On the defect of an analytic disc

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Abstract

Although the concept of defect of an analytic disc attached to a generic manifold of $\mathbb{C}^n$ seems to play a merely technical role, it turns out to be a rather deep and fruitful notion for the extendability of CR functions defined on the manifold.

In this paper we give a new geometric description of defect, drawing attention to the behaviour of the interior points of the disc by infinitesimal perturbations. For hypersurfaces a stronger result holds because these perturbations describe a complex vector space of $\mathbb{C}^n$.

For a big analytic disc the defect does not need to be smaller than the codimension of the manifold. Indeed we show by an example that it can be arbitrarily large independently of the codimension of the manifold.

Nevertheless we also prove that the defect is always finite. In the case of a hypersurface we give a geometric upper bound for the defect.

Introduction.

The concept of defect of an analytic disc attached to a CR manifold $M \subset \mathbb{C}^n$ appeared first in the well known paper of A.E. Tumanov on the

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edge of the wedge theorem [T1]. Although the defect seemed there to play a merely technical role, it turned out later to be a rather deep and fruitful notion.

Besides the study of extendability of CR functions, the defect has interesting applications to propagation phenomena ([Tr]) and CR-maps ([BR], [CR]). Rather than the original definition, we prefer to start sketching two characteristic properties of the defect which are the core of Tumanov’s theorem.

Let \( M \subset \mathbb{C}^n \) be a suitably smooth manifold that we assume to be generic, i.e. the tangent space \( T_p M \) to \( M \) at any \( p \in M \) generates linearly all of \( \mathbb{C}^n \) over \( \mathbb{C} \).

Let \( D \) be the unit disc and \( \Gamma \) its boundary. Consider an analytic disc \( \phi : \bar{D} \to \mathbb{C}^n \) of class \( C^{1,\delta} \), \( 0 < \delta < 1 \), \( \phi \in \mathcal{O}(D) \), with boundary on \( M \) (\( \phi \Gamma \subset M \)), and fix a base point \( p = \phi(\zeta_0) \), \( \zeta_0 \in \Gamma \), at the boundary.

Consider first ”suitably” small discs. The infinitesimal perturbations \( \dot{\phi} \) of \( \phi \), keeping fixed the point \( \phi(\zeta_0) \) and still respecting \( \phi \Gamma \subset M \), form a vector space \( U \).

Now the defect can be described as follows.

a) Fix arbitrarily \( \zeta_1 \in \Gamma \), \( \zeta_1 \neq \zeta_0 \). As \( \dot{\phi} \) runs through all perturbations in \( U \), \( \phi(\zeta_1) \) fills a vector subspace of \( T_{\phi(\zeta_1)} M \). The codimension of this subspace is the defect of \( \phi \).

b) Consider the starting velocity \( \vec{v} \) of the curve \( t \mapsto \phi[(1 - t)\zeta_0] \) at the base point \( p = \phi(\zeta_0) \) when we move along the radius of the disc and fix an arbitrary supplementary vector space \( S \) to \( T_p M \) with projection \( \pi \).

As \( \phi \) undergoes all perturbations in \( U \), \( \pi \vec{v} \) describes a vector-subspace of \( S \) whose codimension is again the defect of \( \phi \).

The main purpose of the present paper is to show that the defect can be also described by perturbations of the interior of the disc in the following way.

c) Fix arbitrarily an interior point \( \zeta_2 \in \bar{D} \) and again subject \( \phi \) to the perturbations \( \dot{\phi} \). Then this differential has an image \( V \) whose span over \( \mathbb{C} \) has a codimension equal to the defect of \( \phi \). Furthermore for hypersurfaces we obtain a stronger result because \( V \) is always a complex vector space. This is the content of our Theorem 2 in section 3.

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1 All CR manifolds are locally CR-equivalent to a generic one.
The actual definition of defect in [T1] is neither a) nor b): it is introduced in
a rather technical way (see [H] in section 1 below) and apparently depends
on the base point p.
However this can be easily reformulated, as we did in [R], in order to make
the defect independent of p, and not to require that the disc is small.
In section 1 we introduce this "reformulated" definition and prove that it is
the same as the original one for small discs. This is the same point of view
of Baouendi-Rothschild-Trépreau in [BRT].

Let us now speak about large discs. In this case Tumanov’s definition of
defect has no meaning. We shall use our definition and prove that the defect
is always finite.
In any case the characterizations a), b), c) no longer hold for large discs.
Indeed, according to those characterizations, the defect obviously cannot
exceed the codimension of M, while in Proposition 1 we show that the unit
disc of the z₁-axis, viewed as analytic disc attached to a particular algebraic
real hypersurface, has defect 2k + 1.
In Proposition 2 we give an upper bound for the defect of an analytic disc
attached to a hypersurface and in Theorem 1 we show that also in higher
codimension the defect is finite.
In order to clarify the geometric construction which leads to Theorem 2,
which is stated and proved in section 3, we gather in section 2 several results
concerning mainly the Hilbert transform and matrix valued functions in the
disc. Although they should be considered as a part of the proof, some of
those results might have some interest in their own.
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§1. The defect of a disc.

In this section we show, as we did in [R], that the original definition
of defect can be reformulated in a geometric way eliminating the particular
equation of M and the base point of the disc.
Let M be a real, generic manifold of class \( C^{2,\epsilon} \), \( 0 < \epsilon < 1 \) in \( \mathbb{C}^n \). We
shall always assume M to be an open, relatively compact subset of a larger
manifold.
The fiber at \( p \in M \) of the holomorphic co-normal bundle \( CM \) of M is the
real vector space of the forms $\omega = \sum_{j=1}^{n} a_j dz_j$ such that $\mathcal{I}m(\omega)$ vanishes on the tangent space $T_p M$.

If $\{\rho_1 = \cdots = \rho_m = 0\}$ are local real equations for $M$, with $\partial \rho_1 \wedge \cdots \wedge \partial \rho_m \neq 0$ on $M$ (by the genericity of $M$), then $CM = i \Re \partial \rho_1 \oplus \cdots \oplus i \Re \partial \rho_m$. Let $\phi$ be an analytic disc of class $C^{1,\delta}$, $0 < \delta < \epsilon$, with boundary on $M$, i.e. $\phi \Gamma \subset M$. A section of the pull-back $\phi^* CM \to \Gamma$ of $CM$ has the form $\omega \circ \phi = \sum_{j=1}^{n} (a_j \circ \phi) dz_j$. We say that $\omega \circ \phi$ extends holomorphically into the disc $D$ if all coefficients $a_j \circ \phi$ extend holomorphically to $D$.

**Definition 1** The defect $d(\phi)$ of a disc $\phi$ is the dimension of the real vector space $E_\phi = \{C^{1,\delta} - \text{sections of } \phi^* CM \text{ which extend holomorphically to } D\}$.

Observe that, if $\alpha$ is an automorphism of the disc, then $E_\phi = E_{\phi \circ \alpha}$. Thus the defect is invariant by right composition with an automorphism of the disc. Define the size of a disc $\phi$ as

$$|\phi| = \inf_{z_0 \in \mathbb{C}^n} ||\phi + z_0||_{1,\delta}.$$

This quantity measures how far is a disc from being a constant disc.

We shall prove in Proposition 3 that, if $|\phi|$ is smaller than a constant depending only on $M$, then $\phi^* CM$ has a moving frame such that the sections which extend holomorphically to $D$ have constant components with respect to this frame. Since $\text{rk}(CM) = \text{codim} M$, this gives

$$d(\phi) \leq \text{codim} M, \text{ for small } |\phi|.$$

On the other hand Tumanov’s characterizations of the defect (a), (b) of the introduction) and our Theorem 2 (or statement c) in the introduction) also obviously imply $d(\phi) \leq \text{codim} M$.

We will now show that, for large discs,

$$d(\phi) > \text{codim} M$$

can also occur, but in any case the defect is finite.

**Proposition 1** The analytic disc $\phi : \zeta \mapsto (\zeta, 0)$, $|\zeta| \leq 1$, as a disc attached to the real hypersurface

$$M = \{(z_1, z_2) \in \mathbb{C}^2, \Re(z_1 z_2) = 0, z_1 \neq 0\}$$

has defect $2k + 1$. 

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Proof. Setting $\rho(z_1, z_2) = 2\mathcal{R}(z_1^k z_2)$, we have $\rho_{z_1}(\zeta, 0) = 0$, $\rho_{z_2}(\zeta, 0) = \zeta^k$. The real vector space $E_\phi$ of the holomorphic functions in the unit disc, whose restrictions to $|\zeta| = 1$ are equal to $\zeta^k$ times a real function, obviously coincides with the set of the functions of the type $\zeta^k(p_k + \bar{p}_k)$, where $p_k$ is any holomorphic polynomial of degree $k$. Thus $d(\phi) \equiv \dim E_\phi = 2k + 1$. □

Remark. In the example above, except for its center, the whole analytic disc is contained in $M$. This is in fact a concidence. Indeed we only need to add the term $(|z_1|^2 - 1)^2$ to the equation of $M$ and have $\phi D \cap M = \emptyset$. The functions $\rho_{z_j}(\zeta, 0)$, $j = 1, 2$, as well as the defect of $\phi$, will not change.

Theorem 1 The defect of an analytic disc attached to a generic manifold is finite.

Proof. If $M$, and hence $\phi^*CM \to \Gamma$ have not global equations, we can reduce to this case taking its pull-back by the map $\Gamma \to \Gamma$ defined by $\sigma \mapsto \sigma^2$.

First we observe that since $M$ is generic the complex codimension of $T^c_pM$ equals the real codimension $m$ of $M$. Now, since $\left\{T^c_{\phi(\sigma)}M, \sigma \in \Gamma\right\}$ is a $C^1$ family of $m$-codimensional complex vector spaces in $\mathbb{C}^n$ which represents a 0-measure set in the corresponding Grassmannian, indeed there exists an open dense set in the Grassmannian where we can choose an $m$-dimensional complex vector space $V$ such that $V \cap T^c_{\phi(\sigma)}M = \{0\}$ for all $\sigma \in \Gamma$.

After a linear change of coordinates we can assume that $V$ is the $\{z_1, \ldots, z_m\}$ plane. Thus the matrix $A(\sigma) := \left(\frac{\partial \phi_{z_j}(\sigma)}{\partial z_i}\right)_{k,j \leq m}$ is non degenerate for all $\sigma \in \Gamma$. An element of $E_\phi$ (see Definition 1) is identified with a $\mathbb{C}^n$ valued function of the type $\gamma(\sigma)\rho_{z^k[\phi(\sigma)]}$, where $\gamma$ and $\rho_z$ are respectively a real $(1,m)$ and a complex $(m,n)$ matrixes.

By definition of $E_\phi$, $\gamma \rho_z$, and hence $\gamma(\sigma)A(\sigma)$, extends holomorphically to the disc. Since the matrix $A$ is non degenerate we shall be done if we prove that if this happens, then $b := \gamma A$, on $\Gamma$, belongs to a finite dimensional vector space.

Since $\gamma = bA^{-1}$ is real valued we have

$$bA^{-1} = \overline{bA^{-1}}, \text{ on } \Gamma.$$ \hspace{1cm} (1)

$^2T^c_pM$ is the complex tangent space to $M$ at $p$. 

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Set \( u(z) := \overline{b(z)} \). \( u \) is a holomorphic function for \( |z| > 1 \) and \( u|_{\Gamma} = \overline{b} \). Then we can write (1) in terms of \( u \)

\[
 bA^{-1} = uA^{-1}, \text{ on } \Gamma.
\]  

(2)

Now, since \( u \in \mathcal{O}(\mathbb{C} - \bar{D}) \) has Hölder trace on \( \Gamma \), by Plemelj formula we have

\[
 \frac{1}{2} u(z) + p.v. \frac{1}{2\pi i} \int_{\Gamma} \frac{u(\zeta)}{\zeta - z} d\zeta = 0, \forall z \in \Gamma.
\]  

(3)

Set \( I(u)(z) := p.v. \frac{1}{2\pi i} \int_{\Gamma} \frac{u(\zeta)}{\zeta - z} d\zeta \). By (2) we have \( u = bA^{-1}\overline{A} \) on \( \Gamma \) and substituting this expression in (3) and setting \( C := A^{-1}\overline{A} \) we obtain

\[
 \frac{1}{2} b + I(bC)C^{-1} = 0, \text{ on } \Gamma.
\]  

(4)

Analogously, since \( b \) is a holomorphic function in \( D \), applying again the Plemelj formula we have

\[
 -\frac{1}{2} b + I(b) = 0, \text{ on } \Gamma.
\]  

(5)

Subtracting (3) from (4) we obtain that \( b \) belongs to the kernel of the Fredholm operator of the second kind (see [V], p.26)

\[
 L(b)(z) := b(z) + p.v. \frac{1}{2\pi i} \int_{\Gamma} b(\zeta) \frac{C(\zeta)C^{-1}(z) - 1}{\zeta - z} d\zeta
\]

and thus it varies in a finite dimensional vector space. \( \square \)

In the case of hypersurface one can give a simple geometric upper bound for \( d(\phi) \) suggested by the example in Proposition 1.

Consider a complex direction \( L \) such that, for all \( \sigma \in \Gamma, T_{\phi(\sigma)}^c M \cap L = \{0\} \). The set of such \( L \) is open dense in \( \mathbb{P}^{n-1}(\mathbb{C}) \). Then \( l(\sigma) = T_{\phi(\sigma)}^c \cap L \) is a real straight line in \( L \) through the origin. When \( \sigma \) turns once in the circle \( \Gamma \), \( l(\sigma) \) will turn \( k \) times in \( L \), \( k \in \mathbb{Z} \).

**Proposition 2** Let \( k \) be the number of times that the real line \( l(\sigma) \) above turns in the complex direction \( L \), while \( \sigma \) turns once in the unit circle. For the defect of \( \phi \) we have the upper bound

\[
 d(\phi) \leq \sup (0, -2k + 1).
\]
The proof will be based on the following elementary

**Lemma** Let \( f : \Gamma \to \mathbb{C} \setminus \{0\} \) be a \( C^\alpha \) map, \( 0 < \alpha < 1 \), with winding number \( s \). Then the real vector space \( V_f \) of the holomorphic functions \( g \in \mathcal{O}(D) \cap C^\alpha(\bar{D}) \) such that \( g|_\Gamma f^{-1} \) is real, has dimension \( \sup (0,2s+1) \).

**Proof.** For \( f(\sigma) = \sigma^s \) we have obviously \( V_{\sigma^s} = \{0\} \) when \( s < 0 \), while \( V_{\sigma^s} \) has \( \{\sigma^s(\sigma^j + \sigma^{-j}), i\sigma^s(\sigma^j - \sigma^{-j}), 0 \leq j \leq s\} \) as a basis when \( s \geq 0 \).

Then \( \dim V_{\sigma^s} = \sup (0,2s+1) \). For a general \( f \) with winding number \( s \), we can write \( f(\sigma) = \sigma^s e^{X(s) + iY(s)} \), where \( X, Y \) are real \( C^\alpha \) functions on \( \Gamma \). Let \( T \) be the Hilbert transform. Set \( r = X + TY \) and \( h = iY - TY \) so that \( h \) is the trace on \( \Gamma \) of a function in \( \mathcal{O}(D) \cap C^\alpha(\bar{D}) \). We have \( f(\sigma) = \sigma^s e^{r(\sigma) + h(\sigma)} \) and \( g|_\Gamma f^{-1} \) is real if and only if \( g|_\Gamma e^{-h(\sigma)^{-s}} \) is real, i.e. \( g \in e^h V_{\sigma^s} \). This gives \( \dim V_f = \dim V_{\sigma^s} = \sup (0,2s+1) \). \( \square \)

**Proof of Proposition 2.** Let \( L \) be a complex direction such that \( T^c_{\phi(\sigma)} M \cap L = \{0\}, \forall \sigma \in \Gamma \). We can assume that \( L \) is the \( z_1 \)-axis so that, if \( \{\rho = 0\} \) is the equation of \( M \), we have \( \rho_{z_1}[\phi(\sigma)] \neq 0, \forall \sigma \in \Gamma \). Let \( s \) be the winding number of \( \sigma \mapsto \rho_{z_1}[\phi(\sigma)] \). (Thus the real line \( l(\sigma) \) above turns \( k = -s \) times.) The first component of a section of \( \phi^* C M \) is a function of the form \( r(\sigma) \rho_{z_1}[\phi(\sigma)] \) with real \( r \). If the section extends holomorphically to \( D \), so will do our function. Thus, by the Lemma, \( d(\phi) \leq \sup (0,2s+1) \). \( \square \)

We shall now only deal with small discs and always assume

\[ |\phi| < R, \quad (6) \]

reducing \( R \) when it is necessary.

In order to refer our definition of defect to the original one, it is necessary to give a cartesian form to \( M \).

First we need the following elementary

**Lemma 1** Let \( M \) be an open, relatively compact subset of a generic manifold of class \( C^{2+} \) and codimension \( m \) in \( \mathbb{C}^n \). Then, for every \( \lambda > 0 \), there exists \( R(\lambda) > 0 \) with the following properties:

for every set \( L \subset M \) with \( \text{diam} L < R(\lambda) \) and \( \forall p \in L \), there exist complex affine coordinates \((z = x + iy, w) \in \mathbb{C}^m \times \mathbb{C}^{n-m}\) with origin at \( p \), such that \( L \) has a neighbourhood in \( M \) which is contained in the set

\[ x = h(w,y), \quad |w| < r, \quad |y| < r, \quad (7) \]
where \( h : \{|w| < r, |y| < r\} \rightarrow \mathbb{R}^m \) is a function of class \( C^{2,\epsilon} \), satisfying
\[
h(0, 0) = dh(0, 0) = 0 \quad \text{and} \quad \|h_y(w, y)\| < \lambda, \forall |w| < r, |y| < r.
\]
Here \( \| \cdot \| \) stands for the matrix norm.

The proof is quite standard and so we omit it.

Now we want to prove that \( d(\phi) \leq \text{codim} M \), for \( R \) sufficiently small and show that our definition and Tumanov’s original definition of the defect are the same. In particular this shows that the latter is independent of the choice of base point.

For this we need to sketch Tumanov’s presentation which is rather technical. Consider the Hilbert transform \( T_1 : C^{1,\delta}(\Gamma) \rightarrow C^{1,\delta}(\Gamma) \) normalized at 1. \( T_1 \) is defined on real functions by the fact that \( f + iT_1f \) extends holomorphically to \( D \) and \( T_1f(1) = 0 \). It is a bounded operator and \( T_1^2f = -f \) whenever \( f(1) = 0 \).

Fix arbitrarily \( p \in \phi \Gamma \). Since the defect is invariant by automorphisms, we can assume \( p = \phi(1) \). Replace \( L = \phi \Gamma \) in Lemma 1 and choose \( R < R(1/\|T_1\|) \) in (3).

There exists a unique \( G : \Gamma \rightarrow GL(m, \mathbb{R}) \) of class \( C^{1,\delta} \) such that \( \sigma \mapsto G(\sigma)(1 + ih_y[\phi(\sigma)]) \) extends holomorphically to \( D \). In fact, since \( \|h_y\| < 1/\|T_1\| \) on \( \phi \Gamma \), we can solve the equation
\[
G = 1 - T_1[G(h_y \circ \phi)]
\]
which is equivalent to the holomorphic extendability of \( G(1 + ih_y) \) to the unit disc and \( G(1) = 1 \).

Since \( (h_y \circ \phi)(1) = 0 \), we have \( T_1G = G(h_y \circ \phi) \). Again using Lemma 1, we can take \( R \) in (3) so small that the norm of the matrix \( h_y \circ \phi \) is smaller than an absolute constant which guarantees that, in addition, the holomorphic extension of \( G(1 + i(h_y \circ \phi)) \) is non-degenerate at all points of \( \bar{D} \). Indeed, as a solution of a fixed point problem, \( G \) depends continuously on \( h_y \) and, for \( h_y = 0 \), we have \( G = 1, G(1 + i(h_y \circ \phi)) = 1 \).

The defect of \( \phi \), with \( |\phi| < R \), was originally defined in [T1] as the dimension of the vector space
\[
V_\phi = \{ c \in \mathbb{R}^m \mid cG(h_w \circ \phi) \text{ extends holomorphically to } D \}.
\]

The next proposition establishes the identity between our definition of defect and the original one.
Proposition 3 If $\phi \Gamma \subset M$ and $|\phi|$ is smaller than a constant depending on $M$, then $d(\phi) = \dim V_\phi$. In particular $d(\phi) \leq m = \text{codim}M$.

Proof. As we have seen, a neighbourhood of $\phi \Gamma$ in $M$ is contained in the manifold $(\mathbb{R}^n)$. Thus we can assume that $M$ has equation $(\mathbb{R}^n)$. Since $d(\phi)$ does not depend on the equations of $M$, we can choose $\rho = x - h(w, y)$ and obtain $\partial \rho = \frac{1}{2}[(1 + ih_y)dz - 2h_wdw]$. In those coordinates we have

$$E_\phi \cong \{ \gamma | \gamma(1 + ih_y \circ \phi) \text{ and } \gamma h_w \circ \phi \text{ extend holomorphically to } D \},$$

where $\gamma$ is a real $(m \times m)$-matrix function of class $C^{1,\delta}$ on $\Gamma$.

Thus, if $c \in V_\phi$, then $\gamma = cG \in E_\phi$. If vice versa $\gamma \in E_\phi$, then $\gamma(1 + ih_y \circ \phi)$ extends to a holomorphic matrix $F$ and $G(1 + ih_y \circ \phi)$ to a holomorphic, nondegenerate matrix $g$.

We have on $\Gamma \gamma G^{-1} = F g^{-1}$ and, since the left hand side is real and the right hand side extends holomorphically, this is a constant real vector $c$. We obtained $\gamma \in E_\phi \Rightarrow \gamma = cG$, for some $c \in \mathbb{R}^m$. Replacing $\gamma = cG$ in the second equation of $E_\phi$, we obtain $c \in V_\phi$. Thus $\gamma \in E_\phi \iff \gamma = cG$ with $c \in V_\phi$. This gives $\text{dim}E_\phi = \dim V_\phi$. \[\square\]

Remark. The disc in Proposition 1 can be done arbitrarily small taking $(\epsilon \zeta, 0)$ instead of $(\zeta, 0)$ without any change in the conclusion. This seems to be in contradiction with the proposition above. But the closure of the hypersurface of Proposition 1 is singular, thus this last cannot be relatively compact in any other manifold, while the manifold in Proposition 3 is assumed to have this property.

§2. Preliminary results.

We now introduce the Hilbert transform $T_0$ normalized at 0, i.e. with the condition $\int_0^{2\pi} T_0 f d\theta = 0$. Thus $T_0^2 f = -f + \frac{1}{2\pi} \int_0^{2\pi} f d\theta$. Again using Lemma 1, we further reduce $R$ in (8) so that $||h_y(w, y)|| < 1/||T_0||$.

The next lemma gives a relation between the matrix $G$ defined by (8) and the unique solution $G_0$ of

$$G_0 = 1 - T_0[G_0(h_y \circ \phi)]. \quad (10)$$
Lemma 2 Let $G$ be the matrix-function defined by (8). Then there exists a constant matrix $C \in GL(m, \mathbb{R})$ such that $G_0 = CG$ is the solution of (10). Furthermore $G_0(1 + ih_y)$ extends holomorphically to $D$ and $T_0G_0 = G_0(h_y \circ \phi) - \frac{1}{2\pi} \int_0^{2\pi} G_0(\sigma)h_y[\phi(\sigma)]d\theta$, $\sigma = e^{i\theta}$.

Proof. By the definition of $G$ we have that $G(1 + ih_y)$ and $[G(1 + ih_y)]^{-1}$ extend holomorphically to $D$. Then the real matrix $C \equiv G_0G^{-1} = G_0(1 + ih_y)[G(1 + ih_y)]^{-1}$ also extends holomorphically to $D$. Thus $C$ is constant. Furthermore, applying to (10) the transform $T_0$, we obtain the last assertion of the lemma. □

Now we give some more results which will be needed in the next section.

Lemma 3 Let $f$ be a function of class $C^\alpha$, $0 < \alpha < 1$, with $f(1) = 0$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} (f + iT_1f)d\theta = -\frac{1}{\pi} \int_0^{2\pi} \frac{f(\sigma)}{\sigma - 1}d\theta, \quad \sigma = e^{i\theta}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} (f - iT_1f)d\theta = -\frac{1}{\pi} \int_0^{2\pi} \frac{f(\sigma)}{\sigma - 1}d\theta, \quad \sigma = e^{i\theta}.\]$$

Proof. Since both sides of those equalities are linear, it is sufficient to prove the lemma for real $f$. Observe also that in this case the second part follows immediately from the first.

Now $f + iT_1f$ is the boundary value of a holomorphic function $F$ vanishing at 1. We must compute $F(0)$. For $\zeta \in D$ set $F_1(\zeta) = \frac{1}{2\pi} p.v. \int_0^{2\pi} \frac{f(\sigma)}{\sigma - \zeta}f(\sigma)d\theta$, with $\sigma = e^{i\theta}$. On $\Gamma$ we have $\mathcal{R}e(F_1) = f = \mathcal{R}e(F)$. Thus $F(\zeta) = F_1(\zeta) - F_1(1) = \frac{\zeta - 1}{\pi} \int_0^{2\pi} \frac{f(\sigma)}{(\sigma - \zeta)(\sigma - 1)}d\theta$, $\sigma = e^{i\theta}$. Now, setting $\zeta = 0$, we have the result. □

Lemma 4 Let $f, g \in C^\alpha$, $0 < \alpha < 1$, be such that $\frac{1}{2\pi} \int_0^{2\pi} f(\sigma)d\theta = 0$ and $g(1) = 0$. Then

$$\int_0^{2\pi} \frac{fg - (T_0f)(T_1g)}{\sigma - 1}d\theta = 0$$

and

$$\int_0^{2\pi} \frac{fg - (T_0f)(T_1g)}{\bar{\sigma} - 1}d\theta = 0.$$

The integrals converge absolutely because $f(1) = 0$.\]
Proof. As in Lemma 3 it is sufficient to only prove the first equality for real \( f \) and \( g \). \( f + iT_0 f \) and \( g + iT_1 g \) are boundary values of holomorphic functions \( F_0 \) and \( F_1 \) and, by the hypothesis on \( f \), \( F_0(0) = 0 \). On \( \Gamma \) we have \( F_0 F_1 = fg - (T_0 f)(T_1 g) + i(fT_1 g + gT_0 f) \) and, since the imaginary part vanishes at 1, we obtain \( fT_1 g + gT_0 f = T_1 [fg - (T_0 f)(T_1 g)] \). Set \( A := fg - (T_0 f)(T_1 g) \) so that \( F_0 F_1 = A + iT_1 A \) on \( \Gamma \) and \( A(1) = 0 \). Since \( F_0 F_1 \) vanishes at 0, an application of Lemma 3 to \( A \) gives the desired equality. \( \square \)

Proposition 4 Let \( G_0 \) be the matrix-function defined by equation (10), and let \( X \) and \( Y \) be functions of class \( C^\alpha \) on \( \Gamma \), \( 0 < \alpha < 1 \), linked by the relation \( Y = T_1 X \). Set \( K = \frac{1}{2\pi} \int_0^{2\pi} G_0(\sigma) h_y[\phi(\sigma)] d\theta \), with \( \sigma = e^{i\theta} \). Then

\[
\int_0^{2\pi} \frac{X - KY}{\sigma - 1} d\theta = \int_0^{2\pi} \frac{G_0(X - h_y Y)}{\sigma - 1} d\theta
\]

and

\[
\int_0^{2\pi} \frac{X - K\bar{Y}}{\bar{\sigma} - 1} d\theta = \int_0^{2\pi} \frac{G_0(X - h_y Y)}{\bar{\sigma} - 1} d\theta.
\]

Proof. It is sufficient to apply the Lemma 4 with \( f = G_0 - 1 \) and \( g = X \), recalling that \( T_0 G_0 = G_0 h_y - K \) by Lemma 2. \( \square \)

Proposition 5 Let \( g \) be a vector-function of type \( 1 \times s \) depending \( C^{1,\delta} \) on \( \sigma \in \Gamma \).

If, for all \( f = (f_1, \ldots, f_s) \in C^{1,\delta}(\bar{D}) \cap \mathcal{O}(D) \) with \( f(0) = f(1) = 0 \), we have

\[
I = \int_0^{2\pi} \left( \frac{a}{\sigma - 1} + \frac{\bar{a}}{\bar{\sigma} - 1} \right) g f d\theta = 0 \quad \sigma = e^{i\theta},
\]

with \( a \in \mathbb{C} \), then \( ag \) and \( \bar{a}g \) extend holomorphically into \( D \).

Proof. By the arbitrary nature of \( f \) we can take \( f(\sigma) = (\sigma - 1)^l e_j \), with \( l \geq 1 \), where \( e_j \) is a vector of the canonical basis of \( \mathbb{R}^s \) (note that this \( f \) satisfies the conditions \( f(0) = 0, f(1) = 0 \)). Then, since \( (\bar{\sigma} - 1)^{-1} = -\sigma(\sigma - 1)^{-1} \), we obtain that \( (a - \bar{\sigma} a)g \) extends into the disc \( D \) as a holomorphic function \( h \).

\[\text{Note that the integrand is continuous because } gf \text{ is of class } C^{1,\delta} \text{ and vanishes at 1.}\]
Suppose $a \neq 0$ and since $a$ is a scalar we obtain that $(a - \sigma \bar{a})$ has no zeros into $D$. Then \( \frac{b}{a-\sigma \bar{a}} \) is a holomorphic function and so $g$ extends holomorphically. In particular $ag$ and $\bar{a}g$ extend holomorphically into $D$. $\square$

§3. The main theorem.

We shall now correctly state and prove the main Theorem. As we saw in Section 1, a disc $\phi$ of class $C^{1,\delta}$ attached to $M$, with $|\phi| < R = R(M)$ has the property that, for each $p \in \phi \Gamma$, $\phi$ is a Bishop-lifting of a unique analytic disc $w(\zeta)$ lying in the complex tangent space $T_p^{c}M \equiv \mathbb{C}^{n-m}_w$ and the Bishop's lifting maps a neighbourhood of $w(\zeta)$ in $[\mathcal{O}(D) \cap C^{1,\delta}(\bar{D})]^{n-m}$ onto a neighbourhood of $\phi$ in the set $\mathcal{M}_p$ of the $C^{1,\delta}$ discs in $\mathbb{C}^n$ satisfying $p \in \phi \Gamma \subset M$, $|\phi| < R$. Since $M$ is of class $C^{2,\epsilon}$, with $\delta < \epsilon < 1$, the lifting $w(\zeta) \mapsto \phi$ is of class $C^1$ and thus $\mathcal{M}_p$ has a natural structure of a $C^1$-manifold. So, refering to the point c) in the introduction, it makes sense to fix $\zeta_2$ in the interior of $D$ and to differentiate the $C^1$ map $\mathcal{M}_p \rightarrow \mathbb{C}^n$ given by the evaluation at $\zeta_2$. Since the group of the automorphisms of $D$ acts nicely on the right on $\mathcal{M}_p$ and preserves the defect, we can add the condition $\phi(1) = p$ to the discs in $\mathcal{M}_p$.

**Theorem 2** The differential of the evaluation map $\mathcal{M}_p \rightarrow \mathbb{C}^n$ given by $\phi \mapsto \phi(\zeta)$ (for fixed $\zeta \in D$), has an image $V$ whose span over $\mathbb{C}$ has complex codimension equal to the defect of the disc $\phi$.

For hypersurfaces a stronger result holds because this image is always a complex vector space.

If and only if the defect is 1, then $\zeta \mapsto V(\zeta)$ as a map $D \rightarrow \mathbb{P}^{n-1}(\mathbb{C}^n)$ is a holomorphic extension of the map $\Gamma \rightarrow \mathbb{P}^{n-1}(\mathbb{C}^n)$ given by $\sigma \mapsto T_{\phi(\sigma)}^{c}M$.

Furthermore we have $d(\phi) = 0$ if and only if $V(\zeta) = \mathbb{C}^n$ for one (and thus all) $\zeta \in D$.

**Proof.** By previous discussion we can consider as point of evaluation the point $\zeta = 0$.

Taking coordinates at $p$ as in Lemma 1, with the restrictions we imposed in the statement of Proposition 3, the element of $\mathcal{M}_p$ corresponding to $w(\zeta)$...
will be \((w(\zeta), z(\zeta))\), where \(z(\zeta) = x(\zeta) + iy(\zeta)\) is defined by its boundary value \(z(\sigma), \sigma \in \Gamma\), with \(x(\sigma)\) determined by
\[
x(\sigma) = h(w(\sigma), y(\sigma)), \quad \sigma = e^{i\theta},
\]
and \(y(\sigma)\) is uniquely determined by the Bishop’s equation
\[
y(\sigma) = T_1 x(\sigma) = T_1 h(w(\sigma), y(\sigma)), \quad \sigma = e^{i\theta}.
\]
From Poisson’s formula we have
\[
z(\zeta) = \frac{1}{2\pi} \int_{\Gamma} \frac{\sigma + \zeta}{\sigma - \zeta} x(\sigma) d\theta + iy(0), \quad \sigma = e^{i\theta}, \quad |\zeta| < 1.
\]
When \(|\zeta| = 1\), the integral must be taken in the sense of a principal value. We must differentiate the composed map
\[
w(\zeta) \mapsto (w(\zeta), z(\zeta)) \mapsto (w(0), z(0)),
\]
which is defined by (11) and (12) and where \(w(\zeta)\) will vary in the Banach space \(W\) of vector functions \(w = w(\zeta) : \bar{D} \to \mathbb{C}^s\) of class \(C^{1,\alpha}(\bar{D})\), holomorphic in \(D\), with the property \(w(1) = 0\).

From Lemma 3 and (11) we obtain
\[
z(0) = -\frac{1}{\pi} \int_0^{2\pi} \frac{x(\sigma)}{\sigma - 1} d\theta = -\frac{1}{\pi} \int_0^{2\pi} \frac{h(w(\sigma), y(\sigma))}{\sigma - 1} d\theta, \quad \sigma = e^{i\theta}.
\]
Now we differentiate this expression with respect to \(w(\zeta) \in W\), taking (11) and (12) into account.
If dot means the differentiation with respect to \(w(\zeta)\), on \(\Gamma\) we have
\[
\begin{align*}
\dot{x} &= h_w \dot{w} + h_w \ddot{w} + h_y \dot{y} \\
\dot{y} &= T_1 \dot{x}
\end{align*}
\]
where \(\dot{x}, \dot{y}\) depend \(\mathbb{R}\)-linearly on \(\dot{w}\). If we set \(X\) and \(Y\) for their \(\mathbb{C}\)-linear parts, we have \(\dot{x} = X + \bar{X}, \dot{y} = Y + \bar{Y}\) with
\[
\begin{align*}
X &= h_w \dot{w} + h_y Y \\
Y &= T_1 X.
\end{align*}
\]
Consider now the case \( m = 1 \).
For \( a \in \mathbb{C} \), \( b \in \mathbb{C}^{n-1} \) we set
\[
l(a, b, \dot{w}) = \mathbb{C} - \text{linear part of } a \dot{z}(0) + \bar{a} \dot{\bar{z}}(0) + b \dot{w}(0) + \bar{b} \dot{\bar{w}}(0)
\]
where "\( \mathbb{C} \)-linear" refers to the dependence on \( \dot{w} \in W \).
A real subspace is a complex subspace of complex codimension \( d \) if and only if its annihilator is complex subspace of dimension \( d \).
So we must only prove that the space
\[
A \equiv \{(a, b) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid l(a, b, \dot{w}) = 0, \ \forall \dot{w} \in W\}
\]
is a complex space of dimension \( d(\phi) \).
The linear parts of the restrictions to \( \Gamma \) of \( \dot{z} \) and \( \dot{\bar{z}} \) are \( X + iY \) and \( X - iY \).
Thus we have
\[
l(a, b, \dot{w}) = \frac{1}{2\pi} \int_0^{2\pi} [a(X + iY) + \bar{a}(X - iY)] d\theta + b \dot{w}(0).
\]
Setting \( a' = a(1 + iK)^{-1} \), where \( K \) is defined in Proposition 4, we have
\[
[a(X + iY) + \bar{a}(X - iY)] = a'[(X - KY) + iT_1(X - KY)] + \bar{a}'[(X - KY) - iT_1(X - KY)],
\]
because \( T_1 Y = -X \). If we apply now the Lemma 3, we obtain
\[
l(a, b, \dot{w}) = -\frac{1}{\pi} \int_0^{2\pi} \left( \frac{a'}{\sigma - 1} + \frac{\bar{a}'}{\bar{\sigma} - 1} \right)(X - KY)d\theta + b \dot{w}(0).
\]
Thus, using the Proposition 4 for \( X = h_w \dot{w} + h_y Y \) and setting \( a'' = Ca' \), where \( C \) is defined in Lemma 2, we have
\[
l(a, b, \dot{w}) = -\frac{1}{\pi} \int_0^{2\pi} \left( \frac{a''}{\sigma - 1} + \frac{\bar{a}''}{\bar{\sigma} - 1} \right) G h_w \dot{w} d\theta + b \dot{w}(0).
\]
Now we choose \( \dot{w} \) such that \( \dot{w}(0) = 0 \) and apply the Proposition 5 with \( g = G h_w \) and \( f = \dot{w} \), and obtain that \( l(a, b, \dot{w}) \) vanishes for all such \( \dot{w} \) if and only if \( a'' \in V_\phi + iV_\phi \), where \( V_\phi \) is defined by (B) in section 1. Hence \( a \in C^{-1}(V_\phi + iV_\phi)(1 + iK) \) and this is a \( \mathbb{C} \)-linear space \( L \) having complex dimension \( \text{dim}_\mathbb{R} V_\phi = d(\phi) \). Thus, if \( (a, b) \in A \), then \( a \in L \).
We now assume \( a \in L \) and attempt to find \( b \).
Since \( \frac{1}{\sigma - 1} = -\frac{\sigma}{\bar{\sigma} - 1} \), the equation \( l(a, b, \dot{w}) = 0 \) can be written in the form
\[
b \dot{w}(0) = \frac{1}{\pi} \int_0^{2\pi} \frac{a'' - \sigma \dot{a}''}{\sigma - 1} G h_w \dot{w} d\theta, \quad \sigma = e^{i\theta}.
\]
We choose \( \dot{w} = (1 - \sigma)e_j \in W \), where \( e_j \) is the canonical basis of \( \mathbb{R}^{n-1} \).

Substituting this expression in (13), we obtain

\[
b = \frac{i}{\pi} \int \frac{\sigma''}{\sigma - 1} G \ h_w d\sigma.
\]

On the other hand, since \((\bar{a}'') + a'')G \ h_w\) extends holomorphically to \(D\), we have

\[
f_{\Gamma} a''G \ h_w d\sigma = -f_{\Gamma} a''G \ h_w d\sigma.
\]

Then \(b\) is given by

\[
b = \frac{d''i}{\pi} \int \frac{\sigma + 1}{\sigma} G \ h_w d\sigma = \frac{i}{\pi} C a(1 + iK)^{-1} \int \frac{\sigma + 1}{\sigma} G \ h_w d\sigma
\]

and this is a \(C\)-linear function of \(a \in L\). Thus \(A\), as the graph of a \(C\)-linear function on the complex \(d(\phi)\)-dimensional space \(L\), is itself a complex \(d(\phi)\)-dimensional space.

In the general case \((m > 1)\) we take \(a \in \mathbb{C}^m\), \(b \in \mathbb{C}^{n-m}\) and set

\[
\lambda_1(a, b, \dot{w}) = \mathbb{C} - \text{linear part of } a \dot{z}(0) + b\dot{w}(0),
\]

\[
\lambda_2(a, b, \dot{w}) = \mathbb{C} - \text{linear part of } \bar{a} \dot{\bar{z}}(0) + \bar{b}\dot{w}(0)
\]

where ”\(C\) - linear” refers again to the dependence on \(\dot{w} \in W\).

As above we have to prove that the space

\[
B \equiv \{(a, b) \in \mathbb{C}^m \times \mathbb{C}^{n-m} \mid \lambda_j(a, b, \dot{w}) = 0, \ \forall \dot{w} \in W \ \text{and} \ j = 1, 2\}
\]

has dimension \(d(\phi)\).

The expressions of \(\lambda_j(a, b, \dot{w})\) are given by

\[
\lambda_1(a, b, \dot{w}) = \frac{1}{2\pi} \int_0^{2\pi} a(X + iY) d\theta + b\dot{w}(0),
\]

\[
\lambda_2(a, b, \dot{w}) = \frac{1}{2\pi} \int_0^{2\pi} \bar{a}(X - iY) d\theta.
\]

Repeating the previous computations we obtain

\[
\lambda_1(a, b, \dot{w}) = -\frac{1}{\pi} \int_0^{2\pi} \frac{a''}{\sigma - 1} G \ h_w \dot{w} d\theta + b\dot{w}(0),
\]

\[
\lambda_2(a, b, \dot{w}) = -\frac{1}{\pi} \int_0^{2\pi} \frac{\bar{a}''}{\sigma - 1} G \ h_w \dot{w} d\theta.
\]
If we choose $\dot{w}$ such that $\dot{w}(0) = 0$, we have $a'' \in V_{\phi} + iV_{\phi}$ and hence also in this case $a \in L$.

Now to show the dependence on $a$ of $b$ it is enough to choose in

$$b(0) = \frac{1}{\pi} \int_{0}^{2\pi} \frac{a'' \cdot \text{Im} \left( \frac{1}{\sigma - 1} \right) G_{w} \dot{w} d\theta, \quad \sigma = e^{i\theta}}$$

$\dot{w} = (1 - \sigma) e_{j}$ obtaining, as above, that $b$ is a $\mathbb{C}$-linear function of $a \in L$. $\Box$

§4. A counterexample in the case of higher codimension.

In this section we will show that the Theorem 2 is not true if the codimension of the manifold is greater than 1.

Consider a manifold $M \subset \mathbb{C}^{3}$ of real codimension 2, having equation

$$x = h(w)$$

with $h \in C^{\infty}(\mathbb{C}, \mathbb{R}^{2})$, $h(0) = dh(0) = 0$.

Call $V_{\phi}$ the image of the differential of the function $M_{p} \ni \phi \mapsto \phi(0) \in \mathbb{C}^{3}$, $p \in M$.

We will show the existence of a sequence of analytic discs $\phi_{\nu} \in M_{p}$ with $d(\phi_{\nu}) = 0$, $||\phi_{\nu}||_{1,\delta} \to 0$ and $V_{\phi_{\nu}} \neq \mathbb{C}^{3}$.

**Proposition 6** There exists a function $h$ defining the manifold $M$ with the properties described above and $h(\frac{\sigma - 1}{\nu}) = 0$, for $|\sigma| = 1$ and $\nu \in \mathbb{N}$, such that

(i) the disc $\phi_{\nu}(\zeta) = (0, \frac{\zeta - 1}{\nu}) \in M_{p}$ has defect 0;

(ii) $V_{\phi_{\nu}} \neq \mathbb{C}^{3}$.

For the proof of the proposition we need the following

**Lemma 5** Set $\Gamma_{\nu} = \{ \frac{\sigma - 1}{\nu} \in \mathbb{C}, \quad |\sigma| = 1 \}$ and let $r_{\nu} \in C^{\infty}(\Gamma_{\nu}, \mathbb{R}^{2})$ be given for each $\nu \in \mathbb{N}$.

Then there exists $h$ as above and for each $\nu$, $f_{\nu} \in C^{\infty}(\Gamma_{\nu}, \mathbb{R})$, $f_{\nu} \neq 0$ such that we have

$$h = 0, \quad D_{\rho} h = f_{\nu} r_{\nu}, \quad \text{on} \quad \Gamma_{\nu}$$

where $D_{\rho}$ stays for the normal derivatives to $\Gamma_{\nu}$.

---

\footnote{Recall that $M_{p}$ is the $C^{1}$ Banach manifold of suitably small analytic discs attached to $M$ through $p$.}
Proof. For fixed $\nu$ we choose $F_\nu \in C^\infty(\mathbb{R}^2)$, $0 \leq F_\nu \leq 1$, which vanishes only on $\bigcup_{j \neq \nu} \Gamma_j \cup \{0\}$ and $F_\nu \equiv 1$ out of a very big compact. If necessary, $F_\nu$ can be changed in a small neighborhood of the point $-2/\nu$ in order to make sure that $D_\rho F_\nu \not\equiv 0$ on $\Gamma_\nu$. Fix also $v_\nu \in C^\infty(\mathbb{C}, \mathbb{R}^2)$ bounded and such that

$$v_\nu = 0, \quad D_\rho v_\nu = r_\nu, \text{ on } \Gamma_\nu.$$ 

If we choose a real sequence $\lambda_\nu$ very rapidly decreasing to 0, then

$$h = \sum_{\nu=1}^{\infty} \lambda_\nu F_\nu v_\nu$$

converges obviously to a smooth function which will be our function. Indeed among the $F_j$’s, only $F_\nu$ is non vanishing on $\Gamma_\nu$, but $v_\nu$ vanishes there, thus $h|_{\Gamma_\nu} = 0$. So we only need to take $f_\nu = \lambda_\nu D_\rho F_\nu$. Also $h$ vanishes with its gradient at 0 because so does each $F_\nu$ (indeed $F_\nu(0) = 0$ and $F_\nu \geq 0$).

Proof of the Proposition 6. Set $a = (1, i) \in \mathbb{C}^2$ and choose in the Lemma 5 $r_\nu(\sigma) = (|1 + \sigma|^2, 2 \text{Im}\sigma) = (1 + \bar{\sigma})(a + \sigma \bar{a}), |\sigma| = 1$.

For $\sigma = e^{i\theta}$ we have

$$0 = D_\theta h(\frac{\sigma - 1}{\nu}) = \frac{i}{\nu} \sigma h_w(\frac{\sigma - 1}{\nu}) - i \frac{\nu}{\nu} \bar{\sigma} h_w(\frac{\sigma - 1}{\nu})$$

thus

$$D_\rho h(\frac{\sigma - 1}{\nu}) = \frac{1}{\nu} \sigma h_w(\frac{\sigma - 1}{\nu}) + \frac{1}{\nu} \bar{\sigma} h_w(\frac{\sigma - 1}{\nu}) = \frac{2}{\nu} \sigma h_w(\frac{\sigma - 1}{\nu}).$$

Hence on $\Gamma_\nu$

$$h_w(\frac{\sigma - 1}{\nu}) = \frac{\nu}{2} \bar{\sigma} D_\rho h = \frac{\nu(1 + \bar{\sigma})}{2} f_\nu(\sigma)(\bar{\sigma} a + \bar{a}). \quad (14)$$

For proving $(i)$ we assume that $ch_w(\frac{\sigma - 1}{\nu})$ extends holomorphically into $D$ for some $c = (c_1, c_2) \in \mathbb{R}^2$. Set $C = c^t \bar{a} = c_1 + ic_2$. We obtain that the scalar function

$$f_\nu(\sigma)(1 + \bar{\sigma})(C \bar{\sigma} + \bar{C}) = \frac{f_\nu(\sigma)}{\sigma^2}(1 + \sigma)(C + \bar{C} \sigma)$$

extends holomorphically.

If $C$ is not zero then $C + \bar{C} \sigma$ only vanishes on $\Gamma$ and thus $\frac{f_\nu(\sigma)}{\sigma^2}$ extends holomorphically. But this is impossible because $f_\nu$ is real and not zero.

Thus $C$, and consequently $c$, vanishes. This proves $(i)$.

For proving $(ii)$ it is sufficient to prove that the form $\omega = adz + \bar{a}d\bar{z} \not\equiv 0$
vanishes on \( \mathcal{V}_{\phi_{\nu}}, \forall \nu. \)

We have \(< \omega, (\dot{z}(0), \dot{w}(0)) >= a\dot{z}(0) + \bar{a}\dot{z}(0).\)

Now, for \( \phi(\zeta) = (z(\zeta), w(\zeta)) \), we have

\[
z(0) = -\frac{1}{\pi} \int_{0}^{2\pi} \frac{h[w(\sigma)]}{\sigma - 1} d\theta,
\]

and thus

\[
\dot{z}(0) = -\frac{1}{\pi} \int_{0}^{2\pi} \frac{h_w \dot{w} + h_{\bar{w}} \dot{\bar{w}}}{\sigma - 1} d\theta, \quad \sigma = e^{i\theta}.
\]

Therefore we can write

\[
a\dot{z}(0) + \bar{a}\dot{\bar{z}}(0) = -\frac{1}{\pi} \int_{0}^{2\pi} \frac{(a - \sigma \bar{a})h[w(\sigma)]}{\sigma - 1} \dot{w} d\theta - \frac{1}{\pi} \int_{0}^{2\pi} \frac{(a - \sigma \bar{a})h_{\bar{w}}[w(\sigma)]}{\sigma - 1} \dot{\bar{w}} d\theta.
\]

Passing to \( \phi_{\nu} \), we take \( w(\sigma) = \frac{2\nu}{\nu} \) and by [14] obtain \( (a - \sigma \bar{a})h_w[w(\sigma)] = 0 \)
and \( (a - \sigma \bar{a})h_{\bar{w}}[w(\sigma)] = 0 \) because \( (a - \sigma \bar{a})(\bar{a} + \sigma a) = 0 \). Thus \( a\dot{z}(0) + \bar{a}\dot{\bar{z}}(0) = 0. \)

\[\square\]

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