Covariance Matrix Estimation under Total Positivity for Portfolio Selection

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Abstract

Selecting the optimal Markowitz portfolio depends on estimating the covariance matrix of the returns of $N$ assets from $T$ periods of historical data. Problematically, $N$ is typically of the same order as $T$, which makes the sample covariance matrix estimator perform poorly, both empirically and theoretically. While various other general purpose covariance matrix estimators have been introduced in the financial economics and statistics literature for dealing with the high dimensionality of this problem, we here propose an estimator that exploits the fact that assets are typically positively dependent. This is achieved by imposing that the joint distribution of returns be multivariate totally positive of order 2 (MTP\textsubscript{2}). This constraint on the covariance matrix not only enforces positive dependence among the assets, but also regularizes the covariance matrix, leading to desirable statistical properties such as sparsity. Based on stock-market data spanning over thirty years, we show that estimating the covariance matrix under MTP\textsubscript{2} outperforms previous state-of-the-art methods including shrinkage estimators and factor models.

1. Introduction

Given a universe of $N$ assets, what is the optimal way to select a portfolio? When “optimal” refers to selecting the portfolio with minimal risk or variance for a given level of expected return, then the solution, commonly known as the Markowitz optimal portfolio, depends on two quantities: the vector of expected returns $\mu^*$ and the covariance matrix of the returns $\Sigma^*$ \cite{Mar52}. In practice, $\mu^*$ and $\Sigma^*$ are unknown and must be estimated from historical returns. Since $\Sigma^*$ requires estimating $O(N^2)$ parameters while $\mu^*$ only requires estimating $O(N)$ parameters, the main challenge lies in estimating $\Sigma^*$. A naive strategy is to use the sample covariance matrix $S$ to estimate $\Sigma^*$. However, this estimator is known to have poor properties \cite{MP67, Wac78, BY93, Joh01, JLN09}, as can be seen by the following degrees-of-freedom argument (see also \cite[Section 3.1]{ELW17}): as is common when daily or monthly returns are used, the number of historical data points $T$ is of the order of 1000 while the number of assets $N$ typically ranges between 100 and 1000. Since in this case $T \ll N^2$, then only $O(1)$ effective samples are used to estimate each entry in the covariance matrix, making the sample covariance matrix perform poorly out-of-sample \cite{LW04, LW12, ELW17}.

Given the importance and the statistical challenges of covariance matrix estimation in the high-dimensional setting, this problem has been widely studied in the statistics...
and financial economics literature. In the statistical literature, a number of estimators have been proposed based on banding or soft-thresholding the entries of $S$ [BL08, WP09, CZZ10]. Such estimators, which are equivalent to selecting the covariance matrix closest to $S$ in Frobenius norm subject to the covariance matrix lying within a specified $L_1$ ball, were proven to be minimax optimal with respect to the Frobenius norm and spectral norm loss [CZZ10]. However, such estimators may not output a covariance matrix estimate that is positive definite, which is required for the Markovitz portfolio selection problem. Moreover, while such estimators are optimal in a minimax sense for the Frobenius and spectral norm loss, these losses may not be relevant to measure the excess risk that results from using an estimate of $\Sigma^*$ instead of $\Sigma^*$ itself to compute the Markovitz portfolio; see [ELW17, Section 4.1] for details.

Another reason to consider estimators beyond those in [BL08, WP09, CZZ10] is that these methods do not exploit some of the structure that often holds in $\Sigma^*$. In particular, the eigenspectrum of $\Sigma^*$ is often structured; we expect to find several important “directions” (i.e., eigenvectors) that well-approximate $S$. For example, under the capital asset pricing model [BJS72], the eigenspectrum of $\Sigma^*$ contains a dominant eigenvector corresponding to the market; as a consequence, $S$ could be well-approximated by a rank one matrix. More generally, covariance matrix estimators based on low-rank approximations of $S$ are advantageous statistically since such estimators have smaller variance. In practice, low-rank covariance estimates are based on explicitly provided factors [FF93b, FF15, BJS72], or data-driven factors learned by performing principal component analysis (PCA) on $S$ [FLM13, JYM11]. Another related popular strategy for estimating $\Sigma^*$ is based on the assumption that the eigenvalues of $\Sigma^*$ are well-behaved, and exploit results from random matrix theory [EK08, MP67]. In particular, various methods considered regularizing the eigenvalues of $S$ [LW04, LW12, ELW17, JM03, DMUN13]; collectively, these methods can be regarded as particular instances of empirical Bayesian shrinkage [Hal80, LW04, Ste56]. Finally, a number of papers have proposed covariance estimators based on the assumption that the precision matrix is sparse [FHT08, RWRY11]. Such a constraint is motivated by the fact that a sparse precision matrix implies that the induced undirected graphical model associated with the joint distribution is sparse, which is desirable both for better interpretability and robustness properties.

In this paper, we propose a new type of covariance matrix estimator for portfolio selection based on the assumption that the underlying distribution is multivariate totally positive of order 2 (MTP$_2$), which exploits a particular type of structure in the covariance matrix. MTP$_2$ was first studied in [FKG71, KR80a, Bol82, KR83] from a purely theoretical perspective and later also in the context of statistical modeling, in particular graphical models, in [SH14, FLS+17, LUZ19a, LUZ19b]. MTP$_2$ is a strong form of positive dependence that can be used in combination with the above methods for covariance estimation. The structure we exploit is motivated by the observation that asset returns are often positively correlated since assets typically move together with the market. As an illustration, consider the sample correlation matrix $S$ and its inverse $S^{-1}$ based on the 2016 monthly returns of global stock markets shown in Figure 1. Note that all correlations (i.e., off-diagonal entries of $S$) are positive, and moreover, also all partial correlations (i.e., negative of the off-diagonal entries of $S^{-1}$) are positive. In fact, since

\footnote{If the covariance matrix estimator has rank $M$, then the effective number of parameters estimated is $O(NM)$ instead of $O(N^2)$ where $M \ll N$.}
Figure 1: The sample correlation matrix of global stock markets based on the 2016 daily returns. Notice that the covariance matrix contains all positive entries and the precision matrix is an M-matrix which implies that the joint distribution is MTP2 (see Section 3.2 for details).

$$S^{-1}$$ is a symmetric $M$-matrix, i.e. $(S^{-1})_{ij} \leq 0$ for all $i \neq j$, it implies that all entries of $S$ are non-negative [Ost37, DMSM14].

A multivariate Gaussian distribution with mean $\mu$ and positive definite covariance matrix $\Sigma$ is MTP2 if and only if the precision matrix $\Sigma^{-1}$ is a symmetric M-matrix [Bøl82, KR80a] and hence all correlations and partial correlations are non-negative. This implies that the sample distribution of the 2016 daily returns of global stocks considered in Figure 1 is MTP2. This is quite remarkable, since uniformly sampling correlation matrices, e.g. using the method described in [Joe06], shows that less than 0.001% of all $5 \times 5$ correlation matrices satisfy the MTP2 constraint. Since factor analysis models with a single factor are MTP2 when each observed variable has a positive dependence on the latent factor [WM14], the capital asset pricing model implies MTP2, which further motivates studying MTP2 in the context of portfolio selection.

The remainder of this paper is organized as follows: In Section 2 we review the Markowitz portfolio selection problem and explain how it relates to covariance matrix estimation and discuss various covariance estimation techniques.

2. Background

After introducing some notation, we will review the Markowitz portfolio selection problem and explain how it relates to covariance matrix estimation and discuss various covariance estimation techniques.
2.1. Notation
We assume throughout that we are given $N$ assets, which we index using the subscript $i$, from $T$ dates (e.g. days), which we index using the subscript $t$. We let $r_{i,t}$ denote the observed return for asset $i$ at date $t$ for $1 \leq i \leq N$ and $1 \leq t \leq T$. The vector $\mathbf{r}_t := (r_{1,t}, \ldots, r_{N,t})^T$ consists of the returns of each asset on day $t$. Finally, $\mu_t := E[r_t]$ and $\Sigma_t := \text{Cov}(r_t)$ denote the expected returns and the covariance matrix of the returns for day $t$, respectively.

2.2. Optimal Markowitz Portfolio Allocation
Markowitz portfolio theory concerns the problem of assigning weights $w \in \mathbb{R}^N$ to a universe of $N$ possible assets in order to minimize the variance of the portfolio for a specified level of expected returns $R$. More precisely, the optimal portfolio weights $w \in \mathbb{R}^N$ on day $t$ are found by solving

$$
\begin{align*}
\text{minimize} \quad & w^T \Sigma_t^* w \\
\text{subject to} \quad & w^T \mu_t^* = R \quad \text{and} \quad \sum_{i=1}^N w_i = 1,
\end{align*}
$$

where $\mu_t^*$ and $\Sigma_t^*$ denote the true expected returns and covariance matrix of the returns for day $t$. In practice, $\mu_t^*$ and $\Sigma_t^*$ are unknown and must be estimated from historical returns. Since the main difficulty lies in estimating $\Sigma_t^*$ (it requires estimating $O(N^2)$ parameters as compared to $O(N)$ for $\mu_t^*$), a widely used tactic to specifically evaluate the quality of a covariance matrix estimator is by finding the global minimum variance portfolio, which does not require estimating $\mu^*$ [HB91, JM03]. Such a portfolio can be found by solving

$$
\begin{align*}
\text{minimize} \quad & w^T \Sigma_t w \\
\text{subject to} \quad & \sum_{i=1}^N w_i = 1,
\end{align*}
$$

where $w$ is chosen to minimize the variance of the portfolio. Replacing the unknown true covariance matrix of returns $\Sigma_t^*$ by some estimator $\hat{\Sigma}_t$ yields the following analytical solution for Eq. (2):

$$
\hat{\Sigma}_t := \frac{\hat{\Sigma}_t^{-1}}{1^T \hat{\Sigma}_t^{-1} 1}.
$$

2.3. Covariance Estimators
As discussed in Section 1, the sample covariance matrix is a poor estimator of the true covariance matrix, particularly in the high-dimensional setting when the number of assets $N$ exceeds the number of periods $T$ (the sample size). Although the sample covariance matrix is an unbiased estimator of the true covariance matrix, in the high-dimensional setting it is not invertible, has high variance, and is not consistent (e.g., the eigenvectors of $S$ do not converge to those of $\Sigma^*$ [MP67, Joh01, Wac78, BY93, JLN+09]). Making structural assumptions about the true covariance matrix allows the construction of estimators that have lower variance with only a small increase in bias. In the following, we review several models and techniques for covariance matrix estimation that are commonly used in a financial context.
2.3.1. Factor Models

A common modeling assumption in financial applications is that the returns for day $t$ are given by a linear combination of a (small) collection of latent factors $f_{k,t}$ for $1 \leq k \leq K$, which are either explicitly provided or estimated from the data. In such a factor model, the returns are modeled as

$$r_{i,t} = \alpha_i + \beta_i^T f_t + u_{i,t}, \quad f_t := (f_{1,t}, \ldots, f_{K,t}),$$

(4)

where $u_{i,t}$ is the idiosyncratic error term for asset $i$ that is uncorrelated with $f_t$. Letting $B \in \mathbb{R}^{K \times N}$ be the matrix whose $i$th column is $\beta_i$, the covariance matrix of the returns can be expressed as

$$\Sigma_t = B^T \Sigma_{f,t} B + \Sigma_{u,t}, \quad \text{for } 1 \leq t \leq T,$$

where $\Sigma_{f,t} := \text{Cov}(f_t)$ and $\Sigma_{u,t} := \text{Cov}(u_t)$. In practice, $K \ll N$ factors are selected, making $B^T \Sigma_{f,t} B$ low-rank. This low-rank structure makes estimating $\Sigma_t$ easier since $\Sigma_{f,t}$ and $B$ only have $O(K^2)$ and $O(NK)$ free parameters, respectively. When $K \ll N$, and $K \ll T^2$, then by standard concentration of measure results, $\Sigma_{f,t}$ can be estimated well by $\hat{\Sigma}_{f,t}$, the sample covariance matrix of the factors. Similarly, by Eq. (4), the $i$th row of $B$ can be estimated by regressing the returns of asset $i$ on the $K$ latent factors, for example using ordinary least-squares. In this case, $\hat{\beta}_i \approx \beta_i$ and hence the error $u_{i,t}$ is approximately equal to the residual $\hat{u}_{i,t} := r_{i,t} - \hat{\beta}_i^T f_t - \hat{\alpha}_i$. Thus $\Sigma_{u,t}$ can be approximated by a covariance matrix estimate $\hat{\Sigma}_{u,t}$ based on the residuals.

Several different types of factor models of varying complexity have been considered in the literature: The general model in Eq. (4) is known as a dynamic factor model. A static factor model assumes that the covariance matrices $\Sigma_{u,t}$ and $\Sigma_{f,t}$ are time-invariant, i.e., $\Sigma_{u,t} = \Sigma_u$ and $\Sigma_{f,t} = \Sigma_f$ do not depend on $t$. An exact factor model furthermore assumes that the covariance matrix $\Sigma_u$ is diagonal, whereas an approximate factor model assumes that $\Sigma_u$ has bounded $L^1$ or $L^2$ norm. In this paper, we concentrate on static estimators. The following static factor-based covariance matrix estimators are popularly used in financial applications.

- **POET:** is based on an approximate factor model and was first proposed in [FLM13]. POET estimates $B^T \Sigma_{f,t} B$ by a rank $K$ truncated singular value decomposition (SVD) of the sample covariance matrix $\hat{\Sigma}$, which we denote by $\hat{\Sigma}_K$. $\hat{\Sigma}_u$ is estimated by soft-thresholding the off-diagonal entries of the residual covariance matrix $\hat{S}_u = \hat{\Sigma} - \hat{\Sigma}_K$ based on the method in [BL08].

- **EFM:** is an estimator based on the exact factor model using the Fama-French factors [FF93a] such as the 1-factor or 5-factor model consisting of all 5 Fama-French factors. In either case, $\hat{\Sigma}_f$ equals the sample covariance matrix of the factors $\{f_t\}$ and $\hat{\Sigma}_u$ equals the diagonal of $\hat{S}_u$.

- **AFM-POET:** is an estimator based on an approximate factor model using the Fama-French factors. $\hat{\Sigma}_f$ is obtained as in EFM, whereas $\hat{\Sigma}_u$ is obtained by soft-thresholding $\hat{S}_u$ as in POET.
2.3.2. Shrinkage of Eigenvalues

Another way to impose structure on the covariance matrix is through assumptions on the eigenvalues of the covariance matrix. Assuming that the true covariance matrix is well-conditioned, then the extreme eigenvalues of the sample covariance matrix are generally too small/large as compared to the true covariance matrix [MP67, BY93]. This motivates the development of covariance matrix estimators such as linear shrinkage [LW04] and extensions thereof [LW12, ELW17] that shrink the eigenvalues of the sample covariance matrix for better statistical properties.

To be more precise, let

\[ S = \sum_{i=1}^{N} \lambda_i v_i v_i^T, \]

be the eigendecomposition of the sample covariance matrix \( S \), where \( \lambda_i \) denotes the \( i \)-th eigenvalue of \( S \) and \( v_i \) the corresponding eigenvector. Then the linear shrinkage estimator is given by

\[ \hat{\Sigma}_{LS} = \sum_{i=1}^{N} \gamma_i v_i v_i^T, \]

where \( \gamma_i = \rho \lambda_i + (1 - \rho) \bar{\lambda} \) with \( \bar{\lambda} \) denoting the average of the eigenvalues of \( S \) and \( 0 < \rho < 1 \) a tuning parameter that determines the amount of shrinkage. Note that \( \hat{\Sigma}_{LS} \) can equivalently be expressed as

\[ \hat{\Sigma}_{LS} = \rho S + (1 - \rho) \bar{\lambda} I_N, \quad (5) \]

where \( I_N \in \mathbb{R}^{N \times N} \) denotes the identity matrix (Eq. (5) follows from the uniqueness of the eigenvalue decomposition). Thus \( \hat{\Sigma}_{LS} \) is obtained by shrinking the sample covariance matrix towards a multiple of the identity, which from a Bayesian point of view can also be interpreted as using the identity matrix as a prior for the true covariance matrix [LW04]. The shrinkage estimator \( \hat{\Sigma}_{LS} \) is asymptotically efficient given a particular choice of \( \rho \) that depends on the sample covariance matrix \( S \), its dimension \( N \) (i.e., the number of assets) and the number of samples \( T \) (i.e., the number of dates) [LW04].

An extension of linear shrinkage, known as non-linear shrinkage, considers non-linear transforms of the eigenvalues according to the Marcenko-Pastur distribution, which describes the asymptotic distribution of eigenvalues of random matrices, and has been shown to out-perform the linear-shrinkage estimator empirically [LW12]. In addition, the shrinkage estimators described here can be combined with the factor model estimators from Section 2.3.1 by using linear or non-linear shrinkage applied to \( \hat{\Sigma}_u \), the sample covariance matrix of the error terms, to obtain the estimate \( \hat{\Sigma}_u \). For example, \( AFM-LS \) and \( AFM-NLS \) apply linear shrinkage and non-linear shrinkage respectively, to estimate \( \Sigma_u \) for the approximate factor model using the Fama-French factors.

2.3.3. Regularization of the Precision Matrix

Another common technique for covariance matrix estimation is to assume that the true unknown inverse covariance matrix \( K^* := (\Sigma^*)^{-1} \), also known as the precision matrix, is sparse, i.e. the number of non-zero entries in \( K^* \) is bounded by an integer
$\kappa > 0$. Since estimating $K$ under the constraint

$$\|K\|_0 := \sum_{i \neq j} I[K_{ij} \neq 0] \leq \kappa$$  \hspace{1cm} (6)$$

is computationally intractable as it involves solving a difficult combinatorial optimization problem, a standard approach is to replace the $L_0$ constraint in Eq. (6) by an $L_1$ constraint. In particular, assuming that the data follows a multivariate Gaussian distribution, then the $L_1$-regularized maximum likelihood estimator (also known as graphical lasso) can be used to estimate $K$ [FHT08, RWRY11]. Maximum likelihood estimation under the the $L_1$ constraint leads to the following convex optimization problem:

$$\hat{K} := \text{arg max}_{K \succeq 0} \log \det K - \text{trace}(KS) \quad \text{subject to} \quad \|K\|_1 \leq \lambda,$$  \hspace{1cm} (7)

where $\lambda \geq 0$ is a tuning parameter. Instead of maximizing the log-likelihood, the popular CLIME estimator [LHZ12] finds a sparse estimate of the precision matrix by solving

$$\hat{K} := \text{arg min}_K \|K\|_1 \quad \text{subject to} \quad \|SK - I_N\|_\infty \leq \lambda.$$  \hspace{1cm} (8)

and has similar consistency guarantees as graphical lasso in the Gaussian setting.

To overcome the restrictive Gaussian assumption, recent work suggested replacing the sample covariance matrix $S$ in Eq. (7) and Eq. (8) by Kendall’s tau correlation matrix $S_\tau$ with $(S_\tau)_{ij} := \sin(\frac{\pi}{2} \hat{\tau}_{ij})$, where

$$\hat{\tau}_{ij} := \frac{1}{T^2} \sum_{1 \leq t \leq t' \leq T} \text{sign}(X_{it} - X_{it'}) \text{sign}(X_{jt} - X_{jt'}).$$  \hspace{1cm} (9)

Interestingly, the resulting estimators can also be used for data from heavy-tailed distributions (including elliptical distributions such as the $t$-distribution) with almost no loss in efficiency [LHZ12, BK18]; see also Section 3.3.

3. Covariance Matrix Estimation under MTP$_2$

So far, we have reviewed different methods for high-dimensional covariance matrix estimation assuming a low-rank factor model, regularity in the eigenspectrum, and/or sparsity in the underlying inverse covariance matrix. In the following, we propose a new structure for modeling asset returns data, namely by exploiting that assets are typically positively dependent. In particular, we consider distributions that are MTP$_2$.

**Definition 3.1** ([FKG71, KR80b]). A distribution on $\mathcal{X} \subseteq \mathbb{R}^M$ is multivariate totally positive of order 2 (MTP$_2$) if its density function $p$ satisfies

$$p(x)p(y) \leq p(x \wedge y)p(x \vee y) \quad \text{for all} \quad x, y \in \mathcal{X},$$

where $\wedge, \vee$ denote the coordinate-wise minimum and maximum, respectively.
MTP\textsubscript{2} is a strong form of positive dependence that implies most other known forms including e.g. positive association; see for example [CSS05] for a recent overview. Note that when \( p(x) \) is a strictly positive density, then Definition 3.1 is equivalent to \( p(x) \) being log-supermodular. Log-supermodularity has a long history in economics, in particular in the context of complementarity and comparative statics [Top78, MR90, MS94, Top98, Ath02, Cos09].

In Fig. 1, we provided an example of 5 global stocks, where the sample distribution is MTP\textsubscript{2}. To further motivate studying MTP\textsubscript{2} as a constraint for covariance matrix estimation for portfolio selection we discuss its connection to latent tree models in Section 3.1. In particular, we show that the capital asset pricing model implies that the resulting joint distribution is MTP\textsubscript{2}. Then in Section 3.2, we discuss how to perform covariance matrix estimation under MTP\textsubscript{2} in the Gaussian setting. Finally, in Section 3.3, we propose how to extend this estimator to heavy-tailed distributions.

3.1. Latent Tree Models

A powerful framework to model complex data such as stock-market returns is through models with latent variables. Factor models, which are widely used for covariance estimation for portfolio selection (see Section 2.3.1) are examples thereof. A latent tree model is an undirected graphical model on a tree (where every node represents a random variable that may or may not be observed and any two nodes are connected by a unique path). For financial applications, latent tree models have been used, for example, for unsupervised learning tasks, such as clustering similar stocks, or for modeling and learning the dependence structure among asset returns [CTAW11, Man99]. A factor analysis model with a single factor is a particular example of a latent tree model consisting of an unobserved root variable that is connected to all the observed variables; see Fig. 2 for a concrete example of a single-factor analysis model and a more general latent tree model. The prominent capital asset pricing model (CAPM) is a single-factor analysis.
The return of stock $i$ is modeled as

$$r_i = r_f + \beta_i (r_m - r_f)$$

where $r_f$ is known as the risk-free rate of return and $r_m$ is the market return. Typically, the parameters $\beta_i$ are positive, which explains why the covariance between stock returns is usually positive. Non-negative correlation is in general necessary but not sufficient to imply MTP$^2$. The following theorem states that for latent tree models non-negative correlation already implies MTP$^2$. The proof follows from [LUZ19a, Theorem 5.4].

**Theorem 3.2.** Let $X \in \mathbb{R}^M$ follow a multivariate Gaussian distribution that factorizes according to a tree. If $\text{Cov}(X) \geq 0$, then $X$ is MTP$^2$ and any marginal of $X$ is MTP$^2$.

While working with CAPM is convenient from a theoretical perspective, its simplicity often comes at the expense of underfitting. In particular, there commonly are additional sector-level factors that drive returns. Identifying these factors is an active area of research; for instance, CAPM was recently extended to include three and then five new factors [FF93b, FF15]. However, identifying relevant factors is in general a challenging task; for example, learning the structure of a latent tree model from data is known to be NP-hard [Coo90]. We here propose to instead take a structure-free approach by constraining the joint distribution over the observed variables to be MTP$^2$. This approach provides more flexibility than modeling stock returns using latent tree models and at the same time allows overcoming the computational bottleneck of fitting a latent tree model. In particular, we show in Section 3.2 that an MTP$^2$ covariance matrix estimator can be computed by solving a convex optimization problem.

### 3.2. MTP$^2$ Covariance Matrix Estimation Assuming Multivariate Gaussian Returns

For multivariate Gaussian distributions, a necessary and sufficient condition for a distribution to be MTP$^2$ is that the precision matrix $K := \Sigma^{-1}$ is an M-matrix, i.e., $K_{ij} \leq 0$ for all $i \neq j$; or equivalently, all partial correlations are nonnegative. [KR80a]. Following [LUZ19a], we consider the maximum likelihood estimator (MLE) of $K$ subject to $K$ being an M-matrix.

Recall that the log-likelihood function $\mathcal{L}$ of $K$ given data $D := \{r_t\}_{t=1}^T \overset{\text{i.i.d}}{\sim} N(0, K)$ is, up to additive and multiplicative constants, given by

$$\mathcal{L}(K; D) = \log \det K - \text{trace}(KS), \quad (10)$$

where $S \in \mathbb{R}^{N \times N}$ denotes the sample covariance matrix of the returns $\{r_t\}_{t=1}^T$ or log-returns. Without the MTP$^2$ constraint, the MLE of $K$ is obtained by maximizing $\mathcal{L}(K; D)$ over the set of all positive semidefinite matrices and is given by $S^{-1}$ when $N \leq T$ (i.e., the dimension of the covariance matrix is less than the number of samples). Note that when $N > T$, the MLE does not exist, i.e., the log-likelihood function is unbounded above. Remarkably, by adding the constraint that $K$ is an M-matrix (i.e., that the distribution is MTP$^2$), then the MLE

$$K = \arg \max_{K \geq 0} \log \det K - \text{trace}(KS) \quad \text{subject to} \quad K_{ij} \leq 0 \quad \forall i \neq j, \quad (11)$$

Note that when $97\%$ of the entries of the sample covariance matrix of 1000 assets (based on daily returns from 1980-2015) are positive.
exists with probability 1 when $T \geq 2$ for any dimension $N$ [SH14, LUZ19a]. Similarly, the CLIME estimator in Eq. (8) could be extended to the MTP$_2$ setting by adding the constraints $K_{ij} \leq 0$ for all $i \neq j$ and it would be of interest to understand its properties.

The fact that a unique solution exists for Eq. (11) for any $N$ when $T \geq 2$ suggests that the MTP$_2$ constraint adds considerable regularization for covariance matrix estimation. In addition, the problem in Eq. (11) is a convex optimization problem and computationally efficient coordinate-descent algorithms have been described for computing $\hat{K}$ [LUZ19a, SH14]. Finally, another desirable property is that the MTP$_2$ covariance matrix estimator $\hat{K}$ in Eq. (11) is usually sparse [LUZ19a, Corallary 2.9], which reduces the intrinsic dimensionality of the model and hence reduces the variance of the estimator. Note that in Eq. (11) this is achieved without the need of any tuning parameter, an immediate advantage over methods that explicitly add sparsity-inducing $L_1$ penalties such as the graphical lasso [FHT08, RWRY11] discussed in Section 2.3.3.

### 3.3. Extensions to Heavy-Tailed Distributions

It is common to log-transform data on stock returns before estimating the covariance matrix. However, even after log-transformation the data may still be heavy-tailed. In this case, the Gaussian assumption made for estimating the covariance matrix in Section 3.2 may be problematic. Transelliptical distributions form a convenient class of distributions that contain the Gaussian distribution as well as heavy-tailed distributions such as the $t$-distribution. In the following, we provide an extension of the estimator in Eq. (11) to transelliptical distributions.

A random vector $X$ with density function $p(x)$, mean $\mu \in \mathbb{R}^M$ and covariance matrix $\Sigma \in \mathbb{R}^{M \times M}$ follows an elliptical distribution if its density function can be expressed as

$$g((x - \mu)^T \Sigma^{-1} (x - \mu))$$

for some function $g$. More generally, $X$ follows a transelliptical distribution if there exist monotonically increasing functions $f_i$, $i = 1, \ldots, M$, such that $(f_1(X_1), \ldots, f_M(X_M))$ follows an elliptical distribution. We denote the covariance matrix of this elliptical distribution by $\Sigma_f$. The following result provides a necessary condition for a transelliptical distribution to be MTP$_2$.

**Theorem 3.3.** Suppose that the joint distribution of $(X_1, \cdots, X_M)$ is MTP$_2$ and transelliptical, i.e., there exist increasing functions $f_i$, $i = 1, \ldots, M$, such that the density function of $(f_1(X_1), \cdots, f_M(X_M))$ can be written as $g((x - \mu)^T \Sigma_f^{-1} (x - \mu))$. Then, $\Sigma_f^{-1}$ is an M-matrix.

The proof of this theorem requires the following simple lemma.

**Lemma 3.4.** Suppose $g(x)$ is differentiable, non-negative, and $\int_{-\infty}^{\infty} g(x) dx = 1$. Then, for any $\delta, M > 0$, there exists an $x^* > M$ such that $g(\cdot)$ is strictly decreasing on the interval $(x^*, x^* + \delta)$.

**Proof.** Let $I = \{x : g'(x) > 0\}$. Then, the Lebesgue measure of $I$ is finite since $g(\cdot)$ is non-negative and integrates to one. Suppose towards a contradiction that there was no such $x^*$. Then, for any $x > M$, $g(\cdot)$ is not monotonically decreasing on $(x, x + \delta)$. Hence, by continuity of $g(\cdot)$, there exists an interval $I_x$ of length $\Delta_x$ contained in $(x, x + \delta)$ such
that \( g(\cdot) \) is monotonically increasing on \( I_x \). Let \( \bigcup_{j=1}^\infty I_{x_j} \) be some disjoint covering of \( \{x : x > M\} \), where \( I_{x_j} := (x_j, x_j + \delta) \). Then, by our previous argument, \( I_{x_j} \) contains an interval of length \( \Delta_{x_j} \) where \( g(\cdot) \) is monotonically increasing. By assumption, \( \inf_j \Delta_{x_j} > 0 \) and \( \lim_{j \to \infty} \Delta_{x_j} > 0 \). Hence, \( \sum_j \Delta_{x_j} = \infty \) which contradicts that \( I \) has finite Lebesgue measure.

**Proof of Theorem 3.3.** Note that by [KR80a, Equation 1.13], if \( X \) is MTP₂, then so is \((f_1(X), \cdots, f_M(X))\). Hence \( \Sigma_{ij} \geq 0 \) for all \( i \neq j \). To complete the proof, we need to show that \( (\Sigma^{-1})_{ij} \leq 0 \) for all \( i \neq j \). Without loss of generality, we assume that \( \mu = 0 \). We consider the two points \( x = s_1 e_i - s_2 e_j \) and \( y = -x \), where \( e_k \in \mathbb{R}^M \) denotes the \( k \)-th unit vector and \( s_i \in \mathbb{R} \). For ease of notation, let \( \Sigma_{i,i} = a \), \( \Sigma_{j,j} = b \), and \( \Sigma_{i,j} = \Sigma_{j,i} = c \).

Notice that

\[
p(x) = p(y) = g(s_1^2 a + s_2^2 b - 2s_1 s_2 c) \quad \text{and} \quad p(x \lor y) = (x \land y) = g(s_1^2 a + s_2^2 b + 2s_1 s_2 c).
\]

Hence, since \((f_1(X_1), \cdots, f_M(X_M))\) is MTP₂, it holds that

\[
g(s_1^2 a + s_2^2 b - 2s_1 s_2 c)^2 \leq g(s_1^2 a + s_2^2 b + 2s_1 s_2 c)^2,
\]

which simplifies to \( g(s_1^2 a + s_2^2 b - 2s_1 s_2 c) \leq g(s_1^2 a + s_2^2 b + 2s_1 s_2 c) \).

Let \( s_2 = \frac{1}{2} \) and \( \delta = 4|c| \). If \( c = 0 \), the claim trivially holds. Therefore, suppose \( |c| > 0 \). Then, Lemma 3.4 implies that there exists an \( x^* \) such that \( g(\cdot) \) is monotonically decreasing on \( (x^*, x^* + 4|c|) \). Since the range of the function \( h(s) = as^2 + \frac{b}{s^2} \) is \((M, \infty)\) for some \( M > 0 \), then by Lemma 3.4 there must exist \( s_1 \in \mathbb{R} \) such that \( x^* = s_1^2 a + \frac{b}{s_1^2} \). Since \( g(x^* - 2c) \leq g(x^* + 2c) \), then

\[
x^* - 2c \geq x^* + 2c
\]

by monotonicity, which implies \( c < 0 \) as desired. □

While Theorem 3.3 shows that the covariance matrix of any elliptical distribution is an inverse M-matrix, the following example shows that, unlike in the Gaussian setting, this is not a sufficient condition for MTP₂.

**Example 3.5.** Suppose \( X \) is a two-dimensional \( t \)-distribution with one degree of freedom and precision matrix

\[
\Sigma^{-1} = \begin{bmatrix}
1 & -0.1 \\
-0.1 & 1
\end{bmatrix}.
\]

Then \( X \) is not MTP₂, since for \( x = (-1, 1) \) and \( y = (0, 0) \) its density function \( p(\cdot) \) satisfies \( p(x)p(y) > p(x \lor y)p(x \land y) \).

This shows that for transelliptical distributions, the constraint that \( \Sigma^{-1} \) is an M-matrix is a relaxation of MTP₂. In terms of covariance matrix estimators for transelliptical distributions (without the MTP₂ constraint), it was shown recently that replacing the sample covariance matrix \( S \) in Eq. (7) and Eq. (8) by Kendall’s tau correlation matrix \( S_r \) defined in Eq. (9) yields consistent estimators of \( \Sigma_f \) [LHZ12, BK18]. This is quite remarkable, since it does not involve any changes to the objective function apart from replacing \( S \) by \( S_r \). Motivated by these results, we propose to extend the MTP₂ covariance matrix estimator from Section 3.2 to heavy-tailed distributions using the covariance matrix estimator in Eq. (11) by simply replacing the sample covariance matrix \( S \) by \( S_r \).
4. Empirical Evaluation

In this section, we first describe both the data used for the evaluation and our experimental setup, which closely follows [DNLW18]. We then present our empirical evaluation of the various methods discussed in this paper based on the global minimum variance portfolio problem and the full Markowitz portfolio problem.

4.1. Data

We use daily stock returns data from the Center for Research in Security Prices (CRSP), starting in 1975 and ending in 2015. We restrict our attention to stocks from the NYSE, AMEX and NASDAQ stock exchanges, and consider different portfolio sizes $N \in \{100, 200, 500\}$. As in [DNLW18], 21 consecutive trading days constitute one ‘month’. To account for distribution shift over time, we use a rolling out-of-sample estimator. That is, for each month in the out-of-sample period, we estimate the covariance matrix using the most recent $T$ daily returns, and update the portfolio monthly. We vary $T$ with $N$ to evaluate how sensitive different covariance estimators are with respect to increasing dimensionality. In particular, for a given $N$, we vary $T$ such that the ratio $N/T \in \{\frac{1}{2}, 1, 2, 4\}$. We also include $T = 1260$ (which corresponds to 5 years of market data) in order to replicate the results in [DNLW18]. We consider 360 months for evaluation, starting from 01/06/1978 and ending on 12/31/2015, using the portfolio and covariance updating strategy described above. We index each of these 360 investment periods by $h \in \{1, \ldots, 360\}$.

For each investment period and portfolio size, we vary the investment universe because many stocks do not have data for the entire period and the most relevant stocks (i.e. by market capitalization or volume) naturally vary over time. We use the same procedure as in [DNLW18] to construct the investment universe. Specifically, we consider the set of stocks that have (1) an almost complete return history over the most recent $T = 1260$ days and (2) a complete return ‘future’ in the next 21 days (which is the investment period). Next, we remove one stock in each pair of highly correlated stocks, defined as those with sample correlation exceeding 0.95. More precisely, for each pair we remove the stock with the lower market capitalization for period $h$. Finally, we pick the largest $N$ stocks (as measured by their market capitalization on the investment date $h$) for the subsequent analysis. We use $I_{h,N}$ to denote this investment universe, where the subscripts emphasize the dependence on $N$ and $h$.

4.2. Competing Covariance Matrix Estimators

We compare the performance of the proposed MTP$_2$ covariance matrix estimator to the estimators described in Section 2.3. In addition, as a baseline, we also consider the equally weighted portfolio denoted by $1/N$. We evaluate each estimator both in terms of its out-of-sample standard deviation (see Section 4.3) and information ratio (see Section 4.4). These results are summarized in Table 1 and Table 2. In the following, we provide details regarding the implementation of the various covariance matrix estimators included in our empirical analysis.

- **LS**: linear shrinkage, as described in Section 2.3.2, applied to the sample covariance matrix.
• **NLS:** non-linear shrinkage, as described in Section 2.3.2, applied to the sample covariance matrix; we used the implementation in the R package `shrink` [DSH16].

• **AFM-LS:** approximate factor model, as described in Section 2.3.1, with 5 Fama-French factors and linear shrinkage applied to estimate the covariance matrix of the residuals.

• **AFM-NLS:** approximate factor model, as described in Section 2.3.1, with 5 Fama-French factors and non-linear shrinkage applied to estimate the covariance matrix of the residuals.

• **POET (k=3):** POET, as described in Section 2.3.1, using the top 3 principal components; we used the implementation in the R package `POET`.

• **POET (k=5):** POET, as described in Section 2.3.1, using the top 5 principal components; we used the implementation in the R package `POET`.

• **GLASSO:** graphical lasso, as described in Section 2.3.3, using the Python implementation in `sklearn` [PVG+11]; cross-validation is used to select the hyperparameter $\lambda$; we used the default parameters, i.e. using 3-fold cross-validation and testing $\lambda$ on a grid of 4 points refined 4 times (the parameter values for $\alpha$ and $\mu_{\text{iter}}$ respectively). We note that this results in a biased estimator due to the $\ell_1$-penalty.

• **CLIME:** as described in Section 2.3.3; we used the implementation in the R package `CLIME` with hyperparameter $\lambda = \sqrt{\log(p)/n}$, which is asymptotically optimal; the CLIME estimator using this hyperparameter only exists when $T \geq N$ and hence we only benchmarked CLIME in this range.

• **CLIME-KT:** CLIME estimator as described above but using Kendall’s tau correlation matrix instead of the sample correlation matrix. Since Kendall’s tau correlation matrix is not singular, the CLIME-KT estimator exists even when when $T \leq N$.

• **MTP2:** our method, as described in Section 3.2. We used the implementation from [SH14], a computationally efficient coordinate-descent algorithm implemented in Matlab.

• **MTP2-KT:** MTP2 estimator as described above but using Kendall’s tau correlation matrix instead of the sample correlation matrix; see Section 3.3.

4.3. Evaluation on the Global Minimum Variance Portfolio Problem

For each fixed portfolio size $N$, lookback duration $T$, and investment period $h$, we let $\hat{\Sigma}_{T,h}(I_{h,N})$ denote the estimated covariance matrix between the assets in universe $I_{h,N}$ obtained using estimator $\mathcal{M}$. We then computed the portfolio weights $\hat{\omega}_h^\mathcal{M}$ via Eq. (3) and the corresponding returns $r_h^\mathcal{M}$ for $h = 1, \ldots, 360$. We estimated the portfolio standard deviation from these 360 returns for each estimator and multiplied each standard deviation by $\sqrt{12}$ to annualize. Note that a smaller standard deviation implies a lower variance portfolio, and hence better empirical performance.

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3The implementation can be found at [https://sites.google.com/site/slawskimartin/code](https://sites.google.com/site/slawskimartin/code).
Table 1 summarizes the results for each estimator. Each row corresponds to a particular choice of $N$ (size of investment universe) and $T$ (lookback period). Each column corresponds to a different covariance matrix estimator. The best performing estimator is highlighted in blue in each row. While no estimator outperforms all other estimators across all $N$ and $T$, Table 1 shows that the MTP2, non-linear shrinkage (NLS) and POET estimators perform consistently well in all settings.

As discussed in Section 3.3, to deal with the heavy-tailed nature of the distribution

| N  | T  | 1/N LS | NLS | AFM-LS | POET (k=3) | POET (k=5) |
|----|----|--------|-----|--------|------------|------------|
| 100| 50 | 18.724 | 13.452 | 12.976 | 12.617     |            |
| 100| 100| 18.724 | 13.695 | 13.111 | 12.348     |            |
| 200| 1200| 18.724 | 12.451 | 12.347 | 12.707     |            |
| 400| 200 | 18.724 | 12.560 | 12.347 | 12.344     | 12.255     |
| 1260| 400 | 18.724 | 12.151 | 12.122 | 12.130     | 12.722     |
| 200| 100 | 18.134 | 12.583 | 12.320 | 12.348     | 12.255     |
| 200| 200 | 18.134 | 11.881 | 11.603 | 11.495     | 12.455     |
| 400| 400 | 18.134 | 11.656 | 11.431 | 11.469     | 12.559     |
| 800| 800 | 18.134 | 11.670 | 11.424 | 11.449     | 12.593     |
| 1260| 1260| 18.134 | 11.665 | 11.534 | 11.568     | 12.898     |
| 500| 250 | 17.925 | 11.140 | 10.516 | 10.517     | 11.203     |
| 500| 500 | 17.925 | 11.934 | 10.856 | 10.816     | 11.917     |
| 1000| 1000| 17.925 | 11.373 | 10.838 | 10.816     | 11.917     |
| 1260| 1260| 17.925 | 11.469 | 10.943 | 11.005     | 12.395     |

| N  | T  | GLASSO | CLIME | CLIME-KT | MTP2 | MTP2-KT |
|----|----|--------|-------|----------|------|---------|
| 100 | 50 | 13.594 | nan   | 15.484   | 12.655 | 12.623  |
| 100 | 100| 13.822 | nan   | 15.024   | 12.327 | 12.049  |
| 200 | 200| 13.985 | 14.945| 15.140   | 11.858 | 11.742  |
| 400 | 400| 13.607 | 15.127| 15.223   | 12.294 | 12.114  |
| 1260| 1260| 13.631| 15.253| 15.316   | 12.087 | 12.087  |
| 200 | 200| 13.522 | nan   | 14.983   | 11.803 | 11.445  |
| 200 | 200| 13.719 | nan   | 14.344   | 11.586 | 11.442  |
| 400 | 400| 13.920 | 14.563| 14.964   | 11.880 | 11.905  |
| 800 | 800| 14.096 | 14.778| 14.862   | 11.635 | 11.661  |
| 1260| 1260| 13.958| 15.013| 15.013   | 11.710 | 11.749  |
| 500 | 500| 13.855 | nan   | 15.677   | 10.455 | 10.512  |
| 500 | 500| 14.171 | nan   | 20.896   | 11.009 | 11.261  |
| 1000| 1000| 14.283| 15.523| 14.330   | 11.031 | 11.273  |
| 1260| 1260| 14.290| 14.776| 14.962   | 11.187 | 11.422  |

Table 1: For each combination of $N$ (portfolio size), $T$ (lookback period), and covariance matrix estimator, we report the out-of-sample standard deviation of the returns of the portfolio. The most competitive value in each row is highlighted in blue.
of returns, Kendall’s tau correlation matrix can be used instead of the sample correlation matrix in the CLIME and MTP2 estimators which assume Gaussianity. Columns CLIME-KT and MTP2-KT in Table 1 indicate that while using Kendall’s tau correlation matrix usually does not make a significant difference in the performance, it can give a slight boost for the MTP2 estimator in particular when \( N = 100 \) or 200.

Instead of comparing the covariance matrix estimators only based on one number, the standard deviation of the returns of the resulting portfolios across the entire out-of-sample period, it is also of interest to examine the performance of each estimator throughout the out-of-sample period. Figure 3 shows the standard deviation of the returns of the different estimators for \( N \in \{100, 200, 500\} \) and \( T = 1260 \) when varying the out-of-sample period from 100 to 360 (where 360 is the maximal number of total out-of-sample months). Note that the ordering between the different estimators is relatively consistent over time, indicating that the conclusions from the comparison of the different estimators in Table 1 would remain unchanged even when varying the length of the out-of-sample period.

4.4. Evaluation on Full Markowitz Portfolio Problem with Momentum Signal

We also benchmarked the different covariance matrix estimators based on the performance of the portfolios selected by solving Eq. (1), where \( \Sigma^* \) is replaced by the estimator. A standard performance metric is the information ratio, which is the ratio between the expected portfolio returns \( R \) and the standard deviation (i.e., risk) of the portfolio. Hence, a higher information ratio indicates better performance.

We selected the desired expected returns level \( R \) as in [DNLW18]. Namely, we considered the EW-TQ portfolio which places equal weight on each of the top 20% of assets (based on expected returns). We then set \( R \) equal to the expected return of the EW-TQ portfolio. In addition, since the true vector of expected returns \( \mu^* \) is unknown, we estimated it from the data. We do this using the momentum factor [JT93], as in [DNLW18], which for a given investment period \( h \) and stock is the geometric average of returns of the previous year excluding the past month.

The out-of-sample information ratios of each estimator are shown in Table 2. As in Table 1, each row corresponds to a different choice of \( N \) and \( T \) and each column corresponds to a different estimator. The best performing estimator for each row is highlighted in blue. This analysis shows that the MTP2 estimator achieves the best performance for almost all choices of \( N \) and \( T \). Although the results are similar, comparing MTP2 to MTP2-KT indicates that it is recommendable to use Kendall’s tau correlation matrix instead of the sample correlation matrix with the MTP2 estimator when \( N = 100 \) or 200.
Similarly as in Figure 3, in Figure 4 we show the information ratio of the returns of the different estimators for $N \in \{100, 200, 500\}$ and $T = 1260$ when varying the out-of-sample period from 100 to 360. Note that while the ordering between the different estimators is still relatively consistent over time, it varies more than for the standard deviation plotted in Figure 3 and could provide additional valuable information regarding each estimator that is not captured in Table 2.

| N  | T  | EQ-TW | LS   | NLS  | AFM-LS | AFM-NLS | POET (k=3) | POET (k=5) |
|----|----|-------|------|------|-------|---------|------------|------------|
| 100| 50 | 0.694 | 0.625| 0.648| 0.617 | 0.621   | 0.760      | 0.791      |
| 100| 100| 0.694 | 0.600| 0.682| 0.628 | 0.620   | 0.797      | 0.690      |
| 200| 200| 0.694 | 0.670| 0.720| 0.691 | 0.675   | 0.802      | 0.706      |
| 400| 400| 0.694 | 0.736| 0.772| 0.803 | 0.776   | 0.824      | 0.753      |
| 1260|1260| 0.694 | 0.831| 0.864| 0.832 | 0.831   | 0.841      | 0.831      |
| 200| 100| 0.757 | 0.719| 0.735| 0.715 | 0.728   | 0.766      | 0.762      |
| 200| 200| 0.757 | 0.812| 0.793| 0.796 | 0.790   | 0.747      | 0.764      |
| 400| 400| 0.757 | 0.864| 0.885| 0.888 | 0.892   | 0.825      | 0.820      |
| 800| 800| 0.757 | 0.967| 0.961| 0.962 | 0.967   | 0.747      | 0.870      |
| 1260|1260| 0.757 | 0.906| 0.907| 0.913 | 0.906   | 0.773      | 0.770      |
| 500| 250| 0.764 | 0.985| 0.995| 0.997 | 0.993   | 0.869      | 1.030      |
| 500| 500| 0.764 | 0.940| 0.955| 0.995 | 1.003   | 0.849      | 1.027      |
| 1000|1000| 0.764 | 0.918| 0.976| 0.993 | 0.980   | 0.772      | 0.861      |
| 1260|1260| 0.764 | 0.920| 0.967| 0.984 | 0.982   | 0.806      | 0.909      |

| N  | T  | GLASSO | CLIME | CLIME-KT | MTP2 | MTP2-KT |
|----|----|--------|-------|----------|------|---------|
| 100| 50 | 0.858  | nan   | 0.788    | 0.849| 0.905   |
| 100| 100| 0.885  | nan   | 0.837    | 0.896| 0.975   |
| 200| 200| 0.855  | 0.830 | 0.882    | 0.899| 0.950   |
| 400| 400| 0.877  | 0.852 | 0.823    | 0.892| 0.924   |
| 1260|1260| 0.878  | 0.778 | 0.767    | 0.890| 0.855   |
| 200| 100| 0.887  | nan   | 0.844    | 0.829| 0.918   |
| 200| 200| 0.859  | nan   | 0.896    | 0.885| 0.919   |
| 400| 400| 0.865  | 0.916 | 0.821    | 0.886| 0.893   |
| 800| 800| 0.862  | 0.860 | 0.805    | 0.970| 0.945   |
| 1260|1260| 0.887  | 0.845 | 0.885    | 0.955| 0.931   |
| 500| 250| 0.908  | nan   | 0.596    | 1.112| 1.133   |
| 500| 500| 0.887  | nan   | 0.511    | 1.045| 1.005   |
| 1000|1000| 0.897  | 0.828 | 1.101    | 1.061| 0.993   |
| 1260|1260| 0.896  | 0.858 | 0.806    | 1.034| 0.958   |

Table 2: For each combination of $N$ (portfolio size), $T$ (lookback period), and covariance matrix estimator, we report the out-of-sample information ratio (ratio of the average return to the standard deviation of return) of the portfolio. The most competitive value in each row is highlighted in blue.
5. Conclusion

In this paper, we proposed a new covariance matrix estimator for portfolio selection based on the assumption that returns are MTP\textsubscript{2}, which is a strong form of positive dependence. While the MTP\textsubscript{2} assumption is strong, this constraint adds considerable regularization, thereby reducing the variance of the resulting covariance matrix estimator. Empirically, the added bias of MTP\textsubscript{2} is outweighed by the reduction in variance. In particular, the proposed MTP\textsubscript{2} estimator outperforms previous state-of-the-art covariance matrix estimators in terms of out-of-sample information ratio.

In our empirical evaluation we observed that using Kendall tau’s correlation matrix instead of the sample covariance matrix in the MLE under MTP\textsubscript{2} performed particularly well for a portfolio size of 100 or 200. It would therefore be of interest to analyze the theoretical properties of such covariance matrix estimators including MLE or CLIME under MTP\textsubscript{2} for heavy-tailed distributions. In addition, while we only considered static covariance matrix estimators in this paper, the MTP\textsubscript{2} estimator naturally extends to the dynamic setting, where the covariance matrix evolves over time. Specifically, we may adapt the techniques developed in [ELW17] to obtain a dynamic estimator under MTP\textsubscript{2}. In future work, it would be interesting to compare the resulting estimator to other state-of-the-art dynamic covariance matrix estimators. Another interesting future direction is the theoretical analysis of the spectrum of symmetric M-matrices in the high-dimensional setting. If the MTP\textsubscript{2} constraint already implicitly regularizes the spectrum sufficiently, then shrinkage methods such as those developed in [LW04, LW12, ELW17, JM03, DMUN13] may be unnecessary under MTP\textsubscript{2}. Alternatively, covariance matrix estimators under MTP\textsubscript{2} could be combined with shrinkage methods to potentially achieve even better performance.

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