Application of Rank Likelihood Ratio Scanning Method in Multiple Mean Changes in Long Memory Time Series with Heavy Tail

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Abstract. In this paper, a rank-likelihood ratio scanning method for long memory time series with heavy tail is proposed to solve the problem that when likelihood ratio scanning method is used to estimate the mean change points in the long memory time series with heavy tail, the estimation accuracy decreases rapidly with the decrease of the heavy tail index. Numerical simulation and analysis of real data demonstrate the effectiveness and practicability of the rank-likelihood ratio scanning method.

1. Introduction
The change-point test has always been a hot research topic in statistics and finance. In addition to directly constructing statistics to detect change points in the sequence, it is also an effective method to find change points by optimizing a specific objective function, such as the least square criterion, Bayesian criterion, BIC criterion, etc. for the research results of change point detection using such criteria, please refer to the literature Yao (1987), Horváth (1993), However, it is not easy to find the change point by optimizing the objective function. It is more difficult to get the exact number of change points with the increase of sample size, the combination of observations or data that may be regarded as change points will become larger and larger, with an exponential growth trend, which undoubtedly increases the computing burden of the computer and the computational complexity of the objective function.

In recent years, many scholars have proposed some methods to solve the computer burden and the computational complexity of the objective function, such as the genetic algorithm proposed by Davis et al. (2006), Pelt algorithm proposed by Killick et al. (2012) compared with the traditional methods of enumeration, these algorithms have greatly improved the computing speed, but they still can not solve the problem of the huge amount of computer computation and the increased complexity of objective function computation caused by the exponential growth of the combination of suspected change points with the increase of sample size, especially the genetic algorithm proposed by Davis, which involves many tuning parameters and the complexity is not reduced. Killick's pelt algorithm, the number of fixed change points, with the increase of sample size, the amount of computer calculation is also as much as possible. In order to effectively solve the computational complexity problem when optimizing the objective function, in 2016, Yau and Zhao (2016) proposed the likelihood ratio scanning method to test the variable point problem in the stationary piecewise auto-regressive series, and greatly reduced the amount of computer calculation. In this paper, the simulation results of the likelihood ratio scanning method and the other three methods (AP, MSML, WBS) are compared. The results show that LRSM (likelihood ratio scanning method) has the highest accuracy in detecting the number of change points. After that, Ng et al. (2017) extended the application of likelihood ratio scanning method in Linear
Autoregression to the detection of variable points in nonlinear sequences. Li et al. (2017) studied the detection of variable points with partial information at the variable points on the basis of likelihood ratio scanning method. Eichinger et al. (2018) estimated the problem of variable points with randomness using MOSUM process. Zhang et al. (2018) studied the detection of multiple points with a fast screen shape recognition algorithm. Xu Qiongyao et al. (2019) applied the likelihood ratio scanning method to the detection and estimation of the mean variation points in the long memory time series.

This paper mainly studies the application of likelihood ratio scanning method in the detection of the variable points of the mean of long memory time series. It is found by numerical simulation that direct application of this method to the long memory time series with heavy tail innovation can not accurately detect the variable points. The accuracy of the method of likelihood ratio scanning for rank, the estimation of the number of the variable points and the detection of the position of the variable points is improved. After that, the validity of the improved method is proved by numerical simulation and actual data analysis. The main framework of this paper is as follows: in Section 2, the likelihood ratio scanning method is introduced in detail, in Section 3, the rank likelihood ratio scanning method is introduced, in Section 4, the likelihood ratio scanning method and the rank likelihood ratio scanning method are intuitively compared by numerical simulation, which are applied to the long memory time series with thick tail innovation to estimate the number of change points and the accuracy of position. The analysis of actual data is placed in Section 5.

2. Likelihood Ratio Scanning Method

2.1 Basic Settings and Assumptions

Suppose that the observation sequence \( \{X_t\}_{t=1}^{\Delta} \) can be partitioned into \( m+1 \) stationary AR processes. For \( j=1, \ldots, m \), the \( j \)th change point \( \tau_j \) is the position at which the \( j \)th segment of the AR process abruptly change to the \((j+1)\)-segment. Set \( \tau_0 = 0 \) and \( \tau_{m+1} = n \). The \( j \)th stationary segment of the series is given by

\[
Y_{t,j} = X_t, \quad \tau_{j-1} < t < \tau_j,
\]

where \( \{Y_{t,j}\} \) is a stationary AR process satisfying

\[
Y_{t,j} = \phi_{j,0} + \phi_{j,1} Y_{t-1,j} + \Lambda + \phi_{j,p} Y_{t-p,j} + \epsilon_t
\]

\( \epsilon_t \) is an independent white noise sequence with the same distribution, and the mean value is 0, and the variance is 1.

Suppose 2.1.1
Denote the parameter vector of \( j \)th segment by \( \theta_j := (\phi_{j,0}, \phi_{j,1}, \Lambda, \phi_{j,p}, 1) \), which is assumed to be an interior point of the compact space \( \Theta_j \).

Suppose 2.1.2
The order of the AR process in each segment is assumed to be bounded by an integer \( p_{\text{max}} \).

Suppose 2.1.3
The segments \( \{Y_{t,j}\}_{j=1,2,\ldots,m+1} \) are assumed to be independent.

In the next section, we introduce three steps of likelihood ratio scanning.

2.2 Three Steps to Estimate the Change Point with Likelihood Scanning Method

Step 1:
Define the scanning window at \( t \):

\[
W_t(h) = \{t - h + 1, \Delta, t + h\}
\]
Corresponding observations as 

\[ X_{W(h)} = (X_{t-h+1}, \ldots, X_{t+h}) \]

Where \( t = h, \ldots, n-h \) and \( h \) is called the window radius. When the sample size is less than or equal to 800, the value is 25. When the sample size is greater than 800, \( h \geq 2 \log(n^2) \), here \( n \) is the sample size. When the sample size is greater than 800, this paper takes \( h = 2 \log(n^2) \).

To detect a change point in a scanning window, we propose a rank likelihood ratio statistic. Given a sample \( Z = \{z_1, \ldots, z_n\} \) define the quasi-likelihood to be 

\[ L(\theta) = \sum_{i=1}^{n} l_i(\theta) = \sum_{i=1}^{n} \log(f_{\theta}(z_i/z_{i-2}, \ldots, z_{i-1}, \ldots, z_{i+p}, z_{i-p})) \]

\[ f_{\theta}(z_i/z_{i-2}, \ldots, z_{i-1}, \ldots, z_{i+p}, z_{i-p}) = \exp \left\{ \left( z_i - \phi_0 - \phi_1 z_{i-2} - \ldots - \phi_p z_{i-p} - \Lambda - \phi_0 z_{i-1} \right)^2 / (2\sigma^2) \right\} \sqrt{(2\pi)\sigma} \]

is the conditional density of \( z_i \) given previous observations and \( z_s = 0 \) for \( s \leq 0 \), then, the rank likelihood ratio scan statistic for the scanning window \( W(h) \) is defined by 

\[ S_h(t) = \frac{1}{h} L_{1h}(t, \hat{\theta}_1) + \frac{1}{h} L_{2h}(t, \hat{\theta}_2) + \frac{1}{h} L_{3h}(t, \hat{\theta}_3) \]

\[ L_{1h}(t, \hat{\theta}_1), \quad L_{2h}(t, \hat{\theta}_2), \quad \text{and} \quad L_{3h}(t, \hat{\theta}_3) \]

are defined similarly to \( L(\theta) \) in expression.

Scanning the observed time series by using \( S_h(t) \) we can obtain a set of potential change points from the local change point estimates, defined by 

\[ \hat{J}^{(i)} = \left\{ m \in \{h, h+1, \ldots, n-h\} : S_h(m) = \max_{t \in \{m-h, m+h\}} S_h(t) \right\} \]

Where \( S_h(t) = 0 \) for \( t < h \) and \( t > n-h \), i.e., for each \( m \in \{h, h+1, \ldots, n-h\} \), we regard \( m \) as a local change point estimator if \( S_h(m) \) is the maximum over the window \( \{m-h+1, m+h\} \) centering at \( m \).

**Step 2:**

As there may be over estimation problem in the first step, a minimization criterion should be selected in the second step to optimize the number and location of change points obtained in the first step. The minimization criterion adopted in this paper is the minimum description length (MDL) criterion with better experience ability, which is given by the following formula:

\[ \text{MDL}(m, J, p) = \log(m) + (m + 1) \log(n) + \sum_{j=1}^{m+1} \log(p_j) + \sum_{j=1}^{m+1} \frac{p_j + 2}{2} \log(n_j) - \sum_{j=1}^{m+1} L_{j}(\hat{\theta}_j) \]

\( J = (r_1, \ldots, r_m) \) is the set of change points, \((n_1, \ldots, n_{m+1}) \) and \((p_1, \ldots, p_{m+1}) \) are the lengths and AR orders of all segments. The optimized variable points can be obtained by minimizing the variable points in the set. The expression is as follows:

\[ \left( \hat{m}^{(2)}, \hat{J}^{(2)}, \hat{p}^{(2)} \right) = \arg\min_{m=|J|, \ldots, m_{\text{max}}} \text{MDL}(m, J, p) \]

**Step 3:**

Final estimate of the change point.
Define the extended local window
\[ E_j(h) = \left\{ \hat{f}_j^{(2)} - 2h + 1, \hat{f}_j^{(2)} + 2h \right\} \]

Corresponding observations for the \( j \) th estimate change point \( \hat{f}_j^{(2)} \)
\[ X_{E_j(h)} = \left( X_{\hat{f}_j^{(2)}-2h+1}, \ldots, X_{\hat{f}_j^{(2)}+2h} \right) \]

Let
\[ L_j(\tau, \theta_1, \theta_2) = \sum_{r=\hat{f}_j^{(2)}-2h+1}^{\hat{f}_j^{(2)}+2h} l_1(\theta_1) + \sum_{r=\hat{f}_j^{(2)}+2h+1}^{\hat{f}_j^{(2)}+2h} l_2(\theta_2) \]

For \( j = 1, \ldots, m^{(2)} \), define the final estimate as
\[ \hat{f}_j = \arg \max_{\tau \in \hat{f}_j^{(2)}-2h+1} \sum_{r=\hat{f}_j^{(2)}-2h+1}^{\hat{f}_j^{(2)}+2h} l_1(\theta_1) \]
\[ \hat{\theta}_j = \hat{\theta}_j(\tau) = \arg \max_{\tau \in \hat{f}_j^{(2)}-2h+1} \sum_{r=\hat{f}_j^{(2)}-2h+1}^{\hat{f}_j^{(2)}+2h} l_1(\theta_1) \]
\[ \text{and } \hat{\theta}_{j+1} \text{ is defined analogously.} \]

3. Rank Likelihood Ratio Scanning Method

3.1 Models and Assumptions
Consider the long memory time series model with heavy tail,
\[ \Phi(L)(1-L)^{d_0} X = \Psi(L) \varepsilon \]

Where \( \Phi(L) \) is AR polynomial, Ma polynomial is \( \Psi(L) \), \( L \) is lag operator and \( d_0 \) is long memory parameter. \( X \) is a stationary long memory time series, \( \varepsilon \) is a heavy tail innovation process satisfying hypothesis 3.1.1.

**Hypothesis 3.1.1** Random process \( \varepsilon \) is a strictly stable sequence with one-dimensional symmetric edge distribution. The range of the heavy tail index \( k \in (1,2) \), and \( \varepsilon \) satisfies \( E(\varepsilon) = 0 \).

3.2 Model Calculation Steps
This section mainly describes the simulation process of rank likelihood ratio scanning method.
1. Random generation of a group of long memory time data \( X = \{ x_1, \ldots, x_n \} \) with mean variation points in R language program.
2. Rank the generated long memory time series \( R_i = \text{rank}(X_i) = \sum_{j=1}^{n} 1_{x_j \leq x_i}, i = 1, \ldots, n \)
3. The residual sequence in the long memory time series is estimated directly by the function in the package fracdiff, and then the rank likelihood function \( L(\hat{\theta}) \) is constructed by the estimated residual sequence.
4. Define the scan window. The length of the scan window is \( 2h \). The size depends on the size of the sample size. In general, \( h = 210(n^2) \), if sample size is less than 800, \( h = \max \{ 25, 2\log(n^2) \} \)
5. Using the rank likelihood function, the scan statistics based on the scan window is constructed.
6. Using the rank likelihood ratio scan statistics, scan the whole long memory time series to get all local variation points.
7. The MDL method is used to select the variable points in the set to get a more accurate set of
variable points.

8. Finally, by expanding the scanning window to twice of the original time series and re-scanning the long memory time series, the final set of change point estimates is obtained.

4. Numerical Simulation

All simulation experiments are implemented by R language program version 3.5.0, all data are generated by the model $ARFIMA(p,d,q)$. $n = 600,800,1000$ $d = 0.1,0.2,0.3,0.4$ $\kappa = 1.2,1.5,1.8,2$.

$\tau = [\lambda * n]$ represents the position of the change point in the sample. The number of change points is expressed in $m$. in this simulation experiment, the number of change points is set to no change point, with one change point and two change points. Before and after the change point position, the jumping degree is set to 1. When a change point is set in the long memory time series, the value of $\lambda$ is 0.5; when two change points are set in the long memory time series, the value of $\lambda$ is 0.5 and 0.75 respectively. S.D is used to represent the standard deviation, while MSE is used to represent the mean square error. Here, all the simulation results are obtained through 100 cycles of experiments. Consider the length of the article, only the simulation results with the sample size of 1000 are given in this paper, as shown in table 1 ~ table 3. From table 1 to table 3, it can be concluded that with the increase of memory parameters and the decrease of heavy tail index, the likelihood ratio scanning method can not accurately estimate the number and position of change points in the long memory time series with heavy tail innovation process. In contrast, the rank likelihood ratio scanning method can accurately estimate the number and position of change points in the long memory time series with thick tail innovation process. The overall accuracy of the estimation is high, which shows the effectiveness of the rank likelihood ratio scanning method.

Table 1 simulation results of likelihood ratio scanning before and after taking rank under no change points

| $d_0$ | $k = 2$ | $k = 1.8$ | $k = 1.5$ | $k = 1.2$ | $k = 2$ | $k = 1.8$ | $k = 1.5$ | $k = 1.2$ |
|-------|--------|---------|---------|---------|--------|---------|---------|---------|
| 0.1   | 99     | 42      | 4       | 2       | 99     | 99      | 96      | 74      |
| 0.2   | 95     | 37      | 4       | 2       | 96     | 94      | 84      | 53      |
| 0.3   | 94     | 37      | 5       | 2       | 90     | 87      | 61      | 35      |
| 0.4   | 85     | 33      | 3       | 1       | 82     | 68      | 37      | 18      |
Table 2 simulation results of likelihood ratio scanning before and after taking rank under one change point

| $d_0$ | LRSM | RLRSM |
|-------|------|-------|
|       | $k = 2$ | $k = 1.8$ | $k = 1.5$ | $k = 1.2$ | $k = 2$ | $k = 1.8$ | $k = 1.5$ | $k = 1.2$ |
| 0.1   | 99    | 35    | 11    | 3     | 98    | 96    | 83    | 55    |
| $\lambda$ | 0.493 | 0.533 | 0.480 | 0.652 | 0.504 | 0.505 | 0.498 | 0.536 |
| S.D   | 0.047 | 0.159 | 0.314 | 0.341 | 0.50 | 0.042 | 0.090 | 0.134 |
| MSE   | 0.002 | 0.026 | 0.090 | 0.101 | 0.003 | 0.002 | 0.008 | 0.019 |
| 0.2   | 76    | 29    | 10    | 6     | 79    | 53    | 30    | 19    |
| $\lambda$ | 0.494 | 0.561 | 0.537 | 0.411 | 0.493 | 0.504 | 0.501 | 0.570 |
| S.D   | 0.095 | 0.234 | 0.357 | 0.354 | 0.086 | 0.109 | 0.188 | 0.296 |
| MSE   | 0.009 | 0.056 | 0.116 | 0.112 | 0.007 | 0.012 | 0.034 | 0.087 |
| 0.3   | 33    | 28    | 9     | 7     | 36    | 24    | 23    | 26    |
| $\lambda$ | 0.518 | 0.567 | 0.633 | 0.400 | 0.496 | 0.473 | 0.577 | 0.575 |
| S.D   | 0.163 | 0.318 | 0.350 | 0.324 | 0.170 | 0.208 | 0.275 | 0.303 |
| MSE   | 0.026 | 0.102 | 0.126 | 0.100 | 0.028 | 0.042 | 0.078 | 0.094 |
| 0.4   | 22    | 31    | 10    | 6     | 24    | 22    | 27    | 16    |
| $\lambda$ | 0.508 | 0.558 | 0.657 | 0.414 | 0.511 | 0.537 | 0.526 | 0.505 |
| S.D   | 0.211 | 0.343 | 0.346 | 0.353 | 0.252 | 0.323 | 0.309 | 0.353 |
| MSE   | 0.042 | 0.117 | 0.132 | 0.111 | 0.061 | 0.101 | 0.093 | 0.117 |
Table 3. Simulation results of likelihood ratio scanning before and after taking rank under two change points

| \(d_0\) | LRSM | RLRSM |
|-------|------|-------|
|       | \(k = 2\) | \(k = 1.8\) | \(k = 1.5\) | \(k = 1.2\) | \(k = 2\) | \(k = 1.8\) | \(k = 1.5\) | \(k = 1.2\) |
| 0.1   | 97   | 43   | 17   | 14   | 94   | 87   | 51   | 18   |
| \(\hat{\lambda}_1\) | 0.497 | 0.478 | 0.410 | 0.283 | 0.498 | 0.498 | 0.503 | 0.463 |
| \(\hat{\lambda}_2\) | 0.750 | 0.765 | 0.734 | 0.742 | 0.749 | 0.748 | 0.745 | 0.697 |
| S.D.  | 0.029 | 0.120 | 0.208 | 0.151 | 0.020 | 0.031 | 0.063 | 0.087 |
| S.D.  | 8.8e-3 | 0.063 | 0.157 | 0.157 | 0.006 | 0.012 | 0.015 | 0.121 |
| MSE   | 8.6e-4 | 0.015 | 0.049 | 0.068 | 4.2e-4 | 9.3e-4 | 0.004 | 0.009 |
| MSE   | 7.8e-5 | 0.004 | 0.023 | 0.023 | 3.7e-5 | 1.4e-4 | 2.5e-4 | 0.017 |
| 0.2   | 51   | 38   | 19   | 13   | 55   | 35   | 14   | 17   |
| \(\hat{\lambda}_1\) | 0.486 | 0.425 | 0.393 | 0.288 | 0.498 | 0.496 | 0.410 | 0.460 |
| \(\hat{\lambda}_2\) | 0.749 | 0.738 | 0.706 | 0.707 | 0.746 | 0.748 | 0.700 | 0.673 |
| S.D.  | 0.058 | 0.150 | 0.199 | 0.150 | 0.049 | 0.061 | 0.133 | 0.176 |
| S.D.  | 0.017 | 0.122 | 0.175 | 0.173 | 0.015 | 0.027 | 0.105 | 0.152 |
| MSE   | 0.004 | 0.028 | 0.049 | 0.066 | 0.002 | 0.004 | 0.024 | 0.031 |
| MSE   | 2.7e-4 | 0.015 | 0.031 | 0.030 | 2.2e-4 | 7.2e-4 | 0.013 | 0.027 |
| 0.3   | 20   | 31   | 22   | 13   | 24   | 16   | 17   | 22   |
| \(\hat{\lambda}_1\) | 0.479 | 0.394 | 0.378 | 0.268 | 0.494 | 0.525 | 0.390 | 0.354 |
| \(\hat{\lambda}_2\) | 0.753 | 0.688 | 0.675 | 0.711 | 0.754 | 0.726 | 0.626 | 0.619 |
| S.D.  | 0.113 | 0.197 | 0.220 | 0.152 | 0.088 | 0.127 | 0.162 | 0.198 |
| S.D.  | 0.042 | 0.161 | 0.197 | 0.179 | 0.019 | 0.110 | 0.175 | 0.208 |
| MSE   | 0.013 | 0.049 | 0.061 | 0.075 | 7.5e-3 | 0.016 | 0.037 | 0.059 |
| MSE   | 0.003 | 0.029 | 0.042 | 0.031 | 3.9e-4 | 0.012 | 0.044 | 0.058 |
| 0.4   | 4    | 25   | 20   | 11   | 16   | 10   | 25   | 24   |
| \(\hat{\lambda}_1\) | 0.490 | 0.410 | 0.389 | 0.309 | 0.509 | 0.480 | 0.392 | 0.388 |
| \(\hat{\lambda}_2\) | 0.801 | 0.664 | 0.681 | 0.681 | 0.749 | 0.658 | 0.599 | 0.676 |
| S.D.  | 0.206 | 0.177 | 0.224 | 0.127 | 0.185 | 0.136 | 0.170 | 0.201 |
| S.D.  | 0.063 | 0.209 | 0.202 | 0.120 | 0.147 | 0.153 | 0.198 | 0.214 |
| MSE   | 0.032 | 0.038 | 0.060 | 0.051 | 0.032 | 0.017 | 0.039 | 0.051 |

5. Real Data Analysis
This section will use a set of actual data to illustrate the effectiveness of the new method. Because the stock data with heavy tail and long memory, we choose the data of 2483 stock returns of Shanghai stock index from January 2, 1992 to December 29, 2000. After getting the data, we first log the data to get the data of the yield rate. After taking the log data, we can refer to the figure See Figure 1. The likelihood ratio scanning method is used to detect the change points of this group of data, and the results show that there are change points (red line) in the fifth sample points of 93,978,1165,1735 and 2329, respectively. Similarly, the rank likelihood ratio scanning method is used to detect the change
points of the reorganization sequence again, and the detection results show that there are change points (blue line) in the positions of 323 and 1713. It is clear from Figure 1 that the result without change point detected after sample point 1713 is more consistent with the specific situation, which shows that the likelihood ratio scanning method may have the problem of over estimation, and the rank likelihood ratio scanning method proposed is more effective, practical and reliable in testing the mean change point of long memory time series.

![Figure 1. Shanghai composite index on January 2, 1992 solstice yield on December 29, 2000](image)

6. Conclusion
Based on the likelihood ratio scanning method, this paper studies the rank likelihood ratio scanning method, which is used to detect and estimate the mean variation of long memory time series with heavy tail innovation. From the simulation experiment, the rank likelihood ratio scanning method is more ideal than the original method. From the analysis of actual data, the rank likelihood ratio scanning method is more ideal than the original method. The results of detection and estimation are more in line with the actual situation, which shows the effectiveness and practicability of the rank likelihood ratio scanning method.

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7. Reference
[1] Yao, Y. C. Approximating the distribution of the maximum likelihood estimate of the change point in a sequence of independent random variables. Ann. Statist., 1987, 15: 1321-1328.
[2] Horváth, L. The maximum likelihood method of testing changes in the parameters of normal observations. Ann. Statist., 1993, 21: 671-680.
[3] Davis, R. A., Lee, T. C. M. and Rodriguez-Yam, G. A. Structural break estimation for non-stationary time series models. Am. Statist. Ass., 2006, 101: 223-239.
[4] Killick, R., Fearnhead, P. and Eckley, I. A. Optimal detection of change points with a linear computational cost. Am. Statist. Ass., 2012, 107: 1590-1598.
[5] Yau, C., and Zhao, Z. Inference for multiple change points in time series via likelihood ratio scan statistics. Journal of the Royal Statistical Society: Series B, 2016, 78: 895-916.
[6] Ng, W., Pan, S., Yau, C. Inference for Multiple change-points in Linear and Non-linear Time Series. arXiv:1703.00647v1.
[7] Li, Y., Lund, R., Hewaarachchi, A. Multiple change point Detection with Partial information on change point times. arXiv:1511.07238v3.
[8] Eichinger, B., and Kirch, C. A MOSUM procedure for the estimation of multiple random change points. Bernoulli , 2018, 24(1):526-564.
[9] Zhuang, D., and Liu, Y. A fast screen and shape recognition algorithm for multiple change point detection. Mathematical Problems in Engineering, 2018, ID: 8371085.
[10] Qiongyao Xu, Yuhong Xing. The application of likelihood ratio scanning method in long memory time series. Data Mining, 2019,9(2):9-17.