Analysis on the imaging properties of a left-handed material slab

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We investigate in this paper the imaging properties of an absorptive left-handed material (LHM) slab. For a line source, a geometric explanation to the reason of the thickness limitation on an ideal lossless slab is given. For a lossy slab, the imaging properties are determined by the wavelength, the slab thickness, the distance from the source to the nearer boundary of the slab, and the absorption effect. Varying the ratios between these quantities, the image width can be changed from wavelength to subwavelength scale. In the former situation, the energy density is mainly concentrated at the two image spots. In the later case, though image of subwavelength width appears on the focal plane, however, most energy is located at about the two boundaries of the slab. The relations between the subwavelength imaging and uncertainty principle are also discussed.

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I. INTRODUCTION

Negative refraction of electromagnetic waves by a left-handed material (LHM), first proposed in 1960s by Veselago [1], has attracted strong research interests [2, 3, 4, 5, 6] and generated heated debate [7, 8, 9, 10, 11]. Among all the phenomena that could happen in a LHM, the most fascinating one may be the possibility of “superlensing effect” proposed by Pendry [2]: that is, a slab made of uniform and isotropic LHM with both the permittivity \( \varepsilon = -1 \) and the permeability \( \mu = -1 \) acquires a negative refractive index \( n = -1 \), which makes this slab a perfect lens. It can capture both the propagating and the evanescent waves emitted from a point source placed in front of the slab and refocuses them into two point images, one inside and the other behind the slab.

Recently, this superlensing effect was questioned by a number of authors [7, 8, 9, 10, 11]. In Ref. [9], the authors augured that negative refraction of energy flow implies the violation of causality principle, and a little amount of absorption will largely deform the waves. In Ref. [10], the authors showed that although there is amplification of evanescent waves in an ideal lossless left-handed medium, however, to avoid the divergence of the field energy inside the lens, it must be limited to a thickness smaller than the distance between the line source and the nearer boundary of the slab, thus perfect imaging is impossible. In addition, a little absorption may destroy the negative refraction effect completely. In Ref. [11], the recovery rate for a lossy slab was studied, and the author showed that the image quality can be significantly affected by the absorption effect. In Ref. [12, 13], the authors showed that the energy flow indeed goes to the “negative way” when passing through the surface of an absorptive and dispersive LHM. In [14], a slab lens of photonic crystal was considered, and the simulation showed that negative refraction of energy flow does not contradict the causality principle. Further in [15, 16], the concept of “constant frequency curves” introduced in [4] were used to study the refraction behavior of the waves in the medium. Most interestingly, in [16] an all-angle negative refraction photonic crystal slab lens was designed to focus the light into a subwavelength region.

Although the focusing effect of a LHM slab lens has already been studied by a number of authors, however, in most previous studies researchers used some Finite-Difference-Time-Domain (FDTD) method. The method is easy to implement but the physical meanings of the simulation results are not easy to be extracted. In some other studies the authors used frequency domain method, however, they usually considered only one single Fourier component of the fields. To get a definite result, one has to sum over these Fourier components.

In this paper we study the imaging problem using a spectrum decomposition method. We first decompose the cylindrical wave emitted by a line source into a series of plane waves of different transverse wave numbers. By considering the boundary conditions at the source point and the two boundaries of the slab lens, we then can determine the transmission and reflection coefficients for each plane wave. These quantities are utilized to construct the field function in every space region.

Our method does not adopt complicated numerical skills, thus makes us easier to get the physical insight. We also give a very simple geometrical explanation to the reason of the thickness limitation for the ideal slab lens (the
The continuity conditions for the tangential components of the 
E fields, which leads to the boundary condition for the

\[ \varepsilon = \mu = n = -1 \text{ case} \]  

Finally, we found that the imaging mechanism for a negative refraction lens system is subtler than that of the conventional lens system.

II. MODEL AND METHOD

We first describe the setup of the slab system. In this paper we consider only the E-polarized wave, which means that the wave propagation direction is parallel to the XZ plane. The x axis is parallel to the two boundaries of the slab, and the boundary near the source is the \( z = 0 \) plane. A current line source \( J(r)e^{-i\omega t} = \hat{y}J_0 \delta(r-r_0)e^{-i\omega t} \) located at \( r_0 = (x_0, z_0) = (0, z_0) \), \( z_0 < 0 \), emits monochromatic waves of angular frequency \( \omega \), thus both the E and H fields get a time factor \( e^{-i\omega t} \). The E field wave radiated from it is \( E_{rad}(r)e^{-i\omega t} = \hat{y}A_0 H_0^{(1)}(k|r-r_0|) e^{-i\omega t} \), which satisfies

\[
(\nabla^2 + k^2) E_{rad}(r) = -i\frac{4\pi\omega}{c^2} J(r).
\]

Here \( H_0^{(1)}(x) \) is the zeroth order Hankel function of the first kind, \( J_0 \) and \( A_0 = -\pi J_0 / \omega c \) are two constants proportional to each other, \( r \) is the observation point, and \( k = \omega / c \) and \( c \) are the wave number of the cylindrical wave and the speed of light in vacuum (outside of the slab), respectively.

To calculate the total \( E(r) \) field, we first introduce the Green’s function satisfying

\[
(\nabla^2 + k^2(z)) G(r, r') = -\delta^{(2)}(r - r'),
\]

then the E field is given by

\[
E(r) = i\frac{4\pi\omega}{c^2} \int d^2 r' G(r, r') J(r').
\]

Here \( k^2(z) = k^2 = \omega^2 / c^2 \) in the regions outside the slab, and \( k^2(z) = \varepsilon \mu \omega^2 / c^2 \) if \( 0 \leq z \leq d \). \( \varepsilon \) and \( \mu \) are the permittivity and permeability in the slab, respectively.

To proceed further, the waves have to be decomposed into various Fourier components. Each component has a definite \( k_x \). It is a plane wave with either a real \( k_z = \sqrt{\omega^2 / c^2 - k_x^2} \), if \( |k_x| \leq \omega / c \), or an imaginary \( k_z \), if \( |k_x| > \omega / c \). In the former case we have a propagating wave, and in the later case the wave is evanescent.

Write \( G(r, r_0) \) as

\[
G(r, r_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_x e^{ik_x z} g(z, k_x),
\]

then we have

\[
\left[ \frac{d^2}{dz^2} + k^2(z) - k_x^2 \right] g(z, k_x) = -\delta(z - z_0),
\]

which leads to the boundary condition for \( g \) at \( z = z_0 \):

\[
g'(z, k_x) |_{z_0+} - g'(z, k_x) |_{z_0-} = -1.
\]

The continuity conditions for the tangential components of the E and H fields at the two boundaries of the slab lead to

\[
g(z, k_x) |_{\text{outside}} = g(z, k_x) |_{\text{inside}},
\]

\[
g'(z, k_x) |_{\text{outside}} = \frac{1}{\mu} g'(z, k_x) |_{\text{inside}}.
\]

Define

\[
\kappa_0 = \sqrt{k_x^2 - k_z^2}, \quad \kappa = \sqrt{k_x^2 \varepsilon \mu - k_z^2},
\]
the solution for \( g \) is given by
\[
g = \begin{cases} 
  e^{i\kappa_0 |z-z_0|} + R e^{i\kappa_0 (|z_0|-z)}, & z < 0 \\
  e^{i\kappa_0 |z|} T \left[ \cos \kappa (z-d) + i \frac{\kappa_0}{\kappa} \sin \kappa (z-d) \right], & 0 \leq z \leq d \\
  T e^{i\kappa_0 (d-|z|)}, & z > d.
\end{cases}
\] (10)

Here
\[
T = \frac{1}{\cos \kappa d - \frac{i}{2} \left( \frac{\kappa}{\mu_0} + \frac{\mu_0}{\kappa} \right) \sin \kappa d}
\] (11)

and
\[
R = \frac{\frac{i}{2} \left( \frac{\kappa}{\mu_0} - \frac{\mu_0}{\kappa} \right) \sin \kappa d}{\cos \kappa d - \frac{i}{2} \left( \frac{\kappa}{\mu_0} + \frac{\mu_0}{\kappa} \right) \sin \kappa d}
\] (12)

are the transmission and reflection coefficients, calculated from the transfer matrix method [11, 17].

III. AN IDEAL SLAB

We now turn to the discussion of an ideal slab lens. For an ideal slab we mean that we can find a frequency \( \omega_0 \) such that for a dispersive medium slab lens medium with frequency dependent permittivity \( \varepsilon(\omega) \) and permeability \( \mu(\omega) \) and zero absorption effect we have \( \varepsilon(\omega_0) = \mu(\omega_0) = -1 \). Pendry pointed out in [2] that a slab lens of this kind is a perfect lens with \( n = -1 \). It focuses the propagating waves and amplifies the evanescent waves, thus can recover all the information carried by the wave emitted from the line source. Although Pendry in his derivation showed that for a single Fourier component the lens indeed amplifies the evanescent wave and thus the amplitude of the wave can be completely recovered, however, he did not sum over these Fourier components to get a result of the total field. In [10], the authors showed that if the thickness \( d \) of the lens is greater than \( d_1 = |z_0| \), then the total field will diverge inside of the lens. On the other hand, if \( d < d_1 \), there will be no image at all. Thus perfect imaging is impossible.

Although the thickness limitation discussed in [10] for an ideal LHM slab lens is correct, however, it is hard to believe that there is some physical principle that can restrict the slab thickness, if a thinner one can be made. To resolve this puzzle, here we give a simple geometrical explanation to the reason of this restriction (See Fig. 1). Our explanation shows that the origin of the restriction comes from the boundary conditions.

![FIG. 1: The field patterns as a function of x and z for an ideal slab with \( \varepsilon = \mu = -1 \). The arrows indicate the directions of energy flows. Inside the slab, energy flows in the direction opposite to that of the wave vector \( \mathbf{k} \). In (a) the slab has a thickness \( d \) shorter than the distance \( d_1 = |z_0| \) between the source point and the left surface of the slab. The waves inside the slab region can be viewed of as radiated from a virtual source behind the slab, whereas the waves in the region behind the slab can be viewed of as radiated from a virtual source inside the slab. In (b) the slab has a thickness \( d \) larger than \( d_1 = |z_0| \). No stationary solution can exist in the empty (question marks) region.](image)

Since the ideal slab does not reflect light at all [2], thus the field inside and behind the the slab are
\[
E_{\text{inside}}(\mathbf{r}, t) = A_0 H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_1|) e^{-i\omega t},
\] (13)

and
\[
E_{\text{behind}}(\mathbf{r}, t) = A_0 H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_2|) e^{-i\omega t},
\] (14)

respectively. Here \( \mathbf{r}_1 = (0, -z_0) \) and \( \mathbf{r}_2 = (0, 2d + z_0) \) are the positions of the two images predicted by the geometric optics. Now, if \( d < d_1 \), then \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) are respectively located outside and inside of the slab, respectively; that is, they are virtual images (virtual line sources). In this case the fields are finite everywhere except at the source point. However, if we increase the slab thickness to \( d > d_1 \), then both images become real, and this contradicts the
boundary conditions. More specifically, a real image means a delta function term, i.e., a line source. Since there is no other line source except the original one that located at \( r_0 \), we conclude that the perfect imaging is impossible.

Put it in another way. The time-averaged Poynting vector \( \mathbf{S} \) must satisfy the divergenceless condition \( \nabla \cdot \mathbf{S} = 0 \), thus there should be no singular point satisfying |\( \mathbf{S} \)| = \( \infty \) except the source point. Since in the slab the wave vector \( \mathbf{k} \) and \( \mathbf{S} \) are antiparallel to each other, thus the waves propagating in the \( 0 < z < -z_0 \) and \( -z_0 < z < d \) regions must be “radiated from” and “absorbed by” the image inside the slab. This leads to the wave phase mismatch at \( z = -z_0 \) if \( A_0 \neq 0 \). From these consideration we conclude that the thickness limitation is a restriction originating from the boundary conditions of this system, and it implies that the stationary state (monochromatic waves) cannot satisfy these boundary conditions simultaneously. In other words, there is no stationary state.

This result is consistent with the time domain results in Ref.18 where the source was treated as a driving force and the two surface plasmon modes were two coupled oscillators. As one can see in Fig.3 of Ref.18, the time evolution of the modulation amplitude \( A(t)/A_{\text{stat}}(\tau \rightarrow \infty) \) oscillates with a period \( T_{\text{osc}} = 4\pi/\Delta \omega k \), where \( \Delta \omega k \) is the frequency difference between the symmetric and antisymmetric surface plasmon modes. When the absorption of the slab goes to zero, these two modes become degenerate, which leads to \( T_{\text{osc}} \rightarrow \infty \) and \( A_{\text{stat}}(\tau \rightarrow \infty) \rightarrow \infty \). This case corresponds to the problem of driven oscillation without damping term. Therefore, the stationary state will not appear, and the field energy inside the slab grows to a larger and larger value without limitation.

IV. A LOSSY SLAB

Now we turn to the discription of the numerical results for a lossy slab. The permittivity and permeability of the slab are chosen as \( \varepsilon = -1 + i\delta_\varepsilon \) and \( \mu = -1 + i\delta_\mu \); both \( \delta_\varepsilon \) and \( \delta_\mu \) are small positive real numbers. With these parameters, the \( g \) function can be calculated. We then calculate the integral of Eq.11 numerically as a sum. We first let \( k_s = k \tan \theta \), with \( -f\pi/2 < \theta < f\pi/2 \). Here \( \theta \) is a reference angle, and \( 0 < f < 1 \) gives the cutoff of \( k_s \): \( (k_s)_{\text{max}} = (\omega/c) \tan(f\pi/2) \). In this paper we choose \( f = 0.96 \), which gives us a \( (k_s)_{\text{max}}/k \approx 16 \), large enough and numerically implementable to give us meaningful results about subwavelength imaging. The range \( (-f\pi/2, f\pi/2) \) is then being discretized to \( n_s = 3000 \) intervals, and the \( dk_x \) is replaced by \( k \sec^2(\theta)d\theta \), with \( d\theta = f\pi/n_s \).

FIG. 2: (a1) The field strength pattern as a function of \( x \) and \( z \). In this case \( z_0 = -1, d = 2, \lambda = 0.3, \varepsilon = \mu = -1 + 0.001i \). The images have widths of the wavelength scale. The two straight lines represent the boundaries of the slab. (a2) The field strength at the focal plane as a function of \( x \). (a3) The field strength on the \( x=0 \) plane as a function of \( z \). The three straight lines represent the slab boundaries and the focal plane. (b1) to (b3) are for the case of subwavelength images. In this case \( z_0 = -1, d = 2, \lambda = 2, \) and \( \varepsilon = \mu = -1 + 0.000001i \).

Figure 2. shows two typical cases for the imaging problem. In case A (Fig.2(a1) to (a3)) the lens system creates two images, one inside and one outside of the slab, and they have widths of the wavelength scale. Here we have chosen \( z_0 = -1, d = 2, \lambda = 2\pi/k = 0.3 \), and \( \varepsilon = \mu = -1 + 0.001i \). We observe clearly that the largest field strength locates at the two images. However, there is also some surface resonance effect near the boundaries. As we decrease the degree of the absorption, a stronger surface resonance effect is observed. In case B (Fig.2(b1) to (b3)) we choose \( z_0 = -1, d = 2, \lambda = 2\pi/k = 2 \), and \( \varepsilon = \mu = -1 + 0.000001i \). In this case, the images become subwavelength scale. It is also clear that the field strength is very large at the two boundaries of the slab. This implies that surface-plasmon-polariton (SPP) plays important roles in this case. It is interesting to note that, although on the focal plane the field strength indeed has a peak along the \( x \)-direction, however, the field strength does not have a local maximum around the image, and in the \( z \)-direction the wave strength decays from the second slab boundary. In this example the field strength at the focal plane is about only 1% of that at the boundaries. A closer observation find that the field strength at the image point is the same order as that around the source. This implies that if we turn on a line source, then the system has to spend a long time (several hundreds of 2

An approximate image size can be obtained by analyzing the transmission coefficient \( T \). For \( |k_x| \gg \omega/c \), and \( \varepsilon = \mu = -1 + i\delta, \delta \ll 1 \), we have

\[
T \approx \frac{1}{e^{-|k_x|d + \frac{\delta^2}{4}e^{|k_x|d}}},
\]

(15)
which is a hyperbolic secant function with a peak value

$$T_{\text{max}} \approx \frac{1}{\delta} \quad (16)$$

at the transverse wave number

$$\bar{k}_x = (1/d) \ln(2/\delta). \quad (17)$$

Thus the image size is given by

$$W = \frac{2\pi d}{\ln(2/\delta)} \quad (18)$$

A similar result has already been given by Merlin in Ref. 20.

For the case B of Fig. 2, we have $W \approx 0.43\lambda$, which is indeed a subwavelength focusing. However, the actual size of the image is in fact a little larger than that given by Eq. (18). The reason is that the $g(z, k_x)$ function in the $z > d$ region contains a factor $e^{i\kappa_0(z-d+|z_0|)/(-2i\kappa_0)}$, and thus in the integral the contributions from Fourier component with $|k_x| < \bar{k}_x$ cannot be neglected. It seems that the near field excitations (evanescent surface waves) and small enough absorption play the most important roles in the subwavelength imaging process.

V. CONCLUSION

In conclusion, we have studied the imaging properties of a negative-refraction slab lens, using a spectrum decomposition method. We have also given a simple geometrical explanation to the reason of the slab thickness limitation for an ideal negative refraction lens. For a slab with appropriate amount absorption, we found that both the wavelength size and subwavelength size images can be formed.

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