A reliable algorithm based on the shifted orthonormal Bernstein polynomials for solving Volterra–Fredholm integral equations

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ABSTRACT
This paper deals with the numerical solution of Volterra–Fredholm integral equations. In this work, we approximate the unknown functions based on the shifted orthonormal Bernstein polynomials, in conjunction with the least-squares approximation method. The method is using a simple computational manner to obtain a quite acceptable approximate solution. The merits of this method lie in the fact that, on the one hand, the problem will be reduced to a system of algebraic equations. On the other hand, the efficiency and accuracy of the method for solving these equations are high. The convergence analysis of proposed method have been discussed through some theorems. Moreover, we will obtain an estimation of error bound for this algorithm. Finally, some examples are given to show the capability of presented method in comparison with four well-known algorithms in the literature namely the Legendre collocation method, Taylor collocation method, Taylor polynomial method and Lagrange collocation method.

1. Introduction
Applications of integral equation within various areas of science, engineering, bio-engineering, finance, viscoelasticity, electrochemistry, control and electromagnetic applied mathematics and others become nowadays wide and flourishing [1,2]. Hence, a number of approximate methods for solving numerically various classes of integral equations have been developed [3–7]. Consequently, much attention have been paid to the integral equations during last decades. The linear and nonlinear Volterra–Fredholm integral equations can be solved by utilizing many different basic functions and some of which have been used recently to estimate the most appropriate solutions. Some of these well-known methods are reproducing kernel Hilbert space method [8], the Legendre wavelets method [9], the homotopy perturbation method [10], the composite collocation method [11], the rationalized Haar functions method [12], the variational iteration method [13], the collocation method based on the radial basis functions [14], Sinc method [15] and Legendre collocation method [16].

The Bernstein polynomials have been used for solving linear and nonlinear integral equations such as nonlinear Fredholm–Volterra integro-differential equations [17], Fredholm–Volterra–Hammerstein integral equations [18], Volterra integral equations [19]. Spectral method based on Bernstein polynomials was presented for solving coupled system of Fredholm integral equations [20]. In [21], the Bernstein polynomials and hybrid Bernstein Block–Pulse functions have been used to solve system of Volterra–Fredholm integral equations. The shifted orthonormal Bernstein polynomials were applied to approximate the solution of generalized pantograph equations [22].

In this paper, we consider the Fredholm–Volterra integral equation in the following form [23,24]:

\[
A(x)y(x) + B(x)y(h(x)) = g(x) + \lambda_1 \int_0^h k_1(x,t)y(t) \, dt + \lambda_2 \int_0^t k_2(x,t)y(h(t)) \, dt,
\]

where \(k_1(x,t)\) and \(k_2(x,t)\) are known as kernel functions on \([0, L] \times [0, L]\). Also \(A(x)\) and \(B(x)\) are continuous on interval \([0, L]\), \(h(x)\) and \(g(x)\) are known functions defined on this interval and \(0 \leq h(x) \leq L\). The function \(y(x)\) is unknown while \(\lambda_1\) and \(\lambda_2\) are real constants such that \(\lambda_1^2 + \lambda_2^2 \neq 0\). When \(h(x)\) is a first-order polynomial, Equation (1) is a functional integral equation with the proportional delay. Some numerical methods such as the Legendre collocation method [24], Taylor collocation method [25], Taylor polynomial method [26] and Lagrange collocation method [27] have been applied for solving Equation (1). In this work, we will present the shifted orthonormal Bernstein polynomials.
method to approximate the solution of Equation (1). Furthermore, convergence analysis and an estimation of error bound for this method will be given. Finally, we apply this method to several examples to show the efficiency of our method. These results show that the proposed method is very effective and more accurate than the Legendre collocation method, Taylor collocation method, Taylor polynomial method and Lagrange collocation method. The rest of this article is organized as follows.

In Section 2, some preliminaries are presented which will be used hereafter. The method for approximating the solution of Equation (1) will be discussed in Section 3. Section 4 is devoted to the convergence analysis of the current method. In Section 5, we give an error estimation for the presented method. Section 6 offers some numerical examples to illustrate the efficiency of this method. Finally, some brief conclusions are given in Section 7.

2. Shifted orthonormal Bernstein polynomials

In this section, some basic preliminaries and notations which have fundamental importance in this paper will be illustrated.

**Definition 2.1:** [28] The well-known Bernstein polynomials of degree \( n \) are defined on the interval \([0,1]\) and can be determined with the aid of the following formula:

\[
B_{j,n}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad j = 0, 1, 2, \ldots, n. \tag{2}
\]

If \( j < 0 \) or \( j > n \), then we let \( B_{j,n}(x) = 0 \). By using the binomial expansion \((1-x)^{n-j} = \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} x^i\), one obtains

\[
B_{j,n}(x) = \sum_{i=0}^{n-j} (-1)^i \binom{n}{j} \binom{n-j}{i} x^{i+j}, \quad x \in [0,1]. \tag{3}
\]

The orthonormal Bernstein polynomials of \( n \)th degree are defined on the interval \([0,1]\) as [29]:

\[
\psi_j(x) = \sqrt{2(n-j) + 1} (1-x)^{n-j} \sum_{k=0}^{\frac{j}{2}} \binom{\frac{j}{2}}{k} \binom{\frac{j}{2}+1}{\frac{j}{2}-k} x^{\frac{j}{2}-k}, \quad \text{for } j = 0, 1, \ldots, n. \tag{4}
\]

for \( j = 0, 1, \ldots, n \). Using the original non-orthonormal Bernstein basis functions, Equation (4) can be displayed in the following form [29]:

\[
\psi_j(x) = \sqrt{2(n-j) + 1} \sum_{k=0}^{\frac{j}{2}} (-1)^k \binom{\frac{j}{2}}{k} \binom{\frac{j}{2}+1}{\frac{j}{2}-k} B_{j-k,n-k}(x). \tag{5}
\]

The orthogonality property of this polynomials is given by

\[
\int_0^1 \psi_i(x) \psi_j(x) = \delta_{ij}, \quad i, j = 0, 1, \ldots, n,
\]

where \( \delta_{ij} \) is the Kronecker delta function. Using the transformation \( t = x/L \), the so-called shifted orthonormal Bernstein polynomials of degree \( n \) on the interval \([0, L]\) will be defined as

\[
\phi_j(x) = \frac{1}{\sqrt{L}} \psi_j \left( \frac{x}{L} \right), \quad j = 0, 1, \ldots, n. \tag{6}
\]

We use shifted orthonormal Bernstein polynomials as basis functions to estimate the solution of Equation (1).

Let \( H = L^2[0, L] \) be a Hilbert space with the following inner product:

\[
\langle y, u \rangle = \int_0^L y(x) u(x)^T \, dx \tag{7}
\]

and

\[
\Phi_n = \text{span} \{ \phi_0(x), \phi_1(x), \ldots, \phi_n(x) \}. \tag{8}
\]

where \( \Phi_n \) is a finite-dimensional subspace of \( H \), so it is closed and convex. Therefore, for all \( y \in H \), there exists a unique best approximation out of \( \Phi_n \) such as \( v \), such that [28]

\[
\| y - v \| \leq \| y - u \|, \quad \forall u \in \Phi_n. \tag{9}
\]

Using shifted orthonormal Bernstein polynomials, a function \( y(x) \in H \) can be represented as

\[
y(x) = \sum_{l=0}^{\infty} a_l \phi_l(x), \tag{10}
\]

where the coefficients \( a_l \) are given by

\[
a_l = \frac{1}{L} \int_0^L \phi_l(x) y(x) \, dx, \quad l = 0, 1, \ldots, n. \tag{11}
\]

Considering the first \( (n+1) \)-terms of shifted orthonormal Bernstein polynomials, one obtains

\[
y(x) \approx \sum_{l=0}^{n} a_l \phi_l(x). \tag{12}
\]

3. Description of the method

In this section, we approximate the solution of Equation (1) by the shifted orthonormal Bernstein polynomials method. To do this, at first we consider the
Fredholm–Volterra integral equation (1) in the form:

\[
A(x)y(x) + B(x)y(h(x)) = g(x) + \lambda_1 \int_0^h k_1(x, t)y(t) \, dt \\
+ \lambda_2 \int_0^L k_2(x, t)y(h(t)) \, dt.
\]

For any fixed positive integer \( n \), a function \( y(x) \) defined on the interval \([0, L]\) can be presented by the shifted orthonormal Bernstein series expansion as follows:

\[
y_n(x) = \sum_{i=0}^n a_i \phi_i(x),
\]

where \( a_i, l = 0, 1, \ldots, n \) are the unknowns coefficients which should be determined. Also, one can consider that

\[
y_n(h(x)) = \sum_{i=0}^n a_i \phi_i(h(x)).
\]

Substituting Equations (14) and (15) in Equation (1), the following integral equation is resulted:

\[
A(x)y_n(x) + B(x)y_n(h(x)) = g(x) + \lambda_1 \int_0^h k_1(x, t)y_n(t) \, dt \\
+ \lambda_2 \int_0^L k_2(x, t)y_n(h(t)) \, dt,
\]

so, we get

\[
A(x) \sum_{i=0}^n a_i \phi_i(x) + B(x) \sum_{i=0}^n a_i \phi_i(h(x)) = g(x) + \lambda_1 \int_0^h k_1(x, t) \sum_{i=0}^n a_i \phi_i(t) \, dt \\
+ \lambda_2 \int_0^L k_2(x, t) \sum_{i=0}^n a_i \phi_i(h(t)) \, dt.
\]

For simplicity one can rearrange this equation as

\[
\sum_{i=0}^n a_i \left\{ A(x)\phi_i(x) + B(x)\phi_i(h(x)) \right\} - \lambda_1 \int_0^h k_1(x, t)\phi_i(t) \, dt \\
- \lambda_2 \int_0^L k_2(x, t)\phi_i(h(t)) \, dt = g(x).
\]

We define

\[
\Xi(x, y_n(x)) = \sum_{i=0}^n a_i \Pi_i(x) - g(x),
\]

in which

\[
\Pi_i(x) = A(x)\phi_i(x) + B(x)\phi_i(h(x)) \\
- \lambda_1 \int_0^h k_1(x, t)\phi_i(t) \, dt \\
- \lambda_2 \int_0^L k_2(x, t)\phi_i(h(t)) \, dt.
\]

In our method, we use the the least-squares approximation to find the unknown coefficients \( a_0, a_1, \ldots, a_n \). For this purpose, we set

\[
\chi = \chi(x, a_0, a_1, \ldots, a_n) = \int_0^L \Xi^2(x, y_n(x)) \, dx.
\]

A necessary condition that \( \chi \), for \( k = 0, 1, \ldots, n \) be a minimizer of \( \chi(x, a_0, a_1, \ldots, a_n) \) is that its first variation vanishes at \( a_k \), i.e., [30–32]

\[
\frac{\partial \chi}{\partial a_k} = 0, \quad k = 0, 1, \ldots, n.
\]

Using Equation (21), one obtains

\[
\frac{\partial \chi}{\partial a_k} = \frac{\partial}{\partial a_k} \int_0^L \Xi^2(x, y_n(x)) \, dx \\
= 2 \int_0^L \Xi(x, y_n(x)) \frac{\partial \Xi(x, y_n(x))}{\partial a_k} \, dx \\
= 2 \int_0^L \sum_{i=0}^n a_i \Pi_i(x) - g(x) \left\{ \sum_{i=0}^n a_i \Pi_i(x) - g(x) \right\} \, dx \\
= 2 \int_0^L \sum_{i=0}^n a_i \Pi_i(x).\Pi_k(x) - \int_0^L g(x).\Pi_k(x) \, dx = 0.
\]

So, one can conclude that

\[
\sum_{i=0}^n a_i \int_0^L \Pi_i(x).\Pi_k(x) \, dx = \int_0^L g(x).\Pi_k(x) \, dx, \quad k = 0, 1, 2, \ldots, n.
\]

This system consists \((n + 1)\) equations which can be solved for the unknown coefficients. To find \( y_n(x) \), the \((n + 1)\) linear equations must be solved for the \((n + 1)\) unknowns \( a_i \). Therefore, system (24) can be written as

\[
GE = F,
\]

where

\[
G = \begin{bmatrix}
(\Pi_0, \Pi_0) & (\Pi_0, \Pi_1) & \cdots & (\Pi_0, \Pi_n) \\
(\Pi_1, \Pi_0) & (\Pi_1, \Pi_1) & \cdots & (\Pi_1, \Pi_n) \\
\vdots & \vdots & \ddots & \vdots \\
(\Pi_n, \Pi_0) & (\Pi_n, \Pi_1) & \cdots & (\Pi_n, \Pi_n)
\end{bmatrix}, \quad \Pi_0 = \Pi_1 = \cdots = \Pi_n
\]

\[
E = [a_0, a_1, \ldots, a_n]^T.
\]
and
\[ F = \left[ (g, \Pi_0), (g, \Pi_1), \ldots, (g, \Pi_n) \right]^T. \] (28)

Solving Equation (25), the unknown values \( \hat{\mathbf{e}} = [\hat{a}_0, \ldots, \hat{a}_n]^T \) will be obtained, where \( \hat{\mathbf{e}} \) is the approximate values of matrix \( \mathbf{E} \) and can be written in the following form:
\[
\hat{\mathbf{e}} = \mathbf{G}^{-1} \mathbf{F}. \] (29)

Let us assume that the functions \( \Pi_l(x), l = 0, 1, \ldots, n \), are also linearly independent functions on the interval \([0, L] \). If Gram matrix \( \mathbf{G} \) be a nonsingular matrix, then Equation (25) has a unique solution. Therefore, the approximate solution of Equation (1) will be determined as follows:
\[
\hat{y}_n(x) = \sum_{l=0}^{n} \hat{a}_l \phi_l(x). \] (30)

**Definition 3.1:** For every \( \varepsilon > 0 \), if \( \chi(x, a_0, a_1, \ldots, a_n) \leq \varepsilon \), then \( \hat{y}_n(x) \) is called an \( \varepsilon \)-approximate solution of Equation (1).

**Definition 3.2:** If Equation (25) has a unique solution \( \eta = \eta^0 = [a_0^0, a_1^0, \ldots, a_n^0]^T \), then \( \hat{y}_n(x) = \sum_{l=0}^{n} a_l^0 \phi_l(x) \) is called an optimal squared approximate solution of Equation (1) defined on the set \( \Phi_n \).

**Remark 3.1:** If \( \lim_{n \to \infty} \chi(x, a_0, a_1, \ldots, a_n) = 0 \), then the optimal squared approximate solution \( \hat{y}_n(x) \) converges to the exact solution \( y(x) \) of the Fredholm–Volterra integral equation (1).

### 4. Convergence analysis

In this section, we are going to prove the convergence of the current method. Based on the process in Section 3, in the following theorem we prove that as \( n \to \infty \), the approximate solution \( \hat{y}_n(x) \) will converge to the exact solution \( y(x) \) of Equation (1).

**Theorem 4.1:** Suppose that \( y(x) \) and \( \hat{y}_n(x) \) are the exact solution and the approximate solution obtained by the proposed method for Equation (1) defined on \([0, L] \), respectively. If there is \( P_n(x) = \sum_{j=0}^{n} a_j x^j, a_j \in \mathbb{R} \), such that for all \( x \in [a, b] \), \( \hat{y}_n(x) \) converges to the exact solution \( y(x) \) of Equation (1), then
\[
\lim_{n \to \infty} \chi(x, a_0, a_1, \ldots, a_n) = 0. \] (31)

**Proof:** Considering
\[
\mathcal{N}(x, a_0, a_1, \ldots, a_n) = \int_a^b \Xi^2(x, P_n(x)) \, dx, \] (32)
we can easily find that \( 0 \leq \chi(x, a_0, a_1, \ldots, a_n) \leq \mathcal{N}(x, a_0, a_1, \ldots, a_n) \), then, one can write
\[
0 \leq \lim_{n \to \infty} \chi(x, a_0, a_1, \ldots, a_n) \leq \lim_{n \to \infty} \mathcal{N}(x, a_0, a_1, \ldots, a_n). \] (33)

Since \( P_n(x) \) converges to the exact solution \( y(x) \), so
\[
\lim_{n \to \infty} \mathcal{N}(x, a_0, a_1, \ldots, a_n) = 0. \] (34)

Therefore,
\[
\lim_{n \to \infty} \chi(x, a_0, a_1, \ldots, a_n) = 0, \] (35)
and the proof is completed. ■

### 5. Error estimate

In this section, we will obtain an estimation error bound for our numerical method.

Let us define \( ||A||_{\infty} = \max_{1 \leq l \leq n} \sum_{i=1}^{n} |a_{i,l}| \) as the infinity norm of matrix \( A \).

**Theorem 5.1:** Let \( y \) be an exact solution of Equation (1) in \( C^{n+1}[0, L] \) and \( \hat{y}_n \) be the best approximation of \( y \) out of \( \Phi_n \), then we have
\[
||y(x) - \hat{y}_n(x)||_{\infty} \leq \frac{M_n Q^{n+1}}{(n+1)!}, \] (36)

where \( M_n = \max_{x \in [0, L]} |y^{(n+1)}(x)| \) and \( Q = \max\{L - x_0, x_0\} \).

**Proof:** We consider the Taylor polynomials interpolation as follows:
\[
p_n(x) = y(x_0) + y'(x_0)(x - x_0) + y''(x_0) \frac{(x - x_0)^2}{2} + \cdots + y^{(n)}(x_0) \frac{(x - x_0)^n}{n!}, \] (37)
then, there exists \( \eta \in (0, L) \) such that
\[
y(x) - p_n(x) = \frac{1}{(n+1)!} y^{(n+1)}(\eta)(x - x_0)^{n+1}. \] (38)

Since \( \hat{y}_n \) is the best approximation for \( y \), so we have
\[
||y(x) - \hat{y}_n(x)||_{\infty} \leq ||y(x) - p_n(x)||_{\infty} \leq \frac{M_n (x - x_0)^{n+1}}{(n+1)!} \]
\[
\leq \frac{M_n Q^{n+1}}{(n+1)!}, \] (39)
and the proof is completed. ■

In the following theorem, we perform a bound on \( ||y(x) - \hat{y}_n(x)||_{\infty} \).

**Theorem 5.2:** Consider the Volterra–Fredholm integral equation (1). Let \( y(x) \in C^{n+1}[0, L] \) be the exact solution
and \( \hat{y}_n(x) = \sum_{i=0}^{n} \hat{a}_i \phi_i(x) \) be the approximate solution obtained by the proposed method. Then, we have
\[
\| y(x) - \hat{y}_n(x) \|_\infty \leq \frac{M_n Q^{n+1}}{(n+1)!} + N_n k(\mathbf{G}) \| \mathbf{E} \|_\infty O(\varepsilon),
\]
(40)
where \( M_n = \max_{x \in [0,1]} | y^{(n+1)}(x) | \), \( N_n = \max_{x \in [0,1]} \sum_{i=0}^{n} \phi_i(x), Q = \max \{ L - x_0, x_0 \} \), \( k(\mathbf{G}) \) is the condition number of \( \mathbf{G} \) and \( \mathbf{E} = (\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_n)^T \) is the solution of Equation (25) computed through the presented method.

**Proof:** We can write
\[
\| y(x) - \hat{y}_n(x) \|_\infty = \| y(x) - \hat{y}_n(x) + \hat{y}_n(x) - \hat{y}_n(x) \|_\infty \\
\leq \| y(x) - \hat{y}_n(x) \|_\infty + \| \hat{y}_n(x) - \hat{y}_n(x) \|_\infty,
\]
(41)
where \( \hat{y}_n(x) \) is the best approximation of \( y(x) \). According to Theorem 5.1, the first term in the right-hand side is achieved. To obtain an upper bound for \( \| \hat{y}_n(x) - \hat{y}_n(x) \|_\infty \), we see that
\[
\| \hat{y}_n(x) - \hat{y}_n(x) \|_\infty = \left| \sum_{i=0}^{n} (a_i - \hat{a}_i) \phi_i(x) \right| \\
\leq \max_{0 \leq i \leq n} | a_i - \hat{a}_i | \max_{x \in [0,1]} \sum_{i=0}^{n} \phi_i(x).
\]
(42)
Considering \( N_n = \max_{x \in [0,1]} \sum_{i=0}^{n} \phi_i(x) \), relation (42) implies that
\[
\| \hat{y}_n(x) - \hat{y}_n(x) \|_\infty \leq N_n \max_{0 \leq i \leq n} | a_i - \hat{a}_i | = N_n \| \mathbf{E} - \hat{\mathbf{E}} \|_\infty.
\]
(43)
Note that \( \mathbf{E} = (a_0, a_1, \ldots, a_n)^T \) and \( \hat{\mathbf{E}} = (\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_n)^T \) are the exact and approximate solutions for \( \mathbf{G} \mathbf{E} = \mathbf{F} \), respectively. Also, suppose that the perturbation matrix \( \mathbf{E} \) is such that \( \| \mathbf{E} + \mathbf{I} \mathbf{E} = \mathbf{F} \). It can be shown that
\[
\| \mathbf{E} - \hat{\mathbf{E}} \|_\infty \leq k(\mathbf{G}) \| \mathbf{E} \|_\infty \| \hat{\mathbf{E}} \|_\infty,
\]
(44)
where \( k(\mathbf{G}) \) is the condition number of \( \mathbf{G} \). It is known that the Gaussian elimination with partial pivoting is almost numerically stable. Therefore, if we use this method for solving the linear system \( \mathbf{G} \mathbf{E} = \mathbf{F} \), then \( \| \mathbf{I} \|_\infty / \| \mathbf{G} \|_\infty \) is close to the machine precision \( \varepsilon \). This implies that
\[
\| \mathbf{E} - \hat{\mathbf{E}} \|_\infty \leq k(\mathbf{G}) \| \hat{\mathbf{E}} \|_\infty O(\varepsilon).
\]
(45)
Using relations (43) and (45), one obtains
\[
\| \hat{y}_n(x) - \hat{y}_n(x) \|_\infty \leq N_n k(\mathbf{G}) \| \hat{\mathbf{E}} \|_\infty O(\varepsilon).
\]
(46)
From Equations (36), (41) and (46), the following relation is resulted
\[
\| y(x) - \hat{y}_n(x) \|_\infty \leq \frac{M_n Q^{n+1}}{(n+1)!} + N_n k(\mathbf{G}) \| \hat{\mathbf{E}} \|_\infty O(\varepsilon),
\]
(47)
and this completes the proof.

These results can be generalized in the following theorem for the case of an arbitrary function \( h(x) \).

**Theorem 5.3:** Let \( y(x) \) be the exact solution and \( \hat{y}_n(x) \) be the approximate solutions of Equation (1), then
\[
\| y(h(x)) - \hat{y}_n(h(x)) \|_\infty \leq \gamma + p \left( \frac{M_n Q^{n+1}}{(n+1)!} + N_n k(\mathbf{G}) \| \hat{\mathbf{E}} \|_\infty O(\varepsilon) \right),
\]
(48)
where \( \gamma = \sup_{x \in [0,1]} | h(x) | \int_{0}^{h(x)} | k_1(x,t) | \, dt, \beta = \sup_{x \in [0,1]} | k_2(x,t) | \, dt, p = \max_{x \in [0,1]} | k_2(x,t) |, q = \min_{x \in [0,1]} | h(x) | \) and \( q - \beta > 0 \).

**Proof:** Substituting the approximate solutions (14) and (15) in Equation (1), we get
\[
A(x) \hat{y}_n(x) = B(x) \hat{y}_n(h(x)) \\
= g + \lambda_1 \int_{0}^{h(x)} k_1(x,t) \hat{y}_n(t) \, dt + \lambda_2 \int_{0}^{L} k_2(x,t) \hat{y}_n(h(t)) \, dt.
\]
(49)
From Equations (1) and (49), one obtains
\[
- p \cdot \| y(h(x)) - \hat{y}_n(h(x)) \|_\infty \leq | \int_{0}^{h(x)} | k_1(x,t) | \, dt \cdot \| y(x) - \hat{y}_n(x) \|_\infty | \\
\leq | \lambda_1 | \int_{0}^{h(x)} | k_1(x,t) | \, dt \cdot \| y(x) - \hat{y}_n(x) \|_\infty + | \lambda_2 | \int_{0}^{L} | k_2(x,t) | \, dt \cdot \| y(h(x)) - \hat{y}_n(h(x)) \|_\infty.
\]
(50)
So, we have
\[
q \cdot \| y(h(x)) - \hat{y}_n(h(x)) \|_\infty - p \cdot \| y(h(x)) - \hat{y}_n(h(x)) \|_\infty \leq \gamma + q \cdot \| y(x) - \hat{y}_n(x) \|_\infty + \beta \cdot \| y(h(x)) - \hat{y}_n(h(x)) \|_\infty,
\]
(51)
this implies that
\[
(q - \beta) \cdot \| y(h(x)) - \hat{y}_n(h(x)) \|_\infty \leq (\gamma + p) \cdot \| y(x) - \hat{y}_n(x) \|_\infty.
\]
(52)
Using Equation (47), we conclude that
\[
\| y(h(x)) - \hat{y}_n(h(x)) \|_\infty \leq \gamma + p \left( \frac{M_n Q^{n+1}}{(n+1)!} + N_n k(\mathbf{G}) \| \hat{\mathbf{E}} \|_\infty O(\varepsilon) \right),
\]
(53)
which completes the proof.
6. Numerical experiments

In this section, the numerical results of proposed scheme on some test problems will be presented and we compare them with the results by four other efficient methods. These four methods are LECM [24], TCM [25], TPM [26] and LACM [27]. In these examples, one can observe that the error of presented method is less than the errors obtained by other methods. In order to show the error, we introduce the following notations:

\[ e_n(x) = |y(x) - \hat{y}_n(x)|, \quad (54) \]

\[ \|e_n\|_2 = \left( \int_0^L e_n^2(x) \, dx \right)^{1/2}, \quad (55) \]

\[ \|\eta_n\| = \|y(x) - \hat{y}_n(x)\|_{\infty}, \quad (56) \]

\[ \rho_n = \frac{M_n Q^{n+1}}{(n + 1)!}, \quad (57) \]

where \( y(x) \) and \( \hat{y}_n(x) \) are the exact solution and the solution obtained by presented method, respectively. In all examples, \( L = 1 \) and we use the notations \( M_n, N_n, \kappa(G) \) and \( O(\varepsilon) \) which were defined in Section 5. To obtain the numerical solutions \( \hat{y}_n(x) \) the CPU times (s) are also given.

**Example 6.1:** [24] Consider the following Volterra-Fredholm integral equation:

\[ (\sin x) \, y(x) + (\cos x) \, y(e^x) = f(x) + \int_0^x e^{x+t} y(t) \, dt - \int_1^x e^{x+t} y(e^t) \, dt, \quad (58) \]

where

\[ f(x) = \frac{1}{3} e^x (-1 + e^3) + e^x \left\{ 2 - e^x \left[ 2 + e^x (-2 + e^x) \right] \right\} + e^{2x} \cos x + x^2 \sin x. \quad (59) \]

The exact solution of this equation is \( y(x) = x^2 \). Tables 1, 2 and Figures 1–3 show the numerical results for this example. Table 1 exhibits the values of \( \|e_n\|_2 \) for the proposed method, LECM [24], TCM [25], TPM [26] and LACM [27] with \( n = 2, 3 \) and 4. These results confirm that the current method is very effective and more accurate than LECM, TCM, TPM and LACM. From Table 2, we conclude that Theorem 5.2 can be applied to this example. Figure 1 shows a comparison between the obtained results by our method with the results of other mentioned methods. Figure 2 displays the approximate solution and the exact solution by this method with \( n = 1, 2 \). We see that by increasing \( n \), the solutions become more and more accurate. The condition numbers \( \kappa(G) \) of the matrices \( G \) by this method with different values of \( n \) are depicted in Figure 3.

![Figure 1. Plot of \( e_n(x) \) with \( n = 2, 3 \) and 4 for Example 6.1.](image-url)
Figure 2. Plot of the exact solution and approximate solutions with $n = 1, 2$ for Example 6.1.

Figure 3. Plot of the condition numbers $\kappa(G)$ for $n = 2, 3, 4, 10, 100, 500, 1000$ in Example 6.1.

Example 6.2: [24] For the second example consider the following Volterra–Fredholm integral equation:

$$x^2 y(x) + e^x y(2x) = f(x) + \int_0^{2x} e^{x+t} y(t) \, dt - \int_0^1 e^{x-2t} y(2t) \, dt,$$

where

$$f(x) = -\frac{e^x}{4} - \frac{e^{-2+x}}{4} \cos 2 + \frac{e^{3x}}{2} \cos 2x$$

$$- \frac{e^{-2+x}}{4} \sin 2 + x^2 \sin x$$

$$+ e^x \sin 2x - \frac{e^{3x}}{2} \sin 2x.$$ (61)

The exact solution is $y(x) = \sin(x)$. The numerical results are displayed in Tables 3, 4 and Figures 4, 5. Table 3 exhibits the values of $\|e_n\|_2$ for the proposed method, LECM, TCM, TPM and LACM with different values of $n$. This table reveals that our method can provide more accurate results in comparison with LECM, TCM, TPM and LACM. From Table 4, we conclude that Theorem 5.2 can be applied to this example. We compare the error of our method with the errors of LECM [24], TCM [25], TPM [26] and LACM [27] for $n = 2, 5, 8$ and 9 in Figure 4. One can see that, as $n$ is increased, the error is decreased. Figure 5 depicts the condition numbers $\kappa(G)$ of the matrices $G$ for this method with $n = 2, 5, 8, 9$. 

| $n$ | Our method | LECM | TCM | TPM | LACM |
|-----|------------|------|-----|-----|------|
| 2   | $2.14 \times 10^{-3}$ | $1.46 \times 10^{-2}$ | $7.87 \times 10^{-2}$ | $3.41 \times 10^{-2}$ | $7.87 \times 10^{-2}$ |
| 5   | $8.23 \times 10^{-4}$ | $2.93 \times 10^{-3}$ | $6.23 \times 10^{-3}$ | $6.23 \times 10^{-3}$ | $6.23 \times 10^{-3}$ |
| 8   | $1.36 \times 10^{-10}$ | $3.94 \times 10^{-9}$ | $1.69 \times 10^{-8}$ | $1.69 \times 10^{-8}$ | $1.69 \times 10^{-8}$ |
| 9   | $2.41 \times 10^{-11}$ | $2.29 \times 10^{-9}$ | $2.35 \times 10^{-8}$ | $3.46 \times 10^{-7}$ | $7.21 \times 10^{-6}$ |
Table 4. Numerical results for Example 6.2.

| $n$ | $M_n$ | $\rho_n$ | $N_n$ | $\kappa(G)$ | $\|e_n\|$ | $O(\varepsilon)$ | CPU, s |
|-----|------|---------|------|-------|--------|--------|-------|
| 2   | 1.0000 | $1.6670 \times 10^{-1}$ | 3 | 25.193 | $5.6810 \times 10^{-3}$ | $4.2711 \times 10^{-5}$ | 1.1036 |
| 5   | 0.8414 | $1.1686 \times 10^{-3}$ | 6 | 99.245 | $1.2543 \times 10^{-5}$ | $1.9223 \times 10^{-6}$ | 3.0409 |
| 8   | 1.0000 | $2.7577 \times 10^{-6}$ | 9 | 157.45 | $7.4519 \times 10^{-10}$ | $2.3016 \times 10^{-12}$ | 5.8927 |
| 9   | 0.8414 | $2.3186 \times 10^{-7}$ | 10 | 189.32 | $8.7839 \times 10^{-11}$ | $1.2619 \times 10^{-13}$ | 6.9654 |

Example 6.3: [24] Consider the following Volterra–Fredholm integral equation:

\[
y(x) = f(x) + \int_0^{Ln(x+1)} e^{x+t}y(t) \, dt - \int_0^1 e^{x+Ln(t+1)}y(Ln(t+1)) \, dt,
\]

where

\[
f(x) = e^{-x} - e^x (Ln(x+1) - 1).
\]

The exact solution of this problem is $y(x) = e^{-x}$.

The numerical results are displayed in Tables 5, 6 and Figures 6, 7. Comparison of the error norm $\|e_n\|_2$ between our method with the errors of LECM, TCM,
Table 5. Comparison of $\|e_n\|_2$ between our method with the other methods in Example 6.3.

| n   | Our method | LECM | TCM | TPM | LACM |
|-----|------------|------|-----|-----|------|
| 2   | $7.32 \times 10^{-4}$ | $1.99 \times 10^{-3}$ | $3.27 \times 10^{-3}$ | $3.59 \times 10^{-2}$ | $3.27 \times 10^{-2}$ |
| 5   | $8.14 \times 10^{-8}$ | $2.56 \times 10^{-7}$ | $4.30 \times 10^{-7}$ | $3.05 \times 10^{-6}$ | $4.30 \times 10^{-7}$ |
| 8   | $3.57 \times 10^{-9}$ | $2.78 \times 10^{-8}$ | $5.96 \times 10^{-8}$ | $5.61 \times 10^{-7}$ | $5.78 \times 10^{-7}$ |
| 9   | $3.26 \times 10^{-10}$ | $2.35 \times 10^{-8}$ | $8.84 \times 10^{-8}$ | $1.41 \times 10^{-7}$ | $1.86 \times 10^{-5}$ |

Table 6. Numerical results for Example 6.3.

| n   | $M_n$  | $\rho_n$ | $N_n$ | $\kappa(G)$ | $\|e_n\|$ | $O(\epsilon)$ | CPU, sec. |
|-----|--------|----------|-------|-------------|------------|-------------|-----------|
| 2   | 2.7183 | 4.5304 $\times 10^{-1}$ | 3     | 27.361     | $2.1403 \times 10^{-3}$ | 5.2834 $\times 10^{-5}$ | 1.1365 |
| 5   | 2.7183 | 3.7753 $\times 10^{-1}$ | 6     | 54.298     | $1.0531 \times 10^{-5}$ | 1.2064 $\times 10^{-8}$ | 2.5036 |
| 8   | 2.7183 | 7.4908 $\times 10^{-6}$ | 9     | 143.67     | $8.7198 \times 10^{-9}$ | 3.0276 $\times 10^{-11}$ | 6.5832 |
| 9   | 2.7183 | 7.4908 $\times 10^{-7}$ | 10    | 156.95     | $9.6305 \times 10^{-10}$ | 6.1616 $\times 10^{-12}$ | 7.2071 |

Figure 6. Comparison between the exact solution and approximate solutions with $n = 1, 2$ for Example 6.3.

Figure 7. Plot of the condition numbers $\kappa(G)$ for $n = 2, 5, 8, 9$ in Example 6.3.

TPM and LACM for different values of $n$ are presented in Table 5. From these results, it is evident that the presented method provides a good approximate solutions in comparison with LECM, TCM, TPM and LACM. From Table 6, we conclude that Theorem 5.2 can be applied to this example. Figure 6 displays the exact solution and the approximate solutions with $n = 1, 2$. In addition, Figure 7 depicts the obtained condition numbers $\kappa(G)$ of the matrices $G$ by this method for different values of $n$.

Example 6.4: As a final test problem, consider the following Volterra–Fredholm integral equation:

$$
(x^3 + 2x^2)y(x) - \left(6x^2 + \frac{1}{5}x\right)y\left(\frac{x}{3}\right) = f(x) + \int_0^\frac{x}{2} xty(t)\, dt - \int_0^1 (x-t)y\left(\frac{t}{3}\right)\, dt,
$$

(64)
Table 7. Absolute errors of our method with $n = 3$ for Example 6.4.

| $n$ | Exact solution | Numerical solution | Absolute error |
|-----|----------------|-------------------|----------------|
| 0   | 1.0000000      | 0.9999887         | $1.1209615 \times 10^{-5}$ |
| 0.2 | 1.0328374      | 1.0328303         | $7.0547814 \times 10^{-6}$ |
| 0.4 | 1.3627307      | 1.3627280         | $2.6839560 \times 10^{-6}$ |
| 0.6 | 2.0368182      | 2.0368119         | $6.2279045 \times 10^{-6}$ |
| 0.8 | 3.1023200      | 2.1023146         | $5.3380454 \times 10^{-6}$ |
| 1.0 | 4.6065306      | 4.6065274         | $3.1703582 \times 10^{-6}$ |

where

$$f(x) = \frac{17}{18} x^6 + \frac{515}{108} x^5 + x^3 e^{-x/2} - \frac{539}{135} x^4 - 2x^2 e^{-x/2}$$

with the exact solution $y(x) = x^3 + 3x^2 + e^{-x/2}$. The numerical results of this example are displayed in Tables 7, 8 and Figures 8, 9. Table 7 exhibits the results of absolute errors between the exact solution and our numerical solution with $n = 3$. From Table 8, we conclude that Theorem 5.2 can be applied to this example. Figure 8 depicts the absolute errors of this algorithm.

Table 8. Numerical results for Example 6.4.

| $n$ | $M$  | $\rho$ | $N$  | $\kappa(G)$ | $\|\eta_n\|$ | $O(\varepsilon)$ | CPU, s |
|-----|------|--------|------|-------------|----------------|-----------------|--------|
| 3   | 0.0625 | 2.6041 $\times 10^{-3}$ | 4 | 40.180 | $1.1209 \times 10^{-5}$ | $5.6627 \times 10^{-6}$ | 1.7359 |
| 4   | 0.0313 | 2.0952 $\times 10^{-4}$ | 5 | 64.203 | $1.1625 \times 10^{-5}$ | $7.8049 \times 10^{-7}$ | 3.0409 |
| 5   | 0.0156 | 2.1666 $\times 10^{-5}$ | 6 | 106.38 | $3.0904 \times 10^{-8}$ | $2.4481 \times 10^{-9}$ | 3.8770 |
| 6   | 0.0078 | 1.5476 $\times 10^{-6}$ | 7 | 162.44 | $3.0891 \times 10^{-8}$ | $6.8240 \times 10^{-10}$ | 4.6319 |

Figure 8. Plot of $e_n(x)$ with $n = 4, 5$ and 6 for Example 6.4.

Figure 9. Plot of the condition numbers $\kappa(G)$ for $n = 3, 4, 5, 6, 10, 50, 100$ in Example 6.4.
with different values of $n$. One can see that, as $n$ is increased, the error is decreased. Figure 9 depicts the condition numbers $\kappa(G)$ of the matrices $G$ with $n = 4, 5, 6, 10, 50, 100$. So, not only because of its performance in providing more efficient and accurate results, but also because of its computing times, our method is preferable.

7. Conclusion

In this paper, we have presented an efficient algorithm based on the shifted orthonormal Bernstein polynomials for the numerical solution of Volterra–Fredholm integral equations. The properties of this method were used to convert the equation into a system of algebraic equations which could be solved more easily. The convergence and error estimation of current method have been discussed through some theorems. The obtained results showed that the proposed method for solving Volterra–Fredholm integral equation was very effective with a high accuracy in comparison with some other well-known methods such as the Legendre collocation method, Taylor collocation method, Taylor polynomial method and Lagrange collocation method.

Disclosure statement

No potential conflict of interest was reported by the authors.

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