EXISTENCE OF INVARIANT NORMS IN $p$-ADIC REPRESENTATIONS OF $GL_2(F)$ OF LARGE WEIGHTS

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ABSTRACT. In [BS07] Breuil and Schneider formulated a conjecture on the equivalence of the existence of invariant norms on certain $p$-adically locally algebraic representations of $GL_n(F)$ and the existence of certain de-Rham representations of $Gal(F/F)$, where $F$ is a finite extension of $Q_p$. In [Bre03b, DI13] Breuil and de Ieso proved that in the case $n = 2$ and under some restrictions, the existence of certain admissible filtrations on the $\phi$-module associated to the two-dimensional de-Rham representation of $Gal(F/F)$ implies the existence of invariant norms on the corresponding locally algebraic representation of $GL_2(F)$. In [Bre03b, DI13], there is a significant restriction on the weight - it must be small enough. In [CEG+13] the conjecture is proved in greater generality, but the weights are still restricted to the extended Fontaine-Laffaille range. In this paper we prove that in the case $n = 2$, even with larger weights, under some restrictions, the existence of certain admissible filtrations implies the existence of invariant norms.

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1. Introduction, Notation and Main Results

1.1. Introduction. Let $p$ be a prime number. Let $F$ be a finite extension of $\mathbb{Q}_p$, and let $C$ be a finite extension of $\mathbb{Q}_p$ which is “large enough” in a precise way to be
defined in Section 2. This paper lies in the framework of the $p$-adic local Langlands programme, whose goal is to associate to certain $n$-dimensional continuous $p$-adic representations of $\text{Gal}(\overline{F}/F)$, certain representations of $G = \text{GL}_n(F)$.

If $F = \mathbb{Q}_p$ and $n = 2$, then this is essentially well understood - one has a correspondence $V \mapsto \Pi(V)$ ([Col10], [Pas13], [CDP13]) associating to a 2-dimensional $C$-representation $V$ of $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$, a unitary admissible representation of $\text{GL}_2(\mathbb{Q}_p)$. This correspondence is compatible with the classical local Langlands correspondence and with completed Tate cohomology ([Eme10]).

Other cases seem somewhat more delicate. In particular, Breuil and Schneider have formulated in [BS07] a conjecture, generalizing a previous conjecture of Schneider and Teitelbaum [STUS06], which reveals a deep connection between the category of $n$-dimensional continuous de-Rham representations of $\text{Gal}(\overline{F}/F)$, and certain locally algebraic representations of $\text{GL}_n(F)$.

By the theory of Colmez and Fontaine ([CF00]), one knows that a de-Rham representation of $\text{Gal}(\overline{F}/F)$, $V$, is equivalent to a vector space, $D = D_{dR}(V)$, equipped with an action of the Weil-Deligne group of $F$ and a filtration, such that the filtration and the action satisfy a certain relation called weak admissibility. To this object, called the filtered $(\phi,N)$-module attached to $V$, one can associate a smooth representation $\pi$ of $\text{GL}_n(F)$ by a slight modification of the classical local Langlands correspondence ([BS07], p. 16-17). On the other hand, the Hodge-Tate weights of the filtration give rise to an irreducible algebraic representation of $\text{GL}_n(F)$, which we denote by $\rho$. The Breuil-Schneider conjecture essentially says that the existence of a weakly admissible filtration on $D$ must be equivalent to the existence of a $\text{GL}_n(F)$-invariant norm on the locally algebraic representation $\rho \otimes \pi$. We mention that partial results, in this generality, have been obtained by Hu ([Hu09]), who proved that the existence of an invariant norm on $\rho \otimes \pi$ implies the existence of a weakly admissible filtration on $D$, and Sorensen ([Sor13]), who proved the equivalence when $\pi$ is essentially discrete series.

In this paper we consider the particular case where $n = 2$, and the representation of the Galois group is crystalline. Let $D$ be a $\phi$-module of rank 2 over $F \otimes_{\mathbb{Q}_p} C$, equipped with a weakly admissible filtration. Imposing some additional technical restrictions on the weights of the filtration and on the smooth part, we show in this paper that the locally algebraic representation $\Pi(D)$ associated to $D$ according to the above process admits a $G$-invariant norm. The methods we employ in order to prove this result are well-known and were previously employed by Breuil ([Bre03b]) and de Ieso ([DI13]). The novelty of this paper is the extension of these methods to larger weights, even though this is accompanied by a substantial restriction on the smooth representation, $\pi$.

We remark that in [CEG+13], the authors have proved many cases of the conjecture formulated by Breuil and Schneider, using global methods. However, the results we obtain in this paper are not included in their work, as they restrict the weights to be in the extended Fontaine-Laffaille range, which, for $n = 2$, means that the weight is small.

1.2. Notation. Let $p$ be a prime number. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and a finite extension $F$ of $\mathbb{Q}_p$, contained in $\overline{\mathbb{Q}}_p$. Denote by $\mathcal{O}_F$ the ring of integers of $F$, by $\mathfrak{p}_F$ its maximal ideal, and by $\kappa_F = \mathcal{O}_F/\mathfrak{p}_F$ its residue field. We also fix a uniformizer $\varpi = \varpi_F \in \mathfrak{p}_F$. 

Denote by $C$ a finite extension of $\mathbb{Q}_p$, satisfying $|S| = [F : \mathbb{Q}_p]$, where $S := Hom_{alg}(F, C)$, and containing a square root of $\sigma(\varpi)$ for every $\sigma \in S$.

Denote by $\mathcal{O}_C$ the ring of integers of $C$, by $p_C$ its maximal ideal, and by $\kappa_C = \mathcal{O}_C/p_C$ its residue field. We also fix a uniformizer $\varpi = \varpi_C \in p_C$.

We denote $f = |\kappa_F : \mathbb{F}_p|$, $q = p^f$ the size of the residue field, and by $e$ we denote the ramification index of $F$ over $\mathbb{Q}_p$, so that $[F : \mathbb{Q}_p] = ef$ and $\kappa_F \simeq \mathbb{F}_q$. We denote by $F_0 = \text{Frac}(W(\kappa_F))$ the maximal unramified subfield of $F$, and by $\varphi_0$ the absolute Frobenius of degree $p$ in $\text{Gal}(F_0/\mathbb{Q}_p)$. We denote by $\text{Gal}(\overline{F}/F)$ the Galois group of $F$ and by $W(\overline{F}/F)$ its Weil group. Class field theory gives rise to a homomorphism $\text{rec} : W(\overline{F}/F)^{ab} \to F^\times$ (Artin reciprocity map) which we normalize by sending the coset of the arithmetic Frobenius to $\varpi^{-1}\mathcal{O}_F^\times$.

Denote by $v = v_F$ the $p$-adic valuation on $\mathbb{Q}_p$ normalized by $v_F(\varpi) = 1$. If $x \in \overline{F}$, we let $|x| = q^{-e v(x)}$. If $\lambda \in \kappa_F$, we denote by $[\lambda]$ the Teichmüller representative of $\lambda$ in $\mathcal{O}_F$. If $\mu \in C^\times$, we denote by $\text{nr}(\mu) : F^\times \to C^\times$ the unramified character sending $\varpi$ to $\mu$.

Denote by $G$ the algebraic group $\text{GL}_2$ defined over $\mathcal{O}_F$, and let $G = G(F)$ be its $F$-points.

Let $B$ be the Borel subgroup of $G$ consisting of upper triangular matrices, and let $B = B(F)$ be its $F$-points.

Let $N$ be the unipotent radical of $B$, and let $N = N(F)$ be its $F$-points.

Let $K$ be the group $\text{GL}_2(\mathcal{O}_F)$, which is, up to conjugation, the unique maximal compact subgroup of $G$. Let $I$ be the Iwahori subgroup of $K$ corresponding to $B$, and let $I(1)$ be its pro-$p$-Iwahori.

Recall that the reduction mod $p_F$ induces a surjective homomorphism

$$\text{red} : K \to G(\kappa_F)$$

and that $I = \text{red}^{-1}(B(\kappa_F))$ and $I(1) = \text{red}^{-1}(N(\kappa_F))$.

We denote by $Z \simeq F^\times$ the center of $G$, and denote

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \alpha w = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}.$$

If $\lambda \in \mathcal{O}_F$, we denote

$$w_\lambda = \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}.$$

If $n = (n_\sigma)_{\sigma \in S}, m = (m_\sigma)_{\sigma \in S}$ are elements of $\mathbb{Z}_+^S$, we write:

(i) $n^! = \prod_{\sigma \in S} n_\sigma!$,
(ii) $|n| = \sum_{\sigma \in S} n_\sigma$,
(iii) $n - m = (n_\sigma - m_\sigma)_{\sigma \in S}$,
(iv) $n \leq m$ if $n_\sigma \leq m_\sigma$ for all $\sigma \in S$,
(v) $\frac{n}{m} = \frac{\prod n_\sigma}{\prod m_\sigma}$,
(vi) If $z \in \mathcal{O}_F$, we write $z^n = \prod_{\sigma \in S} \sigma(z)^{n_\sigma}$.

1.3. **Main Results.** We fix $(\lambda_1, \lambda_2) \in C^\times \times C^\times$ such that $\lambda_1 \lambda_2^{-1} \notin \{q^2, 1\}$ and $\underline{k} \in \mathbb{Z}_+^S$. Denote

$$S^+ = \{ \sigma \in S \mid k_\sigma \neq 0 \} \subseteq S$$

We also fix some $i \in S$, and partition $S^+$ according to the action of $\sigma \in S^+$ on the residue field. More precisely, for each $l \in \{0, \ldots, f - 1\}$, denote

$$J_l = \{ \sigma \in S^+ \mid \sigma([\zeta]) = i \circ \varphi_0^l([\zeta]) \} \quad \forall \zeta \in \kappa_F \}.$$
For example, if $F$ is unramified, then $|J_l| \leq 1$ for all $l$.
If $i \in \mathbb{Z}$, we denote by $\overline{i}$ the unique representative of $i \mod f$ in $\{0, \ldots, f-1\}$. For $\sigma \in J_l$, we denote
$$v_{\sigma} = \inf \{ i \mid 1 \leq i \leq f, \ J_{l+i} \neq \emptyset \}$$
that is, the smallest power of Frobenius $\varphi_0$ that is needed to pass from $J_l$ to another, nonempty $J_k$.
We denote by $\chi : GL_2(F) \to F^\times$ the character defined by
$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto \varphi_0^{-v_F(ad-bc)}$$
For $k \in \mathbb{Z}_{\geq 0}$, we denote by $\rho_k$ the irreducible algebraic representation of $G$ of highest weight $\text{diag}(x_1, x_2) \mapsto x_k^2$ with respect to $B$, the Borel subgroup of upper triangular matrices.
We regard it also as a representation of $G = G(F)$, and for any $\sigma \in S$, denote by $\rho_k^\sigma$ the base change of $\rho_k$ to a representation of $G \otimes_{F, \sigma} C$.
Also, for any $\sigma \in S$, we fix a square root of $\sigma(\varphi)$ and write $\rho_k^\sigma = \rho_k^\sigma \otimes_C (\sigma \circ \chi)^k$.
For $k \in \mathbb{Z}_{\geq 0}$, we write
$$\rho_k = \bigotimes_{\sigma \in S} \rho_k^\sigma, \quad \rho_k^\sigma = \bigotimes_{\sigma \in S} \rho_k^\sigma$$
Let $T$ be the standard maximal torus of $B$ consisting of diagonal matrices, and let $T = T(F)$.

**Definition 1.1.** Let $\theta : T \to C^\times$ be a $C$-character of $T$ inflated to $B$, via $T \simeq B/N$.
The smooth principal series representation corresponding to $\theta$ is
$$\text{Ind}_{\theta}^G(\theta) = \left\{ f : G \to C \mid \exists U_f \text{ open s.t. } f(bgk) = \theta(b)f(g) \right\}$$
with the group $G$ acting by right translations, namely $(gf)(x) = f(xg)$ for all $x, g \in G$ and $f \in \text{Ind}_{\theta}^G(\theta)$.
Finally, we denote by
$$\pi = \text{Ind}_{\theta}^G(\text{nr}(\lambda_1^{-1}) \otimes \text{nr}(\lambda_2^{-1}))$$
the smooth unramified parabolic induction.
Note that the hypothesis on $(\lambda_1, \lambda_2)$ assures us that $\pi$ is irreducible.
We shall from now on consider the irreducible locally algebraic representations of the form $\rho_k \otimes \pi$.
Note that $\rho_k$ is not the most general irreducible algebraic representation of $G$, as it can be twisted by a power of the determinant.
However, for the purpose of existence of $G$-invariant norms, a twist by a power of the determinant is equivalent to a twist by a power of $\chi$, which can be then absorbed by $\pi$ into the values of $\lambda_1, \lambda_2$.
The Breuil-Schneider conjecture can be reformulated as follows (see [DT13]).

**Conjecture 1.1.** The following two statements are equivalent:

(i) The representation $\rho_k \otimes \pi$ admits a $G$-invariant norm, i.e. a $p$-adic norm such that $\|gv\| = \|v\|$ for all $g \in G$ and $v \in \rho_k \otimes \pi$.

(ii) The following inequalities are satisfied:
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• $v_F(\lambda_1^{-1}) + v_F(\lambda_2^{-1}) + |k| = 0$
• $v_F(\lambda_2^{-1}) + |k| \geq 0$
• $v_F(q\lambda_1^{-1}) + |k| \geq 0$

The implication $(ii) \Rightarrow (i)$ of Conjecture 1.1 follows from the work of Hu, which shows it in full generality (for $GL_n(F)$) in [Hu09], using a result of Emerton ([E+05], Lemma 1.6).

It remains to show $(ii) \Rightarrow (i)$. The case $\lambda_1 \in O_C^\times$ (resp. $q\lambda_2 \in O_C^\times$) is treated in [DI13, Prop. 4.10] hence we may assume that $\lambda_1, q\lambda_2 \notin O_C^\times$.

In [Bre03b, DI13] Breuil and de Ieso represent $\rho_k \otimes \pi$ as a quotient of a compact induction.

We briefly recall the definition of locally algebraic compact induction.

**Definition 1.2.** Let $G$ be a topological group, and let $H$ be a closed subgroup. Let $R$ be either $O_C$ or $C$. Let $(\pi, V)$ be an $R$-linear representation of $H$ over a free $R$-module of finite rank $V$. We denote by $\text{ind}_H^G\pi$ or by $\text{ind}_H^GV$ the locally algebraic compact induction of $(\pi, V)$ from $H$ to $G$. The space of the representation is

1. $\text{ind}_H^G\pi = \left\{ f : G \to V \mid f(hg) = \pi(h)f(g) \quad \forall h \in H \right\}$

and $G$ acts on $\text{ind}_H^G\pi$ by right translation, i.e. $(gf)(x) = f(xg)$ for all $g, x \in G$.

Then

$$\rho_k \otimes \pi \simeq \frac{\text{ind}_K^Z\rho_k}{(T-a)\text{ind}_K^Z\rho_k} =: \Pi_{k,a}$$

where $a = \lambda_1 + q\lambda_2 \in p_C : \rho_k^0$ is an $O_C$-lattice in $\rho_k$, $\text{ind}_K^Z\rho_k$ denotes the compact induction, and $T$ is the usual Hecke operator [BL+94].

We then have a natural map

$$\theta : \frac{\text{ind}_K^Z\rho_k^0}{(T-a)\text{ind}_K^Z\rho_k^0} \to \Pi_{k,a}$$

whose image is denoted by $\Theta_{k,a}$.

This is a sub-$O_C[K]$-module of finite type which generates $\rho_k \otimes \pi$ over $C$.

Proving Conjecture 1.1 is then equivalent to proving that $\Theta_{k,a}$ is separated, i.e. does not contain a $C$-line (see [E+05, Prop. 1.17]). In this paper, we prove that this is the case, for some additional values of $k$ and $a$.

This generalizes the previous works of Breuil and de Ieso in [Bre03b, DI13], using similar methods.

In fact, de Ieso proves the following theorem:

**Theorem 1.2.** We follow the preceding notations. The morphism $\theta$ is injective if and only if the following two conditions are satisfied:

(i) For all $l \in \{0, \ldots, f-1\}$, $|J_l| \leq 1$.
(ii) For all $\sigma \in J_l$

$$k_\sigma + 1 \leq p^{|\sigma|}.$$
As a corollary, it follows that under these conditions $\Theta_{k,a}$ is separated.

In this paper, we prove that even in some cases where $\theta$ is not injective, the lattice $\Theta_{k,a}$ is still separated. Namely, we prove the following theorem:

**Theorem 1.3.** We follow the preceding notations. Assume that $|S^+| = 1$, denote by $\sigma$ the unique element in $S^+$, and let $k = k_\sigma = d \cdot q + r$, with $0 \leq r < q$. Assume that one of the following three conditions is satisfied:

1. $k \leq \frac{1}{2}q^2$ with $r < q - d$ and $v_F(a) \in [0,1]$.
2. $k \leq \frac{1}{2}q^2$ with $2v_F(a) - 1 \leq r < q - d$ and $v_F(a) \in [1,e]$.
3. $k \leq \min (p \cdot q - 1, \frac{1}{2}q^2)$, $d - 1 \leq r$ and $v_F(a) \geq d$.

Then $\Theta_{k,a}$ is separated.

Therefore, these conditions on $k,a$ ensure the existence of a $G$-invariant norm on $\mathcal{O}_F \otimes \pi$, establishing new cases of Conjecture 1.1.

**Example 1.4.** Here are a couple of explicit examples for the established new cases:

1. Let $p \neq 2$, $k = \frac{1}{2}(q^2 - 1)$ and $v_F(a) \in [0, \min(e, 2^{q+1})]$. Then, as $k = \frac{1}{2}(q - 1)q + \frac{1}{2}(q - 1)$, we see that $d = r = \frac{1}{2}(q - 1)$, hence $2v_F(a) - 1 \leq 2 \cdot \frac{q + 1}{4} - 1 = \frac{1}{2}(q - 1) = r < q - d = \frac{1}{2}(q + 1)$, so either (i) or (ii) in Theorem 1.3 is satisfied, showing that the lattice $\Theta_{k,a}$ is separated in this case.

2. Let $q = p \neq 2$, $k = \frac{1}{2}(p^2 - 1)$ and $v_F(a) \geq \frac{1}{2}(p - 1)$. As in the previous example, $d = r = \frac{1}{2}(p - 1)$, hence $d - 1 \leq r$, and $v_F(a) \geq d$. This shows that condition (iii) in Theorem 1.3 is satisfied, showing that the lattice $\Theta_{k,a}$ is separated in this case.

2. Preliminaries

2.1. The Bruhat-Tits Tree. We refer to [Bre03a] and [Ser80] for further details concerning the construction and properties of the Bruhat-Tits tree of $G$.

Let $\mathcal{T}$ be the Bruhat-Tits tree of $G$: its vertices are in equivariant bijection with the left cosets $G/KZ$.

The tree $\mathcal{T}$ is equipped with a combinatorial distance, and $G$ acts on it by isometries.

We denote by $s_0$ the standard vertex, corresponding to the trivial class $KZ$.

Equivalently, as the vertices are in equivariant bijection with homothety classes of lattices in $F^2$, $s_0$ corresponds to the homothety class of the lattice $\mathcal{O}_F \oplus \mathcal{O}_F$.

For $n \geq 0$, we call the collection of vertices in $\mathcal{T}$ at distance $n$ from the standard vertex $s_0$, the circle of radius $n$.

Recall that we have the Cartan decomposition

$$G = \prod_{n \in \mathbb{N}} KZ \alpha^{-n} KZ = \left( \prod_{n \in \mathbb{N}} IZ \alpha^{-n} KZ \right) \prod_{n \in \mathbb{N}} IZ \beta \alpha^{-n} KZ.$$

In particular, for any $n \in \mathbb{N}$, the classes of $KZ \alpha^{-n} KZ / KZ$ correspond to vertices $s_i$ of $\mathcal{T}$ such that $d(s_i, s_0) = n$. Denote $I_0 = \{0\}$, and for any $n \in \mathbb{N}_{>0}$

$$I_n = \{ [\mu_0] + \mathcal{O}_F | [\mu_1] + \mathcal{O}_F | \ldots \mathcal{O}_F | [\mu_{n-1}] + \mathcal{O}_F | (\mu_0, \ldots, \mu_{n-1}) \in k_F^1 \} \subseteq \mathcal{O}_F$$

is a set of representatives for $\mathcal{O}_F / \mathcal{O}_F$. 

For \( n \in \mathbb{N} \) and \( \mu \in I_n \), we denote :
\[
g^0_{n,\mu} = \begin{pmatrix} \varpi^n & \mu \\ 0 & 1 \end{pmatrix}, \quad g^1_{n,\mu} = \begin{pmatrix} 1 & 0 \\ \varpi \varpi^{n+1} & 1 \end{pmatrix}.
\]

We note that \( g^0_{0,0} \) is the identity matrix, \( g^1_{0,0} = \alpha \) and that, for all \( n \in \mathbb{N} \) and any \( \mu \in I_n \), we have \( g^1_{n,\mu} = \beta g^0_{n,\mu} \). Then, \( g^0_{n,\mu} \) and \( g^1_{n,\mu} \) define a system of representatives for \( G/KZ \):
\[
G = \left( \prod_{n \in \mathbb{N}, \mu \in I_n} g^0_{n,\mu} KZ \right) \bigg/ \left( \prod_{n \in \mathbb{N}, \mu \in I_n} g^1_{n,\mu} KZ \right).
\]

For \( n \in \mathbb{N} \) we denote
\[
S^0_n = IZ\alpha^{-n}KZ = \prod_{\mu \in I_n} g^0_{n,\mu} KZ, \quad S^1_n = IZ\beta^{-n}KZ = \prod_{\mu \in I_n} g^1_{n,\mu} KZ
\]
and we let \( S_n = S^0_n \bigcup S^1_n \) and \( B_n = B^0_n \bigcup B^1_n \), where \( B^0_n = \bigsqcup_{m \leq n} S^0_m \) and \( B^1_n = \bigsqcup_{m \leq n} S^1_m \).

In particular, we have \( S_0 = KZ \bigcup I \alpha KZ \).

**Remark 2.1.** Recall, as in [Bre83, DIT3], that \( S^0_n \bigcup S^1_{n-1} \) (resp. \( B^0_n \bigcup B^1_{n-1} \)) is the collection of vertices in \( T \) at distance \( n \) (resp. at most \( n \)) from \( s_0 \). Similarly, \( S^1_n \bigcup S^0_{n-1} \) (resp. \( B^1_n \bigcup B^0_{n-1} \)) is the collection of vertices in \( T \) at distance \( n \) (resp. at most \( n \)) from \( \alpha s_0 \).

We denote by \( R \) either the field \( C \) or its ring of integers \( \mathcal{O}_C \). Let \( \sigma \) be a continuous \( R \)-linear representation of \( KZ \) on a free \( R \)-module of finite rank \( V_\sigma \). We denote by \( ind^G_{KZ} \sigma \) the \( R \)-module of functions \( f : G \to V_\sigma \) compactly supported modulo \( Z \), such that
\[
f(\kappa g) = \sigma(\kappa) f(g) \quad \forall \kappa \in KZ, g \in G
\]
with \( G \) acting by right translations, i.e. \( (g \cdot f)(g') = f(g'g) \).

As in [BL+94], for \( g \in G, \ v \in V_\sigma \), we denote by \( [g, v] \) the element of \( ind^G_{KZ} \sigma \) supported on \( KZg^{-1} \) and such that \( [g, v](g^{-1}) = v \).

Then we have
\[
\forall g, g' \in G, \ v \in V_\sigma \quad g \cdot [g', v] = [gg', v]
\]
\[
\forall g \in G, \ \kappa \in KZ, \ v \in V_\sigma \quad [g \kappa, v] = [g, \sigma(\kappa)v]
\]

We can think of \( ind^G_{KZ} \sigma \) as a vertex coefficient system on \( T \), having \( \sigma \) as the module on each vertex, identifying \( [g, v] \) with the vector \( v \) at the vertex corresponding to \( g \), i.e. identifying vertex \( g \) with \( KZg^{-1} \). Note that the choice of representative for \( gKZ \) affects the choice of vector \( v \in \sigma \).

Recall the following result ([BL+94], §2), which gives a basis for the \( R[G] \)-module \( ind^G_{KZ} \sigma \).

**Proposition 2.1.** Let \( B \) be a basis for \( V_\sigma \) over \( R \), and let \( \mathcal{G} \) be a system of representatives for left cosets of \( G/KZ \). Then the family of functions \( \mathcal{I} := \{ [g, v] \mid g \in G, v \in B \} \) forms a basis for \( ind^G_{KZ} \sigma \) over \( R \).

**Remark 2.2.** The representation \( ind^G_{KZ} \sigma \) is isomorphic to the representation of \( G \) given by the \( R[G] \)-module \( R[G] \otimes_{R[KZ]} V_\sigma \). More precisely, if \( g \in G \) and \( v \in V_\sigma \), then the element \( g \otimes v \) corresponds to the function \([g, v] \).
From proposition 2.1 and the decomposition 3, any function \( f \in \text{ind}^G_K \sigma \) can be written uniquely as a finite sum of the form
\[
f = \sum_{n=0}^{n_0} \sum_{\mu \in I_n} (g^0_{n,\mu} v^0_{n,\mu} + g^1_{n,\mu} v^1_{n,\mu})
\]
with \( v^0_{n,\mu}, v^1_{n,\mu} \in V_\sigma \), and where \( n_0 \) is a non-negative integer, which depends on \( f \).
We call the support of \( f \) the collection of \( g^i_{n,\mu} \) such that \( v^i_{n,\mu} \neq 0 \). We write \( f \in S_n \) (resp. \( B_n, S^0_n, \) etc.) if the support of \( f \) is contained in \( S^n \) (resp. \( B_n, S^0_n, \) etc.).
We write \( f \in B_0 \) if the support of \( f \) is contained in \( B_0 \) for some \( n \), and \( f \in B_1 \) if the support of \( f \) is contained in \( B_1 \) for some \( n \).
Let \( \pi \) be a continuous \( R \)-linear representation of \( G \) over an \( R \)-module. From [BL+94], we have a canonical isomorphism of \( R \)-modules
\[
\text{Hom}_R(G(\text{ind}^G_K \sigma, \pi) \cong \text{Hom}_R(K \sigma, \pi |_{KZ})
\]
which translates to the fact that the functor of compact induction \( \text{ind}^G_K \) is left adjoint to the restriction functor, and is called compact Frobenius reciprocity.

2.2. Hecke Algebras. Let \( \sigma \) be a continuous \( R \)-linear representation of \( KZ \) over a free \( R \)-module \( V_\sigma \) of finite rank. The Hecke algebra \( \mathcal{H}(KZ, \sigma) \) associated to \( KZ \) and \( \sigma \) is the \( R \)-algebra defined by
\[
\mathcal{H}(KZ, \sigma) = \text{End}_R(G(\text{ind}^G_K \sigma))
\]
We can interpret \( \mathcal{H}(KZ, \sigma) \) as a convolution algebra. In fact, denote by \( \mathcal{H}_{KZ}(\sigma) \) the \( R \)-module of functions \( \varphi : G \to \text{End}_R(V_\sigma) \) compactly supported modulo \( Z \), such that
\[
\forall \kappa_1, \kappa_2 \in KZ, \ \forall g \in G, \ \varphi(\kappa_1 g \kappa_2) = \sigma(\kappa_1) \circ \varphi(g) \circ \sigma(\kappa_2).
\]
This is a unitary \( R \)-algebra with the convolution product defined, for all \( \varphi_1, \varphi_2 \in \mathcal{H}_{KZ}(\sigma) \) and all \( g \in G \), by the following formula:
\[
\varphi_1 \ast \varphi_2(g) = \sum_{xKZ \in G/KZ} \varphi_1(x) \circ \varphi_2(x^{-1} g).
\]
It admits as a unit element the function \( \varphi_c = [1, id] \) defined by
\[
\varphi_c(g) = \begin{cases} \sigma(g) & g \in KZ \\ 0 & \text{else} \end{cases}
\]
One may verify that the bilinear map
\[
\mathcal{H}_{KZ}(\sigma) \times \text{ind}^G_K \sigma \rightarrow \text{ind}^G_K \sigma
\]
\( (\varphi, f) \mapsto T_\varphi(f) := \sum_{xKZ \in G/KZ} \varphi(x)(f(x^{-1} g)) \)
equips \( \text{ind}^G_K \sigma \) with the structure of a left \( \mathcal{H}_{KZ}(\sigma) \)-module, which commutes with the action of \( G \).
The following Lemma is well known, see e.g. [DI13, Lemma 2.4].

Lemma 2.2. The map
\[
\mathcal{H}_{KZ}(\sigma) \rightarrow \mathcal{H}(KZ, \sigma)
\]
\( \varphi \mapsto T_\varphi(f) \)
is an isomorphism of $R$-algebras. In particular, if $g \in G$, and if $v \in V_\sigma$, the action of $T_\varphi$ on $[g,v]$ is given by

$$T_\varphi([g,v]) = \sum_{xKZ \in G/KZ} [gx, \varphi(x^{-1})(v)].$$

We assume now that $R = C$. Denote by the trivial representation of $KZ$ and assume that $\sigma$ is the restriction to $KZ$ of a locally analytic representation (in the sense of [ST03, ST02]) of $G$ on $V_\sigma$. By [STUS06], the map

$$\iota_\sigma : H_{KZ}(1) \rightarrow H_{KZ}(\sigma) \quad \varphi \mapsto \varphi(g) := \varphi(g)\sigma(g)$$

is then an injective homomorphism of $C$-algebras. Before we state a condition assuring the bijectivity of $\iota_\sigma$, we recall the existence of a $Q_p$-linear action of the Lie algebra $g$ of $G$ on the space $V_\sigma$ defined by

$$\forall x \in g, \forall v \in V_\sigma, \quad xv = \frac{d}{dt}\exp(tx)v \big|_{t=0}$$

where $\exp : g \rightarrow G$ denotes the exponential map defined locally in the neighbourhood of $0$ ([ST02, §2]).

This action is extended to an action of the Lie algebra $g \otimes Q_p$, and allows de Ieso to obtain the following result: (see [DI13, Lemma 4.2.5])

**Lemma 2.3.** If the $g \otimes Q_p$-module $V_\sigma$ is absolutely irreducible, then the map $\iota_\sigma$ is bijective.

### 3. Representations of $GL_2(F)$

#### 3.1. $Q_p$-algebraic representations of $GL_2(F)$

For $k \in \mathbb{N}$, we denote by $\rho_k$ the irreducible algebraic representation of $G$ of highest weight $\text{diag}(x_1^k, x_2^k)$ with respect to $B$, and we consider it also as a representation of $G = G(F)$.

For $\sigma \in S$, we denote by $\rho_{k,\sigma}$ the base change of $\rho_k$ to a representation of $G \otimes_{F,\sigma} C$.

We denote by $\chi : GL_2(F) \rightarrow F^\times$ the character defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c^{\nu_F(ad-bc)}.$$

Also, choose a square root of $\sigma(\pi)$ in $C$, and let

$$\rho_{k,\sigma}^{\chi} = \rho_{k,\sigma} \otimes_C (\sigma \circ \chi)^{\frac{1}{2}}.$$

We denote by $\chi : GL_2(F) \rightarrow F^\times$ the character defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c^{\nu_F(ad-bc)}.$$

For $\sigma \in S$ and $k \in \mathbb{N}$, we identify $\rho_{k,\sigma}^{\chi}$ with the representation of $G$ given by the $C$-vector space

$$\bigoplus_{i=0}^{k} C \cdot x_\sigma^k y_{\sigma}^i$$

of homogeneous polynomials of degree $k$ in $x_\sigma, y_\sigma$ with coefficients in $C$, on which $G$ acts by the following formula:

$$\rho_{k,\sigma}^{\chi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (x_\sigma^k y_{\sigma}^i) =$$

$$\left(\sigma \circ \chi \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{\frac{1}{2}} (\sigma(a)x_\sigma + \sigma(c)y_\sigma)^k (\sigma(b)x_\sigma + \sigma(d)y_\sigma)^i.$$
If \( w_\sigma \in \rho^\sigma \) and if \( g \in G \), we denote simply \( gw_\sigma \) for the vector obtained from letting \( g \) act on \( w_\sigma \).

**Remark 3.1.** The formula \( \varrho \) assures, in particular, that for every \( w_\sigma \in \rho^\sigma \)

\[
\left( \begin{array}{cc}
\varrho & 0 \\
0 & \varrho
\end{array} \right) w_\sigma = w_\sigma.
\]

Fix \( \mathbf{k} = (k_\sigma)_{\sigma \in S} \in \mathbb{N}^S \), and let

\[
I_\mathbf{k} = \left\{ i = (i_\sigma)_{\sigma \in S} \in \mathbb{N}^S, \quad 0 \leq i_\sigma \leq k_\sigma \quad \forall \sigma \in S \right\}.
\]

We denote by \( \rho_\mathbf{k} \) (resp. \( \rho^\mathbf{k} \)) the representation of \( G \) on the following vector space

\[
V_{\rho_\mathbf{k}} := \bigotimes_{\sigma \in S} \rho^\sigma_{k_\sigma} \quad \text{(resp. } V_{\rho^\mathbf{k}} := \bigotimes_{\sigma \in S} \rho^\sigma)\]

on which an element \( \left( \begin{array}{cccc}
a & b \\
c & d
\end{array} \right) \in G \) acts componentwise. In particular, for all \( \bigotimes_{\sigma \in S} w_\sigma \in V_{\rho_\mathbf{k}} \) we have:

\[
\rho_{\mathbf{k}} \left( \begin{array}{cccc}
a & b \\
c & d
\end{array} \right) \left( \bigotimes_{\sigma \in S} w_\sigma \right) = \bigotimes_{\sigma \in S} \left( \left( \begin{array}{cccc}
a & b \\
c & d
\end{array} \right) w_\sigma \right).
\]

These are two absolutely irreducible representations of \( G \) which remain absolutely irreducible even when we restrict them to the action of an open subgroup of \( G \) ([BS07, §2]).

For all \( i \in I_{\mathbf{k}} \), we let:

\[
e_{\mathbf{k},i} := \bigotimes_{\sigma \in S} e_{k_\sigma,i_\sigma}
\]

where, for any \( \sigma \in S \), \( e_{k_\sigma,i_\sigma} \) denotes the monomial \( x^{k_\sigma} y^{i_\sigma} \). We then denote by \( U_{\mathbf{k}} \) the endomorphism of \( V_{\rho_\mathbf{k}} \) defined by

\[
U_{\mathbf{k}} := \bigotimes_{\sigma \in S} U_{\rho^\sigma_{k_\sigma}}
\]

where \( U_{\rho^\sigma_{k}} \) denotes, for all \( \sigma \in S \) and \( k \in \mathbb{N} \), the endomorphism of \( \rho^\sigma_{k} \) given, with respect to the basis \( (e_{k,i})_{i=0}^k \), by the diagonal matrix

\[
U_{\rho^\sigma_{k}} = \begin{pmatrix}
\sigma(\varrho)^k & 0 & \cdots & 0 \\
0 & \sigma(\varrho)^{k-1} & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}
\]

In [DI13, Lemma 3.2], de Ieso proves the following Lemma.

**Lemma 3.1.** There exists a unique function \( \psi : G \to \text{End}_C(V_{\rho_\mathbf{k}}) \) supported in \( KZ\alpha^{-1}KZ \) such that:

(i) For any \( \kappa_1, \kappa_2 \in KZ \) we have \( \psi(\kappa_1 \alpha^{-1} \kappa_2) = \rho_{\mathbf{k}}(\kappa_1) \circ \psi(\alpha^{-1}) \circ \rho_{\mathbf{k}}(\kappa_2) \).

(ii) \( \psi(\alpha^{-1}) = U_{\mathbf{k}} \).
We remark that in fact, \( \psi = \rho_k|_{KZ^{-1}KZ} \), since
\[
U_k = \rho_k(\alpha^{-1})
\]
(7)

By Lemma 2.2, we know that the Hecke algebra \( H(KZ, \rho_k) \) is naturally isomorphic to the convolution algebra \( H_{KZ}(\rho_k) \) of functions \( \varphi : G \to \text{End}_C(V_{\rho_k}) \) compactly supported modulo \( Z \), such that
\[
\forall \kappa_1, \kappa_2 \in KZ, g \in G, \quad \varphi(\kappa_1 g \kappa_2) = \rho_k(\kappa_1) \circ \varphi(g) \circ \rho_k(\kappa_2).
\]

It follows that the map \( \psi \) from Lemma 3.1 corresponds to an operator \( T \in H_{KZ}(\rho_k) \) whose action on the elements \([g, v]\) for \( g \in G \) and \( v \in V_{\rho_k} \) is given by the formula (5).

Moreover,

**Remark 3.2.** A simple argument using the Bruhat-Tits tree of \( G \) shows that \( T \) is injective on \( \text{ind}_{KZ}^G \rho_k \).

### 3.2. Lattices.

We keep the notations of Section 3.1 and denote by \( \rho^\sigma,0_k \), for \( \sigma \in S \) and \( k \in \mathbb{N} \), the representation of the group \( KZ \) on the \( \mathcal{O}_C \)-module
\[
\bigoplus_{i=0}^k \mathcal{O}_C \cdot x_{\sigma}^{k-i} y_{\sigma}^i
\]
of homogeneous polynomials of degree \( k \), on which an element \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \) acts by
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x_{\sigma}^{k-i} y_{\sigma}^i) = (\sigma(a)x_{\sigma} + \sigma(c)y_{\sigma})^{k-i} (\sigma(b)x_{\sigma} + \sigma(d)y_{\sigma})^i
\]
and the matrix \( \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} \) \( \in Z \) acts as the identity. If \( w_{\sigma} \in \rho^\sigma,0_k \) and if \( g \in G \), we simply denote by \( gw_{\sigma} \) the vector obtained from letting \( g \) act on \( w_{\sigma} \).

**Definition 3.1.** Let \( V \) be a \( C \)-vector space. A lattice \( \mathcal{L} \) in \( V \) is a sub-\( \mathcal{O}_C \)-module of \( V \), such that, for any \( v \in V \), there exists a nonzero element \( a \in \mathcal{O}_C^\times \) such that \( av \in \mathcal{L} \). A lattice \( \mathcal{L} \) is called separated if \( \bigcap_{n \in \mathbb{N}} \varpi^n \mathcal{L} = 0 \), which is equivalent to demanding that it contains no \( C \)-line.

**Example 3.2.** The \( \mathcal{O}_C \)-module \( \rho^\sigma,0_k \) is a separated lattice of \( \rho^\sigma_k \), which is moreover stable under the action of \( KZ \).

**Remark 3.3.** There are many choices of possible separated lattices in \( \rho^\sigma_k \), which are stable under the action of \( KZ \). Another natural choice (and in some sense even more natural than ours), as pointed out by C. Breuil, is the lattice
\[
\bigoplus_{i=0}^k \mathcal{O}_C \cdot \frac{x_{\sigma}^{k-i} y_{\sigma}^i}{(k-i)! \cdot i!}
\]
which, in the case \( q > p \), is different from \( \rho^\sigma,0_k \). However, as using this lattice facilitates some of the technical aspects, others become more difficult. In particular, we strongly use the divisibility by powers of \( p \) of certain binomial coefficients, which is not possible when using this alternative lattice. Therefore, we have not been able to use different lattices in order to prove more cases of the conjecture. We have
further hypothesized the possibility of using different lattices for different values of $v_F(a)$, but this as well did not yield any results.

**Example 3.3.** We denote by $\rho^0_\alpha$ the representation of $KZ$ on the following space

$$V_{\rho^0_\alpha} = \bigotimes_{\sigma \in S} \rho^0_\sigma$$

on which an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in KZ$ acts via

$$\rho^0_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left( \bigotimes_{\sigma \in S} w_\sigma \right) = \bigotimes_{\sigma \in S} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} w_\sigma \right)$$

The example 3.2 assures us that the $O_C$-module $V_{\rho^0_\alpha}$ is a separated lattice of the space $V_{\rho^0_\sigma}$ constructed in Section 3.1. Therefore, the $O_C$-module $\text{ind}_{KZ}^G \rho^0_\alpha$ is also a separated lattice of $\text{ind}_{KZ}^G \rho^0_\sigma$, and is, by construction, stable under the action of $G$.

By Remark 2.2, we can deduce the existence of an injective map $H(KZ,\rho^0_\alpha) \rightarrow H(KZ,\rho^0_\sigma)$. Moreover, one verifies that the operator $T \in H(KZ,\rho^0_\alpha)$ defined in Section 3.1, induces by restriction a $G$-equivariant endomorphism of $\text{ind}_{KZ}^G \rho^0_\alpha$, which we again denote by $T$.

The following Lemma is proved in [DI13, Lemma 3.3], but for sake of completeness we include here a proof of both isomorphisms.

**Lemma 3.4.** There are isomorphisms of $O_C$-algebras $H_{\rho^0_\alpha}(KZ,G) \simeq O_C[T]$ and $H_{\rho^0_\sigma}(KZ,G) \simeq C[T]$.

**Proof.** The space $V_{\rho^0_\alpha}$ is an absolutely irreducible $g \otimes Q_\alpha$-module, hence by Lemma 2.3, $\iota_{\rho^0_\alpha}$ is an isomorphism of $C$-algebras. Lemma 2.2 shows that there exists a unique morphism of $C$-algebras $u_{\rho^0_\alpha} : H_C(KZ,G) \rightarrow H_{\rho^0_\alpha}(KZ,G)$ making the following diagram commute

$$
\begin{array}{ccc}
H_{KZ}(C) & \simeq & H_C(KZ,G) \\
\iota_{\rho^0_\alpha} & & u_{\rho^0_\alpha} \\
H_{KZ}(\rho^0_\alpha) & \simeq & H_{\rho^0_\alpha}(KZ,G) \\
\end{array}
$$

By construction, this morphism is an isomorphism of $C$-algebras. Denote by $T_1 \in H_C(KZ,G)$ the element corresponding to $1_{KZ^n,KZ} \in H_{KZ}(C)$ by Frobenius reciprocity.

If $\varphi \in H_{KZ}(C)$, then as it has compact support, by the Cartan decomposition [2], it is supported on $\bigcap_{i=0}^n KZ \alpha^{-i} KZ$ for some integer $n$. As $\varphi$ is $KZ$-bi-invariant (recall that $C$ is the trivial representation), its restriction to each $KZ \alpha^{-i} KZ$ is constant, hence we may write $\varphi = \sum_{i=0}^n \varphi_i \cdot 1_{KZ^n,KZ}$. Let $T_i \in H_C(KZ,G)$ be the operator corresponding to $1_{KZ^n,KZ}$ by Frobenius reciprocity. Then we see that the $T_n$’s span $H_C(KZ,G)$ over $C$. Geometrically, $T_n$ is the operator associating to a vertex $s$ the sum of the vertices at distance $n$ from $s$: this is because

$$1_{KZ^n,KZ} = \sum_{KZx \in KZ \backslash KZ \alpha^{-n} KZ} 1_{KZx} =$$
and then the \( x^{-1}s_0 \) are all distinct and give all vertices \( s' \in T_0 \) such that \( s' \) is \( KZ \)-equivalent to \( s_n = \alpha^{-n}s_0 \). This means that \((s_0, s')\) is equivalent to \((s_0, s_n)\), which is precisely our assertion. From the geometrical description of \( T_n \), one gets directly, since the tree \( T \) is \((q + 1)\)-regular, that

\[
T_1^2 = T_2 + (q + 1)Id
\]

\[
T_1T_{n-1} = T_n + qT_{n-2} \quad \forall n \geq 3
\]

It follows that for all \( n \), \( T_n \in \mathcal{O}_C[T_1] \) is monic of degree \( n \). In particular, \( \mathcal{H}_C(KZ, G) \cong C[T_1] \). Since \( u_n(T_1) = T \), it follows that \( \mathcal{H}_C(KZ, G) \cong C[T] \).

Let us show that the restriction of this isomorphism to \( \mathcal{H}_C(KZ, G) \) has image \( \mathcal{O}_C[T] \).

As \( T \in \mathcal{H}_C(KZ, G) \), clearly \( \mathcal{O}_C[T] \) is contained in the image. Let \( p(T) \in C[T] \) be a polynomial corresponding to an element in \( \mathcal{H}_C(KZ, G) \).

Assume \( \text{deg}(p) = n \), and let \( a_n \) be the leading coefficient, i.e. \( p(T) = a_nT^n + p_{n-1}(T) \), where \( \text{deg}(p_{n-1}) = n - 1 \). It follows that \( p(T) = a_nT_n + q_{n-1}(T) \), for some \( q \) with \( \text{deg}(q_{n-1}) = n - 1 \).

We recall that \( T_n \) is the image under the natural isomorphisms of \( 1_{KZ^\alpha^{-n}KZ} \in H_{KZ}(C) \), which maps to \( 1_{KZ^\alpha^{-n}KZ} \cdot \mu_\kappa \in H_{KZ}(\mu_\kappa) \), finally mapping to

\[
T_n([g, v]) = \sum_{x \in \mathcal{K} \cap \mathcal{G}/\mathcal{K}Z} [gx, 1_{KZ^\alpha^{-n}KZ}(x^{-1})\mu_\kappa(x^{-1})(v)] = \sum_{x \in \mathcal{K} \cap \mathcal{K}Z^\alpha^{-n}KZ/\mathcal{K}Z} [gx, \mu_\kappa(x^{-1})(v)]
\]

Since \( \alpha^n \in KZ^\alpha^{-n}KZ \), and polynomials of order less than \( n \) are supported on \( \prod_{i=0}^{n-1} KZ^\alpha^{-i}KZ \), it follows that for any \( v \in \mu_\kappa \), one has

\[ (p(T)([1, v])(\alpha^n) = (a_nT_n([1, v]))(\alpha^n) = a_n\mu_\kappa(\alpha^{-n})(v) = a_nU_{KZ}^n(v) \]

where the rightmost equality follows from \( \mathcal{U} \).

In particular, taking \( v = \bigotimes_{x:F \rightarrow C} y_{\alpha}^{h_{x,F}} \), we see that \( v \in \mu_\kappa^0 \), hence \( [1, v] \in \text{ind}_{KZ\mu_\kappa}^C \mu_\kappa^0 \). As we assume \( p(T) \in \mathcal{H}_C(KZ, G) = \text{End}_{C[C]}(\text{ind}_{KZ\mu_\kappa}^C \mu_\kappa^0) \), it follows that \( p(T)([1, v]) \in \text{ind}_{KZ\mu_\kappa}^C \mu_\kappa^0 \), hence \( a_nU_{KZ}^n(v) = (p(T)([1, v]))(\alpha^n) \in \mu_\kappa^0 \). But, by definition of \( U \), we see that \( U_{KZ}(v) = v \), hence \( a_nv \in \mu_\kappa^0 \).

However, by definition of \( \mu_\kappa^0 \), this is possible if and only if \( a_n \in \mathcal{O}_C \). Therefore, we see that \( a_nT^n \in \mathcal{O}_C[T] \), and it suffices to prove the claim for \( p(T) - a_nT^n = p_{n-1}(T) \), which is a polynomial of degree less than \( n \).

Proceeding by induction, where the induction basis consists of constant polynomials, which can be integral if and only if they belong to \( \mathcal{O}_C \), we conclude that \( p(T) \in \mathcal{O}_C[T] \). \( \Box \)
3.3. **Formulas.** We keep the notations of Sections 3.1 and 3.2. For $0 \leq m \leq n$, we denote by $[\cdot]_m : I_n \to I_m$ the “truncation” map, defined by:

$$
\left[ \sum_{i=0}^{n-1} \varpi^i [\mu_i] \right]_m = \begin{cases} 
\sum_{i=0}^{m-1} \varpi^i [\mu_i] & m \geq 1 \\
0 & m = 0
\end{cases}
$$

For $\mu \in I_n$, we denote

$$
\lambda_\mu = \mu - \frac{[\mu]_{n-1}}{\varpi^{n-1}} \in I_1
$$

so that if $\mu = \sum_{i=0}^{n-1} \varpi^i [\mu_i]$, then $\lambda_\mu = [\mu_{n-1}]$.

We then have the following two results (see [Bre03b, DI13]), where $\psi$ denotes the function defined in Lemma 3.3.

**Lemma 3.5.** Let $n \in \mathbb{N}, \mu \in I_n$, and let $v \in V_{\psi}^\phi$. We have:

$$
T \left( \left[ g^0_{n,\mu}, v \right] \right) = T^+ \left( \left[ g^0_{n,\mu}, v \right] \right) + T^- \left( \left[ g^0_{n,\mu}, v \right] \right)
$$

where

$$
T^+ \left( \left[ g^0_{n,\mu}, v \right] \right) := \sum_{\lambda \in I_1} \left[ g^0_{n+1,\mu+\varpi^{n} \lambda} \left( \rho_k (w) \circ \psi (\alpha^{-1}) \circ \rho_k (w_\lambda) \right) (v) \right]
$$

and

$$
T^- \left( \left[ g^0_{n,\mu}, v \right] \right) := \begin{cases} 
\left[ g^0_{n-1,\mu-n+1} \left( \rho_k (w_{w-\lambda_n}) \circ \psi (\alpha^{-1}) \right) (v) \right] & n \geq 1 \\
0 & n = 0
\end{cases}
$$

**Lemma 3.6.** Let $n \in \mathbb{N}, \mu \in I_n$, and let $v \in V_{\psi}^\phi$. We have:

$$
T \left( \left[ g^1_{n,\mu}, v \right] \right) = T^+ \left( \left[ g^1_{n,\mu}, v \right] \right) + T^- \left( \left[ g^1_{n,\mu}, v \right] \right)
$$

where

$$
T^+ \left( \left[ g^1_{n,\mu}, v \right] \right) := \sum_{\lambda \in I_1} \left[ g^1_{n+1,\mu+\varpi^{n} \lambda} \left( \psi (\alpha^{-1}) \circ \rho_k (w_\lambda w) \right) (v) \right]
$$

and

$$
T^- \left( \left[ g^1_{n,\mu}, v \right] \right) := \begin{cases} 
\left[ g^1_{n-1,\mu-n+1} \left( \rho_k (w_{w-\lambda_n}) \circ \psi (\alpha^{-1}) \circ \rho_k (w) \right) (v) \right] & n \geq 1 \\
\left[ Id \left( \rho_k (w) \circ \psi (\alpha^{-1}) \circ \rho_k (w) \right) (v) \right] & n = 0
\end{cases}
$$

By using the equality $g^1_{n,\mu} = \beta g^0_{n,\mu} w$, these two Lemmata yield the following two equalities:

$$
T^+ \left( \left[ g^1_{n,\mu}, v \right] \right) = \beta T^+ \left( \left[ g^0_{n,\mu}, \rho_k (w) (v) \right] \right)
$$

$$
T^- \left( \left[ g^1_{n,\mu}, v \right] \right) = \beta T^- \left( \left[ g^0_{n,\mu}, \rho_k (w) (v) \right] \right)
$$

and also the following result

**Corollary 3.7.** Let $n \in \mathbb{N}, \mu, \lambda \in I_n, i, j \in \{0, 1\}$ and $v_1, v_2 \in V_{\psi}^\phi$. If $i \neq j$ or if $\mu \neq \lambda$, then $T^+ \left( \left[ g^i_{n,\mu}, v_1 \right] \right)$ and $T^+ \left( \left[ g^j_{n,\lambda}, v_2 \right] \right)$ have disjoint supports.

The following Lemma is a simple generalization of [Bre03b, Lemma 2.2.2].
Lemma 3.8. Let \( v = \sum_{0 \leq i \leq k} c_i e_{k,i} \in V_{\rho_k} \) and \( \lambda \in \mathcal{O}_F \). We have:

\[
(\rho_k(w) \circ \psi(\alpha^{-1}) \circ \rho_k(w_\lambda))(v) = \sum_{0 \leq i \leq k} \left( \sum_{j \leq i \leq k} c_j \left( \frac{i}{j} \right)(-\lambda)^{i-j} \right) e_{k,j} \tag{10}
\]

\[
(\rho_k(ww_\lambda) \circ \psi(\alpha^{-1}))(v) = \sum_{0 \leq j \leq i \leq k} \left( \sum_{k \leq i \leq j} \sum_{k \leq j \leq i} c_k \left( \frac{i}{j} \right)(-\lambda)^{i-j} \right) e_{k,j} \tag{11}
\]

\[
\psi(\alpha^{-1})(v) = \sum_{0 \leq j \leq k} \sum_{i} \left( \frac{i}{j} \right)(-\lambda)^{i-j} e_{k,j} \tag{12}
\]

Proof. Equation (10) is proved in [DI13] and equation (12) is immediate. For equation (11), we note that by equation (6), we have for any \( \sigma \in S \) and any \( 0 \leq i_\sigma \leq k_\sigma \):

\[
(w \circ w_\lambda \circ U_{k_\sigma})(e_{k_\sigma,i_\sigma}) = \begin{pmatrix} 1 & -\lambda & 0 \end{pmatrix} \begin{pmatrix} \sigma(x) \end{pmatrix}^{k_\sigma-i_\sigma} = \begin{pmatrix} \sigma \sigma(x+y+(-\lambda)x)^{i_\sigma} = \sum_{j_\sigma=0}^{i_\sigma} \left( i_\sigma \right)(-\lambda)^{i_\sigma-j_\sigma} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \sigma(x) \end{pmatrix}^{k_\sigma-i_\sigma} = \begin{pmatrix} \sigma \sigma(x+y+(-\lambda)x)^{i_\sigma} = \sum_{j_\sigma=0}^{i_\sigma} \left( i_\sigma \right)(-\lambda)^{i_\sigma-j_\sigma} \end{pmatrix} \]

Using equation (8), we deduce that

\[
(\rho_k(ww_\lambda) \circ \psi(\alpha^{-1}))(v) = \sum_{0 \leq j \leq k} \sum_{i} \left( \frac{i}{j} \right)(-\lambda)^{i-j} e_{k,j} \]

This leads to the following corollary, which is a simple generalization of [Bre03b], Corollary 2.2.3.

Corollary 3.9. Let \( m \in \mathbb{Z}_{>0}, a \in C \), and for any \( \mu \in I_m \) (resp. \( \mu \in I_{m-1} \), resp. \( \mu \in I_{m+1} \), \( v_\mu = \sum_{0 \leq i \leq k} c_{m,i} \cdot e_{k,i} \) (resp. \( v_{m-1} = \sum_{0 \leq i \leq k} c_{m-1,i} \cdot e_{k,i} \), resp. \( v_{m+1} = \sum_{0 \leq i \leq k} c_{m+1,i} \cdot e_{k,i} \)) an element of \( \rho_k \). We denote

\[
f_m = \sum_{\mu \in I_m} [g_{m,\mu}, v_\mu]
\]
Then

\[ T^-(f_{m+1}) + T^+(f_{m-1}) - af_m = \sum_{\mu \in I} g_0^{m,\mu} \sum_{0 \leq \lambda \leq k} \gamma_0^{m,\mu \lambda} \cdot [\lambda]^{k-\lambda} + \lambda^{-1} \cdot \sum_{0 \leq \lambda \leq k} c_{m,\mu - 1}^{m,\mu - 1} \left( \frac{i}{j} \right) (-\lambda_j)^{k-\lambda} - ac_{m,\mu}^{m,\mu} \]

where

\[ c_{m,\mu}^{m,\mu} = \sum_{0 \leq \lambda \leq k} \gamma_0^{m,\lambda} \cdot \sum_{0 \leq \lambda \leq k} \left( \frac{i}{j} \right) (-\lambda_j)^{k-\lambda} - ac_{m,\mu}^{m,\mu} \]

4. A Criterion for Separability

4.1. The main result. We adhere to the notations of Sections 3.2 and 3.3 and fix an embedding \( \iota : F \hookrightarrow C \). Denote

\[ S^+ = \{ \sigma \in S \mid k_\sigma \neq 0 \} \subseteq S \]

We partition \( S^+ \) with respect to the action of \( \sigma \in S^+ \) on the residue field of \( F \). More precisely, for any \( l \in \{0, \ldots, f - 1\} \), we let

\[ J_l = \{ \sigma \in S^+ \mid \sigma(\lambda) = \iota \circ \varphi_0^l(\lambda) \quad \forall \lambda \in I_1 \} \]

where \( I_1 = \{ [\zeta] \mid \zeta \in \kappa_F \} \). In particular, we remark that

\[ \prod_{l=0}^{f-1} J_l = S^+, \quad \forall l \in \{0, \ldots, f - 1\}, \quad |J_l| \leq e. \]

For any integer \( i \in \mathbb{Z} \), we denote by \( \overline{i} \) the unique representative of \( i \mod f \) in \( \{0, \ldots, f - 1\} \). We also let, for any \( \sigma \in J_1 \), \( \gamma_\sigma := l \) and

\[ v_\sigma = \inf \{ i \mid 1 \leq i \leq f, J_{i+1} \neq \emptyset \} \]

that is the minimal power of Frobenius \( \varphi_0 \) needed to pass from \( J_l \) to another nonempty \( J_k \).

Let \( a \in \mathfrak{p}_C \). We let

\[ \Pi_{K,a} = \frac{ind_{\rho_K}^{\mathcal{G}_K}}{(T-a)(ind_{\rho_K}^{\mathcal{G}_K})}. \]

This is a locally algebraic representation of \( G \), which can be realized as the tensor product of an algebraic representation with a smooth representation. More precisely, we have the following result, which is stated in [DI13].

**Proposition 4.1.** Let \( u_\sigma = \frac{k_\sigma}{q} \) for any \( \sigma \in S \).

(i) If \( a \notin \{ \pm(q+1)\varphi_0 \} \), then \( \Pi_{K,a} \) is algebraically irreducible and

\[ \Pi_{K,a} \simeq \rho_\lambda \otimes Ind_{\mathcal{G}}^G(nr(\lambda_1^{-1}) \otimes nr(\lambda_2^{-1})) \]

where \( \lambda_1, \lambda_2 \) satisfy

\[ \lambda_1 \lambda_2 = \varphi, \quad \lambda_1 + q \lambda_2 = a \]
(ii) If \(a \in \{\pm ((q + 1)\omega)\}^{n}\), then we have a short exact sequence
\[
0 \to \bigoplus_{k} \otimes St_{G} \otimes (nr(\delta) \circ \det) \to \Pi_{k,a} \to \bigoplus_{k} \otimes (nr(\delta) \circ \det) \to 0
\]
where \(St_{G} = C^{0}(\mathbb{P}^{1}(F), C)/\{\text{constants}\}\) denotes the Steinberg representation of \(G\) and where \(\delta = (q + 1)/a\).

As in [DI13], we define
\[
\Theta_{k,a} = \text{Im} \left( \text{ind}_{KZ}^{G} p^{0}_{k} \to \Pi_{k,a} \right)
\]
which is the same as
\[
\Theta_{k,a} = \frac{\text{ind}_{KZ}^{G} p^{0}_{k}}{\text{ind}_{KZ}^{G} p^{0}_{k} \cap (T - a)(\text{ind}_{KZ}^{G} p^{0}_{k})}.
\]
This is a lattice in \(\Pi_{k,a}\) and, since \(\text{ind}_{KZ}^{G} p^{0}_{k}\) is a finitely generated \(\mathcal{O}_{C}[G]\)-module, we see that \(\Theta_{k,a}\) is also a finitely generated \(\mathcal{O}_{C}[G]\)-module.

Now, the Breuil Schneider conjecture [11] asserts that \(p_{k} \otimes \pi\) admits a \(G\)-invariant norm.

By [E+05 Prop. 1.17], this is equivalent to the existence of a separated lattice, and even to any finitely generated lattice being separated.

The following conjecture is then a restatement of the Breuil-Schneider conjecture.

**Conjecture 4.2.** The \(\mathcal{O}_{C}\)-module \(\Theta_{k,a}\) does not contain any \(C\)-line (it is separated).

We also recall that Breuil, in [Bre03a] proves the conjecture for \(F = \mathbb{Q}_{p}\) and \(k < 2p - 1\), and that de Ieso, in [DI13], proves it when \(|J_{l}| \leq 1\) for all \(l \in \{0, \ldots, f - 1\}\) and for any \(\sigma \in S^{+}\), \(k_{\sigma} + 1 \leq p^{\sigma}\).

The idea, as in [Bre03a], is to reduce the problem to a statement which we can prove inductively, sphere by sphere.

As we shall use that idea repeatedly, we introduce a related definition. Abusing notation, we denote by \(B_{N} = B_{N}^{0} \bigcup_{N - 1} B_{N - 1}^{1}\), where \(B_{N}^{0} = \bigcup_{M \leq N} S_{M}^{0}\), \(B_{N}^{1} = \bigcup_{M \leq N} S_{M}^{1}\), and we have defined
\[
S_{M}^{0} = I_{\alpha^{-M}} KZ, \quad S_{M}^{1} = I_{\beta_{M}} KZ
\]
We also recall that \(B_{0}^{0}, B_{1}^{1}\) denote the sets of functions supported on \(\bigcup_{N} B_{N}^{0}, \bigcup_{N} B_{N}^{1}\), respectively.

**Definition 4.1.** Let \(k \in \mathbb{N}^{S}\), and let \(a \in \mathcal{O}_{C}\). We say that the pair \((k, a)\) is separated if for all \(N \in \mathbb{Z}_{\geq 0}\) large enough, there exists a constant \(\epsilon \in \mathbb{Z}_{\geq 0}\) depending only on \(N, k, a\) such that for all \(n \in \mathbb{Z}_{\geq 0}\), and all \(f \in B_{0}^{0}\)
\[
(T - a)(f) \in B_{N} + \omega^{n} \text{ind}_{KZ}^{G} p^{0}_{k} \Rightarrow f \in B_{N - 1} + \omega^{n - \epsilon} \text{ind}_{KZ}^{G} p^{0}_{k}
\]

**Remark 4.1.** We slightly abuse notation here, as \(\omega \notin C\), but as \(v_{F}(\sigma(\omega)) = v_{F}(\omega) = 1\) for all \(\sigma \in S\), one may choose any embedding \(\sigma : F \to C\), and consider \(\sigma(\omega)^{n}\) instead.

The upshot is that we have the following result.

**Theorem 4.3.** Let \(k \in \mathbb{N}^{S}\), let \(a \in \mathcal{O}_{C}\). If \((k, a)\) is separated, then \(\Theta_{k,a}\) is separated.
Proof. First, note that if $[13]$ holds for all $f \in \text{ind}_{KZ}^G \mathcal{P}_k$, then the proof of $[\text{Bre03b}]$, Corollary 4.1.2 shows that $\Theta_{k,a}$ is separated.

Next, for an arbitrary $f \in \text{ind}_{KZ}^G \mathcal{P}_k$, write $f = f^0 + f^1$ with $f^0 \in B^0$ and $f^1 \in B^1$. Then by the formulas in Lemma 3.5 and Lemma 3.6 it follows that

$$\text{supp} \left( (T - a)(f^0) \right) \cap \text{supp} \left( (T - a)(f^1) \right) \subseteq S_0 = B_0 \subseteq B_N$$

If we assume that

$$(T - a)(f^0) + (T - a)(f^1) = (T - a)(f) \in B_N + \varpi^n \text{ind}_{KZ}^G \mathcal{P}_k^0$$

it follows that both $(T - a)(f^0) \in B_N + \varpi^n \text{ind}_{KZ}^G \mathcal{P}_k^0$ and $(T - a)(f^1) \in B_N + \varpi^n \text{ind}_{KZ}^G \mathcal{P}_k^0$.

Since $f^0 \in B^0$ and $(k, a)$ is separated, it follows that $f^0 \in B_{N-1} + \varpi^n \text{ind}_{KZ}^G \mathcal{P}_k^0$.

Moreover, since $T$ is $G$-equivariant, and $\varpi^n \cdot Id$ acts trivially, we see that

$$\beta(T - a)(\beta f^1) = (T - a)(f^1) \in B_N + \varpi^n \text{ind}_{KZ}^G \mathcal{P}_k^0$$

Since $\beta$ acts by translation, it does not affect the values of the function, and since $\beta B_N = B_N$, it follows that

$$(T - a)(\beta f^1) \in B_N + \varpi^n \text{ind}_{KZ}^G \mathcal{P}_k^0$$

with $\beta f^1 \in B^0$. Since $(k, a)$ is separated, we get $\beta f^1 \in B_{N-1} + \varpi^n \text{ind}_{KZ}^G \mathcal{P}_k^0$, hence $f^1 \in B_{N-1} + \varpi^n \text{ind}_{KZ}^G \mathcal{P}_k^0$.

In conclusion

$$f = f^0 + f^1 \in B_{N-1} + \varpi^n \text{ind}_{KZ}^G \mathcal{P}_k^0$$

as claimed. \qed

It therefore remains to show that certain pairs $(k, a)$ are separated.

In this section, we will prove the following theorem:

**Theorem 4.4.** Assume that $|S^+| = 1$, denote by $\sigma$ the unique element in $S^+$, and let $k = k_\sigma = d \cdot q + r$, with $0 \leq r < q$. Assume that one of the following three conditions is satisfied:

(i) $k \leq \frac{1}{2} q^2$ with $r < q - d$ and $v_F(a) \in [0, 1]$.

(ii) $k \leq \frac{1}{2} q^2$ with $2v_F(a) - 1 \leq r < q - d$ and $v_F(a) \in [1, e]$.

(iii) $k \leq \min \left( p \cdot q - 1, \frac{1}{2} q^2 \right)$, $d - 1 \leq r$ and $v_F(a) \geq d$.

Then $(k, a)$ is separated.

**Corollary 4.5.** Under the above conditions, $\Theta_{k,a}$ is separated, hence $\Pi_{k,a}$ admits an invariant norm.

Since our assumptions include the fact that $|S^+| = 1$, we may proceed with the following notational simplifications.

We assume that $C$ contains $F$, and let $\iota : F \hookrightarrow C$ be the natural inclusion.

We may further let $k = k_\sigma$ stand for the multi-index $k$ corresponding to $k$, and similarly for all multi-indices.
4.2. Preparation. Before we prove the theorems, let us first prove the following useful lemmata, which we will employ later on.

Lemma 4.6. Let $\kappa$ be a finite field of characteristic $p$ containing $\mathbb{F}_q$. Consider a polynomial $h \in \kappa[x]$, such that

$$h(x + \lambda) \in x^j \cdot \kappa[x] \quad \forall \lambda \in \mathbb{F}_q$$

Then

$$h(x) \in (x^q - x)^j \cdot \kappa[x]$$

Proof. We will prove the Lemma by induction on $j$. For $j = 1$, $h(x + \lambda) \in x \cdot \kappa[x]$ implies that $h(\lambda) = 0$ for all $\lambda \in \mathbb{F}_q$, hence $x^q - x \mid h(x)$, as claimed.

In general, $h(x + \lambda) \in x^{j'} \cdot \kappa[x] \subseteq x \cdot \kappa[x]$ for all $\lambda \in \mathbb{F}_q$, hence $h(x) = (x^q - x) \cdot g(x)$ for some $g(x) \in \kappa[x]$, by the $j = 1$ case. But $\text{gcd}(x^q - x, x^j) = x$, hence we get

$$h(x + \lambda) = (x^q - x) \cdot g(x + \lambda) \in x^j \cdot \kappa[x] \Rightarrow g(x + \lambda) \in x^{j-1} \cdot \kappa[x]$$

for all $\lambda \in \mathbb{F}_q$.

By the induction hypothesis, it follows that $g(x) \in (x^q - x)^{j-1} \cdot \kappa[x]$, hence $h(x) \in (x^q - x)^j \cdot \kappa[x]$. □

Lemma 4.7. Let $k, d \in \mathbb{N}$. Let $h(x) = \sum_{i=0}^k c_i x^i \in \mathcal{O}_C[x]$ be such that for all $0 \leq j \leq d$, and all $\lambda \in \mathbb{F}_q$, we have

$$\sum_{i=j}^k \binom{i}{j} c_i [\lambda]^{i-j} \in \mathcal{O}_{C}$$

where $[\lambda] \in \mathcal{O}_F \mapsto \mathcal{O}_C$ is the Teichmüller representative of $\lambda$. Then $h(x) \in (x^q - x)^{d+1} \cdot \mathcal{O}_C[x] + \mathcal{O}_{C} \cdot \mathcal{O}_C[x]$.

Proof. By our assumption, since

$$h(x + [\lambda]) = \sum_{i=0}^k c_i (x + [\lambda])^i = \sum_{i=0}^k c_i \sum_{j=0}^i \binom{i}{j} x^j [\lambda]^{i-j} =$$

$$= \sum_{j=0}^k \left( \sum_{i=j}^k \binom{i}{j} c_i [\lambda]^{i-j} \right) x^j$$

we see that

$$h(x + [\lambda]) \in (x^{d+1}, \mathcal{O}_C) \quad \forall \lambda \in \mathbb{F}_q$$

Equivalently, considering the image in $k_C = \mathcal{O}_C/\mathcal{O}_{C} \cdot \mathcal{O}_C$, we have $\overline{h} \in k_C[x]$ of degree at most $d$, satisfying

$$\overline{h}(x + \lambda) \in (x^{d+1}) \text{ for all } \lambda \in \mathbb{F}_q.$$  

By Lemma 4.6, we see that $\overline{h}(x) \in (x^q - x)^{d+1} \cdot k_C[x]$, hence

$$h(x) \in ((x^q - x)^{d+1}, \mathcal{O}_C).$$

This establishes the Lemma. □

Lemma 4.8. Let $n, k, d \in \mathbb{N}$. Let $f(x) = \sum_{i=0}^k c_i x^i \in \mathcal{O}_C[x]$ be such that for all $0 \leq j \leq d$ and all $\lambda \in \mathbb{F}_q$ we have

$$\sum_{i=j}^k \binom{i}{j} c_i [\lambda]^{i-j} \in \mathcal{O}_{C}$$
where $[\lambda] \in \mathcal{O}_F \mapsto \mathcal{O}_C$ is the Teichmüller representative of $\lambda$. Then $f(x) \in (x^q - x)^{d+1} \cdot \mathcal{O}_C[x] + \varpi^n_C \cdot \mathcal{O}_C[x]$.

**Proof.** By induction on $n$. For $n = 1$, this is Lemma 4.7. Assume it holds for $n - 1$, and let us prove it for $n$.

Since $\varpi^n_C \mathcal{O}_C \subseteq \varpi^{n-1}_C \mathcal{O}_C$, the induction hypothesis implies that $f(x) \in ((x^q - x)^{d+1}, \varpi^{n-1}_C)$, so we may write

$$f(x) = (x^q - x)^{d+1} \cdot g(x) + \varpi^{n-1}_C \cdot h(x)$$

By (15), our assumption implies that

$$f(x + [\lambda]) \in (x^{d+1}, \varpi_C^n) \quad \forall \lambda \in \mathbb{F}_q$$

substituting in the above equation, we get

$$((x + [\lambda])^q - (x + [\lambda]))^{d+1} \cdot g(x + [\lambda]) + \varpi^{n-1}_C \cdot h(x + [\lambda]) \in (x^{d+1}, \varpi_C^n)$$

But

$$(x + [\lambda])^q - (x + [\lambda]) = \sum_{i=0}^{q} \binom{q}{i} [\lambda]^{q-i} x^i - x = \sum_{i=1}^{q} \binom{q}{i} [\lambda]^{q-i} x^i - x \in \mathcal{O}_C[x]$$

since $[\lambda]^q = [\lambda]$ for all $\lambda \in \mathbb{F}_q$. This shows that $((x + [\lambda])^q - (x + [\lambda]))^{d+1} \subseteq (x^{d+1}) \subseteq (x^{d+1}, \varpi_C^n)$, hence

$$\varpi^{n-1}_C \cdot h(x + [\lambda]) \in (x^{d+1}, \varpi_C^n) \quad \forall \lambda \in \mathbb{F}_q$$

which implies that

$$h(x + [\lambda]) \in (x^{d+1}, \varpi_C^n) \quad \forall \lambda \in \mathbb{F}_q$$

Considering the reduction modulo $\varpi_C$, by Lemma 4.6 it follows that $h(x) \in ((x^q - x)^{d+1}, \varpi_C)$, hence

$$f(x) \in ((x^q - x)^{d+1} + \varpi^{n-1}_C \cdot ((x^q - x)^{d+1}, \varpi_C)) = ((x^q - x)^{d+1}, \varpi_C^n)$$

establishing the claim. $\square$

**Lemma 4.9.** Let $n \in \mathbb{Z}$, $k, d \in \mathbb{N}$. Let $f(x) = \sum_{i=0}^{k} c_i x^i \in \mathcal{C}[x]$ be such that for all $0 \leq j \leq d$ and all $\lambda \in \mathbb{F}_q$ we have

$$\sum_{i=j}^{k} \binom{i}{j} c_i [\lambda]^{i-j} \in \varpi^n_C \mathcal{O}_C$$

where $[\lambda] \in \mathcal{O}_F \mapsto \mathcal{O}_C$ is the Teichmüller representative of $\lambda$. Then $f(x) \in (x^q - x)^{d+1} \cdot \mathcal{C}[x] + \varpi^n_C \cdot \mathcal{O}_C[x]$.

**Proof.** Let $L = \min_{0 \leq j \leq k} v_C(c_j)$. Consider $g(x) = \varpi_C^{-L} \cdot f(x) \in \mathcal{O}_C[x]$. If $n \leq L$, then as $f(x) \in \varpi_C^L \mathcal{O}_C[x] \subseteq \varpi^n_C \mathcal{O}_C[x]$, we are done.

Else, $g(x)$ satisfies for all $0 \leq j \leq d$ and all $\lambda \in \mathbb{F}_q$

$$\sum_{i=j}^{k} \binom{i}{j} \varpi_C^{-L} \cdot c_i [\lambda]^{i-j} \in \varpi^{n-L}_C \mathcal{O}_C$$

with $n - L \geq 1$, hence by Lemma 4.8, $g(x) \in ((x^q - x)^{d+1}, \mathcal{O}_C[x] + \varpi^{n-L}_C \cdot \mathcal{O}_C[x])$, hence $f(x) \in ((x^q - x)^{d+1}, \mathcal{C}[x] + \varpi^n_C \cdot \mathcal{O}_C[x])$. $\square$
Lemma 4.10. Let $n \in \mathbb{Z}$ and let $k \in \mathbb{N}$. Let $d = \lfloor k/q \rfloor$. Let $(c_i)_{i=0}^{k}$ be a sequence in $C$ such that for all $0 \leq j \leq d$, and all $\lambda \in \mathbb{F}_q$, we have

$$
\sum_{i=j}^{k} \binom{i}{j} c_i [\lambda]^{i-j} \in \pi^n_C \mathcal{O}_C
$$

where $[\lambda] \in \mathcal{O}_F \hookrightarrow \mathcal{O}_C$ is the Teichmuller representative of $\lambda$. Then $c_i \in \omega^n_C \mathcal{O}_C$ for all $0 \leq i \leq k$.

Proof. By Lemma 4.9 we see that $f(x) = \sum_{i=0}^{k} c_i x^i \in (x^q - x)^{d+1} \cdot C[x] + \omega^n_C \mathcal{O}_C[x]$, but $\deg(f) \leq k < q(d+1)$, hence $f(x) \in \omega^n_C \mathcal{O}_C[x]$. This establishes the Lemma. □

Lemma 4.11. Let $k, d \in \mathbb{N}$. Let $f(x) = \sum_{i=0}^{k} c_i x^i \in C[x]$ and let $n \in \mathbb{Z}$. Assume that for all $0 \leq j \leq d$, and all $\lambda \in \mathbb{F}_q$, we have

$$
\sum_{i=j}^{k} \binom{i}{j} c_i [\lambda]^{i-j} \in \omega^n_C \mathcal{O}_C
$$

where $[\lambda] \in \mathcal{O}_F \hookrightarrow \mathcal{O}_C$ is the Teichmuller representative of $\lambda$. Then $c_i \in \omega^n_C \mathcal{O}_C \ \forall 0 \leq i \leq d$

where

$$
\sum_{l=d}^{d+1} \binom{l}{d} \cdot c_{j+l(q-1)} \in \omega^n_C \mathcal{O}_C \ \forall d + 1 \leq j \leq d + q - 1
$$

Proof. By Lemma 4.9 we see that $f(x) \in (x^q - x)^{d+1} \cdot C[x] + \omega^n_C \mathcal{O}_C[x]$. We proceed by reducing $f(x)$ modulo $(x^q - x)^{d+1}$.

In order to do so, we first have to understand the reduction of a general monomial $x^t$ modulo $(x^q - x)^{d+1}$.

We prove, by induction on $s$, that for every $0 \leq s \leq \left\lfloor \frac{t - d - 1}{q - 1} \right\rfloor - d - 1$ and every $t \geq q(d + 1)$ we have

$$
x^t \equiv \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d+1+s}{l+s} \cdot \binom{l+s-1}{s} x^{t-(l+s)(q-1)} \mod (x^q - x)^{d+1}
$$

Indeed, for $s = 0$, this is simply a restatement of the binomial expansion, as

$$
x^t = x^{t-(d+1)q} \cdot x^{(d+1)q} \equiv x^{t-(d+1)q} \cdot (x^q)^{(d+1)q} = x^{t-(d+1)q} \cdot \left( \sum_{l=0}^{d+1} (-1)^l \binom{d+1}{l} \cdot (x^q)^{(d+1)q-l} \cdot x^l \right)
$$

$$
= x^{t-(d+1)q} \cdot \left( \sum_{l=1}^{d+1} (-1)^l \binom{d+1}{l} x^{(d+1)q-l} \cdot x^l \right)
$$

$$
= \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d+1}{l} x^{t-l(q-1)} \mod (x^q - x)^{d+1}
$$

Assume it holds for $s - 1$, and let us prove it holds for $s$.

By the induction hypothesis

$$
x^t \equiv \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d+s}{l+s-1} \binom{l+s-2}{s-1} x^{t-(l+s-1)(q-1)} \mod (x^q - x)^{d+1}
$$

Therefore, we have

$$
x^t \equiv \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d+s}{l+s-1} \binom{l+s-2}{s-1} x^{t-(l+s-1)(q-1)} \mod (x^q - x)^{d+1}
$$
Since $s \leq \left\lfloor \frac{t-d-1}{q-1} \right\rfloor - d - 1$, we see that

$$(q - 1)(d + 1 + s) \leq t - (d + 1) \Rightarrow t - s(q - 1) \geq q(d + 1)$$

This implies, by the case $s = 0$, that

$$x^{t - s(q - 1)} \equiv \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d + s}{l} x^{t - (l + s)(q - 1)} \mod (x^q - x)^{d+1}$$

Substituting in (18) we get

$$x^t \equiv \left( \binom{d + s}{s} \right) \sum_{l=1}^{d+1} (-1)^{l+1} \binom{d + 1}{l} x^{t - (l + s)(q - 1)} +$$

$$+ \sum_{l=2}^{d+1} (-1)^{l+1} \binom{d + s}{l + s - 1} \binom{l + s - 2}{s - 1} x^{t - (l + s - 1)(q - 1)} =$$

$$= \sum_{l=1}^{d+1} (-1)^{l+1} \left( \binom{d + s}{s} \binom{d + 1}{l} - \binom{d + s}{l + s} \binom{l + s - 1}{s - 1} \right) x^{t - (l + s)(q - 1)}$$

Calculation yields

$$\binom{d + s}{s} \binom{d + 1}{l} - \binom{d + s}{l + s} \binom{l + s - 1}{s - 1} =$$

$$= \frac{(d + s)!(d + 1)!}{s!l!(d + 1 - l)!} - \frac{(d + s)!(l + s - 1)!}{(l + s)!(d - l)!l!(s - 1)!} =$$

$$= \frac{(d + s)!(d + 1) \cdot (l + s)}{(l + s) \cdot s!l!(d + 1 - l)!} - \frac{(d + s)! \cdot s \cdot (d + 1 - l)}{(l + s) \cdot (d + 1 - l)!l!s!} =$$

$$= \frac{(d + s)!}{(l + s) \cdot s!l!(d + 1 - l)!} \cdot ((d + 1)l + (d + 1)s - (d + 1)s + sl) =$$

$$= \frac{(d + s + 1)!}{(l + s) \cdot s!(l - 1)!(d + 1 - l)!} =$$

$$= \frac{(d + 1 + s)!}{(l + s)!(d + 1 - l)!} \cdot \frac{(l + s - 1)!}{s!(l - 1)!} = \binom{d + 1 + s}{l + s} \binom{l + s - 1}{s}$$

establishing the identity (17).
It now follows from (17), by letting \( t = j + l(q - 1) \) and \( s = l + d - 1 \), that

\[
f(x) = \sum_{i=0}^{k} c_i x^i = \sum_{i=0}^{d} c_i x^i + \sum_{j=d+1}^{d+q-1} \sum_{l=0}^{\left\lfloor \frac{k-j}{q-1} \right\rfloor} c_{j+i(q-1)} x^{j+l(q-1)} \equiv \]

\[
\equiv \sum_{i=0}^{d} c_i x^i + \sum_{j=d+1}^{d+q-1} \sum_{l=0}^{\left\lfloor \frac{k-j}{q-1} \right\rfloor} c_{j+i(q-1)} x^{j+l(q-1)} \]

\[
+ \sum_{j=d+1}^{d+q-1} \left( \sum_{l=0}^{d} c_{j+l(q-1)} x^{j+l(q-1)} + \sum_{l=d+1}^{d+q-1} \sum_{r=1}^{q-1} \gamma_{r,l,d} \cdot x^{j+(r-1)(q-1)} \right) = \]

\[
= \sum_{i=0}^{d} c_i x^i + \sum_{j=d+1}^{d+q-1} \sum_{l=0}^{d} \left( c_{j+l(q-1)} + \sum_{m=d+1}^{\left\lfloor \frac{k-j}{q-1} \right\rfloor} \delta_{m,l,d} \cdot c_{j+m(q-1)} \right) x^{j+l(q-1)} \mod (x^q - x)^{d+1}
\]

where

\[
\gamma_{r,l,d} = (-1)^{r+1} \binom{l}{r + l - d - 1} \binom{r + l - d - 2}{l - d - 1}
\]

and

\[
\delta_{m,l,d} = (-1)^{d-l} \binom{m}{l} \binom{m - l - 1}{d - l}.
\]

As this is a polynomial of degree less than \( q(d+1) \), and we know that \( f(x) \in (x^q - x)^{d+1} \cdot C[x] + \mathcal{O}_C[x] \), it follows that it must lie in \( \mathcal{O}_C[x] \).

In particular, \( c_i \in \mathcal{O}_C \) for all \( 0 \leq i \leq d \), and looking at the coefficient of \( x^{d(q-1)} \) yields (18), as claimed. \( \square \)

**Lemma 4.12.** Let \( q \) be a power of a prime number \( p \). Let \( k = d \cdot q + r \) be such that \( d < q \) and \( 0 \leq r < q - d \). Then for any \( 0 \leq i \leq r \), any \( 0 \leq j \leq d \) and any \( j + 1 \leq l \leq d \), one has \( p \mid \binom{k-i}{k-j-l(q-1)} \).

**Proof.** Since \( i \leq r < q \), we know that \( 0 \leq r - i < q \) and \( d < q \), so that \( k - i = d \cdot q + (r - i) \) is the base \( q \) representation of \( k - i \).

Since \( j + 1 \leq l \leq d \), one has \( 1 \leq r + 1 \leq r + l - j \leq r + l \leq r + d < q \) and it follows that

\[
k - j - l(q-1) = d \cdot q + r - l \cdot q + l - j = (d-l) \cdot q + (r + l - j)
\]

is the base \( q \) representation of \( k - j - l(q-1) \).

Finally, by Kummer’s Theorem on binomial coefficients, as for any \( l \geq j + 1 \) and any \( i, j \geq 0 \),

\[
r + l - j \geq r + 1 > r \geq r - i
\]

there is at least one digit in the base \( p \) representation of \( r + l - j \), which is larger than the corresponding one in the base \( p \) representation of \( r - i \), hence

\[
p \mid \binom{k-i}{k-j-l(q-1)}
\]
Lemma 4.13. Let $a \in \mathbb{Z}$. The matrix $A = A_m(a) \in \mathbb{Z}^{m \times m}$ with entries $(A_{li})_{i=1}^{m} = \binom{a+l}{i-1}$ satisfies $\det A = 1$.

Proof. We prove it by induction on $m$. For $m = 1$, this is the matrix (1), which is nonsingular.

Note that for any $2 \leq l \leq m$, and any $1 \leq i \leq m$, one has

$$\binom{a+l}{i-1} - \binom{a+l-1}{i-1} = \binom{a+l-1}{i-2}$$

where $\binom{k}{1} = 0$.

Therefore, subtracting from each row its preceding row, we obtain the matrix $B$, with $B_{li} = A_{li}$ for all $1 \leq i \leq m$, and $B_{li} = \binom{a+l-1}{i-2}$.

By the induction hypothesis, the matrix $(B_{li})_{m}^{li}$ is in fact $A_{m-1}(a)$, and $\det(B_{li})_{m}^{li} = 1$. But, as $B_{l1} = 0$ for all $l \geq 2$ and $B_{11} = 1$, it follows that $\det A = \det B = 1$. □

Corollary 4.14. Let $a \in \mathbb{Z}$, $m \in \mathbb{N}$. Let $t \in \{2, \ldots, m\}$. Consider the matrix $A \in \mathbb{Z}^{m \times m}$ with entries

$$A_{li} = \begin{cases} \binom{a+l}{i-1} & t \leq i \leq m \\ \binom{a+l+1}{i-1} & 1 \leq i < t \\ \forall l \in \{1, 2, \ldots, m\} \end{cases}$$

Then $\det A = 1$.

Proof. This matrix is obtained from the one in Lemma 4.13 by adding each of the first $t - 2$ columns to its subsequent column, since

$$\binom{a+l+1}{i-1} = \binom{a+l}{i-1} + \binom{a+l}{i-2}$$

As these operations do not affect the determinant, the result follows. □

Corollary 4.15. Let $k \in \mathbb{N}$. Write $k = d \cdot q + r$, with $1 \leq d < p$, $0 \leq r < q$ and assume that $d - 1 \leq r$. Let $1 \leq m \leq d$. Then the matrix $A \in \mathbb{F}_p^{m \times m}$ with entries $(A_{li})_{m}^{li} = \binom{k-i+1}{m+l(q-1)}$, is nonsingular.

Proof. For any $1 \leq l \leq m$, we note that $m+l(q-1) = lq + (m-l)$, hence (as $d < q$ and $r - i + 1 \geq r - d + 1 \geq 0$) by Lucas’ Theorem

$$\binom{k-i+1}{m+l(q-1)} = \binom{dq+r-i+1}{l} \cdot \binom{r-i+1}{m-l} \mod p$$

Since $1 \leq l \leq d < p$, we get that the $\binom{d}{l}$ are nonzero mod $p$, hence we can divide the $l$-th column by the appropriate multiplier without affecting the singularity of $A$, call the resulting matrix $B$.

Then $B_{li} = \binom{r-i+l}{m-l}$, which up to rearranging rows and columns, is the matrix from Lemma 4.13 hence nonsingular. □
Theorem 4.16. Let $0 \leq k \leq \min\left( p \cdot q - 1, \frac{q^2}{2} \right)$. Assume further that $k = dq + r$ with $d - 1 \leq r < q$. Let $a \in O_C$ be such that $v_F(a) \geq d$, and let $N \in \mathbb{Z}_{>0}$. There exists a constant $c \in \mathbb{Z}_{>0}$ depending only on $N, k, a$ such that for all $n \in \mathbb{Z}_{\geq 0}$, and all $f \in \mathrm{ind}_{KZ}^G \rho_k$,

$$(T - a)(f) \in B_N + \mathcal{w}^n \mathrm{ind}_{KZ}^G \rho_k^0 \Rightarrow f \in B_{N-1} + \mathcal{w}^{n-c} \mathrm{ind}_{KZ}^G \rho_k^0$$

Proof. As before, we may assume that $f = \sum_{m=0}^{M} f_m$ where $f_m \in S_{N+m}^0$, and denote $f_m = 0$ for $m > M$. Looking at $S_{N+m}$, we have the equations

$$T^{-}(f_{m+1}) + T^{+}(f_{m-1}) - af_m \in \mathcal{w}^n \mathrm{ind}_{KZ}^G \rho_k^0$$

We shall prove the theorem with $c = d$.

Assume, by descending induction on $n$, that $f_m, f_{m+1} \in \mathcal{w}^{n-d} \mathrm{ind}_{KZ}^G \rho_k^0$. We will show that $f_{m-1} \in \mathcal{w}^{n-d} \mathrm{ind}_{KZ}^G \rho_k^0$.

By the above equations, we immediately obtain from (13) (note that $af_m \in \mathcal{w}^n \mathrm{ind}_{KZ}^G \rho_k^0$, since $v_F(a) \geq d$)

$$(20) \quad \sum_{i=j}^{k} \binom{i}{j} c_{i,\mu}^{m-1} |\lambda|^{i-j} \in \mathcal{w}^{n-d-j} O_C$$

for all $\mu \in I_{m-1}$, all $\lambda \in \mathbb{F}_q$, and all $0 \leq j \leq d$.

By Lemma 4.10 it follows that for all $i, c_{i,\mu}^{m-1} \in \mathcal{w}^{n-2d} O_C$.

Next, for any $1 \leq j \leq d$, consider the formulas for $C_{j+(q-1),\mu}^{m}$ for any $1 \leq l \leq d$.

Note that $j + l(q-1) \leq d + d(q-1) = dq \leq k$.

Since $k \leq q^2/2$, one has

$$d = \left\lfloor \frac{k}{q} \right\rfloor \leq \frac{k}{q} \leq \frac{q}{2} \Rightarrow 2d \leq q$$

so that $n - 2d + q \geq n$.

Therefore, we get that

$$\mathcal{w}^{j+l(q-1)} \left( \binom{i}{j+l(q-1)} c_{i,\mu}^{m-1} \right) \in \mathcal{w}^q c_{i,\mu}^{n-1} O_C \subseteq \mathcal{w}^{n-2d+q} O_C \subseteq \mathcal{w}^n O_C$$

for all $j, l$. Since for $i \leq k - d$, $\mathcal{w}^d \mid \mathcal{w}^{k-i}$ and $c_{i,\mu+\mathcal{w}^m|\lambda}^{m+1} \in \mathcal{w}^{n-d} O_C$, it follows that

$$(21) \quad C_{j+(q-1),\mu}^{m} \equiv \sum_{i=k-d+1}^{k} \mathcal{w}^{k-i} \left( \binom{i}{j+l(q-1)} \sum_{\lambda \in \kappa_F} c_{i,\mu+\mathcal{w}^m|\lambda}^{m+1} |\lambda|^{i-j-l(q-1)} \right) \equiv 0 \mod \mathcal{w}^n O_C$$

Since $k = d \cdot q + r$, with $r \geq d$, we see that $k - d + 1 - d \cdot q = r + 1 - d \geq 1$, showing that for any $1 \leq l \leq d$, any $k - d + 1 \leq i \leq k$, we get $i - j - l(q-1) \geq 1$, hence for any $\lambda \in \kappa_F$, $|\lambda|^{i-j-l(q-1)} = |\lambda|^{i-j}$. (Had $i - j - l(q-1)$ been 0, this is violated when $\lambda = 0$).
By the induction hypothesis, we know that \( c_{k-i,\mu}^{m+1} \in \mathcal{O}_C \). Write, for \( 0 \leq i \leq d - 1 \) and \( 1 \leq j \leq d \), \( \sum_{\lambda \in \mathbb{F}_p} c_{k-i,\mu}^{m+1} \cdot [\lambda]^{k-i-j} = \mathcal{O}_C \). Then the above equations for \( 1 \leq l \leq d \) yield

\[
\sum_{i=0}^{d-1} \mathcal{O}^i \left( \frac{k-i}{j + l(q-1)} \right) \cdot x_{ij} \equiv 0 \mod \mathcal{O}^d
\]  

(22)

Let us prove that that \( x_{ij} \in \mathcal{O}_C \) for all \( 1 \leq j \leq d \) and all \( 0 \leq i \leq j \). Note that for \( i = j \), it is trivial, so we will prove it for \( 0 \leq i \leq j - 1 \).

Indeed, fix \( j \). Then, looking modulo \( \mathcal{O}^j \), and setting \( y_{ij} = \mathcal{O}^i x_{ij} \), one obtains the equations (for all \( 0 \leq i \leq j - 1 \) and all \( 1 \leq l \leq j \))

\[
\sum_{i=0}^{j-1} \left( \frac{k-i}{j + l(q-1)} \right) \cdot y_{ij} \equiv 0 \mod \mathcal{O}^j.
\]

By Corollary \[4.15\] with \( m = j \), we see that the matrix of coefficients here is nonsingular modulo \( p \), hence also invertible modulo \( \mathcal{O}^j \), and it follows that \( y_{ij} \in \mathcal{O}_C \) for all \( 0 \leq i \leq j - 1 \). But this precisely means that \( x_{ij} = \mathcal{O}^{-i} y_{ij} \in \mathcal{O}_C \).

Therefore,

\[
\mathcal{O}^{-i} \sum_{\lambda \in \mathbb{F}_q} c_{k-i,\mu}^{m+1} \cdot [\lambda]^{k-i-j} = \mathcal{O}^i \cdot \mathcal{O}_C^{n-d} x_{ij} \in \mathcal{O}_C^{n-d+j}.
\]

Considering now the formulas for \( C_{j,\mu}^m \), with \( 1 \leq j \leq d \), we get

\[
\sum_{i=j}^{k} \binom{i}{j} c_{i,\mu}^{m-1} [\lambda]^{i-j} \in \mathcal{O}_C^{-n-d}.
\]

This also holds when \( j = 0 \) trivially as a consequence of \( (20) \).

Hence, applying once more Lemma \[4.10\]

\[
c_{i,\mu}^{m-1} \in \mathcal{O}_C^{-n-d}
\]

as claimed. Therefore, in this case, taking \( e = d \) suffices. \( \square \)

4.4. The case \( 0 < v_F(a) \leq \epsilon \). In this subsection, we will prove the following theorem. Since the case \( v_F(a) = 0 \) is covered by [11-43] Prop. 4.10, it establishes (i) and (ii) in Theorem [4.3] for that case.

**Theorem 4.17.** Let \( 0 \leq k \leq q^2 / 2 \). Assume further that \( k = dq + r \) with \( 0 \leq r < q - d \). Let \( a \in \mathcal{O}_C \) be such that \( 0 < v_F(a) \leq \epsilon \). Assume either that \( 0 < v_F(a) \leq 1 \) or that \( 2v_F(a) - 1 \leq r \). Then \( (k,a) \) is separated.

We prove the theorem by considering two cases.

We shall first prove the case where \( \max(2v_F(a) - 1, 1) \leq r \), and then the case \( r = 0, v_F(a) \leq 1 \).

Unfortunately, we have not been able to provide a proof for the case \( 0 \leq r < 2v_F(a) - 1 \).
Proof. Let \( f \in \text{ind}_K^G \mathbb{Z} \mathbb{L}_k \) be such that \((T - a)f \in B_N + \omega^n \text{ind}_K^G \mathbb{Z} \mathbb{L}_k^0\). We may assume that \( f = \sum_{m=0}^M f_m \) where \( f_m \in S_{N+m} \), and denote \( f_m = 0 \) for \( m > M \).

Looking at \( S_{N+m} \), we have the equations

\[
(T^-(f_{m+1}) + T^+(f_{m-1}) - af_m \in \omega^n \text{ind}_K^G \mathbb{Z} \mathbb{L}_k^0) \quad 1 \leq m \leq M + 1
\]

(23)

Our proof will be based on descending induction on \( m \), showing that if \( f_m, f_{m+1} \) are highly divisible, so must be \( f_{m-1} \).

We will initially obtain some bound for the valuation of \( f_{m-1} \) using \( f_m \) and \( f_{m+1} \), and then we will use that initial bound to bootstrap and obtain better bounds on the valuation of \( f_m, f_{m+1} \) and, in turn, \( f_{m-1} \).

Moreover, we may assume that \( f_m \in S_{N+m}^0 \), using \( G \)-equivariance.

We refer the reader to the definition of the coefficients \( c_{m,\mu}^i \) in Corollary 3.9 and to formula (13).

As under our assumptions \(|S^+| = 1\), we will usually replace the multi-index notation \( \hat{j} \) by \( j = j_\nu \).

The idea of this part of the proof is as follows - the contribution from the \( T^+ \) part (the inner vertex) has high valuation when \( j \) is large, while the contribution from the \( T^- \) part (the outer vertices) has high valuation when \( j \) is small.

Let us introduce the statements \( \mathcal{A}_m, \mathcal{B}_m, \mathcal{C}_m, \mathcal{D}_m \) for the rest of the proof.

The assumptions \( \mathcal{A}_m \) are made to ensure that for small values of \( j \), the contribution from \( T^+ \) is of high enough valuation, hence we can infer something about its preimage (by the previous Lemmata). These give us the initial bound for the valuation of \( f_{m-1} \).

In the bootstrapping part, this bound shows that for large values of \( j \), the main contribution comes from \( T^- \), whence we must use \( \mathcal{B}_m \) in order to obtain better bounds on the valuation of \( f_m \). These bounds for large values of \( j \) can improve our bounds for small values of \( j \) by using the assumption \( \mathcal{C}_m \), which is a linear relation involving one small value of \( j \), while all the others are large.

Finally, this is used to obtain a better bound on the valuation of \( f_{m-1} \), establishing the theorem.

\[
\begin{align*}
\mathcal{A}_m : \quad & c_{m,\mu}^j \in \frac{\omega^n - j}{a} \cdot O_C \quad \forall 0 \leq j \leq d, \quad c_{m,\mu}^i \in \frac{\omega^n - d}{a} \cdot O_C \quad \forall 0 \leq i \leq k \quad \forall \mu \in I_m \\
\mathcal{B}_m : \quad & c_{m,\mu}^{k-j} \in \frac{\omega^n - j}{a} \cdot O_C \quad \forall 0 \leq j \leq d, \quad c_{m,\mu}^i \in \frac{\omega^n - d}{a} \cdot O_C \quad \forall 0 \leq i \leq k \quad \forall \mu \in I_m \\
\mathcal{C}_m : \quad & \sum_{s=j}^{\lfloor \frac{i+s}{j} \rfloor} \binom{s}{j} c_{m,\mu}^{s(j-1)} \in \frac{\omega^n - j}{a} \cdot O_C \quad \forall j + 1 \leq i \leq j + q - 1, \quad \forall 0 \leq j \leq d \\
\mathcal{D}_m : \quad & c_{m,\mu}^i \in \frac{\omega^n}{a^2} \cdot O_C \quad \forall 0 \leq i \leq k 
\end{align*}
\]

Assume, by descending induction on \( m \), that \( \mathcal{A}_m, \mathcal{B}_m, \mathcal{B}_{m+1}, \mathcal{C}_m \) hold for all \( \mu, \lambda \).

Note that, as \( f_{M+1} = f_{M+2} = 0 \), they trivially hold for \( m = M + 1 \). We will prove that \( \mathcal{A}_{m-1}, \mathcal{B}_{m-1}, \mathcal{B}_m, \mathcal{C}_{m-1} \) hold.

For this, we make use of the subsequent Lemma 4.18.

We assume \( \mathcal{A}_m, \mathcal{B}_m, \mathcal{B}_{m+1}, \mathcal{C}_m \), hence by Lemma 4.18 we know that \( \mathcal{A}_{m-1}, \mathcal{B}_{m-1}, \mathcal{D}_m \) also hold.

It remains to show that \( \mathcal{B}_{m-1} \) holds. In fact, we need only to show that \( c_{k-j,\mu}^{m-1} \in \frac{\omega^n - j}{a} O_C \) for all \( 0 \leq j \leq d \).
Note that since \( D_m \) holds, by applying Lemma 4.13 to \( m - 1 \), we see that \( D_{m-1}, G_{m-2} \) hold as well, and so does \( D_{m-1} \).

Next, we see from \( D_{m-1} \) that we have \( c^{-1}_{k-j,\mu} \in \frac{\omega^m}{a} O_C \subseteq \frac{\omega^{m-j}}{a} O_C \) for all \( v_F(a) \leq j \leq d \), which we get “for free”. Therefore, it remains to show that \( c^{-1}_{k-j,\mu} \in \frac{\omega^{m-j}}{a} O_C \) for all \( 0 \leq j < \min(v_F(a), d) \).

Fix some \( 0 \leq j < \min(v_F(a), d) \).

Now, since by Lemma 4.12 \( \omega \) \( p | (k-j-l(q-1)) \) for all \( k - 2v_F(a) < i \leq k \) and all \( j + 1 \leq l \leq d \) (here we use \( 2v_F(a) - 1 \leq r < q - d \)), and by \( D_m, c^{-1}_{i,\mu} \in \frac{\omega^{n-k+i}}{a} O_C \) for all \( k - 2v_F(a) < i \leq k \), we get (as \( \omega \mid p \)) that

\[
\omega k-i \cdot \left( \binom{i}{k-j-l(q-1)} \right) \cdot c^{-1}_{i,\mu} \in \omega k-i+e \cdot \frac{\omega^{n-k+i}}{a} O_C = \frac{\omega^{n+e}}{a} O_C \subseteq \omega^n O_C
\]

where the last inclusion follows from \( v_F(a) \leq e \).

Furthermore, since we have shown \( D_m \), we know that \( c^{-1}_{i,\mu} \in \frac{\omega^n}{a} O_C = \omega^{n-2v_F(a)} O_C \) for all \( 0 \leq i \leq k \), hence for \( i \leq k - 2v_F(a) \), we get

\[
\omega k-i \cdot c^{-1}_{i,\mu} \in \omega^{2v_F(a)} \cdot \omega^{n-2v_F(a)} O_C = \omega^n O_C.
\]

At this point we make use of the hypothesis (23).

It then follows from equation (13) for \( C^{-1}_{k-j-l(q-1)} \), and equations (24), (25) that for all \( \mu \in I_{m-1} \)

\[
\omega k-j-l(q-1) \cdot \sum_{i=k-j-l(q-1)}^{k} \left( \binom{i}{k-j-l(q-1)} \right) \cdot c^{-1}_{i+1,\mu} \subseteq \omega^{n+e} O_C.
\]

But recall that \( l \leq d \), so that

\[
k-j-l(q-1) = (d-l) \cdot (q-1) \cdot r + (d-j) \cdot r + (d-j) \cdot d + \max(1, 2v_F(a) - 1) - j
\]

where in the last inequality we use our assumption that \( r \geq 1 \).

Since we have established \( D_{m-2} \), we know that \( c^{-1}_{i,\mu} \in \frac{\omega^{n-d}}{a} O_C \), hence

\[
\omega k-j-l(q-1) \cdot c^{-1}_{i,\mu} \in \frac{\omega^{n+max(1, 2v_F(a) - 1) - j}}{a} O_C \subseteq \omega^{n-j} O_C.
\]

Therefore, we obtain that \( a \cdot c^{-1}_{k-j-l(q-1),\mu} \in \omega^{n-j} O_C \), hence

\[
c^{-1}_{k-j-l(q-1),\mu} \in \frac{\omega^{n-j}}{a} O_C \quad \forall j + 1 \leq l \leq d.
\]

We shall now use \( G_{m-1} \) to infer from the divisibility of these coefficients, the divisibility of the coefficient \( c^{-1}_{k-j,\mu} \) by \( \frac{\omega^{n-j}}{a} \) as desired. This shall be done as follows.

Let \( i \) be the unique integer satisfying \( j + 1 \leq i \leq j + q - 1 \) such that \( i \equiv k-j \mod (q-1) \), and let \( l_0 = \left\lfloor \frac{k-j}{q-1} \right\rfloor \), so that \( k-j = i + l_0(q-1) \). (Recall that \( k-j+q-1 \geq k-d+q-1 > k \).)

If \( i < q \), we let \( A \in \mathbb{Z}^{(j+1) \times (j+1)} \) be the matrix with entries \( A_{il} = \binom{l_0-l}{t} t_{il} \).

If \( i \geq q \), we let \( A \) be the matrix with entries

\[
A_{il} = \begin{cases} \binom{l_0-l}{t} & i - q < t \leq j \\ \binom{l_0-l+1}{t} & 0 \leq t \leq i - q \\ 0 & \text{otherwise} \end{cases} \quad \forall l \in \{0, 1, 2, \ldots, j\}
\]

In each of the cases, \( A \in GL_{j+1}(\mathbb{Z}) \), either by Lemma 4.13 or by Corollary 4.14.
Therefore, there exists a non-trivial $\mathbb{Z}$-linear combination of its rows, some $\alpha_t \in \mathbb{Z}$ such that for all $0 \leq i \leq j$

$$\sum_{t=0}^{j} \alpha_t A_t = \delta_{i,0}. \quad (27)$$

For $t > i - q$, substituting in $\mathcal{F}_{m-1}$ the value $t$ for $j$, we obtain for all $\mu \in I_{m-1}$

$$\Xi_t := \sum_{s=t}^{i_0} \binom{s}{t} \cdot c_{i+s(q-1),\mu}^{m-1} \in \frac{\omega_{n-t}}{a} \cdot \mathcal{O}_C \subseteq \frac{\omega_{n-j}}{a} \cdot \mathcal{O}_C. \quad (28)$$

Note that indeed $t + 1 \leq j + 1 \leq i \leq t + q - 1$, as required.

For $0 \leq t \leq i - q$, substituting in $\mathcal{F}_{m-1}$ the value $t$ for $j$ and the value $i - (q - 1)$ for $i$, we obtain for all $\mu \in I_{m-1}$

$$\Xi_t := \sum_{s=t}^{i_0} \binom{s}{t} \cdot c_{i+s(q-1),\mu}^{m-1} \leq \frac{\omega_{n-t}}{a} \cdot \mathcal{O}_C \subseteq \frac{\omega_{n-j}}{a} \cdot \mathcal{O}_C. \quad (29)$$

Note that indeed $t + 1 \leq i - (q - 1) \leq j \leq d - 1 \leq q - 1 \leq t + q - 1$, as required.

Considering the linear combination $\sum_{t=0}^{j} \alpha_t \Xi_t$, we see that

$$\sum_{i-q}^{i-q} \sum_{s=t-1}^{i_0} \alpha_t \binom{s+1}{t} \cdot c_{i+s(q-1),\mu}^{m-1} + \sum_{t=i-q+1}^{j} \sum_{s=t}^{i_0} \alpha_t \binom{s}{t} \cdot c_{i+s(q-1),\mu}^{m-1} = \sum_{t=0}^{j} \alpha_t \Xi_t \in \frac{\omega_{n-j}}{a} \cdot \mathcal{O}_C$$

which, reindexing, is the same as

$$\sum_{t=0}^{i-q} \sum_{l=0}^{i-q} \alpha_t \binom{l_0 - l + 1}{t} \cdot c_{k-j-l(q-1),\mu}^{m-1} + \sum_{t=0}^{i-q} \sum_{l=0}^{i-q} \alpha_t \binom{l_0 - l}{t} \cdot c_{k-j-l(q-1),\mu}^{m-1} = \frac{\omega_{n-j}}{a} \cdot \mathcal{O}_C. \quad (29)$$

which lies in $\frac{\omega_{n-j}}{a} \cdot \mathcal{O}_C$.

Since we assumed that $r < q - d$ we have

$$k - j - (d + 1)(q - 1) \leq k - (d + 1)(q - 1) = d \cdot q + r - (dq + q - d - 1) = r - (q - d - 1) \leq 0 < j + 1 \leq i = k - j - l_0(q - 1)$$
Assume that for some \( \text{Lemma 4.18} \). From (23) and (13) we see that for any \( 0 \leq j \leq l \), by (26) we have \( c_{k-j-l(q-1),\mu}^m \in \frac{\omega^{n-j}}{a} \mathcal{O}_C \), so that (29) yields

\[
\sum_{l=0}^{j} \left( \sum_{t=0}^{n} \alpha_t A_t \right) \cdot c_{k-j-l(q-1),\mu}^m =
\]

\[
= \sum_{l=0}^{j} \left( \sum_{t=0}^{n} \alpha_t \left( l_0 - l + 1 \right) \right) + \sum_{l=i-q+1}^{j} \alpha_t \left( l_0 - l \right) \cdot c_{k-j-l(q-1),\mu}^m \in \frac{\omega^{n-j}}{a} \mathcal{O}_C.
\]

Now we apply (27) to see that this is no more than

\[
\sum_{l=0}^{j} \left( \sum_{t=0}^{n} \alpha_t \right) (l_0 - l + 1) + \sum_{l=i-q+1}^{j} \alpha_t (l_0 - l) \cdot c_{k-j-l(q-1),\mu}^m \in \frac{\omega^{n-j}}{a} \mathcal{O}_C.
\]

Furthermore, for any \( 0 \leq j \leq l \), we know that \( c_{j,\mu}^m \in \frac{\omega^{n-j}}{a} \mathcal{O}_C \), hence for every \( (k, a) \) is separated.

**Lemma 4.18.** Assume that for some \( m \), \( \mathcal{A}_m \), \( \mathcal{B}_m \), \( \mathcal{C}_m \) hold.

Then \( \mathcal{A}_{m-1}, \mathcal{B}_{m-1}, \mathcal{C}_{m} \) hold as well.

**Proof.** From (24) and (13) we see that for any \( 0 \leq j \leq d \)

\[
C_{j,\mu}^m = \sum_{i=j}^{k} \omega^{k-i} \binom{i}{j} \sum_{\lambda \in \mathcal{K}_F} \omega^{m \cdot \lambda} \cdot \frac{n-j}{a} \mathcal{O}_C
\]

(30)

\[
+ \omega^j \sum_{i=j}^{k} \omega^{m-1} \binom{i}{j} (-\lambda_\mu)^{i-j} - ac_{j,\mu}^m \in \omega^n \mathcal{O}_C
\]

where \( \lambda_\mu = \frac{\mu - [\mu]_{m-1}}{\omega^{n-m}} \).

By the hypothesis \( \mathcal{B}_{m+1} \), for any \( k-d < i \leq k \) (and any \( \mu \)), we have \( c_{i,\mu}^{m+1} \in \frac{\omega^{n-k+i}}{a} \mathcal{O}_C \), hence \( \omega^{k-i} \cdot c_{i,\mu}^{m+1} \in \frac{\omega^{n}}{a} \mathcal{O}_C \).

Also, for any \( 0 \leq i \leq k - d \), by \( \mathcal{B}_{m+1} \), we have \( c_{i,\mu}^m \in \frac{\omega^{n-d}}{a} \mathcal{O}_C \), hence \( \omega^{k-i} \cdot c_{i,\mu}^{m+1} \in \omega^{d} \omega^{n-d} \mathcal{O}_C = \frac{\omega^{n}}{a} \mathcal{O}_C \).

We conclude that for any \( 0 \leq i \leq k \), one has

\[
\omega^{k-i} \cdot c_{i,\mu}^{m+1} \in \frac{\omega^{n}}{a} \mathcal{O}_C.
\]

(31)

This implies that the first sum in (30) lies in \( \frac{\omega^n}{a} \mathcal{O}_C \), hence

\[
\omega^j \sum_{i=j}^{k} c_{i,\mu}^{m+1} \binom{i}{j} (-\lambda_\mu)^{i-j} - ac_{j,\mu}^m \in \omega^n \mathcal{O}_C
\]

(32)

Furthermore, for any \( 0 \leq j \leq d \), by \( \mathcal{A}_m \), we know that \( c_{j,\mu}^m \in \frac{\omega^{n-j}}{a} \mathcal{O}_C \), hence

\[
ac_{j,\mu}^m \in \omega^{n-j} \mathcal{O}_C
\]

(33)
Therefore, we have established (35) that
\[ 0 \leq c_i \] for all \( i \).

If \( j \leq \nu_F(a) \), we see that \( \nu^{n-j} \in \frac{\nu^n}{a} \cdot O_C \), so we get from (31), (33), and (30) that
\[ \nu^j \sum_{i=j}^{k} \left( \binom{i}{j} \right) c_{i,\mu}^{m-1} |\lambda|^{i-j} \in \frac{\nu^n}{a} \cdot O_C \Rightarrow \sum_{i=j}^{k} \left( \binom{i}{j} \right) c_{i,\mu}^{m-1} |\lambda|^{i-j} \in \frac{\nu^n}{a} \cdot O_C. \]

In particular, by Lemma 4.10, we see that if \( \nu_F(a) \leq d \), then \( c_{i,\mu}^{m-1} \in \frac{\nu^n}{a} \cdot O_C \) for all \( 0 \leq i \leq k \), and if \( \nu_F(a) \geq d \), then \( c_{i,\mu}^{m-1} \in \frac{\nu^n}{a} \cdot O_C \) for all \( 0 \leq i \leq k \).

Substituting \( \lambda = 0 \) in (35), we get \( c_{j,\mu}^{m-1} \in \frac{\nu^n}{a} \cdot O_C \).

Therefore, if \( \nu_F(a) \geq d \), we have already established \( \mathcal{A}_{m-1} \). In this case, since \( \frac{\nu^n}{a} \in \frac{\nu^n}{a} \cdot O_C \), \( \mathcal{D}_m \) trivially holds.

If \( \nu_F(a) < d \), we consider the coefficients \( C_{m,2d,j}, C_{m,2d+1,j}, \ldots, C_{m,k,j} \). By (30) and (31), using the fact that \( c_{j,\mu}^{m-1} \in \frac{\nu^n}{a} \cdot O_C \) for all \( i \), we get \( a c_{j,\mu}^{m} \in \frac{\nu^n}{a} \cdot O_C \) for all \( j \geq 2d \).

In particular, since, by assumption, \( q \geq 2k/q \geq 2d \), we get that for any \( 1 \leq j \leq 2d - 1 \) and any \( 1 \leq l \),
\[ j + l(q - 1) \geq 1 + (q - 1) = q \geq 2d \]
hence \( c_{j+l(q-1),\mu}^{m} \in \frac{\nu^n}{a} \cdot O_C \).

By the assumption \( \mathcal{C}_m \) (substituting \( i \) for \( i \) and \( 0 \) for \( j \)), it follows also that \( c_{j,\mu}^{m} \in \frac{\nu^n}{a} \cdot O_C \). Therefore \( a c_{j,\mu}^{m} \in \frac{\nu^n}{a} \cdot O_C \) for all \( 1 \leq j \leq 2d - 1 \), hence for all \( 0 \leq j \leq k \), establishing \( \mathcal{A}_m \). Note that the case \( j = 0 \) is given by \( \mathcal{A}_m \).

We may now consider once more the equations for \( C_{m,1,j}, \ldots, C_{m,k,j} \), and get from (31), (30) and \( \mathcal{D}_m \) that for all \( 1 \leq j \leq d \)
\[ \nu^j \sum_{i=j}^{k} \left( \binom{i}{j} \right) c_{i,\mu}^{m-1} |\lambda|^{i-j} \in \frac{\nu^n}{a} \cdot O_C \Rightarrow \sum_{i=j}^{k} \left( \binom{i}{j} \right) c_{i,\mu}^{m-1} |\lambda|^{i-j} \in \frac{\nu^n}{a} \cdot O_C. \]

When \( j = 0 \), this holds by (55). By Lemma 4.10, it follows that \( c_{j,\mu}^{m-1} \in \frac{\nu^n}{a} \cdot O_C \) for all \( i \). Also, it shows that \( c_{j,\mu}^{m-1} \in \frac{\nu^n}{a} \cdot O_C \) for \( 1 \leq j \leq d \), by substituting \( \lambda = 0 \). Therefore, we have established \( \mathcal{A}_{m-1} \) in this case as well.

Finally, for any \( 0 \leq j \leq d \), and for any \( 0 \leq t \leq j \)
\[ \sum_{i=t}^{k} \left( \binom{i}{j} \right) c_{i,\mu}^{m-1} |\lambda|^{i-t} \in \frac{\nu^n}{a} \cdot O_C \leq \frac{\nu^n}{a} \cdot O_C \]
for all \( \lambda \in \kappa_F \). Thus, by Lemma 4.11, substituting \( j \) for \( d \) and \( i \) for \( j \), we get \( \mathcal{A}_{m-1} \).

We now consider the case \( 0 < \nu_F(a) \leq 1 \) and \( r = 0 \), using a different argument.
Theorem 4.19. Let $k = dq$, and assume $1 \leq d < \frac{k}{2}$ (note that this excludes $q = 2$). Let $a \in \mathcal{O}$ be such that $0 < \nu_F(a) \leq 1$, and let $N \in \mathbb{Z}_{\geq 0}$. There exists a constant $\epsilon \in \mathbb{Z}_{\geq 0}$ depending only on $N, k, a$ such that for all $n \in \mathbb{Z}_{\geq 0}$, and all $f \in \text{ind}^G_{KZ} \mathbb{Z}_k^0$:

$$(T - a)(f) \in B_N + \omega^n \text{ind}^G_{KZ} \mathbb{Z}_k^0 \Rightarrow f \in B_{N-1} + \omega^{n-1} \text{ind}^G_{KZ} \mathbb{Z}_k^0$$

Proof. We may assume that $f = \sum_{m=0}^M f_m$, where $f_m \in \mathbb{Z}_{N+m}$, and denote $f_m = 0$ for $m > M$. Looking at $S_{N+m}$, we have the equations

$$T^-(f_{m+1}) + T^+(f_{m-1}) - af_m \in \omega^n \text{ind}^G_{KZ} \mathbb{Z}_k^0 \quad 1 \leq m \leq M + 1$$

Assume, by descending induction on $m$, that the following hold:

$$(36) \quad c^{m+1}_{k, \mu} \in \omega^n a \mathcal{O}, \quad c^{m+1}_{k-j, \mu} \in \omega^n a \mathcal{O} \quad \forall 0 < j \leq d$$

$$c^{m+1}_{i, \mu} \in \omega^n a \mathcal{O} \quad \forall 0 \leq i \leq k \quad \forall \mu \in I_{m+1}$$

$$(36) \quad c^m_{0, \mu} \in \omega^n a \mathcal{O}, \quad \sum_{l=0}^{\frac{1}{d} - 1} c^m_{j+l(q-1), \mu} \in \omega^n a \mathcal{O} \quad \forall 1 \leq j \leq d$$

$$c^m_{i, \mu} \in \omega^n a \mathcal{O} \quad \forall 0 \leq i \leq k \quad \forall \mu \in I_m$$

$$\sum_{\lambda \in \kappa_F} c^m_{k, \mu + \omega^m \lambda} \lambda^d \in \omega^n a \mathcal{O}, \quad \forall \lambda \in \{0, 1, 2, \ldots, d, q - 1\} \quad \forall \mu \in I_m$$

We will show that the same formulas hold for $m - 1$, hence establish that they hold for all $0 \leq m \leq M + 1$.

First, for $\mu \in I_{m+1}$ and $\lambda \in \kappa_F$, consider the formula for $C^m_{0, \mu + \omega^m \lambda}$ : see $(13)$. By $(36)$ with $l = d$, using the fact that $|\lambda|^q = |\lambda|$ for all $\lambda \in \kappa_F$, we know that

$$\sum_{\lambda \in \kappa_F} c^{m+1}_{k, \mu + \omega^m \lambda} \lambda^d = \sum_{\lambda' \in \kappa_F} c^{m+1}_{k, \mu + \omega^m \lambda'} \lambda'^d \in \omega^n a \mathcal{O}$$

which is the first summand in the first sum in $(13)$ with $j = 0$.

For $i \leq k - d$, since we assume $c^{m+1}_{i, \mu + \omega^m \lambda} \lambda^d \in \omega^n a \mathcal{O}$, we see that

$$\omega^{k-i} \cdot \sum_{\lambda' \in \kappa_F} c^{m+1}_{i, \mu + \omega^m \lambda'} \lambda'^d \in \omega^n a \mathcal{O}$$

Also, for $k - d \leq i < k$, since we assume $c^m_{i, \mu + \omega^m \lambda} \lambda^d \in \omega^n a \mathcal{O}$, we get

$$\omega^{k-i} \cdot \sum_{\lambda' \in \kappa_F} c^m_{i, \mu + \omega^m \lambda'} \lambda'^d \in \omega^n a \mathcal{O}$$

This shows that the entire first sum in $(13)$ with $j = 0$ lies in $\frac{\omega^n}{a} \mathcal{O}$. In addition, we have assumed that $c^m_{0, \mu + \omega^m \lambda} \lambda^d \in \omega^n a \mathcal{O}$. Therefore

$$\sum_{i=0}^{k} c^m_{i, \mu} \lambda^i \in \omega^n a \mathcal{O}$$

Next, we consider the formulas for $C^m_{j, \mu + \omega^m \lambda}$ with $1 \leq j \leq d$. By $(36)$ with $l = d - j$ for $j \neq d$ and $l = q - 1$ for $j = d$, using the fact that $|\lambda|^q = |\lambda|$ for all
Therefore, by lemma 4.10 we have
\[ c_{k,\mu+\varpi^m-\lambda} \sum_{\lambda' \in \kappa_F}^k c_{k,\mu+\varpi^m-\lambda} \xi |\lambda'|^{dq-j} = \]
\[ = \left( \binom{k}{j} \sum_{\lambda' \in \kappa_F}^k c_{k,\mu+\varpi^m-\lambda} \xi |\lambda'|^{dq-j} \right) \in \varpi^n_a \cdot \mathcal{O}_C \subseteq \varpi^n \cdot \mathcal{O}_C \]
which is the first summand in the first sum in (13).

Since for all \( i \), we have \( c_{i,\mu+\varpi^m-\lambda} \in \varpi^n_a \cdot \mathcal{O}_C \), when considering \( i < k \) we also have \( 1 \leq k - i \), hence
\[ \varpi^{k-i} \sum_{i=k-j}^{k} c_{i,\mu+\varpi^m-\lambda} \xi |\lambda'|^{i-j} \in \varpi^n_{\mu} \cdot \mathcal{O}_C \subseteq \varpi^n \cdot \mathcal{O}_C \]
where the last inclusion holds as \( v_F(a) \leq 1 \). This shows that the entire first sum in (13) lies in \( \varpi^n \cdot \mathcal{O}_C \).

Since we also have \( c_{i,\mu+\varpi^m-\lambda} \in \varpi^n_a \cdot \mathcal{O}_C \), by (13) we see that for all \( 1 \leq j \leq d \)
\[ \sum_{i=j}^{k} c_{i,\mu+\varpi^m-\lambda} \xi |\lambda'|^{i-j} \in \varpi^n_{\mu} \cdot \mathcal{O}_C \]
Therefore, by lemma 4.10 we have \( c_{j,\mu+\varpi^m-\lambda} \in \varpi^{n-2d} \cdot \mathcal{O}_C \) for all \( i \).

Let \( 0 < j \leq d \). Looking at the formula for \( C_{k-j,\mu} \), using the fact that \( k - j \geq dq-d = d(q-1) \geq 2d \) (recall \( q \neq 2 \)), we see that the second sum satisfies
\[ \varpi^{k-j} \sum_{i=k-j}^{k} c_{i,\mu+\varpi^m-\lambda} \xi |\lambda'|^{i-j} \in \varpi^{2d} \cdot \varpi^{n-2d} \cdot \mathcal{O}_C = \varpi^n \cdot \mathcal{O}_C \]
Also, we deduce from the hypothesis \( (39) \) with \( l = j \) that
\[ \left( \binom{k}{k-j} \sum_{\lambda' \in \kappa_F}^k c_{k,\mu+\varpi^m-\lambda} \xi |\lambda'|^{k-j} \right) = \]
\[ = \left( \binom{k}{k-j} \sum_{\lambda' \in \kappa_F}^k c_{k,\mu+\varpi^m-\lambda} \xi |\lambda'|^{k-j} \right) \cdot \varpi^n_{a} \cdot \mathcal{O}_C \subseteq \varpi^n \cdot \mathcal{O}_C \]
since \( p \mid \binom{k}{k-j} = \binom{d(q-1)}{d} \) by Kummer’s theorem, and \( v_F(a) \leq 1 \).

For \( i < k-d \), since we assume \( c_{i,\mu+\varpi^m-\lambda} \in \varpi^{n-2d} \cdot \mathcal{O}_C \), we see that
\[ \varpi^{k-i} \sum_{\lambda' \in \kappa_F}^k c_{i,\mu+\varpi^m-\lambda} \xi |\lambda'|^{i-j} \in \varpi^{d+1} \cdot \varpi^{n-d} \cdot \mathcal{O}_C = \varpi^{n+1} \cdot \mathcal{O}_C \subseteq \varpi^n \cdot \mathcal{O}_C \]
Also, for \( k-d \leq i < k \), since we assume \( c_{i,\mu+\varpi^m-\lambda} \in \varpi^n \cdot \mathcal{O}_C \), and \( 1 \leq k-i \), we get
\[ \varpi^{k-i} \sum_{\lambda' \in \kappa_F}^k c_{i,\mu+\varpi^m-\lambda} \xi |\lambda'|^{i-j} \in \varpi \cdot \varpi^n \cdot \mathcal{O}_C = \varpi^n \cdot \mathcal{O}_C \]
This shows that both sums in (13) lie in \( \varpi^n \cdot \mathcal{O}_C \), hence also
\[ a \cdot c_{j+\varpi^m-\lambda} \in \varpi^n \cdot \mathcal{O}_C \]
Furthermore, for any \( 1 \leq j \leq d \), looking at the formulas for
\[ C_{j+\varpi^m+1,\mu}, \ldots, C_{j+\varpi^m+d,\mu}, \ldots \]

as \( j + l(q - 1) \geq j + q - 1 \geq q > 2d \), by the same reasoning, we deduce from the hypothesis \( 36 \) with \( l = d - j \) that

\[
a \cdot c_{j+l(q-1),\mu}^m \in \frac{\varpi^n}{a}O_C
\]

Since \( \sum_{l=0}^{\lfloor \frac{j}{q} \rfloor} c_{j+l(q-1),\mu}^m \in \frac{\varpi^n}{a}O_C \) for \( 1 \leq j \leq d \), this shows that \( a \cdot c_{j,\mu}^m \in \frac{\varpi^n}{a}O_C \) for all \( 0 \leq j \leq k \).

We also note that \( p \mid \binom{dq}{d+l(q-1)} \) for all \( 1 \leq l < d \), by Kummer's theorem, therefore showing that

\[
a \cdot c_{d+l(q-1),\mu}^m \in \frac{\varpi^n}{a}O_C
\]

Since \( \sum_{l=0}^{d} c_{d+l(q-1),\mu}^m \in \frac{\varpi^n}{a}O_C \), we deduce that

\[
(37) \quad c_{d,\mu}^m + c_{d+q,\mu}^n \in \frac{\varpi^n}{a}O_C
\]

Therefore, we have established that

\[
c_{k,\mu}^m \in \frac{\varpi^n}{a^2}O_C, \quad c_{k-j,\mu}^m \in \frac{\varpi^n}{a}O_C \quad \forall 0 < j \leq d, \quad c_{i,\mu}^m \in \frac{\varpi^{n-d}}{a}O_C \quad \forall 0 \leq i \leq k
\]

Returning to the formulas for \( C_{0,\mu}^m, C_{1,\mu}^m, \ldots, C_{d,\mu}^m \), we see that for all \( \lambda \in \mathbb{F}_q \) one has

\[
(38) \quad \sum_{i=0}^{k} c_{i,\mu}^{m-1} \lambda^i \in \frac{\varpi^n}{a}O_C,
\]

\[
\sum_{i=1}^{k} i c_{i,\mu}^{m-1} \lambda^{i-1} \in \frac{\varpi^{n-1}}{a}O_C,
\]

\[
\vdots
\]

\[
\sum_{i=d}^{k} \binom{i}{d} c_{i,\mu}^{m-1} \lambda^{i-d} \in \frac{\varpi^{n-d}}{a}O_C
\]

Therefore, by Lemma \( 3.10 \) we have \( c_{i,\mu}^{m-1} \in \frac{\varpi^{n-d}}{a}O_C \) for all \( i \). Moreover, we see that

\[
(39) \quad c_{0,\mu}^{m-1} \in \frac{\varpi^n}{a}O_C, \quad \sum_{l=0}^{\lfloor \frac{k}{q} \rfloor} c_{j+l(q-1),\mu}^m \in \frac{\varpi^n}{a}O_C \quad \forall 1 \leq j \leq d
\]

\[
c_{i,\mu}^{m-1} \in \frac{\varpi^{n-d}}{a}O_C \quad \forall 0 \leq i \leq k
\]

It remains to establish \( 36 \) for \( m \). Looking at the equation for \( C_{d,\mu}^m \), we see that for all \( \mu \) we have

\[
a \cdot \sum_{\lambda \in \mathbb{F}_q} c_{d,\mu+\varpi^{m-1}}^m [\lambda] \lambda^j - \varpi^d \cdot \sum_{\lambda \in \mathbb{F}_q} \binom{i}{d} c_{i,\mu}^{m-1} \sum_{\lambda \in \mathbb{F}_q} [\lambda]^{i-l-d} \in \varpi^n O_C
\]

since \( p \mid \binom{dq}{d} \). Fixing \( \mu \in I_{m-1} \) and summing over all \( \lambda \in \mathbb{F}_q \), we get

\[
a \cdot \sum_{\lambda \in \mathbb{F}_q} c_{d,\mu+\varpi^{m-1}}^m [\lambda] \lambda^j - \varpi^d \cdot \sum_{\lambda \in \mathbb{F}_q} \binom{i}{d} c_{i,\mu}^{m-1} \sum_{\lambda \in \mathbb{F}_q} [\lambda]^{i-l-d} \in \varpi^n O_C
\]
for any $l \in \{0, 1, 2, \ldots, d, q-1\}$.

However, as

$$\sum_{\lambda \in \mathbb{F}_q} [\lambda]^l \equiv \begin{cases} -1 & \text{if } i \mid q - 1, \\ 0 & \text{else} \end{cases} \mod p$$

we obtain

$$a \cdot \sum_{\lambda \in \mathbb{F}_q} c_{d, \mu + w^{-1}[\lambda]}\left[\frac{d - l + h(q - 1)}{d}\right] c_{d - l + h(q - 1), \mu} \in \mathcal{O}_C$$

Fix some $l \in \{0, 1, \ldots, d\}$. Note that for all $h \leq d - l$, one has

$$p \mid \frac{(d - l + h(q - 1))}{d} = \left(\frac{h \cdot q + (d - l - h)}{d}\right)$$

This means we have

$$a \cdot \sum_{\lambda \in \mathbb{F}_q} c_{d, \mu + w^{-1}[\lambda]}\left[\frac{d - l + h(q - 1)}{d}\right] c_{d - l + h(q - 1), \mu} \in \mathcal{O}_C$$

For $l = 0$, this already implies

$$\sum_{\lambda \in \mathbb{F}_q} c_{d, \mu + w^{-1}[\lambda]} \in \mathcal{O}_C$$

hence by (45)

$$\sum_{\lambda \in \mathbb{F}_q} c_{d_{dq-l, \mu} + w^{-1}[\lambda]} \in \mathcal{O}_C$$

For arbitrary $l$, we proceed as follows.

Consider the formulas for $C_{0, \mu}^{m-1}, C_{1, \mu}^{m-1}, \ldots, C_{d, \mu}^{m-1}$. We obtain as before that

$$c_{i, \mu}^{m-2} \in \mathcal{O}_C$$

for all $i$.

We may now consider the formulas for $C_{d_{dq-l, \mu}, \mu}^{m-1}, C_{(d-1)_{dq-l+1, \mu}}^{m-1}, \ldots, C_{(d-l+1)_{dq-1, \mu}}^{m-1}$.

Since

$$(d - l + 1)q - 1 \geq q - 1 \geq 2d$$

we get

$$\left(\frac{d_{dq-l, \mu} + w^{-1}[\lambda]}{d - l + h(q - 1)}\right) \cdot \sum_{\lambda \in \mathbb{F}_q} c_{d_{dq-l, \mu} + w^{-1}[\lambda]} \in \mathcal{O}_C$$

for all $d - l + 1 \leq h \leq d$. Substituting back in (40), we get

$$a \cdot \sum_{\lambda \in \mathbb{F}_q} c_{d_{dq-l, \mu} + w^{-1}[\lambda]} \in \mathcal{O}_C$$

where

$$A = \left(\frac{a + 1}{a} \sum_{h=d-1+1}^{d} \mathcal{O}_{\mathcal{O}_C} \left(\frac{d_{dq-l, \mu} + w^{-1}[\lambda]}{d - l + h(q - 1)}\right) \cdot \left(\frac{d - l + h(q - 1)}{d}\right)\right)$$
Note that
\[
\nu_F \left( \frac{d^q}{a} \cdot \frac{dq}{d - l + h(q - 1)} \cdot \left( \frac{d - l + h(q - 1)}{d} \right) \right) \geq d + 1 - \nu_F(a)
\]
But, as \( \nu_F(a) \leq 1 < \frac{1+d}{2} \), it follows that
\[
\nu_F(a) < d + 1 - \nu_F(a)
\]
so that we must have
\[
a \cdot \sum_{\lambda \in \mathbb{F}_q} c_{\ell_d, \lambda}^{m-1}|\lambda|^d \in \mathbb{W}^n \mathcal{O}_C
\]
as claimed.

Finally, looking at the formulas for \( C_{dq}^{m-1}, \ldots, C_{d+q-1}^{m-1} \), we have
\[
\left( \frac{dq}{d + h(q - 1)} \right) \cdot \sum_{\lambda \in \mathbb{F}_q} c_{\ell_d, \lambda}^{m-1}|\lambda|^{q-1} + a \cdot c_{\ell_d, \lambda}^{m-1} \in \mathbb{W}^n \mathcal{O}_C
\]
for all \( 1 \leq h \leq d - 1 \), and
\[
\sum_{\lambda \in \mathbb{F}_q} c_{\ell_d, \lambda}^{m-1}|\lambda| + a \cdot c_{\ell_d, \lambda}^{m-1} \in \mathbb{W}^n \mathcal{O}_C
\]
Substituting in (40), and recalling that \( \sum_{h=0}^{d} c_{\ell_d, \lambda}^{m-1}|\lambda|^{q-1} \in \mathbb{W}^n \mathcal{O}_C \), we obtain
\[
\left( a + \frac{1}{a} \cdot \mathbb{W}^d \sum_{h=0}^{d-1} \left( d + h(q - 1) \right) \left( \frac{dq}{d} \right) \left( d + h(q - 1) \right) \right) \sum_{\lambda \in \mathbb{F}_q} c_{\ell_d, \lambda}^{m-1}|\lambda|^{q-1} \in \mathbb{W}^n \mathcal{O}_C
\]
since \( \mathbb{W}^2 | \mathbb{W}^d \cdot \left( \frac{dq}{d} \right) \).
Since \( \nu_F(a) \leq 1 < \frac{d+1}{2} \), this is only possible if \( \sum_{\lambda \in \mathbb{F}_q} c_{\ell_d, \lambda}^{m-1}|\lambda|^{q-1} \in \mathbb{W}^n \mathcal{O}_C \). Therefore, we are done, and \( \epsilon = d + \nu_F(a) \) suffices.
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