PERIODIC LINEAR MOTIONS WITH MULTIPLE COLLISIONS
IN A FORCED KEPLER TYPE PROBLEM

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Abstract. In [7] the author proved the existence of multiple periodic linear motions with collisions for a periodically forced Kepler problem. We extend this result obtaining periodic solutions with multiple collisions for a forced Kepler type problem. In order to do that we apply the Poincaré-Birkhoff theorem.

1. Introduction. Isaac Newton related the acceleration of a body with the forces exerted over it. Making use of Kepler’s laws of planetary motion he understood that the gravitational force exerted by a mass over another was proportional to the inverse square of their distances. Being \( r \) the position of one of the masses relative to the other, this gives

\[
\dddot{r} \propto -\frac{r}{|r|^3}.
\]

(1)

However, if we modify a little bit this equation, i.e., if we make a perturbation, the properties of its solutions are not fully understood, not even in the one dimensional case.

In [7], R. Ortega considered the equation

\[
\dddot{u} = -\frac{1}{u^2} + p(t), \quad u > 0,
\]

(2)

where \( p : \mathbb{R} \to \mathbb{R} \) is a \( 2\pi \)-periodic continuously differentiable function and proved the existence of generalized periodic solutions. The generalized solutions are solutions with collisions, i.e., solutions which may attain the singularity in a discrete set of instants. The author proved the existence of two types of periodic solutions: solutions with exactly one collision in the interval \( [0, 2\pi] \) and, for each \( N > 1 \), solutions with exactly one collision in the interval \( [0, 2\pi] \) and no collision on \( [2\pi, 2N\pi[ \). In a recent paper [11], L. Zhao proved the existence of generalized quasi-periodic solutions for (2) when \( p \) is sufficiently regular.

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In this work we consider the equation
\[ \ddot{u} = -\frac{1}{u^\alpha} + p(t), \quad u > 0, \quad \alpha = 2 \vee \alpha \geq 4, \] (3)
where \( p : \mathbb{R} \to \mathbb{R} \) is a \( 2\pi \)-periodic \( C^2 \)-differentiable function and generalize the results obtained in [7] proving, for each \( M \) and \( n \) the existence of positive periodic solutions with \( n \) collisions in the interval \([0, 2M\pi]\). In particular we can find periodic solutions with any number of collisions in the interval \([0, 2\pi]\). Our equation models the linear motions of a particle under an attractive central force given by \( -\frac{r}{|r|^{\alpha+1}} \).

Both in the cases of attractive and repulsive forces, that is for forces given by \( \pm \frac{r}{|r|^{\alpha+1}} \), it was proved in [1] the existence of periodic solutions for some forcing terms \( p \). Now we analyze the existence of periodic solutions with collisions, a problem much richer. However, we must restrict the power of the central force to the above specified values of \( \alpha \). In our argument it is essential that the vector field associated with the regularized system corresponding to equation (3) is \( C^2 \)-differentiable everywhere. This is due to the fact that we used an extension theorem (see [6, Lemma 2.4. and the remarks above]) which is valid under this restriction, probably our result is true for a larger class of \( \alpha \).

We applied the Poincaré-Birkhoff theorem to the iterates of a successor map. A version of the Poincaré-Birkhoff theorem was employed in [7] in order to obtain fixed points of a map, unfortunately this version of the theorem cannot be used in the case of its iterates. Hence we opted to apply the classical theorem for the cylinder (see for example [5]). To do that we argued as in [6]. We could have applied other versions of the theorem (see for example the survey [2]), but the one in [5] seems more adequate to deal with this problem.

The plan of the paper is the following: in Section 2 we give the preliminaries necessary to obtain the main result. As a corollary of these preliminaries a result about the existence of solutions with one collision in the interval \([0, 2M\pi]\) is given. As for the proof of this result we do not need to apply the extension theorem mentioned above, less regularity is needed and the result holds for \( \alpha \geq 2 \). In Section 3 we prove the existence of solutions with multiple collisions. Finally in Section 4, the Appendix, we present some auxiliary proofs.

2. Existence of solutions with one collision. In this section we give some preliminaries essential in the following. Some of these preliminaries have proofs analogous to others obtained in [7] in the case \( \alpha = 2 \) and hence we omit them in this paper but they can be found in [8]. The proofs we opted to give can be found in the Appendix. At the end of this section we give a result on the existence of periodic solutions of (3) with exactly one collision in the interval \([0, 2M\pi]\), for some \( M \in \mathbb{N} \).

We are dealing with a non-autonomous differential equation. Let us define the energy function as the Hamiltonian of the equation without perturbations and denote it by \( h \). Therefore, define
\[ h(u, v) = \frac{1}{2} v^2 - \frac{1}{\alpha - 1} \frac{1}{u^{\alpha-1}}. \] (4)
Given a solution \( u \) of (3), we will denote \( h(t) = h(u(t), \dot{u}(t)) \).

Following the work in [7], we will consider solutions with collisions, hence we deal with a wide set of solutions. We recall the concept of bouncing solution.
Definition 2.1. A bouncing solution of equation (3) is a continuous function $u : \mathbb{R} \rightarrow [0, +\infty]$ satisfying

1. The set of instants of collision $Z = \{ t \in \mathbb{R} : u(t) = 0 \}$ is discrete.
2. For any open interval $A \subseteq \mathbb{R} \setminus Z$ the function $u$ is a classical solution of (3).
3. For each $t_0 \in Z$ the limit $\lim_{t \to t_0} h(t)$ exists.

Given suitable conditions, it is possible to state an existence and uniqueness theorem concerning these solutions. Below we will introduce the concepts of collision conditions and of generalized Cauchy problem. Bouncing solutions are also called generalized solutions.

Proposition 1. Let $u(\cdot)$ be a classical solution of (3) defined in a maximal interval $[t_0, t_1]$ and $t_0 \in \mathbb{R}$. Then we have the following asymptotic expansions:

$$u(t) = \left[\frac{(\alpha + 1)^2}{2(\alpha - 1)}\right]^\frac{\alpha+1}{\alpha-1} (t - t_0)^\frac{\alpha+1}{\alpha-1} + O((t - t_0)^{\frac{\alpha+1}{\alpha-1}}), \; t \downarrow t_0$$

$$\dot{u}(t) = \frac{2}{\alpha+1} \left[\frac{(\alpha + 1)^2}{2(\alpha - 1)}\right]^\frac{\alpha+1}{\alpha-1} (t - t_0)^\frac{\alpha-1}{\alpha+1} + O((t - t_0)^{\frac{\alpha-1}{\alpha+1}}), \; t \downarrow t_0.$$  

(5)  

Also we have

$$\lim_{t \to t_0^+} u(t) = 0,$$  

$$\exists h_0 \in \mathbb{R} : \lim_{t \to t_0^+} h(t) = h_0.$$  

(7)  

If $t_1 \in \mathbb{R}$ a similar result holds when the solution approaches $t_1$.

The limit (8) is a consequence of the asymptotic expansions (5) - (6) of the solution (see the deduction of these expansions in the Appendix).

Using the Sundman integral for the $\alpha - 1$-potential

$$S(t) = \int_{t_0}^{t} \frac{1}{u(s)^{\frac{\alpha}{2}}} \; ds$$

we can remove the singularity in (3). Notice that from (5) we conclude that this is a convergent integral. Let us denote by $T$ the inverse of $S$. If we consider a solution $u$ of (3), the functions $U(s) = u(T(s))$ and $H(s) = h(T(s))$ then $U, T$ and $H$ satisfy

$$\begin{cases}
U' = V \\
V' = \frac{1}{\alpha-1} + p(T)U^\alpha + \alpha U^{\alpha-1}H \\
T' = U^\frac{\alpha}{2} \\
H' = p(T)V
\end{cases}.$$  

(10)

Additionally, when $U \geq 0$, this system admits the first integral

$$I = U^\alpha H - \frac{1}{2} V^2 + \frac{1}{\alpha-1} U.$$  

(11)

By construction, a solution of (3) satisfying (7) and (8) gives rise to a solution of system (10), contained in $I^{-1}(0)$ and satisfying the initial conditions

$$U(0) = 0, \; V(0) = 0, \; T(0) = t_0, \; \text{and} \; H(0) = h_0.$$  

(12)

Conversely, a nonnegative solution of (10) contained in $I^{-1}(0)$ and satisfying (12) leads to a nonnegative solution of (3) for $t \geq t_0$ satisfying (7) and (8). The details are easy to check and are analogous to those in [7]. In particular the fact that $V(0) = 0$ follows from Lemma 4.1 or from Proposition 1. Notice that system
(10) may not be defined in \( \mathbb{R}^4 \) for some values of \( \alpha \), which could possibly be an obstacle to the application of existence and uniqueness theorem to conditions (12). However, we can easily avoid the situation by considering the absolute value of \( U \) in the regularized system (10). Also we would define \( I \) with absolute values on \( U \).

After that the problem would be overcome and we could proceed analogously, since solutions of equation (3) correspond to functions \( U \) that are positive and therefore satisfy system (10). Conversely a solution of (10) in an interval where \( U \) is positive satisfies (10) and corresponds to a solution of (3).

Note that, as \( \alpha = 2 \) or \( \alpha \geq 4 \), if \( p \) is a \( C^2 \) function then the vector field associated with system (10) is \( C^2 \)-differentiable in all its domain. We remark that if we had prescribed a value of \( \alpha \) lying in \( (2, 4) \) then this vector field would not be \( C^2 \)-differentiable everywhere anymore.

As a consequence of the existence and uniqueness for the Cauchy problem associated to system (10), we can prove that given \( t_0, h_0 \in \mathbb{R} \) there exists a unique classical solution of (3) satisfying (7) and (8).

**Definition 2.2.** We will say that (3) - (7) and (8) is a *generalized Cauchy problem* and that (7) and (8) are the *collision conditions*.

By gluing together consecutive classical solutions and imposing compatibility conditions on collision instants, it is possible to construct a bouncing solution. Of course it is necessary to exclude the possibility that the zeros accumulate at finite time. The solution obtained is by construction unique. This is the content of the following theorem:

**Theorem 2.3.** If \( p \) is Lipschitz-continuous, given \( t_0, h_0 \in \mathbb{R} \) there exists a unique bouncing solution of the generalized Cauchy problem (3) - (7) and (8).

From now on, we will denote by \( u(\cdot; t_0, h_0) \) the unique solution described by the theorem.

Now we introduce the successor map which we will denote by \( \mathcal{P} \). This map is analogous to the Poincaré map associated to a dynamical system, in the sense that for each \( (t_0, h_0) \) such that \( \mathcal{P}^n(t_0, h_0) = (t_0 + 2\pi M, h_0) \) we can associate a \( 2\pi M \)-periodic bouncing solution of (3) with \( n \) zeros in \([0, 2\pi M]\).

Let

\[ D = \{(t_0, h_0) \in \mathbb{R}^2 : t_1(t_0, h_0) < +\infty\} \]

where \( t_1(t_0, h_0) \) is the first instant of collision of the solution \( u(\cdot; t_0, h_0) \).

If this value is finite, then by Proposition 1 the energy function of the solution has a finite value at time \( t_1(t_0, h_0) \), which we will denote by \( h_1(t_0, h_0) \). The successor map is defined precisely using these two functions:

\[ \mathcal{P} : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \mathcal{P}(t_0, h_0) = (t_1(t_0, h_0), h_1(t_0, h_0)). \]

It is fairly easy to show that this map satisfies the compatibility conditions

\[ t_1(t_0 + 2\pi, h_0) = t_1(t_0, h_0) + 2\pi \quad \text{and} \quad h_1(t_0 + 2\pi, h_0) = h_1(t_0, h_0), \]

(13)
due to periodicity of equation (3). Additionally, from existence and uniqueness of bouncing solutions we deduce that \( \mathcal{P} \) is one-to-one. Moreover, we have the following result

**Proposition 2.** There exists a function \( \psi : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\} \) such that the domain of \( \mathcal{P} \) is characterized by

\[ D = \{(t_0, h_0) \in \mathbb{R}^2 : h_0 < \psi(t_0)\}. \]
The function $\psi$ is $2\pi$-periodic, lower semi-continuous and
\[ \min_{\mathbb{R}} \psi \geq -\frac{\alpha}{\alpha - 1} \|p\|_\infty. \]
Finally for each $t_0 \in \mathbb{R}$, the function $t_1(t_0, \cdot)$ is strictly increasing in $]-\infty, \psi(t_0)[$, 
lim_{h_0 \to -\infty} t_1(t_0, h_0) = t_0 and lim_{h_0 \to \psi(t_0)} t_1(t_0, h_0) = +\infty.

![Figure 1. The restriction of a possible (according to Proposition 2) set $D$ to the strip $0 \leq t_0 \leq 2\pi$.](image)

In the proof of this proposition Lemma 4.6 in the Appendix is fundamental. Also comparison with some autonomous equations is used and hence also Lemma 4.3 is applied. The theorem implies that $P$ is a twist map.

If $p$ is a $C^k$ function and $\alpha \geq 2k$, then $P$ is $C^k$-differentiable. This fact has a very delicate proof. Essentially, the idea is to work with the regularized system
\[
\begin{align*}
U' &= V \\
V' &= \frac{1}{\alpha - 1} + p(T)|U|^{\alpha} + \alpha|U|^{\alpha - 1}H \\
T' &= |U|^{\frac{2}{\alpha}} \\
H' &= p(T)V
\end{align*}
\]
and to find the first return instant to the surface
\[ \{(U, V, T, H) \in \mathbb{R}^4 : U = 0\} \]
by means of the implicit function theorem. Then $C^k$-differentiability of $P$ is a consequence of $C^k$-differentiability of this first return map, which in turn is a consequence of $C^k$-regularity of (14). An analogous proof for the case $k = 1$ and $\alpha = 2$ is given in [7].

In [8] it is proved, with an analogous proof to the one in [7] for the particular case, that $P$ is exact symplectic, i.e., the differential form $r_1 d\theta_1 - rd\theta$ is exact on the cylinder. This means there exists some function $V \in C^2(D)$ such that
\[ dV = r_1 d\theta_1 - rd\theta \quad \text{and} \quad V(\theta + 2\pi, r) = V(\theta, r), \quad \forall (\theta, r) \in D. \]

We are now in position to apply a version of the Poincaré-Birkhoff theorem given in [7] and used there to prove the main theorem, which generalizes for $\alpha \geq 2$:

**Theorem 2.4.** Suppose $p$ is $2\pi$-periodic and of class $C^1$. Given $M \in \mathbb{N}$, equation (3) has at least two bouncing solutions of period $2M\pi$ and having exactly one collision in the interval $[0, 2\pi]$ and none in $[2\pi, 2M\pi]$. 
3. **Main results.** The previous discussion leads immediately to a naïve question. Is it possible, by applying similar methods, fixing $M, n \in \mathbb{N}$ to deduce the existence of periodic solutions with $n$ collisions in the interval $[0, 2M\pi]$? The answer to this question is the ultimate goal of this discussion.

In order to prove there are periodic solutions with period $2M\pi$ and containing $n$ collisions in $[0, 2M\pi]$, where $n, M \in \mathbb{N}$, it suffices to prove that the equation:

$$\mathcal{P}^n(t_0, h_0) = (t_0 + 2M\pi, h_0)$$

(15)

has a solution in $\mathbb{R}^2$.

Now, the idea behind the proof is to apply a version of the Poincaré-Birkhoff theorem to the iterates $\mathcal{P}^n$ of the successor map. However, two different questions come up. The first one is if we can apply the version of the Poincaré-Birkhoff theorem used to obtain the previous result. The second one is more delicate. What is the domain where these iterates are well-defined (see Figure 1)?

As it turns out, the answer to the first question is no, we cannot apply the same version of the Poincaré-Birkhoff theorem. The reason for this is that we cannot guarantee anymore that $\mathcal{P}^n$ is a twist map in the points where it is well-defined. In fact, for example for $n = 2$, denoting $\mathcal{P}^2 = (t_2(t_0, h_0), h_2(t_0, h_0))$ we cannot guarantee that for fixed $t_0$ we have that $h_0 \rightarrow h_2(t_0, h_0)$ is strictly increasing as we cannot guarantee that for $h_0 < h_0^*$ we have $t_1(t_0, h_0) = t_1(t_0, h_0^*)$ in order to apply the monotonicity property.

Instead, we will use the the Poincaré-Birkhoff theorem in the cylinder (see for example [5]).

**Theorem 3.1.** Consider a map $\mathcal{S} : B \longrightarrow B$ given by

$$\mathcal{S}(\theta, r) = (\theta_1, r_1) = (T(\theta, r), R(\theta, r)), \text{ where } B = \mathbb{R} \times [-b, b] \ (b > 0)$$

which is the lift of an area preserving homeomorphism $\widetilde{\mathcal{S}} : \widetilde{B} \longrightarrow \widetilde{B}$, where $\widetilde{B} = S^1 \times [-b, b]$. Assume that $\widetilde{\mathcal{S}}$ is isotopic by homeomorphisms to the identity in the cylinder and that

$$T(\theta, b) - \theta > 0, \quad \theta \in [0, 2\pi], \quad T(\theta, -b) - \theta < 0, \quad \theta \in [0, 2\pi].$$

Then $\mathcal{S}$ has at least two fixed points.

We recall that we say that two $C^k$-embeddings are $C^k$-isotopic if there exists an homotopy between these two maps, $\mathcal{H} : \widetilde{B} \times I \rightarrow \widetilde{B}$ such that, for each $\lambda \in I$, $\mathcal{H}_\lambda$ is a $C^k$-embedding. We will always denote the unit interval $[0, 1]$ by $I$ throughout the text.

We will follow the steps used by Maró in [6] to construct a map $\mathcal{S}$ to which we apply Theorem 3.1 in order to deduce the existence of periodic bouncing solutions with multiple collisions. More precisely we will construct a function $\mathcal{S}$ defined in a strip $B$ and which is an extension of $\mathcal{P}|_A$ where $A$ is a strip contained in $B$.

To do that we will need to consider a family of equations

$$\ddot{u} = -\frac{1}{u^\alpha} + p_\lambda(t), \quad (16)$$

where $p_\lambda$ depend continuously on $\lambda$ and state some lemmas on continuous dependence of their solutions on $\lambda$. As the continuous dependence on $\lambda$ is just a consequence of continuous dependence in the regularized system for (16), the proofs of these lemmas are analogous to the ones in [7] for the case without $\lambda$ and hence we omit them.
Lemma 3.2. The Sundman integral

\[ S(t; t_0, h_0, \lambda) = \int_{t_0}^{t} \frac{ds}{u^2(s; t_0, h_0, \lambda)} \]

is a continuous function of the four variables in the set

\[ D = \{(t; t_0, h_0, \lambda) \in \mathbb{R}^3 \times I : t_0 < t < t_1\} . \]

As a simple corollary of the last lemma, we have the continuity of solutions of (20) on \( \lambda \).

Lemma 3.3. The map

\[ (t; t_0, h_0, \lambda) \in D \mapsto (u(t; t_0, h_0, \lambda), \dot{u}(t; t_0, h_0, \lambda)) \in \mathbb{R}^2 \]

is continuous.

Also we need the following

Lemma 3.4. Let \( u \) be a classical solution of (3) defined in \( ]t_0, t_1[ \), \( t_0, t_1 \in \mathbb{R} \). Suppose there exists \( \tau \in ]t_0, t_1[ \) such that

\[ 0 < u(\tau) < \frac{1}{\sqrt{\| p \|_\infty}}, \quad \dot{u}(\tau) < 0. \]

Then, the solution satisfies:

1. \( \ddot{u}(t) < 0, \quad \forall \ t \in ]\tau, t_1[ \);  
2.  
\[ t_1 < \tau - \frac{u(\tau)}{\dot{u}(\tau)}. \quad (17) \]

To meet the theorem’s conditions we will first prove the following lemma

Lemma 3.5. The function \( P \) is isotopic to the inclusion in its domain.

Proof. We could prove this fact in different ways, we will proceed by construction.

First of all, we note that the successor map associated to an autonomous equation is always isotopic to the inclusion. In fact, given the autonomous equation

\[ \ddot{u} = -\frac{1}{u^\alpha} + P, \quad P \in \mathbb{R}, \quad (18) \]

its successor map, which we will denote by \( \mathcal{A}_P(t_0, h_0) \), is given by the expression

\[ \mathcal{A}_P(t_0, h_0) = (t_0 + \tau(h_0, P), h_0). \quad (19) \]

This map is easily shown to be isotopic to the inclusion in \( \mathbb{R}^2 \), it suffices to consider the isotopy that “stops” time:

\[ \mathcal{G}(t_0, h_0, \lambda) = (t_0 + \lambda \tau(h_0, P), h_0). \]

Going back to equation (3), the idea is that its successor map \( \mathcal{P} \) is isotopic in its domain \( D \) to \( \mathcal{A}_P \) where we choose \( P = -\|p\|_\infty \). Then the conclusion that \( \mathcal{P} \) is isotopic to the inclusion immediately follows.

Note that by Lemma 4.5, the domain of the successor map \( \mathcal{H}_\lambda \) of the equation

\[ \ddot{u} = -\frac{1}{u^\alpha} + p_\lambda(t), \quad (20) \]

where \( p_\lambda(t) = \lambda p(t) + (1 - \lambda)(-\|p\|_\infty) \) and \( \lambda \in I \), contains \( D \). In what follows \( \mathcal{H}_\lambda \) denotes its restriction to \( D \). Observe that \( \mathcal{H}_1 = \mathcal{P} \) and \( \mathcal{H}_0 = \mathcal{A}_P \) and that
\[ \|p_\lambda\|_\infty \leq 2\|p\|_\infty \] for all \( \lambda \in I \). We will prove that \( \mathcal{H} \) is an isotopy between \( \mathcal{P} \) and \( \mathcal{A}_\mathcal{P} \).

First, let us discuss the continuity of \( \mathcal{H} \) in all the three variables. We begin by proving that \( t_1(t_0, h_0, \lambda) \) is a continuous function.

Let \( \{(t_0, h_0, \lambda_n)\}_{n \in \mathbb{N}} \subseteq D \) be a sequence converging to \((t_0, h_0, \lambda) \in D\) and remember that the solutions \( u(\cdot; t_0, h_0, \lambda_n) \) converge uniformly to \( u(\cdot; t_0, h_0, \lambda) \) on compact intervals inside \([t_0, t_1]\).

For simplicity let us denote \( t_{1n} := t_1(t_0, h_0, \lambda_n), \; t_1 := t_1(t_0, h_0, \lambda), \; u_n := u(\cdot; t_0, h_0, \lambda_n) \) and \( u := u(\cdot; t_0, h_0, \lambda) \).

Fix \( \delta > 0 \) and let \( \tau \in \left[ t_1 - \frac{\delta}{2}, t_1 \right] \) satisfy
\[
0 < u(\tau) < \frac{1}{\sqrt{2\|p\|_\infty}}, \quad \dot{u}(\tau) < 0 \quad \text{and} \quad \left| \frac{u(\tau)}{\dot{u}(\tau)} \right| < \frac{\delta}{2}. \tag{21}
\]

Notice that it is possible to choose \( k \in \mathbb{N} \) in order that \( u_n(\tau) \) and \( \dot{u}_n(\tau) \) satisfy (21), \( \forall \; n \geq k \). Thus, by Lemma 3.4, we conclude that
\[
|t_{1n} - t_1| \leq |t_{1n} - \tau| + |\tau - t_1| < \delta, \quad \forall \; n \geq k.
\]

Now it is left to check the continuity of the function \( h_1(t_0, h_0, \lambda) \). Consider a point \((t_0, h_0, \lambda) \) and a sequence \( \{(t_0, h_0, \lambda_n)\} \) in the same conditions as before. By simplicity, set \( h_{1n} := h_1(t_0, h_0, \lambda_n) \) and \( h_1 := h_1(t_0, h_0, \lambda) \).

Fix \( \delta > 0 \), as the function \( h(\cdot; t_0, h_0, \lambda) \) is continuous in \([t_0, t_1]\) we can choose \( \tau \in \]t_0,t_1\[ \) such that
\[
\|p\|_\infty \cdot u(\tau) < \frac{\delta}{4}, \tag{22}
\]
and
\[
|h(\tau; t_0, h_0, \lambda) - h_{1n}| < \frac{\delta}{4}. \tag{23}
\]

By continuous dependence, we can choose a number \( k_1 \in \mathbb{N} \) such that
\[
|h(\tau; t_0, h_0, \lambda_n) - h(\tau; t_0, h_0, \lambda)| < \frac{\delta}{4}, \quad \forall \; n \geq k_1. \tag{24}
\]

Combining (23) and (24) we get
\[
|h(\tau; t_0, h_0, \lambda_n) - h_{1}| < \frac{\delta}{2}, \quad \forall \; n \geq k_1. \tag{25}
\]

To establish the continuity of \( h_1(t_0, h_0, \lambda) \) we must prove that there is a number \( k \in \mathbb{N} \) such that
\[
|h_{1n} - h_{1}| < \delta, \quad \forall \; n \geq k.
\]

Therefore, since we already know the inequality (25), now it suffices to prove that there is a number \( k_2 \in \mathbb{N} \) such that
\[
|h_{1n} - h(\tau; t_0, h_0, \lambda_n)| < \frac{\delta}{2}, \quad \forall \; n \geq k_2.
\]

If we apply the fundamental theorem of calculus and the mean value theorem, taking into account that by Lemma 3.4 we have \( \dot{u}(t) < 0 \) for each \( t \in [\tau, t_{1n}] \), we obtain
\[
|h_{1n} - h(\tau; t_0, h_0, \lambda_n)| = \left| -p_{\lambda_n}(\xi) \; u_n(\tau) \right|, \quad \xi \in [\tau, t_{1n}],
\]
and so from here it follows that
\[
|h_{1n} - h(\tau; t_0, h_0, \lambda_n)| \leq 2\|p\|_\infty \; u_n(\tau).
\]
Since, by continuous dependence, there is a \( k_2 \in \mathbb{N} \) such that
\[
\|p\|_{\infty} u_n(\tau) < \frac{\delta}{4}, \quad \forall \ n \geq k_2,
\]
The proof of the continuity is completed.

Now, let us notice that for each \( \lambda \in I \), the map \( \mathcal{H}_\lambda \) is one-to-one, continuous and differentiable since it is a successor map. Moreover, it is a differentiable embedding since its inverse is also a successor map. Therefore, \( \mathcal{H} \) is an isotopy between \( \mathcal{P} \) and \( \mathcal{A}_\mu \).

\[ \square \]

In order to apply the Poincaré-Birkhoff theorem we will consider, as previously said, the successor map restricted to a suitable set. To define this set we will need the following lemma.

**Lemma 3.6.** There exist a \( C^\infty \) 2\(\pi\)-periodic function \( c : \mathbb{R} \to \mathbb{R}, \ t_0 \mapsto c(t_0) \) and \( a_0 > 0 \) such that \( t_1(t_0,c(t_0)) > t_0 + 2M\pi + 1/2 \) and
\[
\begin{cases}
    h_1(t_0,h_0) - h_0 < \frac{1}{n}, \\
    t_1(t_0,h_0) - t_0 < \frac{1}{\pi},
\end{cases}
\]
for all \( h_0 \leq -a_0 + 1 \).

**Proof.** By Proposition 2 we know that for each \( t_0 \) there exists an unique \( c_1(t_0) \) such that \( t_1(t_0,c_1(t_0)) = t_0 + 2M\pi + 1 \). Also it is clear that \( c_1 \) is \( T \)-periodic and it is easy to prove that it is continuous. Either \( t_0 \mapsto c_1(t_0) \) is \( C^\infty \) or it is not but by the Fejer-Cesaro theorem we can approximate it by a \( C^\infty \) 2\(\pi\)-periodic function \( c \) satisfying the desired property. The existence of \( a_0 \) follows from the same results and also from the fact that for each \( t_0 \) the
\[
\lim_{h_0 \to -\infty} (h_1(t_0,h_0) - h_0) = 0.
\]
This last property comes from the fact that, as in [7, Proof of Proposition 3.1.],
\[
h_1(t_0,h_0) - h_0 = -\int_{t_0}^{t_1} p(s)u(s)ds
\]
and the result easily follows using the properties of the solution of the autonomous equation (40) associated with \( \mathcal{P} = \|p\|_{\infty} \).

We can now consider the successor map \( \mathcal{P} \) and the region
\[
A = \{ (t_0, h_0) \in \mathbb{R}^2 : -a_0 - 1 \leq h_0 \leq c(t_0) \},
\]
contained in the domain of \( \mathcal{P} \).

The set \( A \) can be transformed into a strip via the area-preserving diffeomorphism
\[
\mathcal{C}(t_0,h_0) = \left( \frac{\int_0^{t_0} c(s)ds + (a_0 + 1) t_0}{\mu + a_0 + 1}, -a_0 - 1 + (\mu + a_0 + 1) \frac{h_0 + a_0 + 1}{c(t_0) + a_0 + 1} \right),
\]
where \( \mu = \frac{1}{2\pi} \int_0^{2\pi} c(s)ds \). Therefore we will assume that \( A \) is the strip \( \mathbb{R} \times [-a_0 - 1, \mu] \) and keep the letter \( \mathcal{P} \) to denote the composition \( \mathcal{CPC}^{-1} \). Note that points which verify (15) for this new \( \mathcal{P} \) lead to points which verify (15) for the previous one.

Set \( B = \mathbb{R} \times [-b, b] \), where we have chosen \( b > \max \{ \mu, a_0 + 1 \} \) and such that \( \mathcal{P}(A) \subseteq B \).

Since \( \mathcal{P} \) is of class \( C^2 \) we can apply [6, Lemma 2.4. and the remarks above] and extend \( \mathcal{P}|_A : A \to B \) to an area-preserving diffeomorphism \( g_1 : B \to B \) which is isotopic to the identity, restricted to \( \partial B \) is the identity and it is the lift of a
diffeomorphism in the cylinder $\mathcal{B} = S^1 \times [-b, b]$. Then we can alter it on the regions where its values do not agree with $\mathcal{P}$. Let us write $g_1$ as

$$g_1(t_0, h_0) = (t_0 + \varphi(t_0, h_0), \xi(t_0, h_0)),$$

where $\varphi : B \to \mathbb{R}$ is a real function. As $g_1$ is a continuous extension of $\mathcal{P}$, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\begin{align*}
\varphi(t_0, h_0) &> 2M \pi, \quad \mu \leq h_0 \leq \mu + \varepsilon_1 \\
\varphi(t_0, h_0) &< \frac{1}{n}, \quad -a_0 - 1 - \varepsilon_2 \leq h_0 \leq -a_0 - 1 \\
\xi(t_0, h_0) - h_0 &< \frac{1}{n}, \quad -a_0 - 1 - \varepsilon_2 \leq h_0 \leq -a_0 - 1.
\end{align*}$$

(27)

Consider two smooth positive functions on $\mathbb{R}$ such that

$$\psi_1(h_0) = \begin{cases} 0, & \text{if } h_0 \leq \mu \\ 1, & \text{if } h_0 \geq \mu + \varepsilon_1 \end{cases}$$

(28)

and also the following translations defined in $B$

$$\begin{align*}
Q_1(t_0, h_0) &= (t_0 + L\psi_1(h_0), h_0) \\
Q_2(t_0, h_0) &= (t_0 - L\psi_2(h_0), h_0),
\end{align*}$$

where

$$L = \sup_{(t_0, h_0) \in B} |\varphi(t_0, h_0)| + 2M \pi$$

and define $f(t_0, h_0) = (T(t_0, h_0), R(t_0, h_0)) = g_1 \circ Q_2 \circ Q_1(t_0, h_0)$.

Lemma 3.7. The map $f : B \to B$ defined above is an area-preserving diffeomorphism such that the points satisfying the property

$$f^n(t_0, h_0) = (t_0 + 2M \pi, h_0)$$

(30)

verify (15).

Proof. The function $f$ is the composition of area-preserving diffeomorphisms. Hence the only thing we need to check is that the points satisfying property (30) are such that their orbit along $n$-iterates of $f$, $\text{Orb}_n(x) = \{ y \in B : y = f^k(x), \ k = 0, \ldots, n \} \subset A$.

Let us examine what is the orbit $\text{Orb}_n(x)$ of points outside $A$.

First we look at the image of these points by the map $f$. In the strip $\mathbb{R} \times [\mu + \varepsilon_1, b]$, which we will refer to as region $D_1$ (see Figure 2),

$$f(t_0, h_0) = (t_0 + L + \varphi(t_0 + L, h_0), R(t_0 + L, h_0)),$$

therefore, $T(t_0, h_0) > t_0 + 2M \pi$.

Analogously, on the region $\mathbb{R} \times [-b, -a_0 - 1 - \varepsilon_2]$, which we will denote by $D_4$,

$$f(t_0, h_0) = (t_0 - L + \varphi(t_0 - L, h_0), R(t_0 - L, h_0)),$$

and thus $T(t_0, h_0) < t_0$.

On the strip $D_2 := \mathbb{R} \times [\mu, \mu + \varepsilon_1]$, we do not know the precise value of $\psi_1$. However, by (27), as $\varphi(t_0 + L\psi_1(h_0), h_0) > 2M \pi$ on this interval, then $T(t_0, h_0) > t_0 + 2M \pi$.
On the strip $D_3 := \mathbb{R} \times [-a_0 - 1 - \varepsilon_2, -a_0 - 1]$, as $\varphi(t_0 - L\psi_2(h_0), h_0) < \frac{1}{n}$ we conclude $T(t_0, h_0) < t_0 + \frac{1}{n}$. Moreover, also by (27), points on this region are subject to the condition
\[ R(t_0, h_0) - h_0 < \frac{1}{n}. \]

Figure 2. Cylinder B

Now, let us denote $f^n(t_0, h_0) = (T_n(t_0, h_0), R_n(t_0, h_0))$. Points in region $D_4$ such that their orbit remains in $D_4$, cannot satisfy property (30) as $T_n(t_0, h_0) < t_0$ in this set of points. Therefore points in $D_4$ can satisfy property (30) only if their orbit leaves $D_4$. We will return to this case in a moment but first consider what happens to points in the remaining regions. A point such that its orbit remains in $D_3$ cannot satisfy property (30) because $T_n(t_0, h_0) > t_0 + 1 < t_0 + 2M\pi$. The same happens trivially in regions $D_1$ and $D_2$ because $T(t_0, h_0) > t_0 + 2M\pi$ in these regions. Therefore we must examine orbits that leave its original region.

Observe that $T_n(t_0, h_0) > 0$ in region $A$. Therefore we may conclude that points in regions $D_1$ and $D_2$ could only satisfy property (30) if its orbit intersected regions $D_3$ or $D_4$. However, we will show that points in regions $D_3$ and $D_4$ neither satisfy property (30) nor can their orbits intersect sets $D_1$ and $D_2$ in less than $n$-iterates.

By (27) and the definition of $a_0$, we see that points in $D_3$ do not leave the region $\mathbb{R} \times [-b, -a_0]$. And so points in $D_3 \cup D_4$ cannot reach regions $D_1$ and $D_2$ in less than $n$-iterates. As, by (26), points in $\mathbb{R} \times [-b, -a_0]$ satisfy
\[ T_n(t_0, h_0) < t_0 + 1, \]
we conclude that there are no points in $D_3 \cup D_4$ satisfying property (30).

Finally points in $A$ which satisfy (30) have $n$-orbits which cannot intersect $D_i$ for each $i$. In fact it is immediate to see that they cannot intersect $D_1$ and $D_2$. Also they cannot intersect $D_3$ and $D_4$, in fact in order that the orbits intersect these sets the points should lie in $\mathbb{R} \times [-a_0 - 1, -a_0]$ and in this case by Lemma 3.6 they cannot satisfy (30).

We are finally in position to state and prove our main theorem.
Theorem 3.8. Suppose $p$ is $2\pi$-periodic and of class $C^2$. Given $n, M \in \mathbb{N}$, equation (3) has at least two bouncing solutions of period $2M\pi$ and having exactly $n$ collisions in the interval $[0, 2M\pi[$.

Proof. We apply Theorem 3.1 to the map $S : B \to B$ defined by
$$S(t_0, h_0) = f^n(t_0, h_0) - (2M\pi, 0).$$

This map is isotopic to the identity, it is area-preserving and satisfies the boundary twist conditions. Now by Lemma 3.7 the result follows. \hfill \square

4. Appendix. In this appendix we give the proofs of auxiliary results.

4.1. Asymptotic expansions near a collision. In order to obtain the asymptotic expansions near collisions for the solutions of equation (3), we adapted the analogous proof in [10]. We give here the details. Let us begin with two auxiliary lemmas.

Lemma 4.1. Consider a solution $u(\cdot)$ of equation (3) defined in its maximal interval of existence $[t_0, t_1]$. Then, if $t_0 \in \mathbb{R}$,
$$\lim_{t \to t_0^+} \frac{u}{\pi} \dot{u} = 0.$$

Proof. First we prove that $\dot{u}$ maintains the sign in a neighbourhood of $t_0$. Note that, since $u(t)$ is positive in the interval $[t_0, t_1]$ and it has a collision at $t_0$, there exists $\varepsilon > 0$ such that $\dot{u}(t) \leq 0$, $\forall t \in [t_0, t_0 + \varepsilon]\subset [0, t_1]$. Also,
$$\exists t^* \in (t_0, t_0 + \varepsilon]: \quad \dot{u}(t^*) > 0.$$

Then
$$\dot{u}(t^*) - \dot{u}(t) = \int_{t}^{t^*} \dot{u}(s) \, ds < 0, \quad \forall t \in [t_0, t^*],$$
and so $\dot{u}(t) > \dot{u}(t^*) > 0$, $\forall t \in (t_0, t^*]$. We have just proved that there is a neighbourhood of $t_0$ in which $\dot{u}$ is positive. Multiplying equation (3) by $\dot{u}(t)$ and integrating we find that
$$\frac{1}{2} \dot{u}^2(t^*) - \frac{1}{2} \dot{u}^2(t) = -\left[ \frac{1}{1 - \alpha u^{\alpha - 1}} \right]_t^{t^*} + \int_{t}^{t^*} u(s)p(s) \, ds,$$
where $t \in [t_0, t^*]$. To the last integral we apply the mean value theorem and conclude that
$$\frac{1}{2} \dot{u}^2(t^*) - \frac{1}{2} \dot{u}^2(t) = -\left[ \frac{1}{1 - \alpha u^{\alpha - 1}} \right]_t^{t^*} + p(\xi) \int_{t}^{t^*} \dot{u}(s) \, ds, \quad \text{for some } \xi \in [t, t^*],$$
and so, noting that $\xi$ depends on $t$, we find
$$\dot{u}^2(t) = \frac{2}{1 - \alpha u^{\alpha - 1}(t^*)} - \frac{2}{1 - \alpha u^{\alpha - 1}(t)} - 2p(\xi(t)) (u(t^*) - u(t)) + \dot{u}^2(t^*).$$

So in fact we have
$$\dot{u}^2(t) = -\frac{2}{1 - \alpha u^{\alpha - 1}(t)} + 2p(\xi(t)) u(t) - 2p(\xi(t)) u(t^*) + A,$$
where $A$ is a constant depending just on $t^*$.

Therefore,
$$u^\alpha(t) \dot{u}^2(t) = -\frac{2}{1 - \alpha} u(t) + 2p(\xi(t)) u^{\alpha + 1}(t) - 2p(\xi(t)) u(t^*) u^\alpha(t) + A u^\alpha,$$
As \( \lim_{t \to t_0} u(t) = 0 \) and \( p \) is bounded it follows

\[
\lim_{t \to t_0} u^\alpha(t) \dot{u}^2(t) = 0.
\]

Finally, taking square roots we end up with

\[
\lim_{t \to t_0} u^2(t) \dot{u}(t) = 0.
\]

Lemma 4.2. Consider again a solution \( u(\cdot) \) of equation (3) defined in its maximal interval of existence \([t_0, t_1]\). Then, if \( t_0 \in \mathbb{R} \), the energy function \( h \) of the system is bounded in a neighbourhood of \( t_0 \).

Proof. We have that

\[
\dot{h}(t) = \dot{u}(t)p(t)
\]

is the derivative of the energy function. As before, let us choose a point \( t^* \) in a neighbourhood of \( t_0 \), say \([t_0, t_0 + \varepsilon]\), in which \( \dot{u} \) is positive. Integrating the derivative of the energy between some \( t \in [t_0, t^*] \) and \( t^* \) and then applying the mean value theorem we conclude

\[
h(t^*) - h(t) = \int_t^{t^*} \dot{u}(s)p(s) \, ds = p(\xi)(u(t^*) - u(t)), \quad \xi \in [t, t^*].
\]

As \( p \) is a bounded function and \( u \) stays bounded as \( t \) approaches \( t_0 \) we conclude that \( h(t) \) remains bounded as \( t \) approaches \( t_0 \).

Now we are ready to obtain the asymptotic expansions.

Proof of (5), (6). First note that

\[
\frac{d^2}{dt^2}(u^\alpha) = \alpha(\alpha - 1)u^{\alpha-2}\ddot{u}^2 + \alpha u^{\alpha-1}\dddot{u}.
\]

We drop the argument on \( u \) and its derivatives for simplicity. We know that

\[
\ddot{u}^2 = 2h(t) + \frac{2}{(\alpha - 1)u^{\alpha-1}}
\]

and that

\[
\dddot{u} = -\frac{1}{u^\alpha} + p(t).
\]

Inserting (33) and (34) in (32) we get

\[
\frac{d^2}{dt^2}(u^\alpha) = 2\alpha(\alpha - 1)u^{\alpha-2}h(t) + \frac{\alpha}{u} + \alpha u^{\alpha-1}p(t)
\]

Let us define \( R = u^\alpha \) and \( b(t) = 2\alpha(\alpha - 1)u^{\alpha-2}h(t) + \alpha u^{\alpha-1}p(t) \). Note that \( b \) is a bounded function in a neighbourhood of \( t_0 \).

Then we find that

\[
\dddot{R} = \frac{\alpha}{R^{\frac{\alpha}{2}}} + b(t).
\]

Multiplying this equation by \( \dddot{R} \) and integrating we get

\[
\int_{t_0}^t \frac{d}{ds} \left( \frac{1}{2} \dddot{R}^2 \right) \, ds = \int_{t_0}^t \frac{\alpha}{R^{\frac{\alpha}{2}}} \dddot{R} \, ds + \int_{t_0}^t b(s) \dddot{R} \, ds.
\]
We have to be careful with the integration. As \( \lim_{t \to t_0} R(t) = 0 \) and
\[
\lim_{t \to t_0} \dot{R}(t) = \lim_{t \to t_0} \alpha u^{\alpha-1} \dot{u}(t) = \lim_{t \to t_0} \alpha u^{2-1} u^2 \dot{u}(t) = 0,
\]
by Lemma 4.1 and as \( \frac{2}{\alpha} - 1 \geq 0 \), then it follows that
\[
\dot{R}^2(t) = \frac{2\alpha^2}{\alpha - 1} R^{\frac{\alpha-1}{\alpha}}(t) + 2 \int_{t_0}^{t} b(s) \dot{R}(s) \, ds.
\]
To the second integral, we apply the mean value theorem to conclude that there exists \( \xi(t) \in [t_0, t] \) such that
\[
\int_{t_0}^{t} b(s) \dot{R}(s) \, ds = b(\xi(t)) \int_{t_0}^{t} \dot{R}(s) \, ds,
\]
Therefore, defining \( b(t) = b(\xi(t)) \) we obtain
\[
\dot{R} = \left( \frac{2\alpha^2}{\alpha - 1} R^{\frac{\alpha-1}{\alpha}}(t) + 2b(t)R(t) \right)^{\frac{1}{2}}. \tag{35}
\]
Again let us operate another change of variables. This time let
\[
z = R^{\frac{\alpha+1}{2\alpha}}.
\]
Therefore, substituting in equation (35) we get
\[
\dot{z} = \frac{\alpha + 1}{2\alpha} \left[ \frac{2\alpha^2}{\alpha - 1} + 2b(t)z^{\frac{\alpha}{\alpha+1}} \right]^{\frac{1}{2}}. \tag{36}
\]
Integrating between \( t_0 \) and \( t \) and using the fact that \( \lim_{t \to t_0} z(t) = 0 \) it follows
\[
z(t) = \frac{\alpha + 1}{2\alpha} \int_{t_0}^{t} \left[ \frac{2\alpha^2}{\alpha - 1} + 2b(s)z^{\frac{\alpha}{\alpha+1}} \right]^{\frac{1}{2}} \, ds. \tag{37}
\]
Notice that
\[
\left| \frac{z(t)}{t - t_0} \right| = \left| \frac{z(t) - z(t_0)}{t - t_0} \right| = |\dot{z}(\xi)|, \text{ with } \xi \in [t_0, t].
\]
Furthermore, \( |\dot{z}(\xi)| \) is bounded in a neighbourhood of \( t_0 \) as the function \( b \) is also bounded and as \( z \) converges to 0 as \( t \) approaches \( t_0 \). Therefore we conclude that
\( z(t) = O(t - t_0) \).
Going back to the formula (37) we find
\[
z(t) = \frac{\alpha + 1}{2\alpha} \int_{t_0}^{t} \left[ \frac{2\alpha^2}{\alpha - 1} + 2b(s)O \left( (s - t_0)^{\frac{\alpha}{\alpha+1}} \right) \right]^{\frac{1}{2}} \, ds.
\]
Then, as \( b \) is a bounded function,
\[
z(t) = \frac{\alpha + 1}{2\alpha} \int_{t_0}^{t} \left[ \frac{2\alpha^2}{\alpha - 1} + O \left( (s - t_0)^{\frac{\alpha}{\alpha+1}} \right) \right]^{\frac{1}{2}} \, ds.
\]
Expanding the integrand we deduce
\[
z(t) = \frac{\alpha + 1}{2\alpha} \int_{t_0}^{t} \left[ \left( \frac{2\alpha^2}{\alpha - 1} \right)^{\frac{1}{2}} + O \left( (s - t_0)^{\frac{\alpha}{\alpha+1}} \right) \right] \, ds. \tag{38}
\]
And again expanding the integral, it follows
\[
z(t) = \frac{\alpha + 1}{2\alpha} \left( \frac{2\alpha^2}{\alpha - 1} \right)^{\frac{1}{2}} (t - t_0) \left[ 1 + O \left( (t - t_0)^{\frac{2}{\alpha + 1}} \right) \right].
\] (39)

Finally, as \( u = z^{\frac{2}{\alpha + 1}} \) we get after expanding the power
\[
u(t) = \left[ \frac{\alpha + 1}{2\alpha} \left( \frac{2\alpha^2}{\alpha - 1} \right)^{\frac{1}{2}} \right] \left( t - t_0 \right)^{\frac{2}{\alpha + 1}} + O \left( (t - t_0)^{\frac{4}{\alpha + 1}} \right).
\]

We have found the Taylor expansion of \( u \) around \( t_0 \).

Now, we know that \( \dot{u} = \frac{2}{\alpha + 1} \alpha^\alpha \dot{z} \). Following the same procedure we used to obtain equation (38), we can write equation (36) in the form
\[
\dot{z}(t) = \frac{\alpha + 1}{2\alpha} \left( \frac{2\alpha^2}{\alpha - 1} \right)^{\frac{1}{2}} (t - t_0) \left[ 1 + O \left( (t - t_0)^{\frac{2}{\alpha + 1}} \right) \right].
\]

Using equation (39) and expanding the power we get
\[
\dot{u} = \frac{2}{\alpha + 1} \left( \frac{\alpha + 1}{\sqrt{2(\alpha - 1)}} \right)^{\frac{1-\alpha}{\alpha + 1}} (t - t_0)^{\frac{1-\alpha}{\alpha + 1}} \left[ 1 + O \left( (t - t_0)^{\frac{2}{\alpha + 1}} \right) \right] \cdot L(t),
\]

where \( L(t) = \frac{\alpha + 1}{\sqrt{2(\alpha - 1)}} + O \left( (t - t_0)^{\frac{2}{\alpha + 1}} \right) \).

Let us denote
\[
A = \frac{2}{\alpha + 1} \left( \frac{\alpha + 1}{\sqrt{2(\alpha - 1)}} \right)^{\frac{1-\alpha}{\alpha + 1}}
\]
\[
B = \frac{\alpha + 1}{\sqrt{2(\alpha - 1)}}.
\]

Then
\[
\dot{u} = A(t - t_0)^{\frac{1-\alpha}{\alpha + 1}} \left[ B + O \left( (t - t_0)^{\frac{2}{\alpha + 1}} \right) + BO \left( (t - t_0)^{\frac{2}{\alpha + 1}} \right) \right] +
\]
\[
+ A(t - t_0)^{\frac{1-\alpha}{\alpha + 1}} O \left( (t - t_0)^{\frac{2}{\alpha + 1}} \right) O \left( (t - t_0)^{\frac{2}{\alpha + 1}} \right).
\]

Therefore,
\[
\dot{u} = AB(t - t_0)^{\frac{1-\alpha}{\alpha + 1}} + A(t - t_0)^{\frac{1-\alpha}{\alpha + 1}} O \left( (t - t_0)^{\frac{3}{\alpha + 1}} \right).
\]

Hence, we get
\[
\dot{u} = \frac{2}{\alpha + 1} \left( \frac{\alpha^2 + 2\alpha + 1}{2(\alpha - 1)} \right)^{\frac{1}{\alpha + 1}} (t - t_0)^{\frac{1-\alpha}{\alpha + 1}} + O \left( (t - t_0)^{\frac{3}{\alpha + 1}} \right).
\]

\[\square\]

4.2. Autonomous equation and comparison lemmas. Consider now \( P \in \mathbb{R} \) and the autonomous equation
\[
\ddot{u} = -\frac{1}{u^\alpha} + P.
\] (40)
This reduces to a first order Hamiltonian system, with Hamiltonian function

\[ E(u, v) = \frac{1}{2} v^2 + \frac{1}{\alpha - 1} u^{\alpha - 1} - Pu, \]

which is a first integral of the system. It is the sum of the kinetic energy \( K(v) \) and the potential energy \( V(u) \). If \( P \) is negative, then each classical solution is defined in a bounded maximal interval. But this is not the case when \( P \) is non-negative. If \( P \) is positive, classical solutions are defined in a bounded maximal interval if

\[ E \leq -\frac{\alpha}{\alpha - 1} P^{\frac{\alpha}{\alpha - 1}} \]

and

\[ u \leq P^{-\frac{1}{\alpha}}. \]

Finally, if \( P = 0 \), classical solutions are defined in a bounded maximal interval if \( E < 0 \).

In every case, if the classical solution is defined in a bounded maximal interval then it has a unique maximum \( u_{max} \) attained at the midpoint of its interval domain. The length of the domain is given by the positive function

\[ \tau(E, P) = 2 \int_0^{u_{max}} \frac{du}{\sqrt{2(E - V(u))}}. \]  

(41)

Also we have the following

**Lemma 4.3.** Consider a classical solution of equation (40) with a bounded maximal interval of definition. If \( P < 0 \), then

\[ \lim_{h_0 \rightarrow -\infty} \tau(h_0, P) = 0, \quad \lim_{h_0 \rightarrow +\infty} \tau(h_0, P) = +\infty. \]

If \( P > 0 \), then

\[ \lim_{h_0 \rightarrow -\infty} \tau(h_0, P) = 0. \]

**Proof.** Given \( t_0, h_0 \in \mathbb{R} \), let \( u \) be a classical solution of equation (40), satisfying the collision conditions (7) and (8) and having a bounded maximal interval of definition.

First suppose that \( h_0 \) and \( P \) are both negative. Let us define

\[ u_0 := \frac{u_{max}}{2} \]  

(42)

and recall that \( u_{max} \) is the maximum value of \( u \) in its interval of definition. As, in this case, \( V \) is an increasing function we have

\[ -V(u_0) > -V(u_{max}) = -h_0, \]  

(43)

and then, by definition of \( V \),

\[ 1 + (\alpha - 1)Pu_0^{\alpha} + (\alpha - 1)u_0^{\alpha - 1}h_0 > 0. \]

Let us split \( \tau(h_0, P) \) into two integrals

\[ \tau(h_0, P) = 2 \int_0^{u_0} \frac{du}{\sqrt{2(h_0 + Pu + \frac{1}{\alpha - 1} u^{\alpha - 1})}} + 2 \int_{u_0}^{u_{max}} \frac{du}{\sqrt{2(h_0 + Pu + \frac{1}{\alpha - 1} u^{\alpha - 1})}}. \]
Let us prove that the first integral, which we will refer to simply as $A$, vanishes when $h_0$ is sufficiently small. For simplicity, we will omit the constant $\frac{2}{\sqrt{2}}$ for it will not take part in what follows.

\[
A = \int_0^{u_0} \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha-1}u^{\alpha-1}}} = \int_0^{u_0} \frac{\sqrt{(\alpha - 1)u^{\alpha-1}}}{\sqrt{(\alpha - 1)u^{\alpha-1}h_0 + (\alpha - 1)Pu^{\alpha} + 1}}
\]

Therefore,

\[
A \leqslant \int_0^{u_0} \frac{\sqrt{(\alpha - 1)u^{\alpha-1}}}{\sqrt{(\alpha - 1)u^{\alpha-1}h_0 + (\alpha - 1)Pu^{\alpha} + 1}}
\]

\[
= u_0 \frac{\sqrt{(\alpha - 1)u^{\alpha-1}}}{\sqrt{(\alpha - 1)u^{\alpha-1}h_0 + (\alpha - 1)Pu^{\alpha} + 1}}
\]

The fact that $-V(u_{\text{max}}) = -h_0$ implies that

\[
1 + (\alpha - 1)P u_{\text{max}}^\alpha + (\alpha - 1)h_0u_{\text{max}}^{\alpha-1} = 0. \tag{44}
\]

Then, using (42) and (44) we conclude that

\[
(\alpha - 1)h_0u_{\text{max}}^{\alpha-1} + (\alpha - 1)Pu_{\text{max}}^\alpha + 1 = 1 - \frac{1}{2^{\alpha-1}} - (\alpha - 1)P \left(\frac{u_{\text{max}}}{2}\right)^\alpha.
\]

So

\[
A \leqslant \left(\frac{u_{\text{max}}}{2}\right) \frac{\sqrt{(\alpha - 1)\left(\frac{u_{\text{max}}}{2}\right)^\alpha}}{\sqrt{1 - \frac{1}{2^{\alpha-1}} - (\alpha - 1)P \left(\frac{u_{\text{max}}}{2}\right)^\alpha}}.
\]

It is not difficult to prove that when $P$ is negative the following hold

\[
h_0 \to +\infty \Rightarrow u_{\text{max}} \to +\infty, \quad h_0 \to -\infty \Rightarrow u_{\text{max}} \to 0. \tag{45}
\]

Therefore,

\[
\lim_{h_0 \to -\infty} A = 0.
\]

Now let us prove that the second integral, to which we will simply refer to as $B$, also vanishes when $h_0$ is sufficiently small.

Substituting $h_0$ by $V(u_{\text{max}})$ in $B$ it follows

\[
B = \int_{u_0}^{u_{\text{max}}} \frac{du}{P(u - u_{\text{max}}) + \frac{1}{\alpha-1} \left(\frac{u_{\text{max}}^{\alpha-1} - u^{\alpha-1}}{u_{\text{max}}^{\alpha-1} - u^{\alpha-1}}\right)}.
\tag{46}
\]

Applying Lagrange’s Theorem to the function $f(u) = u^{\alpha-1}$ we conclude

\[
u^{\alpha-1} = u^{\alpha-1} + (\alpha - 1)\xi_u^{\alpha-2}(u - u_{\text{max}}), \quad \text{with } \xi_u \in \left[u, u_{\text{max}}\right]. \tag{47}
\]

Hence, as $P(u - u_{\text{max}}) \geq 0$ in $[u_0, u_{\text{max}}]$ and by (47) we have

\[
B \leqslant \int_{u_0}^{u_{\text{max}}} \frac{du}{\sqrt{\frac{1}{\alpha-1} \left(\frac{u_{\text{max}}^{\alpha-1} - u^{\alpha-1}}{u_{\text{max}}^{\alpha-1} - u^{\alpha-1}}\right)}}
\]

\[
= \int_{u_0}^{u_{\text{max}}} \frac{\xi_u^{\alpha-2} + (\alpha - 1)(u - u_{\text{max}})u^{\alpha-1}}{u_{\text{max}} - u} du.
\]
Also, as \( u - u_{\text{max}} \leq 0 \) in \([u_0, u_{\text{max}}]\), we obtain a much simpler inequality

\[
B \leq \int_{u_0}^{u_{\text{max}}} \sqrt{\frac{c^2 - \alpha}{c^2 - \alpha + 1} \frac{u_{\text{max}} - u}{u_{\text{max}}}} \, du.
\]

Finally, \( c^2 - \alpha \leq u_0^2 - \alpha = (\frac{u_{\text{max}}}{c^2 - \alpha})^2 - \alpha \) in \([u_0, u_{\text{max}}]\). Thus,

\[
B \leq \int_{u_0}^{u_{\text{max}}} \sqrt{\frac{(\frac{u_{\text{max}}}{2})^{2 - \alpha}}{u_{\text{max}} - u} \frac{u_{\text{max}} - u}{u_{\text{max}}}} \, du = \sqrt{2^{2 - \alpha} - \alpha} \int_{u_0}^{u_{\text{max}}} \sqrt{\frac{u_{\text{max}}^2}{u_{\text{max}} - u}} \, du = 2\sqrt{2^{\alpha - 2} - \alpha} \sqrt{\frac{u_{\text{max}}^2}{u_{\text{max}} - u}}.
\]

Therefore, it is obvious that \( \lim_{h_0 \to -\infty} B = 0 \). Hence, \( \lim_{h_0 \to -\infty} \tau(h_0, P) = 0 \).

Suppose now that \( h_0 \) is positive and consider some \( c \in [0, u_{\text{max}}] \). Again, as \( V \) is increasing, we have the following inequality for all \( u \in [0, u_{\text{max}}] \)

\[
h_0 + \frac{1}{\alpha - 1} \frac{1}{u^\alpha} > h_0 + Pu + \frac{1}{\alpha - 1} \frac{1}{u^\alpha} > 0.
\]

Therefore,

\[
C := \int_0^c \left. \frac{1}{\sqrt{h_0 + Pu + \frac{1}{\alpha - 1} \frac{1}{u^\alpha}}} \right] \, du \geq \frac{e^{\frac{\alpha + 1}{2}}}{\sqrt{(\alpha - 1)h_0^\alpha + 1}}.
\]

When \( h_0 \to +\infty \) this integral approaches zero. Substituting \( h_0 \) by the expression \( V(u_{\text{max}}) \) as in (46) and using (47), we get

\[
D := \int_c^{u_{\text{max}}} \left. \frac{1}{\sqrt{P(u - u_{\text{max}}) + \frac{1}{\alpha - 1} \left( \frac{1}{u^\alpha} - \frac{1}{u_{\text{max}}^\alpha} \right)}} \right] \, du
\]

\[
= \int_c^{u_{\text{max}}} \left. \frac{1}{\sqrt{\left( \frac{c^\alpha - 2}{c^\alpha - 1} \frac{1}{u^\alpha - 1} \right) \left( u_{\text{max}} - u \right)}} \right] \, du
\]

As \( u \) is being integrated between \( c \) and \( u_{\text{max}} \) it follows

\[
\frac{c^\alpha - 2}{c^\alpha - 1} u_{\text{max}} - 1 \leq \frac{u_{\text{max}}^\alpha - 2}{c^\alpha - 1} \frac{1}{u_{\text{max}}^\alpha} = \frac{1}{c^\alpha - 1} u_{\text{max}}^\alpha
\]

and, hence,

\[
D \geq \int_c^{u_{\text{max}}} \left. \frac{1}{\sqrt{\left( \frac{1}{c^\alpha - 1} \frac{1}{u_{\text{max}}^\alpha} - P \right) \left( u_{\text{max}} - u \right)}} \right] \, du = \frac{2^{\alpha - 2} \sqrt{u_{\text{max}}^\alpha - c}}{\sqrt{c^\alpha - 1} u_{\text{max}}^\alpha - P}
\]

Letting \( h_0 \to +\infty \) we know that \( u_{\text{max}} \to +\infty \). Therefore,

\[
\lim_{h_0 \to +\infty} \frac{2^{\alpha - 2} \sqrt{u_{\text{max}}^\alpha - c}}{\sqrt{c^\alpha - 1} u_{\text{max}}^\alpha - P} = +\infty,
\]

which implies that \( \lim_{h_0 \to +\infty} \tau(h_0, P) = +\infty \).

Now suppose that \( P \) is positive. Let us choose

\[
h_0 \leq V_{\text{max}} := -\frac{\alpha}{\alpha - 1} P^{\frac{\alpha - 1}{\alpha}} < 0.
\]
Recall that as $u$ has bounded maximal interval, then $u \leq P^{-\frac{1}{\alpha}}$, and note that $V$ is strictly increasing on the interval $[0, P^{-\frac{1}{\alpha}}]$.

Define the function

$$\tilde{V}(u) := -\frac{1}{\alpha - 1} \frac{1}{u^{\alpha - 1}}.$$

As $P$ is positive,

$$V(u) < \tilde{V}(u), \forall u \in [0, +\infty].$$

Moreover this function is strictly increasing on $[0, +\infty]$ as its derivative is strictly positive on this interval.

Therefore there exists a unique $u_0 \in [0, +\infty]$ such that

$$\tilde{V}(u_0) = h_0. \quad (48)$$

However as $h_0 = V(u_{\text{max}})$ and $V(u_0) < \tilde{V}(u_0)$, it follows that

$$u_0 < u_{\text{max}}. \quad (49)$$

From the definition of $u_0$ and the fact that the function $V$ attains the value $h_0$ at $u_{\text{max}}$, we can extract formulas for $u_0^{\alpha - 1}$ and $u_{\text{max}}^{\alpha - 1}$, respectively.

Dividing these formulas, we get

$$\frac{u_0^{\alpha - 1}}{u_{\text{max}}^{\alpha - 1}} = 1 + \frac{Pu_{\text{max}}}{h_0}.$$

By letting $h_0 \to -\infty$ and noting that the second limit in (45) still holds, we conclude that

$$\frac{u_0^{\alpha - 1}}{u_{\text{max}}^{\alpha - 1}} \to 1. \quad (50)$$

We prove now that the following integral vanishes when $h_0$ is sufficiently small.

Let

$$E := \int_0^{u_0/2} \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha - 1} u^{\alpha - 1}}} \leq \int_0^{u_0/2} \frac{du}{\sqrt{h_0 + \frac{1}{\alpha - 1} u^{\alpha - 1}}},$$

taking into account that the function on the integrand is defined for all $u \in [0, u_0[$.

Then

$$E \leq \int_0^{u_0/2} \frac{du}{\sqrt{h_0 + \frac{1}{\alpha - 1} \frac{u^{\alpha - 1}}{u_0^{\alpha - 1}}}} = \frac{\sqrt{(\alpha - 1)\left(u_0/2\right)^{\alpha - 1}}}{\sqrt{(\alpha - 1)h_0\left(u_0/2\right)^{\alpha - 1} + 1}} \frac{u_0}{2} = \sqrt{\frac{\alpha - 1}{1 - \frac{1}{2^{\alpha - 1}}} \left(\frac{u_0}{2}\right)^{\frac{\alpha + 1}{\alpha - 1}}},$$

where the last equality uses (48).

By letting $h_0 \to -\infty$ and noting (49) we conclude that

$$\sqrt{\frac{\alpha - 1}{1 - \frac{1}{2^{\alpha - 1}}} \left(\frac{u_0}{2}\right)^{\frac{\alpha + 1}{\alpha - 1}}} \to 0.$$

Therefore, $E$ vanishes. Finally, consider the integral

$$F := \int_{u_0/2}^{u_{\text{max}}} \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha - 1} u^{\alpha - 1}}}.$$
Again, substituting \( h_0 \) by \( V(u_{\text{max}}) \) as in (46) and using (47) the integral becomes
\[
F = \int_{u_0/2}^{u_{\text{max}}/2} \frac{du}{\sqrt{\left(\frac{\xi u^{-2}}{u_{\text{max}}^{\alpha-1}} - P\right)(u_{\text{max}} - u)}}.
\]

Notice that, for all \( u \in [u_0, u_{\text{max}}] \), \( \xi u^{-2} \geq \frac{u_0}{2} \) and \( u < u_{\text{max}} \),
\[
\frac{\xi u^{-2}}{u_{\text{max}}^{\alpha-1}} - P \geq \frac{1}{2^{\alpha-2}} \frac{u_0^{\alpha-2}}{u_{\text{max}}^{\alpha-2}} \frac{1}{u_{\text{max}}^{\alpha-1}} - P > \frac{1}{2^{\alpha-2}} \frac{u_0^{\alpha-1}}{u_{\text{max}}^{\alpha-1}} \frac{1}{u_{\text{max}}} - P.
\] (51)

By (50) and the second limit in (45), we see that (51) diverges when \( h_0 \) is sufficiently small. Therefore we can fix \( K > 0 \) such that
\[
\frac{\xi u^{-2}}{u_{\text{max}}^{\alpha-1}} - P \geq K > 0,
\]
for all \( h_0 \) sufficiently small. Consequently
\[
F \leq \int_{u_0/2}^{u_{\text{max}}/2} \frac{du}{\sqrt{K(u_{\text{max}} - u)}} = \frac{2}{\sqrt{K}} \left( u_{\text{max}} - \frac{u_0}{2} \right)^{\frac{1}{2}} \rightarrow 0,
\]
where the limit is taken when \( h_0 \rightarrow -\infty \). Hence, we have proven that
\[
\lim_{h_0 \rightarrow -\infty} \tau(h_0, P) = 0.
\]

\( \square \)

In the case \( \alpha = 2 \), it is proved in [7] that classical solutions of (3) satisfying the collisions conditions (7) and (8) are well approximated by classical solutions satisfying initial conditions sufficiently close, in the sense that, if \( u \) satisfies (7) and (8) is defined in \([t_0, t_1]\) and \( u^\varepsilon \) is the solution of the Cauchy problem satisfying
\[
u(t_0) = \varepsilon, \quad \dot{u}(t_0) = \sqrt{2\left(h_0 + \frac{1}{\alpha - 1} \frac{1}{\varepsilon^{\alpha-1}}\right)},
\]
then \( u^\varepsilon \) converges to \( u \) in compact intervals contained in \([t_0, t_1]\). A similar result for the general case holds:

**Lemma 4.4.** Suppose that \( p \) is Lipschitz continuous and that \( J \subseteq [t_0, t_1] \) is a compact interval. Then there is an \( \varepsilon_J > 0 \) such that if \( 0 < \varepsilon < \varepsilon_J \) then the solution \( u^\varepsilon \) is well defined and positive on \( J \) and
\[
\lim_{\varepsilon \rightarrow 0} u^\varepsilon(t) = u(t), \quad \lim_{\varepsilon \rightarrow 0} \dot{u}^\varepsilon(t) = \dot{u}(t)
\]
uniformly on \( J \).

This lemma is very useful to prove properties that are preserved by uniform convergence. In fact, it is used in the proof of the last of the two following lemmas on comparison of solutions. This lemma is essential in the proof of Proposition 2. In fact, in order to prove that proposition a comparison with an autonomous equation is used.

**Lemma 4.5.** Suppose \( u_1 \) and \( u_2 \) are solutions of
\[
\ddot{u} = -\frac{1}{u^\alpha} + p_i(t)
\]
for \( i = 1, 2 \), respectively, with \( p_1, p_2 \) Lipschitz-continuous and bounded functions satisfying
\[
p_1(t) \leq p_2(t)
\]
for each \( t \in \mathbb{R} \) and \( u_1 \) and \( u_2 \) defined on maximal intervals \( I_1 = ]t_0, t_1[ \) and \( I_2 = ]t_0^*, t_1^*[ \) such that \( I_1 \cap I_2 \neq \emptyset \). Suppose that for some \( \tau \in I_1 \cap I_2 \)
\[
u_1(\tau) \leq u_2(\tau), \quad \dot{\nu}_1(\tau) \leq \dot{u}_2(\tau).
\]
Then
\[
t_1 \leq t_1^*, \quad u_1(t) \leq u_2(t), \quad \dot{u}_1(t) \leq \dot{u}_2(t), \quad \forall \; t \in [\tau, t_1[.
\]
Lemma 4.6. Suppose \( p_i \) are as in the previous lemma and \( u_i \) are solutions of
\[
\ddot{u} = -\frac{1}{u^\alpha} + p_i(t)
\]
for \( i = 1, 2 \), defined in the maximal intervals \( I_i = ]t_0, t_i[ \), respectively, with \( t_0 \) finite. Let
\[
h_{0i} = \lim_{t \to t_i^+} \left[ \frac{1}{2} \dot{u}_{i}^2(t) - \frac{1}{\alpha - 1} u_{i}^{\alpha-1}(t) \right], \quad i = 1, 2,
\]
be the energy function associated to the solutions \( u_i \), respectively, at \( t_0 \). If \( h_{01} \leq h_{02} \) then
\[
t_1 \leq t_2, \quad u_1(t) \leq u_2(t), \quad \dot{u}_1(t) \leq \dot{u}_2(t), \quad \forall \; t \in ]t_0, t_1[.
\]
In the case that the inequality of energies is strict, that is if \( h_{01} < h_{02} \), then \( t_1 < t_2 \).

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