1. Introduction

Anyons are two-dimensional particles with arbitrary statistics interpolating between bosons and fermions [1,2] (for reviews see for example [3,4] and references therein). In the past few years they have attracted considerable interest especially in connection with the interpretation of certain condensed matter phenomena, most notably the fractional quantum Hall effect [5]. Since the first seminal papers on the subject [1,6], it was clear that anyons are deeply connected with the braid group of which they are abelian representations, just like bosons and fermions are abelian representations of the permutation group. The appearance of the braid group in place of the permutation group is a peculiar feature of two dimensional systems; in fact as is well known, in three or more dimensions only bosons and fermions can exist so that the manybody wavefunctions are either symmetric or antisymmetric in the exchange of any two identical particles. In two dimensions instead, when one exchanges two identical particles, it is no longer enough to compare their final ordering to the original one (i.e. to specify their permutation) but it is necessary to specify also how the exchanging trajectories of the two particles wind or braid around each other. These braiding properties make the quantum mechanics of anyons extremely difficult.

One possibility to shed some light on this problem could be to explore and study the characteristic symmetries of anyon systems. The natural candidates for these appear to be the quantum groups \(^1\), which are deformations of ordinary Lie algebras [7-9]. One of the fundamental features of quantum groups is that their centralizer is the braid group, just like the permutation group is the centralizer of the ordinary Lie algebras. In other words quantum groups are endowed with a comultiplication \(^2\) which is not cocommutative but involves suitable braiding factors [7-13].

The fundamental role played by the braid group both in the theory of quantum groups and in the theory of anyons suggests that a deep relation between these two subjects might exist. In this paper we show that this is indeed the case. It is well known that bosonic or fermionic oscillators, characterized by commutative or anti-commutative Heisenberg algebras, can be combined à la Schwinger to construct non-abelian Lie algebras with the permutation group as their centralizer [14]. Similarly one can think of using anyonic oscillators with braid group properties and \(q\)-deformed commutation relations to build non-abelian Lie algebras with the braid group as their centralizer, i.e. to construct non-abelian quantum groups. So far, this connection between anyons and quantum groups has not been investigated in the literature despite intensive studies in both fields. In this paper we elaborate on this idea and show how to construct quantum groups from anyons.

Actually, oscillators with \(q\)-deformed commutation relations have been intro-

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\(^1\) Here and in the following, we adopt the commonly used terminology. However more properly, one should speak of quantum enveloping algebras.

\(^2\) The comultiplication is the operation which is used to make tensor products of representations.
duced a few years ago and are known as \( q \)-oscillators [15-18]. Later, the standard bosonic and fermionic Schwinger constructions for \( SU(2) \) have been generalized to \( q \)-oscillators yielding the quantum group \( SU(2)_q \). We would like to stress that despite the many formal analogies, our anyonic construction does not have anything to do with that of the \( q \)-oscillators. This is so for several reasons. First of all, the \( q \)-oscillators can be defined in any dimensions and are not related to the braid group, whereas anyons are strictly two dimensional objects. Secondly, the \( q \)-oscillators are local operators, whilst anyons are intrinsically non-local due to their braiding properties [19]. This non-locality, which is essential to distinguish whether anyons are exchanged clockwise or anticlockwise, allows also to define a natural ordering among the particles, which in turn is essential to define a non-cocommutative comultiplication like the one of quantum groups.

This paper is organized as follows. In section 2, we review the standard bosonic and fermionic Schwinger construction of \( SU(2) \) and briefly discuss its \( q \)-oscillator extension. In section 3, we introduce anyonic oscillators of statistics \( \nu \) on a two dimensional square lattice by means of the Jordan-Wigner construction [20], and extensively discuss their generalized commutation relations. The choice of working on a lattice is dictated by the need of having a discrete set of particles so that later we will be able to define a discrete comultiplication with no ambiguities. However, to have simple deformed commutation relations for the anyonic oscillators, it is convenient to take a sort of a continuum limit that can be realized by embedding the original lattice into one with an infinitesimal spacing. In section 4, we use the anyonic oscillators of statistics \( \nu \) to construct the generators of the quantum group \( SU(2)_q \) where the deformation parameter is \( q = \exp(i\pi\nu) \). Finally, in section 5 we present our conclusions.

2. Bosonic and Fermionic Constructions of \( SU(2) \)

It is very well known that the \( SU(2) \) algebra can be explicitly realized in several ways. One of the simplest of these is the Schwinger construction [14]. Given a pair of bosonic harmonic oscillators such that

\[
\left[ a_i , a_j^\dagger \right] = \delta_{ij}
\]  

(2.1)

for \( i, j = 1, 2 \), the three generators of \( SU(2) \) are realized as bilinears in \( a \) and \( a^\dagger \).
according to

\[ j^+ = a_1^\dagger a_2 , \]
\[ j^0 = \frac{1}{2} \left( a_1^\dagger a_1 - a_2^\dagger a_2 \right) , \]
\[ j^- = a_2^\dagger a_1 . \]

Indeed, using (2.1) it follows immediately that

\[ [j^+, j^-] = 2j^0 , \]
\[ [j^0, j^\pm] = \pm j^\pm . \] (2.3)

It is interesting to point out that using a single pair of bosonic oscillators we can obtain not only an algebraic realization of SU(2) but also the full set of its (unitary) representations. To see this, let us define the (normalized) states

\[ |j, m\rangle = \frac{1}{\sqrt{(j+m)!(j-m)!}} \left( a_1^\dagger \right)^{j+m} \left( a_2^\dagger \right)^{j-m} |0\rangle \] (2.4)

where \( j \) and \( m \) are arbitrary integers or half-integers such that \( j \pm m \in \mathbb{Z}_+ \cup \{0\} \), and the state \( |0\rangle \) is the “vacuum” which satisfies

\[ a_i |0\rangle = 0 \] (2.5)

for \( i = 1, 2 \). Using (2.1), it is easy to verify that

\[ j^0 |j, m\rangle = m |j, m\rangle , \]
\[ j^\pm |j, m\rangle = \sqrt{(j \pm m)(j \pm m + 1)} |j, m \pm 1\rangle . \] (2.6)

Thus \( |j, m\rangle \) are the familiar angular momentum states in which \( j \) labels the “total” spin and \( m \) its third component. Since \( j \) can take arbitrary positive integer or half-integer values, all possible unitary representations of SU(2) are spanned by the states (2.4) on which the corresponding generators are simply given by (2.2).

The Schwinger construction can be realized also using fermions instead of bosons. In fact let us consider a pair of fermionic oscillators obeying the following anticommutation relations

\[ \{ c_i , c_j^\dagger \} = \delta_{ij} \] (2.7)

for \( i, j = 1, 2 \). Then, in analogy with (2.2) let us define

\[ j^+ = c_1^\dagger c_2 , \]
\[ j^0 = \frac{1}{2} \left( c_1^\dagger c_1 - c_2^\dagger c_2 \right) , \]
\[ j^- = c_2^\dagger c_1 . \] (2.8)
It is again straightforward to verify that these generators indeed close the $SU(2)$ algebra. However, in contrast with the bosonic case it is now impossible to recover the full set of the unitary representations of $SU(2)$. In fact, denoting by $|0\rangle$ the fermionic vacuum ($c_i |0\rangle = 0$ for $i = 1, 2$), we can construct only the following four states

$$|0\rangle, \ c_1^\dagger |0\rangle, \ c_2^\dagger |0\rangle, \ c_1^\dagger c_2^\dagger |0\rangle,$$

because the anticommutation relations prevent to put more than one fermion of each kind, and thus only the $j = 0$ and $j = 1/2$ representations are realized. To overcome this problem we must consider many copies of pairs of fermionic oscillators, or more properly we must consider a two-component Pauli spinor field

$$c(x) = \left( c_1(x) \\ c_2(x) \right)$$

where $x$ belongs to some manifold $\Omega$ to be specified later (we can think of $x$ as the label that distinguishes the different copies of the fermionic oscillators). The adjoint of $c(x)$ is given by

$$c^\dagger(x) = \left( c_1^\dagger(x) , c_2^\dagger(x) \right)$$

where $c_1^\dagger(x)$ can be interpreted as the operator creating a fermion of type 1 ("spin up") at the point $x \in \Omega$, and $c_2^\dagger(x)$ as the operator creating a fermion of type 2 ("spin down") at $x$. Moreover we assume the standard anticommutation relations, i.e.

$$\{ c_i(x), c_j^\dagger(y) \} = \delta_{ij} \delta(x,y)$$

where $\delta(x,y)$ is the delta function on $\Omega$. Then we can define the local operators

$$j^+(x) = c_1^\dagger(x)c_2(x)$$

$$j_0^0(x) = \frac{1}{2} \left( c_1^\dagger(x)c_1(x) - c_2^\dagger(x)c_2(x) \right)$$

$$j^-(x) = c_2^\dagger(x)c_1(x)$$

which obey the following commutation relations

$$[j^+(x), j^-(y)] = 2j_0^0(x) \delta(x,y)$$

$$[j_0^0(x), j_\pm(y)] = \pm j_\pm(x) \delta(x,y).$$

Eqs. (2.14) indicate that a “local” $SU(2)$ algebra is realized at each point of $\Omega$. Obviously these local algebras have only the spin 0 and spin 1/2 representations which we may call “local” representations.

A global algebra can be readily constructed by combining local ones with a repeated use of comultiplication. In fact using (2.14), it is easy to check that the operators

$$J^\pm \equiv \sum_x J^\pm(x) = \sum_x 1 \otimes \cdots \otimes 1 \otimes j^\pm(x) \otimes 1 \otimes \cdots 1,$$

$$J^0 \equiv \sum_x J^0(x) = \sum_x 1 \otimes \cdots \otimes 1 \otimes j^0(x) \otimes 1 \otimes \cdots 1$$

(2.15)
generate a global $SU(2)$ algebra. The symbol $\otimes$ in (2.15) denotes the direct product so that the operators $J^\pm(x)$ and $J^0(x)$ act as the identity at all points other than $x$ and as $j^\pm(x)$ and $j^0(x)$, respectively, at $x$. By combining the local spin 0 and spin 1/2 representations it is possible to construct all the unitary representations of the global $SU(2)$ algebra. For example the spin 1 representation is carried by the space spanned by the following three states

$$c_1^\dagger(x_1)c_1^\dagger(x_2)|0\rangle, \quad \frac{1}{\sqrt{2}} \left( c_1^\dagger(x_1)c_2^\dagger(x_2) + c_2^\dagger(x_1)c_1^\dagger(x_2) \right) |0\rangle, \quad c_2^\dagger(x_1)c_2^\dagger(x_2)|0\rangle,$$

where $x_1$ and $x_2$ are two arbitrary distinct points in $\Omega$. All other representations can be realized in a similar way.

The Schwinger construction of $SU(2)$ has been recently generalized to the so-called $q$-oscillators [15-18]. These are deformations of the ordinary harmonic oscillators characterized by the following generalized commutation relations

$$\tilde{a}_i \tilde{a}_i^\dagger - q^{-1} \tilde{a}_i^\dagger \tilde{a}_i = q \tilde{N}_i,$$  \hspace{1cm} (2.16)

where $q$ is the deformation parameter and $\tilde{N}_i$ is the number operator. Notice that $\tilde{N}_i$ is not $\tilde{a}_i^\dagger \tilde{a}_i$, but it nevertheless satisfies the usual relations with $\tilde{a}_i$ and $\tilde{a}_i^\dagger$, namely

$$\left[ \tilde{N}_i, \tilde{a}_i^\dagger \right] = \tilde{a}_i^\dagger, \quad \left[ \tilde{N}_i, \tilde{a}_i \right] = -\tilde{a}_i. \hspace{1cm} (2.17)$$

Only in the limit $q \to 1$, we retrieve the standard bosonic Heisenberg algebra of the harmonic oscillator and $\tilde{N}_i$ becomes $\tilde{a}_i^\dagger \tilde{a}_i$. Using $\tilde{a}_i$ and $\tilde{a}_i^\dagger$ for $i = 1, 2$ in the Schwinger approach, one can define the following operators

$$J^+ = \tilde{a}_1^\dagger \tilde{a}_2, \quad J^- = \tilde{a}_2^\dagger \tilde{a}_1, \quad J^0 = \frac{1}{2} \left( \tilde{N}_1 - \tilde{N}_2 \right),$$

and check that they satisfy [15,16]

$$\left[ J^0, J^\pm \right] = \pm J^\pm, \quad \left[ J^+, J^- \right] = \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}}.$$ \hspace{1cm} (2.19)

These are the commutators of the quantum group $SU(2)_q$ [7-9], and thus one can say that the Schwinger construction for $q$-oscillators naturally leads to a quantum group with $q$ as deformation parameter.

It is worthwhile to mention that the standard Schwinger construction of $SU(2)$ can be generalized also in a different way by using fermionic oscillators together with a non cocommutative comultiplication [11]. For this to be possible however, it is
necessary to order the fermions for example by putting them on a line. Then one can define

\[ J_{q}^\pm = \sum_x J_{q}^{\pm}(x) = \sum_x \left( \prod_{y<x} q^{-2j^0(y)} j^{\pm}(x) \prod_{z>x} q^{2j^0(z)} \right) , \]

\[ J_{q}^0 = \sum_x J_{q}^0(x) = \sum_x j^0(x) , \]

(2.20)

where \( j^{\pm}(x) \) and \( j^0(x) \) are given by eq. (2.13), \( x \) is the coordinate of a one dimensional chain and \( q \) is an arbitrary complex number. From the local algebra (2.14) it is easy to check that

\[ [J_q^0, J_{q}^{\pm}] = \pm J_{q}^{\pm} , \]

\[ [J_q^+, J_q^-] = \frac{q^{2J_q^0} - q^{-2J_q^0}}{q - q^{-1}} . \]

(2.21)

These are again the commutators of \( SU(2)_q \). It is interesting to observe that since \((j^+(x))^\dagger = j^-(x)\) and \((j^0(x))^\dagger = j^0(x)\), we have

\[ [J_q^+(x)]^\dagger = J_q^-(x) \]

(2.22)

for any \( q \). This construction appears naturally in several one-dimensional quantum spin systems (like for example the XXZ model) of which \( J_{q}^{\pm} \) and \( J_{q}^0 \) turn out to be symmetry operators [11].

In the next sections we will present a construction of \( SU(2)_q \) which, even though still inspired by the Schwinger approach, is completely different from the ones we have just mentioned. Indeed our construction will exploit anyonic operators, which contrarily to the \( q \)-oscillators, carry a representation of the braid group and are intrinsically non-local objects. Moreover, in our case the non cocommutativity of the comultiplication will be an automatic consequence of the statistics of the anyonic operators.

### 3. Lattice Angle Function and Anyonic Oscillators

In the following two sections we are going to generalize the Schwinger construction to the case of anyonic oscillators of statistics \( \nu \) which continuously interpolate between
bosons and fermions. As is well known, anyons can exist only in two space dimensions where the braid group replaces the permutation group in the classification of all possible statistics [4]. Therefore despite of the many formal analogies between them, anyonic oscillators do not have to be confused with the q-oscillators mentioned in the previous section which in principle can exist in any dimension. Since we will be interested in the anyonic case, from now on we will work only with objects defined on a two dimensional manifold $\Omega$ whose points will be denoted by $x = (x_1, x_2)$. Moreover for reasons which will be clear later, we take $\Omega$ to be a two dimensional lattice (for definiteness a square lattice) with spacing $a = 1$.

The first step of our analysis is the construction of anyonic oscillators on $\Omega$. Several recent papers have already analyzed this problem [21-26], but nevertheless we are going to review it again to set the notations and above all to point out a few important subtleties that have been overlooked in the literature. Our general strategy is to implement on the lattice $\Omega$ the Jordan-Wigner transformation [20] which allows to transmute for example fermions into bosons in any dimension and fermions into anyons of arbitrary statistics in two space dimensions. We remark that in this case the Jordan-Wigner transformation is inspired by the Chern-Simons construction of anyons [2,21,27,28] to which it is intrinsically related. An essential ingredient of such transformation is the so called angle function. In the continuous plane $\mathbb{R}^2$ the angle function is a rather familiar object. Formally it can be defined through the Green function of the Laplace operator, i.e. through the function

$$ G(x, y) = \ln |x - y| $$

which satisfies

$$ \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i} G(x, y) = 2\pi \delta(x - y) . $$

Then, if we introduce the vector field $f(x, y) = (f^1(x, y), f^2(x, y))$ according to

$$ f^i(x, y) = -\epsilon^{ij} \frac{\partial}{\partial x^j} G(x, y) = -\epsilon^{ij} \frac{x_j}{|x - y|^2} $$

where $\epsilon^{ij}$ is the completely antisymmetric symbol ($\epsilon^{12} = -\epsilon^{21} = 1$), the angle function $\Theta(x, y)$ is defined by

$$ \frac{\partial}{\partial x^i} \Theta(x, y) = f_i(x, y) , $$

and satisfies

$$ \epsilon^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \Theta(x, y) = 2\pi \delta(x - y) . $$

A solution of eq. (3.4) is

$$ \Theta(x, y) = \tan^{-1} \left( \frac{y_2 - x_2}{y_1 - x_1} \right) , $$

which is indeed the naive definition of the angle between two points measured from the positive $x$-axis. In this formula $x_i$ and $y_i$ are on the same footing (the right hand
side of eq. (3.6) is indeed symmetric under \( x_i \leftrightarrow y_i \) while they are not in eq. (3.4); therefore to remove the ambiguity one has to say e.g. that \( \Theta(x, y) \) is the angle under which the point \( x \) is seen from \( y \). Furthermore, the function \( \Theta \) is multivalued, and hence a cut has to be chosen. For example one can take \(-\pi \leq \Theta(x, y) < \pi\). With this choice it is not difficult to verify that

\[
\Theta(x, y) - \Theta(y, x) = \begin{cases} 
\pi \text{sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2, \\
\pi \text{sgn}(x_1 - y_1) & \text{for } x_2 = y_2.
\end{cases} 
\]  

(3.7)

The angle function can be defined unambiguously also on a two-dimensional lattice; however some care must be used in this generalization [22,23,25,26]. First of all, let us recall that there are two lattice derivative operators, \( \partial_i \) and \( \tilde{\partial}_i \), which for any function \( f(x) \) are defined through

\[
\partial_i f(x) = f(x + \hat{i}) - f(x), \\
\tilde{\partial}_i f(x) = f(x) - f(x - \hat{i}),
\]

(3.8)

where \( \hat{i} \) is the unit vector in the positive \( i \)-direction \( (i = 1, 2) \). In terms of \( \partial_i \) and \( \tilde{\partial}_i \) the correct lattice version of eq. (3.2) turns out to be

\[
\partial_i \tilde{\partial}_i G(x, y) = 2\pi \delta(x, y),
\]

(3.9)

where \( \delta(x, y) \) is the lattice delta function \( (\delta(x, y) = 0 \text{ if } x \neq y; \delta(x, y) = 1 \text{ if } x = y) \). The solution of eq. (3.9) is explicitly given by

\[
G(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d^2p \frac{[1 - \cos p \cdot (x - y)]}{2 \sum_{i=1}^{2}(1 - \cos p_i)}.
\]

(3.10)

This is a function only of the difference \( (x - y) \) and can be regarded as the lattice version of the continuum Green function given in eq. (3.1). Thus, to define the lattice angle function we can repeat steps similar to those in eqs. (3.3-6). We first define the vector field \( f(x, y) \) as

\[
f^i(x, y) = -\epsilon^{ij} \tilde{\partial}_j G(x, y);
\]

(3.11)

then from eq. (3.9) it follows that

\[
f^2(x + \hat{1}, y) - f^2(x, y) - f^1(x + \hat{2}, y) + f^1(x, y) \\
= \epsilon^{ij} \partial_i f^j(x, y) = \partial_i \tilde{\partial}_i G(x, y) = 2\pi \delta(x, y).
\]

(3.12)

One possible representation of \( f \) satisfying (3.12) is provided by

\[
f^i(x, y) = \varphi(x, y^*, x + \hat{i})
\]

(3.13)

where \( \varphi(x, y^*, x + \hat{i}) \) is the angle under which the oriented link between \( x \) and \( x + \hat{i} \) is seen from the point \( y^* = (y_1 + 1/2; y_2 + 1/2) \) as shown in Fig. 1. Notice that the point
\( y^* \) belongs to the dual lattice \( \Omega^* \). With this choice, \( f^i(x,y) \) is unambiguously defined also when \( x = y \); on the other hand if \( x \) and \( y \) are very far apart from each other the difference between \( y \) and \( y^* \) becomes negligible; moreover using the representation (3.13), we can easily realize that

\[
f^i(x,y) \rightarrow 0 \quad \text{if} \quad |x - y| \rightarrow \infty.
\] (3.14)

Let us now return to the general properties of \( f \). Recalling that \( f^i(x,y) \) is defined on the lattice link between \( x \) and \( x + \hat{i} \), we can rewrite eq. (3.12) as

\[
\oint_{\Gamma_x} f(x,y) = 2\pi \delta(x,y) \quad \text{(3.15)}
\]

where \( \Gamma_x \) is the positively oriented boundary of the elementary plaquette \( A_x \) whose lower left corner is \( x \) (see Fig. 2). Thus, for any closed curve \( \Gamma \) encircling \( k \) times the point \( y^* \), we have

\[
\oint_{\Gamma} f(x,y) = 2\pi k \quad \text{(3.16)}
\]

Given the vector field \( f \), it is possible to define unambiguously the angle between two lattice points \( x \) and \( y \). To this aim let us consider a path \( P_x \) which, following the lattice bonds, starts from a base point \( B \) (eventually moved to infinity) and ends at the point \( x \); then the function

\[
\Theta_{P_x}(x,y) = \int_{P_x} f(x,y)
\] (3.17)

is the lattice angle function. If we use the explicit representation of \( f \) given in eq. (3.13), we can describe \( \Theta_{P_x}(x,y) \) as the angle between the base point \( B \) and the point \( x \) measured from the point \( y^* \in \Omega^* \) along the curve \( P_x \) as shown in Fig. 3. This function has all the properties of any angle. In fact we easily see that

\[
\varepsilon^{ij}\partial_i \partial_j \Theta_{P_x}(x,y) = 2\pi \delta(x,y),
\] (3.18)

which is the lattice analogue of eq. (3.5); moreover it is multivalued because if the path \( P_x \) winds counterclockwise around the point \( y^* \), \( \Theta_{P_x}(x,y) \) increases of \( 2\pi \) like any angle centered in \( y^* \). Actually, using eq. (3.16) we have

\[
\Theta_{P_x}(x,y) - \Theta_{P'_x}(x,y) = \oint_{P_x \cdot P'_x^{-1}} f(x,y) = 2\pi k
\] (3.19)

where \( k \) is the winding number of the closed loop \( P_x \cdot P'_x^{-1} \) around the point \( y^* \).

The lattice extension of eq. (3.7) is not immediate; it has been determined in [26] in quite great generality. Here we present our version and make some general comments. Let us choose as base point \( B \) the point at infinity of the positive \( x \)-axis, and let us associate to each point \( x \) the straight lattice path \( C_x \), parallel to the \( x \)-axis
from \( B \) to \( x \) (see Fig. 4). Given any two distinct points \( x \) and \( y \) and their associated paths \( C_x \) and \( C_y \), it is possible to show that

\[
\Theta_{C_x}(x, y) - \Theta_{C_y}(y, x) = \begin{cases} 
\pi \text{sgn}(x_2 - y_2) + \xi(x, y) & \text{for } x_2 \neq y_2 \\
\pi \text{sgn}(x_1 - y_1) + \xi(x, y) & \text{for } x_2 = y_2 \end{cases},
\] (3.20)

where the function \( \xi(x, y) \) is given in terms of the vector field \( f \) by [26]

\[
\xi(x, y) = -\frac{1}{2} \left[ f^1(x, y) + f^2(x, y) + f^1(x + \hat{2}, y) + f^2(x + \hat{1}, y) \right].
\] (3.21)

Comparing eq. (3.20) with eq. (3.7), we notice that the extra term \( \xi(x, y) \) appears in the right hand side. Such a term is a genuine lattice feature arising from the fact that the angles are measured from points of the dual lattice. This is clearly seen in our explicit representation (3.13) where

\[
\xi(x, y) = \varphi(x + \hat{1} + \hat{2}, y^*, x).
\] (3.22)

The right hand side represents the angle between \( x \) and \( x + \hat{1} + \hat{2} \) as seen from \( y^* \) which is equal to the angle between the lines \((x, y^*)\) and \((x^*, y)\), as shown in Fig. 5. A few remarks are in order. Eq. (3.14) implies that if \( x \) and \( y \) are very far apart from each other, \( \xi(x, y) \) is negligible and hence the left-hand side of eq. (3.20) simplifies considerably. However this is not the case if \( x \) and \( y \) are close to each other. A possible way of removing the \( \xi \)-term from eq. (3.20) even when \( x \) and \( y \) are not far apart, is to embed the lattice \( \Omega \) into another lattice \( \Lambda \) whose spacing \( \varepsilon \) is taken to be much smaller than one. Then, since \( \Omega \) becomes a sublattice of \( \Lambda \), the quantities defined on \( \Omega \) can be viewed as the restriction to \( \Omega \) of quantities defined on \( \Lambda \). Under this assumption, for all points \( x, y \in \Omega \subset \Lambda \) the vector field \( f \) can be represented by

\[
f^i(x, y) = \varphi(x, y^i, x + \varepsilon \hat{i})
\] (3.23)

where \( \varphi(x, y^i, x + \varepsilon \hat{i}) \) is the angle under which the oriented link of \( \Lambda \) between \( x \) and \( x + \varepsilon \hat{i} \) is seen from the point \( y' = (y_1 + \varepsilon/2, y_2 + \varepsilon/2) \) belonging to the dual lattice \( \Lambda^* \). If we use the realization (3.23), \( \Theta_{C_x}(x, y) \) represents the angle between \( B \) and \( x \) measured from \( y' \) along the curve \( C_x \), while the function \( \xi(x, y) \) becomes

\[
\xi(x, y) = \varphi(x + \varepsilon \hat{1} + \varepsilon \hat{2}, y^i, x).
\] (3.24)

In the limit \( \varepsilon \to 0 \), two distinct points \( x \) and \( y \) are always far apart from each other (from the standpoint of \( \Lambda \)) and the function \( \xi(x, y) \) is always negligible. Therefore eq. (3.20) simplifies to

\[
\Theta_{C_x}(x, y) - \Theta_{C_y}(y, x) = \begin{cases} 
\pi \text{sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2 \\
\pi \text{sgn}(x_1 - y_1) & \text{for } x_2 = y_2 \end{cases},
\] (3.25)

which is the exact analogue of eq. (3.7) valid in the continuum plane. In this respect we remark that the choice of the curves \( C_x \) is in fact equivalent to fix for any \( x \) a cut
\( \gamma_x \) from \( \mathbf{x}' \in \Lambda' \) to \(-\infty\) along \( x \)-axis (see Fig. 6) in such a way that \(-\pi \leq \Theta_{\mathbf{c}_x} < \pi\). This is the same fundamental interval in which the continuum angle was defined. Thus one can say that any point \( \mathbf{x} \in \Omega \) is characterized either by a curve \( \mathcal{C}_x \) made of lattice bonds or by a cut \( \gamma_x \) made of dual bonds.

We now use eq. (3.25) to establish an ordering relation among the points of the lattice which will later on be essential to define anyonic operators. First of all, let us introduce the notation \( \mathbf{x}_c \) to denote the point \( \mathbf{x} \in \Omega \) with its curve \( \mathcal{C}_x \). Then, given two distinct points \( \mathbf{x} \) and \( \mathbf{y} \) we can posit
\[
\mathbf{x}_c > \mathbf{y}_c \iff \Theta_{\mathcal{C}_x}(\mathbf{x}, \mathbf{y}) - \Theta_{\mathcal{C}_x}(\mathbf{y}, \mathbf{x}) = \pi \ .
\] (3.26)

Using eq. (3.25), we can rewrite this relation more explicitly as follows
\[
\mathbf{x}_c > \mathbf{y}_c \iff \left\{ \begin{array}{l}
x_2 > y_2 \\
x_2 = y_2, \ x_1 > y_1
\end{array} \right.
\] (3.27)

Obviously, if \( x_2 < y_2 \), or if \( x_2 = y_2 \) and \( x_1 < y_1 \), then \( \mathbf{x}_c < \mathbf{y}_c \). This definition is unambiguous and endows the lattice \( \Omega \) with an ordering relation enjoying all the correct properties. However this ordering is not unique. It crucially depends on the choice of the curves \( \mathcal{C}_x \). If we chose other types of curves, we could clearly change the ordering relation. A more fundamental change in the ordering can be obtained by modifying the definition of the angle function. This is possible if we introduce a new vector field \( \mathbf{f}(\mathbf{x}, \mathbf{y}) \) through
\[
\tilde{f}^i(\mathbf{x}, \mathbf{y}) = -\epsilon^{ij} \partial_j G(\mathbf{x}, \mathbf{y}) \ .
\] (3.28)

This equation differs from that defining the old field \( \mathbf{f} \) because the derivative \( \tilde{\partial}_j \) has been replaced by \( \partial_j \) (cf eq. (3.11)). Moreover, considering \( \Omega \) as embedded into \( \Lambda \), \( \tilde{f}^i(\mathbf{x}, \mathbf{y}) \) is defined on the link between \( \mathbf{x} \) and \( \mathbf{x} - \epsilon \hat{i} \), in contrast with \( f^i(\mathbf{x}, \mathbf{y}) \) which is defined on the link between \( \mathbf{x} \) and \( \mathbf{x} + \epsilon \hat{i} \). Keeping this in mind, it is easy to show that
\[
\tilde{f}^i(\mathbf{x}, \mathbf{y}) - \tilde{f}^i(\mathbf{x} - \epsilon \hat{2}, \mathbf{y}) - \tilde{f}^i(\mathbf{x}, \mathbf{y}) + \tilde{f}^i(\mathbf{x} - \epsilon \hat{1}, \mathbf{y}) = \epsilon^{ij} \tilde{\partial}_l \tilde{f}^j(\mathbf{x}, \mathbf{y}) = \tilde{\partial}_l \partial_i G(\mathbf{x}, \mathbf{y}) = 2\pi \delta(\mathbf{x}, \mathbf{y}) \ ,
\] (3.29)

and hence
\[
\oint_{\tilde{\Gamma}_x} \tilde{f}(\mathbf{x}, \mathbf{y}) = 2\pi \delta(\mathbf{x}, \mathbf{y})
\] (3.30)

where \( \tilde{\Gamma}_x \) is the positively oriented boundary of the elementary plaquette \( \tilde{A}_x \) of \( \Lambda \) whose upper right corner is the point \( \mathbf{x} \) (see Fig. 7). One possible representation of \( \tilde{f} \) is the following
\[
\tilde{f}^i(\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{x}, \tilde{\mathbf{y}}, \mathbf{x} - \epsilon \hat{i})
\] (3.31)

where \( \varphi(\mathbf{x}, \tilde{\mathbf{y}}, \mathbf{x} - \epsilon \hat{i}) \) is the angle under which the oriented link of \( \Lambda \) between \( \mathbf{x} \) and \( \mathbf{x} - \epsilon \hat{i} \) is seen from the point \( \tilde{\mathbf{y}} = (y_1 - \epsilon/2, y_2 - \epsilon/2) \in \Lambda^* \). Then, given an arbitrary path \( P_x \) from a base point \( \tilde{B} \) to \( \mathbf{x} \), we can define a new angle function through
\[
\tilde{\Theta}_{P_x}(\mathbf{x}, \mathbf{y}) = \int_{P_x} \tilde{f}(\mathbf{x}, \mathbf{y})
\] (3.32)

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Analogously to eq. (3.17), if we use the representation (3.31) we can interpret \( \tilde{\Theta}_{P_x}(x,y) \) as the angle between \( \tilde{B} \) and \( x \) measured from \( \tilde{y} \) along the curve \( P_x \). This angle function clearly satisfies

\[
e^{ij} \tilde{\gamma}_i \tilde{\gamma}_j \tilde{\Theta}_{P_x}(x,y) = 2\pi \delta(x,y) ,
\]

which is another lattice version of eq. (3.5), and

\[
\tilde{\Theta}_{P_x}(x,y) - \tilde{\Theta}_{P_x'}(x,y) = \oint_{P_x P_x'^{-1}} \tilde{f}(x,y) = 2\pi k \quad (3.34)
\]

where \( k \) is the winding number of the closed loop \( P_x P_x'^{-1} \) around the point \( \tilde{y} \in \Lambda^* \).

To write relations similar to eqs. (3.20) or (3.25) for the function \( \tilde{\Theta} \) we must again remove all possible ambiguities; for instance this can be done if we choose as base point \( \tilde{B} \) the point at infinity of the negative \( x \)-axis, and associate to each point \( x \) the straight lattice path \( D_x \), parallel to the \( x \)-axis from \( \tilde{B} \) to \( x \) (see Fig. 8). Then, with this choice and in the limit \( \varepsilon \to 0 \), one can prove that for any two distinct points \( x \) and \( y \)

\[
\tilde{\Theta}_{D_x}(x,y) - \tilde{\Theta}_{D_y}(y,x) = \begin{cases} -\pi \text{sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2 \\ -\pi \text{sgn}(x_1 - y_1) & \text{for } x_2 = y_2 \end{cases} .
\]

(3.35)

The relation (3.35) can be used to define a new ordering among the points of the lattice. In fact, if we denote by \( x_D \) the point \( x \) with its associated curve \( D_x \), in analogy with eq. (3.26) we can posit

\[
x_D > y_D \iff \tilde{\Theta}_{D_x}(x,y) - \tilde{\Theta}_{D_y}(y,x) = \pi ,
\]

(3.36)

that is

\[
x_D > y_D \iff \begin{cases} x_2 < y_2 \\ x_2 = y_2, x_1 < y_1 \end{cases} .
\]

(3.37)

Comparing with eq. (3.27) we can easily realize that the ordering defined by (3.37) is exactly the opposite of the ordering induced by the curves \( C_x \). Thus we have

\[
x_C > y_C \iff x_D < y_D
\]

(3.38)

For later convenience, we now establish a direct relation between the \( C \)- and the \( D \)-angles. A moment thought reveals that, if \( x \neq y \)

\[
\tilde{\Theta}_{D_x}(x,y) - \Theta_{C_x}(x,y) = \begin{cases} -\pi \text{sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2 \\ -\pi \text{sgn}(x_1 - y_1) & \text{for } x_2 = y_2 \end{cases} .
\]

(3.39)

From eqs. (3.39) and (3.35) it follows that

\[
\tilde{\Theta}_{D_x}(x,y) - \Theta_{C_y}(y,x) = 0
\]

(3.40)
for all $x \neq y$.

Actually, eq. (3.40) is formally correct also for $x = y$. Indeed, $\tilde{\Theta}_D(x, x) = -\frac{3\pi}{4}$ and $\Theta_{C_x}(x, x) = -\frac{3\pi}{4}$ so that

$$\tilde{\Theta}_D(x, x) - \Theta_{C_x}(x, x) = 0 \quad (3.41)$$

We can summarize our findings by saying that the set of the lattice points is doubled into points of the $C$-type (i.e. $x_C$ for $x \in \Omega$) and into points of the $D$-type (i.e. $x_D$ for $x \in \Omega$) which are ordered among themselves according to eqs. (3.27) and (3.37) respectively. As we will show in the next section, this doubling of the lattice points is crucial in the construction of the quantum group structure.

After this discussion on the lattice angle function and on the ordering relations induced by it, we are finally ready to introduce anyonic operators on $\Omega$. These are defined by means of a (generalized) Jordan-Wigner transformation from the fermionic operators $c_1(x), c_2(x)$ and their adjoints which were considered in the previous section. Roughly speaking, the Jordan-Wigner transformation amounts to simply stick a disorder operator to the fermions $c_i(x)$ [21,28,29]. Such a disorder operator can be written as the exponential of the local fermion density $c_i^\dagger(x)c_i(x)$ summed over all lattice points and weighted with the angle function. However, since we have defined two lattice angle functions, we should expect two types of disorder operators, and hence two types of lattice anyons (type $C$ and type $D$).

Let us first define the anyons of type $C$ according to

$$a_i(x_C) = K_i(x_C) c_i(x) \quad (3.42a)$$

and

$$a_i^\dagger(x_C) = c_i^\dagger(x) K_i^\dagger(x_C) \quad (3.42b)$$

where

$$K_i(x_C) = e^{i\nu \sum_{y \in \Omega} \Theta_{C_x}(x, y) c_i^\dagger(y)c_i(y)}$$

$$K_i^\dagger(x_C) = e^{-i\nu \sum_{y \in \Omega} \Theta_{C_x}(x, y) c_i^\dagger(y)c_i(y)} = K_i(x_C)$$

are disorder operators [21,28] $^3$. In these formulas $i = 1, 2$ for “spin-up” and “spin-down” respectively, and $\nu$ is a real parameter which, as we will see, represents the statistics (our conventions are such that for $\nu = 0$ we have fermionic statistics, whereas for $\nu = 1$ we have bosonic statistics).

Using the canonical commutation relations of the fermionic operators (cf eq. (2.12)), it is very easy to show that

$$K_i(x_C) c_i(y) = e^{-i\nu \Theta_{C_x}(x, y)} c_i(y) K_i(x_C)$$

$$K_i(x_C) c_i^\dagger(y) = e^{i\nu \Theta_{C_x}(x, y)} c_i^\dagger(y) K_i(x_C)$$

$$K_i(x_C) K_i(y_C) = K_i(y_C) K_i(x_C) \quad (3.44)$$

$^3$ Notice that when plugging the expressions (3.43) into eqs. (3.42), the term with $y = x$ in the sum of the exponent does not contribute since $[c_i(x)]^2 = [c_i^\dagger(x)]^2 = 0$; we remark however that $\Theta_{C_x}(x, y)$ is defined also for $y = x$. 

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for all \( x \) and \( y \).

To see that the operators \( a_i(x_C) \) and \( a_i^\dagger(x_C) \) are indeed anyons of statistics \( \nu \), let us compute their “commutation” relations. Let us also simplify the notations and introduce the symbol \( x > y \) to mean \( x_C > y_C \) (and hence also \( x_D < y_D \)). Thus, for \( x > y \) we have

\[
a_i(x_C) a_i(y_C) = K_i(x_C) c_i(x) K_i(y_C) c_i(y) \\
= -e^{-i\nu[\Theta_{c_1}(x,y) - \Theta_{c_2}(y,x)]} K_i(y_C) c_i(y) K_i(x_C) c_i(x) \\
= -e^{-i\nu\pi} a_i(y_C) a_i(x_C)
\]

where to derive the last line we used eqs. (3.44) and (3.26). This formula can be written also as

\[
a_i(x_C) a_i(y_C) + q^{-1} a_i(y_C) a_i(x_C) = 0
\]

where

\[
q = e^{i\nu\pi}.
\]

For \( \nu = 0 \pmod 2 \) eq. (3.45a) is an anticommutator signaling fermionic statistics, whilst for \( \nu = 1 \pmod 2 \) it is a commutator signaling bosonic statistics. The embedding of the lattice \( \Omega \) into \( \Lambda \) and the limiting procedure \( \varepsilon \to 0 \) which allowed us to eliminate the \( \xi \)-term from eq. (3.20), are essential to get a generalized commutator with a constant \( q \)-factor; otherwise one would obtain eq. (3.46a) with \( q \) depending on the distance \( x - y \). For an alternative procedure to remove this dependence and the \( \xi \)-term see [26]. With similar calculations we can derive also the following generalized commutation relations

\[
a_i^\dagger(x_C) a_i^\dagger(y_C) + q^{-1} a_i^\dagger(y_C) a_i^\dagger(x_C) = 0, \\
a_i(x_C) a_i(y_C) + q a_i(y_C) a_i(x_C) = 0, \\
a_i^\dagger(x_C) a_i(y_C) + q a_i(y_C) a_i^\dagger(x_C) = 0,
\]

for all \( x > y \). We notice that eqs. (3.46b) and (3.46d) are the hermitian conjugate of eqs. (3.46a) and (3.46c) respectively, since \( q^* = q^{-1} \). For completeness we recall that

\[
\left[a_i(x_C)\right]^2 = \left[a_i^\dagger(x_C)\right]^2 = 0,
\]

and

\[
\{a_1(x_C), a_2(y_C)\} = \{a_1^\dagger(x_C), a_2^\dagger(y_C)\} \\
= \{a_1^\dagger(x_C), a_2(y_C)\} = \{a_1(x_C), a_2^\dagger(y_C)\} = 0
\]

for all \( x \) and \( y \) and for any value of \( \nu \). Eq. (3.48a) enforces the Pauli exclusion principle; thus for \( \nu = 1 \pmod 2 \) the oscillator \( a_i(x_C) \) represents a boson with a hard core. The (anti)commutation relations of \( a_i \) and \( a_i^\dagger \) in the same point deserve
particular attention. In fact, using the definitions (3.42) and (3.43) and the relations (3.44), we have

\[ a_i(x_C) a_i^\dagger(x_C) = K_i(x_C) c_i(x) c_i^\dagger(x) K_i^\dagger(x_C) \]

\[ = K_i(x_C) \left( -c_i^\dagger(x) c_i(x) + 1 \right) K_i^\dagger(x_C) \]

\[ = -e^{i\nu c_i(x,y)} c_i^\dagger(x) K_i(x_C) e^{-i\nu c_i(x,y)} K_i^\dagger(x_C) c_i(x) + 1 \]

\[ = -a_i^\dagger(x_C) a_i(x_C) + 1 \]  

(3.49)

In conclusion we find

\[ a_i(x_C) a_i^\dagger(x_C) + a_i^\dagger(x_C) a_i(x_C) = 1 \]  

(3.50)

without any phase factor. Therefore, contrarily to several statements in the literature [21,25,26,28], we see that the fermionic based operators \( a_i \) and \( a_i^\dagger \) with anyonic statistics \( \nu \) obey standard anticommutation relations at the same point \(^4\). Furthermore, since anyons carry a representation of the braid group (see for instance [4]), when we exchange two of them it is essential to specify the orientation of the exchanging trajectories (i.e. their braidings), and on the lattice this can be done unambiguously only by exploiting the ordering induced by the lattice angle function. In fact the exchange of two anyons located in \( x \) and \( y \) can be realized by a half-circle rotation of \( y \) around \( x \) followed by a rigid translation. The orientation of such rotation can be uniquely defined by requiring that \( y \) does not cross the cut \( \gamma_x \). One can easily get convinced that the rotation is counterclockwise for \( x > y \) and clockwise for \( x < y \) anyons (see Fig. 10). This aspect has not been sufficiently emphasized in the previous literature on this subject [21,25,26,28], but after all, also in the continuum theory one has to specify if the anyons are exchanged counterclockwise or clockwise to define unambiguously their statistics!

Let us now define the anyon operators of type \( D \). They are given by

\[ a_i(x_D) = K_i(x_D) c_i(x) , \]  

(3.51a)

and

\[ a_i^\dagger(x_D) = c_i^\dagger(x) K_i^\dagger(x_D) \]  

(3.51b)

where the type \( D \) disorder operators are

\[ K_i(x_D) = e^{i\nu \sum_{y \in \Omega} \hat{\Theta}_{D_x}(x,y) c_i^\dagger(y) c_i(y)} \]

\[ = K_i^{-1}(x_D) \]  

(3.52)

\[ K_i^\dagger(x_D) = e^{-i\nu \sum_{y \in \Omega} \hat{\Theta}_{D_x}(x,y) c_i^\dagger(y) c_i(y)} \]

\[ = K_i^{-1}(x_D) \]  

\(^4\) We remark that one consistently obtains the same result (3.50) also if one defines \( K_i(x_C) \) by excluding the point \( y = x \) from the sum at exponent in eq. (3.43); in such a case then, \( K_i(x_C) \) would commute with both \( a_i(x_C) \) and \( a_i^\dagger(x_C) \).
Clearly these disorder operators obey the same relations as in eq. (3.44) with $x_C$ and $\Theta_{C_x}$ replaced by $x_D$ and $\Theta_{D_x}$ respectively. With manipulations similar to those that led to eqs. (3.46), it is easy to show that

\begin{align}
a_i(x_D) a_i(y_D) + q a_i(y_D) a_i(x_D) &= 0, \quad (3.53a) \\
a_i^\dagger(x_D) a_i^\dagger(y_D) + q a_i^\dagger(y_D) a_i^\dagger(x_D) &= 0, \quad (3.53b) \\
a_i(x_D) a_i^\dagger(y_D) + q^{-1} a_i^\dagger(y_D) a_i(x_D) &= 0, \quad (3.53c) \\
a_i^\dagger(x_D) a_i(y_D) + q^{-1} a_i(y_D) a_i^\dagger(x_D) &= 0, \quad (3.53d)
\end{align}

for all $x > y$. Notice that $x > y$ means $x_C > y_C$ and hence $x_D < y_D$ (cf eq. (3.38)). Furthermore

\begin{equation}
\{ a_i(x_D), a_i^\dagger(x_D) \} = 1, \quad (3.54)
\end{equation}

\textit{i.e.} they satisfy standard anticommutation relations at the same point. Eqs. (3.53) must be interpreted by saying that $a_i(x_D)$ and $a_i^\dagger(x_D)$ are again anyons of statistics $\nu$. However we see that these generalized commutation relations differ from the corresponding ones for the type $C$ operators (cf eqs. (3.46)) because $q$ has been replaced by $q^{-1}$. This should not come as a surprise because the $C$ and the $D$ orderings are inverse to one another. More precisely, one can say that the $D$ ordering can be obtained from the $C$ ordering with a parity transformation which, as well known, changes the braiding phase $q$ into $q^{-1}$ (see for instance [4]). Indeed Fig. 11 clearly shows that the opposite orientation of the cuts $\delta_x$ with respect to the cuts $\gamma_x$ reverses also the orientation of the exchanging trajectories.

It is now interesting to establish a direct relation between the $C$ and the $D$ operators. Repeatedly using their definitions and eq. (3.44) with its analogue for the $D$ operators, we find

\begin{equation}
a_i(x_D) a_i(y_C) = K_i(x_D) c_i(x) K_i(y_C) c_i(y) \\
= -e^{-i\nu[\Theta_{\delta_x}(x,y) - \Theta_{\gamma_x}(y,x)]} a_i(y_C) a_i(x_D).
\end{equation}

The exponent in the last line actually vanishes for all $x$ and $y$ because of eq. (3.40). Thus we get

\begin{equation}
\{ a_i(x_D), a_i(y_C) \} = 0 \quad \forall \ x, \ y. \quad (3.56a)
\end{equation}

Similarly we have

\begin{equation}
\{ a_i(x_D), a_i^\dagger(y_C) \} = 0 \quad \forall \ x \neq y. \quad (3.56b)
\end{equation}

By taking the hermitian conjugate of these expressions, we obtain

\begin{equation}
\{ a_i^\dagger(x_D), a_i^\dagger(y_C) \} = 0 \quad \forall \ x, \ y. \quad (3.56c)
\end{equation}

\begin{equation}
\{ a_i^\dagger(x_D), a_i(y_C) \} = 0 \quad \forall \ x \neq y. \quad (3.56d)
\end{equation}
Things are not so simple in the anticommutation relation of \( a_i(x_D) \) and \( a_i^\dagger(x_C) \), i.e. at the same point. Indeed we have

\[
a_i(x_D) a_i^\dagger(x_C) = K_i(x_D) c_i(x) c_i^\dagger(x) K_i^\dagger(x_C)
\]

\[
= K_i(x_D) \left( -c_i^\dagger(x) c_i(x) + 1 \right) K_i^\dagger(x_C)
= -e^{i\nu [\Theta_{D_2}(x,x) - \Theta_{C_2}(x,x)]} a_i^\dagger(x_C) a_i(x_D) + K_i(x_D) K_i^\dagger(x_C)
= -a_i^\dagger(x_C) a_i(x_D) + K_i(x_D) K_i^\dagger(x_C)
\]

(3.57)

where in the final step we made use of eq. (3.41). Inserting the explicit expressions of the disorder operators, we can simplify the last term and get

\[
K_i(x_D) K_i^\dagger(x_C) = e^{i\nu \sum_{y \in \Omega} \left[ \Theta_{D_2}(x,y) - \Theta_{C_2}(x,y) \right]} c_i^\dagger(y) c_i(y)
\]

\[
= e^{-i\nu \pi \left[ \sum_{y < x} - \sum_{y > x} \right]} c_i^\dagger(y) c_i(y)
\]

(3.58)

where eqs. (3.39) and (3.41) have been taken into account. Combining the last two equations we obtain

\[
\left\{ a_i(x_D) , a_i^\dagger(x_C) \right\} = q \left[ \sum_{y < x} - \sum_{y > x} \right] c_i^\dagger(y) c_i(y) ;
\]

(3.59)

finally, taking its hermitian conjugate yields

\[
\left\{ a_i(x_C) , a_i^\dagger(x_D) \right\} = q \left[ \sum_{y < x} - \sum_{y > x} \right] c_i^\dagger(y) c_i(y) .
\]

(3.60)

This concludes our discussion of anyon oscillators on the lattice; in the next section we will use them to realize the quantum group \( SU(2)_q \) with the Schwinger construction, and will find that in order to close correctly the quantum algebra it is essential the employ both the type \( C \) and the type \( D \) operators. Therefore these two types of lattice anyons which, as we have mentioned, are related to one another by a parity transformation, will find an algebraic application in a quite natural way.
4. The Schwinger Construction with Anyons and the Quantum Group $SU(2)_q$

In this section we are going to show that the Schwinger construction of $SU(2)$ which was previously discussed using bosons and fermions, can be generalized to anyons of statistics $\nu$ in a quite direct though non-trivial way. However in the case of anyons, the Schwinger construction will not realize the ordinary group $SU(2)$ but rather its quantum deformation $SU(2)_q$ with $q = \exp(i\pi\nu)$.

In analogy with the bosonic formula (2.2), or even better with the fermionic ones (2.13) and (2.15), we start by introducing the local step-up operator

$$J^+(x) = a_1^\dagger(x_C) a_2(x_C), \quad (4.1)$$

which, as an immediate consequence of eqs. (3.46), satisfies

$$J^+(y) \ J^+(x) = q^2 \ J^+(x) \ J^+(y) \quad (4.2)$$

for $x > y$. This formula is quite important for several reasons. First of all, it shows that these operators have braiding properties just like their constituent factors. Therefore, in discussing their generalized commutation relations it is crucial to specify the ordering of the points; indeed the explicit $q$-factor in the right hand side changes if we change the braiding orientation, \emph{i.e.} the ordering of $x$ and $y$. Secondly, and also in view of the last observation, it should be clear that if we used type $D$ anyons instead of type $C$ ones, we would make a parity transformation reversing the ordering of the points and thus we would change $q$ into $q^{-1}$. In fact the operator

$$\tilde{J}^+(x) = a_1^\dagger(x_D) a_2(x_D) \quad (4.3)$$

satisfies

$$\tilde{J}^+(y) \ \tilde{J}^+(x) = q^{-2} \ \tilde{J}^+(x) \ \tilde{J}^+(y) \quad (4.4)$$

for $x > y$, as one can check using the generalized commutation relations (3.53).

Finally, we remark that the behaviour of $J^+(x)$ exhibited in eq. (4.2) is the same as the one of the local densities of quantum group generators. By this we mean that if $J^+_q = \sum_x J^+_q(x)$ is a generator of the quantum group $SU(2)_q$ obtained by repeated use of comultiplication starting from the local operators $j^+_q(x)$ and $j^0_q(x)$, then $J^+_q(y) \ J^+_q(x) = q^2 J^+_q(x) J^+_q(y)$ for $x > y$ (see for example [9,10]). This is a significant hint that actually the step-up operator (4.1) can be somehow considered as the local density of a quantum group generator, \emph{i.e.}

$$J^+(x) \simeq J^+_q(x), \quad (4.5)$$

Similarly from eq. (4.4) one may say that

$$\tilde{J}^+(x) \simeq J^+_{q^{-1}}(x), \quad (4.6)$$
\( J_{q^{-1}} \) being the generator of the quantum group \( SU(2)_{q^{-1}} \).

If we pursue this conjecture further on, we should expect that the step-down operator \( J^{-}(x) \) be related to the step-up operator \( J^{+}(x) \) like the local densities of the quantum group generators, namely like

\[
J^{-}(x) = \left[ J_{q^{*}}^{+}(x) \right]^{\dagger}
\]

(see for example [9] and eq. (2.22)). In our case \( q^{*} = q^{-1} \), and thus we are led to posit

\[
J^{-}(x) = \left[ J_{q}^{+}(x) \right]^{\dagger} = a_{1}^{\dagger}(x_{D}) a_{1}(x_{D}).
\]

(4.8)

Using eqs. (3.53), one easily proves that

\[
J^{-}(y) J^{-}(x) = q^{-2} J^{-}(x) J^{-}(y)
\]

for \( x > y \), as expected.

Inspired by the ordinary Schwinger construction (cf eqs. (2.2) and (2.13)), we may define the Cartan generator \( J_{q}^{0}(x) \) according to

\[
J_{q}^{0}(x) = \frac{1}{2} \left( a_{1}^{\dagger}(x_{C}) a_{1}(x_{C}) - a_{2}^{\dagger}(x_{C}) a_{2}(x_{C}) \right).
\]

(4.10)

It is interesting to realize that in this expression we could have used the anyon oscillators of type \( D \) without any change; in fact one may check that the disorder operators cancel out yielding

\[
a_{i}^{\dagger}(x_{D}) a_{i}(x_{D}) = a_{i}^{\dagger}(x_{C}) a_{i}(x_{C}) = c_{i}^{\dagger}(x) c_{i}(x).
\]

(4.11)

Thus, \( J_{q}^{0}(x) \) does not depend on \( q \) and is the same as in the fermionic realization (see eq. (2.13)). In conclusion, one may say that the anyonic generalization of the Schwinger construction leads to consider the following three local operators

\[
J^{+}(x) = a_{1}^{\dagger}(x_{C}) a_{2}(x_{C}),
\]

\[
J^{0}(x) = \frac{1}{2} \left( a_{1}^{\dagger}(x_{C}) a_{1}(x_{C}) - a_{2}^{\dagger}(x_{C}) a_{2}(x_{C}) \right)
\]

\[
= \frac{1}{2} \left( a_{1}^{\dagger}(x_{D}) a_{1}(x_{D}) - a_{2}^{\dagger}(x_{D}) a_{2}(x_{D}) \right),
\]

\[
J^{-}(x) = a_{2}^{\dagger}(x_{D}) a_{1}(x_{D}).
\]

(4.12)

What remains to be discussed is what algebra, if any, such operators close. We have already noticed and conjectured a sort of relation between these operators and the local densities of quantum group generators; hereinafter we are going to prove that this conjecture is correct.

With a straightforward application of eqs. (3.46), (3.53) and (3.56), we can easily check that

\[
[J^{0}(x), J^{\pm}(y)] = \pm J^{\pm}(x) \delta(x, y),
\]

\[
[J^{+}(x), J^{-}(y)] = 0 \quad \forall x \neq y.
\]

(4.13)

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The commutation relation of \( J^+(x) \) with \( J^-(x) \) \( (i.e. \) at the same point) is slightly more complicated because it involves the anticommutators of anyons of type \( C \) with anyons of type \( D \). However, using eqs. (3.59) and (3.60), it is not difficult to show that

\[
\left[ J^+(x) , J^-(x) \right] = q \left[ \sum_{y < x} - \sum_{y > x} \right] c_1^+(y) c_2^-(y) \, a_1^+(x_C) \, a_1^-(x_D) \\
- \left[ \sum_{y < x} - \sum_{y > x} \right] c_2^+(y) c_1^-(y) \, a_2^+(x_D) \, a_2^-(x_C) .
\]

(4.14a)

If we insert the explicit definition of anyon oscillators in the right hand side and then use eq. (3.39), this commutator can be rewritten in a more useful form as follows

\[
\left[ J^+(x) , J^-(x) \right] = \prod_{y < x} \frac{q^{-2J^0(y)} 2J^0(x)}{q^{2J^0(z)} \prod_{z > x}} .
\]

(4.14b)

After these preliminaries we are now in the position of defining the global generators. These are given by

\[
J^\pm = \sum_{x \in \Omega} J^\pm(x) , \\
J^0 = \sum_{x \in \Omega} J^0(x) ,
\]

(4.15)

and close the \( SU(2)_q \) algebra. In fact, from eqs. (4.13) and (4.14) one easily obtains

\[
\left[ J^0 , J^\pm \right] = \pm J^\pm ,
\]

(4.16a)

\[
\left[ J^+ , J^- \right] = \sum_{x \in \Omega} \left( \prod_{y < x} q^{-2J^0(y)} 2J^0(x) \prod_{z > x} q^{2J^0(z)} \right) .
\]

(4.16b)

These are precisely the defining commutation relations of \( SU(2)_q \) when \( J^0(x) \) is in the spin 0 or spin 1/2 representation for any \( x \). In the literature on quantum groups one usually finds a different expression for the last commutator, namely

\[
\left[ J^+ , J^- \right] = \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}}
\]

(see for instance [7-9]). Despite the appearance, there is actually no difference between eqs. (4.16b) and (4.17); in fact one can prove that in our case

\[
\sum_{x \in \Omega} \left( \prod_{y < x} q^{-2J^0(y)} 2J^0(x) \prod_{z > x} q^{2J^0(z)} \right) = \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}} .
\]

(4.18)

To prove this equality we use the method of complete induction. First of all one has to realize that our operator \( J^0(x) \) admits only the eigenvalues 0 and \( \pm 1/2 \) for any \( x \).
This fact is a direct consequence of the Pauli exclusion principle for anyon operators, which, in this respect, behave like ordinary fermions (cf eq. (3.48a)). Therefore, the following identity holds

$$2J^0(x) = \frac{q^{2J^0(x)} - q^{-2J^0(x)}}{q - q^{-1}} \quad (4.19)$$

for any $x$. If the lattice $\Omega$ has only one site, say the point $a$, clearly $J^0 = J^0(a)$ and thus the equality (4.18) is true by virtue of the identity (4.19).

Let us now assume that eq. (4.18) is correct for a lattice $\Omega$ of $N$ sites and then prove that it remains true when an extra point, say $b$, is added. We suppose that

$$b > x_i$$

for $i = 1, ..., N$, so that eq. (4.18) for the lattice with $N + 1$ sites becomes

$$\sum_{i=1}^{N} \left[ \prod_{j<i} q^{-2J^0(x_j)} \prod_{k>i} q^{2J^0(x_k)} \right] q^{2J^0(b)} + q^{-2J^0} q^{2J^0(b)} = \frac{1}{q - q^{-1}} \left( q^{2J^0} q^{2J^0(b)} - q^{-2J^0} q^{-2J^0(b)} \right),$$

where $J^0 = \sum_{i=1}^{N} J^0(x_i)$. Eq. (4.20) is easily proved using eq. (4.18) to replace the term in square brackets in the left hand side with

$$\frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}},$$

so that one is left with

$$q^{-2J^0} q^{2J^0(b)} = \frac{q^{-2J^0}}{q - q^{-1}} \left( q^{2J^0(b)} - q^{-2J^0(b)} \right),$$

which is true because of the identity (4.19). The same result can be obtained also when $b < x_i$. This concludes our proof of eq. (4.18).

Therefore we have explicitly shown that the operators $J^\pm$ and $J^0$ built out of anyon oscillators by means of the generalized Schwinger construction do close the algebra of $SU(2)_q$ where the deformation parameter $q$ is directly related to the statistics $\nu$ of the anyon oscillators by $q = \exp(i\pi\nu)$. 

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5. Conclusions

We conclude this paper with a few comments. If \( \nu = 1 \), i.e. \( q = -1 \), the anyonic oscillators have ordinary bosonic statistics and no braiding phases appear in their commutation relations. Therefore one should expect that the Schwinger construction in this case yields a standard Lie algebra. This is precisely what happens, because \( SU(2)_{-1} \) is equivalent to the non-compact Lie algebra \( SU(1, 1) \) \([30]\). It is worthwhile to stress that even though the oscillators have bosonic statistics when \( \nu = 1 \), they are not ordinary bosons because they satisfy a hard-core condition, or equivalently a Pauli exclusion principle (cf eq. (3.48a)), like the fermions from which they originate via the Jordan-Wigner transformation. As a matter of fact, for any value of \( \nu \), such a hard-core constraint is an essential ingredient of our construction since it is heavily used in showing that the generators made out of anyons actually close the standard quantum group commutators. Indeed the proof of the equivalence between eqs. (4.16a) and (4.17) is based on the fact that in any site \( x \) of the lattice, the operator \( J^0(x) \) admits only the eigenvalues 0 and \( \pm 1/2 \). Therefore, for our purposes it is essential to use fermion based anyonic oscillators.

It is well known that the Schwinger construction of \( SU(2) \) can be easily generalized to \( SU(N) \) by using \( N \) sets of oscillators instead of two. Thus, we expect that our construction of \( SU(2)_q \) can be extended to \( SU(N)_q \) with no difficulty \([31]\). It could be interesting also to extend our construction to the case in which anyons are defined on a continuum space instead of a lattice. In such a case, one should replace all discrete sums with suitably defined integrals both in the disorder operators and, more generally, in the definition of the comultiplication. Finally, it seems even more interesting to find dynamical systems of anyons in 2+1 dimensions (either on a lattice or in the continuum) which are endowed with this quantum group symmetry, and in particular to study the physical consequence of this rich algebraic structure.

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Figure Captions

Fig. 1 Representation of the elementary lattice angle $\varphi(x, y^*, x + \hat{2})$ under which the link between $x$ and $x + \hat{2}$ is seen from the point $y^*$ of the dual lattice.

Fig. 2 The elementary plaquette $A_x$ whose lower left corner is the point $x$.

Fig. 3 Representation of the lattice angle $\Theta_{P_x}(x, y)$ between the base point $B$ and $x$ measured along the curve $P_x$ from the point $y^*$.

Fig. 4 Examples of the curves $C_x$ for a few points of the lattice.

Fig. 5 The angle between $x$ and $x + \hat{1} + \hat{2}$ centered in $y^*$ is equal to the angle between the lines $(x y^*)$ and $(x^* y)$.

Fig. 6 Examples of the cuts $\gamma_x$ for a few points of the lattice.

Fig. 7 The elementary plaquette $\tilde{A}_x$ whose upper right corner is the point $x$.

Fig. 8 Examples of the curves $D_x$ for a few points of the lattice.

Fig. 9 Examples of the cuts $\delta_x$ for a few points of the lattice.

Fig. 10 Exchanging trajectories for type $C$ anyons. In order not to cross the cut in $x$, the particle in $y$ has to move counterclockwise if $x > y$ and clockwise if $x < y$.

Fig. 11 Exchanging trajectories for type $D$ anyons. In order not to cross the cut in $x$, the particle in $y$ has to move clockwise if $x > y$ and counterclockwise if $x < y$. Notice that $x > y$ means $x_C > y_C \iff x_D < y_D$ (cf eq. (3.38)).
ANYONS AND QUANTUM GROUPS *

Alberto Lerda¹ and Stefano Sciuto

Dipartimento di Fisica Teorica
Università di Torino, and I.N.F.N. Sezione di Torino
Via P. Giuria 1, I-10125 Torino, Italy

Abstract

Anyonic oscillators with fractional statistics are built on a two-dimensional square lattice by means of a generalized Jordan-Wigner construction, and their deformed commutation relations are thoroughly discussed. Such anyonic oscillators, which are non-local objects that must not be confused with $q$-oscillators, are then combined à la Schwinger to construct the generators of the quantum group $SU(2)_q$ with $q = \exp(i\pi\nu)$, where $\nu$ is the anyonic statistical parameter.

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¹ Also at Institute for Theoretical Physics, S.U.N.Y. at Stony Brook, Stony Brook, N.Y. 11794, U.S.A.