ORDER-AUTOMORPHISMS OF THE SET OF BOUNDED OBSERVABLES

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Running title:
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ABSTRACT

Let $H$ be a complex Hilbert space and denote by $B_s(H)$ the set of all self-adjoint bounded linear operators on $H$. In this paper we describe the form of all bijective maps (no linearity or continuity is assumed) on $B_s(H)$ which preserve the order $\leq$ in both directions.
I. Introduction and Statements of the Results

In the Hilbert space framework of quantum mechanics the bounded observables are represented by self-adjoint bounded linear operators. If $H$ denotes the underlying Hilbert space, then these operators form the set $B_s(H)$ on which usually several operations and relations are considered. The automorphisms of $B_s(H)$ with respect to those operations and/or relations are, just as with any algebraic structure in mathematics, of remarkable importance.

First of all, $B_s(H)$ with the usual addition, scalar multiplication and Jordan product forms a Jordan algebra. It is a well-known result that the corresponding automorphisms of $B_s(H)$ are implemented by unitary or antiunitary operators of $H$ (see, for example, Ref. [2], where the automorphisms of some other important structures appearing in the probabilistic aspects of quantum mechanics are also treated).

The aim of this paper is to determine another class of automorphisms of $B_s(H)$. Namely, we equip the set $B_s(H)$ with the usual order among self-adjoint operators. That is, for any $A, B \in B_s(H)$, we write $A \leq B$ if $\langle Ax, x \rangle \leq \langle Bx, x \rangle$ holds for every $x \in H$. Alternatively, in the language of quantum mechanics, the bounded observable $A$ is said to be less than or equal to the bounded observable $B$ if the expected value of $A$ in any state is less than or equal to the expected value of $B$ in
the same state. The relation $\leq$ is no doubt an important one among observables.

In what follows we determine all the automorphisms of $B_s(H)$ as a partially ordered set with the relation $\leq$ (this is done in the main result of the paper Theorem 2) and also present some corollaries (Corollary 3, Corollary 4) that we believe have physical meaning.

We begin with the following proposition on which the proof of our main result rests. Let $H$ be a complex Hilbert space and let $B(H)^+$ denote the cone of all positive operators on $H$ (that is, the set of all $A \in B_s(H)$ for which $\langle Ax, x \rangle \geq 0$ holds for every $x \in H$). Our first result describes the form of all bijective maps on $B(H)^+$ which preserve the order $\leq$ in both directions.

**Theorem 1.** Assume that $\dim H > 1$. Let $\phi : B(H)^+ \rightarrow B(H)^+$ be a bijective map with the property that

$$A \leq B \iff \phi(A) \leq \phi(B)$$

holds whenever $A, B \in B(H)^+$. Then there exists an invertible bounded either linear or conjugate-linear operator $T : H \rightarrow H$ such that $\phi$ is of the form

$$\phi(A) = TAT^* \quad (A \in B(H)^+).$$
After having proved this result, it will be easy to deduce the main result of the paper that follows.

**Theorem 2.** Suppose that \( \dim H > 1 \). Let \( \phi : B_s(H) \to B_s(H) \) be a bijective map with the property that

\[
A \leq B \iff \phi(A) \leq \phi(B)
\]

holds whenever \( A, B \in B_s(H) \). Then there exists an operator \( X \in B_s(H) \) and an invertible bounded either linear or conjugate-linear operator \( T : H \to H \) such that \( \phi \) is of the form

\[
\phi(A) = TAT^* + X \quad (A \in B_s(H)).
\]

This result has some corollaries that seem worth mentioning. In the first one we determine the form of all bijective transformations on \( B_s(H) \) which preserve the order and the commutativity in both directions (in quantum mechanics, instead of commutativity they usually use the word compatibility for this important concept).

**Corollary 3.** Assume that \( \dim H > 1 \). Let \( \phi : B_s(H) \to B_s(H) \) be a bijective map which preserves the order and the commutativity in both directions. Then there is an either unitary or antiunitary operator \( U : H \to H \), a positive scalar \( \lambda \), and a real number \( \mu \) such that \( \phi \) is of
the form

$$\phi(A) = \lambda UAU^* + \mu I \quad (A \in B_s(H)).$$

The next corollary describes all the bijective maps on $B_s(H)$ which preserve the order and the complementarity in both directions (two observables are called complementary if the range of any nontrivial projection from the range of the spectral measure of the first observable has zero intersection with the range of any nontrivial projection from the range of the spectral measure of the second observable). Although this latter concept is in some sense opposite to compatibility, as it turns out below we still have the same form for $\phi$ as above.

**Corollary 4.** Suppose that $\dim H > 1$. Let $\phi : B_s(H) \to B_s(H)$ be a bijective map which preserves the order and the complementarity in both directions. Then there is an either unitary or antiunitary operator $U : H \to H$, a positive scalar $\lambda$, and a real number $\mu$ such that $\phi$ is of the form

$$\phi(A) = \lambda UAU^* + \mu I \quad (A \in B_s(H)).$$

Finally, our last corollary characterizes those bijective maps on $B_s(H)$ which preserve the order and the orthogonality in both directions (two operators $A, B \in B_s(H)$ are called orthogonal if $AB = 0$ which is just equivalent to the mutual orthogonality of the ranges of $A$ and $B$).
Corollary 5. Assume that $\dim H > 1$. Let $\phi : B_+(H) \to B_+(H)$ be a bijective map which preserves the order and the orthogonality in both directions. Then there is an either unitary or antiunitary operator $U : H \to H$, and a positive scalar $\lambda$ such that $\phi$ is of the form

$$\phi(A) = \lambda U A U^* \quad (A \in B_+(H)).$$

Closing this section we note that all the above statements could be conversed, that is, if a map $\phi$ is of any of the above forms, then it necessarily preserves the corresponding properties.

II. Proofs

This section is devoted to the proofs of our results. We begin with the following auxiliary results. If $A$ is a bounded linear operator, then $\text{rng} \ A$ denotes its range. The rank of $A$ is, by definition, the algebraic dimension of $\text{rng} \ A$ and it is denoted by $\text{rank} \ A$. For any $A \in B(H)^+$, $\sqrt{A}$ stands for the unique positive linear operator whose square is $A$.

Lemma 1. Let $A, B \in B(H)^+$ be such that $\text{rank} \ A = 1, \text{rank} \ B < \infty$. We have $\lambda A \leq B$ for some positive scalar $\lambda$ if and only if $\text{rng} \ A \subset \text{rng} \ B$.

Proof. We recall the following nice result of Busch and Gudder (Ref. [1, Theorem 3]): if $B \in B(H)^+$, $x \in H$, and $P$ is the rank-1 projection
projecting onto the subspace generated by $x$, then we have $\lambda P \leq B$ for some positive scalar $\lambda$ if and only if $x$ is in the range of $\sqrt{B}$. As in our case $B$ is a finite rank operator, it follows from the spectral theorem that $\text{rng} B = \text{rng} \sqrt{B}$. Since the positive rank-1 operators are exactly the positive scalar multiples of rank-1 projections, we obtain the assertion.

\begin{proof}

Lemma 2. Let $A \in B(H)^+$ and $n \in \mathbb{N}$. We have $\text{rank} A > n + 1$ if and only if there are operators $E, F \in B(H)^+$ such that $E, F \leq A$, $\text{rank} E = n$, $\text{rank} F > 1$ and there is no $G \in B(H)^+$ of rank 1 with $G \leq E, F$.

Proof. Suppose that $\text{rank} A > n + 1$. We assert that there exists a finite rank operator $A' \in B(H)^+$ such that $A' \leq A$ and $\text{rank} A' > n + 1$. In case $A$ is of finite rank, this is trivial. If $A$ is compact and not of finite rank, then by the spectral theorem of compact self-adjoint operators we can verify our claim very easily. Finally, if $A$ is non-compact, then using the spectral theorem of self-adjoint operators and the properties of the spectral integral, we can find an infinite rank projection $P$ on $H$ and a positive scalar $\lambda$ such that $\lambda P \leq A$ from which the existence of an appropriate operator $A'$ follows.
Clearly, $A'$ can be written as the sum of positive scalar multiples of pairwise orthogonal rank-1 projections. Let $E$ be the sum of the first $n$ terms in this sum and let $F$ be the sum of the remaining part. It is easy to see that $E, F$ have the required property. In fact, the non-existence of $G$ follows from Lemma I.

To prove the converse, suppose that there are operators $E, F \in B(H)^+$ with the properties formulated in the lemma. It follows from the relation $E, F \leq A$ that $E, F$ are of finite rank and $\text{rng } E, \text{rng } F \subset \text{rng } A$. As there is no positive rank-1 operator $G$ with $G \leq E, F$, by Lemma I we have $\text{rng } E \cap \text{rng } F = \{0\}$. So, $\text{rng } A$ contains two subspaces with trivial intersection the sum of whose dimensions is greater than $n + 1$. This shows that $\text{rank } A > n + 1$, completing the proof of the lemma.

Now, we are in a position to prove our first theorem.

Proof of Theorem I. We first remark that our proof is based on a beautiful result of Rothaus (Ref. [5]) concerning the automatic linearity of bijective maps between closed convex cones in normed spaces preserving order in both directions. In the paper Ref. [5] conclusions of that kind were reached under some quite restrictive assumptions. In our present situation, that is, when the normed space in question is an
operator algebra, those assumptions are fulfilled exactly when the under-
lying Hilbert space $H$ is finite dimensional. Accordingly, the main
point of our proof is to reduce the problem to the finite dimensional
case. This is in fact what we are going to do below.

Clearly, $\phi(0) = 0$. We prove that $\phi$ preserves the rank of operators.
In fact, we show that the assertion that

$$\text{rank } A = k \iff \text{rank } \phi(A) = k$$

$(k = 1, \ldots, n)$ holds for every $n \in \mathbb{N}$. To begin, as for the case $n = 1$,
we remark that a nonzero operator $A \in B(H)^+$ is of rank 1 if and
only if the operator interval $[0, A]$ is total under the partial ordering
$\leq$, that is, every two elements of it are comparable. Suppose that our
assertion is true for some $n \in \mathbb{N}$. We show that in that case it holds
also for $n + 1$. Let $A \in B(H)^+$ be of rank $n + 1$. By our assumption of
induction, it follows that the rank of $\phi(A)$ is at least $n + 1$. Suppose
that $\text{rank } \phi(A) > n + 1$. Using Lemma 2 and the order preserving
property of $\phi$ we obtain that $\text{rank } A > n + 1$ which is a contradiction.
Therefore, we have $\text{rank } \phi(A) = n + 1$. Referring to the fact that $\phi^{-1}$
shares the same properties as $\phi$, we obtain the desired assertion.

We now prove that if $A_1, \ldots, A_n \in B(H)^+$ are of rank 1, then their
ranges are linearly independent if and only if so are the ranges of
\( \phi(A_1), \ldots, \phi(A_n) \). (A system of 1-dimensional subspaces in \( H \) of \( n \) members is called linearly independent if they cannot be included in an \((n - 1)\)-dimensional subspace.) This statement is clear for \( n = 1 \).

Suppose that it holds for \( n \) and prove that it then necessarily holds also for \( n + 1 \). Let \( A_1, \ldots, A_n, A_{n+1} \) be rank-1 operators with linearly independent ranges and assume that this is not the case with the ranges of \( \phi(A_1), \ldots, \phi(A_n), \phi(A_{n+1}) \). Then these ranges can be included in an at most \( n \)-dimensional subspace implying that there is a rank-\( n \) operator \( B \in B(H)^+ \) such that \( \phi(A_1), \ldots, \phi(A_{n+1}) \leq B \).

By the rank-preserving property of \( \phi \) we have a rank-\( n \) operator \( A \in B(H)^+ \) such that \( A_1, \ldots A_n, A_{n+1} \leq A \). By Lemma [lemma] this implies that \( \text{rng} A_1, \ldots, \text{rng} A_{n+1} \subset \text{rng} A \) and it follows that the ranges of \( A_1, \ldots, A_{n+1} \) can be included in an \( n \)-dimensional subspace of \( H \) which is a contradiction. This verifies our claim.

Fix rank-1 operators \( A_1, \ldots, A_n \in B(H)^+ \) with linearly independent ranges which generate the \( n \)-dimensional subspace \( H_n \) of \( H \). Denote by \( H'_n \) the \( n \)-dimensional subspace of \( H \) generated by the ranges of \( \phi(A_1), \ldots, \phi(A_n) \). We assert that an operator \( T \in B(H)^+ \) acts on \( H_n \) if and only if \( \phi(T) \) acts on \( H'_n \). (We say that an operator \( T \) acts on the closed subspace \( M \) of \( H \) if \( M \) is an invariant subspace of \( T \) and \( T \) is zero
on the orthogonal complement of $M$.) This will follow from the following observation: the positive finite rank operator $T$ acts on $H_n$ if and only if for every rank-1 operator $A$ for which the ranges of $A_1, \ldots , A_n, A$ are linearly independent we have $A \not\preceq T$. To see this, suppose that $T$ acts on $H_n$. If $A \preceq T$, then we have $\text{rng } A \subset \text{rng } T \subset H_n$ implying that the ranges of $A_1, \ldots , A_n, A$ cannot be linearly independent. This gives us the necessity. As for the sufficiency, suppose that $T$ does not act on $H_n$. Then there exists a unit vector $x$ in the range of $T$ which does not belong to $H_n$. On the other hand, as $x \in \text{rng } T$, by Lemma 1 it follows that a positive scalar multiple of the rank-1 projection projecting onto the subspace generated by $x$ is less than or equal to $T$. This gives us a rank-1 operator $A$ for which the ranges of $A_1, \ldots , A_n, A$ are linearly independent and we have $A \preceq T$. This proves our claim.

So, for any $n$-dimensional subspace $H_n$ of $H$, there exists an $n$-dimensional subspace $H'_n$ of $H$ such that for every $T \in B(H)^+$, $T$ acts on $H_n$ if and only if $\phi(T)$ acts on $H'_n$. This gives rise to a bijective transformation $\psi$ on the cone $M_n(\mathbb{C})^+$ of all positive $n \times n$ complex matrices which preserves the order in both direction. (Here positivity is used in the operator theoretical sense, so our concept of positivity is just the same as positive semidefiniteness in matrix theory.)
Since $\phi$ preserves the rank, it follows that $\psi$ preserves the rank-$n$ matrices in both directions. The set of all such matrices is just the interior of $M_n(\mathbb{C})^+$ in the real normed space of all $n \times n$ Hermitian matrices. Now, the result Ref. [5, Proposition 2] of Rothaus on the linearity of order preserving maps can be applied and it gives us that $\psi$ is linear on the set of all rank-$n$ elements in $M_n(\mathbb{C})^+$. We show that $\psi$ is linear on the whole set $M_n(\mathbb{C})^+$. Pick $A, B \in M_n(\mathbb{C})^+$. Then there are sequences $(A_k), (B_k)$ of rank-$n$ elements in $M_n(\mathbb{C})^+$ which are monotone decreasing with respect to the order $\leq$ and $A_k \to A, B_k \to B$. It is clear that the equalities $A = \inf_k A_k, B = \inf_k B_k$ and $A + B = \inf_k (A_k + B_k)$ hold in the partially ordered set $M_n(\mathbb{C})^+$. By the order preserving property of $\psi$ we obtain that $\psi(A) = \inf_k \psi(A_k), \psi(B) = \inf_k \psi(B_k)$ and $\psi(A + B) = \inf_k \psi(A_k + B_k)$. The sequences $\psi(A_k), \psi(B_k), \psi(A_k + B_k)$ are monotone decreasing and bounded below. By Vigier’s theorem (Ref. [4, 4.1.1. Theorem]) they necessarily converge (strongly) to their infima. Now, by the partial additivity property of $\psi$ which has been obtained above as a consequence of Rothaus’s result, we have

$$\psi(A + B) = \lim_k \psi(A_k + B_k) = \lim_k \psi(A_k) + \lim_k \psi(B_k) = \psi(A) + \psi(B).$$

So, $\psi$ is additive on $M_n(\mathbb{C})^+$ and one can prove in the same way that it is positive homogeneous as well. Since every pair of finite rank elements
in $B(H)^+$ can be embedded into a matrix space $M_n(\mathbb{C})^+$, we deduce that $\phi$ is additive and positive homogeneous on the set of all finite rank elements in $B(H)^+$.

Since every finite sum $\sum_i \lambda_i P_i$, where the $\lambda_i$’s are positive numbers and the $P_i$’s are projections of not necessarily finite rank, is the strong limit of a monotone increasing net of finite rank elements in $B(H)^+$, one can prove in a very similar way as above that $\phi$ is additive and positive homogeneous on the set of all such finite sums. Finally, using the fact that every operator in $B(H)^+$ is the norm limit of a monotone increasing sequence of operators of the form $\sum_i \lambda_i P_i$ (this follows form the spectral theorem), repeating the above argument once again, we obtain that $\phi$ is additive and positive homogeneous.

Extend $\phi$ from $B(H)^+$ to $B(H)_s$ in the obvious way, that is, define $\tilde{\phi}(T) = \phi(A) - \phi(B)$ for every $T \in B_s(H)$ and $A, B \in B(H)^+$ for which $T = A - B$. It is easy to check that $\tilde{\phi} : B(H)_s \to B(H)_s$ is a linear transformation which preserves the order in both directions. To see the less trivial part of this last assertion, suppose that $T \in B_s(H)$, $T = A - B$, $A, B \in B(H)^+$ are such that $0 \leq \tilde{\phi}(T) = \phi(A) - \phi(B)$. This implies that $\phi(B) \leq \phi(A)$ which yields $B \leq A$, that is, we have $0 \leq T$. The linear transformation $\tilde{\phi}$ is surjective since $B(H)^+$ is included in its range. Moreover, it is injective as well which follows from the fact that
preserves the order in both directions. Now, if one further extends \( \tilde{\phi} \) to a linear transformation on the algebra \( B(H) \) of all bounded linear operators on \( H \), one gets a linear bijection of the \( C^* \)-algebra \( B(H) \) which preserves the order in both directions. Due to a well-known result of Kadison (Ref. [3, Corollary 5]) every such transformation sending the identity to itself is a Jordan *-automorphism. Therefore, the linear transformation

\[
A \mapsto \sqrt{\phi(I)^{-1}} \phi(A) \sqrt{\phi(I)^{-1}} \]

is a Jordan *-automorphism of \( B(H) \). But these transformations of \( B(H) \) are well-known to be implemented by unitary-antiunitary operators (see, for example, Ref. [2]). It is now easy to infer that \( \phi \) is of the desired form. This completes the proof of the theorem. \( \square \)

Our main result is now easy to prove.

Proof of Theorem 2. Let \( X = \phi(0) \) and consider the transformation

\[
\psi : A \mapsto \phi(A) - X.
\]

Clearly, \( \psi \) is a bijection of \( B_s(H) \) preserving the order in both directions. So, without loss of generality we can assume that \( \phi(0) = 0 \). Now, restricting \( \phi \) onto \( B(H)^+ \) we have a bijection of \( B(H)^+ \) which preserves the order in both directions. So, we can apply Theorem [1] and obtain
that there exists an invertible bounded either linear or conjugate-linear
operator $T : H \to H$ for which we have

$$\phi(A) = TAT^* \quad (A \in B(H)^+). \quad (1)$$

It remains to show that this equality holds also for every $A \in B_+(H)$. Let $B \in B_+(H)$ be arbitrary but fixed. Then there exists a constant $K \in \mathbb{R}$ such that $K \leq B$ (for example, one can choose $K = -\|B\|$).

Consider the transformation

$$A \mapsto \phi(A + K) - \phi(K)$$

on $B(H)^+$. Just as above, this transformation is a bijective map on

$B(H)^+$ which preserves the order in both directions. Therefore, there
exists an invertible bounded either linear or conjugate-linear operator

$S : H \to H$ such that

$$\phi(A + K) - \phi(K) = SAS^* \quad (A \in B(H)^+). \quad (2)$$

If $A \geq -K, 0$, then by (1) we have

$$T(A + K)T^* - \phi(K) = SAS^*. \quad (3)$$

Considering this equality for another $A'$ with $A' \geq -K, 0$, we see that

$$T(A - A')T^* = S(A - A')S^*.$$
As the difference $A - A'$ can be an arbitrary self-adjoint operator, we obtain that $TCT^* = SCS^*$ holds for every $C \in B_s(H)$. It now follows from (3) that

$$T(A + K)T^* - \phi(K) = SAS^* = TAT^*$$

where $A \in B_s(H)$, $A \geq -K, 0$. This yields $\phi(K) = TKT^* = SKS^*$. We deduce from (2) that

$$\phi(A + K) = SAS^* + \phi(K) = TAT^* + TKT^* = T(A + K)T^*$$

holds for every $A \in B(H)^+$. Choosing $A = B - K \geq 0$, we have

$$\phi(B) = TBT^*.$$

This completes the proof. \qed

We now turn to the proofs of the corollaries.

Proof of Corollary 3. By Theorem 2 there is an invertible bounded either linear or conjugate-linear operator $T$ on $H$ such that

$$\phi(A) = TAT^* + \phi(0) \quad (A \in B_s(H))$$

Since 0 is commuting with every $A \in B_s(H)$, the same is true for $\phi(0)$. This gives us that $\phi(0)$ is a scalar operator, that is, there is a $\mu \in \mathbb{R}$ such that $\phi(0) = \mu I$. Similarly, we have a constant $\lambda \in \mathbb{R}$ such that $TT^* = \phi(I) - \phi(0) = \lambda I$. It is trivial that $\lambda$ is necessarily
positive and then we obtain that the operator \( T/\sqrt{\lambda} \) is either unitary or antiunitary.

\( \square \)

**Proof of Corollary 4.** It is easy to see that \( A \in B_s(H) \) is complementary with every \( B \in B_s(H) \) if and only if \( A \) is scalar. Hence, \( \phi \) preserves the scalar operators and one can apply the argument in the proof of Corollary [4] to get the desired form of \( \phi \).

\( \square \)

In the proof of Corollary 5 we make use the following notation. If \( x, y \in H \), then \( x \otimes y \) denotes the operator defined by \( (x \otimes y)z = \langle z, y \rangle x \) \((z \in H)\).

**Proof of Corollary 5.** Since 0 is the only operator in \( B_s(H) \) which is orthogonal to every operator, we infer that \( \phi(0) = 0 \). By Theorem 2 we have an invertible bounded either linear or conjugate-linear operator \( T \) on \( H \) such that \( \phi(A) = TAT^* \) holds for every \( A \in B_s(H) \). Without serious loss of generality we can suppose that \( T \) is linear. It now follows that for every \( A, B \in B_s(H) \) with \( AB = 0 \) we have \( \phi(A)\phi(B) = 0 \) which implies that \( AT^*TB = 0 \). Choosing nonzero orthogonal vectors \( x, y \in H \), for \( A = x \otimes x \) and \( B = y \otimes y \) we get \( x \otimes Tx \cdot Ty \otimes y = 0 \) which yields \( \langle T^*Tx, y \rangle = \langle Tx, Ty \rangle = 0 \). So, we have \( \langle T^*Tx, y \rangle = 0 \) whenever \( \langle x, y \rangle = 0 \). This clearly implies that for every \( x \in H \) there is a scalar \( \lambda_x \) such that \( T^*Tx = \lambda_x x \). In another expression, the operators \( T^*T \)
and \( I \) are locally linearly dependent. It is a folk result (whose proof requires only elementary linear algebra) that in that case the operators \( T^*T \) and \( I \) are necessarily linearly dependent, that is, there exists a scalar \( \lambda \in \mathbb{R} \) such that \( T^*T = \lambda I \). Now, the proof can be completed as in the proof of Corollary 3.

\[ \square \]

III. Acknowledgements

This research was supported from the following sources: (1) Hungarian National Foundation for Scientific Research (OTKA), Grant No. T030082, T031995, (2) Ministry of Education, Hungary, Grant No. FKFP 0349/2000, (3) Joint Hungarian-Slovene Research Project, Reg. No. SLO-3/00.
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