On the Associativity of Star Product in Systems with Nonlinear Constraints

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Abstract

The noncommutative star product of phase space functions is, by construction, associative for both non-degenerate and degenerate case (involving only second class constraints) as has been shown by Berezin, Batalin and Tyutin. However, for the latter case, the manifest associativity is lost if an arbitrary coordinate system is used but can be restored by using an unconstrained canonical set. The existence of such a canonical transformation is guaranteed by a theorem due to Maskawa and Nakajima. In terms of these new variables, the Kontsevich series for the star product reduces to an exponential series which is manifestly associative. We also show, using the star product formalism, that the angular momentum of a particle moving on a circle is quantized.

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1 Introduction

Noncommutative geometry\footnote{Groenewold} is one of the most important areas of investigation these days. It is also related to deformation quantization\footnote{Berezin, Batalin and Tyutin} where ordinary commutative multiplication between c-number valued functions of classical phase space variables is replaced by noncommutative star product which is supposed to be associative in nature. For a simple classical system like a particle moving on a real line, the construction of the star product can be motivated by considering the set of Weyl ordered phase space operators and its isomorphism to the set of classical phase space functions. (See \footnote{Groenewold} for a review.) The rule of composition of the phase space operators is ordinary multiplication of Weyl ordered operators whereas it is given by the star product in the space of c-number valued classical functions. This has been shown by Groenewold\footnote{Groenewold} and later through a geometric approach by Berezin\footnote{Berezin}, Batalin and Tyutin\footnote{Batalin and Tyutin}. Recent studies in string theory also indicate that at a very small length scale the nature of space-time co-ordinates may become noncommutative\footnote{Recent studies in string theory also indicate that at a very small length scale the nature of space-time co-ordinates may become noncommutative}.
One problem that plagues the definition of star product is that it is not manifestly associative. This is despite the fact that the star product, by construction, is taken to be associative both in the non-degenerate and degenerate cases (i.e., in presence of second class constraints) in the geometric approach of Batalin and Tyutin \cite{7}. Since the star product is of fundamental importance in defining noncommutative space-time, it is essential to make a reappraisal of the problem of associativity of this product. Also it becomes important to compare the non-geometrical approach of Groenewold \cite{3, 4} with the geometrical approach \cite{7}.

In the above mentioned example of a particle moving on the real line $\mathbb{R}^1$ the classical phase space is given by the 2-d space $\mathbb{R}^2$. As has been shown in \cite{3}, the star product between two phase space functions $f(x, p)$ and $g(x, p)$ can be expressed in a Fourier representation by an exponential factor involving the area of a triangle so that the associative nature becomes manifest. However, this property of associativity becomes more involved for systems involving nonlinear constraints, as happens, for the motion of a particle on a $n$-sphere($S^n$). For such systems, Kontsevich discovered an elaborate graphical rule for the generation of the appropriate associative star product as a series in $\hbar$ \cite{12}. In this expansion, using Cartesian coordinates, one can see easily that the series deviates from the pure exponential form right from $O(\hbar^2)$ term onwards. This indicates that for these systems simple exponentiation of the symplectic bracket kernel, having a noncanonical structure, will not yield an associative star product. This is in contrast with the above discussed unconstrained case of a particle moving on a real line where simple exponentiation of the Poisson bracket kernel yields an associative star product.

On the other hand, it has been shown in \cite{7} that for any degenerate case, involving only second class constraints, it is still possible to express the star product through an exponential factor involving the area of a geodesic triangle which reduces to a rectilinear triangle, as in the non-degenerate case, on the constraint surface.

At this stage, one can therefore ask whether it is possible to make a suitable canonical(point) transformation where the Dirac bracket reduces to the usual Poisson bracket in terms of the physical(generalized) coordinates and their conjugate momenta with the symplectic matrix $J$ taking the canonical form. An answer to this question is provided by a well known theorem of Maskawa and Nakajima(MN)\cite{13}, which states that for any system described by a set of canonical variables $\Psi_a, \Pi_a$ ($a = 1, 2, ...N$) and governed by second class constraints, it is possible by a canonical transformation, to construct two sets of independent variables $Q_n, \bar{Q}_r$ ($n = 1, 2, ...M$, $r = M + 1, ...N$) and their respective canonical conjugates $P_n, \bar{P}_r$, such that the constraints read $\bar{Q}_r \approx 0, \bar{P}_r \approx 0$. Then the symplectic(Dirac) brackets of the system calculated with respect to the entire set of variables reduce to Poisson brackets with respect to the unconstrained physical variables $Q_n, P_n$ of the system. A possible implication of this theorem would be that, when expressed in terms of the unconstrained variables, the noncanonical Dirac bracket kernel reduces to a canonical Poisson bracket kernel and a simple exponentiation of which should then yield an associative star product.

The objective of this paper is to study the similarities and dissimilarities of the above mentioned non-geometric \cite{4, 5} and geometric \cite{7} approaches through the construction of the MN variables explicitly for a few simple constrained systems. In this paper we consider the examples of the Landau problem, particles constrained to move on a circle($S^1$) and a sphere($S^2$).
The organization of the paper is as follows. Section 2 provides a review of the historical origin, definition and certain properties of star product that are relevant to this work. A brief description of MN theorem is also provided. Section 3 discusses the Landau problem. Here the constraints are linear and it is straightforward to find the canonical transformation that reduces the Dirac bracket to the Poisson bracket using the Maskawa-Nakajima theorem. A simple exponentiation leads to an associative star product. Section 4 analyses the problem of the motion of a particle on a circle($S^1$) and a sphere($S^2$). As is well known, here the constraints are nonlinear leading to variable dependent symplectic or Dirac brackets. A naive exponentiation of the bracket kernel does not yield an associative star product. A canonical transformation, which is actually the transition from the Cartesian to the spherical polar basis, is employed. Finally, after passing to the constraint shell, we show that the bracket kernel simplifies to a canonical structure. Now a suitable exponentiation yields an associative star product, exactly as happens in the unconstrained case. As a byproduct of this analysis, we get the usual quantization of angular momentum, modulo a half factor, in the case of a particle moving on a circle. Our conclusions are given in section 5.

2 Review of Star Product Formalism

In this section we briefly review the basic ideas of star product formalism essentially following [3, 4]. The historical origin of the definition of star product can be traced back to Groenewold’s work [4] wherein it was shown that the space of classical phase space functions $f(x, p)$ and the corresponding space of Weyl ordered operators $\hat{f}(\hat{x}, \hat{p})$ can have an isomorphism, provided the classical functions compose through the star product. The function $f(x, p)$ is called the classical kernel of $\hat{f}(\hat{x}, \hat{p})$. Let us consider the case of a particle moving on a real line $\mathbb{R}^1$ as an illustrative example. Clearly the classical phase space $(x, p)$ is the two dimensional space $\mathbb{R}^2$. An arbitrary phase space function $f(x, p)$ can be written as

$$f(x, p) = \int_{-\infty}^{+\infty} dx' dp' \delta(x - x') \delta(p - p') f(x', p')$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dx' dp' d\tau d\sigma e^{i(\tau(x-x') + \sigma(p-p'))} f(x', p')$$

where the integral representation

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau e^{i\tau(x-x')}$$

of the Dirac delta function $\delta(x - x')$ and a similar representation for $\delta(p - p')$ are used. Here $\delta(x - x')$ (3) represents the completeness property of the exponential functions $e^{i\tau x}$ (which are eigenfunctions of the Laplacian operator in $\mathbb{R}^1$) characterized by $\tau$ while $\delta(p - p')$ represents the orthonormality. In this case, the phase space variables and hence all the integration variables have noncompact ranges. At the quantum level, the operator analogues $\hat{x}, \hat{p}$ of $x, p$ obey the Heisenberg-Weyl Lie algebra

$$[\hat{x}, \hat{p}] = i\hbar, \quad [\hat{x}, \hat{x}] = 0, \quad [\hat{p}, \hat{p}] = 0$$

(3)
and \( \exp[i(\tau \hat{x} + \sigma \hat{p})] \) is a particular element of the corresponding Lie group.

Weyl’s prescription [14] for arriving at the operator \( \hat{f}(\hat{x}, \hat{p}) \) corresponding to the kernel \( f(x, p) \) (taken to have a polynomial form) consists of rewriting \([1] \) with the replacements \( x \to \hat{x}, p \to \hat{p} \) to get

\[
\hat{f}(\hat{x}, \hat{p}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dx' dp' d\tau d\sigma e^{i[\tau(\hat{x} - x') + \sigma(\hat{p} - p')]} f(x', p').
\]  

(4)

An equivalent prescription due to Batalin and Tyutin [7] is to define

\[
\hat{f}(\hat{x}, \hat{p}) = e^{[\hat{x}\partial_x + \hat{p}\partial_p]} f(x, p)|_{x=p=0}
\]

(5)

We are however continuing with the prescription \([1] \) for the time being. The quantum operator \( \hat{f} \), regarded as a power series in \( \hat{x} \) and \( \hat{p} \), is first ordered in a completely symmetrized manner using \([1] \) and a term with \( m \) powers of \( \hat{x} \) and \( n \) powers of \( \hat{p} \) is then given by the coefficient of \( (\tau^m\sigma^n) \) in the expansions of \( (\tau \hat{x} + \sigma \hat{p})^{m+n} \).

Using the mapping \([1] \), Groenewold later obtained the classical kernel of the operator product \( \hat{f}\hat{g} \) of two phase space operators \( \hat{f} \) and \( \hat{g} \) from the corresponding kernels \( f \) and \( g \) respectively. For that one has to express \( \hat{g}(\hat{x}, \hat{p}) \) just in the manner of \( \hat{f} \) in \([4] \). One can then write

\[
\hat{f}\hat{g} = \frac{1}{(2\pi)^4} \int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x', p') g(x'', p'') \times \exp i(\xi(\hat{p} - p') + \eta(\hat{x} - x')) \exp i(\xi'(\hat{p} - p'') + \eta'(\hat{x} - x''))
\]

(6)

\[
= \frac{1}{(2\pi)^4} \int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x', p') g(x'', p'') \exp i((\xi + \xi')\hat{p} + (\eta + \eta')\hat{x}) \times \exp i\left(-\xi p' - \eta x' - \xi' p'' - \eta' x'' + \frac{\hbar}{2}(\xi\eta' - \eta\xi')\right).
\]

Changing integration variables to

\[
\xi' \equiv \frac{2}{\hbar}(x - x'), \quad \xi \equiv \tau - \frac{2}{\hbar}(x - x'), \quad \eta' \equiv \frac{2}{\hbar}(p' - p), \quad \eta \equiv \sigma - \frac{2}{\hbar}(p' - p),
\]

(7)

reduces the above integral to

\[
\hat{f}\hat{g} = \frac{1}{(2\pi)^2} \int d\tau d\sigma dp dp' dp'' dx dx' dx'' f(x', p') g(x'', p'') \left[ \frac{1}{(\pi\hbar)^2} \exp \left(-\frac{2i}{\hbar} (p(x' - x'') + p'(x'' - x) + p''(x - x'))\right) \right] \}
\]

(8)

The choice of the origin \( x = p = 0 \) for evaluating \( \hat{f}(\hat{x}, \hat{p}) \) is not mandatory. The operator \( \hat{f}(\hat{x}, \hat{p}) \), evaluated at different points, are in fact related by canonical transformations \([1] \).
In the above equation, consider the exponential inside the square bracket,

\[
\frac{1}{(\pi\hbar)^2} \exp \left( \frac{-2i}{\hbar} \left( p(x' - x'' + p'(x'' - x) + p''(x - x')) \right) \right)
\]

\[
= \frac{1}{(\pi\hbar)^2} \exp \left( \frac{i}{\hbar} \left( -2(p' - p)(x'' - x) + 2(x' - x)(p'' - p) \right) \right)
\]

\[
= \frac{1}{(2\pi)^2} \int d\lambda d\mu \delta(x' - x - \frac{\mu\hbar}{2}) \delta(p' - p + \frac{\lambda\hbar}{2}) \exp \left( i(\lambda(x'' - x) + \mu(p'' - p)) \right)
\]

\[
= \frac{1}{(2\pi)^4} \int d\lambda d\mu d\alpha d\beta \exp \left[ i[\alpha(x' - x) + \beta(p' - p)] \right] \exp \left( \frac{i\hbar}{2} (\hat{\partial}_x \hat{\partial}_p - \hat{\partial}_p \hat{\partial}_x) \exp \left( i(\lambda(x'' - x) + \mu(p'' - p)) \right) \right)
\]

where the representation (2) is used. With the aid of the above relation one can write the integral in the curly bracket in (8) as,

\[
\frac{1}{(2\pi)^4} \int d\lambda d\mu d\alpha d\beta \exp \left[ i[\alpha(x' - x) + \beta(p' - p)] \right] \exp \left( \frac{i\hbar}{2} (\hat{\partial}_x \hat{\partial}_p - \hat{\partial}_p \hat{\partial}_x) \exp \left( i(\lambda(x'' - x) + \mu(p'' - p)) \right) \right)
\]

F\times g(x', p') g(x'', p'')
\]

\[
= f(x, p) e^{\frac{i\hbar}{2} (\hat{\partial}_x \hat{\partial}_p - \hat{\partial}_p \hat{\partial}_x)/2} g(x, p).
\]

Hence the composition rule is given by,

\[
\hat{f} \hat{g} = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \exp \left[ i(\tau(\hat{p} - p) + \sigma(\hat{x} - x)) \right] (f \ast g)(x, p)
\]

where the \ast product is defined as,

\[
f(x, p) \ast g(x, p) \equiv f(x, p) e^{\frac{i\hbar}{2} (\hat{\partial}_x \hat{\partial}_p - \hat{\partial}_p \hat{\partial}_x)} g(x, p).
\]

Thus Groenewold showed that \( f(x, p)^{\ast} g(x, p) \) is the classical kernel of \( \hat{f} \hat{g} \). But this demonstration holds only for systems where particles are moving in \( \mathbb{R}^n \) and corresponds to a non-degenerate case. The same results have been obtained in \( \mathbb{R}^n \) using a geometrical approach which unlike this derivation does not make use of any of the correspondences given in (4) or even (3). i.e., they do not invoke the operators \( \hat{x} \) and \( \hat{p} \). They determine the \ast product just by demanding associativity, i.e.,

\[
(f \ast g) \ast h = f \ast (g \ast h),
\]

neutrality of unity i.e.,

\[
1 \ast f = f \ast 1 = f
\]

and the classical correspondence

\[
\lim_{\hbar \to 0} (f \ast g) = f \cdot g
\]
where \([f, g]_* = f \ast g - g \ast f\) is the Moyal bracket. Eq (14) clearly demonstrates that \(\ast\) multiplication is really a deformation of the ordinary multiplication. The latter is restored when the deformation parameter \(\hbar \to 0\). Since the star product involves exponentials of derivative operators, it is possible in certain cases to write it as a translation in function arguments;

\[
f(x, p) \ast g(x, p) = f \left( x + \frac{i\hbar}{2} \partial_p, \ p - \frac{i\hbar}{2} \partial_x \right) g(x, p)
\]

so that, as a corollary, we get the Moyal brackets

\[
[x, p]_* = i\hbar, \quad [x, x]_* = [p, p]_* = 0
\]

which is the counterpart of the Heisenberg-Weyl Lie algebra (3). Clearly these Moyal brackets trivially satisfy the conditions (14) and (15). Conversely, using the defining relation (11) it is easy to cross-check algebraically that the star product is indeed associative owing to the form of the exponential involved in the relation. Although the ranges of the phase space variables \((x, p)\) is \((-\infty, +\infty)\), this algebraic demonstration does not depend on this. The associativity of the star product in this case (1-dimensional free particle) also follows from a geometrical approach\[3\]. The expression multiplying \((-2i/\hbar)\) in the second exponential of (8) can be written as

\[
2A(r'', r', r) = (r' - r) \times (r - r'') = r'' \times r' + r' \times r + r \times r''
\]

where \(A(r'', r', r)\) is the area of the phase space triangle \((r'', r', r)\), with \(r \equiv (x, p)\). Thus \(f \ast g\) can be expressed in a more compact form as,

\[
f \ast g(r) = \frac{1}{(\pi \hbar)^2} \int dr'dr'' f(r') g(r'') \exp \left( \frac{4}{i\hbar} A(r'', r', r) \right).
\]

This same form for the exponent involving the area of a triangle in phase space was obtained in [7] in a coordinate independent manner. There it involved the integral of a closed symplectic 2-form

\[
J = \frac{1}{2} J_{ij} dz^i \land dz^j; \quad dJ = 0
\]

along a surface spanned by the contour of a geodesic triangle in a flat (i.e., vanishing Riemann tensor\[4\]) even-dimensional phase space having \(z^i\) as coordinates. The sides of the geodesic triangle are determined relative to the symmetric connection which provides a covariant constancy of the symplectic 2-form. As a matter of notation the matrix \(\{J^{ij}\}\) - the inverse of the matrix \(\{J_{ij}\}\) occurring in (20):

\[
J^{ij} J_{jk} = \delta^i_k
\]

is used to define Poisson bracket as

\[
\{f, g\} = f \partial g \equiv f \left( \sum_{i,j} \overset{\leftrightarrow}{\partial_i} J^{ij} \overset{\leftrightarrow}{\partial_j} \right) g
\]

5Note that the phase space is not endowed with any metric as such.
and Λ is referred to as the Poisson bracket kernel. Since the phase space is taken to be flat, Darboux theorem ensures the existence of a suitable coordinate system, where \{J_{ij}\} becomes a constant matrix and the connection vanishes so that the above mentioned geodesic triangle reduces to a rectilinear triangle. By suitable transformation involving scaling etc, this matrix can be further reduced to the following canonical form

\[ \{J_{ij}\} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \]  \hspace{1cm} (23)

Note that the star product defined in (11) is nothing but the exponential of the Poisson bracket kernel Λ sandwiched between the functions \( f \) and \( g \) with the matrix \( \{J_{ij}\} \) given in (23).

Now the associativity property of the star product can be easily seen from this representation because of the appearance of the area of a phase space triangle in the exponent. For triple star product involving functions \( f, g \) and \( h \) we have

\[
(f \star g) \star h = \frac{1}{\hbar^4 \pi^4} \int d\mathbf{r} d\mathbf{r}' d\mathbf{r}'' d\mathbf{r}''' f(\mathbf{r}') g(\mathbf{r}'') h(\mathbf{r}''') \exp \left[ \frac{4}{i\hbar} \left( A(\mathbf{r}'', \mathbf{r}', \mathbf{r}) + A(\mathbf{r}''', \mathbf{r}'', \mathbf{r}) \right) \right].
\]  \hspace{1cm} (24)

The integrations can be exploited to obtain corresponding \( \delta \)-functions and the product can be simplified to

\[
((f \star g) \star h)(\mathbf{r}) = \frac{1}{\hbar^2 \pi^2} \int d\mathbf{r}' d\mathbf{r}'' d\mathbf{r}''' f(\mathbf{r}') g(\mathbf{r}'') h(\mathbf{r}''') \times \delta(\mathbf{r} - \mathbf{r}' + \mathbf{r}'' - \mathbf{r}''') \exp \left( \frac{4}{i\hbar} A(\mathbf{r}''', \mathbf{r}'', \mathbf{r}', \mathbf{r}) \right).
\]  \hspace{1cm} (25)

The argument of the delta functions in the above expression ensures that \( A(\mathbf{r}''', \mathbf{r}'', \mathbf{r}', \mathbf{r}) \) is the area of the parallelogram with vertices \((\mathbf{r}''', \mathbf{r}'', \mathbf{r}', \mathbf{r})\). Therefore, the order in which the functions \( f, g \) and \( h \) are associated becomes immaterial thus providing pictorial proof of the associativity.

The Groenewold derivation and the area interpretation following from it, as discussed above relied crucially on the Fourier expansion (1) and is related to the fact that the simple exponential functions are eigenfunctions of the Laplacian operator \( (\partial^2 / \partial x^2) \) in \( \mathbb{R}^1 \). These exponential functions provide a complete orthonormal basis for the space of functions defined on \( \mathbb{R}^1 \). Therefore, this may not necessarily be implementable in the case of physical systems where the eigenfunctions of the Laplacian operator, although form a complete set of basis, are not given by simple exponential functions. For example, in the case of a particle constrained to \( S^2 \), to be discussed subsequently, the expansion corresponding to (1) should be obtained from an appropriate harmonic analysis where the spherical harmonics \( Y_{lm}(\theta, \phi) \) are the eigenfunctions of the Laplacian operator and provide the complete set of basis. Here the \( \theta \) dependence is not given by the exponential functions anymore. Nevertheless, the geometrical approach of Batalin and Tyutin \[7\] shows that here too one gets a geodesic triangle in the exponent of the star product, thus demonstrating the superiority of the geometrical approach.

It is also very clear from the above analysis that the star product between classical phase space functions incorporates some of the quantum features right at the classical level. This gives rise to the possibility that quantum mechanics can be formulated in terms of the classical phase
space variables [3]. This is because the classical phase functions must compose, as we have seen, through the star product in order to have an isomorphism between the set of classical phase space functions and Weyl ordered operators. We shall see another beautiful demonstration of this in section 4, where we will be dealing with a particle moving on a circle.

Coming to the degenerate case, i.e., for systems with nonlinear second class constraints the star product is to be obtained by the exponentiation of the Dirac bracket kernel as has been shown in [7]. But unlike the Poisson bracket kernel, the Dirac bracket kernel may not in general lead to a manifestly associative star product if Cartesian coordinates are used. This is because the $J_{ij}$ are now dependent on the phase space variables. To circumvent this difficulty, Kontsevich[12] devised detailed graphical rules wherein the star product can be expressed as a series expansion in $\bar{\hbar}$.

\[
f \star g = f[1 + i\hbar \Hat{\partial}_i J_{ij} \Hat{\partial}_j - \frac{\hbar^2}{2} \left(\Hat{\partial}_i \Hat{\partial}_k J_{ij} J_{kl} \Hat{\partial}_j \Hat{\partial}_l \right) - \frac{\hbar^2}{3} \left(\Hat{\partial}_i \Hat{\partial}_k J_{ij} J_{kl} \Hat{\partial}_j \Hat{\partial}_l - \Hat{\partial}_k J_{ij} J_{kl} \Hat{\partial}_i \Hat{\partial}_l \right) + O(\hbar^3)]g. \tag{26}\]

One can see that the series deviates from the exponential right from the $O(\hbar^2)$ term onwards. However associativity is restored. One may also notice that if the $J_{ij}$’s are constants(as in the case of the Poisson bracket kernel in Darboux coordinates), the extra terms vanish thereby restoring the exponential form of the series. Associativity of the star product is then automatically guaranteed.

It is here that the Maskawa-Nakajima theorem [13] provides a clue as to how one can obtain a manifestly associative star product for second class constraint systems without using the Kontsevich series because of the possibility of reducing the Dirac bracket kernel to Poisson bracket kernel by MN’s appropriate canonical transformations. We now give a brief outline of the method [16]. According to Dirac’s prescription [17], for a system with second class constraints the Poisson bracket of two phase space functions $A$ and $B$ should be generalized to the Dirac bracket(DB):

\[
\{A, B\}_{DB} = \{A, B\} - \{A, \Omega_i\}(C^{-1})_{ij}\{\Omega_j, B\} \tag{27}\]

where $\Omega_i \approx 0$ are the constraints of the system and

\[
C_{ij} \equiv \{\Omega_i, \Omega_j\} \tag{28}\]

is the matrix of the Poisson brackets of the constraints. The Maskawa-Nakajima theorem [13] states that for a second class constraint system there exists a set of independent canonical variables $(Q_n, \bar{Q}_r, P_n, \bar{P}_r)$ where $P_n, \bar{P}_r$ are canonical conjugates of $Q_n, \bar{Q}_r$ respectively, such that the constraints of the system are

\[
\Omega_{1r} = \bar{Q}_r \approx 0, \quad \Omega_{2r} = \bar{P}_r \approx 0 \tag{29}\]

After the rearrangement, the elements of the matrix $C$ read

\[
C_{1r,2s} = \{\bar{Q}_r, \bar{P}_s\} = \delta_{rs} \quad C_{1r,1s} = \{\bar{Q}_r, \bar{Q}_s\} = 0 \quad C_{2r,2s} = \{\bar{P}_r, \bar{P}_s\} = 0 \tag{30}\]
and Poisson brackets of any function $A$ with the constraints now become

$$\{A, \Omega_1\} = -\frac{\partial A}{\partial P_r}, \quad \{A, \Omega_2\} = \frac{\partial A}{\partial Q_r}$$  \hspace{1cm} (31)

Since the $C$ matrix has the property $C^{-1} = -C$, the Dirac bracket can be written as

$$\{A, B\}_{DB} = \{A, B\} + \{A, \Omega_1\}\Omega_2 - \{A, \Omega_2\}\Omega_1$$

$$ = \{A, B\} - \frac{\partial A}{\partial Q_r} \frac{\partial B}{\partial P_r} + \frac{\partial B}{\partial Q_r} \frac{\partial A}{\partial P_r}$$  \hspace{1cm} (32)

$$= \frac{\partial A}{\partial Q_n} \frac{\partial B}{\partial P_n} - \frac{\partial B}{\partial Q_n} \frac{\partial A}{\partial P_n}$$  \hspace{1cm} (33)

That is, the Dirac bracket is equal to a Poisson bracket computed with respect to the reduced set of unconstrained physical variables $(Q_n, P_n)$. Since the Poisson bracket kernel can be exponentiated to yield an associative star product, the problem of constructing a manifestly associative star product for a second class system reduces to that of finding out the correct set of canonical variables in terms of which the Dirac brackets become equal to Poisson brackets. The existence of such canonical variables is guaranteed by the Maskawa-Nakajima theorem though the actual task of obtaining those variables for a given system may be highly nontrivial. Once the proper variable have been identified the relevant bracket kernel, after passing to the constraint shell, assumes the standard form with the $J$ becoming a constant symplectic matrix. Associativity of the star product then follows naturally.

Here we would like to mention that the Maskawa-Nakajima theorem, strictly speaking, ensures the existence of only a local coordinate system that separates constrained and unconstrained variables. But the Fourier transform reflects the global topology, as we have discussed earlier with the example of $S^2$. As we shall see later (in section 4.3), the polar coordinates $(\theta, \phi)$ can be identified with the physical variables for $S^2$. Clearly these variables are not defined globally, as the north and south poles have to be excluded to define these coordinates unambiguously. In fact, a single coordinate chart which covers the entire $S^2$ does not exist at all. Also the choice of spherical harmonics $Y_{lm}(\theta, \phi)$ as the basis of expansion, satisfying both orthonormality and closure relations, although reflects the global topology of the base manifold $S^2$, can only be defined in a coordinate neighborhood.

However the absence of a single coordinate chart covering the entire $S^2$ is not a serious problem, as one can construct an atlas comprising of several compatible charts covering the entire $S^2$. Since the transition from one chart to another amounts to a point canonical transformation, it is clear that the final Poisson bracket kernel will remain invariant under such a transformation. We show this through a simple example in section 4.1, where the canonical form of the bracket kernel (in $S^1$) is preserved as one makes a transition from polar to stereographically projected coordinates. Besides, the algebraic demonstration of associativity of the star product for a degenerate (second class constrained) system through order by order ($\hbar$) expansion becomes exactly similar to the non-degenerate case once the Dirac bracket kernel
is reduced to the Poisson bracket kernel through the identification of the physical variables, whose existence is ensured by MN theorem. The curvilinear nature of these coordinates then become completely inconsequential. Also one does not need to invoke the “triangle argument”.

For simple second class systems like the Landau problem, a particle constrained to move along a circle or on the surface of a sphere etc., it is possible to obtain the exact form of these unconstrained physical variables from symmetry considerations. Usually, star products even for these systems are defined in a complicated manner using the Kontsevich series expansion method. In the subsequent sections we demonstrate with the help of these systems, how one can arrive at a manifestly associative star product, although their Dirac brackets in the Cartesian system do not reveal this.

3 Landau Problem

The Lagrangian for the classical Landau problem of a spinless charged particle moving on a two dimensional plane, in a constant background magnetic field $B$, can be written as,

$$L = \frac{m}{2} \dot{x}^2 + \frac{B}{2} \dot{x} \times \dot{x} - \frac{k}{2} x^2,$$  

(34)

where a harmonic oscillator potential is chosen. When the magnetic field is very strong we can suppress the kinetic term (which is equivalent to the limit $m \to 0$) and the Lagrangian then describes a first order constrained system. Writing the kinetic term in the component form, we get,

$$L = \frac{B}{2} x_i \epsilon_{ij} \frac{dx_j}{dt} - \frac{k}{2} x^2$$

(35)

The canonically conjugate momenta are constrained to the respective transverse coordinates,

$$\Omega_i = p_i + \frac{B}{2} \epsilon_{ij} x_j \approx 0$$

(36)

giving the two second class constraints. The corresponding Dirac brackets are given by

$$\{x_i, x_j\}_{DB} = -\frac{\epsilon_{ij}}{B}, \quad \{x_i, p_j\}_{DB} = \frac{\delta_{ij}}{2}, \quad \{p_i, p_j\}_{DB} = -\frac{B}{4} \epsilon_{ij}.$$  

(37)

Thus the two perpendicular directions $x_1$ and $x_2$ do not commute (in the sense of classical Dirac brackets), rather they behave as canonical conjugates to each other. The Dirac bracket for two phase space functions $f$ and $g$ can be written as

$$\{f, g\}_{DB} = f(\partial_{x_i} \frac{1}{2} \partial_{p_i} - \partial_{p_i} \frac{1}{2} \partial_{x_i} - \epsilon_{ij} \frac{1}{B} \partial_{x_j} - \partial_{p_i} \frac{1}{4} \partial_{p_j})g \equiv f \Lambda g$$

with

$$f \Lambda g = f \partial_{\mu} J^{\mu \nu} \partial_{\nu} g.$$  

(38)
where the symplectic matrix is now given by

\[
\{J^{\mu \nu}\} = \begin{pmatrix}
0 & -\frac{1}{B} & \frac{1}{2} & 0 \\
\frac{1}{B} & 0 & 0 & \frac{1}{2} \\
-\frac{1}{2} & 0 & 0 & -\frac{B}{4} \\
0 & -\frac{1}{2} & \frac{B}{4} & 0
\end{pmatrix}.
\]  

(39)

In the example considered, the space where the antisymmetric matrix \(\{J^{\mu \nu}\}\), \((\mu, \nu = 1, \ldots, 4)\) acts is 4-dimensional (in the order \(x_1, x_2, p_1, p_2\)). Although \(\{J^{\mu \nu}\}\) does not have the canonical form (23) the exponentiation of this Dirac bracket kernel gives an associative star product as one can see immediately. This is because the \(J\) matrix is a constant matrix arising from the linear form of the constraints (36). The corresponding phase space coordinates \((x_i, p_j)\) can therefore already be identified with Darboux coordinates. This, however, will not be true for systems involving nonlinear constraints. There one has to isolate the physical degrees of freedom, following a systematic technique.

To see the power of this technique of working with the physical variables, we apply it to the present case and show that the associativity becomes even more transparent, in the sense that \(\{J^{\mu \nu}\}\) matrix (39) takes the canonical form (23) and it becomes similar to an unconstrained system. To that end, make a canonical transformation from \((x_1, x_2, p_1, p_2)\) variables to a new set of variables given by \((\bar{X}, X, \bar{P}, P)\), where \((X, P)\) and \((\bar{X}, \bar{P})\) are the new independent canonical pairs,

\[
\begin{align*}
\bar{X} &= \frac{p_1}{2} + \frac{B}{4} x_2 \\
\bar{P} &= \frac{2}{B} p_2 - x_1 \\
X &= x_1 + \frac{2}{B} p_2 \\
P &= \frac{p_1}{2} - \frac{B}{4} x_2.
\end{align*}
\]

(40)  (41)  (42)  (43)

It may be noted that the new variables \(\bar{X}, \bar{P}\) are proportional to the constraints (36) of the system \((\bar{X} = \frac{1}{2} \Omega_1, \bar{P} = \frac{1}{2} \Omega_2)\) while the physical variables are \(X\) and \(P\). Its implication for the bracket kernel is now discussed. In terms of the new basis \((\frac{\partial}{\partial X}, \frac{\partial}{\partial \bar{X}}, \frac{\partial}{\partial p_i}, \frac{\partial}{\partial P})\) for the tangent space of the phase space, the old ones are given by

\[
\begin{align*}
\frac{\partial}{\partial x_1} &= \frac{\partial}{\partial X} - \frac{\partial}{\partial P} \\
\frac{\partial}{\partial x_2} &= \frac{B}{4} \left( \frac{\partial}{\partial X} - \frac{\partial}{\partial P} \right) \\
\frac{\partial}{\partial p_1} &= \frac{1}{2} \left( \frac{\partial}{\partial X} + \frac{\partial}{\partial P} \right) \\
\frac{\partial}{\partial p_2} &= \frac{2}{B} \left( \frac{\partial}{\partial X} + \frac{\partial}{\partial P} \right).
\end{align*}
\]
The bracket kernel, in the new variables, is given by
\[ \Lambda = \sum \partial_{\mu} J^{\mu\nu} \partial_{\nu} = \partial_{X} \partial_{P} - \partial_{P} \partial_{X} \] (44)
which is the standard canonical form. Hence the associativity of the star product follows trivially, being exactly identical to the unconstrained case. Note that the 4-dimensional phase space where the original kernel was defined reduces to the physical two dimensional space, when using the new variables. The two dimensional constraint sector just cancels out.

4 Particle on \( n \)-Sphere \( (S^n) \)

The Lagrangian of a particle restricted to move on the surface \( S^n \) of unit radius is given by
\[ L = \frac{1}{2}(\dot{x}^2) - \lambda(x^2 - 1) \quad x \in \mathbb{R}^{n+1} \] (45)
where \( \lambda \) is a Lagrange multiplier enforcing the primary constraint
\[ \Omega_1 = x^2 - 1 \approx 0. \] (46)
Time conservation of the primary constraint,
\[ \{\Omega_1, H_T\} = 0 \] (47)
leads to the secondary constraint
\[ \Omega_2 = x \cdot p \approx 0 \] (48)
with the total Hamiltonian \( H_T \) being given by
\[ H_T = H_c + \lambda \Omega_1 \] (49)
where \( H_c \) is the canonical Hamiltonian obtained from (45). The pair \( \Omega_1, \Omega_2 \) form a second class system of constraints. The corresponding Dirac brackets in component form are
\[ \{x_i, x_j\}_{DB} = 0, \quad \{x_i, p_j\}_{DB} = \delta_{ij} - x_i x_j, \quad \{p_i, p_j\}_{DB} = x_j p_i - x_i p_j, \] (50)
and the Dirac bracket kernel is,
\[ \Lambda = \partial_{X} \partial_{P} - \partial_{P} \partial_{X} - \partial_{P} \partial_{X} \partial_{P} + \partial_{P} \partial_{P} \partial_{X} \partial_{X} + \partial_{P} \partial_{X} \partial_{P} \partial_{X} \] (51)
The exponentiation of this kernel does not yield a manifestly associative star product. However, the Maskawa-Nakajima theorem helps us to write it in a manifestly associative form. We illustrate this first with the example of a particle constrained to move along a circle \( S^1 \) and then in the case of a particle on a 2-sphere \( S^2 \).
4.1 Reduction of the Bracket Kernel for $S^1$ and Associativity

The simplest case of a phase space with the Dirac brackets (50) is that of particle moving along a unit circle described by (45) with $i = 1, 2$. The symplectic matrix associated with the bracket kernel of this system is given by

$$\{J^{\mu \nu}\} = \begin{pmatrix}
0 & 0 & 1 - x_1^2 & -x_1 x_2 \\
0 & 0 & -x_1 x_2 & 1 - x_2^2 \\
-(1 - x_1^2) & x_1 x_2 & 0 & x_2 p_1 - x_1 p_2 \\
x_1 x_2 & -(1 - x_2^2) & -(x_2 p_1 - x_1 p_2) & 0
\end{pmatrix}$$

(52)

which acts on the phase space spanned by the coordinates $(x_1, x_2, p_1, p_2)$.

We now consider the following new set of phase space coordinates which can be obtained from the set $(x_1, x_2, p_1, p_2)$ by a canonical transformation;

$$\bar{x} = \frac{1}{\sqrt{2}}(x_i^2 - 1)$$

(53)

$$\theta = \tan^{-1} \frac{x_2}{x_1}$$

(54)

$$\bar{p} = \frac{1}{\sqrt{2}} x_i p_i$$

(55)

$$p_\theta = x_1 p_2 - x_2 p_1$$

(56)

where $(\bar{x}, \bar{p})$ and $(\theta, p_\theta)$ are the new independent canonical pairs. Observe that the canonical pair $(\bar{x}, \bar{p})$ corresponds respectively to the constraints (43) and (44) of the system and normalized suitably to yield $\{\bar{x}, \bar{p}\} = 1$. Also note that $\theta$ is the angle coordinate and $p_\theta$ is the angular momentum. Indeed, the above canonical transformation is the usual transformation that maps from a Cartesian to polar basis. Correspondingly, the tangent space basis of the phase space $(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial p_i})$ undergo a suitable transformation. The old tangent space basis in terms of the new one is given by

$$\frac{\partial}{\partial x_1} = \sqrt{2} \cos \theta \frac{\partial}{\partial \bar{x}} + \frac{1}{\sqrt{2}} \left( \sqrt{2} \cos \theta \bar{p} - \sin \theta \ p_\theta \right) \frac{\partial}{\partial \bar{p}}$$

$$\frac{\partial}{\partial x_2} = \sqrt{2} \sin \theta \frac{\partial}{\partial \bar{x}} + \frac{1}{\sqrt{2}} \left( \cos \theta \ p_\theta + \sqrt{2} \sin \theta \bar{p} \right) \frac{\partial}{\partial \bar{p}}$$

$$\frac{\partial}{\partial p_1} = \cos \theta \frac{\partial}{\sqrt{2} \partial \bar{p}} - \sin \theta \frac{\partial}{\partial p_\theta}$$

$$\frac{\partial}{\partial p_2} = \sin \theta \frac{\partial}{\sqrt{2} \partial \bar{p}} + \cos \theta \frac{\partial}{\partial p_\theta}$$
Making use of the above set of relations one can show that the bracket kernel (51) can be expressed as,

$$\Lambda = \frac{\bar{\theta}}{\partial_p} \frac{\bar{p}}{\partial \theta} + \frac{\bar{x}}{\sqrt{2}} \left( \frac{\bar{x}}{\partial \bar{x}} - \frac{\bar{p}}{\partial \bar{p}} \right)$$  

(57)

If we now pass to the constraint shell $\bar{x} \approx 0$, the second term in (57) vanishes so that

$$\Lambda = \frac{\bar{\theta}}{\partial_p} \frac{\bar{p}}{\partial \theta}$$  

(58)

This is the standard canonical form. In contrast to the Landau problem, the constraint part does not simply cancel by itself. It is necessary to pass to the constraint shell at the end of the computation to obtain the desired canonical structure for $\Lambda$. Using this bracket kernel one defines [7] the star product as an exponential series in $\bar{h}$ just as (11) to yield

$$f(\theta, p_\theta) \star g(\theta, p_\theta) = f(\theta, p_\theta) \exp \left[ \frac{i\bar{h}}{2} \left( \frac{\bar{\theta}}{\partial_p} \frac{\bar{p}}{\partial \theta} - \frac{\bar{p}}{\partial_p} \frac{\bar{\theta}}{\partial \theta} \right) \right] g(\theta, p_\theta)$$  

(59)

Clearly this has the same form as (11) except that the $\theta$ variable occurring here has a compact range ($0 \leq \theta < 2\pi$). As far as algebraic demonstration of the associativity is concerned, this is inconsequential. Associativity is demonstrated exactly as happens in the unconstrained case.

If one uses a coordinate system different from $\theta$, then also the Dirac bracket kernel takes the canonical form of the Poisson bracket kernel. An illustrative example is the stereographic projection of $S^1$ to $R^1$ given by,

$$X = \cos \theta$$  

(60)

The momentum conjugate to $X$ is

$$P_X = (1 - \sin \theta)p_\theta.$$  

(61)

The transition from $(\theta, p_\theta)$ to $(X, P)$ is a canonical transformation and it is easy to see that, in terms of these new variables ((60) and (61)), the kernel (58) takes the canonical form

$$\Lambda = \frac{\bar{X}}{\partial P_X} \frac{\bar{P_X}}{\partial \bar{X}}$$  

(62)

which obviously gives a manifest associativity for the star product.

4.2 Geometrical Proof of Associativity and Angular Momentum Quantization

The associativity of the star product in the preceding example can also be seen from the triangle interpretation. However, in the present case, the coordinate variable $\theta$ has the range $0 \leq \theta < 2\pi$ while its conjugate momentum varies continuously(classically) over a noncompact range ($-\infty, +\infty$). Hence the classical phase space of the system has the geometry of an infinite cylinder of unit radius. Therefore, one has to make appropriate modifications in the Fourier
representation of \( f(\theta, p_0) \) in order to verify the associativity using the triangle interpretation. That is, one writes the phase space function \( f \) as

\[
f(\theta, p_0) = \int_{-\pi}^{+\pi} d\theta' \int_{-\infty}^{+\infty} dp' \delta_p(\theta - \theta') \delta(p_0 - p') f(\theta', p')
\]

\[
= \frac{1}{(2\pi)^2} \sum_n \int_{-\pi}^{+\pi} d\theta' \int_{-\infty}^{+\infty} dp' \phi e^{i[n(\theta - \theta') + \phi(p_0 - p')]} f(\theta', p').
\]

(63)

where \( n \) takes integer values and

\[
\delta_p(\theta - \theta') = \frac{1}{2\pi} \sum_n e^{in(\theta - \theta')}
\]

(64)
is the periodic delta function. Note that the periodic delta function is incorporated on account of the periodicity of \( \theta \) whereas the usual Dirac delta function suffices in the case of \( p_0 \) which has a noncompact range. Therefore the star product of two functions \( f(\theta, p_0) \) and \( g(\theta, p_0) \) using this Fourier representation is to be written as

\[
f \star g = \frac{1}{(2\pi)^2} \sum_{m,n} \int_{-\pi}^{+\pi} d\theta'' \int_{-\infty}^{+\infty} dp'' d\phi d\chi
\times e^{i[n(\theta'' - \theta') + \phi(p'' - p')]} e^{i\left[\phi \frac{\partial}{\partial \theta} \phi \frac{\partial}{\partial p} \phi \frac{\partial}{\partial \phi} \phi \frac{\partial}{\partial \chi}\right]} e^{i[m(\theta'' - \theta''') + \chi(p'' - p''')]} f(\theta', p') g(\theta'', p''').
\]

(65)

Note at this stage that the \( c \)-number valued exponential functions appearing on either side of the \( \star \) operator are eigenfunctions of \( \partial_\theta \) and \( \partial_{p_0} \). Therefore, one can combine all these exponential factors into a single one and a subsequent integration over \( \phi \) and summation over \( n \) yields,

\[
f \star g = \frac{1}{(2\pi)^2} \sum_{m,n} \int_{-\pi}^{+\pi} d\theta'' \int_{-\infty}^{+\infty} dp'' d\chi \delta(p_0 - p'' + \frac{\hbar m}{2}) \delta(p - p' + \frac{\hbar n}{2}) f(\theta', p') g(\theta'', p''')
\times e^{i[m(\theta'' - \theta''') + \chi(p'' - p''')]} f(\theta', p') g(\theta'', p''').
\]

This involves a pair of delta functions, one of which is periodic. We next perform the integration over \( \chi \) and use the properties of the periodic delta functions\(^6\) to rewrite the integral in terms of the ordinary (non periodic) Dirac delta functions as follows;

\[
f \star g = \frac{1}{(2\pi)^2} \sum_{m,n} \int_{-\pi}^{+\pi} d\theta' \int_{-\infty}^{+\infty} dp' dp''
\]

\(^6\)Notice that \( [\boxed{1}] \) is the completeness relation for the exponential functions \( e^{in\theta} \) \( (n \in \mathbb{Z}) \) which provide a complete set of basis in the space of functions defined on \( \mathbb{S}^1 \). Also, any function \( \tilde{f}(\theta) \) defined on \( \mathbb{S}^1 \) satisfies the usual property of a delta function, viz., \( \tilde{f}(\theta) = \int_{-\pi}^{\pi} d\theta' \delta(\theta - \theta') \tilde{f}(\theta') \).

\(^7\)For any function \( f(x) \) defined over the real line \( \mathbb{R}^1 \), the periodic delta function satisfies \( \int_{-\infty}^{+\infty} dx \delta_p(\theta - x) f(x) = \sum_n e^{in\theta} \tilde{f}(n) \) where \( \tilde{f}(n) = \tilde{f}(k)|_{k=n} \) and \( \tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx e^{-ikx} f(x) \) is the Fourier transform of \( f(x) \).
\[ \delta(p_\theta - p' + \frac{\hbar m}{2})\delta(p_\theta - p'' - \frac{\hbar n}{2})e^{\frac{\pi i}{\hbar}[(p_\theta - p')(\theta - \theta') + (p_\theta - p'')(\theta - \theta'')]} f(\theta', p') g(\theta'', p''). \tag{66} \]

Here we notice the presence of a set of delta functions in the integral which restrict the difference between any pair among the variables \(p_\theta, p', p''\) to integral multiples of \(\frac{\hbar}{2}\).

\[ p' - p_\theta = \frac{\hbar n_1}{2}, \quad p'' - p_\theta = \frac{\hbar n_2}{2}, \quad p' - p'' = \frac{\hbar n_3}{2} \tag{67} \]

where \(n_1, n_2, n_3\) are some integers. General solutions to the above set of three equations, consistent with the chiral symmetry of the system (clockwise ↔ anticlockwise interchange in this case), are given by,

\[ p_\theta = \frac{\hbar m_1}{2}, \quad p' = \frac{\hbar m_2}{2}, \quad p'' = \frac{\hbar m_3}{2} \tag{68} \]

with \(m_1, m_2, m_3\) are a new set of integers. Thus, up to a factor of half this gives the usual quantization rule for the angular momentum. In other words, the star product automatically incorporates the quantization condition even at the classical level. Here too the exponent in (66) can be written as the area of a phase space triangle, in the cylindrical phase space mentioned earlier, with vertices \(\mathbf{r} = (\theta, p_\theta), \mathbf{r}' = (\theta', p')\) and \(\mathbf{r}'' = (\theta'', p'')\), the only difference with the case of a particle on a line being that the momentum variables vary over a discrete spectrum. That is

\[ -2A(\mathbf{r}, \mathbf{r}', \mathbf{r}'') = p(\theta'' - \theta') + p'(\theta - \theta'') + p''(\theta' - \theta). \tag{69} \]

Thus the star product in this case can be written as

\[ f \star g = \frac{1}{(2\pi)^2} \sum_{m,n} \int_{-\pi}^{\pi} d\theta' d\theta'' \int_{-\infty}^{\infty} dp' dp'' \delta(p_\theta - p' + \frac{\hbar m}{2})\delta(p_\theta - p'' - \frac{\hbar n}{2})e^{\frac{\pi i}{\hbar}A(\mathbf{r}, \mathbf{r}', \mathbf{r}'')} f(\mathbf{r}') g(\mathbf{r}''). \tag{70} \]

Now we make a redefinition of the momentum variables as follows.

\[ P_\theta = \frac{2p_\theta}{\hbar}, \quad P' = \frac{2p'}{\hbar}, \quad P'' = \frac{2p''}{\hbar}. \tag{71} \]

Notice that this is not a canonical transformation. In terms of the new variables we can write \(f \star g\) as

\[ f \star g = \frac{1}{(2\pi)^2} \sum_{m,n} \int_{-\pi}^{\pi} d\theta' d\theta'' \int_{-\infty}^{\infty} dP' dP'' \delta(P_\theta - P' + m)\delta(P_\theta - P'' + n)e^{\frac{-2\pi i A(\hat{\mathbf{r}}, \hat{\mathbf{r}}', \hat{\mathbf{r}}'')}} f(\hat{\mathbf{r}}') g(\hat{\mathbf{r}}''). \tag{72} \]

Here \(\hat{\mathbf{r}}\) stands for \((\theta, P)\) now. The form of the delta functions appearing in the above expression for star product enables one to replace the pair of Dirac delta functions with suitable Kronecker deltas and simultaneously changing integrations over \(P', P''\) with summations to yield

\[ f \star g = \frac{1}{(2\pi)^2} \sum_{m,n, P', P''} \int_{-\pi}^{\pi} d\theta' d\theta'' \delta_{P', P_\theta + m} \delta_{P'', P_\theta - n} e^{\frac{-2\pi i A(\hat{\mathbf{r}}, \hat{\mathbf{r}}', \hat{\mathbf{r}}'')}} f(\hat{\mathbf{r}}') g(\hat{\mathbf{r}}''). \tag{73} \]
Now on account of the restrictions imposed by the Kronecker deltas, the summation over $P', P''$ implies the $m, n$ summations also. Hence we obtain

\[
(f * g)(\tilde{r}) = \frac{1}{(2\pi)^2} \sum_{p', p''} \int_{-\pi}^{+\pi} d\theta' d\theta'' e^{-2iA(\tilde{r}, \tilde{r}', \tilde{r}'')} f(\tilde{r}') g(\tilde{r}'')
\]

(74)

This expression is same as (19), which was obtained for the star product for a 1-d particle except that in place of the continuous variables $p', p''$, here we have the discrete variables $P', P''$ (defined in (71) with a built-in factor of $(2/\pi)$) and hence the corresponding integrations are replaced by summations. Although the associativity is a built in property in the definition of star product, the verification here through the “triangle” approach to indicate the similarities and differences of the case, where the particle is moving on the real line. The first step towards this is to express the product $(f * g) * h$ in terms of the area of a phase space parallelogram. To achieve this end, exploiting (74) we write,

\[
((f * g) * h)(\tilde{r}) = \frac{1}{(2\pi)^2} \sum_{p', p''} \int_{-\pi}^{+\pi} d\theta' d\theta'' e^{-2iA(\tilde{r}, \tilde{r}', \tilde{r}'')} (f * g)(\tilde{r}') h(\tilde{r}'')
\]

\[
= \frac{1}{(2\pi)^4} \sum_{p', p''} \sum_{p'''}, p''' \int_{-\pi}^{+\pi} d\theta' d\theta'' d\theta''' d\theta'''' e^{-2i[A(\tilde{r}, \tilde{r}', \tilde{r}'') + A(\tilde{r}, \tilde{r}', \tilde{r}'')] \frac{1}{(2\pi)^2} \sum_{p, p''} \int_{-\pi}^{+\pi} d\theta' d\theta'' d\theta''' d\theta''''}
\]

(75)

The exponent appearing in the above equation can be written as

\[
2A(\tilde{r}, \tilde{r}', \tilde{r}'') + 2A(\tilde{r}, \tilde{r}', \tilde{r}'') = \tilde{r}' \times [-\tilde{r} + \tilde{r}'' + \tilde{r}''' - \tilde{r}'''] + \tilde{r}'' \times \tilde{r} + \tilde{r}''' \times \tilde{r}'''.
\]

(76)

Using (70) and the relation

\[
\sum_{p'} \int_{-\pi}^{+\pi} d\theta' e^{i\theta' \cdot [\tilde{r} + \tilde{r}' + \tilde{r}'' - \tilde{r}''']} = \sum_{p'} \int_{-\pi}^{+\pi} d\theta' e^{i[\theta' \cdot (P_0 + P'''' - P'') - P' \cdot (\theta'' + \theta''' - \theta''')]} = (2\pi)^2 \delta_{\theta'' + \theta''' - \theta'''} \delta_p(\theta')
\]

(77)

we can write (73) as

\[
((f * g) * h)(\tilde{r}) = \frac{1}{(2\pi)^2} \sum_{p', p'', p'''} \int_{-\pi}^{+\pi} d\theta' d\theta'' d\theta''' d\theta'''' \delta_{\theta'' + \theta''' - \theta'''} \delta_p(\theta')
\]

\[
\times e^{-i[\tilde{r}'' \cdot \tilde{r} + \tilde{r}''' \cdot \tilde{r}''']} f(\tilde{r}'') g(\tilde{r}''') h(\tilde{r}''').
\]

(78)

Comparing with (25), we notice here the occurrence of a product of a Kroenecker delta and a periodic delta function instead of a product of two Dirac delta functions. The Kroenecker delta present in the above product ensures that the $P$ sector of the summation variables indeed form the vertices of a parallelogram though the same cannot be said of the $\theta$ part because of the periodic delta function, as it cannot strictly enforce the requirement $(-\theta + \theta'' + \theta''' - \theta''') = 0$. Nevertheless, the formal appearance of the above expression is enough to prove the associative of the star product. Similar calculation for the product $f * (g * h)$ indeed gives an identical result thereby proving associativity.
4.3 Particle on the Surface of a Sphere $S^2$

We now consider the case of a particle constrained to the surface of a sphere. The analysis in this case proceeds exactly as that of the particle on a circle discussed before. The Lagrangian of the system and the corresponding constraints are given by (13), (16) and (18) with $i = 1, 2, 3$. Similarly the Dirac bracket and its kernel are obtainable from (50) and (51) respectively. In the present case, the $\{ J^{\mu \nu} \}$ matrix involved in the kernel (51) acts on the 6-dimensional phase space spanned by $(x_i, p_i)$ and is given by

$$\{ J^{\mu \nu} \} = \begin{pmatrix}
0 & 0 & 0 & 1 - x_1^2 & -x_1 x_2 & -x_1 x_3 \\
-1 - x_2^2 & x_1 x_2 & x_1 x_3 & 0 & x_2 p_1 - x_1 p_2 & x_3 p_1 - x_1 p_3 \\
x_1 x_3 & x_2 x_3 & x_1 p_2 - x_2 p_1 & 0 & x_3 p_2 - x_2 p_3 \\
0 & 0 & 0 & -x_1 x_3 & -x_2 x_3 & 1 - x_3^2
\end{pmatrix}. \tag{79}$$

As was done in the previous examples, we make a transformation from the $(x_i, p_i)$ coordinate system to a new system of independent canonical variables $(\bar{x}, \phi, \bar{p}, p_\phi, p_\theta)$,

$$\bar{x} = \frac{1}{\sqrt{2}}(x_i^2 - 1), \quad \bar{p} = \frac{1}{\sqrt{2}}(x_i p_i) \tag{80}$$

$$\phi = \tan^{-1} \frac{x_2}{x_1}, \quad p_\phi = x_1 p_2 - x_2 p_1 \tag{81}$$

$$\theta = \cos^{-1} x_3, \quad p_\theta = \frac{x_3(x_p a) - p_3(x^2)}{\sqrt{x_1^2 + x_2^2}}. \tag{82}$$

The canonical pair $(\bar{x}, \bar{p})$ are proportional to the constraints and the unconstrained variables are $(\phi, \theta, p_\phi, p_\theta)$. The variables $\phi$ and $\theta$ are nothing but the azimuth and polar angles respectively and hence have the compact ranges $0 \leq \phi < 2\pi$ and $0 \leq \theta \leq \pi$. We introduce the notation $\phi = \theta_1, \theta = \theta_2$ and proceed just as in the case of a particle on a circle, in the bracket kernel, we eliminate the partial differential operators with respect to the Cartesian coordinates in favor of those with respect to the new set of variables $\bar{x}, \bar{p}$ and $\theta_a, p_{\theta_a}$ with $a = 1, 2$. The corresponding bracket kernel reduces to the following simple form

$$\Lambda = \sum_{a=1,2} \left( \hat{\partial}_{\theta_a} \hat{\partial}_{p_{\theta_a}} - \hat{\partial}_{p_{\theta_a}} \hat{\partial}_{\theta_a} \right) - \sqrt{2} \bar{\mathcal{E}} \left( \hat{\partial}_{\bar{x}} \hat{\partial}_{\bar{p}} - \hat{\partial}_{\bar{p}} \hat{\partial}_{\bar{x}} \right) \tag{83}$$

Upon implementing the constraint $1 - x_1^2 \approx 0$ strongly, one finds that the variables $\bar{x}$ and $\bar{p}$ disappear from the bracket kernel (83) and it reduces to the canonical form

$$\Lambda = \sum_{a=1,2} \left( \hat{\partial}_{\theta_a} \hat{\partial}_{p_{\theta_a}} - \hat{\partial}_{p_{\theta_a}} \hat{\partial}_{\theta_a} \right), \tag{84}$$

which involves only the physical variables. This is again in conformity with the Maskawa-Nakajima theorem. Now it is obvious that this kernel, when exponentiated, produces a perfectly well behaved associative star product. Here we would like to emphasize once again, that
algebraic verification of associativity can be carried out in each order of \( \hbar \) just as for the particle moving on the real line, as the algebraic structure of the bracket kernel in either case takes a similar canonical form and this is regardless of the nature of these coordinates. Also, the fact that the coordinate chart \((\theta, \phi)\) does not cover the entire \( S^2 \) is of no consequence as we can define new polar coordinates with respect to an \( SO(3) \) rotated axes or stereographic variables as was done for \( S^1 \) to define another chart. Two or more compatible charts can be used to provide an atlas for the whole of \( S^2 \). And the bracket kernel can be shown to take a similar canonical form in all the charts. We have demonstrated this in the case of \( S^1 \). It can be easily demonstrated for \( S^2 \) also.

Finally let us make the following observations. At the classical level, the spherical polar coordinate variables \( \theta \) and \( \phi \) having the compact ranges \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi < 2\pi \) respectively, coordinatize only a neighborhood of the manifold \( S^2 \) whereas their conjugate momenta \( p_\theta, p_\phi \) varying from \(-\infty \) to \( +\infty \) constitute a plane \( R^2 \). Thus the phase space of the system given by \((\theta, \phi, p_\theta, p_\phi)\) cover only a neighborhood of the whole phase space. Besides, the basis for the expansion (analogous to \((1)\) and \((63)\)) of functions in \( S^2 \) are the spherical harmonics \( Y_{lm}(\theta, \phi) \) while in \( R^2 \) the usual exponential functions provide the basis. Though the \( \phi \) dependence of \( Y_{lm}(\theta, \phi) \) is via the usual exponential functions, its \( \theta \) dependence is through the associated Legendre polynomials \( P_{lm}(\cos \theta) \) which are not eigenfunctions of \( \partial_\theta \). This suggests that carrying out a naive Fourier/harmonic analysis may not yield any quantization condition for the angular momentum as could be done for \( S^1 \). The coordinate independent Fourier transform of \([7]\) may be useful for this purpose. This nontrivial point requires further investigation.

Unlike the previous examples, the Groenewold method of reduction of the star product to an integral involving the exponentiated area of a phase space triangle (i.e. analogous to \((18)\) and \((74)\)) is not viable for particle on \( S^2 \). The source of this difficulty can be traced to the nontrivial geometry and/or topology of the phase space. But as has been shown in \([7]\), the construction of the star product has built-in associativity and triangle interpretation is also available, where one gets a geodesic triangle which reduces to a rectilinear triangle (as in the non-degenerate case) in terms of the coordinates on the constraint surface. In the phase space approach of Weyl-Groenewold \([4, 14]\), demonstration of associativity is done by using the MN variables as discussed here.

## 5 Conclusion

In this paper we studied various aspects of the construction of star product for certain physical systems. The construction and properties of star product were reviewed in the case of a particle moving on a real line. We have shown how an associative star product can be constructed for systems involving nonlinear constraints. As is well known, the Dirac brackets in this case are variable dependent and a naive exponentiation of the bracket kernel does not yield a manifestly associative star product. By exploiting a theorem due to Maskawa and Nakajima (MN) \([13]\), we were able to extract the physical unconstrained set of variables. The bracket kernel, computed with respect to these variables, simplified to the standard canonical symplectic structure, once
the constraints were imposed. Associativity of the star product is then manifestly evident.

Our method was illustrated with the examples of Landau problem, particle on a circle and a particle restricted to be on the surface of a sphere. The Kontsevich series expansion of the star product was shown to reduce to an exponential series when expressed in terms of the unconstrained variables of the systems. Similar to the case of a 1-dimensional free particle, we showed that the associativity of the star product for a particle on the circle can be given the triangle (in the cylindrical phase space) interpretation with the difference that the angular momentum taking integral values up to a factor of \((\hbar/2)\). We thus obtained the usual quantization condition of the angular momentum (up to a factor of half) in the case of a particle moving on the circle.

The quantization condition of the angular momentum, however, could not be reproduced for a particle moving on the surface of a sphere. Also, MN variables can be defined only locally for systems with topologically more complicated phase spaces (particle constrained on a sphere for example). Also, in the case of \(S^2\) (as for most degenerate systems), the star product cannot be given the triangle interpretation in the Weyl-Groenewold approach. Nevertheless, it must be emphasized that the lack of such an interpretation or the local nature of the MN variables are of no consequence in demonstrating the associativity of the star product using MN variables. The reason for this is that we employ a purely algebraic method for demonstrating associativity rather than a geometric one. This was illustrated clearly in the case of the examples considered in this paper, viz., Landau problem, particles moving on a circle and on a sphere.

It may be mentioned that, in the coordinate free formulation of Batalin and Tyutin [7], the star product is defined in such a way as to satisfy associativity property automatically. Thus it is always possible to give a triangle interpretation to the star product irrespective of the geometry the base manifold. For degenerate systems with second class constraints, the star product can be written in terms of the exponential of the area of a geodesic triangle which reduces to a rectilinear triangle on the constraint surface. Thus, a geometric interpretation of the associativity of star product is always available in the coordinate independent approach for both degenerate and non-degenerate systems.

Finally, we feel that the coordinate independent Fourier transform of [7] would be helpful in resolving the difficulties mentioned in section 4.3 for giving an area interpretation for the associativity of star product (phase space approach) and obtaining the angular momentum quantization condition for a particle on the surface of a sphere. This problem is under further investigation and will be reported elsewhere.

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