A Quasi-3D Higher-Order Theory for Bending of FG Nanoplates Embedded in an Elastic Medium in a Thermal Environment

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Abstract: This paper presents the effects of temperature and the nonlocal coefficient on the bending response of functionally graded (FG) nanoplates embedded in an elastic foundation in a thermal environment. The effects of transverse normal strain, as well as transverse shear strains, are considered where the variation of the material properties of the FG nanoplate are considered only in thickness direction. Unlike other shear and deformations theories in which the number of unknown functions is five and more, the present work uses shear and deformations theory with only four unknown functions. The four-unknown normal and shear deformations model, associated with Eringen nonlocal elasticity theory, is used to derive the equations of equilibrium utilizing the principle of virtual displacements. The effects due to nonlocal coefficient, side-to-thickness ratio, aspect ratio, normal and shear deformations, thermal load and elastic foundation parameters, as well as the gradation in FG nanoplate bending, are investigated. In addition, for validation, the results obtained from the present work are compared to ones available in the literature.

Keywords: nonlocal theory; FG nanoplates; thermal load; four-unknown normal and shear deformations theory; elastic foundations

1. Introduction

Nanotechnology is the study of small objects and their applications and has many uses in scientific fields, such as physics, materials science, engineering, chemistry, and biology. For centuries, nanotechnology has been used even though modern nanoscience and nanotechnology are modern. Aifantis [1] discussed the interpretation of size effects using the strain gradient theory. Reddy [2] studied the bending, buckling, and vibration of beams utilizing nonlocal theories. Hashemi and Samaei [3] discussed the buckling of micro/nanoscale plates used the nonlocal elasticity theory. Zenkour and Sobhy [4] discussed the thermal buckling of nanoplates resting on Winkler–Pasternak foundations utilizing the nonlocal elasticity theory. The thermo-mechanical bending and free vibration behavior of single-layered graphene sheets lying on elastic foundations were studied by Sobhy [5].

Functionally graded materials (FGMs) consist of a mixture of metal and ceramic materials, which range from one material to the other following the law of volume fractions of the two materials through the thickness of the nanoplate [6–8]. Due to their distinct physical and thermal properties, the FGMs are preferable in many real-life applications. Maintaining the structural reliability of FGMs in a high thermal gradient environment is one of the advantages of using FGMs [9–13]. Consequently, many studies about the applications FG nanoplates/nanobeams can be found in the literature [14–19]. Zenkour et al. [20,21]
investigated the deflection and stresses of laminated plates resting on Winkler–Pasternak foundations in thermal and hygrothermal environments, respectively.

Due to the importance of designing foundations, various methods have been developed to study the response of beams and plates that are resting on elastic foundations such as Winkler’s soil model [22] and Pasternak’s model [23–28] and many other studies that are available in the literature [29–31]. However, most of the available shear and deformations theories used in the analysis involve five, six, and more unknown functions.

A refined four-unknown higher-order normal and shear deformations theory (RHT) for bending analysis of FG nanoplates embedded in elastic foundations is presented in this work where only four independent known functions are used. The equations of equilibrium are then analytically solved for bending and deflections of simply supported nanoplates to investigate the influence of the nonlocal parameter in which the material properties are influenced by the variation of temperature. The effects of foundation parameters, temperature, transverse normal deformation, plate aspect ratio, side-to-thickness ratio, nonlocal coefficient, and volume fraction on deflections and stresses are also investigated.

2. Geometrical Formulation

A rectangular \((a \times b)\) FG nanoplate is considered with thickness of \(h\), as shown in Figure 1. The FG nanoplate is embedded in an elastic foundation and exposed to a distributed transverse load \(q(x, y)\), as well as temperature \(T(x, y, z)\). According to two gradation models (Equations (1) and (2)), the material properties \(P\) such as the modulus of elasticity \(E\) and the thermal expansion coefficient \(\alpha\) of the FG nanoplate with simply-supported edges in thermal environments, might be assumed:

\[
P_1(z) = P_m + P_{cm}V_{\beta}, \quad V_{\beta} = \left(\frac{2z + h}{2h}\right)^{1/2}, \quad (1)
\]

\[
P_2(z) = P_m\left(\frac{P_c}{P_m}\right)^{1/2}, \quad (2)
\]

where \(P_m\) is the property of the metal, \(P_{cm} = P_c - P_m\), \(P_c\) is the property of the ceramic and \(\beta\) is the FG parameter. In addition, Equations (1) and (2) implies that the upper surface of FG nanoplate \((z = \frac{h}{2})\) is ceramic-rich, while the lower surface \((z = -\frac{h}{2})\) of FG nanoplate is metal-rich. The Poisson’s ratio \(\nu\) is generally assumed constant throughout the plate thickness and equal to 0.3. Based on the two gradation models, the variation of the modulus of elasticity \(E\) across the thickness of FG nanoplate for different values of the parameter \(\beta\) is shown in Figure 2.

![Figure 1. A rectangular FG nanoplates embedded in an elastic medium.](image-url)
2.1. Nonlinear Thermal Conditions

For thermal-structural analysis, only linearly varying across the thickness temperature distribution $T(x, y, z) = T_1(x, y) + \frac{z}{h} T_2(x, y)$ and nonlinear variation through the thickness temperature distribution $T(x, y, z) = \frac{1}{h} \Psi(z) T_3(x, y)$ and a combination of both are defined as $\[20,21\]$

$$T(x, y, z) = T_1(x, y) + \frac{z}{h} T_2(x, y) + \frac{1}{h} \Psi(z) T_3(x, y), \quad (3)$$

where $\Psi(z) = -\frac{z}{4} \left[1 - \frac{5}{3} \left(\frac{z}{h}\right)^2\right]$.

2.2. Displacements and Strains

The in-plane displacements, which are denoted as $v_1$ and $v_2$ and the transverse displacement $v_3$ in FG nanoplate are assumed according to a modified four-unknown normal and shear deformations theory (see in $[32-38]$):

$$v_1(x, y, z) = u - z \partial_x w - \Psi(z) \partial_x \phi,$$
$$v_2(x, y, z) = v - z \partial_y w - \Psi(z) \partial_y \phi,$$
$$v_3(x, y, z) = w + \left[1 + \zeta \Phi(z)\right] \phi. \quad (4)$$

The function $\Psi(z)$ in the present theory should be odd function of $z$ and $\Phi(z) = 1 - \Psi^0$. The prime (′) represent differentiation with respect to $z$. The strain components compatible with the above displacement are given as

$$\begin{align*}
\varepsilon_{xx} &= \varepsilon_x^0 + z \varepsilon_x^1 + \Psi(z) \varepsilon_x^2, \\
\varepsilon_{yy} &= \varepsilon_y^0 + z \varepsilon_y^1 + \Psi(z) \varepsilon_y^2, \\
\gamma_{xy} &= \gamma_{xy}^0 + z \gamma_{xy}^1 + \Psi(z) \gamma_{xy}^2, \\
\gamma_{iz} &= (1 + \zeta) \Phi(z) \gamma_{iz}^0, \\
\varepsilon_{zz} &= -\zeta \Psi^0 \varepsilon_{zz}^0, \quad (i = x, y),
\end{align*} \quad (5)$$
where

\[
\begin{align*}
\epsilon_x^0 &= \partial_x u, \quad \epsilon_y^0 = \partial_y v, \quad \gamma_{xy}^0 = \partial_x v + \partial_y u, \\
\epsilon_x^1 &= -\partial_{xx}^2 w, \quad \epsilon_y^1 = -\partial_{yy}^2 w, \quad \gamma_{xy}^1 = -2\partial_{xy}^2 w, \\
\epsilon_x^2 &= -\partial_{xx}^2 \phi, \quad \epsilon_y^2 = -\partial_{yy}^2 \phi, \quad \gamma_{xy}^2 = -2\partial_{xy}^2 \phi, \\
\gamma_{iz}^0 &= \partial_i \phi, \quad \epsilon_i^0 = \phi, \quad (i = x, y).
\end{align*}
\]

(6)

2.3. Constitutive Equations

For Eringen nonlocal elasticity theory [39–42], the nonlocal constitutive relations of an FG nanoplate in thermal environment are given as

\[
\begin{align*}
\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix} &= \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} \epsilon_{xx} - a(z)T \\ \epsilon_{yy} - a(z)T \\ \epsilon_{zz} - a(z)T \end{pmatrix}, \\
\begin{pmatrix} \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix},
\end{align*}
\]

(7)

in which \( \mathcal{R} = 1 - \mu^2 \nabla^2 \) is the nonlocal operator and \( \mu = c_0 \ell \) is the small scale effect in nanostructures (i.e., the nonlocal coefficient), where \( c_0 \) is a constant and \( \ell \) is an internal characteristic length. The constitutive constants \( c_{ij} \) may be expressed as

\[
\begin{align*}
c_{11}(z) &= c_{22}(z) = c_{33}(z) = \frac{(1 - \nu)E(z)}{(1 - 2\nu)(1 + \nu)}, \\
c_{12}(z) &= c_{13}(z) = c_{23}(z) = \frac{\nu E(z)}{(1 - 2\nu)(1 + \nu)}, \\
c_{ij}(z) &= G(z) = \frac{E(z)}{2 + 2\nu}, \quad (j = 4, 5, 6).
\end{align*}
\]

(8)

2.4. Governing Equations

In this section, we will use the principle of virtual displacements to get the equilibrium equations, that is,

\[
\int_{-h/2}^{h/2} \int_{\Omega} \left[ \sigma_{xx} \delta \epsilon_{xx} + \sigma_{yy} \delta \epsilon_{yy} + \sigma_{zz} \delta \epsilon_{zz} + \sigma_{xy} \delta \gamma_{xy} + \sigma_{yz} \delta \gamma_{yz} + \sigma_{xz} \delta \gamma_{xz} \right] d\Omega dz + \int_{\Omega} V d\Omega = 0,
\]

(9)

where \( V = (\Gamma - q)\delta v_3 \) and \( \Gamma = K_1 v_3 - K_2 \left( \partial_{xx}^2 + \partial_{yy}^2 \right) v_3 \) is the virtual work done by elastic foundations and \( K_1 \) and \( K_2 \) are the Winkler-type and Pasternak-type foundations, respectively. Substitute Equations (5)–(7) into Equation (9) and integrate Equation (9) over the thickness of FG nanoplate:

\[
\int_{\Omega} \left[ N_x \delta \epsilon_x^0 + N_y \delta \epsilon_y^0 + N_z \delta \epsilon_z^0 + N_{xy} \delta \gamma_{xy}^0 + M_x \delta \epsilon_x^1 + M_y \delta \epsilon_y^1 + M_{xy} \delta \gamma_{xy}^1 + S_x \delta \epsilon_x^2 + S_y \delta \epsilon_y^2 + S_{xy} \delta \gamma_{xy}^2 + Q_{xz} \delta \epsilon_{xz}^0 + Q_{yz} \delta \epsilon_{yz}^0 + V \right] d\Omega = 0,
\]

(10)

The stress resultants \( N, M, S, \) and \( Q \) can be expressed as
\[
\begin{pmatrix}
N_x \\
N_y \\
M_x \\
M_y \\
S_x \\
S_y \\
S_z
\end{pmatrix} =
\begin{pmatrix}
D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} & D_{17} \\
D_{22} & D_{23} & D_{24} & D_{25} & D_{26} & D_{27} & D_{28} \\
D_{33} & D_{34} & D_{35} & D_{36} & D_{37} & D_{38} & D_{39} \\
D_{44} & D_{45} & D_{46} & D_{47} & D_{48} & D_{49} & D_{50} \\
D_{55} & D_{56} & D_{57} & D_{58} & D_{59} & D_{60} & D_{61} \\
D_{66} & D_{67} & D_{68} & D_{69} & D_{70} & D_{71} & D_{72}
\end{pmatrix}
\begin{pmatrix}
e_0^1 \\
e_0^2 \\
e_0^3 \\
e_1^1 \\
e_1^2 \\
e_1^3
\end{pmatrix} - \begin{pmatrix}
N_x^T \\
N_y^T \\
M_x^T \\
M_y^T \\
S_x^T \\
S_y^T \\
S_z^T
\end{pmatrix},
\]

(11)

The elements \( D_{ij} \) and \( A_{ij} \) appeared in Equation (11) are given in Appendix A. The thermal stress and moment resultants \( N_x^T, M_x^T \) and \( S_x^T \) are defined by

\[
\begin{aligned}
\{N_x^T, M_x^T, S_x^T\} &= \int_{-h/2}^{h/2} (c_{11} + c_{12} + c_{13})(1, z, \Psi(z))\alpha T \, dz, \\
\{N_y^T, M_y^T, S_y^T\} &= \int_{-h/2}^{h/2} (c_{12} + c_{22} + c_{23})(1, z, \Psi(z))\alpha T \, dz, \\
N_x^T &= -\xi \int_{-h/2}^{h/2} \Psi''(z)(c_{13} + c_{23} + c_{33})\alpha T \, dz.
\end{aligned}
\]

According to Equation (10) the equilibrium equations can be written as

\[
\begin{aligned}
\delta u : \quad \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} &= 0, \\
\delta v : \quad \frac{\partial N_y}{\partial x} + \frac{\partial N_y}{\partial y} &= 0, \\
\delta w : \quad \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_x}{\partial y \partial x} + \frac{\partial^2 M_y}{\partial y^2} + \left(1 - \mu^2 \nabla^2\right)(q - \Gamma) &= 0, \\
\delta \phi : \quad \frac{\partial^2 S_x}{\partial x^2} + 2\frac{\partial^2 S_x}{\partial y \partial x} + \frac{\partial^2 S_y}{\partial y^2} + \frac{\partial Q_{xy}}{\partial y} + \frac{\partial Q_{xy}}{\partial x} - N_z + \left(1 - \mu^2 \nabla^2\right)(q - \Gamma) &= 0.
\end{aligned}
\]

(13)

Substituting Equation (11) into Equation (13) yields a system of simultaneous algebraic equations:

\[
[K] \{\delta\} = \{f\},
\]

(14)

where the elements \( K_{ij} = K_{ji} \) are the differential operators and given in Appendix B. The vector \( \{f\} = \{f_1, f_2, f_3, f_4\}^T \), while \( \{\delta\} = \{u, v, w, \psi\}^T \). The components of the force vector \( \{f\} \) are given as

\[
\begin{aligned}
f_1 &= \partial_x N_x^T, \\
f_2 &= \partial_y N_y^T, \\
f_3 &= \partial_{xx} M_x^T + \partial_{yy} M_y^T - \left(1 - \mu^2 \nabla^2\right)q, \\
f_4 &= \partial_{xx} S_x^T + \partial_{yy} S_y^T + N_z^T - \left(1 - \mu^2 \nabla^2\right)q.
\end{aligned}
\]

(15)

3. Closed-Form Solution

The external force and the thermal loads proposed by Navier are used to solve the operator Equation (14), which are given as
where
\[ q(x, y) = \sum_{m,n=1,3,5,...}^{\infty} q_{mn}(\lambda_m x) \sin(\gamma_n y), \quad q_{mn} = \frac{16q_0}{mn\pi^2}, \]
(16)
\[ T_s = t_s \sin(\lambda_m x) \sin(\gamma_n y), \quad s = 1, 2, 3, \]
and for the simply-supported boundary conditions at the side edges for the FG nanoplate are imposed as
\[ u = v = w = \partial_y \psi = N_x = M_x = S_x = 0 \quad \text{at} \quad x = 0, a, \]
\[ u = v = w = \partial_x \psi = N_y = M_y = S_y = 0 \quad \text{at} \quad y = 0, b. \]
(17)
and \( \lambda_m = \frac{m\pi}{a}, \gamma_n = \frac{n\pi}{b}, t_s \) are constants. At \( m = n = 1 \) then the sinusoidal load is considered and \( q_{11} = q_0 \). According to the given boundary conditions, the Navier solution for \( u, v, w, \) and \( \psi \) is assumed as
\[
\begin{pmatrix}
 u \\
 v \\
 w \\
\psi
\end{pmatrix} =
\begin{pmatrix}
 U_{mn}^1 \cos(\lambda_m x) \sin(\gamma_n y) \\
 U_{mn}^2 \sin(\lambda_m x) \cos(\gamma_n y) \\
 U_{mn}^3 \sin(\lambda_m x) \sin(\gamma_n y) \\
 U_{mn}^4 \sin(\lambda_m x) \sin(\gamma_n y)
\end{pmatrix}
\]
(18)
where \( U_{mn}^1, U_{mn}^2, U_{mn}^3, \) and \( U_{mn}^4 \) are arbitrary parameters. Substituting Equation (18) into Equation (14) leads to a system of simultaneous algebraic equations, which can be expressed in a compact form as
\[ [\mathcal{H}] \{\Delta\} = \{\mathcal{F}\}, \]
(19)
where \( \{\Delta\} \) and \( \{\mathcal{F}\} \) represent the columns:
\[ \{\Delta\} = \{U_{mn}^1, U_{mn}^2, U_{mn}^3, U_{mn}^4\}^T, \]
\[ \{\mathcal{F}\} = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4\}^T, \]
(20)
in which
\[ \mathcal{F}_1 = \lambda_m \sum_{j=1}^{3} \left( e_{ij}^1 t_j \right) \left[ 1 + \mu^2 \left( \lambda_m^2 + \gamma_n^2 \right) \right], \]
\[ \mathcal{F}_2 = \gamma_n \sum_{j=1}^{3} \left( e_{ij}^2 t_j \right) \left[ 1 + \mu^2 \left( \lambda_m^2 + \gamma_n^2 \right) \right], \]
\[ \mathcal{F}_3 = -\left( \sum_{j=1}^{3} \left( \lambda_m^2 \left( e_{ij}^3 t_j \right) + \gamma_n^2 \left( e_{ij}^4 t_j \right) \right) + q_0 \right) \left[ 1 + \mu^2 \left( \lambda_m^2 + \gamma_n^2 \right) \right], \]
\[ \mathcal{F}_4 = -\left( \sum_{j=1}^{3} \left( \lambda_m^2 \left( e_{ij}^3 t_j \right) + \gamma_n^2 \left( e_{ij}^4 t_j \right) \right) - \left( e_{ij}^2 t_j \right) \right) \left[ 1 + \mu^2 \left( \lambda_m^2 + \gamma_n^2 \right) \right]. \]
(21)
The elements \( \mathcal{H}_{ij} = \mathcal{H}_{ji} \) of the coefficient matrix \([\mathcal{H}]\) and \( e_{ij} \) are given in Appendix C.

4. Numerical Results

The numerical results are calculated to verify the accuracy of the present theory in predicting the effects of the nonlocal coefficient on the bending response of the simply-supported FG nanoplates embedded in elastic foundations under thermal load. The upper surface \( (z = \frac{h}{2}) \) of FG nanoplate is Titanium, while the lower surface \( (z = -\frac{h}{2}) \) of FG nanoplate is Zirconia. In the case of mechanical bending, only the nanoplate is made from alumina (Al2O3) and aluminum (Al). Table 1 gives the material properties of the FG nanoplate. For verification purposes, the present outcomes are compared well to various plate theories, and a good agreement is observed. It is found that the best value of \( \xi \) that provides accurate and efficient results is \( \xi = 2/15 \). The following fixed data are
\( q_0 = 100, \beta = 1.5, a = 10h, a = b, t_1 = 0, a = 10nm \) (unless otherwise stated). The following dimensionless deflection, stresses, and foundation parameters are applied as:

\[
\bar{w} = \frac{10^2 D}{q_0 a^4} v_3 \left( \frac{a}{2}, \frac{b}{2}, 0 \right), \quad w^* = \frac{10h^3 E_c}{q_0 a^4} v_3 \left( \frac{a}{2}, \frac{b}{2}, 0 \right), \quad \kappa_1 = \frac{a^4}{D} K_1, \\
\sigma_1 = \frac{h}{q_0 a} \sigma_{xx} \left( \frac{a}{2}, \frac{b}{2}, z \right), \quad \sigma_2 = \frac{h}{q_0 a} \sigma_{yy} \left( \frac{a}{2}, \frac{b}{2}, \frac{h}{2} \right), \quad \kappa_2 = \frac{a^2}{D} K_2, \\
\sigma_4 = -\frac{h}{q_0 a} \sigma_{yx} \left( \frac{a}{2}, 0, 0 \right), \quad \sigma_5 = -\frac{h}{q_0 a} \sigma_{zx} \left( 0, \frac{b}{2}, 0 \right), \quad \sigma_6 = \frac{h}{q_0 a} \sigma_{xy} \left( 0, 0, -\frac{h}{2} \right), \\
\sigma_3 = -\frac{h}{q_0 a} \sigma_{zz} \left( a^2, b^2, z \right).
\]

where \( D = \frac{h^3 E_{12}}{12(1-\nu^2)} \). The deflection and stresses due to the thermal bending for FG nanoplates resting on Winkler–Pasternak foundations are presented. Results are reported in Tables 2–8 and Figures 3–9, where the results in Tables 2–4 and 7 are obtained by using the first gradation model given in Equation (1); however, the results in Tables 5, 6, and 8 and Figures 3–9 are obtained by using the second gradation model given in Equation (2).

| Properties | Mechanical Bending | Thermal Bending |
|------------|--------------------|-----------------|
|            | Aluminum | Alumina | Titanium | Zirconia |
| \( E \) (GPa) | 70        | 380     | 66.2     | 117      |
| \( \nu \) | 0.3       | 0.3     | 1/3      | 1/3      |
| \( \alpha \) \( (10^{-6} / \degree \text{C}) \) | —        | —       | 10.3     | 7.11     |

Table 2. Nondimensionalized deflection \( w^* \) of and the in-plane normal stress \( \sigma_1 (h/3) \) in FG square plates under sinusoidal loads.

| \( \beta \) | Theory | \( a/h = 4 \)  | 10 | 100 | \( a/h = 4 \)  | 10 | 100 |
|-------------|--------|----------------|----|-----|----------------|----|-----|
| 1           | Ref. [43] | 0.729          | 0.589 | 0.563 | 0.806          | 2.015 | 20.150 |
|             | Ref. [44] | 0.717          | 0.588 | 0.563 | 0.622          | 1.506 | 14.969 |
|             | Ref. [45] | 0.700          | 0.585 | 0.562 | 0.593          | 1.495 | 14.969 |
|             | Present | 0.6929         | 0.5685 | 0.5462 | 0.5795          | 1.4647 | 14.549 |
| 4           | Ref. [43] | 1.113          | 0.874 | 0.829 | 0.642          | 1.605 | 16.049 |
|             | Ref. [44] | 1.159          | 0.882 | 0.829 | 0.488          | 1.197 | 11.923 |
|             | Ref. [45] | 1.118          | 0.875 | 0.829 | 0.440          | 1.178 | 11.932 |
|             | Present | 1.0945         | 0.8411 | 0.7933 | 0.4204          | 1.1241 | 11.3919 |
| 10          | Ref. [43] | 1.318          | 0.997 | 0.936 | 0.480          | 1.199 | 11.990 |
|             | Ref. [44] | 1.375          | 1.007 | 0.936 | 0.370          | 0.897 | 8.908  |
|             | Ref. [45] | 1.349          | 0.875 | 0.829 | 0.323          | 1.178 | 11.932 |
|             | Present | 1.3247         | 0.9786 | 0.9139 | 0.3089          | 0.8438 | 8.5898 |
Table 3. Comparison of non-dimensional deflection and stresses of FG square plate under sinusoidal distributed load ($a = 10h$).

| $\beta$ | Theory | $w^*$ | $\sigma_1$ | $\sigma_2$ | $\sigma_6$ | $\sigma_4$ | $\sigma_5$ |
|---------|--------|-------|------------|------------|------------|------------|------------|
| ceramic | Ref. [46] | 0.2960 | 1.9955 | 1.3121 | 0.7065 | 0.2132 | 0.2462 |
| present | 0.2936 | 2.0211 | 1.3240 | 0.6932 | 0.2428 | 0.2731 |
| 1 | Ref. [46] | 0.5889 | 3.0870 | 1.4894 | 0.6110 | 0.2622 | 0.2462 |
| present | 0.5684 | 3.1022 | 1.4647 | 0.5618 | 0.2985 | 0.2731 |
| 2 | Ref. [46] | 0.7573 | 3.6094 | 1.3954 | 0.5441 | 0.2763 | 0.2265 |
| present | 0.7224 | 3.6032 | 1.3509 | 0.4944 | 0.2758 | 0.2202 |
| 3 | Ref. [46] | 0.8377 | 3.8742 | 1.2748 | 0.5525 | 0.2715 | 0.2107 |
| present | 0.7977 | 3.8407 | 1.2218 | 0.5026 | 0.2429 | 0.1837 |
| 4 | Ref. [46] | 0.8819 | 4.0693 | 1.1783 | 0.5667 | 0.2580 | 0.2029 |
| present | 0.8411 | 4.0129 | 1.1241 | 0.5184 | 0.2149 | 0.1647 |
| 5 | Ref. [46] | 0.9118 | 4.2488 | 1.1029 | 0.5755 | 0.2429 | 0.2017 |
| present | 0.8720 | 4.1760 | 1.0510 | 0.5292 | 0.1941 | 0.1569 |
| 6 | Ref. [46] | 0.9356 | 4.4244 | 1.0417 | 0.5803 | 0.2296 | 0.2041 |
| present | 0.8974 | 4.3405 | 0.9934 | 0.5365 | 0.1797 | 0.1556 |
| 7 | Ref. [46] | 0.9562 | 4.5971 | 0.9903 | 0.5834 | 0.2194 | 0.2081 |
| present | 0.9199 | 4.5062 | 0.9460 | 0.5419 | 0.1704 | 0.1575 |
| 8 | Ref. [46] | 0.9750 | 4.7661 | 0.9466 | 0.5856 | 0.2121 | 0.2124 |
| present | 0.9407 | 4.6712 | 0.9062 | 0.5462 | 0.1648 | 0.1608 |
| 9 | Ref. [46] | 0.9925 | 4.9303 | 0.9092 | 0.5875 | 0.2072 | 0.2164 |
| present | 0.9602 | 4.8334 | 0.8723 | 0.5501 | 0.1619 | 0.1648 |
| 10 | Ref. [46] | 1.0089 | 5.0890 | 0.8775 | 0.5894 | 0.2041 | 0.2198 |
| present | 0.9786 | 4.9916 | 0.8438 | 0.5536 | 0.1609 | 0.1689 |
| metal | Ref. [46] | 1.6070 | 1.9955 | 1.3121 | 0.7065 | 0.2132 | 0.2462 |
| present | 1.5938 | 2.0211 | 1.3240 | 0.6932 | 0.2428 | 0.2731 |

Table 4. Comparison of non-dimensional deflection $10\bar{w}$ of square plate subjected to uniformly distributed load.

| $a/lh$ | $\kappa_1$ | $\kappa_2$ | Ref. [47] | Ref. [26] | Present | Ref. [47] | Ref. [26] | Present |
|--------|-------------|-------------|------------|------------|----------|------------|------------|----------|
| $1$ | | | 10 | 5 | 3.3455 | 3.3455 | 3.16463 | 3.2200 | 3.2000 | 3.21954 |
| | | | 15 | 2.3331 | 2.3331 | 2.21865 | 2.2763 | 2.2763 | 2.27599 |
| | | | 20 | 2.0244 | 2.0244 | 1.92843 | 1.9834 | 1.9834 | 1.98315 |
| $3^4$ | | | 10 | 5 | 2.8422 | 2.8421 | 2.69617 | 2.7552 | 2.7552 | 2.75481 |
| | | | 15 | 2.3983 | 2.3983 | 2.28056 | 2.3390 | 2.3390 | 2.33863 |
| | | | 20 | 2.0730 | 2.0730 | 1.97479 | 2.0306 | 2.0306 | 2.03035 |
| | | | 20 | 1.8245 | 1.8244 | 1.74054 | 1.7932 | 1.7932 | 1.79296 |
| $5^4$ | | | 10 | 5 | 1.3785 | 1.3785 | 1.32246 | 1.3688 | 1.3688 | 1.36864 |
| | | | 15 | 1.2615 | 1.2615 | 1.21104 | 1.2543 | 1.2543 | 1.25412 |
| | | | 20 | 1.1627 | 1.1627 | 1.11682 | 1.1572 | 1.1572 | 1.15710 |
| | | | 20 | 1.0782 | 1.0782 | 1.03612 | 1.0740 | 1.0740 | 1.07389 |
Table 5. Effects of the nonlocal coefficient, FG parameter and foundation parameters on the deflection $10 w^*$ of and in-plane normal stress $\sigma_1$ in the FG square nanoplate ($a/h = 10$).

| $\beta$ | Theory | $e_z$ | $(0,0)$ | $(100,0)$ | $(100,100)$ | $(0,0)$ | $(100,0)$ | $(100,100)$ |
|---------|--------|-------|---------|-----------|-------------|---------|-----------|-------------|
| $10 w^*$ | Ref. [48] | $= 0$ | 2.9603 | 2.3290 | 0.4470 | 5.2977 | 3.5671 | 0.4789 |
| present | $\neq 0$ | 2.9359 | 2.3183 | 0.4499 | 5.2539 | 3.5577 | 0.4825 |
| 0.5 | Ref. [48] | $= 0$ | 5.4971 | 3.6564 | 0.4805 | 9.8374 | 5.1752 | 0.4998 |
| present | $\neq 0$ | 5.3352 | 3.5937 | 0.4828 | 9.5477 | 5.1133 | 0.5029 |
| 2.5 | Ref. [48] | $= 0$ | 8.8382 | 4.8847 | 0.4969 | 15.8166 | 6.4599 | 0.5096 |
| present | $\neq 0$ | 8.4675 | 4.7865 | 0.4996 | 15.1532 | 6.3769 | 0.5129 |
| 5.5 | Ref. [48] | $= 0$ | 10.0219 | 5.2259 | 0.5003 | 17.9350 | 6.7874 | 0.5115 |
| present | $\neq 0$ | 9.7162 | 5.1633 | 0.5038 | 17.3878 | 6.7447 | 0.5156 |
| 10.5 | Ref. [48] | $= 0$ | 11.1361 | 5.5135 | 0.5028 | 19.9288 | 7.0545 | 0.5130 |
| present | $\neq 0$ | 10.9327 | 5.4889 | 0.5069 | 19.5648 | 7.0506 | 0.5175 |

The superscript $^*$ denotes $\mu = 0$ and $^{**}$ denotes $\mu = 2$.

Table 6. Effects of the nonlocal coefficient and FG parameter on transverse shear stress $\sigma_5$ and in-plane tangential stress $\sigma_6$ in the FG square nanoplate for different values of the foundation parameters ($a/h = 10$).

| $\beta$ | Theory | $e_z$ | $(0,0)$ | $(100,0)$ | $(100,100)$ | $(0,0)$ | $(100,0)$ | $(100,100)$ |
|---------|--------|-------|---------|-----------|-------------|---------|-----------|-------------|
| $\sigma_5$ | Ref. [48] | $= 0$ | 2.4618 | 1.9368 | 3.0133 | 35.7108 | 2.9664 | 0.3983 |
| present | $\neq 0$ | 2.7311 | 2.1566 | 3.0973 | 36.1685 | 2.9416 | 0.3729 |
| 0.5 | Ref. [48] | $= 0$ | 2.9544 | 1.9725 | 2.5922 | 35.7108 | 2.9664 | 0.3983 |
| present | $\neq 0$ | 3.2017 | 2.0936 | 2.6950 | 36.1685 | 2.9416 | 0.3729 |
| 2.5 | Ref. [48] | $= 0$ | 4.1345 | 2.3121 | 2.3522 | 47.2686 | 3.0577 | 2.4120 |
| present | $\neq 0$ | 4.3104 | 2.3438 | 2.4369 | 47.2686 | 3.0577 | 2.4120 |
| 5.5 | Ref. [48] | $= 0$ | 5.0438 | 2.3004 | 2.5177 | 59.2620 | 3.1591 | 2.5744 |
| present | $\neq 0$ | 5.1957 | 2.3324 | 2.5969 | 59.2620 | 3.1591 | 2.5744 |
| 10.5 | Ref. [48] | $= 0$ | 6.0491 | 2.3261 | 2.7959 | 71.9982 | 3.2753 | 2.8160 |
| present | $\neq 0$ | 5.9369 | 2.3076 | 2.7978 | 71.9982 | 3.2753 | 2.8160 |

The superscript $^*$ denotes $\mu = 0$ and $^{**}$ denotes $\mu = 2$. 

| $\beta$ | Theory | $e_z$ | $(0,0)$ | $(100,0)$ | $(100,100)$ | $(0,0)$ | $(100,0)$ | $(100,100)$ |
|---------|--------|-------|---------|-----------|-------------|---------|-----------|-------------|
| $\sigma_6$ | Ref. [48] | $= 0$ | 10.7450 | 8.4534 | 1.3013 | 19.2289 | 12.9475 | 1.7383 |
| present | $\neq 0$ | 10.5389 | 8.3218 | 1.3615 | 18.8601 | 12.7712 | 1.7322 |
| 0.5 | Ref. [48] | $= 0$ | 4.4493 | 2.9595 | 0.3889 | 7.9624 | 4.1888 | 0.4045 |
| present | $\neq 0$ | 4.1639 | 2.8048 | 0.3768 | 7.4517 | 3.9908 | 0.3925 |
| 2.5 | Ref. [48] | $= 0$ | 7.5813 | 4.1900 | 0.4263 | 13.5671 | 5.5412 | 0.4371 |
| present | $\neq 0$ | 7.0295 | 3.9736 | 0.4147 | 12.5797 | 5.2939 | 0.4258 |
| 5.5 | Ref. [48] | $= 0$ | 8.1778 | 4.2642 | 0.4082 | 14.6345 | 5.5383 | 0.4173 |
| present | $\neq 0$ | 7.7237 | 4.1045 | 0.4005 | 13.8222 | 5.3616 | 0.4098 |
| 10.5 | Ref. [48] | $= 0$ | 8.5915 | 4.2537 | 0.3879 | 15.3751 | 5.4425 | 0.3957 |
| present | $\neq 0$ | 8.2471 | 4.1405 | 0.3824 | 14.7587 | 5.3186 | 0.3903 |

The superscript $^*$ denotes $\mu = 0$ and $^{**}$ denotes $\mu = 2$. 

Table 7. Effects of the FG parameter $\beta$ and thermal loads on the transverse normal stress $\sigma_3$ and transverse shear stress $\sigma_5$ of a sinusoidal distributed loaded FG plate resting on elastic foundations $(a = 10h)$.

| $\beta$ | $t_2$ | $t_3$ | $\kappa_1$ | $\kappa_2$ | $a = b$ | $a = 3b$ | $a = b$ | $a = 3b$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 10 | 0 | 10 | 0 | 0.48746 | 0.38594 | 0.60708 | 0.33318 |
| | | | | | 0.32324 | 0.34652 | 0.94541 | 0.40092 |
| | | | | | 1.28029 | 1.63318 | 5.41912 | 2.17549 |
| | | | | | 1.87987 | 1.81455 | 4.18383 | 1.86379 |
| 50 | 0 | 10 | 0 | 1.27622 | 1.62833 | 5.39699 | 2.14016 |
| 50 | 0 | 10 | 0 | 1.87576 | 1.81042 | 4.16179 | 1.82721 |
| | | | | | 1.4373 | 1.38628 | 3.62410 | 1.68999 |
| | | | | | 1.27622 | 1.62833 | 5.39699 | 2.14016 |
| 3 | 10 | 0 | 10 | 0 | 0.39812 | 0.29807 | 0.52052 | 0.30127 |
| | | | | | 0.23874 | 0.26176 | 0.82659 | 0.36535 |
| | | | | | 0.90245 | 1.22039 | 4.72572 | 1.98281 |
| | | | | | 1.45633 | 1.39903 | 3.61608 | 1.66221 |
| | | | | | 0.91362 | 1.23191 | 4.72064 | 1.95719 |
| 5 | 10 | 0 | 10 | 0 | 0.30534 | 0.23133 | 0.50168 | 0.29444 |
| | | | | | 0.18281 | 0.20146 | 0.80330 | 0.35847 |
| | | | | | 1.26797 | 1.07195 | 3.50757 | 1.62887 |
| | | | | | 0.68188 | 0.93579 | 4.38793 | 1.94566 |
| | | | | | 1.17707 | 1.06916 | 3.50444 | 1.62887 |
| | | | | | 0.67731 | 0.93184 | 4.58685 | 1.92322 |
| 10 | 10 | 0 | 10 | 0 | 0.20486 | 0.15336 | 0.50286 | 0.29515 |
| | | | | | 0.11918 | 0.13195 | 0.81804 | 0.36261 |
| | | | | | 0.74464 | 0.70768 | 3.54313 | 1.66206 |
| | | | | | 0.43925 | 0.61024 | 4.66649 | 1.96915 |
| | | | | | 0.71976 | 0.68308 | 3.54437 | 1.63893 |
| | | | | | 0.41302 | 0.58472 | 4.67273 | 1.94892 |

Table 8. Effects of the nonlocal coefficient and thermal parameters on the deflection $\bar{w}$ and transverse normal stress $\sigma_3$ in the FG square nanoplate embedded in an elastic medium ($\kappa_1 = \kappa_2 = 10$, $a/h = 10$, $\beta = 2$).

| $\mu$ | $t_2$ | $t_3$ | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|---|---|
| $\bar{w}$ | 10 | 10 | 1.21750 | 1.24068 | 1.30230 | 1.38399 | 2.54608 |
| | 50 | 1.21900 | 1.28133 | 1.34504 | 1.42952 | 1.51561 |
| | 20 | 2.11913 | 2.15908 | 2.26507 | 2.40491 | 3.35866 |
| | 50 | 2.15900 | 2.19973 | 2.30779 | 2.45043 | 2.59450 |
| | 50 | 4.82403 | 4.91429 | 5.15336 | 5.46765 | 5.78275 |
| | 100 | 4.86389 | 4.95494 | 5.19609 | 5.51317 | 5.83117 |
| | 100 | 9.33219 | 9.50629 | 9.96717 | 10.57220 | 11.17721 |
| | 50 | 9.37206 | 9.54695 | 10.00990 | 10.61773 | 11.22563 |
| $\sigma_3$ | 10 | 10 | 0.13637 | 0.16288 | 0.24665 | 0.39753 | 0.62948 |
| | 50 | 0.05472 | 0.07783 | 0.15156 | 0.28608 | 0.49165 |
| | 50 | 0.45713 | 0.51432 | 0.69335 | 1.01145 | 1.48607 |
| | 50 | 0.37547 | 0.42928 | 0.59826 | 0.89999 | 1.35223 |
| | 50 | 1.41939 | 1.56865 | 2.03345 | 2.85321 | 4.06783 |
| | 50 | 1.33773 | 1.48360 | 1.93836 | 2.74175 | 3.93399 |
| | 50 | 3.02316 | 3.32587 | 4.26694 | 5.92280 | 8.37076 |
| | 50 | 2.94150 | 3.24082 | 4.17185 | 5.81135 | 8.23692 |
Figure 3. Effects of (a) FG parameter $\beta$ and (b) nonlocal coefficient $\mu$ on the deflection $\bar{w}$ through-the-thickness of the FG square nanoplates embedded in an elastic medium ($a = 10h, t_2 = t_3 = 200, \kappa_1 = \kappa_2 = 10$).

Figure 4. Effects of the nonlocal coefficient and thermal loads versus the side-to-thickness ratio $a/h$ on the deflection $\bar{w}$ of the FG square nanoplates embedded in Winkler elastic medium (a) $t_2 = t_3 = 0$ and (b) $t_2 = t_3 = 50$ ($z/h = 0, \kappa_1 = 10, \kappa_2 = 0$).
Figure 5. Effect of the nonlocal coefficient $\mu$ on the deflection $\bar{w}$ of the FG nanoplate versus aspect ratio $a/b$ (a) $\kappa_1 = \kappa_2 = 0$ and (b) $\kappa_1 = \kappa_2 = 10$ ($z/h = 0, a/h = 10, t_2 = t_3 = 50$).

Figure 6. (a) Effect of the nonlocal coefficient $\mu$ on the deflection $\bar{w}$ ($z/h = 0$) and (b) effect of the thermal loads $t_2$ and $t_3$ on the transverse normal stress $\sigma_3$ through the thickness in FG square nanoplates ($\kappa_1 = \kappa_2 = 0, a/h = 10$).
Figure 7. Effect of the nonlocal coefficient $\mu$ on the transverse normal stress $\sigma_3$ through-the-thickness of FG square nanoplates (a) $t_2 = 100$, $t_3 = 0$ and (b) $t_2 = 0$, $t_3 = 100$ ($a = 10h$, $\kappa_1 = \kappa_2 = 10$).

Figure 8. Effect of the nonlocal coefficient $\mu$ on the in-plane normal stress $\sigma_1$ through-the-thickness of FG square nanoplates (a) $t_2 = 50$, $t_3 = 0$ and (b) $t_2 = t_3 = 50$ ($a = 10h$, $\kappa_1 = \kappa_2 = 10$).
4.1. Comparison Analyses

To check the reliability and accuracy of the present theory and formulations, five comparison studies were carried out (see Tables 2–6). The first comparison analysis is performed between the in-plane normal stress $\sigma_1(h/3)$ and the deflection $w^*(0)$ in the FG square plates obtained using the proposed theory and those obtained by Carrera et al. [43,44] and Neves et al. [45], as shown in Table 2. The present model gives good results compared to Carrera et al. [43,44] and Neves et al. [45].

Table 3 shows the deflection and stresses are compared to those depicted by Thai and Vo [46]. A good agreement is achieved for all the values of the FG parameter $\beta$. As the third example, the deflection $10\bar{w}$ of the square plate under uniformly load is computed and listed in Table 4. The results of the present theories are compared to those presented in Han and Liew [47] and Thai and Choi [26].

The final two comparison analyses (see Tables 5 and 6) are performed between the deflection and stresses obtained by the present theory and the data presented by Sobhy [48] in two cases ($\mu = 0$) and ($\mu = 2$) for the FG square nanoplate embedded in elastic foundations for different values $\beta$. The local plate is more stiffened than the nonlocal one so the nonlocal theory always over predicts the magnitude of stresses and deflection.

4.2. Benchmark Results

Table 7 shows the effects of the FG parameter $\beta$ and thermal loads on stresses of a sinusoidally distributed loaded FG plate lying on elastic foundations. It can be seen that the deviation of the deflection caused by the foundation parameter $\kappa_2$ is greater than that caused by the spring’s parameter $\kappa_1$. The deflection is increasing by increasing the thermal parameters $t_2$ and $t_3$, but it is decreasing by increasing the parameter $\beta$. Table 8 demonstrates the impact of nonlocal parameter $\mu$ and thermal loads on the deflection $\bar{w}$ and transverse normal stress $\sigma_3$ of a sinusoidally distributed loaded FG square nanoplate embedded in an elastic medium ($\kappa_1 = \kappa_2 = 10, a/h = 10$). It is established that the deflection $\bar{w}$ and stress $\sigma_3$ increase by increasing the nonlocal coefficient $\mu$ and the thermal parameters. Due to the increase in thermal parameter $t_3$ only the transverse normal stress $\sigma_3$ decreases.

Effects of (a) FG parameter $\beta$ and (b) nonlocal coefficient $\mu$ on the deflection $\bar{w}$ through-the-thickness of the FG square nanoplates embedded in an elastic medium ($a = 10h$, $t_2 = t_3 = 200$), $(\kappa_1 = \kappa_2 = 10)$, is shown in Figure 3. It is clear that the deflection increases
as the nonlocal coefficient $\mu$ increases but it is decreasing as the FG parameter $\beta$ increases.

Figure 4 displays the effects of the nonlocal coefficient and thermal loads versus the side-to-
thickness ratio $a/h$ on the deflection $\ddot{w}$ of the FG square nanoplates embedded in Winkler
elastic medium (a) $t_2 = t_3 = 0$ and (b) $t_2 = t_3 = 50$ ($z/h = 0, k_1 = 10, k_2 = 0$). The
deflection $\ddot{w}$ is decreasing with the increase of ratio $a/h$, and it is rapidly increasing with
inclusion of the thermal parameters. Figure 5 shows the effect of the nonlocal coefficient
$\mu$ on the deflection $\ddot{w}$ of the FG nanoplate versus aspect ratio $a/b$ (a) $k_1 = k_2 = 0$ and (b) $k_1 = k_2 = 10$ ($z/h = 0, a/h = 10, t_2 = t_3 = 50$). It is clear that the deflection decreases as the
parameters $k_1$ and $k_2$ increase while it increases by increasing the aspect ratio $a/b$ and the
nonlocal coefficient $\mu$. Figure 6 shows (a) the effect of the nonlocal coefficient $\mu$ on the
deflection $\ddot{w}$ $(z/h = 0)$ and (b) effect of the thermal loads $t_2$ and $t_3$ on the transverse normal
stress $\sigma_3$ through the thickness in FG square nanoplates ($k_1 = k_2 = 0, a/h = 10$). The
deflection is linearly directly proportional to the thermal load $t_2$. In addition, the deflection
increases as the thermal load $t_2$ increases; it also increases with the inclusion of the nonlocal
coefficient $\mu$. Figure 7 shows the effect of the nonlocal coefficient $\mu$ on the transverse normal stress $\sigma_3$ through-the-thickness of FG square nanoplates (a) $t_2 = 100, t_3 = 0$ and (b) $t_2 = 0, t_3 = 100$ ($a = 10h, k_1 = k_2 = 10$). The tensile stress $\sigma_3$ occurs along the upper half-
plane, while the compressive stress $\sigma_3$ occurs along the lower half-plane of the FG nanoplate.
The transverse normal stress $\sigma_3$ decreases with the increase of the nonlocal coefficient $\mu$ in the lower half-plane while, it increases with the increase of the nonlocal coefficient $\mu$ in the upper half-plane in the case of neglecting the thermal parameter $t_3$. In the case of neglecting
the thermal parameter $t_2$ the maximum value of the transverse normal stress $\sigma_3$ occurs at the
upper surface of the FG nanoplate. The transverse normal stress $\sigma_3$ increases by increasing
the nonlocal coefficient $\mu$ in the two intervals $0.4 \leq z/h \leq 0.5$ and $-0.4 \leq z/h \leq 0.0$, while it
decreases by increasing the nonlocal coefficient $\mu$ in the two intervals $0.0 \leq z/h \leq 0.4$ and
$-0.5 \leq z/h \leq -0.4$. Figure 8 displays the Effect of the nonlocal coefficient $\mu$ on the in-plane normal stress $\sigma_1$ through-the-thickness of FG square nanoplates (a) $t_2 = 50, t_3 = 0$ and (b) $t_2 = t_3 = 50$ ($a = 10h, k_1 = k_2 = 10$). The tensile stress $\sigma_1$ increases by increasing the nonlocal coefficient $\mu$ in the interval $-0.5 \leq z/h \leq -0.1$, while it decreases by increasing
the nonlocal coefficient $\mu$ in the interval $-0.1 \leq z/h \leq 0.5$, in the case of neglecting the
thermal parameter $t_3$. The tensile stress $\sigma_1$ increases by increasing the nonlocal coefficient $\mu$ in the interval $-0.5 \leq z/h \leq -0.25$ while it decreases by increasing the nonlocal coefficient $\mu$ in the interval $-0.25 \leq z/h \leq 0.5$, in the case of the inclusion of the thermal parameters
$t_2$ and $t_3$.

Finally, Figure 9 shows the effect of (a) the nonlocal coefficient $\mu$ and (b) thermal loads
$t_2$ and $t_3$ on the transverse shear stress $\sigma_0$ of FG square nanoplates ($a = 10h, k_1 = k_2 = 10$).
It is observed that the shear stress $\sigma_0$ increases with the increase in all parameters.

5. Conclusions
A refined plate theory is used for the nonlinear and linear thermal analyses of FG
nanoplates resting on an elastic medium under thermal loading using two power-law
distributions. The present theory shows a satisfaction of the stress boundary conditions on the
upper and lower surfaces of the FG nanoplate by considering both normal and shear
deformations by a higher-order variation of all displacements throughout the thickness.
The effects of the nonlocal coefficient on the material properties, temperature, and the
elastic medium parameters are included in the present numerical results. The effects of
several parameters $\mu, \beta, a/h, a/b, t_2, t_3, k_1$ and $k_2$ are all investigated. The present work
shows a good agreement of the results with the ones available in the literature, which
demonstrates the accuracy of the results along with the simplicity of the present model
in solving the static behavior of the FG nanoplates embedded in an elastic medium in a
thermal environment.
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Appendix A

The elements \( D_{ij} \) and \( A_{ij} \) presented in Equation (11) are given by

\[
\begin{align*}
\{(D_{11}, D_{13}, D_{15}), (D_{12}, D_{14}, D_{16})\} &= \int_{-h/2}^{h/2} (1, z, \Psi(z))\{c_{11}, c_{12}\}dz, \\
\{D_{22}, D_{24}, D_{26}\} &= \int_{-h/2}^{h/2} c_{22}\{1, z, \Psi(z)\}dz, \\
\{D_{44}, D_{45}, D_{46}\} &= \int_{-h/2}^{h/2} z\{zc_{22}, \Psi(z)c_{12}, \Psi(z)c_{22}\}dz, \\
\{(D_{33}, D_{35}), (D_{34}, D_{36})\} &= \int_{-h/2}^{h/2} z(z, \Psi(z))\{c_{11}, c_{12}\}dz, \\
\{D_{55}, D_{56}, D_{66}\} &= \int_{-h/2}^{h/2} \Psi^2(z)\{c_{11}, c_{12}, c_{22}\}dz, \\
D_{17} &= D_{37} = D_{57} = -\xi \int_{-h/2}^{h/2} c_{13}\Psi^h(z)dz, \\
D_{27} &= D_{47} = -\xi \int_{-h/2}^{h/2} zc_{23}\Psi^h(z)dz, \\
\{D_{25}, D_{25}\} &= \int_{-h/2}^{h/2} c_{12}\{z, \Psi(z)\}dz, \\
D_{67} &= -\xi \int_{-h/2}^{h/2} c_{23}\Psi^h(z)dz, \\
D_{77} &= \xi^2 \int_{-h/2}^{h/2} c_{33}\Psi^h(z)dz, \\
\{A_{11}, A_{12}, A_{13}\} &= \int_{-h/2}^{h/2} c_{66}\{1, z, \Psi(z)\}dz, \\
\{A_{22}, A_{23}, A_{33}\} &= \int_{-h/2}^{h/2} c_{66}\{z^2, z\Psi(z), \Psi^2(z)\}dz, \\
\{A_{44}, A_{55}\} &= (\xi^2 + 1) \int_{-h/2}^{h/2} (\Psi^h)^2\{c_{55}, c_{44}\}dz.
\end{align*}
\]

Appendix B

The elements \( K_{ij} = K_{ji} \) presented in Equation (14) are given by
Appendix C

The elements $\mathcal{H}_{ij} = \mathcal{H}_{ji}$ presented in Equation (19) are given by

\[ \mathcal{H}_{11} = -D_{11}\lambda_m^2 - A_{11}\gamma_n^2, \]
\[ \mathcal{H}_{12} = -\lambda_m\gamma_n(D_{12} + A_{11}), \]
\[ \mathcal{H}_{13} = \lambda_m\left[\lambda_m^2 D_{13} + \gamma_n^2(D_{14} + 2A_{12})\right], \]
\[ \mathcal{H}_{14} = \lambda_m\left[\lambda_m^2 D_{15} + \gamma_n^2(D_{16} + 2A_{13}) + D_{17}\right], \]
\[ \mathcal{H}_{22} = -\lambda_m^2 A_{11} - \gamma_n^2 D_{22}, \]
\[ \mathcal{H}_{23} = \gamma_n\left[\lambda_m^2 D_{24} + \lambda_m^2(D_{25} + 2A_{12})\right], \]
\[ \mathcal{H}_{24} = \gamma_n\left[\lambda_m^2 D_{26} + \lambda_m^2(D_{25} + 2A_{13}) + D_{27}\right], \]
\[ \mathcal{H}_{33} = -\gamma_n^2\left[K_2\left(1 + \mu^2\left(\lambda_m^2 + \gamma_n^2\right)\right) + D_{44}\gamma_n^2 + 2\lambda_m^2(D_{34} + 2A_{22})\right] - \lambda_m^2\left(D_{33}\lambda_m^2 + K_2\left(1 + \mu^2\left(\lambda_m^2 + \gamma_n^2\right)\right)\right) - K_1\left(1 + \mu^2\left(\lambda_m^2 + \gamma_n^2\right)\right), \]
\[ \mathcal{H}_{34} = -\gamma_n^2\left[K_2\left(1 + \mu^2\left(\lambda_m^2 + \gamma_n^2\right)\right) + D_{47} + D_{46}\gamma_n^2 + \lambda_m^2(D_{36} + D_{45} + 4A_{23})\right] - \lambda_m^2\left(D_{33}\lambda_m^2 + D_{37} + K_2\left(1 + \mu^2\left(\lambda_m^2 + \gamma_n^2\right)\right)\right) - K_1\left(1 + \mu^2\left(\lambda_m^2 + \gamma_n^2\right)\right), \]
\[ \mathcal{H}_{44} = -\lambda_m^2\left[K_2\left(1 + \mu^2\left(\lambda_m^2 + \gamma_n^2\right)\right) + A_{44} + 2D_{75} + D_{55}\lambda_m^2 + 2\gamma_n^2(D_{56} + 2A_{33})\right] - \lambda_m^2\left[K_2\left(1 + \mu^2\left(\lambda_m^2 + \gamma_n^2\right)\right) + D_{66}\gamma_n^2 + A_{55} + 2D_{67}\right] - K_1\left(1 + \mu^2\left(\lambda_m^2 + \gamma_n^2\right)\right). \]

The elements $e_i^f$ presented in Equation (21) are given by
\{c_{11}, c_{12}, c_{13}\} = \frac{1}{h} \int_{-h/2}^{h/2} \left( c_{11} + c_{12} + c_{13} \right) \{h, z, \Psi(z)\} \alpha(z)dz,
\{c_{21}, c_{22}, c_{23}\} = \frac{1}{h} \int_{-h/2}^{h/2} \left( c_{12} + c_{22} + c_{23} \right) \{h, z, \Psi(z)\} \alpha(z)dz,
\{c_{31}, c_{32}, c_{33}\} = \frac{1}{h} \int_{-h/2}^{h/2} z \left( c_{11} + c_{12} + c_{13} \right) \{h, z, \Psi(z)\} \alpha(z)dz,
\{c_{41}, c_{42}, c_{43}\} = \frac{1}{h} \int_{-h/2}^{h/2} z \left( c_{12} + c_{22} + c_{23} \right) \{h, z, \Psi(z)\} \alpha(z)dz,
\{c_{51}, c_{52}, c_{53}\} = \frac{1}{h} \int_{-h/2}^{h/2} \Psi(z) \left( c_{11} + c_{12} + c_{13} \right) \{h, z, \Psi(z)\} \alpha(z)dz,
\{c_{61}, c_{62}, c_{63}\} = \frac{1}{h} \int_{-h/2}^{h/2} \Psi(z) \left( c_{12} + c_{22} + c_{23} \right) \{h, z, \Psi(z)\} \alpha(z)dz,
\{c_{71}, c_{72}, c_{73}\} = -z \psi \left( c_{13} + c_{23} + c_{33} \right) \{h, z, \Psi(z)\} \alpha(z)dz.

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