Stationary viscoelastic wave fields generated by scalar wave functions

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Abstract

The usual Helmholtz decomposition gives a decomposition of any vector valued function into a sum of gradient of a scalar function and rotation of a vector valued function under some mild condition. In this paper we show that the vector valued function of the second term i.e. the divergence free part of this decomposition can be further decomposed into a sum of a vector valued function polarized in one component and the rotation of a vector valued function also polarized in the same component. Hence the divergence free part only depends on two scalar functions. Further we show the so called completeness of representation associated to this decomposition for the stationary wave field of a homogeneous, isotropic viscoelastic medium. That is by applying this decomposition to this wave field, we can show that each of these three scalar functions satisfies a Helmholtz equation. Our completeness of representation is useful for solving boundary value problem in a cylindrical domain for several partial differential equations of systems in mathematical physics such as stationary isotropic homogeneous elastic/viscoelastic equations of system and stationary isotropic homogeneous Maxwell equations of system. As an example, by using this completeness of representation, we give the solution formula for torsional deformation of a pendulum of cylindrical shaped homogeneous isotropic viscoelastic medium.

1 Introduction

The Helmholtz decomposition has played an important role to solve boundary value problems in a rectangle parallelepiped, sphere and cylinder for several partial differential equations of systems in mathematical physics such as stationary isotropic homogeneous elastic/viscoelastic equations of system and stationary isotropic Maxwell equations of system.
The aim of this paper is to provide a special Helmholtz decomposition for solutions of an isotropic homogeneous elastic/viscoelastic equations of system in a finite cylindrical domain. The special Helmholtz decomposition enables to represent solutions of this homogeneous viscoelastic system in a finite cylindrical shaped domain by scalar wave functions via a special Helmholtz decomposition associated to the cylindrical coordinates. This is called the completeness of representation. More precisely let $\Omega$ be a homogeneous isotropic viscoelastic medium in $\mathbb{R}^3$ with complexified Lamé moduli $\lambda, \mu$ such that their real parts $\lambda_R, \mu_R$ and imaginary parts $\lambda_I, \mu_I$ satisfy the strong convexity condition:

$$\mu_R, \mu_I > 0, \quad 3\lambda_R + 2\mu_R, \quad 3\lambda_I + 2\mu_I > 0$$

(1.1)

(see [9]). The shape of $\Omega$ could be either a ball or cylinder. Define the stationary homogeneous isotropic viscoelastic operator $L_{\lambda,\mu}$ with density $\rho > 0$ and angular frequency $\omega > 0$ by

$$L_{\lambda,\mu} u := (\lambda + \mu)\text{grad div} u + \mu \Delta u + \rho \omega^2 u.$$  

(1.2)

Then the completeness of representation is given as follows. That is for $u$ satisfying $L_{\lambda,\mu} u = 0$ in $\Omega$, there exists scalar functions $\varphi, \psi, \chi$ such that $u$ has a special Helmholtz decomposition given as

$$u = \nabla \varphi + \nabla \times (\psi e) + \nabla \times \nabla \times (\chi e)$$

(1.3)

and $\varphi, \psi, \chi$ satisfy the following Helmholtz type equations:

$$\begin{cases}
(\Delta + \frac{\rho \omega^2}{\lambda + 2\mu}) \varphi = 0, \\
(\Delta + \frac{\rho \omega^2}{\mu}) \psi = 0, \\
(\Delta + \frac{\rho \omega^2}{\mu}) \chi = 0,
\end{cases}$$

(1.4)

where $e$ is a unit vector in the axial direction and direction of $z$ axis for the case $\Omega$ is a ball and cylinder, respectively. If there is a global orthogonal curvilinear coordinates $(x^1, x^2, x^3)$ in a neighborhood of $\overline{\Omega}$, then there is a question about the completeness of representation when $e$ is replaced by $w(x^3)e_3$ with unit vector $e_3$ say in the $x^3$ direction and some auxiliary function $w(x^3)$ for the separation of variables.

This completeness of representation was rigorously proved in [3] when a medium has a shape of ball. After that in [5], a more general discussion was made for the completeness of representation. It included the case that the medium has a shape of cylinder. However it did not give any rigorous proof for this case. To the best of our knowledge, it seems that the proof for this case has been left opened for more than 40 years.

In this paper our main aim is to give a rigorous proof for the completeness of representation for the case $\Omega$ is a finite cylinder. Hence in the rest of the paper, $\Omega$ denotes an open finite cylinder.

Needless to say that this completeness of representation is very useful for giving a solution formula to solve given boundary value problem. As an example, we apply
the torque boundary condition (see (4.2)) and find the solution which satisfies our representation (1.3). In fact we used the solution formula to identified the complex shared modulus of the sample using a digitized commercial base pendulum-type viscoelastic spectrometer (PVS) (see [6] for its details).

The rest of this paper is organized as follows. In Section 2 we give a special type of Helmholtz decomposition. Then we give the completeness of representation for this decomposition in Section 3. The last section is devoted to an application of the completeness of representation. That is we apply it to solve a boundary value problem for a cantilever homogeneous isotropic viscoelastic cylinder with a given torque at its bottom by using the completeness of representation.

2 Special Helmholtz Decomposition

For \( l, m \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}, \ l \geq m \) and \( p \geq 2 \), we define a function space
\[
v \in W^{l,m}_{p,loc}(\mathbb{R}^3) \iff \sum_{|\alpha|+j \leq l, j \leq m} \int_{\mathbb{R}^3} |\partial_x^\alpha \partial_z^j (\eta v)(x',z)|^p dx' dz < \infty
\]
for all \( \eta \in C_0^\infty(\mathbb{R}^3) \), where we used \( x = (x,y,z) \in \mathbb{R}^3 \), \( x' = (x,y) \). We also define \( W^{l,m}_p(\Omega) \) by deleting \( \eta \) and replacing \( \mathbb{R}^3 \) by \( \Omega \) in the above definition. For \( p = 2 \), we use the notations \( H^{l,m}_{loc}(\mathbb{R}^3) \), \( H^{l,m}(\Omega) \). Also, we can infer that
\[
\partial_x^\alpha \partial_z^j v \in W^{l-|\beta|-k, \min\{m-k, l-|\beta|-k\}}_{p,loc}(\mathbb{R}^3)
\]
which follows from
\[
\left\{ |\alpha| + |\beta| + j + k \leq l, \ j + k \leq m \right\} \iff \left\{ |\alpha| + j \leq l - |\beta| - k, \ j \leq m - k. \right\}
\]

**Theorem 2.1.** Assume that \( u \in W^6_p(\mathbb{R}^3), p \geq 2 \) and it has a compact support. Then \( u \) admits the representation
\[
u = \nabla \varphi + \nabla \times (\psi e_z) + \nabla \times \nabla \times (\chi e_z),
\]
where \( \varphi \in W^7_{p,loc}(\mathbb{R}^3), \psi \in W^{7,4}_{p,loc}(\mathbb{R}^3), \chi \in W^{8,5}_{p,loc}(\mathbb{R}^3). \) Note that the regularities of \( \psi, \chi \) given above imply \( \nabla \times (\psi e_z), \nabla \times \nabla \times (\chi e_z) \in W^{6,4}_{p,loc}(\mathbb{R}^3). \)

**Proof.** First of all, it is well known that for any given \( u \in W^6_p(\mathbb{R}^3) \) with compact support, we can have the following Helmholtz decomposition [4]:
\[
u = \nabla \varphi + \nabla \times A \text{ in } \mathbb{R}^3 \text{ with}
\]
where \( \varphi \) and \( A \) are given by
\[
\varphi(x) = -(4\pi)^{-1} \int \frac{\nabla u(y) dy}{|y-x|} \in W^7_{p,loc}(\mathbb{R}^3),
A(x) = (4\pi)^{-1} \int \frac{\nabla \times u(y) dy}{|y-x|} \in H^7_{p,loc}(\mathbb{R}^3).
\]
where the integration are take over $\mathbb{R}^3$.

Next in order to decompose $\nabla \times A$ into two potentials, we prepare the solvability of the equation for $\chi$ which satisfies

$$
\Delta' \chi = -e_z \cdot (\nabla \times A) : = f \in W^6_{p,loc}(\mathbb{R}^3) \text{ in } \mathbb{R}^2,
$$

(2.4)

where $\Delta'$ is the Laplacian with respect to $x' = (x_1, x_2) = (x, y)$. Since the inner product $(A(x), b)$ of $A(x)$ with any constant vector $b$ is given as

$$
(\nabla \times A(x), b) = (4\pi)^{-1} \int (\nabla_y \times u(y), \nabla_x \times (|y - x|^{-1}b) \, dy
$$

$$
= -(4\pi)^{-1} \int (u(y), \nabla_y \times \nabla_x \times (|y - x|^{-1}b) \, dy,
$$

(2.5)

we have $\partial^\alpha_x (\nabla \times A(x)) = O((1 + |x|)^{-3-|\alpha|}) (|x| \to \infty)$ for $|\alpha| \leq 1$. Hence fixing $z \in \mathbb{R}$, we have

$$
\partial^\alpha_x (\nabla \times A(x', z)) = O((1 + |x'|)^{-3-|\alpha|}) (|x'| \to \infty)
$$

(2.6)

for $|\alpha| \leq 1$. In the rest of this proof we will omit $(|x| \to \infty)$ and $(|x'| \to \infty)$ for simplicity.

Now we will apply the following result in [7] for the case $n = 2$ in which we are interested. That is, if $p$, $\delta$ satisfies the condition

$$
p > 2, -\frac{2}{p} - 1 < \delta < -\frac{2}{p} \quad \text{except } \delta + \frac{2}{p} \text{ or } -(\delta + \frac{2}{p}) \in \mathbb{Z}_+,
$$

(2.7)

then $\Delta' : M^2_{p,\delta} \to L^2_{\delta+2}$ is surjective and the kernel of $\Delta'$ is $\mathbb{C}$. Here $L^p_{\delta+2} = L^p_{\delta+2}(\mathbb{R}^2)$ and $M^2_{p,\delta}$ is the completion of $C^\infty_0(\mathbb{R}^2)$ in the norm

$$
\sum_{|\alpha| \leq 2} \| (1 + |x'|^2)^{(\delta+|\alpha|)/2} \partial^\alpha_x h \|_{L^p(\mathbb{R}^2)}
$$

(2.8)

with $x = (x', z) = (x, y, z)$, $\partial^\alpha_x = \partial^i_{x_i} \partial^j_{x_j}$, $i + j = \alpha$ for nonnegative $i, j$ and $L^p_{\delta+2}$ is the $L^2(\mathbb{R}^2)$ with weight $(1 + |x'|^2)^{(\delta+2)/2}$. Let $\delta$ satisfy (2.7), then we have $e_z \cdot (\nabla \times A(x)) \in L^p_{\delta+2}$ because

$$
O\{(1 + |x'|)^{\delta+2}(e_z \cdot (\nabla \times A(x', z)))) = O((1 + |x'|)^{-1+\delta})
$$

and $p(-1+\delta) < -2$ since $\delta < -2/p$. Therefore, the solution $\chi \in M^2_{p,\delta}$ of (2.4) exists.

Let $\mathbb{M}_{p,\delta} = \{ \zeta \in M^2_{p,\delta} : \int_{|x'| \leq R} \zeta(x') \, dx' = 0 \}$ with a large enough fixed $R > 0$. Then, since the kernel of $\Delta'$ is $\mathbb{C}$,

$$
\Delta' : \mathbb{M}_{p,\delta} \to L^p_{\delta+2}
$$

is a bi-continuous isomorphism by Banach’s closed graph theorem and open mapping theorem. Since the mapping

$$
\mathbb{R} \ni z \to e_z \cdot (\nabla \times A(\cdot, z)) \in L^p_{\delta+2}
$$
is locally continuous, the solution $\chi = \chi(\cdot, z) \in M_{p, \delta}$ depends continuously on $z \in \mathbb{R}$.

Furthermore, we can show $\partial_x' \chi \in W^{7,5}_{p, \text{loc}}(\mathbb{R}^3)$. By using the fundamental solution for $\Delta'$, we have following representation \cite{7}, that

$$
\partial_x' \chi(x', z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x' - y'|) \partial_y' f(y', z) dy'.
$$

Above representation holds because $\partial_x' f(x', z) \in L^p_{\delta + 1}(\mathbb{R}^3) \subset L^p(\mathbb{R}^3)$ when we take $\delta + 1 > 0$. So, since $\partial_x' f(x', z) \in W^{5}_{p, \text{loc}}(\mathbb{R}^3)$, we have $\partial_x' \chi \in W^{7,5}_{p, \text{loc}}(\mathbb{R}^3)$. Therefore, we have $\chi \in W^{8,5}_{p, \text{loc}}(\mathbb{R}^3)$.

In a similar way, we can easily check that the existence of a solution $\psi \in M^2_{2, \delta}$ of

$$
\Delta' \psi = - e_z \cdot (\nabla \times \nabla \times A(x)) := g \in W^5_{p, \text{loc}}(\mathbb{R}^3) \quad \text{in} \quad \mathbb{R}^2 \quad (2.9)
$$

and its properties which implies $\partial_x' \psi \in W^{6,4}_{p, \text{loc}}(\mathbb{R}^3)$, where $p, \delta$ satisfy \cite{2.7} and $\delta + 1 > 0$. So, we have $\psi \in W^{7,4}_{p, \text{loc}}(\mathbb{R}^3)$.

Now define $V$ by

$$
V = \nabla \times A - \nabla \times \nabla \times (\chi e_z) - \nabla \times (\psi e_z). \quad (2.10)
$$

Then, we have the following lemma.

**Lemma 2.2.**

$$(\Delta' V)(x', z) = 0 \quad \text{in} \quad \mathbb{R}^2_{x'}, \quad \text{for any} \quad z \in \mathbb{R}. \quad (2.11)$$

**Proof.** Observe that for $V = V_1 e_x + V_2 e_y + V_3 e_z$, we have

$$
\begin{align*}
\text{div} V &= 0, \\
e_z \cdot V &= e_z \cdot (\nabla \times A) - e_z \cdot (\nabla \times \nabla \times (\chi e_z)) = e_z \cdot (\nabla \times A) + \Delta' \chi = 0, \\
e_z \cdot (\nabla \times V) &= e_z \cdot (\nabla \times \nabla \times A) - e_z \cdot (\nabla \times \nabla \times (\psi e_z)) = e_z \cdot (\nabla \times \nabla \times A) + \Delta' \psi = 0.
\end{align*}
$$

(2.12)

Then the proof can be finished by just observing that the first and third equations give the Cauchy Riemann system of equations for $V - (e_z \cdot V) e_z = V_1 e_x + V_2 e_y$. This completes the proof of Lemma 2.2.

To proceed further, we recall the following result in \cite{7}. That is if $\delta'$ satisfies the condition

$$
-\frac{2}{p} < \delta' < -\frac{2}{p} + 1 \quad \text{except} \quad \delta' + \frac{2}{p} \quad \text{or} \quad -(\delta' + \frac{2}{p}) \in \mathbb{Z}_+,
$$

(2.13)

then $\Delta' : M^2_{p, \delta'} \rightarrow L^p_{\delta' + 2}$ is injective. We will consider $\delta' = \delta + 1$ for $\delta$ satisfying \cite{2.7}.

Now, we show that $V \in M^2_{2, \delta'}$ for each fixed $z \in \mathbb{R}$. Since $\nabla \times A = O((1 + |x'|)^{-3})$, we need $p(\delta' - 3) < -2$. It holds from the condition $\delta' < 0$. Hence $\nabla \times A \in M^2_{2, \delta'}$.

To examine the second term of $V$, we need a following lemma in \cite{7}.

**Lemma 2.3.** *If* $u \in L^p_{\delta'}$, $\Delta' u \in L^p_{\delta' + 2}$, *then* $u \in M^2_{p, \delta'}$.
For any constant vector \( b \), we have from (2.3) that
\[
\Delta'(b \cdot \nabla \times \nabla \times (\chi e_z)) = b \cdot \nabla \times \nabla \times ((-e_z \cdot \nabla \times A)e_z) = O((1 + |x'|)^{-5}).
\]
We can easily check that \( p(-5 + \delta' + 2) < -2 \). Hence \( \Delta'(b \cdot \nabla \times \nabla \times (\chi e_z)) \in L^p_{\delta' + 2} \).
Also, for the potential \( \chi \) of (2.4), we have \( \chi \in M^{2, \delta} \) and by the definition of \( M^{2, \delta} \), we have \( b \cdot \nabla \times \nabla \times (\chi e_z) \in L^2_{\delta' + 2} \subset L^2_{\delta'} \) with the \( \delta \) of (2.7). Therefore by Lemma 2.3, we have \( \nabla \times \nabla \times (\chi e_z) \in M^{2, \delta'} \).

As we did in the second term, for the third term of \( V \), we have
\[
\Delta'(b \cdot \nabla \times (\psi e_z)) = b \cdot \nabla \times ((-e_z \cdot \nabla \times \nabla \times A)e_z) = O((1 + |x'|)^{-5})
\]
and hence \( \Delta'(b \cdot \nabla \times (\psi e_z)) \in L^p_{\delta' + 2} \). Also, from \( \psi \in M^{2, \delta} \) with \( \delta \) of (2.7), we have \( \nabla \times (\psi e_z) \in L^2_{\delta'} \). Hence by Lemma 2.2, we have \( \nabla \times (\psi e_z) \in M^{2, \delta'} \).

Therefore we have shown \( V = \nabla \times A - \nabla \times \nabla \times (\psi e_z) - \nabla \times (\chi e_z) \in M^{2, \delta'} \) which together with Lemma 2.2 immediately gives \( V = 0 \). This completes the proof of Theorem 2.1.

\section{Completeness of representation}

In this section we will show the completeness of representation based on the special Helmholtz decomposition. From now on for our convenience, we will use the convention embedding \( L^p \)-spaces into \( L^2 \)-spaces for \( p > 2 \) and bounded domains. For example, \( W^7_p(\Omega) \subset H^7(\Omega), W^{7,5}_p(\Omega) \subset H^{7,5}(\Omega) \). If we don’t use this convention, it will be stated.

We first prepare a Boggio’s type result as lemma for further arguments.

**Lemma 3.1.** Let \( \varphi \in H^{m+2}(\Omega) \), \( m \geq 0 \) be a solution for \( \nabla^2 (\Box_1 \varphi) = 0 \) in \( \Omega \) where
\[
\Box_1 = \nabla^2 + c^2_1, \ c \text{ is constant.}
\]
Then \( \varphi \) can be decomposed into \( \varphi = \varphi_1 + \varphi_2 \) with \( \varphi_1, \varphi_2 \in H^m(\Omega) \) such that \( \nabla^2 \varphi_1 = 0 \) and \( \Box_1 \varphi_2 = 0 \) in \( \Omega \).

**Proof.** Let \( \varphi = \varphi_1 + (\varphi - \varphi_1) \) such that \( \varphi_1 = \frac{1}{c_1} \Box_1 \varphi, \varphi_1 \in H^m(\Omega) \). Then we can easily check that \( \nabla^2 \varphi_1 = 0 \) and \( \Box_1 (\varphi - \varphi_1) = \Box_1 (-\frac{1}{c_1} \nabla^2 \varphi) = 0 \) in \( \Omega \). Hence we can finish the proof by setting \( \varphi_2 = \varphi - \varphi_1 \in H^m(\Omega) \). Thus we have completed the proof of Lemma 3.1.

Now using the lemma given above and Theorem 2.1, we have the following completeness result for homogeneous viscoelastic system. Let \( \Omega \subset \mathbb{R}^3 \) be the cylinder given by \( \Omega = \{(r, \theta, z) : 0 \leq r \leq a, 0 \leq \theta \leq 2\pi, 0 \leq z \leq h\} \) in terms of cylindrical coordinates \( (r, \theta, z) \), where \( a, h \) are positive constants.

**Theorem 3.2.** If \( u \in W^5_p(\Omega) \) is a solution of \( L^p_{\lambda, \mu} u = 0 \) in \( \Omega \), then \( u = \nabla \varphi_1 + \nabla \times (\psi_1 e_z) + \nabla \times \nabla \times (\chi_1 e_z) \), where \( e_z \) is the unit vector in the \( z \) direction, and \( \varphi_1, \psi_1, \chi_1 \) are solutions of

\[
\begin{align*}
(\Delta + \frac{\rho_0^2}{\lambda + 2\mu}) \varphi_1 &= 0, \\
(\Delta + \frac{\rho_0^2}{\mu}) \psi_1 &= 0, \\
(\Delta + \frac{\rho_0^2}{\mu}) \chi_1 &= 0
\end{align*}
\]
in $\Omega$, where $\varphi_1 \in H^5(\Omega)$ and $\psi_1 \in H^{5,2}(\Omega)$, $\chi_1 \in H^{6,3}(\Omega)$.

**Proof.** Since the solution $u$ is in $W_p^6(\Omega)$, we have $u = \nabla \varphi + \nabla \times (\psi_2) + \nabla \times \nabla \times (\chi_2)$ in $\Omega$ with $\varphi \in H_{p,loc}^7(\mathbb{R}^3)$, $\psi \in H_{p,loc}^{7,4}(\mathbb{R}^3)$, $\chi \in H_{p,loc}^{8,5}(\mathbb{R}^3)$ by extending $u$ to $W_p^6(\mathbb{R}^3)$ with compact support (see [10] about this extension) and Theorem 2.1. Then $L_{\lambda,\mu}u = 0$ in $\Omega$ has the form

$$(\lambda + 2\mu)(\nabla \Box_1 \varphi) + \mu \nabla \times (\Box_2 \psi_2) + \mu \nabla \times \nabla \times (\Box_2 \chi_2) = 0 \text{ in } \Omega$$

(3.2)

where $\Box_1 = \Delta + \frac{\rho \omega^2}{\lambda + 2\mu}$, $\Box_2 = \Delta + \frac{\rho \omega^2}{\mu}$. In the rest of the proof we will omit "in $\Omega$" for simplicity. Taking the divergence of (3.2), we have $(\lambda + 2\mu)(\nabla^2 \Box_1 \varphi) = 0$. Here we decompose $\varphi$ using Lemma 3.1 into $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1, \varphi_2 \in H^5(\Omega)$ such that $\Box_1 \varphi_1 = 0$, $\nabla^2 \varphi_2 = 0$.

On the other hand, taking curl of (3.2), we have $\mu \nabla \times \nabla \times (\Box_2 \psi_2) + \nabla \times (\Box_2 \chi_2) = 0$. Taking one more curl there, we have $-\mu \nabla \times \nabla \times (\Box_2 \psi_2) + \nabla \times (\Box_2 \chi_2) = 0$ from the identity $\nabla \times \nabla \times \nabla \times = -\nabla \times \nabla^2$. Let $\Psi := \nabla \times (\psi_2) + \nabla \times \nabla \times (\chi_2)$. Then $\Psi \in H^{6,4}(\Omega)$ and $\nabla^2 \Box_2 \Psi = 0$. Now we decompose $\Psi$ using Lemma 3.1 into $\Psi = \Psi_1 + \Psi_2$ with $\Psi_1, \Psi_2 \in H^{4,2}(\Omega)$ such that

$$\Box_2 \Psi_1 = 0, \quad \nabla^2 \Psi_2 = 0$$

(3.3)

and $\Psi_1 = -\mu(\rho \omega^2)^{-1}\nabla^2 \Psi$, $\Psi_2 = \mu(\rho \omega^2)^{-1}\Box_2 \Psi$. That is $\Psi_1, \Psi_2$ are given by

$$\Psi_1 = -\mu(\rho \omega^2)^{-1}\nabla^2 (\nabla \times (\psi_2) + \nabla \times (\chi_2))$$

$$\Psi_2 = \mu(\rho \omega^2)^{-1}\Box_2 (\nabla \times (\psi_2) + \nabla \times (\chi_2))$$

Hence by defining

$$\psi_1 := -\mu(\rho \omega^2)^{-1}\nabla^2 \psi \in H_{loc}^{5,2}(\mathbb{R}^3), \quad \chi_1 := -\mu(\rho \omega^2)^{-1}\nabla^2 \chi \in H_{loc}^{6,3}(\mathbb{R}^3),$$

(3.4)

$$\psi_2 := -\mu(\rho \omega^2)^{-1}\Box_2 \psi \in H_{loc}^{5,2}(\mathbb{R}^3), \quad \chi_2 := -\mu(\rho \omega^2)^{-1}\Box_2 \chi \in H_{loc}^{6,3}(\mathbb{R}^3),$$

(3.5)

we clearly have

$$\nabla \times (\Box_2 \psi_1 + \nabla \times (\Box_2 \chi_1)) = 0,$$

(3.6)

$$\nabla \times \nabla^2 (\psi_2 + \nabla \times (\chi_2)) = 0.$$

(3.7)

Put $\Xi := \psi_1 + \nabla \times (\chi_1) \in H_{loc}^{5,2}(\mathbb{R}^3)$. Then from (3.6), we have $\nabla \times \Box_2 \Xi = 0$ and hence $\Box_2 \Xi = \nabla h$ for some scalar potential $h \in H^{4,1}(\mathbb{R}^3)$. Here we consider the solvability for the equation $\Box_2 \Xi = \nabla h \in H_{loc}^{3,0}(\mathbb{R}^3)$. We can show that $\Xi_2 := \Box_2^{-1}(\nabla h) \in H^{5,2}(\mathbb{R}^3)$ (see [12]). Define $\Xi_1 := \Xi - \Xi_2 \in H^{5,2}(\mathbb{R}^3)$. Then, we can have a Boggio type decomposition for $\Xi$ as $\Xi = \Xi_1 + \Xi_2$ with $\Box_2 \Xi_1 = 0$ and $\nabla \times \Xi_2 = 0$.

Taking curl on $\Xi_1$, we have

$$\nabla \times \Xi_1 = \nabla \times (\Xi - \Xi_2)$$

(3.8)
\[
\n\nabla \times (\psi e_z) + \nabla \times \nabla \times (\chi e_z)
\]

with

\[
\begin{align*}
\Delta' \psi_1 &= -e_z \cdot (\nabla \times \nabla \times \Xi_1), \\
\Delta' \chi_1 &= -e_z \cdot (\nabla \times \Xi_1).
\end{align*}
\]

(3.9)

Taking \(\Box_2\) on (3.9), we have

\[
\begin{align*}
\Box_2 \Delta' \psi_1 &= 0, \\
\Box_2 \Delta' \chi_1 &= 0.
\end{align*}
\]

Here, we will use the uniqueness for \(\Delta'\) in Theorem 2.1. To do so, we have to show \(\Box_2 \psi_1, \Box_2 \chi_1 \in M_{2,\delta}(\mathbb{R}^2)\) where \(-1 < \delta < 0\) as in the condition (2.13). From (3.4), we have \(\Box_2 \psi_1 \in H^{4,0}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)\) and \(\Box_2 \chi_1 \in H^{4,1}_{loc}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)\). Since \(\delta < 0\), we have \(L^2(\mathbb{R}^3) \subset L^2_{\delta}(\mathbb{R}^3)\). So, we have for fixed \(z \in \mathbb{R}\), \(\Box_2 \psi_1(x', z), \Box_2 \chi_1(x', z) \in L^2_{\delta}(\mathbb{R}^2)\).

Obviously, we have \(\Delta' \Box_2 \psi_1 = \Delta' \Box_2 \chi_1 = 0 \in L^2_{\delta+2}(\mathbb{R}^2)\). By applying the lemma 2.3 for any fixed \(z \in \mathbb{R}\), we have \(\Box_2 \psi_1(x', z), \Box_2 \chi_1(x', z) \in M_{2,\delta}(\mathbb{R}^2)\). So, the uniqueness for \(\Delta'\) give us that

\[
\begin{align*}
\Box_2 \psi_1 &= 0, \\
\Box_2 \chi_1 &= 0.
\end{align*}
\]

Then it follows from the decompositions on \(\varphi, \psi, \chi\) and (3.2) that we have

\[
0 = (\lambda + 2\mu)(\nabla \Box_1 \varphi) + \mu \nabla \times (\Box_2 \psi e_z) + \mu \nabla \times \nabla \times (\Box_2 \chi e_z)
\]

\[
= (\lambda + 2\mu)(\nabla \Box_1 (\varphi_1 + \varphi_2)) + \mu \nabla \times (\Box_2 (\psi_1 + \psi_2) e_z) + \mu \nabla \times \nabla \times (\Box_2 (\chi_1 + \chi_2) e_z)
\]

\[
= (\lambda + 2\mu)(\nabla \Box_1 \varphi_2) + \mu \nabla \times (\Box_2 \psi_2 e_z) + \mu \nabla \times \nabla \times (\Box_2 \chi_2 e_z).
\]

And using \(\nabla^2 \varphi_2 = 0\) and (3.7), we have

\[
\rho \omega^2 (\nabla \varphi_2) + \rho \omega^2 \nabla \times (\psi_2 e_z) + \rho \omega^2 \nabla \times \nabla \times (\chi_2 e_z) = 0.
\]

It follows that our solution \(u\) has the form

\[
u = \nabla \varphi + \nabla \times (\psi e_z) + \nabla \times \nabla \times (\chi e_z)
\]

\[
\begin{align*}
u &= \nabla \varphi_1 + \nabla \times (\psi_1 e_z) + \nabla \times \nabla \times (\chi_1 e_z) \\
&+ \nabla \varphi_2 + \nabla \times (\psi_2 e_z) + \nabla \times \nabla \times (\chi_2 e_z)
\end{align*}
\]

\[
\begin{align*}
u &= \nabla \varphi_1 + \nabla \times (\psi_1 e_z) + \nabla \times \nabla \times (\chi_1 e_z),
\end{align*}
\]

where the three scalar functions \(\varphi_1, \psi_1\) and \(\chi_1\) satisfy (3.1) in \(\Omega\). Thus we have completed the proof of Theorem 3.2.

4 Solution formula to torsional deformation
In this section we will give the solution for a torsional deformation of cylinder. For that we consider again the viscoelastic equations of system

\[ L_{\lambda,\mu}u = 0 \quad \text{in} \quad \Omega, \quad (4.1) \]

where \( \Omega \subset \mathbb{R}^3 \) is the cylinder given by \( \Omega = \{(r, \theta, z) : 0 \leq r \leq a, 0 \leq \theta \leq 2\pi, 0 \leq z \leq h\} \). Decompose the boundary \( \partial \Omega \) of \( \Omega \) into \( D = \{(r, \theta) | 0 \leq r < a, \ 0 \leq \theta \leq 2\pi\} \), \( aD_0 = D \times \{0\} \) and \( D_h = D \times \{h\} \) which are the bottom and top parts of the boundary \( \partial \Omega \), respectively. Then the mixed type boundary condition for torsional deformation is given as

\[
\begin{cases}
t_{rr} = 0 & \text{on } \partial \Omega \ \setminus (D_0 \cup D_h), \\
t_{r\theta} = 0 & \text{on } (D_0 \cup D_h), \\
t_{rz} = 0 & \text{on } \partial \Omega \ \\t_{zr} = 0 & \text{on } D_0, \\
t_{z\theta} = f(r) & \text{on } D_0, \\
u(r, \theta, z) = 0 & \text{on } D_h. \quad (4.2)
\end{cases}
\]

Here we used the notations \( t_{ij}'s \) to denote tractions in cylindrical coordinates whose definitions can be found in [1, p. 74-75] (see the the next page for precise definitions). The function \( f \) represents the torque given at the bottom surface of the cylinder. We assume that it is smooth enough and depends only on the radial variable.

The aim of this section is to give the form of solution \( u \) of (4.1) satisfying the mixed type boundary type boundary condition (4.2) by using Theorem 3.2. To be precise, we can see from (4.21) given later that the solution \( u \) has a form of

\[ u = \nabla \times (\psi(r, z)e_z) \]

for some scalar function \( \psi(r, z) \) satisfying (3.11). This type is well known as torsional wave which involves a displacement in the circumferential direction only (see [2]).

We recall the definitions of \( t_{ij}'s \). For a vector valued function \( u = (u_1, u_2, u_3) \), we define \( u_r, u_\theta \) and \( u_z \) by

\[ u_r := u_1 \cos \theta + u_2 \sin \theta, \quad u_\theta := -u_1 \sin \theta + u_2 \cos \theta, \quad u_z := u_3. \quad (4.3) \]

Then the components of the traction, denoted by \( t_{ij} \) in (4.2), are given by the following formulae:

\[
\begin{align*}
t_{rr} &= \lambda \nabla \cdot u + 2\mu \partial_r u_r, \\
t_{r\theta} &= \mu \left( \partial_r u_\theta - r^{-1} u_\theta + r^{-1} \partial_\theta u_r \right), \\
t_{rz} &= t_{zr} = \mu \left( \partial_z u_r + \partial_r u_z \right), \\
t_{z\theta} &= \mu \left( \partial_z u_\theta + r^{-1} \partial_\theta u_z \right), \\
t_{zz} &= \lambda \nabla \cdot u + 2\mu \partial_z u_z.
\end{align*}
\]

Now we define a basis function for series solution of (4.1) satisfying (4.2). Let \( J_1 \) be the Bessel function of order 1, i.e., the solution to

\[ r^2 J_1''(r) + r J_1'(r) + (r^2 - 1)J_1(r) = 0, \quad (4.5) \]
and let $0 < k_2 < k_3 < \cdots$ be positive numbers such that
\[ k_n J_1'(k_n) = J_1(k_n). \]  
(4.6)

Define $\varphi_n$ by
\[ \varphi_n(r) = \begin{cases} c_1 r, & n = 1, \\ c_n J_1(k_n r), & n \geq 2, \end{cases} \]  
(4.7)

where $c_n$ is a normalization constant such that
\[ \| \varphi_n \|^2 := \int_0^1 |\varphi_n(r)|^2 r dr = 1. \]  
(4.8)

We note that $\varphi_n$ ($n \geq 2$) satisfies
\[ r^2 \varphi_n''(r) + r \varphi_n'(r) + (k_n^2 r^2 - 1) \varphi_n(r) = 0, \]  
(4.9)

and
\[ \varphi_n'(1) = \varphi_n(1). \]  
(4.10)

It is known that $\{ \varphi_n \}_{n=1}^{\infty}$ forms a complete orthonormal system in $L^2((0,1), r dr)$ (see [11]).

Let $a$ be the radius of the cylinder, and define
\[ \varphi_n^a(r) := a^{-1} \varphi_n(a^{-1} r). \]  
(4.11)

Then $\{ \varphi_n^a \}_{n=1}^{\infty}$ is a complete orthonormal system in $L^2((0,a), r dr)$. So, the torque $f$ can be expanded as
\[ f(r) = \sum_{n=1}^{\infty} f_n \varphi_n^a(r). \]  
(4.12)

Moreover, one can see from (4.9) that the following holds:
\[ r^2 (\varphi_n^a)''(r) + r (\varphi_n^a)'(r) + \left( \frac{k_n}{a} \right)^2 r^2 - 1 \varphi_n^a(r) = 0. \]  
(4.13)

Let $k^2 = \rho \omega^2 / \mu$ and define $\gamma_n$ by
\[ \gamma_n^2 = \begin{cases} -k^2, & n = 1, \\ k_n^2 / a^2 - k^2, & n \geq 2. \end{cases} \]  
(4.14)

Also let
\[ q_n(z) = \frac{e^{\gamma_n (h-z)} - e^{\gamma_n (z-h)}}{e^{-\gamma_n h} + e^{\gamma_n h}}, \quad n = 1, 2, \cdots. \]  
(4.15)

Then we have the following lemma and theorem (see [6, Section2] for their proofs).
Lemma 4.1. For $n = 1, 2, \cdots$ let

$$v_n(r, z) = \frac{1}{\mu \gamma_n} q_n(z) \varphi_n^a(r)$$

(4.16)

and

$$\Theta(\theta) := \sin \theta e_1 - \cos \theta e_2,$$

(4.17)

where $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$. Then $w_n(r, \theta, z) := v_n(r, \theta, z) \Theta(\theta)$ is the solution of mixed type boundary value problem (4.1), (4.2) with $f$ replace by $f_n \varphi_n^a(r)$.

Theorem 4.2. Let $f = \sum_{n=1}^{\infty} f_n \varphi_n^a(r)$ satisfy

$$\sum_{n=1}^{\infty} \frac{|f_n|^2}{n} < \infty,$$

(4.18)

or equivalently $f(r, \theta) := f(r) \Theta(\theta)$ belong to $H^{-1/2}(D_0)$. Then $u$ defined by

$$u = \sum_{n=1}^{\infty} f_n w_n = \sum_{n=1}^{\infty} f_n v_n(r, \theta) \Theta(\theta)$$

(4.19)

is the unique solution of the mixed type boundary value problem (4.1), (4.2) in $H^1(\Omega)$. Further $u$ satisfies the estimate

$$\|u\|_{H^1(\Omega)} \leq C \|f\|_{H^{-1/2}(D_0)}.$$

(4.20)

with some constant $C$ independent of $f$.

We remark on Theorem 4.2 that if we define $\psi$ by

$$\psi(r, z) := \sum_{n=1}^{\infty} \left( \int_{0}^{r} \varphi_n^a(s) ds \right) \frac{f_n}{\mu \gamma_n} q_n(z),$$

then $u$ can be expressed as

$$u = \nabla \times (\psi e_z).$$

(4.21)

where $\psi$ satisfies (3.1). This is really giving the formula of solution based on the completeness of representation.

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