not necessary, since it introduces only an overall phase factor).

The number of emitted particles follows a Poisson distribution with average value

\[ \langle n \rangle = \|F\Omega\|^2. \]

This is just the quantity given by eq. (3.12).

In general, the function \( f \) that describes the source is not arbitrary; it may be subject to some conservation law or to some other constraint arising from the field equations. For instance, it is not clear if a point particle has to be described by a one-dimensional trajectory in \( \tilde{\mathcal{P}} \) or by a manifold with higher dimension as it is discussed in ref. 31. The formulation of possible constraints requires a deeper understanding of the theory and we disregard this problem in the following exercise. We consider a source that is bound to an accelerated frame, obtained from an initial frame by means of the following one-parameter group of Poincaré transformations:

\[ t \to \left((a^{-1}\sinh(at), 0, 0, a^{-1}(\cosh(at) - 1)), a_3(at)\right). \]

An infinitesimal transformation of this group is the product of a time translation by an infinitesimal amount \( dt \) and a boost along the \( z \) axis with infinitesimal velocity \( adt \), where \( a \) represents a constant acceleration. The parameter \( t \) is the proper time of the accelerated frame.

Since a point source is too singular, we consider a source concentrated on a disk lying in the \( x^1, x^2 \) plane of the accelerated frame; then we have:

\[ F = \int P(r)Q(t) \times \]

\[ \times \psi \left((a^{-1}\sinh(at), r \cos \phi, r \sin \phi, a^{-1}(\cosh(at) - 1)), a_3(at)\right) rdr d\phi dt, \]

where \( P(r) \) is the “density” of the disk and \( Q(t) \) is a function equal to one in an interval of length \( T \) and going smoothly to zero outside this interval. We expect that when \( T \) is large, \( \langle n \rangle \) is proportional to \( T \).

Then from eq. (10.3) we obtain

\[ \langle n \rangle = \]

\[ = \int P(r_1)P(r_2)Q(t_1)Q(t_2)W(x, a_3(at_2 - at_1)) r_1 dr_1 d\phi_1 r_2 dr_2 d\phi_2 dt_1 dt_2, \]

where

\[ x^0 = a^{-1}\sinh(at), \quad x^3 = a^{-1}(\cosh(at) - 1), \quad t = t_2 - t_1, \]

\[ x^1 = r_2 \cos \phi_2 - r_1 \cos \phi_1, \quad x^2 = r_2 \sin \phi_2 - r_1 \sin \phi_1. \]
Note that $W$ depends on the quantities (4.5), that take the form

\begin{equation}
\rho^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\phi_2 - \phi_1),
\end{equation}

(10.8)

\begin{equation}
\sigma = 2a^{-1} \sinh(\frac{1}{2}at).
\end{equation}

(10.9)

We can also write

\begin{equation}
\langle n \rangle = 2\pi \int P(\rho) \hat{Q}(t) W(x, a_3(\beta)) \rho d\rho dt,
\end{equation}

where

(10.10)

\begin{equation}
x = (\sigma, \rho, 0, 0), \quad \beta = at,
\end{equation}

(10.11)

\begin{equation}
\hat{P}(\rho) = \int P(r) P(\rho^2 + r^2 + 2\rho r \cos \phi) rdr d\phi,
\end{equation}

(10.12)

\begin{equation}
\hat{Q}(t) = \int Q(t_1) Q(t + t_1) dt_1.
\end{equation}

(10.13)

When the proper time interval $T$ is large, we have

\begin{equation}
T^{-1} \hat{Q}(t) \simeq 1 - |t| T^{-1}.
\end{equation}

(10.14)

It follows that in the same limit the average number of produced particles per unit of proper time is given by

\begin{equation}
T^{-1} \langle n \rangle = 2\pi \int_{-\infty}^{\infty} dt \int_{0}^{\infty} \hat{P}(\rho) W(x, a_3(\beta)) \rho d\rho,
\end{equation}

if the integral converges. If we take eq. (3.29) into account, we can write

\begin{equation}
T^{-1} \langle n \rangle = 4\pi \int_{0}^{\infty} dt \int_{0}^{\infty} \hat{P}(\rho) \text{Re} W(x, a_3(\beta)) \rho d\rho.
\end{equation}

(10.15)

(10.16)

Now we consider the theory defined in Sec. IX and we look for singularities of the integral (10.16). Since the singularities arise for small values of $t$ and $\rho$, we approximate the function $W$ by means of eqs. (9.26)-(9.29), namely we use the formula

\begin{equation}
W(x, a_3(\beta)) \simeq \lambda_1^{-1} \lambda_2^{-1} (\lambda_1 + \lambda_2)^{-1} = (\lambda_1^2 - \lambda_2^2)^{-1} (\lambda_2^{-1} - \lambda_1^{-1}) =
\end{equation}

\begin{equation}
= (4\rho \beta)^{-1} \left( ((\rho - \beta)^2 - \sigma^2)^{-\frac{1}{2}} - ((\rho + \beta)^2 - \sigma^2)^{-\frac{1}{2}} \right),
\end{equation}

(10.17)
where $\beta$ and $\sigma$ are given as functions of $t$ by eqs. (10.9) and (10.11).

After some calculation, we obtain

\[
\begin{align*}
\int \hat{P}(\rho) \text{Re} W(x,a_3(\beta)) \rho d\rho &= (4\beta)^{-1} \int_{\sigma}^{\infty} (\hat{P}(r + \beta) - \hat{P}(|r - \beta|))(r^2 - \sigma^2)^{-\frac{1}{2}} dr + \\
&+ \theta(\beta - \sigma)(2\beta)^{-1} \int_{\sigma}^{\beta} \hat{P}(\beta - r)(r^2 - \sigma^2)^{-\frac{1}{2}} dr.
\end{align*}
\]

If $\hat{P}(\rho)$ has a bounded derivative, the first integral in the right hand side has at most a logarithmic singularity for small $t$. The second integral is present only if $a > 1$ and for small $t$ it behaves as

\[
\hat{P}(0)(2at)^{-1} \log \left( a + (a^2 - 1)^{\frac{1}{2}} \right).
\]

As a consequence, if $a > 1$, the integral over $t$ in eq. (10.16) diverges. In other words, if the acceleration $a$ is larger than a critical value, conventionally taken equal to one, the number of particles and the energy radiated per unit proper time become infinite.

**Appendix A: Properties of the functions $w^{M_{\infty}}(a)$.**

The quantities defined by eqs. (5.14) or (5.18) are elementary functions, but for large values of $j$ they are too complicated and it is preferable to introduce some integral representations. We start from the realization of the representation $D^M_{\infty}$ by means of operators acting on the square integrable functions defined on $SU(2)$ which have the covariance property

\[
f(u_3(\phi)u) = \exp(-iM\phi)f(u).
\]

An orthonormal basis in this Hilbert space is given by the functions

\[
f_{jm}(u) = (2j + 1)^{\frac{1}{2}} R^j_m(u).
\]

The invariant measure $d^3u$ on $SU(2)$ is normalized in such a way that the measure of the whole group is one.

We consider the decomposition

\[
a = k(a)a_0,
\]

where

\[
a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL(2,\mathbb{C}),
\]

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(A.5) \[ k(a) = \begin{pmatrix} (p(a))^{-1} & q(a) \\ 0 & p(a) \end{pmatrix}, \quad p(a) > 0, \]

(A.6) \[ a_0 = \begin{pmatrix} \frac{\alpha}{-\beta} \\ \frac{\beta}{\alpha} \end{pmatrix} \in SU(2), \quad |\alpha|^2 + |\beta|^2 = 1. \]

We see that

(A.7) \[ p(a) = (|a_{21}|^2 + |a_{22}|^2) \frac{1}{2}, \]

(A.8) \[ \alpha = (p(a))^{-1}a_{22}, \quad \beta = -(p(a))^{-1}a_{21}. \]

We shall use the properties

(A.9) \[ p(au) = p(a), \quad (au)_0 = a_0 u, \quad u \in SU(2), \]

(A.10) \[ p(u_3(\phi)a) = p(a), \quad (u_3(\phi)a)_0 = u_3(\phi)a_0. \]

The representation operator is defined by

(A.11) \[ [D_{M^c}^M(a)f](u) = (p(ua))^{2c-2}f((ua)_0) \]

and the matrix elements we need are given by

(A.12) \[ D_{j mj m}^{M^c}(a) = (f_{jm}, D_{M^c}^M(a)f_{jm}) = \]
\[ = (2j + 1) \int_{SU(2)} \overline{R}_{Mm}^j(u)(p(ua))^{2c-2}R_{Mm}^j((ua)_0) d^3u. \]

If we sum over \( m \) and we use eq. (A.9), we get the required formula

(A.13) \[ w_{M^c j}^M(a) = \int_{SU(2)} (p(ua))^{2c-2}R_{M M}^j((uau^{-1})_0) d^3u. \]

We can also use the integral representation

(A.14) \[ R_{M M}^j(u) = (2\pi)^{-1} \int_0^{2\pi} F_{M j}^M(u_3(-\phi)u u_3(\phi)) d\phi, \]

where

(A.15) \[ F_{M j}^M(u) = (\alpha - \beta)^{j+M}(\overline{\alpha} + \beta)^{j-M} \]
and we have used the expression (A.6) for the matrix $u$. If we substitute eq. (A.14) into eq. (A.13) and we use the properties (A.9) and (A.10), we obtain

\[(A.16) \quad w^{Mcj}(a) = \int_{SU(2)} (p(uu))^2e^{-2F^Mj((uu^{-1})_0)}d^3u\]

or, more explicitly,

\[(A.17) \quad w^{Mcj}(a) = \int_{SU(2)} (|b_{21}|^2 + |b_{22}|^2)^{c-1-j} (\overline{b}_{22} + b_{21})^j + M (b_{22} - \overline{b}_{21})^j - M d^3u,\]

where

\[(A.18) \quad b = uu^{-1}.\]

The exponential mapping can be written in the form

\[(A.19) \quad a = \exp\left(\frac{1}{2}(x'' - ix') \cdot \sigma\right) = \cosh \chi + \frac{1}{2} \chi^{-1} \sinh \chi (x'' - ix') \cdot \sigma,\]

where

\[(A.20) \quad \chi^2 = \frac{1}{4}(x'' - ix') \cdot (x'' - ix').\]

If we indicate by $R(u)$ the $SO(3)$ rotation matrix corresponding to the element $u \in SU(2)$ and we put

\[(A.21) \quad y' = R(u)x', \quad y'' = R(u)x'',\]

we have

\[(A.22) \quad b = uu^{-1} = \cosh \chi + \frac{1}{2} \chi^{-1} \sinh \chi (y'' - iy') \cdot \sigma.\]

In particular, it is

\[(A.23) \quad b_{21} = \frac{1}{2}(y''_1 - iy'_1 + iy''_2 + y'_2)\chi^{-1} \sinh \chi,\]

\[b_{22} = \cosh \chi + \frac{1}{2}(-y''_3 + iy'_3)\chi^{-1} \sinh \chi.\]

If we substitute these expressions into eq. (A.17) and we disregard terms of order higher that the second in the variables $x'$ and $x''$, we can perform the integral by means of the formulae

\[(A.24) \quad \int_{SU(2)} y' d^3u = 0,\]
(A.25) \[ \int_{SU(2)} y_r y_s \, d^3 u = \frac{1}{3} \delta_{rs} x' \cdot x'' \]

and other similar consequences of eq. (A.21). The result is

(A.26) \[ w^{M, c j}(a) \simeq \]

\[ \simeq 1 - \frac{1}{6} j(j + 1)(x')^2 - \frac{1}{6} (j(j + 1) + 1 - c^2 - M^2)(x'')^2 + \frac{1}{3} i M c \cdot x' \cdot x''. \]

Now we want to compare the integral (A.17) with the integral (8.20). We can expand the exponential in eq. (8.20) keeping terms up to the second order in the variables \( x' \) and \( x'' \) and perform the integral by means of eqs. (A.24) and (A.25). We obtain

(A.27) \[ I(k', k'', x', x'') \simeq 1 - \frac{1}{6} (k')^2(x')^2 - \frac{1}{6} (k'')^2(x'')^2 + \frac{1}{3} k' \cdot k'' \cdot x' \cdot x'' \]

and eq. (8.25) follows immediately.

Another interesting limit can be derived from eqs. (A.17) and (A.23):

(A.28) \[ \lim_{n \to \infty} w^{n M, n c, n j} (\exp((2n)^{-1}(x'' - i x') \cdot \sigma)) = \]

\[ = \int_{SU(2)} \exp((j - c)y_3') \exp(\frac{1}{2}(j + M)(-y_3'' - iy_3' + y_1' - iy_1'' + iy_2') \times \]
\[ \times \exp(\frac{1}{2}(j - M)(-y_3'' + iy_3' - y_1' + iy_1'' - iy_2')) \, d^3 u = I(k', k'', x', x''), \]

where

(A.29) \[ k' = (-j, -iM, -M), \quad k'' = (iM, -j, -ic). \]

The quantity \( I(k', k'', x', x'') \) defined in eq. (8.20) is an entire analytic function of its arguments and it depends on \( k', k'' \) through the invariants

(A.30) \[ (k')^2 = j^2, \quad (k'')^2 = j^2 - c^2 - M^2, \quad k' \cdot k'' = i M c. \]

If we indicate by \( k'_n, k''_n \) the coordinates of a representative point of the orbit defined by the parameters \( n M, n c, n j, \) we have

(A.31) \[ \lim_{n \to \infty} v^{n M, n c, n j} (n^{-1} x', n^{-1} x'') = \lim_{n \to \infty} I(k'_n, k''_n, n^{-1} x', n^{-1} x') = \]

\[ = \lim_{n \to \infty} I(n^{-1} k'_n, n^{-1} k''_n, x', x') = I(k', k'', x', x'), \]

where

(A.32) \[ k' = \lim_{n \to \infty} (n^{-1} k'_n), \quad k'' = \lim_{n \to \infty} (n^{-1} k''_n). \]

Since these limits satisfy eq. (A.30), we see that the limits (A.28) and (A.31) are equal. This result can be used to generalize the treatment of Sec. IX to a larger class of theories with a broken higher symmetry.
Appendix B: Geometry of the vector spaces $\mathcal{T}$ and $\mathcal{T}^*$.

In this Appendix we summarize some results of refs. 2, 3, 7. We use the Dirac matrices with the properties

\begin{equation}
\gamma_i \gamma_k + \gamma_k \gamma_i = 2g_{ik}, \quad \gamma_k^T = -C^{-1} \gamma_k C, \quad C^T = -C.
\end{equation}

We adopt the Majorana representation in which the Dirac matrices are real and we put $C = \gamma_0$. The vectors of $\mathcal{T}$ and $\mathcal{T}^*$ can be labelled, respectively, by means of the real symmetric $4 \times 4$ matrices

\begin{equation}
\hat{x} = \frac{1}{2} x^k C^{-1} \gamma_k - \frac{1}{4} x^{rs} C^{-1} \gamma_r \gamma_s,
\end{equation}

\begin{equation}
\hat{k} = -\frac{1}{2} k_k \gamma^k C + \frac{1}{4} k_{rs} \gamma^r \gamma^s C.
\end{equation}

The closed cones $\mathcal{T}^+$ and $\mathcal{T}^{*+}$ contain the elements labelled by positive semidefinite matrices.

The following formulas are useful:

\begin{equation}
\text{Tr} (\hat{k} \hat{x}) = -k_k x^k + \frac{1}{2} k_{rs} x^{rs} = k^0 x^0 - \mathbf{k} \cdot \mathbf{x} + \mathbf{k}' \cdot \mathbf{x}' - \mathbf{k}'' \cdot \mathbf{x}'',
\end{equation}

\begin{equation}
A = \text{Tr} (C \hat{x})^2 = x_k x^k - \frac{1}{2} x_{rs} x^{rs} = -(x^0)^2 + (\mathbf{x})^2 - (\mathbf{x}')^2 + (\mathbf{x}'')^2,
\end{equation}

\begin{equation}
s^2 = \text{Tr} (\hat{k} C^{-1})^2 = k_k k^k - \frac{1}{2} k_{rs} k^{rs} = -(k^0)^2 + (k)^2 - (k')^2 + (k'')^2,
\end{equation}

\begin{equation}
B = 16 \det \hat{x} = ((x^0)^2 - (\mathbf{x})^2 - (\mathbf{x}')^2 - (\mathbf{x}'')^2)^2 - 4||\mathbf{x} \times \mathbf{x}'||^2 - 4||\mathbf{x} \times \mathbf{x}''||^2 - 8x^0 \mathbf{x}'' \cdot \mathbf{x}' \times \mathbf{x}
\end{equation}

and a similar formula for $\det \hat{k}$. Note that the parameter $s$ which appears in eqs. (B.6) and (9.10) was indicated by $2s$ in ref. 30.

A vector of $\mathcal{T}$ belongs to $\mathcal{T}^+$ when $x^0$ is larger or equal to the largest root of the equation $\det \hat{x} = 0$ and a similar statement holds for $\mathcal{T}^{*+}$. For $k = 0$ and $k^0 = \mu > 0$, we have

\begin{equation}
16 \det \hat{k} = (\mu^2 - (k')^2 - (k'')^2)^2 - 4||\mathbf{k}' \times \mathbf{k}''||^2
\end{equation}

and the condition for belonging to $\mathcal{T}^{*+}$ is just eq. (9.1).

The quantities $\pm \lambda_{1,2}$ introduced in Sec. IX are the eigenvalues of the matrix $2C \hat{x}$ (the factor 2 was not present in ref. 30). They are given by eq. (9.27), where $A$ and $B$ are given by eqs. (B.5) and (B.7). If we assume eqs. (4.5) and (8.27), we get eq. (9.28). It has been shown in ref. 30 that in a flat-space theory invariant under $Sp(4, \mathbb{R})$ the commutator vanishes unless one of the quantities $\lambda_{1,2}^2$ is real negative. This means that the commutator vanishes if $A^2 - B < 0$ or if $A > 0$, $B > 0$. These conditions are satisfied if $\sigma^2 < 0$ or if eq. (9.3) holds.
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