A causal statistical family of dissipative divergence type fluids

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October 6, 2018

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Abstract

In this paper we investigate some properties, including causality, of a particular class of relativistic dissipative fluid theories of divergence type. This set is defined as those theories coming from a statistical description of matter, in the sense that the three tensor fields appearing in the theory can be expressed as the three first momenta of a suitable distribution function. In this set of theories the causality condition for the resulting system of hyperbolic partial differential equations is very simple and allow to identify a subclass of manifestly causal theories, which are so for all states outside equilibrium for which the theory preserves this statistical interpretation condition. This subclass includes the usual equilibrium distributions, namely Boltzmann, Bose or Fermi distributions, according to the statistics used, suitably generalized outside equilibrium. Therefore this gives a simple proof that they are causal in a neighborhood of equilibrium. We also find a bigger set of dissipative divergence type theories which are only pseudo-statistical, in the sense that the third rank tensor of the fluid theory has the symmetry and trace properties of a third momentum of an statistical distribution, but the energy-momentum tensor, while having the form of a second momentum distribution, it is so for a different distribution function. This set also contains a subclass (including the one already mentioned) of manifestly causal theories.

1 Introduction

In the last few years there has been a considerable effort to understand dissipation in relativistic theories of fluids. Straight forward generalizations to relativity of the Navier-Stokes scheme resulted in systems with an ill-posed initial value formulation. Even if these generalizations would have worked, they would have yielded a parabolic system, well-posed in the mathematical sense, but unacceptable on physical grounds due to the presence of infinite propagation velocities. Thus, alternative theories were proposed resulting in a formalism having an extended number of dynamical variables and where dissipation is at the microscopic level completely different to the standard parabolic dissipation of Navier-Stokes equations, but which behaves at measurable scales in the same way. Due to the problems encountered on the generalizations alluded above, one of the basic requirements to be checked on these alternative theories was their relativistic causality.

Among the alternative theories, we specialize to the case of theories of divergence type, that is theories with the following structure

\begin{align*}
\nabla_a N^a &= 0 \\
\nabla_a T^{ab} &= 0 \\
\nabla_a A^{abc} &= I^{bc}
\end{align*}

where the tensors $A^{abc}$ (tensor of fluxes) and $I^{bc}$ (dissipation-source tensor) are supposed to be functions on a smaller set of fields, which in principle can be taken to be the direct observables of the theory, namely the particle number $N^a$ and the energy-momentum $T^{ab}$ of the fluid. The tensor $I^{ab}$ represents the non-equilibrium interactions in the fluid, and vanishes at the local equilibrium fluid configurations, i.e. at the momentarily static configurations. We focus on this particular type of theories, not just because their structure is simpler, but also because: a) They already contain enough degrees of freedom to account for the usual description of nonrelativistic dissipative fluids. b) Since these theories generically develop shock waves, it is desirable their structure has this form in order for the shock wave solutions to make sense.

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The requirement of an entropy law, that is the existence of an extra vector field whose divergence, by the sole virtue of the above equations, is a function of the basic fields at the point (and not of any of its derivatives), puts severe restrictions in the theory. In particular it implies the existence of a single generating function $\chi$, and variables $(\zeta, \zeta^a, \zeta^{ab})$, with $\zeta$ negative, $\zeta^a$ future directed timelike, and $\zeta^{ab}$ symmetric and trace free, such that

$$
N^a = \frac{\partial^2 \chi}{\partial \zeta \partial \zeta^a}
$$

$$
T_{ab} = \frac{\partial^2 \chi}{\partial \zeta^a \partial \zeta^b}
$$

$$
A_{abc} = \frac{\partial^2 \chi}{\partial \zeta^{ab} \partial \zeta^c}
$$

That is, this generating function determines completely the principal part, in the sense of the theory of partial differential equations, of the above system, and so its causal properties.

Non-dissipative fluids can also be described with divergence type theories by simply choosing a generating function of the form $\chi(\zeta, \zeta^a)$, and setting to zero the tensor $I^{ab}$. This kind of generating function describes a perfect fluid theory, with $\chi$ related with the fluid’s equation of state, as explained in section II and Appendix A.

In this paper we first define a subclass of non-dissipative divergence type theories that we call statistical. This perfect fluid theories can be characterized as those having a statistical origin, in the sense that the particle-number current and the stress-energy tensor can be expressed as suitable linear combination of the first two momenta of some distribution function. This requirement singles out a generating function for the perfect fluid theory as a functional of the corresponding distribution function, which determines completely the dynamics, thus in particular its causal properties. We obtain a very simple sufficient condition on the associated distribution function, to ensure causality of the whole perfect fluid theory. We call the theories satisfying this condition manifestly causal theories. They include as particular case the Boltzmann, Bose, and Fermi gasses.

From the statistical non-dissipative divergence theory we can easily define a dissipative divergence theory, keeping its statistical origin. We shall say in this case that the dissipative theory thus obtained is a natural extension of the non-dissipative one. This extension, has an important property: If the original non-dissipative theory is manifestly causal, then its dissipative extension is also manifestly causal. In particular, since the theories of equilibrium Boltzmann, Bose, and Fermi gasses can be cast in the form of manifestly causal non-dissipative theories, we conclude their natural extensions are also manifestly causal, thus considerably generalizing and simplifying the works of [8], [9], [10], [11].

In spite of its naturality, this extension has some drawbacks, as explained in Section II, it is only well defined for some extensions which are also manifestly causal.

Finally, for completeness, we include two appendices, one is a short review of the relativistic dissipative theories of divergence form, and the other of relativistic statistical mechanics.

2 Statistical Theories

The divergence type fluid theories are briefly reviewed in Appendix A. The simplest particular case of these theories is when the generating function $\chi$ does not depend on dissipative variables, and therefore is non-dissipative. In this case the theory describes a perfect fluid, with the particle-number current and the stress-energy tensor having the usual structure, namely,

$$
N^a = -\chi \zeta^a u^a,
$$

$$
T^{ab} = \chi_{,\mu\nu} u^a u^b - \frac{\chi_{,\mu}}{\mu} q^{ab}
$$
where \( q^{ab} = g^{ab} + u^a u^b \), and \( \zeta^a = \mu u^a \), with \( u^a u_a = -1 \). So we have the particle-number of the fluid \( n = -\chi, \zeta, \mu \), the energy density \( \rho = \chi, \mu \), and pressure \( p = -\frac{\mu}{\zeta} \) in terms of \( \chi \). It can be seen that in this theory \( \mu \) is one over the temperature of the fluid, and \( \zeta \) is a chemical potential per unit temperature. The choice \( \chi = \chi(\zeta, \zeta^a) \) implies that the tensor \( A^{abc} \) is identically zero, and this is consistent with the choice \( f^{ab} = 0 \), for the dissipation-source tensor. In these theories the expression for the entropy current given in the Appendix reduces to the usual one, \( S^a = nsu^a \), with \( s = (\rho + p)/(nT) - \zeta \), and it satisfies \( \nabla_a S^a = 0 \).

We say that a perfect fluid theory is of statistical type, if its generating function \( \chi(\zeta, \zeta^a) \) can be written in the following way

\[
\chi(\zeta, \zeta^a) = \int f(\zeta + p_a \zeta^a) \, d\omega
\]  

(7)

where the integral is on the future mass shell \( p^a p_a = -m^2 \), and \( f \) is a smooth scalar function on the negative real line, which for large (negative) values decays fast enough as to make the integral well defined and \( \chi(\zeta, \zeta^a) \) smooth in all its variables. Notice that \( \zeta < 0 \), since it is a chemical potential, while \( p_a \zeta^a < 0 \), since it is the scalar product of two future directed time-like vectors.

The simplest example is when \( f(x) = k^2 e^{x/k} \), where \( k \) is Boltzmann’s constant, namely Boltzmann’s gas. Indeed, the expressions for the number-current and the stress-energy tensors are in this case given by

\[
N_a = \int p_a \, e^{(\zeta + p_a \zeta^a)/k} \, d\omega,
\]

\[
T_{ab} = \int p_a p_b \, e^{(\zeta + p_a \zeta^a)/k} \, d\omega.
\]

That is, we interpret \( N_a \) and \( T_{ab} \) as the two first momenta of the Boltzmann’s equilibrium distribution function.

For an arbitrary statistical perfect fluid it is easy to see that the \( \chi \) defined as in (7) corresponds to a statistical theory with a distribution function given by \( f'' \).

An interesting property of this integral representation is that causality is easy to analyze. Indeed following (8) (see Appendix A), we say that a perfect fluid theory is causal if:

\[
t_a E^a = \frac{1}{2} t_a M_{AB}^a Z^A Z^B < 0
\]

for all perturbations \( Z^A = (\delta \zeta, \delta \zeta^a) \) and all future-directed time-like vectors \( t_a \). But for statistical perfect fluid theories, using the expression of \( \chi \) in terms of mass shell integrals, this condition becomes:

\[
t_a E^a = \frac{1}{2} \int (t_a p^a) f'''(\delta \zeta + p_a \delta \zeta^a)^2,
\]

(8)

for all perturbations \( (\delta \zeta, \delta \zeta^a) \). Thus we can isolate from the set of all statistical theories a subclass of manifestly causal theories, namely those having \( f'' \neq 0 \), \( f''' \geq 0 \). Notice that Boltzmann’s gas is inside this subclass, which also includes Bose and Fermi’s gases, since for them we have

\[
f'' = \frac{1}{e^{-(\zeta + p_a \zeta^a)/k - \epsilon}}
\]

(according of the value of \( \epsilon \), 0, 1 or \(-1\), we have respectively Boltzmann, Bose or Fermi gases), and it is easy to see that for all of them \( f''' \geq 0 \). The generating functions \( \chi \) for Bose and Fermi distribution functions are obtained by straightforward integration.

We now turn to the problem of extending these theories outside equilibrium.

The extension outside equilibrium we propose is the following: Given a statistical perfect fluid theory characterized by a function \( f \), we define a dissipative divergence theory of statistical type by the generating function

\[
\chi(\zeta, \zeta^a, \zeta^{ab}) = \int f(\zeta + p_a \zeta^a + p_a p_b \zeta^{ab}) \, d\omega.
\]

(9)

Note that this extension is only valid for values of \( \zeta^{ab} \) such that the argument of \( f \) is negative, while in general \( \zeta^{ab} \) can take values for which the argument is positive. Since \( \zeta \) and \( p_a \zeta^a \) are already negative for all \( p_a \) in the future mass shell, the values of \( \zeta^{ab} \) for which the argument is negative for all values of the momentum in the future mass shell form a cone, \( C^- = \{ \zeta^{ab} | \zeta^{ab} = \zeta^{ba}, \, g_{ab} \zeta^{ab} = 0, \, l_a b_a \zeta^{ab} \leq 0, \, \forall \, l_a \text{ null} \} \), of maximal dimension. Thus this extension is not unique, the equilibrium behavior of the statistical fluid only gives information encoded in an \( f \) which is only defined for negative values of the real line, while here we seem to need an \( f \) defined on the whole line. But the situation here is even worse, as shown in the following lemma.

1Note that there are theories which are not manifestly causal, but nevertheless are causal.

2Near equilibrium the theory should behave, at least in some respects, like Eckart’s theory. But in this theory there is a relation between \( \zeta^{ab} \) and the derivatives of the flow velocity which renders totally unphysical any imposition on the sign of \( \zeta^{ab} \).
Particular examples of this theorem are the dissipative extensions of Boltzmann, Bose and Fermi’s gases, given by

\[ F(c) = \int_0^\infty f(-x + cx^2)dx, \]

where \( f \) is any smooth, positive definite function, which is of compact support or decays faster than \( x^{-4} \) is discontinuous at \( c = 0 \), having there a finite limiting value from the left and diverging from the right.

**Proof**

To prove that the limiting value from the left is finite just note that if \( c \) is negative, then the argument of \( f \) just grows in absolute value and so the decay assumptions on \( f \) imply convergence. To prove the statement about the the limiting value from the left, take a double step function (a small square), \( sc(x) \), smaller than \( f \), then we have,

\[ F(c) \geq G(c) = \int_0^\infty sc(-x + cx^2)dx. \]

So it suffices to prove the divergence on the limit for the function \( G(c) \). Assume for simplicity that \( sc(x) \) is different from zero only in the interval \([-1, 1]\), and that there its value is 1. For positive values of \( c \), the integral above has contributions from two intervals. If \( c \) is small, those intervals can be calculated up to first order in \( c \) obtaining \([1 + c, 0]\) and \([\frac{1}{c} + 1, \frac{1}{c} - 1]\). The contribution to the integral from the first one is finite for all values of \( c \), while the second can be estimated using the mean value theorem to be bigger than \( \frac{1}{c} + 1 + O(c) \) and so we see that it diverges for \( c \) going to zero from positive values.

Q.E.D.

This Lemma gives a strong argument against the possibility of extending smoothly the definition of \( \chi \) to values of \( \zeta^{ab} \) such that \( p_a p_b \zeta^{ab} > 0 \), by simply extending the definition of \( f \) to positive values, no matter how smooth or how strong a decay condition we impose on the extension. But of course, there exist a lot of smooth extensions of the generating function \( \chi \) to the presently forbidden values of \( \zeta^{ab} \). Any of those, essentially arbitrary, smooth extensions will be assumed to have been made in what follows, in particular we require that extension in a neighborhood of equilibrium, that is, in a neighborhood of the apex of the cone \( C^- \). The results on causality near equilibrium do not depend on the particular extension chosen outside \( C^- \).

For these statistically extended dissipative fluids theories also a simple expression for the causality condition is easy to obtain, namely:

\[ t_a E^a = \frac{1}{2} \int (t_a p^a) f''' (\delta \zeta + p_a \delta \zeta^a + p_a p_b \delta \zeta^{ab})^2 \]

So we have have the following result:

**Theorem 1** Let function \( f : R^+ \to R \) defining a statistical perfect fluid be \( C^3 \), and such that the equilibrium theory is well defined. If the equilibrium theory is manifestly causal, i.e. \( f''' > 0 \), then the extended dissipative theory is also causal in a neighborhood of equilibrium.

**Proof**

The above expression shows causality for all values of \( (\zeta, \zeta^a, \zeta^{ab}) \) such that \( \zeta^{ab} \in C^- \). Since the cone is of maximal dimension, partial derivatives along directions inside the cone suffices to determine completely the differentials of \( \chi \) at the apex of the cone, that is, at equilibrium. Thus the smooth extension outside the cone can not change the causality properties of the equilibrium configuration. The result extends to a neighborhood of equilibrium trivially by noticing that in our setting causality is a continuous property.

Q.E.D.

Particular examples of this theorem are the dissipative extensions of Boltzmann, Bose and Fermi’s gases, given by

\[ f''' = \frac{1}{e^{-(\zeta + p_a \zeta^a + p_a p_b \zeta^{ab})/k} - \epsilon} \]

according of the value of \( \epsilon \) is 0, 1 or \(-1\) respectively. In all cases is easy to see that \( f''' > 0 \), so they are smooth and manifestly causal for all values of the parameters for which the integral expression converges, in and off equilibrium. This generalize and simplify our previous work [11], which in turn was a generalization of other results [8], [10].

### 3 Pseudostatistical Theories

There is an even larger set of theories defined in terms of the integral of certain function \( f \) on a future mass shell, for which it is straightforward to find sufficient conditions on \( f \) such that the resulting divergence theory is causal. We
say that a dissipative fluid of divergence form is of pseudo-statistical origin, if its generating function $\chi$ can be written as
\[ \chi(\zeta, \zeta', \zeta'') = \int f(\zeta + p_0 p \zeta', p_0 \zeta'') \, d\omega \]
where the integral is on the future mass shell, and $f = f(x, y)$ is a scalar function of two variables, $x = \zeta + p_0 p \zeta'$ and $y = p_0 \zeta''$. We call them pseudo-statistical because of the tensors $N^a$ and $A^{abc}$ can be thought as coming from a distribution function $f_{xy}$ while the stress-energy tensor $T^{ab}$ can be thought as coming from a different distribution function, $f_{yy}$, in fact,
\[ N_a = \int p_a f_{xy} \, d\omega \]
\[ T_{ab} = \int p_0 p_b f_{yy} \, d\omega \]
\[ A_{abc} = \int p_a p_b p_c f_{xy} \, d\omega. \]
Here the values of the variables $(\zeta, \zeta')$ have the same restrictions as in section above. The lemma 1 does not apply here and there are cases, like Boltzmann, where one can obtain a pseudo-statistical extension for all values of $\zeta''$ starting from an equilibrium statistical theory.

The pseudo-statistical theories are interesting since again the causality condition is simple and it is easy to impose a sufficient condition such that the theory is manifestly causal. Indeed, it is easy to see that in this case the causality condition is,
\[ t_a E^a = \frac{1}{2} \int (t_a p^a) \left[ f_{xxy}(\delta \zeta)^2 + f_{yy}(p_0 \delta \zeta')^2 + f_{xxy}(p_{<dpe}> \delta \zeta')^2 + 2 f_{xxy} \delta \zeta' \right] \, d\omega. \]
Rearranging terms we have
\[ t_a E^a = \frac{1}{2} \int (t_a p^a) \left[ f_{xxy}(\delta \zeta + p_{<dpe> \delta \zeta'})^2 + f_{yy}(p_0 \delta \zeta')^2 + 2 f_{xxy}(\delta \zeta + p_{<dpe> \delta \zeta'}) \right] \, d\omega, \]
and defining $\delta x = \delta \zeta + p_{<a> \delta \zeta'}$ and $\delta y = p_a \delta \zeta'$,
\[ t_a E^a = \frac{1}{2} \int (t_a p^a) \left[ f_{xxy}(\delta x)^2 + f_{yy}(\delta y)^2 + 2 f_{xxy} \delta x \delta y \right] \, d\omega. \]
Thus, if we assume $f_{xxy} \geq 0$, $f_{yy} \geq 0$ and $(f_{xxy})^2 \leq f_{yy} f_{xxy}$, we obtain,
\[ t_a E^a \leq \frac{1}{2} \int (t_a p^a) \left( \sqrt{f_{xxy}} |\delta x| - \sqrt{f_{yy}} |\delta y| \right)^2 \, d\omega, \]
and again isolate a subclass of dissipative theories (containing the one previously described) which are manifestly causal, namely those which satisfy,
\[ f_{xxy} \geq 0 \quad f_{yy} \geq 0 \quad (f_{xxy})^2 \leq f_{yy} f_{xxy}. \]

We thus have an obvious generalization of the previous theorem.

**Theorem 2** Let function $f : R^\times \times R^\times \to R$ defining a perfect (equilibrium) pseudostatistical fluid be $C^3$, and such that the equilibrium theory is well defined. If the equilibrium theory is manifestly causal, then the extended dissipative theory is also causal in a neighborhood of equilibrium.

Remark: There are causal $f$’s whose derivatives does not satisfies conditions above. Even more, if we extend the theory in a pseudostatistical way for positive arguments, then the decay properties would in general imply the above condition can not be imposed, so even in this case, causality for naturally extended theories seems to hold only in a restricted domain of possible field variables.

The generating functions $\chi$ associated with a statistical or a pseudo-statistical dissipative divergence type theories, can be thought as particular solution of equation (30) of Appendix B. This equation is obtained by imposing some conditions on the divergence type theories, related with the symmetry and trace of the third rank tensor, $A^{abc}$, of the theories. Those conditions come naturally from kinetic theory and are explained in that Appendix. From the above subclass of theories it seems then that there are globally defined, and nicely behaved solutions to those equations, but they are not statistical, as one would have liked them to be, they are at most pseudostatistical.\footnote{Here the subindices indicate partial derivatives with respect to the argument signaled by the index.}
As seen in the introduction and Appendix A, the requirement of an entropy law is a very strong condition which substantially constraints the kinematics and dynamics of the fluid theories, since imposes that it can only depend on a freely given generating function, \( \chi \), (essentially an equation of state), and a basically freely given dissipation source tensor \( T^{ab} \). Even more, it gives a fixed formula for the entropy as a functional of the arbitrary generating function defining particular theories. This entropy is of dynamical origin and in principle it is not related to the entropy concept coming from information theory as applied to equilibrium configurations. In fact, the entropy law property can be divided in two conditions the more stringent one is the existence of an extra vector field whose divergence is, as a consequence of the other equations, a local function of the dynamical field (and not of their derivatives), and a weaker one which requires this last function to be non-negative and to vanish at equilibrium. The first condition is the one which restricts the theory, the one which gives its rigidity, and is the one which determines completely the entropy functional, while the other condition, in this picture, just puts mild conditions on the dissipation-source tensor. We shall see in what follows how these conditions work for the natural extensions of statistical theories.

In the theories of divergence type the entropy is given by, (see Appendix A.)

\[
S_a = \frac{\partial \chi}{\partial \zeta^a} - \zeta N_a - \zeta^b T_{ab} - \zeta^{bc} A_{abc} \\
= \chi_a - \zeta^A \frac{\partial \chi}{\partial \zeta^A} 
\]  

(11)

where \( \zeta^A = (\zeta, \zeta^a \zeta^{ab}) \). For the naturally extended statistical theories this translate into, (See Appendix B.)

\[
S_a = \int p_a (f' - \zeta^A P_A f'') \, d\omega
\]

where \( P_A \equiv (1, p_a, p_a p_b) \).

We observe that this entropy is a linear functional of the distribution function, and so there is no convexity property a this stage. Its form does not depends on any statistical counting of states assumption, once the distribution function \( f'' \) is given, the entropy is determined.

The relation with the usual concepts and formulae comes about when one identifies, via Boltzmann equation, the tensor \( T^{ab} \) with a collision integral. That is we pretend that the dependence of this tensor with \( \zeta^A \) is only through the distribution function. In that case, the weaker condition we had in the dynamical picture, i.e. the vanishing of the divergence of the entropy current to second order in the dissipation variables implies, via the collision functional, that the global equilibrium states have the Gibbs form, which are thus linked to the statistics considerations used to deduce the form of that functional. To see how this works we assume the above setting and compute the entropy for the global equilibrium states, (they enter only in the choice of collision functional used to define \( T^{ab} \)), that is the examples considered above. Thus assume that the associated distribution function to the fluid theory is of the form

\[
f'' = F = \frac{\eta}{e^X - \epsilon},
\]

where \( \eta \) is a constant proportional to \( 1/h^3 \) with \( h \) the Plank’s constant and for simplicity we denote \( X = -P_d \zeta^A/k = -(x + y)/k = -(\zeta + p_a \zeta^a + p_a p_b \zeta^{ab})/k \). It is direct to see that

\[
f' = -\frac{\eta k}{\epsilon} \left[ \ln(\epsilon^X - \epsilon) - X \right].
\]

If we define \( \Delta = 1 + \epsilon F/\eta \), then we can write \( e^X = (\eta \Delta)/F \); and

\[
f' = \frac{\eta k}{\epsilon} \ln(\Delta).
\]

So the entropy current density functional \( S^a \) can be written as

\[
S_a = \int p_a \left( f' + kX f'' \right) \, d\omega \\
= k \int p_a \left( \frac{\eta}{\epsilon} \ln(\Delta) + \ln \left( \frac{\eta \Delta}{F} \right) \right) \, d\omega \\
= -k \int p_a \left( F \ln \left( \frac{F}{\eta} \right) - \frac{\eta}{\epsilon} \Delta \ln(\Delta) \right) \, d\omega.
\]

The last expression correspond to the usual entropy functional defined in relativistic kinetic theory whose equilibrium states corresponds to Bose or Fermi’s equilibrium distribution function if \( \epsilon \) is 1 or \( -1 \), respectively. The case \( \epsilon = 0 \)
corresponds to the Boltzmann’s distribution function, and is easy to see that $S^a$ given by (11) has the usual form

$$S_a = -k \int p_a F \ln \left( \frac{F}{\eta} \right) d\omega$$

Thus the usual formulae for the entropy functionals is recuperated, but they are only valid for global equilibrium distribution functions, that is the properties usually assigned to entropies are only relations among certain distribution functions, and do not seems to hold outside that set, even at local equilibrium states.

5 Conclusions

We have presented a variety of dissipative divergence type theories with a statistical origin, in the sense that the tensors of the theory can be expressed as appropriate functions of the three first momenta of a suitable distribution function. This represents a relation with kinetic theory, which is manifest in the integral expression for the resulting generating function $\chi$. From this integral expression we could easily derive a simple condition on the associated distribution function for the resulting theory to be causal, even for some states even far momentarily equilibrium states. This condition had been easily verified for the natural extensions to non-equilibrium states of equilibrium distribution functions associated with Boltzmann, Bose and Fermi’s statistics. The results obtained for these particular examples represent a great simplification of our previous work [11], which was a generalization of former works [8], [10].

The dynamical entropy defined in the divergence type theories has not been related, in principle, to the entropy concept coming from a statistical theory as applied to equilibrium distribution functions. This dynamical entropy is just a vector field (constructed from the fluid fields) whose divergence is a pointwise function in the fields and its definition does not use anything about equilibrium configurations. This expression has a surprising form, for in the theories here considered it is linear on the associated distribution function. A remarkable fact for this dynamical entropy is that for momentarily equilibrium configurations corresponding to the distribution functions associated with the usual Boltzmann, Bose and Fermi statistics; it takes the familiar form encounters in statistical mechanics when applied to equilibrium configurations. Thus, from this point of view, the expressions usually used are only valid for a very restricted set of distribution functions and when so, only for some regions of the non-equilibrium configuration variables manifold, namely that where the natural extension holds.

There exists a serious limitation for this integral representation of the statistical theories, as has been shown with the Lemma 1. This Lemma says that we can not extend smoothly the definition of $\chi$, to values of $\zeta^{ab}$ such that $\zeta^{ab} p_a p_b > 0$, simply by extending the definition of $f$ to such values. In other words, the statistical interpretation of $\chi$ can not be extended. We can, of course, extend smoothly the definition of $\chi$ to the forbidden values of $\zeta^{ab}$ in an almost arbitrary way. All those extensions will be, by continuity, causal in a neighborhood of the region where $p_a p_b \zeta^{ab}$ is non-negative, in particular in a neighborhood of equilibrium. Does the limitation in the integral representation, have any physical interpretation? The condition $\int \zeta^{ab} \zeta^{ab} \leq 0, \forall \eta$ null seems to be unphysical if we look near equilibrium, for there the theory should resemble Eckart, and in that theory it is known that there does not exist any restricting condition on the dissipative variable $\zeta^{ab}$. So the problem, if this is really a problem, should be in the fluid approximation, which in this case is the pretention that we can describe the evolution of a distribution function via Boltzmann’s equation via the evolution of a finite set of variables, namely $\zeta^b = (\zeta, \zeta^a, \zeta^{ab})$. While this seems to work very well at equilibrium, it might be that it fails completely away from equilibrium [11] and for an appropriate description an infinite of field variables are needed. In that case a natural extension seems to be possible, provided the distribution function is suitably extended for positive values of its argument. This solution is of the type, add more dynamical fields in order to get some sort of cutoff. Other type of possible extension is to abandon the direct relation between distribution momenta and dynamical variables outside equilibrium, but still retain the flavor of a statistical theory in the sense of the representation as integral over mass shells. This is easily implemented by changing the argument of the distribution function in a non-linear way, for instance: $x = \zeta + p_a \zeta^a + p_a p_b \zeta^{ab} + \lambda(p_a p_b \zeta^{ab})^2).$ This solution is of the type, add a constant, in order to get a non-linear cutoff. Both sound interesting and perhaps the solution lies in between.

It has been possible to extend the proof of causality given for the statistical theories to a bigger set of divergence theories. For this set, the three tensors fields, $N^a$, $T^{ab}$, and $A^{abc}$ are also averages over a future mass shell, but for different (although related) distribution functions. We do not have any application for this larger class, but believe it should be of relevance for describing some physical phenomena. In this case extensions seems to be possible for all values of the non-equilibrium variables, but such extensions in general are not causal for all such values, and so do not seem to be very physical. Are there special extensions which from a manifestly causal equilibrium statistical theory yield a causal pseudo-statistical non-equilibrium theory?

\footnote{It is not clear in this context what it means expressions like near equilibrium and its relation with, for instance, the the size of $\zeta^{ab}$.}
Appendix A: Review of dissipative fluid theories of divergence type

Following [5], [7], [9], we define a Dissipative Fluid Theory of Divergence Type as a theory having the following three properties:

1) The dynamical variables can be taken to be the particle-number current $N^a$, and the (symmetric) stress-energy tensor $T^{ab}$.  

2) The dynamical equations are

$$\nabla_a N^a = 0, \quad (12)$$

$$\nabla_a T^{ab} = 0, \quad (13)$$

$$\nabla_a A^{abc} = I^{bc}, \quad (14)$$

where the tensors $A^{abc}$ (tensor of fluxes) and $I^{ab}$ (dissipation-source tensor) are local functions of the dynamical variables $N^a$ and $T^{ab}$, and are trace free and symmetric in the last two indices.

3) There exist and entropy current $S^a$ (local function of $N^a$ and $T^{ab}$) which, as a consequence of the dynamical equations, must satisfy

$$\nabla_a S^a = \sigma \quad (15)$$

where $\sigma$ is some positive function of $N^a$ and $T^{ab}$.

It is proved in [5], [7], that a theory having these three properties is determined by specifying a single scalar generating function $\chi$ and the tensor $I^{ab}$ as functions of a new set of dynamical variables $\zeta, \zeta^a, \zeta^{ab}$ (with the later trace free and symmetric). The dynamical equations for these variables are (1-3), with

$$N_a = \frac{\partial^2 \chi}{\partial \zeta \partial \zeta^a} \quad (16)$$

$$T^{ab} = \frac{\partial^2 \chi}{\partial \zeta^a \partial \zeta^b} \quad (17)$$

$$A^{abc} = \frac{\partial^2 \chi}{\partial \zeta^a \partial \zeta^{bc}} \quad (18)$$

while the entropy current is given by

$$S_a = \frac{\partial \chi}{\partial \zeta^a} - \zeta N_a - \zeta^b T_{ab} - \zeta^{bc} A_{abc} \quad (19)$$

while its divergence is

$$\sigma = -\zeta^{ab} I_{ab}. \quad (20)$$

It is helpful to represent the collection of dynamical variables as, $\zeta^A = (\zeta, \zeta^a, \zeta^{ab})$, and the dissipation-source tensor as, $I_A = (0, 0, I^{ab})$. Equations (1-3) can then be written in this notation as

$$M_{AB}^a \nabla_a \zeta^B = I_A \quad (21)$$

with

$$M_{AB}^a = \frac{\partial^3 \chi}{\partial \zeta_a \partial \zeta^A \partial \zeta^B}$$

The system of equations (21) is automatically symmetric, since partial derivatives commute. We say that a symmetric system is hyperbolic if the vector

$$E^a = \frac{1}{2} M_{AB}^a Z^A Z^B$$

lies to the future of some space-like three-dimensional sub-space of the tangent space, for all non-vanishing $Z^A$. The system is called space-time causal if furthermore $E^a$ lies within the future light cone (i.e., if $E^a$ is a future-directed timelike vector), for all non-vanishing $Z^A$. The property of hyperbolicity ensures that system (21) has a well-posed initial-value formulation, while space-time causality ensures that no fluid signals can propagate faster than light. Because the underlining statistical mechanics origin of fluids, it is natural to demand causality even for its dissipative versions, this implies conditions on the generating function $\chi$.

Following [5] we call a state, $\zeta^A$, i.e. the value of $\zeta^A$ at a given time, a momentarily equilibrium state, if at that time the entropy production vanishes, i.e. $\sigma = 0$. In the generic case this implies $\zeta^{ab} = 0$. i.e. $\zeta_{me} = (\zeta, \zeta^a, 0)$. An equilibrium state or strict equilibrium state is a time reversible state. In that case it can be seen that generically not only $\zeta^A = (\zeta, \zeta^a, 0)$, and $I^{ab}(\zeta^A) = 0$, but $\zeta$ is constant, and $\zeta^a$ is Killing. For these strictly equilibrium states the modulus of $\zeta^a$, that is $\mu = \sqrt{-\zeta^a \zeta_a}$, can be associated with one over the temperature of the fluid, and its direction $u^a \equiv \zeta^a/\mu$ with the 4-velocity of the fluid. The variable $\zeta$ can be associated with a chemical potential per unit temperature of the fluid.
We consider a distribution of identical particles in a given, fixed, spacetime. The particles interact via short-range forces, idealized as point collisions. A distribution function $F$ is defined by the statement that

$$ F = \frac{\mathcal{P}_a}{m} d\Sigma_a \ d\omega $$

is the number of world-lines cutting an element of 3-surface $d\Sigma_a$ and having 4-momenta $p_a$ which terminate on a cell of 3-area $d\omega$ on the mass shell $p_a p^a = -m^2$. This distribution function is the solution of the relativistic transport equation

$$ p^a \nabla_a F(x, p^d) = C(x, p^d) $$

where the derivative is along a curve in phase space which is geodesic, starting at the point where we evaluate it and having as tangent vector $p^a$. $C$ is a collision term defined requiring that

$$ C(x, p_d) \frac{d\omega}{m} \sqrt{-g} d^4 x $$

be the number of particles in the momentum range $d\omega$ around $p_d$ which are created by collisions in the 4-volume $\sqrt{-g} d^4 x$, around the point $x$. Given an expression for the collision term, in general a functional of $F$, Boltzmann’s equation determines the dynamics of $F$. This collision term can not be an arbitrary function of $F$, it has to satisfy certain physical properties. Usually the following general properties are required:

1. The form of $C$ is consistent with 4-momentum and particle number conservation at collisions.

2. The collision term $C$ yields a non-negative expression for the entropy production.

To express these two conditions in a formal way, it is necessary to relate the information on the distribution function with macroscopic quantities, such as particle-number current or the stress-energy tensor. To do this it is useful to introduce, given a distribution function, $F$ the set of moments associated to this distribution, which is the following hierarchy of totally symmetric tensors

$$ J_{a_1 \cdots a_n} = \int p_{a_1} \cdots p_{a_n} F \ d\omega, \quad n = 0, \ldots, \infty $$

where the integral is on the future mass shell, $p^a p_a = -m^2$.

These moments are not independent quantities because they satisfy the following relations

$$ J_{a_1 \cdots a_j \cdots a_i \cdots a_{n-2}} g^{a_j a_i} = -m^2 J_{a_1 \cdots a_{n-2}} \quad i, j = 1 \cdots n \quad i \neq j. \quad (22) $$

By virtue of transport equation, the $(n+1)$ momentum satisfies,

$$ \nabla_{a_j} J_{a_1 \cdots a_{n-1}} = I_{a_1 \cdots a_n}, \quad (23) $$

with the source tensor $I_{a_1 \cdots a_n}$ defined as,

$$ I_{a_1 \cdots a_n} = \int p_{a_1} \cdots p_{a_n} C \ d\omega. $$

If we identify the first momentum with the particle-number current, that is $N_a = J_a$; and the second momentum with the stress-energy tensor of the fluid, that is $T_{ab} = J_{ab}$; then condition 1) on the collision term can be written as

$$ \int C \ d\omega = 0 \quad \int p_a C \ d\omega = 0 \quad (24) $$

The statistical entropy current density is defined as a functional of $F$ as follows:

$$ \tilde{S}_a(x) \equiv -\frac{1}{m} \int \phi(F) p_a d\omega \quad \text{with} \quad \phi(F) = \left( F \ln(h^3 F) - \frac{\omega}{\hbar^3} \Delta \ln(\Delta) \right) $$

and

$$ \Delta(x, p_a) = 1 + \frac{\hbar^3}{\omega} F(x, p_a) $$

5 This statistical entropy is not the dynamical entropy we are using. They only coincident for equilibrium states.
where \( h \) the Planck’s constant, \( \omega \) is the spin-weight (number of available states per quantum phase-cell) and \( \epsilon \) is 1 for Bosons and −1 for Fermions. Then condition 2) on the collision term can be written as,

\[
\nabla_a \vec{S}^a = -\frac{1}{m} \int \phi'(F) C(F) \, d\omega \geq 0 \quad \text{where} \quad \phi'(F) = \ln \left( \frac{\hbar^3 F}{\Delta} \right).
\]

Finally, local equilibrium distributions (or states), which are defined requiring that for them the entropy production vanishes, namely \( \nabla_a S^a = 0 \). By first property imposed on the collision term, a sufficient condition is that for this \( F \)'s

\[
\phi = \frac{\zeta}{k} + \frac{1}{k} \zeta^a p_a.
\]

This conclude a brief review of the basics Kinetic Theory we need.

To relate Kinetic theory with the divergence type fluid theories it is necessary to make some identifications between the basic tensor fields of both theories. As we did above, is direct to associate the two first moments \( J^a \) and \( J^{ab} \) with the particle-number current \( N^a \) and the stress-energy tensor \( T^{ab} \). These relations do not imply any restriction on the generating function \( \chi \), at least from the symmetry and number of linearly independent elements of these tensors. Such a restriction on \( \chi \) comes if one wishes to relate the third momentum of a distribution, \( J^{abc} \), with the tensor \( A^{abc} \) of the divergence type theories. This is suggested because, in analogy with the divergence type dissipative fluids the first three momenta satisfy

\[
\begin{align*}
\nabla_a J^a &= 0 \quad \text{(26)} \\
\nabla_a J^{ab} &= 0 \\
\nabla_a J^{a(bc)} &= (bc)
\end{align*}
\]

where the symbol \( ( ) \) means symmetrization and trace free. But this analogy is only superficial and involves a very stringent assumption. Indeed, the above momenta equations are not a closed system of equations, for \( I^{ab} \) is not a function of some finite number of variables –for instance the previous two momenta–, but it is a function of the distribution function, which in general is a functional of all momenta. Thus systems are analogous only for those situations which can be described with a distribution function which is to a certain degree of approximation a function of only the first two momenta. At present there is not convincing argument that this is ever the case, nevertheless we proceed here with the analogy. Since \( J^{ab} = -m^2 J_a = -m^2 N_a \), while, \( A^{ab} = 0 \), and assuming that \( I^{ab} = I_{ab} \), the only possible relation between these tensors such that both dynamical system are equivalent is:

\[
J^{abc} = A^{abc} - \frac{m^2}{4} N_a g_{bc}
\]

Since \( J^{abc} \) is totally symmetric, this condition imposes conditions in \( A^{abc} \) and \( N_a \) (such that the right hand side of (29) be totally symmetric), which in turn impose conditions on the generating function \( \chi \). These conditions are equivalent to the following system of equations for \( \chi \)

\[
\frac{\partial^2 \chi}{\partial \zeta^a \partial \zeta^b/c} - \frac{m^2}{4} \frac{\partial^2 \chi}{ \partial \zeta \partial \zeta^a} g_{bc} = 0
\]

Thus, for this identification to make sense the generating function must be a solution of (30). If we obtain such a solution, we can then describe a fluid by the corresponding divergence type theory, which thus has a statistical origin as given by the above identification, even away from equilibrium. A family of solutions of this system of equations is given by

\[
\chi(\zeta, \zeta^a, \zeta^{ab}) = \int f(x, y) \, d\omega,
\]

where \( x = \zeta + p_a \zeta^a \) and \( y = p_a \zeta^a \), the integral is on the all future-directed momenta \( p_a \), and \( f \) is assumed to behave asymptotically in such a way that the integral converges. With subindices in \( f \) indicating differentiation, we have

\[
\frac{\partial^2 \chi}{\partial \zeta^a \partial \zeta^{bc}} = \int p_a p_b p_{bc} f_{xy} \, d\omega
\]

\[
= \int p_a \left( p_b p_{c} + \frac{m^2}{4} g_{bc} \right) f_{xy} \, d\omega
\]

\[
= \int p_a p_b p_{c} f_{xy} \, d\omega + \frac{m^2}{4} g_{bc} \frac{\partial^2 \chi}{\partial \zeta \partial \zeta^a}
\]

where \( p_{<b} p_{c} = p_b p_{c} + \frac{m^2}{4} g_{bc} \) means symmetrization and trace free. So have

\[
\frac{\partial^2 \chi}{\partial \zeta^a \partial \zeta^b/c} - \frac{m^2}{4} \frac{\partial^2 \chi}{ \partial \zeta \partial \zeta^a} g_{bc} = \int d\omega \, p_{<a} p_{b} p_{c} f_{xy} = 0
\]
As we have seen in the section II, there seems to be that there are not pure statistical solutions, i.e. solutions of the type: \( f(x, y) = f(x + y) \), with the appropriate differentiability conditions on \( \chi \) they generate. Thus the analogy discussed above seems to be unjustified.

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