Cosmic strings

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Abstract

The topic of cosmic strings provides a bridge between the physics of the very small and the very large. They are predicted by some unified theories of particle interactions. If they exist, they may help to explain some of the largest-scale structures seen in the Universe today. They are ‘topological defects’ that may have been formed at phase transitions in the very early history of the Universe, analogous to those found in some condensed matter systems—vortex lines in liquid helium, flux tubes in type-II superconductors, or disclination lines in liquid crystals. In this review, we describe what they are, why they have been hypothesized and what their cosmological implications would be. The relevant background from the standard models of particle physics and cosmology is described in section 1. In section 2, we review the idea of symmetry breaking in field theories, and show how the defects formed are constrained by the topology of the manifold of degenerate vacuum states. We also discuss the different types of cosmic strings that can appear in different field theories. Section 3 is devoted to the dynamics of cosmic strings, and section 4 to their interaction with other fields. The formation and evolution of cosmic strings in the early Universe is the subject of section 5, while section 6 deals with their observational implications. Finally, the present status of the theory is reviewed in section 7.

This review was received in November 1994.
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1. Introduction

1.1. Outline

There have been rapid and exciting developments over the last few years on the interface between particle physics and cosmology. Particle physicists pursuing the goal of unification would like to test their theories at energy scales far beyond those available now or in the future in terrestrial accelerators. An obvious place to look is to the very early Universe, where conditions of extreme temperature and density obtained. Meanwhile, cosmologists have sought to understand features of the Universe currently observed by tracing their history back to that very early period.

The early Universe was an intensely violent environment, so it is not easy to find direct traces of very early events. There are, however, some special events that may have left traces still visible today—in particular, phase transitions. If our present ideas about unification of forces and about the Big Bang are correct, then the Universe, in the first fraction of a second after its birth, underwent a series of phase transitions. Like more familiar transitions in condensed matter systems, these may have led to the formation of defects of one kind or another—domain walls, strings or vortices, monopoles, or combinations of these. In many cases, such defects are stable for topological or other reasons and may therefore survive, a few of them even to the present day. If defects exist, they constitute a uniquely direct connection to the highly energetic events of the early Universe.

Cosmic strings in particular have very intriguing properties. They are very massive objects and may have played an important role in structure formation, perhaps providing at least some of the density inhomogeneities from which galaxies eventually grew. Some of their observational signatures are quite distinctive, and within observational reach.

In this review, our aim is to explain what cosmic strings are, how they are formed, how they move and evolve, and what their cosmological implications might be.

In this introductory section, we review the necessary background material, the basic ideas about phase transitions, defects and the Big Bang. We begin with the concept of symmetry and spontaneous symmetry breaking, and discuss the restoration of symmetry at high temperatures. Next we examine the conditions for defect formation at a phase transition. Then we review briefly the standard cosmological model, the Hot Big Bang, and explain why, in conjunction with ideas about unification, it suggests a sequence of phase transitions in the very early Universe. The section ends with a summary of the properties of cosmic strings.

In section 2, we discuss the topological conditions under which string defects are formed in field theories. The dynamics of strings are reviewed in section 3 and their interaction with other constituents of the Universe in section 4. Section 5 is devoted to the formation and evolution of defects, especially strings in the early Universe, and section 6 to their observational implications. The present status of the theory is summarized in section 7.

1.2. Symmetry, spontaneous symmetry breaking and symmetry restoration

The concept of symmetry in particle physics runs very deep. In the space available we can only sketch the issues involved, and refer readers to reference books (for a selection see [1]) for more detail. The idea that physical laws should be invariant under groups
of transformations is a powerful one, as the theories of relativity attest. In the context of quantum field theories, symmetries group particles together and relate their scattering probability amplitudes. The importance of symmetry in the current understanding of the fundamental structure of nature is such that particle physics can be characterized as the search for a single underlying symmetry behind the interactions of particles and fields.

The symmetries of quantum field theory come in two classes: space–time and internal. The former are those of the space–time through which the field propagates; for example, empty space has the Poincaré group of symmetries, consisting of translations, rotations and Lorentz boosts. Internal symmetries relate fields to one another. To implement these symmetries, a field must have some well-defined transformation properties under the action of the symmetry group: that is, it must form a representation of the group. The labels of different representations are often called quantum numbers. For example, the representations of the Poincaré group are labelled by their mass and their spin. Internal symmetries transform fields into one another, and (with the exception of fundamental string theory—not directly related to cosmic strings) there are only finite numbers of fields. The corresponding groups are finite dimensional and compact. If they are not discrete groups then they are Lie groups; thus the study of Lie groups and their representations assumes fundamental importance in particle physics. Generally, a relativistic quantum field theory is specified by the representations of the fields which comprise it (i.e. their masses, spins and internal quantum numbers), which are then assembled into an invariant Lagrangian.

The most powerful type of symmetry is the gauge (or ‘local’) symmetry, where the Lagrangian is invariant under a symmetry transformation which may be different at every point. (If the invariance exists only for transformations which are constant in space and time, the symmetries are distinguished by being called ‘global’ or ‘rigid’.) For internal symmetries, this requires a spin-1 field, the gauge field, of which the electromagnetic field is the prototypical example. Gravity can also be formulated as a gauge theory of local Lorentz transformations: the gauge field is the gravitational field itself, which has spin 2.

Lastly, there is supersymmetry (see for example [2]). This is a powerful symmetry which relates particles of different spins and, in a sense, combines internal and space–time aspects. The prospect of uniting the particles of matter (spin-$\frac{1}{2}$ quarks and leptons) with those of the known forces (spin-1 gauge bosons) is clearly an exciting one, which is so far unrealized, for none of the known particles fit together into representations of supersymmetries, called supermultiplets. Perhaps the superpartners of today’s particles will be discovered at the Large Hadron Collider currently under construction at CERN.

The problem with invoking all these symmetries is how to hide or ‘break’ them. If one proposes a symmetric Lagrangian, as well as a symmetric ground state for the theory, then there is a deep theorem which states that the existence of a gauge symmetry implies that the associated spin-1 particles are massless. Only one such particle is known: the photon. The resolution of the problem is the breaking of the symmetry of the ground state by what is known as the Higgs mechanism. This works by introducing spin-0 fields (denoted $\phi$) transforming non-trivially under the symmetry group $G$, and constructing for them an energy density which is minimized at some non-zero value $\phi_0$. The theory then has a ground state which is invariant only under the subgroup of $G$ that leaves $\phi_0$ unchanged. We say that the symmetry $G$ is broken to $H$.

It is often useful to draw analogies with condensed matter physics, where similar mechanisms occur. A good example is a nematic liquid crystal [3], which consists of rod-like or disk-like molecules. The free-energy density of this system is invariant under the group of spatial rotations, SO(3). This is a global symmetry: the thermodynamics should not depend on the orientation of the sample. However, at low temperatures and at high
pressures the molecules prefer to line up: thus in any particular sample the full rotational
symmetry will not be respected by the direction of alignment. The rotational symmetry is
still there, for there is no preferred alignment direction, but it is broken by the equilibrium
state, which has all the molecules pointing one way. A subgroup of the rotation group
is left unbroken, which consists of rotations around the alignment direction and rotations
through \( \pi \) about axes in the orthogonal plane (the interactions between the molecules do
not distinguish between their ends). This group is \( O(2) \), called \( D_\infty \) in the condensed matter
literature. This breaking is usually denoted

\[
SO(3) \rightarrow O(2). \tag{1.1}
\]

The degree and direction of alignment of the molecules, when averaged over many molecular
spacings, can be described by a director field \( \Phi(\omega) \). This field must transform appropriately
under \( SO(3) \) to describe a 'headless vector': it turns out to be a traceless symmetric tensor.
In the nematic phase,

\[
\Phi_{ij} = \lambda(n_in_j - \frac{1}{3}\delta_{ij}) \tag{1.2}
\]

where \( \pm n_i \) is the alignment axis of the molecules, known as the director field.

A good description of the transition between the disordered \( (\Phi = 0) \) and the ordered
\( (\Phi \neq 0) \) phases can be obtained by postulating that the free-energy density for constant
fields takes the form

\[
f(\Phi) = \alpha + \beta \operatorname{tr} \Phi^2 + \gamma \operatorname{tr} \Phi^3 + \delta(\operatorname{tr} \Phi^2)^2 + \ldots. \tag{1.3}
\]

The value of \( \Phi \) at the minimum of \( f \) depends on the coefficients \( \alpha - \delta \), which are temperature
and pressure dependent. In particular, the transition occurs when the sign of \( \beta \) changes.
The form of the free-energy function can tell us whether the transition is first or second
order: if \( \gamma \neq 0 \), the equilibrium free energy changes discontinuously from one minimum
to another, so that we have a first-order transition; only if \( \gamma = 0 \) is this transition second
order.

In the quantum field theory of scalar fields at non-zero (often called finite) temperature
the free-energy density for constant fields is known as the finite-temperature effective
potential \( V_T(\phi) \) (see [4] and section 5.1). This is in principle calculable from the
zero-temperature classical potential in a perturbative expansion in powers of \( \hbar \), the loop
expansion. One finds that broken symmetries in quantum field theories are almost always
restored at high enough temperatures.

### 1.3. Defect formation

In some phase transitions it happens that the order parameter does not take up its equilibrium
magnitude everywhere, and indeed may vanish inside objects known as topological defects.
Defects may be two, one or zero dimensional, and their origin lies in the topological
properties of the set of equilibrium states, i.e. the minima of the free energy or the effective
potential.

As an example, let us take a theory of a single real scalar field \( \phi \), with a reflection
symmetry \( \phi \leftrightarrow -\phi \). The energy functional for static fields has the form

\[
\mathcal{H} = \frac{1}{2}(\nabla \phi)^2 + V(\phi) \tag{1.4}
\]
where the zero-temperature classical potential is $V(\phi) = \frac{1}{8}\lambda(\phi^2 - \eta^2)^2$. The equilibrium states have $\phi = \phi_{\pm} = \pm \eta$. Suppose that in one region of space we find the field at $\phi_+$, and nearby at $\phi_-$. Then along any path joining these two regions the field must necessarily pass through zero at least once. The set of points where the field vanishes form a two-dimensional surface, or domain wall, which separates domains of $\phi = \phi_+$ from domains of $\phi = \phi_-$. There is energy associated with the wall, because the potential energy density inside is higher than its equilibrium value, and also because the field is not constant.

It is the disconnected nature of the set of equilibrium states of the theory which allows domain walls to exist. Other types of defect have different topological requirements, which we shall discuss in section 2. Here we will merely note that our example system, the nematic liquid crystal, allows both line and point defects [5]. (In the condensed matter literature, such defects, arising from broken rotational symmetries, are called disclinations.) Around the line disclinations the molecules change their orientation by a rotation through angle $\pi$ around an orthogonal axis, while in a point defect the molecules are in a 'hedgehog' configuration, directed away from a central point. Along the central lines and points of the disclinations, the molecules cannot have any preferred alignment direction and so the order parameter vanishes.

![Figure 1](image.png)

Figure 1. Defect formation at the isotropic–nematic phase transition in a liquid crystal. As the sample cools, bubbles of the ordered, nematic, phase nucleate (a), grow (b), and merge (c), resulting in the formation of a network of line disclinations at the bubble boundaries (d). The scale of the network then increases as strings straighten and loops collapse (e). (Reprinted with permission from Bowick M J, Chandar L, Schiff E A and Srivastava A M 1994 *Science* 264 943. Copyright 1994 American Association for the Advancement of Science.)

In section 5.2 we consider in detail how strings (line defects) are formed in cosmological phase transitions [6]. However, the analogy between the nematic liquid crystal and a field
theory makes it plausible that what happens in the laboratory in a cooling sample of the liquid crystal also happens in the early Universe. Figure 1 shows a sequence of frames taken from a video of an isotropic–nematic phase transition in action [7]. The transition is first order, and we see that it proceeds by the nucleation of bubbles of the low-temperature, nematic phase. These bubbles grow, so reducing the total energy of the system, and then finally merge. Out of the mess emerges a network of strings. What happens is that the director field takes up random directions in each bubble when it nucleates. When a set of bubbles nucleates there is an appreciable probability that the field will be so misaligned that disclinations are formed at the interstices.

1.4. The Big Bang

The Standard Model of the early Universe is based on two observational pillars: the recession of galaxies and the cosmic microwave background (CMB).

Hubble observed in 1926 that recession velocities determined from redshifts are roughly proportional to distance [8]:

\[ v = Hr \]

where \( H \) is the Hubble parameter,

\[ H = 100h \text{ km s}^{-1} \text{Mpc}^{-1}. \]

The dimensionless parameter \( h \), in the range \( 0.5 \lesssim h \lesssim 1 \), encodes our present uncertainty. The characteristic expansion time is

\[ H^{-1} = 10^{10}h^{-1} \text{yr}. \]

The microwave background radiation, first observed by Penzias and Wilson [9], is the redshifted relic of radiation emitted when the Universe was dense and hot. It has a very accurate blackbody spectrum, with a temperature of \( 2.726 \pm 0.010 \text{ K} \) [10].

It is often convenient to use comoving coordinates, so that (neglecting small random velocities) each galaxy retains the same fixed coordinates. The true distance \( r(t) \) to a galaxy is related to its coordinate distance \( x \) by a universal scale factor \( a(t) \):

\[ r(t) = a(t)x. \]

Hubble's law, (1.5), then follows, with

\[ H = \dot{a}/a. \]

The redshifts of distant objects are given by \( 1+z = a(t_0)/a(t) \), where \( t_0 \) is the current time and \( t \) is the time when the radiation was emitted by the object.

The evolution of the scale factor \( a \) is described according to general relativity by the Friedmann equation,

\[ \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho - \frac{K}{a^2} \]

where \( G \) is Newton's constant, \( \rho \) the mass density in the Universe (assumed homogeneous on the scales of interest), and \( K \) is a constant, the (uniform) curvature of the spatial sections.
If $K > 0$, the Universe is spatially closed, assuming zero cosmological constant; it will expand to a maximum radius and then contract, eventually reaching the 'big crunch'. If $K \leq 0$, it is open and will continue expanding for ever. The condition for closure may also be expressed in terms of the density: the Universe is closed if $\Omega > 1$, where

$$\rho_c = \frac{\rho}{\rho_c} = \frac{3H^2}{8\pi G} = 1.88h^2 \times 10^{-26} \text{ kg m}^{-3}$$

(1.10)

$\rho_c$ being the critical density. Observation tells us that $\Omega$ lies in the range $0.1 < \Omega < 2$. Theoretical prejudice suggests that $\Omega = 1$, as predicted if the Universe underwent an early inflationary phase (see section 1.6).

The Friedmann equation (1.9) must be supplemented by the energy equation,

$$\dot{\rho} + 3\frac{a}{a}(\rho + p) = 0$$

(1.11)

and also by an equation of state giving the pressure $p$ in terms of the density $\rho$. The most important cases are a universe dominated by radiation, or extremely relativistic particles, with $p = \frac{1}{3}\rho$, and one dominated by dust, or very non-relativistic matter, for which $p = 0$. The energy equation (1.11) tells us that in these cases $\rho \propto a^{-4}$ and $\rho \propto a^{-3}$, respectively. In either case, $\rho$ varies much more rapidly than the curvature term in (1.9), so at early times it is always a good approximation to neglect $K$. The solution is then very simple: $a \propto t^{1/2}$ in the radiation-dominated universe and $a \propto t^{2/3}$ in the case of dust.

The observed large-scale homogeneity and isotropy of the Universe implies that to a good approximation space-time may be described by the Robertson-Walker metric. When $K$ is negligible, we have the spatially flat Einstein-de Sitter universe, with metric

$$ds^2 = dt^2 - a^2(t)dx^2.$$  

(1.12)

The Universe today is probably matter dominated, although it has been suggested recently that we are entering a phase of vacuum energy domination ($\rho \approx \text{constant}$), based on indirect arguments about galaxy correlations [11]. There is compelling evidence that much of the matter whose gravitational effects we can detect, for example through measuring rotation curves of galaxies, is invisible [12]. Moreover, studies of the synthesis of helium and other light elements in the Big Bang [13] place firm limits on the density of ordinary baryonic matter (protons and neutrons), of

$$0.01 < \Omega_B h^2 < 0.015.$$ 

It is therefore probable that most of the matter in the Universe is of some other type, known as 'dark matter'. Dark matter candidates are generally classified as either 'hot' or 'cold', depending on whether the particles were still relativistic when they decoupled from the rest of the matter in the Universe.

Because $\rho_{\text{rad}}$ falls faster than $\rho_{\text{mat}}$, the Universe was initially radiation dominated, until the epoch $t_{\text{eq}}$ of equal radiation and matter densities. If $\Omega = 1$, this occurred at a redshift $z_{\text{eq}} \approx 24,000h^2$, when the Universe was about $1000h^{-4}$ years old. Shortly afterwards, the Universe became sufficiently cool for the electrons and the nuclei to 're'-combine into neutral gas. This happened when the temperature was around 0.3 eV, at a redshift $z_{\text{rec}} \approx 1100$.

These are important milestones, for density perturbations in cold dark matter can start to grow only when the Universe becomes matter dominated, while radiation pressure prevents the growth of perturbations in the baryons until recombination.
The period when cosmic strings may have been formed is very early in the history of the Universe, when it was certainly radiation dominated. The Hubble parameter was then $H = 1/2t$. Since $K$ was negligible, $\rho$ was very nearly equal to $\rho_c$. Thus the density was, by (1.9),

$$\rho = 3/32\pi Gt^2.$$  

(1.13)

One can also find a useful relation between time and temperature. The density of a relativistic gas is given by

$$\rho = \frac{\pi^2}{30}\frac{g_s}{g}T^4$$  

(1.14)

where $g_s$ is the effective number of distinct spin states of relativistic particles (one for particles of zero spin and two for higher spins, multiplied by $\frac{7}{8}$ for fermions) [8]. Equating these two expressions for the density, we find

$$T^2t = \sqrt{\frac{45}{\pi^4 g_s}} \frac{M_{\text{Pl}}}{4\pi} = \frac{2.42}{g_s^{1/2}} \text{MeV}^2 \text{s}.$$  

(1.15)

Here $M_{\text{Pl}}$ is the Planck mass,

$$M_{\text{Pl}} = G^{-1/2} = 1.22 \times 10^{28} \text{eV}.$$  

(We use units in which $c = \hbar = k_B = 1$.)

1.5. Unification

It is now very well established that two of the four fundamental interactions of elementary particles, the electromagnetic and weak interactions, can be described by a unified gauge theory, the Weinberg-Salam theory, based on the gauge group $SU(2) \times U(1)$. At low energies, the two interactions appear very different, but the true underlying symmetry emerges above the characteristic energy scale of the unified theory, around 100 GeV (the rest energy of the W and Z gauge bosons). The theory exhibits a phase transition at a temperature of this order.

The strong interactions too are described by a very successful gauge theory, quantum chromodynamics, based on the group $SU(3)$. This theory may also exhibit a phase transition, with a critical temperature of order 100 MeV, associated with quark confinement. (Above this temperature, we have a dense soup of quarks and gluons; below it we have individual hadrons, like protons, neutrons and pions.)

The low-energy physics of elementary particles is thus described by a gauge theory with three independent coupling constants, $g_3$, $g_2$ and $g_1$, associated with the three groups in the low-energy symmetry $SU(3) \times SU(2) \times U(1)$. These couplings are energy dependent, though only logarithmically. Extrapolating from their low-energy behaviour, it appears that all three will become roughly equal at an energy scale of around $10^{15}$ to $10^{16}$ GeV, only three or four orders of magnitude below the Planck mass. It is therefore very plausible to suppose that there is a grand unified theory, encompassing all three interactions. This grand symmetry would be manifest only above that characteristic energy scale.

If this is correct, there is probably a grand unification phase transition, with a critical temperature of about $10^{15}$ GeV. There is some reason to believe that this is not a single
transition but involves two or more separate transitions. In the simplest models with only
a single transition, the extrapolated couplings do not in fact quite meet. A better fit
can be obtained in models with more than one transition, for example in a model with
supersymmetry broken at some intermediate scale.

Thus the picture emerges of a sequence of phase transitions occurring in the very
early evolution of the Universe. The corresponding times can be estimated from (1.15).
The expected number of spin states \( g_s \), depends on the precise particle-physics model, but
during the relevant period is roughly of order 100. Thus we see that the electroweak and
quark–hadron transitions occur when the age of the Universe is about \( 10^{-11} \) s and \( 10^{-5} \) s,
respectively, while a grand unification transition would occur at between \( 10^{-39} \) and \( 10^{-37} \) s.

Of course one can legitimately question whether we really understand enough of the
basic physics to make reliable predictions of the behaviour of the Universe so close to the
Big Bang. So far as electroweak and quark–hadron transitions are concerned, we do now
have a rather firm understanding of the basic physics. Grand unification does admittedly
represent a very considerable extrapolation from the region where we have reliable tests.
On the other hand, our theories of fundamental particle interactions work extremely well
at ordinary energies and there is no intrinsic reason to expect them to break down until
we reach the Planck energy, the energy at which gravity also becomes strong and quantum
gravity effects necessarily enter. So although we should be properly cautious, we do have
good reasons to think that our basic understanding can be pushed this far. Naturally, if our
confidence turns out to be misplaced, that would be an even more dramatic and interesting
discovery.

1.6. Inflation

Some discussion is in order regarding the relation between the concepts of inflation and of
cosmic strings. There are rival theories of large-scale structure formation based on these
concepts, which at first sight appear incompatible, but can in fact be reconciled if one really
wishes to do so.

The idea of inflation (for a much more complete exposition see [8]) was invented to solve
a number of cosmological puzzles, particularly the horizon problem, the flatness problem
and the monopole problem.

Since the Universe is of finite age, there is at any epoch \( t \) a particle horizon; no signal
from beyond the horizon can yet have reached us. In the radiation-dominated era of the
standard Friedmann–Robertson–Walker (FRW) universe, the particle horizon at time \( t \) is
of radius \( 2t \). When we observe the cosmic microwave background radiation coming to
us from opposite directions in the sky, we are seeing light emitted from regions that were
then separated by nearly 100 times the horizon distance at the time. It therefore seems very
surprising that the temperature of the radiation coming from opposite directions is the same:
no causal process could have established thermal equilibrium between such distant parts of
the Universe. This is the horizon problem.

Another puzzle, the flatness problem, is why \( \Omega \) today, as defined by (1.10), is so close
to unity. It is easy to verify that \( \Omega = 1 \) is an unstable point of the FRW evolution equation
(1.9). To have \( \Omega \) close to 1 today, it must have been very close indeed at early times—
seemingly requiring fine tuning by many orders of magnitude.

Almost all grand unified theories predict the existence of ultra-heavy stable magnetic
monopoles. Once formed, they are very hard to eliminate and so would rapidly come to
dominate the energy density of the Universe, exceeding by many orders of magnitude the
energy density in baryons. This is the monopole problem.
All these problems can be cured by invoking the idea of inflation: a very early period of rapid expansion in which the energy density is dominated by ‘vacuum energy’. This is achieved by introducing a scalar inflaton field $\sigma$ with an appropriately chosen potential $V(\sigma)$. The energy density and pressure due to such a field in a FRW background are given by

$$
\rho = \frac{1}{2} \dot{\sigma}^2 + \frac{1}{2} a^{-2} (\nabla \sigma)^2 + V(\sigma),
$$

$$
p = \frac{1}{2} \dot{\sigma}^2 - \frac{1}{2} a^{-2} (\nabla \sigma)^2 - V(\sigma).
$$

The trick is to arrange that in some sufficiently large region these expressions are dominated by the potential term. Thus we have the strange equation of state $p \approx -\rho$. It follows that $\rho \approx$ constant, and hence, by (1.9), that $a(t)$ increases exponentially. Actually, $\rho$ is not exactly constant. The field $\sigma$ does evolve slowly, eventually reaching a point where the conditions for inflation are no longer satisfied, bringing the period of rapid expansion to an end. During the period of accelerating expansion $\Omega$ tends towards 1 rather than away from it, thus solving the flatness problem. The horizon problem is also cured during inflation, the causal horizon distance increases by a huge factor, so that the entire presently visible Universe was originally well within a single horizon volume. Finally, the expansion dilutes the density of any previously existing monopoles to an undetectably low level.

Unfortunately, inflation also dilutes any other topological defects, in particular cosmic strings. At first sight, therefore, the two ideas are indeed mutually incompatible. However, it is possible to reconcile the two with models in which strings or other defects are formed during the late stages of inflation. We discuss them briefly at the end of section 5.2.

### 1.7. Cosmic strings

In the following sections, we shall deal with many different properties of cosmic strings in detail. To set the scene, it may be useful, however, to review their most important characteristics here.

Cosmic strings are linear defects, analogous to flux tubes in type-II superconductors, or to vortex filaments in superfluid helium. They might have been formed at a grand unification transition, or, conceivably, much later, at the electro-weak transition, or somewhere in between. The standard electroweak model does not predict stable strings, but some quite simple generalizations of it, entirely compatible with available data, do.

These strings have enormous energy. In the simplest, canonical type of string, the energy per unit length, $\mu$, and the string tension are equal. Equivalently, the characteristic speed of waves on the string is the speed of light. This equality is a consequence of local invariance of the field configuration around a string under Lorentz boosts along its direction. Roughly speaking, one expects that for strings produced at a phase transition at $T_c$, $\mu \sim T_c^2$.

An important number in the theory is the dimensionless quantity $G\mu \sim (T_c/M_p)^2$, which characterizes the strength of the gravitational interaction of strings. For grand unified strings, $G\mu$ is of order $10^{-6}$ or $10^{-7}$; it is useful to define a parameter $\mu_6$ by

$$
G\mu = 10^{-6}\mu_6.
$$

Translated into more ordinary units, $\mu$ is very large indeed; for example the mass per unit length is

$$
\mu = 1.35 \times 10^{21} \mu_6 \text{ kg m}^{-1} = 2.09 \times 10^7 \mu_6 M_{\odot} \text{ pc}^{-1}.
$$

\(1.16\)
A grand unified string of length equal to the solar diameter would be as massive as the Sun, while such a length of string formed at the electroweak scale would weigh only 10 mg. The gravitational effects of the latter are essentially negligible, though such strings may still be of great interest, because of other types of interactions. In particular, cosmic strings may behave like thin superconducting wires, with a critical current $\sim T_c$. Superconducting strings interact strongly with magnetic fields in the Universe, and even light ones may have significant astrophysical effects.

The parameter $G\mu$ plays a crucial role in discussions of the observational consequences of such heavy strings, for the size of the gravitational perturbations induced by the strings is $O(G\mu)$; $G\mu$ is then the order of magnitude both of the seed density perturbations for galaxy formation, and of the induced fluctuations in the cosmic microwave background. In section 6 we shall see that $\mu_6 \sim 1$ provides fluctuations of the order of $10^{-5}$, just the right magnitude to explain the observed features of the CMB spectrum and the matter distribution in the Universe today. The fact that grand unified strings give the right amplitude for the gravitational perturbations is a good feature of string-based theories of structure formation. In theories invoking the amplification of quantum fluctuations during inflation, the figure $10^{-5}$ is not easily explained [8].

2. Strings in field theories

2.1. Global strings

The simplest theory exhibiting string solutions is that of a complex scalar field $\phi(x)$, described by the Lagrangian density

$$\mathcal{L} = \partial_{\mu}\phi^{*}\partial^{\mu}\phi - V(\phi)$$

\[ V = \frac{1}{2}\lambda(|\phi|^2 - \frac{1}{2}\eta^2)^2 \tag{2.1} \]

which has a global U(1) symmetry, under the transformation $\phi \rightarrow \phi e^{i\alpha}$, with $\alpha$ constant.

The Euler–Lagrange equations that follow from (2.1) are

$$[\partial^2 + \lambda(|\phi|^2 - \frac{1}{2}\eta^2)]\phi = 0. \tag{2.2}$$

The ground state, or vacuum, solution is $\phi = (\eta/\sqrt{2}) \exp(i\alpha_0)$ with $\alpha_0$ constant, which has zero energy. Since the energy is bounded below by zero this solution (unlike $\phi = 0$) is clearly stable. It is not invariant under the U(1) symmetry transformation: the symmetry is said to be broken by the vacuum. The mass $m_\phi$ of the scalar particle in the symmetry-breaking vacuum is given by $m_\phi^2 = \lambda\eta^2$. There is also a massless particle, the Nambu–Goldstone boson, which is associated with the broken global symmetry. It corresponds to space-dependent oscillations in the phase of $\phi$.

Besides the vacuum, there are also static solutions with non-zero energy density. Let us make the following cylindrically symmetric ansatz:

$$\phi = \frac{\eta}{\sqrt{2}} f(m_\phi \rho) e^{in\varphi} \tag{2.3}$$

where $\{\rho, \varphi, z\}$ are cylindrical polar coordinates, and $n$ is an integer. The field equations then reduce to a single non-linear ordinary differential equation

$$f'' + \frac{1}{\xi} f' - \frac{n^2}{\xi^2} f - \frac{1}{2}(f^2 - 1) f = 0 \tag{2.4}$$
where $\xi \equiv m_s \rho$. As $\xi \to 0$, continuity of $\phi$ requires that $f \to 0$. At infinity, $f \to 1$, so that the field approaches its ground state $|\phi| = \eta/\sqrt{2}$. Writing $f = 1 - \delta f$, it is not hard to show that, at large $\xi$, $\delta f \sim n^2/\xi^2$. Figure 2 displays $f$ over the whole range of $\xi$, along with the energy density

$$\mathcal{E} = |\phi|^2 + |\nabla \phi|^2 + V(\phi).$$

Although $\mathcal{E}$ is well localized near the origin, it has a $\xi^{-2}$ tail at large $\xi$, which comes from the angular part of the gradient term. This means that the energy per unit length of this solution is infinite: inside a cylinder of radius $R \gg m_\ast^{-1}$ it is approximately $\pi n^2 \eta^2 \ln(m_\ast R)$.

Figure 2. Field (a) and energy density (b) profiles of the global string. The shading in (a) shows the phase of the complex scalar field changing from 0 (lightest) to $2\pi$ (darkest) around the string. The sharp point in the energy density is an artefact of the discretization.

These solutions are known as global strings or vortices [14,15]. They are closely related to the vortices in superfluid $^4$He [16,17], where the complex scalar field represents the wavefunction of the condensed $^4$He atoms, and $-i\phi^* \frac{\partial}{\partial k} \phi / 2 |\phi|^2$ is proportional to the superfluid velocity, which has a circulation around the vortex core. In particle cosmology, the most commonly considered global strings are those associated with a spontaneously broken axial $U(1)$ symmetry, the axion strings [14,18].
2.2. Local or gauge strings

Let us now consider what happens when the internal symmetry is a local one. This requires the introduction of a vector field $A_\mu$. The Lagrangian density is then

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - V(\phi)$$  \hspace{1cm} (2.6)

where $D_\mu = \partial_\mu + i e A_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This is the Abelian Higgs model [1], the prototypical example of a gauge field theory with spontaneous symmetry breaking. The U(1) invariance is realized by the transformations

$$\phi \rightarrow \phi e^{i \Lambda(x)} \quad A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \Lambda(x)$$  \hspace{1cm} (2.7)

where $\Lambda$ is a real single-valued function. The field equations are

$$[D^2 + \lambda(|\phi|^2 - \frac{1}{2} \eta^2)] \phi = 0$$

$$\partial_\nu F^{\mu\nu} + ie (\phi^* D^\mu \phi - D^\mu \phi^* \phi) = 0.$$  \hspace{1cm} (2.8)

The particle spectrum still has the Higgs with mass $m_s = \sqrt{\lambda} \eta$, but the Nambu–Goldstone boson is incorporated into the vector field, which gains a mass $m_v = \eta e$. A vortex solution still exists [19], but its properties are quite different from those of the global vortex. Now the energy per unit length is finite, because the dangerous angular derivative of the field is replaced by a covariant derivative, which can vanish faster than $\rho^{-1}$ at infinity. To be more specific, let us choose the radial gauge $A_\rho = 0$. Then we may write the general cylindrically symmetric ansatz for the gauge field as

$$\phi = \frac{\eta}{\sqrt{2}} f(m_v \rho) e^{im_\phi} \quad A^i = \frac{n}{\epsilon \rho} \phi^i a(m_v \rho).$$  \hspace{1cm} (2.9)

The resulting coupled ordinary differential equations (ODEs) do not have solutions in terms of known functions, but it is straightforward to obtain their asymptotic behaviour:

$$f \simeq \begin{cases} f_0 \xi^n & \text{as } \xi \rightarrow 0 \\ 1 - f_1 \xi^{-1/2} \exp(-\sqrt{\beta} \xi) & \text{as } \xi \rightarrow \infty \end{cases}$$

$$a \simeq \begin{cases} a_0 \xi^2 - \frac{|n| f_0^2}{4(|n| + 1)^2} \xi^{2|n|+2} & \text{as } \xi \rightarrow 0 \\ 1 - a_1 \xi^{1/2} \exp(-\xi) & \text{as } \xi \rightarrow \infty \end{cases}$$  \hspace{1cm} (2.10)

Here, $\xi = m_v \rho$ and $\beta = \lambda/e^2 = (m_s/m_v)^2$. (In the case $\beta > 4$, $\xi^{-1/2} \exp(-\sqrt{\beta} \xi)$ is replaced by $\xi^{-1} \exp(-2\xi)$.) Note that the energy density is much more localized than in the global string.

The local string also contains a tube of magnetic flux, with quantized flux

$$\int d^2 x \mathbf{B} \cdot \hat{z} = \int_{S^1_\infty} dx^1 A^1 = \frac{2\pi n}{e}$$  \hspace{1cm} (2.11)

where $S^1_\infty$ denotes a circle of infinite radius centred on the string. This flux quantization is a result of the vanishing of the covariant derivative, which determines $A_1$ in terms of
derivatives of $\phi$. The phase of $\phi$ must change by an integer multiple of $2\pi$, which forces the flux to be quantized.

Local vortices are stable for any $n$ if $\beta < 1$ [20,21]. If $\beta > 1$, a flux $2\pi n/e$ vortex with $|n| > 1$ is unstable with respect to splitting into $n$ vortices carrying the elementary unit of flux $2\pi/e$. At the boundary value, $\beta = 1$, the multiply-charged vortex is neutrally stable with respect to dissociation: it has $n$ zero modes, or fluctuations with eigenvalue zero. One can interpret these results in terms of forces acting between the vortices. The gauge field generates a repulsive force, because lines of magnetic flux repel each other, while the scalar field produces an attractive force, essentially because it is energetically favourable to minimize the area over which the potential energy density is non-zero. The range of these forces is controlled by the Compton wavelength of the mediating boson, and whichever has longer range dominates. For example, when $\beta > 1$ the scalar boson is heavier, the gauge force dominates and vortices repel. If $n$ vortices are sitting on top of one another, as in the $n$-unit vortex, any perturbation is likely to disturb the cylindrical symmetry and enable it to break apart.

The vortices in the Abelian Higgs model have condensed matter analogues: flux tubes in superconductors [22]. There are differences [23,24], stemming from the fact that Nielsen-Olesen vortices exist in a vacuum background, whereas superconductor vortices live in a background of charged bosons, the Cooper pairs.

The energy per unit length of the local string is $\mu = \int \rho \, d\rho \, d\varphi \, E(\rho)$. Substituting the cylindrically symmetric ansatz for the fields into the energy functional, it is not hard to see that it must have the form

$$\mu = \pi \eta^2 \epsilon(\beta).$$

(2.12)

Remarkably, it is possible to show analytically that $\epsilon(1) = 1$ [25]. Numerical studies [20,26] show that $\epsilon$ increases monotonically with $\beta$, albeit rather slowly, going as $\log \beta$ for $\beta > 1$.

2.3. Vortices and topology

In order to decide whether a theory with a larger symmetry group than $U(1)$ possesses stable vortex solutions, we need to examine the topology of the vacuum manifold, the set of minima of the potential [27]-[33].

Let us consider a theory of a scalar field $\phi$ which transforms under some representation of a compact Lie group $G$, with Hermitean generators $T_a$ satisfying

$$[T_a, T_b] = i f_{abc} T_c.$$

We are interested in finding vortex solutions, or finite-energy static solutions in $\mathbb{R}^2$ to the field equations. The static energy functional is

$$E = \int d^2 x \left( \frac{1}{4} F_{ij}^a F_{ij}^a + |D_k \phi|^2 + V(\phi) \right)$$

(2.13)

where $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a - ef^{abc} A_i^b A_j^c$ and $D_k = \partial_k + ieA_k$, with $A_k = A_k^a T_a$. The potential $V$ is some $G$-invariant quartic polynomial in the fields $\phi$. By the addition of a constant we can set $V = 0$ at its minima, thus ensuring that $E$ is non-negative.

Let us denote the vacuum manifold by $\mathcal{M}$, and let $\phi_0$ be one point on $\mathcal{M}$. Then for any $g \in G$, $g\phi_0$ is also on $\mathcal{M}$. But many different elements $g$ may yield the same point. It is
convenient to introduce the isotropy group (or little group) \( H \) of \( \phi_0 \), the set of all elements \( h \in G \) such that \( h\phi_0 = \phi_0 \). Then clearly \( g\phi_0 = g'\phi_0 \) if and only if \( g' = gh \), with \( h \in H \).

In other words, the points of \( M \) are in one-to-one correspondence with the left cosets of \( H \) in \( G \); we write \( M = G/H \).

For a finite-energy solution, each term in (2.13) must tend sufficiently rapidly to zero as \( \rho = |r| \to \infty \). As before let us choose the radial gauge \( A_\rho = 0 \), in which the fields tend to definite values as \( \rho \to \infty \). To make the potential term finite, we require that in any direction \( \varphi \), \( \phi(\rho, \varphi) \to \tilde{\phi}(\varphi) \in \mathcal{M} \). Let us choose \( \tilde{\phi}(0) = \phi_0 \). To make the first, magnetic, term finite, we require that \( A \) tend to a pure gauge field, i.e.

\[
\textrm{i} e A_k(x) \to -\delta_k g(\varphi) g^{-1}(\varphi) \quad \text{as} \quad \rho \to \infty
\]  

(2.14)

for some \( g(\varphi) \in G \). Moreover, we can always choose \( g(0) = 1 \), the identity. Finally, to make the gradient term finite, we require that \( D_\varphi \phi \to 0 \) as \( \rho \to \infty \). It follows at once that\( \delta_k [g^{-1}(\varphi) \phi(\varphi)] = 0 \). Since \( g^{-1}(0) \tilde{\phi}(0) = \phi_0 \), this means that

\[
\tilde{\phi}(\varphi) = g(\varphi) \phi_0.
\]  

(2.15)

Now the fields \( A_k^\varphi \) and \( \phi \) must be single valued and continuous, but \( g(\varphi) \), though continuous, need not be single valued; \( g(2\pi) \) need not coincide with \( g(0) \), though we do require \( g(2\pi) \in H \).

We may regard \( \tilde{\phi}(\varphi) \) as defining a loop in \( \mathcal{M} \), a map from the circle \( S^1 \) into \( \mathcal{M} \), based at \( \phi_0 \). Whether or not there is a vortex solution depends on the topological characterization of this loop. If it is contractible, i.e. can be smoothly shrunk to a point within \( \mathcal{M} \), then it is possible to find a zero-energy solution with this asymptotic value, so we should not expect to find a stable vortex.

Non-contractible loops are classified by the elements of the fundamental group, or first homotopy group of \( \mathcal{M} \), denoted \( \pi_1(\mathcal{M}, \phi_0) \). Two loops based at \( \phi_0 \) are homotopic if one can be smoothly deformed into the other without leaving \( \mathcal{M} \). This is an equivalence relation; the equivalence classes, or homotopy classes of loops, are the elements of \( \pi_1(\mathcal{M}, \phi_0) \). These classes have a group structure: the identity is the class of contractible loops, homotopic to the trivial loop which remains at \( \phi_0 \); the inverse is the class comprising the same loops traversed in the reverse sense; and the product is defined by traversing two loops in succession. It is intuitively clear that, if \( \mathcal{M} \) is connected, then \( \pi_1(\mathcal{M}, \phi_0) \) does not depend on the base point \( \phi_0 \), and so the first homotopy group is often denoted simply by \( \pi_1(\mathcal{M}) \). If \( \pi_1(\mathcal{M}) \) is trivial, comprising the identity element only, then \( \mathcal{M} \) is said to be simply connected. A necessary, but not sufficient, condition for the existence of stable vortices is that \( \pi_1(\mathcal{M}) \) be non-trivial, or multiply connected. Figure 3 depicts a multiply-connected manifold: the dashed loop is in the identity class, while the solid loop is not. Hence \( \pi_1(\mathcal{M}) \) has more than one element, and is non-trivial.

If \( G \) is chosen to be simply connected (which can always be done by going to the universal covering group’, for example replacing \( \text{SO}(3) \) by its two-fold covering group \( \text{SU}(2) \)), then an equivalent condition is that \( H \) contains disconnected pieces. In fact, if \( g(2\pi) \) belongs to the connected component, \( H_0 \) say, of \( H \), then \( g(\varphi) \) can be smoothly deformed, without changing \( \tilde{\phi} \), so that \( g(2\pi) \) becomes \( 1 \). But then, since any loop in \( G \) is contractible by hypothesis, so is \( \tilde{\phi} \). In this case, the group \( \pi_1(\mathcal{M}) \) is isomorphic to the quotient group \( H/H_0 \), also denoted by \( \pi_0(H) \), whose order is the number of disconnected components of \( H \). Figure 4 shows a path joining two disconnected parts of \( H \). Such a path would be mapped to a non-trivial path in \( G/H \), such as the one the solid loop represents in figure 3.
The Abelian case discussed earlier may easily be included in this picture. To do so, we have to replace $U(1)$ by its covering group, the additive group of reals, $R$. Then the isotropy subgroup is the group $Z$ of integers (transformations with phase equal to a multiple of $2\pi$). Here $\mathcal{M} = R/Z = S^1$, a circle, and $\pi_1(\mathcal{M}) = \pi_0(Z) = Z$.

If $\phi$ is non-contractible, we may suspect the existence of stable vortices, but there is certainly no guarantee. For example, in the Abelian theory, all loops with non-zero winding number $n$ are non-contractible, but for $\beta > 1$ and $|n| > 1$ there are no stable vortices. In any $n$-vortex configuration, the vortices will repel each other; the energy can always be lowered by expanding the spatial scale. We shall encounter a more interesting counter-example below.

To illustrate these ideas, let us consider the simplest non-Abelian gauge theory, with $G = SO(3)$ and a scalar field $\phi = (\phi^a)$ in the vector representation. We take $V = \frac{1}{8}\lambda(\phi^2 - \eta^2)^2$, so here $\mathcal{M}$ is the sphere in $\phi$ space of radius $\eta$. This is consistent because the isotropy group of a fixed vector $\phi$ is $H = SO(2)$ and so $\mathcal{M} \simeq SO(3)/SO(2) \simeq S^2$. Since all loops on the 2-sphere are contractible, $\pi_1(\mathcal{M})$ is trivial and there are no vortex solutions.
Cosmic strings

It is not difficult, however, to extend the model to accommodate vortices. Let us suppose there are two scalar fields \( \phi_1 \) and \( \phi_2 \) each in the vector representation, and that the potential is chosen so that at the minima, \( |\phi_1| \) and \( |\phi_2| \), are fixed and \( \phi_1 \cdot \phi_2 = 0 \). Then the group \( G \) is completely broken, i.e. \( H \) consists of the identity element only, so \( \mathcal{M} \simeq \text{SO}(3) \). To apply the general criterion, however, we should replace \( G \) by its covering group \( \tilde{G} = \text{SU}(2) \). Then the isotropy group of a fixed pair of vacuum fields \((\phi_1, \phi_2)\) is \( \tilde{H} = \mathbb{Z}_2 = \{1, -1\} \). So \( \pi_1(\mathcal{M}) \simeq \pi_0(\mathbb{Z}_2) \simeq \mathbb{Z}_2 \). The single class of non-contractible loops in \( \mathcal{M} \) comprises paths from the identity to a \( 2\pi \) rotation (paths to a \( 4\pi \) rotation are trivial).

More detailed study reveals that there are two distinct vortex solutions in this theory \([34] - [36]\). The solutions have the asymptotic forms

\[
\tilde{\phi}_A(\varphi) = e^{i\omega M} \tilde{\phi}_A(0) \quad A^k = M \tilde{\phi}^k / e^\rho
\]  

(2.16)

where \( M \) is an element of the \( \text{SO}(3) \) algebra in the adjoint representation. For simplicity, let us take the case where \( |\phi_1| = |\phi_2| = \eta \). If we set \( \Phi = (\phi_1 + i\phi_2)/\sqrt{2} \), we can resolve \( \Phi \) into eigenvectors of \( M \):

\[
\Phi(\rho, \varphi) = \eta \sum_{n=-1}^{1} f_{An}(\rho) e^{i\eta\varphi} e_n
\]

(2.17)

where \( e_n \cdot e_n^* = 1 \). It can be shown that the ansatz (2.16) allows two solutions with different forms:

(i) \( \Phi = \eta f_{+1}(\rho) e^{i\varphi} e_{+1} \);

(ii) \( \Phi = \eta [f(\rho)(e^{i\varphi} e_{+1} - e^{-i\varphi} e_{-1}) + i f_0(\rho) e_0] \),

as well as their charge conjugates. In the first case the equations reduce to those of the Abelian string (2.8). Thus we find a solution which is an embedding of this string in the \( \text{SO}(3) \) model: the scalar field changes phase by \( 2\pi \) at infinity, and vanishes at the origin. In the second case, there are two independent scalar fields, \( f(\rho) = f_{+1}(\rho) = -f_{-1}(\rho) \), and \( f_0(\rho) \). The boundary conditions are \( f(\infty) = 1/2 \) and \( f_0(\infty) = 1/\sqrt{2} \), while at the origin only \( f(\rho) \) need vanish.

We can picture the two types of solution by denoting the magnitudes and directions in internal ‘isospin’ space of \( \phi_1 \) and \( \phi_2 \) by arrows, as in figure 5. The solid arrows represent \( \phi_1 \), and dotted ones \( \phi_2 \). This makes the difference between the two solutions especially clear: in (i) both fields \( \phi_1 \) and \( \phi_2 \) rotate around the origin, while only \( \phi_1 \) does in (ii). This makes it plausible that vortex (ii) has lower energy, which is indeed the case in the regions of parameter space that have been investigated numerically \([36]\). This is because \( \phi_2 \) does not have to vanish at the core.

2.4. Semilocal and electroweak vortices

So far we have assumed that either all or none of the symmetries of the scalar field theory are gauged. Let us now suppose that only a subgroup is a gauge symmetry. The local and global symmetries must commute, and so the general symmetry-breaking pattern \( G \to H \) has the form

\[
[G_1 \times G_2] / D_G \to [H_1 \times H_2] / D_H
\]

(2.18)

where the subscripts indicate whether the symmetry is local or global, and \( D_G \) and \( D_H \) are possible discrete subgroups in common between the local and global groups. We can
move the scalar field to any point on its vacuum manifold $\mathcal{M}$ by a transformation in $G$, and factoring out the little group we have as before $\mathcal{M} \cong G/H$. The vacuum manifold need not separate into local and global parts, $\mathcal{M}_i \cong G_i/H_i$ and $\mathcal{M}_g \cong G_g/H_g$. If

$$\mathcal{M} \cong [\mathcal{M}_1 \times \mathcal{M}_g]/D$$

(2.19)

with $D$ a discrete subgroup, the theory is called semilocal. If there are vortex solutions they show quite surprising behaviour, at odds with the intuition gained from pure gauge vortices.

Let us illustrate some of these points with the original semilocal theory [37]. This has a complex scalar doublet $\Phi$ with components $\phi_1$ and $\phi_2$. The Lagrangian is constructed so that it has total symmetry $U(2) \cong [SU(2) \times U(1)]/Z_2$, but only the Abelian part is gauged:

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + |D_{\mu} \Phi|^2 - \frac{1}{2} \lambda (\Phi^2 - \frac{1}{2} \eta^2)^2$$

(2.20)

where $D_{\mu} = \partial_{\mu} + ieA_{\mu}$ and $A_{\mu}$ is the Abelian gauge field. This is essentially the bosonic sector of the electroweak theory, in the limit in which the Weinberg angle is $\pi/2$, so that the $W$ field decouples [38]. In the ground state, the potential ensures that $\Phi$ gains an expectation value, which we can choose to be $(0, 1)i/\sqrt{2}$. This leaves unbroken a global $U(1)$ symmetry, generated by $(\tau_3 + 1)/2$. The symmetry-breaking pattern is therefore

$$[SU(2)_g \times U(1)_g]/Z_2 \rightarrow U(1)_g.$$ 

(2.21)

The gauge symmetry is fully broken, so the gauge orbit space $G_1/H_1$ is isomorphic to a circle, as for the ordinary Abelian Higgs model. The full vacuum manifold is the 3-sphere, defined by $|\phi_1|^2 + |\phi_2|^2 = \frac{1}{2} \eta^2$. Now, although $S^3$ contains circles, it is not the direct product of a circle with any other space and thus, by the above criterion, the theory is semilocal.

Recalling the discussion of section 2.3, the gauge orbits are topologically non-trivial with respect to the first homotopy group, and thus we should expect to find finite-energy
vortex solutions. From the equations of motion, we can see immediately that one vortex solution is

$$\Phi = \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 \\ f(\rho) e^{i\varphi} \end{pmatrix} \quad A^k = \frac{\phi^k}{\epsilon} \frac{a(\rho)}{\epsilon \rho}$$

(2.22)

where \(f\) and \(a\) are the functions introduced in (2.9) for the Nielsen–Olesen vortex. Others are obtained by global SU(2) rotations.

We issued a warning earlier that the topological condition did not guarantee the existence of a stable vortex solution. In the case at hand, \(M\) contains \(M_1\) but is not identical to it, and it happens that non-contractible loops in \(M_1\) are contractible in \(M\). In that case it is possible, by allowing the upper component of \(\Phi\) to be non-zero, to construct field configurations which lie in \(M\) everywhere in the plane (still reaching \(M_1\) at infinity). These configurations have vanishing potential energy density, resulting in a tendency to spread out in order to minimize the total gradient and magnetic field energies [39].

The result of a detailed analysis of this spreading instability [40, 41] is that the embedded Nielsen–Olesen vortex is stable only for \(\beta < 1\). If \(\beta > 1\) the string is unstable and the vortex spreads out to infinity. At the borderline, where \(\beta = 1\), the vortices can be any size, and there are also stable multivortex solutions. We refer the reader to [42, 43] for more information.

The significance of semilocal strings is that they violate the canonical law about gauged topological defects: that stable defects exist if the relevant homotopy group is non-trivial. This has implications for cosmology, for one has to careful in predicting whether a particular theory will make strings in the early Universe. It has also been shown that the perturbative stability of the \(\beta < 1\) string persists when one gauges the SU(2) symmetry, provided the Weinberg angle remains close to \(\frac{\pi}{2}\) [44], although for physical values of the Weinberg angle these electroweak strings are unstable [45]. However, they do have some features in common with the sphaleron [46, 47], another unstable defect of the Standard Model, so electroweak strings may still have an important role to play.

2.5. Composite defects: strings and domain walls

Realistic theories have several stages of symmetry breaking, each associated with a Higgs field gaining an expectation value. As the early Universe cooled it would have gone through a series of phase transitions (see section 5.1) at which the symmetry was progressively reduced, so a symmetry-breaking sequence arranged in order of decreasing symmetry can be thought of as a sequence in time. This section is concerned with studying situations where \(H\) is not the final unbroken symmetry.

Let us first consider the sequence

$$G \xrightarrow{\eta} H \xrightarrow{\nu} H'$$

(2.23)

where the letter over each arrow denotes the field which breaks the symmetry, and the letter underneath the scale of the symmetry breaking. We assume that \(\pi_1(G/H)\) is non-trivial. Without loss of generality we may take \(G\) to be simply connected, in which case \(H\) must be disconnected, and \(\pi_1(G/H) \simeq \pi_0(H) \simeq H/H_0\). For simplicity we also suppose that \(\nu \ll \eta\), so that when \(\chi\) gets an expectation value, any change in \(\phi\) can be ignored.

The direction of \(\chi\) in the vacuum is partly determined by its coupling to \(\phi\) in the potential \(V(\phi, \chi)\). Let us suppose that \(V\) is minimized at \((\phi_0, \chi_0)\), and normalized to zero.
at the minimum. Then \( V(\phi_0, g \chi_0) \neq V(\phi_0, \chi_0) \) unless \( g \in H \) or unless there were an enlarged symmetry\(^\dagger\). In the string background

\[
\phi(\varphi) = g(\varphi)\phi_0 \quad A^k = -i g \partial^k \varphi^{-1} / e = M \tilde{\phi}^k / e \rho
\]

(2.24)

and it follows that there will be a quadratic divergence in the potential energy per unit length unless \( \chi \) 'follows' \( \phi \) around the string, or

\[
\chi(\varphi) = g(\varphi)\chi_0.
\]

(2.25)

Consider for example the model whose symmetry-breaking pattern is

\[
\text{SO}(3) \xrightarrow{5 \eta} \text{O}(2) \xrightarrow{3 \nu} \text{SO}(2)
\]

(2.26)

where the boldface numbers indicate the dimension of the representation. The five-dimensional representation of \( \text{SO}(3) \) is a traceless symmetric tensor \( \Phi \), while the 3 is just the adjoint representation of \( \text{SO}(3) \) vectors. The sequence can be arranged with a suitable potential containing couplings \( \chi^T \Phi \chi \) and \( \chi^T \Phi^2 \chi \), such that

\[
\Phi_0 = \eta \text{diag}(-1, -1, 2)/\sqrt{6} \quad \chi_0 = \nu(0, 0, 1).
\]

(2.27)

The intermediate \( \text{O}(2) \) then consists of matrices \( \exp(i\alpha T_3) \) and \( \exp(i\pi T_1) \exp(i\alpha T_3) \). The disconnected component of \( \text{O}(2) \) changes the sign of \( \chi_0 \), and so the final symmetry is \( \text{SO}(2) \).

After the first symmetry breaking, strings will form. A typical string configuration at large distances may be written

\[
\Phi = e^{i \varphi T_1 / 2} \Phi_0 e^{-i \varphi T_1 / 2} \quad A^k = T_1 \tilde{\phi}^k / 2e \rho.
\]

(2.28)

Thus, as \( \chi \) follows \( \Phi \) around the string it comes back to \( \exp(i\pi T_1) \chi_0 = -\chi_0 \) [48]–[50]. The problem is a discontinuity in \( \chi \) at \( \varphi = 2\pi \), where it must change from \(-\chi_0 \) back to \( \chi_0 \). The best that can be done is to allow \( \chi \) to leave its vacuum manifold, and possibly vanish, on a plane of constant \( \varphi \), creating a structure called a \textit{domain wall} [51, 52]. Domain wall solutions appear in theories with disconnected vacuum manifolds, and can be characterized by two quantities: their thickness, which is the Compton wavelength of \( \chi \) in the vacuum, \( m_{\chi}^{-1} \); and their surface energy \( \sigma \sim V_w m_{\chi}^{-1} \), where \( V_w \) is the potential energy density at the centre of the wall.

Topologically, the string exists because \( \pi_1(\text{SO}(3)/\text{O}(2)) \cong \mathbb{Z}_2 \). The existence of the domain wall depends on the properties of the element \( \tilde{h} = g(2\pi) \in H \); in this case \( \tilde{h} = \text{diag}(1, -1, -1) \). Domain walls appear if \( \tilde{h} \) is not equivalent to the identity under the action of \( H' \), and thus lies in a non-trivial class of \( H/H' \). This condition is satisfied here, for \( \tilde{h} \) is invariant under \( H' \), which in this case is the \( \text{SO}(2) \) generated by \( T_1 \), and clearly inequivalent to the identity.

Similar symmetry-breaking sequences occur in \( \text{SO}(10) \) models, where there is a discrete charge conjugation symmetry at intermediate energies, analogous to \( \tilde{h} \), which is broken at a lower scale [52].

\(^\dagger\) If one could make separate transformations on both fields, this would enlarge the symmetry group to \( G_1 \times G_5 \), or some subgroup thereof. In this case the string becomes "frustrated" [48].
2.6. Composite defects: strings and monopoles

Let us now suppose that $G$ is not the full symmetry group of the theory. There will then be a sequence

$$G' \xrightarrow{\sigma} G \xrightarrow{\phi} H$$

(2.29)

and we shall assume that the scales $s$ and $\eta$ are well separated. If we were unaware of the existence of the larger symmetry, we would predict strings if $\pi_1(G/H) \neq \{1\}$. The topological stability of these strings ultimately depends on the fundamental group of the true vacuum manifold, $\pi_1(G'/H)$. By analogy with the above discussion of domain walls and strings, we might guess that it might be possible to break strings corresponding to trivial elements of $\pi_1(G'/H)$ by creating a pair of point defects, or monopoles [53].

It can be shown that if $G'$ is simply connected then monopoles exist if and only if $G$ is not. To see heuristically how this comes about, let us again choose the radial gauge $\tilde{A}_k = 0$, in which the scalar field $\sigma$ has a well-defined value on the 2-sphere at infinity $S^2_{\infty}$, and maps it into the vacuum manifold $M' \simeq G'/G$. The monopole field at infinity can be written

$$\sigma(\theta, \varphi) = g'(\theta, \varphi) \sigma(N) \quad A_k = -ig'\tilde{A}_k g'^{-1}/e$$

(2.30)

where $N$ is the north pole $\theta = 0$. The image of $g'$ in $G'$ can be thought of as a smooth set of loops $g'(\tau, \psi)$ $(\tau, \psi \in [0, 1])$, fixed at $N$ for $\psi = 0, 1$. These loops sweep over the whole sphere as $\tau$ and $\psi$ vary over their ranges (see figure 6). Without loss of generality, we can choose $g'(0, \psi) = 1$. Then, in order for $\sigma(\theta, \varphi)$ to be continuous at $N$, the loop $\tilde{g}(\psi) = g'(1, \psi)$ must lie in $G$. If $G$ is multiply connected, and $\tilde{g}$ is non-contractible in $G$, $g'(\tau, \psi)$ cannot be continuously deformed to the trivial constant configuration without $g'(1, \psi)$ leaving $G$. Therefore, one cannot eliminate the monopole without introducing discontinuities in the field.

When the symmetry group $G$ is broken further to $H$ by the field $\phi$, the potential forces $\phi$ to follow $\sigma$:

$$\phi(\theta, \varphi) = g'(\theta, \varphi) \phi(N).$$

(2.31)

However, if $\tilde{g}(\psi)$ is not in $H$, $\phi$ must fall into a string configuration near $N$ to maintain continuity. In this way monopoles become attached to strings in subsequent stages of symmetry breaking. Conversely, strings can break by the creation of monopole–antimonopole pairs [53,56,57].

As an instructive example, let us take the SO(3) model of section 2.3, and separate the scales of the fields $\phi_1$ and $\phi_2$, so that the breaking sequence expressed in terms of the covering group is

$$\text{SU}(2) \xrightarrow{\phi_1} \text{U}(1) \xrightarrow{\phi_2} \mathbb{Z}_2$$

(2.32)

where $\mathbb{Z}_2 = \{1, -1\}$. After the first stage the theory supports monopoles, since $\pi_1(\text{U}(1)) \simeq \mathbb{Z}$. There are strings of the Nielsen–Olesen type associated with the second stage, because

† Monopoles [54,29,30] play a similar role in (3+1)-dimensional Yang–Mills–Higgs theories to vortices in 2+1 dimensions. Gauge monopoles have finite energy, while the energy of global monopoles is linearly divergent [55].
Figure 6. Monopoles joined by strings. Monopoles exist if $G$ is multiply connected: the diagram shows the gauge transformations on a large sphere around the monopole. If $H$ is disconnected, as here, the monopoles may get joined by more than one string.

$\pi_1(G/H) \simeq \mathbb{Z}$. The significance of the unbroken $\mathbb{Z}_2$ is that strings with winding number $n = 1$ are stable while those with $n = 2$ (or any even $n$) can be broken by creating monopole-antimonopole pairs. A flux-2 string is in the same topological class as two flux-1 strings: this leads one to envisage a configuration in which the monopole is attached to two flux-1 strings at the north and south poles. This topologically stable configuration, in which the string 'threads' the monopole, is known as a bead [34]. This situation is depicted in figure 6: the non-contractible loop in $G$, which defines the monopole, passes through the two disconnected components of $H$. This path can be divided into two segments, each of which is a closed loop in $G/H$, and thus represents a string.

Beads can also be thought of as kinks [58] interpolating between topologically distinct string solutions. At first sight there appears to be only one type of string, since there is only one non-trivial class in $\pi_1(G'/H)$. However, the vacuum possesses a discrete symmetry which is broken by the string solution, given for this model in section 2.3. If we write the vacuum $\Phi_0 = (\eta_1, \eta_2, 0)/\sqrt{2}$, then this discrete symmetry is $\Phi_0 \rightarrow \text{diag}(1, 1, -1)\Phi_0$. This is violated by type (ii) strings, but not by type (i). This broken symmetry means that there are two distinct type (ii) strings which can be smoothly deformed into each other via a type (i) configuration, which has higher energy. This is the bead. The number of different string solutions is clearly related to the number of discrete vacuum symmetries broken by the string solution. If there are more than two, then the bead can be the junction of several strings [34, 59, 60].

2.7. Superconducting strings: bosonic currents

Models with extra fields can often support currents in the core of the string [61], which can have quite dramatic dynamical effects, as we shall see in section 3.4. If the currents are
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electromagnetic, then the string behaves as a thin superconducting wire, with an enormous
critical current by terrestrial standards.
Consider an Abelian U(1) × U(1) model with Lagrangian
\[ L = |D_\mu \phi|^2 + |D_\mu \chi|^2 - V(\phi, \chi) - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]  
(2.33)
where \( D_\mu \phi = (\partial_\mu + ieB_\mu)\phi, D_\mu \chi = (\partial_\mu + iqA_\mu)\chi, F_{\mu\nu}, G_{\mu\nu} \) are field strengths associated
with \( A_\mu \) and \( B_\mu \), respectively, and the potential is given by
\[ V(\phi, \chi) = \frac{1}{2}\lambda_1(|\phi|^2 - \frac{1}{2}v^2)^2 + \frac{1}{2}\lambda_2|\chi|^4 + \lambda_3(|\phi|^2 - \frac{1}{2}v^2)|\chi|^2. \]  
(2.34)
Let us suppose that the parameters are chosen so that the minimum of \( V \) is at \( |\phi| = \eta/\sqrt{2}, \)
\( |\chi| = 0 \), so that the U(1) symmetry associated with \( A_\mu \) remains unbroken; this we consider
to be the electromagnetic U(1). (This minimum is ensured by having \( v^2 < \eta^2 \) and
\( \lambda_3^2v^4 < \lambda_1\lambda_2\eta^4 \) ) As in the case of the semilocal vortex, there is a solution in which \( \phi \)
and \( B_\mu \) make up a Nielsen-Olesen vortex, and \( \chi \) and \( A_\mu \) vanish.
However, in large regions of parameter space, a solution is preferred in which \( \chi \neq 0 \) in
the core \([26, 62, 63]\). For given couplings, there is a critical value of the ratio \( r = m_\chi^2/m_\phi^2 \),
where \( m_\chi^2 = \frac{1}{2}\lambda_3(\eta^2 - v^2) \) and \( m_\phi^2 = \lambda_1\eta^2 \), below which a \( \chi \) condensate exists. This
ratio tends to 1 as \( \lambda_3/\lambda_1 \to \infty \), which is when the fields are strongly coupled to each
other. Given a condensate, there is, in fact, a family of solutions, for if \( \chi = X(\rho)/\sqrt{2} \)
is a solution to the field equations then so is \( \chi = X(\rho)e^{ik\rho}/\sqrt{2} \). This phase constitutes an
extra, 'internal' degree of freedom for the string. This is the Nambu–Goldstone boson of
the electromagnetic U(1) symmetry, broken in the core of the string.
Suppose the phase varies linearly with time and position, i.e. that \( \alpha = k\rho - \omega t \). Then
the string carries an electromagnetic current
\[ J^\mu = +iq\chi^*\delta^{\mu\nu} \chi - 2q^2A^\mu|\chi|^2 = qX^2(\omega - qA^0, 0, 0, k - qA^3). \]  
(2.35)
Now let us compute \( \partial_\rho J^3 \) in the gauge \( A_0 = 0 \). Providing the current is sufficiently small
for us to ignore time-dependent backreaction terms on \( X \), we have
\[ \partial_\rho J^3 / \partial t = q^2X^2E^3. \]  
(2.36)
Thus the current increases linearly in time in proportion to the electric field: this is the
London equation, characteristic of a superconductor \([64]\). Borrowing from our knowledge
of superconductors, we see that if the penetration depth \( \sim (qX(0))^{-1} \) is much greater than
the width of the condensate, the applied electric field changes very little over the cross
section of the string, so we may integrate equation (2.36) to get
\[ dI / dt = 2q^2\kappa E^3 \]  
(2.37)
where \( I \) is the total current and \( \kappa = \frac{1}{2}\int dx dy X^2 \). The current cannot continue increasing
indefinitely: just as in ordinary superconductors there is a critical current above which the
string goes 'normal'. Substituting \( \chi = Xe^{ik\rho}/\sqrt{2} \), which corresponds to a current \( \sim qk\kappa \),
the effective mass-squared ratio is increased:
\[ r \to r_{\text{eff}} = r + l^2/4q^2\kappa^2m_\phi^2. \]  
(2.38)
Thus, if \( I^2 \) becomes too large, we leave the region of parameter space where the \( \chi \)
condensate exists. Since there is an implicit dependence on \( X \) in this equation, it is not
straightforward to calculate the critical current $I_c$ from the parameters of the potential. Roughly speaking, we would expect $I_c \sim qKm_φ$. Numerical work [62, 63], [65]–[68] bears this out, showing that the implicit dependence of $I/κ$ on $X(ρ)$ causes the quenching to occur very rapidly at $I_c$ [63].

Current can also be lost by quantum tunnelling [61, 69, 62]. The rate per unit length of this process is estimated to be [62]

$$dΓ/dz \sim m_χ^2 e^{-2πκ}. \quad (2.39)$$

Thus strings with long-lived currents must have large values of $κ \sim v^2/m_χ^2$, meaning that the condensate must be large and wide. Stability against quantum tunnelling then turns out to favour the region of parameter space where $λ_2$ is small [62].

To summarize, a bosonic superconducting string in this $U(1) \times U(1)$ model must be constructed from fields which are weakly self-coupled but strongly coupled to each other, and preferably with a large difference between the masses of the string field $φ$ and the current-carrying field $χ$.

More general cases can be analysed. The underlying point is that the vortex solution need not preserve the vacuum symmetry $H$. If it does not, there then exist transformations, generalizations of $χ \rightarrow χe^{iθ}$, which generate new vortex solutions [70, 50, 71]. A supercurrent is set up on the string when the transformations are space and time dependent. The analysis is complicated if $H$ does not commute with the generator of flux on the string, $M$, for it is then conjugated by $g(φ) = \exp(iφM)$ around the string, giving a position-dependent isotropy group:

$$H(φ) = g(φ)H(0)g^{-1}(φ).$$

It is certainly true that $H(2π) = H(0)$, and hence $h = g(2π)$ is an inner automorphism of $H(0)$. If it is trivial, so that $h = g(2π)$ is an inner automorphism of $H(0)$, then the isotropy group is well defined around the string and it is possible to show that there are bosonic zero modes corresponding to every generator of $H(0)$ acting non-trivially on the string fields [70, 71]. If the automorphism is not trivial, then $H(0)$ is said to be globally unrealizable, and the vortices are so-called ‘Alice’ strings [72, 73].

As one might expect from their name, Alice strings have curious properties. Any particle transported around the string comes back conjugated by $i$. In many cases (the $SO(3)$ string of section 2.5 is one of them), $h$ is actually the charge conjugation operator, and so particles come back as their own antiparticles. Charge conservation is maintained by the string’s acquiring a balancing charge, although it is charge of a rather peculiar nature, for it cannot be localized anywhere (continuing the theme, this has been dubbed ‘Cheshire’ charge [71]). The reason it cannot be pinned down is that it is actually a non-Abelian charge, and thus is not gauge invariant [74]. Alice strings can also carry magnetic charge [75], and a closed loop can collapse to form a monopole (indeed, this process can be observed in nematic liquid crystals [76]). It is perhaps unfortunate for students of the bizarre that there can be no Alice strings in the Universe today, for charge conjugation is not a symmetry of the Standard Model of particle physics.

2.8. Superconducting strings: fermionic currents

It is also possible for currents to be carried by fermionic degrees of freedom confined to the core of the string [61]. Let us extend our bosonic $U(1) \times U(1)$ symmetric Lagrangian to include a fermionic sector

$$\mathcal{L}_f = \bar{ψ}_i \overline{D}_t ψ_i + \bar{ψ}_i \overline{D}_t ψ_i - gφ\bar{ψ}_i ψ_i - gφ^*\bar{ψ}_i ψ_i \quad (2.40)$$
where $\psi_r$ and $\psi_l$ are Weyl spinors with chirality $\pm 1$. The coupling to the $\phi$ field constrains the couplings to $B_\mu$: they must differ by an integer multiple of $e$. We choose $\pm e/2$, and we will also suppose that the electromagnetic charges are both $q/2$.\[^\dagger\]

In the vacuum background $|\phi| = \eta/\sqrt{2}$, this Lagrangian describes a Dirac fermion with mass $m_f = g\eta/\sqrt{2}$. In the vortex background it appears that the fermion mass is position dependent, and vanishes at the core of the string. Thus there could be massless states confined to the string. There is indeed a solution to the transverse Dirac equation with zero eigenvalue \[77,78\],

\[\begin{align*}
    i\gamma^A D_A \psi_1 - g\phi \psi_r &= 0 \\
    i\gamma^A D_A \psi_r - g\phi^* \psi_l &= 0.
\end{align*}\] (2.41)

It has the form $\psi_1 = \zeta_1 P[\bar{\phi}, \bar{B}]$, where $\zeta_1$ is a constant right-handed spinor satisfying $i\gamma^1 \gamma^2 \zeta_1 = \zeta_1$, and $P$ is a function of the background vortex fields $\bar{\phi}$ and $\bar{B}$. At large $\rho$, $P$ goes as $\exp(-m_f \rho)$, showing the localization of the state.

Now, we can generate new space- and time-dependent solutions $\psi_{r,l} \exp(i\beta(t,z))$, provided

\[\begin{align*}
    i[\gamma^0 \partial_t + \gamma^3 \partial_z] \psi_{r,l} \exp(i\beta(t,z)) &= 0.
\end{align*}\] (2.42)

Using $\gamma^0 \gamma^3 \zeta_r = \zeta_r$, we see that $\beta = \beta(t - z)$ solves this equation. Thus the string supports modes moving at the speed of light in the $+z$ direction only, which we call right-moving. If we want a left-moving fermion, we should couple $\psi_l \psi_r$ to $\phi^*$ instead of $\phi$.

These massless fermionic solutions are often called zero modes. Their existence is actually a consequence of the topology of the background field: it is possible to show by quite general methods \[79\] that in the winding number $n$ sector of the field configuration space there are at least $2|n| - 1$ zero modes.

Let us now consider the behaviour of our string-fermion system under an applied electric field. In the ground state, all levels of the Dirac sea up to zero energy are occupied, much like a 1D metal with zero Fermi momentum $k_F$. In the presence of an electric field $E^3$ applied along the string, the occupied states move under the Coulomb force, so that after time $\Delta t$ the Fermi surface is at $\frac{1}{2} q \int_{\Delta t} E^3(t) dt$. Depending on which way the surface moves, fermions or antifermions are created. The 1D density of states is $\frac{1}{2\pi} \sqrt{\lambda}$, so the result is the appearance of a current $I$ at a rate

\[\frac{dI}{dt} = \frac{1}{2\pi} \left( \frac{q}{2} \right)^2 E^3.\] (2.43)

This is identical in form to (2.37), the equation for the current on a bosonic superconducting string in an electric field. However, it is not strictly fair to call this a superconductor: it is more precisely a perfect conductor. A metal would also behave in this way if there were no impurities to scatter fermions from one side of the Fermi surface to the other.

It may have been noticed that we are creating a charge density on the string at the same rate as the current increase. This type of violation of charge conservation at the quantum level is known as an anomaly \[80\]. The classical field theory with Lagrangian (2.40) has two conserved currents coupled to the two gauge fields, but adding the fermion

\[^\dagger\] This theory is anomalous: however, since the gauge symmetry is violated in an instructive way, we will tackle the problem later.
gives anomalies to both [61, 81, 82]. We may restore charge conservation to the theory by adding another fermion with charges \((-e/2, q/2)\), a procedure known as ‘cancelling’ the anomaly. This fermion couples to \(\phi^*\) where the first coupled to \(\phi\), and so produces a set of left-moving zero modes. When the electric field is applied along the string, we create (assuming both \(q\) and \(E^3\) positive) left-moving antiparticles of charge \(-q/2\) at the same rate as the right-moving particles, and thus charge is conserved. The antiparticles contribute to the current with the same sign as the particles, so the rate of current increase is doubled: 

\[
dl/dt = q^2 E^3 / 4 \pi^2.
\]

Another way to ensure a consistent theory is to set \(e = 0\). This means that we are considering fermions in the background of a global string (section 2.1). There still seems to be a violation of charge conservation, but a careful calculation of the anomaly in the global theory shows that the scalar field contributes an extra piece to the current [83]:

\[
j^\mu = \frac{1}{2} q \langle \bar{\psi} \gamma^\mu \psi \rangle - \frac{q}{32 \pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\nu\rho} \phi^* \frac{\nabla_\sigma \phi}{|\phi|^2}.
\]

In the string background, \(-i \phi^* \nabla_\sigma \phi / 2|\phi|^2 \) is in the azimuthal direction, so an electric field along the string induces an inward radial current of just the right magnitude to keep the charge conserved on the string. This is an important case, for it models the interaction of an axion string with an ordinary fermion of the Standard Model [84].

It was mentioned above that fermionic superconducting strings should really be called perfect conductors. If there were a way for right-moving fermions to scatter into left-moving states, then the string would behave like an ordinary conductor and exhibit resistance: the current would stop growing when the scattering rate matches the accelerating effect of the electric field. In our fully gauged model, charge conservation ensures that this cannot happen. The only way that current can be lost is if the zero modes scatter off each other with enough energy to produce a fermion with sufficient energy to escape the string. The threshold is when the sum of the Fermi momenta equals the sum of the fermion masses. This is the analogue of the critical current in the bosonic superconductor. Models can be constructed which do exhibit resistance [85], although they are somewhat complicated. Whether or not this complexity is in some sense generic to grand unified theories, and fermionic superconducting strings are therefore rare, is another question.

In general, one might expect to see both bosonic and fermionic superconductivity present at once. The possibility then arises that the bosonic condensate \(\chi\) couples left- and right-movers together [86, 87, 88]. Remarkably, in the light of the above argument, this does not destroy the superconductivity, unless \(\chi\) is electrically neutral and the sum of the electric charges of the zero modes is zero [88].

### 2.9. Strings in unified theories

As mentioned above, the electroweak theory does not possess any topologically stable string solutions, because its gauge orbit space is isomorphic to \(S^3\), which is simply connected. Thus if cosmic strings exist in nature, they must arise from the breaking of as yet unknown symmetries, perhaps those of a grand unified theory or GUT [89]. In a GUT, where the gauge interactions are unified in a simple compact Lie group, stable gauge strings require an unbroken discrete subgroup at low energies.

Minimal SU(5) grand unification [90] does not incorporate any such discrete symmetries: the smallest group to do so is SO(10), or rather its simply connected covering group Spin(10) [91]. In this scheme, each family of fermions is supplemented by a left-handed antineutrino and assembled into a spinorial 16. A possible discrete symmetry \(D\) is then just \(-1\) in
Spin(10). In order to leave $D$ unbroken, subsequent symmetry breakings must be performed by scalar fields invariant under this element. Such representations can be constructed from the tensor product of an even number of spinor $16s$ and $\bar{16}s$. The natural representation to use is the $126$, which is in the symmetrized product $16 \times 16$, for it alone gives the left-handed antineutrinos Majorana masses at tree level via the ‘see-saw’ mechanism [92].

The symmetry-breaking scheme $\text{Spin}(10) \rightarrow SU(5) \times Z_2$ is closely analogous to the breaking $SU(2)^3 \rightarrow Z_2$ discussed earlier. The $\text{Spin}(10)$ string is indeed very similar to the $SO(3)$ string of section 2.3 [36,93]. From a phenomenological point of view, this $\text{Spin}(10)$ theory is similar to $SU(5)$ unification, and so supersymmetry seems to be necessary for the consistency of the scheme [94]. There are many other types of $SO(10)$ unification [91]: non-supersymmetric ones tend to go via $S[O(6) \times O(4)]$ at the grand unification scale, with a subgroup $SU(2)_R$ surviving to lower energies. The strings produced at the first symmetry breaking do not survive to low energies [52] and would therefore not be important for producing density perturbations (see section 6), while stable strings produced much below $10^{15}$ GeV could only be astrophysically important if they were superconducting [95]–[99].

There may also be spontaneously broken global symmetries in nature. The most commonly considered is an axial $U(1)_A$ or Peccei–Quinn symmetry [100]–[102], which rotates the phases of the left- and right-handed fermions in the opposite sense. This can only be accommodated in a model with at least two Higgs doublets. Its breaking allows global strings of the type discussed in section 2.1. However, axion strings are in a category of their own, because the axial symmetry is anomalous. Under normal circumstances, the axion field $a$, which is essentially the phase of a complex scalar field, would be a massless Goldstone boson. At the quantum level however, it gains a temperature-dependent potential, via its anomalous coupling to instantons, which are topologically non-trivial configurations of the gluon field [103]. This potential has the form

$$V(a) = \Omega(T)[1 - \cos(Na/f_a)]$$

where $N$ is a model-dependent integer, and $f_a$ is the expectation value of the field breaking the Peccei–Quinn symmetry, also known as the axion decay constant. $\Omega(T)$ increases with decreasing temperature, giving the axion a mass-squared $m_a^2 = N^2\Omega(T)/f_a^2$. At temperatures which are low in comparison to the QCD scale of $\sim 100$ MeV, a current algebra calculation gives the mass directly: $m_a = m_\pi f_\pi/f_a$, where $m_\pi$ and $f_\pi$ are the pion mass and decay constant, respectively. The axion decay constant also controls the strength of the coupling of the axion to ordinary fermions. This coupling is axial, of the form $i(m/f_a)\bar{\psi}\gamma_5\psi$. Thus, the larger the scale $f_a$, the smaller the couplings to fermions. There are very stringent astrophysical and cosmological bounds on $f_a$. If $f_a$ is too low, the couplings to ordinary fermions become sufficiently large to affect calculations of stellar models [104]. The strongest astrophysical constraint comes from SN1987a [105], which puts $f_a \gtrsim 10^9$–$10^{10}$ GeV. Radiation from axion strings gives a cosmological upper bound of about $10^{11}$ GeV [106]–[108]. We shall say more about this in section 5.

Larger global symmetries, such as family symmetry [109], have been proposed. If there are three families of fermions, the symmetry can only be $SU(2)$ or $SU(3)$, with a possible $U(1)$ factor for the overall phase of the fermions. There are many possibilities for topological defects in models with family unification [110], and several models with strings have been proposed [111]. Perhaps the most intriguing proposal is for quaternionic strings [112], where the first homotopy group is the non-Abelian group of quaternions. This results in strings with rather unusual properties: for example, they can come together in three-way
junctions. Although such strings are seen in biaxial nematic liquid crystals, very little is known about their cosmological effects.

3. String dynamics

3.1. Equations of motion

So far we have found a rather limited class of solutions to the field equations: straight, static strings. Here, we seek an effective action whose extrema give moving string solutions. The idea is to start from the field theory action and to reduce the number of degrees of freedom from those of a four-dimensional field theory down to the coordinates of the two-dimensional string worldsheet. We consider only the case in which the curvature of the worldsheet is small when compared to the inverse width of the string.

We begin by defining the position of the string as the coordinates of the zeros of the Higgs field $\phi(x)$, which we denote $X^a(\sigma^a)$ ($a = 0, 1$). (We ignore technical complications posed by non-Abelian theories whose strings do not necessarily have a zero of the Higgs field.) There is a metric $\gamma_{ab}$ induced on this surface by the embedding in the background space–time, given by

$$\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}. \quad (3.1)$$

The Lagrangian density for a static solution in a flat background space–time [19, 113] is just the negative of the energy density. In terms of the energy per unit length, $\mu$, the action for a straight string on the $z$ axis is therefore the Nambu–Goto action [114, 115]

$$S_0 = -\mu \int dt \sqrt{\gamma} = -\mu \int d^2\sigma \sqrt{-\gamma}. \quad (3.2)$$

This last expression is generally covariant both in two and in four dimensions, so it holds for any background metric $g_{\mu\nu}$ and for any embedding $X^\mu(\sigma)$, providing the string remains close to being straight. This is the first term in the expansion of the effective action for a string in powers of the curvature; we shall consider the next term briefly below.

Let us first study the equations of motion of the pure Nambu–Goto string. Two of the three degrees of freedom of the worldsheet metric may be removed by using 2D reparametrization (general coordinate) invariance [116]. That is, we can always make a coordinate transformation $\sigma^a \rightarrow \tilde{\sigma}^a(\sigma)$ such that the metric takes the form

$$\gamma_{ab} = \Omega^2(\sigma)\eta_{ab} \quad (3.3)$$

where $\eta_{ab} = \text{diag}(1, -1)$. This is known as the conformal gauge. Let us call the timelike worldsheet coordinate $\tau$ and the spacelike one $\sigma$, and denote differentiations of $X$ with respect to $\tau$ and $\sigma$ by $\dot{X}$ and $\hat{X}$, respectively. Then the conformal gauge imposes the constraints

$$\dot{X}^2 + \hat{X}^2 = 0 \quad \dot{X} \cdot \hat{X} = 0 \quad (3.4)$$

and the equations of motion obtained by varying $S_0$ are

$$\dddot{X}^\mu - \dddot{X}^\mu + \Gamma^\mu_{\nu\rho}(\dot{X}^\nu \dot{X}^\rho - \hat{X}^\nu \hat{X}^\rho) = 0. \quad (3.5)$$
The string stress tensor $T^{\mu\nu}(x)$ is obtained by variation of the action with respect to $g_{\mu\nu}$, giving

$$\sqrt{-g}T^{\mu\nu}(x) = \mu \int d^2\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \delta_4(x - X). \quad (3.6)$$

If the string curvature is small but no longer negligible, we may consider an expansion in powers of the curvature. The relevant measure of curvature is given by the extrinsic curvature tensor $K^A_{ab}$ ($A = 1, 2$), which is computed in terms of an orthogonal pair of spacelike unit normals $n^\mu_A$, satisfying

$$n_A \cdot n_B = -\delta_{AB} \quad \partial_a X \cdot n_A = 0. \quad (3.7)$$

Then

$$K^A_{ab} = -\partial_a n^A_{\mu} \partial_b X^\mu = n^A_{\mu} \partial_a \partial_b X^\mu. \quad (3.8)$$

In two dimensions, the Ricci curvature scalar $R$ is a function of the extrinsic curvature, namely,

$$R = K^{ab}K^A_{ab} - K^A K^A \quad (3.9)$$

where $K^A = \gamma^{ab}K^A_{ab}$.

It can be shown that to second order in $K$, the effective action takes the form

$$S = -\int d^2\sigma \sqrt{-\gamma} (\mu - \alpha K^A K^A + \beta R) \quad (3.10)$$

where $\alpha$ and $\beta$ are dimensionless numbers. By the Gauss–Bonnet theorem [117], the integral of $R$ over the worldsheet is $4\pi$ times its Euler characteristic, which is a topological invariant. Thus $\beta$ does not affect the equations of motion.

Both the sign and the magnitude of the constant $\alpha$ have been subjects of some disagreement in the literature [118]–[125], due in part to some ambiguity in the definition of the problem. A simple energy argument appears to show that $\alpha$ is in fact positive [118]. However, the correction term proportional to $\alpha$ is proportional to the extrinsic curvature $K^A$, which vanishes identically when the lowest-order Nambu–Goto equation (3.5) is satisfied. Thus, any solution of the Nambu–Goto equation is actually also a solution of the corrected equation. To this order, $\alpha$ too does not affect the equations of motion.

### 3.2. Strings in Minkowski space

The conformal gauge condition still leaves considerable freedom. To analyse the solutions of the equations of motion, it is convenient to fix the gauge further. In Minkowski space, with metric $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, the usual choice is the temporal gauge [114], in which one identifies the worldsheet time $\tau$ with the Minkowski time $X^0$. The constraints and the equations of motion become

$$\dot{X}^2 + \vec{X} \cdot \vec{X} = 1 \quad \dot{X} = 0 \quad \vec{X} = 0. \quad (3.11)$$

This choice has two happy results: the velocity is orthogonal to the string tangent vector; and $\sigma$ measures equal energy intervals along the string. From (3.6), we have

$$T^0_0(x) = \mu \int d\sigma \delta_3(x - X). \quad (3.12)$$
The general solution to the wave equation and constraints (3.11) is

\[ X = \frac{1}{2}[a(\sigma - t) + b(\sigma + t)] \quad \alpha^2 = 1 = \beta^2. \]  

(3.13)

Thus we solve the Nambu–Goto equations by specifying two curves on the unit sphere [126].

If we require a length \( L \) of string to form a closed loop, there is an additional constraint on the curves:

\[ \int_0^L d\sigma X = \frac{1}{2} \int_0^L d\sigma (\alpha + b) = 0. \]  

(3.14)

Furthermore, in the centre of momentum frame

\[ \int_0^L d\sigma X = \frac{1}{2} \int_0^L d\sigma (-\alpha + b) = 0. \]  

(3.15)

Thus the curves \( \alpha \) and \( b \) are both centred on the origin for stationary loops. From the periodicity of \( \alpha \) and \( b \) it follows that:

\[ X(t + L/2, \sigma + L/2) = \frac{1}{2} [a(\sigma - t) + b(\sigma + t + L)] = X(t, \sigma). \]  

(3.16)

Thus although the defining curves have period \( L \) the loop itself has period \( L/2 \).

Another property of loops in Minkowski space is that their mean square velocity is \( \frac{1}{2} \) in their rest frame [127], as may easily be verified from (3.11).

There can be points on the worldsheet where the string moves at the speed of light. Through the constraints we see that, if \( |X| = 1 \), the string tangent vector vanishes, and so at that point

\[ \alpha(\sigma - t) + b(\sigma + t) = 0. \]  

(3.17)

Providing \( \vec{X} \) is not zero at that point, the string takes the appearance of a cusp. These cusps are in a loose sense generic, for \( b \) and \( -\alpha \) are both closed curves centred on the origin, so it is intuitively ‘likely’ that there will be points of intersection. However, this need not happen: we could, for example, wrap the curves rather like seams on a tennis ball [128]. There is also no formal need for \( \alpha \) and \( b \) to be continuous [128]. The string then has discontinuities in the tangent vector, or ‘kinks’, moving in one or other direction at the speed of light. It is relatively easy to construct kinky loops with no cusps. A simple example is a loop made up entirely of kinks and straight segments:

\[ \alpha(u) = \begin{cases} e_1 & n\ell \leq u < (n + \frac{1}{2})L \\ -e_1 & (n + \frac{1}{2})L \leq u \leq nL \end{cases} \]

\[ \beta(v) = \begin{cases} e_2 & n\ell \leq v < (n + \frac{1}{2})L \\ -e_2 & (n + \frac{1}{2})L \leq v < nL \end{cases} \]  

(3.18)

where \( e_1 \cdot e_2 = 0 \). Over one period, this solution changes from a square to a doubled line in the \( e_1 + e_2 \) direction, back to a square, and then to a doubled line again, but in the \( e_1 - e_2 \) direction.
Figure 7. A simple loop trajectory over one period of oscillation. Note the cusps at \( t = 0 \) and \( t = \pi \).

In the simplest class of continuous solutions, \( \dot{a} \) and \( \dot{b} \) are great circles at an angle \( \psi \) [129]:

\[
\dot{a}(u) = \cos(2\pi Mu/L)e_1 + \sin(2\pi Mu/L)e_2 \\
\dot{b}(u) = \cos(2\pi Nu/L)e_1 + \sin(2\pi Nu/L)(\cos \psi e_2 + \sin \psi e_3)
\]

(3.19)

with \( M \) and \( N \) relatively prime. In figure 7 we display a sequence of projections of the string over one period for \( M = 1 \) and \( N = 2 \). This trajectory has two cusps, at \( t = 0 \) and \( t = \pi \), on opposite sides of the loop.

In an infinite universe we may also consider strings which do not join up on themselves and do not therefore satisfy \( \int d\sigma \dot{X} = 0 \). The simplest such solution is of course just a stationary straight string \( \dot{X} = 0, \dot{X} = \hat{n} \). There is also a simple solution describing helical standing waves [130]:

\[
\dot{a}(u) = OA[\cos(\omega u)e_1 + \sin(\omega u)e_2] + (1 - \omega^2 A^2)^{1/2} e_3 \\
\dot{b}(u) = OA[\cos(\omega u)e_1 + \sin(\omega u)e_2] + (1 - \omega^2 A^2)^{1/2} e_3
\]

(3.20)

where \( A \) is the amplitude, and the pitch of the helix is \( (1 - \omega^2 A^2)^{1/2}/2\pi \omega \).

The temporal gauge has the advantage of being easy to interpret, although solutions in closed form are hard to find because of the non-linear constraints. There is another gauge which does solve the constraints: the light-front gauge [116,131]. Instead of the condition \( \tau = X^0 \), we identify \( \tau \) with a parameter labelling a family of null 3-surfaces, or

\[
\tau = X^+ \equiv X^0 + X^3.
\]

(3.21)

Thus a constant-\( \tau \) slice of the worldsheet is simply its intersection with a family of null geodesics with 4-velocity \((1; 0, 0, -1)\). It is exactly as if we were looking at the shadow of the string illuminated by a distant source and projected on an imaginary screen. The
independent degrees of freedom are the two transverse coordinates $X^A$, which satisfy a wave equation

$$\ddot{X}^A - \ddot{X}^A = 0. \quad \text{(3.22)}$$

The constraints are now solvable in terms of these coordinates, for

$$\dot{X}^- = (\dot{X}^A)^2 + (\dot{X}^A)^2 \quad \dot{X}^- = 2\dot{X}^A \dot{X}^A. \quad \text{(3.23)}$$

The general solution in the light-front gauge is thus given by any two curves $\dot{X}_1^A(\sigma - \tau)$ and $\dot{X}_2^A(\sigma + \tau)$. There is in fact one further constraint if one wishes to consider only closed loops of string, for not merely do we have to make the transverse coordinates periodic, but $X^3$ must be periodic too. Hence $\int d\sigma \dot{X}^A \dot{X}^A = 0$.

We shall see in section 5.2 that when strings are formed in a phase transition, they are random walks. The mean square distance is given by

$$\langle (X(\sigma) - X(0))^2 \rangle \simeq \xi^2 (\sigma/\xi)^{2v} \quad \text{(3.24)}$$

where $\xi$ is the persistence length. The exponent $v$ is $\frac{1}{3}$ for $\sigma/\xi \gg 1$, and $\sim 1$ for small $\sigma/\xi$. Numerical and analytic studies, discussed in section 5.5, indicate that, as the string network evolves, $\xi$ settles down to a 'scaling' solution where it grows in proportion to the time $t$. On small scales, $v$ approaches 1 rather slowly as $\sigma/\xi \to 0$. This slow approach towards the value for a straight string indicates that there is a significant amount of 'wigglesness' on the string network on scales well below its persistence length.

We would like to have some description of the string dynamics which ignores the details of the wiggles while keeping track of their effects. Consider for example a string along the $z$ axis, which is straight apart from small wiggles. For the two functions in the general solution (3.13) we take

$$\alpha(u) = ku\hat{z} + a_\perp(u) \quad \beta(u) = ku\hat{z} + b_\perp(u) \quad \text{(3.25)}$$

where $k$ is a constant in the range $0 < k < 1$ and $a_\perp$ and $b_\perp$ lie in the $x$-$y$ plane. The gauge constraints require

$$a_\perp^2 = b_\perp^2 = 1 - k^2. \quad \text{(3.26)}$$

We may define an effective stress-energy tensor for the string, by averaging over a length scale $\Delta$ which is large compared to the wiggles:

$$T^{\mu\nu}_{\text{eff}}(t, x_\perp, z) = \theta^{\mu\nu}(t, z) \delta^2(x_\perp) \quad \text{(3.27)}$$

where

$$\theta^{\mu\nu} = \frac{1}{\Delta^2} \int_{t-\Delta/2}^{t+\Delta/2} dt' \int_{z-\Delta/2}^{z+\Delta/2} dz' \int d^2x_\perp T^{\mu\nu}(t', x_\perp, z'). \quad \text{(3.28)}$$

We then find that the only two non-vanishing components of $\theta^{\mu\nu}$ are the coarse-grained energy per unit length and the coarse-grained tension,

$$\mu_{\text{eff}} = \theta^{00} = \mu_0/k \quad T_{\text{eff}} = -\theta^{33} = k\mu_0. \quad \text{(3.29)}$$

Note that the product of the effective energy density and tension remains constant:

$$\mu_{\text{eff}} T_{\text{eff}} = \mu^2 \quad \text{(3.30)}$$

a result first derived for wiggly or 'noisy' strings by Carter [133], and then demonstrated for straight strings with wiggles by Vilenkin [132].
Cosmic strings

3.3. Strings with damping

The motion of strings in the early Universe is damped for two reasons: the Hubble expansion and the friction due to interaction with other particles. Let us first consider the effect of expansion.

We shall limit ourselves to a spatially flat Friedmann–Robertson–Walker cosmology (a very good approximation in the early Universe) for which

\[ ds^2 = dt^2 - a^2(t)dx^i dx^i = a^2(\eta)(d\eta^2 - dx^i dx^i) \]  

where \( \eta \) is the 'conformal time'. Then the temporal and conformal gauge conditions are no longer compatible; one must be sacrificed. It is very convenient for numerical purposes to keep the identification of worldsheet time with the background (conformal) time, so instead we drop the condition \( \dot{X}^2 + \dot{X}^2 = 0 \). We define \( \epsilon \) by \( \epsilon^2 = -\dot{X}^2/\dot{X}^2 = \dot{X}^2/(1-\dot{X}^2) \), where dots now denote \( d/d\eta \), and find that the equations of motion become [134]

\[ \ddot{\epsilon} + 2h \dot{X}^2 \epsilon = 0 \]

\[ \dot{X} + 2h(1-\dot{X}^2)X - \frac{1}{\epsilon} \partial_\sigma \left( \frac{1}{\epsilon} \dot{X} \right) = 0 \]  

with \( h = \dot{a}/a = aH \) (not to be confused with the dimensionless parameter \( h \) defined in section 1.4). The effect of the expansion of the Universe is to provide a damping term. The significance of \( \epsilon \) is seen if we compute the total energy of the string:

\[ E = \mu a^{-2} \int d\sigma \epsilon. \]  

We see that \( \epsilon \) is simply a dimensionless measure of the linear mass density of the string, which in Minkowski space we could set to be unity for all time.

A rough idea of string dynamics in an expanding background is obtained by considering two opposite limits for waves on infinite string, set by whether the curvature radius is large or small compared with the comoving Hubble length \( h^{-1} \) [135, 134]. For very-long-wavelength modes we can neglect the \( \sigma \) derivatives, to obtain

\[ \ddot{X} + 2h(1-\dot{X}^2)X = 0 \]  

which has an attractor at \( \dot{X} = 0 \). Thus the comoving velocity of the string is zero, and we infer that long-wavelength configurations are conformally stretched as the Universe expands. Conversely, we can neglect the Hubble damping for short wavelengths, to obtain approximately Minkowski equations of motion

\[ \dot{X} - \partial_\sigma^2 X = 0 \]  

where \( ds = \epsilon d\sigma \). This is also true of small loops of string.

We now turn to the effect of friction. In the early Universe the strings are moving through a hot medium of particles with which they interact, resulting in a frictional force [6, 136]. In the rest frame of the string, the force per unit length \( f \) exerted by a fluid of density \( \rho \) moving with velocity \( v \) is, in the conformal temporal gauge,

\[ f = \sigma \rho [v - (\dot{X} \cdot v)\dot{X}/\dot{X}^2] \]
where $\sigma$ is the scattering cross section per unit length (the calculation of $\sigma$ will be discussed in section 4.6). A covariant generalization of (3.33) is \[ f^\mu = \sigma \rho (u^\mu - \gamma^{ab} \partial_a X^\mu \partial_b X^\nu v_\nu). \] (3.34)

Thus the frictional effect of a fluid with velocity $u^\mu(x)$ is to change the equations of motion to

\[ \Box X^\mu + \Gamma^\mu_{\nu\rho} v^a \partial_a X^\nu \partial_b X^\rho = \left( \frac{\sigma \rho}{\mu} \right) [u^\mu(X) - \gamma^{ab} \partial_a X^\mu \partial_b X^\nu v_\nu(X)]. \] (3.35)

In a radiation-dominated FRW universe, the fluid velocity is $u^\mu = (a^{-1} \dot{x}, 0, 0, 0)$, and it is found that the temporal gauge equations of motion become \[ \dot{X} + \left( 2h + a \sigma \rho / \mu \right) X^2 \dot{\epsilon} = 0 \]

\[ \ddot{X} + \left( 2h + a \sigma \rho / \mu \right) (1 - X^2) \ddot{X} - \frac{1}{\epsilon} \partial_\sigma \left( \frac{1}{\epsilon} \dot{X} \right) = 0. \] (3.36)

Thus friction acts in exactly the same way as the damping due to the expansion of the universe. Associated with the friction there is a length scale $l_f = \mu / \sigma \rho$ which acts in the same way as the Hubble length. If $l_f < 2h/a$, frictional damping dominates over expansion; the network is conformally stretched on scales greater than $l_f$, while short wavelengths evolve as for Minkowski space. We will see (in section 5.3) that the cross section per unit length $\sigma$ goes as the inverse momentum of the scattering particles, so $\sigma \propto T^{-1} \propto t^{1/2}$. The density $\rho$ goes as $t^{-2}$. Thus, although friction is important at early times, it eventually becomes subdominant to expansion damping at (physical) time $t_*$ given by \[ t_* = \sigma (t_e) \rho (t_e) / \mu \sim (G \mu)^{-1} t_c \] (3.37)

where $t_c$ is the time at which the Universe was at the string critical temperature $T_c \simeq \mu^{1/2}$.

### 3.4. Superconducting string effective action

Superconducting strings carry currents in the worldsheet, which have a similar effect to small-amplitude waves in that they change the effective tension and linear mass density \[ [138]-[140]. \] If these currents are sources of a long-range gauge field, then there are additional complications due to the self-interaction of the string. Let us first neglect the gauge field, and consider a straight static string with a bosonic condensate $\chi = X e^{i \alpha} / \sqrt{2}$. This condensate contributes an extra piece to the action

\[ S_\chi = \int d\tau d^2 x \sqrt{g} \frac{1}{2} \{ (\partial_\mu \alpha)^2 - (\partial_\nu \alpha)^2 \} - (\partial_\mu X)^2 - (\partial_\nu X)^2 \}. \] (3.38)

To zeroth order in the curvature, the first two terms are constants, unimportant for the string dynamics. Introducing a pair of coordinates $\rho^A$ in the plane orthogonal to the worldsheet, we may write

\[ S_\chi \simeq \int d^2 \rho \sqrt{-\gamma} \kappa \gamma^{ab} \partial_\rho \alpha \partial_b \alpha \] (3.39)

where, as in section 2.7, $\kappa = \frac{1}{2} \int d^3 \rho X^2$. The phase of the scalar field is therefore a massless bosonic degree of freedom on the worldsheet. In principle, $\kappa$ is a function of $(\partial \alpha)^2$, since
the current $j_a = \delta_a \alpha$ reacts back on the condensate; we shall return to this point below. So, for small currents and curvatures, the equations of motion for the superconducting string are

$$\partial_a [\sqrt{-g} (\mu \gamma^{ab} + \theta^{ab}) \partial_b X^\mu] = 0$$

$$\partial_a (\sqrt{-g} \gamma^{ab} \kappa j_a) = 0$$

(3.40)

where $X^\mu$ now represents the string worldsheet position and is unrelated to the field $X$ above. Here $\theta_{ab}$ is the worldsheet energy–momentum tensor, given by

$$\theta_{ab} = \kappa (2 j_a j_b - \gamma_{ab} j^2).$$

(3.41)

By virtue of the equations both the current and the stress tensor are covariantly conserved; moreover, $\theta_{ab}$ is traceless.

Let us examine the dynamical effect of a constant current $j_a = (\omega, k)$ on a string in a Minkowski background. The stress tensor is

$$\theta_{ab} = \kappa \left( \begin{array}{ccc} \omega^2 + k^2 & 2\omega k & 0 \\ 2\omega k & \omega^2 + k^2 & 0 \\ 0 & 0 & \mu^2 \end{array} \right).$$

(3.42)

In principle, it would seem to be possible to construct a stationary loop solution by arranging that $\mu - \kappa (\omega^2 + k^2) = 0$ [96, 139, 140]. (It is easy to see that (3.40) reduces to this single condition for a static loop with constant current.) However, one has to be careful in pushing the equations of motion this far, because the stabilization of the loop would require a current of order $(\mu/\kappa)^{1/2}$. One would not then expect backreaction effects on $\kappa$ to be negligible. Furthermore, if $\mu - \kappa (\omega^2 + k^2)$ were to become negative, the loop would become unstable to small transverse perturbations [141]. The existence of classically stable loops has cosmological relevance, for such objects would have a relatively long lifetime and could contribute to the mass density of the Universe, even to the extent of ruling the theory out [139, 62, 142]. The crucial issue is whether or not the critical current, at which the condensate is ‘quenched’ and the superconductivity of the string is lost, is higher than the current which stabilizes the loop.

From numerical investigations of superconducting cosmic strings [62, 63], [65]–[68], it has emerged that null currents ($j^2 = 0$) have no backreaction, while timelike currents actually increase the size of the condensate. Loops with null currents ('vortons' [63]) seem to be the best bet for forming stable stationary configurations, although it should be emphasized that all stability studies have been carried out for straight strings. A general framework for the analysis of the stability of current-carrying strings has been developed by Carter [141]. The variation of the linear mass density $\mu$, the tension $T$ and the physical current $J_a = \kappa j_a$ with the worldsheet current $j_a$ gives the string an equation of state $T = T(\mu)$, which can be computed numerically for the model field theory of section 2.7 [67]. This equation of state allows one to calculate the speeds of propagation of transverse and longitudinal waves, which are respectively given by

$$c_T^2 = \frac{T}{\mu} \quad c_L^2 = -\frac{dT}{d\mu}. \quad (3.43)$$

The longitudinal waves are disturbances of varying current density, which involve the magnitude of the condensate. If $dT/d\mu$ becomes positive an instability develops in the condensate, and so the string cannot support currents beyond the threshold $dT/d\mu = 0$. 
We display the generic behaviour of \( \mu \), \( T \) and \( I = J_i \) for spacelike worldsheet currents in figure 8. Classically stable loop configurations can exist only if the tension can vanish before the critical current \( I_c \) is reached. Numerical studies indicate that this is not possible for spacelike currents without the extra help of a gauge field \([63,68]\).

So, let us now consider the dynamics when a massless gauge field \( A_\mu \), which couples to the worldsheet current, is included. The condensate field \( \chi \) now contributes

\[
S_\chi = \frac{1}{2} \int \text{d}t \text{d}z \text{d}x \text{d}y \left[ (\partial_t \alpha + q A_t)^2 - (\partial_z \alpha + q A_z)^2 \right]
\]

(3.44)

to the action. Provided we are interested only in gauge field configurations with wavelengths much greater than the string width, we may approximate \( A_\mu(x) \) across the string by its value at the centre, \( A_\mu(X) \), and write

\[
S_\chi = \int \text{d}^2 \sigma \sqrt{-\gamma} \chi \gamma^{ab}(\partial_a \alpha + q A_a)(\partial_b \alpha + q A_b)
\]

(3.45)

where \( A_a = \partial_a X^\mu A_\mu(X) \) is the (unnormalized) projection of the gauge field into the worldsheet. The string carries a gauge current \( J_\mu = -\delta S_\chi / \delta A_\mu \), which in turn acts as a source for the gauge field. The equations of motion for the superconducting string interacting with the gauge field are then

\[
\begin{align*}
\partial_a [\sqrt{-\gamma} (\gamma^{ab} + \theta^{ab}) \partial_b X^\mu] &= -2q\kappa \sqrt{-\gamma} F^{\mu\nu}(X) \partial^a X^\nu j_a \\
\partial_a (\sqrt{-\gamma} \gamma^{ab} \kappa j_b) &= 0 \\
\partial_\sigma F^{\mu\nu} &= 2q\kappa \int \text{d}^2 \sigma \sqrt{-\gamma} \partial^a X^\mu j_a \delta_4 (x - X(\sigma))
\end{align*}
\]

(3.46)

with \( j_a = \partial_a \alpha + q A_a \).

These equations first appeared in a dimensionally reduced classical string theory \([143,144]\). They arise naturally from a string propagating in a five-dimensional space–time, where one of the spatial dimensions is compactified to a circle. This is the original Kaluza–Klein construction \([145]\), which results in a theory of gravity and electromagnetism. When strings propagate in this background, the periodic coordinate in the compactified direction
behaves in the same way as $\alpha$, the phase of the scalar field. There is also a parameter playing the same role as $\kappa$, which is essentially the radius squared of the compactified dimension in units of the inverse string tension.

The self-interaction of the current-carrying string now makes true solutions hard to find, although perturbative solutions exist for small $q$. The principal effects of the gauge field are, firstly, to contribute a logarithmic term to the stress tensor and, secondly, to cause the string to lose energy through radiation. We consider radiation in section 4.4: here we find the logarithmic term. Formally, the gauge field from the string source is

$$A_\mu(x) = \int d^4x' G_{\alpha\beta}(x-x')J_\mu(x')$$

$$= \frac{q}{\pi} \int d^2\sigma \sqrt{-\gamma} \partial^\alpha X_\mu j_\alpha \delta((x-X)^2).$$

The leading logarithmic divergence can be extracted [146], and we find that the gauge field on the string is

$$A_\mu(X) = \frac{q}{\pi} \ln(L/w) j_\mu(X)$$

where the upper and lower cut-offs $L$ and $w$ are interpreted as a local curvature radius and a string width. This is quite familiar for a wire carrying current and charge. Thus if $j_\mu = (\omega, k)$ as before, the worldsheet stress tensor becomes

$$\theta_{ab} = \kappa(2j_a j_b - \gamma_{ab} j^2)[1 + (q^2/\pi) \ln(L/w)^2].$$

The effect of the gauge field is to reduce the effective tension and increase the effective mass density still further. It was found in [62] that zero effective tension (including the gauge field) could be reached with purely spacelike currents in a straight string for some parameter values in the simple model of section 2.7. However, relatively large values of $L/w$ were necessary, and so it is the pressure in the magnetic field which resists the collapse of a loop. The local tension in the string remains positive [68].

### 3.5. Global string effective action

The global string differs markedly from the gauge string in that it is surrounded by a long-range Nambu–Goldstone field $\phi$, which falls off as $r^{-1}$ away from the string. The string is a source for this field and interacts with it, making a general solution to the equations of motion impossible. Even perturbative solutions are not without problems, for there is no small coupling constant in which to expand.

Let us consider once again the action of a straight string on the $z$ axis, this time separating out the Nambu–Goldstone mode $\alpha$. By writing $\phi = f e^{i\sigma}/\sqrt{2}$, we have

$$S = \int dt dz \int dx dy [\frac{1}{2}(\partial f)^2 - \frac{1}{8} \lambda (f^2 - \eta^2)^2] + \int d^4x \frac{1}{2} f^2 (\partial \alpha)^2.$$ (3.50)

We know from section 2.1 that the last term is logarithmically divergent, but the first term is finite, and gives $-\mu_0 \int dt dz$. The action of the massless field $\alpha$ generally has two contributions: a multivalued part from the string itself, and single-valued propagating Nambu–Goldstone modes. Both can be described by a single-valued antisymmetric tensor field $B_{\mu\nu}$ and its accompanying field strength $H_{\mu
u\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ [147]–[149].
Locally, the fields are related by \( H_{\mu\nu} = \eta \varepsilon_{\mu\nu\rho} \delta^p \alpha \), (where \( \eta \) is the expectation value of the field away from the string), but in the presence of the string there are extra terms in the action describing the coupling between the string and the antisymmetric tensor field. Davis and Shellard [150] have found the canonical transformation which takes the action (3.50) into the dual form

\[
S^* = -\mu_0 \int d^2 \sigma \sqrt{-g} + 2\pi \eta \int d^2 \sigma \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} + \frac{1}{6} \int d^4 x H^2. \tag{3.51}
\]

The equations of motion that follow from this action are

\[
\mu_0 \Box X^\mu = 2\pi \eta H^{\mu\nu\rho} \partial_a X^\nu \partial_b X^\rho \varepsilon^{ab} \tag{3.52}
\]

\[
\partial_\rho H^{\mu\nu\rho} = 2\pi \eta \int d^2 \sigma \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu \delta_4(x - X(\sigma)).
\]

In the temporal gauge it is not hard to check that the straight string on the \( z \) axis gives rise to a field

\[
H_{\mu\nu} = \eta \varepsilon_{\mu\nu\rho} \phi \tag{3.53}
\]

which is consistent with \( \alpha = \varphi \). The strength of the coupling between the string and the antisymmetric tensor field is in fact fixed by this equality [149], which amounts to a consistency condition: we must always have \( \oint C dx^\mu \partial_\mu \alpha = 2\pi \) when taken on some curve \( C \) around the string. By Stokes’ theorem, this implies that \( \eta^{-1} \int d\Sigma^{\mu\nu} \partial_\rho H_{\mu\nu\rho} = 2\pi \) over a surface whose boundary is \( C \), and hence the value for this coupling of \( 2\pi \eta \). We can think of the field generated by the string as a Coulomb-type field arising from a ‘charge’ per unit length of \( 2\pi \eta \). This is not small compared with the string scale \( 1/\eta \), for \( \mu_0 \) is also fixed in terms of \( \eta^2 \) (and is independent of \( \lambda \)). Thus the string coupling is always strong. We shall amplify the discussion of global string self-interactions when we come to discuss radiation in section 4.

3.6. Intercommuting

The Nambu action described in section 3.1 is a good approximation for string segments which are well separated. However, a crucial dynamical question lies outside the framework: what happens when strings cross. To answer this question fully, one has to go to numerical solutions of the underlying classical field theory [151]–[155]. However, some qualitative and heuristic remarks serve to set the scene [156].

In figure 9, two segments of string, assumed Abelian, have just crossed. For our purposes it is irrelevant whether they are global or local. If we examine the planes \( A \) and \( B \), we see that the total winding number in \( A \) is 2, while in \( B \) it is zero. Thus there is no topological reason for plane \( B \) to be pierced by any string at all. It is almost as if plane \( B \) contains a vortex and an antivortex which, if they get sufficiently close, can annihilate into radiation. By the same token, plane \( A \) contains two vortices, whose total flux is conserved. Thus the evolution of the fields subsequent to figure 9 seems likely to eliminate the strings passing through \( B \), so that end 1 joins end 2' and vice versa. The strings are said to ‘intercommute’.

Numerical simulations of this process for both global [151] and local [152]–[154] strings, as well as for superconducting ones [155], confirm that this picture is correct. The string ends exchange partners, with the creation of sharp bends in each string. Each bend is resolved into two kinks moving in opposite directions at the speed of light (figure 10).
Intercommuting can even be observed in the laboratory, between line disclinations in a liquid crystal [76], although in this system the strings are heavily damped and so we do not see kinks.

One possible complication is if the strings are carrying currents. Suppose string 1' carries current $I_1$ and 2' carries $I_2$. Then, if reconnection takes place, charge must build up on the bridging segment on string 12', at a rate $(I_2 - I_1)$. Since each kink is moving at the speed of light, the length of that segment is increasing at a rate equal to $2c$, and...
so the charge density between the kinks is \( p = (I_2 - I_1)/2 \) in natural units, where \( c = 1 \). However, it might be argued that if \( I_1 \) and \( I_2 \) are oppositely oriented, the momentum in the current will act to preserve the connectivity 1', 22'. Nonetheless, numerical simulations by Laguna and Matzner [155] find that reconnection nearly always takes place, despite the build-up of charge on the reconnected strings.

Finally, we note that this discussion has been entirely about Abelian strings. The original argument in favour of intercommutation clearly does not apply to \( Z_n \) strings, where neither plane has any topologically conserved flux. Given that strings cannot break, reconnection must occur, but it can happen between 1 and 2 as well. This may be energetically disfavoured if, for example, it leads to the creation of a pair of beads (section 2.6). If \( \pi_1 \) is non-Abelian, then there is an additional possibility: the two strings may correspond to elements \( \tilde{h}_1, \tilde{h}_2 \), of the unbroken subgroup \( H \) which do not commute. In that case they pass through each other and create a third which joins the two intersection points, associated with the element \( \tilde{h}_1 \tilde{h}_2 \tilde{h}_1^{-1} \tilde{h}_2^{-1} \) [5].

4. String interactions

The observational effects of strings (see section 6) are the results of their interactions with the various particles and fields of nature. To some extent, the interactions are model dependent: if the strings are global they radiate Goldstone bosons; if superconducting, electromagnetic waves. They also interact with particles through scattering off the strong fields in the core of the string. However, all strings are the source of a gravitational field, through which they become a candidate for the origin of primordial density perturbations.

4.1. Gravity

The energy density of a gauge string is concentrated in its core, where we therefore expect a region of high curvature. Rather than working with the energy–momentum tensor of the field directly, it is often more convenient to approximate the string as a \( \delta \)-function source. For a gauge string obeying the Nambu action (3.21, the energy–momentum tensor for a string moving through a general background space-time was given in (3.6). In Minkowski space, where we can use the conformal temporal gauge (3.11), a straight string on the z axis has

\[
T^{\mu\nu} = \mu \, \text{diag}(1, 0, 0, -1) \delta(x) \delta(y). \tag{4.1}
\]

The string is a line source with equal tension and mass density, a result which can be derived purely from Lorentz invariance along the string, together with energy–momentum conservation [157]. Although the space–time curvature on the string is formally infinite in this approach, outside the string we may use the linearized Einstein equations for the perturbation \( h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \). In the harmonic gauge these are

\[
\Box h_{\mu\nu} = -16\pi G (T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T). \tag{4.2}
\]

With the source (4.1)) the inhomogeneous part of the solution is

\[
h_{\mu\nu} = 8G \mu \ln(\rho/\rho_0) \, \text{diag}(0, 1, 1, 0) \tag{4.3}
\]

where \( \rho^2 = x^2 + y^2 \), and \( \rho_0 \) is an arbitrary scale. The linearized solution cannot be correct everywhere, since the logarithm blows up for large and small \( \rho \). However,
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This solution can be matched onto an exact solution by the coordinate transformation
$$\left[1 - 8\pi G \mu \ln(\rho/\rho_0)\right] \rho^2 = (1 - 4G\mu)^2 R^2.$$ To order $G^2\mu^2$, the metric in the new coordinates is
$$ds^2 = dt^2 - dz^2 - dR^2 - (1 - 4G\mu)^2 R^2 d\varphi^2.$$ (4.4)

With a new angular coordinate $\tilde{\varphi} = (1 - 4G\mu)\varphi$ the space–time is seen to be flat everywhere (except at $R = 0$), but with the angular coordinate running from 0 to $2\pi - \delta$, with $\delta = 8\pi G\mu$. Thus it has the form of a cone in the plane transverse to the string, with an angle deficit $\delta$. Being flat, it clearly satisfies the full Einstein equations everywhere where $T^{\mu\nu} = 0$. It can be shown that the solution inside the string can be matched on to the asymptotic conical metric [158]-[160]. In the resulting metric, the angle deficit is no longer quite proportional to the mass per unit length, there being $O(G^2\mu^2)$ corrections [161,162]. Strings may also be embedded in other space–times, including FRW [163,164]. All that is required is that the space–time have an axis of symmetry: then the string space–time can be created by removing a wedge and then gluing together the resulting edges.

As mentioned above, the form of the string metric was dictated by Lorentz invariance along the string. As we have seen, there are string solutions which break this symmetry: superconducting strings and wiggly strings (see sections 2 and 3). Their energy–momentum tensors have the approximate form
$$T^{\mu\nu} = \text{diag}(\mu, 0, 0, -T) \delta(x) \delta(y)$$ (4.5)
where $T$ is the string tension. (There are also radial and azimuthal stresses, dictated by energy–momentum conservation, but they are $O(G\mu(\mu - T))$ and therefore negligible to first order [165,162].) The linearized metric becomes, after a redefinition of the radial coordinate,
$$ds^2 = [1 + 2\psi(R)] dt^2 - [1 - 2\psi(R)] dz^2 - dR^2 - [1 - 2G(\mu + T)]^2 R^2 d\varphi^2$$ (4.6)
where $\psi(R) = 4G(\mu - T) \ln(R/R_0)$. This can also be matched to a static, cylindrically symmetric exact solution [165,162]. It is conical perpendicular to the string, with angle deficit $4\pi G(\mu + T)$, but there is also a non-zero Newtonian potential due to the $h_{00}$ term, which results in an attractive force
$$F = 2G(\mu - T)/R$$ (4.7)
toward the string.

Around a global string, the energy–momentum tensor does not vanish. As we saw in section 2.1, there is a contribution from the Goldstone mode (the phase of the field) which decreases as $r^{-2}$ away from the string core, and leads to a logarithmically divergent mass per unit length. Near the string, however, a weak-field solution to the metric exists, which is [166]
$$ds^2 = [1 - 4G\mu \ln(R/R_0)](dt^2 - dz^2) - dR^2 + [1 - 8G\mu \ln(R/R_0)] R^2 d\varphi^2.$$ (4.8)
Remarkably, the straight global string has a repulsive gravitational force $2G\mu/R$. The angle deficit $\delta$ increases with distance, which causes problems at $\delta = 2\pi$, or $R = R_0 \exp(1/8G\mu)$. Unlike the gauge string, this is not just a problem with the coordinates: there is a genuine singularity at large $R$ [167,168]. However, this is at such a large distance that the singularity will not occur in physically realistic string configurations—we do not expect to find a completely isolated, straight string in the early Universe.
4.2. Gravitational radiation from loops

An oscillating string radiates gravity waves. The power in gravitational radiation produced by an isolated loop of length $L$ can be estimated using the quadrupole formula [169]

$$P \propto G(\mathcal{F}_{ij})^2$$  \hspace{1cm} (4.9)

where $\mathcal{F}_{ij}$ is the trace-free part of the quadrupole moment. On dimensional grounds, if $L$ is the only length scale on the loop, $\mathcal{F}_{ij} \sim \mu L^3$, while each time derivative brings in a factor of $L^{-1}$. Thus we find that

$$P = \Gamma G \mu^2$$  \hspace{1cm} (4.10)

where $\Gamma$ is a constant for each loop. That is, the power is independent of its size. In order to calculate $\Gamma$, we have to go to a full weak-field calculation. The average flux in radiation of frequency $\omega$ and momentum $k$ is [169]

$$\frac{dP}{d\omega} = \frac{G\omega^2}{\pi} \left[ T^+_{\mu\nu} (\omega_0, k) T^{\mu\nu} (\omega_0, k) - \frac{1}{2} |T^\nu_\nu (\omega_0, k)|^2 \right]$$  \hspace{1cm} (4.11)

where $T^{\mu\nu} (\omega_0, k)$ is the Fourier transform of the energy-momentum tensor. The string has a period $L/2$, so the frequencies $\omega$ are multiples of this fundamental: $\omega_n = 4\pi n/L$. In the conformal temporal gauge

$$T^{\mu\nu} (\omega_n, k) = -\mu \frac{2}{L} \int du dv (\partial_\mu X^\mu \partial_\nu X^\nu + \partial_\nu X^\mu \partial_\mu X^\nu) e^{ik \cdot X}$$  \hspace{1cm} (4.12)

where $u = \sigma - t$ and $v = \sigma + t$. Resolving the string into left- and right-movers, $X^\mu = X^\mu_R(u) + X^\mu_L(v)$, we find that the Fourier transform can be written

$$T^{\mu\nu} = -2\mu L (U^\mu V^\nu + V^\mu U^\nu)$$  \hspace{1cm} (4.13)

where

$$U^\mu (\omega_n, k) = \int_0^L \frac{du}{L} \partial_\mu X^\mu (u) e^{ik \cdot X_L} \hspace{1cm} V^\mu (\omega_n, k) = \int_0^L \frac{dv}{L} \partial_\mu X^\mu (v) e^{ik \cdot X_L}.$$  \hspace{1cm} (4.14)

Thus the source of the gravitational radiation is the interaction of left- and right-movers on the string. A loop has to be constructed from both, so a loop must always radiate.

Besides energy, the gravity waves also take away momentum and angular momentum [170]-[172]. Again, dimensional arguments indicate that the rate of loss of each must go as $\Gamma P G \mu^2$ and $\Gamma L G \mu^2 L$, respectively, where $\Gamma P$ and $\Gamma L$ are dimensionless constants. The radiation of momentum is particularly interesting, since it means that the loop will be accelerated in the opposite direction—the so-called 'rocket' effect [170]. Although each impulse is small, of order $G \mu^2 L$, the net effect over many oscillations can build up so that the loop is moving with speeds comparable to $c$ towards the end of its life.

The constants $\Gamma$, $\Gamma P$ and $\Gamma L$ can be worked out exactly for simple trajectories [171, 129, 172], for which it is found that $\Gamma$ is in the range 50-100, while $\Gamma P$ and $\Gamma L$ are somewhat smaller, around 5 [172]. For some special cases, the total power actually diverges, due to the presence of persistent cusps.
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Cusps, we recall, are points on the string where the tangent vector vanishes and the string formally reaches the velocity of light, usually just for an instant. In the weak-field approximation this results in the metric diverging along a null line originating at the cusp, and pointing in the direction of the 3-velocity of the string at the cusp [130]. This divergence is not very strong [173], and for loops with isolated cusps the power is still finite. The only sign of it is that the power falls very slowly with the mode number $n$: $P_n \propto n^{-4/3}$ [171]. However, if the cusp persists throughout the loop’s period, the power becomes infinite.

One should take into account the backreaction of the gravitational field on the string [174, 175]. What happens is entirely analogous to Dirac’s computation of the self-force of the electron. There is a logarithmically divergent self-energy which can be absorbed into a renormalization of the string tension, while the finite parts contribute corrections to the Nambu–Goto equations of motion at third and higher order in the derivatives. For trajectories with isolated cusps, the weak-field self-force equations can be solved numerically. This involves computing the impulse on each point of the string due to the force exerted over one period by all the others, which is finite and $O(G\mu)$. Quashnock and Spergel [175] have followed the trajectories of several cuspy and kinky loops, finding that the kinks seem to be rounded off, while cusps change their shape and arrive later, but do not disappear, in accordance with the arguments of Thompson [131].

An intriguingly different tack was taken by Mitchell et al [176], who went to the first quantized string theory in order to compute the decay rate of highly excited string states, which are in some sense quasi-classical. They found that, in four dimensions, string states with level number $N$, which in the classical limit corresponded to rotating rods of length $\sqrt{N}$, decayed primarily into states at level $N - 1$ and 1, with a finite rate proportional to $\sqrt{N}$. This is a nice result for three reasons: the level-1 states correspond to the massless graviton, dilaton and antisymmetric tensor fields; the energy in the emitted massless states goes as $1/\sqrt{N}$ so, as in the classical process, the total power is independent of the length; and the classical power from the rotating rod is infinite, whereas Mitchell et al obtain

$$P \approx 360G\mu^2.$$  (4.15)

It is not entirely clear, however, that this can be regarded as the solution to the backreaction problem for a gauge string, since the connection between the semi-classical gauge string and the quantized fundamental string is obscure.

4.3. Gravitational radiation from infinite strings

Small-amplitude waves on infinite strings are also the source of gravitational waves [177, 178]. Cosmic string simulations indicate that there is a substantial amount of this small-scale structure on the string network (see section 5), and it cannot be neglected as a source of radiation. For a string on the $z$ axis with small perturbations we can write

$$X^\mu = (t, X^\perp, X^3).$$  (4.16)

The flux per unit length in waves with frequency $\omega$ and $z$ component of momentum $k$ through a large cylinder enclosing the string is then [178, 179]

$$\frac{dP}{dz}(\omega, k) = 32\pi G\mu^2 \omega(|U^\perp|^2|V^\perp|^2 + |U^\perp \cdot V^\perp|^2 - |U^{\perp*} \cdot V^{\perp*}|^2)$$  (4.17)

where $U^\perp$ and $V^\perp$ are ordinary Fourier transforms of the right- and left-moving derivatives of the perturbations:

$$U^\perp(\omega, k) = \int_0^L \frac{du}{L} \partial_u X^\perp e^{-i(\omega + k)u/2} \quad V^\perp(\omega, k) = \int_0^L \frac{du}{L} \partial_u X^\perp e^{i(\omega - k)u/2}.  (4.18$$
Here, $L$ is a large repeat length, needed to render integrals finite. Again we see that it is the interaction of right- and left-movers which provides the radiation. Note that, in this small-amplitude approximation, perturbations of frequency $\omega$ also produce waves of frequency $\omega$. In loops, the interactions also produce higher harmonics, which decrease only as a power law. On the straight string the higher harmonics are exponentially damped and are usually negligible, unless some special symmetry forbids the lowest harmonic [179]. A further difference from loops is that there can be waves travelling in one direction only, which do not radiate, for the gravitational wave moves along with the perturbation [180]. In fact, there are exact travelling-wave solutions [181], which can be thought of as ordinary gravitational plane-wave solutions with a wedge cut out of the space around the direction of propagation.

The gravity waves remove energy from the string, which must result in the decrease in the amplitude of the perturbations. We see from (4.17) that the power radiated is proportional to the frequency: thus high-frequency modes decay more quickly. This has implications for the small-scale structure, which is made up of many small-angle kinks, for the high frequencies in the kinks will disappear first, rounding them off (see section 5).

### 4.4. Pseudoscalar and electromagnetic radiation

In section 2 we found that strings may be the sources of other massless fields. If the string is a global one, such as an axion string, it is surrounded by the field of the pseudoscalar Goldstone boson associated with the symmetry breaking. If it is superconducting, it can produce a long-range electromagnetic field. When in motion the string then radiates coherent Goldstone bosons [182] or electromagnetic waves as well as gravitational radiation. As we saw in section 3.6, we can treat global strings as sources for an antisymmetric tensor field $B_{\mu\nu}$, which is equivalent to the Goldstone boson. In this representation one can derive the equations for the emitted power per unit solid angle in much the same way as for gravitational radiation, to obtain [149]

$$\frac{dP}{d\Omega}(\omega, k) = 2\omega^2 J^*_{\mu\nu}(\omega, k)J^{\mu\nu}(\omega, k)$$

(4.19)

where $\omega$ is quantized in units of $4\pi/L$, and $J^{\mu\nu}$ is the Fourier transform of the source distribution for the antisymmetric tensor field, or

$$J^{\mu\nu}(\omega, k) = \frac{\eta}{L} \int dudv \left( \partial_\mu X^\nu \partial_\nu X^\mu - \partial_\nu X^\mu \partial_\mu X^\nu \right) e^{ik.x}.$$ 

(4.20)

An analogous formula exists for infinite strings [183]. On dimensional grounds, the total power from a loop is independent of size and proportional to $\eta^2$, where the constant of proportionality tends to be about 50–100 for the trajectories for which it can be computed (which are the same as those for which the gravitational power is known) [149].

Compared to the gravitational case, the coupling between the string and the field is very strong, so it is legitimate to ask whether the power calculated in this way really represents the true radiation. Indeed, it has been argued that the string is overdamped, and should dissipate its energy without oscillating at all [107, 184]. However, a careful investigation by Battye and Shellard [179] has compared the antisymmetric tensor formulation with the radiation in a full numerical simulation of the complex scalar field theory represented by the Lagrangian (2.1), and found them to be in good agreement. The string is heavily damped—for example, kinks are very quickly smoothed out—but not overdamped, for trajectories oscillate several
times before losing their energy. Settling this question is extremely important for axion strings, for a very strong bound on the symmetry-breaking scale comes from cosmological limits on the energy density in coherent axion modes (see section 5.10).

A string carrying a current will also produce electromagnetic radiation, with luminosity

$$\frac{dP}{d\Omega}(\omega, k) = -2\omega^2 J^*_\mu(\omega, k) J^\mu(\omega, k)$$  \hspace{1cm} (4.21)$$

where

$$J^\mu(\omega, k) = \frac{2}{L} \int du dv (j_u \partial_v X^\mu + j_v \partial_u X^\mu) e^{ik \cdot X}$$  \hspace{1cm} (4.22)$$

which is the Fourier transform of the current distribution on the string. The power of a string with constant current $I$ can be estimated with the formula for dipole radiation: the average power emitted by an oscillating magnetic dipole $m$ is roughly $(m^2)$. If the string has length $L$, the dipole moment is $\sim eIL^2$ and, since the period is $2/L$, we find

$$P = \Gamma_{em} e^2 I^2.$$  \hspace{1cm} (4.23)$$

$\Gamma_{em}$ is a constant for each trajectory, and as for the gravitational radiation can be computed in special cases [128, 146]. Again, typical values are 50–100.

4.5. Particle emission

A vibrating string can also be the source of other fields besides the massless ones considered above. For example, it is likely that the scalar field which makes the string will also couple to the conventional Higgs of the Standard Model. However, it has been shown by Srednicki and Theisen [185] that the power radiated in massive particles is generally not significant compared to gravitational radiation. The essential point is that particles of total energy $E$ are produced only in regions of the string where the fields are changing very rapidly compared with the scale $E$, which turn out to be small regions near cusps.

For simplicity let us consider only a single real field $h$, which is coupled to the string field $\phi$ through terms in the Lagrangian such as $\frac{1}{2}\lambda |\phi|^4 h^2$. Let $\phi$ have expectation value $\eta/\sqrt{2}$ in the vacuum. It can be shown that, if backreaction on the string state $|S\rangle$ is neglected, then the amplitude for the emission of two particles with momenta $k_1$ and $k_2$ is

$$\langle S'; k_1 k_2 |S\rangle = \lambda' \int d^4x e^{ik_1 \cdot x} \langle S|(|\phi|^2 - \frac{1}{2} \eta^2)|S\rangle.$$  \hspace{1cm} (4.24)$$

The expectation value of $|\phi|^2 - \frac{1}{2} \eta^2$ in the string state $|S\rangle$ can be evaluated by changing to string-centred coordinates, much as we did in section 3 to calculate the string effective action, and we find (in the conformal temporal gauge)

$$\langle S|(|\phi|^2 - \frac{1}{2} \eta^2)|S\rangle \simeq \frac{1}{\lambda} \int dt d\sigma X^2 \delta^{(3)}(x - X(\sigma, t))$$  \hspace{1cm} (4.25)$$

where $\lambda$ is the self-coupling of the field $\phi$, so that the string width is $M^{-1} \simeq (\sqrt{\lambda} \eta)^{-1}$.

Using this approach, Srednicki and Theisen [185] found that ultrarelativistic particles are emitted in a narrow cone around the cusp velocity of angle $(\omega L)^{-1/3}$, where $L$ is the length of the loop. The sum over modes is apparently divergent, but in fact the neglect of
backreaction must fail when $\omega \sim M$, so $M$ acts as a cut-off. The emitted power is found to be

$$P \sim \frac{\lambda'^2 \mu}{\lambda M L}. \quad (4.26)$$

Comparing with the gravitational power $P_G \sim G \mu^2$, we see that (for $\lambda' = \lambda$) loops larger than about $(G \mu)^{-1} M^{-1}$ emit an insignificant fraction of the energy in two-particle decays. Equally, loops smaller than this critical value quickly end their lives in a burst of particle emission. The reason for graviton emission being much more likely is the dimensionful coupling and the one-particle final state, which yields a convergent mode sum

$$P_G \sim G \mu^2 \sum n^{-4/3}. \quad (4.27)$$

Note that the fall-off with $n$ is just that advertised for loops with cusps in section 4.2.

These methods are unable to give much insight into the emission of particles with the same mass scale as the string, especially those which actually make up the string. For particles of mass $\sim M$ the backreaction is not negligible, and the replacement of $|S'\rangle$ by $|S\rangle$ is no longer valid. The prime site for particle emission is the cusp, because the fields are changing very rapidly there. The cusp is also a region where the string changes direction, and oppositely oriented segments get very close. In fact, they overlap in a region of length $\sigma_c \sim (M|\vec{X}_0|)^{-1/3}$, where the subscript 0 indicates the value at the cusp. Thus there is the possibility of a classical annihilation of this region, with the energy being transferred into propagating modes of the scalar and gauge fields which make up the string [186]. This 'cusp annihilation' can result in the emission of energy $\mu \sigma_c \sim \mu L (ML)^{-1/3}$. If this happened every period, the power would be $(ML)^{-1/3} \mu$, which dominates light particle emission and also gravitational radiation for loops smaller than $\sim (G \mu)^{-2} M^{-1}$. However, it is not clear that the cusp does recur: the replacement of a segment of string of length $\Delta \sigma \sim L (ML)^{-1/3}$ with one of length $\Delta \sigma \sim M^{-1}$ could also result in $|\vec{X}_0|$ becoming very large. The centre of the string now takes a short cut of physical distance $M^{-1}$, so $|\vec{X}_0|$ is now of order $M$, and the new overlap region contains energy $M$—much less than the original $\mu L (ML)^{-1/3}$.

Small amounts of energy are also available for conversion into particles when strings self-intersect and reconnect, especially at high relative velocities, where the scalar field can be seen to be highly excited in the intersection region after the strings have moved away [151]–[155]. However, none of these particle emission processes challenges gravitational radiation as the dominant decay mode of the gauge string, or coherent Goldstone boson radiation for the global string, except when the loops are smaller than $(G \mu)^{-1}$ times their width.

4.6. Particle scattering

A perturbative calculation of the scattering cross section per unit length for a scalar particle coupled as in the last section reveals that it takes the form

$$\left( \frac{d \sigma}{d\ell} \right)_b \sim \left( \frac{\lambda'}{\lambda} \right)^2 \frac{1}{E} \quad (4.28)$$

where $E$ is the energy of the incident particle. The cross section is therefore determined by the Compton wavelength of the incident boson. A nice physical argument for this behaviour
has been given by Brandenberger et al [187]. The string effectively consists of a chain of scatterers of size $M^{-1}$, with geometric cross sections $M^{-2}$. A boson of energy $E$ will scatter coherently off $M/E$ of them, thus amplifying the cross section of a length $E^{-1}$ by a factor $(M/E)^2$. The final cross section per unit length is therefore $M^{-2}(M/E)^2 E \sim E^{-1}$.

Fermions have different couplings and a different phase space. With a coupling $g\phi^*\tilde{\psi}\psi$, the cross section per unit length goes as [187]

\[
\frac{d\sigma}{dl} \sim \left( \frac{g\eta^2}{M^2} \right)^2 E \sim \left( \frac{g}{\lambda} \right)^2 \frac{E}{M^2}.
\]

This is again explicable in terms of our chain of scatterers: fermion wavefunctions do not add coherently, so in the length $E^{-1}$ the fermion scatters from only one point with cross section $M^{-2}$. Thus $(d\sigma/dl)_F \sim M^{-2}E$.

These estimates are perturbative estimates, and are only good if they represent small corrections to the probability of the particle being unaffected. Hence the coupling of the scattering particle to the string must be small compared with the string self-coupling. Otherwise, one must take into account the distortion of the wavefunctions of the scattering particles by the string core. This in general leads to amplification of the scattering amplitude by a factor $A \sim |\psi(M^{-1})|/|\psi_0(M^{-1})|$, where $\psi_0$ is the free wavefunction [187]. The amplification occurs because the expansion in angular momentum eigenstates of the exterior part of the true wavefunction can contain modes which diverge at small radius. In the free case, the condition of regularity at the origin means that they cannot be present, but when the string is there they can match onto the solution in the core. Precisely which divergent modes are excited determines the amplification.

A crucial factor in the calculation of the cross section for gauge strings is the charge of the scattering particle under the generator of the broken $U(1)$, say $q$. If the charge of the string field is $e$, then the cross section depends very strongly on the ratio $q/e$. If it is an integer, then it is found (for both bosons [136] and fermions [188]) that

\[
\frac{d\sigma}{dl} = \frac{4}{k \ln^2(k/M)}
\]

where $k$ is the transverse momentum of the incident particle. If, however, $q/e$ is fractional, the particle experiences an Aharonov-Bohm effect [49]. The wavefunction of a particle passing the string picks up a relative phase factor $e^{i2\pi q/e}$ between the parts which pass on either side of the string, and so they can interfere very strongly. In that case it is found that the cross section is just the usual Aharonov-Bohm form

\[
\frac{d\sigma}{dl} = \frac{1}{2\pi k} \frac{\sin^2(\pi q/e)}{\cos^2 \theta/2}.
\]

The cross section does in fact have some dependence on the details of the fields in the core of the string [189]. It can depart from the Aharonov-Bohm form if, for example, the flux in the core changes direction with radius (which might happen in strings in theories with several stages of symmetry breaking).

There is another effect, gravitational in origin, which contributes to particle scattering. We recall that the space–time around a straight string has an angle deficit $\delta = 8\pi G \mu$, so if we place a charge in this background, we must satisfy the boundary condition that the electric field is the same at $\theta = \pi + \delta/2$ and $\theta = \pi - \delta/2$. By solving the Poisson equation
with these modified periodicity requirements, it can be shown [190] that there is a repulsive force

\[ F = \frac{G \mu q^2}{16r^2} \hat{r} \]  

(4.32)

where \( \hat{r} \) is the unit vector from the nearest point on the string to the particle. This results in a Coulomb-like scattering cross section [191].

4.7. Baryon-number violation

In the interior of a string formed after the breaking of a grand unified symmetry are fields which carry both baryon and lepton number. Thus if a quark gets to the core of the string it can interact with the background core fields and emerge a lepton, and vice versa. Strings therefore have the potential for mediating baryon-number-violating processes. This is very reminiscent of the Rubakov–Callan effect [192], where quarks and leptons change into each other in the interior of a monopole. The cross section for this process is not the geometric cross section: rather, it depends on the Compton wavelength of the scattering particle, going as \( E^{-2} \). For strings, we have seen how the elastic scattering cross section depends crucially on the ratio \( q/e \), since it controls the amplification of the scattering wavefunction at the core of the string. Since any \( B \)-violating processes happen in the string core, the same is also true of the \( B \)-violating cross section [193,188]. The cross section also depends on the type of interaction: that is, whether it is mediated by a scalar or by a vector field. A vector field produces its greatest effect for integer \( q/e \), when the \( B \)-violating cross section per unit length can reach the scattering cross section \( \sim E^{-1} \) [188]. The \( B \)-violating cross section from a core scalar field is greatest for half-integer \( q/e \), when it can also reach \( \sim E^{-1} \) [193,188].

There have also been suggestions that there can be \( B \)-violation produced by a kind of Aharonov–Bohm effect [194,195]. If the generator of the gauge field of the string does not commute with the generator of \( B \) or \( L \), then the adiabatic transport of a particle around the string can result in its returning with different \( B \) or \( L \) quantum numbers: this is then apparently a \( B \)- or \( L \)-violating process. In a scattering experiment, the outgoing wavefunction resulting from an incident beam of pure quarks could contain a mixture of quarks and leptons, in different proportions depending on the scattering angle \( \theta \). However, it has been shown that this mixture is an illusion [196]. While it is true that the quark and lepton fields in the original basis at \( \theta = 0 \) are apparently mixed as we move around the string, the definition of what mixture actually constitutes a quark or lepton field also changes. The worst that can happen is that the definition at \( \theta = 2\pi \) differs by a phase from that at \( \theta = 0 \): but this is just the original Aharonov–Bohm effect, without any peculiar baryon-number violation.

The physical consequences of \( B \)-violation are greatest when the string density is greatest, soon after the phase transition which forms them. Any existing \( B \)-asymmetry in the Universe can relax to zero via scattering off strings [197]. If \( \xi(t) \) is the correlation length of the string network, and \( \sigma_B \) is the \( B \)-violating cross section per unit length, then

\[ \frac{dn_B}{dt} \approx -\bar{v}\sigma_B \frac{1}{\xi^2} n_B \]  

(4.33)

where \( n_B \) is the baryon number density and \( \bar{v} \) is the thermally averaged relative velocity of the particles and the strings, which is of order 1. The cross section per unit length goes as
where $T$ is the temperature and $c$ a dimensionless constant, while the correlation length increases at $t^p$. When the network is scaling (see section 5.3) $p = \frac{1}{2}$, but just after the phase transition it has the value $\frac{3}{4}$. Using the relation $T \sim (t/M_{pl})^{-1/2}$, we find

$$n_B(t) \simeq n_B(t_0) \exp \left( C \left[ \frac{(t_0/t)^{2p+1/2} - 1}{\xi(t_0)} \right] \right)$$

(4.34)

where

$$C \sim \frac{c}{(t_0/M_{pl})^{1/2}} \left( \frac{t_0}{\xi(t_0)} \right).$$

(4.35)

Thus if the baryon number is generated when the string is scaling, that is $\xi \sim t$, the reduction factor $\exp(-C)$ is insignificant. However, if it is generated at or before the string-forming phase transition, when $\xi(t_0)/t_0 \sim T_0/\lambda M_{pl}$, there is potentially a very large suppression.

Where there is scattering there is also emission, so strings may also be a source of baryon number. As Sakharov pointed out [198], generating a baryon asymmetry in the early Universe requires not only $B$-violation but also $C P$-violation, as well as a departure from thermal equilibrium. If quarks and leptons have $C P$-violating couplings to the strings, or to a string decay product, then emission from strings constitutes an injection of baryon number into the equilibrium distribution of particles in the early Universe [199]–[202]. This requires a high string density, in order to produce a sufficient asymmetry. However, the tendency of the strings to reduce the baryon-number density by scattering has not been taken into account in existing estimates.

5. Strings in the early Universe

If cosmic strings exist, they were formed in the first fraction of a second after the Big Bang. Any effects they have on observable cosmological features occur at much later epochs. To follow the causal chain from one to the other, we have to study how they were formed and how the network of strings evolves over this immense span of time.

5.1. High-temperature field theory

To study the formation of strings, we need to consider the effects of high temperature on a field theory, since the early Universe was very hot.

The most important effect is to replace the potential $V(\phi)$ in the Lagrangian by a high-temperature effective potential, $V_T(\phi)$ [203,204,4]. This is the free-energy density in a state with the prescribed value of $\phi$. When $T$ is much larger than all the masses involved (i.e. $T \gg m(T)$, the temperature-dependent mass), the leading correction terms are easily identifiable with the free-energy density of an ideal relativistic gas:

$$V_T(\phi) = V(\phi) - \frac{g_s \pi^2}{90} T^4 + \frac{M^2(\phi)}{24} T^2 + O(T).$$

(5.1)

Here $g_s$ is the effective number of spin states of relativistic particles (with a factor of $\frac{7}{8}$ for fermions) and $M^2(\phi)$ is the sum of squared masses of particle excitations about the chosen state (with a factor of $\frac{1}{2}$ for fermions). The $T^4$ term is independent of $\phi$, and hence does not affect the symmetry breaking. The important correction is the $T^2$ term, which yields symmetry restoration at high temperature.
Consider for example the model with Lagrangian (2.6) and \( V(\phi) = \frac{1}{2} \lambda (|\phi|^2 - \frac{1}{2} \eta^2)^2 \). The zero-temperature symmetry breaking is signalled by the fact that \( V \) has a maximum at \( \phi = 0 \). Here we find \( g_* = 4 \) and \( M^2(\phi) = 3e^2|\phi|^2 + \lambda (2|\phi|^2 - \eta^2) + \frac{1}{12} (2\lambda + 3e^2) T^2 \). Thus at high temperature, the coefficient of \(|\phi|^2\) in \( V_T \) becomes positive. Then \( V_T \) takes its minimum value at \( \phi = 0 \). In this approximation, we have a second-order phase transition. The critical temperature above which the symmetry is restored is given by

\[
T_c^2 = \frac{6\lambda \eta^2}{2\lambda + 3e^2}, \tag{5.2}
\]

The \( O(T) \) term in (5.1) can change this analysis: formally, it is \( -T \sum_j m_j^2(\phi)/12\pi \), where \( j \) labels the particle species. However, it turns out that this term can only be trusted if \( \lambda \ll e^2 \). Its effect is to give the potential a shallow minimum at \( \phi = 0 \), which suggests that for heavy vector particles there should be a first-order phase transition.

For the electroweak transition, the high-temperature approximation appears to be reasonable provided that \( \lambda \), or equivalently the Higgs mass \( m_H \), is not too large [205]. In practice, values of the Higgs mass below about 65 GeV can probably now be ruled out. For reasonable values of \( m_H \), the critical temperature is around 200 GeV, substantially larger than \( m_W \) and \( m_Z \). However, the conclusion concerning the order of the transition is still far from certain. There are infrared divergences in higher-order correction terms which invalidate the perturbation calculation in the immediate neighbourhood of the critical point, the effects of which are not wholly understood [4]. It has been argued that non-perturbative effects may yield a first-order transition even in the case of large Higgs mass [206].

In grand unified theories, there is a phase transition (or transitions) at a critical temperature of \( 10^{15} \) or \( 10^{16} \) GeV. In this case, the estimated critical temperature is roughly comparable with the induced particle masses, so a first-order transition is rather more likely. This is mainly because the number of contributing light particles is much larger.

There is still much work to be done before it can be said that we have a completely reliable understanding of the high-temperature phase transitions in field theory.

It should be mentioned that there is probably another important transition in the early Universe, considerably later than those mentioned so far. This is the quark–hadron transition, at which the dense gas of free quarks and gluons condenses into individual hadrons. It occurs at a temperature of perhaps 200 MeV, and may be associated with the breaking of chiral symmetry. This transition, while not generating cosmic strings itself, is an important stage in the evolution of axion strings, as we shall describe in section 5.10.

5.2. Formation of strings

Because the scales of interest are mostly very large compared to the thickness of a cosmic string, the long-term evolution of a string network is almost independent of the details of the field-theory model in which the strings appear—though we have to distinguish local from global strings and superconducting from non-superconducting ones, and of course we have to know the symmetry-breaking scale. So, for simplicity, let us consider the simplest model, the Abelian Higgs model described by the Lagrangian (2.6), which predicts non-superconducting, local strings, with tension \( \mu \approx \eta^2 \) (c.f. (2.7)). We shall also concentrate on thermal phase transitions, which happen as a result of the cooling of the Universe in a conventional radiation-dominated FRW universe, reserving a brief discussion of the formation of defects during inflation for the end of this section.

Initially, when the Universe is very hot, it is in the symmetric phase and no strings are present. When it cools to below the critical temperature \( T_\mathrm{c} \), the Higgs field \( \phi(x) \) will tend to
Cosmic strings

settle down into the valley containing the circle of minima of the potential \( V_T \). Eventually, as \( T \to 0 \), the system will tend towards one of the vacuum states described by the points on this circle.

Of course, this is a quantum system, so it might also be in a superposition of these states. If we were dealing with a single particle then, as every student of quantum mechanics knows, the true ground state would be a symmetric superposition; there would be no symmetry breaking. However, a field theory is different, because it involves an infinite number of degrees of freedom. To get from one to another of the degenerate vacua, one must rotate the phase of the Higgs field at an infinite number of points. Thus a quantum superposition of states with different values of the phase is for all practical purposes equivalent to a classical superposition. No local observable has matrix elements between the degenerate states. Because of gauge invariance, the different vacuum states are physically indistinguishable; the overall phase of \( \phi \) is not an observable.

In fact, even the relative phase of \( \phi \) between different points is not an observable, unless we have chosen a gauge. For example, we might work in the Coulomb gauge. Then, provided the remaining gauge ambiguity is removed by imposing a suitable boundary condition, the relative phase between the \( \phi \) values at different points is an observable.

Now what happens when the temperature in the expanding Universe falls below the critical temperature? Once it has fallen well below \( T_c \), the Higgs field in most regions will have acquired a non-zero average value, but for causality reasons the phases in widely separated regions will be uncorrelated. This must be true even in a gauge theory, once the gauge ambiguity has been removed. We should of course also consider the coupling to the gauge field. However, the currents that generate such fields arise from phase fluctuations only once the magnitude of \( \phi \) has become significantly non-zero. It therefore seems unlikely that this effect would make a major difference to the probability of string formation. (However, for a different view, see [207].)

The phase transition may be first order, in which case it will proceed by bubble nucleation [208]–[210] or by spinodal decomposition [211]; alternatively it may be a continuous, second-order process. The order depends, as we have seen, on details such as the ratios of coupling constants. The net effect is much the same in both cases, though one may perhaps expect a rather different initial scale for the resulting string network.

In the bubble-nucleation case, we should expect the phase in each bubble to be a more or less independent random variable, although if the phase is determined by the random values of other fields, one might expect the phases in neighbouring bubbles to have some correlation. When two bubbles meet, the values of \( \phi \) across the boundary will tend to interpolate between those in the two bubbles [212]. When three bubbles meet, a string may be trapped inside their mutual boundary, if the net phase change around it is \( 2\pi \) [6] (see figures 11). The scale size of the resulting network is related to the separation of nucleation centres, and to the probability of trapping a string at a boundary, which is typically of order 0.3 [213,214]. The scale would be larger if there were indeed a correlation between the phases of neighbouring bubbles—the separation of nucleation centres would be replaced by the overall phase correlation length.

In the alternative case of a second-order transition, or of a first-order transition proceeding by spinodal decomposition, the Higgs field everywhere will tend to move away from zero—the peak of the potential hump—at about the same time. It is of course characteristic of a second-order transition that the values of \( \phi \) will be correlated over long distances, though the correlation length will never actually become infinite if the rate of change of temperature is finite. Immediately after the transition, when the hump is still small, fluctuations taking \( \phi \) over it will be quite probable, so the phase is not really fixed. However,
as the temperature falls further, such fluctuations become increasingly improbable. When
one reaches the Ginzburg temperature $T_G$ [215], at which the energy of a fluctuation over the hump on the scale of a correlation volume is equal to $T$, these fluctuations effectively cease and the phase is 'frozen in'. (Remember that in our units $k_B = 1$.) In this case too, string defects will be trapped in places where the phase change around a loop is $2\pi$. The initial length scale of the resulting network will be determined, as before, by the probability of string trapping and by the correlation length at the Ginzburg temperature.

In either case the net result is a random network of strings with some characteristic scale, which we shall call $\xi$; $\xi$ is certainly $< t$ and probably $<< t$. The strings cannot have free ends and must either form closed loops or be infinitely long. Numerical simulations of string formation on a cubic lattice performed by Vachaspati and Vilenkin [213] suggest that initially about 20% would be in the form of loops with the remaining 80% in 'long' strings, defined as strings which crossed the periodic box of the simulation. This figure, however, depends on the choice of lattice. A tetrahedral lattice (which is preferable because it avoids the ambiguity associated with having more than one string passing through a cubic cell) gives 37% in loops [216]. It is interesting that for $\mathbb{Z}_2$ strings on a cubic lattice the proportion in loops appears to be much less, about 6% [217].

The precise details of this structure are not important, because the subsequent evolution leads, as we shall see, to a final state which is more or less independent of this early structure.

As we mentioned in the introduction, there are models in which strings or other defects can be formed during the late stages of inflation [15,218,219]. This can be achieved by using an appropriate type of coupling between the inflaton field $\sigma$ and the field $\phi$ responsible for string formation. For example [219], if there is a term in the potential of the form $\lambda (|\phi|^2 + v^2 - \alpha \sigma^2)^2$, where $\lambda$, $v$ and $\alpha$ are constants, then, when $\sigma$ is small, $\phi$ will remain near zero, but as it evolves it will eventually reach the point where $\sigma > \sqrt{\alpha v}$, and it is then energetically favourable for $\phi$ to acquire a non-zero average value. Thus the string-forming phase transition occurs during inflation. Many variations on this theme are possible.

A notable example is that of Yokoyama [220], who pointed out that there would in general be a coupling between $\phi$ and the scalar curvature $R$, of the form $\frac{1}{2} \xi R |\phi|^2$. Thus $R$ behaves exactly like the square of the temperature in the effective potential (5.1), and as it decreases towards the end of inflation, a phase transition in the $\phi$ field can happen, if $\xi$ is positive and $V(\phi)$ possesses the usual symmetry-breaking form of equation (2.1).

5.3. Early evolution

After a thermal phase transition, the string tension $\mu$ acquires its zero-temperature value of order $\eta^2$ once the temperature becomes small compared to $T_c$. Immediately after their formation, however, the strings have smaller tension (in the simple approximation used in section 5.1, it is reduced by a factor $[1 - (T^2 / T_c^2)]$). At that time, the strings are moving in a very dense environment, causing their motion to be heavily damped.

As we saw in section 3.3, there is a length scale $l_t = \mu \sigma / \rho$ associated with friction, where $\rho$ is the energy density of the surrounding matter and $\sigma$ the average linear cross section for particle scattering (strictly speaking, the momentum-transfer cross section). According to (1.14), $\rho \sim T^4$ while, as we saw in section 4.6, $\sigma$ is (in very rough terms, neglecting logarithmic factors) of order $1/k$ for particles of momentum $k$. Since typically $k \sim T$, we may set $\sigma \sim 1/T$. Thus $l_t \sim \mu / T^3$. Recall also that $\mu \sim T_c^2$. Initially, therefore, when $T$ is close to $T_c$, $l_t \sim 1/T$. We then expect $l_t < \xi < t$. The question is: how fast does this length scale $\xi$ of the string network grow in response to the effect of friction?

Think of a segment of string with radius of curvature $r$, initially at rest. Because of the string tension $\mu$, it experiences a transverse accelerating force $\mu / r$ per unit length. When
moving with velocity \( v \) it suffers a damping force equal to \( \rho \sigma v \) per unit length, so the expected limiting velocity of our string segment is \( \mu / r \rho \sigma \sim l_t / r \). Typically, \( r \sim \xi \), so the expected timescale for straightening kinks is \( t_d \sim \xi^2 / l_t \).

This is also the timescale on which \( \xi \) increases, i.e. \( \xi / \xi \sim 1 / t_d \sim l_t / \xi^2 \). Since we are in the radiation-dominated era, \( l_t \sim \mu / T^3 \propto t^{3/2} \). Thus (apart from an initial transient) the correlation length behaves like \( \xi \propto t^{3/4} \), increasing faster than the horizon distance. This process continues [221] until all three lengths, \( l_t, \xi \) and \( t \), are of the same order. This happens when the temperature of the Universe is \( T = T_* \sim T_\xi^3 / M_{pl} \). For example, if \( T_\xi \) corresponds to the temperature of the supposed grand unification transition, namely \( T_\xi \sim 10^{15} \text{ GeV} \), then \( T_* \sim 10^{12} \text{ GeV} \), which still of course occurs at a very early time, in fact when \( t \sim 10^{-31} \text{ s} \).

As \( T \) falls further, the damping becomes less and less important. Once \( T \ll T_* \), it is negligible, and the motion of the strings can be considered essentially independently of anything else in the Universe. They soon acquire relativistic speeds.

What happens thereafter to the characteristic length scale \( \xi \) of the string network? As a matter of fact there may be several different length scales, so in order to answer the question we have to define it more precisely. One way of defining \( \xi \) is simply in terms of the overall string density (excluding small loops, for reasons to be explained later); essentially, \( \xi \) is the length such that within any volume \( \xi^3 \) we expect to find on average a length \( \xi \) of string.

This is equivalent to saying that the string density \( \rho_s \) is

\[
\rho_s = \mu / \xi^2.
\] (5.3)

For causality reasons, \( \xi \) can never grow larger than \( t \), so there are two logical possibilities: either the network approaches a 'scaling' regime in which \( \xi \) remains a fixed fraction of \( t \), or it grows more slowly, in which case the ratio \( t / \xi \) increases.

Now in the radiation-dominated era, the total density \( \rho_{tot} \) of the Universe, given by (1.13), scales like \( 1 / t^2 \). Thus in the scaling regime, the strings constitute a fixed fraction of the total energy; in fact, if we define \( \gamma \) to be the ratio of the Hubble distance to \( \xi \), then in the radiation era

\[
\gamma = 2t / \xi \rightarrow \text{constant}.
\] (5.4)

The fraction of the total density in the form of strings then becomes

\[
\frac{\rho_s}{\rho_t} = \frac{8\pi}{3} \gamma^2 G \mu
\] (5.5)

so the strings constitute an interesting though small fraction.

On the other hand, if \( \xi \) grows more slowly than \( t \), it is inevitable that the strings will eventually come to dominate the total energy density. Such a universe would be very different from ours, at any rate if the strings are heavy, with a characteristic scale anywhere near that associated with grand unification. It is just possible that the universe might be string dominated if the strings were very much lighter and so formed at a much later stage in the history of the universe [222,223]; however, this does not seem very likely.

5.4. The role of loops

There are, then, these two quite different final states. What determines the choice? Fundamentally, the answer is: the efficiency of the energy-loss mechanism.
Suppose for a moment that the string configuration simply stretches with the Hubble expansion. Then, of course, the total length of string in any comoving volume would grow in proportion to the cosmic scale factor $a$, so the string density would go like $1/a^2$. In other words, $\xi \propto a \propto t^{1/2}$. Then the strings would quite rapidly come to dominate the energy density.

So clearly, if this is not to happen, there must be some efficient mechanism for transferring energy from the strings to other forms of matter and radiation. Strings of course lose energy by radiating particles of all kinds. But non-superconducting strings generally do not carry non-zero charges of any type, so their interaction with most fields is rather weak ([224], and see section 4). On the other hand, for grand unified strings, their characteristic energy scale is not many orders of magnitude below the Planck scale, so their gravitational interactions, at least over a long time, are non-negligible.

Strings do lose a significant amount of energy in gravitational radiation, but that in itself would not be enough. An important feature of the energy-loss mechanism is the formation of small closed loops [225]. As we discussed in section 3.6, whenever two strings intersect, they reconnect or intercommute (see figures 9 and 10). When a string intersects itself, it breaks off a closed loop. Once formed, such a loop will oscillate quasi-periodically, gradually losing energy by gravitational radiation. This is a very slow process; a loop of length $l$ has a lifetime of $l/TG\mu$, which for grand unified strings is about $10^4l$. But this is irrelevant to the fate of the long-string network; once formed, the loop is effectively lost—unless it rejoins the long-string network by a second intercommuting event.

Large loops quite often rejoin the main network. Smaller loops, however, usually survive without again encountering any long strings. It is not too difficult to estimate what ‘long’ and ‘short’ must mean here. The probability that any particular segment of string, of length $l$ say, intersects another piece of string in a short time interval $\delta t$ is clearly proportional to $l$, to $\delta t$, and to the overall density of long strings, i.e. to $1/\xi^2$. Hence it is $\chi l \delta t/\xi^2$, where $\chi$ is a dimensionless constant [226], now estimated to be about 0.1 [227]. The probability of intercommuting is a rapidly decreasing function of time, because the string density is falling. Assuming that $\xi$ at least approximately scales, i.e. that $\xi \propto t$, we find that the probability that a loop of length $l$ formed at time $t$ will survive without reconnection is

$$\exp\left(-\int_t^\infty \frac{dl}{\xi^2} \right) \sim \exp\left(-\frac{\chi lt}{\xi^2}\right). \quad (5.6)$$

Note that the assumption about scaling is a very weak one: if $\xi$ decreases as some other power of $t$, the only difference will be an additional numerical factor in the exponent. We may conclude therefore that strings longer than $\xi^2/\chi t$ are very likely to reconnect, while much shorter strings are unlikely to do so.

5.5. Simulations of string evolution

Given that strings do have this energy-loss mechanism, how can we establish whether it is adequate to ensure that the string network does indeed approach a scaling regime?

Much of the evidence we have bearing on this question comes from computer simulations of an evolving string network (in a spatially flat FRW universe) performed by three different groups, referred to here as AT [228, 229], BB [230]–[232] and AS [233]–[234] (see figures 12). There are some disagreements of detail between the groups, but all three find evidence for scaling of the long-string network; in other words, they find that the
density of long strings eventually behaves like $1/t^2$, as scaling predicts. However, this is very far from a conclusive answer to the question, for several reasons.

Firstly, the simulations can only be run for relatively limited times, perhaps twenty Hubble times. So if there were, for example, logarithmic departures from scaling, they would certainly not show up.

Secondly, there certainly are features of the network that have not yet reached a scaling limit. One feature that has shown up very clearly in the work of BB and AS, as well as in the more recent simulations of AT, is that on small scales the strings are extremely wiggly. This comes about because of the intercommuting process: every intercommuting event introduces four new kinks, two on each of the strings, propagating away from each other at the speed of light. The kink angles do decrease, due to the Universal expansion, but only very slowly, typically as $t^{-0.1}$ [127, 126], and the kinks are slowly rounded by the backreaction of gravitational radiation. But because these processes are so slow, the number of kinks on the strings grows throughout the period covered by any of the simulations. Whether this small-scale structure does eventually scale is a crucial unsolved problem, to which we shall return below.

Finally, there is an unresolved disagreement about the actual density at which strings scale, i.e. the value of the constant $\gamma$ introduced above. In the radiation-dominated era, AT find $\gamma \approx 14$, while both BB and AS predict a value of $\gamma \approx 7$. This indicates a difference in the predicted scaling density of a factor of 4.

The origin of this disagreement appears to lie in the treatment of the very small-scale structure. Sharp-angled kinks are very hard to treat in numerical simulations. All the simulations use cut-offs or smoothing of one kind or another and the way in which this is done affects the results. In particular, there is also a significant disagreement about the typical size of the loops that are formed. The initial naive guess, which formed the basis of early work on the cosmic string scenario, was that loops should typically have a size comparable with the scale length $\xi$. This is the prediction of what has been described as a 'one-scale model' [127]. It is now agreed by all authors that that initial guess was quite wrong: typically loops are much smaller; the length satisfies $l \ll \xi$. But the various simulations do not agree about how much smaller they are: typical loops in the simulations of BB and AS are smaller than those of AT (even though AT's value of $\xi$ is less).
It is obviously important to resolve these issues, and one alternative approach to studying the dynamics of string networks is to do the numerical simulations in Minkowski space [235,236]. This has the advantage that there is an exact algorithm for evolving the string [235], and so there is no problem with kinks—they remain sharp, being evolved in accordance with the true equations of motion. However, the damping from the expansion of the Universe is neglected, and it is unclear how good an approximation that is. Ultimately, though, what is needed to complement numerical simulations is an analytic treatment.

5.6. Analytic treatment of string evolution

The first attempts at an analytic treatment, by Kibble [127] and by Bennett [237,238], were founded, as mentioned, on a ‘one-scale model’ of string evolution, based on assumptions about how the rates of various processes depended on the scale length $\xi$. These processes are stretching (due to the Hubble expansion), intercommuting and loop production. (Gravitational radiation was assumed to have a negligible effect on the evolution of the long-string network itself, though being crucial in the decay of loops.)

It soon became apparent, however, that this was a very crude description of the behaviour revealed by the simulations. In particular, the recognition that the typical loop size is much smaller than $\xi$ called in question the whole idea of a one-scale model.

Kibble and Copeland, and later with Austin [239,226] then developed a ‘two-scale model’ intended to accommodate this extra structure. The second scale, $\bar{\xi}$, was defined essentially as the persistence length along a string—more precisely, along the left-moving string. Recall that the motion of a string can be resolved into ‘left-moving’ and ‘right-moving’ modes (see equation (3.13)), with tangent vectors $p = a'$ and $q = b'$, respectively. In an expanding Universe (see equation (3.29)), there is a small coupling between $p$ and $q$—negligibly small for wavelengths much shorter than the Hubble radius. The definition of $\bar{\xi}$ is

$$\bar{\xi} = \int_0^\infty ds \langle p(0) \cdot p(s) \rangle$$

where the angle brackets denote averaging over an ensemble of string distributions and $s$ is the length along the string. (Thus if the angular correlation function $\langle p(0) \cdot p(s) \rangle$ is assumed to have an exponential dependence on $s$, then it would be $e^{-s/\bar{\xi}}$.) Of course, one could equally use $q$ rather than $p$ in the definition.

The approach adopted was to try to derive equations for the rates of change of the length scales $\xi$ and $\bar{\xi}$ due to the processes of stretching, intercommuting and of loop formation. A rather different, but in some ways complementary, treatment has been developed by Embacher [240].

As an illustration of the approach let us consider for example the effect of intercommuting. The probability that a small segment of string of length $l$ undergoes an intercommuting event during a time interval $\delta t$ is $\chi l \delta t / \bar{\xi}^2$, where $\chi$ is the dimensionless parameter introduced above.

Now, what is the effect of intercommuting on $\xi$ and $\bar{\xi}$? In this particular case, the answer is rather simple. Intercommuting does not change the total length of string, so $\xi$ is unaffected. On the other hand, intercommuting introduces new kinks onto the string, so its effect would be to decrease $\bar{\xi}$. If one thinks of $\bar{\xi}$ very roughly as representing the mean distance between large-angle kinks on the string, one can see that the result should be to increase $1/\bar{\xi}$ by an amount $\chi \delta t / \bar{\xi}^2$. Thus the expected contribution to $d\bar{\xi}/dt$ due to this effect is in fact $-\chi \bar{\xi}^2 / \bar{\xi}^2$. (A more detailed calculation suggests that there should be a numerical factor, of order unity [227].)
Using similar arguments, one can derive expressions for the rates of change of $\xi$ and $\tilde{\xi}$ due to stretching and loop formation. To look for scaling, it is convenient to introduce new variables, $\gamma$ (already defined above) and $\tilde{\gamma}$:

$$\gamma = 2t/\xi \quad \tilde{\gamma} = 2t/\tilde{\xi}.$$  \hfill (5.8)

(We consider only the radiation-dominated era for simplicity.)

The equations obtained [239] (written in a slightly modified notation) were

$$t \frac{d\gamma}{dt} = 1 + \frac{\alpha}{4} \gamma - \frac{c}{4} \gamma \tilde{\gamma}$$

$$t \frac{d\tilde{\gamma}}{dt} = \left( 1 - \frac{3\alpha}{2} \right) \tilde{\gamma} - \frac{q - 1}{2} c \tilde{\gamma}^2 + \frac{x}{2} \gamma^2.$$  \hfill (5.9)

There are three new dimensionless parameters here: $\alpha$ is defined by $\alpha = -\langle p(0) : q(0) \rangle$, or equivalently in terms of the mean square transverse velocity of the strings, $\alpha = 1 - 2 \langle v^2 \rangle$; $c$ is a measure of the efficiency of loop formation; while $q$ corresponds to a more subtle effect, relating to the expected 'kinkiness' of excised loops. (Regions of string with more than the usual number of kinks are more likely to produce loops than smoother regions; consequently loops will tend to have more than the average number of kinks on them.)

'Scaling' is achieved if $\gamma$ and $\tilde{\gamma}$ approach definite finite limits as $t \to \infty$, i.e. if the equations (5.9) have a stable fixed point. In the most naive version of the model, the three dimensionless parameters were taken to be constants. Then it is easy to see that scaling will be achieved provided that $q$ exceeds a critical value, in fact

$$q > \frac{3 - 2\alpha}{1 + \alpha}.$$  

According to BB [232], $\alpha \approx 0.14$, so the required condition is $q \gtrsim 2.4$. In general terms, this conclusion has been confirmed in a more sophisticated version of the model [226], in which the fact that $\alpha$ and $c$ actually depend on the ratio of the length scales is taken into account.

\subsection{5.7. Inclusion of small-scale structure}

There are two major problems, however, with the model just described. Firstly, there seems to be no obvious way of reliably estimating the magnitude of the most important parameter, $q$. Even more critically, though the model was originally intended to provide some understanding of the small-scale structure seen in the simulations, it does nothing of the sort. For reasonable values of the parameters, the two length scales $\xi$ and $\tilde{\xi}$ obstinately turn out to be of the same order of magnitude.

The trouble here is that the definition (5.7) of $\tilde{\xi}$ is dominated by long-range correlations and has rather little to do with the small-scale structure. What the simulations reveal is a very complex structure. Viewed on a sufficiently coarse scale, the strings are roughly straight on a scale $\langle \xi \rangle$ quite comparable with the typical inter-string distance $\langle \xi \rangle$; but looking at the finer detail, one sees many wiggles on each of these 'straight' segments. The typical inter-kink distance $\langle \xi \rangle$, say) is very much less than $\tilde{\xi}$.

To deal with this problem, the model has been further refined [227], now becoming a three-scale model. The definition of the third scale must involve the short-distance behaviour of the angular correlation function. It can be taken to be

$$\frac{1}{\xi} = -\left. \frac{\partial}{\partial s} \langle p(0) \cdot p(s) \rangle \right|_{s=0}.$$  \hfill (5.10)
The full set of equations describing the evolution of all three length scales is too complicated to quote here. The conclusions of the analysis, however, are fairly easy to describe.

If we include only the effects discussed so far, the equations do not scale (unless one of the parameters is larger than seems at all probable). Initially, if all the length scales start out roughly comparable, $\xi$ and $\tilde{\xi}$ start to grow, approximately proportionally to $t$ as scaling would require. However, $\xi$ does not grow, but remains roughly constant. So as their overall dimensions grow, the strings would become more and more kinky. To put this another way, let us define another scaling parameter $\epsilon$, by analogy with (5.8), by

$$\epsilon = 2t/\xi.$$  

We then find that, while $\gamma$ and $\tilde{\gamma}$ approach constant scaling values, $\epsilon$ continues to grow throughout the evolution.

But there is another effect which we must include, namely gravitational radiation. It is actually relatively straightforward to do so, using results derived by Hindmarsh [178]. Several authors [241]-[243] have already argued that it is the backreaction of gravitational radiation that will eventually cause the small-scale structure on strings to scale.

This hypothesis is confirmed by the analysis of the evolution equations [227]. Under reasonable conditions, the string network does eventually reach a full scaling regime in which all three length scales grow in proportion to the time, but with a large ratio between $\xi$ and $\tilde{\xi}$ on the one hand and $\xi$ on the other: in fact, one expects $\xi/\tilde{\xi} \sim \Gamma_8 G \mu$. Here, $\Gamma_8$ is a numerical constant, analogous to our earlier constant $\Gamma$, which characterizes the rate of energy loss by gravitational radiation from long strings, rather than from loops. We expect $\Gamma_8$ to be of order 10.

This conclusion still rests on the values of some undetermined constants, and in particular on a parameter $\tilde{C}$ that governs the effectiveness of gravitational backreaction in smoothing small-scale kinkiness on the strings. This must exceed a critical value if scaling is to be achieved. Various physical arguments suggest that it is likely that this condition is fulfilled (if it were not, the amount of small-scale structure would continue to grow, apparently without limit).

One important conclusion from this work is that when the gravitational-radiation effect switches on and the full scaling regime is reached, the value of $\tilde{\gamma}$ will drop, as will that of $\gamma$, though by a smaller factor. This means that the final scaling values of these parameters are likely to be smaller than those found in the simulations, where gravitational radiation has not yet been taken into account.

5.8. Scaling configuration

In terms of observational consequences, the most important questions concern the likely scales of the string structures.

Assuming that scaling has been achieved, the mean inter-string distance, $\xi$, at the present time is likely to be a sizeable fraction of the horizon distance, determined by the parameter $\gamma$ above. The simulation of Bennett and Bouchet, for example, [231] predicts that in the matter-dominated era $\gamma \approx 2.8 \pm 0.7$. As remarked above, it is likely that the simulations have if anything over-estimated $\gamma$, so a reasonable guess is that $\gamma$ is today of order 2. This would mean that the typical inter-string distance now is perhaps $1500h^{-1}$ Mpc; the nearest long string to us might have a redshift of perhaps 0.3.

The persistence length $\tilde{\xi}$ along the strings is of similar order of magnitude, but the scale $\xi$ characteristic of the small-scale structure is much less, perhaps a few hundred kiloparsecs.
For many purposes, it is not the strings present today that are important, but those that were present around the time of decoupling of matter and radiation. Decoupling occurs not very long after the time of equal matter and radiation densities, so $\gamma$ was probably still somewhat above its final matter-era scaling value. (The string density in the radiation-dominated era was proportionately much larger than it is now: a reasonable estimate of $\gamma$ in the radiation-dominated universe, based on the value of 7 from the simulations, might be around 5.) Allowing for subsequent expansion by a factor of $10^3$, the comoving size of the inter-string distance at decoupling could be about $25h^{-1}$ Mpc, comparable with the scales of the large-scale structures.

String loops also play a very significant role. At one time, it was thought that loops might act as seeds on which galaxies could grow. However, the relatively small size of the loops now envisaged means that that is no longer a viable scenario. Nevertheless, loops may still have very important effects.

Typically, the lengths of the loops produced at any given time are probably a few times the small-scale length $\xi$, which is itself of order $\Gamma_{ls}G\mu t$. We write the mean size of a loop born at $t_b$ as $(\kappa - 1)\Gamma G\mu t_b$. This parametrization is convenient because the size of this loop at a later time $t$ is then given by

$$l = \Gamma G\mu (\kappa t_b - t).$$

The loop finally disappears when $t = \kappa t_b$. The parameter $\kappa$ is not at present well determined, but is expected to lie between 2 and 10. So any loops we see today were probably born in the fairly recent past.

The total number density of loops in the matter dominated era is [244]

$$n_{\text{loops}}(t) = \frac{\nu}{\Gamma G\mu t^3}$$

(5.11)

where $\nu$ can be expressed in terms of the various scaling parameters, and is probably of order 0.1. (In the radiation era, the expression is slightly more complicated, involving also a function of $\kappa$.)

The typical separation between loops would then be

$$n_{\text{loops}}^{-1/3} \approx \left( \frac{\Gamma}{100} \frac{\mu_s}{\nu} \right)^{1/3} 90h^{-1} \text{Mpc}$$

where as before $\mu_s = 10^6 G\mu$. The nearest loop to us is probably at about half this distance. Thus the expected number of loops with redshift less than 0.3, say, would be a few hundred, so it is quite likely that some loops could be detectable. A typical size for a loop might be hundreds of kiloparsecs; if so, by (1.17), their masses would be comparable with those of large galaxies.

5.9. Superconducting strings

In several important ways superconducting strings behave quite differently. In particular, as we saw in section 3.4, the string tension, $T$ say, and the energy per unit length, $\mu$, are no longer constants—they depend on the current in the string. Even more significantly, they are no longer equal.

The fact that $T \neq \mu$ means of course that the invariance under local Lorentz transformations along the direction of the string is broken. The string can have a longitudinal
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momentum; a distinguished rest frame is defined, in which the longitudinal momentum vanishes. By definition, $\mu$ is the energy per unit length, and $T$ the tension, in that frame.

In terms of the effect on the evolution of a string network the most significant effect is that loops of string may be dynamically stabilized. Even non-rotating loops might be stable; they have been called springs [139]. However, there is some doubt about whether this can occur at currents less than the critical current, beyond which superconductivity disappears. On the other hand, when the angular momentum is non-zero, there are definitely configurations, termed vortons [63], which are classically stable, although susceptible to decay by quantum tunnelling. A stable vorton is a circular loop rotating in its own plane, a configuration which would not radiate gravitationally. Therefore, if vortons can be formed, they are likely to be very long-lived; they will then come to dominate the energy density of the Universe.

In the case of grand unified strings, where the characteristic energy scale is large, the result would be catastrophic: at a relatively early stage in its history the Universe becomes dominated by stable vorton loops. This effectively rules out theories in which high-energy superconducting strings are formed, unless quantum tunnelling is rapid [62].

On the other hand, if the energy scale is much lower, say the electroweak scale, things are very different. Very light non-superconducting strings are not of much cosmological interest; their contribution to the energy density would be so small as to be negligible. However, light superconducting strings would have several interesting effects. The vortons they generate could eventually come to dominate the energy density, but only at a rather late epoch. They could indeed constitute the dark matter [245]. Superconducting strings interact strongly with magnetic fields, which exist both within and between galaxies. A segment of string moving through a galactic magnetic field gains an induced current, and could become a strong source of synchrotron radiation [95]. Very light strings would still be evolving towards scaling, and so their density could be quite high. For example, electroweak scale strings would have a correlation length $\xi \sim (t_0/t_{ew})^{5/4} \xi(T_0)$. Depending on the self-coupling of the relevant scalar field, this could be as low as 10 kpc. In fact, it has been argued [246] that the density of these lightest strings could be even higher, because of their strong coupling to the cosmic plasma. The coupling might even be so strong that the strings are frozen in to the plasma. Turbulence in the plasma would then actually stretch them and increase the total length of string. In this way the interstellar medium could be filled with very light ($\mu_6 \sim 10^{-28}$) strings. Even the centres of stars could become sites for the generation of string through turbulence.

A theory of structure formation in the Universe involving GUT scale superconducting strings was proposed some years ago by Ostriker et al [96]. It required a primordial magnetic field to induce very large currents on the strings, which could then move matter around with the pressure of the resulting low-frequency electromagnetic radiation. This is essentially a variant of the explosive galaxy formation scenario [247], using strings instead of supernovae as a power source. However, it is thought that this class of theories would induce large microwave background fluctuations, and so they have fallen from favour [248]. Massive superconducting strings were also proposed as power sources for quasars [98], although once again a primordial magnetic field is required to induce sufficiently large currents.

5.10. Axion strings

Axion strings are formed at the Peccei–Quinn phase transition, which, if it happens at all, is quite tightly constrained to take place at around $10^{10}$–$10^{11}$ GeV. Axion strings are global strings, which are thought to evolve towards a scaling solution in much the same way as
gauge strings. The efficiency of the energy-loss mechanism for global strings is if anything higher because of the strong coupling between the strings and the axion field. Early on in the evolution of the axion string network the axions are essentially massless, and redshift as radiation. However, axions are in fact pseudo-Goldstone bosons: they have a small temperature-dependent mass $m_a(t)$, supplied by QCD instanton effects, which ensures that they become non-relativistic shortly before the QCD phase transition. The energy density in axions, $\rho_a$, then redshifts more slowly, and may eventually come to dominate the total energy density. The critical density forms an approximate upper bound on the total energy density in the Universe, and so from this an upper bound can be derived on the PQ symmetry-breaking scale $f_a$ through the dependence of the mass and the density of the axions on $f_a$ [106].

Let us outline the history of the axion field. The axion string network is formed at around $t_s \sim (Gf_a^2)^{-1}T_H$, when the temperature is $\sim f_a$. As for gauge strings, the network is thought initially to lose energy mostly by friction, until $t_r \sim (Gf_a^2)^{-1}T_r$, when the strings can all move freely and radiate axions. Thus the energy density in the string network is gradually transferred into axion radiation. However, although the axion is very light, its mass is increasing as the temperature decreases, while the characteristic radiation frequency $\xi$ decreases as the network coarsens. Eventually, the Compton wavelength becomes smaller than $\xi$, and the network can no longer radiate. This happens at a time $\tilde{t}$ defined by $m_a(\tilde{t}) = 1/\xi(\tilde{t})$. If $\xi \sim t$, this turns out to be at a temperature of around 1 GeV [106].

We recall from section 2.9 that the QCD instanton effects introduce a potential into the Lagrangian for the axion field $\alpha$, of the form $\Omega(T)[1 - \cos(N\alpha/f_a)]$, where $\Omega(T) \propto T^{-2} \ln^5(\pi T/\Lambda)$ [103, 249] and $\Lambda \sim 200$ MeV is the QCD scale. One can think of the growth of $\Omega(T)$ as a ‘buckling’ of the circle of minima of the potential of the underlying complex scalar field. At temperatures high compared to the QCD scale, all points on the circle have equal free energy. However, as QCD effects become important, only the points at $\alpha = 2\pi f_a/N$ remain unaffected [250]. If $N = 1$ there is only one minimum left, at $\alpha = 0$, and we can imagine the potential ‘tilting’. Since $\alpha/f_a$ is identified with the CP-violating $\theta$-parameter [1] of QCD, we have an explanation (originally due to Peccei and Quinn [100]) of why that parameter is so small.

The tilt or buckling of the axion potential greatly changes the evolution of the string network, for each string becomes attached to $N$ domain walls. The domain walls appear as a result of the energy density at $\alpha/f_a = \pi/N$, for it is energetically favourable to concentrate the change in the axion field $\alpha$ in $N$ sheets rather than having it spread around the string (this is essentially the same argument as in section 2.5). If $N = 1$, then every string forms the boundary of a wall, and the walls’ surface tension acts to draw the strings together so that they can annihilate [14, 251]. If $N > 1$, then disaster ensues, for annihilation cannot take place. Thus, if the Universe was ever as hot as $f_a$, any viable axion model must have $N = 1$ [18].

Following the disappearance of the string/wall system, its energy lives on in the form of axions, most of which are non-relativistic. They are very weakly interacting, for their self-coupling is $O(\Lambda^4/f_a^4)$, and their couplings to fermions are $O(m_f/f_a)$. Hence they cannot annihilate, and the comoving axion number density $a^3 n_a$ is conserved. The energy density in axions is therefore $a^3 m_a n_a$, which grows relative to the energy density in the rest of the particle species in the Universe. This is required to be not much greater than the critical density today. Both $m_a$ and $n_a$ depend on $f_a$, and as a result one can derive an upper bound on $f_a$: $f_a \lesssim 10^{11}$ GeV [106, 108, 179].

It should be mentioned that Sikivie and coworkers [107, 184] arrive at a higher bound, $f_a \lesssim 10^{12}$ GeV. This is essentially the same as that obtained if the axion field is
homogeneous [252]–[254], which would be the case after a period of inflation. The difference arises through an argument that an axion string releases its energy very quickly, and without any oscillation at all. This results in the energy density of the network being transferred into fewer, higher-momentum axions, and hence in a decrease in the axion energy density at late times. The careful numerical simulations of Battye and Shellard [179], however, do not bear out this contention.

6. Observational consequences

The concept of cosmic strings, like any scientific hypothesis, must ultimately be tested by examining its observational implications. In this section we shall examine the various ways in which the notion of cosmic strings might be tested. We confine our attention to the case of non-superconducting gauge strings with a large characteristic energy scale, around $10^{15}$ or $10^{16}$ GeV, so that the dimensionless parameter $G \mu$ is roughly of order $10^{-6}$ or $10^{-7}$.

6.1. Gravitational lensing

The most important interactions of ordinary strings are the gravitational ones. The gravitational field of a string is quite remarkable: around a straight, static string, the gravitational acceleration vanishes. This is a consequence of the exact equality of the energy per unit length, $\mu$, and the tension, $T$. In linearized gravitation theory, tension acts as a negative source of the gravitational field and, when $T = \mu$, the two effects exactly cancel.

As we saw in section 4, the space–time near a string is locally, but not globally, flat. The space around the string is cone-shaped, as though a wedge of angle $\delta$ had been removed and the faces stuck together [157]. The curvature is entirely localized within the string. The deficit angle $\delta$ is related to the string tension:

$$\delta = 8 \pi G \mu = 5.18 \mu_6''$$

(6.1)

where $\mu_6 = 10^6G\mu$.

The most immediate observational prediction from the conical space–time around a string is of gravitational lensing [157,255,158,256]. If we look at a distant quasar, for example (see figure 13) we may see two images, one on either side of the string.
Suppose the comoving distances to the string and the galaxy are \( x_s \) and \( x_g \), respectively. Let \( \mathbf{n} \) be the unit vector along the line of sight from us to the string, and suppose the string is moving with transverse peculiar velocity \( v \).

It is best to transform to the rest frame of the string, in which both the observer and the galaxy are moving with velocity \(-v\). If we work in terms of the angular coordinate \( \bar{\phi} \) defined after (4.4), then light will propagate along straight lines, but there is a missing wedge of space of angle \( \delta \) around the string (see figure 13). It is then easy to show [257] that the angular separation between the two images of the galaxy, to first order in \( \delta \), is

\[
\psi = \frac{\delta \sin \theta}{\gamma_\nu (1 - \mathbf{v} \cdot \mathbf{n})} \frac{x_g - x_s}{x_g}
\]

where of course \( \gamma_\nu = (1 - v^2)^{-1/2} \).

The probability that a randomly chosen galaxy with redshift \( z_g \) will be lensed by a long string is then found to be

\[
\text{probability} = \frac{64\pi}{3} \gamma^2 G\mu \left( \frac{\sqrt{1 + z_g} + 1}{\sqrt{1 + z_g} - 1} \ln \frac{\sqrt{1 + z_g} + 1}{\sqrt{1 + z_g} - 2} \right)
\]

\[
= \frac{8\pi}{9} \gamma^2 G\mu (z_g^2 - z_g^3 + \ldots)
\]

Including the effects of loops is straightforward; this does not change the result very much, because the total energy in loops is almost certainly substantially less than that in long strings in the matter era.

As we have seen, a reasonable value for \( \gamma \) in the matter-dominated era is around 2, so for, say, \( z_g = 0.4 \) and \( G\mu = 10^{-6} \), only about one galaxy in \( 10^6 \) is lensed.

There are of course many other cosmological objects that act as gravitational lenses, such as giant galaxies and rich clusters (see e.g. [258]). What is special about strings is the linear nature of the lens. A long string stretched across our field of view would be likely to induce many lensed pairs [255] arranged in a roughly linear array. A loop would give several lensed pairs within a small region of the sky. There has in fact been one report of a small field (less than one arcminute across) containing seven apparent lensed pairs [259,260], but it is unclear whether all these are in fact genuine double images, rather than merely accidental pairs of similar galaxies. Neither is the signal of a line of lensed pairs likely to be obvious, because the density of string is too low to make them stand out clearly above the 'noise' consisting of line-of-sight galaxy pairs [261]. The alignment of the position angles of the pairs could help in filtering out the noise, but the small-scale structure on the string tends to reduce the correlations between them. It remains an open question as to whether it is possible to search for string this way.

An important feature is that, because strings are so narrow and fast moving, their effects can change rapidly with time. Patterns of magnification produced by gravitational lenses generically exhibit caustics along which the magnification is formally infinite. Pairs of images coalesce and disappear as one crosses a caustic. (In reality the magnification never becomes actually infinite because of the finite size of the objects being imaged.) Hogan and Narayan [262] pointed out that in the case of lensing by very small loops, objects may cross these caustics in a relatively short space of time, of the order of months or less. Unfortunately, most loops are probably not small enough to produce significant magnification of images. Nevertheless, pairs of images may appear or disappear within a timescale comparable with the light travel time across the source.
Another special feature that, if found, would be a very clear signal of cosmic strings is the appearance of sharp edges on images [263]. One of the images of an extended object seen behind a cosmic string might well exhibit such sharp edges. Even if the object is too far away for its size to be resolved, this effect would lead to a pair of images of different magnitude, and even of different colour if, for example, only the arms of a spiral galaxy were lensed.

6.2. Effect on microwave background

Possibly the most definitive test of the idea of cosmic strings with GUT energy scale will come from observations of the predicted discontinuities in the temperature of the cosmic microwave background (CMB) [264].

Returning to figure 13, let us suppose that, in the absence of the string, the light would have an observed angular frequency \( \omega \). The effect of the moving string is that the light is redshifted in the beam ahead of the string and blueshifted behind it. The difference in frequency is given by [257, 180]

\[
\frac{\Delta \omega}{\omega} = n \cdot t \wedge \nu \delta = \nu \delta \sin \theta \sin \alpha
\]  

where \( t \) is the unit tangent vector on the string and \( \alpha \) is the azimuth angle of \( \nu \). Clearly there will be a discontinuity in the observed microwave background temperature, with \( \Delta T/T = \Delta \omega/\omega \).

To compare with observation, one needs to calculate the expected pattern of temperature fluctuations generated by the string distribution. Typically, the observations compare the temperature in two directions in the sky separated by a given angle. To distinguish cosmic strings from other sources of perturbation, observations on small angular scales are particularly important, because it is only on small scales that the stringy nature of the perturbing source will be visible.

Fortunately, calculating the temperature fluctuations on angular scales that are small compared to the horizon at decoupling is simpler than one might expect, because it turns out that a Minkowski-space treatment is sufficient [265, 266]. Consider an initially parallel beam of photons, propagating towards us with unperturbed momentum \( p^\mu = (E, 0, 0, E) \). In the presence of a weak gravitational field the perturbation in momentum is given to first order by

\[
\Delta p_\mu(x) = -\frac{1}{2} \int_0^1 d\lambda h_{\nu\rho,\mu}(x(\lambda)) p^\nu p^\rho
\]  

where the unperturbed trajectory of the photon arriving at our position \( x^\mu \) is \( x^\mu(\lambda) = x^\mu + \lambda p^\mu \). It is convenient to choose the harmonic gauge for the perturbation \( h_{\mu\nu} \) around the Minkowski metric, i.e. \( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = 0 \). Then by applying a transverse two-dimensional Laplacian \( \nabla^2_\perp \) and dropping an unimportant surface term, one finds a direct relation between \( \Delta p_\mu \) and the perturbing stress-energy tensor, namely

\[
\nabla^2_\perp \Delta p_\mu = -8\pi G \partial_\mu \int_0^1 d\lambda T_{\nu\rho}(x(\lambda)) p^\nu p^\rho.
\]  

This formula can readily be applied to the case where \( T_{\mu\nu} \) is the stress tensor of a cosmic string (3.6). The calculation is simplified by using the conformal light-front gauge, in which

\[
X^+(\tau, \sigma) \equiv X^0 + X^3 = \tau
\]  

\[ (6.7) \]
and as usual
\[ \dot{X} \cdot \dot{X} = 0 \quad \dot{X}^2 + \dot{X}^2 = 0. \]  
(6.8)

Then, using the fact that \( \Delta T/T = \Delta E/E \), one easily finds
\[ \nabla^2 \frac{\Delta T}{T} = -8\pi G \mu \int d\sigma \dot{X} \cdot \nabla \delta^{(2)}(x_\perp - X_\perp(x^+, \sigma)). \]  
(6.9)

This result means that, in Minkowski space, the temperature distortion depends only on the apparent positions of the strings and their apparent velocities, not on the entire history of the worldsheet. From here one can go on to calculate the power spectrum \( \langle |\Delta_k|^2 \rangle \) of the temperature perturbation, where \( \Delta_k \) is the two-dimensional Fourier transform of \( \Delta T/T \).

This same problem must be tackled numerically in an expanding universe, using the string distributions obtained from numerical simulations. Comparison with observation yields information on the magnitude of the string parameter \( G\mu \). Bouchet et al [267] have used equation (6.9) (expressed in the temporal gauge) to compute the fluctuations from a set of numerically generated string configurations. From this they obtained an upper bound on \( G\mu \) from the small angular scale (\( \sim 10' \)) OVRO NCP data [268]. However, it turns out that the temporal gauge formula was incorrect [269]. A recent reanalysis [266] yields the 95\% confidence limit
\[ \gamma G \mu < 6 \times 10^{-6}. \]  
(6.10)

The bound is expressed on \( \gamma G \mu \) because of the systematic and statistical uncertainties in the string density parameter \( \gamma \), which we defined in section 5.3.

Deriving bounds on larger angular scales is more problematic, because one must take the rest of the matter in the Universe into account. Scales of around 1° are particularly complicated: one must incorporate gravitational potential fluctuations at recombination, as well as the motions of the coupled electron-photon fluid. On larger angular scales one only has to worry about the general relativistic equations for the gravitational field and the density field of the dark matter.

Bennett et al [270] argued that the dark matter perturbations could be neglected on large angular scales, and found that the COBE first year results [271, 272] gave
\[ G\mu = (1.5 \pm 0.5) \times 10^{-6}. \]  
(6.11)

However, this used the old, incorrect, formula, which underestimates the string-induced CMB perturbations. Coulson et al [273], using a Minkowski-space approximation for the string network, but otherwise incorporating general relativistic effects in their code, quote
\[ G\mu = (2.0 \pm 0.5) \times 10^{-6}. \]  
(6.12)

They were also able to incorporate the effect of the electron motion at last scattering, by assuming that the electrons were subject to the same perturbations as the dark matter. This is appropriate for a reionized universe in which last scattering was at \( z \sim 10^2 \).

Useful information can be obtained from other statistical measures besides the power spectrum. On small scales, strings are expected to generate distinctly non-Gaussian perturbations. In particular, because of the localized nature of the source, the phases of the various Fourier components \( \Delta_k \) are not independent random variables. It has been argued [274] that a sensitive test of the cosmic string picture will be to measure the kurtosis of the distribution of the gradient of \( \Delta T/T \) (the ratio of the fourth moment to the square of the second moment). The distribution predicted by strings has higher peaks and a longer tail than a purely random distribution.
6.3. Density perturbations

One of the great unresolved problems of cosmology is the determination of the origin of the initial density perturbations from which galaxies eventually condensed. There are two main contenders at present: quantum fluctuations during an early epoch of inflation, and perturbations due to topological defects of one kind or another, such as strings or textures.

The cosmic string scenario in its original form \[225,275\] placed the main emphasis on the role of loops. It was supposed that loops would be formed with a characteristic size equal to a substantial fraction of the horizon size. Such large loops would live for a long time, so that there would always be a large population of loops present. The idea was that these loops would form seeds on which galaxies would start to condense. Much of the early excitement generated by the idea of cosmic strings was due to the apparent success of the scenario in predicting the correlation function of Abell clusters \[276\].

More recent results from the simulations have shown that loops are born with much smaller size and therefore live a much shorter time than was originally supposed. It now seems that they are too small and do not survive long enough to perform the function of seeding condensations. Strings may still have a very important role to play in generating density perturbations, but the centre of attention has shifted from the loops to the long strings. This is not to say that the loops are unimportant; although they probably represent a smaller fraction of the total density of the Universe, they are far more uniformly distributed than the long strings, so their effects may well be significant.

The first indication of the role of long strings in creating density perturbations was due to Silk and Vilenkin \[277\], who pointed out that moving strings will leave behind them an over-dense wake. This is another consequence of the conical structure of the space around a cosmic string. Consider a straight string moving through the matter in the Universe with transverse velocity \(v\). In the rest-frame of the string, other matter is streaming past it with velocity \(-v\). Let us suppose the matter is collision free, so that it will follow geodesics around the string. In terms of the coordinate \(\varphi\), these are straight lines, so in terms of the original coordinate \(\varphi\), every particle is deviated inwards as it passes the string by an angle \(\delta/2 = 4\pi G\mu\). In other words, every particle acquires an inward velocity component, of magnitude \(v\gamma,\delta/2\). Note that this is independent of the impact parameter. The result is (see figure 14) that in the region behind the string matter from both sides converges, leaving an over-dense wake.

![Figure 14](image-url) The formation of a wake by a string. In the string rest-frame, matter (assumed collisionless) approaches with velocity \(-v\). In the conical coordinate system, the particles move in straight lines. When projected onto a plane, the particle trajectories are bent behind the string into a wake with opening angle \(8\pi G\mu\).
In fact, however, as we have seen, the strings are by no means straight. There is a
great deal of small-scale structure, so that viewed from a large scale the effective energy
per unit length, $\mu$, is increased, while the effective tension $T$ is decreased by the same
factor. This means that it is no longer true that space–time around a string is locally flat.
In linear gravitation theory, there is no longer a cancellation between the effects of $T_{00}$ and
$T_{33}$. A kinky string, therefore, acts as an ordinary gravitational attractor (equation (4.7)),
so that even a string at rest will generate a density perturbation. In fact, a particle moving
past a kinky string will be deviated from its path by two separate effects, the conical defect
angle and the normal gravitational attraction. The resulting velocity perturbation is given
by [278]

$$\Delta v = 4\pi G\mu v\gamma_0 + \frac{2\pi G(\mu - T)}{v\gamma_0}.$$ \hspace{1cm} (6.13)

Clearly, the first of these two terms is more important for fast-moving strings, while
the second is relevant for slower-moving ones. Fast-moving string will generate planar
structures in the surrounding matter, whereas with slower strings the shape will be more
nearly linear. We might expect to see a combination of these planar and linear features in
the Universe, but there is an important question, to which we shall return shortly, about
the extent to which such ‘stringy’ features will be visible above the general stochastic
background.

The simulations suggest [234] that the r.m.s. velocity of strings on a small scale is large
(about 0.6c), but that the coherent velocity of long, roughly straight, sections of string is
much less, no more than 0.1c or 0.2c. For this reason, the second term in (6.13) is usually
more important than the first. On the whole, linear features should predominate over planar
ones.

Quantitative predictions are hard to come by in the cosmic string scenario. In particular,
we would like to know the power spectrum of the density perturbations in the dominant
matter component,

$$P(k) = |\delta_k|^2$$ \hspace{1cm} (6.14)

where $\delta_k = (8\pi^3V)^{-1} \int d^3k \delta(\mathbf{x}) e^{ik\cdot\mathbf{x}}$, the (comoving) Fourier transform of the fractional
density perturbation $\delta(x) = \delta\rho(x)/\rho$.

A comprehensive account of cosmological perturbations in the presence of strings and
other ‘stiff’ sources has been given by Veeraraghavan and Stebbins [279], who derived the
linearized Einstein equations in a flat FRW model for the perturbations in the metric and in
the matter, given a string energy–momentum tensor $T_{\mu\nu}^S(\eta, \mathbf{x})$. A flavour of their analysis
can be gained from considering a universe consisting of nothing but cold dark matter and
strings. Then the density perturbation obeys the equation

$$\ddot{\delta} + \frac{a}{\dot{a}} \dot{\delta} - \frac{3}{2} \left(\frac{a}{\dot{a}}\right)^2 \delta = 4\pi G(T_{00}^S + T_{ii}^S)$$ \hspace{1cm} (6.15)

in the synchronous gauge. This is a coordinate choice, which is defined by setting the
time–time and the space–space components of the metric perturbation to zero [169, 8].

Equation (6.15) is an inhomogeneous second-order ordinary differential equation, subject
to boundary conditions on $\delta$ and $\dot{\delta}$ at some initial time $\eta_i$. It is thus solvable by a Green’s
function method. One finds that the solution splits into two parts: $\delta^I(\eta)$, which depends on
the boundary conditions, and $\delta^S(\eta)$, which depends only on $T_{00}^S = T_{ii}^S$ Veeraraghavan
Cosmic strings

and Stebbins termed these the ‘initial compensation’ and the ‘subsequent compensation’. The combined energy–momentum of the matter and the strings cannot be affected on scales greater than the horizon: the large-scale \((k \eta \ll 1)\) density perturbations in the strings are compensated by perturbations in the matter. On these large scales the compensation obeys a conservation law \([279]\)

\[
8\pi G (a^2 \rho \delta + T_0^0) + 2 \left( \frac{\dot{a}}{a} \right) \delta = 0.
\] (6.16)

The initial compensation apparently contributes a white-noise term proportional to \(k^0\) to the power spectrum, since energy density has been taken from the matter in order to create the string network. This is partly cancelled by the subsequent compensation, because the strings preferentially seed overdensities where the underdensities are. This cancellation fails when the strings move away from their original locations. Since the strings move relativistically, inhomogeneities are generated on a scale \(k \sim \eta^{-1}\). The net effect is to produce density perturbations whose behaviour at late times \((\eta \gg k^{-1})\) is

\[
P(k) \sim k.
\] (6.17)

This is the Harrison–Zel’dovich form of the power spectrum, which is scale invariant: that is, the r.m.s. mass fluctuation in spheres the size of the horizon is constant \([8]\). The power spectrum of galaxy number density fluctuations is consistent with an approximately scale-invariant spectrum, so this is a prerequisite for any successful theory of structure formation. The scale invariance of the spectrum is in fact a direct consequence of the scaling behaviour of the network. If the total length of string per horizon volume is proportional to the horizon length, then the string density goes as \(\mu \eta^{-2}\), which, as we saw in section 5.3, is a constant fraction of the total density, \(1/(6\pi G \rho \zeta)\) in the matter era. The amplitude of the density perturbations is proportional to the amount of string, and so \(\delta \sim G \mu\).

The full cold dark matter (CDM) power spectrum has been studied by Albrecht and Stebbins \([280]\), using the AT simulations, who also incorporated theoretical uncertainties about network evolution by modelling strings with different amounts of small-scale structure. The resulting power spectra for one of their models are displayed in figure 15, along with the same authors’ calculation of the string-seeded hot dark matter (HDM) spectrum \([281]\). For comparison, the standard inflationary spectra are also displayed \([282]\). All are conventionally normalized so that \(\sigma_8\), the variance of the mass fluctuations in spheres of \(8h^{-1}\) Mpc, is 1. We shall say more about this below.

Albrecht and Stebbins also looked for specifically ‘stringy’, planar or linear, features. They concluded that they would not show up clearly in CDM; they would be swamped by the general background of perturbations generated by strings on smaller scales. The situation is very different, however, in the case of a HDM universe \([281]\). There they found that stringy features would indeed stand out rather clearly, essentially due to the ‘free streaming’ of the neutrinos on scales less than \(20h^{-1}\) Mpc.

The parameter \(\sigma_8\) is a common measure of the density fluctuations on galactic scales. This is the scale on which the density perturbations are just now going non-linear; that is to say, the observed value \(\sigma_{8,\text{gal}}\) based on measurements of the galaxy correlation function is unity. However, this is not a direct measurement of \(\sigma_8\) per se. It seems likely that galaxy formation will occur preferentially near the highest peaks of the density distribution, in which case the galaxy distribution will be more correlated that the mass distribution. In other words, the observed fluctuation in the galaxy distribution is expected to be larger than
that in the (mainly dark matter) mass distribution. It is usual to define a bias parameter \( b \) by

\[
\sigma_{8,\text{gal}} = b \sigma_8. \tag{6.18}
\]

Using the COBE normalization to calculate \( \sigma_8 \) then gives an estimate of the required bias, namely

\[
b = \sigma_8^{-1}. \tag{6.19}
\]

The value of the dimensionless parameter \( G\mu \) required to yield the right magnitude of the fluctuations depends on the assumptions made about string evolution. But assuming reasonable values for the parameters, Albrecht and Stebbins found, in the cold dark matter case,

\[
\mu_6 \approx 1.8 \text{ (} h = 1 \text{)} \quad \mu_6 \approx 2.8 \text{ (} h = 0.5 \text{)} \tag{6.20}
\]

where \( \mu_6 = 10^6 G\mu \), in reasonable agreement with the COBE normalizations quoted above ((6.11) and (6.12)). (These figures come from one of three possible scenarios analysed by Albrecht and Stebbins.) In the case of hot dark matter, the corresponding values were

\[
\mu_6 \approx 2.0 \text{ (} h = 1 \text{)} \quad \mu_6 \approx 4.0 \text{ (} h = 0.5 \text{)}. \tag{6.21}
\]

Thus at \( h = 1 \) the bias parameter at \( 8h^{-1} \text{ Mpc} \) is approximately 1 for both CDM and HDM, but at \( h = 0.5 \) CDM requires a bias in the range 1–2, while HDM needs 2–4. However, these figures should be treated with caution, for there is still considerable uncertainty about the correct basis of the calculations of the power spectrum.

6.4. Gravitational waves

As we have seen, the primary mechanism for energy loss from loops is gravitational radiation. Long strings also radiate significantly. Although gravitational waves cannot as
yet be detected directly, this radiation may have measurable indirect effects. Consideration of the known limits on the intensity of gravitational radiation in the Universe has yielded an important bound on the string parameter $G\mu$.

There are in fact two distinct ways of bounding $G\mu$ by considering the emitted gravitational radiation: from Big Bang nucleosynthesis and from studies of the millisecond pulsar.

The primordial abundance of helium in the Universe, observed to be around 23%, is a key datum in the Big Bang nucleosynthesis scenario. The amount of helium produced depends on a balance between the rates of the various processes involved and the rate of the Hubble expansion, which in turn depends on the energy density. If, for example, there were one extra species of neutrino, that would increase the rate of expansion and so increase the helium abundance, to an unacceptably high value—this is the cosmological evidence for the limit on the number of neutrino species [13].

Energy in the form of gravitational waves would have exactly the same effect. The validity of the nucleosynthesis scenario thus places a limit on the total energy density of gravitational radiation present in the Universe at the time of nucleosynthesis. This may be expressed as a fraction of the critical density [283],

$$\Omega_{\text{gr,nuc}} < 0.05.$$  (6.22)

More recent estimates [284,285] give

$$\Omega_{\text{gr,nuc}} < 0.016 \text{ or } 0.007$$  (6.23)

respectively.

If the gravitational radiation is generated by cosmic strings, this translates into a limit on $G\mu$. Bennett and Bouchet [286] obtained the limit $G\mu < 6 \times 10^{-6}$, or a weaker bound if the QCD transition is inhomogeneous. Caldwell and Allen [283] have done a careful numerical study and shown that the limit may be reduced if the typical loop-size parameter is large (in our notation, if $\kappa \gg 1$). For $\kappa > 10$, say, the bound could be significant. On the other hand, the recent conclusions about the expected scaling configuration [227], in particular the fact that the parameters $\gamma$ and $\tilde{\gamma}$ are likely to be smaller in the final scaling regime than the simulations would suggest, imply some weakening of the constraint.

It is easy to find an approximate expression for the gravitational-wave energy density. Let us consider first the contribution of loops. We saw in section 4.2 that each individual loop contributes the same power, $\Gamma G\mu^2$, throughout its lifetime. Thus what we have to calculate is the total number of loops present at any given time.

During the radiation-dominated era, energy in the form of gravitational radiation scales in exactly the same way as the total density (except for a small numerical factor due to the change in the total number of massless particle states). Thus the total gravitational-wave energy gradually builds up throughout this era, giving a logarithmic time dependence. The power radiated depends on the total number density of loops, given by (5.11). The total energy emitted into gravitational radiation per unit volume between times $t_1$ and $t_1 + dt_1$ is then $\Gamma G\mu^2 n_{\text{loops}}(t_1) dt_1$. The gravitational radiation emitted at time $t_1$ is then redshifted by the expansion of the Universe, so that its density at a later time $t$ is reduced by the factor $(t_1/t)^2$. Integrating over $t_1$ yields

$$\Omega_{\text{gr,nuc}} \approx \frac{64\pi}{3} G\mu v A(\kappa) \left( \frac{g_{*,\text{nuc}}}{g_{*,\text{form}}} \right)^{1/3} \ln \frac{\sigma_{\text{nuc}}}{\sigma_{\text{form}}}$$  (6.24)
where

\[ A(\kappa) = \frac{2 \kappa^{3/2} - 1}{3 \kappa - 1}. \]  

(6.25)

\( a_{\text{nuc}} \) and \( a_{\text{from}} \) are the values of the scale factor at nucleosynthesis and at the epoch of formation of cosmic strings, respectively, while \( g_{*,\text{nuc}}, g_{*,\text{form}} \) are the corresponding values of the effective number of spin states of relativistic particles. Since \( \ln(a_{\text{nuc}}/a_{\text{from}}) \approx 40 \), and \( g_{*,\text{nuc}}/g_{*,\text{form}} \approx 0.1 \), this yields the bound

\[ G_{\mu \nu \alpha} (\kappa) < 4 \times 10^{-5} \]  

(6.26)

or, using the bounds (6.23),

\[ G_{\mu \nu \alpha} (\kappa) < 1.4 \times 10^{-5} \text{ or } 6 \times 10^{-6} \]  

(6.27)

respectively. In general, even the latter bound is not very restrictive, though if \( v \) and \( \kappa \) are both fairly large, say \( v \approx 2 \) and \( \kappa \approx 10 \), it becomes significant. It may become much tighter in the near future as further results on nucleosynthesis become available.

In principle, we should also include the gravitational waves emitted by the long strings. However, in the radiation era, the loops emit proportionately much more gravitational radiation, except perhaps when \( \kappa \) is small, in which case the bound is in any case very weak. At the very most, this could not change the bound by more than a factor of two.

The second bound imposed by consideration of gravitational waves derives from the astonishing precision of timing measurements of the millisecond pulsars, believed to be extremely rapidly spinning neutron stars. The first of these pulsars has now been followed for over 10 years [287]. The rate of emission of pulses is extremely regular, though gradually slowing. Now, any gravitational waves propagating in the space between us and the pulsar would cause fluctuations in the timing of the observed pulses, called timing residuals.

The experiment is complicated by the fact that there are other sources of fluctuations, such as intrinsic noise in the pulsar, interstellar propagation effects, and even instabilities in the clock used to time the pulsar. However, the observed residuals do place an important upper bound on the total amount of gravitational radiation that can be present in the Universe today.

Unlike the previous case, this is not a limit on the total gravitational-wave energy density, but rather a limit on the waves within a particular wavelength interval near \( \lambda \approx T \) where \( T \) is the period of observation. Let the energy density in gravitational waves with wavelength between \( \lambda \) and \( \lambda + d\lambda \) be \( \rho_{g\lambda}(\lambda)d\lambda/\lambda \). We also write, as usual, \( \Omega_{g\lambda} = \rho_{g\lambda}(\lambda)/\rho_c \). Gravitational waves of cosmological origin are expected to have a scale-invariant spectrum, that is constant \( \Omega_{g\lambda} \). In this case the current limit is [287]

\[ \Omega_{g\lambda} < 6 \times 10^{-8}h^{-2} \]  

(95% confidence).

(6.28)

For comparison, the limit in 1990 was [287]

\[ \Omega_{g\lambda} < 4 \times 10^{-7}h^{-2}. \]  

(6.29)

With reasonable assumptions about the spectrum, this yields a much lower bound on the total energy density in gravitational radiation than the one at the time of nucleosynthesis. The
reason is that once we enter the era of matter domination, the energy density of gravitational radiation, like that of electromagnetic radiation, rapidly becomes a small fraction of the total. Since the epoch, \( t_{eq} \), of equal radiation and matter densities, the contribution to \( \Omega_{gr} \) of radiation generated during the radiation-dominated era has been decreasing like \( a^{-1} \), giving an overall reduction by a factor of \( 7.5 \times 10^{-5} h^{-2} \).

To calculate the expected power in the relevant wavelength interval, one needs to know how loops radiate. A small loop of length \( L \) oscillates quasi-periodically, with period \( L/2 \). It therefore radiates gravitational waves at the fundamental frequency \( 2/L \) and at all the harmonics \( 2n/L \). In the early studies it was assumed that essentially all the power is radiated at the fundamental. But studies by Vachaspati and Vilenkin [171] (see also [288]) showed that in fact a lot of power goes into high harmonics; the power \( P_n \) dies off with \( n \) as \( n^{-9} \), where \( q \approx \frac{4}{3} \). This implies that the dominant contribution to the \( \Omega_{gr}(T) \) comes from radiation emitted in the fairly recent past and, for \( q < 2 \), at rather high values of \( n \).

From (6.28), Bennett and Bouchet [286], assuming that all the power was radiated in the lowest mode, derived the bound \( G\mu < 4 \times 10^{-5} \). Caldwell and Allen [283] found that allowing for the dependence of \( P_n \) on \( n \) has a large effect. For \( h = 1 \), they derived a much tighter bound \( G\mu < 2 \times 10^{-6} \), or a lower figure for large values of \( \kappa \). Using a more recent value \( (\Omega_{gr} < 10^{-7} h^{-2}) \), they obtained an even lower bound,

\[
G\mu < 7 \times 10^{-7} \quad (h = 1).
\]  

(6.30)

However, for \( h = 0.5 \) their limits are less stringent by a factor of 10.

As in the case of the previous bound, however, the recent reassessment of the scaling parameters [227] will significantly weaken the bound. For \( q = \frac{4}{3} \), the revised limit is

\[
\mu_6 \lesssim 0.1 \left( \frac{0.1}{v} \right)^{3/2} \left( \frac{\Gamma}{100} \right)^{1/2} \left( \frac{10}{\kappa} \right)^{1/2} \left( \frac{\kappa - 1}{\kappa} \right)^{1/2} \frac{1}{h^{7/2}}.
\]  

(6.31)

If \( h \approx 0.5 \), this is quite easy to satisfy, but for \( h \approx 1 \) it could become a significant bound.

All these limits are derived assuming a scaling solution back to the earliest time that gravitational waves with period 1–10 yr today could have been emitted. This would have been between redshifts of approximately \( 10^4 \) and \( 10^6 h^{-2} \). In models where strings form at the end of inflation, such as those discussed in section 5.2, it is conceivable that the strings re-enter the horizon at around that time, or even afterwards, and still have a small enough correlation length to seed the galaxy-scale perturbations [289]. This means that the amount of gravitational radiation at these periods is reduced, and so the pulsar bound is relaxed or even eliminated.

6.5. Cosmic rays

Another of the outstanding mysteries of astrophysics is the origin of the most energetic cosmic rays, with energies in excess of \( 10^{20} \) eV. The known mechanisms for accelerating particles, for example by shock-wave acceleration in active galactic nuclei, do not seem capable of generating such large energies [290].

Some years ago it was suggested [291] that 'evaporation' of particles from cusps on cosmic strings might provide the explanation. As mentioned in section 3.2, cusps occur where \( X \) vanishes instantaneously and, correspondingly, \( X^2 = 1 \); the string momentarily reaches the speed of light. Near a cusp, there are two overlapping, oppositely oriented segments. It seems likely that the interactions of the underlying fields would convert some of the energy of the string into quanta of these gauge or scalar fields. Such particles are
of course unstable and would eventually decay into quarks and leptons. The quarks in turn undergo the usual QCD fragmentation process, ending up as jets of hadrons, mostly pions with a few nucleons. Since the decaying particles have masses in excess of \(10^{24} \text{ eV}\), and are moving with relativistic speeds, some of the decay products may easily have energies well in excess of \(10^{20} \text{ eV}\).

The processes involved are extremely complex and it is difficult to calculate the expected high-energy particle flux with any precision, but estimates [291,292] suggest that the mechanism could at least be a significant contributor to the cosmic ray spectrum above \(10^{19} \text{ eV}\), though this has recently been questioned [293]. (It has also been suggested [294] that superconducting strings could be responsible for the puzzling gamma-ray bursters.)

The key question is: what is the fraction \(f_{\alpha}\) of the initial energy of a cosmic string loop that is converted in this way into high-energy particles? Comparison with the observed limits yields an upper bound,

\[
G\mu v f_{\alpha} \lesssim 1.7 \times 10^{-9}.
\]  

(6.32)

If cosmic strings are the explanation, then this should be an equality. Recalling that for GUT strings, \(G\mu \sim 10^{-6}\) and \(v \sim 0.1\), this means \(f_{\alpha} \approx 10^{-2}\).

This seems a rather large fraction. Firstly, it obviously requires that the actual energy emitted in the form of energetic particles by each cusp should scale in the same way as the total length of string. For a string of fixed width, this is not what one would expect [293]. It also requires cusp evaporation to be a repeated event. Ignoring energy-loss mechanisms, the motion of a small loop is periodic and any cusp will recur once every period. It is not clear, however, what effect the energy loss by evaporation will have on the cusp. If it causes it to disappear permanently, so that the loop evaporation process occurs only once or a few times, then it is hardly conceivable that \(f_{\alpha}\) could be as large as \(10^{-2}\). On the other hand, it is possible that the effect is only to shift the position of the cusp, in which case each oscillation produces a new cusp; the cusp gradually eats into the string. Even so, a fraction \(10^{-2}\) is hard to attain.

7. Conclusions

In this final section, we aim to assess the current state of the theory of cosmic strings.

Cosmic strings come in many different guises. As we discussed in section 2, they are a feature of many unified theories of fundamental forces. They may appear at a thermal phase transition with critical temperature \(T_c\) anywhere between the grand unification scale, a few orders of magnitude below the Planck mass, and the electroweak transition at a few hundred GeV. They may even appear in the late stages of an inflationary epoch. The tension \(T\) and the energy per unit length \(\mu\) are both of order \(T_c^2\), and in simple cases \(T = \mu\). Thus the possible values of \(\mu\) range over nearly thirty orders of magnitude. As yet, there is no firm indication that cosmic strings of any type do actually exist, but equally they are far from being ruled out and in some contexts they provide a very natural explanation for the observations.

7.1. GUT-scale strings

The cosmic strings which have attracted the most attention are those produced at a grand unification transition, for which the dimensionless parameter \(G\mu\) is of order \(10^{-6}\) (i.e. \(\mu_6 \sim 1\)). Such strings naturally generate density perturbations and fluctuations in the
microwave background temperature of roughly the observed order of magnitude. As we emphasized in section 1, this is one of the great attractions of the cosmic string scenario. In the most popular alternative theory, based on inflation, the density perturbations are generated by quantum fluctuations during the inflationary era. In that case, to achieve the right order of magnitude requires fine tuning of a coupling constant. The simplest inflationary models with a single scalar inflaton field require that the self-coupling $\lambda$ of the inflaton satisfy typically $\lambda \lesssim 10^{-10}$ [295]–[297]. It is possible to avoid this requirement in extended theories involving two scalar fields [298]–[301], but even then there is a fine-tuning requirement for the coupling-constant ratios [302].

The cosmic string scenario is thus a very attractive one. The question is: does it stand up to more detailed scrutiny? Can it provide a quantitative as well as a qualitative understanding of structure formation in the Universe? How do its predictions compare with those of alternative models, particularly the inflationary model?

In their simplest forms, both the inflationary and cosmic string models have essentially a single adjustable parameter, the inflaton coupling $\lambda$ or $G\mu$, respectively. One important test is whether it is possible to fit both the COBE results [271,303] on the fluctuations in the microwave background and the density fluctuations on galactic scales as determined by observations of the galaxy distribution.

Both models predict essentially the Harrison–Zel’dovich spectrum of perturbations: the power spectrum $P(k)$ of fluctuations behaves as $P(k) \propto k$ for small $k$. This is a consequence of the scale invariance of the physics responsible for generation of the perturbations. In the inflationary model, the magnitude $\delta\rho/\rho$ of the density perturbations on each scale is the same at the epoch when the appropriate wavelength is equal to the horizon distance. Similarly, in the cosmic string scenario, the magnitude of the perturbations generated on each scale, once it has come inside the horizon, is the same. Once inside, the behaviour of perturbations depends on how the Universe is expanding. Before $t_{eq}$, the time of equal densities of matter and radiation, their amplitude remains constant. After $t_{eq}$, the perturbations grow in proportion to the scale factor. This introduces a characteristic scale in the problem, namely the comoving size of the Universe at the time $t_{eq}$. This corresponds to a wavelength $\lambda \sim 40 h^{-1}$ Mpc. On scales less than this, where the wavelength was equal to the horizon distance at some time before matter–radiation equality, the perturbations cannot grow. Thus for larger values of $k$, the power spectrum $P(k)$ starts to bend, eventually reaching $P(k) \propto k^{-3}$. This happens in both the inflationary and cosmic string models, but for cosmic strings the characteristic scale is related to the scale size of the string network at $t_{eq}$, rather than the horizon distance, and is thus somewhat smaller. The curvature of $P(k)$ does not then become significant until we reach lower scales. For a given value of $\sigma_8$, the perturbation on very large scales is a little smaller, while there is quite a lot more power on small scales, especially with HDM.

It is becoming common to normalize the perturbation spectrum not by $\sigma_8$, but by fitting the COBE r.m.s. at angular scales of $10^\circ$. In principle this fixes the inflaton coupling or, in the cosmic string case, the magnitude of $G\mu$. The calculations still have rather large errors, but the range of values predicted is generally from 1 to $3 \times 10^{-6}$. For example, Perivolaropoulos [304] gives a value

$$G\mu = (1.7 \pm 0.7) \times 10^{-6}$$

(7.1)

while Bennett et al [270] give

$$G\mu = (1.5 \pm 0.5) \times 10^{-6}.$$  

(7.2)
The most recent and thorough calculations, albeit based on a Minkowski space string simulation [273], give

\[ G\mu = (2.0 \pm 0.5) \times 10^{-6}. \]  

(7.3)

On the large scales probed by COBE, the main contribution to the microwave background perturbations, both from strings and inflation, comes from the Sachs–Wolfe effect. In this case, there are simple relations between the temperature fluctuation \( \delta T/T \) and the perturbation in the Newtonian potential \( \Phi \), or equivalently the density perturbation, on the same scale. In the inflationary scenario, the density perturbations are generated at very early times and are of 'adiabatic' type (so called because there is no perturbation in the entropy per baryon). This means that there is a fixed relation between the density perturbations in radiation and matter, namely \( (\delta \rho/\rho)_{\text{rad}} = \frac{4}{3}(\delta \rho/\rho)_{\text{mat}} \). Thus, once the matter component comes to dominate the energy density, we have

\[ \frac{\delta T}{T} = \frac{1}{3} \frac{\delta \rho}{\rho}. \]  

(7.4)

For strings, the perturbations are generated while the CMB radiation is travelling through them, and the relation is [305]

\[ \frac{\delta T}{T} = \frac{1}{\gamma} \frac{\delta \rho}{\rho}, \]  

(7.5)

where, we recall, \( \gamma \) is the ratio between the Hubble radius and the scale size of the string network. Thus, if the scale size were the horizon distance, strings (and other defects) could give as much as six times the temperature fluctuation for a given amplitude of \( \delta \rho/\rho \). However, for strings \( \gamma \) seems to be between 1.5 and 2.5, so the relation between the temperature and density fluctuations is much the same for strings and for inflation.

This rough equivalence shows up in the calculations of the bias for the two scenarios. We recall that the bias \( b \) was defined as the ratio between the r.m.s. fluctuations in the number of galaxies in \( 8h^{-1}\text{Mpc} \) spheres, or \( \sigma_{\text{gal}} \), and the mass fluctuations \( \sigma_8 \) in the same volume. The COBE normalized inflationary cold dark matter (CDM) theory predicts

\[ b = (0.5 \pm 0.1)h^{-1}. \]  

(7.6)

For a Hubble constant of 50 km s\(^{-1}\) Mpc\(^{-1}\) (\( h = 0.5 \)), this is approximately 1. The values predicted by the cosmic string scenario are slightly larger, but also subject to relatively large errors because of the shakiness of the calculations. For the case \( h = 0.5 \), Perivolaropoulos [304] quotes,

\[ b = 1.1 \text{ to } 2.9 \quad \text{(CDM)} \]

\[ b = 1.4 \text{ to } 4 \quad \text{(HDM)}. \]  

(7.7)

More stringent tests are set by studying CMB perturbations on a range of angular scales. Although inflation and cosmic strings both predict an approximately scale-invariant spectrum of density perturbations, their CMB spectra are different. For example, the standard inflationary CDM model predicts a flat spectrum of CMB perturbations on scales above about 2\(^o\), whereas the string spectrum is expected to fall slightly, corresponding to a tilted model with power spectrum \( P(k) \sim k^{1.4} \) [270,304]. The quadrupole term is expected
to be very much lower than the r.m.s. prediction from inflation, as the density fluctuation
responsible for it has not yet been fully generated. The COBE data is unfortunately not
good enough to distinguish between $n = 1$ and $n = 1.4$. The measured quadrupole [306], at
$6 \pm 3 \mu K$, is however rather lower than can be comfortably accounted for by cosmic variance
around the prediction from an inflationary Harrison–Zel’dovich spectrum normalized at $10^3$, $Q_{\text{rms-ps}} = 19.9 \pm 1.6 \mu K$ [307]. However, the measurement of the quadrupole is subject to
severe contamination by galactic emission and so this discrepancy may not be significant.

Measurements have been reported of $\delta T/T$ on the $1^\circ$ angular scale. The most accurate
of these is the MAX experiment, who report [308]

$$\left. \frac{\delta T}{T} \right|_{1^\circ} \simeq (3.6 \pm 0.2) \times 10^{-5}. \quad (7.8)$$

This is in good accord with COBE normalized inflationary CDM, with the standard HZ
scale-invariant power spectrum. The only calculation on these scales for the string scenario
[273] assumed that the Universe was reionized some time after $t_d$, for example by an early
generation of very massive stars. This has the effect of smoothing the perturbations on $1^\circ$
scales, so direct comparison between strings and inflation over this result is not yet possible.

In terms of distinguishing between inflation and cosmic strings, it is currently more
useful to look at the spectrum of density perturbations. If both models are normalized
to the same value of $\sigma_8$, the cosmic string model will predict substantially more power
on small scales and less on large scales. In the case of a CDM universe, the power on
large scales predicted by inflation is already somewhat too low, particularly for matching
the observations of large-scale structure [309]. Clearly the cosmic string model is worse.
Similarly, because of the excess power on small scales, it would predict random small-scale
velocities larger than those observed. It seems clear that the combination of cosmic strings
and CDM is not a viable model. The standard inflationary CDM model is itself in some
difficulty, which has led to various proposed alternatives, such as mixed hot and cold dark
matter [310]–[312], or more complex inflationary models with tilted spectra [313]; it is in
any case certain that cosmic strings and CDM do not offer an improvement here.

The situation is very different, however, in the case of hot dark matter (HDM), where
the addition of cosmic strings definitely does improve the agreement with observation [281].
Cosmic strings and HDM seem to provide a promising alternative to inflation and CDM,
although there are potential problems with the high bias required and a deficit of very-
small-scale power. Unfortunately, the uncertainties involved in the calculations are still too
large to allow definitive predictions, primarily because of the difficulty of tying down the
parameters of an evolving string network.

7.2. Lighter strings

Once we get below the grand unification scale, it is really misleading to talk of the cosmic
string scenario: there are many different possible scenarios depending on the type of string—
global or local, superconducting or not, Abelian or non-Abelian—and on the symmetry-
breaking scale.

One of the most attractive ideas, in the sense that it does least violence to the Standard
Model, is the axion string scenario. This is rather tightly constrained, in that the relevant
symmetry, the Peccei–Quinn global $U(1)$ symmetry, has to be broken at a scale in the range
$10^{10}$–$10^{11}$ GeV. The strings would have all annihilated by today, having been joined by
domain walls at the QCD phase transition, but they would have left an important relic in
the form of a population of non-relativistic axions, possibly with critical density. Thus
the axion strings could have generated the dark matter. Experiments to detect dark matter axions are in progress [314]. However, the difficulty of detecting such weakly interacting particles suggests that confirmation or otherwise will be some time coming.

Another set of models are those involving relatively light, superconducting strings, formed at a transition not far above the electroweak scale. Peter [315] has emphasized that superconductivity is a rather generic feature of cosmic string models. Such strings are probably not relevant to large-scale structure formation, but may have significant astrophysical effects [95,246]. Massive cosmic strings (superconducting or not) may have important roles to play in the generation of large-scale magnetic fields [316,317] and perhaps in creating the net baryon number in the Universe, i.e. the matter–antimatter asymmetry [201,202].

There are many exciting possibilities in this area, but as yet it is difficult to point to definitive tests of the models. Much more theoretical work is needed, in particular on the evolution of a network of superconducting or global strings, where the dynamics is influenced by the long-range forces between strings.

7.3. Summary

Cosmic strings are predicted by many unified theories of fundamental forces. If they exist, they provide one of the few direct links between the observational features of our present Universe and the dramatic events of the immediate aftermath of the Big Bang.

They can be formed at a phase transition anywhere between the GUT and electroweak scales. Light cosmic strings would only be observable if superconducting, since their gravitational interactions are so weak. Axion strings, formed at an intermediate scale, have the exciting possibility of generating a substantial density of cold dark matter in the form of axions.

The best developed of the many cosmic string scenarios concerns GUT-scale strings, which generate both density perturbations and perturbations in the microwave background of roughly the observed order of magnitude. In a CDM universe, they do not seem very attractive, but the combination of strings and HDM is a very promising alternative to the popular inflationary models. The idea of combining both hot and cold dark matter has been put forward to improve inflationary models with a Harrison–Zel’dovich spectrum, but this type of mixed dark matter (MDM) has yet to be tried with strings. If the predictions can be tied down more accurately by improved theoretical understanding of the problem of network evolution, then definitive tests of this model should be possible soon.

Acknowledgments

We are grateful to Mark Bowick for sending us a copy of figure 1, to Michael Goodband for providing figures 2 and to Paul Shellard for figure 12. We thank Albert Stebbins for helpful comments, and Catherine Wolfe for reading the manuscript. We also wish to acknowledge the hospitality of the Isaac Newton Institute where this review was completed.

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