PROBABILISTIC GLOBAL WELL-POSEDNESS OF THE VISCOUS NONLINEAR WAVE EQUATION WITH A DEFOCUSING QUINTIC NONLINEARITY

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Abstract. We continue the study of low regularity behavior of the viscous nonlinear wave equation (vNLW) on $\mathbb{R}^2$, initiated by Čanić and the first author (2021). In this paper, we focus on the defocusing quintic nonlinearity and, by combining a parabolic smoothing with a probabilistic energy estimate, we prove almost sure global well-posedness of vNLW for initial data in $H^s(\mathbb{R}^2)$, $s > -\frac{1}{5}$, under a suitable randomization.

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1. Introduction

1.1. Viscous nonlinear wave equation. In this paper, we consider the following nonlinear wave equation (NLW) on $\mathbb{R}^2$, augmented by viscous effects:

$$
\begin{align*}
\begin{cases}
\partial^2_t u - \Delta u + 2\mu D \partial_t u + |u|^{p-1}u = 0 \\
(u, \partial_t u)|_{t=0} = (u_0, u_1)
\end{cases}
\end{align*}
\tag{1.1}
$$

where $\mu > 0$ is a constant, $D = |\nabla| = \sqrt{-\Delta}$, and $\mathbb{R}_+ = [0, \infty)$. In the following, we refer to (1.1) as the viscous nonlinear wave equation (vNLW). Here, the viscosity term comes from the Dirichlet-Neumann operator typically arising in fluid-structure interaction problems in three dimensions.

The viscous nonlinear wave equation arises from a prototypical model for fluid-structure interaction, and models wave dynamics under the influence of viscous regularizing effects. Fluid-structure interaction (FSI) is a physical phenomenon involving the coupled dynamical interaction between a solid and a fluid, where the solid is for instance, deformable with elastic or viscoelastic properties. Such problems feature mathematical difficulties, in terms of the coupling between the solid and fluid equations, and additional geometric nonlinearities that appear in problems in which the fluid domain evolves over time, giving rise to a moving boundary problem. As such, the nonlinear viscous wave equation gives a simplified prototypical model in which one can study well-posedness and ill-posedness of the FSI system, by isolating the effects of fluid viscosity and nonlinear effects on the elastic structure.

The mathematical study of FSI is an area of extensive research, motivated by the presence of numerous physical applications. In recent years, many FSI problems motivated by real-life physical systems have been considered in the mathematics literature, including blood flow in a curved compliant artery with a coronary stent [67], the dynamics of floating objects on bodies of water [39], and a system consisting of an elastic membrane and a gas as a model for problems arising in aeroelasticity [19].

The nature of the coupling between the partial differential equations (PDEs) describing the dynamics of the fluid and the elastodynamics of the structure is crucial in the study of FSI. In mathematical models of FSI, one can consider either linear coupling or nonlinear coupling. The mathematical study of FSI first involved models with the assumption of linear coupling, for example between an elastic solid surrounded by a viscous fluid, as in [24, 1, 2, 38]. Even though FSI problems are often moving boundary problems in which the deformation or displacement of the structure affects the fluid domain so that the fluid domain is not known a priori, linear coupling assumes as an approximation that the fluid-structure interface can be taken to be fixed. Other works have considered the more general case of nonlinear coupling, in which the fluid domain is not known a priori. Thus, one must solve a moving boundary problem, which introduces additional geometric nonlinearities into the problem. See for example, [3, 42, 29, 14, 27, 47, 48, 41, 21, 22, 37, 16, 15, 32, 62, 33, 51, 50, 49, 28, 67].

The viscous nonlinear wave equation, first considered in [35], arises naturally as a prototypical model for linearly coupled fluid-structure interaction between an elastic structure with nonlinear forcing effects and a fluid that isolates the interaction between the viscous effects of the fluid, nonlinear effects, and the hyperbolic dynamics of the elastic structure.
This model features an elastic membrane and a fluid modeled by the stationary Stokes equations, coupled together. Even though this FSI model has two components (the structure and the fluid), the various assumptions in the model, including the linear coupling assumption, give rise to a self-contained equation for the elastodynamics of the structure without reference to the fluid, which is the viscous nonlinear wave equation (1.1).

The viscous nonlinear wave equation (1.1), which arises from this fluid-structure interaction model, is given by the usual nonlinear wave equation, considered for example in [65], augmented by the fractional Laplacian operator $D = \sqrt{-\Delta}$ acting on the structure velocity $\partial_t u$. The operator $D$ arises naturally as the Dirichlet-Neumann operator for the lower half-plane in $\mathbb{R}^d$ (see, for example, [12] for more information about the fractional Laplacian operator). The presence of this operator in the viscous nonlinear wave equation represents the (parabolic) regularizing effects of the fluid viscosity on the structure dynamics. For this reason, we study the viscous nonlinear wave equation (1.1) by making use of both the dispersive and dissipative properties of the dynamics.

We now recall the derivation of the equation, which can be found in full detail in [35]. We consider an infinite elastic prestressed membrane, whose reference configuration is given by the infinite plane

$$\Gamma = \{(x, y, 0) \in \mathbb{R}^3\},$$

interacting dynamically with an incompressible, viscous Newtonian fluid residing in the lower half plane, which we will denote by

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : z < 0\}.$$

See Figure 1. We will make the assumption of linear coupling. This means that even though the elastic structure will displace from its reference configuration $\Gamma$, we will assume for the purposes of the fluid-structure coupling that the fluid-structure interface is fixed at $\Gamma$.

For the structure subproblem, we model the displacement of the elastic structure from its reference configuration $\Gamma$ using the wave equation. We assume that the structure only experiences displacement in the $z$ direction so that the scalar-valued function $u$ denotes the vertical displacement of the structure from $\Gamma$. Thus, the elastic structure is modeled by

$$\partial^2_t u - \Delta u = f, \quad (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}_+,$$  \hspace{1cm} (1.2)

where $f$ is the external loading force on the elastic membrane.

For the fluid subproblem, we model the fluid as an incompressible, viscous Newtonian fluid on the fluid domain $\Omega$, which is fixed in time due to the assumption of linear coupling. In order to isolate the effect of the fluid viscosity on the structure, we consider the case in which viscous effects in the fluid dominate over inertial effects. Hence, we model the fluid velocity $v = (v_1, v_2, v_3)$ and pressure $\pi$ by the stationary Stokes equations for an incompressible, viscous, Newtonian fluid,

$$\begin{cases}
\nabla \pi = \mu \Delta v \\
\n\nabla \cdot v = 0
\end{cases} \quad \text{on } \Omega,$$  \hspace{1cm} (1.3)

where the constant, $\mu > 0$, denotes the fluid viscosity. See Figure 1.

We couple the structure and fluid subproblems using two coupling conditions:

(1) The *kinematic coupling condition* is a no-slip condition which states that particles of the fluid on the boundary, assumed to be fixed by the linear coupling assumption,
move with the same velocity as the corresponding point on the structure. Since the structure displaces only in the \( z \) direction, the kinematic coupling condition reads

\[
v = (\partial_t u) e_z, \quad \text{on } \Gamma,
\]

where \( e_z = (0, 0, 1) \).

2. The dynamic coupling condition gives the effect of the fluid load on the structure, hence explicitly giving the external loading force \( f \) in (1.2) in terms of the Cauchy stress tensor of the fluid,

\[
\sigma = -\pi \text{Id} + 2\mu D(v),
\]

and any additional forces on the structure, which we denote by \( F_{\text{ext}} \). The dynamic coupling condition specifically states that

\[
f = -\sigma e_z \cdot e_z |_{\Gamma} + F_{\text{ext}},
\]

so that the structure equation with the dynamic coupling condition reads

\[
\partial_t^2 u - \Delta u = -\sigma e_z \cdot e_z |_{\Gamma} + F_{\text{ext}}.
\]

The nonlinear viscous wave equation is derived by using the stationary Stokes equations for the fluid (1.3) along with the kinematic coupling condition (1.4) and the dynamic coupling condition (1.5) to express the term \(-\sigma e_z \cdot e_z\) in the equation (1.6) entirely in terms of \( u \) and its derivatives, which will hence give a self-contained equation for the dynamics of the structure without reference to the fluid. To do this, we note that

\[
-\sigma e_z \cdot e_z |_{\Gamma} = \left( \pi - 2\mu \frac{\partial v_3}{\partial z} \right) |_{\Gamma} = \pi |_{\Gamma}.
\]

Here, the second equality follows from

\[
\frac{\partial v_3}{\partial z} |_{\Gamma} = 0,
\]
which is a consequence of the incompressibility condition and the fact that \( v_1 = v_2 = 0 \) on \( \Gamma \). Hence, we obtain
\[
\partial_t^2 u - \Delta u = \pi|_{\Gamma} + F_{\text{ext}}. \tag{1.8}
\]
One can then solve the stationary Stokes equations (1.3) with the boundary condition (1.4) on \( \Gamma \) for \( \pi \) using a Fourier transform argument, to obtain the final result that
\[
\pi|_{\Gamma} = -2\mu \partial_t u. \tag{1.9}
\]
We will only sketch the main steps of this derivation in the following, and refer readers to the full derivation in [35] for the remaining details.

From (1.8), we see that the goal is to express \( \pi|_{\Gamma} \) in terms of the structure displacement \( u \) and its derivatives. We use the fact that \( \pi \) and \( \nu \) satisfy the stationary Stokes equations (1.3) with a boundary condition provided by the kinematic coupling condition (1.4). We impose a boundary condition at infinity that \( \nu \) is bounded and \( \pi \) decreases to zero at infinity. By taking the divergence of the first equation in the stationary Stokes system (1.3) and using the incompressibility condition on \( \nu \), we observe that \( \pi \) is a harmonic function on \( \Omega \). Furthermore, by taking an inner product of the first equation in (1.3) with \( e_z \) and evaluating on \( \Gamma \), we have
\[
\partial \pi \partial_z \bigg|_{\Gamma} = \left( \mu \Delta_{x,y} v_3 + \mu \partial^2 v_3 \partial z^2 \right) \bigg|_{\Gamma}. \tag{1.10}
\]
Hence, we can find \( \pi|_{\Gamma} \) in (1.9) by inverting the Dirichlet-Neumann operator on the lower half space \( \Omega \).

Our main goal is to express \( v_3 \) in terms of \( u \) and its derivatives, as this will give the Neumann boundary condition for the harmonic function \( \pi \) in (1.10). By taking the Laplacian of (1.10) and recalling that \( \pi \) is harmonic, we see that \( v_3 \) satisfies the biharmonic equation:
\[
\Delta^2 v_3 = 0 \tag{1.11}
\]
with boundary conditions given by the kinematic boundary condition (see (1.4)):
\[
\nu_3|_{\Gamma} = \partial_t u \tag{1.12}
\]
and (1.7). Furthermore, there is a boundary condition at infinity that \( v_3 \) must be bounded in the lower half space \( \Omega \).

By taking the Fourier transform of (1.11) in the \( x \) and \( y \) variables but not the \( z \) variable, and solving the resulting ODE in \( z \) together with the boundary conditions mentioned above, we obtain
\[
\hat{v}_3(\xi, z) = \hat{\partial_t u}(\xi)e^{i|\xi|z} - i|\xi|\hat{\partial_{\xi} u}(\xi)ze^{i|\xi|z}, \tag{1.13}
\]
where \( \xi \) denotes the frequency corresponding to the \( x \) and \( y \) variables. For more details, see the explicit calculation in [35]. Then, by taking the Fourier transform of (1.10) in the \( x \) and \( y \) variables together with (1.12) and (1.13), we obtain
\[
\left. \frac{\partial \pi}{\partial z} \right|_{\Gamma} = -2\mu |\xi|^2 \hat{\partial_{\xi} u}(\xi). \tag{1.14}
\]
By taking the inverse Fourier transform, this gives the Neumann boundary condition for the harmonic function \( \pi \) in terms of (derivatives) of \( u \). Recall that the Dirichlet-Neumann operator for the lower half plane with the vanishing boundary condition at infinity is given by \( D = \sqrt{-\Delta} \); see [12]. By inverting this operator, we see that the Neumann-Dirichlet operator with the same boundary condition at infinity is given by the Riesz potential.
\[ D^{-1} = (-\Delta)^{-\frac{1}{2}} \] with a Fourier multiplier \(|\xi|^{-1}\). Therefore, by applying the Neumann-Dirichlet operator to (the inverse Fourier transform of) (1.14), we obtain the desired result in (1.9). We remark that the use of the biharmonic equation to study properties of the stationary Stokes equations is a classical approach; see, for example, [63] for more details.

The nonlinear viscous wave equation incorporates nonlinear effects on the structure. This is motivated by the fact that nonlinear effects in FSI systems appear in physical applications, such as in the study of blood flow in a curved compliant artery with a stent in [67], in which nonlinear elastic effects appear due to the curved cylindrical geometry of the artery, in the form of a cubic nonlinearity. In addition, such external forcing on the structure in fluid-structure interaction has been described in the context of compliant arteries in the human cardiovascular system, as forces exerted by surrounding tissues on arterial walls [46]. As a model for nonlinear restoring external forcing effects, we consider a defocusing power nonlinearity of the form

\[ F_{\text{ext}}(u) = -|u|^{p-1}u, \]  

for positive integers \(p > 1\). Such a power-type nonlinearity has been studied extensively for dispersive equations such as the nonlinear Schrödinger equations and the nonlinear wave equations; see, for example, [65]. Combining (1.8), (1.9), and (1.15) gives the final form of the viscous nonlinear wave equation, as stated in (1.1), in dimension \(d = 2\). Although \(d = 2\) corresponds to the situation described in this fluid-structure interaction model, the equation (1.1) can be stated in full generality for arbitrary dimension \(d\).

Let us now turn to analytical aspects of the viscous NLW (1.1). When \(\mu \geq 1\), this equation is purely parabolic, where the general solution to the homogeneous linear equation

\[ \partial_t^2 u - \Delta u + 2\mu \partial_t u = 0 \]

with initial data \((u, \partial_t u)|_{t=0} = (u_0, u_1)\), is given by

\[ u(t) = e^{-\mu|\nabla|\sqrt{\mu^2 - 1}} f_1 + e^{-\mu|\nabla|\sqrt{\mu^2 - 1}} f_2. \]

Noting that \(-\mu|\xi| + \sqrt{\mu^2 - 1}|\xi|^2 \sim \mu^{-1}|\xi|\) in this case \((\mu \geq 1)\), the solution theory can be studied by simply using the Schauder estimate for the Poisson kernel (see Lemma 2.5 below) in a straightforward manner. We will not pursue this direction in this paper. Instead, our main interest in this paper is to study the combined effect of the dissipative-dispersive mechanism, appearing in (1.1). As such, we will restrict our attention to \(0 < \mu < 1\) in the following. Without loss of generality, we set \(\mu = \frac{1}{2}\) as in [35] and focus on the following version of vNLW:

\[
\begin{cases}
\partial_t^2 u - \Delta u + D \partial_t u + |u|^{p-1}u = 0 \\
(u, \partial_t u)|_{t=0} = (u_0, u_1).
\end{cases}
\]  

As in the case of the usual NLW:

\[ \partial_t^2 u - \Delta u + |u|^{p-1}u = 0, \]  

the viscous NLW in (1.16) enjoys the following scaling symmetry. If \(u(x, t)\) is a solution to (1.16), then \(u^\lambda(x, t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda t)\) is also a solution to (1.16) for any \(\lambda > 0\). This
induces the critical Sobolev regularity $s_{\text{crit}}$ on $\mathbb{R}^d$ given by

$$s_{\text{crit}} = \frac{d}{2} - \frac{2}{p - 1}$$

such that the homogeneous Sobolev norm on $\mathbb{R}^2$ remains invariant under this scaling symmetry. This scaling heuristics provides a common conjecture that an evolution equation is well-posed in $H^s$ for $s > s_{\text{crit}}$, while it is ill-posed for $s < s_{\text{crit}}$. Indeed, for many dispersive PDEs, ill-posedness below a scaling critical regularity is known. In particular, the following form of strong ill-posedness, known as *norm inflation*, is established for many dispersive PDEs, including NLW; see [18, 10, 13, 34, 52, 59, 17, 60, 66, 56, 25]. Norm inflation in the case of the wave equation on $\mathbb{R}^d$ states the following; given any $\varepsilon > 0$, there exist a solution $u$ to (1.17) and $t_\varepsilon \in (0, \varepsilon)$ such that

$$\| (u, \partial_t u)(0) \|_{H^s} < \varepsilon \quad \text{but} \quad \| (u, \partial_t u)(t_\varepsilon) \|_{H^s} > \varepsilon^{-1},$$

where

$$H^s(\mathbb{R}^d) = H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d).$$

In [35], Čanić and the first author studied this issue for $v$NLW (1.16). Due to the presence of the viscous term in (1.16), which induces some smoothing property, one may expect to have a different ill-posedness result but this was shown not to be the case. More precisely, Čanić and the first author proved norm inflation for $v$NLW (1.16) in $H^s(\mathbb{R}^d)$ for $0 < s < s_{\text{crit}}$ (for any odd integer $p \geq 3$) as in the case of the usual NLW. Moreover, they showed that the viscous contribution has the potential to slow down the speed of the norm inflation. See [35] for details. It is of interest to see if norm inflation in negative Sobolev spaces for the usual NLW [18, 56, 25] carries over to the viscous NLW. See [23].

This norm inflation for $v$NLW (1.16) shows that the equation is ill-posed in $H^s(\mathbb{R}^d)$ for $0 < s < s_{\text{crit}}$, showing that there is no hope in studying well-posedness in this low regularity in a deterministic manner. There is, however, a recent growing interest in studying nonlinear dispersive equations with randomized initial data [8, 10, 20, 45, 11, 5, 4, 61, 53, 6], which allows us to go beyond the limit of deterministic analysis. See also a survey paper [7] in this direction. In [35], Čanić and the first author considered the Cauchy problem (1.16) with $p = 5$ and dimension $d = 2$, and proved almost sure local well-posedness for randomized initial data in $H^s(\mathbb{R}^2)$ for $s > -\frac{1}{6}$. Our main goal in this paper is to extend globally in time these random solutions constructed in [35]. In the next subsection, we first go over a specific randomization that we consider for our problem.

### 1.2. Wiener randomization

Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\text{supp} \psi \subset [-1, 1]^d$, $\psi(-\xi) = \overline{\psi(\xi)}$, and

$$\sum_{n \in \mathbb{Z}^d} \psi(\xi - n) \equiv 1 \quad \text{for all } \xi \in \mathbb{R}^d.$$

Then, any function $f$ on $\mathbb{R}^d$ can be written as

$$f = \sum_{n \in \mathbb{Z}^d} \psi(D - n)f,$$

where $\psi(D - n)$ denotes the Fourier multiplier operator with symbol $\psi(\cdot - n)$. Hence, $\psi(D - n)f$ localizes $f$ in the frequency space around the frequency $n \in \mathbb{Z}^d$ over a unit scale.
We recall a particular example of Bernstein’s inequality:
\[
\|\psi(D - n)f\|_{L^p}\|L^q}\|_p \lesssim \|\psi(D - n)f\|_{L^p}\|L^q}\|_p
\] (1.19)
for any \(1 \leq p \leq q \leq \infty\). This classical inequality follows from the localization in the frequency space due to the compact support of \(\psi\), and Young’s convolution inequality (see, for example, Lemma 2.1 in [15]).

We now introduce a randomization adapted to the uniform decomposition (1.18). For \(j = 0, 1\), let \(\{g_{n,j}\}_{n \in \mathbb{Z}^d}\) be a sequence of mean-zero complex-valued random variables on a probability space \((\Omega, \mathcal{F}, P)\) such that
\[
g_{-n,j} = \overline{g_{n,j}}
\] (1.20)
for all \(n \in \mathbb{Z}^d, j = 0, 1\). In particular, \(g_{0,j}\) is real-valued. Moreover, we assume that \(\{g_{0,j}, \text{Re} \, g_{0,j}, \text{Im} \, g_{0,j}\}_{n \in \mathcal{I}, j=0,1}\) are independent, where the index set \(\mathcal{I}\) is defined by
\[
\mathcal{I} = \bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{Z}_+ \times \{0\}^{d-k-1}.
\]
Note that \(\mathbb{Z}^d = \mathcal{I} \cup (-\mathcal{I}) \cup \{0\}\). Then, given a pair \((u_0, u_1)\) of functions on \(\mathbb{R}^d\), we define the Wiener randomization \((u_0^\omega, u_1^\omega)\) of \((u_0, u_1)\) by
\[
(u_0^\omega, u_1^\omega) = \left(\sum_{n \in \mathbb{Z}^d} g_{n,0}(\omega)\psi(D - n)u_0, \sum_{n \in \mathbb{Z}^d} g_{n,1}(\omega)\psi(D - n)u_1\right).
\] (1.21)
See [68 [15] [5] [4]. We emphasize that thanks to (1.20), this randomization has the desirable property that if \(u_0\) and \(u_1\) are real-valued, then their randomizations \(u_0^\omega\) and \(u_1^\omega\) defined in (1.21) are also real-valued.

We make the following assumption on the probability distributions \(\mu_{n,j}\) for \(g_{n,j}\); there exists \(c > 0\) such that
\[
\int e^{\gamma \cdot x} d\mu_{n,j}(x) \leq e^{c|\gamma|^2}, \quad j = 0, 1,
\] (1.22)
for all \(n \in \mathbb{Z}^d\), (i) all \(\gamma \in \mathbb{R}\) when \(n = 0\), and (ii) all \(\gamma \in \mathbb{R}^2\) when \(n \in \mathbb{Z}^d \setminus \{0\}\). Note that (1.22) is satisfied by standard complex-valued Gaussian random variables, standard Bernoulli random variables, and any random variables with compactly supported distributions.

It is easy to see that, if \((u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^d)\) for some \(s \in \mathbb{R}\), then the Wiener randomization \((u_0^\omega, u_1^\omega)\) is almost surely in \(\mathcal{H}^s(\mathbb{R}^d)\). Note that, under some non-degeneracy condition on the random variables \(\{g_{n,j}\}\), there is almost surely no gain from randomization in terms of differentiability (see, for example, Lemma B.1 in [10]). Instead, the main feature of the Wiener randomization (1.21) is that \((u_0^\omega, u_1^\omega)\) behaves better in terms of integrability. More precisely, if \(u_j \in L^2(\mathbb{R}^d), j = 0, 1\), then the randomized function \(u_j^\omega\) is almost surely in \(L^p(\mathbb{R}^d)\) for any finite \(p \geq 2\). See [5].

1.3. Main results. In the remaining part of this paper, we restrict our attention to \(\mathbb{R}^2\), as this is the dimension corresponding to the fluid-structure interaction problem from which the viscous nonlinear wave equation arises. Fix \((u_0, u_1) \in \mathcal{H}^{s}(\mathbb{R}^2)\) for some \(s \in \mathbb{R}\) and let
(\(u_0^\omega, u_1^\omega\)) denote the Wiener randomization of \((u_0, u_1)\) defined in (1.21). We then consider the following defocusing quintic vNLW on \(\mathbb{R}^2\) with the random initial data:

\[
\begin{aligned}
&\partial_t^2 u - \Delta u + D\partial_t u + v^5 = 0 \\
&(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)
\end{aligned}
\]  

(1.23)

In [35], Čanić and the first author proved almost sure local well-posedness of (1.23) for \(s > -\frac{1}{5}\). The following theorem extends this almost sure local well-posedness result to \(s > -\frac{1}{5}\).

**Theorem 1.1.** Let \(s > -\frac{1}{5}\). Then, the quintic vNLW (1.23) is almost surely locally well-posed with respect to the Wiener randomization \((u_0^\omega, u_1^\omega)\) as initial data. More precisely, there exist \(C, c, \gamma > 0\) and \(0 < T_0 \ll 1\) such that for each \(0 < T \leq T_0\), there exists a set \(\Omega_T \subset \Omega\) with the following properties:

(i) \(P(\Omega_T^c) < C \exp\left(-\frac{c}{T^\gamma \| (u_0, u_1) \|_{H^s}}\right)\),

(ii) For each \(\omega \in \Omega_T\), there exists a (unique) local-in-time solution \(u\) to (1.23) with \((u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)\) in the class

\[
V(t)(u_0^\omega, u_1^\omega) + C([0, T]; H^{s_0}(\mathbb{R}^2)) \cap L^{5+\delta}([0, T]; L^{10}(\mathbb{R}^2))
\]

for some \(s_0 = s_0(s) > \frac{3}{5}\), sufficiently close to \(\frac{3}{5}\), and small \(\delta > 0\) such that \(s_0 \geq 1 - \frac{1}{5+\delta} - \frac{2}{10}\). Here, \(V(t)\) denotes the linear propagator for the viscous wave equation defined in (2.2).

**Remark 1.2.** Let \(k_0\) be the smallest integer such that \(k_0 \geq T_0^{-1}\). Then, by setting

\[
\Sigma = \bigcup_{k=k_0}^{\infty} \Omega_{k-1},
\]

we have (i) \(P(\Sigma) = 1\) and (ii) for each \(\omega \in \Sigma\), there exist a (unique) local-in-time solution \(u\) to (1.23) with \((u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)\) on the time interval \([0, T_\omega]\) for some \(T_\omega > 0\). More specifically, for \(\omega \in \Omega_{k-1}\), the random local existence time \(T_\omega\) is given by \(T_\omega = k^{-1}\).

As in [35], the proof of Theorem 1.1 is based on the first order expansion [8] [10] [20] [5] [4] [35]:

\[
u = z + v,
\]

where \(z = z^\omega\) denotes the random linear solution given by

\[
z(t) = V(t)(u_0^\omega, u_1^\omega).
\]

(1.25)

Then, we can rewrite (1.23) as

\[
\begin{aligned}
&\partial_t^2 v - \Delta v + D\partial_t v + (v + z)^5 = 0 \\
&(v, \partial_t v)|_{t=0} = (0, 0)
\end{aligned}
\]  

(1.26)

and we study the fixed point problem (1.26) for \(v\). In [35], the proof of almost sure local well-posedness was based on the Strichartz estimate for the viscous wave equation (Lemma 2.1) with the diagonal Strichartz space \(L^{6}([0, T]; L^{6}(\mathbb{R}^2))\). In proving Theorem 1.1, we instead rely on the Schauder estimate for the Poisson kernel (Lemma 2.5) and work in a non-diagonal space \(L^{5+\delta}([0, T]; L^{10}(\mathbb{R}^2))\). See Section 3 for details.
Next, we turn our attention to the global-in-time aspect.

**Theorem 1.3.** Let $s > -\frac{1}{5}$. Then, the defocusing quintic vNLW (1.23) is almost surely globally well-posed with respect to the Wiener randomization $(u_0^\omega, u_1^\omega)$ as initial data. More precisely, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) = 1$ such that, for each $\omega \in \Sigma$, there exists a (unique) global-in-time solution $u$ to (1.23) with $(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)$ in the class

$$V(t)(u_0^\omega, u_1^\omega) + C(\mathbb{R}_+; H^{s_0}(\mathbb{R}^2))$$

for some $s_0 > \frac{3}{5}$.

Here, the uniqueness holds in the following sense. Given any $t_0 \in \mathbb{R}_+$, there exists a random time interval $I(t_0, \omega) \ni t_0$ such that the solution $u = u^\omega$ constructed in Theorem 1.3 is unique in

$$V(t)(u_0^\omega, u_1^\omega) + C(I(t_0, \omega); H^{s_0}(\mathbb{R}^2)) \cap L^{5+\delta}(I(t_0, \omega); L^{10}(\mathbb{R}^2)),$$

where $s_0 > \frac{3}{5}$ and $\delta > 0$ are as in Theorem 1.1.

The main idea of the proof of Theorem 1.3 is based on the energy estimate as in [10, 57]. With $\vec{v} = (v, \partial_t v)$, a smooth solution $\vec{v}$ to the defocusing vNLW (1.26) (with $z \equiv 0$ and general initial data) satisfies monotonicity of the energy (for the usual NLW):

$$E(\vec{v}) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_t v)^2 dx + \frac{1}{6} \int_{\mathbb{R}^2} v^6 dx. \quad (1.27)$$

Indeed a simple integration by parts with (1.26) (with $z \equiv 0$) shows

$$\partial_t E(\vec{v}) = -\|\partial_t v\|_{H^1}^2 \leq 0.$$

For our problem, we proceed with the first order expansion (1.24) and thus the residual term $v = u - z$ only satisfies the perturbed vNLW (1.26). As such, the monotonicity of the energy $E(\vec{v})$ no longer holds. Nonetheless, by using the time integration by parts trick introduced by the second author and Pocovnicu [57], we establish a Gronwall type estimate for $E(\vec{v})$ to prove almost sure global well-posedness.

One important point to note is that as it is written, the local theory (Theorem 1.1) does not provide a sufficient regularity (i.e. $H^1(\mathbb{R}^2)$) for $\vec{v}$ to guarantee finiteness of the energy $E(\vec{v})$. By using the Schauder estimate (Lemma 2.5), however, we can show that the residual term $\vec{v}(t)$ is smoother and indeed lies in $H^1(\mathbb{R}^2)$ for strictly positive times. It is at this step that the dissipative nature of the equation plays an important role in this globalization argument. See Subsection 4.1 for details.

**Remark 1.4.** (i) In [35], Čanić and the first author also proved a kind of probabilistic continuous dependence, a notion introduced by Burq and Tzvetkov [11]. See also [61]. While we expect that this probabilistic continuous dependence extends to the range $s > -\frac{1}{5}$, we omit details.

(ii) It is also possible to establish almost sure global well-posedness with respect to the Wiener randomization for the defocusing vNLW (1.16) on $\mathbb{R}^2$ with a general defocusing nonlinearity $|u|^{p-1}u$ for $p < 5$, provided that $s > -\frac{1}{p}$. For $p \leq 3$, a straightforward Gronwall type argument by Burq and Tzvetkov [11] applies. See also [61]. For $3 < p < 5$, one can adapt the argument in Sun and Xia [64] which interpolates the $p = 3$ case [11] and
the $p = 5$ case \cite{57} in the context of the usual NLW. See Remark 4.2 (ii) for a discussion on the $p > 5$ case.

(iii) By considering a random external forcing $F_{ext}$ in \eqref{14}, the derivation discussed above leads to a stochastic version of vNLW. In \cite{36}, Canić and the first author studied the following stochastic vNLW on $\mathbb{R}^2$ with a multiplicative space-time white noise forcing:

$$\partial_t^2 u - \Delta u + 2\mu D\partial_t u = F(u)\xi,$$  \hfill (1.28)

where $\xi$ denotes a space-time white noise on $\mathbb{R}^2 \times \mathbb{R}_+$. Under a suitable assumption on $F$, they proved global well-posedness of \eqref{1.28}. See \cite{44, 43} for well-posedness of stochastic vNLW with a (singular) additive noise on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$.

### 2. Basic lemmas

In this section, we go over the deterministic and probabilistic linear estimates.

#### 2.1. Notations.

We first introduce some notations. We write $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some $C > 0$. Similarly, we write $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when we have $A \leq cB$ for small $c > 0$.

We define the operators $D$ and $\langle \nabla \rangle$ by setting

$$D = |\nabla| = \sqrt{-\Delta} \quad \text{and} \quad \langle \nabla \rangle = \sqrt{1 - \Delta},$$  \hfill (2.1)

viewed as Fourier multiplier operators with multipliers $|\xi|$ and $\langle \xi \rangle$, respectively.

#### 2.2. Linear operators and the relevant linear estimates.

By writing \eqref{1.23} in the Duhamel formulation, we have

$$u(t) = V(t)(u_0^a, u_1^a) - \int_0^t W(t - t')u_5(t')dt',$$

where the linear propagator $V(t)$ is defined by

$$V(t)(u_0, u_1) = e^{-\frac{\Delta}{2}t}\left(\cos \left(\frac{\sqrt{3}}{2}Dt\right) + \frac{1}{\sqrt{3}} \sin \left(\frac{\sqrt{3}}{2}Dt\right)\right)u_0 + e^{-\frac{\Delta}{2}t}\frac{\sin \left(\frac{\sqrt{3}}{2}Dt\right)}{\sqrt{3}D}u_1,$$  \hfill (2.2)

and $W(t)$ is defined by

$$W(t) = e^{-\frac{\Delta}{2}t}\frac{\sin \left(\frac{\sqrt{3}}{2}Dt\right)}{\sqrt{3}D}.$$  \hfill (2.3)

By letting

$$P(t) = e^{-\frac{\Delta}{2}t},$$  \hfill (2.4)

denote the Poisson kernel (with a parameter $\frac{\Delta}{2}$) and

$$S(t) = \frac{\sin \left(\frac{\sqrt{3}}{2}Dt\right)}{\sqrt{3}D},$$

we have

$$W(t) = P(t) \circ S(t).$$
By defining \( U(t) \) by
\[
U(t)(u_0, u_1) = \left( \cos \left( \frac{\sqrt{3}}{2} Dt \right) + \frac{1}{\sqrt{3}} \sin \left( \frac{\sqrt{3}}{2} Dt \right) \right) u_0 + \sin \left( \frac{\sqrt{3}}{2} Dt \right) \frac{\sqrt{3}}{2} D u_1,
\]
we have
\[
V(t) = P(t) \circ U(t). \tag{2.5}
\]

We first recall the Strichartz estimates for the homogeneous linear viscous wave equation (Theorem 3.2 in [35]). Given \( \sigma > 0 \), we say that a pair \((q, r)\) is \( \sigma \)-admissible if \( 2 \leq q, r \leq \infty \) with \((q, r, \sigma) \neq (2, \infty, 1)\) and
\[
\frac{2}{q} + \frac{2\sigma}{r} \leq \sigma. \tag{2.6}
\]

**Lemma 2.1.** Given \( \sigma > 0 \), let \((q, r)\) be a \( \sigma \)-admissible pair with \( r < \infty \). Then, a solution \( u \) to the homogeneous linear wave equation on \( \mathbb{R}^d \):
\[
\begin{align*}
\partial_t^2 u - \Delta u + D \partial_t u &= 0, \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1)
\end{align*}
\]
satisfies
\[
\|(u, \partial_t u)\|_{L^\infty(\mathbb{R}_+; H^s_2(\mathbb{R}^d))} + \|u\|_{L^q(\mathbb{R}_+; L^r(\mathbb{R}^d))} \lesssim \|(u_0, u_1)\|_{H^s(\mathbb{R}^d)}, \tag{2.7}
\]
provided that the following scaling condition holds:
\[
\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s. \tag{2.8}
\]

**Remark 2.2.** In view of the scaling condition \(2.8\), if a pair \((q, r)\) satisfies \(2.8\) for some \( s \geq 0 \), then it is \( \sigma \)-admissible with \( \sigma = d \).

**Remark 2.3.** We remark that the bounding constant in the estimate \(2.7\) depends only on \( \sigma > 0 \). See [35] for details.

**Remark 2.4.** In the usual Strichartz estimates for the homogeneous wave equation, one must impose an additional restriction on \( s \) that \( 0 \leq s \leq 1 \). This is not present in the corresponding estimate for the homogeneous viscous wave equation in Lemma 2.1. Although a restriction on \( s \) is not explicitly stated in Lemma 2.1, \( s \) does have a limited range of possible values, due to the constraints imposed by the fact that \( \sigma > 0 \), \( 2 \leq q, r \leq \infty \), with \((q, r, \sigma) \neq (2, \infty, 1)\) in \(2.6\), and the scaling condition \(2.8\). The exponent \( s \) can take values in the range \(-\frac{1}{2} < s \leq \frac{d}{2}\) depending on the choice of parameters. One attains the lower end of the range by taking \( q = 2 \) and taking \( r \) arbitrarily close to 2, while one attains the upper endpoint of the range by taking \( q, r = \infty \) for \( s = \frac{d}{2} \).

Next, we state a Schauder-type estimate for the Poisson kernel \( P(t) \), which allows us to exploit the dissipative nature of the dynamics.

**Lemma 2.5.** Let \( 1 \leq p \leq q \leq \infty \) and \( \alpha \geq 0 \). Then, we have
\[
\|D^\alpha P(t)f\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\alpha - \left(\frac{d}{p} - \frac{1}{2}\right)}\|f\|_{L^p(\mathbb{R}^d)} \tag{2.9}
\]
for any \( t > 0 \).
Proof. Let $K_t(x)$ denote the kernel for $P(t)$, whose Fourier transform is given by $\hat{K}_t(\xi) = e^{-\frac{|\xi|^2}{2}t}$. Then, we have

$$K_t(x) = t^{-d}K_1(t^{-1}x),$$

(2.10)

where $K_1(x)$ satisfies

$$K_1(x) = \frac{c_1}{(c_2 + |x|^2)^{d+1}},$$

for some $c_1, c_2 > 0$. In particular, we have $K_1 \in L^r(\mathbb{R}^d)$ for any $1 \leq r \leq \infty$.

We first consider the case $\alpha = 0$. For $1 \leq r \leq \infty$ with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} + 1$, it follows from (2.10) that

$$\|K_t\|_{L^r} = t^{-d(1 - \frac{1}{r})}\|K_1\|_{L^r} = C_r t^{-d(\frac{1}{q} - \frac{1}{p} + 1)}.$$  

(2.11)

Then, (2.9) follows from Young’s inequality and (2.11).

Next, we consider the case $\alpha > 0$. Noting that $D^\alpha P(t)f = (D^\alpha K_t) * f$, we need to study the scaling property of $D^\alpha K_t$. On the Fourier side, we have

$$\hat{D^\alpha K_t}(\xi) = |\xi|^\alpha e^{-\frac{|\xi|^2}{2}t} = t^{-\alpha}(|t\xi|^\alpha e^{-\frac{|t\xi|^2}{2}}) = t^{-\alpha} \hat{D^\alpha K_1}(t\xi).$$

Namely, we have

$$D^\alpha K(t,x) = t^{-d-\alpha}(D^\alpha K_1)(t^{-1}x),$$

(2.12)

Then, proceeding as before, the bound (2.9) follows from Young’s inequality and (2.12). □

2.3. Probabilistic estimates. In this subsection, we establish certain probabilistic Strichartz estimates. See also Lemma 5.3 in [35].

We first recall the following probabilistic estimate. See [10] for the proof.

Lemma 2.6. Given $j = 0,1$, let $\{g_{n,j}\}_{n \in \mathbb{Z}^d}$ be a sequence of mean-zero complex-valued, random variables, satisfying (1.22), as in Subsection 1.2. Then, there exists $C > 0$ such that

$$\left\| \sum_{n \in \mathbb{Z}^d} g_{n,j}(\omega)c_n \right\|_{L^p(\Omega)} \leq C \sqrt{p}\|c_n\|_{\ell^2(\mathbb{Z}^d)}$$

for any $j = 0,1$, any finite $p \geq 2$, and any sequence $\{c_n\} \in \ell^2(\mathbb{Z}^d)$.

We now establish the first probabilistic Strichartz estimate.

Proposition 2.7. Given $(u_0, u_1) \in \mathcal{H}^0(\mathbb{R}^d)$, let $(u_0^\omega, u_1^\omega)$ be its Wiener randomization defined in (1.21), satisfying (1.22). Then, given any $2 \leq q, r < \infty$ and $\alpha \geq 0$, satisfying $q\alpha < 1$, there exist $C, c > 0$ such that

$$P(\|D^\alpha V(t)(u_0^\omega, u_1^\omega)\|_{L^q([0,T];L_r^\omega)}> \lambda) \leq C \exp\left(-c \frac{\lambda^2}{T^{\frac{q}{q-2\alpha}(u_0, u_1)_{\mathcal{H}^0}}}\right)$$

(2.13)

for any $T > 0$ and $\lambda > 0$.

Remark 2.8. (i) From (2.13), we conclude that

$$P(\|D^\alpha V(t)(u_0^\omega, u_1^\omega)\| \leq \lambda) \longrightarrow 1,$$

as $\lambda \rightarrow \infty$ for fixed $T > 0$, or as $T \searrow 0$ for fixed $\lambda > 0$. 
(ii) Let $\alpha_0 \geq 0$ and $q\alpha_0 < 1$. Then, by applying Proposition 2.7 with $\alpha = 0$ and $\alpha = \alpha_0$, we have

$$P\left(\|\langle \nabla \rangle^\alpha V(t)(u_0^\omega, u_1^\omega)\|_{L^q([0,T]; L^p)} > \lambda\right) \leq C \exp\left(-\frac{\lambda^2}{T^{\frac{2}{q} - 2\alpha_0} \|\langle u_0, u_1 \rangle\|_{H^0}^2}\right)$$

(2.14)

for any $0 < T \leq 1$ and $\lambda > 0$, where $\langle \nabla \rangle = \sqrt{1 - \Delta}$ is as in (2.1). We also have

$$P\left(\|\langle \nabla \rangle^\alpha V(t)(u_0^\omega, u_1^\omega)\|_{L^q([0,T]; L^p)} > \lambda\right) \leq C \exp\left(-\frac{\lambda^2}{T^{\frac{2}{q}} \|\langle u_0, u_1 \rangle\|_{H^0}^2}\right)$$

(2.15)

for any $T \geq 1$ and $\lambda > 0$.

See also Lemma 5.3 in [35], where the case $q = r = 6$ was treated. The proof of Proposition 2.7 follows the usual proofs of the probabilistic Strichartz estimates via Minkowski’s integral inequality [10, 20, 5] but also utilizes the Schauder estimate (Lemma 2.5).

**Proof.** From (2.5) and Lemma 2.5 followed by Minkowski’s integral inequality, we have

$$\left\|D^\alpha V(t)(u_0^\omega, u_1^\omega)\right\|_{L^p([0,T]; L^p)} \leq \left\|t^{-\alpha} \|U(t)(u_0^\omega, u_1^\omega)\|_{L^p([0,T])}\right\|_{L^p(\Omega)}$$

(2.16)

for any finite $p \geq \max(q, r)$. By Lemma 2.6, Minkowski’s integral inequality, Bernstein’s unit-scale inequality (1.19), and the boundedness of $U(t)$ from $H^0(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$, we obtain

$$\text{(2.16)} \lesssim \sqrt{p} \left\|t^{-\alpha} \|\psi(D - n)U(t)(u_0, u_1)\|_{H^0(\mathbb{R}^d)}\right\|_{L^p(\mathbb{R}^d)(0,T)}$$

$$\leq \sqrt{p} \left\|t^{-\alpha} \|\psi(D - n)U(t)(u_0, u_1)\|_{H^0(\mathbb{R}^d)}\right\|_{L^p(\mathbb{R}^d)(0,T)}$$

(2.17)

where we used $q\alpha < 1$ in the last step. Then, the tail estimate (2.13) follows from (2.17) and Chebyshev’s inequality. See the proof of Lemma 3 in [35].

In establishing almost sure global well-posedness, we need to introduce several additional linear operators. Define $\bar{V}(t)$ by

$$\bar{V}(t)(u_0, u_1) = \langle \nabla \rangle^{-1} \partial_t V(t)$$

$$= -\frac{2\sqrt{3}}{3} \langle \nabla \rangle D e^{-\frac{D^2 t}{2}} \sin\left(\frac{\sqrt{3}}{2} Dt\right) u_0$$

$$+ e^{-\frac{D^2 t}{2}} \left(-\frac{1}{2} \langle \nabla \rangle \sin\left(\frac{\sqrt{3}}{2} Dt\right) + \cos\left(\frac{\sqrt{3}}{2} Dt\right)\right) u_1.$$  

\(^1\)Lemma 2.2 in the arXiv version.
Then, defining $\tilde{U}(t)$ by

$$\tilde{U}(t)(u_0, u_1) = -\frac{2\sqrt{3}}{3} \frac{D}{\langle \nabla \rangle} \sin \left( \frac{\sqrt{3}}{2}Dt \right) u_0$$

$$+ \left( -\frac{1}{2} \frac{D}{\langle \nabla \rangle} \sin \left( \frac{\sqrt{3}}{2}Dt \right) \frac{\sqrt{3}}{2} + \cos \left( \frac{\sqrt{3}}{2}Dt \right) \right) u_1,$$

we have

$$\tilde{V}(t) = P(t) \circ \tilde{U}(t).$$

(2.19)

Next, we state a probabilistic estimate involving the $L^\infty_t$-norm, which plays an important role in establishing an energy bound for almost sure global well-posedness. The proof is based on an adaptation of the proof of Proposition 3.3 in [57] combined with the Schauder estimate (Lemma 2.5).

**Proposition 2.9.** Given a pair $(u_0, u_1)$ of real-valued functions defined on $\mathbb{R}^2$, let $(u_0^\omega, u_1^\omega)$ be its Wiener randomization defined in (1.21), satisfying (1.22). Fix $T \gg 1 \geq T_0 > 0$ and let $V^*(t) = V(t)$ or $\tilde{V}(t)$ defined in (2.2) and (2.18), respectively. Then, given any $2 \leq r \leq \infty$, $\alpha \geq 0$, and $\varepsilon > 0$, there exist $C, c > 0$ such that

$$P\left( \left\{ \| D^\alpha V^*(t)(u_0^\omega, u_1^\omega) \|_{L^r([T_0, T]; L^{\infty}_x)} > \lambda \right\} \right) \leq C \exp \left( -c \frac{\lambda^2}{T^{2T_0^{-2\alpha}} \| (u_0, u_1) \|_{\mathcal{H}^\varepsilon}^2} \right)$$

for any $\lambda > 0$.

**Proof.** Let $U^*(t) = P(-t) \circ V^*(t)$. Then, from Lemma 2.5 with (2.5) or (2.19), we have

$$\| D^\alpha U^*(t)(u_0^\omega, u_1^\omega) \|_{L^\infty([T_0, T]; L^r_\omega)} \lesssim \| t^{-\alpha} \| U^*(t)(u_0^\omega, u_1^\omega) \|_{L^\infty} \| U^*(t)(u_0^\omega, u_1^\omega) \|_{L^r([T_0, T])}$$

$$\leq T_0^{-\alpha} \| U^*(t)(u_0^\omega, u_1^\omega) \|_{L^\infty([T_0, T]; L^r_\omega)}$$

As in the proof of Proposition 3.3 in [57], the rest follows from Lemma 3.4 in [57], which established similar $L^\infty_t$-bounds for the half-wave operators $e^{±itD}$.

**Remark 2.10.** It is also possible to prove Proposition 2.9 using the Garsia-Rodemich-Rumsey inequality ([26, Theorem A.1]). See, for example, Lemma 2.3 in [31] in the context of the stochastic nonlinear wave equation.

3. LOCAL WELL-POSEDNESS

In this section, we present the proof of Theorem 1.1. Instead of (1.26) with the zero initial data, we study (1.26) with general (deterministic) initial data $(v_0, v_1)$:

$$\left\{ \begin{array}{l}
\partial_t^3 v - \Delta v + D \partial_t v + (v + z)^5 = 0 \\
(v, \partial_t v)\big|_{t=0} = (v_0, v_1).
\end{array} \right.$$

(3.1)

We recall from (1.25) and (2.2) that $z = V(t)(u_0^\omega, u_1^\omega)$ is the random linear solution with the randomized initial data $(u_0^\omega, u_1^\omega)$ which is the result of the Wiener randomization (1.21) performed on the given deterministic initial data $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^2)$.

**Theorem 3.1.** Let $s > -\frac{1}{5}$. Fix $(v_0, v_1) \in \mathcal{H}^{s_0}(\mathbb{R}^2)$ for some $s_0 = s_0(s) > \frac{3}{5}$ sufficiently close to $\frac{3}{5}$. Then, there exist $C, c, \gamma > 0$ and $0 < T_0 \ll 1$ such that for each $0 < T \leq T_0$, there exists a set $\Omega_T \subset \Omega$ with the following properties:
(i) The following probability bound holds:

\[ P(\Omega_T^c) < C \exp \left( - \frac{c}{T^\gamma \| (u_0, u_1) \|_{H^s}^2} \right). \]  

(3.2)

(ii) For each \( \omega \in \Omega_T \), there exists a (unique) solution \((v, \partial_t v)\) to (3.1) with \((v, \partial_t v)|_{t=0} = (v_0, v_1)\) in the class

\[(v, \partial_t v) \in C([0, T]; H^{s_0}(\mathbb{R}^2)) \quad \text{and} \quad v \in L^{5+\delta}([0, T]; L^{10}(\mathbb{R}^2)) \]  

for small \( \delta > 0 \) such that \( s_0 \geq 1 - \frac{1}{5+\delta} - \frac{2}{10} \).

In Subsection 3.1, we first state several linear estimates. We then present the proof of Theorem 3.1 in Subsection 3.2.

3.1. Linear estimates. In this subsection, we establish several nonhomogeneous linear estimates, which are slightly different from those in Theorem 3.3 in [35].

Lemma 3.2. Let \( W(t) \) be as in (2.3). Then, given sufficiently small \( \delta > 0 \), we have

\[ \left\| \int_0^t W(t - t') F(t') dt' \right\|_{L^{5+\delta}_t([0, T]; L^{10}_x(\mathbb{R}^2))} \lesssim \| F \|_{L^1([0, T]; L^2_x(\mathbb{R}^2))} \]  

for any \( 0 < T \leq 1 \).

Proof. Let \( P_{\leq 1} \) be a smooth projection onto spatial frequencies \( \{ |\xi| \leq 1 \} \) and set \( P_{> 1} = \text{Id} - P_{\leq 1} \). In the following, we separately estimate the contributions from \( P_{\leq 1} F \) and \( P_{> 1} F \).

Let us first estimate the low frequency contribution. By Minkowski’s integral inequality and Bernstein’s unit-scale inequality (1.19) with \( \sin x \leq x \) for \( x \geq 0 \), we have

\[ \left\| \int_0^t W(t - t') P_{\leq 1} F(t') dt' \right\|_{L^{5+\delta}_t([0, T]; L^{10}_x(\mathbb{R}^2))} \]

\[ \leq \left\| \int_0^t |1_{[0,t]}(t') W(t - t') P_{\leq 1} F(t')| dt' \right\|_{L^{5+\delta}_t([0, T])} \]

\[ \leq \left\| \int_0^t (t - t') |1_{[0,t]}(t') P_{\leq 1} F(t')| dt' \right\|_{L^{5+\delta}_t([0, T])} \]

\[ \lesssim T^\theta \| F \|_{L^1([0, T]; L^2_x(\mathbb{R}^2))} \]  

for some \( \theta > 0 \).

Next, we estimate the high frequency contribution. Note that the pair \((5 + \delta, 10)\) is \( \sigma \)-admissible for \( \sigma \geq \frac{1}{2} \) in the sense of (2.6). Let

\[ s_0 = 1 - \frac{1}{5 + \delta} - \frac{2}{10} = \frac{3}{5} + \delta_0 \]  

(3.6)

Namely, given by a smooth Fourier multiplier.
for some small $\delta_0 = \delta_0(\delta) > 0$. Then, by Minkowski’s integral inequality and the homogeneous Strichartz estimate (Lemma 2.1), we have
\[
\left\| \int_0^t W(t - t')P_{\geq 1}F(t')dt' \right\|_{L_t^{5+\delta}(L_x^{10})} \leq \int_0^T \|1_{[0,T]}(t')W(t - t')P_{\geq 1}F(t')\|_{L_t^{5+\delta}(L_x^{10})}dt' \\
\lesssim \int_0^T \|P_{\geq 1}F(t')\|_{H_x^{7-1}}dt' \\
\lesssim \|F\|_{L^1([0,T];L_x^2)}.
\]
(3.7)
The desired bound (3.4) then follows from (3.5) and (3.7).

Lemma 3.3. Let $W(t)$ be as in (2.3). Then, given $0 \leq s \leq 1$, we have
\[
\left\| \int_0^t W(t - t')F(t')dt' \right\|_{C([0,T];H_x^s(R^2))} \lesssim \|F\|_{L^1([0,T];L_x^2(R^2))},
\]
(3.8)
and
\[
\left\| \partial_t \int_0^t W(t - t')F(t')dt' \right\|_{C([0,T];H_x^{s-1}(R^2))} \lesssim \|F\|_{L^1([0,T];L_x^2(R^2))},
\]
(3.9)
for any $0 < T \leq 1$.

Proof. The first estimate (3.8) follows from Minkowski’s integral inequality with (2.3). As for the second estimate (3.9), we first note from (2.3) that
\[
\partial_t \int_0^t W(t - t')F(t')dt' = \int_0^t \partial_t W(t - t')F(t')dt',
\]
where
\[
\partial_t W(t) = e^{-\frac{Dt}{2}} \left( \cos \left( \frac{\sqrt{3}}{2}Dt \right) - \frac{1}{\sqrt{3}} \sin \left( \frac{\sqrt{3}}{2}Dt \right) \right).
\]
Then, the second estimate (3.9) follows from Minkowski’s integral inequality and the boundedness of $\partial_t W(t)$ on $H^{s-1}(R^2)$.

3.2. Local well-posedness. We now present the proof of Theorem 3.1.

Proof of Theorem 3.1. Fix $s > -\frac{1}{2}$ and $(u_0, u_1) \in \mathcal{H}^s(R^2)$. Then, there exists small $\delta > 0$ such that
\[
s > -\frac{1}{5 + \delta},
\]
(3.10)
and we fix this choice of $\delta > 0$ for the remainder of the proof.

Fix $C_0 > 0$ and define the event $\Omega_T = \Omega_T(C_0)$ by setting
\[
\Omega_T = \{ \omega \in \Omega : \|z\|_{L_t^{5+\delta}(L_x^{10})} \leq C_0 \}.
\]
(3.11)
Then, from the probabilistic Strichartz estimate (Proposition 2.7) (see also 2.14) with (1.25), (1.21), and (3.10) (which guarantees $\alpha_0 q < 1$ in invoking (2.14) with $\alpha_0 = -s$ and $q = 5 + \delta$), we have
\[
P(\Omega_T^c) \leq C \exp \left( -c \frac{C_0^2}{T^{\frac{2}{q} + 2s} \|(u_0, u_1)\|_{\mathcal{H}^s}^2} \right)
\]
(3.12)
for any $0 < T \leq 1$. We remark that the choice of $C_0 > 0$ does not matter, and that the specific value of $C_0 > 0$ affects only the size of $T_0 \ll 1$ and the constants in the estimate \eqref{3.2}.

By writing \eqref{3.1} in the Duhamel formulation, we have
\[
v(t) = \Gamma((v_0, v_1); z) := V(t)(v_0, v_1) - \int_0^t W(t - t')(v + z)^5(t') \, dt'.
\]

For simplicity, we set $\Gamma = \Gamma((v_0, v_1); z)$. Let $\Gamma(v) = (\Gamma(v), \partial_t \Gamma(v))$. Let $s_0 = s_0(\delta) = \frac{3}{5} + \delta_0$ as in \eqref{3.6}. Then, given $T > 0$, define the solution space $Z(T)$ by setting
\[Z(T) = X(T) \times Y(T),\]
where $X(T)$ and $Y(T)$ are defined by
\[X(T) = C([0, T]; H^{s_0}(\mathbb{R}^2)) \cap L^{5+\delta}([0, T]; L^{10}(\mathbb{R}^2))\]
\[Y(T) = C([0, T]; H^{s_0-1}(\mathbb{R}^2)).\]

In order to prove Theorem 3.1 we show that there exists small $0 < T_0 \ll 1$ such that $\Gamma : (v, \partial_t v) \mapsto (\Gamma(v), \partial_t \Gamma(v))$ is a strict contraction on an appropriate closed ball in $Z(T)$ for any $0 < T \leq T_0$ and for any $\omega \in \Omega_T$, where $\Omega_T$ is as in \eqref{3.11}. The probability estimate \eqref{3.2} on $\Omega_T$ follows from \eqref{3.12}.

Fix arbitrary $\omega \in \Omega_T$ for $0 < T \leq T_0$, where $T_0$ is to be determined later. Recall $\bar{\Gamma}(v) = (\Gamma(v), \partial_t \Gamma(v))$. Note that the ordered pair $(5 + \delta, 10)$ is $\sigma$-admissible for $\sigma \geq \frac{1}{2}$ in the sense of \eqref{2.6} and furthermore, it satisfies the scaling condition \eqref{2.8} with $s_0$ as defined in \eqref{3.6}. Then, by Lemmas 2.1 3.2 and 3.3 with \eqref{3.11}, we have
\[\|\bar{\Gamma}(v)\|_{Z(T)} \lesssim \|(v_0, v_1)\|_{H^{s_0}} + \|(v + z)^5\|_{L^1([0, T]; L^2_x)} + \|v\|_{L^{5+\delta}([0, T]; L^{10}_x)} + \|z\|_{L^{5+\delta}([0, T]; L^{10}_x)}\]
\[\lesssim \|(v_0, v_1)\|_{H^{s_0}} + T^\theta \left(\|\bar{v}\|_{Z(T)} + C_0\right),\]
for some $\theta > 0$, where $\bar{v} = (v, \partial_t v)$.

A similar computation yields the following difference estimate:
\[\|\bar{\Gamma}(v) - \bar{\Gamma}(w)\|_{Z(T)} \lesssim \|(v + z)^5 - (w + z)^5\|_{L^4([0, T]; L^4_x)} + \|v - w\|_{L^{5+\delta}([0, T]; L^{10}_x)} + \|z - w\|_{L^{5+\delta}([0, T]; L^{10}_x)}\]
\[\lesssim T^\theta \left(\|\bar{v}\|_{Z(T)} + \|\bar{v}\|_{Z(T)} + C_0\right)\|\bar{v} - \bar{w}\|_{Z(T)}.
\]

Hence by choosing $T_0 > 0$ sufficiently small, depending on the initial choice of $C_0 > 0$ and $\|(v_0, v_1)\|_{H^{s_0}}$, we see that $\bar{\Gamma} = \bar{\Gamma}((v_0, v_1); z)$ is a strict contraction on the ball in $Z(T)$ of radius $\sim 1 + \|(v_0, v_1)\|_{H^{s_0}}$, whenever $\omega \in \Omega_T$ and $0 < T \leq T_0$. This proves almost sure local well-posedness of \eqref{3.1} \eqref{1.23} for $s > -\frac{1}{6}$. This concludes the proof of Theorem 3.1 \eqref{3.1} (and hence of Theorem \ref{1.1}). \hfill $\Box$

Let us conclude this section by stating some corollaries and remarks. Given $N \in \mathbb{N}$, let $P_{\leq N}$ denote a smooth projection onto the (spatial) frequencies $\{||\xi|| \leq N\}$. Then, consider
the following perturbed vNLW:
\[
\begin{aligned}
& \frac{\partial^2_r v_N}{\partial t^2} - \Delta v_N + D \partial_t v_N + (v_N + z_N)^5 = 0 \\
& (v_N, \partial_t v_N)_{|t=0} = (P_{\leq N} v_0, P_{\leq N} v_1),
\end{aligned}
\]  
(3.13)
where \(z_N\) denotes the truncated random linear solution defined by
\[
z_N(t) = V(t)(P_{\leq N} u_0^\omega, P_{\leq N} u_1^\omega).
\]

Then, a slight modification of the proof of Theorem 3.1 yields the following approximation result.

**Corollary 3.4.** Let \(s > -\frac{1}{5}\) and \(s_0 > \frac{3}{5}\) be as in Theorem 3.1. Fix \((v_0, v_1) \in H^{s_0}(\mathbb{R}^2)\). Let \(\Omega_T\) be as in Theorem 3.1. Furthermore, for each \(\omega \in \Omega_T\), let \((v, \partial_t v)\) be the solution to (3.1) on \([0, T]\) with \((v, \partial_t v)_{|t=0} = (v_0, v_1)\) constructed in Theorem 3.1. By possibly shrinking the local existence time \(T\) by a constant factor (while keeping the definition (3.11) of \(\Omega_T\) unchanged), for each \(\omega \in \Omega_T\), the solution \((v_N, \partial_t v_N)\) to (3.13) converges to \((v, \partial_t v)\) in the class (3.3) as \(N \to \infty\).

Next, consider the following perturbed vNLW:
\[
\begin{aligned}
& \frac{\partial^2_r v}{\partial t^2} - \Delta v + D \partial_t v + (v + f)^5 = 0 \\
& (v, \partial_t v)_{|t=0} = (v_0, v_1),
\end{aligned}
\]  
(3.14)
where \(f\) is a given deterministic function. As a corollary to the proof of Theorem 3.1 we have the following local well-posedness result of (3.14).

**Corollary 3.5.** Let \(s > -\frac{1}{5}\), \(s_0 > \frac{3}{5}\), and small \(\delta > 0\) be as in Theorem 3.1. Fix \((v_0, v_1) \in H^{s_0}(\mathbb{R}^2)\) and fix \(t_0 \in \mathbb{R}_+\). Suppose that
\[
f \in L^{5+\delta}([t_0, t_0 + 1]; L^{10}(\mathbb{R}^2)).
\]
Then, there exists \(T = T(\|v_0\|_{H^{s_0}}, \|f\|_{L^{5+\delta}([t_0, t_0 + T]; L^{10})}) > 0\) and a (unique) solution \((v, \partial_t v)\) to (3.14) on the time interval \([t_0, t_0 + T]\) with \((v, \partial_t v)_{|t=t_0} = (v_0, v_1)\) in the class \((v, \partial_t v) \in C([t_0, t_0 + T]; H^{s_0}(\mathbb{R}^2))\) and \(v \in L^{5+\delta}([t_0, t_0 + T]; L^{10}(\mathbb{R}^2))\).

**Remark 3.6.** (i) In terms of the current approach based on the first order expansion (1.24), the threshold \(s = -\frac{1}{5}\) seems to be sharp. Since we need to measure the quintic power in \(L^1\) in time, this forces us to measure the random linear solution essentially in \(L^5\) in time. In view of Proposition 2.7 local-in-time integrability of \(t^s\) in \(L^5\) requires \(s > -\frac{1}{5}\). It is worthwhile to note that the regularity restriction \(s > -\frac{1}{5}\) comes only from the temporal integrability and does not have anything to do with the spatial integrability.

With a \(p\)th power nonlinearity \(|u|^{p-1}u\), \(p > 1\), (in place of the quintic power \(u^5\)), a similar argument shows almost sure local well-posedness of (1.23) for \(s > -\frac{1}{p}\), which is essentially sharp (in terms of the first order expansion). For \(p \notin 2\mathbb{N} + 1\), the nonlinearity is not algebraic and thus we need to proceed as in [53], where probabilistic well-posedness of the nonlinear Schrödinger equations with non-algebraic nonlinearities was studied. See [43] for details. See also Remark 1.2.

(ii) It would be of interest to investigate if higher order expansions, such as those in [6], [58], give any improvement over Theorem 3.1 on almost sure local well-posedness. One may also adapt the paracontrolled approach used for the stochastic NLW [30, 51, 9, 55] to study vNLW with random initial data.
4. Global well-posedness

In this section, we prove almost sure global well-posedness of \(1.23\). As noted in \[20\]14, it suffices to prove the following “almost” almost sure global well-posedness result.

**Proposition 4.1.** Let \(s > -\frac{1}{5}\). Given \((u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^2)\), let \((\bar{u}_0^\omega, \bar{u}_1^\omega)\) be its Wiener randomization defined in \(1.21\), satisfying \(1.22\). Then, given any \(T, \varepsilon > 0\), there exists a set \(\Omega_{T, \varepsilon} \subset \Omega\) such that

(i) \(P(\Omega_{T, \varepsilon}) < \varepsilon\),

(ii) For each \(\omega \in \Omega_{T, \varepsilon}\), there exists a (unique) solution \(u\) to \(1.23\) on \([0, T]\) with \((u, \partial_t u)|_{t=0} = (\bar{u}_0^\omega, \bar{u}_1^\omega)\).

It is easy to see from the Borel-Cantelli lemma that almost sure global well-posedness (Theorem 1.3) follows once we prove “almost” almost sure global well-posedness stated in Proposition 4.1 above. See \[20, 4\]. Hence, the remaining part of this section is devoted to the proof of Proposition 4.1.

Fix \(T \gg 1\). In order to extend our local-in-time result for the initial value problem \(1.26\) to a result on \([0, T]\) for arbitrary \(T > 0\), we consider \(3.14\) with \(f = z = V(t)(u_0^\omega, u_1^\omega)\), given explicitly by

\[
\begin{align*}
\partial_t^2 v - \Delta v + D \partial_t v + (v + z)^5 &= 0 \\
(v, \partial_t v)|_{t=0} &= (v_0, v_1),
\end{align*}
\]

(4.1)

where \(t_0 \in \mathbb{R}^+\). In view of Corollary 3.5 and almost sure boundedness of the \(L^{5+\delta}([0, T]; L^2_x(\mathbb{R}^2))\)-norm of the random linear solution \(z(t) = V(t)(u_0^\omega, u_1^\omega)\) thanks the probabilistic Strichartz estimate (Proposition 2.7), it suffices to control the \(H^s\)-norm of the remainder term \(\bar{v} = (v, \partial_t v)\), where \(s_0 = \frac{3}{5} + \delta_0\) as in \(3.6\) and the remainder term \(v\) satisfies the initial value problem \(1.26\). In the next subsection, we first show a gain of regularity such that \((v(t), \partial_t v(t))\) indeed belongs to \(H^1(\mathbb{R}^2)\) as soon as \(t > 0\). Then, the problem is reduced to controlling the growth of the energy \(E(\bar{v})\) in \(1.27\), associated with the standard nonlinear wave equation, since the energy \(E(\bar{v})\) controls the \(H^1\)-norm (and hence \(H^s\)-norm) of the remainder term \((v, \partial_t v)\), as needed to establish the result. See Subsection 4.2.

4.1. Gain of regularity. Consider the initial value problem \(4.1\). Fix \(s > -\frac{1}{5}\) and \(s_0 = \frac{3}{5} + \delta_0\) with small \(\delta_0 > 0\) as in (the proof of) Theorem 3.1. Let \((\bar{u}_0^\omega, \bar{u}_1^\omega)\) be the Wiener randomization of a given deterministic pair \((u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^2)\) and fix \((v_0, v_1) \in \mathcal{H}^{s_0}(\mathbb{R}^2)\).

Let \(T \gg 1\) and let \(z(t) = V(t)(u_0^\omega, u_1^\omega)\) be the random linear solution. Then, it follows from the probabilistic Strichartz estimate (Proposition 2.7) (see also \(2.15\)) that there exists an almost surely finite random constant \(C_\omega = C_\omega(T) > 0\) such that

\[
\|z\|_{L^{5+\delta}([0, T]; L^2)} \leq C_\omega.
\]

(4.2)

Fix a good \(\omega \in \Omega\) such that \(C_\omega\) in \(4.2\) is finite. Then, from Corollary 3.5, we see that there exist \(\tau_\omega > 0\) and a unique solution \(\bar{v} = (v, \partial_t v)\) to \(4.1\) on the time interval \([t_0, t_0 + \tau_\omega]\) with \((v, \partial_t v)|_{t=t_0} = (v_0, v_1)\) in the class

\[
(v, \partial_t v) \in C([t_0, t_0 + \tau_\omega]; \mathcal{H}^{s_0}(\mathbb{R}^2)) \quad \text{and} \quad v \in L^{5+\delta}([t_0, t_0 + \tau_\omega]; L^2(\mathbb{R}^2)).
\]
We show that the solution $\tilde{v} = (v, \partial_t v)$ to (4.1) in fact belongs to $C([t_0, t_0 + T]; \mathcal{H}^1(\mathbb{R}^2))$ thanks to the smoothing due to the Poisson kernel $P(t)$ in (2.4). Fix $t > t_0$. By (2.5) and Lemma 2.5 we have
\[
\|V(t - t_0)(v_0, v_1)\|_{\mathcal{H}^1} \lesssim (t - t_0)^{-1+\delta_0}\|(v_0, v_1)\|_{\mathcal{H}^0}.
\]
Then, from (4.3), Lemma 3.3 with $s = 1$, and (4.2), we have, for any $t_0 < t \leq t_0 + \tau$, 
\[
\|\tilde{v}(t)\|_{\mathcal{H}^1} \lesssim (t - t_0)^{-1+\delta_0}\|(v_0, v_1)\|_{\mathcal{H}^0} + \|(v + z)^5\|_{L^1([t_0, t_0 + \tau]; L^2_x)} \\
\lesssim (t - t_0)^{-1+\delta_0}\|(v_0, v_1)\|_{\mathcal{H}^0} + \tau\delta_0\|v\|_{L^{5+\delta}([t_0, t_0 + \tau]; L^{10}_x)} + C_\omega \lesssim \infty.
\]
This proves the gain of regularity for $\tilde{v} = (v, \partial_t v)^3$. In the following, our main goal is to control the $\mathcal{H}^1$-norm of $\tilde{v}(t)$ on $[0, T]$ for any given $T \gg 1$.

4.2. Energy bound. Fix $\varepsilon > 0$. Then, it follows from Theorem 3.1 that there exists $\Omega_{T_0}$ with sufficiently small $T_0 = T_0(\varepsilon) > 0$ such that
\[
P(\Omega^c_{T_0}) < \frac{\varepsilon}{2}
\]
and, for each $\omega \in \Omega_{T_0}$, the local well-posedness of (1.23) holds on $[0, T_0]$.

Fix a large target time $T \gg 1$. In the following, by excluding further a set of small probability, we construct the solution $\tilde{v} = (v, \partial_t v)$ on the time interval $[T_0, T]$ and hence on $[0, T]$. We achieve this goal by controlling the growth of the $\mathcal{H}^1$-norm of $\tilde{v}(t)$ on $[T_0, T]$. In view of the fundamental theorem of calculus:
\[
\|v(t)\|_{L^2_x} = \left\| \int_0^t \partial_t v(t') dt' \right\|_{L^2_x} \leq T\|\partial_t v\|_{L_t^\infty L_x^2}
\]
for $0 < t \leq T$, it suffices to control the homogeneous Sobolev $\mathcal{H}^1(\mathbb{R}^2)$-norm, where $\mathcal{H}^1(\mathbb{R}^2) := \dot{H}^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$.

For this purpose, we estimate the growth of the energy $E(\tilde{v})$ in (1.27). In the case of the cubic nonlinearity, one can follow the Gronwall argument by Burq and Tzvetkov [11]. See Remark 1.4 (i). In the current quintic case, however, this argument fails. To overcome this difficulty, we employ the integration-by-parts trick introduced by Pocovnicu and the second author [57] in studying almost sure global well-posedness of the energy-critical defocusing quintic NLW on $\mathbb{R}^3$.

Let $z(t) = V(t)(u_0^\omega, u_1^\omega)$ be the random linear solution defined in (1.25). With $\tilde{V}(t)$ as in (2.18), define $\tilde{z}$ by
\[
\tilde{z}(t) = \langle \nabla \rangle^{-1}\partial_t z(t) = \tilde{V}(t)(u_0^\omega, u_1^\omega).
\]
Then, given $0 < T_0 < T$, we set $A(T_0, T)$ as
\[
A(T_0, T) = 1 + \|\tilde{z}\|_{L_t^\infty([T_0, T]; L_x^\infty)}^2 + \|\tilde{z}\|_{L_t^6([T_0, T]; L_x^6)}^6 + \|\tilde{z}\|_{L_t^6([T_0, T]; L_x^6)}^6 \\
+ \|\langle \nabla \rangle^{s_1}\tilde{z}\|_{L_t^\infty([T_0, T]; L_x^\infty)}^2,
\]
where $s_1 > \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$ (to be chosen later).

\[3\]Here, we did not show the continuity in time of $\tilde{v}$ in $\mathcal{H}^1(\mathbb{R}^2)$ but this can be done by a standard argument, which we omit.
By differentiating $E(\vec{v})$ in time and using the equation (1.26), we have
\[
\frac{d}{dt}E(\vec{v})(t) = \int_{\mathbb{R}^2} \partial_t v (\partial_t^2 v - \Delta v + v^5)dx
\]
\[
= - \int_{\mathbb{R}^2} (D_t \partial_t v)^2 dx - \int_{\mathbb{R}^2} \partial_t v ((v + z)^5 - v^5)dx 
\]
\[
\leq - \int_{\mathbb{R}^3} \partial_t v (5zv^4 + \mathcal{N}(z,v)) dx,
\]
where $\mathcal{N}(z,v)$ denotes the lower order terms in $v$:
\[
\mathcal{N}(z,v) = 10z^2v^3 + 10z^3v^2 + 5z^4v + z^5.
\]
By integrating (4.7) in time from $T_0$ to $t$, we have
\[
E(\vec{v})(t) - E(\vec{v})(T_0) \leq - \int_{T_0}^t \int_{\mathbb{R}^2} \partial_t v(v(t')) \left(5z(t')v(t')^4 + \mathcal{N}(z,v)(t')\right) dx dt'
\]
\[
= - \int_{T_0}^t \int_{\mathbb{R}^2} z(t') \partial_t v(v(t')^5) dt' dx - \int_{T_0}^t \int_{\mathbb{R}^2} \partial_t v(t') \mathcal{N}(z,v)(t') dx dt' 
\]
\[
=: I(t) + II(t),
\]
for any $t \in [T_0, T]$. By Young’s inequality, we have
\[
|\mathcal{N}(z,v)(t')| \lesssim |z(t')^2v(t')^3| + |z(t')|^5.
\]
Thus, by Cauchy’s inequality with (1.27) and (4.6), we have
\[
|II(t)| \lesssim \int_{T_0}^t \|\partial_t v(t')\|_{L^2} \|z(t')\|_{L^2}^2 \|v(t')\|_{L^2}^3 dt' + \int_{T_0}^t \|\partial_t v(t')\|_{L^2} \|z(t')\|_{L^2}^5 dt'
\]
\[
\lesssim (1 + \|z\|_{L^\infty([T_0,T];L^2)}^2) \int_{T_0}^t E(\vec{v})(t') dt' + \|z\|_{L^1([T_0,T];L^6)}^{10} 
\]
\[
\leq A(T_0,T) \int_{T_0}^t E(\vec{v})(t') dt' + A(T_0,T).
\]
Next, we control the term $I(t)$ in (4.8). By integration by parts in time, we have
\[
I(t) = - \int_{\mathbb{R}^2} z(t') v(t')^5 dx \bigg|_{T_0}^t + \int_{T_0}^t \int_{\mathbb{R}^2} \partial_t z(t') v(t')^5 dt' dx =: I_1(t) \bigg|_{T_0}^t + I_2(t).
\]
As for the first term $I_1$, we bound it by
\[
|I_1(t) - I_1(T_0)| \lesssim \sum_{t' \in \{T_0,t\}} \left(\varepsilon_0^{-6}\|z(t')\|_{L^2}^6 + \varepsilon_0^{6} \|v(t')\|_{L^6}^6\right) 
\]
\[
\lesssim \varepsilon_0^{-6}\|z\|_{L^\infty([T_0,T];L^6)}^6 + \varepsilon_0^{6} E(\vec{v})(t) + \varepsilon_0^{6} E(\vec{v})(T_0) 
\]
for some small constant $\varepsilon_0 > 0$ (to be chosen later).
Next, we consider the second term $I_2$ in (4.10). While we closely follow the argument in [57], we present details here for readers’ convenience. From (4.5), we have
\[
I_2(t) = \int_{\mathbb{R}^2} \int_{T_0}^t \partial_t z(t') v(t')^5 dt' dx = \int_{T_0}^t \int_{\mathbb{R}^2} \langle \nabla \rangle \bar{z}(t') \cdot v(t')^5 dx dt'.
\]
Given dyadic $M \geq 1$, let $Q_M$ denote the (nonhomogeneous) Littlewood-Paley projector onto the (spatial) frequencies $\{||\xi|| \sim M\}$. Then, define $I(t)$ by

$$I(t) := \int_{\mathbb{R}^2} \langle \nabla \rangle \tilde{z}(t) \cdot v(t)^5 \, dx$$

$$\sim \sum_{M \geq 1} \sum_{k=-1}^{1} M \int_{\mathbb{R}^2} Q_M 2^k \tilde{z}(t) Q_M (v(t)^5) \, dx$$

$$=: \sum_{M \geq 1} I^M(t)$$

with the understanding that $Q_{2^{-1}} = 0$. We also set

$$I^{M \geq 2}(t) = I(t) - I^1(t).$$

- **Case 1:** $M = 1$. In this case, we can bound the contribution to $|I_2(t)|$ by Young’s and Bernstein’s inequalities as

$$\left| \int_{T_0}^{t} I^1(t') \, dt' \right| \lesssim \parallel \tilde{z} \parallel_{L^6([T_0,T];L^6_x)}^6 + \int_{T_0}^{t} \|v(t')\|_{L^6_x}^6 \, dt'$$

$$\lesssim A(T_0, T) + \int_{T_0}^{t} E(\tilde{v})(t') \, dt'.$$

- **Case 2:** $M \geq 2$. By applying the Littlewood-Paley decomposition, we can write

$$v^5 = \sum_{M_j \geq 1} \prod_{j=1}^{5} Q_{M_j} v.$$

Without loss of generality, we assume that $M_1 \geq M_2 \geq \cdots \geq M_5$. Noting that $Q_M (v(t)^5) = 0$ unless $M_1 \gtrsim M$, it follows from Hölder’s inequality that

$$|I^{M \geq 2}(t)| \lesssim \sum_{k=-1}^{1} \sum_{M \geq 2} \sum_{M_1, \ldots, M_5 \geq 1} \parallel \langle \nabla \rangle^{s_2 - \theta} Q_{2^k} \tilde{z}(t) \parallel_{L^\infty} \parallel M_1^{-s_2 + \theta} \parallel_{L^1_x} \prod_{j=1}^{5} Q_{M_j} v(t) \parallel$$

for $s_2 > 0$ (to be determined later) and small $\theta > 0$. By summing over dyadic $M, M_1, \ldots, M_5$ (with a slight loss in a power of $M_1$) and applying Bernstein’s and Young’s inequalities, we

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4Namely, $Q_1$ is a smooth projector onto the (spatial) frequencies $\{||\xi|| \lesssim 1\}$. 
have

$$\left| \mathcal{I}^{M\geq 2}(t) \right| \lesssim \sup_{M_1, \ldots, M_5 \geq 1} \left\| \langle \nabla \rangle^{s_2-\theta} z(t) \right\|_{L^\infty_x M_1^{1-s_2+2\theta}} \left\| \sum_{j=2}^5 Q_{M_j} v(t) \right\|_{L^1_x}$$

$$\lesssim \sup_{M_1, \ldots, M_5 \geq 1} \left\| \langle \nabla \rangle^{s_2-\theta} z(t) \right\|_{L^\infty_x} \left( \left\| M_1^{1-s_2+2\theta} Q_{M_1} v(t) \right\|_{L^3_x}^3 + \left\| \sum_{j=2}^5 Q_{M_j} v(t) \right\|_{L^3_x}^{\frac{3}{2}} \right)$$

$$\lesssim \sup_{M_1, \ldots, M_5 \geq 1} \left\| \langle \nabla \rangle^{s_2-\theta} z(t) \right\|_{L^\infty_x} \left( M_1^{3(1-s_2+2\theta)} \left\| Q_{M_1} v(t) \right\|_{L^3_x}^3 + \left\| \sum_{j=2}^5 Q_{M_j} v(t) \right\|_{L^3_x}^{\frac{3}{2}} \right)$$

$$\lesssim \sup_{M_1 \geq 1} \left\| \langle \nabla \rangle^{s_2-\theta} z(t) \right\|_{L^\infty_x} \left( M_1^{3(1-s_2+2\theta)} \left\| Q_{M_1} v(t) \right\|_{L^3_x}^3 + \left\| \sum_{j=2}^5 Q_{M_j} v(t) \right\|_{L^3_x}^{\frac{3}{2}} + \left\| v(t) \right\|_{L^6_x}^6 \right).$$

Finally, by interpolating $L^3$ between $L^2$ and $L^6$ and then applying Young’s and Bernstein’s inequalities with (1.27), we obtain

$$\left| \mathcal{I}^{M\geq 2}(t) \right| \lesssim \sup_{M_1 \geq 1} \left\| \langle \nabla \rangle^{s_2-\theta} z(t) \right\|_{L^\infty_x} \left( \left\| M_1^{2(1-s_2+2\theta)} Q_{M_1} v \right\|_{L^2_x}^{\frac{3}{2}} \left\| Q_{M_1} v \right\|_{L^6_x}^{\frac{3}{2}} + E(v) \right)$$

$$\lesssim \sup_{M_1 \geq 1} \left\| \langle \nabla \rangle^{s_2-\theta} z(t) \right\|_{L^\infty_x} \left( \left\| M_1^{2(1-s_2+2\theta)} Q_{M_1} v \right\|_{L^2_x}^{\frac{3}{2}} \left\| Q_{M_1} v \right\|_{L^6_x}^{\frac{3}{2}} + \left\| Q_{M_1} v \right\|_{L^6_x}^6 + E(v) \right)$$

$$\lesssim \left\| \langle \nabla \rangle^{s_2-\theta} z(t) \right\|_{L^\infty_x} E(v),$$

provided that

$$2(1-s_2+2\theta) \leq 1.$$  \hspace{1cm} (4.16)

Hence, by setting $s_1 = s_2 - \theta$, it follows from (4.6) and (4.15) that

$$\left| \mathcal{I}_2(t) \right| \lesssim \left\| z \right\|_{L^6([T_0, T]; L^6_x)}^6 \left( 1 + \left\| \langle \nabla \rangle^{s_2-\theta} z \right\|_{L^\infty([T_0, T]; L^\infty_x)} \right) \int_{T_0}^t E(v)(t') dt'$$

$$\lesssim A(T_0, T) + A(T_0, T) \int_{T_0}^t E(v)(t') dt'.$$  \hspace{1cm} (4.18)

Finally, putting (4.8), (4.9), (4.10), (4.11), and (4.18) together and choosing sufficiently small $\varepsilon_0 > 0$ in (4.11), we obtain

$$E(v)(t) \lesssim E(v)(T_0) + A(T_0, T) + A(T_0, T) \int_{T_0}^t E(v)(t') dt',$$  \hspace{1cm} (4.19)
for any $t \in [T_0, T]$.

**Remark 4.2.** (i) In order to justify the formal computation in this subsection, we need to proceed with the smooth solution $(v_N, \partial_t v_N)$ associated with the frequency truncated random initial data (for example, to guarantee finiteness of the term $-\int_{\mathbb{R}^2} (D^{\frac{3}{2}} \partial_t v)^2 dx$ in (4.7)) and then take $N \to \infty$, using the approximation argument (Corollary 3.4). This argument, however, is standard and thus we omit details. See, for example, [57].

(ii) In this section, we followed the argument in [57] to obtain an energy bound in the quintic case. In this argument, the first term after the first inequality in (4.9) provides the restriction $p \leq 5$ on the degree of the nonlinearity $|u|^{p-1}u$. For $p > 5$, we will need to apply the integration by parts trick to lower order terms as well. See for example [40] in the context of the standard NLW. In a recent preprint [43], Liu extended Theorems 1.1 and 1.3 to the super-quintic case ($p > 5$) and proved almost sure global well-posedness of the defocusing vNLW (1.1) in $H^s(\mathbb{R}^2)$ for $s > -\frac{1}{p}$.

### 4.3. Proof of Proposition 4.1

Fix a target time $T \gg 1$ and small $\varepsilon > 0$. Then, let $T_0$ be as in (4.4). With $A(T_0, T)$ as in (4.6), set

$$A_\lambda = \{ \omega \in \Omega : A(T_0, T) < \lambda \}$$

for $\lambda > 0$. From Proposition 2.9, there exists $\lambda_0 \gg 1$ such that

$$P(A^c_{\lambda_0}) < \frac{\varepsilon}{2}. \tag{4.20}$$

Now, set $\Omega_{T, \varepsilon} = \Omega_{T_0} \cap A_{\lambda_0}$. Then, from (4.4) and (4.20), we have $P(\Omega_{T, \varepsilon}^c) < \varepsilon$.

Let $\omega \in \Omega_{T, \varepsilon}$. From (4.6) and H"older’s inequality, we see that $A(T_0, T)$ controls the $L^{5+\delta}_t(\mathbb{R}^2)$ norm of $z$:

$$\|z\|_{L^{5+\delta}_t([T_0, T]; L^{\infty}_x)} \lesssim T^\theta \lambda_0^{\frac{1}{3}}$$

for some $\theta > 0$, where $\delta > 0$ is as in (the proof of) Theorem 3.1. Then, together with the energy bound (4.19), we can iteratively apply Corollary 3.5 (see also the discussion right after Proposition 4.1) and construct a solution $u = z + v$ to (1.23) on $[0, T]$ with $(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)$ for each $\omega \in \Omega_{T, \varepsilon}$. This proves Proposition 4.1 and hence almost sure global well-posedness (Theorem 1.3).

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