Structure of States Which Satisfy Strong Subadditivity of Quantum Entropy with Equality

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Abstract: We give an explicit characterisation of the quantum states which saturate the strong subadditivity inequality for the von Neumann entropy. By combining a result of Petz characterising the equality case for the monotonicity of relative entropy with a recent theorem by Koashi and Imoto, we show that such states will have the form of a so-called short quantum Markov chain, which in turn implies that two of the systems are independent conditioned on the third, in a physically meaningful sense. This characterisation simultaneously generalises known necessary and sufficient entropic conditions for quantum error correction as well as the conditions for the achievability of the Holevo bound on accessible information.

I. Introduction

The von Neumann entropy [13]

\[ S(\rho) = -\text{Tr}\rho \log \rho, \]

of a density operator \( \rho \) on a finite dimensional Hilbert space \( \mathcal{H} \) shares many properties with its classical counterpart, the Shannon entropy

\[ H(P) = -\sum_{x \in \mathcal{X}} P(x) \log P(x) \]

of a probability distribution \( P \) on a discrete set \( \mathcal{X} \). (All logarithms in this work are understood to be to base 2. Also, we will use the terms “state” and “density operator” interchangeably.) For example, both are nonnegative, and equal to 0 if and only if the state (distribution) is an extreme point in the set of all states (distributions), i.e. if \( \rho \) is pure (\( P \) is a point mass). Both are concave and, moreover, both are subadditive: for a state \( \rho_{AB} \) on a composite system \( \mathcal{H}_A \otimes \mathcal{H}_B \) with reduced states

\[ \rho_A = \text{Tr}_B(\rho_{AB}), \quad \rho_B = \text{Tr}_A(\rho_{AB}), \]
it holds that
\[ S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B). \]
A directly analogous inequality holds for a distribution over a product set and its marginals. (Many more properties of \( S \) are collected in the review by Wehrl [25] and in the monograph [14].)

We shall view von Neumann entropy as a generalisation of Shannon entropy [19] in the following precise way: if the set \( \mathcal{X} \) labels an orthonormal basis \( \{|x\rangle : x \in \mathcal{X}\} \) of \( \mathcal{H}_X \) we can construct the state
\[ \rho_P = \sum_{x} P(x) |x\rangle \langle x| \]
corresponding to the distribution \( P \). This clearly defines an affine linear map from distributions into states. It is then straightforward to check that
\[ S(\rho_P) = H(P), \]
so all properties of von Neumann entropy of a single system also hold for Shannon entropy of a single distribution.

Similarly, for a distribution \( P \) on a cartesian product \( \mathcal{X} \times \mathcal{Y} \), we use the tensor product basis \( \{|xy\rangle = |x\rangle \otimes |y\rangle : x \in \mathcal{X}, y \in \mathcal{Y}\} \) to define the state \( \rho_P \) on \( \mathcal{H}_X \otimes \mathcal{H}_Y \). Again, it is straightforward to check that reduced states correspond to taking marginals:
\[ \text{Tr}_Y(\rho_P) = \rho_{P|X}, \quad \text{Tr}_X(\rho_P) = \rho_{P|Y}. \]
Hence all entropy relations for bipartite states also hold for bipartite distributions.

In [9] Lieb and Ruskai proved the remarkable relation
\[ S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_{ABC}) + S(\rho_B), \]
with a tripartite state \( \rho_{ABC} \) on the system \( \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \). It clearly generalises the previous subadditivity relation, which is recovered for a trivial system \( B: \mathcal{H}_B = \mathcal{C} \). In fact, this inequality plays a crucial role in nearly every nontrivial insight in quantum information theory, from the famous Holevo bound [5] and the properties of the coherent information [3, 18] to the recently proved additivity of capacity for entanglement–breaking channels [20].

The present investigation aims to resolve the problem of characterising the states which satisfy this relation with equality: the main result is Theorem 6. Roughly speaking, the strong subadditivity inequality expresses the fact that discarding a subsystem of a quantum system is a dissipative operation, in the sense that it can only destroy correlations with the rest of the world. Our work, therefore, can be interpreted as providing a detailed description of the conditions under which the act of discarding a quantum system can be locally reversed on a particular input. We restrict ourselves to finite dimensional systems in this paper. The question of whether a similar result holds in infinite dimension is left open.

The rest of the paper is organised as follows. In Sect. II we will review the case of probability distributions: there the solution to our problem is easy to obtain, and in fact well–known. This will provide the intuitive basis for understanding our main result. After that, in Sect. III we review quantum relative entropy and the relation of its monotonicity