A Geometric Interpretation of Reference Frames and Transformations: dq0, Clarke and Park

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Abstract—Transformations between abc, stationary dq0 (αβ0) and rotating dq0 reference-frames are used extensively in the analysis and control of three-phase technologies such as machines and inverters. Previous work on deriving the matrices describing these transformations follows one of two approaches. The first approach derives Clarke’s matrix by modifying symmetrical components. Park’s matrix can be subsequently found from a rotation matrix. The second approach derives Park’s matrix using trigonometric projection by interpreting the transformation as a rotation in the plane of the cross-section of a machine. Then, Clarke’s matrix can be found trivially using a reference angle of zero in Park’s matrix. This paper presents a third approach to deriving the Clarke and Park transformation matrices: a geometric interpretation. The approach exploits properties of the linear transformation using the Cartesian representation. We introduce the locus diagram of a three-phase quantity and show how these diagrams have applications in power quality. We show that, unlike a phasor diagram, a single locus diagram can fully represent a three-phase system with harmonics.

Keywords—dq0, Park, Clarke, reference-frame theory, locus diagram, power quality

I. INTRODUCTION

Transformations between abc and dq0 reference-frames were originally used to assist in electrical machine analysis and modelling [1]. Currently, dq0 based models are used in a wide variety of applications including: modelling and control of electric machines and drives [1], [2], multimachine modelling [3], multi-inverter modelling [4], microgrid simulation [5]–[7], phase-locked loops (PLLs) [8] and active power filters [9]. In many of these examples, the dq components no longer refer to the direct and quadrature axes of a machine. This motivates an alternative interpretation of reference-frames that is detached from any specific technology or application. This paper provides one such alternative perspective which we refer to as the “geometric interpretation”.

Considering the wealth of literature available on the subject of reference frames, it is essential that this paper positions its contributions in the context of previous work. To this end, we must first study the history of these transformations, taking care to cite the most notable references. We review Krause’s arbitrary reference-frame, and how this perspective relates to each transformation. Following this, we present the geometric approach to deriving the Clarke and Park transformation matrices. We distinguish the contributions of this paper from previous work, and the advantages and disadvantages of the geometric approach are discussed.

A. Review of the Clarke and Park Transformations

Fig. 1 provides an overview of the transformations. The Clarke transformation converts three-phase abc quantities to αβ0 (ie stationary dq0). The Park transformation converts abc quantities to dq0 and can be thought of as applying the Clarke transformation first, followed by the αβ0 to dq0 transformation. Here we refer to the latter as simply the “dq0 transformation” for simplicity of subscript notation. Later we will discuss how this corresponds to the “frame-to-frame-transformation” as described in [1].

Fig. 1. Relationships between the Park and Clarke transformations. Note that the term “dq0 transform” as defined in this paper refers to a transformation from αβ0 to dq0 and is therefore not equivalent to the Park transformation.

Fig. 2. Clarke and Park transformations applied to three-phase 50 Hz voltages 1 under three conditions: (i) Balanced fundamental frequency with a phase shift (ii) Unbalanced fundamental (iii) balanced with harmonics (1\textsuperscript{st}, 5\textsuperscript{th}, 7\textsuperscript{th}).

Fig. 2 shows the affect of applying the standard Clarke and Park transformations under three different conditions: (i) Balanced voltages result in equal magnitudes for $v_\alpha$ and $v_\beta$ and constant values of $v_d$ and $v_q$. (ii) Unbalanced voltages result in unequal magnitudes for $v_\alpha$ and $v_\beta$ and time-varying $v_d$ and $v_q$ at the 2\textsuperscript{nd} harmonic. $v_0$ is a zero-sequence component at the fundamental and is always identical in both transformations.

Condition (iii) in Fig. 2 illustrates the affect of harmonics. Each phase voltage includes fundamental, 5\textsuperscript{th} and 7\textsuperscript{th} harmonics, with balanced voltages at each harmonic. These particular harmonics appear as a 6\textsuperscript{th} harmonic in $v_d$ and $v_q$. There is no zero-sequence component for these particular harmonics. The voltage $v_0$ is equivalent to $v_0$; and $v_\beta$ has a different harmonic profile to $v_\alpha$ due to a 180\degree phase-shift on its positive sequence components. In Section V, each of the conditions (i), (ii) and (iii) in Fig. 2 are explained using the geometric interpretation.

1) History of the Clarke Transformation: In the 1930s, Clarke made a series of modifications to symmetrical components [10], [11]. These modifications simplified the calculations for certain classes of unbalanced three-phase problems [11], [12]. The $\alpha$, $\beta$ and 0 components were one set of these innovations [11], and were particularly useful as they did not require the $q$ operator (1/120\degree) or complex numbers.
reference speeds for their dq0 axes, to assist in the analysis of induction machines. Eliminating time-varying inductance in addition to achieving a diagonalised inductance matrix were primary objectives [1].

In 1965, Krause described in [20] that all of the different reference-frames used in [11], [13], [17]–[19] are specific applications of the “arbitrary reference-frame”. They all refer to dq0 axes that rotate at a specified \( \omega \). A list of commonly used reference-speed frames are given below [1]:

- \( \omega \). The dq0 axes rotate at an arbitrary speed. [20].
- \( \omega = \omega_r \). The dq0 axes rotate at the rotor speed [13].
- \( \omega = \omega_e \). The dq0 axes rotate at the synchronous speed.
- \( \omega = 0 \). The dq0 axes are stationary (Clarke transform).

All of the reference-frames listed can be described by Park’s transformation matrix, except that each uses a different rotation speed \( \omega \) for the dq0 axes. We refer the reader to [1] for an extensive discussion of the various reference-frames.

One of the goals of this work is to provide an alternative derivation of Park’s transformation matrix, which describes all of the listed reference-frames (when the appropriate \( \omega \) is inserted into this matrix). Therefore, we shall take a general approach and consider an arbitrary reference speed \( \omega \) whenever we are referring to Park’s transformation matrix. We will still refer to this matrix as “Park’s transformation”, even though we are referring to Park’s transformation matrix. One is free to choose any reference speed they wish.

C. Contributions

The contributions of this paper are summarised below. The remainder of this section elaborates on these points:

1) We group the previous approaches to deriving the matrices describing the Park and Clarke transformations into two approaches. This paper presents a third approach to deriving the Clarke and Park transformation matrices: a geometric approach.

2) We introduce the “locus diagram” of a three-phase quantity and how this locus changes in the presence of unbalance and harmonics.

The first contribution of this paper is to provide an alternative approach to derive the Park and Clarke transformations. Previous work on deriving these transformation matrices follows one of two approaches:

(i) The Clarke transformation matrix is derived from symmetrical components [11] as shown in Fig. 3. The Park transformation matrix can be subsequently derived using a rotation matrix such as Eq. (29).

(ii) The Park transformation matrix is derived trigonometrically by interpreting the transformation as a rotation in the plane of the cross-section of a machine. The abc axes are coplanar stationary axes that lie 120° apart and dq quantities can be projected onto the abc axes in a manner shown in Fig. 4(b). A third transformation variable is introduced to satisfy the change of variables. This is chosen to be the zero component, which is added separately. Many authors trigonometrically project in the opposite manner; from abc to dq and are thus required to specify scaling factors \( k_d \) and \( k_q \), normally equal to either \( 2/3 \) or \( \sqrt{2/3} \); see [21]. These projections describe the approach taken by the majority of authors such as [1], [13], [17]–[24]. The Clarke transformation matrix can then be derived trivially by setting \( \omega = 0 \) in Park’s transformation matrix. One should note that the coplanar abc axes are usually considered to have a physical meaning relating to the magnetic axes in the cross-section of a machine as in Fig. 4(a), but this
physical interpretation of the abc axes is not necessary to derive the matrix [1].

This paper presents a third approach to deriving the Clarke and Park transformation matrices: a geometric interpretation. This geometric approach uses the Cartesian representation: three-phase quantities are represented by vectors in \( \mathbb{R}^3 \), where each orthogonal component of the vector corresponds to the instantaneous value of one of the three phases. The first appearance of the Cartesian representation applied to three-phase quantities was given by Lipo in [25]. Other work that uses this representation includes [26], [27]. More recently, Montanari and Gole use a three-dimensional perspective to introduce a new transformation termed the “\( m \)-\( m \)-transform” [28]. The \( m \)-\( m \)-transformation assists in the calculation of instantaneous real and reactive power for systems containing four-wire inverters. This enables the mitigation of power oscillations that normally occur when such systems are unbalanced [28]. Although others have utilised the Cartesian representation in [25]–[28], this paper is unique as the representation is used to derive the matrices describing the Clarke and Park transformations.

The geometric approach is explained step-by-step in Section III and Section IV. A summary of the derivations provided by the geometric view is given by Fig. 5. Each transformation is interpreted as a combination of vector rotation and scaling in \( \mathbb{R}^3 \). The abc axes are orthogonal stationary axes that lie 90° apart and have basis vectors that span \( \mathbb{R}^3 \). We exploit the linearity property of matrix transformations, and derive each transformation matrix by observing how each transformation affects the orthonormal basis vectors of the vector space.

The geometric approach has many advantages when compared to the two traditional approaches listed previously. These include:

- When trigonometrically deriving the Park transformation such as in Fig. 4(b), zero-sequence components are treated separately in the derivation. The \( d \) and \( q \) components are found from a projection operation whereas the 0 components are added separately. The geometric approach finds all \( dq0 \) components in a unified manner via Eq. (4).

- The previous approaches interpret the Clarke transformation as either a manipulation of symmetrical components as in Fig. 3, or as a specialised case of the arbitrary reference-frame with stationary \( dq0 \) axes [1]. The geometric approach interprets the power-invariant Clarke transformation as a single rotation in \( \mathbb{R}^3 \), which some readers may find to be a simpler explanation (see Fig. 8). The standard (amplitude-invariant) Clarke transformation is shown in Fig. 10 to be a combination of rotation and scaling in \( \mathbb{R}^3 \).

- Similarly, previous approaches interpret the Park transformation as either a manipulation of symmetrical components [11] combined with a rotation matrix, or as a projection onto coplanar abc axes [13]. The geometric approach interprets the power-invariant Park transformation as two consecutive rotations in \( \mathbb{R}^3 \), which some readers may find to be more intuitive (see Fig. 5). The standard (amplitude-invariant) Park transformation is interpreted as first applying the standard Clarke transform (rotation and scaling) followed by a pure rotation in \( \mathbb{R}^3 \) given by Fig. 12.

- The orthogonality (\( A^T = A^{-1} \)) of the power-invariant forms of both transformations can be easily seen from all three approaches via matrix manipulation. The geometric interpretation illustrates this orthogonal property: orthogonal transformations preserve vector length and can thus be visualised as pure rotations in \( \mathbb{R}^3 \) [29].

The disadvantages of the geometric interpretation, compared with the two traditional approaches include:

- The geometric derivation is more involved. This can be seen by comparing Fig. 5 to the two traditional approaches illustrated by Fig. 3 and Fig. 4(b).

- The diagrams required to explain the geometric view are more complex to draw as they are three-dimensional.

The second contribution of this paper involves the “locus diagram” of a three-phase quantity and how this locus changes in the presence of unbalance and harmonics. This contribution consists of the following:

- In Section II-C we show that for balanced systems, the locus corresponds to a circle in \( \mathbb{R}^3 \). We derive Eq. (12) which shows that this circle has a radius of \( V \sqrt{3/2} \) where \( V \) is the voltage magnitude on each phase.

- In Section V we extend the locus diagram to cases of harmonics and unbalance. Systems with purely positive and negative sequence will have a locus that lies within the \( \alpha \beta \)-plane. The locus of a zero-sequence component is a line segment perpendicular to the \( \alpha \beta \)-plane.

- We show that a single locus diagram can fully represent a three-phase quantity containing harmonics in Fig. 18. This is not possible using a single phasor diagram.

II. TRANSFORMATIONS, CARTESIAN REPRESENTATIONS & THE LOCUS OF THREE-PHASE QUANTITIES

This section introduces the mathematical tools that are used to develop geometric derivations of the Clarke and Park transformation matrices. Firstly, we review the fundamentals of linear transformations. Secondly, we describe the Cartesian representation and compare it with phasor notation. Thirdly, we introduce the concept of the “locus of a three-phase quantity, and show that the locus of a balanced set of signals is a circle in \( \mathbb{R}^3 \).

A. Review of Linear Transformations

Transformations are functions that operate on vectors. This section derives a basic method to finding a unique matrix \( A \) that fully describes a linear transformation \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \).

Any vector \( \vec{v} \in \mathbb{R}^n \) can be written as a linear combination of the standard basis unit vectors \( \{ \hat{\epsilon}_1, \hat{\epsilon}_2, \ldots, \hat{\epsilon}_n \} \).

\[
\vec{v} = v_1 \hat{\epsilon}_1 + v_2 \hat{\epsilon}_2 + \ldots + v_n \hat{\epsilon}_n
\]

To find \( T(\vec{v}) \in \mathbb{R}^m \) we apply the linear transformation \( T \) to \( \vec{v} \) and impose the additivity and homogeneity constraints of linearity:

\[
T(\vec{v}) = v_1 T(\hat{\epsilon}_1) + v_2 T(\hat{\epsilon}_2) + \ldots + v_n T(\hat{\epsilon}_n)
\] (2)

\( T(\vec{v}) \) in Eq. (2) is now expressed in terms of transformed standard basis vectors scaled by the components of \( \vec{v} \). Such a linear combination of column vectors can always be written as a matrix-vector product:

\[
T(\vec{v}) = A \vec{v}
\]

(3)

Eq. (3) says that any linear transformation \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) can be expressed as a matrix-vector product \( A \vec{v} \).
B. Cartesian Representation of Three-Phase Voltages

Three-phase quantities such as voltages, currents and flux linkages are often expressed using phasor notation. This section introduces the Cartesian representation and compares it with phasor notation.

1) Phasor Representation: Eq. (5) is an example of a set of three-phase voltages with no harmonics. For now these voltages may or may not be balanced, where “balanced” would require \( \phi_a = \phi_b = \phi_c = 0 \) and \( V_a = V_b = V_c \).

\[
\begin{aligned}
v_a(t) &= V_a \cos(\omega t + \phi_a) \\
v_b(t) &= V_b \cos(\omega t - \frac{2\pi}{3} + \phi_b) \\
v_c(t) &= V_c \cos(\omega t + \frac{2\pi}{3} + \phi_c)
\end{aligned}
\] (5)

Each of the three sinusoidal voltages in Eq. (5) can be represented by a unique phasor. Phasor notation is the use of a single complex known as a phasor to store the two parameters of magnitude \( V \) and phase \( \phi \). The magnitude of the phasor \( V \) represents the RMS value of \( v_1(t) \) and the phase \( \phi \) corresponds to the angle of the voltage \( v_1(t) \).

Note that the expression for each sinusoidal voltage in Eq. (5) is actually defined by three parameters: voltage magnitude \( V \), phase \( \phi \) and frequency \( \omega \). A known frequency must be assumed, which is one limitation of the phasor representation. In addition, the phasor representation cannot be used to represent signals containing more than one frequency component, such as signals with harmonics.

Eq. (6) expresses the voltages in Eq. (5) as three phasors \( V_a, V_b \) and \( V_c \). These three phasors can be drawn on a single complex plane in a phasor diagram. Fig. 6(a) draws a balanced case.

\[
\begin{aligned}
V_a &= \frac{1}{\sqrt{3}} V_o e^{j\phi_a} \\
V_b &= \frac{1}{\sqrt{2}} V_o e^{j(-\frac{2\pi}{3} + \phi_b)} \\
V_c &= \frac{1}{\sqrt{2}} V_o e^{j(\frac{2\pi}{3} + \phi_c)}
\end{aligned}
\] (6)

Each voltage phasor in Eq. (6) can be converted back to a function of time using Euler’s relation as shown in Eq. (7).

\[
v_1(t) = \sqrt{2} \Re \{ V e^{j\omega t} \}
\] (7)

2) Cartesian Representation: We use the notation \( \overrightarrow{v_{abc}} \) to signify the Cartesian representation of a set of three-phase voltages. Previous work that uses the Cartesian representation applied to three phase quantities includes: [25]-[28]. \( \overrightarrow{v_{abc}} \) is a single vector in \( \mathbb{R}^3 \) and has three components corresponding to three orthogonal \( abc \) axes:

\[
\overrightarrow{v_{abc}} = v_a \hat{e}_a + v_b \hat{e}_b + v_c \hat{e}_c
\] (8)

The components of Eq. (8) vary with time. Thus \( \overrightarrow{v_{abc}} \) is a vector that moves in \( \mathbb{R}^3 \) over time as seen in Eq. (9):

\[
\begin{bmatrix}
\overrightarrow{v_{acl}}(t) \\
\overrightarrow{v_{bc}}(t) \\
\overrightarrow{v_{cc}}(t)
\end{bmatrix} =
\begin{bmatrix}
V_a \cos(\omega t + \phi_a) \\
V_b \cos(\omega t - \frac{2\pi}{3} + \phi_b) \\
V_c \cos(\omega t + \frac{2\pi}{3} + \phi_c)
\end{bmatrix}
\] (9)
Fig. 6. Three-phase voltage representations: (a) Phasor representation (b) Cartesian representation at time $t_1$.

Fig. 7. Locus diagram of balanced three-phase voltages.

Eq. (11) can be rewritten using trigonometric identities to give:

$$\|\vec{v}_{abc}(t)\| = V \sqrt{\frac{3}{2} \sin^2(\omega t) + \frac{3}{2} \cos^2(\omega t)}$$

Eq. (12) shows that the locus of $\vec{v}_{abc}$ is a circle in $\mathbb{R}^3$ for a balanced set of three-phase voltages, as the vector length is constant. This circle is shown in Fig. 7 and has a radius of $V\sqrt{3/2}$ where $V$ is the voltage magnitude on each phase.

This exercise of finding the length of $\vec{v}_{abc}$ also illustrates another important concept: the length of $\vec{v}_{abc}$ is not equivalent to the peak phase voltage, even when the voltages are balanced. It is scaled by $\sqrt{3/2}$. This geometric analysis explains why the power-invariant Clarke and Park transformations have such scaling terms, as will be discussed in sections III and IV.

III. GEOMETRIC DERIVATION OF THE CLARKE TRANSFORMATION

There are two versions of the Clarke transformation: the standard (amplitude-invariant) transformation and the power-invariant transformation. The derivation introduced by Clarke as shown in Fig. 3 is the amplitude-invariant form, which is the most commonly used version. It is convenient because the magnitude of $\vec{v}_a$ is the same as the magnitude of $\vec{v}_c$ when the voltages are balanced. Previous approaches to deriving the Clarke transformation either rely on a manipulation of symmetrical components [11], or use the arbitrary reference-frame with stationary axes [11]. In this section, we use the geometric approach to derive both the standard and power-invariant Clarke transformations. The power-invariant version is derived first, as it is geometrically simpler.

A. Power-Invariant Clarke Transformation Derivation

The power-invariant Clarke transformation is a pure rotation, such that the locus of a balanced three-phase quantity lies in the $ab$-plane. Fig. 8 illustrates the locus diagrams for the geometric power-invariant Clarke transformation.

All transformations can be visualised as either a coordinate (axes) transformation or as a vector transformation. Fig. 8(a) is the axes transformation where the vector is fixed and the $abc$ axes are rotated such that the locus of a balanced system lies in the rotated $ab$-plane (ie the $\alpha\beta$-plane). Fig. 8(b) is the vector transformation, where the axes are fixed and the vector rotates such that its locus lies in the $ab$-plane.

Note that there are infinite transformations that can achieve a locus that lies in the $ab$-plane, but only one of these anchor the $\alpha$-axis so that it is in line with the balanced Cartesian voltage when the phase angle is zero ($\theta = \omega t + \phi = 0$), as seen in Fig. 8. We will see later that this family of infinite transformations is given by Park’s matrix where substituting a value of theta anchors the $\alpha$-axis at a different location in the plane.

Using the constraints of bringing the locus into the $ab$-plane and anchoring the $\alpha$-axis appropriately we can determine the matrix associated with the power-invariant Clarke transformation. In Section II-A we derived Eq. (4) which describes steps to find a matrix (e.g. $A_{\alpha\beta}$) that represents a linear transformation ($T_{\alpha\beta}$). These steps require one to know how each orthonormal basis vector of a given space is affected by a transformation.

The inverse Clarke transformation $T^{-1}_{\alpha\beta}$ is more convenient to derive geometrically than $T_{\alpha\beta}$. One can visualise how $T^{-1}_{\alpha\beta}$ transforms vectors by reading Fig. 8 from right to left. The inverse transformation rotates the unit vectors $\hat{e}_a$ and $\hat{e}_b$ such that they lie in the plane of $\vec{v}_{abc}$. Thus, these transformed unit
Each step involves a rotation of a unit vector. We apply these three steps of Eq. (14) are shown graphically in Fig. 9.

We showed in Section II-C that balanced three-phase systems have circular loci with a radius given by Eq. (12). Substituting Eq. (12) into Eq. (15) gives:

$$T_c^{-1}(\hat{e}_\alpha) = \frac{-\vec{v}_{abc}}{||\vec{v}_{abc}||}$$ (15)

Fig. 9(b) illustrates how the unit vector $\hat{e}_\beta$ is rotated to align with the Cartesian voltage when the angle is $\pi/2$.

$$T_c^{-1}(\hat{e}_\beta) = \frac{-\vec{v}_{abc}}{||\vec{v}_{abc}||}$$ (17)

In Fig. 9(c) we see how $\hat{e}_0$ is rotated such that it is perpendicular to the plane of a balanced locus. Mathematically, this can be thought of as pointing in the direction of the cross product of $\hat{e}_\alpha$ and $\hat{e}_\beta$, as given by the right-hand rule:

$$T_c^{-1}(\hat{e}_0) = \frac{-\vec{v}_{abc}}{||\vec{v}_{abc}||} \times \frac{-\vec{v}_{abc}}{||\vec{v}_{abc}||}$$ (18)

$$T_c^{-1}(\hat{e}_0) = \sqrt{\frac{2}{3}} \left[ \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right]$$

The three steps given by Eq. (17), Eq. (18) and Eq. (19) are combined with Eq. (14) to find $A_c^{-1}$.

$$A_c^{-1} = \sqrt{\frac{2}{3}} \left[ \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right]$$ (20)

The matrix $A_c^{-1}$ is an orthogonal matrix because it is associated with a pure rotation. This means its transpose is equal to its inverse, $A_c = (A_c^{-1})^\top$.

B. Standard Clarke Transformation Derivation

The standard (amplitude-invariant) Clarke transformation was originally derived by Clarke in a manner shown in Fig. 3. This section geometrically derives the amplitude-invariant Clarke transformation which has become the standard version.

The standard Clarke transformation is a rotation and scaling, such that the locus of a balanced three-phase quantity lies in the $ab$-plane with a radius equal to the phase magnitude. It can be thought of as first applying the pure rotation described by the power-invariant Clarke transformation followed by a scaling operation. Eq. (12) shows that the locus of a balanced three-phase voltage is a circle of radius $V\sqrt{3/2}$. The standard Clarke transformation scales this locus, such that the circle has a radius of $V$.
The vector’s magnitude of $\sqrt{3}/2$ is scaled by $1/\sqrt{3}$ in order to agree with the definition of zero-sequence. Whatever voltage exists on the $0$-axis will appear with the same magnitude on the $a$, $b$, and $c$ axes.

Fig. 10(b) is the vector transformation, where the axes are fixed and the vector rotates such that its locus lies in the $ab$-plane. The vector’s $\alpha$ and $\beta$ components are scaled by $\sqrt{2}/\sqrt{3}$, meaning the locus of a balanced Cartesian vector will appear as a circle with a radius of $V$ when referenced to the $\alpha\beta$-axes. The $0$-axis is scaled by $1/\sqrt{3}$ to match the symmetrical components definition of zero-sequence.

Just as with the power-invariant transformation, there are infinite transformations that can achieve a locus that lies in the $ab$-plane with the scaling described as above. However, only one of these ensure that the $\alpha$-axis is in line with the balanced Cartesian voltage when the phase angle is zero ($\theta = \omega t + \phi = 0$).

We follow the same procedure as the power-invariant derivation. Once again, we find the matrix $A^{-1}_c$ associated with the inverse transformation $T_{c}^{-1}$ using Eq. (14). Please refer to Section III-A for a discussion on why we choose to derive the inverse Clarke transformation.

The three steps described by Eq. (14) are shown graphically in Fig. 11. They involve transforming each of the three unit vectors under $T_{c}^{-1}$. Fig. 11(a) shows how $\hat{e}_\alpha$ is transformed under the inverse standard Clarke transformation. $\hat{e}_\alpha$ is rotated and stretched by $\sqrt{3}/2$, making it equivalent to the per-unit Cartesian voltage when the angle is zero.

$$T_{c}^{-1}(\hat{e}_\alpha) = \frac{\bar{v}_{abc}}{V} = \begin{bmatrix} V \cos\theta \\ V \cos(\theta - \frac{2\pi}{3}) \\ V \cos(\theta + \frac{2\pi}{3}) \end{bmatrix}$$

For $\theta = 0$ and $\theta = \pi/2$, we have:

$$\hat{e}_\alpha = T_{c}^{-1}(\hat{e}_\alpha) = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \sqrt{3} \end{bmatrix}^T$$

Similarly, Fig. 11(b) shows that $\hat{e}_\beta$ is rotated and stretched by $\sqrt{3}/2$, making it equivalent to the per-unit Cartesian voltage when the angle is $\pi/2$.

$$T_{c}^{-1}(\hat{e}_\beta) = \frac{\bar{v}_{abc}}{V} = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ -\sqrt{3}/2 \end{bmatrix}^T$$

Fig. 11(c) explains how $\hat{e}_0$ is transformed. It points perpendicular to the plane in which the locus of $\bar{v}_{abc}$ lies, and is scaled by $\sqrt{3}$. The scaling is necessary so that the $0$-component agrees with the $0$-sequence as defined by symmetrical components.

$$T_{c}^{-1}(\hat{e}_0) = \sqrt{3} \begin{bmatrix} \bar{v}_{abc} \bigg|_{\theta=0} \times \bar{v}_{abc} \bigg|_{\theta=\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The three transformed unit vectors given by Eq. (21), Eq. (22) and Eq. (23) are combined with Eq. (14) to find $A^{-1}_c$.

$$A^{-1}_c = \begin{bmatrix} 1 & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ -\frac{1}{2} & -\frac{3}{2} & 1 \end{bmatrix}$$

We find $A_c$ by taking the inverse of the matrix $A^{-1}_c$.

IV. GEOMETRIC DERIVATION OF THE PARK TRANSFORMATION

There are two versions of the Park transformation: the standard (amplitude-invariant) transformation and the power-invariant transformation. The derivation introduced by Park in Fig. 4(b) is the amplitude-invariant form, which is the most commonly used version. It is convenient because the magnitude of $v_d$ is the same as the magnitude of $v_a$ if two conditions are met: the voltages are balanced and the reference signal is in phase with phase $\alpha$.

Previous approaches to deriving the Park transformation either use: trigonometric projection with coplanar abc axes [13] or modifying symmetrical components to obtain Clarke’s matrix [11] and applying a rotation matrix. This section derives
the Park transformation matrix using the geometric approach. We utilise the relationship between the Park and Clarke transformations as shown in Fig. 1. The Park transformation can be decomposed into two consecutive transformations: the Clarke transformation followed by the Park transformation matrix using the geometric approach. Section III details the geometric derivation of the Clarke transformation. This section completes the Park transformation matrix derivation by first deriving the \( \alpha \beta \) to \( dq \) transformation. Then the Park transformation matrix is obtained by simple matrix multiplication. The overall geometric interpretation of the Park transformation is summarised in Fig. 5.

### A. Transformation between Reference-Frames: \( \alpha \beta \) to \( dq \) Transformation Derivation

The “transformation between reference-frames” or simply “frame-to-frame transformation” in [1] is used in multimachine [3] and multi-inverter modelling [4]. Each device is modelled in its own \( dq \) reference-frame, and each \( dq \) frame may have a different angle \( \theta \) with respect to a common reference-frame. All devices can be translated to the common reference-frame using the transformation between two rotating \( dq \) frames [4]. The matrix describing this transformation has the same form as one that transforms from a stationary to a rotating \( dq \) reference-frame. The transformation between two rotating \( dq \) frames is equivalent to this paper’s \( \alpha \beta \) to \( dq \) transformation. We use Eq. (4) to derive this transformation, whereas the “transformation between reference-frames” is derived in an alternative manner, using matrix multiplication: see section 3.10 of [1].

The \( \alpha \beta \) to \( dq \) transformation can be geometrically interpreted in \( \mathbb{R}^3 \) as a pure rotation about the \( 0 \)-axis by a specified angle \( \theta_{ref} \). Fig. 12 illustrates the axes and vector transformation loci diagrams for the \( \alpha \beta \) to \( dq \) transformation.

Fig. 12(a) is the axes transformation where the \( \alpha \beta \) axes are rotated counterclockwise (CCW) about the \( 0 \)-axis by an angle \( \theta_{ref} \). It is helpful to visualise the motion of the axes and vectors to understand the \( \alpha \beta \) to \( dq \) transformation. Balanced systems have a Cartesian vector \( \vec{v}_{\alpha\beta} \) that lies in the \( \alpha \beta \)-plane and rotates CCW about the \( 0 \)-axis at speed \( \omega \). Note that \( \vec{v}_{\alpha\beta} \) has an arbitrary angle \( \theta_0 \) with respect to the \( \alpha \)-axis (\( \theta_0 = \omega t + \phi_0 \)). The \( \alpha \beta \) axes are stationary and \( \theta_0 \) increases with time.

The \( dq \) axes of Fig. 12(a) are not stationary, unlike the \( \alpha \beta \) axes. These \( dq \) axes rotate CCW about the \( 0 \)-axis at an angle \( \theta_{ref} = \omega_{ref} t + \phi_{ref} \). \( \vec{v}_{dq} \) is the Cartesian vector referenced to \( dq \) coordinates. If \( \omega_{ref} = \omega \) then \( \vec{v}_{dq} \) will have \( v_d \) and \( v_q \) components which appear constant as the \( dq \) axes are rotating at the same speed as the Cartesian vector \( \vec{v}_{\alpha\beta} \). This case is illustrated by condition (i) of Fig. 2.

Fig. 12(b) is the vector transformation, where the axes are fixed and the Cartesian vector \( \vec{v}_{\alpha\beta} \) is rotated clockwise (CW) about the \( 0 \)-axis by an angle \( \theta_{ref} \). Thus, the vector has a net CCW angle of \( \theta_0 - \theta_{ref} \) relative to the \( d \)-axis. \( \vec{v}_{\alpha\beta} \) is rotating CCW at an angular velocity \( \omega \) when referenced to the \( \alpha \beta \) axes. When referenced to the \( dq \) axes, the vector \( \vec{v}_{dq} \) has a CCW angular velocity of \( \omega - \omega_{ref} \). If \( \omega_{ref} = \omega \) then \( \vec{v}_{dq} \) will appear stationary on the \( dq \) axes.

The matrix \( A_{dq0} \) in Eq. (25) is found using Eq. (4). We apply \( T_{dq0} \) to each basis vector \( \{\hat{e}_d, \hat{e}_q, \hat{e}_0\} \) as shown in Fig. 13. \( T_{dq0} \) rotates the vectors \( \hat{e}_d \) and \( \hat{e}_q \) CW about the \( 0 \)-axis by \( \theta_{ref} \). The components of \( T_{dq0}(\hat{e}_d) \) and \( T_{dq0}(\hat{e}_q) \) can be found using trigonometric relations.

\[
T_{dq0}(\hat{e}_d) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ sin \theta & \cos \theta & 0 \end{bmatrix}^T
\]

\[
T_{dq0}(\hat{e}_q) = \begin{bmatrix} \sin \theta & \cos \theta & 0 \\ -\sin \theta & \cos \theta & 0 \end{bmatrix}^T
\]

Fig. 13 shows how \( \hat{e}_0 \) is preserved under \( T_{dq0} \).

\[
T_{dq0}(\hat{e}_0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T
\]

The three transformed unit vectors given by Eq. (26), (27) and (28) are combined with Eq. (4) to find \( A_{dq0} \). The inverse transformation is found readily as the matrix is orthogonal (\( A_{dq0}^{-1} = A_{dq0}^\dagger \)).

\[
A_{dq0} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

### B. Power-Invariant Park Transformation Derivation

We derive Park’s transformation utilising the relationships between the transformations in Fig. 1. The Park transformation is decomposed into the Clarke and \( \alpha \beta \) to \( dq \) transformations in Eq. (30).

\[
\vec{v}_{dq0} = T_{dq0}(T_{e}(\vec{v}_{abc})) = A_{dq0}A_{e}\vec{v}_{abc} = A_{p}\vec{v}_{abc}
\]

The power-invariant Park transformation is constructed using the power-invariant Clarke transformation of Eq. (20) and the \( \alpha \beta \) to \( dq \) transformation in Eq. (29). Please refer to Section III-A for a comprehensive derivation of the power-invariant Clarke transformation.

\[
A_{p} = A_{dq0}A_{e} = A_{dq0} \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\]

\[
A_{p} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos \theta & \cos (\theta - \frac{2\pi}{3}) & \cos (\theta + \frac{2\pi}{3}) \\ -\sin \theta & -\sin (\theta - \frac{2\pi}{3}) & \sin (\theta + \frac{2\pi}{3}) \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
A_{p} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\]
C. Standard Park Transformation Derivation

The standard Park transformation can be determined in the same way as the power-invariant transformation using the relationships between the transformations (see Eq. 1) and Eq. (30). The difference is that we substitute the standard Clarke transformation of Eq. (24) for \( A_c \). \( A_{dq0} \) is given by Eq. (29). Please refer to Section III-B for a comprehensive derivation of the standard Clarke transformation.

\[
A_p = A_{dq0} A_c = A_{dq0} \begin{bmatrix} 2 \cos \theta & \cos \left( \theta - \frac{2\pi}{3} \right) & \cos \left( \theta + \frac{2\pi}{3} \right) \\ 0 & \sqrt{3} & -\sqrt{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \]

\[
A_p = \frac{2}{3} \begin{bmatrix} \cos \theta & \cos \left( \theta - \frac{2\pi}{3} \right) & \cos \left( \theta + \frac{2\pi}{3} \right) \\ -\sin \theta & -\sin \left( \theta - \frac{2\pi}{3} \right) & -\sin \left( \theta + \frac{2\pi}{3} \right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} 
\]  

(32)

D. Standard Park Transformation Derivation: A Direct Geometric Approach

In this paper we decoupled the Park transformation into two operations, as shown in Fig. 5. Understanding the Park transformation as two consecutive operations highlights the geometric relationship between the Clarke, Park and frame-to-frame transformations.

Alternatively, one can use the geometric approach to directly derive the transformation from \( abc \) to \( dq0 \), without considering an intermediate \( ab0 \) reference frame. In this section, we show this direct derivation for the standard Park transformation using the approach outlined in Section II-A and given by Eq. (4). The power-invariant Park transformation can also be found directly using a similar approach.

Fig. 14 illustrates the standard \( abc \) to \( dq0 \) transformation. This is plotted for the case where the \( d \)-axis lines up with the vector \( \vec{v}_{dq0} \). Refer to Fig. 12 for the case where the \( d \)-axis may not be in line with \( \vec{v}_{dq0} \). Fig. 14(a) shows the axis transformation, where we see the axes rotating and stretching so that \( \vec{v}_{dq0} \) traces out a circle of radius \( V \) in the \( dq \)-plane.

Fig. 14(b) shows the standard Park transformation from a different perspective. Instead of rotating and stretching the axes, we do the opposite. The axes are fixed and we rotate and scale the vector so that \( \vec{v}_{dq0} \) has a length of \( V \) and moves CCW about the \( 0 \)-axis at a net speed of \( \omega - \omega_{ref} \) relative to the \( dq \)-axes. \( \vec{v}_{dq0} \) will appear stationary relative to the \( dq \)-axes, if we apply Park’s matrix using a reference signal at the same frequency as the phase voltages \( (\omega - \omega_{ref}) \).

The inverse Park transformation \( T_p^{-1} \) is more convenient to derive geometrically than \( T_p \) (analogous to why we derived \( T_{dq}^{-1} \) in Section III). One can visualise how \( T_p^{-1} \) transforms vectors by reading Fig. 14 from right to left. The inverse transformation rotates the unit vectors \( \hat{e}_d \) and \( \hat{e}_q \) such that they lie in the plane of \( \vec{v}_{abc} \). Thus, these transformed unit vectors \( T_p^{-1}(\hat{e}_d), T_p^{-1}(\hat{e}_q) \) have a direction given by \( \vec{v}_{abc} \) at angles of \( \theta \) and \( \theta + \pi/2 \) respectively. Whereas \( T_p \) rotates the \( abc \) unit vectors \( \hat{e}_a, \hat{e}_b, \hat{e}_c \) to a location that is inconvenient to determine.

We derive Park’s matrix by applying Eq. (4), which requires finding all three transformed unit vectors. These three steps are shown graphically by Fig. 15, where each of the unit vectors \( \hat{e}_d, \hat{e}_q, \hat{e}_0 \) are transformed under \( T_p^{-1} \). Fig. 15(a) shows how \( \hat{e}_d \) is transformed under the inverse Park transformation. \( \hat{e}_d \) is rotated such that it lines up with \( \vec{v}_{abc} \) at angle \( \theta \) and stretched by \( \sqrt{3}/2 \).

\[
T_p^{-1}(\hat{e}_d) = \vec{v}_{abc} \begin{bmatrix} V \cos \theta & V \cos \left( \theta - \frac{2\pi}{3} \right) & V \cos \left( \theta + \frac{2\pi}{3} \right) \end{bmatrix}^{\top} = \begin{bmatrix} V \cos \theta & V \cos \left( \theta - \frac{2\pi}{3} \right) & V \cos \left( \theta + \frac{2\pi}{3} \right) \end{bmatrix}^{\top} \]

\[
T_p^{-1}(\hat{e}_q) = \vec{v}_{abc} \begin{bmatrix} V \cos \left( \theta + \frac{2\pi}{3} \right) & V \cos \left( \theta - \frac{2\pi}{3} \right) & V \cos \left( \theta + \frac{2\pi}{3} \right) \end{bmatrix}^{\top} = \begin{bmatrix} \cos \left( \theta + \frac{2\pi}{3} \right) & \cos \left( \theta - \frac{2\pi}{3} \right) & \cos \left( \theta + \frac{2\pi}{3} \right) \end{bmatrix}^{\top} \]

\[
T_p^{-1}(\hat{e}_0) = \vec{v}_{abc} \begin{bmatrix} V \cos \left( \theta + \frac{2\pi}{3} \right) & V \cos \left( \theta - \frac{2\pi}{3} \right) & V \cos \left( \theta + \frac{2\pi}{3} \right) \end{bmatrix}^{\top} = \begin{bmatrix} \cos \left( \theta + \frac{2\pi}{3} \right) & \cos \left( \theta - \frac{2\pi}{3} \right) & \cos \left( \theta + \frac{2\pi}{3} \right) \end{bmatrix}^{\top} \]

(33)

Fig. 15(c) explains how \( \hat{e}_0 \) is transformed. It points perpendicular to the plane of the locus and is scaled by \( \sqrt{3} \), in order to correspond to the definition of zero-sequence. Mathematically, \( T_p^{-1}(\hat{e}_0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\top} \) is found by taking the cross-product of the other two transformed unit vectors, and scaled by \( \sqrt{3} \), just as we saw in Eq. (23).

All three transformed unit vectors are combined using Eq. (4) \( A_p^{-1} = [T_p^{-1}(\hat{e}_d), T_p^{-1}(\hat{e}_q), T_p^{-1}(\hat{e}_0)] \) to give the standard inverse Park transformation matrix. We can take the inverse of Eq. (35) to obtain the transformation from \( abc \) to \( dq0 \), which will result in Eq. (32).
Harmonics generate positive, negative and zero sequence components in an interesting pattern [30]. The positive sequence harmonics are 1\textsuperscript{st}, 4\textsuperscript{th}, 7\textsuperscript{th} and so on. The harmonics 2\textsuperscript{nd}, 5\textsuperscript{th} and 8\textsuperscript{th} etc appear as negative sequence. Triplen harmonics (3\textsuperscript{rd}, 6\textsuperscript{th}, 9\textsuperscript{th} etc.) appear as zero-sequence in the abc domain. The first three harmonics are illustrated in Fig. 17(a).

The locus diagrams of Fig. 17(b) provide an intuitive means to understand how harmonics appear in dq0. The relative velocity of a vector and the dq0-axes determines what frequency a harmonic appears at. Positive sequence components rotate in the same direction as the dq0 axes. Thus, \( \vec{v}_{dq04} \) and \( \vec{v}_{dq07} \) have \( d \) and \( q \)-components containing the 3\textsuperscript{rd} and 6\textsuperscript{th} harmonics respectively. Negative sequence components appear faster relative to the dq0 axes, such that \( \vec{v}_{dq02} \) and \( \vec{v}_{dq05} \) have \( d \) and \( q \)-components with the 3\textsuperscript{rd} and 6\textsuperscript{th} harmonics respectively.

Finally, a single locus diagram can fully represent a signal containing harmonics. This is not possible with a phasor diagram. Fig. 18 shows an example similar to case (iii) of Fig. 2. The vector \( \vec{v}_{abc} \) contains fundamental, 5\textsuperscript{th} and 7\textsuperscript{th} harmonics. Both harmonics appear as a 6\textsuperscript{th} harmonic in dq0 as given by the shape of the locus of Fig. 18, which has 6 lobes.

### Appendix

#### A. Summary of Transformations and Instantaneous Power Calculations

Table I summarises the standard and power-invariant forms of the Clarke and Park transformations. Note that there are many conventions for the direction of the \( d \) and \( q \) axes, with some even having a \( q \)-axis lagging the \( d \)-axis. This paper follows the \( dq0 \) convention of Kundur and others in [21], [24] (see Table I). The angle \( \theta \) is referenced with respect to the \( d \)-axis as shown in Fig. 4(a). There exists an alternative convention where the angle is referenced with respect to the \( q \)-axis. This convention (\( qd0 \)) is used by Krause and others [1], [3].

Table II shows the instantaneous real and reactive power calculations in each of the \( \alpha\beta0 \) and \( dq0 \) domains. We use the definition of instantaneous reactive power given by [31].

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Fig. 16. Symmetrical components applied to an unbalanced 3-phase system with no harmonics. (a) phasor diagrams (b) locus diagrams in abc coordinates (c) locus diagrams in rotating dq0 coordinates.

**Table I**

| Standard (Amplitude-Invariant) | Power-Invariant |
|--------------------------------|-----------------|
| **Clarke abc to αβ0** | **Park abc to dq0** |
| | |
| \[
\begin{bmatrix}
1 & -1/2 & -1/2 \\
0 & \sqrt{3}/2 & -\sqrt{3}/2 \\
1/2 & 1/2 & 1/2
\end{bmatrix}
\] | \[
\begin{bmatrix}
1 & -1/2 & -1/2 \\
0 & \sqrt{3}/2 & -\sqrt{3}/2 \\
1/2 & 1/2 & 1/2
\end{bmatrix}\sqrt{2/3}
\] |
| | |
| **Inverse Clarke αβ0 to abc** | **Inverse Park dq0 to abc** |
| | |
| \[
\begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 1 \\
\sin(\theta) & \cos(\theta - 2\pi/3) & 1 \\
\sin(\theta + 2\pi/3) & -\cos(\theta - 2\pi/3) & 1
\end{bmatrix}
\] | \[
\begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 1/\sqrt{3} \\
\sin(\theta - 2\pi/3) & \cos(\theta - 2\pi/3) & 1/\sqrt{3} \\
\sin(\theta + 2\pi/3) & -\cos(\theta + 2\pi/3) & 1/\sqrt{3}
\end{bmatrix}
\] |

**Table II**

| Standard (Amplitude-Invariant) | Power-Invariant |
|--------------------------------|-----------------|
| **αβ0** | **dq0** |
| | |
| \[
p(t) = \frac{a}{2}(v_{a}i_{a} + v_{b}i_{b} + 2v_{0}i_{0})
\] | \[
p(t) = \frac{a}{2}(v_{a}i_{a} + v_{b}i_{b} + 2v_{0}i_{0})
\] |
| \[
q(t) = \frac{b}{2}(v_{b}i_{a} - v_{a}i_{b})
\] | \[
q(t) = v_{b}i_{a} - v_{a}i_{b}
\] |
| \[
p(t) = \frac{q}{2}(v_{a}i_{d} + v_{b}i_{q} + 2v_{0}i_{0})
\] | \[
p(t) = v_{a}i_{d} + v_{b}i_{q} + 2v_{0}i_{0}
\] |
| \[
q(t) = \frac{a}{2}(v_{b}i_{d} - v_{a}i_{q})
\] | \[
q(t) = v_{b}i_{d} - v_{a}i_{q}
\] |

**Fig. 17.** Locus diagrams of first three harmonics in: (a) abc coordinates (b) dq0 coordinates.

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