The Universal Composable Security of Quantum Message Authentication with Key Recycling

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Barnum, Crépeau, Gottesman, Tapp, and Smith [1] proposed methods for authentication of quantum messages. The first method is an interactive protocol (TQA) based on teleportation. The second method is a noninteractive protocol (QA) in which the sender first encrypts the message using a protocol QEnc and then encodes the quantum ciphertext with an error correcting code chosen secretly from a set (a purity test code (PTC)). Encryption was shown to be necessary for authentication.

We augment the protocol QA with an extra step which recycles the entire encryption key provided QA accepts the message. We analyze the resulting integrated protocol for quantum authentication and key generation, which we call QA+KG. Our main result is a proof that QA+KG is universal composable (UC) secure in the Ben-Or–Mayers model [2]. More specifically, this implies the UC-security of (a) QA, (b) recycling of the encryption key in QA, and (c) key-recycling of the encryption scheme QEnc by appending PTC. For an m-qubit message, encryption requires 2^m bits of key; but PTC can be performed using only O(log m) + O(log \epsilon) bits of key for probability of failure \epsilon. Thus, we reduce the key required for both QA and QEnc, from linear to logarithmic net consumption, at the expense of one bit of back communication which can happen any time after the conclusion of QA and before reusing the key. UC-security of QA also extends security to settings not obvious from [1].

Our security proof structure is inspired by and similar to that of [1], reducing the security of QA to that of TQA . In the process, we define UC-secure entanglement, and prove the UC-security of the entanglement generating protocol given in [1], which could be of independent interest.

1 Context, results and related work

Encryption and authentication of quantum messages

Barnum, Crépeau, Gottesman, Tapp, and Smith [1] studied authentication of quantum messages. Their first proposed method is an interactive protocol (TQA) based on teleportation. Entanglement is first established between the sender, Alice, and the receiver, Bob, via an insecure quantum channel, using a method called the purity test protocol (PTP). If that is successful, the quantum message is teleported. A two-way authenticated classical channel is assumed. Their second proposed method is a noninteractive protocol (QA) in which the sender first encrypts the message (using a protocol called QEnc [3,4] and 2m bits of key for an m-qubit message) and then encodes the quantum ciphertext with an error correcting code chosen secretly from a set. QA rejects/accepts if an error is/not detected. The set of possible error correcting codes is called the purity test code (PTC). Each code uses s = s' + 1/2 − log(m/s) extra qubits of communication, and takes 2s' + 2 + 2log(m/s') key-bits to choose secretly, in order to achieve a probability of failure (as defined below) of \epsilon ≤ 2^{-s'}. Unlike authentication of classical messages, which can be done without encryption and with key size O(log(m)), [1] proved the necessity of encryption in quantum authentication. Thus, in the noninteractive setting, the key length must be at least 2m required for encryption [3].
Key recycling – intuition and early ideas

The protocol QA is somewhat analogous to the classical scheme due to Wegman and Carter [5]. The latter requires a large key but most of it can be reused so that only a logarithmic sized key is actually consumed. A natural question is whether it is possible to reuse part of the key required in QA. For quantum messages, successful eavesdropping necessarily causes disturbance [6]. This insight [7,8], which even then suggested the possibility of key recycling, led to the original discovery of quantum key distribution (QKD) [9]. In [10], encryption using a quantum key with recycling was proven secure, and the question arose whether the classical key in QEnc could likewise be securely recycled. Using two-way classical discussion to implement a form of quantum authentication before it was formalized in [1], some security statements were obtained. Qualitatively, it is unlikely for a quantum message to be authenticated and accepted if it has been attacked and the key been compromised. This opens the tantalizing possibility of reusing the key whenever the message is accepted. However, a proper security statement for key recycling can be hard to formulate, let alone be obtained, because it requires an analysis of the most general joint quantum attack on quantum authentication together with the scheme that subsequently uses the recycled key. To complicate matters further, the usual security measure for a key in terms of Eve’s classical mutual information was found to be highly unstable with respect to additional classical information on the key (see [11], the motivation for [12] and [13]).

The universal composability approach

To resolve these questions in a robust way, we analyze the security of QA and key recycling in the framework of universal composability. This also proves that QA has some additional nontrivial security features.

Composability is concerned with the security of composing cryptographic primitives in a possibly complex manner. The simplest example is the security of using a cryptographic primitive as a subroutine in another application. We will follow the universal composability (UC) approach: For a specific task (functionality), a primitive that realizes the task is defined to be universally composable if it cannot be distinguished (up to a bias which is the security parameter) from the ideal functionality (augmented with a simulator) by any “environment” that controls the input, retains a purification, provides it to the adversary, directs the adversarial attacks, and receives the state possessed by the adversary as well as all user outputs of the protocol. Any application using the primitive (as a subroutine) is provably essentially as secure as one using the ideal functionality. Also, a recursive argument for security holds for a composite protocol with acyclic modular structure, and the resulting security parameter is at most additive. A security definition that ensures universal composability was recently proposed by Canetti [14] in the classical setting. A simpler model in the quantum setting and a corresponding universal composable security definition were reported in [15,16]. (An alternative approach to composability was obtained in [17,18].)

Since we are concerned with unconditional security, the analysis is particularly simple – it suffices to show that the actual primitive and the ideal functionality (augmented with a simulator) cannot be distinguished by any physical process. Universal composability provides a systematic, general and robust framework for analyzing the security of recycled key, even in the presence of subtleties including entanglement and collective attacks.
Our techniques, proof structure, and results

In [1], security for quantum authentication is defined in terms of the probability of failing to reject in the presence of a detectable error. The authors consider a protocol TQA which is similar to TQA’ except Bob never tells Alice whether the entanglement is accepted or rejected. In TQA, the purity test code (PTC) is only used in a subroutine to establish entanglement (ebits) between Alice and Bob. The authors proved the security of TQA and the equivalence of the security of QA and TQA.

To analyze the security of key recycling in the UC framework, we consider an augmented protocol, QA+KG, which recycles the $2m$ key-bits used in the QEnc step if QA accepts. Note that key recycling requires Alice to know whether QA accepts or rejects. We model our ideal functionality for key generation for the non-interactive protocol QA+KG, such that if Alice further receives one bit of back communication from Bob, she can complete the ideal key recycling step. With this in mind:

1. We show QA+KG and TQA+KG are indistinguishable to any environment. Thus the two protocols QA and TQA still have equivalent securities even with key recycling and in the UC framework.

2. We also formalize how TQA uses a subroutine “EBIT[PTC]” which generates entanglement via insecure channel using PTC as a subroutine. TQA+KG teleports [19] the quantum message using EBIT[PTC] and a perfect encrypted and authenticated classical channel denoted by $C_I$. After using the classical message to complete teleportation, it is output as a key. In other words, the protocol TQA+KG can be interpreted as $(TP+KG)[EBIT[PTC],C_I]$ where TP stands for teleportation.

3. Following [1], and applying results from there, we show that EBIT[PTC] cannot be distinguished from a different protocol EBIT[PTP] for generating entanglement.

4. We show that EBIT[PTP] and the ideal functionality EBIT for generating entanglement cannot be distinguished by any environment with bias better than $2\sqrt{\epsilon}^{1/3}$ where $\epsilon$ is the probability of failure in PTP. This proves that EBIT[PTP] is a UC secure method to generate entanglement.

5. We apply (2)-(3) and the composable theorem to show that TP[EBIT[PTC],C_I] and the ideal channel $Q_I=TP[EBIT_I,C_I]$ cannot be distinguished with bias greater than $2\sqrt{\epsilon}^{1/3}$.

Finally, we show that $(TP+KG)[EBIT[PTC],C_I]$ and $Q_I+KD_I$ (where KD_I denotes an ideal key generating functionality) cannot be distinguished by any environment with bias greater than $2\sqrt{\epsilon}^{1/3}$. The intuition is that, replacing EBIT[PTC] by EBIT_I in $(TP+KG)[EBIT[PTC],C_I]$ also protects the classical teleportation message which then can be reused as a key.

Together, QA+KG is distinguishable from $Q_I+KD_I$ with bias at most $2\sqrt{\epsilon}^{1/3}$.

We thus prove that QA+KG is UC secure in the Ben-Or-Mayers model [2]. More specifically, this implies the UC-security of (a) QA, (b) recycling of the encryption key in QA, and (c) key-recycling of QEnc by appending PTC. We reduce the key required for both QA and QEnc, from linear in the message size to a logarithmic net consumption (if QA accepts), at the expense of one bit of back communication which can happen any time after the conclusion of QA and before using the
recycled key. Furthermore, UC-security of QA implies it can be used securely in other cryptographic tasks. In particular, parallel composition is secure against joint attacks, and QA is still secure if the adversary possesses the purifying system of the message to be authenticated. These are not immediate consequences of the analysis in [1].

In the process, we define UC security for entanglement generation and prove the UC-security for the protocol EBIT[PTC] proposed in [1], which is of independent interest.

Our result does not contradict earlier lower bounds on the key size, which applies to noninteractive protocols and is concerned with the initial key needed. Another nice aspect of our results is that one can simply reuse the encryption key without further privacy amplification, in contrast to quantum key distribution and earlier proposals for key recycling.

Prior and related work

We have discussed background results leading to this investigation (which started 2003) – the QEnc and QA protocols in [3,1], UC security [2] and some early investigations of key recycling [10]. Our proof steps are similar to those in [1], but we resolve definitional ambiguities in [1] and with the composability framework obtain more precise and stronger security results (UC security of QA, QEnc (by adding PTC), EBIT[PTC], EBIT[PTP], and key recycling). Throughout, we may emphasize the modular structure of a protocol $\mathcal{P}$ calling a subroutine $\sigma$ by writing $\mathcal{P}[\sigma]$.

We now discuss other related works since this project started.

The security of key recycling in QA was studied independently by M. Horodecki and Oppenheim [20] in 2003. However [20] does not address the security of QA, and it assumes an adversary who does not possess the purification. For that reason, we believe their claim to UC security, even if it holds, requires a nontrivial proof, but none was given.

In 2005, Damgard, Pedersen, and Salvail [21] proposed key recycling for the encryption of classical messages by using the Wegman-Carter classical authentication scheme followed by a quantum encryption scheme based on key uncertainty or locking [11]. Encryption of quantum messages was said to be possible in the introduction but no proof of this assertion was given in the text. Regardless, the results in [21] are quite different from ours because encryption and authentication of classical messages are much weaker tasks computationally. Also, locking is highly non-composable when a quantum adversary has quantum memory and delays measurements. It is unclear how the analysis in [21] fits into the composable security framework, despite a claim (without formal definition or proof) of the composable security of the regenerated key. (We detail the differences here since an earlier version of our paper was rejected in 2007 by a referee who assumed this work to be similar to [21].)

In this paper, we emphasize the necessity of considering the composable security of the regenerated key, since the entire purpose of recycling the key is to use it later. Furthermore, universal composable security is precisely what allows the key to be used in a yet-to-be-specified and unrestricted manner. Without such an assurance, the security of key recycling is ill-defined.

This paper has had an unusually long gestation. We presented a preliminary version of our results at QIP 2004 and a draft has informally circulated since 2008. An updated version appeared in QCRYPT 2011. (The full submission was provided to the authors of related works [22,23,24] prior to their appearing in the arXiv.)
Until this year, ours remained the only proof of the composable security of QA and of key recycling in QA and QEnc. A flurry of recent activity in the area by other authors prompted us to produce and submit this final version of our draft.

First, Garg, Yuen, and Zhandry [23] gave a new definition of quantum authentication called “total authentication” that they showed to permit composably secure key recycling. They further exhibited several new protocols satisfying the definition. Our work implied that QA proposed in [1] satisfies total authentication but with a very small key leakage.

More recently, Portmann [24] has established proofs of both of our main results in the framework of abstract cryptography [25]. Furthermore, partial key recycling is proven secure when authentication fails. His work formally considers impersonation attack, whereas all other work implicitly assumes this is a special case of the substitution attack. His work also explicitly considers communication over noisy channels. We have added a short discussion to our paper that illustrates how secure authentication (with key recycling) over noisy channels is an immediate corollary of composable security of QA+KG.

One feature that slightly distinguishes [23,24] from ours is that they demonstrate that the entire key can be recycled whereas we sacrifice a vanishing fraction of the key. While interesting theoretically, the distinction is not practically important because, in our case, additional key to make up for the small loss can be added to the message with negligible additional cost. Furthermore, some of the schemes that allow total key recycling require substantially more initial key (while QA is key-optimal up to an additive logarithmic amount, which we believe can be reduced to a constant in view of results in [20]).

In another recent contribution, somewhat closing the circle, Fehr and Salvail [27] proved that secret key could be securely recycled in a quantum protocol for authenticating classical messages. Their protocol is a slightly modified version of one first proposed by Bennett, Brassard, and Breidbart in 1982 [7] and within reach using current experimental techniques.

The current manuscript differs from our QCRYPT’11 submission in four ways. (1) We found a mis-statement of the adversarial power in the QCRYPT’11 submission which is corrected here – the adversary should be given the purification of the message (full quantum side information) for the attack. Our proof is independent of whether the adversary is given this purification or not. (2) We simplified the last step of the proof (and as a bonus reduced the insecurity parameter by a factor of 3). (3) In view of [22], we removed claims of proof of the composability of the Wegman-Carter authentication scheme for classical messages in this paper. Our claim was based on a simple (but slightly mistaken) proof in a half-page appendix. We decide to keep the appendix for readers who want a quick main idea, but refer to the detailed subsequent result in [22]. (4) We revived an appendix on authentication of pure quantum states (which was removed in QCRYPT’11 due to page limit). Finally, as mentioned earlier, we briefly discussed the case for transmission through noisy channel, and made other minor changes.

Comparison with other methods

There are two other quantum methods that provide similar security for quantum message encryption and authentication using only a small key. We now compare other costs, such as the amount of (forward) quantum communication, forward and backward classical communication, and the round/interaction complexity.
If the original message has \( m \) qubits, QA+KG consumes a little more than \( m \) qubits of communication, one bit of back communication (which can be delayed until right before reusing the key), and a little more than \( 2m \) key-bits; \( 2m \) of them can be regenerated if authentication accepts.

A first alternative to this approach is to use QKD to securely expand the classical key before running QA without key recycling. This requires only a small initial key (not just the amortized one). The drawback is that the QKD protocol itself needs at least \( 2m \) qubits, two rounds of classical back communication, and a linear amount of forward classical communication. The first round of back communication is to acknowledge the receipt of the quantum states by Bob, followed by 1- or 2-way public discussion (that itself has to be authenticated) and finally more back communication to finalize the output key size. Then, \( m \) more qubits have to be sent in QA. Thus, this method consumes substantially more quantum communication and forward classical communication, and one extra round of back communication. Furthermore, the back communication has to be performed during the protocol. Running QA before QKD requires the long \( 2m \)-bit initial key, but the back communication can be delayed until the QKD is run (but before the application using the key produced).

A second alternative is TQA – teleport the quantum message using ebits obtained by potentially insecure means in addition to an insecure forward classical channel that needs classical message authentication. (Since we prove the composable security of EBIT[PTC] and the Wegman-Carter scheme is composably secure [22], this method is composably secure.) Classical message authentication requires a long key, but most of it can be reused securely regardless of the authentication result. EBIT[PTC] uses a small key and back communication, and generates a quantum key. (Thus, back communication is needed during the protocol itself, unlike for QA+KG.) Compared to QA+KG, this scheme uses a similar amount of quantum communication, more initial key and forward classical communication, in addition to a similar amount of classical back communication. But TQA offers two advantages over QA+KG. First, failing PTC when generating ebits does not destroy the quantum message itself (so the message is not only authenticated, but protected). Second, the classical authentication key is always recycled.

We emphasize that these methods and QA+KG are incomparable and interesting for different reasons. Also, we are concerned not only with the key requirement, but security definitions and composability of protocols like QA and EBIT[PTC].

**Organization of the paper**

We will discuss background materials concerning the security setting, quantum mechanics, quantum encryption, quantum authentication, and quantum universal composability in Sect. 2 and prove the security of QA+KG in Sect. 3. Other results and open questions will be discussed in Sect. 4. A glossary, the quantum UC model, the extended transpose trick, a simple (but slightly mistaken) proof of the quantum UC-security of the Wegman-Carter scheme, and the security of authentication for pure quantum states with half of the initial key cost are given in the appendices.
2 Setting, notation, definitions, and background

Setting. The sender, the receiver, and the adversary are called Alice, Bob, and Eve, respectively. We consider unconditional security, i.e., security against an Eve whose capabilities are only limited by quantum mechanics.

Basic elements of quantum mechanics. A quantum system with \( d \) distinguishable states is associated with the \( d \)-dimensional complex Hilbert space \( \mathbb{C}^d \). The set of linear and unitary operators acting on \( \mathbb{C}^d \) are denoted by \( \mathcal{B}(\mathbb{C}^d) \) and \( \mathcal{U}(\mathbb{C}^d) \) respectively. Composite systems are associated with tensor product Hilbert spaces.

The state of a quantum system is represented by a positive semidefinite density matrix \( \rho \in \mathcal{B}(\mathbb{C}^d) \) of unit trace. It is a convex combination (or probabilistic mixture) of rank-1 projectors (commonly called pure states). Pure states can be represented as unit vectors \( |\psi\rangle \in \mathbb{C}^d \), up to a physically unobservable phase, and we write the corresponding density matrix \( |\psi\rangle \langle \psi| \) simply as \( \psi \). Throughout, we denote an ebit by \( |\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \) and its density matrix by \( \Phi \).

A measurement \( \mathcal{M} \) is specified by a POVM — a set of positive semidefinite operators \( \{O_k\} \) such that \( \sum_k O_k = I \). If the state is initially \( \rho \), the measurement \( \mathcal{M} \) yields the outcome \( k \) with probability \( \text{Tr}(O_k\rho) \) and changes the state to \( \sqrt{O_k} \rho \sqrt{O_k} / \text{Tr}(O_k\rho) \), without loss of generality. \( \mathcal{M} \) is said to be along a basis \( \{|k\}\} \) if \( \{O_k\} = \{|k\rangle \langle k|\} \). Measuring an unknown state generally disturbs it.

The most general evolution of a state is given by a trace-preserving completely-positive (TCP) linear map \( \mathcal{E} \) acting on \( \mathcal{B}(\mathbb{C}) \). (See [28], for various representations.) Discarding a (sub)system is given by the (partial) trace operation. Every state \( \rho \in \mathcal{B}(\mathbb{C}^d) \) can be written as the partial trace of some pure state \( |\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \). In other words, \( \rho = \text{Tr}_2(\psi) \) and \( |\psi\rangle \) is called its purification, and the extra system is called the purifying system.

Subscripts of states and operations often (though not always) label the system being acted on.

We mention two distance measures for quantum states. The first measure is the trace distance \( \frac{1}{2} \|\rho_1 - \rho_2\|_1 \) between two density matrices \( \rho_1 \) and \( \rho_2 \), where \( \|\cdot\|_1 \) denotes the Schatten 1-norm. The maximum probability of distinguishing the two states drawn randomly is given by \( \frac{1}{2} + \frac{1}{2} \|\rho_1 - \rho_2\|_1 \).

The second measure is the fidelity, \( F(\rho_1, \rho_2) = \max_{|\psi_1\rangle, |\psi_2\rangle} |\langle \psi_1 | \psi_2 \rangle|^2 \), where \( \rho_{1,2} \in \mathcal{B}(\mathbb{C}), \langle \psi_1, \psi_2 \rangle \in \mathbb{C} \otimes \mathbb{C} \) are purifications of \( \rho_{1,2} \) and \( \langle \cdot | \cdot \rangle \) is the inner product in \( \mathbb{C} \otimes \mathbb{C} \). Note that we have an additional square in the fidelity compared to other references such as [28].

We denote by \( \sigma_{10} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( \sigma_{01} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), and \( \sigma_{11} = \sigma_{10} \sigma_{01} \) the Pauli matrices acting on 1 qubit. The Pauli group acting on \( m \) qubits is generated multiplicatively by \( \sigma_{10}, \sigma_{01} \) acting on each qubit.

The interested reader can consult [28] for a more comprehensive review.

2.1 Quantum encryption

Definition

The cryptographic task of quantum encryption can be described as follows. Alice and Bob share a key \( K \) in which the realization \( k \) occurs with probability \( p_k \). To send a message \( \rho \), Alice transmits \( \mathcal{E}_k(\rho) \) and Bob applies \( \mathcal{D}_k \) to retrieve \( \rho \). A quantum encryption scheme should satisfy two properties:
Completeness: ∀k \( D_k \mathcal{E}_k = I \), the identity operation.

Soundness: \( \mathcal{R}(\rho) := \sum_k p_k \mathcal{E}_k(\rho) = \rho_0 \), where \( \rho_0 \) is a constant.

The soundness condition is an exact security statement that, without knowledge of the key, a specimen of the encrypted message \( \mathcal{R}(\rho) \) is independent of the actual message \( \rho \). If the message system \( M \) is entangled with other systems, let \( R \) be its purifying system, and \( |\psi\rangle_{MR} \) the purification. By linearity \( (\mathcal{R}_M \otimes \mathcal{I}_R)(\psi) = \rho_0 \otimes \text{Tr}_M(\psi) \) which means that the transmission is still completely useless to the strongest eavesdropping adversary who already possesses all the correlations with \( M \) contained in \( R \). A natural approximate security condition is, \( \forall |\psi\rangle_{MR}, \| (\mathcal{R} \otimes \mathcal{I})(\psi) - \rho_0 \otimes \text{Tr}_M(\psi) \|_1 \leq \epsilon \), a small security parameter. Note \( \| \mathcal{R}(\rho) - \rho_0 \|_1 \leq \epsilon \) is generally too weak for a security definition \[29\], unless the adversary is restricted to not having the purifying system \( R \). A scheme that satisfies this last condition will be called an approximate encryption scheme (with security parameter \( \epsilon \)).

**Known constructions**

A special case is \( \mathcal{E}_k(\rho) = U_k \rho U_k^\dagger \) with each \( U_k \) unitary. In particular, exact encryption can be achieved by taking \( K \) to be a random \( 2m \)-bit string, and for \( k = (x_1, z_1, \ldots, x_m, z_m) \), \( U_k = \sigma_{x_1 z_1} \otimes \cdots \otimes \sigma_{x_m z_m} =: \sigma_{xz} \). We call this specific protocol \( \text{QEnc} \). There are approximate encryption schemes with certain \( U_k \) and \( K \) of size \( m + O(\log m) + 2 \log(1/\epsilon) \) bits \[29,30,31\] (improved to \( m + 2 \log(1/\epsilon) + 8 \) bits in \[32\]). We focus on the scheme in \[29,32\] and call it \( \approx \text{QEnc} \).

**Relation of \( \text{QEnc} \) to teleportation and remote state preparation, and lower bounds**

In teleportation (TP) \[19\] of 1 qubit, Alice and Bob share one ebit in systems \( A \) and \( B \). The message \( \rho \) in system \( M \) is transmitted by Alice measuring \( MA \) in the Bell basis \( \{ I \otimes \sigma_{xz} |\Phi\rangle \}_{x,z} \). Conditioned on the outcome \( x, z \), the state in system \( B \) is \( \sigma_{xz} \rho \sigma_{xz} \). Thus, if Bob knows \( x, z \) (sent to him from Alice by a classical channel), he can recover the message \( \rho \). An \( m \)-qubit message can be teleported qubit-wise.

Thus, there is a one-to-one correspondence between the protocols for \( \text{QEnc} \) and TP; the key in \( \text{QEnc} \) translates to the measurement outcome, and thus the communicated message, in TP. A similar correspondence holds between any quantum encryption scheme and a generalized teleportation protocol that sends quantum states using classical communication and entanglement \[33\]. Composing generalized teleportation with superdense coding \[34\] to transmit \( 2m \) classical bits proves that \( 2m \) bits is a lower bound on the communication cost of generalized teleportation and thus a lower bound on the key cost in any exact encryption as well.

Likewise, there is a one-to-one correspondence between approximate quantum encryption protocols and a class of schemes \[29,35\] for remote state preparation (RSP) \[36\]. In these schemes, Alice has a classical description of the message, applies a measurement to her half of the ebits, and sends the outcome to Bob. In particular, the POVM can be chosen to contain the operators \( \{ \frac{1}{M} (U_k \rho U_k^\dagger) T \}_k \) for \( M = \| \sum_k U_k \rho U_k^\dagger \|_\infty \), and conditioned on receiving the outcome \( k \), Bob’s half of the ebits becomes \( U_k \rho U_k^\dagger \). (See also Appendix \[C\]) When \( k \) takes \( 2^{m+2 \log(1/\epsilon) + 8} \) values, RSP succeeds with probability at least \( 1 - \epsilon \) \[29,32\], while using only about half of the communication required by teleportation. Furthermore, this communication cost is optimal \[35,29\]. As a result, approximate encryption requires about half of the key needed for exact quantum encryption. One can interpret this result as follows. Exact quantum encryption breaks all possible correlations with a purifying system, while approximate encryption does not. The decorrelation in exact quantum encryption requires the extra key.

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2.2 Quantum authentication

Definition

Alice and Bob share a key $K$ with distribution $\{p_k\}_k$, and the realization is $k$. Alice applies an encoding map $E_k$ that takes the $m$-qubit message system $M$ to an $(m+l)$-qubit system $T$, which is transmitted to Bob. After Bob receives the possibly altered system $T$, he applies a decoding map $D_k$ which outputs an $m$-qubit message in $M$ and one extra qubit $V$ with two states labelled by $|\text{acc}\rangle$, $|\text{rej}\rangle$. The security conditions apply to any purification $|\psi\rangle_{RM}$ (with reference system $R$) of the message state in system $M$.

Completeness: $\forall k: (I_R \otimes (D_k E_k)_M)(|\psi\rangle_{RM}) = |\psi\rangle_{RM} \otimes \text{ACC}$

Soundness: Let the adversarial attack be a TCP map given by $O$. Then the output of the protocol is $\hat{\rho}_{RMV} = \sum_k p_k (I_R \otimes (D_k O E_k)_M)(|\psi\rangle_{RM})$. The scheme is said to have security parameter $\epsilon$ if $\text{Tr}\left[(I - |\psi\rangle_{RM} \otimes \text{ACC}_V) \hat{\rho}_{RMV}\right] < \epsilon$.

Intuitively, the above conditions say that quantum authentication should accept and transmit a message perfectly in the absence of tampering, and reject with high probability otherwise.

Unlike quantum encryption, it is intrinsically impossible to achieve perfect soundness. The issue of approximate security involving purifications is subtle but it was not explicitly dealt with in [1]. (See endnote [37].) The above conditions take into account purifications which captures all possible correlations between the message $M$ and other systems (though not obviously composable).

Known constructions

We first describe the quantum authentication scheme QA constructed in [1] in detail in the following. It has two main subroutines, the quantum encryption scheme $\text{QEnc}$ described in the previous subsection, and quantum purity test codes (PTC), which are closely related to quantum purity test protocols (PTP).

Consider a set of quantum stabilizer codes $\{Q_t\}$ [38,39] encoding $m$ qubits into $n$ qubits. The set $\{Q_t\}$ is said to be a stabilizer purity test code with error $\epsilon$ if, for any nontrivial $n$-qubit Pauli error $E$, at least a fraction $1 - \epsilon$ of the codes detect it.

A purity test protocol with error $\epsilon$ is a superoperator $\mathcal{T}$ which can be implemented with local operations and classical communication (LOCC), and which maps $2n$ qubits, half held by Alice and half by Bob, to $2m + 1$ qubits satisfying the following two conditions (here $n = m + l, l \geq 0$):

Completeness: $\mathcal{T}(\Phi^\otimes n) = \Phi^\otimes m \otimes \text{ACC}$.

Soundness: $\forall \rho \text{ Tr} \left[ \mathcal{T}(\rho) \left[ (I - \Phi^\otimes n) \otimes \text{ACC} \right] \right] < \epsilon$.

Each purity test code $\{Q_t\}$ gives rise to a purity test protocol $\mathcal{T}$ as follows [1]. Each of Alice and Bob measures the syndrome of $Q_t$ on his/her $n$ qubits, for the same random $t$. If their syndromes agree, they accept and then perform the decoding procedure for $Q_t$; otherwise they reject. If the purity test code $\{Q_t\}$ has error $\epsilon$, then $\mathcal{T}$ is a purity test protocol with error $\epsilon$. An efficient purity test code was constructed in [1], such that $l = s$ and $\epsilon = 2 \frac{1 + m/s}{1 + 2s}$ for message length $m$ and any chosen $s$. The shared random variable $t$ should be independent of the $2n$-qubit input of the purity test protocol. Throughout this paper, $t$ is a secret key inaccessible to the adversary to ensure the independence condition. In the LOCC setting it can be generated by one party and communicated to the other party.
The noninteractive protocol QA with security parameter \( \epsilon \) consists of first applying QEnc to the \( m \)-qubit message, followed by using additional secret key to further encode with a purity test code and then apply an operation corresponding to a random syndrome (all parameters as described above). Formally, QA=\( \text{QA}[\text{QEnc},\text{PTC},\text{KD}] \). It requires \( m+s \) qubits of quantum communication and \( 2m+s+\log_2(2^s+1) \) key-bits. Both costs are asymptotically optimal as quantum encryption is necessary for quantum encryption \([1]\). The security of QA is reduced to that of an interactive protocol TQA’ in which a purity test protocol is first used to establish a \((2m+1)\)-qubit state, the first \( 2m \) qubits are used to teleport the message, and the last is in the ACC or REJ state. QA satisfies the completeness and soundness security conditions stated above. (See endnote \([37]\).)

See Sect. 1 (under “related works”) for some additional very recent constructions \([23]\).

### 2.3 Quantum Universal Composability Theorem

Throughout the paper, we denote the associated ideal functionality of a protocol by adding a subscript I. Different protocols can have the same ideal functionality. A protocol \( \mathcal{P} \) calling a sub-protocol \( \sigma \) is denoted as \( \mathcal{P}[\sigma] \). Two conjoining protocols (implemented by a joint circuit) are written as \( \mathcal{P}_1+\mathcal{P}_2 \). Two protocols \( \mathcal{P}_1, \mathcal{P}_2 \) implemented with the same circuit are said to be equal \( \mathcal{P}_1=\mathcal{P}_2 \); the circuit can be interpreted in two ways.

In the universal composability (UC) approach \([14] [15] [2]\):

1. A UC security definition for a primitive is one that can be stated for a single execution of the primitive but nonetheless guarantees security of composition with any other properly defined system. This definition involves a description of some ideal functionality of the primitive. The goal is to preserve security in a basic composition. More concretely, we want a security definition such that, if \( \sigma \) is a secure realization of an ideal subroutine \( \sigma_I \), and a protocol \( \mathcal{P} \) using \( \sigma_I \), written as \( \mathcal{P}[\sigma_I] \), is a secure realization of \( \mathcal{P}_I \) (the ideal functionality of \( \mathcal{P} \)), then \( \mathcal{P}[\sigma] \) is also a secure realization of \( \mathcal{P}_I \).

2. A prescription for how to securely perform basic composition recursively allows any complex protocol to be built out of secure components.

A simplified model appropriate to our setting is described in Appendix \([3]\) (See also \([12]\).) In essence, the UC security condition for \( \mathcal{P} \) expresses that \( \mathcal{P} \) and \( \mathcal{P}_I \) are indistinguishable by any adversarial attack. It does so by defining an “environment” \( \mathcal{E} \) that includes the actual adversary and any adversarial protocol that calls \( \mathcal{P} \). The environment controls the protocol’s input and receives its output, and ultimately itself outputing a binary random variable \( \Gamma \). For this \( \mathcal{E} \), extend \( \mathcal{P}_I \) by a simulator \( \mathcal{S} \) to an extended ideal protocol and denote the conjoining protocols as \( \mathcal{P}_I+\mathcal{S}. \) \( \mathcal{E} \) still controls the input/output of the unit (out of the control of \( \mathcal{S} \)) but insecure channels and other insecurities of \( \mathcal{P} \) are “simulated” by \( \mathcal{S} \). (See Fig. 1 in Appendix \([3]\).) The random variable \( \Gamma_1 \) output in this case generally differs from \( \Gamma \), and their statistical difference quantifies the security – the smaller the statistical difference the higher the security. This motivates the following definition of universal composable security:

**Definition 1:** \( \mathcal{P} \) is said to \( \epsilon \)-securely realize \( \mathcal{P}_I \) (shorthand \( \mathcal{P} \epsilon \text{-s.r.} \mathcal{P}_I \)) if

\[
\forall \mathcal{E} \ \exists \mathcal{S} \ \text{ s.t. } \|\Gamma - \Gamma_1\|_1 \leq \epsilon. \tag{1}
\]

We call \( \epsilon \) in \([1]\) the distinguishability-advantage between \( \mathcal{P} \) and \( \mathcal{P}_I \). It has a simple operational meaning. The entire interaction between the environment \( \mathcal{E} \) (including the adversary) and the protocol \( \mathcal{P} \) can be described by a circuit of gates and channels, as can the interaction between \( \mathcal{E} \)
and $\mathcal{P}_I + \mathcal{S}$. For a given environment, each interaction results in a corresponding final state. The environment makes the best quantum measurement to distinguish which one of the two final states it has, and the output distributions for the two interactions are $\Gamma$ and $\Gamma_I$ respectively. Due to a result by Helstrom (see [40,41]) the maximum value of $\| \Gamma - \Gamma_I \|_1$ is simply the trace distance between the two possible final states (before the measurement). Thus $\epsilon$ is an upper bound to the trace distance between the possible final states, maximized by the environment and minimized by the simulator.

This security definition (in the model described) is useful because security of basic composition follows “by definition” [15,2].

**Theorem 1.** Suppose a protocol $\mathcal{P}$ calls a subroutine $\sigma$. If $\sigma \epsilon_{\sigma} \cdot s.r. \sigma_I$ and $\mathcal{P}[\sigma_I] \epsilon_{\mathcal{P}} \cdot s.r. \mathcal{P}_I$, then $\mathcal{P}[\sigma] \epsilon \cdot s.r. \mathcal{P}_I$ for $\epsilon \leq \epsilon_{\mathcal{P}} + \epsilon_{\sigma}$.

Theorem 1 can be generalized to any arbitrary protocol with a proper modular structure, as defined in Appendix B. An example of an improper modular structure is one with a security deadlock, but the protocols we analyze in this paper all generate proper modular structures. The idea is to represent the protocol as a tree and then apply Theorem 1 recursively to the leaves of the tree. Roughly speaking, to build the tree, represent any arbitrary protocol $\mathcal{P}$ using subprotocols $\{\sigma_i\}$ by a 1-level tree, with $\mathcal{P}$ being the parent and $\{\sigma_i\}$ the children. Recursively replace these children by trees until the leaves are the basic primitives subject to analysis, and call this the associated tree of $\mathcal{P}$. (More general modular structures, represented by acyclic directed graphs, can be transformed into trees [2].) Then the security of $\mathcal{P}$ can be stated in terms of that of the components in the tree:

**Theorem 2.** Let $\mathcal{P}$ be a protocol and $T_\mathcal{P}$ its associated tree. For each vertex $v$ in $T_\mathcal{P}$, let $\mathcal{M}_v$ be the subprotocol corresponding to $v$ with its own subprotocols $\{\mathcal{N}_i\}_{i=1,\ldots,l}$. (This can be an empty set if $v$ is a leave.) Then, if $\mathcal{M}_v[\mathcal{N}_{i_1}, \ldots, \mathcal{N}_{i_l}] \epsilon_{\mathcal{M}_v} \cdot s.r. \mathcal{M}_I$, we have $\mathcal{P} \epsilon \cdot s.r. \mathcal{P}_I$ for $\epsilon \leq \sum_v \epsilon_{\mathcal{M}_v}$.

Theorem 2 is obtained by the recursive use of Theorem 1 and the triangle inequality, replacing each subprotocol by its ideal functionality, from the highest to the lowest level (from the leaves toward the root). The distinguishability-advantage between $\mathcal{P}$ and $\mathcal{P}_I$ is upper bounded by the sum of all the individual distinguishability-advantages for the replacements.

It is worth mentioning that there is an alternative to Definition 1 above for the universal composability security definition:

**Definition 2:** $\mathcal{P}$ is said to $\epsilon$-securely realize $\mathcal{P}_I$ (shorthand $\mathcal{P} \epsilon \cdot s.r. \mathcal{P}_I$) if

$$\exists \mathcal{S} \text{ s.t. } \forall \mathcal{E}, \| \Gamma - \Gamma_I \|_1 \leq \epsilon.$$  \(2\)

In other words, the order of the quantifiers has been exchanged in this alternative. Definition 2 offers stronger security than Definition 1. The basic composition law holds for each definition – UC-secure primitives satisfying Definition 1 give composition with like security, and similarly for Definition 2. However, when composing protocols with mixed security definitions, the composition generally satisfies the weaker definition.

Definition 1 is used in the Ben-Or–Mayers model. Their analysis still holds for definition 2 for composition involving a constant number of components.

In our work, we prove UC-security for EBIT[PTP], TQA[EBIT[PTP],CI] and QA+KG according to definition 2. Thus simple applications of these protocol will inherit the stronger security.
3 Universal Composable Security for QA+KG

We now show the UC-security of QA+KG in the universal composability framework. As discussed after Definition 1, we can describe the interaction between the environment $\mathcal{E}$ and the protocol $\mathcal{P}$ (or $\mathcal{P}_1\mathcal{S}$) by a circuit of gates and channels. The distinguishability advantage is just the trace distance between the possible final states, maximized by the environment and minimized by the simulator. The circuit representation of the interaction is a very concise summary of the state at each stage of the interaction. Moreover, if we replace one circuit component by another, we can capture the difference induced on the state right after that component, and additional circuit elements cannot increase the trace distance (by the monotonicity of the trace distance under quantum operations).

With the above in mind, our proof consists of the following steps:

(1) Show that for any environment, the interactions with QA+KG and with TQA+KG result in the same final state, and they are therefore completely indistinguishable to the environment. In other words, QA+KG $\circ$s.r. TQA+KG.

Recall that QA uses $\text{QEnc}$ and $\text{PTC}$ as subroutines, as well as secret keys (given by ideal key distribution boxes $\text{KD}_I$). In QA+KG, the encryption key is recycled if QA accepts the message. So we may express the first protocol QA+KG as $(\text{QA+KG})[\text{QEnc},\text{PTC},\text{KD}_I]$. The second protocol TQA+KG first creates entanglement using PTC via the insecure channel (but Bob never tells Alice if the entanglement is accepted or rejected), next teleports the quantum message from Alice to Bob (using an ideal classical channel which is authenticated, encrypted, and hidden) and finally outputs the Bell measurement outcome in teleportation as a new key, if PTC accepts. Thus, TQA+KG can be expressed as $(\text{TP+KG})[\text{EBIT}[\text{PTC}],\text{C}_I]$.

(2) Re-write the circuit for $\text{EBIT}[\text{PTC}]$ as a circuit for $\text{EBIT}[\text{PTP}]$, a protocol creating ebits using a purity test protocol, such that the two circuits are completely indistinguishable to the environment. Therefore, $\text{EBIT}[\text{PTC}] \circ$s.r. $\text{EBIT}[\text{PTP}]$.

(3) Show that $\text{EBIT}[\text{PTP}]$ $(2\sqrt{2}\varepsilon^1/3)$-s.r. $\text{EBIT}_1$ using the soundness condition of $\text{EBIT}[\text{PTP}]$. Here, $\varepsilon \leq 2^{-s'}$ is the upper bound of probability of failure in PTP.

(4) By (2) and (3), applying Theorem 1 with $\mathcal{P}=\text{TP}$, $\sigma=\text{EBIT}[\text{PTC}]$, it follows that $\text{TP}[\text{EBIT}[\text{PTC}],\text{C}_I]$ $(2\sqrt{2}\varepsilon^1/3)$-s.r. $\text{TP}[\text{EBIT}_1,\text{C}_I]$ which 0-s.r. $Q_I$, the ideal functionality of the perfectly authenticated and encrypted quantum channel.

(5) Show that $(\text{TP+KG})[\text{EBIT}[\text{PTC}],\text{C}_I] \ 2\sqrt{2}\varepsilon^1/3$-s.r. $(\text{TP+KG})[\text{EBIT}_1,\text{C}_I]$ 0-s.r. $Q_I+\text{KD}_I$, where $\text{KD}_I$ is the ideal functionality for generating a key of a certain size between Alice and Bob.

Overall result:

Putting (1), (5), and (4) together, $(\text{QA+KG})[\text{QEnc},\text{PTC},\text{KD}_I] \ 0$-s.r. $(\text{TP+KG})[\text{EBIT}[\text{PTC}],\text{C}_I] \ 2\sqrt{2}\varepsilon^1/3$-s.r. $(\text{TP+KG})[\text{EBIT}_1,\text{C}_I] \ 0$-s.r. $Q_I+\text{KD}_I$.

So, $(\text{QA+KG})[\text{QEnc},\text{PTC},\text{KD}_I] \ 2\sqrt{2}\varepsilon^1/3$-s.r. $Q_I+\text{KD}_I$. 

We now prove these steps. Consider Circuit Diagram 1 below, with a schematic diagram for QA+KG and its interaction with the environment:

![Circuit Diagram 1](image)

Our circuit diagrams use the following conventions. (See [28] for more detail.) Time runs from left to right. The box around the environment denotes what is accessible to it. Single and double lines represent quantum and classical information (moving in time or space) respectively. Additional arrows explicitly indicate the direction of information flow. Small boxes with input/output information denote operations. Such diagrams are concise descriptions of the protocols and summarizes how the states evolve. They will be used to present a significant part of our proofs. To help readers gain familiarity with this representation, we go through the above diagram in detail.

We consider the most general environment allowed in the universal composability model. The environment chooses an arbitrary state $|\psi\rangle_{RM}$. The system $M$ carries the quantum message to be authenticated, and $R$ carries all possible correlations to the quantum message. The environment supplies the register $M$ as input to the analyzed protocol (in particular, as input to QA), by communicating $M$ to Alice. See the far left of circuit diagram 1. In the security definition for quantum authentication in Section 2.2, the system $R$ is left unchanged and is used as a reference for checking that the correlations to the message $M$ are preserved. However, when considering universal composability, the environment can share data with the adversary. Therefore, in our analysis, the system $R$ is given to the adversary who can use it as quantum side information when attacking the transmission in QA. After the attack, the adversary passes all data back to the environment (labeled $\Box$).

We now turn to the analyzed protocol QA+KG. We model the perfect keys used by the protocol by including an ideal key distribution box labeled KD$_I$, which distributes a perfect key (with 4 parts, $x, z, t, y$) between Alice and Bob that is not accessible to the environment. As a side remark, note that we can model components of a protocol mathematically with perfect devices that need not be realized physically. It simplifies further analysis when the perfect key comes from an actual protocol; we only need to check the universal composable security of the latter [12,13] (and apply Theorem 1).
As described in Section 2.2, Alice encrypts the message in system $M$ with $\vec{\sigma}_{xz}$ (the $m$-qubit Pauli operator specified by the two $m$-bit keys $x$ and $z$) and then applies the purity test code by encoding in an error correcting code indexed by key $t$ and injecting an error syndrome indexed by key $y$. Without loss of generality, this encoding operation is a unitary transformation $C_t$ that acts jointly on the input logical state and the syndrome $y$. The encrypted and encoded state is then transmitted to the adversary (part of the environment here), which can attack it jointly with $R$ using any physical process. After that, the possibly altered state is received by Bob. He first applies $D_t$, which reverses $C_t$ to return some quantum state $\psi$ and a syndrome $y'$. Then $y$ and $y'$ are compared in the “Eq” operation, which outputs in register $V$ the state $\text{ACC}$ if $y = y'$ and $\text{REJ}$ otherwise. Bob also decrypts the quantum output of $D_t$ (5) with $\vec{\sigma}_{xz}$, producing a final quantum message in system $M$. Finally, the $\text{REJ}$ state will trigger the filter to replace the final quantum message and the keys $x, z$ by error symbols, else the systems $VM$ will be the output for QA, and the keys $x, z$ will be the output for KG.

**Step (1):** Showing QA+KG 0-s.r. $\text{TQA+KG=} (\text{TP+KG})[\text{EBIT}[\text{PTC}],[\text{CI}])$.

Consider Circuit Diagram 2 below, with a schematic diagram for TQA+KG and its interaction with the environment.

Circuit diagrams 1 and 2 (for the interaction between the environment and QA+KG and TQA+KG) differ in only two places. First, encryption in QA+KG is done in TQA+KG by Alice preparing $m$ ebits $\Phi \otimes m$ (all $2m$ qubits with her) and applying Bell measurements on the $m$ halves of the ebits and the incoming message $M$. Having measurement outcomes $x, z$ in TQA+KG is the same as applying $\vec{\sigma}_{xz}$ directly to the message in QA+KG. The states labeled $\circled{3}$ are identical in both protocols, though prepared differently. More specifically, $(x, z)$ arises differently, but it is completely random in both protocols, and for a given $(x, z)$, the postmeasurement states in $\circled{3}$ are identical. Second, the measurement outcomes $x, z$ are communicated to Bob by a hidden channel $C_I$ whose execution and content are unknown by anyone except for Alice and Bob, and the content is transmitted exactly. Such an unrealistic resource can be part of an ideal or partially ideal functionality against which
we are comparing the actual protocol. Neither change will affect the final state of the environment, and thus the two interactions are identical from the point of view of the environment.

Now, in TQA+KG, Alice’s Bell measurement can formally be delayed until after the “Eq” operation. Thus, the dashed box in Circuit Diagram 2 with input 3 and outputs 4, 5 is only used in transmitting half of $\Phi^{\otimes m}$ (the other half being 6). So, TQA+KG can be interpreted as teleportation TP calling EBIT[PTC] in addition to the hidden channel $C_1$ which is perfectly encrypted and authenticated classical channel as subroutines. We write TQA+KG = (TP+KG)[EBIT[PTC],$C_1$], to emphasize the modular structure and the potentially insecure components. Note that in this version of EBIT[PTC], Alice does not know whether Bob accepts or rejects, since we want to make a correspondence with the noninteractive QA.

**Step (2):** Re-expressing the circuit for EBIT[PTC] as a circuit for EBIT[PTP].

Consider EBIT[PTC], the components in TQA+KG that start with the state labeled by 3⑥ and led to the state labeled by 4⑤6. There is no input message, and it simply creates $m$ ebits using the purity test code. We extract it as the left diagram in the following (Circuit Diagram E1) for the analysis of its UC security. We include in EBIT[PTC] the register $V$ that holds the measurement result ACC or REJ, which is known to Bob. Alice will not know if Bob accepts or rejects in our application, but our analysis also holds for interactive protocols. The filter operation replaces 5 by an error symbol if $V$ is in the state REJ. (For interactive protocols, 6 will also be replaced by an error symbol.) We use $A$ and $B$ to denote the registers holding the final output “EPR pairs” (systems ⑤ and ⑥ in EBIT[PTC]). The circuit components of EBIT[PTP] (in Circuit Diagram E2) are defined similarly, except $y$ is now a measurement outcome that is communicated from Alice to Bob using a perfect classical channel. The subroutine EBIT[PTP] is only used in the analysis and the requirement to send $y$ does not play a big role, so, we omit the explicit subroutine label $C_1$ for simplicity.

Now, the states labeled by 4⑤6 are the same in EBIT[PTC] (circuit diagram E1, left) and EBIT[PTP] (circuit diagram E2, right) shown above. This was proved in Appendix E of [1] and we provide an elementary proof in Appendix C.
Step (3): Showing EBIT[PTP] \( (2\sqrt{2}e^{1/3}) \)-s.r. EBIT1.

We now analyze EBIT[PTP] in the UC framework against the ideal protocol EBIT1 defined as follows. EBIT1 takes an input in the state ACC or REJ and outputs m perfect ebits or an error state ERR accordingly. In this paper, the state ERR is a tensor product of a maximally mixed state on system A and an error symbol on system B. (For interactive protocols, both AB can be in error symbols.) This choice for ERR will minimize the distinguishability advantage between EBIT[PTP] and EBIT1.

The simulator \( S \) runs a “dummy” execution of EBIT[PTP] and takes the dummy ACC or REJ state and feeds it into EBIT1. Note that this \( S \) is independent of the environment. So, the quantifiers are as given by (3) in Definition 2.

The distinguishability advantage can be upper bounded by the trace distance between the two states held by the environment after the executions of EBIT[PTP] and \( S+E'BIT1 \). Let these states be denoted by \( \eta^{PT} \) and \( \eta^I \). Then we can write

\[
\eta^{PT} = p_{ACC} \xi_{ABE} \otimes ACC + p_{REJ} ERR_{AB} \otimes \mu_{E} \otimes REJ \\
\eta^I = p_{ACC} \Phi^{\otimes m} \otimes \xi_{E} \otimes ACC + p_{REJ} ERR_{AB} \otimes \mu_{E} \otimes REJ
\]

where \( \xi_{E} = Tr_{AB} \xi_{ABE} \) (likewise, \( \xi_{AB} = Tr_{E} \xi_{ABE} \), and similarly for \( \eta \)), and ERR_{AB} is the tensor product of a maximally mixed state on A and an error symbol on B, chosen so that the REJ terms are identical in \( \eta^{PT} \) and \( \eta^I \). Note that \( p_{ACC}, p_{REJ}, \xi_{ABE}, \) and \( \mu_{E} \) are determined both by the adversarial attack and the PTP. Our choice of the simulator ensures that the reduced states on \( E \), conditioned on each of ACC and REJ, are the same for \( \eta^{PT} \) and \( \eta^I \).

From (3), \( \| \eta^{PT} - \eta^I \|_1 = p_{ACC} \| \xi_{ABE} - \Phi_{AB}^{\otimes m} \otimes \xi_{E} \|_1 \). We upper bound this for \( p_{ACC} \leq e^{1/3} \) and \( p_{ACC} > e^{1/3} \) separately. If \( p_{ACC} \leq e^{1/3} \), then, the bound is \( 2p_{ACC} \leq 2e^{1/3} \) since the trace distance is at most 2. If \( p_{ACC} > e^{1/3} \), we seek an upper bound for \( \| \xi_{ABE} - \Phi_{AB}^{\otimes m} \otimes \xi_{E} \|_1 \). This is equivalent to finding a lower bound for the fidelity \( F(\xi_{ABE}, \Phi_{AB}^{\otimes m} \otimes \xi_{E}) \). Since the fidelity is the maximum overlap squared between all purifications, any specific purifications of \( \xi_{ABE} \) and \( \xi_{E} \) give a lower bound.

Any purification of \( \xi_{ABE} \) can be expressed as

\[
|\xi_{ABE}\rangle = \sqrt{1-\alpha} |\Phi^{\otimes m}_{AB}\rangle \otimes |a\rangle_{ER} + \sum_i \sqrt{\alpha_i} |\Psi_i\rangle_{AB} \otimes |b_i\rangle_{ER}
\]

where \( |\Phi^{\otimes m}_{AB}\rangle \) and \{\( |\Psi_i\rangle\}_i \) form a basis for \( (C^2)^{\otimes 2m} \), \( |a\rangle \) and \( |b_i\rangle \) are unit vectors, and \( \alpha = \sum_i \alpha_i \). Furthermore, \( \xi_{ER} = (1-\alpha)|a\rangle \langle a| + \sum_i \alpha_i |b_i\rangle \langle b_i| \), so, \( F(|a\rangle \langle a|, \xi_{ER}) = \langle a|\xi_{ER}|a\rangle \geq 1-\alpha \). Now, \( F(Tr_{R}|a\rangle \langle a|, \xi_{E}) \geq F(|a\rangle \langle a|, \xi_{ER}) \geq 1-\alpha \), so, there exists a purification \( |a^*\rangle \) of \( \xi_E \) such that \( \langle a^*|a\rangle \geq 1-\alpha \). Using this and (4), we obtain \( \langle a^*|a\rangle \|\Phi_{AB}^{\otimes m} \otimes |a^*\rangle_{ER} \|_2^2 \geq (1-\alpha)^2 \). We now show that \( \alpha \leq e^{2/3} \). To do so, note that the soundness condition of the purity test protocol in this context can be expressed as \( Tr \left[ (Tr_{\Phi}(\eta^{PT}) (I-\Phi^{\otimes m}_{AB}) \otimes ACC) \right] \leq \epsilon \). A direct substitution of (3) into the condition gives
\[
    \text{Tr} \left[ \xi_{AB} \left( I - \Phi \otimes m \right) \right] \leq \epsilon/p_{\text{acc}} < \epsilon^{2/3}. \tag{5}
\]

We can obtain \(\xi_{AB}\) from (4) by tracing out \(ER\) in \(|\xi\rangle\)_{ABER}; substituting this \(\xi_{AB}\) into (5) gives \(\alpha \leq \epsilon^{2/3}\), as claimed. Together,

\[
    \|\xi_{AB} - \Phi_{AB} \otimes \xi_{E}\|_1 \leq 2 \sqrt{1 - F(\xi_{AB}, \Phi_{AB} \otimes \xi_{E})} \leq 2 \sqrt{2} \alpha \leq 2 \sqrt{2} \epsilon^{1/3} \tag{12}
\]

Thus, EBIT[PTP] (2\(\sqrt{2}\epsilon^{1/3}\))-s.r. EBIT\(_I\).

**Step (4):** By theorem 11 and the above, we conclude that TP[EBIT[PTC],\(_C\)] (2\(\sqrt{2}\epsilon^{1/3}\))-s.r. TP[EBIT\(_I\),\(_C\)]. Note that TP[EBIT\(_I\),\(_C\)] in turn can serve as our definition of the ideal functionality \(Q_I\). The ideal functionality \(Q_I\) uses EBIT\(_I\) and \(C_I\) as subroutines, takes input \(M\) (the quantum message) and has two outputs \(V\) and \(M\). If the subroutine EBIT\(_I\) outputs \(\text{rej}\), \(Q_I\) also outputs \(\text{rej}\) in \(V\) and an error message in \(M\). If EBIT\(_I\) outputs \(\text{acc}\), then, \(Q_I\) provides an encrypted, authenticated, and hidden channel that transmits \(M\).

**Step (5):** Showing (TP+KG)[EBIT[PTC],\(_C\)] (2\(\sqrt{2}\epsilon^{1/3}\))-s.r. (TP+KG)[EBIT\(_I\),\(_C\)] 0-s.r. \(Q_I+KD_I\).

Consider Circuit Diagram 2 again, and replace the subroutine EBIT[PTC] by EBIT\(_I\)+\(S\) so the analyzed protocol becomes (TP+KG)[EBIT\(_I\),\(_C\)]. By step (3), (TP+KG)[EBIT[PTC],\(_C\)] (2\(\sqrt{2}\epsilon^{1/3}\))-s.r. (TP+KG)[EBIT\(_I\),\(_C\)]. We now show that (TP+KG)[EBIT\(_I\),\(_C\)] 0-s.r. \(Q_I+KD_I\). In \(Q_I+KD_I\), if Bob outputs \(\text{rej}\), Alice will hold a uniformly random variable in her key output system while Bob will hold an error message in his key output system and also in \(M\). (If Bob can send a bit to Alice, she will replace her random variable by an error message as well.)

First, the \(\text{acc}\) event occurs with the same probabilities in both (TP+KG)[EBIT\(_I\),\(_C\)] and \(Q_I+KD_I\), and similarly for the \(\text{rej}\) event. When Bob outputs \(\text{rej}\), for both protocols, we have the reduced state in (1), tensor product with an error message in \(M\), a uniformly random output on Alice’s key system, and an error message on Bob’s key system. When Bob outputs \(\text{acc}\), \(M\) is teleported perfectly in (TP+KG)[EBIT\(_I\),\(_C\)]. Furthermore, the measurement outcome \((x, z)\) is uncorrelated with \(RM\) after the operation \(\vec{\sigma}_{x, z}\) and uncorrelated with everything else the environment has, so it is indistinguishable from an ideal key. Thus (TP+KG)[EBIT\(_I\),\(_C\)] 0-s.r. \(Q_I+KD_I\).

**Overall result:**
Putting (1), (5), and (4) together, (QA+KG)[QEnc,PTC,KD\(_I\)] 0-s.r. (TP+KG)[EBIT[PTC],\(_C\)] (2\(\sqrt{2}\epsilon^{1/3}\))-s.r. (TP+KG)[EBIT\(_I\),\(_C\)] 0-s.r. \(Q_I+KD_I\).

So, (QA+KG)[QEnc,PTC,KD\(_I\)] 2\(\sqrt{2}\epsilon^{1/3}\)-s.r. \(Q_I+KD_I\).

### 4 Discussion and open problems

**Other results:** See appendices 14 and 15 for proofs.

First, a variant of QA+KG called PSQA+KG is UC secure when the message is known to be pure, such as when Alice prepares it herself. PSQA uses the approximate encryption scheme \(\approx\text{QEnc}\) in place of QEnc, and \(\approx\text{QEnc}\) uses only half the key needed in QEnc. The proof involves a variant of TQA+KG in which a remote state preparation (RSP) scheme is used in place of TP.

The Wegman-Carter classical authentication scheme is UC secure in the quantum composability framework. This is important in the light of the frequent need of authenticated classical channels.
in many quantum cryptographic protocols. We provide a short informal proof, and refer to [22] for a more complete discussion.

Discussion of our results and further open questions

Using 1 bit of back communication, and in the absence of a detected attack, the key costs of authentication can be made negligible. We have also discussed two other methods to reduce the key cost in Sect. 1. In terms of communication cost, both QA+KG and TQA+KG are superior to the QKD-based methods discussed in Section 1. The main drawback of both QA+KG and TQA+KG is that they require a large initial key. This can be circumvented at the expense of a slowly growing round complexity, however – one can divide the \( m \)-qubit messages into \( \sqrt{m} \) groups of \( \sqrt{m} \) qubits each and apply QA+KG to each sequentially. Besides the \( m \) qubits of quantum communication, all other resources, including the initial key and all classical communication, will be sublinear in \( m \). Similarly, TQA+KG can be used instead, but an extra \( 2m \) classical bits of forward communication is needed to gain the extra data protection.

A remaining question is the extent to which recycling can be done in QA+KG when the authentication output is rej. Ref. [21], which considers encryption of classical messages, proved that at least \( m-1 \) key-bits have to be discarded. Because of superdense coding, our intuition is that one may need to discard roughly \( 2m \) key-bits for unknown quantum messages. Surprisingly, partial key recycling is recently proved to be possible in [24].

At the other extreme, [21] found that the entire key, even the authentication tag, can be recycled in their scheme (by using much more key to start with) in the case of acc. More recently, [23] proposed quantum authentication scheme with the same feature (though their schemes also require more key than QA), and [24] proved that the entire key can be recycled in QA. While qualitatively interesting, this question is not of practical importance; we could easily make up for this small extra recycled key without it by having Alice append that number of ebit-halves to the message and keeping the other halves herself. Upon passing the authentication test, this will make up for the consumed keys \( t,y \) and more if so desired. Meanwhile, the number of ebits is sublinear in the message size so the resource counting is unaffected.

In fact, Bennett posed to us the question of using QA+KG as a simple means to perform QKD. QKD can resist very high transmission noise and eavesdropping. The main challenge here is that noise is not dealt with efficiently by PTC, which is an error detecting code rather than an error correcting code. The solution turns out very simple and there are two equivalent ways to see it.

More generally, consider the case of quantum message authentication through uses of a noisy channel \( \mathcal{N} \). This information can be given to Alice and Bob, or they can make reasonable estimates and assumptions about the underlying channel. The adversary can further tamper with the transmission. Alice and Bob should agree on an error correcting code. Let \( \mathcal{N}' \) denote the composition of the encoding, transmission by \( \mathcal{N} \), and decoding. We require \( \mathcal{N}' \) to approximate \((m+l)\)-qubit of noiseless communication in the diamond norm. (This is the usual requirement for transmission through noisy channel even in the absence of adversarial attack.) We keep all the steps in QA+KG, except the \((m+l)\)-qubit message transmission is replaced by \( \mathcal{N}' \). In other words, we apply QA+KG to \( \mathcal{N}' \) and additional adversarial attack on \( \mathcal{N} \) is transformed to an effective attack on \( \mathcal{N}' \). Thus the security of this new protocol reduces to that of QA+KG. Note that \( l \) is sublinear in \( m \) in QA, so this method has communication rate similar to the case without an adversary, and does not require any extra
key. In the end, regular error correction handles the regular channel noise, and QA handles the remaining adversarial attack.

A second way to see this solution is to note that PTP can be composed with a mixed-state entanglement purification protocol (this solution was informally suggested to us by Anne Broadbent). For TQA', any composably secure ebits (with likewise secure classical channel) gives a composably secure interactive quantum authentication scheme. There is no difference if these ebits are established via a noisy channel and are purified, as long as the resulting ebits are tested by PTP. To obtain a noninteractive quantum authentication scheme, the additional purification protocol has to be equivalent to an error correcting code as described in [43], or [1], or in step (2).

In similar spirits, the entanglement testing scheme in [26] can produce composably secure ebits using a constant amount of secure quantum communication that only depends on the desired accuracy. This provides yet another secure authentication scheme with slightly different initial resource requirement in the interactive setting. This new entanglement test can be further adapted to an noninteractive by replacing the quantum communication by trusted entanglement (in turns by using PTC with constant key size) and authentication of classical messages (again of constant size).

Finally, when the transmitted state is known to the sender, the lower bound for the key remains open.

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A Notation

We gather notation used frequently in the paper, roughly in order of first appearance:

- $\Phi$: A perfect EPR pair $\frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$
- ebit: A unit for entanglement contained in $\Phi$
- QEnc: The particular encryption scheme that applies a random $m$-qubit Pauli matrix $\vec{\sigma}_{xx}$ to an $m$-qubit message. This requires $2m$ bits of key.
- QA: The particular noninteractive scheme proposed in [1] which in turn, applies QEnc, a quantum purity test code (PTC), and then a secret syndrome.
- PTC: A purity test code with error $\epsilon$ is a set of quantum codes such that given any nontrivial Pauli error, at least a fraction $1 - \epsilon$ of the codes detect it.
- PTP: A purity test protocol with error $\epsilon$ is an LOCC scheme that, with probability less than $\epsilon$, outputs a quantum state tagged $\text{acc}$ but orthogonal to $m$ ebits.
TP: Teleportation

TQA': The interactive scheme to achieve quantum authentication using TP, in which PTP is used to establish entanglement.

TQA: a modification of TQA' in which Bob never tells Alice whether the entanglement is accepted or rejected.

KD₁: In ideal key distribution box that simply provides Alice and Bob a perfect, secret, shard key. A variant was considered in [12] that takes an auxiliary input bit, conditioned on which either a key or an error message will be output.

C₁: A perfect classical channel, encrypted, authenticated, and hidden.

EBIT₁: The ideal functionality for generating ebits, where an input acc vs rej will control whether the output is a number of perfect ebits or an error symbol.

Q₁: Our model of a perfect quantum channel which is TP[EBIT₁,C₁].

QKD: Quantum key distribution

QA+KG: QA augmented with recycling of the key used for QEnc if authentication passes. It is treated as a pair that performs QA and key generation, while consuming a key from an ideal KD box KD₁.

Alice and Bob: Two honest parties trying to communicate

Eve: An active adversary

Capitalized letters often (though not always) denote random variables and the corresponding uncapsulated letters denote particular outcomes.

log: Logarithm in base 2

ρ: Generic symbol for a density matrix

|·⟩: A vector in a Hilbert space, with label “·”

|·⟩⟨·|: The projector onto the subspace spanned by |·⟩, also known as “outer-product” of the ket |·⟩ and the bra ⟨·|. We will simply write “·” in place of the bra and ket.

|ψ⟩: A unit vector or pure state. Its density matrix is given by \( ρ = |ψ⟩⟨ψ| \).

Tr(·): The trace

Tr_{H₁}(·): The partial trace over the system H₁. Let \( ρ_{12} \) be the density matrix for a joint state on H₁ and H₂. Tr_{H₁}(ρ_{12}) is the state after H₁ is discarded.

∥·∥₁: The Schatten 1-norm.

F: The fidelity. For two states \( ρ_1, ρ_2 \) in \( H \), \( F(ρ_1, ρ_2) = \max_{|ψ_1⟩, |ψ_2⟩} |⟨ψ₁|ψ₂⟩|^2 \) where \( |ψ_{1,2}⟩ \in H⊗H' \) are “purifications” of \( ρ_{1,2} \) (i.e., \( Tr_{H'}(|ψ_{1,2}⟩⟨ψ_{1,2}|) = ρ_{1,2} \)), and \( ⟨·|·⟩ \) is the inner product. Here, we can take \( \dim(H') = \dim(H) \).

σ, P, σ₁, P₁: σ and P are generic labels for protocols, with σ possibly used as a subroutine. The symbol of a protocol with a subscript I denotes the ideal functionality of the protocol.

• P[σ]: A protocol P calling a subroutine σ.

• P₁+P₂: Conjoining two protocols P₁ and P₂.

Note that a subroutine σ receives an input from the main protocol P and returns an output to it. In contrast, for conjoining protocols, each may have its own input and output. The protocols may be run in parallel or sequentially. In particular, information are generally exchanged between the two, and each protocol may provide an input to the other.

• P₁ = P₂ if they have the same circuit (but different interpretations and/or modular structures).

• E, S: The environment and the simulator. These are sets of registers and operations and they are sometimes personified in our discussion.
– $\Gamma, \Gamma_1$: The random variables describing output bits of $E$ when interacting with $P$ and $P_1 + S$ respectively.
– $\epsilon$-s.r.: $P$ $\epsilon$-securely realizes $P_1$ (see mathematical definition in (11)). $\epsilon$ is called the distinguishability-advantage between $P$ and $P_1$.
– $T_P$: The associated tree for a protocol $P$

B  The Simplified Universal Composability (Ben-Or-Mayers) model

Our current setting is simpler than that considered in [15] in two ways. First, we are concerned with unconditional security only. Second, there is no unknown corruption of any party – Alice and Bob are honest and Eve is adversarial. We do not use the formal corruption rules.

We consider the acyclic quantum circuit model (see, for example, [44,45]), with an important extension [2] (see also the endnotes [46]). Throughout the paper, we only consider circuits in the extended model.

1. Structure of a protocol. A (cryptographic) protocol $P$ can be viewed as a quantum circuit in the extended model [2,46], consisting of inputs, outputs, a set of registers, and some partially ordered operations.

A protocol may consist of a number of subprotocols and parties. Each subprotocol consists of smaller units called “unit-roles,” within which the operations are considered “local.” For example, the operations and registers of each party in each subprotocol form a unit-role. Communications between unit-roles within a subprotocol represent internal communications; those between unit-roles in different subprotocols represent input/output of data to the subprotocols. A channel is modeled by an ordered pair of operations by the sender and receiver on a shared register. The channel available for the communication determines its security features.

2. The game: security in terms of indistinguishability from the ideal functionality. Let $P_1$ denote the ideal functionality of $P$. Intuitively, $P$ is secure (in a sense defined by $P_1$) if $P$ and $P_1$ behave similarly under any adversarial attack. “Similarity” between $P$ and $P_1$ is modeled by a game between an environment $E$ and a simulator $S$. These are sets of registers and operations to be defined, and they are sometimes personified in our discussion. In general, $P$ and $P_1$ have very different internal structures and are very distinguishable, and the simulator $S$ is added to $P_1$ to make an extended ideal protocol $P_1 + S$ that is less distinguishable from $P$. $E$ consists of the adversaries that act against $P$ and an application protocol that calls $P$ as a subprotocol. At the beginning of the game, $P$ or $P_1 + S$ are picked at random. $E$ will interact with the chosen protocol (running it and attacking its vulnerabilities), and will output a bit $\Gamma$ at the end of the game. The similarity between $P$ and $P_1 + S$ (or the lack of it) is captured in the statistical difference in the output bit $\Gamma$. See Fig. 1 for a summary of the game.

3. Valid $E$. The application and adversarial strategy of $E$ are first chosen. (These cannot depend on whether $E$ is interacting with $P$ or $P_1 + S$.) $E$ has to obey quantum mechanics, but is otherwise unlimited in computation power. If $P$ is chosen in the game, $E$ can (i) control the input/output of $P$, (ii) attack insecure internal communication as allowed by the channel type, (iii) direct the adversarial parties to interact with the honest parties in $P$. $E + P$ has to be an acyclic circuit in the extended model [2,46]. Without loss of generality, an adversary can be modeled to only forward messages between the environment and the protocol, and the actual attack is executed by the environment. (See Lemma 12 in [17].)
4. Valid $P_1$ and $S$. If $P_1 + S$ is chosen in the game, $E$ (i) controls the input/output of $P_1$ as before. However, the interaction given by (ii) and (iii) above will now occur between $E$ and $S$ instead. ($S$ is impersonating or simulating $P_1$.) The strategy of $S$ can depend on the strategy of $E$. $P_1$ should have the same input/output structure as $P$, but is otherwise arbitrary. (Of course, the security definition is only useful if $P_1$ carries the security features we want to prove for $P$.) In particular, $P_1$ may be defined with internal channels and adversaries different from those of $P$. $S$ can (ii)′ attack insecure internal communication of $P_1$ and (iii)′ simulate the adversarial parties when interacting with the honest parties in $P_1$. Thus, $P_1$ exchanges information with $S$, and this can modified the security features of $P_1$. To $E$, $S$ acts like part of $P_1$, “padding” it to look like $P$, while to $P_1$, $S$ acts like part of $E$. It is amusing to think of $S$ as making a “man-in-the-middle” attack between $E$ and $P_1$. Finally, $E + P_1 + S$ has to be an acyclic circuit in the extended circuit model [2, 46]. Let the output bit be $\Gamma_1$ in this case. See Fig. 1 for a summary of the rules.

Fig. 1. The game defining the composable security definition. The curved region in $E$ represents the adversaries against $P$, and the curved region in $S$ represents the adversaries against $P_1$. We label the types of interactions as described in the text.

We now restate the universal composable security definition and give a slightly more extended description of the fundamental composability theorems.

**Definition 1:** $P$ is said to $\epsilon$-securely realize $P_1$ (shorthand $P$ $\epsilon$-s.r. $P_1$) if

$$\forall E \exists S \text{ s.t. } ||\Gamma - \Gamma_1||_1 \leq \epsilon.$$  

We call $\epsilon$ in (6) the distinguishability-advantage between $P$ and $P_1$. This security definition (in the model described) is useful because security of basic composition follows “by definition” [15,2]. We have the following simple version of a universal composability theorem.

**Theorem 1.** Suppose a protocol $P$ calls a subroutine $\sigma$. If $\sigma \epsilon_\sigma$-s.r. $\sigma_1$ and $P[\sigma_1] \epsilon_P$-s.r. $P_1$, then $P[\sigma]$ $\epsilon$-s.r. $P_1$ for $\epsilon \leq \epsilon_P + \epsilon_\sigma$.

Theorem 1 can be generalized to any arbitrary protocol with a proper modular structure. An example of an improper modular structure is one with a security deadlock, in which the securities of two components are interdependent.

Proper modular structures can be characterized as follows. Let $P[\sigma_1, \sigma_2, \ldots]$ be any arbitrary protocol using a number of subprotocols. This can be represented by a 1-level tree, with $P$ being the parent and $\{\sigma_i\}$ the children. For each $\sigma_i$ that uses other subprotocols, replace the corresponding node by an appropriate 1-level subtree. This is done recursively, until the highest-level subprotocols (the leaves) call no other subprotocols. These are the primitives. It was proved in [2] that more general modular structures, represented by an acyclic directed graph, can be transformed to a tree.
The following composability theorem relates the security of a protocol $P$ to the security of all the components in the tree.

**Theorem 2.** Let $P$ be a protocol and $T_P$ its associated tree. For each vertex $v$ in $T_P$, let $M_v$ be the subprotocol corresponding to $v$ with its own subprotocols $\{N_i\}_{i=1,\ldots,\ell}$. (This can be an empty set if $v$ is a leaf.) Then, if $M_v[N_1,\ldots,N_\ell] \in \epsilon_{M_v}$-s.r. $M_1$, we have $P \in \epsilon$-s.r. $P_1$ for $\epsilon \leq \sum_v \epsilon_{M_v}$.

Theorem 2 is obtained by the recursive use of Theorem 1 and the triangle inequality, replacing each subprotocol by its ideal functionality, from the highest to the lowest level (from the leaves toward the root). The distinguishability-advantage between $P$ and $P_1$ is upper bounded by the sum of all the individual distinguishability-advantages between pairs of protocols before and after each replacement.

C The extended transpose trick

One of the most useful tricks in quantum information theory is that a transformation acting on one half of a maximally entangled state can be implemented by applying a different transformation acting on the other half.

We give an extended version of this trick allowing changes in the dimensions.

Let $M = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} M_{ji} |i\rangle\langle j|$ be a possibly rectangular matrix, and $M^T = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} M_{ji} |i\rangle\langle j|$ be its transpose.

**Lemma 1:** $(M^T \otimes I) \sum_{j=1}^{d_2} |j\rangle|j\rangle = (I \otimes M) \sum_{i=1}^{d_1} |i\rangle|i\rangle$ ($\in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$).

**Proof:** LHS $= \sum_{j=1}^{d_2} (M^T |j\rangle) |j\rangle = \sum_{j=1}^{d_2} \sum_{i=1}^{d_1} M_{ji} |i\rangle |j\rangle = \sum_{i=1}^{d_1} |i\rangle (\sum_{j=1}^{d_2} M_{ji} |j\rangle) = \sum_{i=1}^{d_1} |i\rangle (M |i\rangle)$

$= \text{RHS}$.

Note that we are considering states differing from the maximally entangled states by a relevant normalization.

Let $U$ be a square matrix acting on systems 1 and 2 of $d$ and $d_2$ dimensions. Then,

$$(U_{12} \otimes I_3) \left[ \sum_{j=1}^{d_2} |j\rangle_1 \langle j|_2 |j\rangle_3 \right] = (I_{12} \otimes \langle y|_4 U_{43}^T \right) \sum_{i=1}^{d_2} |i\rangle_{12} |i\rangle_{43}.$$  \hspace{1cm} (7)

**Proof:** The LHS can essentially be interpreted as the state obtained by applying $U_{12}|y\rangle_1$ to system 2, where $U_{12}|y\rangle_1$ is the rectangular block of $U_{12}$ corresponding to $y$ ($d_2$ contiguous columns). Applying the first claim with $U_{12}|y\rangle_1 \rightarrow M^T$ and $d_1 \rightarrow d_2$, the resulting state is given by $(I_{12} \otimes M_{43}) \sum_{i=1}^{d_1} |i\rangle_{12} |i\rangle_{43}$ exactly as claimed.

The LHS has the interpretation that we take $U_{12}$ as a real, unitary matrix that encodes the logical state and the syndrome $y$ into the codeword acted on by error consistent with $y$.

The RHS, with $U_{12}^T = U_{12}^\dagger$ has the interpretation as a decoding into the logical space and the syndrome, with postselection on outcome $y$.

The equality in lemma 2 exactly proves the equivalence between EBIT[PTC] and EBIT[PTP].

D Security of PSQA+KG

Recall from Section 2.1 that in quantum encryption, Alice and Bob share a key $K$ in which the realization $k$ occurs with probability $p_k$. To send a message $\rho$, Alice transmits $E_k(\rho)$ and Bob applies
\[ D_k \text{ to retrieve } \rho. \text{ The approximate soundness condition is given by} \]
\[ \forall |\psi\rangle_{MR} \left\| (R \otimes I)(\psi) - \rho_0 \otimes \text{Tr}_M(\psi) \right\|_1 < \delta \]
where \( \rho_0 \) is independent of \( \psi \) and \( \delta \) is a vanishing security parameter. We focus on \( E_k(\rho) = U_k \rho U_k^\dagger \) and uniform \( p_k \).

The particular protocol \( \text{QEnc} \) has \( k = (x_1, z_1, \cdots, x_m, z_m) \) and \( U_k = \sigma_{x_1z_1} \otimes \cdots \otimes \sigma_{x_mz_m} =: \sigma_{xz} \), with \( 2^{2m} \) values of \( k \), or key size \( 2m \) bits. In \cite{29}, another scheme called \( \approx \text{QEnc} \) is found (existentially) such that only \( 134m^2/\delta^2 \) values of \( k \) are used, but the unitaries \( U_k \) are more complicated. It has weaker security, in that \( \forall \rho, \left\| \frac{1}{m} \sum_k U_k \rho U_k^\dagger - \frac{1}{2m} I \right\|_\infty \leq \frac{\delta}{2m} \), and it satisfies the inequality of the soundness condition only if the adversary does not have the purification of \( \rho \) (a powerful form of quantum side information), such as when \( \rho \) is pure.

Recall from Section 2.2 that the specific protocol \( \text{QA} \) given in \cite{1} first applies \( \text{QEnc} \) to the \( m \)-qubit message, followed by encoding with a purity test code (choose a code \( C_t \) from a set based on a random value of \( t \)) and then applying an operation corresponding to a random syndrome \( y \) (all parameters as described above).

Naturally, a question arises, whether one can replace \( \text{QEnc} \) by \( \approx \text{QEnc} \) if the input for quantum message authentication is promised to be pure. We call the resulting protocol \( \text{PSQA} \) (standing for Pure State QA), and again, we can append key recycling as an additional step.

In this section, we prove the composable security of \( \text{PSQA+KG} \). We believe the proof techniques are of independent interest.

The main challenge is to model the promise that a pure state is given to Alice to be transmitted. We handle this by imposing a restriction on the environment, and call this restricted set of environments \( E^X \) in the analysis of \( \text{PSQA+KG} \). Fix a mapping between a set of classical labels and a set of \( m \)-qubit pure quantum states \( x \rightarrow |\psi_x\rangle \). The label can be real-valued and \( |\psi_x\rangle \) unrestricted. For each environment \( E \in E^X \), \( E \) can choose a value \( x \) and this results in Alice receiving an input \( |\psi_x\rangle \) which is unknown to her. One possible way this can happen is that a trusted party receives \( x \) and then prepares \( |\psi_x\rangle \) and gives it to Alice.

Recall from Section 2.3 that composable security can be proved directly if for each environment \( E \) (which is then fixed), there exists a simulator, \( S \), such that \( \text{PSQA+KG} \) is indistinguishable from an appropriately chosen ideal functionality conjoining \( S \). For each \( E \), this simulator \( S \) for \( \text{PSQA+KG} \) can be chosen to be the simulator for a different protocol \( \text{PSQA}^X \) (to be defined), against the same environment \( E \). In \( \text{PSQA}^X \), an input \( x \) is given to Alice, who prepares \( |\psi_x\rangle \) and then runs \( \text{PSQA+KG} \). Since the two protocols, \( \text{PSQA+KG} \) and \( \text{PSQA}^X \), are exactly indistinguishable to each environment \( E \), a simulator in the analysis of latter gives the same distinguishability advantage for the former. It therefore suffices to prove composable security for \( \text{PSQA}^X \) against all \( E \in E^X \).

The security proof for \( \text{PSQA}^X \) is similar to that for \( \text{QA+KG} \), and can be done by defining a sequence of protocols, the first being \( \text{PSQA}^X \) and the last an ideal protocol, such that each protocol is similar to the next.

The first protocol is \( \text{PSQA}^X \), the second is \( \text{PSRQA}^X \) which is the analogue of \( \text{TQA+KG} \) in which remote state preparation (RSP) is used in place of teleportation, i.e., \( \text{PSQA}^X = (\text{RSP+KG})[\text{EBIT}[\text{PTC}], C_1] \). \( \text{PSRQA}^X \) uses ebits prepared by insecure means. The third protocol is \( (\text{RSP+KG})[\text{EBIT_1}, C_1] \). It has a small probability of failure \( \delta \) that is not caused by any
adversary, and provides an ideal functionality for the analysis. The analysis is identical to that for \(QA+KG\), except for step (1). It thus remains to show the similarity between \(PSQA^X+KG\) and \(PSRQA^X+KG\).

Recall from Section 2.1 that there is a one-to-one correspondence between \(\approx QEnc\) and RSP. In RSP, Alice and Bob shares \(m\) ebits, and to transmit a state \(\rho\), Alice applies a measurement \(M\) to her half of the ebits, and sends the outcome to Bob. The POVM has the form \(\{\frac{1}{M} (U_k \rho U_k^\dagger)^T\}_k\) for \(M = \| \sum_k U_k \rho U_k^\dagger \|_\infty\), together with an extra POVM element \(F = I - \frac{1}{M} (\sum_k U_k \rho U_k^\dagger)^T\) (we say that \(k = f\)). Conditioned on an outcome \(k \neq f\), Bob’s half of the ebits becomes \(U_k \rho U_k^\dagger\) (this can be proved by the transpose trick in the previous appendix), but if \(k = f\), RSP fails. Ref. [29] shows that taking \(134m2^m/\delta^2\) different \(U_k\)’s is sufficient to ensure that \(\Pr(f) < \delta\).

Once again, consider schematic diagrams, now for \(PSQA^X+KG\) and \(PSRQA^X+KG\):

![Circuit Diagram 3](image1.png)

![Circuit Diagram 4](image2.png)
In PSRQA\textsuperscript{X}+KG, Alice’s measurement $\mathcal{M}$ is defined with $\rho = \psi_x$ and this requires her knowledge of $x$. This is why we focus on PSQA\textsuperscript{X}+KG and use its composable security to infer that of PSQA+KG. If $k \neq f$, the states label $\oplus$ in the two protocol are identical. This introduces an additional contribution of $\delta$ to the distinguishability advantage. The rest of the analysis is identical to that of QA+KG. An additional $\delta$ results from the difference between $\text{RSP}[^{EBIT}_I,C_I]$ and $Q_I$.

Putting everything together, PSQA+KG $(2\sqrt{2e^{1/3} + 2\delta})$-s.r. $Q_I$+KD$_I$.

### E Quantum universal composable security of the Wegman-Carter scheme

We consider a Wegman-Carter type of authentication scheme (WC) that does the following. Let $\mathcal{H} = \{h_k\}_k$ be an $\epsilon$-almost-strongly universal family of hash functions from the set of messages $\mathcal{M}$ to the set of authentication tags $\mathcal{T}$. Let $k, t$ be the value of the shared key. If the message is $x$, Alice transmits $(x, h_k(x) \oplus t)$ to Bob, where $\oplus$ represents the bitwise xor. In other words, the hash value of the message is one-time-padded with the key $t$. Usually, when executing this scheme WC, the same hash function is reused for subsequent messages.

Here, we analyze WC+KG, which runs WC and recycles the key $k$, in a way similar to but much simpler than QA+KG. In particular, we let $k, t$ be the output from an internal KD$_I$ box that provides the keys to WC. Note that recycling of the key $k$ is more general than reusing of the hash function (universal vs non-universal composability).

The ideal functionality has two parts. The first part is a magic authentication box that sends the message $x$ via an insecure channel, and the received message is $x'$. The output is $(x, \text{acc})$ if $x = x'$, and $(\text{rej})$ if $x \neq x'$. The second part is KD$_I$ which outputs a perfect key $k$ between Alice and Bob.

The simulator runs the ideal authentication box to transmit $x$ to the environment, but appends a random tag $h$, and receives $x', h'$. The simulator checks if $h = h'$ and feeds $x'$ into the ideal authentication box, which then outputs $(x, \text{acc})$ or $(\text{rej})$. If $t = t'$ and the output is $(x, \text{acc})$, the simulator makes the final output $(x, t, \text{acc}, k)$, else, the simulator outputs $(\text{rej}, k)$.

WC+KG and the ideal functionality differ only if $(x, h) \neq (x', h')$ and the former accepts. This happens with probability less than $\epsilon$, which then upper bounds the distinguishability advantage.

Due to [22], we are now aware of a problem that in principle, an adversary can guess a key value and tamper with the message accordingly, and the subsequent acc or rej output leaks information on the key to be recycled. This introduces an additional contribution to the distinguishability advantage. We leave the work of correcting the above proof for a later version of this manuscript.

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36. In [3] and [1], the security definition and security proofs are given without explicitly considering the purifying system. When exact security definitions are achievable (as in [3]), the former implies the latter by linearity. However, [29] shows that an approximate condition holding for all pure states need not imply the same condition in the presence of purifications. In particular, for quantum encryption, security in the presence of purifications doubles the key size. For the QA protocol in [1], the security is proved explicitly for the security definition without the purification, though it is not difficult to check that QA satisfies the stronger security definition with purifications. (This also follows immediately from our proofs.) Meanwhile, QA uses QEnc as a subroutine, and the lower bound for the keysize comes from the most general quantum encryption. It follows from [29] that the correct lower bound is $m$ bits for pure state schemes and $2m$ bits in the presence of purifications, instead of the $2m$ bits stated in [1]). Appendix [D] shows that the lower bound for pure state is attainable asymptotically. To keep our discussion simple, we stick to the stronger security definitions including purifications, except for in Appendix [D].
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46. An acyclic circuit is a partially ordered set of gates. However, associating the circuit with constraints on the timing of the adversarial attack is a delicate issue. Suppose the circuit contains conditional gates controlled by random public classical registers. The gates on the target may or may not be applied depending on the values of the control registers. When the gates are not applied, the associated time-constraints of the adversarial attack disappear. In the extension to the usual acyclic circuit model, we consider all possible values of the control registers and the resulting sets of nontrivial partially ordered operations, and the corresponding constraints on the adversarial attack.

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