SPECTRAL THEORY OF SPIN SUBSTITUTIONS

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Dedicated to our late friend and colleague, Uwe Grimm

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Abstract. We introduce substitutions in $\mathbb{Z}^m$ which have non-rectangular domains based on an endomorphism $Q$ of $\mathbb{Z}^m$ and a set $\mathcal{D}$ of coset representatives of $\mathbb{Z}^m/Q\mathbb{Z}^m$, which we call digit substitutions. Using a finite abelian ‘spin’ group we define ‘spin digit substitutions’ and their subshifts $(\Sigma, \mathbb{Z}^m)$. Conditions under which the subshift is measure-theoretically isomorphic to a group extension of an $m$-dimensional odometer are given, inducing a complete decomposition of the function space $L^2(\Sigma, \mu)$. This enables the use of group characters in $\hat{G}$ to derive substitutive factors and analyze the spectra of specific subspaces. We provide general sufficient criteria for the existence of pure point, absolutely continuous, and singular continuous spectral measures, together with some bounds on their spectral multiplicity.

1. Introduction. In this work, we define and develop the theory of digit substitutions, which are a type of multidimensional morphisms on a finite alphabet $A$ wherein the geometry is based on digit tilings; see [47, 48]. Given an expanding endomorphism $Q$ of $\mathbb{Z}^m$ and a full set $\mathcal{D}$ (our digit set) of coset representatives of $\mathbb{Z}^m/Q\mathbb{Z}^m$ we can partition $\mathbb{Z}^m$ into subsets of the form $Q\mathbb{Z}^m + \vec{j}$ with $\vec{j} \in \mathcal{D}$. There are already interesting questions on ‘tiling the integers’ (see numerous references from [18, 27]), but our construction is a type of discrete finite automaton instead. The choice of a digit set $\mathcal{D}$ completely determines a sequence $\{\mathcal{D}(n)\}$ of full sets of coset representatives for $\mathbb{Z}^m/Q^n\mathbb{Z}^m$ for all $n \in \mathbb{N}$, which is suitable in building the

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support of the $n$-supertiles of a digit substitution; see [15] for a recent account which focuses on algebraic invariants. This generalizes the hierarchical structure present in constant-length substitutions in one dimension and block substitutions in higher dimensions, compare [41, 23, 12, 7]. Like these cases, we study the subshift $\Sigma$ of $\mathbb{Z}^m$ constructed using $n$-supertiles as a pre-language.

Once the issues regarding the underlying geometry are settled, it becomes clear that one can use the same techniques in constructing analogous variants in our setting (e.g. substitutions with coincidences, bijective substitutions, etc.). For the rest of the paper, we then focus on a method for defining spin digit substitutions, which is inspired by the construction of the Rudin–Shapiro sequence as an iterated morphism. These substitutions are special types of digit substitutions having additional invariance properties from action of a finite abelian group $G$. The resulting subshifts allow a relatively complete analysis that generalizes known families algebraically, geometrically and spectrally; see [42, 2, 24, 1, 16]. The construction works in any dimension, allows the existence of disconnected supertiles and holds for any choice of (finite) abelian group. Further novel aspects of this work include (i) explicit formulation of such systems as skew products (ii) substitutive factors defined by the characters and their relation to the spectrum and (iii) analysis via Lyapunov exponents on certain subspaces. Under additional assumptions, a subshift with the same spectral features can be defined even when the group is no longer finite, but is still compact, yielding new and interesting phenomena.

The alphabet for a spin substitution consists of ‘spinned’ versions of the digits, where the spins are provided by a finite abelian spin group $G$. A $|D| \times |D|$ matrix $W$ over $G$ completely determines a substitution on this alphabet by defining the spin allocation for ‘spin-free’ letters and extending by multiplication in $G$. The geometry of $D$ governs the placement of the letters in $\mathbb{Z}^m$. So constructed, the substitution subshift commutes with group multiplication, which in turn allows for spectral analysis.

Some comments on nomenclature are in order. We selected the term digit for our substitutions because they are supported on digit sets; see [47, 48]. The usage of the term spin is inspired by spin models in statistical mechanics, for instance in the Ising model where the group of spins is $\{-1, 1\}$. Such a choice (i.e., the spin group being a finite abelian group) is natural even in that context, as there have been generalized spin models with the group of spins being a finite abelian group; see [10, 19].

The paper is organized as follows. In Section 2 we recall standard notions in the theory of tilings and digit tilings in $\mathbb{Z}^m$. In Section 3, we define what a digit substitution is and discuss the dynamical properties of the subshift that arises from it, which it inherits from an associated (self-similar) tiling dynamical system (with an $\mathbb{R}^m$-action). Here it is made clear that the connectedness of the supertiles is not a prerequisite to define the subshift. We discuss the corresponding odometer in Section 3.3 and then specialize to spin substitutions in Section 3.4. The rest of the paper then focuses on this subclass.

Section 4 begins with a brief survey of notions in spectral theory, followed by a discussion of the group action by $G$ on the subshift. In Theorem 4.2, we prove that a spin substitution subshift is isomorphic to a group extension and give an explicit formula for the cocycle. This induces a splitting of the function space into orthogonal subspaces associated to each character $\chi \in \hat{G}$ and enables one to handle each subspace independently. Proposition 5 states that every subspace is spectrally
pure. The main characterization result given in Theorem 4.5 gives sufficient conditions for the presence of certain spectral types. We will write $\chi(W)$ for the matrix with entries $\chi(W_{ij})$. Under mild dynamical assumptions, Theorem 4.5 shows:

- If $\chi$ is the trivial character, then the subspace $H^\chi$ is pure point.
- If $\frac{1}{\sqrt{|D|}}\chi(W)$ is unitary, then $H^\chi$ is absolutely continuous.
- If $\chi(W)$ is rank-1, then $H^\chi$ is either pure point or purely singular continuous.

Note that these spectral results are independent of the geometry and only rely on $W$. These statements are proved in different sections where the appropriate arguments are developed separately. Section 4.5 contains the proof of the second statement, which relies on computing the Fourier coefficients of certain spectral measures explicitly. The method of proof for this subcase is constructive and generalizes that of [24].

In Section 5, we exploit known connections between the diffraction spectrum and the spectral measures of specific functions that we know are in $H^\chi$ to prove singularity results. Proposition 10 describes the properties of substitutive factors, which always exist for spin substitutions, and how the spectral measures they admit restrict the spectral type of $H^\chi$. Proposition 11 and the discussion following it finish the proof of the third claim on singularity. For this we need the construction of the factor substitution induced by a character $\chi \in \hat{G}$; when $\chi(W)$ is rank-1, this substitution is bijective.

Beyond the conditions given in Theorem 4.5, one can use a renormalization approach via Lyapunov exponents to obtains singularity results, which we discuss in Section 5.2. To be more precise, the corresponding matrix cocycle $B$ is unitarily block-diagonalizable, where each block $B_\chi$ corresponds to a character $\chi$ and its subspace $H^\chi$. Proposition 14 provides a numerically-verifiable singularity criterion for $H^\chi$.

Several examples with particular spectral properties are then presented in Section 6. We end with some open questions and an appendix on diffraction measures.

2. Tilings.

2.1. General notions. We define a tile in $\mathbb{R}^m$ to be a pair of the form $(C, a)$, where the support $C$ is a compact subset of $\mathbb{R}^m$ that is the closure of its interior, and the type $a$ is in some finite alphabet $\mathcal{A}$. Two tiles are equivalent if their supports are translates of one another and they are the same type. A finite set $\mathcal{P}$ of nonequivalent tiles on a given alphabet $\mathcal{A}$ is called a prototile set. Copies of prototiles are used to form tilings of $\mathbb{R}^m$.

Suppose $T$ is a set of tiles, all of which are equivalent to tiles in $\mathcal{P}$. We call $T$ a tiling if the interiors of any two nonequal tiles are disjoint and the supports’ union is $\mathbb{R}^m$. If $X$ is a set of tilings that is invariant under translation by $G = \mathbb{Z}^m$ or $\mathbb{R}^m$, and is closed under the local topology, we call $(X, G)$ a tiling dynamical system.

Let $A: \mathbb{R}^m \to \mathbb{R}^m$ be an expansive map. An inflate-and-subdivide rule $\phi$ on a finite prototile set $\mathcal{P}$ with map $A$ is one where, for every $(C, a) \in \mathcal{P}$, there exist a finite set $F$ and translation vectors $\vec{d}_i \in \mathbb{R}^m$ such that $\phi(C, a) = \bigcup_{i \in F} (C_i + \vec{d}_i, a_i)$ and $AC = \bigcup_{i \in F} (C_i + \vec{d}_i)$, where $(C_i, a_i) \in \mathcal{P}$ for all $i \in F$. If $A$ is a similarity transformation, one calls $\phi$ a self-similarity or a stone inflation. A tiling which is invariant under such a map is called a self-similar tiling.
2.2. Digit tilings. A **digit system** \((Q, D)\) (see [47]) is an expansive endomorphism \(Q\) of \(\mathbb{Z}^m\) (i.e., \(|\lambda| > 1\) for all eigenvalues \(\lambda\) of \(Q\)) along with a complete set \(D\) of coset representatives of \(\mathbb{Z}^m/Q\mathbb{Z}^m\) called a **digit set**. Since \(\mathbb{Z}^m = Q\mathbb{Z}^m + D\), \(D\) (seen as a direct Minkowski sum) is a set that covers \(\mathbb{Z}^m\) periodically and comes with a natural rescaling map \(Q\). Those properties allow digit systems to support the underlying geometry of a substitution rule in \(\mathbb{Z}^m\) or \(\mathbb{R}^m\).

Every digit system \((Q, D)\) of the type we consider gives rise to an iterated function system with contraction maps \(c_{\vec{d}}(x) = Q^{-1}(x + \vec{d})\) for each \(\vec{d} \in D\). The attractor is the **digit tile**

\[
t = \left\{ \sum_{k=1}^{\infty} Q^{-k} \vec{d}_k \mid \vec{d}_k \in D \right\} = \lim_{k \to \infty} Q^{-k} D^{(k)}.
\]  

(1)

Its existence and properties are explored for broad classes of digit systems. Gröchenig and Haas considered these digit tiles and asked when they tile \(\mathbb{R}^n\) via some lattice [28]. Vince calls digit systems “radix systems” in [48] where he provided several algorithms to check whether all lattice elements admit radix expansions. Lagarias and Wang have a series of papers on digit tiles in \(\mathbb{R}^m\), including [33] on general tiling and geometric properties of digit tiles and [32], where they focused on the case when \(Q\) is an integer matrix. A recent and comprehensive summary in this area is Vince’s survey article [47].

**Example 1.** Let

\[
Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad D = \{(0, 0), (1, 0), (0, 1), (-1, -1)\}.
\]

The digit tile \(t\) corresponding to \((Q, D)\) looks like this:

![Digit Tile Example](image)

The one-tile prototile set \(\{t\}\) can form tilings of \(\mathbb{R}^m\) in two possibly nonequivalent ways. A ‘self-replicating tiling’ arises from the fact that \(Qt = \bigcup_{\vec{d} \in D} t + \vec{d}\), which gives rise to a stone inflation. It can be shown [44] that there is a tiling of \(\mathbb{R}^m\) that is invariant under expansion by \(Q\) followed by subdivision.

The digit tile can also tile \(\mathbb{R}^m\) using a lattice \(L\) in the sense that

\[
\{t + \vec{x}, \vec{x} \in L\}
\]

covers \(\mathbb{R}^m\) and the tiles intersect only at boundaries. There are situations where \(L\) is a proper, full-rank sublattice of \(\mathbb{Z}^m\) and we need to avoid these situations; compare with Section 3.2 below. The relevant properties needed in what follows are summarized next.

**Proposition 1.** (see [28, 33, 32, 48]) Suppose \(Q\) is an expanding endomorphism of \(\mathbb{Z}^m\) and \(D\) is a complete set of coset representatives of \(\mathbb{Z}^m/Q\mathbb{Z}^m\). Then

(i) \(t\) is the closure of its interior,

(ii) the Lebesgue measure \(\mu_L(t)\) is a positive integer, and

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1This is called a **standard digit set** in [32].
(iii) $\mu_L(t) = 1$ if and only if $t$ lattice-tiles $\mathbb{R}^m$ as in Eq. (2) with $L = \mathbb{Z}^m$.

**Definition 2.1.** We call $(Q, D)$ a unit digit system if its digit tile has Lebesgue measure 1.

Geometric and topological questions surrounding digit systems in general are quite subtle (see [28, 33, 32, 48]) and are related to the possibility of a digit system $(Q, D)$ giving rise to non-empty subshifts.

3. Digit and spin substitutions.

3.1. Digit substitutions.

**Definition 3.1.** Let $(Q, D)$ be a digit system and let $A$ be a finite set called the alphabet. Let $A^D$ be the set of finite words over $A$ whose support is $D$. A digit substitution is a map $S = S^1 : A \to A^D$. We write $S^n(a)$ to mean the 1-supertile of type $a$. For $\vec{d} \in D$, we denote by $S_{\vec{d}}(a)$ to be the letter at position $\vec{d}$ in the 1-supertile of type $a$. The $n$-supertiles are defined recursively as

$$S^{n+1}(a) = \bigcup_{\vec{d} \in D} S^n(S_{\vec{d}}(a)) + Q^n(\vec{d}).$$

(3)

For each $n \in \mathbb{N}$ we call the set $D^{(n)} = QD^{(n-1)} + D$ the domain of the $n$-supertile, where $D^{(0)} := \{0\} \subset \mathbb{Z}^m$. With this, $S^{n+1}_d(a)$ is well defined for $\vec{d} \in D^{(n)}$.

**Remark 1.** There are two equivalent ways to view an $(n+1)$-supertile: fusion and substitution (see [22]). The substitutive approach for computing $S^{n+1}(a)$ is to apply $S$ to each element in $S^n(a)$. This results in the (equivalent) formulation

$$S^{n+1}(a) = \bigcup_{\vec{d} \in D^{(n)}} S(S^n_{\vec{d}}(a)) + Q^n(\vec{d}).$$

The fusion approach seen in our definition is that $S^{n+1}(a)$ is assembled from $n$-supertiles of the same types as those in $S(a)$ translated by $Q^n$ times their corresponding vectors. This viewpoint is needed for the majority of our proofs.

**Example 2.** We present an example of a digit substitution on two letters, whose geometry is based from Example 1. Let

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad D = \{(0,0), (1,0), (0,1), (-1, -1)\}, \quad \text{and} \quad A = \{a, b\}.$$ 

Let $S(a)$ take the four digits to $a, a, a$ and $b$ respectively, pictured at the left in Figure 1 with $a$ and $b$ represented as pink and blue squares with their lower left corners at digits. Let $S(b)$ take the digits to $b, b, b, a$, the opposite of $S(a)$. (This type of substitution is called bijective). Figure 1 shows the first three iterations of an $a$ under this digit substitution. (Note that a copy of $S(b)$ appears at the lower left of the 2-supertile pictured in the center).

3.2. Digit substitution dynamical systems. Although individual supertiles in a digit substitution may not be connected, sometimes they contain arbitrarily large rectangular words. Whether or not a subshift exists depends entirely on whether or not they do. This in turn depends entirely on properties of its underlying digit system $(Q, D)$. The dynamics of the subshift depend both on the digit system and the supertile labels from $A$. 
In the example just given, we see that any \( n \)-supertile contains several \( 2 \times 2 \) words if \( n \geq 2 \). All of the iterates of these \( 2 \times 2 \) words under the digit substitution are therefore subwords of supertiles at some level as well. Figure 2 shows a later iteration of a word with domain \( \{(0, 0), (0, -1), (-1, -1), (-1, 0)\} \).

This figure suggests that it is not necessary to have connectivity conditions on supertiles to define a subshift. Therefore, in line with standard practice, we use our supertiles as a sort of pre-language to make the following definition.

**Definition 3.2.** Let \( \mathcal{S} : \mathcal{A} \rightarrow \mathcal{A}^D \) be a digit substitution in \( \mathbb{Z}^m \), and let \( R \) be a rectangular subset of \( \mathbb{Z}^m \). A patch \( P \in \mathcal{A}^R \) is **admitted by \( \mathcal{S} \)** if there is an \( N \in \mathbb{N} \) and a letter \( a \in \mathcal{A} \) such that a shift of \( P \) appears in \( \mathcal{S}^N(a) \).

An element \( \tau \in \mathcal{A}^{\mathbb{Z}^m} \) is **admitted by \( \mathcal{S} \)** if every rectangular subword of \( \tau \) is admitted by \( \mathcal{S} \). If nonempty, the set \( \Sigma \) of admitted sequences in \( \mathcal{A}^{\mathbb{Z}^m} \) under the action of the shift is the **substitution subshift** \( (\Sigma, \mathbb{Z}^m) \).

Sufficient conditions for a substitution subshift to be non-empty are not undertaken in this work but can be checked in examples. To determine when digit substitution dynamical systems are minimal, and to understand the nature of their invariant measures, it is necessary to delve into some of these topological questions. Our arguments will rely on existing results from tiling dynamical systems theory, so we review these briefly as well.
Suppose \((\Sigma, \mathbb{Z}^m)\) is a non-empty subshift given by a digit substitution over a unit digit system \((\mathbb{Q}, \mathcal{D})\). We construct two tiling spaces for \(\Sigma\) that are topologically conjugate as dynamical systems over \(\mathbb{R}^m\). Let \(\mathbf{u} = [0, 1]^m\) be the unit cube and let \(\mathbf{t}\) be the digit tile for \((\mathbb{Q}, \mathcal{D})\). Without loss of generality, we may assume that 0 \(\in \mathcal{D}\) so that 0 is in both \(\mathbf{u}\) and \(\mathbf{t}\). Let \(A_u = \{(\mathbf{u}, a) : a \in A\}\) and \(A_t = \{(\mathbf{t}, a) : a \in A\}\) be two prototile sets, the former for constructing a tiling space \(\Omega_u\) and the latter for constructing \(\Omega_t\). For the substitution of Example 2 we have two fractal prototiles \((\text{fractiles})\) that look like this:

For each \(\tau \in \Sigma\), define the tilings

\[
\tau_u = \bigcup_{\mathbf{j} \in \mathbb{Z}^m} (\mathbf{u} + \mathbf{j}, \tau(\mathbf{j})) \quad \text{and} \quad \tau_t = \bigcup_{\mathbf{j} \in \mathbb{Z}^m} (\mathbf{t} + \mathbf{j}, \tau(\mathbf{j})),
\]

the latter forming a tiling since \((\mathbb{Q}, \mathcal{D})\) is a unit digit system. The tilings \(\tau_u\) and \(\tau_t\) can be translated by any element of \(\mathbb{R}^m\), not just elements of \(\mathbb{Z}^m\). We construct the tiling spaces \(\Omega_u\) and \(\Omega_t\) as the orbit closure under translation in \(\mathbb{R}^m\) of all such tilings.

**Claim 3.3.** *The tiling dynamical systems \((\Omega_u, \mathbb{R}^m)\) and \((\Omega_t, \mathbb{R}^m)\) are topologically conjugate.*

There are two ways to see that this is true. The first is to note that the spaces are mutually locally derivable; see [6, Sec. 5.2]. To do this, a local map must be constructed that identifies, for each tiling in \(\Omega_u\) and each \(x \in \mathbb{R}^m\), exactly what tile to place at \(x\) in the corresponding tiling in \(\Omega_t\). A local map from \(\Omega_t\) to \(\Omega_u\) must also be constructed. In our situation, the local code simply replaces a copy of \(u\) with a copy of \(t\) at the same location (or vice versa), of course keeping the same tile label. Thus the tiling dynamical systems \((\Omega_u, \mathbb{R}^m)\) and \((\Omega_t, \mathbb{R}^m)\) are mutually locally derivable, which in particular means they are topologically conjugate.

The second way to prove the claim is to construct a homeomorphism from \(\Omega_u\) to \(\Omega_t\) directly. The easiest way to do this is to define the homeomorphism first on the transversal, the set of tilings for which the tile at the origin is the trivial shift of a prototile. The homeomorphism between the transversals is given by Eq. (4). The homeomorphism extends to \(\mathbb{R}^m\) since every tiling in either space is the translate of some element of the transversal.

The fact that \(Q \mathbf{t} = \bigcup_{\mathbf{d} \in \mathcal{D}} \mathbf{t} + \mathbf{d}\) leads to an inflate-and-subdivide rule for \(\Omega_t\), which is given by

\[
\phi(t, a) = \bigcup_{\mathbf{d} \in \mathcal{D}} (t + \mathbf{d}, S_{\mathbf{d}}(a)).
\]

For Example 2, the inflate-and-subdivide rule for the pink fractile is pictured on the left of Figure 3, along with its 2- and 3-supertiles. This figure should be compared to Figure 1. The location of the origin should be noticed in both the square tile and the fractile, because under the digit substitution the origin of a tile is moved to the elements of \(\mathcal{D}\). The color of each tile is specified by \(S\).

Figure 4 shows a large rectangular word in some \(\tau \in \Sigma\) embedded in \(\Omega_u\) and in \(\Omega_t\). The image of \(\tau\) in either space is constructed by moving the origin of the
prototiles to the same location, regardless whether the prototiles are based on u or t. Since we have two dynamical systems which are topologically conjugate, we can transfer ergodic properties of the self-affine tiling space to its counterpart which is not self-affine.

**Proposition 2.** Let $S$ be a primitive digit substitution for which $(Q, D)$ is a unit digit system. If the substitution subshift $(\Sigma, \mathbb{Z}^m)$ is nonempty, it is strictly ergodic and the unique invariant measure is given by the patch frequency measure.

**Proof.** Construct the tiling dynamical systems $(\Omega_u, \mathbb{R}^m)$ and $(\Omega_t, \mathbb{R}^m)$ for $(\Sigma, \mathbb{Z}^m)$. The fact that $\Omega_t$ is a self-affine tiling dynamical system follows from the presence of the inflate-and-subdivide rule $\phi$: all finite patches in any tiling in $\Omega_t$ appear in an $n$-supertile for some sufficiently large $n$. By [44], $(\Omega_t, \mathbb{R}^m)$ is strictly ergodic, meaning that $(\Omega_u, \mathbb{R}^m)$ is as well. Since $(\Omega_u, \mathbb{R}^m)$ is a suspension of $(\Sigma, \mathbb{Z}^m)$, we conclude that $(\Sigma, \mathbb{Z}^m)$ is also strictly ergodic.

As the transition matrix is the same for both the substitution $S$ and the inflate-and-subdivide rule $\phi$, its right Perron–Frobenius eigenvector contains the frequencies of elements of $A$.

3.3. **Digit odometers.** The general framework for odometers in $\mathbb{Z}^m$ was introduced in [17], which generalizes both the notion of an $L$-adic odometer in one dimension and also products of $L_i$-adic odometers for $i \in \{1, \ldots, m\}$. We will show that a digit substitution with expansion matrix $Q$ has an underlying $Q$-adic odometer per [17]. Such $Q$-adic odometers have been used by [34] in the context of substitution Delone sets on lattices and by [47] in the context of digit tilings. We recall some notions below and discuss the explicit formulation for digit substitutions.

Let $\mathbb{Z}^m = Z_1 \supset Z_2 \supset \cdots$ be a sequence of subgroups isomorphic to $\mathbb{Z}^m$. Since $Z_i \supseteq Z_{i+1}$, we have that $\mathbb{Z}^m/Z_i \subseteq \mathbb{Z}^m/Z_{i+1}$ and this inclusion gives rise to the inverse limit

$$\mathcal{O} = \lim_{\leftarrow i} \mathbb{Z}^m/Z_i \subseteq \prod_{i \geq 1} \mathbb{Z}^m/Z_i.$$ 

An element $k = (\tilde{k}_i)_{i \geq 1} \in \mathcal{O}$ satisfies $\tilde{k}_i \equiv \tilde{k}_{i+1} \pmod{Z_i}$. This set forms a group under coordinate-wise modular addition, defined for $k, j \in \mathcal{O}$ to be

$$k \oplus j = \left( \tilde{k}_i + \tilde{j}_i \pmod{Z_i} \right)_{i \geq 1}.$$
The image of some $\tau \in \Sigma$ embedded into the tiling spaces $\Omega_u$ (top) and $\Omega_t$ (bottom).

The action of $\mathbb{Z}^m$ on $\mathcal{O}$ is given by extending $\oplus$ to elements $\vec{j} \in \mathbb{Z}^m$. For $k \in \mathcal{O}$, define

$$k \oplus \vec{j} = \left( \vec{k}_i + \vec{j} \pmod{Z_i} \right)_{i \geq 1}.$$ 

This can be thought of as embedding $\mathbb{Z}^m$ into $\mathcal{O}$ and then adding using the existing group addition. The group $\mathbb{Z}^m$ acts freely on $\mathcal{O}$ if and only if $\cap_{i \geq 0} Z_i = \{0\}$. The dynamical system $(\mathcal{O}, \mathbb{Z}^m)$ is called a $\mathbb{Z}^m$-odometer.

Let $\mathcal{S}$ be a digit substitution with expansive map $Q$ and digit set $\mathcal{D}$. For each $i$, let $Z_i = Q^i \mathbb{Z}^m$ and construct the odometer $\mathcal{O} = \lim_{i \to -\infty} Z_i / Q^i \mathbb{Z}^m$. Addition $\oplus$ is well-defined regardless of any specific choice for the equivalence classes modulo $Q^i \mathbb{Z}^m$. Since $Q$ is expansive, we have that $\cap_{i \geq 0} Q^i \mathbb{Z}^m = \{0\}$, so we know addition acts freely on $\mathcal{O}$. 
Since $D^{(i)}$ is a complete residue system for $Q^i \mathbb{Z}^m$, it provides a convenient set of coset representatives. We thus identify
\[
\mathcal{O} = \lim_{\leftarrow i} \mathbb{Z}^m / Q^i \mathbb{Z}^m \cong \lim_{\leftarrow i} D^{(i)}.
\]

We endow $\mathcal{O}$ with the product topology and choose a measure $\nu$ to be the frequency measure of $\mathcal{O}$ as follows. For each $M \in \mathbb{N}$ and $\vec{d} \in D^{(M)}$ define a cylinder set
\[
Z_{\vec{d}}^{(M)} = \left\{ (\vec{k})_{i \geq 1} \in \mathcal{O} \text{ such that } \vec{k}_M \equiv \vec{d} \pmod{D^{(M)}} \right\}.
\]
The frequency measure $\nu$ is defined on all such $M$ and $d$ to be
\[
\nu(Z_{\vec{d}}^{(M)}) = \frac{1}{|D|},
\]
which extends to Borel subsets of $\mathcal{O}$ since cylinder sets form a basis of the product topology.

**Definition 3.4.** The digit odometer associated to a digit substitution with expansion matrix $Q$ is the odometer $\mathcal{O} = \lim_{\leftarrow i} D^{(i)}$ endowed with the product topology and digit frequency measure $\nu$.

For the next result, we need the notion of a digit being at the boundary. Let $P = \bigcup_{\vec{d} \in D} (u + \vec{d})$ be the patch in $\mathbb{R}^m$ defined by the digit set $D$. We say that $\vec{d} \in D$ is a **boundary digit** if there is no $\varepsilon > 0$ such that the ball of radius $\varepsilon$ lies entirely in $P$ and contains the unit cube corresponding to $\vec{d}$. As an example, all digits in the digit set in Example 2 are boundary digits. Digits which are not boundary digits are called **interior digits**. We denote by $B(D)$ the set of all boundary digits of $D$.

We call a digit set $D$ **thick** if for any $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that the patch $P^{(n)}$ corresponding to $D^{(n)}$ contains a ball of radius $\varepsilon$ in its interior. Thickness implies that larger supertiles admit larger connected regions. This condition is related to the equivalent conditions in Proposition 1, and can easily be confirmed for examples.

**Proposition 3.** The subshift of an aperiodic digit substitution factors measure-theoretically onto its digit odometer.

**Proof.** Since the subshift exists, we can assume that the digit set is thick. In particular, we can choose an $n \in \mathbb{N}$ such that $D^{(n)}$ admits at least one interior digit and consider $\mathcal{S} := \mathcal{S}^n$. Aperiodicity implies that we have the unique composition property \[45\], which means that for every $T \in \Sigma$ and every $M \in \mathbb{N}$, there exist a unique $T^{(M)} \in \Sigma$ and $\vec{j}_M \in D^{(M)}$ such that
\[
T = S^M (T^{(M)}) - \vec{j}_M.
\]

We write $j := (\vec{j}_M)_{M \geq 1}$. Let $\mathcal{O}'$ be the set of all elements of $\mathcal{O}$ whose expansion $(\vec{j}_M)_{M \geq 1}$ does not have a tail consisting only of boundary digits. Note that this is a full measure subset of $\mathcal{O}$. Let $\Sigma'$ be the elements of the subshift, whose sequence of translation vectors in Eq. (6) does not have a tail consisting only of boundary digits. We now define the coding $\Theta$ of $\Sigma$ onto the odometer as
\[
\Theta : \Sigma \to \mathcal{O}', \quad \Theta(T) = j,
\]
j being the unique sequence of translation vectors in Eq. (6), where every $i$th component is seen as an element of $\mathbb{Z}^m / Q^i \mathbb{Z}^m$. This map is surjective since every sequence in $\mathcal{O}$ gives rise to an element $T$ in $\Sigma$. Now let $\vec{v} \in \mathbb{Z}^m$ be fixed. Note that the sequences which map onto the exceptional set $\mathcal{O} \setminus \mathcal{O}'$ are those which are unions of infinite level supertiles. We show that $\vec{v} \circ \Theta = \Theta \circ \vec{v}$ in $\Sigma'$. 
Fix $T \in \Sigma'$ whose odometer coding is $j := (\tilde{j}_M)_{M \geq 1}$. For a fixed $M$, there exists $T^{(M)} \in \Sigma$ such that $T - \bar{\bar{v}} = S^M(T^{(M)}) - (\tilde{j}_M + \bar{\bar{v}})$ mod $Q^MZ^m$, where $T^{(M)}$ is an appropriate translate of $T^{(M)}$ in Eq. (7). We then get $(\Theta(T - \bar{\bar{v}}))_M = (\tilde{j}_M + \bar{\bar{v}})$ mod $Q^MZ^m = (\bar{\bar{v}} \circ \Theta(T))_M$, since the projection map commutes with addition for every $M$.

**Remark 2.** The exceptional set $\Sigma \setminus \Sigma'$ in the previous proposition can also be viewed as the set of sequences $T$ such that there exists $\bar{\bar{v}} \in Z^m$ for which there exists no $M \in \mathbb{N}$ such that $T(0)$ and $T(\bar{\bar{v}})$ belong to the same level-$M$ supertile. We work again with the full-measure subset $\Sigma'$ in the proof of absolutely continuity in Theorem 4.5(b) below and in the skew product representation in Theorem 4.2.

### 3.4. Spin substitutions

Spin substitutions are digit substitutions that are completely defined by three things: a digit system $(Q, D)$, a finite abelian group $G$ called the **spin group**, and a spin matrix $W : D \times D \rightarrow G$.

**Notation.**

1. The alphabet is defined to be $A = G \times D$, where a letter $a$ is written as a formal product $a = g\bar{\bar{d}}$.
2. When $d = e\bar{\bar{d}}$, where $e$ is the identity of $G$, we call it **spin-free**. The subalphabet $\{e\} \times D$ is the set of all spin-free letters.
3. From here onwards, we follow the following convention:
   - $d$ is a (spin-free) element of the alphabet, i.e., $d \in \{e\} \times D$
   - $\bar{\bar{d}}$ specifies a spatial location, i.e., $\bar{\bar{d}} \in \mathbb{Z}^m$
4. By abuse of notation, we also denote the subalphabet $\{e\} \times D$ by $D$. It will be clear from the context whether $D$ is used as a spatial object (i.e., a subset of $\mathbb{Z}^m$), or as a subalphabet.

Note that $G$ acts on $A$ via $g(\bar{\bar{d}}) = (gg)d$. The canonical projections $\pi_G : A \rightarrow G$ and $\pi_D : A \rightarrow D$ on $A$ are called the **spin map** and **digit map**, respectively. These projections can be extended to maps on the subshift. In this case, the spin map commutes with multiplication in $G$ but forgets its location, while the digit map forgets the spin. Letters in the alphabet therefore keep track of both spatial and algebraic information.

**Definition 3.5.** The **spin substitution** $S : A \rightarrow A^D$ given by $(Q, D, G, W)$ is defined for spin-free letters $d = e\bar{\bar{d}}$ in $A$ to be

$$S\bar{\bar{g}}(d) = W(\bar{\bar{d}}, \bar{\bar{d}}')d', \text{ for all positions } \bar{\bar{d}}' \in D.$$  

Here, $d'$ is the spin-free letter associated to $\bar{\bar{d}}'$. This extends to a rule for the rest of $A$ via

$$S\bar{\bar{g}}(gd) = gW(\bar{\bar{d}}, \bar{\bar{d}}')d' = gS\bar{\bar{g}}(d).$$

Note that $S\bar{\bar{g}}(a)$ is indeed always an element of $A$ since $\pi_G(a)$ and $W(\pi_D(a), \bar{\bar{d}})$ are both in $G$. Moreover, $S\bar{\bar{g}}(a)$ is an element of the equivalence class of letters

$$[d] = \{gd \mid g \in G\} \subset A.$$

Put another way, no matter which $a \in A$ is substituted, the letter at $\bar{\bar{d}}$ in $S(a)$ will be in $[d]$. In this way, spin substitutions keep track of their underlying digit structure.

To summarize, spin substitutions satisfy two key properties. For each $\bar{\bar{d}} \in D$ and $a \in A$
One obtains the $n$-supertiles $S^n(a)$ via Eq. (3). Note that property (R2) extends to $n$-supertiles, i.e., to get the supertile $S^n(ga)$, one only needs to multiply the spin of each letter of $S^n(a)$ by $g$. This form of compatibility with the spin group $G$ is crucial in our spectral characterization of these systems; see Sections 4.2 and 4.3 below.

Example 3 (Rudin–Shapiro as spin substitution). The one-dimensional Rudin–Shapiro substitution is the spin substitution given by $(Q, D, G, W)$, where $Q = 2$, $D = \{0, 1\}$, $G = C_2 = \{1, -1\}$, and $W = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. More explicitly, we have

$$S_{RS}: \begin{cases} 0 \mapsto 01, & 0 \mapsto \bar{0}1, \\ 1 \mapsto \bar{0}1, & 1 \mapsto 01. \end{cases}$$

Here, the barred letters correspond to letters with spin $-1$. This is a mere recoding of the Rudin–Shapiro substitution given by $\varrho_{RS}: a \mapsto ab, b \mapsto ad, c \mapsto cd, d \mapsto cb$ via the identification $a\hat{=}0, b\hat{=}1, c\hat{=}\bar{0}$ and $d\hat{=}\bar{1}$.

Example 4 (Triomino substitution). This is one of the simplest examples that generalizes Rudin–Shapiro type substitutions to three spins and digits while having reasonable geometry. The digit system is given by $Q = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ with digit set $D = \{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$, with $\vec{d}_1 = (0, 0), \vec{d}_2 = (1, 0)$, and $\vec{d}_3 = (0, 1)$.

Choosing $G = C_3 = \{1, \omega, \omega^2\}$ with $\omega = \exp(2\pi i/3)$, one gets the nine-letter alphabet $A = \{a = \omega^j \vec{d}_i, 0 \leq j \leq 2, 1 \leq i \leq 3\}$.

The letters in $A$ are depicted in Figure 5 as colored unit squares whose position will be given by the lower left corner, which is assumed to be in $\mathbb{Z}^2$. The spins are associated with colors, with spin-free letters appearing in shades of red in the leftmost column. The spin $\omega$ is represented in shades of green, and $\omega^2$ in blue.

![Figure 5. The alphabet. Spins are depicted as colors that vary in shade by digit.](image-url)
The map $W$ is taken to be $W: (\vec{d}_i, \vec{d}_j) \mapsto \omega^{(i-1)(j-1)}$, the (Vandermonde) matrix

$$W = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. $$

The substitution for the spin-free letters are given by the rows of $W$:

$$S_{\text{tri}}(d_1) = \{d_1, d_2, d_3\}$$

$$S_{\text{tri}}(d_2) = \{d_1, \omega d_2, \omega^2 d_3\},$$

$$S_{\text{tri}}(d_3) = \{d_1, \omega^2 d_2, \omega d_3\},$$

from which the rest of the substitution is defined via $(R2)$.

The 1- and 2-supertiles are shown in Figure 6, arranged in the same way as in Figure 5. The leftmost column in each grid represents the supertiles for the spin-free digits. This column comes directly from the corresponding equations (8). The remaining columns come from multiplication by $\omega$ and $\omega^2$.

Figure 7 shows $S_{10}^{(d_1)}$, where $d_1$ is the tile in the upper left corner of Figure 5.

**Remark 3.** No successful attempt was made to have the spin colors reflect multiplication in $G$, but still there is a reason to associate the colors with spins rather than digits. The spin map $\pi_G$ produces a natural factor map that forgets the digits. In this case, a single shade of each color represents tilings in $\pi_G(\Sigma)$. If instead we choose the transposed color scheme wherein the digits are associated with colors and the shades depend on the spin, this factor map yields periodic tilings regardless of choice of $W$.

**Remark 4** (Spin substitutions vs bijective substitutions). We briefly mention that there are other substitutive systems called bijective substitutions which have additional group structure. In fact, like primitive spin substitutions, some of them are also skew products over the corresponding odometer. We note, however, that this does not tell anything about the spectral type. As an example, both the Thue–Morse substitution and the Rudin–Shapiro substitution are $\mathbb{C}_2$-extensions of the dyadic odometer $\mathbb{Z}_2$, but Thue–Morse has purely singular dynamical spectrum whereas Rudin–Shapiro admits an absolutely continuous component; see Example 11.

4. Dynamical spectrum of spin substitutions.

4.1. Spectral theory on $L^2(\Sigma, \mu)$. We briefly recall some notions in the spectral theory of dynamical systems. Let $(\Sigma, \mathbb{Z}^m, \mu)$ be a measure-theoretic dynamical system with $\mathbb{Z}^m$-invariant measure $\mu$. For $\vec{j} \in \mathbb{Z}^m$, we have the unitary operator $U_{\vec{j}}: L^2(\Sigma, \mu) \to L^2(\Sigma, \mu)$ given by $(U_{\vec{j}} f)(T) = f(T - \vec{j})$. The **spectral or Fourier coefficients** of $f \in L^2(\Sigma, \mu)$ are defined for each $\vec{j} \in \mathbb{Z}^m$ to be

$$\hat{f}(\vec{j}) = \langle U_{\vec{j}} f | f \rangle = \int_{\Sigma} f(T - \vec{j})\overline{f(T)} \, d\mu(T).$$

Since the function $\hat{f}: \vec{j} \mapsto \langle U_{\vec{j}} f | f \rangle$ is positive definite, by the Herglotz–Bochner theorem, there exists a unique positive measure $\sigma_f \in \mathcal{M}^+(\mathbb{T}^d)$ with

$$\hat{f}(\vec{j}) = \int_{\mathbb{T}^d} z^\vec{j} \, d\sigma_f(z).$$

The measure $\sigma_f$ is called the **spectral measure** of $f$. For $f \in L^2(\Sigma, \mu)$, its associated cyclic subspace $Z(f)$ is defined as $Z(f) = \text{span}\{(U_{\vec{j}} f) | \vec{j} \in \mathbb{Z}^m\}$, which
extends to $\Z(\langle f_1, \ldots, f_r \rangle), \{ f_1, \ldots, f_r \}$ being a finite set. The positive measure $\sigma_f$ admits a generalized Lebesgue decomposition

$$\sigma_f = (\sigma_f)_{pp} + (\sigma_f)_{ac} + (\sigma_f)_{sc}.$$ 

into its pure point, absolutely continuous, and singularly continuous components, not all of which are trivial. An eigenvalue of the $\Z^m$-action of a measure-theoretic dynamical system $(\Sigma, \Z^m, \mu)$ is an $\alpha \in \T^m$ for which there exists $f \in L^2(\Sigma, \mu)$ with $f \neq 0$ that satisfies $U^\jmath f = e^{2\pi i(\langle \alpha | \jmath \rangle)}f$ for all $\jmath \in \Z^m$. An eigenvalue is called continuous if it has a continuous eigenfunction.

4.2. Spin group representation. Spin substitutions have an automorphism on $\Sigma$ given by $(gT)(\jmath) = g(T(\jmath))$ that commutes with the shift by (R2). This induces
an action on $L^2(\Sigma, \mu)$ given by

$$(U_g f)(\mathcal{T}) = f(g \mathcal{T}) \text{ for all } \mathcal{T} \in \Sigma \text{ and } g \in G. \quad (9)$$

Let $\mathcal{G}$ be the dual group of $G$, i.e. the group of continuous characters $\chi : G \to S^1$. We say $f \in L^2(\Sigma, \mu)$ is an eigenfunction for $U_g$ if there exists $\chi \in \mathcal{G}$ such that

$$(U_g f)(\mathcal{T}) = \chi(g) f(\mathcal{T}) \text{ for all } \mathcal{T} \in \Sigma \text{ and } g \in G. \quad (10)$$

This helps provide spectral information through its own eigenfunctions, as we shall see below. Every function $f \in L^2(\Sigma, \mu)$ admits the spectral decomposition

$$f(\mathcal{T}) = \frac{1}{|G|} \sum_{\chi \in \mathcal{G}} \sum_{g \in G} \chi(g) f(g \mathcal{T}) = \sum_{\chi \in \mathcal{G}} f^\chi(\mathcal{T}), \quad (11)$$

where $f^\chi$ is an eigenfunction of $U_g$ with eigenvalue $\chi(g)$.

For $gd \in \mathcal{A}$, define $1_{gd} \in L^2(\Sigma, \mu)$ via $1_{gd}(\mathcal{T}) = 1$ whenever $\mathcal{T}(0) = gd$ and zero otherwise. Recalling that $[d]$ is the equivalence class of elements of $\mathcal{A}$ that share the same digit as $d$, we define the indicator function of $[d]$ as

$$1_{[d]} = \sum_{g \in G} 1_{gd}.$$  

Given a character $\chi \in \mathcal{G}$, define

$$f_\chi^d = \chi(\pi_G(\mathcal{T})) 1_{[d]}.$$
We will see that the spectral analysis on $L^2(\Sigma, \mu)$ can be narrowed down to these functions.

**Example 5.** Let $S$ be the Triomino substitution of Example 4. For a fixed spin-free letter $d \in A$, we have
\[
\begin{align*}
    f_d^x &= 1_d + 1_{\omega_d} + 1_{\omega^2_d} \\
    f_d^y &= 1_d + \omega 1_{\omega_d} + \omega^2 1_{\omega^2_d} \\
    f_d^z &= 1_d + \omega^2 1_{\omega_d} + \omega 1_{\omega^2_d},
\end{align*}
\]
where $\chi_i \in \mathbb{C}_3$.

4.3. **Skew product representation.** We show in this section that a spin digit substitution is measure-theoretically isomorphic to a group extension of the odometer $O$ if it is aperiodic and primitive. First, we recall the definition of a skew product and construct one which fits the setting of spin substitutions.

**Definition 4.1.** Let $(X, G, \nu)$ be a measure-theoretic dynamical system with a group action induced by $G$. Let $G$ be a compact abelian group equipped with Haar measure $\mu_G$. Consider the product space $(X \times G, \nu \times \mu_G)$ and let $(x, g) \in X \times G$.

From a measurable map $\phi : G \times X \to G$ and $v \in G$, one can build $\Psi_v : X \times G \to X \times G$ via
\[
\Psi_v(x, g) = (v(x), \phi(v, x)g).
\]
This map $\Psi$ is called a **skew product** and the action $\phi$ on the second coordinate is called a **cocycle**. The tuple $(X \times G, G, \nu \times \mu_G)$ is called a **compact abelian group extension**, where the action of $G$ on $X \times G$ is induced by $\Psi$. This is a special type of a skew product dynamical system.

Let $S = (Q, D, G, W)$ be an aperiodic primitive spin digit substitution in $\mathbb{Z}^m$ with subshift $\Sigma$ and underlying odometer $O$. Write $j = \sum_{n=0}^\infty Q^n \cdot (j^{(n)})$ and $\bar{j} + \bar{v} = \sum_{n=0}^\infty Q^n \cdot (j + v)^{(n)}$. Construct the map $\phi : \mathbb{Z}^m \times O \to G$ with
\[
\phi(\bar{v}, j) = \left( \prod_{\ell=0}^{M-1} W((j^{(\ell)} + v)^{(\ell)}), (j + v)^{(\ell)} \right) \bigg| \prod_{\ell=0}^{M-1} W((j^{(\ell)} + v)^{(\ell)}), (j + v)^{(\ell)} \right) \bigg) (12)
\]
where
\[
M := \max \left\{ n \in \mathbb{N} : j^{(n)} = (j + v)^{(n)} \text{ and } j^{(n-1)} \neq (j + v)^{(n-1)} \right\}. (13)
\]
This map defines the cocycle $\Psi_{\bar{v}}$ via
\[
\Psi_{\bar{v}}(j, g) = (j + \bar{v}, \phi(\bar{v}, j)g). (14)
\]
One can check that this satisfies the cocycle property $\Psi_{\bar{v} + \bar{w}} = \Psi_{\bar{v}} \circ \Psi_{\bar{w}}$.

**Theorem 4.2.** The dynamical system $(\Sigma, \mathbb{Z}^m, \mu)$ is measure-theoretically isomorphic to the group extension $(O \times G, \mathbb{Z}^m, \nu \times \mu_G)$, where the action of $\mathbb{Z}^m$ is induced by $\Psi$ whose cocycle action $\phi$ on the second coordinate is defined in Eq. (14).

The proof proceeds in a similar manner as the proof for certain bijective substitutions, which are also skew products over odometers; see also [1] for a similar construction for chained sequences. We again stress that even though both spin and bijective substitutions can be recovered as group extensions, they can be spectrally different.
Lemma 4.3. For any letter $S$, lemma, which describes the cocycle structure for the spins of level-$M$ super tiles $\Sigma$.

Proof. We do this by induction. For $M = 1$, this is directly satisfied from the definition of $W$. We now assume it is true for $M$ and prove that it then holds for $M + 1$. Let $\vec{i} \in D^{(M)}$ and $\vec{i} = (i^{(M)}, \ldots, i^{(1)}, i^{(0)})$ be the $(Q, D)$-expansion of $\vec{i}$. One has $S^{M+1}(\vec{d}) = S^M(S(\vec{d}))$. Since $S$ commutes with $\pi_G$, one then has

$$
\pi_G(S^{M+1}(\vec{d})) = \pi_G(S^M(S(\vec{d}))) = W(\vec{d}, i^{(M)}) \pi_G(S^M(i^{(M)})),
$$

with $\vec{i} = (i^{(M-1)}, \ldots, i^{(0)})$. The claim then follows by induction. \hfill \square

Proof of Theorem 4.2. Let $T \in \Sigma$ with unique odometer coding $\vec{j}$ and consider a translation vector $\vec{v} \in \mathbb{Z}^m$. Again, $\Sigma'$ consists of all sequences whose coding does not have a tail of boundary digits. The odometer coding of $T - \vec{v}$ is given by $\vec{j} + \vec{v}$, which accounts for the first coordinate of $\Psi(T - \vec{v})$. By Property (R2), we know that $\psi_{j} \gamma \in \mathcal{A}$. For $M \in \mathbb{N}$, $\vec{j} \in D^{(M)}$ and a spin-free letter $\vec{d}$, $\pi_G(S_j^M(\vec{d}))$ is given by Eq. (15) from Lemma 4.3.

We know that $g = \pi_G(T(0)) = \pi_G(S_j^{M-1}(\vec{d}))$ for some $h \in G$. By Property (R2), one has $h = \varphi \cdot \pi_G(S_j^{M-1}(\vec{d}))^{-1}$, where the second term is completely determined by $W$ and the $(Q, D)$-expansion of $\vec{j}^{(M)}$.

The spin-free type $\vec{d}$ of the level-$M$ supertile at the origin is given by $\vec{j}^{(M)}$. With this, we have

$$
\pi_G(T - \vec{v})(0) = \pi_G(S_j^{M-1}(\vec{d}^{(M)})) = h \cdot \pi_G(S_j^{M-1}(\vec{d}^{(M)}))^{-1} \cdot \pi_G(S_j^{M-1}(\vec{d}^{(M)}))
$$

This means that the spin at zero and the odometer coding completely determines $T(\vec{v})$ for all $\vec{v} \in \mathbb{Z}^m$, hence $T$ itself. Combining with Eq. (15), the previous equality yields $\pi_G(T - \vec{v})(0) = \phi(v, j)g$.

The metric isomorphism $\Gamma: \Sigma \to \mathcal{O} \times G$ is then given by

$$
\Gamma(T) = (\Theta(T), \pi_G(T(0))),
$$

which satisfies $\Gamma \circ U_\vec{v} = \Psi_\vec{v} \circ \Gamma$. It remains to show that $\Gamma$ is measure-preserving, i.e., $\mu(\Gamma^{-1}(V)) = (\nu \times \mu_H)(V)$ for $V \subset \mathcal{O} \times G$ measurable. It suffices to show this for the cylinder set $V = Z^M_d \times \{g\}$, where $d \in D^{(M)}$ for some $M$ and $g \in G$. To this end, we have $(\nu \times \mu_H)(V) = \frac{1}{|D|^m} \cdot \frac{1}{|G|}$. The set $\Gamma^{-1}(V) \subset \Sigma$ is the set of all $T$.

For bijective substitutions satisfying some additional assumption, almost every $T \in \Sigma$ is completely determined by its odometer coding and its letter $T(0)$ at the origin; see [23, Sec. 2.3]. For spin substitutions, we show that, for a subset of $\Sigma$ of full $\mu$-measure, every $T$ is completely determined by its coding $j \in \mathcal{O}$ and its spin $\pi_G(T(0))$ at the origin. Before continuing with the proof, we need the following lemma, which describes the cocycle structure for the spins of level-$M$ super tiles $S^M(d)$, where $d$ is a spin-free letter.
whose first \( M \) terms of the odometer coding coincides with that of \( \vec{d} \) and such that the spin of \( T \) at the origin is \( g \).

Let \( \vec{d}_M = (d^{(M-1)}, \ldots, d^{(0)}) \) and with \( \vec{d}_M \in D^{(M)} \). For \( 0 \leq \ell \leq M - 1 \), the cylinder set of all tilings with a level-\( \ell \) supertile of spin-free tile type \( d^{(0)} \) has measure \( \frac{1}{|D|} \). Multiplying all these contributions for all \( 0 \leq \ell \leq M - 1 \) yields that the cylinder set of tilings with the specified supertile structure (up to level \( M \)) has measure \( \frac{1}{|D|} \). Finally, since there are \( |G| \) choices for the spin at the origin, we get \( \mu(\Gamma^{-1}(V)) = \frac{1}{|D|M} \cdot \frac{1}{|G|} \), as desired. \( \Box \)

The isomorphism \( \Gamma \) in Eq. (16) induces a unitary isomorphism of the corresponding function spaces. It then suffices to consider \( L^2(O \times G, \nu \times \mu_H) \). The following result is well known for compact abelian group extensions with discrete group actions \([23, \text{Thm. 4.2}]\), \([43, \text{Thm. 2}]\), \([40]\) and \([20, \text{Sec. 3}]\).

**Proposition 4.** One has \( \tilde{H} = L^2(O \times G, \nu \times \mu_H) = \bigoplus \tilde{H}^x \) with
\[
\tilde{H}^x = \left\{ f \mid f(j, g) = \chi(g)\tilde{f}(j), \tilde{f} \in L^2(O, \nu) \right\} = L^2(O, \nu) \otimes \chi.
\]
Moreover, \( \tilde{H}^x \) is the eigenspace of \( U_g \) to the eigenvalue \( \chi(g) \). Each \( \tilde{H}^x \) is \( \Psi \)-invariant and is generated by functions of the form \( \{ \chi(\cdot) \mathbb{1}_{[d]} \} \), where \( d \in D \).

**Proof.** The splitting follows since \( U_g \) (as a representation) splits into a direct sum of one-dimensional irreducible representations. The invariance of \( \tilde{H}^x \) can be shown by a direct computation. For the last claim, note that the subspace \( H^{x_0} \) where \( \chi_0 \) is the trivial character, is composed of functions which only depend on the odometer component. The span of the indicator functions \( \{ \mathbb{1}_{[d]} \} \) of cylinder sets is dense in \( L^2(O, \nu) \) and hence \( H^{x_0} = L^2(O, \nu) = Z\left( \langle \mathbb{1}_{[d]} \rangle \right) \). The claim for other characters follows immediately since \( \tilde{H}^x = L^2(O, \nu) \otimes \chi \). \( \Box \)

Note that the subrepresentation \( U_{\Psi, \varphi}|_{\tilde{H}^x} \) is unitarily equivalent to the operator \( V_{\vec{d}, \chi} : L^2(O, \nu) \to L^2(O, \nu) \) given by \( V_{\vec{d}, \chi}(\tilde{f})(j) = \chi(\phi_{\vec{d}}(j))\tilde{f}(j + \vec{d}) \). We now have the following spectral purity result, which is a generalization of Helson’s result for cocycles in \( S^1 \). We omit the proof here and refer the reader to \([30, 41]\) for the version of the proof for \( \mathbb{Z} \)-actions; see also \([20]\). The proof involves showing that each of the subspaces \( \tilde{H}_\alpha \) with \( \alpha \in \{ pp, ac, sc \} \) corresponding to functions in \( L^2(O, \nu) \) with pure point, absolutely continuous or singular continuous spectral measures (with respect to \( V_{\vec{d}, \chi} \)) are all invariant under the operators \( R_\rho : f \mapsto f \otimes \rho \), for all \( \rho \in \tilde{O} \), the latter being the Pontryagin dual of \( O \).

**Proposition 5.** The subrepresentation \( U_{\Psi, \varphi}|_{\tilde{H}^x} \) is spectrally pure. Equivalently, for a fixed \( \chi \in \tilde{G} \), all functions in \( \tilde{H}^x \) have the same Lebesgue spectral type, which is either pure point, purely absolutely continuous or purely singular continuous.

4.4. **Statement of results.** We now have our main results regarding the dynamical spectrum of \((\Sigma, \mathbb{Z}^m, \mu)\), to be proved in the sections that follow. Part (b) of Theorem 4.5 is a generalization of \([24, \text{Thm. 4.1}]\) where the spin matrices are real Hadamard matrices and \( G = C_2 \).

**Proposition 6.** Let \((\Sigma, \mathbb{Z}^m, \mu)\) be the measure-theoretic dynamical system induced by the subshift of a primitive, aperiodic spin digit substitution \( S \) arising from \((Q, D)\)
with spin matrix $W$ and spin group $G$. Then, one has
\[ L^2(\Sigma, \mu) = \bigoplus_{\chi \in \hat{G}} H^\chi, \quad \text{where} \quad H^\chi = Z(\langle f^\chi \rangle). \]

This follows as a corollary of Theorem 4.2 and Proposition 4 above. Note that the function $f^\chi$ in Eq. (11) is always in $H^\chi$, for all $f \in L^2(\Sigma, \mu)$. Proposition 6 means we can do a separate spectral analysis on $H^\chi$ for each character $\chi \in \hat{G}$. The following is immediate from Proposition 5.

**Proposition 7.** The subrepresentation $U_{\hat{\sigma}}|_{H^\chi}$ is spectrally pure. Equivalently, for a fixed $\chi \in \hat{G}$, all functions in $H^\chi$ have the same Lebesgue spectral type, i.e., they are all either pure point, purely absolutely continuous or purely singular continuous.

**Remark 5.** Proposition 7 has a very strong implication in the determination of the spectral type of $H^\chi$. Since the type is constant in $H^\chi$, it is enough to find one function $f \in H^\chi$ of known spectral type in order to determine the Lebesgue spectral type of the entire subspace $H^\chi$. This makes the use of diffraction-based techniques possible; see Section 5.

The next theorem provides sufficient conditions under which $H^\chi$ contains pure point, purely absolutely continuous or purely singular continuous spectral measures and bounds on the corresponding spectral multiplicities. We first define the following notions which depend on the spin map $W$. For $\chi \in \hat{G}$, we define $\chi(W) \in \text{Mat}(S^1, |D|)$ via $\chi(W)(\bar{d}, \bar{d}') = \chi(W(d, \bar{d}'))$.

**Definition 4.4.** Let $S$ be a digit substitution with digit set $D$ and spin matrix $W$, and let $\chi \in \hat{G}$. We call $S$

1. $\chi$-**unitary** if $\frac{1}{\sqrt{|D|}} \chi(W)$ is a unitary matrix
2. $\chi$-**rank-1** if $\chi(W)$ is a rank-1 matrix.

**Remark 6.** The condition of being $\chi$-unitary is the generalization of property (H) in [24] and is equivalent to the following: For every $\bar{d}_i \neq \bar{d}_j \in D$, one has
\[ \sum_{d \in D} \chi(\pi_G(S_{\bar{d}_i}(d))) \chi(\pi_G(S_{\bar{d}_j}(d))) = 0. \] (17)

The normalization constant $\sqrt{|D|}$ comes from the fact that the sum in Eq. (17) is equal to $|D|$ when $\bar{d}_i = \bar{d}_j$. For the second condition, we briefly note here that every $\chi \in \hat{G}$ defines a factor substitution $S^{(\chi)}$ on a possibly smaller alphabet; see Proposition 10 below. Being $\chi$-rank-1 implies that the factor substitution $S^{(\chi)}$ is a bijective substitution.

As we shall see next, the two conditions in Definition 4.4 have explicit consequences for the spectral types present in the corresponding subspace $H^\chi$. We emphasize that these spectral results neither depend on the geometry of supertiles, nor on the dimension of the space $\mathbb{Z}^m$ where the elements of $\Sigma$ live.

**Theorem 4.5.** Let $S$ be an aperiodic primitive spin digit substitution in $\mathbb{Z}^m$ with digit set $D$ and spin matrix $W$. Fix $\chi \in \hat{G}$.

(a) If $\chi = \chi_0$ is the trivial character, every function $f \in H^{\chi_0}$ has pure point spectral measure. The corresponding group of eigenvalues is
\[ E_\mathcal{O} = \bigcup_{i \geq 0} (Q^T)^{-i} \mathbb{Z}^m, \]
which are the eigenvalues of the corresponding odometer.

(b) If $S$ is $\chi$-unitary, the associated subspace $H^\chi$ decomposes as

$$H^\chi = \bigoplus_{d \in D} Z(f^\chi_d)$$

and the functions $\{f^\chi_d\}, d \in D$ all have $\frac{1}{|D|}\mu_L$ as their spectral measure, where $\mu_L$ is Lebesgue measure on $\mathbb{T}^m$. In this case, the spectral multiplicity of $H^\chi$ is exactly $|D|$.

(c) If $S$ is $\chi$-rank-1, $\chi$ induces a factor map to a bijective substitution $S^{(\chi)}$ on the alphabet $G/\ker(\chi)$.

(i) Every $f \in H^\chi$ has singular spectral measure (either pure point or purely singular continuous). Moreover, the spectral multiplicity of $H^\chi$ is bounded from above by $|G/\ker(\chi)|$.

(ii) If further $S^{(\chi)}$ is aperiodic, $H^\chi$ has purely singular continuous spectrum that is generated by generalized Riesz products.

We split the proof of this main spectral result in several sections. We prove part (a) in this section. Part (b), which takes advantage of the cocycle structure satisfied by the spin, will be proved in Section 4.5.

For the proof of part (c), which includes the presence of singular continuous components, we will need the analysis via substitutive factors in Section 5. The complete proof of part (c) is then provided in Section 5.1.

Remark 7. The spectral classification result in Theorem 4.5 extends to the case when $G$ is an arbitrary compact Hausdorff abelian group, but there one would need to assume that the digit substitution is recognizable and uniquely ergodic. In this setting, we get that the corresponding subshift is of infinite local complexity [26], and then it is possible that aperiodicity no longer guarantees recognizability. An example in one dimension is given by $D = \{0, 1\}, G = S^1$ and where the spin matrix is

$$W = \begin{pmatrix} 1 & -1 \\ \alpha & \alpha \end{pmatrix},$$

where $\alpha \in S^1$ is irrational. Here, $\hat{G} \simeq \mathbb{Z}$. This example has countably infinite Lebesgue components, countably infinite singular continuous components and a pure point component corresponding to the dyadic odometer. We refer the reader to [38] for details regarding this example.

We now focus on the subspace $H^\chi_0$ associated to the trivial character. By Proposition 4, one can identify this with $L^2(O, \nu)$. Since the odometer is equicontinuous, it has pure point dynamical spectrum with respect to the translation action. The following result gives the structure of the group of eigenvalues.

 Proposition 8 ([17, Prop. 1]). Let $O = \lim_{\tau \to -i} (\mathbb{Z}^m / Z_i, \pi_i)$ be a $\mathbb{Z}^m$-odometer. The set of eigenvalues of $O$ is given by

$$E_O := \bigcup_{i > 0} \{ \alpha \in \mathbb{R}^m \mid \langle \alpha \mid z \rangle \in \mathbb{Z} \text{ for all } z \in Z_i \} \subset \mathbb{Q}^m.$$

Moreover, every eigenvalue of $O$ is continuous.

Proof of Theorem 4.5(a). From Proposition 3, every eigenfunction $\tilde{f} \in L^2(O, \nu)$ gives rise to an eigenfunction of $U \tilde{\varphi}$ in $L^2(S, \mu)$ via $\hat{f} = \tilde{f} \circ \Theta$. Since $L^2(O, \nu)$ has pure point spectrum, it has a complete system of eigenfunctions, and hence so does
We prove that Eq. (18) is satisfied if the level-1 version is. Denote by
\[ f \]
and let \( S \) be an aperiodic primitive spin digit substitution with spin matrix \( W \) and let \( \chi \in \hat{G} \). Suppose \( S \) is \( \chi \)-unitary. Then, \( S^M \) is also \( \chi \)-unitary, i.e.
\[
\sum_{d \in D} \chi(\pi_G(S^M_i(d))) \overline{\chi(\pi_G(S^M_j(d)))} = 0, \tag{18}
\]
for all \( \vec{i}, \vec{j} \in D^M \) whose corresponding \((M - 1)\)st digits in the \((Q, D)\)-adic expansion are distinct.

**Proof.** We prove that Eq. (18) is satisfied if the level-1 version is. Denote by \((i^{(M-1)}, \ldots, i^{(1)}, i^{(0)})\) and \((j^{(M-1)}, \ldots, j^{(1)}, j^{(0)})\) the \((Q, D)\)-adic expansion of \( \vec{i} \) and \( \vec{j} \), respectively, where we assume that \( i^{(M-1)} \neq j^{(M-1)} \). Inserting Eq. (15), and using the multiplicativity of \( \chi \), one gets
\[
\alpha_{\vec{i}} \alpha_{\vec{j}} \sum_{d \in D} \chi(W(d, i^{(M-1)})) \overline{\chi(W(d, j^{(M-1)}))} = 0,
\]
which is a direct consequence of the level-1 version since \( i^{(M-1)}, j^{(M-1)} \in D \) with \( i^{(M-1)} \neq j^{(M-1)} \), thus proving the claim. Here, \( \alpha_{\vec{i}} = \prod_{\ell=1}^{M-1} \chi(W(i^{(\ell)}, i^{(\ell-1)})) \).

**Proof of Theorem 4.5(b).** We first show that the associated spectral measure to each \( f^X_d \) is a constant multiple of Lebesgue measure. It is clear that \( \hat{f}^X_d(0) = \frac{1}{|D|} \).

We need to show that, for all \( \vec{j} \neq 0 \) and \( d = d' \),
\[
\langle U_{\vec{j}} f^X_d | f^X_{d'} \rangle = \int_{\Sigma} f^X_d(T - \vec{j}) \overline{f^X_{d'}(T)} d\mu(T) = 0. \tag{19}
\]

Note that it suffices to consider \( \vec{j} \in Q\mathbb{Z}^m \), since, for any other \( \vec{j} \), the functions \( f^X_d \) and \( U_{\vec{j}} f^X_d \) cannot simultaneously be non-zero due to property (R1). Let \( \vec{j} \in \mathbb{Z}^m \) and \( d \in D \) be fixed. We can restrict to the set of \( T \in \Sigma \) with \( T(0) \in [d'] \), \( T(x) \in [d] \). For each \( M \in \mathbb{N} \), define \( \Sigma^{(d,d')} (M) \) to be the set of all elements \( T \in \Sigma \) which satisfy \( T(0) \in [d'], T(\vec{j}) \in [d] \), and for which \( M \) is the smallest integer such that \( T(0) \) and \( T(\vec{j}) \) are contained in the same level-\( M \) supertile. Set \( \Sigma(M) := \Sigma^{(d,d')} (M) \).

One then has
\[
\int_{\Sigma} \mathbb{1}_{[d]}(T - \vec{j}) \mathbb{1}_{[d]}(T)d\mu(T) = \sum_{M=1}^{\infty} \mu(\Sigma(M)).
\]

Now consider the following subset of \( D^M \):
\[
P(M) = \left\{ \vec{p} \in D^M \mid \vec{p} + \vec{j} \in D^M, \vec{p} \text{ and } \vec{p} + \vec{j} \text{ are not in the same level-} (M-1) \text{ supertile} \right\}.
\]
Assuming that \( S^M_{\vec{p}}(a), S^M_{\vec{p}+\vec{j}}(a) \in [d] \), with \( \vec{p} \in P(M) \) and \( a \in A \), construct the set
\[
\Sigma(a, M, \vec{p}) = \{ T \in \Sigma(M) \mid T(0) = S^M_{\vec{p}}(a) \}.
\]
This is the set of all elements of \( \Sigma \) where the \( M \)-supertile at the origin is of type \( a \) at position \( \vec{p} \). This yields the decomposition \( \Sigma(M) = \bigcup_{\vec{p} \in P(M)} \bigcup_{a \in A} \Sigma(a, M, \vec{p}) \).

Going back to the Fourier coefficients, observe that the functions \( f^M_{\vec{a}} \) are constant over \( \Sigma(a, M, \vec{p}) \), where one has \( f^M_{\vec{a}}(T) = \chi(\pi_G(S^M_{\vec{p}}(a))) \) for any \( T \in \Sigma(a, M, \vec{p}) \). Using this decomposition and \( f^M_{\vec{a}}(\vec{T} - \vec{\jmath}) = \chi(\pi_G((\vec{T} - \vec{\jmath})(0))) \mathbb{1}_{[d]}(\vec{T} - \vec{\jmath}) \), we find
\[
\hat{f}^M_{\vec{a}}(\vec{\jmath}) = \sum_{M=0}^{\infty} \sum_{\vec{p} \in P(M)} \sum_{a \in A} \int_{\Sigma(a, M, \vec{p})} \chi(\pi_G((\vec{T} - \vec{\jmath})(0))) \overline{\chi(\pi_G(\Sigma(0)))} \, d\mu(T)
\]
\[
= \sum_{M=0}^{\infty} \sum_{\vec{p} \in P(M)} \sum_{a \in A} \chi(\pi_G(S^M_{\vec{p}+\vec{j}}(a))) \overline{\chi(\pi_G(S^M_{\vec{p}}(a)))} \mu(\Sigma(a, M, \vec{p}))
\]
\[
= \sum_{M=0}^{\infty} \sum_{\vec{p} \in P(M)} \chi(\pi_G(S^M_{\vec{p}+\vec{j}}(d))) \overline{\chi(\pi_G(S^M_{\vec{p}}(d)))} = 0.
\]
Note that the product \( \chi(\pi_G(S^M_{\vec{p}+\vec{j}}(a))) \overline{\chi(\pi_G(S^M_{\vec{p}}(a)))} \) is constant for all \( a \in [d] \) by property (R2). Moreover, \( \mu(\Sigma(a, M, \vec{p})) \) only depends on \( M \) and \( \vec{p} \). This means that there are \( |G| \) copies of the same summand in the second equality, which implies the third one. There, one has \( C(M, \vec{p}) = |G| \mu(\Sigma(d, M, \vec{p})) \), where \( \mu(\Sigma(d, M, \vec{p})) \) does not depend on \( d \). The last equality then follows from Proposition 9.

We next show that different spin-free letters generate orthogonal cyclic subspaces, i.e., \( Z(f^M_{\vec{a}}) \perp Z(f^{d'}_{\vec{a}'}), \) for \( d \neq d' \) whenever \( S \) is \( \chi \)-unitary. To show this, one has to prove that Eq. (19) holds for \( d \neq d' \). One can then proceed as in the previous proof except that we now consider \( \Sigma^{(d,d')} = \bigcup_{\vec{p} \in P(M)} \bigcup_{a \in A} \Sigma^{(d,d')} \), where \( \Sigma^{(d,d')} = \bigcup_{a \in A} \Sigma^{(d,d')} \). Here, \( S^M_{\vec{p}}(a) \in [d'] \) and \( S^M_{\vec{p}+\vec{j}}(a) \in [d] \), and
\[
\Sigma^{(d,d')} = \{ T \in \Sigma(M) \mid T(0) = S^M_{\vec{p}}(a) \text{ and } (\vec{T} - \vec{\jmath})(0) = S^M_{\vec{p}+\vec{j}}(a) \}.
\]
One can check that \( \chi \)-unitarity, together with this decomposition, lead to the inner product in Eq. (19) being \( \int_{\Sigma} f^M_{\vec{a}}(\vec{T} - \vec{\jmath}) f^{d'}_{\vec{a}'}(\vec{T}) \, d\mu(T) = 0 \), for all \( \vec{\jmath} \in \mathbb{Z}^m \), which proves the claim.

5. Singularity results. Another spectral notion associated to tilings and point sets is the diffraction \( \hat{\gamma} \), which is a positive measure on \( \mathbb{R}^m \). It is well known that there is a connection between the spectral measure of functions in \( L^2(\Sigma, \mu) \) and the admissible diffraction measures; see Appendix A for more details. In Section 5.1, we complete the proof of Theorem 4.5. For cases not covered by Theorem 4.5, it is still possible to prove singularity results via Lyapunov exponents, which we discuss in Section 5.2.

5.1. Substitutive factors and singular continuous spectrum. Here, we describe how non-trivial characters \( \chi \in \hat{G} \) reveal factors of \( (\Sigma, \mathbb{Z}^m) \) which are relevant in determining the spectral type of \( H^\chi \) when \( S \) is not \( \chi \)-unitary.

Proposition 10. Let \( S \) be an aperiodic primitive spin digit substitution in \( \mathbb{Z}^m \), with spin matrix \( W \) and spin group \( G \). For each \( \chi \in \hat{G} \), there exists a substitutive factor...
\(S^{(x)}\) over the alphabet \(A^{(x)} = G/\ker(\chi) \times D\). Moreover, the maximal spectral type of \(H^x\) is absolutely continuous with respect to the maximal spectral type of \(S^{(x)}\).

**Proof.** Let \(gd, g'd \in A\). If \(g\) and \(g'\) belong to the same coset of \(\ker(\chi)\), the corresponding supertiles satisfy
\[
\chi(S(gd)) = \chi(S(g'd))
\]
by Property (R2). This gives rise to an equivalence relation \(\sim\) in \(A\) where one has \(gd \sim g'd \iff g\ker(\chi) = g'\ker(\chi)\). Using this relation, one can now define the factor digit substitution \(S^{(x)}\) via the matrix \(W' = \chi(W)\). Since \(G/\ker(\chi)\) is cyclic [39], the entries of \(W'\) are now roots of unity which generate the cyclic group \(G/\ker(\chi)\).

The factor map is given by \(\pi: \Sigma \to \Sigma^{(x)}\) with \(\pi(T(\hat{j})) = \chi(\pi_G(T(\hat{j}))\pi_D(T(\hat{j})))\) for \(\hat{j} \in \mathbb{Z}^m\). The last claim follows since we have \(\chi(\pi_G(T(0))) = \pi_{G/\ker(\chi)}(T'(0))\), for some \(T'\) in the factor shift generated by \(S^{(x)}\), and hence the spectral measure of \(\sum_{d \in D} f_d^x \in H^x\) is the spectral measure of a function defined over the factor subshift \(\Sigma^{(x)}\) which forgets the digits and retains the spin component in \(G/\ker(\chi)\). This specific spectral measure, by the discussion in Example 12, is recoverable from a diffraction measure of the factor substitution \(S^{(x)}\).

The following is a consequence of Proposition 10 and Proposition 7.

**Corollary 1.** If the factor substitution \(S^{(x)}\) has singular spectrum, then either \(H^x\) is pure point or purely singular continuous.

We note that these factor substitutions exist for all spin digit substitutions. When the matrix \(\chi(W)\) is not of full rank, it is possible that letters in \(A\) with *different* digits get identified as well and hence \(S^{(x)}\) becomes a substitution on an alphabet *smaller* than \(G/\ker(\chi) \times D\). In these cases, they may no longer be spin substitutions but will remain digit substitutions supported on the same digit set.

We now show that, in the extreme case where \(\chi(W)\) is a rank-1, one obtains a much simpler factor substitution. We define a *bijective digit substitution* to be one where the map \(S^d_j: A \to A\) is a bijection for all \(d \in D\).

**Proposition 11.** Let \(S\) be an aperiodic primitive spin digit substitution in \(\mathbb{Z}^m\), with spin matrix \(W\) and spin group \(G\). If \(\chi(W)\) is a rank-1 matrix, the associated factor substitution \(S^{(x)}\) is a bijective digit substitution over the alphabet \(G/\ker(\chi)\).

**Remark 8.** The reader may wish to gain intuition by considering, for instance, the Vierdrachen example in Example 8 with character \(\chi_2 \in C_2 \times C_2\). The proof below makes precise the identifications that one sees in such examples.

**Proof.** Since \(\chi(W)\) is rank-1, there exist vectors \(v\) and \(w \in (S^1)^{|D|}\) which satisfy \(\chi(W) = v \otimes w\). This induces another equivalence relation on the alphabet as follows. Let \(d_i, d_j\) be spin-free letters in \(A\) which correspond to the rows \(W_i\) and \(W_j\) of the spin matrix. Then, one has \(d_i \sim g d_j \iff \chi(W_i) = \chi(g) \chi(W_j)\). This relation induces a partition of the alphabet satisfying the following properties.

1. The set of equivalence classes is indexed by \(G/\ker(\chi)\). More explicitly, we have
\[
A = \bigcup_{g \in G/\ker(\chi)} [gd_0],
\]
where each equivalence class corresponds to a coset of \(\ker(\chi)\).
2. Because \( \chi(W) \) is rank-1, each equivalence class contains at least one element of \( [d] \) for each \( d \in D \).

3. It is possible for two letters with the same spin to be in the same equivalence class (this happens when \( \chi(W_i) = \chi(W_j) \)).

The substitution \( S^{(x)} \) induced by \( \chi \) on the equivalence classes is given by

\[
S^{(x)}: [d_0] \mapsto [g_0d_0][g_1d_1]\cdots[g_{\ell-1}d_{\ell-1}]
\]

where \( \ell = |D| \). From the second statement, we know that each equivalence class \( [g_i d_j] \) contains at least one element whose spin-free projection is \( d_0 \), which means \( [g_d d_j] = [g'_d d_0] \) for some \( g'_d \). Using \( [gd_j] = g[d_j] \), Eq. (20) then simplifies to

\[
S^{(x)}: [d_0] \mapsto g_0[d_0]g'_1[d_0]\cdots g'_{\ell-1}[d_0],
\]

which one can then write as a substitution on the alphabet \( G/\ker(\chi) \) as

\[
S^{(x)}: g \mapsto L_{g_0}(g)L_{g'_1}(g)\cdots L_{g'_{\ell-1}}(g)
\]

for \( g \in G/\ker(\chi) \), where \( L_g \) is left multiplication by \( g \). Our alphabet now is \( G/\ker(\chi) \) and one can easily check that \( S^{(x)} \) is a bijective substitution whose 1-tiles are still supported on \( D \).

\( \square \)

**Proof of Theorem 4.5(c).** Since \( G/\ker(\chi) \) is cyclic [39], the factor substitution \( S^{(x)} \) is abelian. Moreover, \( S^{(x)} \) is primitive, because \( S \) is primitive, and hence all equivalence classes \( [gd_0] \) appear in \( \left( S^{(x)} \right)^n \) for \( n \) large enough. It then follows from several singularity results for bijective abelian substitutions [12, 5] that \( S^{(x)} \) can only give rise to singular diffraction for arbitrary weights on the alphabet, and hence singular spectral measures for functions in \( H^x \). In particular, the measures which generate the maximal spectral type of \( S^{(x)} \) are Riesz products generated by polynomials in \( \mathbb{Z}^m \) of the form \( P(\vec{x}) = \sum_{d \in D} \rho(g'_d)e^{2\pi i (\vec{d} \cdot \vec{x})} \), where \( g'_d \in G/\ker(\chi) \) and \( \rho \in G/\ker(\chi) \). The singularity of such Riesz products is established in [12, Sec. 3.5]. Alternatively, one can conclude singularity by computing bounds for the a.e. value of the Lyapunov exponents and showing that they are strictly less than \( \log \sqrt{|\det(Q)|} \); see [5, Sec. 4] or Theorem 5.3 below. This establishes the second claim.

For the last claim, note that a higher-dimensional generalization of Dekking’s result states that an aperiodic substitution with trivial height lattice has pure discrete spectrum if and only if it has a column which is a coincidence. It follows that an aperiodic bijective substitution on \( \mathbb{Z}^m \) must have a non-trivial continuous component (which in this case must be singular continuous); compare [23, 12].

\( \square \)

**Remark 9.** In the setting of Corollary 1, either \( H^x \) is singular continuous or pure point. Both cases are possible; see Examples 8 and 9.

If \( \chi(W) \) is not of rank 1, one can still analyze the spectral type of \( S^{(x)} \) via several singularity criteria. For certain subclasses, one can apply the following sufficient criterion for singularity due to Berlinkov and Solomyak.

**Theorem 5.1 ([13, Thm. 1.1]).** Let \( \varrho \) be a one-dimensional substitution of constant-length \( L \). Then, if the substitution matrix \( M \) of \( \varrho \) does not have an eigenvalue of modulus \( \sqrt{L} \), the dynamical spectrum of \( \varrho \) is purely singular.
5.2. Analysis via Lyapunov exponents. An important quantity that reveals the scaling behavior and spectral type of diffraction measures of substitution dynamical systems is the Lyapunov exponent of the associated matrix cocycle; see [5, 14]. Below, we present the properties of the matrix cocycle associated with spin digit substitutions.

Let $\mathcal{S}$ be a spin digit substitution with digit set $\mathcal{D}$, spin group $G$ and spin matrix $W$. Let $a_i = gd$ and $a_j = g'd'$, with $g, g' \in G$ and $d, d' \in \mathcal{D}$. One has the decomposition $\mathcal{D} = \bigcup T_{ij}$, where the sets $T_{ij}$ consists of the positions of tiles of type $a_i$ in $\mathcal{S}(a_j)$. Note that every level-1 supertile contains at most one spun version of each spin-free letter, i.e., $\text{card}(T_{ij}) \in \{0, 1\}$, for all $1 \leq i, j \leq |\mathcal{D}| |G|$. One can build the matrix $T = (T_{ij})$, which we call the displacement matrix. The displacement matrix $T$ and the expansive map $Q$ completely determine the hierarchical structure of $\mathcal{S}$, i.e., the structure of the $n$-supertiles. In general, one has

$$S^n(a_j) = \bigcup_{m=1}^{|\mathcal{D}| |G|} \left(QT_{mj}^{(n-1)} + S(a_m)\right),$$

where $T^{(1)} = T$ and $T^{(n)}$ is the displacement matrix for the power $S^n$. Note that the matrix of sets $T^{(n)}$ is related to $D^{(n)}$ via $D^{(n)} = \bigcup_{ij} T^{(n)}_{ij}$.

From this, one can build a matrix-valued function $B: \mathbb{R}^m \to \text{Mat}(|\mathcal{D}| |G|, \mathbb{C})$, which is called the Fourier matrix, via $B(k)_{ij} := \sum_{\xi \in T_{ij}} e^{2\pi i \langle \xi, k \rangle}$. Now, consider the map $\psi: \mathbb{R}^m \to \mathbb{R}^m$ given by $\kappa \mapsto Q^\dagger \kappa$. The matrix-valued function $B$, together with $\psi$, induces a skew product $F: \mathbb{R}^m \times \mathbb{C}^{|\mathcal{D}| |G|} \to \mathbb{R}^m \times \mathbb{C}^{|\mathcal{D}| |G|}$ given by $F(\kappa, w) = (\psi(\kappa), wB(\kappa))$. The iterated action on the second coordinate becomes the matrix cocycle $B^{(n)}(\kappa) := B(\kappa)B(Q^\dagger \kappa) \cdots B((Q^\dagger)^{n-1} \kappa)$ which we call the Fourier cocycle. We are interested in the asymptotic exponential growth rate of the norm of this cocycle for $\kappa \in \mathbb{R}^m$, which is measured by the Lyapunov exponent. We define the (maximal) Lyapunov exponent of this cocycle to be

$$\lambda^{B}(\kappa) := \limsup_{n \to \infty} \frac{1}{n} \log \|B^{(n)}(\kappa)\|$$

for $\kappa \in \mathbb{R}^m$.

**Example 6 (Triomino substitution).** Recall the Triomino substitution from Example 4. Fix an ordering of $C_3$ as $\{1, \omega, \omega^2\}$. From Property (R2), one can directly verify that the displacement matrix is given by $T = \phi(1) \otimes T_1 + \phi(\omega) \otimes T_\omega + \phi(\omega^2) \otimes T_{\omega^2}$ with

$$T_1 = \begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{e_1\} & \emptyset & \emptyset \\ \{e_2\} & \emptyset & \emptyset \end{pmatrix}, \quad T_\omega = \begin{pmatrix} \emptyset & \emptyset & \emptyset \\ \emptyset & \{e_1\} & \emptyset \\ \emptyset & \emptyset & \{e_2\} \end{pmatrix}, \quad T_{\omega^2} = \begin{pmatrix} \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset \end{pmatrix}$$

and $\phi: C_3 \to \text{GL}(3, \mathbb{Z})$ is the representation via permutation matrices. It then follows that, for this substitution, the Fourier matrix is given by

$$B(\kappa) = \phi(1) \otimes Z_1 + \phi(\omega) \otimes Z_\omega + \phi(\omega^2) \otimes Z_{\omega^2}$$

with

$$Z_1 = \begin{pmatrix} 1 & 1 & 1 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}, \quad Z_\omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & y & 0 \end{pmatrix}, \quad Z_{\omega^2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix},$$

where $x = e^{2\pi i (\kappa_{e_1})}$ and $y = e^{2\pi i (\kappa_{e_2})}$ with $\kappa \in \mathbb{R}^2$. 

The following result relates \( \lambda^B(\vec{k}) \) to the presence of absolutely continuous diffraction.

**Theorem 5.2 [5]***. Let \( S \) be a primitive stone inflation in \( \mathbb{R}^m \), with expansive map \( Q \). Suppose \( \det(B(\vec{k})) \neq 0 \). If the diffraction \( \hat{\gamma} \) has a non-trivial absolutely continuous component, the Lyapunov exponent \( \lambda^B(\vec{k}) \) associated to the Fourier cocycle generated by \( B(\vec{k}) \) with base dynamics \( Q^\top \) must satisfy

\[
\lambda^B(\vec{k}) = \log \sqrt{|\det Q|},
\]

for a set of \( \vec{k} \in \mathbb{R}^m \) of positive Lebesgue measure.

We have the following complementary condition which proves the singularity of the diffraction; see [14] for the analogue for spectral measures.

**Theorem 5.3 [5]***. Let \( S \) be a primitive stone inflation in \( \mathbb{R}^m \), with expansive map \( Q \). Suppose \( \det(B(\vec{k})) \neq 0 \). If there exists an \( \varepsilon > 0 \) such that the Lyapunov exponent \( \lambda^B(\vec{k}) \) associated to the Fourier cocycle generated by \( B(\vec{k}) \) with base dynamics \( Q^\top \) satisfies

\[
\lambda^B(\vec{k}) < \log \sqrt{|\det Q|} - \varepsilon,
\]

for a.e. \( \vec{k} \in \mathbb{R}^m \), then, for any choice of complex weights, the diffraction \( \hat{\gamma} \) is purely singular with respect to Lebesgue measure.

**Remark 10.** Digit substitutions are generally not stone inflations because the level-1 supertiles are not merely expanded squares. Nevertheless, since it is topologically conjugate to a stone inflation (given by \( \phi \) in Eq. 5), the pair correlations are well defined as limits of Birkhoff averages and are constant in \( \Sigma \), and hence the arguments involving Lyapunov exponents in Theorems 5.2 and 5.3 follow through for these cases; compare [5, Rem. 5.6].

We now present the general structure of Fourier cocycles associated to spin digit substitutions, and prove that they satisfy a modified version of Theorem 5.2 whenever \( S \) is \( \chi \)-unitary. This is consistent with the presence of absolutely continuous components shown in Theorem 4.5(b).

**Proposition 12.** Let \( S = (Q, D, G, W) \) be an aperiodic primitive spin digit substitution in \( \mathbb{Z}^m \). The Fourier matrix \( B(\vec{k}) \) is unitarily block diagonalizable into

\[
B(\vec{k}) \cong \bigoplus_{\chi \in \hat{G}} B_{\chi}(\vec{k}),
\]

for all \( \vec{k} \in \mathbb{R}^m \). Moreover, if \( S \) is \( \chi \)-unitary, the subblock \( B_{\chi} \) is of the form

\[
\sqrt{|\det(Q)|} U,
\]

where \( U \) is a unitary matrix, which implies \( \lambda^B = \log \sqrt{|\det(Q)|} \) for all \( \vec{k} \in \mathbb{R}^m \).

**Proof.** It follows from the construction and from Property \( (R2) \) of the substitution \( S \) that the Fourier matrix \( B(\vec{k}) \) is of the form

\[
B(\vec{k}) = \sum_{g \in G} \phi(L_g) \otimes Z_g
\]

where \( \phi : S_{|\mathcal{G}|} \to \text{GL}(d, \mathbb{Z}) \) is the permutation representation, and \( \sum Z_g \) is a rank-1 matrix whose rows are of the form \( r_\ell = x_\ell(1, \ldots, 1) \), with \( x_\ell = e^{2\pi i \langle \vec{k}, \vec{d} \rangle} \) for some \( \vec{d} \in D \).
For the proof of the block diagonal structure, one invokes the property of the permutation matrices \(\phi(L_g)\) being simultaneously diagonalisable. The subspaces \(V_j = v_j \otimes \mathbb{C}^{|D|}\) are invariant with respect to \(B(\tilde{k})\), where \(\{v_j\}\) are the (normalised) shared eigenvectors of the matrices \(\{\phi(L_g)\}\). One can then choose the columns of the similarity transformation \(S\) to be \(v_j \otimes e_\ell\), where \(e_\ell\) are the canonical basis vectors of \(\mathbb{C}^{|D|}\). The corresponding blocks will then be of the form \(B_\chi(\tilde{k}) = \sum_{g \in G} \chi(g) Z_g(\tilde{k})\), for \(\chi \in \hat{G}\).

To prove the last claim, note that \(B_\chi(\tilde{k})\) also admits the representation
\[
B_\chi(\tilde{k}) = \begin{pmatrix}
p_1(\tilde{k}) \chi(W)_1 \\
p_2(\tilde{k}) \chi(W)_2 \\
\vdots \\
p_K(\tilde{k}) \chi(W)_K,
\end{pmatrix}
\]

where \(p_\ell(\tilde{k}) = e^{2\pi i \langle \tilde{k}, \tilde{a}_\ell \rangle}\) with \(\tilde{a}_\ell \in D\) and \(\chi(W)_\ell\) is the \(\ell\)th row of the matrix \(\chi(W)\). From the unitarity of \(\frac{1}{\sqrt{|D|}} \chi(W)\) we get
\[
(B_\chi(\tilde{k})B_\chi^\dagger(\tilde{k}))_{ij} = p_i(\tilde{k}) \overline{p_j(\tilde{k})}|D|\delta_{ij}
\]
which yields \(B_\chi(\tilde{k})B_\chi^\dagger(\tilde{k}) = |D||D|^{-1}\) for all \(\tilde{k} \in \mathbb{R}^m\). As a direct consequence, we get that the cocycle induced by this block satisfies \(B_\chi(\tilde{k}) = |D|^{2} U^{(\nu)}(\tilde{k})\), where \(U^{(\nu)}(\tilde{k})\) is a unitary matrix for all \(\tilde{k} \in \mathbb{R}^m\), which further implies that \(B_\chi(\tilde{k})\) has trivial Lyapunov spectrum, i.e., all exponents exist and are equal to \(\lambda = \log \sqrt{|D|}\).

**Example 7.** For the Triomino substitution in Example 4, we get that the Fourier matrix \(B(\tilde{k})\) can be block-diagonalised into the blocks
\[
B_{\chi_0}(\tilde{k}) = \begin{pmatrix} 1 & 1 & 1 \\ x & x & x \\ y & y & y \end{pmatrix}, \quad B_{\chi_1}(\tilde{k}) = \begin{pmatrix} 1 & 1 & 1 \\ x & \omega x & \omega^2 x \\ y & \omega^2 y & \omega y \end{pmatrix}, \quad B_{\chi_2}(\tilde{k}) = \begin{pmatrix} 1 & 1 & 1 \\ x & \omega^2 x & \omega x \\ y & \omega y & \omega^2 y \end{pmatrix},
\]

It can easily be checked that \(S\) is \(\chi_1\)- and \(\chi_2\)-unitary, which by Proposition 12 implies that \(B(\tilde{k})\) has Lyapunov exponent \(\lambda = \log \sqrt{3}\) of multiplicity 6. In this case, we have exactly \(|D| - 1\) blocks with this exponent.

**Remark 11** (Non-invertibility of \(B(\tilde{k})\)). Note that, in Example 7, \(B_{\chi_0}(\tilde{k})\) is a rank-1 matrix, which implies that, for all \(\tilde{k}\), \(B_{\chi_0}(\tilde{k})\) (and consequently \(B(\tilde{k})\)) has zero as an eigenvalue of multiplicity at least \(|D| - 1\). This \(a\ posteriori\) proves that \(\det(B(\tilde{k})) = 0\) for all \(\tilde{k} \in \mathbb{R}^m\) for any spin digit substitution \(S\), so Theorem 5.2 cannot directly be invoked. However, when one restricts to the invariant subspace acted upon by \(B_\chi(\tilde{k})\), one recovers an analogous criterion which is satisfied by all of these examples. This is exactly the same mechanism for Rudin–Shapiro, which belongs to this family of substitution rules; see [37].

As mentioned in the previous remark, Proposition 12 suggests that our tilings could have absolutely continuous diffraction spectrum, which is something we directly confirm as a corollary of our main result regarding the spectral measures in the next section. Note however that Proposition 12 is interesting in its own right because so far, we are not aware of an example of a substitution tiling with expansive map \(Q\) with Lyapunov exponent equal to \(\lambda = \log \sqrt{|\det(Q)|}\) that does not
contain an absolutely continuous spectral component, which suggests that a variant of the criterion in Theorem 5.2 might be sufficient as well.

The growth rate of the norm of the blocks $B_\chi(\hat{k})$ prescribes the scaling behavior of the spectral measures in $H^\chi$. It also determines the properties of diffraction measures where the weight functions are given by the characters in $G$. Together with Theorem A.1, one can look at the Lyapunov exponents of $B_\chi(\hat{k})$ to rule out the presence of absolutely continuous components in $H^\chi$ when $\frac{\lambda}{\sqrt{|D|}}\chi(W)$ is not unitary but is still of full rank.

**Proposition 13.** Let $S$ be an aperiodic primitive spin substitution in $\mathbb{Z}^m$ with spin matrix $W$. If $\chi(W)$ is of full rank, $B_\chi(\hat{k})$ is invertible for a.e. $\hat{k} \in \mathbb{R}^m$. Moreover, the Lyapunov exponent $\lambda^{B^\chi}_W(\hat{k})$ exists as a limit and is constant for a.e. $\hat{k} \in \mathbb{R}^m$.

**Proof.** Almost everywhere invertibility follows from $\chi(W) = B_\chi(0)$ being of full rank, and from the analyticity of $B_\chi(\hat{k})$. Moreover, $B(\hat{k})$ is $\mathbb{Z}^m$-periodic, i.e., $B(\hat{k}) = B(\hat{j} + \hat{k})$ for $\hat{j} \in \mathbb{Z}^m$ and $\hat{k} \in \mathbb{R}^m$. Since the map $\hat{k} \mapsto Q^T\hat{k}$ is ergodic with respect to the Haar measure of $\mathbb{T}^m$, the almost sure existence of the Lyapunov exponent follows from Kingman’s subadditive ergodic theorem; see [11, 46].

**Proposition 14.** Let $S$ be an aperiodic primitive spin substitution in $\mathbb{Z}^m$ with spin matrix $W$ and suppose $\chi(W)$ is of full rank. If the almost sure value of the Lyapunov exponent $\lambda^{B^\chi}_W$ is strictly less than $\log \sqrt{|D|}$, the spectral measure $\sigma_f$ is singular for all $f \in H^\chi$ and is either pure point or purely singular continuous.

**Proof.** Let $T \in \Sigma$ and let $\omega_\chi = \sum_{\hat{j} \in \mathbb{Z}^m} \chi(\pi_G(T - \hat{j}))\delta_{\hat{j}}$ and let $\hat{\gamma} := \hat{\gamma}_\chi$ be the corresponding (fundamental) diffraction measure. From Example 12, we know that $\hat{\gamma}$ is the spectral measure $\sigma_f$ of the function $f = \sum_{d \in D} f_d^\chi$. From the spectral purity result in Proposition 7, it suffices to determine the spectral type of $\sigma_f$ to know the spectral type of $H^\chi$. The idea is then to use the block diagonal structure of the Fourier matrix in Proposition 12 and a slight modification of the proof of Theorem 5.3.

Let $\hat{\gamma}_{ac} = h(\hat{k})\mu_L$ be the absolutely continuous component of the diffraction (now viewed on $\mathbb{R}^m$), where $\mu_L$ is Lebesgue measure in $\mathbb{R}^m$ and $h(\hat{k})$ is the corresponding Radon–Nikodym density. One can decompose $h(\hat{k})$ into

$$h(\hat{k}) = \sum_{d,d' \in D, g,g' \in G} \chi(g)\chi(g')h_{a,a'}(\hat{k})$$

where $a = gd$ and $a' = gd'$. The functions $h_{a,a'}(\hat{k})$ satisfy $h_{a,a'}(-\hat{k}) = h_{a',a}(\hat{k}) = h_{a,a'}(\hat{k})$. This allows one to rewrite $h_{a,a'}(\hat{k})$ as

$$h_{a,a'}(\hat{k}) = \sum_{\ell=1}^s \chi(g)(v_\ell(\hat{k}))_a(v_\ell(\hat{k}))_{a'}^T\chi(g')$$

It then suffices to look at the growth rate of the vectors in the subspace

$$\Xi_\chi = (\chi(g_1), \ldots, \chi(g_{|G|})) \otimes \mathbb{C}^{|D|}$$

under the matrix cocycle $B^{(n)}(\hat{k})$. From Proposition 12, we know that the restriction of $B(\hat{k})$ on this subspace is determined by the action of $B_\chi(\hat{k})$ on the $\mathbb{C}^{|D|}$-component. Now suppose the almost sure value of the Lyapunov exponent $\lambda^{B^\chi}_W < \log \sqrt{|D|}$. This implies that, for all nonzero starting vector $v \in \mathbb{C}^{|D|}$,
\[ \|vB_{\chi}^{(n)}(\vec{k})\| \text{ grows exponentially fast.} \]
This contradicts the translation boundedness of \( h(\vec{k}) \) unless \( h(\vec{k}) = 0 \) for Lebesgue-a.e. \( \vec{k} \); compare [5, Thm. 3.28]. From this, one gets \( \tilde{\gamma}_{ac} = 0 \) and hence the diffraction, and equivalently, \( \sigma_f \) must be singular, which completes the proof.

When \( \chi(W) \) is neither of full rank, nor rank-1, one can still construct the Fourier cocycle for \( S^{(\chi)} \) and compute its Lyapunov exponent. If it satisfies the singularity condition in Theorem 5.3, one can invoke Proposition 10 to conclude that \( H^x \) only admits singular spectral measures.

6. Examples.

6.1. Planar example with all spectral types.

Example 8 (Vierdrachen substitution). Let \( Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \) with the associated digit set \( D = \{(0,0), (1,0)\} = \{d_0, d_1\} \). Choose the group of spins to be the Klein-4 group \( G = C_2 \times C_2 = \{e, a, b, ab\} \) and the spin matrix to be \( W = \begin{pmatrix} e & a \\ e & ab \end{pmatrix} \). We call the spin digit substitution \( S \) with these defining data the Vierdrachen substitution, alluding to the geometry of the twin dragon tiling [47] and the spin group being the Vierergruppe.

This table shows the colors representing the alphabet:

\[
\begin{array}{c|ccc}
(0,0) & a & b & ab \\
(1,0) & e & a & ab \\
\end{array}
\]

Figure 8 shows the 1- and 2-supertiles organized in the same way.

Figure 8. The level -1 and -2 supertiles of the Vierdrachen substitution.

The left of Figure 9 shows the 9-supertile of type \( d_0 \), the middle shows the periodic tiling you get when the spins are identified, and the right shows its image under the factor map taking a letter to its spin. (This “forget-the-digits” map, which is a single-block code on \( \Sigma \), likely represents a topological conjugacy for many spin substitutions. The argument would be adapted from the proof of equivalence of the Rudin–Shapiro substitution space and the original RS sequence space on \( A = \{-1, 1\} \).)
Figure 9. The level-9 supertile for $d_0$ and its image under the forget-the-spins and the forget-the-digits maps.

Let $\hat{G} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$, where the corresponding kernels for $\chi_i$, $0 \leq i \leq 3$ are $G, \langle a \rangle, \langle b \rangle$ and $\langle ab \rangle$, respectively. The corresponding matrices are then

$$
\chi_0(W) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \chi_1(W) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\
\chi_2(W) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \chi_3(W) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
$$

From Theorem 4.5(a), $\chi_0$ corresponds to the pure point component of the spectrum $H^{x_0}$ arising from the odometer. The factor substitution for $\chi_0$ is periodic, and a large supertile is shown in the middle of Figure 9. It is easy to check that $\frac{1}{\sqrt{2}} \chi_1(W)$ and $\frac{1}{\sqrt{2}} \chi_3(W)$ are unitary, which means $H^{x_1}$ and $H^{x_2}$ each decompose into two orthogonal cyclic subspaces, all having Lebesgue spectral measure by Theorem 4.5(b). These two substitutions appear the more disordered in Figure 10. Lastly, the matrix $\chi_2(W)$ is rank-1, which means that the factor substitution is bijective. Thus $H^{x_3}$ comprises of singular continuous components of multiplicity at most 2 by Theorem 4.5(c).

Figure 10. The image of $S^9(d_0)$ under the single-block codes given by the nontrivial characters. From (L) to (R), the corresponding character $\chi$ and the spectral type of $H^{x_1}$: $\chi_1$ (ac), $\chi_2$ (sc) and $\chi_3$ (ac).
This is the planar analog of the one-dimensional substitution in [4], which has the same spin matrix as the Vierdrachen and has $Q = 2$ and $D = \{0, 1\}$.

6.2. Aperiodic example with a non-trivial periodic factor.

Example 9. Let $Q = 3$, $D = \{0, 1, 2\}$ and $G = C_4 = \langle i \rangle$. Consider the spin matrix

$$W = \begin{pmatrix} 1 & i & 1 \\ 1 & -i & -1 \\ 1 & i & -1 \end{pmatrix}.$$ 

One can verify that the corresponding spin substitution $S$ is primitive and aperiodic via the substitution matrix and the existence of non-trivial proximal pairs, respectively. Moreover, $\chi_2(W)$ is a rank-1 matrix, where $\chi_2(g) = g^2$ for $g \in C_4$. From Theorem 4.5, $\chi_2$ produces a factor substitution $S^{(\chi_2)}$. Here, $S^{(\chi_2)}$: $a \mapsto aba, b \mapsto bab$, which is periodic. This means that the set of eigenvalues contains $\mathbb{Z}[\sqrt{3}] \times C_2$. One can also check that the substitution matrix $M$ does not have any eigenvalue of modulus $\sqrt{3}$, and hence the spectrum of $L^2(\Sigma, \mu)$ is purely singular by Theorem 5.1. There are also analogues of this example in higher dimensions.

6.3. Spin digit substitutions from Kronecker products. It is natural to ask whether one can build a new spin digit substitution using two old ones by combining their spin matrices. Given two spin digit substitutions $S_1 = (D_1, W_1)$ and $S_2 = (D_2, W_2)$ with underlying spin groups $G_1$ and $G_2$, respectively, one can define $S := (D, W_1 \otimes W_2)$, where $|D| = |D_1| \cdot |D_2|$ with spin group $G = G_1 \times G_2$ as follows

1. Index the elements of the new digit set $D$ via elements of $D_1 \otimes D_2$, i.e.,
   $$\vec{d} = \vec{d}_1 \otimes \vec{d}_2$$
   and order them using the lexicographic order in $D_1 \otimes D_2$.
2. For $\vec{d}_1, \vec{d}_2 \in D_1$, define $W(\vec{d}_1, \vec{d}_2) := (W_1(\vec{d}_1, \vec{d}_1), W_2(\vec{d}_2, \vec{d}_2)) \in G_1 \times G_2$.

Some comments on the notation are in order. We emphasize that the tensor product $\vec{d}_1 \otimes \vec{d}_2$ does not have any spatial interpretation whatsoever. It is only used as bookkeeping in order to assign the right spin in the supertiles of the new digit set, which is done in the second line above. We also point out that this substitution is not unique. In fact, our only restriction is on the cardinality of $D$, and we have the freedom to choose the expansive map $Q$, the digit set $D$ itself and even the dimension of the space where the substitution is defined. For such an $S$, one has the following properties.

Proposition 15. Let $S_1$ and $S_2$ be defined as above and $G = G_1 \times G_2$ be the underlying spin group. Let $S = (D, W_1 \otimes W_2)$ be a spin digit substitution arising from $S_1$ and $S_2$ via the construction above. Then, the following holds.

1. $S$ is primitive if and only if $S_1$ and $S_2$ are both primitive.
2. Let $\chi = \chi_1 \otimes \chi_2 \in G_1 \times G_2$. Then, $S$ is $\chi$-unitary if and only if $S_1$ and $S_2$ are $\chi_1$- and $\chi_2$-unitary, respectively. Similarly, $S$ is $\chi$-rank-1 if and only if $S_1$ and $S_2$ are rank-1 with respect to $\chi_1$ and $\chi_2$.

Proof. The first claim follows from the fact that the substitution matrix of $S$ inherits the Kronecker product structure, i.e., $M_S = M_{S_1} \otimes M_{S_2}$, hence if one chooses $n = \text{lcm}(n_1, n_2)$, where $n_i$ is the index of primitivity of $M_{S_i}$, one gets that $M^n_S = M^n_{S_1} \otimes M^n_{S_2}$ is a positive matrix. The second claim follows from the fact that Kronecker products of unitary matrices are unitary and that rank is multiplicative under $\otimes$.  \[\square\]
6.4. Singular subspaces via Lyapunov exponents.

Example 10. Let $G = C_4 = \langle i \rangle$ and $D = \{0, 1\}$. Consider the map

$$W = \begin{pmatrix} 1 & i \\ 1 & -1 \end{pmatrix}.$$ 

Let $\chi_n : g \mapsto g^n$ be a character in $\hat{C}_4$. For $n = 0$, we get the odometer factor. For $n = 2$, we get that $\frac{1}{\sqrt{2}} \chi_2(W)$ is unitary, so $H^{\chi_2}$ is of Lebesgue type. For $n = 1, 3$, we do not get smaller substitutive factors since $\ker(\chi_1) = \ker(\chi_3) = \{1\}$, but $\chi_1(W)$ and $\chi_3(W)$ are both of full rank, which means we can use Lyapunov exponents to prove the singularity of the spectrum by invoking Proposition 14. The corresponding (complex-valued) matrix cocycles are

$$B_{\chi_1}(\vec{k}) = \begin{pmatrix} 1 & i \\ e^{2\pi i \vec{k}} & -e^{2\pi i \vec{k}} \end{pmatrix} \quad \text{and} \quad B_{\chi_3}(\vec{k}) = \begin{pmatrix} 1 & -i \\ e^{2\pi i \vec{k}} & -e^{2\pi i \vec{k}} \end{pmatrix}.$$ 

A sequence of almost sure upper bounds of the Lyapunov exponent is given by

$$f(N) = \frac{1}{2N} \int_0^1 \log \| B^{(N)}_{\chi}(\vec{x}) \|^2 d\vec{x} \quad (23)$$

which can be computed numerically for each $N \in \mathbb{N}$. The expansion factor for this example is $|\det(Q)| = |D| = 2$. If, for some $N$, this upper bound in Eq. (23) is strictly less than $\log \sqrt{2}$, one gets that $\lambda^{B_{\chi}}(\vec{k}) < \log \sqrt{2}$ for a.e. $\vec{k} \in \mathbb{R}$; compare [3, 5].

| $N$ | 10 | 11 | 12 | 13 |
|-----|----|----|----|----|
| $\frac{1}{N} \int_T \log \| B_{\chi_1}^{(N)}(\vec{x}) \|^2_F$ | 0.703953 | 0.695342 | 0.688005 | 0.682035 |

Table 1. Numerical values for upper bounds for $2f(N)$ for $B_{\chi_1}$. Here $\| \cdot \|_F$ stands for the Frobenius norm. All numerical errors are less than $10^{-3}$.

Since $\log(2) \approx 0.693147$, we get that $\lambda^{B_{\chi}}(\vec{k}) \leq f(12) < \log \sqrt{2}$ for a.e. $\vec{k} \in \mathbb{R}$, which implies $H^{\chi_1}$ is singular. One can carry out the same computation for $\chi_3$ which leads to the same result.

7. Summary and outlook. We can summarize the spectral analysis we have for spin digit substitutions as follows. Proposition 6 tells us that $L^2(X, \mu)$ splits into a direct sum of orthogonal subspaces $H^\chi$ each corresponding to a $\chi \in \hat{G}$. To analyze the spectral properties of $H^\chi$ we ask:

1. Is $\chi(W)$ of full rank?
   (a) Is $\frac{1}{|D|} \chi(W)$ unitary? If yes, $H^\chi$ has $|D|$ orthogonal absolutely continuous components. (Theorem 4.5(b))
   (b) If $\frac{1}{|D|} \chi(W)$ is of full rank but not unitary, use Lyapunov exponents to confirm singularity (Proposition 13)

2. Is $\chi(W)$ of rank 1?
   (a) If yes, then $H^\chi$ is singular with at most $|G/\ker(\chi)|$ orthogonal components. (Theorem 4.5(c))
(b) If $S^{(x)}$ is aperiodic, then $H^x$ must be purely singular continuous. Otherwise, it is pure point.

3. Is $\chi(W)$ of rank strictly between 1 and $|D|$?
   (a) Check whether the factor substitution $S^{(x)}$ has singular spectrum (e.g. via Lyapunov exponents). If it does, then $H^x$ must be singular. (Proposition 10)

**Question 7.1.** Let $S$ be a primitive spin digit substitution arising from a digit set $\mathcal{D}$ and group $G$. Is it true that the following are equivalent?

1. There are exactly $n$ characters $\chi \in \hat{G}$ for which $S$ is $\chi$-unitary.
2. The substitution matrix $M$ has exactly $n|D|$ eigenvalues of modulus $\sqrt{|D|}$.
3. The Fourier matrix $B(\mathbf{k})$ has Lyapunov exponent $\log \sqrt{|D|}$ of multiplicity $n|D|$.
4. $L^2(\Sigma, \mu)$ has a Lebesgue component of multiplicity $n|D|$.

Equivalently, is there any other mechanism for $H^x$ to be absolutely continuous apart from $\chi$-unitarity?

**Question 7.2.** All of the examples we considered have even Lebesgue multiplicity. Is there a primitive spin digit substitution $S$ whose Lebesgue multiplicity is odd? This requires $|D|$ to be odd if the equivalences in Question 7.1 hold.

Note that our spectral characterization does not impose further conditions on the digit set, aside from the fact that it gives rise to an aperiodic subshift. The means that two spin substitutions $S = (Q, \mathcal{D}, G, W)$ and $S' = (Q, \mathcal{D}', G, W)$ which have the same data except for the digit set are measure-theoretically isomorphic from Theorem 4.2. This implies that certain spin substitutions give rise to families of strictly ergodic dynamical systems with mixed spectrum which satisfy a Halmos–von Neumann type result, i.e., they are measure-theoretically isomorphic and they have the same maximal spectral type; see [29]. What is not clear is whether changing the digit set $\mathcal{D}$ while keeping the same spin matrix $W$ can result to moving from an aperiodic subshift to a periodic one.

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**Appendix A. Spectral measures via diffraction measures.** In this appendix, we expound on the connection between the dynamical spectrum and the diffraction spectrum. For systems with pure point spectra, both notions are equivalent, with very little assumptions on ergodicity and local complexity [8]. For systems with mixed spectra, it is more complicated, and one usually needs a (possibly infinite) family of diffraction measures to characterize the entire dynamical spectrum [9].

Given an element $T$ of the suspension $\Omega_u$, one can derive from it a colored point set $A_T$ by choosing control points for each tile. We can pick the control point to be a vertex or the center of the cube such that the underlying point set becomes $\mathbb{Z}^m$ and $A_T = \bigcup_{g \in G} \bigcup_{d \in \mathcal{D}} A_{gd}$. From $A_T$, one can then build a weighted Dirac
comb \( \omega \) by considering a complex-valued function \( w: A_T \to \mathbb{C} \), which yields 
\[
\omega = \sum_{\vec{x} \in \mathbb{Z}^m} w(\vec{x}) \delta_\vec{x}.
\]

In our situation, the autocorrelation of \( \omega \) is known to be 
\[
\gamma = \sum_{\vec{j} \in \mathbb{Z}^m} \eta(\vec{j}) \delta_{\vec{j}}, \quad \text{where} \quad \eta(\vec{j}) = \lim_{N \to \infty} \frac{1}{(2N+1)^d} \sum_{|\vec{x}| \leq N} w(\vec{x}) \overline{w(\vec{x} - \vec{j})}.
\]

The existence of the autocorrelation coefficients \( \eta(\vec{j}) \) is guaranteed due to unique ergodicity. The Fourier transform \( \widehat{\gamma} \) is the diffraction measure. Like spectral measures, the diffraction admits a decomposition into pure point, absolutely continuous, and singular continuous components. We refer the reader to [6] for a thorough introduction on diffraction theory.

Recently, Lenz has shown in [36] that all spectral measures of a dynamical system with an action of a locally compact, \( \sigma \)-compact abelian group \( G \) are recoverable as diffraction measures. Recall from above that the functions \( f_\vec{x}^\delta \) generate \( L^2(\Sigma, \mu) \). We show below that there is an explicit weighted Dirac comb whose diffraction is exactly the spectral measure \( \sigma_f^\hat{\gamma} \). To this end, we need the following notions; see [36].

Let \((\Sigma, \mathbb{Z}^m, \mu)\) be an ergodic dynamical system and \( U \) be the corresponding Koopman operator for the \( \mathbb{Z}^m \)-action. Let \( f \in L^2(\Sigma, \mu) \). Consider the map \( \mathcal{N}: C_0(\mathbb{Z}^m) \to L^2(\mathbb{Z}, \mu) \) given by \( \mathcal{N} f(\phi) := \sum_{\vec{x} \in \mathbb{Z}^m} \phi(\vec{x}) U f, \phi \in C_c(\mathbb{Z}^m) \), which is linear and \( U \)-equivariant. To \( \mathcal{N} f \), one can canonically associate a diffraction measure \( \widehat{\gamma}_f \), where the autocorrelation \( \gamma_f \) is of the form 
\[
\gamma_f = \sum_{\vec{j} \in \mathbb{Z}^m} \eta(\vec{j}) \delta_{\vec{j}},
\]
with 
\[
\eta(\vec{j}) = \eta_f(\vec{j}) := \left\langle U f \mathcal{N} f(1_\mathbb{Z} \mathcal{N} f(1_\mathbb{Z}) \mid \mathcal{N} f(1_\mathbb{Z}) \right\rangle.
\]

Here, \( \langle \cdot | \cdot \rangle \) is the inner product in \( L^2(\Sigma, \mu) \) and \( 1_\mathbb{Z} \) is the lookup function at the origin. One then has the following result.

**Theorem A.1** ([9, Thm. 4],[36, Thm. 2.1]). Let \( f \in L^2(\Sigma, \mu) \). Then, the spectral measure \( \sigma_f \) of \( f \) is the diffraction measure \( \widehat{\gamma}_f \).

In particular, when \( f \in L^2(\Sigma, \mu) \) is a hyperlocal function, i.e., \( f(T) \) only depends on \( T(0) \), the spectral measure \( \sigma_f \) is the diffraction of the weighted Dirac comb 
\[
\omega = \sum_{\vec{x} \in \mathbb{Z}^m} f(T(\vec{x})) \delta_\vec{x},
\]
for any \( T \in \Sigma \) whenever \((\Sigma, \mathbb{Z}^m)\) is strictly ergodic, which always holds for primitive digit substitutions; see Proposition 2.

**Remark 12.** Strictly speaking, the diffraction of a weighted Dirac comb supported on \( \mathbb{Z}^m \) will be a measure on \( \mathbb{R}^m \) which is \( \mathbb{Z}^m \)-periodic, i.e., \( \widehat{\gamma} = \delta_{\mathbb{Z}^m} * \widehat{\gamma}_{FD} \), where \( \widehat{\gamma}_{FD} \) is a measure on \( \mathbb{T}^m \) and is called the fundamental diffraction in [9]. By abuse of notation, and to be consistent with Theorem A.1, we will simply refer to \( \widehat{\gamma}_{FD} \) as the diffraction.

**Example 11.** Consider the Thue–Morse substitution \( \rho : a \mapsto ab, b \mapsto ba \) and choose \( f \) to be \( f = 1_s(T(0)) \in L^2(\Sigma, \mu) \). It follows from Theorem A.1 that the spectral measure \( \sigma_f \) is the diffraction of the weighted Dirac comb on any element on the Thue–Morse hull with weights \( w_a = 1 \) and \( w_b = 0 \). There is a closed form for the diffraction of such a Dirac comb for Thue–Morse, which is given by 
\[
\widehat{\gamma}_\omega = \left| \frac{w_a + w_b}{2} \right|^2 \delta_0 + \left| \frac{w_a - w_b}{2} \right|^2 \widehat{\gamma}_{TM},
\]
where \( \widehat{\gamma}_{TM} \) is the purely singular continuous Thue–Morse measure; see [6]. Substituting \( w_a = 1 \) and \( w_b = 0 \) yields \( \sigma_f = \frac{1}{4} \delta_0 + \frac{1}{4} \widehat{\gamma}_{TM} \); compare [41].
For spin digit substitutions, we have the following correspondence.

**Example 12.** Let $S$ be a spin substitution with spin group $G$ and digit set $D$. Consider the function $f = \chi_{d} \circ S := \chi(\pi_{G}(T(0))) I_{d}$. Here one has $U_{S} f I_{d} = \chi(\pi_{G}(T - \bar{x})) U_{\bar{x}} I_{d}$. Moreover, the diffraction $\widehat{\gamma}_{f}$ is the diffraction of the weighted Dirac comb $\omega_{f} = \sum_{\bar{x} \in \mathbb{Z}^{m}} \omega(\bar{x}) \delta_{\bar{x}}$, where the weights are given by

$$
\omega(\bar{x}) := \begin{cases} 
\chi(\pi_{G}(T - \bar{x})), & \text{if } \pi_{G}(T - \bar{x}) \in [d], \\
0, & \text{otherwise}.
\end{cases}
$$

From Theorem A.1 and the discussion before it, one has $\sigma_{f_{d}} = \gamma_{f}$.

When $S$ is $\chi$-unitary, we recover the following implication for the diffraction measure $\gamma$ for a specific choice of weight functions.

**Corollary 2.** Let $\Lambda$ be a colored point set arising from a spin digit substitution. Consider the weight functions $w^{\chi} : \Lambda \rightarrow \mathbb{C}$, $w^{\chi}(\bar{x}) := \chi(\pi_{G}(T(\bar{x})))$, and the corresponding weighted Dirac comb $\omega^{\chi} = \sum_{\bar{x} \in \Lambda} w^{\chi}(\bar{x}) \delta_{\bar{x}}$. Then, if $S$ is $\chi$-unitary, the associated diffraction $\gamma$ is Lebesgue measure $\mu_{L}$ in $\mathbb{R}^{m}$.

**Proof.** Note that, from Theorem A.1, the diffraction $\gamma$ of $\omega^{\chi}$ is the spectral measure of the function $F^{\chi} = \sum_{\bar{d} \in \mathcal{D}} f_{d}^{\chi}$. From Theorem 4.5(b), the functions $f_{d}^{\chi}$ define orthogonal cyclic subspaces, and hence one has $\sigma_{F^{\chi}} = \sum_{\bar{d} \in \mathcal{D}} \sigma_{F_{d}^{\chi}}$. The claim follows since each of the summands is $\frac{1}{|\mathcal{D}|} \mu_{L}$. $\square$

Theorem A.1 allows one to derive restrictions on the maximal spectral type of $f_{d}^{\chi}$ by looking at the possible spectral types of $\gamma_{f}$, which is what is done in Section 5.

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