Uniformly locally o-minimal open core

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This paper discusses sufficient conditions for a definably complete expansion of a densely linearly ordered abelian group to have uniformly locally o-minimal open cores of the first/second kind and strongly locally o-minimal open core, respectively.

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1 Introduction

The open core of an expansion of a linear order is its reduct generated by its definable open sets. Miller & Speisssegger [11] first introduced the notion of open core and Dolich, Miller & Steinhorn [2] gave a sufficient condition for the structure having an o-minimal open core. Definably complete expansions of ordered groups enjoying the uniform finiteness property have o-minimal open core by [2, Theorem A]. By definition, a densely linearly ordered structure enjoys the uniform finiteness property if we can take $I = \mathbb{R}$ and $U = \mathbb{R}^m$ in Definition 1.1.

Locally o-minimal structures are defined and investigated in [12]. Fornasiero [5] also investigated necessary and sufficient conditions for a definably complete expansion of an ordered field to have a locally o-minimal open core. A uniformly locally o-minimal structure was first introduced in [9] and a systematic study was made in [6]. These notions are reviewed in § 2. The purpose of this paper is to give sufficient conditions for structures having uniformly locally o-minimal open cores of the first/second kind and having a strongly locally o-minimal open core, respectively.

We first introduce local uniform finiteness conditions.

**Definition 1.1** (Local uniform finiteness) For any set $X \subseteq \mathbb{R}^{m+n}$ definable in a structure $\mathcal{R} = (\mathbb{R}, \ldots)$ and for any $x \in \mathbb{R}^n$, the notation $X_x$ denotes the fiber defined as $\{y \in \mathbb{R}^n \mid (x, y) \in X\}$. We use this notation throughout the paper.

A densely linearly ordered structure $\mathcal{R} = (\mathbb{R}, <, \ldots)$ satisfies local uniform finiteness of the second kind—for short, $\mathcal{R} \models \text{LUF}_2$ if, for any definable subset $X \subseteq \mathbb{R}^{m+n}$, any $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$, there exist

1. a positive integer $N$,
2. a closed interval $I$ with $a \in \text{int}(I)$ and
3. an open box $U \subseteq \mathbb{R}^n$ with $b \in U$

such that $|X_x \cap J| = \infty$ or $|X_x \cap J| \leq N$ for all $x \in U$ and all closed intervals $J$ with $a \in \text{int}(J) \subseteq I$.

An easy induction shows that $\mathcal{R} \models \text{LUF}_2$ if and only if, for any definable subset $X \subseteq \mathbb{R}^{m+n}$ and $(b, a) \in \mathbb{R}^{m+n}$, there exist a positive integer $N$, a closed box $B$ with $a \in \text{int}(B)$ and an open box $U$ with $b \in U$ such that $|X_x \cap B'| = \infty$ or $|X_x \cap B'| \leq N$ for all $x \in U$ and all closed boxes $B'$ with $a \in \text{int}(B') \subseteq B$.

The structure $\mathcal{R} = (\mathbb{R}, <, \ldots)$ satisfies local uniform finiteness of the first kind—for short, $\mathcal{R} \models \text{LUF}_1$ if $\mathcal{R} \models \text{LUF}_2$ and we can take $U = \mathbb{R}^n$ in the definition of LUF$_2$. The structure $\mathcal{R} = (\mathbb{R}, <, \ldots)$ satisfies strong local uniform finiteness—for short, $\mathcal{R} \models \text{SLUF}$ if $\mathcal{R} \models \text{LUF}_1$ and we can take $I$ independently of the definable set $X \subseteq \mathbb{R}^{m+n}$.

The properties LUF$_2$, LUF$_1$ and SLUF are axiomatizable properties; hence, the above notations such as $\mathcal{R} \models \text{LUF}_2$ make sense.

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A densely linearly ordered structure \( \mathcal{M} = (M, <, \ldots) \) is definably complete if every definable subset of \( M \) has both a supremum and an infimum in \( M \cup \{\pm \infty\} \); cf. [10]. The definable completeness is also an axiomatizable property. The notation \( \mathcal{R} \models DC \) means that the structure \( \mathcal{R} \) is definably complete in this paper. The following theorem is our main theorem.

**Theorem 1.2** Consider an expansion of a densely linearly ordered abelian group \( \mathcal{R} = (R, <, +, 0, \ldots) \) with \( \mathcal{R} \models DC \).

1. If \( \mathcal{R} \models SLUF \), the structure \( \mathcal{R} \) has a strongly locally o-minimal open core.
2. If \( \mathcal{R} \models LUF_1 \), the structure \( \mathcal{R} \) has a uniformly locally o-minimal open core of the first kind.
3. If \( \mathcal{R} \models LUF_2 \), the structure \( \mathcal{R} \) has a uniformly locally o-minimal open core of the second kind.

We can prove the theorem basically following the same strategy as the proof of [2, Theorem A]. We first review the notion of \( D_{\Sigma} \)-families and \( D_{\Sigma} \)-sets used in the proof of [2, Theorem A] in § 3. We know in general that (1) \( D_{\Sigma} \)-sets are closed under projections, finite unions and finite intersections, (2) constructible definable sets are \( D_{\Sigma} \)-sets and (3) constructible definable sets are closed under complements. The remaining task is to demonstrate that \( D_{\Sigma} \)-sets are constructible. The key lemma is that, for a \( D_{\Sigma} \)-family \( \{X(r, s)\}_{r,s>0} \), the set \( X(r, s) \) has a nonempty interior for some \( r > 0 \) and \( s > 0 \) when the \( D_{\Sigma} \)-set \( X = \bigcup_{r,s>0} X(r, s) \) has a non empty interior. This lemma holds true both for the o-minimal open core case [2, 3.1] and our case. However, a new investigation is necessary to demonstrate the lemma in our case. Basic lemmas including the above lemma are proved in § 4. We cannot use the same definition of dimension of \( D_{\Sigma} \)-sets as [2]. We give a new definition of dimension in § 3. We finally prove the theorem in § 5 using this concept.

We introduce the terms and notations used in this paper. The term ‘definable’ means ‘definable in the given structure with parameters’. For a linearly ordered structure \( \mathcal{R} = (R, <, \ldots) \), an open interval is a definable set of the form \( \{x \in R \mid a < x < b\} \) for some \( a, b \in R \). It is denoted by \( (a, b) \). We define a closed interval in the same manner and it is denoted by \( [a, b] \). An open box in \( R^n \) is the direct product of \( n \) open intervals. A closed box is defined similarly. A CBD set is a closed, bounded and definable set. A CDD set is a closed, discrete and definable set. Let \( A \) be a subset of a topological space. The notations \( \text{int}(A) \) and \( \overline{A} \) denote the interior and the closure of the set \( A \), respectively. The notation \( |S| \) denotes the cardinality of a set \( S \). We use the same notation for the absolute value of an element, but this abuse will not confuse readers.

## 2 Definitions

We review the definitions introduced in the previous studies. A constructible set is a finite boolean combination of open sets. Note that every constructible definable set is a finite boolean combination of open definable sets by [3]. We review the definitions of local o-minimality and their relatives.

**Definition 2.1** A densely linearly ordered structure \( \mathcal{M} = (M, <, \ldots) \) is locally o-minimal if, for every definable subset \( X \) of \( M \) and for every point \( a \in M \), there exists an open interval \( I \) containing the point \( a \) such that \( X \cap I \) is a finite union of points and open intervals. A locally o-minimal structure \( \mathcal{M} = (M, <, \ldots) \) is strongly locally o-minimal if, for every point \( a \in M \), there exists an open interval \( I \) containing the point \( a \) such that \( X \cap I \) is a finite union of points and open intervals for every definable subset \( X \) of \( M \).

A locally o-minimal structure \( \mathcal{M} = (M, <, \ldots) \) is a uniformly locally o-minimal structure of the first kind if, for any positive integer \( n \), any definable set \( X \subseteq M^{n+1} \) and \( a \in M \), there exists an open interval \( I \) containing the point \( a \) such that the definable sets \( X_i \cap I \) are finite unions of points and open intervals for all \( y \in M^n \).

A locally o-minimal structure \( \mathcal{M} = (M, <, \ldots) \) is a uniformly locally o-minimal structure of the second kind if, for any positive integer \( n \), any definable set \( X \subseteq M^{n+1} \) and \( a \in M \) and \( b \in M^n \), there exist an open interval \( I \) containing the point \( a \) and an open box \( B \) containing \( b \) such that the definable sets \( X_i \cap I \) are finite unions of points and open intervals for all \( y \in B \).

An \( \omega \)-saturated uniformly locally o-minimal structure of the first kind is strongly locally o-minimal by [9, Proposition 7]. Similar assertion holds true for Definition 1.1.
Locally o-minimal structures and strongly locally o-minimal structures are defined and investigated in [12]. Uniformly locally o-minimal structures of the first/second kind are studied in [6, 7, 9]. A uniformly locally o-minimal structure of the first kind is called a uniformly locally o-minimal structure in [9].

We give two remarks on relations between local uniform finiteness conditions and local o-minimality.

**Remark 2.2** A definably complete uniformly locally o-minimal structure of the second kind satisfies locally uniform finiteness by [6, Theorem 4.2].

**Remark 2.3** Consider a locally o-minimal expansion of the group of reals \( \tilde{\mathbb{R}} = (\mathbb{R}, <, 0, +, \ldots) \). The following assertion is [7, Theorem 4.3]:

For any definable subset \( X \) of \( \mathbb{R}^{n+1} \), there exist a positive element \( r \in \mathbb{R} \) and a positive integer \( K \) such that, for all \( a \in \mathbb{R}^n \), the definable set \( X \cap ([a] \times (-r, r)) \) has at most \( K \) connected components.

By reviewing its proof, it is easy to check that the above \( r \) can be taken independently of \( X \) because \( \tilde{\mathbb{R}} \) is strongly locally o-minimal by [12, Corollary 3.4]. Therefore, we have \( \mathbb{R} \models \text{SLUF} \).

We give an example of a structure which has a strongly locally o-minimal open core, but is not locally o-minimal.

**Example 2.4** We first consider the language \( L_1 = \{0, 1, +, -, <\} \). We define \( L_1 \)-structures

\[
\tilde{\mathbb{Q}} = (\mathbb{Q}, 0, 1, +, -, <) \quad \text{and} \quad \tilde{\mathbb{R}} = (\mathbb{R}, 0, 1, +, -, <)
\]

naturally. It is easy to demonstrate that both \( \tilde{\mathbb{Q}} \) and \( \tilde{\mathbb{R}} \) have quantifier elimination. We can also easily demonstrate that \( \tilde{\mathbb{Q}} \) is an elementary substructure of \( \tilde{\mathbb{R}} \) using the Tarski-Vaught test. They are both o-minimal structures. The notation \( \mathcal{M} = (\tilde{\mathbb{R}}, \tilde{\mathbb{Q}}) \) denotes their dense pair. The definition of dense pairs is found in [4]. The dense pair \( \mathcal{M} \) is an \( L_1 \)-structure whose universe is the set of reals \( \mathbb{R} \), where \( L_2 = \{0, 1, +, -, <, P_{\tilde{\mathbb{Q}}}\} \). Here, \( P_{\tilde{\mathbb{Q}}} \) is a unary predicate and we interpret as follows:

\[
\mathcal{R} \models P_{\tilde{\mathbb{Q}}}^\mathbb{R}(x) \text{ if and only if } x \in \mathbb{Q}.
\]

The dense pair \( \mathcal{M} \) enjoys uniform finiteness property by [4, Corollary 4.5]. That is, for any definable set \( S \) in \( \mathbb{R}^{m+n} \), there exists a positive integer \( N \) such that, for all \( x \in \mathbb{R}^m \), we have \( |S_x| \leq N \) whenever the fibers \( S_x \) are finite. The structure \( \mathcal{M}_{(0,1)} \) is the restriction of \( \mathcal{M} \) to the set \( [0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \) defined in [9, Definition 2]. Consider the simple product \( \mathcal{R} \) of \( \mathcal{Z} \) and \( \mathcal{M}_{(0,1)} \). The simple product of two structures are defined in [9]. Its universe is the Cartesian product of the universes of two structures. Roughly speaking, a set definable in it is a boolean combination of the Cartesian products of definable sets. Cf. [9, §4] for its precise definition. The universe of \( \mathcal{R} \) is \( \mathbb{Z} \times [0, 1) \). It is identified with \( \mathbb{R} \) via the bijection \( (m, t) \mapsto m + t \). Under this identification, the order and the sum in \( \mathbb{Z} \times [0, 1) \) coincide with the natural order and the sum in \( \mathbb{R} \), respectively, by [9, Example 15.3].

Let \( L = \{0, 1, +, -, <, P_{\mathcal{Z}}, P_{\tilde{\mathbb{Q}}} \} \) be a language, where \( P_{\mathcal{Z}} \) and \( P_{\tilde{\mathbb{Q}}} \) are unary predicates. Consequently, the structure

\[
\mathcal{R} = (\mathbb{R}, 0^\mathbb{R}, 1^\mathbb{R}, +^\mathbb{R}, -^\mathbb{R}, <^\mathbb{R}, P_{\mathcal{Z}}^\mathbb{R}, P_{\tilde{\mathbb{Q}}}^\mathbb{R})
\]

is an \( L \)-structure defined as follows:

1. \( \mathcal{R} \models P_{\mathcal{Z}}^\mathbb{R}(x) \text{ if and only if } x \in \mathbb{Z} \);
2. \( \mathcal{R} \models P_{\tilde{\mathbb{Q}}}^\mathbb{R}(x) \text{ if and only if } x \in \mathbb{Q} \).

This structure is also discussed in [1, §3.1]. We obtain the following claim by [9, Lemma 17]:

**Claim** Let \( X \) be a subset of \( \mathbb{R}^n \) definable in \( \mathcal{R} \). There exist finite subsets \( X_1, \ldots, X_k \) of \( (0,1) \) definable in \( \mathcal{M}_{(0,1)} \) and a map \( \iota : \mathbb{Z}^n \to \{1, \ldots, k\} \) such that, for any \( z = (z_1, \ldots, z_n) \in \mathbb{Z}^n \), we have

\[
X \cap \left( \prod_{i=1}^n [z_i, z_i + 1] \right) = z + X_{\iota(z)},
\]

where \( \{c, d) = \{x \in R \mid c \leq x < d\} \) and \( z + X_{\iota(z)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1 - z_1, \ldots, x_n - z_n) \in X_{\iota(z)} \} \).
The structure \( \mathcal{R} \) is not locally o-minimal because the set \( \mathbb{Q} \) is definable in \( \mathcal{R} \). We have \( \mathcal{R} \models \text{SLUF} \) by the above claim because \( \mathcal{M} \) satisfies uniform finiteness. We also have \( \mathcal{R} \models \text{DC} \) because the universe \( \mathbb{R} \) is complete. The structure \( \mathcal{R} \) has a strongly locally o-minimal open core by Theorem 1.2.

### 3 D\(_{\Sigma}\)-sets and their dimension

We can apply the strategy of Dolich, Miller & Steinhorn [2] to our problem. They used the notion of D\(_{\Sigma}\)-sets, which play an important role also in this paper.

**Definition 3.1 (D\(_{\Sigma}\)-sets)** Consider an expansion of a densely linearly ordered abelian group \( \mathcal{R} = (\mathbb{R}, <, +, 0, \ldots) \). A parameterized family of definable sets \( \{X(x)\}_{x \in S} \) is the family of the fibers of a definable set; i.e., \( S \) is a definable set and there is a definable set \( X \) with \( X(x) = X \_x \) for all \( x \in S \). A parameterized family \( \{X(x, s)\}_{r > 0, s > 0} \) of CBDF subsets \( X(r, s) \) of \( \mathbb{R}^n \) is called a D\(_{\Sigma}\)-family if \( X(r, s) \subseteq \mathcal{R}(r', s') \) and \( X(r', s') \subseteq X(r, s) \) whenever \( r < r' \) and \( s < s' \). Note that \( X(r, s) \) and \( X(r', s') \) are not necessarily strictly contained in \( X(r', s') \) and \( X(r, s) \), respectively. A definable subset \( X \) of \( \mathbb{R}^n \) is a D\(_{\Sigma}\)-set if it is a parameterized family \( \{X(x, s)\}_{r > 0, s > 0} \).

The following two lemmas are found in [2, 10].

**Lemma 3.2** Consider an expansion of a densely linearly ordered abelian group \( \mathcal{R} \) with \( \mathcal{R} \models \text{DC} \). The following assertions are true:

1. The projection image of a D\(_{\Sigma}\)-set is D\(_{\Sigma}\).
2. Fibers, finite unions and finite intersections of D\(_{\Sigma}\)-sets are D\(_{\Sigma}\).
3. Every constructible definable set is D\(_{\Sigma}\).

**Proof.** (1) Immediate from [10, Lemma 1.7]. (2) Cf. [2, 1.9(1)]. (3) Cf. [2, 1.10(1)].

**Lemma 3.3** Let \( \mathcal{R} = (\mathbb{R}, <, +, 0, \ldots) \) be an expansion of a densely linearly ordered abelian group with \( \mathcal{R} \models \text{DC} \). A CBDF set \( X \subseteq \mathbb{R}^{n+1} \) has a nonempty interior if the CBDF set

\[
\{x \in \mathbb{R}^n \mid X_i \text{ contains a closed interval of length } s\}
\]

has a nonempty interior for some \( s > 0 \).

**Proof.** Cf. [2, 2.8(2)]

The notion of dimension used for o-minimal open cores in [2] is not appropriate for our setting. We give a new definition of dimension of a D\(_{\Sigma}\)-set. The dimension of a set definable in a locally o-minimal structure admitting local definable cell decomposition is defined in [6, § 5]. In a definably complete uniformly locally o-minimal structure of the second kind, the dimension defined below coincides with the dimension defined in [6, § 5] by [6, Corollary 5.3].

**Definition 3.4** (Dimension) Let \( \mathcal{R} = (\mathbb{R}, <, \ldots) \) be an expansion of a densely linearly ordered structure. Recall that \( \mathbb{R}^0 \) is a singleton. Consider a D\(_{\Sigma}\)-subset \( X \) of \( \mathbb{R}^n \) and a point \( x \in \mathbb{R}^n \). The local dimension \( \dim_x \) of \( X \) at \( x \) is defined as follows:

1. \( \dim_x X = -\infty \) if there exists an open box \( B \) with \( x \in B \) and \( B \cap X = \emptyset \).
2. Otherwise, \( \dim_x X \) is the supremum of nonnegative integers \( d \) such that, there exists a coordinate projection \( \pi : \mathbb{R}^n \to \mathbb{R}^d \) such that the image \( \pi(B \cap X) \) has a nonempty interior for any open box \( B \) with \( x \in B \).

The dimension of \( X \) is defined by \( \dim X = \sup \{\dim_x X \mid x \in \mathbb{R}^n\} \). The projective dimension \( \text{proj. dim} \) of \( X \) is defined as follows:

1. \( \text{proj. dim} X = -\infty \) if \( X \) is an empty set.
2. Otherwise, \( \text{proj. dim} X \) is the supremum of nonnegative integers \( d \) such that the image \( \pi(X) \) has a nonempty interior for some coordinate projection \( \pi : \mathbb{R}^n \to \mathbb{R}^d \).

The following lemma illustrates that the dimension and the projective dimension coincides in some open box.
Lemma 3.5 Let $\mathcal{R} = (R, <, +, 0, \ldots)$ be an expansion of a densely linearly ordered structure. Consider a $D_{\mathcal{R}}$-subset $X$ of $R^n$ of dimension $d$. Take a point $x \in R^n$ with $\dim X = d$. We have $\dim (X \cap B) = \text{proj.} \dim (X \cap B) = d$ for any sufficiently small open box $B$ in $R^n$ with $x \in B$.

Proof. Since $\dim X = d$, the projection image $\pi (B \cap X)$ has an empty interior for any coordinate projection $\pi : R^n \to R^{d+1}$ when $B$ is a sufficiently small open box with $x \in B$. In particular, $\text{proj.} \dim B \cap X \leq d$. It is obvious that $d = \dim X \leq \dim B \cap X \leq \dim X$. We have shown that $\dim B \cap X = d$ and $\text{proj.} \dim B \cap X \leq \dim B \cap X$. The opposite inequality $\dim B \cap X \leq \text{proj.} \dim B \cap X$ is obvious from the definition.

4 Basic lemmas

We introduce basic lemmas in this section. We first prove two lemmas by localizing the arguments in [2, 2.4].

Lemma 4.1 Let $\mathcal{R} = (R, <, +, 0, \ldots)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$.

For any definable set $X \subseteq R^{n+m}$ and a point $(b, a) \in R^n \times R^m$, there exist a positive integer $N$, a closed box $B$ with $a \in \text{int}(B)$ and an open box $U$ containing the point $b$ such that, if $X \cap B$ is discrete, we have $|X \cap B| \leq N$ for any $x \in U$.

In addition, we can take $U = R^n$ if $\mathcal{R} \models LUF_1$. We can take $B$ independently of $X \subseteq R^{n+m}$ if $\mathcal{R} \models SLUF$.

Proof. We first demonstrate the following claim:

Claim. Let $X$ be a definable subset of $R^{n+m}$. Assume that the fiber $X_x$ is CDD for any $x \in R^n$. For any $(b, a) \in R^n \times R^m$, there exists a closed box $B$ with $a \in \text{int}(B)$ and an open box $U$ containing the point $b$ such that $X \cap B$ is a finite set for any $x \in U$. In addition, we can take $U = R^n$ if $\mathcal{R} \models LUF_1$. We can take $B$ independently of $X \subseteq R^{n+m}$ if $\mathcal{R} \models SLUF$.

Proof of Claim. Assume the contrary. We can find a point $(b, a) \in R^n \times R^m$ such that, for any closed box $B$ with $a \in \text{int}(B)$ and any open box $U$ with $b \in U$, $X \cap B$ is infinite for some $x \in U$. Consider the set

$$Y = \{(x, y_1, y_2) \in R^n \times R^m \times R^m \mid (x, y_1) \in X, (x, y_2) \in X, y_1 \geq y_2\}$$

in the lexicographic order.

Since $\mathcal{R} \models LUF_2$, there exists a positive integer $N$, a closed box $B$ with $a \in \text{int}(B)$ and an open box $V$ with $(b, a) \in V = U_1 \times U_2 \subseteq R^n \times R^m$ such that, for any $(x, y_1) \in V$, we have $|Y_{(x, y_1)} \cap B| = \infty$ or $|Y_{(x, y_1)} \cap B| \leq N$. Shrinking $B$ if necessary, we may assume that $B$ is bounded and contained in $U_2$. Fix such a closed box $B$. Take a point $x \in U_1$ such that $X_x \cap B$ is infinite. Such a point $x$ exists by the assumption. Note that $X_x \cap B$ is CDD. We construct a point $z_k \in B$ with $|Y_{(x, z_k)} \cap B| = k$ inductively. Take $z_1 = \text{lexmin}(X_x \cap B)$, then $Y_{(x, z_1)} \cap B = \{z_1\}$. The notation $\text{lexmin}(A)$ denotes the lexicographic minimum defined in [10], which is the smallest element in $A$ in the lexicographic order. The lexicographic minimum $\text{lexmin}(A)$ always exists for any CBD set $A$ by [10, p. 1785]. The point $z_1$ exists because $X_x \cap B$ is CBD. The set $X_x \cap B \setminus \{z_1\}$ is CDD and bounded because $X_x \cap B$ is CDD and bounded. It means that $X_x \cap B \setminus \{z_1\}$ is CBD. Take $z_2 = \text{lexmin}(X_x \cap B \setminus \{z_1\})$, then $Y_{(x, z_2)} \cap B = \{z_1, z_2\}$. We can take $z_3, z_4, \ldots$ in this manner. We have $|Y_{(x, z_{n+1})} \cap B| = N + 1$, which is a contradiction.

It is obvious that $U = R^n$ if $\mathcal{R} \models LUF_1$ and $B$ is common to all $X$ if $\mathcal{R} \models SLUF$.

We return to the proof of the lemma. Take a bounded closed box $C$ with $b \in \text{int}(C)$. Set $D = \{(x, s, y) \in R \times C \times R^m \mid (\prod_{i=1}^m [y_i - s, y_i + s]) \cap X = \{y\}\}$, where $y = (y_1, \ldots, y_n)$ and the notation $\prod_{i=1}^m [y_i - s, y_i + s]$ denotes the product of $n$ closed intervals. The following assertions are trivial.

1. The fiber $D_{(s, x)}$ is CDD.
2. $\bigcup_{s > 0} D_s = \{(x, y) \mid y \text{ is a discrete point in } X_x\}$.

Apply the claim to the set $D$. We can take an open box $U$ with $b \in U$ and a closed box $B$ with $a \in \text{int}(B)$ such that $D_{(s, x)} \cap B$ are finite sets for all $x \in U$ and all sufficiently small $s > 0$. Since $\mathcal{R} \models LUF_2$, shrinking $B$ and $U$ if necessary, we have $|D_{(s, x)} \cap B| \leq N$ for some positive integer $N$, any $x \in U$ and any sufficiently small $s > 0$. Since $D_{(s, x)} \subseteq D_{(s', x)}$ for all $s > s' > 0$, we have $|\bigcup_{s > 0} D_{(s, x)} \cap B| \leq N$. The ‘in addition’ parts of the lemma are obvious.

Lemma 4.2 Let $\mathcal{R} = (R, <, +, 0, \ldots)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$. 

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For any definable set $X \subseteq \mathbb{R}^{n+1}$ and a point $(b, a) \in \mathbb{R}^n \times R$, there exist positive integers $N_1, N_2$, an open interval $I$ with $a \in I$ and an open box $U$ containing the point $b$ such that, for any $x \in U$,

1. the open set $\text{Int}(X) \cap I$ is the union of at most $N_1$ open intervals in $I$;
2. the closed set $\overline{X} \cap I$ is the union of at most $N_1$ points and $N_2$ closed intervals in $I$.

In addition, we can take $U = \mathbb{R}^n$ if $\mathcal{R} \models LUF$. We can take $I$ independently of $X \subseteq \mathbb{R}^{n+1}$ if $\mathcal{R} \models SLUF$.

Proof. The assertion on the closure follows from the assertion on the interior by considering $\mathbb{R}^{n+1} \setminus X$ in place of $X$. We only prove the latter. We may assume that $X_1$ is open for any $x \in \mathbb{R}^n$ without loss of generality. Consider the set

$$C = \{(r, x, y) \in R \times \mathbb{R}^n \times R \mid r > 0, \exists \varepsilon > 0,$$

the open interval $(y - \varepsilon, y + \varepsilon)$ is contained in $X \cap (a - r, a + r)$ and

two points $y - \varepsilon$ and $y + \varepsilon$ are contained in $\partial(X_1 \cap (a - r, a + r)),$

where $\partial(X_1 \cap (a - r, a + r))$ denotes the boundary of the set $X_1 \cap (a - r, a + r)$ in $R$. The fiber $C_{(r, x)}$ is discrete. By Lemma 4.1, there exist a positive integer $N_1$, a positive element $s > 0$, a closed interval $I$ with $a \in \text{Int}(I)$ and an open box $U$ containing the point $b$ such that $|C_{(r, x)} \cap I| \leq N_1$ for all $x \in U$ and $0 \leq r < s$. Take a sufficiently small $r > 0$ with $[a - r, a + r] \subseteq I$ and set $I = (a - r, a + r)$. The definable set $X_1 \cap I$ consists of at most $N$ open intervals for any $x \in U$. □

Lemma 4.3 is the key lemma introduced in § 1. We prove the lemma combining the arguments of [2] and [8]. Lemma 4.4 is another key lemma corresponding to [6, Theorem 3.3]. They are proved simultaneously.

**Lemma 4.3** Let $\mathcal{R} = (R, <, +, 0, \ldots)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC$, $LUF_2$. Let $\{X(r, s) \subseteq \mathbb{R}^n_{r, s > 0, x > 0}\}$ be a $D_\Sigma$-family. Set $X = \bigcup_{r, s} X(r, s)$. One of the following conditions is satisfied:

1. The $D_\Sigma$-set $X$ has an empty interior.
2. The CBD set $X(r, s)$ has a nonempty interior for some $r > 0$ and $s > 0$.

When $n = 1$, the set $X$ is discrete and closed in the former case.

**Lemma 4.4** Let $\mathcal{R} = (R, <, +, 0, \ldots)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC$, $LUF_2$. Consider a $D_\Sigma$-subset $X$ of $\mathbb{R}^n$ with a nonempty interior. Let $X = X_1 \cup X_2$ be a partition into two $D_\Sigma$-sets. At least one of $X_1$ and $X_2$ has a nonempty interior.

Proof. We prove the lemmas by induction on $n$. We first consider the case in which $n = 1$. We first demonstrate Lemma 4.3. Assume that $\text{Int}(X(r, s)) = \emptyset$ for all $r > 0$ and $s > 0$. We have only to show that $X$ is discrete and closed. Fix an arbitrary point $a \in R$ and $r > 0$. Set $X(r) = \bigcup_{r > 0} X(r, s)$. There exist a closed interval $I$ with $a \in \text{Int}(I)$, a positive element $t \in R$, positive integers $N_1$ and $N_2$ such that the intersection $I \cap X(r, s)$ is the union of at most $N_1$ points and $N_2$ closed intervals for any $0 < s < t$ by Lemma 4.2. The set $I \cap X(s)$ consists of at most $N_1$ points because $\text{Int}(X(r, s)) = \emptyset$. We have $|X(r) \cap I| \leq N_1$ because $\{X(r, s)_{r > 0}\}$ is a decreasing sequence. The set $X(r)$ is a CDD set.

We show that $X = \bigcup_{r > 0} X(r)$ is a CDD set. Assume the contrary. There exists a point $a \in R$ such that, for any open interval $I$ containing a point $a$, the definable set $I \cap X$ is an infinite set. We may assume that $\{x > a\} \cap I \cap X$ is an infinite set for any open interval $I$ containing the point $a$ without loss of generality.

Consider the definable function $f : \{r \in R \mid r > 0\} \to \{x \in R \mid x > a\}$ defined by $f(r) = \inf\{x > a \mid x \in X(r)\}$. As in the proof of [8, Theorem 4.3], we can prove the following assertions:

1. The definable function $f$ is a decreasing function and $\lim_{r \to \infty} f(r) = a$.
2. Consider the image $\text{Im}(f)$ of the function $f$. For any $b \in \text{Im}(f)$, there exists a point $b_1 \in \text{Im}(f)$ such that $b < b_1$ and the open interval $(b, b_1)$ has an empty intersection with $\text{Im}(f)$.

We may assume that the intersection $\overline{\text{Im}(f)} \cap I$ of the closure of the image with $I$ consists of finite points and finite closed intervals by Lemma 4.2 shrinking the open interval $I$ if necessary. We lead to a contradiction assuming that it contains a closed interval $J$. Take an arbitrary point $b \in \text{Im}(f)$ in the interior of the closed interval $J$. The open
interval \((b, b_1)\) has an empty intersection with \(\text{Im}(f)\) for some \(b_1 \in R\). It is a contradiction. We have shown that \(I \cap \text{Im}(f)\) is a finite set. It is a contradiction to the fact that \(\lim_{r \to \infty} f(r) = a\). We have shown that \(X = \bigcup_{r>0} X(r)\) is a CDD set. It is obviously discrete and closed. We have demonstrated Lemma 4.3 when \(n = 1\).

Lemma 4.4 is immediate from Lemma 4.3 when \(n = 1\).

We next consider the case in which \(n > 1\). We first demonstrate Lemma 4.3. We can prove that \(X_r = \bigcup_{s>0} X(r, s)\) have empty interiors if \(X(r, s)\) have empty interiors for all \(s > 0\) in the same way as [8, Lemma 3.3]. Now we can get Lemma 4.3 in the same way as the proof of [8, Lemmas 4.1 & 4.2] using Lemma 4.4 for \(n - 1\) instead of [6, Theorem 3.3].

The remaining task is to prove Lemma 4.4 when \(n > 1\). Take \(D_\Sigma\)-families \(\{X_i(r, s)\}_{r,s}\) with \(X_i = \bigcup_{r,s} X_i(r, s)\) for \(i = 1, 2\). Set \(X(r, s) = X_1(r, s) \cup X_2(r, s)\). It is a CBD set. We have \(\text{int}(X(r, s)) \neq \emptyset\) for some \(r > 0\) and \(s > 0\) by Lemma 4.3 for \(n\) because \(X = \bigcup_{r,s} X(r, s)\). If at least one of \(X_1(r, s)\) and \(X_2(r, s)\) has a nonempty interior, at least one of \(X_1\) and \(X_2\) has a nonempty interior. Therefore, we may assume that \(X_1\) and \(X_2\) are CBD sets. Let \(B\) be a closed box contained in \(X\). We have \(B = (X_1 \cap B) \cup (X_2 \cap B)\). If the lemma is true for \(B\), the lemma is also true for the original \(X\). Hence, we may assume that \(X\) is a closed box.

Let \(\pi\) be the coordinate projection forgetting the last coordinate. For \(i = 1, 2\) and \(s > 0\), we set

\[S_i(s) = \{x \in \pi(X_i) \mid \text{the fiber } (X_i)_s \text{ contains a closed interval of length } s\}.\]

They are CBD sets. Set \(T_i = \bigcup_{s>0} S_i(s)\), which is \(D_\Sigma\). It is obvious that \(T_1 = \{x \in \pi(X_1) \mid (X_1)_s \text{ contains an open interval}\}\). Since \(X\) is a closed box, we have \(\pi(X) = T_1 \cup T_2\) by Lemma 4.4 for \(n = 1\). At least one of \(T_1\) and \(T_2\) has a nonempty interior by the induction hypothesis. We may assume that \(\text{int}(T_i) \neq \emptyset\) without loss of generality. We have \(\text{int}(S_i(s)) \neq \emptyset\) for some \(s > 0\) by Lemma 4.3 for \(n - 1\). The CBD set \(X_1\) has a non-empty interior by Lemma 3.3. We have finished the proof of Lemma 4.4.

We finally demonstrate that a definable map whose graph is \(D_\Sigma\) is continuous on a dense set.

**Lemma 4.5** Let \(\mathcal{R} = (R, <, +, 0, \ldots)\) be an expansion of a densely linearly ordered abelian group with \(\mathcal{R} \models \text{DC, LUF}_2\). Consider a definable map \(f : U \to R^n\) defined on an open set \(U\) whose graph is a \(D_\Sigma\)-set. Then \(f\) is continuous on a dense definable open subset of \(U\).

**Proof.** We can prove the lemma in the same way as [8, Lemma 5.1]. We use Lemma 4.3 instead of [8, Lemma 3.4]. We omit the proof.

## 5 Uniformly locally o-minimal open core

We demonstrate Theorem 1.2 in this section. The following lemma claims that all \(D_\Sigma\)-subsets of \(R\) are constructible.

**Lemma 5.1** Let \(\mathcal{R} = (R, <, +, 0, \ldots)\) be an expansion of a densely linearly ordered abelian group with \(\mathcal{R} \models \text{DC, LUF}_2\). Consider a definable subset \(X\) of \(R\). The following are equivalent:

1. The set \(X\) is a \(D_\Sigma\)-set.
2. For any \(x \in R\), there exists an open interval \(I\) such that \(x \in I\) and \(X \cap I\) is a finite union of points and open intervals.
3. The set \(X\) is constructible.

**Proof.** (1) \(\implies\) (2): The difference \(X \setminus \text{int}(X)\) is \(D_\Sigma\) by Lemma 3.2, and it is discrete and closed by Lemma 4.3. For any \(x \in R\), the intersection \(\text{int}(X) \cap I\) is a finite union of open intervals for some open interval \(I\) with \(x \in I\) by Lemma 4.2. We get the assertion (2).

(2) \(\implies\) (3): The difference \(X \setminus \text{int}(X)\) is discrete and closed by the assertion (2). It means that \(X\) is constructible.

(3) \(\implies\) (1): Immediate from Lemma 3.2(3).
Lemma 5.2 Let $\mathcal{R} = (\mathbb{R}, <, +, 0, \ldots)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models$ DC, LUF$_2$. Consider a $D_2$-subset $X$ of $\mathbb{R}^{n+1}$. For any $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$, there exist positive integers $N_1$, $N_2$, an open interval $I$ with $a \in I$ and an open box $B$ with $b \in B$ such that $X_s \cap I$ is the union of at most $N_1$ points and $N_2$ open intervals for any $x \in B$.

Furthermore, we can take $B = \mathbb{R}^n$ if $\mathcal{R} \models$ LUF$_1$. We can take $I$ independently of $X \subseteq \mathbb{R}^{n+1}$ if $\mathcal{R} \models$ SLUF.

Proof. Let $\mathcal{R} = (\mathbb{R}, <, +, 0, \ldots)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models$ DC, LUF$_2$. A $D_2$-set of dimension zero is discrete and closed, hence constructible.

Proof. Consider a $D_2$ subset $X$ of $\mathbb{R}^n$ of dimension zero. Let $\pi_i$ be the projections onto the $i$-th coordinate for all $1 \leq i \leq n$. Let $x \in \mathbb{R}^n$ be an arbitrary point. Take a sufficiently small open box $B$ with $x \in B$. The projection images $\pi_i(X \cap B)$ have empty interiors for all $i$ because $\dim X \leq 0$. By Lemma 5.1, we may assume that $\pi_i(X \cap B)$ is empty or a singleton by shrinking $B$ if necessary. It means that $X$ is discrete and closed.

We investigate $D_2$-sets of dimension zero.

Lemma 5.3 Let $\mathcal{R} = (\mathbb{R}, <, +, 0, \ldots)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models$ DC, LUF$_2$. Consider a $D_2$-subset $X$ of $\mathbb{R}^{n+1}$. Set

$I(X) = \{ x \in \mathbb{R}^n \mid \text{the fiber } X_x \text{ contains an open interval} \}$.

Then $I(X)$ is a $D_2$-set and we have $\proj \dim(I(X)) < \proj \dim(X)$.

Proof. Let $\{X(r,s)\}_{r>0,s>0}$ be a $D_2$-family with $X = \bigcup_{r>0,s>0} X(r,s)$. Set $Y(r,s) = \{ x \in \mathbb{R}^n \mid \exists r \in \mathbb{R}, [t-s, t+s] \subseteq (X(r,s))_x \}$. The set $Y(r,s)$ is CBD. The family $\{Y(r,s)\}_{r>0,s>0}$ is obviously a $D_2$-family. We show that $I(X) = \bigcup_{r,s} Y(r,s)$. It is obvious that $\bigcup_{r,s} Y(r,s) \subseteq I(X)$. We demonstrate the opposite inclusion. Take an arbitrary point $x \in I(X)$. We have $\int(X_x) \neq \emptyset$. We get $\int(X(r,s))_x \neq \emptyset$ for some $r > 0$ and $s > 0$ by Lemma 4.3. There exist $t \in \mathbb{R}$ and $s_1 > 0$ with $[t - s_2, t + s_2] \subseteq (X(r,s_1))_x$, because the set $(X(r,s_1))_x$ contains an open interval. Set $s = \min\{s_1, s_2\}$. We have $[t - s, t + s] \subseteq [t - s_2, t + s_2] \subseteq (X(r,s_1))_x \subseteq (X(r,s))_x$ by the definition of a $D_2$-family. Hence, we get $x \in Y(r,s)$. We have demonstrated that $I(X) = \bigcup_{r,s} Y(r,s)$. In particular, $I(X)$ is a $D_2$-set.

We next demonstrate that $\proj \dim(I(X)) < \proj \dim(X)$. It is obvious when $\int(X) \neq \emptyset$ because $\proj \dim(I(X)) \leq n$ and $\proj \dim(X) = n+1$ by the definition. We consider the case in which $\int(X) = \emptyset$. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the coordinate projection forgetting the last coordinate. We have $\proj \dim(I(X)) \leq \proj \dim(X) \leq \proj \dim X$ because $I(X) \subseteq \pi(X)$. Assuming that $\proj \dim(I(X)) = \proj \dim X$ we lead to a contradiction. Set $d = \proj \dim X = \proj \dim I(X)$. Take a coordinate projection $\pi_1 : \mathbb{R}^n \to \mathbb{R}^d$ with $\int(\pi_1(I(X))) \neq \emptyset$. The coordinate projection $\pi_2 : \mathbb{R}^{n+1} \to \mathbb{R}^d$ is the composition of $\pi_1$ with $\pi$. We have $\int(\pi_2(X)) = \emptyset$ because $\pi_1(I(X)) \subseteq \pi(X)$. The notation $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denotes the coordinate projection onto the last coordinate. The coordinate projection $\Pi = (\pi_2, \pi_1) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is given by $\Pi(x) = (\pi_2(x), \pi_1(x))$. Consider the set $T = \{ x \in \mathbb{R}^d \mid \Pi(x)_s \text{ contains an open interval} \}$. We have $\pi_1(I(X)) \subseteq T$. In fact, take $x \in I(X)$ and open interval $J \subseteq X_s$. The set $\pi_1(x) \times J$ is contained in $I(X)_x$. It means that $\pi_1(x) \in T$. We get $\int(T) \neq \emptyset$ because $\pi_1(I(X))$ has a nonempty interior. Set $T(r,s) = \{ x \in \mathbb{R}^d \mid \exists r \in \mathbb{R}, [t-s, t+s] \subseteq (\Pi(X(r,s)))_x \}$. The set $T$ is $D_2$ and $T = \bigcup_{r,s} T(r,s)$ as demonstrated previously. We have $\int(T(r,s)) \neq \emptyset$ for some $r > 0$ and $s > 0$ by Lemma 4.3. We get $\int(\Pi(X(r,s))) \neq \emptyset$ by Lemma 3.3 and we obtain $\int(\Pi(X)) \neq \emptyset$. It is a contradiction to the assumption that $\proj \dim X = d$.

Lemma 5.5 Let $\mathcal{R} = (\mathbb{R}, <, \ldots)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models$ DC, LUF$_2$. Let $X$ be a $D_2$-subset of $\mathbb{R}^n$ of $\proj \dim(X) = d$. Take a coordinate projection $\pi : X \to \mathbb{R}^d$ such that $\pi(X)$ has a nonempty interior. Then, there exists a $D_2$-subset $Z$ of $\mathbb{R}^d$ such that $Z$ has an empty interior and the fiber $X \cap \pi^{-1}(x)$ is discrete and closed for any $x \in \mathbb{R}^d \setminus Z$.
Proof. For all $1 \leq i \leq n - d$, we can take coordinate projections $\pi_i : R^{n-i+1} \to R^{n-i}$ with $\pi = \pi_{n-d} \circ \cdots \circ \pi_1$. We may assume that $\pi$ are the coordinate projections forgetting the last coordinate without loss of generality. Set $\Pi_i = \pi_i \circ \cdots \circ \pi_1$ and $\Phi_i = \pi_{n-d} \circ \cdots \circ \pi_{i-1}$. Consider the sets $T_i = \{x \in R^{n-i} \mid \pi_i^{-1}(x) \cap \Pi_i^{-1}(X) \text{ contains an open interval}\}$. The sets $T_i$ are $D_2$ and we have $\text{proj. dim}(T_i) < \text{proj. dim} \Pi_i^{-1}(X) = \text{proj. dim} X = d$ by Lemma 5.4. Set $U_i = \Phi_i(T_i) \subseteq R^d$ for all $1 \leq i \leq n - d$. The projection images $U_i$ are $D_2$-sets by Lemma 3.2(1). We get $\text{int}(U_i) = \emptyset$ because $\text{proj. dim}(T_i) < d$. Set $Z = \bigcup_{i=1}^{n-d} U_i$. It also has an empty interior by Lemma 4.4.

The fiber $X \cap \pi_i^{-1}(x)$ is discrete and closed for any $x \in R^d \setminus Z$. In fact, let $y \in R^d$ be an arbitrary point with $x = \pi(y)$. Set $y_0 = y$ and $y_1 = \Pi_i(y)$ for $1 \leq i \leq n - d$. We have $y_{n-d} = x$ by the definition. We construct an open box $B_i$ in $R^{n-d-i}$ for $0 \leq i \leq n - d$ such that $y_i \in B_i$ and $(\{x \times B_i\} \cap \Pi_i(X)$ consists of at most one point in decreasing order. When $i = n - d$, the open box $B_{n-d} = R^d$. When $(\{x \times B_i\} \cap \Pi_i(X) = \emptyset$, set $B_{n-i} = B_i \times R$. We have $(\{x \times B_{n-i}\} \cap \Pi_i^{-1}(Y)) = \emptyset$. When $(\{x \times B_i\} \cap \Pi_i(X) \neq \emptyset$, the fiber $\Pi_{n-i}^{-1}(x) \cap \pi_i^{-1}(y)$ is discrete and closed by Lemma 5.1. Therefore, there exists an open box $B_{n-i}$ in $R^{n-d+i}$ such that $\pi_i(B_{n-i}) = B_i$, $y_{n-d-i} \in B_{n-i}$ and $(\{x \times B_{n-i}\} \cap \Pi_i^{-1}(Y))$ consists of at most one point. We have constructed the open boxes $B_i$ in $R^{n-d-i}$ for all $0 \leq i \leq n - d$. The existence of $B_0$ implies that $X \cap \pi_i^{-1}(x)$ is discrete and closed. \hfill □

We let $1 \leq i_1 < \ldots < i_d \leq n$ and let $\pi : R^d \to R^d$ be the coordinate projection given by $\pi(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_d})$. Let $B$ be an open box in $R^d$. Take the increasing sequence of positive integers $j_1, \ldots, j_{n-d}$ in $\{1, \ldots, n\}$ such that $\{j_1, \ldots, j_d\} \cap \{j_1, \ldots, j_{n-d}\} = \emptyset$. The map $\sigma_\pi : R^d \to R^d$ is defined by $\sigma_\pi(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_d}, x_{j_1}, \ldots, x_{j_{n-d}})$. By abuse of the word "graph," a definable set $S$ is called the graph of a definable continuous map $f$ defined on $\pi(B)$ if the image $\sigma_\pi(S)$ is the graph of the definable continuous map $f$ defined on $\pi(B)$ in the following lemma.

Lemma 5.6 Let $\mathcal{R} = (R, <, \ldots)$ be a densely linearly ordered structure. Let $X$ be a definable subset of $R^n$ of dimension $d < n$. Consider the set

$$G(X) = \{x \in X \mid \text{there exist a coordinate projection } \pi : R^d \to R^d \text{ and an open box } B \text{ with } x \in B \text{ such that } X \cap B \text{ is the graph of a continuous map defined on } \pi(B)\}.$$ 

It is definable and constructible. Furthermore, we have $\text{dim}(X \setminus G(X)) < d$ if the following conditions are all satisfied:

1. $\mathcal{R} = (R, <, +, 0, \ldots)$ is an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models \text{DC, LUF} \ldots$;
2. $X$ is a $D_2$ set;
3. Any $D_2$ set of dimension smaller than $d$ is constructible.

Proof. It is obvious that the set $G(X)$ is definable. We show that $G(X)$ is constructible. Fix an arbitrary coordinate projection $\pi : R^d \to R^d$. Consider the set $G_\pi(X)$ of points $x \in R^d$ such that $X \cap B$ is the graph of a continuous map defined on $\pi(B)$ for some open box $B$ containing the point $x$. The set $G_\pi(X)$ is locally closed because $G_\pi(X)$ is locally the graph of a continuous map. Therefore, it is constructible. The set $G(X)$ is also constructible because we have $G(X) = \bigcup G_\pi(X)$.

The differences $Y = X \setminus G(X)$ and $Y_\pi = X \setminus G_\pi(X)$ are $D_2$ by Lemma 3.2. We next demonstrate that $\text{dim}(X \setminus G(X)) < d$ under the given conditions. When $d = 0$, $X = G(X)$ by Lemma 5.3. The lemma is obvious.

Consider the case in which $d > 0$. Note that we always have $G_\pi(X) \setminus U = G_\pi(X \setminus U)$ and $G_\pi(X) \setminus U = G_\pi(X \setminus U)$ for any definable open subset $U$ of $R^n$ by the definitions of $G(X)$ and $G_\pi(X)$. We also have $U \cap (X \setminus G(X)) = (X \setminus U) \setminus G(X \setminus U)$. We use this fact without mentioning.

We lead to a contradiction assuming that $\text{dim}(Y) = d$. Take a point $y \in R^d$ with $y = d$. We can take a coordinate projection $\pi : R^d \to R^d$ such that $\pi(Y \cap U)$ has a nonempty interior for any open box $U$ containing $y$. We fix this coordinate projection $\pi$ in the proof. We may assume that this projection is the projection onto the first $d$ coordinates without loss of generality. We fix a sufficiently small open box $B$ in $R^n$ with $y \in B$. Note that $\text{dim} X = d$ because $d = \text{dim}, Y \leq \text{dim}, X \leq \text{dim}, X = d$. We can prove by an easy induction that there exists a positive integer $N_i$ with $|X \cap B \cap \pi_i^{-1}(x)| \leq N_i$ or $|X \cap B \cap \pi_i^{-1}(x)| = \infty$ for all $x \in \pi(X \cap B)$ using Lemma 5.2.

We omit the proof. We also have

$$\text{dim}(X \cap B) = \text{dim}(Y \cap B) = \text{proj. dim}(X \cap B) = \text{proj. dim}(Y \cap B) = d.$$
by Lemma 3.5. We may assume that
1. $\pi(Y)$ has a nonempty interior,
2. $|X \cap \pi^{-1}(x)| \leq N_1$ or $|X \cap \pi^{-1}(x)| = \infty$ for all $x \in \pi(X)$ and
3. $d = \dim(X) = \dim(Y) = \text{proj. dim}(X) = \text{proj. dim}(Y)$,
considering $X \cap B$ instead of $X$. Under these assumptions, we also have $\dim(Y_\pi) = \text{proj. dim}(Y_\pi) = d$ and $\pi(Y_\pi)$ has a nonempty interior because $Y \subseteq Y_\pi \subseteq X$.

We can take a $D_\Sigma$-subset $Z$ of $R^d$ such that $Z$ has an empty interior and $\pi^{-1}(x) \cap X$ is discrete and closed for any $x \in R^d \setminus Z$ by Lemma 5.5. Since $\dim(Z) < d$, $Z$ is a constructible set by the assumption. Therefore, the set $\pi(Y_\pi) \setminus Z$ is a $D_\Sigma$-set by Lemma 3.2(3). The set $\pi(Y_\pi) \setminus Z$ has a nonempty interior by Lemma 4.4. We may further assume that
1. $\pi(Y_\pi) = \pi(X) = B'$ for some open box $B'$ in $R^d$ and
2. $|X \cap \pi^{-1}(x)| \leq N_1$ for all $x \in \pi(X)$
considering $X \cap \pi^{-1}(B')$ instead of $X$, where $B'$ is an open box contained in $\pi(Y_\pi) \setminus Z$.

We can reduce to the case in which there exists a positive integer $N$ with $|Y_\pi \cap \pi^{-1}(x)| = N$ for all $x \in \pi(Y_\pi)$. We need the following claim to prove it.

**Claim** There exists a positive integer $N$ with $N \leq N_1$ such that the set $E = \{x \in R^d \mid |\pi^{-1}(x) \cap Y_\pi| = N\}$ is $D_\Sigma$ and $\text{int}(E) \neq \emptyset$.

**Proof of Claim** Let $1 \leq i \leq N_1$, consider the sets $C_i = \{x \in R^d \mid |\pi^{-1}(x) \cap Y_\pi| = i\}$ and $D_i = \{x \in R^d \mid |\pi^{-1}(x) \cap Y_\pi| \geq i\}$. The set $D_i$ is $D_\Sigma$. In fact, $D_i$ is the projection image of the $D_\Sigma$-set $\{(x_1, \ldots, x_i) \in (R^d)^i \mid \pi(x_1) = \cdots = \pi(x_i), x_j \neq x_k \text{ for all } j \neq k, x_j \in Y_\pi \text{ for all } j\}$. We demonstrate the claim by induction on $N_1$. When $N_1 = 1$, we have nothing to prove. We have $N = 1$ and $E = \pi(Y_\pi)$. When $N_1 > 1$, the set $C_{N_1} = D_{N_1}$ is $D_\Sigma$. If $\text{int}(C_{N_1}) \neq \emptyset$, set $N = N_1$ and $E = C_{N_1}$. Otherwise, we have $\dim(C_{N_1}) < d$. The definable set $C_{N_1}$ is constructible by the assumption. By the induction hypothesis, we can get $N < N_1$ such that $E = \{x \in R^d \mid |\pi^{-1}(x) \cap (Y_\pi \setminus (C_{N_1} \times R^{m-d}))| = N\}$ is $D_\Sigma$ and $\text{int}(E) \neq \emptyset$. It is obvious that $E = \{x \in R^d \mid |\pi^{-1}(x) \cap Y_\pi| = N\}$.

Let $E$ be the $D_\Sigma$-set in the claim and take an open box $B''$ contained in $E$. Set $X' = X \cap \pi^{-1}(B'')$. We may assume that $|Y_\pi \cap \pi^{-1}(x)| = N$ for all $x \in \pi(X)$ and $\pi(X)$ is an open box by considering $X'$ in place of $X$. Applying the same argument to the new $X$, we can reduce to the following case:

1. We have $\pi(Y_\pi) = \pi(X) = V$ for some open box $V$ in $R^d$;
2. There exist positive integers $N$ and $N'$ such that $|Y_\pi \cap \pi^{-1}(x)| = N$ and $|X \cap \pi^{-1}(x)| = N'$ for any $x \in V$.

We demonstrate that the closure of $G_\pi(X)$ has an empty intersection with $Y_\pi$. We lead to a contradiction assuming that there is a point $y$ in the intersection. We use two facts. Firstly, the function $\text{dist} : R^n \rightarrow R$ given by $\text{dist}(u, v) = \min_{1 \leq i \leq n} |u_i - v_i|$ is a distance function on $R^n$, where $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. Secondly, the group $(R, \times)$ is divisible by [10, Proposition 2.2]. Note that $|G_\pi(X) \cap \pi^{-1}(x)| = M$ for all $x \in V$, where $M = N' - N$. Let $\{y_1, \ldots, y_M\}$ be the fiber $G_\pi(X) \cap \pi^{-1}(y)$. We have $y \neq y_i$ for all $1 \leq i \leq M$ and $y_j \neq y_i$ for $i \neq j$. Set $\varepsilon = \min\{|\text{dist}(y_i, y_j)| \mid 1 \leq i \leq M\} > 0$. Since $\pi(X)$ is locally the graph of a continuous function, there are open boxes $W_i$ containing the point $y_i$ such that $X \cap W_i$ is the graph of a definable continuous map $f_i : U_i = \pi(W_i) \rightarrow R^{n-d}$ for any $1 \leq i \leq M$. The map $g_i : U_i \rightarrow W_i$ is given by $g_i(x) = (x, f_i(x))$. Shrinking $U_i$ if necessary, we may assume that $\text{dist}(g_i(x), y_i) = \text{dist}(f_i(x), f_i(\pi(y_i))) < \varepsilon/3$ for all $x \in U_i$ and all $1 \leq i \leq M$ because $f_i$ is continuous. Set $U = \bigcap_{i=1}^M U_i$. The set $U$ is an open set containing the point $\pi(y)$.

On the other hand, we can take a point $y' \in G_\pi(X)$ such that $\pi(y') \in U$ and $\text{dist}(y', y) < \varepsilon/3$ because $y$ is a point of the closure of $G_\pi(X)$. We obtain $y' \neq g_i(\pi(y'))$ for all $1 \leq i \leq M$. In fact, if we have $y' = g_i(\pi(y'))$ for some $1 \leq i \leq M$, we have $\text{dist}(y, y_i) \leq \text{dist}(y, g_i(\pi(y'))) + \text{dist}(y', y) < 2\varepsilon/3$. It contradicts the definition of $\varepsilon$. It means that $G_\pi(X) \cap \pi^{-1}(y')$ contains at least $(M + 1)$ points $g_1(\pi(y')), \ldots, g_M(\pi(y'))$ and $y'$. Contradiction to the fact that $|G_\pi(X) \cap \pi^{-1}(y')| = M$. We have demonstrated that $G_\pi(X) \cap Y_\pi = \emptyset$.

Set

$$Y_i = \{x \in R^n \mid x \text{ is the } i\text{-th minimum in } Y_\pi \cap \pi^{-1}(\pi(x)) \text{ in the lexicographic order}\}$$
for all $1 \leq i \leq N$. Consider the $D_\Sigma$-set
\[
Z = \{(x_1, \ldots, x_N) \in (\mathbb{R}^n)^N \mid \pi(x_1) = \cdots = \pi(x_N), \; x_1 < x_2 < \cdots < x_N \}
\]
in the lexicographic order, $x_i \in Y_\pi$ for all $1 \leq i \leq N$.

The definable set $Y_i$ is the projection image of the $D_\Sigma$-set $Z$, and is hence $D_\Sigma$ by Lemma 3.2(1). The projection image $\pi(Y_i)$ is an open box because $\pi(Y_1) = \pi(Y_N) = V$. Consequently, $Y_i$ is simultaneously a $D_\Sigma$-set and the graph of a definable map defined on an open box for any $1 \leq i \leq N$. Applying Lemma 4.5 iteratively to $Y_i$, we can find a nonempty open box $W$ such that $Y_i \cap \pi^{-1}(W)$ is the graphs of definable continuous maps defined on $W$ for all $1 \leq i \leq N$. Since the closure of $G_\pi(X)$ has an empty intersection with $Y_\pi$, we have $B \cap Y_\pi = \emptyset$ for any point $y \in Y_\pi$ and any sufficiently small open box $B$ containing the point $y$. Therefore, we have $Y_\pi \cap \pi^{-1}(W) \subseteq G_\pi(X)$. Contradiction to the definition of $Y_\pi$. □

We finally get the following theorem.

**Theorem 5.7** Let $\mathcal{R} = (\mathbb{R}, <, +, 0, \ldots)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$. Any $D_\Sigma$-set is constructible. In particular, any set definable in the open core of $\mathcal{R}$ is constructible.

**Proof.** Let $X$ be a $D_\Sigma$-subset of $\mathbb{R}^d$ of dimension $d$. We show that $X$ is constructible by the induction on $d$. When $d = 0$, it is clear from Lemma 5.3. When $d > 0$, consider the constructible set $G(X)$ defined in Lemma 5.6. The difference $X \setminus G(X)$ is a $D_\Sigma$-set of dimension smaller than $d$ by Lemma 5.6. It is constructible by the induction hypothesis. Consequently, $X$ is also constructible.

It is obvious that any set definable in the open core of $\mathcal{R}$ is constructible because the projection image of a constructible set is again constructible by the assertion we have just proven and Lemma 3.2(1). □

**Proof of Theorem 1.2.** Obvious by Lemma 3.2(3), Lemma 5.2 and Theorem 5.7. □

**References**

[1] A. Dolich and J. Goodrick, Strong theories of ordered abelian groups, Fund. Math. **236**, 269–296 (2017).
[2] A. Dolich, C. Miller, and C. Steinhorn, Structure having o-minimal open core, Trans. Amer. Math. Soc. **362**, 1371–1411 (2010).
[3] A. Dougherty and C. Miller, Definable boolean combinations of open sets are boolean combination of open definable sets, Illinois J. Math. **45**, 1347–1350 (2001).
[4] L. van den Dries, Dense pairs of o-minimal structures, Fund. Math. **157**, 61–78 (1998).
[5] A. Fornasiero, Locally o-minimal structures and structures with locally o-minimal open core, Ann. Pure Appl. Log. **164**, 211–229 (2013).
[6] M. Fujita, Uniformly locally o-minimal structures and locally o-minimal structures admitting local definable cell decomposition, Ann. Pure Appl. Log. **171**, 102756 (2020).
[7] M. Fujita, Uniform local definable cell decomposition for locally o-minimal expansion of the group of reals, arXiv:1912.05782 (2019).
[8] M. Fujita, Dimension inequality for a uniformly locally o-minimal structure of the second kind, J. Symb. Log. **65**, 1654–1663 (2020).
[9] T. Kawakami, K. Takeuchi, H. Tanaka, and A. Tsuboi, Locally o-minimal structures, J. Math. Soc. Japan **64**, 783–797 (2012).
[10] C. Miller, Expansions of dense linear orders with the intermediate value property, J. Symb. Log. **66**, 1783–1790 (2001).
[11] C. Miller and P. Speissegger, Expansions of the real line by open sets: o-minimality and open cores, Fund. Math. **162**, 193–208 (1999).
[12] C. Toffalori and K. Vozoris, Notes on local o-minimality, Math. Log. Q. **55**, 617–632 (2009).