Ginsparg-Wilson operators and a no-go theorem

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Abstract

If one uses a general class of Ginsparg-Wilson operators, it is known that CP symmetry is spoiled in chiral gauge theory for a finite lattice spacing and the Majorana fermion is not defined in the presence of chiral symmetric Yukawa couplings. We summarize these properties in the form of a theorem for the general Ginsparg-Wilson relation.

The Ginsparg-Wilson relation provides a convenient framework for the analyses of chiral symmetry in lattice theory. It has been recently pointed out by Hasenfratz that the overlap operator has a conflict with CP symmetry in chiral gauge theory for any finite lattice spacing. We pointed out that the lattice chiral symmetry of the Ginsparg-Wilson operator has a certain conflict with the definition of the Majorana fermion in the presence of Yukawa couplings. (The breaking of CP symmetry in a different context was also mentioned in ). These analyses are either based on the simple form of the Ginsparg-Wilson relation and its explicit solution, or on the generalized forms of Ginsparg-Wilson relation but still on their explicit solutions. It may be useful to formulate these properties in a more abstract and general setting to understand the general features of these complications. In this paper we provide such an analysis.

We deal with a hermitian lattice operator defined by

\[ H = a\gamma_5 D = H^\dagger = aD^\dagger \gamma_5 \]  \hspace{1cm} (1)

where \( D \) stands for the lattice Dirac operator and \( a \) is the lattice spacing. We analyze the general Dirac operator defined by the algebraic relation

\[ \gamma_5 H + H\gamma_5 = 2H^2 f(H^2) \]  \hspace{1cm} (2)
where \( f(H^2) \) is assumed to be a regular function of \( H^2 \) and \( f(H^2)^\dagger = f(H^2) \): To be definite, we assume that \( f(x) \) is monotonous and non-decreasing for \( x \geq 0 \), and \( f(H^2) = 1 \) corresponds to the conventional Ginsparg-Wilson relation\([2]\). We also assume that the operator \( H \) is local in the sense that it is analytic in the entire Brillouin zone. One can confirm the relation

\[
\gamma_5 H^2 = (\gamma_5 H + H \gamma_5)H - H(\gamma_5 H + H \gamma_5) + H^2 \gamma_5 = H^2 \gamma_5
\]

which implies \( H^2 = \gamma_5 H^2 \gamma_5 \) and thus \( DH^2 = H^2 D \). The above defining relation (2) is written in a variety of ways such as

\[
\gamma_5 \Gamma_5 \hat{\gamma}_5 = \gamma_5 \Gamma_5 \gamma_5 D + D \Gamma_5 = 0, \quad \gamma_5 H + H \hat{\gamma}_5 = 0, \quad \gamma_5 D + D \hat{\gamma}_5 = 0
\]

and \( \hat{\gamma}_5^2 = 1 \) where

\[
\Gamma_5 = \gamma_5 - H f(H^2), \quad \hat{\gamma}_5 = \gamma_5 - 2 H f(H^2).
\]

We also note the relation

\[
\gamma_5 \Gamma_5 \hat{\gamma}_5 = \gamma_5 2 \Gamma_5^2 - \gamma_5 \Gamma_5 \gamma_5 = \gamma_5 (\gamma_5 \Gamma_5 + \Gamma_5 \gamma_5) - \gamma_5 \Gamma_5 \gamma_5 = \gamma_5 (\gamma_5 \Gamma_5).
\]

We now examine the action defined by

\[
S = \int d^4 x \bar{\psi} D \bar{\psi} \equiv \sum_{x,y} \bar{\psi}(x)D(x,y)\psi(y)
\]

which is invariant under the lattice chiral transformation

\[
\delta \psi = i \epsilon \hat{\gamma}_5 \psi, \quad \delta \bar{\psi} = \bar{\psi} i \epsilon \gamma_5.
\]

If one considers the field re-definition

\[
\psi' = \gamma_5 \Gamma_5 \psi, \quad \bar{\psi}' = \bar{\psi}
\]

the above action is written as

\[
S = \int d^4 x \bar{\psi}' D \frac{1}{\gamma_5 \Gamma_5} \psi'
\]

which is invariant under the naive chiral transformation (by using (6))

\[
\delta \psi' = \gamma_5 \Gamma_5 \delta \psi = \gamma_5 \Gamma_5 i \epsilon \hat{\gamma}_5 \psi = i \epsilon \gamma_5 \psi',
\]

\[
\delta \bar{\psi}' = \bar{\psi}' i \epsilon \gamma_5.
\]

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This chiral symmetry implies the relation
\[
\{\gamma_5, D \frac{1}{\gamma_5 \Gamma_5} \} = 0.
\] (12)

We here recall the conventional no-go theorem in the form of Nielsen and Ninomiya\[3\], which states in view of (10) and (12) that
(i) If the operator \(D\) is local and if \(1/(\gamma_5 \Gamma_5)\) is analytic in the entire Brillouin zone, the operator \(D\) contains the species doubling. The simplest choice \(f(H^2) = 0\) and thus \(\Gamma_5 = \gamma_5\) is included in this case.
(ii) If the operator \(D\) is local and free of species doubling, then the operator \(\gamma_5 \Gamma_5\) is also local by its construction. But the operator \(1/(\gamma_5 \Gamma_5)\) cannot be analytic in the entire Brillouin zone, which in turn suggests that
\[
\Gamma_5^2 = 1 - H^2 f^2(H^2) = 0
\] (13) has solutions inside the Brillouin zone. These properties are proved for vanishing gauge field.

Since we are interested in the local and doubler-free operator \(D\), we can summarize the analysis so far in the form of a theorem.

**Theorem:** For any lattice operator \(D\) defined by the algebraic relation (2), which is local (i.e., analytic in the entire Brillouin zone) and free of species doubling, the operator \(1/(\gamma_5 \Gamma_5)\) is singular inside the Brillouin zone and \(\Gamma_5^2 = 1 - H^2 f^2(H^2)\) has at least one zero inside the Brillouin zone.

These properties are known\[3\] for the specific case \(f(H^2) = H^{2k}\) with a non-negative integer \(k\) by using explicit solutions\[3\]. Here we learn that these properties are of more general validity and intrinsically related to the basic notions of locality and species doubling.

In the following we discuss the implications of the above theorem. We first summarize the representation of the algebra (2). See, for example, ref.[7]. Let us consider the eigenvalue problem
\[
H \varphi_n(x) = \lambda_n \varphi_n(x), \quad (\varphi_n, \varphi_m) = \delta_{nm}.
\] (14)
We first note \(H \Gamma_5 \varphi_n(x) = -\Gamma_5 H \varphi_n(x) = -\lambda_n \Gamma_5 \varphi_n(x)\), and
\[
(\Gamma_5 \varphi_n, \Gamma_5 \varphi_m) = [1 - \lambda_n^2 f^2(\lambda_n^2)] \delta_{nm}.
\] (15)
These relations show that eigenfunctions with \(\lambda_n \neq 0\) and \(\lambda_n f(\lambda_n^2) \neq \pm 1\) come in pairs as \(\lambda_n\) and \(-\lambda_n\) (when \(\lambda_n = 0\), \(\varphi_0(x)\) and \(\Gamma_5 \varphi_0(x)\) are not necessarily linearly independent).

We can thus classify eigenfunctions as follows:
(i) \(\lambda_n = 0\) \((H \varphi_0(x) = 0)\). For this one may impose the chirality on \(\varphi_0(x)\) as
\[
\gamma_5 \varphi_0^\pm(x) = \Gamma_5 \varphi_0^\pm(x) = \pm \varphi_0^\pm(x).
\] (16)
We denote the number of \(\varphi_0^+ (x)\) \((\varphi_0^- (x))\) as \(n_+ (n_-)\).
(ii) \( \lambda_n \neq 0 \) and \( \lambda_n f(\lambda_n^2) \neq \pm 1 \). As shown above,
\[
H \varphi_n(x) = \lambda_n \varphi_n(x), \quad H \bar{\varphi}_n(x) = -\lambda_n \bar{\varphi}_n(x),
\]
where
\[
\bar{\varphi}_n(x) = \frac{1}{\sqrt{1 - \lambda_n^2 f^2(\lambda_n^2)}} \Gamma_5 \varphi_n(x).
\]
We have
\[
\Gamma_5 \varphi_n(x) = \sqrt{1 - \lambda_n^2 f^2(\lambda_n^2)} \bar{\varphi}_n(x), \quad \Gamma_5 \bar{\varphi}_n(x) = \sqrt{1 - \lambda_n^2 f^2(\lambda_n^2)} \varphi_n(x),
\]
and
\[
\gamma_5 \varphi_n(x) = \sqrt{1 - \lambda_n^2 f^2(\lambda_n^2)} \bar{\varphi}_n(x) + \lambda_n f(\lambda_n^2) \varphi_n(x),
\gamma_5 \bar{\varphi}_n(x) = \sqrt{1 - \lambda_n^2 f^2(\lambda_n^2)} \varphi_n(x) - \lambda_n f(\lambda_n^2) \bar{\varphi}_n(x).
\]
(iii) \( \lambda_n f(\lambda_n^2) = \pm 1 \), or
\[
H \Psi_\pm(x) = \pm \Lambda \Psi_\pm(x), \quad \Lambda f(\Lambda^2) = 1.
\]
We see
\[
\Gamma_5 \Psi_\pm(x) = 0,
\]
and
\[
\gamma_5 \Psi_\pm(x) = \pm \Lambda f(\Lambda^2) \Psi_\pm(x) = \pm \Psi_\pm(x).
\]
We denote the number of \( \Psi_+(x) \) (\( \Psi_-(x) \)) as \( N_+ \) (\( N_- \)). From the relation \( Tr \gamma_5 = 0 \) valid on the lattice, one can derive the chirality sum rule\[8\]
\[
n_+ - n_- + N_+ - N_- = 0.
\]

We next recall the charge conjugation properties of various operators. We employ the convention of the charge conjugation matrix \( C \)
\[
C \gamma^\mu C^{-1} = - (\gamma^\mu)^T, \quad C \gamma_5 C^{-1} = \gamma_5^T, \quad C^\dagger C = 1, \quad C^T = -C.
\]
We then have\[5\] \[9\]
\[
CD C^{-1} = D^T, \quad C \gamma_5 \Gamma_5 C^{-1} = (\gamma_5 \Gamma_5)^T, \quad C \gamma_5 H \gamma_5 C^{-1} = H^T, \quad CH^2 C^{-1} = (H^2)^T, \quad C(\gamma_5 \Gamma_5 \gamma_5 / \Gamma) C^{-1} = (\Gamma_5 / \Gamma)^T
\]
where
\[
\Gamma = \sqrt{\Gamma_5^2} = \sqrt{(\gamma_5 \Gamma_5 \gamma_5)^2} = \sqrt{1 - H^2 f^2(\Lambda^2)}.
\]
Here we imposed the relation \( CDC^{-1} = D^T \) or \((CD)^T = -CD\) which is consistent with the defining algebraic relation (2): To be precise, we need to perform simultaneously a suitable charge conjugation of gauge field in gauge theory\(^1\). In the following this charge conjugation of gauge field is implicitly assumed when we deal with theories with gauge field.

We now examine the CP symmetry in chiral gauge theory

\[
\mathcal{L} = \bar{\psi}_L D \psi_L
\]  
(30)

where we defined the projection operators

\[
D = \bar{P}_L P_L + \bar{P}_R P_R, \quad \psi_{L,R} = P_{L,R} \psi, \quad \bar{\psi}_{L,R} = \bar{\psi} \bar{P}_{L,R}.
\]  
(31)

If the algebra (2) is applicable to strong interactions it is natural that the parity operation is realized in the standard way, and we concentrate on the charge conjugation. The proper transformation property under CP is then ensured by

\[
CP L C^{-1} = \bar{P}_R^T, \quad CP C^{-1} = P_R^T
\]  
(32)

which transforms \( \mathcal{L} \to \mathcal{L}^c = \bar{\psi}_R D \psi_R \) under the charge conjugation \( \bar{\psi} \to \psi^T C \) and \( \psi \to -C^{-1} \psi^T \). One can confirm that the following projection operators

\[
P_{L,R} = \frac{1}{2}(1 \mp \Gamma_5/\Gamma), \\
\bar{P}_{L,R} = \frac{1}{2}(1 \pm \gamma_5 \Gamma_5 \gamma_5/\Gamma).
\]  
(33)

satisfy the requirement (32). One can in fact prove that these projection operators give the unique solutions by using the following statement.

**Statement**: If the operators \( \bar{U} \) and \( V \) are regular in \( \gamma_5 \) and \( H \) and satisfy

\[
\bar{U} H + HV = 0,
\]  
(34)

and (with \( B = C \gamma_5, B^T = -B \))

\[
B \bar{U} B^{-1} = V^T, \quad BV B^{-1} = \bar{U}^T,
\]  
(35)

for a generic \( H \), then they are of the form

\[
\bar{U} = V = \Gamma_5 h(H^2),
\]  
(36)

with a regular function \( h(H^2) \).

**Proof**: Since \( \gamma_5 = \Gamma_5 + H f(H^2) \), \( \bar{U} \) and \( V \) are regular also in \( \Gamma_5 \) and \( H \). Noting \( \Gamma_5^2 = 1 - H^2 f^2(H^2) \), the most general form of \( \bar{U} \) reads

\[
\bar{U} = g(H) + \Gamma_5 h(H^2) + \Gamma_5 H k(H^2).
\]  
(37)

\(^1\)If one defines the CP operation by \( W = C \gamma_0 = \gamma_2 \) with hermitian \( \gamma_2 \) and the CP transformed gauge field by \( U^{CP} \), one has \( WD(U^{CP}) W^{-1} = D(U)^T \). If the parity is realized in the standard way, one has \( CD(U^{CP}) C^{-1} = D(U)^T \).
This implies
\[ B\tilde{U}B^{-1} = [g(H) + \Gamma_5 h(H^2) - \Gamma_5 Hk(H^2)]^T, \]  
from \( BHB^{-1} = HT, \ B\Gamma_5B^{-1} = \Gamma_5^T \) and \( \Gamma_5H + HT\Gamma_5 = 0 \). Eqs. (35) and (38) imply
\[ V = g(H) + \Gamma_5 h(H^2) - \Gamma_5 Hk(H^2), \]  
and thus eq. (34) imposes
\[ Hg(H) + \Gamma_5 H^2k(H^2) = 0. \]  
(40)

The matrix element of this equation between \( \tilde{\varphi}_n(x) \) and \( \varphi_n(x) \) reads
\[ ([\varphi_n, [Hg(H) + \Gamma_5 H^2k(H^2)]]\varphi_n) = \sqrt{1 - \lambda_n^2 f^2(\lambda_n^2) \lambda_n^2 k(\lambda_n^2)} = 0. \]  
(41)

This shows that \( k(x) \) must have zero at \( x = \lambda_n^2 \), but this is impossible for a generic \( H \) unless \( k(x) = 0 \). Similarly, we have \( g(H) = 0 \) and obtain eq. (36).

On the basis of this statement, one can construct the modified chiral operators
\[ \Gamma_5/\Gamma, \ \gamma_5\Gamma_5\gamma_5/\Gamma \]  
(42)
with \( (\Gamma_5/\Gamma)^2 = 1 \) and \( (\gamma_5\Gamma_5\gamma_5/\Gamma)^2 = 1 \). In this construction, we assumed that \( h(H^2) \) exhibits the most favorable property, namely, has no zeroes. These projection operators (33) however inevitably contain singularities in the modified chiral operators \( \Gamma_5/\Gamma \) and \( \gamma_5\Gamma_5\gamma_5/\Gamma \), as is specified by our theorem. These projection operators (33) also become singular in the presence of topologically non-trivial gauge fields, since the massive modes \( N_\pm \) in (22) inevitably appear as is indicated by the chirality sum rule (24). This generalizes the analysis of Hasenfratz[3] in a more abstract setting.

As for the analysis of the Majorana fermion, we start with
\[ \mathcal{L} = \bar{\psi}_R D\psi_R + \bar{\psi}_L D\psi_L + m[\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R] \]
\[ + 2g[\bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^\dagger \psi_L] \]
\[ = \bar{\psi} D\psi + m\bar{\psi}_\gamma_5\Gamma_5 \psi + \frac{g}{\sqrt{2}} \bar{\psi}[A + (\gamma_5\Gamma_5\gamma_5/\Gamma)A(\Gamma_5/\Gamma) + i(\gamma_5\Gamma_5\gamma_5/\Gamma)B + iB(\Gamma_5/\Gamma)] \psi \]
where we used \( \phi = (A + iB)/\sqrt{2} \). We then make the substitution
\[ \psi = (\chi + i\eta)/\sqrt{2}, \]
\[ \bar{\psi} = (\chi^T C - i\eta^T C)/\sqrt{2} \]
and obtain\(^2\)
\[ \mathcal{L} = \frac{1}{2}\chi^T C D\chi + \frac{1}{2}m\chi^T C\gamma_5\Gamma_5 \chi \]

\(^2\)If \( (CO)^T = -CO \) for a general operator \( O \), the cross term vanishes \( \eta^T CO\chi - \chi^T CO\eta = 0 \) by using the anti-commuting property of \( \chi \) and \( \eta \). In the presence of background gauge field, we assume that the representation of gauge symmetry is real.
$+ \frac{g}{2\sqrt{2}} \chi^T C [A + (\gamma_5 \Gamma_5 / \Gamma) A (\Gamma_5 / \Gamma) + i(\gamma_5 \Gamma_5 \gamma_5 / \Gamma) B + iB(\Gamma_5 / \Gamma)] \chi$

$+ \frac{1}{2} \eta^T C D \eta + \frac{1}{2} m \eta^T C \gamma_5 \Gamma_5 \eta$

$+ \frac{g}{2\sqrt{2}} \eta^T C [A + (\gamma_5 \Gamma_5 \gamma_5 / \Gamma) A (\Gamma_5 / \Gamma) + i(\gamma_5 \Gamma_5 \gamma_5 / \Gamma) B + iB(\Gamma_5 / \Gamma)] \eta$. (45)

One can then define the Majorana fermion $\chi$ (or $\eta$) and the resulting Pfaffian. But this formulation of the Majorana fermion\cite{5} inevitably suffers from the singularities of the modified chiral operators $\Gamma_5 / \Gamma$ and $\gamma_5 \Gamma_5 \gamma_5 / \Gamma$ in the Brillouin zone without gauge fields or from the singularities of these chiral operators caused by the massive modes $N_\pm$ (22) in the presence of topologically non-trivial gauge fields.

We note that the condition (32), which is required by the consistent $CP$ property in (30), is directly related to the condition of the consistent Majorana reduction for the term containing scalar field $A(x)$,

$$C(\gamma_5 \Gamma_5 \gamma_5 / \Gamma) A(x) (\Gamma_5 / \Gamma) = -[C(\gamma_5 \Gamma_5 \gamma_5 / \Gamma) A(x) (\Gamma_5 / \Gamma)]^T$$ (46)

in the Yukawa coupling, if one recalls that the difference operators in $\Gamma_5$ and $\Gamma$ do not commute with the field $A(x)$. In other words, if one uses the projection operators which do not satisfy the condition (32) the consistent Majorana reduction is not realized. For the chiral symmetric Yukawa couplings such as in supersymmetry the Majorana reduction is thus directly related to the condition (32) (and consequently to the $CP$ invariance of (43)), provided that parity properties are the standard ones\cite{3}.

It should be noted that our analysis does not show how serious the complications associated with $CP$ symmetry and Majorana reduction are in the actual applications of lattice regularization. As for the breaking of $CP$ symmetry, it could be similar to the breaking of Lorentz symmetry for finite lattice spacing $a$; it may well be restored in a suitable continuum limit. Nevertheless it must be useful to keep in mind that exact and manifest $CP$ is not implemented for a general Ginsparg-Wilson operator. As for the absence of Majorana reduction, a reliable analysis of supersymmetry in some of supersymmetric theories is not possible; practically this could be serious since the divergence cancellation in supersymmetric theory, for example, is very sensitive to the precise implementation of supersymmetry algebra.

In conclusion, we have shown that both of the consistent definitions of $CP$ symmetry in chiral gauge theory\cite{3} and the Majorana fermion in the presence of chiral symmetric Yukawa couplings\cite{3} are based on the same condition (32), and that the construction of projection operators (33) inevitably suffers from the singularities in the modified chiral operators for any Dirac operator $D$ satisfying the algebraic relation (2). Our analysis is based on several key assumptions including the algebraic relation (2) itself, though those assumptions appear to be natural in the framework of Ginsparg and Wilson. We find it quite interesting that the breaking of $CP$ symmetry and a conflict with Majorana reduction are directly related to the basic notions of locality and species doubling in lattice

\footnote{The Yukawa coupling in the Higgs mechanism on the lattice in general is also constrained by $CP$ invariance. In this sense, the constraint arising from $CP$ symmetry is more universal.}
theory. A detailed analysis of the possible implications of the breaking of CP symmetry and the absence of Majorana reduction will be given elsewhere.

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