THE AUTOMORPHISM GROUPS
OF THE GROUPS OF ORDER $8p^2$

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Abstract. The automorphism groups for the groups of orders $8p$ and $8p^2$ are given. The calculations were done using the programming language CAYLEY. Explicit presentations for both the groups of these orders and their automorphism groups are given.

1. Introduction

In the early development of group theory much effort was devoted to the determination of the number of groups of a certain type, e.g., the number of groups of degree 8 say, or the number of groups of a certain order, e.g., $72, 16p$ ($p$ being any odd prime), etc. This is not as active an endeavor as it was about 100 years ago, but some work is still being done in this area. (See references in Appendix 0 of [1] and the book of M. W. Short [2].) Some of these authors obtained explicit presentations for all of the groups of orders such as $8p, 16p, 16p^2, \ldots$ for $p$ being an odd prime. There has been very little work devoted to the explicit determination of the automorphism groups of these groups. It seemed to the authors that if explicit presentations for these groups could be found, then one might be able to do the same for their automorphism groups. This has in fact turned out to be the case.

Knowledge of the automorphism groups of groups is useful in a variety of contexts; e.g., in the general problem of group extensions, $\text{Aut}(G)$ is one more invariant for the group $G$ which might serve to distinguish one group from a second nonisomorphic one. There are other questions that could also be asked, e.g., what relation if any exists between the structure of the group $G$ and that of its automorphism group, $\text{Aut}(G)$.

Desktop computers are now being used in many different areas where the tried and true methods of the field are paper and pencil types of proof. Modern algebra, and especially group theory is one such field. The use of computers to solve problems in group theory has come about because of the work of individuals such as Drs. J. Cannon and J. Neubueser and their computing systems (CAYLEY/MAGMA and GAP respectively) [3]. The development of computing algorithms per say, as useful as these computing procedures may be by themselves does not contribute to the solution of typical problems in abstract algebra. The intelligent use of the computer for providing a large number of examples and counterexamples can lead to the formulation of conjectures (would-be theorems if you will) that point the way to further progress in many areas. This work attempts to show how one can use computers to formulate theorems that deal with the automorphism groups of finite
groups. Additional questions involving automorphism groups that may be investigated by means of computers can be found in the article by D. MacHale [4].

The calculations reported upon below were done using the programming language CAYLEY. For the groups of order $8p^2$, the calculations were explicitly done for the primes $p = 3, 5$ and $7$, as well as for some higher orders, e.g., $p = 17$, when the systematic behavior for these higher orders is not clear from the lower-order calculational results. The tables list the calculational results as a function of $p$, for any odd prime. In some cases, for example cyclic groups, dihedral and dicyclic groups, the results were proven a long time ago using alternate, and more traditional, group-theoretic methods and can be found in many group theory textbooks. In other cases the general expressions given are deductions (or conjectures if you will) as to what the general behavior of these automorphism groups will be based upon the explicit calculations done for the cases of the smaller primes. Each entry in the tables below can be looked upon as a theorem to be proven in the more traditional theorem-proof approach to group theory problems.

The calculational results to be presented below almost certainly can be determined by other non-computer methods, but explicit calculational methods, where they exist, are difficult to find; e.g., the standard reference on the automorphism groups for abelian groups is a 1928 article in German by K. Shoda, and the standard work on automorphism group towers is the 1939 article by H. Wielandt [5], also in German. With the large number of group theory textbooks on the market it is surprising that most of them contain only a brief discussion of automorphism groups and even less on methods for their calculation [5].

We hope that the general patterns found in our calculations of these automorphism groups will encourage others to develop other, more standard group-theoretic methods for the calculation of the automorphism groups of finite groups. The structural patterns found in the order $8p$ and $8p^2$ groups’ automorphism groups are present in almost all of the other cases that the author has investigated. The results of these other investigations for orders $16p$, $16p^2$, $32p$, $8p^3$, . . . will be published elsewhere.

A major reason for this publication is to encourage others to consider the study of automorphism groups of finite groups as a field ripe with possibilities for producing very general theorems — a structural theory of automorphism groups, if you will. The relationship between the group and its automorphism group is also a good area to look at, since in many cases the automorphism groups $[Aut(C_p)@T]$ take the form $Hol(C_p) \times$ (invariant factor). The “invariant factor” should be determinable from the group $T$ alone.

In what follows we will need some information on the groups of orders 8 and $8p$ as well as their automorphism groups. This material will be summarized below for the reader’s convenience. Much of this material is available in standard books on group theory, e.g., Carmichael [7] and Burnside [8], among others. Some of the material on the automorphism groups for the groups of order $8p$ may be found in the books by Coxeter and Moser [9], and Wood and Thomas [10]. We have not been
able to obtain a complete “closed form” expression for certain of the automorphism
groups for the orders $8p^2$, namely those for the groups in which the action of the
2-group on the $p$-group is by a $Q_2$ image in which $p \equiv 7 \mod(8)$. The problem
is determining a representation of the group $< 2, 3, 4 >$ of order 48 in the groups
$GL(2, p)$ for $p \equiv 7 \mod(8)$ (see appendix 2). Section 3 contains a brief discussion
of the automorphism groups of the groups of order $8p^2$ and some possible general-
izations. Section 4 contains additional material on the $C_8$ extensions. Certain
questions on determining when two groups are isomorphic or distinct may provide
for interesting classroom exercises in elementary group theory. One can certainly
use some fancy programming system (e.g., GAP) to do this, but it might be useful
to see why/how these various groups of order $8 \times 17^2$ differ.

2. Background material on groups of order 8 and $8p$
and their automorphism groups

The five groups of order 8 and their automorphism groups are:

| group $g$ | $C_8$ | $C_4 \times C_2$ | $C_2 \times C_2 \times C_2$ | $D_4$ | $Q_2$ |
|-----------|-------|-----------------|-----------------|-------|-------|
| aut($g$)  | $C_2 \times C_2$ | $D_4$ | $GL(3, 2)$ | $D_4$ | $S_4$ |

The group $GL(3, 2)$ is the simple group of order 168, and $S_4$ (the symmetric
group of order 24) is a complete group.

A complete group is a group all of whose automorphisms are inner. An alternate
definition of a complete group is a group with a trivial center and whose automor-
phism group is isomorphic to the group itself.

The Lattice Structure of $D_4$

To obtain the groups of order $8p$ where $p$ is an odd prime we need to know the
normal subgroup structure of the groups of order 8. This information is readily
available in many textbooks, e.g., the Hall-Senior tables [11] and in [10]. A typical
example is the one for $D_4$ shown in Figure 1. From this figure we see that we have
two different quotient groups of order 2, one with a kernel $C_4$ and the second one
with the kernel $C_2 \times C_2$. For the groups of order 8 there are seven such distinct
normal subgroups, and each such $C_2$ quotient gives rise to a distinct group of order
All of these groups will appear in every order for which there exist groups of order $8p$. There are two groups of order 8 that have a quotient group $C_4$; these two groups will give rise to an additional pair of groups of order $8p$ for those primes that are equal to 1 mod(4). There is one additional group, coming from $C_8$, that will give rise to an additional group when $p \equiv 1 \mod(8)$. There are two groups of order 8 that have an automorphism of order 3, namely $C_2 \times C_2 \times C_2$ and $Q_2$. For the order 24 these two groups will give rise to an additional pair of groups (namely $A_4 \times C_2$ and SL$(2,3)$). In order 24 then we have the following groups:

| 5 direct products of the form $C_4 \times$ group of order 8 |
|-------------------------------------------------------------|
| 7 $C_2$-type extensions                                     |
| 2 groups having a normal sylow 2-subgroup:                  |
| ($A_4 \times C_2$ and SL$(2,3)$)                           |
| 1 group without a normal sylow subgroup, $S_4$              |

For order 40 ($p = 5$ so $p$ is of the form 1 mod(4)), we have the five direct products plus the 7 $C_2$-type extensions plus the two coming from the $C_4$ action on $C_5$. There are no automorphisms of order 5 in the automorphism groups of order 8; so there are no groups of order 40 with a normal sylow 2-subgroup. All of the groups of order 40 have a normal sylow 5-subgroup. For $p = 7$ we have one extra group, namely $(C_2 \times C_2 \times C_2) \otimes C_7$, the Frobenius group of order 56.

Presentations for the groups of order 8 are listed in Table 1. The non-direct-product groups of order $8p$ along with their automorphism group factors are also listed in Table 1. A simple illustration will show the reader the meaning of the entries in this table. Consider the extensions coming from the group $D_4$. From Table 1, $D_4$ has the presentation $a^4 = b^2 = a^b \ast a = 1$. The group $D_4$ has two distinct normal subgroups of order 4 ($C_2 \times C_2$ and $C_4$), giving rise to quotient groups of order 2. The two possible actions of $D_4$ on $C_p$ are:

(1) a) $e^p = a^4 = b^2 = a^b \ast a = c^b \ast c = (b,c) = 1$ [case a in Table 1]
for the case of the $C_2 \times C_2$ kernel and

(2) b) $e^p = a^4 = b^2 = a^b \ast a = (a,c) = c^b \ast c = 1$ [case b in Table 1]
for the case where the kernel is $C_4$.
Case a) has an automorphism group isomorphic to

(3) (entry in table) $\times Hol(C_p)$

or

(4) $C_2 \times C_2 \times Hol(C_p)$

and case b) has an automorphism group isomorphic to

(5) $D_4 \times Hol(C_p)$,
where $Hol(C_p)$ is the holomorph of the group $C_p$.

The remaining entries in this table follow the same pattern. The “entry in table” factor will appear in all $C_2$ extensions of $p$-groups by the same order 8 group.
3. Groups of order $8p^2$ and their automorphism groups

Let us start by considering the case $(C_p \times C_p)@D_4$. This is a group of order 72 for $p = 3$. In fact, there are several possible groups of this form. Let us for now only concern ourselves with the $C_2$ actions of $D_4$ on the $p$-group. From the above comments we know there are two possible $C_2$ actions that can arise from $D_4$, but each of these actions can act in two different ways on $C_p \times C_p$, viz:

$$c^p = d^p = (c, d) = a^4 = b^2 = a^b \ast a =$$

or

$$c^p = d^p = (c, d) = a^4 = b^2 = a^b \ast a =$$

(6) \hspace{1cm} (a, c) = (a, d) = (b, c) = d^b \ast d = 1 \hspace{1cm} C_p \times (C_p@D_4),

or

$$c^b \ast c = (a, d) = a^b \ast d = 1 \hspace{1cm} (C_p \times C_p)@D_4. \tag{7}$$

The other case arises from interchanging the actions of $a$ and $b$ on the $p$-group, viz:

$$c^b = d^b = (c, d) = a^4 = b^2 = a^b \ast a =$$

or

$$c^a \ast c = (b, c) = d^a \ast d = (b, d) = 1 \hspace{1cm} (C_p \times C_p)@D_4. \tag{8}$$

The automorphism groups for the four different groups of order $8p^2$ are all different but easily specified for all primes $p$:

(10) \hspace{1cm} Hol(C_p) \times C_{(p-1)} \times D_4,

(11) \hspace{1cm} Hol(C_p \times C_p) \times D_4,

(12) \hspace{1cm} Hol(C_p) \times C_{(p-1)} \times C_2 \times C_2,

(13) \hspace{1cm} Hol(C_p \times C_p) \times C_2 \times C_2.

In fact, the general case looks like this:

(14) \hspace{1cm} c^p = c^2 = \cdots = c^p = a^4 = b^2 = a^b \ast a = \cdots = 1

with $a$ commuting with the generators $c_1, \ldots, c_n$ and $b$ acting as an element of order 2 on the first $m$ generators $c_1, \ldots, c_m$, and commuting with the last $n - m$ generators of order $p$. In this case the automorphism group of this group is

$$Hol((C_p \times C_p) \times \cdots \times C_p) \times GL((n - m), p) \times D_4, \tag{15}$$

where we have $m C_p$'s in the holomorph. If the generator $b$ were commuting with the generators $c_1, \ldots, c_n$ and the generator $a$ were acting on the generators $c_1, \ldots, c_n$ instead, then the resulting automorphism group would be obtained by just replacing the $D_4$ factor by the factor $C_2 \times C_2$ in (15). In a like manner one can find similar patterns in other groups or with other actions, e.g., $C_2 \times C_2$, $C_4$ or an order 8 group.

The groups of order $8p^2$ that arise by a $C_2$ action of the 2-group on the $p$-group can be easily written out, and their automorphism groups also follow a very simple pattern. Namely, for the automorphism groups, we determine the $C_2$ action on the $p$-group, pull out the corresponding entry in Table 1, and the “$p$-factor” is now either

$$C_{(p-1)} \times Hol(C_p) \tag{16}$$
if the $C_2$ acts only on one of the $C_p$’s or
\begin{equation}
Hol(C_p \times C_p)
\end{equation}
if the $C_2$ acts on both of the $C_p$’s.

When the $p$-group is cyclic, the groups behave just like the case of order $8p$. Namely, just use the “invariant factor” in Table 1 with $Hol(C_{(p^2)})$ and you have the automorphism groups for these cases.

The cases involving an order 4 or higher-order action are listed in Table 2. In the case of $C_2 \times C_2$ actions the “$p$-factor” is one of two types: $Hol(C_p) \times Hol(C_p)$, or a group that can be represented as a wreath product \[12\]. The automorphism groups coming from a $C_2 \times C_2$, $C_4$ or a $C_4 \times C_2$ action are explicitly written out in Table 2. The cases arising from the $D_4$ or $Q_2$ actions depend upon $p$ and are given in the notes to Table 2 (Appendix 1).

Table 2 also gives in an abbreviated form the presentations of the groups of order $8p^2$. To obtain the required presentation from this table a few examples will show how to reconstruct the presentations. Look at the $(C_2 \times C_2)$ image from $C_4 \times C_2$, for the case $[ab,b]$. The group whose automorphism group is \[Hol(C_p) \times C_2] wr C_2\] has the presentation:
\begin{equation}
a^4 = b^2 = (a,b) = c^p = d^p = (c,d) = c^a * c = c^b * c = (a,d) = d^p * d = 1.
\end{equation}
For the group $(C_4 \times C_2)$ with an order 8 group action, Table 2 gives $[a,ab]$ (with automorphism group $Hol(C_p) wr C_2$) giving the presentation:
\begin{equation}
a^4 = b^2 = (a,b) = c^p = d^p = (c,d) = c^a * c^x = d^a * d = (b,c) = d^b * d = 1,
\end{equation}
where $x^4 \equiv 1 \mod(p)$ (i.e., $x$ is a fourth root of unity \[20\]).

The groups without a normal sylow $p$-subgroup are treated in Table 3.

For the case of $p \equiv 1 \mod(2)$ but not $1 \mod(4)$ or $1 \mod(8)$ we have only two cases of a $C_4$ and one case of a $C_8$ extension. For $p = 7$ we have extensions in which the action is full (i.e., by a group of order 8) involving only the groups $C_8$, $D_4$, and $Q_2$. The automorphism group of $(C_p \times C_p)@Q_2$ is always a complete group \[13\]. The automorphism group of the group with a full $C_8$ action is also a complete group, with order 4704 (for $p = 7$). The $D_4$ extension’s automorphism group is not complete, but has order 2352 (for $p = 7$).

In connection with the study of “higher” orders, e.g., 2-groups of orders 16, 32, and especially of order 64 the following observation is very useful: if the group associated with the extension is a characteristic subgroup of index 2 for the given 2-group, then the automorphism group arising from this extension is given by \[14\]
\begin{equation}
Hol(C_p) \times \text{Automorphism group(2-group)}.
\end{equation}
In connection with the groups of order 192 with a normal sylow 3-subgroup, this means that about 700 of the automorphism groups of this type in this order are already known, without the need for performing any calculations whatsoever \[15\]. In fact since these groups have a “natural extension” to groups of order $64p$, $p$ any odd prime, this whole class of automorphism groups is already known without the
need to do any calculations! General results like this considerably reduce computer
time in the calculation of automorphism groups, and at the same time provide a
link between the structure of a group and its automorphism group. One should also
note that this simplification does not extend to the cases of the form \( C_q \otimes X \), where
\( X \) is a \( p \)-group (\( p \) an odd prime, \( q \equiv 1 \mod(p) \)) and \( X \) acts on \( C_q \) as an operator
of order \( p \). An interesting exercise would be to see what happens both in this case
and in the cases for 2-groups where the group associated with the extension is a
characteristic subgroup of index greater than 2.

4. Groups of order \( 8p^2 \). The \( C_8 \) Extensions

The most recent discussion of the groups of order \( 8p^2 \) is that given by Zhang
Yuanda [16]. Following Zhang Yuanda [16] the groups of order \( 8p^2 \) with the \( p \)-group
being \( C_p \times C_p \) and with the 2-group being cyclic give rise to the cases listed in Table
4.

All of the numbers in Table 4 refer to the listing of the groups as given in the ar-
ticle by Zhang Yuanda [16]. The relations for the groups in Table 4 are given below.

For the purpose of determining the number of groups of order \( 8p^2 \) of the form
\((C_p \times C_p) \otimes C_8\) for a particular prime \( p \), it is more convenient to use the following
breakdown:

| Number of groups of the form \((C_p \times C_p) \otimes C_8\) |
|---------------|
| prime \( p \) | number of groups |
| image of 2-group | \( C_2 \) \( C_4 \) \( C_8 \) |
| if \( p \equiv 1 \ \text{mod}(2) \) | \( 2 + 1 + 1 \) |
| if \( p \equiv 1 \ \text{mod}(4) \) | \( 2 + 4 + 1 \) |
| if \( p \equiv 1 \ \text{mod}(8) \) | \( 2 + 4 + 8 \) |

The relations for the \( C_4 \) action when \( p \) is not equal to \( 1 \ \text{mod}(4) \) can be written
as

\[(21) \quad a^p = b^p = (a, b) = c^8 = a^c \ast b^{-1} = b^c \ast a = 1.\]

The relations that Zhang Yuanda gives for a \( C_8 \) action when \( p \equiv 3, 5 \) and 7 \( \text{mod}(8) \)
are:

| number in ref. [16] | relation for \( s \) and \( t \) | \( (p,s,t) \) | \( Z(g) \) |
|---------------------|------------------|------------|--------|
| 16. for \( p \equiv 3 \ \text{mod}(8) \) | \( s^2 \equiv -2 \ \text{mod}(p) \), \( t = 1 \) | \( (11,3,1) \) | I |
| 7. for \( p \equiv 1 \ \text{mod}(4) \) | \( s = 0, t^2 \equiv -1 \ \text{mod}(p) \) | \( (13,0,8) \) | I |
| 17. for \( p \equiv 7 \ \text{mod}(8) \) | \( s^2 \equiv 2 \ \text{mod}(p) \), \( t = -1 \) | \( (7,3,-1) \) | I |
Explicit relations for the first few cases of \((C_p \times C_p) \times C_8\) are given by:

\[(22) \quad a^p = b^p = (a, b) = c^8 = a^w * b^x = b^y * a^v * b^z = 1,\]

where the entries in Table 5 are \((w, x; y, z)\). The automorphism groups for these groups are given in Table 6.

The following two sets of presentations yield groups that are not distinguishable by their class structure and automorphism groups alone:

| [-2,2], [-2,8], [-2,9], \(\rightarrow\)’s 7, 12, 13 in Table 6 |
|---------------------------------------------------------------|
| number of elements | order of elements | number of classes |
|---------------------|------------------|------------------|
| 289                 | 2                | 1                |
| 578                 | 4                | 2                |
| 1156                | 8                | 4                |
| 288                 | 17               | 36               |

Here we are using the notation \([-2,a] = (-2,0:0,a)\).

| [-2,4], [-2,13] \(\rightarrow\)’s 11 and 10 in Table 6 |
|---------------------------------------------------------------|
| number of elements | order of elements | number of classes |
|---------------------|------------------|------------------|
| 17                  | 2                | 1                |
| 578                 | 4                | 2                |
| 1156                | 8                | 4                |
| 288                 | 17               | 38               |
| 272                 | 34               | 4                |

This poses the interesting question of how to determine when two groups with different presentations might be isomorphic. Clearly in many cases one can compare the conjugacy classes of the two groups to show that they are not isomorphic. In other cases when this does not distinguish between two groups the automorphism groups of these groups may be different. Within each of the sets above neither of these methods enables one to be able to distinguish between these nonisomorphic groups. (An interesting exercise for the reader would be to develop a computational method which would enable one to distinguish between these different groups \([17]\).) Such methods obviously exist. It would be helpful to be able to list the group invariant that enables one to distinguish between these groups. Group theory programming systems such as GAP have a routine that enables its users to determine whether two groups are isomorphic or not.

5. Conclusions

We were told by some group theorists when we started this work around 1980 that the study of the automorphism groups would be very difficult since there were very few general results known \([18]\). We felt at that time if one could determine general presentations for the groups of a particular type (e.g., of order \(8p\)), then one should probably be able to determine their automorphism groups with equal precision. As this work shows, we have in fact been able to obtain explicit expressions for the automorphism groups for the groups of orders \(8p\) and \(8p^2\) for all odd primes \(p\) (modulo a representation theory problem for the \(p \equiv 7 \mod(8)\) cases). In this work the “factor” that is independent of “\(p\)” in Aut\((G)\) appears to be a new group
invariant associated with the group $G$ (or $\text{Aut}(G)$). More details on this aspect of
the work will be presented in a paper on the automorphism groups for the groups
of order $32p$.

The situation arising with the matrix representations for the group $\langle 2, 3, 4 \rangle$
discussed in Appendix 2 also shows the limitations of a purely computational ap-
proach to the subject. If one had restricted the calculations to primes $p < 103,$
then one would have been led to believe that one could always find a matrix rep-
resentation for the group $\langle 2, 3, 4 \rangle$ using matrices of the form $\text{[13]}$ in Appendix
2. As is apparent from the calculations mentioned in Appendix 2 this is not the case.

The use of computers in mathematical research is widespread. The real value of
a computer in many areas of mathematics, like group theory among others, seems
to us to be in providing a large number of examples, or conjectures if you will, to
enable the mathematician to formulate new theorems which may be proven by other
noncomputational means. In the case of group theory this is certainly a desirable
goal. The computer like all finite machines has its limits. In some of the computer
runs to be reported in other papers the time taken to compute one automorphism
group may be as much as 24 hours. Clearly as one goes to larger and larger orders
other less brute force methods become more and more relevant. The insight pro-
vided by the lower-order calculations should lead one to a better understanding of
just what is likely to be true and what is false. A single example can show that a
“theorem” is false, but no finite number of cases can show that it is true. Hence the
intelligent interactions of the mathematician and the computer hopefully provide a
guide to the formulation and solution of otherwise difficult problems. The authors’
work on the computation of the automorphism groups is an attempt to provide the
first step in showing just how useful the computation of automorphism groups can
be in providing a guide as to what the general structure of these automorphism
groups is. The author hopes the conjectures or “theorems” proposed above show
just how useful this interaction of computers with the more standard methods of
group theory can be for future research.
Mainly presentations for automorphism groups of $D_4$ and $Q_2$ images.

The group [144] in Table 2, for $p > 3$ means

$$\text{Aut}\{(C_p \times C_p) \oplus C_4\}$$

when $p \equiv 3, 7 \mod(8)$, or $H(p^2)$. Here $H(p^n)$ is the group of all mappings $x \mapsto a \cdot x^t + b$ where $a$ (different from zero) and $b$ are elements of the Galois field, $\text{GF}(p^n)$, and $t$ is an element of the Galois group of $\text{GF}(p^n)$. Here $p$ is a prime number. The order of the group $H(p^n)$ is $p^n \cdot (p^n - 1) \cdot n$. (See D. Robinson [19].)

Notes for $C_4$ column:

For $p \equiv 1 \mod(4)$ we have the following four cases for a $C_4$ action on the $p$-group, and the [144] factor being replaced by the factors below arising from the four nonisomorphic cases, which in lowest order ($p = 5$, and with the $C_4$ action in $C_4 \times C_2$) take the form:

| Group ($p = 5$ case) | Automorphism groups for $p > 5$ |
|----------------------|----------------------------------|
| $(C_5 \oplus C_4) \times C_5$ $\text{Hol}(C_5) \times C_5$ | $\text{Aut}(g) = \text{Hol}(C_p) \times C_{(p-1)}$ |
| $(C_5 \oplus C_5) \oplus C_4$ | $\text{Aut}(g) = \text{Hol}(C_p) \times \text{Hol}(C_p)$ |
| $(C_5 \oplus C_5) \oplus C_4$ | $\text{Aut}(g) = \text{Hol}(C_p) \times C_p$ |
| $(C_5 \oplus C_5) \oplus C_4$ | $\text{Aut}(g) = \text{Hol}(C_p) \wr C_2$ |

The last two groups have the same class (order) structure and so cannot be distinguished from their conjugacy classes alone. As can be seen above though, they can be distinguished from the fact that they have different automorphism groups.

The sequence of groups of order $4p^2$ with the presentations

(23) $a^p = b^p = (a, b) = c^4 = a^c * (b^{-1}) = b^c * a = 1$

have complete groups for their automorphism groups for $p = (3, 5, 7, 11)$. For primes $p < 23$ the automorphism groups and their orders are:

| $p$ | order of group | automorphism group |
|-----|----------------|--------------------|
| 3   | $144 = 2^4 \cdot 3^2$ | $H(3^2)$ |
| 5   | $800 = 2^5 \cdot 5^2$ | $\text{Hol}(C_5) \wr C_2$ |
| 7   | $4704 = 2^8 \cdot 3 \cdot 7^2$ | $H(7^2)$ |
| 11  | $29,040 = 2^4 \cdot 3^3 \cdot 5 \cdot 11^2$ | $H(11^2)$ |
| 13  | $48,072 = 2^8 \cdot 3^2 \cdot 13^2$ | $\text{Hol}(C_{13}) \wr C_2$ |
| 17  | $147,968 = 2^9 \cdot 17^2$ | $\text{Hol}(C_{17}) \wr C_2$ |
| 19  | $259,920 = 2^4 \cdot 3^2 \cdot 5 \cdot 19^2$ | $H(19^2)$ |
If \( p \equiv 1 \) mod(4), the automorphism group of the group whose presentation is

\[
(24) \quad a^p = b^p = (a, b) = c^4 = a^c * (b^{-1}) = b^c * a = 1
\]

is \( Hol(C_p)wrC_2 \); otherwise, it is \( H(p^2) \) of order \( p^2 * (p^2 - 1) * 2 \). In either case, Aut\((g)\) is complete.

Note. Not all products of the form \( (Hol(C_p)wrC_2) \) are complete groups. The group \( Hol(C_3)wrC_2 \) has order 72 and is not complete, but \( Hol(C_7)wrC_2 \) is a complete group.

Notes for order 8 column:

a). For the \( C_8 \) cases see Table 4.

b). \( D_4 \) cases.
The presentation for this set of groups is:

\[
\begin{align*}
 a^4 = b^2 & = a^b * a = c^p = d^p = (c, d) = \\
 c^a * d & = d^a * (c^{-1}) = c^b * c = (b, d) = 1.
\end{align*}
\]

The automorphism groups for this sequence of groups have the following orders:

| \( p \) | order Aut\((g)\) | comments |
|---|---|---|
| 3 | 144 | complete group |
| 5 | 800 | complete group |
| 7 | 2,352 | not complete (1) |
| 11 | 9,680 | complete group |
| 13 | 16,228 | complete group |
| 17 | 36,992 | not complete (2) |
| 19 | 51,984 | complete |
| 23 | 93,104 | not complete (3) |

(1) Aut\((2,352)\) has order 4,704.

(2) 36,992 \( \rightarrow \) 73,984 \( \rightarrow \) \( Hol(C_{17})wrC_2 \) (complete).

(3) Aut here has order 186,208.

The orders of these Aut\((g)\)'s follow the pattern \( 8 * p^2 * (p - 1) \), the \( p \equiv 3 \) or 5 mod (8) being complete. If \( p \equiv 1 \) or 7 mod(8), then the automorphism group is not a complete group. The relations for the \( p = 3, 5, 7, \) and 1 mod 8 are:
If we replace the term $a^b * a^5$ in the $p \equiv 3 \mod(8)$ presentation and the term $a^b * a$ in the $p \equiv 7 \mod(8)$ presentation above with $a^b * a^s$ with $s$ given by

$$p \equiv (8 - s) \mod(8),$$

we then have just one form that is valid for both the $p \equiv 3 \mod(8)$ and the $p \equiv 7 \mod(8)$ presentations.
A four-generator permutation representation of this group for \( p = 13 \), of degree 26 is:

\[
\begin{align*}
    a &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13); \\
    b &= (1, 8, 12, 5)(2, 3, 11, 10)(4, 6, 9, 7); \\
    c &= (1, 14)(2, 15)(3, 16)(4, 17)(5, 18)(6, 19)(7, 20) \\
        (8, 21)(9, 22)(10, 23)(11, 24)(12, 25)(13, 26); \\
    d &= (2, 10, 4)(3, 6, 7)(5, 11, 13)(8, 12, 9) \\
        (15, 23, 17)(16, 19, 20)(18, 24, 26)(21, 25, 22).
\end{align*}
\]

For the \( p \equiv 1 \mod(8) \) case we have only been able to verify the presentation for the first case, namely that for \( p = 17 \),

\[
\begin{align*}
    a^{17} &= b^{16} = a^{b \ast a^3} = c^4 = d^2 = (a, c) = (b, c) = (b, d) = \\
    (a \ast d)^2 \ast ((a^{-1} \ast d)^2 &= b^4 \ast ((d^{-1} \ast (c^{-1}))^2 = 1.
\end{align*}
\]

We conjecture that the general matrix representation of the group of order 8\((p – 1)\) is given by

\[
\begin{align*}
    b &= \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \\
    c &= \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \\
    d &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{align*}
\]

where

\[
\begin{align*}
    x^4 &\equiv 1 \mod(p), \\
    z^{(p-1)} &\equiv 1 \mod(p).
\end{align*}
\]

The matrix \( b \) is the matrix representation for the center of the group \( GL(2,p) \) (see [20]), and the group generated by \(< c, d >\) is \( C_4 \wr C_2 \). The group \( < b, c, d > \) appears to be the same group arising in the automorphism groups for the groups of order 16\(p^2\) with a \( D_4 \) or \( Q_2 \) action on the group \( C_{17} \times C_{17} \).

c). \( Q_2 \) image case.

The presentations of these groups of order 8\(p^2\) can be read from the representation of \( Q_2 \) in \( GL(2,p) \).

A matrix representation of \( Q_2 \) for the presentation

\[
\begin{align*}
    a^4 &= b^4 = a^2 \ast b^2 = a^b \ast a = 1
\end{align*}
\]

is

\[
\begin{align*}
    a &= \begin{pmatrix} x & y \\ y & -x \end{pmatrix}, \\
    b &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\end{align*}
\]

where \((x^2 + y^2) \equiv -1 \mod(p)\).

The automorphism groups of these groups have the structure [21]:

\[
(C_p \times C_p)@\text{(group of order } 24(p – 1) \).
\]
The group of order $24(p - 1)$, in this sequence of automorphism groups, depends upon the prime $p$ as follows:

| prime $p$ | quotient group $q = (p - 1)/2$ | $q = (p - 1)/4$ | $q = (p - 1)/2$ |
|-----------|--------------------------------|-----------------|-----------------|
| $p \equiv 3 \mod(8)$ | $GL(2,3) \times C_q$ | $GL(2,3) \times C_q$ | $SL(2,3) \times C_4$ |
| $p \equiv 5 \mod(8)$ | $SL(2,3) \times C_4$ | $SL(2,3) \times C_4$ | $SL(2,3) \times C_4$ |
| $p \equiv 7 \mod(8)$ | $< 2, 3, 4 > \times C_q$ | $< 2, 3, 4 > \times C_q$ | $< 2, 3, 4 > \times C_q$ |
| $p \equiv 1 \mod(8)$ | $GL(2,3) \times C_4$ | $GL(2,3) \times C_4$ | $GL(2,3) \times C_4$ |

In the cases where $p \equiv 1, 3, \text{or } 5 \mod(8)$ these groups of order $24(p - 1)$ can be written in the form:

$$SL(2,3) \times C(p - 1).$$

with the presentation:

$$a^2 = b^3 = (a \ast b \ast a \ast b \ast a \ast (b^{-1}))^2 = (a, c) = (b, c) = 1,$$

$$c^x = c^y \ast (a \ast b)^4 = 1,$$

where $x = (p - 1)$ and $y = x/2$. A matrix representation for this series of groups is given by:

$$a = \begin{pmatrix} 0 & s \\ t & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix},$$

where the matrix $c$ is just the center of the group $GL(2,p)$ and $H = < a, b >$ is a representation of $GL(2,3)$ by $2 \times 2$ matrices over the field $GF(p)$. For the $p \equiv 1 \mod(8)$ case we have for the first few cases:

$$p = (17, 41, 73, 89, 97, 113);$$
$$s = (2, 3, 10, \ldots);$$
$$t = (9, 14, 22, \ldots);$$
$$z = (3, 6, 5, 3, 5, 3, \ldots).$$

In the general case we have $s, t$ and $z$ being given by the following relations:

$$s^8 \equiv 1 \mod(p),$$
$$t = (p + 1)/s,$$

and

$$z^{(p-1)} \equiv 1 \mod(p)$$

(see [20]). In the other case, the group $GL(2,3)$ is replaced by the group $< 2, 3, 4 >$. In this case we have the following presentation:

$$(a^{-2}) \ast b^3 = (a^{-2}) \ast c^4 = (a^{-1}) \ast b \ast c =$$

$$d^x = d^y \ast a^2 = (a, d) = (b, d) = (c, d) = 1.$$

Explicit presentations for the $Q_2$ image cases:

Nota bene in the presentations below that the roles of $a$ and $b$ are interchanged in the $p \equiv 3$ and $5 \mod(8)$ cases.
Here the matrices are

\[
\begin{pmatrix}
-1 & 1 \\
-1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
v & x \\
y & w
\end{pmatrix}.
\]

Here \(a\) and \(b\) generate the group \(\langle 2, 3, 4 \rangle\).
An alternate representation for this automorphism group is based upon the group \(< 2, 3, 4 > \times C_q\) and is generated by matrices of the form \([b]\), and is (for \(p = 7\))

\[
\begin{align*}
    a^3 & = b^{(4q)} = (a, b^2) = (a * (b^{-1}))^4 * (b^{-(p-5)}) = \\
    c^p & = d^p = (c, d) = c^a * c * (d^{-1}) = d^a * c = \\
    c^b * ((c^{-1}) * (d^{-x}) & = d^b * d * (c^{-y}) = \\
    (a * b)^4 * b^2 & = 1,
\end{align*}
\]

and for \(p > 7\) we have

\[
\begin{align*}
    a^3 & = b^{(4q)} = (a, b^2) = (a * (b^{-1}))^4 * (b^{-(p-5)}) = \\
    c^p & = d^p = (c, d) = c^a * c * (d^{-1}) = d^a * c = \\
    c^b * ((c^{-1}) * (d^{-x}) & = d^b * d * (c^{-y}) = \\
    (a * b)^2 * (a^{-1}) * (b^{-1}) * (a * (b^{-1}))^2 * (a^{-1}) * b & = 1.
\end{align*}
\]

In this form \([b]\) has order \(4 * q = 4 * (p - 1)/2\). For many (but not all!) primes of the form \(p \equiv 7 \mod(8)\) one can represent the generator \([b]\) for the presentation \([39]\) with \(v = w = 1\), and \(x * y = -2 \mod(p)\). One can also find representations for \([b]\) for the form \([49]\) with \(v = -w = 1\). I do not have closed form expressions for the values of \(x\) and \(y\), for either form of the above presentations.

This is a problem in the representation theory of \(< 2, 3, 4 >\) in terms of \(2 \times 2\) matrices over the field GF(\(p\)). See Appendix 2 for details.

7. Appendix 2

Question on Matrix Representations of the Coxeter Group \(< 2, 3, 4 >\) by \(2 \times 2\) Matrices over GF(\(p\)).

The Coxeter group \(< 2, 3, 4 >\) arises in connection with getting the presentations for the automorphism groups of the groups

\[
(C_p \times C_p)@Q_2 \quad \text{when } p \equiv 7 \mod(8).
\]

For many cases the following matrices will give a matrix representation for the Coxeter group \(< 2, 3, 4 >:\)

\[
\begin{align*}
    a & = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, &
    b & = \begin{pmatrix} 1 & x \\ y & -1 \end{pmatrix}.
\end{align*}
\]

Here the matrix \(a\) has order 3 and \(b\) is of order 4. The presentation for this matrix representation is

\[
\begin{align*}
    a^3 & = b^4 = (a, b^2) = a * b * (a * (b^{-1}))^3 = 1
\end{align*}
\]

when the entries in the matrix \(b\) obey the relation:

\[
\begin{align*}
    x * y & = -2 \mod(p).
\end{align*}
\]

Problem. Find an algebraic method for determining the values of \(x\) and \(y\) in the above matrix \(b\). The following are the values found by trial and error (actually a computer run for various primes).
Matrix representations of the form \((xy)\) do not exist for the primes 103, 127, 151 and 263. The values listed in the above table are apparently the only possible values for \(x\) and \(y\) yielding the above representation.

Alternate forms are required for the primes 103, 127, 151, and 263 and probably others larger than 263. A few possible choices for the matrix \(b\), which together with \(a\) above obeys the relations (44), are:

\[
b = \begin{pmatrix} v & x \\ y & w \end{pmatrix},
\]

where \((v, x, y, w)\) are given in the following table:

\[
\begin{array}{|c|c|c|c|}
\hline
\text{prime} & (x,y) & \text{prime} & (x,y) \\
\hline
7 & 1,5 & 103 & none \\
23 & 2,6 & 127 & none \\
31 & 12,5 & 151 & none \\
47 & 18,26 & 167 & 54,68 \\
71 & 7,20 & 191 & 8,143 \\
79 & 19,29 & 199 & 12,33 \\
\hline
\end{array}
\]

The values \((v, x, y, w)\) given are just a sample; there are a large number of quadruples that will work here. They were found by using the matrix representation for \(<2,3,4>\times C_q\) given below and then raising the matrix \(b\) to the power \(q\).

For the primes 103, 127, 151, and 263 (as well as for the other primes \(p = 23, 31, \ldots\)), alternate matrix representations were found for the group \(<2,3,4>\times C_q\). A selected sample of the values for \((x, y)\) that appear in the matrix \(b\) (which now has order \(4(p-1)/2\)) follows:

\[
b = \begin{pmatrix} 1 & x \\ y & -1 \end{pmatrix},
\]

where \(b\) has order \(4\left(\frac{p-1}{2}\right)\).

Some choices for \((x, y)\) are given in the following table:
The numbers in the square brackets indicate the number of solutions found. A * means the run was truncated before all solutions were found. In the case of $p = 103$ all were found but it was a very large number and was not counted. No computer runs were done for the other primes.

The presentations for the group $<2, 3, 4> \times C_q$ are given by (for $p = 7$):

$$a^3 = (a, b^2) = (a * b)^4 * b^2 = (a * (b^{-1}))^4 * b^{-2} \quad \text{for } p = 7,$$

and for $p > 7$,

$$a^3 = (a, b^2) = (a * (b^{-1}))^4 * (b^{-x}) = (a * b)^2 * (a^{-1}) * (b^{-1}) * (a * (b^{-1}))^2 * (a^{-1}) * b = 1.$$

Here $x = p - 5$. These presentations for the group $<2, 3, 4> \times C_q$ have been checked out for primes up to 151 and for $p = 263$.

This thus poses the interesting question in the theory of group representations for the group $<2, 3, 4>$:

What algebraic relationship does the pair $(x, y)$ obey such that the presentation (44) (or 10) is satisfied? More generally, what conditions on the elements of the following order 4 matrix

$$b' = \begin{pmatrix} v & x \\ y & w \end{pmatrix}$$

are required in order for $<a, b'>$ to obey the presentation or relations (44)?

| prime | $(x, y)$ | prime | $(x, y)$ |
|-------|----------|-------|----------|
| 7     | (1,4)    | 79    | (15,22)  |
|       | (3,3)    |       | (1,44)   |
|       | (3,6)    |       | (100,44) |
|       | (4,4)    |       |          |
| 23    | (1,9)    | 127   | (1,56)   |
|       | (1,16)   |       | (2,26)   |
|       | (2,7)    |       | (24,2)   |
| 31    | (3,14)   | 151   | (4,125)  |
|       | (4,26)   |       | (25,95)  |
| 47    | (1,19)   |       | (52,193) |
|       | (1,39)   |       | (108,208)|
|       | (12,22)  | 263*  | (1,29)   |
|       | (17,3)   |       | (29,200) |
| 71    | (17,13)  |       |          |
|       | (61,43)  |       |          |
If one has access to a programming system such as Maple or Mathematica one may be able investigate this problem rather easily; otherwise, it could be a fairly messy algebra problem. Remember what you are looking for is a representation valid for all primes $p \equiv 7 \mod(8)$.

8. ACKNOWLEDGEMENTS

This work was started at Michigan State University around 1980. Additional computations were carried out during the authors’ stay at the University of Rhode Island and during several summers at Syracuse University’s High Energy Particle Groups’ Vax cluster, and was completed at Brown University with the Department of Linguistics and Cognitive Sciences DEC computers. The order 72 groups were done at Michigan State University, the orders 200 and 392 were done at URI and in Syracuse. The higher-order cases were done at Brown University.

We would like to thank Drs. M. Goldberg and G. C. Moneti for making available time on the Syracuse University DEC cluster, and Mr. Carl Brown and later Ms. Judith Reed for assistance in getting CAYLEY running there. At Brown University, Dr. James Anderson has allowed us to use a DEC 6000-510 to finish the work reported here as well as a great deal of additional work that will be reported on in other papers. Ms. Margaret Doll at Brown University has been extremely helpful in getting the various versions of CAYLEY up and running on the DEC computers at Brown University and assisting in clearing up other computer problems as they arose in the course of this work.

A special expression of thanks must go to Dr. John Cannon for making available to us the programming system CAYLEY, without which none of this work could have been done. We very much regret that we (Dr. Cannon and the authors) live so very far away from each other. Dr. Cannon is a very fine gentleman and it would have been a great pleasure to have been able to share the results (and problems found in this work using CAYLEY) with him on a more personal basis.

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[1] Becker, W., “A Preliminary Survey of a Computer Study of Finite Groups and Their Automorphism Groups”. Unpublished, 1994.

[2] Short, M. W., “The Primitive Soluble Permutation Groups of Degree Less than 256”, Lecture Notes in Mathematics # 1519, Springer-Verlag, N. Y. (1992).

[3] In addition to these large general purpose systems there are a number of smaller more teaching-oriented programs for teaching abstract algebra. The groups discussed here are not of very high order and may be able to be run on many of these smaller educationally oriented group theory systems.

[4] MacHale, D., “Minimum Counterexamples in Group Theory”, Math. Mag. vol. 54 p. 23 (1981). In many cases, one knows examples where the given statements contained in this article are false, but the question is what is the smallest order group showing the incorrectness of the “theorem”.

[5] The book by Fuchs on “Abelian Groups” is one of the few exceptions. This book does have a few pages devoted to the discussion of the automorphism groups of abelian groups. This
book also contains references to the articles of K. Shoda on the automorphism groups of abelian groups. See also the brief discussion on automorphism groups of abelian groups in Carmichael’s textbook on finite groups [6].

[6] Wielandt, H. “Eine Verallgemeinerung der Invarianten Untergruppe” Math. Zeit. vol. 45 pp. 209-244 (1939). This paper is briefly discussed in Appendix G of H. Zassenhaus’s book “Theory of Groups” 2nd edition (Chelsea reprint 1958).

[7] Carmichael, Robert D., “The Theory of Groups of Finite Order”, Original 1930. Reprint Dover (1956).

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[10] Thomas, A. D. and Wood, G. V., “Group Tables”, Shiva Publishing Limited. Orpington, Kent, U.K. (1980).

[11] Hall, M. and Senior, J. K. “Tables of Groups of Order 2^n for n ≤ 6”, MacMillan (1964).

[12] The wreath product of the group $T$ with $C_p$ ($T_{wr}C_p$) can be described as follows. Write the generators of the group $T$ in terms of permutations of some degree, say $n$. Then the group $C_p$ is expressed in the form: $C_p$: $(1,n+1,2n+1,\ldots,(p-1)n+1)$ $(2,n+2,2n+2,\ldots,(p-1)n+2)$ $\ldots$ $(n,2n,3n,\ldots,p\cdot n)$.

That is, we have $n$ cycles of length $p$ in this representation. In the particular cases below of $(Hol(C_p) \times C_2)_{wr}C_2$ we have for $p = 3$ the following: $a = (1,2,3)$, $b = (2,3)$, $c = (4,5)$, $d = (1,6)(2,7)(3,8)(4,9)(5,10)$. Here the permutations $a$, $b$ and $c$ generate the group $Hol(C_3) \times C_2$.

[13] Laue, R., Private communication.

[14] We know of no published proof of this statement, but this seems to be a known result by some group theorists. It was pointed out to us around 1980 by Dr. M. F. Fry while we were all at Michigan State University.

[15] The way to determine the exact number of cases here is to look at the normal subgroup lattices for the groups of order 64 given in the Hall-Senior book [11] and actually count the number of different $C_2$ quotient groups listed there. There are 1120 such cases; 700 of these are associated with a characteristic subgroup. As one can readily see this is a rather tedious undertaking.

[16] Yuanda, Zhang, “The Structures of Groups of Order $23 \cdot p^2$” Chinese Annals of Mathematics Series B, vol. 4, pp. 77–93 (1983). This is not the original determination of these groups. The results date back to the early 1900’s. This is simply the most recent discussion of groups of these orders.

[17] To do this for an arbitrary group (i.e., to give a finite list of $n$ quantities, no two sets of which are the same for nonisomorphic groups) appears to be a very hard problem in group theory.

[18] Comments made to us in discussions with some members of Michigan State University’s Mathematics Department.
[19] Derek Robinson (Letter) June 6, 1985.

[20] Relations of the form $x^t \equiv 1 \mod(p)$ occur in several places in the text and the appendix. In each case, $x$ (or $q$ or $z$) is a $t$-th root of unity (here $t = 3, 4, \ldots, (p - 1)$).

[21] See also the work of Michael Smith (Ph.D. thesis ANU in Canberra). Dr. Newman sent the author some pages from this thesis describing how one might compute this automorphism group by hand. The automorphism group(s) are represented by means of the actions of the generators of the automorphism group upon the generators of the group $(C_p \times C_p)@Q_2$. Presentations for the automorphism group itself could probably be derived from such a representation, but as far as we know this has not been done. In this case, however, things are simpler for this group than they might be for other groups since the group $(C_p \times C_p)@Q_2$ appears as a characteristic subgroup in its own automorphism group. This of course means that $Aut[(C_p \times C_p)@Q_2]$ is a complete group.
### Table 1
Groups of order 8 and their automorphism group factors

| 2-group | Normal sylow $p$ cases; image of 2-group | Normal sylow 2-subgroup cases |
|---------|------------------------------------------|-----------------------------|
| $C_8$   | $a$ $C_2 \times C_2$                     | $a$ $C_2$ $1$              |
| $C_4 \times C_2$ | $a$ $D_4$ | $a$ $C_2$ |
| $C_2 \times C_2 \times C_2$ | $b$ $C_2$ | $b$ $D_4$ |
| $D_4$   | $a$ $C_2 \times C_2$                     | $b$ $D_4$ |
| $Q_2$   | $b$ $D_4$                                | $b$ $D_4$ |

**Notes for Table 1**

* This order 168 group is the complete group of this order.

This group is isomorphic to a degree 8 permutation group with generators:

$a = (1,2)(3,6,7,4,5,8)$ \quad $b = (1,7,2,6,4,5)(3,8)$

and presentation:

$a^6 = (a * b^{-1})^3 = b^6 = a^2 * b^{-1} * a * b * a^{-2} * b = (a^2 * b)^2 * a^{-1} * b^{-2} = 1$

There is one additional case without a normal sylow subgroup, namely $S_4$.

$\text{Aut}(S_4) = S_4$.

The automorphism groups of the groups of order 8 with a normal sylow $p$-subgroup are obtained by forming the direct product of $\text{Hol}(C_p)$ with the entry in the above table.

**Relations used for the groups of order 8:**

$C_8 : a^8 = 1$ \quad $C_4 \times C_2 : a^4 = b^2 = (a, b) = 1$ \quad $D_4 : a^4 = b^2 = ab * a = 1$

$C_2 \times C_2 : a^2 = b^2 = c^2 = (a, b) = (a, c) = (b, c) = 1$

$Q_2 : a^4 = b^4 = a^2 * b^2 = ab * a = 1$
## Table 2
Groups of order $8 * p^2$ and their automorphism groups; cases with a normal sylow $p$-subgroup

| 2-group       | image of 2-group                                                                 | order 8 cases                                                                 |
|---------------|----------------------------------------------------------------------------------|------------------------------------------------------------------------------|
| $C_8$         | $[a] \ C_2 \times [144]^*$                                                       | $[a] \ [144]^*$                                                             |
|               | for $p \equiv 1 \mod(8)$                                                         | see table 6                                                                  |
| $C_4 \times C_2$ | $[a,b] \ G = Q_p \times D_p$                                                      | $[a,b] \ G = Hol(C_p) \times D_p$                                           |
|               | $C_2 \times C_2 \times Hol(C_p) \times Hol(C_p)$                                | $C_2 \times [144]^*$                                                        |
|               | $[ab,b]$                                                                         | For $p \equiv 1 \mod(4)$                                                    |
|               | $(Hol(C_p) \times C_2)wrC_2$                                                     | $[a,ab] \ Hol(C_p)wrC_2$                                                    |
| $C_2 \times C_2 \times C_2$ | $[a,b] \ G = D_p \times D_p \times C_2$                                         |                                                                              |
|               | $(Hol(C_p) \times C_2)wrC_2$                                                     |                                                                              |
| $D_4$         | $[a,b]$                                                                          | for $p = 3$, $G = S_3 wr C_2$                                               |
|               | $C_2 \times C_2 \times Hol(C_p) \times Hol(C_p)$                                | and $\text{Aut}(G)$ is a                                                    |
|               | $[ab,a]$                                                                         | is a complete group                                                         |
|               | $(Hol(C_p) \times C_2)wrC_2$                                                     | of order $144 = [144]^*$                                                    |
| $Q_2$         | $[ab,a]$                                                                         |                                                                              |
|               | $(Hol(C_p) \times C_2)wrC_2$                                                     | $p = 3 \ Hol(C_3 \times C_3)$                                              |
|               | $p > 3$                                                                           | $p > 3$ complete group                                                       |
|               |                                                                                | not $Hol(C_p \times C_p)$                                                   |

* See Notes for Table 2 contained in Appendix 1: for the $D_4$ case see Table D, for the $Q_2$ case see Table Q.
| 2-group | $p$-group | Aut($G$) |
|----------|-----------|----------|
| $C_2 \times C_2 \times C_2$ | $C_3 \times C_3$ | $A_4 \times C_3 \times C_2$ |
| $C_9$ | $(C_2 \times C_2)@C_9 \times C_2$ | $S_4 \times C_3$ |
| $C_7 \times C_7$ | $(C_2 \times C_2 \times C_2)@C_7 \times C_7$ | $[168] \times Hol(C_7)$ ** |
| $C_{49}$ | $(C_2 \times C_2 \times C_2)@C_{49}$ | $[168] \times C_6$ ** |
| $Q_2$ | $C_3 \times C_3$ | $SL(2,3) \times C_3$ |
| $C_9$ | $Q_2@C_9$ | $S_4 \times C_3$ |

** [168] is the complete group of this order; see Table 1 for details.

Non-normal sylow subgroup types: $p = 3$.

- $S_4 \times C_3 \rightarrow \text{Aut}(G) = S_4 \times C_2$
- $A_4 \times S_3 \rightarrow \text{Aut}(G) = S_4 \times S_3$
- $(C_2 \times C_2)@D_9 \rightarrow \text{Aut}(G) = \text{complete group of order 216}$
- $(A_4 \times C_3)@C_2$ [even permutations in $S_3 \times S_4$].

$\text{Aut}(G)$ = complete group of order 432 with 20 conjugacy classes.

A presentation for the complete group of order 216 is:

\[
c^2 = b^2 = c^3 = (a, c) = (a \ast d)^2 = (b, c) = c \ast d^2 \ast (c^{-1}) \ast d
\]
\[
= a \ast b \ast a \ast d \ast b \ast (d^{-1}) = (a \ast b)^2 \ast (d^{-1}) \ast b \ast d = 1;
\]

A presentation for the complete group of order 432 is:

\[
a^4 = b^2 = c^3 = d^2 = (a, c) = (a, d) = (a, d) = (b, d) = (c \ast d)^2 = (a \ast b)^3
\]
\[
= (b \ast c)^2 \ast (b \ast (c^{-1}))^2 = a \ast b \ast a \ast c \ast b \ast c \ast a \ast b \ast c = 1;
\]
Table 4

$C_8$ extensions from Yuanda’s work [16]

| prime         | $C_2$ image | $C_4$ image | $C_8$ image               |
|---------------|-------------|-------------|---------------------------|
| $p \equiv 1 \text{ mod}(8)$ | (2,6)       | (3,5,8,9)   | (4,7,10,11,12, 13,14,15) |
| $p \equiv 3 \text{ mod}(8)$ | (2,6)       | (5)         | (16)                      |
| $p \equiv 5 \text{ mod}(8)$ | (2,6)       | (3,5,8,9)   | (7)                       |
| $p \equiv 7 \text{ mod}(8)$ | (2,6)       | (5)         | (17)                      |
Table 5
Explicit representations for \((C_p \times C_p)@C_8\)
for some small primes

| prime | \(C_2\) image | \(C_4\) image | \(C_8\) image |
|-------|---------------|---------------|---------------|
| \(p = 3\) | \((1,0;0,-1)\) | \((0,-1;1,0)\) | \((16)=(0,1;1,1)\) |
| \(p = 5\) | \((1,0;0,-1)\) | \((2,0;0,-1)\) | \((7)=(0,1;2,0)\) |
| \(p = 7\) | \((1,0;0,-1)\) | \((0,-1;1,0)\) | \((17)=(0,1;-1,3)\) |
| \(p \equiv 1 \mod(8)\) | \((1,0;0,-1)\) | \((4,0;0,-1)\) | \((4)=(-2,0;0,-2)\) |
| \(p = 17\) | \((1,0;0,1)\) | \((4,0;0,1)\) | \((7)=(0,-1;4,0)\) |
| \(p \equiv 1 \mod(8)\) | \((1,0;0,1)\) | \((4,0;0,4)\) | \((10)=(-2,0;0,-4)\) |
| \(p = 17\) | \((4,0;0,-4)\) | \((11)=(-2,0;0,4)\) | \((12)=(-2,0;0,8)\) |
| \(p \equiv 1 \mod(8)\) | \((1,0;0,1)\) | \((13)=(-2,0;0,-8)\) | \((14)=(-2,0;0,1)\) |
| \(p = 17\) | \((4,0;0,-4)\) | \((15)=(-2,0;0,-1)\) | \((15)=(-2,0;0,-1)\) |

* Zhang Yuanda says number 15 has a center of order 1. This is not the case. This group has the form \((C_{17}@C_{16}) \times C_{17}\).
Table 6
Automorphism groups for the groups given in Table 5

| Prime | $C_2$ Image | $C_4$ Image | $C_8$ Image |
|-------|-------------|-------------|-------------|
| $p = 3$ | $\text{Hol}(C_3) \times C_2$ | $H(3^2) \times C_2$ | 16) = $H(3^2)$ |
| | $\times C_2 \times C_2$ | | |
| | $\times \text{Hol}(C_3 \times C_3)$ | | |
| | $\times C_2 \times C_2$ | | |
| $p = 5$ | $\text{Hol}(C_5) \times C_4$ | $\text{Hol}(C_5) \times C_4 \times C_2$ | 7) = $H(5^2)$ |
| | $\times C_2 \times C_2$ | $\text{Hol}(C_5) \times \text{Hol}(C_5) \times C_2$ | |
| | $\times C_2 \times C_2$ | $\text{Hol}(C_5) \times C_5 \times C_2$ | |
| | | $(\text{Hol}(C_5) \times \text{wr}C_2) \times C_2$ | |
| $p = 7$ | $\text{Hol}(C_7) \times C_6$ | $H(7^2) \times C_2$ | 17) = $H(7^2)$ |
| | $\times C_2 \times C_2$ | | |
| | $\times \text{Hol}(C_7 \times C_7)$ | | |
| | $\times C_2 \times C_2$ | | |
| $p \equiv 1 \mod(8)$ (e.g., $p = 17$). | See comments below for the $C_2$ and $C_4$ image cases. | 4) = $\text{Hol}(C_p \times C_p)$ | |
| | | 7) = $\text{Hol}(C_p) \times \text{wr}C_2$ | |
| | | 10) = $\text{Hol}(C_p) \times \text{Hol}(C_p)$ | |
| | | 11) = $\text{Hol}(C_p) \times \text{Hol}(C_p)$ | |
| | | 12) = $\text{Hol}(C_p) \times \text{wr}C_2$ | |
| | | 13) = $\text{Hol}(C_p) \times \text{wr}C_2$ | |
| | | 14) = $\text{Hol}(C_p) \times \text{Hol}(C_p)$ | |
| | | 15) = $\text{Hol}(C_p) \times C_{(p-1)}$ | |

$H(p^n)$ is the group of all mappings $x \mapsto a \times x^t + b$, where $a \neq 0$ and $b$ are elements of the Galois group of $F = \text{GF}(p^n)$ and $t$ is a field automorphism of $F$.

The order of $H(p^n)$ is $p^n \times \phi(p^n)$, where $\phi$ is Euler’s totient function.

For the $p \equiv 1 \mod(8)$ and $p \equiv 7 \mod(8)$ cases, the entries in Table 6 have the following generalizations:

In the $C_2$ image cases just make the following substitutions:
- $\text{Hol}(C_3) \times C_2$ by $\text{Hol}(C_3) \times C_{p-1}$
- $\text{Hol}(C_3 \times C_3)$ by $\text{Hol}(C_3 \times C_3)$

In the $C_4$ image cases just make the following substitutions:
- [144] (Table 2) by Aut[$(C_p \times C_p) \times C_4]$ = $H(p^2)$.

The analogue, or recurrence, for the $p \equiv 1 \mod(4)$ case is $\text{Hol}(C_p) \times \text{wr}C_2$.

In the $C_8$ image cases just make the following substitutions:
- $H(3^2)$ by $H(7^2)$ respectively by $H(p^n)$.

For $p \equiv 1, 5 \mod(8)$ make the following changes in the $p = 5$ cases:
- In the $C_2$ image cases just make the following substitutions:
  - $C_5$ by $C_p$ and $C_4$ by $C_{p-1}$.
- In the $C_4$ image cases just make the following substitutions:
  - $C_5$ by $C_p$ and $C_4$ by $C_{p-1}$.
- In the $C_8$ image cases just make the following substitutions:
  - $H(5^2)$ by $H(p^2)$ [for $p = 5 \mod(8)$ here].