Entanglement, Coherence, and Extractable Work in Quantum Batteries

Hai-Long Shi,1,2,3† Shu Ding,1∗ Qing-Kun Wan,2,3 Xiao-Hui Wang,1,4,5‡ and Wen-Li Yang6,4,5∗

1School of Physics, Northwest University, Xi’an 710127, China
2State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, APM, Chinese Academy of Sciences, Wuhan 430071, China
3University of Chinese Academy of Sciences, Beijing 100049, China
4Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xi’an 710127, China
5Peng Huanwu Center for Fundamental Theory, Xi’an 710127, China
6Institute of Modern Physics, Northwest University, Xi’an 710127, China

(Dated: September 26, 2022)

We investigate the connection between quantum resources and extractable work in quantum batteries. We demonstrate that quantum coherence in the battery or the battery-charger entanglement is a necessary resource for generating nonzero extractable work during the charging process. At the end of the charging process, we also establish a tight link of coherence and entanglement with the final extractable work: coherence naturally promotes the coherent work while coherence and entanglement inhibit the incoherent work. We also show that obtaining maximally coherent work is faster than obtaining maximally incoherent work. Examples ranging from the central-spin battery and the Tavis-Cummings battery to the spin-chain battery are given to illustrate these results.

PACS numbers: 03.67.-a, 03.65.-w

Introduction.— The rapid development of modern technology to probe and manipulate physical systems at the qubit level requires a new reconstruction of three laws of thermodynamics by including dominant quantum features [1–11]. One of the most serious tasks is to identify the impact of quantum resources, such as entanglement [12–18], coherence [19–26], and discord [27–29], on extractable work. Quantum resource theories [30–34] provide a possible route to deal with it, but troublesome couplings among different quantum resources that appear in finite-time nonequilibrium thermodynamic processes make the problem complicated and intractable. Different quantifiers of extractable work based on free energy, ergotropy, or process-dependent work operators may cause further confusion [10, 16, 30, 35–39]. Recent studies on quantum batteries have contributed greatly to the investigation of these problems [13, 17, 18, 24, 40, 41].

Quantum batteries (QBs) are designed to store energy efficiently for subsequent use by exploiting various quantum resources. A paradigmatic model of a quantum battery has been experimentally implemented based on an organic microcavity [42]. Subject to the second law of thermodynamics, not all energy injected into the battery can be extracted, and the useful part is called extractable work. Up to now, considerable attention has been mostly focused on identifying useful quantum resources that speed up energy charging dynamics or improve the storage of extractable work. Distinguished from individual QB cells, collective QB cells do display a quantum correlation-induced speedup in the energy charging process [43–48]. Recent works focusing on the Tavis-Cummings (TC) battery and the central-spin (CS) battery have illustrated an unfavorable effect of the battery-charger entanglement on the storage of extractable work [13] [18]. Another effort is to isolate the quantum correlation contributions to ergotropy [24] [29]. Despite such progress, it remains vital to establish some general connections between entanglement, coherence, and extractable work.

In this Letter, we shed some light on the clarification of the role of entanglement and coherence on extractable work by providing a model-independent analysis for general QBs. We present Theorem 1 to emphasize that generating nonzero extractable work requires nonvanishing entanglement or coherence during the charging process. By restricting to incoherent QBs, whereby only diagonal elements are involved, we prove an inverse relationship between the final extractable work and the battery-charger entanglement; see Theorem 2. For a general QB, inspired by Refs. [24] [29], we treat extractable work as a sum of incoherent (extractable) work and coherent (extractable) work. We show that the former is limited by the diagonal entropy of the battery, i.e., the sum of entanglement and coherence, while the latter is promoted by the battery’s coherence. Furthermore, we argue that achieving maximally coherent work is faster than achieving maximally incoherent work. We illustrate our results in some concrete models: the CS battery, the TC battery, and the XXZ battery. These results hold regardless of using free energy work or ergotropy as the quantifier of extractable work. Our work also deepens the understanding of the role of quantum resources played in energy transfer and work extraction.

Preliminaries.— A general Hamiltonian for describing
the charging process can be written as
\[ \mathcal{H}(t) = \mathcal{H}_b + \mathcal{H}_c + \mathcal{V}(t), \]
where the local Hamiltonians \( \mathcal{H}_b \) and \( \mathcal{H}_c \) characterize the battery part and the charger part, respectively. The charging operator \( \mathcal{V}(t) \) incorporates all terms that control the switch of the energy injection, such as battery-charger interactions or some external driven fields. At time \( t = 0 \), the whole system is prepared in a product state \( \rho_{bc}(0) \) with the battery being the ground state \( |\varepsilon_n\rangle \) of \( \mathcal{H}_b \). We then suddenly turn on \( \mathcal{V} \) and aim to inject as much energy as possible into the battery for a finite time interval \([0, T]\). Such a time interval \( T \) is called the charging time. The evolving state is given by \( \rho_{bc}(t) = U(t)\rho_{bc}(0)U^\dagger(t) \) with \( U(t) = \mathcal{T} \exp(-i \int_0^t \mathcal{H}(t) dt) \).

An important physical quantity is the energy that can be extracted from the battery state \( \rho_{bc}(t) \) and a specified thermal state \( \tau_\beta = \exp(-\beta \mathcal{H}_b)/Z \)
\[ W_f(t) = F(\rho_{bc}(t)) - F(\tau_\beta), \]
where \( \beta \) is the inverse temperature and the free energy is \( F(\rho_{bc}) = \text{Tr}(\rho_{bc}\mathcal{F}) \) with \( \mathcal{F} = \mathcal{H}_b + \beta^{-1} \log \rho_{bc} \).

The first extracted work quantifier is given by the free-energy difference between \( \rho_{bc}(t) \) and a specified thermal state \( \tau_\beta = \exp(-\beta \mathcal{H}_b)/Z \)
\[ W_c(t) = E(\rho_{bc}(t)) - E(\rho_{bc}(t)) \]
where \( E(\rho_{bc}) = \text{Tr}(\rho_{bc}\mathcal{F}) \) and \( \mathcal{U}_c \) denotes the set of all cyclic unitary transformations. A close expression for this expression can be obtained in terms of the passive state \( \tilde{\rho}_{bc}(t) = \sum_n r_n |\varepsilon_n\rangle \langle \varepsilon_n| \)
\[ W_c(t) = E(\rho_{bc}(t)) - E(\tilde{\rho}_{bc}(t)) \]
where the eigenstates of \( \mathcal{H}_b = \sum_n |\varepsilon_n\rangle \langle \varepsilon_n| \) and \( \tilde{\rho}_{bc}(t) = \sum_n r_n |\varepsilon_n\rangle \langle \varepsilon_n| \) are reordered so that \( r_0 \geq r_1 \geq r_2 \geq \ldots \)
and \( 0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \ldots \). This kind of quantum work has been experimentally measured recently in a single-atom heat engine \[59\] and a spin heat engine \[61\].

Quantum coherence and entanglement are two fundamental quantum features that reflect different manifestations of the same principle—the superposition principle. We will focus on them to explore their connections with extractable work. Coherence of the battery is characterized by the minimum relative entropy \( S(\rho_{bc}(t)|\delta) \) of \( \rho_{bc}(t) \) with respect to all incoherent states \( \delta \in \mathcal{I} \)
\[ C(t) = \min_{\delta \in \mathcal{I}} S(\rho_{bc}(t)|\delta) = S_{\Delta}(t) - S(t), \]
where the minimum is obtained by the dephased state \( S_{\Delta}[\rho_{bc}(t)] = \sum_n |\varepsilon_n\rangle \langle \varepsilon_n| \langle \varepsilon_n| \langle \varepsilon_n| \) and \( S_{\Delta}(t) = -\text{Tr}[\Delta(\rho_{bc}(t)) \log_2(\Delta(\rho_{bc}(t)))] \) is the diagonal entropy. For a bipartite pure state \( \rho_{bc}(t) \), the Von Neumann entropy of the battery’s reduced density matrix \( S(t) = -\text{Tr}[\rho_{bc}(t) \log_2(\rho_{bc}(t))] \) characterizes its battery-charger entanglement \[53\]. Therefore, from Eq. \[4\] we see that the diagonal entropy \( S_{\Delta}(t) \) quantifies the sum of entanglement and coherence. Henceforth, entanglement always refers to the battery-charger entanglement.

Necessity of nonzero entanglement or coherence.—As proven in Ref. \[10\], the rate of extractable work changes is given by \( dW_f/dt = -\text{Tr}([\rho_{bc}(t), \mathcal{F}]\mathcal{V}) \), where \( \mathcal{V} \) describes interactions between the battery and the charger. If during the charging process \( \rho_{bc}(t) \) keeps a product state without coherence in its battery part, then \( \rho_{bc}(t), \mathcal{F} = 0 \) and thus \( W_f(t) = W_f(0) = 0 \). Here we always assume that there is no extractable work at \( t = 0 \). For general extractable work quantifiers, we can find the same result:

Theorem 1.—For a product pure state \( \rho_{bc}(0) \otimes \rho_c(0) \) as the initial state, if no battery’s coherence and no battery-charger entanglement appear during the charging process then extractable work is always zero.

The core idea behind this theorem is that quantum dynamics is heavily restricted by entanglement and coherence. In an extreme case (no entanglement and no coherence), our proof \[54\] reveals that the battery state \( \rho_{bc}(t) \) cannot be evolved into a high-energy state and thus possesses no extractable work. This theorem emphasizes the necessity of entanglement and coherence for generating nonzero extractable work. Meanwhile, it also implies that it is possible to charge a battery only using entanglement or only using coherence. The following examples (the CS battery without coherence and the XXZ battery) will confirm this possibility. In this sense, entanglement and coherence should be placed on an equal footing to treat.

So how do these resources specifically determine the behavior of extractable work? Firstly, we will focus on incoherent QBs to uncover the role of entanglement.

Incoherent quantum batteries.—Because of the vanishing coherence, the evolution of the density matrix of the incoherent battery can be parametrized as \( \rho_{bc}(t) = \sum_n r_n(t)|\varepsilon_n\rangle \langle \varepsilon_n| \) and \( r_n(T) \) denotes the population of the \( n \)-th energy level. The charging process is characterized by exciting low-energy states to high-energy states. At the end of the charging process, it is thus expected that the population of low-energy levels is not greater than that of high-energy levels, i.e., \( p_n(T) \leq p_{n+1}(T) \), for an “excellent” QB. Under this ideal assumption, we find that the final extractable work is enslaved to the battery-charger entanglement in incoherent QBs.

Theorem 2.—The final extractable work (whether defined by \( W_c(T) \) or \( W_f(T) \)) is negatively related to the entanglement \( S(T) \) for all incoherent QBs.

We now use Fig. 1(a) to explain the essence of the rigorous proof given in the Supplemental Material \[54\]. Figure 1(a) plots two energy-level configurations for charged battery states. Compared with the left configuration, the right configuration has a larger population at high-energy levels. This difference will lead to \( S(T) > S'(T) \) but \( W_{c_{\text{eff}}}(T) < W_{c_{\text{eff}}}'(T) \), which proves the theorem.

This theorem indicates that vanishing entanglement at
the end of the charging process is helpful to improve the final extractable work. It seems to contradict the previous theorem, Theorem 1. In fact, although the final entanglement is harmful, it is still necessary during the charging process to generate excited states. The continuing excitation from low-energy states to high-energy states strengthens the battery-charger entanglement until the population of the battery part displays a nearly balanced distribution. Further excitation will induce the population inversion and lead to a decrease in entanglement. Therefore, at the end of the charging process, the vanishing entanglement characterizes that the battery is located in a high-energy state and thus possesses optimal extractable work.

Example 1: Central-spin battery. – The CS quantum battery consists of $N_b$ battery cells and $N_c$ charging units, which are described by collective spin operators $S^x = \sum_{i=1}^{N_b} \sigma_i^x/2$ and $J^z = \sum_{i=1}^{N_c} \sigma_i^z/2$ ($\alpha = x, y, z$), respectively. The charging Hamiltonian reads as

$$H_{\text{CS}} = \omega S^z + \omega J^z + A(S^+J^- + S^-J^+),$$

where $A$ characterizes the flip-flop interaction and $S^\pm = S^x \pm iS^y$. We prepare the battery in the ground state $|\downarrow\downarrow\cdots\downarrow\rangle_b$ of $H_b = \omega S^z$ ($\omega > 0$) and the charger in a Dicke state $|m\rangle_c$ with $m$ spin-ups. Because of the flip-flop interaction, the battery will be excited to high-energy states at the cost of decreasing the number of spin-up charging units. An analytical treatment for this model can be found in Ref. [18]. In Fig. 1(b), we plot the evolutions of entanglement and extractable work for three chargers, namely, $m = 2, 3, 6$. We see that from $m = 2$ to $m = 6$ the final extractable work $W_{e,f}(T)$ increases, while the final entanglement $S(T)$ decreases, which is what our Theorem 2 stated. Moreover, the almost zero $S(T)$ for $m = 6$ indicates almost optimal work, i.e., $W_{e,f}(T) \simeq W_{e,f}(|\uparrow\uparrow\rangle) = 2\omega$.

General quantum batteries. – To emphasize the role of coherence, it is instructive to isolate the incoherent contribution to the ergotropy, namely, incoherent work [24],

$$W^c_e(t) = E(\Delta[\rho_b(t)]) - E(\Delta[\tilde{\rho}_b(t)]),$$

(5)

which is defined as the energy difference between the dephased state and its passive counterpart. Coherent work is naturally given by

$$W^c_f = W_e(t) - W^c_e(t) = E(\Delta[\rho_b(t)]) - E(\tilde{\rho}_b(t)).$$

(6)

An important bound uncovering the role of coherence is given by [24]

$$|W^c_e(t) - \beta^{-1}C(t)| \leq \beta^{-1}D(\tilde{\rho}_b(t)||\tau_\beta),$$

(7)

where $C(t)$ is the battery’s coherence defined in Eq. (4), $\tilde{\rho}_b(t)$ is the passive state, $\tau_\beta$ is an arbitrary thermal state with the inverse temperature $\beta$, and $D(\rho||\sigma) = \text{Tr}(|\log \rho - \log \sigma|)$ is the quantum relative entropy. If the passive state can be rewritten as a thermal state, then coherent work is exactly the coherence up to a coefficient $\beta^{-1}$. This condition naturally holds for QBs with only two energy levels, but it is not satisfied for general QBs. Although it has been shown that $W^c_e$ is not a coherence monotone in general [23], we still find that the battery’s coherence can qualitatively describe the change of coherent work. Figures 2(c) and 2(d) show that coherent work $W^c_e(t)$ is positively related to coherence $C(t)$ in the TC battery and the XXZ battery.

To unify the different extractable work quantifiers, we take the same strategy to define the incoherent work for the free energy:

$$W^f_e(t) = F(\Delta[\rho_b(t)]) - F(\tau_\beta).$$

(8)

Corresponding coherent work is given by

$$W^f_f(t) = W_f(t) - W^f_e(t) = F(\rho_b(t)) - F(\Delta[\rho_b(t)]).$$

(9)

By noticing that $\rho_b(t)$ and $\Delta[\rho_b(t)]$ have the same energy, we immediately see an exact correspondence between free-energy-based coherent work $W^f_f(t)$ and the battery’s coherence $C(t)$

$$W^f_f(t) = \beta^{-1}C(t).$$

(10)

Equations (7) and (10) indicate that quantum coherence can improve coherent work. So what quantum resource is responsible for incoherent work?

From Eqs. (5) and (6), we see that incoherent work can be understood as the extractable work from the dephased state $\Delta[\rho_b(t)]$. Therefore, Theorem 2 is applicable to analyze incoherent work by replacing the von Neumann entropy $S(T)$ with the diagonal entropy $S^c(\Delta[T])$. Explicitly, the smaller the diagonal entropy $S^c(\Delta[T])$, the larger the incoherent work $W^c_{e,f}(T)$. Recall from what we discussed earlier that diagonal entropy equals entanglement plus...
coherence. So coherence is now detrimental to incoherent work. The distinct effects of coherence on coherent work and incoherent work explain why it is so difficult to clarify the role of coherence in extractable work. The negative relation between incoherent work and the diagonal entropy also reveals that different quantum features will couple to each other and determine the performance of QBs together.

To obtain optimally incoherent work, it is thus expected that the battery after charging has vanishing coherence and vanishing entanglement, i.e., $S_\Delta(T) = 0$. Thus, incoherent quantum batteries are promising candidates, e.g., the CS battery. We also observe from the definitions of extractable work [Eqs. (2) and (3)] that optimal work can be achieved when $\rho_b(T)$ is located in the highest energy state, which also requires that $S_\Delta(T) = 0$. So we identify optimally incoherent work with optimal work. On the other hand, to obtain optimally coherent work, it is naturally expected that $\rho_c(T)$ is in a maximally coherent state. Because of the complementarity between maximal coherence and entanglement [55], we give priority to QBs that are charged by only some external fields since no chargers ensures no entanglement. Next we will give concrete examples to achieve optimally coherent and incoherent work.

Example 2: TC battery and XXZ battery. The charging Hamiltonian of the TC battery reads as $\mathcal{H}_{TC} = \omega S^z + \omega a^\dagger a + g(S^+a + S^-a^\dagger)$.

Different from the CS battery, the TC battery uses photons as the charger, which is characterized by the creation and annihilation operators $a^\dagger$ and $a$. Performance of the TC battery heavily relies on the probability distribution of the optical field initial state in the number states since the whole Hilbert space can be split into a direct sum of invariant subspaces, each containing a component of the initial state $|\psi\rangle_b = \sum_{n=0}^{\infty} \alpha_n | - N_b/2 \rangle_n \rangle_c$. Here the quantum states of the battery part and the charger part are expressed in terms of the Dicke states and the Fock states, respectively. We introduce the probability amplitudes $\alpha_n^c = e^{-N_c/2} N_c^{n/2} / \sqrt{n!}$ and $\alpha_n^b = \delta_{n,N_b}$ to denote the coherent state and the Fock state with $N_c$ photons, respectively. The analytic form for the time evolution operator within the invariant subspaces is given in Ref. 50. 57.

For a two-cell TC battery ($N_b = 2$), time evolution of the population of the highest energy state $|1\rangle_b$ is given by

$$\rho_{11}(t) = \sum_{n=2}^{\infty} \alpha_n^b \left[ \frac{\sqrt{b_n b_{n-1}}}{b_n + b_{n-1}} - \frac{\sqrt{b_{n-1} b_n}}{b_n + b_{n-1}} \cos(\Delta_n t) \right]^2$$

where $b_n = 2n$ and $\Delta_n = g\sqrt{b_n + b_{n-1}}$. It immediately follows that the optimal work, i.e., $W_{e,f}(T) \simeq 2\omega$ or $\rho_{11}(T) = 1$, can be achieved when we use the Fock state as the charger and assume the number of photons $N_c \gg 1$. We see from Fig. 2(a) that this result holds even for $N_c = 8$ and the optimal work corresponds exactly to the disappearance of diagonal entropy. Different from the Fock state case, however, coherent work originating from the off-diagonal elements will exist in the coherent state case; see Fig. 2(c). Although coherence vanishes after the charging process, the optimal work still cannot be achieved due to the nonvanishing diagonal entropy, see Figs. 2(b) and 2(c).

Figure 2(c) also shows that the battery’s coherence reaches its maximum but is not maximally coherent. To achieve the optimally coherent work, we consider the XXZ quantum battery [17]. The corresponding Hamiltonian reads as

$$\mathcal{H}_{XXZ}(t) = \mathcal{H}_b + \mathcal{V}(t), \quad \mathcal{H}_b = \frac{\omega}{2} \sum_{j=1,2} \sigma_j^z,$$

$$\mathcal{V}(t) = J (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \Omega \sum_{j=1,2} [\cos(\omega t) \sigma_j^x + \sin(\omega t) \sigma_j^y].$$

Notice that this QB is charged only via an external field instead of a charger, and thus no entanglement exists. Thus, the diagonal entropy is precisely the coherence. As shown in Fig. 2(d), the maximally coherent state with coherence $C = 2$ is achieved at $t = T/2$, and the coherent work $W_e$ is half of the optimal work $W(|\uparrow\uparrow\rangle) = 2$. The shortened time comes from the fact that the optimally coherent work only requires a balanced distribution of population instead of the battery being the highest energy state.

Discussion and conclusion. We have demonstrated
the role of the battery-charger entanglement and the battery’s coherence played in extractable work for general QBs. We find that nonvanishing entanglement or coherence during the charging process is necessary for nonzero extractable work. Instead, at the end of the charging process, entanglement and coherence become obstacles to the final extractable work. For general QBs, extractable work can be decomposed into incoherent and coherent parts. The former is negatively related to not only the battery-charger entanglement but also the battery’s coherence, which can be characterized by the diagonal entropy. The latter is positively related to the coherence in the battery part. Therefore coherent work requires a nearly balanced population to ensure large overlaps of different energy levels, but incoherent work only requires a population inversion. Recent work is devoted to realizing such controls [58]. Our model-independent analysis applies to extractable work, whether it is defined by free energy or ergotropy. Our work deepens the understanding of quantum correlations and extractable work in QBs.

Acknowledgments

This work was supported by the NSFC key Grant No. 12134015, the NSFC (Grants No. 12275215, No. 11875220, No. 11874393, No. 12121004, No. 12047502, No. 11975183 and No. 12175178), the Major Basic Research Program of Natural Science of Shaanxi Province (Grant No. 2021JJCW-19), and the Double First-Class University Construction Project of Northwest University.

[1] F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso, *Thermodynamics in the Quantum Regime*, (Springer Nature Switzerland, Cham, 2018).
[2] G. N. Hatsopoulos and E. P. Gyftopoulos, *Found. Phys.* 6, 127 (1976).
[3] S. Toyabe, T. Sagawa, M. Ueda, E. Munevuki, and M. Sano, *Nat. Phys.* 6, 988 (2010).
[4] J. Baugh, O. Moussa, C. A. Ryan, A. Nayak, and R. Laflamme, *Nature (London)* 438, 470 (2005).
[5] A. Alemany and F. Ritort, *Europhysics News* 41, 27 (2010).
[6] F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, *Proc. Natl. Acad. Sci. U.S.A.* 112, 3275 (2015).
[7] M. P. Müller, *Phys. Rev. X* 8, 041051 (2018).
[8] P. Cwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, *Phys. Rev. Lett.* 115, 210403 (2015).
[9] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, *Phys. Rev. X* 5 021001 (2015).
[10] P. Skrzypczyk, A. J. Short, S. Popescu, *Nat. Commun.* 5, 4185 (2014).
[11] M. N. Bera, A. Riera, M. Lewenstein, and A. Winter, *Nat. Commun.* 8, 2180 (2017).
[12] M. Perarnau-Llobet, K. V. Hovhannisyan, M. Huber, P. Skrzypczyk, N. Brunner, and A. Acín, *Phys. Rev. X* 5, 041011 (2015).
[13] G. M. Andolina, M. Keck, A. Mari, M. Campisi, V. Giovannetti, and M. Polini, *Phys. Rev. Lett.* 122, 047702 (2019).
[14] R. Alicki and M. Fannes, *Phys. Rev. E* 87, 042123 (2013).
[15] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. Lett.* 89, 180402 (2002).
[16] A. E. Allahverdyan, R. Balian, and Th. M. Nieuwenhuizen, *Europhys. Lett.* 67, 565 (2004).
[17] F. H. Kamin, F. T. Tabesh, S. Salimi, and A. C. Santos, *Phys. Rev. E* 102, 052109 (2020).
[18] J.-X. Liu, H.-L. Shi, Y.-H. Shi, X.-H. Wang, and W.-L. Yang, *Phys. Rev. B* 104, 245418 (2021).
[19] M. Lostaglio, D. Jennings, and T. Rudolph, *Nat. Commun.* 6, 6383 (2015).
[20] R. Uzdin, A. Levy, and R. Kosloff, *Phys. Rev. X* 5, 031044 (2015).
[21] K. Korzekwa, M. Lostaglio, J. Oppenheim, and D. Jennings, *New J. Phys.* 18, 023045 (2016).
[22] F. Caravelli, B. Yan, L. P. García-Pintos, A. Hamma, *Quantum* 5, 505 (2021).
[23] B. Cakmak, *Phys. Rev. E* 102 042111 (2020).
[24] G. Francica, F. C. Binder, G. Guarnieri, M. T. Mitchison, J. Goold, and F. Plastina, *Phys. Rev. Lett.* 125, 180603 (2020).
[25] J. Monsel, M. Fellous-Asiani, B. Huard, and A. Auffèves, *Phys. Rev. Lett.* 124, 130601 (2020).
[26] S. F. E. Oliviero, L. Leone, F. Caravelli, and A. Hamma, *SciPost Phys.* 10, 076 (2021).
[27] G. Manzano, F. Plastina, and R. Zambrini, *Phys. Rev. Lett.* 121, 120602 (2018).
[28] G. Francica, J. Goold, F. Plastina, and M. Paternostro, *npj Quantum Inf.* 3, 12 (2017).
[29] G. Francica, *Phys. Rev. E* 105, L052101 (2022).
[30] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, *Phys. Rev. Lett.* 111, 250404 (2013).
[31] E. Chitambar and G. Gour, *Rev. Mod. Phys.* 91, 025001 (2019).
[32] A. Streltsov, G. Adesso, and M. B. Plenio, *Rev. Mod. Phys.* 89, 041003 (2017).
[33] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* 81, 865 (2009).
[34] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, *J. Phys. A: Math. Theor.* 49, 143001 (2016).
[35] W. Niedenzu, M. Huber, E. Boukobza, *Quantum* 3, 195 (2019).
[36] J. Åberg, *Phys. Rev. Lett.* 113, 150402 (2014).
[37] A. E. Allahverdyan and Th. M. Nieuwenhuizen, *Phys. Rev. E* 71, 066102 (2005).
[38] P. Talkner, E. Lutz, and P. Hänggi, *Phys. Rev. E* 75, 05102(R) (2007).
[39] M. Perarnau-Llobet, E. Bäumer, K. V. Hovhannisyan, M. Huber, and A. Acín, *Phys. Rev. Lett.* 118, 070601 (2017).
[40] L. P. García-Pintos, A. Hamma, and A. del Campo, *Phys. Rev. Lett.* 125, 040601 (2020).
[41] F. Barra, *Phys. Rev. Lett.* 122, 210601 (2019).
[42] J. Q. Quach, K. E. McGhee, L. Ganzer, D. M. Rouse, B. W. Lovett, E. M. Gauger, J. Keeling, G. Cerullo, D. G.
[43] F. Campaioli, F. A. Pollock, F. C. Binder, L. Céleri, J. Goold, S. Vinjanampathy, and K. Modi, *Phys. Rev. Lett.* **118**, 150601 (2017).
[44] J.-Y. Gyhm, D. Šafránek, and D. Rosa, *Phys. Rev. Lett.* **128**, 140501 (2022).
[45] D. Ferraro, M. Campisi, G. M. Andolina, V. Pellegrini, and M. Polini, *Phys. Rev. Lett.* **120**, 236402 (2018).
[46] D. Ferraro, M. Campisi, G. M. Andolina, V. Pellegrini, and M. Polini, *Phys. Rev. Lett.* **125**, 236402 (2020).
[47] F. C. Binder, S. Vinjanampathy, K. Modi, and J. Goold, *New J. Phys.* **17**, 075015 (2015).
[48] T. P. Le, J. Levinsen, K. Modi, M. M. Parish, and F. A. Pollock, *Phys. Rev. A* **97**, 022106 (2018).
[49] Y.-H. Shi, H.-L. Shi, X.-H. Wang, M.-L. Hu, S.-Y. Liu, W.-L. Yang, H. Fan, *J. Phys. A: Math. Theor.* **53**, 085301 (2020).
[50] N. V. Horne, D. Yum, T. Dutta, P. Hänggi, J. Gong, D. Poletti, and M. Mukherjee, *npj Quantum Inf.* **6**, 37 (2020).
[51] D. von Lindenfels, O. Gräb, C. T. Schmiegelow, V. Kaushal, J. Schulz, M. T. Mitchison, J. Goold, F. Schmidt-Kaler, and U. G. Poschinger, *Phys. Rev. Lett.* **123**, 080602 (2019).
[52] T. Baumgratz, M. Cramer, and M. B. Plenio, *Phys. Rev. Lett.* **113**, 140401 (2014).
[53] M. A. Nielsen and I. Chuang, *Quantum computation and quantum information*, *Cambridge University Press* (2000); For bipartite pure states $\rho_{bc}$, the Von Neumann entropy $S(\rho_b)$ is a necessary and sufficient criterion for separability, that is to say, $S(\rho_b) = 0$ if and only if $\rho_{bc}$ is non-entangled. Moreover, $S(\rho_b)$ reaches its maximum value if $\rho_{bc}$ is a maximally entangled state. Satisfying the criterion proposed by [V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, *Phys. Rev. Lett.* **78**, 2275 (1997).] makes $S(\rho_b)$ a good entanglement measure for bipartite pure states.
[54] See Supplemental Material for proof of theorem 1 and 2.
[55] U. Singh, M. N. Bera, H. S. Dhar, and A. K. Pati, *Phys. Rev. A* **91**, 052115 (2015).
[56] X. Zhang and M. Blaauboer, [arXiv:1812.10139](http://arxiv.org/abs/1812.10139).
[57] A. Delmonte, A. Crescente, M. Carrega, D. Ferraro, and M. Sassetti, *Entropy* **23**, 612 (2021).
[58] M. T. Mitchison, J. Goold, and J. Prior, *Quantum* **5**, 500 (2021).
Supplemental materials for “Entanglement, Coherence, and Extractable Work in Quantum Batteries”

I. PROOF OF THEOREM 1 AND 2

Theorem 1.– For a product pure state $\rho_b(0) \otimes \rho_c(0)$ as the initial state, if no battery’s coherence and no battery-charger entanglement appear during the charging process then extractable work is always zero.

Proof.– Since we always assume that there is no extractable work for the initial state then it is enough to show that the battery’s state after charging is the same as the initial state $\rho_b(0)$ if no entanglement and no coherence appear during the charging dynamics. The condition of vanishing entanglement implies that the charging operator $\mathcal{V}$ involves no battery-charger interactions. Furthermore, considering that $\rho_b(0)$ is a pure state, we can write the evolving state of the battery as $|\psi(t)\rangle_b = \sum_n \alpha_n(t)|\varepsilon_n\rangle$ where $|\varepsilon_n\rangle$ are the eigenstates of $\mathcal{H}_b$. The vanishing coherence requires that, for a given $t$, only one $\alpha_n(t) = 1$ and the rest are 0. Without loss of generality, we may assume that $\alpha_0(0) = 1$ and $\alpha_n \neq 0(0) = 0$. To show $|\psi_b(T)\rangle = |\psi_b(0)\rangle$, it is enough to show that $\alpha_0(T) = 1$ where $T$ is the charging time. If this were not the case, then $\alpha_0(T) = 0$. Due to the analyticity of $\alpha_0(t)$ there must be $0 < t_* < T$ such that $0 < \alpha_0(t_*) < 1$, which implies that the battery is coherent at $t_*$. Thus we arrive at a contradiction.

Theorem 2.– The final extractable work (no matter defined by $W_e(T)$ or $W_f(T)$) is negatively related to the entanglement $S(T)$ for all incoherent QBs.

Proof.– As discussed in the main text, we only consider the “excellent” QBs whose population of low-energy levels is not greater than that of high-energy levels at the end of the charging process. Under this assumption and without loss of generality, we consider two energy level configurations for charged battery states: one is $\rho_b(T) = \sum_{n=0}^N p_n|\varepsilon_n\rangle \langle \varepsilon_n|$ with $p_0 \leq p_1 \leq \ldots \leq p_N$ and another is $\rho_b'(T) = \sum_{n=0}^N p'_n|\varepsilon_n\rangle \langle \varepsilon_n|$ with $p'_0 = p_0 - \delta$, $p'_1 = p_1 + \delta$, $p'_m = p_m$ ($m \geq 2$) for some $\delta > 0$ such that $p'_n \leq p_{n+1}$ ($n \geq 0$). Here $|\varepsilon_n\rangle$ is the $n$-th energy eigenstate of $\mathcal{H}_b$, and energy levels $\{\varepsilon_n\}$ are ordered so that $\varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_N$. It is enough to show that

$$\Delta S \equiv S(\rho_b'(T)) - S(\rho_b(T)) < 0, \quad \Delta W_{c/f} \equiv W_{c/f}(\rho_b'(T)) - W_{c/f}(\rho_b(T)) > 0,$$

for completing the proof. A simple calculation gives

$$\Delta S = -(p_0 - \delta) \log_2(p_0 - \delta) - (p_1 + \delta) \log_2(p_1 + \delta)$$
$$+ p_0 \log_2 p_0 + p_1 \log_2 p_1 < 0, \quad \Delta W_f = \delta(\varepsilon_1 - \varepsilon_0) - \beta^{-1} \Delta S > 0, \quad \Delta W_e = \delta(\varepsilon_1 - \varepsilon_0) + \delta(\varepsilon_N - \varepsilon_{N-1}) > 0.$$

The first inequality is derived by noticing that $d\Delta S/d\delta < 0$ and $\Delta S(\delta = 0) = 0$. 