Interaction induced phase fluctuations in a guided atom laser

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In this paper, we determine the magnitude of phase fluctuations caused by atom-atom interaction in a one-dimensional beam of bosonic atoms. We imagine that the beam is created with a large coherence length, and that interactions only act in a specific section of the beam, where the atomic density is high enough to validate a Bogoliubov treatment. The magnitude and coherence length of the ensuing phase fluctuations in the beam after the interaction zone are determined.

Recent progress on micro-fabricated magnetic traps and guides for atoms have stimulated the efforts to realize guided atom-laser beams with possible applications for high precision atom interferometry. The phase coherence is a crucial property of such beams and interferometers, and interactions between atoms may influence the coherence. So far, the effect of interactions on the condensate phase were mainly studied in condensates of finite spatial extent, where both degradation and squeezing of the phase have been studied. In these cited works, the main effect can be ascribed to the number fluctuations in the condensate and the dependence of the mean field energy on atom number due to interactions. The goal of the present work is to determine the intensity of phase fluctuations in a one-dimensional beam of bosonic atoms. We imagine that the beam is created with a large coherence length, and that interactions only act in a specific section of the beam, where the atomic density is high enough to validate a Bogoliubov treatment. The magnitude and coherence length of the ensuing phase fluctuations in the beam after the interaction zone are determined.

The system is described by the effective one dimensional Hamiltonian

\[ H = \int dx \psi^{\dagger}(x)h_0\psi(x) + \frac{g}{2} \int_0^L dx \psi^{\dagger}(x)\psi^{\dagger}(x)\psi(x)\psi(x), \]

where \( h_0 = -\hbar^2/(2m)d^2/dx^2 \), and the coupling constant \( g = \frac{2\hbar \omega a_\perp}{m} \), assuming that the size of the transverse ground state in the guide \( a_\perp = \sqrt{2\hbar/m\omega} \) is much larger than the s-wave scattering length \( a \).

We always consider the case where \( na_\perp^2/a >> 1 \) so that correlations between atoms are small, and the state of the system can be described by a mean field wave function and a noise term

\[ \psi(x) = \sqrt{n}e^{iKx} + \delta\psi(x). \]

The fluctuations represented by \( \delta\psi(x) \) are probe by interference with a reference beam in a coherent state \( \psi_{ref}(x) = ae^{iKx} \). The quantity measured is the number of atoms on each side of the beam splitter. In an actual experiment, this beam splitter may be realized by tunneling between two guides, or the atomic beams may be physically overlapping, but with two different internal states, which can be coupled by Raman laser beams. In either case, the operators giving the density of atoms in the two output states are

\[ n_\pm(x) = \frac{1}{2} (\psi_{ref}^{\dagger}(x) \pm \psi^{\dagger}(x)) (\psi_{ref}(x) \pm \psi(x)) \]

For \( a = \sqrt{n_0e^{i\pi}/2} \), the mean density is identical in the two output beams and we define the local phase operator as

\[ \theta(x) = \frac{n_+(x) - n_-(x)}{\sqrt{n_0n}}. \]
The fluctuations in $\theta(x)$ are given by the correlation function
\begin{equation}
\langle \theta(x)\theta(x') \rangle = \frac{1}{n} \left\{ -(\delta \psi(x')\delta \psi(x))e^{-iK(x+x')} - c.c. \right. \\
+ (\delta \psi(x')\delta \psi(x))e^{iK(x'-x)} + c.c. \\
+ \delta(x'-x) \},
\end{equation}
and it is our goal to present a quantitative analysis of these fluctuations. After averaging over a distance $X$, the fluctuations of the phase is given by
\begin{equation}
\Delta \theta^2 = \left\langle \left( \frac{1}{X} \int_X \theta(x)dx \right)^2 \right\rangle .
\end{equation}
In the absence of interactions between atoms, only the last $\delta$-correlated quantum noise term is present in Eq.(5), and we find $\Delta \theta = 1/\sqrt{nX}$ as expected from the usual number and phase uncertainties in a coherent state.

In the presence of interactions between atoms, a condensate will experience phase diffusion which is due to the spread of chemical potential over the Poisson distribution of number states which form the coherent state. Let us assume that the atomic beam is in fact a very long wavepacket with length $L$. When a superposition of number states $|N\rangle$ of this wavepacket passes the interaction zone the interaction perturbs the energy of each number state by the amount $\Delta\theta$. The passage time is $T = L/(\hbar K/m)$ and hence the number state component $|N\rangle$ experiences a phase shift $\theta_N = \frac{\sqrt{nmL}}{m\hbar} N^2$. The phase of a coherent state is the derivative $\partial \theta/dN$ of the phase of the Fock components. After passage of the interacting zone, this derivative depends on $N$ and the spread of the phase over the width $\sqrt{N}$ of the Fock state distribution is
\begin{equation}
\Delta \theta = \sqrt{N} \frac{\sqrt{nmL}}{\hbar K/m} = \frac{\sqrt{nmL}}{\hbar K/m} .
\end{equation}
The relative number fluctuations vanish in the limit of very large coherence length $L$ of the incident beam, and in that limit the Bogoliubov excitations will give the dominant contribution to phase diffusion.

If the expression for the field operator $\hat{\psi}$ is introduced in the second quantized Hamiltonian $\hat{H}$, and the noise terms $\delta \psi(x), \delta \psi^\dagger(x)$ are truncated above second order, $\hat{H}$ can be diagonalized by a Bogoliubov transformation, $\hat{H} = \hat{H}_0 + \sum_\omega \omega \alpha_\omega^\dagger \alpha_\omega$, where the field operator $\delta \psi(x)$ is expanded on the bosonic operators $\alpha_\omega, \alpha_\omega^\dagger$ as $\delta \psi(x) = \sum_k \left( u_k(x) \alpha_k - v_k^\dagger(x) \alpha_k^\dagger \right)$. The wavefunctions $u_k$ and $v_k$ solve the equations
\begin{equation}
\begin{pmatrix}
\hbar_0 - \mu + 2g|\varphi|^2 \\
-\hbar_0 + \mu - 2g|\varphi|^2
\end{pmatrix}
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix}
= \omega_k
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix}
\end{equation}
where $g$ takes the value zero outside the interaction interval, and $u_k$ and $v_k$ fulfill the normalization condition $\int dx (u_k(x)u_{k'}(x)^* - v_k(x)v_{k'}(x)^*) = \delta(k-k')$.

We neglect reflection of the mean field wave function at the entrance and exit of the interaction region, which is a good approximation as long as $gn \ll \hbar^2 K^2/2m$. When we write the mean field wave function in the interaction zone as $\varphi(x) = \sqrt{n}e^{iKx}$, the chemical potential is $\mu = \hbar^2 K^2/2m + gn$, and Eq.(8) has the plane wave solutions
\begin{equation}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= \begin{pmatrix}
\bar{e}^{i(k+K)x}U \\
\bar{e}^{i(k-K)x}V
\end{pmatrix} .
\end{equation}
Inserting this expansion in Eq.(8) we obtain
\begin{equation}
\begin{pmatrix}
\frac{\bar{h}^2}{2m}(k+K)^2 - \mu + 2gn \\
\frac{\bar{h}^2}{2m}(k-K)^2 - \mu + 2gn
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= \omega
\begin{pmatrix}
u \\
v
\end{pmatrix}
\end{equation}
Outside the interaction zone, where $g = 0$, $U$ and $V$ are independent with energies $\omega_{\nu/v} = \hbar K \pm k^2/2$. Inside the interaction zone where $g \neq 0$, we multiply the first equation of Eq.(10) by $\frac{\bar{h}^2}{2m}(k-K)^2 - \mu + 2gn$ and obtain a second order equation for $\omega$ which leads to
\begin{equation}
\omega = \frac{\bar{h}^2}{m}kK \pm \sqrt{\frac{\bar{h}^2 k_0^2}{2m} \left( \frac{\bar{h}^2 k_0^2}{2m} + 2gn \right)}
\end{equation}
The spectrum has two branches corresponding for high energies to particle and hole-like excitations, respectively. For the two branches, the ratios $V/U$ of the solutions are
\begin{equation}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
= \begin{pmatrix}
\frac{v_1}{U_1} \\
\frac{v_2}{U_2}
\end{pmatrix}
= 1 + \frac{\bar{h}^2}{mgn} \left( k_0^2 + \sqrt{k_0^2 \left( k_0^2 + 2mgn/\bar{h}^2 \right)} \right)
\end{equation}
In each region along the atomic beam, four solutions are possible, corresponding in the noninteracting region to particles or holes propagating in one direction or the other, and the global solutions should be continuous with continuous derivatives for both the $u$ and $v$ functions. Thus 8 equations relate the 12 parameters and for each energy 4 solutions exist. Like in normal scattering theory, our choice of boundary conditions serves to identify the relevant basis for the solution to the problem. Excitations with an escaping hole (removal of an incident particle) in either direction are not physically relevant, and we thus study the elementary excitations with incoming particle components only.

Let us first consider the excitation corresponding to an incoming particle $u(x) \propto e^{i(k+K)x}$ from the right, $(k < -K)$. This particle has a momentum which differs from that of the condensate by more than $K$. If we assume that rather than being an exact delta-function, the atomic interaction has a momentum cut off which is smaller than $K$, the coupling to the $v(x)$ function in the Bogoliubov equations is suppressed, and this mode does not contribute to phase fluctuations in the Bogoliubov vacuum.
Another set of Bogoliubov modes corresponds to incoming particles from the left with momentum \( k + K \) and amplitude \( U_k = \frac{1}{\sqrt{2\pi}} \) (the condition \( [\delta \psi(x), \delta \psi^*(x')] = \delta(x - x') \) serves to normalize the Bogoliubov mode since \( \int_k dU_k(k) e^{i(K + k)x} U_k^*(k') e^{-i(K + k')x'} = \delta(x - x') \) on the left side where the Bogoliubov modes contain no holes). Eq. (17) has four solutions, but we will consider only momentum components \( k + k_1 \) and \( k + k_2 \) close to \( K \) inside the interaction zone, since the effective momentum cut-off in the interaction prevents coupling to waves with very different momenta. Continuity of the solution at the left entrance to the interaction zone implies

\[
\begin{align*}
U_1 &= U_1 + U_2 \\
0 &= V_1 + V_2
\end{align*}
\]  

(13)

Eq. (12) dictates the ratio between the \( U \) and \( V \) amplitudes, and the continuity of the derivatives of \( u \) and \( v \) cannot be fulfilled without allowing for reflection. This reflection will, however, be very small as \( k \ll K \), and we make an insignificant error by requiring only continuity of the Bogoliubov mode functions. All amplitudes in the interaction zone are hence given by the incident amplitude

\[
U_i = \frac{1}{\sqrt{2\pi}}.
\]

At the right of the interaction zone, we have to match to the independent solutions \( U e^{i(k + K)x} \) and \( V e^{-i(k - K)x} \) of same Bogoliubov energy \( \omega \), i.e., the amplitudes \( U \) and \( V \) are given by the continuity relations

\[
\begin{align*}
U e^{i(k + K)L} &= U_1 e^{i(k_1 + K)L} + U_2 e^{i(k_2 + K)L} \\
V e^{-i(k' - K)L} &= V_1 e^{-i(k_1 - K)L} + V_2 e^{-i(k_2 - K)L}
\end{align*}
\]  

(14)

Using the relations (12), this gives

\[
\begin{align*}
U &= U_1 e^{-ikL} \left( \frac{y_2}{y_1 - y_2} e^{ik_1 L} + \frac{y_1}{y_1 - y_2} e^{ik_2 L} \right) \\
V &= U_1 e^{-ik_L} \left( \frac{y_1}{y_1 - y_2} e^{-ik_1 L} + \frac{y_2}{y_1 - y_2} e^{-ik_2 L} \right)
\end{align*}
\]  

(15)

The phase fluctuations given in Eq. (12) can now be expressed as

\[
(\theta(x)\theta(x')) = \frac{1}{n} \int dk \left\{ 2Re(V(k)^*U(k)e^{ikx'} e^{-ikx} + 2|V(k)|^2 \cos(k'(x' - x)) \right\},
\]

(17)

where the \( \delta \)-function part has been omitted.

We now present a number of simplifications, which are valid under reasonable assumptions and which will lead to a better understanding of the dependence of the phase fluctuations on the physical parameters of the problem. First, as long as \( k_1 \simeq k_2 \simeq k \ll K \) we can use the expressions of \( y_1 \) and \( y_2 \) for the single value \( k \). We cannot, however, replace \( k_1 \) and \( k_2 \) generally by \( k \) in the exponential factors in (16). Instead, we look at limiting cases. If \( k \gg mgn/h \), the energy \( \omega = kK + k^2/2 \) leads to the same dispersion law as that of a non interacting gas. This dispersion law leads to the expressions, \( k_1 \simeq k + k^2/2K \) and \( k_2 \simeq k \), and the expressions for \( y_1 \) and \( y_2 \) are in this limit approximated by \( y_1 \simeq \frac{hK}{mgn} \) and \( y_2 \simeq \frac{h}{mgn} \). The suppression of the \( V(k) \) amplitude in Eq. (16) by the factor \( \frac{y_1}{y_1 - y_2} \simeq \frac{mgn}{hK} \ll 1 \) results in only negligible contributions to phase fluctuations from these momentum components. If instead, \( |k| \ll \sqrt{mgn}/h \), the Bogoliubov dispersion law approximately yields \( k_1 \simeq k + \frac{mgn}{hK} \) and \( k_2 \simeq k - \frac{mgn}{hK} \), and the expressions for \( y_1 \) and \( y_2 \) are in this limit approximated by \( y_1 \simeq 1 + \frac{h|k|}{\sqrt{mgn}} \) and \( y_2 \simeq 1 - \frac{h|k|}{\sqrt{mgn}} \). We then find

\[
U(k) \simeq V(k) \simeq -\frac{e^{ikL}}{\sqrt{2\pi}} \sqrt{\frac{mgn}{2\pi k}} e^{ikx} \sin \left( \frac{k y_1 hK}{\sqrt{mgn}} L \right).
\]

(18)

In Eq. (17) \( k' \) is the momentum associated with the \( v \)-component in the noninteracting region and, as long as \( |k| \ll K \), it writes \( k' \simeq k + \frac{k^2}{2K} \), and the cross term in \( V(k)^*U(k) \) can thus be written,

\[
I(x, x') = \frac{2}{n} \int dk |V(k)|^2 Re(e^{ik(x'-x)}e^{ikx/2}/K).
\]

(19)

Approximating \( k' \) by \( k \) the \( |V(k)|^2 \) term in Eq. (17) can be written

\[
J(x, x') = \frac{2}{n} \int dk |V(k)|^2 \cos(k(x - x')).
\]

(20)

For large \( x \), \( I(x, x') \) becomes wider as a function of \( y = (x' - x) \). If we label \( x/K \) as \( t/2m \) in the last exponential we recover the expression for the spreading wave packet of a massive particle, given by the momentum spread \( \Delta k \) of \( |V(k)|^2 \), i.e., \( \Delta y = h\Delta k \frac{2}{\sqrt{mgn}} \). At distances far from the interacting region \( I(x, x') \) gives thus a negligible contribution to local phase fluctuations, but when integrated over intervals larger than \( \Delta y \), the conserved 'norm' of the spreading wave packet implies a result comparable to the contribution from the \( |V(k)|^2 \) term, estimated below.

Fig. 2a) shows the value of \( (\theta(x)\theta(x')) \) for different values of \( x \) and as functions of \( x' \). The curves are obtained by a numerical integration of Eq. (14) with the proper expressions for \( U(k) \) and \( V(k) \). For large \( x \), we both see the spreading \( I(x, x') \) and the narrow \( J(x, x') \) component, cf. Fig 2b).

We are now in position to obtain the essential scaling of the phase fluctuations with the physical parameters of the problem: If we focus on the narrow part \( J(x, x') \), at distances from the interaction zone larger than \( L^2 gn/K \), the phase fluctuation have a finite coherence length, estimated as the inverse of the width in momentum space of
\[ V(k), \]

\[ \Delta x \approx \frac{\sqrt{mgnL}}{hK}. \]  \hspace{1cm} (21)

The dependence on the size of the interacting region \( L \) is due to a phase matching conditions: momentum and energy conservation cannot be both fulfilled in a collision which creates excitations and the violation of momentum conservation becomes more severe as \( L \) increases. The amplitude of the fluctuations increases linearly with \( L \),

\[ \langle \theta(x) \rangle^2 = \frac{1}{n} \left( \frac{mgn}{\hbar^2} \right)^{3/2} \frac{L}{K}. \]  \hspace{1cm} (22)

and to recover a well defined phase, one has to integrate over a finite detector region in space. If one detects atoms on a length \( X \) larger than \( \Delta x \), the phase precision is given by the \( k = 0 \) component of the fourier transform of the phase fluctuations,

\[ \Delta \theta^2 = \frac{4\pi}{n} |V(0)|^2 = \frac{2}{nX} \left( \frac{mgnL}{\hbar^2 K} \right)^2 = 2 \langle \theta(x) \rangle^2 \frac{X}{\Delta x}. \]  \hspace{1cm} (23)

If \( X \) becomes larger than the length scale \( \Delta y \), the integral of \( J(x, x') \) contributes with the same amount, i.e., Eq. \((23)\) is multiplied by a factor of 2. This analysis is confirmed in Fig. 2c), showing as a dashed line the analytical result \((23)\) and as a solid line the results of the full numerical calculation. The dot-dashed line in the figure shows the value \((22)\), applicable for short intervals \( X < \Delta x \).

Let us conclude with a numerical example for a beam of Rubidium atoms with velocity \( \hbar K / m = 80 \text{ mm/s} \), and a linear density of \( n = 10^6 \) atoms/m, subject to an interacting region with a perpendicular confinement of \( a_\perp = 0.5 \mu \text{m} \) and a length of \( L = 1 \text{ cm} \). With the value \( a = 5 \text{ nm} \) for the scattering length we then obtain

\[ \Delta x \approx 26 \mu \text{m} \quad \text{and} \quad \langle \theta(x) \rangle^2 \approx 2. \]  \hspace{1cm} (24)

To have a phase uncertainty \( \Delta \theta \) as small as 0.1, one has to count the Rb atoms in this example over a length of order 2 cm. Note that the phase uncertainty is two orders of magnitude larger than the standard result for a perfect coherent state on that interval, and if we assume a coherence length \( L \) of the beam exceeding 2 cm, the contribution from number fluctuations \((7)\) is smaller than the Bogoliubov excitation result by at least a factor of two.

To summarize, we have quantified the phase fluctuations in an atomic beam, and found that they manifest themselves at different observational length scales: If \( X \sim 1/n \), the number of atoms observed is of order unity and the phase is always ill-defined. The phase uncertainty falls off when \( X \) is increased, but in the presence of interactions, until \( X \) reaches the length scale \( \Delta x = \sqrt{mgnL}/\hbar K \), it levels off to a value given by Eq. \((22)\). For \( X \) larger than \( \Delta x \), it decreases according to Eq. \((23)\) or Eq. \((23)\) multiplied by a factor two if \( X \) is also larger than \( \Delta y = \hbar x_0 / (L \sqrt{mgn}) \), where \( x_0 \) is the location of the detection interval. If the coherence length \( L \) of the beam is short, there is an extra contribution \((2)\) to phase fluctuations. Finally, it should be mentioned that other sources of phase fluctuations may be important in a guided atomic beam. For example, the decoherence and heating of the atomic beam due to the presence of the surfaces of the macroscopic elements that provide the guiding potential has been analysed in \([8]\).

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FIG. 2: Phase fluctuations calculated for an example with numerical parameters $L = 10^4/K$ and $\gamma n = 0.02h^2K^2/m$. (a): Phase fluctuations given by Eq. (17). For large $x$ the curves clearly separate in a central feature and a broad pedestal due to the $|V(k)|^2$ and the $V(k)^*U(k)$ contributions, respectively. (b): Phase correlations around $x_0 = 20L^2gn/K$. The $J(x, x')$ term is shown as a dashed line. (c): Phase fluctuations averaged over intervals of variable length $X$. The curves show $\langle (\int_{x_0}^{x_0+X} dx \theta(x)/X)^2 \rangle$ for $x_0 = 50L^2gn/K$. The full line is the result of a numerical calculation, the dashed line is the analytical expression (23), and the horizontal dot-dashed line is the value (22) expected for $X$ smaller than the coherence length $\Delta x = \sqrt{\gamma nmL/hK}$. 