S1. Lattice function and norms. We will consider only Bravais lattices in this work, which is denoted as \( L \). Let \( \{a_j\}_{j=1}^d \subset \mathbb{R}^d \) be the basis vectors of \( L \), and \( d \) be the dimension,

\[
L = \left\{ x \in \mathbb{R}^d \mid x = \sum_j n_j a_j, n \in \mathbb{Z}^d \right\}.
\]

Let \( \{b_j\}_{j=1}^d \subset \mathbb{R}^d \) be the reciprocal basis vectors satisfying \( a_j \cdot b_k = 2\pi \delta_{jk} \), where \( \delta_{jk} \) is the standard Kronecker delta symbol. The reciprocal lattice \( L^* \) is

\[
L^* = \left\{ x \in \mathbb{R}^d \mid x = \sum_j n_j b_j, n \in \mathbb{Z}^d \right\}.
\]

We take a computational domain

\[
\Omega = \left\{ \sum_j x_j a_j \mid x \in [0,1)^d \right\},
\]

and let \( \Omega_{\varepsilon} \) be a grid mesh in \( \Omega \) with mesh size \( \varepsilon = 1/(2N) \), \( N \in \mathbb{Z}_+ \):

\[
\Omega_{\varepsilon} = \left\{ x_{\nu} = \varepsilon \sum_j \nu_j a_j \mid \nu \in \mathbb{Z}^d, 0 \leq \nu_j < 2N \right\}.
\]

Using the reciprocal basis \( \{b_j\} \), we define

\[
L_{\varepsilon}^* = \left\{ \xi = \sum_j k_j b_j \mid k \in \mathbb{Z}^d, -N \leq k_j < N \right\}.
\]

We will identify functions defined on \( \Omega_{\varepsilon} \) with their periodic extensions in this work, i.e., we consider the periodic boundary condition. General boundary conditions will be left for future work.

For \( \mu \in \mathbb{Z}^d \), we define the translation operator \( T_{\varepsilon}^\mu \) as

\[
(T_{\varepsilon}^\mu u)(x) = u(x + \varepsilon \mu_j a_j) \quad \text{for} \quad x \in \mathbb{R}^d,
\]

where the index summation convention is used. We define the forward and backward difference operators as

\[
D_{\varepsilon,\mu}^+ = \varepsilon^{-1}(T_{\varepsilon}^\mu - I) \quad \text{and} \quad D_{\varepsilon,\mu}^- = \varepsilon^{-1}(I - T_{\varepsilon}^{-\mu}),
\]
where $I$ denotes the identity operator. We say $\alpha$ is a multi-index, if $\alpha \in \mathbb{Z}^d$ and $\alpha \geq 0$. We will use the notation $|\alpha| = \sum_{j=1}^{d} \alpha_j$. For a multi-index $\alpha$, the difference operator $D_\varepsilon^\alpha$ is given by

$$D_\varepsilon^\alpha = \prod_{j=1}^{d} (D_{\varepsilon, e_j}^+)^{\alpha_j},$$

where $\{e_j\}_{j=1}^{d}$ are the canonical basis of $\mathbb{R}^d$ (columns of a $d \times d$ identity matrix).

We will use the notation $|\alpha| = \sum_{j=1}^{d} \alpha_j$. For a multi-index $\alpha$, the difference operator $D_\varepsilon$ is given by

$$D_\varepsilon = \prod_{j=1}^{d} (D_{\varepsilon, e_j}^+) = \prod_{j=1}^{d} (D_{\varepsilon, e_j}^-),$$

where $\{e_j\}_{j=1}^{d}$ are the canonical basis of $\mathbb{R}^d$ (columns of a $d \times d$ identity matrix).

We will use various norms for functions defined on $\Omega_\varepsilon$. For integer $k \geq 0$, define the difference norm $\|u\|_{L^k_\varepsilon,k} = \sum_{0 \leq |\alpha| \leq k} \varepsilon^d \sum_{x \in \Omega_\varepsilon} |(D_\varepsilon^\alpha u)(x)|^2$. It is clear that $\|\cdot\|_{L^k_\varepsilon,k}$ is a discrete analog of Sobolev norm associated with $H^k(\Omega)$. Hence, we denote the corresponding spaces of lattice functions as $H^k_\varepsilon(\Omega)$ and $L^2_\varepsilon(\Omega)$ when $k = 0$. We also need the uniform norms on $\Omega_\varepsilon$, which are defined as

$$\|u\|_{L^\infty_\varepsilon} = \max_{x \in \Omega_\varepsilon} |u(x)|,$$

$$\|u\|_{W^{k,\infty}_\varepsilon} = \sum_{0 \leq |\alpha| \leq k} \max_{x \in \Omega_\varepsilon} |(D_\varepsilon^\alpha u)(x)|.$$

Recall that we identify lattice function $u$ with its periodic extension to function defined on $\varepsilon L$, and hence differences of the lattice functions are well-defined. These norms may be extended to vector-valued functions as usual. For $k > d/2$, we have the discrete Sobolev inequality $\|u\|_{L^\infty_\varepsilon} \lesssim \|u\|_{L^k_\varepsilon,k}$. Here and throughout this paper, we denote $A \lesssim B$ if $A \leq CB$ with $C$ an absolute constant.

The discrete Fourier transform for a lattice function $u$ is given for $\xi \in \mathbb{L}_\varepsilon^*$ by

$$\hat{u}(\xi) = \left(\frac{\varepsilon}{2\pi}\right)^d \sum_{x \in \Omega_\varepsilon} e^{-i\xi \cdot x} u(x).$$

By the Fourier inversion formula, for $x \in \Omega_\varepsilon$,

$$u(x) = \sum_{\xi \in \mathbb{L}_\varepsilon^*} e^{i\xi \cdot x} \hat{u}(\xi).$$

We will use a symbol introduced by Nirenberg in [26], which plays the same role for the difference operators as $\Lambda^2(\xi) = 1 + \Lambda^2_0(\xi) = 1 + |\xi|^2$ for the differential operators. For $\varepsilon > 0$, let

$$\Lambda_{j,\varepsilon}(\xi) = \frac{1}{\varepsilon} \left| e^{ix_j \xi_j} - 1 \right|, \quad j = 1, \ldots, d,$$

and

$$\Lambda^2_\varepsilon(\xi) = 1 + \Lambda^2_\varepsilon(\xi) = 1 + \sum_{j=1}^{d} \Lambda^2_{j,\varepsilon}(\xi) = 1 + \sum_{j=1}^{d} \frac{4}{\varepsilon^2} \sin^2 \left(\frac{\varepsilon \xi_j}{2}\right).$$

It is not hard to check for any $\xi \in \mathbb{L}_\varepsilon^*$, there holds

$$c\Lambda^2(\xi) \leq \Lambda^2_\varepsilon(\xi) \leq \Lambda^2(\xi),$$

where the positive constant $c$ depends on $\{b_j\}$. 

S2. Proof of Lemma 3.2. Let us first recall the following consistency lemma proved in [22, Section 2] (proofs of these results do not depend on the smoothness of \( \theta \)).

Lemma S2.1 (Consistency). For any \( u \) smooth, we have

\[
\| F_{at}[u] - F_{CB}[u] \|_{L^\infty} \leq C \varepsilon^2 \| u \|_{W^{18,\infty}},
\]
(\text{S2.1})

\[
\| F_{\varepsilon}[u] - F_{CB}[u] \|_{L^\infty} \leq C \varepsilon^2 \| u \|_{W^{18,\infty}},
\]
(\text{S2.2})

\[
\| F_{hy}[u] - F_{CB}[u] \|_{L^\infty} \leq C \varepsilon^2 \| u \|_{W^{18,\infty}},
\]
(\text{S2.3})

where the constant \( C \) depends on \( V \) and \( \| u \|_{L^\infty} \), but is independent of \( \varepsilon \).

Proof. [Proof of Lemma 3.2] The proof for (3.6) and (3.7) are analogous, and hence we will only prove the latter. By definition, for \( 1 \leq j, k \leq d \),

\[
(h_{at})_{jk}(\xi) = e^{-ix \cdot \xi} (H_{at}(ekf_\xi))_j(x),
\]

\[
(h_{hy})_{jk}(x, \xi) = e^{-ix \cdot \xi} (H_{hy}(ekf_\xi))_j(x),
\]

where \( f_\xi(x) = e^{ix \cdot \xi} \) for \( x \in \Omega \). Taking difference of the above two equations, we obtain the bound

\[
\| h_{at}(\xi) - h_{hy}(x, \xi) \| \leq C \sup_{1 \leq j, k \leq d} \| H_{at}(ekf_\xi) - H_{hy}(ekf_\xi) \|_{L^\infty}.
\]

Note that by the definition of linearized operators \( H_{at} \) and \( H_{hy} \), we have

\[
H_{at}(ekf_\xi) - H_{hy}(ekf_\xi) = \lim_{t \rightarrow 0^+} \frac{1}{t} (F_{at}[t(ekf_\xi)] - F_{hy}[t(ekf_\xi)]).
\]

Hence,

\[
\| H_{at}(ekf_\xi) - H_{hy}(ekf_\xi) \|_{L^\infty} = \lim_{t \rightarrow 0^+} \frac{1}{t} \| F_{at}[t(ekf_\xi)] - F_{hy}[t(ekf_\xi)] \|_{L^\infty}
\]

\[\lesssim \varepsilon^2 \| e_\xi f_\xi \|_{W^{18,\infty}} \lesssim \varepsilon^2 \| e_\xi f_\xi \|_{H^s} \lesssim \varepsilon^2 (1 + |\xi|^2)^{s/2},\]

where \( s \) is chosen so that the Sobolev inequality \( \| f \|_{W^{18,\infty}(\Omega)} \leq C \| f \|_{H^s(\Omega)} \) holds for any \( f \in H^s(\Omega) \) (\( s \) depends on the dimension). Here, we have used Lemma S2.1 in the first inequality, noticing that \( \| e_\xi f_\xi \|_{L^\infty} \) is uniformly bounded for \( \xi \) as \( t \rightarrow 0 \). This concludes the proof. \( \square \)

S3. Additional details for Example 1. Lemma S3.1. \( z_1, z_2 \) and \( z_3 \) are distinct roots.

Proof. It is clear that

\[
z_2 = w_2 \zeta^{1/2} \quad \text{and} \quad z_3 = w_3 \zeta^{1/2}
\]

with \(-1 < w_3 < 0 < w_2 < 1\), this implies \( z_2 \neq z_3 \).

A direct calculation gives

\[
z_1 = \frac{6 - \zeta - \zeta \sqrt{(4 - \zeta - \zeta)(2 - \zeta - \zeta)}}{2(1 + \zeta)}
\]

\[= \frac{6 - \zeta - \zeta \sqrt{(4 - \zeta - \zeta)(2 - \zeta - \zeta)}}{2(1 + \zeta)(1 + \zeta)}(1 + \zeta)
\]

\[= \frac{6 - \zeta - \zeta \sqrt{(4 - \zeta - \zeta)(2 - \zeta - \zeta)}}{2(2 + \zeta + \zeta)}(\zeta^{1/2} + \zeta^{1/2}) \zeta^{1/2}.
\]
Recalling \( \zeta = e^{i\theta} \) with \( \theta \in (-\pi, \pi) \), and we may write
\[
z_1 = \frac{2 \cos(\theta/2)}{3 - \cos \theta + \sqrt{(7 - \cos \theta)(1 - \cos \theta)}} e^{i\theta/2}.
\]
Note that
\[
\frac{2 \cos(\theta/2)}{3 - \cos \theta + \sqrt{(7 - \cos \theta)(1 - \cos \theta)}} > 0 > w_3,
\]
this implies \( z_1 \neq z_3 \).

It remains to prove \( z_1 \neq z_2 \). Note that
\[
z_2 = \frac{1}{2} \left( B - \sqrt{B^2 - 4} \right) e^{i\theta/2}
\]
with
\[
B = -A/2 + \sqrt{A^2/4 + 14 - (\zeta + \bar{\zeta})}.
\]
Using
\[
A = \zeta + \bar{\zeta} + \zeta^3 + \bar{\zeta}^3 = (\zeta + \bar{\zeta}) (\zeta^2 + \bar{\zeta}^2) = 4 \cos(\theta/2) \cos \theta,
\]
we write
\[
A^2/4 + 14 - (\zeta + \bar{\zeta}) = 16 \cos^2(\theta/2) \cos^2 \theta + 14 - 2 \cos \theta = 16 \cos^2(\theta/2) \cos^2 \theta + 14 - 2(2 \cos^2(\theta/2) - 1) = 16 - 4 \cos^2(\theta/2) \sin^2 \theta.
\]
This gives
\[
B = 2 \sqrt{4 - \cos^2(\theta/2) \sin^2 \theta - 2 \cos(\theta/2) \cos \theta}.
\]
To prove \( z_1 \neq z_2 \), it remains to show \( |z_1| \neq |z_2| \), i.e.,
\[
\frac{1}{2} \left( B - \sqrt{B^2 - 4} \right) \neq \frac{2 \cos(\theta/2)}{3 - \cos \theta + \sqrt{(7 - \cos \theta)(1 - \cos \theta)}}.
\]
Actually, we shall prove that for \( \theta \in (-\pi, \pi) \) and \( \theta \neq 0 \), there holds
\[
\frac{1}{2} \left( B - \sqrt{B^2 - 4} \right) > \frac{2 \cos(\theta/2)}{3 - \cos \theta + \sqrt{(7 - \cos \theta)(1 - \cos \theta)}}. \quad \text{(S3.1)}
\]
The above inequality is equivalent to
\[
3 - \cos \theta + \sqrt{(7 - \cos \theta)(1 - \cos \theta)} > \cos(\theta/2) \left( B + \sqrt{B^2 - 4} \right). \quad \text{(S3.2)}
\]
Denote by \( t = \cos(\theta/2) \), we write the above inequality as
\[
2 - t^2 + \sqrt{(4 - t^2)(1 - t^2)} > t \left( g(t) + \sqrt{g^2(t) - 1} \right), \quad t \in [0, 1), \quad \text{(S3.3)}
\]
where

\[ g(t): = t - 2t^3 + 2\sqrt{1 - t^4 + t^6}. \]

To prove \( S3.3 \), we firstly prove

\[ 2 - t^2 > tg(t) \quad t \in [0, 1). \]  

(S3.4)

A direct calculation gives

\[
2 - t^2 - tg(t) = 2(1 - t^2) + 2t \left( t^3 - \sqrt{1 - t^4 + t^6} \right)
= 2(1 - t^2) + \frac{2t(t^4 - 1)}{\sqrt{1 - t^4 + t^6 + t^3}}
= 2(1 - t^2) \left( 1 - \frac{t + t^3}{\sqrt{1 - t^4 + t^6 + t^3}} \right).
\]

Note that

\[ \sqrt{1 - t^4 + t^6} > t, \]
which follows from \((1 - t^2)(1 - t^4) > 0\). Combining the above two inequalities, we obtain \( S3.4 \).

Next, by \( S3.4 \) and note \( g(t) \geq 0 \), we obtain

\[ (4 - t^2)(1 - t^2) = (2 - t^2)^2 - t^2 \geq t^2(g^2(t) - 1). \]

A direct calculation gives that \( g(t) \geq 1 \). Therefore,

\[ \sqrt{(4 - t^2)(1 - t^2)} \geq t\sqrt{g^2(t) - 1}, \]
which together with \( S3.4 \) gives \( S3.3 \). This implies \( z_1 \neq z_2 \) and completes the proof. □