Spacetime Information

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Abstract

In usual quantum theory, the information available about a quantum system is defined in terms of the density matrix describing it on a spacelike surface. This definition must be generalized for extensions of quantum theory which neither require, nor always permit, a notion of state on a spacelike surface. In particular, it must be generalized for the generalized quantum theories appropriate when spacetime geometry fluctuates quantum mechanically or when geometry is fixed but not foliable by spacelike surfaces. This paper introduces a four-dimensional notion of the information available about a quantum system’s boundary conditions in the various sets of decohering, coarse-grained histories it may display. This spacetime notion of information coincides with the familiar one when quantum theory is formulable in terms of states on spacelike surfaces but generalizes this notion when it cannot be so formulated. The idea of spacetime information is applied in several contexts: When spacetime geometry is fixed the information available through alternatives restricted to a fixed spacetime region is defined. The information available through histories of alternatives of general operators is compared to that obtained from the more limited coarse-grainings of sum-over-histories quantum mechanics that refer only to coordinates. The definition of information is considered in generalized quantum theories. We consider as specific examples time-neutral quantum mechanics with initial and final conditions, quantum theories with non-unitary evolution, and the generalized quantum frameworks appropriate for quantum spacetime. In such theories complete information about a quantum system is not necessarily available on any spacelike surface but must be searched for throughout spacetime. The information

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loss commonly associated with the “evolution of pure states into mixed states” in black hole evaporation is thus not in conflict with the principles of generalized quantum mechanics.

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I. INTRODUCTION

In the usual quantum theory of a system of matter fields in a fixed background spacetime, the state of the fields on a spacelike Cauchy surface is as complete a description of the system as it is possible to give. When the state is specified, the missing information is zero. If only probabilities $\pi_i$ for the state to be one of a set of states $\{|\psi_i(\sigma)\rangle\}$ on a Cauchy surface $\sigma$ are specified, then the missing information is greater. The system may then be described by a density matrix $\rho(\sigma)$

$$\rho(\sigma) = \Sigma_i |\psi_i(\sigma)\rangle \pi_i \langle \psi_i(\sigma)|,$$

and the missing information is

$$S(\sigma) = -\Sigma_i \pi_i \log \pi_i = -Tr[\rho(\sigma) \log \rho(\sigma)].$$

The unitary evolution of $\rho(\sigma)$ through a foliating family of Cauchy surfaces ensures that $S(\sigma)$ defined by (1.2) is independent of $\sigma$. Complete information about a system is obtainable on any Cauchy surface in the foliating family and that information is the same on one surface as on any other.

However, when quantum fluctuations of spacetime geometry are taken into account, as in any quantum theory of gravity, it is difficult to formulate quantum theory in terms of states on spacelike surfaces. This, not least, because there is no fixed geometry to give a meaning to “spacelike”. Similar difficulties exist when spacetime geometry is fixed but not foliable by spacelike surfaces, as in spacetimes with closed timelike curves (e.g., [5]). A possible approach to such situations is to generalize the quantum framework for prediction so that is in fully spacetime form and does not require a notion of “state on a spacelike surface”. How does one discuss information when quantum mechanics is in spacetime form and does not necessarily have a notion of state on a spacelike surface with which to define (1.2)? This paper proposes an answer to this question.

In a quantum theory fully in spacetime form it is appropriate to take a four-dimensional, spacetime approach to the definition of information. This is the guiding principle of this paper. Applying ideas of M. Gell-Mann and the author [7], we implement this principle in Sections II–IV in the usual formulation of the quantum mechanics of a closed system. In Sections V–VIII we consider spacetime information in generalized quantum theories [8,6,9].

In the usual formulation of quantum mechanics of a closed system with a fixed background spacetime, probabilities for decohering sets of histories are determined from an initial condition represented by a Heisenberg-picture density matrix $\rho$. In Section II we define the information available about this initial condition in any set of decoherent histories. The minimum missing information among all such sets defines the complete information available about the system. We show that in the usual formulation this spacetime notion of complete information coincides with (1.2) and is available on every spacelike surface.

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1 For lucid reviews of the difficulties see [1–4].

2 As in [6] where references to the earlier literature may be found.
The information available in a spacetime region may be defined by considering sets of histories representing alternatives that are restricted to that region. This is discussed in Section III. A spacelike surface is one kind of spacetime region and the information available on it coincides with the usual definition (1.2).

In Section IV we compare the information available in a quantum theory that allows alternative values of all Hermitian operators (coordinates, momenta, etc.) with that available in a sum-over-histories formulation of quantum mechanics that allows only alternatives defined by paths in a set of generalized coordinates. Infinitely less information is available on one spacelike surface in sum-over-histories quantum mechanics than can be obtained by utilizing all possible observables. However, much greater information is available in histories in sum-over-histories quantum mechanics than is available on a spacelike surface. We discuss situations when this can become complete information about the system.

The spacetime approach to defining the complete information about a closed quantum system reproduces the familiar (1.2) when quantum theory can be formulated in terms of states on spacelike surfaces. When quantum theory cannot be so formulated the spacetime approach generalizes (1.2). Section V describes this more general notion of information in time-neutral generalized quantum theories with initial and final conditions [10,8,11]. The general case is discussed in Section VI including generalized quantum theories of quantum spacetime [8]. This is applied to spacetimes with closed, timelike curves in Section VII.

Section VIII contains some brief remarks on the implications of the spacetime approach to information for black hole evaporation. When quantum theory is not formulable in terms of states on a spacelike surface complete information about a system is not necessarily to be found on a given spacelike surface (even when “spacelike” can be defined). Rather, one must search among all possible decoherent sets of spacetime histories for those which give complete information. In black hole evaporation these may refer to alternatives on a spacelike surface after the hole has evaporated as well as to alternatives near to or inside the horizon. The “evolution of pure states into mixed states” that is often discussed [12] is thus not a violation of the principles of generalized quantum mechanics. It is only at conflict with the idea that the evolution of the system can be completely described by states on spacelike surfaces.

II. FIXED BACKGROUND SPACETIMES

We begin by considering spacetime information in the approximation in which spacetime geometry is fixed, foliable by spacelike surfaces, and quantum theory concerns particles or fields moving in this given background. This is an excellent approximation on accessible scales for epochs later than the Planck era. As throughout this paper, we consider a closed quantum system most generally the universe as a whole. In this Section and in Sections III–IV we shall restrict attention to the usual formulation of quantum theory in which probabilities for alternative, coarse-grained histories are determined by a initial condition in the far past represented by a density matrix $\rho$ together with an action or Hamiltonian summarizing the dynamics of particles or fields in the fixed geometry. We shall consider generalizations of this standard framework in Sections V–VIII. To make our assumptions more precise, and to introduce the notation we use, we now very briefly review the elements
of the quantum mechanics of closed systems. We follow the treatment in [13, 8, 14] where more detailed expositions as well as references to the earlier literature may be found.

We assume that the fixed spacetime is foliable by spacelike surfaces and pick a particular foliation, labeling the surfaces by a time coordinate, $t$. A set of alternative coarse-grained histories, of the closed system may be described by giving sets of “yes-no” alternatives at a sequence of times $t_1, \ldots, t_n$. The alternatives at a particular time $t_k$ are represented by a set of Heisenberg-picture projection operators $\{P_{\alpha_k}(t_k)\}$. In this notation, $\alpha_k$ is an index specifying the particular alternative in the set and the superscript $k$ indicates that there may be different sets at different times. The projections satisfy

$$P_{\alpha_k}(t_k) P_{\beta_k}(t_k) = \delta_{\alpha_k\beta_k} P_{\alpha_k}(t_k), \quad \sum_{\alpha_k} P_{\alpha_k}(t_k) = I$$

which show that the alternatives are exclusive and exhaustive. The operators $P_{\alpha_k}(t_k)$ obey the Heisenberg equation of motion. An individual history is a particular sequence of alternatives $(\alpha_1, \ldots, \alpha_n) \equiv \alpha$ at the times $t_1, \ldots, t_n$. It is represented by the corresponding chain of Heisenberg-picture projections:

$$C_\alpha = P_{\alpha_n}(t_n) \cdots P_{\alpha_1}(t_1).$$

The set of all possible sequences gives a set of alternative histories $\{C_\alpha\}$. Evidently,

$$\sum_\alpha C_\alpha = I.$$

A set of histories $\{C_\alpha\}$ may be coarse-grained by partitioning it into mutually exclusive classes. The class operators representing the individual histories $\bar{C}_\alpha$ in the coarser-grained set are the sums of the $C_\alpha$ over the classes. The general form of the class operators representing an individual history in a set of exclusive alternative ones is therefore

$$C_\alpha = \sum_{(\alpha_1, \ldots, \alpha_n) \in \alpha} P_{\alpha_n}(t_n) \cdots P_{\alpha_1}(t_1)$$

with (2.3) continuing to hold. Even more generally, the set of histories may be branch dependent with the sets at later times depending on the sets, times, and specific alternatives at earlier times although we have not extended the notation for the $P$’s to indicate this dependence explicitly.

The decoherence functional $D(\alpha', \alpha)$ assigns a complex number to every pair of histories in a set of alternative ones that measures the coherence between that pair. It is defined by

$$D(\alpha', \alpha) = \text{Tr} \left( C_{\alpha'} \rho C_{\alpha}^\dagger \right)$$

where $\rho$ is the Heisenberg picture density matrix representing the initial condition of the closed system. The set of histories is said to decohere if the “off-diagonal” elements of $D(\alpha', \alpha)$ are sufficiently small. Quantum mechanics predicts probabilities only for decoherent sets of histories whose probabilities obey the sum rules of probability theory as a consequence of decoherence. The probabilities $p(\alpha)$ of the individual histories in a decoherent set are the diagonal elements of the decoherence functional.

The probabilities of decohering sets of alternative histories provide information about the system’s initial condition. The rest of this section reviews the construction of this missing
information by M. Gell-Mann and the author [13, 14]. The missing information \( S(\{C_\alpha\}) \) in any particular set of histories \( \{C_\alpha\} \) is defined by employing a generalization of the Jaynes “maximum-entropy” construction [15]. The entropy functional of a density matrix \( \tilde{\rho} \) is defined to be

\[
S(\tilde{\rho}) = -Tr (\tilde{\rho} \log \tilde{\rho}) .
\]  

(2.6)

This is the standard information measure on density matrices; in Section V we shall provide a justification of this formula from a more general point of view. The missing information \( S(\{C_\alpha\}) \) is the maximum of \( S(\tilde{\rho}) \) over all density matrices \( \tilde{\rho} \) that contain the information available about the system through the histories \( \{C_\alpha\} \) and no more than that information.

The information available through \( \{C_\alpha\} \) consists roughly of two parts, (i) the decoherence of the set and (ii) the probabilities of the individual histories in the set. A \( \tilde{\rho} \) that reproduces this information should reproduce the decoherence functional for the set of histories \( \{C_\alpha\} \):

\[
Tr \left( C_{\alpha'} \tilde{\rho} C_\alpha^\dagger \right) = Tr \left( C_{\alpha'} \rho C_\alpha^\dagger \right) .
\]  

(2.7)

Thus,

\[
S(\{C_\alpha\}) = \max_{\tilde{\rho}} \left[ S(\tilde{\rho}) \right]_{Tr(\tilde{\rho} C_\alpha^\dagger) = D(\alpha', \alpha)} .
\]  

(2.8)

The definition (2.8) incorporates any standard of approximate decoherence that may be enforced. If the off-diagonal elements of \( D(\alpha', \alpha) \) are required to be zero to some accuracy, the density matrix \( \tilde{\rho}_{\text{max}} \) that determines \( S(\{C_\alpha\}) \) will reproduce decoherence and probabilities with that accuracy for histories \( \{C_\alpha\} \). The definition (2.8) is consistent with the notion of physically equivalent sets of histories described in [16]. Sets of histories whose class operators \( \{C_\alpha\} \) differ by a reassignment of the times in (2.4) or by a constant unitary transformation that preserves the initial \( \rho \) are physically equivalent. Since physically equivalent sets have the same decoherence functional they will also have the same missing information through (2.8).

The density matrix \( \tilde{\rho}_{\text{max}} \) that maximizes \( S(\tilde{\rho}) \) subject to the constraints (2.7) may be found by the method of Lagrange multipliers. One first extremizes \( S(\tilde{\rho}) \) with respect to all operators \( \tilde{\rho} \) (not just density matrices) enforcing (2.7). The result is

\[
\tilde{\rho}_{\text{max}} = \exp \left( -\sum_{\alpha\alpha'} \lambda^{\alpha\alpha'} C_\alpha^{\dagger} C_{\alpha'} \right) .
\]  

(2.9)

The Lagrange multipliers \( \lambda^{\alpha\alpha'} \) are determined by (2.7) and there is a solution with \( \lambda^{\alpha\alpha'} = (\lambda^{\alpha'} \alpha)^* \). Thus \( \tilde{\rho}_{\text{max}} \) is Hermitian. It is also normalized because summing both sides of (2.7) over \( \alpha \) and \( \alpha' \) gives \( Tr(\tilde{\rho}_{\text{max}}) = Tr(\rho) = 1 \). The operator \( \tilde{\rho}_{\text{max}} \) is therefore a density matrix. The missing information \( S(\{C_\alpha\}) \) is easily expressed directly in terms of the multipliers and the decoherence functional

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3 It would be possible to define a missing information in the probabilities alone simply by reproducing only the diagonal elements of \( D \) in (2.8), with that accuracy for histories \( \{C_\alpha\} \). The missing information so defined would, of course, be greater than that including decoherence.
\[
S(\{C_\alpha\}) = \sum_{\alpha \alpha'} \lambda^{\alpha \alpha'} D(\alpha', \alpha) \approx \sum_\alpha \lambda^{\alpha \alpha} p(\alpha) ,
\]
the last equality holding when the set \(\{C_\alpha\}\) decoheres.

The conditions \((2.7)\) are in general difficult to solve for the multipliers \(\lambda^{\alpha \alpha'}\), but there is one useful case where the solution may be obtained explicitly. That is when the histories consist of alternatives at a single moment of time so that the \(\{C_\alpha\}\) are a set of orthogonal projections \(\{P_\alpha\}\). Decoherence is then automatic from the cyclic property of the trace in \((2.5)\) and the orthogonality of the \(\{P_\alpha\}\) [\textit{cf.} \((2.1)\)]. The conditions \((2.7)\) reduce to
\[
p(\alpha) = \text{Tr} \left[ P_\alpha \exp \left( -\sum_\beta \lambda^\beta P_\beta \right) \right] \quad (2.11)
\]
whose solution is
\[
\lambda^\alpha = -\log \left[ p(\alpha)/\text{Tr} (P_\alpha) \right] .
\]
(2.12)

The result for \(\tilde{\rho}_{\max}\) in \((2.9)\) is
\[
\tilde{\rho}_{\max} = \sum_\alpha \left[ p(\alpha) P_\alpha / \text{Tr}(P_\alpha) \right] ,
\]
and then
\[
S(\{P_\alpha\}) = -\sum_\alpha p(\alpha) \log p(\alpha) + \sum_\alpha p(\alpha) \log \text{Tr}(P_\alpha) . \quad (2.14)
\]
In simple cases the expression \((2.14)\) gives the familiar entropy of statistical mechanics \([15,4]\). Suppose \(\rho\) is an eigenstate of the total energy — the “microcanonical ensemble” and let \(\{P_\alpha\}\) be projections down on ranges of energy of width \(\Delta E\). Then clearly \(p(\alpha) = 0\) for all ranges except that containing the energy of \(\rho\) and so
\[
S(\{P_\alpha\}) = \log \text{Tr}(P_\alpha) = \log N \quad (2.15)
\]
where \(N\) is the number of states with energy \(E\) in the range \(\Delta E\). This is the usual Boltzmann entropy.

The missing information in any coarse grainings of the set \(\{C_\alpha\}\) is greater than the missing information in \(\{C_\alpha\}\) itself. To see this note that the class operators for a coarser-grained set \(\tilde{C}_\tilde{\alpha}\) are \textit{sums} of the class operators for \(\{C_\alpha\}\):
\[
\tilde{C}_\tilde{\alpha} = \sum_{\alpha \tilde{\alpha}} C_\alpha . \quad (2.16)
\]
Correspondingly, the conditions \((2.7)\) for \(\tilde{C}_\tilde{\alpha}\) are sums of the conditions for \(\{C_\alpha\}\). They are therefore \textit{weaker} conditions, and the maximum in \((2.8)\) can only be greater for \(\{\tilde{C}_\tilde{\alpha}\}\) than it is for \(\{C_\alpha\}\). Thus,
\[
S(\{\tilde{C}_\tilde{\alpha}\}) \geq S(\{C_\alpha\}) . \quad (2.17)
\]
Requiring that the density matrix \(\tilde{\rho}\) reproduce the full decoherence functional \(\rho\) rather than, say, just the diagonal elements which are the probabilities means that \(\tilde{\rho}\) functions as an initial condition not only for \(\{C_\alpha\}\) but for all coarser grainings of it. In particular the probabilities for coarser-grained sets of histories are given by
\[ p(\bar{\alpha}) = Tr \left( \bar{C}_{\bar{\alpha}} \tilde{\rho} \bar{C}_{\bar{\alpha}}^\dagger \right) \] (2.18)

because the set \{\bar{C}_{\bar{\alpha}}\} decoheres with respect to \tilde{\rho}.

We have defined the missing information in a particular set of alternative histories \{C_\alpha\}. Given a class of sets of histories, one member may yield more information about the system than another. We therefore define the missing information in a class \mathcal{C} of sets of alternative decohering histories as

\[ S(\mathcal{C}) = \min_{\{C_\alpha\} \in \mathcal{C}} S\left(\{C_\alpha\}\right). \] (2.19)

Computing \(S(\mathcal{C})\) for different classes enables one to understand different ways information about a quantum system can be obtained. For example we could study whether the same information is available in sum-over-histories quantum mechanics as using a general operator formulation, and whether the same information is available in homogeneous histories which are chains of \(P\)'s as in (2.2), vs. the more general inhomogeneous histories which are sums of chains as in (2.4).

The minimum of \(S(\{C_\alpha\})\) over the class of all decohering histories is the least missing information about the system — the complete information. We write this as \(S_{\text{compl}}\) without an argument

\[ S_{\text{compl}} = \min_{\text{decohering}} S\left(\{C_\alpha\}\right). \] (2.20)

In the usual quantum mechanics under discussion, no more information is available about the system in any set of histories than is contained in the initial density matrix by the measure (2.8). To see this first note the general inequality\(^{5}\) for any pair of density matrices \(\rho_1\) and \(\rho_2\).

\[-Tr(\rho_1 \log \rho_2) \geq -Tr(\rho_1 \log \rho_1).\] (2.21)

Then note that from the explicit form (2.3) it follows that\(^{6}\)

\[ S(\{C_\alpha\}) \equiv -Tr(\tilde{\rho}_{\text{max}} \log \tilde{\rho}_{\text{max}}) = -Tr(\rho \log \tilde{\rho}_{\text{max}}). \] (2.22)

Finally, using (2.21), we have

\[ S(\{C_\alpha\}) \geq S(\rho). \] (2.23)

The bound (2.23) is realized for a set of histories in which \(C_\alpha = P_\alpha\) where the \(P_\alpha\) are projections onto a complete set of states that diagonalize \(\rho\). That is because decoherence is

\(^{4}\)Following the terminology of Isham \[9\].

\(^{5}\)For a convenient proof see \[17\].

\(^{6}\)This is one of the requirements that \(\tilde{\rho}_{\text{max}}\) be a coarse-graining of \(\rho\), see \[6\].
automatic for sets of histories that are projections, as a consequence of the cyclic property of the trace in (2.3). Thus,
\[ S_{\text{compl}} = S(\rho) \equiv -Tr(\rho \log \rho). \] (2.24)

Since in the Heisenberg picture every operator corresponds to some quantity at any time the \( \{P_\alpha\} \) may be regarded as residing on any spacelike surface. Thus, on a spacelike surface it is always possible to recover complete information, \( S_{\text{compl}} \), about the initial condition through a suitable choice of decoherent histories consisting of a single set of projections.

### III. INFORMATION IN SPACETIME REGIONS

An interesting and useful example of the missing information in a class of histories is the missing information in the class of histories that refer only to a particular spacetime region. To illustrate the idea we consider a quantum field theory with a single scalar field \( \phi(x) \). The generalization to a realistic panoply of gauge, spinor, and tensor fields should be straightforward. Given a spacetime region \( R \) at time \( t \), we can define operators \( \mathcal{O}(R, t) \) that refer only to the region \( R \) as functions of the fields and their conjugate momenta *inside* \( R \) at that time, viz.

\[ \mathcal{O}(R, t) = \mathcal{O}(\phi(x, t), \pi(x, t)), \quad x \in R \text{ at } t. \] (3.1)

Alternative range of values of such variables are represented by sets of projection operators. We denote these by \( \{P_{\alpha}^{\mathcal{O}(R, t)}(t)\} \) where the discrete index \( \alpha \) runs over an exhaustive set of mutually exclusive ranges of \( \mathbf{R} \). (The two occurrences of \( t \) are redundant; we include them for clarity.) Sequences of such sets of alternatives \( \mathcal{O}_1(R, t_1), \ldots, \mathcal{O}_n(R, t_n) \) at a series of times \( t_1, \ldots, t_n \) that are contained within the span of \( R \) define examples of sets of alternative histories that refer only to \( R \). Individual histories correspond to particular sequences of alternatives \( \alpha \equiv (\alpha_1, \ldots, \alpha_n) \) at the definite moments of time and are represented by the corresponding chain of projection operators

\[ C_{\alpha}(R) = P_{\alpha_n}^{\mathcal{O}_n(R, t_n)}(t_n) \cdots P_{\alpha_1}^{\mathcal{O}_1(R, t_1)}(t_1). \] (3.2)

More generally, we may consider partitions of such histories into mutually exclusive classes \( \{c_\alpha\} \) and consider the limit where there is one alternative at each and every time. The resulting class operators have the form

\[ C_{\alpha}(R) = \lim_{n \to \infty} \sum_{(\alpha_1 \cdots \alpha_n) \in \alpha} P_{\alpha_n}^{\mathcal{O}_n(R, t_n)}(t_n) \cdots P_{\alpha_1}^{\mathcal{O}_1(R, t_1)}(t_1). \] (3.3)

This is the most general notion of a set of alternative histories that refers to a spacetime region \( R \), and operators of this form define the class of histories \( \mathcal{C}_R \) that refer only to the spacetime region \( R \).

A simple example may be helpful. Consider the average of the field \( \phi(x) \) over the region \( R \):

\[ \phi(R) = \frac{1}{V(R)} \int_R d^4x \phi(x) \] (3.4)
where \( V(R) \) is the four-dimensional volume of \( R \). Of course, this average can be written
\[
\phi(R) = \frac{1}{V(R)} \int_R dt \phi(R, t) \tag{3.5}
\]
where by \( \phi(R, t) \) we mean the spatial average over the intersection of \( R \) with the constant time surface labeled by \( t \). Take \( \phi(R, t) \) for the \( \mathcal{O}(R, t) \) at a discrete series of times \( t_1, \ldots, t_n \) equally spaced by a small interval \( \epsilon \). Take a set of small intervals of \( R \) of equal size \( d \) for the ranges labeled by \( \alpha \). Then the history in which the spacetime average \( \phi(R) \) lies in the range \( \Delta \) is represented by a sum of the form (3.3). The sum is over all ranges \( \alpha_1, \ldots, \alpha_n \) of \( \phi(R, t_1), \ldots, \phi(R, t_n) \) such that their central values, \( \phi_{\alpha_1}, \ldots, \phi_{\alpha_n} \) satisfy
\[
\frac{1}{V(R)} \sum_k \epsilon \phi_{\alpha_k} \in \Delta . \tag{3.6}
\]
As \( \epsilon \to 0, n \to \infty, \) and \( d \to 0 \) the formula (3.3) gives the class operator corresponding to the history in which \( \phi(R) \) lies in the range \( \Delta \).

We can now define the missing information, \( S(R) \) associated with a spacetime region \( R \) as the minimum of the missing information in the class \( \mathcal{C}_R \) of decoherent sets of histories \( \{C(R)_{\alpha}\} \) that refer only to \( R \). In symbols,
\[
S(R) = \min_{\text{decohering}} \min_{\{C_{\alpha}(R)\}} S(\{C_{\alpha}(R)\}) . \tag{3.7}
\]
Thus \( S(R) \) is a measure of how far we are from having complete information about a system if we only have access to alternatives inside a spacetime region \( R \). Since \( S(R) \) is defined for spacetime regions it is a fully four-dimensional notion of information.

We can illustrate this idea with three examples shown in Figure 1. The region in Fig. 1a contains a spacelike surface \( \sigma \). Complete information is therefore available through the set of histories represented by projections onto the basis which diagonalizes \( \rho(\sigma) \). The missing information is
\[
S(R_a) = -Tr(\rho \log \rho) . \tag{3.8}
\]

Note that this representation of \( \phi(R) \) in the range \( \Delta \) is not the projection operator of the average of Heisenberg fields onto the range \( \Delta \). The class operator defined by (3.6) is not generally a projection operator. Ranges of values of time averages of Heisenberg operators certainly describe other alternatives, but to be incorporated into a quantum framework that deals with histories they would have to be assigned a time (cf. [8], Section IV.1). That is certainly possible since every Heisenberg picture projection may be interpreted as the projection into the value of some quantity at any time. However, when the region \( R \) extends over time there is no natural value for this time. The present construction does not require such a specification and corresponds to the sum-over-histories definition of alternatives as partitions of field histories (see e.g. [6]). The different operator representations of the same classical quantity reflects the usual factor ordering ambiguity in quantum mechanics, arising in this case because field values at different times generally do not commute.
Complete information cannot be available for regions like those of Fig 1b because the initial state may contain wave packets that never cross \( R \), at least if it has a finite extent in the time direction. Thus

\[
S(R_b) \geq -Tr(\rho \log \rho) .
\]  

The question would become more interesting if space were closed so that any wave packet heading away from \( R_b \) would inevitably return.

In Figure 1c the region \( R \) is the domain of dependence of a region \( L \) of a spacelike surface \( \sigma \). The field equations (or equivalently the Heisenberg equations of motion) permit every \( \phi(x), x \in R \) to be expressed in terms of \( \phi(x), x \in L \) on the spacelike surface. Thus every \( \{C_\alpha(R)\} \) can be so reexpressed. Further, every \( \{C_\alpha(L)\} \) certainly refers to the spacetime region \( R \). Thus the missing information in \( R \) is the same as the missing information in the region \( L \) of \( \sigma \), \( S(R_c) = S(L) \). The missing information \( S(L) \) may be calculated if we assume a regularization of the field degrees of freedom so that it is meaningful to speak of a Hilbert space \( \mathcal{H}_L \otimes \mathcal{H}_\bar{L} \) that is a tensor product of a factor for the degrees of freedom inside \( L \) and another for those outside. \( S(L) \) is then the minimum of \((2.14)\) over sets of projections of the form \( P_\alpha \otimes I \) that refer only to the region inside \( L \). A straightforward calculation shows:

\[
S(R_c) = S(L) = -tr[Sp(\rho) \log(Sp(\rho)/Sp(I))] \]  

where \( Sp \) denotes the trace over \( \mathcal{H}_\bar{L} \) and \( tr \) is the trace over \( \mathcal{H}_L \). Of course, from the general result \((2.23)\)

\[
S(R_c) = S(L) \geq -Tr(\rho \log \rho) .
\]  

The notion of missing information in a spacetime region \( R \) is a fully four-dimensional way of discussing the localization of information in a fixed background spacetime. In Section VIII we shall apply it to a discussion of information in black hole spacetimes.

**IV. INFORMATION IN SUM-OVER-HISTORIES QUANTUM MECHANICS**

Feynman’s sum-over-histories formulation of quantum mechanics is an alternative, spacetime, formulation of quantum theory. It agrees with the usual quantum theory formulated in terms of states, operators, etc. for alternatives that can be described in terms of the coordinates of configuration space. However, it differs from usual quantum theory in that it is restricted to such configuration space alternatives. Alternative values of momentum, for example, are not defined at an instant of time, but, only approximately, in terms of configuration space alternatives at several moments of time \[13\]. It is, therefore, an interesting question whether complete information, in the sense we have defined it, is available in sum-over-histories quantum mechanics from its more limited class of alternatives. We examine that question in this section.

The missing information in sum-over-histories quantum mechanics is defined, as in \((2.19)\),

\[
S(\text{soh}) = \min_{\text{decohering}} S\left(\{C_\alpha\}\right) .
\]  

\[11\]
To understand more precisely what this formula means, and to take some first steps in its evaluation, let us consider the model of a single, non-relativistic particle moving in \( \nu \)-dimensions.

The fine-grained histories of a non-relativistic particle are paths \( q'(t) \) which are single-valued functions of the time, \( t \), say, on an interval \([0, T]\). The general, spacetime notion of a set of alternatives for which quantum theory might predict probabilities is a partition of this set of fine-grained paths into mutually exclusive classes \( \{ c_\alpha \} \), \( \alpha = 1, 2, \cdots \). The totality of all such partitions defines the class of histories of sum-over-histories form \( C_{\text{soh}} \). The class operators representing such alternatives have matrix elements given by

\[
\langle q''|C_\alpha|q'\rangle = \int_{[q', q'']} \delta q e^{iS[q(\tau)]} .
\]

Here, \( S[q(\tau)] \) is the action functional, units are used where \( \hbar = 1 \), and the sum is over paths which start at \( q' \) at \( t = 0 \), end at \( q'' \) at \( t = T \), and lie in the class \( c_\alpha \). Coordinate indices have been expressed for compactness. As shown by Caves and others [19,20] the operators \( \{ C_\alpha \} \) may be expressed as a limit of forms like (2.4) where the times become dense in the interval \([0, T]\) and the projections are onto ranges of position. The decoherence functional is given by (2.5) and the missing information in a set \( \{ C_\alpha \} \) by (2.8). The missing information in all histories of sum-over-histories form is then given by (4.1) where the minimum is over partitions of the paths \( \{ c_\alpha \} \) that decohere.

As the simplest example, consider a partition of paths by which of a set of intervals \( \{ \Delta^1_\alpha \} \) the particle passes through at time \( t_1 \). The path integral (4.2) can be rewritten

\[
\langle q''|C_\alpha|q'\rangle = \int_{\Delta^1_\alpha} dq \left( \int_{[q', q'']} \delta q e^{iS[q(\tau)]} \right) \left( \int_{[q', q]} \delta q e^{iS[q(\tau)]} \right)
\]

where the individual path integrals in (4.3) are over unrestricted paths from \((q', 0)\) to \((q, t_1)\) and \((q, t_1)\) to \((q'', T)\) respectively. The unrestricted path integral from \((q_1, t_1)\) to \((q_2, t_2)\) defines a propagator according to

\[
\int_{[q_1, q_2]} \delta q e^{iS[q(\tau)]} = \langle q_2 t_2|q_1 t_1 \rangle = \langle q_2 |e^{-iH(t_2-t_1)}|q_1 \rangle .
\]

Combining (4.3) and (4.4) one has

\[
C_\alpha = e^{-iHT} P^1_\alpha (t_1) ,
\]

where \( P^1_{\alpha_1}(t_1) \) is the Heisenberg-picture projector onto the region \( \Delta^1_\alpha \) at time \( t_1 \). [We are thus, in this section, using a convenient normalization of the class operators such that \( \Sigma_\alpha C_\alpha = \exp(-iHT) \) rather than (2.3). The value of the decoherence functional is unaffected by this choice.]

\( S(\{ C_\alpha \}) \) for histories consisting of a projection at a single moment of time was calculated in eq (2.14). \( S(\{ C_\alpha \}) \) for the histories (4.5) is the same since the factor \( \exp(-iHT) \) does not affect the value of the decoherence functional in (2.5) by the cyclic property of the trace. The probabilities \( p_\alpha = Tr[ P^1_\alpha (t_1) \rho] \) will be finite for reasonable \( \rho \), but the traces of the projection operators in (2.14) will diverge.
where \( V(\Delta) \) is the configuration-space volume of the region \( \Delta \). Thus, there is an infinite amount of missing information in histories that are partitions of the paths at a single moment of time.

The situation is already much improved with two times. Suppose the paths are partitioned by their behavior with respect to sets of regions \( \{\Delta_{\alpha_1}^1\} \) at time \( t_1 \) and \( \{\Delta_{\alpha_2}^2\} \) at time \( t_2 \). By the straightforward extension of the argument given above, the class operators are

\[
C_\alpha = e^{-iHT}P_{\alpha_2}^2(t_2)P_{\alpha_1}^1(t_1) \tag{4.7}
\]

where the projections are onto the regions \( \Delta_{\alpha_2}^2 \) at \( t_2 \) and \( \Delta_{\alpha_1}^1 \) at \( t_1 \) respectively. The missing information in such a set is no longer given by the simple formula (2.14) but now must by calculated from (2.11). The \( \lambda^{\alpha\alpha'} \) are determined by the condition.

\[
D(\alpha', \alpha) = Tr \left[ C_\alpha^1 C_{\alpha'} \exp \left( -\sum_{\beta\beta'} \lambda^{\beta\beta'} C_\beta^1 C_{\beta'} \right) \right] \tag{4.8}
\]

with \( C \)'s of the form (4.7). Expanding the exponential in a power series one finds a series of terms with coefficients of the form

\[
Tr \left[ P_{\mu_1}^1(t_1)P_{\mu_2}^2(t_2)P_{\mu_3}^1(t_1)P_{\mu_4}^2(t_2) \cdots P_{\mu_n}^2(t_2) \right], \tag{4.9}
\]

that is, strings of alternating \( P^1(t_1) \)'s and \( P^2(t_2) \)'s. These traces are all finite. For example, the simplest one is

\[
Tr \left[ P_{\mu_1}^1(t_1)P_{\mu_2}^2(t_2) \right] = \int_{\Delta_{\mu_1}^1} dq_1 \int_{\Delta_{\mu_2}^2} dq_2 \langle q_1 t_1 | q_2 t_2 \rangle \langle q_2 t_2 | q_1 t_1 \rangle. \tag{4.10}
\]

The propagators are not divergent if \( t_1 \neq t_2 \) and the \( q \)-integrals are over finite ranges. For example, if the propagators were those of a free particle of mass \( m \) in \( \nu \)-dimensions

\[
Tr \left[ P_{\mu_1}^1(t_1)P_{\mu_2}^2(t_2) \right] = \frac{V(\Delta_{\mu_1}^1)V(\Delta_{\mu_2}^2)}{(2\pi/m)^{\nu}|t_1 - t_2|^\nu}. \tag{4.11}
\]

Of course, as \( t_2 \to t_1 \), the propagators diverge and we recover the infinity of (4.6). Mere finiteness of the coefficients in the expansion of the equation [(4.8)] that determines the multipliers \( \lambda^{\alpha\alpha'} \) does not imply that the solutions will be finite but is at least consistent with it. By contrast a similar expansion of (2.11) in the case of alternatives at a single time yields divergent coefficients and a divergent solution (2.12).

It is therefore plausible that, while the missing information in partitions of paths at a single time is infinite, the missing information in partitions that involve several times is finite. There is thus an infinite improvement in passing from a single time to many. That is perhaps not so very surprising. The alternatives usually defined by momentum operators, for example, are not available at a single moment of time in sum-over-histories quantum
mechanics. However, they are available approximately through models of time of flight determinations of momentum involving two or more times $[15]$. We are thus led to the interesting question of whether the complete information in sum-over-histories quantum mechanics is greater than that in the usual operator formulation or coincides with it. Even if they differ, the above arguments suggest that they differ by only a finite amount for finite dimensional configuration spaces. The value of this finite difference would itself be interesting.

It is already known that the decoherence functional for general alternatives can be recovered from the sum-over-histories decoherence functional through suitable transformations $[23]$. Were the complete information available in sum-over-histories quantum mechanics the same as in the operator versions of the theory, that would be another argument for the sufficiency of a sum-over-histories formulation for prediction in physics.

V. INFORMATION IN TIME-NEUTRAL GENERALIZED QUANTUM MECHANICS

A. Quantum Mechanics with Initial and Final Conditions

We now turn to information in generalized quantum theories $[8,9]$. A general discussion will be given in the next section, but as an introduction we consider one of the simplest examples in this section — quantum mechanics with both initial and final conditions $[24,10,8,11]$. This is a quantum theory whose notions fine- and coarse-grained histories coincide with those of the usual formulation in Section II. Individual members of a set of alternative histories continue to be represented by class operators $C_\alpha$ of the general form $[2.4]$ in a Hilbert space $\mathcal{H}$. Only the decoherence functional differs from the usual $[2.5]$ by incorporating both an initial condition represented by a positive Hermitian matrix $\rho_i$ and a final condition represented by a positive Hermitian matrix $\rho_f$. This decoherence functional is

$$D(\alpha', \alpha) = N \ Tr \left( \rho_f C_{\alpha'} \rho_i C^\dagger_\alpha \right), \quad (5.1a)$$

where $N$ is determined so that $\Sigma_{\alpha', \alpha} D(\alpha', \alpha) = 1$, specifically,

$$N^{-1} = Tr \left( \rho_f \rho_i \right). \quad (5.1b)$$

A set of histories is said to (medium) decohere when the off-diagonal elements of $(5.1a)$ are sufficiently small; the approximate probabilities of the histories are then the diagonal elements. These probabilities are consistent with the rules of probability theory as a consequence of decoherence.

---

8A. Connes (private communication) has shown that the two notions of complete information do not coincide for some simple models with finite dimensional Hilbert spaces, where restrictions to projections on a particular basis is the analog of the restriction to configuration space histories in a sum-over-histories formulation.
Quantum theory based on the decoherence functional (5.1) is a generalization of usual quantum theory described in Section II. The decoherence functional of the usual formulation (2.5) is the special case of (5.1) with $\rho^f = I$ and $\rho^i = \rho$. In contrast to the usual formulation, which incorporates an arrow of time, the generalized quantum theory based on (5.1) is time neutral. The decoherence functional (2.5) distinguishes the ends of the histories. At one end (conventionally called the past) there is a density matrix. At the other end (conventionally called the future) there is the trace. The generalized form (5.1) treats the ends symmetrically and $\rho^f$ and $\rho^i$ can be interchanged using the cyclic properties of the trace. In a quantum cosmology based on the time-neutral (5.1) and time-symmetric dynamical laws, all observed arrows of time arise from differences between $\rho^i$ and $\rho^f$.

Usual quantum mechanics can be formulated in terms of an evolving state on a spacelike surface that summarizes the past for the purposes of future prediction – an essentially time-asymmetric notion. Clearly, time-neutral generalized quantum theory cannot be so formulated, but is fully predictive as we have described. For a fuller discussion of this and other features of time-neutral generalized quantum theory see [10,11]. We shall now discuss the appropriate generalizations of the notions of information that were described for the usual theory in Section II.

B. Information in the Initial and Final Conditions

In Section II we defined the missing information in a set of histories making use of the information measure $S(\rho) = -Tr(\rho \log \rho)$ on density matrices. We posited this measure; we did not derive it. To define the analogous notions of information in time-neutral generalized quantum mechanics we need a measure of information in pairs of positive, Hermitian operators $\tilde{\rho}^i$ and $\tilde{\rho}^f$. We now derive that measure.

A general approach to the definition is to define the missing information $S(\tilde{\rho}^f, \tilde{\rho}^i)$ in $\tilde{\rho}^i$ and $\tilde{\rho}^f$ as $-\sum_\alpha [p(\alpha) \log p(\alpha)]$ for some set of probabilities $\{p(\alpha)\}$ determined by $\tilde{\rho}^i$ and $\tilde{\rho}^f$. These probabilities are naturally the probabilities of some set of decoherent histories as determined by the decoherence functional (5.1). However, we cannot use histories that are too coarse grained or the measure will be trivial. For example, if we use the maximally coarse grained set consisting of the single history $P = I$, then it has probability 1 and $-p \log p = 0$. Put more informally, the form $-\sum_\alpha [p(\alpha) \log p(\alpha)]$ contains no penalty for asking stupid questions. Rather, in order to define the missing information in $\tilde{\rho}^i$ and $\tilde{\rho}^f$, we should consider the probabilities of only some standard class of very fine-grained decoherent sets. The natural candidate for this standard class in the case of time-neutral quantum mechanics is the class of completely fine-grained decoherent sets, since such sets exist, as we shall now show. (We shall return to a discussion of this standard class in the general discussion in Section VI.) We thus define

\[
S(\tilde{\rho}^f, \tilde{\rho}^i) \equiv \min_{\text{fine-grained decoherent } \{C_\alpha\}} \left[ -\sum_\alpha p(\alpha) \log p(\alpha) \right] \tag{5.2}
\]

where the minimum is over the fine-grained decoherent sets $\{C_\alpha\}$ for which

\[
D(\alpha', \alpha) = N \ Tr \left( \tilde{\rho}^f C_{\alpha'} \tilde{\rho}^i C_{\alpha}^\dagger \right) = \delta_{\alpha'\alpha} p(\alpha) . \tag{5.3}
\]
We now compute the measure $S(\tilde{\rho}'', \tilde{\rho}')$ so defined as an explicit functional of $\tilde{\rho}''$ and $\tilde{\rho}'$.

Fine-grained histories consist of sequences of sets of one-dimensional projections onto a basis for $\mathcal{H}$ (a complete set of states) at each and every time. To keep the notation manageable let us consider for a moment a finite sequence of times $t_1, \ldots, t_n$. The $\{C_\alpha\}$ representing the finest-grained histories at these times may be written

$$C_\alpha = P^n_{\alpha_1} \cdots P^1_{\alpha_n}, \quad P^k_{\alpha_k} = |k, \alpha_k\rangle \langle k, \alpha_k| \quad (5.4)$$

where $\{|k, \alpha_k\rangle\}$ are a set of basis vectors at time $t_k$ as $\alpha_k$ ranges over a set of discrete indices. To compress the notation we shall write $\{|\alpha_k\rangle\}$ for these basis vectors at each time, remembering that there may be different sets at different times.

The condition that a set of the form (5.4) decoheres is, from (5.3),

$$\langle \alpha_n | \tilde{\rho}'' | \alpha_n \rangle \langle \alpha'_n | \alpha'_n \rangle \cdots \langle \alpha'_2 | \alpha'_2 \rangle \langle \alpha'_1 | \tilde{\rho}' | \alpha_1 \rangle \langle \alpha_1 | \alpha_2 \rangle \cdots \langle \alpha_{n-1} | \alpha_n \rangle = 0 \quad (5.5)$$

whenever any $\alpha'_k \neq \alpha_k$. The probabilities of the individual histories in this decoherent set are

$$p(\alpha_n, \ldots, \alpha_1) = \langle \alpha_n | \tilde{\rho}'' | \alpha_n \rangle |\langle \alpha_n | \alpha_{n-1} \rangle|^2 \cdots |\langle \alpha_2 | \alpha_1 \rangle|^2 \langle \alpha_1 | \tilde{\rho}' | \alpha_1 \rangle. \quad (5.6)$$

To compute the minimum in (5.2) that defines $S(\tilde{\rho}'', \tilde{\rho}')$, we should choose the bases $\{|\alpha_k\rangle\}$ so as to satisfy (5.7) and minimize

$$s(p) \equiv -\sum_{\alpha_n, \ldots, \alpha_1} p(\alpha_n, \ldots, \alpha_1) \log p(\alpha_n, \ldots, \alpha_1). \quad (5.7)$$

Less distributed sets of probabilities have smaller values of $s(p)$. More precisely, consider two probability distributions $p(\alpha_n, \ldots, \alpha_1)$ and $p'(\alpha_n, \ldots, \alpha_1)$ which differ only in that the probability of $\alpha_k$ is distributed in the first, but exactly correlated with, say, $\alpha_{k-1}$ in the latter. That is, consider

$$p'(\alpha_n, \ldots, \alpha_1) = \delta_{\alpha_k, \alpha_{k-1}} \sum_{\alpha_k} p(\alpha_n, \ldots, \alpha_1). \quad (5.8)$$

Then

$$s(p) - s(p') = -\sum_{\alpha_n, \ldots, \alpha_1} p(\alpha_n, \ldots, \alpha_1) \log \left[ \frac{p(\alpha_n, \ldots, \alpha_1)}{\sum_{\alpha_k} p(\alpha_n, \ldots, \alpha_1)} \right]. \quad (5.9)$$

Since the $p$'s are positive numbers, this shows that

$$s(p') \leq s(p). \quad (5.10)$$

Thus, we reduce $s(p)$ computed from the probabilities (5.6) by aligning the bases as much as possible so that they are exactly correlated from one time to the next. At the minimum the intermediate bases coincide with either $\{|\alpha_1\rangle\}$ or $\{|\alpha_n\rangle\}$. This yields a decoherent set if $\tilde{\rho}'$ is diagonal in $\{|\alpha_1\rangle\}$ and $\tilde{\rho}''$ is diagonal in $\{|\alpha_n\rangle\}$.

The sum $-\sum p(\alpha) \log p(\alpha)$ is the same for a set of histories and a finer-grained set with alternatives *exactly* correlated with those of the first and this is true for an arbitrary number of times $n$. Thus, for the purposes of computing $S(\tilde{\rho}'', \tilde{\rho}')$, completely fine-grained histories may be replaced by histories of the form
\[
C_\alpha = P_{\alpha f}^f P_{\alpha i}^i \tag{5.11}
\]

where \(P_{\alpha f}^f\) are projections onto a basis \(\{|\alpha_f\}\) diagonalizing \(\hat{\rho}^f\) and \(P_{\alpha i}^i\) are projections onto a basis \(\{|\alpha_i\}\) diagonalizing \(\hat{\rho}^i\). Such sets of histories are exactly decoherent. A simple expression for the probabilities of the histories (5.11) can be found by summing (5.3) over \(\alpha\).

Then

\[
p(\alpha_f, \alpha_i) = \mathcal{N} \text{Tr} \left( \hat{\rho}^f P_{\alpha f}^f P_{\alpha i}^i \hat{\rho}^i \right)
= \frac{\tilde{\pi}_{\alpha f}^f \tilde{\pi}_{\alpha i}^i |\langle \alpha_f | \alpha_i \rangle|^2}{\sum_{\alpha_f \alpha_i} \tilde{\pi}_{\alpha f}^f \tilde{\pi}_{\alpha i}^i |\langle \alpha_f | \alpha_i \rangle|^2}
\tag{5.12b}
\]

where \(\tilde{\pi}_{\alpha f}^f\) and \(\tilde{\pi}_{\alpha i}^i\) are the eigenvalues of \(\hat{\rho}^f\) and \(\hat{\rho}^i\) respectively. Thus

\[
\mathcal{S}(\hat{\rho}^f, \hat{\rho}^i) = \min_{\{|\alpha_f\}\} \min_{\{|\alpha_i\}\} \left[ -\sum_{\alpha_f \alpha_i} p(\alpha_f, \alpha_i) \log p(\alpha_f, \alpha_i) \right]
\tag{5.13}
\]

where the minimum is taken over bases \(\{|\alpha_f\}\) that diagonalize \(\hat{\rho}^f\) and bases \(\{|\alpha_i\}\) that diagonalize \(\hat{\rho}^i\). Such a minimum is still necessary for the definition because there will be several different bases that diagonalize \(\hat{\rho}^f\) and/or \(\hat{\rho}^i\) if the \(\{\tilde{\pi}_{\alpha f}^f\}\) or \(\{\tilde{\pi}_{\alpha i}^i\}\) are degenerate. We shall illustrate in what follows.

Eq (5.13) gives an explicit form for \(\mathcal{S}(\hat{\rho}^f, \hat{\rho}^i)\) in terms of the eigenvalues of \(\hat{\rho}^f\) and \(\hat{\rho}^i\) and the bases that diagonalize them. It is thus completely determined by \(\hat{\rho}^f\) and \(\hat{\rho}^i\). We can illustrate the construction with two special cases:

We first consider the case \(\hat{\rho}^f = I\) and \(\hat{\rho}^i \equiv \tilde{\rho}\), a density matrix. This is the case of a final condition of indifference with respect to final state (which is no condition at all) and an initial density matrix. It coincides with usual quantum mechanics of Section II. Any basis will diagonalize \(\hat{\rho}^f = I\) and \(\tilde{\pi}_{\alpha f}^f = 1\). From (5.12) we then have

\[
p(\alpha_f, \alpha_i) = \tilde{\pi}_{\alpha i} |\langle \alpha_f | \alpha_i \rangle|^2 \equiv \tilde{\pi}_{\alpha i} q_{\alpha f}^{\alpha i}\n\tag{5.14}
\]

where \(\tilde{\pi}_{\alpha i}\) are the probabilities which are the diagonal elements of the density matrix \(\tilde{\rho}\). Note that for each \(\alpha_i\), the \(q_{\alpha f}^{\alpha i}\) are themselves a set of probabilities, and we can write

\[
- \sum_{\alpha_f \alpha_i} p(\alpha_f, \alpha_i) \log p(\alpha_f, \alpha_i) = s(\tilde{\pi}) + \sum_{\alpha_i} \tilde{\pi}_{\alpha i} s(q^{\alpha i}) . \tag{5.15}
\]

To find \(\mathcal{S}(I, \tilde{\rho})\) we minimize (5.13) over all bases \(\{|\alpha_f\}\}. Since \(s(q^{\alpha i}) \geq 0\), the minimum is obtained by choosing \(\{|\alpha_f\}\} to coincide with \(\{|\alpha_i\}\}. All the probabilities \(q_{\alpha f}^{\alpha i}\) are then either zero or one and \(s(q^{\alpha i}) = 0\) for each \(\alpha_i\). Thus,

\[
\mathcal{S}(I, \tilde{\rho}) = - \sum_{\alpha_i} \tilde{\pi}_{\alpha i} \log \tilde{\pi}_{\alpha i} = - \text{Tr} (\tilde{\rho} \log \tilde{\rho}) \tag{5.16}
\]

In this way we derive the usual information measure on single density matrices as the least missing information in fine-grained decoherent sets of histories.

We should point out, however, that the limit \(\hat{\rho}^f \rightarrow I\) is not smooth. Consider \(\hat{\rho}^f = I + \epsilon B\) where \(\epsilon\) is a small parameter and \(B\) is a Hermitian operator with non-degenerate eigenvalues
in a basis \{ | \beta \rangle \}. Then, following through the above calculation, we find in the limit \( \epsilon \to 0 \) that

\[
\lim_{\epsilon \to 0} S(I + \epsilon B, \tilde{\rho}) = -Tr(\tilde{\rho} \log \tilde{\rho}) + \sum_{\alpha_i} \tilde{\pi}_{\alpha_i} s(q^{\alpha_i}) \tag{5.17}
\]

where the \( q^{\alpha_i} = |\langle \alpha_i | \beta \rangle|^2 \) are now fixed. The limit of \( S(\tilde{\rho}^f, \tilde{\rho}^i) \) as \( \tilde{\rho}^f \to I \) therefore depends on the direction that \( I \) is approached in the space of operators \( \tilde{\rho}^f \). The least of these limits is \( (5.16) \). The largest might be as large as \( -Tr(\tilde{\rho} \log \tilde{\rho}) + \log N \), where \( N \) is the dimension of the Hilbert space since \( s(p) \leq \log N \). Similar statements will apply in approaching any \( \tilde{\rho}^f \) or \( \tilde{\rho} \) with degenerate eigenvalues.

The origin of this direction dependence may be intuitively understood as follows: \( S(\tilde{\rho}^f, \tilde{\rho}^i) \), as defined by \( (5.13) \), measures not only how distributed the probabilities \( \tilde{\pi}^f_{\alpha_i} \) of the initial state are, and how distributed the probabilities \( \tilde{\pi}^i_{\alpha_i} \) of the final state are, but also how distributed the probabilities are of the final states given an initial state — the quantities \( |\langle \alpha_f | \alpha_i \rangle|^2 \). As long as \( \epsilon \) is finite those quantities are fixed as \( \epsilon \to 0 \). When \( \epsilon \) is strictly 0, \( \tilde{\rho}^f = I \) no longer singles out any basis. A compression of information is possible from that needed to specify a particular basis \( \{|\alpha_f\rangle\} \) relative to the initial one to the trivial statement that all bases are equivalent.

Another interesting case of the measure \( S(\tilde{\rho}^f, \tilde{\rho}^i) \) occurs when \( \tilde{\rho}^f \) and \( \tilde{\rho}^i \) commute. Then, assuming no degeneracy, there is a unique common basis in which they are diagonal and

\[
p(\alpha_f, \alpha_i) = \delta_{\alpha_f \alpha_i} \sum_{\beta} \tilde{\pi}^f_{\alpha_f} \tilde{\pi}^i_{\alpha_i} \tag{5.18}
\]

(Cases where \( \tilde{\rho}^f \) and/or \( \tilde{\rho}^i \) are degenerate may be discussed with arguments similar to those following \( (5.17) \) with the result that \( (5.18) \) provides the minimizing probabilities). The diagonal elements of \( (5.13) \) are those of the density matrix

\[
\tilde{\rho} = \frac{\tilde{\rho}^f \tilde{\rho}^i}{Tr(\tilde{\rho}^f \tilde{\rho}^i)} \tag{5.19}
\]

Thus, when \( \tilde{\rho}^f \) and \( \tilde{\rho}^i \) commute,

\[
S(\tilde{\rho}^f, \tilde{\rho}^i) = -Tr(\tilde{\rho} \log \tilde{\rho}) \tag{5.20}
\]

This result might have been expected classically. In classical physics, where there is no non-commutation, there is a deterministic correlation between initial and final conditions. A restriction on histories by a distribution of phase space initial conditions and a distribution of final conditions is equivalent to a more restrictive distribution of initial conditions and no restriction at all on final conditions. That new initial distribution is the product of the old one with the final condition evolved back to the initial time. In the Heisenberg picture we are using, that product is the analog of \( (5.19) \) when \( \tilde{\rho}^f \) and \( \tilde{\rho}^i \) commute.

The maximum possible value of \( S(\tilde{\rho}^f, \tilde{\rho}^i) \) is attained when \( \tilde{\rho}^f = I + \epsilon B \) and \( \tilde{\rho}^i = I + \epsilon E \) in the limit as \( \epsilon \to 0 \) where \( B \) and \( E \) are operators whose diagonal bases are maximally skewed \( |\langle \alpha_f | \alpha_i \rangle|^2 = \) constant. Then, in the limit \( \tilde{\pi}^f_{\alpha_i} = 1/N \) and \( \tilde{\pi}^i_{\alpha_i} = 1/N \) where \( N \) is the dimension of the Hilbert space, the probabilities \( p(\alpha_f, \alpha_i) = 1/N^2 \) are as distributed as possible and
\[ S_{\text{max}} = 2 \log N . \] (5.21)

The minimum possible value of \( S \) is zero since \(- \sum p \log p \) is positive or zero. Pairs of positive, Hermitian operators for which there is no missing information are of special interest. In order for \( S(\hat{\rho}^f, \hat{\rho}^i) \) defined by (5.13) to vanish, all but one of the probabilities \( p(\alpha_f, \alpha_i) \) must vanish. Call the labels of that probability \((\hat{\alpha}_f, \hat{\alpha}_i)\). Then, from (5.12)

\[ N \pi^f_{\hat{\alpha}_f} \pi^i_{\hat{\alpha}_i} |\langle \alpha_f | \alpha_i \rangle|^2 = \delta_{\alpha_f \hat{\alpha}_f} \delta_{\alpha_i \hat{\alpha}_i} . \] (5.22)

Two extreme cases illustrate some of the ways of satisfying the conditions (5.22). First suppose that \( \langle \alpha_f | \alpha_i \rangle \neq 0 \) for all \((\alpha_f, \alpha_i)\). Then \( \pi^f_{\alpha_f} = \delta_{\alpha_f \hat{\alpha}_f} \) and \( \pi^i_{\alpha_i} = \delta_{\alpha_i \hat{\alpha}_i} \). The initial and final conditions are both pure states. This is not a very interesting case because the decoherence functional (5.1) factors and only trivial sets of histories can decohere.

At the opposite extreme the bases \( \{ |\alpha_i \rangle \} \) and \( \{ |\alpha_f \rangle \} \) may coincide so that \( \rho^f \) and \( \rho^i \) commute. The condition (5.22) is then satisfied when \( \bar{\rho} \) given by (5.18) is pure. This can happen when either the initial or final state is pure. Thus, another example is:

\[ S(\rho, |\psi\rangle \langle \psi|) = 0 \text{ if } [\rho, |\psi\rangle \langle \psi|] = 0 . \] (5.23)

This includes the familiar case

\[ S(I, |\psi\rangle \langle \psi|) = 0 \] (5.24)

that arises in usual quantum mechanics in (5.16). There are many other ways of satisfying (5.22).

\section*{C. The Missing Information in a Set of Histories}

The information measure \( S(\hat{\rho}^f, \hat{\rho}^i) \) in pairs of positive, Hermitian operators may now be used to define the missing information in a decoherent set of histories \( \{C_\alpha\} \) in time-neutral generalized quantum mechanics in analogy with the construction of Section II. We assume we are given initial and final operators \( \rho^f \) and \( \rho^i \) that define a decoherence functional through (5.1). We define the missing information in the set \( \{C_\alpha\} \) by

\[ S(\{C_\alpha\}) = \max_{\rho^f, \rho^i} \left[ S(\rho^f, \rho^i) \right] \tilde{D}(\alpha') = D(\alpha') . \] (5.25)

The maximum is taken over positive, Hermitian operators \( \hat{\rho}^f \) and \( \hat{\rho}^i \) that preserve the value of the decoherence functional for the histories \( \{C_\alpha\} \) according to

\[ \tilde{D}(\alpha', \alpha) = \tilde{N} \text{ Tr} \left( \hat{\rho}^f C_{\alpha'} \hat{\rho}^i C_{\alpha}^\dagger \right) = D(\alpha', \alpha) = N \text{ Tr} \left( \rho^f C_{\alpha'} \rho^i C_{\alpha}^\dagger \right) . \] (5.26)

If \( \{C_\alpha\} \) is a coarse graining of \( \{C_\alpha\} \) in the sense of (2.16), then the conditions for preserving the decoherence functional of the coarser-grained set are linear combinations of the conditions for preserving the finer-grained set. Thus

\[ S(\{\bar{C}_\alpha\}) \geq S(\{C_\alpha\}) . \] (5.27)
Evidently $\tilde{\rho}^f = \rho^f$ and $\tilde{\rho}^i = \rho^i$ preserve the decoherence functional so that

$$S(\{C_\alpha\}) \geq S(\rho^f, \rho^i).$$

(5.28)

The missing information in any set of histories is always greater than the information measure of the operators defining the initial and final conditions.

We define complete information about the system as the least missing information in any set of decoherent histories

$$S_{\text{compl}} = \min_{\text{decoherent}} S(\{C_\alpha\}).$$

(5.29)

From (5.28) the minimum cannot be less than $S(\rho^f, \rho^i)$, and so

$$S_{\text{compl}} \geq S(\rho^f, \rho^i).$$

(5.30)

For generic initial and final operators $\rho^f$ and $\rho^i$ whose non-zero eigenvalues are non-degenerate, we expect (5.30) to be an equality because preserving the decoherence functional for the fine-grained set $\{C_\alpha = P_{\alpha_f}^f P_{\alpha_i}^i\}$, where $\{P_{\alpha_f}^f\}$ and $\{P_{\alpha_i}^i\}$ are projections onto bases, in (5.26) uniquely determines $\tilde{\rho}^f$ and $\tilde{\rho}^i$ up to trivial rescalings. However, for those situations where the condition does not determine $\tilde{\rho}^f$ and $\tilde{\rho}^i$ uniquely, the maximum in (5.25) may be larger than the minimum in (5.29), and (5.30) be only an inequality. We illustrate with an example:

Consider a set of histories $\{C_\alpha\}$ and the case of usual quantum mechanics with $\rho^f = I$ in an $N$-dimensional Hilbert space with $N$ even. In absence of further argument we have no reason to suppose that the $(\tilde{\rho}^f, \tilde{\rho}^i)$ that provide the maximum in (5.25) are of the form $(I, \tilde{\rho}^i)$, but suppose that to be the case. Then $S$ is given by (5.17) with the choice of the arbitrary basis $\{|\beta\rangle\}$ such as to maximize its value. The maximum values of $s(q^{\alpha_i})$ are each $\log N$ and this can be realized since, when $N$ is even, there is a unitary matrix all of whose elements have the same absolute value. Under these assumptions

$$S(\{C_\alpha\}) = S_{\text{usual}}(\{C_\alpha\}) + \log N$$

(5.31)

where $S_{\text{usual}}(\{C_\alpha\})$ is the missing information in $\{C_\alpha\}$ calculated according to the rules of usual quantum mechanics as in Section II. Were (5.31) to hold for every set of decohering histories, it would follow that the complete information would be

$$S_{\text{compl}} = -Tr(\rho^i \log \rho^i) + \log N.$$

(5.32)

This is larger than $S(I, \rho^i)$ by the addition of $\log N$ [c.f. (5.16)], so that (5.30) is only an inequality.

Eq (5.32) is enough to show that even when $\rho^f = I$ the notion of missing information in time-neutral quantum mechanics does not necessarily coincide with that of usual quantum mechanics. That is because time-neutral quantum mechanics utilizes a notion of information that involves both initial and final conditions and the relation between them. Loosely speaking there is more information to be missing. However, were the two notions compared when $\rho^f = I$ the difference in information between sets of histories would be the same in both formulations and that is what is needed to discriminate between sets of histories by their information content.
D. Missing Information on Spacelike Surfaces

The time-neutral generalized quantum mechanics we have been discussing cannot generally be reformulated in terms of states on spacelike surfaces. However, we can discuss the information available in alternatives on a spacelike surface and whether that information is the same on one spacelike surface as on another.

Consider for simplicity a spacelike surface of constant time $t$ in a particular Lorentz frame. Alternatives at that moment of time are represented by sets of orthogonal Heisenberg picture projection operators $\{P_\alpha(t)\}$. We define the missing information on the surface of constant $t$ by

$$S(t) \equiv \min_{\text{decoherent } \{P_\alpha(t)\}} S(\{P_\alpha(t)\}) .$$

(5.33)

That is, the missing information at $t$ is the least of that missing in all the alternatives at that time.

Whatever its value, $S(t)$ as defined by (5.33) is conserved. That is because, in the Heisenberg picture, any set of projections $\{P_\alpha\}$ may be regarded as projections on some quantity at any time. Thus the same sets of projection operators are available on any surface. The minimum is therefore the same on all surfaces:

$$S(t') = S(t'') .$$

(5.34)

Is complete information available on any surface? That is the question of whether $S(t)$ defined by (5.33) coincides with $S_{\text{compl}}$ as defined by (5.29).

Consider for simplicity the case when neither $\rho^i$ nor $\rho^f$ have degenerate, non-zero eigenvalues and do not commute. Then $S_{\text{compl}}$ is $S(\rho^i, \rho^f)$. Complete information would be available on any spacelike surface if there were a set of projections $\{P_\alpha\}$ defining decoherent alternatives such that $S(\{P_\alpha\}) = S(\rho^i, \rho^f)$. However, $S(\rho^i, \rho^f)$ is realized by fine-grained histories that are sequences of at least two sets of projections [cf. (5.11)]. One might choose $\{P_\alpha\}$ to coincide with one or the other of these but not both. Therefore, generically $S(t)$ is greater than $S_{\text{compl}}$. Complete information about the initial and final conditions is not available on any one spacelike surface in time-neutral quantum mechanics. Complete information is available through histories that involve alternatives on at least two spacelike surfaces, although these may be separated by only an infinitesimal time.

VI. INFORMATION IN GENERALIZED QUANTUM THEORIES

The discussion of spacetime information for time-neutral generalized quantum mechanics in the preceding section suggests how measures of information could be constructed in arbitrary generalized quantum theories. We describe that construction in this section.

A generalized quantum theory of a closed system consists of three elements [19]: (1) the sets of fine-grained histories which are the most refined possible description of the system; (2) the allowed coarse-grained sets of alternative histories, which generally are partitions of some fine-grained set into mutually exclusive classes $\{c_\alpha\}$; and (3) a decoherence functional $D(\alpha', \alpha)$ measuring interference between pairs of histories in a coarse-grained set. The
decoherence functional is a complex valued functional on pairs of histories which is (i) Hermitian, (ii) positive, (iii) normalized, and (iv) consistent with the principle of superposition in the specific senses described in [19]. Given these three elements a set of coarse-grained alternative histories (approximately, medium) decoheres when the “off-diagonal” elements of \( D(\alpha', \alpha) \) are sufficiently small. The diagonal elements give the probabilities of the individual histories in the decoherent set. The rules both for which sets of histories may be assigned probabilities and for the values of those probabilities are thus summarized by the fundamental formula:

\[
D(\alpha', \alpha) \approx \delta_{\alpha'\alpha} p(\alpha) .
\] (6.1)

Two examples of generalized quantum theories have been described in previous sections. The first is usual quantum theory. Its fine-grained histories are defined by sequences of sets of one-dimensional projections, one set at each time, with individual histories represented by (continuous) chains of projections, one from each set. Coarse-grained histories are represented by class operators which are sums of these, as in (2.4). The decoherence functional is given by (2.5). The time-neutral generalized quantum theory of Section V is a second example. The fine-grained and coarse-grained sets of histories are the same as in usual quantum mechanics, but the decoherence functional (5.1) is different, incorporating both an initial and final condition. Other examples are the generalized quantum field theory in fixed background spacetimes with closed timelike curves to be discussed in the next section and the generalized quantum mechanics of dynamical spacetime geometry described in [6].

What all these examples have in common is a decoherence functional constructed from certain elements which represent histories and their evolution and other elements which represent the quantum boundary conditions. Examples of the former are the projections, Hamiltonian action, etc. in the examples discussed. Examples of the elements specifying boundary conditions are the positive matrices \( \rho_i \) and \( \rho_f \) representing initial and final conditions in (5.1). Assuming such a division of the elements entering the decoherence functional we can construct information measures as follows:

We first define the information content of the boundary conditions. One way to do this would be to simply choose an information measure on the elements of the decoherence functional that define the boundary conditions. We did this in Section II when we chose the measure \( S(\tilde{\rho}) = -Tr(\tilde{\rho} \log \tilde{\rho}) \) for density matrices. However, a more satisfactory approach is to define the measure, \textit{intrinsically}, in terms of the probabilities of a standard class, \( C_{\text{stand}} \), of very fine-grained, decoherent, sets of histories that probe these boundary conditions. Specifically, we define the missing information in the boundary conditions, \( S(D) \), as the least missing information in the probabilities of the sets of histories \( \{c_\alpha\} \) in the class \( C_{\text{stand}} \)

\[
S(D) = \min_{\{c_\alpha\} \in C_{\text{stand}}} \left[ - \sum_\alpha p(\alpha) \log p(\alpha) \right] .
\] (6.2)

A natural choice for the class \( C_{\text{stand}} \) is the class of decoherent completely fine-grained sets of histories. This was used to define the measure \( S(\tilde{\rho}; \tilde{\rho}^f) \) for the time neutral quantum mechanics discussed in Section V. However, for some generalized quantum theories there may be no completely fine-grained sets that decohere. For instance, this is likely to be the case in any sum-over-histories generalized quantum theory. Another possibility for \( C_{\text{stand}} \) would be the class of finest-grained sets of histories that decohere that is, the class of sets
which decohere but for which no finer graining decoheres. However, this is a more difficult class to compute and it is not even clear that the this choice would lead to the standard measure $-Tr(\hat{\rho}\log \hat{\rho})$ in usual quantum mechanics.

Once class $\mathcal{C}_{\text{stand}}$ is chosen, or the measure $S(D)$ otherwise fixed, the Jaynes construction can be implemented to define the missing information in a set of coarse-grained decoherent histories $\{c_\alpha\}$. The missing information is the maximum of the information content of those boundary conditions which reproduce the decoherence and probabilities of the set $\{c_\alpha\}$. That is,

$$S(\{c_\alpha\}) = \max_D \left[ S(\hat{D}) \right]_{\hat{D}(\alpha',\alpha)=D(\alpha',\alpha)}$$

(6.3)

where the maximum is over all boundary conditions. Evidently

$$S(\{c_\alpha\}) \geq S(D) .$$

(6.4)

The missing information in a class $\mathcal{C}$ of sets of histories is then straightforwardly defined as

$$S(\mathcal{C}) = \min_{\text{decoherent}} S(\{c_\alpha\}) .$$

(6.5)

The least missing information in the class of all histories defines the complete information:

$$S_{\text{compl}} = \min_{\text{decoherent}} S(\{c_\alpha\}) .$$

(6.6)

Clearly,

$$S(\{c_\alpha\}) \geq S_{\text{compl}} \geq S(D) .$$

(6.7)

If there is at least one set of histories for which the condition $\hat{D}(\alpha',\alpha) = D(\alpha',\alpha)$ implies that $\hat{D} = D$, then $S_{\text{compl}}$ equals $S(D)$. That was the case for the usual quantum mechanics of Section II, but not the case for the time-neutral quantum mechanics of Section V.

It is easily seen that the definitions (6.2), (6.3), and (6.5) coincide with the specific examples discussed in previous sections, but provide a general and abstract framework which we shall illustrate with another example in the next section.

**VII. FIXED SPACETIMES WITH NON-CHRONAL REGIONS**

Spacetime must be foliable by spacelike surfaces for the quantum mechanics of matter fields to be formulated in terms of the unitary evolution and reduction of a state vector defined on such surfaces. However, not all spacetimes permit a foliation by spacelike surfaces. Examples are spacetimes with closed timelike curves, such as might be produced by the relative motions of wormhole mouths [25]. For such spacetimes a more general formulation of quantum mechanics is required and a number have been discussed [26], [27], [28], [29], [30], [31], [32], [33], [34]. In this section we apply the general notions of information described in the previous sections to generalized quantum theories suitable for spacetimes with such non-chronal regions.
The generalized quantum theories we shall discuss were described in [5] and [31] whose notation we follow. We briefly review them here. We consider spacetimes with a fixed background geometry having a compact non-chronal region $NC$. We consider an initial region before $NC$ that is foliable by spacelike surfaces and a final region after $NC$ also foliable by spacelike surfaces. The region in between, however, is not so foliable. We consider the quantum mechanics of a single scalar field $\phi(x)$ moving in this background geometry. The characteristic feature of the theories we consider is that the transition matrix between a state of definite spatial field configuration $\phi'(x)$ on a spacelike surface $\sigma'$ before $NC$ and a similar state of definite spatial field configuration $\phi''(x)$ after $NC$ is generally non-unitary.

Transition matrices may be defined by sums-over-field histories between $\sigma'$ and $\sigma''$

$$\langle \phi''(x), \sigma'' | \phi'(x), \sigma' \rangle = \int_{[\phi',\phi'']} \delta \phi \exp(iS[\phi(x)])$$ \hspace{1cm} (7.1)

where $S[\phi(x)]$ is the action functional for the scalar field. As suggested by Klinkhammer and Thorne [27], and demonstrated by Friedman, Papastamatiou, and Simon [28], the transition matrix defined by (7.1) is non-unitary for an interacting field theory, order by order in perturbation theory.

We can construct generalized quantum theories incorporating such non-unitary evolution as follows: For the set of fine-grained histories we take sequences of sets of one-dimensional projections on every member of a foliating set of spacelike surfaces before $NC$ and every member of a foliating family of spacelike surfaces after $NC$. Simple examples of coarse-grained sets may be represented in a Heisenberg-like picture by chains of projections before $NC$

$$C_\alpha = P_{\alpha k}^k (\sigma_k) \cdots P_{\alpha_1}^1 (\sigma_1), \quad \sigma_i < \sigma_-, \hspace{1cm} (7.2a)$$

and chains of projections after $NC$

$$C_\beta = P_{\beta n}^n (\sigma_n) \cdots P_{\beta_{k+1}}^{k+1} (\sigma_{k+1}), \quad \sigma_i > \sigma_+, \hspace{1cm} (7.2b)$$

where $\sigma_-$ is a spacelike surface just before $NC$ and $\sigma_+$ is a spacelike surface just after $NC$. The projections in (7.2) evolve unitarily both before and after $NC$. A non-unitary operator $X$, derived from (7.1), connects the alternatives before $NC$ to those after $NC$ so that a whole history consisting of sets of alternatives at sequences of times is represented by

$$C_\beta XC_\alpha \hspace{1cm} (7.3)$$

with $C_\alpha$ and $C_\beta$ as given by (7.2). More general coarse grainings are obtained by partitioning such histories into mutually exclusive classes with class operators which are the corresponding sums of those of (7.3).

Two different forms of the decoherence functional give two distinct generalized quantum theories incorporating non-unitary evolution. The original proposal of [5] was to take, for histories of the form (7.3),

$$D(\beta', \alpha'; \beta, \alpha) = Tr \left[ C_{\beta'} X C_{\alpha'} \rho C_{\alpha}^\dagger X^\dagger C_{\beta}^\dagger \right] / Tr \left( X \rho X^\dagger \right) \hspace{1cm} (7.4)$$
where $\rho$ is a density matrix representing the initial condition of the closed system. An attractive alternative, proposed by Anderson [31], is to take

$$D(\beta',\alpha';\beta,\alpha) = Tr \left[ X^{-1}C_{\beta'}XC_{\alpha'}\rho X^\dagger C_{\beta}(X^\dagger)^{-1} \right]$$

(7.5)

assuming $X$ is invertible. Both (7.4) and (7.5) are easily seen to satisfy the general requirements (i)–(iv) for decoherence functionals mentioned in Section VI. The definition (7.5) has the advantages that, in contrast to (7.4), it is linear in the initial $\rho$ and does not lead to acausal effects in which the non-chronal regions in the future can effect the probabilities of present alternatives. Which, if any, form emerges from a more fundamental quantum theory of spacetime is an open question. In this section we shall concentrate on implementing the notions of information described in Section VI using Anderson’s (7.5). A similar but not coincident discussion could be given on the basis of (7.4).

A word is in order concerning the relation of the generalized quantum theories under discussion to the sum-over-field-histories formulations described in [5]. In a sum-over-histories formulation the set of fine-grained histories are the possible four-dimensional, spacetime field configurations, $\phi(x)$, and coarse grainings are restricted to partitions of these into mutually exclusive classes. Partitions by the values of spatial field configuration $\phi(x)$ on a spacelike surface outside of $NC$ defines one kind of coarse graining of field histories whose class operators can be represented by projection operators as in (7.2). However, partitions of the fields can also be used to define alternatives inside $NC$, for instance a partition by ranges of values of the field averaged over a spacetime region inside $NC$. The class operators of such spacetime alternatives, defined by functional integrals over the appropriate class of fields, are not generally projection operators or even of the form (7.3). The generalized quantum theories we are discussing in this section are both more restricted and more general than such sum-over-histories formulations. They are more restricted because they do not deal with alternatives inside $NC$ but only on spacelike surfaces outside of $NC$. On the other hand, outside $NC$, the alternatives are more general. The alternatives of a sum-over-field-histories formulation would be restricted to projections onto ranges of spatial field configurations. By contrast, the present discussion considers all the alternatives available by transformation theory provided they are defined on spacelike surfaces outside $NC$. We consider information of this particular example of generalized quantum theory, not because it is more general or more fundamental than the sum-over-field-histories formulation, but because its closer connection with the usual quantum theory discussed in Section II makes it a more useful example.

In the language of Section VI, $\rho$ is the element in the decoherence functionals (7.4) and (7.5) representing the boundary condition, and the $P$’s, $X$, $H$, etc. are the elements representing the histories and their evolution. Our first task, therefore, is to find the measure of missing information in a density matrix $\tilde{\rho}$ in the presence of the non-unitary evolution $X$. This is defined by (6.2) and denoted by $S_X(\tilde{\rho})$. We take the class of all fine-grained sets of histories of the form (7.2) for the class $C_{\text{stand}}$. Then, specifically,

$$S_X(\tilde{\rho}) = \min_{\text{fine-grained decoherent} \{C_{\beta}, C_{\alpha}\}} \left[ -\sum_{\alpha\beta} \tilde{p}(\beta,\alpha) \log \tilde{p}(\beta,\alpha) \right].$$

(7.6)

The fine-grained sets of histories are chains of sets of one-dimensional projections, one on
each spacelike surface outside $NC$. Extending the analysis of Section V one easily sees that for the purpose of computing $S_X(\rho)$ one may consider histories of the form

$$C_\alpha = P^i_\alpha = |\alpha\rangle\langle\alpha|; \quad C_\beta = P^f_\beta = |\beta\rangle\langle\beta|$$

(7.7)

where $\{|\alpha\rangle\}$ is a basis in which $\rho$ is diagonal and $\{|\beta\rangle\}$ is a basis which $XX^\dagger$ is diagonal. Decoherence requires all other finer-grained alternatives to be exactly correlated with these.

The probabilities of the decoherent set (7.7) are

$$\tilde{p}(\beta, \alpha) = \tilde{\pi}_\alpha |\langle \beta |X|\alpha\rangle|^2 \langle \beta |(XX^\dagger)^{-1}|\beta\rangle \equiv \tilde{\pi}_\alpha q^\alpha_\beta .$$

(7.8)

It is easily seen that, for fixed $\alpha$, the numbers $q^\alpha_\beta$ are probabilities and that, similarly to (5.15),

$$-\sum_{\alpha, \beta} \tilde{p}(\beta, \alpha) \log \tilde{p}(\beta, \alpha) = s(\tilde{\pi}) + \sum_{\alpha} \tilde{\pi}_\alpha s(q^\alpha) .$$

(7.9)

Thus, we have for the information measure of the initial condition in $\tilde{\rho}$ the presence of a non-unitary $X$.

$$S_X(\tilde{\rho}) = -Tr(\tilde{\rho} \log \tilde{\rho}) + \min_{\{|\beta\rangle\}} \sum_{\alpha} \tilde{\pi}_\alpha s(q^\alpha)$$

(7.10)

where the minimum is over bases that diagonalize $XX^\dagger$. If there is a unique basis that diagonalizes $XX^\dagger$, then (7.10) gives an explicit formula for $S_X(\tilde{\rho})$.

Were $X$ unitary, we could pick the orthogonal basis $\{|\beta\rangle\}$ to be $\{X^\dagger|\alpha\rangle\}$ thereby making $q^\alpha_\beta = \delta^\alpha_\beta$ and $S_X(\tilde{\rho}) = S(\tilde{\rho})$. However, when $X$ is non-unitary we have only

$$S_X(\tilde{\rho}) \geq S(\tilde{\rho}) = -Tr(\tilde{\rho} \log \tilde{\rho}) .$$

(7.11)

Eq (7.11) shows that $S_X(\tilde{\rho})$ generally does not coincide with $S(\tilde{\rho})$. In the presence of a domain non-unitary evolution somewhere in the spacetime, the missing information in a density matrix is greater than it would be if the domain had not been present. That is because the missing information in $\tilde{\rho}$ has been defined in terms of the probabilities of the finest-grained decoherent histories which it predicts, and those histories extend over the whole of spacetime — both before and after any non-chronal region.

With the definition of the information content of an initial $\rho$ in hand, the missing information in a set of histories $\{C_\beta, C_\alpha\}$ of the form (7.3) can be straightforwardly defined from the general schema (6.3)

$$S_X(\{C_\beta\}, \{C_\alpha\}) \equiv \max_{\tilde{\rho}} [S_X(\tilde{\rho})]_{D(\alpha', \alpha) = D(\alpha', \alpha)} .$$

(7.12)

Missing information in a class of histories is defined by (7.5) and complete information, $S_{compl}$, by the minimum of (7.12) over all decoherent sets of histories. Evidently, $S_{compl} \geq S_X(\rho)$, but we shall show in the following that

$$S_{compl} = S_X(\rho) ,$$

(7.13)

by exhibiting one example for which the equality is satisfied.
This generalized quantum mechanics of fields in non-chronal backgrounds cannot be
reformulated in terms of states on a spacelike surface, their unitarily evolution between
such surfaces, and their reduction at them. However, we may still investigate how much
information about the system is available in histories that are confined to a spacelike surface
σ. Clearly there are two types of surfaces — those before the non-chronal region NC and
those after it. We define
\[ S_{\text{before}}(\sigma) = \min_{\text{decoherent} \{P_{\alpha}(\sigma)\}} S_X(I, \{P_{\alpha}(\sigma)\}) , \]  
(7.14a)
for \( \sigma \leq \sigma_- \), and similarly, for \( \sigma \geq \sigma_+ \),
\[ S_{\text{after}}(\sigma) = \min_{\text{decoherent} \{P_{\beta}(\sigma)\}} S_X(\{P_{\beta}(\sigma)\}, I) . \]  
(7.14b)

In the Heisenberg picture, a set of projection operators \( \{P_{\alpha}\} \) is a projection onto ranges
of the values of some quantity on any surface. Therefore, the minimum (7.14a) will be the
same on all surfaces \( \sigma \) before NC. Similarly for (7.14b) after NC. Thus, missing information
\( S_{\text{before}}(\sigma) \) is conserved before NC and missing information \( S_{\text{after}}(\sigma) \) is conserved after NC.
It is not immediately obvious, however, whether information is conserved in passing from
before to after NC. We now show that it is for a reasonably generic set of cases, and that
complete information is available on each spacelike surface outside NC.

We consider the case where \( XX^\dagger \) is independent of \( X\rho X^\dagger \) in a sense made precise below.
For surfaces before NC consider the missing information in a set of projections \( P_{\alpha} = |\alpha\rangle\langle\alpha| \)
on onto a basis that diagonalizes \( \rho \). According to (7.12) this is the maximum of \( S_X(\tilde{\rho}) \) over all
\( \tilde{\rho} \) which reproduce the decoherence functional for this set of projections. The condition that
the decoherence functional be reproduced is
\[ \tilde{D}(\alpha', \alpha) = \langle \alpha'|\tilde{\rho}|\alpha \rangle = D(\alpha', \alpha) = \langle \alpha'|\rho|\alpha \rangle = D_{\alpha\alpha} \]  
(7.15)
Thus, \( \tilde{\rho} = \rho \) is the unique density matrix which reproduces the decoherence functional. From (7.12)
\[ S_X(I, |\alpha\rangle\langle\alpha|) = S_X(\rho) = S_{\text{compl}} . \]  
(7.16)
This one example is enough to demonstrate the equality in (7.13). Thus, complete informa-
tion is available on every spacelike surface before NC.

For surfaces after NC consider sets of projections \( P_{\beta} = |\beta\rangle\langle\beta| \) onto a basis which diagonalizes
the Hermitian operator \( X\rho X^\dagger \). This set is decoherent. The condition that a density
matrix \( \tilde{\rho} \) reproduce the decoherence functional for this set is
\[ \tilde{D}(\beta', \beta) = \langle \beta'(XX^\dagger)^{-1}|\beta'\rangle\langle \beta'|X\tilde{\rho}X^\dagger|\beta \rangle = D(\beta', \beta) = \delta_{\beta\beta'} \langle \beta'(XX^\dagger)^{-1}|\beta\rangle\langle \beta|X\rho X^\dagger|\beta \rangle . \]  
(7.17)
If we assume that \( \langle \beta'(XX^\dagger)^{-1}|\beta' \rangle \) has no non-vanishing matrix elements, then we can con-
clude that \( X\tilde{\rho}X^\dagger = X\rho X^\dagger \). Since \( X \) is assumed invertible we have \( \tilde{\rho} = \rho \) for the unique \( \tilde{\rho} \)
that reproduces the decoherence functional of \( \rho \) for the set \( \{P_{\beta}\} \). Thus, from (7.12)
\[ S_X(\beta \langle \beta \rangle, I) = S_X(\rho) = S_{\text{compl}}. \] (7.18)

Thus, complete information is available about the quantum system on every spacelike surface after NC.

Taken together, (7.16) and (7.18) show that, despite the absence of states on a spacelike surface, complete information about the quantum system is available on every spacelike surface outside of NC. Complete information is conserved; it is the same \( S_X(\rho) \) on all spacelike surfaces, both before and after NC. However, the conserved, complete, missing information \( S_X(\rho) \) on any surface in a spacetime with a non-chronal region is greater than the missing information \( S(\rho) = -Tr(\rho \log \rho) \) in a spacetime without such a non-chronal region. That is the case even though the predictions of Anderson’s generalized quantum mechanics coincide exactly with the usual theory for spacelike surfaces before NC. The reason is that the missing information \( S_X(\hat{\rho}) \) has been defined in terms the probabilities of fine-grained sets of decoherent histories of the closed system. These extend arbitrarily far into future and thus are affected by the existence of any non-chronal region. The probabilities of these finest-grained decoherent histories are more distributed because of the non-unitary evolution arising from the non-chronal region than they would be without [cf. (7.8)]. Thus, the missing information \( S_X(\rho) \) is greater than \( S(\rho) \). One might be tempted to define missing information in \( \rho \) before NC by the usual \(-Tr(\rho \log \rho)\) since the predictions of this generalized quantum mechanics coincide with the usual theory there. That, however, would lead to an unexplained loss of information in passing from before the non-chronal region to after it. Here, we have consistently adopted a spacetime approach to information with the result that information is conserved in passing from one spacelike surface to another.

VIII. BLACK HOLES

Hawking’s 1974 prediction of a steady flux of thermal radiation in test fields from black hole background spacetimes raised the possibility that black holes could evaporate completely. As a consequence, information as usually defined in terms of states on spacelike surfaces, would be permanently lost in the process of evaporation. The questions of the outcome of the Hawking process and its consistency with the basic principles of quantum mechanics has been of intense interest since. In this section we discuss these questions utilizing the notions of spacetime information developed in this paper in the context of a generalized quantum theory of spacetime geometry.

As yet we have no theory of quantum gravity adequate for predicting the history of an evaporating black hole when it has shrunk to less than Planck scale dimensions, despite a number of interesting models. No improvement on this situation is offered there. Rather, we describe information in the black hole evaporation process assuming that the black hole evaporates completely. We do this in the kinematical framework for a generalized quantum mechanics developed in [6]. The qualitative information theoretic issues we shall discuss are probably insensitive to the details in this framework but it provides a reasonably concrete, if formal, setting in which to consider them. We briefly recall some of its relevant features:

As mentioned in Section VI, there are three elements in a generalized quantum theory: (1) the fine-grained histories, (2) the allowed coarse-grained sets of histories, and (3)
a decoherence functional defining the notion of interference between pairs of histories in coarse-grained sets. For a theory of black hole evaporation, we take the fine-grained histories (1) to be a class of asymptotically flat spacetime geometries with matter fields whose Penrose diagrams have the form shown in Figure 1a. We are thus dealing with a sum-over-histories generalized quantum theory. The precise nature of the geometries interior to the asymptotic region – how differentiable they are, and what kinds of singularities are permitted, etc – are central issues in the specification of a complete theory of quantum spacetime. We do not resolve these issues here, but we assume that the class of fine-grained histories at least includes those (Figure 2b) whose Penrose diagram are of the kind commonly taken to describe the complete evaporation of a black hole. The coarse-grained histories (2) are partitions of these fine-grained histories into mutually, exclusive, diffeomorphism-invariant classes \( \{c_\alpha\} \). In calculating the analogs of transition amplitudes, for example, we are typically interested in partitions of the histories by invariant descriptions of their asymptotic geometries and matter fields on \( I^-, I^- \) and \( I^+, I^+ \), leaving the interior fields and geometry unrestricted. The remaining element (3) is the decoherence functional. This is specified first by constructing amplitudes for the individual histories \( c_\alpha \) in a coarse-grained set by sums over the corresponding class of histories of the schematic form

\[
\int_{c_\alpha} \delta g \delta \phi \exp(iS[g, \phi])
\]  

where \( S[g, \phi] \) is the action of gravity coupled to matter fields \( \phi(x) \) and the integral is over geometries and fields in the class \( c_\alpha \) with additional restrictions necessary to incorporate the boundary conditions. The decoherence functional \( D(\alpha', \alpha) \) is a bilinear combination of these amplitudes analogous to (5.1).

We have deliberately been brief in sketching the details of such a putative quantum kinematics of spacetime geometry because we wish to make only one point: Such a generalized quantum mechanics of spacetime geometry is not formulated in terms of states on spacelike surfaces or their unitary evolution between such surfaces. Neither is it likely that it can be so formulated since the fine-grained histories single out no set of spacelike surfaces to supply the preferred time usually required in such theories. The notions of information, and of complete information in particular, must be reexamined in such a theory. The discussion in Section VI provides a general framework for doing so.

The information measure of the boundary conditions, \( S(D) \), is the least missing information in the probabilities of the sets of histories is the class \( C_{\text{stand}} \) which, for definiteness, we may take to be the class of finest-grained decoherent sets of histories. (The argument is not strongly dependent on the choice for \( C_{\text{stand}} \).) Coarser-grained sets of histories will generally have more missing information. Complete information is the least missing information among all sets of decohering histories. Complete information is available in some set of histories, but complete information may not be available from histories defined by alternatives on a single spacelike surface. Further, the most nearly complete information available on one spacelike surface is not guaranteed to be the same as that available on another spacelike surface.

Suppose the fine-grained histories describing a theory of transitions between asymptotically flat regions contain those with Penrose diagrams as in Figure 2b commonly assumed to describe the complete evaporation of a black hole. Then it seems especially likely that...
the same information is not available on the surface $I^+$ as it was on $I^-$. Plausibly some information has gone down the black hole.

When quantum mechanics is in spacetime form and information formulated in terms of spacetime histories complete information may not be available on any particular spacelike surface. Rather, it may be necessary to search about the four-dimensional spacetime to find complete information. Histories defined by partitions near any horizon as well as partitions near infinity may be necessary. Thus, the absence of complete information on a spacelike surface after the complete evaporation of a black hole is not a violation of the principles of quantum mechanics suitably generally stated. Rather it becomes an interesting example of the utility of formulating the theory in fully spacetime form.

**IX. CONCLUSIONS**

Complete descriptions of quantum systems in terms of a state on a spacelike surface can be expected only in situations when there is an unambiguous notion of “spacelike surface” and the notion of causality implicit in such a state apply. In more general cases, where spacetime geometry fluctuates, as in a quantum theory of gravity, or where it is fixed, but lacking a foliating family of spacelike surfaces, as in spacetimes with non-chronal regions, or when the boundary conditions are inconsistent with usual causality, as in the time-neutral formulation, we cannot expect to formulate quantum mechanics in terms of states on spacelike surfaces. Rather a more general formulation of quantum mechanics is needed. In these circumstances it seems natural to formulate quantum mechanics in fully spacetime form both with respect to dynamics and alternatives. For such generalizations, the usual quantum mechanical notions of information must also be generalized. This paper has provided one such generalization. We have provided a general schema for defining the complete information content in the boundary conditions of a generalized quantum theory. We have defined the information available about these boundary conditions in a set of alternative histories of the closed system and in classes of such sets. These notions of information are in fully spacetime form. Complete information may not be available on any spacelike surface. Rather it may be distributed about spacetime and available only through histories that are not specific to any one spacelike surface.

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Fig 1: Three examples of spacetime regions. The region $R_a$ contains a Cauchy surface $\sigma$. Complete information is therefore available in $R_a$. Parts of a wave packet moving away from region $R_b$ will never intersect it at any later time. Complete information is therefore unlikely to be available in $R_b$. The region $R_c$ is the domain of dependence of a region $L$ of a Cauchy surface $\sigma$. The missing information in $R_c$ is the same as the missing information in the reduced density matrix for $L$, i.e., the density matrix $\rho(\sigma)$ traced over all field variables outside $L$.

Fig 2: In a generalized quantum mechanics describing black hole evaporation the fine-grained histories are a class of asymptotically flat geometries with Penrose diagrams of the form shown in (a). If the class contains geometries with Penrose diagrams like that of (b), that are usually said to describe the evaporation, then it is unlikely that the generalized quantum theory can recast as a theory of evolving states on spacelike surfaces. Complete information therefore may not be available on any one spacelike surface and in particular not on the surface $I^+$, because, plausibly information has gone down the black hole.
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Fig 1

(a) $R$

(b) $R$

(c) $R$, $L$
This figure "fig1-2.png" is available in "png" format from:

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Fig 2