On linear $q$-ary completely regular codes with $\rho = 2$ and dual antipodal

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Abstract

We characterize all linear $q$-ary completely regular codes with covering radius $\rho = 2$
when the dual codes are antipodal. These completely regular codes are extensions of

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linear completely regular codes with covering radius 1, which are all classified. For \( \rho = 2 \), we give a list of all such codes known to us. This also gives the characterization of two weight linear antipodal codes.

1 Introduction and Terminology

Let \( \mathbb{F}_q = GF(q) \) be the Galois field with \( q \) elements, where \( q \) is a prime power. \( \mathbb{F}_q^n \) is the \( n \)-dimensional vector space over \( \mathbb{F}_q \). Let \( wt(v) \) be the Hamming weight of a vector \( v \in \mathbb{F}_q \), and \( d(v, u) = wt(v - u) \) denotes the Hamming distance between two vectors \( v, u \in \mathbb{F}_q^n \). Say that vectors \( v \) and \( u \) are neighbors if \( d(v, u) = 1 \).

A \( q \)-ary code \( C \) of length \( n \) is a subset of \( \mathbb{F}_q^n \). If \( C \) is a \( k \)-dimensional linear subspace of \( \mathbb{F}_q^n \), then \( C \) is a \( q \)-ary linear code, denoted by \( [n, k, d]_q \), where \( d \) is the minimum distance between any pair of codewords, assuming \( k \geq 1 \). If any pair of distinct codewords are at distance \( d \), then the code \( C \) is equidistant. A linear code \( C \) will be called antipodal if it contains a vector of weight equal to the length of the code. Say that a linear \( [n, k, d]_q \) code is trivial, if \( k \leq 1 \) or \( k \geq n - 1 \).

For a \( q \)-ary code \( C \) with minimum distance \( d \) denote by \( e = \lfloor (d-1)/2 \rfloor \) its packing radius. Given any vector \( v \in \mathbb{F}_q^n \), its distance to the code \( C \) is

\[
d(v, C) = \min_{x \in C} \{d(v, x)\}
\]

and the covering radius of the code \( C \) is

\[
\rho = \max_{v \in \mathbb{F}_q^n} \{d(v, C)\}.
\]

Clearly \( e \leq \rho \) and \( C \) is perfect if and only if \( e = \rho \). It is well known that the only nontrivial linear perfect codes for \( e = \rho = 1 \) are the Hamming codes, which have length \( n = (q^m - 1)/(q - 1) \), dimension \( k = n - m \) and minimum distance 3. We denote by \( H_m \) a parity check matrix of a Hamming code, that is, any vector \( v \) is a codeword if and only if \( H_m \cdot v^t = 0^t \), where \( 0^t \) is the all-zero column vector.
For any \( x \in \mathbb{F}_q^n \), let \( D = C + x \) be a coset of \( C \). A leader of \( D \) is a minimum weight vector in \( D \).

For a given \( q \)-ary code \( C \) with covering radius \( \rho = \rho(C) \) define
\[
C(i) = \{ x \in \mathbb{F}_q^n : d(x, C) = i \}, \quad i = 1, 2, ..., \rho.
\]

**Definition 1.1** A code \( C \) is completely regular, if for all \( l \geq 0 \) every vector \( x \in C(l) \) has the same number \( c_l \) of neighbors in \( C(l - 1) \) and the same number \( b_l \) of neighbors in \( C(l + 1) \). Also, define \( a_l = (q - 1)n - b_l - c_l \) and note that \( c_0 = b_\rho = 0 \). The intersection array of \( C \) is \((b_0, \ldots, b_{\rho - 1}; c_1, \ldots, c_\rho)\).

A linear automorphism of \( \mathbb{F}_q^n \) is a coordinate permutation together with a product by a nonzero scalar value at each position. Such an automorphism \( \sigma \) can be represented by a \( n \times n \) monomial matrix \( M \) such that \( xM = \sigma(x) \), for all \( x \in \mathbb{F}_q^n \). Two codes, \( C \) and \( C' \), are equivalent if there is a linear automorphism of \( \mathbb{F}_q^n \), say \( \sigma \), such that \( C' = \sigma(C) \). From now on, if \( C \subseteq \mathbb{F}_q^n \) is a linear code, the full automorphism group of \( C \), denoted \( \text{Aut}(C) \), is the group of linear automorphisms of \( \mathbb{F}_q^n \) that leaves \( C \) invariant. We say that \( \text{Aut}(C) \) is transitive if all one weight vectors in \( \mathbb{F}_q^n \) are in the same orbit. For a linear code \( C \), the group \( \text{Aut}(C) \) acts on the set of cosets of \( C \) in the following way: for all \( \phi \in \text{Aut}(C) \) and for every vector \( v \in \mathbb{F}_q^n \) we have \( \phi(v + C) = \phi(v) + C \).

**Definition 1.2** ([30, 18]) Let \( C \) be a linear \( q \)-ary code with covering radius \( \rho \). Then \( C \) is completely transitive if \( \text{Aut}(C) \) has \( \rho + 1 \) orbits when acts on the cosets of \( C \).

Since two cosets in the same orbit should have the same weight distribution, it is clear that any completely transitive code is completely regular.

**Lemma 1.3** ([30]) Let \( C \) be a \([n, k, d]_q \) code with covering radius \( \rho = 1 \). If \( \text{Aut}(C) \) is transitive, then \( C \) is completely transitive.
Proof. Obvious, since all cosets of $C$, different of $C$, have leaders of weight 1. Thus, all such cosets are in the same orbit. □

The next known result follows from Singer theorem [29].

Lemma 1.4 Let $\mathcal{H}$ be a $[n,k,3]_q$ Hamming code. Then, $\text{Aut}(\mathcal{H})$ is transitive. Hence, any such code is completely transitive.

It has been conjectured [23] for a long time that if $C$ is a completely regular code and $|C| > 2$, then $e \leq 3$. For the case of binary linear completely transitive codes, the problem of existence is solved: it is proven in [5, 6] that for $e \geq 4$ such nontrivial codes do not exist. The conjecture is also proven for the case of perfect codes ($e = \rho$) [31, 33] and quasi-perfect ($e + 1 = \rho$) uniformly packed codes [28, 32] (defined and studied also in [1, 13, 19]).

When $e \leq 3$, there are many well known completely regular codes and, recently, we have presented new constructions of binary and non-binary completely regular codes [7, 24, 25]. However, there does not exist a general classification of completely regular codes with $e \leq 3$. Since $d \in \{2e + 1, 2e + 2\}$, $e \leq \rho$ and any perfect code has odd $d$, we have that the minimum distance of a completely regular code with $\rho = 1$ is $d \leq 3$ and for the codes with $\rho = 2$ is $d \leq 5$. In this paper we classify all linear $q$-ary completely regular codes with $\rho = 1$ and we also characterize the structure of linear completely regular codes with $\rho = 2$, whose dual is antipodal. We also list all such codes we know.

After submission of [8] we found [20], where a large class of so called arithmetic completely regular codes has been classified. In particular, in [20], it also appears the classification of all linear $q$-ary completely regular codes with $\rho = 1$. The approach in [20] is based on known results on classification of distance regular graphs in Hamming schemes. Our approach here is self contained and based only on classical results on perfect and uniformly packed codes. Both approaches are interesting from the point of view of classification of completely regular codes with small covering radius $\rho$, in particular, for enumeration of all completely regular codes with small parameters $qn \leq 48$, suggested by Neumaier [23]. We emphasize that,
unlike the linear case, the classification of nonlinear completely regular codes with $\rho = 1$ as well as of nonlinear equidistant codes are hard problems (see, for example, [16, 17] for the first object and [3], and references there, for the second).

The paper is organized as follows. In Section 2 we construct for any prime power $q$ and any integers $n$ and $k$, such that $n \geq q + 1$ and $2 \leq k \leq n - 2$, all nontrivial, nonequivalent, linear $q$-ary completely regular $[n, k, d]_q$ codes with covering radius $\rho = 1$. This also gives the construction of all the linear $q$-ary equidistant codes. We prove that the constructed codes are the only linear $q$-ary completely regular codes with $\rho = 1$ and show that all such completely regular codes are completely transitive. In Section 3 we consider linear $q$-ary nontrivial completely regular $[n, k, d]_q$ codes with covering radius $\rho = 2$, whose dual is antipodal. We give necessary and sufficient conditions for the shape of the parity check matrices of such codes known to us and we give their intersection arrays. Also, we point out some remarkable properties (e.g. self-duality) and equivalence to the existence of some two weight codes.

2 Classification of linear $q$-ary completely regular codes with $\rho = 1$

We consider nontrivial linear $[n, k, d]_q$ codes, i.e. $k$ is in the region $2 \leq k \leq n - 2$.

Definition 2.1 Let $C$ be a $q$-ary code of length $n$ and let $\rho$ be its covering radius. We say that $C$ is uniformly packed in the wide sense, i.e. in the sense of [1], if there exist rational numbers $\beta_0, \ldots, \beta_{\rho}$ such that for any $v \in \mathbb{F}_q^n$

$$\sum_{k=0}^{\rho} \beta_k \alpha_k(v) = 1,$$

where $\alpha_k(v)$ is the number of codewords at distance $k$ from $v$.

Note that the case $\rho = e + 1$ and $\beta_{\rho - 1} = \beta_{\rho}$ corresponds to strongly uniformly packed codes [28] and the case $\rho = e + 1$ corresponds to uniformly packed codes [19].
For a \([n, k, d]_q\) code \(C\) let \((\eta_0^\perp, \ldots, \eta_n^\perp)\) be the weight distribution of its dual code \(C^\perp\), assume \((\eta_0^\perp, \ldots, \eta_n^\perp)\) has \(s = s(C)\) nonzero components \(\eta_i^\perp\) for \(1 \leq i \leq n\). Following to Delsarte [15], we call \(s\) the external distance of \(C\).

\textbf{Lemma 2.2} Let \(C\) be a code with minimum distance \(d\), packing radius \(e = \lfloor \frac{d-1}{2} \rfloor\), covering radius \(\rho\) and external distance \(s\). Then:

(i) [15] \(\rho \leq s\).

(ii) \([2]\) \(\rho = s\) if and only if \(C\) is uniformly packed in the wide sense.

(iii) \([9]\) If \(C\) is completely regular, it is uniformly packed in the wide sense.

(iv) \([19, 28]\) If \(C\) is uniformly packed in the wide sense and \(\rho = e + 1\), then it is completely regular.

The next fact follows from Lemma 2.2. Earlier, it was mentioned for \(q = 2\) in [21] and for \(q \geq 2\) (for example, in [27]).

\textbf{Lemma 2.3} Let \(C\) be a nontrivial linear \(q\)-ary code with \(d = 3\) and let \(C^\perp\) be its dual. Then \(C\) is a Hamming code, if and only if \(C^\perp\) is equidistant.

The next statement is a simple generalization of Lemma 2.3 to the case of arbitrary completely regular codes with \(\rho = 1\).

\textbf{Lemma 2.4} Let \(C\) be a nontrivial linear \(q\)-ary code and \(C^\perp\) be its dual. Then \(C\) is completely regular with \(\rho = 1\) if and only if \(C^\perp\) is equidistant.

So, to classify all linear completely regular codes with \(\rho = 1\) we have to classify all linear equidistant codes. It has been done in [4] when a code does not contain trivial (zero) positions, but this is not enough for completely regular codes with \(\rho = 1\).

\textbf{Definition 2.5} (Construction I(u)). Let \(C\) be a \([n, k, d]_q\) code with a parity check matrix \(H\). Define a new code \(C^+_u\) with parameters \([n + u, k + u, 1]_q\) as the code with parity check matrix \(H^+_u\), obtained by adding \(u > 0\) zero columns to \(H\).
The following statement follows directly from the definition of $C^{+1}$.

**Lemma 2.6** Let $C$ be a $[n, k, d]_q$ code and let $C^{+1}$ be obtained from $C$ by Construction I(1). Let $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ be at distance $i$ apart from $\alpha_i$ codewords in $C$ and at distance $i - 1$ apart from $\alpha_{i-1}$ codewords in $C$. Then, for any $x_{n+1} \in \mathbb{F}_q$ the vector $x' = (x_1, \ldots, x_n \mid x_{n+1})$ is at distance $i$ apart from exactly $\alpha_i + (q-1)\alpha_{i-1}$ codewords in $C^{+1}$.

**Proposition 2.7** Codes $C$ and $C^+u$ have the same covering radius and, moreover, $C$ is completely regular if and only if $C^{+u}$ is completely regular. In this case, both codes have the same intersection numbers, i.e.

$$a'_i = a_i + (q-1)u, \quad b'_i = b_i, \quad c'_i = c_i, \quad i = 0, 1, \ldots, \rho.$$

**Proof.** It is enough to consider the case $u = 1$. For any vector $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ denote by $x' = (x_1, \ldots, x_n \mid x_{n+1})$ the corresponding $q$ vectors from $\mathbb{F}_q^{n+1}$. Let $y = (y_1, \ldots, y_n) \in C$ be a codeword at distance $\rho$ from $x$. Then $y' = (y_1, \ldots, y_n \mid x_{n+1})$ is a codeword in $C^{+1}$ at the same distance $\rho$ from $x'$. Therefore, $C$ and $C^{+1}$ have the same covering radius $\rho$.

Assume $C$ is completely regular. For any vector $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ at distance $t \leq \rho$ from $C$, denote by $\alpha_{i,t}$ the number of codewords in $C$ at distance $i$ from $x$ ($0 \leq i \leq n$). As $C$ is completely regular, $\alpha_{i,t}$ does not depend on $x$, but just on $t$ and $i$. Take a vector $x' = (x_1, \ldots, x_n \mid x_{n+1}) \in \mathbb{F}_q^{n+1}$, which is at distance $t$ from $C^{+1}$. It is easy to see that the number of codewords in $C^{+1}$ at distance $i$, say $\alpha'_{i,t}$, depends only on $t$ and $i$. Indeed, by Lemma 2.6 we have $\alpha'_{i,t} = \alpha_{i,t} + (q-1)\alpha_{i-1,t}$, for all $i = 0, \ldots, n$, and $\alpha'_{n+1,t} = (q-1)\alpha_{n,t}$.

Conversely, assume that $C$ is not completely regular. Let $x, y \in \mathbb{F}_q^n$ be such that $d(x, C) = d(y, C) = t > 0$ and, for $0 \leq i \leq n$, let $\alpha_{i,t}(x)$ (respectively, $\alpha_{i,t}(y)$) denote the number of codewords at distance $i$ from $x$ (respectively, from $y$). Since $C$ is not completely regular, we can select $x$ and $y$ such that $\alpha_{i,t}(x) \neq \alpha_{i,t}(y)$ for some $i \geq t$. Let $i$ be the minimum possible of such values, that is $\alpha_{i-1,t}(x) = \alpha_{i-1,t}(y)$. Then, by Lemma 2.6 for the corresponding
vectors \( \mathbf{x}' \) and \( \mathbf{y}' \), we have \( \alpha_{i,t}'(\mathbf{x}') \neq \alpha_{i,t}'(\mathbf{y}') \). Consequently, \( C^{+1} \) is not completely regular.

\[ \square \]

As a summary, starting with any completely regular code, we obtain an infinite family of completely regular codes with the same covering radius.

**Definition 2.8** (Construction \( II(\ell) \)). Let \( C \) be a \([n, k, d]_q \) code with parity check matrix \( H \). Let \( C \times \ell \) be the code with parameters \([n \ell, k + (\ell + 1)n, 2]_q \), whose parity check matrix, denoted \( H^{\times \ell} \), is \( \ell \) times the repetition of \( H \) (or nonzero multiples of \( H \)), i.e.

\[
H^{\times \ell} = [H^{(1)} | H^{(2)} | \ldots | H^{(\ell)}],
\]

where \( H^{(i)} \) is scalar nonzero multiple of \( H \), for all \( i = 1, \ldots, \ell \).

**Proposition 2.9** A \([n, k, d]_q \) code \( C \) is completely regular with covering radius \( \rho = 1 \) if and only if \( C \times \ell \) is completely regular with covering radius \( \rho' = 1 \).

**Proof.** If \( C \) is completely regular with \( \rho = 1 \), its parity check matrix \( H \) is the generator matrix of an equidistant code since, by Lemma 2.2, the external distance \( s \) equals to \( \rho = 1 \). The matrix \( H^{\times \ell} \) generates such a code too. Hence, \( C^{\times \ell} \) has an external distance \( s = 1 \) and, by Lemma 2.2, the covering radius \( \rho' = 1 \). We deduce, again by Lemma 2.2, that \( C^{\times \ell} \) is completely regular. The converse statement follows by using the same arguments, if we take into account the shape of the matrix \( H^{\times \ell} \). \( \square \)

Finally, we summarize the main results of this section.

**Theorem 2.10** Let \( C \) be a nontrivial \([n, k, d]_q \) code with covering radius \( \rho = 1 \). Then, \( C \) is completely regular if and only if its parity check matrix is of the form

\[
H = ((H_m^{\times \ell})^+)^u,
\]

(up to column permutations), where \( H_m \) is the parity check matrix of a Hamming code of length \( n_m = (q^m - 1)/(q - 1) \). The length and dimension of \( C \) are \( n = n_m \ell + u \) and \( k = n - m \), respectively.

Furthermore
(i) \( d = 3 \), if and only if \( u = 0, \ell = 1, n = n_m \) and \( C \) is a Hamming code.

(ii) \( d = 2 \), if and only if \( u = 0, \ell \geq 2, n = n_m \ell \).

(iii) \( d = 1 \), if and only if \( u > 0, \ell \geq 1 \).

(iv) The code \( C \) has the intersection numbers:

\[
a_0 = (q - 1)u, \quad b_0 = (q - 1)\ell n_{n-k}, \quad c_1 = \ell, \quad a_1 = (\ell n_{n-k} + u)(q - 1) - \ell.
\]

(v) The code \( C \) is completely transitive.

**Proof.** The “if part” is clear combining Constructions \( I(u) \) and \( II(\ell) \).

For the “only if part”, since \( \rho = 1 \) we deduce that \( d \in \{1, 2, 3\} \).

We separate these three cases:

(i) If \( d = 1 \), then \( H \) has zero columns. Thus, using Construction \( I(u) \), \( C \) can be obtained from a completely regular code with minimum distance greater than 1 and covering radius 1.

(ii) If \( d = 2 \), since \( 2 \leq k \leq n - 2 \) and \( \rho = 1 \), the matrix \( H \) generates the equidistant \([n, n-k, d^\perp]_q\) code \( C^\perp \). As \( d = 2 \) this matrix \( H \) contains repeated columns and does not contain zero columns.

First, we prove that every column of \( H \) occurs the same number, say \( \ell \) times, where \( \ell \geq 2 \) (counting includes, of course, the columns, obtained multiplying by scalar elements from \( \mathbb{F}_q \)). Assume that each column \( h \) occurs \( \ell_h \) times. Since \( C \) is completely regular the intersection number \( c_1 \) is the same for any vector \( x \in C(1) \). Take such a vector of weight 1 with its nonzero position at the column \( h \). Then \( c_1(h) \) is equal to \( \ell_h + 1 \) (we take into account the zero codeword). We conclude that the number \( \ell_h \) should be the same for every column in the matrix \( H \), i.e. \( \ell_h = \ell \).

The last equality implies that \( n \) should be divisible by \( \ell \). Denote \( n' = n/\ell \). Present the matrix \( H \) in the form

\[
H = [H' \ldots | H'],
\]
where all columns in $H'$ are different.

Now, we claim that $n' = n_m = (q^m - 1)/(q - 1)$ for some $m \geq 2$, i.e. the matrix $H'$ is the parity check matrix of a Hamming $[n_m, n_m - m, 3]_q$ code and so, the matrix $H$ generates an equidistant code. Since each row of $H$ is $\ell$ times the repetition of the same vector, the matrix $H'$ also generates an equidistant code, say $E$ of length $n'$. Since the dual code of $E$ has minimum distance $d \geq 3$, covering radius $\rho = 1$ and external distance $s = 1$ (Lemma 2.2), it is a perfect code of the length $n_m = (q^m - 1)/(q - 1)$ for some $m \geq 2$ (Lemma 2.3). If $m = 1$, then $H$ consists of only one row, implying that $d' = 2$.

Finally, we conclude that $k = n - m$ and $C$ is obtained by Construction $II(\ell)$ from the perfect (Hamming) $[n_m, n_m - m, 3]_q$ code.

(iii) If $d = 3$, since $\rho = 1$, $C$ is a perfect code by definition.

(iv) It is straightforward to find the intersection numbers.

(v) If $d \in \{2, 3\}$, using Lemma 1.4, $C$ is equivalent to a code $C'$ such that $\text{Aut}(C')$ is transitive. Thus, $\text{Aut}(C)$ is transitive and, by Lemma 1.3, $C$ is completely transitive.

If $d = 1$, then let $D$ be the ‘reduced’ code, that is, the code obtained from $C$ by doing the reverse operation of Construction $I(u)$. Since both, the covering radius of $C$ and $D$ are 1, we have that $C \neq \mathbb{F}_q^n$ and, by Proposition 2.7, $D$ is a completely regular code with $d > 1$. Hence, $D$ is a completely transitive code. This means that we can choose a set of $q^{n-k} - 1$ coset leaders of weight one such that they are in the same orbit of $\text{Aut}(D)$. But $C$ and $D$ have the same number of cosets and we can choose, in both $C$ and $D$, the same coset leaders. Since $\text{Aut}(D) \subseteq \text{Aut}(C)$, we have that these coset leaders are in the same orbit. Therefore, all the cosets different of $C$ are in the same orbit and $C$ is a completely transitive code.

The next statement follows directly from Theorem 2.10 and it has been obtained in [4], for the case of codes without trivial (zero) positions.

**Corollary 2.11** Given an equidistant $[n, k, d]_q$ code we have that $n = n_{n-k} \ell + u$ for some $\ell \geq 1$ and some $u \geq 0$. Furthermore, a generator matrix is obtained from $H_{n-k}$ (a parity check matrix of the Hamming code) by repeating this matrix $\ell$ times and then adding $u$ trivial
3 Linear $q$-ary completely regular codes with $\rho = 2$

In this section we deal with linear $q$-ary nontrivial completely regular $[n, k, d]_q$ codes with covering radius $\rho = 2$, whose dual is antipodal and we show a characterization of these codes using the ones with covering radius one.

Now, we recall some facts on extension of codes. For a $q$-ary code $C$ of length $n$, the extended code $C^*$ of length $n + 1$ is obtained by adding an overall parity check position. This means that for any codeword $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in C^*$, we have $\sum_{i=1}^{n+1} x_i = 0$ (where the sum is in $\mathbb{F}_q$). We say that such an extension works if $d^* = d + 1$, where $d^*$ is the minimum distance in $C^*$ and $d$ is the minimum distance in $C$. Generally speaking, extension of equivalent codes can result in different codes, but if an extension works for two equivalent codes, then the resulting codes would have the same parameters. The following result is well known and can be found, for example, in [22].

**Lemma 3.1** Let $\mathcal{H}_m$ be a $[n_m, k, 3]_q$ Hamming code. Then, the extended code $\mathcal{H}_m^*$ has minimum distance 4 if

(i) $q = 2$ and $m \geq 2$, or

(ii) $q = 2^r \geq 4$ and $m = 2$, i.e. $n_m + 1 = q + 2$ and $k = q - 1$.

Denote by $H_m^*$ the parity check matrix of the $[n_m + 1, k, 4]$ code, obtained as the extended Hamming code $\mathcal{H}_m$ for a case when extension works. In this case, denote such code by $\mathcal{H}_m^*$. Denote by $D_m$ the matrix of size $(m+1) \times q^m$ whose columns are all the $q^m$ vectors of length $m \geq 1$ with an extra $(m+1)$th position equal to 1; (this matrix, without the $(m+1)$th row, generates a difference matrix [12]; this is why we denote it by $D_m$). We remark that this matrix $D_m$ can also be obtained by repeating the $q - 1$ different multiples of the matrix $H_m$ and by adding, first, a zero column and, finally, the all-one row of length $q^m$. 

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We recall a result in [14]. For a given \([n, k, d]_q\) code \(C\) with parity check matrix \(H\) define its complementary \([n_{n-k} - n, k, \bar{d}]\) code \(\bar{C}\), whose parity check matrix \(\bar{H}\) is obtained from the matrix \(H_{n-k}\) by removing all the columns of \(H\) and multiples of them. Recall an important property of complementary codes: to any codeword of weight \(w\) in a \([n, k, d]_q\) code \(C\) corresponds a codeword of weight \(\bar{w} = q^{n-k-1} - w\) in the complementary code \(\bar{C}\). As a corollary of this fact above we have the next lemma.

**Lemma 3.2** [14] A linear projective \([n, k, d]_q\) code \(C\) with covering radius \(\rho = 2\), which is not a difference-matrix code, does exist simultaneously with its complementary projective code \(\bar{C}\) with the same covering radius \(\bar{\rho} = 2\).

**Theorem 3.3** Let we have a nontrivial \([n, k, d]_q\) code \(C\). Let \(H\) be its parity check matrix. Then, \(C\) is completely regular with covering radius \(\rho = 2\) and the dual code \(C^\perp\) is antipodal if and only if the matrix \(H\) looks, up to equivalence, as follows:

\[
H = \begin{bmatrix}
1 & \cdots & 1 \\
M
\end{bmatrix},
\]

where \(M\) generates an equidistant code \(E\) with the following property: for any nonzero codeword \(v \in E\), every symbol \(\alpha \in \mathbb{F}_q\), which occurs in a coordinate position of \(v\), occurs in this codeword exactly \(n - \tilde{d}\) times, where \(\tilde{d}\) is the minimum distance of \(E\). Moreover, up to equivalence, \(C\) is the extension of a completely regular code \(C'\) with covering radius \(\rho' = 1\).

**Proof.** The code \(C^\perp\) is antipodal if and only if \(H\) contains the all-one row, up to equivalence. Now, by Lemma [2.2] \(C\) is completely regular with \(\rho = 2\) if and only if the external distance is \(s = 2\). Equivalently, \(H\) generates a code, the dual code \(C^\perp\), with two different weights and \(M\) generates an equidistant code \(E\) such that every symbol \(\alpha \in \mathbb{F}_q\), which occurs in a coordinate position of a codeword \(v \in E\), occurs in this codeword exactly \(n - \tilde{d}\) times, where \(\tilde{d}\) is the minimum distance of \(E\). Notice that if \(M\) has this property, we can add any multiple of the all-one row to any row of \(M\) and we do not change this property.
Also, up to equivalence, we can rewrite $H$ in the following form:

\[
\begin{bmatrix}
1 & \cdots & 1 & 1 \\
H' & 0
\end{bmatrix},
\]

where $0$ is the zero column of length $n - k - 1$. Hence, $H'$ generates words of only one weight. This means that $H'$ is a parity check matrix for a $[n - 1, k, d']_q$ code $C'$ (which is obtained by puncturing the last coordinate of $C$) with external distance $s' = 1$ and, by Lemma 2.2, covering radius $\rho' = 1$. Therefore, $C'$ is a completely regular code and it must be one of the cases of Theorem 2.10.

We conclude that any nontrivial completely regular code with covering radius 2, whose dual code is antipodal, is obtained from some code with covering radius 1 by adding the overall parity checking position. \qed

The following statement is a direct corollary of Theorem 3.3.

**Corollary 3.4** Let we have a nontrivial two-weight $[n, k, d]_q$ code $C$ with weights $w_1$ and $w_2 = d$ and with generator matrix $G$. If $w_1 = n$, then

\[
G = \begin{bmatrix}
1 & \cdots & 1 & 1 \\
M & 0
\end{bmatrix},
\]

where $M$ generates an equidistant $[n - 1, k - 1, d]_q$ code $E$ with the following property: for every codeword $v \in E$, every symbol $\alpha \in \mathbb{F}_q$ which occurs in $v$, occurs in $v$ exactly $n - d$ times.

Now, we enumerate the completely regular codes with $\rho = 2$, whose dual is antipodal, that we know. We also compute the intersection array for all the enumerated codes. Some of these codes were mentioned in [19]. Dual of these codes are two-weight antipodal codes mostly due to Delsarte [14], studied by many other authors (see a nice survey of two-weight codes in [11]). We do not know if the list is exhaustive but any other such code, according to Theorem 3.3, would also be an extended completely regular code with $\rho = 1$. 13
Proposition 3.5 The following codes are completely regular with covering radius $\rho = 2$ and their dual codes are antipodal.

(i) The binary extended perfect $[n, k, 4]_2$ code $H^*_m$ of length $n = 2^m$, where $k = n - m - 1$ and $m \geq 2$. Its intersection array is

$$(n, n - 1; 1, n).$$

(ii) The extended perfect $[n, k, 4]_q$ code $H^*_m$ of length $n = q + 2$ with $k = q - 1$, where $q = 2^r \geq 4$, and $m = 2 \quad [10, 14]$ (the family $TF1$ in [11]). Its intersection array is

$$( (q + 2)(q - 1), q^2 - 1; 1, q + 2 ).$$

(iii) The difference-matrix $[n, k, 3]_q$ code of length $n = q^m$, dimension $k = n - (m + 1)$ with parity check matrix $D_m$, where $m \geq 1$, and $q \geq 3$ is any prime power (the dual code generated by the matrix $D_m$ has been given in [27]). The complementary code of this code is the Hamming code $H_m$ and its intersection array is

$$(n(q - 1), n - 1; 1, n(q - 1)).$$

(iv) Latin-square $[n, n - 2, 3]_q$ code of length $n$, with parity check matrix $H$, obtained from $D_1$ by deleting any $q - n$ columns, where $3 \leq n \leq q$ and $q \geq 3$ is any prime power [14]. Its intersection array is

$$(n(q - 1), (q - n + 1)(n - 1); 1, n(n - 1)).$$

(v) A $[n = q(q - 1)/2, k = n - 3, 4]_q$ code for $q = 2^r \geq 4 \quad [14]$ (the complementary code belongs to $TF1^d$, i.e. it is the projective dual code to the code (ii) (family $TF1$ in [11])). Its intersection array is

$$( (q - 1)n, (q - 2)(q + 1)(q + 2)/4; 1, q(q - 1)(q - 2)/4 ).$$
(vi) A \([n = 1 + (q + 1)(h - 1), k = n - 3,4]_q \) code, where \(1 < h < q\) and \(h\) divides \(q\), for \(q = 2^r \geq 4\) (the family \(TF2\) in \([11]\)). Its intersection array is
\[
\left( (q - 1)n, (q + 1)(h - 1)(q - h + 1); 1, (h - 1)n \right).
\]

(vii) A \([n = q(q - h + 1)/h, k = n - 3,4]_q \) code, where \(1 < h < q\) and \(h\) divides \(q\), for \(q = 2^r \geq 4\) (the complementary belongs to the family \(TF2^d\) \([11]\)). Its intersection array is
\[
\left( (q - 1)n, (q + 1)(q - h)(q(h - 1) + h)/h^2; 1, q(q - h)(q - h + 1)/h^2 \right).
\]

Cases (i) and (ii) correspond to extended codes of case (i) in Theorem \([2.10]\). Case (iii) corresponds to an extended code \(C\) of case (ii) in Theorem \([2.10]\) where \(\ell = q - 1\) and the parity check matrix \(H\) of \(C\) is
\[
H = [H^{(1)}_m | \ldots | H^{(q-1)}_m],
\]
where each \(H^{(i)}_m\) is a different scalar multiple of \(H_m\). Case (iv) corresponds to an extended trivial completely regular code of co-dimension 1 and covering radius 1. Finally, cases (v) - (vii) correspond to extension of several (multiple) copies of completely regular codes of co-dimension 2 and covering radius 1.

Notice, we can apply Construction \(II(\ell)\) to any of the codes above, obtaining a completely regular code with \(\rho = 2\) and \(d = 2\). Also, we can apply Construction \(I(u)\) to anyone of these codes, including those obtained by Construction \(II(\ell)\), obtaining a code with \(\rho = 2\) and \(d = 1\).

To finish this section it is proper to emphasize one interesting class of codes, which belong to the family (iv).

Let \(C\) be a \(q\)-ary linear code of length \(n\) and parity check matrix \(H\). For any integer \(r \geq 1\), we define the lifted code \([26]\) \(C' \subset \mathbb{F}^n_q\) as the linear code which has parity check matrix \(H\). Two nice properties of lifted codes are the following.
Lemma 3.6 Let $C$ be a $[n, k, d]_q$ code and let $C' \subset \mathbb{F}_q^n$ be any lifted code. Then, $C$ is self-dual ($C = C^\perp$) if and only if $C'$ is self-dual.

Proof. Note that $C$ is self-dual if and only if $n = 2k$ and the rows of its parity check matrix $H$ are orthogonal. But $n$, $k$ and $H$ do not change for $C'$ and, since $\mathbb{F}_q^n$ is a subspace of $\mathbb{F}_{q^r}$, the rows of $H$ are orthogonal vectors in $\mathbb{F}_{q^r}$ if and only if they are orthogonal in $\mathbb{F}_q$. \qed

Proposition 3.7 Let $C' \subset \mathbb{F}_q^n$ be a lifted code from a Hamming $[n, n-m, 3]_q$ perfect code $\mathcal{H}_m$. Then $C'$ is self-dual if and only if $\mathcal{H}_m$ is a ternary $[4, 2, 3]_3$ Hamming code.

Proof. If $C'$ is self-dual, then $|C'| = |(C')^\perp|$ implying that $n - m = m$, i.e. $n = 2m$. But the length of a $q$-ary Hamming code is

$$n = \frac{q^m - 1}{q-1} = \sum_{i=1}^{m} q^{m-i}.$$ 

Hence, we immediately obtain that $m = 2$, $q = 3$ and $n = 4$. A parity check matrix for a Hamming $[4, 2, 3]_3$ $\mathcal{H}_2$ code is

$$H_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$ 

Since the rows of $H_2$ are orthogonal, $\mathcal{H}_2$ is a self-dual code. Finally, by Lemma 3.6, if $C'$ is a lifted code from $\mathcal{H}_2$, then $C'$ is self-dual. \qed

Notice that the lifted perfect $[q+1, q-1, 3]_{q^r}$ codes (see [26]), with parity check matrix $H_2$ over $\mathbb{F}_q$ are particular cases of the family (iv) in Proposition 3.5 for $r > 1$. So, these codes are completely regular with intersection array

$$((q+1)(q^r-1), q^2(q^{r-1} - 1); 1, q(q+1)).$$

According to Proposition 3.7, the case $q = 3$ corresponds to a self-dual code, for any $r$. Actually, self-dual codes with $\rho = 2$ exist for any prime power $q \geq 4$.  

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Proposition 3.8 Let \( q \geq 4 \) be any prime power and let \( \mathbb{F}_q = \{0, 1, \xi_2, \ldots, \xi_{q-1}\} \). Let the matrix \( D_1^{q-4} \),

\[
D_1^{q-4} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & \xi_i & \xi_j \\
\end{bmatrix}
\]

be a parity check matrix for the code \( C \) and a generator matrix for the code \( C^\perp \), where \( \xi_i, \xi_j \in \mathbb{F}_q^* \) are two different elements such that \( \xi_i + \xi_j + 1 = 0 \). Then \( C \), as well as \( C^\perp \), is a linear antipodal completely regular \([4,2,3]_q\) code with covering radius \( \rho = 2 \) and with intersection array \((4(q-1),3(q-3);1,12)\). Furthermore, for the case \( q = 2^r \geq 4 \), these two equivalent codes coincide: \( C = C^\perp \), i.e. \( C \) is self-dual.

Proof. It is straightforward that, when \( q = 2^r \geq 4 \), the equation \( \xi_i^2 + \xi_j^2 + 1 = 0 \) is always satisfied when \( \xi_i + \xi_j + 1 = 0 \). \( \Box \)

We notice that there are nonlinear completely regular \( q \)-ary Latin-square codes with length \( n = 4 \), cardinality \( q^2 \), minimum distance \( d = 3 \); covering radius \( \rho = 2 \) and with the same intersection array \((4(q-1),3(q-3);1,12)\) that the codes in Proposition 3.8. The existence is guaranteed for any integer \( q \geq 3 \) with one exception for \( q = 6 \) \([12]\); since two orthogonal Latin squares of order 6 do not exist.

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