Rough differential equation in Banach space driven by weak geometric $p$-rough path

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Abstract

By using an explicit ordinary differential equation to approximate the exponential solution flow, we extend the universal limit theorem to rough differential equation in Banach space driven by weak geometric rough path, and give the quantitative dependence of solution in term of the initial value, vector field and driving rough path.

1 Introduction

Rough paths theory gives meaning to systems driven by rough signals, and provides a robust solution which is continuous with respect to the driving signal in rough path metric. There are several formulations of rough paths theory. In his original paper, Lyons [22] developed the theory of rough paths. He works with rough differential equation driven by geometric $p$-rough path in Banach space, treats rough differential equation as a special case of rough integral equation, and proves that the solution exists uniquely and is robust. Gubinelli [15,16] and Davie [10] define a continuous path to be a solution if its increment on small intervals is close to high order Euler expansion. Gubinelli introduces the notions of controlled rough path [15] and branched rough path [16], and he can solve differential equation driven by general (geometric & non-geometric) rough paths. Davie works with general rough paths when $p < 3$, and gives sharp conditions on vector field for the existence and uniqueness of solution with some impressive examples delimiting the sharpness. Inspired by Davie’s work, Friz and Victoir [13] define solution of rough differential equation (RDE) as the limit of solutions of ordinary differential equations (ODE), and extend the formulation to general $p \geq 1$ by using geodesic approximation. There are also alternative approaches by Feyel & La Pradelle [11] and Hu & Nualart [19]. For more systematic treatment of the theory, see Lyons & Qian [25], Lejay [23], Lyons, Caruana & Lévy [24], Friz & Victoir [14] and Friz & Hairer [12].

For rough path theory, there is a considerable gap between lower $p$ and arbitrary large $p$, and between finite dimensional space and infinite dimensional space. (So far, only Lyons’ approach can deal with rough paths in infinite dimensional space with arbitrary roughness.) In comparison with the case when signals are moderate oscillatory, as the roughness of system increases, there should be some algebraic structure coming in to streamline the otherwise complicated (if at all possible) calculation. This leads to a more complete theory and provides an unified resolution. For finite dimensional space, one can use Arzelà-Ascoli theorem to prove that the solution exists when the vector field is $\operatorname{Lip}(\gamma)$ for $\gamma > p - 1$, but solution may not exist in general Banach space when $\gamma \in (p - 1, p)$. Indeed, according to Shkarin [32], for a large family of Banach spaces (including $L_p[0,1]$, $1 \leq p < \infty$, and $C[0,1]$) and for any $\alpha \in (0, 1)$, there exists an $\alpha$-Hölder continuous $f$ such that, the ordinary differential equation $\dot{x} = f(x)$ has no solution in any interval of the real line. In finite dimensional space, one can use Jacobi matrix to prove continuity of solutions in initial value and use Whitney’s theorem to treat locally Lipschitz vector fields, but Jacobi matrix and Whitney’s theorem are not available in Banach space. Moreover, in finite dimensional space, the set of signatures of continuous bounded variation paths explore the truncated Lie group easily, and every weak geometric $p$-rough path is a geometric $p'$-rough path for any $p' > p$. While for general Banach space, Rashevskii-Chow Theorem [9,20] no longer holds, and there is a part of the algebraic structure which can not be reached by continuous bounded variation paths.

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We work with RDE in Banach space driven by weak geometric rough paths, and we use an explicit ODE to approximate the truncated exponential solution flow. Chen [8] prove that the logarithm of the signature of a continuous bounded variation path is a Lie series. Castell and Gaines [7] use an ODE, whose vector field is a Lie polynomial, to approximate the truncated exponential solution flow for stochastic differential equations. Boutaib et al [7] use similar ODE to approximate the (first level) RDE solution in Banach space. We modify the ODE in [7] and use its solution to recover the truncated solution flow on small intervals. The method of our analysis is based on Davie [10] and Friz & Victoir [13,14]—basically by comparing the increment of RDE solution on an interval with the solution of an ODE and building up mathematical induction on the length of the interval. Another independent work in this direction is Bailleul [2,3].

We prove that the solution of rough differential equation driven by weak geometric $p$-rough path exists uniquely when the vector field is $\text{Lip} (\gamma)$ for $\gamma > p$. Since being a weak geometric rough path is easier to check (as the authors assumed) than being a geometric rough path, this moderate extension of Lyons’ original theorem could provide certain convenience when one works in Banach space. As a consequence of our theorem, the solution of rough differential equation in the sense of Lyons, Gubinelli, Davie and Friz & Victoir coincide if the following two conditions are satisfied:

1. For integer $n \geq 1$, the symmetric group $S_n$ acts by isometries on $V^{\otimes n}$, that is

$$\| \sigma v \| = \| v \|, \ \forall \sigma \in S_n, \forall v \in V^{\otimes n}. \quad (1)$$

2. The tensor product has norm 1, that is,

$$\| u \otimes v \| \leq \| u \| \| v \|, \ \forall u \in V^{\otimes m}, \forall v \in V^{\otimes n}, \forall m, n \geq 1.$$

For example, injective and projective tensor norms are admissible norms, see [31].

**Definition 1 (admissible norm)** We say that the tensor product of $V$ is endowed with an admissible norm, if the following two conditions are satisfied:

1. For integer $n \geq 1$, the symmetric group $S_n$ acts by isometries on $V^{\otimes n}$, that is

$$\| \sigma v \| = \| v \|, \ \forall \sigma \in S_n, \forall v \in V^{\otimes n}. \quad (1)$$

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For example, injective and projective tensor norms are admissible norms, see [31].

**Definition 2 ($V^{\otimes n}$ and $[V]^n$)** When $n = 1$, $V^{\otimes 1} := V$ and $[V]^1 := V$. For $n \geq 2$, we select an admissible norm and define $V^{\otimes n}$ and $[V]^n$ respectively as the closure of (with $[u, v] := u \otimes v - v \otimes u$)

$$\left\{ \sum_{k=1}^{m} v_1^k \otimes \cdots \otimes v_{n-1}^k \otimes v_n^k \middle| \{v_i^k\} \subset V, \ m \geq 1 \right\},$$

$$\left\{ \sum_{k=1}^{m} [v_1^k, \cdots, [v_{n-1}^k, v_n^k]] \middle| \{v_i^k\} \subset V, \ m \geq 1 \right\},$$

w.r.t. the norm selected.

**Notation 3** For integers $n \geq k \geq 1$, $\pi_k$ denotes the projection of $\mathbb{R} \oplus V \oplus \cdots \oplus V^{\otimes n}$ to $V^{\otimes k}$.
Definition 4 \((L^n(V))\) For integer \(n \geq 1\), \(L^n(V)\) denotes the Banach space
\[
L^n(V) := \mathbb{R} \oplus V \oplus \cdots \oplus V^\otimes n,
\]
equipped with the norm
\[
\|l\| := \sum_{k=0}^{n} \|\pi_k(l)\|, \forall l \in L^n(V).
\]

Definition 5 \((T^n(V))\) For integer \(n \geq 1\), define
\[
T^n(V) := 1 \oplus V \oplus \cdots \oplus V^\otimes n.
\]
For \(g, h \in T^n(V)\), define \(g \otimes h\) and \(g^{-1}\) by
\[
g \otimes h := \sum_{k=0}^{n} \sum_{j=0}^{k} \pi_j(g) \otimes \pi_{k-j}(h),
\quad g^{-1} := 1 + \sum_{k=1}^{n} \pi_k \left( \sum_{j=1}^{n} (-1)^j (g^{-1})^\otimes j \right),
\]
and equip \(T^n(V)\) with \(\|\cdot\|\) defined by
\[
\|t\| := \sum_{k=1}^{n} \|\pi_k(t)\|^\frac{1}{k}, \forall t \in T^n(V).
\]
Then \(T^n(V)\) is a nilpotent topological group.

Definition 6 For \(\lambda > 0\) and integer \(n \geq 1\), define the dilation operator \(\delta_\lambda : T^n(V) \rightarrow T^n(V)\) by
\[
\delta_\lambda g := \sum_{k=0}^{n} \lambda^k \pi_k(g), \forall g \in 1 \oplus V \oplus \cdots \oplus V^\otimes n.
\]
\(T^n(V)\) is nilpotent because \([t^n, \cdots [t^2, t^1]] = 0, \forall \{t^i\}_i^{n} \subset T^n(V)\). \(\|\cdot\|\) defined at (2) is homogeneous w.r.t. dilation, but is not a norm because it is not sub-additive. While \(\|\cdot\|\) is equivalent to a norm up to a constant depending on \(n\) (see Exercise 7.38 [14] where the equivalency extends naturally to Banach spaces).

Definition 7 Define \(\exp : V \oplus \cdots \oplus V^\otimes n \rightarrow 1 \oplus V \oplus \cdots \oplus V^\otimes n\) by
\[
\exp (a) := 1 + \sum_{k=1}^{n} \pi_k \left( \sum_{j=1}^{n} \left( -1 \right)^j (a \otimes j)^1 \right), \forall a \in V \oplus \cdots \oplus V^\otimes n.
\]
Define \(\log : 1 \oplus V \oplus \cdots \oplus V^\otimes n \rightarrow V \oplus \cdots \oplus V^\otimes n\) by
\[
\log (g) := \sum_{k=1}^{n} \pi_k \left( \sum_{j=1}^{n} \left( -1 \right)^{j+1} j (g^{-1})^\otimes j \right), \forall g \in 1 \oplus V \oplus \cdots \oplus V^\otimes n.
\]
Then it can be checked that \(\log (\exp (t-1)) = t-1\) and \(\exp (\log (t)) = t\), \(\forall t \in T^n(V)\).

Definition 8 \((G^n(V))\) For integer \(n \geq 1\), \(\text{with } |V|^k \text{ in Definition 2}\) we define
\[
G^n(V) := \left\{ \exp (a) | a \in |V|^1 \oplus |V|^2 \oplus \cdots \oplus |V|^n \right\}.
\]
Then \(G^n(V)\) is a subgroup of \(T^n(V)\) (based on Baker–Campbell–Hausdorff formula), called the step-\(n\) nilpotent Lie group of degree \(n\).

For more about nilpotent Lie group, please refer to e.g. [30].
2.2 Vector Field and Differential Operator

Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be Banach spaces.

**Definition 9** For $\gamma > 0$, we say $r : \mathcal{V} \rightarrow \mathcal{U}$ is $\text{Lip}(\gamma)$ and denote $r \in C^\gamma(\mathcal{V}, \mathcal{U})$, if $r$ is $\lfloor \gamma \rfloor$-times Fréchet differentiable ($\lfloor \gamma \rfloor$ denotes the largest integer which is strictly less than $\gamma$), and

$$|r|_\gamma := \left( \max_{k=0,1,\ldots,\lfloor \gamma \rfloor} \|D^k r\|_\infty \right) \vee \|D^{\lfloor \gamma \rfloor} r\|_{(\gamma - \lfloor \gamma \rfloor)-\text{Hölder}} < \infty,$$

where $\|\cdot\|_\infty$ denotes the uniform norm and $\|\cdot\|_{(\gamma - \lfloor \gamma \rfloor)-\text{Hölder}}$ denotes the $(\gamma - \lfloor \gamma \rfloor)$-Hölder norm.

Denote by $C^0(\mathcal{V}, \mathcal{U})$ the space of bounded measurable mappings from $\mathcal{V}$ to $\mathcal{U}$.

Denote by $C^{\gamma, \text{loc}}(\mathcal{V}, \mathcal{U})$ the space of locally $\text{Lip}(\gamma)$ mappings.

**Definition 10** $L(\mathcal{W}, C^\gamma(\mathcal{V}, \mathcal{U}))$ denotes the space of linear mappings from $\mathcal{W}$ to $C^\gamma(\mathcal{V}, \mathcal{U})$. Define

$$|f|_\gamma := \sup_{w \in \mathcal{W}, \|w\|=1} |f(w)|_\gamma, \forall f \in L(\mathcal{W}, C^\gamma(\mathcal{V}, \mathcal{U})).$$

Similarly, $L(\mathcal{W}, C^{\gamma, \text{loc}}(\mathcal{V}, \mathcal{U}))$ denotes the space of linear mappings from $\mathcal{W}$ to $C^{\gamma, \text{loc}}(\mathcal{V}, \mathcal{U})$.

For $r \in C^{k, \text{loc}}(\mathcal{U}, \mathcal{U})$ and $j = 0, \ldots, k$, $D^j r \in L(\mathcal{U}^{\otimes j}, C^{k-j, \text{loc}}(\mathcal{U}, \mathcal{U}))$.

**Notation 11** ($D^k(\mathcal{U})$) For integer $k \geq 0$, denote by $D^k(\mathcal{U})$ the set of locally bounded $k$th order differential operators (on $C^{k, \text{loc}}(\mathcal{U}, \mathcal{U})$). More specifically, $p \in D^k(\mathcal{U})$ if and only if $p : C^{k, \text{loc}}(\mathcal{U}, \mathcal{U}) \rightarrow C^{0, \text{loc}}(\mathcal{U}, \mathcal{U})$ and there exist locally bounded $p_j \in C^{0, \text{loc}}(\mathcal{U}, \mathcal{U}^{\otimes j})$, $j = 0, 1, \ldots, k$, with $p_k \neq 0$, such that

$$p \circ (r) (u) = \sum_{j=0}^k (D^j r) (p_j (u)) (u), \forall u \in \mathcal{U}, \forall r \in C^{k, \text{loc}}(\mathcal{U}, \mathcal{U}).$$

We define the norm $|\cdot|_k$ on $D^k(\mathcal{U})$ by

$$|p|_k := \max_{j=0,1,\ldots,k} \sum_{n=1}^{\infty} \left( \sup_{\|u\| \leq n} \|p_j (u)\| \right) 2^n, \forall p \in D^k(\mathcal{U}).$$

Then $D^k(\mathcal{U})$ is a Banach space (with the natural addition and scalar multiplication).

**Definition 12** (composition) Let $p^1 \in D^{j_1}(\mathcal{U})$ and $p^2 \in D^{j_2}(\mathcal{U})$ for integers $j_1 \geq 0$, $j_2 \geq 0$. When the components of $p^2$ are locally $\text{Lip}(j_2)$, we define the composition of $p^1$ and $p^2$, $p^1 \circ p^2 \in D^{j_1 + j_2}(\mathcal{U})$, by

$$(p^1 \circ p^2) (r) := p^1 \circ (p^2 (r)), \forall r \in C^{j_1 + j_2, \text{loc}}(\mathcal{U}, \mathcal{U}).$$

For $p \in D^j(\mathcal{U})$, $j \geq 0$, when the components of $p$ are locally $\text{Lip}((k-1) \times j)$ for integer $k \geq 1$, we define the differential operator $p^{(k)} \in D^{k \times j}(\mathcal{U})$ by

$$p^{(k)} := p \text{ and } p^{(k)} := p \circ p^{(k-1)}, k \geq 2.$$

Compositions of differential operators are associative, i.e. $(p^1 \circ p^2) \circ p^3 = p^1 \circ (p^2 \circ p^3)$.

**Definition 13** ($F^{(k)}$) Suppose $F \in L(\mathcal{V}, C^{\gamma, \text{loc}}(\mathcal{U}, \mathcal{U}))$ for some $\gamma \geq 0$. Then for any $v \in \mathcal{V}$, we define $F^{(1)}(v) \in D^1(\mathcal{U})$ by

$$F^{(1)}(v) (r) (u) := (Dr) (F (v) (u)) (u), \forall u \in \mathcal{U}, \forall r \in C^{1, \text{loc}}(\mathcal{U}, \mathcal{U}).$$

For integer $k \in 1, 2, \ldots, \lfloor \gamma \rfloor + 1$ and $\{v_j\}_{j=1}^k \subset \mathcal{V}$, we define $F^{(k)}(v_1 \otimes \cdots \otimes v_k) \in D^k(\mathcal{U})$ by

$$F^{(k)}(v_1 \otimes \cdots \otimes v_k) := (F^{(1)}(v_1)) \circ (F^{(1)}(v_2)) \circ \cdots \circ (F^{(1)}(v_k)).$$

Then we denote by $F^{(k)} \in L(\mathcal{V}^{\otimes k}, D^k(\mathcal{U}))$ the unique continuous linear operator which satisfies $\square$. 
2.3 Rough Differential Equation

Recall $\|\cdot\| := \sum_{k=1}^{\lceil p \rceil} \|\pi_k(\cdot)\|^\frac{1}{k}$ defined at (2). $\lceil p \rceil$ denotes the largest integer which is less or equal to $p$.

Definition 14 For $p \geq 1$, suppose $X : [0, T] \to G^{[p]}(\mathcal{V})$ is continuous. Define $p$-variation of $X$ by $(X_{s,t} := X_{s-1} \otimes X_t)$

$$\|X\|_{p\text{-var},[0,T]} := \sup_{D \subseteq [0,T]} \left( \sum_{j,t_j \in D} \|X_{t_j,t_{j+1}}\|^p \right)^\frac{1}{p},$$

where the supremum is taken over all finite partitions $D = \{t_j\}_{j=0}^n$, $0 = t_0 < t_1 < \cdots < t_n = T$. Let $C^{p\text{-var}}([0,T], G^{[p]}(\mathcal{V}))$ denote the set of continuous paths with finite $p$-variation.

Definition 15 (weak geometric rough path) For $p \geq 1$, $X : [0, T] \to G^{[p]}(\mathcal{V})$ is called a weak geometric $p$-rough path if $X \in C^{p\text{-var}}([0,T], G^{[p]}(\mathcal{V}))$.

Recall Banach space $L^n(\mathcal{U}) = \mathbb{R} \oplus \mathcal{U} \oplus \cdots \oplus \mathcal{U}^\otimes n$ defined in Definition 4. For $l \in L^n(\mathcal{U})$ and $u \in \mathcal{U}$, $l \otimes u \in L^n(\mathcal{U})$ is defined by $\pi_0(l \otimes u) = 0$ and $\pi_k(l \otimes u) = \pi_{k-1}(l) \otimes u$, $k = 1, 2, \ldots, n$.

Notation 16 For $\gamma \geq 0$, suppose $f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))$ and $\eta \in \mathcal{U}$. Denote $f(\cdot + \eta) \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))$ by

$$f(\cdot + \eta)(v)(u) := f(v)(u + \eta), \forall v \in \mathcal{V}, \forall u \in \mathcal{U}. \quad (7)$$

Denote $F(f) \in L(\mathcal{V}, C^{\gamma,\text{loc}}(L^n(\mathcal{U}), L^n(\mathcal{U})))$ by

$$F(f)(v)(l) := l \otimes (f(v)(\pi_1(l))), \forall v \in \mathcal{V}, \forall l \in L^n(\mathcal{U}). \quad (8)$$

Gubinelli [13,16] and Davie [10] define a continuous path $Y$ to be a solution, if the increment of $Y$ on small interval is comparable to high order Euler expansion. (Gubinelli’s formulation is more algebraic, but his solution could be stated in this way.)

For $f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))$ and $\eta \in \mathcal{U}$, denote $f(\cdot + \eta) \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))$ as at (7), denote $F(f(\cdot + \eta)) \in L(\mathcal{V}, C^{\gamma,\text{loc}}(L^n(\mathcal{U}), L^n(\mathcal{U})))$ as at (8) and define $F(f(\cdot + \eta))^{\otimes k} \in L(\mathcal{V}^{\otimes k}, D^k(L^n(\mathcal{U})))$ as in Definition 13.

Denote $\text{Id}_{L^n(\mathcal{U})}$ the identity function on $L^n(\mathcal{U})$, i.e. $\text{Id}_{L^n(\mathcal{U})}(l) = l$, $\forall l \in L^n(\mathcal{U})$.

Definition 17 (Gubinelli/Davie) For $\gamma > p \geq 1$, suppose $X \in C^{p\text{-var}}([0,T], G^{[p]}(\mathcal{V}))$, $f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))$ and $\xi \in G^{[p]}(\mathcal{U})$. Then $Y : [0, T] \to T^{[p]}(\mathcal{U})$ is said to be a solution of the rough differential equation

$$dY = f(Y) dX, \ Y_0 = \xi,$$

if there exists a function $\theta : \{0 \leq s \leq t \leq T\} \to \mathbb{R}^+$ satisfying

$$\lim_{D \subseteq [0,T], \|D\| \to 0} \sum_{j,t_j \in D} \theta(t_j,t_{j+1}) = 0,$$

such that, for all sufficiently small $[s,t] \subseteq [0,T]$, (with $Y_{s,t} := Y_{s-1} \otimes Y_t$ and $1 \in L^{[p]}(\mathcal{U})$)

$$\left\|Y_{s,t} - \sum_{k=1}^{[p]} F(f(\cdot + \pi_1(Y_s)))^{\otimes k} \pi_k(X_{s,t}) \left(\text{Id}_{L^{[p]}(\mathcal{U})}(1)\right)\right\| \leq \theta(s,t).$$

As will be apparent in the proofs, the shuffle product (used in [22]) is hidden in $F(f(\cdot + \pi_1(Y_s)))^{\otimes k} \pi_k(X_{s,t})$.

3 Main Result

Definition 18 $\omega : \{0 \leq s \leq t \leq T\} \to \mathbb{R}^+$ is called a control, if $\omega$ is continuous, vanishing on the diagonal, and is sub-additive i.e.

$$\omega(s,u) + \omega(u,t) \leq \omega(s,t), \forall 0 \leq s \leq u \leq t \leq T.$$
Theorem 19 For $\gamma > p \geq 1$, suppose $X \in C^{p, \var} ([0, T], G^{[p]} (V))$, $f \in L (V, C^{\gamma} (U, U))$ and $\xi \in G^{[p]} (U)$. Then the rough differential equation

$$dY = f (Y) \, dX, \, Y_0 = \xi,$$

has a unique solution (denoted as $Y$) in the sense of Definition 17, which is a continuous path taking values in $G^{[p]} (U)$. If define control $\omega$ by

$$\omega (s, t) := |f|_p^\gamma \|X\|_{p, \var}^p [s, t], \forall 0 \leq s \leq t \leq T,$$

then there exists a constant $C_p$ such that, for any $0 \leq s \leq t \leq T$,

$$\|Y\|_{p, \var}^p [s, t] \leq C_p \left( \omega (s, t) \frac{1}{p} \vee \omega (s, t) \right).$$

Moreover, for $0 \leq s \leq t \leq T$, if let $y^{s,t} : [0, 1] \rightarrow L^{[p]} (U)$ denote the solution of the ordinary differential equation

$$dy^{s,t}_u = \sum_{k=1}^{[p]} F (f \cdot + \pi_1 (Y_s)) \odot \pi_k (\log X_{s,t}) \left( Id_{L^{[p]} (U)} \right) (y^{s,t}_u) du, \, u \in [0, 1], \, y^{s,t}_0 = 1,$$

then $y^{s,t}$ takes value in $G^{[p]} (U)$, and there exists a constant $C_p$, such that, $(Y_{s,t} := Y^{s,1})$

$$\|Y_{s,t} - y^{s,t}_1\| \leq C_p \left( \omega (s, t) \frac{[p]+1}{p} \vee \omega (s, t) \right),$$

and

$$\|Y_{s,t} - y^{s,t}_1\| \leq C_p \left( \omega (s, t) \frac{[p]+1}{p} \vee \omega (s, t) \right).$$

The proof of Theorem 19 starts from 24

Remark 20 The solution of (9) is defined in Gubinelli/Davie’s sense. Based on Universal Limit Theorem and Theorem 24 below, when the vector field is Lip ($\gamma$) for $\gamma > p$, the solutions in Lyons [22] and in Friz & Victoir [14] coincide with our solution.

Remark 21 Based on Euler expansion of solution of ODE (25) in Lemma 30 below and the definition of RDE solution (Definition 17), the solution of the ODE (11) coincides with the solution of the RDE:

$$dY = f (Y) \, dX^{s,t}, \, Y_0 = \xi,$$

with $X^{s,t} \in C^{p, \var} ([0, 1], G^{[p]} (V))$ defined by $X^{s,t}_u = \exp (u \log X^{s,t})$, $u \in [0, 1]$.

Notation 22 Suppose $\omega : [0 \leq s \leq t \leq T] \rightarrow \mathbb{R}^+$ is a control. For $\alpha \in (0, 1)$, define control

$$\omega^\alpha (s, t) := \sup_{D \subseteq [s, t] \omega (t_j, t_{j+1}) \leq \alpha} \sum_{j \in D} \omega (t_j, t_{j+1}), \forall 0 \leq s \leq t \leq T.$$

Suppose $X^i \in C^{p, \var} ([0, T], G^{[p]} (V)), i = 1, 2$. For control $\omega$, integer $n = 1, 2, \ldots , [p], [s, t] \subseteq [0, T]$ and $\alpha \in (0, 1)$, denote

$$d^n_{p, \omega} (X^1, X^2) := \left( \sup_{D \subseteq [s, t] \omega (t_j, t_{j+1}) \leq \alpha} \sum_{j \in D} \left\| \pi_n \left( X^1_{t_j, t_{j+1}} \right) - \pi_n \left( X^2_{t_j, t_{j+1}} \right) \right\|^2 \right)^{\frac{1}{p}},$$

and

$$d^n_{p, \alpha} (X^1, X^2) := \left( \sup_{D \subseteq [s, t], \omega (t_j, t_{j+1}) \leq \alpha} \sum_{j \in D} \left\| \pi_n \left( X^1_{t_j, t_{j+1}} \right) - \pi_n \left( X^2_{t_j, t_{j+1}} \right) \right\|^2 \right)^{\frac{1}{p}}.$$

Theorem 23 For $i = 1, 2$ and $\gamma > p \geq 1$, suppose $X^i \in C^{p, \var} ([0, T], G^{[p]} (V))$, $f^i \in L (V, C^{\gamma} (U, U))$ and $\xi^i \in G^{[p]} (U)$. Let $Y^i : [0, T] \rightarrow G^{[p]} (U)$ be the solution of the rough differential equation

$$dY^i = f^i (Y^i) \, dX^i, \, Y^i_0 = \xi^i.$$

Define control $\omega$ by

$$\omega (s, t) := |f^1|_p^\gamma \|X^1\|_{p, \var}^p [s, t] + |f^2|_p^\gamma \|X^2\|_{p, \var}^p [s, t], \forall 0 \leq s \leq t \leq T.$$
For $\alpha \in (0, 1]$, define $\omega^\alpha$ and $d_n^{\alpha,\alpha}$ based on $\omega$ as at (13) and (15). Then there exists $C_{p,\gamma}$ (which only depends on $p$ and $\gamma$) such that, for $\alpha \in (0, 1]$, $[s, t] \subseteq [0, T]$ and $k = 1, 2, \ldots, [p],$
\[
d_k^{p, [s, t]} (Y^1, Y^2) \leq C_{p,\gamma} \exp \left( C_{p,\gamma} \alpha^{-1} \omega^\alpha (s, t) \right) \times \left( \frac{\omega^\alpha (s, t)^k}{n} \left( \| \pi_1 (Y^1_s) - \pi_1 (Y^2_s) \| + \left| f_1 \right|_{\gamma}^{p} - f_2 \right)_{\gamma}^{\prime} \right) + \sum_{n=1}^{[p]} \omega^\alpha (s, t)^{\frac{[p] - n}{p}} \cdot d_{p, [s, t]} (\delta_{[f_1]}, X^1, \delta_{[f_2]}, X^2). \tag{16}\]

The proof of Theorem 23 starts from [12].
Based on Lemma 31 below and sub-additivity of a control, (16) holds with $X_i$ replaced by $\log X_i$, $i = 1, 2$.
According to Cass, Litterer & Lyons [5], for a large family of Gaussian processes (including fractional Brownian motion when $H > 4^{-1}$) and any $\alpha \in (0, 1]$, $\exp \left( C_{p,\gamma} \alpha^{-1} \omega^\alpha (s, t) \right)$ has finite moments of all orders.

**Corollary 24** For $\gamma > p \geq 1$, suppose $X \in C^{p-var} \left( [0, T], G^{[p]} (V) \right)$, $f \in L (V, C^\gamma (U, U))$ and $\xi \in G^{[p]} (U)$. Let $Y : [0, T] \to G^{[p]} (U)$ be the solution of the rough differential equation
\[
d Y = f (Y) \, dX, \quad Y_0 = \xi.
\]
For finite partition $D = \{ t_j \}_{j=0}^n$ of $[0, T]$, let $y^D : [0, T] \to G^{[p]} (V)$ be the solution of the ordinary differential equation
\[
d y^D = \sum_{k=1}^{[p]} F (f)^{\alpha_k} \pi_k (\log X_{t_j, t_{j+1}}) \left( Id_{L^{\infty} (U)} \right) (y^D) \frac{du}{t_{j+1} - t_j}, \quad u \in [t_j, t_{j+1}], \quad y_0^D = \xi. \tag{17}\]
Denote control $\omega$ by $\omega (s, t) := \| f \|_{p-var} \| X \|_{p-var, [s, t]}$ and denote $\omega^\alpha$ based on $\omega$ as at (13). Then $y^D$ takes value in $G^{[p]} (U)$, and there exists $C_{p,\gamma}$ such that, for any $\alpha \in (0, 1]$, (with $\alpha_0 := \max_{t_j \in D} \omega (t_j, t_{j+1}))$
\[
\| Y_T - y_T \| \leq C_{p,\gamma} \| \xi \| \exp \left( C_{p,\gamma} \left( \omega^\alpha (0, T) + \alpha^{-1} \omega^\alpha (0, T) \right) \right) \left( \sum_{j=0}^{[p]} \omega (t_j, t_{j+1})^{\frac{[p]}{p}} \right). \tag{18}\]

The proof of Corollary 24 starts from [13].
By similar arguments, (18) holds if $y^D$ is replaced by concatenated Euler approximation. If we only consider $\| \pi_1 (Y_T) - \pi_1 (y_T^D) \|$, then we can drop $\omega (t_j, t_{j+1})^{[p]}$ in (18).

4 Proofs
We specify the dependence of coefficients (e.g. $C_{p,\gamma}$), but their exact values may change from line to line.
For $\gamma > 0$, let $\lfloor \gamma \rfloor$ denote the largest integer which is strictly less than $\gamma$, and denote $\{ \gamma \} := \gamma - \lfloor \gamma \rfloor$.

4.1 Preparation
For Banach space $U$, denote $Id_U$ as the identity function on $U$, i.e. $Id_U (u) = u, \forall u \in U$.
We define ordered shuffle as in [21] (p73-74).

**Definition 25 (ordered shuffle)** For integer $k \geq 1$, denote by $S_k$ the symmetric group of order $k$. For $j_1 + \cdots + j_n = k, j_i \geq 1$, define ordered shuffle $OS (j_1, \ldots, j_n)$ to be the set of $\sigma \in S_k$ which satisfy
\[
\sigma (1) < \sigma (2) < \cdots < \sigma (j_1), \quad \sigma (j_1 + 1) < \cdots < \sigma (j_1 + j_2), \quad \sigma (j_1 + \cdots + j_n-1+1) < \cdots < \sigma (j_1 + \cdots + j_n), \quad \sigma (j_1) < \sigma (j_1 + j_2) < \cdots < \sigma (j_1 + \cdots + j_n).
\]

**Notation 26** For $f \in L (V, C^{\gamma} (U, U))$ and $j_1 + \cdots + j_n = k, j_i = 1, \ldots, \lfloor \gamma \rfloor$, denote by $f^{\circ j_1} \otimes \cdots \otimes f^{\circ j_n} \in L (V^{\otimes k}, C (\max_{j_1}^{\gamma} (U, U), C^0 (U, U^{\otimes n})))$ the unique continuous linear operator which satisfies that,
\[
\forall \{ v_j \}_{j=1}^k \subset V, \forall r \in C^{\max_{j_1}^{\gamma} (U, U), \forall u \in U. \]

\[
(f^{\circ j_1} \otimes \cdots \otimes f^{\circ j_n}) (v_k \otimes \cdots \otimes v_1) (r) (u) = (f^{\circ j_1} (v_k \otimes \cdots \otimes v_{k-j_1+1}) (r) (u)) \otimes \cdots \otimes (f^{\circ j_1} (v_j \otimes \cdots \otimes v_1) (r) (u)).
\]
Recall the Banach space $L^n(U) := \mathbb{R} \oplus U \oplus \cdots \oplus U^{\otimes n}$ in Definition 4 on p.3, $F(f)(y) := y \otimes f(\pi_1(y))$ as denoted at S on p.5 and $F(f)^{\circ k}$ defined in Definition 13 on p.4. For $\sigma \in S_k$, denote by $\sigma : \mathcal{V}^\otimes k \rightarrow \mathcal{V}^\otimes k$ the unique continuous linear operator which satisfies
\[ \sigma(v_k \otimes \cdots \otimes v_1) = v_{\sigma(k)} \otimes \cdots \otimes v_{\sigma(1)}, \forall \{v_j\}_{j=1}^k \subset \mathcal{V}. \]

**Lemma 27** Suppose $f \in L(V, C^\gamma(U, U))$. Then for $k = 1, \ldots, \lceil \gamma \rceil + 1$ and any $v \in \mathcal{V}^\otimes k$,
\[
F(f)^{\circ k}(v) (Id_{L^n(U)}) (y)
= y \otimes \left( \sum_{j_1 + \cdots + j_n = k, j_i \geq 1} \sum_{\sigma \in OS(j_1, \ldots, j_n)} (f^{\circ j_n} \otimes \cdots \otimes f^{\circ j_1})(\sigma)(v) \right) (Id_U)(\pi_1(y)).
\]
In particular, for $g \in C^{[p]}(V)$, if let $y : [0, 1] \rightarrow L^{[p]}(U)$ denote the solution to the ODE
\[
dy_u = \sum_{k=1}^{[p]} F(f)^{\circ k} \pi_k (\log g) (Id_{L^{[p]}(U)}) (y_u) du, \quad u \in [0, 1], \quad y_0 = 1,
\]
then, with $y^k := \pi_k(y)$, $k = 0, 1, \ldots, [p]$, we have
\[
y^k_0 = 1,
\]
\[
y^k_u = \sum_{j=1}^{k} \int_0^t y^{k-j}_u \otimes \left( \sum_{l_1 + \cdots + l_n = k, l_i \geq 1} \sum_{\sigma \in OS(l_1, \ldots, l_n)} ((f^{\circ l_n} \otimes \cdots \otimes f^{\circ l_1})(\sigma)(\pi_l(\log g))) (Id_U)(y_u) \right) du.
\]

**Proof.** (19) can be proved by mathematical induction when $v = v_k \otimes \cdots \otimes v_1$ for $\{v_j\}_{j=1}^k \subset \mathcal{V}$. Then by using linearity and continuity in $\mathcal{V}^\otimes k$, (19) holds for any $v \in \mathcal{V}^\otimes k$. (20) follows from (19). □

**Lemma 28** For $\gamma > p \geq 1$, suppose $f \in L(V, C^\gamma(U, U))$, $g, h \in C^{[p]}(V)$ and $\xi \in L^{[p]}(V)$. Let $y : [0, 2] \rightarrow L^{[p]}(U)$ be the solution to the ordinary differential equation:
\[
dy_u = \left\{ \sum_{k=1}^{[p]} F(f)^{\circ k} \pi_k (\log g) (Id_{L^{[p]}(U)}) (y_u) du, \quad u \in [0, 1], \quad y_0 = \xi. \right\}
\]
For $j_1 + \cdots + j_n \leq [p]$, $j_i \geq 1$, denote two mappings in $C^{1+(\gamma) \cdot \text{loc}}(L^{[p]}(U), L^{[p]}(U))$ by
\[
F((\log g)^{(j_1, \ldots, j_n)}) := F((f)^{(j_1, \ldots, j_n)}) (\pi_{j_1} (\log g) \otimes \cdots \otimes \pi_{j_n} (\log g)) (Id_{L^{[n]}(U)}),
\]
\[
F((\log g)^{(j_1, \ldots, j_n-1)} h^{(j_n)}) := F((f)^{(j_1, \ldots, j_n-1)}) (\pi_{j_1} (\log g) \otimes \cdots \otimes \pi_{j_{n-1}} (\log g) \otimes \pi_{j_n} (h)) (Id_{L^{[n]}(U)}).
\]
Then we have
\[
y_2 - \xi - \sum_{k=1}^{[p]} F(f)^{\circ k} \pi_k (g \otimes h) (Id_{L^{[n]}(U)}) (\xi)
= \sum_{j_1 + \cdots + j_n = [p], j_i \geq 1} \int \cdots \int \left( F((\log h)^{(j_1, \ldots, j_n)}) (y_{u_1}) - F((\log h)^{(j_1, \ldots, j_n)}) (y_1) \right) du_1 \cdots du_n
+ \sum_{j_1 + \cdots + j_n = [p], j_i \geq 1} \int \cdots \int \left( F((\log g)^{(j_1, \ldots, j_n-1)} h^{(j_n)}) (y_{u_1}) - F((\log g)^{(j_1, \ldots, j_n-1)} h^{(j_n)}) (\xi) \right) du_1 \cdots du_n
+ \sum_{j_1 + \cdots + j_n = [p], j_i \geq 1, n \geq 1} \int \cdots \int \left( F((\log g)^{(j_1, \ldots, j_n-1)} h^{(j_n)}) (y_{u_1}) - F((\log g)^{(j_1, \ldots, j_n-1)} h^{(j_n)}) (\xi) \right) du_1 \cdots du_n
\]
are first order differential operators, and for integer

\[ y \]

Then by subtraction and using the fact that \( k \)

\[ \gamma > p \]

Indeed, (24) holds clearly when \( k = 0 \). Then by using \( \| \sigma (\pi_k (\log g)) \| = \| \pi_k (\log g) \| \) (tensor norm is symmetric as at (1)), for \( k = 1, \ldots, [p] \), (since \( \| g \| \leq 1, |f|_\gamma = 1 \))

\[
\sup_{\tau \in [0,1]} \left\| y^k_\tau \right\| \leq C_p \| g \|^k, \quad k = 0, 1, \ldots, [p].
\]  (24)
Then we prove that $y$ takes value in $G^{[p]}(\mathcal{U})$. For $i = 1, 2$, let $\mathcal{V}^i$ be Banach spaces, and $F^i \in L (\mathcal{V}^i, \mathcal{D}^{k_i} (L^{[p]}(\mathcal{U})))$. Denote by $[F^2, F^1] \in L (\mathcal{V}^2 \otimes \mathcal{V}^1, \mathcal{D}^1 (L^{[p]}(\mathcal{U})))$ the unique continuous linear operator which satisfies

\[
[F^2, F^1] (v^2 \otimes v^1) (r) := (Dr) (F^2 (v_2) \circ F^1 (v_1) - F^1 (v_1) \circ F^2 (v_2)) (Id_{L^{[p]}(\mathcal{U})}),
\]  
\[
\forall v^1 \in \mathcal{V}^1, \forall v^2 \in \mathcal{V}^2, \forall r \in C^{1,\text{loc}} (L^{[p]}(\mathcal{U}), L^{[p]}(\mathcal{U})).
\]  

(25)

For integer $k = 1, \ldots, [p]$, with $F (f)^{\circ 1} \in L (\mathcal{V}, \mathcal{D}^1 (L^{[p]}(\mathcal{U})))$ defined at (5) (on 4), we define

\[
[F (f)]^{\circ 1} := [F (f)^{\circ 1}, [F (f)^{\circ 1}]].
\]

Then based on Lemma 21 in [4] (whose proof applies to locally Lipschitz vector fields), for $k = 1, \ldots, [p]$, \( \{v_i\}_{i=1}^k \subset \mathcal{V} \) and \( r \in C^{1,\text{loc}} (L^{[p]}(\mathcal{U}), L^{[p]}(\mathcal{U})) \),

\[
F (f)^{\circ k} [v_k, \ldots, v_2, v_1] (r) = [F (f)]^{\circ k} (v_k \otimes \cdots \otimes v_1) (r)
\]

(26)

\[
= (Dr) \left( [F (f)]^{\circ k} (v_k \otimes \cdots \otimes v_1) (Id_{L^{[p]}(\mathcal{U})}) \right).
\]

We want to prove that, for $k = 2, \ldots, [p]$, there exist \( \{G^{*,j,k}_{i} \} \subset C^1 (\mathcal{U}, \mathcal{U}) \), such that, for any $y \in L^{[p]}(\mathcal{U})$,

\[
[F (f)]^{\circ 2} (v_k \otimes \cdots \otimes v_1) (Id_{L^{[p]}(\mathcal{U})}) (y)
\]

(27)

\[
= y \otimes \left( \sum_{j=2}^{k} \sum_{s=1}^{l_j} \left[ G^{*,j,k}_j, \ldots, \left[ G^{*,j,k}_j, G^{*,j,k}_{k-1} \right] \ldots \ldots \right] (\pi_1 (y)) + f^{\circ 2} ([v_k, \ldots, [v_2, v_1] \ldots]) (Id_{\mathcal{U}})(\pi_1 (y)) \right).
\]

When $k = 2$, by using the definition of $[F (f)]^{\circ 2}$, we have

\[
[F (f)]^{\circ 2} (v_2 \otimes v_1) (Id_{L^{[p]}(\mathcal{U})}) (y)
\]

\[
= \left( \left( [F (f)]^{\circ 1} (v_2) \right) \circ [F (f)]^{\circ 1} (v_1) \right) - \left( [F (f)]^{\circ 1} (v_1) \right) \circ \left( [F (f)]^{\circ 1} (v_2) \right) \right) (Id_{L^{[p]}(\mathcal{U})}) (y)
\]

\[
= y \otimes \left( \left( [F (f)]^{\circ 1} (v_2) \right) \otimes \left( [F (f)]^{\circ 1} (v_1) \right) \right) - \left( [F (f)]^{\circ 1} (v_1) \right) \otimes \left( [F (f)]^{\circ 1} (v_2) \right) \right) (Id_{\mathcal{U}})(\pi_1 (y))
\]

\[
= y \otimes \left( \left( [F (f)]^{\circ 1} (v_2) \right) (Id_{\mathcal{U}}) \right) \left( [F (f)^{\circ 1} (v_1) \right) (Id_{\mathcal{U}})(\pi_1 (y)) + f^{\circ 2} ([v_2, v_1]) (Id_{\mathcal{U}})(\pi_1 (y)).
\]

Then (27) holds when $k = 2$ with $l_2 = 1$ and $G^{*,2,2}_i = [F^{\circ 1} (v_1)] (Id_{\mathcal{U}})$, $i = 1, 2$. Suppose (27) holds for $k$. Then for $k + 1$, by using (25), the second equality in (26) and inductive hypothesis (27), we have

\[
[F (f)]^{\circ (k+1)} (v_{k+1} \otimes v_k \otimes \cdots \otimes v_1) (Id_{L^{[p]}(\mathcal{U})}) (y)
\]

\[
= \left( \left( [F (f)]^{\circ 1} (v_{k+1}) \right) \circ [F (f)]^{\circ k} (v_k \otimes \cdots \otimes v_1) \right) - \left( [F (f)]^{\circ k} (v_k \otimes \cdots \otimes v_1) \right) \circ \left( [F (f)]^{\circ 1} (v_{k+1}) \right) \right) (Id_{L^{[p]}(\mathcal{U})}) (y)
\]

\[
= D \left( [F (f)]^{\circ k} (v_k \otimes \cdots \otimes v_1) (Id_{L^{[p]}(\mathcal{U})}) \right) \left( F (f)^{\circ 1} v_{k+1} \right) (Id_{L^{[p]}(\mathcal{U})}) (y)
\]

\[
- D \left( [F (f)]^{\circ 1} v_{k+1} \right) \left( Id_{L^{[p]}(\mathcal{U})} \right) \left( F (f)^{\circ k} (v_k \otimes \cdots \otimes v_1) \right) (Id_{L^{[p]}(\mathcal{U})}) (y)
\]

\[
= y \otimes \left( \sum_{j=2}^{k} \sum_{s=1}^{l_j} \left[ [f^{\circ 1} (v_{k+1}) \right] (Id_{\mathcal{U}}) \right)
\]

\[
+ y \otimes \left( \sum_{j=2}^{k} \sum_{s=1}^{l_j} \left[ G^{*,j,k}_j, \ldots, \left[ G^{*,j,k}_j, G^{*,j,k}_{k-1} \right] \ldots \ldots \right] (\pi_1 (y)) \right)
\]

\[
+ y \otimes \left( \sum_{j=2}^{k} \sum_{s=1}^{l_j} \left[ [f^{\circ 1} (v_{k+1}) \right] (Id_{\mathcal{U}}) \right)
\]

\[
+ y \otimes \left( \sum_{j=2}^{k} \sum_{s=1}^{l_j} \left[ G^{*,j,k}_j, \ldots, \left[ G^{*,j,k}_j, G^{*,j,k}_{k-1} \right] \ldots \ldots \right] (\pi_1 (y)) \right)
\]

\[
As a result, by choosing \{G^{*,j,k+1}_i \} properly, (27) holds for $k + 1$.\]
Then based on (26) and (27), we have that, there exists a function \( L \) on \( \mathcal{U} \) taking values in Lie polynomials \([\mathcal{U}]^1 \oplus \cdots \oplus [\mathcal{U}]^p\) (with \([\mathcal{U}]^n\) in Definition 28, 29), such that the ODE (22) can be re-written as
\[
du{y_u} = y_u \otimes (L(\pi_1(y_u))) \, du, \quad u \in [0,1].
\]
As a result, if we denote
\[
\gamma_t := \int_0^t L(\pi_1(y_u)) \, du,
\]
Then \( \gamma \) is differentiable, taking value in Lie polynomials of degree \([p]\), and
\[
dy{y_u} = y_u \otimes d\gamma_u, \quad u \in [0,1].
\]
Then it can be checked that \( y \) takes values in \( G^{[p]}(\mathcal{U}) \).

**Lemma 30** For \( \gamma > p \geq 1 \), suppose \( f \in L (\mathcal{V}, C^\gamma (\mathcal{U}, \mathcal{U})) \) with \( |f|_\gamma = 1 \), and \( g, h \in G^{[p]}(\mathcal{U}) \) satisfying \( \|g\| \vee \|h\| \leq 1 \). Let \( y_0^g : [0,1] \to G^{[p]}(\mathcal{U}) \) and \( y_0^{g,h} : [0,1] \to G^{[p]}(\mathcal{U}) \) be the solution to the ODE:
\[
dy{g} = \sum_{k=1}^{[p]} F(f)^{\otimes k} \pi_k (\log g) \left( Id_{L^{[p]}(\mathcal{U})} \right) (y_0^g) \, du, \quad u \in [0,1], \quad y_0^g = 1.
\]
\[
du{y_u}{g,h} = \left\{ \begin{array}{ll}
\sum_{k=1}^{[p]} F(f)^{\otimes k} \pi_k (\log g) \left( Id_{L^{[p]}(\mathcal{U})} \right) (y_u^g) \, du, & u \in [0,1] \\
\sum_{k=1}^{[p]} F(f)^{\otimes k} \pi_k (\log h) \left( Id_{L^{[p]}(\mathcal{U})} \right) (y_u^g) \, du, & u \in [1,2], \quad y_0^{g,h} = 1.
\end{array} \right.
\]
Then
\[
\|y_1^g - 1 - \sum_{k=1}^{[p]} F(f)^{\otimes k} \pi_k (g) \left( Id_{L^{[p]}(\mathcal{U})} \right)(1)\| \leq C_p \|g\|^{[p]+1},
\]
and
\[
\|y_2^{g,h} - y_1^{g,h}\| \leq C_p \left( \|g\| \vee \|h\| \right)^{[p]+1}.
\]
**Proof.** Based on explicit Euler expansion of \( y_1^g \) in Lemma 28 and sup \( u \in [0,1] \|\pi_k (y_u^g)\| \leq C_p \|g\|^{[p]} \) at (23) in Lemma 29 (28) holds; again based on Lemma 28 and using (25), (29) holds. ■

**Lemma 31** For \( i = 1, 2 \), suppose \( g_i \in G^{[p]}(\mathcal{V}) \) satisfying \( \delta := \|g_1\| \vee \|g_2\| \leq 1 \). Then for \( \beta \geq 0 \),
\[
\sum_{n=1}^{[p]} \delta^{(\beta-n)} \|\pi_n (\log g_1) - \pi_n (\log g_2)\| \quad \text{and} \quad \sum_{n=1}^{[p]} \delta^{(\beta-n)} \|\pi_n (g_1) - \pi_n (g_2)\|,
\]
are equivalent up to a constant \( C_p \).

**Proof.** For \( n = 1, 2, \ldots, [p] \),
\[
\delta^{(\beta-n)} \|\pi_n (\log g_1) - \pi_n (\log g_2)\| \leq C_p \delta^{(\beta-n)} \|\pi_n (g_1) \otimes \cdots \otimes \pi_n (g_2)\|.
\]
Then by using that
\[
a_1 \otimes \cdots \otimes a_l - b_1 \otimes \cdots \otimes b_l = \sum_{i=1}^{l-1} a_1 \otimes \cdots \otimes a_i \otimes (a_{i+1} - b_{i+1}) \otimes b_{i+2} \otimes \cdots \otimes b_l,
\]
we have (\( \delta := \|g_1\| \vee \|g_2\| \))
\[
\sum_{j_1+\cdots+j_l=n, j_i \geq 1} \|\pi_{j_1} (g_1) \otimes \cdots \otimes \pi_{j_l} (g_1) - \pi_{j_1} (g_2) \otimes \cdots \otimes \pi_{j_l} (g_2)\| \leq C_p \sum_{j=1}^{[p]} \delta^{n-j} \|\pi_j (g_1) - \pi_j (g_2)\|,
\]
and
\[
\delta^{(\beta-n)} \|\pi_n (\log g_1) - \pi_n (\log g_2)\| \leq C_p \sum_{j=1}^{[p]} \delta^{(\beta-n)} \|\pi_j (g_1) - \pi_j (g_2)\| \leq C_p \sum_{j=1}^{[p]} \delta^{(\beta-j)} \|\pi_j (g_1) - \pi_j (g_2)\|.
\]
The proof for the other direction is similar. ■
Lemma 32 For $i = 1, 2$ and $\gamma > p \geq 1$, suppose $f^i \in L(V, C^\gamma(U, U))$, $X^i \in C^p$-var $[0, T], G^p(U)$ and $g^i \in G^p(V)$. Let $y^i : [0, 1] \to G^p(U)$ be the solution to the ODE

$$dy^i_u = \sum_{k=1}^{[p]} F(f^i)^{\circ k} \pi_k (\log g^i) \left( Id_{L^p(U)} \right) (y^i_u) du, \quad u \in [0, 1], \quad y^i_0 = 1.$$  

We further assume that $|f^i|_\gamma = 1$, $i = 1, 2$, and $\delta := \|g^1\| \vee \|g^2\| \leq 1$. Then for $k = 1, \ldots, [p],$

$$\|\pi_k (y^1_u) - \pi_k (y^2_u)\| \leq C_p \left( \delta^k |f^1 - f^2|_{[p]-1} + \delta^{(k-n)\vee 0} \sum_{n=1}^{[p]} \|\pi_n (g^1) - \pi_n (g^2)\| \right). \tag{30}$$

Proof. For $i = 1, 2$ and $k = 0, 1, \ldots, [p]$, denote $y^{i,k} := \pi_k (y^i)$. Based on (20) in Lemma 27 on 18 we have

$$y^{1,1}_t - y^{2,1}_t = \sum_{n=1}^{[p]} \int_0^t \left( (f^1)^{\circ n} \pi_n (\log g^1) (Id_U) (y^{1,1}_u) - (f^2)^{\circ n} \pi_n (\log g^2) (Id_U) (y^{2,1}_u) \right) du.$$  

Then ($\delta \leq 1$)

$$\left\| y^{1,1}_t - y^{2,1}_t \right\| \leq C_p \left( \delta |f^1 - f^2|_{[p]-1} + \sum_{n=1}^{[p]} \|\pi_n (\log g^1) - \pi_n (\log g^2)\| + \delta \int_0^t \|y^{1,1}_u - y^{2,1}_u\| \right).$$

By using Gronwall’s inequality, we have

$$\sup_{t \in [0,1]} \left\| y^{1,1}_t - y^{2,1}_t \right\| \leq C_p \left( \delta |f^1 - f^2|_{[p]-1} + \sum_{n=1}^{[p]} \|\pi_n (\log g^1) - \pi_n (\log g^2)\| \right). \tag{31}$$

Then for $j = 1, \ldots, k - 1$, based on (20) in Lemma 27 on 18 we have, ($\delta \leq 1$)

$$\left\| y^{1,k}_t - y^{2,k}_t \right\| \leq C_p \sum_{j=1}^k \delta^j \sup_{t \in [0,1]} \left\| y^{1,k-j}_t - y^{2,k-j}_t \right\|

+ C_p \sum_{j=1}^k \sup_{t \in [0,1]} \left\| y^{2,k-j}_t \right\| \left( \delta^j |f^1 - f^2|_{[p]-1} + \sum_{l=j}^{[p]} \|\pi_l (\log g_1) - \pi_l (\log g_2)\| + \delta^j \left\| y^{1,1}_t - y^{2,1}_t \right\| \right).$$

Then by using $\sup_{t \in [0,1]} \left\| y^{2,k-j}_t \right\| \leq C_p \delta^{k-j}$ as at (23) in Lemma 29 and using inductive hypothesis (31),

$$\left\| y^{1,k}_t - y^{2,k}_t \right\| \leq C_p \left( \delta^k |f^1 - f^2|_{[p]-1} + \sum_{n=1}^{[p]} \delta^{(k-n)\vee 0} \|\pi_n (\log g_1) - \pi_n (\log g_2)\| \right).$$

Then combined with Lemma 31 one can replace $\log g_i$ by $g_i$ (up to a constant depending on $p$). \hfill \blacksquare

Lemma 33 Suppose $f^1$ and $f^2$ are Lip ($\beta$) for some $\beta \in (1, 2)$. Then

$$\left\| f^1 (u_1) - f^1 (u_2) - (f^2 (v_1) - f^2 (v_2)) \right\|$$

$$\leq |f^1|_\beta \left\| u_1 - u_2 - (v_1 - v_2) \right\| + \left( \left\| u_1 - u_2 \right\| + \|v_1 - v_2\| \right)^{\beta-1} \left( |f^1|_\beta \left\| u_2 - v_2 \right\| + |f^1 - f^2|_{\beta-1} \right).$$
Lemma 34 Suppose $|p| + 1 \geq \gamma > p \geq 1$. For $i = 1, 2$, let $X^i \in C^{p\text{-var}}([0, T], G^{[p]}(V))$, $f^i \in L(V, C^\gamma(U, U))$ and $\xi^i \in G^{[p]}(U)$. Assume $|f^i| \leq 1$ and $\|\xi^i\| \leq 1$. Define control $\omega : \{(s, t) | 0 \leq s \leq t \leq T\} \to \mathbb{R}^+$ by

$$\omega(s, t) := \|X^1\|_{p\text{-var}, [s, t]}^p + \|X^2\|_{p\text{-var}, [s, t]}^p.$$

For $(s, t) \in [0, T]$ satisfying $\omega(s, t) \leq 1$, let $y^{i, s, t} : [0, 1] \to G^{[p]}(U)$ be the solution of the ODE

$$dy^{i, s, t}_r = \sum_{k=1}^{[p]} F(f^i)^{\otimes k} \pi_k (\log X^i_r) (Id_{L^{[p]}(U)})(y^{i, s, t}) dr,$$

$r \in [0, 1]$.

For $u \in [s, t]$, let $y^{i, s, u, t} : [0, 2] \to G^{[p]}(U)$ be the solution of the ODE

$$dy^{i, s, u, t}_r = \left\{ \begin{array}{ll}
\sum_{k=1}^{[p]} F(f^i)^{\otimes k} \pi_k (\log X^i_u) (Id_{L^{[p]}(U)})(y^{i, s, u, t}) dr, & r \in [0, 1], \\
\sum_{k=1}^{[p]} F(f^i)^{\otimes k} \pi_k (\log X^i_t) (Id_{L^{[p]}(U)})(y^{i, s, u, t}) dr, & r \in [1, 2],
\end{array} \right.$$

and $y^{i, s, u, t} = \xi^i$.

Then there exists a constant $C_p$ such that, for any $(s, t) \subseteq [0, T]$ satisfying $\omega(s, t) \leq 1$ and any $u \in [s, t]$, (with $d^n_{p, [s, t]}(X^1, X^2)$ defined at (14) on p.9)

$$\|y^{1, s, u, t}_2 - y^{2, s, u, t}_2 - (y^{1, s, t}_2 - y^{2, s, t}_2)\| \leq C_p \left( \omega(s, t)^\gamma \left( |f^1 - f^2|_{\gamma-1} + \|\xi^1 - \xi^2\| \right) \right),$$

Proof. Fix $(s, t) \subseteq [0, T]$ satisfying $\omega(s, t) \leq 1$. Since $\|\xi^i\| \leq 1$ and $\omega(s, t) \leq 1$, (based on (23) in Lemma 29 on p.9) there exists $C_p$ such that

$$\max_{i = 1, 2} \|y^{i, s, u, t}\|_\infty \leq \|y^{i, s, t}\|_\infty \leq C_p,$$

(32)

Based on Lemma 28 (explicit remainder of Euler expansion of ODE) and Lemma 33 and using (32) and Lemma 31 (replacing log-signature by signature), we have

$$\|y^{1, s, u, t}_0 - y^{2, s, u, t}_0 - (y^{1, s, t}_0 - y^{2, s, t}_0)\| \leq C_p (I + II + III + IV),$$

(33)

where $(\delta := \omega(s, t) \leq 1)$

$$I = \delta^{[p]} \left( \sup_{r \in [0, 1]} \|y^{1, s, u, t}_r - y^{1, s, t}_r - y^{2, s, u, t}_r + y^{2, s, t}_r\| \right)$$

$$+ \left( \sup_{r \in [0, 1]} \|y^{1, s, u, t}_r - y^{1, s, t}_r\| + \|y^{2, s, u, t}_r - y^{2, s, t}_r\| \right)^{\gamma} \left( \delta^{[p]} \left( \|y^{1, s, u, t}_r - \xi^1 - y^{2, s, u, t}_r + \xi^2\| \right) \right),$$

$$II = \delta^{[p]} \left( \sup_{r \in [0, 1]} \|y^{1, s, u, t}_r - y^{1, s, t}_r + y^{2, s, u, t}_r - \xi^1 - y^{2, s, u, t}_r + \xi^2\| \right)$$

$$+ \left( \sup_{r \in [0, 1]} \|y^{1, s, u, t}_r - \xi^1\| + \|y^{2, s, u, t}_r - \xi^2\| \right)^{\gamma} \left( \delta^{[p]} \|\xi^1 - \xi^2\| + \delta^{[p]} |f^1 - f^2|_{\gamma-1} + \sum_{n=1}^{[p]} \delta^{[p] - n} d^n_{p, [s, t]}(X^1, X^2) \right),$$

$$III = \delta^{[p]} \left( \sup_{r \in [0, 1]} \|y^{1, s, u, t}_r - \xi^1 - y^{2, s, u, t}_r + \xi^2\| \right)$$

$$+ \left( \sup_{r \in [0, 1]} \|y^{1, s, t}_r - \xi^1\| + \|y^{2, s, t}_r - \xi^2\| \right)^{\gamma} \left( \delta^{[p]} |\xi^1 - \xi^2| + \delta^{[p]} |f^1 - f^2|_{\gamma-1} + \sum_{n=1}^{[p]} \delta^{[p] - n} d^n_{p, [s, t]}(X^1, X^2) \right),$$

$$IV = \delta^{[p]} \left( \sup_{r \in [0, 1]} \|y^{1, s, u, t}_r - y^{1, s, t}_r + y^{2, s, u, t}_r - \xi^1 - y^{2, s, t}_r + \xi^2\| \right)$$

$$+ \left( \sup_{r \in [0, 1]} \|y^{1, s, t}_r - \xi^1\| + \|y^{2, s, t}_r - \xi^2\| \right)^{\gamma} \left( \delta^{[p]} |\xi^1 - \xi^2| + \delta^{[p]} |f^1 - f^2|_{\gamma-1} + \sum_{n=1}^{[p]} \delta^{[p] - n} d^n_{p, [s, t]}(X^1, X^2) \right),$$
\[ IV = \delta|p|^{+1} \left( \sup_{r \in [0, 1]} \left\| y_{r_1}^{1, s, u, t} - y_{r_2}^{2, s, u, t} \right\| + \sup_{r \in [0, 1]} \left\| y_{r_1}^{1, s, t} - y_{r_2}^{2, s, t} \right\| + \left| f^{1} - f^{2} \right|_{\gamma - 1} \right) + \sum_{n=1}^{[p]} \delta^{p+1-n} d_{p, t}^{n} \left( X^{1}, X^{2} \right). \]

Based on Lemma 32 (continuous dependence of ODE solution), we have

\[ \left\| y^{1, s, u, t} - y^{2, s, u, t} \right\|_{\infty, [0, 1]} \leq C_{p} \left( \left\| \xi^{1} - \xi^{2} \right\| + \left| f^{1} - f^{2} \right|_{\gamma - 1} + \sum_{n=1}^{[p]} d_{p, t}^{n} \left( X^{1}, X^{2} \right) \right), \]

and

\[ \sup_{r \in [0, 1]} \left( \left\| y_{r_1}^{1, s, u, t} - y_{r_1}^{1, s, u, t} - y_{r_1}^{2, s, u, t} \right\| \right) \leq C_{p} \left( \left\| \xi^{1} - \xi^{2} \right\| + \left| f^{1} - f^{2} \right|_{\gamma - 1} \right) + \sum_{n=1}^{[p]} d_{p, t}^{n} \left( X^{1}, X^{2} \right). \]

Based on Lemma 33 (error of high order Euler expansion), we have

\[ \sup_{r \in [0, 1]} \left( \left\| y^{i, s, u, t} - y^{i, s, u, t} \right\| \right) \leq C_{p} \delta. \]

By substituting these estimates into (33), we have

\[ \left\| y_{2}^{i, s, u, t} - y_{2}^{i, s, u, t} - \left( y_{1}^{i, s, t} - y_{1}^{i, s, t} \right) \right\| \leq C_{p} \left( \left\| \xi^{1} - \xi^{2} \right\| + \left| f^{1} - f^{2} \right|_{\gamma - 1} \right) + \sum_{n=1}^{[p]} \delta^{p+1-n} d_{p, t}^{n} \left( X^{1}, X^{2} \right). \]

\[ \mathbf{\text{4.2 RDE driven by weak geometric rough path in Banach space}} \]

\[ \textbf{Notation 35} \text{ Suppose } \omega : \{(s, t) | 0 \leq s \leq t \leq T \} \rightarrow \mathbb{R}^{r} \text{ is a control. For integer } n \geq 0, \text{ let } D^{n} = \{ t_{j}^{n} \}_{j=0}^{2^{n}} \text{ be a sequence of nested finite partitions of } [0, T], \text{ defined recursively as } \]

\[ t_{0}^{0} = 0, t_{1}^{0} := T, t_{2}^{n+1, j} := t_{j}^{n}, j = 0, 1, \ldots, 2^{n}, n \geq 0, \]

\[ \text{and } t_{2}^{n+1, j+1} \in (t_{2}^{n, j}, t_{2}^{n+1, j}) \text{ satisfying } \]

\[ \omega (t_{2}^{n+1, j}, t_{2}^{n+1, j+1}) = \omega (t_{2}^{n, j}, t_{2}^{n, j+1}) \leq \frac{1}{2} \omega (t_{2}^{n, j}, t_{2}^{n+1, j}), j = 0, \ldots, 2^{n} - 1, n \geq 0. \]

\[ \text{Denote } \Lambda (n) := \{ t_{j}^{n} | j = 0, \ldots, 2^{n} \}, n \geq 0. \]

\[ \text{We call } [t_{j}^{n}, t_{j+1}^{n}] \text{ a dyadic interval of level } n \text{ and call } t_{j}^{n} \text{ a dyadic point of level } n. \]

As a result, when the level of dyadic intervals increases, their "length" decreases. Then we decompose an interval as union of dyadic intervals. (The decomposition is in the same spirit as 4.1.1 in [25] or Lemma 28 in [24].)

\[ \textbf{Lemma 36} \text{ For integer } n \geq 0 \text{ and } \{ s, t \} \subseteq \Lambda (n), \text{ denote by } n_{0} \text{ the level of biggest dyadic interval in } [s, t]. \text{ Then we can decompose } [s, t] \text{ as union of dyadic intervals in such a way that, there exists a dyadic point } p \in [s, t] \text{ of level } n_{0} - 1, \text{ such that the level of dyadic intervals to the left/right of } p \text{ is strictly increasing.} \]

\[ \textbf{Proof.} \text{ We recursively cut out the biggest dyadic interval in } [s, t], \text{ and decompose } [s, t] \text{ as union of dyadic intervals. Denote the level of the biggest dyadic interval in } [s, t] \text{ by } n_{0}. \text{ Then } n_{0} \leq n, \text{ and there could be one level } n_{0} \text{ dyadic interval or two adjacent level } n_{0} \text{ dyadic intervals in } [s, t], \text{ but there can not be more than two of them. Indeed, if there are more than two level } n_{0} \text{ dyadic intervals, then (since } [s, t] \text{ is connected) two of them will compose a level } n_{0} - 1 \text{ dyadic interval, which contradicts with our assumption that the biggest dyadic interval is of level } n_{0}. \text{ Let } I_{l} / I_{r} \text{ denote the interval on the left/right side of level } n_{0} \text{ dyadic interval(s) in } [s, t]. \text{ Since we cut out the level } n_{0} \text{ dyadic interval(s) in } [s, t], \text{ } I_{l} / I_{r} \text{ is strictly contained in a level } n_{0} \text{ dyadic interval, with its right/left boundary point a level } n_{0} \text{ dyadic point. Thus, by recursively cutting out the biggest dyadic interval in } I_{l} / I_{r}, \text{ we decompose } I_{l} / I_{r} \text{ as the union of dyadic intervals which are strictly monotone in their level.} \]
In this way, we decompose \([s, t]\) as the union of dyadic intervals. If there are two level \(n_0\) dyadic intervals in \([s, t]\) (denoted as \(I^1\) and \(I^2\)), we select \(p\) as the point between \(I^1\) and \(I^2\), so \(p\) is a level \(n_0 - 1\) dyadic point. If there is only one level \(n_0\) dyadic interval (denoted as \(I\)), we select \(p\) as the boundary point of \(I\) which is of level \(n_0 - 1\). Based on our construction, the level of dyadic interval(s) to the left/right of \(p\) is strictly increasing.

Lemma 37 and Lemma 38 extend estimates on dyadic intervals to general intervals.

**Lemma 37** Suppose \(\omega\) is a control with dyadic partition \(\Lambda(n) = \{t^n_j\}_j\) as in Notation 35. Suppose \(\gamma : [0, T] \to U\) is a continuous path, and for some \(\theta > 0\) and some integer \(n \geq 1\),

\[\|\gamma_t - \gamma_s\| \leq \omega(s, t)^\theta, \forall [s, t] = [t^l_j, t^{l+1}_j] \subseteq [0, T], \ l = 0, 1, \ldots, n.\]

Then there exists \(C_0\) such that

\[\|\gamma_t - \gamma_s\| \leq C_0 \omega(s, t)^\theta, \forall [s, t] \subseteq [0, T], \ \{s, t\} \subseteq \Lambda(n).\]

**Proof.** Fix \(\{s, t\} \subseteq \Lambda(n)\). Denote by \(n_0\) the level of biggest dyadic interval in \([s, t]\). Then we decompose \([s, t]\) as union of dyadic intervals as in Lemma 38, so there exists a level \(n_0 - 1\) dyadic point \(u \in [s, t]\) such that the level of dyadic intervals to the left/right of \(u\) is strictly increasing. We estimate \([u, t]\) as an example. The estimation of \([s, u]\) is similar. Suppose the dyadic decomposition of \([u, t]\) is \([t_0, t_1] \cup \cdots \cup [t_{l-1}, t_l]\). Since the level of \([t_j, t_{j+1}]\) is strictly increasing as \(j\) increases and \(u\) is a dyadic point of level \(n_0 - 1\), we have, the level of \(t_j\) is strictly lower than the level of \([t_j, t_{j+1}], j = 0, 1, \ldots, l - 1\). Then,

\[\omega(t_j, t_{j+1}) = \omega(t_j, t_l) \leq \frac{1}{2} \omega(t_{j-1}, t_l) \leq \cdots \leq \frac{1}{2^j} \omega(t_0, t_l) = \frac{1}{2^j} \omega(u, t).\]

Thus,

\[\|\gamma_t - \gamma_u\| \leq \sum_{j=0}^{l-1} \|\gamma_{t_{j+1}} - \gamma_{t_j}\| \leq \sum_{j=0}^{l-1} \omega(t_j, t_{j+1}) \leq \left(\sum_{j=0}^{l-1} \frac{1}{2^j}\right) \omega(u, t)^\theta \leq C_0 \omega(u, t)^\theta.\]

Similarly, we get

\[\|\gamma_t - \gamma_u\| \leq C_0 \omega(s, u)^\theta.\]

Then for \(\{s, t\} \subseteq \Lambda(n), [s, t] \subseteq [0, T]\), we have

\[\|\gamma_t - \gamma_s\| \leq \|\gamma_u - \gamma_s\| + \|\gamma_t - \gamma_u\| \leq C_0 \omega(s, u)^\theta + C_0 \omega(u, t)^\theta \leq C_0 \omega(s, t)^\theta.\]

**Lemma 38** For \(\gamma > p \geq 1\), suppose \(X \in \mathcal{C}^{p-\text{var}}([0, T], G^{[p]}(U))\) and \(f \in L(V, C^\gamma(U, U))\). Define control

\[\omega(s, t) := |f|^p X ||X||_{p-\text{var}, [s, t]}, \forall 0 \leq s \leq t \leq T.\]

For some \(C_p > 0\) and \(\{s_0, t_0\} \subseteq \Lambda(n)\) satisfying \(\omega(s_0, t_0) \leq 1\), suppose \(y : [s_0, t_0] \to G^{[p]}(U)\) is continuous and satisfies

\[\sup_{r \in [s_0, t_0]} \|y^r\| \leq C_p.\]

For \(\{s, t\} \subseteq \Lambda(n)\), \([s, t] \subseteq [s_0, t_0]\), let \(y^{s, t} : [0, 1] \to G^{[p]}(U)\) be the solution of the ODE

\[dy^{s, t}_r = \sum_{k=1}^{[p]} F(f)^{\circ k} \left(\pi_k (\log X_{s,t})\right) \left(Id_{L^{[p]}(U)}\right) (y^{s, t}_r) \, dr, \ r \in [0, 1], \ y^{s, t}_0 = y_s.\]

Suppose for some \(C_p > 0\) and \(\theta > 0\), we have

\[\|y_t - y^{s, t}_1\| \leq C_p \omega(s, t)^\theta, \forall [s, t] = [t^l_j, t^{l+1}_j] \subseteq [s_0, t_0], \ l = 0, 1, \ldots, n.\]

Then for any \(\{s, t\} \subseteq \Lambda(n), [s, t] \subseteq [s_0, t_0]\), we have

\[\|y_t - y^{s, t}_1\| \leq C_p \omega(s, t)^\theta.\]
Proof. When \([s, t] \subseteq [s_0, t_0]\) is non-dyadic, we decompose \([s, t]\) as the union of dyadic intervals as in Lemma 36. Denote the level of biggest dyadic interval in \([s, t]\) by \(n_0\). Then based on Lemma 36 there exists a level \(n_0 - 1\) dyadic point \(u \in [s, t]\), such that the level of dyadic intervals to the left/right of \(u\) is strictly increasing. Then

\[
\|y_e - y_{1,e}t\| \leq \|y_e - y_{1,u}t\| + \|y_u - y_{1,ut}\| + \left\| \left( y_{1,e}t - y_u \right) - \left( y_{1,u}t - y_u \right) \right\|.
\]

(35)

We take \(\|y_e - y_{1,u}t\|\) as an example. The estimation of \(\|y_u - y_{1,ut}\|\) is similar.

Denote the dyadic decomposition of \([u, t]\) (as in Lemma 36) by \([t_0, t_1] \cup \cdots \cup [t_{l-1}, t_l]\). For \(j = 1, \ldots, l - 1\), let \(y_{j-1,j+1}\) denote the solution of the ODE:

\[
dy_{j-1,j+1} = \left\{ \begin{array}{ll}
\sum_{k=1}^{[p]} F \left( f \circ \pi_k \log X_{t_{j-1}, t_j} \right) \left( Id_{L^p(\mu)} \right) \left( y_{j-1,j+1} \right) dr, & r \in [0, 1] \\
\sum_{k=1}^{[p]} F \left( f \circ \pi_k \log X_{t_{j+1}, t_j} \right) \left( Id_{L^p(\mu)} \right) \left( y_{j+1,j} \right) dr, & r \in [1, 2]
\end{array} \right.
\]

(36)

Then

\[
\|y_e - y_{1,u}t\| = \|y_e - y_u - \left( y_{1,u}t - y_u \right)\|
\]

\[
\leq \sum_{j=0}^{l-1} \left\|y_{t_{j+1}} - y_{t_j} - \left( y_{1,j+1} - y_{t_j} \right)\right\|
\]

\[
+ \sum_{j=1}^{l-1} \left\| \left( y_{1,j+1} - y_{t_{j-1}} \right) + \left( y_{t_j} - y_{t_{j-1}} \right) - \left( y_{1,j+1} - y_{t_{j-1}} \right) \right\|
\]

\[
\leq \sum_{j=0}^{l-1} \left\|y_{t_{j+1}} - y_{t_j} - \left( y_{1,j+1} - y_{t_j} \right)\right\|
\]

\[
+ \sum_{j=1}^{l-1} \left\| \left( y_{2,j+1} - y_{1,j} \right) + \left( y_{1,j} - y_{t_{j-1}} \right) \right\| + \left\| y_{1,j+1} - y_{t_{j-1}} \right\|
\]

(37)

Since the level of \(t_j\) is strictly lower than the level of \([t_j, t_{j+1}]\) and the level of \([t_j, t_{j+1}]\) is strictly increasing as \(j\) increases, we have, for \(j = 0, \ldots, l - 1\),

\[
\omega (t_j, t_{j+1}) \leq \omega (t_j, t_l) \leq \cdots \leq \frac{1}{2} \omega (u, t).
\]

(38)

Then since \([[t_j, t_{j+1}]]\) are dyadic, by using assumption 34, we have

\[
\sum_{j=0}^{l-1} \left\|y_{t_{j+1}} - y_{t_j} - \left( y_{1,j+1} - y_{t_j} \right)\right\| \leq C_p \sum_{j=0}^{l-1} \omega (t_j, t_{j+1})^\theta
\]

\[
\leq C_p \left( \sum_{j=0}^{\infty} \left( \frac{1}{2} \right)^j \right) \omega (u, t)^\theta \leq C_p \omega (u, t)^\theta.
\]

(39)

On the other hand, since \(\sup_{r \in [u_0, t_0]} \|y_r\| \leq C_p\), by using Lemma 32 on \([u, t]\) (continuous dependence on initial value) and using the assumption on dyadic intervals \([t_{j-1}, t_j]\) at \([u, t]\), we have

\[
\left\| \left( y_{2,j+1} - y_{1,j} \right) + \left( y_{1,j} - y_{t_{j-1}} \right) \right\| \leq C_p \omega (t_j, t_l)^\frac{1}{p} \left\| y_{1,j+1} - y_{t_j} \right\| \leq C_p \omega (t_{j-1}, t_l)^\theta + \frac{1}{p}.
\]

(40)

Based on 29 in Lemma 30 on \([u, t]\) (error between two-steps ODE and one-step ODE) and \(\sup_{r \in [u_0, t_0]} \|y_r\| \leq C_p\),

\[
\left\| y_{1,j+1} - y_{t_j} \right\| \leq C_p \omega (t_{j-1}, t_l)^\frac{j+1}{p}.
\]

(41)
Then, combining (40), (41) and (38), we have \((\omega (u, t) \leq \omega (s_0, t_0) \leq 1)\)
\[
\sum_{j=1}^{l-1} \left( \| y_{j-1, j, t_j} - y_{j-1, t_j} \| + \| y_{j, t_j, t_j} - y_{j-1, t_j} \| \right) \leq C_p \sum_{j=1}^{l-1} \omega (t_{j-1}, t_j) \left( \frac{\theta + 1}{p} \right) \left( \frac{p+1}{p} \right) \omega (u, t) \left( \frac{\theta + 1}{p} \right) \left( \frac{p+1}{p} \right)
\]
\[
\leq C_p, \omega (u, t) \left( \frac{\theta + 1}{p} \right) \left( \frac{p+1}{p} \right)
\]
As a result, combined with (35), we have, for any \(C\),
\[
\| y_{t} - y_{s, t} \| \leq C_{p, \omega} (s, u) \theta \left( \frac{p+1}{p} \right).
\]
Similarly, we have
\[
\| y_{u} - y_{s, u} \| \leq C_{p, \omega} (s, u) \theta \left( \frac{p+1}{p} \right).
\]
On the other hand, with \(y^{s, u, t}\) defined at (39) and by using (43), similar estimate as at (40) and (41) lead to
\[
\| \left( y^{s, u, t}_1 - y_s \right) - \left( y^{s, u, t}_1 - y_u \right) - \left( y^{s, u, t}_1 - y_s \right) \| \leq \| y^{s, u, t}_2 - y^{s, u, t}_1 \| + \| y^{s, u, t}_2 - y^{s, u, t}_1 \| \leq C_p, \omega (u, t) \left( \frac{p+1}{p} \right)
\]
As a result, combined with (35), we have, for any \(\{ s, t \} \in \Lambda (n), \{ s, t \} \subseteq [s_0, t_0],\)
\[
\| y_{t} - y_{s, t} \| \leq C_{p, \omega} (s, u) \theta \left( \frac{p+1}{p} \right).
\]

Lemma 39 For \(\gamma > p \geq 1\), suppose \(X \in C^{p,-\varphi} ([0, T], G^{[p]} (V))\), \(f \in L (V, C^\gamma (U, U))\) and \(\xi \in G^{[p]} (U)\). Define control
\[
\omega (s, t) := \| f^p \|_{p, \varphi, (s, t)} \| X \|_{p, \varphi, \gamma}, \forall 0 \leq s \leq t \leq T.
\]
Based on \(\omega\), define dyadic partitions \(\Lambda (n) := \{ t^n_j \}, n \geq 0\), as in Notation 37. For \(n \geq 0\), let \(y^n : [0, T] \rightarrow G^{[p]} (U)\) be the solution of the ordinary differential equation
\[
y^n_0 = \xi, \quad dy^n_j = \sum_{k=1}^{[p]} F (f)^{\pi_k} (\log X_{t^n_j, t^n_{j+1}}) (Id_{L^{[p]} (U)}) (y^n_j) \frac{dt}{t^n_j}, t \in [t^n_j, t^n_{j+1}].
\]
Then there exists \(C_p\) such that, \((y^n_{s, t} := (y^n_{s, t})^{-1} \otimes y^n_{s, t})\)
\[
\| y^n_{s, t} \| \leq C_{p, \omega} (s, t) \left( \frac{p+1}{p} \right), \forall \{ s, t \} \subseteq \Lambda (n), \omega (s, t) \leq 1, \forall n \geq 1.
\]
Moreover, for \(n \geq 1\), \(\{ s_0, t_0, s, t \} \subseteq \Lambda (n)\) satisfying \(\{ s, t \} \subseteq [s_0, t_0] \) and \(\omega (s, t) \leq 1\), let \(y^{n, s, t} : [0, 1] \rightarrow G^{[p]} (U)\) denote the solution of the ordinary differential equation
\[
dy^{n, s, t}_u = \sum_{k=1}^{[p]} F (f (\cdot + \pi_1 (y^{n, s, t}_u))) \pi_k (\log X_{s, t}) (Id_{L^{[p]} (U)}) (y^{n, s, t}_u) du, u \in [0, 1], y^{n, s, t}_0 = y^{n, s, t}_{s_0, s}.
\]
Then there exists \(C_p\) such that,
\[
\| y^n_{s_0, t} - y^{n, s, t}_{1, t} \| \leq C_{p, \omega} (s, t) \left( \frac{p+1}{p} \right), \forall \{ s_0, t_0, s, t \} \subseteq \Lambda (n), \{ s, t \} \subseteq [s_0, t_0], \omega (s_0, t_0) \leq 1, \forall n \geq 1.
\]
Proof. We assume $|f|_\gamma = 1$. Otherwise, we replace $f$ and $X$ by $|f|_{\gamma}^{-1} f$ and $\delta_{|f|_{\gamma}^{-1} X}$ respectively. In that case, both $y^n$ and $y^{n,s,u,t}$ will stay unchanged.

Fix $n \geq 1$ and $s_0, t_0 \in \Lambda (n)$ satisfying $\omega (s_0, t_0) \leq 1$. For $\{s, u, t\} \subseteq \Lambda (n)$, $s_0 \leq s \leq u \leq t \leq t_0$, let $y^{n,s,u,t} : [0, 2] \rightarrow C^{[p]} (U)$ be the solution of the ordinary differential equation

$$
dy^{n,s,u,t} = \left\{ \sum_{k=1}^{[p]} F \left( f \left( \cdot + \pi_1 (y^n_{s,u}) \right) \right) \pi_k \left( \log X_{s,u} \right) \left( Id_{L[1]} (U) \right) \left( y^{n,s,u,t} \right) dr, \quad r \in [0, 1] \right\}, \quad y^{n,s,u,t} = y^{n,s,u,t}_{s_0, t_0}.
$$

(49)

To simply the notation, we omit $n$ and denote

$$
y^{s,t} := y^{n,s,t} \quad \text{and} \quad y^{s,u,t} := y^{n,s,u,t}.
$$

Yet the coefficients below are all independent of $n$. For $\{s, t\} \subseteq \Lambda (n)$, $\{s, t\} \subseteq [s_0, t_0]$, denote

$$
\Gamma^{s,t} := y^{n,s,t}_{s_0, t_0} - y^{n,s,t} = y^{n,s,t}_{s_0, t_0} - \left( y^{n,s,t}_{s, t} \right).
$$

(50)

Based on the definition of $y^n$ at (45) and the definition of $y^{s,t}$ at (14), it can be checked that, on any level-$n$ dyadic interval $[t^n_j, t^n_{j+1}] \subseteq [s_0, t_0]$, \[ \Gamma^{n,t}_{j,j+1} = y^{n,t}_{s_0, t_0} - y^{n,t}_{s, t} = 0. \]

(51)

Suppose $u \in (s, t)$, $u \in \Lambda (n)$, then

$$
\Gamma^{s,u} + \Gamma^{u,t} - \Gamma^{s,t} = y^{s,t} - y^{s,u}_s - \left( y^{n,u}_s - y^{n,u}_{s_0, t_0} \right) - \left( y^{n,t}_1 - y^{n,t}_{s_0, u} \right) = \left( y^{s,t}_{1} - y^{s,u}_2 \right) + \left( y^{u,t}_2 - y^{u,t}_1 \right) - \left( y^{s,t}_1 - y^{s,t}_2 \right) - \left( y^{s,t}_2 - y^{s,t}_1 \right).
$$

(52)

For $k \in \{0, 1, \ldots, [p]\}$, denote

$$
M_k := \max_{j=0, 1, \ldots, k} \sup_{\omega (s_0, s) \leq 1, s \in \Lambda (n)} \sup_{n \geq 1} \| \pi_j (y^n_{s,t}) \|^\frac{1}{k+1}.
$$

(53)

We first prove that $M_{[p]} \leq C_p$. It is clear that $M_0 = 1$. Based on Lemma 30 on (11) (error between two-steps ODE and one-step ODE), we have,

$$
\|\pi_k (y^{s,u,t}_2 - y^{s,t}_1)\| \leq C_{p} M_{k-1} \omega (s, t)^{\frac{[p]+1}{k+1}}.
$$

(54)

On the other hand,

$$
y^{s,u,t}_2 - y^{s,u}_1 - \left( y^{n,u}_1 - y^{n,u}_{s_0, t_0} \right) = y^{s,u}_1 \otimes \left( \left( (y^{n,u}_1)^{-1} \otimes y^{s,u,t}_2 \right) - \left( y^{n,u}_2 \otimes \left( (y^{n,u}_2)^{-1} \otimes y^{n,u}_1 \right) \right).\n$$

(55)

Based on the explicit expression of $F (f)^\pi_k$ in Lemma 27 on (25) $(y^{n,u}_1)^{-1} \otimes y^{s,u,t}_2 - 1$ and $(y^{n,u}_2)^{-1} \otimes y^{s,u,t}_2 - 1$, $r \in [0, 1]$, are respectively the solution of the ODE

$$
dy_r = \sum_{k=0}^{[p]} F \left( f \left( \cdot + \pi_1 (y^{n,u}_1) + \pi_1 (y^{n,u}_2) \right) \pi_k \left( \log X_{s,u} \right) \left( Id_{L[1]} (U) \right) \left( y^{n,u}_r \right) dr, \quad r \in [0, 1] \right\}, \quad y_0 = 0.
$$

\[ dy_r = \sum_{k=1}^{[p]} F \left( f \left( \cdot + \pi_1 (y^{n,u}_1) + \pi_1 (y^{n,u}_2) \right) \pi_k \left( \log X_{s,u} \right) \left( Id_{L[1]} (U) \right) \left( y^{n,u}_r \right) dr, \quad r \in [0, 1] \right\}, \quad y_0 = 0.

It is clear that

$$
\pi_0 \left( \left( y^{n,u}_1 \right)^{-1} \otimes y^{s,u,t}_2 - 1 \right) = \pi_0 \left( y^{n,u}_2 \otimes y^{n,u}_1 \right)^{-1} - 1 = 0.
$$

Based on Lemma 32 on (12) we have, for $k = 1, \ldots, [p]$,

$$
\| \pi_k (y^{n,u}_1 \otimes y^{s,u,t}_2 - 1) - \pi_k (y^{n,u}_2 \otimes y^{n,u}_1 \otimes y^{n,u}_1 - 1) \|
$$

(56)

$$
\leq C_p \omega (u, t)^\frac{k}{k} \left[ f \left( \cdot + \pi_1 (y^{n,u}_1) + \pi_1 (y^{n,u}_2) \right) - f \left( \cdot + \pi_1 (y^{n,u}_1) \right) \right]_{[p]-1} \leq C_p \omega (u, t)^\frac{k}{k} \| \left( y^{s,u}_1 \right) - \pi_1 (y^{n,u}_1) \| = C_p \omega (u, t)^\frac{k}{k} \| \pi_1 (y^{s,u}_1) \|.
$$

(57)
and based on (23) on Lemma 29
\[ \left\| \pi_k \left( (y_1^{s,t})^{-1} \otimes y_2^{s,u,t} - 1 \right) \right\| \leq C_p \omega \left( u, t \right)^{1/2}. \] (57)

Then by combining (50), (53), (55), (56) and (57), we have
\[ \left\| \pi_k \left( y_2^{s,u,t} - y_1^{s,u} - y_2^{s,u,t} - y_1^{s,u} \right) \right\| \]
\[ \leq \sum_{j=1}^{k-1} \left\| \pi_j \left( y_1^{s,u} - y_1^{s,u} \right) \right\| \left\| \pi_{k-j} \left( (y_1^{s,u})^{-1} \otimes y_2^{s,u} - 1 \right) \right\| 
\[ + \sum_{j=0}^{k-1} \left\| \pi_j \left( y_1^{s,u} \right) \right\| \left\| \pi_{k-j} \left( (y_1^{s,u})^{-1} \otimes y_2^{s,u} - 1 \right) \right\| 
\leq C_p \sum_{j=1}^{k-1} \omega \left( u, t \right)^{k-j} + C_p \sum_{j=0}^{k-1} M_j \omega \left( u, t \right)^{k-j} \left\| \pi_1 \left( (\Gamma^{s,u}) \right) \right\| 
\leq C_p \sum_{j=1}^{k-1} \omega \left( u, t \right)^{k-j} \left\| \pi_j \left( (\Gamma^{s,u}) \right) \right\| + C_p M_{k-1} \omega \left( u, t \right)^{1/2} \left\| \pi_1 \left( (\Gamma^{s,u}) \right) \right\| 
\] (58)

As a result, combining (52), (51) and (58), we have
\[ \left\| \pi_k \left( \Gamma^{s,t} \right) - \pi_k \left( (\Gamma^{s,u}) \right) - \pi_k \left( (\Gamma^{u,t}) \right) \right\| \]
\[ \leq C_p \left( M_{k-1} \omega \left( s, t \right)^{\left\lceil \frac{\log p}{p} \right\rceil} + M_{k-1} \omega \left( u, t \right)^{1/2} \left\| \pi_1 \left( (\Gamma^{s,u}) \right) \right\| + \sum_{j=1}^{k-1} \omega \left( s, t \right)^{k-j} \left\| \pi_j \left( (\Gamma^{s,u}) \right) \right\| \right). \] (59)

In particular, since \( M_0 = 1 \), we have
\[ \left\| \pi_1 \left( (\Gamma^{s,t}) \right) - \pi_1 \left( (\Gamma^{s,u}) \right) - \pi_1 \left( (\Gamma^{u,t}) \right) \right\| \leq C_p \left( \omega \left( s, t \right)^{\left\lceil \frac{\log p}{p} \right\rceil} + \omega \left( u, t \right)^{1/2} \left\| \pi_1 \left( (\Gamma^{s,u}) \right) \right\| \right). \]

When \([s, t] \subseteq [s_0, t_0]\) is dyadic, by recursively bisecting \([s, t]\), we have
\[ \left\| \pi_1 \left( \Gamma^{s,t} \right) \right\| \leq \left( \sum_{k=0}^{n} \prod_{j=0}^{k} \left( 1 + 2^{-j} C_p \omega \left( s, t \right)^{1/2} \right) \left( \frac{1}{2} \right)^{\left\lfloor \frac{\log p}{p} \right\rfloor} \right) \omega \left( s, t \right)^{\left\lfloor \frac{\log p}{p} \right\rfloor} 
\[ + \left( \prod_{j=0}^{n} \left( 1 + 2^{-j} C_p \omega \left( s, t \right)^{1/2} \right) \right) \sum_{t_0, t_0+1 \subseteq [s,t]} \left\| \pi_1 \left( \Gamma^{s,t}_j \right) \right\| \right). \]

By using \( \Gamma^{s,t}_j = 0 \) as at (51) and that \( \omega \left( s, t \right) = \omega \left( s_0, t_0 \right) \leq 1 \), we have
\[ \left\| \pi_1 \left( \Gamma^{s,t} \right) \right\| \leq \left( \sum_{k=0}^{n} \prod_{j=0}^{k} \left( 1 + 2^{-j} \omega \left( s, t \right)^{1/2} \right) \left( \frac{1}{2} \right)^{\left\lfloor \frac{\log p}{p} \right\rfloor} \right) \omega \left( s, t \right)^{\left\lfloor \frac{\log p}{p} \right\rfloor} \]
\[ \leq \exp \left( \frac{2^{1/p}}{2^{1/p} - 1} \right) \left( \frac{2^{1/p - 1} - 1}{2^{1/p} - 1} \right) \omega \left( s, t \right)^{\left\lfloor \frac{\log p}{p} \right\rfloor} = C_p \omega \left( s, t \right)^{\left\lfloor \frac{\log p}{p} \right\rfloor}. \] (60)

Thus, for any dyadic interval \([s, t] \subseteq [s_0, t_0]\), we have
\[ \left\| \pi_1 \left( \Gamma^{s,t}_j \right) \right\| \leq C_p \omega \left( s, t \right)^{\left\lfloor \frac{\log p}{p} \right\rfloor}. \]

Then we prove \( M_1 \leq C_p \). Based on the explicit expression of \( F (f) \) as in Lemma 27 \((y_{s_0,s}^{n})^{-1} \otimes y_{s,t}^{n} - 1\) is the solution of the ODE
\[ dy_r = \sum_{k=1}^{\left\lfloor \frac{\log p}{p} \right\rfloor} F \left( f \left( \cdot + \pi_1 \left( y_{s_0,s}^{n} \right) + \pi_1 \left( y_{s_0}^{n} \right) \right) \right)^{\pi_k} \pi_k \left( \log X_{s,t} \right) \left( \text{Id}_{L^{1/2} \left( \{ t \} \right)} \right) \left( y_r \right) dr, r \in [0,1], y_0 = 0. \]
Then similar estimate holds for non-dyadic intervals based on Lemma 37, and we have, for any
\[
\pi_0 \left( \left( y_{n,0}^n \right)^{-1} \otimes y_{s,t}^n \right) - 1 \right) \right\| \leq C_{p,\omega} (s, t)^\frac{1}{p} \, , \, k = 1, 2, \ldots, [p] \, . \tag{61}
\]
As a result, we have, for any dyadic interval \([s, t] \subseteq [s_0, t_0]\),
\[
\| \pi_1 \left( y_{n,0}^n \right) - \pi_1 \left( y_{n,0}^n \right) \right\| \leq \| \pi_1 (\Gamma_{s,t}^n) \| + \| \pi_1 \left( y_{s,t}^n - y_{s,0}^n \right) \| \leq C_{p,\omega} (s, t)^\frac{1}{p} \, . \tag{62}
\]
Then by using the inductive relationship of \(\pi_k (\Gamma_{s,t}^n)\) as at (59), we have
\[
\| \pi_k (\Gamma_{s,t}^n) - \pi_k (\Gamma_{s,t}^n) \right\| \leq C_{p,\omega} (s, t)^\frac{1}{p} \, , \, j = 1, 2, \ldots, k - 1. \tag{63}
\]
Then by using the inductive relationship of \(\pi_k (\Gamma_{s,t}^n)\) as at (59), we have
\[
\| \pi_k (\Gamma_{s,t}^n) - \pi_k (\Gamma_{s,t}^n) \right\| \leq C_{p,\omega} (s, t)^\frac{1}{p} \leq C_{p,\omega} (s, t)^\frac{1}{p} \, . \tag{64}
\]
Moreover, based on (61) and (64), for any dyadic interval \([s, t] \subseteq [s_0, t_0]\), we have
\[
\| \pi_k \left( y_{n,0}^n - y_{s,0}^n \right) \right\| \leq \| \pi_k \left( y_{s,t}^n - y_{s,0}^n \right) \right\| + \| \pi_k (\Gamma_{s,t}^n) \| \leq C_{p,\omega} (s, t)^\frac{1}{p} \, . \tag{65}
\]
As a result,
\[
M_k \leq M_{k-1} + \sup_{\omega(s,0) \leq 1, \omega(s) \in \Lambda(n)} \sup_{n \geq 1} \| \pi_k \left( y_{n,0}^n \right) \right\| \leq C_p \, . \tag{66}
\]
Based on (64), for any dyadic \([s, t] \subseteq [s_0, t_0]\), \(\| \Gamma_{s,t}^\omega \| \leq C_{p,\omega} (s, t)^\frac{1}{p} \). Similar estimate holds for non-dyadic intervals based on Lemma 38 and we have, for any \([s, t] \subseteq [s_0, t_0]\),
\[
\| \Gamma_{s,t}^\omega \| \leq C_{p,\omega} (s, t)^\frac{1}{p} \, . \tag{67}
\]
In particular, when \([s, t] = [s_0, t_0], y^{s, t} : [0, 1] \to G^{[p]}(V)\) (defined at (17)) is the solution of the ODE

\[
d y^{s, t}_r = \sum_{k=1}^{[p]} F(f(\cdot + \pi_1(y^s_r))) \circ \pi_k(\log X_{s,t}) (Id_{L^{[p]}(U)}) (y^{s,t}_r) dr, \ r \in [0, 1], \ y^{s,t}_{0} = 1.
\]

Then based on (23) in Lemma 29 on p9 we have, for \(k = 1, 2, \ldots, [p],\)

\[
\|\pi_k(y^{s,t}_r)\| \leq C_p \omega(s, t)^{\frac{k}{p}}.
\]

Then, based on (66) and (67), there exists constant \(C_p,\) such that, for any \(n \geq 1,\) any \(s, t \in \Lambda(n),\) \(\omega(s, t) \leq 1,\) and \(k = 1, 2, \ldots, [p],\)

\[
\|\pi_k(y^{n,s,t}_r)\| \leq \|\pi_k(\Gamma^{s,t})\| + \|\pi_k(y^{s,t}_1)\| \leq C_p \omega(s, t)^{\frac{k}{p}}.
\]

\[\Box\]

**Lemma 40** For \([p] + 1 \geq \gamma > p \geq 1,\) suppose \(X \in C_p^{\vartheta}([0, T], G^{[p]}(V)),\) \(f \in L(V, C^\gamma(U, U))\) and \(\xi^i \in G^{[p]}(U),\ i = 1, 2.\) Define control \(\omega\) by

\[
\omega(s, t) := |f|^p |X|^p_{C^{\vartheta}_{p-\vartheta}(s, t)}, \ \forall 0 \leq s \leq t \leq T.
\]

Based on \(\omega,\) define dyadic partition \(\Lambda(n) = \{t^n_i\}\) as in Notation 39. For \(i = 1, 2,\) let \(y^i : [0, T] \to G^{[p]}(U)\) be the solution of the ODE (with different initial value)

\[
y^0_0 = \xi^i, \\
y^j_r = \sum_{k=1}^{[p]} F(f(\cdot + \pi_1(y^i_r))) \circ \pi_k(\log X_{s,t}) (Id_{L^{[p]}(U)}) (y^i_r) dr, \ r \in [0, 1], \ y^0_0 = y^{s,t}_{s_0,t_0}.
\]

Then, there exist \(C_p, \gamma > 0\) and \(\delta_{p, \gamma} \in (0, 1],\) such that, for any \(s, t \in \Lambda(n)\) satisfying \(\omega(s, t) \leq \delta_{p, \gamma},\)

\[
\|y^1_r - y^2_r\| \leq C_p \gamma \|y^1_r - y^2_r\|^2.
\]

**Proof.** We assume \(|f| > 1.\) Otherwise, we replace \(f\) and \(X\) by \(|f|^{-1} f\) and \(\delta_{|f|} X\) respectively. Fix \(s_0, t_0 \in \Lambda(n)\) satisfying \(\omega(s_0, t_0) \leq 1.\) Then based on Lemma 39 there exists constant \(C_p,\)

\[
\max_{i=1, 2} \sup_{s \in \Lambda(n), t \in [0, 1]} \|y_i^{s,t}_{s_0,t_0}\| \leq C_p < \infty.
\]

For \(i = 1, 2,\) \(s, u, t, \in \Lambda(n)\) satisfying \(s_0 \leq s \leq u \leq t \leq t_0,\) let \(y^{i,s,t} : [0, 1] \to G^{[p]}(U)\) and \(y^{i,s,u,t} : [0, 2] \to G^{[p]}(U)\) be the solution to the ODE

\[
y^{i,s,t}_r = \sum_{k=1}^{[p]} F(f(\cdot + \pi_1(y^{i,s}_r))) \circ \pi_k(\log X_{s,t}) (Id_{L^{[p]}(U)}) (y^{i,s,t}_r) dr, \ r \in [0, 1], \ y^{i,s,t}_{0} = y^{i,s,u,t}_{s_0,t_0}.
\]

For \(s, t \in \Lambda(n),\) \(s_0 \leq s \leq t \leq t_0,\) denote

\[
y_t := y^{1}_{s_0,t} - y^{2}_{s_0,t}, \ y^{s,t}_1 := y^{1}_{s,t} - y^{2}_{s,t} \ \text{and} \ \Gamma^{s,t} := y_t - y^{s,t}_1.
\]

Then it can be checked that,

\[
\Gamma^{j+1}_{s_{j+1}, t_{j+1}} = 0, \ j = 0, 1, \ldots, 2^n - 1.
\]

For \(u \in \Lambda(n), \ u \in (s, t),\) denote

\[
y^{s,u,t} := y^{1,s,u,t} - y^{2,s,u,t}.
\]

Then, it can be computed that,

\[
\|\Gamma^{s,u} + \Gamma^{u,t} - \Gamma^{s,t}\| = \|y^{1,u}_{s} - y_s + y^{u,t}_{1} - y_u - (y^{s,t}_1 - y_s)\|
\]

\[
\leq \|y^{u,t}_{1} - y_u - (y^{s,u,t}_2 - y^{s,t}_1)\| + \|y^{s,u,t} - y^{s,t}_1\|.
\]
Denote \( \delta := \omega(s, t) \frac{2}{2^\delta - 1} \), then \( \delta \leq \omega(s_0, t_0) \frac{2}{2^\delta - 1} \leq 1 \). Based on Lemma 34 (on p12), we have

\[
\| y_2^{s,u,t} - y_1^{s,u,t} \| \leq C_p \delta \| y_u \|. 
\] (73)

On the other hand, we want to prove

\[
\| y_2^{s,u,t} - y_u - (y_2^{s,u,t} - y_1^{s,u}) \| \leq C_p \delta \| \Gamma_{s,u} \| + C_p \delta \| y_u \|. 
\]

Based on (70), we have \( \max_{i=1,2} \| y_{s,u}^{i} \| \geq \| y_{s,u}^{i} \| \leq C_p \). Then by using Lemma 33 (on p12) and estimating the two cases \( k = 1, \ldots, [p] - 1 \) and \( k = [p] \) separately, we get

\[
\| y_2^{s,u,t} - y_u - (y_2^{s,u,t} - y_1^{s,u}) \| \leq C_p \delta \| \Gamma_{s,u} \| + C_p \delta \| y_u \|. 
\] (74)

By using \( \| y_{s,u}^{i} \| \leq C_p \) as at (70), and based on Lemma 32 on p12 (continuity in initial value) and 13 in Lemma 39 on p17, we have,

\[
\sup_{r \in [0,1]} \| y_r^{i,u,t} - y_r^{c,s,u,t} \| \leq C_p \| y_{s,u}^{i} - y_1^{s,u} \| \leq C_p \omega(s, u)^{\frac{[p]+1}{R}} \leq C_p \delta^{[p]+1}. 
\] (75)

By using \( \| y_{s,u} \| \leq C_p \), and based on Lemma 32 (continuity in initial value),

\[
\sup_{r \in [0,1]} \| y_r^{1,u,t} - y_r^{2,u,t} \| \leq C_p \| y_{s,u}^{1} - y_2^{s,u,t} \| = C_p \| y_u \|. 
\] (76)

As a result, combining (74), (75) and (76), we have (\( \delta \leq 1 \))

\[
\| y_r^{u,t} - y_u - (y_r^{u,t} - y_1^{s,u}) \| \leq C_p \delta \int_0^r \| y_v^{u,t} - y_{v+1}^{s,u,t} \| dv + C_p \delta \| y_u \|. 
\]

Then by using Gronwall’s inequality (\( \delta \leq 1 \)), we have

\[
\sup_{r \in [0,1]} \| y_r^{u,t} - y_r^{s,u,t} \| \leq C_p \| y_u - y_1^{s,u} \| + C_p \delta \| y_u \| = C_p \| \Gamma_{s,u} \| + C_p \delta \| y_u \| ,
\]

and

\[
\sup_{r \in [0,1]} \| y_r^{u,t} - y_r^{s,u,t} \| \leq C_p \| \Gamma_{s,u} \| + C_p \delta \| y_u \|. 
\] (77)

As a result, combining (72), (73) and (77), we have, for \( \{ s, t \} \in \Lambda(n) \), \( [s, t] \subseteq [s_0, t_0] \), \( \omega(s, t) \leq 1 \),

\[
\| \Gamma^{s,u} + \Gamma^{u,t} - \Gamma^{s,u,t} \| \leq C_p \left( \omega(s, t)^{\frac{2}{2^\delta - 1}} \| \Gamma^{s,u} \| + \omega(s, t)^{\frac{2}{2^\delta - 1}} \left( \sup_{r \in [s,t]} \| y_r \| \right) \right). 
\]

If \( [s, t] \subseteq [s_0, t_0] \) is a dyadic interval, then after a sequence of bisections, we have \( \Gamma^{s,t}_{r+1} = 0 \) as at (71)

\[
\| \Gamma^{s,t} \| \leq \left( \prod_{k=0}^{n} \left( 1 + 2^{-2^k \omega(s, t)^{\frac{2}{2^\delta - 1}}} \left( \frac{2^{n-1}}{2} \right)^k \right) \right) \omega(s, t)^{\frac{2}{2^\delta - 1}} \left( \sup_{r \in [s,t]} \| y_r \| \right) 
\]

\[
\leq \exp \left( \frac{2^\delta}{2^\delta - 1} \right) \frac{2^\delta - 1}{2^\delta - 1} \omega(s, t)^{\frac{2}{2^\delta - 1}} \left( \sup_{r \in [s,t]} \| y_r \| \right). 
\]
As a result, when \([s, t] \subseteq [s_0, t_0]\) is a dyadic interval, we have
\[
\|y_t - y_t^{s,t}\| \leq C_{p,\gamma} \omega(s, t)^{\hat{p}} \sup_{r \in [s, t]} \|y_r\|.
\]

Based on Lemma 38 on p15, we can pass similar estimate to non-dyadic intervals and get, for any \(\{s, t\} \in \Lambda(n), [s, t] \subseteq [s_0, t_0]\),
\[
\|y_t - y_t^{s,t}\| \leq C_{p,\gamma} \omega(s, t)^{\hat{p}} \sup_{r \in [s, t]} \|y_r\|. \quad (78)
\]

Let \([s, t] = [s_0, t_0]\), then \(y^{s,t}\) is the solution to the ODE:
\[
dy^{s,t} = \sum_{k=1}^{[p]} F\left(f \left(\cdot + \pi_k(y^{s,t})\right)\right) \omega_k \left(\log X_{s,t}\right) (Id_{L^p([t])})(y^{s,t}) \, dr, \quad r \in [0, 1], \ y_0^{s,t} = 1.
\]

Based on Lemma 32 on p12, \(\delta \leq \|y^{s,t}\|\), for \(k = 1, 2, \ldots, [p]\),
\[
\|\pi_k(y^{s,t})\| = \|\pi_k(y^{1,s,t}) - \pi_k(y^{2,s,t})\| \leq C_{p,\gamma} \omega(s, t)^{\hat{p}} \|\pi_1(y^{1}) - \pi_1(y^{2})\|. \quad (79)
\]

Combining (78) and (79), we get, for \(k = 1, 2, \ldots, [p]\),
\[
\|\pi_k(y^{1}) - \pi_k(y^{2})\| = \|\pi_k(y_t)\|
\leq C_{p,\gamma} \omega(s, t)^{\hat{p}} \|\pi_1(y^{1}) - \pi_1(y^{2})\| + C_{p,\gamma} \omega(s, t)^{\hat{p}} \left(\sup_{u \in [s, t]} \|y^{1}_{s,u} - y^{2}_{s,u}\|\right).
\]

In particular, we have
\[
\sup_{u \in [s, t]} \|y^{1}_{s,u} - y^{2}_{s,u}\| \leq C_{p,\gamma} \omega(s, t)^{\hat{p}} \|\pi_1(y^{1}) - \pi_1(y^{2})\| + C_{p,\gamma} \omega(s, t)^{\hat{p}} \left(\sup_{u \in [s, t]} \|y^{1}_{s,u} - y^{2}_{s,u}\|\right).
\]

Choose \(\delta_{p,\gamma} \in (0, 1]\) satisfying \(C_{p,\gamma} \delta_{p,\gamma}^2 \leq 2^{-1}\). Then for \([s, t]\) satisfying \(\omega(s, t) \leq \delta_{p,\gamma}\), we have
\[
\sup_{u \in [s, t]} \|y^{1}_{s,u} - y^{2}_{s,u}\| \leq C_{p,\gamma} \omega(s, t)^{\hat{p}} \|\pi_1(y^{1}) - \pi_1(y^{2})\|.
\]

Combined with (81),
\[
\|\pi_k(y^{1}) - \pi_k(y^{2})\| \leq C_{p,\gamma} \omega(s, t)^{\hat{p}} \|\pi_1(y^{1}) - \pi_1(y^{2})\|. \quad (82)
\]

As a result, by combining (81) with \(\|\pi_j(y^{s,t})\| \leq C_{p,\gamma} \omega(s, t)^{\hat{p}} \leq C_p\) as at (16) on p14, there exists \(\delta_{p,\gamma} \in (0, 1]\) such that for any \([s, t]\) satisfying \(\omega(s, t) \leq \delta_{p,\gamma}\), we have \((\|y^{1}\| \geq \|y_0^{1}\| = 1)\)
\[
\|\pi_k(y^{1}) - \pi_k(y^{2})\|
\leq \sum_{j=1}^{k} \|\pi_{k-j}(y^{1})\| \|\pi_{j}(y^{1}) - \pi_{j}(y^{2})\| + \sum_{j=1}^{k} \|\pi_{j}(y^{1}) - \pi_{j}(y^{2})\| \|\pi_{k-j}(y^{2})\|
\leq C_{p,\gamma} \|y^{2}\| \|y^{2}_{s,t} - y^{2}_{s,t}\|.
\]

Then we re-state Theorem 19 and give a proof.

**Theorem 19** For \(\gamma > p \geq 1\), suppose \(X \in C^{p-var}([0, T], G^{[p]}(\mathcal{V})), f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))\) and \(\xi \in G^{[p]}(\mathcal{U})\). Then the rough differential equation
\[
dY = f(Y) \, dX, \quad Y_0 = \xi,
\]
has a unique solution (denoted as \(Y\)) in the sense of Definition 17, which is a continuous path taking values in \(G^{[p]}(\mathcal{U})\). If define control \(\omega\) by
\[
\omega(s, t) := |f|^{p} \|X\|^{p}_{p-var, [s, t]}, \quad \forall 0 \leq s \leq t \leq T;
\]

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then there exists a constant $C_p$ such that, for any $0 \leq s \leq t \leq T$,

$$\|Y\|_{p-\text{var},[s,t]} \leq C_p \left( \omega(s,t)^{\frac{n}{p}} + \omega(s,t) \right).$$

Moreover, for $0 \leq s \leq t \leq T$, if let $y^{s,t} : [0,1] \to L^{[p]}(U)$ denote the solution of the ordinary differential equation

$$dy^{s,t}_u = \sum_{k=1}^{[p]} F\left(f(\cdot + \pi_1_Y(u))\right) \cdot k \cdot \pi_k \left(\log X_{s,t}\right) \left(Id_{L^{[p]}(U)}\right) \left(y^{s,t}_u\right) du, u \in [0,1], \ y^{s,t}_0 = 1,$$

then $y^{s,t}$ takes value in $G^{[p]}(U)$, and there exists a constant $C_p$, such that,

$$\|Y_{s,t} - y^{s,t}_1\| \leq C_p \left( \omega(s,t)^{\frac{n+1}{p}} + \omega(s,t)^{[p]} \right),$$

$$\left\|Y_{s,t} - \sum_{k=1}^{[p]} F\left(f(\cdot + \pi_1_Y(u))\right) \cdot k \cdot \pi_k \left(\log X_{s,t}\right) \left(Id_{L^{[p]}(U)}\right) \left(y^{s,t}_u\right) \right\| \leq C_p \left( \omega(s,t)^{\frac{n+1}{p}} + \omega(s,t)^{[p]} \right).$$

**Proof.** Firstly, with $\delta_{p,\gamma} \in (0,1]$ selected in Lemma 40, we assume $\omega(0,T) \leq \delta_{p,\gamma}$ and prove existence and uniqueness. Denote dyadic partitions $\Lambda(n) = \{t^n\}$ of $\omega$ as in Notation 35 on 14. Let $y^n : [0, T] \to G^{[p]}(U)$ be the solution to the ODE

$$dy^n_u = \sum_{k=1}^{[p]} F\left(f(\cdot + \pi_1_Y(u))\right) \cdot k \cdot \pi_k \left(\log X_{s,t}\right) \left(Id_{L^{[p]}(U)}\right) \left(y^n_u\right) du, u \in [0,1], \ y^n_0 = \xi.$$

For $[s, t] \subseteq [0, T]$, let $y^{n,s,t} : [0,1] \to G^{[p]}(U)$ be the solution of the ODE

$$dy^{n,s,t}_u = \sum_{k=1}^{[p]} F\left(f(\cdot + \pi_1_Y(u))\right) \cdot k \cdot \pi_k \left(\log X_{s,t}\right) \left(Id_{L^{[p]}(U)}\right) \left(y^{n,s,t}_u\right) du, u \in [0,1], \ y^{n,s,t}_0 = 1.$$

Based Lemma 39 (uniform bound on concatenated dyadic ODEs), we have, $(\omega(0,T) \leq \delta_{p,\gamma})$

$$\sup_{n \geq 1} \sup_{u \in [0,T]} \|y^n_u\| \leq C_p.$$

Suppose $m \geq n \geq 1$. For $j = 0, 1, \ldots, 2^n$, as in the proof of Thm 2.3 by Davie [10], we let $Z_j$ be the solution of the ODE (the ODE approximation w.r.t. $\Lambda(m)$ starting at $t^{n,1}_j$ from point $y^{n,1}_j$)

$$dZ^n_j = \sum_{k=1}^{[p]} F\left(f(\cdot + \pi_1_Y(u))\right) \cdot k \cdot \pi_k \left(\log X_{s,t}\right) \left(Id_{L^{[p]}(U)}\right) \left(Z^n_j\right) \frac{dt}{t^{m+1} - t^m}, t \in \left[t_j^{m+1}, t^m_j\right], \ l \geq j, \ Z^n_j = y^n_j.$$

Then $Z^n_j = y^n_j$ and $Z^n_j = y^n_j$. Moreover, based on [48] in Lemma 39 on 17, for $j = 0, 1, \ldots, 2^n - 1$,

$$\left\|Z^n_{l,j} - y^{n,l+1}_{l+1,j} \right\| \leq C_p \left( t^{n,1}_j, t^{n,1}_j \right)^{\frac{n+1}{p}}.$$

Combined with Lemma 39 (continuity in initial value of dyadic ODE approximations) and using (50), we have, for $k = 0, 1, \ldots, 2^n$,

$$\left\|y^n_k - y^n_k \right\| \leq \sum_{j=0}^{k-1} \left\|Z^n_{l,k} - Z^n_{l,k} \right\| \leq C_p \sum_{j=0}^{k-1} \left\|Z^n_{l,j} - Z^n_{l,j} \right\| = C_p \sum_{j=0}^{k-1} \left\|y^n_k \otimes \left(t^{n,1}_j, t^{n,1}_j \right)^{\frac{n+1}{p}} - Z^n_{l,j} \right\| \leq C_p \sum_{j=0}^{k-1} \omega \left(t^{n,1}_j, t^{n,1}_j \right)^{\frac{n+1}{p}} \omega(0,T)^{\frac{n+1}{p}} \to 0 as \ n \to \infty.$$
where \( y^{s,t} : [0, 1] \to G^{[p]} (\mathcal{U}) \) is the solution of the ODE

\[
dy^{s,t}_{u} = \sum_{k=1}^{[p]} F \left( f \left( \cdot + \pi_1 (Y_u) \right) \phi_k \left( \log X_{s,t} \right) (Id_{L^{[p]}(\mathcal{U})}) \right) \left( y^{s,t}_u \right) du, \quad u \in [0, 1], \quad y^{s,t}_0 = 1.
\]

Combined with the Euler expansion of \( y^{s,t}_1 \) in Lemma \[30\] on \[11\], we have

\[
\left\| Y_{s,t} - \sum_{k=1}^{[p]} F \left( f \left( \cdot + \pi_1 \left( \tilde{Y}_u \right) \right) \phi_k \left( \log X_{s,t} \right) (Id_{L^{[p]}(\mathcal{U})}) \right) \right\| \leq C_p \omega \left( s, t \right)^{\frac{[p]+1}{p}}.
\]

Thus, based on the definition of solution (Definition \[17\]), \( Y \) is a solution to the RDE \[32\]. Based on Lemma \[24\] on \[11\], \( y^n \) takes value in \( G^{[p]} (\mathcal{U}) \), so \( Y \) takes value in \( G^{[p]} (\mathcal{U}) \).

Then we prove that the solution is unique. Suppose \( \check{Y} \) is another solution. Then, by combining the definition of solution and the Euler expansion of ODE as in Lemma \[30\] on \[11\] for some \( \theta : [0 \leq s \leq t \leq T] \to \mathbb{R}^+ \) satisfying

\[
\lim_{|D| \to 0} \sum_{j \in D} \theta (t_j, t_{j+1}) = 0,
\]

we have, for all sufficiently small \([s, t] \subseteq [0, T] \),

\[
\left\| \check{Y}_{s,t} - \tilde{y}^{s,t}_1 \right\| \leq C_p \omega \left( s, t \right)^{\frac{[p]+1}{p}} + \theta \left( s, t \right),
\]

where \( \tilde{y}^{s,t} : [0, 1] \to L^{[p]} (\mathcal{U}) \) is the solution of the ODE

\[
d\tilde{y}^{s,t}_{u} = \sum_{k=1}^{[p]} F \left( f \left( \cdot + \pi_1 \left( \tilde{Y}_u \right) \right) \phi_k \left( \log X_{s,t} \right) (Id_{L^{[p]}(\mathcal{U})}) \right) \left( \tilde{y}^{s,t}_u \right) du, \quad u \in [0, 1], \quad \tilde{y}^{s,t}_0 = 1.
\]

For integer \( n \geq 1 \) and \( j = 0, 1, \ldots, 2^n \), denote \( \check{Z}^j \) as the solution of the ODE (the ODE approximation w.r.t. \( \Lambda (n) \) starting at time \( t^n_j \) from point \( \tilde{Y}^n_j \) ),

\[
d\check{Z}^j_u = \sum_{k=1}^{[p]} F \left( f \right) \phi_k \left( \log X^{t^n_j, t^n_{j+1}} \right) \left( Id_{L^{[p]}(\mathcal{U})} \right) \left( \check{Z}^j_u \right) \frac{dt}{t^n_{j+1} - t^n_j}, \quad t \in [t^n_j, t^n_{j+1}], \quad j \geq 0, \quad \check{Z}^j_0 = \tilde{Y}^n_j.
\]

Then \( \check{Z}^0_n = y^n_0 \) and \( \check{Z}^n_j = \tilde{Y}^n_j \). Based on \[37\] and that \( \omega (0, T) \leq \delta_{p, \gamma} \leq 1 \), \( \check{Y} \) is bounded. Then by using Lemma \[10\] (continuity in initial value of dyadic ODE approximations) and \[37\], we have, for \( k = 0, 1, \ldots, 2^n \),

\[
\left\| \check{Y}^n_{t_j} - \check{Y}^n_{t_k} \right\| \leq \sum_{j=0}^{2^n-1} \left\| \check{Z}^j_{t_k} - \check{Z}^j_{t_j} \right\| \leq C_{p, \gamma} \sum_{j=0}^{k-1} \left\| \tilde{Y}^n_j \otimes \left( \tilde{Y}^n_{t_j, t_{j+1}} - \check{Y}^n_{t_j, t_{j+1}} \right) \right\|
\]

\[
\leq C_{p, \gamma} \omega \theta \sum_{j=0}^{2^n-1} \left( \omega (t^n_j, t^n_{j+1}) \right)^{\frac{[p]+1}{p}} \left( \omega (t^n_j, t^n_{j+1}) \right)^{\frac{[p]+1}{p}} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

As a result, \( \check{Y} \) is the uniform limit of \( y^n \), so coincides with \( Y \).

Then we prove \[33\]. Since \( Y \) is the uniform limit of \( y^n \), based on \[16\] in Lemma \[30\] on \[11\] for \([s, t] \subseteq [0, T] \) satisfying \( \omega \left( s, t \right) \leq 1 \), we have

\[
\left\| Y_{s,t} \right\| \leq \omega \left( s, t \right)^{\frac{1}{p}}, \text{ so by using sub-additivity of } \omega, \left\| Y \right\|_{p-var, [s, t]} \leq \omega \left( s, t \right)^{\frac{1}{p}}.
\]

When \( \omega \left( s, t \right) \geq 1 \), as in Prop 5.10 \[12\], we decompose \([s, t] = \bigcup_{j=0}^{n-1} [t_j, t_{j+1}] \) with \( \omega \left( t_j, t_{j+1} \right) = 1 \), \( j = 0, \ldots, n-2 \), \( \omega \left( t_{n-1}, t_n \right) \leq 1 \). Then

\[
n = 1 + \sum_{j=0}^{n-2} \omega \left( t_j, t_{j+1} \right) \leq 2 \omega \left( s, t \right),
\]
and (since $\|\cdot\|$ is equivalent to an additive norm up to a constant depending on $p$, Exer 7.38 [14]) we have

$$
\|Y\|_{p-\text{var},[s,t]} \leq C_p \sum_{j=0}^{n-1} \|Y\|_{p-\text{var},[t_j,t_{j+1}]} \leq C_p n \leq C_p \omega(s,t). \quad (88)
$$

As a result, we have

$$
\|Y\|_{p-\text{var},[s,t]} \leq C_p \left( \omega(s,t)^{\frac{1}{p}} \vee \omega(s,t) \right).
$$

Then we prove (85). When $\omega(s,t) \leq 1$, by using Lemma 39 (uniform estimate of dyadic approximations) and that $y^n$ converge uniformly to $Y$, we have, (with $y_i^{s,t}$ defined at (84))

$$
\left\| Y_{s,t} - y_i^{s,t} \right\| \leq C_p \omega(s,t)^{\frac{|p|+1}{p}}.
$$

Combined with Lemma 30 (high order Euler expansion of solution of ODE), we have

$$
\left\| Y_{s,t} - \sum_{k=1}^{[p]} F(f(\cdot + \pi_1 (Y_s)))^{\circ k} \pi_k (X_{s,t})(Id_{L^p(U)})(1) \right\| \leq C_p \omega(s,t)^{\frac{|p|+1}{p}}.
$$

When $\omega(s,t) \geq 1$, based on (83),

$$
\|Y_{s,t}\| = \sum_{p=1}^{[p]} \|\pi_k (Y_{s,t})\| \leq \omega(s,t)^{[p]}.
$$

On the other hand, based on Lemma 27 (explicit expression of $F(f)^{\circ k}$), it can be proved inductively that,

$$
\sup_{u \in [0,1]} \|\pi_k (y_u^{\cdot,t})\| \leq C_p \omega(s,t)^{\frac{k}{2p}}, \quad k = 1, 2, \ldots, [p], \text{ so } \|y_i^{s,t}\| \leq C_p \omega(s,t)^{\frac{2}{p}} \leq C_p \omega(s,t)^{[p]}.
$$

Then

$$
\|Y_{s,t} - y_i^{s,t}\| \leq \|Y_{s,t}\| + \|y_i^{s,t}\| \leq C_p \omega(s,t)^{[p]}.
$$

For high order Euler expansion, when $\omega(s,t) \geq 1$,

$$
\left\| Y_{s,t} - \sum_{k=1}^{[p]} F(f(\cdot + \pi_1 (Y_s)))^{\circ k} \pi_k (X_{s,t})(Id_{L^p(U)})(1) \right\| \leq C_p \omega(s,t)^{[p]} + C_p \omega(s,t)^{\frac{|p|}{p}} \leq C_p \omega(s,t)^{[p]}.
$$

4.3 Continuity of solution in initial value, vector field and driving noise

**Proof of Theorem 23** We assume $|f_i|_{\gamma} = 1$, $i = 1, 2$. Otherwise, we replace $f^i$ and $X^i$ by $|f^i|_{\gamma}^{-1} f^i$ and $\delta_{|f^i|_{\gamma}^{-1} X^i}$, and the solution $Y^i$ will stay unchanged based on the definition of solution of RDE (in Definition 17).

Replace $\gamma$ by $\gamma \wedge ([p]+1)$, so $[p] + 1 \geq \gamma > p$.

Fix interval $[s_0, t_0]$ satisfying $\omega(s_0, t_0) \leq 1$. For $i = 1, 2$ and $[s, t] \subseteq [s_0, t_0]$, let $y^{i,s,t}: [0, 1] \rightarrow G^{[p]}(U)$ be the solution of the ODE

$$
d y^{i,s,t}_{r} = \sum_{k=1}^{[p]} F(f^i(\cdot + \pi_1 (Y^{i}_{s_0}))^{\circ k} \pi_k (X^{i}_{s,t})(Id_{L^p(U)})(1)r \in [0, 1], \quad y^{i,s,t}_0 = Y^{i}_{s_0,s}.
$$

Denote

$$
Y_{t} := Y^{1,\cdot,s,t}_{s_0,t} - Y^{2,\cdot,s,t}_{s_0,t}, \quad y^{1,\cdot,s,t}_{t} := y^{1,\cdot,s,t}_{s_0,t} - y^{2,\cdot,s,t}_{s_0,t},
$$

$$
\Gamma^{i,s,t} := Y^{i}_{s_0,s} - y^{i,s,t}_{s_0,s} - (y^{i,s,t}_{s_0,s} - y^{i,s,t}_{s_0,t}),
$$

$$
\Gamma^{s,t} := \Gamma^{1,s,t} - \Gamma^{2,s,t} = Y_{t} - y^{1,s,t}_{t}.
$$

For $s_0 \leq s \leq u \leq t \leq t_0$, let $y^{i,s,u,t}: [0, 2] \rightarrow G^{[p]}(U)$ be the solution of the ODE

$$
d y^{i,s,u,t}_{r} = \left\{ \begin{array}{ll}
\sum_{k=1}^{[p]} F(f^i(\cdot + \pi_1 (Y^{i}_{s_0}))^{\circ k} \pi_k (X^{u,s}_{s,t})(Id_{L^p(U)})(1)r \in [0, 1], & y^{i,s,u,t}_0 = Y^{i}_{s_0,s}, \\
\sum_{k=1}^{[p]} F(f^i(\cdot + \pi_1 (Y^{s}_{s_0}))^{\circ k} \pi_k (X^{u,s}_{s,t})(Id_{L^p(U)})(1)r \in [1, 2], & y^{i,s,u,t}_0 = Y^{i}_{s_0,s},
\end{array} \right.
$$

(91)
and denote \( y^{s,u,t} := y^{1,s,u,t} - y^{2,s,u,t} \).

Then, we have
\[
\| \Gamma^{s,u} + \Gamma^{u,t} - \Gamma^{s,t} \| = \| y^{s,u}_1 - y_1 - (y^{s,t}_1 - Y_s) \| \leq \| y^{u,t}_1 - Y_u - (y^{s,u,t}_1 - y^{s,u}_1) \| + \| y^{s,u,t}_2 - y^{s,u}_1 \| \tag{92}
\]

Based on Lemma 33 on p.13, we have
\[
\| y^{s,u,t}_2 - y^{s,u}_1 \| \leq C_p \left( \omega (s,t)^\gamma \| y_s \| + \| f_1 - f^2 \|_{\gamma-1} \right) + \sum_{n=1}^{[p]} \omega (s,t) \frac{2-\gamma}{2} d_{p,[s,t]}^n (X^1, X^2). \tag{93}
\]

On the other hand, since \( y^{u,t}_1 = y^{1,u,t} - y^{2,u,t} \) and \( y^{s,u,t} = y^{1,s,u,t} - y^{2,s,u,t} \), based on the definition of \( y^{i,s,u} \) (at (59)) and \( y^{i,s,u,t} \) (at (61)), we have, for \( r \in [0,1] \),
\[
\| y^{u,t}_1 - Y_u - (y^{s,u,t}_1 - y^{s,u}_1) \| \leq \sum_{k=1}^{[p]} \int_0^r \left| F (f_1) \right| \left| \pi_k (\log X^{1,u,t}_1) (I_{L_p}^{[\gamma]} (\| u \|) (y^{1,u,t}_1 - y^{s,u,t}_1) \right| 
\times \left| (f_2) \right| \left| \pi_k (\log X^{2,u,t}_1) (I_{L_p}^{[\gamma]} (\| u \|) (y^{2,u,t}_1 - y^{s,u,t}_1) \right| dv. \tag{94}
\]

Since \( \omega (s_0,t_0) \leq 1 \), we have \( \max_{i=1,2} \sup_{r \in [0,1]} \| y^{i,u,t}_r \| \leq \sup_{r \in [0,1]} \| y^{i,s,u,t}_r \| \leq C_p \). Then by using Lemma 33 on p.12 and estimating the two cases \( k = 1, \ldots, [p] - 1 \) and \( k = [p] \) separately, we have
\[
\| y^{u,t}_r - Y_u - (y^{s,u,t}_r - y^{s,u}_1) \| \leq C_p \left( \omega (s,t)^\gamma \| y_s \| + \| f_1 - f^2 \|_{\gamma-1} \right) + \sum_{n=1}^{[p]} \omega (s,t) \frac{2-\gamma}{2} d_{p,[s,t]}^n (X^1, X^2). \tag{95}
\]

Based on Lemma 32 on p.12 (continuity of ODE solutions in initial value) and Theorem 19 on p.9 (difference between RDE solution and ODE solution), we have, for \( i = 1,2 \),
\[
\sup_{r \in [0,1]} \| y^{i,u,t}_r - y^{i,s,u}_r \| \leq C_p \left( \| Y_u \| + \omega (s,t)^\gamma \| f_1 - f^2 \|_{\gamma-1} + \sum_{n=1}^{[p]} \omega (s,t) \frac{2-\gamma}{2} d_{p,[s,t]}^n (X^1, X^2) \right). \tag{96}
\]

Then combining (94), (95) and (96), we get
\[
\| y^{u,t}_r - Y_u - (y^{s,u,t}_r - y^{s,u}_1) \| \leq C_p \left( \omega (s,t)^\gamma \| y_s \| + \| f_1 - f^2 \|_{\gamma-1} \right) + C_p \sum_{n=1}^{[p]} \omega (s,t) \frac{2-\gamma}{2} d_{p,[s,t]}^n (X^1, X^2). \tag{97}
\]
Then by using Gronwall’s inequality, we have

\[
\sup_{r \in [0,1]} \|y^{u,t}_r - y^{s,u}_r\| \\
\leq C_p \left(\|Y_u - y^{s,u}_1\| + \omega(s,t)\frac{3}{2} \left(\|Y_u\| + |f_1 - f_2|_{\gamma - 1}\right) + \sum_{n=1}^{[p]} \omega(s,t) \frac{2-n}{p} \sup_{[p]} \|X^n\| \right),
\]

and combined with (97), we have \((\Gamma^{s,u} := Y_u - y^{s,u}_1)\)

\[
\|y^{u,t}_1 - Y_u - (y^{s,u}_2 - y^{s,u}_1)\| \\
\leq C_p \left(\omega(s,t)\frac{3}{2} \|\Gamma^{s,u}\| + \omega(s,t)\frac{3}{2} \left(\|Y_u\| + |f_1 - f_2|_{\gamma - 1}\right) + \sum_{n=1}^{[p]} \omega(s,t) \frac{2-n}{p} \sup_{[p]} \|X^n\| \right). 
\] (98)

As a result, combining (92), (93) and (98), we have

\[
\|\Gamma^{s,u} + \Gamma^{u,t} - \Gamma^{s,t}\| \\
\leq C_p \left(\omega(s,t)\frac{3}{2} \|\Gamma^{s,u}\| + \omega(s,t)\frac{3}{2} \left(\sup_{r \in [s,t]} \|Y_r\| + |f_1 - f_2|_{\gamma - 1}\right) + \sum_{n=1}^{[p]} \omega(s,t) \frac{2-n}{p} \sup_{[p]} \|X^n\| \right). 
\]

Therefore, if we denote

\[
\tilde{\omega}(s,t) := \omega(s,t) \left(\sup_{r \in [s,t]} \|Y_r\| + |f_1 - f_2|_{\gamma - 1}\right)^{\frac{3}{2}} + \sum_{n=1}^{[p]} \omega(s,t) \frac{2-n}{p} \left(\sup_{[p]} \|X^n\| \right)^{\frac{3}{2}},
\]

then \(\tilde{\omega}\) is a control, i.e. \(\tilde{\omega}\) is continuous, vanishes on the diagonal and

\[
\tilde{\omega}(s,u) + \tilde{\omega}(u,t) \leq \tilde{\omega}(s,t), \forall s \leq u \leq t.
\]

Indeed, since

\[
d^n_{p,[s,t]} (X^1, X^2) \tilde{\omega} = \sup_{D \subset [s,t]} \sum_{j: j \in D} \left\|\pi_n \left(X^{j,t}_{j+1}\right) - \pi_n \left(X^{j+1}_{j+1}\right)\right\|^{\tilde{\omega}},
\]

is a control, and \(\frac{2-n}{\gamma} + \frac{3}{2} = 1\), by using Hölder inequality (or based on (iii) in Exer. 1.9 [14]),

\[
\omega(s,t) \frac{2-n}{p} \left(\sup_{[p]} \|X^n\| \right)^{\frac{3}{2}}
\]

is another control.

Then we have

\[
\|\Gamma^{s,u} + \Gamma^{u,t} - \Gamma^{s,t}\| \leq C_\omega \omega(s,t) \tilde{\omega} \left(\|\Gamma^{s,u}\| + C_\omega \tilde{\omega}(s,t)\right)\tilde{\omega}.
\]

Since \(\omega\) and \(\tilde{\omega}\) are two different controls, we select the dyadic partition in such a way that, for some integers

\[
1 \leq m \leq M \text{ satisfying } 1 > \frac{m}{M} > \frac{p}{\gamma},
\]

we let

\[
I_0^0 := [s,t] \text{ and } I_k^n = I_{2k}^n \cup I_{2k+1}^n \text{ where }
\]

\[
\tilde{\omega}(I_{2k}^n) = \frac{1}{2} \tilde{\omega}(I_k^n) \text{ for } k = 0, 1, \ldots, 2^n - 1 \text{ and } n = lM + s, s = 1, \ldots, M, l \in \mathbb{N},
\]

\[
\omega(I_{2k}^n) = \frac{1}{2} \omega(I_k^n) \text{ for } k = 0, 1, \ldots, 2^n - 1 \text{ and } n = lM + s, s = m + 1, \ldots, M, l \in \mathbb{N}.
\]

Therefore, we have

\[
\sup_{k=0,\ldots,2^n-1} \tilde{\omega}(I_k^n) \leq 2^{-\frac{m}{2^n}} \tilde{\omega}(s,t) \leq 2^{-\frac{m}{2^n}} \omega(s,t) \leq 2^{-\frac{m}{2^n}} \omega(s,t),
\]

\[
\sup_{k=0,\ldots,2^n-1} \omega(I_k^n) \leq 2^{-\frac{m}{2^n}} \omega(s,t) \leq 2^{-\frac{m}{2^n}} \omega(s,t).
\]
By recursively bisecting \([s, t]\) in this way, we have,

\[
\|\Gamma^{s,t}\| \leq \lim_{n \to \infty} \sum_{k=0}^{n} \left( \prod_{j=0}^{k} \left( 1 + \max_{i=0,1,\ldots,2^{j}-1} \omega \left( I_{i}^{j} \right)^{\frac{1}{p}} \right) \left( \sum_{j=0}^{2^{n}-1} \left\| \Gamma_{j,t}^{s,n} \right\| \right) \right).
\]

Then since \(1 > \frac{n}{m} > \frac{p}{\gamma}\) and \(\|\Gamma_{j,t}^{s,n} + 1\| \leq C_{p, \omega} \left( t_{n}^{s,t}, t_{n}^{s,t} + 1 \right)^{\frac{|j+1|}{p}}\) in Theorem 13 on \(\Phi\), we have

\[
\|\Gamma^{s,t}\| \leq \exp \left( \frac{2^{\frac{1}{p}}}{1 - 2^{-\frac{2n}{m^{p}p}}} \omega \left( s, t \right)^{\frac{1}{p}} \right) \frac{2^{\frac{1}{p}}}{1 - 2^{-\frac{2n}{m^{p}p}}} \omega \left( s, t \right)^{\frac{1}{p}}
\]

As a result, for any \([s, t] \subseteq [s_{0}, t_{0}]\), \(\omega \left( s_{0}, t_{0} \right) \leq 1\),

\[
\|Y_{t} - y_{1}^{s,t}\| \leq C_{p, \gamma} \omega \left( s, t \right)^{\frac{1}{p}}.
\]

In particular, if we let \([s, t] = [s_{0}, t_{0}]\),

then \(y_{1}^{s,t}\) is the solution of the ODE

\[
dy_{r}^{s,t} = \sum_{k=1}^{[p]} F \left( f^{t} \cdot + \pi_{1} \left( Y_{s, t}^{1} \right) \right)^{\frac{1}{p}} \pi_{k} \left( \log \left( X_{s, t} \right) \right) \left( Id_{L^{p}[\Omega]} \right) \left( y_{r}^{s,t} \right) \, dr, \ r \in [0, 1], \ y_{0}^{s,t} = 1.
\]

Based on Lemma 22 on \(\Phi\) for \(k = 1, 2, \ldots, [p]\),

\[
\left\| \pi_{k} \left( y_{r}^{s,t} \right) \right\| = \left\| \pi_{k} \left( y_{r}^{1,s,t} \right) - \pi_{k} \left( y_{1}^{s,t} \right) \right\|
\]

\[
\leq C_{p} \left( \omega \left( s, t \right)^{\frac{1}{p}} \left( 1 - f_{1}^{2} \right) + \omega \left( Y_{s}^{1} \right) - \pi_{1} \left( Y_{s}^{2} \right) \right) + \sum_{n=1}^{[p]} \omega \left( s, t \right)^{\frac{1}{p}} d_{p,[s,t]} \left( X^{1}, X^{2} \right).
\]

Combined with (100), we get \((Y_{t} = Y_{s,t}^{1} - Y_{s,t}^{2})\)

\[
\left\| \pi_{k} \left( Y_{s,t}^{1} \right) - \pi_{k} \left( Y_{s,t}^{2} \right) \right\| \leq C_{p, \gamma} \omega \left( s, t \right)^{\frac{1}{p}} \left( 1 - f_{1}^{2} \right) + \omega \left( Y_{s}^{1} \right) - \pi_{1} \left( Y_{s}^{2} \right) \right) + C_{p, \gamma} \left( \sum_{n=1}^{[p]} \omega \left( s, t \right)^{\frac{1}{p}} d_{p,[s,t]} \left( X^{1}, X^{2} \right) + \omega \left( s, t \right)^{\frac{1}{p}} \left( \sup_{u \in [s,t]} \left\| Y_{s, u}^{1} - Y_{s, u}^{2} \right\| \right) \right).
\]

Recall control \(\omega \left( s, t \right) := \left\| f_{\gamma}^{1} \right\|_{[P, X_{p, \vartheta}, [s, t]} + \left\| f_{\gamma}^{2} \right\|_{[P, X_{p, \vartheta}, [s, t]} \). For \(\alpha \in (0, 1]\), denote \(\omega_{\alpha} \left( s, t \right)\) as at (13) on \(\Phi\) and denote \(d_{p,[s,t]}^{p, \omega_{\alpha}} \left( X^{1}, X^{2} \right)\) as at (13) on \(\Phi\). Then based on (101), by using sub-additivity of control, we
have $(\omega^\alpha(s,t) \leq \omega(s,t) \leq 1)$

$$\left\| \pi_k \left( Y_{s,t}^1 \right) - \pi_k \left( Y_{s,t}^2 \right) \right\| \leq d_{p,|s,t|}^k (Y^1, Y^2)$$

(102)

$$\leq C_{p,\gamma} \omega^\alpha(s,t)^{\frac{\beta}{p}} \left( |f^1 - f^2|_{\gamma^{-1}} + \sup_{u \in [s,t]} \left| \pi_1 \left( Y_u^1 \right) - \pi_1 \left( Y_u^2 \right) \right| \right) + C_{p,\gamma} \left( \sum_{n=1}^{|p|} \omega^\alpha(s,t)^{\frac{(k-n)\alpha}{p}} d_{p,|s,t|}^{n,\alpha} (X^1, X^2) \right)$$

$$= C_{p,\gamma} \omega^\alpha(s,t)^{\frac{\beta}{p}} \left( |f^1 - f^2|_{\gamma^{-1}} + \sup_{u \in [s,t]} \left| \pi_1 \left( Y_u^1 \right) - \pi_1 \left( Y_u^2 \right) \right| \right) + C_{p,\gamma} \left( \sum_{n=1}^{|p|} \omega^\alpha(s,t)^{\frac{(k-n)\alpha}{p}} d_{p,|s,t|}^{n,\alpha} (X^1, X^2) \right).$$

As a result,

$$\sup_{u \in [s,t]} \left| Y_{s,u}^1 - Y_{s,u}^2 \right| \leq C_{p,\gamma} \omega^\alpha(s,t)^{\frac{\beta}{p}} \left( |f^1 - f^2|_{\gamma^{-1}} + \sup_{u \in [s,t]} \left| \pi_1 \left( Y_u^1 \right) - \pi_1 \left( Y_u^2 \right) \right| \right) + C_{p,\gamma} \left( \sum_{n=1}^{|p|} \omega^\alpha(s,t)^{\frac{(k-n)\alpha}{p}} d_{p,|s,t|}^{n,\alpha} (X^1, X^2) \right),$$

and combined with (102), we have, for any $[s,t]$ satisfying $\omega(s,t) \leq \delta_{p,\gamma}$ and any $\alpha \in (0, \delta_{p,\gamma}]$,

$$\left\| \pi_k \left( Y_{s,t}^1 \right) - \pi_k \left( Y_{s,t}^2 \right) \right\| \leq C_{p,\gamma} \omega^\alpha(s,t)^{\frac{\beta}{p}} \left( |f^1 - f^2|_{\gamma^{-1}} + \sup_{u \in [s,t]} \left| \pi_1 \left( Y_u^1 \right) - \pi_1 \left( Y_u^2 \right) \right| \right) + C_{p,\gamma} \left( \sum_{n=1}^{|p|} \omega^\alpha(s,t)^{\frac{(k-n)\alpha}{p}} d_{p,|s,t|}^{n,\alpha} (X^1, X^2) \right).$$

For $[s_0, t_0]$ satisfying $\omega(s_0, t_0) \geq \delta_{p,\gamma}$ and $\alpha \in (0, \delta_{p,\gamma}]$, we decompose $[s_0, t_0] = \cup_{j=0}^{m-1} [t_j, t_{j+1})$, $m \geq 2$, with $\omega(t_j, t_{j+1}) = \alpha$, $j = 0, 1, \ldots, m-2$, and $\omega(t_{m-1}, t_m) \leq \alpha$. Then $(m-1)\alpha = \sum_{j=0}^{m-2} \omega(t_j, t_{j+1})$, so

$$m = \alpha^{-1} \left( \sum_{j=0}^{m-2} \omega(t_j, t_{j+1}) \right) + 1 \leq 2\alpha^{-1} \left( \sum_{j=0}^{m-1} \omega(t_j, t_{j+1}) \right) \leq 2\alpha^{-1} \omega^\alpha(s,t).$$

(104)

Firstly, we estimate $\sum_{j=0}^{m-1} \left\| \pi_k \left( Y_{t_j,t_{j+1}}^1 \right) - \pi_k \left( Y_{t_j,t_{j+1}}^2 \right) \right\|$ for $k = 1, 2, \ldots, |p|$. By using (103) and denote $\gamma_k := \alpha^{\frac{k-1}{p}}$, $\beta := C_{p,\gamma} \alpha^{\frac{\beta}{p}}$ and $\lambda_k := C_{p,\gamma} \alpha^{\frac{(k-n)\alpha}{p}}$, we have, $(\omega^\alpha(t_j, t_{j+1}) = \omega(t_j, t_{j+1}) \leq \alpha, \forall j)$

$$\sum_{j=0}^{m-1} \left\| \pi_k \left( Y_{t_j,t_{j+1}}^1 \right) - \pi_k \left( Y_{t_j,t_{j+1}}^2 \right) \right\| \leq \alpha^{\frac{k-1}{p}} \sum_{j=0}^{m-1} C_{p,\gamma} \alpha^{\frac{\beta}{p}} \left( |f^1 - f^2|_{\gamma^{-1}} + \sup_{u \in [s,t]} \left| \pi_1 \left( Y_u^1 \right) - \pi_1 \left( Y_u^2 \right) \right| \right) + \sum_{j=0}^{m-1} \sum_{n=1}^{|p|} C_{p,\gamma} \alpha^{\frac{(k-n)\alpha}{p}} d_{p,|s,t|}^{n,\alpha} (X^1, X^2)$$

$$= \gamma_k \sum_{j=0}^{m-1} \beta \left( |f^1 - f^2|_{\gamma^{-1}} + \sup_{u \in [s,t]} \left| \pi_1 \left( Y_u^1 \right) - \pi_1 \left( Y_u^2 \right) \right| \right) + \sum_{j=0}^{m-1} \sum_{n=1}^{|p|} \lambda_k d_{p,|s,t|}^{n,\alpha} (X^1, X^2).$$
Then by repeatedly using (103), we have

\[ \sum_{j=0}^{m-1} \pi_k \left( Y_{t_j, t_{j+1}}^1 \right) - \pi_1 \left( Y_{t_j, t_{j+1}}^2 \right) \leq \gamma_k \left( m \beta \left| \sum_{j=0}^{m-1} p_{[t_j, t_{j+1}]} \right| + \sum_{j=0}^{m-1} \sum_{n=1}^{[p]} \lambda_{k,n} d_{p_{[t_j, t_{j+1}]}}^{m,n} \left( X^1, X^2 \right) \right) \]

+ \gamma_k \sum_{j=0}^{m-2} \left( \pi_1 \left( Y_{t_j}^1 \right) - \pi_1 \left( Y_{t_j}^2 \right) \right) + \beta \left( \sum_{j=0}^{m-1} \sum_{n=1}^{[p]} \lambda_{k,n} d_{p_{[t_j, t_{j+1}]}}^{m,n} \left( X^1, X^2 \right) \right)

\[ \leq \cdots \leq \gamma_k \left( m \beta + (m-1) \beta^2 + \cdots + \beta^{m-2} \right) \left( \sum_{j=0}^{m-1} \sum_{n=1}^{[p]} \lambda_{k,n} d_{p_{[t_j, t_{j+1}]}}^{m,n} \right) \left( X^1, X^2 \right) \]

When \( p \) is not an integer, for \( n = 1, \ldots, [p] \), by using Hölder inequality,

\[ \sum_{j=0}^{m-1} \left( 1 + \beta \right)^{m-1-j} d_{p_{[t_j, t_{j+1}]}}^{m,n} \left( X^1, X^2 \right) \leq \left( \sum_{j=0}^{m-1} \left( 1 + \beta \right)^{m-1-j} \right) \left( \sum_{j=0}^{m-1} d_{p_{[t_j, t_{j+1}]}}^{m,n} \left( X^1, X^2 \right) \right)^{p} \]

\[ \leq C_p \left( \frac{1 + \beta}{\beta} \right)^{\frac{m}{p}} \left( \frac{m^{p}}{p} \right)^{\frac{1}{p}} \left( \sum_{j=0}^{m-1} d_{p_{[t_j, t_{j+1}]}}^{m,n} \left( X^1, X^2 \right) \right). \]

When \( p \) is an integer, for integer \( n < p \), (105) holds; for \( n = p \),

\[ \sum_{j=0}^{m-1} \left( 1 + \beta \right)^{m-1-j} d_{p_{[t_j, t_{j+1}]}}^{m,n} \left( X^1, X^2 \right) \leq \left( \frac{1 + \beta}{\beta} \right)^{m-1} \left( d_{p_{[t_j, t_{j+1}]}}^{m,n} \left( X^1, X^2 \right) \right). \]

Then by using \( (1 + \beta)^m - 1 \leq m (1 + \beta)^{m-1} \beta \), we have \( (1, \beta) \) is strictly less than \( p \)

\[ \sum_{j=0}^{m-1} \pi_k \left( Y_{t_j, t_{j+1}}^1 \right) - \pi_1 \left( Y_{t_j, t_{j+1}}^2 \right) \leq C_p \left( \frac{1 + \beta}{\beta} \right)^{\frac{m}{p}} \left( \frac{m^{p}}{p} \right)^{\frac{1}{p}} \left( \sum_{j=0}^{m-1} d_{p_{[t_j, t_{j+1}]}}^{m,n} \left( X^1, X^2 \right) \right). \]

Combined with (104), we have \( (\gamma_k := \alpha \frac{k-1}{p}, \beta := C_{p, \gamma} \alpha^{\frac{1}{p}}, \lambda_{k,n} := C_{p, \gamma} \alpha^{\frac{(k-n)q}{p}}, \alpha \in (0, \delta_{p, \gamma}] with \delta_{p, \gamma} \leq 1) \)

\[ \sum_{j=0}^{m-1} \pi_k \left( Y_{t_j, t_{j+1}}^1 \right) - \pi_1 \left( Y_{t_j, t_{j+1}}^2 \right) \leq C_{p, \gamma} \exp \left( \frac{C_{p, \gamma} \alpha^{-1} \omega_{p} (s, t)}{\beta} \right) \left( \frac{1 + \beta}{\beta} \right)^{\frac{m}{p}} \left( \frac{m^{p}}{p} \right)^{\frac{1}{p}} \left( \sum_{j=0}^{m-1} d_{p_{[t_j, t_{j+1}]}}^{m,n} \left( X^1, X^2 \right) \right). \]
Then, it can be checked that,

$$
\| \pi_k (Y_{s,t}^1) - \pi_k (Y_{s,t}^2) \| \\
\leq \sum_{j=0}^{m-1} \| \pi_k (Y_{t_j,t_{j+1}}^1) - \pi_k (Y_{t_j,t_{j+1}}^2) \| \\
+ \sum_{i_0 + \ldots + i_{m-1} = k, \ i_s = 0, 1, \ldots, m-1} \| \pi_{i_0} (Y_{s,t_1}^1) \otimes \ldots \otimes \pi_{i_{m-1}} (Y_{t_{m-1},t_m}^1) - \pi_{i_0} (Y_{s,t_1}^2) \otimes \ldots \otimes \pi_{i_{m-1}} (Y_{t_{m-1},t_m}^2) \|. 
$$

Based on (10) in Theorem 19 on p6, for $j = 0, 1, \ldots, m-1$ and $s = 1, 2, \ldots, [p]$, $\| \pi_s (Y_{t_j,t_{j+1}}^i) \| \leq C_{p} \omega (t_j, t_{j+1})^{(k-1)/p}$. Then by replacing $\omega$ by $C_{p} \omega$, we have

$$
\| \pi_s (Y_{t_j,t_{j+1}}^1) \| \leq \frac{\omega (t_j, t_{j+1})^{(k-1)/p}}{(s/p)!}, \ s = 1, \ldots, [p], \ j = 0, 1, \ldots, m-1, \ i = 1, 2.
$$

Then

$$
\sum_{i_0 + \ldots + i_{m-1} = k, \ i_s \leq k-1} \| \pi_{i_0} (Y_{s,t_1}^1) \otimes \ldots \otimes \pi_{i_{m-1}} (Y_{t_{m-1},t_m}^1) - \pi_{i_0} (Y_{s,t_1}^2) \otimes \ldots \otimes \pi_{i_{m-1}} (Y_{t_{m-1},t_m}^2) \| \\
\leq \sum_{j=0}^{m-1} \sum_{s=1}^{[p]} \| \pi_s (Y_{t_j,t_{j+1}}^1) - \pi_s (Y_{t_j,t_{j+1}}^2) \| \left( \sum_{i \neq j} \omega (t_i, t_{i+1})^{(k-1)/p} \left( \frac{s}{p} \right)! \right).
$$

By using neo-classical inequality (Exer 3.9 [23], proved by Hara and Hino [17]): for $x_1, x_2, \ldots, x_m \geq 0,

$$
\sum_{s_1 + \ldots + s_m = k, s_i \geq 0} \frac{x_1^{s_1}}{(s_1/p)!} \ldots \frac{x_m^{s_m}}{(s_m/p)!} \leq p^{2m-2} (x_1 + \ldots + x_m)^{k/p},
$$

we have

$$
\sum_{i_0 + \ldots + i_{m-1} = k, \ i_s \leq k-1} \| \pi_{i_0} (Y_{s,t_1}^1) \otimes \ldots \otimes \pi_{i_{m-1}} (Y_{t_{m-1},t_m}^1) - \pi_{i_0} (Y_{s,t_1}^2) \otimes \ldots \otimes \pi_{i_{m-1}} (Y_{t_{m-1},t_m}^2) \| \\
\leq C_{p} p^{2m} \sum_{s=1}^{k-1} \left( \omega (s,t)^{k-s} \left( \sum_{j=0}^{m-1} \| \pi_s (Y_{t_j,t_{j+1}}^1) - \pi_s (Y_{t_j,t_{j+1}}^2) \| \right) \right).
$$

Then, by using (104) and (106), we have

$$
\| \pi_k (Y_{s,t}^1) - \pi_k (Y_{s,t}^2) \| \\
\leq C_{p} p^{2m} \sum_{s=1}^{k} \left( \omega (s,t)^{k-s} \left( \sum_{j=0}^{m-1} \| \pi_s (Y_{t_j,t_{j+1}}^1) - \pi_s (Y_{t_j,t_{j+1}}^2) \| \right) \right) \\
\leq C_{p,\gamma} \exp \left( C_{p,\gamma} \alpha^{-1} \omega (s,t) \right) \left( \alpha^{(k-1)/p} \left( \| f^1 - f^2 \|_{\gamma-1} + \| \pi_1 (Y_{s_1}^1) - \pi_1 (Y_{s_1}^2) \| \right) + \sum_{n=1}^{[p]} \omega (s,t)^{(k-n)\alpha/p} \left( \| X^1 - X^2 \|_{p,s,t} \right) \right).
$$

Thus, for any $\alpha \in (0, \delta_{p,\gamma}]$ and any $[s, t]$ satisfying $\omega (s,t) > \alpha$, we have

$$
\| \pi_k (Y_{s,t}^1) - \pi_k (Y_{s,t}^2) \| \\
\leq C_{p,\gamma} \exp \left( C_{p,\gamma} \alpha^{-1} \omega (s,t) \right) \left( \alpha^{(k-1)/p} \left( \| f^1 - f^2 \|_{\gamma-1} + \| \pi_1 (Y_{s_1}^1) - \pi_1 (Y_{s_1}^2) \| \right) + \sum_{n=1}^{[p]} \omega (s,t)^{(k-n)\alpha/p} \left( \| X^1 - X^2 \|_{p,s,t} \right) \right),
$$

(107)
On the other hand, based on (103), (107) also holds when $\omega(\alpha, s, t) \leq \alpha$ for $\alpha \in (0, \delta_{p, \gamma})$. To extend (107) to all $\alpha \in (0, 1]$, let $\alpha = \delta_{p, \gamma}$ in (107) and using $\omega(s, u) + \omega(u, t) \leq \omega(s, t)$, $\forall s \leq u \leq t$, we have, for $[s, t]$ satisfying $\omega(s, t) \leq 1$,

$$
\left\| \pi_k \left( Y_{s,t}^1 \right) - \pi_k \left( Y_{s,t}^2 \right) \right\| \\
\leq C_{p, \gamma} \left( \omega(s, t) \frac{k}{p} \left( \| f^1 - f^2 \|_{\gamma-1} + \| \pi_1 \left( Y_{s,t}^1 \right) - \pi_1 \left( Y_{s,t}^2 \right) \| \right) \right) + \sum_{n=1}^{[p]} \omega(s, t) \left( \frac{k}{(k-n)\gamma n} \right) d_{p, [s,t]}^n \left( X^1, X^2 \right).
$$

Then by using definition of control $\omega(\alpha)$, it can be proved that (103) holds for any $[s, t]$ satisfying $\omega(s, t) \leq 1$ and for any $\alpha \in (0, 1]$. Then by following the same argument after (103), we have (107) holds for any $\alpha \in (0, 1]$ and any $[s, t] \subseteq [0, T]$. By using sub-additivity of a control, (107) holds with $\left\| \pi_k \left( Y_{s,t}^1 \right) - \pi_k \left( Y_{s,t}^2 \right) \right\|$ replaced by $d_{p, [s,t]}^k \left( Y^1, Y^2 \right)$. \hfill \blacksquare

**Proof of Corollary 24.** Fix $D = \{ t_j \}_{j=0}^n \subset [0, T]$. That $y^D$ takes value in $G^{[p]}(\mathcal{U})$ follows from Lemma 29 on p.9. We assume $\xi = 1$. Otherwise, we replace $Y$ and $y^D$ by $\xi^{-1} \odot Y$ and $\xi^{-1} \odot y^D$, and replace $f$ by $f(\cdot + \pi_1(\xi))$.

For $j = 0, 1, \ldots, n$, let $Z^j : [t_j, T] \to G^{[p]}(\mathcal{V})$ be the solution to the RDE

$$
dZ^j = f \left( Z^j \right) dX, \quad Z^j_{t_j} = y^D_{t_j}.
$$

Then $Z^j_0 = Y_{t_j}, Z^j_i = y^D_{t_j}$, and

$$
Y_{t_j} - y^D_{t_j} = \sum_{i=0}^{j-1} \left( Z^i_{t_j} - Z^i_{t_j+1} \right) = \sum_{i=0}^{j-1} \left( Z^i_{t_{i+1}, t_j} - Z^i_{t_{i+1}, t_j+1} \right) = \sum_{i=0}^{j-1} \left( Z^i_{t_{i+1}, t_j} - Z^i_{t_{i+1}, t_j+1} \right) + y^D_{t_{i+1}, t_j} = \sum_{i=0}^{j-1} \left( Z^i_{t_{i+1}, t_j} - Z^i_{t_{i+1}, t_j+1} \right) + y^D_{t_{i+1}, t_j}.
$$

Based on (12) in Theorem 19 on p.8 (difference between RDE solution and ODE solution), we have

$$
\left\| Z^i_{t_{i+1}, t_j} - y^D_{t_{i+1}, t_j} \right\| \leq C_p \left( \omega(t_i, t_{i+1}) \odot \omega(t_i, t_{i+1}) \right)^{[p]}, \tag{109}
$$

Based again on Theorem 19 for the bound on RDE solution (by decomposing big interval as the union of small intervals, similar as at [SS] on p.20, for any $\alpha \in (0, 1]$, we have

$$
\left\| Z^i_{t_{i+1}, t_j} \right\| \leq C_p \omega(0, T)^{\frac{j}{p}} \odot \left( \alpha^{-1} \omega(0, T) \right)^{[p]}.
$$

According to Theorem 24 on p.10 (continuous dependence of RDE solution on initial value) and (109),

$$
\left\| Z^i_{t_{i+1}, t_j} - Z^i_{t_{i+1}, t_j+1} \right\| \leq C_{p, \gamma} \exp \left( C_{p, \gamma} \alpha^{-1} \omega(0, T) \right) \left\| \pi_1 \left( Z^i_{t_{i+1}} \right) - \pi_1 \left( Z^i_{t_{i+1}, t_j} \right) \right\| \tag{111}
$$

$$
\leq C_{p, \gamma} \exp \left( C_{p, \gamma} \alpha^{-1} \omega(0, T) \right) \left\| \pi_1 \left( Z^i_{t_{i+1}, t_j} \right) - \pi_1 \left( y^D_{t_{i+1}, t_j} \right) \right\| \leq C_{p, \gamma} \exp \left( C_{p, \gamma} \alpha^{-1} \omega(0, T) \right) \left( \omega(t_i, t_{i+1}) \odot \omega(t_i, t_{i+1}) \right)^{[p]}.
$$

For $k = 1, 2, \ldots, [p]$, denote

$$
M_k := \max_{l=0,1,\ldots,k} \max_{j=0,1,\ldots,n} \left\| \pi_l \left( y^D_{t_j} \right) \right\|.
$$

Combining (108), (109), (110) and (111), we have, for $k = 1, 2, \ldots, [p]$,

$$
\max_{j=0,1,\ldots,n} \left\| \pi_k \left( Y_{t_j} \right) - \pi_k \left( y^D_{t_j} \right) \right\| \leq C_{p, \gamma} M_{k-1} \exp \left( C_{p, \gamma} \alpha^{-1} \omega(0, T) \right) \left( \sum_{j=0}^{n-1} \omega(t_j, t_{j+1}) \odot \omega(t_i, t_{i+1}) \right)^{[p]}.
$$

Then by mathematical induction, we prove

$$
M_{[p]} \leq C_{p, \gamma} \exp \left( C_{p, \gamma} \alpha^{-1} \omega(0, T) \right) \left( \sum_{j=0}^{n} \omega(t_j, t_{j+1}) \odot \omega(t_i, t_{i+1}) \right)^{[p]} .
$$

Then by mathematical induction, we prove

$$
M_{[p]} \leq C_{p, \gamma} \exp \left( C_{p, \gamma} \alpha^{-1} \omega(0, T) \right) \left( \sum_{j=0}^{n} \omega(t_j, t_{j+1}) \odot \omega(t_i, t_{i+1}) \right)^{[p]} .
$$
It is clear that $M_0 = 1$. For $k = 1, \ldots, [p]$, suppose

$$M_{k-1} \leq C_{p, \gamma} \exp \left( C_{p, \gamma} \left( \omega^{\alpha_0} (0, T) + \alpha^{-1} \omega^{\alpha} (0, T) \right) \right). \quad (114)$$

Based on similar estimates as that leads to (110), we have, for $\alpha \in (0, 1]$ and $k = 1, 2, \ldots, [p]$, $(\xi = 1)$

$$\max_{j=0,1,\ldots,n} \left\| \pi_k (Y_{j, t}) \right\| \leq C_{p} \left( \omega^{\alpha} (0, T) \right)^{\frac{p}{k}} \vee (\alpha^{-1} \omega^{\alpha} (0, T))^k, \quad (115)$$

Then by combining (112) with (114), we have, with $\alpha_0 := \max_{t_j \in D} \omega (t_j, t_{j+1})$, $(M_{k-1} \geq M_0 = 1)$

$$\max_{j=0,1,\ldots,n} \left\| \pi_k \left( y_{j, t}^D \right) \right\| \leq C_{p, \gamma} M_{k-1} \exp \left( C_{p, \gamma} \alpha^{-1} \omega^{\alpha} (0, T) \right) \left( 1 + \sum_{j=0}^{n-1} \omega (t_j, t_{j+1}) \frac{\lvert i+j+1 \rvert}{p} \vee \omega (t_j, t_{j+1}) \right)$$

$$\leq C_{p, \gamma} \exp \left( C_{p, \gamma} \left( \omega^{\alpha_0} (0, T) + \alpha^{-1} \omega^{\alpha} (0, T) \right) \right) \prod_{j=0}^{n-1} \exp \left( C_{p} \omega (t_j, t_{j+1}) \right)$$

$$\leq C_{p, \gamma} \exp \left( C_{p, \gamma} \left( \omega^{\alpha_0} (0, T) + \alpha^{-1} \omega^{\alpha} (0, T) \right) \right).$$

Then

$$M_k \leq M_{k-1} + \max_{j=0,1,\ldots,n} \left\| \pi_k \left( y_{j, t}^D \right) \right\| \leq C_{p, \gamma} \exp \left( C_{p, \gamma} \left( \omega^{\alpha_0} (0, T) + \alpha^{-1} \omega^{\alpha} (0, T) \right) \right).$$

Combining (112) with (113), we have

$$\left\| Y_T - y_T^D \right\| \leq C_{p, \gamma} \exp \left( C_{p, \gamma} \left( \omega^{\alpha_0} (0, T) + \alpha^{-1} \omega^{\alpha} (0, T) \right) \right) \left( \sum_{j=0}^{n-1} \omega (t_j, t_{j+1}) \frac{\lvert i+j+1 \rvert}{p} \vee \omega (t_i, t_{i+1}) \right).$$

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