Abstract. Additive symmetric Lévy noise can induce directed transport of overdamped particles in a static asymmetric potential. We study, numerically and analytically, the effect of an additional dichotomous random flashing in such a Lévy ratchet system. For this purpose we analyze and solve the corresponding fractional Fokker–Planck equations and we check the results with Langevin simulations. We study the behavior of the current as a function of the stability index of the Lévy noise, the noise intensity and the flashing parameters. We find that flashing allows us both to enhance and to diminish, over a broad range, the static Lévy ratchet current, depending on the frequencies and asymmetry of the multiplicative dichotomous noise, and on the additive Lévy noise parameters. Our results thus extend those for dichotomous flashing ratchets with Gaussian noise to the case of broadly distributed noises.

Keywords: driven diffusive systems (theory), stochastic particle dynamics (theory), Brownian motion, diffusion
1. Introduction

The study of noise induced transport in anisotropic spatially periodic systems is a relevant subject for many problems in physics [1, 2] and biology [3], and is also acquiring an increasing interest due to possible applications in the development of technological devices [2].

In general, the basic models consider a particle in an external spatially periodic potential with broken symmetry (the ratchet potential), and subject to an additional signal or source of fluctuations which generates the nonequilibrium condition necessary for the emergence of directional transport [2]. Thermal noise is also usually considered, although depending on the nature of the nonequilibrium forces it may not be a fundamental ingredient. Ratchet models can be classified into two main classes according to the way in which the nonequilibrium forcing affects the particle dynamics. Simple additive nonequilibrium forces lead to the so called rocking ratchets, while, when the signal enters as a multiplicative modulation of the ratchet potential, we speak of flashing ratchets. Systems combining the two kinds of forcings can also be considered [4].

Recently, two simultaneous papers [5, 6] have shown that a minimal setup for producing directional transport is obtained by considering a simple static ratchet potential and an additive white symmetric Lévy [5]–[7] noise as the only two ingredients. Here we will refer to such a system as the static or non-flashing Lévy ratchet. The preferred direction of motion for such a system is found to be toward the steepest slope of the potential. The effect was explained in [5] as a consequence of the large Lévy jumps, which lead the particles to the flatter zones of the potential with larger probability than to the steeper zones. In the limit case in which the stability parameter $\alpha$ [5, 6] defining the
Lévy noise is set to be equal to 2, the Lévy distribution becomes Gaussian [5, 6] and the equilibrium situation with vanishing current is recovered.

After the aforementioned pioneering papers on the static Lévy ratchet, several works have appeared providing further analysis of the system and studying different generalizations. In [8] it has been shown that an inversion of current can be obtained by considering a time periodic modulation of the chirality of the Lévy noise. Reference [9] studies inertial effects and proposes a way of measuring the rectification efficiency in Lévy ratchets. In [10], the weak noise limit is analyzed. In [11], the related problem of the spatially tempered fractional Fokker–Planck equation is studied. Reference [12] analyzes the competition between Lévy forcing and a periodic ac driving in a ratchet system, while [13] studies the coexistence of Lévy flights and subdiffusion. The increasing interest in ratchet systems influenced by Lévy noises is due to various facts. Firstly, there is an intrinsic theoretical significance in the generalizations of previous ratchet models to account for cases where fluctuations present long tailed probability distributions, giving rise to anomalously large particle displacements. Moreover, since Lévy noises may induce divergencies in the moments of the velocity distributions [5], there is an additional challenge in providing the appropriate quantities for measuring currents [5, 11], particle dispersion [5, 11] and efficiencies [9]. From another point of view, Lévy ratchets may be of direct interest for atomic transport in dissipative optical lattices. Cold atom ratchets have indeed been recently studied experimentally [15]–[19] and theoretically [20, 21], whereas anomalous dynamics described by Lévy statistics were also reported for the same system [22, 23].

In this paper we study the dynamics of a ratchet system influenced by two different signals inducing nonequilibrium conditions: a dichotomic multiplicative noise and an additive Lévy forcing. We analyze both the Langevin and Fokker–Planck approaches, focusing mainly on the latter. In different limit situations, the model generalizes various previous systems found in the literature. As we will see, for fast, slow and null flashing we recover different versions of the static Lévy ratchet, while for $\alpha \to 2$ we recover a version of the well known flashing ratchet with thermal noise [1, 2]. In particular we find that flashing allows us both to enhance and to diminish, over a broad range, the static ratchet current depending on the frequencies of flashing. Our results thus extend those for dichotomous flashing ratchets with Gaussian noise to the case of broadly distributed noise.

It is worth mentioning that a dichotomous flashing ratchet with Lévy noise was briefly analyzed in the past as part of other studies on the influence of superdiffusion on directional motion [24]. However, only heuristic arguments were given, by considering an adiabatic approximation which assumes relatively slow flashing. Interestingly, as the study preceded the findings in [5, 6], the contribution to directional transport during the ‘on’ stage of the flashing was naturally ignored.

The organization of the paper is as follows. In section 2 we introduce the model. In section 3 we give the solution of the Fokker–Planck equation. In section 4 we analyze some limit situations which connect our results with previous results for other systems studied in the literature. In section 5 we present a detailed numerical analysis of the results considering a particular but standard ratchet potential. Section 6 is devoted to our conclusions and some final remarks.

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2. The model

We consider the one-dimensional overdamped motion of a particle at position \( X(t) \) subject to a randomly fluctuating ratchet potential \( f(t) V(x) \), and to an additive Lévy random force \( \xi(t) \). This situation is described by the Langevin equation

\[
\frac{dX}{dt} = -f(t)V'(X) + \xi(t).
\] (1)

We consider \( \xi(t) \) as a symmetric \( \alpha \)-stable Lévy noise with \( 1 < \alpha \leq 2 \) and intensity \( \chi \), described by the characteristic function \( \langle \exp[i k \int_t^{t+\Delta t} \xi(t') \, dt'] \rangle = \exp[-\chi |k|^{\alpha} \Delta t] \). We consider a ratchet potential with spatial period \( L \) and amplitude \( \Delta V \), and the multiplicative noise \( f(t) \) as a dichotomous Markov process taking the value of 1 (mode A) or 0 (mode B), switching between these modes at rates \( \Gamma_{AB} \) and \( \Gamma_{BA} \), respectively. Equation (1), thus, models a particle in a ratchet potential \( V(x) \) that switches between ‘on’ and ‘off’ states with transition rates \( \Gamma_{AB} \) and \( \Gamma_{BA} \) respectively (see figure 1). We are interested in the steady-state directional transport that can be generated in this system. This can be quantified by the long-time limit of the average velocity \( \langle dX/dt \rangle \), where \( \langle \cdots \rangle \) stands for the average over all possible realizations of \( \xi(t) \) and \( f(t) \). Note that we only consider the domain \( 1 < \alpha \leq 2 \) since it was shown that transport is highly inefficient in Lévy ratchets for \( \alpha < 1 \) due to the divergencies of the mean value of Lévy distributions [5,9]. Moreover, the analysis in such a parameter region demands special care for defining the mean velocity and current [5]. Note that our parameter region of interest includes the case \( \alpha = 2 \), for which the Lévy noise becomes Gaussian white noise and (1) describes the dichotomous flashing ratchet studied in [27]. On the other hand, for \( \Gamma_{AB} = 0 \), equation (1) corresponds to the static or non-flashing Lévy ratchet mentioned in section 1 and recently analyzed in [5,6].

The Langevin equation (1) was solved by using a standard algorithm (see [5] and in [5,6]) for which the numbers distributed according to Lévy laws were obtained by the method explained in [28].

As mentioned in section 1, most of our analysis will be done using the Fokker–Planck approach to the problem. In order to present it, we first consider a more general situation in which the potential in state B can be non-vanishing, and we introduce the probability densities \( P^A = P^A(x,t) \) and \( P^B = P^B(x,t) \) of finding the particle at position \( x \) and time \( t \) in the A and B modes respectively. For such a system we have the following coupled
Fokker–Planck equations for the probability densities:

\[
\begin{align*}
\partial_t P^A &= \partial_x (P^A \partial_x V^A) + \chi \partial_x^\alpha P^A + \Gamma_B P^B - \Gamma_A P^A \\
\partial_t P^B &= \partial_x (P^B \partial_x V^B) + \chi \partial_x^\alpha P^B + \Gamma_A P^A - \Gamma_B P^B.
\end{align*}
\]

Here, \(\partial_x^\alpha \equiv \partial^\alpha / \partial |x|^\alpha\) stands for the Riesz–Feller fractional derivative of order \(\alpha\) [29], while \(V^A(x)\) and \(V^B(x)\) are the external potentials considered in states A and B. For \(V^B(x) = 0\), the system (2) is equivalent to the Langevin equation (1). We will limit to such a case when performing our analysis of transport and presenting our numerical results. However, the method of solution of the Fokker–Planck equation will be introduced (up to some point) for an arbitrary potential \(V^B(x)\), since this leads to more general and symmetrical expressions.

Within the Fokker–Planck framework, the probability current \(J(x, t) = \langle (dX(t)/dt) \delta(x - X(t)) \rangle\) is obtained from the continuity equation \(\partial_t P = -\partial_x J\), where \(P \equiv P^A + P^B \equiv \langle \delta(x - X(t)) \rangle\) is the probability density regardless of the mode. In the long-time limit \(\partial_t P = \partial_t P^A = \partial_t P^B = 0\) and the current reaches the spatially constant value \(J\) characterizing the average directed motion or ratchet response. Since the steady-state solutions that we are interested in, \(P^A(x)\) and \(P^B(x)\), are periodic with the period \(L\), we can directly solve (2) for \(\partial_t P^A = \partial_t P^B = 0\) in the interval \(0 \leq x \leq L\) with periodic boundary conditions. The steady-state current is then simply related to the average velocity in the Langevin description, \(J = \langle dX(t)/dt \rangle / L\).

3. Solution of the Fokker–Planck equation

We exploit the fact that the steady-state solution that we seek is periodic with the period \(L\) of the ratchet potential by using discrete Fourier transforms, \(f(x) = \sum_q \hat{f}_q \exp(iqx)\), where \(q \equiv 2\pi n_q / L\) with \(n_q\) an integer and \(\hat{f}_k = \int_0^L (dx/L) f(x) \exp(-ikx)\) denotes the corresponding Fourier amplitude. In particular, the prescription \([\partial_x^\alpha f(x)]_k = -|k|^\alpha \hat{f}_k\) for the Fourier transform of the fractional derivative, valid in an infinite or periodic support, greatly simplifies our method as it avoids the complications due to the nonlocal nature of the fractional Laplacian operator that can arise for different boundary conditions that break translational invariance in bounded domains [30]. Fourier transforming (2) gives

\[
\begin{align*}
\partial_t \hat{P}_k^A &= -k \sum_q \hat{V}_q^A \hat{P}_{k-q}^A - \chi |k|^\alpha \hat{P}_k^A + \Gamma_B \hat{P}_k^B - \Gamma_A \hat{P}_k^A \\
\partial_t \hat{P}_k^B &= -k \sum_q \hat{V}_q^B \hat{P}_{k-q}^B - \chi |k|^\alpha \hat{P}_k^B + \Gamma_A \hat{P}_k^A - \Gamma_B \hat{P}_k^B,
\end{align*}
\]

where we have used (and use from now on) the wavevector name, e.g. \(k\), to denote both its value, \(k = 2\pi n_k / L\), and its associated integer, \(n_k\), to avoid excessive notation. By construction, equations (3) only admit periodic initial conditions with period \(L\). This is not important as we will be interested in the steady-state limit \(\partial_t \hat{P}_k^{A,B} = 0\), whose solution is unique and thus independent of the initial condition. Equations (3) must be solved with the constraint \(\hat{P}_0^A + \hat{P}_0^B = 1\), arising from the normalization of the entire probability function \(\int_0^L dx \ P = 1\). This leads immediately to the zero-mode steady-state
As indicated in section 2, the current in the stationary regime is a constant. This can be
seen from (10), which for 
\[ \partial_t \tilde{P}_k(t) = -ik \tilde{J}_k(t), \]
where \( \tilde{P}_k(t) = \tilde{P}_k^A(t) + \tilde{P}_k^B(t) \). Adding the two equations (3) together, we can identify:
\[ \tilde{J}_k = -i \sum_q q \tilde{V}_q \tilde{P}_{k-q} - i \chi |k|^{\alpha-1} \text{sgn}(k) (\tilde{P}_k^A + \tilde{P}_k^B). \]
As indicated in section 2, the current in the stationary regime is a constant. This can be seen from (10), which for \( \partial_t \tilde{P}_k(t) = 0 \) implies \( J_k = 0 \) for \( k \neq 0 \). Thus, assuming \( \alpha > 1 \),
the stationary current is simply

\[ J \equiv J_0 = -i \sum_{q} q \tilde{V}_q \tilde{P}_q. \] (12)

Note that the solution for the general case \( V^B(x) \neq 0 \) demands dealing with the whole system (5). Clearly, although the procedure is a little bit more intricate, the equations can also be written in matrix form and solved for the variables \( \tilde{P}_q^A \) and \( \tilde{P}_q^B \), enabling the calculation of the current.

4. Dimensional analysis, relevant parameters and limit situations

Considering a potential \( V(x) \) of amplitude \( \Delta V \) and period \( L \), a straightforward dimensional analysis of (7) shows that the Fokker–Planck solutions for the probability distribution and current are of the form

\[ P = f \left( \frac{x}{L}; \delta, \frac{\Gamma_{AB}L^2}{\Delta V}, \frac{\chi L^{2-\alpha}}{\Delta V} \right), \quad J = g \left( \alpha, \delta, \frac{\Gamma_{AB}L^2}{\Delta V}, \frac{\chi L^{2-\alpha}}{\Delta V} \right) \frac{\Delta V}{L}, \] (13)

where \( f \) and \( g \) are dimensionless functions. Thus, there are essentially four relevant parameters. Note that the scaled ‘on’ rate \( \Gamma_{BA}L^2/\Delta V \) can be considered as a relevant parameter instead of \( \delta \) or instead of \( \Gamma_{AB}L^2/\Delta V \). In the following we will speak alternatively of the three parameters depending on the feature we want to stress. Note that for \( \delta = 1 \), (7) reduces to the equation for the standard, non-flashing, Lévy ratchet [6,11].

The relaxation time for a particle in the potential \( V(x) \) is expected to be a relevant characteristic time of the system. According to the deterministic part of (1), considering a typical length \( L \) and a velocity \( (\Delta V)/L \), we get \( L^2/\Delta V \) as the typical relaxation time. It is interesting to note, thus, that the scaled parameters \( \Gamma_{AB}L^2/\Delta V \) and \( \Gamma_{BA}L^2/\Delta V \) are simply measures of the switching rates in the time scale of the inverse of the relaxation time. Equivalently, \( L^\alpha/\chi \) can be interpreted as a superdiffusion time which in the solution of the Fokker–Planck equation appears weighted to the relaxation time. Note that in the Gaussian limit we get the usual expression for the diffusion time \( \equiv L^2/\chi \).

By modifying \( \Gamma_{AB} \) and \( \Gamma_{BA} \) but keeping their ratio constant (i.e. keeping \( \delta \) constant), we can analytically study the limits of infinitely slow and infinitely fast switching without changing the relative residence times. Using (7) the calculation of these limits is very simple and instructive.

4.1. Infinitely slow switching

In the infinitely slow switching case we have \( \Gamma_{AB} \to 0 \) and \( \Gamma_{BA} \to 0 \), keeping \( \delta \) constant. Replacing this in (7) and using (4), we can write

\[ -k \sum_{q \neq k} q \tilde{V}_q (\tilde{P}_q^A/\delta) - \chi |k|^\alpha (\tilde{P}_k^A/\delta) = k^2 \tilde{V}_k \quad (k \neq 0) \]

\[ (\tilde{P}_0^A/\delta) = 1. \] (14)

These equations are identical to the non-flashing Lévy ratchet equations for \( \tilde{P}_k^A/\delta \). If we call \( P_{st}(V) \) the real-space static Lévy ratchet solution in a potential \( V \), we have, in this
slow switching limit,

\[ P^A_{\text{slow}} = P^\text{st}(V) \delta, \quad J_{\text{slow}} = J^\text{st}(V) \delta, \quad (15) \]

where the second equation follows from (12). Intuitively, the idea is that the rates are so large that the system reaches the steady state when the potential is 'on' and when it is 'off', and having the transients between these modes a negligible contribution, the total current is just an average of the two corresponding steady-state currents. According to (7) the result is expected to be a very good approximation when the following relations hold between the scaled parameters: \( \Gamma_{AB} L^2 / \Delta V \ll \chi L^{2-\alpha} \) and \( \Gamma_{BA} L^2 / \Delta V \ll \chi L^{2-\alpha} \) (i.e. \( \Gamma_{AB} \ll \chi / L^\alpha \) and \( \Gamma_{BA} \ll \chi / L^\alpha \)). We will check this in section 5 when analyzing transport for the case of a standard ratchet potential.

4.2. Infinitely fast switching

In the infinitely fast switching case we have \( \Gamma_{AB} \to \infty \) and \( \Gamma_{BA} \to \infty \), keeping \( \delta \) constant. Replacing this in (7) the bracketed term becomes \( 1/\delta \). We can thus write

\[
-k \sum_{q \neq k} q \tilde{V}_q \delta (\tilde{P}^A_{k-q} / \delta) - \chi |k|^\alpha (\tilde{P}^A_k / \delta) = k^2 \tilde{V}_k \delta \quad (k \neq 0)
\]

\[ (\tilde{P}^A_0 / \delta) = 1. \quad (16) \]

which again corresponds to the non-flashing ratchet equations for the variable \( \tilde{P}^A_{k-q} / \delta \) but in the renormalized potential \( V_\delta \). Thus, the solution in this fast switching limit is

\[ P^A_{\text{fast}} = P^\text{st}(V \delta) \delta, \quad J_{\text{fast}} = J^\text{st}(V \delta). \quad (17) \]

In this case, the current for the flashing Lévy ratchet is the same as for a system where the same potential is always 'on' but with an intensity rescaled by \( \delta \). This is intuitively evident: the potential is being switched so fast that the particle can only 'feel' the temporal average of the potential (i.e. \( V(x) \delta \)). According to (7) the limit results in (17) are expected to be valid for \( \Gamma_{AB} \gg \chi / L^\alpha \) and \( \Gamma_{BA} \gg \chi / L^\alpha \). We have verified this using a standard differentiable ratchet potential, as we will show in section 5.

Note however that, at variance with what happens in the slow limit, the fast limit is not valid for all the Fourier modes. The denominator of the bracketed term in (7) shows that the fast approximation is only valid for Fourier modes that satisfy \( |k|^\alpha \ll \Gamma_{AB} / \chi \). Nevertheless, if the potential is an analytic function the probability density function, \( P(x) \), should also be an analytic function, which implies that the coefficients of its Fourier expansion decay exponentially with \( k \). Therefore we expect the higher Fourier modes that are not well approximated in the fast limit not to change the overall limit of the probability density.

4.3. Perturbative analysis for fast and slow switching

Using a simple perturbation analysis in (7) we can obtain the lowest order corrections to the infinitely slow and fast switching limits described above, with \( \Gamma_{AB} \) and \( \Gamma_{BA}^{-1} \) the small parameters, respectively, for a fixed value of \( \delta \).

Approaching the infinitely slow limit a natural ansatz to propose is \( P^A_{\text{fast}}(x) \sim P^A_{\text{slow}}(x) + \Gamma_{AB} \tilde{W}(x) \), with \( P^A_{\text{slow}} \) given by (15). Inserting this ansatz in (7) we get the

\[ P^A_{\text{slow}} = P^\text{st}(V) \delta, \quad J_{\text{slow}} = J^\text{st}(V) \delta, \quad (15) \]

where the second equation follows from (12). Intuitively, the idea is that the rates are so large that the system reaches the steady state when the potential is ‘on’ and when it is ‘off’, and having the transients between these modes a negligible contribution, the total current is just an average of the two corresponding steady-state currents. According to (7) the result is expected to be a very good approximation when the following relations hold between the scaled parameters: \( \Gamma_{AB} L^2 / \Delta V \ll \chi L^{2-\alpha} / \Delta V \) and \( \Gamma_{BA} L^2 / \Delta V \ll \chi L^{2-\alpha} / \Delta V \) (i.e. \( \Gamma_{AB} \ll \chi / L^\alpha \) and \( \Gamma_{BA} \ll \chi / L^\alpha \)). We will check this in section 5 when analyzing transport for the case of a standard ratchet potential.

4.2. Infinitely fast switching

In the infinitely fast switching case we have \( \Gamma_{AB} \to \infty \) and \( \Gamma_{BA} \to \infty \), keeping \( \delta \) constant. Replacing this in (7) the bracketed term becomes \( 1/\delta \). We can thus write

\[
-k \sum_{q \neq k} q \tilde{V}_q \delta (\tilde{P}^A_{k-q} / \delta) - \chi |k|^\alpha (\tilde{P}^A_k / \delta) = k^2 \tilde{V}_k \delta \quad (k \neq 0)
\]

\[ (\tilde{P}^A_0 / \delta) = 1. \quad (16) \]

which again corresponds to the non-flashing ratchet equations for the variable \( \tilde{P}^A_{k-q} / \delta \) but in the renormalized potential \( V_\delta \). Thus, the solution in this fast switching limit is

\[ P^A_{\text{fast}} = P^\text{st}(V \delta) \delta, \quad J_{\text{fast}} = J^\text{st}(V \delta). \quad (17) \]

In this case, the current for the flashing Lévy ratchet is the same as for a system where the same potential is always ‘on’ but with an intensity rescaled by \( \delta \). This is intuitively evident: the potential is being switched so fast that the particle can only ‘feel’ the temporal average of the potential (i.e. \( V(x) \delta \)). According to (7) the limit results in (17) are expected to be valid for \( \Gamma_{AB} \gg \chi / L^\alpha \) and \( \Gamma_{BA} \gg \chi / L^\alpha \). We have verified this using a standard differentiable ratchet potential, as we will show in section 5.

Note however that, at variance with what happens in the slow limit, the fast limit is not valid for all the Fourier modes. The denominator of the bracketed term in (7) shows that the fast approximation is only valid for Fourier modes that satisfy \( |k|^\alpha \ll \Gamma_{AB} / \chi \). Nevertheless, if the potential is an analytic function the probability density function, \( P(x) \), should also be an analytic function, which implies that the coefficients of its Fourier expansion decay exponentially with \( k \). Therefore we expect the higher Fourier modes that are not well approximated in the fast limit not to change the overall limit of the probability density.

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Using a simple perturbation analysis in (7) we can obtain the lowest order corrections to the infinitely slow and fast switching limits described above, with \( \Gamma_{AB} \) and \( \Gamma_{BA}^{-1} \) the small parameters, respectively, for a fixed value of \( \delta \).

Approaching the infinitely slow limit a natural ansatz to propose is \( P^A_{\text{fast}}(x) \sim P^A_{\text{slow}}(x) + \Gamma_{AB} \tilde{W}(x) \), with \( P^A_{\text{slow}} \) given by (15). Inserting this ansatz in (7) we get the
following result for the Fourier-transformed first-order correction for $k \neq 0$:

$$-k \sum_{q \neq k} q \tilde{V}_q \tilde{W}_{k-q} - \chi |k|^\alpha \tilde{W}_k = [\tilde{P}_A]_{\text{slow}} k.$$  \hspace{1cm} (18)

For various particular potentials we confirm numerically that (18) has solutions with finite currents. As we will see, this is also consistent with the slow switching behavior found in the full numerical solution discussed in section 5. We therefore conclude that in the absence of particular symmetries, the first-order correction generically does not vanish. We thus predict

$$J - J_{\text{slow}} \sim \Gamma_{AB}$$  \hspace{1cm} (19)

for small enough $\Gamma_{AB}$ ($\Gamma_{BA}$) at fixed $\delta$, with $J_{\text{slow}}$ given by (15).

Similarly, approaching the infinitely fast switching limit, a natural ansatz to propose is $P^A(x) \sim P_{\text{fast}}(x) + \Gamma_{AB}^{-1} W(x)$, with $P_{\text{fast}}(x)$ given by (17). Inserting this ansatz in (7) we get the following result for the Fourier-transformed first-order correction for $k \neq 0$:

$$-k \sum_{q \neq k} q \tilde{V}_q \tilde{W}_{k-q} - \chi |k|^\alpha (\tilde{W}_k/\delta) = -[(\delta^{-1} - 1)\chi |k|^\alpha]^2 [\tilde{P}_A]_{\text{fast}} k.$$  \hspace{1cm} (20)

As before, in the absence of particular symmetries this equation yields generically solutions with a finite current. This is also confirmed by direct numerical evaluation of the last equation for particular potentials, and is also consistent with the fast switching behavior of the full numerical solution presented in the following sections. We thus predict a non-vanishing first-order correction:

$$J - J_{\text{fast}} \sim \Gamma_{AB}^{-1}$$  \hspace{1cm} (21)

for large enough $\Gamma_{AB}$ ($\Gamma_{BA}$) at fixed $\delta$, with $J_{\text{fast}}$ given by (17).

It is important to note that the asymptotic behaviors of the currents that we have found (equations (19) and (21)) are also valid for the case of Gaussian noise ($\alpha = 2$). However, they do not coincide with the corresponding limits found for a randomly flashed triangular ratchet [27] and for a periodically pulsed ratchet with a frequency $\Omega$ [2]. In this last case, the corrections analytically found in the slow and fast limits are $J \sim \Omega^2$ and $J \sim -\Omega^{-2}$ respectively. The discrepancy is most likely due simply to the difference between a stochastic and a deterministic switching of the potential.

In the case of [27], however, the discrepancy is more significant because the system analyzed is also a (dichotomous) stochastically flashed ratchet. Using a heuristic argument, Astumian and Bier provide an expression for the current as a function of what we have called $\Gamma_{AB}$ (using $\delta = 0.5$), which in the slow and fast limits behaves as $J \sim \Gamma_{AB}^{3/2}$ and $J \sim \exp(-\Gamma_{AB})$. One of the possible reasons for this difference could be that, in the case of a non-analytic potential, the Fourier coefficients do not decay fast enough for our arguments to be valid. Another possible reason could be that the qualitative argument in [27] is not correct in the limits of fast and slow switching.
5. Results for a standard ratchet potential

Now we consider a standard ratchet potential [1]

\[ V(x) = \frac{1}{2\pi} \left[ \sin(2\pi x) + \frac{1}{4} \sin(4\pi x) \right], \]  

(22)

with period \( L = 1 \) and barriers of amplitude \( \Delta V \equiv V_{\text{max}} - V_{\text{min}} \approx 0.35 \). For this choice of the potential, we analyze the dependence of the current on the parameters \( \chi \) and \( \alpha \) characterizing the Lévy noise, and on the transition rates \( \Gamma_{AB} \) and \( \Gamma_{BA} \). The results of this section correspond, thus, to the system (1) considering the potential \( V(x) \) given in (22) or, equivalently, to the Fokker–Planck equations (2) with \( V_A(x) = V(x) \) and \( V_B(x) = 0 \). The results for the current given throughout this section correspond mostly to the Fokker–Planck formalism. However, in order to show the complete agreement between the two formalisms we also include Langevin results for some specific cases.

Concerning the solution of the Fokker–Planck equation, the Fourier transform of \( V(x) \) is simply

\[ \tilde{V}_k = \frac{1}{4\pi} \left[ \delta_{k,k+1} + \delta_{k,k-1} + \frac{1}{4} (\delta_{k,k+2} + \delta_{k,k-2}) \right], \]  

(23)

where \( \delta_{i,j} \) is the Kronecker delta. Thus, the matrix \( \mathbf{M} \) given in (9) is penta-diagonal. Despite this simplification, the system (8) can only be solved numerically, for what we typically consider a number of Fourier modes ranging from \( N = 1000 \) for \( \alpha \sim 2 \) to \( N = 2000 \) for \( \alpha \) close to 1. This guarantees that our solutions closely approximate those of the continuum limit if the Lévy noise intensity satisfies \( \chi < \Gamma_{BA}/|k_{\text{max}}|^\alpha \), with \( k_{\text{max}} = 2\pi N/L \) the maximum wavevector.

5.1. Symmetrical transition rates

We first consider the case of equal rates of transition between A and B states. This means that the potential is ‘on’ and ‘off’ for the same fraction of time, on average. We thus have \( \delta = 0.5 \) and we define \( \Gamma \equiv \Gamma_{AB} = \Gamma_{BA} \).

In figure 2 we show the current as a function of the noise power \( \chi \) for several values of the transition rate \( \Gamma \) and two different values of \( \alpha \). As usual in most ratchet systems, in all the cases studied we see that the current attains a maximum at an optimal noise intensity, while it decreases to zero for low and large intensities. The optimal value of \( \chi \) depends only slightly on \( \Gamma \). As we can see, it decreases by a factor of order 1/2 when changing \( \Gamma \) by six orders of magnitude. We can also see that it decreases with increasing \( \alpha \). In all the cases studied the optimal value of \( \chi \) remains in the interval between \( \Delta V/10 \) and \( \Delta V \).

Concerning the weak noise limit \( \chi \to 0 \), our numerical results seem to indicate that the current behaves as \( J \sim \chi \), in agreement with the findings for non-flashing Lévy ratchets in [10]. However, we have not been able to obtain the exact analytical limit law within our Fokker–Planck approach.

Note that, for the two values of \( \alpha \) analyzed in figure 2, we find that \( \Gamma = 2 \) produces the largest value of \( J \) for almost all values of \( \chi \), while larger and smaller transition rates lead almost always to smaller currents. The results in figure 3 confirm the existence of an optimum value of \( \Gamma \) maximizing \( J \), which is almost independent of \( \alpha \) and grows with \( \chi \).
Figure 2. Current $J$ as a function of Lévy noise intensity for different transition rates ($\Gamma = 2 \times 10^{-3}, 2 \times 10^{-1}, 2 \times 10^{0}, 2 \times 10^{1}, 2 \times 10^{3}$). (a) Stability index $\alpha = 1.4$. (b) Stability index $\alpha = 1.8$. All the continuous curves correspond to Fokker–Planck results, while the open circles were obtained from Langevin simulations for $\Gamma_{AB} = 2 \times 10^{0}$.

Figure 3 also shows the validity of the asymptotic formulas found in section 4 for the limits of small and large $\Gamma$, and of the perturbative analysis close to them. Another fact that can be observed in figure 3 is that, for a fixed flashing mechanism (fixed $\Gamma$) and a fixed noise intensity, Lévy noise leads almost always to larger values of $J$ than Gaussian noise.

Finally, in figure 4 we analyze further the dependence of the current on the noise power and the stability index. In order to identify a relevant noise region, we plot $J$ as a function of $\alpha$ and $\chi$ for an intermediate (near optimal) value of the transition rate. For $\alpha$ small, the range of noise intensities that maximizes the current is centered around $\chi \sim 0.4$. As the stability index increases, the optimal noise intensity shifts to smaller values. At the same time, the optimal noise range narrows down as we move to $\alpha = 2$. It is interesting to note that a given current value can be obtained by combining $\chi$ and $\alpha$ in different ways. For instance, it is sometimes possible that a small noise power with a large value of $\alpha$ gives the same current as a large noise power with a small $\alpha$.

5.2. Non-symmetrical transition rates

Now we study the general case of different rates of transition between ‘on’ and ‘off’ states.

In figure 5 we show the current as a function of $\delta$ for different values of $\alpha$ and $\Gamma_{AB}$, considering a relatively small noise intensity $\chi = 0.05$. We see that the current vanishes for $\delta \sim 0$. This is because, at that limit, the potential remains most of the time ‘off’. In contrast, for $\delta \to 1$, the potential is ‘on’ most of the time and we get the static or non-flashing Lévy ratchet limit. In such situation, the current depends on $\alpha$ and vanishes only in the case $\alpha \to 2$, when the system approaches the equilibrium situation of a non-flashing ratchet with Gaussian noise.

The different panels of figure 5 show us that the transition probability $\Gamma_{AB}$ plays a significant role in the current behavior (i.e. $\delta$ alone does not determine the dynamics). For intermediate values of $\Gamma_{AB}$ (figures 5(b) and (c)) the current has a maximum at an intermediate duty ratio for almost all values of $\alpha$. In contrast, in the case of very large...
Figure 3. Current as a function of the switching rate for different values of $\alpha$ considering $\chi = 0.05$ (a), $\chi = 0.2$ (b) and $\chi = 0.6$ (c). In panels (a), (b) and (c), the symbols at the extreme values of $\Gamma$ indicate the infinitely slow (equations (15)) and infinitely fast (equations (17)) limit approximations showing complete agreement with the exact solutions. Panel (d) shows $(J - J_{\text{slow}})$ versus $\Gamma$ in the slow switching range confirming the validity of the asymptotic formula of equation (19). The segment indicates a linear dependence on $\Gamma$ for reference. Analogously, the results for $(J - J_{\text{fast}})$ in panel (e) shows the validity of the perturbative analysis at fast switching (equation (21)). The segment indicates a $1/\Gamma$ dependence. Results in panels (d) and (e) are for $\chi = 0.05$.

Figure 4. $J$ versus $\alpha$ and $\chi$ for a fixed value of the transition rate $\Gamma = 2 \times 10^{-1}$. doi:10.1088/1742-5468/2011/08/P08025
Figure 5. Current versus duty ratio for different values of $\alpha$ and fixed $\chi = 5 \times 10^{-2}$ considering $\Gamma_{AB} = 2 \times 10^{-3}$ (a), $\Gamma_{AB} = 2 \times 10^{-1}$ (b), $\Gamma_{AB} = 2 \times 10^{0}$ (c) and $\Gamma_{AB} = 2 \times 10^{2}$ (d).

Figure 6. Current versus duty ratio for a fixed value of the noise power $\chi = 0.2$ and different values of $\Gamma_{AB}$. Calculations for $\alpha = 1.2$ (a), $\alpha = 1.6$ (b), and $\alpha = 1.8$ (c).

or very small $\Gamma_{AB}$ (figures 5(a) and (d)), except for $\alpha$ equal or very close to $\alpha = 2$, the current is a monotonic function of the duty ratio, and reaches the maximum for $\delta = 1$. Note that the linear dependence of $J$ on $\delta$ observed in figure 5(a) corresponds to the limit of slow switching indicated in section 4. Namely, $J \simeq \delta \times J_{st}(V)$.

Figure 6 analyzes the dependence of $J$ on $\delta$ considering a larger noise intensity, and sweeping wide ranges of $\alpha$ and $\Gamma_{AB}$. We see that in most cases the maximum current
is achieved close to $\delta = 1$. The exceptions occur for values of $\alpha$ close enough to 2 (for instance that in figure 6(c)) and intermediate values of $\Gamma_{AB}$.

We can summarize the results of our analysis of $J$ as a function of $\delta$ at fixed $\Gamma_{AB}$ as follows. When considering small enough $\chi$ or large enough $\alpha$, the inclusion of an appropriate flashing mechanism improves the performance of the static ratchet. In contrast, for large $\chi$ or small $\alpha$, the largest value of $J$ is obtained considering a slow flashing mechanism close to the static ratchet (i.e. $\delta = 1$). We thus see that the currents depend separately and richly on both $\Gamma_{AB}$ and $\delta$, allowing us both to enhance and to decrease the static Lévy ratchet currents for fixed noise parameters.

6. Conclusions and final remarks

We have studied the combined action of a Lévy additive noise and a random dichotomic flashing in a ratchet system. Our results provide a complete generalization of previous studies on ‘non-flashed’ Lévy ratchets and on standard flashing ratchets with Gaussian noises.

We have presented a complete analysis of the two-variable fractional Fokker–Planck equation associated with the system. Our Fourier treatment allowed us to analytically convert the system of partial differential equations into an infinite-matrix linear system that can be easily solved numerically considering an appropriate truncation. Moreover, we were able to provide analytical asymptotic laws for slow and fast flashing that behave respectively as $J - J_{\text{slow}} \sim \Gamma$ and $J - J_{\text{fast}} \sim 1/\Gamma$, where $\Gamma$ is the flashing frequency. The solution of the Fokker–Planck equation was given for an arbitrary periodic potential, and indicated also for the case in which the system switches randomly between two different potentials.

Considering a standard ratchet potential we have systematically investigated the behavior of the current as a function of the stability index of the Lévy noise, the noise intensity and the ‘on’ and ‘off’ rates of the flashing mechanisms. The Fokker–Planck results for the current were checked by means of Langevin calculations. We have found that random dichotomic flashing can produce a rich behavior of the ratchet current. It allows us both to enhance and to diminish the static Lévy ratchet current appreciably, depending on the magnitude and relative magnitude of the flashing frequencies, and on the Lévy noise parameters. A general statement to note is that for small enough noise intensity or large enough stability index, a flashing mechanism can enhance the current of the static ratchet. Another relevant result indicates that, for a fixed flashing mechanism, the Lévy noise gives larger current than the Gaussian noise in almost all situations.

Our work thus contributes with quite general results and procedures to the understanding of the mechanisms of transport on ratchets, and in particular, to the rapidly growing new field of Lévy ratchets.

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