Classification of Dark Solitons via Topological Vector Potentials

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Dark soliton is one of the most interesting nonlinear excitations in physical systems, manifesting a spatially localized density “dip” on a uniform background accompanied with a phase jump across the dip. However, the topological properties of the dark solitons are far from fully understood. Our investigation for the first time uncovers a vector potential underlying the nonlinear excitation whose line integral gives the striking phase jump. More importantly, we find that the vector potential field has a topological configuration in analogous to the Wess-Zumino term in a Lagrangian representation. It can induce some point-like magnetic fields scattered periodically on a complex plane, each of them has a quantized magnetic flux of elementary $\pi$. We then calculate the Euler characteristic of the topological manifold of the vector potential field and classify all known dark solitons according to the index.

Introduction—Dark soliton is one of the most commonly nonlinear excitation emerged in both quantum and classical systems, including optics [1, 2], ultracold Bose-Einstein condensates [3, 4], polariton fluid [5–7], water waves [8] and the plasmas [9, 10]. The generation of dark solitons are controllable and manipulatable in some situations [11], allowing for many important applications [12], such as optical communication, observation of negative mass effect [13], atomic matter-wave interferometers [14], quantum switches and splitters [4], etc.

Dark soliton serving as an exact solution for nonlinear differential equations has many implications in physics. If we stand on a moving soliton to investigate its behavior, the dark soliton (or anti-dark soliton) will represent a kind of transmission waves that can pass through a nonlinear potential well (or barrier for anti-dark soliton) without any reflection. More interestingly, in contrast to its bright counterpart, there is a phase jump (or shift) during the process depending on the soliton’s velocity. In the limit of zero velocity, i.e., for a stationary dark soliton, the phase jump usually takes $\pi$ value in many cases [1]. It is thus reckoned that the $\pi$ phase jump is the signal of topological excitation and the stationary dark soliton is in several respects the one-dimensional counterpart to vortices [15, 16]. However, recent studies indicates that the phase jump can be greater than $\pi$ for a saturable nonlinear media [17, 18], for an anti-dark soliton it becomes zero and tend to $-\pi/2$ when the soliton velocity approaches sound speed [19]. Moreover, the phase jump for the dark soliton in a derivative nonlinear system is found to be $\pi/2$ that is independent of the soliton’s velocity [20]. Thus, the topological properties underlying the striking phase jumps are far from fully understood.

On the other aspect, topology has emerged in real space to manifest some important physical effects such as Aharonov-Bohm effect [21] to or demonstrate some topological structures of vortex [15, 16, 22, 23], skyrmions [24, 25], and knots [26, 27]. It can also emerge in parameter space such as Berry phase theory or momentum space such as topological energy band theory, to reveal bizarre virtual particles and characterize new forms of matter including topological insulators [28], Weyl fermion semimetal [29], and even to promote the quantum computing [30, 31]. In this paper, we investigate the topological properties of dark solitons in a complex space to address the striking phase jumps. We obtain an area theorem that can associate the phase jumps with an area on a plane of the amplitude vs local phase of a soliton solution. With exploiting analytic extension of complex function, we uncover a topological vector potential in analogous to the Wess-Zumino term [32], whose line integral gives the striking phase jump. The vector potential corresponds to some point-like magnetic fields with magnetic flux of elementary $\pi$. Our result indicates that, even though the dark soliton moving in real axis can not see any magnetic fields, the point-like magnetic fields on the complex plane can actually affect the dark soliton’s motion with assigning a phase variation. We then have made a topological classification for all known dark solitons according to their Euler characteristic of the vector potential field.

Area theorem and topological vector potential—A dark soliton is a spatially localized density “dip” on top of a finite uniform background, accompanied with a phase jump through the dip [1–4]. A finite phase step emerges at soliton center when the dark soliton is stationary. For a moving dark soliton, there is a continuous phase shift across the soliton. The solution for a dark soliton can be depicted explicitly by a complex function $\psi(x, v, t)$, where $x$ is the spatial coordinate, $v$ is the soliton’s moving velocity, and $t$ is the evolution time. If we choose the soliton’s center as the reference to investigate the dark soliton’s evolution, the soliton will be always stationary.

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but its background admits a uniform density flow with a velocity $v$. In this frame, the dark soliton can be expressed as an eigenstate solution of form $\psi(x, v)$, with the boundary conditions: $\lim_{x \to +\infty} \bar{\psi}(x, v) = \sqrt{|I|} e^{i v x}$, and $\lim_{x \to -\infty} \bar{\psi}(x, v) = \sqrt{|I|} e^{i v x} + \Delta \phi_{ds}$, where $I$ is the background density, $\Delta \phi_{ds}$ is the phase jump across the dark soliton.

The local phase of the wave function $\psi(x, v)$ can be written as $\phi(x) = \phi_{ds}(x) + v x$, where $v x$ term is from the extended background flow and $\phi_{ds}(x)$ denotes the phase of the localized dark soliton solution. This analysis implies that a plane wave can propagate from $-\infty$ to $+\infty$ without any reflection through an effective quantum well induced by a dark soliton, the total phase jump can be expressed as $\Delta \phi_{ds} = \int_{-\infty}^{+\infty} \frac{d\phi_{ds}(x)}{dx} dx$.

The stationary wavefunction $\psi(x, v)$ can be viewed as an one-dimensional steady flow, satisfying the flow conservation of $|\bar{\psi}(x, v)|^2 \frac{d\phi_{ds}(x)}{dx} = Iv$. We calculate $\int_{-\infty}^{+\infty} \bar{\psi}^* (-i \frac{\partial}{\partial x}) \bar{\psi} dx = \int_{-\infty}^{+\infty} \bar{\psi}(x, v) \frac{d\phi_{ds}(x)}{dx} dx = \int_{-\infty}^{+\infty} Iv dx$, and then we have $\Delta \phi_{ds} = \int_{-\infty}^{+\infty} \frac{d\phi_{ds}(x)}{dx} dx = \int_{-\infty}^{+\infty} (1 - |\bar{\psi}(x, v)|^2 / I) d\phi$. The above formula implied that the the total phase shift of the dark soliton exactly corresponds to an area on the amplitude-phase plane. We term it as area theorem and sketch it in Fig. 1 (a). Its validity has been verified by our numerical simulations as shown in Fig. 1 (b). The area theorem is rather general, not only applicable to the simple scalar dark soliton (or anti-dark soliton) as discussed, but also other complicated vector solitons, such as dark-bright soliton [33-35], spin soliton [36], magnetic soliton [37] and dark-bright-bright soliton [38]. The area on the plane of amplitude vs. phase in fact corresponds to a classical canonical action, which has a close relation to the Aharonov-Anandan phases of nonadiabatic evolutions [39].

The area theorem can help us to uncover the topological properties of the dark solitons. We introduce a function $F[z]$, which is the analytic extension of $F[x] = (1 - |\bar{\psi}(x, v)|^2 / I) d\phi(x)/dx$, with replacing $x$ by $z = x + iy$. The phase jump of dark soliton can be described by an integral of the vector potential $A$ along real $x$ coordinate, which is introduced by considering a circle integral in the complex plane, i.e., $\oint_C F[z] dz = \oint_C (u[x,y] + iw[x,y]) (dx + idy) = \oint_C (u[x,y] - w[x,y]) \cdot dr + i \oint_C (w[x,y], u[x,y]) \cdot dr = \oint A \cdot dr$, where $A = u[x,y]e_x - w[x,y]e_y$, and $dr = dx e_x + dy e_y$. From the area theorem, we see that $F[z]$ might have some singularities of $z_N = x_N + iy_N \ (N \text{ is an integer})$ on complex plane corresponding to the divergence of flow velocity or the zero point of the density amplitude. According to the Cauchy integral formula, a meromorphic function can be expressed in terms of these singularities, that is, $F[z] = F(\infty) - \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw = \sum_N \frac{\text{Res}[F[z_N]]}{z-z_N}$. (1)

Here, $A$ term takes the form of Wess-Zumino topological term, i.e., a closed differential 1-form and its differential 2-form is zero in the Lagrangian representation [32]. This implies that corresponding magnetic field will be zero everywhere in whole complex plane except for those singular points. That is,

$$B = e_z \sum_N \pm \Omega \delta |r - r_N|.$$ (2)

The ± represent the magnetic flux direction.

It is interesting to compare with the Aharonov-Bohm effect [21] that predicts a topological phase when an electron moving on a close path around a solenoid. A dark soliton solution moving on real axis can not see those magnetic fields scattered on the complex plane, however it will acquire a phase jump due to the presence of the vector potential. In this sense, the phase jump for the dark soliton can be viewed as 1D counterpart to the famous Aharonov-Bohm phase. However, the phase shift usually does not simply equal to the magnetic flux, because the integral path for dark soliton is not a closed path around these singularities. Moreover, we find that the flux has a quantized magnetic flux of elementary $\Omega = \pi$, corresponding to a monopole with a charge $1/2$ [41]. In contrast to 2D topological excitation of vortex that admits a zero density core and a topological singularity of velocity field [15, 16, 22, 23], a moving dark soliton does not have a zero density core and its fluid field is continuous.

**Topological properties of scalar dark solitons**—We first demonstrate our theory with a simple scalar soliton. The scalar nonlinear Schrödinger equation (NLS) of form $i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi$, has wide applications in non-
FIG. 2: (a) The topological vector potential $A$ for a scalar dark soliton. (b) The corresponding magnetic field $B$ and (c) the phase jump induced by the magnetic field. The soliton speed is set to be $v = 0.5c_s$. (d) The vector potential $A$ for an anti-dark soliton \[19\]. (e) The corresponding magnetic field $B$ and (f) the phase jump induced by the magnetic field. The soliton speed is set to be $v = 0.5c_s$.

linear fiber [1], water wave [8], plasma [9], and Bose-Einstein condensate \[4\]. It has a dark soliton solution with a uniform flowing background, $\bar{\psi}(x,v) = (iv + \sqrt{1 - v^2}\tanh[\sqrt{1 - v^2}x])e^{ivx}$, with an eigenvalue of $\mu = 1 + v^2/2$. According to area theorem, we can calculate the phase jump of the dark soliton from following integral expression, $\Delta \phi = \int_{-\infty}^{+\infty} (1 - |\bar{\psi}|^2) \frac{d\phi}{dx} dx = \int_{-\infty}^{+\infty} \frac{e(1-v^2)}{\cosh^2(\sqrt{1-v^2}x)+v^2-1} dx$. The results are shown in Fig. 1 (b). It indicates that the area theorem can precisely predicts the phase jump.

The phase variation of dark soliton can be also understood from the WKB approximation. The amplitude of dark soliton serves as an effective quantum well potential. The classical action is $\int_{-\infty}^{+\infty} (\sqrt{2k(x)} - v) dx$, where $k(x) = \mu - |\bar{\psi}|^2$. In WKB approximation, the action should correspond to a quantum phase. The results are shown in Fig. 1 (b). Interestingly, the WKB phase agrees well with the area theorem in the limit when the soliton’s velocity tends to sound speed, while they deviate dramatically in the low velocity limit. This can be understood from the quantum-classical correspondence \[41, 42\].

We then can calculate the vector potential by determining the singularity locations in the complex plane, i.e.,

$$Z_N : x_N = 0, y_N = \pm y_0 + NT, \quad (N = 0, \pm 1, \ldots)$$

with $y_0 = \arccos(\sqrt{1-v^2})$ and $T = \pi/\sqrt{1-v^2}$ is the period. The vector potential is shown in Fig. 2 (a). The corresponding magnetic field $B = \nabla \times A$ can be obtained.

FIG. 3: (a) The phase distribution of the dark soliton phase and its corresponding magnetic field $B$ on the complex plane for a dark-bright-bright soliton for which dark soliton admits symmetric double-valley. The soliton speed is $v = 0.22c_s$. The singularity locations are $(\pm 0.61, \pm 0.35)$ with a period of $\pi$ on the $y$-axis. The subgraphs show the phase variations induced by the singular magnetic fields on the two separate lines, respectively. (b) The phase distribution of dark soliton and its corresponding magnetic field $B$ for a dark-bright-bright soliton for which dark soliton admits asymmetric double-valley. The singularity locations are $(\pm 0.26, \pm 0.33)$, $(\pm 0.04, \pm 2.21)$, $(0.30, \pm 2.49)$, $(1.54, \pm 2.59)$, and $(1.67, \pm 0.41)$ with a period of $2\pi$ on the $y$-axis. The soliton speed is $v = 0.19c_s$. The subgraphs show the phase variations induced by the singular magnetic fields on the five separate lines, respectively.
accordingly, which is shown in Fig. 2 (b). The corresponding phase jump for dark soliton is shown in Fig. 2 (c). We see that the positive and negative magnetic flux emerge in pairs and locate periodically on the imaginary axis. Both the distance between the singularities and the period depends explicitly on the velocity of soliton. For instance, in the limit of \( v \to 0 \), the magnetic fields at \( \pm y_0 \) tends to merge each other at origin. With increasing the moving velocity to sound speed (here \( c_s = 1 \)), they tend to merge with other two magnetic flux at \( y = \mp y_0 \pm T \), respectively. In this limit, the dark soliton degenerates into a plane wave.

We now calculate the merge process when \( v \to 0 \) to understand \( \pi \) jump of the soliton from the vector potential perspective. When \( v \to 0 \), \( |y_0| \to 0 \). In this limit,

\[
\lim_{v \to 0} A \equiv \lim_{|y_0| \to 0} \left( \frac{\pi (x e_y + |y_0| e_x)}{2\pi (x^2 + |y_0|^2)} - \frac{-\pi (x e_y - |y_0| e_x)}{2\pi (x^2 + |y_0|^2)} \right) = \pi \delta(x) e_x.
\]

Note that other singularities will merge each other in the process, and then \( \Delta \phi_{ds} = \int_{-\infty}^{\infty} A \, dx = \pi \), indicating that the \( \pi \) jump arises from the quantized magnetic flux corresponding to a monopole.

The phase jump for the anti-dark soliton \([19]\) is also of interest. It is zero for the static solution and tend to be \(-\pi/2\) when the moving speed approaching sound speed. This behavior can be explained by underlying topological potential. The vector potentials and magnetic fields are shown Fig. 2 (d) and (e). In the limit of \( v \to 0 \), the paired magnetic fields tends to merge each other and gives zero phase jump. With increasing the soliton velocity to sound speed, they tend to locate at imaginary axis separately. Comparing Fig. 2 (e) with the Fig. 2 (b), we find that the directions of the corresponding flux are reversed that leads to a negative phase jump (see Fig. 2 (f)).

**Topological properties of vector dark solitons**— We now extend our discussions to a more complicated three-component coupled NLS system. With using the developed Darbox transformation \([43, 44]\) and after lengthy deductions, we obtain a kind of dark-bright-bright soliton (DBBS)\([45]\), for which there is a dark soliton with corresponding magnetic fields and the dark soliton phase \( \phi_{ds} \) is in the regime \([0, \pi]\) component is in the regime \([\frac{\pi}{2}, \pi]\) and \([-\frac{\pi}{2}, 0]\) correspond to dark and anti-dark soliton components, respectively. \(-\pi/2\) means corresponding quantities can not be calculated due to the absence of exact explicit expressions.

\[
TABLE I: \Delta \phi_{ds} \ the \ interval \ of \ phase \ jump; \ T \ is \ the \ period \ in \ y \ axis \ where \ c_s \ is \ the \ sound \ speed; \ g \ refers \ to \ the \ number \ of \ the \ singularities \ in \ one \ period; \ M \ is \ topological \ manifold \ space; \ Z_1 \ represents \ singular \ points; \ \chi = 0 - 2g \ is \ the \ Euler \ characteristic \ number. \ The \ solutions \ of \ DBBS1, \ DBBS2 \ and \ DBBS3 \ are \ obtained \ from \ present \ work\([45]\). \ For \ MS \([37]\), \ the \ phase \ jump \ regimes \([\frac{\pi}{2}, \pi]\) and \([-\frac{\pi}{2}, 0]\) correspond to dark and anti-dark soliton components, respectively. \(-\pi/2\) means corresponding quantities can not be calculated due to the absence of exact explicit expressions.

\[
\text{Types} & \quad \Delta \phi_{ds} & \quad T & \quad g & \quad M & \quad \chi \\
\hline
\text{DS} \ [1] & \[0, \pi\] & \pi & 2 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -4 \\
\text{DS} \ [19] & \[\frac{\pi}{2}, \pi\] & \sqrt{\xi^2 + \nu^2} & 2 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -4 \\
\text{A-DS} \ [19] & \[-\frac{\pi}{2}, 0\] & \sqrt{\xi^2 + \nu^2} & 2 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -4 \\
\text{SS} \ [36] & \[0, \pi\] & \pi & 2 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -4 \\
\text{DS} \ [46] & \[0, \pi\] & \sqrt{\xi^2 + \nu^2} & 2 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -4 \\
\text{DS} \ [20] & \frac{\pi}{2} & \pi & 2 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -4 \\
\text{DBS} \ [35] & \[0, \pi\] & \pi & 2 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -4 \\
\text{DBBS} \ [38] & \frac{\pi}{2} & \pi & 2 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -4 \\
\text{DS} \ [17, 18] & \[0, 2\pi\] & \pi & 4 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -8 \\
\text{MS} \ [37] & \[\frac{\pi}{2}, \pi\] & \pi & 6 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -12 \\
\text{DBBS1} & \[0, 2\pi\] & \pi & 4 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -8 \\
\text{DBBS2} & \[0, 2\pi\] & 2\pi & 10 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -20 \\
\text{DBBS3} & \[0, 3\pi\] & \pi & 6 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -12 \\
\text{DBBS4} & \[0, 3\pi\] & \pi & 6 & S^1 \times \mathbb{R}/\mathbb{Z}_1 & -12 \\
\end{array}
\]

From Table I, we see that the dark soliton in a derivative nonlinear system is very special, in which the phase jump is \( \frac{\pi}{2} \) independent of the soliton velocity \([20]\). However, the singularities of vector potential locate at \( \pm i \frac{\pi}{2} + iNT \) with \( T = 4\pi/v \), whose distribution obviously depends on the velocity. Nevertheless, after a simple scaling transformation, an equivalent topological vector potential can be derived in a velocity independent form, i.e., \( A = \sum \frac{\pm |xe_y - (y - y_0)e_x|}{2|x^2 + (y - y_0)^2|} \) with the singularities of \( y_0 = \pm \frac{\pi}{2} + 2\pi N \), whose line integral will gives a definite \( \pi/2 \) phase jump.

**Conclusion**— We show that the dark solitons serving as a kind of simple nonlinear excitations can demonstrate very interesting topological properties. The phase jump can be viewed as the 1D counterpart to the famous Aharonov-Bohm phase, where the solenoid that might carry an arbitrary flux is replaced by a monopole with
a quantized magnetic flux of elementary $\pi$. Underlying vector fields demonstrate the topology of Wess-Zumino term. Our investigations have resolved a long-standing puzzle on the topological origin of dark solitons and provides a possibility to investigate topological vector potential via the generation of dark solitons that are controllable in current BEC, optic and microcavity polariton condensates experiments.

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