Vortical structures interaction

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Abstract. As a rule, a fluid flow near solid bodies is corresponded by vortical structures generation. Mathematical methods for study of vortical flows depend on the vortical structures genesis and scale. Four problems connected with the extended vortical structures interaction with each other and with solid bodies are solved in this paper.

1. Exact solution for the non-symmetrical flow near the symmetrical slender body under the symmetrical uniform flow
Asymmetry of vortical structures arising on solid surfaces at high angles of attack is known to cause the side force acting on symmetrical slender body. If the Reynolds number based on the body size and the far stream velocity is high, the flow can be considered within the Euler equations.

Consider the inviscid incompressible fluid flow near the low aspect ratio wing that assumed to be curved with the parabolic law and has the parabolic shape in plan with the partition in the symmetry plane and the partition height grows with the parabolic law (figure 1).

\[
\begin{align*}
\ell(x) &= 2\alpha \varepsilon \sqrt{x} , \\
y(x) &= -\varepsilon \beta \sqrt{x} , \\
h(x) &= \varepsilon \gamma \sqrt{x} 
\end{align*}
\]  

(1)

where \(\ell(x)\) is the dependence of the wing width on the streamwise coordinate \(x\), \(y(x)\) is the intersection of the wing and the partition, \(h(x)\) is the partition height, \(\varepsilon\) is the dimensional parameter, \(\alpha\), \(\beta\), \(\gamma\) are the dimensionless constants, \(0 \leq x < \infty\).

Figure 1. Parabolic wing geometry.

Due to the unsteady analogy \([1-3]\) the steady flow near the wing is equivalent to the expanding plate with the partition. The special geometrical shape (1) of the wing leads to arising the discrete vortical filaments instead of the vortical sheets that satisfy the Chaplygin–Zhukovsky conditions, i.e.
the finite velocity on the sharp edges [4]. The integro-differential equations of the vortical sheets evolution are reduced to algebraic equations and the problem of the separated flow near the slender parabolic wing admits the exact solution. If there are only two vortex filaments the flow is symmetrical. Two vortex filaments satisfy Chaplygin–Zhukovsky conditions on the side sharp edges of the wing and on the partition edges because of the symmetry. If the flow is non-symmetrical three vortex filaments satisfy three Chaplygin–Zhukovsky conditions on three sharp edges. It appears there is the range of the numbers $\beta = \beta / \alpha$, $\gamma = \gamma / \alpha$, when only the symmetrical flow is possible and there is the range of the numbers when the symmetrical and the non-symmetrical flows are possible. The example of two solutions with $\beta = 5$, $\gamma = 1.5$, are shown in figures 2 and 3. Thus, the concrete solution realization depends on the initial conditions, i.e. the conditions of the vortical structure generation in the vertex of the wing.

**Figure 2.** Non-symmetrical solution.

**Figure 3.** Symmetrical solution.

2. **Two counter-swirling jets interaction problem**

Viscosity influences the vortical structures interacting with each other as well as with the solid bodies. Consider the interaction of two counter-swirling Loitsyanskiy jets [5]. The statement of the problem is as follows. Two jets of a viscous incompressible fluid flow in submerged space from two parallel tubes. The internal surfaces are assumed to rotate in opposite directions and external ones are immobile. The last condition provides circulation to be zero for the jet in the tube exit unlike the two vortices diffusion problem [6]. For definiteness, the plane of the tube axes is horizontal. The rotation of the left tube is counter-clockwise and the rotation of the right tube is clockwise if we look upwind. The subject of the study is the space evolution of two jets scenario along the streamwise coordinate $x$ (the tube exit cross sections lay in the plane $x = 0$) and the confluence of two jets into one. Let $y$ axis be vertical and $z$ axis be orthogonal to the symmetry plane. Let $u$, $v$, $w$ be the velocity vector projections on the $x$, $y$, $z$ axes respectively and $p$ be the pressure.

The total streamwise momentum flux through every cross section of jets $J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u^2 + p) \ dy \ dz$ is conserved in the flow beyond the tube exits. The vertical momentum flux $J_y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uv \ dy \ dz$ and the streamwise angular momentum flux $L = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\gamma w - zw) \ dy \ dz$ are also conserved in the boundary layer approximation. Both these integrals equal to zero.

The problem was solved by the numerical integration of Navier–Stokes equations. The similarity parameters provided by these equations are the Reynolds number $Re$ based on the parameters of tube flow, the swirl number $S$ or the ratio of the maximal azimuthal velocity and the maximal streamwise
velocity in the tube, the ratio $\ell$ of the distance between the tube axes to the tube radius. They are taken $Re = 100$, $S = 0.5$, $\ell = 8$ in the calculation procedure.

The tangential discontinuity of the azimuthal velocity beyond the tube exit provides the velocity circulation around a loop enclosed the jet cross section to be zero. Further, while two jets interact weakly there are zones of streamwise vorticity with different signs inside every jet so the total vorticity flux through the jet cross section is zero. After the jet outflowing from a tube the tangential discontinuity are diffused into the vortex “coat” that rounds the vortex core of the jet with the vorticity of the opposite sign. In addition, the vortex zones of the jets are attracted due to the viscous diffusion and the suction (figure 4 (a)), the external vortex zone starts to annihilate and the vortex coat are dropped from the vortex core (figure 4 (b)). Further, the vortex cores are attracted and partially annihilate whereas the vortex coats are concentrated to the single vortices (figure 4 (c)). Thereon two jets confluence into one as the point with maximal streamwise velocity is in the symmetry plane. Thus, there are four vortices with alternating signs inside the expanding in the $x$ direction jet. With the further growth of the coordinate $x$ the upper vortices become almost symmetric with the lower vortices (figure 4 (d)). During the evolution of two jets the vorticity flux through the whole cross section is zero due to the symmetry but it is not zero for the flux through the left (or right) half plane of the cross section. The diffusion and the dissipation of vorticity determine the change of the vorticity flux through the left (or right) half plane of the jet cross section that is the velocity circulation on the contour catching this zone (figure 5). The circulation is zero at $x = 0$ because the external surfaces of tubes are immobile. Then the circulation grows due to the vortex coat annihilation. Circulation is maximum at $x = 16$ and begins to decrease due to the vortex cores annihilation. Circulation is zero at $x = 70$ when two jets confluence into one. Then circulation becomes negative and achieves the minimum at $x = 120$ and then its absolute value slowly decreases.

![Figure 4](image1.png)  
**Figure 4.** Streamwise vorticity at the different cross sections of two jets.

![Figure 5](image2.png)  
**Figure 5.** Vorticity flux through left half-plane of the flow cross section.

### 3. Problem of compressible fluid flow around rotating cylinder

A solitary rectilinear vortex filament velocity field corresponds to infinite kinetic energy per unit length, with an infinite contribution of either velocity field in the vicinity of the filament and velocity field at unlimited distance from it. However, the generation of the filament-like velocity field within a finite zone using a rotating cylinder is possible. In the case of incompressible fluid, such velocity field occurs due to the whirling fluid interaction with the cylinder surface by means of the friction force. What is the compressibility affect on the velocity field? It turned out that compressibility modifies the circumferential velocity field substantially so that the velocity circulation around certain closed curves can exceed the velocity circulation around a circle corresponding to the cylinder surface. A considerable influence of the moving surface speed on the produced velocity field is observed [7, 8].
Let us turn to the problem statement. The circular infinitely extended cylinder starts to rotate around its axis at the moment $t = 0$. The radius of cylinder is $r_\infty$, the speed of rotation is $w_\infty = \Gamma_\infty / 2\pi r_\infty$, the temperature of the cylinder surface is $T_\infty$. Undisturbed fluid is described by density $\rho_0$, pressure $p_0$, temperature $T_\infty$ and dynamic viscosity coefficient $\mu_0$. We assume the fluid to be an ideal gas, the Prandtl number $Pr$ to be constant and the dynamic viscosity coefficient depending on temperature linearly $\mu / T = \text{const}$.

The problem solution is determined by numerical integration of the Navier-Stokes equations. The axial symmetry is assumed. The temperature of cylinder surface $T_\infty$ is assumed equal to $T_0$. It turned out that the velocity circulation $\Gamma$ around a circle of radius $r$ is not monotonically decreasing function of $r$, in contrast to the incompressible case. The function $\Gamma(r)/\Gamma_\infty$ for the cases when the Mach number $M_* = w_\infty / a_0$, where $a_0$ is the undisturbed gas sound speed, is equal to 5 and 7 is presented in the figure 6. In close proximity of the cylinder surface, the velocity circulation falls, after that it starts to increase and reaches the value which is much higher than the velocity circulation around the cylinder surface. This nonmonotonic solution is caused by the temperature-dependent viscosity coefficient. The behavior of temperature is the sharp increase in a small zone near cylinder and then the monotonic approaching to undisturbed solution (see the figure 7). It is shown that the maximal values of temperature and circulation increase proportionally to $M_*^2$ as $M_*$ tends to infinity.

![Figure 6](image1.png) ![Figure 7](image2.png)

4. Vortex stretching in turbulent flow

In contrast to a two-dimensional case, in a three-dimensional flow vortical lines stretching is feasible. This process ensures a transfer of energy from the small scales to the large scales in a turbulent flow as well as the curvature of vortex tubes. The vortex tubes stretching happens due to the non-uniform external velocity field induced by other vortices, leading to the intensification of vorticity because vorticity flux through the vortex tube section is constant. While considering vortex stretching, we commonly imagine vortex tubes of circular or near-circular section. However, the shape of the vortex tube in the section where the stretching occurs can depend on the vortex structures scale.

Let us declare the problem statement. We consider a lattice of infinitely thin vortex filaments with the chequerwise alternating circulations $\pm \Gamma_1$. The distances $h$ between horizontal and vertical rows of vortices are equal. The positive vortices coordinates are $x = 2ih$, $y = 2jh$ and $x = (2i + 1)h$, $y = (2j + 1)h$, while the negative ones are $x = 2ih$, $y = (2j + 1)h$ and $x = (2i + 1)h$, $y = 2jh$. For any vortex $-\infty < z < \infty$ and $i, j$ are integers from $-\infty$ to $\infty$. The vortices of the first lattice are placed...
between the vortices of the second lattice. If the first lattice vortex axis is directed along the \( z \) axis then the second one is directed along the \( x \). The circulations of the second lattice vortices \( \pm \Gamma_2 \) are also chequerwise alternating. The positive vortices coordinates are \( y = (2j + 0.5)h \), \( z = 2kh \) and \( y = (2j + 1.5)h \), \( z = (2k + 1)h \), while the negative ones are \( y = (2j + 1.5)h \), \( z = 2kh \) and \( y = (2j + 0.5)h \), \( z = (2k + 1)h \). For any vortex \(-\infty < x < \infty \) and \( j,k \) are integers from \(-\infty \) to \( \infty \). In the framework of the inviscid fluid equations, all the vortices are unmoving even if the interaction of lattices is incorporated. The velocity induced by the lattices at the points belonging to a vortex is parallel to this vortex axis. Without loss of generality, in order to determine this velocity we consider the vortex at the point \( y = 0.5h \), \( z = 0 \) of the second lattice. The cumulative velocity induced by the whole lattice is

\[
\mathbf{u}(x) = \frac{\Gamma}{2h} \sum_{k=-\infty}^{\infty} (1)^{k+1} \sec h \left( \frac{\pi}{h} \left( \frac{x}{h} - k \right) \right).
\]

If vortices with non-zero thickness, i.e. structured vortices, are considered, either stretching or contraction are possible. The stretching of vortices in special cases of viscous flow at infinitely large times is outlined in [9]. In the present work, the problem of the stretching of the vortex at specific finite times is considered. We consider the diffusion of the second lattice vortex in the vicinity of the point \( x = 0.5h \), \( y = 0.5h \), \( z = 0 \). The velocity component \( u \) along the axis \( x \) vanishes but its derivative is non-zero, which results to stretching of the vortex

\[
\frac{\partial u}{\partial x} = \beta \approx 1.094 \frac{\Gamma}{h^3}.
\]

The stretching in the vicinity of the thin vortex generates the velocity along the \( y \) axis

\[
v = -\beta(y - 0.5h) = -\beta y_+.
\]

The thin vortex vorticity diffusion obeys the Helmholtz equation

\[
\frac{\dot{\omega}_x}{\dot{t}} + u \frac{\partial \omega_x}{\partial x} + v \frac{\partial \omega_x}{\partial y} + w \frac{\partial \omega_x}{\partial z} - \omega_y \frac{\partial u}{\partial y} - \omega_z \frac{\partial u}{\partial z} = \nu \left( \frac{\partial^2 \omega_x}{\partial x^2} + \frac{\partial^2 \omega_x}{\partial y^2} + \frac{\partial^2 \omega_x}{\partial z^2} \right),
\]

where \( \omega_x, \omega_y, \omega_z \) are vorticity components, \( \nu \) is kinematic coefficient of viscosity. At the initial moment, only the component \( \omega_z \) is non-zero. Because of the symmetry condition and the thinness of vortices, in the section \( x = 0.5h \) we assume longitudinal velocity \( u \) to vanish, \( \partial u / \partial x \) to obey the relation (2), \( \omega_x = \omega_z = 0 \), \( \partial u / \partial y = \partial u / \partial z = 0 \), \( \partial^2 \omega_x / \partial x^2 \) to be much smaller than the transverse derivatives. So, the equation (3) takes the form

\[
\frac{\partial \omega_x}{\partial t} + \left( \frac{\partial \nu}{\partial z} - \beta y_+ \right) \frac{\partial \omega_x}{\partial y} - \beta \omega_z = \nu \left( \frac{\partial^2 \omega_x}{\partial y^2} + \frac{\partial^2 \omega_x}{\partial z^2} \right).
\]

Here \( \nu = \partial \nu / \partial z \), \( w = -\partial \nu / \partial y \).

Imagine that suchlike lattices is a model of fully developed turbulent flow. The quantities in the equation (4) are nondimensionalized by its specific scales [10]

\[
y_1 = h^{-1} y, \quad z_1 = h^{-1} z, \quad t_1 = \nu^{1/3} h^{-2/3} t, \quad \nu_1 = \nu^{-1/3} h^{4/3} \nu, \quad \omega_1 = \nu^{-1/3} h^{2/3} \omega,
\]

where \( \nu \) is energy dissipation rate per unit mass. Moreover, we define two dimensionless quantities

\[
v_1 = \nu^{1/3} h^{-4/3} \nu, \quad B_1 = \nu^{-1/3} h^{2/3} \beta.
\]

The equation (4) in the new variables takes the form

\[
\frac{\partial \omega_{1x}}{\partial t_1} + \left( \frac{\partial \nu_1}{\partial z_1} - \beta_1 y_1 \right) \frac{\partial \omega_{1x}}{\partial y_1} - \beta_1 \omega_{1z} = \nu_1 \left( \frac{\partial^2 \omega_{1x}}{\partial y_{1}^2} + \frac{\partial^2 \omega_{1x}}{\partial z_{1}^2} \right).
\]

The stream function obeys the equation
\[
\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\omega_{z1}.
\]

The solution and, particularly, the vortical zone geometry depends on three parameters: \( \beta_1 \), \( \nu_1 \) and the second vortex reduced circulation \( \Gamma_{2,1} = e^{-1/3} h^{-4/3} \Gamma_2 \). In addition, the solution depends on the initial distribution of vorticity \( \omega_{z1}(y_1, z_1) \), which is

\[
\omega_{z1} = \frac{\Gamma_{2,1}}{m_2} e^{-r^2/m_2^2}.
\]

The dependence on \( \Gamma_1 \) disappears because \( \beta_1 = 1.094 \Gamma_{1,1} \). If \( \Gamma_1 = \Gamma_2 \), we obtain \( \beta_1 \approx \Gamma_{2,1} \). The order of \( \beta_1 \) and \( \nu_1 \) depends on the order of \( h \). It is known that \( \nu \sim e^{1/3} \lambda_0^{4/3} \) at the Kolmogorov scale \( \lambda_0 \), so that \( \nu_1 \sim (\lambda_0/h)^{4/3} \). The reduced kinematic viscosity \( \nu_1 \sim O(1) \) at the Kolmogorov scale while at large scales \( \nu_1 \) is small. If \( \Gamma_{2,1} = 1 \) then \( u \sim e^{1/3} h^{1/3} \), \( \beta \sim e^{1/3} h^{-2/3} \), \( \beta_1 \sim O(1) \). The fields of vorticity corresponding \( \nu_1 = 10^{-3} \), \( \beta_1 = 10 \), \( \nu_1 = 0.5 \) at different initial conditions \( r_0 = 0.2 \) and \( r_0 = 0.4 \) are shown in the figures 8, 9.

**Figure 8.** The fields of vorticity corresponding \( \nu_1 = 10^{-3} \), \( \beta_1 = 10 \), \( r_0 = 0.2 \), \( t_1 = 0.5 \)

**Figure 9.** The fields of vorticity corresponding \( \nu_1 = 10^{-3} \), \( \beta_1 = 10 \), \( r_0 = 0.4 \), \( t_1 = 0.5 \)

The shape of the stretched vortex can be similar to axisymmetric or to tangential discontinuity, depending on conditions. At Kolmogorov scales, when diffusion of the vortex is dominating, the vortex is elliptic. In the figure 10 the vorticity field corresponding \( \nu_1 = 10^{-1} \), \( \beta_1 = 10 \), \( r_0 = 0.4 \), \( t_1 = 0.5 \) is presented. Consequently, the scenario of vortex stretching depends on the vortex scale.

**Figure 10.** The fields of vorticity corresponding \( \nu_1 = 10^{-1} \), \( \beta_1 = 10 \), \( r_0 = 0.4 \), \( t_1 = 0.5 \)
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