We propose and study a model for $N$ hard-core bosons which allows for the interpolation between one- and high-dimensional behavior by variation of just a single external control parameter $s/t$. It consists of a ring-lattice of $d$ sites with a hopping rate $t$ and an extra site at its center. Increasing the hopping rate $s$ between the central site and the ring sites induces a transition from the regime of a quasi-condensed number $N_0$ of bosons proportional to $\sqrt{N}$ to complete condensation with $N_0 \simeq N$. In the limit $s/t \rightarrow 0, d \rightarrow \infty$ with $\tilde{s} = (s/t)\sqrt{d}$ fixed the low-lying excitations follow from an effective ring-Hamiltonian. An excitation gap makes the condensate robust against thermal fluctuations at low temperatures. These findings are supported and extended to the full parameter regime by large scale density matrix renormalization group computations. We show that ultracold bosonic atoms in a Mexican-hat-like potential represent an experimental realization allowing to observe the transition from quasi to complete condensation by creating a well at the hat’s center.

Introduction.— The existence of phase transitions and the occurrence of long range order depends sensitively on the spatial dimension $D$. In contrast to one-dimensional systems (with short range interactions) interacting systems in higher dimensions always exhibit phase transitions, if $D$ is large enough. This $D$ dependence has intensively been explored in spin systems (see, e.g., Ref. [11]) or for particles (see, e.g., Refs. [2–7]) on a hypercubic lattice by varying the underlying dimension $D$. Quite in contrast to such theoretical and numerical studies, $D$ cannot be changed in experiments. This raises the question to which extent dimensional crossover from $D = 1$ to $D \gg 1$ can be experimentally simulated, e.g., by the variation of a single controllable parameter. Moreover, the presence of long range order also depends on the density of the low-lying excitations. In particular the existence of an excitation gap makes long range order in the ground state robust against thermal fluctuations. It is the goal of our work to propose a model which facilitates such a dimensional crossover. Its comprehensive solution will reveal and illustrate a distinctive mechanism for generating a gap.

The focus of the present work is on Bose-Einstein condensation (BEC), one of the most striking quantum phenomena in nature (see, e.g., Refs. [11]). For an ideal gas of $N$ bosons in a $D$-dimensional box of volume $V$ the system undergoes for $D \geq 3$ a transition at a temperature $T_0 > 0K$ from a normal fluid to a low-temperature phase where a finite number $N_0(n)$ of the bosons are condensed. $T_0$ depends on the density $n = N/V$. At zero temperature (i.e., in the ground state) noninteracting bosons even exhibit BEC in one and two dimensions. Since the experimental discovery of BEC for ultracold gases [12–14] the study of BEC has become a particularly active field of research. This has also stimulated the theoretical investigation of BEC for trapped particles (see the review Ref. [15] and references therein).

One of the major challenges has been to explore how far BEC persists in the presence of interactions. For a diluted homogeneously Bose gas in $D = 3$ Bogoliubov theory [16] yields $N_0(n) \simeq N[1 - \frac{8}{\gamma N} \sqrt{\mu n^3}]$ with $a_s$ the s-wave scattering length. This result was confirmed by perturbation theory [17–23], simulations [24], and experimentally only very recently [25]. The Gross-Pitaevskii theory [26–28] is a classical approach using the order parameter field. Since it is based on a mean field approximation its validity is restricted to weakly correlated bosons.

Our work is concerned with bosons on a lattice with hard-core interaction. These hard-core bosons (HCBs) were originally introduced as a model for liquid Helium II in order to investigate superfluidity [29, 30]. Let us consider a one-dimensional lattice with lattice constant $a$ and $d$ sites. In the continuum limit $d \rightarrow \infty$, $a \rightarrow 0$ with $L = ad$ fixed one obtains the Tonks-Girardeau gas of impenetrable (spinless) bosons [31]. This system was realized experimentally by ultracold gases, as demonstrated first in Ref. [32]. The verification of BEC follows either from the largest eigenvalue of the one-particle reduced density matrix $\gamma(x, x')$ [33] or by its off-diagonal long range order [34]. For the Tonks-Girardeau gas it is $\gamma(x, x') \sim 1/[|x - x'|^{1/2} + 1]$ for $|x - x'| \rightarrow \infty$ [35, 36] (see also [37]). This implies for the number of condensed bosons $N_0(N) \sim \sqrt{N}$ for $N \gg 1$ [35, 36, 38, 40]. This result remains valid in the presence of a trap [40, 45]. Particularly, for a ultracold Bose gas in a cigar-shaped trap the $\sqrt{N}$ dependence was observed experimentally [32]. While still no BEC is present in one dimension even at $T = 0K$ interesting phase behavior can occur [46]. In the following we will propose and solve a model which allows for the transition from quasi-condensation with $N_0(N) \sim \sqrt{N}$ to complete condensation with $N_0(N) \propto N$ by increasing a single controllable parameter.
The model.—We consider $N$ HCBs on a lattice. The main part of that lattice consists of a ring with $d$ sites and lattice constant $a$ (see left part of Fig. 1). The HCBs can hop between nearest neighbor sites with a rate $t$ (right part of Fig. 1). Their only interaction comes from the hard-core condition. The scattering of two bosons with momenta $q_1$ and $q_2$ only interchanges these momenta, since this model is integrable [47]. This changes drastically if a central site is added (middle part of Fig. 1). Indeed, bosons can then hop to the central site and during that process exchange momentum with other bosons. Accordingly, the central site acts like an impurity and the lattice and additional interactions [58]. For finite values of $a^{-1}$ it is $q_{\mu} = (\pi/d)(2\mu + 1)$ for $N$ even and $q_{\mu} = (\pi/d)(2\mu)$ for $N$ odd [56, 66, 67]. The corresponding eigenvalues are $E_{n}^\mu(N) = -2 \sum_{n=1}^{N} \cos q_{\mu n}$.

A general normalized $N$-HCB state takes the form

$$|\phi_N\rangle = \alpha|\phi_N\rangle_r \otimes |0\rangle_c + \beta|\varphi_{N-1}\rangle_r \otimes |1\rangle_c,$$

since the existence of the central site couples the (normalized) ring-states $|\varphi_{N-1}\rangle_r$ and $|\phi_N\rangle_r$ with $N - 1$ and $N$ particles, respectively. $|0\rangle_c$ and $|1\rangle_c$ denote the vacuum of the ring-sites and central site, and $|1\rangle_c = h_0|0\rangle_c$. The sectors with $N - 1$ and $N$ particles can be decoupled by expanding $|\varphi_{N-1}\rangle_r$ and $|\phi_N\rangle_r$ with respect to the unperturbed eigenstates:

$$|\phi_N\rangle_r = \sum_{\nu} A_{\nu} |\psi_{\nu}^{0}(N)\rangle , \quad |\varphi_{N-1}\rangle_r = \sum_{\mu} a_{\mu} |\psi_{\mu}^{0}(N - 1)\rangle .$$

We consider the frame in which the center of mass is at rest (see also Ref. [68]). Hence the summations in Eq. (4) are restricted to total momentum $Q = 0$. For $d \to \infty$ the dependence of the perturbed eigenvalues and eigenstates on $s$ becomes nonanalytical. Therefore, for arbitrary small values of $s$ all unperturbed eigenstates contribute in Eq. (3). Substitution of Eqs. (2) and (3) into $H|\Psi_N\rangle = E|\Psi_N\rangle$ leads to (see Appendix A)

$$[E - E_0^{\mu}(N)]A_{\nu} = \sum_{\nu'} M_{\nu\nu'}(E)A_{\nu'},$$

$$[E - E_0^{\mu}(N - 1)]a_{\mu} = \sum_{\mu'} m_{\mu\mu'}(E)a_{\mu'}.$$

Without solving Eq. (4) explicitly, it already allows us to characterize qualitatively the $N$-particle spectrum of $\tilde{H}$. For $d \to \infty$ the unperturbed eigenvalues of $N - 1$ and $N$ HCBs form a band with lower band edges $E_0^{\mu}(N, d) < E_0^{\mu}(N - 1, d)$ (for $N/d < 1/2$; this is not a restriction due to the particle-hole symmetry). In Appendix A it is shown that for $s \neq 0$ the band between $E_0^{\mu}(N, d)$ and $E_0^{\mu}(N - 1, d)$ persists. Below $E_0^{\mu}(N, d)$ a discrete spectrum occurs exhibiting an excitation gap.

Let us discuss the solution of Eq. (4) below $E_0^{\mu}(N, d)$. In Appendix A we show that $M_{\nu\nu'}(E)$ and $m_{\mu\mu'}(E)$ strongly simplify in two regimes which is (i) the scaling limit $d \to \infty, s \to 0$ with $s = sv_{\tilde{H}}$ and $N$ fixed, and (ii) the strong coupling limit $s \to \infty$ for finite density $n$. In these limits the spectrum of $\tilde{H}$ below $E_0^{\mu}(N, d)$ follows from the solution of

$$H_{N}^{eff}|\phi_{N}\rangle_r = E_{eff}|\phi_{N}\rangle_r , \quad h_{N-1}^{eff}|\phi_{N-1}\rangle_r = E_{eff}|\varphi_{N-1}\rangle_r$$

with the effective Hamiltonians

$$H_{N}^{eff} = s^2(1/d) \sum_{i,j=1}^{d} h_i^h j , \quad h_{N}^{eff} = s^2(1/d) \sum_{i,j=1}^{d} h_i^j h_j .$$
The eigenvalues of $H_{N}^{eff}$ and $h_{N-1}^{eff}$ are identical and are related to $E$ by $E_{N}^{eff} = [E - (E_{0}^{low} - E_{F})][E - E_{0}^{0}]$. The Fermi energy, $E_{F}(N, d)$, plays a role here due to the equivalence of HCBs and spinless fermions on the unperturbed ring-lattice. The ground state eigenvalue $E_{0}(N, d; \tilde{s})$ follows from the largest eigenvalue $E_{N,d}^{max}(N, n; \tilde{s}) \approx \tilde{s}^{2}N(1 - n)$ (see Appendix A). This yields $(n = N/d) \approx E_{0}(N, d; \tilde{s}) \approx E_{0}^{0}(N, d) - E_{F}(N, d)/2 + \sqrt{(E_{F}(N, d)/2)^{2} + \tilde{s}^{2}N(1 - n)}$. (7)

In Appendix A, the corresponding ground states $|\phi_{N}\rangle_{r}$ and $|\varphi_{N-1}\rangle_{r}$ of $H_{N}^{eff}$ and $h_{N-1}^{eff}$ are presented and the existence of an excitation gap for $\tilde{s} \neq 0$ is proven.

The number $N_{0} = (1/d)\langle \Psi_{N} | \sum_{i,j=1}^{d} h_{ij} h_{ij} | \Psi_{N} \rangle$ of condensed particles is identical to $\langle \Psi_{N} | \hat{H}_{N}^{eff} | \Psi_{N} \rangle/\tilde{s}^{2}$. The calculation of this expectation value is performed in Appendix A. One gets for the ground state

$$N_{0}(N, n; \tilde{s}) \approx N[(1 - n) - |\beta(N, n; \tilde{s})|^{2}(1 - 2n)N^{-1}]$$

$$|\beta(N, n; \tilde{s})|^{2} \approx \frac{1}{2} \tilde{s}^{2}N(1 - n) \left\{ \left( E_{F}/2 \right)^{2} + \tilde{s}^{2}N(1 - n) \right\}^{-1}$$

The second term in the square bracket in the first line of Eq. (8) is a correction due to finite $N$.

The results for $E_{0}(N, d; \tilde{s})$ and $N_{0}(N, d; \tilde{s})$ for finite $d$ are valid (i) in the scaling limit if $\tilde{s} \gg 2\sqrt{2\pi}/\sqrt{d}$ and (ii) in the strong coupling limit if $\tilde{s} \gg 4\sqrt{n/(1 - n)}\sqrt{d}$ (see Appendix A). Since the scaling limit involves $d \rightarrow \infty$ for $N$ fixed it also implies the low density limit $n \rightarrow 0$.

**Results from DMRG.** In order to check the range of validity of the results above and to extend those for finite $d$ to small and intermediate coupling strengths $\tilde{s}$, we have performed large scale density matrix renormalization group computations (DMRG) for various system sizes and number of particles using optimization tools based on concepts of quantum information theory. Besides calculating energy eigenvalues and the one-particle reduced density matrices we have also determined one- and two-site von Neumann entropies and the two-site mutual information, $I_{ij}$. More details on our DMRG approach can be found in Appendix C. Since we are mostly interested in BEC we only present the results for $N_{0}(N, n; \tilde{s})$ as a function of $\log(\tilde{s})$. Analysis of the mutual information is summarized in Appendix C. Part (a) of Fig. 2 shows $N_{0}$ in the low density regime for $n \approx 0.05 \ll 1$ fixed, whereas in part (b) $d = 199$ is fixed and $n$ takes various values.

Three main observations can be made. First, the DMRG-results for $(N, d)$ fixed exhibit the crossover from quasi-BEC to BEC in all cases. Of course, the transition from $N_{0} \sim \sqrt{N}$ to $N_{0} \sim N$ becomes more pronounced with increase of $N$. Second, in the regime of quasi-BEC $(\log(\tilde{s}) < 0)$ the agreement between the DMRG-results and the result for impenetrable bosons [40] is very good, for low densities. This holds because in the limit $d \rightarrow \infty$ with $N$ fixed the ground state of HCBs becomes identical to that of impenetrable bosons. In this case, $N_{0}(N)$ for small $N$ follows from the numerical exact computation of the Toeplitz determinant [40]. Yet, for $N \rightarrow \infty$, $d \rightarrow \infty$ with $n$ finite the HCBs on the ring-lattice differ from impenetrable bosons in one dimension. Therefore, both results in Fig. 2(b) deviate more and more from each other as $n$ increases. Third, in the regime of BEC $(\log(\tilde{s}) > 0)$ the DMRG results also fit well with the analytical one (Eq. (8)) for all densities. Even the non-monotonic $\tilde{s}$-dependence stemming from the finite-$N$ correction in Eq. (8) is reproduced for small $N$ (see, e.g., the result in Fig. 2(a) for $d = 39$, corresponding to $N = 2$). With increasing $N$ the DMRG-result approaches the asymptotic value $N(1 - n)$ (full circles in Fig. 2(b) at $\log(\tilde{s}) = 4$).

**Experimental realization.** As an experimental realization of the ‘wheel’ model we suggest to load $N$ ultracold bosonic atoms into a Mexican-hat-like potential with $d$ local wells (left of Fig. 3). Such a ring-type confinement was already realized experimentally (see, e.g., Refs. [76–80]). Tuning the pair interaction [81–84] and the ring-well geometry...
such that multiple occupancy of a well is excluded one obtains a realization of HCBs on a ring-lattice. Then measuring the number of HCBs in their zero-momentum ring-state should yield a quasi-condensate with \( N_0(N) \sim \sqrt{N} \) \([40, 44, 45]\), which was already observed for impenetrable bosons in a cigar-shaped confinement \([32]\).

Next, creation of a local well at the hat’s center (right of Fig. 3) and increasing its depth more and more will allow the HCBs in the ring-wells to overcome each other by making transitions back and forth between any ring-well and the central one. This will significantly change the physical behavior since BEC will occur with \( N_0(N) \sim N \). In order for this to happen for finite \( d \) it must be \( s/t \gg 1/d \) (see Appendix A). The hopping occurs due to tunneling between the corresponding wells. Let \( (V_r, l_r = a) \) and \( (V_c, l_c = ad/(2\pi)) \) denote the potential barrier and tunneling distance, respectively, between two adjacent ring-wells and between a ring-well and the central one. Use of the WKB tunneling rate yields the estimate \( s/t \approx \frac{\gamma_{c/r}}{\sqrt{\gamma_c}} \exp[-\sqrt{\frac{ma^2}{\hbar^2} \sqrt{V_r/d} (2\pi \gamma_c - \sqrt{V_r})}] \) with \( m \) the particle’s mass and \( \gamma_{a,\alpha} = c, r \) the so-called attempt frequency related to the zero-point oscillation frequency in the corresponding well. For instance, if \( d = 79 \) and \( N = 4 \) (Fig. 2a) ‘BEC’-like behavior should occur for \( s/t > 1 \). This can be satisfied for all \( m \) and \( a \) if \( V_c/V_r \approx (2\pi/d)^2 \) provided \( \gamma_c/\gamma_r \approx 1 \). Note, log\((s/t)\sqrt{d}\) (used in Fig. 2) is directly related to the barrier heights.

Of course, there is no macroscopic number of condensed particles for \( N \) small. But the crossover is already visible for small \( N \), as demonstrated by the DMRG result. \( N \gg 1 \) requires \( d \gg 1 \). Which maximum values for \( d \) can be realized is not yet clear. However, generating a true Mexican-hat potential with continuous rotational invariance would realize the Tonks-Girardeau gas of impenetrable bosons. Since this corresponds to the continuum limit \( a \rightarrow 0, d \rightarrow \infty \) with \( ad \) fixed, the only condition for BEC would be \( s > 0 \), requiring the generation of a central well.

**Summary and conclusions.** We have presented and discussed a model which allows realization of the dimensional crossover from \( D = 1 \) to \( D \gg 1 \) by variation of a single control parameter. This feature is related to its toroidal topology. The model consists of a ring-lattice with \( d \) sites and an extra site at its center. Since our main focus is on BEC, we have investigated \( N \) hard-core bosons (HCBs) with nearest neighbor hopping rate \( t \) on the ring and a hopping rate \( s \) between the ring and the central site. The latter hopping drastically changes the behavior of the HCBs. Varying for large but finite \( d \) the ratio \( s/t \) from \( s/t \ll 1 \) to \( s/t \gg 1 \) induces a transition from a quasi-BEC regime with a number of condensed HCBs \( N_0 \sim \sqrt{N} \) to a BEC regime with \( N_0 \sim N \). The transition is particularly pronounced for macroscopic \( N \). However, the crossover already becomes visible for small \( N \), as clearly demonstrated by the large scale DMRG computations (cf. Fig. 2). As argued above ultracold bosonic atoms in a Mexican-hat-like potential should allow the experimental observation of this dimensional crossover for BEC.

The model is also interesting from a different point of view since it presents a mechanism creating an excitation gap. The unperturbed spectrum consists of an \((N - 1)\)- and \( N \)-particle band. For \( n \leq 1/2 \) the former is a subset of the latter. Turning on the hopping to the central site a band of scattering states occurs with lower and upper band edge identical to those of the unperturbed one. Below (and above) that band the coupling between the unperturbed \((N - 1)\)- and \( N \)-particle rings generates a discrete spectrum with an excitation gap. It is exactly this gap, which makes the BE-condensate robust against thermal fluctuations. The discrete spectrum follows in the limit \( s/t \rightarrow 0, d \rightarrow \infty \) with \( \delta = (s/t)\sqrt{d} \) from an effective range-Hamiltonian (cf. Eq. (9)) with ‘infinite’ range hopping. The variable \( \delta \) also occurs for electrons on a lattice in high dimensions \([2, 4, 6, 7]\), for the Hubbard model with infinite range hopping \([5]\) and for the fermionic Hubbard star \([8]\). Nevertheless, our approach is qualitatively different. The former models reduce to an effective one-site model whereas we obtain a model on the ring with renormalized hopping. This highlights the crucial difference to models with a conventional mean-field character and thus described, e.g., by the Gross-Pitaevskii theory \([26–28]\).

Finally, extending the ‘wheel’ model to interacting fermions and spins would be interesting, as well. This would allow one to elaborate on the analogous dimensional crossover for fermions and spins. For instance for spins, increasing the ratio of the corresponding exchange constants should induce long range magnetic order on the ring.

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The unperturbed eigenstates (i.e., s=0) for
\[ E \text{band with lower band edge} \]
\[ n \]
\[ \text{with} \ E \] and the Fermi energy
\[ \text{we note that the result for} \]
The normalized, totally symmetric ‘wave functions’
\[ \Psi \]
\[ \text{constructed from the one-particle states} \]
\[ N \text{with} \]
\[ \mu \]
\[ \text{highlights the well-known equivalence of spinless fermions and hard-core bosons in} \]
\[ 1d \] and one has
\[ \mu \]
\[ \text{one gets} \]
\[ \text{unperturbed eigenstates are labelled by} \]
\[ \mu \]
\[ \text{odd} \ [56, 66, 67] . \] Since
\[ \text{the unperturbed eigenstates are complete we have(cf. also Eq. (3) of the main text)} \]
\[ \text{Appendix A: Derivation of the effective Hamiltonian} \]
\[ \text{The Hamiltonian} \ H \text{is given by Eq. (1). Using} \]
\[ \hat{H} = \hat{H}_0 + \hat{H}_1 \] (with \( \hat{H}_0 \) the ring-Hamiltonian) and
\[ |\Psi_N\rangle = \alpha|\phi_N\rangle \otimes|0\rangle_c + \beta|\varphi_{N-1}\rangle \otimes|1\rangle_c, \] (A1)
the eigenvalue equation \( \hat{H}|\Psi_N\rangle = E|\Psi_N\rangle \) becomes
\[ \alpha\hat{H}_0|\phi_N\rangle - \beta s \sum_{i=1}^{d} h_i^\dagger|\varphi_{N-1}\rangle = \alpha E|\phi_N\rangle \]
\[ -\alpha s \sum_{i=1}^{d} h_i|\phi_N\rangle + \beta H_0|\varphi_{N-1}\rangle = \beta E|\varphi_{N-1}\rangle. \] (A2)
The unperturbed eigenstates (i.e., s=0) for \( N \) HCBs can be represented as
\[ |\psi^0_\mu(N)\rangle = \sum_{1 \leq n_1 < \cdots < n_N \leq d} \psi^0_\mu(n_1, \cdots, n_N) h^\dagger_{n_1} \cdots h^\dagger_{n_N}|0\rangle. \] (A3)
The normalized, totally symmetric ‘wave functions’ \( \psi^0_\mu(n_1, \cdots, n_N) \) are given for \( 1 \leq n_1 < \cdots < n_N \leq d \) by the determinant constructed from the one-particle states \( \exp(iq_{\mu_k}n_l) \) \[ |\psi^0_\mu(n_1, \cdots, n_N)\rangle = \mathcal{N} \sum_{P \in S_N} \text{sgn}(P) \exp \left( i \sum_{k=1}^{N} q_{\mu(P_k)}n_k \right), \] (A4)
with \( \mathcal{N} = d^{-N/2} \). \( S_N \) denotes the permutation group of the integers (1, 2, \cdots, \( N \)) and \( \text{sgn}(P) \) its signature. The form (A4) highlights the well-known equivalence of spinless fermions and hard-core bosons in 1d and one has \( \mu_1 < \mu_2 < \cdots < \mu_d \). The unperturbed eigenstates are labelled by \( \mu = (\mu_1, \cdots, \mu_N) \) and \( \mu \) determines the wave number \( q_\mu = (\pi/d)(2\mu + 1) \) for \( N \) even and \( q_\mu = (\pi/d)2\mu \) for \( N \) odd \[ |\psi^0_\mu(N)\rangle \]. Since \( q_\mu \) is restricted to the first Brillouin zone \( \mu \) takes the values \( -d/2 + 1, -d/2 + 2, \cdots, -1, 0, 1, \cdots, d/2 - 1, d/2 \) for even and \( -d/2 - 1, -(d-1)/2 + 1, \cdots, -1, 0, 1, \cdots, (d-1)/2 - 1, (d-1)/2 \) for odd. The corresponding unperturbed eigenvalues are given by
\[ E^0_\mu(N, d) = -2 \sum_{k=1}^{N} \cos(q_{\mu_k}). \] (A5)
The unperturbed ground state energy, \( E^0_{\text{low}}(N, d) \), is easily calculated. Using \( \cos(x) = [\exp(ix) + \exp(-ix)])/2 \) it is straightforward to calculate the sum in Eq. (A5). As a result one gets
\[ E^0_{\text{low}}(N, d) = -2 \sin(\pi/dN)/\sin(\pi/d) = -2 \frac{d}{\pi} \sin(\pi/dN), \] (A6)
and the Fermi energy \( E_F(N, d) = E^0_0(N, d) - E^0_0(N-1, d) \) becomes
\[ E_F(N, d) = -2 \left[ \tan(\pi/d) \sin(\pi/dN) + \cos(\pi/dN) \right] \simeq -2 \cos(\pi n) \] (A7)
with \( n = N/d \) the particle density. For \( d \to \infty, N \to \infty \) with \( n = N/d \) fixed the unperturbed \( N \)-particle spectrum is a single band with lower band edge \( E^0_{\text{low}}(N, n) \simeq -2N \sin(\pi n)/(\pi n) \) and band width \( W(N, n) = 4N \sin(\pi n)/(\pi n) \propto 4N \). First we note that the result for \( E^0_{\text{low}}(N, n) \) holds for \( N \) even and odd and second that the ground state lies in the subspace with total momentum \( Q = \sum_{k=1}^{N} q_{\mu_k} \) equal to zero. \( Q \) is a good quantum number due to the invariance of \( H \) under lattice translations on the ring.
Since the unperturbed eigenstates are complete we have(cf. also Eq. (3) of the main text)
\[ |\phi_N\rangle = \sum_\mu A_\mu |\psi^0_\mu(N)\rangle \]
\[ |\varphi_{N-1}\rangle = \sum_\mu a_\mu |\psi^0_\mu(N-1)\rangle. \] (A8)
with \( \sum_{\nu} |A_{\nu}|^2 = 1 \) and \( \sum_{\mu} |a_{\mu}|^2 = 1 \). The summations in Eq. (A8) are restricted such that \( Q = \sum_{j=1}^{N} q_{\nu} = \sum_{j=1}^{N-1} q_{\mu} \) is fixed (mod 2\( \pi \)). In the following we choose \( Q = 0 \), i.e., we consider the HCBs in the frame where the center of mass of the HCBs is at rest (see also [68]). Substituting the ansatz (A8) into Eq. (A2) leads to a decoupling of the \((N-1)\)-particle and the \(N\)-particle sector:

\[
[E - E_{\nu}^0(N, d)] A_{\nu} = s^2 \sum_{\nu'} M_{\nu\nu'}(E) A_{\nu'},
\]

\[
[E - E_{\mu}^0(N - 1, d)] a_{\mu} = s^2 \sum_{\mu'} m_{\mu\mu'}(E) a_{\mu'}.
\]  

(A9)

Here we used that \( E_{\mu}^0(N - 1, d) \) and \( E_{\nu}^0(N, d) \) are the corresponding unperturbed eigenvalues of \( |\psi_\nu^0(N-1)\rangle \) and \( |\psi_\nu^0(N)\rangle \), respectively. The matrix elements \( M_{\nu\nu'}(E) \) and \( m_{\mu\mu'}(E) \) depend only on the unperturbed eigenstates and eigenvalues and are given by

\[
M_{\nu\nu'}(E) = \sum_{\mu'} (b^\dagger)_{\nu\mu'} (E - E_{\mu}^0(N - 1, d))^{-1} b_{\nu'\nu},
\]

\[
m_{\mu\mu'}(E) = \sum_{\nu'} b_{\mu\nu'} (E - E_{\nu'}^0(N, d))^{-1} b^\dagger_{\nu'\nu}.
\]  

(A10)

The crucial quantity is the matrix \( b \) with elements

\[
b_{\mu\nu} = \langle \psi_\mu^0(N - 1) | \sum_{i=1}^{d} h_i | \psi_\nu^0(N) \rangle.
\]  

(A11)

Having solved Eq. (A9) one obtains from Eq. (A2) with Eq. (A8) the coefficients \( \alpha \) and \( \beta \).

Operating with \( \sum_{\nu} A_{\nu}' [E - E_{\nu}^0(N, d)]^{-1} \) and \( \sum_{\mu} a_{\mu} [E - E_{\mu}^0(N - 1, d)]^{-1} \), respectively, on the 1st and 2nd line of Eq. (A9) and taking the normalization of \( \{A_{\nu}\} \) and \( \{a_{\mu}\} \) into account the eigenvalue equations take the form

\[
1 = s^2 f^{(N-1)}_d(E; \{a_{\mu}\})
\]

\[
1 = s^2 F^{(N)}_d(E; \{A_{\nu}\}),
\]  

(A12)

with

\[
f^{(N-1)}_d(E; \{a_{\mu}\}) = \sum_{\mu\nu} A_{\mu} (E - E_{\mu}^0(N - 1, d))^{-1} b_{\mu\nu} [E - E_{\nu}^0(N, d)]^{-1} (b^\dagger)_{\nu\mu} a_{\nu}.
\]  

(A13)

and

\[
F^{(N)}_d(E; \{A_{\nu}\}) = \sum_{\nu\nu'} A_{\nu'} (E - E_{\nu'}^0(N, d))^{-1} (b^\dagger)_{\nu'\nu} [E - E_{\mu}^0(N - 1, d)]^{-1} b_{\nu'\nu} A_{\nu'}. 
\]  

(A14)

The unperturbed eigenfunctions can always be chosen to be real, since the unperturbed Hamiltonian is real. Therefore \( b_{\mu\nu} \) and \( (b^\dagger)_{\nu\mu} \) are real. Furthermore, \( \{a_{\mu}\} \) and \( \{A_{\nu}\} \) can also be chosen to be real since the Hamiltonian \( \hat{H} \) is real as well. Therefore the functions \( f^{(N-1)}_d(E; \{a_{\mu}\}) \) and \( F^{(N)}_d(E; \{A_{\nu}\}) \) are real.

Eq. (A12) together with Eqs. (A13) and (A14) already allows to obtain some qualitative information on the low-energy part of the perturbed \( N \)-particle spectrum. A crucial observation is that \( f^{(N-1)}_d(E; \{a_{\mu}\}) \) and \( F^{(N)}_d(E; \{A_{\nu}\}) \) have poles at the unperturbed \((N-1)\)- and \(N\)-particle eigenvalues. As discussed above the unperturbed spectrum of \((N-1)\) and \(N\) HCBs form a band with lower band edge \( E_{\nu}^0(N - 1, d) \) and \( E_{\mu}^0(N, d) \), respectively. It is \( E_{\nu}^0(N, d) = E_{\nu}^0(N - 1, d) + E_F(N, d) \) with the Fermi energy from Eq. (A7). For \( n < 1/2 \) it follows \( E_F(N, d) < 0 \), Note, this is not a restriction due to the particle-hole duality. Therefore, \( E_{\nu}^0(N, d) < 0 \) and \( E_{\nu}^0(N - 1, d) \) persists. The discrete spectrum exhibits an excitation gap even for \( d = \infty \).

Let us choose \( E \) between \( E_{\nu}^0(N, d) \) and \( E_{\nu}^0(N - 1, d) \) and let us denote the increasingly ordered unperturbed eigenvalues \( \{E_{\nu}^0(N, d)\} \) in this interval by \( E_{\nu}^{k}(N, d) \), \( k \geq 1 \). Due to \( E < E_{\nu}^{k}(N - 1, d) \) the denominators \( [E - E_{\nu}^{k}(N - 1, d)]^{-1} \) in Eq. (A13) are negative for all \( \mu \), i.e., they do not change sign. Then, under variation of \( E \) between \( E_{\nu}^0(N, d) \) and \( E_{\nu}^{k+1}(N, d) \), the function \( f^{(N-1)}_d(E; \{a_{\mu}\}) \) varies continuously from \( \pm \infty \) at \( E = E_{\nu}^{k}(N, d) \) to \( \mp \infty \) at \( E = E_{\nu}^{k+1}(N, d) \), independent of \( \{a_{\mu}\} \). Accordingly, for arbitrary \( s \neq 0 \) the first equation of (A12) has always a solution \( E_{\nu}^s(N, d; s; \{a_{\mu}\}) \) which is between...
\[ E^{0}_{\nu}(N, d) \] and \( E'^{0}_{\nu_{\pm 1}}(N, d) \). Substituting \( E_{\nu r}(N, d; \{a_{\nu}\}) \) into the second line of Eq. (A9) yields \( \{a^{(\nu r)}_{\nu}\} \) which in turn leads to the perturbed eigenvalues \( E_{\nu r}(N, d; s) = E_{\nu r}(N, d; \{a^{(\nu r)}_{\nu}\}) \), \( k \geq 1 \). For \( d \to \infty \) these perturbed eigenvalues \( \{E_{\nu r}(N, d; s)\} \) form a band with lower band edge \( E^{0}_{\nu}(N, d) \) and upper edge \( E^{0}_{\nu}(N-1, d) \).

For \( E \geq E^{0}_{\nu}(N-1, d) \) there exist pairs \( \nu', \mu \) such that there is no unperturbed eigenvalue between \( E^{0}_{\nu}(N, d) \) and \( E^{0}_{\nu}(N-1, d) \). In that case \( [E - E^{0}_{\nu}(N, d)]^{-1}[E - E^{0}_{\nu}(N-1, d)]^{-1} \) in Eq. (A13) changes from \( \pm \infty \) at \( E^{0}_{\nu}(N, d) \) to \( \pm \infty \) at \( E^{0}_{\nu}(N-1, d) \) under varying \( E \) between \( E^{0}_{\nu}(N, d) \) and \( E^{0}_{\nu}(N-1, d) \). Therefore, \( f^{(N-1)}_{d}(E; \{a_{\mu}\}) \) does not necessarily change sign and the first equation of (A12) may only have a solution for \( s \) small enough. In case that the solution between \( E^{0}_{\nu}(N, d) \) and \( E^{0}_{\nu}(N-1, d) \) disappears if \( s \) becomes large enough, a perturbed eigenvalue must appear below(or above) the lower(upper) band edge \( -[E^{0}_{\nu}(N, d)]/[E^{0}_{\nu}(N-1, d)] \), since the total number of eigenvalues does not depend on \( s \).

Finally, let us discuss \( E < E^{0}_{\nu}(N, d) \). In that case the product of both denominators in Eq. (A13) is always positive. For \( E \to E^{0}_{\nu}(N, d) \) from below \( f^{(N-1)}_{d}(E; \{a_{\mu}\}) \) will diverge to \( +\infty \). Because \( f^{(N-1)}_{d}(E; \{a_{\mu}\}) \to 0 \) for \( E \to -\infty \) there must exist at least one solution \( E(N, d; s, \{a_{\mu}\}) \) of the first equation of Eq. (A12) for all \( s \neq 0 \). \( E(N, d; s, \{a_{\mu}\}) \) will have a gap to the lower band edge \( E^{0}_{\nu}(N, d) \). Depending on \( s^{2} \) and \( \{a_{\mu}\} \) there may exist more than one solution. Substitution them into the first line of Eq. (A9) yields a discrete spectrum. The same qualitative discussion can be done for \( E^{0}_{\nu}(N, d) \) (Eq. (A14)) in combination with the second equation in (A12). We have checked the correctness of these qualitative results on the perturbed spectrum for \( N = 2 \). There is little doubt that they become incorrect for \( N > 2 \).

Now we describe how the discrete part of the perturbed spectrum and the corresponding eigenstates below \( E^{0}_{\nu}(N, d) \) can be obtained exactly in two limiting cases. In these two cases \([E - E^{0}_{\nu}(N, d)]^{-1} \) and \([E - E^{0}_{\nu}(N, d)]^{-1} \) can be replaced by \([E - E^{0}_{\nu}(N, d) + E_{F}(N, d)]^{-1} \) and \([E - E^{0}_{\nu}(N, d)]^{-1} \), respectively. Then, the matrices \( E_{\mu \nu}(E) \) and \( m_{\mu \nu}(E) \) strongly simplify since the sums in Eq. (A10) can be performed using Eq. (A11) and the completeness relations \( \sum_{\mu} \langle \psi_{\nu}(N)| \psi_{\nu}(N) \rangle = 1 \) for any \( N \). For \( N \to \infty \), \( A_{\mu} \) and \( \{a_{\mu}\} \) become the identity operator, respectively, in the \( (N-1) \) and \( N \) particle subspace. Then it follows

\[ M_{\mu \nu}(E) \approx [E - E^{0}_{\nu}(N, d)]^{-1} \psi_{\nu}(N)| \sum_{ij} h_{ij} | \psi_{\nu}(N) \rangle \]

and Eq. (A9) simplifies to

\[ E^{\text{eff}}_{\mu}(N, d) \approx s^{2} \sum_{\nu} \langle \psi_{\nu}(N)| \sum_{ij} h_{ij} | \psi_{\nu}(N) \rangle A_{\nu} \]

\[ E^{\text{eff}}_{\mu}(N, d) \approx s^{2} \sum_{\nu} \langle \psi_{\nu}(N)| \sum_{ij} h_{ij} | \psi_{\nu}(N) \rangle a_{\mu} \],

(A15)

with

\[ E^{\text{eff}}(N, d) = [E - E^{0}_{\nu}(N, d) + E_{F}(N, d)] [E - E^{0}_{\nu}(N, d)] \).

(A16)

The eigenvalue equations (A15) are identical to the eigenvalue equations following from

\[ \hat{H}_{N}^{\text{eff}}|\phi_{N}\rangle = E^{\text{eff}}(N, d)|\phi_{N}\rangle \]

\[ \hat{H}_{N-1}^{\text{eff}}|\varphi_{N-1}\rangle = E^{\text{eff}}(N, d)|\varphi_{N-1}\rangle \).

(A17)

with the effective Hamiltonians

\[ \hat{H}_{N}^{\text{eff}} = s^{2} \frac{1}{d} \sum_{i,j=1}^{d} h_{ij} h_{ij} \]

\[ \hat{H}_{N-1}^{\text{eff}} = s^{2} \frac{1}{d} \sum_{i,j=1}^{d} h_{ij} \]

(A18)

and \( |\phi_{N}\rangle, |\varphi_{N-1}\rangle \) from Eq. (A8). Note, the eigenvalue \( E^{\text{eff}}(N, d) \) is identical for \( \hat{H}_{N}^{\text{eff}} \) and \( \hat{H}_{N-1}^{\text{eff}} \). \( \hat{s} = (s/t)\sqrt{d} \) denotes the scaled dimensionless coupling constant. We remind the reader that we used \( t = 1 \).

In the following it is more convenient to use the equivalence \( h_{ij} = S_{i}^{-} \), \( h_{i} = S_{i}^{+} \), \( (1 - 2h_{ij}h_{ij}) = 2S_{i}^{z} \) between the hard-core Bose operators and the spin-one-half operators. The commutation relations of the latter read

\[ [S_{i}^{+}, S_{j}^{-}] = 2\delta_{ij} S_{i}^{-}, \]

\[ [S_{j}^{z}, S_{i}^{z}] = \pm \delta_{ij} S_{i}^{z} \].

(A19)
The effective Hamiltonians become
\[ \hat{H}^{\text{eff}}_N = \hat{S}^2/2 \sum_{i,j=1}^{d} S_i^+ S_j^+ \]
\[ \hat{H}_N^{\text{eff}} = \hat{s}^2/2 \sum_{i,j=1}^{d} S_i^+ S_j^- , \]
\[ (A20) \]
\[ (A21) \]

Let \( \hat{S} = \sum_{i=1}^{d} \hat{S}_i \) be the spin operator of the total spin. Because \( \hat{H}_N^{\text{eff}} \) and \( \hat{H}_N^{\text{eff}} \) commute with \( \hat{S}^2 \) and \( S^2 \) all its eigenstates can be chosen such that they are also eigenstates of \( \hat{S}^2 \) and \( S^2 \) with eigenvalues \( S(S + 1) \) and \( M \), respectively. They will be denoted by \( |S, M\rangle \). \( M \) is related to the particle number by \( M = d/2 - N \) and for fixed \( N \) the total spin quantum number takes the values \( S = d/2 - N, d/2 - N + 1, \ldots, d/2 \). The corresponding eigenvalues of \( \hat{H}^{\text{eff}}_N \) are given by
\[ E^{\text{eff}}(S, M; \hat{s}) = \hat{s}^2/2 [S(S + 1) - M(M + 1)] . \]
\[ (A22) \]

The ground state eigenvalue \( E_0(N, d; \hat{s}) \) of \( \hat{H} \) follows from the largest eigenvalue \( E^{\text{max}}_N(N, d; \hat{s}) \) of \( \hat{H}^{\text{eff}}_N \) which corresponds to \( S_{\text{max}} = d/2 \). Then we obtain from Eq. \( (A22) \) in the thermodynamic limit \( N \to \infty, d \to \infty \) with density \( n = N/d \) fixed
\[ E^{\text{eff}}(N, d; \hat{s}) \approx \hat{s}^2 N(1 - n) . \]
\[ (A23) \]

The corresponding eigenstates are given by
\[ |\phi^{\text{eff}}_N\rangle = \left( \begin{array}{c} d \\ N \end{array} \right)^{-1/2} \left( \sum_{i=1}^{d} h_i^\dagger \right)^N |0\rangle \]
\[ |\phi^{\text{eff}}_{N-1}\rangle = \left( \begin{array}{c} d \\ N-1 \end{array} \right)^{-1/2} \left( \sum_{i=1}^{d} h_i^\dagger \right)^{N-1} |0\rangle . \]
\[ (A24) \]

Using the spin analogy it is \( |\phi^{\text{eff}}_N\rangle = |d/2, d/2 - N\rangle \) and \( |\phi^{\text{eff}}_{N-1}\rangle = |d/2, d/2 - (N - 1)\rangle \). Note, these eigenstates belong to the subspace with total spin \( Q = 0 \).

Substitution of \( E^{\text{eff}}_{\text{max}}(N, d; \hat{s}) \) from Eq. \( (A23) \) into Eq. \( (A16) \) leads to the perturbed ground state eigenvalue
\[ E_0(N, d; \hat{s}) \approx E_{\text{low}}^0(N, d) - E_F(N, d)/2 - \sqrt{\left[ E_F(N, d)/2\right]^2 + \hat{s}^2 N(1 - n)} . \]
\[ (A25) \]

We remind the reader that Eq. \( (A7) \) implies \( E_F(N, d) \leq 0 \) for \( 0 \leq n \leq 1/2 \).

The energy, \( E_1(N, d; \hat{s}) \), of the first excitation follows from Eq. \( (A16) \) for the second largest effective eigenvalue. Using the spin analogy the latter corresponds to \( S = S_{\text{max}} - 1 \equiv d/2 - 1 \) which yields \( E^{\text{eff}}(N, d; \hat{s}) = E^{\text{eff}}_{\text{max}}(N, d; \hat{s}) - \hat{s}^2 \). Accordingly we obtain
\[ E_1(N, d; \hat{s}) \approx E_{\text{low}}^0(N, d) - E_F(N, d)/2 - \sqrt{\left[ E_F(N, d)/2\right]^2 + \hat{s}^2 [N(1 - n) - 1]} . \]
\[ (A26) \]

It is easy to see that the excitation gap \( \Delta E(N, d; \hat{s}) = E_1(N, d; \hat{s}) - E_0(N, d; \hat{s}) \) is finite for \( \hat{s} \neq 0 \) and all \( d \), including \( d = \infty \). The higher excitation energies \( E_n(N, d; \hat{s}) \) for \( n = 2, \ldots, N - 1 \) follow similarly using \( S = d/2 - n \). Therefore, the eigenvalues of \( \hat{H} \) below \( E_{\text{low}}^0(N, d) \) form a discrete spectrum of \( N \) eigenvalues.

Let us summarize: The unperturbed spectrum in the subspace \( Q = 0 \) consists of two bands. One band, \( B_{N-1}^0 \), of \( K_{N-1} \) eigenvalues, \( \{ E^0_0(N - 1, d) \} \), and the other band, \( B_N^0 \), with \( K_N \) eigenvalues, \( \{ E^0_0(N, d) \} \). These two bands correspond to \( (N - 1) \) and \( N \) HCBS on the ring-lattice. The band edges of \( B_{N-1}^0 \) are at \( \pm E_{\text{low}}^0(N - 1, d) \) and those of \( B_N^0 \) at \( \pm E_{\text{low}}^0(N, d) \). For \( d \) finite, both sets \( \{ E^0_0(N - 1, d) \} \) and \( \{ E^0_0(N, d) \} \) are disjoint, and for density \( n = N/d \leq 1/2 \) \( B_{N-1}^0 \) is a subset within the interval \([-|E_{\text{low}}^0(N, d)|, +|E_{\text{low}}^0(N, d)|]\). Turning on \( \sigma \) leads to a coupling between these two bands. Part of these two bands persist. The lower band edge of the perturbed band coincides for \( d = \infty \) with the lower band edge of the unperturbed band. Below that band a discrete spectrum of maximally \( N \) eigenvalues occurs exhibiting an excitation gap. The number of discrete eigenvalues may change with \( \hat{s} \).

The number \( N_0 = (1/d) \langle \Psi_N | \sum_{i,j=1}^{d} h_i^\dagger h_j | \Psi_N \rangle \) of condensed particles in the state \( |\Psi_N \rangle \) is easily obtained since \( (1/d) \langle \Psi_N | \sum_{i,j=1}^{d} h_i^\dagger h_j | \Psi_N \rangle \approx \langle \Psi_N | \hat{H}_N^{\text{eff}} | \Psi_N \rangle / \hat{s}^2 \). Substitution of \( |\Psi_N \rangle \) from Eq. \( (A1) \) leads to \( N_0 = \).
\[ [\alpha^2 (\phi_{\alpha}^* f | \hat{E} \hat{H}^2 \hat{f} | \phi_{\alpha}^* f) + |\beta|^2 (\phi_{\beta}^* f | \hat{E} \hat{H}^2 \hat{f} | \phi_{\beta}^* f)] \equiv \frac{s^2}{d^2} \sum_{i=1}^{d} (1 - 2\hat{n}_i). \] (A27)

With \(|\alpha|^2 + |\beta|^2 = 1\), \(\sum_{i=1}^{d} \hat{n}_i |\phi_{\alpha}^* f\rangle = (N - 1) |\phi_{\alpha}^* f\rangle\), Eqs. (A17), (A18) and (A23) we obtain for \(d \to \infty\) the final result
\[ N_0(N; \hat{s}) \simeq N \left[ (1 - n) - |\beta|^2 (1 - 2n) N^{-1} \right]. \] (A28)

Since \(|\beta|^2 \leq 1\) the second term on the r.h.s. of Eq. (A28) is a negative correction for \(n < 1/2\) to the leading order term \((1-n)\) which is of order \(O(1/N)\).

As discussed above the mapping of the original model to an effective one is valid if one is allowed to replace \(E_{\nu}^0(N, d)\) by the unperturbed ground state energy eigenvalue \(E_{\nu}^0(N, d)\). This is equivalent to the replacement of \(\hat{H}\) in the first line of Eq. (A2) by \(E_{\nu}^0(N, d)\). With this replacement and that of \((|\phi_N\rangle, |\varphi_{N-1}\rangle)\) by \((|\phi_{\alpha}^* f\rangle, |\phi_{\beta}^* f\rangle)\) from Eq. (A24) one can solve the linear equation for \(\alpha, \beta\). With use of \(E_{\nu}(N, d; \hat{s}) - E_{\nu}^0(N, d)\) from Eq. (A25) and the normalization condition \(|\alpha|^2 + |\beta|^2 = 1\) one obtains for the ground state
\[ |\alpha|^2 \simeq 1 - \frac{1}{2} \hat{s}^2 N (1 - n) \left\{ \left[ (E_F/2)^2 + \hat{s}^2 N (1 - n) \right] - \left( E_F/2 \right)^2 \right\}^{-1} \] (A29)
\[ |\beta|^2 \simeq \frac{1}{2} \hat{s}^2 N (1 - n) \left\{ \left[ (E_F/2)^2 + \hat{s}^2 N (1 - n) \right] - \left( E_F/2 \right)^2 \right\}^{-1}. \] (A30)

Now we discuss the validity of the above mapping of the original eigenvalue problem to an effective one. The simplest limiting case under which the mapping becomes exact is the strong coupling limit \(s \to \infty\). For the ground state energy \(E = E_0(N, d; \hat{s})\) the denominators in Eq. (A10) can be rewritten as follows
\[ [E_0(N, d; \hat{s}) - E_{\nu}^0(N, d)] = [E_0(N, d; \hat{s}) - E_{\nu}^0(N, d)] \left\{ 1 + [E_{\nu}^0(N, d) - E_{\nu}^0(N, d)]/[E_0(N, d; \hat{s}) - E_{\nu}^0(N, d)] \right\}^{-1}. \] (A31)

and similar for \([E_0(N, d; \hat{s}) - E_{\mu}^0(N - 1, d)]\). From Eqs. (A5) and (A6) we get 0 \(\leq [E_{\nu}^0(N, d) - E_{\nu}^0(N, d)] \leq 4N\) for all \(\nu\). The mapping becomes exact if
\[ [E_{\nu}^0(N, d) - E_{\nu}^0(N, d)]/[E_0(N, d; \hat{s}) - E_{\nu}^0(N, d)] \to 0. \] (A32)

Substituting \([E_{\nu}^0(N, d) - E_{\nu}^0(N, d)]\) from Eq. (A25) and taking the upper bound, \(4N\), for \([E_{\nu}^0(N, d) - E_{\nu}^0(N, d)]\) into account leads for \(N = O(d)\), i.e. for finite \(n\), to the condition
\[ s \gg 4 \sqrt{n} \sqrt{1 - n}. \] (A33)

Note that \(n \leq 1/2\), and \(\hat{s} = s\sqrt{d}\) is used.

The reason why the mapping becomes exact in the scaling limit \(s \to 0, d \to \infty\) with \(\hat{s} = s\sqrt{d}\) and \(N\) fixed, is more subtle. In that case the variation with \(\mu'\) of the numerator \(\hat{b}_{\mu'} b_{\mu' \nu'}\) and of the denominator \([E - E_{\mu'}^0(N - 1, d)]\) in the first line of Eq. (A10) plays the essential role. In Appendix B we prove that \(b_{\mu' \nu'} \sim \sqrt{d}\) for \(\mu, \nu\) fixed and \(d \to \infty\) whereas \(b_{\mu} = O(1)\) if \(\mu_k \in \mu\) and \(\nu_k \in \nu\) of \(O(d)\). This means that \(b_{\mu' \nu'}\) decreases fast with increasing \(\mu_k\) and \(\nu_k\). Therefore the main contributions in the sums in Eq. (A10) for \(N\) arbitrary large but fixed comes from \(\mu'\), \(\nu'\) with \(\mu_k\) and \(\nu_k\) arbitrary large but fixed. Therefore restricting the sums in Eq. (A10), e.g., over \(\mu'\) to \(|\mu'_k| \leq \sqrt{d}\) for all \(k\) does not change the result if \(d\) becomes very large. Due to this restriction of \(|\mu_k|\) we obtain with \(E_{\nu}^0(N, d) \simeq -2N\) and \(\cos (2\pi \mu_k / d) \leq 1 - 2\pi^2 / d\) from (A5) the upper bound \([E_{\nu}^0(N, d) - E_{\nu}^0(N, d)] \ll 4\pi^2 N / d\) for the numerator in Eq. (A32). Substituting this upper bound and \([E_{\nu}^0(N, d) - E_{\nu}^0(N, d)]\) from Eq. (A25) into Eq. (A32) leads for \(N\) fixed and \(d \gg 1\) to the condition
\[ s \gg 2\sqrt{2\pi / d}. \] (A34)

Here we also used \(E_F(N, d) \approx -2N\) because \(n \approx 0\) for \(N\) fixed and \(d \gg 1\) (cf. Eq. (A7)).
Appendix B: Behavior of $b_{\mu,\nu}$ for $N$ fixed and $d \to \infty$

To study the behavior of $b_{\mu,\nu}$ for $N$ fixed and $d \to \infty$ we first observe that the translational invariance on the ring implies that Eq. (A11) becomes $b_{\mu,\nu} = d \langle \psi_{\mu}^0(N-1) | h_1 | \psi_{\nu}^0(N) \rangle$. Substituting $|\psi_{\mu}^0(N-1)\rangle, |\psi_{\nu}^0(N)\rangle$ from Eq. (A3) and taking advantage of the ordering $1 \leq n_1 < \cdots < n_N \leq d$ one arrives at

$$b_{\mu,\nu} = d \sum_{2 \leq m_2 < \cdots < m_N \leq d} \psi_{\mu}^0(m_2, \cdots, m_N) \psi_{\nu}^0(1, m_2, \cdots, m_N). \quad (B1)$$

Introducing new variables $n_i = m_{i+1} - 1$ and taking the translational invariance into account this yields

$$b_{\mu,\nu} = d \sum_{1 \leq n_1 < \cdots < n_{N-1} \leq d-1} \psi_{\mu}^0(n_1, \cdots, n_{N-1}) \psi_{\nu}^0(0, n_1, \cdots, n_{N-1}). \quad (B2)$$

Substituting the normalized ‘wave functions’ from Eq. (A4) leads to

$$b_{\mu,\nu} = d d^{-(N-\frac{1}{2})} \sum_{P \in S_{N-1}} \sum_{P' \in S_N} \text{sgn}(P) \text{sgn}(P') \sum_{1 \leq n_1 < \cdots < n_{N-1} \leq (d-1)} \exp \left[ -i \sum_{k=1}^{N-1} \sum_{q_{\mu,\nu}(k)} - q_{\nu,\nu}(k+1) n_k \right]. \quad (B3)$$

The crucial quantity is the 2nd line of Eq. (B3). This sum can be written as $\sum_{n_1=1}^{d-N+1} \sum_{n_2=1}^{d-N+2} \cdots \sum_{n_{N-1}=1}^{d-N-1} \sum_{n_{N-1}=n_{N-2}+1}^{d}$ (each single sum generates a denominator of the form $(1 - \exp[-i \sum_{j=1}^{N-1} a_j q_{\mu,\nu} - \sum_{j=1}^{N} b_j q_{\nu,\nu}])$ where the integers $\{a_j\}$ and $\{b_j\}$ take values 0, \pm 1. There is a product of $(N-1)$ such denominators. For $\mu, \nu$ fixed and for $d \to \infty$ this product is proportional to $d^{N-1}$. Performing the sums in the 2nd line of Eq. (B3) also generates numerators of the form $(1 - \exp[-i \sum_{j=1}^{N-1} a_j q_{\mu,\nu} - \sum_{j=1}^{N} b_j q_{\nu,\nu}])$ where $\{a_j\}$ and $\{b_j\}$ take values 0, \pm 1. Therefore some of the numerators vanish and some do not. The latter take the value 2. Accordingly, for $d \to \infty$ the contribution of these terms in the 2nd line of Eq. (B3) is of order $d^{N-1}$. The contribution of all the other terms are of $O(d^{N-2})$. Taking the prefactor $d^{-(N-\frac{1}{2})}$ on the r.h.s. of Eq. (B3) into account one obtains for $(\mu, \nu)$ arbitrary but fixed and $d \to \infty$ in leading order in $d$

$$b_{\mu,\nu} \sim \sqrt{d}, \quad (B4)$$

which we wanted to prove.

Appendix C: Details of the DMRG calculations

The DMRG calculations were performed for $d \leq 199$ and $N \leq 98$. In the DMRG procedure we have performed calculations using the dynamic block state selection approach [24] and by fixed bond dimension. We have set a tight error bound on the diagonalization procedure, i.e., we set the residual error of the Davidson method to $10^{-5}$ and used ten DMRG sweeps. We have checked that the various quantities of interest are practically insensitive for $M \geq 1024$. In the regime of validity of the analytical results, e.g., the ground state energy $E_0(N, d; \hat{s})$ (Eq. (7)) and $N_0(N, n; \hat{s})$ (Eq. (8)) agree with the corresponding DMRG-results better than one percent.

Besides calculating energy eigenvalues and the one-particle reduced density matrices we have also determined one- and two-site von Neumann entropies ($s_i$ and $s_{ij}$, respectively) and the two-site mutual information, $I_{ij}$, given as $I_{ij} = s_i + s_j - s_{ij}$. Here $s_i = -\text{Tr} \rho_i \ln \rho_i$ and $s_{ij} = -\text{Tr} \rho_{ij} \ln \rho_{ij}$ where $\rho_i (\rho_{ij})$ is the reduced density matrix of site $i$ (sites $i$ and $j$), which is derived from the density matrix of the total system by tracing out the configurations of all other sites.

In Fig. 4, we show mutual information for various pairs of sites as a function of $\log(\hat{s})$ for $d = 199$ and $n \approx 0.05$. The change in the correlation pattern related to the crossover from quasi-BEC to BEC is also clearly visible through the mutual information. The correlation between the central and a ring site, $I_{i|c}$, vanishes for small $\hat{s}$ coupling while it saturates to a finite value in the strong coupling limit when the model exhibits infinite-range hopping. Similarly, for arbitrary two ring sites, $I_{i|i+\ell}$, saturates to a constant value for large $\hat{s}$ when the hopping along the ring is mediated by the central site. For $\hat{s} = 0$ $I_{i|i+\ell}$ decays algebraically with increasing $\ell$ while it becomes exponential as the gap opens and saturate to finite value for very large $\ell$ values.
FIG. 4. Two-site mutual information, $I_{i|c}$, measured between the central site and a ring site (red star symbol), and $I_{i|i+\ell}$ for various two sites on the ring separated by distance $\ell = 1, 3, 10, 99$ for $d = 199$ and $n \simeq 0.05$. 