Dynamical percolation transition in the two-dimensional ANNNI model

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Abstract

The dynamical percolation transition of the two-dimensional axial next nearest-neighbor Ising model due to a pulsed magnetic field has been studied by finite size scaling analysis (by Monte Carlo simulation) for various values of frustration parameters, pulse width and temperature (below the corresponding static transition temperature). It has been found that the size of the largest geometrical cluster shows a transition for a critical field amplitude. Although the transition points shift, the critical exponents remain invariant for a wide range of frustration parameters. They are also the same as those obtained for the 2d Ising model. This suggests that although the static phase diagrams of these two models differ significantly in various aspects, the dynamical percolation transitions of these models belong to the same universality class.

1. Introduction

The response of magnetic systems to time-dependent external magnetic fields has been of current interest in statistical physics [1–12]. Magnetization reversal dynamics is an important subject for many technological applications of magnetic materials. In the magnetic recording industry, fast magnetization reversal in storage media is essential for a high data transfer rate [13]. These spin systems driven by time-dependent external magnetic fields have basically got a competition between two time scales: the time period of the driving field and the relaxation time of the driven system, giving rise to interesting non-equilibrium phenomena. Magnetization reversal by a pulsed magnetic field has been studied extensively [1–12]. The magnetization reversal was studied below the corresponding static transition temperature ($T_c$), where the system remains ordered. A pulse of finite strength ($h_p$) and finite duration ($\Delta t$), applied along the opposite direction to the prevalent order, tends to reverse the magnetization. Depending on the temperature ($T$) and pulse duration ($\Delta t$), there will be a critical field strength ($h_c$) at which the system will be forced to change from one ordered state (magnetization $m_0$) to another ordered state (magnetization $-m_0$). Below this critical strength, the magnetization will eventually remain unchanged as the field pulse is taken off. This transition is called a ‘magnetization reversal transition’.

A percolation transition is also associated with this magnetization reversal induced by the external magnetic field. By percolation transition we mean the transition of the largest ‘geometrical cluster’ (consisting of nearest-neighbor parallel spins) size from a large value to zero. In the initially ordered state, large clusters of a particular spin orientation (‘up’ or ‘down’) exist. However, owing to the pulsed field directed along the opposite direction, the spins gradually flip (depending on the magnitude and duration) and for a critical value of the field or pulse duration the pre-existing large clusters vanish, giving rise to oppositely oriented clusters. Thus, if we measure the largest ‘geometrical cluster’ size for a particular orientation at a certain temperature (below the critical temperature) and finite duration of field pulse, it shows a transition at a critical field magnitude. Although the thermal percolation transition of the ‘geometrical’ clusters in the 2d Ising model has been studied extensively [14–17], very little is known about the field induced dynamical percolation transition [18]. It was found numerically (based on Monte Carlo simulation) that for the pure two-dimensional Ising model at finite temperature the field induced magnetization transition and the percolation transition occur for the same critical magnetic field. Moreover, the critical exponents were...
found (using finite size scaling analysis) to be different from those of the thermal percolation transition, indicating the dynamical percolation transition to belong to a separate universality class [18]. This transition was also studied for the 2d Ising model with diagonal next-nearest-neighbor interaction for a typical value of the frustration parameter [18] and it was found that the critical exponents remain unchanged.

Here we intend to study the field induced percolation transition for the 2d axial next-nearest-neighbor Ising (ANNNI) model, which is one of the simplest classical Ising models with a tunable frustration and has been studied for a long time (see [19–22] for review). The 2d ANNNI model is a square lattice Ising model with a nearest-neighbor ferromagnetic interaction along both the axial directions and a second-nearest-neighbor anti-ferromagnetic interaction along one axial direction. The Hamiltonian is

$$\mathcal{H} = -J \sum_{x,y} s_{x,y}[s_{x+1,y} + s_{x,y+1} - \kappa s_{x+2,y}]$$

(1)

where the sites \((x, y)\) run over a square lattice, the spins \(s_{x,y}\) have binary states \((\pm 1)\), \(J\) is the nearest-neighbor interaction strength and \(\kappa\) is a parameter of the model. For positive values of \(\kappa\), the second-nearest neighbor interaction introduces a competition or frustration. The \(T-\kappa\) phase diagram for this model shows a rich behavior. It is easy to prove analytically that [19–22] at zero temperature, the system is in a ferromagnetic state for \(\kappa < 0.5\), and in antiphase \((+++---+---\cdots\text{along the } x\text{ direction, and all like spins along the } y\text{ direction})\) for \(\kappa > 0.5\) with a ‘multiphase’ state at \(\kappa = 0.5\). (The multiphase state comprises all possible configurations that have no domain of length 1 along the \(x\) direction.) However, the finite temperature phase diagram is yet to be solved analytically. There had been many predictions from various kinds of numerical calculations. Here we present the qualitative phase diagram (figure 1) that has been obtained by extensive Monte Carlo studies [23, 24]. As the previously conjectured floating phase was not found in [23, 24], it has not been shown in figure 1.

The phase diagram consists of a ferromagnetic phase (for \(\kappa < 0.5\)) and an antiphase (for \(\kappa > 0.5\)) at low temperature along with a paramagnetic phase at high temperature (for all \(\kappa\) values). Thus, for both \(\kappa < 0.5\) and \(\kappa > 0.5\) there is no transition temperature depending upon the value of \(\kappa\). For \(\kappa < 0.5\), the study of the percolation transition is done in a similar way as that for the unfrustrated 2d Ising model. However, for \(\kappa > 0.5\), the cluster cannot be defined as we have for \(\kappa < 0.5\), because for \(\kappa > 0.5\) below the static transition temperature we mainly have striped states with like spins oriented along the \(y\)-axis and a periodic modulation along the direction of frustration. Here we propose an alternative definition of a cluster to study the percolation transition. By a cluster we mean a set of nearest-neighbor spins which is in the same state as it was in its initial condition. Starting from an exactly antiphase state (initial cluster size is the system size) we achieve a steady state at a particular temperature (due to thermal agitation some of the spins flip and thus patches of clusters form). Now we apply a field pulse which varies spatially along the system such that the field acts on the spin along the direction opposite to that with which it started the dynamics i.e. \(\text{sgn}(h_p) = -\text{sgn}(s_{x,y}(0))\), where \(h_p\) is the applied field strength at site \(x, y\) and \(s_{x,y}(0)\) is the initial spin value at lattice site \(x, y\). This means that if we start with a ‘+−−+−−−−’ spin configuration along the \(x\) direction, the applied field will be along the downward direction for the first two columns and upward along the next two columns and this pattern continues over the entire system. As soon as we switch on the field pulse (applied along the reverse direction of the initial spin orientation), the spins will tend to orient along the field direction and the spins which are at a state identical to that of its initial state may flip (depending on the field strength and duration) and thus the cluster sizes eventually decrease giving rise to a percolation transition. If instead of the spatially modulated pulse, we would have applied a homogeneous pulse (for \(\kappa > 0.5\)), then the role of the applied field would have been different for alternate stripes of the antiphase. Some of the spins would have been favored, while others would have been opposed by the applied external field pulse. On the other hand for \(\kappa < 0.5\), even if we apply a homogeneous field, all of the spins are equally treated (tends to orient along the field). Thus, to keep similarity in the nature of the transition, we have applied a spatially modulated field for \(\kappa > 0.5\).

In this paper we study the percolation transition for a wide range of frustration parameters \((0.0 < \kappa < 1.0)\) and find how the critical field strength varies with the frustration parameter for a particular pulse duration. We have used Monte Carlo simulation to obtain the numerical results and a fully periodic boundary condition throughout our entire study. We have calculated the critical exponents numerically by finite size scaling analysis to determine whether the exponents are affected by the frustration parameter. In section 2, we discuss the model studied. In section 3, we present the results obtained for the finite size scaling analysis of the percolation order parameter and in section 4 the fractal dimension and the scaling analysis of the Binder cumulant. In the final section we discuss some consequences related to our study. The main observation of this work is that the critical exponents are independent of the frustration parameter and also the same as those of the normal 2d Ising model.

![Schematic phase diagram of the two-dimensional ANNNI model according to recent studies [23, 24].](image-url)
2. The model

The model studied here is the 2d ANNNI model under a time-dependent external magnetic field pulse, described by the Hamiltonian

\[ \mathcal{H} = -J \sum_{x,y} s_{x,y} [s_{x+1,y} + s_{x,y+1} - \kappa s_{x+2,y}] - h(t) \sum_{x,y} s_{x,y} \]

where the sites \((x, y)\) run over a square lattice, the spins \(s_{x,y}\) are \(\pm 1\), \(J\) is the nearest-neighbor interaction strength (we keep \(J = 1.0\) here on) and \(\kappa\) is the competition or frustration parameter. The static critical temperature \((T^c)\) depends on the value of \(\kappa\) (figure 1). For \(\kappa = 0\), \(T^c = 2/\ln(1 + \sqrt{2}) \approx 2.269\ldots\). For \(\kappa < 0.5\), the field pulse applied is spatially uniform (directed opposite to that of the initial spin configuration we start with) having a time dependence:

\[ h(t) = \begin{cases} -h_p, & t_0 \leq t \leq t_0 + \Delta t \\ 0, & \text{otherwise} \end{cases} \]

where \(\Delta t\) is the duration of the field pulse applied (in terms of the number of Monte Carlo sweeps). For \(\kappa > 0.5\), the field pulse applied is such that the field direction is opposite to that of the starting configuration (antiphase order) of the spins, but the time dependence is the same as that of equation (3). In our simulation we ensure that at the time \(t_0\) at which the pulse is applied, the system reaches its equilibrium configuration at that temperature \(T (< T_c)\).

The spins are selected randomly and are flipped following the normal Metropolis algorithm. The energy difference \((\Delta E)\) to flip the spin is calculated and flipped with probability \(\min[1, \exp(-\Delta E/T)]\). At \(t = t_0\), the system achieves a steady state at the corresponding temperature (in the absence of any field). As soon as the field pulse is switched on, a large number of spins which were intact in their initial state, flip during the pulse duration. Thus, the percolation order parameter \(P_{\text{max}} = S_t/L^2\) (where \(S_t\) is the size of the largest cluster and \(L\) is the linear size of the system) decreases during this duration and for a particular combination of \(T, \kappa\) and \(\Delta t\), for a particular value of \(h_p^c\), the system undergoes a percolation transition (figure 2).

It is found that for a particular temperature for \(\kappa < 0.5\), as the value of \(\kappa\) is increased, the transition field \(h_p^c\) decreases, but for \(\kappa > 0.5\), \(h_p^c\) increases with \(\kappa\) (figure 3). For \(\kappa < 0.5\), as \(\kappa\) is increased, the increasing frustration itself destabilizes the order and thus the pulse amplitude needed for transition decreases. However, for \(\kappa > 0.5\), the situation is different. The increasing frustration parameter stabilizes the antiphase more and more, thus requiring a higher pulse amplitude for the percolation transition (figure 3).

Typical domain structures for a \(100 \times 100\) lattice both for \(\kappa < 0.5\) (\(\kappa = 0.2\)) and \(\kappa > 0.5\) (\(\kappa = 0.8\)) at \(h_p \ll h_p^c\), \(h_p = h_p^c\) and \(h_p \gg h_p^c\) are shown in figures 4 and 5 respectively. It can be seen from both figures that for \(h_p \ll h_p^c\) and \(h_p \gg h_p^c\), ordered structure prevails. Only the spin orientation changes. This change (transition) occurs at \(h_p = h_p^c\).

3. Finite size scaling analysis of the percolation order parameter

In this section we discuss the percolation transition behavior. For a particular combination of \(\kappa, T\) and \(\Delta t\), we study the variation of \(P_{\text{max}}\) with \(h_p\). The percolation transition is characterized by several exponents which also determine the universality class. The order parameter i.e., the relative size of the largest cluster, varies as

\[ P_{\text{max}} \sim (h_p^c - h_p)^\beta. \]

The correlation length diverges near the percolation transition point as

\[ \xi \sim (h_p^c - h_p)^{-v}, \]

where \(h_p^c\) is the critical field amplitude. The other exponents can be calculated from various scaling relations [25]. However, owing to finite size effects it is difficult to
Figure 4. Typical domain structures of a $100 \times 100$ lattice for $h_p \ll h^c_p$, $h_p = h^c_p$ and $h_p \gg h^c_p$ for $\kappa = 0.2$, $\Delta t = 4$ and $T = 0.2$. The shaded regions represent the domains corresponding to up spins.

Figure 5. Typical domain structures of a $100 \times 100$ lattice for $h_p \ll h^c_p$, $h_p = h^c_p$ and $h_p \gg h^c_p$ for $\kappa = 0.8$, $\Delta t = 4$ and $T = 0.2$. The shaded regions represent the domains corresponding to up spins.
determine these exponents precisely. We have determined these exponents from the finite size scaling relations. $P_{\text{max}}$ follows the scaling form

$$P_{\text{max}} = L^{-\beta/\nu} F\left[L^{1/\nu}(h_p^c - h_p)\right], \quad (6)$$

where $F$ is a suitable scaling function. This scaling function shows that the largest cluster grows with system size and applied field. It also indicates that at the critical field the scaled largest cluster has a constant value. By tuning $\beta/\nu$, all of the $P_{\text{max}}L^{\beta/\nu} - h_p$ curves can be made to cross at a single point and this point determines the critical field amplitude ($h_p^c$). Now, once $\beta/\nu$ and $h_p^c$ has been estimated, by tuning $1/\nu$, one can collapse all of the $P_{\text{max}}L^{\beta/\nu} - (h_p^c - h_p)L^{1/\nu}$ curves on one another, and thus $\nu$ is determined. We have shown both these plots for $\kappa = 0.2$ ($\kappa < 0.5$) and $\kappa = 0.7$ ($\kappa > 0.5$) in figures 6 and 7, respectively. For both the curves we get the same values for $\beta/\nu = 0.20 \pm 0.05$ and $1/\nu = 0.85 \pm 0.05$ and more importantly this value is the same as that we obtained for the normal 2d Ising model found earlier [18], which indicates that the dynamical percolation transition of the 2d ANNNI model also belongs to the same universality class as that of the normal 2d Ising model. Only the critical field amplitude changes (as shown in figure 3). We have repeated the same exercise for some other combination of $\kappa$, $T$ and $\Delta t$, but the exponents remain unchanged. This suggests that the magnitude of frustration does not have any drastic effect on the dynamical percolation phenomenon except the quantitative shift of the transition point.

To have a picture of the effect of various parameters on the critical field amplitude we present two graphs, one for $\kappa < 0.5$ (figure 8) and the other for $\kappa > 0.5$ (figure 9), each showing the variation of $h_p^c$ with $\Delta t$ for different sets of $\kappa$ and $T$. It is quite clear from figure 8 that, for any combination of $\kappa$ and $T$, $h_p^c$ decreases with $\Delta t$. Also for any combination of $\kappa$ and $T$, $h_p^c$ decreases with $T$, because thermal excitation always helps in percolation transition and thus reducing the necessary external field amplitude. However, the frustration parameter plays opposite roles in the two halves of the $\kappa$ regime. For a fixed pair of $T$ and $\Delta t$, when we are below $\kappa < 0.5$, with an increase in $\kappa$ we need less field to reach the transition point because increasing the frustration parameter itself reduces the parallel alignment between the adjacent spins and thus facilitates the transition. However, above $\kappa = 0.5$, the increase in the frustration parameter supports the antiphase structure (with which we begin the dynamics for $\kappa > 0.5$), and thus we have to apply more field amplitude to flip the spins to the opposite direction. Thus, the frustration parameter plays opposite roles in cluster formation in the two regimes.
Figure 9. Plot of $h_c^\kappa$ against $\Delta t$ for three different combinations of $(\kappa, T)$ for the regime $\kappa > 0.5$.

Figure 10. Variation of the largest cluster size with the linear system size for two different values of $\kappa = 0.2$ and $0.7$ at the critical field for $T = 0.2$ and $\Delta t = 4$. The log–log plot gives the fractal dimension to be $D = 1.82 \pm 0.01$ for both the values of $\kappa$.

4. Fractal dimension and Binder cumulant

At the transition point, the percolation clusters become self-similar. The largest cluster depends on the linear system size and it varies as $S_L \sim L^D$, where $S_L$ is the largest cluster and $D$ is the fractal dimension. The fractal dimension measured (figure 10) is found to be independent of the value of $\kappa$ and the value is $D = 1.82 \pm 0.01$, which is again the same as that we found in the case of the normal 2d Ising model.

For further verification of the critical points and the exponents, we have also studied the Binder cumulant of the order parameter, defined as [26],

$$U = 1 - \frac{(P_{\text{max}}^4)}{3(P_{\text{max}}^2)^2},$$

where $P_{\text{max}}$ is the percolation order parameter and the angular brackets denote the ensemble average. The crossing point of the curves $U-h_p$ for different system sizes gives the critical field amplitude ($h_c^\kappa$). The Binder cumulant has also been used to determine the correlation length exponent as it follows the scaling form

$$U = U((h_c^\kappa-h_p)L^{1/\nu})$$

where $U$ is a suitable scaling function. Having obtained $h_c^\kappa$ from the crossing point, if we plot $U$ with the scaled field amplitude ($h_c^\kappa-h_p)L^{1/\nu}$, for a suitable value of $\nu$ we will obtain a data collapse and that value of $\nu$ determines the correlation length exponent. We have studied these results both for $\kappa < 0.5$ (figure 11) and $\kappa > 0.5$ (figure 12).
5. Discussions

Spatially modulated phases of metallic alloys can be investigated in terms of the ANNNI model and considering the application of these alloys in the magnetization reversal phenomenon, we have studied the magnetic field pulse induced percolation phenomenon in the 2d ANNNI model. We have focused on the role of the frustration parameter in affecting the percolation transition of the 2d ANNNI model. By applying a magnetic field pulse of finite duration ($\Delta t$) for a certain frustration parameter ($\kappa$) and temperature ($T$) (below the corresponding static transition temperature), we have determined the critical field strength ($h_c^\kappa$) for which the transition occurs. Although we have found that the frustration parameter highly enriches the static phase diagram of the 2d ANNNI model (various novel phases and transitions appear due to $\kappa$), quite surprisingly it does not have any large impact on the dynamical percolation transition apart from shifting the transition points. We have restricted our study to the regime $0 < \kappa < 1.0$. From finite size scaling analysis we have showed that both the percolation order parameter and the Binder cumulant lead to the same values of critical exponents as we have found for the normal 2d Ising model (thus, it belongs to the same universality class as that of the dynamical percolation transition of the 2d Ising model). The values of exponents determined are $\beta/\nu = 0.20 \pm 0.05$ and $1/\nu = 0.85 \pm 0.05$. The fractal dimension $D = 1.82 \pm 0.01$ also confirms the relations between the exponents.

Although we have not presented the details, in this context we have also investigated the transition behavior of the magnetization (for $\kappa < 0.5$) and sub-lattice magnetization (for $\kappa > 0.5$). The sub-lattice magnetization is defined as $x$-direction; the sub-lattice magnetization is defined by $m_x = \frac{1}{L} \sum_{y=0}^{L-1} m_{x y} = \frac{1}{L} \sum_{x=0}^{L-1} m_{x y}$, where $L$ is the length of the system. The transition points were found to be the same as those of the percolation transition (for a certain set of values of $\kappa$, $T$ and $\Delta t$). The study of finite size scaling of the order parameters also gives the same critical exponents ($\beta/\nu$ and $1/\nu$). Only the value of the crossing point of the Binder cumulant for magnetization shows non-universal behavior for different temperatures. This value ($U^* = 0.62 \pm 0.01$) is robust in the case of percolation transition. This study may be useful in the application of magnetization reversal of spatially modulated magnetic materials where the only change necessary is the value of the magnitude of the critical field, keeping the nature of the transition intact. Although we have limited our regime of investigation to below $\kappa = 1.0$, one can also explore this transition beyond $\kappa = 1.0$ or other modulated structures.

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