COMPLEX SOLUTIONS TO MAXWELL’S EQUATIONS

SACHIN MUNSHI* AND RONGWEI YANG

Abstract. This paper provides a view of Maxwell’s equations from the perspective of complex variables. The study is made through complex differential forms and the Hodge star operator in $\mathbb{C}^2$ with respect to the Euclidean and the Minkowski metrics. It shows that holomorphic functions give rise to nontrivial solutions, and the inner product between the electric and the magnetic fields is considered in this case. Further, it obtains a simple necessary and sufficient condition regarding harmonic solutions to the equations. In the end, the paper gives an interpretation of the Lorenz gauge condition in terms of the codifferential operator.

1. Introduction

Named after the physicist and mathematician James C. Maxwell, Maxwell’s equations form the foundation of classical electromagnetism, optics, and electrodynamics (see for instance [1, 7, 13]). They are a set of partial differential equations that describe the interactions between the electric and magnetic fields that emerge from distributions of electric charges and currents, and of course, how these fields change in time. Maxwell’s equations are the differential equations

\begin{align}
\nabla \cdot \mathbf{B} &= 0, \quad (1.1a) \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0, \quad (1.1b) \\
\n\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0}, \quad (1.1c) \\
\n\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{J}, \quad (1.1d)
\end{align}

where $\mathbf{E} = (E_1, E_2, E_3)$ is the electric field, $\mathbf{B} = (B_1, B_2, B_3)$ is the magnetic field, the scalar $\rho$ is the electric charge density, and the vector $\mathbf{J}$ is the electric current density vector.

2010 Mathematics Subject Classification. Primary 35Q61, 78A25; Secondary 32A10.

Key words and phrases. Maxwell’s equations, differential forms, Hodge star operator, harmonic functions, holomorphic functions, Lorenz gauge.

* Corresponding author.
Moreover, $\epsilon_0$ is the *vacuum permittivity*, $\mu_0$ is the *vacuum permeability*, and $c := 1/\sqrt{\epsilon_0\mu_0}$ is the *speed of light* in vacuum. For ease of use, most modern physicists and mathematicians simply set $c = 1$, as shall we throughout this paper. For convenience, we fix the notation $x_0 = ct = t$. Then in a more compact form, equations (1.1a)-(1.1d) can be written in differential forms over Minkowski space-time as

\begin{align*}
    dF &= 0, \\
    \ast d\ast F &= J,
\end{align*}

where

\begin{equation}
    F = -dx_0 \wedge (E_1dx_1 + E_2dx_2 + E_3dx_3)
    - B_1dx_2 \wedge dx_3 + B_2dx_1 \wedge dx_3 - B_3dx_1 \wedge dx_2,
\end{equation}

and it is referred to as the *Faraday 2-form*. Here, $J$ is the *current 1-form*, $d$ is the *exterior differential operator*, and $\ast$ is the *Hodge star operator* ([1]). These will be described in detail in this paper. By the Poincaré lemma, Equation (1.2) implies that locally $F = d\omega$ for some differentiable 1-form $\omega = \eta_0dx_0 + \eta_1dx_1 + \eta_2dx_2 + \eta_3dx_3$. This paper investigates the case when $\eta_j$, $0 \leq j \leq 3$, are all harmonic functions in $(x_0, x_1, x_2, x_3)$ with respect to the Euclidean or Minkowski metric. If one takes advantage of the identification $\mathbb{R}^4 \simeq \mathbb{C}^2$, i.e.

$$\mathbb{R}^4 \ni (x_0, x_1, x_2, x_3) \mapsto (x_0 + ix_1, x_2 + ix_3) := (z_1, z_2) \in \mathbb{C}^2,$$

then $\omega$ can be written as $\omega(z) = f_1dz_1 + f_2dz_2 + f_1d\bar{z}_1 + f_2d\bar{z}_2$. The following is the main result.

**Theorem 1.1.** Let $f_j, \bar{f}_j$, $j = 1, 2$ be harmonic functions on $\mathbb{C}^2$. Then the complex differential form $F_\omega := d\omega$ is a solution to the source-free Maxwell’s equations in the Euclidean metric if and only if $\bar{\partial}_1f_1 + \bar{\partial}_2f_2 + \partial_1f_1 + \partial_2f_2$ is constant.

A parallel theorem holds with respect to the Minkowski metric on $\mathbb{R}^4 \simeq \mathbb{C}^2$. A solution $F_\omega$ to the Maxwell’s equations with respect to the Minkowski metric is said to be *wavelike* if each of the functions $f_j, \bar{f}_j$, $j = 1, 2$, is a solution to the d’Alembertian equation (or wave equation)

$$\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right) u = 0.$$ 

Known solutions to the Maxwell’s equations are all wavelike, so the next result seems surprising.
Theorem 1.2. There exist non-wavelike solutions to the source-free Maxwell’s equations with respect to the Minkowski metric.

The paper is organized as follows.

Contents

1. Introduction 1
2. Preliminaries 3
   2.1. Complex Differential Forms 3
   2.2. Hodge Star Operator $\star$ 5
   2.3. Self-dual and Anti-self-dual Forms 8
   2.4. The Hodge Laplacian 8
3. Maxwell’s Equations 9
   3.1. Classical Version 9
   3.2. Differential Forms Version 10
4. Harmonic Solutions To Maxwell’s Equations 11
   4.1. Euclidean Metric Case 14
   4.2. Minkowski Metric Case 19
5. On the Lorenz gauge 23
6. Concluding Remarks 26
References 26

2. Preliminaries

Recall that $\mathbb{R}^4$ is just Euclidean space of real dimension 4. As a vector space, a point in $\mathbb{R}^4$ may be considered as a row vector $\mathbf{x} = (x_0, x_1, x_2, x_3), x_i \in \mathbb{R}$. Note that in the Lorentzian signature, $\mathbb{R}^4$ is denoted as $\mathbb{R}^{1,3}$, and is referred to as Minkowski space-time, with $x_0$ as the time variable (denoted $ct$), where $c$ is the speed of light, and $x_1, x_2, x_3$ are the space variables.

Going back to viewing $\mathbb{R}^4$ in the Euclidean metric, we have the identification $\mathbb{R}^4 \simeq \mathbb{C}^2$, i.e.

$$\mathbb{R}^4 \ni (x_0, x_1, x_2, x_3) \mapsto (x_0 + ix_1, x_2 + ix_3) := (z_1, z_2) \in \mathbb{C}^2.$$ 

2.1. Complex Differential Forms. We first introduce some preliminaries on complex differential forms. Consider $\mathbb{C}^n$ with points given by coordinates $z = (z_1, z_2, \ldots, z_n)$. The
tangent space of $\mathbb{C}^n$ is
\[ T(\mathbb{C}^n) = \text{span}\left\{ \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_k} : 1 \leq k \leq n \right\}, \]
and the cotangent space is given by
\[ T^* (\mathbb{C}^n) = \text{span}\left\{ dz_k, d\bar{z}_k : 1 \leq k \leq n \right\}. \]

**Definition 2.1.** Let $f$ be a smooth function on a domain $\mathcal{M} \subset \mathbb{C}^n$. Consider the linear operators $\partial, \bar{\partial}, d$ defined to act on $f$ as follows:
\[ \partial f = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} dz_k, \quad \bar{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k, \quad df = (\partial + \bar{\partial}) f. \]
Here, $d$ is called the complex exterior differential operator, while $\partial, \bar{\partial}$ are called the Dolbeault operators.

Recall that a smooth function $f$ on a domain $\mathcal{M}$ is said to be holomorphic if $\bar{\partial} f = 0$ everywhere on $\mathcal{M}$. Clearly, this means that $f$ is analytic in each variable. Note that if $f$ is holomorphic, then $df = \partial f$. The following fact is well-known.

**Fact 2.2.** $\partial^2 = \bar{\partial}^2 = d^2 = 0$.

For a multi-index $I = i_1 i_2 \cdots i_p$ we assume $i_1 < i_2 < \cdots < i_p$ and define its length $|I| = p$.

**Definition 2.3.** The space of complex $(p, q)$-forms on $\mathcal{M}$ is defined as
\[ \Omega^{p,q} (\mathcal{M}) := \left\{ \sum_{|I| = p, |J| = q} f_{I,J} dz_I \wedge d\bar{z}_J : p + q \leq 2n, f_{I,J} \in C^\infty (\mathcal{M}) \right\}. \]
We may drop the $\mathcal{M}$ from the definition for convenience. Clearly, $\Omega^{1,0}$ is the space of complex differential forms containing only the $dz_k$ terms, and $\Omega^{0,1}$ is the space of forms containing only the $d\bar{z}_k$ terms. Then in terms of the exterior product on differential forms we have
\[ \Omega^{p,q} = \Omega^{1,0} \wedge \cdots \wedge \Omega^{1,0} \wedge \Omega^{0,1} \wedge \cdots \wedge \Omega^{0,1}. \]
Slightly abusing notation, we shall adhere to the following definition throughout this paper.

**Definition 2.4.** $\Omega^k := \bigoplus_{p+q=k} \Omega^{p,q}$ is the space of all complex differential forms of total degree $k = p + q$.

**Definition 2.5.** For each $p > 0$, the forms given by $\sum_{|I|=p} f_I dz_I$, where each $f_I$ is holomorphic, are called holomorphic $p$-forms, and they form a holomorphic section of $\Omega^{p,0}$.
Note that if \( \eta = \sum_{|I| = p} f_I d\bar{z}_I \), with \( f_I \) holomorphic, then \( \bar{\partial} \eta = 0 \).

2.2. **Hodge Star Operator \( \star \)**. We briefly go over some basics of Hodge theory withholding any discussion on topology or manifold theory. We identify \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \) with the representation \( z_k = x_{2k-2} + ix_{2k-1} \), \( k = 1, 2, ..., n \). Given a nondegenerate self-adjoint \( 2n \times 2n \) matrix \( g = (g_{st}) \), it induces a sesquilinear form \( \langle \cdot, \cdot \rangle \) on the tangent space \( T(\mathbb{R}^{2n}) \) with complex coefficients by the evaluations

\[
\langle \alpha \partial/\partial x_s, \beta \partial/\partial x_t \rangle_g := \alpha \beta g_{st},
\]

where \( \alpha \) and \( \beta \) are complex numbers and \( 0 \leq s, t \leq 2n - 1 \). Then on the cotangent space \( T^*(\mathbb{R}^{2n}) \) one has the corresponding sesquilinear form given by

\[
\langle \alpha dx_s, \beta dx_t \rangle_g := \alpha \beta g^{st},
\]

where \( (g^{st}) = g^{-1} \). Using the fact \( z_k = x_{2k-2} + ix_{2k-1} \) above and the following representation

\[
\partial/\partial z_k = \frac{1}{2} \left( \frac{\partial}{\partial x_{2k-2}} - i \frac{\partial}{\partial x_{2k-1}} \right), k = 1, 2, ..., n,
\]

one may regard the aforementioned sesquilinear form \( \langle \cdot, \cdot \rangle_g \) as a sesquilinear form on \( \mathbb{C}^n \).

Further, it can be extended to a sesquilinear form on \( \Omega^p \) such that for \( \eta = \eta_1 \wedge \cdots \wedge \eta_p \), \( \xi = \xi_1 \wedge \cdots \wedge \xi_p \) one has

\[
\langle \eta, \xi \rangle_g = \det[\langle \eta_s, \xi_t \rangle_g]_{s,t=1}^p, \quad \eta, \xi \in \Omega^p.
\]

One observes that if the matrix \( g \) is positive definite then \( \langle \cdot, \cdot \rangle_g \) is an inner product on the set of constant \( p \)-forms for each \( 1 \leq p \leq 2n \).

**Definition 2.6.** The Hodge star operator \( \star : \Omega^p(\mathbb{C}^n) \to \Omega^{2n-p}(\mathbb{C}^n) \) with respect to the bilinear form \( \langle \cdot, \cdot \rangle_g \) is a linear operator such that for \( \eta, \xi \in \Omega^p(\mathbb{C}^n) \) one has

\[
\eta \wedge \star \xi = \langle \eta, \xi \rangle_g \text{vol}_g,
\]

where \( \text{vol}_g = \left( \frac{i}{2} \right)^n \sqrt{|\det g|} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \) is the volume form (11).

**Example 2.7.** For the Euclidean metric on \( \mathbb{R}^{2n} \) we have \( g = I_{2n} \). Let \( \Omega^p(\mathbb{R}^{2n}) \) denote the space of real differential \( p \)-forms over \( \mathbb{R}^{2n} \). Then it is well-known that

\[
\star (dx_0 \wedge dx_2 \wedge \cdots \wedge dx_{p-1}) = dx_p \wedge dx_{p+2} \wedge \cdots \wedge dx_{2n-1}.
\]

And for a permutation \( \sigma = (i_0, i_1, \ldots, i_{2n-1}) \), we have

\[
\star (dx_{i_0} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}}) = (-1)^\sigma dx_{i_p} \wedge dx_{i_{p+1}} \wedge \cdots \wedge dx_{i_{2n-1}}
\]
depending on the parity of \( \sigma \). In particular, \( \star(dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{2n-1}) = 1 \).

Moreover, for a \( p \)-form \( \omega \),

\[
\star^2 \omega = (-1)^{p(2n-p)} \omega.
\]

This is only true in the Euclidean metric. Generally, \( \star^2 \omega = (-1)^{p(2n-p)} s \omega \), where \( s \) is the parity of the signature of the inner product defined by the metric.

The next two examples give the Hodge dual on \( \mathbb{C}^2 \) with respect to the Euclidean metric and the Minkowski metric, respectively, and they will be used later. Using the earlier identification \( z_1 = x_0 + i x_1, z_2 = x_2 + i x_3 \), we have

\[
dz_1 = dx_0 + i dx_1, dz_2 = dx_2 + i dx_3, d\bar{z}_1 = dx_0 - i dx_1, d\bar{z}_2 = dx_2 - i dx_3,
\]

and these complex \((1,0)\)- and \((0,1)\)-forms span the complex cotangent space \( T^* (\mathbb{C}^2) \). Clearly, \( T^* (\mathbb{C}^2) \) is also spanned by the real forms \( dx_k, 0 \leq k \leq 3 \) with complex coefficients.

**Example 2.8.** One may directly use Definition 2.6 to compute the Hodge dual of complex differential forms, or one may first write them as real forms, apply Example 2.7 and then convert back to complex forms. We shall use the latter approach.

Let’s start with \( \star dz_1 \). Since \( dz_1 = dx_0 + i dx_1 \), it follows that

\[
\star dz_1 = \star dx_0 + i \star dx_1 \\
= (dx_1 \wedge dx_2 \wedge dx_3) - i (dx_0 \wedge dx_2 \wedge dx_3) \\
= (dx_1 - i dx_0) \wedge (dx_2 \wedge dx_3) \\
= (dx_1 - i dx_0) \wedge \frac{i}{2} (dz_2 \wedge d\bar{z}_2) \\
= (dx_0 + i dx_1) \wedge \frac{1}{2} (dz_2 \wedge d\bar{z}_2) \\
= \frac{1}{2} (dz_1 \wedge dz_2 \wedge d\bar{z}_2),
\]

where the fourth line follows from the fact that

\[
dz_2 \wedge d\bar{z}_2 = (dx_2 + i dx_3) \wedge (dx_2 - i dx_3) = -2 i dx_2 \wedge dx_3.
\]

Likewise one verifies that

\[
\star dz_2 = -\frac{1}{2} (dz_1 \wedge dz_2 \wedge d\bar{z}_1), \star d\bar{z}_1 = \frac{1}{2} (dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2), \star d\bar{z}_2 = -\frac{1}{2} (dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2),
\]

\(^1dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{2n-1} \) is the volume form.
and

\[ * (dz_1 \wedge d\bar{z}_1) = dz_2 \wedge d\bar{z}_2, \quad * (dz_2 \wedge d\bar{z}_2) = dz_1 \wedge d\bar{z}_1, \quad * (dz_1 \wedge d\bar{z}_2) = -dz_1 \wedge d\bar{z}_2, \]
\[ * (dz_2 \wedge d\bar{z}_1) = -dz_2 \wedge d\bar{z}_1, \quad * (dz_1 \wedge d\bar{z}_1) = dz_1 \wedge d\bar{z}_2, \quad * (dz_1 \wedge d\bar{z}_2) = dz_2 \wedge d\bar{z}_1. \]

The Hodge star of 3-forms can be computed using the fact that \(*^2 \omega = (-1)^{p(n-p)} \omega\), and one has

\[ * (dz_1 \wedge dz_2 \wedge d\bar{z}_1) = 2dz_2, \quad * (dz_1 \wedge dz_2 \wedge d\bar{z}_2) = -2dz_1, \]
\[ * (dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2) = 2d\bar{z}_2, \quad * (dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2) = -2d\bar{z}_1. \]

Finally, for the unique \((2,2)\)-form, we have

\[ * (dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2) = 4. \]

**Example 2.9.** The Minkowski metric on \( \mathbb{R}^{1,3} \) has the signature \((+ - - -)\) represented by the metric matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

In this case, the inner product of complex 1-forms is determined by the facts:

\[ \langle dz_1, dz_1 \rangle = \langle dz_1, dz_2 \rangle = \langle dz_1, d\bar{z}_2 \rangle = \langle dz_2, d\bar{z}_2 \rangle = 0 \]

and

\[ \langle dz_1, d\bar{z}_1 \rangle = \langle dz_2, dz_2 \rangle = 2. \]

Calculation of the Hodge star operator on \( \mathbb{C}^2 \) with respect to the Minkowski metric is similar to that in Example 2.8. Here we only list its action on 2-forms and 3-forms for later use:

\[ * (dz_1 \wedge dz_2) = -dz_2 \wedge d\bar{z}_1, \quad * (dz_1 \wedge d\bar{z}_2) = -d\bar{z}_1 \wedge dz_2, \quad * (dz_2 \wedge d\bar{z}_1) = dz_1 \wedge d\bar{z}_2, \]
\[ * (d\bar{z}_1 \wedge d\bar{z}_2) = dz_1 \wedge d\bar{z}_2, \quad * (dz_1 \wedge d\bar{z}_1) = -dz_2 \wedge d\bar{z}_2, \quad * (dz_2 \wedge d\bar{z}_2) = dz_1 \wedge d\bar{z}_1; \]

and

\[ * (dz_1 \wedge dz_2 \wedge d\bar{z}_1) = 2dz_2, \quad * (dz_1 \wedge dz_2 \wedge d\bar{z}_2) = 2dz_1, \]
\[ * (dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2) = 2d\bar{z}_2, \quad * (dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2) = 2d\bar{z}_1. \]
2.3. **Self-dual and Anti-self-dual Forms.** We now focus on the Hodge star operator on the complex differential forms on \( \mathbb{C}^2 \) with respect to the Euclidean metric and the Minkowski metric.

1. Under the Euclidean metric, we have \( \star^2 \omega = \omega \) for every \( \omega \in \Omega^2(\mathbb{C}^2) \). Self-dual and anti-self-dual forms are the eigenvectors corresponding to the eigenvalues 1 and \(-1\), respectively, of the Hodge star operator on \( \mathbb{C}^2 \).

**Definition 2.10.** A differential form \( \omega \) is said to be self-dual if it is equal to its Hodge dual, i.e. \( \star \omega = \omega \). If \( \star \omega = -\omega \), then \( \omega \) is said to be anti-self-dual.

Out of the complex differential forms we considered in the previous subsection, six of them correspond to the pair \((p, q)\) such that \( p + q = 2 \). Let \( \Omega^2_+, \Omega^2_- \) denote the bases of self-dual and anti-self-dual forms, respectively, in \( \Omega^2 \). Then by the computations in Example 2.8 we have that

\[
\Omega^2_+ = \text{span} \{ dz_1 \wedge d\bar{z}_2, dz_1 \wedge d\bar{z}_2, dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 \}, \tag{2.3}
\]

\[
\Omega^2_- = \text{span} \{ dz_1 \wedge d\bar{z}_2, dz_2 \wedge d\bar{z}_1, dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2 \}.
\]

2. Under the Minkowski metric, we have \( \star^2 \omega = -\omega \) for any \( \omega \in \Omega^2(\mathbb{C}^2) \). Hence \( \star \) has eigenvalues \( i, -i \). With a bit of abuse of terminology, the eigenspaces corresponding to them are often also called self-dual and anti-self-dual forms, respectively. For consistency we shall also denote them by \( \Omega^2_+ \) and \( \Omega^2_- \), respectively. Then by the computations in Example 2.9 we have

\[
\Omega^2_+ = \text{span} \{ dz_1 \wedge d\bar{z}_2 + idz_2 \wedge d\bar{z}_1, dz_1 \wedge d\bar{z}_2 + idz_2 \wedge d\bar{z}_1 + dz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2 \}, \tag{2.4}
\]

\[
\Omega^2_- = \text{span} \{ dz_1 \wedge d\bar{z}_2 - idz_2 \wedge d\bar{z}_1, dz_1 \wedge d\bar{z}_2 - idz_2 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2 \}.
\]

2.4. **The Hodge Laplacian.** The Hodge star operator gives rise to a Hermitian bilinear form on compactly supported differentiable \( p \)-forms defined by

\[
(\eta, \xi)_{g} := \int_{\mathbb{C}^n} \eta \wedge \bar{\xi} = \int_{\mathbb{C}^n} \langle \eta, \xi \rangle_{g} \text{vol}_{g}.
\]

The differential operators \( d : \Omega^{p-1}(\mathbb{C}^n) \to \Omega^{p}(\mathbb{C}^n) \), where \( 1 \leq p \leq 2n \), have the following natural adjoint with respect to the bilinear form \((\cdot, \cdot)_{g}\).

**Definition 2.11.** The co-differential operator \( d^{*} : \Omega^{p}(\mathbb{C}^n) \to \Omega^{p-1}(\mathbb{C}^n) \) is defined by

\[
d^{*} \omega := (-1)^{n(p+1)+1} \star d \star \omega, \tag{2.5}
\]

where \( \omega \) is any differential \( p \)-form. In particular, for \( \mathbb{C}^2 \) we have \( d^{*} \omega = -\star d \star \omega \).
Definition 2.12. A differential $p$-form $\omega$ is said to be Hodge-Laplace harmonic (HL-harmonic for short) if

$$\Delta \omega := (dd^* + d^*d) \omega = 0,$$

where $\Delta$ is referred to as the Hodge Laplacian (or the Laplace-de Rham operator).

For more information on complex differential forms we refer readers to [1, 2, 9, 19].

3. Maxwell’s Equations

Maxwell equations have several equivalent formulations ([1, 6, 13, 18]). From the viewpoint of physics, there are versions of Maxwell’s equations based on electric and magnetic potentials that allow one to solve the equations as a boundary value problem within the realms of classical physics and quantum mechanics. Quantum mechanics is not in the scope of this paper, so we shall only consider Maxwell’s equations in the classical sense. Moreover, the space-time formulations of Maxwell’s equations are primarily used in high-energy physics and gravitational physics, in conjunction with Einstein’s theories of special relativity and general relativity. From the viewpoint of mathematics, Maxwell’s equations are important in vector calculus, potential field theory, gauge theory, differential geometry, topology, and many other studies ([1, 7, 9]).

3.1. Classical Version. Let $E = (E_1, E_2, E_3)$ and $B = (B_1, B_2, B_3)$ be electric and magnetic fields, respectively, in a convex region of $\mathbb{R}^4$ with Lorentzian signature $+-+-$, where the $E_j, B_j$ are scalar-valued functions of time and space. In this vector space, we represent points as coordinate column vectors $(x_0, x_1, x_2, x_3)$, where $x_0 = ct$ ($c$ being the speed of light) is the time variable and $x_1, x_2, x_3$ are the space variables. Maxwell’s equations are given by the following partial differential equations:

$$\nabla \cdot B = 0,$$  \hspace{1cm} (3.1a)

$$\nabla \times E + \frac{\partial B}{\partial t} = 0,$$  \hspace{1cm} (3.1b)

$$\nabla \cdot E = \rho,$$  \hspace{1cm} (3.1c)

$$\nabla \times B - \frac{\partial E}{\partial t} = J,$$  \hspace{1cm} (3.1d)

---

2This is essentially the Minkowski space-time $\mathbb{R}^{1,3}$.

3For simplicity, we often set $c = 1$. 
where the scalar $\rho$ is the electric charge density and the vector $J$ is the electric current density vector. The equations (3.1a) and (3.1b) are homogeneous, while the equations (3.1c) and (3.1d) are inhomogeneous. From vector calculus, $\nabla \cdot (\nabla \times A) = 0$ for any smooth vector field $A = (A_1, A_2, A_3)$ and $\nabla \times (\nabla \phi) = 0$ for any scalar function $\phi$.\footnote{From physics viewpoint, a scalar field.} Now by Poincaré’s lemma, in $\mathbb{R}^3$, if $\nabla \cdot B = 0$, then $B = \nabla \times A$ for some vector field $A$. This leads us to the potential field theory aspect of electromagnetism. So in this context, one considers the magnetic vector potential $A$ and the electric scalar potential $\phi$ such that

$$B = \nabla \times A, \quad (3.2)$$

$$E = -\frac{\partial A}{\partial t} - \nabla \phi. \quad (3.3)$$

Putting (3.2) and (3.3) in equations (3.1c) and (3.1d), we obtain

$$\rho = -\frac{\partial}{\partial t} (\nabla \cdot A) - \Delta \phi, \quad (3.4)$$

where $\Delta$ in (3.4) is the Laplacian in $\mathbb{R}^3$.

So if we let $A' = A - \nabla \psi, \phi' = \phi + \frac{\partial \psi}{\partial t}$, for any scalar field $\psi$, then the 4-vector $(\phi', A')$ solves (3.2) and (3.3), but not uniquely. In particular, this gives one the gauge freedom to choose $\phi$ such that $(\phi, A)$ satisfies the Lorenz\footnote{Not to be confused with Lorentz!} gauge:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot A = 0, \quad (3.5)$$

which implies that

$$\rho = \left( \frac{\partial^2}{\partial t^2} - \Delta \right) \phi, \quad (3.6)$$

$$J = \left( \frac{\partial^2}{\partial t^2} - \Delta \right) A, \quad (3.7)$$

where $\frac{\partial^2}{\partial t^2} - \Delta$ is the d’Alembertian, or wave operator. Since Lorentz transformations keep the Minkowski metric invariant, the d’Alembertian gives a Lorentz scalar. Further, Maxwell’s equations are Lorentz invariant and gauge invariant.

3.2. Differential Forms Version. Set $\omega = \phi dx_0 - A_1 dx_1 - A_2 dx_2 - A_3 dx_3$, where $A = (A_1, A_2, A_3)$ is a time-dependent smooth vector field in $\mathbb{R}^3$. This 1-form $\omega$ is often referred to
as the magnetic potential 1-form. With the Lorenz gauge from (3.5), we assume \( \omega \) satisfies the following normalization:

\[
\frac{\partial \phi}{\partial x_0} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = 0.
\]  (3.8)

Now set

\[
J = \rho dx_0 + J_1 dx_1 + J_2 dx_2 + J_3 dx_3.
\]  (3.9)

Define the 2-form \( F_\omega = d\omega \). This is called the Faraday field strength or simply the Faraday 2-form. Using (3.2) and (3.3), we have

\[
F_\omega = -dx_0 \wedge (E_1 dx_1 + E_2 dx_2 + E_3 dx_3)
\]  (3.10)

\[- B_1 dx_2 \wedge dx_3 + B_2 dx_1 \wedge dx_3 - B_3 dx_1 \wedge dx_2.\]

With the exterior derivative \( d \) and the Hodge star operator \( * \), the Maxwell’s equations take the concise form

\[
dF_\omega = 0, \]  (3.11)

\[
* d* F_\omega = J, \]  (3.12)

where (3.11) is referred to as the Bianchi identity. Equation (3.11) is equivalent to the homogeneous Maxwell’s equations (3.1a) and (3.1b); while (3.12) is equivalent to the inhomogeneous Maxwell’s equations (3.1c) and (3.1d). We may also refer to (3.11) and (3.12) as the exterior differential form of Maxwell’s equations. And we say \( \omega \), or alternatively \( F_\omega \), is a solution to the Maxwell’s equations if (3.11) and (3.12) are satisfied.

It is now apparent that solutions to the Maxwell’s equations are not unique, since if \( \omega \) is a solution then \( \omega' = \omega + d\psi \), where \( \psi \) is any smooth function. This fact gives the gauge freedom of choosing \( \phi \) that satisfies the Lorenz gauge condition (3.5).

This differential form formulation of Maxwell’s equations also makes the following remark apparent.

**Remark 3.1.** In vacuum, that is, when \( \rho = 0, J = 0 \), it follows from (3.11) and (3.12) that every self-dual or anti-self-dual 2-form \( F_\omega \) is a solution to the source-free Maxwell’s equations.

4. Harmonic Solutions To Maxwell’s Equations

In this section we would like to consider Maxwell’s equations in vacuum, and study some complex solutions and, in particular, harmonic solutions to those equations. But before
doing this, we shall first construct a complex differential form solution to Maxwell’s equations initially not in vacuum. So let’s begin by considering smooth functions \( f_1, f_2, f_1, f_2 \) in the two variables \( z_1, z_2 \). Define the complex differential form

\[
\omega (z) = f_1 dz_1 + f_2 dz_2 + f_1 d\bar{z}_1 + f_2 d\bar{z}_2. \tag{4.1}
\]

The form \( \omega \) acts as a potential, much like the magnetic potential discussed in Section 3.2. In a more general context, it is often referred to as a connection form. The associated curvature form or curvature field is defined as \( F_\omega := d\omega + \omega \wedge \omega \). Since \( \omega \) is scalar-valued, we have \( \omega \wedge \omega = 0 \) and therefore \( F_\omega = d\omega \). Direct computation shall verify that

\[
F_\omega = (\partial_1 f_2 - \partial_2 f_1) dz_1 \wedge dz_2 + (\bar{\partial}_1 f_2 - \bar{\partial}_2 f_1) d\bar{z}_1 \wedge d\bar{z}_2 \\
+ (\partial_1 f_2 - \bar{\partial}_1 f_1) dz_1 \wedge d\bar{z}_2 + (\bar{\partial}_2 f_1 - \partial_1 f_2) dz_2 \wedge d\bar{z}_1 \\
+ (\partial_1 f_1 - \bar{\partial}_1 f_1) dz_1 \wedge dz_1 + (\bar{\partial}_2 f_2 - \bar{\partial}_2 f_2) dz_2 \wedge d\bar{z}_2. \tag{4.2}
\]

Now in terms of real 2-forms, we can write (4.2) in the following way:

\[
F_\omega = -2i (\partial_1 f_1 - \bar{\partial}_1 f_1) dx_0 \wedge dx_1 \\
+ ((\partial_1 f_2 - \partial_2 f_1) + (\partial_1 f_2 - \partial_2 f_1)) dx_0 \wedge dx_2 \\
+ (\partial_1 f_2 - \bar{\partial}_2 f_1) - (\partial_2 f_1 - \bar{\partial}_1 f_2)) dx_0 \wedge dx_3 \\
- (\partial_1 f_2 - \bar{\partial}_2 f_1) - (\partial_2 f_1 - \bar{\partial}_1 f_2)) dx_1 \wedge dx_3 \\
- 2i (\partial_2 f_2 - \partial_2 f_2) dx_2 \wedge dx_3 \\
+ ((\partial_1 f_2 - \partial_2 f_1) - (\partial_1 f_2 - \partial_2 f_1)) dx_1 \wedge dx_3 \\
+ (\partial_1 f_2 - \partial_2 f_1) - (\partial_2 f_1 - \partial_1 f_2)) dx_1 \wedge dx_2. \tag{4.3}
\]

One can easily determine the electric and magnetic components in view of (3.10):

\[
E_1 = 2i (\partial_1 f_1 - \bar{\partial}_1 f_1), \\
E_2 = - ((\partial_1 f_2 - \partial_2 f_1) + (\partial_1 f_2 - \partial_2 f_1) + (\partial_1 f_2 - \partial_2 f_1) - (\partial_2 f_1 - \partial_1 f_2)), \\
E_3 = -i ((\partial_1 f_2 - \partial_2 f_1) - (\partial_1 f_2 - \partial_2 f_1) - (\partial_1 f_2 - \partial_2 f_1) - (\partial_2 f_1 - \partial_1 f_2)), \\
B_1 = 2i (\partial_2 f_2 - \partial_2 f_2),
\]
\[ B_2 = (\partial_1 f_2 - \partial_2 f_1) - (\partial_1 f_2 - \partial_2 f_1) + (\partial_1 f_2 - \partial_2 f_1) - (\partial_2 f_1 - \partial_1 f_2), \]

\[ B_3 = -i ((\partial_1 f_2 - \partial_2 f_1) - (\partial_1 f_2 - \partial_2 f_1) + (\partial_1 f_2 - \partial_2 f_1) + (\partial_2 f_1 - \partial_1 f_2)). \]

In general, since \( E \) and \( B \) are complex vectors in \( \mathbb{C}^3 \), the “electromagnetic dynamics” occurs in 7-dimensional space \((\dim \mathbb{C}^3 + 1)\). The spatial inner product is defined by

\[ \langle E, B \rangle = E_1 \overline{B}_1 + E_2 \overline{B}_2 + E_3 \overline{B}_3. \]

To facilitate the computation of this inner product and the energy density \( \frac{1}{2}(|E|^2 + |B|^2) \), we write \((4.2)\) as

\[ F_\omega = F_{12} dz_1 \wedge dz_2 + F_{13} d\bar{z}_1 \wedge d\bar{z}_2 + \sum_{j,k=1,2} F_{jk} dz_j \wedge d\bar{z}_k. \tag{4.4} \]

Then the following can be verified by direct computation:

\[ \langle E, B \rangle = 4F_{11}\overline{F}_{22} + 2 [(F_{12} - F_{21})(\overline{F}_{12} + \overline{F}_{21}) + (F_{12} + F_{21})(\overline{F}_{12} - \overline{F}_{21})]. \tag{4.5} \]

The energy density of the electromagnetic dynamics can be computed as

\[ \frac{1}{2}(|E|^2 + |B|^2) = 2(|F_{12}|^2 + |F_{12}|^2 + \sum_{j,k=1,2} |F_{jk}|^2). \tag{4.6} \]

If \( g \) is the Euclidean metric on \( \mathbb{C}^2 \), then one can also verify that

\[ |E|^2 + |B|^2 = \langle F_\omega, F_\omega \rangle. \]

In particular, one observes that if \( f_i = \bar{f}_i, i = 1,2 \) then both the electric field \( E \) and the magnetic field \( B \) are real vector-fields in \( \mathbb{R}^3 \). For example, one can write

\[ E_1 = 2iF_{11} = 2i \left( \overline{\partial_1 f_1} - \partial_1 f_1 \right) \]

and see that \( E_1 \) is 4 times the imaginary part of \( \overline{\partial_1 f_1} \). Other components can be checked similarly. Alternatively, one may observe that since \( \omega \) is real in this case, the curvature field \( F_\omega = d\omega \) must be real. In this case, one has \( F_{12} = \overline{F}_{12} \) and \( F_{12} = -\overline{F}_{21} \), and it follows that

\[ \langle E, B \rangle = 4 \left( F_{11}\overline{F}_{22} + |F_{12}|^2 - |F_{21}|^2 \right). \tag{4.7} \]

If \( E \) and \( B \) are real vectors, then \( \cos^{-1} \frac{\langle E, B \rangle}{|E||B|} \) is the angle between the electric field and the magnetic field. But when \( E \) and \( B \) are complex vectors, the physical meaning of \( \langle E, B \rangle \) is less clear.

Although the original Maxwell’s equations were formulated under the Minkowski metric, the differential form formulation \((3.11)\) and \((3.12)\) makes good sense under other metrics. In
In the sequel we shall consider the Maxwell’s equations in this formulation under two different metrics.

4.1. Euclidean Metric Case. Under the Euclidean metric, applying the Hodge star operator to (4.2) (cf. Example 2.8), one has

\[
\star F_\omega = (\partial_1 f_2 - \bar{\partial}_2 f_1) \, dz_1 \land dz_2 + (\bar{\partial}_1 f_2 - \partial_2 f_1) \, \bar{dz}_1 \land d\bar{z}_2 \\
- (\partial_1 f_2 - \bar{\partial}_2 f_1) \, dz_1 \land d\bar{z}_2 - (\partial_2 f_1 - \bar{\partial}_1 f_2) \, dz_2 \land d\bar{z}_1 \\
+ (\partial_2 f_2 - \bar{\partial}_2 f_2) \, dz_1 \land d\bar{z}_1 + (\bar{\partial}_1 f_1 - \bar{\partial}_1 f_1) \, dz_2 \land d\bar{z}_2.
\]

(4.8)

It is immediate that \( F_\omega \) is self-dual if and only if

\[
\partial_1 f_1 - \bar{\partial}_1 f_1 = \partial_2 f_2 - \bar{\partial}_2 f_2,
\]

\[
\partial_1 f_2 - \bar{\partial}_2 f_1 = \partial_2 f_1 - \bar{\partial}_1 f_2 = 0,
\]

and it is anti-self-dual if and only if

\[
\partial_1 f_1 - \bar{\partial}_1 f_1 = -(\partial_2 f_2 - \bar{\partial}_2 f_2),
\]

\[
\partial_1 f_2 - \bar{\partial}_2 f_1 = \bar{\partial}_1 f_2 - \bar{\partial}_2 f_1 = 0.
\]

The following fact follows immediately from (4.5).

**Corollary 4.1.** Let \( F_\omega \) be a self-dual solution to the Maxwell equations in vacuum with respect to the Euclidean metric on \( \mathbb{C}^2 \). Then \( \langle E, B \rangle \geq 0 \) holds with equality only if \( \omega \) is a trivial solution.

**Proof.** In fact, Equations (4.9) shows that \( F_{11} = F_{22} \) and \( F_{12} = F_{21} = 0 \). Hence by (4.5) one has

\[
\langle E, B \rangle = 2(2|F_{11}|^2 + |F_{12}|^2 + |F_{21}|^2) \geq 0.
\]

If \( \langle E, B \rangle = 0 \) then all six coefficients of the 2-forms in \( F_\omega \) are 0 and hence \( \omega \) is a trivial solution. \( \square \)

Since \( F_\omega = d\omega \), \( F_\omega \) is exact. Hence if \( F_\omega \) is self-dual or anti-self-dual then

\[
d \star F_\omega = \pm dF_\omega = \pm d^2 \omega = 0.
\]

Now if \( f_1, f_2 \) are holomorphic and \( f_1, f_2 \) are conjugate holomorphic, then the equations in (4.9) are automatically satisfied. We thus have the following fact.
Proposition 4.2. The form $F_\omega$, determined by $\omega$ in (4.1) with $f_1, f_2$ holomorphic and $f_1, f_2$ conjugate holomorphic, is a self-dual solution for the source-free Maxwell’s equations (4.2) and (4.8) under the Euclidean metric. Further, in this case

$$F_\omega = (\partial_1 f_2 - \partial_2 f_1) dz_1 \wedge dz_2 + (\bar{\partial}_1 f_2 - \bar{\partial}_2 f_1) d\bar{z}_1 \wedge d\bar{z}_2.$$  

(4.11)

Consequently, we have

Corollary 4.3. If $\omega$ is of the form (4.1) and $f_1, f_2$ are holomorphic and $f_1, f_2$ are conjugate holomorphic, then

$$E_1 = B_1 = 0,$$

$$E_2 = B_2 = -\left((\partial_1 f_2 - \partial_2 f_1) + (\bar{\partial}_1 f_2 - \bar{\partial}_2 f_1)\right),$$

$$E_3 = B_3 = -i \left((\partial_1 f_2 - \partial_2 f_1) - (\bar{\partial}_1 f_2 - \bar{\partial}_2 f_1)\right).$$

It is surprising that in this case the electric field and the magnetic field coincide, or in other words they are mathematically indistinguishable. The same phenomenon occurs later in Example 4.6 on Dirac monopole.

Of course self-duality (resp. anti-self-duality) of the differential form $F_\omega$ does not necessarily require $f_j, f_j, j = 1, 2$ being holomorphic (resp. conjugate holomorphic). For instance, $f_1 = z_1 + \bar{z}_1, f_2 = z_2 + \bar{z}_2, f_1 = \bar{z}_1 - z_1, f_2 = z_2 - z_2$ satisfy (4.9), while $f_1 = z_1 - \bar{z}_2, f_2 = z_2 - \bar{z}_1, f_1 = \bar{z}_1 - z_1, f_2 = z_2 - \bar{z}_2$ satisfy (4.10). The next example exhibits a 1-dimensional electromagnetic dynamics.

Example 4.4. Let $\tau(z) = f_1(dz_1 + d\bar{z}_1) + f_2(dz_2 + d\bar{z}_2)$, where $f_1$ and $f_2$ are holomorphic in $z_1$ and $z_2$. We claim that $F := d\tau$ is self-dual if only if $F = m(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ for some constant $m$. First, it is clear that if $F = m(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ then it is self-dual. Conversely, since $f_1, f_2$ are holomorphic, using the definition of $F$, after some simplification we have

$$F = \partial_1 f_1 dz_1 \wedge d\bar{z}_1 + (\partial_1 f_2 - \partial_2 f_1) dz_1 \wedge dz_2 + \partial_1 f_2 dz_1 \wedge d\bar{z}_2$$

$$+ \partial_2 f_1 dz_2 \wedge d\bar{z}_1 + \partial_2 f_2 dz_2 \wedge d\bar{z}_2.$$  

(4.12)

After setting $F = *F$ and comparing coefficients, it follows that

$$\partial_1 f_1 = \partial_2 f_2, \quad \partial_1 f_2 = 0 = \partial_2 f_1.$$  

(4.13)

The second relation in (4.13) implies $f_1$ is independent of $z_2$, and likewise $f_2$ is independent of $z_1$. From the first relation, since $\partial_1 f_1 = \partial_2 f_2$, $f_1$ and $f_2$ must be constant. Therefore, we
can write \( f_1 = mz_1 + c_1, f_2 = mz_2 + c_2 \) for constants \( m, c_1, c_2 \in \mathbb{C} \). Substituting the conditions in (4.13) into (4.12), we have

\[
F = m (dz_1 \land d\bar{z}_1 + dz_2 \land d\bar{z}_2),
\]

\[
= -2im (dx_0 \land dx_1 + dx_2 \land dx_3). \tag{4.14}
\]

One can read off the electric and magnetic components in (4.14) by comparing with (3.10):

\[
E_1 = B_1 = -2im,
\]

\[
E_2 = B_2 = E_3 = B_3 = 0,
\]

which shows that the electromagnetic dynamics in this case is 1-dimensional.

We now continue with the discussion on the general case that \( f_i, f_j, i = 1, 2 \) are smooth functions on \( \mathbb{C}^2 \). After applying the exterior derivative to (4.8) and some simplification we obtain

\[
d \ast F_\omega = (-\partial_2 \bar{\partial}_1 f_1 + (2 \partial_1 \bar{\partial}_1 + \partial_2 \bar{\partial}_2) f_2 - \partial_1 \bar{\partial}_2 f_1 - \partial_2^2 f_2) dz_1 \land dz_2 \land d\bar{z}_1 \\
+ (- (\partial_1 \bar{\partial}_1 + 2 \partial_2 \bar{\partial}_2) f_1 + \partial_1 \bar{\partial}_2 f_2 + \partial_2^2 f_1 + \partial_1 \bar{\partial}_2 f_2) dz_1 \land dz_2 \land d\bar{z}_2 \\
+ (-\partial_1 \bar{\partial}_2 f_1 - \partial_2^2 f_2 - \bar{\partial}_2 \partial_1 f_1 + (2 \bar{\partial}_1 \partial_1 + \bar{\partial}_2 \partial_2) f_2) dz_1 \land d\bar{z}_1 \land d\bar{z}_2 \\
+ (\partial_1^2 f_1 + \bar{\partial}_1 \partial_2 f_2 - (\bar{\partial}_1 \partial_1 + 2 \bar{\partial}_2 \partial_2) f_1 + \bar{\partial}_1 \partial_2 f_2) dz_2 \land d\bar{z}_1 \land d\bar{z}_2. \tag{4.15}
\]

Applying the Hodge star operator to (4.15) and rearranging terms, we have

\[
\ast d \ast F_\omega = 2 ((\partial_1 \bar{\partial}_1 + 2 \partial_2 \bar{\partial}_2) f_1 - \partial_1 \bar{\partial}_2 f_2 - \partial_2^2 f_1 - \partial_1 \bar{\partial}_2 f_2) dz_1 \\
+ 2 (-\partial_2^2 f_1 - \partial_1 \bar{\partial}_2 f_2 + (\bar{\partial}_1 \partial_1 + 2 \bar{\partial}_2 \partial_2) f_1 - \bar{\partial}_1 \partial_2 f_2) d\bar{z}_1 \\
+ 2 (-\partial_2 \bar{\partial}_1 f_1 + (2 \bar{\partial}_1 \partial_1 + \bar{\partial}_2 \partial_2) f_2 - \partial_1 \partial_2 f_1 - \bar{\partial}_2^2 f_2) dz_2 \\
+ 2 (-\bar{\partial}_1 \partial_2 f_1 - \partial_2^2 f_2 - \bar{\partial}_2 \partial_1 f_1 + (2 \bar{\partial}_1 \partial_1 + \bar{\partial}_2 \partial_2) f_2) d\bar{z}_2. \tag{4.16}
\]

The RHS of (4.16) can be viewed as a complex current form that we may denote by \( J \). For the sake of simplicity, we can rewrite (4.16) as

\[
\ast d \ast F_\omega = J := P_1 dz_1 + P_1 d\bar{z}_1 + P_2 dz_2 + P_2 d\bar{z}_2, \tag{4.17}
\]

where the coefficients \( P_j, P_j, j = 1, 2 \) can be easily read off from (4.16). Likewise, we can write the RHS of (4.16) in terms of real 1-forms in view of (3.9):

\[
J = (P_1 + P_1) dx_0 + i (P_1 - P_1) dx_1 + (P_2 + P_2) dx_2 + i (P_2 - P_2) dx_3.
\]
Again, if \( f_i = \tilde{f}_i, i = 1, 2 \) then \( J \) is real. Here \( P_1 + P_1 \) can be considered as a scalar electric charge density \( \rho \), while the last three coefficients for the above current 1-form correspond to the last three components of the electric current density vector \( J = (\rho, J_1, J_2, J_3) \) in Section 3.1. So to summarize, with smooth complex-valued functions \( f_1, f_2, f_1, f_2 \) in two variables \( z_1, z_2 \), we determined from the complex differential form \( \omega \) in (4.1), a solution \( F_\omega \) (in (4.2)) to a complex analogue of Maxwell’s equations \( dF_\omega = 0 \) and \( \ast d \ast F_\omega = J \).

Let \( \nabla^2 := 4 \left( \partial_1 \bar{\partial}_1 + \partial_2 \bar{\partial}_2 \right) \) be the Laplacian on \( \mathbb{C}^2 \) in the Euclidean metric. Then a complex function \( f \) is said to be \textit{harmonic} if \( \nabla^2 f = 0 \) on \( \mathbb{C}^2 \). The following is the main result of this subsection.

**Theorem 4.5.** Let \( f_j, f_j, j = 1, 2 \) be harmonic functions. Then the complex differential form \( \omega \) is a solution to the source-free Maxwell’s equations in the Euclidean metric if and only if \( \bar{\partial}_1 f_1 + \bar{\partial}_2 f_2 + \partial_1 f_1 + \partial_2 f_2 \) is constant.

**Proof.** First, we can rewrite (4.15) as

\[
d \ast F_\omega = (\nabla^2 f_2 - \partial_2 \bar{\partial}_1 f_1 - \partial_2^2 f_2 - \partial_1 \partial_2 f_1 + \partial_1 \bar{\partial}_2 f_2) dz_1 \wedge d\bar{z}_2 \wedge dz_2 \wedge d\bar{z}_1
\]

\[
+ \left( \nabla^2 f_1 + \partial_1 \bar{\partial}_2 f_2 + \partial_1 \partial_2 f_1 \right) dz_2 \wedge d\bar{z}_2 \wedge dz_2 \wedge d\bar{z}_1
\]

\[
\ast (\nabla^2 f_2 - \partial_1 \bar{\partial}_2 f_1 - \partial_2^2 f_2 - \partial_2 \bar{\partial}_1 f_1) dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_2
\]

\[
\ast \left( \nabla^2 f_1 + \partial_2 \bar{\partial}_1 f_2 + \partial_1 \partial_2 f_1 \right) dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_1 \wedge dz_2
\]

Since \( f_j, f_j, j = 1, 2 \) are harmonic, we have \( \nabla^2 f_j = 0 \) and \( \nabla^2 f_j = 0 \), which implies

\[
d \ast F_\omega = (-\partial_2 \bar{\partial}_1 f_1 - \partial_2^2 f_2 - \partial_1 \partial_2 f_1 - \partial_2 \bar{\partial}_2 f_2) dz_1 \wedge d\bar{z}_2 \wedge dz_2 \wedge d\bar{z}_1
\]

\[
+ (\partial_1 \bar{\partial}_2 f_2 + \partial_1 \partial_2 f_1 + \partial_1 \bar{\partial}_1 f_1) dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_2
\]

\[
+ (\partial_2 \bar{\partial}_1 f_1 - \partial_2^2 f_2 - \partial_2 \bar{\partial}_2 f_2 - \partial_2 \bar{\partial}_2 f_1) dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_2
\]

\[
+ (\partial_2 \bar{\partial}_2 f_1 + \partial_1 \bar{\partial}_2 f_2 + \partial_1 \bar{\partial}_2 f_1) dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_1 \wedge dz_2.
\]

For \( F_\omega \) to satisfy the source-free Maxwell’s equations, we require \( d \ast F_\omega = 0 \), which we may now write in matrix form as

\[
\text{diag}\{-\partial_2, \partial_1, -\bar{\partial}_2, \bar{\partial}_1\} \begin{pmatrix} \bar{\partial}_1 & \bar{\partial}_2 & \partial_1 & \partial_2 \\ \bar{\partial}_1 & \bar{\partial}_2 & \partial_1 & \partial_2 \\ \bar{\partial}_1 & \bar{\partial}_2 & \partial_1 & \partial_2 \\ \bar{\partial}_1 & \bar{\partial}_2 & \partial_1 & \partial_2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_1 \\ f_2 \end{pmatrix} = 0,
\]
where “diag” stands for a diagonal matrix and $0$ denotes the column 4-vector of zeroes. Clearly, the above is true if and only if $\bar{\partial}_1 f_1 + \bar{\partial}_2 f_2 + \partial_1 f_1 + \partial_2 f_2$ is a constant. 

Example 4.6. With $z_1 = x_0 + ix_1$, $z_2 = x_2 + ix_3$, consider the form

$$
\omega(z) = i (x_0 dx_1 - x_1 dx_0 + x_2 dx_3 - x_3 dx_2)
$$

$$
= \frac{1}{2} \left( \eta(z) - \bar{\eta}(z) \right),
$$

where $\eta(z) = \bar{z}_1 dz_1 + \bar{z}_2 dz_2$. In this case, we have $f_1 = \frac{1}{2} \bar{z}_1$, $f_2 = \frac{1}{2} \bar{z}_2$, $f_1 = -\frac{1}{2} z_1$, $f_2 = -\frac{1}{2} z_2$, where $f_j, f_j, j = 1, 2$ are not holomorphic (resp. conjugate holomorphic) but are all clearly harmonic. Moreover, we have the associated curvature form

$$
F_\omega = d\omega = -(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2),
$$

which is self-dual in the Euclidean metric. Now observe that

$$
\partial_1 f_1 - \bar{\partial}_1 f_1 = -1 = \partial_2 f_2 - \bar{\partial}_2 f_2,
$$

$$
\partial_1 f_2 = 0 = \bar{\partial}_1 f_2,
$$

$$
\partial_2 f_1 = 0 = \bar{\partial}_2 f_1.
$$

The above conditions clearly satisfy the self-duality of $F_\omega$. Moreover, $\bar{\partial}_1 f_1 + \bar{\partial}_2 f_2 + \partial_1 f_1 + \partial_2 f_2 = 0$.

The above example is related to the Dirac monopole ([17]), which is a hypothetical magnetic charge. The original idea was proposed in a 1931 paper by Paul Dirac ([3]). Evidently, the above example shows that the existence of Dirac monopoles does not conflict with Maxwell’s equations in vacuum (away from the magnetic monopole). See [3, 15] and the references therein for more background on this rather intriguing subject. A notable fact here is that $E = B$. It is easy to compute that $E = B = (-2i, 0, 0)$, and therefore

$$
\langle E, B \rangle = \frac{1}{2}(|E|^2 + |B|^2) = 4.
$$

Theorem 4.5 leads to easy constructions of non-self-dual solutions to Equations (3.11) and (3.12) in vacuum.

Example 4.7. In fact, a simple working example that satisfies the conditions in Theorem 4.5 but fails the conditions for $F_\omega$ to be self-dual nor anti-self-dual, is $f_1 = 2 \bar{z}_1 - z_2, f_2 = z_1 + 2 \bar{z}_2, f_1 = z_1 + \bar{z}_1, f_2 = z_2 + \bar{z}_2$. 


4.2. Minkowski Metric Case. We believe that much of the work in Section 4.1 can be done in a parallel manner with respect to other bilinear forms $\langle \cdot , \cdot \rangle_g$ on $\mathbb{R}^4$, where $g$ is a nondegenerate constant $4 \times 4$ self-adjoint matrix. But since the Minkowski metric on $\mathbb{R}^{1,3}$ is more conforming with our reality, and it is indeed where the Maxwell’s equations were initially studied, we shall work it out in details in this subsection.

Recall that the d’Alembertian in the Minkowski metric is given by

$$\square = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} = \nabla^2,$$

where again $\nabla^2$ is the Laplacian in $\mathbb{R}^3$. Using the identities

$$\partial_1 = \frac{1}{2} \left( \frac{\partial}{\partial x_0} - i \frac{\partial}{\partial x_1} \right), \quad \bar{\partial}_1 = \frac{1}{2} \left( \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} \right),$$
$$\partial_2 = \frac{1}{2} \left( \frac{\partial}{\partial x_2} - i \frac{\partial}{\partial x_3} \right), \quad \bar{\partial}_2 = \frac{1}{2} \left( \frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_3} \right),$$

we have the following.

**Lemma 4.8.** In terms of complex variables, the d’Alembertian in the Minkowski metric is given by

$$\square = 2 \left( \partial_1^2 + \bar{\partial}_1^2 - 2 \partial_2 \bar{\partial}_2 \right).$$

**Proof.** Using the first two identities in (4.21), note that

$$\partial_1 + \bar{\partial}_1 = \frac{\partial}{\partial x_0}, \quad i (\partial_1 - \bar{\partial}_1) = \frac{\partial}{\partial x_1}.$$ 

Then the first two terms of the d’Alembertian in (4.20) are

$$\left( \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} \right) \left( \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1} \right) = (\partial_1 + \bar{\partial}_1 + i (\partial_1 - \bar{\partial}_1)) (\partial_1 + \bar{\partial}_1 - i (\partial_1 - \bar{\partial}_1))$$

$$= (\partial_1 + \bar{\partial}_1)^2 + (\partial_1 - \bar{\partial}_1)^2$$

$$= 2 \left( \partial_1^2 + \bar{\partial}_1^2 \right).$$

Similarly, using the last two identities in (4.21), it follows that the last two terms of the d’Alembertian in (4.20) are $-4 \partial_2 \bar{\partial}_2$. Therefore, $\square = 2 \left( \partial_1^2 + \bar{\partial}_1^2 - 2 \partial_2 \bar{\partial}_2 \right)$. $\square$

**Definition 4.9.** A function $f$ is said to be $M$-harmonic if $\square f = 0$.

Here the “$M$” refers to the Minkowski metric. It is well-known that $M$-harmonic functions $\psi$ describe waves propagating in $\mathbb{R}^{1,3}$. A smooth 1-form as defined in (4.1) is said to be wavelike if the functions $f_j, \bar{f}_j, j = 1, 2$ are all $M$-harmonic. Likewise, the curvature field $F_\omega$
is said to be wavelike if its coefficient functions $F_{12}$, $F_{21}$, and $F_{jk}$, $j, k = 1, 2$ in (4.4) are all $M$-harmonic. It is easy to check that if $\omega$ is wavelike then $F_\omega$ is also wavelike, for instance,

$$\square F_{12} = \square (\partial_1 f_2 - \partial_2 f_1) = \partial_1 \square f_2 - \partial_2 \square f_1 = 0,$$

and other coefficients are checked similarly. In particular, this implies that $\square E_j = \square B_j = 0, 1 \leq j \leq 3$, i.e., $E$ and $B$ are waves in the space $\mathbb{C}^3$. However, as we will see a bit later, there exists non-wavelike $\omega$ for which $F_\omega$ is wavelike.

Now let $f_j, f_j, j = 1, 2$ be complex smooth functions as before. Then $F_\omega$ is still the same as in (4.2). However in the Minkowski metric, by Example 2.9 we have

$$\star F = \partial_1 f_2 - \bar{\partial}_2 f_1$$

provided $\langle \\bar{\omega} \rangle$ is either 0 or purely imaginary.

In the Minkowski metric on $\mathbb{C}^2$, the curvature form $F_\omega$ is self-dual (resp. anti-self-dual) provided $\star F_\omega = \pm i F_\omega$ (15). So self-duality of $F_\omega$ requires

$$\begin{align*}
\partial_2 f_1 - \bar{\partial}_1 f_2 &= i (\partial_1 f_2 - \partial_2 f_1), \\
\bar{\partial}_1 f_2 - \bar{\partial}_2 f_1 &= i (\partial_1 f_2 - \partial_2 f_1), \\
\partial_2 f_2 - \bar{\partial}_2 f_2 &= i (\partial_1 f_1 - \bar{\partial}_1 f_1).
\end{align*}$$

On the other hand, anti-self-duality of $F_\omega$ requires

$$\begin{align*}
\partial_2 f_1 - \bar{\partial}_1 f_2 &= -i (\partial_1 f_2 - \partial_2 f_1), \\
\bar{\partial}_1 f_2 - \bar{\partial}_2 f_1 &= -i (\partial_1 f_2 - \bar{\partial}_2 f_1), \\
\partial_2 f_2 - \bar{\partial}_2 f_2 &= -i (\partial_1 f_1 - \bar{\partial}_1 f_1).
\end{align*}$$

The following fact is immediate.

**Corollary 4.10.** If $\omega$ is a self-dual or anti-self-dual solution to the Maxwell’s equations in vacuum with respect to the Minkowski metric, then $\langle E, B \rangle$ is either 0 or purely imaginary. In particular, if $\omega$ is a real self-dual or anti-self-dual solution then $\langle E, B \rangle = 0$.

**Proof.** If $\omega$ is self-dual, then (4.25) indicates that

$$F_{21} = i F_{12}, \quad F_{12} = i F_{21}, \quad F_{22} = i F_{11}.$$
Applying these relations to (4.5), one has
\[
\langle \mathbf{E}, \mathbf{B} \rangle = -4i |F_{11}|^2 + 2(1 - i)^2 |F_{12}|^2 + 2(1 + i)^2 |F_{12}|^2
\]
\[
= -4i \left( |F_{11}|^2 + |F_{12}|^2 - |F_{12}|^2 \right).
\]

In the case \( \omega \) is anti-self-dual, parallel computations yield
\[
\langle \mathbf{E}, \mathbf{B} \rangle = 4i \left( |F_{11}|^2 + |F_{12}|^2 - |F_{12}|^2 \right).
\]

If \( \omega \) is real then \( \langle \mathbf{E}, \mathbf{B} \rangle \) is real and therefore it must be equal to 0.

To proceed, as in the previous subsection we assume \( F_\omega \) is neither self-dual nor anti-self-dual in the Minkowski metric. So for the source-free Maxwell’s equations to be satisfied, we require \( d \ast F_\omega = 0 \). In this case, we have
\[
d \ast F_\omega = (\partial_1 \partial_2 f_1 - (\partial_1^2 + \partial_2^2 - \partial_2 \bar{\partial}_2) f_2 + \partial_2 \bar{\partial}_1 f_1 - \partial_1^2 f_2)dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_2
\]
\[
+ (\partial_1 \bar{\partial}_1 f_1 - \bar{\partial}_1 \partial_2 f_2 - (\partial_2^2 - 2\partial_2 \bar{\partial}_2) f_1 - \partial_2 \bar{\partial}_1 f_2)dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_2
d + (\partial_1 \bar{\partial}_2 f_1 - \bar{\partial}_2 \partial_2 f_2 + \bar{\partial}_2 \bar{\partial}_1 f_1 - (\partial_2^2 + \partial_1^2 - \partial_2 \bar{\partial}_2) f_2)dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2
\]
\[
+ (- (\partial_2^2 - 2\partial_2 \bar{\partial}_2) f_1 - \partial_1 \bar{\partial}_2 f_2 + \partial_1 \bar{\partial}_1 f_1 - \partial_2 \partial_2 f_2)dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2.
\]

Observe that for \( f_j, j = 1, 2 \) holomorphic and respectively conjugate holomorphic, equations (4.24) and (4.25) imply that all the coefficients of the 2-forms in \( F_\omega \) are 0. Hence there is no nontrivial self-dual or anti-self-dual solution to the source-free Maxwell equations in this case. However, it follows from the above computation that
\[
d \ast F_\omega = (\partial_1 \partial_2 f_1 - \partial_1^2 f_2)dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_1 + (\bar{\partial}_1 \bar{\partial}_2 f_1 - \bar{\partial}_1^2 f_2)dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2.
\]

Hence \( d \ast F_\omega = 0 \) if and only if
\[
\partial_1 \partial_2 f_1 - \partial_1^2 f_2 = \partial_1 \bar{\partial}_2 f_1 - \bar{\partial}_1^2 f_2 = 0.
\]

Further, if \( f_j = \overline{f_j}, j = 1, 2 \) then the above two equations are the same. One thus obtains the following fact.

**Proposition 4.11.** Let \( f_1 \) and \( f_2 \) be holomorphic functions and \( f_3 = \overline{f_j}, j = 1, 2 \). Then \( \omega \) is a solution to the Maxwell’s equations in vacuum with respect to the Minkowski metric if and only if \( \partial_2 f_1 - \partial_1 f_2 \) is independent of the variable \( z_1 \).
Similar to Proposition 4.2 and Corollary 4.3 in this case we have
\[ F_\omega = (\partial_1 f_2 - \partial_2 f_1) dz_1 \wedge dz_2 + (\overline{\partial_1 f_2} - \overline{\partial_2 f_1}) d\overline{z}_1 \wedge d\overline{z}_2, \]
and consequently,
\[ E_1 = B_1 = 0, \]
\[ E_2 = B_2 = 2\Re((\partial_1 f_2 - \partial_2 f_1)), \]
\[ E_3 = B_3 = -2\Im((\partial_2 f_1 - \partial_1 f_2)), \]
where \(\Re(a)\) and \(\Im(a)\) stand for the real, and respectively, imaginary part of a complex number \(a\). Observe that \(E = B\) in this case, which resembles the Dirac monopole example we examined earlier.

**Example 4.12.** There are plenty of holomorphic functions \(f_1\) and \(f_2\) that satisfy the condition in the above proposition. For instance, let
\[ f_1 = z_1^2 h(z_2) + g(z_2), \quad f_2 = \frac{z_1^3}{3} \partial_2 h(z_2), \]
where \(g\) and \(h\) are arbitrary one-variable entire functions. Then \(\partial_2 f_1 - \partial_1 f_2 = \partial_2 g(z_2)\), which is independent of \(z_1\). Further, since in this case
\[ \Box f_1 = 2h(z_2), \quad \Box f_2 = 2z_1 \partial_2 h(z_2), \]
which can be nonzero, the 1-form \(\omega\) may not be wavelike. Further, it is easy to see that \(\partial_2 g(z_2)\) is M-harmonic and hence \(F_\omega\) is wavelike.

We state this observation as follows.

**Corollary 4.13.** There are real analytic non-wavelike solutions to the Maxwell’s equations in vacuum.

**Remark 4.14.** Corollary 4.3 and the above observations also indicate that, under both the Euclidean metric and the Minkowski metric, the Maxwell’s equations in vacuum have solutions in which the electric field and the magnetic field are mathematically indistinguishable. However, it is not clear if such solutions exist in nature.

Now coming back to our familiar wavelike solutions we have the following fact. Its proof is similar to that of Theorem 4.5
Theorem 4.15. Assume \( \omega \) as in (4.11) is wavelike. Then it is a solution to the Maxwell’s equations in vacuum under the Minkowski metric if and only if \( \partial_1 f_1 - \check{\partial}_2 f_2 + \check{\partial}_1 f_1 - \partial_2 f_2 \) is constant.

Proof. Let \( 0 \) denote the column 4-vector of zeroes. Since \( \omega \) is wavelike, we have \( \Box f_j = \Box f_j = 0, j = 1, 2 \), which means

\[
(\partial_1^2 + \check{\partial}_2^2 - 2 \partial_2 \check{\partial}_2) f_j = 0, \quad (\partial_1^2 + \check{\partial}_1^2 - 2 \check{\partial}_1 \partial_2) f_j = 0.
\] (4.28)

Plugging (4.28) into (4.26) and rearranging terms, we have

\[
d \star F_\omega = (\partial_1 \partial_2 f_1 - \partial_2 \check{\partial}_2 f_2 + \partial_2 \check{\partial}_1 f_1 - \partial_2 \check{\partial}_2 f_2) \, dz_1 \wedge dz_2 \wedge d\bar{z}_1
+ (\partial_1 \check{\partial}_1 f_1 - \check{\partial}_1 \check{\partial}_2 f_2 + \check{\partial}_1 \partial_2 f_1 - \partial_2 \check{\partial}_2 f_2) \, dz_1 \wedge dz_2 \wedge d\bar{z}_2
+ (\partial_2 \check{\partial}_2 f_1 - \partial_2 \check{\partial}_2 f_2 + \partial_2 \check{\partial}_1 f_1 - \partial_2 \check{\partial}_2 f_2) \, dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2
+ (\check{\partial}_1^2 f_1 - \partial_1 \check{\partial}_2 f_2 + \partial_1 \check{\partial}_1 f_1 - \partial_1 \partial_2 f_2) \, dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2.
\]

To satisfy the source-free Maxwell’s equations, we need \( d \star F_\omega = 0 \), which in matrix form is

\[
\text{diag}\{\partial_2, \check{\partial}_1, \check{\partial}_2, \partial_1\} \begin{pmatrix}
\partial_1 & -\check{\partial}_2 & \check{\partial}_1 & -\partial_2 \\
\partial_1 & -\check{\partial}_2 & \check{\partial}_1 & -\partial_2 \\
\check{\partial}_1 & -\check{\partial}_2 & \check{\partial}_1 & -\partial_2 \\
\check{\partial}_1 & -\check{\partial}_2 & \check{\partial}_1 & -\partial_2
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
f_1 \\
f_2
\end{pmatrix} = 0.
\]

Clearly, this is true if and only if \( \partial_1 f_1 - \check{\partial}_2 f_2 + \check{\partial}_1 f_1 - \partial_2 f_2 \) is constant and this completes the proof. \( \square \)

5. On the Lorenz gauge

It was indicated in Section 3.2 that given a smooth 4-vector \( (\phi, A_1, A_2, A_3) \) one can associate with it the magnetic potential 1-form \( \omega = \phi dx_0 - A_1 dx_1 - A_2 dx_2 - A_3 dx_3 \). The Lorenz gauge condition (3.5) stipulates the normalization (3.8) regarding the sum of partial derivatives, namely,

\[
\frac{\partial \phi}{\partial x_0} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = 0.
\]

Theorems 4.5 and 4.15 indeed give a mathematical explanation as to why the Lorenz gauge matters. Here we give a unified treatment.

Corollary 5.1. Let \( \omega \) be a smooth 1-form as defined in (4.11). Then the sum of partial derivatives appearing in Theorem 4.5 is \(-\frac{1}{2} d^* \omega\), and that in Theorem 4.15 is \(\frac{1}{2} d^* \omega\).
Proof. First, recall that in the Euclidean metric over \( \mathbb{C}^2 \) we have that \( d^\ast \omega = - \ast d \ast \omega \). Then using the calculations in Example 2.8 one easily verifies that

\[
\ast \omega = \frac{1}{2} \left( f_1 dz_1 \wedge d\bar{z}_2 + f_1 dz_2 \wedge d\bar{z}_1 + f_2 dz_1 \wedge d\bar{z}_2 - f_2 dz_2 \wedge d\bar{z}_1 - f_3 dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \right).
\]

It follows that

\[
\bar{\partial}_1 f_1 + \bar{\partial}_2 f_2 + \partial_1 f_1 + \partial_2 f_2 = -\frac{1}{2} d^\ast \omega.
\]

The sums of partial derivatives appearing in (3.8) is the real variable version of that in Theorem 4.15. In the Minkowski metric over \( \mathbb{C}^2 \), we have

\[
d^\ast \omega = \ast d \ast \omega.
\]

(5.1)

Under the Minkowski metric, using Example 2.9 we have

\[
\ast \omega = \frac{1}{2} \left( f_1 dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 + f_2 dz_1 \wedge d\bar{z}_2 + f_3 dz_1 \wedge d\bar{z}_2 \right)
\]

(5.2)

Now applying the exterior derivative to (5.2), after simplifying and rearranging terms, we have

\[
d \ast \omega = \frac{1}{2} \left( \partial_1 f_1 - \bar{\partial}_2 f_2 + \bar{\partial}_1 f_1 - \partial_2 f_2 \right) (dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2)
\]

\[
= -2 \left( \partial_1 f_1 - \bar{\partial}_2 f_2 + \bar{\partial}_1 f_1 - \partial_2 f_2 \right) (dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3).
\]

(5.3)

Since \( \ast (dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3) = -1 \) in the Minkowski metric, it follows that

\[
d^\ast \omega = 2 \left( \partial_1 f_1 - \bar{\partial}_2 f_2 + \bar{\partial}_1 f_1 - \partial_2 f_2 \right).
\]

(5.4)

\[\square\]

In the case we write \( \omega \) in the real form \( \phi dx_0 - A_1 dx_1 - A_2 dx_2 - A_3 dx_3 \), then (5.4) implies

\[
d^\ast \omega = - \left( \frac{\partial \phi}{\partial x_0} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \right).
\]

If \( d^\ast \omega \) is a constant, say \( k \), then one can easily modify \( f_i, f_i \) \( i = 1, 2 \) such that \( d^\ast \omega \) becomes 0. For example, in the Euclidean metric we can replace \( f_1 \) by \( f_1 + \frac{k}{2} \bar{z}_1 \) and keep other functions unchanged. Similar modification can be done in the Minkowski metric. In this view, the Lorenz gauge condition is just a trivial strengthening of the condition \( d^\ast \omega \) being constant. Therefore, we shall say that a smooth 1-form \( \omega \) satisfies the Lorenz gauge condition if \( d^\ast \omega \) is constant.
With the foregoing observation, in the Euclidean metric case one can write (4.15) as

\[
\begin{align*}
\ast dF & = \left( 2\nabla^2 f_2 + \frac{1}{2} \partial_2 d^* \omega \right) dz_1 \wedge dz_2 \wedge d\bar{z}_2 - \left( 2\nabla^2 f_1 + \frac{1}{2} \partial_1 d^* \omega \right) dz_1 \wedge dz_2 \wedge d\bar{z}_1 \\
& + \left( 2\nabla^2 f_2 + \frac{1}{2} \partial_2 d^* \omega \right) d\bar{z}_1 \wedge d\bar{z}_2 - \left( 2\nabla^2 f_1 + \frac{1}{2} \partial_1 d^* \omega \right) dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2.
\end{align*}
\]

And in the Minkowski metric case, one can write (4.26) as

\[
\begin{align*}
2\ast dF & = \left( -\Box f_2 + \partial d^* \omega \right) dz_1 \wedge dz_2 \wedge d\bar{z}_2 - \left( -\Box f_1 + \bar{\partial} d^* \omega \right) dz_1 \wedge dz_2 \wedge d\bar{z}_1 \\
& + \left( -\Box f_2 + \bar{\partial} d^* \omega \right) d\bar{z}_1 \wedge d\bar{z}_2 - \left( -\Box f_1 + \partial d^* \omega \right) dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2.
\end{align*}
\]

Moreover, if \( \omega \) is a solution to the Maxwell’s equations, we have \( d^* d\omega = d^* F_\omega = 0 \). Hence \( \Delta \omega = (dd^* + d^* d) \omega = 0 \) if and only if \( dd^* \omega = 0 \), i.e., \( d^* \omega \) is a constant, or in other words \( \omega \) satisfies the Lorenz gauge condition. We summarize Theorem 4.15, Theorem 4.15, and the foregoing observations in the following corollary.

**Corollary 5.2.** Let \( \omega \) be a solution to the Maxwell’s equations in vacuum under the Euclidean or Minkowski metric. Then the following are equivalent.

(a) \( \omega \) satisfies the Lorenz gauge condition.
(b) \( \omega \) is harmonic or respectively wavelike.
(c) \( \omega \) is HL-harmonic.

If \( \omega \) as defined in (4.1) is a solution to the Maxwell’s equations in vacuum under the Euclidean or Minkowski metric, then for every smooth function \( u \) the form \( \omega' = \omega + du \) is also a solution because

\[
F_{\omega'} = d\omega' = d\omega + d^2 u = d\omega = F_\omega.
\]

This is the gauge invariance of the Maxwell’s equations in differential forms. We write the above gauge transformations as

\[
f'_j = f_j + \partial_j u, \quad f'_j = f_j + \bar{\partial}_j u, \quad j = 1, 2.
\]

Then with respect to Theorem 4.15 direct computations give

\[
\bar{\partial}_1 f'_1 + \bar{\partial}_2 f'_2 + \partial_1 f'_1 + \partial_2 f'_2 = \bar{\partial}_1 f_1 + \bar{\partial}_2 f_2 + \partial_1 f_1 + \partial_2 f_2 + \frac{1}{2} \nabla^2 u, \quad (5.5)
\]

and likewise with respect to Theorem 4.15 we have

\[
\begin{align*}
\partial_1 f'_1 - \bar{\partial}_2 f'_2 + \bar{\partial}_1 f'_1 - \partial_2 f'_2 = \partial_1 f_1 - \bar{\partial}_2 f_2 + \bar{\partial}_1 f_1 - \partial_2 f_2 + \frac{1}{2} \Box u.
\end{align*}
\]
It is known that ([4, 8]) for every smooth function \( h \) on \( \mathbb{R}^4 \), the equations

\[
\nabla^2 u = h, \quad \text{and} \quad \Box u = h
\]

both have solutions (non-unique). Hence there exists a smooth function \( u \) such that \( \omega' = \omega + du \) satisfies the Lorenz gauge condition with respect to the Euclidean metric (or the Minkowski metric). Hence by Corollary 5.2 the curvature field \( F_{\omega} = F_{\omega'} = d\omega' \) is harmonic (or respectively wavelike). We summarize this observation as follows.

**Corollary 5.3.** Let \( F_{\omega} \) be a solution to the Maxwell’s equations in vacuum under the Euclidean or Minkowski metric. Then \( F_{\omega} \) is harmonic, or respectively wavelike.

In particular, this indicates that there is no non-wavelike solution to the Maxwell’s equations in vacuum.

6. **Concluding Remarks**

Complex analysis is a core component in mathematics, and it has also played an increasingly important role in modern physics. It is thus meaningful to reinterpret some fundamental theories in physics from a complex perspective, for instance special relativity, Maxwell’s equations, and Yang-Mills equations, whose original formulations were in real variables. This reinterpretation will not only provide a complex formulation of the theories, but also give rise to new and natural observations from this point of view. The exploration in this direction has been made in literature, see for example [10, 12], but it is far from being complete. This paper shall serve as a starting point for the authors to explore greater applications of complex analysis to physics theories.

**Acknowledgments.** The authors would like to thank Marius Beceanu and Oleg Lunin for valuable comments on the initial draft of this paper. This paper is in part based on the first author’s doctoral dissertation ([14]) submitted to SUNY at Albany, and he is grateful to the Department of Mathematics and Statistics for providing him an opportunity to pursue his research interests.

**References**

[1] J. Baez and J. P. Muniain: *Gauge Fields, Knots, and Gravity*, vol. 4, World Scientific, London 1994.

[2] R. W. R. Darling: *Differential Forms and Connections*, 1st ed., Cambridge University Press, New York 1994.
[3] P. A. M. Dirac: Quantised singularities in the electromagnetic field, *Proc. R. Soc. Lond. A, Containing Papers of a Mathematical and Physical Character* **133**, no. 821, 60-72 (1931).

[4] L. C. Evans: *Partial Differential Equations*, 2nd ed., vol. 19, American Mathematical Society, Providence, RI 2010.

[5] B. Felsager: *Geometry, Particles, and Fields*, Springer Science & Business Media, New York 2012.

[6] D. Fleisch: *A Student’s Guide to Maxwell’s Equations*, Cambridge University Press, Cambridge, UK 2008.

[7] T. A. Garrity: *Electricity and Magnetism for Mathematicians: A Guided Path from Maxwell’s Equations to Yang-Mills*, Cambridge University Press, New York 2015.

[8] S. Hassani: *Mathematical Physics: A Modern Introduction to its Foundations*, 2nd ed., Springer Science & Business Media, New York 2013.

[9] D. D. Holm: *Geometric Mechanics: Dynamics and Symmetry*, vol. 1, Imperial College Press, London 2008.

[10] C. Hoyos, N. Sircar, and J. Sonnenschein: New knotted solutions of Maxwell’s equations, *J. Phys. A: Math. Theor.* **48**, no. 25, 255204 (2015).

[11] D. Huybrechts: *Complex Geometry: An Introduction*, Springer Science & Business Media, Heidelberg, Germany 2006.

[12] F. Kleefeld: Complex covariance, *arXiv:1209.3472v1*, 2012.

[13] J. C. Maxwell: Vi. a dynamical theory of the electromagnetic field, *Philos. Trans. R. Soc. Lond.* **155**, 459-512 (1865).

[14] S. Munshi: Maxwell’s equations and Yang-Mills equations in complex variables: New perspectives, ProQuest Dissertations Publishing, 1-69 (2020).

[15] J. L. Pinfold: Dirac’s dream—the search for the magnetic monopole, *AIP Conf. Proc.* **1304**, 234-239 (2010).

[16] R. M. Range: *Holomorphic Functions and Integral Representations in Several Complex Variables*, vol. 108, Springer Science & Business Media, New York 2013.

[17] W. G. Ritter: Gauge theory: Instantons, monopoles, and moduli spaces, *arXiv:math-ph/0304026v1*, 2003.

[18] M. S. Swanson: *Path Integrals and Quantum Processes*, Dover Publications Inc. (Courier Corporation), Mineola, NY 2014.

[19] L. W. Tu: *Differential Geometry, Connections, Curvature, and Characteristic Classes*, Springer, Cham, Switzerland 2017.

(Sachin Munshi) DEPARTMENT OF MATHEMATICS AND STATISTICS, SUNY AT ALBANY, ALBANY, NY 12222, U.S.A.

*Email address*, Sachin Munshi: sacmun86@gmail.com, smunshi@albany.edu
(Rongwei Yang) Department of Mathematics and Statistics, SUNY at Albany, Albany, NY 12222, U.S.A.

Email address, Rongwei Yang: ryan@albany.edu