The quantum measurement problem enhanced

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Abstract

The quantum measurement problem as formalised by Bassi and Ghirardi [Phys. Lett. A 275 (2000)373] without taking recourse to sharp apparatus observables is extended to cover impure initial states.

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1 Introduction

In a recent letter [1] Bassi and Ghirardi have developed a strong and still very simple formulation of the quantum measurement problem (QMP) within standard quantum theory. They derived the nonoccurrence of definite apparatus configurations from assumptions considerably weaker than the ones of von Neumann’s original QMP [2]. They carefully justified their assumptions from a standard quantum theoretical perspective and removed certain oversimplifications from von Neumann’s treatment. The latter had been used by some authors (quoted in [1]) as loopholes to deny the QMP’s very existence.

Bassi and Ghirardi take the premeasurement state as a tensor product of two pure states. One for the microsystem to be measured and one for the environment incorporating the apparatus. If each individual quantum system indeed possesses a pure state, then the Bassi Ghirardi QMP is sufficiently strong, since if individual systems do not develop definite apparatus configurations, an ensemble of different pure states cannot do so either. Yet standard quantum theory suggests that individual systems in general have impure states. The argument is as follows. If a composite system with Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ has a pure state density operator $\rho$, its subsystem 1 has the induced density operator $\rho_1$ defined through: $\text{Tr}(\rho (A \otimes id_2)) = \text{Tr}(\rho_1 A)$ for all linear continuous $A : \mathcal{H}_1 \to \mathcal{H}_1$. In general $\rho_1$ is impure. This fact leads to the idea that impure states should be considered as equally fundamental as pure ones. They should not be associated with ensembles only, but with individual systems too. Therefore from the perspective of standard quantum theory a weakness of the Bassi Ghirardi QMP lies in its limitation to pure states and it seems necessary to investigate whether the argument of Bassi and Ghirardi can be generalised to impure premeasurement states.

A sequence of successively more general formulations of the QMP for impure states already exists. These works by Wigner [3], d’Espagnat [4], Fine [5], Shimony [6], Busch and Shimony [7], Busch [8], they all derive from varying assumptions that the postmeasurement state $\rho$ does not equal a convex sum $\sum_i p_i \rho_i$ (i.e. $\sum_i p_i = 1$, $p_i > 0$) of mutually exclusive states $\rho_i$ with definite apparatus configurations. (Here the apparatus configuration of $\rho_i$ is assumed different from the one of $\rho_j$ for $i \neq j$.) Since the apparatus configuration is definite in the trivial case of a one term sum, i.e. $\rho = \rho_i$, only, the postmeasurement apparatus configuration is indefinite. Note that

\footnote{1} melodramatised by Schrödinger’s cat
even if \( \rho \) would equal a convex sum of the above type with at least two terms, the apparatus configuration still were indefinite - at least if impure states are associated with individual systems. Therefore the nonexistence of the above convex representation proves much more than is needed for having a measurement problem, because for the QMP it is completely irrelevant whether the postmeasurement state is diagonal with respect to the apparatus observable or not. The QMP arises whenever the apparatus observable has nonzero variance.

In all the above impure state formulations of the QMP different definite apparatus configurations are associated with mutually orthogonal subspaces of the problem’s Hilbert space. It is assumed that the ranges of two density operators \( \rho_1 \) and \( \rho_2 \) with different apparatus configurations are orthogonal to each other. Yet it is exactly this conception which has been abandoned by Bassi and Ghirardi for pure states. Therefore the question still stands whether the argument of Bassi and Ghirardi can be generalised to impure premeasurement states.

I shall present here a QMP which also covers the case of impure states but avoids associating different apparatus configurations with orthogonal subspaces. This realises what is called unsharp pointer reading or unsharp objectification in \footnote{The diagonalisability obtains relevance if states are associated with ensembles exclusively. Then a convex decomposition of a state is understood as a mixture of other ensembles. Such a strict ensemble interpretation, however, is unable to associate a state with an ensemble if the ensemble is understood as a single composite system. Or you may ask how many members needs an ensemble to be an ensemble.}

The debate on whether the decoherence program has resolved the QMP continues without emerging consensus \footnote{The debate on whether the decoherence program has resolved the QMP continues without emerging consensus \cite{10}. Clearly, any proposed resolution of a specific QMP should, as a first consistency check, exhibit which of the QMP’s assumptions are violated, in order to circumvent its conclusion. Bohmian mechanics for instance harshly violates the assumption 2 of section 3 and therefore is able to resolve the QMP in a surprisingly simple and eye-opening way \cite{11}, \cite{12}, \cite{13}. The apparatus configuration is determined by Bohmian positions and has little to do with the wave function part of the state. Almost all of the above discussion is absurd from a Bohmian point of view. Another proposal to resolve the QMP violates assumption 3 of section 3: stochastic state reduction models give up the unitary time evolution \cite{14}. Since the decoherence program employs the minimal quantum structures only it necessarily seems incapable to resolve the QMP.}

In order to better display the weakening of assumptions, I shall distill in section 2 the findings of \cite{1} into a tight proposition. Here I shall not duplicate the ample discussion, which Bassi and Ghirardi gave in order to support their assumptions. The proof of their QMP, however, will be redone. In section 3 I shall state and prove the stronger QMP covering impure states. This proposition contains the one of section 2 as a special case. Finally I speculate on the impact which the QMP might bring to the future development of quantum theory.

## 2 QMP for pure states

Bassi and Ghirardi made the following assumptions. Let \( \mathcal{H}_s \) be the microsystem's Hilbert space from which two vectors \( \psi_1, \psi_2 \) are chosen such that \( \langle \psi_i, \psi_j \rangle = \delta_{i,j} \). The environment (including the measurement apparatus and the observer) has the Hilbert space \( \mathcal{H}_E \). The unit sphere in the total space \( \mathcal{H}_s \otimes \mathcal{H}_E \) is assumed to contain subsets \( A_1, A_2 \) such that the observer perceives the apparatus in the distinct configuration \( i \) if and only if the total system’s pure state is represented by a vector belonging to \( A_i \). The sets \( A_i \) are supposed to be separated from each other in the following sense. For some fixed positive value \( \varepsilon < 1/2 \) the inequality

\[
\langle x, y \rangle^2 < \varepsilon
\]

holds for all \( x \in A_1 \) and for all \( y \in A_2 \). The initial state unit vectors are assumed to be given by \( \psi \otimes e \), where \( e \) belongs to the set

\[
P = \{ e \in \mathcal{H}_E \mid |e| = 1 \text{ and } \psi \otimes e \in A_1 \text{ for all unit vectors } \psi \in \mathcal{H}_s \}.
\]
Thus initially the observer perceives apparatus configuration 1. The measurement interaction results in a unitary operator $U$ on $\mathcal{H}_s \otimes \mathcal{H}_E$ giving the time evolution of states to an instant of time at which this interaction has come to an end. Definite apparatus configurations are supposed to result from $U$ for the initial states $\psi_i \otimes e$, i.e. $U(\psi_i \otimes e) \in A_i$ is assumed to hold for all $e \in P$ and for $i = 1, 2$.

**Proposition 1** The vector $(\psi_1 + \psi_2)/\sqrt{2}$ has a neighborhood $\mathcal{D}$ in $\mathcal{H}_s$ such that for all unit vectors $\psi \in \mathcal{D}$ and for all $e \in P$ holds $U(\psi \otimes e) \notin A_1 \cup A_2$.

**Remark 2** An initial state $\psi \otimes e$ with apparatus configuration 1 and $\psi$ sufficiently close to $(\psi_1 + \psi_2)/\sqrt{2}$ does not develop a definite postmeasurement configuration (either 1 or 2) under $U$.

Proof.

$$\left| \left\langle U\left(\frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \otimes e\right), U(\psi_1 \otimes e)\right\rangle \right|^2 =$$

$$= \frac{1}{2} |\langle (\psi_1 + \psi_2) \otimes e, \psi_1 \otimes e \rangle|^2 = \frac{1}{2} |\langle \psi_1, \psi_1 \rangle|^2 = \frac{1}{2} > \varepsilon.$$ 

Since $\psi \mapsto \langle \psi, \psi \rangle$ is continuous, $|\langle U(\psi \otimes e), (\psi_1 \otimes e) \rangle|^2 > \varepsilon$ for all $\psi$ in a neighborhood $\mathcal{D}$ of $(\psi_1 + \psi_2)/\sqrt{2}$ in $\mathcal{H}_s$. Because of $U(\psi_1 \otimes e) \in A_1$, the vector $U(\psi \otimes e)$ for any $\psi \in \mathcal{D}$ is closer to $A_1$ than any vector from $A_2$. Therefore $U(\psi \otimes e) \notin A_2$. Similarly one obtains $U(\psi \otimes e) \notin A_1$ for all $\psi \in \mathcal{D}$. 

# 3 QMP for impure states

As in the former section the total system is divided into a microsystem and an environment. The environment comprises the observer and the measurement apparatus. Since the environment may be entangled with rest of the world, which is ignored in the measurement interaction, it may not have a pure state even at the level of individual systems. According to standard quantum theory its state is generally given by an induced density operator, which is obtained by restricting the expectation value functional to those observables which are sensitive to the environment’s degrees of freedom only. Imperfections in the preparation of the microsystem may result in impure states for individual microsystems as well. Thus one has to address initial states which factorise into two density operators. Density operators are Hilbert Schmidt operators. I shall make use of this fact and therefore I recall some properties of Hilbert Schmidt operators now.

A linear continuous operator $a$ on a separable Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$ belongs to the set $\mathcal{C}_{HS}(\mathcal{H})$ of Hilbert Schmidt operators if and only if $\text{Tr}(a^*a) < \infty$. The set $\mathcal{C}_{HS}(\mathcal{H})$ is a complex vector space with respect to the addition of operators. The Hilbert Schmidt scalar product of $\mathcal{C}_{HS}(\mathcal{H})$ is defined as $\langle a, b \rangle_{HS} := \text{Tr}(a^*b)$. The associated Hilbert Schmidt norm is

$$\|a\|_{HS} := \sqrt{\text{Tr}(a^*a)}.$$ 

$\mathcal{C}_{HS}(\mathcal{H})$ is complete with respect to the norm $\|\cdot\|_{HS}$. Thus $\langle \cdot, \cdot \rangle_{HS}$ makes $\mathcal{C}_{HS}(\mathcal{H})$ into a Hilbert space. For all $a \in \mathcal{C}_{HS}(\mathcal{H})$ and for all $b \in \mathcal{C}(\mathcal{H})$ (the algebra of linear continuous operators on $\mathcal{H}$) there holds $ab \in \mathcal{C}_{HS}(\mathcal{H})$ and $ba \in \mathcal{C}_{HS}(\mathcal{H})$. Thus $\mathcal{C}_{HS}(\mathcal{H})$ is a left/right ideal of $\mathcal{C}(\mathcal{H})$. 


For $\psi \in \mathcal{H} \setminus \{0\}$ the orthogonal projection onto $\mathbb{C} \cdot \psi$ is denoted as $P_\psi$. It holds

$$\langle P_{\psi_1}, P_{\psi_2} \rangle_{HS} = \frac{1}{\|\psi_1\|^2 \cdot \|\psi_2\|^2} \text{Tr} \left( \psi_1 \langle \psi_1, \psi_2 \rangle \langle \psi_2, \cdot \rangle \right)$$

$$= \frac{|\langle \psi_1, \psi_2 \rangle|^2}{\|\psi_1\|^2 \cdot \|\psi_2\|^2} \in [0, 1].$$

Thus $\|P_\psi\|^2_{HS} = 1$. More generally, since $\|\rho\|^2_{HS} = \text{Tr}(\rho^* \rho) = \text{Tr}(\rho^2) \leq 1$, the set of density operators on $\mathcal{H}$ is a subset of $C_{HS}(\mathcal{H})$. The equality $\|\rho\|_{HS} = 1$ holds if and only if $\rho^2 = \rho$. Observe that $\|\sqrt{\rho}\|_{HS} = 1$ for any density operator, i.e. the square root of a density operator belongs to the unit sphere in $C_{HS}(\mathcal{H})$. Here $\sqrt{\rho}$ denotes that unique positive operator which obeys $(\sqrt{\rho})^2 = \rho$. For pure states holds $\sqrt{\rho} = \rho$.

I shall address the measurement problem now.

- **Assumption 1:** The microsystem has the (separable) Hilbert space $\mathcal{H}_s$. Two vectors $\psi_1, \psi_2 \in \mathcal{H}_s$ obeying $\langle \psi_i, \psi_j \rangle = \delta_{i,j}$ are chosen. The environment has the (separable) Hilbert space $\mathcal{H}_E$. The total Hilbert space is $\mathcal{H} := \mathcal{H}_s \otimes \mathcal{H}_E$.

- **Assumption 2:** The set of density operators on the total Hilbert space is supposed to contain two subsets $A_1, A_2$ with the property that the apparatus is (perceived) in configuration $i = 1, 2$ if and only if the total system’s density operator $\rho$ belongs to $A_i$. The sets $A_i$ are assumed to be separated from each other in the following sense, which also makes them disjoint. There exists a positive real number $\varepsilon < 1/2$ such that for all $\rho_1 \in A_1, \rho_2 \in A_2$ the inequality

$$|\langle \sqrt{\rho_1}, \sqrt{\rho_2} \rangle_{HS}| < \varepsilon < \frac{1}{2} \quad (2)$$

holds. In case of pure states this condition specialises to the analogous distance condition \(\|\|\) of section 2.

**Remark 3** Inequality (2) is weaker than the usual assumption that the apparatus configurations $i = 1, 2$ are associated with two orthogonal projections $P_i$, projecting onto mutually orthogonal subspaces of $\mathcal{H}$, eigenspaces of some apparatus observable. Thus $P_i P_j = \delta_{i,j} P_i$ holds. A density operator $\rho$ with apparatus configuration $i$ is assumed to give the projection $P_i$ the expectation value 1. Thus $\text{Tr}(\rho P_i) = 1$ holds for $\rho$ having apparatus configuration $i$. Now $\text{Tr}(\rho P_i) = 1$ is equivalent to $P_i \rho P_i = \rho$. From this then follows for density operators $\rho_i$ with apparatus configurations $i$ that

$$\langle \sqrt{\rho_1}, \sqrt{\rho_2} \rangle_{HS} = \left\langle \sqrt{P_1 \rho_1 P_1}, \sqrt{P_2 \rho_2 P_2} \right\rangle_{HS}$$

$$= \left\langle P_1 \sqrt{\rho_1} P_1, P_2 \sqrt{\rho_2} P_2 \right\rangle_{HS} = \text{Tr} \left( P_1 \sqrt{\rho_1} P_1 P_2 \sqrt{\rho_2} P_2 \right) = 0.$$

Thus the usual treatment with sharp apparatus observables \(\|\|\) amounts to imposing

$$\langle \sqrt{\rho_1}, \sqrt{\rho_2} \rangle_{HS} = 0$$

for all $\rho_1 \in A_1$, and for all $\rho_2 \in A_2$.

**Remark 4** In \(\|\|\) unsharp apparatus observables (of a restricted type to allow for definite apparatus configurations) are considered. In the present case of distinguishing two apparatus configurations only, this amounts to introducing a positive operator valued measure on the
measure space \( \{1, 2\} \). Thus the projections \( P_i \) are replaced by effects, i.e. linear continuous operators \( E_i : \mathcal{H} \rightarrow \mathcal{H} \) with \( 0 \leq E_i \leq \text{id}_\mathcal{H} \) and \( E_1 + E_2 = \text{id}_\mathcal{H} \). The effects \( E_i \) are assumed to have the eigenvalue \( 1 \) and a state \( \rho \) with definite apparatus position \( i \) is assumed to obey \( \text{Tr}(E_i \rho) = 1 \). Introducing an orthonormal basis of eigenvectors \( e_\alpha \) of \( \rho \) one obtains the spectral representation

\[
\rho = \sum_{\alpha \in I} \lambda_\alpha P_\alpha
\]

with \( \sum_{\alpha \in I} \lambda_\alpha = 1 \) and \( \lambda_\alpha > 0 \) for all \( \alpha \in I \). Now \( 0 \leq E_i \leq \text{id}_\mathcal{H} \) implies \( 0 \leq \langle e_\alpha, E_i e_\alpha \rangle \leq 1 \) and from

\[
1 = \text{Tr}(E_i \rho) = \sum_{\alpha \in I} \lambda_\alpha \langle e_\alpha, E_i e_\alpha \rangle
\]

one infers \( \langle e_\alpha, E_i e_\alpha \rangle = 1 \) for all \( \alpha \in I \). Therefore \( E_i e_\alpha = e_\alpha \), i.e. every eigenvector of \( \rho \) with nonzero eigenvalue is an eigenvector of \( E_i \) with eigenvalue \( 1 \). Let now \( \rho_1 \) and \( \rho_2 \) obey \( \text{Tr}(\rho_i E_i) = 1 \) and let \( x \) and \( y \) be eigenvectors of \( \rho_1 \) and \( \rho_2 \) for nonzero eigenvalues respectively. Thus \( E_1 x = x \) and \( E_2 y = y \) follows. From this and \( E_1 + E_2 = \text{id}_\mathcal{H} \) we obtain

\[
\langle x, y \rangle = \langle x, (E_1 + E_2) y \rangle = \langle x, E_1 y \rangle + \langle x, E_2 y \rangle = \langle E_1 x, y \rangle + \langle E_2 y, y \rangle = 2 \langle x, y \rangle.
\]

Thus \( \langle x, y \rangle = 0 \) follows. From this we obtain \( \sqrt{\rho_1} x = 0 \) and from the spectral representation \( \rho_1 = \sum_{\alpha \in I} \lambda_\alpha P_\alpha \) finally

\[
\langle \sqrt{\rho_1}, \sqrt{\rho_2} \rangle_{HS} = \sum_{\alpha \in I} \sqrt{\lambda_\alpha} \langle e_\alpha, \sqrt{\rho_2} e_\alpha \rangle = 0
\]

follows. Thus also Busch’s treatment \(^3\) with unsharp apparatus observables amounts to imposing \( \langle \sqrt{\rho_1}, \sqrt{\rho_2} \rangle_{HS} = 0 \) for all \( \rho_1 \in A_1 \), and for all \( \rho_2 \in A_2 \).

- Assumption 3: The time evolution under the measurement interaction from a premeasurement instant of time to a postmeasurement one is assumed to be given by some unitary operator \( U : \mathcal{H} \rightarrow \mathcal{H} \). The evolution of density operators is then the mapping \( u : \mathcal{C}_{HS}(\mathcal{H}) \rightarrow \mathcal{C}_{HS}(\mathcal{H}), a \mapsto U a U^* \).

**Remark 5** The linear mapping \( u \) is a unitary algebra automorphism, i.e.

\[
u(ab) = u(a)u(b) \quad \text{and} \quad \langle u(a), u(b) \rangle_{HS} = \langle a, b \rangle_{HS}.
\]

Thus for a density operator \( \rho \) there holds \( \sqrt{u(\rho)} = u(\sqrt{\rho}) \) and furthermore

\[
\langle \sqrt{u(\rho_1)}, \sqrt{u(\rho_2)} \rangle_{HS} = \langle u(\sqrt{\rho_1}), u(\sqrt{\rho_2}) \rangle_{HS} = \langle \sqrt{\rho_1}, \sqrt{\rho_2} \rangle_{HS}.
\]

**Proposition 6** Let the density operator \( E \) from \( \mathcal{H}_E \) be such that \( \rho_s \otimes E \in A_i \) for all density operators \( \rho_s \) on \( \mathcal{H}_s \). Assume \( (P_{\psi_i} \otimes E) \in A_i \) for \( i = 1, 2 \). Then the pure state \( P_{\psi_i} \) with \( \phi := \frac{1}{\sqrt{2}} (\psi_1 + \psi_2) \) has a neighborhood \( \mathcal{D} \) in the set of density operators on \( \mathcal{H}_s \) such that for all \( \rho_s \in \mathcal{D} \) holds \( u(\rho_s \otimes E) \notin A_1 \cup A_2 \).

**Remark 7** The proposition demonstrates that if the apparatus (irrespective of the microsystem’s state) is in configuration 1 before the measurement and if for the initial states \( P_{\psi_i} \otimes E \) the unitary measurement dynamics \( u \) results in states with apparatus configuration \( i \) then for an initial state \( \rho_s \otimes E \) with \( \rho_s \) sufficiently close to \( P_{\psi_i} \otimes E \) the postmeasurement density operator \( u(\rho_s \otimes E) \) does not possess a definite postmeasurement apparatus configuration.
Proof. Due to remark 5 we have
\[
\langle \sqrt{u(P_{\phi} \otimes E)}, \sqrt{u(P_{\psi_i} \otimes E)} \rangle_{HS} = \langle \sqrt{P_{\phi} \otimes \sqrt{E}}, \sqrt{P_{\psi_i} \otimes \sqrt{E}} \rangle_{HS}
\]
\[
= \langle \sqrt{P_{\phi}}, \sqrt{P_{\psi_i}} \rangle_{HS} \langle \sqrt{\sqrt{E}}, \sqrt{\sqrt{E}} \rangle_{HS}
\]
\[
= \langle P_{\phi}, P_{\psi_i} \rangle_{HS} = |\langle \phi, \psi_i \rangle|^2 = \frac{1}{2}.
\]
Thus \(\left| \langle \sqrt{u(P_{\phi} \otimes E)}, \sqrt{u(P_{\psi_i} \otimes E)} \rangle_{HS} \right| > \varepsilon\). Since \(u(P_{\psi_i} \otimes E) \in A_i\), the vector \(u(P_{\phi} \otimes E)\) neither belongs to \(A_1\) nor to \(A_2\). Because of the continuity of \(\langle \cdot, \cdot \rangle_{HS}\), for all \(\rho_s\) in a neighborhood \(D\) of \(P_{\phi}\) also \(u(\rho_s \otimes E) \notin A_1 \cup A_2\) holds.

4 Conclusion

The QMP shows that so far standard quantum theory has no rule of how to ascribe internal properties to general states of closed systems (comprising observers) such that our definite everyday sensations find an explanation. Standard quantum theory seems to need a splitting of the world into a quantum part and a dynamically unresolved environment from which the quantum part is observed and in relation to which properties can be induced by observation. Depending on the way how the environment observes, such external observation forces the quantum part into assuming properties by means of stochastic quantum jumps. These jumps either lead into the specific set of states which have the property under consideration or into the set of states which definitely do not have this property. (Since the union of these two subsets is unequal to the set of all states, complementarity of properties emerges.) All this seems to tell that standard quantum theory cannot constitute a consistent theoretical framework for describing arbitrarily large systems.

Bohmian mechanics enriches the conceptual framework of standard quantum mechanics by degrees of freedom ("hidden variables"), such that with the states of closed systems definite properties can be associated without making reference to any external agent. Quantum jumps and state reduction do not occur. States and their properties vary continuously and deterministically with time. The QMP completely disappears.\([\text{11}]\) If the Bohmian program could be extended to the realm of relativistic quantum fields, a quantum frame work with the potential for universal validity were found.

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