Holographic anomaly in 3D $f(\text{Ric})$ gravity

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Abstract

By applying the holographic renormalization method to the metric formalism of $f(\text{Ric})$ gravity in three dimensions, we obtain the Brown–York boundary stress tensor for backgrounds which asymptote to the locally AdS 3 solution of Einstein gravity. The logarithmic divergence of the on-shell action can be subtracted by a non-covariant cut-off-independent term which exchanges the trace anomaly for a gravitational anomaly. We show that the central charge can be determined by means of Banados–Teitelboim–Zanelli holography or in terms of the Hawking effect of a Schwarzschild black hole placed on the boundary.

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1. Introduction

Black hole physics is the essential ingredient of any quantum theory of gravity. In the context of AdS$_3$/CFT$_2$ correspondence, the CFT partition function of a Banados–Teitelboim–Zanelli (BTZ) black hole [1, 2] can be identified via a modular transformation in terms of the free energy of the vacuum which corresponds to the thermal AdS$_3$ [3], and the Cardy formula [4] reproduces the Bekenstein–Hawking black hole entropy [5]. The Virasoro algebra of the dual CFT is initially identified as the asymptotic symmetry algebra of the AdS$_3$ spacetime [6]. For Einstein gravity, the corresponding central charge can be determined in terms of the holomorphic Weyl anomaly [7]. In [8, 9], the holographic stress–energy tensor is identified in terms of the Brown–York tensor [10].

Inspired by the Brown–Henneaux approach to the AdS/CFT correspondence [6], it is natural to seek the extension of the duality to higher derivative gravity in AdS$_3$. Since in three dimensions, the Riemann tensor can be given in terms of the metric $G_{\mu\nu}$ and the Ricci tensor $R_{\mu\nu}$, $f(R_{\mu\nu})$ gravity, in which $f$ is a polynomial in $R_{\mu\nu}$, is quite interesting. Massive gravity studied in [11] is an example of such models.

The first step towards holography is identifying the CFT stress tensor. Following [12], the AdS/CFT correspondence implies that the expectation value of the stress–energy tensor of the dual CFT can be identified with the Brown–York tensor [9]. In order to obtain the Brown–York tensor, one needs to identify the surface terms which are needed to make the action stationary.
given only $\delta R_{\mu\nu} = 0$ on the boundary. For the two-derivative Einstein–Hilbert action, the surface term is the Gibbons–Hawking term [13].

For $f(R)$ models, in which $R$ denotes the Ricci scalar, one needs to cancel surface terms that depend on $\delta R$. In [14], the authors argue that no such boundary terms exist in general. It is known that $f(R)$ gravity is equivalent to the Einstein gravity coupled to a scalar field. Of course, this equivalence relies on a conformal transformation which can be in general singular [15]. More precisely, the $f(R)$ model in the metric formalism is equivalent to $\omega = 0$ the Brans–Dicke theory [16]. From this point of view, $R$ carries the scalar degree of freedom and $\psi \equiv f'(R)$ is christened scalaron [17]. So it is reasonable to set $\delta R = 0$ on the boundary [18]. In the GR limit $f(R) \rightarrow R$, the scalar field decouples from theory [19] and consequently there is no need to make any assumption on $\delta R|_\partial$ in General Relativity (GR).

In the more general case of $f(R^n)$ gravity, different approaches have been considered. For example, in [20] the surface terms are determined for general Euler density actions; in [21], these terms are given in a first-order formulation of the theory, and in [22], the surface terms are obtained in an on-shell perturbative approach, i.e. one considers the higher derivative terms as perturbations to the Einstein–Hilbert action and uses the field equations to compute the necessary boundary term.

In order to find the Brown–York tensor, one also needs to determine the counter-terms which holographically renormalize the action, i.e. make the action finite for asymptotically locally AdS backgrounds. For Einstein gravity, these terms are computed in [7–9]. In [23, 24], this method is generalized to $R^n$ models and in [25], the corresponding counter-terms are obtained in the second-order formulation involving an auxiliary tensor field. We intend to generalize these results to arbitrary $f(R^n)$ models in three dimensions.

Actually, Ostrogradski’s theorem implies that $f(R^n)$ models are in general instable [26]. This instability is explicitly shown e.g. in [27] and is extensively studied in the case of massive gravity [11]. We are not going to study the stability of $f(R^n)$ models here. Our goal is to obtain the holographically renormalized Brown–York tensor for $f(R^n)$ gravity in backgrounds which asymptote to the locally AdS$_3$ solution of Einstein gravity,

$$R_{\mu\nu} = -2\ell^{-2}\tilde{G}_{\mu\nu}. \quad (1.1)$$

In principle, if AdS/CFT correspondence can be generalized to higher derivative gravity, then the instability of the $f(R^n)$ model can be realized in the dual CFT. So, in principle, the issue of stability could deepen our understanding of holography.

The central charge of the dual CFT can be identified in terms of the Weyl anomaly [7]. In [28], a universal formula for the so-called type A anomalies is obtained for $f(R)$ gravity. In particular, in three dimensions, the value of the central charge computed by this method equals the value obtained in [29, 30] which generalizes the results of [7] to higher derivative models of gravity. By using these methods, one can determine the central charge without necessarily obtaining the stress tensor. The central charge appears to be given by the Brown–Henneaux formula, in which Newton’s constant $G$ is screened by $\Omega$ defined by [25, 29, 30]

$$f^{\mu}_{\nu}|_{\partial} = \Omega \delta^{\mu}_{\nu}, \quad f'_{\mu} = \frac{df}{dR^n}. \quad (1.2)$$

In this paper, we apply the holographic renormalization method to the $f(R^n)$ model in backgrounds that asymptote to locally AdS$_3$ spacetimes (1.1). In the second-order formulation given by the action [27]

1 In [14], it is shown that the assumption $\delta R|_\partial = 0$ can be relaxed if the spacetime is assumed to be maximally symmetric, i.e. assuming $K^{\mu\nu}\delta K_{\mu\nu}|_{\partial} = 0$ and $\delta(\nu^{\mu} \nabla_{\mu} K) |_{\partial} = 0$, where $K_{\mu\nu}$ is the traceless part of the extrinsic curvature $\mathcal{K}_{\mu\nu}$, $n^\mu$ is the unit normal to the boundary and $\nabla_{\mu}$ denotes the covariant derivative with respect to the Levi-Civita connection corresponding to $\mathcal{G}_{\mu\nu}$.
in which \( f_\nu \) stands for \( \int d^{d+1}x \sqrt{\mathcal{G}} \) and \( \chi^{\mu}_\nu \) is an auxiliary tensor field, one can simply follow the method of [25]. In this formulation, \( \delta \chi^{\mu}_\nu \) is assumed to be vanishing on the boundary, and the method of [7, 8] can be used, where, effectively, the Gibbons–Hawking term is given in terms of the screened Newton’s constant.

The higher derivative formulation of the \( f(R^\mu_\nu) \) model is given by the action

\[
S = \int_V f(R^\mu_\nu) .
\]  
(1.4)

In this case, one needs to add a counter-term to compensate for the \( \delta R \)-dependent surface terms. As we discuss in section 4.2, such a boundary term is accessible in asymptotically locally AdS3 backgrounds where the traditional Fefferman–Graham expansion [31] is available. We show that the resulting stress tensor is essentially equivalent to the one obtained in the second-order formulation.

We then turn to the on-shell value of the action, which following [12] is an essential ingredient of holography, as it gives the leading term in the CFT partition function. It is known that there is a logarithmic divergence in the on-shell value of the action, which can be subtracted by a cut-off-dependent covariant counter-term [7, 8]. Here, we examine a cut-off-independent term which appears to be not covariant. After adding this term, the trace anomaly disappears and a gravitational anomaly materializes instead. It is known that in two dimensions, gravitational anomaly and trace anomaly can be switched by adding a local counter-term [32]. Here, we show that the value of the central charge can be determined in terms of the gravitational anomaly, by means of the holography of BTZ black holes or in terms of the Hawking effect of a Schwarzschild black hole placed on the boundary.

The organization of the paper is as follows. In section 2, following [28] we compute the Weyl anomaly in the \( f(R^\mu_\nu) \) model by studying bulk diffeomorphisms corresponding to the Weyl transformation of the boundary metric. In section 3, we review the holographic renormalization in Einstein gravity [7, 8], and extend it to \( f(R^\mu_\nu) \) gravity in section 4. In section 5, we study the gravitational anomaly that appears when the logarithmic divergence is subtracted by means of a cut-off-independent counter-term. Section 6 is devoted to a short discussion about the CFT dual to \( f(R^\mu_\nu) \) gravity. Some technical details are relegated to appendices.

2. Weyl anomaly in the \( f(R^\mu_\nu) \) model

Assume a general gravitational action,

\[
S = \int_V f(R^\mu_\nu) .
\]  
(2.1)

We consider \( f(R^\mu_\nu) \) as a function of \( R^\mu_\nu = \mathcal{G}^{\mu\nu} R_{\mu\nu} \), with all contractions made between raised and lowered indices, so that the metric does not enter explicitly [27]. Under a bulk diffeomorphism, this action is invariant up to a boundary term [33]

\[
\delta \xi S = \int d^{d+1}x \partial_\alpha \left[ \sqrt{\mathcal{G}} f(R^\mu_\nu) \xi^\alpha \right] = - \int_B n_\alpha \xi^\alpha f(R^\mu_\nu) ,
\]  
(2.2)

in which \( \int_B \) stands for \( \int d^d x \sqrt{\gamma} \), where \( \gamma \) is the induced metric on the boundary and \( n_\alpha \) is the inward pointing unit normal to the boundary. For an asymptotically locally AdS solution \( \mathcal{G}_{\mu\nu} = \mathcal{G}_{\mu\nu} \), the Weyl anomaly is given by this boundary term for PBH (Penrose–Brown–Henneaux) transformation [28, 33]. Details of this transformation are not important for us.
What we are going to show is that, the Weyl anomaly of the \( f(\mathcal{R}_{\mu\nu}) \) model for an asymptotically locally AdS solution \( \mathcal{G}_{\mu\nu} = \bar{\mathcal{G}}_{\mu\nu} \) equals the Weyl anomaly of the Einstein–Hilbert action with a cosmological constant term corresponding to the AdS background \( \mathcal{G}^{\text{AdS}} \) describing the asymptotic geometry of \( \bar{\mathcal{G}}_{\mu\nu} \) and a screened Newton’s constant \( G/\Omega \). To see this, one needs to compute the Taylor expansion of \( f(\mathcal{R}_{\mu\nu}) \) around \( \bar{\mathcal{R}}_{\mu\nu} \), with the Ricci tensor corresponding to \( \bar{\mathcal{G}}_{\mu\nu} \):

\[
f(\mathcal{R}_{\mu\nu}) = f(\bar{\mathcal{R}}_{\mu\nu}) + \frac{df}{d\mathcal{R}_{\mu\nu}}(\mathcal{R}_{\mu\nu} - \bar{\mathcal{R}}_{\mu\nu}) + \mathcal{O}(\mathcal{R}_{\mu\nu} - \bar{\mathcal{R}}_{\mu\nu})^2. \tag{2.3}
\]

Thus,

\[
\delta \xi |_{\mathcal{G}=\bar{\mathcal{G}}} = -\left[ \Omega \int_B n.\xi (R - 2\Lambda) \right]_{\mathcal{G}=\mathcal{G}^{\text{AdS}}}, \tag{2.4}
\]

in which \( \Omega \) is given by equation (1.2) and

\[
2\Lambda = [R - \Omega^{-1}f(\mathcal{R}_{\mu\nu})]_{\mathcal{G}=\mathcal{G}^{\text{AdS}}}. \tag{2.5}
\]

In other words,

\[
\delta \xi |_{\mathcal{G}=\bar{\mathcal{G}}} = \delta \xi |_{\text{EH}}|_{\mathcal{G}=\mathcal{G}^{\text{AdS}}}, \tag{2.6}
\]

where

\[
\delta \xi |_{\text{EH}} = \frac{\Omega}{16\pi G} \int_V (R - 2\Lambda). \tag{2.7}
\]

This result confirms that the Weyl anomaly in \( f(\mathcal{R}_{\mu\nu}) \) gravity on asymptotically locally AdS backgrounds is given by the Brown–Henneaux formula [6] with a screened Newton’s constant [29].

### 3. Holographic renormalization in pure Einstein gravity

In this section, we review the holographic renormalization of Einstein gravity in asymptotically locally AdS3 spacetimes [7, 8].

The AdS3 solution of the Einstein field equation with a negative cosmological constant \( \Lambda = -\ell^{-2} \),

\[
\Pi_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \quad \mu, \nu = 0, 1, 2, \tag{3.1}
\]

is given by

\[
dx^2 = \frac{\ell^2 \, dr^2}{4r^2} + r^{-1}(-dr^2 + d\theta^2), \tag{3.2}
\]

in which \( t = \ell^{-1}t_{\text{AdS}} \). An asymptotically locally AdS solution in \textit{normal} coordinates is given by

\[
dx^2 = \frac{\ell^2 \, dr^2}{4r^2} + \gamma_{ij} \, dx^i \, dx^j, \quad \mu, \nu = 0, 1, 2, \tag{3.3}
\]

where, using the \textit{traditional} Fefferman–Graham asymptotic expansion [31],

\[
\gamma_{ij} = r^{-1} g_{ij} = r^{-1} S_{ij}^{(0)} + g_{ij}^{(2)} + h_{ij}^{(2)} \ln r + \mathcal{O}(r). \tag{3.4}
\]

In these coordinates, the boundary is located at \( r = 0 \). The extrinsic curvature of the boundary is given by

\[
K_{\mu\nu} = \nabla_\mu n_\nu, \tag{3.5}
\]
in which $\nabla_\mu$ denotes the covariant derivative with respect to the Levi-Civita connection corresponding to the metric (3.3) and

$$n^\mu = \left( \frac{2r}{\ell}, 0, 0 \right),$$

(3.6)
is the inward pointing surface-forming normal vector. The components of the extrinsic curvature are

$$K_{\mu} = 0, \quad K_{ij} = \frac{r}{\ell} \gamma_{ij}.$$  

(3.7)

Equation (3.1) implies that \cite{8}

$$h^{(2)}_{ij} = 0,$$  

(3.8)

$$D_\mu t_{ij} = 0,$$  

(3.9)

$$R^{(0)} = -2\ell^{-2} \text{tr} g^{(2)},$$ 

(3.10)

where $\gamma_{ij}$ and its inverse are used to lower and raise the Latin indices, while the trace operator ‘tr’ is defined in terms of $g^{(2)}_{ij}$. The covariant derivative $D_i$ is defined with respect to the Levi-Civita connection corresponding to $\gamma_{ij}$:

$$D_i = D^{(0)}_i + O(r),$$

(3.11)
in which $D^{(0)}_i$ is defined with respect to $g^{(0)}_{ij}$ and $R^{(0)}$ is the corresponding scalar curvature. Finally,

$t_{ij} = K_{ij} - (K + \ell^{-1}) \gamma_{ij} = \ell^{-1} \left( g^{(2)}_{ij} - g^{(0)}_{ij} \text{tr} g^{(2)} \right) + O(r).$  

(3.12)

For example, equation (3.9) is given by the field equation

$$0 = R_{ij} = \gamma^{\nu}_i [\nabla_\nu, \nabla_i] n^\mu = D_i t_j,$$

(3.13)
The last equality is obtained by noting that

$$D_i K^i_j = \gamma^{\nu}_i \gamma^j_\mu \gamma^\nu_\rho \nabla_\nu K^\rho_p = \left( \delta^\nu_p - n^\nu n_p \right) \gamma^j_\mu \nabla_\nu K^\rho_p,$$

(3.14)

where, in order to obtain the first equality, we have used lemma 10.2.1 of \cite{34}. The second equality is obtained by noting that $n \cdot \nabla n = n_\mu K^\mu = 0$.

The Einstein–Hilbert action is given by

$$S_{\text{EH}} = \frac{1}{2\kappa^2} \int_V (R - 2\Lambda),$$

(3.15)
in which $\kappa^2 = 8\pi G$. The variation of the action with respect to $\delta G_{\mu\nu}$ is given by

$$\delta S_{\text{EH}} = \frac{1}{2\kappa^2} \int_V (G^{\mu\nu} \delta R_{\mu\nu} + \Pi_{\mu\nu} \delta G^{\mu\nu}) = \frac{1}{2\kappa^2} \int_V (G^{\mu\nu} \delta K_{\mu\nu} + \delta K) - \frac{1}{2\kappa^2} \int_V \Pi^{\mu\nu} \delta G_{\mu\nu},$$

(3.16)

where we have used equations (D.2), (D.3), (B.13) and (B.14). The second term gives the Einstein field equation (3.1) and is vanishing on-shell. Henceforth, we drop this term. The first term depends on $n \cdot \nabla \delta G_{\mu\nu}$ and can be removed by adding the Gibbons–Hawking term

$$S_{\text{GH}} = -\frac{1}{\kappa^2} \int_B K.$$  

(3.17)
Thus,

$$\delta S = -\frac{1}{2\kappa^2} \int_B (K_{\mu\nu} - K G_{\mu\nu}) \delta G^{\mu\nu},$$

(3.18)
in which $S = S_{EH} + S_{GH}$. The Brown–York tensor is defined by

$$T_{ij} = -\frac{2}{\sqrt{\gamma}} \frac{\delta S}{\delta \gamma_{ij}} \bigg|_{\text{on-shell}},$$

(3.19)

where $\delta \gamma_{\mu \nu}$ is the variation of the induced metric on the boundary which obeys the constraint $n^\mu \delta \gamma_{\mu \nu} = 0$. Furthermore, one assumes that $\delta n^\mu = 0$. The minus sign in equation (3.19) reflects the fact that one defines the energy–momentum tensor in terms of $\delta \gamma_{ij}$. Here, noting that $\delta \gamma_{ij} \sim O(r)$ and $\sqrt{\gamma} \sim O(r^{-1})$, we have given the Brown–York tensor in terms of $\delta \gamma_{ij}$.

The idea is to identify $T_{ij}$, after renormalization, with the expectation value of the stress–energy tensor of the dual CFT [9],

$$\langle T_{ij} \rangle_{\text{CFT}} = T_{ij}^{\text{ren}},$$

(3.20)

where on the boundary, the indices are raised and lowered by $g_{ij}^{(0)} = r \gamma_{ij}|_B$. Using equation (3.18), one obtains

$$\kappa^2 T_{ij}^{\text{ren}} = K_{ij} - K_{ij} = \ell^{-1} \gamma_{ij} + t_{ij}.$$  

(3.21)

The first term is singular on the boundary and can be removed by adding a counter-term to the Gibbons–Hawking term [9]

$$S_{\text{reg}}^{\text{GH}} = -\frac{1}{\kappa^2} \int_B (K + \ell^{-1}).$$

(3.22)

Thus, the action is

$$2\kappa^2 S = \int_V (R - 2\Lambda) - 2 \int_B (K + \ell^{-1}),$$

(3.23)

and the regularized Brown–York tensor is

$$T_{ij}^{\text{ren}} = \kappa^{-2} t_{ij}.$$  

(3.24)

We still need to remove a logarithmic divergence in the on-shell value of the action [8]. Recall that the on-shell value of the action gives the tree-level contribution to the free-energy of the boundary CFT [12]. Since,

$$\sqrt{\mathcal{G}} = \frac{\ell}{2r} \sqrt{\gamma} = \frac{\ell \sqrt{g^{(0)}}}{2} \left(1 + \frac{\text{tr} g^{(2)}}{2r} + \cdots\right),$$

(3.25)

one verifies that

$$\int_V (R - 2\Lambda) = -\lim_{\epsilon \to 0} \frac{2}{\ell} \int d^2 x \sqrt{g^{(0)}} \left(\frac{1}{\epsilon} - \frac{1}{2} \text{tr} g^{(2)} \ln \epsilon\right) + \text{finite}. $$

(3.26)

The regularized Gibbons–Hawking term $S_{\text{reg}}^{\text{GH}}$ removes the $\epsilon^{-1}$ term. Thus, using equation (3.10), one obtains [7, 8]

$$2\kappa^2 S_{\log-term}^{\text{log}} = -2\pi \ell \chi \lim_{\epsilon \to 0} \ln \epsilon,$$

(3.27)

in which $\chi$ is the Euler characteristic of the boundary,

$$\chi = \frac{1}{4\pi} \int d^2 x \sqrt{g^{(0)}} R^{(0)}.$$  

(3.28)

Thus, the counter-term is a topological term and it does not contribute to the Brown–York stress tensor [8].

Equation (3.9) implies that

$$D^a T_{ij}^{\text{ren}} = 0,$$

(3.29)

and equation (3.10) gives

$$\text{tr} T^{\text{ren}} = \frac{c}{24\pi} R^{(0)},$$

(3.30)

in which $c = 3\ell/2G$ is the Brown–Henneaux central charge [6]. It is important to note that the logarithmic divergence of the on-shell action (3.27) is given by the central charge [8, 29]

$$S_{\log-term} = -\frac{c}{12} \lim_{\epsilon \to 0} \ln \epsilon.$$  

(3.31)
4. Holographic renormalization of the $f(R^\mu_{\nu})$ model

In the previous section, we studied the renormalization of the on-shell Einstein–Hilbert action and the corresponding Brown–York stress tensor for asymptotically locally AdS$_3$ spacetimes. In this section, we study this problem in the $f(R^\mu_{\nu})$ model of gravity. In section 4.1, we discuss the generalization of the Gibbons–Hawking term in the second-order formulation and in the higher derivative formulation of $f(R^\mu_{\nu})$ gravity. In section 4.2, we obtain the surface terms for asymptotically locally AdS$_3$ spacetimes and study the holographic renormalization of the corresponding Brown–York tensor.

4.1. Surface terms

The higher derivative formulation of $f(R^\mu_{\nu})$ gravity is given by the action (1.4) which is classically equivalent to a second-order action given by equation (1.3) [35]. The field equation for $\chi$ gives

$$\frac{df^\mu_{\nu}}{dx^\mu} (\chi^\nu - R^\nu_{\mu}) = 0, \quad (4.1)$$

implying that $\chi^\nu_{\mu} = R^\nu_{\mu}$ whenever $\det \frac{df^\mu_{\nu}}{dx^\mu} \neq 0$ [27]. It should be noted that this field equation does not depend on $\delta \chi^\nu_{\mu} |_{B}$. As far as the auxiliary field is considered as an independent field, one can assume that $\delta \chi^\nu_{\mu}$ is vanishing on the boundary [25].

In the both formulations, one supplements the action with a boundary term

$$\int_B (\mathcal{L}_{GH} + \mathcal{L}_{ct}), \quad (4.2)$$

in which $\mathcal{L}_{GH}$ is the Gibbons–Hawking term and $\mathcal{L}_{ct}$ is a counter-term that subtracts the infinite terms into the on-shell action and the Brown–York tensor. We will discuss the counter-term later. The Gibbons–Hawking term is added in such a manner that $\delta S$ does not depend on the normal derivative of $\gamma^\mu_{\nu}$.

We begin by studying the higher derivative formulation. In this case,

$$\delta \int_V f(R^\mu_{\nu}) = \int_V \mathcal{X}^\mu_{\nu} \delta G^\mu_{\nu} + \delta S^1_B + \delta S^2_B, \quad (4.3)$$

where [27]

$$\mathcal{X}^\mu_{\nu} = f^\nu_{\alpha} R^\alpha_{\mu} - \frac{1}{2} f^\nu_{\alpha} \tilde{G}^\alpha_{\mu \nu} + \frac{1}{2} (\tilde{G}^\mu_{\nu \alpha} \nabla_{\alpha} + \tilde{G}^\alpha_{\mu \nu \beta} \nabla_{\beta} - \tilde{G}^\mu_{\nu \alpha} \nabla_{\beta} + \tilde{G}^\alpha_{\mu \nu \beta} \nabla_{\mu} - \tilde{G}^\mu_{\nu \alpha} \nabla_{\nu} - \tilde{G}^\alpha_{\mu \nu \beta} \nabla_{\nu}) f^\nu_{\beta}, \quad (4.4)$$

in which $\Box = \nabla^\mu \nabla_{\mu}$. Henceforth, we drop the first term on the right-hand side of equation (4.3). The surface terms are

$$\delta S^1_B = -\frac{1}{2} \int_B f^\nu_{\alpha} \{ \nabla_{\nu} \delta \tilde{G}_{\alpha \beta} + \nabla_{\beta} \delta \tilde{G}_{\nu \alpha} - \nabla_{\alpha} \delta \tilde{G}_{\nu \beta} - \nabla_{\beta} \delta \tilde{G}_{\nu \alpha} \} - \eta_{\alpha \beta} \nabla_{\nu} \delta g^\nu_{\beta}, \quad (4.5)$$

$$= \int_B (f^\nu_{\alpha} \delta K_{\mu \nu} + s \delta K) + \frac{1}{2} \int_B h^\mu_{\nu} \gamma^\rho_{\sigma} \nabla_{\mu} \delta G^\rho_{\sigma}, \quad (4.6)$$

where, inspired by equation (B.2), we have defined

$$s = n_{\mu} n_{\nu} f^\mu_{\nu}, \quad h^\mu = \gamma^\mu_{\nu} H^\nu, \quad H^\mu = n_{\nu} f^\mu_{\nu}$$

and

$$\delta S^2_B = \frac{1}{2} \int_B \{ n_{\nu} \delta \tilde{G}_{\alpha \nu} + \eta_{\alpha \beta} \delta \tilde{G}_{\nu \alpha} - n_{\alpha} \delta \tilde{G}_{\nu \beta} \} - \eta_{\alpha \beta} \nabla_{\nu} \delta g^\nu_{\beta}, \quad (4.7)$$

$$= \frac{1}{2} \int_B \{ n_{\nu} \delta \tilde{G}_{\alpha \nu} + \eta_{\alpha \beta} \delta \tilde{G}_{\nu \alpha} - n_{\alpha} \delta \tilde{G}_{\nu \beta} \} \nabla_{\nu} - \eta_{\alpha \beta} \nabla_{\nu} \delta g^\nu_{\beta}.$$
Thus, on-shell
\[
\delta \int_V f(R_{\mu\nu}^{\alpha}) = \delta S_B^1 + \delta S_B^2.
\] (4.9)
in which
\[
\delta S_B^1 = \int_B (f_{\mu\nu} \delta K_{\mu\nu} + s \delta K)
\] (4.10)
and
\[
\delta S_B^2 = \delta S_B^1 - \frac{1}{2} \int_B \gamma^{\mu\nu} \delta \gamma_{\mu\nu} \mathcal{D}_x h^x.
\] (4.11)
The generalized Gibbons–Hawking term should be added in such a manner that it removes the \(n^\mu \partial_\mu \delta \gamma_{\mu\nu}\)-dependent terms in \(\delta S_B^1\). A covariant choice is
\[
S_{GH} = - \int_B (f_{\mu\nu} K_{\mu\nu} + s K).
\] (4.12)
This term has been derived in [25] for \(D = 3\) massive gravity.

Using the normal coordinates,
\[
\Delta s^2 = \sqrt{\gamma} \frac{d^2 s^2}{\Delta \gamma_{ij}} = \gamma_{ij} (dx^i dx^j),
\] (4.13)
the Brown–York tensor is defined by
\[
T_{ij} = \frac{2}{\sqrt{\gamma}} \delta \mathcal{S} \bigg|_{\text{on-shell}} = T_{ij}^1 + T_{ij}^2 + T_{ij}^{ct}.
\] (4.14)
\(T_{ij}^{ct}\) comes from the counter-terms, to be discussed later,
\[
T_{ij}^1 = - \frac{2}{\sqrt{\gamma}} \frac{1}{\gamma_{ij}} \left( \int_B K_{ab} \delta (f^{ab} \sqrt{\gamma}) + K \delta (s \sqrt{\gamma}) \right)
\] (4.15)
and
\[
T_{ij}^2 = 2 \int_B \delta S_B^1 = n^\mu \nabla f^{ij} - n^\mu \nabla f^{ij} - \gamma^{ij} (n^\alpha \nabla f^{\alpha\nu} + \mathcal{D}_x h^x) \]
\[
= - K_{ij} f^{ik} - n^\nu \nabla f^{ij} - \nabla f^{ij} + \nabla f^{ij} K_{ab} f^{ab} + \nabla f^{ij} - \gamma^{ij} (n^\alpha \nabla f^{\alpha\nu} + \mathcal{D}_x h^x).
\] (4.16)
in which
\[
\nabla^i H^j = \mathcal{D}_i h^j + K_{ij} s, \quad \nabla^a H^\alpha = \mathcal{D}_a h^\alpha + (\mathcal{D}_r + K) s,
\] (4.17)
where \(\mathcal{D}_r = n^\mu \partial_\mu\). Furthermore, \(\gamma_{ij,k} = 2n_i K_{ij}\) and consequently,
\[
n^\mu \nabla f^{ij} = \mathcal{D}_r f^{ij} + K_{ij} f^{ik}.
\] (4.18)

So far, our results are valid in both the second-order and higher derivative formulations. If one assumes that \(\delta f_{\nu}^{\alpha} |_{B} = 0\) which is a legitimate assumption in the second-order formulation, then
\[
\delta f_{\nu}^{\alpha} |_{B} = f_{k}^{\alpha} \delta \gamma^{kj}, \quad \delta s = 0.
\] (4.20)
In this case,
\[
T_{ij}^{ct} = K_{ij} f^{ik} - (K_{ab} f^{ab} + s K) \gamma^{ij},
\] (4.21)
and \(T_1 + T_2\) reproduces the stress tensor derived in [25] for the massive gravity.

In contrast, the assumption \(\delta f_{\nu}^{\alpha} |_{B} = 0\) cannot be taken for granted in the higher derivative formulation of the \(f(R_{\mu\nu}^{\alpha})\) model, and the contribution from \(\delta R_{\nu}^{\alpha} |_{B}\) has to be taken into account [22, 23].
Since we are interested in asymptotically locally AdS spacetimes, we simplify the problem by assuming that
\[ f^{\mu\nu}|_B = \Omega G^{\mu\nu}, \] 
where \( \Omega \) is a constant. In this case, in the both formulations, \( \delta \tilde{S}^2 \) does not contribute to the Brown–York tensor, i.e. \( \mathcal{T}_{ij}^2 = 0 \), as can be verified by evaluating equation (4.16). Furthermore, the Gibbons–Hawking term (4.12) simplifies to
\[ S_{\text{GH}} = -2 \int_B \Omega (K + \ell^{-1}), \] 
where we have added a counter-term similar to equation (3.22). Noting that for such backgrounds,
\[ \delta \Omega = \frac{1}{d+1} \delta^\rho_\nu \delta^\mu_\mu = \Upsilon \delta R, \] 
which follows from equation (A.8), one verifies that
\[ \delta \Omega = \frac{1}{d+1} \delta^\rho_\nu \delta^\mu_\mu = \Upsilon \delta R, \] 
Thus,
\[ \delta S = - \int_B \Omega t_{ij} \delta y^{ij} - 2 \int_B \Upsilon (K + \ell^{-1}) \delta R, \] 
in which \( t_{ij} \) is defined in equation (3.12). \( \delta R \) is given by equation (B.17) and depends on the normal derivative of \( \delta \gamma_{ij} \). In principle, one seeks a surface term which removes this term. In [14], it is argued that no such surface term exists in general. In the next section, we obtain the corresponding surface term for the asymptotically locally AdS3 spacetimes given by equation (1.1).

4.2. Asymptotically locally AdS Einstein solutions

Henceforth, we restrict ourselves to backgrounds which asymptote to the locally AdS3 solution and use the traditional Fefferman–Graham asymptotic expansion of the metric given by equations (3.3) and (3.4).\(^2\)

In the Fefferman–Graham coordinates, \( \delta R \) is given by equation (C.10). In this case, the unwanted term in equation (4.26) is encapsulated in \( \delta \mathcal{P} \). Furthermore,
\[ K = \frac{2}{\ell} + \mathcal{O}(r), \] 
i.e. \( K \) is constant on the boundary located at \( r = 0 \). Consequently, one can use the following counter-term in order to remove \( \delta \mathcal{P} \) in equation (4.26),
\[ S^b_a = - \frac{2}{\ell} \int_B \Upsilon \mathcal{P}_a, \quad \mathcal{P}_a = (1 - \alpha) \mathcal{R} + \mathcal{P}, \] 
where \( \alpha \in \mathbb{R} \) is arbitrary, and \( \Upsilon \) is defined in equations (4.24) and (4.25). Note that \( \mathcal{P}_a \sim \mathcal{O}(r) \) and \( \mathcal{P}_{a, \text{on-shell}} = -\alpha r \mathcal{R}^{(0)} + \mathcal{O}(r^2) \). This changes equation (4.26) to
\[ \delta S = \int_B \Omega t_{ij} \delta y^{ij} + \frac{2}{\ell} \int_B \Upsilon \left( \alpha \delta \mathcal{R} + \frac{1}{2} \mathcal{P}_a \gamma_{ij} \delta y^{ij} \right) - \frac{2}{\ell} \int_B \mathcal{P}_a \delta \Upsilon. \] 
\(^2\) For an asymptotically AdS spacetime, the metric asymptotes to the exact AdS metric at the boundary. In an asymptotically locally AdS spacetime, the boundary metric is treated as a free field and one can use the Fefferman–Graham expansion. It is known that, in general, this expansion is insufficient to describe all solutions of \( f(R^{(3)}) \) models, in particular at the critical point \( \Omega = 0 \); see [36] and references therein.
Equations (C.7) and (C.10) imply that $P_\alpha \delta \gamma \sim \mathcal{O}(r^2)$ and, consequently, the last term in equation (4.29) is vanishing\(^3\). Therefore, no further counter-term is needed in order to make the variational principle well defined. The second term in equation (4.29) is vanishing on-shell, because
\[
\int_B \gamma^{ij} \delta R_{ij} = \int_B (\mathcal{D}_i \mathcal{D}_j - \gamma^{ij} \mathcal{D}_k \mathcal{D}_k) \delta \gamma^{ij} = 0 \tag{4.30}
\]
and
\[
R_{ij} = \frac{1}{2} R^{(0)}_{ij} \gamma^{(0)} + \mathcal{O}(r). \tag{4.31}
\]
In summary, we have verified that the variational principle is well defined for the action
\[
S = \int_V f - 2 \int_B \Omega (K + \ell^{-1}) - \frac{2}{\ell} \int_B \gamma \mathcal{P}_a, \tag{4.32}
\]
and the corresponding Brown–York tensor is
\[
T_{\text{ren}}^{ij} = 2 \Omega \mathcal{P}_{ij}. \tag{4.33}
\]
We still need to determine another counter-term which subtracts the logarithmic divergence into the on-shell value of the action (4.32) \[^8\]. Using equation (3.25), one obtains
\[
\int_V f \bigg|_{\text{on-shell}} = \lim_{\epsilon \to 0} \frac{\ell f_0}{2} \int d^2 x \sqrt{g^{(0)}} \left( \frac{1}{\epsilon} - \frac{\text{tr} g^{(2)}}{2 \ln \epsilon} \right) + \text{finite}, \tag{4.34}
\]
where $f_0$ denotes the (asymptotic) on-shell value of $f(R^a_\mu)$. Furthermore,
\[
-2 \int_B \Omega (K + \ell^{-1}) \bigg|_{\text{on-shell}} = \frac{2 \Omega}{\ell} \int d^2 x \sqrt{g^{(0)}} \epsilon^{-1} + \text{finite} \tag{4.35}
\]
and
\[
- \frac{2}{\ell} \int_B \gamma \mathcal{P}_a \bigg|_{\text{on-shell}} = \frac{2}{\ell} (4 \pi \chi) \gamma \alpha = \text{finite}, \tag{4.36}
\]
where $\chi$ is the Euler characteristic of the boundary given by equation (3.28). For an AdS\(_3\) solution, $\Omega$ in equation (4.22) is a constant, and the equation of motion $\Xi_{\mu \nu} = 0$ implies that
\[
f_0 + 4 \ell^{-2} \Omega = 0. \tag{4.37}
\]
Consequently, the $\epsilon^{-1}$-terms in equations (4.34) and (4.35) cancel out and
\[
S_{\text{on-shell}} = - \frac{\ell \Omega}{4 \pi \chi} \lim_{\epsilon \to 0} \ln \epsilon + \text{finite}. \tag{4.38}
\]

The parameter $\alpha$ in equation (4.28) remains arbitrary. This reflects the fact that classically, one can arbitrarily add or remove the Euler characteristic to the action. Since this is a finite term, holographic renormalization is also ignorant of it. In principle, $\alpha$ can be determined by AdS/CFT correspondence, since the on-shell value of the action gives the leading term in the CFT partition function \[^12\].

Formula (4.29) is obtained in the higher derivative formulation given by the action (1.4). By simply omitting the $\gamma$-terms, one obtains the corresponding formula in the second-order formulation (1.3).

\(^3\) Recall that $\delta \gamma^{ij} \sim \mathcal{O}(r)$ and $\sqrt{\gamma} \sim \mathcal{O}(r^{-1})$.\[^10\]
Brown–York stress tensor

Since the log counter-term is a topological term, it will not contribute to the Brown–York stress tensor (4.33). Using equation (3.9), one verifies that
\[ D^iT_{ij}^\text{ren} = 0. \] (4.39)
Furthermore,
\[ \text{tr}T^\text{ren} = \ell \Omega R^{(0)} = \frac{c}{24\pi} R^{(0)}. \] (4.40)
Thus,
\[ c = \frac{3\ell}{2G} (16\pi G \Omega), \] (4.41)
which is the central charge obtained in [25, 29]. Equation (4.38) implies that, similar to equation (3.31), tr\(T\) is given by the logarithmic divergence of the action [7, 8, 29]
\[ S_{\text{log-term}} = -\frac{c\chi}{12} \ln \epsilon. \] (4.42)

5. A non-covariant cut-off-independent counter-term

By the AdS/CFT correspondence, the leading term in the CFT partition function is given by the finite term of the classical gravity action [12]
\[ \left\langle \exp \int_B \phi^{(0)} \mathcal{O} \right\rangle_{\text{CFT}} = \exp(-S(\phi^{(0)})), \] (5.1)
in which \(\phi^{(0)}\) denotes the boundary value of the classical field \(\phi^{(0)}\), and the expectation value of the stress–energy tensor of the dual CFT is identified with the Brown–York tensor [9].

The finite term in the gravity action (4.38) is the sum of the finite terms in the bulk term (4.34) and the boundary terms (4.35) and (4.36). The contribution from the boundary terms is given by
\[ 4\pi \chi \left( \frac{\ell \Omega}{2} + \frac{2\alpha \Upsilon}{\ell} \right), \] (5.2)
where \(\chi\) is the Euler characteristic of the boundary. It is a topological term and, consequently, the boundary data \(g^{(0)}_{ij}\) are obscured in this term.

A closely related problem is the value of the divergence of the stress tensor. The argument in [28] reviewed in section 2, as well as the method of [29] cannot determine the divergence of the stress tensor. Since \(f(R_G)\) gravity is parity-preserving, there is no room for a gravitational anomaly in the dual CFT given by
\[ D^iT_{ij} = \beta \epsilon_i^j \partial_i R^{(0)}, \] (5.3)
i.e. \(\beta = 0\). Nevertheless, one can still add boundary local terms which induce a gravitational anomaly given by
\[ D^iT_{ij} = \frac{b}{24\pi} \partial_j R^{(0)}. \] (5.4)
The holographic renormalization can produce such an anomaly, depending on the counter-term one uses to subtract the logarithmic divergence in the on-shell value of the action given by equations (3.31) and (4.42).

The prescription in [7, 8] is subtracting the ‘covariant’ cut-off-dependent counter-term
\[ S_{\text{log}} = -S_{\text{log-term}} \] given in equation (4.42). This results in equation (5.2). One can instead use another counter-term which is independent of the cut-off,
\[ S_{\text{log}} = -\frac{c}{48\pi} \int_B R \sqrt{g} \ln \sqrt{g} = -\frac{c}{48\pi} \int_B R^{(0)} \sqrt{g^{(0)}} (-\ln \epsilon + \ln \sqrt{g^{(0)}}). \] (5.5)
This counter-term is not covariant. Its contribution to the on-shell value of the classical action is

$$- \frac{c}{48\pi} \int_B R^{(0)} \sqrt{g^{(0)}} \ln \sqrt{g^{(0)}}.$$

(5.6)

which, unlike the topological term (5.2), inherits the boundary data. Furthermore, it adds a new term to the Brown–York tensor,

$$T_{\log,ij}^{ct} = - \frac{c}{48\pi} R \gamma_{ij} \bigg|_{r=0} = - \frac{c}{48\pi} R^{(0)} g_{ij}^{(0)}.$$

(5.7)

In this scenario, the renormalized Brown–York tensor is

$$T_{ij} = - \frac{c}{12\pi \ell} t_{ij} - \frac{c}{48\pi} R^{(0)} g_{ij}^{(0)}.$$

(5.8)

Consequently,

$$\text{tr} T = 0, \quad D_i T_{ij} = - \frac{c}{48\pi} \partial_j R^{(0)},$$

(5.9)

which is similar to the case studied in [32]. This observation motivates us to consider a more general situation where

$$T_{ij} = \left( a - 2b \right) \frac{1}{12\pi \ell} t_{ij} + b \frac{24}{48\pi} R^{(0)} g_{ij}^{(0)}.$$

(5.10)

which gives

$$T_i^i = \frac{a}{24\pi} R^{(0)}, \quad \nabla_j T_j^i = b \frac{24}{48\pi} \partial_i R^{(0)}.$$

(5.11)

For the covariant subtraction \((a, b) = (c, 0)\) and for the cut-off-independent subtraction \((a, b) = (0, -c/2)\).

### 5.1. Hawking effect of a 2D Schwarzschild black hole

In the following, we show that the true value of the central charge \(c = a - 2b\) can be recognized via the Hawking effect of an asymptotically flat two-dimensional black hole located on the boundary [37]. Consider a Schwarzschild black hole,

$$ds^2 = -u(x) dt^2 + \frac{dx^2}{u(x)},$$

(5.12)

where \(u(x)\) has a simple zero at \(x_h\) indicating the event horizon and

$$\lim_{x \to \infty} u(x) = 1.$$

(5.13)

The non-vanishing Christoffel symbols are

$$\Gamma^t_{tx} = - \Gamma^x_{tx} = \frac{u'}{2u}, \quad \Gamma^t_{tt} = \frac{uu'}{2},$$

(5.14)

and \(R^{(0)} = -u''(x).\) Equation (5.11) reads,

$$\Gamma^x_{tx} + T_x^t = - \frac{a}{24\pi} u'', \quad \partial_x T_x^t + \frac{u'}{2u} \left( T_x^x - T_t^t \right) = - \frac{b}{24\pi} u''', \quad \partial_i T_i^x = 0.$$

(5.15)

These equations can be solved and the integration constants can be determined by requiring that: \((a)\) \(T_t^t\) and \(T_x_t\) are finite at the horizon [37] and \((b)\) asymptotically,

$$T_t^t = c_+ \frac{\pi}{6} T_H^2, \quad T_x_t = c_- \frac{\pi}{6} T_H^2.$$

(5.16)

in which \(T_H^t = g'(x_h)/4\pi\) is the Hawking temperature of the black hole and \(c_{\pm} = (c_L \pm c_R)/2\). Finiteness of \(T_x_t\) at the horizon implies that \(c_- = 0\) and consequently no gravitational anomaly is detected by the Hawking effect, i.e. \(c_L = c_R\). Finiteness of \(T_t^t\) at the horizon gives

$$c_+ = a - 2b.$$

(5.17)
5.2. BTZ black hole

It is interesting to note that the true value of the central charge can also be recognized by studying BTZ black holes. The boundary of a static BTZ black hole is a flat torus, i.e., both the trace anomaly and the gravitational anomalies (5.11) are vanishing in this case. Thus, the BTZ black hole can be used to verify, via holography, whether $c_+$ defined by equation (5.17) is the correct central charge or not.

The BTZ geometry,

$$dx^2 = -(r^2 - 8GM\ell^2)\,dt^2 + \frac{\ell^2 dr^2}{r^2 - 8GM\ell^2} + r^2 d\phi^2,$$  \hspace{1cm} (5.18)

in the Fefferman–Graham coordinates is given by

$$g_{ij}^{(0)} = \eta_{ij}, \quad g_{ij}^{(2)} = 4\frac{GM\ell^2}{\beta^2} \delta_{ij},$$  \hspace{1cm} (5.19)

in which $\eta_{ij} = \text{diag}(-1, 1)$, $\delta_{ij} = \text{diag}(1, 1)$, and the Hawking temperature $\frac{1}{\beta}$ gives the torus complex structure $\tau = \frac{i}{\beta/2\pi}$. Since $R_{(0)} = -2\ell^{-2}\text{tr}g^{(2)} = 0$, equation (5.10) gives

$$T_{ij} = \frac{\pi c}{6\beta^2} \delta_{ij} - \frac{c}{24\pi^2} \beta \delta_{ij},$$  \hspace{1cm} (5.20)

To see why this result is important recall that the CFT free energy of a BTZ black hole can be obtained by a modular transformation $\tau \rightarrow -\frac{1}{\tau}$ from the free energy of the vacuum, which corresponds to the thermal AdS [3, 29],

$$I_{\text{BTZ}}(\tau, \bar{\tau}) = -\frac{i}{12} \left( \frac{c_L}{\tau} - \frac{c_R}{\bar{\tau}} \right).$$  \hspace{1cm} (5.21)

Consequently, the corresponding CFT weights are

$$\Delta = -\frac{1}{2\pi i} \frac{\partial I}{\partial \tau} = \frac{c_L}{24\pi^2}, \quad \tilde{\Delta} = \frac{1}{2\pi i} \frac{\partial I}{\partial \bar{\tau}} = \frac{c_R}{24\pi^2}.$$  \hspace{1cm} (5.22)

Thus, $\Delta + \tilde{\Delta}$ is equivalent to the Brown–York mass of the black hole:

$$M_{\text{BY}} = \int_0^{2\pi} d\phi T_{00} = \frac{\pi^2 c}{3\beta^2}.\hspace{1cm} (5.23)$$

Note that the time coordinate $t$ in equation (5.18) equals, $\ell^{-1}t_{\text{BTZ}}$, and, consequently, $M_{\text{BY}} = \ell M_{\text{BTZ}}$. The Cardy formula gives [5]

$$S_{\text{Cardy}} = 2\pi \sqrt{\frac{c_L \Delta}{6} + 2\pi} \sqrt{\frac{c_R \tilde{\Delta}}{6}} = \frac{c}{6\ell} A_{\text{BTZ}},$$  \hspace{1cm} (5.24)

where $A_{\text{BTZ}}$ is the area of the event horizon:

$$A_{\text{BTZ}} = 2\pi\ell \left( \frac{2\pi}{\beta} \right).$$  \hspace{1cm} (5.25)

6. Discussion

For backgrounds in which the traditional Fefferman–Graham expansion is available, we found the Gibbons–Hawking term in the higher derivative formulation of $f(R)$ gravity and determined the corresponding counter-terms. The resulting Brown–York tensor appeared to be equivalent to the one obtained in the second-order formulation, in which an auxiliary field is used.

We also verified that the logarithmic divergence of the on-shell action can be subtracted either by a cut-off-dependent covariant counter-term quite similar to the one used in [7, 8]
or by a cut-off-independent non-covariant counter-term. In the former case, one obtains a trace anomaly equivalent to the one obtained in [29, 30]. In the latter case, the Weyl anomaly is vanishing and one encounters a gravitational anomaly instead, which can be exchanged for the familiar Weyl anomaly by adding a local surface term. We verified that, keeping the gravitational anomaly, one can determine the value of the central charge in terms of the Hawking effect of a Schwarzschild black hole placed on the boundary, or by means of BTZ holography.

The CFT dual to \( f(R) \) gravity should address various phenomena which are absent in GR. For example, Ostrogradski’s theorem implies that \( f(R) \) theories are in general instable [26]. From this point of view, \( f(R) \) models in which \( f \) is an algebraic function of an undifferentiated Ricci scalar are viable models [26]. Of course, in these models, the positivity of the screened Newton’s constant requires that \( \Omega \sim f' > 0 \). This condition is also necessary for the unitarity of the boundary CFT as it implies that the central charge given by the holographic Weyl anomaly is positive. Unitary \( f(R) \) gravities in three dimensions and their CFT duals are widely studied; see e.g. [38] and references therein.

In the context of \( f(R) \) gravity, Ricci stability also imposes \( \Upsilon \sim f''(R) > 0 \) [39], which should be addressed in the dual CFT. Furthermore, there is vDVZ discontinuity [40] in \( f(R) \) gravity models [41], since \( f(R) \) gravity models are essentially equivalent to GR with an additional scalar. Thus, it is necessary to realize the vDVZ discontinuity in the CFT dual. We could not trace these effects in the holographic renormalization of the theory, since both the Brown–York stress tenor and the on-shell action appeared to be insensitive to such details.

### Appendix A. \( f(R) \) as a Polynomial in \( R \)

In this appendix, we compute \( f^\mu_\nu \) and \( d f^\beta_\alpha / dR^\mu_\nu \). Assuming that \( f \) is a polynomial in \( R^\mu_\nu \),

\[
f(R^\mu_\nu) = \sum_{[n_1 \ldots n_k]} c_{n_1 \ldots n_k} R^{n_1}_\mu \ldots R^{n_k}_\nu,
\]

where

\[
R^a = (R^a)_\mu^\mu, \quad (R^{a+1})_v^\nu = R^a_{\alpha_1} R^a_{\alpha_2} \ldots R^a_{\alpha}, \quad 1^\mu_\nu = \delta^\mu_\nu,
\]

one verifies that

\[
\delta f = \sum_{[n_1 \ldots n_k]} c_{n_1 \ldots n_k} \sum_{i=1}^k R^{n_i} \ldots \delta R^{n_i} \ldots R^{n_k},
\]

in which

\[
\delta R^a = n(R^{a-1})^\mu_\mu \delta R^a_\mu.
\]

Thus,

\[
f^\beta_\alpha = \sum_{[n_1 \ldots n_k]} c_{n_1 \ldots n_k} \sum_{i=1}^k n_i R^{n_i} \ldots \hat{R}^{n_i} \ldots R^{n_k} (R^{n_i-1})^\beta_\alpha,
\]

where the term with a hat is replaced by 1, e.g., \( x^\hat{z} = xz \). In order to compute \( \delta f^\beta_\alpha \), one needs to compute

\[
\delta \left[ \left( \prod_{i=1}^k R^{n_i} \right) (R^{n_k})^\beta_\alpha \right],
\]
which is given by equation (A.4) and
\[
\delta(R^n)_{\mu} = \sum_{k=0}^{n-1} (R^k)_{\mu} (R^{n-k-1})_{\mu} \delta R^k_{\mu}.
\] (A.7)

Consequently,
\[
\frac{df_{\alpha}^{\beta}}{dR^\rho} = \sum_{\{n_1 \cdots n_k\}} c_{n_1 \cdots n_k} \left[ \sum_{i \neq j} n_i R^{n_i} \cdots (R^{n_j-1})_{\rho} (R^{n_k-1})_{\rho} + \sum_{j=0}^{n-2} \frac{n_i R^{n_i} \cdots (R^{n_j-1})_{\rho}}{n_j} \right].
\] (A.8)

Appendix B. Induced geometry on the boundary

In this paper, we assume that the spacetime given by the metric \( G_{\mu\nu} \) is surrounded by a space-like boundary \( B \) given by a continuous and surface-forming vector field \( n_\mu \) \cite{14},
\[
n_\mu n^\mu = 1, \quad \nabla_{[\alpha} n_{\beta]} = 0. \] (B.1)

Furthermore, we assume that this vector field is ‘inward’ pointing normal to the boundary. The induced metric on the boundary is given by
\[
\gamma_{\mu\nu} = G_{\mu\nu} - n_\mu n_\nu, \] (B.2)

where
\[
n^\mu \gamma_{\mu\nu} = 0, \quad \gamma_{\mu\nu} \gamma^{\mu\nu} = \delta^\nu_{\mu} - n_\mu n^\nu. \] (B.3)

The extrinsic curvature of the boundary is defined by
\[
K_{\mu\nu} = \nabla_\mu n_\nu. \] (B.4)

It is useful to recall that in the ADM decomposition,
\[
dx^2 = N^2 dr^2 + \gamma_{ij}(dx^i + N^i \, dr)(dx^j + N^j \, dr), \] (B.5)

\( n_\mu = (0', N) \) and \( n^\rho = (-N^{-1}N', N^{-1}) \). Furthermore, \( \delta G_{ij} = \delta \gamma_{ij} \) and \( \delta G_{ij} = N^j \delta \gamma_{ij} \). One defines the Brown–York tensor with respect to \( \delta \gamma_{ij} \), assuming that \( \delta N = 0 \). It is clear that \( \delta n_{\mu} = 0 \) and \( n^\rho \delta G_{\mu\rho} = 0 \). See also appendix A of \cite{25}.

Following section 10 of \cite{34} and noting that here \( \gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \), i.e. \( n^2 = 1 \), one verifies that
\[
R_{ijkl} = \gamma_{\mu}^{\mu} \gamma_{\nu}^{\nu} R_{ijkl}^{\gamma_{\mu}^{\mu} \gamma_{\nu}^{\nu} A_\mu A_\nu} + K_{ik} K_{jk} - K_{ik} K_{jk}, \] (B.6)

where \( R_{ijkl} \) denotes the Riemann tensor defined with respect to \( \gamma_{ij} \), the metric induced on the boundary. Similar to \cite{34}, our curvature convention is \( [\nabla_{\mu}, \nabla_{\sigma}] A^\rho = R_{\nu \rho \sigma} A^\rho \) and \( R_{\mu\nu} = R_{\mu \rho \nu} \). Consequently,
\[
R_{ij} = \gamma_{\mu}^{\mu} \gamma_{\nu}^{\nu} (R_{\mu\nu} - n_\sigma n_\sigma R_{\mu\nu \sigma\rho} + KK_{ij} - K_{ik} K_{kj}). \] (B.7)

Since, using equation (B.1),
\[
n^\alpha R_{\alpha\nu\mu\beta} = n_\alpha [\nabla_{\nu}, \nabla_{\beta}] n_\mu = -K_{\beta}^\alpha K_{\rho \mu} - n \cdot \nabla K_{\mu\nu}, \] (B.8)

one verifies that
\[
R_{ij} = \gamma_{\mu}^{\mu} \gamma_{\nu}^{\nu} R_{\mu\nu} + n \cdot \nabla K_{ij} + KK_{ij}. \] (B.9)
This gives, in particular [14],
\[ R = R + K_{ij}K^{ij} + K^2 + 2n \cdot \nabla K, \] (B.10)
where, we have used
\[ R_{\mu\nu}n^\mu n^\nu = n^\rho [\nabla_\rho, n_\sigma]n^\sigma = -K_{\mu
u}K^{\mu
u} - n^\mu \partial_\mu K. \] (B.11)

In order to obtain the Gibbons–Hawking term, one needs to compute the surface terms that appear in the variation of the action with respect to the metric. Assuming that the boundary \( B \) is fixed [14], i.e. \( \delta n_\rho = 0 \) and \( \delta \gamma_{\mu\nu} \) is tangential,
\[ n^\mu \delta \gamma_{\mu\nu} = 0, \quad \delta n^\rho = 0, \quad \gamma_{\mu\nu} \delta \gamma_{\mu\nu} = -\gamma^{\mu\nu} \delta \gamma_{\mu\nu}. \] (B.12)
one obtains, using equation (D.2),
\[ \delta K_{\mu\nu} = -\frac{1}{2} n^\rho \left( \nabla_\mu \delta \gamma_{\rho\nu} + \nabla_\nu \delta \gamma_{\rho\mu} - \nabla_\rho \delta \gamma_{\mu\nu} \right). \] (B.13)
Consequently,
\[ \delta K = \frac{1}{2} \gamma^{\mu\nu} n^\rho \nabla_\rho \delta \gamma_{\mu\nu}, \] (B.14)
\[ \delta (K^{\mu\nu} K_{\mu\nu}) = K^{\mu\nu} n^\rho \nabla_\rho \delta \gamma_{\mu\nu}, \] (B.15)
\[ \delta (2n^\mu \partial_\mu K) = \gamma^{\mu\nu} n^\rho \nabla_\rho \nabla_\nu \delta \gamma_{\mu\nu}. \] (B.16)

Therefore,
\[ \delta R|_B = \delta R - [(K^{\mu\nu} + K^{\nu\mu})(n.\nabla) + \gamma^{\mu\nu}(n.\nabla)^2] \delta \gamma_{\mu\nu}. \] (B.17)

**Appendix C. Curvature in Fefferman–Graham coordinates**

The asymptotically locally AdS\(_3\) backgrounds,
\[ ds^2 = \frac{\ell^2}{4} \left( \frac{dr}{r} \right)^2 + g_{ij}(x) dx^i dx^j, \quad \gamma_{ij} = r^{-1} g_{ij}, \] (C.1)
where the boundary is located at \( r = 0 \), can be given in terms of the Fefferman–Graham expansion [7, 31]:
\[ g_{ij} = \sum_{n=0}^{d/2} g_{ij}^{(2n)}(x) r^n + h_{ij}^{(d/2)} \ln r + O(r^{d/2+1}). \] (C.2)
For \( d = 2 \), one obtains
\[ K_{ij} = -\frac{1}{\ell^2} \left( g_{ij}^{(0)} - rh_{ij} + O(r^2) \right), \] (C.3)
\[ K = \ell^{-1} [-2 + r[tr g^{(2)}] + (1 + \ln r)tr h] + O(r^2), \] (C.4)
where the trace operator is defined with respect to \( g_{ij}^{(0)} \); e.g. \( tr h = g_{ij}^{(0)} h_{ij} \), and we have used the equality
\[ \gamma^{ij} = r(g_{ij}^{(0)} - r g_{ij}^{(2)}) = r \ln rh^{ij} + O(r^2), \] (C.5)
in which the \( i \) and \( j \) indices are raised and lowered by \( g_{ij}^{(0)} \). Some other useful identities are
\[ K^2 = 4\ell^{-2} [1 - r[tr g^{(2)}] + (1 + \ln r)tr h] + O(r^2), \]
\[ K_{ij}K^{ij} = 2\ell^{-2} [1 - r[tr g^{(2)}] + (1 + \ln r)tr h] + O(r^2), \]
\[ n.\nabla K = 2\ell^{-2} [tr g^{(2)}] + (2 + \ln r)tr h] + O(r^2). \] (C.6)
which, using equation (B.10), give
\[ R = -\frac{6}{\ell^2} + r R^{(0)} + \mathcal{P} + O(r^2), \quad \mathcal{P} = \frac{6}{\ell} \left( K + \frac{2}{\ell} \right) - 2n \cdot \nabla K \sim O(r). \] (C.7)

Here, \( R^{(0)} \) denotes the scalar curvature defined with respect to \( g^{(0)}_{ij} \). One verifies that
\[ R_{ij} = R^{(0)}_{ij} + O(r), \] (C.8)

where \( R^{(0)}_{ij} \) is the Ricci tensor corresponding to \( g^{(0)}_{ij} \) and, consequently,
\[ R = r R^{(0)} + O(r^2). \] (C.9)

Thus,
\[ \delta R = r \delta R^{(0)} + \delta \mathcal{P} + O(r^2). \] (C.10)

Since \( \text{tr}\ h = 0 \) on-shell \cite{8}, one obtains
\[ P_{\text{on-shell}} = \frac{2r}{\ell^2} \text{tr} \, g^{(2)} + O(r^2) = -r R^{(0)} + O(r^2). \] (C.11)

Appendix D. Some useful identities

In sections 3 and 4, we have used the following identities:
\[ \delta \Gamma^\alpha_{\beta\rho} = \frac{1}{2} \left( \nabla_\beta \delta G^\alpha_{\rho\sigma} + \nabla_\rho \delta G^\alpha_{\beta\sigma} - \nabla_\sigma \delta G^\alpha_{\beta\rho} \right) + \Gamma^\sigma_{\rho\beta} \delta G^\alpha_{\sigma\alpha}. \] (D.1)

where, \( \Gamma^\alpha_{\beta\rho} = g_{\alpha\sigma} \Gamma^\sigma_{\beta\rho} \). Consequently,
\[ \delta \Gamma^\alpha_{\beta\rho} = \frac{1}{2} G^\alpha_{\mu\nu} \left( \nabla_\nu \delta G^\mu_{\beta\rho} + \nabla_\rho \delta G^\mu_{\beta\alpha} - \nabla_\alpha \delta G^\mu_{\beta\rho} \right). \] (D.2)

This identity can be used to show that
\[ G^{\mu\nu} \delta R_{\mu\nu} = G^{\mu\nu} \left( \nabla_\mu \delta \Gamma^\rho_{\mu\nu} - \nabla_\nu \delta \Gamma^\rho_{\beta\mu} \right) = (-\nabla_\mu \nabla_\nu + G^{\mu\nu} \Box) \delta G^{\mu\nu}. \] (D.3)

References

[1] Banados M, Teitelboim C and Zanelli J 1992 Phys. Rev. Lett. 69 1849 (arXiv:hep-th/9204099)
[2] Banados M, Henneaux M, Teitelboim C and Zanelli J 1993 Phys. Rev. D 48 1506 (arXiv:gr-qc/9302012)
[3] Maldacena J M and Strominger A 1998 J. High Energy Phys. JHEP12(1998)005 (arXiv:hep-th/9804085)
[4] Cardy J L 1986 Nucl. Phys. B 270 186
[5] Strominger A 1998 J. High Energy Phys. JHEP02(1998)009 (arXiv:hep-th/9712251)
[6] Brown J D and Henneaux M 1986 Commun. Math. Phys. 104 207
[7] Henningsson M and Skenderis K 1998 J. High Energy Phys. JHEP07(1998)023 (arXiv:hep-th/9806087)
[8] de Haro S, Solodukhin S N and Skenderis K 2001 Commun. Math. Phys. 217 595 (arXiv:hep-th/0002230)
[9] Balasubramanian V and Kraus P 1999 Commun. Math. Phys. 208 413 (arXiv:hep-th/9902012)
[10] Brown J D and York Jr J W 1993 Phys. Rev. D 47 1407 (arXiv:gr-qc/9209012)
[11] Bergshoeff E A, Hohm O and Townsend P K 2009 Phys. Rev. Lett. 102 201301 (arXiv:0901.1766 [hep-th])
[12] Bergshoeff E A, Hohm O and Townsend P K 2009 Phys. Rev. D 79 124042 (arXiv:0905.1259 [hep-th])
[13] Witten E 1998 Adv. Theor. Math. Phys. 2 253 (arXiv:hep-th/9802150)
[14] Gibbons G W and Hawking S W 1977 Phys. Rev. D 15 2752
[15] Madsen M S and Barrow J D 1989 Nucl. Phys. B 323 242
[16] Hawking S W and Luttrell J C 1984 Nucl. Phys. B 247 250
[17] Whitt B 1984 Phys. Lett. B 145 176
[18] Dyer E and Hinterbichler K 2009 Phys. Rev. D 79 024028 (arXiv:0809.4033 [gr-qc])
[19] Olmo G J 2007 Phys. Rev. D 75 023511 (arXiv:gr-qc/0612047)
[20] Myers R C 1987 Phys. Rev. D 36 392
[21] Fukuma M, Matsuura S and Sakai T 2001 Prog. Theor. Phys. 105 1017 (arXiv:hep-th/0103187)
[22] Cremonini S, Liu J T and Szepietowski P 2010 J. High Energy Phys. JHEP03(2010)042 (arXiv:0910.5159 [hep-th])
[23] Nojiri S ’i and Odintsov S D 2000 Phys. Rev. D 62 064018 (arXiv:hep-th/9911152)
[24] Nojiri S ’i and Odintsov S D 2000 Int. J. Mod. Phys. A 15 413 (arXiv:hep-th/9903033)
[25] Holm O and Tomm E 2010 J. High Energy Phys. JHEP04(2010)093 (arXiv:1001.3598 [hep-th])
[26] Woodard R P 2007 Lect. Notes Phys. 720 403 (arXiv:astro-ph/0601672)
[27] Hindawi A, Ovrut B A and Waldram D 1996 Phys. Rev. D 53 5597 (arXiv:hep-th/9509147)
[28] Imbimbo C, Schwimmer A, Theisen S and Yankielowicz S 2000 Class. Quantum Grav. 17 1129 (arXiv:hep-th/9910267)
[29] Kraus P and Larsen F 2005 J. High Energy Phys. JHEP09(2005)034 (arXiv:hep-th/0506176)
[30] Saida H and Soda J 2000 Phys. Lett. B 471 358 (arXiv:gr-qc/9909061)
[31] Fefferman C and Graham C R 1985 Conformal invariants Elie Cartan et les Mathématiques d’Aujourd’hui (Astérisque vol H S) (Paris: Soc. Math. France) p 95
[32] Karakhanian D R, Manvelyan R P and Mkrtchian R L 1994 Phys. Lett. B 329 185 (arXiv:hep-th/9401031)
[33] Schwimmer A and Theisen S 2008 Nucl. Phys. B 801 1 (arXiv:0802.1047 [hep-th])
[34] Wald R M 1984 (Chicago, IL: University of Chicago Press) ISBN 0-226-87033-2
[35] Balcerzak A and Dabrowski M P 2009 J. Cosmol. Astropart. Phys. 0109018 (arXiv:0804.0855 [hep-th])
[36] Cunliff C 2013 J. High Energy Phys. JHEP04(2013)141 (arXiv:1301.1347 [hep-th])
[37] Solodukhin S N 2006 Phys. Rev. D 74 024015 (arXiv:hep-th/0509148)
[38] Gullu I, Sisman T C and Tekin B 2011 Phys. Rev. D 83 024033 (arXiv:1011.2419 [hep-th])
[39] Sotiriou T P and Faraoni V 2010 Rev. Mod. Phys. 82 451 (arXiv:0805.1726 [gr-qc])
[40] van Dam H and Veltman M J G 1970 Nucl. Phys. B 22 397
Zakharov V I 1970 Zh. Eksp. Teor. Fiz. 12 447
[41] Myung Y S 2011 Eur. Phys. J. C 71 1550 (arXiv:1012.2153 [gr-qc])