Abstract. In this paper we study congruences on sums of products of binomial coefficients that can be proved using properties of the Jacobi polynomials. Special attention is given to polynomial congruences containing Catalan numbers, second-order Catalan numbers, the sequence \((A176898)\)

\[ S_n = \binom{6n}{3n} \binom{3n}{n} 2^{n+1}, \]

and the binomial coefficients \(\binom{3n}{n}\) or \(\binom{4n}{2n}\). As an application, we address several conjectures of Z. W. Sun on congruences of sums involving \(S_n\) and we prove a cubic residuacy criterion in terms of sums of the binomial coefficients \(\binom{3n}{n}\) conjectured recently by Z. H. Sun.

1. Introduction

In this paper, building on our previous work with Tauraso [3], we continue to apply properties of the Jacobi polynomials \(P_n^{(\pm 1/2, \pm 1/2)}(x)\) for proving polynomial and numerical congruences containing sums of binomial coefficients. In particular, we derive polynomial congruences for sums involving binomial coefficients \(\binom{3n}{n}\), \(\binom{4n}{2n}\), Catalan numbers

\[ C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}, \quad n = 0, 1, 2, \ldots, \]

second-order Catalan numbers

\[ C_n^{(2)} = \frac{1}{2n+1} \binom{3n}{n} - 2 \binom{3n}{n-1}, \quad n = 0, 1, 2, \ldots, \]

and the sequence \((A176898)\)

\[ S_n = \frac{\binom{6n}{3n}}{2^{n+1}}, \quad n = 0, 1, 2, \ldots, \]  

arithmetical properties of which have been studied very recently by Sun [12] and Guo [2].

Recall that the Jacobi polynomials \(P_n^{(\alpha, \beta)}(x)\) are defined by

\[ P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} F(-n, n + \alpha + \beta + 1; \alpha + 1; (1-x)/2), \quad \alpha, \beta > -1, \]

where

\[ F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k, \]

is the Gauss hypergeometric function and \((a)_0 = 1\), \((a)_k = a(a+1) \cdots (a+k-1)\), \(k \geq 1\), is the Pochhammer symbol.

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The polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the three term recurrence relation \[13\] Section 4.5
\[
2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x)
= [(2n+\alpha+\beta+1)(\alpha^2-\beta^2) + (2n+\alpha+\beta)3x]P_n^{(\alpha,\beta)}(x)
- 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(x)
\]
with the initial conditions $P_0^{(\alpha,\beta)}(x) = 1$, $P_1^{(\alpha,\beta)}(x) = (x(\alpha+\beta+2) + \alpha - \beta)/2$.

While in \[3\] we studied binomial sums arising from the truncation of the series
\[
\arcsin(z) = \sum_{k=0}^{\infty} \frac{(2k)!}{4k(2k+1)!} z^{2k+1}, \quad |z| \leq 1,
\]
the purpose of the present paper is to consider a quadratic transformation of the Gauss hypergeometric function given by (see \[8\] p. 210)
\[
\sin(a \arcsin(z)) = zF \left( \frac{1+a}{2}, \frac{1-a}{2}; \frac{3}{2}; z^2 \right), \quad |z| \leq 1,
\]
which essentially can be regarded as a generalization of series \[4\]. Note that letting $a$ approach zero in \[4\] yields \[4\]. On the other side, identity \[5\] serves as a source of generating functions for some special sequences of numbers including those mentioned above. Namely, for $a = 1/2, 1/3, 2/3$, we have
\[
\sin \left( \frac{\arcsin(z)}{2} \right) = 2 \sum_{k=0}^{\infty} C_{2k} \left( \frac{z}{4} \right)^{2k+1}, \quad |z| \leq 1,
\]
\[
\arcsine \left( \frac{\arcsin(z)}{3} \right) = \sum_{k=0}^{\infty} \frac{C(2)}{3} \left( \frac{4z^2}{27} \right)^k, \quad |z| \leq 1,
\]
\[
\sin \left( \frac{2}{3} \arcsin(z) \right) = \frac{4z}{3} \sum_{k=0}^{\infty} S_k \left( \frac{z^2}{108} \right)^k, \quad |z| \leq 1.
\]

In this paper we develop a unified approach to the calculation of polynomial congruences modulo a prime $p$ arising from the truncation of the series \[3\]-\[8\] and polynomial congruences involving binomial coefficients $(\binom{3k}{k})$, $(\binom{4k}{2k})$ and also the sequence $(2k+1)S_k$ within various ranges of summation depending on a prime $p$.

Note that the congruences involving binomial coefficients $(\binom{3k}{k})$, $(\binom{4k}{2k})$ have been studied extensively from different points of view (see, for example \[8\], \[9\], \[10\], \[11\], \[15\]). Z. H. Sun \[9\], \[10\] studied congruences for the sums $\sum_{k=1}^{[p/3]} (\binom{3k}{k}) t^k$ and $\sum_{k=1}^{[p/4]} (\binom{4k}{2k}) t^k$ using congruences for Lucas sequences and properties of the cubic and quartic residues. Sun \[8\] also investigated interesting connections between values of $\sum_{k=1}^{[p/3]} (\binom{3k}{k}) t^k \pmod{p}$, solubility of cubic congruences and cubic residuacity criteria. First congruences for the sums $\sum_{k=1}^{[p-1]} (\binom{3k}{k}) t^k$ and $\sum_{k=1}^{[p-1]} (\binom{4k}{2k}) t^k$ at $t = 2$ were obtained by Zhao, Pan and Z. W. Sun \[15\] with the help of some combinatorial identity and then extended by Z. W. Sun \[11\] who used properties of third-order recurrences and cubic residues to give explicit congruences for $t = -4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$.

Our approach is based on reducing values of the sums discussed above modulo a prime $p$ to values of the Jacobi polynomials $P^{(\pm 1/2, \pm 1/2)}(x)$, which is done in Section 2, and then investigating congruences for the Jacobi polynomials in the subsequent sections. In Section 3 we deal with polynomial congruences involving binomial coefficients $(\binom{4k}{2k})$ and even-indexed Catalan numbers $C_{2k}$. In Section 4 we study polynomial congruences containing binomial coefficients $(\binom{3k}{k})$ and
second-order Catalan numbers $C_n^{(2)}$. In Sections 5, 6 we apply the theory of cubic residues developed in [7] to study polynomial congruences of the form

$$
\sum_{k=(p+1)/2}^{[2p/3]} \binom{3k}{k} t^k, \quad \sum_{k=1}^{p-1} \binom{3k}{k} t^k, \quad \sum_{k=1}^{p-1} C_k^{(2)} t^k, \quad \sum_{k=0}^{p-1} S_k t^k, \quad \sum_{k=0}^{p-1} (2k+1)S_k t^k.
$$

As a result, we prove several cubic residuacity criteria in terms of these sums, one of which, in terms of $\sum_{k=(p+1)/2}^{[2p/3]} \binom{3k}{k} t^k$, confirms a question posed by Z. H. Sun [8, Conj. 2.1].

In Section 6, we derive polynomial congruences for the sums $\sum_{k=0}^{[p/6]} S_k t^k$, $\sum_{k=0}^{p-1} S_k t^k$, $\sum_{k=0}^{[p/6]} (2k+1)S_k t^k$, $\sum_{k=0}^{p-1} (2k+1)S_k t^k$ and also give many numerical congruences which are new and have not appeared in the literature before. In particular, we show that

$$
\sum_{k=0}^{p-1} S_k \equiv \frac{1}{2} \left( \frac{3}{p} \right) \pmod{p}
$$

confirming a conjecture of Z. W. Sun [12, Conj. 2]. Finally, in Section 7 we prove a closed form formula for a companion sequence of $S_n$ answering another question of Sun [12, Conj. 4].

2. Main Theorem

For a non-negative integer $n$, we consider the sequence $w_n(x)$ defined in [3, Section 3] by

$$
w_n(x) := (2n+1)F(-n, n+1; 3/2; (1-x)/2) = \frac{n!}{(1/2)_n} P_{n/2,-1/2}(x). \quad (9)
$$

From (3) it follows that $w_n(x)$ satisfies the second order linear recurrence with constant coefficients

$$
w_{n+1}(x) = 2xw_n(x) - w_{n-1}(x)
$$

and initial conditions $w_0(x) = 1$, $w_1(x) = 1 + 2x$. This yields the following formulae:

$$
w_n(x) = \begin{cases} 
\frac{(\alpha + 1)\alpha^n - (\alpha^{-1} + 1)\alpha^{-n}}{\alpha - \alpha^{-1}} & \text{if } x \neq \pm 1, \\
2n+1 & \text{if } x = 1, \\
(-1)^n & \text{if } x = -1,
\end{cases} \quad (10)
$$

where $\alpha = x + \sqrt{x^2 - 1}$. Note that for $x \in (-1, 1)$ we also have an alternative representation

$$
w_n(x) = \cos(n \arccos x) + \frac{x + 1}{\sqrt{1 - x^2}} \sin(n \arccos x). \quad (11)
$$

By the well-known symmetry property of the Jacobi polynomials

$$
P_n^{(\alpha, \beta)}(x) = (-1)^\alpha P_n^{(\beta, \alpha)}(-x)
$$

and formula (2), we get one more expression of $w_n(x)$ in terms of the Gauss hypergeometric function

$$
w_n(x) = (-1)^n F(-n, n+1; 1/2; (1+x)/2). \quad (12)
$$

For a given prime $p$, let $D_p$ denote the set of those rational numbers whose denominator is not divisible by $p$. Let $\varphi(m)$ be the Euler totient function and $(\frac{2}{p})$ be the Legendre symbol. We put $(\frac{a}{p}) = 0$ if $p|a$. For $c = a/b \in D_p$ written in its lowest terms we define $(\frac{c}{p}) = (\frac{ab}{p})$ in view that the congruences $x^2 \equiv c \pmod{p}$ and $(bx)^2 \equiv ab \pmod{p}$ are equivalent. It is clear that $(\frac{c}{p})$ has all the formal properties of the ordinary Legendre symbol. For any rational number $x$, denote by $v_p(x)$ the $p$-adic order of $x$. Let $[x]$ be the greatest integer not exceeding $x$. 


Theorem 2.1. Let \( m \) be a positive integer with \( \varphi(m) = 2 \), i.e., \( m \in \{3, 4, 6\} \) and \( p > 3 \) be a prime. Then for any \( t \in D_p \), we have

\[
\sum_{k=0}^{[p/m]} \frac{\left(\frac{1}{m}\right)_k \left(\frac{m-1}{m}\right)_k}{(2k+1)!!} t^k \equiv \frac{1}{1 + 2[p/m]} W_{\frac{p}{m}}(1 - t/2) \pmod{p},
\]

(13)

\[
\sum_{k=0}^{[p/m]} \frac{\left(\frac{1}{m}\right)_k \left(\frac{m-1}{m}\right)_k}{(2k+1)!!} t^k \equiv (-1)^{[p/m]} W_{\frac{p}{m}}(t/2 - 1) \pmod{p},
\]

(14)

\[
\sum_{k=(p-1)/2}^{[m-1]/p} \frac{\left(\frac{1}{m}\right)_k \left(\frac{m-1}{m}\right)_k}{(2k+1)!!} t^k \equiv \frac{-1}{m(1 + 2[p/m])} \left( W_{\frac{m-1}{m}}(1 - t/2) + W_{\frac{p}{m}}(1 - t/2) \right) \pmod{p},
\]

(15)

\[
\sum_{k=(p+1)/2}^{[m-1]/p} \frac{\left(\frac{1}{m}\right)_k \left(\frac{m-1}{m}\right)_k}{(2k+1)!!} t^k \equiv (-1)^{[p/m]} m \left( W_{\frac{m-1}{m}}(t/2 - 1) - W_{\frac{p}{m}}(t/2 - 1) \right) \pmod{p}.
\]

Corollary 2.1. Let \( m \) be a positive integer with \( \varphi(m) = 2 \), i.e., \( m \in \{3, 4, 6\} \) and \( p > 3 \) be a prime. Then for any \( t \in D_p \), we have

\[
\sum_{k=0}^{p-1} \frac{\left(\frac{1}{m}\right)_k \left(\frac{m-1}{m}\right)_k}{(2k+1)!!} t^k \equiv \frac{1}{m(1 + 2[p/m])} \left( (m-1)W_{\frac{m}{p}}(1 - t/2) - W_{\frac{m-1}{m}}(1 - t/2) \right) \pmod{p},
\]

\[
\sum_{k=0}^{p-1} \frac{\left(\frac{1}{m}\right)_k \left(\frac{m-1}{m}\right)_k}{(2k+1)!!} t^k \equiv (-1)^{[p/m]} m \left( W_{\frac{m-1}{m}}(t/2 - 1) + W_{\frac{m-1}{m}}(t/2 - 1) \right) \pmod{p}.
\]

Proof of Theorem 2.1. Let \( m \in \{3, 4, 6\} \), i.e., \( \varphi(m) = 2 \). Suppose \( p \) is an odd prime greater than 3 and \( p \equiv r \pmod{m} \), where \( r \in \{1, m-1\} \). Put \( n = \frac{p-r}{m} \), then \( p = mn + r \) and from (9) we have

\[
w_n(x) = \frac{2p - 2r + m}{m} F(-n, n+1; 3/2; (1-x)/2) = \frac{2p - 2r + m}{m} \sum_{k=0}^{n} \frac{(r-m)k (m-r+m)k}{(\frac{3}{2})_k k!} \left(1 - \frac{x}{2}\right)^k.
\]

Since \( (\frac{3}{2})_k = \frac{3\cdot5\cdots(2k+1)}{2^k} \) and \( 2k + 1 \leq 2n + 1 \leq (p+1)/2 \), the denominators of the summands are coprime to \( p \) and we have

\[
w_n(x) \equiv \frac{m - 2r}{m} \sum_{k=0}^{[p/m]} \frac{(r-m)k (m-r+m)k}{(\frac{3}{2})_k k!} \left(1 - \frac{x}{2}\right)^k \equiv \frac{m - 2r}{m} \sum_{k=0}^{[p/m]} \frac{(1)k (m-1)k}{(\frac{3}{2})_k k!} \left(1 - \frac{x}{2}\right)^k \pmod{p}
\]

or

\[
w_n(x) \equiv \frac{m - 2r}{m} \sum_{k=0}^{[p/m]} \frac{(1)k (m-1)k}{(2k+1)!} \left(2(1-x)\right)^k \pmod{p}.
\]

Replacing \( x \) by \( 1 - t/2 \), we get (13).

Applying formula (12) to \( w_n(x) \), similarly as before, we get

\[
w_n(x) = (-1)^n F(-n, n+1; 1/2; (1+x)/2) = (-1)^n \sum_{k=0}^{n} \frac{(r-p)k (p-r+m)k}{(\frac{3}{2})_k k!} \left(1 + \frac{x}{2}\right)^k
\]
or

\[ w_n(x) \equiv (-1)^n \sum_{k=0}^{n} \frac{\binom{1}{m}}{k!} \left( \frac{m-1}{m} \right)_k \left( \frac{1 + x}{2} \right)^k = (-1)^n \sum_{k=0}^{n} \frac{\binom{1}{m}}{(2k)!} (2(1 + x))^k \pmod{p}. \]

Substituting \( t = 2(1 + x) \), we obtain (14).

To prove the other two congruences, we consider \((m-1)p\) modulo \( m \). It is clear that \((m-1)p \equiv r \pmod{m} \), where \( r \in \{1, m-1\} \). Put \( n = \frac{(m-1)p - r}{m} \), then \((m-1)p = mn + r \) and from (9) we have

\[ w_n(x) = \frac{2(m-1)p - 2r + m}{m} \sum_{k=0}^{n} \frac{\binom{r-(m-1)p}{m}}{k!} \left( \frac{(m-1)p + m - r}{m} \right)_k \left( \frac{1 - x}{2} \right)^k. \]  

(16)

Note that \( p \) divides \( \left( \frac{3}{2} \right)_k \) if and only if \( k \geq (p-1)/2 \). Moreover, \( p^2 \) does not divide \( \left( \frac{3}{2} \right)_k \) for any \( k \) from the range of summation. Similarly, we have

\[ \left( \frac{r-(m-1)p}{m} \right)_k = \prod_{l=0}^{k-1} \frac{r + ml - (m-1)p}{m}. \]

All possible multiples of \( p \) among the numbers \( r + ml \), where \( 0 \leq l \leq k-1 \leq \frac{(m-1)p - r}{m} \), could be only of the form \( r + ml = jp \) with \( 1 \leq j \leq m-2 \). This implies \( jp \equiv r \equiv -p \pmod{m} \) or \((j+1)p \equiv 0 \pmod{m} \), which is impossible since \( (p, m) = 1 \) and \( j+1 < m \). So \( p \) does not divide \( \frac{r-(m-1)p}{m} \). Considering

\[ \left( \frac{(m-1)p + m - r}{m} \right)_k = \prod_{l=1}^{k} \left( \frac{(m-1)p + ml - r}{m} \right), \]

we see that \( p \) divides \( \left( \frac{(m-1)p + m - r}{m} \right)_k \) if and only if \( k \geq \frac{p+r}{m} \). Moreover, \( p^2 \) does not divide \( \left( \frac{p+r}{4} \right)_k \) for any \( k \) from the range of summation. Indeed, if we had \( ml - r = jp \) for some \( 1 < j \leq m-1 \), then \( p \equiv -r \equiv jp \pmod{m} \) and therefore \( p(j-1) \equiv 0 \pmod{m} \), which is impossible. From the divisibility properties of the Pochhammer’s symbols above and (16) we easily conclude that

\[ w'_{\frac{(m-1)p}{m}}(x) \equiv \frac{m - 2r}{m} \left( \sum_{k=0}^{\lceil p/m \rceil} \sum_{k=(p-1)/2}^{\lceil (m-1)p/m \rceil} \frac{\binom{r-(m-1)p}{m}}{k!} \left( \frac{(m-1)p + m - r}{m} \right)_k \left( \frac{1 - x}{2} \right)^k \pmod{p} \]

and therefore,

\[ w'_{\frac{(m-1)p}{m}}(x) \equiv \frac{m - 2r}{m} \left( \sum_{k=0}^{\lceil p/m \rceil} \sum_{k=(p-1)/2}^{\lceil (m-1)p/m \rceil} \frac{\binom{r}{m}}{k!} \left( \frac{(m-1)p + m - r}{m} \right)_k \left( \frac{1 - x}{2} \right)^k \pmod{p}, \]

where for the second sum, we employed the congruence

\[ \left( \frac{(m-1)p + m - r}{m} \right)_k = \frac{(m-1)p + m - r}{m} \cdot \frac{(m-1)p + 2m - r}{m} \cdots \frac{(m-1)p + m - r}{m} \cdot \frac{(m-1)p + m - r}{m} \cdot \cdots \frac{(m-1)p + mn - r}{m} \equiv m \frac{(m-r/m)}{k!} \left( \frac{3}{2} \right)_k \pmod{p} \]
valid for \((p - 1)/2 \leq k \leq [(m - 1)p/m]\). Now by \([13]\), we obtain

\[
\frac{m}{m - 2r} \frac{w_{[(m-1)p/m]}(x)}{x} \equiv \frac{1}{1 + 2[p/m]} \frac{w_{[(m-1)p/m]}\left(\frac{1}{m}\right)}{x} + m \sum_{k=(p-1)/2}^{[(m-1)p/m]} \left(\frac{1}{m}\right)k \left(\frac{m-1}{m}\right)k \frac{(k+1)!}{(2k+1)!} \frac{(1)(2(1-x))^k}{(mod\ p)}.
\]

Taking into account that \(\left\lfloor \frac{(m-1)p}{m} \right\rfloor = p - 1 - \left\lfloor \frac{p}{m} \right\rfloor\) and replacing \(x\) by \(1 - t/2\), we get the desired congruence \([15]\).

Finally, applying formula \([12]\) and following the same line of arguments as for proving \([15]\), we have

\[
w_{[(m-1)p/m]}(x) = (-1)^n F(-n, n + 1; 1/2; (1 + x)/2)
\]

\[
= (-1)^{[(m-1)p/m]} \left[ \sum_{k=0}^{[(m-1)p/m]} \frac{(m-1)p+r}{m} \left(\frac{1}{k}\right)k! \right] \left(\frac{1+x}{2}\right)^k
\]

\[
\equiv (-1)^{[(m-1)p/m]} \left( \sum_{k=0}^{[(m-1)p/m]} \frac{r}{m} + \sum_{k=0}^{[(m-1)p/m]} \frac{(m-1)p+r}{m} \left(\frac{1}{k}\right)k! \right) \left(\frac{1+x}{2}\right)^k
\]

\[
\equiv (-1)^{[(m-1)p/m]} \left( \sum_{k=0}^{[(m-1)p/m]} \frac{r}{m} + \sum_{k=(p+1)/2}^{[(m-1)p/m]} \left(\frac{1}{m}\right)k \left(\frac{m-1}{m}\right)k \frac{(k+1)!}{(2k)!} \frac{(2(1+x))^k}{(mod\ p)}\right)
\]

Now by \([15]\), we obtain

\[
(-1)^{[(m-1)p/m]} w_{[(m-1)p/m]}(x) \equiv (-1)^{[\frac{r}{m}]} w_{[\frac{r}{m}]}(x) + m \sum_{k=(p+1)/2}^{[(m-1)p/m]} \left(\frac{1}{m}\right)k \left(\frac{m-1}{m}\right)k \frac{(k+1)!}{(2k)!} \frac{(2(1+x))^k}{(mod\ p)}
\]

and after the substitution \(x = t/2 - 1\) the last congruence follows.

**Proof of Corollary 2.1** Let \(p = mn + r\), where \(r \in \{1, m - 1\}\). If \(n + 1 = [p/m] + 1 \leq k \leq \frac{p-3}{2}\), then \(v_p((2k + 1)! - 0)\) and \(v_p((\frac{r}{m})_k \geq 1\), since the product \(\prod_{l=0}^{k-1} (r + lm)\) is divisible by \(p\).

If \((m-1)n+r = [(m-1)p/m] + 1 \leq k \leq p-1\), then it is easy to see that \(v_p((2k+1)! - v_p((2k)! = 1, v_p((\frac{r}{m})_k \geq 1\), and the product \(\prod_{l=1}^{k} (lm - r)\) contains the factor \((m - 1)p\). This implies that \(v_p((\frac{m-1}{m})_k \geq 1\) and therefore we have

\[
\sum_{k=0}^{p-1} \frac{1}{m}k \frac{(m-1)}{m}k \frac{1}{2k+1}! \] \[\equiv \sum_{k=0}^{[\frac{p}{m}]} \frac{1}{m}k \frac{(m-1)}{m}k \frac{1}{2k+1}! \] \[\equiv \sum_{k=(p-1)/2}^{[(m-1)p/m]} \left(\frac{1}{m}\right)k \left(\frac{m-1}{m}\right)k \frac{(k+1)!}{(2k)!} \frac{(2(1+x))^k}{(mod\ p)}.
\]

\[
\sum_{k=(p+1)/2}^{[(m-1)p/m]} \left(\frac{1}{m}\right)k \left(\frac{m-1}{m}\right)k \frac{(k+1)!}{(2k)!} \frac{(2(1+x))^k}{(mod\ p)}.
\]

which by Theorem 2.1 proves the Corollary.

**3. Polynomial congruences involving \(\binom{4n}{2n}\) and \(C_{2n}\)**

In this Section, we consider applications of Theorem 2.1 when \(m = 4\). In this case, we get polynomial congruences involving even-indexed Catalan numbers (A048990) \(C_{2n}\) and binomial coefficients \(\binom{4n}{2n}\).
Theorem 3.1. Let $p$ be an odd prime and $t \in D_p$. Then
\[
\sum_{k=0}^{[p/4]} C_{2k} t^k \equiv 2(-1)^{p-1} w_{n/4}(1 - 32t) \pmod{p},
\]
\[
\sum_{k=(p-1)/2}^{[3p/4]} C_{2k} t^k \equiv \frac{(-1)^{p+1}}{2} w_{3p/4}(1 - 32t) + w_{4/4}(1 - 32t) \pmod{p},
\]
\[
\sum_{k=0}^{[p/4]} \binom{4k}{2k} t^k \equiv \left(\frac{-2}{p}\right) w_{3p/4}(32t - 1) \pmod{p},
\]
\[
\sum_{k=(p+1)/2}^{[3p/4]} \binom{4k}{2k} t^k \equiv \frac{1}{4} \left(\frac{-2}{p}\right) w_{3p/4}(32t - 1) - w_{4/4}(32t - 1) \pmod{p}.
\]

Corollary 3.1. Let $p$ be an odd prime and $t \in D_p$. Then
\[
\sum_{k=0}^{p-1} C_{2k} t^k \equiv \frac{1}{2} \left(\frac{-1}{p}\right) \left(3w_{3p/4}(1 - 32t) - w_{4/4}(1 - 32t)\right) \pmod{p},
\]
\[
\sum_{k=0}^{p-1} \binom{4k}{2k} t^k \equiv \frac{1}{4} \left(\frac{-2}{p}\right) \left(w_{3p/4}(32t - 1) + 3w_{4/4}(32t - 1)\right) \pmod{p}.
\]

Evaluating values of the sequences $w_{3p/4}(x)$ and $w_{4/4}(x)$ modulo $p$, we get numerical congruences for the above sums. Here are some typical examples.

Corollary 3.2. Let $p$ be a prime greater than 3. Then
\[
\sum_{k=0}^{p-1} \frac{C_{2k}}{16^k} \equiv \left(\frac{2}{p}\right) \pmod{p},
\]
\[
\sum_{k=[p/4]}^{p-1} \frac{C_{2k}}{16^k} \equiv 2 \left(\frac{2}{p}\right) \pmod{p},
\]
\[
\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}}{16^k} \equiv \frac{1}{4} \left(\frac{2}{p}\right) \pmod{p},
\]
\[
\sum_{k=[3p/4]}^{p-1} \frac{\binom{4k}{2k}}{16^k} \equiv -\frac{1}{4} \left(\frac{2}{p}\right) \pmod{p},
\]
\[
\sum_{k=0}^{[p/4]} \frac{C_{2k}}{32^k} \equiv 2 \left(\frac{-1}{p}\right) (-1)^{[p/8]} \pmod{p},
\]
\[
\sum_{k=[p/4]}^{p-1} \frac{\binom{4k}{2k}}{32^k} \equiv \left(\frac{-2}{p}\right) (-1)^{[p/8]} \pmod{p},
\]
\[
\sum_{k=0}^{p-1} \frac{C_{2k}}{32^k} \equiv \begin{cases} 
\left(-1\right)^{(p-1)/2+[p/8]} \pmod{p} & \text{if } p \equiv \pm1 \pmod{8}, \\
2\left(-1\right)^{(p-1)/2+[p/8]} \pmod{p} & \text{if } p \equiv \pm3 \pmod{8},
\end{cases}
\]
\[
\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}}{32^k} \equiv \begin{cases} 
\left(-1\right)^{[p/8]} \pmod{p} & \text{if } p \equiv \pm1 \pmod{8}, \\
\frac{1}{2} \left(-1\right)^{[p/8]} \pmod{p} & \text{if } p \equiv \pm3 \pmod{8}.
\end{cases}
\]

Proof. The proof easily follows from the facts that
\[
w_n(-1) = (-1)^n, \quad w_n(0) = (-1)^{[n/2]}, \quad \text{and} \quad w_n(1) = 2n + 1. \quad (17)
\]

□
Corollary 3.3. Let $p$ be a prime greater than 3. Then

$$\frac{2}{p} \sum_{k=0}^{p-1} \frac{C_{2k}}{64^k} \equiv \begin{cases} 1 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{12}, \\ -7/2 \pmod{p} & \text{if } p \equiv \pm 5 \pmod{12}, \end{cases}$$

$$\frac{2}{p} \sum_{k=0}^{p-1} \frac{(4k)}{64^k} \equiv \begin{cases} 1 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{12}, \\ 1/4 \pmod{p} & \text{if } p \equiv \pm 5 \pmod{12}, \end{cases}$$

$$\sum_{k=0}^{p-1} C_{2k} \left(\frac{3}{64}\right)^k \equiv \begin{cases} 1 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{12}, \\ 1/2 \pmod{p} & \text{if } p \equiv \pm 5 \pmod{12}, \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{(4k)}{2k} \left(\frac{3}{64}\right)^k \equiv \begin{cases} 1 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{12}, \\ -5/4 \pmod{p} & \text{if } p \equiv \pm 5 \pmod{12}, \end{cases}$$

Proof. Noticing that by (11),

$$w_n(1/2) = 2 \cos \left(\frac{\pi(n-1)}{3}\right) \quad \text{and} \quad w_n(-1/2) = \frac{2}{\sqrt{3}} \sin \left(\frac{\pi(2n+1)}{3}\right), \quad (18)$$

we obtain

$$w_{[4]}(1/2) \equiv \begin{cases} (-1)^{[p/4]} \pmod{p} & \text{if } p \equiv \pm 1 \pmod{12}, \\ -2(-1)^{[p/4]} \pmod{p} & \text{if } p \equiv \pm 5 \pmod{12}, \end{cases}$$

$$w_{[4]}(-1/2) \equiv \begin{cases} (-1)^{(p-1)/2} \pmod{p} & \text{if } p \equiv \pm 1 \pmod{12}, \\ 0 \pmod{p} & \text{if } p \equiv \pm 5 \pmod{12}, \end{cases}$$

and $w_{[3p/4]}(1/2) \equiv (-1)^{[p/4]}$, $w_{[3p/4]}(-1/2) \equiv (-1)^{(p-1)/2} \pmod{p}$. Combining this with Corollary 3.1 and the equality $(-1)^{(p-1)/2+[p/4]} = \left(\frac{2}{p}\right)$, we get the desired congruences. $\square$

Lemma 3.1. For any $x \neq \pm 1$,

$$w_n(2x^2 - 1) = \frac{\alpha^{2n+1} - \alpha^{-2n-1}}{\alpha - \alpha^{-1}}, \quad \text{where} \quad \alpha = x + \sqrt{x^2 - 1}.$$ 

Proof. By (10), we have

$$w_{2n}(x) = \frac{(\alpha + 1)\alpha^{2n} - (\alpha^{-1} + 1)\alpha^{-2n}}{\alpha - \alpha^{-1}}$$

$$= \frac{\alpha^2 + 1}{\alpha^2 - \alpha^{-2}} \alpha^{2n} - (\alpha^{-2} + 1)\alpha^{-2n} + (\alpha + \alpha^{-1})(\alpha^{2n} - \alpha^{-2n})$$

$$= w_n(2x^2 - 1) + \frac{\alpha^{2n} - \alpha^{-2n}}{\alpha - \alpha^{-1}}.$$ 

This implies

$$w_n(2x^2 - 1) = w_{2n}(x) - \frac{\alpha^{2n} - \alpha^{-2n}}{\alpha - \alpha^{-1}} = \frac{\alpha^{2n+1} - \alpha^{-2n-1}}{\alpha - \alpha^{-1}}$$

and the lemma follows. $\square$
Lemma 3.2. Let \( p \) be a prime, \( p > 3 \), \( x \in D_p \). Then
\[
w_{\frac{p}{4}}(2x^2 - 1) = \frac{1}{2} \left( \frac{-2}{p} \right) \left( \left( \frac{1 - x}{p} \right) + \left( \frac{1 + x}{p} \right) \right) \pmod{p},
\]
\[
w_{\frac{3p}{4}}(2x^2 - 1) = \frac{1}{2} \left( \frac{-2}{p} \right) \left( \left( \frac{1 - x}{p} \right) (1 + 2x) + \left( \frac{1 + x}{p} \right) (1 - 2x) \right) \pmod{p}.
\]

Proof. First suppose that \( x \neq \pm 1 \). Then by Lemma 3.1 if \( p \equiv 1 \pmod{4} \), we have
\[
w_{\frac{p}{4}}(2x^2 - 1) = w_{\frac{p-1}{4}}(2x^2 - 1) = \frac{\alpha^{\frac{p+1}{2}} - \alpha^{-\frac{p+1}{2}}}{\alpha - \alpha^{-1}}
\]
\[
= \frac{1}{2 \sqrt{x^2 - 1}} \sum_{k=1}^{\frac{p}{2}} \left( \frac{p}{k} \right) \left( \frac{x - 1}{2} \right)^{\frac{k}{2}} \left( \frac{x + 1}{2} \right)^{\frac{p-k}{2}}
\]
\[
+ \frac{1}{2 \sqrt{x^2 - 1}} \sum_{k=0}^{\frac{p-1}{2}} \left( \frac{p}{k} \right) \left( \frac{x - 1}{2} \right)^{\frac{k+1}{2}} \left( \frac{x + 1}{2} \right)^{\frac{k-1}{2}}
\]
\[
\equiv \frac{(x - 1)x^1}{2^{p+1}} + \frac{(x + 1)x^1}{2^p} \equiv \frac{1}{2} \left( \frac{2}{p} \right) \left( \left( \frac{x - 1}{p} \right) + \left( \frac{x + 1}{p} \right) \right) \pmod{p}.
\]

If \( p \equiv 3 \pmod{4} \), then by Lemma 3.1, we have
\[
w_{\frac{p}{4}}(2x^2 - 1) = w_{\frac{p-1}{4}}(2x^2 - 1) = \frac{\alpha^{\frac{p-1}{2}} - \alpha^{-\frac{p-1}{2}}}{\alpha - \alpha^{-1}}
\]
\[
= \frac{1}{2 \sqrt{x^2 - 1}} \sum_{k=1}^{\frac{p}{2}} \left( \frac{p}{k} \right) \left( \frac{x - 1}{2} \right)^{\frac{k+1}{2}} \left( \frac{x + 1}{2} \right)^{\frac{p-k}{2}}
\]
\[
= \frac{1}{2 \sqrt{x^2 - 1}} \sum_{k=0}^{\frac{p-1}{2}} \left( \frac{p}{k} \right) \left( \frac{x - 1}{2} \right)^{\frac{k-1}{2}} \left( \frac{x + 1}{2} \right)^{\frac{p+1}{2}}
\]
\[
\equiv \frac{(x - 1)x^1}{2^{p+1}} + \frac{(x + 1)x^1}{2^p} \equiv \frac{1}{2} \left( \frac{2}{p} \right) \left( \left( \frac{x - 1}{p} \right) - \left( \frac{x + 1}{p} \right) \right) \pmod{p},
\]
and the first congruence of the lemma follows. Similarly, to prove the second congruence, we consider two cases. If \( p \equiv 1 \pmod{4} \), then we get
\[
w_{\frac{3p}{4}}(2x^2 - 1) = w_{3(p-1)/4}(2x^2 - 1) = \frac{\alpha^{\frac{3p-1}{2}} - \alpha^{-\frac{3p-1}{2}}}{\alpha - \alpha^{-1}}
\]
\[
= \frac{1}{2 \sqrt{x^2 - 1}} \sum_{k=1}^{\frac{3p-1}{2}} \left( \frac{3p}{k} \right) \left( \frac{x - 1}{2} \right)^{\frac{k+1}{2}} \left( \frac{x + 1}{2} \right)^{\frac{3p-k}{2}}
\]
\[
= \frac{1}{2 \sqrt{x^2 - 1}} \sum_{k=0}^{\frac{3p-1}{2}} \left( \frac{3p}{k} \right) \left( \frac{x - 1}{2} \right)^{\frac{k-1}{2}} \left( \frac{x + 1}{2} \right)^{\frac{3p+1}{2}}
\]
\[
\equiv \frac{(x - 1)x^1}{2^{3p+1}} + \frac{(x + 1)x^1}{2^p} \equiv \frac{1}{2} \left( \frac{2}{p} \right) \left( \left( \frac{x - 1}{p} \right)^3 - \left( \frac{x + 1}{p} \right)^3 \right) \pmod{p}.
\]
Simplifying the right-hand side modulo \( p \), we obtain
\[
\frac{1}{2} \left( \frac{x - 1}{2} \right)^{\frac{3p-1}{2}} - \frac{1}{2} \left( \frac{x + 1}{2} \right)^{\frac{3p-1}{2}} + \frac{3}{2} \left( \frac{x - 1}{2} \right)^{\frac{p-1}{2}} \left( \frac{x + 1}{2} \right)^p - \frac{3}{2} \left( \frac{x + 1}{2} \right)^{\frac{p-1}{2}} \left( \frac{x - 1}{2} \right)^p
\]
and therefore,
\[
w_{\frac{3p}{4}}(2x^2 - 1) \equiv \frac{1}{2} \left( \frac{2}{p} \right) \left( \left( \frac{x - 1}{p} \right)^{2} + \left( \frac{x + 1}{p} \right)^{2} \right) \pmod{p}.
\]
If \( p \equiv 3 \pmod{4} \), then
\[
w_{\frac{3p}{4}}(2x^2 - 1) = w_{\frac{3p-1}{4}}(2x^2 - 1) = \frac{\alpha^{\frac{3p+1}{2}} - \alpha^{-\frac{3p+1}{2}}}{\alpha - \alpha^{-1}} = \frac{\left( \sqrt{x + \frac{1}{2}} \right)^{3p+1} - \left( \sqrt{x - \frac{1}{2}} \right)^{3p+1}}{2\sqrt{x^2 - 1}}.
\]
Simplifying the right-hand side modulo \( p \), we get
\[
\frac{1}{2} \left( \frac{x - 1}{2} \right)^{\frac{3p-1}{2}} + \frac{1}{2} \left( \frac{x + 1}{2} \right)^{\frac{3p-1}{2}} + \frac{3}{2} \left( \frac{x - 1}{2} \right)^{\frac{p-1}{2}} \left( \frac{x + 1}{2} \right)^p + \frac{3}{2} \left( \frac{x + 1}{2} \right)^{\frac{p-1}{2}} \left( \frac{x - 1}{2} \right)^p
\]
and therefore,
\[
w_{\frac{3p}{4}}(2x^2 - 1) \equiv \frac{1}{2} \left( \frac{2}{p} \right) \left( \left( \frac{x - 1}{p} \right)^{2} + \left( \frac{x + 1}{p} \right)^{2} \right) \pmod{p},
\]
as required. If \( x = \pm 1 \), then by (10) we have \( w_{\frac{3p}{4}}(1) = 2^{\frac{p-1}{4}} + 1 \equiv (-1)^{(p-1)/2} \pmod{p} \) and \( w_{\frac{3p}{4}}(1) = 2^{\frac{p-1}{4}} + 1 \equiv (-1)^{(p+1)/2} \pmod{p} \), which completes the proof of the lemma.

From Lemma 3.2 and Corollary 3.1 we immediately deduce the following result.

**Theorem 3.2.** Let \( p \) be a prime, \( p > 3 \) and \( t \in D_p \). Then
\[
\sum_{k=0}^{p-1} C_{2k} \left( \frac{1 - t^2}{16} \right)^k \equiv \frac{1}{2} \left( \frac{2}{p} \right) \left( \left( \frac{1 - t}{p} \right)(1 - t) + \left( \frac{1 + t}{p} \right)(1 + t) \right) \pmod{p},
\]
\[
\sum_{k=0}^{p-1} \frac{4k}{2k} t^{2k} \equiv \frac{1}{2} \left( \left( \frac{1 - 4t}{p} \right)(1 + 2t) + \left( \frac{1 + 4t}{p} \right)(1 - 2t) \right) \pmod{p}.
\]

**Proof.** From Corollary 3.1 we have
\[
\sum_{k=0}^{p-1} C_{2k} \left( \frac{1 - t^2}{16} \right)^k \equiv \frac{1}{2} \left( \frac{-1}{p} \right) \left( 3w_{\frac{3p}{4}}(2t^2 - 1) - w_{\frac{3p}{4}}(2t^2 - 1) \right) \pmod{p},
\]
and
\[
\sum_{k=0}^{p-1} \frac{4k}{2k} t^{2k} \equiv \frac{1}{4} \left( \frac{-2}{p} \right) \left( 3w_{\frac{3p}{4}}(32t^2 - 1) + w_{\frac{3p}{4}}(32t^2 - 1) \right) \pmod{p}
\]
Now by Lemma 3.2 with \( x \) replaced by \( 4t \) for the second congruence we conclude the proof.
Theorem 3.3. Let \( p \) be a prime, \( p > 3 \), \( a, b \in \mathbb{Z}, ab \not\equiv 0 \pmod{p} \) and \( a \not\equiv b \pmod{p} \). Then

\[
\sum_{k=0}^{p-1} C_{2k} \frac{(a-b)^{2k}}{(-64ab)^k} \equiv \begin{cases} \\
\frac{(ab)^{p-1}}{2(a-b)} & (3a + b) \left( \frac{b}{p} \right) - (3b + a) \left( \frac{a}{p} \right) \quad \text{(mod } p\text{)} \quad \text{if } p \equiv 1 \pmod{4}, \\
\frac{(ab)^{p+1}}{2(a-b)} & (3a + b) \left( \frac{b}{p} \right) - \frac{3b + a}{b} \left( \frac{a}{p} \right) \quad \text{(mod } p\text{)} \quad \text{if } p \equiv 3 \pmod{4}, \\
\end{cases}
\]

Proof. By Corollary 3.1 we have

\[
\sum_{k=0}^{p-1} C_{2k} \frac{(a-b)^{2k}}{(-64ab)^k} \equiv \begin{cases} \\
\frac{(ab)^{p-1}}{4(a-b)} & (3a - b) \left( \frac{b}{p} \right) - (3b - a) \left( \frac{a}{p} \right) \quad \text{(mod } p\text{)} \quad \text{if } p \equiv 1 \pmod{4}, \\
\frac{(ab)^{p+1}}{4(b-a)} & (3a - b) \left( \frac{b}{p} \right) - \frac{3b - a}{b} \left( \frac{a}{p} \right) \quad \text{(mod } p\text{)} \quad \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

From (10) we obtain

\[
w_n \left( \frac{a^2 + b^2}{2ab} \right) = \frac{a(a/b)^n - b(b/a)^n}{a-b}.
\]

If \( p \equiv 1 \pmod{4} \), then

\[
w_{\left[ \frac{n}{2} \right]} \left( \frac{a^2 + b^2}{2ab} \right) = w_{\frac{n-1}{4}} \left( \frac{a^2 + b^2}{2ab} \right) = \frac{a(a/b)^{n/4} - b(b/a)^{n/4}}{a-b} = \frac{a \left( \frac{b}{p} \right) - b \left( \frac{a}{p} \right)}{a-b} \quad \text{(mod } p\text{)};
\]

\[
w_{\left[ \frac{3n}{4} \right]} \left( \frac{a^2 + b^2}{2ab} \right) = \frac{a(a/b)^{3(n-1)/4} - b(b/a)^{3(n-1)/4}}{a-b} = \frac{a \left( \frac{b}{p} \right) - b \left( \frac{a}{p} \right)}{a-b} \quad \text{(mod } p\text{)}.
\]

If \( p \equiv 3 \pmod{4} \), then we have

\[
w_{\left[ \frac{n}{2} \right]} \left( \frac{a^2 + b^2}{2ab} \right) = w_{\frac{n-1}{4}} \left( \frac{a^2 + b^2}{2ab} \right) = \frac{a(a/b)^{n/4} - b(b/a)^{n/4}}{a-b} = \frac{a \left( \frac{b}{p} \right) - b \left( \frac{a}{p} \right)}{a-b} \quad \text{(mod } p\text{)};
\]

\[
w_{\left[ \frac{3n}{4} \right]} \left( \frac{a^2 + b^2}{2ab} \right) = \frac{a(a/b)^{3(n-1)/4} - b(b/a)^{3(n-1)/4}}{a-b} = \frac{a \left( \frac{b}{p} \right) - b \left( \frac{a}{p} \right)}{a-b} \quad \text{(mod } p\text{)}.
\]

Now substituting the above congruences in (19) and (20), we conclude the proof. \( \square \)

4. Polynomial congruences involving \( \binom{3n}{n} \) and \( C_n^{(2)} \)

In this Section we will deal with a particular case of Theorem 2.1 when \( m = 3 \). This case leads to congruences containing second-order Catalan numbers \( C_n^{(2)} \) and binomial coefficients \( \binom{3n}{n} \).
Theorem 4.1. Let $p$ be a prime greater than 3 and $t \in D_p$. Then

\[
\begin{align*}
\sum_{k=0}^{[p/3]} C_k^{(2)} t^k &\equiv 3 \left( \frac{p}{3} \right) w_{\frac{p}{3}} (1 - 27t/2) \pmod{p}, \\
\sum_{k=(p-1)/2}^{[2p/3]} C_k^{(2)} t^k &\equiv - \left( \frac{p}{3} \right) \left( w_{\frac{p}{3}} (1 - 27t/2) + w_{\frac{p}{3}} (1 - 27t/2) \right) \pmod{p}, \\
\sum_{k=0}^{[p/3]} \left( \frac{3k}{k} \right) t^k &\equiv \left( \frac{p}{3} \right) w_{\frac{p}{3}} (27t/2 - 1) \pmod{p}, \\
\sum_{k=(p+1)/2}^{[2p/3]} \left( \frac{3k}{k} \right) t^k &\equiv \frac{1}{3} \left( \frac{p}{3} \right) \left( w_{\frac{p}{3}} (27t/2 - 1) - w_{\frac{p}{3}} (27t/2 - 1) \right) \pmod{p}.
\end{align*}
\]

Corollary 4.1. Let $p$ be a prime greater than 3 and $t \in D_p$. Then

\[
\begin{align*}
\sum_{k=0}^{p-1} C_k^{(2)} t^k &\equiv \left( \frac{p}{3} \right) \left( 2w_{\frac{p}{3}} (1 - 27t/2) - w_{\frac{p}{3}} (1 - 27t/2) \right) \pmod{p}, \\
\sum_{k=0}^{p-1} \left( \frac{3k}{k} \right) t^k &\equiv \frac{1}{3} \left( \frac{p}{3} \right) \left( 2w_{\frac{p}{3}} (27t/2 - 1) + w_{\frac{2p}{3}} (27t/2 - 1) \right) \pmod{p}.
\end{align*}
\]

Using the exact values of $w_n$ from [17] and [18], we immediately get numerical congruences at the points $t = 4/27, 2/27, 1/27, 1/9$.

Corollary 4.2. Let $p$ be a prime greater than 3. Then

\[
\begin{align*}
\sum_{k=0}^{p-1} C_k^{(2)} \left( \frac{4}{27} \right)^k &\equiv 1, \quad \sum_{k=0}^{[p/3]} C_k^{(2)} \left( \frac{4}{27} \right)^k \equiv 3 \pmod{p}, \\
\sum_{k=0}^{p-1} \left( \frac{3k}{k} \right) \left( \frac{4}{27} \right)^k &\equiv \frac{1}{9}, \quad \sum_{k=0}^{[p/3]} \left( \frac{3k}{k} \right) \left( \frac{4}{27} \right)^k \equiv \frac{1}{3} \pmod{p},
\end{align*}
\]

\[
\begin{align*}
\sum_{k=0}^{p-1} C_k^{(2)} \left( \frac{2}{27} \right)^k &\equiv 2 \left( \frac{3}{p} \right) - 1, \quad \sum_{k=0}^{[p/3]} C_k^{(2)} \left( \frac{2}{27} \right)^k \equiv 3 \left( \frac{3}{p} \right) \pmod{p}, \\
\sum_{k=0}^{p-1} \left( \frac{3k}{k} \right) \left( \frac{2}{27} \right)^k &\equiv \frac{2}{3} \left( \frac{3}{p} \right) + \frac{1}{3}, \quad \sum_{k=0}^{[p/3]} \left( \frac{3k}{k} \right) \left( \frac{2}{27} \right)^k \equiv \frac{3}{p} \pmod{p}.
\end{align*}
\]

Remark 4.1.1. Note that another proof of the first congruence in (24) based on third-order recurrences can be found in [11] Theorem 3.1. The second congruence in (24) as well as the second congruence in (25) was also proved by Z.H. Sun [10] Remark 3.1 with the help of Lucas sequences.
Corollary 4.3. Let \( p \) be a prime, \( p > 3 \). Then

\[
\sum_{k=0}^{\lfloor p/3 \rfloor} \frac{C_k^{(2)}}{27^k} = \begin{cases} 
-6 \pmod{p} & \text{if } p \equiv \pm 4 \pmod{9}, \\
3 \pmod{p} & \text{otherwise}, 
\end{cases}
\]

\[
\sum_{k=0}^{p-1} \frac{C_k^{(2)}}{27^k} = \begin{cases} 
1 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{9}, \\
4 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{9}, \\
-5 \pmod{p} & \text{if } p \equiv \pm 4 \pmod{9}, 
\end{cases}
\]

\[
\sum_{k=0}^{p-1} \binom{3k}{k} \frac{1}{27^k} = \begin{cases} 
1 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{9}, \\
-2/3 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{9}, \\
-1/3 \pmod{p} & \text{if } p \equiv \pm 4 \pmod{9}. 
\end{cases}
\]

Corollary 4.4. Let \( p \) be a prime, \( p > 3 \). Then

\[
\sum_{k=0}^{p-1} \frac{C_k^{(2)}}{9^k} = \begin{cases} 
-2 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{9}, \\
1 \pmod{p} & \text{otherwise}, 
\end{cases}
\]

\[
\sum_{k=0}^{\lfloor p/3 \rfloor} \frac{C_k^{(2)}}{9^k} = \begin{cases} 
3 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{9}, \\
-3 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{9}, \\
0 \pmod{p} & \text{if } p \equiv \pm 4 \pmod{9}, 
\end{cases}
\]

\[
\sum_{k=0}^{p-1} \binom{3k}{k} \frac{1}{9^k} = \begin{cases} 
1 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{9}, \\
0 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{9}, \\
-1 \pmod{p} & \text{if } p \equiv \pm 4 \pmod{9}. 
\end{cases}
\]

Remark 4.1.2. Congruences (26), (27) were also proved in [11, Theorem 1.5] by using cubic residues and third-order recurrences.

Lemma 4.1. For any \( x \neq 1, -1/2 \),

\[
w_n(4x^3 - 3x) = \frac{\alpha^{3n+2} - \alpha^{-3n-1}}{\alpha^2 - \alpha^-1}, \quad \text{where} \quad \alpha = x + \sqrt{x^2 - 1}.
\]

Proof. Starting with \( w_{3n}(x) \), by (10) we have

\[
w_{3n}(x) = \frac{(\alpha + 1)\alpha^{3n} - (\alpha^{-1} + 1)\alpha^{-3n}}{\alpha - \alpha^{-1}}
\]

\[
= \frac{(\alpha^3 + 1)\alpha^{3n} - (\alpha^{-3} + 1)\alpha^{-3n}}{\alpha^3 - \alpha^{-3}} \cdot \frac{\alpha^3 - \alpha^{-3}}{\alpha - \alpha^{-1}}
\]

\[
+ \frac{\alpha^{3n+1} - \alpha^{3n+3} - \alpha^{-3n-1} + \alpha^{-3n-3}}{\alpha - \alpha^{-1}}
\]

\[
= w_n(4x^3 - 3x)(\alpha^2 + 1 + \alpha^{-2})
\]

\[
+ \frac{\alpha^{3n+1} - \alpha^{3n+3} - \alpha^{-3n-1} + \alpha^{-3n-3}}{\alpha - \alpha^{-1}}.
\]

Comparing right and left-hand sides, we get

\[
w_n(4x^3 - 3x)(\alpha^2 + 1 + \alpha^{-2}) = \frac{\alpha^{3n} + \alpha^{3n+3} - \alpha^{-3n} - \alpha^{-3n-3}}{\alpha - \alpha^{-1}} = \frac{(\alpha^3 + 1)(\alpha^{3n} - \alpha^{-3n-3})}{\alpha - \alpha^{-1}}
\]
and therefore,
\[ w_n(4x^3 - 3x) = \frac{(\alpha^3 + 1)(\alpha^{3n} - \alpha^{-3n-3})}{\alpha^3 - \alpha^{-3}} = \frac{\alpha^3(\alpha^{3n} - \alpha^{-3n-3})}{\alpha^3 - 1} = \frac{\alpha^{3n+2} - \alpha^{-3n-1}}{\alpha^2 - \alpha^{-1}}. \]

Lemma 4.2. Let \( p \) be a prime, \( p > 3 \) and \( x \in D_p \). Then
\[ (2x + 1) \cdot w_{\frac{p}{3}}(4x^3 - 3x) \equiv \left( \frac{p}{3} \right) x + \left( \frac{x^2 - 1}{p} \right) (x + 1) \pmod{p}, \]
\[ (2x + 1) \cdot w_{\frac{2p}{3}}(4x^3 - 3x) \equiv \left( \frac{p}{3} \right) (1 - 2x^2) + 2 \left( \frac{x^2 - 1}{p} \right) x(x + 1) \pmod{p}. \]

Proof. First we suppose that \( x \not\equiv 1, -1/2 \pmod{p} \). Then by Lemma 4.1, if \( p \equiv 1 \pmod{3} \) we have
\[ w_{\frac{p}{3}}(4x^3 - 3x) = w_{\frac{p-1}{3}}(4x^3 - 3x) = \frac{\alpha^{p+1} - \alpha^{-p}}{\alpha^2 - \alpha^{-1}}. \quad (28) \]
For \( p \)-powers of \( \alpha \) and \( \alpha^{-1} \), we easily obtain
\[ \alpha^{\pm p} = (x \pm \sqrt{x^2 - 1})^p \equiv x^p \pm (\sqrt{x^2 - 1})^p \equiv x \pm \sqrt{x^2 - 1}(x^2 - 1)^{p-1} \]
\[ \equiv x \pm \left( \frac{x^2 - 1}{p} \right) \sqrt{x^2 - 1} \pmod{p}. \quad (29) \]
Substituting (29) into (28) and simplifying, we get
\[ w_{\frac{p}{3}}(4x^3 - 3x) \equiv \frac{(x + \sqrt{x^2 - 1})(x + (\frac{x^2 - 1}{p})\sqrt{x^2 - 1}) - x + (\frac{x^2 - 1}{p})\sqrt{x^2 - 1}}{(2x + 1)(x - 1 + \sqrt{x^2 - 1})} \]
\[ \equiv \frac{x + (\frac{x^2 - 1}{p})(x + 1)}{2x + 1} \pmod{p}. \]
If \( p \equiv 2 \pmod{3} \), then by Lemma 4.1 and (29), we have
\[ w_{\frac{p}{3}}(4x^3 - 3x) = w_{\frac{p-2}{3}}(4x^3 - 3x) = \frac{\alpha^p - \alpha^{1-p}}{\alpha^2 - \alpha^{-1}} \]
\[ \equiv \frac{x + (\frac{x^2 - 1}{p})\sqrt{x^2 - 1} - (x + \sqrt{x^2 - 1})(x - (\frac{x^2 - 1}{p})\sqrt{x^2 - 1})}{(2x + 1)(x - 1 + \sqrt{x^2 - 1})} \]
\[ \equiv \frac{-x + (\frac{x^2 - 1}{p})(x + 1)}{2x + 1} \pmod{p} \]
and the first congruence of the lemma follows. Similarly, if \( p \equiv 1 \pmod{3} \), then we have
\[ w_{\frac{2p}{3}}(4x^3 - 3x) = w_{\frac{2(p-1)}{3}}(4x^3 - 3x) = \frac{\alpha^{2p} - \alpha^{1-2p}}{\alpha^2 - \alpha^{-1}} \]
\[ \equiv \frac{(x + (\frac{x^2 - 1}{p})\sqrt{x^2 - 1})^2 - (x + \sqrt{x^2 - 1})(x - (\frac{x^2 - 1}{p})\sqrt{x^2 - 1})^2}{(2x + 1)(x - 1 + \sqrt{x^2 - 1})} \pmod{p}. \]
Simplifying, we easily find
\[ w_{\frac{2p}{3}}(4x^3 - 3x) \equiv \frac{1 - 2x^2 + 2(\frac{x^2 - 1}{p})x(x + 1)}{2x + 1} \pmod{p}. \]
Lemma 4.3. Let \( p \) be a prime, then
\[
\frac{w_{\left[\frac{p}{3}\right]}(4x^3 - 3x)}{w_{\left[\frac{p-1}{3}\right]}(4x^3 - 3x)} = \frac{\alpha^{2p+1} - \alpha^{-2p}}{\alpha^2 - \alpha^{-1}}
\]
\[
\equiv \frac{(x + \sqrt{x^2 - 1})(x + (\frac{x^2 - 1}{p}) \sqrt{x^2 - 1})^2 - (x - (\frac{x^2 - 1}{p}) \sqrt{x^2 - 1})^2}{(2x + 1)(x - 1 + \sqrt{x^2 - 1})} \pmod{p}
\]
and after simplification we get
\[
\frac{w_{\left[\frac{p}{3}\right]}(4x^3 - 3x)}{w_{\left[\frac{p-1}{3}\right]}(4x^3 - 3x)} \equiv \frac{2x^2 - 1 + 2(\frac{x^2 - 1}{p})x(x + 1)}{2x + 1} \pmod{p}
\]
as desired.

Finally, if \( x \equiv 1 \pmod{10} \), we have \( 3w_{\left[\frac{p}{3}\right]}(1) = 3(2p/3 + 1) \equiv \left(\frac{5}{3}\right) \pmod{p} \) and \( 3w_{\left[\frac{p}{3}\right]}(1) = 3(2p/3 + 1) \equiv -\left(\frac{5}{3}\right) \pmod{p} \), which coincide with the right-hand sides of the required congruences when \( x \equiv 1 \pmod{p} \).

If \( x \equiv -1/2 \pmod{p} \), then the congruences become trivial and the proof is complete.

**Lemma 4.3.** Let \( p \) be a prime, \( p > 3 \), \( x \in D_p \). Then the following congruences hold modulo \( p \):

\[
(2x + 1) \cdot w_{\left[\frac{p}{3}\right]}(4x^3 - 3x) \equiv \left(\frac{2x - 2}{p}\right) x + \left(\frac{-6x - 6}{p}\right) (x + 1),
\]
\[
(2x + 1) \cdot w_{\left[\frac{p}{3}\right]}(4x^3 - 3x) \equiv \left(\frac{2x - 2}{p}\right) x(4x^2 + 2x + 1) \frac{-6x - 6}{p} (x + 1)(4x^2 - 2x - 1).
\]

**Proof.** First suppose that \( x \not\equiv 1, -1/2 \pmod{p} \). If \( p \equiv 1 \pmod{6} \), then by Lemma 4.1 we have
\[
w_{\left[\frac{p}{3}\right]}(4x^3 - 3x) = w_{\left[\frac{p-1}{3}\right]}(4x^3 - 3x) = \frac{\alpha^{p+1} - \alpha^{-p+1}}{\alpha^2 - \alpha^{-1}}.
\]
Substituting \( \alpha = x + \sqrt{x^2 - 1} = (\sqrt{x + 1}/2 + \sqrt{x - 1}/2)^2 \), we have
\[
w_{\left[\frac{p}{3}\right]}(4x^3 - 3x) = \frac{(x + \sqrt{x^2 - 1})(\sqrt{x + 1} + \sqrt{x - 1})^{p+1} - (\sqrt{x + 1} - \sqrt{x - 1})^{p+1}}{2^{p+1}/2(\alpha^2 - \alpha^{-1})}
\]
\[
= \frac{(x + \sqrt{x^2 - 1})(\sqrt{x + 1} + \sqrt{x - 1})^p - 1/2(\sqrt{x + 1} - \sqrt{x - 1})^{p+2}}{2^{p+1}/2(2x + 1)\sqrt{x - 1}}
\]
\[
\equiv \frac{(x + \sqrt{x^2 - 1})(\sqrt{x + 1} + \sqrt{x - 1})^p - 1/2(\sqrt{x + 1} - \sqrt{x - 1})^{p+2}}{2^{p+1}/2(2x + 1)\sqrt{x - 1}}
\]
\[
\equiv \frac{x(x - 1)^{p-1} + (x + 1)^{p+1}}{2^{p+1}/2(2x + 1)} \equiv \frac{2x^2 - 1 + 2(\frac{x^2 - 1}{p})x(x + 1)}{2x + 1} \pmod{p}.
\]
Since \( \left(\frac{3}{p}\right) = \left(\frac{5}{3}\right) = 1 \), we get the desired congruence in this case.

If \( p \equiv 5 \pmod{6} \), then we have
\[
w_{\left[\frac{p}{5}\right]}(4x^3 - 3x) = w_{\left[\frac{p-3}{5}\right]}(4x^3 - 3x) = \frac{\alpha^{p+1} - \alpha^{-p+1}}{\alpha^2 - \alpha^{-1}}.
\]
and therefore,
\[
w_{\left[\frac{p}{6}\right]}(4x^3 - 3x) = \frac{(x - \sqrt{x^2 - 1})(x + \frac{\sqrt{x^2 - 1}}{x}) - (x + \sqrt{x^2 - 1})(x + \frac{\sqrt{x^2 - 1}}{x})}{2(\sqrt{x^2 - 1}/2)(2x + 1)\sqrt{x^2 - 1}}
\]
\[
\equiv \frac{x(x - 1)^{\frac{p-1}{2}} - (x + 1)^{\frac{p-1}{2}}}{2(\sqrt{x^2 - 1}/2)(2x + 1)} \equiv \frac{(2x-2)\alpha(x + 1)}{2x + 1} (\mod p)
\]
as desired in view of the fact that \( \frac{-\sqrt{3}}{p} = \left(\frac{2}{p}\right) = -1 \).

The similar analysis can be applied for evaluating \( w_{\left[\frac{2p}{6}\right]}(4x^3 - 3x) \) modulo \( p \). If \( p \equiv 1 \) (mod 6), then
\[
w_{\left[\frac{2p}{6}\right]}(4x^3 - 3x) = w_{\left[\frac{p}{6}\right]}(4x^3 - 3x) = \frac{\alpha^{\frac{p-1}{2}} - \alpha^{\frac{p-3}{2}}}{\alpha^{2} - \alpha^{1}}.
\]
Simplifying, we obtain
\[
w_{\left[\frac{2p}{6}\right]}(4x^3 - 3x) = \frac{(x - \sqrt{x^2 - 1})(\sqrt{x + 1} + \sqrt{x - 1})^{5p} - (x + \sqrt{x^2 - 1})(\sqrt{x + 1} - \sqrt{x - 1})^{5p}}{2(\sqrt{x^2 - 1}/2)(2x + 1)\sqrt{x^2 - 1}}
\]
\[
\equiv \frac{(x - \sqrt{x^2 - 1})(x + \frac{\sqrt{x^2 - 1}}{x}) - (x + \sqrt{x^2 - 1})(x + \frac{\sqrt{x^2 - 1}}{x})}{2(\sqrt{x^2 - 1}/2)(2x + 1)}
\]
\[
\equiv \frac{(2x-2)\alpha(x + 1)(4x^2 - 2x - 1)}{2x + 1} (\mod p),
\]
as desired. If \( p \equiv 5 \) (mod 6), then
\[
w_{\left[\frac{2p}{6}\right]}(4x^3 - 3x) = \frac{(x - \sqrt{x^2 - 1})(\sqrt{x + 1} + \sqrt{x - 1})^{5p + 2} - (\sqrt{x + 1} - \sqrt{x - 1})^{5p + 2}}{2(\sqrt{x^2 - 1}/2)(2x + 1)\sqrt{x^2 - 1}}
\]
\[
\equiv \frac{(x + \sqrt{x^2 - 1})(x + \frac{\sqrt{x^2 - 1}}{x}) - (x - \sqrt{x^2 - 1})(x + \frac{\sqrt{x^2 - 1}}{x})}{8 \cdot 2(\sqrt{x^2 - 1}/2)(2x + 1)\sqrt{x^2 - 1}}
\]
\[
\equiv \frac{(2x-2)\alpha(x + 1)(4x^2 - 2x - 1)}{2x + 1} (\mod p),
\]
and the congruence is true in this case. If \( x \equiv 1 \) (mod 6), then by (10) we have \( 3w_{\left[\frac{p}{6}\right]}(1) = 3(2[6/p] + 1) = 2(p/3) \) (mod \( p \)) and \( 3w_{\left[\frac{2p}{6}\right]}(1) = 3(2[5p/6] + 1) = -2(p/3) \) (mod \( p \)), which proves the lemma in this case too. Finally, if \( x \equiv -1/2 \) (mod \( p \)), we get the trivial congruences \( 0 \equiv 0 \), and the proof is complete. \( \square \)

**Theorem 4.2.** Let \( p \) be a prime, \( p > 3 \) and \( t \in D_p \).

If \( t \not\equiv 0 \) (mod \( p \)), then
\[
\sum_{k=1}^{\left[\frac{p}{3}\right]} C_k(2)^k (t^2(t + 1))^k \equiv \frac{1 + t}{2t} - \frac{1 - 3t}{2t} \left(\frac{(1 + t)(1 - 3t)}{p}\right) (\mod p),
\]
\[
\sum_{k=1}^{p-1} C_k(2)^k (t^2(t + 1))^k \equiv \frac{(1 + t)(1 - 3t)}{2t} \left(1 - \left(\frac{(1 + t)(1 - 3t)}{p}\right)\right) (\mod p). \quad (30)
\]
If $3t + 2 \not\equiv 0 \pmod{p}$, then
\[
\sum_{k=1}^{\lfloor p/3 \rfloor} \binom{3k}{k} (t^2(t+1))^k \equiv \frac{3(t+1)}{2(3t+2)} \left( \left( \frac{1+t}{p} \right)(1-3t) - 1 \right) \pmod{p},
\]
and
\[
\sum_{k=0}^{p-1} \binom{3k}{k} (t^2(t+1))^k \equiv \frac{3(t+1)^2}{2(3t+2)} \left( \left( \frac{1+t}{p} \right)(1-3t) - 1 \right) \pmod{p}.
\]

**Proof.** From (21), Corollary 4.1 and Lemma 4.2 we have
\[
\sum_{k=0}^{\lfloor p/3 \rfloor} C_k^{(2)} \left( \frac{2(1-x)(2x+1)^2}{27} \right)^k \equiv 3 \left( \frac{p}{3} \right) w_{\frac{x}{3}}(4x^3 - 3x) = \frac{3x}{2x+1} + \frac{3x + 3}{2x + 1} \left( \frac{3 - 3x^2}{p} \right) \pmod{p}
\]

and
\[
\sum_{k=0}^{p-1} C_k^{(2)} \left( \frac{2(1-x)(2x+1)^2}{27} \right)^k \equiv \left( \frac{p}{3} \right) \left( 2w_{\frac{x}{3}}(4x^3 - 3x) - w_{\frac{x}{3}}(4x^3 - 3x) \right) = \frac{2x^2 + 2x - 1}{2x + 1} + \frac{2(1-x^2)}{2x + 1} \left( \frac{3 - 3x^2}{p} \right) \pmod{p}
\]
for any $x \in D_p$ such that $2x + 1 \not\equiv 0 \pmod{p}$. Replacing $x$ by $(-1 - 3t)/2$ with $t \not\equiv 0 \pmod{p}$, we get the first two congruences of the theorem.

Similarly, from (22), Corollary 4.1 and Lemma 4.2 for any $x \in D_p$ with $2x + 1 \not\equiv 0 \pmod{p}$, we have
\[
\sum_{k=0}^{\lfloor p/3 \rfloor} \binom{3k}{k} \left( \frac{2(x+1)(2x-1)^2}{27} \right)^k \equiv \left( \frac{p}{3} \right) w_{\frac{x}{3}}(4x^3 - 3x) = \frac{x}{2x+1} + \frac{x + 1}{2x + 1} \left( \frac{3 - 3x^2}{p} \right) \pmod{p},
\]
and
\[
\sum_{k=0}^{p-1} \binom{3k}{k} \left( \frac{2(x+1)(2x-1)^2}{27} \right)^k \equiv \frac{1}{3} \left( \frac{p}{3} \right) \left( 2w_{\frac{x}{3}}(4x^3 - 3x) + w_{\frac{x}{3}}(4x^3 - 3x) \right) = \frac{1 + 2x - 2x^2}{3(2x+1)} + \frac{2(x+1)^2}{3(2x+1)} \left( \frac{3 - 3x^2}{p} \right) \pmod{p}.
\]

This implies that
\[
\sum_{k=1}^{\lfloor p/3 \rfloor} \binom{3k}{k} \left( \frac{2(x+1)(2x-1)^2}{27} \right)^k \equiv \frac{x + 1}{2x + 1} \left( \left( \frac{3 - 3x^2}{p} \right) - 1 \right) \pmod{p},
\]
and
\[
\sum_{k=1}^{p-1} \binom{3k}{k} \left( \frac{2(x+1)(2x-1)^2}{27} \right)^k \equiv \frac{2(x+1)^2}{3(2x+1)} \left( \left( \frac{3 - 3x^2}{p} \right) - 1 \right) \pmod{p}.
\]
Replacing $x$ by $(3t + 1)/2$, we derive the other two congruences of the theorem. \(\square\)

**Remark 4.2.1.** Note that Z. H. Sun \(8\) Theorem 2.3 proved congruence (31) by another method using cubic congruences. If we put $t = -c/(c+1)$ in (30) and (32), we recover corresponding congruences of Z.W. Sun \(11\) Theorem 1.1] proved by applying properties of third-order recurrences.
5. Cubic residues and non-residues and their application to congruences

We begin with a brief review of basic facts from the theory of cubic residues that will be needed later in this section. Let $\omega = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$. We consider the ring of the Eisenstein integers $\mathbb{Z}[\omega] = \{a + b\omega : a, b \in \mathbb{Z}\}$. To define the cubic residue symbol, we recall arithmetic properties of the ring $\mathbb{Z}[\omega]$ including description of its units and primes (see [2, Chapter 9]).

If $\alpha = a + b\omega \in \mathbb{Z}[\omega]$, the norm of $\alpha$ is defined by the formula $N(\alpha) = a\overline{a} - ab + b^2$, where $\overline{\alpha} = a + b\overline{\omega} = a + b\omega^2 = (a - b) - b\omega$ is the complex conjugate of $\alpha$. Note that the norm is a nonnegative integer always congruent to 0 or 1 modulo 3. It is well known that $\mathbb{Z}[\omega]$ is a unique factorization domain. The units of $\mathbb{Z}[\omega]$ are $\pm 1, \pm \omega, \pm \omega^2$.

Let $p$ be a prime in $\mathbb{Z}$, then $p$ in $\mathbb{Z}[\omega]$ falls into three categories [1, Prop. 4.7]: (i) if $p = 3$, then $3 = -\omega^2(1 - \omega)$, where $1 - \omega$ is prime in $\mathbb{Z}[\omega]$ and $N(1 - \omega) = (1 - \omega)(1 - \omega^2) = 3$; (ii) if $p \equiv 2 \pmod{3}$, then $p$ remains prime in $\mathbb{Z}[\omega]$ and $N(p) = p^2$; (iii) if $p \equiv 1 \pmod{3}$, then $p$ splits into the product of two conjugate nonassociate primes in $\mathbb{Z}[\omega]$, $p = \pi\overline{\pi}$ and $N(\pi) = N(\overline{\pi}) = p$. Moreover, every prime in $\mathbb{Z}[\omega]$ is associate to one of the primes listed in (i) – (iii).

An analog of Fermat’s Little Theorem is true in $\mathbb{Z}[\omega]$: if $\pi$ is a prime and $\pi \nmid \alpha$, then

$$\alpha^{N(\pi)-1} \equiv 1 \pmod{\pi}.$$  

Note that if $\pi$ is a prime such that $N(\pi) \neq 3$, then $N(\pi) \equiv 1 \pmod{3}$ and the expression $\alpha^{\frac{N(\pi)-1}{3}}$ is well defined in $\mathbb{Z}[\omega]$, i.e., $\alpha^{\frac{N(\pi)-1}{3}} \equiv \omega^j \pmod{\pi}$ for a unique unit $\omega^j$. This leads to the definition of the cubic residue character of $\alpha$ modulo $\pi$ (see [2, p. 112]):

$$\left(\frac{\alpha}{\pi}\right)_3 = \begin{cases} 0 & \text{if } \pi \nmid \alpha, \\ \omega^j & \text{if } \alpha^{\frac{N(\pi)-1}{3}} \equiv \omega^j \pmod{\pi}. \end{cases}$$  (33)

The cubic residue character has formal properties similar to those of the Legendre symbol [1, Prop. 9.3.3]:

(i) The congruence $x^3 \equiv \alpha \pmod{\pi}$ is solvable in $\mathbb{Z}[\omega]$ if and only if $\left(\frac{\omega}{\pi}\right)_3 = 1$, i.e., iff $\alpha$ is a cubic residue modulo $\pi$;

(ii) $\left(\frac{\alpha \beta}{\pi}\right)_3 = \left(\frac{\beta}{\pi}\right)_3 \left(\frac{\alpha}{\pi}\right)_3$;

(iii) $\left(\frac{\omega}{\pi}\right)_3 = \left(\frac{\overline{\omega}}{\pi}\right)_3$;

(iv) If $\pi$ and $\theta$ are associates, then $\left(\frac{\omega}{\pi}\right)_3 = \left(\frac{\omega}{\theta}\right)_3$;

(v) If $\alpha \equiv \beta \pmod{\pi}$, then $\left(\frac{\alpha}{\pi}\right)_3 = \left(\frac{\beta}{\pi}\right)_3$.

Let $\pi = a + b\omega \in \mathbb{Z}[\omega]$. We say that $\pi$ is primary if $\pi \equiv 2 \pmod{3}$, that is equivalent to $a \equiv 2 \pmod{3}$ and $b \equiv 0 \pmod{3}$. If $\pi \in \mathbb{Z}[\omega]$, $N(\pi) > 1$ and $\pi \equiv \pm 2 \pmod{3}$, we may decompose $\pi = \pm \pi_1 \ldots \pi_r$, where $\pi, \ldots, \pi_r$ are primary primes (see [2, p. 135]). For $\alpha \in \mathbb{Z}[\omega]$, the cubic Jacobi symbol $\left(\frac{\alpha}{\pi}\right)_3$ is defined by

$$\left(\frac{\alpha}{\pi}\right)_3 = \left(\frac{\alpha}{\pi_1}\right)_3 \ldots \left(\frac{\alpha}{\pi_r}\right)_3.$$  

Now let $p$ be a prime. We define a cubic residue symbol modulo $p$ in $\mathbb{Z}$. We say that $m \in \mathbb{Z}$ is a cubic residue modulo $p$ if the congruence $x^3 \equiv m \pmod{p}$ has an integer solution, otherwise $m$ is called a cubic non-residue modulo $p$. If $p = 3$, then by Fermat’s little theorem, $m^3 \equiv m \pmod{3}$ for all integers $m$, so $x^3 \equiv m \pmod{3}$ always has a solution. If $p \equiv 2 \pmod{3}$, then every integer $m$ is a cubic residue modulo $p$. Indeed, we have $2p - 1 \equiv 0 \pmod{3}$ and by Fermat’s little theorem, $m^{2p-1} \equiv (m^{\frac{2p-1}{3}})^3 \pmod{p}$. So the only interesting case which remains is when a prime $p \equiv 1 \pmod{3}$. 

If a prime $p \equiv 1 \pmod{3}$, then it is well known that there are unique integers $L$ and $|M|$ such that $4p = L^2 + 27M^2$ with $L \equiv 1 \pmod{3}$. In this case, $p$ splits into the product of primes of $\mathbb{Z}[\omega]$, $p = \pi \overline{\pi}$, where we can write $\pi$ in the form

$$\pi = \frac{1}{2}(L + 3M \sqrt{-3}) = \frac{L + 3M}{2} + 3M \omega.$$

It is easy to see that $\left( \frac{L}{\pi} \right)^2 \equiv -3 \pmod{p}$ and therefore for any integer $m$ coprime to $p$ by Euler’s criterion (see [5, 14]), we have one of the three possibilities

$$m(p-1)/3 \equiv 1, \quad (-1 - L/(3M))/2 \quad \text{or} \quad (-1 + L/(3M))/2 \pmod{p}.$$

Moreover, $m(p-1)/3 \equiv 1 \pmod{p}$ if and only if $m$ is a cubic residue modulo $p$. When $m$ is a prime and a cubic non-residue modulo $p$, K. S. Williams [14] found a method how to choose the sign of $M$ so that $m(p-1)/3 \equiv (-1 - L/(3M))/2 \pmod{p}$. To classify cubic residues and non-residues in $\mathbb{Z}$, Z. H. Sun [7] introduced three subsets

$$C_j(m) = \left\{ c \in D_m \left| \left( \frac{c + 1 + 2\omega}{m} \right)_3 = \omega^j \right. \right\}, \quad j = 0, 1, 2, \ m \in \mathbb{N}, \ m \neq 0 \pmod{3},$$

of $D_m$, which possess the following properties:

(i) $C_0(m) \cup C_1(m) \cup C_2(m) = \{ c \in D_m \mid (c^2 + 3, m) = 1 \}$,

(ii) $c \in C_0(m)$ if and only if $-c \in C_0(m)$,

(iii) $c \in C_1(m)$ if and only if $-c \in C_2(m)$.

(iv) If $c, c' \in D_m$ and $cc' \equiv -3 \pmod{m}$, then $c \in C_j(m)$ if and only if $c' \in C_j(m)$.

Using these sets, Z. H. Sun proved the following criterion of cubic residuacity in $\mathbb{Z}$: Let $p$ be a prime of the form $p \equiv 1 \pmod{3}$ and hence $4p = L^2 + 27M^2$ for some $L, M \in \mathbb{Z}$ and $L \equiv 1 \pmod{3}$. If $q$ is a prime with $q | M$, then $q^{(p-1)/3} \equiv 1 \pmod{p}$. If $q \nmid M$ and $j \in \{0, 1, 2\}$, then

$$q^{(p-1)/3} \equiv ((-1 - L/(3M))/2)^j \pmod{p} \quad \text{if and only if} \quad L/(3M) \in C_j(q). \quad (34)$$

Sun [8] gave a simple criterion for $c \in C_j(p)$ in terms of values of the sum $\sum_{k=1}^{[p/3]} (3k) \left( \frac{4}{9(c^2 + 3)} \right)^k$ modulo $p$ and conjectured a similar criterion in terms of the sum $\sum_{k=(p+1)/2}^{[2p/3]} (3k) t^k$. In this section, using our formulas from Theorem 4.1, we address this question of Sun (see Theorem 5.2 below). First, we prove the following criterion.

**Theorem 5.1.** Let $p > 3$ be a prime, $c \in D_p$ and $c^2 \not\equiv -3 \pmod{p}$. Then

$$c \sum_{k=(p+1)/2}^{[2p/3]} (3k) \left( \frac{4}{9(c^2 + 3)} \right)^k \equiv \begin{cases} 0 \pmod{p} & \text{if} \ c \in C_0(p), \\ 1 \pmod{p} & \text{if} \ c \in C_1(p), \\ -1 \pmod{p} & \text{if} \ c \in C_2(p). \end{cases}$$

To prove the theorem, we will need the following statement.

**Lemma 5.1.** ([7, Lemma 2.2]) Let $p \not\equiv 3 \pmod{3}$ be a prime and $c \in D_p$.

(i) If $p \equiv 1 \pmod{3}$ and so $p$ splits into the product of primes, $p = \pi \overline{\pi}$ with $\pi \in \mathbb{Z} \{\omega\}$ and $\pi \equiv 2 \pmod{3}$, then

$$\left( \frac{c + 1 + 2\omega}{p} \right)_3 = \left( \frac{c^2 + 3(c - 1 - 2\omega)}{\pi} \right)_3,$$

$$\left( \frac{c - 1 - 2\omega}{p} \right)_3 = \left( \frac{c + 1 + 2\omega}{p} \right)_3^{-1} = \left( \frac{c^2 + 3(c + 1 + 2\omega)}{\pi} \right)_3.$$
Now by Lemma 5.1, we have
\[
\left( \frac{c+1+2\omega}{p} \right)^3 \equiv (c^2+3)^{(p-2)/3}(c+1+2\omega)^{(p+1)/3} \quad (\text{mod } p),
\]
\[
\left( \frac{c-1-2\omega}{p} \right)^3 = \left( \frac{c+1+2\omega}{p} \right)^{-1} \equiv (c^2+3)^{(p-2)/3}(c-1-2\omega)^{(p+1)/3} \quad (\text{mod } p).
\]

**Proof of Theorem 5.1.** By (23), we have

\[
\left[ \frac{\omega}{1} \right] = \sum_{k=(p+1)/2}^{[2p/3]} \binom{3k}{k} \binom{4}{9(c^2+3)}^k \equiv \frac{1}{3} \left( \frac{p}{3} \right) \left( w_{[\pi]} \left( \frac{3-c^2}{3+c^2} \right) - w_{[\pi]} \left( \frac{3-c^2}{3+c^2} \right) \right) \quad (\text{mod } p). \tag{35}
\]

From (10) it easily follows that

\[
w_{n} \left( \frac{3-c^2}{3+c^2} \right) = \frac{(-1)^n}{2c(c^2+3)^{p-1/3}} \left( (c-1-2\omega)^{2(p-1)/3} + (c+1+2\omega)^{2(p-1)/3} \right). \tag{36}
\]

If \( p \equiv 1 \pmod{3} \), then \( p \) splits into the product of primes in \( \mathbb{Z}[\omega] \), \( p = \pi \bar{\pi} \) with \( \pi \equiv 2 \pmod{3} \) and by (36), we have

\[
w_{[\pi]} \left( \frac{3-c^2}{3+c^2} \right) = \frac{1}{2c(c^2+3)^{(p-1)/3}} \left( (c-1-2\omega)^{2(p-1)/3} + (c+1+2\omega)^{2(p-1)/3} \right). \tag{37}
\]

By (33) and Lemma 5.1 we have

\[
(c^2+3)^{(p-1)/3}(c-1-2\omega)^{(p-1)/3} \equiv \left( \frac{(c^2+3)(c-1-2\omega)}{\pi} \right)^2 = \left( \frac{c+1+2\omega}{p} \right)^2 \quad (\text{mod } \pi) \tag{38}
\]

and

\[
(c^2+3)^{(p-1)/3}(c+1+2\omega)^{(p-1)/3} \equiv \left( \frac{(c^2+3)(c+1+2\omega)}{\pi} \right)^2 = \left( \frac{c+1+2\omega}{p} \right)^2 \quad (\text{mod } \pi). \tag{39}
\]

Substituting (38), (39) into (37), we get

\[
w_{[\pi]} \left( \frac{3-c^2}{3+c^2} \right) \equiv \frac{1}{2c} \left( (c-1-2\omega) \left( \frac{c+1+2\omega}{p} \right)^2 + (c+1+2\omega) \left( \frac{c+1+2\omega}{p} \right)^{-2} \right) \quad (\text{mod } \pi)
\]

and therefore,

\[
w_{[\pi]} \left( \frac{3-c^2}{3+c^2} \right) \equiv \begin{cases} 1 \pmod{\pi} & \text{if } c \in C_0(p), \\ -\frac{3+c}{2c} \pmod{\pi} & \text{if } c \in C_1(p), \\ \frac{3-c}{2c} \pmod{\pi} & \text{if } c \in C_2(p). \end{cases} \tag{40}
\]

Since both sides of the above congruence are rational, the congruence is also true modulo \( p = \pi \bar{\pi} \). Similarly, if \( p \equiv 2 \pmod{3} \), then

\[
w_{[\pi]} \left( \frac{3-c^2}{3+c^2} \right) = \frac{-1}{2c(c^2+3)^{(p-2)/3}} \left( (c-1-2\omega)^{2(p+1)/3-1} + (c+1+2\omega)^{2(p+1)/3-1} \right). \tag{41}
\]

Now by Lemma 5.1 we have

\[
(c+1+2\omega)^{(p+1)/3} \equiv (c+1+2\omega)^{2(p-2)/3} \left( \frac{c+1+2\omega}{p} \right)^2 \quad (\text{mod } p)
\]
Applying the similar argument to evaluation of $w$ and therefore, $w$ and $p$

Combining congruences (40) and (42), we obtain that for all primes $p$,

Substituting the above congruences into (41) and noticing that $c^2 + 3 = (c + 1 + 2\omega)(c - 1 - 2\omega)$, we get

and therefore,

Combining congruences (40) and (42), we obtain that for all primes $p > 3$,

Applying the similar argument to evaluation of $w_{\frac{3}{2p}}(\frac{3 - c^2}{3 + c^2})$, we see that if $p \equiv 1 \pmod{3}$, then $2p \equiv 2 \pmod{3}$ and therefore,

which implies

If $p \equiv 2 \pmod{3}$, then $2p \equiv 1 \pmod{3}$ and we obtain

and therefore,
Combining (44) and (45), we see that for all primes $p > 3$,
\[
\left(\frac{p^3}{3}\right) w_3^2 \left(\frac{p^3}{3} + 2^2\right) = \begin{cases} 
1 & \text{if } c \in C_0(p), \\
\frac{3-c}{2c} & \text{if } c \in C_1(p), \\
\frac{3+c}{2c} & \text{if } c \in C_2(p).
\end{cases}
\] (46)

Now by (43), (46) and (35), the congruence of the theorem easily follows. □

From Theorem 5.1 and criterion (34) we deduce the following result confirming a question of Sun [8, Conj. 2.1].

**Theorem 5.2.** Let $q$ be a prime, $q \equiv 1 \pmod{3}$ and so $4q = L^2 + 27M^2$ with $L, M \in \mathbb{Z}$ and $L \equiv 1 \pmod{3}$. Let $p$ be a prime with $p \neq 2, 3, q$ and $p \nmid LM$. Then
\[
\sum_{k=(p+1)/2}^{[2p/3]} \binom{3k}{k} \frac{M^{2k}}{q^k} \equiv \begin{cases} 
0 & \text{if } p^{2-1} \equiv 1 \pmod{q}, \\
\pm \frac{3M}{L} & \text{if } p^{2-1} \equiv -1+9M/L \pmod{q}
\end{cases}
\] and
\[
\sum_{k=(p+1)/2}^{[2p/3]} \binom{3k}{k} \frac{L^{2k}}{(27q)^k} \equiv \begin{cases} 
0 & \text{if } p^{2-1} \equiv 1 \pmod{q}, \\
\pm \frac{L}{9M} & \text{if } p^{2-1} \equiv -1+L/(3M) \pmod{q}
\end{cases}
\]

**Proof.** To prove the first congruence, put $c = \frac{L}{3M}$ in Theorem 5.1. Then $c(c^2 + 3) \equiv 0 \pmod{p}$, $\frac{4}{9(c^2 + 3)} = \frac{M^2}{q}$ and we have
\[
\sum_{k=(p+1)/2}^{[2p/3]} \binom{3k}{k} \frac{M^{2k}}{q^k} \equiv \begin{cases} 
0 & \text{if } L/(3M) \in C_0(p), \\
\frac{3M}{L} & \text{if } L/(3M) \in C_1(p), \\
-\frac{3M}{L} & \text{if } L/(3M) \in C_2(p).
\end{cases}
\]

Now applying (44) and taking into account that $L/(3M) \equiv -9M/L \pmod{q}$, we get the result.

To prove the second congruence, we put $c = -9M/L$ in Theorem 5.1. Then $c(c^2 + 3) \equiv 0 \pmod{p}$, $\frac{4}{9(c^2 + 3)} = \frac{L^2}{27q}$ and we have
\[
\sum_{k=(p+1)/2}^{[2p/3]} \binom{3k}{k} \frac{L^{2k}}{(27q)^k} \equiv \begin{cases} 
0 & \text{if } -9M/L \in C_0(p), \\
-\frac{L}{9M} & \text{if } -9M/L \in C_1(p), \\
\frac{L}{9M} & \text{if } -9M/L \in C_2(p).
\end{cases}
\] (47)

By (iv), we know that $-9M/L \in C_j(p)$ if and only if $L/(3M) \in C_j(p)$. This together with (47) and (34) implies the required congruence. □

From Corollary 4.1 and formulas (43) and (46) we get the following statement.

**Theorem 5.3.** Let $p > 3$ be a prime, $c \in D_p$ and $c^2 \equiv -3 \pmod{p}$. Then
\[
\sum_{k=0}^{p-1} \binom{3k}{k} \frac{4}{9(c^2 + 3)}^k \equiv \begin{cases} 
1 & \text{if } c \in C_0(p), \\
\frac{1+c}{2c} & \text{if } c \in C_1(p), \\
\frac{1-c}{2c} & \text{if } c \in C_2(p)
\end{cases}
\]

and
\[
\sum_{k=0}^{p-1} \binom{4c^2}{27(c^2 + 3)}^k \equiv \begin{cases} 
1 & \text{if } c \in C_0(p), \\
\frac{9+c}{2c} & \text{if } c \in C_1(p), \\
\frac{9-c}{2c} & \text{if } c \in C_2(p)
\end{cases}
\]
From Theorem 5.3 and criterion (34) we get the following congruences.

**Theorem 5.4.** Let \( q \) be a prime, \( q \equiv 1 \pmod{3} \) and so \( 4q = L^2 + 27M^2 \) with \( L, M \in \mathbb{Z} \) and \( L \equiv 1 \pmod{3} \). Let \( p \) be a prime with \( p \neq 2, 3, q \) and \( p \nmid LM \). Then

\[
\sum_{k=0}^{p-1} \binom{3k}{k} \frac{M^{2k}}{q^k} \equiv \begin{cases} 
1 \pmod{p} & \text{if } p^{q-1} \equiv 1 \pmod{q}, \\
\pm 3M - L \pmod{2L} & \text{if } p^{q-1} \equiv -\frac{1+L/3M}{2} \pmod{q}, 
\end{cases}
\]

and

\[
\sum_{k=0}^{p-1} \binom{3k}{k} \frac{L^{2k}}{(27q)^k} \equiv \begin{cases} 
1 \pmod{p} & \text{if } p^{q-1} \equiv 1 \pmod{q}, \\
\pm L - 9M \pmod{18M} & \text{if } p^{q-1} \equiv -\frac{1+9M/L}{2} \pmod{q}, 
\end{cases}
\]

and

\[
\sum_{k=0}^{p-1} C_k^{(2)} \frac{M^{2k}}{q^k} \equiv \begin{cases} 
1 \pmod{p} & \text{if } p^{q-1} \equiv 1 \pmod{q}, \\
\pm L - M/2M \pmod{2M} & \text{if } p^{q-1} \equiv -\frac{1+9M/L}{2} \pmod{q}, 
\end{cases}
\]

and

\[
\sum_{k=0}^{p-1} C_k^{(2)} \frac{L^{2k}}{(27q)^k} \equiv \begin{cases} 
1 \pmod{p} & \text{if } p^{q-1} \equiv 1 \pmod{q}, \\
\pm 27M - L \pmod{2L} & \text{if } p^{q-1} \equiv -\frac{1+9M/L}{2} \pmod{q}, 
\end{cases}
\]

**Proof.** Substituting consequently \( c = L/(3M) \) and then \( c = -9M/L \) in Theorem 5.3 and following the same line of reasoning as in the proof of Theorem 5.2, we get the above congruences. \( \square \)

In particular, setting \( q = 7, 19, 31, 37 \) in Theorem 5.4, we get the following numerical congruences.

**Corollary 5.1.** Let \( p \) be a prime, \( p \neq 2, 3, 7 \). Then

\[
\sum_{k=0}^{p-1} \binom{3k}{k} \frac{1}{189^k} \equiv \begin{cases} 
-2 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{7}, \\
1 \pmod{p} & \text{otherwise},
\end{cases}
\]

and

\[
\sum_{k=0}^{p-1} C_k^{(2)} \frac{1}{189^k} \equiv \begin{cases} 
1 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{7}, \\
-14 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{7}, \\
13 \pmod{p} & \text{if } p \equiv \pm 3 \pmod{7}.
\end{cases}
\]

**Corollary 5.2.** Let \( p \) be a prime, \( p \neq 2, 3, 7, 19 \). Then

\[
\sum_{k=0}^{p-1} \binom{3k}{k} \frac{1}{19^k} \equiv \begin{cases} 
1 \pmod{p} & \text{if } p \equiv \pm 1, \pm 7, \pm 8 \pmod{19}, \\
-2/7 \pmod{p} & \text{if } p \equiv \pm 2, \pm 3, \pm 5 \pmod{19}, \\
-5/7 \pmod{p} & \text{if } p \equiv \pm 4, \pm 6, \pm 9 \pmod{19}.
\end{cases}
\]

**Corollary 5.3.** Let \( p \) be a prime, \( p \neq 2, 3, 31 \). Then

\[
\sum_{k=0}^{p-1} \binom{3k}{k} \binom{4}{31}^k \equiv \begin{cases} 
1 \pmod{p} & \text{if } p \equiv \pm 1, \pm 2, \pm 4, \pm 8, \pm 15 \pmod{31}, \\
-5/4 \pmod{p} & \text{if } p \equiv \pm 3, \pm 6, \pm 7, \pm 12, \pm 14 \pmod{31}, \\
1/4 \pmod{p} & \text{if } p \equiv \pm 5, \pm 9, \pm 10, \pm 11, \pm 13 \pmod{31}.
\end{cases}
\]
Corollary 5.4. Let \( p \) be a prime, \( p \neq 2, 3, 11, 37 \). Then
\[
\sum_{k=0}^{[p/6]} \left(\binom{3k}{k} \frac{1}{37^k}\right) \equiv \begin{cases} 
1 \pmod{p} & \text{if } p \equiv 1, \pm 6, \pm 8, \pm 10, \pm 11, \pm 14 \pmod{37}, \\
-4/11 \pmod{p} & \text{if } p \equiv 2, \pm 9, \pm 12, \pm 15, \pm 16, \pm 17 \pmod{37}, \\
-7/11 \pmod{p} & \text{if } p \equiv 3, \pm 4, \pm 5, \pm 7, \pm 13, \pm 18 \pmod{37}, 
\end{cases}
\]

6. Polynomial congruences involving \( S_n \)

In this Section we will deal with a particular case of Theorem 2.1 when \( m = 6 \). In this case we get polynomial congruences containing the sequences \( S_k \) and \((2k + 1)S_k\).

Theorem 6.1. Let \( p \) be a prime greater than 3 and \( t \in D_p \). Then
\[
\sum_{k=0}^{[p/6]} S_k t^k \equiv \frac{3}{4} \left(\frac{p}{3}\right) w_{[2p]}(1 - 216t) \pmod{p},
\]
\[
\sum_{k=(p-1)/2}^{[5p/6]} S_k t^k \equiv -\frac{1}{8} \left(\frac{p}{3}\right) \left( w_{[2p]}(1 - 216t) + w_{[p]}(1 - 216t) \right) \pmod{p},
\]
\[
\sum_{k=(p+1)/2}^{[p/6]} (2k + 1) S_k t^k \equiv \frac{1}{12} (-1)^{\frac{p-1}{2}} \left( w_{[2p]}(216t - 1) - w_{[p]}(216t - 1) \right) \pmod{p},
\]
\[
\sum_{k=(2k+1)/2}^{[5p/6]} (2k + 1) S_k t^k \equiv \frac{1}{12} (-1)^{\frac{p-1}{2}} \left( w_{[2p]}(216t - 1) - w_{[p]}(216t - 1) \right) \pmod{p}.
\]

Corollary 6.1. Let \( p \) be a prime greater than 3 and \( t \in D_p \). Then
\[
\sum_{k=0}^{p-1} S_k t^k \equiv \frac{1}{8} \left(\frac{p}{3}\right) \left( 5 w_{[2p]}(1 - 216t) - w_{[2p]}(1 - 216t) \right) \pmod{p},
\]
\[
\sum_{k=0}^{p-1} (2k + 1) S_k t^k \equiv \frac{1}{12} (-1)^{\frac{p-1}{2}} \left( w_{[2p]}(216t - 1) + 5 w_{[p]}(216t - 1) \right) \pmod{p}.
\]

Taking into account (17), we get the following explicit congruences. Note that the first congruence below confirms a conjecture of Z.W. Sun [12, Conj. 2].

Corollary 6.2. Let \( p \) be a prime greater than 3. Then
\[
\sum_{k=0}^{p-1} S_k \equiv \frac{1}{2} \left(\frac{3}{p}\right), \quad \sum_{k=0}^{[p/6]} S_k \equiv \frac{3}{4} \left(\frac{3}{p}\right) \pmod{p},
\]
\[
\sum_{k=0}^{p-1} S_k \equiv \frac{1}{2} \left(\frac{2}{p}\right), \quad \sum_{k=0}^{[p/6]} S_k \equiv \frac{3}{4} \left(\frac{2}{p}\right) \pmod{p},
\]
Corollary 6.3. Let $p > 3$ be a prime. Then

$$
\sum_{k=0}^{p-1} \frac{(2k+1)S_k}{108^k} \equiv \frac{2}{9} \left( \frac{3}{p} \right), \quad \sum_{k=0}^{[p/6]} \frac{(2k+1)S_k}{108^k} \equiv \frac{1}{3} \left( \frac{3}{p} \right) \pmod{p},
$$

$$
\sum_{k=0}^{p-1} \frac{(2k+1)S_k}{216^k} \equiv \frac{1}{2} \left( \frac{6}{p} \right), \quad \sum_{k=0}^{[p/6]} \frac{(2k+1)S_k}{216^k} \equiv \frac{1}{2} \left( \frac{6}{p} \right) \pmod{p}.
$$

From Corollary 6.1 and (18) we get the following congruences.

**Corollary 6.3.** Let $p > 3$ be a prime. Then

$$
\frac{3}{p} \sum_{k=0}^{p-1} \frac{S_k}{432^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv 1 \pmod{9}, \\
-11/8 \pmod{p} & \text{if } p \equiv 2 \pmod{9}, \\
7/8 \pmod{p} & \text{if } p \equiv 4 \pmod{9}, 
\end{cases}
$$

$$
\sum_{k=0}^{p-1} S_k \left( \frac{3}{432} \right)^k \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv 1 \pmod{9}, \\
1/8 \pmod{p} & \text{if } p \equiv 2 \pmod{9}, \\
-5/8 \pmod{p} & \text{if } p \equiv 4 \pmod{9}, 
\end{cases}
$$

$$
\left( \frac{p}{3} \right) \sum_{k=0}^{p-1} \frac{(2k+1)S_k}{432^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv 1 \pmod{9}, \\
-1/12 \pmod{p} & \text{if } p \equiv 2 \pmod{9}, \\
-5/12 \pmod{p} & \text{if } p \equiv 4 \pmod{9}, 
\end{cases}
$$

$$
\sum_{k=0}^{p-1} \frac{(2k+1)S_k}{432^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv 1 \pmod{9}, \\
-3/4 \pmod{p} & \text{if } p \equiv 2 \pmod{9}, \\
1/4 \pmod{p} & \text{if } p \equiv 4 \pmod{9}.
\end{cases}
$$

The following theorem provides two families of polynomial congruences.

**Theorem 6.2.** Let $p$ be a prime, $p > 3$ and $t \in D_p$.

If $t \not\equiv 0 \pmod{p}$, then the following congruences hold modulo $p$:

$$
\sum_{k=0}^{[p/6]} S_k(t^2(4t+1))^k \equiv \frac{1+12t}{32t} \left( \frac{1+4t}{p} \right) - \frac{1-12t}{32t} \left( \frac{1-12t}{p} \right),
$$

$$
\sum_{k=0}^{p-1} S_k(t^2(4t+1))^k \equiv \frac{(1+12t)(1+4t)(1-6t)}{32t} \left( \frac{1+4t}{p} \right) - \frac{(1-12t)(24t^2+6t+1)}{32t} \left( \frac{1-12t}{p} \right).
$$

If $6t+1 \not\equiv 0 \pmod{p}$, then we have modulo $p$,

$$
\sum_{k=0}^{[p/6]} (2k+1)S_k(t^2(4t+1))^k \equiv \frac{1+12t}{8(1+6t)} \left( \frac{1-12t}{p} \right) + \frac{3(1+4t)}{8(1+6t)} \left( \frac{1+4t}{p} \right),
$$

$$
\sum_{k=0}^{p-1} (2k+1)S_k(t^2(4t+1))^k \equiv \frac{(1+12t)(24t^2+6t+1)}{8(1+6t)} \left( \frac{1-12t}{p} \right) + \frac{3(1-6t)(1+4t)^2}{8(1+6t)} \left( \frac{1+4t}{p} \right).$$
Proof. From (48), Corollary 6.1 and Lemma 4.3 for any \( x \in D_p \) with \( 2x + 1 \not\equiv 0 \pmod{p} \) we have
\[
\sum_{k=0}^{[p/6]} S_k \left( \frac{(1-x)(2x+1)^2}{216} \right) \equiv \frac{3}{4} \left( \frac{p}{3} \right) w_{[\frac{p}{6}]}(4x^3 - 3x) \\
= \frac{3(x+1)}{4(2x+1)} \left( \frac{2x+2}{p} \right) + \frac{3x}{4(2x+1)} \left( \frac{6-6x}{p} \right) \pmod{p},
\]
\[
\sum_{k=0}^{p-1} S_k \left( \frac{(1-x)(2x+1)^2}{216} \right) \equiv \frac{1}{8} \left( \frac{p}{3} \right) (5w_{[\frac{p}{6}]}(4x^3 - 3x) - w_{[\frac{p}{6}]}(4x^3 - 3x)) \\
= \frac{(x+1)(2x^2 - 2x + 2)}{4(2x+1)} \left( \frac{2x+2}{p} \right) - \frac{x(x-1)(2x+3)}{4(2x+1)} \left( \frac{6-6x}{p} \right) \pmod{p}.
\]

Now replacing \( x \) by \((-12t-1)/2\), we get the first two congruences of the theorem.
Similarly, from (49), Corollary 6.1 and Lemma 4.3 for any \( x \in D_p \) such that \( 2x+1 \not\equiv 0 \pmod{p} \), we have
\[
\sum_{k=0}^{[p/6]} (2k+1) S_k \left( \frac{(x+1)(2x-1)^2}{216} \right) \equiv \frac{1}{2} (-1)^{\frac{x-1}{2}} w_{[\frac{p}{6}]}(4x^3 - 3x) \\
= \frac{x}{2(2x+1)} \left( \frac{2-2x}{p} \right) + \frac{x+1}{2(2x+1)} \left( \frac{6x+6}{p} \right) \pmod{p},
\]
\[
\sum_{k=0}^{p-1} (2k+1) S_k \left( \frac{(x+1)(2x-1)^2}{216} \right) \equiv \frac{1}{12} (-1)^{\frac{x-1}{2}} (5w_{[\frac{p}{6}]}(4x^3 - 3x) + w_{[\frac{p}{6}]}(4x^3 - 3x)) \\
= \frac{x(2x^2 + 2x + 2)}{6(2x+1)} \left( \frac{2-2x}{p} \right) - \frac{(x+1)(2x-3)}{6(2x+1)} \left( \frac{6x+6}{p} \right) \pmod{p}.
\]
Replacing \( x \) by \((12t + 1)/2\), we conclude the proof. \( \square \)

The next theorem gives a criterion for \( c \in C_j(p) \) in terms of values of the sums \( \sum_{k=0}^{p-1} S_k t^k \) and \( \sum_{k=0}^{p-1} (2k+1) S_k t^k \) modulo \( p \).

**Theorem 6.3.** Let \( p > 3 \) be a prime, \( c \in D_p \) and \( c^2 \not\equiv -3 \pmod{p} \). Then
\[
\left( \frac{3(c^2 + 3)}{p} \right) \cdot \sum_{k=0}^{p-1} S_k \left( \frac{c^2}{108(c^2 + 3)} \right) \equiv \begin{cases} 
1/2 & \pmod{p} \text{ if } c \in C_0(p), \\
9 - 2c & \pmod{p} \text{ if } c \in C_1(p), \\
-9 + 2c & \pmod{p} \text{ if } c \in C_2(p)
\end{cases}
\]
and
\[
\left( \frac{c^2 + 3}{p} \right) \cdot \sum_{k=0}^{p-1} (2k+1) S_k \left( \frac{c^2}{36k(3+c^2)^k} \right) \equiv \begin{cases} 
1/2 & \pmod{p} \text{ if } c \in C_0(p), \\
2 - c & \pmod{p} \text{ if } c \in C_1(p), \\
-2 + c & \pmod{p} \text{ if } c \in C_2(p)
\end{cases}
\]
Proof. From Corollary 6.1 we have

\[
\sum_{k=0}^{p-1} S_k \left( \frac{c^2}{108(3 + c^2)} \right)^k \equiv \frac{1}{8} \left( \frac{p}{3} \right) \left( 5w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) - w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) \right) \quad (\text{mod } p),
\]

(50)

\[
\sum_{k=0}^{p-1} \frac{(2k + 1)S_k}{36k(3 + c^2)} \equiv \frac{(-1)^{\frac{p-1}{2}}}{12} \left( 5w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) + w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) \right) \quad (\text{mod } p).
\]

(51)

If \( p \equiv 1 \) (mod 6), then \( p \) splits into the product of primes in \( \mathbb{Z}[\omega] \), \( p = \pi \bar{\pi} \) with \( \pi \equiv 2 \) (mod 3) and by (36), we easily obtain

\[
w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) = \frac{(-1)^{\frac{p-1}{2}}}{2c(c^2 + 3)} \left( (c - 1 - 2\omega)^{\frac{p-1}{3}+1} + (c + 1 + 2\omega)^{\frac{p-1}{3}+1} \right).
\]

Applying Lemma 5.1 we have

\[
w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) \equiv \frac{(-1)^{\frac{p-5}{6}}}{2c(c^2 + 3)} \left( (c - 1 + 2\omega) \left( \frac{c + 1 + 2\omega}{p} \right)^3 \right) \]

modulo \( \pi \) and therefore,

\[
(-1)^{\frac{p-1}{2}} \left( \frac{c^2 + 3}{p} \right) \cdot w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) \equiv \begin{cases} 1 \quad (\text{mod } p) & \text{if } c \in C_0(p), \\ \frac{3-c}{2c} \quad (\text{mod } p) & \text{if } c \in C_1(p), \\ -\frac{3+c}{2c} \quad (\text{mod } p) & \text{if } c \in C_2(p). \end{cases}
\]

(52)

If \( p \equiv 5 \) (mod 6), then

\[
w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) = \frac{(-1)^{\frac{p-5}{6}}}{2c(c^2 + 3)} \left( (c - 1 - 2\omega)^{\frac{p+1}{3}-1} + (c + 1 + 2\omega)^{\frac{p+1}{3}-1} \right).
\]

Now by Lemma 5.1 we get

\[
w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) \equiv \frac{(-1)^{\frac{p-5}{6}}}{2c(c^2 + 3)} \left( (c + 1 + 2\omega) \left( \frac{c + 1 + 2\omega}{p} \right)^3 \right) \]

modulo \( p \) and therefore,

\[
(-1)^{\frac{p-5}{6}} \left( \frac{c^2 + 3}{p} \right) \cdot w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) \equiv \begin{cases} 1 \quad (\text{mod } p) & \text{if } c \in C_0(p), \\ \frac{3+c}{2c} \quad (\text{mod } p) & \text{if } c \in C_1(p), \\ -\frac{3-c}{2c} \quad (\text{mod } p) & \text{if } c \in C_2(p). \end{cases}
\]

(53)

Comparing (52) and (53), we get that (52) holds for all primes \( p > 3 \).

Applying the similar argument for evaluation of \( w_{[\frac{p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) \), we see that if \( p \equiv 1 \) (mod 6), then \( 5p \equiv 5 \) (mod 6) and we have

\[
w_{[\frac{5p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) = \frac{(-1)^{\frac{p-1}{6}}}{2c(c^2 + 3)} \left( (c - 1 - 2\omega)^{\frac{5(p-1)}{3}+1} + (c + 1 + 2\omega)^{\frac{5(p-1)}{3}+1} \right).
\]

Now by Lemma 5.1 we easily find

\[
w_{[\frac{5p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) \equiv \frac{(-1)^{\frac{p-1}{6}}}{2c(c^2 + 3)} \left( (c - 1 - 2\omega) \left( \frac{c + 1 + 2\omega}{p} \right)^5 \right) \]

modulo \( p \) and therefore,

\[
(-1)^{\frac{p-1}{6}} \left( \frac{c^2 + 3}{p} \right) \cdot w_{[\frac{5p}{6}]} \left( \frac{3 - c^2}{3 + c^2} \right) \equiv \begin{cases} 1 \quad (\text{mod } p) & \text{if } c \in C_0(p), \\ \frac{3+c}{2c} \quad (\text{mod } p) & \text{if } c \in C_1(p), \\ -\frac{3-c}{2c} \quad (\text{mod } p) & \text{if } c \in C_2(p). \end{cases}
\]
modulo \( \pi \), which implies

\[
(-1)^{\frac{p-1}{2}} \left( \frac{c^2 + 3}{p} \right) \cdot w(\frac{c}{p}) \left( \frac{3 - c^2}{3 + c^2} \right) \equiv \begin{cases} 1 \pmod{p} & \text{if } c \in C_0(p), \\ -3 + c & 2c \pmod{p} & \text{if } c \in C_1(p), \\ 3 - c & 2c \pmod{p} & \text{if } c \in C_2(p). \end{cases}
\]

Similarly, if \( p \equiv 5 \pmod{6} \), then \( 5p \equiv 1 \pmod{6} \) and we have

\[
w(\frac{2p}{5p}) \left( \frac{3 - c^2}{3 + c^2} \right) = \frac{(-1)^{\frac{5p-1}{6}}}{2c(c^2 + 3)^{\frac{5p-1}{6}}} \left( (c - 1 - 2\omega)^{\frac{5p+2}{3}} + (c + 1 + 2\omega)^{\frac{5p+2}{3}} \right).
\]

By Lemma 5.1, we readily get

\[
w(\frac{2p}{5p}) \left( \frac{3 - c^2}{3 + c^2} \right) = \frac{(-1)^{\frac{5p-1}{6}}}{2c(c^2 + 3)^{\frac{5p-1}{6}}} \left( (c + 1 + 2\omega)^{-\frac{5}{3}} + (c - 1 - 2\omega)^{\frac{5}{3}} \right)
\]

modulo \( p \) and therefore after simplification we obtain that (54) holds for all primes \( p > 3 \). Finally, substituting (52) and (54) into (50) and (51), we get the congruences of the theorem. \( \square \)

From Theorem 6.3 with \( c = -9M/L \) and \( c = L/3M \) and criterion (54) we get the following congruences.

\textbf{Theorem 6.4.} Let \( q \) be a prime, \( q \equiv 1 \pmod{3} \) and so \( 4q = L^2 + 27M^2 \) with \( L, M \in \mathbb{Z} \) and \( L \equiv 1 \pmod{3} \). Let \( p \) be a prime with \( p \not\equiv 2, 3, q \) and \( p \nmid LM \). Then

\[
\left( \frac{2}{p} \right)^{p-1} \sum_{k=0}^{p-1} S_k \frac{M^{2k}}{(16q)^k} \equiv \begin{cases} 1/2 \pmod{p} & \text{if } p^{\frac{q-1}{3}} \equiv 1 \pmod{q}, \\ \pm \frac{1}{2} \frac{8M - 2L}{8M} \pmod{p} & \text{if } p^{\frac{q-1}{3}} \equiv -\frac{1+9M/L}{2} \pmod{q}. \end{cases}
\]

\[
\left( \frac{3q}{p} \right)^{p-1} \sum_{k=0}^{p-1} S_k \frac{L^{2k}}{(432q)^k} \equiv \begin{cases} 1/2 \pmod{p} & \text{if } p^{\frac{q-1}{3}} \equiv 1 \pmod{q}, \\ \pm \frac{1}{2} \frac{27M - 2L}{8L} \pmod{p} & \text{if } p^{\frac{q-1}{3}} \equiv -\frac{1+9M/L}{2} \pmod{q}. \end{cases}
\]

and

\[
\left( \frac{2}{p} \right)^{p-1} \sum_{k=0}^{p-1} (2k + 1)S_k \frac{M^{2k}}{(16q)^k} \equiv \begin{cases} 1/2 \pmod{p} & \text{if } p^{\frac{q-1}{3}} \equiv 1 \pmod{q}, \\ \pm \frac{1}{4} \frac{6M - L}{4L} \pmod{p} & \text{if } p^{\frac{q-1}{3}} \equiv -\frac{1+9M/L}{2} \pmod{q}. \end{cases}
\]

\[
\left( \frac{3q}{p} \right)^{p-1} \sum_{k=0}^{p-1} (2k + 1)S_k \frac{L^{2k}}{(432q)^k} \equiv \begin{cases} 1/2 \pmod{p} & \text{if } p^{\frac{q-1}{3}} \equiv 1 \pmod{q}, \\ \pm \frac{1}{36} \frac{2L - 9M}{36M} \pmod{p} & \text{if } p^{\frac{q-1}{3}} \equiv -\frac{1+9M/L}{2} \pmod{q}. \end{cases}
\]

For example, if \( q = 7 \), then \( 4q = L^2 + 27M^2 \) with \( L = M = 1 \) and by Theorem 6.4 we get
Corollary 6.4. Let \( p \) be a prime, \( p \neq 2, 3, 7 \). Then
\[
\left( \frac{7}{p} \right)^{p-1} \sum_{k=0}^{p-1} \frac{S_k}{112^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{7}, \\
-3/8 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{7}, \\
-1/8 \pmod{p} & \text{if } p \equiv \pm 3 \pmod{7}, 
\end{cases}
\]

\[
\left( \frac{21}{p} \right)^{p-1} \sum_{k=0}^{p-1} \frac{S_k}{3024^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{7}, \\
25/8 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{7}, \\
-29/8 \pmod{p} & \text{if } p \equiv \pm 3 \pmod{7}, 
\end{cases}
\]

\[
\left( \frac{7}{p} \right)^{p-1} \frac{(2k+1)S_k}{112^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{7}, \\
5/4 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{7}, \\
-7/4 \pmod{p} & \text{if } p \equiv \pm 3 \pmod{7}, 
\end{cases}
\]

\[
\left( \frac{21}{p} \right)^{p-1} \frac{(2k+1)S_k}{3024^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{7}, \\
-11/36 \pmod{p} & \text{if } p \equiv \pm 2 \pmod{7}, \\
-7/36 \pmod{p} & \text{if } p \equiv \pm 3 \pmod{7}. 
\end{cases}
\]

Similarly, setting \( q = 13, 19, 31 \) in Theorem 6.3, we obtain the following congruences.

Corollary 6.5. Let \( p \) be a prime, \( p \neq 2, 3, 5, 13 \). Then
\[
\left( \frac{13}{p} \right)^{p-1} \sum_{k=0}^{p-1} \frac{S_k}{208^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm 1, \pm 5 \pmod{13}, \\
-7/8 \pmod{p} & \text{if } p \equiv \pm 2, \pm 3 \pmod{13}, \\
3/8 \pmod{p} & \text{if } p \equiv \pm 4, \pm 6 \pmod{13}, 
\end{cases}
\]

\[
\left( \frac{39}{p} \right)^{p-1} \sum_{k=0}^{p-1} S_k \frac{25}{5616^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm 1, \pm 5 \pmod{13}, \\
17/40 \pmod{p} & \text{if } p \equiv \pm 2, \pm 3 \pmod{13}, \\
-37/40 \pmod{p} & \text{if } p \equiv \pm 4, \pm 6 \pmod{13}, 
\end{cases}
\]

\[
\left( \frac{13}{p} \right)^{p-1} \frac{(2k+1)S_k}{208^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm 1, \pm 5 \pmod{13}, \\
1/20 \pmod{p} & \text{if } p \equiv \pm 2, \pm 3 \pmod{13}, \\
-11/20 \pmod{p} & \text{if } p \equiv \pm 4, \pm 6 \pmod{13}, 
\end{cases}
\]

\[
\left( \frac{39}{p} \right)^{p-1} \frac{(2k+1)S_k}{5616^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm 1, \pm 5 \pmod{13}, \\
-19/36 \pmod{p} & \text{if } p \equiv \pm 2, \pm 3 \pmod{13}, \\
1/36 \pmod{p} & \text{if } p \equiv \pm 4, \pm 6 \pmod{13}. 
\end{cases}
\]

Corollary 6.6. Let \( p \) be a prime, \( p \neq 2, 3, 7, 19 \). Then
\[
\left( \frac{19}{p} \right)^{p-1} \sum_{k=0}^{p-1} \frac{S_k}{304^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm 1, \pm 7, \pm 8 \pmod{19}, \\
5/8 \pmod{p} & \text{if } p \equiv \pm 2, \pm 3, \pm 5 \pmod{19}, \\
-9/8 \pmod{p} & \text{if } p \equiv \pm 4, \pm 6, \pm 9 \pmod{19}, 
\end{cases}
\]

\[
\left( \frac{19}{p} \right)^{p-1} \frac{(2k+1)S_k}{304^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm 1, \pm 7, \pm 8 \pmod{19}, \\
-13/28 \pmod{p} & \text{if } p \equiv \pm 2, \pm 3, \pm 5 \pmod{19}, \\
-1/28 \pmod{p} & \text{if } p \equiv \pm 4, \pm 6, \pm 9 \pmod{19}. 
\end{cases}
\]
Corollary 6.7. Let \( p \) be a prime, \( p \neq 2, 3, 31 \). Then

\[
\left( \frac{31}{p} \right)^{p-1} \sum_{k=0}^{p-1} \frac{S_k}{124^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm1, \pm2, \pm4, \pm8, \pm15 \pmod{31}, \\
-1/2 \pmod{p} & \text{if } p \equiv \pm3, \pm6, \pm7, \pm12, \pm14 \pmod{31}, \\
0 \pmod{p} & \text{if } p \equiv \pm5, \pm9, \pm10, \pm11, \pm13 \pmod{31},
\end{cases}
\]

\[
\left( \frac{93}{p} \right)^{p-1} \sum_{k=0}^{p-1} \frac{S_k}{837^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm1, \pm2, \pm4, \pm8, \pm15 \pmod{31}, \\
23/16 \pmod{p} & \text{if } p \equiv \pm3, \pm6, \pm7, \pm12, \pm14 \pmod{31}, \\
-31/16 \pmod{p} & \text{if } p \equiv \pm5, \pm9, \pm10, \pm11, \pm13 \pmod{31},
\end{cases}
\]

\[
\left( \frac{31}{p} \right)^{p-1} \sum_{k=0}^{p-1} \frac{2k+1}{124^k} \equiv \begin{cases} 
-1 \pmod{p} & \text{if } p \equiv \pm5, \pm9, \pm10, \pm11, \pm13 \pmod{31}, \\
1/2 \pmod{p} & \text{otherwise},
\end{cases}
\]

\[
\left( \frac{93}{p} \right)^{p-1} \sum_{k=0}^{p-1} \frac{2k+1}{837^k} \equiv \begin{cases} 
1/2 \pmod{p} & \text{if } p \equiv \pm1, \pm2, \pm4, \pm8, \pm15 \pmod{31}, \\
-13/36 \pmod{p} & \text{if } p \equiv \pm3, \pm6, \pm7, \pm12, \pm14 \pmod{31}, \\
-5/36 \pmod{p} & \text{if } p \equiv \pm5, \pm9, \pm10, \pm11, \pm13 \pmod{31}.
\end{cases}
\]

7. Closed form for a companion sequence of \( S_n \).

As we noticed in the Introduction, the sequence \( S_n \) can be defined explicitly by formula (1) or by the generating function (8). Sun [12] considered a companion sequence of \( S_n, T_n \), whose definition comes from a conjectural series expansion of trigonometric functions (see [12, Conj. 4]): there are positive integers \( T_1, T_2, T_3, \ldots \) such that

\[
\sum_{k=0}^{\infty} S_k x^{2k+1} + \frac{1}{24} - \sum_{k=1}^{\infty} T_k x^{2k} = \frac{1}{12} \cos \left( \frac{2}{3} \arccos \left( \frac{6 \sqrt{3} x}{5} \right) \right)
\]

(55)

for all real \( x \) with \( |x| \leq 1/(6 \sqrt{3}) \). The first few values of \( T_n \) are as follows:

1, 32, 1792, 122880, 9371648, 763363328, \ldots.

In this Section, we give an exact formula for \( T_n \). It easily follows from the companion series expansion to (5) (see [6, p. 210(12)])

\[
\cos(a \arcsin(z)) = F \left( -\frac{a}{2}, \frac{a}{2}; \frac{1}{2}; z^2 \right), \quad |z| \leq 1.
\]

(56)

Proposition 1. The coefficients \( T_k, k \geq 1 \), in expansion (55) are given by

\[
T_k = \frac{16^{k-1}}{k} \left( \frac{3k-2}{2k-1} \right) = 16^{k-1} \left( 2 \binom{3k-2}{k-1} - \binom{3k-2}{k} \right).
\]

Proof. Combining formulas (5) and (56) with the obvious trigonometric identity

\[
\arcsin(z) + \arccos(z) = \frac{\pi}{2},
\]

we get a transformation formula connecting both hypergeometric functions from (5) and (56):

\[
\cos \left( \frac{\pi a}{2} \right) F \left( -\frac{a}{2}, \frac{a}{2}; \frac{1}{2}; z^2 \right) + \sin \left( \frac{\pi a}{2} \right) a z F \left( 1 + \frac{a}{2}, 1 - \frac{a}{2}; \frac{3}{2}; z^2 \right) = \cos(a \arccos(z)), \quad |z| \leq 1.
\]

Plugging in \( a = 2/3 \), we get

\[
\frac{1}{2} F \left( -\frac{1}{3}, \frac{1}{3}; \frac{1}{2}; z^2 \right) + \frac{z}{\sqrt{3}} F \left( \frac{1}{6}, \frac{1}{6}; \frac{3}{2}; z^2 \right) = \cos \left( \frac{2}{3} \arccos(z) \right), \quad |z| \leq 1.
\]
Replacing $z$ by $z = 6\sqrt{3}x$ with $|x| \leq 1/(6\sqrt{3})$ and taking into account that
\[
F \left( \frac{1}{6}, \frac{5}{6}, \frac{3}{2}; 108x^2 \right) = 2 \sum_{k=0}^\infty S_k x^{2k},
\]
we obtain
\[
\frac{1}{24} F \left( -\frac{1}{3}, \frac{1}{3}, \frac{1}{2}; 108x^2 \right) + \sum_{k=0}^\infty S_k x^{2k+1} = \frac{1}{12} \cos \left( \frac{2}{3} \arccos(6\sqrt{3}x) \right),
\]
which gives the following generating function for the companion sequence $T_n$:
\[
\frac{1}{24} - \sum_{k=1}^\infty T_k x^{2k} = \frac{1}{24} F \left( -\frac{1}{3}, \frac{1}{3}, \frac{1}{2}; 108x^2 \right),
\]
Comparing coefficients of powers of $x^2$, we get a formula for $T_k$,
\[
T_k = -\frac{1}{24} (-1/3)_k (1/3)_k 108^k = -\frac{16^{k-1}}{k} \binom{3k-2}{2k-1} = -\frac{16^{k-1}}{k} \left( 2 \binom{3k-2}{k-1} - \binom{3k-2}{k} \right),
\]
which shows that $T_k \in \mathbb{N}$ for all positive integers $k$. □

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