EXPLICIT RATIONALITY OF SOME CUBIC FOURFOLDS

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Abstract. Recent results of Hassett, Kuznetsov and others pointed out countably many divisors \( C_d \) in the open subset of \( \mathbb{P}^5 = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(3))) \) parametrizing all cubic 4-folds and lead to the conjecture that the cubes corresponding to these divisors should be precisely the rational ones. Rationality has been proved by Fano for the first divisor \( C_{14} \) and in [RS17] for the divisors \( C_{26} \) and \( C_{38} \). In this note we describe explicit birational maps from a general cubic fourfold in \( C_{14} \), in \( C_{26} \) and in \( C_{38} \) to \( \mathbb{P}^4 \), providing concrete geometric realizations of the more abstract constructions in [RS17]. We shall also point out some relations with other explicit birational maps onto four dimensional linear sections of Grassmannians of lines analogous to Fano’s example. We also construct some new irreducible surfaces in \( \mathbb{P}^5 \) admitting a congruence of \((3e-1)\)-secant curves of degree \( e \) for \( e = 3, 5 \), such as a remarkable octic scroll with six nodes and the first example of a non rational surface of this kind.

Introduction

The rationality of smooth cubic hypersurfaces in \( \mathbb{P}^5 \) is an open problem on which a lot of new and interesting contributions and conjectures appeared in the last decades. The classical work by Fano in [Fan43], correcting some wrong assertions in [Mor40], has been the only known result about the rationality of cubic fourfolds for a long time and, together with a great amount of recent theoretical work on the subject (see for example the survey [Has16]), lead to the expectation that the very general cubic fourfold should be irrational. More precisely, in the moduli space \( \mathcal{C} \) the locus \( \text{Rat}(\mathcal{C}) \) of rational cubic fourfolds is the union of a countable family of closed subsets \( T_i \subseteq \mathcal{C}, i \in \mathbb{N} \), see [dFF13, Proposition 2.1] and [KT17, Theorem 1].

Hassett defined in [Has99, Has00] (see also [Has16]) via Hodge Theory infinitely many irreducible divisors \( C_d \) in \( \mathcal{C} \) and introduced the notion of admissible values \( d \in \mathbb{N} \), i.e. those even integers \( d > 6 \) not divisible by 4, by 9 and nor by any odd prime of the form \( 2 + 3m \). More recent contributions by Kuznetzsov via derived categories in [Kuz10, Kuz16] (see also [AT14, Has16]) fortified the conjecture that

\[
\text{Rat}(\mathcal{C}) = \bigcup_{d \text{ admissible}} C_d.
\]

The first admissible values are \( d = 14, 26, 38, 42 \) and Fano showed the rationality of a general cubic fourfold in \( C_{14} \), see [Fan43, BRS15]. The main results of [RS17] are summarised in the following:

**Theorem.** Every cubic fourfold in the irreducible divisors \( C_{26} \) and \( C_{38} \) is rational.

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The geometrical definition of $C_d$ can be also given as the (closure of the) locus of cubic fourfolds $X \subset \mathbb{P}^5$ containing an explicit surface $S_d \subset X$. These surfaces are obviously not unique and a standard count of parameters shows that in specific examples the previous locus is a divisor (the degree and self-intersection of $S_d$ determine the value $d$ via the formula $d = 3 \cdot S^2 - \deg(S)^2$), see [Has99, Has00] for more details on the Hodge theoretical definition of $C_d$. For example, the divisor $C_{14}$ can be described either as the closure of the locus of cubic fourfolds containing a smooth quintic del Pezzo surface or, equivalently, a smooth quartic rational normal scroll, see [Fan43, BRS15] and also [Nue15] for other descriptions with $12 \leq d \leq 44$, $d \neq 42$ or [Lai16] for $d = 42$.

Fano proved that the restriction of the linear system of quadrics through a smooth quintic del Pezzo surface, respectively a smooth quartic rational normal scroll, to a general cubic through the surface defines a birational maps to $\mathbb{P}^4$, respectively onto a smooth four dimensional quadric hypersurface. Indeed, a general fiber of the map $\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ given by quadrics through a quintic del Pezzo surface is a secant line to it, yielding the birationality of the restriction to $X$ (the other case is similar). The extension of this explicit geometrical approach to rationality for other (admissible) values $d$ appeared to be impossible because there are no other irreducible surfaces with one apparent double point contained in a cubic fourfold, see [CR11, BRS15].

In [RS17] we discovered irreducible surfaces $S_d \subset \mathbb{P}^5$ admitting a four-dimensional family of 5-secant conics such that through a general point of $\mathbb{P}^5$ there passes a unique conic of the family (congruences of 5-secant conics to $S_d$) for $d=14, 26$ and 38. From this we deduced the rationality of a general cubic in $|H^0(I_{S_d}(3))|$, showing that it is a rational section of the universal family of the congruence of 5-secant conics.

Here we come back to Fano’s method and we propose an explicit realisation of the previous abstract approach. Some simplifying hypotheses, based on the known examples in [RS17], suggest that rationality might be related to linear systems of hypersurfaces of degree $3e - 1$ having points of multiplicity $e \geq 1$ along a right surface $S_d$ contained in the general $X \in C_d$, see Section 1. Clearly the problem is to find the right $S_d$, prove that the above map is birational and, if the dimension of the linear system is bigger than four, describe the image (which might be highly non trivial). This expectation is motivated by the remark that if the surface $S_d \subset \mathbb{P}^5$ admits a congruence of $(3e - 1)$-secant curves of degree $e \geq 1$ generically transversal to the cubics through $S_d$ (see Section 1 for precise definitions), then the above linear systems contract the curves of the congruence. So if the general fiber of the map is a curve of the congruence, then a general cubic through $S_d$ is birational to the image of the associated map. We shall see that, quite surprisingly, this really occurs for the first three admissible values $d = 14, 26, 38$ and, even more surprisingly, that these linear systems provide by restriction explicit birational maps from a general cubic fourfold in $C_d$ for $d = 14, 26, 38$ to $\mathbb{P}^4$ (or to a four dimensional linear section of a $\mathbb{G}(1,3)$, respectively $\mathbb{G}(1,4)$, respectively $\mathbb{G}(1,5)$), a fact which was not known before at least for $d = 26, 38$.

Therefore, for $d = 26, 38$ we have couples of surfaces that behave as the smooth quintic del Pezzo surface and the smooth quartic rational normal scroll, providing birational maps to $\mathbb{P}^4$ and, respectively, to a four dimensional linear section of $\mathbb{G}(1,4)$ and $\mathbb{G}(1,5)$. In particular for $d = 14$ and $d = 38$ we have an interesting phenomenon via linkage in notable scrolls of lines: the Segre 3-fold in the first case; a particular singular Bordiga scroll in the second case. For $d = 38$ we discovered via linkage an unknown remarkable rational octic scroll with six nodes.
The existence of a series of examples for every \( d = 14 + 12k \) mapping a general cubic in \( C_d \) to a four dimensional linear section \( Y_k^4 \) of \( G(1,3+k) \) is unknown for \( k \geq 3 \). For \( k \geq 4 \) the variety \( Y_k^4 \) is irrational unless it becomes singular, see Section 3 for more details and speculations.

To analyze the algebraic and geometric properties of the surfaces \( S_d \subset \mathbb{P}^5 \) involved as well the particular linear systems of hypersurfaces of degree \( 3e - 1 \) having points of multiplicity \( e \) along the \( S_d \)'s we used Macaulay2 [GS18] together with some standard semicontinuity arguments to pass from a particular verification to the general case.

We conclude by constructing some new examples of irreducible surfaces \( S \subset \mathbb{P}^5 \) admitting a congruence of \((3e - 1)\)-secant curves of degree \( e \) for \( e = 3, 5 \), see Section 4. For \( e = 3 \) we produce an example of a non rational surface of this kind which is the projection of a K3 surface of genus 8 and degree 14 from three general points on it. For \( e = 5 \) we describe two examples coming from special Cremona transformations of \( \mathbb{P}^6 \), which show very interesting connections with other interesting rare phenomena. Some of the examples we are aware of are collected in Tables 1 and 2 below.

**Acknowledgements.** We wish to thank János Kollár for asking about the maps defined by the linear systems of quintics singular along surfaces admitting a congruence of 5-secant conics and for his interest in our subsequent results.

| \( d \) | Surface \( S \subset \mathbb{P}^5 \) | 2-secant lines | 5-secant conics | 8-secant twisted cubics | \( h^0(\mathcal{I}_S/\mathbb{P}^5(3)) \) | \( h^0(N_S/\mathbb{P}^5) \) | \( h^0(N_S/X) \) |
|---|---|---|---|---|---|---|---|
| 14 | Rational normal scroll of degree 4 | 1 | 0 | 0 | 28 | 29 | 2 |
| 14 | Smooth del Pezzo surface of degree 5 | 1 | 0 | 0 | 25 | 35 | 5 |
| 14 | Isomorphic projection of a smooth surface in \( \mathbb{P}^6 \) of degree 8 and sectional genus 3, obtained as the image of \( \mathbb{P}^3 \) via the linear system of quartic curves with 8 general base points | 7 | 1 | 0 | 13 | 49 | 7 |
| 14 | Projection from 3 internal points of a K3 surface of degree 14 and sectional genus 8 | 13 | 10 | 1 | 9 | 60 | 14 |
| 26 | Projection of a smooth del Pezzo surface of degree 7 in \( \mathbb{P}^7 \) from a line intersecting the secant variety in one general point | 5 | 1 | 0 | 14 | 42 | 1 |
| 26 | Rational scroll of degree 7 with 3 nodes | 7 | 1 | 0 | 13 | 44 | 2 |
| 38 | Smooth surface of degree 10 and sectional genus 6, obtained as the image of \( \mathbb{P}^2 \) via the linear system of curves of degree 10 with 10 general triple points | 7 | 1 | 0 | 10 | 47 | 2 |
| 38 | Rational scroll of degree 8 with 6 nodes | 9 | 4 | 1 | 10 | 47 | 2 |
| 38 | Projection of a smooth del Pezzo surface of degree 8 in \( \mathbb{P}^8 \) from a plane intersecting the secant variety in 3 general points | 7 | 6 | 1 | 10 | 44 | 0 |

**Table 1.** Surfaces \( S \subset \mathbb{P}^5 \) contained in a cubic fourfold \( [X] \in C_d \) and admitting a congruence of \((3e - 1)\)-secant rational normal curves of degree \( e \leq 3 \).
1. Explicit rationality via linear systems of hypersurfaces of degree $3e - 1$ having points of multiplicity $e$ along a right surface

Let us recall the following definitions introduced in [RS17, Section 1]. Let $\mathcal{H}$ be an irreducible proper family of (rational or of fixed arithmetic genus) curves of degree $e$ in $\mathbb{P}^5$ whose general element is irreducible. We have a diagram

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\pi} & \mathcal{H} \\
\downarrow & & \downarrow \psi
\end{array}
$$

where $\pi : \mathcal{D} \rightarrow \mathcal{H}$ is the universal family over $\mathcal{H}$ and where $\psi : \mathcal{D} \rightarrow \mathbb{P}^5$ is the tautological morphism. Suppose moreover that $\psi$ is birational and that a general member $[C] \in \mathcal{H}$ is $(re - 1)$-secant to an irreducible surface $S \subset \mathbb{P}^5$, that is $C \cap S$ is a length $re - 1$ scheme, $r \in \mathbb{N}$. We shall call such a family $\mathcal{H}$ a congruence of $(re - 1)$-secant curves of degree $e$ to $S$. Let us remark that necessarily $\dim(\mathcal{H}) = 4$.

An irreducible hypersurface $X \in |H^0(\mathcal{I}_S(r))|$ is said to be transversal to the congruence $\mathcal{H}$ if the unique curve of the congruence passing through a general point $p \in X$ is not contained in $X$. A crucial result is the following.

**Theorem 1.** [RS17, Theorem 1] Let $S \subset \mathbb{P}^5$ be a surface admitting a congruence of $(re - 1)$-secant curves of degree $e$ parametrized by $\mathcal{H}$. If $X \in |H^0(\mathcal{I}_S(r))|$ is an irreducible hypersurface transversal to $\mathcal{H}$, then $X$ is birational to $\mathcal{H}$.

If the map $\Phi = \Phi_{|H^0(\mathcal{I}_S(r))|} : \mathbb{P}^5 \dashrightarrow \mathbb{P}(H^0(\mathcal{I}_S(r)))$ is birational onto its image, then a general hypersurface $X \in |H^0(\mathcal{I}_S(r))|$ is birational to $\mathcal{H}$.

Moreover, under the previous hypothesis on $\Phi$, if a general element in $|H^0(\mathcal{I}_S(r))|$ is smooth, then every $X \in |H^0(\mathcal{I}_S(r))|$ with at worst rational singularities is birational to $\mathcal{H}$.

\[\begin{array}{|c|c|c|c|c|c|c|}
\hline
| d | e & Multidegree & $Y^4$ & $\delta$ & $\deg(\mathcal{B})$ & $g(\mathcal{B})$ & $\deg(Supp(\mathcal{B}))$ & $g(Supp(\mathcal{B}))$
\hline
| 14 | 1 & $3, 6, 8, 6, 2$ & $\mathbb{G}(1, 3) \subset \mathbb{P}^5$ & 3 & 10 & 7 & 10 & 7
| 14 | 1 & $3, 6, 7, 4, 1$ & $\mathbb{P}^4$ & 4 & 9 & 8 & 9 & 8
| 14 | 2 & $3, 15, 19, 9, 1$ & $\mathbb{P}^4$ & 9 & 52 & 256 & 32 & 106
| 14 | 3 & $3, 24, 42, 24, 3$ & cubic hyp. in $\mathbb{P}^5$ & 8 & 117 & 577 & 62 & 164
| 26 | 2 & $3, 15, 31, 25, 5$ & $\mathbb{G}(1, 4) \cap \mathbb{P}^4 \subset \mathbb{P}^4$ & 5 & 77 & 212 & 43 & 73
| 26 | 2 & $3, 15, 20, 9, 1$ & $\mathbb{P}^4$ & 9 & 51 & 246 & 31 & 100
| 38 | 2 & $3, 15, 27, 9, 1$ & $\mathbb{P}^4$ & 9 & 42 & 165 & 18 & 39
| 38 | 3 & $3, 24, 80, 70, 14$ & $\mathbb{G}(1, 5) \cap \mathbb{P}^{10} \subset \mathbb{P}^{10}$ & 5 & 204 & 633 & ? & ?
| 38 | 3 & $3, 24, 41, 22, 2$ & $\mathbb{G}(1, 3) \subset \mathbb{P}^5$ & 11 & 150 & 1010 & 61 & ?
\hline
\end{array}\]

Table 2. Birational maps from a cubic fourfold $[X] \in C_d$ to a fourfold $Y^4$ defined by the restrictions to $X$ of the linear systems $|H^0(\mathcal{I}_S^e(3e - 1))|$, where $S \subset \mathbb{P}^5$ are the surfaces in Table 1 admitting a congruence of $(3e - 1)$-secant rational normal curves of degree $e \leq 3$. Here, $\mathcal{B}$ denotes the base locus of the inverse map, which is a 2-dimensional scheme; $g$ stands for the sectional arithmetic genus; $\delta$ is the degree of the forms defining the inverse map.
Thus, when a surface $S \subset \mathbb{P}^5$ admits a conguence of $(3e - 1)$-secant curves of degree $e \geq 1$ which is transversal to a general cubic fourfold through it, such a cubic fourfold is birational to $\mathcal{H}$ via $\pi \circ \psi^{-1}$; see the proof of the above result. Obviously, the difficult key point is to find a congruence as above with $\mathcal{H}$ rational (or irrational, depending on the application).

Since $\psi : D \to \mathbb{P}^5$ is birational, we also have a rational map

$$\varphi = \pi \circ \psi^{-1} : \mathbb{P}^5 \dashrightarrow \mathcal{H},$$

whose general fiber through $p \in \mathbb{P}^5$, $F = \varphi^{-1}(\varphi(p))$, is the unique curve of the congruence passing through $p$. On the contrary, if there exists a map $\varphi : \mathbb{P}^5 \dashrightarrow Y \subseteq \mathbb{P}^N$ with $Y$ a four dimensional variety, whose general fiber $F$ is an irreducible curve of degree $e$ which is $(3e - 1)$-secant to a surface $S \subset \mathbb{P}^5$, then we have found a congruence for $S$, a birational realization of the variety $\mathcal{H}$ and a concrete representation of the abstract map $\pi \circ \psi^{-1}$.

It is natural to ask what linear systems on $\mathbb{P}^5$ can give maps $\varphi : \mathbb{P}^5 \dashrightarrow Y$ as above. Since a linear system in $|H^0(\mathcal{I}_S(3e - 1))|$ contracts the fibers of $\varphi$, these (complete) linear systems appear as natural potential candidates, as remarked by János Kollár. A posteriori we shall see that, quite surprisingly, this really occurs with $Y = \mathbb{P}^4$ (or with $Y$ a linear section of a Grassmannian of lines) for the first three admissible values $d = 14, 26, 38$ and, even more surprisingly, that these linear systems provide the explicit rationality of a general cubic fourfold through $C_d$ for $d = 14, 26, 38$.

Let $E \subset \text{Bl}_S \mathbb{P}^5$ be the exceptional divisor of the blow-up of $\mathbb{P}^5$ along $S$, let $H \subset \text{Bl}_S \mathbb{P}^5$ be the pull back of a hyperplane in $\mathbb{P}^5$, let $F'$ be the strict transform of $F$ and let $\tilde{\varphi} : \text{Bl}_S \mathbb{P}^5 \dashrightarrow \mathcal{H}$ be the rational map induced by $\varphi$. Then $E \cdot F' = 3e - 1$ by hypothesis and $(3H - E) \cdot F' = 1$. The last condition translates both that a general cubic through $S$ is mapped birationally onto $\mathcal{H}$ by $\varphi$, both the fact that the linear system $|H^0(\mathcal{I}_S(3))|$ sends a general $F$ into a line contained in the image of the corresponding map $\psi : \mathbb{P}^5 \dashrightarrow \mathbb{P}(H^0(\mathcal{I}_S(3)))$ and, last but not least, also that the congruence is transversal to a general cubic fourfold through $S$ (see [RS17] for a systematic use of these key remarks).

For $e = 1$ one should consider linear systems of quadric hypersurfaces through $S$; for $e = 2$ quintics having double points along $S$; for $e = 3$ hypersurfaces of degree 8 having triple points along $S$ and so on.

For $e = 1$ we have a unique secant line to $S$ passing through a general point of $\mathbb{P}^5$, which is a very strong restriction. Indeed, such a $S$ is a so called surface with one apparent double point. These surfaces are completely classified in [CR11] and those contained in a cubic fourfold are only quintic del Pezzo’s and smooth quartic rational normal scrolls. Cubic fourfolds through these surfaces describe the divisor $C_{11}$ as it was firstly remarked by Fano in [Fan43] (see also [BRS15, Theorem 3.7] for a modern account of Fano’s original arguments using deformations of quartic scrolls).

Let $D \subset \mathbb{P}^5$ be an arbitrary smooth quintic del Pezzo surface. Then $|H^0(\mathcal{I}_D(2))| = \mathbb{P}^4$ and this linear system determines a dominant rational map $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$, whose general fiber $F$ is a secant line to $D$. Then the restriction of $\varphi$ to a cubic fourfold through $D$ yields a birational map $\psi : X \dashrightarrow \mathbb{P}^4$ and hence the rationality of a general $X \in C_{14}$, as firstly remarked by Fano in [Fan43].

Let $T \subset \mathbb{P}^4$ be a smooth quartic rational normal scroll. Then $|H^0(\mathcal{I}_T(2))| = \mathbb{P}^5$ and this linear system determines a dominant rational map $\varphi : \mathbb{P}^5 \dashrightarrow Q \subset \mathbb{P}^5$, whose general fiber $F$
is a secant line to $T$ and with $Q$ a smooth quadric hypersurface. Then the restriction of $\varphi$ to a cubic fourfold through $T$ yields a birational map $\psi : X \dasharrow Q$ and another proof of the rationality of a general $X \in \mathcal{C}_{14}$, see [Fan43].

We shall mainly consider the surfaces $S_d \subset \mathbb{P}^5$ admitting a congruence of 5-secant conics parametrised by a rational variety studied in [RS17] (but also other new examples) to determine explicitly the rationality of a general $X \in \mathcal{C}_d$ for $d = 14, 26, 38$ with $e = 2$ (or also for other values $e \geq 3$). To this aim we shall summarise some well known facts in the next subsection.

1.1. Linear systems of quintics with double points along a general $S_d \in \mathcal{S}_d$. Let $S_d$ be an irreducible component of the Hilbert scheme of surfaces in $\mathbb{P}^5$ with a fixed Hilbert polynomial $p(t)$ and such that

$$\mathcal{C}_d = \{ [X] \in \mathcal{C} \text{ for which } \exists [S_d] \in \mathcal{S}_d : S_d \subset X \}.$$ 

One can verify explicitly the previous equality by comparing the Hodge theoretic definition on the left with the geometrical description on the right. A modern count of parameters usually shows that the right side is at least a divisor in $\mathcal{C}$ so that equality holds because $\mathcal{C}_d$ is an irreducible divisor if not empty. The hard problem is to compute the dimension of the family of $S_d$’s contained in a fixed (general) $X$ belonging to the set on the right side above, see [Nue15, RS17] for some efficient computational arguments based on semicontinuity.

For every $a, b \in \mathbb{N}$ the functions $h^0(\mathcal{I}_{S_d}^b(a))$ are upper semicontinuous on $\mathcal{S}_d$. In particular there exists an open non empty subsets $U \subseteq \mathcal{S}_d$ on which $h^0(\mathcal{I}_{S_d}^b(a))$ attains a minimum value $m = m(a, b)$.

We shall be mainly interested in the case $a = 5$ and $b = 2$, that is the computation of the dimension of the linear system $|H^0(\mathcal{I}_{S_d}^2(5))|$ for $S_d \in \mathcal{S}_d$ general. To this aim we consider the exact sequence

$$(1) \quad 0 \to \mathcal{I}_{S_d}^2(5) \to \mathcal{I}_{S_d}(5) \to N^*_{S_d/\mathbb{P}^5}(5) \to 0.$$ 

Suppose that we know $h^0(N^*_{S_d/\mathbb{P}^5}(5)) = y$ and $h^0(\mathcal{I}_{S_d}(5)) = x$ for the general $S_d \in \mathcal{S}_d$ via standard exact sequences (or also computationally) or for some geometrical property of the surfaces. From (1) we deduce $h^0(\mathcal{I}_{S_d}^2(5)) \geq x - y$ for a general $S_d$. By the upper semicontinuity of $h^0(\mathcal{I}_{S_d}^2(5))$ it will be sufficient to find a surface $S \in \mathcal{S}_d$ with $h^0(\mathcal{I}_{S_d}^2(5)) = x - y$ to deduce that the same holds for a general $S_d \in \mathcal{S}_d$.

If $\pi : \chi_\mathcal{S}_d \to \mathcal{S}_d$ is the universal family and if a general $[S_d] \in \mathcal{S}_d$ is smooth, let $V \subseteq \mathcal{S}_d$ be the non empty open set of points $[S] \in \mathcal{S}_d$ such that $S \subset \mathbb{P}^5$ is a smooth surface. Then $\pi^{-1}(V) \to V$ is a smooth morphism and the function $h^0(N^*_{S_d/\mathbb{P}^5}(a))$ is upper semicontinuous on $V$ for every $a \in \mathbb{N}$. In particular, if there exists $[S] \in V$ such that $h^0(N^*_{S_d/\mathbb{P}^5}(a)) = z$, then, for a general $[S_d] \in \mathcal{S}_d$, we have $h^0(N^*_{S_d/\mathbb{P}^5}(a)) \leq z$ and hence $h^0(\mathcal{I}_{S_d}^2(a)) \geq m(a, 2) - z$. If moreover $h^0(\mathcal{I}_{S_d}^2(a)) = m(a, 2) - z$ for a $[S'] \in \mathcal{S}_d$, then the same holds for a general $[S_d] \in \mathcal{S}_d$. These standard and well known remarks will be useful in our analysis of the examples in the next sections, where we shall also deal with the case $e = 3$ and the linear systems $|H^0(\mathcal{I}_{S_d}^3(8))|$ for $d = 14, 38$. 


1.2. Computations via Macaulay2. To study surfaces in $\mathbb{P}^5$ admitting congruences of $(3e - 1)$-secant curves of degree $e$, the rational maps given by hypersurfaces of degree $3e - 1$ having points of multiplicity $e$ along these surfaces and also the lines contained in the images of $\mathbb{P}^5$ via the linear system of cubics through these surfaces we mostly used Macaulay2 [GS18].

Our proofs of various claims exploit the fact that the irreducible components $S_d$ of the Hilbert schemes considered here are unirational. Therefore, by introducing a finite number of free parameters, one can explicitly construct the generic surface in $S_d$ in function of the specified parameters. Adding more parameters one can also take the generic point of $\mathbb{P}^5$, and then one can for instance compute the generic fiber of the map defined by the cubics through the generic surfaces can get an experimental proof that a certain property holds or not for the generic specialization commutes with this type of computation. So, using a common computer one to the original field via a generic specialization of the parameters and, above all, the generic

Theorem 2. We verified that the closure of a general fiber of $S$ a general linear system determines a rational map $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ whose general fiber is a 5-secant conic parametrized by its symmetric product and transversal to $X$.

Proof. A general $S_1 \subset S$ is $k$-normal for every $k \geq 2$, see [AR02, Example 3.8], yielding $h^0(\mathcal{I}_{S_{14}}(5)) = 141$. For a particular smooth $S \in S_{14}$ we verified that $h^0(N_{S/\mathbb{P}^5}(5)) = 136$ so that for a general $S_{14} \in S_{14}$ we have $h^0(N_{S_{14}/\mathbb{P}^5}(5)) \leq 136$. By (1) $h^0(\mathcal{I}_{S_{14}}^2(5)) \geq 5$ for a general $S_{14} \in S_{14}$. Since in the example we studied $h^0(\mathcal{I}_S^2(5)) = 5$, the same holds for a general $S_{14} \in S_{14}$, see Subsection 1.1.

Let $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ be the rational map associated to $|H^0(\mathcal{I}_{S_{14}}^2(5))| = \mathbb{P}^4$ with $S_{14}$ general. We verified that the closure of a general fiber of $\varphi$ is a 5-secant conic to $S_{14}$, concluding the proof.

Remark 3. Letting notation be as in Section 1, on $\text{Bl}_{S_{14}} \mathbb{P}^5$ we have $((5H - 2E))^5 \neq 0$ for an arbitrary $S_{14} \subset \mathbb{P}^5$ so that the induced rational map $\bar{\varphi} : \text{Bl}_{S_{14}} \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ is not a morphism. This phenomenon happens in every example treated in this note. Moreover, the base locus scheme of the restriction to a general cubic may be non reduced and its support may contain other irreducible components besides the fixed surface.

Let $S_{26} \subset \mathbb{P}^5$ be a rational septimic scroll with three nodes recently considered by Farkas and Verra in [FV18], where they also proved that a general $X \in C_{26}$ contains a surface of
this kind. Also these surfaces admit a congruence of 5-secant conics transversal to $X$ and parametrized by a rational variety, see [RS17, Remark 6].

Theorem 4. For a general surface $S_{26} \subset \mathbb{P}^5$ as above we have $|H^0(I_{S_{26}}(5))| = \mathbb{P}^4$ and this linear system determines a rational map $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ whose general fiber is a 5-secant conic to $S_{26}$. In particular, the restriction of $\varphi$ to a general cubic through $S_{26}$ is birational.

Proof. For a general $S_{26} \in S_{26}$ as above, we have $h^0(I_{S_{26}}(5)) = 144$, $h^0(N^*_{S_{26}/\mathbb{P}^5}(5)) = 139$ and in an explicit example of $S \in S_{26}$ we verified that $h^0(I_{S_{26}}(5)) = 5$. Thus $|H^0(I_{S_{26}}(5))| = \mathbb{P}^4$ for a general $S_{26}$, see Subsection 1.1. Let $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ be the rational map associated to $|H^0(I_{S_{26}}(5))| = \mathbb{P}^4$ with $S_{26}$ general. We verified that a general fiber of the corresponding $\varphi$ is a 5-secant conic to $S_{26}$, concluding the proof. \hfill $\square$

Let $S_{38} \subset \mathbb{P}^5$ be a general degree 10 smooth surface of sectional genus 6 obtained as the image of $\mathbb{P}^2$ by the linear system of plane curves of degree 10 having 10 fixed triple points. As shown by Nuer in [Nue15], these surfaces are contained in a general $[X] \in C_{38}$. In [RS17, Theorem 4] we proved that a general $S_{38}$ admits a congruence of 5-secant conics transversal to $X$ and parametrised by a rational variety.

Theorem 5. For a surface $S_{38} \subset \mathbb{P}^5$ as above we have $|H^0(I_{S_{38}}(5))| = \mathbb{P}^4$ and this linear system defines a rational map $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ whose general fiber is a 5-secant conic to $S_{38}$. In particular, the restriction of $\varphi$ to a general cubic through $S_{38}$ is birational.

Proof. A general $S_{38} \in S_{38}$ has ideal generated by 10 cubic forms and is thus 5-normal by [BEL, Proposition 1], yielding $h^0(I_{S_{38}}(5)) = 126$ for a general $S_{38}$ (a fact which can also be verified by a direct computation). For a particular smooth $S \in S_{38}$ we verified that $h^0(N^*_{S/\mathbb{P}^5}(5)) = 121$ so that for a general $S_{38} \in S_{38}$ we have $h^0(N^*_{S_{38}/\mathbb{P}^5}(5)) \leq 121$. From (1) we deduce $h^0(I_{S_{38}}(5)) \geq 5$ for a general $S_{38} \in S_{38}$. Since in the previous explicit example we also have $h^0(I_{S_{26}}(5)) = 5$, the same holds for a general $S_{38} \in S_{38}$, see Subsection 1.1. We verified that a general fiber of the corresponding rational map $\varphi$ is a 5-secant conic to $S_{38}$, concluding the proof. \hfill $\square$

3. Explicit birational maps to linear sections of $\mathbb{G}(1, 3 + k)$ for cubics in $C_{14+12k}$ for $k = 1, 2$

In this section we analyse some examples and look at them as suitable generalisation of those considered by Fano. A smooth quintic del Pezzo surface $D \subset \mathbb{P}^5$ can be realized as a divisor of type $(1, 2)$ on the Segre 3-fold $Z = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ while a smooth quartic rational normal scroll $T \subset \mathbb{P}^5$ can be realized (also) as a divisor of type $(2, 1)$. Moreover, a general cubic through $D$ will cut $Z$ along $D$ and a smooth divisor of type $(2, 1)$, that is a smooth rational normal scroll $T$ (and viceversa).

One might wonder if something similar happens for the next admissible values $d = 26$ and $d = 38$ or if, at least, in these cases there exist surfaces $S_d$ giving explicit birational maps to four dimensional (smooth) linear sections of $\mathbb{G}(1, 3 + k)$, $d = 14 + 12k$. We shall see that very surprisingly this is the case although the linkage phenomenon described above appears again only for $d = 38$. We shall speculate more on this below.
Let $S \subset \mathbb{P}^6$ be a septimic surface with a node, which is the projection of a smooth del Pezzo surface of degree seven in $\mathbb{P}^7$ from a general point on its secant variety. Let $S'_{26} \subset \mathbb{P}^5$ be the projection of $S$ from a general point outside the secant variety $\text{Sec}(S) \subset \mathbb{P}^6$. These surfaces admit a congruence of 5-secant conics parametrised by a rational variety and a general cubic in $C_{26}$ contains such a surface, see [RS17, Theorem 4].

**Theorem 6.** For a general surface $S'_{26} \subset \mathbb{P}^5$ as above we have $|H^0(\mathcal{I}_{S'_{26}}^2(5))| = \mathbb{P}^7$ and this linear system determines a rational map $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ with $\mathbb{P}^4$ a smooth linear section of $\mathbb{G}(1,4) \subset \mathbb{P}^9$. A general fiber of $\varphi$ is a 5-secant conic to $S'_{26}$ and the restriction of $\varphi$ to a general cubic through $S'_{26}$ is birational.

**Proof.** For a general $S'_{26} \in S'_{26}$ as above we verified that $h^0(\mathcal{I}_{S'_{26}}^2(5)) = 8$ and that the closure of the image of $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^7$ is a smooth four dimensional linear section $Y^4$ of $\mathbb{G}(1,4)$. Moreover, a general fiber of the corresponding map $\varphi$ is a 5-secant conic to $S'_{26}$, concluding the proof. □

The two surfaces $S_{26}$ and $S'_{26}$ are not linked in a variety of dimension three via cubics because the sum of their degrees is 14, which is not divisible by 3.

Studying the rational map $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ treated in Theorem 5 we realized that its base locus contains an irreducible component of dimension three $B \subset \mathbb{P}^5$ of degree 6 and sectional genus 3. This variety has 7 singular points and it has homogeneous ideal generated by four cubics. So $B \subset \mathbb{P}^5$ is a degeneration of the so called Bordiga scroll, which is a threefold given by the maximal minors of a general $3 \times 4$ matrix of linear forms on $\mathbb{P}^5$. The variety $B$ contains the surface $S_{38} \subset \mathbb{P}^5$ and a general cubic through $S_{38}$ cuts $B$ along $S_{38}$ and an octic rational scroll $S'_{38} \subset \mathbb{P}^5$ with 6 nodes belonging to the singular locus of $B$. The scroll $S'_{38}$ is a projection of a smooth octic rational normal scroll $S \subset \mathbb{P}^8$ from a special $\mathbb{P}^3$ cutting the secant variety to $S$ in six points.

As far as we know this octic rational scroll with six nodes has not been constructed before and, in this context, it is the right generalization of the smooth quartic rational normal scroll considered by Fano. Moreover, it is remarkable also because it does not come from the diagonal construction via the associated $K3$ surfaces as for the Farkas-Verra and Lai scrolls (see [FV18, Lai16] also for more details on this construction). Let us now describe some geometrical properties of the octic rational scroll $S'_{38} \subset \mathbb{P}^5$.

**Theorem 7.** A general $|X| \in C_{38}$ contains an octic rational scroll $S'_{38} \subset \mathbb{P}^5$ with 6 nodes. Moreover, for a general surface $S'_{38} \subset \mathbb{P}^5$ we have $|H^0(\mathcal{I}_{S'_{38}}^3(8))| = \mathbb{P}^{10}$ and this linear system determines a rational map $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^{10}$ with $\mathbb{P}^{10}$ a linear section of $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$. The general fiber of $\varphi$ is an 8-secant twisted cubic to $S'_{38}$ and the restriction of $\varphi$ to a general cubic through $S'_{38}$ is birational. In particular, an octic rational scroll $S'_{38} \subset \mathbb{P}^5$ admits a congruence of 8-secant cubics.

**Proof.** A general octic scroll $S'_{38} \subset \mathbb{P}^5$ depends on 47 parameters and it has homogeneous ideal generated by 10 cubics forms. In an explicit example $S \subset S'_{38}$ we verified that $h^0(N_{S/\mathbb{P}^5}) = 47$ and that $S$ is contained in smooth cubic hypersurfaces. Therefore $S'_{38}$ is generically smooth of dimension 47 and the natural incidence correspondence in $S'_{38} \times C$ above the open subset of $S'_{38}$ where $h^0(\mathcal{I}_S(3)) = 10$ has an irreducible component $\widehat{C}_{38}$ of dimension 56. Since $(S'_{38})^2 = 34$...
to prove that a general \([X] \in \mathcal{C}_{38}\) contains such a surface, we verified in an explicit general example that \(h^0(N_{S_{38}^I/X}) = 2\).

For a general \(S'_{38} \in \mathcal{S}_{38}'\) we have \(h^0(T_{S_{38}}^3, (8)) = 11\) and the closure of the image of the associated rational map \(\varphi : \mathbb{P}^5 \dasharrow \mathbb{P}^{10}\) is a linear section \(Y^4 \subset \mathbb{P}^{10}\) of \(G(1, 5) \subset \mathbb{P}^{14}\). We verified that a general fiber of \(\varphi\) is an 8-secant twisted cubic to \(S'_{38}\), concluding the proof. \(\square\)

**Remark 8.** For \(d = 26, 38\) with \(e = 2\), respectively \(e = 3\), we have a phenomenon analogous to the case \(d = 14\) and \(e = 1\) studied by Fano in [Fan43]. The analogy is more transparent for \(d = 38\) than for \(d = 26\) due to the linkage between the two involved surfaces.

One might ask if there exists a series of examples for every \(d = 14 + 12k\) (or for \(d = 14 + 12 \cdot 2r\)) mapping a general cubic in \(C_d\) to a four dimensional linear section \(Y^4_k\) of \(G(1, 3 + k)\). If \(Y^4_k\) is a general four dimensional linear section of \(G(1, 3 + k)\), then \(Y^4_k\) is not rational for \(k \geq 4\) (and probably also for \(k = 3\)). So if a series of examples of the previous kind exists for every \(k \geq 3\), then either a cubic fourfold through the corresponding surface \(S \subset \mathbb{P}^5\) is irrational or \(Y^4_k\) is singular and becomes rational. We have constructed examples for \(k = 1, 2\) but we are unable to continue for every \(k \geq 3\) due to the difficulty of the problem, also from the computational point of view.

As far as we know these birational representations of four dimensional linear sections of Grassmannians of lines have not been studied before although there was a lot of classical work on special representations of (linear sections of) Grassmannians (of sufficiently high dimensions), see [Sm, Roo, Rot]. To understand how subtle the problems involved can be, let us recall that a general four dimensional linear section \(Y^4 \subset G(1, 5)\) is rational while a general three dimensional linear section \(W^3 \subset G(1, 5)\) is birational to a cubic threefold—a fact firstly shown by Fano—and hence it is irrational. Continuing, it is well known that a general 6-dimensional linear section of \(G(1, 7)\) is rational, while the general five dimensional linear section seems to be birational to a mildly singular quartic hypersurface in \(\mathbb{P}^5\), see [Roo, pg. 197] (in [Rot, pg. 107] it is claimed that the quartic is smooth). We plan to come back to this interesting topic elsewhere.

4. **Some examples of surfaces admitting a congruence of (3e - 1)-secant curves of degree e with e = 3, 5**

This section will be even more discursive than the previous ones and we shall mainly describe some new remarkable examples.

4.1. **The congruence of 8-secant twisted cubics to an octic surface with three nodes.** Let \(D_8 \subset \mathbb{P}^8\) be a smooth del Pezzo surface of degree 8, which is the quadratic embedding of \(\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3\), and let \(\tilde{S}_{38} \subset \mathbb{P}^5\) be a projection of \(D_8\) from a plane intersecting the secant variety to \(D_8\) in three general points. Then \(\tilde{S}_{38} \subset \mathbb{P}^5\) is an octic surface with three nodes of sectional genus 1, has homogeneous ideal generated by 10 cubic forms and is contained in a smooth cubic hypersurface. On such a cubic fourfold we have \((\tilde{S}_{38})^2 = 34\) and \(d = 3 \cdot 34 - 64 = 38\). By counting the parameters we verified that the closure of the locus of smooth cubics containing a surface \(\tilde{S}_{38}\) has codimension two in \(\mathcal{C}\), i.e. it is only a divisor in \(\mathcal{C}_{38}\).

One verifies that \(h^0(T_{\tilde{S}_{38}}^3, (8)) = 6\) and that the associated rational map \(\varphi : \mathbb{P}^5 \dasharrow \mathbb{P}^5\) maps \(\mathbb{P}^5\) onto a smooth quadric hypersurface \(Q \subset \mathbb{P}^5\). A general fiber of \(\varphi\) is an 8-secant twisted
cubic to \( S_{38} \) so that the restriction of \( \varphi \) to a general cubic fourfold through \( S_{38} \) is a birational map onto \( Q \) and \( S_{38} \subset \mathbb{P}^5 \) admits a congruence of 8-secant twisted cubics.

This last property can be verified also in this way. The linear system \( |H^0(I_{S_{38}}(3))| \) defines a map \( \psi : \mathbb{P}^5 \dasharrow \mathbb{P}^9 \) which is birational onto the image \( Z = \overline{\psi(\mathbb{P}^5)} \subset \mathbb{P}^9 \). The variety \( Z \) has degree 17 and it is defined by 3 quadratic forms and 4 cubic forms. Through a general point \( z = \psi(p) \in Z \) there passes 14 lines contained in \( Z \). From these 7 are images of the seven secant lines to \( S_{38} \) passing through \( p \). The remaining 7 come from six 5-secant conics to \( S_{38} \) passing through \( p \) and a single 8-secant twisted cubic to \( S_{38} \) passing through \( p \).

4.2. The congruence of 8-secant twisted cubics to a degree 11 smooth surface, projection from three internal points of a \( K \) surface of degree 14 and sectional genus 8. Let \( S \subset \mathbb{P}^8 \) be a smooth \( K3 \) surface of degree 14 and sectional genus 8, which is a general linear section of \( G(1,5) \subset \mathbb{P}^{14} \). Let \( S_{14} \subset \mathbb{P}^5 \) be a projection of \( S \) from a general trisecant plane, that is the projection of \( S \) from the plane generated by three general points on it. Then \( S_{14} \subset \mathbb{P}^5 \) is a smooth surface of degree 11 and sectional genus 8 with ideal generated by 9 cubic forms and which is contained in smooth cubic hypersurfaces. On such a cubic fourfold we have \((S_{14})^2 = 45 \) and \( d = 3 \cdot 45 - 121 = 14 \). One verifies that the surfaces \( S_{14} \) depend on sixty parameters and that there exists a cubic fourfold containing a 14-dimensional family of such surfaces. From this we can deduce that the closure of the locus of smooth cubics containing the surfaces \( S_{14} \)'s is \( C_{14} \), providing another different geometric description of this divisor.

Then \( h^0(I_{S_{14}}^3(8)) = 6 \) and moreover the associated rational map \( \varphi : \mathbb{P}^5 \dasharrow \mathbb{P}^5 \) maps \( \mathbb{P}^5 \) onto a smooth cubic hypersurface \( X = \overline{\varphi(\mathbb{P}^5)} \subset \mathbb{P}^5 \). Since a general fiber of \( \varphi \) is an 8-secant twisted cubic to \( S_{14} \), the surface \( S_{14} \) admits a congruence of 8-secant twisted cubics and the restriction of \( \varphi \) to a general cubic \( X \) through \( S_{14} \) is a birational map onto \( X \).

The existence of the congruence of 8-secant twisted cubics to \( S_{14} \) can be also verified in this way. The linear system \( |H^0(I_{S_{14}}(3))| \) defines a map \( \psi : \mathbb{P}^5 \dasharrow \mathbb{P}^8 \) which is birational onto the image \( Z = \overline{\psi(\mathbb{P}^5)} \subset \mathbb{P}^8 \), which is a complete intersection of two quadric hypersurfaces and a cubic hypersurface. Through the general point \( z = \psi(p) \in Z \) there passes 24 lines contained in \( Z \). Of these 13 are images of the thirteen secant lines to \( S_{14} \) passing through \( p \). The remaining 11 come from ten 5-secant conics to \( S_{14} \) passing through \( p \) and a single 8-secant twisted cubic to \( S_{14} \) passing through \( p \).

As far as we know, this seems to be the first example of a non rational surface admitting a congruence of \((3e-1)\)-secant curves of degree \( e \). Let us recall that for \( e = 1 \) such a surface is necessarily rational because its symmetric product, which birationally parametrizes secant lines, is rational (it is clearly birational to a general hyperplane of \( \mathbb{P}^5 \)), see also [CR11] for another direct proof via projection from a general tangent plane.

4.3. Congruences of 14-secant quintic rational normal curves coming from special Cremona transformations of \( \mathbb{P}^6 \). Let \( \Phi : \mathbb{P}^6 \dasharrow \mathbb{P}^6 \) be a Cremona transformation not defined along a smooth 3-fold \( B \subset \mathbb{P}^6 \), that is a so called special Cremona transformation of \( \mathbb{P}^6 \) with three dimensional base locus. These transformations has been recently completely classified by the second author in [Sta15], solving the long standing question of the classification of all special Cremona transformations in \( \mathbb{P}^N \) with \( N \leq 6 \).
The results in [Sta15] yield that $B \subset \mathbb{P}^6$ is one of the following:
(1) a threefold of degree 14, sectional genus 15 with trivial canonical bundle which is Pfaffian, i.e. given by the Pfaffians of a skew–symmetric matrix;
(2) a conic bundle over $\mathbb{P}^2$, embedded in $\mathbb{P}^6$ as a threefold of degree 13 and sectional genus 12.

Let $S \subset \mathbb{P}^5$ be a general hyperplane section of a $B \subset \mathbb{P}^6$ as above. Then the restriction of $\Phi$ to $H$ determines a birational map $\tilde{\Phi} : H \dashrightarrow Z \subset \mathbb{P}^6$ with $Z$ a quintic hypersurface since the Cremona transformation $\Phi$ is of type $(3,5)$. Through a general point $z = \tilde{\Phi}(p)$ there passes $120 = 5!$ lines $L_i \subset Z$. Let $B' \subset Z$ be the base locus of $\tilde{\Phi}^{-1}$. Every line $L_i$ is mapped by $\tilde{\Phi}^{-1}$ onto a smooth curve of degree $e = 5 - \text{length}(L_i \cap B')$ which is $(3e - 1)$-secant to $S$ being mapped back to the line $L_i$ by $\tilde{\Phi}$. In both examples there is a unique line $L$ through $z$ with $L \cap B' = \emptyset$, yielding that these surfaces admit a congruence of 14-secant rational normal curves of degree 5.

We also proved that the cubic fourfolds through both types of surfaces exhaust $\mathcal{C}_14$, we calculated $h^0(\mathcal{I}_6^2(14))$ and we verified that the general fiber of the associated rational map is a rational normal curve of degree 5 which is 14-secant to $S$. The whole analysis from this point of view of the last two examples is very difficult due to the high degree of the linear systems involved and also due to the high multiplicity along the surfaces.

5. Computations

The aim of this section is to show how one can ascertain the contents of Theorems 2, 4, 5, and similar results in specific examples using the computer algebra system Macaulay2 [GS18].

We begin to observe that given the defining homogeneous ideal of a subvariety $X \subset \mathbb{P}^n$, the computation of a basis for the linear system $|H^0(\mathcal{I}_X^d)|$ of hypersurfaces of degree $d$ with points of multiplicity at least $e$ along $X$ can be performed using pure linear algebra. This approach is implemented in the Macaulay2 package Cremona (see [Sta18]), which turns out to be effective for small values of $d$ and $e$. In practice, in any Macaulay2 session with the Cremona package loaded, if $I$ is a variable containing the ideal of $X$, we get a rational map defined by a basis of $|H^0(\mathcal{I}_X^d)|$ by the command¹ $\text{rationalMap}(I,d,e)$.

Now we consider a specific example related to Theorem 5. In the following code, we first load the file ExplicitRationality.m2 provided as an ancillary file to our arXiv submission (this also makes the tools of the Cremona package available), and then we produce a pair $(f,\varphi)$ of rational maps: $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5$ is a birational parameterization of a smooth surface $S = S_{38} \subset \mathbb{P}^5$ of degree 10 and sectional genus 6 as in Theorem 5, and $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^9$ is a rational map defined by a basis of cubic hypersurfaces containing $S$ (see also Section 5 of [RS17]). Here we work over the finite field $\mathbb{F}_{10000019}$ for speed reasons.

Macaulay2, version 1.12
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage "ExplicitRationality";
i2 : (f,phi) = example(38,ZZ/10000019);

¹For all the examples treated in this paper, the linear system of hypersurfaces of degree $d$ with points of multiplicity at least $e$ along $X \subset \mathbb{P}^n$ coincides with the homogeneous component of degree $d$ of the saturation with respect to the irrelevant ideal of $\mathbb{P}^n$ of the $e$-power of the homogeneous ideal of $X$. So one can also compute it using the code: $\text{gens image basis}(d,\text{saturate}(I^e))$. 
We now compute the rational map $\psi$ defined by the linear system of quintic hypersurfaces of $\mathbb{P}^5$ which are singular along $S$. From the information obtained by its projective degrees we deduce that $\psi$ is a dominant rational map onto $\mathbb{P}^4$ with generic fibre of dimension 1 and degree 2 and with base locus of dimension 3 and degree $5^2 - 19 = 6$.

Next we compute a special random fibre $F$ of the map $\psi$.

Finally, the following two lines of code tell us that the restriction of $\psi$ to a random cubic fourfold containing $S$ is a birational map whose inverse map is defined by forms of degree 9 and has base locus scheme of dimension 2 and degree $9^2 - 27 = 54$.

For the convenience of the reader, we have included in the file the examples mentioned in this paper of birationality between cubic fourfolds and other rational fourfolds. One of the examples can be obtained as follows:

The above command produces the following 6 rational maps:

(1) the same parameterization $f : \mathbb{P}^2 \dasharrow \mathbb{P}^5$ obtained above of the surface $S = S\text{,}38 \subset \mathbb{P}^5$ of degree 10 and sectional genus 6;
(2) the rational map $\psi : \mathbb{P}^5 \dasharrow \mathbb{P}^4$ defined by the quintic hypersurfaces with double points along $S$;
(3) the restriction of $\psi$ to a cubic fourfold $X$ containing $S$, which is a birational map;
(4) the linear projection of a smooth scroll surface of degree 8 in $\mathbb{P}^9$ from a linear 3-dimensional subspace intersecting the secant variety of the scroll in 6 points, so that the image is a scroll surface $T \subset \mathbb{P}^5$ of degree 8 with 6 nodes; moreover, we have the relation: $T \cup S = X \cap \text{top}(\text{Bs}(\psi))$, where $\text{top}(\text{Bs}(\psi))$ denotes the top component of the base locus of $\psi$;
(5) the rational map $\eta : \mathbb{P}^5 \dasharrow Z \subset \mathbb{P}^{10}$ defined by the octic hypersurfaces with triple points along $T$, and where $Z \subset \mathbb{P}^{10}$ is a 4-dimensional linear section of $\text{G}(1, 5) \subset \mathbb{P}^{14}$;
the restriction of $\eta$ to the cubic fourfold $X$, which is a birational map onto $Z$.

Now we can quickly get information on the maps, e.g. on the inverse of the last one:

```
$ii3 : g = last oo;
$ii3 : RationalMap (birational map from hypersurface in PP^5 to 4-dimensional subvariety of PP^10)
$ii4 : describe inverse g
$ii4 = rational map defined by forms of degree 5
  source variety: 4-dimensional variety of degree 14 in PP^10 cut out by 15
  hypersurfaces of degree 2
  target variety: smooth cubic hypersurface in PP^5
  birationality: true
  projective degrees: {14, 70, 80, 24, 3}
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