Uniformization and Skolem Functions in the Class of Trees.

BY

SHMUEL LIFSCHES and SAHARON SHELAH*

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

ABSTRACT

The monadic second-order theory of trees allows quantification over elements and over arbitrary subsets. We classify the class of trees with respect to the question: does a tree $T$ have definable Skolem functions (by a monadic formula with parameters)? This continues [LiSh539] where the question was asked only with respect to choice functions. Here we define a subclass of the class of tame trees (trees with a definable choice function) and prove that this is exactly the class (actually set) of trees with definable Skolem functions.

1. Introduction: The Uniformization Problem

Definition 1. The monadic second-order logic is the fragment of the full second-order logic that allows quantification over elements and over monadic (unary) predicates only. The monadic version of a first-order language $L$ can be described as the augmentation of $L$ by a list of quantifiable set variables and by new atomic formulas $t \in X$ where $t$ is a first order term and $X$ is a set variable. The monadic theory of a structure $\mathcal{M}$ is the theory of $\mathcal{M}$ in the extended language where the set variables range over all subsets of $|\mathcal{M}|$ and $\in$ is the membership relation.

Definition 2. The \textit{monadic language of order} $L$ is the monadic version of the language of order $\{<\}$. For simplicity, we add to $L$ the predicate $\text{sing}(X)$ saying “$X$ is a singleton” and use only formulas with set variables. Thus the meaning of $X < Y$ is: $X = \{x\}$ & $Y = \{y\}$ & $x < y$.

Definition 3. Let $T$ be a tree and $\bar{P} \subseteq T$.

1. $\varphi$ is an $(n, l)$-formula if $\varphi = \varphi(X, Y, \bar{P})$ with $\text{dp}(\varphi) = n$ and $l(\bar{P}) = l$.

2. $\varphi = \varphi(X, Y, \bar{P})$ is \textit{potentially uniformizable in} $T$ (p.u) if $T \models (\forall Y)(\exists X)\varphi(X, Y, \bar{P})$.

* The second author would like to thank the U.S.–Israel Binational Science Foundation for partially supporting this research. Publ. ***
2. Tame Trees

Definition 2.1. A tree is a partially ordered set $(T, \triangleleft)$ such that for every $\eta \in T$, $\{\nu : \nu \triangleleft \eta\}$ is linearly ordered by $\triangleleft$.

Note, a chain $(C, <^*)$ and even a set without structure $I$ is a tree.

Branch, Sub-branch, Initial segment.

Definition 2.2. (1) $(C, <^*)$ is a scattered chain iff ...

(2) For a scattered chain $(C, <^*)$ $Hdeg(C)$ is defined inductively by:

$Hdeg(C) = 0$ iff ...

$Hdeg(C) = \alpha$ iff ...

$Hdeg(C) \geq \delta$ iff ...

Theorem 2.3. $Hdeg(C)$ exists for every scattered chain $C$.

Lemma 2.4. $Hdeg(C) < \omega$ then $C$ has a definable well ordering.

Proof. See A1 in the appendix

Definition 2.5. $\sim_0^A, \sim_1^A$ (from [LiSh539] 4.1)

Definition 2.6. (1) A tree $T$ is called wild if either

(i) $\sup\{|top(A) / \sim_0^A| : A \subseteq T \text{ an initial segment}\} \geq \aleph_0$ or

(ii) There is a branch $B \subseteq T$ and an embedding $f : Q \rightarrow B$ or

(iii) All the branches of $T$ are scattered linear orders but $\sup\{Hdeg(B) : B \text{ a branch of } T\} \geq \omega$.

(iv) There is an embedding $f : \omega > 2 \rightarrow T$

(2) A tree $T$ is tame for $(n^*, k^*)$ if the value in (i) is $\leq n^*$, (ii) does not hold and the value in (iii)

is $\leq k^*$

(3) A tree $T$ is tame if $T$ is tame for $(n^*, k^*)$ for some $n^*, k^* < \omega$.

The following is the content of [LiSh539], (2) $\Rightarrow$ (3) is given in theorem A2 in the appendix.

Theorem 2.7. The following are equivalent:

1. $T$ has a definable choice function.
2. $T$ has a definable well ordering.
3. $T$ is tame.

3. Composition Theorems

Notations. $x, y, z$ denote individual variables, $X, Y, Z$ are set variables, $a, b, c$ elements and $A, B, C$

sets. $\bar{a}, \bar{A}$ are finite sequences and $\lg(\bar{a}), \lg(\bar{A})$ their length. We write e.g. $\bar{a} \in C$ and $\bar{A} \subseteq C$ instead

of $\bar{a} \in \text{lg}(\bar{a}) C$ or $\bar{A} \in \text{lg}(\bar{A}) P(C)$

Definition 3.1. For any chain $C$, $\bar{A} \in \text{lg}(\bar{A}) P(C)$, and a natural number $n$, define by induction

$t = Th^n(C; \bar{A})$
for \( n = 0 \):
\[
\{\phi(\bar{X}) : \phi(\bar{X}) \in L, \phi(\bar{X})\text{ quantifier free, } C \models \phi(\bar{A})\}.
\]

for \( n = m + 1 \):
\[
\{\text{Th}^m(C; \bar{A}) : B \in \mathcal{P}(C)\}.
\]

We may regard \( \text{Th}^n(C; \bar{A}) \) as the set of \( \varphi(\bar{X}) \) that are boolean combinations of monadic formulas of quantifier depth \( \leq n \) such that \( C \models \varphi(\bar{A}) \).

**Definition 3.2.** \( T_{n,l} \) is the set of all formally possible \( \text{Th}^n(C; \bar{P}) \) where \( C \) is a chain and \( \lg(P) = l \). \( T_{n,l} \) is \( |T_{n,l}| \).

**Fact 3.3.**
(A) For every formula \( \psi(\bar{X}) \in L \) there is an \( n \) such that from \( \text{Th}^n(C; \bar{A}) \) we can effectively decide whether \( C \models \psi(\bar{X}) \). If \( n \) is minimal with this property we will write \( \text{dp}(\psi) = n \).
(B) If \( m \geq n \) then \( \text{Th}^m(C; \bar{A}) \) can be effectively computed from \( \text{Th}^n(C; \bar{A}) \).
(C) For every \( t \in T_{n,l} \) there is a monadic formula \( \psi_t(\bar{X}) \) with \( \text{dp}(\psi) = n \) such that for every \( \bar{A} \in \mathcal{P}(C), \ C \models \psi_t(\bar{A}) \iff \text{Th}^n(C; \bar{A}) = t \).
(D) Each \( \text{Th}^n(C; \bar{A}) \) is hereditarily finite, and we can effectively compute the set \( T_{n,l} \) of formally possible \( \text{Th}^n(C; \bar{A}) \).

**Proof.** Easy.

\( \Box \)

**Definition 3.4.** If \( C, D \) are chains then \( C + D \) is any chain that can be split into an initial segment isomorphic to \( C \) and a final segment isomorphic to \( D \).

If \( \langle C_i : i < \alpha \rangle \) is a sequence of chains then \( \sum_{i<\alpha} C_i \) is any chain \( D \) that is the concatenation of segments \( D_i \), such that each \( D_i \) is isomorphic to \( C_i \).

**Theorem 3.5 (composition theorem for linear orders).**
(1) If \( \lg(\bar{A}) = \lg(\bar{B}) = \lg(\bar{A}') = \lg(\bar{B}') = l \), and
\[
\text{Th}^m(C; \bar{A}) = \text{Th}^m(C'; \bar{A}') \quad \text{and} \quad \text{Th}^m(D; \bar{B}) = \text{Th}^m(D'; \bar{B}')
\]
then
\[
\text{Th}^m(C + D; A_0 \cup B_0, \ldots, A_{l-1} \cup B_{l-1}) = \text{Th}^m(C' + D'; A'_0 \cup B'_0, \ldots, A'_{l-1} \cup B'_{l-1}).
\]
(2) If for \( i < \alpha \), \( \text{Th}^m(C_i; \bar{A}_i) = \text{Th}^m(D_i; \bar{B}_i) \) where \( \bar{A}_i = \langle A'_0, \ldots, A'_{i-1} \rangle, \bar{B}_i = \langle B'_0, \ldots, B'_{i-1} \rangle \), then
\[
\text{Th}^m\left(\sum_{i<\alpha} C_i; \cup_{i<\alpha} A^i_0, \ldots, \cup_{i<\alpha} A^i_{l-1}\right) = \text{Th}^m\left(\sum_{i<\alpha} D_i; \cup_{i<\alpha} B^i_0, \ldots, \cup_{i<\alpha} B^i_{l-1}\right)
\]

**Proof.** By [Sh] Theorem 2.4 (where a more general theorem is proved), or directly by induction on \( m \).

\( \Box \)

**Definition 3.6.**
(1) \( t_1 + t_2 = t_3 \) means:
for some \( m,l < \omega \), \( t_1,t_2,t_3 \in T_{m,l} \) and if
\[
t_1 = \text{Th}^m(C; A_0, \ldots, A_{l-1}) \quad \text{and} \quad t_2 = \text{Th}^m(D; B_0, \ldots, B_{l-1})
\]
then
\[
t_3 = \text{Th}^m(C + D; A_0 \cup B_0, \ldots, A_{l-1} \cup B_{l-1}).
\]

By the previous theorem, the choice of \( C \) and \( D \) is immaterial.
(2) \( \sum_{i<\alpha} \text{Th}^m(C_i; \bar{A}_i) \) is \( \text{Th}^m(\sum_{i<\alpha} C_i; \cup_{i<\alpha} A^i_0, \ldots, \cup_{i<\alpha} A^i_{l-1}) \).

3
**Notation 3.7.**
(1) \( \text{Th}^n(C; \bar{P}, \bar{Q}) \) is \( \text{Th}^n(C; \bar{P} \wedge \bar{Q}) \).
(2) If \( D \) is a subchain of \( C \) and \( X_1, \ldots, X_{l-1} \) are subsets of \( C \) then \( \text{Th}^m(D; X_0, \ldots, X_{l-1}) \) abbreviates \( \text{Th}^m(D; X_0 \cap D, \ldots, X_{l-1} \cap D) \).
(3) For \( C \) a chain, \( a < b \in C \) and \( \bar{P} \subseteq C \) we denote by \( \text{Th}^n((a, b); \bar{P} \cap [a, b]) \) the theory \( \text{Th}^n((a, b); \bar{P}) \mid_{[a, b]} \).
(4) We will use abbreviations as \( \bar{P} \cup \bar{Q} \) for \( \langle P_0 \cup Q_0, \ldots \rangle \) and \( \cup_i \bar{P}_i \) for \( \langle \bigcup_i P_i, \ldots \rangle \) (of course we assume that all the involved sequences have the same length).
(5) We shall not always distinguish between \( \text{Th}^n((C; \bar{P}, \emptyset) \) and \( \text{Th}^n(C; \bar{P}) \).

**Theorem 3.8.** For every \( n, l < \omega \) there is \( m = m(n, l) < \omega \), effectively computable from \( n \) and \( l \), such that whenever \( I \) is a chain, for \( i \in I \) \( C_i \) is a chain, \( \bar{Q}_i \subseteq C_i \) and \( \log((Q_i)) = l \), if \( (C; \bar{Q}) = \sum_{i \in I} (C_i; \bar{Q}_i) := (\sum_{i \in I} C_i; \cup_{i \in I} \bar{Q}_i) \) and if for \( t \in \mathcal{T}_{n,l} \) \( P_t := \{ i \in I : \text{Th}^n(C_i; \bar{Q}_i) = t \} \) and \( \bar{P} := \langle P_t : t \in \mathcal{T}_{n,l} \rangle \) then from \( \text{Th}^m(\bar{I}; \bar{P}) \) we can effectively compute \( \text{Th}^n(C; \bar{Q}) \).

**Proof.** By [Sh] Theorem 2.4. \( \heartsuit \)

**Definition 3.9.**
(1) Let \( T_0, T_1 \) be disjoint trees with \( \eta_0 = \text{root}(T_0) \). Define a tree \( T \) to be the ordered sum of \( T_0 \) and \( T_1 \) by:
\[
T = T_0 \bigoplus T_1 \text{ iff } T = T_0 \cup T_1 \text{ where the partial order on } T, \prec_T, \text{ is induced by the partial orders of } T_0 \text{ and } T_1 \text{ and the (only) additional rule:}
\[
\sigma \in T_1 \Rightarrow \eta_0 \prec \sigma.
\]
(2) If \( T_0 \) doesn’t have a root then \( \prec_T \) is the disjoint union \( \prec_{T_0} \cup \prec_{T_1} \) (So \( \tau \in T_0 \& \sigma \in T_1 \Rightarrow \tau \perp \sigma \)).
(3) When \( I \) is a chain and \( T_i \) are pairwise disjoint trees for \( i \in I \) we define \( T = \bigoplus_{i \in I} T_i \) by \( T = \cup_{i \in I} T_i \) with similar rules on \( \prec = \prec_T \), namely
\[
\sigma, \tau \in T_i \Rightarrow [\sigma \prec \tau \iff \sigma \prec_{T_i} \tau]
\]
\[
[\sigma = \text{root}(T_i), i < j, \tau \in T_j] \Rightarrow \sigma \prec \tau
\]
\[
[\sigma \in T_i, \sigma \neq \text{root}(T_i), i \neq j, \tau \in T_j] \Rightarrow \sigma \perp \tau
\]

**Theorem 3.10** (composition theorem along a complete branch).
For every \( n < \omega \) there is an \( m = m(n) < \omega \), effectively computable from \( n \), such that if \( I \) is a chain and \( T_i \) are trees for \( i \in I \) then \( \langle \text{Th}^m(T_i) : i \in I \rangle \) and \( \text{Th}^m(\langle \eta_i : i \in I \rangle) \) (which is a theory of a chain) determine \( \text{Th}^m(\bigoplus_{i \in I} T_i) \).

**Proof.** See theorem 3.14. \( \heartsuit \)

Given a tree \( T \), we would like to represent it as a sum of subtrees, ordered by a branch \( B \subseteq T \). Sometimes however we may have to use a chain \( B \) that embeds \( B \).
Definition 3.11. Let $T$ be a tree $T$, $B \subseteq T$ a branch $\nu \in T$, $\eta \in B$ and $X \subseteq B$ be an initial segment without a last element.

(a) $\nu$ cuts $B$ at $\eta$ if $\eta \tau \nu$ and for every $\tau \in B$, if $-\tau \epsilon \eta$ then $-\tau \epsilon \nu$. (In particular, $\eta$ cuts $B$ at $\eta$).

(b) $\nu$ cuts $B$ at $X$ if $\eta \epsilon \nu$ for every $\eta \in X$ and $-\tau \epsilon \nu$ for every $\tau \in B \setminus X$.

(c) $B^t \subseteq P(B)$ is defined by $X \in B^t$ if $[X \ni \eta]$ for some $\eta \in B$ or $[X \subseteq B$ is an initial segment without a last element and there is $\nu \in T \setminus B$ that cuts $B$ at $X]$.

(d) Define a linear order $\leq \leq_{B^t}$ on $B^t$ by $X_0 \leq X_1$ iff $[X_0 = [\eta_0], X_1 = [\eta_1]$ and $\eta_0 \epsilon \eta_1]$ or $[X_0 \subseteq X_1]$.

Note that the statements $X \in B^t$ and $X_0 \leq_{B^t} X_1$ are expressible by monadic formulas $\psi(x, B)$ and $\psi(x, X_0, X_1, B)$.

(e) For $X \in B^t$ define $T_X := \{\nu \in T : \nu$ cuts $B$ at $X\}$.

Now $B^t$ has the disadvantage of not being a subset of $T$ and (at the small cost of adding a new parameter) we shall replace the chain $(B^t, \leq_{B^t})$ by a chain $(B, \leq_B)$ where $B \subseteq T$.

Definition 3.12. $B \subseteq T$ is obtained by replacing every $X \in B^t$ by an element $\eta_x \in T$ in the following way: if $X = [\eta]$ then $\eta_x = \eta$ and if $X \subseteq B$ is an initial segment then $\eta_x$ is a favourite element from $T_X$. $\leq_B$ is defined by $\eta_x \leq_B \eta_{x'}$ if $X_1 \leq_{B^t} X_2$ and $B^t \subseteq T$ will be $B \setminus \{\eta_x : X = [\nu], \nu \in B\}$, $(B \setminus B^t, \leq_B) \cong (B, \leq_B)$). For $\eta \in B$ let $T_\eta$ be $T_{\{\eta\}}$ as defined in (e) above, and for $\eta = \eta_x \in B^t$ let $T_\eta = T_X$ as above (in this case $T_\eta$ is $\{\nu \in T : \nu \sim_B \eta\}$ as in definition 2.5).

Fact 3.13. $\leq_B$ is definable from $B$ and $B^c$, $T_\eta$ is definable from $\eta$, $B$ and $B^c$ and $T = \bigoplus_{\eta \in B} T_\eta$ in accordance with definition 3.9.

Theorem 3.14 (Composition theorems for trees).

Assume $T$ is a tree, $B \subseteq T$ a branch and $\bar{Q} \subseteq T$ with $\text{lg}(\bar{Q}) = l$. Let $B$ and $B^c$ be defined as above, for $\eta \in B$ $T_\eta$ is defined as above (so $T = \bigoplus_{\eta \in B} T_\eta$) and $S_\eta$ is $T_\eta \setminus B$ (so, abusing notations, $T = B \cup \bigoplus_{\eta \in B} S_\eta$). Then:

1) Composition theorem on a branch: for every $n < \omega$ there is $k = k(n, l) < \omega$, effectively computable from $n$ and $l$, such that $T^k(B; B, B^c, \bar{P})$ determines $T^n(T; \bar{Q})$ where for $t \in T_{\eta, l}$, $P_t := \{\eta \in B : T^n(T_\eta; \bar{Q} \cap T_\eta) = t\}$ and $\bar{P} := \langle P_t : t \in T_{\eta, l}\rangle$.

2) Composition theorem along a branch: for every $n < \omega$ there is $k = k(n, l) < \omega$, effectively computable from $n$ and $l$, such that $T^k(B; \bar{Q})$ and $T^n(S_\eta; B, B^c, \bar{Q}) : \eta \in B)$ determine $T^n(T; \bar{Q})$.

Proof. By Theorem 1 in [GuSh]\S2.4.

Definition 3.15. Additive colouring....

Theorem 3.16 (Ramsey theorem for additive colourings). ...

Proof. By [Sh] Theorem 1.1. ✽
4. Well Orderings of Ordinals

A chain is tame iff it is scattered of Hausdorff degree $< \omega$. We will define for a tame chain $C$, $\log(C)$ and show later (in proposition 4.8) that this function is well defined.

**Definition 4.1.** Let $\log:\{\text{tame chains}\} \to \omega \cup \{\infty\}$ be defined by:

- $\log(C) = \infty$ iff there is $\varphi(x,y,\bar{P})$ that defines a well ordering on the elements of $C$ of order type $\geq \omega^\omega$.
- $\log(C) = k$ iff there is $\varphi(x,y,\bar{P})$ that defines a well ordering on the elements of $C$ of order type $\alpha$ with $\omega^k \leq \alpha < \omega^{k+1}$.

**Fact 4.2.** A tame chain $C$ has a reconstrutible well ordering i.e. there is a formula $\varphi(x,y,\bar{P})$ ($\bar{P} \subseteq C$) that defines a well ordering on the elements of $C$ of order type $\alpha$ and there is a formula $\psi(x,y,\bar{Q})$ ($\bar{Q} \subseteq \alpha$) that defines a linear order $<^*$ on the elements of $\alpha$ such that $(\alpha, <^*) \cong (C, <)$.

**Proof.** By induction on $\hdeg(\alpha)$, using the proof of Theorem A1 in the appendix.

**Definition 4.3.** Let $\alpha, \beta$ be ordinals. $\alpha \to \beta$ means the following: “there is $\varphi(x,y,\bar{P})$ that defines a well ordering on the elements of $\alpha$ of order type $\beta$”.

**Claim 4.4.**

1) $\alpha \to \beta$ & $\beta \to \gamma \Rightarrow \alpha \to \gamma$.
2) $\alpha \to \gamma$ & $\gamma \geq \alpha \cdot \omega \Rightarrow \alpha \to \alpha \cdot \omega$.

**Proof.** Straightforward.

**Notation.** Suppose $\alpha \to \beta$ holds by $\varphi(x,y,\bar{P})$. Define a bijection $f: \alpha \to \beta$ by $f(i) = j$ iff $i$ is the $j$'th element in the well order defined by $\varphi$.

**Lemma 4.5.** For any ordinal $\alpha$, $\alpha \not\to \alpha \cdot \omega$.

**Proof.** Assume that $\alpha$ is minimal such that $\alpha \to \alpha \cdot \omega$. It follows that:

- (i) $\alpha \geq \omega$,
- (ii) $\alpha$ is a limit ordinal (by $\alpha \to \alpha+1$ and 2.7),
- (iii) for $\beta < \alpha$, $\{f(i): i < \beta\}$ does not contain a final segment of $\alpha \cdot \omega$ (otherwise clearly $\beta \to \alpha \cdot \omega$ hence by 2.7 $\beta \to \alpha \cdot \omega$ but $\alpha$ is minimal).

So let $\varphi(x,y,\bar{P})$ define a well order of $\alpha$ of order type $\alpha \cdot \omega$ and let $Q \subseteq \alpha$ be the following subset:

$x \in Q$ iff for some $k < \omega$, $\alpha \cdot 2k \leq f(x) < \alpha \cdot (2k+1)$. Let $E$ an equivalence relation on $\alpha$ defined by $xEy$ if for some $l < \omega$, $f(x)$ and $f(y)$ belong to the segment $[\alpha \cdot l, \alpha \cdot (l+1))$. Clearly there is a monadic formula $\psi(x,y,\bar{P},Q)$ that defines $E$ moreover, some monadic formula $\theta(X,Y,Z)$ expresses the statement “$\bigvee_{i < \omega} (X = Q_i)$” where $\langle Q_i : i < \omega \rangle$ are the $E$-equivalence classes.

Let $n := \max\{dP(\varphi), dP(\psi), dP(\theta)\} + 5$, and

Let $\delta = \text{cf}(\alpha)$ and $\{x_i\}_{i < \delta}$ be stricly increasing and cofinal in $\alpha$. By [Sh]Theorem 1.1 applied to the colouring $h(i,j) = \text{Th}^n(\alpha; \bar{P}, Q, x_i, x_j)$ we get a cofinal subsequence $\{j_i\}_{i < \delta}$ such that $\text{Th}^n(\alpha; \bar{P}, Q, \beta_{j_i}, \beta_{j_i})$ is constant for $j_1 < j_2 < \delta$. Note that it follows

- $\uparrow$ the theories $\text{Th}^n(\alpha; \bar{P}, Q) \mid_{[0,\beta_{j_1}]}$, $\text{Th}^n(\alpha; \bar{P}, Q) \mid_{[\beta_{j_1}, \alpha]}$, and $\text{Th}^n(\alpha; \bar{P}, Q, \beta_{j_1}) \mid_{[\beta_{j_1}, \beta_{j_2}]}$ are constant for every $j < \delta$ and for every $j_1 < j_2 < \delta$.  

\[ \vspace{1cm} \]
Note that each $E$-equivalence class $Q_i$ is unbounded in $\alpha$ since if some $\beta < \alpha$ contains some $E$-equivalence class $Q_i$, it would easily follow that $\beta \to \alpha$ contradicting fact (iii).

Fix some $1 < j < \delta$ let $x < \beta_j$ and let $Q_i(x)$ be the $E$-equivalence class containing $x$. Since $Q_i(x)$ is unbounded in $\alpha$ there is some $j < l < \delta$ such that $[\beta_j, \beta_l) \cap Q_i(x) \neq \emptyset$. This statement is expressible by $\text{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_l)$ which is equal to

$$\text{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_l) |_{[0, \beta_j)} + \text{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_l) |_{[\beta_j, \beta_l)} + \text{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_l) |_{[\beta_l, \alpha)} = \text{Th}^n(\alpha; \bar{P}, Q, x, 0, \emptyset, \emptyset) |_{[0, \beta_j)} + \text{Th}^n(\alpha; \bar{P}, Q, x, 0, \emptyset, \emptyset) |_{[\beta_j, \beta_l)} + \text{Th}^n(\alpha; \bar{P}, Q, x, 0, \emptyset, \emptyset) |_{[\beta_l, \alpha)}.$$

By (i) we may replace the second theory by $\text{Th}^n(\alpha; \bar{P}, Q, \emptyset, \emptyset, \emptyset) |_{[\beta_j, \beta_l)}$ and the third theory by $\text{Th}^n(\alpha; \bar{P}, Q, \emptyset, \emptyset, \emptyset) |_{[\beta_l, \alpha)}$, and conclude:

$$\text{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_l) = \text{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_{l+1})$$

Therefore for every $x < \beta_j$, $[\beta_j, \beta_{j+1}) \cap Q_i(x) \neq \emptyset$.

Finally, let $j < \delta$ be such that the segment $[0, \beta_j)$ intersects $m + 1$ different $E$-equivalence classes, say $Q_{i_0}, \ldots, Q_{i_m}$. By the previous argument we have $[\beta_j, \beta_{j+1} \cap Q_{i_l} \neq \emptyset$ for every $l \leq m$.

By the choice of $m$ there are different $a, b \in \{i_0, \ldots, i_m\}$ such that

*(+) $\text{Th}^n(\alpha; \bar{P}, Q, Q_a) |_{[\beta_j, \beta_{j+1})} = \text{Th}^n(\alpha; \bar{P}, Q, Q_b) |_{[\beta_j, \beta_{j+1})}.$

Let $\beta \in \alpha$ be $((0, \beta_j) \cap Q_a) \cup ((\beta_j, \beta_{j+1}) \cap Q_a) \cup (\beta_{j+1}, \alpha) \cap Q_a)$. Now $\text{Th}^n(\alpha, \bar{P}, Q, R) =$

$$\text{Th}^n(\alpha; \bar{P}, Q, Q_a) |_{[0, \beta_j)} + \text{Th}^n(\alpha; \bar{P}, Q, Q_a) |_{[\beta_j, \beta_{j+1})} + \text{Th}^n(\alpha; \bar{P}, Q, Q_a) |_{[\beta_{j+1}, \alpha)} = \text{Th}^n(\alpha; \bar{P}, Q, Q_a) |_{[0, \beta_j)} + \text{Th}^n(\alpha; \bar{P}, Q, Q_a) |_{[\beta_j, \beta_{j+1})} + \text{Th}^n(\alpha; \bar{P}, Q, Q_a) |_{[\beta_{j+1}, \alpha)} = \text{Th}^n(\alpha; \bar{P}, Q, Q_a) |_{[0, \beta_j)} + \text{Th}^n(\alpha; \bar{P}, Q, Q_a) |_{[\beta_j, \beta_{j+1})} + \text{Th}^n(\alpha; \bar{P}, Q, Q_a) |_{[\beta_{j+1}, \alpha)} = \text{Th}^n(\alpha; \bar{P}, Q, Q_a).

But $Q_a$ is an $E$-equivalence class while $R$ is not. Since $\text{Th}^n(\alpha, \bar{P}, Q, Z)$ computes the statement “$Z$ is $E$-equivalence class” we get a contradiction from $\text{Th}^n(\alpha; \bar{P}, Q, R) = \text{Th}^n(\alpha, \bar{P}, Q, Q_a)$.

$\diamond$

**Claim 4.6.** If $\alpha \to \beta$ and $\beta < \alpha$ then $(\exists \gamma_1, \gamma_2)((\gamma_1 + \gamma_2 = \alpha) \& (\gamma_2 + \gamma_1 = \beta))$.

**Proof.** Let’s prove first:

Subclaim: $\omega + \omega \not\to \omega$.

Proof of the subclaim: Assume that $\varphi(x, y, \bar{P})$ well orders $\omega + \omega$ of order type $\omega$ and that $dp(\varphi) = n$, $l(\bar{P}) = l$. Let $x <^* y$ mean $(\omega + \omega, <) \models \varphi(x, y, \bar{P})$. Using Ramsey theorem (and as $<^*$ is well founded) we may assume that $i_1 < i_2 \Rightarrow x_{i_1} <^* x_{i_2}$ and $j_1 < j_2 \Rightarrow y_{j_1} <^* y_{j_2}$.

We will show now that for $0 < i < \omega$ and $0 < j < \omega$, $\text{Th}^n(\omega + \omega; x_i, y_j, \bar{P})$ is constant. Indeed,
Now proceed as before: choose a contradiction.

\( \beta \)

then by (\( \ast \)) \( \gamma + \beta = \gamma + \gamma \) and \( \alpha \) does not witness the weirdness of \( \beta \), so \( \alpha \geq \beta + \beta \).

Let \( \varphi(x, y, P) \) well order \( \alpha \) of order type \( \beta \) with \( \text{dp}(\varphi) = n \) and \( l(P) = l \). As above \( x <^* y \) means \( (\alpha, <) \models \varphi(x, y, P) \) and finally let \( \delta = \text{cf}(\beta) \).

Now \( \text{otp}(\alpha, <^* ) = \beta \) but what is \( \text{otp}(0, \beta), <^*|_{[0, \beta)} ) \)? Clearly, as \( \text{Th}^n(\alpha, P) = \text{Th}^n(\alpha, P) \upharpoonright_{[0, \beta)} + \text{Th}^n(\alpha, P) \upharpoonright_{[\beta, \alpha)} \) we have \( \beta \to \text{otp}(0, \beta), <^*|_{[0, \beta)} \) hence \( \beta = \text{otp}(0, \beta), <^*|_{[0, \beta)} \) (otherwise, by (\( \ast \)), \( \text{otp}(0, \beta), <^*|_{[0, \beta)} \) is weird and \( < \beta \)). Similarly we can show that \( \text{otp}(\beta, \beta + \beta), <^*|_{[\beta, \beta + \beta)} = \beta \).

\( \to \) Insert Ramsey theorems

Now proceed as before: choose \( \{ x_i \}_{i < \delta} \subseteq [0, \beta) \) and \( \{ y_j \}_{j < \delta} \subseteq [\beta, \beta + \beta) \) that are homogeneous unbounded and \( <^* \) unbounded and use them to show that \( \text{otp}(\alpha, <^* ) \geq \beta + 1 \).

Second case: (\( \ast \ast \)) holds i.e. \( \beta = \gamma + \gamma \).

Call \( \epsilon \) quite weird if for some \( k < \omega \) \( \epsilon \cdot k \) is weird. Let \( \epsilon \leq \gamma \) be the first quite weird ordinal. Let \( k_1 \) be the first such that \( \epsilon \cdot k_1 \) is weird. Look at \( \gamma \): if \( \gamma = \gamma_1 + \gamma_2 \) and \( \gamma_2 + \gamma_1 < \gamma \) we would have \( \alpha \to \beta = \gamma + \gamma \to \gamma + \gamma_2 + \gamma_1 < \beta \) and a contradiction. Hence either \( \gamma_1 < \gamma \Rightarrow (\gamma_1 + \gamma_1 < \gamma) \) and in this case \( \gamma = \epsilon \) or \( \gamma = \gamma_1 + \gamma_1 \). Repeat the same argument to get \( \gamma_1 = \epsilon \) or \( \gamma_1 = \gamma_2 + \gamma_2 \). After finitely many steps we are bound to get \( \beta = \epsilon \cdot 2k \) where \( 2k = k_1 \) and \( \epsilon_1 < \epsilon \Rightarrow \epsilon_1 \cdot \omega \leq \epsilon \) and of course \( \epsilon_1 < \epsilon \Rightarrow \epsilon \neq \epsilon_1 \).

Let \( \varphi(x, y, P) \) and \( <^* \) be as usual and \( \delta := \text{cf}(\beta) = \text{cf}(\epsilon) \). Let \( \alpha = \beta + \epsilon^* \) if \( \epsilon^* < \epsilon \) then \( \epsilon^* + \beta = \beta \) and \( \alpha \) doesn’t witness weirdness, therefore \( \epsilon^* \geq \epsilon \).

Proceed as before: choose \( \{ x_i^0 \}_{i < \delta}, \{ x_i^1 \}_{i < \delta}, \ldots, \{ x_i^l \}_{i < \delta} \) with \( \{ x_i^l \}_{i < \delta} \subseteq [\epsilon \cdot l, \epsilon(l + 1)) \), homogeneous, unbounded and \( <^* \) increasing.

By the composition theorem it will follow that \( \text{otp}(\epsilon \cdot l, \epsilon(l + 1)), <^* ) \geq \epsilon \) and by homogeneity we will have, for \( 0 < i, j < \omega \) and \( l \leq k \), \( x_i^l <^* x_j^{l+1} \). It follows that \( \text{otp}(\alpha, <^* ) \geq (\epsilon \cdot k) + 1 = \beta + 1 \) and a contradiction.
Theorem 4.7. Well ordering of ordinals are obtained only by the following process:

let \((P_0, P_1, \ldots, P_{n-1})\) be a partition of \(\alpha\) and

\[ i <^* j \iff [(\exists k < n)(i \in P_k \& j \in P_k \& i < j)] \lor [i \in P_k, \& j \in P_{k_2} \& k_1 < k_2]. \]

Proposition 4.8. \(\Log(C)\) is well defined.

Proof. Let \((C, <^*)\) be a scattered chain and let \((\alpha, <)\) and \((\beta, <)\) be results of a definable well orderings of \((C, <^*)\) where in addition (by 4.2) there is \(\psi(x, y, \bar{Q})\) that defines \(C\) in \(\alpha\). So \(\alpha \rightarrow \beta\) and by 4.5 and 4.6 \(\alpha < \omega^\omega \iff \beta < \omega^\omega\ and \alpha \in [\omega^k, \omega^{k+1}) \iff \beta \in [\omega^k, \omega^{k+1}).\)

5. \((\omega^\omega, <)\) and longer chains

The following lemma is a part of Theorem 3.5(B) in [Sh]:

Lemma 5.1. Let \(I\) be a well ordered chain of order type \(\geq \omega^k\). Let \(f: I^2 \rightarrow \{t_0, t_1, \ldots, t_{l-1}\}\) be an additive colouring and assume that for \(\alpha < \beta \in I\), \(f(\alpha, \beta)\) depends only on the order type in \(I\) of the segment \([\alpha, \beta)\).

Then there is \(i < l\) such that for some \(p \leq l\), for every \(r \geq p\), if \(\otp([\alpha, \beta)) = \omega^r\) then \(f(\alpha, \beta) = t_i\). Moreover, \(t_i + t_i = t_i\).

Proof. To avoid triviality assume \(k > l\). For \(\alpha < \beta\) in \(I\) with \(\ otp([\alpha, \beta)) = \delta\), denote \(f(\alpha, \beta)\) by \(t(\delta)\) (makes sense by the assumptions).

By the pigeon-hole principle there are \(1 \leq p \leq l, s > p\) and some \(t_i\) with \(t(\omega^p) = t(\omega^s) = t_i\). Now \(\omega^{p+2} = \sum_{i<\omega}(\omega^{p+1} + \omega^p)\) and by the additivity of \(f\):

\[ t(\omega^{p+2}) = t(\sum_{i<\omega}(\omega^{p+1} + \omega^p)) = \sum_{i<\omega} t(\omega^{p+1} + \omega^p) = \sum_{i<\omega} (t(\omega^{p+1}) + t(\omega^p)) = \sum_{i<\omega} t(\omega^{p+1}) + t(\omega^p) = t(\sum_{i<\omega} \omega^p) = t(\omega^{p+1}). \]

Hence

\[ t(\omega^{p+2}) = t(\omega^{p+1}). \]

Using this and as \(\omega^{p+3} = \sum_{i<\omega}(\omega^{p+2} + \omega^{p+1})\) we have

\[ t(\omega^{p+3}) = t(\sum_{i<\omega}(\omega^{p+2} + \omega^{p+1})) = \sum_{i<\omega} t(\omega^{p+2} + \omega^{p+1}) = \sum_{i<\omega} (t(\omega^{p+2}) + t(\omega^{p+1})) = \]

\[ \sum_{i<\omega} t(\omega^{p+1}) + t(\omega^{p+1}) = \sum_{i<\omega} t(\omega^{p+1}) = t(\sum_{i<\omega} \omega^{p+1}) = t(\omega^{p+2}). \]

Hence

\[ t(\omega^{p+3}) = t(\omega^{p+2}). \]
So for every \( j > 0 \), \( t(\omega^{p+1}) = t(\omega^{p+j}) \) and in particular \( t(\omega^{p+1}) = t(\omega^p) = t_i \).

This proves the first part of the lemma. As for the moreover clause, since \( \omega^{p+1} = \omega^p + \omega^{p+1} \) we have
\[
t_i = t(\omega^{p+1}) = t(\omega^p + \omega^{p+1}) = t(\omega^p) + t(\omega^{p+1}) = t_i + t_i.
\]

\[\blacklozenge\]

**Proposition 5.2.** The formula \( \varphi(X, Y) \) saying “if \( Y \) is without a last element then \( X \subseteq Y \) is an \( \omega \)-sequence unbounded in \( Y \) (and if not then \( X = \emptyset \))” can not be uniformized in \( (\omega^\omega, <) \).

Moreover, if \( \psi_m(X, Y, \bar{P}_m) \) uniformizes \( \varphi \) on \( \omega^m \) then one of the sets \( \{dp(\psi_m) : m < \omega\} \) or \( \{lg(\bar{P}_m) : m < \omega\} \) is unbounded.

**Proof.** Suppose the second statement fails, then:

(†) there is a formula \( \psi(X, Y, Z) \) such that for an unbounded set \( I \subseteq \omega \), for every \( m \in I \) there is \( \bar{P}_m \subseteq \omega^m \) such that \( \psi(X, Y, \bar{P}_m) \) uniformizes \( \varphi \) on \( \omega^m \).

Let \( \bar{P}_m = \bar{P} \) let \( n = dp(\bar{V}) + 1 \) and \( M := |\{Th^n(C; X, Y, Z) : C \text{ a chain }, X, Y, Z \subseteq C, lg(Z) = lg(\bar{P})\}| \).

Let \( m \in I \) be large enough \( (m > 2M + 3 \text{ will do}) \), and let’s show that \( \psi \) doesn’t work for \( \omega^m \) and a subset \( Y_k \) that will be defined now.

If \( \alpha < \omega^m \) then \( \alpha = \omega^{m-1}k_{m-1} + \omega^{m-2}k_{m-2} + \ldots + \omega k_1 + k_0 \). Let \( k(\alpha) := \min\{i : k_i \neq 0\} \) and let \( A_k := \{\alpha < \omega^m : k(\alpha) = k\} \). Note that \( otp(A_k) = \omega^{m-k} \).

For \( k \in \{1, 2, \ldots, m-1\} \) we will choose \( Y_k \subseteq A_k \) with \( otp(Y_k) = otp(A_k) = \omega^{m-k} \) such that for \( \alpha < \beta \) in \( Y_k \):

\[(*) \quad Th^n(\omega^m; \bar{P}, Y_k) \upharpoonright_{[\alpha, \beta)} \text{ depends only on otp}((\alpha, \beta) \cap Y_k)\]

we will start with \( k = m - 1 \) and proceed by inverse induction:

Let \( A_{m-1} = \{\alpha_j : j < \omega\} \). Let for \( l < p < \omega \), \( h(l, p) := Th^n(\omega^m; \bar{P}, \alpha_l) \upharpoonright_{[\alpha_l, \alpha_p]} \). Let \( J \subseteq \omega \) be homogeneous with respect to this colouring namely, for some fixed theory \( t_{m-1} \), for every \( l < p \) in \( J \),

\[Th^n(\omega^m; \bar{P}, \alpha_l) \upharpoonright_{[\alpha_l, \alpha_p]} = t_{m-1} \bigl.\bigr|_{[\alpha_l, \alpha_p]}\]

By the composition theorem, for every \( l < p \) in \( J \),

\[Th^n(\omega^m; \bar{P}, Y_{m-1}) \upharpoonright_{[\alpha_l, \alpha_p]} = t_{m-1} \cdot |Y_{m-1} \cap [\alpha_l, \alpha_p]|\]

and this proves \( (*) \) for \( Y_{m-1} \).

Rename \( Y_{m-1} \) by \( (\alpha_i) : i < \omega \). In each segment \( [\alpha_i, \alpha_{i+1}) \) choose \( \langle \beta_i^l : 0 < l < \omega \rangle \subseteq A_{m-2} \) increasing and cofinal such that for every \( l < p < \omega \) the theory \( Th^n(\omega^m; \bar{P}, \beta_i^l) \upharpoonright_{[\beta_i^l, \beta_i^p]} \) is constant.

Returning to \( Y_{m-1} \), for \( i < j < \omega \) let

\[h_1(i, j) := \langle Th^n(\omega^m; \bar{P}) \upharpoonright_{[\alpha_i, \beta_j^{i-1}]}, Th^n(\omega^m; \bar{P}, \beta_j^{i-1}) \upharpoonright_{[\beta_j^{i-1}, \beta_j^j]} \rangle\]

w.l.o.g. (by thinning out and re-renaing and noting that we don’t harm \( (*) \)) \( Y_{m-1} \) is homogeneous with respect to this colouring.

Hence, for some theories \( t^* \) and \( t_{m-2} \), for every \( i < j < \omega \) we have

\[h_1(i, j) = \langle t^*, t_{m-2} \rangle\]
Let $Y_{n-2} := \langle \beta^n : 0 < l < \omega, i < \omega \rangle$, clearly \text{otp}(Y_{n-2}) = \omega^2$. Let's check (*) for $Y_{n-2}$:

Firstly, note that for $l < p < \omega$,

$$\text{Th}^n(\omega^m; \bar{P}, Y_{n-2}) |_{[\beta_l, \beta_p]} = t_{m-2} \cdot (p - l).$$

Secondly, for $i < j < \omega$ \text{Th}^n(\omega^m; \bar{P}, Y_{n-2}) |_{[\beta_i, \beta_j]} = \text{Th}^n(\omega^m; \bar{P}, Y_{n-2}) |_{[\alpha_i, \alpha_{i+1}]} + \text{Th}^n(\omega^m; \bar{P}, Y_{n-2}) |_{[\alpha_{i+1}, \alpha_{i+2}]} + \cdots + \text{Th}^n(\omega^m; \bar{P}, Y_{n-2}) |_{[\alpha_{j-1}, \alpha_j]} + \text{Th}^n(\omega^m; \bar{P}, Y_{n-2}) |_{[\alpha_j, \beta_j]}$$

where the first theory is equal to $t_{m-2} \cdot \omega$, the last theory is $t^* + t_{m-2} \cdot (p - l)$, and the middle theories are $t^* + t_{m-2} \cdot \omega$. These observations prove (*) for $Y_{n-2}$.

For defining $Y_{n-3}$ let's restrict ourselves to a segment $[\alpha_i, \alpha_{i+1}]$ where $\alpha_i, \alpha_{i+1} \in Y_{n-1}$. In this segment we have defined $[\beta_l : 0 < l < \omega] \subseteq Y_{n-2}$. Now choose in each $[\beta_l, \beta_{l+1}]$ an increasing cofinal sequence $\langle \gamma_j^i : 0 < j < \omega \rangle$ such that for $j < p < \omega$, $\text{Th}^n(\omega^m; \bar{P}, \gamma_j^i) |_{[\gamma_j^i, \beta_j^i]}$ is constant.

For $0 < l < p < \omega$ let

$$h_1^l(l, p) := \langle \text{Th}^n(\omega^m; \bar{P}) |_{[\beta_l, \gamma_j^l]}, \text{Th}^n(\omega^m; \bar{P}, \gamma_j^{l-1}) |_{[\gamma_j^l, \beta_j^l]} \rangle$$

and again w.l.o.g we may assume that $\langle \beta_l^i : 0 < l < \omega \rangle$ is homogeneous with respect to $h_1^l$.

Next, for $i < j < \omega$ define

$$h_2^l(i, j) := \langle \text{Th}^n(\omega^m; \bar{P}) |_{[\alpha_i, \gamma_j^{i-1}]}, \text{Th}^n(\omega^m; \bar{P}, \gamma_j^{i-1}) |_{[\gamma_j^{i-1}, \beta_j^{i-1}]} \rangle$$

by thinning out and renaming we may assume that $Y_{n-1}$ is homogeneous with respect to $h_2$, now $Y_{n-2}$ is also thinned out but each new $\langle \beta_l^i : 0 < l < \omega \rangle$ which is some old $\langle \beta_l^i : 0 < l < \omega \rangle$ is still homogeneous.

As a result we will have, for some theories $t^*, t^{***}, t_{m-3}$:

$$(\forall i < j < \omega)(\forall 0 < l < p < \omega)[h_1^l(l, p) = \langle t^*, t_{m-3} \rangle \& h_2^l(i, j) = \langle t^{***}, t_{m-3} \rangle].$$

Let $Y_{m-3} := \{ \gamma_j^i : i < \omega, 0 < l < \omega, 0 < j < \omega \}$, as before (*) holds by noting that if for example $i_1 < i_2 < \omega$ and $1 < l_2$ then

$$\text{Th}^n(\omega^m; \bar{P}, \gamma_j^{i_1,i_2}) |_{[\gamma_j^{i_1,i_2}, \gamma_j^{i_2,i_2}]} = t_{m-3} \cdot \omega + (t^* + t_{m-3} \cdot \omega) \cdot \omega + [t^{***} + (t^* + t_{m-3} \cdot \omega) \cdot \omega] \cdot (i_2 - i_1 - 1) + t^{***} + t_{m-3} \cdot \omega + (t^* + t_{m-3} \cdot \omega)(l_2 - 1) + t^* + t_{m-3} \cdot (j_2 - 1)$$

and similarly for the other possibilities.

$Y_{m-4}, Y_{m-5}, \ldots, Y_1$ are defined by using the same prescription i.e. $Y_{n-1}$ is defined by taking a homogenous sequence between two successive elements of $Y_{n-1}$ then homogenous sequences between two successive elements of $Y_{n-2}$ by using colouring of the form $h_1, h_2, \ldots$. The thinning out and w.l.o.g’s for already defined $Y_{n-k}$’s are not necessary but they ease notations considerably.

We will show now that $\psi$ doesn’t choose an unbounded $\omega$-sequence in $Y_1$ that is, for every $\omega$-sequence $X \subseteq Y_1$ there is an $\omega$-sequence $X' \subseteq Y_1$ such that $\text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) = \text{Th}^n(\omega^m; \bar{P}, Y_1, X')$. 

11
By (\ast), for \( \alpha < \beta \) in \( Y_1 \) the additive colouring \( f(\alpha, \beta) := \text{Th}^n(\omega^m; \bar{P}, Y_1) \upharpoonright_{(\alpha, \beta)} \) depends only on otp\((\alpha, \beta) \cap Y_1) \) hence we can apply lemma 5.1 and conclude that for some \( p \leq m/2 \), for every \( r \geq p \), Th\(^n(\omega^m; \bar{P}, Y_1) \upharpoonright_{(\alpha, \beta)} \) is equal to some fixed theory \( t \) whenever otp\((\alpha, \beta) \cap Y_1) = \omega^r \). (Remember that \( f \) has at most \( M \) possibilities and that \( m > 2M \)). Moreover, we know that \( t + t = t \).

Assume now that for some \( X \subseteq Y_1 \), \( \psi(X, Y, \bar{P}) \) holds, so \( X \) is a cofinal \( \omega \)-sequence. Let \( X = \{ \delta_i : i < \omega \} \). As otp\((Y_1) = \omega^{m-1} \) for unboundedly many \( i \)'s we have otp\((\delta_i, \delta_{i+1}) \cap Y_1) \geq \omega^{m-2} > \omega^p \).

Let \( \beta_i := \text{otp}(\delta_i, \delta_{i+1}) \cap Y_1 \) and denote by \( t(\epsilon) \) the theory Th\(^n(\omega^m; \bar{P}, Y_1) \upharpoonright_{(\alpha, \beta)} \) when otp\((\alpha, \beta) \cap Y_1) = \epsilon \) (by (\ast) it doesn’t matter which \( \alpha \) and \( \beta \) we use).

We are interested in Th\(^{n-1}(\omega^m; \bar{P}, Y_1, X) \) which is

\[
\text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, 0) \upharpoonright_{[0, \delta_0)} + \text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_0) \upharpoonright_{[\delta_0, \delta_1)} + \text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_1) \upharpoonright_{[\delta_1, \delta_2)} + \ldots
\]

As \( \delta_i \) is the first element in \( [\delta_i, \delta_{i+1}) \cap Y_1 \), Th\(^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \upharpoonright_{[\delta_i, \delta_{i+1})} \) is determined by Th\(^n(\omega^m; \bar{P}, Y_1) \upharpoonright_{(\delta_i, \delta_{i+1})} = t(\beta_i) \) and abusing notations we will say

\[
(**) \quad \text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) \simeq t(\delta_0) + \sum_{i < \omega} t(\beta_i).
\]

Let \( i < \omega \) be such that \( \beta_i \geq \omega^{m-2} \) and let \( j > i \) be the first with \( \beta_j \geq \omega^{m-2} \).

First case: \( i+1 = j \).

Let \( \beta_i := \text{otp}(\delta_i, \delta_{i+1}) \cap Y_1) = \omega^{m-2} \cdot k_1 + \epsilon_1 \) and \( \beta_{i+1} = \text{otp}(\delta_{i+1}, \delta_{i+2}) \cap Y_1) = \omega^{m-2} \cdot k_2 + \epsilon_2 \) where \( k_1, k_2 \geq 1 \) and \( \epsilon_1, \epsilon_2 < \omega^{m-2} \).

Define \( \gamma := \text{the } \omega^{m-2} \cdot k_1 + \omega^{m-3} + \epsilon_1 \text{'th successor of } \delta_i \text{ in } Y_1 \). So \( \delta_{i+1} < \gamma < \delta_{i+2} \) but otp\((\delta_{i+1}, \delta_{i+2}) \cap Y_1) = \beta_{i+1} \) hence

\[
\text{Th}^n(\omega^m; \bar{P}, Y_1) \upharpoonright_{[\gamma, \delta_{i+2})} = \text{Th}^n(\omega^m; \bar{P}, Y_1) \upharpoonright_{[\delta_{i+1}, \delta_{i+2})} = t(\beta_{i+1})
\]

hence

\[
\text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \gamma) \upharpoonright_{[\gamma, \delta_{i+2})} = \text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_{i+1}) \upharpoonright_{[\delta_{i+1}, \delta_{i+2})}.
\]

On the other hand,

\[
\text{Th}^n(\omega^m; \bar{P}, Y_1) \upharpoonright_{[\delta_i, \gamma)} = t(\omega^{m-2} \cdot k_1) + t(\omega^{m-3}) + t(\epsilon_1)
\]

but \( m - 3 \geq p \) hence \( t(\omega^{m-3}) = t(\omega^{m-2}) = t \) moreover \( t + t = t \) and it follows that

\[
t(\omega^{m-2} \cdot k_1) + t(\omega^{m-3}) = t(\omega^{m-2}) \cdot k_1 + t(\omega^{m-3}) = t(\omega^{m-2}) \cdot (k_1 + 1) = t(\omega^{m-2}) \cdot (k_1) = t(\omega^{m-2} \cdot k_1)
\]

hence

\[
\text{Th}^n(\omega^m; \bar{P}, Y_1) \upharpoonright_{[\delta_i, \gamma)} = t(\omega^{m-2} \cdot k_1) + t(\epsilon_1) = \text{Th}^n(\omega^m; \bar{P}, Y_1) \upharpoonright_{[\delta_i, \delta_{i+1})} = t(\beta_{i+1})
\]

hence

\[
\text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \upharpoonright_{[\delta_i, \gamma)} = \text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \upharpoonright_{[\delta_i, \delta_{i+1})}.
\]

Now all other relevant theories are left unchanged therefore, letting \( X' := X \setminus \{ \delta_{i+1} \} \cup \{ \gamma \} \) we get \( X \neq X' \) but

\[
\text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) = \text{Th}^n(\omega^m; \bar{P}, Y_1, X')
\]

General case: \( j = i + l \).
Look at \( \delta_{i+1}, \delta_{i+2}, \ldots, \delta_{i+t-1}, \delta_{i+t} = \delta_j \). We’ll define \( \gamma_1, \gamma_2, \ldots, \gamma_t \) with \( \delta_{i+k} < \gamma_k < \delta_{i+k+1} \) for \( 0 < k < l \) and \( \gamma_l = \delta_i + 1 = \delta_j \). This will be done by ‘shifting’ the \( \delta_{i+k} \)’s by \( \omega^{m-3} \) (remember that \( \beta_{i+k} < \omega^{m-2} \) for \( 0 < k < l \)).

Assume as before that \( \beta_i = \text{otp}(\{\delta_i, \delta_{i+1}\} \cap Y_1) = \omega^{m-2} \cdot k_1 + \epsilon_1 \) where \( k_1 \geq 1 \) and \( \epsilon_1 < \omega^{m-2} \).

Define \( \gamma_1 := \omega^{m-2} \cdot k_1 + \omega^{m-3} + \epsilon_1 \)'th successor of \( \delta_i \) in \( Y_1 \), \( \gamma_2 := \beta_{i+1} \)'th successor of \( \gamma_1 \) in \( Y_1 \), \( \gamma_3 := \beta_{i+2} \)'th successor of \( \gamma_2 \) in \( Y_1 \) and so on, \( \gamma_i \) will clearly be equal to \( \delta_j \).

As before we have for \( 1 < k \leq l \), (by preserving the order types)

\[
\text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \gamma_k) |_{[\gamma_k, \gamma_{k+1})} = \text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_{i+k}) |_{[\delta_{i+k}, \delta_{i+k+1})} .
\]

and (using \( t + t = t \))

\[
\text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) |_{[\delta_i, \gamma_1)} = \text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) |_{[\delta_i, \delta_{i+1})} .
\]

Letting \( X' := X \setminus \{\delta_{i+1}, \delta_{i+2}, \ldots, \delta_{j-1}\} \cup \{\gamma_1, \gamma_2, \ldots, \gamma_{l-1}\} \) we get \( X \not= X' \) but

\[
\text{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) = \text{Th}^n(\omega^m; \bar{P}, Y_1, X') .
\]

Since \( \text{dp}(\psi) = n - 1 \), \( X \) is not the unique \( \omega \)-sequence chosen by \( \psi \) from \( Y_1 \). Therefore, \( \psi \) does not uniformize \( \phi \) on \( \omega^m \), a contradiction.

[complete, using composition theorem, for \( \omega^n \)]

\[\heartsuit\]

**Theorem 5.3.** If \( C \) has the uniformization property then \( \text{Log}(C) < \omega \).

\[\heartsuit\]

### 6. Very Tame Trees

**Proposition 6.1.** If the ordinals \( \alpha \) and \( \beta \) have the uniformization property then so do \( \alpha + \beta \) and \( \alpha \cdot \beta \).

**Proof.** \( \alpha + \beta \) is similar to \( \alpha + \alpha = \alpha \cdot 2 \) and we leave it to the reader. We shall prove that \( \alpha \cdot \beta \) has the uniformization property.

Let \( \varphi(X, Y, \bar{Q}) \) be p.n in \( \alpha \beta \) with \( \text{dp}(\varphi) = n \) and \( \text{lg}(\bar{Q}) = l \). Let \( \langle t_0, \ldots, t_{a-1} \rangle \) be an enumeration of the the theories in \( T_{a, l+2} \). For \( i < a \) and \( X, Y \subseteq \alpha \beta \) define \( P_i(X, Y, \bar{Q}) \subseteq K := \{\alpha \gamma : \gamma < \beta\} \) by

\[
P_i(X, Y, \bar{Q}) := \{\alpha \gamma : \text{Th}^n(\alpha \beta; X, Y, \bar{Q}) |_{[\alpha \gamma, \alpha \gamma + \alpha)} = t_i\}
\]

it follows that, for every \( X, Y \subseteq \alpha \beta \) \( \bar{P} = \bar{P}(X, Y, \bar{Q}) = \langle P_0(X, Y, \bar{Q}), \ldots, P_{a-1}(X, Y, \bar{Q}) \rangle \) is a partition of \( K \) that is definable from \( X, Y, \bar{Q} \) and \( K \).

\( \alpha \cdot \beta = \sum_{\gamma < \beta}[\alpha \gamma, \alpha \gamma + \alpha] \) and by theorem 3.8 there is \( m = m(n, l) \) such that \( \text{Th}^n(K; \bar{P}(X, Y, \bar{Q})) \) determines \( \text{Th}^n(\alpha \beta; X, Y, \bar{Q}) \).

Let \( R = \{r_0, \ldots, r_{c-1}\} \) be the set of theories that satisfy, for every \( X, Y \subseteq \alpha \beta \):

\( \text{Th}^n(K; \bar{P}(X, Y, \bar{Q})) \in R \Rightarrow \alpha \beta \models \varphi(X, Y, \bar{Q}) \).
Now let \( \langle s_0, \ldots, s_{b-1} \rangle \) be an enumeration of the the theories in \( T_{n+1,l+1} \). For \( i < b \) and \( Y \subseteq \alpha \beta \) define \( R^0_i(Y, \bar{Q}) \subseteq K \) by

\[
R^0_i(Y, \bar{Q}) := \{ \alpha \gamma : \text{Th}^{n+1}(\alpha \beta; Y, \bar{Q}) |_{[\alpha \gamma, \alpha \gamma + \alpha]} = s_i \}
\]
as before, for every \( Y \subseteq \alpha \beta \), \( \bar{R}^0 = \bar{R}^0(\alpha \beta; Y, \bar{Q}) := (R^0_0(Y, \bar{Q}), \ldots, R^0_{b-1}(Y, \bar{Q})) \) is a partition of \( K \) that is definable from \( Y, \bar{Q} \) and \( K \).

Now let \( \bar{R}^1 = \langle R^1_0, \ldots, R^1_{a-1} \rangle \) be any partition of \( K \). We will say that \( \bar{R}^0(Y, \bar{Q}) \) and \( \bar{R}^1 \) are coherent if

1. \( \alpha \gamma \in (R^0_i \cap R^1_j) \) implies that for every chain \( C, B \subseteq C \) and \( \bar{D} \subseteq C \) of length \( l \):
   \[
   \text{if } \text{Th}^{n+1}(C; B, \bar{D}) = s_i \text{ then } (\exists A \subseteq C)[\text{Th}^n(C; A, B, \bar{D}) = t_j],
   \]
2. \( \text{Th}^n(K; \bar{R}^1) \in \mathcal{R} \).

Since \( a, b \) and \( c \) are finite, there is a formula \( \theta_1(\bar{U}, \bar{W}) \) (with \( \lg(\bar{U}) = b \) and \( \lg(\bar{W}) = a \)) such that for any \( \bar{R}^0, \bar{R}^1 \subseteq K \),
\( K \models \theta_1(\bar{R}^0, \bar{R}^1) \) iff \( \bar{R}^0 \) and \( \bar{R}^1 \) are coherent partitions of \( K \).

Moreover, as \( K \cong \beta \) and \( \beta \) has the uniformization property, there exists \( S \subseteq K \) and a formula \( \theta_2(\bar{U}, \bar{W}) \) such that for every \( \bar{R}^0 \subseteq K \),
\( \text{if } (\exists W)\theta_1(\bar{R}^0, \bar{W}) \text{ then } (\exists W)\theta_2(\bar{R}^0, \bar{W}, S). \)

Let \( \theta(\bar{U}, \bar{W}, \bar{S}) := \theta_1 \land \theta_2 \).

Now let \( Y \subseteq \alpha \beta \), let \( \bar{R}^0 = \bar{R}^0(\alpha \beta; Y, \bar{Q}) \) and suppose that \( \bar{R}^0 \) and some \( \bar{R}^1 \) are coherent partitions of \( K \). When \( \alpha \gamma \in (R^0_0 \cap R^1_1) \), we know by the first clause in the definition of coherence that
\( (\exists X \subseteq \alpha \beta) [\text{Th}^n(\alpha \beta; X, \bar{Y}, \bar{Q}) |_{[\alpha \gamma, \alpha \gamma + \alpha]} = t_j]. \)

Now as \( [\alpha \gamma, \alpha \gamma + \alpha] \cong \alpha \) and \( \alpha \) has the uniformization property, there is \( \bar{T}_\gamma \subseteq [\alpha \gamma, \alpha \gamma + \alpha] \) and a formula \( \psi_j^\gamma(X, Y, \bar{T}_\gamma) \) (of depth \( k(n,l) \) that depends only on \( n \) and \( l \)) that uniformizes the formula that says \( \text{Th}^n(\alpha \beta; X, \bar{Y}, \bar{Q}) |_{[\alpha \gamma, \alpha \gamma + \alpha]} = t_j \). It follows that when \( \psi_j^\gamma(X, Y, \bar{T}_\gamma) \) holds, \( X \cap [\alpha \gamma, \alpha \gamma + \alpha] \) is unique.

W.l.o.g all \( \bar{T}_\gamma \) have the same length and (by taking prudent disjunctions) \( \psi_j^\gamma(X, Y, \bar{T}_\gamma) = \psi_j(X, Y, \bar{T}_\gamma) \) and let \( \bar{T} = \cup_{\gamma < \beta} \bar{T}_\gamma \) (the union is disjoint). We are ready to define \( U(X, Y, \bar{Q}, \bar{T}, \bar{S}) \) that uniformizes \( \varphi(X, Y, \bar{Q}): \)
\( U(X, Y, \bar{Q}, \bar{T}, \bar{S}) \) says: “for every partition \( \bar{R}^0 \) of \( K \) that is equal to [the definable] \( \bar{R}^0(\alpha \beta; Y, \bar{Q}) \) every \( \bar{R}^1 \) that is a [in fact the only] partition that satisfies \( \theta(\bar{R}^0, \bar{R}^1, \bar{S}) \), if \( \alpha \gamma \in R^1_j \) and \( D = [\alpha \gamma, \alpha \gamma + \alpha] \) and \( \alpha \gamma \) and \( \alpha \gamma + \alpha \) are two successive elements of \( K \) then \( D \models \psi_j(X \cap D, Y \cap D, \bar{Q} \cap D, \bar{T} \cap D) \).”

Check that \( U(X, Y, \bar{Q}, \bar{T}, \bar{S}) \) does the job: clause (1) in the definition of coherence and the \( \psi_j \)’s guarantee that \( X \) is unique, clause (2) guarantees that \( U(X, Y, \bar{Q}, \bar{T}, \bar{S}) \Rightarrow \varphi(X, Y, \bar{Q}) \).

\( \heartsuit \)

**Fact 6.2.** Every finite chain has the uniformization property.

\( \heartsuit \)

**Theorem 6.3.** \( (\omega, <) \) has the uniformization property.

**Corollary 6.4.** An ordinal \( \alpha \) has the uniformization property iff \( \alpha < \omega^\omega \).

**Definition 6.5.** \( (T, <) \) is very tame if
1. \( T \) is tame
2. \( \text{Sup}\{\text{Log}(B) : B \subseteq T, \ B \text{ a branch}\} < \omega \)

**Lemma 6.6.** If \( (T, <) \) is not very tame then \( (T, <) \) does’nt have the uniformization property.
Proof. If $T$ is not tame then by theorem 2.7 it doesn’t have even a definable choice function.
If $T$ is tame then either there is a branch $B \subseteq T$ with $\log(B) = \infty$ or it has branches of unbounded $\log$. By 3.14(3) and 5.2 and using the definable well ordering of $T$, there is a formula $\varphi(X, Y, Z)$ that can’t be uniformized.

\[ \Box \]

Theorem 6.7. $(T, \prec)$ has the uniformization property iff $(T, \prec)$ is very tame.

Proof. Assume $T$ is $(l^*, n^*, k^*)$ very tame and let $\varphi(X, Y, \bar{Q})$ be p.u in $T$ with $dp(\varphi) = n$ and $\log(\bar{Q}) = l$.

As $T$ is $(n^*, k^*)$ tame it can be well ordered $T$ in the following way [the full construction is given in theorem A.2 in the appendix]: partition $T$ into a disjoint union of sub-branches, indexed by the nodes of a well founded tree $\Gamma$ and reduce the problem of a well ordering of $T$ to a problem of a well ordering of $\Gamma$. At the first step we pick a branch of $T$ in theorem A.2 in the appendix: partition $\bar{\eta}$ well ordering of $\Gamma$ and the well ordering of each second step we pick a branch chain with $\log(\bar{\eta})$ nodes of a well founded tree $\Gamma$ and reduce the problem of a well ordering of $\Gamma$ we will define partitions $\bar{\eta}$ won’t always mention $\bar{\eta}$.

To get started let $T = A(\}) \cup \bigoplus_{\eta \in \{\}^+} T_\eta$. The union $\bar{\eta}$ won’t always mention $\bar{\eta}$ and we want to define some $\bar{\eta}$ won’t always mention $\bar{\eta}$.

The sequence $\bar{\eta}$ won’t always mention $\bar{\eta}$ and we want to define some $\bar{\eta}$ won’t always mention $\bar{\eta}$.

What we’ll do here in order to uniformize $\varphi(X, Y, \bar{Q})$ is the following: given $\bar{Q} \subseteq T$ we will use the decomposition $T = \bigcup_{\eta \in \Gamma} A_\eta$ and the fact that each $A_\eta$ is a scattered chain with $\log(A_\eta) < l^*$, (hence satisfies the uniformization property), to define a unique $X_\bar{\eta} \subseteq A_\eta$. This will be done in such a way that when we glue the parts letting $X^* = \bigcup_{\eta \in \Gamma} X_\eta$ we will still get $T = \varphi(X, Y, \bar{Q})$.

We will use the set of representatives $\bar{\eta}$ and the fact that $A_\eta$ and $T_\eta$ are defined from $u_\eta$ but we won’t always mention $\bar{\eta}$. We will also rely on the fact that $\Gamma$ is well founded (in fact, we only need to know that $\Gamma$ does not have a branch of order type $\geq \omega + 1$).

So let $\eta \in \Gamma$ and we want to define some $X^* = X^*(Y, \bar{Q}) \subseteq T$. The proof will go as follows: for each $\eta \in \Gamma$ we will define partitions $P^1(Y, \bar{Q})_\eta$ and $P^2(Y, \bar{Q})_\eta$ of $K_{\eta^+} := \{u_\nu : \nu \in \eta^+\}$ then, using the composition theorem 3.14 and similarly to the proof of proposition 6.1, we will define a notion of coherence and let $\bar{R}^1(Y, \bar{Q})_\eta$ and $\bar{R}^2(Y, \bar{Q})_\eta$ be a pair that is coherent with $P^1(Y, \bar{Q})_\eta$ and $P^2(Y, \bar{Q})_\eta$.

The union $\bar{R}^1(Y, \bar{Q}) = \bigcup_{\eta \in \Gamma} \bar{R}^1(Y, \bar{Q})_\eta$ is a partition of $K$ and $\Th^\eta(A_\eta; X_\eta, Y \cap A_\eta, \bar{Q} \cap A_\eta)$ will be determined by the unique member of $\bar{R}^1(Y, \bar{Q})$ to which $u_\eta$ belongs. Moreover, we will be able to choose $X_\eta$ uniquely and by coherence $X^* = \bigcup_{\eta \in \Gamma} X_\eta$ will satisfy $\varphi(X, Y, \bar{Q})$.

To get started let $T = A(\}) \cup \bigoplus_{\eta \in \{\}^+} T_\eta$. Now as in definition 3.12 $K_{\{\}^+}$ has a natural structure of a chain with $\log(K_{\{\}^+}) = \log(A_\{\}) < l^*$ and by theorem 3.14(2) there is some $m = m(n, l)$ such that when $X \subseteq T$ is given, from $\Th^m(A_\{\}; X, Y, \bar{Q})$ and $\langle \Th^m(T_\eta; X, Y, \bar{Q}) : \eta \in \{\}^+\rangle$ we can compute $\Th^m(T; X, Y, \bar{Q})$.

Let $\langle s_0, \ldots, s_{n-1} \rangle$ be an enumeration of the the theories in $T_{n+1, l+1}$.

15
Define \( \tilde{P}^1(Y, \bar{Q})_{(i)} = \langle P^1_Y(Y, \bar{Q}), \ldots, P^1_{b-1}(Y, \bar{Q})_{(i)} \rangle \) a partition of \( K_{(i)}^+ \) by

\[
\eta \in \mathcal{P}^1_Y(Y, \bar{Q}_{(i)}) \iff \text{Th}^{n+1}(T_\eta; Y, \bar{Q}) = s_i
\]

By the previous remarks \( \tilde{P}^1(Y, \bar{Q})_{(i)} \) is definable from \( u_{(i)}, K, Y, \bar{Q} \) (and \( \bar{K}_0 \)).

Define \( \tilde{P}^2(Y, \bar{Q})_{(i)} = \langle P^2_Y(Y, \bar{Q}), \ldots, P^2_{b-1}(Y, \bar{Q})_{(i)} \rangle \) a partition of \( K_{(i)}^+ \) by

\[
\eta \in \mathcal{P}^1_Y(Y, \bar{Q})_{(i)} \iff \text{Th}^{n+1}(A_\eta; Y, \bar{Q}) = s_i
\]

Again, \( \tilde{P}^2(Y, \bar{Q})_{(i)} \) is definable from \( u_{(i)}, K, Y, \bar{Q} \) and \( \bar{K}_0 \).

Let \( \langle t_0, \ldots, t_{a-1} \rangle \) be an enumeration of the the theories in \( \mathcal{T}_{n,t+2} \).

A partition of \( K_{(i)}^+ \), \( \bar{R}^1 = \langle R^1_0, \ldots, R^1_{a-1} \rangle \) is coherent with \( \tilde{P}^1(Y, \bar{Q})_{(i)} \) if \( P^1_Y(Y, \bar{Q})_{(i)} \cap R^1_j \neq \emptyset \) implies “for every tree \( S \) and \( B, \bar{C} \subseteq S \) with \( \lg(\bar{C}) = l \), if \( \text{Th}^{n+1}(S; B, \bar{C}) = s_i \) then there is \( A \subseteq S \) such that \( \text{Th}^n(S; A, B, \bar{C}) = t_j \).”

Similarly a partition of \( K_{(i)}^+ \), \( \bar{R}^2 = \langle R^2_0, \ldots, R^2_{a-1} \rangle \) is coherent with \( \tilde{P}^2(Y, \bar{Q})_{(i)} \) if \( P^2_Y(Y, \bar{Q})_{(i)} \cap R^2_j \neq \emptyset \) implies “for every chain \( S \) and \( B, \bar{C} \subseteq S \) with \( \lg(\bar{C}) = l \), if \( \text{Th}^{n+1}(S; B, \bar{C}) = s_i \) then there is \( A \subseteq S \) such that \( \text{Th}^n(S; A, B, \bar{C}) = t_j \).”

Finally, a pair of partitions of \( K_{(i)}^+ \), \( \langle \bar{R}^1, \bar{R}^2 \rangle \) is \( t^* \)-coherent with the pair \( \langle \tilde{P}^1(Y, \bar{Q})_{(i)}, \tilde{P}^2(Y, \bar{Q})_{(i)} \rangle \) if

1. \( \bar{R}^1 \) is coherent with \( \tilde{P}^1(Y, \bar{Q})_{(i)} \),
2. \( \bar{R}^2 \) is coherent with \( \tilde{P}^2(Y, \bar{Q})_{(i)} \), and
3. For every \( X \subseteq T \), if \( \text{Th}^n(A_{\eta}; X, Y, \bar{Q}) = t^* \) and if for every \( \eta \in \langle \eta \rangle^+ \) \( \text{Th}^n(T_\eta; X, Y, \bar{Q}) = t_i \) \( \iff u_\eta \in R^1_1 \), then \( T \models \varphi(X, Y, \bar{Q}) \).

As \( T = (\exists X) \varphi(X, Y, \bar{Q}) \) there are \( t^* \) (that will be fixed from now on), \( \bar{R}^1 \) and \( \bar{R}^2 \) such that \( \langle \bar{R}^1, \bar{R}^2 \rangle \) is \( t^* \)-coherent with the pair \( \langle \tilde{P}^1(Y, \bar{Q})_{(i)}, \tilde{P}^2(Y, \bar{Q})_{(i)} \rangle \).

Moreover, \( \langle \bar{R}^1, \bar{R}^2 \rangle \) is \( t^* \)-coherent with the pair \( \langle \tilde{P}^1(Y, \bar{Q})_{(i)}, \tilde{P}^2(Y, \bar{Q})_{(i)} \rangle \)

is determined by \( \text{Th}^n(K_{(i)}^+; \bar{R}^1, \bar{R}^2, \tilde{P}^1(Y, \bar{Q})_{(i)}, \tilde{P}^2(Y, \bar{Q})_{(i)}) \) where \( k \) depends only on \( n \) and \( l \).

The first two clauses are clear (since \( a \) and \( b \) are finite) and for the third clause use theorem 3.14(2).

So the statement is expressed by a p.u formula \( \psi^1(\bar{R}^1, \bar{R}^2, \tilde{P}^1(Y, \bar{Q})_{(i)}, \tilde{P}^2(Y, \bar{Q})_{(i)}) \) of depth \( k \).

As by a previous remark \( \text{Log}(K_{(i)}^+) < t^* \) there is \( \bar{S}_i \subseteq K_{(i)}^+ \) and a formula \( \psi^1(\bar{U}_1, \bar{U}_2, \bar{W}_1, \bar{W}_2, \bar{S}_i) \) that uniformizes \( \psi^1 \).

To conclude the first step use \( \text{Log}(A_{\eta}) < t^* \) to define, by a formula \( \theta_{(i)}(X, Y \cap A_{\eta}, \bar{Q} \cap A_{\eta}, \bar{O}_{(i)}) \) and a sequence of parameters \( \bar{O}_{(i)} \subseteq A_{\eta} \), a unique \( X_{(i)} \subseteq A_{\eta} \) that will satisfy \( \text{Th}^n(A_{\eta}; X_{(i)}, Y, \bar{Q}) = t^* \).

The result of the first step is the following:

a) we have defined \( X_{(i)} \subseteq A_{\eta} \) using \( \bar{O}_{(i)} \subseteq A_{\eta} \) and \( \theta_{(i)} \). \( X_{(i)} \) is the intesection of the eventual \( X^* \) with \( A_{\eta} \).

b) we have chosen \( \bar{R}^1_{(i)} \), \( \bar{R}^2_{(i)} \subseteq K_{(i)}^+ \) using \( \psi \) and \( \bar{S}_{(i)} \).

c) \( \bar{R}^1_{(i)} \) and \( \bar{R}^2_{(i)} \) tell us what are (for \( \eta \in \langle \eta \rangle^+ \) the theories \( \text{Th}^n(T_\eta; X^*, Y, \bar{Q}) \) and \( \text{Th}_n(A_\eta; X_\eta, Y, \bar{Q}) \) respectively: if \( u_\eta \in R^1_1 \) then the eventual \( X^* \cap T_\eta \subseteq T_\eta \) will satisfy \( \text{Th}^n(T_\eta; X^* \cap T_\eta, Y, \bar{Q}) = t_i \) and if \( u_\eta \in R^2_1 \) then the soon to be defined \( X_\eta \subseteq A_\eta \) will satisfy \( \text{Th}^n(A_\eta; X_\eta, Y, \bar{Q}) = t_j \).

We will proceed by induction on the level of \( \eta \) in \( \Gamma \) (remember, all the levels are \( < \omega \)) to define \( \bar{S}_\eta, \bar{O}_\eta \subseteq A_\eta \) and \( \bar{R}^1_{\eta^+, \bar{R}^2_{\eta^+} \subseteq K_{\eta^+} \) and \( X_\eta \subseteq T_\eta \).

The induction step:
We are at $\nu \in \Gamma$ where $\nu \in \eta^+$ and we want to define $S^\nu, O^\nu \subseteq A^\nu$, $R^1_\nu, R^2_\nu \subseteq K^\nu$ and $X^\nu \subseteq T^\nu$.

Now as $R^1_{\eta^+}$ and $R^2_{\eta^+}$ are defined, $u^\nu$ belongs to one member of $R^1_{\eta^+}$ say the $i_1$'th and to one member of $R^2_{\eta^+}$ say the $i_2$'th. This implies that there is some $X^\nu \subseteq T^\nu$ such that $Th^\eta(T^\nu; X^\nu, Y, Q) = t_{i_1}$ and $Th^\eta(A^\nu; X^\nu \cap A^\nu, Y, Q) = t_{i_2}$.

Let $P^1(Y, Q)_\nu$ and $P^2(Y, Q)_\nu$ be partitions of $K_{\nu^+}$ that are defined as in the first step by saying, for $\tau \in \nu^+$, what are $Th^{\eta^+}[(\tau; Y, Q)$ and $Th^{\eta^+}[(A^\tau; Y, Q)$. $(R^1_{\nu^+}, R^2_{\nu^+}) \subseteq K_{\nu^+}$ will be a pair that is $t_{i_1}, t_{i_2}$-coherent with $(P^1(Y, Q)_\nu, P^2(Y, Q)_\nu)$ that is:

(1) $R^1_{\nu^+}$ is coherent with $P^1(Y, Q)_\nu$,
(2) $R^2_{\nu^+}$ is coherent with $P^2(Y, Q)_\nu$, and
(3) For every $X \subseteq T^\nu$, if $Th^n(A^\nu; X, Y, Q) = t_{i_2}$ and for every $\tau \in \nu^+$ \[Th^n(T^\nu; X, Y, Q) = t_i \iff u^\tau \in \text{the } i^\text{th member of } R^1_{\nu^+}\], then $Th^n(T^\nu; X, Y, Q) = t_{i_1}$.

Using Log$(K_{\nu+}) < l^*$ choose $S^\nu \subseteq K_{\nu^+}$ and $\psi_{i_1,i_2}(R^1, R^2, P^1(Y, Q)_\nu, P^2(Y, Q)_\nu, S^\nu)$ that uniformizes the formula that says "$(R^1, R^2)$ is $t_{i_1}, t_{i_2}$-coherent with $(P^1(Y, Q)_\nu, P^2(Y, Q)_\nu)$". We may assume that $\psi_{i_1,i_2}$ depends only on $i_1$ and $i_2$ and that Log$(S^\nu)$ is constant.

Let $\eta = \cup_{\nu \in \Gamma} O^\nu$, $\bar{S} = \cup_{\nu \in \Gamma} S^\nu$. The uniformizing formula $U(X, Y, Q, O, \bar{S}, K, \bar{K}_0)$ says:

"$X \cap A^\nu$ is defined as in the first step, and for every pair of partitions $(P^1, P^2)$ of $K$ that agrees on each $K_{\nu^+}$ with [the definable] $(P^1_{\eta^+}(Y, Q), P^2_{\eta^+}(Y, Q))$, and $(\text{agrees with } (P^1_{\eta^+}(Y, Q), P^2_{\eta^+}(Y, Q)) \text{ on } K_{(\eta^+)}$, and for every $(R^1, R^2)$ that is a [in fact the only] pair of partitions that satisfies for every $u^\eta \in K$: if $u^\eta \in P^1_{i_1} \cap P^2_{i_2}$ then $\psi_{i_1,i_2}(R^1 \cap K^\eta, R^2 \cap K^\eta, P^1 \cap K^\eta, P^2 \cap K^\eta, \bar{S} \cap K^\eta)$ holds, and $(\text{agrees with } (R^1_{(\eta^+)}, R^2_{(\eta^+)}) \text{ on } K_{(\eta^+)}$.

for every $u^\eta \in K$ if $u^\eta \in R^2_{i_1}$ then $\psi_{i_1,i_2}(X \cap A^\eta, Y \cap A^\eta, Q \cap A^\eta, O \cap A^\eta)$ holds."

$U(X, Y, \bar{Q}, O, \bar{S}, K, \bar{K}_0)$ does the job because it defines $X \cap A^\eta$ uniquely on each $A^\eta$ and because, (by the conditions of coherence) the union of the parts, $X$, satisfies $\varphi(X, Y, \bar{Q})$. Note also that $U$ does not depend on $Y$.

\[ \heartsuit \]

7. Hopelessness of General Partial Orders

**Theorem 7.1.** Every partial order $P$ can be embedded in a partial order $Q$ in which $P$ is first-order-definably well orderable.

**Proof.**

\[ \heartsuit \]

**Appendix**
Lemma A.1. Let $C$ be a scattered chain with $\text{Hdeg}(C) = n$. Then there are $P \subseteq C$, $\lg(P) = n-1$, and a formula (depending on $n$ only) $\varphi_n(x, y, \bar{P})$ that defines a well ordering of $C$.

Proof. By induction on $n = \text{Hdeg}(C)$:

$n \leq 1$: $\text{Hdeg}(C) \leq 1$ implies $(C, <_C)$ is well ordered or inversely well ordered. A well ordering of $C$ is easily definable from $<_C$.

$\text{Hdeg}(C) = n + 1$: Suppose $C = \sum_{i \in I} C_i$ and each $C_i$ is of Hausdorff degree $n$. By the induction hypothesis there are a formula $\varphi_n(x, y, \bar{Z})$ and a sequence $\langle \bar{P}^i : i \in I \rangle$ with $\bar{P}^i \subseteq C_i$, $\bar{P}^i = \langle P^i_1, \ldots, P^i_{n-1} \rangle$ such that $\varphi_n(x, y, \bar{P}^i)$ defines a well ordering of $C_i$.

Let for $0 < k < n$, $P_k := \bigcup_{i \in I} P^i_k$ (we may assume that the union is disjoint) and $P_n := \bigcup \{ C_i : i \text{ even} \}$.

We will define an equivalence relation $\sim$ by $x \sim y$ iff $\bigwedge_i (x \in C_i \iff y \in C_i)$. 

$\sim$ and $[x]$, (the equivalence class of an element $x$), are easily definable from $P_n$ and $<_C$. We can also decide from $P_n$ if $I$ is well or inversely well ordered (by looking at subsets of $C$ consisted of nonequivalent elements) and define $<'$ to be $<$ if $I$ is well ordered and the inverse of $<$ if not.

$\varphi_{n+1}(x, y, P)$ will be defined by:

$$
\varphi_{n+1}(x, y, \bar{P}) \iff \left[ x \not< y \land x <' y \right] \lor \left[ x < y \land \varphi_n(x, y, P_1 \cap [x], \ldots, P_{n-1} \cap [x]) \right]
$$

$\varphi_{n+1}(x, y, \bar{P})$ well orders $C$.

Theorem A.2. Let $T$ be a tame tree. If $\omega + 2$ is not embeddable in $T$ then there are $\bar{Q} \subseteq T$ and a monadic formula $\varphi(x, y, \bar{Q})$ that defines a well ordering of $T$.

Proof. Assume $T$ is $(n^*, k^*)$ tame, recall definitions 4.1 and 4.2 and remember that for every $x \in T$, $rk(x)$ is well defined (i.e. $< \infty$). We will partition $T$ into a disjoint union of sub-branches, indexed by the nodes of a well founded tree $\Gamma$ and reduce the problem of a well ordering of $T$ to a problem of a well ordering of $\Gamma$.

Step 1. Define by induction on $\alpha$ a set $\Gamma_{\alpha} \subseteq {\alpha}^{\text{Ord}}$ (this is a our set of indices), for every $\eta \in \Gamma_{\alpha}$ define a tree $T_\eta \subseteq T$ and a branch $A_\eta \subseteq T_\eta$.

$\alpha = 0$: $\Gamma_0 = \{ \langle \rangle \}$, $T_\emptyset$ is $T$ and $A_\emptyset$ is a branch (i.e. a maximal linearly ordered subset) of $T$.

$\alpha = 1$: Look at $(T \setminus A_\emptyset)/\sim_{A_\emptyset}$, it’s a disjoint union of trees and name it $\langle T_{(i)} : i < i^* \rangle$, let $\Gamma_1 := \{ \langle i : i < i^* \rangle : i \in I \}$ and for every $i \in \Gamma_1$ let $A_{(i)}$ be a branch of $T_{(i)}$.

$\alpha = \beta + 1$: For $\eta \in \Gamma_\beta$ denote $(T_\eta \setminus A_\eta)/\sim_{A_\eta}$ by $\langle T_{\eta,i} : i < i_\eta \rangle$, let $\Gamma_{\eta,i} := \{ \langle \eta, i : \eta \in \Gamma_\beta, i < i_\eta \} \} and choose $A_{\eta,i}$ to be a branch of $A_{\eta,i}$.

$\alpha$ limit: Let $\Gamma_\alpha = \{ \eta \in {\alpha}^{\text{Ord}} : \land_{\beta < \alpha} \langle \beta \in \Gamma_\beta, \land_{\beta < \alpha} T_{\eta,\beta} \neq \emptyset \rangle \}$, let for $\eta \in \Gamma_\alpha$ $T_\eta = \cap_{\beta < \alpha} T_{\eta,\beta}$ and $A_\eta$ a branch of $T_\eta$.

Now, at some stage $\alpha \leq |T|$ we have $\Gamma_\alpha = \emptyset$ and let $\Gamma = \cup_{\beta < \alpha} \Gamma_\beta$. Clearly $\{ A_\eta : \eta \in \Gamma \}$ is a partition of $T$ into disjoint sub-branches.

Notation: having two trees $T$ and $\Gamma$, to avoid confusion, we use $x, y, s, t$ for nodes of $T$ and $\eta, \nu, \sigma$ for nodes of $\Gamma$.

Step 2. We want to show that $\Gamma_\omega = \emptyset$ hence $\Gamma$ is a well founded tree. Note that we made no restrictions on the choices of the $A_\eta$’s and we add one now in order to make the above statement true. Let $\land_{\eta,i} \in \Gamma$ define $A_{\land_{\eta,i}}$ to be the sub-branch $\{ t \in A_\eta : (\forall s \in A_{\land_{\eta,i}}) [rk(t) \leq rk(s)] \}$ and $\gamma_{\eta,i}$ to be $\overline{rk(t)}$ for some $t \in A_{\land_{\eta,i}}$. By 5.5(1) and the inexistence of a strictly decreasing sequence of ordinals, $A_{\land_{\eta,i}} \neq \emptyset \} and $\gamma_{\eta,i}$ is well defined. Note also that $s \in A_{\land_{\eta,i}} \Rightarrow rk(s) \leq \gamma_{\eta,i}$.

Proposition: For every $\eta \in \Gamma$ and $i < i_\eta$ the sub-branch $A_{\land_{\eta,i}}$ contains every $s \in T_{\land_{\eta,i}}$ with $rk(s) = \gamma_{\eta,i}$.
Following this we claim: “Γ does not contain an infinite, strictly increasing sequence”. Otherwise let \( \{ \eta_i \}_{i \in \omega} \) be one, and choose \( s_n \in A_{\eta_i} \cdot \sigma_{n+1} \) (so \( s_n \in A_{\eta_i} \)). Clearly \( rk(s_n) \geq rk(s_{n+1}) \) and by the proviso we get
\[
rk(s_n) = rk(s_{n+1}) \Rightarrow rk(s_{n+1}) > rk(s_{n+2})
\]
therefore \( \{ rk(s_n) \}_{n \in \omega} \) contains an infinite, strictly decreasing sequence of ordinals which is absurd.

Step 3. Next we want to make “\( x \) and \( y \) belong to the same \( A_\eta \)” definable.

For each \( \eta \in \Gamma \) choose \( s_\eta \in A_\eta \), and let \( Q \subseteq T \) be the set of representatives. Let \( h: T \rightarrow \{ d_0, \ldots, d_{n-1} \} \) be a colouring that satisfies: \( h \mid A_\eta = d_0 \) and for every \( \eta, i \in \Gamma \), \( h \mid A_{\eta,i} \) is constant and, when \( j < i \) and \( s_{\eta,j} \sim A_\eta s_{\eta,i} \) we have \( h \mid A_{\eta,i} \neq h \mid A_{\eta,j} \). This can be done as \( T \) is \((n^*, d^*)\) tame.

Using the parameters \( D_0, \ldots, D_{n-1} \) (\( x \in D_i \) iff \( h(x) = d_i \)), we can define \( \forall \eta x, y \in A_\eta \) by “\( x, y \) are comparable and the sub-branch \( [x, y] \) (or \([y, x]) \) has a constant colour”.

Step 4. As every \( A_\eta \) has Hausdorff degree at most \( k^* \), we can define a well ordering of it using parameters \( P^0_1, \ldots, P^0_k \) and by taking \( \tilde{\mathcal{P}} \) to be the (disjoint) union of the \( P^0 \)'s we can define a partial ordering on \( T \) which well orders every \( A_\eta \).

By our construction \( \eta \sim \nu \) if and only if there is an element in \( A_\nu \) that ‘breaks’ \( A_\eta \) i.e. is above a proper initial segment of \( A_\eta \). (Caution, if \( T \) does not have a root this may not be the case for \( \langle \rangle \) and a \( n^* \) number of \( \langle i \rangle \)'s and we may need parameters for expressing that). Therefore, as by step 3 “being in the same \( A_\eta \)” is definable, we can define a partial order on the sub-branches \( A_\eta \) (or the representatives \( s_\eta \)) by \( \eta \sim \nu \Rightarrow A_\eta \leq A_\nu \).

Next, note that “\( \nu \) is an immediate successor of \( \eta \) in \( \Gamma \)” is definable as a relation between \( s_\nu \) and \( s_\eta \) hence the set \( A_\eta^+ := A_\eta \cup \{ s_{\eta,i} \} \) is definable from \( s_\eta \). Now the order on \( A_\eta \) induces an order on \( \{ s_{\eta,i} \sim A_\eta \} \) which is can be embedded in the compilation of \( A_\eta \) hence has \( \text{Hdeg} \leq k^* \). Using additional parameters \( Q^0_1, \ldots, Q^0_k \), we have a definable well ordering on \( \{ s_{\eta,i} \sim A_\eta \} \). As for the ordering on each \( \sim A_\eta \) equivalence class (finite with \( \leq n^* \) elements), define it by their colours (i.e. the element with the smaller colour is the smaller according to the order).

Using \( \tilde{\mathcal{D}}, \tilde{\mathcal{P}}, Q \) and \( \tilde{Q} = \cup \eta \tilde{Q}^0 \) we can define a partial ordering which well orders each \( A_\eta^+ \) in such a way that every \( x \in A_\eta \) is smaller then every \( s_{\eta,i} \).

Summing up we can define (using the above parameters) a partial order on subsets of \( T \) that well orders each \( A_\eta \), orders sub-branches \( A_\eta \), \( A_\nu \) when the indices are comparable in \( \Gamma \) and well orders all the “immediate successors” sub-branches of a sub-branch \( A_\eta \).

Step 5. The well ordering of \( T \) will be defined by \( x < y \iff \)
\( a) x \) and \( y \) belong to the same \( A_\eta \) and \( x < y \) by the well order on \( A_\eta \); or
\( b) x \in A_\eta, y \in A_\nu \) and \( \eta \sim \nu \); or
\( c) x \in A_\eta, y \in A_\nu, \) \( \eta \sim \nu \) in \( \Gamma \) (defined as a relation between sub-branches), \( \eta \sim \nu \) in \( \Gamma \) (defined as a relation between sub-branches), \( \eta \sim \nu \) and \( s_{\eta,i} < s_{\eta,j} \) in the order of \( A_\eta^+ \).

Note, that \( x < y \) is a linear order on \( T \) and every \( A_\eta \) is a convex and well ordered sub-chain. Moreover \( < \) is a linear order on \( \Gamma \) and the order on the \( s_\eta \)'s is isomorphic to a lexicographic order on \( \Gamma \).

Why is the above (which is clearly definable with our parameters) a well order? Because of the above note and because a lexicographic ordering of a well founded tree is a well order, provided that immediate successors are well ordered. In detail, assume \( X = \{ x_i \}_{i \in \omega} \) is a strictly decreasing sequence of elements of \( T \). Let \( \eta_i \) be the unique node in \( \Gamma \) such that \( x_i \in A_{\eta_i} \) and by the above note w.l.o.g \( i \neq j \Rightarrow \eta_i \neq \eta_j \). By the well foundedness of \( \Gamma \) and clause (b) we may also assume w.l.o.g.
that the $\eta_i$'s form an anti-chain in $\Gamma$. Look at $\nu_i := \eta_1 \land \eta_i$ which is constant for infinitely many $i$'s and w.l.o.g equals to $\nu$ for every $i$. Ask:

(*): Is there is an infinite $B \subseteq \omega$ such that $i, j \in B \Rightarrow x_i \sim^0_{A_\nu} x_j$?

If this occurs we have $\nu_1 \neq \nu$ with $\nu \triangleleft \nu_1$ such that for some infinite $B' \subseteq B \subseteq \omega$ we have $i \in B' \Rightarrow \nu_1 \triangleleft \eta_i$. (use the fact that $\sim^1_{A_\nu}$ is finite). W.l.o.g $B' = \omega$ and we may ask if (*) holds for $\nu_1$. Eventually, since $\Gamma$ does not have an infinite branch, we will have a negative answer to (*). We can conclude that w.l.o.g there is $\nu \in \Gamma$ such that $i \neq j \Rightarrow x_i \not\sim^0_{A_\nu} x_j$ i.e. the $x_i$'s “break” $A_\nu$ in “different places”.

Define now $\nu_i$ to be the unique immediate successor of $\nu$ such that $\nu_i \triangleleft \eta_i$. The set $S = \{s_{\nu_i} \}_{i<\omega} \subseteq A^+_{\nu}$ is well ordered by the well ordering on $A^+_{\nu}$ and by clause (c) in the definition of $\nu_i > x_j$ is an infinite strictly decreasing subset of $A^+_{\nu}$ — a contradiction.

This finishes the proof that there is a definable well order of $T$.

$\heartsuit$