GLOBAL WELL-POSEDNESS OF 3-D ANISOTROPIC NAVIER-STOKES SYSTEM WITH LARGE VERTICAL VISCOUS COEFFICIENT

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Abstract. In this paper, we first prove the global well-posedness of 3-D anisotropic Navier-Stokes system provided that the vertical viscous coefficient of the system is sufficiently large compared to some critical norm of the initial data. Then we shall adapt the proof to show the global well-posedness of the classical 3-D Navier-Stokes system with the initial data varying fast enough in the vertical direction and the third component of the initial velocity being sufficiently small.

Keywords: Navier-Stokes system, anisotropic Littlewood-Paley theory, well-posedness.

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1. Introduction

In this paper, we first investigate the global well-posedness of the following 3-D anisotropic Navier-Stokes system provided that the vertical viscous coefficient is large enough:

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v - \Delta_\nu v + \nabla p &= 0, \\
\text{div } v &= 0, \\
v|_{t=0} &= v_0 = (v^h_0, v^3_0),
\end{aligned}
\]

where \( v = (v^h, v^3) \) with \( v^h = (v^1, v^2) \) stands for the velocity of the incompressible fluid flow and \( p \) for the scalar pressure function, which guarantees the divergence free condition of the velocity field, \( \nu^2 \) denotes the vertical viscous coefficient, \( \Delta_\nu \equiv \Delta_h + \nu^2 \partial_3^2 \) with \( \Delta_h \equiv \partial_1^2 + \partial_2^2 \).

In what follows, we shall always denote the system \((NS_1)\) by \((NS)\).

When \( \nu = 1 \), \((NS_\nu)\) is exactly the classical Navier-Stokes system, whereas when \( \nu = 0 \), \((NS_\nu)\) reduces to the anisotropic Navier-Stokes system arising from geophysical fluid mechanics (see [2]). The main motivation for us to study Navier-Stokes system with large vertical viscous coefficient comes from the study of Navier-Stokes system on thin domains (see (2.4) of [16] for instance), which we shall present more details later on.

In the seminal paper [13], Leray proved the global existence of finite energy weak solutions to \((NS)\). Yet the uniqueness and regularity to such weak solutions are big open questions in the field of mathematical fluid mechanics except the case when the initial data have special structure. For instance, with axi-symmetric initial velocity and without swirl component, Ladyzhenskaya [12] and independently Ukhovskii and Yudovich [18] proved the existence of weak solution along with the uniqueness and regularity of such solution to \((NS)\). When the initial data \( v_0 \) has a slow space variable, Chemin and Gallagher [3] (see also [6]) can also prove the global well-posedness of such a system.

On the other hand, Fujita and Kato [10] proved the global well-posedness of \((NS)\) when the initial data \( v_0 \) is sufficiently small in the homogeneous Sobolev space, \( H^\frac{7}{2} \). Due to \( \text{div } v_0 = 0 \), Zhang [19] and Paicu and the second author [15] improved Fujita and Kato result by requiring

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only two components of the initial velocity being sufficiently small in some critical Besov space even when \( \nu = 0 \) in \((NS_\nu)\). Lately, Chemin and Zhang \cite{7} proved that if \( T^* \) is the lifespan to the Fujita and Kato solution of \((NS)\) and \( T^* \) is finite, then for any \( p \in ]4,6[ \) (see \cite{8} for the extension of \( p \in ]4,\infty[ \)), there holds
\[
\int_0^{T^*} \| v \cdot e \|_{H_t^{\frac{1}{p} + \frac{2}{p}}}^p \, dt = \infty,
\]
where \( e \) is any unit vector of \( \mathbb{R}^3 \). This result ensures that a critical norm to one component of the velocity field controls the regularity of Fujita and Kato solution to \((NS)\). Yet we still do not know whether or not the classical Navier-Stokes system is globally well-posed with one component of the initial velocity being sufficiently small. As a toy model, we are going to prove this type of result for \((NS_\nu)\) provided that \( \nu \) is sufficiently large. Then by a modification of the proof, we shall prove the global well-posedness of the classical 3-D Navier-Stokes system when the initial data vary fast enough in the vertical direction and the third component of the initial velocity is sufficiently small (see Theorem 1.3 below).

Before we present the main result of this paper, let us recall the following anisotropic Sobolev space from \cite{7}.

**Definition 1.1.** For \((s,s')\) in \( \mathbb{R}^2 \), \( H^{s,s'} \) denotes the space of homogeneous tempered distribution a such that
\[
\| \alpha \|_{H^{s,s'}}^2 = \int_{\mathbb{R}^3} |\xi_h|^{2s} |\xi_3|^{2s'} |\hat{\alpha}(\xi)|^2 \, d\xi < \infty \quad \text{with} \quad \xi_h = (\xi_1,\xi_2).
\]
And for any \( \theta \) in \([0,\frac{1}{2}[, \) we denote \( H^\theta \defeq H^{-\frac{1}{2}+\theta,-\theta} \).

Our first result of this paper states as follows:

**Theorem 1.1.** Let \( p \in ]4,6[ \) and \( \theta \in \left[ \frac{1}{2} - \frac{2}{p}, \frac{1}{6} \right] \). Given initial data \( v_0 \) with \( \Omega_0 = \text{curl} \, v_0 \in L^\frac{3}{2} \) and \( \text{div} \, v_0 = 0 \), we denote \( \omega_0 \defeq \partial_1 v_0^2 - \partial_2 v_0^1 \) and \( M_0 \defeq \| \omega_0 \|_{L^\frac{3}{2}}^2 + \| \partial_3 \omega_0 \|_{H^\theta}^2 + \| v_0^3 \|_{H^\frac{1}{2}, \theta}^2 \). Then there exists some positive constant \( C_1 \) so that if
\[
\nu \geq C_1 (M_0^\frac{1}{3} + M_0^\frac{1}{2}),
\]
\((NS_\nu)\) has a unique global solution \( v \in C([0,\infty[; H^\frac{3}{2}) \cap L^2_{\text{loc}}([0,\infty[; H^\frac{3}{2}) \). Furthermore, for any \( t > 0 \), there holds
\[
M(t) + \int_0^t \left( \| \nabla_{\nu} \omega_3 (t') \|_{L^2}^2 + \| \nabla_{\nu} \partial_3 \omega^3 (t') \|_{H^\theta}^2 + \| \nabla_{\nu} v^3 (t') \|_{H^\frac{1}{2}, \theta}^2 \right) \, dt' \leq 2M_0.
\]

Here and in all that follows, we always denote \( \omega \defeq \partial_1 v^2 - \partial_2 v^1 \), \( \omega_\alpha \defeq \frac{\omega}{|\omega|} |\omega|^\alpha \), \( M(t) \defeq \| \omega_3 (t) \|_{L^2}^2 + \| \partial_3 \omega^3 (t) \|_{H^\theta}^2 + \| v^3 (t) \|_{H^\frac{1}{2}, \theta}^2 \) and \( \nabla_{\nu} \defeq (\partial_1, \partial_2, \nu \partial_3) \).

**Remark 1.1.** If \( \| \Omega_0 \|_{L^\frac{3}{2}} \) is small enough, it follows from (4.1) below that the Condition (1.1) holds for \( \nu = 1 \). Then Theorem 1.1 ensures the global well-posedness of the classical 3-D Navier-Stokes system \((NS)\) with initial vorticity \( \Omega_0 \) satisfying \( \| \Omega_0 \|_{L^\frac{3}{2}} \) being sufficiently small.

With a small modification to the proof of the above theorem, we also have
Theorem 1.2. Let \( p \in [4, 6[, \theta \in ] \frac{1}{2} - \frac{2}{p}, \frac{1}{2} [ \). Given initial data \( v_0 \) with \( \Omega_0 = \text{curl} v_0 \in L^\frac{2}{p} \) and \( \text{div} v_0 = 0 \), we denote \( L_0 \overset{\text{def}}{=} \max(1, \| \omega_0 \|_{L^\frac{2}{p}}, \| \partial_3 v_0^3 \|_{L^\frac{2}{p}}) \). Then there exist some small enough positive constant \( c_0 \) and some large enough positive constant \( C_2 \) so that if

\[
\| v_0^3 \|_{H^\frac{1}{2}, 0} \leq c_0 \quad \text{and} \quad C_2 L_0^\frac{5}{4} \| v_0^3 \|_{H^\frac{1}{2}, 0}^\frac{2}{5} \leq \nu,
\]

\((NS_\nu)\) has a unique global solution \( v \in C([0, \infty]; H^\frac{3}{4}) \cap L^2_{\text{loc}}((0, \infty]; H^\frac{3}{4}). \) Furthermore, for any \( t > 0 \), there hold

\[
L(t) + \int_0^t \left( \| \nabla \omega_\frac{3}{4}(t') \|_{L^2}^2 + \| \nabla \partial_3 v^3(t') \|_{L^2}^2 \right) \, dt' \leq 2L_0,
\]

\[
\| v^3(t) \|_{H^\frac{1}{2}, 0}^2 + \int_0^t \| \nabla v^3(t') \|_{L^2}^2 \, dt' \leq 2 \| v_0^3 \|_{H^\frac{1}{2}, 0}^2,
\]

with \( L(t) \overset{\text{def}}{=} \| \omega_\frac{3}{4}(t) \|_{L^2}^2 + \| \partial_3 v^3(t) \|_{L^2}^2 \).

We mention that in Theorem 1.2, the smallness condition for \( \| v_0^3 \|_{H^\frac{1}{2}, 0} \) does not depend on \( v_0^3 \). The main difference between Theorem 1.1 and Theorem 1.2 is that, Theorem 1.2 can guarantee that \( \| v^3(t) \|_{H^\frac{1}{2}, 0} \) keeps being small for all the time provided that \( \| v_0^3 \|_{H^\frac{1}{2}, 0} \) is sufficiently small. We point out that the main idea used to prove Theorem 1.2 can also be applied to study the global well-posedness of the classical 3-D Navier-Stokes equations with a fast variable:

\[
\begin{cases}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla \Pi = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\text{div} u = 0, \\
u|_{t=0} = v_{0, \nu}(x_h, \nu x_3) = (v_{0, \nu}^h(x_h, \nu x_3), v_{0, \nu}^3(x_h, \nu x_3)).
\end{cases}
\]

Let \( u(t, x) \overset{\text{def}}{=} v(t, x_h, \nu x_3) \) and \( \Pi(t, x) \overset{\text{def}}{=} p(t, x_h, \nu x_3) \). Then \((v, p)\) verifies

\[
\begin{cases}
\partial_t v + v \cdot \nabla v - \Delta v + \nabla p = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\nabla \cdot v = 0, \\
u|_{t=0} = v_{0, \nu} = (v_{0, \nu}^h, v_{0, \nu}^3),
\end{cases}
\]

which is exactly the same system as the rescaled Navier-Stokes system (see (2.4) of [16]) arising from the study of 3-D Navier-Stokes system on thin domains, \( [0, L_1] \times [0, L_2] \times [0, 1/\nu] \). In [16] (see also [9, 11, 17]), Raugel and Sell proved the global well-posedness of (1.6) in a periodic domain, \( [0, L_1] \times [0, L_2] \times [0, 1], \) provided that \( \nu \) is sufficiently large compared to the initial data. The main ideas in [16, 9, 11, 17] is to decompose the solution \( v \) of (1.6) as

\[
v = M(v) + w \quad \text{with} \quad M(v)(t, x_h) \overset{\text{def}}{=} \int_0^1 v(t, x_h, x_3) \, dx_3 \quad \text{and} \quad w \overset{\text{def}}{=} v - M(v).
\]

Then the authors exploited the fact that: 2-D classical Navier-Stokes system is globally well-posed for any data in \( L^2 \), and the fact that: 3-D Navier-Stokes system is globally well-posed with small regular initial data, to prove that the solutions of (1.6) can be split as the sum of a 2-D large solution and a 3-D small solution of (1.6).

We remark that in the whole space case, we do not know how to define the average of the velocity field on the vertical variable. Thus it is not clear how to apply the ideas in [16, 9, 11, 17] to solve (1.6). Our principle result concerning the well-posedness of the system (1.6) is as follows:
Theorem 1.3. Let $p \in [4, 6]$ and $\theta \in \left[\frac{1}{3}, \frac{2}{3}\right]$. Given initial data $v_{0, \nu}$ with $\Omega_{0, \nu} = \text{curl} v_{0, \nu} \in L^\frac{3}{2}$ and $\nabla v : v_{0, \nu} = 0$, we denote $\omega_{0, \nu} \overset{\text{def}}{=} \partial_1 v_{0, \nu}^3 - \partial_2 v_{0, \nu}^1$ and $L_{0, \nu} \overset{\text{def}}{=} \max(1, \|\omega_{0, \nu}\|_{L^\frac{3}{2}} + \|\nu \partial_3 v_{0, \nu}^3\|_{L^\frac{3}{2}})$. Then there exist small enough positive constants $c_0, c_1, c_2$ and some $\eta \in [0, 1]$ so that if
\begin{equation}
\|v_{0, \nu}^3\|_{H^\frac{3}{2}, 0} \leq c_0, \quad \nu^{1 - \frac{2}{p}} L_{0, \nu}^{\frac{2}{p}} \|\omega_{0, \nu}\|_{H^\frac{3}{2}, 0} \leq c_1 \quad \text{and} \quad \nu^{-1} \|v_{0, \nu}^3\|_{H^\frac{3}{2}, 0}^{2\eta} L_{0, \nu}^\frac{2}{p} \leq c_2,
\end{equation}
(1.8) has a unique global solution $v \in C([0, \infty[; H^\frac{1}{2}) \cap L^2_{\text{loc}}([0, \infty[; H^\frac{3}{2})$. Furthermore, for any $t > 0$, there hold
\begin{equation}
L_{\nu}(t) + \int_0^t \left( \|\nabla v\|_{L^2}^2 + \|\nu \partial_3 v^3(t')\|_{H^\frac{3}{2}}^2 \right) dt' \leq 2L_{0, \nu},
\end{equation}
(1.9)
where $L_{\nu}(t) \overset{\text{def}}{=} \|\omega(t)\|_{L^\frac{3}{2}}^\frac{2}{3} + \|\nu \partial_3 v^3(t)\|_{H^\frac{3}{2}}^2$.

Remark 1.2. (1) More precise smallness condition than (1.8) will be given by (5.23) below.
(2) Theorem 1.3 guarantees that the System (1.5) with initial data:
\begin{equation}
u_0(x) = \left( v_0^h(x_h, \nu x_3), \frac{1}{\nu} v_0^3(x_h, \nu x_3) \right) \quad \text{with} \quad \text{div} v_0 = 0 \quad \text{and} \quad \Omega_0 = \text{curl} v_0 \in L^\frac{3}{2}
\end{equation}
is globally well-posed provided that $\nu$ is large enough. In fact, the System (1.5) with initial data (1.10) corresponds to the System (1.6) with initial data $v_{0, \nu}(x) = (v_0^h(x), \frac{1}{\nu} v_0^3(x))$. It is easy to check that (1.8) is satisfied for any $\eta \in \left\{ 1 - \frac{2}{p}, \frac{1}{2} \right\}$ as long as $\nu$ is so large that
\begin{equation}
\nu^{-\frac{2}{p}} L_{0}^{\frac{2}{3}} \|\omega_{0, \nu}\|_{H^\frac{3}{2}, 0} \leq c_1 \quad \text{and} \quad \nu^{-1 + 2\eta} \|v_{0, \nu}^3\|_{H^\frac{3}{2}, 0}^{2\eta} L_{0}^\frac{2}{p} \leq c_2,
\end{equation}
where $L_{0}$ is given by Theorem 1.2.
(3) Chemin, Gallagher and Paicu [4] proved the global well-posedness of (1.5) on $\mathbb{T}^2 \times \mathbb{R}$ with initial data given by (1.10) under the assumptions that $\nu$ is sufficiently small and $v_0$ is small enough in some analytic space.

Let us complete this section by a sketch of this paper.

Sketch of the paper. The main idea of the proof to Theorems 1.1 to 1.3 is motivated by [7]. We recall from [7] that: let $v^h \overset{\text{def}}{=} (v_1^h, v_2^h)$ be the horizontal components of the velocity field, $v$, and $\omega \overset{\text{def}}{=} \partial_1 v^2 - \partial_2 v^1$ be the third component of the vorticity field, $\Omega$. Then $(NS\nu)$ can be equivalently reformulated as
\begin{align}
\partial_t \omega + v \cdot \nabla \omega - \Delta_\nu \omega &= \partial_3 v^3 \omega + \partial_2 v^3 \partial_3 v^1 - \partial_1 v^3 \partial_2 v^2, \\
\partial_t v^3 + v \cdot \nabla v^3 - \Delta_\nu v^3 &= -\partial_3 \Delta^{-1} \left( \sum_{l,m=1}^3 \partial_\nu \omega^l \partial_m v^l \right),
\end{align}
(1.11)
where $\omega|_{t=0} = \omega_0$, $v^3|_{t=0} = v_0^3$. Then due to \text{div}_h v^h = -\partial_3 v^3$, given $(\omega, v^3)$, by Biot-Savart’s law, we write
\begin{equation}
v^h = v^h_{\text{curl}} + v^h_{\text{div}}, \quad \text{where} \quad v^h_{\text{curl}} = \nabla_h \Delta_h^{-1} \omega \quad \text{and} \quad v^h_{\text{div}} = -\nabla_h \Delta_h^{-1} \partial_3 v^3,
\end{equation}
(1.12)
where $\nabla_h \overset{\text{def}}{=} (\partial_1, \partial_2)$, $\nabla_h^\perp \overset{\text{def}}{=} (-\partial_2, \partial_1)$.

In view of (1.12), in order to obtain a critical norm estimate of the horizontal velocity, $v^h$, one needs to handle the estimates of the third component of the vorticity field as well as the vertical derivative to the third component of the velocity field. In [7], Chemin and the second author first derived separately the estimates of $\|\omega^3(t)\|_{L^2}$ and $\|\partial_3 v^3(t)\|_{\mathcal{H}_\theta}$. They provided with the estimate of $\|v^3\|_{L^p_t((H^0)^2)}$, they could manage to close the estimates of $\|\omega^3(t)\|_{L^2}$ and $\|\partial_3 v^3(t)\|_{\mathcal{H}_\theta}$. Hence to prove the global well-posedness of $(NS_\nu)$ and (1.6), it remains to show that $\|v^3\|_{L^p_t((H^0)^2)}$ is finite for any $T < \infty$ and $p \in [4, 6]$.

In Section 3, we shall first derive the energy estimates for $\|\omega^3(t)\|_{L^2}$ and $\|\partial_3 v^3(t)\|_{\mathcal{H}_\theta}$, namely Proposition 3.1 and Proposition 3.2. We point out that compared with the arguments in Sections 5 and 6 of [7], in order to handle the global well-posedness of $(NS_\nu)$ and (1.6), here we need a much more delicate argument.

On the other hand, Lemma 2.7 below claims that

$$\|v^3\|_{H^{3/4, 3/2}_t} \leq \|\partial_3 v^3\|_{\mathcal{H}_\theta}^{2(1 - \theta)} \|\omega^3\|_{H^0}^{\frac{1}{2} - \theta} \|\partial_3 v^3\|_{\mathcal{H}_\theta}^{2} + \|v^3\|_{H^{3/4, 3/2}_t}^{1 - \theta} \|\nabla_h v^3\|_{H^{3/4, 3/2}_t}^{\frac{2}{3}}.$$  

So that in order to derive the estimate of $\|v^3\|_{L^p_t((H^0)^2)}$, it remains to perform the $H^{3/4, 3/2}_t$ estimate for $v^3$, which will be the object of Proposition 3.3.

In Section 4, we shall provide the a priori estimates for $\|\omega^3(t)\|_{L^2}$, $\|\partial_3 v^3(t)\|_{\mathcal{H}_\theta}$ and $\|v^3(t)\|_{H^{3/4, 3/2}_t}$ under the assumptions (1.1). This along with (1.13) ensures Theorem 1.1. With a small modification of the argument, we shall then present the proof of Theorem 1.2.

Finally in Section 5, we shall adapt the arguments in Sections 3 and 4 to prove Theorem 1.3.

## 2. Preliminaries

We first recall some basic facts on anisotropic Littlewood-Paley theory from [1]. Let us recall the following dyadic operators:

$$\Delta_j a = \mathcal{F}^{-1}(\varphi(2^{-j} |\xi|)\hat{a}), \quad \Delta^h_j a = \mathcal{F}^{-1}(\varphi(2^{-k} |\xi_h|)\hat{a}), \quad \Delta^v_j a = \mathcal{F}^{-1}(\varphi(2^{-\ell} |\xi_v|)\hat{a}),$$

$$S_j a = \mathcal{F}^{-1}(\chi(2^{-j} |\xi|)\hat{a}), \quad S^h_j a = \mathcal{F}^{-1}(\chi(2^{-k} |\xi_h|)\hat{a}), \quad S^v_j a = \mathcal{F}^{-1}(\chi(2^{-\ell} |\xi_v|)\hat{a}),$$

where $\xi = (\xi_1, \xi_2)$, $\mathcal{F} a$ and $\hat{a}$ denote the Fourier transform of the distribution $a$, $\chi(\tau)$ and $\varphi(\tau)$ are smooth functions such that

$$\text{Supp} \varphi \subset \left\{ \tau \in \mathbb{R} / \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \tau) = 1,$$

$$\text{Supp} \chi \subset \left\{ \tau \in \mathbb{R} / |\tau| \leq \frac{4}{3} \right\} \quad \text{and} \quad \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j} \tau) = 1.$$

**Definition 2.1.** Let us define the space $(B^{s_1}_{p_1, q_1})_h(B^{s_2}_{p_2, q_2})_v$ (with usual adaptation when $q_1$ or $q_2$ equal $\infty$) as the space of homogenous tempered distributions $u$ so that

$$\|u\|_{(B^{s_1}_{p_1, q_1})_h(B^{s_2}_{p_2, q_2})_v} \overset{\text{def}}{=} \left( \sum_{k \in \mathbb{Z}} 2^{q_1 k s_1} \left( \sum_{\ell \in \mathbb{Z}} 2^{q_2 \ell s_2} \|\Delta^h_k \Delta^v_\ell u\|_{L^p_h(L^{q_2}_v)} \right)^{q_1/q_2} \right)^{1/q_1}$$

is finite.
We remark that \((B_{2,2}^{s_1})_h (B_{2,2}^{s_2})_v\), coincides with the anisotropic Sobolev space \(H^{s_1,s_2}\). Let us also remark that in the case when \(q_1\) is different from \(q_2\), the order of summation is important.

For the convenience of the reader, we make the following anisotropic Bernstein type lemma from [5, 14]:

**Lemma 2.1.** Let \(B_h\) (resp. \(B_v\)) a ball of \(\mathbb{R}^2_h\) (resp. \(\mathbb{R}^2_v\)), and \(C_h\) (resp. \(C_v\)) a ring of \(\mathbb{R}^2_h\) (resp. \(\mathbb{R}^2_v\)); let \(1 \leq p_2 \leq p_1 \leq \infty\) and \(1 \leq q_2 \leq q_1 \leq \infty\). Then there holds:

If the support of \(\tilde{a}\) is included in \(2^k B_h\), then
\[
\left\| \partial_x^\alpha a \right\|_{L^{p_1}_h (L^{q_1}_v)} \lesssim 2^{k(|\alpha|+2(1/p_2-1/p_1))} \left\| a \right\|_{L^{p_2}_h (L^{q_2}_v)}.
\]
If the support of \(\tilde{a}\) is included in \(2^k B_v\), then
\[
\left\| \partial_x^\alpha a \right\|_{L^{p_1}_h (L^{q_1}_v)} \lesssim 2^{(\beta+2(1/q_2-1/q_1))} \left\| a \right\|_{L^{p_2}_h (L^{q_2}_v)}.
\]
If the support of \(\tilde{a}\) is included in \(2^k C_h\), then
\[
\left\| a \right\|_{L^{p_1}_h (L^{q_1}_v)} \lesssim 2^{-kn} \sup_{|\alpha|=N} \left\| \partial_x^\alpha a \right\|_{L^{p_1}_h (L^{q_1}_v)}.
\]
If the support of \(\tilde{a}\) is included in \(2^k C_v\), then
\[
\left\| a \right\|_{L^{p_1}_h (L^{q_1}_v)} \lesssim 2^{N} \left\| \partial_x^N a \right\|_{L^{p_1}_h (L^{q_1}_v)}.
\]
Here and in all that follows, \(a \lesssim b\) means that there exists a uniform constant \(C\) so that \(a \leq Cb\).

We also need the following two Lemmas from [7]:

**Lemma 2.2** (Lemma 4.3 of [7]). For any \(s\) positive and any \(\theta\) in \(]0, s[\), we have
\[
\left\| f \right\|_{(B_{p,q}^{s-\theta})_h(B_{p,1}^{\theta})_v} \lesssim \left\| f \right\|_{B_{p,q}^s}.
\]

**Lemma 2.3** (Lemma 4.5 of [7]). Let \(q \geq 1\), \(p_1 \geq p_2 \geq 1\) with \(\frac{1}{p_2} + \frac{1}{p_1} \leq 1\), and \(s_1 < \frac{q_1}{p_1}\), \(s_2 < \frac{q_2}{p_2}\) (resp. \(s_1 \leq \frac{q_1}{p_1}\), \(s_2 \leq \frac{q_2}{p_2}\) if \(q = 1\)) with \(s_1 + s_2 > 0\). Let \(\sigma_1 < \frac{s_1}{p_1}\), \(\sigma_2 < \frac{s_2}{p_2}\) (resp. \(\sigma_1 \leq \frac{s_1}{p_1}\), \(\sigma_2 \leq \frac{s_2}{p_2}\) if \(q = 1\)) with \(\sigma_1 + \sigma_2 > 0\). Then for \(a\) in \((B_{p_1,q}^{s_1})_h (B_{p_1,q}^{s_2})_v\) and \(b\) in \((B_{p_2,q}^{s_1})_h (B_{p_2,q}^{s_2})_v\), the product \(ab\) belongs to \((B_{p_1,q}^{s_1+s_2-2/p_2})_h (B_{p_1,q}^{s_1,2-2/p_2})_v\), and
\[
\left\| ab \right\|_{(B_{p_1,q}^{s_1+s_2-2/p_2})_h (B_{p_1,q}^{s_1,2-2/p_2})_v} \lesssim \left\| a \right\|_{(B_{p_1,q}^{s_1})_h (B_{p_1,q}^{s_2})_v} \left\| b \right\|_{(B_{p_2,q}^{s_1})_h (B_{p_2,q}^{s_2})_v}.
\]

The lemma below can be viewed as an anisotropic version of Lemma 3.2 in [7]:

**Lemma 2.4**. For \(i = 1, 2, 3\), we have
\[
\left\| \partial_i \omega \right\|_{L^{\frac{3}{2}}_x} \leq C \left\| \partial_i \omega \right\|_{L^{2}} \left\| \omega \right\|_{L^{2}}^{\frac{1}{2}}, \quad \left\| \partial_i \omega \right\|_{L^{2}_x} \leq C \left\| \partial_i \omega \right\|_{L^{2}} \left\| \nabla \omega \right\|_{L^{2}}^{\frac{1}{2}},
\]
Moreover, for any \(s\) in \([-\frac{s}{3}, \frac{s}{3}]\), we have
\[
\left\| \omega \right\|_{H^{s}} \leq C \left\| \omega \right\|_{L^{2}} \left\| \nabla \omega \right\|_{L^{2}}^{\frac{s}{2}+\frac{s}{3}}.
\]

**Proof.** By the definition of \(\omega\), we have \(\left| \partial_i \omega \right| = \frac{1}{3} \left| \partial_i \omega_{\frac{3}{2}} \right| \cdot |\omega|^\frac{1}{3}\). So that applying Holder’s inequality and then Sobolev inequality yields
\[
\left\| \partial_i \omega \right\|_{L^{\frac{3}{2}}_x} \leq C \left\| \partial_i \omega \right\|_{L^{2}} \left\| \omega \right\|_{L^{2}}^{\frac{1}{2}}, \quad \left\| \partial_i \omega \right\|_{L^{2}_x} \leq C \left\| \partial_i \omega \right\|_{L^{2}} \left\| \omega \right\|_{L^{2}}^{\frac{1}{2}}, \quad \left\| \partial_i \omega \right\|_{L^{6}} \leq C \left\| \partial_i \omega \right\|_{L^{2}} \left\| \omega \right\|_{L^{2}}^{\frac{1}{2}} \left\| \nabla \omega \right\|_{L^{2}}^{\frac{1}{2}}.
\]

which leads to (2.2).

The Estimate (2.3) is given by Lemma 3.2 of [7].

As an application of the above basic facts, we shall present the estimate of the horizontal components of a divergence free vector field in terms of the third component of its vorticity field and the vertical derivative of the third component of the vector field, which is a modified version of Proposition 4.1 in [7].

**Proposition 2.1.** Let \( \theta \in [0, \frac{1}{2}] \) and \( s < \min \left( \frac{1}{2}, \frac{1}{3} - \theta \right) \), \( \alpha < \min \left( \frac{2}{3}, \frac{1}{3} + \theta \right) \) satisfy \( \alpha + s > 0 \). Let \( v = (v^h, v^3) \) be a divergence free vector field and \( \omega = \partial_1 v^2 - \partial_2 v^1 \). Then one has

\[
\|v^h\|_{L^2(B_{2,1}^{1-s})} \lesssim \left\| \frac{\omega_3}{4} \right\|_{L^2}^{1-s} \left\| \nabla \omega_3 \right\|_{L^2}^{1-s} + \|\partial_3 v^3\|_{H^s}^{s+\alpha} \|\nabla \partial_3 v^3\|_{H^s}^{1-s-\alpha}.
\]

**Proof.** In view of (1.12), we find

\[
\|v^h\|_{L^2(B_{2,1}^{1-s})} \lesssim \left\| \frac{\omega_3}{4} \right\|_{L^2}^{1-s} \left\| \nabla \omega_3 \right\|_{L^2}^{1-s} + \|\partial_3 v^3\|_{H^s}^{s+\alpha} \|\nabla \partial_3 v^3\|_{H^s}^{1-s-\alpha}.
\]

Applying Lemma 2.1 and then Lemma 2.2 gives

\[
\|\omega\|_{L^2(B_{2,1}^{1-s})} \lesssim \|\omega\|_{L^2(B_{2,1}^{1-s})}^{1-s} \|\nabla \omega\|_{L^2}^{1-s} \lesssim \|\omega\|_{L^2(B_{2,1}^{1-s})}.
\]

Yet due to \( s + \alpha < 1 \) and \( s + \alpha > -\frac{1}{3} \), for any integer \( N_1 \), we have

\[
\|\omega\|_{L^2(B_{2,1}^{1-s})} \lesssim \sum_{j \leq N_1} 2^{j(1-s-\alpha)} \|\Delta_j \omega\|_{L^2}^{1-s} + \sum_{j > N_1} 2^{j(-\frac{1}{3}-s-\alpha)} \|\Delta_j \nabla \omega\|_{L^2}^{1-s}
\]

\[
\lesssim 2^{N_1(1-s-\alpha)} \|\omega\|_{L^2}^{1-s} + 2^{N_1(-\frac{1}{3}-s-\alpha)} \|\nabla \omega\|_{L^2}^{1-s}.
\]

Choosing the integer \( N_1 \) so that \( 2^{N_1} \sim \left( \frac{\|\nabla \omega\|_{L^2}}{\|\omega\|_{L^2}} \right)^{\frac{3}{4}} \) in the above inequality and then applying (2.2) gives rise to

\[
\|\omega\|_{L^2(B_{2,1}^{1-s-\alpha})} \lesssim \left\| \frac{\omega_3}{4} \right\|_{L^2}^{1-s+\alpha} \left\| \nabla \omega_3 \right\|_{L^2}^{1-s-\alpha}
\]

\[
\lesssim \|\omega_3\|_{L^2}^{1-s+\alpha} \left\| \nabla \omega_3 \right\|_{L^2}^{1-s-\alpha}.
\]

Along the same line to proof of (2.5), for any integer \( N_2 \), we write

\[
\|\partial_3 v^3\|_{L^2(B_{2,1}^{1-s})} = \sum_{k \leq \ell \leq N_2} 2^{-k}s 2^{(\frac{1}{2}-\alpha)} \|\Delta_k \Delta_\ell \partial_3 v^3\|_{L^2}
\]

\[
+ \sum_{k \leq \ell, \ell > N_2} 2^{-k}s 2^{(\frac{1}{2}+\alpha)} \|\Delta_k \Delta_\ell \partial_3 v^3\|_{L^2}
\]

\[
+ \sum_{\ell < k \leq N_2} 2^{-k}(1+s) 2^{(\frac{1}{2}-\alpha)} \|\Delta_k \Delta_\ell \nabla \partial_3 v^3\|_{L^2}
\]

\[
+ \sum_{\ell < k, k > N_2} 2^{-k(1+s)} 2^{(\frac{1}{2}-\alpha)} \|\Delta_k \Delta_\ell \nabla \partial_3 v^3\|_{L^2}.
\]
Note that $\theta + s < \frac{1}{2}, \alpha - \theta < \frac{1}{2}$ and $s + \alpha > 0$, we get, by applying Definition 1.1, that
\begin{equation*}
\|\partial_3 v^3\|_{(B_{2,1}^\alpha_h(B_{2,1}^\frac{1}{2}v)} \lesssim \|\partial_3 v^3\|_{H_\theta} \sum_{k \leq \ell \leq N_2} 2^k \left(\frac{s}{2} - \theta - s\right) 2^{\ell\left(\frac{1}{2} - s - \alpha + \theta\right)}
+ \|\partial_3^2 v^3\|_{H_\theta} \sum_{k \leq \ell, \ell > N_2} 2^k \left(\frac{s}{2} - \theta - s\right) 2^{-\ell\left(\frac{1}{2} + \alpha - \theta\right)}
+ \|\partial_3 v^3\|_{H_\theta} \sum_{\ell \leq k \leq N_2} 2^k \left(\frac{s}{2} - \theta - s\right) 2^{\ell\left(\frac{1}{2} - s - \alpha + \theta\right)}
+ \|\nabla h \partial_3 v^3\|_{H_\theta} \sum_{\ell < k, k > N_2} 2^{-k} \left(\frac{s}{2} + \theta - s\right) 2^{\ell\left(\frac{1}{2} - s - \alpha + \theta\right)}
\lesssim 2^{N_2(1 - s - \alpha)} \|\partial_3 v^3\|_{H_\theta} + 2^{-N_2(s + \alpha)} \|\nabla \partial_3 v^3\|_{H_\theta}.
\end{equation*}
Taking the integer $N_2$ so that $2^{N_2} \sim \|\nabla \partial_3 v^3\|_{H_\theta}^{-1} \|\nabla v^3\|_{H_\theta}$ in the above inequality leads to
\begin{equation}
\|\partial_3 v^3\|_{(B_{2,1}^\alpha_h(B_{2,1}^\frac{1}{2}v)} \lesssim \|\partial_3 v^3\|_{H_\theta}^{\alpha + s} \|\nabla \partial_3 v^3\|_{H_\theta}^{1 - s - \alpha}.
\end{equation}
Substituting (2.5) and (2.6) into (2.4) completes the proof of this proposition.

Let us complete this section by some interpolation inequalities which shall use in the whole context.

**Lemma 2.5.** Let $s_1 \geq \frac{1}{2}, s_2 \geq 0$ satisfying $s_1 + s_2 \leq \frac{3}{2}$. Then for any $a \in \mathcal{S}(\mathbb{R}^3)$, we have
\begin{equation}
\|a\|_{H^{s_1+s_2}} \leq \|a\|_{H_{\frac{1}{2},0}}^{\frac{2-s_1-s_2}{2}} \|\nabla h a\|_{H_{\frac{1}{2},0}}^{s_1-\frac{1}{2}} \|\partial_3 a\|_{H_{\frac{1}{2},0}}^{s_2}.
\end{equation}

**Proof.** By Definition 1.1, we get, by applying Hölder’s inequality with measure $|\xi_h| |\hat{a}(\xi)|^2 \, d\xi$, that
\begin{align*}
\|a\|^2_{H^{s_1+s_2}} &= \int_{\mathbb{R}^3} |\xi_h|^{2s_1} |\xi_3|^{2s_2} |\hat{a}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^3} 1^{\frac{3}{2} - s_1 - s_2} (|\xi_h|^2)^{s_1-\frac{1}{2}} (|\xi_3|^2)^{s_2} |\hat{a}(\xi)|^2 \, d\xi \\
&\leq \|a\|^{2\left(\frac{3}{2} - s_1 - s_2\right)}_{H_{\frac{1}{2},0}} \|\nabla h a\|^{2\left(s_1-\frac{1}{2}\right)}_{H_{\frac{1}{2},0}} \|\partial_3 a\|^{2s_2}_{H_{\frac{1}{2},0}},
\end{align*}
which implies (2.7).

**Lemma 2.6.** Let $a \in H_\theta$ with $\nabla a \in H_\theta$. Then for any $s \in [-\theta, 1/2]$, one has
\begin{equation}
\|a\|_{H^{s,0}} \leq \|a\|_{H_\theta}^{\frac{1}{2}-s} \|\nabla h a\|^{\frac{1}{2} + s - \theta}_{H_\theta} \|\partial_3 a\|^{\theta}_{H_\theta}, \quad \|a\|_{H^{s,0}} \leq \|a\|_{H_\theta}^{\frac{1}{2} - s} \|\nabla h a\|_{H_\theta}^{\frac{1}{2} + s - \theta} \|\partial_3 a\|_{H_\theta}^{\theta + s}.
\end{equation}

**Proof.** Due to $s \leq \frac{1}{2}$, by applying Hölder’s inequality with measure $|\hat{a}(\xi)|^2 \, d\xi$, we find
\begin{align*}
\|a\|^2_{H^{s,0}} &= \int_{\mathbb{R}^3} |\xi_h|^{2s} |\hat{a}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^3} (|\xi_h|^{-1+2\theta} |\xi_3|^{-2\theta})^{\frac{1}{2}-s} \left(|\xi_h|^{1+2\theta} |\xi_3|^{-2\theta}\right)^{\frac{1}{2} + s - \theta} \left(|\xi_h|^{-1+2\theta} |\xi_3|^{2(1-\theta)}\right)^{\theta} |\hat{a}(\xi)|^2 \, d\xi \\
&\leq \|a\|^{2\left(\frac{1}{2} - s\right)}_{H_\theta} \|\nabla h a\|^{2\left(\frac{1}{2} + s - \theta\right)}_{H_\theta} \|\partial_3 a\|^{2\theta}_{H_\theta},
\end{align*}
which ensures the first inequality in (2.8). The second one can be proved along the same line. \qed

Lemma 2.7. Let $p \in [2, \infty]$ and $\theta \in [0, \frac{1}{2}]$. Then for any $a \in S(\mathbb{R}^3)$, one has
\begin{equation}
\|a\|_{H_{\frac{1}{2}+\frac{1}{q}}^1} \leq \|\partial_3 a\|_{H_0}^{\frac{1}{3}} \|a\|_p^{\frac{1}{3}} \|\Delta a\|_{H_0}^{\frac{2}{3}} \|\partial_3 a\|_p^2 \|\nabla a\|_{H_{\frac{1}{2}+\frac{1}{q}}}^2.
\end{equation}

Proof. We first deduce by a similar proof of (2.7) that
\begin{equation}
\|a\|_{H_{\frac{1}{2}+\frac{1}{q},0}} \leq \|a\|_{H_{\frac{1}{2}+\frac{1}{q}}} \|\nabla a\|_{H_{\frac{1}{2}+\frac{1}{q},0}}.
\end{equation}

While we get, by applying Hölder’s inequality with measure $|\tilde{a}(\xi)|^2 \, d\xi$, that
\begin{align*}
\|a\|^2_{H_{\frac{1}{2}+\frac{1}{q}}} &= \int_{\mathbb{R}^3} |\xi|^4 |\tilde{a}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^3} (|\xi|^4 |\tilde{a}(\xi)|^2) \frac{1}{2} |\xi|^4 |\tilde{a}(\xi)|^2 \, d\xi \\
&\leq \|\partial_3 a\|_{H_0}^{\frac{1}{3}} \|a\|_{H_{\frac{1}{2}+\frac{1}{q},0}}^{\frac{1}{3}} \|\partial_3 a\|_{H_0}^2 \|\nabla a\|_{H_{\frac{1}{2}+\frac{1}{q},0}}^2.
\end{align*}

Together with (2.10) and the fact that $\|a\|_{H_{\frac{1}{2}+\frac{1}{q}}} \leq \|a\|_{H_{\frac{1}{2}+\frac{1}{q},0}} + \|a\|_{H^{\frac{1}{2}+\frac{1}{q},0}}$, we conclude the proof of (2.9). \qed

Lemma 2.8. Let $s \in [0,1]$. Then for any $a \in S(\mathbb{R}^3)$, we have
\begin{equation}
\|a\|_{(H^{\frac{1}{2}})^s(B_{2,1})_v} \leq C\|a\|_{H_{\frac{1}{2},0}}^{1-s} \|\partial_3 a\|^s_{H_{\frac{1}{2},0}}.
\end{equation}

Proof. By Definition 2.1, we write
\begin{equation}
\|a\|^2_{(H^{\frac{1}{2}})^s(B_{2,1})_v} = \sum_{k \in \mathbb{Z}} 2^{ks} \left( \sum_{\ell \leq N} 2^{\ell s} \|\Delta_k^{\ell} a\|_{L^2} \right)^2.
\end{equation}

Due to $s \in [0,1]$, for any integer $N$, we get, by applying Hölder’s inequality and Lemma 2.1, that
\begin{align*}
\sum_{\ell \in \mathbb{Z}} 2^{\ell s} \|\Delta_k^{\ell} a\|_{L^2} &= \sum_{\ell \leq N} \|\Delta_k^{\ell} a\|_{L^2} \cdot 2^{\ell s} + \sum_{\ell > N} 2^{\ell s} \|\Delta_k^{\ell} a\|_{L^2} \\
&\leq \left( \sum_{\ell \leq N} \|\Delta_k^{\ell} a\|_{L^2}^2 \right)^{\frac{1}{2}} \cdot 2^{N s} + \left( \sum_{\ell > N} \|\Delta_k^{\ell} a\|_{L^2}^2 \right)^{\frac{1}{2}} \cdot 2^{N(s-1)} \\
&\leq \|\Delta_k a\|_{L^2} \cdot 2^{N s} + \|\Delta_k a\|_{L^2} \cdot 2^{N(s-1)}.
\end{align*}

Taking the integer $N$ in the above inequality so that
\begin{equation}
2^N \sim \|\Delta_k a\|_{L^2} \|\Delta_k a\|_{L^2}^{-1}
\end{equation}
gives rise to
\begin{equation}
\sum_{\ell \in \mathbb{Z}} 2^{\ell s} \|\Delta_k^{\ell} a\|_{L^2} \leq C\|\Delta_k a\|_{L^2}^{1-s} \|\Delta_k a\|_{L^2}^s.
\end{equation}

Substituting the above inequality into (2.12), we obtain
\begin{align*}
\|a\|^2_{(H^{\frac{1}{2}})^s(B_{2,1})_v} &\leq C \sum_{k \in \mathbb{Z}} 2^{ks} \|\Delta_k a\|_{L^2}^{2(1-s)} \|\Delta_k a\|_{L^2}^{2s} \\
&\leq C \left( \sum_{k \in \mathbb{Z}} 2^{ks} \|\Delta_k a\|_{L^2}^{2(1-s)} \right)^{1-s} \left( \sum_{k \in \mathbb{Z}} 2^{ks} \|\Delta_k a\|_{L^2}^{2s} \right)^{s} \\
&\leq C \|a\|_{H_{\frac{1}{2},0}}^{2(1-s)} \|\partial_3 a\|_{H_{\frac{1}{2},0}}^{2s}.
\end{align*}
Applying Young’s inequality gives
Proof. Substituting the above inequalities into (3.3) and using (2.2), we obtain
\[ \frac{d}{dt}\|\omega_{\frac{1}{2}}(t)\|_{L^2}^2 + \frac{11}{9}\|\nabla\omega_{\frac{1}{2}}(t)\|_{L^2}^2 \]
\[ \leq \frac{1}{9}\left(\|\partial_3^2 v^3\|_{H^\frac{1}{2}}^2 + \|\nabla_h v^3\|_{H^\frac{1}{2}}^2\right) + C\|\omega_{\frac{1}{2}}\|_{L^2}^2\|v^3\|_{H^\frac{1}{2}}^2\|\partial_3\omega_{\frac{1}{2}}\|_{L^2}^2 \]
\[ + C\|\omega_{\frac{1}{2}}\|_{L^2}^2\left(\|v^3\|_{H^1}^{2-\frac{2}{p}} + \|\omega_{\frac{1}{2}}\|_{L^2}^{\frac{2}{1-\theta}}\left(\|\partial_3^2 v^3\|_{L^2}^2 + \|\partial_3 v^3\|_{H^\frac{1}{2}}^2\right)\right)\]
We first get, by using integration by parts and then H"older’s inequality, that
\[ \int_{\mathbb{R}^3} \partial_3 v^3(\omega_{\frac{1}{2}})^\frac{3}{2} dx \leq \frac{3}{2}\int_{\mathbb{R}^3} \partial_3 v^3(\omega_{\frac{1}{2}})^\frac{3}{2} dx + \frac{3}{2}\int_{\mathbb{R}^3} (\partial_2 v^3 \partial_3 v^3 - \partial_1 v^3 \partial_3 v^3) \omega_{\frac{1}{2}} \]
While it is easy to observe by a similar proof of Lemmas 2.5 to 2.7 that
\[ \|a\|_{H^\frac{1}{2}} \leq \|a\|_{L^2}^{\frac{7}{8}}\|\nabla_h a\|_{L^2}^{\frac{1}{8}} \partial_3 a\|_{L^2}^{\frac{1}{8}} \text{ and } \|a\|_{H^\frac{1}{2}} \leq \|a\|_{H^\frac{3}{2}}\|\nabla_h a\|_{H^\frac{1}{2}}^{\frac{8-7\theta}{8-7}} \|a\|_{H^\frac{3}{2}}^{\frac{7\theta}{8-7}} \]
So that applying Sobolev inequality yields
\[ \|\omega_{\frac{1}{2}}\|_{L^6(L^\frac{6}{5})} \leq \|\omega_{\frac{1}{2}}\|_{L^6(L^\frac{6}{5})} \leq \|\omega_{\frac{1}{2}}\|_{L^6(L^\frac{6}{5})} \|\partial_3\omega_{\frac{1}{2}}\|_{L^6(L^\frac{6}{5})} \]
Substituting the above inequalities into (3.3) and using (2.2), we obtain
\[ \int_{\mathbb{R}^3} \partial_3 v^3(\omega_{\frac{1}{2}})^\frac{3}{2} dx \leq \frac{1}{27}\left(\|\nabla_h \omega_{\frac{1}{2}}\|_{L^2}^2 + \|\partial_3^2 v^3\|_{H^\frac{1}{2}}^2 + \|\nabla_h v^3\|_{H^\frac{1}{2}}^2\right) \]
Applying Young’s inequality gives
\[ \int_{\mathbb{R}^3} \partial_3 v^3(\omega_{\frac{1}{2}})^\frac{3}{2} dx \leq \frac{1}{27}\left(\|\nabla_h \omega_{\frac{1}{2}}\|_{L^2}^2 + \|\partial_3^2 v^3\|_{H^\frac{1}{2}}^2 + \|\nabla_h v^3\|_{H^\frac{1}{2}}^2\right) \]
To deal with the second term on the right side of (3.2), we need the following lemma.
Lemma 3.1 (Lemma 5.2 of [7]). For \( \theta \in ]0, \frac{1}{6}[, \sigma \in ]\frac{3}{4}, 1[ \) and \( s = \frac{1}{2} + 1 - \frac{2}{3} \sigma \), we have

\[
\left| \int_{\mathbb{R}^3} \partial_h \Delta_h^{-1} f \partial_h a \omega_{\frac{3}{2}} \, dx \right| \lesssim \| f \|_{L^2} \| a \|_{H^s} \| \omega_{\frac{3}{2}} \|^2_{H^s} \quad \text{and}
\]

\[
\left| \int_{\mathbb{R}^3} \partial_h \Delta_h^{-1} f \partial_h a \omega_{\frac{3}{2}} \, dx \right| \lesssim \| f \|_{\mathcal{H}_0} \| a \|_{H^s} \| \omega_{\frac{3}{2}} \|^2_{H^s}.
\]

By virtue of (1.12), we have

\[
\partial_t v^h = \nabla_h \Delta_h^{-1} \partial_3 \omega - \nabla_h \Delta_h^{-1} \partial_3^3 v^3.
\]

Applying Lemma 3.1 for \( f = \partial_3 \omega \) and \( \partial_3^3 v^3 \) with \( \sigma = 3(\frac{1}{2} - \frac{1}{6}) \), we achieve

\[
\left| \int_{\mathbb{R}^3} (\partial_2 v^3 \partial_3 v^1 - \partial_1 v^3 \partial_3 v^2) \omega_{\frac{3}{2}} \, dx \right| \lesssim \left( \| \partial_3 \omega \|_{L^2} \| \omega_{\frac{3}{2}} \|^2_{H^s} + \| \partial_3^2 v^3 \|_{\mathcal{H}_0} \| v^3 \|_{H^s} \right) \| \omega_{\frac{3}{2}} \|^2_{H^s} \left( \frac{3}{2} \right)
\]

from which, (1.13) and (2.2), we infer

\[
\left| \int_{\mathbb{R}^3} (\partial_2 v^3 \partial_3 v^1 - \partial_1 v^3 \partial_3 v^2) \omega_{\frac{3}{2}} \, dx \right| \lesssim \left( \| \partial_3 \omega \|_{L^2} \| \omega_{\frac{3}{2}} \|^2_{H^s} + \| \partial_3^2 v^3 \|_{\mathcal{H}_0} \right) \| \omega_{\frac{3}{2}} \|^2_{H^s} \left( \frac{3}{2} \right)
\]

Applying Young’s inequality gives rise to

\[
\left| \int_{\mathbb{R}^3} (\partial_2 v^3 \partial_3 v^1 - \partial_1 v^3 \partial_3 v^2) \omega_{\frac{3}{2}} \, dx \right| \leq \frac{1}{27} \left( \| \nabla \omega_{\frac{3}{2}} \|^2_{L^2} + \| \partial_3^2 v^3 \|^2_{\mathcal{H}_0} + \| \nabla v^3 \|^2_{H^s} \right)
\]

\[
+ C \left( \| \omega_{\frac{3}{2}} \|^2_{L^2} \left( \| v^3 \|^2_{H^s} + \| \partial_3 v^3 \|^2_{\mathcal{H}_0} \right) \right) \left( \| \omega_{\frac{3}{2}} \|^2_{L^2} \left( \| \omega_{\frac{3}{2}} \|^2_{H^s} + \| \partial_3^2 v^3 \|^2_{\mathcal{H}_0} \right) \right)
\]

Inserting the Inequalities (3.5) and (3.7) into (3.2) results in (3.1). This completes the proof of this proposition. \( \square \)

Proposition 3.2. Under the assumptions of Proposition 3.1, for any \( t < T^* \), there holds

\[
\frac{d}{dt} \| \partial_3 v^3(t) \|^2_{\mathcal{H}_0} + \frac{3}{2} \| \nabla \omega \partial_3 v^3(t) \|^2_{\mathcal{H}_0} \leq \frac{1}{g} \| \nabla \omega_{\frac{3}{2}} \|^2_{L^2} + \frac{1}{g} \| \nabla v^3 \|^2_{H^s} \frac{3}{2}
\]

\[
+ C \left( \| \partial_3 v^3 \|^2_{L^4} \| v^3 \|^2_{L^4} \right) \left( \| \omega_{\frac{3}{2}} \|^2_{L^4} \| v^3 \|^2_{H^s} \right)
\]

\[
+ \| \partial_3 v^3 \|^2_{\mathcal{H}_0} \| v^3 \|^2_{H^s} \left( \frac{3}{2} \right) \left( \| \omega_{\frac{3}{2}} \|^2_{L^2} \| \omega_{\frac{3}{2}} \|^2_{H^s} \right)
\]

\[
+ C \left( \| \omega_{\frac{3}{2}} \|^2_{L^2} + \| \partial_3 v^3 \|^2_{H^s} \right) \left( \| v^3 \|^2_{H^s} + \| \partial_3 v^3 \|^2_{\mathcal{H}_0} \right)
\]

\[
+ C \left( \| \omega_{\frac{3}{2}} \|^2_{L^2} + \| \partial_3 v^3 \|^2_{\mathcal{H}_0} \right) \left( \| v^3 \|^2_{H^s} + \| \partial_3 v^3 \|^2_{\mathcal{H}_0} \right)
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In what follows, we shall handle term by term the right-hand side of (3.9).

- The estimate of \((P_1(v, v) | \partial_3 v^3)_{\mathcal{H}_\theta}\).

To deal with this term, we need the following lemma:

**Lemma 3.2** (Lemma 6.1 of [7]). Let \(A\) be a bounded Fourier multiplier. If \(p\) and \(\theta\) satisfy

\[
0 < \theta < \frac{1}{2} - \frac{1}{p},
\]

then we have

\[
\left| (A(D)(fg) | \partial_3 v^3)_{\mathcal{H}_\theta} \right| \lesssim \|f\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \|g\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \|v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}}.
\]

Applying Lemma 3.2 and using (1.12) yields

\[
\left| (P_1(v, v) | \partial_3 v^3)_{\mathcal{H}_\theta} \right| \leq C \|v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \left( \|\omega\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} + \|\partial_3 v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \right).
\]

While it follows from the proof of Lemma 2.7 that

\[
\|v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \lesssim \|\partial_3 v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}},
\]

and Lemma 2.6 ensures that

\[
\|\partial_3 v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \leq \|\partial_3 v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}},
\]

Moreover, because of Condition (3.10), we get, by using Lemma 2.2 and Lemma 2.4, that

\[
\|\omega\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \lesssim \|\omega\|_{L^2} \leq \|\omega\|_{L^2} \leq \left( \frac{\omega^2}{L^2} \right)^{\frac{1}{p}},
\]

As a result, it comes out

\[
\left| (P_1(v, v) | \partial_3 v^3)_{\mathcal{H}_\theta} \right| \leq C \|\partial_3 v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \left( \|\omega\|_{L^2} \right)^{\frac{p+1}{2}} \|\nabla \omega\|_{L^2} \|\nabla \omega\|_{L^2} \|\nabla \omega\|_{L^2} \|\nabla \omega\|_{L^2} \|\partial_3 v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}}.
\]

Applying Young’s inequality gives

\[
\left| (P_1(v, v) | \partial_3 v^3)_{\mathcal{H}_\theta} \right| \leq \frac{1}{27} \left( \|\nabla_\theta \omega\|_{L^2} \|\nabla \omega\|_{L^2} \|\omega\|_{L^2} \right)^{\frac{p+3}{2}} + C \left( \|\partial_3 v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \|\nabla \omega\|_{L^2} \|\omega\|_{L^2} \right)^{\frac{p+3}{2}} + C \left( \|\partial_3 v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \|\nabla \omega\|_{L^2} \|\omega\|_{L^2} \right)^{\frac{p+3}{2}} + C \left( \|\partial_3 v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \|\nabla \omega\|_{L^2} \|\omega\|_{L^2} \right)^{\frac{p+3}{2}} + C \left( \|\partial_3 v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \|\nabla \omega\|_{L^2} \|\omega\|_{L^2} \right)^{\frac{p+3}{2}}.
\]

- The estimate of \((P_2(v, v) | \partial_3 v^3)_{\mathcal{H}_\theta}\).

By using integration by parts, we get

\[
(P_2(v, v) | \partial_3 v^3)_{\mathcal{H}_\theta} = A_1 + A_2 \quad \text{with}
\]

\[
A_1 \overset{\text{def}}{=} -2 \left( (\text{Id} + 2 \partial_3 \Delta^{-1})(v^3 \cdot \nabla_\theta v^3) | \partial_3^2 v^3 \right)_{\mathcal{H}_\theta},
\]

\[
A_2 \overset{\text{def}}{=} -2 \left( (\text{Id} + 2 \partial_3 \Delta^{-1})(v^3 \cdot \nabla_\theta v^3) | \partial_3^2 v^3 \right)_{\mathcal{H}_\theta}.
\]

Applying the product of Lemma 2.3, yields

\[
|A_1| \leq C \|v^3 \cdot \nabla_\theta v^3\|_{\mathcal{H}_\theta} \|\partial_3^2 v^3\|_{\mathcal{H}_\theta} \leq C \|v^3 \cdot \nabla_\theta v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \|\partial_3^2 v^3\|_{\mathcal{H}_\theta}.
\]

\[
|A_2| \leq C \left( B_{2,1} \right)_h \left( B_{2,1} \right)_v \|\nabla_\theta v^3\|_{H^{\theta, \frac{1}{2} - \theta - \frac{1}{p}}} \|\partial_3^2 v^3\|_{\mathcal{H}_\theta}.
\]
Notice from Lemma 2.5 that
\[
\|a\|_{H^{\frac{1}{2} + \theta, \frac{p}{2} - \theta}} \leq \|a\|_{H^{\frac{1}{2}, 0}} \|\nabla_h a\|_{H^{\frac{1}{2}, 0}}^{\theta} \|\partial_3 a\|_{H^{\frac{1}{2}, 0}}^{\frac{2}{p} - \theta},
\]
we find
\[
|A_1| \leq C\|v^h\|_{(B^{\frac{1}{2} + \theta, \frac{p}{2} - \theta})_v} \|v\|_{H^{\frac{1}{2} + \theta, \frac{p}{2} - \theta}} \|v\|_{H^{\frac{1}{2}, 0}} \|\nabla_h v\|_{H^{\frac{1}{2}, 0}}^{\theta} \|\partial_3 v^3\|_{H^{\frac{1}{2}, 0}}^{\frac{2}{p} - \theta} \|\partial_3^2 v^3\|_{H^0}.
\]

Whereas observing that
\[
|A_2| \leq \|v^h \cdot \nabla_h \partial_3 v^3\|_{H^{-1 + 2\theta, \frac{1}{2} + \theta}} \|\partial_3 v^3\|_{H^{0, -\frac{1}{2} + \frac{p}{2}}},
\]

and it follows from Lemma 2.6 that
\[
\|a\|_{H^{\frac{1}{2} + \theta, \frac{p}{2} - \theta}} \leq \|a\|_{H^{\frac{1}{2}, 0}}^{\frac{1}{2} + \theta} \|a\|_{H^{0, \frac{1}{2}}}^{\frac{1}{2} - \theta} \leq \|\nabla_h a\|_{H^0} \|\partial_3 a\|_{H^0},
\]
we achieve
\[
|A_2| \leq C\|v^3\|_{H^{0, \frac{1}{2} + \theta} \cap \|v^h\|_{(B^{\frac{1}{2} + \theta, \frac{p}{2} - \theta})_v}\|\nabla_h \partial_3 v^3\|_{H^{0, \frac{1}{2} + \theta}} \|\partial_3^2 v^3\|_{H^0, \frac{1}{2} - \theta},
\]
from which, Proposition 2.1, (3.11) and (3.15), we infer
\[
\left|(P_2(v, v) \mid \partial_3 v^3)_{H^0}\right| \leq C\left(\|\omega_\frac{1}{4}\|_{L^2}^{\frac{1}{2} + \frac{\theta}{2}} \|\nabla \omega_\frac{1}{4}\|_{L^2}^{\frac{1}{2} - \frac{\theta}{2}} + \|\partial_3 v^3\|_{H^0, \frac{1}{2} + \theta} \|\nabla \partial_3 v^3\|_{H^0, \frac{1}{2} - \theta} \right) \times \left(\|v^3\|_{H^{\frac{1}{2}, 0}}^{\frac{1}{2} + \theta} \|\nabla_h v^3\|_{H^{\frac{1}{2}, 0}}^{\frac{2}{p} - \theta} \|\partial_3 v^3\|_{H^{\frac{1}{2}, 0}}^{\frac{2}{p} - \theta} \|\partial_3^2 v^3\|_{H^0} + \|\partial_3 v^3\|_{H^0, \frac{1}{2} + \theta} \|\nabla_h \partial_3 v^3\|_{H^0} \|\partial_3^2 v^3\|_{H^0, \frac{1}{2} - \theta}\right).
\]

Applying Young's inequality yields
\[
\left|(P_2(v, v) \mid \partial_3 v^3)_{H^0}\right| \leq \frac{1}{2}\left(\|\nabla \omega_\frac{1}{4}\|_{L^2}^2 + \|\nabla_h v^3\|_{L^p}^2 + \|\nabla \partial_3 v^3\|_{H^0}^2\right) + C\left(\|\omega_\frac{1}{4}\|_{L^2}^{\frac{2(p+6)}{2(p-4\theta)}} + \|\partial_3 v^3\|_{H^0, \frac{1}{2} + \theta} \|\partial_3 v^3\|_{H^0, \frac{1}{2} - \theta} \right) \times \left(\|v^3\|_{H^{\frac{1}{2}, 0}}^{\frac{1}{2} + \theta} \|\nabla_h v^3\|_{H^{\frac{1}{2}, 0}}^{\frac{2}{p} - \theta} \|\partial_3 v^3\|_{H^{\frac{1}{2}, 0}}^{\frac{2}{p} - \theta} \|\partial_3^2 v^3\|_{H^0} + \|\partial_3 v^3\|_{H^0, \frac{1}{2} + \theta} \|\nabla_h \partial_3 v^3\|_{H^0} \|\partial_3^2 v^3\|_{H^0, \frac{1}{2} - \theta}\right).
\]

The estimate of \((P_3(v, v) \mid \partial_3 v^3)_{H^0}\).

It is easy to observe that the term \((v^h \cdot \nabla_h \partial_3 v^3 \mid \partial_3 v^3)_{H^0}\) shares the same estimate as \(A_2\) given by (3.16). For \((v^3 \partial_3^2 v^3 \mid \partial_3 v^3)_{H^0}\), we first get, by using Hölder’s inequality in the frequency space, that
\[
\left|(v^3 \partial_3^2 v^3 \mid \partial_3 v^3)_{H^0}\right| \leq \|v^3\|_{H^{\frac{1}{2} + \theta, \frac{p}{2} - \theta}} \|\partial_3 v^3\|_{H^{\frac{1}{2}, 0} \cap \|\partial_3^2 v^3\|_{H^0, \frac{1}{2} + \theta, \frac{p}{2} - \theta}}.
\]

Applying the law of product, Lemma 2.3, and then Lemma 2.2 gives rise to
\[
\left|(v^3 \partial_3^2 v^3 \mid \partial_3 v^3)_{H^0}\right| \leq \|v^3\|_{H^{\frac{1}{2}, 0}} \|\partial_3 v^3\|_{H^0} \|\partial_3 v^3\|_{H^{\frac{1}{2} + \theta, \frac{p}{2} - \theta}} \|\partial_3^2 v^3\|_{H^0} \|\partial_3 v^3\|_{H^{\frac{1}{2}, 0}} \|\partial_3 v^3\|_{H^0, \frac{1}{2} + \theta, \frac{p}{2} - \theta}.
\]
which together with the interpolation inequality
\[ \|a\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}} \leq \|a\|_{H^0}^{1-\frac{2}{p}} \|\nabla_h a\|_{H^0}^{\frac{2}{p}}, \]
ensures that
\[
\left|(v^3 \partial_3^2 v^3 | \partial_3 v^3)_{H^0}\right| \leq C \|v^3\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}} \|\partial_3^2 v^3\|_{H^0} \|\partial_3 v^3\|_{H^0}^{2/3} \|\nabla_h \partial_3 v^3\|_{H^0}^{1-\frac{2}{3} \frac{2}{p}}.
\]

By virtue of (1.13), (3.16) and (3.18), we deduce that
\[
\left|(P_3(v, v) | \partial_3 v^3)_{H^0}\right| \leq \frac{1}{27} \left(\|\nabla v^3\|_{L^2}^2 + \|\nabla_h v^3\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}^2 + \|\nabla \partial_3 v^3\|_{H^0}^2\right)
\]
\[+ C \left(\|v^3\|_{L^2}^{2/3} \|\partial_3 v^3\|_{H^0}^{2/3} \|\partial_3 v^3\|_{H^0}^{2/3} \|\nabla_h \partial_3 v^3\|_{H^0}^{1-\frac{2}{3} \frac{2}{p}} + \|\nabla \partial_3 v^3\|_{H^0}^2 \|\partial_3 v^3\|_{H^0}^{2/3} \|\nabla_h \partial_3 v^3\|_{H^0}^{1-\frac{2}{3} \frac{2}{p}}\right) \|\partial_3 v^3\|_{H^0}^2.
\]

Inserting the Estimates (3.13), (3.17) and (3.19) into (3.9) leads to (3.8).

\begin{proposition}
Under the assumptions of Proposition 3.1, for any \( t < T^* \) and any \( \delta > 0 \), there holds
\[
\frac{d}{dt} \|v^3(t)\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}^2 + \frac{3}{2} \|\nabla v^3(t)\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}^2 \leq 4\delta \left(\|\nabla v^3\|_{L^2}^2 + \|\nabla \partial_3 v^3\|_{H^0}^2\right)
\]
\[+ C \|\partial_3 v^3\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}} \left(\|v^3\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}^2 + \delta^{-\frac{2}{3}} \|v^3\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}^{2/3} \|\nabla v^3\|_{L^2}^2 + \|\nabla \partial_3 v^3\|_{H^0}^2\right)
\]
\[+ \delta^{-1} \|v^3\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}^2 \left(\|v^3\|_{L^2}^2 + \|\partial_3 v^3\|_{H^0}^2 + \delta^{-3} \|v^3\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}^2 \|\nabla v^3\|_{L^2}^2 + \|\partial_3 v^3\|_{H^0}^2\right)\).
\]
\end{proposition}

\begin{proof}
By taking the \( H^{1/3+\theta-\frac{3}{p}}_{H^0} \) inner product of the \( v^3 \) equation of (1.11) with \( v^3 \), we write
\[
\frac{d}{dt} \|v^3(t)\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}^2 + 2\|\nabla v^3(t)\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}^2 = \sum_{i=1}^{3} (Q_i(v, v) | v^3)_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}
\]
with
\[
Q_1(v, v) \overset{\text{def}}{=} 2\partial_3 \Delta^{-1}(\partial_3 v^3)^2 + 2\partial_3 \Delta^{-1} \left(\sum_{\ell, m=1}^{2} \partial_\ell v^m \partial_m v^\ell\right),
\]
\[
Q_2(v, v) \overset{\text{def}}{=} 4\partial_3 \Delta^{-1} \left(\sum_{\ell=1}^{2} \partial_\ell v^\ell \partial_3 v^3\right), \quad \text{and} \quad Q_3(v, v) \overset{\text{def}}{=} 2v \cdot \nabla v^3.
\]

In what follows, we shall deal with the estimates of the above terms.

\begin{itemize}
\item The estimate of \((Q_1(v, v) | v^3)_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}\).
\end{itemize}

Notice that
\[
\|\partial_3 \Delta^{-1} f\|_{(H^{1/3+\theta-\frac{3}{p}}_{H^0})_{(B_{2, \infty})}^v} \leq C \|f\|_{(H^{-1/3+\theta-\frac{3}{p}}_{H^0})_{(B_{2, \infty})}^v},
\]
so that in view of (3.21), by applying the law of product, Lemma 2.3, and Lemma 2.8, we find
\[
\left|\left(Q_1(v, v) | v^3\right)_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}\right| \leq \|Q_1(v, v)\|_{(H^{1/3+\theta-\frac{3}{p}}_{H^0})_{(B_{2, \infty})}^v} \|v^3\|_{(H^{1/3+\theta-\frac{3}{p}}_{H^0})_{(B_{2, 1})}^v}
\]
\[\leq 2 \left(\left(\partial_3 v^3\right)^2_{(H^{-1/3+\theta-\frac{3}{p}}_{H^0})_{(B_{2, \infty})}^v} + \sum_{\ell, m=1}^{2} \|\partial_\ell v^m \partial_m v^\ell\|_{(H^{-1/3+\theta-\frac{3}{p}}_{H^0})_{(B_{2, 1})}^v} \|v^3\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}\right)\|\partial_3 v^3\|_{H^{1/3+\theta-\frac{3}{p}}_{H^0}}\right)
\]
2.8, gives

Then applying the product laws in the anisotropic Besov spaces, Lemma 2.3, and Lemma (3.26)

By using integration by parts, we write

C

\begin{align*}
|B_1| \lesssim & \|\partial_3 \Delta^{-1}(v^h \cdot \nabla_h v^3)\|_{H^{1/2},0} \|\partial_3 v^3\|_{H^{1/2},0} \\
\lesssim & \|v^h \cdot \nabla_h v^3\|_{H^{1/2},0} \|\partial_3 v^3\|_{H^{1/2},0} \\
\lesssim & \|v^h\| \left(\left\langle \mathbf{B}_{2,1} \right\rangle, \left(\mathbf{B}_{0,2,1} \right) \right)_v \|\nabla_h v^3\|_{(H^{1/2})_h} \left(\left\langle \mathbf{B}_{2,1} \right\rangle \right)_v \|\partial_3 v^3\|_{H^{1/2},0}.
\end{align*}

Then applying Proposition 2.1 and Lemma 2.8 yields

(3.25) \quad |B_1| \leq \delta \left(\|\nabla \omega^3\|_{L^2}^2 + \|\nabla \partial_3 v^3\|_{H^{1/2}}^2 \right) + C \delta^{-1/2} \|v^3\|_{H^{1/2},0} \left(\|\omega^3\|_{L^2}^2 + \|\partial_3 v^3\|_{H^{1/2}}^2\right) \|\partial_3 v^3\|_{H^{1/2},0}^2.

On the other hand, it is easy to observe from (3.22) that

\begin{align*}
|B_2| \lesssim & \|\partial_3 \Delta^{-1}(v^h \cdot \nabla_h \partial_3 v^3)\|_{(H^{1/2})_h} \left(\left\langle \mathbf{B}_{2,1} \right\rangle \right)_v \|v^3\|_{(H^{1/2})_h} \left(\left\langle \mathbf{B}_{2,1} \right\rangle \right)_v \\
\lesssim & \|v^h \cdot \nabla_h \partial_3 v^3\|_{(H^{1/2})_h} \left(\left\langle \mathbf{B}_{2,1} \right\rangle \right)_v \|v^3\|_{(H^{1/2})_h} \left(\left\langle \mathbf{B}_{2,1} \right\rangle \right)_v.
\end{align*}

Then applying the product laws in the anisotropic Besov spaces, Lemma 2.3, and Lemma 2.8, gives

\begin{align*}
|B_2| \lesssim & \|v^h\| \left(\left\langle \mathbf{B}_{2,1} \right\rangle, \left(\mathbf{B}_{0,2,1} \right) \right)_v \|\nabla_h \partial_3 v^3\|_{H^{1/2},0} \|v^3\|_{H^{1/2},0} \|\partial_3 v^3\|_{H^{1/2},0} \\
\lesssim & \left(\|\omega^3\|_{L^2}^2 + \|\nabla \partial_3 v^3\|_{H^{1/2}}^2 \right) + C \delta^{-1/2} \|v^3\|_{H^{1/2},0} \left(\|\omega^3\|_{L^2}^2 + \|\partial_3 v^3\|_{H^{1/2}}^2\right) \|\partial_3 v^3\|_{H^{1/2},0}^2.
\end{align*}

Applying Young's inequality gives rise to

(3.26) \quad |B_2| \leq \delta \left(\|\nabla \omega^3\|_{L^2}^2 + \|\nabla \partial_3 v^3\|_{H^{1/2}}^2 \right) + C \delta^{-1/2} \|v^3\|_{H^{1/2},0} \left(\|\omega^3\|_{L^2}^2 + \|\partial_3 v^3\|_{H^{1/2}}^2\right) \|\partial_3 v^3\|_{H^{1/2},0}^2.

The estimate of $(Q_2(v, v) \mid v^3)_{H^{1/2},0}$.

By using integration by parts, we write

$(Q_2(v, v) \mid v^3)_{H^{1/2},0} = - B_1 - B_2$, with

(3.24) \quad B_1 \overset{\text{def}}{=} 4\partial_3 \Delta^{-1}(v^h \cdot \nabla_h v^3) \mid \partial_3 v^3)_{H^{1/2},0}$ \quad and \quad $B_2 \overset{\text{def}}{=} 4(\partial_3 \Delta^{-1}(v^h \cdot \nabla_h \partial_3 v^3) \mid v^3)_{H^{1/2},0}.$
Combining the Estimates (3.25) with (3.26), we obtain
\begin{equation}
\left| (Q_2(v, v) \mid v^3)_{H^{1/4}} \right| \leq 2\delta \left( \| \nabla \omega^3 \|_{L^2}^2 + \| \nabla \partial_3 v^3 \|_{H^0}^2 \right) + C\delta^{-1} \| v^3 \|_{H^{5/8}}^2 \left( \| \nabla \omega^3 \|_{L^2}^2 + \| \nabla \partial_3 v^3 \|_{H^0}^2 \right) \leq C \| v^3 \|_{L^2} \left( \| \nabla v^3 \|_{H^{5/8}} + \| \partial_3 v^3 \|_{H^0} \right).
\end{equation}

The estimate of \((Q_3(v, v) \mid v^3)_{H^{1/4}}\).

We first deduce from the law of product, Lemma 2.3, and Lemma 2.8 that
\begin{align}
\left| (v^h \cdot \nabla h v^3 \mid v^3)_{H^{1/4}} \right| &\leq C \left( \| \nabla \omega^3 \|_{L^2}^2 + \| \nabla \partial_3 v^3 \|_{H^0}^2 \right) \\
&\leq C \left( \| \nabla \omega^3 \|_{H^{5/8}}^2 \left( \| \omega^3 \|_{H^{5/8}} \right)^2 + \| \nabla \partial_3 v^3 \|_{H^0}^2 \right) \leq C \delta^{-1} \| \nabla \omega^3 \|_{H^{5/8}}^2 \left( \| \omega^3 \|_{H^{5/8}} \right)^2 + \| \partial_3 v^3 \|_{H^0}^2 \| \partial_3 v^3 \|_{H^0}^2 \left( \| \nabla \omega^3 \|_{H^{5/8}} \right)^2 \leq C \delta^{-1} \| \nabla \omega^3 \|_{H^{5/8}}^2 \left( \| \omega^3 \|_{H^{5/8}} \right)^2 + \| \partial_3 v^3 \|_{H^0}^2 \| \partial_3 v^3 \|_{H^0}^2 \left( \| \nabla \omega^3 \|_{H^{5/8}} \right)^2 \leq C \delta^{-1} \| \nabla \omega^3 \|_{H^{5/8}}^2 \left( \| \omega^3 \|_{H^{5/8}} \right)^2 + \| \partial_3 v^3 \|_{H^0}^2 \| \partial_3 v^3 \|_{H^0}^2 \left( \| \nabla \omega^3 \|_{H^{5/8}} \right)^2
\end{align}

Applying Proposition 2.1 gives
\begin{equation}
\left| (v^h \cdot \nabla h v^3 \mid v^3)_{H^{1/4}} \right| \leq C \left( \| \nabla \omega^3 \|_{L^2}^2 + \| \nabla \partial_3 v^3 \|_{H^0}^2 \right) \leq \delta \| \nabla \omega^3 \|_{H^{5/8}}^2 \left( \| \omega^3 \|_{H^{5/8}} \right)^2 + \| \partial_3 v^3 \|_{H^0}^2 \| \partial_3 v^3 \|_{H^0}^2 \left( \| \nabla \omega^3 \|_{H^{5/8}} \right)^2 \leq C \delta^{-1} \| \nabla \omega^3 \|_{H^{5/8}}^2 \left( \| \omega^3 \|_{H^{5/8}} \right)^2 + \| \partial_3 v^3 \|_{H^0}^2 \| \partial_3 v^3 \|_{H^0}^2 \left( \| \nabla \omega^3 \|_{H^{5/8}} \right)^2
\end{equation}

While we get, by using Lemma 2.3 and 2.8, that
\begin{equation}
\left| (v^3 \partial_3 v^3 \mid v^3)_{H^{1/4}} \right| \leq C \left( \| \nabla \omega^3 \|_{L^2}^2 + \| \nabla \partial_3 v^3 \|_{H^0}^2 \right) \leq \delta \| \nabla \omega^3 \|_{H^{5/8}}^2 \left( \| \omega^3 \|_{H^{5/8}} \right)^2 + \| \partial_3 v^3 \|_{H^0}^2 \| \partial_3 v^3 \|_{H^0}^2 \left( \| \nabla \omega^3 \|_{H^{5/8}} \right)^2 \leq C \delta^{-1} \| \nabla \omega^3 \|_{H^{5/8}}^2 \left( \| \omega^3 \|_{H^{5/8}} \right)^2 + \| \partial_3 v^3 \|_{H^0}^2 \| \partial_3 v^3 \|_{H^0}^2 \left( \| \nabla \omega^3 \|_{H^{5/8}} \right)^2
\end{equation}

As a consequence, it comes out
\begin{equation}
\left| (Q_3(v, v) \mid v^3)_{H^{1/4}} \right| \leq \delta \left( \| \nabla \omega^3 \|_{L^2}^2 + \| \nabla \partial_3 v^3 \|_{H^0}^2 \right) + C \left( \| v^3 \|_{H^{1/4}}^2 + \delta^{-1} \| v^3 \|_{H^{5/8}}^2 \left( \| \omega^3 \|_{H^{5/8}} \right)^2 + \| \partial_3 v^3 \|_{H^0}^2 \right) \| \partial_3 v^3 \|_{H^0}^2 \left( \| \nabla \omega^3 \|_{H^{5/8}} \right)^2
\end{equation}

Substituting the Estimates (3.23), (3.27) and (3.30) into (3.21), we achieve (3.20). \qed

4. The proof of Theorem 1.1 and 1.2

By combining the a priori estimates obtained in Propositions 3.1, 3.2 and 3.3, we are going to complete the proofs of Theorem 1.1 and 1.2 in this section. Let us first recall the following theorems from [7].

Theorem 4.1 (Theorem 1.3 of [7]). Let us consider an initial data \( v_0 \) with vorticity \( \Omega_0 \in L^{3/2} \). Then a unique maximal solution \( v \) of \((NS_v)\) exists in the space \( C([0, T^*], H^{1/4}) \cap L^{3/2}_\text{loc}([0, T^*]; H^{3/2}) \) for some maximal time \( T^* > 0 \), and the vorticity \( \Omega = \text{curl} v \) is in \( C([0, T^*], L^3) \) with \( \| \nabla \Omega \|_{L^\infty} \| \Omega \|_{L^3}^{-1} \in L^2_{\text{loc}}([0, T^*]; L^2) \).
**Theorem 4.2** (Theorem 1.4 of [7]). Let \( p \in [4, 6] \). Consider the maximal solution \( v \) given by Proposition 4.1. If \( T^* < \infty \), then we have
\[
\int_{0}^{T^*} \| v^3(t) \|_{H^{\frac{3}{2}} + \frac{1}{p}}^p dt = \infty.
\]

Now let us first present the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Under the assumptions of Theorem 1.1, we deduce from Theorem 4.1 that \((NS_v)\) has a unique solution \( v \in C([0, T^*], H^{\frac{3}{2}}) \cap L^2_{\text{loc}}([0, T^*]; H^{\frac{3}{2}})\) for some maximal time \( T^* > 0 \). Moreover, due to \( \Omega_0 \in L^\frac{3}{2} \) and \( \text{div} \, v_0 = 0 \), we have
\[
\| \omega_\frac{3}{4}(0) \|_{L^2}^2 \leq \| \Omega_0 \|_{L^\frac{3}{2}}^2, \quad \| v_0 \|_{H^{\frac{3}{2}} + \frac{1}{0}}^2 \leq \| v_0 \|_{H^{\frac{3}{2}}}^2 \leq \| \Omega_0 \|_{L^\frac{3}{2}}^2, \quad \text{and}
\]
\[
\| \partial_3 v_0 \|_{H^\frac{3}{2}}^2 = \int_{|\xi_1| \leq |\xi_3|} |\xi_1|^{-1+2\theta} \xi_3^{-2\theta} |F(\partial_3 v_0^3)(\xi)|^2 d\xi
\]
\[
+ \int_{|\xi_1| \leq |\xi_3|} |\xi_1|^{-1+2\theta} \xi_3^{-2\theta} |F(-\text{div}_h v_0^3)(\xi)|^2 d\xi
\]
\[
\leq \| v_0 \|_{H^{\frac{3}{2}}}^{2} \leq \| \Omega_0 \|_{L^\frac{3}{2}}^2.
\]

This implies \( M_0 \) given by Theorem 1.1 is finite. Let
\[
T^* \overset{\text{def}}{=} \sup \left\{ T \in [0, T^*] : \text{so that (1.2) holds for any } t \in [0, T] \right\}.
\]

We are going to prove that \( T^* = T^* \). Otherwise, if \( T^* < T^* \), then for any \( t \in [0, T^*] \), we get, by summing up (3.1), (3.8) and (3.20) that
\[
\frac{d}{dt} M(t) + \frac{11}{9} \| \nabla \omega_\frac{3}{4}(t) \|_{L^2}^2 + \frac{3}{2} \left( \| \nabla \omega_3 v^3(t) \|_{H^\frac{3}{2}}^2 + \| \nabla v^3(t) \|_{H^\frac{3}{2} + \frac{1}{0}}^2 \right) \leq (4\delta + \frac{1}{9}) \| \nabla \omega_\frac{3}{4}(t) \|_{L^2}^2 + \| \nabla \omega_3 v^3(t) \|_{H^\frac{3}{2}}^2 + \frac{3}{2} \left( \| \nabla v^3(t) \|_{H^\frac{3}{2} + \frac{1}{0}}^2 \right)
\]
\[
+ C \left( M(t) \frac{p}{2} + M(t) \right) \| \partial_3 \omega_\frac{3}{4} \|_{L^2}^2 + C \left( M(t) \frac{p}{2} + M(t) \right) \| \partial_3 v^3 \|_{H^\frac{3}{2}}^2 + C \left( M(t) \frac{p}{2} + M(t) \right) \| \partial_3 v^3 \|_{H^\frac{3}{2} + \frac{1}{0}}^2,
\]
where \( M(t) \overset{\text{def}}{=} \| \omega_\frac{3}{4}(t) \|_{L^2}^2 + \| \partial_3 v^3(t) \|_{H^\frac{3}{2}}^2 + \| v^3(t) \|_{H^\frac{3}{2} + \frac{1}{0}}^2 \). Taking \( \delta = \frac{1}{36} \) in the above inequality results in
\[
\frac{d}{dt} M(t) + \| \nabla \omega_\frac{3}{4}(t) \|_{L^2}^2 + \| \nabla \partial_3 v^3(t) \|_{H^\frac{3}{2}}^2 + \| \nabla v^3(t) \|_{H^\frac{3}{2} + \frac{1}{0}}^2 \leq C \left( M(t) \frac{p}{2} + M(t) \right) \| \partial_3 \omega_\frac{3}{4} \|_{L^2}^2 + C \left( M(t) \frac{p}{2} + M(t) \right) \| \partial_3 v^3 \|_{H^\frac{3}{2}}^2 + C \left( M(t) \frac{p}{2} + M(t) \right) \| \partial_3 v^3 \|_{H^\frac{3}{2} + \frac{1}{0}}^2
\]
\[
+ C \left( M(t) \frac{p}{2} + M(t) \right) \| \partial_3 v^3 \|_{H^\frac{3}{2} + \frac{1}{0}}^2.
\]
Thanks to (4.2), for any \( t \leq T^* \), we get, by integrating the above inequality over \([0, t]\), that

\[
M(t) + \int_0^t \left( \| \nabla v \omega_{\frac{2}{3}} (t') \|_{L^2}^2 + \| \nabla v \partial_3 v^3 (t') \|_{\dot{H}^0}^2 + \| \nabla v v^3 (t') \|_{H^{\frac{1}{2}, 0}}^2 \right) dt' \\
\leq M_0 + C \left( M_0^6 + M_0 \right) \int_0^t \| \partial_3 \omega_{\frac{2}{3}} (t') \|_{L^2}^2 dt' + C \left( M_0^2 + M_0^6 \right) \int_0^t \| \partial_3 v^3 (t') \|_{\dot{H}^0}^2 dt' \\
+ C \left( M_0^2 + M_0^{\max \left\{ \frac{10}{3}, \frac{10}{3} \right\}} \right) \int_0^t \| \partial_3 v^3 (t') \|_{H^{\frac{1}{2}, 0}}^2 dt' \\
\leq M_0 + C \left( M_0^2 + M_0^8 \right) \int_0^t \left( \| \partial_3 \omega_{\frac{2}{3}} (t') \|_{L^2}^2 + \| \partial_3 v^3 (t') \|_{\dot{H}^0}^2 + \| \partial_3 v^3 (t') \|_{H^{\frac{1}{2}, 0}}^2 \right) dt',
\]

(4.3)

where in the last step, we used the fact that \( \frac{p}{2-p\theta} < 6 \) due to \( p < 6 \) and \( \theta < \frac{1}{6} \).

In particular, if \( C_1 \) in (1.1) is so large that \( C_1 \geq 2C \), we find

\[
\frac{1}{4} t^2 \geq C \left( M_0^2 + M_0^8 \right).
\]

Then we deduce from (4.3) that for any \( t \) in \([0, T^*]\)

\[
M(t) + \frac{3}{4} \int_0^t \left( \| \nabla v \omega_{\frac{2}{3}} \|_{L^2}^2 + \| \nabla v \partial_3 v^3 \|_{\dot{H}^0}^2 + \| \partial_3 v^3 (t') \|_{H^{\frac{1}{2}, 0}}^2 \right) dt' \leq M_0.
\]

This in particular implies

\[
M(t) + \int_0^t \left( \| \nabla v \omega_{\frac{2}{3}} \|_{L^2}^2 + \| \nabla v \partial_3 v^3 \|_{\dot{H}^0}^2 + \| \partial_3 v^3 (t') \|_{H^{\frac{1}{2}, 0}}^2 \right) dt' \leq \frac{4}{3} M_0,
\]

which contracts with the definition of \( T^* \) given by (4.2). This in turn shows that \( T^* = T^* \).

It remains to show that \( T^* = \infty \). Indeed, by virtue of (1.13) and (1.2), we infer

\[
\int_0^{T^*} \| v^3 (t) \|_{H^{\frac{1}{2}, \frac{2}{3}}}^p dt \leq \sup_{t \in [0, T^*]} \left( \| \partial_3 v^3 (t) \|_{\dot{H}^0}^{\frac{p-4(1-\theta)}{2(1-\theta)}} \| v^3 (t) \|_{H^{\frac{1}{2}, 0}}^{\frac{p(1-2\theta)}{2(1-\theta)}} \right) \int_0^{T^*} \| \partial_3^2 v^3 (t) \|_{\dot{H}^0}^2 dt \\
+ \sup_{t \in [0, T^*]} \| v^3 (t) \|_H^{p-2} \int_0^{T^*} \| \nabla v^3 (t) \|_{H^{\frac{1}{2}, 0}}^2 dt \\
\leq (1 + \nu^{-2}) M_0^2,
\]

(4.4)

which together with Theorem 4.2 ensures that \( T^* = \infty \). This completes the proof of Theorem 1.1.

Next let us turn to the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Exactly along the same line of the proof of Theorem 1.1, under the assumption of Theorem 1.2, \((NS_v)\) has a unique solution \( v \in C([0, T^*]; H^{\frac{1}{2}}) \cap L^2_{\text{loc}}(0, T^*; H^{\frac{1}{2}})\) for some maximal time \( T^* > 0 \). Let us denote

\[
T^* \overset{\text{def}}{=} \sup \left\{ T \in [0, T^*] : \text{so that (1.4) holds for any } t \in [0, T] \right\},
\]

(4.5)
We are going to prove that $\widetilde{T}^* = \tilde{T}^*$. Otherwise, if $\tilde{T}^* < \widetilde{T}^*$, let 
$L(t) \overset{\text{def}}{=} \|\nabla_\omega^2(t)\|_{L^2}^2 + \|\partial_3 v^3(t)\|_{H^{3/2}}^2$ for any $t \in [0, \tilde{T}^*)$, we get, by summing up (3.1) and (3.8), that

$\frac{d}{dt} L(t) + \frac{10}{9} \|\nabla_\omega^2(t)\|_{L^2}^2 + \frac{4}{3} \|\nabla_\omega \partial_3 v^3(t)\|_{H^0}^2 \leq \frac{2}{9} \|\nabla_\omega^3\|_{H^{3/2}, 0}^2$

$+ C \|\partial_3 \omega^2(t)\|_{L^2}^2 L(t) \frac{1}{2} \|\nabla_\omega v^3(t)\|_{H^3, 0}^2 + L(t) \frac{2}{5} \|v^3\|_{H^{3/2}, 0}^{2-\frac{1}{2}} + L(t) \frac{1}{2} \|v^3\|_{H^{3/2}, 0}^{2-\frac{1}{2}}$

$+ C \|\partial_3^2 v^3\|_{H^0}^2 L(t) \frac{2}{5} \|\nabla_\omega v^3(t)\|_{H^{3/2}, 0}^{2-\frac{1}{2}} + A(t) (1 + L(t)^{-\frac{1}{2}}) + A(t)^{\frac{2}{5}} L(t)^{\frac{1}{5}}$

$+ A(t)^2 (1 + L(t)^{\frac{1}{2}}) + A(t)^{\frac{4}{5}} L(t)^{\frac{4}{5}} \|v^3\|_{H^{3/2}, 0}^{2-\frac{1}{2}}$

$+ C \|\partial_3 v^3\|_{H^{3/2}, 0}^2 L(t) \frac{2}{2-\frac{1}{2}} \|1 + L(t) \frac{2}{2-\frac{1}{2}}\|_{H^{3/2}, 0}^{2-\frac{1}{2}}$, 

where $A(t) \overset{\text{def}}{=} L(t) \|v^3(t)\|_{H^{3/2}, 0}^{1-\frac{1}{2}} \|v^3(t)\|_{H^{3/2}, 0}^{1-\frac{1}{2}}$. By virtue of (4.5), by integrating the above inequality over $[0, t]$ and using the fact that $L_0 \geq 1$, $\|v^3\|_{H^{3/2}, 0} \leq c_0$, we obtain

$L(t) + \int_0^t \left(\frac{10}{9} \|\nabla_\omega^2(t')\|_{L^2}^2 + \frac{4}{3} \|\nabla_\omega \partial_3 v^3(t')\|_{H^0}^2 \right) dt'$

$\leq L_0 + \frac{4}{9} \|v^3\|_{H^{3/2}, 0}^2 + C L_0^{\frac{1}{2}-\frac{1}{2}} \int_0^t \|\partial_3 \omega^2(t')\|_{L^2}^2 dt' + C \int_0^t \|\partial_3^2 v^3(t')\|_{H^0}^2 dt' \times$

$\times \left(L_0^{\frac{2}{5}} + A_0^{\frac{2}{5}} L_0 + A_0^{\frac{2}{5}} L_0 + A_0^{\frac{2}{5}} L_0 + A_0^{\frac{1}{2+\frac{1}{2}}} + C \|\nabla_\omega^3\|_{H^{3/2}, 0}^{2-\frac{1}{2}} \right) \cdot L_0$

where $A_0 \overset{\text{def}}{=} L_0^{\frac{1}{2}-\frac{1}{2}} \|v^3\|_{H^{3/2}, 0}^{1-\frac{1}{2}} \|v^3\|_{H^{3/2}, 0}^{1-\frac{1}{2}}$. Due to $\|v^3\|_{H^{3/2}, 0} \leq c_0 \leq L_0$, one has $A_0 \leq L_0^{\frac{1}{2}-\frac{1}{2}}$. As a result, it comes out

$L(t) + \int_0^t \left(\frac{10}{9} \|\nabla_\omega^2(t')\|_{L^2}^2 + \frac{4}{3} \|\nabla_\omega \partial_3 v^3(t')\|_{H^0}^2 \right) dt'$

$\leq \frac{4}{3} + C \nu^{-2} L_0 \|\nabla_\omega^2\|_{H^{3/2}, 0}^{(1-\frac{1}{2})} \int_0^t \left(\|\nu \partial_3 \omega^2(t')\|_{L^2}^2 + \|\nu \partial_3^2 v^3(t')\|_{H^0}^2 \right) dt'$

Note that $\frac{2(1-\frac{1}{2})}{2(2-\frac{1}{2})} < 3$ and $\frac{2}{3} + \frac{2}{3} < \frac{1}{2}$, if the constant $C_2$ in (1.3) is large enough, we have $C \nu^{-2} L_0^{\frac{2}{3} + \frac{2}{3}} < \frac{1}{3}$ and $C \nu^{-2} L_0^{\frac{2}{3} + \frac{2}{3}} < \frac{1}{7}$. Then (4.6) implies

$L(t) + \int_0^t \left(\|\nabla_\omega^2(t')\|_{L^2}^2 + \|\nabla_\omega \partial_3 v^3(t')\|_{H^0}^2 \right) dt' \leq \frac{5}{3} L_0$

On the other hand, for any $t \in [0, \tilde{T}^*)$, we get, by integrating (3.20) over $[0, t]$, that

$\|v^3(t)\|_{H^{3/2}, 0}^2 + \frac{3}{2} \int_0^t \|\nabla_\omega v^3(t')\|_{H^{3/2}, 0}^2 dt' \leq \|v^3(t)\|_{H^{3/2}, 0}^2 + 8 \delta L_0$

$+ C \left(\|v^3\|_{H^{3/2}, 0}^2 + \delta^{-\frac{1}{2}} \|v^3\|_{H^{3/2}, 0}^2 \right) L_0 + \|v^3\|_{H^0}^2 \left(\delta^{-3} L_0^2 + \delta^{-3} L_0^2 \right) \int_0^t \|\partial_3 v^3(t')\|_{H^{3/2}, 0}^2 dt'$. 
Proposition 5.1. Let
\[ v \in H^{\frac{3}{2} - \theta}, \quad \theta \in \left(0, \frac{1}{2}\right), \quad s < \min\left(\frac{1}{2}, \frac{1}{2} - \theta\right), \quad \alpha < \min\left(\frac{5}{2}, \frac{1}{2} + \theta\right) \]
satisfy \( \alpha + s > 0 \). Then for \( v^h \) given by (5.2), we have
\[
\|v^h\|_{\left(H^{\frac{3}{2} + s + \alpha}_{0, 1}(h) \left(B^{\frac{1}{2} + s}_{2, 1}\right)_v\right)} \lesssim \|\omega^h\|_{L^2}^{\frac{1}{2} + s + \alpha} \|\nabla \omega^h\|_{L^2}^{1 - s - \alpha} + \|\nu \partial_3 v^3\|_{H^\alpha} \|\nu \nabla \partial_3 v^3\|_{H^{\alpha - s}}^{1 - s - \alpha}.
\]
5.1. The estimate of \( \| \omega(t) \|_{L^\frac{4}{3}} \). By taking \( L^2 \) scalar product of \( \omega \) equation in (5.1) with 
\[
\omega_\frac{3}{4} = \frac{\omega}{|\omega|^\frac{3}{4}},
\]
we obtain
\[
\frac{d}{dt} \| \omega_\frac{3}{4}(t) \|_{L^2}^2 + \frac{4}{3} \| \nabla \omega_\frac{3}{4}(t) \|_{L^2}^2 = \frac{3}{2} \nu \int_{\mathbb{R}^3} \partial_3 v^3 |\omega| \frac{3}{4}(\partial_2 v^3 \partial_3 v^1 - \partial_1 v^3 \partial_3 v^2) \omega_\frac{3}{4} \, dx.
\]
In view of the Estimates (3.4) and (3.6), we find
\[
\frac{d}{dt} \| \omega_\frac{3}{4}(t) \|_{L^2}^2 + \frac{4}{3} \| \nabla \omega_\frac{3}{4}(t) \|_{L^2}^2 
\leq \nu^{-\frac{1}{4}} \left( \frac{\omega}{4} \right) \| \omega_\frac{3}{4} \|_{L^2}^2 + \| \nabla \omega_\frac{3}{4} \|_{L^2}^2 + \| \nu \partial_3 \omega_\frac{3}{4} \|_{H^\theta}^2 + \| \nabla \omega_\frac{3}{4} \|_{L^2}^2 + \frac{\nu}{4} \| \omega_\frac{3}{4} \|_{L^2}^2 \right)\-enabled\end{array}
\]
Inserting (1.13) to the above inequality and then applying Young's inequality gives rise to
\[
\frac{d}{dt} \| \omega_\frac{3}{4}(t) \|_{L^2}^2 + \frac{4}{3} \| \nabla \omega_\frac{3}{4}(t) \|_{L^2}^2 \leq \frac{1}{18} \left( \| \omega_\frac{3}{4} \|_{L^2}^2 + \| \nabla \omega_\frac{3}{4} \|_{L^2}^2 + \| \nu \partial_3 \omega_\frac{3}{4} \|_{H^\theta}^2 + \| \nabla \omega_\frac{3}{4} \|_{L^2}^2 \right)
\]
\[
+ C \left( \| \nabla \omega_\frac{3}{4} \|_{L^2}^2 \right)^\frac{1}{2} \| \omega_\frac{3}{4} \|_{L^2}^2 + \| \nabla \omega_\frac{3}{4} \|_{L^2}^2 \right) \| \omega_\frac{3}{4} \|_{L^2}^2 \left( \nu^{-\frac{1}{4}} \right) \| \omega_\frac{3}{4} \|_{L^2}^2 \right) \| \omega_\frac{3}{4} \|_{L^2}^2 \right) \| \omega_\frac{3}{4} \|_{L^2}^2 \right) \| \omega_\frac{3}{4} \|_{L^2}^2 \right)
\]

5.2. The estimate of \( \| \nu \partial_3 v^3(t) \|_{H^\theta} \). We get, by taking \( H^\theta \) inner product of the \( \partial_3 \) to the \( v^3 \) equation of (5.1) with \( \nu^2 \partial_3 v^3 \), that
\[
\frac{d}{dt} \| \nu \partial_3 v^3(t) \|_{H^\theta}^2 + 2 \| \nabla \nu \partial_3 v^3(t) \|_{H^\theta}^2 = - \sum_{i=1}^{3} (P_{1,\nu}(v, v) | \nu \partial_3 v^3)_{H^\theta} \quad \text{with}
\]
\[
P_{1,\nu}(v, v) \stackrel{\text{def}}{=} 2 \left( \text{Id} + (\nu \partial_3)^2 \Delta_\nu^{-1} \right) (\nu \partial_3 v^3)^2 + 2(\nu \partial_3)^2 \Delta_\nu^{-1} \left( \sum_{\ell, m=1}^{2} \partial_\ell v^m \partial_\ell v^m \right),
\]
\[
P_{2,\nu}(v, v) \stackrel{\text{def}}{=} 2 \left( \text{Id} + 2(\nu \partial_3)^2 \Delta_\nu^{-1} \right) (\nu \partial_3 v^h \cdot \nabla_h v^3), \quad \text{and} \quad P_{3,\nu}(v, v) \stackrel{\text{def}}{=} 2 \nu v \cdot \nabla_h v^3.
\]
We first deduce from Lemma 3.2 and (5.2) that
\[
\left( \| P_{1,\nu}(v, v) | \nu \partial_3 v^3 \right)_{H^\theta} \leq \frac{C \nu}{H^\theta} \left( \| H^\theta \|_{L^\theta} \right) \| \nu \partial_3 v^3 \|_{H^\theta} \left( \| v^3 \|_{L^\theta} \right) \| \nu \partial_3 v^3 \|_{H^\theta} \left( \| v^3 \|_{L^\theta} \right) \| \nu \partial_3 v^3 \|_{H^\theta} \left( \| v^3 \|_{L^\theta} \right) \| \nu \partial_3 v^3 \|_{H^\theta} \left( \| v^3 \|_{L^\theta} \right)
\]
from which and a similar derive (3.12), we infer
\[
\left( \| P_{1,\nu}(v, v) | \nu \partial_3 v^3 \right)_{H^\theta} \leq \frac{C \nu^{-\frac{1}{2}}}{H^\theta} \left( \| \nu \partial_3 v^3 \|_{H^\theta} \right) \| \nu \partial_3 v^3 \|_{H^\theta} \left( \| v^3 \|_{L^\theta} \right) \| \nu \partial_3 v^3 \|_{H^\theta} \left( \| v^3 \|_{L^\theta} \right) \| \nu \partial_3 v^3 \|_{H^\theta} \left( \| v^3 \|_{L^\theta} \right)
\]
Applying Young's inequality gives
\[
\left| (P_{1,\nu}(v, v) | \nu \partial_3 v^3)_{H^0} \right| \leq \frac{1}{18} \left( \| \nu \nabla_h \partial_3 v^3 \|_{H^0}^2 + \| \nabla \omega^2_3 \|_{L^2}^2 \right) + C \| (\nu \partial_3 v^3)^2 \|_{H^0}^2 \left( \nu^{-1} \frac{1}{p} \| \nu \partial_3 v^3 \|_{H^0} \| v^3 \|_{H^0}^{1 - \frac{2}{p}} \right) + \nu^{\frac{p - 2}{2(p - 4)}} \| \nu \partial_3 v^3 \|_{H^0}^{2(p - 3)} \| v^3 \|_{H^0}^{\frac{2(1 - 2\theta)}{p + (1 - 2\theta)}}.
\]

To handle the term involving \( P_{2,\nu}(v, v) \), as in (3.14), by using integration by parts, we find
\[
(P_{2,\nu}(v, v) | \nu \partial_3 v^3)_{H^0} = A_{1,\nu} + A_{2,\nu}, \quad \text{with}
\]
\[
A_{1,\nu} \overset{\text{def}}{=} -2 \left( \left( \text{Id} + 2(\nu \partial_3)^2 \Delta_\nu^{-1} \right) (v^h \cdot \nabla_h v^3) \right) (\nu \partial_3 v^3)_{H^0},
\]
\[
A_{2,\nu} \overset{\text{def}}{=} -2 \left( \left( \text{Id} + 2(\nu \partial_3)^2 \Delta_\nu^{-1} \right) (v^h \cdot \nabla_h \partial_3 v^3) \right) (\nu \partial_3 v^3)_{H^0}.
\]

Applying the law of product, Lemma 2.3, yields
\[
|A_{1,\nu}| \leq C \| v^h \cdot \nabla_h v^3 \|_{H^0} \| (\nu \partial_3 v^3)^2 \|_{H^0} \]
\[
\leq C \| v^h \|_{\left( B_{1,1}^{\frac{7}{5}, \frac{2}{5}} \right)} \left( B_{2,1}^{\frac{7}{5}, \frac{2}{5}} \right) \| \nabla_h v^3 \|_{H^0} \| (\nu \partial_3 v^3)^2 \|_{H^0},
\]

which together with Lemma 2.5 ensures that
\[
|A_{1,\nu}| \leq C \| v^h \|_{\left( B_{1,1}^{\frac{7}{5}, \frac{2}{5}} \right)} \left( B_{2,1}^{\frac{7}{5}, \frac{2}{5}} \right) \| \nabla_h v^3 \|_{H^0} \| (\nu \partial_3 v^3)^2 \|_{H^0}.
\]

Whereas it is easy to observe that
\[
|A_{2,\nu}| \lesssim \| \nu v^h \cdot \nabla_h \partial_3 v^3 \|_{H^{-1 + 2\theta, \frac{7}{5} - \frac{2}{\theta} - 2\theta}} \| \nu \partial_3 v^3 \|_{H^{0, -\frac{1}{2} + \frac{1}{\theta}}}
\]
\[
\lesssim \| v^h \|_{\left( B_{1,1}^{\frac{7}{5}, \frac{2}{5}} \right)} \left( B_{2,1}^{\frac{7}{5}, \frac{2}{5}} \right) \| \nu \nabla_h \partial_3 v^3 \|_{H^{-1 + 2\theta, \frac{7}{5} - \frac{2}{\theta} - 2\theta}} \| \nu \partial_3 v^3 \|_{H^{0, -\frac{1}{2} + \frac{1}{\theta}}},
\]
from which, we deduce by the derivation of (3.16) that
\[
|A_{2,\nu}| \leq C \| \nu v^3 \|_{H^{0, \frac{1}{2} + \frac{1}{\theta}}} \| v^h \|_{\left( B_{1,1}^{\frac{7}{5}, \frac{2}{5}} \right)} \left( B_{2,1}^{\frac{7}{5}, \frac{2}{5}} \right) \| \nu \nabla_h \partial_3 v^3 \|_{H^0} \| (\nu \partial_3 v^3)^2 \|_{H^0}.
\]

Together with (3.11) and (5.7), we infer
\[
\left| (P_{2,\nu}(v, v) | \nu \partial_3 v^3)_{H^0} \right| \leq C \left( \| \nabla \omega_3 \|_{L^2} \| \nabla \omega_3 \|_{L^2}^{1 - \frac{2}{p}} + \| \nu \partial_3 v^3 \|_{H^0} \| \nu \nabla \partial_3 v^3 \|_{H^0} \right)
\]
\[
\times \left( \nu^{-\frac{1}{p} + \theta} \| v^3 \|_{H^0} \| \nabla_h v^3 \|_{H^0} \| \nu \partial_3 v^3 \|_{H^0} \| (\nu \partial_3 v^3)^2 \|_{H^0} \right)
\]
\[
+ \nu^{\frac{2(1 - 2\theta)}{p + (1 - 2\theta)}} \| \nu \partial_3 v^3 \|_{H^0}^{2(p - 3)} \| v^3 \|_{H^0} \| \nu \partial_3 v^3 \|_{H^0}^{\frac{1}{2} + \frac{1}{\theta}} \| \nu \nabla_h \partial_3 v^3 \|_{H^0} \right).
\]

Applying Young's inequality yields
\[
\left| (P_{2,\nu}(v, v) | \nu \partial_3 v^3)_{H^0} \right| \leq \frac{1}{18} \left( \| \nabla \omega_3 \|_{L^2}^2 + \| \nabla_h v^3 \|_{H^0}^2 \| \nu \nabla \partial_3 v^3 \|_{H^0}^2 \right) + C \nu^{-\frac{2}{p} + \theta} \| \nu \partial_3 v^3 \|_{H^0}^{\frac{2(p + 6)}{p - 6}} \| \nu \partial_3 v^3 \|_{H^0}^{\frac{2}{p - 6}} \| \nu \partial_3 v^3 \|_{H^0}^{2(p - 3)} \| v^3 \|_{H^0} \| \nu \partial_3 v^3 \|_{H^0}^{\frac{1}{2} + \frac{1}{\theta}} \| \nu \nabla_h \partial_3 v^3 \|_{H^0} \right).
\]
It is easy to observe that the term $\nu \langle v^3 \partial_3 v^3 \rangle_{H^\frac{1}{2}}$ shares the same estimate as $A_{2, \nu}$. Finally, note that

$$\left\|(v^3(\nu \partial_3)^2 v^3 \ | \nu \partial_3 v^3)_{H^\frac{1}{2}}\right\| \lesssim \|v^3(\nu \partial_3)^2 v^3\|_{H^{\frac{1}{2} + \frac{\nu}{6} - \frac{1}{2} - \frac{1}{2}}, H^{\frac{1}{2} + \frac{\nu}{6} - \frac{1}{2} - \frac{1}{2}}).$$

Then we get by a similar derivation of (3.18) that

$$\left\|(v^3(\nu \partial_3)^2 v^3 \ | \nu \partial_3 v^3)_{H^\frac{1}{2}}\right\| \leq C\|v^3\|_{H^{\frac{1}{2} + \frac{\nu}{6} - \frac{1}{2} - \frac{1}{2}}}(\nu \partial_3^2 v^3\|_{H^\frac{1}{2}}\|\nu \partial_3 v^3\|_{H^\frac{1}{2}}\|\nu \partial_3 \partial_3 v^3\|_{H^\frac{1}{2}},$$

from which (1.13) and (5.8), we deduce that

$$\left\|(P_{3, \nu}(v, v) \ | \nu \partial_3 v^3)_{H^\frac{1}{2}}\right\| \leq \frac{1}{18}\left\|(\nabla \omega_3^3)^2 \|_{H^\frac{1}{2}}\|\nabla h^3v^3\|_{H^\frac{1}{2}, 0}\right\| + \|\nu \nabla \nu \partial_3 v^3\|_{H^\frac{1}{2}}$$

$$+ C\|(\nu \partial_3)^3\|_{H^\frac{1}{2}}\left(\left(\nu - \frac{2(1 - \frac{1}{4})}{H^\frac{1}{2} - \frac{1}{2}}\right)\left(\|\nabla \omega_3^3\|_{H^\frac{1}{2}}\|\nabla h^3v^3\|_{H^\frac{1}{2}, 0} + \|\nu \nabla \nu \partial_3 v^3\|_{H^\frac{1}{2}}\|\nu \partial_3 v^3\|_{H^\frac{1}{2}}\|\nu \partial_3 \partial_3 v^3\|_{H^\frac{1}{2}, 0}\right)\right)\right\|_{H^\frac{1}{2}, 0}\right\|.$$ 

Inserting the Estimates (5.5), (5.9), and (5.10) into (5.4) leads to

$$\frac{d}{dt}\|\nu \partial_3 v^3(t)\|_{H^\frac{1}{2}}^2 + \frac{3}{2}\|\nabla \nu \partial_3 v^3(t)\|_{H^\frac{1}{2}}^2 \leq \frac{1}{6}\|\nabla \omega_3^3\|_{H^\frac{1}{2}}^2 + \frac{1}{9}\|\nabla h^3v^3\|_{H^\frac{1}{2}, 0}\right\| + C\|\nu \partial_3^3\|_{H^\frac{1}{2}}\left(\left(\nu - \frac{2(1 - \frac{1}{4})}{H^\frac{1}{2} - \frac{1}{2}}\right)\left(\|\nabla \omega_3^3\|_{H^\frac{1}{2}}\|\nabla h^3v^3\|_{H^\frac{1}{2}, 0} + \|\nu \nabla \nu \partial_3 v^3\|_{H^\frac{1}{2}}\|\nu \partial_3 v^3\|_{H^\frac{1}{2}}\|\nu \partial_3 \partial_3 v^3\|_{H^\frac{1}{2}, 0}\right)\right)\right\|_{H^\frac{1}{2}, 0}\right\|.$$ 

5.3. The estimate of $\|v^3(t)\|_{H^\frac{1}{2}, 0}$. We first get, by taking the $H^\frac{1}{2}, 0$ inner product of the $v^3$ equation of (5.1) with $v^3$, that

$$\frac{d}{dt}\|v^3(t)\|^2_{H^\frac{1}{2}, 0} + 2\|\nabla \nu \partial_3 v^3(t)\|^2_{H^\frac{1}{2}, 0} = \sum_{i=1}^3 \langle Q_{1, \nu}(v, v) \ | v^3 \rangle_{H^\frac{1}{2}, 0}$$

with

$$Q_{1, \nu}(v, v) \overset{\text{def}}{=} 2\nu \partial_3 \Delta_{-1}(\nu \partial_3 v^3)^2 + 2\nu \partial_3 \Delta_{-1}\left(\sum_{\ell, m=1}^2 \partial_\ell v^m \partial_m v^\ell\right),$$

$$Q_{2, \nu}(v, v) \overset{\text{def}}{=} 4\nu \partial_3 \Delta_{-1}\left(\sum_{\ell=1}^2 \nu \partial_3 v^\ell \partial_\ell v^3\right),$$

and $Q_{3, \nu}(v, v) \overset{\text{def}}{=} 2\nu \nabla \nu \partial_3 v^3.$

In what follows, we shall deal with the estimates of the above terms.

- The estimate of $\langle Q_{1, \nu}(v, v) \ | v^3 \rangle_{H^\frac{1}{2}, 0}$.

Corresponding to (3.22), we have

$$\|\nu \partial_3 \Delta_{-1} f\|_{H^\frac{1}{2}, h}(B^\frac{1}{2}, 0) \leq C\|f\|_{H^\frac{1}{2}, h}(B^\frac{1}{2}, 0),$$

so that in view of (5.12), by applying Lemma 2.3, and Lemma 2.8, we find

$$\langle Q_{1, \nu}(v, v) \ | v^3 \rangle_{H^\frac{1}{2}, 0} \leq \|Q_{1, \nu}(v, v)\|_{H^\frac{1}{2}, h}(B^\frac{1}{2}, 0) \|v^3\|_{H^\frac{1}{2}, h}(B^\frac{1}{2}, 0).$$
\[ \leq 2 \left( \| \nu \partial_3 v^3 \|_{(H^{1/2})_h(B_2, \infty)} \right)_v + \sum_{m=1}^{2} \| \partial_1 v^m \partial_3 v^f \|_{(H^{1/2})_h(B_2, \infty)} \| v^3 \|_{H^{1/2}, \partial} \| \partial_3 v^3 \|_{H^{1/2}, \partial} \]
\[ \leq C \left( \| \nu \partial_3 v^3 \|_{H^{1/2}, \partial}^2 + \| \nabla_h v^h \|_{H^{1/2}, \partial} \| v^3 \|_{H^{1/2}, \partial} \| \partial_3 v^3 \|_{H^{1/2}, \partial} \right). \]

Yet, by virtue of (5.2), we get, by applying (2.3) and (2.8), that
\[ \| \nabla_h v^h \|_{H^{1/2}, \partial} \leq \| \omega \|_{H^{1/2}} + \| \nu \partial_3 v^3 \|_{H^{1/2}, \partial} \]
\[ \leq \| \omega \|_{L^2}^1 \| \nabla \omega \|_{L^2}^1 + \| \nu \partial_3 v^3 \|_{H^0}^1 \| \nu \nabla \partial_3 v^3 \|_{H^0}^1. \]

Hence for any \( \delta > 0 \), we achieve
\[ \left| (Q_{1, \nu}(v, v) \mid v^3)_{H^{1/2}, \partial} \right| \leq C \| v^3 \|_{H^{1/2}, \partial} \| \partial_3 v^3 \|_{H^{1/2}, \partial} \left( \| \omega \|_{L^2}^1 \| \nabla \omega \|_{L^2}^1 + \| \nu \partial_3 v^3 \|_{H^0}^1 \| \nu \nabla \partial_3 v^3 \|_{H^0}^1 \right) \leq \delta \left( \| \nabla \omega \|_{L^2}^1 + \| \nu \nabla \partial_3 v^3 \|_{H^0}^1 \right) + C \delta^{-3} \| v^3 \|_{H^{1/2}, \partial}^2 \left( \| \omega \|_{L^2}^1 \| \nabla \omega \|_{L^2}^1 + \| \nu \partial_3 v^3 \|_{H^0}^1 \right) \| \partial_3 v^3 \|_{H^{1/2}, \partial}^2. \]

The estimate of \((Q_{2, \nu}(v, v) \mid v^3)_{H^{1/2}, \partial}\).

Similar to (3.24), by using integration by parts, we obtain
\[ (Q_{2, \nu}(v, v) \mid v^3)_{H^{1/2}, \partial} = -B_{1, \nu} - B_{2, \nu}, \quad \text{with} \]
\[ B_{1, \nu} \overset{\text{def}}{=} 4 \left( \nu \partial_3 \Delta^{-1} \nu v^h \cdot \nabla_h v^3 \right)_{H^{1/2}, \partial} \]
\[ B_{2, \nu} \overset{\text{def}}{=} 4 \left( \nu \partial_3 \Delta^{-1} \nu v^h \cdot \nabla_h \partial_3 v^3 \right)_{v^3} \]

We get, by applying Lemma 2.3, that
\[ |B_{1, \nu}| \leq \| \nu \partial_3 \Delta^{-1} \nu v^h \cdot \nabla_h v^3 \|_{H^{1/2}, \partial} \| \nu \partial_3 v^3 \|_{H^{1/2}, \partial} \]
\[ \leq \| v^h \cdot \nabla_h v^3 \|_{H^{1/2}, \partial} \| \nu \partial_3 v^3 \|_{H^{1/2}, \partial} \]
\[ \leq \| v^h \|_{B^{1/2}_{1, \nu}} \| \nabla_h v^3 \|_{(H^{1/2})_h(B_{2,1}, \infty)} \| \nu \partial_3 v^3 \|_{H^{1/2}, \partial}. \]

Then applying Proposition 2.1 and Lemma 2.5 yields
\[ |B_{1, \nu}| \leq \delta \left( \| \nabla \omega \|_{L^2}^1 + \| \nu \nabla \partial_3 v^3 \|_{H^0}^1 \right) + C \delta^{-3} \| v^3 \|_{H^{1/2}, \partial}^2 \| \nu \partial_3 v^3 \|_{H^{1/2}, \partial}^2. \]

from which, we infer
\[ |B_{1, \nu}| \leq \delta \left( \| \nabla \omega \|_{L^2}^1 + \| \nu \nabla \partial_3 v^3 \|_{H^0}^1 \right) \]
\[ + C \delta^{-3} \| v^3 \|_{H^{1/2}, \partial}^2 \| \nabla \omega \|_{L^2}^1 + \| \nu \partial_3 v^3 \|_{H^0}^1 \nu^{-3/2} \| \nu \partial_3 v^3 \|_{H^{1/2}, \partial}^2. \]

On the other hand, it is easy to observe from (5.13) that
\[ |B_{2, \nu}| \leq \| \nu \partial_3 \Delta^{-1} \nu v^h \cdot \nabla_h \partial_3 v^3 \|_{(H^{1/2})_h(B_{2, \infty}, \partial)} \| v^3 \|_{H^{1/2}, \partial} \]
\[ \leq \| v^h \cdot \nabla_h \partial_3 v^3 \|_{(H^{1/2})_h(B_{2, \infty}, \partial)} \| v^3 \|_{H^{1/2}, \partial} \]
\[ \leq \| v^h \cdot \nabla_h \partial_3 v^3 \|_{(H^{1/2})_h(B_{2,1}, \infty)} \| v^3 \|_{(H^{1/2})_h(B_{2,1})} \].
Then applying the law of product, Lemma 2.3, Lemma 2.8 and then Proposition 2.1 gives

\[
|B_{2,\nu}| \lesssim \|v^h\|_h \left( |B_{2,1}^\nu| + |B_{2,2}^\nu| \right) + \|\nu \nabla_h \partial_3 v^3\|_{H^{\frac{3}{2},0}} \|v^3\|_{H^{\frac{3}{2},0}} \|\partial_3 v^3\|_{H^{\frac{3}{2},0}}^2
\]

\[
\lesssim \nu \left( \|\omega_{\frac{2}{3}}\|_L^2 \|\nabla v^3\|_{H^{\frac{3}{2},0}} + \|\nu \partial_3 v^3\|_{H^0}^2 \right) \|v^3\|_{H^{\frac{3}{2},0}} \|\partial_3 v^3\|_{H^{\frac{3}{2},0}}^2.
\]

Applying Young's inequality leads to

\[
|B_{2,\nu}| \leq \delta \left( \|\omega_{\frac{2}{3}}\|_L^2 + \|\nu \nabla \partial_3 v^3\|_{H^0}^2 \right)
\]

\[
+ C\delta^{-\frac{1}{3}} \nu \|v^3\|_{H^{\frac{3}{2},0}}^2 \left( \|\omega_{\frac{2}{3}}\|_L^2 + \|\nu \partial_3 v^3\|_{H^0}^2 \right) \|\partial_3 v^3\|_{H^{\frac{3}{2},0}}^2.
\]

Combining the Estimates (5.16) with (5.17), we obtain

\[
\|(Q_{2,\nu}(v, v) | v^3)\|_{H^{\frac{3}{2},0}} \leq 2\delta \left( \|\omega_{\frac{2}{3}}\|_L^2 + \|\nu \nabla \partial_3 v^3\|_{H^0}^2 \right)
\]

\[
+ C\delta^{-\frac{1}{3}} \nu \|v^3\|_{H^{\frac{3}{2},0}}^2 \left( \|\omega_{\frac{2}{3}}\|_L^2 + \|\nu \partial_3 v^3\|_{H^0}^2 \right) \|\partial_3 v^3\|_{H^{\frac{3}{2},0}}^2.
\]

\[
\text{The estimate of } (Q_{3,\nu}(v, v) | v^3)\|_{H^{\frac{3}{2},0}}.
\]

It follows from a similar derivation of (3.28) that

\[
2\|\left( v^h \cdot \nabla h v^3 \right) | v^3\|_{H^{\frac{3}{2},0}} \leq C \left( \|\omega_{\frac{2}{3}}\|_L^2 \|\nabla \omega_{\frac{2}{3}}\|_{H^{\frac{3}{2},0}} + \|\nu \partial_3 v^3\|_{H^0}^2 \|\nabla \partial_3 v^3\|_{H^{\frac{3}{2},0}}^2 \right)
\]

\[
\times \|v^3\|_{H^{\frac{3}{2},0}} \|\partial_3 v^3\|_{H^{\frac{3}{2},0}}^2
\]

\[
\leq \delta \left( \|\omega_{\frac{2}{3}}\|_L^2 + \|\nu \partial_3 v^3\|_{H^0}^2 \right) + \frac{1}{8} \|\nabla h v^3\|_{H^{\frac{3}{2},0}}^2 + C\delta^{-\frac{1}{3}} \|v^3\|_{H^{\frac{3}{2},0}}^2 \|\partial_3 v^3\|_{H^{\frac{3}{2},0}}^2.
\]

While we deduce from a similar derivation of (3.29) that

\[
2\|\left( \nu v^3 \partial_3 v^3 \right) | v^3\|_{H^{\frac{3}{2},0}} \leq C\nu \|v^3\|_{H^{\frac{3}{2},0}}^2 \|\nabla h v^3\|_{H^{\frac{3}{2},0}}^2 + \|v^3\|_{H^{\frac{3}{2},0}}^2 \|\partial_3 v^3\|_{H^{\frac{3}{2},0}}^2
\]

\[
\leq \frac{1}{8} \|\nabla h v^3\|_{H^{\frac{3}{2},0}}^2 + C\nu \|v^3\|_{H^{\frac{3}{2},0}}^2 \|\partial_3 v^3\|_{H^{\frac{3}{2},0}}^2.
\]

As a consequence, it comes out

\[
\|\left( Q_{3,\nu}(v, v) | v^3\right)\|_{H^{\frac{3}{2},0}} \leq \delta \left( \|\omega_{\frac{2}{3}}\|_L^2 + \|\nu \partial_3 v^3\|_{H^0}^2 \right) + \frac{1}{4} \|\nabla h v^3\|_{H^{\frac{3}{2},0}}^2
\]

\[
+ C\left( \nu \|v^3\|_{H^{\frac{3}{2},0}}^2 + \nu \|\partial_3 v^3\|_{H^0}^2 \right) \|\nabla \partial_3 v^3\|_{H^{\frac{3}{2},0}}^2 + C\nu \|v^3\|_{H^{\frac{3}{2},0}}^2 \|\partial_3 v^3\|_{H^{\frac{3}{2},0}}^2.
\]

Substituting the Estimates (5.14), (5.18) and (5.19) into (5.12) results in

\[
\frac{d}{dt} \|v^3(t)\|_{H^{\frac{3}{2},0}}^2 + \frac{3}{2} \|\nabla v^3(t)\|_{H^{\frac{3}{2},0}}^2 \leq 4\delta \left( \|\omega_{\frac{2}{3}}\|_L^2 + \|\nu \partial_3 v^3\|_{H^0}^2 \right)
\]

\[
+ C\|\nu \partial_3 v^3\|_{H^{\frac{3}{2},0}}^2 \left( \nu \frac{3}{2} \|v^3\|_{H^{\frac{3}{2},0}}^2 \|\omega_{\frac{2}{3}}\|_L^2 + \|\partial_3 v^3\|_{H^0}^2 \right) + \nu \frac{3}{2} \|v^3\|_{H^{\frac{3}{2},0}}^2
\]

\[
+ \nu \frac{3}{2} \|v^3\|_{H^{\frac{3}{2},0}}^2 \left( \|\omega_{\frac{2}{3}}\|_L^2 + \|\nu \partial_3 v^3\|_{H^0}^2 \right) + \delta^{-\frac{1}{3}} \left( \|\omega_{\frac{2}{3}}\|_L^2 + \|\nu \partial_3 v^3\|_{H^0}^2 \right).
\]
5.4. The proof of Theorem 1.3. Along the same line to the proof of Theorem 1.3 in [7], it is easy to prove that (1.6) has a unique solution \( v \in C([0, T'_\nu], H^{3\over 2}) \cap L^2([0, T'_\nu]; H^{5\over 2}) \) for some maximal time \( T'_\nu > 0 \). Let us denote

\[
T'_\nu \overset{\text{def}}{=} \sup \left\{ t \in [0, T'_\nu] : \text{so that (1.9) holds} \right\}.
\]

We are going to prove that \( T'_\nu = T^*_\nu \). Otherwise, if \( T'_\nu < T^*_\nu \), let \( L_\nu(t) \overset{\text{def}}{=} \|\omega(t)\|_{L^{3\over 2}}^2 + \|\nu \partial_3 v^3(t)\|_{H_\theta}^2 \), for any \( t \in [0, T'_\nu] \), we get, by summing up (5.3) and (5.11) that

\[
\frac{d}{dt} L_\nu(t) + \frac{10}{9} \|\nabla_\nu \omega_3^2(t)\|_{L^2}^2 + \frac{4}{3} \|\partial_3 v^3(t)\|_{H_\theta}^2 \leq C \\|\nu \partial_3 v^3(t)\|_{L^2}^2
\]

\[
\times \left( \nu^{-\frac{2}{3}} \nu^{\frac{5}{2(1-\theta)}} L_\nu(t)^{\frac{1}{4}} \| v^3 \|_{H_\theta}^{2\over 3} + \nu^{-\frac{1}{3}} \| v^3 \|_{H_\theta}^{1\over 2} + \nu^{\frac{2}{3}} \| v^3 \|_{H_\theta}^{\frac{5}{6}} \right) + C \left( (1 + L_\nu(t)) L_\nu(t)^{\frac{5}{2}} \| v^3 \|_{H_\theta}^{2\over 3} + \nu^{-1} (1 + L_\nu(t)) L_\nu(t)^{\frac{1}{4}} \| v^3 \|_{H_\theta}^{1\over 2} \right)
\]

\[
+ C \nu^{-2} (L_\nu(t)^{\frac{5}{2}} + L_\nu(t)^{\frac{5}{2}} \| v^3 \|_{H_\theta}^{2\over 3}) \|\nu \partial_3 v^3(t)\|_{H_\theta}^2.
\]

Integrating the above inequality over \([0, t]\) and using (1.9), we find

\[
L_\nu(t) + \int_0^t \left( \frac{10}{9} \|\nabla_\nu \omega_3^2(t')\|_{L^2}^2 + \frac{4}{3} \|\partial_3 v^3(t')\|_{H_\theta}^2 \right) dt' \leq L_0 \nu + \frac{1}{3} \|v_0^3\|_{H_\theta}^{2\eta}.
\]

Note that \( p \in [4, 6], \theta \in ]\frac{1}{2} - \frac{2}{p}, \frac{1}{2}[, \) and \( \|v_0^3\|_{H_\theta}^{\frac{2}{3}} \leq c_0 \) (see (1.8)), we deduce from the above inequality that

\[
L_\nu(t) + \int_0^t \left( \frac{10}{9} \|\nabla_\nu \omega_3^2(t')\|_{L^2}^2 + \frac{4}{3} \|\partial_3 v^3(t')\|_{H_\theta}^2 \right) dt' \leq \frac{4}{3} L_0 \nu
\]

\[
+ C L_0 \nu \|v_0^3\|_{H_\theta}^{\frac{2}{3}},
\]

where \( A_0 \nu \overset{\text{def}}{=} \nu^{-\frac{1}{3}} \|v_0^3\|_{H_\theta}^{\frac{2}{3}} \|v_0^3\|_{H_\theta}^{\frac{2}{3}} \).

It is easy to observe from (1.8) that

\[
\nu^{-\frac{1}{3}} \|v_0^3\|_{H_\theta}^{\frac{2}{3}} \overset{c_1}{\leq} 1 \quad \text{and} \quad \nu^{-1} \|v_0^3\|_{H_\theta}^{\frac{2}{3}} \overset{c_2}{\leq} 2.
\]
In particular if $c_1$ in (5.23) is so small enough that

$$CA_{0,\nu}^{\frac{p(1-\theta)}{2}} \leq \frac{1}{6} \quad \text{and} \quad CA_{0,\nu} + A_{0,\nu}^{\frac{5}{2} - \frac{2p}{3}} \leq \frac{1}{9},$$

we deduce from (5.22) that, for any $t \in [0, T^*_\nu]$,

$$L_\nu(t) + \int_0^t \left( \left\| \nabla \nu \omega \right\|_{L^2}^2 + \left\| \nu \nabla \nu \partial_3 v^3 \right\|_{L^6}^2 \right) \, dt' \lesssim \frac{3}{2} L_{0,\nu}.$$  

(5.24)

On the other hand, for any $t \in [0, T^*_\nu]$, we get, by integrating (5.20) over $[0, t]$, that

$$\|v^3(t)\|_{L^2}^2 + \frac{3}{2} \int_0^t \|\nabla \nu v^3(t')\|_{L^2}^2 \, dt' \lesssim \frac{3}{2} \|v_0^3\|_{L^2}^2 + 4\delta L_{0,\nu}$$

(5.25)

$$+ C \int_0^t \|\nu \partial_3 v^3(t')\|_{L^2}^2 \, dt' \left( \nu^{-\frac{2}{3}} \|v_0^3\|_{L^2}^{\frac{2n}{3}} \delta^{-\frac{1}{3}} \left( L_{0,\nu}^\frac{7}{7} + L_{0,\nu}^\frac{1}{7} \right) \right.$$ 

$$+ \nu^{-\frac{2}{3}} \|v_0^3\|_{L^2}^{\frac{4n}{3}} + \nu^{-2} \|v_0^3\|_{L^2}^{\frac{4n}{3}} \left[ \delta^{-\frac{1}{3}} \left( L_{0,\nu}^\frac{7}{7} + L_{0,\nu}^\frac{1}{7} \right) \right] \right).$$

Taking $\delta = \frac{\|v_0^3\|_{L^2}^{\frac{2n}{3}}}{8L_{0,\nu}}$ in (5.25) and recalling from (1.8) that $\|v_0^3\|_{L^2} \leq c_0 \leq L_{0,\nu}$, $\nu \gg 1$, we infer

$$\|v^3(t)\|_{L^2}^2 + \frac{3}{2} \int_0^t \|\nabla \nu v^3(t')\|_{L^2}^2 \, dt' \lesssim \frac{3}{2} \|v_0^3\|_{L^2}^2$$

(5.26)

$$+ C \int_0^t \|\nu \partial_3 v^3(t')\|_{L^2}^2 \, dt' \left( \nu^{-2} \|v_0^3\|_{L^2}^{\frac{4n}{3}} L_{0,\nu}^{\frac{16}{3}} + \left( \nu^{-2} \|v_0^3\|_{L^2}^{\frac{4n}{3}} L_{0,\nu}^{\frac{16}{3}} \right)^{\frac{1}{2}} \right).$$

Now if $c_2$ in (5.23) is so small that

$$C \nu^{-2} \|v_0^3\|_{L^2}^{\frac{4n}{3}} L_{0,\nu}^{\frac{16}{3}} + \left( \nu^{-2} \|v_0^3\|_{L^2}^{\frac{4n}{3}} L_{0,\nu}^{\frac{16}{3}} \right)^{\frac{1}{2}} \lesssim \frac{1}{2},$$

we can deduce from (5.26) that

$$\|v^3(t)\|_{L^2}^2 + \int_0^t \|\nabla \nu v^3(t')\|_{L^2}^2 \, dt' \lesssim \frac{3}{2} \|v_0^3\|_{L^2}^2 \quad \text{for any} \quad t \leq T^*_\nu.$$

Together with (5.24), this inequality contradicts with the definition of $T^*_\nu$ given by (5.21). This in turn shows that $T^*_\nu = T^*_\nu$. Moreover, it follows from (1.9) and (1.13) that

$$\int_0^{T^*_\nu} \|v^3(t)\|_{H^\frac{1}{2}, \nu}^p \, dt \lesssim \sup_{t \in [0, T^*_\nu]} \left( \|\partial_3 v^3(t)\|_{H^\frac{1}{2}, \nu}^{\frac{p(1-\theta)}{2}} \|v^3(t)\|_{H^\frac{1}{2}, \nu}^{\frac{p(1-\theta)}{2}} \right) \int_0^{T^*_\nu} \|\partial_3^2 v^3(t)\|_{H^\nu} \, dt$$

(5.27)

$$+ \sup_{t \in [0, T^*_\nu]} \|v^3(t)\|_{H^\frac{1}{2}, \nu}^{p-2} \int_0^{T^*_\nu} \|\nabla \nu v^3(t)\|_{L^2}^2 \, dt \lesssim \nu^{-4} L_{0,\nu}^{\frac{4p(1-\theta)}{3}} \|v_0^3\|_{L^2}^{\frac{p(1-\theta)}{3}} + \|v_0^3\|_{L^2}^{\frac{p\eta}{2}}.$$

This shows that $T^*_\nu = \infty$ and (1.9) holds for any $t < \infty$. This completes the proof of Theorem 1.3.
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