N=2 SUPER - $W_3$ ALGEBRA AND N=2 SUPER BOUSSINESQ EQUATIONS

E.Ivanov\footnote{E-mail address: eivanov@pib1.physik.uni-bonn.de},

Physikalisches Institut, Universitat Bonn
Nussallee 12, D-5300 Bonn 1, Germany

and

Bogoliubov Theoretical Laboratory
JINR – Joint Institute for Nuclear Research
Dubna, Head Post Office, P.O. Box 79
101000 Moscow, Russian Federation

S.Krivonos\footnote{E-mail address: krivonos@ltp.jinr.dubna.su} and R.P.Malik\footnote{E-mail address: malik@theor.jinrc.dubna.su}

Bogoliubov Theoretical Laboratory
JINR – Joint Institute for Nuclear Research
Dubna, Head Post Office, P.O. Box 79
101000 Moscow, Russian Federation

Abstract

We study classical $N = 2$ super-$W_3$ algebra and its interplay with $N = 2$ supersymmetric extensions of the Boussinesq equation in the framework of the nonlinear realization method and the inverse Higgs - covariant reduction approach. These techniques have been previously applied by us in the bosonic $W_3$ case to give a new geometric interpretation of the Boussinesq hierarchy. Here we deduce the most general $N = 2$ super Boussinesq equation and two kinds of the modified $N = 2$ super Boussinesq equations, as well as the super Miura maps relating these systems to each other, by applying the covariant reduction to certain coset manifolds of linear $N = 2$ super-$W_3^\infty$ symmetry associated with $N = 2$ super-$W_3$. We discuss the integrability properties of the equations obtained and their correspondence with the formulation based on the notion of the second hamiltonian structure.

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# 1 Introduction

During the last couple of years, a substantial progress has been achieved in supersymmetrization of $W$ algebras [1-6]. In particular, $N = 2$ classical [5] and quantum [6] super-$W_3$ algebras have been constructed. They have attracted a great deal of interest, mainly in view of their potential applications in $N = 2$ superconformal field theory which is now a subject of intensive studies (see, e.g., [1, 3]). The most characteristic feature of the $N = 2$ super-$W_3$ algebra is that it exists for an arbitrary value of the central charge $c$, in contrast to various minimal $N = 1$ extensions of $W_3$ which can be consistently defined only for specific values of $c$. Thus $N = 2$ super-$W_3$ is actually the first example of a well-defined supersymmetric extension of the nonlinear $W_3$ algebra. The study of its structure and field-theoretical models associated with it may shed more light both on the origin of nonlinear (super)algebras and on the interplay between supersymmetry and $W$ symmetries.

Until now this superalgebra (at the classical level) appeared in the context of $N = 2$ super Toda theories [3], super Lax pair formulation [10] and Polyakov “soldering” procedure [3]. Its realizations on $N = 2$ superfields were discussed in [11, 12]. In ref. [12] the first non-trivial hamiltonian flow on $N = 2$ super-$W_3$ ($N = 2$ super Boussinesq equation) has been constructed. Decisive steps towards building a $N = 2$ super-$W_3$ string model have been undertaken in a recent paper [13].

In the present paper we study $N = 2$ super-$W_3$ in the framework of the nonlinear realizations approach [14, 15], the application of which to the $W$ type symmetries was initiated in ref.[16-20]. One of the most urgent problems encountered while dealing with (super) $W$ algebras is to understand in full their geometric origin and, based on this, to work out convenient general methods for constructing field-theoretical systems with these algebras as underlying symmetries. The closely related problem is to consistently incorporate into a general geometric picture of $W$ algebras the associated hierarchies of integrable equations (such as the KdV, Toda, KP and Boussinesq hierarchies and their superextensions).

There exist several geometric approaches to the $W$ geometry (see, e.g. [21]). In [19, 20] we proposed to treat $W$ symmetries in the universal geometric language of nonlinear (coset space) realizations [14]. In this approach a given nonlinear $W_n$ algebra is replaced by some associate linear infinite-dimensional algebra $W_n^\infty$. The latter is obtained by treating all higher spin composite objects appearing in the commutators of the basic $W_n$ generators (of spins 2 and 3 in the $W_3$ example) as some independent new ones. The linearity of $W_n^\infty$ symmetry allows to apply to it the standard techniques of group realizations in homogeneous spaces and to implement it in a geometric way as a group motion on its appropriately chosen homogeneous manifolds parametrized by $2D$ space-time coordinates and infinite sets of $2D$ fields. After imposing on these fields the covariant inverse Higgs constraints [13] one is left with a finite set of the coset parameter-fields which define a fully geodesic $2D$ surface in the original infinite-dimensional coset space. The standard nonlinear $W_n$ symmetry is recovered as a particular realization of $W_n^\infty$ on this minimal set of fields.

A remarkable feature of the inverse Higgs effect in the case of coset space realizations of $W$ symmetries, besides the fact that it reduces infinite towers of the coset parameters to a few essential fields, is that it also implies some dynamical equations for these fields. This dynamical version of the inverse Higgs effect was called in [16] the covariant reduction. The equations obtained in this way always amount to the vanishing of some curvature and so are integrable. Moreover, the well-known Miura maps relating different integrable equations also turn out to
get a nice geometric interpretation as a part of the inverse Higgs (alias covariant reduction) constraints [2]. This provides a new systematic way to find these maps in an explicit form, starting with the defining relations of given $W$ algebra.

The examples explicitly elaborated so far are: (1) Liouville equation and its various superextensions [10, 17, 18] related to nonlinear realizations of two commuting light-cone copies of $W_2$, i.e. Virasoro symmetry, and superextensions of $W_2$, (2) the $sl_3$ Toda system related to a nonlinear realization of two light-cone copies of $W_3$ [19], and (3) the Boussinesq equation (together with its modified versions obtained via Miura-type transformations) related to a nonlinear realization of one copy of $W_3$ [20]. Applications of the coset space realizations techniques augmented with the inverse Higgs procedure to the cognate linear $w_{1+\infty}$ symmetry and its some generalizations were given in [22, 23] and in a recent preprint [24].

All this geometric machinery can be rather straightforwardly extended to $N=2$ super-$W_3$ symmetry, and this is what we do in the present paper. We show that the covariant reduction approach naturally gives rise to a $N=2$ superextension of Boussinesq equation and its two modified versions related to each other via $N=2$ super Miura maps. The equations obtained amount to the vanishing of some supercurvatures. In addition to our previous works cited above, this work is another step in the direction of our main goal of providing a common geometrical framework for all two-dimensional integrable systems on the basis of nonlinear realizations of the $W$ type symmetries.

The paper is organized as follows. In Sec. 2 we recapitulate the essential ingredients of our work [20] on nonlinear realizations of $W_3$ symmetry which will be of need in our subsequent discussion of $N = 2$ super-$W_3$ symmetry in a similar context. Then, in Sec. 3, we recall the basic facts about $N = 2$ super-$W_3$ algebra in the formulations via supercurrents and component currents. In Sec. 4 we pass to the linear $N = 2$ super-$W_3^\infty$ symmetry and construct its coset space realizations generalizing those of $W_3^\infty$. In Sec. 5 we effect the covariant reduction of these coset spaces and deduce the $N = 2$ super Boussinesq equations and the related super Miura maps as the most essential conditions of the reduction. Sec. 6 is devoted to the comparison with the hamiltonian approach [12] and discussion of the integrability properties of the systems obtained. Sec. 7 contains concluding remarks. In Appendices A - C we quote the basic SOPE’s and OPE’s of $N = 2$ super-$W_3$ algebra, the (anti)commutation relations of $N = 2$ super-$W_3^\infty$ which are used while constructing the coset space realizations of this symmetry in Sec. 4, and some unwieldy formulas required in the process of deducing $N = 2$ super Boussinesq equations in Sec. 5.

## 2 Preliminaries: nonlinear realization of $W_3$ symmetry

In this Section we sketch the key points of the coset space realizations of $W_3$ algebra.

As $W_3$ algebra is nonlinear, it was unclear how to generalize to the $W_3$ case the standard techniques of group realizations in homogeneous spaces [13]. The basic trick invoked in refs. [19], [20] to overcome this difficulty is to pass to a linear infinite-dimensional algebra $W_3^\infty$ in which all the composite higher spin generators appearing in the commutators of $W_3$ are treated as independent generators. For instance, the spin 4 composite generator $J_m^{(4)} = -\frac{8}{c} \sum_n \mathcal{L}_{m-n} \mathcal{L}_n$ appearing in the classical (centrally extended) $W_3$ algebra [23]:

\[
[\mathcal{L}_n, \mathcal{L}_m] = (n - m) \mathcal{L}_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}
\]
\[ [\mathcal{L}_n, \mathcal{W}_m] = (2n - m)\mathcal{W}_{n+m} \quad (2.1) \]
\[ [\mathcal{W}_n, \mathcal{W}_m] = 16(n - m)\mathcal{J}^{(4)}_{n+m} - \frac{8}{3}(n - m) \left[ n^2 + m^2 - \frac{1}{2}nm - 4 \right] \mathcal{L}_{n+m} \]
\[-\frac{c}{9}(n^2 - 4)(n^2 - 1)n \delta_{n+m,0} \]
together with other higher spin composites \( \mathcal{J}^{(s)}_n \) \((s = 5, 6, 7, \ldots)\) extend \( W_3 \) to the \( W^\infty_3 \) algebra as is given below:
\[ W^\infty_3 = \{ \mathcal{L}_n, \mathcal{W}_n, \mathcal{J}^{(4)}_n, \ldots \mathcal{J}^{(s)}_n, \ldots \} \quad (2.2) \]

One of the most important subalgebras of \((2.2)\) which plays a crucial role in the construction of ref. \[19, 20\] contains all the spin \( s \) \((s > 2)\) generators with indices ranging from \(-(s - 1)\) to \( +\infty \). It is distinguished in that the explicit central charge terms drop out from its commutators (though implicit traces of \( c \) still remain: e.g., the term linear in \( \mathcal{L}_n \) in the r.h.s. of the commutator \([\mathcal{W}_n, \mathcal{W}_m]\) is due to \( c \neq 0 \), one should make use of the whole algebra \((2.1)\) while evaluating the commutators with composite generators, etc). When the above \( W_3 \) (or \( W^\infty_3 \)) is realized as a classical symmetry of some \( 2D \) field-theoretical model (e.g. the \( sl_3 \) Toda model), this subalgebra consists of the infinitesimal group variations with the parameter-functions regular at \( x = 0 \). In what follows we will deal just with this truncated version of \( W^\infty_3 \), the “contact” \( W^\infty_3 \) (and its \( N = 2 \) superextension). The generators with indices ranging from \(-(s - 1)\) to \( (s - 1)\) constitute a wedge subalgebra \( W_\wedge \) in \( W^\infty_3 \). All the composite generators in \( W_\wedge \) form an ideal, so that the quotient of \( W_\wedge \) by this ideal is the algebra \( sl(3, R) \) \[19, 20\]

\[ sl(3, R) \sim W_\wedge/\{ \mathcal{J}^{(4)}_n, \ldots \mathcal{J}^{(s)}_m, \ldots \} = \{ \mathcal{L}_0, \mathcal{L}_{\pm 1}, \mathcal{W}_0, \mathcal{W}_{\pm 1}, \mathcal{W}_{\pm 2} \} \quad (2.3) \]

Let us point out that the commutation relations of \( W^\infty_3 \) can be completely restored from the basic \( W_3 \) relations \((2.1)\). Once this is done, one may forget that the higher spin generators of \( W^\infty_3 \) were initially composite and define \( W^\infty_3 \) by its commutation relations. In practice, it turns out necessary to know only the commutators involving a few first higher spin generators.

Since \( W^\infty_3 \) symmetry is linear, one may construct the relevant coset manifolds and define the left action of \( W^\infty_3 \) on them following general scheme of nonlinear (coset space) realizations. Thus, by nonlinear realizations of \( W_3 \) (and \( W_n \)) symmetry we always mean those of the associate \( W^\infty_3 \) (\( W^\infty_n \)) symmetry. In the same sense we will understand nonlinear realizations of \( N = 2 \) super-\( W_3 \) symmetry.

The specificity of the case at hand is that the relevant coset manifolds are infinite-dimensional, they are parametrized by the \( 2D \) space-time coordinates and an infinite number of \( 2D \) fields. However, by imposing an infinite number of covariant inverse Higgs type constraints on the relevant Cartan forms, one can reduce the infinite set of the initial coset parameter-fields to a finite set of some basic fields and simultaneously obtain a kind of integrable equations for the latter.

There exist several coset realizations of \( W^\infty_3 \) which differ in the choice of the stability subgroup and also in whether one deals with two light-cone copies of this symmetry or with one copy. The former possibility \[19\] eventually yields the Lorentz covariant integrable system: the \( sl_3 \) Toda theory. The latter option \[20\] (there are three nonequivalent choices of the relevant realizations) does not respect \( 2D \) Lorentz covariance and yields another type of two-dimensional integrable systems, the Boussinesq equation and two modified Boussinesq equations. Since this case is directly relevant to the subject of the present paper, we will dwell on it in more detail.
We will restrict our study here to the realization with the stability subalgebra \( \mathcal{H}_{(2)} \) containing a minimal set of generators of \( W_3^\infty \):

\[
\mathcal{H}_{(2)} = \{ W_{-1} + 2L_{-1}, \ J_{-3}^{(4)}, \ldots J_{n}^{(4)}, \ldots J_{-s+1}^{(s)}, \ldots \}. \tag{2.4}
\]

Two other realizations constructed in [20] can be obtained from this one by putting equal to zero some of the involved coset parameters or, equivalently, by placing into the stability subalgebra some of the coset generators, namely, \( L_0, W_0 \) and \( L_1, W_1, W_2, \) respectively (following [20], we denote these subalgebras as \( \mathcal{H}_{(1)} \) and \( \mathcal{H} \)). In all cases, the higher spin generators completing \( W_3 \) to \( W_3^\infty \) are placed in the stability subalgebra. Note that they form a subalgebra on their own (it is an ideal in \( \mathcal{H}_{(2)} \subset \mathcal{H}_{(1)} \subset \mathcal{H} \)).

The coset space element \( g \) corresponding to the choice \( (2.4) \) is parametrized in terms of coordinates \( x, t \) and an infinite tower of the parameter-fields \( (u_0, v_0, u_1, v_1, u_2, v_2, u, v, \psi_n, \xi_m, n \geq 3, m \geq 4) \) as follows:

\[
g = e^{iW_{-2}xL_{-1}}e^{uL_2}e^{vW_3}e^{\sum_{n \geq 4} \psi_nL_n}e^{\sum_{n \geq 4} \xi_nW_n}e^{u_1L_1}e^{v_1W_1}e^{v_2W_2}e^{u_0L_0}e^{v_0W_0}. \tag{2.5}
\]

Here the ”time” coordinate \( t \) is linked with the generator \( W_{-2} \) (which so has a meaning of the time translation generator), while the spatial coordinate \( x \) is associated with the generator \( L_{-1} \). All parameter-fields are assumed to be arbitrary functions of \( x \) and \( t \), so at this step we are actually dealing with a two-dimensional surface embedded in the above coset space, with the parameter-fields as the embedding functions. The \( t \) and \( x \) directions on the coset manifold are entirely independent of each other because \( W_{-2} \) commutes with \( L_{-1} \). The \( W_3^\infty \) symmetry is realized as left shifts of the coset element \( (2.3) \).

The fundamental geometric quantity in the coset space approach is the Cartan one-form \( \Omega = g^{-1}dg \) which can be expressed explicitly as a sum over the spin \( s \ (s \geq 2) \) generators with indices ranging from \( -(s - 1) \) to \( +\infty \). Then one accomplishes the covariant reduction, which means that the Cartan form is restricted to some subalgebra \( \tilde{\mathcal{H}}_{(2)} \) containing the stability subalgebra \( (2.4) \):

\[
\Omega \Rightarrow \Omega^{red} \in \tilde{\mathcal{H}}_{(2)},
\]

\[
\tilde{\mathcal{H}}_{(2)} = \{ \mathcal{H}_{(2)}, W_{-2}, L_{-1} \}. \tag{2.6}
\]

This procedure is manifestly covariant with respect to the left action of \( W_3^\infty \). Putting equal to zero all the components of the Cartan form which do not belong to the covariant reduction subalgebra \( \tilde{\mathcal{H}}_{(2)} \) leads to expressing all higher spin parameter-fields in terms of the two essential fields \( u_0 \) and \( v_0 \):

\[
u_1 = \frac{u_0^2}{2}, \quad v_1 = \frac{v_0^2}{3}, \quad v_2 = \frac{1}{12}(v_0^2 + u_0^2v_0'), \quad u = \frac{1}{6}\left[u_0^2 + \frac{1}{2}(u_0')^2 + \frac{8}{3}(v_0')^2\right],
\]

\[
v = \frac{1}{5}\left[\frac{1}{12}v_0'' + \frac{1}{12}u_0^2v_0' + \frac{1}{4}u_0^2v_0'' + \frac{1}{6}(u_0')^2v_0' - \frac{8}{27}(v_0')^3\right] \quad etc,
\tag{2.7}
\]

where prime stands for \( x \) derivative. Simultaneously, for the essential fields there arise the following dynamical equations:

\[
\dot{u}_0 = -\frac{16}{3}[v_0'' + 2u_0'v_0'], \quad \dot{v}_0 = u_0'' - (u_0')^2 + \frac{16}{3}(v_0')^2, \quad \tag{2.8}
\]
where dot denotes \( t \) derivative.

It is obvious from equations (2.5) and (2.7) that, whereas the original coset manifold is parametrized by an infinite number of fields which carry no dynamics, the reduced manifold is characterized by only two essential fields subjected to eq. (2.8). Geometrically, this means that the two-dimensional surface parametrized by \( x, t \) and embedded in the original coset manifold is required to be a geodesic surface [20]. Most essential points are, first, that this surface is singled out in a way manifestly covariant under \( W_3 \) so that the set \((x, t, u_0(x, t), v_0(x, t))\) turns out to be closed under the action of this symmetry, and, second, that among the conditions defining the surface one finds the set of dynamical equations (2.8). Moreover, the latter automatically proves to be equivalent to a zero-curvature condition for the reduced Cartan form \( \Omega^{\text{red}} \) as a consequence of the original kinematical Maurer-Cartan equations for \( \Omega \) and the dynamical inverse Higgs - covariant reduction constraints. Indeed, the reduced Cartan form, modulo an infinite-dimensional ideal formed by all higher spin generators of the stability subalgebra, can be easily found to be

\[
\Omega^{\text{red}} = e^{-2u_0}dt \, W_{-2} + e^{-u_0} \left[ \left( dx + \frac{16}{3} v_0' dt \right) \cosh(4v_0) + (4u_0' dt) \sinh(4v_0) \right] \mathcal{L}_{-1} - \frac{1}{2} e^{-u_0} \left[ \left( dx + \frac{16}{3} v_0' dt \right) \sinh(4v_0) + (4u_0' dt) \cosh(4v_0) \right] W_{-1}. \tag{2.9}
\]

Taking into account that the generators \( W_{-2}, L_{-1} \) and \( W_{-1} \) form a three-dimensional subalgebra of the \( sl(3, \mathbb{R}) \) (2.3) (modulo an infinite-dimensional ideal just mentioned), it is straightforward to check that the Maurer-Cartan equation

\[
d^{\text{ext}} \Omega^{\text{red}} = \Omega^{\text{red}} \wedge \Omega^{\text{red}}
\]

leads just to the equations (2.8).

To clarify the meaning of eqs. (2.8), let us see which equations they entail for the pairs of the composite coset fields \( u_1, v_1 \) and \( u, v \) via the relations (2.7). It turns out that in both cases we are left with the closed sets of integrable equations. The first pair of equations trivially follows by taking \( x \) derivative of (2.8). The second pair reads

\[
\dot{u} = -\frac{160}{3} v', \quad \dot{v} = \frac{1}{10} u''' - \frac{24}{5} u'u, \tag{2.10}
\]

and it is easily recognized (after proper rescalings) as the Boussinesq equation. As was explained in ref. [24], the coset fields \( u, v \) have the correct conformal properties to be identified with the spin 2 conformal stress-tensor and a primary spin 3 current, respectively. Then relations (2.7) are the appropriate Miura maps projecting \( u, v \) onto the sets of the spin 1 currents and the spin 0 scalar fields. With this in mind, eqs. (2.8) (as well as the corresponding equations for \( u_1, v_1 \)) can be called modified Boussinesq equations. Note that eqs. (2.10) and the equations for \( u_1, v_1 \) can be independently obtained by applying the covariant reduction techniques to two other coset manifolds of \( W_3^\infty \) mentioned above, with zero-curvature representations on \( sl(3, \mathbb{R}) \) (2.3) and a five-dimensional Borel subalgebra of the latter (for details see ref.[24]). We also point out that all these equations are covariant by construction under the \( W_3^\infty \) symmetry. As was shown in [20], while applied to the essential coset fields, the \( W_3^\infty \) transformations coincide with those of \( W_3 \) symmetry which thus proves to be a particular realization of \( W_3^\infty \).

To summarize, in the coset space approach the Boussinesq and modified Boussinesq equations as well as the corresponding Miura maps naturally arise within the single geometric
procedure, the covariant reduction on homogeneous spaces of infinite-dimensional linear symmetry $W_3^\infty$ associated in a definite way to $W_3$ algebra. Taking as an input the defining relations of $W_3$ algebra and further employing a number of geometrically motivated prescriptions, we obtain as an output the above equations together with the zero-curvature representation for them and the explicit form of Miura maps. Moreover, the covariance of these equations with respect to $W_3$ symmetry becomes evident and one may explicitly find the $W_3$ transformations of the involved fields. The basic spin 2 and spin 3 $W_3$ currents as well as the related to them via Miura maps spin 1 and spin 0 fields get a novel geometric interpretation as coordinates of the $W_3^\infty$ coset manifolds. In the next sections we will discuss how all this can be generalized to the case of $N=2$ super-$W_3$ symmetry. We will derive $N=2$ superextensions of Boussinesq and modified Boussinesq equations and the relevant $N=2$ superfield Miura maps.

3 $N=2$ super-$W_3$ algebra

In this section, we briefly recapitulate salient features of the classical $N=2$ super-$W_3$ algebra \[\text{Ref.}\] required for the realization of this algebra through the coset superspace construction.

All the basic currents of $N=2$ super-$W_3$ algebra are accommodated by the spin 1 supercurrent $J(Z)$ and the spin 2 supercurrent $T(Z)$, where $Z \equiv (x, \theta, \bar{\theta})$ are coordinates of $N=2$, 1D superspace. Indeed, the components of these $N=2$ superfields carry, respectively, the conformal spins $(1, 3/2, 3/2, 2)$ and $(2, 5/2, 5/2, 3)$, precisely as the currents generating $N=2$ super-$W_3$. The supercurrent $J$ generates $N=2$ super Virasoro algebra, while $T$ can be chosen to be primary with respect to the latter \[\text{Ref.}\]. The closed set of SOPE’s between these supercurrents has been explicitly written in \[\text{Ref.}\]. We quote them in Appendix A.

The component currents appearing in the $\theta, \bar{\theta}$ decomposition of $T$ are related to their counterparts from ref.\[\text{Ref.}\] via a nonlinear redefinition, because the second set of currents is assembled into a non-primary $N=2$ supermultiplet. Explicitly, the relation between the currents present in $J$ and $T$ and the currents of ref. \[\text{Ref.}\] is as follows

\[
\begin{align*}
J| & \equiv J = 4 J , \quad T| \equiv T = T + 4 \bar{T} - \frac{128}{c} J^2, \\
\mathcal{D} J| & \equiv G = \bar{G} , \quad \mathcal{D} T| \equiv \bar{U} = \frac{3}{4} U - \frac{64}{c} J \bar{G}, \\
-\mathcal{D} J| & \equiv G = G , \quad -\mathcal{D} T| \equiv U = \frac{3}{4} U - \frac{64}{c} J G,
\end{align*}
\]

\[
\frac{1}{2} [\mathcal{D}, \bar{\mathcal{D}}] J| \equiv T = T + \bar{T} , \quad \frac{1}{2} [\mathcal{D}, \bar{\mathcal{D}}] T| \equiv W = \frac{3}{4} W + \frac{32}{c} \left( T + 4 \bar{T} - \frac{128}{c} J^2 \right) J + \frac{40}{c} G \bar{G},
\]

where $|$ means restriction to the $\theta, \bar{\theta}$ independent parts and the currents written in calligraphic obey OPE’s of ref. \[\text{Ref.}\]. The covariant spinor derivatives are defined by

\[
\begin{align*}
\mathcal{D}_\theta & = \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \partial_x , \quad \bar{\mathcal{D}}_\theta = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \theta \partial_x, \\
\{ \mathcal{D}, \bar{\mathcal{D}} \} & = -\partial_x , \quad \mathcal{D}^2 = \bar{\mathcal{D}}^2 = 0.
\end{align*}
\]

The explicit form of OPE’s between the currents contained in $J$ and $T$ is given in Appendix A. It seems useful to make here a few comments on them and their relation to those given in ref. \[\text{Ref.}\].
The property that the supercurrent $T$ is primary with respect to the $N = 2$ conformal supercurrent manifests itself as well on the level of OPE’s. For instance, the following OPE’s (the currents in the r.h.s. are always evaluated at the second argument):

\[
T(z_1) \tilde{T}(z_2) = \frac{2\tilde{T}}{(z_1 - z_1)^2} + \frac{\tilde{T}'}{z_1 - z_2}
\]

\[
J(z_1) \tilde{T}(z_2) = 0
\]

(3.3)

demonstrate that $\tilde{T}$ is a spin 2 primary field with respect to $T$ and it carries zero $U(1)$ charge.

One more remark concerns OPE’s of fermionic currents. OPE’s \( \sim G(z_1)G(z_2) \), \( \bar{G}(z_1)\bar{G}(z_2) \) are vanishing but this is not the case for analogous OPE’s involving the currents $U$, $\bar{U}$. These are as follows:

\[
U(z_1)U(z_2) = \frac{1}{c} \left( \frac{80GU - 32GG'}{z_1 - z_2} \right), \quad \bar{U}(z_1)\bar{U}(z_2) = -\frac{1}{c} \left( \frac{80\bar{G}\bar{U} + 32\bar{G}\bar{G}'}{z_1 - z_2} \right).
\]

(3.4)

This fact is essential for validity of the following graded Jacobi identity

\[
\left[ U_p, \{ \bar{G}_r, U_s \} \right] + \text{graded cyclic} = 0
\]

(3.5)

(for definition of the $N = 2$ super-$W_3$ generators see Appendix B). Note that among the OPE’s explicitly written down in [5] analogous important OPE’s for $U$, $\bar{U}$ were missing. Using the relations (3.1), these OPE’s can be found to be

\[
U(z_1)U(z_2) = \frac{64}{c} \left( \frac{GU - \frac{8}{3}G\bar{G}'}{z_1 - z_2} \right) \quad \text{and} \quad \bar{U}(z_1)\bar{U}(z_2) = -\frac{64}{c} \left( \frac{G\bar{U} + \frac{8}{3}\bar{G}\bar{G}'}{z_1 - z_2} \right).
\]

(3.6)

Proceeding from the OPE’s listed in Appendix A, it is straightforward to compute the (anti) commutators of the $N = 2$ super-$W_3$ generators defined as Laurent modes of the component currents. These relations are quoted in Appendix B. They are of primary use while treating $N = 2$ super-$W_3$ symmetry in the framework of the coset space realizations method. This will be the subject of the next Section.

4 Nonlinear realizations of $N = 2$ super-$W_3$

In this Section, following the prescriptions outlined in ref. [13, 20] and Sec. 2, we define a linear infinite-dimensional $N = 2$ super-$W_3^\infty$ algebra from $N = 2$ super-$W_3$ and discuss the choice of the relevant stability subalgebras and construction of the associate infinite-dimensional coset supermanifolds.

4.1 From $N = 2$ super-$W_3$ to $N = 2$ super-$W_3^\infty$

Like in the $W_3$ case, in order to construct coset space realizations of $N = 2$ super-$W_3$ symmetry one should firstly define a linear infinite-dimensional $N = 2$ super-$W_3^\infty$ algebra. Such a superalgebra (hereafter denoted as $sW_3^\infty$) can be obtained by treating as independent all the
higher spin composite generators appearing in the (anti)commutators of the basic $N = 2$ super-$W_3$ generators. Applying to Appendix B, we find that $sW_3^\infty$ is constituted by the following generators:

$$sW_3^\infty = \{ J_n, G_r, \tilde{G}_r, L_n, \tilde{L}_n, U_r, \tilde{U}_r, W_n, B_n^{(2)}, V_r^{(5/2)}, V_r^{(5/2)}, \ldots, \Phi_n^{(s)}, \Xi_n^{(s)}, \Xi_n^{(s)}, \ldots \}. \quad (4.1)$$

Here the generators $J, G, \tilde{G}, L, \tilde{L}, U, \tilde{U}$ and $W$ are the basic generators of $N = 2$ super-$W_3$ (they come from the $\theta$ and $x$ decompositions of the supercurrents $J, T$) and $B_n^{(2)}, V_r^{(5/2)}, V_r^{(5/2)}, \ldots$ are the generators coming from composite currents of conformal spins $(2, 7/2, 7/2, 3)$ the explicit form of which is also given in Appendix B. The letters $(\Phi_n^{(s)}, \Xi_n^{(s)}, \Xi_n^{(s)}, \ldots)$ stand for still higher spin composite generators of $sW_3^\infty$.

Just as in the bosonic case, we will deal not with the whole $sW_3^\infty$ but with its “contact” subalgebra which is singled out by restricting the indices of all spin $s \geq 1$ generators of $sW_3^\infty$ to vary from $-(s - 1)$ to $+\infty$. Like its bosonic prototype, the contact $sW_3^\infty$ contains no explicit central charge terms. In what follows we will, as a rule, omit the adjective “contact”.

The reflection symmetry, $n \rightarrow (-n)$ and $r \rightarrow (-r)$, inherent in the full superalgebra $sW_3^\infty$ guarantees the existence of an infinite-dimensional wedge subalgebra $sW_\Lambda$ in the contact $sW_3^\infty$. It encompasses all the spin $s$ generators $(s \geq 1)$ with indices varying from $-(s - 1)$ to $(s - 1)$.

Inspection of the structure of $sW_\Lambda$ shows that it contains an infinite-dimensional ideal collecting all composite generators. For our further purposes it will be essential that the quotient of $sW_\Lambda$ over this ideal is the superalgebra $sl(3|2)$:

$$sl(3|2) \sim sW_\Lambda/\{ B_n^{(2)}, V_r^{(5/2)}, V_r^{(5/2)}, \ldots \} = \{ J_0, L_0, L_{\pm 1}, \tilde{L}_0, \tilde{L}_{\pm 1}, W_0, W_{\pm 1}, W_{\pm 2}, G_{\pm 1/2}, U_{\pm 1/2}, U_{\pm 3/2}, \tilde{U}_{\pm 1/2}, \tilde{U}_{\pm 3/2} \}. \quad (4.2)$$

It is an obvious generalization of the quotient algebra $sl(3, R)$ (4.3).

It is worthwhile to mention here that, in general, generators associated with the currents of spins 2, 3 and 5/2 can be modified by adding generators coming from the same spin composite currents, such as $J^2, T^2, \tilde{T}^2, J^3, G^2, \bar{G}^2, JG$, and $JG$, without changing the structure of the quotient algebra (4.2). The usefulness of this statement will become more lucid and transparent in Sec.5 in the process of construction of $N = 2$ super Boussinesq equations (see also Subsec.4.3).

### 4.2 Stability subalgebras

In accordance with the discussion in Sec.2, the important step in defining a coset space realization of the $sW_3^\infty$ symmetry is to choose an appropriate stability subgroup. It is beyond the scope of the present paper to list all possible candidates for this role. Like in the $W_3$ case, we require that all the composite generators are in the stability subalgebra. Then there remain a few possibilities which are easy to analyze.

The maximally enlarged subalgebra of this sort is obtained by putting together the composite generators and the $sl(3|2)$ generators (3.2) from $sW_\Lambda$

$$s\tilde{\mathcal{H}} = \{ sl(3|2) \oplus \text{Higher spin generators} \}. \quad (4.3)$$

One may check that the higher spin generators still form an ideal in (3.3), so the $sl(3|2)$ can equivalently be regarded as a quotient of (4.3) by this ideal

$$sl(3|2) \sim s\mathcal{H}/\{ B_n^{(2)}, V_r^{(5/2)}, V_r^{(5/2)}, \ldots \}. \quad (4.4)$$
However, (4.3) is not quite appropriate candidate for the stability subalgebra (like its $W_3$ prototype $sl(3,R)$, [20]) as it contains the translations generators $L_{-1}$ and $W_{-2}$ which should certainly be in the coset. Actually, in order to have a manifest $N=2$ supersymmetry we are led to place in the coset the generators $W_{-2}, L_{-1}, G_{-1/2}, \tilde{G}_{-1/2}$ with which the coordinates of $N=2$, 2D superspace $(t,x,\theta,\bar{\theta}) \equiv (t,Z)$ will be associated as the relevant coset parameters (see Subsec.4.3). After extracting these generators from (4.3) we are left with a set which is not closed; to get a closed set we need to transfer into the coset more in the stability subalgebra, because otherwise we would need to extend the $N=2$ superspace by some new coordinates associated with these generators. The maximally possible closed set which meets this criterion is as follows

\[ s\mathcal{H} \equiv \{ W_{-1} + \frac{3}{2} L_{-1}, L_{-1} - \tilde{L}_{-1}, U_{-3/2}, \bar{U}_{-3/2}, G_{-1/2} + U_{-1/2}, \bar{G}_{-1/2} - \bar{U}_{-1/2}, J_0, L_0, \tilde{L}_0, W_0, G_{1/2} + \frac{1}{2} U_{1/2}, \bar{G}_{1/2} - \frac{1}{2} \bar{U}_{1/2}, L_1 - \frac{1}{4} \tilde{L}_1, W_1, W_2, \text{Higher spin generators} \} . \] (4.5)

This superalgebra is an analog of the stability algebra $\mathcal{H}$ of the $W_3$ case [20] and contains $\mathcal{H}$ as a bosonic subalgebra. However, the analogy is not literal because $\mathcal{H}$ is less than $\tilde{\mathcal{H}} = \{ sl(3,R) \oplus \text{Higher spin } W_3^\infty \text{ generators} \}$ just by the $t$ and $x$ translations generators $W_{-2}$ and $L_{-1}$ while the quotient of $s\mathcal{H}$ by $\mathcal{H}$ includes additional $sl(3|2)$ generators besides those of the $t,x$ translations and $N=2$ supertranslations. Actually, this subtlety is directly related to the structure of the superalgebra $sl(3|2)$ and we will make some further comments on it in Sec. 5.

By comparing the contents of $s\mathcal{H}$ and $sW_3^\infty$, it is evident that the coset superspace associated with this choice is infinite-dimensional similarly to the bosonic case. Since all the higher spin generators have been placed in the stability subgroup, in the coset there remain only the proper sets of generators related to the basic currents of $N=2$ super-$W_3$, i.e. those with spins $(1,3/2,3/2,2)$ and $(2,5/2,5/2,3)$.

Two other relevant stability subalgebras can be obtained as proper truncations of (4.5). The related coset manifolds are also infinite-dimensional. An analog of the stability subalgebra $\mathcal{H}_1$ of the $W_3$ case is $s\mathcal{H}_{(1)}$ defined as follows

\[ s\mathcal{H}_{(1)} \equiv \{ W_{-1} + \frac{3}{2} L_{-1}, L_{-1} - \tilde{L}_{-1}, U_{-3/2}, \bar{U}_{-3/2}, G_{-1/2} + U_{-1/2}, \bar{G}_{-1/2} - \bar{U}_{-1/2}, J_0, L_0, \tilde{L}_0, W_0, \text{Higher spin generators} \} . \] (4.6)

It should be noticed that, similar to the bosonic case [20], just the specific combinations of generators indicated in eq.(4.6) form the subalgebra. Furthermore, it is the maximally possible stability subalgebra which still forms a closed set with the $N=2$ superspace translations generators $W_{-2}, L_{-1}, G_{-1/2}, \tilde{G}_{-1/2}$. Precisely this set of generators is the covariant reduction subalgebra appropriate for the geometric derivation of $N=2$ super Boussinesq and modified super Boussinesq equations, as well as Miura maps of the $N=2$ $W_3$ supercurrents onto the spin $1/2$ supercurrents [12] (we denote this subalgebra with $s\tilde{\mathcal{H}}_{(1)}$). To deduce further Miura maps onto scalar superfields [11, 12] in the coset space approach, one needs to pass to the realization with a smaller stability subalgebra, namely,

\[ s\mathcal{H}_{(2)} = \{ U_{-3/2}, \bar{U}_{-3/2}, G_{-1/2} + U_{-1/2}, \bar{G}_{-1/2} - \bar{U}_{-1/2}, \} . \]

\footnote{$s\tilde{\mathcal{H}}$ can still be chosen as a covariant reduction subalgebra, see Sec. 5.}
Here the factor \( \{ - \text{equations, the relevant super Miura maps} \} \) certainly respect this symmetry.

which is obtained from \( s\mathcal{H}_1 \) by removing the generators \( W_0, L_0, \tilde{L}_0, J_0 \) (which are thus transferred into the coset). It is an analog of \( \mathcal{H}_2 \) of the \( W_3 \) case (eq. (2.4)). This set and the aforementioned \( N = 2 \) superspace translations generators still form a subalgebra \( s\mathcal{H}_2 \), the covariant reduction to which yields the whole set of \( N = 2 \) super Boussinesq equations and Miura maps.

### 4.3 Construction of the coset supermanifolds

As was mentioned in the above discussion, for our ultimate purpose of getting \( N = 2 \) superextensions of the Boussinesq equations (2.8), (2.10) and the relevant super Miura maps it is enough to consider the realization of \( sW_3^\infty \) which is associated with the stability subalgebra \( s\mathcal{H}_2 \) defined in eq. (1.1). We will comment on the realizations corresponding to the stability subalgebras (1.3) and (4.4) in Sec. 5.

An element of the coset supermanifold of \( sW_3^\infty \) symmetry corresponding to the choice of the stability subalgebra (1.7) can be parametrized as follows:

\[
g(2) = e^{iW_2}e^{xL_1}e^{\theta G_{-1/2} + \mu \tilde{G}_{-1/2}}e^{\psi u_2}e^{\phi^2 L_2}e^{\mu_1 G_{1/2} + \mu_1 \tilde{G}_{1/2}}e^{\nu U_{1/2} + \nu_1 \tilde{U}_{1/2}} \ldots
\]

Here the factor \((-1/2)\) before the coset parameters \( \xi, \bar{\xi} \) has been introduced for further convenience. The time coordinate \( t \) with dimension \((cm)^2\) and the spatial coordinate \( x \) with dimension \((cm)^1\), like in the bosonic case, are associated with the \( t \) and \( x \) translation generators \( W_{-2} \) and \( L_{-1} \). A new point is the presence of the Grassmann coordinates \( \theta \) and \( \bar{\theta} \) (of dimension \((cm)^{1/2}\)) which are associated with the fermionic generators \( G_{-1/2} \) and \( \tilde{G}_{-1/2} \) and extend \((t, x)\) to a \( N = 2 \), 2D superspace \((t, Z) \equiv (t, x, \theta, \bar{\theta})\) (the anticommutator of \( G_{-1/2} \) and \( \tilde{G}_{-1/2} \) is just \( L_{-1} \), see Appendix B). An infinite tower of the remaining coset parameters \((\chi, \bar{\chi}, \xi, \bar{\xi}, u, \bar{u}, \phi, v, u_1, \bar{u}_1, \ldots)\), with spins being various integers and half-integers, are assumed to be superfields given on this superspace. The group \( sW_3^\infty \) (to be more precise, its “contact” subgroup, see Subsection 4.1) acts on the element \( g(2) \) as left shifts, which induces an infinite sequence of symmetry transformations of the coset parameters. The important point about these transformations is that in general they mix the \( N = 2 \) superspace coordinates \( t, x, \theta, \bar{\theta} \) with the parameter-superfields, i.e. these coordinates alone do not form an invariant subspace of \( sW_3^\infty \), quite analogously to the \( W_3 \) case. In principle, the \( sW_3^\infty \) transformations of the coset parameters, at least those corresponding to the basic spins generators, can be found explicitly with making use of the basic (anti)commutation relations given in Appendix B and some additional relations involving composite generators. In what follows we will not be interested in the explicit form of these transformations. For our purposes it will be sufficient to know that at each step we preserve the \( sW_3^\infty \) covariance and so the final relations \((N = 2\) super Boussinesq equations, the relevant super Miura maps, ...) certainly respect this symmetry.

Let us offer a few comments concerning the order of the group factors in (4.8) and the choice of the \( t \) translations generator.

Just because of the special arrangement of the \( t, x \) and \( \theta \) exponentials in the element (4.8) the remaining coset parameters behave as scalars under \( t \) translations, \( x \) translations and rigid supertranslations realized as left shifts of (4.8). This is the reason why they can be consistently
treated as $N = 2$ superfields given on the superspace $(t, Z)$. Note that the $t$ translation generator $W_{-2}$ commutes (like in the bosonic case) with all other translation operators, namely $L_{-1}$, $G_{-1/2}$ and $\bar{G}_{-1/2}$. Physically this can be expressed as the statement that, on the coset supermanifold, translations along the $t$ direction are entirely independent of translations along the $x$, $\theta$ and $\bar{\theta}$ directions.

It is interesting that, in contrast to the bosonic case, now there is a freedom in the definition of the $t$ translations generator. Namely, as such a generator one may equally choose the following linear combination of $W_{-2}$ and one of the composite generators present in the stability subalgebra:

$$W_{-2}^{(\alpha)} = W_{-2} + \alpha(TJ - G\bar{G})_{-2}. \quad (4.9)$$

Here $\alpha$ is an arbitrary real parameter. It can be readily checked that this generator, like $W_{-2}$, commutes with all the rest of the coordinate translations generators. Note that the extra term in (4.9) is the unique combination which has the same conformal dimension as $W_{-2}$ and meets the aforementioned commutativity requirement. After substituting $W_{-2}^{(\alpha)}$ for $W_{-2}$ in (4.8) we obtain a one-parameter family of the coset space realizations of $sW_3^\infty$ with the same stability subgroup. Note that the quotient algebra $sl(3|2)$ defined in Subsec. 4.1 remains the same even if we replace $W_{-2}$ by $W_{-2}^{(\alpha)}$ in the wedge superalgebra $sW_\wedge$.

This freedom will be used in Sec. 5 to derive the most general $N = 2$ super Boussinesq equation in the framework of the coset space approach.

### 4.4 Cartan forms

As was mentioned in Sec. 2, the fundamental geometrical quantity which determines curvature, torsion and other characteristics of a (super)group coset manifold is the differential covariant Cartan one-form ($\Omega$). For simplicity, we specialize here to the case $\alpha = 0$. For the coset supermanifold in question, the Cartan form is introduced as follows:

$$\Omega_{(2)} \equiv g_{(2)}^{-1}dg_{(2)} = \sum_{n=-2}^{\infty} w_n W_n + \sum_{n=-1}^{\infty} l_n L_n + \sum_{n=-1}^{\infty} \bar{l}_n \bar{L}_n + \sum_{r=-1/2}^{\infty} g_r G_r + \sum_{r=-1/2}^{\infty} \bar{g}_r \bar{G}_r$$

$$+ \sum_{r=-3/2}^{\infty} f_r U_r + \sum_{r=-3/2}^{\infty} \bar{f}_r \bar{U}_r + \sum_{n=0}^{\infty} j_n J_n + \text{Higher spin contributions}, \quad (4.10)$$

where we have decomposed $\Omega_{(2)}$ over the $sW_3^\infty$ generators. The differentiation in eq.(4.10) is with respect to the coordinates $t, x, \theta, \bar{\theta}$. One may divide $\Omega_{(2)}$ further into the coset and stability subalgebra parts, singling out in the r.h.s. of (4.10) the appropriate combinations of generators. For the covariance of the inverse Higgs - covariant reduction procedure it will be essential that the coefficient superforms associated with coset generators transform homogeneously. Note that all the forms associated with the higher spin generators belong to the stability subalgebra part of $\Omega_{(2)}$ and they will never appear explicitly in the subsequent consideration.

The evaluation of these forms uses the (anti)commutation relations given in Appendix B and is straightforward though a bit tiresome.

First few coset forms, up to finite rotations by the group factors with generators $L_0$, $\bar{L}_0$, $J_0$, $W_0$ standing on the right end of $g_{(2)}$ in (4.8), are as follows:

$$w_{-2} \sim dt,$$
\[ l_{-1} \sim \Delta x + \left( 12v_1 - \frac{3}{2}\xi \bar{\xi} - 6\bar{\chi} \xi - 6\chi \bar{\xi} \right) dt, \]  
\[ \bar{l}_{-1} \sim \left( -2\phi - 3\chi \bar{\chi} - 3v_1 + \frac{3}{4}\xi \bar{\xi} + 3\bar{\chi} \xi + 3\chi \bar{\xi} \right) dt \text{ etc.}, \]  
where  
\[ \Delta x \equiv dx - \frac{\bar{\theta}d\theta + \theta d\bar{\theta}}{2} \]

is the covariant (with respect to \( N = 2 \) superconformal transformations) differential of \( x \). In what follows we will always expand differentials of the coset parameters-superfields over the covariant set \( dt, d\theta, \bar{d}\bar{\theta}, \Delta x \) according to the rule  
\[ d = dt \partial_t + \Delta x \partial_x + d\theta \mathcal{D}_\theta + \bar{d}\bar{\theta} \bar{\mathcal{D}}_{\bar{\theta}} , \]

where the spinor derivatives \( \mathcal{D}, \bar{\mathcal{D}} \) have been defined in eqs. (3.2). Note that the arrangement of the exponentials in eq. (4.1) is such that in the Cartan forms there explicitly appear only the covariant differentials of coordinates and no explicit \( t, x \) and \( \theta \)'s can appear (this is of course a direct consequence of the fact that the coset parameter-superfields behave as scalars under the \( t \) translations and rigid \( N = 2, 1D \) supersymmetry and everything should clearly be covariant with respect to these symmetries). The projections of the whole Cartan form (4.10) and its coefficients on the above set of differentials are defined by (for the moment we omit the subscript (2) of \( \Omega \) as the subsequent relations are of universal validity)

\[ \Omega = \Omega_t dt + \Omega_x \Delta x + \Omega_\theta d\theta + \Omega_{\bar{\theta}} d\bar{\theta} . \]  

The Maurer-Cartan equation for the Cartan form \( \Omega \),

\[ d^{ext} \Omega = \Omega \wedge \Omega , \]

implies a number of useful general identities for these projections, in particular,  
\[ \Omega_x = D\Omega_\theta + D\Omega_{\bar{\theta}} - \{ \Omega_\theta, \Omega_{\bar{\theta}} \} \]

amounts to the fact that the \( \Delta x \) projections are dependent quantities.

As examples of more complicated (super)forms we quote the forms which begin with the differentials of the coset superfields \( u_0, \bar{u}_0, v_0 \) and \( \phi_0 \):

\[
\begin{align*}
\ell_0 &= du_0 + d\theta \bar{\chi} + d\bar{\theta} \chi - 2u\Delta x + dt(\ell_0)_t \\
\bar{\ell}_0 &= d\bar{u}_0 - \frac{1}{2}(d\theta - \chi \Delta x)\bar{\xi} + \frac{1}{2}(d\bar{\theta} - \bar{\chi} \Delta x)\xi - 2\bar{u}\Delta x + dt(\bar{\ell}_0)_t \\
\omega_0 &= dv_0 - 3v_1 \Delta x + \frac{1}{2}\xi \xi \Delta x - 2d\theta - \bar{\chi} \Delta x)\xi - \frac{1}{2}(d\bar{\theta} - \chi \Delta x)\bar{\xi} + dt(\omega_0)_t \\
\j_0 &= d\phi_0 - \phi \Delta x - \frac{1}{2}X \bar{\chi} \Delta x - \frac{1}{2}\xi \xi \Delta x + \frac{1}{2}d\theta \bar{\chi} - \frac{1}{2}d\bar{\theta} \chi + dt(j_0)_t 
\end{align*}
\]

Here \((\ell_0)_t, (\bar{\ell}_0)_t, (\omega_0)_t\) and \((j_0)_t\) are complicated expressions which collect many contributions including those from the generators associated with the composite currents. These expressions are quoted in Appendix C. They play an important role in obtaining \( N = 2 \) super-Boussinesq equations for the superfields \( u_0, \bar{u}_0, v_0 \) and \( \phi_0 \) in the framework of the covariant reduction.
procedure, as is discussed in the next Section. It will be shown that in the realization in question these superfields are the only essential ones in terms of which all higher spin coset superfields can be expressed after employing the inverse Higgs effect.

The expressions for other (super)forms are much more complicated and it is not too enlightening to give them here.

Before ending this Section we mention that in order to obtain the Cartan forms for the realizations with the stability subalgebras (4.6) and (4.5), one should successively set equal to zero the coset superfields $u_0, \tilde{u}_0, v_0, \phi_0$ and $\chi, \tilde{\chi}, u, v_1, v_2$.

5 $N = 2$ Super Boussinesq equations from the covariant reduction

In this section, by applying the inverse Higgs - covariant reduction procedure to the $sW^\infty_3$ coset supermanifolds defined in the previous section, we shall derive the evolution equations ($N = 2$ super Boussinesq equations) for a few essential coset superfields. We will find the complete agreement with the Hamiltonian formulation of ref. [12]. We will also show that the spin 1 and spin 2 supercurrents, the basic ingredients of $N = 2$ super-$W_3$, naturally come out in the nonlinear realization scheme as some coset parameters, in a close analogy with the bosonic $W_3$ case [20]. The $N = 2$ super Miura maps [12, 11] are then recognized as a part of the inverse Higgs algebraic constraints covariantly relating these parameter-superfields to the lower spin ones. To simplify the presentation, here we choose the value of central charge $c$ to be 8. However, all the subsequent equations can be easily promoted to an arbitrary non-zero value of $c$ by a proper rescaling of the involved superfields.

5.1 Covariant reduction constraints

As was explained in ref. [19, 20] and in Sec. 2, the basic idea of the covariant reduction is the imposition of infinite number of covariant constraints on the initial Cartan form $\Omega$, such that it is reduced to a one-form given on an appropriate subalgebra (the covariant reduction subalgebra) of the original (super)algebra. The necessary requirements the covariant reduction subalgebra should obey [16-20] are: (i) it should contain the stability subalgebra and (ii) it should include the generators of (super)translations.

In order to encompass most general situation, we start with the Cartan form $\Omega(2) = g^{(2)}dg^{(2)}$ corresponding to the realization with the most narrow stability subalgebra (4.7). We will perform the covariant reduction of $\Omega(2)$ successively, step by step, first to the subalgebra $s\tilde{\mathcal{H}}$ (1.3) and then to two other covariant reduction subalgebras, $s\tilde{\mathcal{H}}(1)$ and $s\tilde{\mathcal{H}}(2)$, which are contained in (1.3) and correspond to the realizations with the stability subalgebras (1.6) and (1.7), respectively. This chain of reductions can be expressed as follows

$$A. \quad \Omega(2) \Rightarrow \tilde{\Omega}^{\text{red}} \in s\tilde{\mathcal{H}} \subset sW^\infty_3$$

$$B. \quad \tilde{\Omega}^{\text{red}} \Rightarrow \Omega^{\text{red}}_{(1)} \in s\tilde{\mathcal{H}}(1) \subset s\tilde{\mathcal{H}}$$

$$C. \quad \Omega^{\text{red}}_{(1)} \Rightarrow \tilde{\Omega}^{\text{red}}_{(2)} \in s\tilde{\mathcal{H}}(2) \subset s\tilde{\mathcal{H}}(1)$$

The covariant reduction subalgebras $s\tilde{\mathcal{H}}(1)$ and $s\tilde{\mathcal{H}}(2)$ are constituted by the following sets of
generators

\[ \tilde{s}\mathcal{H}(1) = \{ s\mathcal{H}(1), W_{-2}, L_{-1}, G_{-1/2}, G_{-1/2} \}, \quad (5.4) \]

\[ \tilde{s}\mathcal{H}(2) = \{ s\mathcal{H}(2), W_{-2}, L_{-1}, G_{-1/2}, \bar{G}_{-1/2} \}, \quad (5.5) \]

where \( s\mathcal{H}(1) \) and \( s\mathcal{H}(2) \) have been defined in eqs. (4.6) and (4.7). It is easy to check that these generators indeed form closed sets. Recall that (5.4) is the minimally possible (in the framework of the “contact” \( sW^\infty_3 \) we are dealing with) closed extension of \( s\mathcal{H}(1) \) which incorporates the (super)translations generators (see remark after eq. (4.6)). It is also worth mentioning that an ideal consisting of all the \( \Omega \) stating that the components of \( \Omega \) the quotient algebra \( \mathcal{s}\mathcal{l}(3|2) \), eq. (1.2). These sets of the \( \mathcal{s}\mathcal{l}(3|2) \) generators are closed modulo an ideal consisting of all the \( s \geq 5/2 \) higher spin generators and so constitute quotients of \( s\mathcal{H}(1) \) and \( s\mathcal{H}(2) \) by this infinite-dimensional ideal

\[ \mathcal{s}\mathcal{l}(1)(3|2) \sim \tilde{s}\mathcal{H}(1)/\{ B^{(2)}_n, V^{(5/2)}_r, \bar{V}^{(5/2)}_r, \ldots \} = \{ J_0, L_0, \tilde{L}_0, W_0, L_{-1}, \tilde{L}_{-1}, W_{-2}, W_{-1}, G_{-1/2}, \]

\[ \tilde{G}_{-1/2}, \bar{U}_{-1/2}, \bar{U}_{-3/2}, \bar{U}_{-3} \} \],

\[ \mathcal{s}\mathcal{l}(2)(3|2) \sim \tilde{s}\mathcal{H}(2)/\{ B^{(2)}_n, V^{(5/2)}_r, \bar{V}^{(5/2)}_r, \ldots \} = \{ L_{-1}, \tilde{L}_{-1}, W_{-2}, W_{-1}, G_{-1/2}, \tilde{G}_{-1/2}, U_{-1/2}, \]

\[ \bar{U}_{-1/2}, U_{-3/2}, \bar{U}_{-3} \} \].

The equations (5.1) - (5.3) are a concise notation for an infinite sequence of constraints which follow from equating to zero appropriate coefficients in the decomposition of the relevant \( \Omega \)’s over the set of the \( sW^\infty_3 \) generators. At the step A one equates to zero all the parts of \( \Omega(2) \) which lie out of the subalgebra \( s\mathcal{H} \); at the step B there appear additional constraints stating that the components of \( \Omega^{\text{red}} \) associated with the generators which do not belong to \( s\mathcal{H}(1) \subset s\mathcal{H} \) are zero; finally, at the step C, one puts equal to zero also those components of \( \Omega^{\text{red}}(1) \) which are out of \( s\mathcal{H}(2) \subset s\mathcal{H}(1) \subset s\mathcal{H} \). Two types of constraints emerge: kinematical (or algebraic) and dynamical. The former constraints are just akin to the inverse Higgs effect as it was originally formulated in [15], they furnish covariant expressions for the higher spin coset superfields in terms of a finite number of the essential coset superfields and also imply some irreducibility conditions for the latter. On the other hand, the dynamical constraints lead to the dynamical equations for the essential superfields. The covariance of the whole set of constraints is guaranteed by the fact that all the component one-forms equated to zero belong to the coset and so transform homogeneously under \( sW^\infty_3 \), through each other. Note that the varieties of constraints successively obtained by accomplishing the steps (5.1) - (5.3) are covariant in their own right.

### 5.2 Expressing higher spin coset superfields

One should keep in mind that the vanishing of any component coset one-form gives rise to three independent equations for its projections on the differentials \( dt, d\theta \) and \( d\bar{\theta} \) (the projection on \( \Delta x \) is always expressed through the \( d\theta \) - and \( d\bar{\theta} \) - projections by eq. (1.16)). The equations for the \( d\theta \) and \( d\bar{\theta} \) projections basically produce algebraic constraints while the equations for the \( dt \) projections yield the dynamics. Here we will exhaust the algebraic consequences; the dynamical ones will be discussed in the next Subsection.

A thorough inspection based on general arguments of ref. [15] and on our previous experience of working with this kind of nonlinear realizations shows that at the step A the only independent
coset superfields which remain after solving the algebraic part of constraints (5.1) are the spins 1 and 2 real superfields $\phi(t, Z)$ and $\tilde{u}(t, Z)$ associated, respectively, with the generators $J_1$ and $\tilde{L}_2$. For instance, from the constraints

$$(j_1)_\theta = (j_1)_\bar{\theta} = (\tilde{l}_2)_\theta = (\tilde{l}_2)_\bar{\theta} = 0$$

one expresses the spin 3/2 and spin 5/2 coset superfields $\mu, \bar{\mu}$ and $\nu_1, \bar{\nu}_1$

$$\mu = \overline{D}\phi , \quad \bar{\mu} = -D\phi$$
$$\nu_1 = \frac{1}{2}\overline{D}\tilde{u}_2 , \quad \bar{\nu}_1 = -\frac{1}{2}D\tilde{u}_2 ,$$

from the constraints

$$(g_{3/2})_\theta = (g_{3/2})_{\bar{\theta}} = 0$$

and their conjugate one expresses the spin 2 coset superfield $u_2$, etc. It is easy to see that these two basic coset superfields are none other than the basic $N = 2$ super- $W_3$ supercurrents $J$ and $T$ defined in Sec. 3 (up to unessential numerical rescalings). Indeed, $\phi$ and $\tilde{u}_2$ are the only coset superfields which are shifted, respectively, under the action of generators $J_1$ and $\tilde{L}_2$. On the other hand, the $N = 2$ super-$W_3$ transformations of $J$ and $T$ start with precisely the same constant shifts. Actually, one might find the full transformation laws of $\phi$ and $\tilde{u}_2$ and prove that these coincide with the $N = 2$ super-$W_3$ laws of $J$ and $T$. However, it is simpler to show that for $\phi$ and $\tilde{u}_2$ in the present approach there arise the same evolution equations and Miura maps as those for $J$ and $T$ in ref. [12].

To this end, let us continue the covariant reduction process and switch on the step B constraints (5.2). At this level we find that the only independent superfields through which all other coset parameters (including $\phi$ and $\tilde{u}_2$) can be expressed are the complex spin 1/2 superfields $\xi (\tilde{\xi})$ and $\chi (\tilde{\chi})$ associated with the generators $G_{1/2} (\tilde{G}_{1/2})$ and $U_{1/2} (\tilde{U}_{1/2})$. For instance, from the constraints

$$(g_{1/2})_\theta = (g_{1/2})_{\bar{\theta}} = 0 , \quad (f_{1/2})_\theta = (f_{1/2})_{\bar{\theta}} = 0$$

the following expressions emerge for the first few coset superfields:

$$u = \frac{1}{2}(D\chi + \overline{D}\tilde{\chi}) , \quad \tilde{u} = \frac{1}{4}(D\xi - \overline{D}\tilde{\xi} + \chi\tilde{\xi} - \tilde{\chi}\xi) ,$$
$$v_1 = \frac{1}{6}(\xi\tilde{\xi} + \tilde{\chi}\xi + \chi\tilde{\chi} - D\xi - \overline{D}\tilde{\xi}) ,$$
$$\phi = \frac{1}{2}(\overline{D}\tilde{\chi} - D\chi - \xi\tilde{\xi} - \chi\tilde{\chi}) .$$

Furthermore, the superfields $\chi, \tilde{\chi}, \xi, \tilde{\xi}$ turn out to be chiral (anti-chiral) because, also as a consequence of constraints (5.11), we obtain the following irreducibility conditions

$$D\tilde{\chi} = \overline{D}\chi = D\tilde{\xi} = \overline{D}\xi = 0 .$$

2 Analogous chirality conditions appeared in ref. [17] in the process of deducing $N = 2$ super Liouville equation within the covariant reduction procedure applied to $N = 2$ superconformal algebra.
The expressions for other higher spin coset superfields are much more complicated, though all of them can be straightforwardly computed from the conditions of vanishing of the appropriate higher order component one-forms contained in $\Omega$. All these superfields are eventually expressed in terms of $\xi$ and $\chi$. In particular, as one of the consequences of the constraint

$$\bar{l}_1 = 0 ,$$

(5.15)

one obtains the expression for $\bar{u}_2$

$$12 \bar{u}_2 = \partial(D\xi - \bar{D}\bar{\xi}) + 2\partial\xi\bar{\chi} + \partial\xi\bar{\bar{\chi}} + \partial\bar{\xi}\bar{\xi} + 2\partial\bar{\xi}\chi + \partial X\bar{\bar{x}} - \partial\bar{\chi}\xi - 5\xi\bar{\bar{x}}\chi$$

$$+ \bar{D}\xi[D\bar{\xi} - D\chi - D\bar{\chi} + \bar{\chi} - \xi + 2\bar{\xi} - 4\bar{\bar{x}}]$$

$$+ \bar{D}\xi[D\chi + D\bar{\chi} + D\xi - 3\bar{D}\bar{\xi} + 2\bar{\xi} - 4\bar{\bar{x}} + \bar{\chi} - \xi]$$

$$+ D\chi[\xi\bar{\bar{x}} - 2\bar{\xi}\chi - \xi\bar{\bar{x}}] + \bar{D}\chi[2\bar{\bar{x}}\chi - \bar{\bar{\bar{x}}} + \xi].$$

(5.16)

Comparing (5.13) and (5.16) with the super Miura maps relating the $N = 2 W_3$ supercurrents $J$ and $T$ to two spin $1/2$ chiral supercurrents [12] we observe the complete coincidence between them. This confirms the identification of $\phi$ and $\bar{u}_2$ as the $N = 2 W_3$ supercurrents

$$T \equiv 12\bar{u}_2 , \quad J \equiv 2\phi ,$$

(5.17)

and suggests that $\chi$ and $\xi$ can be identified with the above spin $1/2$ supercurrents.

For our further purposes we will need, besides the expressions already presented, explicit expressions only for the coset superfields $u_2, v_2, \phi_2$ entering the functions $(l_0)_t, (\bar{l}_0)_t, (w_0)_t$ and $(j_0)_t$ in eqs.(4.17). These expressions are given in eq. (C.8-C.10). They are obtained from the constraints

$$(g_{3/2})_{\theta} = (g_{3/2})_{\bar{\theta}} = 0 , \quad (f_{3/2})_{\theta} = (f_{3/2})_{\bar{\theta}} = 0 .$$

(5.18)

We do not quote explicit expressions for the Cartan forms $g_{3/2}$ and $f_{3/2}$ in view of their complexity. Note that at the step B the spin 0 coset superfields $u_0, \bar{u}_0, \phi_0$ and $v_0$ are pure gauge and so can be completely gauged away: the constraints (5.1), (5.2) are covariant under arbitrary right gauge shifts of $g_{(2)}$ (4.10) by the elements of the stability subgroup $sH(1)$ associated with the algebra [L9].

At the step C (eq. (5.3)) this gauge invariance gets broken down to the invariance with respect to right $sH(2)$ multiplications, so the scalar coset superfields just mentioned cease to be pure gauge: on the contrary, they become the essential superfields in this case. The set of the step C constraints incorporates all the constraints imposed before and contains four additional ones

$$l_0 = \bar{l}_0 = w_0 = j_0 = 0 ,$$

(5.19)

where the one-forms $j_0, l_0, \bar{l}_0$ and $w_0$ were defined in eq. (1.17). The $d\theta$ and $d\bar{\theta}$ projections of these new constraints express the spin $1/2$ coset superfields $\chi$ and $\xi$ in terms of the spin 0 ones

$$\chi = -\bar{D}\Phi , \quad \bar{\chi} = D\Phi , \quad \xi = -\bar{D}\bar{\nabla} , \quad \bar{\xi} = D\nabla ,$$

(5.20)

and simultaneously imply the chirality conditions for the latter

$$\bar{D}\Phi = D\Phi = 0 , \quad \bar{D}\nabla = D\nabla = 0 .$$

(5.21)
\[ \Phi = -\left( \phi_0 + \frac{1}{2} u_0 \right) , \quad \bar{\Phi} = -\left( \phi_0 - \frac{1}{2} u_0 \right) , \quad \mathcal{V} = v_0 + \tilde{u}_0 , \quad \bar{\mathcal{V}} = -v_0 + \tilde{u}_0 . \]  \hspace{1cm} (5.22)

Thus, in this case the only essential coset parameter-superfields are two complex chiral spin 0 superfields \( \Phi \) and \( \mathcal{V} \). After substitution of (5.20) into (5.13) and (5.16) one recognizes the latter as the Miura maps of the \( N = 2 \) supercurrents onto the scalar chiral \( N = 2 \) superfields [11, 12].

The main conclusion following from the above consideration is that the whole \( sW^\infty_3 \) symmetry can be realized (in a nonlinear way) on the finite-dimensional manifolds:

\[ \{ t, Z \equiv (x, \theta, \bar{\theta}), \chi(t, Z), \bar{\chi}(t, Z), \xi(t, Z), \bar{\xi}(t, Z) \} , \]  \hspace{1cm} (5.23)

and

\[ \{ t, Z, \Phi(t, Z), \bar{\Phi}(t, Z), \mathcal{V}(t, Z), \bar{\mathcal{V}}(t, Z) \} , \]  \hspace{1cm} (5.24)

where the coordinates-superfields \( \chi, \xi \) and \( \Phi, \mathcal{V} \) satisfy the chirality conditions (5.14) and (5.22). Let us point out once again that the \( N = 2 \) superspace coordinates \( (t, Z) \) do not form an invariant subspace in (5.23) and (5.24): the \( sW^\infty_3 \) transformations mix them with the coordinates-superfields and derivatives of the latter (of all orders). We also note that the closure of \( sW^\infty_3 \) on these sets is achieved only by making use of the evolution (\( N = 2 \) Boussinesq) equations for the involved superfields (see next Subsection) because the algebraic and dynamical parts of the above covariant reduction constraints are mixed under the \( sW^\infty_3 \) transformations (like in the \( W_3 \) case [19, 20]).

This is an appropriate place to summarize the group-theoretical and geometric meaning of these realizations and to compare them with the analogous realizations of \( W^\infty_3 \) [20].

The sets (5.23), (5.24) define the covariant embeddings of \( N = 2 \) superspace \( (t, Z) \) into the cosets \( sW^\infty_3 / sH_1 \) and \( sW^\infty_3 / sH_2 \), where the supergroups in the denominator are related to the superalgebras (4.6) and (4.7), respectively. These embeddings are fully specified by the superfields \( \chi, \xi \) and \( \Phi, \mathcal{V} \) which are subjected to the evolution equations to be given below. Note that in both cases the superfields can be regarded as the essential parameters of the cosets of \( sW^\infty_3 \) over the covariant reduction subgroups, i.e. the subgroups with the algebras \( s\mathcal{H}_1 \) and \( s\mathcal{H}_2 \). The \( N = 2, 2D \) superspace itself can be identified with the coset of the supergroups \( SL(1)2\overline{3} \) and \( SL(2)2\overline{3} \) (eqs. (5.6), (5.7)) over their subgroups generated by

\[ \{ W_0, L_0, \tilde{L}_0, J_0, U_{-3/2}, \bar{U}_{-3/2}, G_{-1/2} + U_{-1/2}, \bar{G}_{-1/2} - \bar{U}_{-1/2}, L_{-1} - \tilde{L}_{-1}, W_{-1} + \frac{3}{2} L_{-1} \} \]

and

\[ \{ U_{-3/2}, \bar{U}_{-3/2}, G_{-1/2} + U_{-1/2}, \bar{G}_{-1/2} - \bar{U}_{-1/2}, L_{-1} - \tilde{L}_{-1}, W_{-1} + \frac{3}{2} L_{-1} \} , \]

respectively (recall that these sets are closed modulo higher spin generators). Bosonic analogs of the supermanifolds (5.23) and (5.24) are the manifolds

\[ (t, x, u_1(t, x), v_1(t, x)) \]

and

\[ (t, x, u_0(t, x), v_0(t, x)) \]
which are closed under the action of $W^\infty_3$ (see Sec.2 and ref. [20]). The fields $u_1, v_1$ and $u_0, v_0$ are the essential parameters of the cosets of $W^\infty_3$ over the subgroups associated with the algebras $\mathcal{H}(1)$ and $\mathcal{H}(2)$, while the space-time coordinates $(t, x)$ in both cases parametrize the cosets of the latter subgroups over the relevant stability subgroups. Equivalently, the coordinates $(t, x)$ can be viewed as parametrizing the cosets of appropriate subgroups of the quotient group $SL(3, R)$ with the algebra (2.3).

In the bosonic case there is one more invariant manifold, namely $(t, x, u_2(t, x), v_3(t, x))$, with $u_2$ and $v_3$ being the spin 2 and spin 3 $W_3$ currents [20]. The basic reason why this manifold is closed under $W^\infty_3$ is that it still admits a coset interpretation: the currents are essential parameters of the coset $W^\infty_3 / \tilde{H}$, with generated by $\tilde{H} = \{sl(3, R) + Higher spin generators\}$, whereas $t, x$ are parameters of a two-dimensional coset of $\tilde{H}$, with $W_-2$ and $L_-1$ as the coset generators.

In the supersymmetric case, despite the fact that the covariant reduction (5.1) leaves the supercurrents $\phi, \tilde{u}_2$ as the only essential parameters of the coset $sW^\infty_3 / s\tilde{H}$, the set
\[
\{t, Z, \phi(t, Z), \tilde{u}_2(t, Z)\}
\]
is not closed under the left action of $sW^\infty_3$. The reason has been already mentioned in Subsec. 4.2 and it consists in that the $N = 2$ superspace $(t, Z)$ cannot be regarded as a coset manifold of the supergroup $s\tilde{H}$: such a manifold of minimal dimension contains, besides the $N = 2$ superspace coordinates, also the coset superfield $\tilde{u}$ and a linear combination of $\xi$ and $\chi$, which follows from comparing eqs. (1.3) and (1.3). Thus, only at cost of adding at least this minimal number of extra superfields, the set (5.25) can be promoted to an invariant space of $sW^\infty_3$. Note, however, that the constraints (5.2) are covariant with respect to the $SL(3|2)$ gauge transformations realized as right $SL(3|2)$ shifts of the coset element (4.8) with arbitrary parameter-superfunctions. Using this freedom, one may choose the gauge so as to kill all the additional superfields mentioned above. In other words, the set (5.25) is invariant under modified $sW^\infty_3$ transformations which are closed modulo a compensating gauge $SL(3|2)$ transformation. Here we will not dwell more on this point.

### 5.3 Dynamics

As was already noticed, the equations restricting the $t$ dependence of the essential coset superfields come from the $dt$ projections of the covariant reduction constraints. The dynamical constraints arising at the steps A - C (eqs. (5.1) - (5.3)) are as follows
\[
A. \quad (j_1)_t = (\bar{l}_2)_t = 0 ,
\]
\[
B. \quad (g_{1/2})_t = (f_{1/2})_t = 0 ,
\]
\[
C. \quad (l_0)_t = (\bar{l}_0)_t = (j_0)_t = (w_0)_t = 0 .
\]

After substituting the inverse Higgs expressions for the involved higher spin coset superfields, these constraints yield the evolution equations for the relevant pairs of the essential superfields: $\phi$ and $\tilde{u}_2$, $\chi$ and $\xi$, $\Phi$ and $\mathcal{V}$, respectively. Only taking account of these evolution equations the sets (5.23) and (5.24) are actually closed under $sW^\infty_3$ symmetry.

Before presenting the explicit form of the equations, let us recall that up to now, for simplicity, we assumed that the $t$ translation generator is $W_-2$. However, it is desirable to consider more general situation, identifying the generator of $t$ translations with $W_-^{(a)}$ defined in eq. (4.9)
and placing the latter in the coset element (4.8) instead of \( W_{-2} \). Thereupon, most of formulas actually needed for our purposes, undergo only slight modifications: e.g., in the expressions (4.17) the dependence on the parameter \( \alpha \) appears only in the functions \( (l_0)_t, (\bar{l}_0)_t, (u_0)_t \) and \( (j_0)_t \) (given by eqs. (C.1-C.4) of Appendix C). Using these modified expressions, it is straightforward to find from (5.26), (5.27) and (5.28) the general form of the sought pairs of the evolution equations

\[
\dot{T} \equiv 12 \dot{\tilde{u}}_2 = 2J'' - [\mathcal{T}, \mathcal{D}] T' - 10 \partial (\mathcal{T}J'DJ) + 4J' [\mathcal{T}, \mathcal{D}] J + 2J [\mathcal{T}, \mathcal{D}] J' - 4J^2 J' - (5 - \alpha) \mathcal{D}JDT - (5 - \alpha) DJT - (8 + 2\alpha) JT - (3 + \alpha) JT' \tag{5.29}
\]

\[
\dot{J} \equiv 2\dot{\phi} = -2T' - \alpha \left( [\mathcal{T}, \mathcal{D}] J + 2JJ' \right) \tag{5.30}
\]

\[
\dot{\chi} = 2\partial^2 \xi - 6\partial^2 \chi + (5 - \alpha) \mathcal{D}D(\chi \xi) + \partial \left[ 2\mathcal{D}\xi - 2\alpha \mathcal{D}\chi - (3 + \alpha) \mathcal{D}\xi + 4\mathcal{D}\chi \xi \right] + \partial\chi \left[ 2\mathcal{D}\xi + 2\alpha \mathcal{D}\chi \right] + \partial\xi \left[ 2\mathcal{D}\chi + 4\mathcal{D}\xi \right] \tag{5.31}
\]

\[
\dot{\bar{\chi}} = -\partial^2 \xi - 2\partial^2 \chi + (5 - \alpha) \mathcal{D}D(\bar{\chi}\xi) + \partial \left[ 2\mathcal{D}\bar{\chi} - 2\alpha \mathcal{D}\bar{\chi} - (3 + \alpha) \mathcal{D}\bar{\bar{\chi}} - 4\mathcal{D}\chi \bar{\chi} \right] + \partial\chi \left[ 4\mathcal{D}\chi + (3 + \alpha) \mathcal{D}\bar{\chi} \right] + \partial\xi \left[ (3 + \alpha) \mathcal{D}\bar{\chi} - 2\mathcal{D}\bar{\chi} \right] \tag{5.32}
\]

\[
\dot{\Phi} = 2\partial^2 \nu + 5\partial^2 \Phi + 2\mathcal{D}D\partial\mathcal{D}\nu + 2\alpha \mathcal{D}D\partial\mathcal{D}\Phi + 4\mathcal{D}D\partial\mathcal{D}\Phi + (3 + \alpha) \mathcal{D}D\partial\mathcal{D}\nu + 2\partial\Phi\partial\nu + 4\partial\nu\partial\bar{\Phi} - 2\partial\nu\partial\Phi + 2\partial\nu\partial\nu + \alpha (2\partial\Phi\partial\bar{\Phi} + \partial\Phi\partial\Phi) + (3 + \alpha) \partial\nu\partial\nu + (5 - \alpha) \left( \partial\mathcal{D}\nu\partial\mathcal{D}\Phi - \partial\mathcal{D}\nu\partial\mathcal{D}\Phi \right) \tag{5.33}
\]

\[
\dot{\bar{\Phi}} = 2\partial^2 \nu - 2\partial^2 \Phi + 2\mathcal{D}D\partial\bar{\mathcal{D}}\nu + 2\alpha \mathcal{D}D\partial\bar{\mathcal{D}}\Phi + 4\mathcal{D}D\partial\bar{\mathcal{D}}\Phi + (3 + \alpha) \mathcal{D}D\partial\bar{\mathcal{D}}\nu + 2\partial\Phi\partial\bar{\nu} + 4\partial\bar{\nu}\partial\bar{\Phi} - 2\partial\bar{\nu}\partial\Phi - 2\partial\bar{\nu}\partial\bar{\nu} + \partial\bar{\nu}\partial\bar{\nu} + (5 - \alpha) (\mathcal{D}D\nu\partial\bar{\mathcal{D}}\Phi - \partial\mathcal{D}\nu\partial\mathcal{D}\Phi) \tag{5.34}
\]

The equations for \( \bar{\xi} \) and \( \bar{\chi} \) can be obtained from eqs. (5.31) and (5.32) by applying the same rules as for the generators \( G \) and \( U \) in the Appendix B.

The system (5.29), (5.30) coincides with the \( N = 2 \) Boussinesq equation derived in [12] as a hamiltonian flow on \( N = 2 \) super-\( W_3 \) (the detailed comparison with the hamiltonian approach will be given in the next section). Thus we conclude that this equation can be alternatively derived in a pure geometric way as one of the conditions of embedding the \( N = 2 \) superspace \( (t, Z) \) as a geodesic supersurface into the infinite-dimensional coset supermanifold \( sW_3^{\infty}/s\bar{H} \). As the superfields \( \chi, \xi, \Phi \) and \( \nu \) are related to \( \phi, \tilde{u}_2 \) via super Miura maps (5.13), (5.16), (5.20), the evolution equations for them can naturally be called modified \( N = 2 \) super Boussinesq equations. Their geometric interpretation within the coset space approach is quite similar to that of \( N = 2 \) super Boussinesq equation. It is straightforward to check that the whole set of equations (5.29) - (5.34) is compatible with the super Miura maps: e.g., taking the \( t \) derivative of both sides of (5.20) and using eqs. (5.33), (5.34), one gets for \( \chi, \bar{\chi} \) just eqs. (5.31), (5.32), etc.

It is remarkable that all these three kinds of \( N = 2 \) super Boussinesq equation together with the relevant super Miura maps naturally come out within the single geometric procedure: covariant reduction of the coset supermanifolds of the \( sW_3^{\infty} \) symmetry.

\[^3\text{The equations given in [12] contain some misprints in the numerical coefficients.}\]
6 Comparison with the hamiltonian approach. Integrability properties

It is well-known that the $W_3$ algebra (2.1) provides the second hamiltonian structure for the Boussinesq system (2.10) [25]: the latter can be written in the Hamilton form

$$
\dot{W} = \{W, H\}_{PB}, \quad \dot{T} = \{T, H\}_{PB}, \quad (6.1)
$$

where

$$
H \propto \int dx W(t, x)
$$

and $T(t, x), W(t, x)$ are the spin 2 and spin 3 currents which, at equal time, satisfy the standard Poisson brackets (or, equivalently, OPE’s) of the $W_3$ algebra (2.1).

In a similar way, it has been demonstrated in ref.[12] that for the $N = 2$ super-$W_3$ case the most general supersymmetric form of the hamiltonian is as follows:

$$
H = -\int dZ (T + \alpha J^2), \quad (6.2)
$$

where $dZ \equiv dx d\theta d\bar{\theta}$. Assuming that the supercurrents $J$ and $T$ obey the SOPE’s of $N = 2$ super-$W_3$ algebra as they are given in Appendix A, the evolution equations associated with this Hamiltonian,

$$
\dot{J} = \{J, H\}_{PB}, \quad \dot{T} = \{T, H\}_{PB}, \quad (6.3)
$$

coincide with eqs.(5.29), (5.30) obtained in the geometric framework of the coset space approach.

In order to establish a link between the two approaches, we expand the integrand in (6.2) in Laurent series in $x$ and integrate over $dx$ and $d\theta, d\bar{\theta}$. As a result we obtain

$$
H \propto W_{-2} + \alpha(TJ - G\bar{G})_{-2} = W^{(\alpha)}_{-2}, \quad (6.4)
$$

i.e. just the most general $t$ translations generator (4.3). Thus, the freedom in the choice of this generator within the coset space approach reflects the freedom in the choice of the hamiltonian in the framework of the approach based on the notion of the second hamiltonian structure.

It is straightforward to check that the evolution equation for the spin 1/2 chiral supercurrents $\chi$ and $\xi$ (eqs.(5.31), (5.32)) can be written in the Hamilton form with the same hamiltonian (6.2)

$$
\dot{\chi} = \{\chi, H\}_{PB}, \quad \dot{\xi} = \{\xi, H\}_{PB}, \quad (6.5)
$$

if one assumes that the two-point functions of these supercurrents are given by the standard expressions characteristic of the spin-1/2 chiral superfields, namely:

$$
<\chi(Z_1)\bar{\chi}(Z_2)> = \frac{1}{Z_{12}} + \frac{\theta_{12}\bar{\theta}_{12}}{2Z_{12}^2}, \quad <\xi(Z_1)\bar{\xi}(Z_2)> = \frac{1}{Z_{12}} + \frac{\theta_{12}\bar{\theta}_{12}}{2Z_{12}^2}, \quad (6.6)
$$

(for notation see Appendix A). Analogously, the equations (5.33), (5.34) can be recovered in the hamiltonian formalism assuming that $\Phi, V$ are free chiral $N = 2$ superfields [12]:

$$
<\Phi(Z_1)\bar{\Phi}(Z_2)> = \ln(Z_{12}) - \frac{\theta_{12}\bar{\theta}_{12}}{2Z_{12}^2}, \quad <V(Z_1)\bar{V}(Z_2)> = \ln(Z_{12}) - \frac{\theta_{12}\bar{\theta}_{12}}{2Z_{12}^2}. \quad (6.7)
$$
We also note that the SOPE’s (6.6), (6.7) produce for the Miura expressions (5.13), (5.16) of the supercurrents precisely the SOPE’s ((A.1)-(A.3)) of $N = 2$ super-$W_3$ algebra.

Finally, let us briefly discuss the integrability issues.

As has been noticed in Sec. 2, the covariant reduction procedure automatically yields a zero curvature representation for the dynamical equations obtained, which is a consequence of the covariant reduction constraints and the Maurer-Cartan identity for the original Cartan form. For the reduced Cartan forms arising at different stages of the covariant reduction (eqs. (5.1) - (5.3)) the zero curvature condition reads

$$ d^{\text{ext}}\Omega^{\text{red}} = \Omega^{\text{red}} \wedge \Omega^{\text{red}}, \quad (6.8) $$

where $\Omega^{\text{red}}$ stands for $\tilde{\Omega}^{\text{red}}$, $\tilde{\Omega}^{\text{red}}_{(1)}$, $\tilde{\Omega}^{\text{red}}_{(2)}$. Since the higher spin generators form ideals in the relevant covariant reduction subalgebras, those parts of the reduced Cartan form which are valued in the quotient algebra $sl(3|2)$ satisfy (6.8) in their own right. So, without loss of generality, we may keep in $\Omega^{\text{red}}$ only these parts

$$ \tilde{\Omega}^{\text{red}} \in sl(3|2), \quad \tilde{\Omega}^{\text{red}}_{(1)} \in sl_{(1)}(3|2), \quad \tilde{\Omega}^{\text{red}}_{(2)} \in sl_{(2)}(3|2), \quad (6.9) $$

where the superalgebras in the r.h.s. were defined by eqs. (4.2), (5.6), (5.7). Here we will not give these Cartan forms explicitly, though they can be readily evaluated; let us only recall that, using the appropriate gauge freedom indicated in Subsec. 5.2, all these forms can be written entirely in terms of the relevant essential superfields: $\tilde{\Omega}^{\text{red}}$ via $J$ and $T$, $\tilde{\Omega}^{\text{red}}_{(1)}$ via $\chi$, $\bar{\chi}$, $\xi$, $\bar{\xi}$ and $\tilde{\Omega}^{\text{red}}_{(2)}$ via $\Phi$, $\bar{\Phi}$, $V$, $\bar{V}$.

Recall also that not all of the projections of $\Omega^{\text{red}}$ on the differentials $dt, \Delta x, d\theta, d\bar{\theta}$ are independent, the $\Delta x$ projection is expressed through spinor ones by eq. (4.16). For completeness, we quote here all the independent projections of the zero curvature condition (6.8)

$$ D\Omega^{\text{red}}_t + \dot{\Omega}^{\text{red}}_\theta + [\Omega^{\text{red}}_t, \Omega^{\text{red}}_\theta] = 0, $$
$$ D\Omega^{\text{red}}_q + \dot{\Omega}^{\text{red}}_{\bar{\theta}} + [\Omega^{\text{red}}_q, \Omega^{\text{red}}_{\bar{\theta}}] = 0, $$
$$ D\Omega^{\text{red}}_q - \{\Omega^{\text{red}}_q, \Omega^{\text{red}}_{\bar{\theta}}\} = 0, $$
$$ D\Omega^{\text{red}}_{\bar{\theta}} - \{\Omega^{\text{red}}_q, \Omega^{\text{red}}_{\bar{\theta}}\} = 0. \quad (6.10) $$

After substitution of the appropriate expressions for the involved $sl(3|2)$ valued projections of $\Omega^{\text{red}}$, these relations yield the evolution equations (5.29) - (5.34).

We point out that the above zero curvature representation exists for any choice of the parameter $\alpha$ in eqs. (5.29) - (5.34). However, recently we have found [26] that the $N = 2$ super Boussinesq equation (5.29), (5.30) admits higher order conserved quantities only for three selected values of $\alpha$, $\alpha = (-4, -1, 5)$. This means that it is integrable only in these special cases (cf. $N = 2$ super KdV equation [27]) despite the fact that it possesses a zero curvature representation for any value of $\alpha$ (the same, of course, is true for the modified $N = 2$ super Boussinesq equations). Hence, the existence of such a representation is a weaker requirement than integrability and there arises the question how to understand the aforementioned restrictions on $\alpha$ within the coset space approach.

The only conceivable answer seems to be as follows. In the hamiltonian formalism the higher conserved quantities of the $N = 2$ super Boussinesq equation are the hamiltonians for the higher equations from the $N = 2$ Boussinesq hierarchy. Their characteristic property is that they commute with each other and with the basic hamiltonian (6.2). After performing
the integration over the $N = 2$ superspace coordinates $Z$ they should be recognized as appropriate modes of some composite objects constructed from the supercurrents $J$ and $T$, like $H$ (6.2) has been recognized as the generator $W_{-2}^{(a)}$. Hence, in the $sW_3^\infty$ language, searching for the higher order conserved quantities amounts to singling out the sequences of the higher spin $sW_3^\infty$ generators which commute with the $t$ translation generator $W_{-2}^{(a)}$ and among themselves. Then the result of [20] means that these infinite sequences exist only for the three values of $\alpha$ indicated above. In order to ensure the integrability in the above sense, i.e. as the existence of infinitely many conserved quantities in involution, one is led to pass to another, more general realization of $sW_3^\infty$, with all mutually commuting $t$ translations generators placed in the coset. This will entail introducing infinitely many “time” coordinates and allowing all coset superfields to depend on these coordinates. One may expect that the covariant reduction will still express all these superfields in terms of the two essential ones $J$ and $T$ and simultaneously yield evolution equations for the latter with respect to each “time” coordinate. In this way the whole $N = 2$ super Boussinesq hierarchy could come out within the coset space approach. In other words, we hope that the value of $\alpha$ can be properly fixed in this approach if we will pose the problem of deducing the whole $N = 2$ super Boussinesq hierarchy rather than its first nontrivial representative (5.29), (5.30). We will examine this intriguing possibility in more detail elsewhere.

7 Conclusion

The key ingredient in our study of $W_3$ and $N = 2$ super-$W_3$ symmetries in the framework of the method of coset space realizations is the construction of linear infinite-dimensional algebras $W_3^\infty$ and $sW_3^\infty$ from the standard nonlinear $W_3$ and $N = 2$ super-$W_3$ algebras by treating as independent generators the Laurent modes of all the composite currents present in the enveloping algebra of the basic currents of these algebras [19, 20]. The application of the standard techniques of nonlinear realizations augmented with the ideas of the inverse Higgs effect and the covariant reduction lead to the interpretation of the plane $t,x$ and its superextension, $N = 2$ superspace $t,x,\theta,\bar{\theta}$, as geodesic submanifolds embedded into infinite-dimensional coset manifolds of $W_3^\infty$ and $sW_3^\infty$ symmetries. These embeddings are completely specified by finite numbers of the essential coset parameter-(super)fields which are recognized either as the basic (super)currents generating the original nonlinear algebras or as the lower spin (super)fields related to the former ones via (super)Miura maps. The Miura maps together with the evolution equations for the essential (super)fields (Boussinesq and $N = 2$ super Boussinesq equations as well as their modified versions) naturally come out as the most essential part of the embedding conditions. The $W_3$ and $N = 2$ super-$W_3$ symmetries turn out to be the particular realizations of $W_3^\infty$ and $sW_3^\infty$ preserving the embeddings just mentioned. The characteristic feature of these realizations is that they necessarily mix the coordinates $t,x$ or $t,x,\theta,\bar{\theta}$ with the (super)functions specifying the embedding.

By exploiting the ideas of the coset space approach we have been thus able to provide a common geometric basis to the following $2D$ integrable systems: the Liouville and super Liouville equations [13, 15, 17], the $sl_3$ Toda equations [14], the Boussinesq and modified Boussinesq equations [20], the $N = 2$ super Boussinesq equations. We are sure that a variety of other $2D$ integrable systems, at least those respecting conformal invariance, can be described on similar grounds, by applying the covariant reduction to other nonlinear $W$ type algebras and super-
algebras. Besides new integrable systems and various free-field type representations for them (via the relevant Miura maps), we expect to obtain in this way new intrinsic relations between different systems related to the same $W$ algebra, e.g., between Boussinesq equations and the $sl_3$ Toda equations. An interesting problem is to treat the full quantum $W$ algebras in the same language of passing to linear $W^\infty$ type algebras and to work out convenient geometric methods for deducing associated quantum evolution equations. It would be also of interest to find out possible implications of our geometric approach in $W$ strings, $W$ gravity and related theories.

**Appendix A**

$N = 2$ super-$W_3$ algebra: SOPE’s and OPE’s

In this Appendix we present the formulations of the classical $N = 2$ super-$W_3$ algebra in terms of SOPE’s of the supercurrents $J(Z)$, $T(Z)$ and OPE’s of the component currents.

The full set of the relevant SOPE’s reads

\[ J(Z_1)J(Z_2) = \frac{c}{4Z_{12}^2} + \frac{1}{2Z_{12}} \bar{\theta}_{12} \bar{\bar{T}} J - \frac{1}{2Z_{12}} \bar{\theta}_{12} \bar{\bar{D}} J - \frac{1}{2Z_{12}} \bar{\theta}_{12} \bar{\bar{T}} J - \frac{1}{Z_{12}^2} \bar{\theta}_{12} \bar{\bar{T}} J , \]

\[ J(Z_1)T(Z_2) = \frac{\theta_{12} \bar{\bar{T}} T}{Z_{12}} - \frac{\theta_{12} \bar{\bar{D}} T}{Z_{12}} + \frac{\theta_{12} \bar{\bar{T}} T}{Z_{12}^2} + \frac{\theta_{12} \bar{\bar{T}} T}{Z_{12}^2} , \]

\[ T(Z_1)T(Z_2) = -\frac{3c}{2Z_{12}^2} - \frac{12 \theta_{12} \bar{\bar{D}} J}{Z_{12}^2} + \frac{12 \theta_{12} \bar{\bar{T}} J}{Z_{12}^2} - \frac{12 \bar{\theta}_{12} \bar{\bar{D}} J}{Z_{12}^2} - \frac{12 \bar{\theta}_{12} \bar{\bar{T}} J}{Z_{12}^2} , \]

\[ + \frac{5T - 2 \bar{\bar{T}} J + B^{(2)}}{Z_{12}^2} \left( \theta_{12} \bar{\bar{T}} J + B^{(2)} \right) \]

\[ - \frac{\theta_{12} \bar{\bar{T}} J}{Z_{12}^2} \left( \theta_{12} \bar{\bar{T}} J - B^{(2)} \right) + \bar{\theta}_{12} \bar{\bar{T}} J \left( \frac{3}{2} \bar{\bar{T}} J - 6 \theta J + U^{(3)} \right) \]

\[ + \frac{3 \partial \bar{\bar{T}} J + 3 \partial \bar{\bar{T}} J}{Z_{12}^2} - \bar{\theta}_{12} \left( 3 \partial \bar{\bar{T}} J + 3 \partial \bar{\bar{T}} J + \Psi^{(7/2)} \right) \]

\[ + \frac{\theta_{12} \bar{\bar{T}} J}{Z_{12}^2} \left( 2 \partial J + \partial \bar{\bar{T}} J \right) \left( \frac{1}{2} \partial U^{(3)} + \frac{1}{2} \bar{\bar{T}} J^{(7/2)} + \frac{1}{2} \bar{\bar{T}} J^{(7/2)} - \frac{1}{4} \partial \left[ \bar{\bar{T}} J \right] B^{(2)} \right) \]

\[ + \frac{\theta_{12} \bar{\bar{T}} J}{Z_{12}^2} \left( 2 \partial J + \partial \bar{\bar{T}} J \right) \left( \frac{1}{2} \partial U^{(3)} + \frac{1}{2} \bar{\bar{T}} J^{(7/2)} + \frac{1}{2} \bar{\bar{T}} J^{(7/2)} - \frac{1}{4} \partial \left[ \bar{\bar{T}} J \right] B^{(2)} \right) . \]

Here $B^{(2)}(Z)$, $\Psi^{(7/2)}(Z)$, $\bar{\bar{\Psi}}^{(7/2)}(Z)$, $U^{(3)}(Z)$ are the composite supercurrents of the spins 2, 7/2, 7/2, 3, respectively

\[ B^{(2)}(Z) = \frac{8}{c} J^2 \]

\[ \bar{\bar{\Psi}}^{(7/2)} = \frac{8}{c} \partial (JDJ) - \frac{72}{c} TDJ + \frac{36}{c} \left[ \bar{\bar{T}}, \bar{\bar{D}} \right] JDJ + \frac{8}{c} JDT - \frac{128}{c^2} J^2 DJ + \frac{4}{c} \partial J DJ \]
\[ \Psi^{(7/2)} = -\frac{8}{c} \partial (J \bar{D} J) - \frac{72}{c} T \bar{D} J + \frac{36}{c} [\bar{D}, D] J \bar{D} J + \frac{8}{c} J \bar{D} T - \frac{128}{c^2} J^2 \bar{D} J - \frac{4}{c} \partial J \bar{D} J \]

\[ U^{(3)} = \frac{56}{c} JT - \frac{32}{c} J [\bar{D}, D] J + \frac{128}{c^2} J^3 + \frac{120}{c} \bar{D} J \bar{D} J , \tag{A.4} \]

where

\[ \theta_{12} = \theta_1 - \theta_2 , \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2 , \quad Z_{12} = z_1 - z_2 + \frac{1}{2} (\theta_1 \bar{\theta}_2 - \theta_2 \bar{\theta}_1) \tag{A.5} \]

and other definitions have been given in Sec. 3.1.

Next we quote the OPE’s of the component currents defined by eq. (3.1). We use the following notation for the composite currents and their spinor derivatives

\[ B^{(2)}| = B^{(2)} , \quad DB^{(2)}| = \bar{V}^{(5/2)} , \quad \bar{D} B^{(2)}| = -V^{(5/2)} , \quad \frac{1}{2} [\bar{D}, D] B^{(2)}| = B^{(3)} , \]

\[ \Psi^{(7/2)}| = \Psi^{(7/2)} , \quad D \Psi^{(7/2)}| = \Lambda^{(4)} , \quad \bar{D} \Psi^{(7/2)}| = -\Lambda^{(4)} , \quad \frac{1}{2} [\bar{D}, D] \Psi^{(7/2)}| = \Psi^{(9/2)} , \]

\[ \Psi^{(7/2)}| = \Psi^{(7/2)} , \quad D \Psi^{(7/2)}| = \Delta^{(4)} , \quad \bar{D} \Psi^{(7/2)}| = -\Delta^{(4)} , \quad \frac{1}{2} [\bar{D}, D] \Psi^{(7/2)}| = \Psi^{(9/2)} , \]

\[ U^{(3)}| = U^{(3)} , \quad D U^{(3)}| = \Gamma^{(7/2)} , \quad \bar{D} U^{(3)}| = -\Gamma^{(7/2)} , \quad \frac{1}{2} [\bar{D}, D] U^{(3)}| = U^{(4)} . \tag{A.7} \]

Then the OPE’s implied for the basic component currents by the SOPE’s (A.1) - (A.3) are as follows

\[ T(z_1)T(z_2) = \frac{3c}{8z_{12}^4} + \left( \frac{2}{z_{12}^2} + \frac{\partial}{z_{12}} \right) T , \quad T(z_1)J(z_2) = \left( \frac{1}{z_{12}^2} + \frac{\partial}{z_{12}} \right) J , \]

\[ T(z_1)G(z_2) = \left( \frac{3}{2z_{12}^2} + \frac{\partial}{z_{12}} \right) G , \quad G(z_1)G(z_2) = -\frac{c}{4z_{12}^3} + \left( \frac{1}{z_{12}^2} + \frac{\partial}{z_{12}} \right) J - \frac{1}{z_{12}} T , \]

\[ G(z_1)G(z_2) = 0 , \quad J(z_1)G(z_2) = -\frac{1}{z_{12}} G , \]

\[ J(z_1)J(z_2) = \frac{c}{4z_{12}^2} , \tag{A.8} \]

\[ T(z_1)\bar{T}(z_2) = \left( \frac{2}{z_{12}^2} + \frac{\partial}{z_{12}} \right) \bar{T} , \quad T(z_1)W(z_2) = \left( \frac{3}{z_{12}^2} + \frac{\partial}{z_{12}} \right) W , \]

\[ T(z_1)U(z_2) = \left( \frac{5}{2z_{12}^2} + \frac{\partial}{z_{12}} \right) U , \quad J(z_1)\bar{T}(z_2) = 0 , \]

\[ J(z_1)U(z_2) = -\frac{1}{z_{12}} U , \quad J(z_1)W(z_2) = \frac{2}{z_{12}^2} \bar{T} , \]

\[ G(z_1)\bar{T}(z_2) = \frac{1}{z_{12}} U , \quad G(z_1)U(z_2) = 0 , \]

\[ G(z_1)\bar{U}(z_2) = -\frac{1}{z_{12}} W + \left( \frac{2}{z_{12}^2} + \frac{\partial}{2z_{12}} \right) \bar{T} , \quad G(z_1)W(z_2) = \left( \frac{5}{2z_{12}^2} + \frac{\partial}{2z_{12}} \right) U , \tag{A.9} \]
\( \tilde{T}(z_1)\tilde{T}(z_2) = -\frac{3c}{2z_{12}^4} + \left( \frac{2}{z_{12}^2} + \frac{\partial}{z_{12}} \right) \left( 5\tilde{T} - 4T + B^{(2)} \right), \)

\( U(z_1)\tilde{T}(z_2) = -\left( \frac{12}{z_{12}^4} + \frac{8\partial}{z_{12}^2} + \frac{3\partial^2}{z_{12}} \right) G + \left( \frac{5}{z_{12}^2} + \frac{3\partial}{z_{12}} \right) U + \frac{1}{z_{12}} \Psi^{(7/2)} + \frac{1}{z_{12}^2} V^{(5/2)}, \)

\( W(z_1)\tilde{T}(z_2) = -\left( \frac{12}{z_{12}^4} + \frac{12\partial}{z_{12}^2} + \frac{6\partial^2}{z_{12}} + \frac{2\partial^3}{z_{12}} \right) J + \left( \frac{3}{z_{12}^2} + \frac{2\partial}{z_{12}} \right) W + \left( \frac{1}{z_{12}} + \frac{\partial}{2z_{12}} \right) U^{(3)} \)
\[ + \frac{1}{2z_{12}} \left( \Lambda^{(4)} - \Lambda^{(4)} - \partial B^{(3)} \right), \)

\( U(z_1)U(z_2) = -\frac{1}{z_{12}} \Lambda^{(4)}, \)

\( \overline{U}(z_1)U(z_2) = \frac{3c}{z_{12}^5} + \left( \frac{12}{z_{12}^4} + \frac{6\partial}{z_{12}^2} + \frac{2\partial^2}{z_{12}} + \frac{\partial^3}{z_{12}} \right) J + \left( \frac{20}{z_{12}^3} + \frac{10\partial}{z_{12}^2} + \frac{3\partial^2}{z_{12}} \right) \left( T - \frac{1}{2}\tilde{T} \right) \)
\[ - \left( \frac{2}{z_{12}^2} + \frac{\partial}{z_{12}} \right) B^{(2)} + \left( \frac{2}{z_{12}^2} + \frac{\partial}{z_{12}} \right) \left( W + \frac{1}{2} B^{(3)} - \frac{1}{2} U^{(3)} \right) - \frac{1}{2z_{12}} \left( \Lambda^{(4)} + \Lambda^{(4)} \right), \)

\( W(z_1)U(z_2) = -\left( \frac{30}{z_{12}^5} + \frac{20\partial}{z_{12}^2} + \frac{15\partial^2}{z_{12}} + \frac{\partial^3}{z_{12}} \right) G + \left( \frac{5}{z_{12}^2} + \frac{3\partial}{z_{12}} + \frac{\partial^2}{z_{12}} \right) U \)
\[ + \left( \frac{1}{z_{12}^2} - \frac{\partial^2}{4z_{12}} \right) V^{(5/2)} + \left( \frac{1}{z_{12}^2} + \frac{\partial}{2z_{12}} \right) \Gamma^{(7/2)} + \frac{1}{2} \Psi^{(7/2)} - \frac{1}{2z_{12}} \Psi^{(9/2)}, \)

\( W(z_1)W(z_2) = -\frac{15c}{2z_{12}^6} + \left( \frac{3}{z_{12}} + \frac{3\partial}{z_{12}} - \frac{\partial^3}{8z_{12}} \right) B^{(2)} - \left( \frac{15}{z_{12}^2} + \frac{15\partial}{z_{12}} + \frac{9\partial^2}{4z_{12}} + \frac{2\partial^3}{z_{12}} \right) \left( 4T - \tilde{T} \right) \)
\[ + \left( \frac{1}{2z_{12}^2} + \frac{\partial}{4z_{12}} \right) \left( 2U^{(4)} + \Lambda^{(4)} + \Lambda^{(4)} \right). \]  

\textbf{Appendix B}

\textbf{Basic structure relations of }\textit{sW}^{\infty}_{3}\textbf{ }

Here we present the basic (anti)commutation relations of the superalgebra \textit{sW}^{\infty}_{3} in terms of generators.

The generators are defined in a standard way as Laurent modes of the currents

\[ J_{n}^{(s)} = \frac{1}{2\pi i} \int dxe^{s-1} J^{(s)}(x), \]  

where \( J^{(s)} \) is a current of the spin \( s \). Using this definition, from OPE’s (A.8) - (A.10) one obtains the following (anti)commutation relations

\[ [L_{n}, L_{m}] = (n-m)L_{n+m} + \frac{c}{16} \left( n^3 - n \right) \delta_{n+m,0} \]
\[ [L_{n}, G_{r}] = \left( \frac{n}{2} - r \right) G_{n+r} \]
\[ [L_{n}, J_{m}] = -m J_{n+m} \]
\[
\{G_r, \bar{G}_s\} = -L_{r+s} + \frac{r-s}{2} J_{r+s} - \frac{c}{8} \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0} \\
[J_n, G_r] = -G_{n+r} \\
[J_n, J_m] = \frac{nc}{4} \delta_{n+m,0} \\
\{G_r, G_s\} = \{G_r, G_s\} = 0 , \tag{B.2}
\]

\[
\begin{align*}
[L_n, \bar{L}_m] &= (n-m) \bar{L}_{n+m} \\
[L_n, U_r] &= \left( \frac{3n}{2} - r \right) U_{n+r} \\
[L_n, W_m] &= (2n-m) W_{n+m} \\
[G_r, \bar{L}_n] &= U_{r+n} \\
\{G_r, U_s\} &= 0 \\
\{G_r, \bar{U}_s\} &= -W_{r+s} + \frac{3r-s}{2} \bar{L}_{r+s} \\
[G_r, W_n] &= \left( 2r - \frac{n}{2} \right) U_{r+n} \\
[J_n, \bar{L}_m] &= 0 \\
[J_n, U_r] &= -U_{n+r} \\
[J_n, W_m] &= 2n \bar{L}_{n+m} , \tag{B.3}
\end{align*}
\]

\[
\begin{align*}
[\bar{L}_n, \bar{L}_m] &= (n-m) \left[ 5 \bar{L}_{n+m} - 4 \bar{L}_{n+m} + B^{(2)}_{n+m} \right] - \frac{c}{4} \left( n^3 - n \right) \delta_{n+m,0} \\
\{U_r, U_s\} &= -\bar{\Lambda}^{(4)}_{r+s} \\
\{\bar{U}_r, U_s\} &= \frac{r-s}{4} \left( 2r^2 + 2s^2 - 5 \right) J_{r+s} \left( 3r^2 + 3s^2 - 4rs - \frac{9}{2} \right) \left( L_{r+s} - \frac{1}{2} \bar{L}_{r+s} \right) \\
&+ \frac{r-s}{2} \left( 2W + B^{(3)} - U^{(3)} \right)_{r+s} \left( r + \frac{3}{2} \right) \left( s + \frac{3}{2} \right) B^{(2)}_{r+s} - \frac{1}{2} \left( \Lambda^{(4)} + \bar{\Lambda}^{(4)} \right)_{r+s} \\
&+ \frac{c}{8} \left( r^2 - \frac{1}{4} \right) \left( r^2 - \frac{9}{4} \right) \delta_{r+s,0} \\
[W_n, \bar{L}_m] &= (n-2m) W_{n+m} + 2m (m^2 - 1) J_{n+m} + \frac{n-m+1}{2} U^{(3)}_{n+m} \\
&+ \frac{n+m+3}{2} B^{(3)}_{n+m} + \frac{1}{2} \left( \Lambda^{(4)} - \bar{\Lambda}^{(4)} \right)_{n+m} \\
[U_r, \bar{L}_n] &= (2r - 3n) U_{r+n} - \left( r^2 + 3n^2 - 2rn - \frac{9}{4} \right) G_{n+r} + \Psi^{(7/2)}_{r+n} \left( r + \frac{3}{2} \right) V^{(5/2)}_{r+n} \\
[W_n, W_m] &= (n-m) \left( 4 - n^2 - m^2 + \frac{mn}{2} \right) \left( 2L_{n+m} - \frac{1}{2} \bar{L}_{n+m} \right) + \frac{n-m}{4} \left( 2U^{(4)} + \Lambda^{(4)} + \bar{\Lambda}^{(4)} \right) \\
&- \frac{(n-m) (14 + 9m + m^2 + 9n + 4mn + n^2) B^{(2)}_{n+m}}{8} \left( n^3 - n \right) \left( n^2 - 4 \right) \delta_{n+m,0} \\
[W_n, U_r] &= \left( r^2 + \frac{n^2}{2} - nr - \frac{5}{4} \right) U_{n+r} - \left( \frac{n^3}{2} - 2r^3 + \frac{3}{2} nr^2 - n^2 r - \frac{19}{8} n + \frac{9}{2} r \right) G_{n+r} - \frac{1}{2} \Psi_{n+r}^{(9/2)}
\end{align*}
\]
\[ + \frac{4n^2 - 4r^2 - 24r - 8nr - 19}{16} V_{(n+r)}^{(5/2)} + \frac{2n - 2r + 1}{4} \left( \frac{1}{2} \psi^{(7/2)} + \Gamma^{(7/2)} \right)_{n+r}. \] (B.4)

Any relation involving higher spin composite generators can be evaluated by making use of these basic relations and analogous ones involving the generators \( \bar{G} \) and \( \bar{U} \). These latter relations follow from those with \( G \) and \( U \) via the substitutions

\[
L \rightarrow L, \ J \rightarrow -J, \ W \rightarrow -W, \ \bar{L} \rightarrow \bar{L} \\
G \rightarrow \bar{G}, \ U \rightarrow -\bar{U}. \quad (B.5)
\]

Appendix C

The Cartan form coefficients \((l_0)_t, (\bar{l}_0)_t, (j_0)_t\) and \((w_0)_t\)

\[
\begin{align*}
(l_0)_t &= -8\bar{u}(2\phi + 3\chi\bar{\chi}) + 48v_2 + (4\bar{\mu} + 2\phi\bar{\chi} - 6u\bar{\chi} - 12\bar{u}\bar{\chi} + 6v\bar{\chi} + 6\nu)\xi \\
&\quad + (4\mu - 2\phi\chi - 6u\chi - 12\bar{u}\chi - 6v\chi - 6\nu)\bar{\xi} + 24v_1(u + \bar{u}) + 36(\chi\bar{\nu} + \bar{\chi}\nu) \\
&\quad + \alpha (-4\phi_2 - 2\bar{\mu}\chi + 2\mu\bar{\chi} + 4u\phi), \quad (C.1)
\end{align*}
\]

\[
\begin{align*}
(\bar{l}_0)_t &= -4\phi_2 - 4u\bar{\chi} + 4\mu\bar{\chi} + (2u + 10\bar{u})(2\phi + 3\chi\bar{\chi}) - 6v_1(u + \bar{u}) - 18(\chi\bar{\nu} + \bar{\chi}\nu) \\
&\quad - 12v_2 + 3(u + \bar{u})\xi\bar{\xi} - (6u\chi - 3v_1\chi + 6\nu)\xi - (6u\bar{\chi} + 3v_1\bar{\chi} - 6\nu)\bar{\xi} \\
&\quad - (2\mu + \phi\chi - 3u\chi - 6\bar{u}\chi + 3v\chi + 3\nu)\xi - (2\mu - \phi\chi - 3u\chi - 6u\chi - 3v_1\chi - 3\nu)\bar{\xi} \\
&\quad + \alpha \left( 4\bar{\mu}\phi - (\mu + \phi\chi)\xi - (\bar{\mu} - \phi\bar{\chi})\bar{\xi} \right), \quad (C.2)
\end{align*}
\]

\[
\begin{align*}
(j_0)_t &= 12\bar{u}_2 + 6(\chi\bar{\nu} - \bar{\chi}\nu) - \xi\bar{\xi}(3v_1 - 2\phi - 3\chi\bar{\chi}) \\
&\quad - (2\bar{\mu} + \phi\bar{\chi} - 3u\bar{\chi} - 6\bar{u}\bar{\chi} + 3v_1\bar{\chi} + 3\nu)\xi + (2\mu - \phi\chi - 3u\chi - 6\bar{u}\chi - 3v_1\chi - 3\nu)\bar{\xi} \\
&\quad + \alpha \left( 3(\phi^2 - u_2) + \mu\bar{\chi} + \bar{\mu}\chi + \phi\chi\bar{\chi} + \phi\xi\bar{\xi} \right), \quad (C.3)
\end{align*}
\]

\[
\begin{align*}
(w_0)_t &= -6u_2 - 6\bar{u}_2 - 4\mu\chi - 4u\bar{\chi} + 2\phi\chi\bar{\chi} + 6(u + \bar{u})^2 - \frac{27}{2} v_1^2 + 3v_1(2\phi + 3\chi\bar{\chi}) \\
&\quad + 6(\chi\bar{\nu} - \bar{\chi}\nu) + \frac{3}{2} \xi\bar{\xi}(v_1 - 2\phi - 3\chi\bar{\chi}) - (6u\bar{\chi} - 3v_1\chi + 6\nu)\xi + (6u\bar{\chi} + 3v_1\bar{\chi} - 6\nu)\bar{\xi} \\
&\quad + (2\bar{\mu} + \phi\bar{\chi} - 3u\bar{\chi} - 6\bar{u}\bar{\chi} + 3v_1\bar{\chi} + 3\nu)\xi - (2\mu - \phi\chi - 3u\chi - 6\bar{u}\chi - 3v_1\chi - 3\nu)\bar{\xi} \\
&\quad + \alpha \left( 6v_1\phi - \phi\xi\bar{\xi} + (\bar{\mu} - \phi\bar{\chi})\xi - (\mu + \phi\chi)\bar{\xi} \right). \quad (C.4)
\end{align*}
\]
The expressions for some higher spin coset superfields

\[ u_2 = \frac{1}{4} \left( \partial v_1 + 2 \bar{u} v_1 + 2 u v_1 - \chi \bar{\nu} - \bar{\chi} \nu + \bar{u} \xi \bar{\xi} + \frac{1}{2} (2 \mu - \phi \chi - \partial \chi - u \chi) \xi \right), \tag{C.5} \]

\[ v_2 = \frac{1}{3} (\mathcal{D} \mu + \overline{\mathcal{D}} \bar{\mu} + \phi^2), \tag{C.6} \]

\[ \phi_2 = \frac{1}{2} \partial \phi. \tag{C.7} \]

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