Steady State Radial Flow in Anisotropic and Homogenous in Confined Aquifers

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Abstract. The purpose of this paper is to introduce the analysis of steady-state radial flow in an anisotropic and homogenous hydraulic property and discuss who we can estimate aquifer parameters from field pumping tests.

Keywords: Partial differential equation; Radial Flow; Anisotropic and Homogenous; Pumping tests.

1. Introduction
Pumping tests are one of the most important techniques used to obtain the properties and parameter of flow and transport problems underground (Kruseman and de Ridder, [1]; Batu, [2]). These properties are important because it is used to develop future work to extract groundwater from the ground. Water flow problems towards test wells with steady and unsteady conditions have been studied early. Thiem [3] and Theis [4] studied the groundwater flow in confined aquifers in the steady state.

Hantush in [5] illustrates the anisotropic aquifers are more apparent in nature, and the anisotropy has caused researchers and hydrologist to be worry because of its effect on the flow of water. To study the anisotropy of aquifers and get accurate solution Cihan et al, in [6] described the properties by the tensor property. The upper limit for the ratio of conductivity in anisotropic aquifers presented by Peijun and others [7]. Hantush [8] used the transformation of coordinate with unknown parameter Ki to convert the anisotropic domain to the isotropic domain to get the solutions of the problems.

We will study the following assumptions:
- Confined aquifer (i.e. confined top and bottom).
- A constant rate of flow Q.
- Reached the steady state (do not change in the head).
- Anisotropic and Homogenous hydraulic conductivity.
2. Steady-State Conditions
After very long time of pumping from the well, the water may reach state with no change in level, this state is called the steady state, at this time since the rate of recharge equal to the discharge rate then there is no growth in the cone of depression, in this case, we have \( \frac{\partial h}{\partial t} = 0 \).

3. Flow with Homogenous and Isotropic Hydraulic Conductivities
The isotropic and homogeneous property means hydraulic conductivity \( K_x = K_y = K_z = K \) where \( K \) is constant in space, there are many studies that have discussed this type of flow such that [10-12].

4. Flow with Homogenous and Anisotropic Hydraulic Conductivities
In this paper, the radial flow will be studied in an anisotropic and homogenous property, i.e., hydraulic conductivity \( K_x \neq K_y \neq K_z \) but all \( K \)'s are constant in space, and the equation presented here can be reduced from three dimensions to two by neglecting the vertical flow i.e. \( \partial h/\partial z = 0 \), in this case assume the following:

To radial flow in 2D, as in figure 1,

The standard equations that govern groundwater flow are derived using the principle of continuity and Darcy’s law as follow [9]:

\[
T_x \left( \frac{\partial^2 h}{\partial x^2} \right) + T_y \left( \frac{\partial^2 h}{\partial y^2} \right) + T_z \left( \frac{\partial^2 h}{\partial z^2} \right) = S \frac{\partial h}{\partial t} \tag{1}
\]

Suppose that:

\[
r = \sqrt{x^2 + y^2} \tag{2}
\]

And let

\[
r' = \sqrt{\frac{T_y x^2 + T_x y^2}{T}}, \quad T = \sqrt{T_x T_y} \tag{3}
\]

Assume that \( x' = \sqrt{\frac{T_y}{T_x}} x \) and \( y' = \sqrt{\frac{T_x}{T_y}} y \).

![Figure 1: Illustrate the polar coordinates to the wells](image)
Then
\[
\frac{\partial^2 h}{\partial x^2} = \frac{T_y x}{r} \frac{\partial x}{\partial r} + \frac{\partial^2 h}{\partial y^2} \frac{\partial^2 h}{\partial y^2} \left( \frac{T^{-2}}{T^2} \right) = \frac{T x}{r^3} \frac{\partial h}{\partial r} + \frac{T y}{r^2} \frac{\partial h}{\partial r} + \frac{T x}{r^3} \frac{\partial h}{\partial r} + \frac{T y}{r^2} \frac{\partial h}{\partial r} + \frac{T x}{r^2} \frac{\partial h}{\partial r} + \frac{T y}{r^2} \frac{\partial h}{\partial r}
\]

i.e.,
\[
\frac{T x}{r^2} \frac{\partial h}{\partial r} + \frac{T y}{r^2} \frac{\partial h}{\partial r} = \frac{T}{r} \frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2}
\]

Now substitution equation (5) in equation (1):
\[
T \left[ \frac{1}{r^2} \frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} \right] = \frac{\partial h}{\partial t}
\]

Or
\[
\frac{1}{r^2} \frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} = \frac{S}{r^2} \frac{\partial h}{\partial t}
\]

From the polar coordinates \( x=r \cos(\theta) \) and \( y=r \sin(\theta) \), where \( \theta \) is the angle between the point (studying well) and the positive x-axis, then from equation (3) we have:
\[
r' = \frac{T y^2 + T x^2}{T} = \frac{\sqrt{T y^2 \cos^2(\theta) + T x^2 \sin^2(\theta)}}{T}
\]
\[
\therefore \quad r' = \frac{\sqrt{T y^2 \cos^2(\theta) + T x^2 \sin^2(\theta)}}{T}
\]
Let \( T_\theta = \sqrt{\frac{T_y \cos^2(\theta) + T_x \sin^2(\theta)}{T}} \)

\[ r' = r T_\theta \quad (9) \]

5. Steady Flow in Confined Aquifer Case

In this case \( \frac{\partial h}{\partial r} = 0 \), then equation (7) becomes:

\[ \frac{1}{r'} \frac{\partial}{\partial r'} \left( r' \frac{\partial h}{\partial r'} \right) = 0 \quad (10) \]

Which can we write as:

\[ \frac{1}{r'} \frac{\partial}{\partial r'} \left( r' \frac{\partial h}{\partial r'} \right) = 0 \quad (11) \]

Or

\[ \frac{\partial}{\partial r'} \left( r' \frac{\partial h}{\partial r'} \right) = 0 \quad (12) \]

By integrated the two sides we get:

\[ r' \frac{\partial h}{\partial r'} = C_1 \quad (13) \]

From Darcy's law:

\[ \frac{Q}{2\pi T_r} = r' \frac{\partial h}{\partial r} \quad (14) \]

Where \( T_r \) is the transmissivities in the \( r \) direction which is defined as Hantush, [8]

\[ T_r = \frac{T_y T_x}{T_y \cos^2(\theta) + T_x \sin^2(\theta)} \quad (15) \]

From equation (9) \( \frac{\partial r'}{\partial r} = T_\theta \) and \( r = \frac{r'}{T_\theta} \), so:

\[ \frac{Q}{2\pi T_r} = r' \frac{\partial h}{\partial r} = \frac{T_\theta}{T_r} r' \frac{\partial h}{\partial r'} = \frac{r'}{T_\theta} \frac{\partial h}{\partial r'} T_\theta = \frac{r'}{T_\theta} \frac{\partial h}{\partial r'} \]

\[ \therefore \frac{Q}{2\pi T_r} = r' \frac{\partial h}{\partial r'} = C_1 \quad (16) \]

Now rewrite equation (16) as follows:

\[ \frac{\partial h}{\partial r'} = \frac{Q}{2\pi T_r} \frac{\partial r'}{\partial r'} \quad (17) \]

By integrated the two sides we get:
Alternatively, from equation (9) we can get:

\[ h = \frac{Q}{2\pi T_r} \ln(r^2) + C_2 \quad (18) \]

If the radius of the well is \( r_v \), see figure 2, the hydraulic head at a distance of \( r_v \) from the center of the well and in all directions is equal to \( h_v \), then

\[ h_v = \frac{Q}{2\pi T_{r_v}} \ln(r_v T_{r_v}) + C_2. \]

Since the hydraulic head in all directions is equal to \( h_v \), then we will choose the same direction to the \( r \) and this implies that \( T_r = T_{r_v} \) and \( T_\theta = T_{\theta_v} \), now from (19) we have:

\[ h - h_v = \frac{Q}{2\pi T_r} \left[ \ln(r T_\theta) - \ln(r_v T_{\theta_v}) \right] = \frac{Q}{2\pi T_r} \ln \left( \frac{r}{r_v} \right) \]

\[ \therefore h = h_v + \frac{Q}{2\pi T_r} \ln \left( \frac{r}{r_v} \right) \quad (20) \]

**Figure 2:** Illustrate the system of pumping

Another estimate to the constant \( C_2 \) can get by using boundary condition which specifies \( h = h_o \) at \( r = R \). Where \( R \) represents the largest radius of effect, see 'Figure (2)', then

\[ h_o = \frac{Q}{2\pi T_r} \ln \left( \frac{T_r \cos^2(\theta_o) + T_x \sin^2(\theta_o)}{T} \right) + C_2 \]

This implies too, after we choose the same direction:
Then equation (19) becomes:

\[ C_2 = h_0 - \frac{Q}{2\pi T_r} \ln \left( R \sqrt{\frac{T_x \cos^2(\theta) + T_y \sin^2(\theta)}{T}} \right) \]

Then equation (19) becomes:

\[ h = \frac{Q}{2\pi T_r} \ln \left( r \sqrt{\frac{T_y \cos^2(\theta) + T_x \sin^2(\theta)}{T}} \right) + h_0 - \frac{Q}{2\pi T_r} \ln \left( R \sqrt{\frac{T_y \cos^2(\theta) + T_x \sin^2(\theta)}{T}} \right) \]

i.e.

\[ h_0 - h = \frac{Q}{2\pi T_r} \left[ \ln \left( R \sqrt{\frac{T_y \cos^2(\theta) + T_x \sin^2(\theta)}{T}} \right) - \ln \left( r \sqrt{\frac{T_y \cos^2(\theta) + T_x \sin^2(\theta)}{T}} \right) \right] \]

\[ = \frac{Q}{2\pi T_r} \left[ \ln \left( \frac{R}{r} \right) \right] \quad (21) \]

Where \( h_0 - h \) is the drawdown \( s_r \), for illustration see figure 2, at a radial distance \( r \) from the well.

For Example, if we have the pumping test system with pumping rate \( Q = 1000 \frac{\text{m}^3}{\text{s}} \), \( h_v = 10 \text{ m} \), \( r_v = 1 \text{ m} \), the distance 100 m on all direction and the transmissivity studying in three case as following:

a) \( T_x = 150 \frac{\text{m}^2}{\text{s}} \), \( T_y = 90 \frac{\text{m}^2}{\text{s}} \)  
b) \( T_x = 90 \frac{\text{m}^2}{\text{s}} \), \( T_y = 150 \frac{\text{m}^2}{\text{s}} \)  
c) \( T_x = 150 \frac{\text{m}^2}{\text{s}} \), \( T_y = 150 \frac{\text{m}^2}{\text{s}} \)

By equation (19) we get the following results:
Figure 3. (a,b,c): Illustrate the transmissivity effect on the hydraulic head

From the previous figures, we observe the transmissivity effect on the hydraulic head by the effect on the cone. In Figure (3.a) since $T_x > T_y$, this main axes of the ellipses are parallel to the $x$-axis, while in (3.b) since $T_x < T_y$, then this main axes of the ellipses are parallel to the $y$-axis, but in (3.c) the Conics are circles because of $T_x = T_y$. 
6. Effect Radius

When the flow reaches a steady state, the curve of the hydraulic head is constant, i.e., there is a distance where the hydraulic head is not affected by the pumping process and the head equals to the initial value. This distance is called the effect radius and is used to locate other wells in the same region. We can calculate the effect radius from equation (20) as follows:

Suppose that the initial value of the hydraulic head is $h_0$ and $h_v$ is the hydraulic head of the pumping well, for illustration, see figure 2, then equation (20) becomes:

$$h_0 - h_v = \frac{Q}{2\pi T_r} \ln \left( \frac{r}{r_v} \right) \rightarrow \frac{2\pi T_r(h_0 - h_v)}{Q} = \ln \left( \frac{r}{r_v} \right)$$

$$\rightarrow e^{\frac{2\pi T_r(h_0 - h_v)}{Q}} = \frac{r}{r_v}$$

Thus, $r = r_v e^{\frac{2\pi T_r(h_0 - h_v)}{Q}}$ (22)

Then by equation (22), we can calculate the radius in any direction $\theta$.

For example, if we have the pumping test system with pumping rate $Q=1000 \ \text{m}^3\text{m}^{-1}$, $h_v = 10 \text{ m}$, $r_v = 1 \text{ m}$, initial head $h_0 = 30 \text{ m}$ and the transmissivity studying in three case as following:

a) $T_x = 150 \ \text{m}^2\text{m}^{-1}$, $T_y = 90 \ \text{m}^2\text{m}^{-1}$;  
b) $T_x = 90 \ \text{m}^2\text{m}^{-1}$, $T_y = 150 \ \text{m}^2\text{m}^{-1}$;  
c) $T_x = 150 \ \text{m}^2\text{m}^{-1}$, $T_y = 150 \ \text{m}^2\text{m}^{-1}$.

By equation (22) we get the following results:

![Diagram](image1)

![Diagram](image2)
Figure 4. (a,b,c): Illustrate the radius of effect on the hydraulic head

Now to show the effect of the transmissivity on the radius for Example, if we have the pumping test system with pumping rate $Q= 1000 \frac{m^3}{s}$, $h_v=10 \text{ m}$, $r_v=1$, initial head $h_0=30 \text{ m}$, $T_y=90 \frac{m^2}{s}$ and studying in three case as following: a) $T_x=110 \frac{m^2}{s}$; b) $T_x=130 \frac{m^2}{s}$; c) $T_x=150 \frac{m^2}{s}$. By equation (22) we get the following results:

Figure 5: Illustrate the effect of the transmissivity on the radius

Which means that when increasing the transmissivity, the radius is increased. Similarly, we can study the effect of the transmissivity on $y$-direction.

7. Estimate the Transmissivity

We can estimate the transmissivity by using equation (20) as the following:

Suppose that the hydraulic head at another well, at $r_x$ distance in $x$-direction, is $h_x$ and $h_v$ is the hydraulic head of the pumping well, in this case $\theta=0$ and hence by (15) $T_r = T_x$, for more illustration, see figure 6 and 7, then by equation (20) we have:
**Figure 6:** Illustrate the new well

**Figure 7:** Illustrate the new well in direction

\[ h_x - h_y = \frac{Q}{2\pi T_x} \ln \left( \frac{r_x}{r_y} \right) \]

\[ \therefore T_x = \frac{Q \ln \left( \frac{r_x}{r_y} \right)}{2\pi (h_x - h_y)} \quad (23) \]
From another hand in y-direction we have

\[ T_y = \frac{Q \ln\left(\frac{R_y}{R_p}\right)}{2\pi (h_y - h_p)} \]  

(24)

For example, if we have the same information of the example in section 4 and choose the following data illustrated in table 1, we get:

**Table 1:** Illustrated the data and results to compute \( T_x \)

| \( r_x \) | \( h_x \) | \( T_x \) |
|---|---|---|
| 6 | 13.66204096 | 150 |
| 11 | 15.68249364 | 150 |
| 16 | 16.93147181 | 150 |
| 21 | 17.83791752 | 150 |
| 26 | 18.54983119 | 150 |

The column \( T_x \) is calculated by (23), we can see that the transmissivity \( T_x \) is equal to that in the example in section 4 for all different distance \( r_x \).

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