INTEGRAL CURVATURE AND TOPOLOGICAL OBSTRUCTIONS FOR SUBMANIFOLDS

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Abstract. We provide integral curvature bounds for compact Riemannian manifolds that allow isometric immersions into a Euclidean space with low codimension in terms of the Betti numbers.

1. Introduction

According to Nash’s embedding theorem, every Riemannian manifold can be isometrically immersed into a Euclidean space with sufficiently high codimension. On the other hand, there are results that put several restrictions on existence of isometric immersions of a Riemannian manifold in a Euclidean space, when the codimension is low. A classical theorem due to Chern, Kuiper and Otsuki states that a compact \( n \)-dimensional Riemannian manifold of non-positive sectional curvature cannot be isometrically immersed into \( \mathbb{R}^{2n-1} \). Moore investigated topological restrictions on positively curved manifolds that allow isometric immersions in a Euclidean space with low codimension. In he proved that any compact \( n \)-dimensional Riemannian manifold with positive sectional curvature that allows an isometric immersion in \( \mathbb{R}^{n+2} \) is homeomorphic to the sphere \( S^n \). Moreover, in it was proved that a compact \( n \)-dimensional Riemannian manifold with constant sectional curvature that admits an isometric immersion in \( \mathbb{R}^{2n-1} \) is isometric to a round sphere.

All these results require conditions on the range of the sectional curvature. It is natural to ask whether the above restrictions on isometric immersions are maintained when one only assumes integral curvature bounds. In this paper, we address the following more general problem:

Problem 1. What kind of restrictions, integral curvature bounds impose on isometric immersions in the Euclidean space with low codimension?

We are interested in bounds for the \( L^{n/2} \)-norm of the tensors \( R - kR_1 \) and \( R - \frac{\text{scal}}{n(n-1)} R_1 \) for compact \( n \)-dimensional Riemannian manifolds \( (M^n, g) \). Here \( R \) and \( \text{scal} \) denote respectively the curvature tensor and the scalar curvature of \( g \), \( k \) is a constant, and \( R_1 = -\frac{1}{2} g \cdot g \), where \( \cdot \) stands for the Kulkarni-Nomizu product. Particular case of the above question is the following problem that was posed by Shiohama and Xu.

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Furthermore, Theorem 1.2.

Problem 2. Let $M^n$ be a compact $n$-dimensional Riemannian manifold ($n \geq 3$) which can be isometrically immersed into $\mathbb{R}^{2n-1}$. Does there exist a positive constant $\varepsilon(n)$ depending only on $n$ such that if

$$\int_M \|R - \frac{\text{scal}}{n(n-1)} R_1\|^2 dM < \varepsilon(n),$$

then $M^n$ is homeomorphic to $S^n$?

Shiohama and Xu [18], affirmatively answered this problem for the case of hypersurfaces.

The aim of the paper is to make a contribution to the above problems for compact Riemannian manifolds that allow isometric immersions in a Euclidean space, with low codimension $p \geq 2$, by providing topological bounds for the $L^{n/2}$-norm of the tensors $R - kR_1$ and $R - \frac{\text{scal}}{(n-1)} R_1$. Throughout the paper, all manifolds under consideration are assumed to be connected, without boundary and oriented. We prove the following results.

Theorem 1.1. Given $n \geq 4, k \in \mathbb{R}$ and $\lambda > 0$, there exists a constant $\varepsilon = \varepsilon(n,k,\lambda) > 0$, such that if $(M^n,g)$ is a compact, $n$-dimensional Riemannian manifold that admits an isometric immersion in $\mathbb{R}^{n+p}, 2 \leq p \leq n/2$, so that the scalar curvature and the second fundamental form $\alpha$ satisfy $|\text{scal}| \geq \lambda \|\alpha\|^2$, then

$$\int_M \|R - kR_1\|^2 dM \geq \varepsilon(n,k,\lambda) I,$$

where

$$I := \left\{ \begin{array}{ll}
\sum_{i=p}^{n-p} \beta_i & \text{if } k > 0, \\
\sum_{i=0}^{n} \beta_i & \text{if } k \leq 0,
\end{array} \right.$$

and $\beta_i$ is the $i$-th Betti number of $M^n$ with respect to an arbitrary coefficient field. Furthermore,

(i) If $k > 0$ and

$$\int_M \|R - kR_1\|^2 dM < \varepsilon(n,k,\lambda),$$

then $M^n$ has the homotopy type of a CW-complex with no cells of dimension $i$ for $p \leq i \leq n - p$.

(ii) For each $k \leq 0, \lambda > 0$ and $a > 0$, the class $\mathcal{M}(n,k,\lambda,\alpha)$ of all compact $n$-dimensional Riemannian manifolds with $\int_M \|R - kR_1\|^2 dM < a$, that are isometrically immersed into $\mathbb{R}^{n+p}, 2 \leq p \leq n/2$, so that the scalar curvature and the second fundamental form satisfy $|\text{scal}| \geq \lambda \|\alpha\|^2$, contains at most finitely many homeomorphism types.

Theorem 1.2. Given $n \geq 4$ and $\lambda > 0$, there exists a constant $\varepsilon = \varepsilon(n,\lambda) > 0$, such that if $(M^n,g)$ is a compact, $n$-dimensional Riemannian manifold that admits an isometric immersion in $\mathbb{R}^{n+p}, 2 \leq p \leq n/2$, so that the scalar curvature and the second fundamental form $\alpha$ satisfy $|\text{scal}| \geq \lambda \|\alpha\|^2$, then

$$\int_M \left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\|^2 dM \geq \varepsilon(n,\lambda) \sum_{i=p}^{n-p} \beta_i.$$

In particular, if

$$\int_M \left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\|^2 dM < \varepsilon(n,\lambda),$$
then $M^n$ has the homotopy type of a CW-complex with no cells of dimension $i$ for $p \leq i \leq n - p$. Moreover,

(i) If the scalar curvature of $M^n$ is everywhere non-positive, then

$$\int_M \left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\|^{n/2} dM \geq \varepsilon(n, \lambda) \sum_{i=0}^{n} \beta_i.$$  

(ii) For each $\lambda > 0$ and $a > 0$, the class $\mathcal{M}(n, \lambda, a)$ of all compact $n$-dimensional Riemannian manifolds with $\int_M \left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\|^{n/2} dM < a$, that are isometrically immersed into $\mathbb{R}^{n+p}$, $2 \leq p \leq n/2$, so that the scalar curvature and the second fundamental form satisfy $\text{scal} \leq -\lambda \|\alpha\|^2$, contains at most finitely many homeomorphism types.

The idea for the proofs is to relate the $L^{n/2}$-norm of the tensors $R - kR_1$ and $R - \frac{\text{scal}}{n(n-1)} R_1$ with the Betti numbers using well-known results of Chern and Lashof [6, 7], Morse theory and the Gauss equation. To this purpose, we need two algebraic inequalities for symmetric bilinear forms (see Propositions 2.3 and 2.4 below). The reason for imposing the additional assumption on the bound of the ratio of the scalar curvature to the squared length of the second fundamental form is that these inequalities fail without this condition. Actually, we provide counterexamples that justify the necessity of this assumption.

The main ingredient for the proof of the inequalities is the theory of flat bilinear forms which was introduced by Moore [14, 15] as an outgrowth of Cartan’s theory of exteriorly orthogonal quadratic forms [3]. This theory plays an essential role in the study of isometric immersions (cf. [2, 8, 9, 10]).

Our results can be formulated in terms of the curvature operator $\mathcal{R} : \wedge^2 TM \rightarrow \wedge^2 TM$. In fact, it is easy to see that

$$\| R - kR_1 \| = \| \mathcal{R} - kId \| \quad \text{and} \quad \left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\| = \left\| \mathcal{R} - \frac{\text{scal}}{n(n-1)} Id \right\|,$$

where $Id$ stands for the identity map on $\wedge^2 TM$. In particular, we have the following

**Corollary 1.1.** Let $(M^n, g)$ be a compact $n$-dimensional Riemannian manifold with positive curvature operator $\mathcal{R}$ that admits an isometric immersion in $\mathbb{R}^{n+p}$, $2 \leq p \leq n/2$, so that the scalar curvature and the second fundamental form satisfy $|\text{scal}| \geq \lambda \|\alpha\|^2$, $\lambda > 0$. If

$$\int_M \left\| \mathcal{R} - \frac{\text{scal}}{n(n-1)} Id \right\|^{n/2} dM < \varepsilon(n, \lambda),$$

then $M^n$ is diffeomorphic to the sphere $S^n$.

2. **Algebraic auxiliary results**

This section is devoted to some algebraic results that are crucial for the proofs. Let $V, W$ be real vector spaces, equipped with nondegenerate inner products which by abuse of notation are both denoted by $\langle , \rangle$. The inner product of $V$ is assumed to be positive definite. We consider the space $\text{Hom}(V \times V, W)$ of symmetric $W$-valued bilinear forms on $V$. This space may be viewed as a complete metric space by means of the usual Euclidean norm $\| . \|$. Recall that the Kulkarni-Nomizu product $\varphi \bullet \psi$ of
two bilinear forms \( \varphi, \psi \in \text{Hom}(V \times V, \mathbb{R}) \) is the \((0,4)\)-tensor \( \varphi \circ \psi : V \times V \times V \times V \to \mathbb{R} \) given by

\[
\varphi \circ \psi(x_1, x_2, x_3, x_4) := \varphi(x_1, x_3)\psi(x_2, x_4) + \varphi(x_2, x_4)\psi(x_1, x_3) - \varphi(x_1, x_4)\psi(x_2, x_3) - \varphi(x_2, x_3)\psi(x_1, x_4),
\]

where \( x_1, x_2, x_3, x_4 \in V \).

Using the inner product of \( W \), we extend the Kulkarni-Nomizu product for bilinear forms \( \beta, \gamma \in \text{Hom}(V \times V, W) \) as the \((0,4)\)-tensor \( \beta \circ \gamma : V \times V \times V \times V \to \mathbb{R} \) given by

\[
\beta \circ \gamma(x_1, x_2, x_3, x_4) := \langle \beta(x_1, x_3), \gamma(x_2, x_4) \rangle + \langle \beta(x_2, x_4), \gamma(x_1, x_3) \rangle - \langle \beta(x_1, x_4), \gamma(x_2, x_3) \rangle - \langle \beta(x_2, x_3), \gamma(x_1, x_4) \rangle.
\]

A bilinear form \( \beta \in \text{Hom}(V \times V, W) \) is called flat with respect to the inner product \( \langle \cdot, \cdot \rangle \) of \( W \) if and only if

\[
\langle \beta(x_1, x_3), \beta(x_2, x_4) \rangle - \langle \beta(x_1, x_3), \beta(x_2, x_3) \rangle = 0
\]

for all \( x_1, x_2, x_3, x_4 \in V \), or equivalently if and only if \( \beta \circ \beta = 0 \).

Associated to a bilinear form \( \beta \in \text{Hom}(V \times V, W) \) is the nullity space \( N(\beta) \) defined by

\[
N(\beta) = \{ x \in V : \beta(x, y) = 0 \text{ for all } y \in V \}.
\]

We need the following results on flat bilinear forms, the proofs of which can be found in [4] [5] [8].

**Proposition 2.1.** Let \( \beta \in \text{Hom}(V \times V, W) \) be a flat bilinear form with respect to a positive definite inner product of \( W \). Then

\[
\dim N(\beta) \geq \dim V - \dim W.
\]

**Proposition 2.2.** Let \( \beta \in \text{Hom}(V \times V, W) \) be a flat bilinear form with respect to a Lorentzian inner product of \( W \). Suppose that \( \dim V > \dim W \) and \( \beta(x, x) \neq 0 \) for all \( x \in V, x \neq 0 \). Then there exist a non-zero isotropic vector \( e \in W \), and a real valued bilinear form \( \phi \in \text{Hom}(V \times V, \mathbb{R}) \) such that

\[
\dim N(\beta - \phi e) \geq \dim V - \dim W + 2.
\]

The following lemma is needed for the proof of the algebraic auxiliary results.

**Lemma 2.1.** Let \( \beta \in \text{Hom}(V \times V, W) \) be a bilinear form, where \( V \) and \( W \) are both equipped with positive definite inner products and \( \dim W \leq \dim V - 2 \). If \( \beta \circ \beta = k \langle \cdot, \cdot \rangle \circ \langle \cdot, \cdot \rangle \) for some \( k \neq 0 \), then \( k > 0 \) and there exist a unit vector \( \xi \in W \) and a subspace \( V_1 \subseteq V \) such that

\[
\dim V_1 \geq \dim V - \dim W + 1
\]

and

\[
\beta(x, y) = \sqrt{k} \langle x, y \rangle \xi \text{ for all } x \in V \text{ and } y \in V_1.
\]

**Proof.** From \( \beta \circ \beta = k \langle \cdot, \cdot \rangle \circ \langle \cdot, \cdot \rangle \), we easily deduce that the bilinear form \( \overline{\beta} \in \text{Hom}(V \times V, W \oplus \mathbb{R}) \) given by

\[
\overline{\beta}(x, y) := (\beta(x, y), \langle x, y \rangle), \quad x, y \in V,
\]

is flat with respect to the inner product \( \langle \cdot, \cdot \rangle \) of \( W \oplus \mathbb{R} \) defined by

\[
\langle (\xi, t), (\eta, s) \rangle := \langle \xi, \eta \rangle - kst, \quad (\xi, t), (\eta, s) \in W \oplus \mathbb{R}.
\]
We claim that $k > 0$. Arguing indirectly, we suppose that $k < 0$. Then $\langle \langle \cdot, \cdot \rangle \rangle$ is positive definite and according to Proposition 2.1, we obtain
\[
\dim N(\tilde{\beta}) \geq \dim V - \dim (W \oplus \mathbb{R}),
\]
which contradicts our assumption on the dimensions.

Thus $k > 0$, the inner product $\langle \langle \cdot, \cdot \rangle \rangle$ has Lorentzian signature and $\tilde{\beta}$ fulfills the assumptions of Proposition 2.2. Hence there exist a non-zero isotropic vector $\eta = (e, t) \in W \oplus \mathbb{R}$ and a symmetric bilinear form $\psi \in \text{Hom}(V \times V, \mathbb{R})$ such that
\[
\dim N(\beta - \phi \eta) \geq \dim V - \dim W + 1.
\]

Setting $V_1 := N(\beta - \phi \eta)$, we immediately see that $\langle x, y \rangle = t \phi(x, y)$ and $\beta(x, y) = \phi(x, y)e$ for all $x \in V$ and $y \in V_1$. Using the fact that $\eta$ is isotropic, we obviously obtain $\beta(x, y) = \sqrt{k} \langle x, y \rangle \xi$ for all $x \in V$ and $y \in V_1$, where $\xi = \pm e/|e|$.

Hereafter, $V, W$ will be real vector spaces of dimensions $n$ and $p$ respectively, both equipped with positive definite inner products. For each $\beta \in \text{Hom}(V \times V, W)$, we define the map
\[
\beta^t : W \rightarrow \text{End}(V), \, \xi \mapsto \beta^t(\xi)
\]
such that
\[
\langle \beta^t(\xi)x, y \rangle = \langle \beta(x, y), \xi \rangle, \quad \text{for all } x, y \in V.
\]
Here, $\text{End}(V)$ denotes the set of all selfadjoint endomorphisms of $(V, \langle \cdot, \cdot \rangle)$.

When $2 \leq p \leq n/2$, for each $\beta \in \text{Hom}(V \times V, W)$, we denote by $\Omega(\beta)$ the following subset of the unit $(p - 1)$-sphere $S^{p-1}$ in $W$
\[
\Omega(\beta) := \{ u \in S^{p-1} : p \leq \text{Index}(\beta^t(u)) \leq n - p \}.
\]

Moreover, we define the “scalar curvature” function $\text{sc} : \text{Hom}(V \times V, W) \rightarrow \mathbb{R}$ by
\[
\text{sc}(\beta) := \frac{1}{2} \sum_{i,j=1}^{n} \beta \bullet \beta(e_i, e_j, e_i, e_j),
\]
where $\{e_1, \ldots, e_n\}$ is an arbitrary orthonormal basis of $V$.

We now may state the auxiliary results that are crucial for the proofs of the main results.

**Proposition 2.3.** Given positive integers $2 \leq p \leq n/2$ and numbers $k \in \mathbb{R}, \delta > 0$, there exists a constant $c = c(n, p, k, \delta) > 0$, such that the following inequality holds
\[
\| \beta \bullet \beta - k\langle \cdot, \cdot \rangle \bullet \langle \cdot, \cdot \rangle \|^2 \geq c(n, p, k, \delta) \left( \int_{\Omega_k(\beta)} |\det \beta^t(u)|dS_u \right)^{4/n},
\]
for any $\beta \in \text{Hom}(V \times V, W)$ with $|\text{sc}(\beta)| \geq \delta^2 \|\beta\|^2$, where
\[
\Omega_k(\beta) := \begin{cases} 
\Omega(\beta) & \text{if } k > 0, \\
S^{p-1} & \text{if } k \leq 0.
\end{cases}
\]

**Proof.** We consider the functions $\varphi_k, \psi_k : \text{Hom}(V \times V, W) \rightarrow \mathbb{R}$ defined by
\[
\varphi_k(\beta) := \| \beta \bullet \beta - k\langle \cdot, \cdot \rangle \bullet \langle \cdot, \cdot \rangle \|^2, \quad \psi_k(\beta) := \int_{\Omega_k(\beta)} |\det \beta^t(u)|dS_u,
\]
where $\beta \in \text{Hom}(V \times V, W)$. Moreover, we define the function $\omega_k : U_{k,\delta} \to \mathbb{R}$ by

$$\omega_k(\beta) := \frac{\varphi_k(\beta)}{(\psi_k(\beta))^{1/n}}, \quad \beta \in U_{k,\delta},$$

where

$$U_{k,\delta} := \begin{cases} \left\{ \beta \in \text{Hom}(V \times V, W) : \psi_k(\beta) \neq 0 \quad \text{and} \quad |\text{sc}(\beta)| \geq \delta^2\|\beta\|^2 \right\} & \text{if } k \neq 0, \\
\left\{ \beta \in \text{Hom}(V \times V, W) : \psi_k(\beta) = 1 \quad \text{and} \quad |\text{sc}(\beta)| \geq \delta^2\|\beta\|^2 \right\} & \text{if } k = 0. \end{cases}$$

We shall prove that $\inf \omega_k(U_{k,\delta}) > 0$. Arguing indirectly, we assume that there is a sequence $\{\beta_m\}$ in $U_{k,\delta}$ such that

(2.1) \[ \lim_{m \to \infty} \omega_k(\beta_m) = 0. \]

We note that $\beta_m \neq 0$ for all $m \in \mathbb{N}$, since $\beta_m \in U_{k,\delta}$. Then we may write $\beta_m = \|\beta_m\|\hat{\beta}_m$, where $\|\hat{\beta}_m\| = 1$.

To reach a contradiction, we distinguish two cases.

**Case 1.** Suppose that the sequence $\{\beta_m\}$ is unbounded. We may assume, by taking a subsequence if necessary, that $\lim_{m \to \infty} \|\beta_m\| = +\infty$. Since $\|\hat{\beta}_m\| = 1$, we may also assume that $\{\beta_m\}$ converges to some $\beta \in \text{Hom}(V \times V, W)$ with $\|\beta\| = 1$.

Using the fact that $\psi_k$ is homogeneous of degree $n$, (2.1) yields

$$\lim_{m \to \infty} \frac{\|\hat{\beta}_m \cdot \hat{\beta}_m - \frac{k}{\|\beta_m\|^2} \langle \xi, \cdot \rangle \cdot \langle \cdot, \xi \rangle \|}{(\psi_k(\hat{\beta}_m))^{4/n}} = 0.$$ 

Since $\{\psi_k(\hat{\beta}_m)\}$ is bounded, the above implies that $\hat{\beta}$ is flat. On the other hand, from $\beta_m \in U_{k,\delta}$ we have $|\text{sc}(\hat{\beta}_m)| \geq \delta^2$, and taking the limit for $m \to \infty$ we obtain $|\text{sc}(\hat{\beta})| \geq \delta^2$, contradiction.

**Case 2.** Assume that the sequence $\{\beta_m\}$ is bounded. Then $\{\beta_m\}$ converges to some $\beta \in \text{Hom}(V \times V, W)$, by taking a subsequence if necessary. From (2.1) it follows that $\varphi_k(\beta) = 0$, or equivalently $\beta \cdot \beta = k\langle \cdot, \cdot \rangle$.

At first we assume that $k = 0$. Then $\beta$ is flat and non-zero. In fact, if $\beta = 0$, then $\beta^2(u) = 0$ for all $u \in S^{p-1}$. Since $\beta_m \in U_{0,\delta}$, there exists $\xi_m \in S^{p-1}$ such that

(2.2) \[ |\det \beta_m^2(\xi_m)| \text{Vol}(S^{p-1}) = 1 \quad \text{for all } m \in \mathbb{N}. \]

On account of $|\xi_m| = 1$, we may assume that the sequence $\{\xi_m\}$ converges to some $\xi \in S^{p-1}$, by passing to a subsequence if necessary. Then, from $\lim_{m \to \infty} \beta_m = \beta$, we get $\lim_{m \to \infty} \beta_m^2(\xi_m) = \beta^2(\xi) = 0$, which contradicts (2.2). Thus $\beta \neq 0$. On the other hand, from $\beta_m \in U_{k,\delta}$ we have $|\text{sc}(\beta_m)| \geq \delta^2\|\beta_m\|^2$. Taking the limit for $m \to \infty$, we obtain $|\text{sc}(\beta)| \geq \delta^2\|\beta\|^2$, contradiction since $\beta$ is flat.

Now assume that $k \neq 0$. According to Lemma 2.1, $k > 0$ and there exist a unit vector $\xi \in W$ and a subspace $V_1 \subseteq V$ such that $\dim V_1 \geq n - p + 1$ and

(2.3) \[ \beta(x, y) = \sqrt{k}(x, y)\xi \quad \text{for all } x \in V \text{ and } y \in V_1. \]

By virtue of the fact that $\beta_m \in U_{k,\delta}$, there exists an open subset $U_m$ of $S^{p-1}$ such that

$$U_m \subseteq \{ u \in S^{p-1} : p \leq \text{Index}(\beta_m^2(u)) \leq n - p \}$$

and

$$\det \beta_m^2(u) \neq 0 \quad \text{for all } u \in U_m \text{ and } m \in \mathbb{N}.$$
Let \( \{u_m\} \) be a sequence such that \( u_m \in \mathcal{U}_m \) for all \( m \in \mathbb{N} \). On account of \( |u_m| = 1 \), we may assume hereafter that \( \{u_m\} \) is convergent, by passing if necessary to a subsequence. We set \( u = \lim_{m \to \infty} u_m \). Since \( \lim_{m \to \infty} \delta_m(u_m) = \delta(u) \) and \( u_m \in \mathcal{U}_m \), we deduce that \( \operatorname{Index}(\beta(u)) \leq n - p \). Then from (2.3) we get \( \langle u, \xi \rangle \geq 0 \).

We claim that \( \langle u, \xi \rangle = 0 \). Indeed, if \( \langle u, \xi \rangle > 0 \), then (2.3) implies that \( \beta(u) \) has at least \( n - p + 1 \) positive eigenvalues, and so, for \( m \) large enough, \( \delta_m(u_m) \) has at least \( n - p + 1 \) positive eigenvalues. This, on account of the fact that \( \det \beta_m(u) \neq 0 \) for all \( u \in \mathcal{U}_m \), confirms that \( \beta_m(u_m) \) has at most \( p - 1 \) negative eigenvalues, that is, \( \operatorname{Index}(\beta_m(u_m)) \leq p - 1 \), contradiction, since \( u_m \in \mathcal{U}_m \).

Thus, we have proved that for any convergent sequence \( \{u_m\} \) such that \( u_m \in \mathcal{U}_m \) for all \( m \in \mathbb{N} \), we have \( \lim_{m \to \infty} \delta_m(u, \xi) = 0 \).

Since \( \mathcal{U}_m \) is open, we may choose convergent sequences \( \{u_m^{(1)}\}, \{u_m^{(2)}\}, \ldots, \{u_m^{(p)}\} \) such that \( u_m^{(1)}, u_m^{(2)}, \ldots, u_m^{(p)} \in \mathcal{U}_m \) and span \( W \) for all \( m \in \mathbb{N} \). Then, by virtue of (2.3) and the fact that \( \lim_{m \to \infty} \beta_m(u_m^{(\alpha)}) = 0 \) for all \( \alpha \in \{1, 2, \ldots, p\} \), we infer that the restriction of \( \beta_m \) to \( V_1 \times V_1 \) satisfies

\[
\lim_{m \to \infty} \beta_m|_{V_1 \times V_1} = 0
\]

and consequently

\[
(2.4) \quad \lim_{m \to \infty} (\beta_m \cdot \beta_m)|_{V_1 \times V_1 \times V_1} = 0.
\]

From the obvious inequality

\[
\| (\beta_m \cdot \beta_m - k(\ldots) \cdot \langle \ldots \rangle) \|_{V_1 \times V_1 \times V_1}^2 \leq \varphi_k(\beta_m),
\]

(2.4) and by virtue of \( \lim_{m \to \infty} \varphi_k(\beta_m) = \varphi_k(\beta) = 0 \), we reach a contradiction since \( k > 0 \).

Thus, we have proved that \( \inf \omega_k(U_{k, \delta}) > 0 \). Obviously \( \inf \omega_k(U_{k, \delta}) \) depends only on \( n, p, k \) and \( \delta \) and is denoted by \( c(n, p, k, \delta) \). Then the desired inequality follows immediately.

**Proposition 2.4.** Given positive integers \( 2 \leq p \leq n/2 \) and a number \( \delta > 0 \), there exists a constant \( c = c(n, p, \delta) > 0 \), such the following inequality holds

\[
\left\| \beta \cdot \beta - \frac{\operatorname{sc}(\beta)}{n(n-1)} \langle \ldots \rangle \cdot \langle \ldots \rangle \right\|^2 \geq c(n, p, \delta) \left( \int_{\Lambda(\beta)} |\det \beta(u)| \, dS_u \right)^{4/n},
\]

for any \( \beta \in \operatorname{Hom}(V \times V, W) \) with \( |\operatorname{sc}(\beta)| \geq \delta^2 \|\beta\|^2 \), where

\[
\Lambda(\beta) := \begin{cases} 
\Omega(\beta) & \text{if } \operatorname{sc}(\beta) > 0, \\
S^{p-1} & \text{if } \operatorname{sc}(\beta) \leq 0.
\end{cases}
\]

**Proof.** We consider the functions \( \varphi, \psi : \operatorname{Hom}(V \times V, W) \to \mathbb{R} \) given by

\[
\varphi(\beta) := \left\| \beta \cdot \beta - \frac{\operatorname{sc}(\beta)}{n(n-1)} \langle \ldots \rangle \cdot \langle \ldots \rangle \right\|^2, \quad \psi(\beta) := \int_{\Lambda(\beta)} |\det \beta(u)| \, dS_u,
\]

where \( \beta \in \operatorname{Hom}(V \times V, W) \).

We shall prove that \( \varphi \) attains a positive minimum on \( U_\delta \), where

\[
U_\delta := \left\{ \beta \in \operatorname{Hom}(V \times V, W) : \psi(\beta) = 1 \text{ and } |\operatorname{sc}(\beta)| \geq \delta^2 \|\beta\|^2 \right\}.
\]
There exists a sequence $\{\beta_m\}$ in $U_\delta$ such that

$$\lim_{m \to \infty} \varphi(\beta_m) = \inf \varphi(U_\delta).$$

We observe that $\beta_m \neq 0$ for all $m \in \mathbb{N}$, since $\beta_m \in U_\delta$. Then we may write

$$\beta_m = ||\beta_m|| \hat{\beta}_m, \text{ where } ||\hat{\beta}_m|| = 1.$$

We claim that the sequence $\{\beta_m\}$ is bounded. Assume to the contrary that there exists a subsequence of $\{\beta_m\}$, which by abuse of notation is denoted again by $\{\beta_m\}$, such that $\lim_{m \to \infty} ||\beta_m|| = +\infty$. Since $||\hat{\beta}_m|| = 1$, we may assume, by taking a subsequence if necessary, that $\{\hat{\beta}_m\}$ converges to some $\hat{\beta} \in \text{Hom}(V \times V, W)$ with $||\hat{\beta}|| = 1$. Using the fact that $\psi$ is homogeneous of degree $n$ and since $\{\beta_m\} \in U_\delta$, we get

$$||\beta_m|| = \frac{1}{(\psi(\hat{\beta}_m))^{1/n}}.$$

Thus $\lim_{m \to \infty} \psi(\hat{\beta}_m) = 0$ and consequently $\varphi(\hat{\beta}) = 0$, or equivalently

$$\hat{\beta} \bullet \hat{\beta} = \frac{\text{sc}(\hat{\beta})}{n(n-1)} (\ldots) \bullet (\ldots).$$

From $\{\beta_m\} \in U_\delta$, we have $|\text{sc}(\hat{\beta}_m)| \geq \delta^2$. Taking the limit for $m \to \infty$, we get $\text{sc}(\hat{\beta}) \neq 0$. According to Lemma 2.1, $\text{sc}(\hat{\beta}) > 0$ and there exists a unit vector $\hat{\xi} \in V$ and a subspace $\hat{V}_1 \subseteq V$ such that $\dim \hat{V}_1 \geq n - p + 1$ and

$$\hat{\beta}(x, y) = \left( \frac{\text{sc}(\hat{\beta})}{n(n-1)} \right)^{1/2} \langle x, y \rangle \hat{\xi} \quad \text{for all } x \in V \text{ and } y \in \hat{V}_1.$$

Moreover, since $\beta_m \in U_\delta$, there exists an open subset $\mathring{U}_m$ of $S^{p-1}$ such that

$$\mathring{U}_m \subseteq \Lambda(\hat{\beta}_m) \quad \text{and} \quad \det \hat{\beta}_m^T(u) \neq 0 \quad \text{for all } u \in \mathring{U}_m \text{ and } m \in \mathbb{N}.$$

From $\text{sc}(\hat{\beta}) > 0$, we deduce that $\text{sc}(\hat{\beta}_m) > 0$ and so

$$\mathring{U}_m \subseteq \{ u \in S^{p-1} : p \leq \text{Index}(\hat{\beta}_m^T(u)) \leq n - p \}$$

for $m$ large enough.

Let $\{\hat{u}_m\}$ be a sequence such that $\hat{u}_m \in \mathring{U}_m$ for all $m \in \mathbb{N}$. On account of $|\hat{u}_m| = 1$, we may assume hereafter that $\{\hat{u}_m\}$ is convergent, by passing if necessary to a subsequence. We set $\hat{u} = \lim_{m \to \infty} \hat{u}_m$. Since $\lim_{m \to \infty} \hat{\beta}_m^T(\hat{u}_m) = \hat{\beta}_m^T(\hat{u})$ and $\hat{u}_m \in \mathring{U}_m$, we deduce that $\text{Index}(\hat{\beta}_m^T(\hat{u})) \leq n - p$. Then, from (2.5) we obtain $\langle \hat{u}, \hat{\xi} \rangle \geq 0$. We claim that $\langle \hat{u}, \hat{\xi} \rangle = 0$. Indeed, if $\langle \hat{u}, \hat{\xi} \rangle > 0$, then (2.5) implies that $\hat{\beta}_m^T(\hat{u})$ has at least $n - p + 1$ positive eigenvalues, and so, for $m$ large enough, $\hat{\beta}_m^T(u_m)$ has at least $n - p + 1$ positive eigenvalues. This, on account of the fact that $\det \hat{\beta}_m^T(u) \neq 0$ for all $u \in \mathring{U}_m$, confirms that $\hat{\beta}_m^T(\hat{u}_m)$ has at most $p - 1$ negative eigenvalues, that is, $\text{Index}(\hat{\beta}_m^T(\hat{u}_m)) \leq p - 1$, contradiction, since $\hat{u}_m \in \mathring{U}_m$.

Thus, we have proved that for any convergent sequence $\{\hat{u}_m\}$ such that $\hat{u}_m \in \mathring{U}_m$ for all $m \in \mathbb{N}$, we have $\lim_{m \to \infty} \hat{u}_m, \hat{\xi} = 0$.

Since $\mathring{U}_m$ is open, we may choose convergent sequences $\{\hat{u}_m^{(1)}\}, \{\hat{u}_m^{(2)}\}, \ldots, \{\hat{u}_m^{(p)}\}$ such that $\hat{u}_m^{(1)}, \hat{u}_m^{(2)}, \ldots, \hat{u}_m^{(p)} \in \mathring{U}_m$ and span $W$ for all $m \in \mathbb{N}$. Then, by virtue of
We notice that $\beta_{\text{sc}}(n) \to a$ subsequence. We set $m(2.6)$ and by virtue of $\lim_{m \to \infty} |\langle \beta_m \rangle - \langle \beta \rangle| \to 0$ and the fact that $|\langle \beta_m \rangle - \langle \beta \rangle| \to 0$, we infer that the restriction of $\beta_{\text{sc}}(\beta)$ to $\hat{V}_1 \times \hat{V}_1$ satisfies

$$\lim_{m \to \infty} \beta_{\text{sc}}(\beta_m) |\hat{V}_1 \times \hat{V}_1 = 0$$

and consequently

$$(2.6) \quad \lim_{m \to \infty} (\beta_m \cdot \beta_m) |\hat{V}_1 \times \hat{V}_1 \times \hat{V}_1 = 0.$$

From the inequality

$$\left\| \left( \beta_m \cdot \beta_m - \frac{\text{sc}(\beta_m)}{n(n-1)} \langle \cdot, \cdot \rangle \langle \cdot, \cdot \rangle \right) |\hat{V}_1 \times \hat{V}_1 \times \hat{V}_1 \right\|^2 \leq \varphi(\beta_m),$$

(2.6) and by virtue of $\lim_{m \to \infty} \varphi(\beta_m) = \varphi(\beta) = 0$, we reach a contradiction since $\text{sc}(\beta) \geq 2^2 > 0$.

Thus, the sequence $\{\beta_m\}$ is bounded, and it converges to some $\beta \in \text{Hom}(V \times V, W)$, by taking a subsequence if necessary.

We claim that $\varphi(\beta) > 0$. Assume to the contrary that $\varphi(\beta) = 0$, or equivalently

$$\beta \cdot \beta = \frac{\text{sc}(\beta)}{n(n-1)} \langle \cdot, \cdot \rangle \langle \cdot, \cdot \rangle.$$

We notice that $\beta \neq 0$. Indeed if $\beta = 0$, then $\beta^2(u) = 0$ for all $u \in S^{p-1}$. Since $\beta_m \in U_\delta$, there exists $\xi_m \in \Lambda(\beta_m)$ such that

$$(2.7) \quad |\det \beta_m^2(\xi_m)| \text{Vol}(\Lambda(\beta_m)) = 1 \quad \text{for all} \ m \in \mathbb{N}.$$ 

On account of $|\xi_m| = 1$, we may assume that the sequence $\{\xi_m\}$ converges to some $\xi \in S^{p-1}$, by passing to a subsequence if necessary. Then, from $\lim_{m \to \infty} \beta_m = \beta$, we get $\lim_{m \to \infty} \beta_m^2(\xi_m) = \beta^2(\xi) = 0$, which contradicts (2.7).

Therefore $\beta \neq 0$. From the fact that $\beta_m \in U_\delta$, we have $|\text{sc}(\beta_m)| \geq 2^2 \|\beta_m\|^2$. Taking the limit for $m \to \infty$, we deduce that $|\text{sc}(\beta)| \geq 2^2 \|\beta\|^2 > 0$.

Then, according to Lemma 2.1, there exists a unit vector $\xi \in W$ and a subspace $V_1 \subseteq V$ such that $\text{dim} V_1 \geq n - p + 1$ and

$$(2.8) \quad \beta(x, y) = \left( \frac{\text{sc}(\beta)}{n(n-1)} \right)^{1/2} \langle x, y \rangle \xi \quad \text{for all} \ x \in V \text{ and } y \in V_1.$$

By virtue of the fact that $\beta_m \in U_\delta$, there exists an open subset $U_m$ of $S^{p-1}$ such that

$$U_m \subseteq \Lambda(\beta_m) \text{ and } \det \beta_m^2(u) \neq 0 \text{ for all } u \in U_m \text{ and } m \in \mathbb{N}.$$ 

Since $\text{sc}(\beta) > 0$, we see that

$$U_m \subseteq \{ u \in S^{p-1} : p \leq \text{Index}(\beta_m^2(u)) \leq n - p \}$$

for $m$ large enough.

Let $\{u_m\}$ be a sequence such that $u_m \in U_m$ for all $m \in \mathbb{N}$. On account of $|u_m| = 1$, we may assume hereafter that $\{u_m\}$ is convergent, by passing if necessary to a subsequence. We set $u = \lim_{m \to \infty} u_m$. Since $\lim_{m \to \infty} \beta_m^2(u_m) = \beta^2(u)$ and $u_m \in U_m$, we deduce that $\text{Index}(\beta^2(u)) \leq n - p$. Then, from (2.8), we get $\langle u, \xi \rangle \geq 0$.

We claim that $\langle u, \xi \rangle = 0$. Indeed, if $\langle u, \xi \rangle > 0$, then (2.8) implies that $\beta_m^2(u)$ has at least $n - p + 1$ positive eigenvalues, and so, for $m$ large enough, $\beta_m^2(u)$ has at least $n - p + 1$ positive eigenvalues. This, on account of the fact that $\det \beta_m^2(u) \neq 0$ for all $u \in U_m$, confirms that $\beta_m^2(u)$ has at most $p - 1$ negative eigenvalues, that is, $\text{Index}(\beta_m^2(u)) \leq p - 1$, contradiction, since $u_m \in U_m$. 


Thus, we have proved that for any convergent sequence \( \{u_m\} \) such that \( u_m \in \mathcal{U}_m \) for all \( m \in \mathbb{N} \), we have \( \lim_{m \to \infty} u_m, \xi = 0 \).

Since \( \mathcal{U}_m \) is open, we may choose convergent sequences \( \{u_m^{(1)}\}, \{u_m^{(2)}\}, \ldots, \{u_m^{(p)}\} \) such that \( u_m^{(1)}, u_m^{(2)}, \ldots, u_m^{(p)} \in \mathcal{U}_m \) and span \( W \) for all \( m \in \mathbb{N} \). Then, by virtue of (2.8) and the fact that \( \lim_{m \to \infty} u_m^{(\alpha)}, \xi = 0 \) for all \( \alpha \in \{1, 2, \ldots, p\} \), we infer that the restriction of \( \beta_m \) to \( V_1 \times V_1 \) satisfies

\[
\lim_{m \to \infty} \beta_m |_{V_1 \times V_1} = 0
\]

and consequently

\[
\lim_{m \to \infty} (\beta_m \cdot \beta_m) |_{V_1 \times V_1 \times V_1} = 0.
\]

From the inequality

\[
\left\| \left( \beta_m \cdot \beta_m - \frac{sc(\beta_m)}{n(n-1)} \langle \cdot, \cdot \rangle \right) |_{V_1 \times V_1 \times V_1} \right\|^2 \leq \varphi(\beta_m),
\]

(2.9) and by virtue of \( \lim_{m \to \infty} \varphi(\beta_m) = \varphi(\beta) = 0 \), we reach a contradiction since \( sc(\beta) \geq \delta^2 \|\beta\|^2 > 0 \).

Consequently, our claim is proved, that is, \( \varphi(\beta) > 0 \) and so \( \varphi \) attains a positive minimum on \( \mathcal{U}_\delta \) which obviously depends only on \( n, p, \delta \) and is denoted by \( c(n, p, \delta) \).

Now let \( \beta \in \mathcal{U}_\delta \). Assume that \( \psi(\beta) \neq 0 \) and set \( \tilde{\beta} = \beta / (\psi(\beta))^{1/n} \). Clearly \( \tilde{\beta} \in \mathcal{U}_\delta \), and consequently \( \varphi(\tilde{\beta}) \geq c(n, p, \delta) \). Since \( \varphi \) is homogeneous of degree 4, the desired inequality is obviously fulfilled. In the case where \( \psi(\beta) = 0 \), the inequality is trivial.

Now we argue on the necessity of the assumption on the scalar curvature of bilinear forms in both Propositions 2.3 and 2.4. Actually we provide counterexamples that ensure that this assumption cannot be dropped.

**Example 2.2.** Let \( \{\eta_m\}, \{a_m^{(j)}\}, 2 \leq j \leq n, \{b_m^{(\alpha)}\}, 2 \leq \alpha \leq p, \) be sequences of real numbers that tend to zero as \( m \to \infty \). Furthermore, we consider convergent sequences \( \{\gamma_m^{(1)}\}, \{\gamma_m^{(\alpha)}\}, 2 \leq j \leq n, 2 \leq \alpha \leq p, \) such that

\[
\eta_m^{2(n-1)/n} \sum_{j=2}^n (a_m^{(j)})^2 + \sum_{\alpha=2}^p \sum_{j=2}^n (b_m^{(\alpha)})^2 = 1
\]

and

\[
(\gamma_m^{(1)})^2 = 1 - \eta_m^{2(n-2)/n} \sum_{\alpha=2}^p (b_m^{(\alpha)})^2.
\]

We now define the sequence \( \{\gamma_m\} \) in \( \text{Hom}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^p) \) by

\[
\gamma_m(x, y) = (\gamma_m^{(1)} x_1 y_1 + \eta_m^{2(n-1)/n} \sum_{j=2}^n a_m^{(j)} x_j y_j) \xi_1
\]

\[
+ \eta_m^{2(n-2)/n} \sum_{\alpha=2}^p (b_m^{(\alpha)})^2 x_1 y_1 + \eta_m^{2(n-2)/n} \sum_{\alpha=2}^p \sum_{j=2}^n (b_m^{(\alpha)})^2 x_j y_j) \xi_\alpha,
\]

where \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \) and \( \xi_1, \ldots, \xi_p \) is the standard basis of \( \mathbb{R}^p \). The sequences are chosen so that

\[
p \leq \text{Index} \left( \text{diag}(\gamma_m^{(1)}, \eta_m^{2(n-1)/n} a_m^{(2)}, \ldots, \eta_m^{2(n-1)/n} a_m^{(n)}) \right) \leq n - p
\]
and 
\[ p \leq \text{Index} \left( \text{diag}(\gamma_m^{2/n} \theta_m^{(\alpha,2)}, \ldots, \gamma_m^{2/n} \theta_m^{(\alpha,n)}) \right) \leq n - p \]
for any \( 2 \leq \alpha \leq p \). This implies that there exists an open subset \( \Omega \) of \( S^{p-1} \) such that \( \Omega \subseteq \Omega(\gamma_m) \) for all \( m \in \mathbb{N} \).

A direct computation shows that
\[
\left\| \gamma_m \cdot \gamma_m - \frac{sc(\gamma_m)}{n(n - 1)} \langle \ldots \rangle \cdot \langle \ldots \rangle \right\|^2 = n_m^{4(n-1)/n} \rho_m,
\]
where
\[
\rho_m = 32 \sum_{j=2}^{n} \left( \gamma_m^{(1)} a_m^{(j)} + \sum_{\alpha=2}^{p} \gamma_m^{(\alpha)} \theta_m^{(j,\alpha)} - \frac{2}{n(n - 1)} \sum_{t=2}^{n} \gamma_m^{(1)} a_m^{(t)} \right) - \frac{2}{n(n - 1)} \sum_{t=2}^{n} \sum_{\alpha=2}^{p} \gamma_m^{(\alpha)} \theta_m^{(t,\alpha)}
\]
\[+ 16 \sum_{i \neq j, i, j \geq 2}^{n} \left( \gamma_m^{(i)} \theta_m^{(j,\alpha)} - \frac{2}{n(n - 1)} \sum_{t=2}^{n} \gamma_m^{(1)} \theta_m^{(t,\alpha)} \right) - \frac{2}{n(n - 1)} \sum_{t=2}^{n} \sum_{\alpha=2}^{p} \gamma_m^{(\alpha)} \theta_m^{(t,\alpha)} \right)^2.
\]
Moreover, writing \( u = \sum_{\alpha} u_\alpha \xi_\alpha \), we see that
\[
\int_{\Omega} |\det \gamma_m^\sharp(u)| dS_u = \eta_m^{n-1} \sigma_m,
\]
where
\[
\sigma_m = \int_{\Omega} \left( \left| u_1 \gamma_m^{(1)} + \eta_m^{(n-2)/n} \sum_{\alpha=2}^{p} u_\alpha \prod_{j=2}^{n} \eta_m^{(n-2)/n} \theta_m^{(j,\alpha)} + \sum_{\alpha=2}^{p} u_\alpha \theta_m^{(j,\alpha)} \right) \right) dS_u.
\]
We observe that \( \lim_{m \to \infty} \rho_m = 0 \) and
\[
\lim_{m \to \infty} \sigma_m = \int_{\Omega} |u_1 \prod_{j=2}^{n} \sum_{\alpha=2}^{p} u_\alpha \theta_m^{(j,\alpha)}| dS_u,
\]
where \( \theta_m^{(j,\alpha)} = \lim_{m \to \infty} \theta_m^{(j,\alpha)} \). We may also choose the sequences so that
\[
\int_{\Omega} |u_1 \prod_{j=2}^{n} \sum_{\alpha=2}^{p} u_\alpha \theta_m^{(j,\alpha)}| dS_u > 0.
\]
For instance, we may choose the sequences \( \{\theta_m^{(j,\alpha)}\}, 2 \leq j \leq n, \text{ such that} \lim_{m \to \infty} \theta_m^{(\alpha,2)} \ldots \theta_m^{(\alpha,n)} \neq 0 \text{ for a fixed } 2 \leq \alpha \leq p. \)
Hence
\[
\lim_{m \to \infty} \left\| \gamma_m \cdot \gamma_m - \frac{sc(\gamma_m)}{n(n - 1)} \langle \ldots \rangle \cdot \langle \ldots \rangle \right\|^2 = \lim_{m \to \infty} \left( \int_{\Omega} |\det \gamma_m^\sharp(u)| dS_u \right)^{4/n} = \frac{\rho_m}{(\sigma_m)^{4/n}} = 0,
\]
which implies that

\[(2.10) \lim_{m \to \infty} \frac{\|\gamma_m \cdot \gamma_m - \frac{\text{sc}(\gamma_m)}{n(n-1)} \langle \cdot, \cdot \rangle}{\left( \int_{\Omega(\gamma_m)} |\det \gamma_m^\sharp(u)|dS_u \right)^{4/n}} = 0.\]

Furthermore, we notice that \(\|\gamma_m\|^2 = 1\) and \(\lim_{m \to \infty} \text{sc}(\gamma_m) = 0\).

This together with (2.10) show that there exist no positive constant depending only on \(n, p\) such that the inequality in Proposition 2.4 hold without the condition on the scalar curvature.

**Example 2.3.** We consider a sequence \(\{\gamma_m\}\) in \(\text{Hom}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^p)\) as in the previous example. Arguing as before, we similarly conclude that

\[
\lim_{m \to \infty} \frac{\|\gamma_m \cdot \gamma_m\|^2}{\left( \int_{\Omega(\gamma_m)} |\det \gamma_m^\sharp(u)|dS_u \right)^{4/n}} = 0.
\]

This proves that there exist no positive constant depending only on \(n, p\) such that the inequality in Proposition 2.3 hold for \(k = 0\) without the condition on the scalar curvature.

Now let \(k \neq 0\). We choose all sequences in Example 2.2 so that \(k \text{sc}(\gamma_m) > 0\) for all \(m \in \mathbb{N}\), and we consider the sequence \(\{\beta_m\}\) given by

\[\beta_m = \frac{n(n-1)k}{\text{sc}(\gamma_m)} \gamma_m.\]

We have

\[
\frac{\|\beta_m \cdot \beta_m - k \langle \cdot, \cdot \rangle \cdot \langle \cdot, \cdot \rangle\|^2}{\left( \int_{\Omega_k(\beta_m)} |\det \beta_m^\sharp(u)|dS_u \right)^{4/n}} = \frac{\|\gamma_m \cdot \gamma_m - \frac{\text{sc}(\gamma_m)}{n(n-1)} \langle \cdot, \cdot \rangle \cdot \langle \cdot, \cdot \rangle\|^2}{\left( \int_{\Omega(\gamma_m)} |\det \gamma_m^\sharp(u)|dS_u \right)^{4/n}}
\]

and on account of (2.10), we get

\[
\lim_{m \to \infty} \frac{\|\beta_m \cdot \beta_m - k \langle \cdot, \cdot \rangle \cdot \langle \cdot, \cdot \rangle\|^2}{\left( \int_{\Omega_k(\beta_m)} |\det \beta_m^\sharp(u)|dS_u \right)^{4/n}} = 0.
\]

Furthermore, since \(\|\gamma_m\|^2 = 1\) and \(\lim_{m \to \infty} \text{sc}(\gamma_m) = 0\), we obtain

\[
\lim_{m \to \infty} \frac{\text{sc}(\beta_m)}{\|\beta_m\|^2} = 0.
\]

From these we conclude that there exist no positive constant depending only on \(n, p, k\) such that the inequality in Proposition 2.3 hold without the condition on the scalar curvature.

**Remark 2.4.** We have not explicitly computed the constants that appear in Propositions 2.3 and 2.4. However these constants have the following property:

\[
\liminf_{\delta \to 0^+} c(n, p, k, \delta) = 0 \quad \text{and} \quad \liminf_{\delta \to 0^+} c(n, p, \delta) = 0.
\]
We can easily get upper bounds for them. For instance, applying the inequality in Proposition 2.4 to the bilinear form \( \beta \) in \( \text{Hom}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^p) \) given by

\[
\beta(x, y) = \left( \sum_{i=1}^{l} x_i y_i - \sum_{i=l+1}^{n} x_i y_i \right) \xi,
\]

where \( 1 \leq l \leq n \), \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) and \( \xi \) is a unit vector in \( W \), we find the following estimate

\[
c(n, p, \delta) \leq \frac{2^{4/n}(8n^2(n-1)^2 - ((n-2l)^2 - n)\delta^2)}{4n(n-1)\left( \int_{S^{n-1}} |\langle u, \xi \rangle|^n dS_u \right)^{4/n}} \text{ for } \delta^2 \leq \frac{(n-2l)^2 - n}{n}.
\]

3. Proofs

At first we recall some basic and well known facts about total curvature and how Morse theory provides restrictions on the Betti numbers.

Let \( f : (M^n, g) \to \mathbb{R}^{n+p} \) be an isometric immersion of a compact, connected, oriented \( n \)-dimensional Riemannian manifold \((M^n, g)\) into the \((n+p)\)-dimensional Euclidean space \( \mathbb{R}^{n+p} \) equipped with the usual Riemannian metric \( \langle \cdot, \cdot \rangle \). The normal bundle of \( f \) is given by

\[
N_f = \{(x, \xi) \in f^*(T\mathbb{R}^{n+p}) : \xi \perp df_x(T_x M) \},
\]

where \( f^*(T\mathbb{R}^{n+p}) \) stands for the induced bundle, and the unit normal bundle of \( f \) is defined by

\[
UN_f = \{(x, \xi) \in N_f : |\xi| = 1 \}.
\]

The generalized Gauss map \( \nu : UN_f \to S^{n+p-1} \) is given by \( \nu(x, \xi) = \xi \), where \( S^{n+p-1} \) is the unit \((n+p-1)\)-sphere in \( \mathbb{R}^{n+p} \). For each \( u \in S^{n+p-1} \), we consider the height function \( h_u : M^n \to \mathbb{R} \) defined by \( h_u(x) = \langle f(x), u \rangle \), \( x \in M^n \). Since \( h_u \) has a degenerate critical point if and only if \( u \) is a critical value of the generalized Gauss map, by Sard’s Theorem there exists a subset \( E \subset S^{n+p-1} \) of zero measure such that \( h_u \) is a Morse function for all \( u \in S^{n+p-1} \setminus E \). For every \( u \in S^{n+p-1} \setminus E \), we denote by \( \mu_i(u) \) the number of critical points of \( h_u \) of index \( i \). We also set \( \mu_i(u) = 0 \) for any \( u \in E \). According to Kuiper [11], the total curvature of index \( i \) of \( f \) is given by

\[
\tau_i = \frac{1}{\text{Vol}(S^{n+p-1})} \int_{S^{n+p-1}} \mu_i(u) dS_u,
\]

where \( dS \) denotes the volume element of the sphere \( S^{n+p-1} \).

Let \( \beta_i = \dim H_i(M; \mathcal{F}) \) be the \( i \)-th Betti number of \( M \), where \( H_i(M; \mathcal{F}) \) is the \( i \)-th homology group with coefficients in a field \( \mathcal{F} \). From the weak Morse inequalities [12], we know that \( \mu_i(u) \geq \beta_i \) for every \( u \in S^{n+p-1} \) such that \( h_u \) is a Morse function. Integrating over \( S^{n+p-1} \), we obtain

\[
(3.1) \quad \tau_i \geq \beta_i.
\]

For each \((x, \xi) \in UN_f\), we denote by \( A_\xi \) the shape operator of \( f \) associated with the direction \( \xi \) given by

\[
g(A_\xi(X), Y) = \langle \alpha(X, Y), \xi \rangle,
\]

where \( X, Y \) are tangent to \( M \) and \( \alpha \) is the second fundamental form of \( f \) viewed as a section of the vector bundle \( \text{Hom}(TM \times TM, N_f) \). There is a natural volume
element $d\Sigma$ on the unit normal bundle $UN_f$. In fact, if $dV$ is a $(p-1)$-form on $UN_f$ such that its restriction to a fiber of the unit normal bundle at $(x, \xi)$ is the volume element of the unit $(p-1)$-sphere $S^{p-1}_x$ of the normal space of $f$ at $x$, then $d\Sigma = dM \wedge dV$. Furthermore, we have

$$\nu^*(dS) = G(x, \xi)d\Sigma,$$

where $G(x, \xi) := (-1)^n \det A_\xi$ is the Lipschitz-Killing curvature at $(x, \xi) \in UN_f$.

A well-known formula due to Chern and Lashof [7] states that

$$\int_{UN_f} |\det A_\xi|d\Sigma = \sum_{i=0}^{n} \int_{S^{n+p-1}_x} \mu_i(u)dS_u.$$  

The total absolute curvature $\tau(f)$ of $f$ in the sense of Chern and Lashof is defined by

$$\tau(f) = \frac{1}{\text{Vol}(S^{n+p-1})} \int_{UN_f} |\nu^*(dS)| = \frac{1}{\text{Vol}(S^{n+p-1})} \int_{UN_f} |\det A_\xi|d\Sigma.$$  

The following result is due to Chern and Lashof [6, 7].

**Theorem 3.1.** Let $f : (M^n, g) \rightarrow \mathbb{R}^{n+p}$ be an isometric immersion of a compact, connected, oriented, $n$-dimensional Riemannian manifold $(M^n, g)$ into $\mathbb{R}^{n+p}$. Then the total absolute curvature of $f$ satisfies the inequality

$$\tau(f) \geq \sum_{i=0}^{n} \beta_i.$$  

For each $i \in \{0, \ldots, n\}$, we consider the subset $U_i N_f$ of the unit normal bundle of $f$ defined by

$$U_i N_f = \{(x, \xi) \in UN_f : \text{Index}(A_\xi) = i\}.$$  

Shiohama and Xu [18, Lemma p. 381] refined formula (3.2) as follows

$$\int_{U_i N_f} |\det A_\xi|d\Sigma = \int_{S^{n+p-1}_x} \mu_i(u)dS_u.$$  

We recall that the Riemannian curvature tensor of $(M^n, g)$ is the $(0,4)$-tensor $R$ that is related to the second fundamental form $\alpha$ via the Gauss equation

$$R(X, Y, Z, W) = \langle \alpha(X, W), \alpha(Y, Z) \rangle - \langle \alpha(X, Z), \alpha(Y, W) \rangle,$$

where $X, Y, Z, W$ are tangent vector fields of $M^n$.

By means of the Kulkarni-Nomizu product, the Gauss equation is written as

$$R = -\frac{1}{2} \alpha \bullet \alpha.$$

Moreover, we consider the $(0,4)$-tensor $R_1$ given by

$$R_1 = -\frac{1}{2} g \bullet g.$$

We are now ready to give the proofs of the main results.

**Proof of Theorem 1.1.** Let $f : (M^n, g) \rightarrow \mathbb{R}^{n+p}$ be an isometric immersion whose scalar curvature and the second fundamental form satisfy $|\text{scal}| \geq \lambda \|\alpha\|^2$. Appealing to Proposition 2.3, we have

$$\|\alpha \bullet \alpha - kg \bullet g\|^{n/2}(x) \geq \left(c(n, p, k, \lambda)\right)^{n/4} \int_{\Omega_\lambda(\alpha|_x)} |\det A_\xi|dV_\xi.$$  

for all $x \in M^n$, where

$$\Omega_k(\alpha|_x) = \begin{cases} \{u \in S_x^{p-1}: p \leq \text{Index}(\beta^g(u)) \leq n - p\} & \text{if } k > 0, \\ S_x^{p-1} & \text{if } k \leq 0. \end{cases}$$

On account of (3.4) and (3.5), the above inequality becomes

$$\|R - kR_1\|^{n/2}(x) \geq \left(\frac{1}{4}c(n, p, k, \lambda)\right)^{n/4} \int_{\Omega(\alpha|_x)} |\det A_\xi|dV_\xi$$

for all $x \in M^n$. Integrating over $M^n$, we obtain

$$\int_M \|R - kR_1\|^{n/2}dM \geq \left(\frac{1}{4}c(n, p, k, \lambda)\right)^{n/4} \sum_{i \in J} \int_{U_i|N_j} |\det A_\xi|d\Sigma,$$

where

$$J := \begin{cases} \{p, \ldots, n - p\} & \text{if } k > 0, \\ \{0, \ldots, n\} & \text{if } k \leq 0. \end{cases}$$

Bearing in mind (3.3), the definition of the total curvature of index $i$ and (3.1), we get

(3.6)$$\int_M \|R - kR_1\|^{n/2}dM \geq \varepsilon(n, k, \lambda) \sum_{i \in J} \tau_i \geq \varepsilon(n, k, \lambda) \sum_{i \in J} \beta_i,$$

where the constant $\varepsilon(n, k, \lambda)$ is given by

$$\varepsilon(n, k, \lambda) := \min_{2 \leq p \leq n/2} \left(\frac{1}{4}c(n, p, k, \lambda)\right)^{n/4}\text{Vol}(S^{n+p-1}).$$

Now suppose that $k > 0$ and $\int_M \|R - kR_1\|^{n/2}dM < \varepsilon(n, k, \lambda)$. Then, in view of (3.6), we conclude that $\sum_{i=1}^{n-p} \tau_i < 1$. Thus, there exists $u \in S^{n+p-1}$ such that the height function $h_u: M^n \rightarrow \mathbb{R}$ is a Morse function whose number of critical points of index $i$ satisfies $\mu_i(u) = 0$ for any $p \leq i \leq n - p$. Appealing to the fundamental theorem of Morse theory (cf. [12, Th. 3.5] or [4, Th. 4.10]), we deduce that $M^n$ has the homotopy type of a CW-complex with no cells of dimension $i$ for $p \leq i \leq n - p$.

For $k \leq 0$, from (3.6) it follows that every $M^n \in \mathcal{M}(n, k, \lambda, a)$ admits an isometric immersion into $\mathbb{R}^{n+p}$, $2 \leq p \leq n/2$, and a height function $h$ with at most $a/\varepsilon(n, k, \lambda)$ critical points. Let $b_1 < \cdots < b_r$ be the critical values of $h$. From Morse Theory we know that for any $b_i < t_1 < t_2 < b_{i+1}$, every connected component of $h^{-1}([t_1, t_2])$ is homeomorphic to $S^{n-1} \times [t_1, t_2]$. Hence there exists a number $s$ depending only on $n, k, \lambda, a$ such that the number of $n$-disks needed to cover $M^n$ is at most $s$.

\textbf{Proof of Theorem 1.2.} Let $f: (M^n, g) \rightarrow \mathbb{R}^{n+p}$ be an isometric immersion whose scalar curvature and the second fundamental form satisfy $|\text{scal}| \geq \lambda\|\alpha\|^2$. Appealing to Proposition 2.4, we have

$$\|\alpha \bullet \alpha - \frac{\text{scal}}{n(n-1)}g \bullet g\|^{n/2}(x) \geq \left(\frac{1}{4}c(n, p, \lambda)\right)^{n/4} \int_{\Omega(\alpha|_x)} |\det A_\xi|dV_\xi$$

for all $x \in M^n$. By virtue of (3.4) and (3.5), the above inequality becomes

$$\|R - \frac{\text{scal}}{n(n-1)}R_1\|^{n/2}(x) \geq \left(\frac{1}{4}c(n, p, \lambda)\right)^{n/4} \int_{\Omega(\alpha|_x)} |\det A_\xi|dV_\xi$$
for all \( x \in M^n \). Integrating over \( M^n \), we obtain
\[
\int_M \left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\|^{n/2} dM \geq \left( \frac{1}{4} c(n, p, \lambda) \right)^{n/4} \sum_{i=p}^{n-1} \int_{U_i} | \det A_\xi | d\Sigma.
\]

Then the rest of the proof is the same as in Theorem 1.1.

If the scalar curvature of \( M^n \) is non-positive, then Proposition 2.4 yields
\[
\left\| \alpha \cdot \alpha - \frac{\text{scal}}{n(n-1)} g \cdot g \right\|^{n/2} (x) \geq \left( \frac{1}{4} c(n, p, \lambda) \right)^{n/4} \int_{S^{p-1}} | \det A_\xi | dV_\xi
\]
for all \( x \in M^n \). In view of (3.4) and (3.5), the above inequality becomes
\[
\left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\|^{n/2} (x) \geq \left( \frac{1}{4} c(n, p, \lambda) \right)^{n/4} \int_{S^{p-1}} | \det A_\xi | dV_\xi
\]
for all \( x \in M^n \). Integrating over \( M^n \), we obtain
\[
\int_M \left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\|^{n/2} dM \geq \left( \frac{1}{4} c(n, p, \lambda) \right)^{n/4} \int_{U_i} | \det A_\xi | d\Sigma.
\]
Bearing in mind the definition of the total absolute curvature \( \tau(f) \) of \( f \), we finally get
\[(3.7) \int_M \left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\|^{n/2} dM \geq \varepsilon(n, \lambda) \tau(f), \]
where
\[\varepsilon(n, \lambda) := \min_{2 \leq p \leq n/2} \left( \frac{1}{4} c(n, p, \lambda) \right)^{n/4} \text{Vol}(S^{n+p-1}).\]

Thus, the desired inequality follows immediately from (3.7) and Theorem 3.1.

The proof that the class \( \mathcal{M}(n, \lambda, a) \) contains at most finitely many homeomorphism types is the same as in Theorem 1.1. □

**Proof of Corollary 1.1.** We shall prove that \( M^n \) is simply connected. Assume to the contrary that \( M^n \) is not simply connected. Since the fundamental form \( \pi_1(M^n) \) is finite, it contains a subgroup isomorphic to \( \mathbb{Z}_q \) for some prime \( q \). Let \( \pi : \tilde{M}^n \to M^n \) be the Riemannian covering of \( M^n \) corresponding to \( \mathbb{Z}_q \). Then \( \tilde{M}^n \) is compact, and we may appeal to Theorem 1.2 for the isometric immersion \( f \circ \pi \), to conclude that \( H_p(\tilde{M}^n; \mathbb{Z}_q) = 0 \). According to a deep result due to Böhm and Wilking [1], the universal covering of \( \tilde{M}^n \) is diffeomorphic to the sphere \( S^n \). Then, from the spectral sequence of the covering \( S^n \to \tilde{M}^n \), we deduce that
\[H_p(\tilde{M}^n; \mathbb{Z}_q) \simeq H_p(K(\mathbb{Z}_q, 1); \mathbb{Z}_q) = \mathbb{Z}_q,\]
contradiction. Thus, \( M^n \) is simply connected and according to [1] it is diffeomorphic to the sphere \( S^n \). □

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