Exact Analytical Expression for the Synchrotron Radiation Spectrum in the Gaussian Turbulent Magnetic Field

Evgeny Derishev\textsuperscript{1} and Felix Aharonian\textsuperscript{2,3}

\textsuperscript{1} Institute of Applied Physics RAS, 46 Ulyanov st, 603950 Nizhny Novgorod, Russia
\textsuperscript{2} Dublin Institute for Advanced Studies, 31 Fitzwilliam Place, Dublin 2, Ireland
\textsuperscript{3} Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, D-69117 Heidelberg, Germany

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Abstract

We demonstrate that the exact solution for the spectrum of synchrotron radiation from an isotropic population of monoenergetic electrons in a turbulent magnetic field with a Gaussian distribution of local field strengths can be expressed in the simple analytic form:

\[
\frac{d\omega}{d\omega_c} = \frac{e^2 B}{2\pi m_e c^2} F\left(\frac{\omega}{\omega_c}\right),
\]

where \(\omega_c = \frac{3\gamma^2 eB}{2mc^2} \equiv \frac{3}{2} \gamma^2 \omega_B\).

Here

\[
F(x) = x \int_x^{\infty} K_{5/3}(\xi) d\xi,
\]

and \(K_{5/3}(\xi)\) is modified Bessel function of the second kind; the subscript \(u\) denotes the uniform magnetic field. Here and below we use \(\omega_B = \frac{eB}{mc}\) to simplify the notation.

For a single electron in a turbulent magnetic field, Equation (1) should be averaged over (1) the pitch angles and (2) the distribution of the local strength of the magnetic field. The result is to be convoluted with the electron distribution function. For an isotropic distribution of radiating particles, the procedure (1) allows exact analytic expressions in terms of special functions. In particular, Crusius & Schlickeiser (1986) have derived the spectrum expressed in terms of Whittaker functions, while Aharonian et al. (2010) suggested an alternative formula in terms of modified Bessel functions. Although these solutions are presented in compact and elegant forms, for practical purposes it is convenient to avoid special functions, i.e., to have analytic approximations containing only elementary functions. The simplest approximation is based on the assumption that electrons emit monochromatic synchrotron photons, whose frequency depends on electrons’ energy and the magnetic field strength. This produces reasonably good approximation for featureless, e.g., power-law, electron distributions, but is known to yield the wrong results for distributions with a high-energy cutoff (e.g., Fritz 1989). Instead, Zirakashvili & Aharonian (2007) and Aharonian et al. (2010) have offered simple analytic approximations that deviate from the exact solution less than 3% and 0.2%, respectively.

Previously published calculations of synchrotron spectrum in a turbulent magnetic field dealt with various options for the distribution of local magnetic field strengths, resulting in analytical asymptotic formulas (e.g., Eilek & Arendt 1996; Kelner et al. 2013). For exponential distribution of local field strengths a complex exact expression was derived (Zirakashvili & Aharonian 2010), for which, however, a simple approximation was proposed. The case of a turbulent magnetic field with a Gaussian distribution was previously studied numerically (Kelner et al. 2013).

In this paper we report the finding that both preliminary integrations (1) and (2) can be done analytically for the turbulent magnetic field with a Gaussian distribution of the local field strength and isotropic distribution of electrons over pitch angles. Moreover, the final exact solution is expressed in terms of elementary functions and is much simpler than even the starting expression given by Equation (1).

The Gaussian distribution of a random field is a well-known situation in physics (see, e.g., Rue & Held 2005). Such a distribution for local magnetic field strengths is called upon in the literature to explain, for example, properties of synchrotron radiation from supernova remnants (Bykov et al. 2008) and pulsar wind nebulae (Bykov et al. 2012). Recently, in the context of synchrotron emission of supernova remnants, this issue has been discussed also by Pohl et al. (2015). Note that in

1. Introduction

Synchrotron radiation is the most common nonthermal emission mechanism in astrophysics. Calculation of its spectrum involves several steps. One starts with the expression for the synchrotron spectrum (the power emitted per unit frequency) of an individual relativistic electron, moving perpendicular to the field lines of the uniform magnetic field. This expression can be found in many textbooks (see, e.g., Equation (74.17) in Landau & Lifshitz 1975):

\[
\frac{dL}{d\omega} = \frac{\sqrt{3}}{2\pi} \frac{e^3 B}{m_e c^2} F\left(\frac{\omega}{\omega_c}\right),
\]

where

\[
\omega_c = \frac{3\gamma^2 eB}{2mc^2} \equiv \frac{3}{2} \gamma^2 \omega_B.
\]

and \(K_{5/3}(\xi)\) is modified Bessel function of the second kind; the subscript \(u\) denotes the uniform magnetic field. Here and below we use \(\omega_B = \frac{eB}{mc}\) to simplify the notation.

For a single electron in a turbulent magnetic field, Equation (1) should be averaged over (1) the pitch angles and (2) the distribution of the local strength of the magnetic field. The result is to be convoluted with the electron distribution function. For an isotropic distribution of radiating particles, the procedure (1) allows exact analytic expressions in terms of special functions. In particular, Crusius & Schlickeiser (1986) have derived the spectrum expressed in terms of Whittaker functions, while Aharonian et al. (2010) suggested an alternative formula in terms of modified Bessel functions. Although these solutions are presented in compact and elegant forms, for practical purposes it is convenient to avoid special functions, i.e., to have analytic approximations containing only elementary functions. The simplest approximation is based on the assumption that electrons emit monochromatic synchrotron photons, whose frequency depends on electrons’ energy and the magnetic field strength. This produces reasonably good approximation for featureless, e.g., power-law, electron distributions, but is known to yield the wrong results for distributions with a high-energy cutoff (e.g., Fritz 1989). Instead, Zirakashvili & Aharonian (2007) and Aharonian et al. (2010) have offered simple analytic approximations that deviate from the exact solution less than 3% and 0.2%, respectively.

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In this paper we report the finding that both preliminary integrations (1) and (2) can be done analytically for the turbulent magnetic field with a Gaussian distribution of the local field strength and isotropic distribution of electrons over pitch angles. Moreover, the final exact solution is expressed in terms of elementary functions and is much simpler than even the starting expression given by Equation (1).

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this paper the authors assumed a Gaussian distribution for the total magnetic field strength, unlike the usual approach where all three magnetic field components are Gaussian-distributed.

The Gaussian distribution naturally results from summation of the magnetic field from many independent or nearly independent modes, for example, in the frequently occurring case of quasi-linear turbulence. Isotropic or nearly isotropic distribution of electrons over pitch angles is expected in the case where the cooling length exceeds the electrons’ mean free path, which is also a typical situation.

Since our results are derived from Equation (1), they inherit all its limitations, which are to be applied to the typical magnetic field strength. In addition we require that the radiating particles effectively isotropize over a distance smaller than their cooling length.

The paper is organized as follows. In Section 2 we discuss how distribution of electrons over pitch angles can be convoluted with the distribution of local magnetic field strength to obtain the effective magnetic field distribution and derive this distribution for the case of Gaussian turbulence. In Section 3 we derive the synchrotron spectrum of an individual electron, averaged over pitch angles and over the magnetic field strength distribution. The steps required to evaluate the integral, which expresses this spectrum, are outlined in Appendix A. We move on to obtaining in Section 4 the expression for the spectrum of synchrotron radiation produced in turbulent magnetic field by electrons, whose distribution function is a power law with cutoff. We also derive, in Section 5, the spectrum from simple (uncut) power-law distribution and elaborate on its relation to the spectrum derived in the monochromatic approximation. In Section 6 we compare our expressions for the synchrotron spectrum in turbulent magnetic fields to those for the uniform magnetic field.

2. Effective Distribution of Magnetic Field Strength for Gaussian Turbulence

We assume that the local strength of the magnetic field in the emitting region results from summation of many independent modes that leads to independent Gaussian distribution with zero mean value for all three Cartesian components of the field. Then the probability density for the magnetic field strength is

\[ P_B = \left( \frac{6}{\pi} \right)^{\frac{1}{2}} \frac{3B^2}{B_0^2} \exp\left( -\frac{3B^2}{2B_0^2} \right); \]

\[ \int_0^\infty P_B(B) dB = 1, \]

\[ \langle B^2 \rangle = \int_0^\infty B^2 P_B(B) dB = B_0^2. \]  

This assumption is natural in the case where turbulence is sustained in the quasi-linear regime, and we consider it a conservative (i.e., underestimating the volume occupied by stronger than average magnetic field) assumption in the case of strongly nonlinear turbulence.

A particle moving along a helical trajectory with a pitch angle \( \theta \gg 1/\gamma \) behaves (in terms of radiated power and spectrum) as if it moves perpendicular to field lines of the magnetic field with effective strength \( B_{eff} = B \sin \theta \). In the case where particle distribution over pitch angles does not vary from point to point, there are two ways to calculate the average synchrotron spectrum. One may either integrate locally over pitch angles and then average over the field strength distribution or—equivalently—calculate effective field strength distribution for all particles with the same pitch angle and after that integrate over pitch angles. We follow the second route.

Effective field strength distribution is formally the same as Equation (3), where \( B \) is replaced by \( B_{eff} = B \sin \theta \).

\[ P_0 = \left| \frac{\partial \cos \theta}{\partial B_{eff}^2} \right| = \left| \frac{\partial}{\partial B_{eff}} \sqrt{1 - \left( \frac{B_{eff}^2}{B_0^2} \right)^2} \right| = \frac{B_{eff}}{B_0 \sqrt{B_0^2 - (B_{eff}^2)^2}}. \]  

resulting in

\[ P_B^{eff} = \int_0^B P_B^{eff}(\theta) P_0 dB_{eff} \]

\[ = \frac{3B_{eff}}{B_0^2} \exp\left( -\frac{3(B_{eff}^2)}{2B_0^2} \right); \]

\[ \int_0^\infty P_B^{eff} dB_{eff} = 1, \]  

\[ \langle B_{eff}^2 \rangle = \frac{2}{3} B_0^2. \]  

We perform this integration by making the substitution \( u = \frac{2}{3}(B_{eff}^2) \left( \frac{1}{B_0^2} - \frac{1}{B_{eff}^2} \right) \). After that, the integration with respect to \( u \) results in the gamma function \( \Gamma(1/2) = \sqrt{\pi} \).

3. Average Synchrotron Spectrum for Isotropic Particles in the Turbulent Magnetic Field

For our purposes it is more convenient to express the synchrotron spectrum (Equation (1)) in terms of the number of synchrotron photons emitted per unit frequency, which is

\[ \left( \frac{dN}{d\omega} \right)_t = \frac{1}{\hbar \omega} \frac{dL}{d\omega} = \frac{1}{\gamma^2} \frac{\alpha}{3} N \left( \frac{\omega}{\omega_c} \right). \]

Here \( \alpha \) is the fine-structure constant and

\[ N(x) = \frac{\sqrt{3}}{\sqrt{\pi}} \frac{1}{x} \ F(x) = \frac{\sqrt{3}}{\sqrt{\pi}} \int_x^\infty K_{5/3}(\xi) d\xi. \]

The numerical factor in the definition of \( N(x) \) is chosen to simplify further notation.

Using the effective distribution of the magnetic field (Equation (5)) instead of the actual one, we treat all particles as if they were moving perpendicular to the field lines, so that the distribution of synchrotron photons over frequency, averaged over space and pitch angles, is

\[ \left( \frac{dN}{d\omega} \right)_t = \int_0^B P_B^{eff} (B) \left[ \frac{1}{\gamma^2} \frac{\alpha}{3} \frac{2}{3} \frac{m_e c}{\gamma^2 \epsilon B} \right] dB, \]  

where the subscript \( t \) denotes turbulent magnetic field.

Noting that \( P_B^{eff}(B) dB \) is the exact differential, and integrating Equation (8) by parts using the substitution

\[ \xi = \frac{8}{9} \frac{\omega B_0}{\omega_0 B}, \quad \omega_0 = \frac{4}{3} \frac{\gamma^2 e B_0}{m_e c} \equiv \frac{4}{3} \gamma^2 \omega_{B,0} \]

\[ \xi = \frac{8}{9} \frac{\omega B_0}{\omega_0 B}, \]  

\[ \omega_0 = \frac{4}{3} \frac{\gamma^2 e B_0}{m_e c} \equiv \frac{4}{3} \gamma^2 \omega_{B,0} \]
where

\[
\frac{dN}{d\omega} = \frac{1}{\gamma^3} \int_0^\infty N \left( \frac{8}{9} \frac{\omega}{\omega_0} B_0 \right) dB + \frac{3}{2} \frac{\omega}{\omega_0} B_0 \exp \left( -\frac{3B^2}{2B_0^2} \right) dB + \frac{1}{\gamma^3} \int_0^\infty \exp \left( -2 \frac{(\omega/\omega_0)^2}{\xi^2} \right) K_{(\xi/\omega_0)} dB,
\]

\[
= \frac{1}{\gamma^3} \frac{\alpha}{\pi} \int_0^\infty \exp \left( -\frac{32\xi^2}{27\xi^2} \right) K_{(\xi/\omega_0)} dB = (1 + \frac{1}{\gamma^3}) \exp (-2\xi^{3/2})
\]

(10)
is evaluated in Appendix A. Note that the numerical factor \((4/3)\) in the definition of \(\omega_0\) differs from the numerical factor \((3/2)\) in the definition of \(\omega_c\). This seemingly cumbersome choice is made to simplify the notation in the final expression.

### 4. Synchrotron Spectrum from Power-law Distribution with Cutoff

A rather general approximation for the distribution function of synchrotron-radiating particles is a power law with cutoff,

\[
f_c(\gamma) = \frac{n_c}{\gamma_0} \left( \frac{\gamma}{\gamma_0} \right)^{-p} \exp \left( -\frac{\gamma}{\gamma_0} \right).
\]

(11)

where \(\gamma_0 \gg 1\) and \(\beta > 0\). We are only interested in the part of this distribution where \(\gamma \gg 1\), so that we can formally assume that the distribution (11) extends to \(\gamma = 0\); this simplifies the notation. Note that we do not require the integral \(\int_0^\infty f_c d\gamma\) to converge.

Calculation of spectral distribution of synchrotron photons for the power-law distribution of radiating particles yields

\[
\frac{dN_{PL}}{d\omega}(\omega, B_0) = \int_0^\infty \frac{dN}{d\omega}(\omega, \gamma, B_0)f_c d\gamma
\]

\[
= \frac{\alpha}{3} \frac{n_c}{\gamma_0^3} \int_0^\infty Q \left( \frac{\omega}{\omega_{\text{cut}}} \right) \gamma^{-3/2} \exp \left( -\frac{\gamma}{\gamma_0} \right) d\gamma
\]

\[
\times \exp \left( -\frac{32\xi^2}{27\xi^2} \right) K_{(\xi/\omega_0)} d\xi
\]

\[
\times \int_0^\infty \frac{1}{\xi^2} \xi^{-2-p} \exp \left( -\frac{\omega}{\omega_{\text{cut}}} \right)^{3/2} d\xi
\]

\[
= \frac{\alpha}{3} \frac{n_c}{\gamma_0} Q_{PL}(\omega/\omega_{\text{cut}}).
\]

(12)

Here we changed integration variable to \(\xi = \sqrt{\frac{\omega_{\text{cut}}}{\omega_0}}\), and introduced

\[
\omega_{\text{cut}} = \frac{4}{3} \cos^2 \omega_{\text{II},0}.
\]

(13)

Substituting \(Q\) from Equation (10), we write the function \(Q_{PL}(x)\) explicitly:

\[
Q_{PL}(x) = x^{-(1-p)/2} \int_0^\infty (\xi^{-p-2} + \xi^{-p-2/3}) \exp (\xi^{-2/3} - x^{3/2}\xi^{3/2}) d\xi,
\]

(14)

where

\[
x = \frac{\omega}{\omega_{\text{cut}}} = \frac{3}{4} \frac{\omega}{\gamma_0^2 \omega_{\text{II},0}}.
\]

(15)

For practical purposes it is useful to derive asymptotic forms of the function \(Q_{PL}(x)\) as well as its approximation in terms of elementary functions. Evaluating the asymptotic form in the limit \(x \to \infty\) we note that the main contribution to the integral in Equation (14) comes from \(\xi \to 0\), keep only the smallest power \(\xi\), and then use Laplace’s method (see Appendix B). Here \(g(\xi) = \xi^{-p-2}\) and \(f(\xi) = -2\xi^{-4/3} - x^{3/2}\xi^{3/2}\), so that

\[
\xi_0 = \left( \frac{8}{3\beta} \right)^{3/4}
\]

\[
[f''(\xi_0)] = \frac{8}{3} \left( \frac{3\beta^{3/2}}{8} \right)^{3/4} \left( \beta + \frac{4}{3} \right)
\]

\[
f(\xi_0) = -\left( \frac{3\beta^{3/2}}{8} \right)^{4/3} \left( 2 + \frac{8}{3\beta} \right)
\]

(16)

and the asymptotic form at large arguments is

\[
Q_{PL}(x) = \frac{4\pi}{3\beta} \left( \frac{3\beta}{8} \right)^{3/4} \left( \beta + \frac{4}{3} \right)^{3/4} \exp \left[ -\left( 2 + \frac{8}{3\beta} \right) \left( \frac{3\beta}{8} \right)^{3/4} x^{3/4} \right]
\]

(17)

For the asymptotic form of the function \(Q_{PL}(x)\) in the limit \(x \to 0\) there are three cases depending on the value of \(p\):

\[
Q_{PL}(x) = x^{-(1-p)/2} \left\{ \begin{array}{ll}
I_h(-p-2) + I_h(-p-2/3), & p > 1/3 \\
I_{2h}, & p = 1/3 \\
I_h(-p-2/3), & p < 1/3,
\end{array} \right.
\]

where

\[
I_h(q) = \int_0^\infty \xi^q \exp (-2\xi^{-4/3}) d\xi
\]

\[
= 3 \times \frac{2^{q-1}}{5} \Gamma \left( -\frac{3}{4} + q \right).
\]
where

\[ C_4 = \frac{3\sqrt{\pi}}{2(3\beta + 4)^{1/2}} \left( \frac{3\beta}{8} \right)^{3\beta/4} , \quad (27) \]

\[ C_5 = \left( 2 + \frac{8}{3\beta} \right) \left( \frac{3\beta}{8} \right)^{4/3} . \quad (28) \]

Combining the two asymptotic forms given by Equations (22) and (26) one obtains an approximation that is valid for any \( x \). For example, in the case of \( p > 1/3 \), which covers the vast majority of situations relevant to astrophysics, we arrive at the following approximate and asymptotically exact expression:

\[ Q_{\text{PL}}(x) \simeq (t_1^k + t_2^k)^{1/k} \exp \left( -C_5 x^{3\beta/4} \right) , \quad (29) \]

where

\[ t_1 = C_1 x^{-(1+p)} \quad \text{and} \quad t_2 = C_4 x^{\beta+2+2p} , \quad (30) \]

and \( k \) is the parameter whose value is chosen to minimize the error for each \((p,\beta)\) pair. For \( \beta = 1.2 \) and the arbitrary power-law index \( p \) the value of the \( k \)-parameter can be taken from Figure 1 and the largest relative error is plotted in Figure 2.

5. Synchrotron Spectrum for Simple Power-law Distribution and Connection to the Monochromatic Approximation

At times one is interested only in the low-energy part of the distribution (11), which can be approximated by a simple power law,

\[ f_\epsilon(\gamma) = \frac{n_e}{\gamma_0} \left( \frac{\gamma}{\gamma_0} \right)^{-p} . \quad (31) \]

The synchrotron spectrum for this distribution is given by the expression similar to Equation (14), but without the second term in the exponent, so that Equation (22) is the exact expression for \( Q_{\text{PL}}(x) \) rather than its asymptotic form in the limit \( x \to 0 \). Considering what is typical in the astrophysics case \( p > 1/3 \), we obtain

\[ \frac{dN_{\text{PL}}}{d\omega}(\omega, B_0) = \frac{\alpha}{8} \frac{n_e}{\gamma_0} 2^{\frac{\omega}{\omega_{\text{cut}}}}(3p + 7) \Gamma \left( \frac{3p - 1}{4} \right) \left( \frac{\omega}{\omega_{\text{cut}}} \right)^{-\frac{p+1}{p+1}} . \quad (32) \]

The synchrotron spectrum obtained for a power-law distribution in the monochromatic approximation (i.e., assuming that each electron radiates at a single frequency \( \omega_{\text{cut}} \), proportional to the square of its Lorentz factor) has the same frequency dependence. To ensure that the numerical factor is also the same it is necessary to choose the frequency \( \omega_m \) in an appropriate way:

\[ \omega_m = \frac{\alpha m^2}{2} \omega_{B,0} . \quad (33) \]
and

$$a_m = \frac{2}{3} \left[ 3p + 7 \frac{1}{16} \Gamma \left( \frac{3p - 1}{4} \right) \right]^{\frac{1}{3}}. \quad (34)$$

Note that $a_m(p)$ is a monotonously rising function of $p$; it equals zero at $p = 1/3$ and is continuous at $p = 3$.

6. Discussion

It is instructive to compare spectra of synchrotron radiation, the total emitted power, and the average energy of synchrotron photons in two cases: for an electron moving perpendicular to the field lines of the uniform magnetic field and for an electron in the turbulent magnetic field, which has the same average energy density.
For an electron moving perpendicular to the field lines of uniform magnetic field, the total emitted power is

\[
L_u = \int_0^\infty \hbar \omega \left( \frac{dN}{d\omega} \right)_u d\omega = \frac{\hbar \omega_c^2}{\gamma^2} \frac{1}{3} \alpha \\
\times \int_0^\infty xN(x)dx = \frac{8}{27} \frac{\hbar \omega_c^2}{\gamma^2}
\]

\[= 2\gamma^2 \sigma_T \frac{B_0^2}{8\pi} \tag{35}\]

and the total photon emission rate is

\[
N_u = \int_0^\infty \left( \frac{dN}{d\omega} \right)_u d\omega = \omega_c \frac{1}{\gamma^2} \frac{1}{3} \alpha \\
\times \int_0^\infty N(x)dx = \frac{5}{3\sqrt{3}} \frac{\alpha \omega_c}{\gamma^2}
\]

\[= \frac{5}{2\sqrt{3}} \alpha \omega_B \tag{36}\]

To find the integrals

\[
\int_0^\infty x N(x)dx = \frac{\sqrt{3}}{\pi} \int_0^\infty x K_{3/3}(x) dx = \frac{\sqrt{3}}{\pi} \frac{1}{6} \frac{1}{6} = \frac{5}{3\sqrt{3}}
\]

\[
\int_0^\infty x^2 K_{3/3}(x) dx = \frac{\sqrt{3}}{2\pi} \frac{1}{3} \frac{2}{3} \frac{1}{3} = \frac{8}{9}
\]

we first integrate by parts, then use the general expression (see, e.g., Equation (6.561.16) in Gradsteyn et al. 2007)

\[
\int_0^\infty x^\mu K_{\nu}(ax) dx = 2^{\mu-1} a^{-\mu-1} \Gamma \left( \frac{1 + \mu - \nu}{2} \right) \Gamma \left( \frac{1 + \mu + \nu}{2} \right)
\]

and Euler’s reflection formula

\[
\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}.
\]

From Equations (35) and (36) we find the average energy of synchrotron photons emitted by an electron moving perpendicular to the field lines of the uniform magnetic field:

\[
\langle \epsilon_{\text{sy}} \rangle_u = L_u / N_u = \frac{8}{15} \frac{1}{\sqrt{3}} \frac{\hbar \omega_c}{2} = \frac{4}{5\sqrt{3}} \gamma^2 \frac{\hbar \epsilon_B}{2} \tag{37}
\]
Similarly, for an electron in the turbulent magnetic field, the total emitted power is

\[
L_t = \int_0^\infty \hbar \omega \left( \frac{d\dot{N}}{d\omega} \right)_t \ d\omega = \hbar \omega_0 \left( \frac{1}{\gamma^2} \right) \alpha \\
\times \int_0^\infty \ x \ Q(x) \ dx = \frac{\alpha \hbar \omega_0}{\gamma^2} \frac{\beta_0^2}{8\pi} \gamma^2 \sigma_T e \\
= \frac{4}{3} \gamma^2 \sigma_T e \frac{\beta_0^2}{8\pi} \tag{38}
\]

and the total photon emission rate is

\[
N_t = \int_0^\infty \left( \frac{d\dot{N}}{d\omega} \right)_t \ d\omega = \omega_0 \left( \frac{1}{\gamma^2} \right) \alpha \\
\times \int_0^\infty \ Q(x) \ dx = \frac{5}{8} \sqrt{\frac{\pi}{2}} \frac{\alpha \omega_0}{\gamma^2} \\
= \frac{5}{6} \sqrt{\frac{\pi}{2}} \alpha \omega_{B,0}. \tag{39}
\]

The integrals

\[
\int_0^\infty Q(x) \ dx = \frac{15}{16} \sqrt{2\pi} \quad \text{and} \quad \int_0^\infty x \ Q(x) \ dx = \frac{3}{4}
\]

are calculated in a straightforward way (reduced to the gamma function).

The average energy per synchrotron photon,

\[
\langle \epsilon_{\gamma t} \rangle = L_t / N_t = \frac{2}{5} \sqrt{\frac{\pi}{2}} \hbar \omega_0 = \frac{8}{15} \sqrt{\frac{\pi}{2}} \gamma^2 \hbar \omega_{B,0}. \tag{40}
\]

is approximately equal ($\approx 1.0854$ times smaller) to the value in Equation (37).

Here we may note that the net effect of turbulent magnetic field is to increase the average energy of synchrotron photons by a factor, which approximately compensates the decrease of this energy due to averaging over isotropic pitch-angle distribution.

The synchrotron luminosity for an electron in the turbulent magnetic field is $2/3$ of the value given by Equation (35). This difference is due to the fact that one of three components of the turbulent magnetic field (the one parallel to electron’s momentum) does not contribute to synchrotron radiation. The same factor appears when Equation (35) is averaged over the isotropic pitch-angle distribution.

The synchrotron spectral energy distributions (SEDs) for monoenergetic electrons in turbulent and uniform magnetic fields are compared in Figure 3. Note that for the uniform magnetic field the SED of electrons moving perpendicular to the field lines peaks at $\omega \approx 1.99 \gamma^2 \omega_{B,0}$, while the isotropic population of electrons in the same field produces a SED that peaks at $\omega \approx 1.70 \gamma^2 \omega_{B,0}$. The presence of regions with a stronger field in the case of the Gaussian turbulent magnetic field almost exactly compensates for the decrease of the SED peak frequency due to averaging over the isotropic pitch-angle distribution, so that the turbulent-field SED peaks at $\omega \approx 1.93 \gamma^2 \omega_{B,0}$.

The spectra of synchrotron radiation in the cases of turbulent and uniform magnetic fields are rather similar at low frequencies, below and around the peak, but the difference between them becomes progressively larger at high frequencies. Although the difference exceeds a factor of 2 only at the highest frequencies, where $\approx 1.5\%$ of the emitted power is concentrated, it shows up in electron distributions with a sharp high-energy cutoff.

In Figure 4 we compare synchrotron SEDs from the power-law electron distribution with an exponential cutoff ($\beta = 1$, panel (a)) and with a Gaussian cutoff ($\beta = 2$, panel (b)) for the turbulent magnetic field (solid line) and for the uniform magnetic field in the monochromatic approximation (dashed line) and calculated using the approximate pitch-angle averaged emissivity function from Zirakashvili & Aharonian (2007) (dotted line). The power-law index of the electron distribution is $p = 3$. Vertical axis—$\nu F_\nu$ in arbitrary units (logarithmic scale). Horizontal axis—frequency (logarithmic scale).

Figure 4. Spectral energy distributions ($\nu F_\nu$) of synchrotron radiation for the power-law electron distribution with an exponential cutoff ($\beta = 1$, panel (a)) and with a Gaussian cutoff ($\beta = 2$, panel (b)) for the turbulent magnetic field (solid line) and for the uniform magnetic field in the monochromatic approximation (dashed line) and calculated using the approximate pitch-angle averaged emissivity function from Zirakashvili & Aharonian (2007) (dotted line). The power-law index of the electron distribution is $p = 3$. Vertical axis—$\nu F_\nu$ in arbitrary units (logarithmic scale). Horizontal axis—frequency (logarithmic scale).
quasi-linear magnetic turbulence, but may still be a better approximation in the case of strongly nonlinear magnetic turbulence in addition to providing a simpler to handle expression.

7. Summary

In this paper we find—in terms of elementary functions—the exact expression for the spectrum of synchrotron radiation of an electron in the turbulent magnetic field with Gaussian statistics of local magnetic field strengths.

This expression reads

\[ \left( \frac{dN}{d\omega} \right)_t = \frac{\alpha}{\gamma^2} \left( 1 + \frac{1}{\chi^2} \right) \exp(-2\chi^2/\gamma^2), \]

where

\[ x = \frac{\omega}{\omega_0}, \quad \omega_0 = 4 \gamma^2 eB_0 / m_e c \equiv 4 \gamma^2 \omega_{B,0}. \]

One should note the slower decline at high frequencies, \( \propto \exp(-2\chi^2/\gamma^2) \), compared to the case of the uniform magnetic field, where the decline is exponential, \( \propto \exp(-8\chi/9) \).

Building on the simple expression for the spectrum of an individual electron, we find—in terms of elementary functions—the exact expression for the synchrotron spectrum from the power-law electron distribution in the turbulent magnetic field given by Equation (32) and show that this spectrum can be reproduced using the monochromatic approximation (each electron radiates at a single frequency proportional to the square of its Lorentz factor) with the appropriate choice of frequency (Equations (33) and (34)). We also derive the synchrotron spectrum for the power-law electron distribution with cutoff (Equations (12) and (14)). In the latter case we provide asymptotic expressions in terms of elementary functions both for low (Equation (22)) and high (Equation (26)) frequencies, as well as an approximation valid for any frequency (Equation (29)).

Again, Gaussian magnetic field strength fluctuations result in a slower decline of the synchrotron spectrum beyond the cutoff, \( \propto \exp(-d\chi^2/\chi_0^2,\gamma^2) \) compared to

\[ \propto \exp\left(-b\left[\frac{\chi}{\chi_0}\right]^{3/(\beta+2)}\right) \]

in the case of the constant-strength magnetic field, both assuming the \( \propto \exp(-[\gamma/\chi_0]^{\beta}) \)-type cutoff in the parent electron distribution.\(^4\)

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Appendix A

Derivation of \( Q(x) \)

The simplest way to evaluate the integral in the definition of \( Q(x) \) is by splitting it into two integrals using the recurrence relations for Macdonald functions. Both the

\[ \int_0^\infty x^{a-1} \exp \left( -\frac{p}{x^2} \right) K_\nu(cx) \, dx \]

integrals are given in Prudnikov et al. (1986, Equations (21.6.14) and (21.6.16)) as simplified special cases of a more general integral (Equation (21.6.13)). Unfortunately, both have typos in the numerical coefficients. We therefore start with the general expression (Equation (21.6.13) in Prudnikov et al. 1986) in the form

\[ \int_0^\infty x^{a-1} \exp \left( -\frac{p}{x^2} \right) K_\nu(cx) \, dx \]

\[ = \frac{2^{\alpha-2} e^{a\nu}}{\Gamma\left(\frac{\alpha+\nu}{2}\right) \Gamma\left(\frac{\alpha-\nu}{2}\right)} \times \, \text{\( \times \) oF}_2 \left( \begin{array}{c} 1 - \frac{\alpha+\nu}{2}, 1 - \frac{\alpha-\nu}{2} \\ -\frac{c^2p}{4} \end{array} \right) + \frac{c^2p^{(\alpha+\nu)/2}}{2^{\nu+2} \Gamma(\nu) \Gamma(-\alpha+\nu/2)} \times \, \text{\( \times \) oF}_2 \left( \begin{array}{c} 1 + \frac{\alpha+\nu}{2}, 1 + \nu, -\frac{c^2p}{4} \end{array} \right) \]

\[ \times \, \text{\( \times \) oF}_2 \left( \begin{array}{c} 1 + \frac{\alpha-\nu}{2}, 1 - \nu, -\frac{c^2p}{4} \end{array} \right) \]

Here

\[ \text{\( \text{oF}_2(a_1, a_2; \mu) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{k!} \, \frac{\zeta^k}{k!} \) \]

is a generalized hypergeometric function

\[ (a)_0 = 1, \]

\[ (a)_n = (a+1)(a+2) \ldots (a+n-1) \]

the Pochhammer symbol. In our case \( \alpha = 1, \beta = 1, \nu = 5/3 \) (Equation (42) simplifies to become (Equation (28)) the substitution \( p = 4\mu \)

\[ \int_0^\infty \exp \left( -\frac{4\mu}{x^2} \right) K_{5/3}(\zeta) \, d\zeta \]

\[ = \frac{1}{2} \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{4}{3}\right) \times \text{\( \times \) oF}_2 \left( \frac{1}{3}, \frac{4}{3}, \frac{7}{3} ; \zeta \right) + \frac{\zeta^{1/3}}{2} \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{4}{3}\right) \times \text{\( \times \) oF}_2 \left( \frac{1}{3}, \frac{4}{3}, \frac{7}{3} ; \zeta \right) \]

\[ = \frac{\pi}{\sqrt{3}} \times \text{\( \times \) oF}_2 \left( \frac{1}{3}, \frac{4}{3}, \frac{7}{3} ; \zeta \right) + \frac{\pi}{\sqrt{3}} \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{4}{3}\right) \times \text{\( \times \) oF}_2 \left( \frac{1}{3}, \frac{4}{3}, \frac{7}{3} ; \zeta \right) \]

\[ = \frac{\pi}{\sqrt{3}} \times \text{\( \times \) oF}_2 \left( \frac{1}{3}, \frac{4}{3}, \frac{7}{3} ; \zeta \right) + \frac{\pi}{\sqrt{3}} \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{4}{3}\right) \times \text{\( \times \) oF}_2 \left( \frac{1}{3}, \frac{4}{3}, \frac{7}{3} ; \zeta \right) \]

\[ = \frac{\pi}{\sqrt{3}} \times \text{\( \times \) oF}_2 \left( \frac{1}{3}, \frac{4}{3}, \frac{7}{3} ; \zeta \right) + \frac{\pi}{\sqrt{3}} \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{4}{3}\right) \times \text{\( \times \) oF}_2 \left( \frac{1}{3}, \frac{4}{3}, \frac{7}{3} ; \zeta \right) \]

(45)

where we used the functional equation \( \Gamma(z + 1) = z \Gamma(z) \) to express all three products of gamma functions in terms of \( \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) \) and then the Gauss multiplication formula \( \prod_{k=1}^{n} \Gamma(z_k) = (2\pi)^{(n-1)/2} n^{-1/2} \).

Then we transform the three generalized hypergeometric series from Equation (45).
In Equation (47) we formally added extra term to the series, which identically is equal to 0 for any $z \neq 0$.

After substituting Equations (46)–(48) into (45) we note that the three terms in square brackets represent different parts of a single series, then split this series into two (rearranging terms), and eventually obtain

$$
\int_0^\infty \exp \left( -\frac{4z}{\xi^2} \right) K_{5/3}(\xi) d\xi
= \frac{\pi}{\sqrt{3}} \cdot \sum_{k=0}^{\infty} \frac{3k}{(3k+2)!} (-3z^{1/3})^{3k+1}
+ \frac{\pi}{\sqrt{3}} \cdot \sum_{k=0}^{\infty} \frac{3k-2}{(3k)!} (-3z^{1/3})^{3k-1}
= \frac{\pi}{3\sqrt{3}} \cdot z^{-1/3} \sum_{k=0}^{\infty} \frac{2-k}{k!} (-3z^{1/3})^k
= \frac{\pi}{3\sqrt{3}} (-2z^{-1/3} + 3) \sum_{k=0}^{\infty} \frac{1}{k!} (-3z^{1/3})^k
= \frac{\pi}{\sqrt{3}} \left( \frac{2}{3z^{1/3}} + 1 \right) \exp(-3z^{1/3}).
$$

The final result is derived from Equation (49) by the substitution $z = 8x^2/27:

$$
\frac{\sqrt{3}}{\pi} \int_0^\infty \exp \left( -\frac{32x^2}{27 \xi^2} \right) K_{5/3}(\xi) d\xi
= \frac{2}{3(8x^2/27)^{1/3} + 1} \exp(-3(8x^2/27)^{1/3})
= \frac{1}{x^{2/3}} + 1 \exp(-2x^{2/3}).
$$

### Appendix B

**Laplace’s Method of Estimating Integrals**

Consider the integral

$$
I = \int_0^\infty g(x) \exp(f(x)) \, dx,
$$

such that the exponent has a sharp maximum at the point $x_0$, where the function $f(x)$ reaches its maximum. Approximate integration (by Laplace’s method) can be done by replacing $f(x)$ by two leading terms of its Taylor series expansion in the vicinity of $x = x_0$:

$$
f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2.
$$
The exponent then becomes a Gaussian function, so that

\[ I \simeq \exp(f(x_0)) \int_0^\infty g(x) \exp\left(-\frac{(x-x_0)^2}{2|f''(x_0)|}\right) dx \]

\[ \simeq \exp(f(x_0)) g(x_0) \left(\frac{2\pi}{|f''(x_0)|}\right)^{1/2}. \]  \hspace{1cm} (53)

ORCID iDs

Evgeny Derishev @ https://orcid.org/0000-0002-6761-5515

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