Computable planar curves intersect in a computable point

Klaus Weihrauch
University of Hagen

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Well-known:

**Computable Intermediate Value Theorem**
Every computable function $f : [0; 1] \rightarrow \mathbb{R}$ such that $f(0) < 0$ and $f(1) > 0$ has a computable zero.

The multi-function $f \mapsto x_0$ is not computable.
Suppose $f, g : [0; 1] \rightarrow [0; 1]^2$ and $f(0; 1), g(0; 1) \subseteq (0; 1)^2$

- **Classically:**
  The functions $f$ and $g$ intersect if they are continuous.

- **[Manukyan 1976]**
  There are (Russian-) computable functions $f$ and $g$ which do not intersect.

- Let $f, g$ be (Grzegorczyk-Lacombe-) computable
**Theorem** [Wei 2017]
The functions $f$ and $g$ intersect in a **computable point** if they are (Grzegorczyk-Lacombe-) **computable**.

Obvious?

The possibly chaotic behavior of $f$ and $g$ must be controlled.

**Proof**

**Main idea:** compute a nested convergent sequence of crossings
The function \( f \) entering and leaving a “simple ball” \( B(f(a), r) \), \( a, r \in \mathbb{Q} \)

\[
\begin{align*}
\min(L_r) &= \sup\{ t > a \mid f[a; t] \subseteq B(f(a), r) \} \quad \text{comp. from below} \\
\max(L_r) &= \inf\{ t > a \mid |f(t) - f(a)| > r \} \quad \text{comp. from above} \\
\min(K_r) &= \sup\{ t < a \mid |f(t) - f(a)| > r \} \quad \text{comp. from below} \\
\max(K_r) &= \inf\{ t < a \mid f[t; a] \subseteq B(f(a), r) \} \quad \text{comp. from above}
\end{align*}
\]
For computing the nested sequence of balls we will need “everywhere” balls such that

\[ x_{ar}, y_{ar} \notin \text{range}(g) \]

\[ Q \iff \text{for all } a, r < s \in \mathbb{Q}, \ldots \]

\[ (\exists t \in \mathbb{Q}, r \leq t \leq s) x_{at}, y_{at} \notin \text{range}(g) \]

**A** If \( \neg Q \) then \( f \) and \( g \) intersect in a computable point.

**B** If \( Q \) then \( f \) and \( g \) intersect in a computable point.
Proof for Case (A)

Suppose $\neg Q$: There are $a, r, s \in \mathbb{Q}$, $r < s$ such that

$$(\forall t \in \mathbb{Q}, r \leq t \leq s) \ (x_{at} \in \text{range}(g) \text{ or } y_{at} \in \text{range}(g))$$

$x_{at}$ and $y_{at}$ are in $\text{range}(f)$ but not computable in general.
\[(\forall t \in [r; s]) \left[ f \circ \max(K_t) \in \text{range}(g) \text{ or } f \circ \min(L_t) \in \text{range}(g) \right]\]
For $r < r' < s' < s$:

We can compute $r_i, s_i \in \mathbb{Q}$ such that

\[ r = r_0 < r_1 < r_2 < \ldots < s_2 < s_1 < s_0 = s \]

and nested sequ. $(l_i)_{i \in \mathbb{N}}$ and $(J_i)_{i \in \mathbb{N}}$ of rat. interv., such that

\[ K_{s_i} \cup K_{r_i} \subseteq l_i, \quad L_{r_i} \cup L_{s_i} \subseteq J_i \]

\[ \{p\} := \bigcap l_i, \quad \{q\} := \bigcap J_i \]

$p$ and $q$ are computable. By continuity of $f$,

\[ \{f(p)\} := \bigcap f(l_i) \quad \text{and} \quad \{f(q)\} := \bigcap f(J_i). \]

By $\neg Q$.

\[ f(l_i) \cap \text{range}(g) \neq \emptyset \quad \text{i.o.} \quad \text{or} \quad f(J_i) \cap \text{range}(g) \neq \emptyset \quad \text{i.o.} \]

since $\text{range}(g)$ is compact, hence complete,

\[ f(p) \in \text{range}(g) \quad \text{or} \quad f(q) \in \text{range}(g) \]
Proof for Case (B)

Suppose $Q$: For all $a, r, s \in \mathbb{Q}$, $r < s$

$$(\exists t \in \mathbb{Q}, r \leq t \leq s) \left( x_{at} \notin \text{range}(g) \text{ and } y_{at} \notin \text{range}(g) \right)$$

The balls with $x_{at}, y_{rt} \notin \text{range}(g)$ are “dense”.
barrier: Ball $B(f(a), r)$ with exit points of $f$ (red) not in range($g$)

Case 1: no branch of $g$ crosses the barrier (repellent barrier)
Case 2: some branch of $g$ crosses the barrier (crossing)

Lemma: Every crossing contains a much smaller crossing
A crossing with balls $B(f(a_i), r_i)$ covering $f$

Somewhere $g$ must cross the strip
Times $a_i, b_i \in \mathbb{Q}$ and radii $r_i \in \mathbb{Q}$ can be defined such that $f$ moves through the balls as follows:

The red points are not in $\text{range}(g)$ (using Condition $Q$) and ...
but not like this ...
Suppose all barriers $B(f(a_i), r_i)$ are repellent. $g$ can still cross the strip of balls.
Consider also **lens-shaped barriers** (intersections of two balls). $g$ crosses the lens-shaped barrier. **There must be a ball-shaped or a lens-shaped crossing.**
Lemma
Every crossing (unit square, ball-shaped or lens-shaped) contains a much smaller crossing (ball-shaped or lens-shaped).

The set of crossings is not c.e.
Lemma
Every crossing (ball-shaped or lens-shaped) contains a smaller proper crossing (ball-shaped or lens-shaped).

Lemma
The set of proper crossings is c.e.
**Theorem** [Wei 2017]
The functions $f$ and $g$ intersect in a computable point if they are (Grzegorczyk-Lacombe-) computable.

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