1. Introduction

Let \( k \) be an algebraically closed field. The classification of algebraic subgroups of groups of birational transformations was initiated in [Enriques 1993], where Enriques shows that each connected algebraic subgroup of \( \text{Bir}(\mathbb{P}^2) \) is conjugate to an algebraic subgroup of \( \text{Aut}^\circ(S) \), with \( S \) isomorphic to \( \mathbb{P}^2 \) or to the \( n \)-th Hirzebruch surface \( \mathbb{F}_n \) for \( n \neq 1 \); and these are all maximal, with respect to the inclusion, among the connected algebraic subgroups of \( \text{Bir}(\mathbb{P}^2) \). The connected algebraic subgroups of \( \text{Bir}(\mathbb{P}^2) \) have been classified over \( k = \mathbb{C} \) by Umemura in a series of four papers [Umemura 1980, 1982a, 1982b, 1985] and it follows again from his classification that each connected algebraic subgroup of \( \text{Bir}(\mathbb{P}^2) \) is contained in a maximal one (see also [BFT21a, BFT21b] for a modern approach). However, it is an open problem whether every connected algebraic subgroup of \( \text{Bir}(\mathbb{P}^3) \) is contained in a maximal one when \( n \geq 4 \).

On the other hand, it is proven in [Fong 2021, Theorem C] that there exist connected algebraic subgroups of \( \text{Bir}(C \times \mathbb{P}^1) \) not contained in a maximal one when \( C \) is a smooth curve of positive genus. The proof of this result is based on the existence of infinite increasing sequences of connected algebraic subgroups of \( \text{Bir}(C \times \mathbb{P}^1) \) (see [Fong 2021, Theorem A]), and on the fact that the dimension of a maximal connected algebraic subgroup of \( \text{Bir}(C \times \mathbb{P}^1) \) is bounded by \( 4 \) (see [Fong 2021, Theorem B] and [Mar 1971, Theorem 3]). Our main result in this note is a higher dimensional analogue of [Fong 2021, Theorem C]:

**Theorem A.** Let \( k \) be an algebraically closed field of characteristic 0. Let \( n \geq 1 \) and \( C \) be a smooth curve of positive genus. Then there exists a connected algebraic subgroup of \( \text{Bir}(C \times \mathbb{P}^n) \) which is not contained in a maximal one.

The idea of the proof is to consider the connected algebraic subgroup \( \text{Aut}^\circ(S \times \mathbb{P}^n) \), where \( S \) is a ruled surface such that \( \text{Aut}^\circ(S) \) is not contained in a maximal connected algebraic subgroup of \( \text{Bir}(S \times \mathbb{P}^n) \), and to show that it cannot be contained in a maximal connected algebraic subgroup of \( \text{Bir}(S \times \mathbb{P}^n) \). Since \( \text{Aut}^\circ(S \times \mathbb{P}^n) \cong \text{Aut}^\circ(S) \times \text{PGL}_{n+1}(k) \) by [BSU 2013, Corollary 4.2.7], the existence of infinite increasing sequences of connected algebraic subgroups of \( \text{Bir}(C \times \mathbb{P}^{n+1}) \) is an immediate consequence of [Fong 2021, Theorem A]. From this alone, it is nonetheless insufficient to deduce that one of the connected algebraic subgroups of \( \text{Bir}(C \times \mathbb{P}^{n+1}) \) appearing in the infinite increasing sequences is not contained in a maximal one (see Remark 2.8), and classifying all connected algebraic subgroups of \( \text{Bir}(C \times \mathbb{P}^{n+1}) \) seems out of reach at the moment.

This article is organized as follows. Section 2 contains two results, namely Lemmas 2.6 and 2.7, which are important for the proof of the higher dimensional case. As a consequence of these two lemmas, we also get a new and short proof of the dimension two case (see Proposition 2.9), without using the classification of the maximal connected algebraic subgroups of \( \text{Bir}(C \times \mathbb{P}^1) \) ([Fong 2021, Theorem B]). In Section 3, we prove the higher dimensional case under the extra assumption that \( \text{char}(k) = 0 \), in view of using the machinery of the MMP and the G-Sarkisov program. The latter has been developed by Floris in [Floris 2020], building upon results of Hacon and McKernan in [HM 2013]. More precisely, if \( G \) is a connected algebraic group, then every \( G \)-equivariant birational map between Mori fibre spaces decomposes into \( G \)-Sarkisov links (see [Floris 2020, Theorem 1.2]). We study the possible links in Lemmas 3.4 and 3.5. Combining Proposition 2.9 and Theorem 3.6, we get Theorem A.

It is very natural to also ask whether for all \( n \geq 2 \), there exists a variety \( X \) of dimension \( n \) such that \( \text{Bir}(X) \) contains algebraic subgroups which are not lying in a maximal one, without the connectedness assumption. If \( n = 2 \), the answer is also affirmative (see [Fong 2021, Lemma 3.1, Corollary B]), and the proof is analogous to that of the connected case. Since the G-Sarkisov program is known only for connected algebraic groups, it is not clear if the proof presented in this article could be adapted for the non-connected case in higher dimension.
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2. Some preliminaries and the case of dimension two

From now on, $C$ will always denote a smooth curve of genus $g$ over a field $k$. In this section, $k$ is an algebraically closed field of arbitrary characteristic. The following invariant was used by Maruyama in [Mar70, Mar71] for his classification of ruled surfaces and their automorphisms.

Definition 2.1. Let $V$ be a rank-2 vector bundle over $C$ and $\tau : S = \mathbb{P}(V) \to C$ be a ruled surface. We say that $\tau$ is decomposable if $V$ is the direct sum of two line bundles over $C$. Otherwise, we say that $\tau$ is indecomposable. We define the Segre invariant of $S$ as

$$\mathcal{S}(S) = \min \{ \sigma^2, \sigma \text{ section of } \tau \}.$$ 

Remark 2.2. Let $\tau : S \to C$ be a ruled surface.

(1) Let $p \in S$ and $\sigma$ be a section of $\tau$. Recall that the blow-up of $S$ at $p$ followed by the contraction of the strict transform of the fibre passing through $p$ yields a ruled surface $\tau' : S' \to C$ and a birational map $\epsilon : S' \to S$ called the elementary transformation of $S$ centered at $p$ (see e.g. [Har77, V. Example 5.7.1]). Let $\sigma'$ be the strict transform of $\sigma$ by $\epsilon$. If $p \in \sigma$, then $\sigma'^2 = \sigma^2 - 1$. Else, $\sigma'^2 = \sigma^2 + 1$.

(2) As $S$ is obtained by finitely many elementary transformations from $C \times \mathbb{P}^1$ (see e.g. [Har77, V. Exercise 5.5]) and $\mathcal{S}(C \times \mathbb{P}^1) = 0$ (see e.g. [Fon21b, Lemma 2.14]), it follows that $\mathcal{S}(S) > -\infty$. If moreover $\mathcal{S}(S) < 0$, then there exists a unique section with negative self-intersection number (see e.g. [Fon21a, Lemma 2.10. (1)])

(3) The Segre invariant $\mathcal{S}(S)$ equals $-e$, where $e$ is the invariant defined in [Har77, V. Proposition 2.8]. If $\tau$ is indecomposable, then by [Har77, V. Theorem 2.12. (b)], we get $\mathcal{S}(S) \geq 2 - 2g = -\deg(K_C)$. In particular, if $\mathcal{S}(S) < -\deg(K_C)$, then $\tau$ is decomposable.

We recall the statement of Blanchard’s lemma and its corollary (see [BSU13, Proposition 4.2.1, Corollary 4.2.6]):

Proposition 2.3. Let $f : X \to Y$ be a proper morphism of schemes such that $f_*(O_X) = O_Y$, and let $G$ be a connected group scheme acting on $X$. Then there exists a unique action of $G$ on $Y$ such that $f$ is $G$-equivariant.

Corollary 2.4. Let $f : X \to Y$ be a proper morphism of projective schemes such that $f_*(O_X) = O_Y$. Then $f$ induces a homomorphism of group schemes $f_* : \text{Aut}^\circ(X) \to \text{Aut}^\circ(Y)$.

Remark 2.5. Let $\tau : S \to C$ be a decomposable ruled surface. Assume that $C$ has genus $g = 1$, and $\mathcal{S}(S) \neq 0$, or that $g \geq 2$. Then by [Mar71, Lemma 7], the morphism induced by Blanchard’s lemma $\tau_* : \text{Aut}^\circ(S) \to \text{Aut}^\circ(C)$ is trivial.

In the next two lemmas, we compute $\text{Aut}^\circ(S)$ and its orbits for a ruled surface $\tau : S \to C$ with $\mathcal{S}(S) < -(1 + \deg(K_C))$ (which is decomposable by Remark 2.2 (3)).

Lemma 2.6. Let $C$ be a curve of genus $g \geq 1$. Let $\tau : S = \mathbb{P}(V) \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathcal{S}(S) < -(1 + \deg(K_C))$. Let $\sigma$ be the minimal section of $\tau$ and $L(\sigma)$ be the line subbundle of $V$ associated to $\sigma$. We choose trivializations of $\tau$ such that $\sigma$ is the infinity section. Then the following hold:

(1) The group $\text{Aut}^\circ(S)$ is isomorphic to $\mathbb{G}_m \ltimes \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$, where $\det(V)$ denotes the determinant line bundle of $V$. This isomorphism associates $\alpha \in \mathbb{G}_m$ and $\gamma \in \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$, to the element $\mu_{\alpha, \gamma} \in \text{Aut}^\circ(S)$ obtained by gluing the automorphisms:

$$U_i \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1$$

$$(x, [u : v]) \mapsto (x, [\alpha u + \gamma|_{U_i}(x)v : v]).$$

(2) The $\text{Aut}^\circ(S)$-orbits in $S$ are $\{p\}$ and $\tau^{-1}(\tau(p)) \setminus \{p\}$ for $p \in \sigma$. 
Proof. (1) The proof follows from the computation made in [Mar71, case (b) p.92]. For the sake of self-containness, we recall it below. Since \( \tau \) is decomposable, we can write its transition maps as \( t_{ij} : U_j \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1 \), \( (x, [u : v]) \mapsto (x, [a_{ij}(x)u : b_{ij}(x)v]) \), where \( [u : v] \) denotes the coordinates of \( \mathbb{P}^1 \), \( a_{ij}, b_{ij} \in \mathcal{O}_C(U_i \cap U_j)^* \) denotes the transition maps of the line bundle \( L(\sigma) \) and \( b_{ij} \in \mathcal{O}_C(U_i \cap U_j)^* \). Let \( \mu \in \text{Aut}^e(S) \). The morphism induced by Blanchard’s lemma \( \tau_\mu : \text{Aut}^e(S) \to \text{Aut}^e(C) \) is trivial (Remark 2.5). Moreover, \( \sigma \) is fixed by \( \text{Aut}^e(S) \) as it is the unique minimal section. Therefore, for each trivializing open subset \( U_i \subset C, \mu \) induces an automorphism \( \mu_i : U_i \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1 \), given by \( (x, [u : v]) \mapsto (x, [a_i(x)u + \gamma_i(x)v : v]) \), where \( \alpha \in \mathcal{O}_C(U_i)^* \) and \( \gamma_i \in \mathcal{O}_C(U_i) \). The condition \( \mu(i,j) = \mu(i,j) \) implies \( \alpha_i = \alpha_j = \alpha \in \mathbb{G}_m \) and \( \gamma_i = b_{ij}^{-1}a_{ij}\gamma_j \). Since \( a_{ij}b_{ij} \) are the transition maps of the line bundle \( \text{det}(V) \), and \( a_{ij} \) denote the transition maps of \( L(\sigma) \), it implies that \( \gamma \in \Gamma(C, \text{det}(V)^* \otimes L(\sigma)^{\otimes 2}) \). The data of \( \alpha \in \mathbb{G}_m \) and \( \gamma \in \Gamma(C, \text{det}(V)^* \otimes L(\sigma)^{\otimes 2}) \) uniquely determine the automorphism \( \mu \), this proves that we have an embedding \( \text{Aut}^e(S) \to \mathbb{G}_m \times \Gamma(C, \text{det}(V)^* \otimes L(\sigma)^{\otimes 2}) \). Conversely, one can check that the automorphisms defined in the statement commute with the transition maps, hence their gluing defines an automorphism of \( S \). Because \( \mathbb{G}_m \times \Gamma(C, \text{det}(V)^* \otimes L(\sigma)^{\otimes 2}) \) is also connected, we get that it is isomorphic to \( \text{Aut}^e(S) \).

(2) Since the morphism induced by Blanchard’s lemma \( \tau_\mu : \text{Aut}^e(S) \to \text{Aut}^e(C) \) is trivial (Remark 2.5), each \( \text{Aut}^e(S) \)-orbit is contained in a fibre of \( \tau \). As \( \sigma \) is the unique section with negative self-intersection number, it is fixed pointwise by \( \text{Aut}^e(S) \). It remains to see that \( \text{Aut}^e(S) \) acts transitively on \( \tau^{-1}(\tau(p)) \setminus \{p\} \) for each \( p \) lying on \( \sigma \).

Let \( L = \text{det}(V)^* \otimes L(\sigma)^{\otimes 2} \). It follows from [Fon21b, Proposition 2.15] that \( \text{deg}(L) = -\mathcal{G}(S) > 1 + \text{deg}(K_C) \). Let \( \rho \in \rho \) and let \( \tau(p) = \rho \). We get by Serre duality that

\[
h^1(C, L) = h^0(C, K_C \otimes L') = 0,
\]

where the last equality follows from the fact that \( \text{deg}(K_C \otimes L') < 1 \). Similarly we get the equality \( h^1(C, L \otimes \mathcal{O}_C(z)^{\otimes 2}) = 0 \). By Riemann-Roch, \( h^0(C, L \otimes \mathcal{O}_C(z)^{\otimes 2}) = \text{deg}(L) - g < \text{deg}(L) - g + 1 = h^0(C, L) \). Therefore, \( z \) is not a base point of the complete linear system \( |L| \), i.e. there exists \( \gamma \in H^0(C, L) \) such that \( \gamma(z) \neq 0 \), and the subgroup \( \mathbb{G}_m \simeq \{\mu_{1,\lambda,\gamma} : \lambda \in k\} \) acts transitively on \( \tau^{-1}(z) \setminus \{p\} \) (see (1) for the definition of \( \mu_{1,\lambda,\gamma} \)).

Let \( S \) be a ruled surface as in Lemma 2.6, and \( \phi : S \to S' \) be an \( \text{Aut}^e(S) \)-equivariant birational map. In the following lemma, we compute the fixed points of the action of \( \phi \text{Aut}^e(S) \phi^{-1} \) on \( S' \).

**Lemma 2.7.** Let \( C \) be a curve of genus \( g \geq 1 \). Let \( \tau : S \to C \) be a decomposable \( \mathbb{P}^1 \)-bundle such that \( \mathcal{G}(S) < -(1 + \text{deg}(K_C)) \). If \( \tau : S' \to C \) is a ruled surface and there exists an \( \text{Aut}^e(S) \)-equivariant birational map \( \phi : S \to S' \) which is not an isomorphism, then \( \mathcal{G}(S') < \mathcal{G}(S) \) and \( \phi \text{Aut}^e(S) \phi^{-1} \subset \text{Aut}^e(S) \). The fixed points of the action of \( \phi \text{Aut}^e(S) \phi^{-1} \) on \( S' \) are the points lying on the minimal section of \( \tau' \) and the base points of \( \phi^{-1} \). Moreover, we can write \( \phi \) as a product of \( \text{Aut}^e(S) \)-equivariant elementary transformations centered on the minimal sections.

**Proof.** By [DI09, Theorem 7.7], we can write \( \phi = \phi_n \cdots \phi_1 \) where each \( \phi_i \) is an \( \text{Aut}^e(S) \)-equivariant elementary transformation. Without loss of generality, we can assume that this decomposition is minimal (i.e. the number of elementary transformations \( n \) is minimal among all possible factorizations), and we prove the statement by induction on \( n \geq 1 \).

Let \( \sigma \) be the minimal section of \( \tau \). By Lemma 2.6 (2), the algebraic group \( \text{Aut}^e(S) \) acts transitively on \( \tau^{-1}(\tau(p)) \setminus \{p\} \) for every \( p \in \sigma \). Since \( \phi_i \) are \( \text{Aut}^e(S) \)-equivariant, it follows that \( \phi_i : S \to S_1 \) is an elementary transformation centered on a point \( p_i \). The strict transform of \( \sigma \) by \( \phi_1 \) is the minimal section \( \sigma_1 \) of the ruled surface \( \tau_1 : S_1 \to C \), and so \( \mathcal{G}(S_1) = \mathcal{G}(S) - 1 \). Since the base point \( q_1 \) of \( \phi^{-1} \) does not lie on the minimal section \( \sigma_1 \) of \( \tau_1 \), it follows by Lemma 2.6 (2) that \( q_1 \) is not fixed by \( \phi_1 \). Since \( q_1 \) is fixed by \( \phi_1 \text{Aut}^e(S) \phi_1^{-1} \), we have the strict inclusion \( \phi_1 \text{Aut}^e(S) \phi_1^{-1} \subset \text{Aut}^e(S_1) \). In the complement of the fibres \( f_{p_i} \subset S \) and \( f_{q_i} \subset S_1 \) containing the points \( p_i \) and \( q_1 \) respectively, \( \phi_1 \) is an isomorphism. Therefore, by Lemma 2.6, the only fixed points of \( \phi_1 \text{Aut}^e(S) \phi_1^{-1} \) that lie in the complement of \( f_{q_i} \) are the points on the minimal section \( \sigma_1 \). It remains to check that the only fixed points on \( f_{q_i} \) are the point \( q_i' \in \sigma_1 \) and the base point \( q_1 \) of \( \phi^{-1} \). Let \( U \) be a trivializing open subset of \( \tau \) with \( \tau(p_1) \in U \), and let \( f \in \mathcal{O}_C(U) \) such that \( \text{div}(f)|_U = \tau(p_1) \). We also choose trivializations of \( \tau \) such that \( \sigma \) is the infinity section. Up to isomorphisms at the source and the target, \( \phi_1|_U \) equals \( (x, [u : v]) \mapsto (x, [f(x)u : v]) \). By Lemma 2.6 (1), there is an action of \( \mathbb{G}_m \) on \( S \) given locally by \( (x, [u : v]) \mapsto (x, [au : v]) \). It implies that there is an action of \( \phi_1 \mathbb{G}_m \phi_1^{-1} \) on \( S_1 \), given locally by \( (x, [u : v]) \mapsto (x, [af(x)u : f(x)v]) = (x, [au : v]) \). Therefore, \( \phi_1 \mathbb{G}_m \phi_1^{-1} \subset \text{Aut}^e(S) \) acts transitively on \( f_{q_i} \setminus \{q_1, q_1'\} \). Since \( \phi_1 \text{Aut}^e(S) \phi_1^{-1} \subset \text{Aut}^e(S') \)
acts fibrewise (Remark 2.5) and is connected, we get that $q_1$ and $q'_1$ are the fixed points of the action of $\phi_1 \Aut^e(S)\phi_1^{-1}$ on $f_{q_1}$.

Assume the statement holds for the birational map $\psi = \phi_1 \cdots \phi_1 : S \rightarrow S_i$, for some $i \geq 1$, and where $\tau_i : S_i \rightarrow C$ is a ruled surface with a minimal section $\sigma_i$. We now prove that the statement is then true for $\phi_{i+1} \psi$. By induction, the fixed points of $\psi \Aut^e(S)\psi^{-1}$ on $S_i$ are the points lying on the minimal section $\sigma_i$ and the base points of $\psi^{-1}$. Assume that $\phi_{i+1}$ is centered on a base point of $\psi^{-1}$, which is (the image of) the base point of the inverse of a previous elementary transformation $\phi_j$. A local calculation yields that we may cancel both $\phi_j$ and $\phi_{i+1}$, which contradicts the minimality of the factorization of $\phi$. So $\phi_{i+1}$ is centered on a point lying on the minimal section $\sigma_i$. Hence $\Theta(S_{i+1}) = \Theta(S_1) = 1 < \Theta(S)$ by induction, and $\phi_{i+1} \psi \Aut^e(S)\psi^{-1} \phi_{i+1}^{-1} \subset \Aut^e(S_{i+1})$. The base point of $\phi_{i+1}$ is fixed by $\phi_{i+1} \psi \Aut^e(S)\psi^{-1} \phi_{i+1}^{-1}$, but is not fixed by $\Aut^e(S_1)$ (by Lemma 2.6). Thus, we get the strict inclusion $\phi_{i+1} \psi \Aut^e(S)\psi^{-1} \phi_{i+1}^{-1} \subset \Aut^e(S_{i+1})$.

The infinite increasing sequences of automorphism groups given in [Fon21b, Theorem A] can be obtained from Lemma 2.7, but they do not imply that $\Aut^e(S)$ is not contained in a maximal connected algebraic subgroup. As is explained below, we can get an infinite increasing sequence of connected algebraic subgroups, where each of them is included in a maximal one, which a fortiori cannot be the same for all of them.

**Remark 2.8.** Let $n \geq d \geq 2$. Define the connected algebraic groups

$$G_d = \{ k^2 \rightarrow k^2, (x, y) \mapsto (x, y + p(x)), p \in k[x], d \},$$

acting regularly on $k^2$, and then birationally on $k^2$ via any embedding $k^2 \hookrightarrow k^2$. Then $G_d \subseteq G_{d+1}$ for all $d$. On the other hand, using an explicit description of $\Aut^e(F_n)$ from [Bla09, §4.2], we get for all $n \geq d$ that $G_d$ is a subgroup of $\Aut^e(F_n)$, which is a maximal connected algebraic subgroup of $\Aut^e(k^2)$.

Notice that for any variety $X$, using Remark 2.8, we may produce an infinite increasing sequence of connected algebraic subgroups of $\Aut^e(k^2)$, without using [Fon21b, Theorem B].

**Proposition 2.9.** Let $C$ be a curve of genus $g \geq 1$ and let $\tau : S \rightarrow C$ be a decomposable $\mathbb{P}^1$-bundle such that $\Theta(S) < - (1 + \deg(K_C))$. Then $\Aut^e(S)$ is not contained in a maximal connected algebraic subgroup of $\Aut^e(S)$.

**Proof.** Assume that $\Aut^e(S)$ is contained in a maximal connected algebraic subgroup $G$ of $\Aut^e(S)$. Then $G$ acts regularly on a surface $Y$ by Weil regularization theorem (see [Wei55], or [Zai95, Kra18] for a modern proof). By [Bri17, Corollary 3], we can choose $Y$ to be normal and projective. Using an equivariant resolution of singularities (see [Lip78, Remark B, p.155]), we can also assume $Y$ to be smooth. Then by Blanchard’s lemma (see Proposition 2.3), the successive contractions of the $(-1)$-curves gives rise to a ruled surface $S'$ such that the induced birational morphism $Y \rightarrow S'$ is $-G$-equivariant. Since $G$ is maximal and connected, it follows that $G \cong \Aut^e(S')$. The induced birational map $\phi : S \rightarrow S'$ is $\Aut^e(S')$-equivariant. If $\phi$ is an isomorphism, then $\Theta(S) = \Theta(S')$. Else $\phi$ factorises as product of $\Aut^e(S')$-equivariant elementary transformations centered on the minimal sections and $\Theta(S') < \Theta(S)$ (by Lemma 2.7). In both cases, we have $\Theta(S') \leq \Theta(S)$. Let $\epsilon : S' \rightarrow S''$ be an elementary transformation centered on the minimal section of $\tau' : S' \rightarrow C$. Then again by Lemma 2.7, it follows that $\epsilon \Aut^e(S')\epsilon^{-1} \subset \Aut^e(S'')$, which contradicts the maximality of $G$ as a connected algebraic subgroup of $\Aut^e(S)$.

3. Higher dimensional case

In what follows, we would like to utilize the machinery of the $G$-Sarkisov program for a connected algebraic group $G$. Thus from now on, we furthermore assume that $\operatorname{char}(k) = 0$. The $G$-Sarkisov program is a non-deterministic algorithm that decomposes every $G$-equivariant birational map between two $G$-Mori fibre spaces as a product of simpler maps called $G$-Sarkisov links. Its non-equivariant version was proven by Hacon and McKernan in [HM13] and, building on their result, Floris proved the $G$-equivariant version in [Fle20]. We follow the strategy of the proof of Proposition 2.9, and in view of using $G$-Sarkisov program, we recall first the definition:

**Definition 3.1.** Let $G$ be a connected algebraic group. A $G$-Mori fibre space is a Mori fibre space with a regular action of $G$. Let $\pi_1 : X_1 \rightarrow B_1$ and $\pi_2 : X_2 \rightarrow B_2$ be two birational $G$-Mori fibre spaces. A
The birational map \( \psi = \alpha_2 \chi \alpha_1^{-1} \) between \( X_1 \) and \( X_2 \) is called a \textit{G-Sarkisov link}.

\textbf{Remark 3.2.} Property (2) does not follow directly from the original definition of a \((G-\text{)Sarkisov}\) diagram of [HM13] and [Flo20]. For a proof, see [BLZ21, Proposition 4.25].

In subsequent proofs we are going to make heavy use of the following elementary but useful observation:

\textbf{Remark 3.3.} Let \( Z \) be one of the varieties appearing in a \( G\)-Sarkisov diagram, such that the relative Picard rank \( \rho(Z/R) \) is 2. Then the \( G\)-Sarkisov diagram is uniquely determined by the datum of \( Z \to R \), by a process known as the 2-ray game (see [BLZ21, section 2.F]).

More specifically, the 2-ray game is a deterministic process that assigns to any such \( Z \to R \) a \( G\)-Sarkisov diagram. Moreover any \( G\)-Sarkisov diagram can be recovered by the 2-ray game on any of its relative Picard rank 2 morphisms. Thus, up to orientation of the diagram, there is a unique \( G\)-Sarkisov diagram that contains \( Z \to R \).

\textbf{Lemma 3.4.} Let \( n \geq 1 \) and \( C \) be a curve of genus \( g \geq 1 \). Let \( \tau: S \to C \) be a decomposable \( \mathbb{P}^1\)-bundle such that \( \mathcal{S}(S) < -(1 + \deg(K_C)) \) with minimal section \( \sigma \) and let \( \phi: S \to S' \) be an \( \text{Aut}^+(S)\)-equivariant birational map (possibly the identity) to a \( \mathbb{P}^1\)-bundle \( \tau': S' \to C \). Let \( \pi' = \tau' \times id_{\mathbb{P}^n}: S' \times \mathbb{P}^n \to C \times \mathbb{P}^n \) and \( \pi_1: S' \times \mathbb{P}^n \to S' \) be the projection to the first factor. Then the following hold:
(1) The only non-trivial $\text{Aut}^\circ(S \times \mathbb{P}^n)$-Sarkisov diagrams, where $\pi': S' \times \mathbb{P}^n \to C \times \mathbb{P}^n$ is the LHS Mori fibre space, are the following ones:

\[
\begin{array}{ccc}
T \times \mathbb{P}^n & \to & T \times \mathbb{P}^n \\
\alpha & \downarrow & \beta \\
S' \times \mathbb{P}^n & \to & S'' \times \mathbb{P}^n \\
\pi' & \downarrow & \pi'' \\
C \times \mathbb{P}^n & \to & C \times \mathbb{P}^n \\
\end{array}
\]

\[
\begin{array}{ccc}
S' \times \mathbb{P}^n & \to & S' \times \mathbb{P}^n \\
\pi & \downarrow & \pi' \\
C \times \mathbb{P}^n & \to & C \\
\end{array}
\]

In the first case, the induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow S'' \times \mathbb{P}^n$ is equal to $\psi \times \text{id}_{\mathbb{P}^n}$, where $\psi: S' \to S''$ is an elementary transformation of $\mathbb{P}^1$-bundles whose center $p$ is a point fixed by $\phi\text{Aut}^\circ(S)\phi^{-1}$, and $T$ is the blow-up of $S'$ at $p$. In the second case, the induced Sarkisov link $S' \times \mathbb{P}^n \to S' \times \mathbb{P}^n$ is equal to $\text{id}_{S' \times \mathbb{P}^n}$.

(2) The only non-trivial $\text{Aut}^\circ(S \times \mathbb{P}^n)$-Sarkisov diagrams, where $\pi'_1: S' \times \mathbb{P}^n \to S'$ is the LHS Mori fibre space, are the following ones:

\[
\begin{array}{ccc}
T \times \mathbb{P}^n & \to & T \times \mathbb{P}^n \\
\eta \times \text{id}_{\mathbb{P}^n} & \downarrow & \pi'_1 \\
S' \times \mathbb{P}^n & \to & T \\
\pi & \downarrow & \pi' \\
S' \times \mathbb{P}^n & \to & C \times \mathbb{P}^n \\
\end{array}
\]

\[
\begin{array}{ccc}
S' \times \mathbb{P}^n & \to & S' \times \mathbb{P}^n \\
\eta & \downarrow & \tau \\
S' \times \mathbb{P}^n & \to & C \\
\end{array}
\]

The induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow T \times \mathbb{P}^n$ is equal to $\eta^{-1} \times \text{id}_{\mathbb{P}^n}$ in the former case and $\text{id}_{S' \times \mathbb{P}^n}$ in the latter, where $\eta: T \to S'$ is the blowup of $S'$ at point $p$ fixed by $\phi\text{Aut}^\circ(S)\phi^{-1}$.

**Proof.** (1) We distinguish between two cases depending on the base $R$ of the diagram: if $R = C \times \mathbb{P}^n$ then we have a link of Type I or II and so the first step of the link is an $\text{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant divisorial contraction $\alpha: Y \to S' \times \mathbb{P}^n$. Note that by [BSU13, Corollary 4.2.7], it follows that $(\phi \times \text{id}_{\mathbb{P}^n})\text{Aut}^\circ(S \times \mathbb{P}^n)(\phi \times \text{id}_{\mathbb{P}^n})^{-1} \simeq \phi\text{Aut}^\circ(S)\phi^{-1} \times \text{PGL}_{n+1}(k)$. Let $(q, x) \in S' \times \mathbb{P}^n$ be a point in the center of $\alpha$. If $q$ is not point fixed by $\phi\text{Aut}^\circ(S)\phi^{-1}$, then and by Lemma 2.6 and the description of $\phi\text{Aut}^\circ(S)\phi^{-1}$, the closure of the orbit of $(q, x)$ is a Cartier divisor and thus $\alpha$ is an isomorphism, contradicting the assumption that $\alpha$ is a divisorial contraction.

Thus we may assume that $q$ is fixed by $\phi\text{Aut}^\circ(S)\phi^{-1}$. In that case the orbit of $(q, x)$ is precisely $\{q\} \times \mathbb{P}^n$. Notice that the codimension of $\{q\} \times \mathbb{P}^n$ is 2 and so by [BLZ21, Lemma 2.13]

$$\alpha = (\eta \times \text{id}_{\mathbb{P}^n}): T \times \mathbb{P}^n \to S' \times \mathbb{P}^n,$$

where $\eta: T \to S'$ is the blowup of $S'$ at $q$. By Remark 3.3, the unique Sarkisov diagram containing $T \times \mathbb{P}^n \to C \times \mathbb{P}^n$ is the one given in the statement.

We now consider the case when $R \neq C \times \mathbb{P}^n$. Then we have a contraction $C \times \mathbb{P}^n \to R$ of relative Picard rank 1. Since $\rho(C \times \mathbb{P}^n) = 2$, the cone of curves $\text{NE}(C \times \mathbb{P}^n)$ has two extremal rays and so there are only two such contractions, namely the projections to the two factors: $C \times \mathbb{P}^n \to C$ and $C \times \mathbb{P}^n \to \mathbb{P}^n$. However, by property (1) of Definition 3.1, $C \times \mathbb{P}^n \to \mathbb{P}^n$ would have to be an output of some MMP on a klt pair $(Z, \Phi)$, and thus by [HM07] its exceptional locus would be rationally connected, a contradiction. Thus $R = C$ and again we conclude by Remark 3.3 for $S' \times \mathbb{P}^n \to C \times \mathbb{P}^n$.

(2) We again proceed by a similar distinction of cases. If $R = S'$ then, as in the proof of (1), the first step is an $\text{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant divisorial contraction $\eta \times \text{id}_{\mathbb{P}^n}: T \times \mathbb{P}^n \to S' \times \mathbb{P}^n$, where $\eta: T \to S'$ is the blow-up of a point of $S'$ fixed by $\phi\text{Aut}^\circ(S)\phi^{-1}$, and we conclude by Remark 3.3.

If $R \neq S'$, then $S' \to R$ is one of the two morphisms $S' \to C$ or $S' \to S'$, where the latter is the contraction of the minimal section. Again, by [HM07] we may exclude the latter case since its exceptional locus is not rationally connected. Finally, Remark 3.3, once again, guarantees that the Sarkisov diagram is the one in the statement. 

□
**Lemma 3.5.** Let $n \geq 1$ and $C$ be a curve of genus $g \geq 1$. Let $\tau : S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathcal{G}(S) < -(1 + \deg(K_C))$ with minimal section $\sigma$. Let $\phi : S \dashrightarrow S'$ be an $\Aut^e(S)$-equivariant birational map, with $S'$ being a smooth projective surface which is not minimal. Denote by $\pi^1 : S' \times \mathbb{P}^n \to S'$ the projection to the first factor. Then the only non-trivial $\Aut^e(S \times \mathbb{P}^n)$-Sarkisov diagrams, where $\pi^1 : S' \times \mathbb{P}^n \to S'$ is the LHS Mori fibre space, are the following ones:

\[
\begin{array}{ccc}
T \times \mathbb{P}^n & \xrightarrow{\eta \times id_{\mathbb{P}^n}} & T \times \mathbb{P}^n \\
\pi^1' & \downarrow & \pi^1' \\
S' \times \mathbb{P}^n & \xrightarrow{\pi^1} & S' \times \mathbb{P}^n \\
\pi^1 & \downarrow & \pi^1 \\
S' & \xrightarrow{\eta} & T. \\
\end{array}
\]

In the first case, $\eta : T \to S'$ is the blow-up of a point $p$ fixed by $\phi \Aut^e(S)\phi^{-1}$. In the second case, $\kappa : S' \to T$ is the contraction of a $(-1)$-curve $l$. In both cases, $\pi^1''$ denotes the projection to the first factor.

**Proof.** We again distinguish between two cases depending on the base $R$ of the Sarkisov diagram: if $R = S'$ then the first step of the link is an $\Aut^e(S \times \mathbb{P}^n)$-equivariant divisorial contraction $\alpha : Y \to S' \times \mathbb{P}^n$. We follow the same strategy of the proof of Lemma 3.4: first by [BSU13, Corollary 4.2.7], $(\phi \times id_{\mathbb{P}^n})\Aut^e(S \times \mathbb{P}^n)(\phi \times id_{\mathbb{P}^n})^{-1} = \phi \Aut^e(S)\phi^{-1} \times \PGL_{n+1}(k)$. This again implies that $\alpha$ has to be an extraction with center of the form $\{q\} \times \mathbb{P}^n$, where $q$ is a point fixed by the action of $\phi \Aut^e(S)\phi^{-1}$ on $S'$. Since the center is of codimension 2, again using [BLZ21, Lemma 2.13], we conclude that

\[a = \eta \times id_{\mathbb{P}^n} : T \times \mathbb{P}^n \to S' \times \mathbb{P}^n,
\]

where $\eta : T \to S'$ is the blow-up of $q$. By Remark 3.3, the diagram is the one given in the statement.

If $R \neq S'$, we have a morphism $S' \to R$ of relative Picard rank 1. Since $S'$ is not minimal, its Picard rank is greater or equal to 3 which already implies that $R = T$ is a surface. Again, using Remark 3.3 we may conclude that the diagram is the one proposed in the statement. Moreover, by property (2) of Definition 3.1, $T \times \mathbb{P}^n$ has to have terminal singularities. Thus the singular locus of $T \times \mathbb{P}^n$ has codimension at least 3 [see [KM98, Corollary 5.18]]. If $q \in T$ is singular, then $\{q\} \times \mathbb{P}^n$ is singular and has codimension 2 in $T \times \mathbb{P}^n$. This implies that $T$ is smooth and consequently, $S' \to T$ is the contraction of a $(-1)$-curve. \hfill $\square$

We prove below the higher dimensional analog of Proposition 2.9.

**Theorem 3.6.** Let $n \geq 1$. Let $C$ be a curve of genus $g \geq 1$, let $S$ be a decomposable $\mathbb{P}^1$-bundle over $C$ such that $\mathcal{G}(S) < -(1 + \deg(K_C))$. Then $\Aut^e(S \times \mathbb{P}^n)$ is not contained in a maximal connected algebraic subgroup of $\text{Bir}(S \times \mathbb{P}^n)$.

**Proof.** Assume that $\Aut^e(S \times \mathbb{P}^n)$ is contained in a maximal connected algebraic subgroup $G \subset \text{Bir}(S \times \mathbb{P}^n)$. By [Bri17, Corollary 3], there exists a normal and projective variety $Y$, $G$-birationally equivalent to $S \times \mathbb{P}^n$, and on which $G$ acts regularly. Then we use an equivariant resolution of singularities (see [Kol07, Thm. 3.36, Prop. 3.9.1]) to furthermore assume that $Y$ is smooth. Running an MMP, which is $G$-equivariant by [FLo20, Lemma 2.5], we get an $\Aut^e(S \times \mathbb{P}^n)$-equivariant birational map $\chi : S \times \mathbb{P}^n \dashrightarrow Y$ such that $G \simeq \Aut^e(Y)$ and $Y \to B$ is a Mori fibre space. By [FLo20, Theorem 1.2], $\chi$ decomposes as a product of $\Aut^e(S \times \mathbb{P}^n)$-equivariant Sarkisov links. By Lemmas 3.4 and 3.5, it follows that $Y = T \times \mathbb{P}^n$ for some surface $T$ and $\chi$ is of the form $\psi \times id_{\mathbb{P}^n}$, where $\psi : S \dashrightarrow T$ is an $\Aut^e(S)$-equivariant birational map. Up to possibly performing an extra link of Type IV (namely the RHS link in Lemma 3.4 (1)), we may assume that $B = T$ and $\theta$ is given by the projection to the first factor. Contracting successively all $(-1)$-curves in $T$ yields an $\Aut^e(S \times \mathbb{P}^n)$-equivariant birational map $\phi \times id_{\mathbb{P}^n} : S \times \mathbb{P}^n \dashrightarrow S' \times \mathbb{P}^n$ (by Blanchard’s lemma, see Proposition 2.3), where $\phi$ is $\Aut^e(S)$-equivariant and $S'$ is a ruled surface. Two cases arise: either $\phi$ is an isomorphism and $\mathcal{G}(S) = \mathcal{G}(S')$, or $\phi$ is not an isomorphism and $\mathcal{G}(S') < \mathcal{G}(S)$ by Lemma 2.7. In both cases, $\mathcal{G}(S') \leq \mathcal{G}(S)$ and since $G$ is maximal, $G$ is isomorphic to $\Aut^e(S' \times \mathbb{P}^n) = \Aut^e(S') \times \PGL_{n+1}(k)$ ([BSU13, Corollary 4.2.7]). Let $\phi' : S' \dashrightarrow S''$ be an elementary transformation of $S'$ centered at a point on the minimal section. Then $\phi' \Aut^e(S') \phi'^{-1} \subset \Aut^e(S'')$ by Lemma 2.6. Thus $(\phi' \times id_{\mathbb{P}^n}) \Aut^e(S' \times \mathbb{P}^n)(\phi' \times id_{\mathbb{P}^n})^{-1} \subset \Aut^e(S'' \times \mathbb{P}^n)$, which contradicts the maximality of $G$ as connected algebraic subgroup of $\text{Bir}(S \times \mathbb{P}^n)$. \hfill $\square$
**Proof of Theorem A.** Let \( C \) be a curve of positive genus and \( S \to C \) be a ruled surface. As \( S \) is birational to \( C \times \mathbb{P}^1 \), we get for all \( n \geq 1 \) that \( \text{Bir}(C \times \mathbb{P}^n) \simeq \text{Bir}(S \times \mathbb{P}^{n-1}) \). We conclude with Proposition 2.9 for \( n = 1 \) and Theorem 3.6 for \( n \geq 2 \).

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