Predictive time optimal algorithm for a third-order dynamical system with delay

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Annotation. There are given the results for the analytical synthesis of optimal control algorithm in a closed circuit for the third-order dynamical system, describing by two different groups of state variables. For improvement the quality of time optimal control system is offered to use the predictive output value, expected through the object’s dead-time. A comparative analysis of quality and sensitivity control systems with the time optimal and linear PID or PI controllers is done. Calculations were carried out on the example of the automatic control system of superheated steam temperature of strait-flow boiler TPP-210.

1. Introduction

One of the most attractive optimal control algorithms is an algorithm of maximal fast-acting (time optimal). At the determined disturbing it provides minimum time of transient process, and at random disturbing it gives a small enough value of variance.

However complication of analytical solving forces the developers of such algorithm to be limited by the choice of controlled process models not higher than the second order [5–7]. For the enough good reflection of thermal processes dynamics the rational part of not high order transfer function must be added by an equivalent delay.

The presence of transport delay in the object model results in a necessity to use predictive algorithms [3], when instead of current values of the state variables, related to the controlled output \( y(t) \) and its derivatives, their predictive on the delay models \( \tau \) value are put in the algorithm of regulator. Prediction accuracy depends mainly on the object dynamics, predictive algorithms and disturbance characteristics.

2. Synthesis time optimal algorithm for an object the second order with the servomechanism the constant speed

We will find the maximal-fast-acting control algorithm for a second-order linear object with delay

\[
W_0(p) = \frac{y(p)}{\mu(p)} = \frac{K}{(T_1 p + 1)(T_2 p + 1)} e^{-p\tau},
\]
which is controlled by the non-synchronous electrical engine (servomechanism) of constant-speed, as it usually does in Russia power energetics,

\[ \frac{d\mu}{dt} = S = \text{const}. \]

Taking into account that in the time optimal systems a control lies on the borders of possible area, servomechanism of constant-speed can be presented by an integral element

\[ W_{SM}(p) = \frac{\mu(p)}{u(p)} = \frac{S}{p}, \quad (2) \]

Thus, the control \( u \) we will understand as the direction of servomechanism moving, i.e.

\[ u = \pm 1. \]

A value \( u = +1 \) corresponds opening of controlling organ at a speed of \( S \), a value \( u = -1 \) is its closing.

It is now possible to examine the controlled system “object — servomechanism”

\[ W(p) = \frac{y(p)}{u(p)} = \frac{KS}{p(T_1 p + 1)(T_2 p + 1)} e^{-p \tau} \]

with a limit on the module control \( |u| \leq 1 \).

Thus, the modified task fully corresponds to the standard task of maximal fast-acting [2].

The synthesis of algorithm of optimal control, as is generally known, comes true for the system without a delay with a subsequent substitution in equations of algorithm of predictive on \( \tau \) at output variables of object.

As our research has shone, the analytic solution the task will be possible just for the canonical state variables. We will decompose shot-rational part of transfer function to the sum of first-order fractions:

\[ \tilde{W}(p) = \frac{KS}{p(T_1 p + 1)(T_2 p + 1)} = \frac{b_1}{p} + \frac{b_2}{p + \alpha_1} + \frac{b_3}{p + \alpha_2}, \quad (3) \]

then differential equations of the control system “object — servomechanism” is accepted kind

\[
\begin{align*}
x_1' &= b_1 u, \\
x_2' &= -\alpha_1 x_2 + b_2 u, \\
x_3' &= -\alpha_2 x_3 + b_3 u, \\
y &= x_1 + x_2 + x_3,
\end{align*}
\]

where \( \alpha_1 = \frac{1}{T_1}; \quad \alpha_2 = \frac{1}{T_2}; \quad b_1 = SK; \quad b_2 = -\frac{SKT_1}{T_1 - T_2}; \quad b_3 = \frac{SKT_2}{T_1 - T_2}. \)

We will write down this system for reverse time, substituting \( dt \) by \( -dt \):
\[
\begin{aligned}
    x_1' &= -b_1 u,
    \\
    x_2' &= +a_1 x_2 - b_2 u,
    \\
    x_3' &= +a_2 x_3 - b_3 u.
\end{aligned}
\]

From here we find a transitional states matrix \( \Phi(t,0) \) and vector of coefficients \( B \) at the control \( u \) for equations in reverse time:

\[
\Phi(t,0) = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & e^{a_1 t} & 0 \\
    0 & 0 & e^{a_2 t}
\end{bmatrix}, \quad B = \begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3
\end{bmatrix}.
\]

Common solution of the differential equations system at unzero initial conditions \( X(0) \), taking into account, according to Pontryagin principle minimum [2], the constancy of value of controlling on separate process parts, will look like the following

\[
\begin{aligned}
    x_1(t) &= x_{10} - b_1 u t, \\
    x_2(t) &= x_{20} e^{a_1 t} + \frac{b_2 u}{a_1} (e^{a_1 t} - 1), \\
    x_3(t) &= x_{30} e^{a_2 t} - \frac{b_3 u}{a_2} (e^{a_2 t} - 1).
\end{aligned}
\]  

(4)

In accordance with the Feldbaum theorem [4] in the system of the third order with different simple roots there must be two switching of control \( u \), formative three parts of controlling.

We will designate through \( u_1 \) the value of control on the first in reverse time part, and time of the first switching — through \( t_1 \). Like \( u_2 \) and \( t_2 \) are values of these parameters on the second part of control. For some simplification of records we will execute counting out the time on every part from a zero. We will consider each of parts individually.

**First part:** \( t \in [0,t_1) \), \( X(0) = 0 \), \( u = u_1 \).

At the end of part for \( t = t_1 \) from the system (4) we have:

\[
\begin{aligned}
    x_1(t_1) &= -b_1 u_1 t_1, \\
    x_2(t_1) &= +\frac{b_2 u_1}{a_1} (e^{a_1 t_1} - 1), \\
    x_3(t_1) &= -\frac{b_3 u_1}{a_2} (e^{a_2 t_1} - 1).
\end{aligned}
\]  

(5)

These equations determine the multitude of intersections (touching) of two half-surfaces, formative the surface of switching \( \Pi \).

From the first equation we express \( t_1 = -x_1 / (b_1 u_1) \) and we put in two other:
\[ x_2 = \frac{b_2 u_1}{\alpha_1} \left( e^{-\alpha_1 x_1 / (b_1 u_1) - 1} \right), \]
\[ x_3 = \frac{b_2 u_1}{\alpha_2} \left( e^{-\alpha_2 x_1 / (b_1 u_1) - 1} \right). \]

To get the generalized equation of curve of the first (in reverse time) switching, we will analyze the signs of state variables on the first part:

for \( u_1 = +1 \):
\[ x_1 = -b_1 t < 0, \quad x_2 > 0, \quad \text{because } b_2 \text{ is negative}, \quad x_3 < 0; \]

for \( u_1 = -1 \):
\[ x_1 = +b_1 t > 0, \quad x_2 < 0, \quad x_3 > 0. \]

Because variables are not changed the sign on the part, instead of two equations of switching curve it is enough to take advantage of one of them, i.e. to examine the projection of spatial curve on the surface of coordinates of phase space.

We will take, for example, equation for \( x_2 \) (projection of switching curve on a surface \( x_1, x_2 \)). Then, substituting the first part control by function \( u_1 = \text{sign} \left( x_1 \right) \), we get equation of curve of the last (in direct time) switching of control sign:

\[ x_2 + \text{sign} \left( x_1 \right) \frac{b_2}{\alpha_1} \left( e^{\text{sign} \left( x_1 \right) x_1 / (b_1 u_1) - 1} \right) = 0. \]

The curve projection of the last switching is shown on figure 1.

![Figure 1. The curve projection of the last switching](image)

We will enter the variable quantity of \( R \), characterizing distance of phase-point in some moment of time to the curve of switching:

\[ R = x_2 + \text{sign} \left( x_1 \right) \frac{b_2}{\alpha_1} \left( e^{\text{sign} \left( x_1 \right) x_1 / (b_1 u_1) - 1} \right). \] (6)

A half- surface above the curve \( AOB \) corresponds to motion of projection of phase-point \( (x_1, x_2) \) on the second part of control at \( u_2 = -1 \). In this half-surface \( R > 0 \). Under the curve \( AOB \) the control \( u_2 = +1 \), and \( R < 0 \).
If for comfort to enter denotation

\[ V = \text{sign}(R), \]

that the signs of state and control variables at the end of the first part will be related to \( V \) by next correlation:

\[ \text{sign}(x_1, x_2, x_3) = [-V, +V, -V], \quad u_1 = +V, \quad u_2 = -V. \] (7)

Taking into account it we will consider the system of equations of motion the phase-point on the second part (4), the initial conditions of that are values \( x_1(t_1), x_2(t_1), x_3(t_1) \) at the end of the first part.

**Second part:** \( t \in [0, t_2) \), \( X(0) = X(t_1) \), \( u = u_2 = -\text{sign}(R) = -V \).

At the end of this part for \( t = t_2 \) we have:

\[
\begin{align*}
x_1(t_2) &= x_1(t_1) - b_1u_2t_2; \\
x_2(t_2) &= x_2(t_1) e^{\alpha_1 t_2} + \frac{b_2u_2}{\alpha_1} \left( e^{\alpha_1 t_2} - 1 \right); \\
x_3(t_2) &= x_3(t_1) e^{\alpha_2 t_2} - \frac{b_2u_2}{\alpha_2} \left( e^{\alpha_2 t_2} - 1 \right),
\end{align*}
\]

or after the substitution the initial conditions (5) and replacement \( u_1, u_2 \) in accordance with (7):

\[
\begin{align*}
x_1(t_2) &= -b_1Vt_1 + b_1Vt_2; \\
x_2(t_2) &= +\frac{b_2V}{\alpha_1} \left( e^{\alpha_1 t_1} - 1 \right) e^{\alpha_1 t_2} - \frac{b_2V}{\alpha_1} \left( e^{\alpha_1 t_2} - 1 \right); \\
x_3(t_2) &= -\frac{b_2V}{\alpha_2} \left( e^{\alpha_2 t_1} - 1 \right) e^{\alpha_2 t_2} + \frac{b_2V}{\alpha_2} \left( e^{\alpha_2 t_2} - 1 \right).
\end{align*}
\] (8)

To get equation of switching surface, from the system (8) it is necessary to eliminate the moments of switching \( t_1 \) and \( t_2 \). From the first equation we express

\[ t_1 = \frac{-Vx_1}{b_1} + t_2 \]

put it in two other equations:

\[
\begin{align*}
x_2 &= +\frac{b_2V}{\alpha_1} \left( e^{-\alpha_1 x_1 V/b_1} - 1 \right) e^{\alpha_1 t_2} - \frac{b_2V}{\alpha_1} \left( e^{\alpha_1 t_2} - 1 \right), \\
x_3 &= -\frac{b_2V}{\alpha_2} \left( e^{-\alpha_2 x_1 V/b_1} - 1 \right) e^{\alpha_2 t_2} + \frac{b_2V}{\alpha_2} \left( e^{\alpha_2 t_2} - 1 \right).
\end{align*}
\]

For an exception \( t_2 \) we will enter replacements \( e^{\alpha_1 t_2} = z \), \( e^{\alpha_2 t_2} = z^{\alpha_2/\alpha_1} = \omega \), then the last system of equalizations will be written down in a kind

\[
\begin{align*}
e^{-\alpha_1 x_1 V/b_1} z^2 - 2z + \left( 1 - \frac{\alpha_1 x_2 V}{b_2} \right) = 0, \\
e^{-\alpha_2 x_1 V/b_1} \omega^2 - 2\omega + \left( 1 + \frac{\alpha_2 x_2 V}{b_3} \right) = 0. \tag{9}
\end{align*}
\]
The system (9) is the switching surface equation \( \Pi \).

From the first equation we express \( z \), whereas, \( z = e^{\alpha_1 t_2} > 0 \), we will write down:

\[
z = e^{-\alpha_1 x_V / h_1} \left[ 1 + \sqrt{1 - \left(1 - \frac{\alpha_1 x_V}{b_2} \right) e^{-\alpha_1 x_V / h_1}} \right].
\]

then

\[
\omega = \frac{\alpha_2 / \alpha_1}{e^{-\alpha_2 x_V / h_1} \left[ 1 + \sqrt{1 - \left(1 - \frac{\alpha_1 x_V}{b_2} \right) e^{-\alpha_1 x_V / h_1}} \right]}.
\]

(10)

Now equation of surface of switching can be presented by the second equation of the system (9) and equation (10):

\[
e^{-\alpha_2 x_V / h_1} \omega^2 - 2\omega + \left(1 + \frac{\alpha_2 x_V}{b_3} \right) = 0;
\]

\[
\omega = e^{-\alpha_2 x_V / h_1} \left[ 1 + \sqrt{1 - \left(1 - \frac{\alpha_1 x_V}{b_2} \right) e^{-\alpha_1 x_V / h_1}} \right]^{\alpha_2 / \alpha_1}.
\]

(11)

Thus, if the coordinates of current position of phase-point satisfy to the system (11), then a phase-point is on the surface of switching \( \Pi \) and sign control must be changed on opposite. If a phase-point does not lie on the surface \( \Pi \), then the first equation of the system will not become the zero.

We will enter a variable \( \sigma \), defining the distance from the located phase-point to the surface of switching:

\[
\sigma = e^{-\alpha_2 x_V / h_1} \omega^2 - 2\omega + \frac{\alpha_2 x_V}{b_3} + 1.
\]

(12)

If a phase-point is above the switching surface (we will put, for example, \( x_3 >> 1 \)), then \( \sigma > 0 \), and a control must be negative \( u = -1 \). If a phase-point is under the surface of switching (for example, at \( x_3 << -1 \)), then \( \sigma < 0 \), and a control must be positive \( u = +1 \), i.e.

\[
u = -\text{sign}(\sigma).
\]

(13)

At \( \sigma = 0 \) a phase-point is on the switching surface and sign control must be changed on opposite.

Combining equations (6) for the distance to the curve of the last switch, (11) for the auxiliary quantity \( \omega \), (12) for the distance to the switching surface, (13) for the sign of optimal control, and providing conditions OFF control when the phase-point gets to the origin, we obtain the algorithm of time optimal control:

\[
R = x_2 + \text{sign}(x_1) \frac{b_2}{\alpha_1} \left( e^{\text{sign}(x_1) \alpha_1 x_V / (b_3 h_1)} - 1 \right);
\]

\[
V = \text{sign}(R);
\]

\[
\omega = e^{-\alpha_2 x_V / h_1} \left[ 1 + \sqrt{1 - \left(1 - \frac{\alpha_1 x_V}{b_2} \right) e^{-\alpha_1 x_V / h_1}} \right]^{\alpha_2 / \alpha_1} ;
\]

(14)

\[
\sigma = e^{-\alpha_2 x_V / h_1} \omega^2 - 2\omega + \frac{\alpha_2 x_V}{b_3} + 1;
\]

\[
u = -\text{sign}(\sigma);
\]

\[
u = 0, \text{ if } |x_1| \leq \varepsilon_1, |x_2| \leq \varepsilon_2, |x_3| \leq \varepsilon_3.
\]
3. The predictive control. State space variables choice

In equalizations of the system \((14)\) it is necessary to put predictive on the \(\tau\) values of vector of state variables \(\mathbf{X}(t+\tau)\), calculated on his current values \(\mathbf{X}(t)\), that in turn it is necessary to calculate, using information about the measurements on controlled object. Such variables must be three.

To predict the state variables included in the control algorithm, it's best to use a polynomial function of the form

\[
x(t + \tau) = a_0(t) + a_1 \tau + a_2 \tau^2 + \ldots.
\]

The coefficients of the polynomial it is advisable to calculate the least squares method on a sliding averaging interval. The best value for the averaging interval and order of the polynomial is chosen numerically depending on the type of state variables and character disturbances.

In the simplest case, a numerical linear prediction in two measurements of the process in the current \(x(t) = x(i\Delta t) = x_i\) and previous \(x(t - \Delta t) = x[(i - 1)\Delta t] = x_{i-1}\) time moments polynomial parameters are calculated according to the formulas

\[
a_0 = x_i; \quad a_1 = \frac{x_i - x_{i-1}}{\Delta t}.
\]

Then the predicted value for the time delay \(\tau = r\Delta t\) function \(x(t + \tau)\) can be calculated by the formula

\[
x(t + \tau) = x_{i + r} = x_i + (x_i - x_{i-1}) r.
\]

Practically always the object output \(y(t)\) and position of regulative organ \(\mu(t)\) are measurable. If the third variable is not measured, then instead of her it is possible to use a derivative \(y'(t)\), calculated on the values of exit \(y\). We will consider this case in more detail.

At first we will set intercommunication between state of canonical form of \(\mathbf{X}\) and variables of the normalized form \(\mathbf{Y}^T = [y, y', y^*]\). Let us take for this purpose the similarity transformation:

\[
\mathbf{X} = \mathbf{V}^{-1}\mathbf{Y}.
\]

Transforming Vandermonde matrix \(\mathbf{V}\) and reverse to her for the examined system \((3)\) with the poles of transfer function \(0, -\alpha_1, -\alpha_2\) will have the view:

\[
\mathbf{V} = \begin{bmatrix}
1 & 1 & 1 \\
0 & -\alpha_1 & -\alpha_2 \\
0 & \alpha_1^2 & \alpha_2^2
\end{bmatrix}, \quad \mathbf{V}^{-1} = \begin{bmatrix}
\frac{1}{\alpha_1 \alpha_2} & \frac{1}{\alpha_1} & \frac{1}{\alpha_2} \\
\frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} & \frac{\alpha_1}{\alpha_1 \alpha_2} & \frac{\alpha_2}{\alpha_1 \alpha_2} \\
\frac{\alpha_1 (\alpha_1 - \alpha_2)}{\alpha_1 \alpha_2} & \frac{\alpha_1 (\alpha_1 - \alpha_2)}{\alpha_1 \alpha_2} & \frac{-\alpha_1}{\alpha_1 \alpha_2}
\end{bmatrix}.
\]

Now for canonical variables we can write down:
\[
\begin{align*}
    x_1 &= y + \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} y' + \frac{1}{\alpha_1 \alpha_2} y'', \\
    x_2 &= \frac{\alpha_2}{\alpha_1 (\alpha_1 - \alpha_2)} y' + \frac{1}{\alpha_1 (\alpha_1 - \alpha_2)} y'', \\
    x_3 &= \frac{-\alpha_1}{\alpha_2 (\alpha_1 - \alpha_2)} y' + \frac{-1}{\alpha_2 (\alpha_1 - \alpha_2)} y''.
\end{align*}
\]

From the transfer function of object \( \bar{W}_o(p) \) we have

\[
    K\mu = T_1 T_2 y'' + (T_1 + T_2) y' + y.
\]

From here we will express a second derivative:

\[
    y'' = K\alpha_1 \alpha_2 \mu - (\alpha_1 + \alpha_2) y' - \alpha_1 \alpha_2 y
\]

and we will put her in the system (15):

\[
\begin{align*}
    x_1 &= K\mu, \\
    x_2 &= \frac{K\alpha_2}{\alpha_1 - \alpha_2} \mu - \frac{1}{\alpha_1 - \alpha_2} y' - \frac{\alpha_2}{\alpha_1 - \alpha_2} y, \\
    x_3 &= -\frac{K\alpha_1}{\alpha_1 - \alpha_2} \mu + \frac{1}{\alpha_1 - \alpha_2} y' + \frac{\alpha_1}{\alpha_1 - \alpha_2} y.
\end{align*}
\]

Thus, by means of the system (16) it is possible to find the current values of canonical variables \( x_1(t), x_2(t), x_3(t) \) on the current values of object variables \( \mu(t), y(t), y'(t) \).

4. Comparative analyses

Undoubted interest presents the possible upgrading of control quality with the regulator of maximal fast-acting as compared to typical linear laws.

Calculations were carried out on the example of the superheated steam temperature automatic control system of uniflow boiler TPP-210. As an example on a fig. 2 the transients in close system are shown with the object parameters \( K = 1, \tau = 14 \text{ c}, T_1 = 47 \text{ c}, T_2 = 45 \text{ c} \) \([1]\) at a giving the step in its output.
Curve 1 corresponds to the system with the regulator of maximal fast-acting, curve 2 — to the system with PID regulator, curve 3 — to the system with PI regulator. The chart of moving of regulative organ $\mu(t)$ by time optimal regulator is shown on the same figure.

In theory duration of transient at using algorithm of maximal fast-acting can be decrease in 4—5 times as compared to linear algorithms, however in the real systems from appearance of sliding or oscillating modes this winning will be some less.

The material expounded here shows evidently, as procedure of receiving the time optimal algorithm becomes complicated with the object order increases. We will notice that a decision in an eventual kind for the objects of fourth order is not got by virtue of his complication. Therefore for practical aims it is desirable to use approximation 2—3 order with delay, if necessary.

5. Conclusion
With the help of the canonical state variables succeed to get the analytical time optimal algorithm in a closed circuit for the third-order dynamical system.

There are given the results for synthesis of, describing by two different groups of state variables. A comparative analysis of quality and sensitivity control systems with the time optimal and linear PID or PI controllers is done. Calculations were carried out on the example of the automatic control system of superheated steam temperature of uniflow boiler TPP-210.

The algorithm includes switching surface equation, the equation of the line of contact half-surfaces for different signs of the control $u$, the selection conditions of the sign control and control off at the entrance of the phase-point in the dead zone.

Taking into account the existing real object measuring two types of state variables suggested, allowing by a similarity transformation to calculate the canonical state variables.

It is discussed the problem of accounting object time delay by using the predicted in future state variables $X(t+\tau)$ instead of their current values $X(t)$.

The comparative analysis of quality control for the time optimal system, linear PI and PID systems shown that the time optimal control theoretically at 4—5 times faster than linear control.

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