Response of entanglement to annealed vis-à-vis quenched disorder in quantum spin models

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Abstract – We investigate bipartite entanglement in random quantum $XY$ models at equilibrium. Depending on the intrinsic time scales associated with equilibration of the random parameters and measurements associated with observation of the system, we consider two distinct kinds of disorder, namely annealed and quenched. We conduct a comparative study of the effects of disorder on nearest-neighbor entanglement, when the nature of randomness changes from being annealed to quenched. We find that entanglement properties of the annealed and quenched systems are drastically different from each other. This is realized by identifying the regions of parameter space in which the nearest-neighbor state is entangled, and the regions where a disorder-induced enhancement of entanglement — order-from-disorder — is obtained. We also analyze the response of the quantum phase transition point of the ordered system with the infusion of disorder.

Introduction. – For the past few decades, there has been a continued interest to understand effects of randomness, which either appear naturally or incorporated artificially, in many-body systems, leading to counterintuitive phenomena [1]. Many of the studies are directed towards investigating structural aspects [2] and cooperative phenomena in random networks, such as disorder-induced localization [3], high-$T_c$ superconductivity [4], percolation clusters [5], and rich quantum phase diagrams [6–8]. Quantum spin models with disordered parameters, in their various incarnations, are known to exhibit many of such phenomena, and can interestingly be implemented in a controlled manner in laboratories dealing with ultracold gases [9,10].

In order to obtain a meaningful value of a physical quantity in a disordered system, one must perform a configurational averaging over the disordered parameters. To obtain a general understanding of the response of the different physical quantities due to introduction of disorder, we often consider two contrasting types of disorder, viz. annealed and quenched [1,11–15], which differ by the relative magnitude of two fundamental time-scales in the system. One of them is the characteristic time, $\tau_m$, during which the system is observed, which includes a possible time-dynamics and a subsequent measurement. The other is the characteristic time, $\tau_c$, required by the disorder in the system to equilibrate.

Materials, natural or artificial, for which $\tau_m$ and $\tau_c$, corresponding to a certain disorder parameter, are of same or nearby order, are referred to as having annealed disorder (AD). To obtain an operationally meaningful observable in this case, we first need to perform a configurational average (over different realizations of the disorder) on the partition function itself and then compute the annealed averaged free energy by considering the logarithm of the averaged partition function, which finally gives the annealed averaged physical quantities.

On the other hand, for certain other materials, there can be a drastically different situation, where $\tau_c \gg \tau_m$. For this case, any disorder configuration of the system, after being realized, naturally or artificially, remains effectively frozen throughout the entire observation process. Hence,
the averaging over the random configurations need to be performed after the calculation of the physical quantities, for arbitrary given configurations, and the system under consideration is said to possess quenched disorder (QD).

Ideas from quantum information science, in particular, entanglement [16], often shed new light onto the collective properties of many-body systems, having both fundamental and technological implications [10]. Many of the works dealing with randomness in quantum networks primarily focus on quenched averaging [17–19]. In particular, effects of QD have recently been studied for quantum information characteristics, e.g., for quantum information characteristics, on quantum correlation length [23], monogamy constraints [24], etc. These results crucially depend on the assumption that randomness remains effectively frozen throughout the measurement process. However, given that the time-scale, $\tau_c$, associated with the system, say a spin network, may also be of the order of the measurement time, it is interesting to know how the equilibrium properties of disorder-averaged entanglement are affected due to such a change in the equilibrating dynamics of the disorder in the network. For such cases, as stated before, the entanglement has to be computed via annealed averaging, where the partition function itself has to be averaged over disorder parameters for different random realizations. There is only a limited body of work that attempts to understand the differences in annealed and quenched averaged physical quantities [15,25,26]. In particular, ref. [15] conducts a study on specific heat as a function of temperature in the Ising spin network, in order to find out the changes in equilibrium properties with the change in nature of disorder, and ref. [26] investigates spontaneous magnetization in the joint presence of both kinds of disorder (annealed and quenched). References [27,28] carry out studies on annealed averaged entanglement and its witness within a perturbative approach valid for weak disorder strength. It is of interest to understand the general characteristics of annealed averaged entanglement for an arbitrary disorder strength, and at the same time, to build careful understanding about the changes in entanglement properties as the nature of disorder changes from annealed to quenched. Furthermore, it is interesting to compare the effect of these drastically varying models of disorder with ordered systems.

In this work, our prime interest is to carry out a comparative analysis of the response of entanglement properties in many-body systems to the insertion of disorders, which can be annealed or quenched. We choose the random quantum transverse $XY$ spin chain, where randomness appears either in the interaction part or in the transverse field part. The investigation has been carried out via Jordan-Wigner and Bogoliubov transformation [29], which helps in accessing reasonably large systems. Our analysis shows that the entanglement properties of the system are drastically different depending on the nature of disorders, i.e., whether they are annealed or quenched. Our work identifies entangled vs. separable phases, and “enhanced” phases (disorder-induced enhancement in entanglement). We also discuss the response of quantum phase transition [7] and factorization points [30] due to the presence of the disorder.

**Annealed and quenched disorders.** – We now briefly discuss about the physical basis as well as the mathematical formalism for obtaining annealed as well as quenched averaged values of observables. Consider a Hamiltonian $\mathcal{H} = \mathcal{H}(\{a_i\})$, where $\{a_i\}, i = 1, \ldots, N$, are the disordered system parameters, so that $\{a_i\}$ can be modelled as independent and identically distributed (i.i.d.) random variables following certain probability distributions. The partition function, corresponding to a particular realization of the disordered parameters, is given by

$$Z(\{a_i\}) = \text{Tr}[e^{-\beta \mathcal{H}(\{a_i\})}], \quad (1)$$

where $\beta = 1/(\kappa_B T)$, with $\kappa_B$ and $T$ being the Boltzmann constant and the absolute temperature, respectively. Let us introduce a “functional” partition function $\tilde{Z}(\{a_i\}, \{\lambda_k\}) = \text{Tr}[e^{-\beta(\mathcal{H}(\{a_i\})+\sum_k \lambda_k A_k^i)}]$, again for the particular realization of the disordered parameters, where the additional term $\sum_k \lambda_k A_k^i$ represents a “probe” required for evaluating the expectation values of the physical quantities represented by Hermitian operators, $A^i_k$. Assuming $\tilde{Z}$ to be a sufficiently smooth function of the $\lambda_k$’s, the expectation value, for that particular realization of disorder, of the physical quantity of interest $A^i_l$ is obtained by differentiating $-\kappa_B T \ln \tilde{Z}(\{a_i\}, \{\lambda_k\})$ with respect to $\lambda_l$ at $\lambda_k = 0 \ \forall k$, where $l$ is any particular $k$. For AD, the statistical properties of the system at equilibrium are obtained by taking averages of the “functional” partition function, $\tilde{Z}$, over several random realizations. The “functional” free energy, $\tilde{F}_a$, after performing configurational averaging over the AD parameters, is given by

$$\tilde{F}_a(\{\lambda_k\}) = -\frac{1}{\beta} \ln \left\{ \int_0^\infty \prod_j da_j \mathcal{P}(a_j) \tilde{Z}(\{a_i\}, \{\lambda_k\}) \right\}, \quad (2)$$

where $\mathcal{P}(a_j)$ represents the probability distribution function of randomness in the annealed parameter $a_j$. The “annealed average” in the canonical equilibrium state of a given observable $A_l$ can finally be obtained as $\langle A_l \rangle_a = \frac{\partial \tilde{F}_a}{\partial \lambda_l} \big|_{\{\lambda_k\}=0}$. For the QD scenario, the functional free energy is computed by performing configurational averaging over the QD parameters of the logarithm of the functional partition function instead of the partition function itself, and reads as

$$\tilde{F}_q(\{\lambda_k\}) = -\frac{1}{\beta} \int_0^\infty \prod_j da_j \mathcal{P}(a_j) \ln \left\{ \tilde{Z}(\{a_i\}, \{\lambda_k\}) \right\}. \quad (3)$$

The quenched average $\langle A_l \rangle_q$ of any $A_l$ can be computed directly, via derivatives of the functional free energy, as
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\[(A_i)_q = \frac{\partial F_{p3}}{\partial N} \big|_{(\lambda_i)=0} \]. See the Supplementary Material Supplementarymaterial.pdf (SM) for more details.

Suppose now that we wish to compute the “quenched averaged” value, \(\langle O \rangle_{q}\), of a certain physical characteristic, \(O\), where the value of \(O\) for a particular physical situation is a non-linear function of the values of physical quantities represented by Hermitian operators \(A_i\) for that situation. A physically meaningful quenched averaged value of the observable \(O\) cannot be obtained from the quenched averaged values of the observables \(A_i\). To find \(\langle O \rangle_{q}\), one needs to calculate \(A_i\) and the corresponding \(O\) for each realization of the disorder \(\{a_i\}\) and then performs averaging over all such realizations, given by

\[
\langle O \rangle_q = \int_{-\infty}^{\infty} \prod_j d a_j P(a_j) \langle O \{a_j\} \rangle.
\] (4)

On the other hand, in the case of AD, the averaged out value, \(\langle O \rangle_{a}\), has to be obtained from the annealed averaged values of \(A_i\). For example, to obtain the annealed averaged entanglement, we first find the annealed averaged magnetizations and classical correlations, and construct the annealed averaged state with these values, after which we calculate the annealed entanglement as the entanglement of the annealed state.

**The model.** – In this work, we consider a one-dimensional anisotropic quantum XY model with nearest-neighbor site-dependent interactions in a random transverse magnetic field. The Hamiltonian is given by

\[
\mathcal{H} = \sum_{i=1}^{N} \frac{J_i}{4}(1+\gamma)\sigma_{i+1}^{x}\sigma_{i}^{x} + (1-\gamma)\sigma_{i+1}^{y}\sigma_{i}^{y} - \sum_{i=1}^{N} h_i \sigma_{i}^{z},
\] (5)

where \(N\) is the number of lattice sites, \(J_i\) is proportional to the coupling constant between nearest-neighbor sites \(i\) and \(i+1\), \(h_i\) is proportional to the strength of the transverse field at the \(i\)-th site, and \(\gamma \neq 0\) is the anisotropy parameter. Here \(\sigma_{i}^{\hat{a}} (\hat{a} = x, y, z)\) are the Pauli spin matrices at the \(i\)-th site. For the homogeneous system, \(J_i\) and \(h_i\) are separately equal for all pairs \((i, i+1)\) and for each site, denoted by \(J\) and \(h\), respectively. We consider the periodic boundary condition, i.e., \(\sigma_{N+1} = \sigma_1\).

In the following, we consider two different cases: **Case I:** The coupling strengths \(J_i\) are drawn from independently and identically distributed (i.i.d.) Gaussian distributions. However, the system is subjected to an site-independent uniform field. **Case II:** The interaction strength in this case is constant for all pairs. However, the \(h_i\) are now i.i.d. Gaussian random variables.

**Annealed vs. quenched entanglement in the random XY model.** – In this work, we choose concurrence (see SM) [31] as the measure of bipartite (two-qubit) entanglement for analyzing the behavior of nearest-neighbor states in AD as well as QD systems, for identifying their similarities and differences, affecting their relative utilities. We also compare the physical properties of the disordered systems with the same in the corresponding ordered systems.

The means of the distributions of the random parameters in the disordered systems are adjusted to be identical to the corresponding parameters of the homogeneous system. Moreover, the standard deviations corresponding to different types of disorder for the same physical parameter (e.g., the coupling strength) in different systems are taken to be equal, so that the responses due to the disorders can be compared effectively. Below, for a given observable \(O\), we compare among \(\langle O \rangle_{a}, \langle O \rangle_{q}\), and \(\langle O \rangle\), which, respectively, are averages in the equilibrium state for AD in the coupling, QD in the same, and a constant (site-independent) coupling.

Let us first consider the case in which randomness is present only in the interaction term, while \(h_i = h \forall i\). Our prime interest is to study the changes in the entanglement properties of the system, governed by the Hamiltonian given in eq. (5), as the nature of disorder changes from annealed to quenched. The random interactions, \(\gamma_i\), are chosen to be independently and identically distributed Gaussian probability distributions with mean \(\langle J \rangle\) and standard deviation \(\sigma\), where \(\langle J \rangle\) and \(\sigma\) are site-independent. Note that \(\sigma\) represents the disorder strength. The associated probability density function, \(P(\gamma_i)\), of the disordered parameters, \(\gamma_i\), is given by \(P(\gamma_i) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left[-\frac{1}{2}(\gamma_i - \langle J \rangle)^2/\sigma^2\right]\). In order to keep the disorder-averaged quantities and the corresponding ordered ones on the same footing, the concurrence in the homogeneous system is calculated between the same sites by setting \(\gamma_i = \langle J \rangle \forall i\). To study any disorder-averaged physical quantity, one typically requires a few thousand random realizations in order to obtain convergent values via configurational averaging. Throughout this work, the averaged-out quantities for the disordered systems are calculated by performing averaging over \(10^4\) random realizations.

It is important to stress a technical point about obtaining the annealed averaged magnetization and classical correlators. For obtaining the behavior of, say, the annealed averaged entanglement as a function of \(\mu = \langle J \rangle/h\), we begin by calculating the values of annealed averaged magnetization for a certain chosen set of points in the interval of interest on the \(\mu\)-axis. (The other parameters of the parameter space, given by \((\gamma, \mu, \sigma, \beta)\), are suitably chosen.) These values are then fitted to a profile using spline interpolation. The same is done for all the classical correlators. The annealed entanglements for the above chosen set of points on the \(\mu\)-axis are finally obtained by using the spline-interpolated annealed averaged magnetizations and classical correlators, in the formula for the entanglement in terms of magnetizations and classical correlators for a generic quantum state.

**Case I:** Transverse XY model with random interactions. Figure 1 shows the behavior of concurrence, \(C\), as a function of \(\mu = \langle J \rangle/h\) in the XY spin chain with \(\gamma = 0.5\) in...
the presence of AD (fig. 1(a)) and QD (fig. 1(b)), respectively. On the other hand, for the homogeneous system, μ is site-independent and it is set at \( \langle J \rangle / \hbar \). Let us start with the homogeneous system. The red solid lines in figs. 1(a) and (b) show the trends of concurrence for the homogeneous XY spin chain for the canonical equilibrium state. The states are entangled except at the \( \mu = 0 \) point and the factorization point [30], with the latter being given by \( \mu = \mu_f \equiv 1/\sqrt{1 - \gamma^2} \).

We find that the introduction of AD affects bipartite entanglement significantly, as can be seen in fig. 1(a). While in some region, entanglement increases in the presence of AD, it can get suppressed due to disorder in other regions. Below, we divide the axis of rescaled disorder strength, \( \sigma = \tilde{\sigma}/\hbar \), into two regions depending on whether the annealed averaged entanglement revives once or twice as we move along the \( \mu \)-axis. Here we fix \( \gamma = 0.5 \). A change of \( \gamma \) can quantitatively change the boundaries of these regions, although the qualitative behavior in the \((\mu, \sigma)\)-plane remains the same for \( \gamma \neq 0 \).

**Region 1** (\( \sigma < \sigma_c \)): This region is defined as that in which there are two “revivals” of entanglement after its “collapse” to zero, as we move along the \( \mu \)-axis, for a given value of \( \sigma \). A “revival” is defined as the appearance of (non-zero) entanglement after it becomes zero ("collapse"). The maximum value of \( \sigma \) for which two revivals occur is denoted by \( \sigma_c \). The first revival of entanglement on the \( \mu \)-axis always leads to a shrinking of entanglement with increasing \( \sigma \). The situation for the second revival is richer, and can lead to both shrinking as well as enhancement of entanglement with \( \sigma \). The end of the first revival and the beginning of the second happens in the vicinity of the quantum phase transition point and the factorization point of the corresponding ordered system. For the system under consideration, we find \( \sigma_c \approx 0.6 \). Note that the defining property of the critical disorder strength can equivalently be taken as the number of entangled segments on any line of constant \( \sigma \).

**Region 2** (\( \sigma > \sigma_c \)): This region is the one that is complementary to Region 1, and there is a single revival of entanglement when we walk along the \( \mu \)-axis. Here we obtain both shrinking as well as enhancement of entanglement with increasing \( \sigma \). For \( \sigma \gtrsim 0.8 \), we find that the point of entanglement revival shifts towards the lower \( \mu \) region. The opposite happens for higher \( \sigma \). Note that for a high \( \sigma \), a finite amount of entanglement survives in the presence of the AD only if the system is deep in the AFM phase of the corresponding ordered system.

The above observation is summarized in fig. 2(a), where the tract with blue stripes and that which is white, respectively represent the entangled and separable phases for the AD systems in the \((\mu, \sigma)\)-plane. There are two points on the \( \sigma = 0 \) line (ordered system), namely, \( \mu = 0 \) and \( \mu = \mu_f \), for which the entanglement vanishes in the zero-temperature state. Insertion of AD indicates that we are pushing away from the \( \sigma = 0 \) line on the \((\sigma, \mu)\)-plane, and the separability-entanglement features in the system respond by creating two “rivers” of separable states. More precisely, the two zero-entanglement points on the \( \sigma = 0 \) line develop into two finite intervals of zero entanglement on any line of (non-zero) constant \( \sigma \), as long as the...
constant is less than $\sigma_c$. For $\sigma > \sigma_c$, the two rivers meet to create a “separable sea”.

Let us now move to the case when we consider a QD system. The bipartite entanglement in this case turns out to be drastically different from that in the annealed case. Unlike annealed averaged entanglement, the quenched averaged entanglement between nearest-neighbor sites remains non-zero throughout the entire parametric stretch of $\mu$, irrespective of the strength of disorder, however small (but non-zero). See fig. 1(b). In particular, we observe that with the increase of the rescaled QD strength, $\sigma$, entanglement is generated even at the points having vanishing entanglement in the corresponding ordered Hamiltonian. As we increase $\sigma$, there is a change in the pattern of entanglement with $\mu$ —it gets flattened as a function of $\mu$. For a high enough $\sigma$ (in our case, $\sigma \gtrsim 1.3$), it saturates to a moderate non-zero value for the entire parametric stretch of $\mu$. This “frozen” entanglement with respect to $\mu$ for high enough $\sigma$ is expected, because with increasing $\sigma$, the stretch of the Gaussian distribution of the disordered parameter increases and finally the random configurations, $\mathcal{H}([J]/\hbar)$, are effectively distributed over the entire parametric regime of $\mu$, and thus become independent of the mean of the distribution.

Bipartite entanglement of the zero-temperature state can characterize the transition at $\mu = 1$ by showing a kink in its derivative. In our case, to maintain consistency with the annealed case, we choose the relative inverse temperature at $\beta h = 20$, which mimics the zero-temperature characteristic of the system. However, we find that due to finite temperature ($\beta h = 20$) and finite system size ($N = 50$), the kink in the derivative of nearest-neighbor entanglement in the zero-temperature state of the ordered system shifts to $\mu = 1.0243$ (correct to four decimal places). In figs. 1 and 3, the red solid line corresponds to concurrence or the derivative of concurrence [32] in the homogeneous case. Since we are dealing with finite temperature and finite system size, instead of a sharp kink, we observe a “blunt” minimum, characterizing the QPT, in fig. 3. The transition point at $\mu = 1.0243$ can be found from the minimum point in the derivative of concurrence (red solid line of fig. 3).

Moreover, we find that the QPT, present in the corresponding ordered system, shifts, as the minimum of $\frac{dC}{d\mu}$ indicates. Specifically, for small values of QD strength ($\sigma \lesssim 0.1$), the minimum of $\frac{dC}{d\mu}$ at $\mu = \mu_{\text{min}}$ remains almost a constant as a function of $\sigma$, but afterwards (i.e., for higher $\sigma$), it shifts towards the left (PM phase of the ordered system). See fig. 2(b). For a relatively stronger disorder ($\sigma > 0.4$), the curves for the QD concurrence gets flattened and a prominent minimum is thus unavailable.

**Disorder-induced enhancement of entanglement.** The presence of any weak QD makes the entire system entangled even when in the neighborhood of $\mu = 0$, implying that disorder helps the system to possess a higher amount of entanglement in comparison to the corresponding ordered one. This feature of “disorder-induced enhancement” is also known as “order-from-disorder” phenomenon, and has been elaborately studied in the context of several physical quantities in the past [17,22,23,33,34], especially in QD systems. In fact, there are large parameter stretches over which such phenomenon occurs. We define an “enhanced phase” as one which supports the disorder-induced-enhancement phenomenon. In contrast, “normal phase” is marked by deterioration of entanglement in the presence of randomness. These two phases in the annealed as well as QD systems are presented in fig. 4. For the AD, the enhanced region appears only in the AFM phase of the ordered system (the blue striped region of fig. 4(a)), while for QD, the same occurs in both AFM and PM phases (the maroon striped regions of fig. 4(b)).

In case of the AD system, the enhanced phase is inside the half-plane $\mu > 1$. This corresponds to the AFM phase in the ordered system. Walking along a constant $\sigma$ line, from low to high values of $\mu$, one encounters the enhanced phase at a value of $\mu$ that is greater than unity, and that is almost independent of $\sigma$. Re-entry into the normal phase, however, depends on $\sigma$. See fig. 4(a).

In the QD scenario, enhanced phases appear in both AFM and PM phases of the corresponding ordered system. We find that for a given $\sigma$, the system is in the enhanced phase for $0 < \mu < \mu_1^\sigma$ and $\mu_2^\sigma < \mu < \mu_3^\sigma$ with $\mu_1^\sigma < \mu_2^\sigma < \mu_3^\sigma$. Interestingly, for high enough $\sigma$, $\mu_1^\sigma$ and $\mu_2^\sigma$ are almost independent of $\sigma$. As depicted in fig. 4(b), the length of the enhanced phase on a constant $\sigma$ line, increases with $\sigma$. Our analysis shows that in the enhanced phase, two-site entanglement increases in magnitude with increasing disorder strength before attaining a saturating value.

It is worth mentioning here that although the whole analysis is carried out for $N = 50$, we have checked that the behaviors of entanglement remain unaltered for larger system sizes (see [31] for a scaling analysis). Moreover,
Fig. 4: Phases with order-from-disorder phenomenon separated from normal phase. We present illustrations of enhanced and normal phases in (a) AD and (b) QD systems in the parameter space $(\mu, \sigma)$. The disorders are in the couplings. In both panels, the white region represents the normal phase, where no order-from-disorder phenomenon is present. In these regions, the value of the annealed or quenched averaged entanglement is lower than the same in the corresponding homogeneous system. The (colored) striped regions, named as enhanced phases, correspond to the parametric regimes with order-from-disorder phenomenon for entanglement. The scaling analysis (see SM) suggests that finite-size fluctuation occurs in the third decimal places. The disorder averages are also checked for convergence up to the third decimal place. To be on the safe side, the phase changes are decided with a tolerance of $10^{-2}$ so that the results will still be valid in the thermodynamic limit. All axes in the plot are dimensionless. Entanglement is measured in ebits.

Although the illustrations have been presented for a specific value of the anisotropic constant, viz. $\gamma = 0.5$, for which the factorization point is at $\mu = \mu_f = 1.1547$, similar analysis performed for other values of $\gamma$ confirms that the qualitative features remain unaltered irrespective of the value of the anisotropy.

**Case II:** XY model in random transverse field. Here, $h_i$ have mean $\langle h_i \rangle$, and standard deviation $\tilde{\sigma}^h$. Set $\lambda = \langle h_i \rangle / \langle J \rangle$ and $\sigma^h = \tilde{\sigma}^h / \langle J \rangle$. Unlike for AD in the interaction where there is a separable point at $\mu = 0$, there is no separable point at $\lambda = 0$ for the same here, and so no separable river originates from $\lambda = 0$. Moreover, because of the presence of factorization points at $\lambda = \pm 1$, we expect a hump-like structure between $-1$ and $1$, and fig. 1(a) in the SM shows half of that. The qualitative behavior, therefore, of entanglement in the presence of AD (as well as for the quenched one) here is similar to the case in which the disorder is in the interaction. However, it is interesting to mention here that the critical value of the disorder strength for AD in the interaction as well as the same in the field are both approximately 0.6. Like Case I, we study the phase diagram demarcating separable and entangled phases. We also present the order-from-disorder analysis for the random field XY model. We observe that with the insertion of AD, an enhanced phase emerges only near the critical point of the homogeneous system, and disappears with the increase of $\sigma^h$. However, as we increase the QD strength, the enhanced phase grows and covers up a significant area in both PM and AFM phases of the ordered system. With a high $\sigma^h$ value, the enhanced phase can be found in the entire AFM region and also in the PM region, except in the neighborhood of the critical point. See the SM for more details.

**Conclusion.** – This work aims to understand how the patterns of entanglement in transverse field quantum XY spin chains at equilibrium are affected by the insertion of disorder, and how they depend on the nature of disorder. We analyze two specific types of disorder, viz. annealed and quenched. We present a general formalism for computing disorder-averaged physical quantities for both kinds of disorder. While the formalism is valid for all temperatures, we focus on the physics at low temperatures.

We found that AD gives rise to entangled and separable phases in the system. The factorization point of the corresponding ordered system grows into a separable phase in the presence of AD. Moreover, for AD in the interaction, above a certain critical value of disorder strength, any finite entanglement in the deep paramagnetic phase, of the corresponding ordered system, is washed out, and a finite value survives only in the deep antiferromagnetic phase. The same is valid for an AD in the field, but with the roles of the magnetic phases reversed. However, entanglement is always non-zero in the presence of QD.

The presence of QD as well as AD may exhibit the order-from-disorder phenomenon, by which we mean an enhancement of the bipartite entanglement with the introduction of disorder. There exist wide parameter regions, in both the magnetic phases of the corresponding ordered system, where disorder-induced enhancement occurs, for the QD systems. Such regions are relatively modest in area in the corresponding annealed systems.

We also analyzed the effect on the quantum-critical point in the corresponding ordered system due to the application of disorder. In particular, we found that with the increase of the QD strength, the boundary between the magnetic phases of the ordered system, as quantified by the minimum of the derivative of entanglement, shifts towards the paramagnetic phase of the ordered system.

Our work is relevant to currently available laboratory systems with engineered disorder having controllable disorder strength, particularly within an optical lattice setup [9,10]. Moreover, solid-state systems with effective interaction strengths from a broad distribution, such as dilute magnetic semiconductors [35], dislocation networks in solid $^4$He [36], provide platforms for accessing systems relevant to our work.

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