Potpourri, 4

Stephen William Semmes
Rice University
Houston, Texas

Contents

1 Random polynomials 1
2 Vector-valued functions 3
3 Polynomials, continued 7
4 An integral operator 8

1 Random polynomials

Fix a positive integer $n$, and let us write $\mathcal{B}_{n+1}$ for the set of binary strings of length $n + 1$ written multiplicatively. To be more precise, an element $s$ of $\mathcal{B}_{n+1}$ is an $(n + 1)$-tuple $s = (s_0, \ldots, s_n)$ such that each $s_j$ is either 1 or $-1$. Put $r_j(s) = s_j$ for each $j$, the $j$th Rademacher function on $\mathcal{B}_{n+1}$. Let $a_0, \ldots, a_n$ be complex numbers, so that we get a polynomial

\begin{equation}
\tag{1.1}
p(z) = \sum_{j=0}^{n} a_j z^j,
\end{equation}

where $z^j$ is interpreted as being equal to 1 when $j = 0$, as usual. For each $s \in \mathcal{B}_{n+1}$ consider the polynomial

\begin{equation}
\tag{1.2}
p_s(z) = \sum_{j=0}^{n} a_j r_j(t) z^j.
\end{equation}
Let us write $T$ for the unit circle in the complex plane, which is to say the set of complex numbers $z$ with $|z| = 1$. For each positive integer $j$ we have that
\[(1.3) \quad \int_T z^j |dz| = 0,\]
and hence also
\[(1.4) \quad \int_T \overline{z}^j |dz| = 0,\]
where $\overline{z}$ denotes the complex conjugate of $z$. Of course
\[(1.5) \quad \frac{1}{2\pi} \int_T |dz| = 1.\]
As usual, this leads to
\[(1.6) \quad \frac{1}{2\pi} \int_T |p(z)|^2 |dz| = \sum_{j=0}^n |a_j|^2.\]
Similarly,
\[(1.7) \quad \frac{1}{2\pi} \int_T |p_s(z)|^2 = \sum_{j=0}^n |a_j|^2\]
for all $s \in B_{n+1}$.

Now consider
\[(1.8) \quad 2^{-n-1} \sum_{s \in B_{n+1}} \frac{1}{2\pi} \int_T |p_s(z)|^4 |dz|.\]
By interchanging the order of summation and integration we can rewrite this as
\[(1.9) \quad \frac{1}{2\pi} \int_T 2^{-n-1} \sum_{s \in B_{n+1}} |p_s(z)|^4 |dz|.\]
There is a positive real number $C$ such that
\[(1.10) \quad 2^{-n-1} \sum_{s \in B_{n+1}} \left| \sum_{j=0}^n b_j r_j(s) \right|^4 \leq C \left( \sum_{j=0}^n |b_j|^2 \right)^2\]
for any complex numbers $b_0, \ldots, b_n$, as one can show by expanding
\[(1.11) \quad \left| \sum_{j=0}^n b_j r_j(s) \right|^4\]
as a quadruple sum and then summing over $s$ first. It follows that

\begin{equation}
2^{-n-1} \sum_{s \in B_{n+1}} \frac{1}{2\pi} \int_T |p_s(z)|^4 |dz| \leq C \left( \sum_{j=0}^{n} |a_j|^2 \right)^2.
\end{equation}

More generally, suppose that $m$ is a positive integer, and consider

\begin{equation}
2^{-n-1} \sum_{s \in B_{n+1}} \frac{1}{2\pi} \int_T |p_s(z)|^{2m} |dz|.
\end{equation}

As before there is a positive real number $C(m)$ such that

\begin{equation}
2^{-n-1} \sum_{s \in B_{n+1}} \left| \sum_{j=0}^{n} b_j r_j(s) \right|^{2m} \leq C(m) \left( \sum_{j=0}^{n} |b_j|^2 \right)^m
\end{equation}

for any complex numbers $b_0, \ldots, b_n$. By interchanging the sum with the integral and then applying this fact one obtains that

\begin{equation}
2^{-n-1} \sum_{s \in B_{n+1}} \frac{1}{2\pi} \int_T |p_s(z)|^{4} |dz| \leq C(m) \left( \sum_{j=0}^{n} |a_j|^2 \right)^m.
\end{equation}

\section{Vector-valued functions}

Let $E$ be a nonempty finite set, and let $V$ be a real or complex vector space. The vector space of functions on $E$ with values in $V$ will be denoted $\mathcal{F}(E, V)$. In particular, $\mathcal{F}(E, \mathbb{R})$, $\mathcal{F}(E, \mathbb{C})$ denote the spaces of real and complex-valued functions on $E$, respectively. These are commutative algebras with respect to pointwise multiplication of functions.

Actually, we can think of $\mathcal{F}(E, V)$ as a module over $\mathcal{F}(E, \mathbb{R})$ or $\mathcal{F}(E, \mathbb{C})$, according to whether $V$ is a real or complex vector space. In other words, a function on $E$ with values in $V$ can be multiplied pointwise by a scalar-valued function in a way that is compatible with addition and scalar multiplication. For that matter, scalar multiplication amounts to the same thing as multiplication by constant scalar-valued functions.

We can also think of $\mathcal{F}(E, V)$ as a module over the algebra of linear transformations on $V$, where a linear transformation on $V$ induces a linear transformation on $V$-valued functions on $E$ by acting on the values pointwise. The actions on $\mathcal{F}(E, V)$ by pointwise multiplication by scalar-valued
functions on $E$, and by pointwise action by linear transformations on $V$, obviously commute with each other.

Suppose that $V$ is equipped with a norm $\|v\|_V$. Thus $\|v\|_V$ is a nonnegative real number for each $v \in V$ which is equal to 0 if and only if $v = 0$,
\begin{equation}
|\alpha v|_V = |\alpha| \|v\|_V
\end{equation}
for all real or complex numbers $\alpha$, as appropriate, and all $v \in V$, and
\begin{equation}
\|v + w\|_V \leq \|v\|_V + \|w\|_V
\end{equation}
for all $v, w \in V$. This choice of norm on $V$ leads to a metric $\|v - w\|_V$ on $V$, as usual.

Let $p$ be given, $1 \leq p \leq \infty$. If $f(x)$ is a real or complex-valued function on $E$, put
\begin{equation}
\|f\|_p = \left( \sum_{x \in E} |f(x)|^p \right)^{1/p}
\end{equation}
when $p < \infty$ and
\begin{equation}
\|f\|_{\infty} = \max\{|f(x)| : x \in E\}.
\end{equation}
As is well-known, these define norms on the vector spaces of real and complex-valued functions on $E$. Moreover, if $1 \leq p \leq q \leq \infty$ and $f$ is a real or complex-valued function on $E$, then
\begin{equation}
\|f\|_q \leq \|f\|_p \leq |E|^{(1/p) - (1/q)} \|f\|_q,
\end{equation}
where $|E|$ denotes the number of elements in $E$.

If $f$ is a $V$-valued function on $E$, put
\begin{equation}
\|f\|_{p,V} = \left( \sum_{x \in E} \|f(x)\|_V^p \right)^{1/p}
\end{equation}
when $1 \leq p < \infty$ and
\begin{equation}
\|f\|_{\infty,V} = \max\{\|f(x)\|_V : x \in E\}.
\end{equation}
These define norms on $\mathcal{F}(E, V)$. Once again we have that
\begin{equation}
\|f\|_{q,V} \leq \|f\|_{p,E} \leq |E|^{(1/p) - (1/q)} \|f\|_{q,V}
\end{equation}
when $1 \leq p \leq q \leq \infty$ and $f \in \mathcal{F}(E, V)$.
Let $V$ be a finite-dimensional real or complex vector space. By a *linear functional* on $V$ we mean a linear mapping from $V$ into the real or complex numbers, as appropriate. One can add linear functionals on $V$ and multiply them by scalars in the usual manner, so that the space $V^*$ of linear functionals on $V$ is also a real or complex vector space, as appropriate, called the dual of $V$.

If $v_1, \ldots, v_n$ is a basis for $V$, so that every element of $V$ can be expressed in a unique manner as a linear combination of the $v_j$'s, then a linear functional $\lambda$ on $V$ is determined uniquely by the $n$ scalars $\lambda(v_1), \ldots, \lambda(v_n)$, and for each choice of $n$ scalars there is a linear functional on $V$ whose values on the basis vectors are those scalars. In particular, $V^*$ is also finite-dimensional and has the same dimension as $V$.

Now suppose that $V$ is equipped with a norm $\|v\|_V$. If $\lambda$ is a linear functional on $V$, then there is a nonnegative real number $k$ such that

\begin{equation}
|\lambda(v)| \leq k \|v\|_V
\end{equation}

for all $v \in V$. Indeed, $V$ is isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$ as a vector space, where $n$ is the dimension of $V$, and it is well-known that any norm on $\mathbb{R}^n$, $\mathbb{C}^n$ determines the same topology as the standard Euclidean norm.

As a result,

\begin{equation}
\|\lambda\|_{V^*} = \sup\{|\lambda(v)| : v \in V, \|v\|_V \leq 1\}
\end{equation}

is finite, and it is the same as the smallest nonnegative real number which can be used as $k$ in the previous inequality. This defines a norm on the dual space $V^*$, which is the dual norm associated to $\| \cdot \|_V$.

Note that

\begin{equation}
\|v\|_V = \sup\{|\lambda(v)| : \lambda \in V^*, \|\lambda\|_{V^*} \leq 1\}.
\end{equation}

More precisely, $\|v\|_V$ is greater than or equal to $|\lambda(v)|$ for all linear functionals $\lambda$ on $V$ with dual norm less than or equal to 1 by definition, and there is such a linear functional with $\lambda(v)$ equal to the norm of $v$ by famous duality results.

As a basic example, let $E$ be a nonempty finite set, and let us consider the vector space of real or complex-valued functions on $E$. If $h$ is a real or complex-valued function on $E$, then we get a linear functional $\lambda_h$ on the
vector space of real or complex-valued functions on \( E \), as appropriate, by setting
\[
\lambda_h(f) = \sum_{x \in E} f(x) h(x)
\]
(2.12)
for \( f \) in the vector space. Every linear functional on the vector space of functions on \( E \) arises in this manner.

If \( 1 \leq p, q \leq \infty \) are conjugate exponents, in the sense that \((1/p) + (1/q) = 1\), then Hölder’s inequality implies that
\[
|\lambda_h(f)| \leq \|f\|_p \|h\|_q
\]
(2.13)
for all functions \( f, h \) on \( E \), where \( \|f\|_p, \|h\|_q \) are as in the previous section. With respect to the norm \( \|f\|_p \) on functions on \( E \), the dual norm of the linear functional \( \lambda_h \) is therefore less than or equal to \( \|h\|_q \), and in fact one can check that it is equal to \( \|h\|_q \).

More generally we can consider \( V \)-valued functions on \( E \). If \( h \) is a function on \( E \) with values in \( V^* \), then we can define a linear functional \( \lambda_h \) on the vector space of \( V \)-valued functions on \( E \) by saying that \( \lambda_h(f) \) is obtained by applying \( h(x) \) as a linear functional on \( V \) to \( f(x) \) as an element of \( V \) for each \( x \in E \), and then summing over \( x \in E \). One can check that every linear functional on \( \mathcal{F}(E, V) \) occurs in this way, so that the dual of \( \mathcal{F}(E, V) \) can be identified with \( \mathcal{F}(E, V^*) \).

Using Hölder’s inequality it is easy to check that \( |\lambda_h(f)| \) is less than or equal to the product of \( \|f\|_{p,V} \) and \( \|h\|_{q,V^*} \) for all \( f \in \mathcal{F}(E, V) \) and \( h \in \mathcal{F}(E, V^*) \) when \( 1 \leq p, q \leq \infty \) are conjugate exponents. Furthermore, the dual norm of \( \lambda_h \) with respect to the norm \( \|f\|_{p,V} \) on \( \mathcal{F}(E, V) \) is exactly equal to \( \|h\|_{q,V^*} \).

Let \( V \) be a finite-dimensional real or complex vector space, and let \( E \) be a nonempty finite set. If \( f \) is a \( V \)-valued function on \( E \), then put
\[
\|f\|_{p,V} = \sup \left\{ \left( \sum_{x \in E} |\lambda(f(x))|^p \right)^{1/p} : \lambda \in V^*, \|\lambda\|_{V^*} \leq 1 \right\}
\]
(2.14)
when \( 1 \leq p < \infty \) and
\[
\|f\|_{\infty,V} = \sup \{ \max\{|\lambda(f(x))| : x \in E\} : \lambda \in V^*, \|\lambda\|_{V^*} \leq 1 \}.
\]
(2.15)
One can check that these define norms on \( \mathcal{F}(E, V) \).
For each $v \in V$ we have that $\|v\|_V$ is equal to the maximum of $|\lambda(v)|$ over $\lambda \in V^*$ with $\|\lambda\|_{V^*} \leq 1$, as discussed in the previous section. It is easy to see that

\[ (2.16) \quad \|f\|_{p,\nu} \leq \|f\|_{p,V} \]

when $1 \leq p < \infty$, and that

\[ (2.17) \quad \|f\|_{\infty,\nu} = \|f\|_{\infty,V}. \]

All of these norms reduce to $\|\cdot\|_V$ when $E$ has just one element.

We can also express $\|f\|_{p,\nu}$ as

\[ (2.18) \quad \|f\|_{p,\nu} = \sup \left\{ \left| \sum_{x \in E} h(x) \lambda(f(x)) \right| : \|h\|_q \leq 1, \lambda \in V^*, \|\lambda\|_{V^*} \leq 1 \right\}. \]

Here $q$ is the conjugate exponent associated to $p$, so that $1 \leq q \leq \infty$ and $(1/p) + (1/q) = 1$, and $h$ is a real or complex-valued function on $E$, as appropriate. Of course $\sum_{x \in E} h(x) \lambda(f(x))$ is the same as $\lambda$ applied to $\sum_{x \in E} h(x) f(x)$, and it follows that $\|f\|_{p,\nu}$ is equal to the supremum of the $V$-norm of $\sum_{x \in E} h(x) f(x)$ over all scalar-valued functions $h$ on $E$ with $\|h\|_q \leq 1$.

### 3 Polynomials, continued

Of course a polynomial on the complex plane has the form

\[ (3.1) \quad p(z) = a_n z^n + \cdots + a_1 z + a_0, \]

where $a_0, \ldots, a_n$ are complex numbers.

Alternatively one might consider polynomials in the real and imaginary parts of $z$, which is equivalent to polynomials in $z$ and $\bar{z}$. A special case of this is given by linear combinations of powers of $z$ and of powers of $\bar{z}$, including constant terms, without products of positive powers of $z$ and $\bar{z}$. One might also consider linear combinations of powers of $z$ and of $z^{-1}$, including constant terms. These classes all define the same functions on the unit circle, where $|z|^2 = z \bar{z} = 1$. Let us note that if $f(z)$ is one of these more general kinds of polynomials, then there is a complex polynomial $p(z)$ as in the preceding paragraph such that $|f(z)| = |p(z)|$ for all $z \in \mathbb{C}$ with $|z| = 1$.

Let us restrict our attention to complex polynomials as in (3.1). If $p(z)$ is as in (3.1), put

\[ (3.2) \quad \|p\| = \sup\{|p(z)| : z \in \mathbb{C}, \ |z| = 1\} \]
and
\[(3.3)\]  \[\|p\|_1 = \sum_{j=0}^{n} |a_j| .\]

These define norms on the complex vector space of polynomials, and we have that
\[(3.4)\]  \[\|p\| \leq \|p\|_1 .\]

If \(p, q\) are complex polynomials, then
\[(3.5)\]  \[\|pq\| \leq \|p\| \|q\| .\]

and
\[(3.6)\]  \[\|pq\|_1 \leq \|p\|_1 \|q\|_1 .\]

If \(p(z)\) is as in (3.1), then
\[(3.7)\]  \[\sum_{j=0}^{n} |a_j|^2 = \frac{1}{2\pi} \int_{\mathbf{T}} |p(z)|^2 |dz| \leq \|p\|^2 ,\]

where \(\mathbf{T}\) denotes the unit circle in \(\mathbf{C}\). One can use this to estimate \(\|p\|_1\) in terms of \(\|p\|\) and \(\|p'\|\), where
\[(3.8)\]  \[p'(z) = na_n z^{n-1} + \cdots + a_1 .\]

is the derivative of \(p(z)\). As a result one can show that
\[(3.9)\]  \[\|p\| = \lim_{l \to \infty} \|p'\|_1^{1/l} .\]

4 An integral operator

Let \(V\) denote the vector space of continuous complex-valued functions on the unit interval \([0, 1]\) in the real line. If \(f \in V\), then we put
\[(4.1)\]  \[\|f\| = \sup \{|f(x)| : 0 \leq x \leq 1\} ,\]

which is the usual supremum norm of \(f\). Define a linear operator \(T\) on \(V\) by
\[(4.2)\]  \[T(f)(x) = \int_{0}^{x} f(s) ds .\]
If $f$ happens to be real-valued, then $T(f)$ is real-valued, and if $f(x) \geq 0$ for all $x \in [0, 1]$ too, then $T(f)(x) \geq 0$ for all $x \in [0, 1]$ as well. Notice that

$$\|T(f)\| \leq \|f\|$$

for all $f \in V$, and that equality holds when $f$ is the constant function equal to 1.

For each positive integer $n$ let $T^n$ denote the $n$-fold composition of $T$ on $V$. Equivalently, this is equal to $T$ when $n = 1$, and in general $T^{n+1}(f) = T(T^n(f))$. One can express $T^n(f)$ explicitly as an $n$-fold integral of $f$, and observe that $T^n(f)$ is real-valued when $f$ is real-valued and nonnegative when $f$ is nonnegative. If $f$ is the constant function equal to 1, then $T^n(f)(1) = 1/n!$. Indeed, $T^n(f)(1)$ is equal to the volume of the points $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$. Using this one can check that if $f$ is the constant function equal to 1, then $\|T^n(f)\| = 1/n!$. For any function $f \in V$ we have that $\|T^n(f)\| \leq \frac{1}{n!}\|f\|$.

For each $f \in V$ we also have that

$$\|T(f)\| \leq \int_0^1 |f(y)| \, dy.$$  

As a result, if $f_1, \ldots, f_l$ are elements of $V$, then

$$\sum_{j=1}^l \|T(f_j)\| \leq \left\| \sum_{j=1}^l |f_j| \right\|.$$  

References

[1] R. Beals, *Advanced Mathematical Analysis*, Springer-Verlag, 1973.

[2] J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Springer-Verlag, 1976.

[3] S. Bochner, *Harmonic Analysis and the Theory of Probability*, University of California Press, 1955.

[4] S. Bochner and K. Chandrasekharan, *Fourier Transforms*, Annals of Mathematics Studies 19, Princeton University Press, 1949.

[5] J. Duoandikoetxea, *Fourier Analysis*, translated and revised by D. Cruz-Uribe, SFO, American Mathematical Society, 2001.
[6] P. Duren, *Theory of $H^p$ Spaces*, Academic Press, 1970.

[7] J. García-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, 1985.

[8] R. Goldberg, *Methods of Real Analysis*, 2nd edition, Wiley, 1976.

[9] S. Goldberg, *Unbounded Linear Operators*, Dover, 1985.

[10] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, 1975.

[11] W. Johnson and J. Lindenstrauss, editors, *Handbook of the Geometry of Banach Spaces*, volume 1, North-Holland, 2001.

[12] W. Johnson and J. Lindenstrauss, editors, *Handbook of the Geometry of Banach Spaces*, volume 2, North-Holland, 2003.

[13] F. Jones, *Lebesgue Integration on Euclidean Space*, Jones and Bartlett, 1993.

[14] J.-L. Journé, *Calderón–Zygmund Operators, Pseudodifferential Operators and the Cauchy Integral of Calderón*, Lecture Notes in Mathematics 994, Springer-Verlag, 1983.

[15] Y. Katznelson, *An Introduction to Harmonic Analysis*, 3rd edition, Cambridge University Press, 2004.

[16] S. Krantz, *Real Analysis and Foundations*, CRC Press, 1991.

[17] S. Krantz, *A Panorama of Harmonic Analysis*, Mathematical Association of America, 1999.

[18] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Lecture Notes in Mathematics 338, Springer-Verlag, 1973.

[19] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, I, Sequence Spaces*, Springer-Verlag, 1977.

[20] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II, Function Spaces*, Springer-Verlag, 1979.
[21] M. Marcus and G. Pisier, *Random Fourier Series with Applications to Harmonic Analysis*, Annals of Mathematics Studies 101, Princeton University Press, 1981.

[22] A. Pelczynski, *Banach Spaces of Analytic Functions and Absolutely Summing Operators*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics 30, American Mathematical Society, 1977.

[23] V. Peller, *Hankel Operators and their Applications*, Springer-Verlag, 2003.

[24] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge University Press, 1989.

[25] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976.

[26] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1987.

[27] W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill, 1991.

[28] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[29] E. Stein, *The development of square functions in the work of A. Zygmund*, Bulletin of the American Mathematical Society (New Series) 7 (1982), 359–376.

[30] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, with the assistance of T. Murphy, Princeton University Press, 1993.

[31] E. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, 2003.

[32] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[33] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Mathematics 1381, Springer-Verlag, 1989.
[34] D. Stroock, *Probability Theory: An Analytic View*, Cambridge University Press, 1993.

[35] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, 1986.

[36] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge University Press, 1991.

[37] A. Zygmund, *Trigonometric Series*, Volumes I and II, 3rd edition, Cambridge University Press, 2002.