DARBOUX TRANSFORMS AND SPECTRAL CURVES OF HAMILTONIAN STATIONARY LAGRANGIAN TORI

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Abstract. The multiplier spectral curve of a conformal torus \( f : T^2 \to S^4 \) in the 4-sphere is essentially \([3]\) given by all Darboux transforms of \( f \). In the particular case when the conformal immersion is a Hamiltonian stationary torus \( f : T^2 \to \mathbb{R}^4 \) in Euclidean 4-space, the left normal \( N : M \to S^2 \) of \( f \) is harmonic, hence we can associate a second Riemann surface: the eigenline spectral curve of \( N \), as defined in \([16]\). We show that the multiplier spectral curve of a Hamiltonian stationary torus and the eigenline spectral curve of its left normal are biholomorphic Riemann surfaces of genus zero. Moreover, we prove that all Darboux transforms, which arise from generic points on the spectral curve, are Hamiltonian stationary whereas we also provide examples of Darboux transforms which are not even Lagrangian.

1. Introduction

A submanifold \( X \) of a real \( 2n \)-dimensional symplectic manifold \((Y^{2n}, \omega)\) is defined to be Lagrangian if it is isotropic with respect to the symplectic non degenerate form \( \omega \) and its dimension is maximal, that is \( \dim X = n \). If furthermore \( Y \) is equipped with a Riemannian metric (in particular if \( Y \) is Kähler), a quite natural question is to ask what are the area-minimizing Lagrangian submanifolds. In 1993 Oh \([20]\) (after previous work of Chen–Morvan \([10]\)) introduced a further variational problem intertwining Riemannian and symplectic geometry: A Hamiltonian variation is given by a compactly supported vector field \( Z \) such that \( Z \cdot \omega \) is an exact 1-form, and a Lagrangian submanifold is called Hamiltonian stationary if it is a critical point of the area functional with respect to all Hamiltonian variations. If \( Y \) is Kähler–Einstein, a \( S^1 \)-valued function \( e^{i\beta} \) can be defined along the submanifold \( X \), and \( \beta \) is called the Lagrangian angle of \( X \). It turns out that a submanifold \( X \) is Hamiltonian stationary if and only if \( \beta \) is harmonic.

In this paper we investigate immersions \( f : M \to \mathbb{C}^2 \) of a Riemann surface \( M \) into complex 2-space \( \mathbb{C}^2 \) whose images \( X = f(M) \) are Hamiltonian stationary surfaces in \( \mathbb{C}^2 \). Even in this simplest non trivial case Hamiltonian stationary surfaces display a rich geometry. While no sphere can exist because of the harmonicity of \( \beta \), there exist tori, e.g. the Clifford torus \( S^1 \times S^1 \), and more non-trivial examples were introduced in \([8]\), \([1]\), \([9]\), \([7]\), \([15]\). In \([15]\) it is also shown that all Hamiltonian stationary Lagrangian tori \( f : T^2 \to \mathbb{C}^2 \) are given by Fourier polynomials, and are in this sense completely described. On the other hand, since a Hamiltonian stationary torus \( f : T^2 \to \mathbb{C}^2 \) is given by a harmonicity condition we expect to see a spectral curve description of \( f \) when introducing a spectral parameter. In fact, it is possible to describe Hamiltonian stationary tori in terms of a spectral curve \([15]\), \([14]\), \([18]\) but the construction shows a surprising singular behavior.

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There are further notions of a spectral curve associated with a conformally immersed Hamiltonian stationary torus: a conformal immersion $f : M \to \mathbb{R}^4$ of a Riemann surface $M$ into Euclidean 4–space induces a complex structure $J$ on the trivial $\mathbb{H}$ bundle $\mathbb{H} = M \times \mathbb{H}$ which is harmonic in the case of a Hamiltonian stationary immersion. In particular, one can define $\{d\}$, $\{d\}$, a $\mathbb{C}_r$–family of complex flat connections $d^\mu$ on the trivial $\mathbb{C}^2$ bundle over $M$. In the case when $M = T^2$ is a 2–torus, the set $\text{Eig}$ of eigenvalues of the holonomy of $d^\mu$ is analytic, and the (eigenline) spectral curve $\Sigma_\epsilon$ of the harmonic complex structure $J$ is defined as the normalization of Eig.

On the other hand, for every conformal immersion $f : T^2 \to S^4$ of a 2–torus $T^2 = \mathbb{C}/\Gamma$ into the 4-sphere the (multiplier) spectral curve is defined [3], see also [13], [21]: the complex structure $J$ on $\mathbb{H}$ gives a quaternionic holomorphic structure on $\mathbb{H}$ by $d = d'\mu$ where $d$ is the trivial connection on $\mathbb{H}$ and $d'$ denotes the (0,1)-part of $d$ with respect to the complex structure $J$. We denote by $\mathbb{H}$ the pullback of the trivial bundle under the projection $\mathbb{C} \to \mathbb{C}/\Gamma$, and by $H^\mu_\mathbb{H}(\mathbb{H})$ the set of holomorphic sections $\varphi \in \Gamma(\mathbb{H})$ with multiplier $h \in \text{Hom}(\Gamma, \mathbb{C}_r)$, that is $D\varphi = 0$ and $\gamma^\epsilon \varphi = \varphi h(\gamma)$ for all $\gamma \in \Gamma$. The multiplier spectral curve $\Sigma$ of a conformal torus $f$ is defined as the normalization of the set of all multipliers of holomorphic sections. For a generic multiplier $h \in \Sigma$ the space of holomorphic sections $H^\mu_\mathbb{H}(\mathbb{H})$ with multiplier $h$ is 1–dimensional. The lines $L_h = H^\mu_\mathbb{H}(\mathbb{H})$ extend smoothly to a line bundle, the kernel bundle, over $\Sigma$. Moreover, the prolongation of a holomorphic section $\varphi \in H^\mu_\mathbb{H}(\mathbb{H})$ with multiplier $h$ defines a conformal torus $\tilde{f} : T^2 \to S^4$ which is geometrically a Darboux transform of $f$. Vice versa, every closed Darboux transform gives a holomorphic section with multiplier. In particular, the spectral curve $\Sigma$ of a conformal torus $f$ is essentially given by the set of all closed Darboux transforms of $f$.

The purpose of this paper is to study the geometry of Darboux transforms of a Hamiltonian stationary torus $f : T^2 \to \mathbb{R}^4$ and the relationship between the multiplier and eigenline spectral curve of $f$: since in this case the complex structure $J$, which is induced by $f$ on $\mathbb{H}$, is harmonic and takes values in a unit circle, it is possible to describe the set of multipliers explicitly in terms of the lattice $\Gamma$ with $T^2 = \mathbb{C}/\Gamma$ and the Lagrangian angle $\beta$ of $f$. From this, we get an explicit description of all holomorphic sections with multiplier. In particular, for a generic multiplier $h$ every holomorphic section $\alpha \in H^\mu_\mathbb{H}(\mathbb{H})$ is given by a Fourier monomial. Furthermore, this description also yields a conceptual proof of the result of [15] that every lattice $\Gamma$ and $\beta_0 \in \Gamma^\ast$ uniquely prescribes a family of Hamiltonian stationary tori with Lagrangian angle $\beta = 2\pi (\beta_0, \cdot)$ provided that $\beta_0$ satisfies a non–degeneracy condition; this follows since all Hamiltonian stationary tori $\tilde{f} : T^2 \to \mathbb{R}^4$ with Lagrangian angle $\beta$ are holomorphic sections with trivial multiplier. The family of Hamiltonian stationary tori with the same Lagrangian angle is therefore obtained by projection of a holomorphic curve in $\mathbb{H}P^k$ to $\mathbb{H}P^1$ where $k = \dim_\mathbb{H} H^0(\mathbb{H})$ is the dimension of the space of holomorphic sections.

We call holomorphic sections which are given by a Fourier monomial, and the associated Darboux transforms, monochromatic. We show that all monochromatic Darboux transforms of a Hamiltonian stationary torus are again Hamiltonian stationary, and that in this case the Lagrangian angle is, up to reparametrization, preserved. The space of holomorphic sections with a given multiplier is generically complex 1–dimensional, and only at multiplier with high–dimensional space of holomorphic sections polychromatic holomorphic sections may occur. In particular, for all but a finite set of multipliers, we see that the associated Darboux transforms are Hamiltonian stationary. Discussing the example of homogeneous tori and Castro Urbano tori, we show that there exist however families
of non–Lagrangian, polychromatic Darboux transforms, each family associated with a multiplier with high–dimensional space of holomorphic sections.

On the other hand, the family of flat connections $d^\mu$ on the trivial $\mathbb{C}^2$–bundle gives rise to a subset of Darboux transforms, the so–called $\mu$–Darboux transforms, since it turns out that every parallel section of $d^\mu$ is indeed holomorphic. We show that all (even local) $\mu$–Darboux transforms of a Hamiltonian stationary immersion $f : M \to \mathbb{R}^4$ have harmonic left normal and are thus constrained Willmore. Furthermore, we see that in the case of a Hamiltonian stationary torus $M = T^2$, all $\mu$–Darboux transforms $\hat{f} : T^2 \to \mathbb{R}^4$ are given by monochromatic holomorphic sections, and thus are Hamiltonian stationary. Conversely, every monochromatic holomorphic section $\alpha$ with multiplier gives rise to a unique $\mu \in \mathbb{C}_*$ such that $d^\mu \alpha = 0$. In particular, this correspondence induces a biholomorphism from the eigenline spectral curve to the multiplier spectral curve. This way, we see that the multiplier spectral curve can be compactified to a connected Riemann surface of genus zero. Though the normalizations of the multiplier spectral curve and the spectral curve in [18] are related this also shows that they do not coincide. However, the original harmonic complex structure and the conformal torus can already be recovered from the spectral curve $\Sigma$ and its kernel bundle by appropriate limits as $\mu \to \infty$.

2. Holomorphic sections with multiplier

To discuss the multiplier spectral curve of a Hamiltonian stationary torus we briefly recall the general construction for conformal tori [3]: In the following we will identify a conformal immersion $f : M \to S^4$ with the quaternionic line subbundle $L \subset V$ of the trivial $\mathbb{H}^2$ bundle $V = \mathbb{H}^2$ whose fibers are $L_p = f(p) \in \mathbb{H}^{\mathbb{P}1}$ for $p \in M$. All quaternionic vector spaces are here, and in what follows, quaternionic right vector spaces. Since $f$ is an immersion, the derivative $\delta = \pi d|_L$ of $f$ has no zeros where $d$ is the trivial connection on $V$, and $\pi : V \to V/L$ is the canonical projection. In particular, the conformality of $f$ defines [12, Section 2.5] a complex structure $J \in \Gamma(\text{End}(V/L))$, $J^2 = –1$, on $V/L$ so that

$$
* \delta = J\delta.
$$

The complex structure $J$ and the trivial connection $d$ on $V$ induce [3] a quaternionic holomorphic structure $D : \Gamma(V/L) \to \Gamma(KV/L)$ on $V/L$ via

$$
D \varphi = (\pi d\tilde{\varphi})''
$$

where $\tilde{\varphi} \in \Gamma(V)$ is an arbitrary lift of $\varphi \in \Gamma(V/L)$, that is $\pi \tilde{\varphi} = \varphi$. Moreover, we denote by

$$
\omega' = \frac{1}{2}(\omega – J * \omega) \quad \text{and} \quad \omega'' = \frac{1}{2}(\omega + J * \omega)
$$

the $(1,0)$ and $(0,1)$–part of a 1–form $\omega \in \Omega^1(E)$ with values in a complex vector bundle $(E, J)$. Note that $D$ is independent of the choice of $\tilde{\varphi}$ since $\pi(d\tilde{\varphi})'' = (\delta \tilde{\varphi})'' = 0$ for all $\tilde{\varphi} \in \Gamma(L)$ by (2.1). We call $\varphi \in \ker D$ a (quaternionic) holomorphic section, and denote by $H^0(V/L) = \ker D$ the space of holomorphic sections. Furthermore, we write $\tilde{V}/L$ for the pullback of $V/L$ under the projection $M \to M$ of the universal cover $\tilde{M}$ to $M$.

**Definition 2.1** (see [3]). If $\varphi \in H^0(\tilde{V}/L)$ with $\gamma^* \varphi = \varphi h_\gamma$ and $h_\gamma \in \mathbb{H}_*$ for all $\gamma \in \pi_1(M)$, then we call $h : \pi_1(M) \to \mathbb{H}_*$ the multiplier of the holomorphic section $\varphi$. We denote by $H^0_h(V/L)$ the set of holomorphic sections with multiplier $h$. 
If \( M = T^2 \) is a 2–torus with lattice \( \Gamma \) the image of \( h: \Gamma \rightarrow \mathbb{H}_+ \) lies in an abelian subgroup of \( \mathbb{H}_+ \). By scaling \( \varphi \) with a quaternion we may thus assume that \( h \) takes values in \( \mathbb{C} = \text{Span}\{1, i\} \). If \( f \) has zero normal bundle degree the set of multipliers \( \Spec = \{ h: \Gamma \rightarrow \mathbb{C}_* \mid \text{there exists } \varphi \in H^0(V/L) \text{ with } \gamma^* \varphi = \varphi h_\gamma \text{ for all } \gamma \in \Gamma \} \) is a 1-dimensional analytic variety, and the normalization of \( \Spec \) is a Riemann surface \( \Sigma \) with at most two connected components, each containing a point at infinity, and with possible infinite genus. The normalization \( \Sigma \) of \( \Spec \) is called the multiplier spectral curve of the conformal torus \( f: T^2 \rightarrow S^4 \). Since \( \gamma^* (\varphi j) = (\varphi j) \hat{h}_\gamma \) for a holomorphic section \( \varphi \in H^0(V/L) \) with multiplier \( h \), we see that \( h \in \Spec \) implies \( \rho(h) := \hat{h} \in \Spec \). In fact, \( \rho \) is induced by a fixed point free real structure \( \rho \) on the multiplier spectral curve \( \Sigma \).

Moreover, there is a complex holomorphic line bundle \( \mathcal{L} \) defined over the multiplier spectral curve, the so–called kernel bundle: at a generic point \( h \in \Sigma \) the space of holomorphic sections with multiplier \( h \) is 1–dimensional, and defines a complex line \( \mathcal{L}_h := H^0_h(V/L) \subset \Gamma(V/L) \).

The lines \( \mathcal{L}_h \) extend smoothly into points on the multiplier spectral curve with high–dimensional space of holomorphic sections, and thus define a holomorphic line subbundle \( \mathcal{L} \) of the trivial \( \Gamma(V/L) \) bundle over \( \Sigma \). If \( \Sigma \) has finite genus, then \( \Sigma \) can be compactified and \( \mathcal{L} \) extends smoothly to the compactified spectral curve \( \bar{\Sigma} \). Moreover, the line bundle \( \mathcal{L} \) is compatible with the real structure \( \rho \) that is \( \rho^* \mathcal{L} = \mathcal{L} \).

We now turn to the case when \( f: T^2 \rightarrow \mathbb{R}^4 \) is a conformal immersion into the 4–space with Gauss map

\[
(N, R): M \rightarrow S^2 \times S^2 = \text{Gr}_2(\mathbb{R}^4).
\]

If we identify the Euclidean 4–space \( \mathbb{R}^4 = \mathbb{H} \) with the quaternions then \( S^2 = \{ n \in \text{Im } \mathbb{H} \mid n^2 = -1 \} \) and the left normal \( N \) and the right normal \( R \) satisfy

\[
*df = Ndf = -dfR.
\]

We consider a conformal immersion \( f: M \rightarrow \mathbb{R}^4 \) as a map into \( S^4 = \mathbb{H}P^1 \) via \( \psi: M \rightarrow \mathbb{H}P^1 \) where

\[
\psi = \left( \begin{array}{c} f \\ 1 \end{array} \right): M \rightarrow \mathbb{H}^2.
\]

In other words, the map \( \psi: M \rightarrow \mathbb{H}P^1 \) becomes \( f: M \rightarrow \mathbb{R}^4 \) after the choice of the point \( \infty = e \mathbb{H} \in S^4 \) where \( e = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \). In the case of a conformal immersion \( f: M \rightarrow \mathbb{R}^4 \) by evaluating \( \delta \) on \( \psi \) we see that the complex structure \( J \) on \( V/L \) is given by \( J \pi e = \pi eN \) where \( N \) is the left normal \( N \) of \( f \). In what follows we identify \( V/L \cong e \mathbb{H} \) via the quaternionic isomorphism \( \pi|_{\mathbb{H}}: e \mathbb{H} \rightarrow V/L, e \mapsto \pi(e) \), and trivialize \( e \mathbb{H} \cong \mathbb{H} \) via the constant section \( e \). In particular, \( \mathbb{H} \) inherits the complex structure \( J \in \text{End}(\mathbb{H}) \) which is given by left multiplication by \( N \). Moreover, the holomorphic structure \( D \) on \( \mathbb{H} \) is given by

\[
D\alpha = \frac{1}{2}(d\alpha + N \ast d\alpha)
\]
so that \( \alpha \in H^0(\mathbb{H}) \) is holomorphic if and only if
\[
* d\alpha = N d\alpha.
\]
(2.2)

We shall consider Lagrangian surfaces of \( \mathbb{R}^4 = \mathbb{H} \) where the complex structure on \( \mathbb{H} \), which determines the symplectic structure, is given by the left multiplication by \( j \). In other words, we identify \( \mathbb{H} = \mathcal{C} \oplus \mathcal{C}i \) where \( \mathcal{C} = \text{Span}\{1, j\} \) so that the action of a unitary matrix
\[
U = e^{j\theta} \begin{pmatrix} m & -n \\ n & m \end{pmatrix} \in U(2), \quad m, n \in \mathcal{C}, \theta \in \mathbb{R},
\]
on \( \mathcal{C}^2 \) corresponds to left multiplication with \( e^{j\theta} \) together with right multiplication by \( m + ni \) on \( \mathbb{H} \):
\[
\mathcal{C}^2 = \mathbb{H} \ni v \mapsto Uv = e^{j\theta}v(m + ni) \in \mathbb{H}.
\]

A conformal immersion \( f: M \to \mathbb{H} \) is Lagrangian if and only if
\[
\left( e^{-u} \frac{\partial f}{\partial x}, e^{-u} \frac{\partial f}{\partial y} \right)
\]
is a unitary frame where \( z = x + iy \) is a local conformal coordinate on \( M \) and \( e^u \) is the conformal factor of \( f \). Choosing the couple \((1, i)\) as reference unitary frame, there exists locally a smooth \( U: M \to U(2) \), \( m, n \in \mathcal{C}, \theta \in \mathbb{R} \), such that
\[
\frac{\partial f}{\partial x} = U \cdot 1 \quad \text{and} \quad \frac{\partial f}{\partial y} = U \cdot i,
\]
that is there exist \( \beta: M \to \mathbb{R} \) and \( q: M \to S^3 \) such that
\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = e^{j\beta} (1 dx + idy)e^u q.
\]
Writing \( g = e^u q \) we thus see that \( f: M \to \mathbb{H} \) is conformal and Lagrangian if and only if
\[
df = e^{j\beta} dzg, \quad \text{where } dz = dx + idy,
\]
(2.3)

for some real valued function \( \beta \) and quaternion valued \( g \). If we change the conformal coordinate \( \tilde{z} = ze^{i\theta} \) we see that \( \tilde{g} = e^{-i\theta}g \). In particular, the Lagrangian angle \( \beta \) is defined independently, up to a constant translation, of the choice of the conformal coordinate \( z \).

It is now easily seen from (2.3) that the left normal is
\[
N = e^{j\beta} i
\]
while the right normal is
\[
R = -g^{-1}ig,
\]
(2.5)
where we use the convention that \( *dz = *(dx + idy) = idz = -dy + idx \). Conversely, if \( f: M \to \mathbb{R}^4 \) is a conformal (branched) immersion with left normal \( N = e^{j\beta}i \) with \( \beta: M \to \mathbb{R} \) then \( df \) satisfies (2.3) for some \( q: M \to \mathbb{H} \), and thus \( f \) is a Lagrangian immersion. Since our interest are Hamiltonian stationary immersions from a 2–torus \( T^2 = \mathbb{C}/\Gamma \) into \( \mathbb{R}^4 \), we will consider \( f: \mathbb{C} \to \mathbb{H} \) as a \( \Gamma \)-periodic map. In particular, if \( f \) is Hamiltonian-stationary, then \( \beta \) is harmonic and we may assume, after possible change of the conformal coordinate \( z \), that
\[
\beta(z) = 2\pi \langle \beta_0, z \rangle,
\]
(2.6)
where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( \mathbb{C} \cong \mathbb{R}^2 \) and \( \beta_0 \) is in the dual lattice \( \Gamma^* \).

In the following we discuss the set of all multipliers of holomorphic sections in case of a Hamiltonian stationary torus. Recall that the multiplier takes values in any complex subspace of \( \mathbb{H} \) since the image of \( h: \Gamma \to \mathbb{H}_* \) is abelian. We choose this subspace to be
$\mathbb{C} = \text{Span}\{1,i\}$ instead of $\mathbb{C} = \text{Span}\{1,j\}$ for purely computational reasons. In particular, for every $h \in \text{Spec}$ there exists a pair $(A, B) \in \mathbb{C}^2$ so that

$$h_\gamma = h^{A,B}_\gamma := e^{2\pi i \langle (A, \gamma) - i(B, \gamma) \rangle} \quad \gamma \in \Gamma.$$ 

Note that $(A, B)$ is unique up to translation of $B$ by elements of the dual lattice since

$$(2.7) \quad h^{A,B} = h^{A,B + \delta} \quad \text{for all } \delta \in \Gamma^*.$$ 

To find conditions on $(A, B)$ we rewrite the holomorphicity condition in terms of the gauged section $\tilde{\sigma}$

where $\tilde{\sigma}$

Note that $(2.10)$ is equivalent to $(2.2)$. If $\alpha \in H^0_h(V/L)$ is a holomorphic section with multiplier $h = h^{A,B}$ then $\alpha(z) = \sigma e^{2\pi i \langle (A,z) - i(B,z) \rangle}$ with $\Gamma$-periodic function $\sigma$ so that

$$(2.9) \quad \tilde{\sigma}(z) = \tilde{\sigma} e^{2\pi i \langle (A,z) - i(B,z) \rangle}$$

where $\tilde{\sigma} = e^{-\frac{j\beta}{2}} \sigma$ is $2\Gamma$-periodic. Writing $\tilde{\sigma} = \tilde{u} + j \tilde{v}$ with complex valued functions $\tilde{u}, \tilde{v}$ equation $(2.8)$ then becomes

$$0 = \left\{ 2\tilde{u}_z + 2\pi (A - iB)\tilde{u} - \pi \beta_0 \tilde{v} \right\} dz + j \left\{ 2\tilde{v}_z + 2\pi (\bar{A} - i\bar{B})\tilde{v} + \pi \beta_0 \tilde{u} \right\} dz,$$

and we have to discuss for which $(A, B) \in \mathbb{C}^2$ there exist complex valued $2\Gamma$-periodic functions $\tilde{u}, \tilde{v}$ satisfying

$$(2.10) \quad \left\{ \begin{array}{l}
2\tilde{u}_z + 2\pi (A - iB)\tilde{u} - \pi \beta_0 \tilde{v} = 0 \\
2\tilde{v}_z + 2\pi (\bar{A} - i\bar{B})\tilde{v} + \pi \beta_0 \tilde{u} = 0
\end{array} \right..$$

Since $\sigma = u + jv: T^2 \to \mathbb{H}$ is defined on the torus we have Fourier expansions

$$u(z) = \sum_{\delta \in \Gamma^*} u_\delta e^{2\pi i \langle \delta, z \rangle} \quad \text{and} \quad v(z) = \sum_{\delta \in \Gamma^*} v_\delta e^{2\pi i \langle \delta, z \rangle}$$

with $u_\delta, v_\delta \in \mathbb{C}$. Abbreviating $e_\delta(z) = e^{2\pi i \langle \delta, z \rangle}$ we obtain with $(2.6)$

$$e^{-\frac{j\beta}{2}} = \cos \frac{\beta}{2} - j \sin \frac{\beta}{2} = \frac{e^{\beta_0} + e^{-\beta_0}}{2} - ij \frac{e^{\beta_0} - e^{-\beta_0}}{2}$$

so that $\tilde{u} + j\tilde{v} = e^{-\frac{j\beta}{2}} (u + jv)$ and $je_{\frac{\beta_0}{2}} = e^{-\frac{j\beta_0}{2}} j$ give the Fourier expansions

$$(2.11) \quad \tilde{u} = \sum_{\delta \in \Gamma^* + \frac{\beta_0}{2}} \tilde{u}_\delta e_{\delta} \quad \text{and} \quad \tilde{v} = \sum_{\delta \in \Gamma^* + \frac{\beta_0}{2}} \tilde{v}_\delta e_{\delta}$$

of $\tilde{u}$ and $\tilde{v}$ with Fourier coefficients $\tilde{u}_\delta, \tilde{v}_\delta \in \mathbb{C}$. Therefore, the gauged holomorphicity equations $(2.10)$ can be written equivalently as

$$(2.12) \quad \left\{ \begin{array}{l}
2(i\delta + A - iB)\tilde{u}_\delta = \beta_0 \tilde{v}_\delta \\
2(i\delta + \bar{A} - i\bar{B})\tilde{v}_\delta = -\beta_0 \tilde{u}_\delta \end{array} \right. \quad \text{for all } \delta \in \Gamma^* + \frac{\beta_0}{2}.$$ 

In particular, the vanishing of one of the Fourier coefficients implies the vanishing of the other since $\beta_0 \neq 0$. For $\tilde{u}_\delta \tilde{v}_\delta \neq 0$ the above equations imply

$$(\delta - iA - B)(\delta - i\bar{A} - \bar{B}) = \frac{|\beta_0|^2}{4},$$
or, equivalently,

\[ |\delta - B|^2 - |A|^2 = \frac{|\beta_0|^2}{4} \quad \text{and} \quad \langle \delta - B, A \rangle = 0. \]

In particular, the condition that the set of admissible frequencies

\[ \Gamma_{A,B}^* = \{ \delta \in \Gamma^* + \frac{\beta_0}{2} \mid \delta \text{ satisfies } (2.13) \} \]

is not empty is necessary for \( h^{A,B} \) to be a multiplier of a holomorphic section.

**Theorem 2.2.** A multiplier \( h \in \text{Spec of a holomorphic section } \varphi \in H^0(\tilde{H}) \) determines a unique pair \((A, B) \in \mathbb{C}^2\), up to translation of \( B \) by the dual lattice \( \Gamma^* \), with \( h^{A,B} = h \). Moreover, the set of multipliers of a Hamiltonian stationary torus \( f : T^2 \to S^4 \) is given by

\[ \text{Spec} = \{ h^{A,B} \mid \Gamma_{A,B}^* \neq \emptyset \} \]

In fact, for all \( \delta \in \Gamma_{A,B}^* \) there exists a complex 1–dimensional subspace \( L_{A,\delta - B} \) of the space \( H^0_{A,B} := H^0_{h^{A,B}}(\tilde{H}) \) of holomorphic sections with multiplier \( h^{A,B} \).

**Proof.** We already have seen that \( \text{Spec} \subset \{ h^{A,B} \mid \Gamma_{A,B}^* \neq \emptyset \} \). To show equality, note that the frequencies \( \delta \) with \( \tilde{u}_\delta \tilde{v}_\delta \neq 0 \) are placed at the intersection of the circle \( C_r(B) \) of radius

\[ r = \sqrt{|A|^2 + \frac{|\beta_0|^2}{4}} \]

centered at \( B \), and a line going through \( B \) orthogonal to \( A \), that is

\[ \Gamma_{0,B}^* = \{ \delta \in \Gamma^* + \frac{\beta_0}{2} \mid \delta = B - \frac{\beta_0}{2} e^{it}, t \in [0, 2\pi) \}. \]

In particular, for \( \delta \in \Gamma_{0,B}^* \) we have

\[ \frac{i\beta_0}{2(\delta - B)} = \frac{2i(\delta - B)}{\beta_0} = ie^{it}, \]

so that \((2.9), (2.11), (2.12)\) show that

\[ \alpha_\delta = e^{i\beta}(1 + ke^{it})e^{-\beta_0 e^{it}} \]

is a holomorphic section with multiplier \( h_{0,B}^{0,B} \). In particular, \( \alpha_\delta \) spans a complex 1–dimensional subspace \( \mathcal{L}_{A,\delta - B} \subset H^0_{0,B} \) of the space of holomorphic sections with multiplier \( h_{0,B}^{0,B} \).

If \( \delta \in \Gamma_{A,B}^* \) with \( A \neq 0 \), then \( \delta \) lies on the intersection of a circle of radius \( r = \sqrt{|A|^2 + \frac{|\beta_0|^2}{4}} \) centered at \( B \) and a line going through \( B \) orthogonal to \( A \), that is

\[ \Gamma_{A,B}^* = \left\{ \delta_{\pm} = B \pm i\frac{A}{|A|} \right\} \cap \left( \Gamma^* + \frac{\beta_0}{2} \right). \]

By \((2.13)\) we see that

\[ \frac{-\beta_0}{2(i\delta_{\pm} + A - iB)} = \frac{2(i\delta_{\pm} + A - iB)}{\beta_0} \]
We call a holomorphic section \(\alpha\) with more than one frequency). As we have seen, any monochromatic holomorphic section where
\[|\alpha| = 1,\]
is given by \(\alpha = e^{i\beta B}e^{2\pi i\langle A,\cdot\rangle}\)
with \(\delta \in \Gamma_{A,B}^*\) and (2.15), (2.17)
\[\lambda_\delta := \lambda_{A,\delta-B} := \frac{2}{\beta_0}(\delta - iA - B) = \left\{ \begin{array}{ll} -e^{it} & \text{for } A = 0, \delta = B - \frac{\beta_0}{2}e^{it}, \\ -2i(\beta_0 A) & \text{for } A \neq 0, \delta = \delta_\pm \end{array} \right. \]
where \(r = \sqrt{|A|^2 + \frac{|\delta_\delta|^2}{4}}\). Since every multiplier \(h\) is given by \(h = h_{A,B}^*\) for some pair \((A, B)\) satisfying (2.13), we also obtain:

**Corollary 2.3.** Every holomorphic section \(\alpha \in H^0_{\beta}(\mathbb{H})\) with multiplier \(h\) is given by
\[\alpha = e^{i\beta B} \left( \sum_{\delta \in \Gamma_{A,B}^*} (1 - k\lambda_\delta)\bar{\alpha}_{\delta-B} e^{2\pi i\langle A,\cdot\rangle} \right),\]
where \(h = h_{A,B}^*\), \(\bar{\alpha}_\delta \in \mathbb{C}\), \(e_{\delta-B}(z) = e^{2\pi i\langle\delta-B,\cdot\rangle}\), and \(\lambda_\delta = \frac{2}{\beta_0}(\delta - iA - B)\).

**Remark 2.4.** Note that \(\alpha\) is independent of the choice of \(B\) with \(h_{A,B}^* = h\) since both \(\lambda_\delta\) and \(e_{\delta-B}\) only depend on \(\delta - B\), and \(\Gamma_{A,B+\zeta} = \Gamma_{A,B} + \zeta\) for all \(\zeta \in \Gamma^*\).

Since \(H^0_{A,B}\) is complex \(k\)-dimensional if and only if the Fourier expansion (2.11) allows exactly \(k\) frequencies \(\delta \in \Gamma^* + \frac{\beta_0}{2}\) with \(\bar{\alpha}_\delta \bar{\alpha}_\delta \neq 0\), the complex dimension of the space of holomorphic sections with multiplier \(h^*_{A,B}\) is given by the number of elements in \(\Gamma_{A,B}^*\):
\[\dim_{\mathbb{C}} H^0_{A,B} = |\Gamma_{A,B}^*| .\]

Note that \(A \neq 0\) uniquely determines \(B\), up to translation by the dual lattice, with \(\Gamma_{A,B}^* \neq \emptyset\), and (2.16) shows that \(|\Gamma_{A,B}^*| \leq 2\) when \(A \neq 0\). If \(|\Gamma_{A,B}^*| = 2\) then both \(\delta_{\pm} = B \pm i\frac{A}{|A|} \in \Gamma^* + \frac{\beta_0}{2}\) are in the translated dual lattice (2.16) so that
\[\delta_{\pm} - \delta_{\pm} =: \zeta \in \Gamma^*.\]
On the other hand, \(\zeta = 2iA\sqrt{1 + \frac{|\delta_\delta|^2}{|A|^2}}\) gives \(4|A|^2 = |\zeta|^2 - |\beta_0|^2\), and \(A = -\frac{i}{2}\zeta \sqrt{1 - \frac{|\beta_0|^2}{|\zeta|^2}}\)
with \(|\zeta| > |\beta_0|\).
Lemma 2.5. Let $f$ be a Hamiltonian stationary torus and $h = h^{A,B} \in \text{Spec}$ a multiplier of a holomorphic section with $A \neq 0$. Then the complex dimension of the space of holomorphic sections with multiplier $h$ is at most 2, and $\dim H^0_{A,B} = 2$ if and only if $A$ is a double point that is

$$A = -\frac{i}{2} \zeta \sqrt{1 - \frac{|\beta_0|^2}{|\zeta|^2}}, \quad \zeta \in \Gamma^*, \quad |\zeta| > |\beta_0|.$$ 

For $A = 0$ the situation is more complicated but we still obtain an upper bound for the dimension of the space of holomorphic sections with multiplier.

Theorem 2.6. For a Hamiltonian stationary torus, the space $H^0_{h}(\mathbb{H})$ of holomorphic sections with multiplier $h$ is generically complex 1–dimensional, and its complex dimension is bounded by $N$ where

$$N = |D \cap (\Gamma^* + \frac{\beta_0}{2})|$$

is the number of points of the translated dual lattice $\Gamma^* + \frac{\beta_0}{2}$ in the (closed) disk $D$ around $\frac{\beta_0}{2}$ of radius $|\beta_0|$. Moreover, there exists at least one multiplier $h$, namely the trivial multiplier, with $\dim_{\mathbb{C}} H^0_{h}(\mathbb{H}) \geq 4$.

**Figure 1.** The points $B$ on the inner circle of radius $\frac{|\beta_0|}{2}$ around $\frac{\beta_0}{2}$ give all multipliers $h = h^{0,B}$. Frequencies $\delta \in \Gamma^* + \frac{\beta_0}{2}$ in the disk of radius $|\beta_0|$ are all frequencies such that the circle $C_{|\beta_0|}(\delta)$ meets the inner circle. The corresponding dimension $k$ of the space of holomorphic section with multiplier $h$ is $k = 2$ or $k = 4$, when 2 respectively 4 circles meet at $B$, and $k = 1$ for the remaining points. All points $B$ with $k > 1$ are congruent modulo the dual lattice to one of the three blue points. This setting corresponds to a homogeneous torus, see Section 4.
Proof. In Theorem 2.2 we have seen that the set of multipliers of holomorphic sections is parametrized by pairs \((A, B)\) with \(\Gamma_{A,B}^* \neq \emptyset\). For \(A \neq 0\) we see that the complex dimension of the space of holomorphic sections with multiplier \(h^{A,B}\) is generically 1–dimensional since \(\dim_C H^0_{A,B} = 2\), \(A \neq 0\), occurs only for a discrete set of double points in \(\mathbb{C}^*\). If \(A = 0\), then by (2.14)

\[
\Gamma^*_0,B = \{ \delta \in \Gamma^* + \frac{\beta_0}{2} \mid \delta = B - \frac{\beta_0}{2} e^{it}, t \in [0, 2\pi) \} \neq \emptyset,
\]

and we may assume without loss of generality that \(B = \frac{\beta_0}{2}(1 + e^{it})\) with \(t \in [0, 2\pi)\), and thus \(\frac{\delta_0}{2} \in \Gamma^*_0,B\), since the multiplier \(h = h^{0,B}\) does not change under translation of \(B\) by the dual lattice (2.7). Now \(\delta\) lies in \(\Gamma^*_0,B\) if and only if \(\delta \in \Gamma^* + \frac{\beta_0}{2}\) and the circle \(C_{|\beta_0|^2}(\delta)\) of radius \(|\beta_0|^2\) centered at \(\delta\) passes through \(B\). Obviously such a \(\delta\) lies inside the closed disk \(D\) centered at \(\frac{\beta_0}{2}\) of radius \(|\beta_0|\). Only finitely many points \(N\) of the (translated) dual lattice are contained in \(D\) which gives the upper bound

\[
\dim_C H^0_{0,B} \leq N < \infty.
\]

Furthermore the complex dimension of \(H^0_{0,B}\) is the number of such circles passing through \(B\), which is one except for finitely many values of \(B\).

By (2.2) we see that \(1 \in H^0(\mathbb{H})\) and \(f \in H^0(\mathbb{H})\) are quaternionic independent holomorphic sections with trivial multiplier. Since the space of holomorphic sections \(H^0(\mathbb{H})\) is quaternionic, we thus have \(\dim_C H^0_{p=1}(\mathbb{H}) = 2 \dim_{\mathbb{H}} H^0(\mathbb{H}) \geq 4. \) \(\Box\)

Remark 2.7. Note that \(\delta \in \Gamma^*_0,B\) if and only if the translate of the main circle \(C_{|\beta_0|}(\frac{\beta_0}{2})\) by the frequency \(\delta - \frac{\beta_0}{2} \in \Gamma^*\) intersects the main circle at \(B\). Put differently, the complex dimension of the space of holomorphic sections with multiplier \(h = h^{0,B}\) is given by the number of self intersections on a fundamental domain of the main circle at \(B\).

Figure 2. In the previous example, on a fundamental domain of \(\mathbb{C}/\Gamma^*\) the inner circle \(C_{|\beta_0|}(\frac{\beta_0}{2})\) self intersects at three points, one of which is covered 4 times.
Figure 3. Same as Figure 2 but the torus \( \mathbb{C}/\Gamma^* \) is drawn in \( \mathbb{R}^3 \). Only the point with four intersections and one of the two double points are visible.

Remark 2.8. In [15] it is shown that all Hamiltonian-stationary tori \( f : M \to \mathbb{R}^4 \) with lattice \( \Gamma \) and Lagrangian angle frequency \( \beta_0 \) satisfy

\[ \Gamma_{\beta_0}^* := \left\{ \delta \in \Gamma^* + \frac{\beta_0}{2}, |\delta| = \frac{|\beta_0|}{2}, \delta \neq \pm \frac{\beta_0}{2} \right\} \neq \emptyset. \]

By definition \( \Gamma_{\beta_0}^* \cup \{ \pm \frac{\beta_0}{2} \} = \Gamma_{\beta_0}^* \cup \{ \pm \frac{\beta_0}{2} \} = \Gamma_{\beta_0}^* \), and \( |\Gamma_{\beta_0}^*| \) is even by symmetry. Since \( h_{0,0}^0 \) is the trivial multiplier, this also implies that \( \dim_{\mathbb{C}} H_{0,0}^0 = |\Gamma_{\beta_0}^*| = |\Gamma_{\beta_0}^*| + 2 \geq 4 \). Since the holomorphic sections \( \alpha_{\pm \frac{\beta_0}{2}} = 1 \pm k \) given by the frequencies \( \pm \frac{\beta_0}{2} \) are constant and \( f \in H^0(\mathbb{H}) \) the Hamiltonian stationary torus \( f \) is given up to rotation and translation by any holomorphic section \( \alpha_{\delta} \) with \( \delta \in \Gamma_{\beta_0}^* \) if \( \dim_{\mathbb{C}} H_{0,0}^0 = 4 \).

More generally, we can construct all Hamiltonian stationary tori with Lagrangian angle \( \beta \):

Theorem 2.9 (see [15]). If \( \Gamma^* \) is a lattice with \( \beta_0 \in \Gamma^* \) such that \( \Gamma_{\beta_0}^* \neq \emptyset \) then there exists a \( \mathbb{H}P^n \)–family of non congruent Hamiltonian stationary tori with Lagrangian angle \( \beta = 2\pi \langle \beta_0, \cdot \rangle \) given by

\[ f = e^{\frac{i\beta}{2}} \sum_{\delta \in \Gamma_{\beta_0, +}^*} \left( 1 - k \frac{2\delta}{\beta_0} \right) e_\delta c_\delta \]

where \( c_\delta \in \mathbb{H}, e_\delta(z) = e^{2\pi i \langle \delta, z \rangle}, n + 1 = \dim_{\mathbb{C}} H^0(\mathbb{H}) \) and

\[ \Gamma_{\beta_0, +}^* := \left\{ \delta \in \Gamma_{\beta_0}^* \mid \Im (\delta \beta_0^{-1}) > 0 \right\}. \]

Conversely, every Hamiltonian stationary torus with Lagrangian angle \( \beta_0 \) arises, up to translation, this way.

Proof. We have seen that a given Hamiltonian stationary torus \( f : T^2 \to \mathbb{R}^4 \) with Lagrangian angle \( \beta = 2\pi \langle \beta_0, \cdot \rangle \) has \( \Gamma_{\beta_0}^* \neq \emptyset \) and \( f \in H^0(\mathbb{H}) \), that is \( f \) is given, up to
translation, by a complex linear combination of the holomorphic sections $\alpha_\delta$, $\delta \in \Gamma^*_\beta_0$, with trivial multiplier. Corollary 2.3 thus gives

$$f = e^{\frac{2\beta}{\beta_0}} \sum_{\delta \in \Gamma^*_\beta_0} \left(1 - k\frac{2\delta}{\beta_0}\right) e_\delta \tilde{u}_\delta$$

with $\tilde{u}_\delta \in \mathbb{C}$. Since $\Gamma^*_\beta_0$ is symmetric, in particular $H^0(\mathbb{H}^1)$ is quaternionic, we can use only half of the lattice points in $\Gamma^*_\beta_0$ by replacing the complex Fourier coefficients by quaternionic ones: using $|\beta_0| = 1$ for $\delta \in \Gamma^*_\beta_0$ and $-ke_\delta k = e_{-\delta}$ we have

$$\left(1 - k\frac{2\delta}{|\beta_0|}\right) e_\delta \tilde{u}_\delta + \left(1 + k\frac{2\delta}{|\beta_0|}\right) e_{-\delta} \tilde{u}_{-\delta} = \left(1 - k\frac{2\delta}{\beta_0}\right) e_\delta \left(\tilde{u}_\delta + \frac{\beta_0}{2\delta} k \tilde{u}_{-\delta}\right)$$

which immediately gives (2.20) with $c_\delta = (\tilde{u}_\delta + \frac{\beta_0}{2\delta} k \tilde{u}_{-\delta}) \in \mathbb{H}$.

Conversely, given a lattice $\Gamma$ and a Lagrangian angle $\beta = 2\pi \langle \beta_0, \cdot \rangle$, $\beta_0 \in \Gamma^*$ with $\Gamma^*_\beta_0 \neq \emptyset$, we can define a complex structure on the trivial bundle $T^2 \times \mathbb{H}$ over the torus $T^2 = \mathbb{C}/\Gamma$ by left multiplication by $N = e^{i\beta_0} i$, and a holomorphic structure by $D = d''$. In particular, every holomorphic section of $D$ gives a conformal torus $f : T^2 \to \mathbb{H}$ with left normal $N = e^{i\beta_0} i$, and thus $f$ is Hamiltonian stationary. Since every holomorphic section of torus with trivial multiplier is given by (2.20) and

$$n + 1 = \dim_{\mathbb{H}} H^0(\mathbb{H}^1) = |\Gamma^*_\beta_0| + 1$$

we obtain this way a $\mathbb{H}P^n$-family of non congruent Hamiltonian stationary tori.

\[ \square \]

Remark 2.10. We have recovered the result of [14] without integration of $df = e^{\frac{2\beta}{\beta_0}} dz g$.

Moreover, the previous theorem shows that the family of Hamiltonian stationary tori with the same lattice $\Gamma$ and Lagrangian angle $\beta$ is obtained by projections of a quaternionic holomorphic curve $F : \mathbb{C}/\Gamma \to \mathbb{H}P^k$ to $\mathbb{H}P^1$ whose complex structure, cf. for example [17], is given by $N = e^{i\beta_0} i$.

Theorem 2.11. The real structure $\rho : \text{Spec} \to \text{Spec}$ induces involutions

$$\rho : \mathbb{C}^2 \to \mathbb{C}^2, (A, B) \mapsto (A, -B) \quad \text{and} \quad \rho : \Gamma^* \to \Gamma^*, \delta \mapsto -\delta$$

so that $\rho(\Gamma^*_A, B) = \Gamma^*_A, -B$ and $\mathcal{L}$ is compatible with $\rho$ that is

$$\rho^* \mathcal{L}_{A, \delta - B} := \mathcal{L}_{A, -(\delta - B)} = \mathcal{L}_{A, \delta - B} j.$$

Proof. Note that $\rho(h^A, B) = h^{A, -B}$, and from (2.14) and (2.16) we see that $\delta \in \Gamma^*_A, B$ implies $-\delta \in \Gamma^*_A, -B$. In Theorem 2.2 the complex line $\mathcal{L}_{A, \delta - B}$ is defined as the span of the monochromatic holomorphic section

$$\alpha^A_{\delta, B} = e^{\frac{2\beta}{\beta_0}} (1 - k\lambda_{A, \delta - B}) c_{\delta - B} e^{2\pi \langle A, \cdot \rangle}.$$

By (2.19), (2.13) we see $\lambda_{A, \delta - B} \lambda_{A, -(\delta - B)} = -1$ so that

$$\alpha^A_{\delta, -B} j = \alpha^A_{\delta, B} w$$

with $w = i \lambda_{A, -(\delta - B)} \in \mathbb{C}$, and $\mathcal{L}$ is compatible with $\rho$. \[ \square \]

A real multiplier $h \in \text{Spec}$, that is $h_{\gamma} = \bar{h}_{\gamma}$ for all $\gamma \in \Gamma^*$, has a quaternionic space of holomorphic sections since $\alpha \in H^0_h(\mathbb{H}^1)$ implies $\alpha j \in H^0_h(\mathbb{H}^1)$. In particular,

$$\dim \mathbb{C} H^0_h(\mathbb{H}^1) = 2 \dim \mathbb{H} H^0_h(\mathbb{H}^1) \geq 2,$$
and in fact, real multipliers occur at the double points $A$:

**Corollary 2.12.** A multiplier $h \in \text{Spec}$ is real if and only if $h = h^{A,B}$ for a double point

$$A = -\frac{i}{2} \zeta \sqrt{1 - \frac{|\beta_0|^2}{|\zeta|^2}}, \quad \zeta \in \Gamma^*, \quad |\zeta| > |\beta_0|,$$

or $h = h^{0,B}$ with $B \in \frac{1}{2} \Gamma^*$.

**Proof.** If $h = h^{0,B}$ then $B = \frac{\beta_0}{2}(1 + e^{it_0}) \mod \Gamma^*$ for some $t_0 \in [0, 2\pi)$. But then

$$h^{0,B} = e^{-\pi i(\beta_0(1 + e^{it_0}))}$$

is real if and only if $\beta_0(1 + e^{it_0}) \in \Gamma^*$, that is $B \in \frac{1}{2} \Gamma^*$ in which case $h^{0,B}_\gamma = e^{2i\pi(B, \gamma)} = \pm 1$ for all $\gamma \in \Gamma^*$.

If $A \neq 0$ and $h = h^{A,B}$ is real then $\dim_{\mathbb{C}} H^0_K(\mathbb{H}) \geq 2$ shows that $A$ is a double point. Conversely, if

$$A = -\frac{i}{2} \zeta \sqrt{1 - \frac{|\beta_0|^2}{|\zeta|^2}}, \quad \zeta \in \Gamma^*, \quad |\zeta| > |\beta_0|$$

is a double point then we obtain from (2.16) that $B = \frac{1}{2}(\beta_0 - \zeta) \mod \Gamma^*$ which shows that $B \in \frac{1}{2} \Gamma^*$ and $h^{A,B}_\gamma = \pm e^{2i\pi(A, \gamma)} \in \mathbb{R}$ is real. \hfill \qed

**Corollary 2.13.** Over generic points of the spectral curve $\Sigma$ the kernel bundle $\mathcal{L}$ is given by

$$\mathcal{L}_h = \mathcal{L}_{A,\delta - B},$$

and $\mathcal{L}$ is compatible with the real structure $\rho$, that is $\rho^* \mathcal{L} = \mathcal{L}_j$.

Summarizing the previous results we see that for a generic multiplier $h = h^{A,B}$ there is a 1–dimensional space of holomorphic sections. If we denote

$$\text{Spec}_0 = \{ h \in \text{Spec} \mid \dim H^0_K(\mathbb{H}) = 1 \}$$

we have a well–defined map

$$\lambda : \text{Spec}_0 \to \mathbb{C}_*, \quad h \mapsto \lambda(h) = \lambda_{A,\delta - B},$$

where the multiplier $h = h^{A,B}$ has $\Gamma^*_{A,B} = \{ \delta \}$ and $\lambda_{A,\delta - B}$ is defined by (2.19)

$$\lambda_{A,\delta - B} = -\frac{2i(|A| \mp r)A}{|\beta_0||A|}, \quad r = \sqrt{|A|^2 + |\beta_0|^2/4}.$$  

(2.21)

Here we used that the expression $\delta - B$ is uniquely defined by $h$ since $\Gamma^*_{A,B} = \Gamma^*_{A,B+\zeta} - \zeta$ for all $\zeta \in \Gamma^*$. The map $\lambda$ gives rise to the normalization of Spec:

**Proposition 2.14.** There is a surjective map $\eta : \mathbb{C}_* \to \text{Spec}$ with $\eta(\lambda(h)) = h$ for all $h \in \text{Spec}_0$. The pullback bundle $\eta^* \mathcal{L}$ extends smoothly across $\lambda(\text{Spec} \setminus \text{Spec}_0) \subset \mathbb{C}_*$.

**Proof.** For $\lambda \in \mathbb{C}_*$ we define $\eta(\lambda) := h^{A^{\lambda}, B^{\lambda}}$ where

$$A^{\lambda} = \frac{i\beta_0}{1} (\lambda - \bar{\lambda}^{-1})$$

and

$$B^{\lambda} = \frac{\beta_0}{4} (2 - \lambda^{-1} - \bar{\lambda}).$$

For $h \in \text{Spec}_0$, $h = h^{A,B}$, with $\lambda(h) \notin S^1$ we see from (2.21) that

$$A = \frac{|\lambda(h)|^2 - 1}{4\lambda(h)}.$$
Clearly, this extends to \( h \in \text{Spec}_0 \) with \( \lambda(h) \in S^1 \), that is \( h = h^{A,B} \) with \( A = 0 \), and thus \( A = A_\lambda \) for all \( h \in \text{Spec}_0 \), \( \lambda = \lambda(h) \). Using (2.19) we see
\[
B = \delta - \frac{\beta_0}{2} \lambda(h) + iA = \delta - \frac{\beta_0}{4}(\lambda(h) + \lambda(h)^{-1})
\]
with \( \delta \in \Gamma_{A,B}^* \subset \Gamma^* + \frac{\beta_0}{2} \) so that \( B = B_\lambda \mod \Gamma^* \), and \( \eta(\lambda(h^{A,B})) = h^{A,B} \). Finally, Corollary 2.3 shows that the line bundle \( \eta^* \mathcal{L} \) is well defined over \( \mathbb{C}_* \). \( \square \)

We will see later that \( \mu = \frac{1}{\lambda^2} \) is in fact (5.12) the spectral parameter of the harmonic map given by the left normal \( N \) of \( f \).

### 3. Darboux transforms

In the previous section, we discussed all possible multipliers of holomorphic sections of the quaternionic holomorphic line bundle which is associated to a Hamiltonian stationary torus. Each holomorphic section with multiplier gives rise \([3]\) to a conformal map, a Darboux transform of \( f \). We quickly recall this construction, and show that the Darboux transforms of a Hamiltonian stationary torus which arise from monochromatic holomorphic sections are again Hamiltonian stationary.

Recall that a sphere congruence \( S: M \to S^4 \) assigns to each point \( p \in M \) a 2–sphere \( S(p) \). If the sphere congruence \( S \) passes through a conformal immersion \( f: M \to \mathbb{R}^4 \), that is \( f(p) \in S(p) \), and the Gauss maps of \( f \) and \( S(p) \) coincide at each point \( p \in M \), then \( S \) is said to envelope \( f \). A pair of conformal immersions \( f, f^2: M \to \mathbb{R}^4 \) such that there exists a sphere congruence \( S \) enveloping both \( f \) and \( f^2 \) is called a classical Darboux pair. In this case, both \( f \) and \( f^2 \) are isothermic \([11]\). More generally, a branched conformal immersion \( \hat{f}: M \to S^4 \) is called a Darboux transform of a conformal immersion \( f: M \to S^4 \) if there exists a sphere congruence \( S \) on \( M \) enveloping \( f \) and left–enveloping \( \hat{f} \) over immersed points. The enveloping conditions can be written in terms of complex structures on the trivial \( \mathbb{H}^2 \)–bundle \( V \) over \( M \): \( \hat{f} \) is a Darboux transform of \( f \) if and only if there exists a complex structure \( S \in \Gamma(\text{End}(V)) \) such that \( L \) and \( \hat{L} \) are \( S \)–stable and \( *\delta = S\delta = \delta S \) and \( *\hat{\delta} = S\hat{\delta} \) where \( \delta \) and \( \hat{\delta} \) denote the derivatives of the line bundles \( L \) and \( \hat{L} \) given by \( f \) and \( \hat{f} \) respectively. Note that the global existence of \( S \) already implies that \( \hat{f} \) has only isolated branch points \([12]\). In the case, when \( f: M \to \mathbb{R}^4 \) maps to the Euclidean 4–space a sphere congruence \( S \) passing through \( f \) is left–enveloping \( \hat{f} \) if the left–normals of \( f \) and \( S(p) \) coincide at each point \( p \in M \).

**Lemma 3.1** (see \([3]\)). Every holomorphic section \( \varphi \in H^0(\mathcal{V}/L) \) of the canonical holomorphic bundle of a conformal immersion \( f: M \to S^4 \) has a unique lift \( \tilde{\varphi} \in \Gamma(\hat{V}) \) of \( \varphi \) such that
\[
\pi d\tilde{\varphi} = 0,
\]
where \( \pi: V \to \mathcal{V}/L \) is the canonical projection. This unique lift \( \tilde{\varphi} \) is called the prolongation of \( \varphi \).

Note that the prolongation \( \tilde{\varphi} \) has the same multiplier as \( \varphi \) so that, if \( \varphi \) has no zeros, \( \hat{f} = \tilde{\varphi} \mathbb{H}: M \to S^4 \) defines a map from the Riemann surface \( M \) into the 4–sphere which turns out to be a Darboux transform of \( f \). In the case when \( \varphi \) has zeros, one obtains a conformal map \( \hat{f} \) away from the zeros of \( \varphi \), which is again a Darboux transform on its domain. Such a map \( \hat{f} \) is called a *singular* Darboux transform. Conversely, every
(singular) Darboux transform \( \hat{f} : M \to S^4 \) gives a holomorphic section with multiplier of \( V/L \).

**Definition 3.2.** Let \( f : T^2 \to \mathbb{R}^4 \) be a Hamiltonian stationary torus with associated holomorphic line bundle \( V/L \).

A branched conformal immersion \( \hat{f} : T^2 \to S^4 \) is called a **monochromatic Darboux transform** of \( f \) (respectively **polychromatic**; see Section 2) if it is given by the prolongation of a monochromatic (respectively polychromatic) holomorphic section with multiplier of \( \tilde{V}/L \).

For a conformal immersion \( f : M \to \mathbb{R}^4 \) we identify as before \( V/L = \mathbb{H} \) via the nowhere vanishing section \( \pi e \in \Gamma(V/L) \) where \( e = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \). To compute the prolongation in terms of the trivialization, note that every nowhere vanishing holomorphic section \( \alpha \in \mathcal{H}^0(\tilde{V}/L) \) with multiplier \( h \) defines a unique quaternionic flat connection \( \tilde{d} \) on \( \tilde{V}/L \) by requiring \( \tilde{d} \alpha = 0 \).

Since \( \alpha \) is holomorphic, that is \( \ast d\alpha = Nd\alpha \), we can write

\[
\tilde{d} = d + df \hat{T},
\]

where \( \hat{T} : M \to \mathbb{H} \) is defined on \( M \) rather than on the universal cover \( \tilde{M} \) of \( M \) since \( \alpha \) and \( d\alpha \) have the same multiplier.

**Lemma 3.3.** Let \( f : M \to \mathbb{R}^4 \) be a conformal immersion. The prolongation \( \tilde{\varphi} \) of a nowhere vanishing, non–constant holomorphic section \( \varphi = \pi e \alpha \in \mathcal{H}^0(\tilde{V}/L) \) is given by

\[
\tilde{\varphi} = \psi \nu + e \alpha
\]

where \( \psi = \left( \begin{array}{c} f \\ 1 \end{array} \right) \in \Gamma(L), \nu = \hat{T} \alpha, \) and \( \hat{T} \) is defined by \( df \hat{T} \alpha = -d\alpha \).

In particular, if the Darboux transform \( \tilde{f} : M \to \mathbb{R}^4 \) associated to \( \alpha \) maps into \( \mathbb{R}^4 \) rather than \( S^4 \), that is, if \( \hat{T} \) is nowhere vanishing, then \( \tilde{f} = f + T : M \to \mathbb{R}^4 \) with \( T = \hat{T} \).**

**Proof.** One easily verifies that \( \tilde{\varphi} \) satisfies (3.1). But then, if \( \hat{T} \neq 0 \),

\[
\tilde{f} = \tilde{\varphi} \mathbb{H} = \left( \begin{array}{c} f \nu + \alpha \\ \nu \end{array} \right) \mathbb{H} = \left( \begin{array}{c} f + T \\ 1 \end{array} \right) \mathbb{H}
\]

and \( \tilde{f} = f + T \) is the Darboux transform of \( f \) when choosing the point at infinity \( \infty = e\mathbb{H} \). \( \square \)

**Remark 3.4.** The previous lemma extends in the obvious way to holomorphic sections with zeros and singular Darboux transforms.

From (2.2) we know that \( \text{Span}_{\mathbb{H}} \{ 1, f \} \subset \mathcal{H}^0(\mathbb{H}) \). In particular, if \( \alpha \in \mathcal{H}^0(\mathbb{H}) \) is constant the previous lemma shows that \( \tilde{\varphi} = e \alpha \) is the prolongation of \( \varphi = \pi e \alpha \). In other words, the Darboux transform given by \( \alpha \) is the constant map \( \tilde{f} = \infty \). More generally,

**Corollary 3.5.** If \( f : T^2 \to \mathbb{R}^4 \) is a conformal immersion and \( \alpha \in \text{Span}_{\mathbb{H}} \{ 1, f \} \subset \mathcal{H}^0(\mathbb{H}) \), then the Darboux transform of \( f \) given by \( \alpha \) is a constant map \( \tilde{f} = c \) with \( c \in \mathbb{H} \cup \{ \infty \} \).

In particular, if the complex dimension of the space of global holomorphic sections is minimal, that is \( \dim_{\mathbb{C}} \mathcal{H}^0(\mathbb{H}) = 4 \), then every Darboux transform given by the trivial multiplier is constant.
From (2.18) we see that monochromatic holomorphic sections of a Hamiltonian stationary torus are always nowhere vanishing, in particular, we obtain regular Darboux transforms defined on the torus. Lemma 3.3 allows to compute all Darboux transforms in 4–space of a Hamiltonian stationary torus:

**Theorem 3.6.** Every non–constant Darboux transform \( \hat{f} : T^2 \to \mathbb{R}^4 \) of a Hamiltonian stationary torus \( f \) with \( df = e^{\frac{i\alpha}{2}} dzg \) is given by a nowhere vanishing holomorphic section \( \alpha \in H^0_h(\mathbb{H}) \) with multiplier \( h = h^{A,B} \), and

\[
\hat{f} = f - e^{\frac{i\beta}{2}} \frac{1}{\pi(4|A|^4 + |\beta_0, A|^2)(2|A|^2 - j|\beta_0, A|)Ag}, \quad \text{if } A \neq 0,
\]

or

\[
\hat{f} = f + e^{\frac{i\beta}{2}} \left( \sum_{s,t \in I_B} (1 + ke^{is})u_s u_te^{\frac{\beta_0}{2}(e^{it} - e^{-it})} \frac{1}{r\pi|\beta_0|} g \right)
\]

where the finite set

\[
I_B = \{ t \in [0,2\pi) \mid B - \frac{|\beta_0|}{2} e^{it} \in \Gamma^*_0,B \} \neq \{0,\pi\}
\]

parametrizes the admissible frequencies and \( u_t \in \mathbb{C} \) are chosen so that the map

\[
r = | \sum_{t \in I_B} u_t \sin t e^{-\frac{|\beta_0|}{2} e^{it}} |^2 + | \sum_{t \in I_B} u_t e^{it} \sin t e^{-\frac{|\beta_0|}{2} e^{it}} |^2
\]

is nowhere vanishing. Here we again use \( e_{\delta}(z) = e^{2\pi i (\delta, z)} \) for \( \delta \in \Gamma^* + \frac{\beta_0}{2} \). Equation (3.3) simplifies in the case of a monochromatic holomorphic section to

\[
\hat{f} = f + e^{\frac{i\beta}{2}} k \frac{e^{it}}{\pi|\beta_0|\sin(t)} g, \quad \text{if } A = 0, I_B = \{t\}, t \notin \{0,\pi\}.
\]

**Remark 3.7.** The compatibility of the real structure \( \rho \) with \( \mathcal{L}_{A,\delta-B} \) implies, see Theorem 2.11, that the induced Darboux transforms of \( h^{A,B} \) and \( \rho(h^{A,B}) = h^{A,-B} \) coincide. Therefore, in the case when \( A \neq 0 \) the Darboux transform does not depend on whether \( \Gamma^*_{A,B} = \{\delta_+\} \) or \( \Gamma^*_{A,B} = \{\delta_-\} \). In particular, if \( h = h^{A,B} \) is a multiplier at a double point \( A \), then \( h \) is real, and we obtain only one closed Darboux transform, given by (3.2), associated with the multiplier \( h \).

**Proof of Theorem 3.6.** It is shown in [3] that every Darboux transform is given by a holomorphic section \( \alpha \in H^0_h(\mathbb{H}) \) and we have seen that \( h = h^{A,B} \). To compute globally defined Darboux transforms \( \hat{f} : T^2 \to \mathbb{R}^4 \) with values in Euclidean 4–space, we have to find holomorphic sections \( \alpha \) with multiplier so that the flat quaternionic connection \( \hat{d} = d + df\hat{T} \) with \( d\alpha = 0 \) satisfies \( \hat{T} \neq 0 \). As before it is computationally easier to use the gauged holomorphicity condition for \( \hat{\alpha} = e^{-\frac{i\beta}{2}} \alpha \): the unique flat quaternionic connection with \( d\hat{\alpha} = 0 \) is

\[
\hat{d} = d + \frac{j|\beta|}{2} + dz\hat{T}
\]

with \( \hat{T} = g\hat{T} e^{\frac{i\beta}{2}} \); in particular, it is enough to find \( \hat{\alpha} \) with nowhere vanishing \( \hat{T} \). By Corollary 2.3 all holomorphic sections \( \alpha \) with multiplier \( h = h^{A,B} \) are given by

\[
\alpha = \sigma e^{2\pi((A,\cdot) - j(B,\cdot))}
\]
where \( \tilde{\sigma} = \sum_{\tau \in \Gamma_{A,B}} (1 - k\lambda_\delta)e_\delta \tilde{u}_\delta \) so that
\[
\tilde{d}\tilde{\alpha} e^{-2\pi((A,\cdot) - i(B,\cdot))} = \pi \sum_{\delta \in \Gamma_{A,B}} \left[(1 \cdot k\lambda_\delta) \left((i\delta - i\bar{B} + \bar{A})dz + (i\delta - iB + A)d\bar{z}\right)
+ \frac{j}{2}(\bar{\beta}_0 dz + \beta_0 d\bar{z})(1 - k\lambda_\delta)\right] \tilde{u}_\delta e_\delta + dz\tilde{\sigma}
= \pi d\bar{z} \sum_{\delta \in \Gamma_{A,B}} \left((i\delta - iB + A) - k\lambda_\delta(i\bar{\delta} - i\bar{B} + \bar{A}) + \frac{j}{2}\beta_0(1 - k\lambda_\delta)\right) \tilde{u}_\delta e_\delta
+ dz \left(\pi \sum_{\delta \in \Gamma_{A,B}} \left((i\delta - iB + A) - k\lambda_\delta(i\bar{\delta} - i\bar{B} + \bar{A}) + \frac{j}{2}\beta_0(1 - k\lambda_\delta)\right) \tilde{u}_\delta e_\delta + \tilde{\tau}\tilde{\sigma}\right).
\]
By holomorphicity of \( \alpha \) the \((0,1)\)-part with respect to left multiplication by \( i \)
\[
\pi d\bar{z} \sum_{\delta \in \Gamma_{A,B}} \left((i\delta - iB + A) + j\lambda_\delta(i\bar{\delta} - i\bar{B} + \bar{A}) + \frac{j}{2}\beta_0(1 + j\lambda_\delta)\right) \tilde{u}_\delta e_\delta = 0
\]
vanesishes \((2.13)\), and thus \( \tilde{\tau} \) with \(-d\tilde{\alpha} = \frac{j\beta_0}{2}\tilde{\alpha} + dz\tilde{\alpha} \) satisfies
\[
\tilde{\tau}\tilde{\sigma} = -\pi \sum_{\delta \in \Gamma_{A,B}} \left((i\bar{\delta} - \bar{B}) - \bar{A} + (i\delta - i\bar{B} - \bar{A})k\lambda_\delta + \frac{j}{2}\beta_0(1 - k\lambda_\delta)\right) \tilde{u}_\delta e_\delta.
\]
If \( \tilde{f} \) is given by a monochromatic holomorphic section then \( \tilde{\sigma} = (1 - k\lambda_\delta)e_\delta \tilde{u}_\delta \) is given by a single frequency \( \delta \in \Gamma_{A,B} \) and
\[
\tilde{\tau} = -\pi \left\{ \left(i(\delta - B) + \delta + \frac{j}{2}(\delta - B)\right) + j\left(\frac{\lambda_\delta}{1 + |\lambda_\delta|^2}\right) \right\}
\]
is constant. Moreover, \((2.19)\) yields
\[
1 - |\lambda_\delta|^2 = \begin{cases} 0 & A = 0 \\ \frac{2}{|\lambda_\delta|^2}(-|A| \pm r) & A \neq 0, \ \delta = B \pm ir\frac{A}{|A|} \end{cases}
\]
and
\[
1 + |\lambda_\delta|^2 = \begin{cases} 2 & A = 0 \\ \frac{8}{|\lambda_\delta|^2}r(\mp |A|) & A \neq 0, \ \delta = B \pm ir\frac{A}{|A|} \end{cases},
\]
so that
\[
\tilde{\tau} = \begin{cases} -k\pi\beta_0e^{it} \sin t & \text{if } A = 0, \ \delta = B - \frac{\beta_0}{2}e^{it} \\ -\pi \left( 2\bar{A} + j\frac{\beta(A)}{A} \right) & \text{if } A \neq 0 \end{cases},
\]
If \( A \neq 0 \) or if \( A = 0 \) and \( t \not\in \{0, \pi\} \) then \( \tilde{\tau} \neq 0 \) is a non zero constant, and we obtain the Darboux transforms \( \tilde{f} = f + e^{\frac{j\beta_0}{2}t} \tilde{\tau}^{-1}g : T^2 \to \mathbb{R}^4 \) given by \((3.4)\) and \((3.2)\).

Finally, the only Darboux transforms which arise in families are given by multipliers \( h = h_{0,B} \) with \( \dim H^0_{0,B} > 2 \) or \( \dim H^0_{0,B} = 2 \) and \( h \) not real. In this case, \((3.5)\) simplifies to
\[
\tilde{\tau} = \pi\beta_0 \left( \sum_{t \in I_{B}} (1 - ke^{it}) \sin t \ u_t e^{-\frac{\beta_0}{2}e^{it}} \right) \tilde{\sigma}^{-1}
\]
with \( \tilde{\sigma} = \sum_{t \in I_B} (1 + ke^{i\tau}) u_t e_B - \frac{\partial}{\partial \tau} e^t, \) \( u_t \in \mathbb{C}, \) and we obtain the general formula for \( h = h^{0,B}. \)

If \( f : T^2 \to \mathbb{R}^4 \) be a Hamiltonian stationary torus with Lagrangian angle \( \beta \) and \( df = e^{rac{i\beta}{2}} dz g \) then the left normal of a Darboux transform \( \hat{f} = f + e^{\frac{i\beta}{2}} \tau g \) of \( f \) can be expressed entirely in terms of \( \tau \) and the Lagrangian angle \( \beta \) of \( f \). Lemma 3.3 and Theorem 3.6 show that \( d\alpha = -d\tau^{-1}\alpha \) for some holomorphic section \( \alpha \in H^0_h(\mathbb{H}) \) with multiplier \( h \) where \( \tau = e^{\frac{i\beta}{2}} g. \) Putting \( \nu = \tau^{-1}\alpha \) this equation implies that \( 0 = df \wedge d\nu \) and thus \( *d\nu = -Rd\nu \) where \( R \) is the right normal of \( f \). Therefore, \( d\hat{f} = df + d\tau = df + d\alpha \nu^{-1} - \alpha^{-1}d\nu \nu^{-1} = -T \tau \alpha^{-1}T \)

shows that the left normal of \( \hat{f} \) is given by \( \hat{N} = -T \tau \alpha^{-1}T \) since \( *d\hat{f} = -T \tau \alpha^{-1}T \). But the right normal of \( f \) is given (2.5) by \( R = -g^{-1} i g \) so that we obtain

\[
\hat{N} = -T \tau \alpha^{-1}T = e^{\frac{i\beta}{2} \tau} i \tau^{-1} e^{-\frac{i\beta}{2} \tau}.
\]

If \( \hat{f} \) is a monochromatic Darboux transform, the left normal \( \hat{N} \) of \( \hat{f} \) turns out to be harmonic:

**Theorem 3.8.** Let \( f : T^2 \to \mathbb{R}^4 \) be a Hamiltonian stationary torus. Then every Darboux transform of \( f \), which is given by a holomorphic section in \( H^0_h(\mathbb{H}) \) with generic multiplier \( h \in \Sigma \), is again Hamiltonian stationary.

More precisely, if \( \beta \) is the Lagrangian angle of \( f \) then every monochromatic Darboux transform \( \hat{f} : T^2 \to \mathbb{R}^4 \) of \( f \) has Lagrangian angle \( \beta = \beta + \beta_h \) where \( \beta_h \in \mathbb{R} \) is constant.

**Proof.** If \( \hat{f} \) is a monochromatic Darboux transform of \( f \) then Theorem 3.6 shows that \( \tau = (\tau_0 + j \tau_1) c \) with \( \tau_0, \tau_1 \in \mathbb{R}, c \in \mathbb{C} \). In this case, the equation (3.6) for the left normal \( \hat{N} \) of \( \hat{f} \) simplifies to

\[
\hat{N} = \frac{1}{\tau_0^2 + \tau_1^2} e^{\frac{i\beta}{2} (\tau_0 + j \tau_1) i (\tau_0 - j \tau_1) e^{-\frac{i\beta}{2}}} = e^{i\beta (\tau_0 + j \tau_1)^2} = e^{i\beta_h (\tau_0^2 + \tau_1^2)}.
\]

Since \( \frac{(\tau_0 + j \tau_1)^2}{\tau_0^2 + \tau_1^2} \in S^3 \cap \mathbb{C}, \) \( \mathbb{C} = \text{Span}_\mathbb{R} \{1, j\} \), we can write

\[
\frac{(\tau_0 + j \tau_1)^2}{\tau_0^2 + \tau_1^2} = e^{i\beta_h}
\]

with \( \beta_h \in \mathbb{R} \) constant. This shows both that the left normal \( \hat{N} \) of the Darboux transform \( \hat{f} \) takes values in the unit circle \( S^3 \cap \mathbb{C} i \) and that \( \hat{N} \) is harmonic, in other words, \( \hat{f} \) is Hamiltonian stationary.

**Remark 3.9.** In fact, we have seen that for all multiplier \( h = h^{A,B} \) with \( A \neq 0 \) the corresponding Darboux transforms are Hamiltonian stationary. Moreover, for \( A = 0 \) we may obtain Darboux transforms which are not Hamiltonian stationary only if \( \dim \mathbb{C} H^0_{0,B} > 2 \) or \( \dim \mathbb{C} H^0_{0,B} = 2 \) and \( h = h^{0,B} \) is not real.

In Section 4 we show that there exist Darboux transforms which are not Lagrangian tori, and thus not Hamiltonian stationary, in \( \mathbb{C}^2 \) using the following characterization of Lagrangian Darboux transforms:
Corollary 3.10. Let \( f : T^2 \to \mathbb{R}^4 \) be a Hamiltonian stationary torus. A Darboux transform \( \hat{f} : T^2 \to \mathbb{C}^2 \) of \( f \) is Lagrangian in \( \mathbb{C}^2 \) if and only if \( \hat{f} = f + e^{2\pi j} (\tau_0 + j\tau_1)g \) with \( \text{Im}(\tau_0\tau_1) = 0 \) where \( \tau_0, \tau_1 : M \to \mathbb{C} \).

Proof. From (3.6) we see that the left normal \( \hat{N} = n_0i + n_1 \) of \( \hat{f} \) is given by

\[
n_0 = \frac{|\tau_1|^2 - |\tau_0|^2 - 2j \text{Re}(\tau_0\tau_1)}{|\tau_0|^2 + |\tau_1|^2} e^{j\beta} \in \mathbb{C}, \quad \text{and} \quad n_1 = 2j \frac{\text{Im}(\tau_0\tau_1)}{|\tau_0|^2 + |\tau_1|^2} \in j\mathbb{R}.
\]

But then (2.3) shows that \( \hat{f} \) is Lagrangian in \( \mathbb{C}^2 \) if and only if \( \hat{N} \in S^3 \cap \mathbb{C}i \) which is equivalent to \( n_1 = 0 \). \( \square \)

Given a Hamiltonian stationary torus \( f \) with Lagrangian angle \( \beta \), Theorem 3.8 shows that every monochromatic Darboux transform \( \hat{f} \) of \( f \) has, after reparametrization, again Lagrangian angle \( \beta \). Therefore, the characterization of Hamiltonian stationary tori by there Lagrangian angle and lattice in Theorem 2.9 shows that every monochromatic Darboux transform of \( f \) is, after reparametrization, in the same family of Hamiltonian stationary tori as \( f \). In particular:

Corollary 3.11. Let \( f : \mathbb{C}/\Gamma \to \mathbb{R}^4 \) be a Hamiltonian stationary torus. If the space of global holomorphic sections has complex dimension \( \dim_{\mathbb{C}} H^0(\mathbb{H}) = 4 \) then every monochromatic Darboux transform of \( f \) is, up to rotation, translation and reparametrization, \( f \).

4. Examples

In this section we illustrate the previous results in the case of homogeneous tori and Castro–Urbano tori. Moreover, we shall see that there are examples of Hamiltonian stationary tori which allow families of Darboux transforms which are not Lagrangian surfaces in \( \mathbb{C}^2 \). Each family is obtained from one of the finitely many multipliers \( h = h^{0,B} \in \text{Spec} \) with \( \dim_{\mathbb{C}} H^0(\mathbb{H}) > 1 \).

4.1. Homogeneous tori. We consider homogeneous tori \( f : \mathbb{C}/\Gamma \to \mathbb{C}^2 \),

\[
f(x, y) = \frac{1}{r_1} e^{2\pi j r_1 x} + i \frac{1}{r_2} e^{2\pi j r_2 y},
\]

where the lattice is given by \( \Gamma = \frac{1}{r_1} \mathbb{Z} \oplus \frac{i}{r_2} \mathbb{Z}, \) \( r_1, r_2 > 0 \). Since the derivative of \( f \) can be written as

\[
 df = 2\pi e^{2\pi j (r_1 x - r_2 y)} dz j e^{2\pi j (r_1 x + r_2 y)}
\]

the conformal immersion \( f \) is Hamiltonian stationary with Lagrangian angle

\[
 \beta(z) = 2\pi (r_1 x - r_2 y), \quad \text{that is} \quad \beta_0 = r_1 - r_2 i \in \Gamma^*,
\]

and \( df = e^{2\pi j} dz g \) with \( g = 2\pi j e^{2\pi j (r_1 x + r_2 y)} \). Let us first discuss monochromatic Darboux transforms of homogeneous tori: these are given by

\[
 \hat{f} = e^{2\pi j r_1 x} \lambda_1 + i e^{2\pi j r_2 y} \lambda_2,
\]

where we obtain for \( A = A_0 + i A_1 \neq 0 \) by (3.2)

\[
 \lambda_1 = \frac{1}{r_1} - w A_0, \quad \lambda_2 = \frac{1}{r_2} + \bar{w} A_1 \quad \text{with} \quad w = \frac{2(\beta_0, A) + 4|A|^2 j}{4|A|^4 + (\beta_0, A)^2},
\]
and for $A = 0, B = \frac{\beta_0}{2}(1 + e^{it})$ mod lattice $\Gamma^*$, by (3.4)
\[
\lambda_1 = \frac{r_1^2 - r_2^2 + 2r_1r_2 \cot t}{r_1(r_1^2 + r_2^2)}, \quad \lambda_2 = \frac{r_1^2 - r_2^2 - 2r_1r_2 \cot t}{r_2(r_1^2 + r_2^2)}.
\]

Since $\hat{f}$ is conformal, we see that in both cases $|\lambda_1| = |\lambda_2| = \frac{r_2}{r_1}$ so that $\hat{f}$ is homogeneous:

**Lemma 4.1.** All monochromatic Darboux transforms of a homogeneous torus $f$ are again (after rescaling and reparametrization) the homogeneous torus $f$.

To consider polychromatic Darboux transforms we have to discuss the points lying in the intersection $\mathcal{D} \cap (\Gamma^* + \beta_0^2)$ of the disc of radius $|\beta_0|$ and the translated dual lattice. Since this depends on the ratio of the radii $r_1$ and $r_2$ we will here only discuss the existence of polychromatic Darboux transforms and show that some of these polychromatic Darboux transforms are not Lagrangian in $\mathcal{C}^2$. We will return to the general case and the study of the geometric properties of polychromatic Darboux transforms in a future paper.

4.1.1. The Clifford torus. The Clifford torus is given by $r_1 = r_2 = 1$. The points on the translated dual lattice $\Gamma^* + \frac{1-i}{2}$ lying in the closed disk $\mathcal{D}$ around $\frac{\beta_0}{2} = \frac{1-i}{2}$ of radius $|\beta_0| = \sqrt{2}$ are
\[
\mathcal{E} = \frac{\beta_0}{2} + \{0, \pm 1, \pm i, \pm (1 + i), \pm (1 - i)\}.
\]

![Figure 4](image_url)

The circle $\mathcal{C}_{\frac{\sqrt{2}}{2}}(\epsilon)$ of radius $\frac{\sqrt{2}}{2}$ around $\epsilon \in \mathcal{E} \setminus \left\{\frac{\beta_0}{2}\right\}$ intersects the circle $\mathcal{C}_{\frac{\sqrt{2}}{2}}(\frac{\beta_0}{2})$, and
\[
\mathcal{C}(\epsilon, \frac{\sqrt{2}}{2}) \cap \mathcal{C}(\frac{\beta_0}{2}, \frac{\sqrt{2}}{2}) \subset \{0, 1, -i, 1 - i\}.
\]

Therefore, $\dim_{\mathcal{C}} H^0_{0, B} = 1$ for all $B = \frac{1-i}{2}(1 + e^{it}), t \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$.

The remaining points are $B = 0 \mod \Gamma^*$, that is $h = h^{0,0}$ is the trivial multiplier, and there are exactly four circles intersecting at $B = 0$, so that $\dim H^0_{0, B} = 4$ for
$B \in \{0, 1, -i, 1-i\}$. In particular, the corresponding Darboux transforms are by Corollary 3.5 constant maps, and there are no polychromatic Darboux transforms of the Clifford torus:

**Lemma 4.2.** Every non–constant Darboux transform $\hat{f} : T^2 \to S^4$ of the Clifford torus is the (scaled and reparametrized) Clifford torus.

**Remark 4.3.** Note that we only discuss closed Darboux transforms which are defined on the original torus $T^2 = \mathbb{C}/\Gamma$. If we allow other periods, then the examples of Bernstein [2] show that there are Darboux transforms (on an appropriate covering) of the Clifford torus which are isothermic but not constrained Willmore, in particular not Hamiltonian stationary.

### 4.1.2. The case $r_1 = 2$ and $r_2 = 1$.

The points in the translated dual lattice lying inside the closed disk $D$ around $\frac{2}{\sqrt{5}} = \frac{2}{2+i}$ of radius $|\beta_0| = \sqrt{5}$ are, see Figure 1, $$\mathcal{E} = \frac{\beta_0}{2} + \{0, \pm 2, \pm i, \pm 2i, \pm (2 + i), \pm (2 - i)\}. $$

The circle $C_{\sqrt{5}/2}(e)$ of radius $\sqrt{5}/2$ around $e \in \mathcal{E} \setminus \{\frac{\beta_0}{2}\}$ intersects the circle $C_{\sqrt{5}/2}(\frac{\beta_0}{2})$ in at most two points, taken in $\frac{\beta_0}{2} + \{1 \pm \frac{i}{2}, \frac{1}{2} \pm i, -\frac{1}{2} \pm i, -1 \pm \frac{i}{2}\}$. However each of these points is congruent modulo $\Gamma^*$ to one of $\frac{1+i}{2}$ or $\frac{3+i}{2}$. The point $B = 0$ lies at the intersection of four circles, so $\dim_c H^0_{0,0} = 4$, and again the Darboux transforms given by the constant multiplier $h = h^{0,0}$ are constant maps.

Let now $B = \frac{1+i}{2}$ (the case $B = \frac{3+i}{2}$ can be treated in a completely analogous way). Then $\dim H^0_{0,B} = 2$ since $$\Gamma_{0,\frac{1+i}{2}}^* = \left\{\frac{\beta_0}{2}, \frac{\beta_0}{2} + 2i\right\} \quad \text{with} \quad I_B = \left\{\pi - \arctan \frac{3}{4}, \frac{3\pi}{2}\right\},$$

and every polychromatic Darboux transforms, which is given by the multiplier $h = h^{0,\frac{1+i}{2}}$, can be written (3.3) as $\hat{f} = f + e^{\frac{\beta_0}{2}r} \frac{1}{\tau_{\mathbb{C}\beta_0}} g$, $r = \tau_0 + j\tau_1$ with

$$\begin{align*}
\tau_0 &= (12i - 16)u_1 u_2 e^{-4\pi i y} + 20i\bar{u}_1 u_2 e^{4\pi i y} \\
\tau_1 &= (18 + 24i)|u_1|^2 + 50|u_2|^2 - (32i + 24)u_1 \bar{u}_2 e^{-4\pi i y} + 10\bar{u}_1 u_2 e^{4\pi i y},
\end{align*}$$

where $u_1, u_2 \in \mathbb{C}$ so that

$$r = 18|u_1|^2 + 50|u_2|^2 - 12 \text{Re} \left((1 + 2i)\bar{u}_1 u_2 e^{4\pi i y}\right)$$

is nowhere vanishing. In particular, we see that in general $\text{Im}(\tau_0\tau_1) \neq 0$, so that Corollary 3.10 shows that the family of polychromatic Darboux transforms $\hat{f}$ given by the multiplier $h = h^{0,\frac{1+i}{2}}$ is not Lagrangian, and thus not Hamiltonian stationary, in $\mathcal{E}^2$.

### 4.1.3. The cases $r_1 = m \in \mathbb{N}, m \geq 2$ and $r_2 = 1$.

Analogous to the case $r_1 = 2$, we see that $B = \frac{m-1}{2}(1+i)$ lies at the intersection of two circles, and

$$\Gamma_{0,\frac{m-1}{2}(1+i)}^* = \left\{\frac{\beta_0}{2}, \frac{\beta_0}{2} + mi\right\}$$

so that we obtain a family of polychromatic Darboux transforms by (3.3) with $I_B = \left\{\pi - \arctan \left(\frac{m^2 - 1}{2m}\right), \frac{3\pi}{2}\right\}$. 
A similar computation as for the case $r_1 = 2$ shows that these surfaces are again not Lagrangian surfaces in $\mathbb{C}^2$. Note that there may exist further $\Gamma$ with $|\Gamma_{\beta_0}^*| \geq 2$, e.g. for $m = 3$ and $B = \frac{\beta_0}{2} + i - \frac{\sqrt{6}}{2}$.

4.2. A Castro–Urbano torus. Let us contemplate another example, which is part of a one-homogeneous family found by Castro and Urbano [8] (see section 3.3 of [15] for details). Take $\Gamma = \text{Span}\{1, i\}$ and $\beta_0 = 3 + i$. Then

$$\Gamma_{\beta_0}^* = \left\{ \frac{1 \pm 3i}{2}, \frac{-1 \pm 3i}{2}, \pm \frac{3 + i}{2} \right\}$$

Using Theorem 2.9 we obtain a Hamiltonian stationary torus by taking holomorphic sections without multiplier, e.g.,

$$f = \alpha_1 \frac{1 + 3i}{2} + \alpha_2 \frac{-1 + 3i}{2} = \frac{1}{5} e^{\pi j(3x-y)} \left( (1 + 4j + 3k)e^{2\pi i(3y-x)} + (1 - 3j + 4k)e^{2\pi i(3x+y)} \right)$$

is a 31–Castro–Urbano torus. One easily verifies that $df = e^{\pi j(3x-y)}dzg$ with

$$g = \frac{4\pi}{5}(3 + i + 3j - k)e^{\pi i(3y-x)} + \frac{3\pi}{5}(3 + i + j + 3k)e^{\pi i(3x+y)}.$$
By Theorem 3.11 all monochromatic Darboux transforms of $f$ must be contained in the figure 7.

*Figure 7. Stereographic projection to $\mathbb{R}^3$ of a 31–Castro–Urbano torus*

31–Castro–Urbano family, however in this case the Darboux transforms are not congruent to $f$.

*Figure 8. Two monochromatic Darboux transforms of a 31–Castro–Urbano torus*

Again, there exists polychromatic Darboux transforms which are not Lagrangian surfaces in $\mathbb{C}^2$. Figure 9 shows a polychromatic Darboux transform of $f$ obtained when choosing $B = \frac{\beta_0}{2} + i - \frac{i}{\sqrt{2}}$ and $\delta_1 = \frac{\beta_0}{2}, \delta_2 = \frac{\beta_0}{2} + 2i \in \Gamma_{0,B}^\ast$.

5. $\mu$–Darboux Transforms

The Darboux transformation is defined for all conformal (branched) immersions $f : M \to S^4$ of a Riemann surface $M$ into the 4-sphere. Whereas our previous discussion restricted to the case when $M = T^2$ is a 2–torus we will now turn to the general Darboux transformation of a Hamiltonian stationary immersion $f : M \to \mathbb{R}^4$ of a Riemann surface $M$ into 4–space. In particular, we investigate the link between Darboux transforms and the
Figure 9. Polychromatic Darboux transform of a 31–Castro–Urbano torus

associated family of complex flat connections given by the harmonic left normal \( N = e^{j\beta}i \) of \( f \). Given a Lagrangian immersion \( f : M \rightarrow \mathbb{R}^4 \) with Lagrangian angle \( \beta : M \rightarrow \mathbb{R} \) and \( df = e^{\frac{j\beta}{2}}dzg \), the left normal of \( f \) is given by \( N = e^{j\beta}i \). Since

\[
(dN)' = \frac{1}{2}(dN - N \ast dN) = \frac{j}{2}e^{\frac{j\beta}{2}}(id\beta - *d\beta)e^{-\frac{j\beta}{2}}
\]

and

\[
(dN)'' = \frac{1}{2}(dN + N \ast dN) = \frac{j}{2}e^{\frac{j\beta}{2}}(id\beta + *d\beta)e^{-\frac{j\beta}{2}}
\]

we see that the real function \( \beta \) is harmonic if and only if \( d(dN)' = 0 \), or, equivalently, \( d(dN)'' = 0 \), that is if \( N \) is harmonic. The mean curvature vector \( \mathcal{H} \) of the immersion \( f \) is given [5, Section 7.2] by

\[
\mathcal{H} = N\bar{H} = \bar{H} \bar{R}
\]

where \( (dN)' = -dfH \). But (5.1) simplifies with \( \beta = 2\pi\langle \beta_0, \cdot \rangle \) to \( (dN)' = -\pi e^{\frac{j\beta}{2}}dz\bar{\beta}_0 e^{\frac{j\beta}{2}}k \) and we see that \( H = \pi g^{-1}\beta_0 e^{\frac{j\beta}{2}}k \).

As customary [22], [16], we can introduce a spectral parameter \( \mu \in \mathbb{C}_* \) so that the harmonicity of \( N \) is expressed in terms of the flatness of the family of complex connections \( d^\mu \). In our formulation, we consider the trivial \( \mathbb{H} \)--bundle as a complex \( \mathbb{C}^2 \)--bundle \((M \times \mathbb{H}, I) = M \times \mathbb{C}^2 \) where \( I \) is the complex structure on \( M \times \mathbb{H} \) given by right multiplication with \( i \). Then the \( \mathbb{C}_* \)--family of flat complex connections is given [12] by

\[
d^\mu = d + \frac{1}{2}dfH(N(a - 1) + b)
\]

on \( M \times \mathbb{H} \) where \( \mu \in \mathbb{C}_* \subset \text{Span}\{1, I\} \) and \( a = \mu^+\mu^{-1}, b = \mu^{-1}\mu I \). Note that by definition \( a^2 + b^2 = 1 \). Formally, the family of flat connections is the same as in Moriya’s paper [19], however our immersion is Hamiltonian in \( \mathbb{C}^2 \) rather than in \( \mathbb{C}^2 \) since we choose a different complex structure on \( \mathbb{R}^4 \).

Furthermore, (2.2) shows that every parallel section \( \alpha \) of \( d^\mu \) is holomorphic since \( d\alpha = -\frac{1}{2}dfH(N\alpha(a - 1) + \alpha b) \).
We conclude the proof by showing that every conformal immersion $\hat{f} : M \to S^4$ which is given by the prolongation of a $d\mu$-parallel section $\alpha \in \Gamma(\mathbb{H})$ for some $\mu \in \mathbb{C}$, is called a $\mu$-Darboux transform of a Hamiltonian stationary immersion $f : M \to \mathbb{R}^4$.

Since a $\mu$-Darboux transform $\hat{f} = f + T$ is given by a $d\mu$-parallel section $\alpha$, Lemma 3.3 shows that $T = TH^{-1}$ with
\begin{equation}
T^{-1} = \frac{1}{2}(N\alpha(a - 1)\alpha^{-1} + ab\alpha^{-1}),
\end{equation}
We abbreviate $\tilde{z} = az\alpha^{-1}$ for $z \in \mathbb{C} = \text{Span}\{1, i\}$ so that $T^{-1} = \frac{1}{2}H(N(\tilde{a} - 1) + \tilde{b})$. We now extend Theorem 3.8 to (local) $\mu$-Darboux transforms of a Hamiltonian stationary surface $f : M \to \mathbb{R}^4$ where now $M$ may be an arbitrary Riemann surface.

**Theorem 5.2.** Let $f : M \to \mathbb{R}^4$ be a Hamiltonian stationary immersion from a Riemann surface $M$ into the 4-space. Then the left normal of every (local) $\mu$-Darboux transform of $f$ is harmonic. In particular, every $\mu$-Darboux transform of $f$ is constrained Willmore.

**Proof.** We essentially follow the proof in [6] that the Gauss map of a $\mu$-Darboux transform of a constant mean curvature surface is harmonic: From (3.6) we see that the left normal $\tilde{N}$ of a Darboux transform $\hat{f} = f + T$ of $f$ is given by $\tilde{N} = -TRT^{-1}$ where $R$ is the right normal of $f$. Now (5.2) shows $RH = H\tilde{N}$, and we obtain $\tilde{N} = -\tilde{T}N\tilde{T}^{-1}$, where $\tilde{T}$ is given by a $d\mu$-parallel section $\alpha$ and (5.4). We put $\tilde{\nu} = \frac{1}{2}(N\alpha(a - 1) + ab)$ and compute its derivative, using $a^2 + b^2 = 1$, $d\tilde{N} = (dN)^\prime - dfH$ and $d\alpha = -dfH\tilde{\nu}$, as
\[
d\tilde{\nu} = (dN)^\prime a\frac{a - 1}{2}.
\]
Therefore, we obtain for $\tilde{T}^{-1} = \tilde{\nu}\alpha^{-1}$ the Riccati type equation
\begin{equation}
d\tilde{T}^{-1} = (dN)^\prime a\frac{a - 1}{2} - dfH\tilde{T}^{-1},
\end{equation}
and the derivative of $\tilde{N} = -\tilde{T}N\tilde{T}^{-1}$ can be computed as
\[
d\tilde{N} = (\tilde{N} - N)dfH\tilde{T}^{-1} + \tilde{T}(dN)^\prime \left(\frac{N\tilde{a} - 1}{2} - \tilde{a} - 1\right) - \tilde{T}dN\tilde{T}^{-1}.
\]
The $(1,0)$-part of $d\tilde{N}$ with respect to the complex structure $\tilde{N}$ is
\begin{equation}
(d\tilde{N})^\prime = -\tilde{T}(dN)^\prime \left(\frac{a - 1}{2} \tilde{N} + \frac{\tilde{b}}{2}\right) = -(dT + dfH),
\end{equation}
where we used that $(\tilde{a} - 1)\tilde{N} + \tilde{b} = (1 - a)\tilde{T}$ by the definition (5.4) of $\tilde{T}$, and the Riccati type equation (5.5). In particular, since $\tilde{N}$ is harmonic and $dfH = -(dN)^\prime$ we see from (5.6) that $(d\tilde{N})^\prime$ is closed. In other words, the left normal $\tilde{N}$ of a $\mu$-Darboux transform of $f$ is harmonic.

We conclude the proof by showing that every conformal immersion $f^\sharp : M \to \mathbb{R}^4$ with harmonic left normal $N^\sharp$ is constrained Willmore. From [4] and the explicit formulae for the mean curvature sphere congruence of an immersion in [5, Prop. 15] we see that $f^\sharp$ is constrained Willmore if and only if there exists $\eta \in \Omega^1(\mathbb{H})$ with
\begin{equation}
*\eta = -R^\sharp\eta = \eta N^\sharp \quad \text{and} \quad d\left(\eta + dh^\sharp + R^\sharp \star dh^\sharp + \frac{1}{2}H^\sharp(N^\sharp dN^\sharp - \star dN^\sharp)\right) = 0,
\end{equation}
where $N^\sharp$ and $R^\sharp$ are the left and right normal of $f^\sharp$, and $H^\sharp$ is given by $(dN^\sharp)^\prime = \frac{1}{2}(dN^\sharp - N^\sharp \star dN^\sharp) = -df^\sharp H^\sharp$. Since $N^\sharp$ is harmonic we have $-df^\sharp \wedge dh^\sharp = d(df^\sharp H^\sharp) = 0$.
which implies that \( *dH^2 = -R^2 dH^2 \). Therefore, \( \eta := -\frac{1}{2}H^2(N^2 dN^2 - *dN^2) \) is a 1–form satisfying (5.7), and every branched conformal immersion with harmonic left normal is constrained Willmore. In particular, every \( \mu \)-Darboux transform of a Hamiltonian Lagrangian immersion \( f \) is constrained Willmore. \( \square \)

In the case when \( M = T^2 \) is a 2–torus, the closed \( \mu \)-Darboux transforms of a Hamiltonian stationary torus are exactly the monochromatic Darboux transforms: To obtain parallel sections of \( d^\mu \) we use again the gauge \( \alpha = e^{i\tilde{\alpha}} \tilde{\alpha} \), and perform a similar computation as in [19]. From \( d^\mu \alpha = 0 \) we see \( d\alpha = (dN)^\mu T\alpha \) and (5.1) then shows

\[
d\tilde{\alpha} = -\frac{i}{4}(2d\beta \tilde{\alpha} + (d\beta + i* d\beta) \tilde{\alpha}(a - 1) + (d\beta - id\beta) \tilde{\alpha}b).
\]

Decomposing \( \tilde{\alpha} = \alpha_0 + j\alpha_1 \) we get the differential equation

\[
d \left( \begin{array}{c} \alpha_0 \\ \alpha_1 \end{array} \right) = \frac{1}{4} \begin{pmatrix} 0 & d\beta(\mu + 1) - *d\beta(\mu - 1) \\ -d\beta(\mu - 1) + *d\beta(\mu + 1) & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix},
\]

where we used \( \mu = a + ib \) and \( \mu^{-1} = a - ib \). If \( M = T^2 \) is a 2–torus, the solution space of the differential equation is for each \( \mu \in \mathbb{C}_* \) 2–dimensional: writing \( \beta(z) = 2\pi(\beta_0, z) \) the general solution is

\[
\tilde{\alpha} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \beta_0 \end{pmatrix} e^{2\pi(xu + yv)} + \begin{pmatrix} 1 \\ -i \beta_0 \end{pmatrix} e^{-2\pi(xu + yv)}
\]

with \( \alpha, \beta \in \mathbb{C}, u = \frac{1}{4}(\beta_0 \sqrt{\mu} + \beta_0 \sqrt{\mu}^{-1}) \) and \( v = \frac{1}{4}(\beta_0 \sqrt{\mu}^{-1} - \beta_0 \sqrt{\mu}) \). Putting

\[
A^\mu = \text{Re} u + i \text{Re} v = \frac{i\beta_0}{4} (\sqrt{\mu}^{-1} - \sqrt{\mu})
\]

and

\[
C^\mu = \text{Im} u + i \text{Im} v = \frac{\beta_0}{4} (\sqrt{\mu}^{-1} + \sqrt{\mu}),
\]

we see that

\[
\alpha_\pm^\mu(z) = \frac{e^{i2\pi z}}{4}(1 \mp k \sqrt{\mu}^{-1}) e^{\pm2\pi((A^\mu, z) + i(C^\mu, z))}
\]

are holomorphic sections with multiplier

\[
h_\pm^\mu(\gamma) = e^{\pm2\pi((A^\mu, \gamma) + i(\mp \frac{\beta_0}{4} + C^\mu, \gamma))}.
\]

In other words, every \( \mu \in \mathbb{C}_* \) gives two monochromatic, \( \mathbb{C} \)-independent holomorphic sections \( \alpha_\pm^\mu \) with multipliers \( h_\pm^\mu \) satisfying (2.18) with \( \delta = \frac{\beta_0}{2}, A = \pm A^\mu, \delta - B = \pm C^\mu, \) and

\[
\lambda_\delta = \frac{2}{\beta_0}(\delta - iA - B) = \pm \frac{1}{\sqrt{\mu}}.
\]

Conversely, we see that every monochromatic holomorphic section is parallel for some \( d^\mu \):

**Lemma 5.3.** For every monochromatic holomorphic section \( \alpha \) there is a unique \( \mu \in \mathbb{C}_* \) such that \( \alpha \) is \( d^\mu \)-parallel.

**Proof.** Every monochromatic holomorphic section \( \alpha \in H^0_r(\mathbb{H}) \) with multiplier \( h \) is given (2.18) by a complex scale of

\[
\alpha_\delta = e^{\frac{\beta_0}{2}(1 - k\lambda_\delta)} e^{2\pi(A, \cdot)}
\]
where $\delta \in \Gamma_{A,B}^*$, $\lambda_\delta = \frac{2}{\sqrt{\mu}}(\delta - iA - B)$ and $h = h^{A,B}$. Since $\delta - B$ is independent of the choice of pair $(A, B)$ with $h = h^{A,B}$ both $\lambda_\delta = \frac{2}{\sqrt{\mu}}(\delta - iA - B)$ and $\mu = \frac{1}{\sqrt{\mu}}$ are uniquely defined by $\alpha$. As in the proof of Proposition 2.14 we obtain from (2.18) and (2.19) that

$$\alpha_\delta = e^{i\beta}(1 \mp k\sqrt{\mu})e^{2\pi(A,)}$$

with $A = \pm \frac{i}{\sqrt{\mu}}(\sqrt{\mu - 1} - \sqrt{\mu})$ and $\delta - B = \pm \frac{b}{\sqrt{\mu}}(\sqrt{\mu - 1} + \sqrt{\mu})$. Comparing with (5.10) we see that $\alpha_\delta = \alpha_+ \mu$ or $\alpha_\delta = \alpha_- \mu$. But this proves that every monochromatic holomorphic section $\alpha = \alpha_\delta c$, $c \in \mathbb{C}$, is parallel with respect to $d^\mu$.

**Corollary 5.4.** The monodromy $H^\mu$ of the flat connection $d^\mu$ is a complex multiple of the identity if and only if $\mu \in S^1$ with $\beta_0\sqrt{\mu} \in \Gamma^*$.

**Proof.** For a given $\mu$ we have a 2-dimensional space of parallel sections with the same multiplier if and only if $h_+^\mu = h_-^\mu$ which is equivalent with (5.11), (5.8) and (5.9) to $A^\mu = 0$ and $C^\mu = \frac{2}{\sqrt{\mu}} \in \frac{1}{2}\Gamma^*$. Finally, $A^\mu = 0$ is equivalent to $\mu \in S^1$. \hfill $\square$

From (5.3) we see that for $\mu \in S^1$ the connection $d^\mu$ is in fact quaternionic. In particular, parallel sections are quaternionically dependent. In other words, for all $\mu \in S^1$ every $d^\mu$–parallel section gives the same $\mu$–Darboux transform $\hat{f}$. In particular, even if $H^\mu$ is a multiple of the identity, we can choose either one of $\alpha_\mu$ to get the $\mu$–Darboux transform $\hat{f}$ by prolongation. We summarize the previous discussions:

**Theorem 5.5.** Every closed $\mu$–Darboux transform $\hat{f}: T^2 \to \mathbb{R}^4$ of a Hamiltonian stationary torus $f: T^2 \to \mathbb{R}^4$ is a monochromatic Darboux transform of $f$, and vice versa.

**Remark 5.6.** Again, we should point out that we only consider Darboux transforms closing on the original lattice.

### 6. The Spectral Curve

In this section we show that the multiplier spectral curve $\Sigma$ of a Hamiltonian stationary torus $f: T^2 \to \mathbb{R}^4$ and the spectral curve $\Sigma_e$ of the harmonic complex structure given by $f$ coincide: Given the $\mathbb{C}$–family of complex flat connections $d^\mu$ of a harmonic complex structure $J$, the spectral curve of $J$ is defined to be the normalization $\Sigma_e$ of the set of eigenvalues

$$\operatorname{Eig} = \{ (\mu, h^\mu) \mid \text{there exists } \alpha \in \Gamma(\mathbb{H}) \text{ with } d^\mu \alpha = 0, \gamma^* \alpha = \alpha h^\mu \}.$$ 

of the monodromy $H^\mu$ of the flat connections $d^\mu$. The eigenlines of $H^\mu$ over points where $H^\mu$ is diagonalizable with two different eigenvalues $h_\pm^\mu$ extend [16] to a line bundle $\mathcal{E}$ over $\Sigma_e$. In our situation, when $J$ is the complex structure induced by the harmonic left normal of a Hamiltonian stationary torus, the spectral curve of $J$ has spectral genus zero [19]:

**Theorem 6.1.** Let $f: T^2 \to \mathbb{R}^4$ be a Hamiltonian stationary torus and $J$ the harmonic complex structure on $\mathbb{H}$ given by left multiplication by the left normal $N = e^{3\beta i}$ of $f$. Then the spectral curve $\Sigma_e$ of $J$ compactifies to $\Sigma_e = \mathbb{C}P^1$ by adding points $x_0$ and $x_\infty$ over $\mu = 0$ and $\mu = \infty$ respectively. Moreover, the map $\mu: \Sigma_e \to \mathbb{C}P^1, (\mu, h^\mu) \mapsto \mu$ is a 2–fold covering over $\mathbb{C}P^1$ branched over 0 and $\infty$, and the eigenline bundle $\mathcal{E}$ extends holomorphically to $\Sigma_e$. 
\textbf{Proof.} We denote by
\[ \text{Eig}_0 = \{ (\mu, h^\mu) \mid H^\mu \neq h^\mu \text{ Id} \} \]
the subset of Eig such that every \( x = (\mu, h^\mu) \in \text{Eig}_0 \) has a unique, up to complex scale, \( d^h \)-parallel section with multiplier \( h^\mu \). We already have seen in Corollary 5.4 that \( \text{Eig}_0 = \text{Eig} \setminus \text{Eig}_0 \), where
\[ \text{Eig}_0 = \{ (\mu, h^\mu) \mid \mu \in S^1 \text{ with } \beta_0\sqrt{\mu} \in \Gamma^* \} \]
is finite. By (5.10) the eigenlines \( \mathcal{E}_{(\mu, h^\mu)} = \alpha_{\pm, \mu}^\mu \mathbb{C} \), \( (\mu, h^\mu) \in \text{Eig}_0 \), extend holomorphically to \( \text{Eig}_0 \), and thus define a line bundle \( \mathcal{E} \) over Eig. Furthermore, for \( \mu \in S^1 \) with \( \beta_0\sqrt{\mu} \in \Gamma^* \) we have (5.10) two distinct limiting lines which shows that the normalization of Eig are two distinct copies of \( \mathbb{C} \).

Since parallel sections are holomorphic we have a natural holomorphic map \( \mathcal{E}(\mu, h^\mu) \rightarrow \mathcal{E}_{x_0} := e^{i\frac{\beta}{2}}j \mathbb{C} \), \( (\mu, h^\mu) \rightarrow x_0 \), and
\[ (6.1) \]
and
\[ \mathcal{E}(\mu, h^\mu) \rightarrow \mathcal{E}_{x_\infty} := e^{i\frac{\beta}{2}}\mathbb{C} \), \( (\mu, h^\mu) \rightarrow x_\infty \).
\[ \square \]

\textbf{Remark 6.2.} There is a fixed point free real structure \( \rho(\mu, h^\mu) := (\frac{1}{\mu}, \bar{h}^\mu) \) on \( \bar{\Sigma}_e \) which is compatible with the eigenline bundle \( \mathcal{E} \) since \( d^h(\alpha j) = (d^\mu \alpha)j \).

\textbf{Lemma 6.3.} Let \( \Sigma_e \) be the spectral curve of the harmonic complex structure \( J \) of a Hamiltonian stationary torus \( f \), and \( \mathcal{E} \) the eigenline bundle over \( \Sigma_e \). Then the complex structure \( J \) can be recovered from \( \Sigma_e \) and \( \mathcal{E} \) by quaternionically extending \( J \varphi = \varphi_i, \varphi \in \mathcal{E}_{x_\infty} \). Moreover, the Hamiltonian stationary torus \( f \) is obtained as the limit of the Darboux transforms \( f^x \) when \( x \in \Sigma_e \) goes to \( x_\infty = \mu^{-1}(\infty) \in \Sigma_e \).

\textbf{Proof.} The first statement follows from \( J e^{i\frac{\beta}{2}} = Ne^{i\frac{\beta}{2}} = e^{i\frac{\beta}{2}}i \) and (6.1). For the second, we observe that (5.8) shows that \( |A^\mu| = |\beta| \sqrt{\frac{1}{2} - \sqrt{\mu} - \sqrt{\bar{\mu}}} \rightarrow \infty \) as \( \mu \rightarrow \infty \) and (3.2) therefore gives \( f^x = f + T^x \) with
\[ |T^x| = \frac{|g|}{\pi \sqrt{4|A^\mu|^2 + \langle \beta_0, A^\mu \rangle^2}} \rightarrow 0 \quad \text{as } x \rightarrow x_\infty, \quad \mu = \mu(x) \).
\[ \square \]

Since parallel sections are holomorphic we have a natural holomorphic map
\[ h : \text{Eig} \rightarrow \text{Spec}, (\mu, h^\mu) \mapsto h^\mu. \]

Lemma 5.3 and Theorem 2.2 show that this map is surjective. By definition, every point \( x \in \text{Eig}_0 \) gives, up to complex scale, a unique monochromatic holomorphic section with multiplier \( h = h(x) \). If \( x, \bar{x} \in \text{Eig}_0 \) with \( h(x) = h(\bar{x}) \) then \( \dim_{\mathbb{C}} H^\mu_{h(\bar{x})} > 1 \). On the other hand, \( H^\mu_{h(x)} \) is spanned by monochromatic holomorphic sections, and each monochromatic holomorphic section \( \alpha \) defines by Lemma 5.3 a unique \( \mu \) with \( d^\mu \alpha = 0 \). In particular, for \( h \notin \text{Spec}_0 \), that is \( \dim_{\mathbb{C}} H^\mu_{h(x)} \neq 1 \), the preimage of \( h \) in \( \text{Eig}_0 \) is finite. Since \( \text{Spec} \setminus \text{Spec}_0 \) is a discrete subset in Spec this shows that \( h \mid \text{Eig}_0 \) is injective away from a discrete set, and thus \( h : \text{Eig} \rightarrow \text{Spec} \) is also injective away from a discrete set since
Eig\(^0\) = Eig \setminus Eig_0 is finite. Therefore, the induced map \(h : \Sigma_e \to \Sigma\) on the normalizations of Eig and Spec respectively is biholomorphic.

**Theorem 6.4.** The multiplier spectral curve \(\Sigma\) of a Hamiltonian stationary torus \(f : T^2 \to \mathbb{R}^4\) and the eigenline spectral curve \(\Sigma_e\) of the harmonic left normal \(N\) of \(f\) are biholomorphic. In particular, the multiplier spectral curve compactifies to \(\bar{\Sigma} = \mathbb{CP}^1\).

Moreover, the kernel bundle \(\mathcal{L}\) of \(f\) and the eigenline bundle \(\mathcal{E}\) of \(N\) coincide, and the spectral curve \(\Sigma\) parametrizes a \(\mathbb{CP}^1\)–family of Hamiltonian stationary tori isospectral to \(f\).

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