EXTREMAL ALMOST-KÄHLER METRICS

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ABSTRACT. We generalize the notions of the Futaki invariant and extremal vector field of a compact Kähler manifold to the general almost-Kähler case and show the periodicity of the extremal vector field when the symplectic form represents an integral cohomology class modulo torsion. We also give an explicit formula of the hermitian scalar curvature in Darboux coordinates which allows us to obtain examples of non-integrable extremal almost-Kähler metrics saturating LeBrun’s estimates.

1. INTRODUCTION

Let \((M^{2n}, \omega)\) be a compact symplectic manifold of dimension \(2n\). Recall that an almost-complex structure \(J\) is compatible with \(\omega\) if the tensor field \(g(\cdot, \cdot) = \omega(\cdot, J \cdot)\) defines a Riemannian metric on \(M\); in this case, the triple \((\omega, J, g)\) is referred to as an \((\omega\text{-compatible})\) almost-Kähler structure on \(M\). Any such structure defines, in a canonical way, a hermitian connection \(\nabla\) on the complex tangent bundle \((\mathbb{T}(M), J, g)\). Taking trace and contracting the curvature of \(\nabla\) by \(\omega\), one obtains the hermitian scalar curvature \(s^\nabla\) of \((\omega, J, g)\).

It is well-known [8, 12] that the space of all \(\omega\text{-compatible}\) almost-Kähler structures, here denoted by \(AK_\omega\), is a contractible Fréchet manifold endowed with a formal Kähler structure. The infinite dimensional group \(\text{Ham}(M, \omega)\) of Hamiltonian symplectomorphisms naturally acts on \(AK_\omega\) and a crucial observation of Donaldson [8] (generalizing [12] to the non-integrable almost-Kähler case) is that this action is Hamiltonian with moment map \(\mu: AK_\omega \to (\text{Lie}(\text{Ham}(M, \omega))^*\) given by \(\mu_J(f) = \int_M s^\nabla f \frac{\omega^n}{n!}\), where \(f\) is any smooth function with zero integral on \(M\), viewed also as an element of the Lie algebra of \(\text{Ham}(M, \omega)\). As already observed in [5], this interpretation of \(s^\nabla\) immediately implies that the critical points of the functional \(J \mapsto \int_M (s^\nabla)^2 \frac{\omega^n}{n!}\) over \(AK_\omega\) are the almost-Kähler metrics for which the symplectic gradient of the hermitian scalar curvature is an infinitesimal isometry of the almost-complex structure \(J\). Motivated by striking analogy with the notion of extremal Kähler metric introduced by Calabi [7], we refer to the almost-Kähler metrics verifying the above condition as extremal almost-Kähler metrics.

The above formal picture, restricted to the subspace of diffeomorphic integrable \(\omega\text{-compatible}\) almost-Kähler structures, gives many insights in the theory of extremal Kähler metrics, where the leading conjectures are derived by a considerable scope of analogy with the well-established GIT in the finite dimensional case [11]. It also suggests that extremal almost-Kähler metrics would provide a natural extension of the theory of extremal Kähler metrics to the non-integrable case. In fact, this link has already become explicit in the toric case [9], where the existence of an extremal Kähler metric is conjecturally equivalent to the existence of (infinitely
many) non-integrable extremal almost-Kähler metrics; this link was also used in [4] to find an explicit criterion to test K-stability (and therefore (non) existence of extremal Kähler metrics) on projective plane bundles over a curve and construct explicit examples of extremal almost-Kähler metrics. Besides the now appealing motivation, a systematic study of extremal almost-Käehler metric is still to come (see however [5, 18, 21]).

In this paper, we generalize the notion of the Futaki invariant and extremal vector field to the general almost-Kähler case. This amounts to the observation that fixing a compact subgroup $G \subset \text{Ham}(M, \omega)$ and considering $G$-invariant $\omega$-compatible almost-Kähler structures $(g, J)$, the $L^2$-projection of the hermitian scalar curvature to the finite dimensional space of hamiltonians of elements of $\text{Lie}(G)$ is independent of $(g, J)$. This fact, which easily follows from the formal picture described above and which is certainly known to experts (see e.g. [4, 16]), is established by a direct argument in Sec. 3.2 below. By taking $G$ be a maximal torus $T \subset \text{Ham}(M, \omega)$, we show in Lemma 3.6 that the projection of the hermitian scalar curvature of any $T$-invariant compatible almost-Kähler metric defines a Killing potential, $z^T_\omega$, which must coincide with the hermitian scalar curvature of any $T$-invariant extremal almost-Kähler metric (should it exist). We call the vector field $Z^T_\omega = \text{grad}_\omega z^T_\omega$ the extremal vector field relative to $T$. Its vanishing is an obstruction to the existence of $T$-invariant almost-Kähler metrics of constant hermitian scalar curvature. Noting that $Z^T_\omega$ doesn’t change under a $T$-invariant isotopy of $\omega$, it naturally generalizes the extremal vector field introduced by [14] in the Kähler context.

The main technical input of the paper is proving that $Z^T_\omega$ has closed orbits when $[\omega^2]_{\pi}$ is an integral class modulo torsion. This extends the corresponding result in the Kähler case [25], but we need to adapt the ‘symplectic’ approach of [15] which relies on the localization formula [10]. Yet, the main technical ingredient is our Lemma 2.6 which essentially computes the momentum map of the $T$-action, with respect to the hermitian Ricci form $\rho^\nabla$.

The structure of the paper is as follows. In Sec. 2, we introduce the necessary background of almost-Kähler geometry with special attention to holomorphic vector fields on almost-Kähler manifolds. In particular, we obtain Bochner formulae involving the hermitian Ricci form $\rho^\nabla$ and the so-called $\ast$-Ricci form $\rho^\ast$ and derive some vanishing results (Corollaries 2.5 and 2.7). We define in Sec. 3 the extremal vector field relative to a maximal torus $T \subset \text{Ham}(M, \omega)$ in the general almost-Kähler case and prove its periodicity when $[\omega^2]_{\pi}$ is integral modulo torsion. In Sec. 4, we give an explicit (local) formula of the hermitian scalar curvature in Darboux coordinates which allows us to recast the expressions of the hermitian Ricci form and the scalar curvature in the toric case [11, 9]. We then specialise to the 4-dimensional case and construct infinite dimensional families of non-integrable extremal almost-Kähler metrics by using local toric symmetry; this allows us to obtain examples of non-integrable Hermitian-Einstein almost-Kähler metrics saturating LeBrun’s estimates [20]. Finally, in Sec. 5, we discuss some avenues of further research.

2. Preliminaries

2.1. Almost-Kähler structures. An almost-Kähler structure on a real manifold $M$ of dimension $2n$ is given by a triple $(\omega, J, g)$ of a symplectic form $\omega$, an almost-complex structure $J$ and a Riemannian metric $g$, which satisfy the compatibility
relation
\begin{equation}
(2.1) \quad g(\cdot, \cdot) = \omega(\cdot, J \cdot).
\end{equation}

We say that the almost-complex structure \( J \) is \( \omega \)-compatible if it induces a Riemannian metric via \( (2.1) \). If, additionally, the almost complex structure \( J \) is integrable, then we have a Kähler structure on \( M \). By the Newlander–Nirenberg theorem, the almost-complex structure \( J \) is integrable if and only if the Nijenhuis tensor
\begin{equation}
4N(\cdot, \cdot) = [J, J \cdot] - [\cdot, J J \cdot] - J [J \cdot, \cdot] - J [\cdot, J J \cdot]
\end{equation}
vanishes; here \([\cdot, \cdot]\) stands for the Lie bracket.

The complexified tangent bundle splits as
\[ T(M) \otimes \mathbb{C} = T^{1,0}(M) \oplus T^{0,1}(M), \]
where \( T^{1,0}(M) \) (resp. \( T^{0,1}(M) \)) corresponds to the eigenvalue \( \sqrt{-1} \) (resp. \( -\sqrt{-1} \)) under the \( \mathbb{C} \)-linear action of \( J \). The complex vector bundle \((T(M), J)\) is identified with \( T^{1,0}(M) \) via the map \( X \mapsto X^{1,0} = \frac{i}{2}(X - \sqrt{-1}JX) \) (resp. \( X^{0,1} = \frac{i}{2}(X + \sqrt{-1}JX) \)). The almost-complex structure \( J \) also induces a decomposition of the complexified cotangent bundle
\[ T^*(M) \otimes \mathbb{C} = \wedge^{1,0}(M) \oplus \wedge^{0,1}(M) \]
where \( \wedge^{1,0}(M) \) (resp. \( \wedge^{0,1}(M) \)) is the annihilator of \( T^{0,1}(M) \) (resp. \( T^{1,0}(M) \)).

Any section \( \psi \) of \( \otimes^2 T^*(M) \) (and therefore of \( T^*(M) \otimes T(M) \) which is identified to \( \otimes^2 T^*(M) \) via the metric) admits an orthogonal splitting \( \psi = \psi^+ + \psi^- \), where \( \psi^+ \) is the \( J \)-invariant part and \( \psi^- \) is the \( J \)-anti-invariant part, given by
\[ \psi^+(\cdot, \cdot) = \frac{1}{2}(\psi(\cdot, \cdot) + \psi(J \cdot, J \cdot)) \quad \text{and} \quad \psi^-(\cdot, \cdot) = \frac{1}{2}(\psi(\cdot, \cdot) - \psi(J \cdot, J \cdot)). \]

In particular, the bundle of 2-forms decomposes under the action of \( J \)
\begin{equation}
(2.2) \quad \wedge^2(M) = \mathbb{R} \cdot \omega \oplus \wedge^+_0(M) \oplus \wedge^-_0(M),
\end{equation}
where \( \wedge^+_0(M) \) is the subbundle of the primitive \( J \)-invariant 2-forms (i.e. 2-forms pointwise orthogonal to \( \omega \)) and \( \wedge^-_0(M) \) is the subbundle of \( J \)-anti-invariant 2-forms.

The fact that \( \omega \) is closed implies the following identities (see [19])
\begin{equation}
(2.3) \quad g((D^g_X J)Y, Z) + g((D^g_J J)Z, X) + g((D^g_J J)X, Y) = 0,
\end{equation}
\begin{equation}
(2.4) \quad (D^g_X \omega)(Y, Z) = 2g(JX, N(Y, Z)),
\end{equation}
where \( D^g \) is the Levi-Civita connection with respect to the Riemannian metric \( g \) and \( X, Y, Z \) are any vector fields. Since \( N \) is a \( J \)-anti-invariant 2-form with values in \( T(M) \), it follows from \( (2.4) \) that
\begin{equation}
(2.5) \quad D^g_{JX} J = -JD^g_X J.
\end{equation}
Moreover, we readily deduce from the relation \( (2.4) \) that the Nijenhuis tensor is identically zero if and only if \( \omega \) (or equivalently \( J \)) is \( D^g \)-parallel.
From the exterior derivative \( d \), we can define the twisted exterior differential \( d^c = (-1)^p J d J \) acting on \( p \)-forms (in particular \( d^c f = J df \) for a smooth function \( f \)). A direct computation shows the following relation for any smooth function \( f \)

\[
(2.6) \quad dd^c f + d^c df = 4d^c f (N(\cdot, \cdot)).
\]

It follows that the almost-complex structure \( J \) is integrable if and only if \( d \) and \( d^c \) anticommute.

We denote by \( \Lambda_\omega \) the contraction by the symplectic form \( \omega \), defined for a \( p \)-form \( \psi \) by \( \Lambda_\omega (\psi) = \frac{1}{2} \sum_{i=1}^{2n} \psi (e_i, Je_i, \cdots) \), where \( \{ e_i \} \) is a local \( J \)-adapted orthonormal frame. As noticed by Gauduchon [16] and Merkulov [23], the commutator of \( \Lambda_\omega \) and \( d \) is equal to

\[
(2.7) \quad [\Lambda_\omega, d] = -\delta^c,
\]

where \( \delta^c = (-1)^p J \delta^g J \) is the twisted codifferential acting on \( p \)-forms; here \( \delta^g \) is the codifferential defined as the formal adjoint of \( d \) with respect to the metric \( g \) (\( \delta^g \) also stands for the adjoint of \( D^g \) with respect to \( g \) when it is applied to sections of \( \otimes^p T^*(M) \)). The operator \( \Lambda_\omega \) commutes with \( J \), so the relation (2.7) implies

\[
(2.8) \quad [\Lambda_\omega, d^c] = \delta^g.
\]

It follows from (2.8) that on any almost-Kähler manifold we have [16]

\[
(2.9) \quad \delta^g d^c + d^c \delta^g = 0.
\]

2.2. The hermitian connection. The canonical hermitian connection \( \nabla \) corresponding to \( J \) is defined by

\[
\nabla_X Y = D^g_Y X - \frac{1}{2} J (D^g_Y J) X.
\]

The connection \( \nabla \) preserves \( J \) (i.e. \( \nabla J = 0 \)) and has a \( J \)-anti-invariant torsion. It also induces the Cauchy–Riemann operator \( \bar{\partial} \) on \( T^{1,0}(M) \) [16], where we recall \( (\bar{\partial} Y^{1,0})(X^{0,1}) := [X^{0,1}, Y^{1,0}]^{1,0} \). Indeed,

**Proposition 2.1.** For any vector fields \( X, Y \)

\[
\nabla_{X^{0,1}} Y^{1,0} = [X^{0,1}, Y^{1,0}]^{1,0}.
\]

**Proof.** Using the above definition of \( \nabla \) and the relation (2.5), we have

\[
\nabla_{X^{0,1}} Y^{1,0} = D^g_{X^{0,1}} Y^{1,0} - \frac{1}{2} J (D^g_{X^{0,1}} J) Y^{1,0}
\]

\[
= \frac{1}{4} D^g_{\sqrt{-1} X + J X} (Y - \sqrt{-1} J Y) - \frac{1}{8} J (D^g_{\sqrt{-1} X + J X} J) (Y - \sqrt{-1} J Y)
\]

\[
= \frac{1}{4} D^g_{X^{0,1}} Y + \frac{1}{4} \sqrt{-1} (D^g_{J X} Y - D^g_{X} (J Y))
\]

\[
= \frac{1}{4} (Id - \sqrt{-1} J) \left( D^g_{X^{0,1}} Y + D^g_{J X} (J Y) \right).
\]
On the other hand,
\[
[X^{0,1}, Y^{1,0}]^{1,0} = \frac{1}{4} [X + \sqrt{-1} JX, Y - \sqrt{-1} JY]^{1,0}
\]
\[
= \frac{1}{8} (Id - \sqrt{-1} J) [X + \sqrt{-1} JX, Y - \sqrt{-1} JY]
\]
\[
= \frac{1}{8} (Id - \sqrt{-1} J) \left( [X, Y] + [JX, JY] + J [JX, Y] - J [X, JY] \right)
\]
\[
= \frac{1}{8} (Id - \sqrt{-1} J) \left( D^2_X Y - D^2_Y X + D^2_Y (JY) - D^2_Y (JX) \right)
\]
\[
+ JD^2_X Y - JD^2_Y (JX) - JD^2_X (JY) + JD^2_Y X \right)
\]
\[
= \frac{1}{8} (Id - \sqrt{-1} J) \left( D^2_X Y - D^2_Y X + D^2_Y (JY) + J(D^2_Y J)X - J(D^2_Y J)X \right)
\]
\[
+ JD^2_X Y + D^2_Y J - J(D^2_Y J)X + D^2_X Y + (D^2_Y J)Y + JD^2_Y X \right)
\]
\[
= \frac{1}{4} (Id - \sqrt{-1} J) \left( D^2_X Y + D^2_J X (JY) \right).
\]

The proposition follows. \qed

2.3. Ricci forms. The canonical hermitian connection \( \nabla \) on \( T(M) \) induces a hermitian connection on the anti-canonical bundle \( K^{-1}_M = \wedge^{n,0}(M) \) (equiped with the hermitian structure induced by \( g \)) with curvature \( \sqrt{-1} \rho^\nabla \), where \( \rho^\nabla \) is a closed (real) 2-form, called the hermitian Ricci form. Hence, the 2-form \( \rho^\nabla \) is a deRham representative of \( 2\pi c_1(M, J) \) in \( H^2(M, \mathbb{R}) \) where \( c_1(M, J) \) is the (real) Chern class of \( K^{-1}_M \).

We consider also \( \rho^* = R(\omega) \), the image of the symplectic form \( \omega \) by the (Riemannian) curvature operator \( R \). The 2-form \( \rho^* \) is called the \( * \)-Ricci form. We have the following relation between these Ricci-type tensors (see [4])
\[
(2.10) \quad \rho^\nabla (X, Y) = \rho^* (X, Y) - \frac{1}{4} \text{tr}(JD^2_X J \circ D^2_Y J).
\]

In the Kähler case (i.e. when \( D^g \omega = D^g J = 0 \)), we readily deduce from the relation (2.10) that \( \rho^\nabla = \rho^* \), which is also equal to the usual Ricci form. Note that neither \( \rho^\nabla \) nor \( \rho^* \) is \( J \)-invariant in general. In fact, by (2.10) and (2.5), \( \rho^\nabla \) is \( J \)-invariant if and only if \( \rho^* \) is.

2.4. Holomorphic vector fields on an almost-Kähler manifold. In this section, we study (pseudo-)holomorphic vector fields on an almost-Kähler manifold \( (M^{2n}, \omega, J, g) \). A (real) vector field \( X \) is said holomorphic if \( \mathfrak{L}_X J = 0 \), where \( \mathfrak{L} \) denotes the Lie derivative. It is equivalent to say that \( [X, JY] = J [X, Y] \) for any vector field \( Y \).

**Lemma 2.2.** On an almost-Kähler manifold \( (M^{2n}, \omega, J, g) \), \( X \) is a holomorphic vector field if and only if
\[
(2.11) \quad (D^g X)^* = -\frac{1}{2} D^g_{JX} J.
\]

**Proof.** For any vector field \( X \), we have \( \mathfrak{L}_X J = D^g_X J - [D^g X, J] \). In particular, if \( X \) is holomorphic then \( D^g_X J = [D^g X, J] \). On the other hand \( J [D^g X, J] = 2(D^g X)^* \). The equality (2.11) follows from (2.5). \qed
Lemma 2.3. Let $\alpha$ be a 1-form. We have

$$\delta^g (D^g \alpha^+) - \delta^g (D^g \alpha^-) = \rho^* \left( \alpha^2, J \cdot \right) - \sum_{i=1}^{2n} (D^g_{e_i \alpha})(D^g_{e_i J})(\cdot) ,$$

where $\{e_i\}$ is a local $J$-adapted orthonormal frame, $\delta^g$ the formal adjoint of the Levi-Civita connection $D^g$ with respect to the metric $g$ and $\rho^*$ stands for the isomorphism between $T^*(M)$ and $T(M)$ induced by $g^{-1}$.

Proof. By using the fact that $\delta^g J = - \sum_{i=1}^{2n} (D^g_{e_i J})(e_i) = 0$, we have

$$\left( \delta^g (D^g \alpha)^+ - \delta^g (D^g \alpha)^- \right)(X) = - \sum_{i=1}^{2n} (D^g_{e_i})(D^g \alpha)^+ - (D^g \alpha)^- (e_i, X)$$

$$= - \sum_{i=1}^{2n} \left[ (D^g_{e_i})([D^g_{e_i \alpha}(J X)] - (D^g_{D^g_{e_i}(J e_i \alpha)})(J X) - (D^g_{e_i}(J e_i \alpha))(J X) \right]$$

$$= - \sum_{i=1}^{2n} \left[ (D^g_{e_i}(J e_i \alpha))(J X) + (D^g_{e_i}(J X))(D^g_{J e_i}(J X)) - (D^g_{D^g_{e_i}(J e_i \alpha)})(J X) \right]$$

$$= - \sum_{i=1}^{2n} \left[ (D^g_{e_i}(J e_i \alpha))(J X) - (D^g_{D^g_{e_i}(J e_i \alpha)})(J X) + (D^g_{e_i}(J X))(D^g_{e_i}(J X)) \right]$$

$$= - \sum_{i=1}^{2n} (D^g_{e_i}(J e_i \alpha))(J X) - \sum_{i=1}^{2n} (D^g_{e_i}(J X))(D^g_{e_i}(J X))$$

$$= 1/2 \sum_{i=1}^{2n} (R_{e_i, J e_i \alpha})(J X) - \sum_{i=1}^{2n} (D^g_{e_i}(J X))(D^g_{e_i}(J X))$$

$$= \rho^*_{\alpha^2, J X} - \sum_{i=1}^{2n} (D^g_{J e_i \alpha})(D^g_{e_i}(J X)).$$

Corollary 2.4. Let $\alpha$ be a 1-form such that $X = \alpha^2$ is holomorphic vector field. Then

$$\delta^g (D^g \alpha)^+ - \delta^g (D^g \alpha)^- = \rho^\nabla(\alpha, J \cdot).$$

Proof. Combining Lemmas 2.2 and 2.3 and using the relations 2.3 and 2.10, we have

$$\left( \delta^g (D^g \alpha)^+ - \delta^g (D^g \alpha)^- \right)(Y) = \rho^*_{X, J Y} - \sum_{i=1}^{n} (D^g_{e_i \alpha})(D^g_{e_i J})(Y)$$

$$= \rho^\nabla_{X, J Y} + \frac{1}{4} \text{tr}(J D^g_{X J} \circ D^g_{J Y} J) - \sum_{i=1}^{2n} (D^g_{J e_i \alpha})(D^g_{e_i J})(Y).$$
\begin{align*}
&= \rho_{X,JY} + \frac{1}{4} tr(JD_X^g J \circ D_Y^g J) - \frac{1}{2} \sum_{i=1}^{2n} \left( (D^g \alpha)_{J e_i} \right) \left( (D^g e_i) (Y) \right) \\
&= \rho_{X,JY} + \frac{1}{4} tr(JD_X^g J \circ D_Y^g J) + \frac{1}{2} \sum_{i=1}^{2n} g \left( (D^g \alpha)_J (J e_i), (D^g e_i) (Y) \right) \\
&= \rho_{X,JY} + \frac{1}{4} tr(JD_X^g J \circ D_Y^g J) - \frac{1}{2} \sum_{i=1}^{2n} g \left( (D^g \alpha)_J (e_i), (D^g e_i) (Y) \right) \\
&= \rho_{X,JY} + \frac{1}{4} tr(JD_X^g J \circ D_Y^g J) - \frac{1}{2} \sum_{i,k=1}^{2n} \left( (D^g \alpha)_J (e_i, e_k) \otimes (D^g e_i) (Y, e_k) \right) \\
&= \rho_{X,JY} + \frac{1}{4} tr(JD_X^g J \circ D_Y^g J) - \frac{1}{4} \sum_{i,k=1}^{2n} \left( (D^g \alpha)_J (e_i, e_k) \otimes (D^g e_i) (Y, e_k) \right) \\
&= \rho_{X,JY} + \frac{1}{4} tr(JD_X^g J \circ D_Y^g J) - \frac{1}{4} \sum_{i=1}^{2n} g \left( (D^g \alpha)_J (e_i), (D^g e_i) (Y) \right) \\
&= \rho_{X,JY} - \frac{1}{4} tr(D_X^g J \circ D_Y^g J) + \frac{1}{4} \sum_{i=1}^{2n} g \left( (D^g \alpha)_J (e_i), (D^g e_i) (Y) \right) \\
&= \rho_{X,JY} - \frac{1}{4} tr(D_X^g J \circ D_Y^g J) + \frac{1}{4} \sum_{i=1}^{2n} g \left( (D^g \alpha)_J (e_i), (D^g e_i) (Y) \right)
\end{align*}

Corollary 2.5. Let \((M^{2n}, \omega, J, g)\) be a compact almost-Kähler manifold. Suppose that the tensor \((\rho^*)^+(\cdot, J \cdot)\) is negative-definite. Then, there is no non-trivial holomorphic vector field on \(M\).

Proof. Let \(\alpha\) be the dual (by the metric \(g\)) of a holomorphic vector field \(X\). By Lemma 2.3 we have

\begin{equation}
(2.12) \quad \left( \delta^g (D^g \alpha)^+ - \delta^g (D^g \alpha)^- \right) (X) = \rho^* (X, JX) - \sum_{i=1}^{2n} (D^g_{J e_i} \alpha) \left( (D^g e_i) (X) \right).
\end{equation}

Using Lemma 2.2 we simplify the second term of the right hand side of (2.12) as in the proof of Corollary 2.4

\[
\frac{1}{2} \sum_{i=1}^{2n} g \left( (D^g \alpha)_{J e_i}, (D^g e_i) (X) \right)
= \frac{1}{2} \sum_{i=1}^{2n} g \left( (D^g \alpha)_{J e_i}, (D^g e_i) (X) \right)
= -\frac{1}{2} \sum_{i=1}^{2n} g \left( (D^g \alpha)_{J e_i}, (D^g e_i) (X) \right)
\]
On the other hand (by Lemma 2.2) \((D^g f)_e\) where \(\delta\)
\[\begin{align*}
\text{(2.13)} & \quad \frac{1}{2} \sum_{i=1}^{2n} g(\left(D^g_X J\right)(e_i), (D^g e_i) J(X)) \\
& \quad = \frac{1}{2} \sum_{i,k=1}^{2n} \left( (D^g_X J)(e_i, e_k) \otimes (D^g e_i) J(X, e_k) \right) \\
& \quad = \frac{1}{4} \sum_{i,k=1}^{2n} \left( (D^g_X J)(e_i, e_k) \otimes \left( [D^g e_i J(X, e_k) - (D^g e_i) J(X, e_i)] \right) \right) \\
& \quad = \frac{1}{4} \sum_{i,k=1}^{2n} \left( (D^g_X J)(e_i, e_k) \otimes (D^g e_i) J(e_i, e_k) \right) \\
& \quad = \frac{1}{4} \sum_{i=1}^{2n} \left( (D^g_X J)(e_i), (D^g e_i) J(e_i) \right) .
\end{align*}\]

Integrating \((2.12)\), we obtain
\[\begin{align*}
\left| (D^g \alpha)^+ \right|_{L^2}^2 - \left| (D^g \alpha)^- \right|_{L^2}^2 & \quad = \left( \int_M \rho^*(X, J X) \frac{\omega^n}{m!} \right) - \frac{1}{4} |D^g_X J|^2_{L^2} .
\end{align*}\]

Because of Lemma 2.2 \((D^g \alpha)^- \left|_{L^2}^2 = \frac{1}{4} |D^g_X J|^2_{L^2} . \) Then, \((D^g \alpha)^+ \left|_{L^2}^2 = \left( \int_M \rho^*(X, J X) \frac{\omega^n}{m!} \right) . \)

The corollary follows

**Lemma 2.6.** Let \((M^{2n}, \omega, J, g)\) be an almost-Kähler manifold. If \(X\) is a hamiltonian Killing vector field, then
\[\begin{align*}
(2.13) & \quad - \frac{1}{2} d\Delta^g f^X = \rho^\nabla(X, \cdot) ,
\end{align*}\]

where \(f^X\) is the momentum of \(X\) with respect to \(\omega\) (i.e. \(\omega(X, \cdot) = -df^X\)) and \(\Delta^g\) is the Riemannian Laplacian with respect to \(g\).

**Proof.** By hypothesis, \(X = (d^e f)^{\cdot} \) is a Killing vector field then \(D^g d^e f = \frac{1}{2} dd^e f\).

So, we have \(\delta^g D^g d^e f = \delta^g (D^g d^e f)^+ + \delta^g (D^g d^e f)^- = \frac{1}{2} \delta^g dd^e f\).

Combining this with Corollary \((2.4)\) applied for the holomorphic vector field \(X\), we obtain
\[\begin{align*}
(2.14) & \quad 2\delta^g (D^g d^e f)^- = \frac{1}{2} \delta^g dd^e f - \rho^\nabla(X, J \cdot) .
\end{align*}\]

On the other hand (by Lemma 2.2) \((D^g d^e f)^- = \frac{1}{3} D^g_{(d^e f)^3} \omega\). Then, using the relations \((2.14)\) and \((2.15)\), we obtain \((D^g d^e f)^- = \frac{1}{4} (dd^e f + d^e df)\) and therefore
\[\begin{align*}
(2.15) & \quad \delta^g (D^g d^e f)^- = \frac{1}{4} \delta^g (dd^e f + d^e df) .
\end{align*}\]

Combining \((2.14)\) with \((2.15)\), we obtain, using the relation \((2.14)\)
\[\begin{align*}
\frac{1}{2} \delta^g (dd^e f + d^e df) & \quad = \frac{1}{2} \delta^g dd^e f - \rho^\nabla(X, J \cdot) \\
\frac{1}{2} \delta^g d^e df & \quad = -\rho^\nabla(X, J \cdot) \\
\frac{1}{2} d^e \delta^g df & \quad = -\rho^\nabla(X, J \cdot) \\
\frac{1}{2} d \Delta^g f & \quad = \rho^\nabla(X, \cdot) .
\end{align*}\]
Corollary 2.7. Let \((M^{2n}, \omega, J, g)\) be a compact almost-\(K\ddot{a}hler\) manifold. Suppose that the tensor \((\rho^N)^y(\cdot, J\cdot)\) is negative-semidefinite. Then, there is no non-trivial hamiltonian Killing vector field on \(M\).

Proof. Suppose that \(X = (d^c f)^\sharp\) is a hamiltonian Killing vector field. By Lemma 2.6 we have

\[
-\frac{1}{2} d\Delta g \left( (df)^\sharp \right) = \rho^N \left( (d^c f)^\sharp, (df)^\sharp \right).
\]

By integrating, we obtain

\[
\frac{1}{2} |\Delta g f|_{L^2}^2 = \left( \int_M \rho^N \left( (df)^\sharp, (d^c f)^\sharp \right) \frac{\omega^n}{n!} \right).
\]

If \(\Delta g f\) is identically zero on a compact Riemannian manifold, \(f\) must be constant and thus \(X = 0\). The corollary follows.

2.5. A localization formula. Let \((M^{2n}, \omega)\) be a compact symplectic manifold endowed with a hamiltonian \(S^1\)-action. Let \(X\) be the generator of this action with a momentum \(f^X\) i.e. \(\omega(X, \cdot) = -df^X\). Thus, \(X\) is a hamiltonian Killing vector field with respect to some compatible almost-\(K\ddot{a}hler\) metrics. The fixed points of the action form a finite union of connected symplectic submanifolds \(N_1, \ldots, N_t\) such that \(f^X\) is constant on each \(N_j\). For each \(N_j\), the normal bundle \(E_j\) splits into complex line bundles \(E_j = L_{k_1} \oplus \cdots \oplus L_{k_{m_j}}\) on which \(S^1\) acts with integer weights \(k_1^1, \ldots, k_{m_j}\). Then, we have the following formula (see [10] [22])

\[
(2.16) \quad \int_M e^{-h f^X} \frac{\omega^n}{n!} = \sum_{j=0}^\gamma \left( \int_{N_j} \prod_{i=1}^{m_j} \frac{1}{c_1(L_i^j) + k_i^j h} \right) e^{-h f^X(N_j)},
\]

for every \(h \in \mathbb{C}\). Here, \(c_1(\cdot)\) denote the first Chern class and we take the formal inverse

\[
\frac{1}{c_1(L_i^j) + k_i^j h} = \frac{1}{k_i^j h} \left( \sum_{r=1}^{(\dim N_j)/2} \left( -\frac{c_1(L_i^j)}{k_i^j h} \right)^r \right).
\]

3. Extremal almost-\(K\ddot{a}hler\) metrics

For the rest of the paper, \((M^{2n}, \omega)\) is a compact and connected symplectic manifold. We denote by \([\omega]\) the deRham cohomology class of \(\omega\). Any \(\omega\)-compatible almost-complex structure is identified with the induced Riemannian metric.

3.1. Hermitian scalar curvature as a moment map. We define the hermitian scalar curvature \(s^\nabla\) of an almost-\(K\ddot{a}hler\) structure \((\omega, J, g)\) as the trace of \(\rho^\nabla\) with respect to \(\omega\), i.e.

\[
(3.1) \quad s^\nabla \omega^n = 2n (\rho^\nabla \wedge \omega^{n-1}),
\]

or, in equivalent way,

\[
(3.2) \quad s^\nabla = 2\Lambda_\omega \rho^\nabla.
\]

Denote by \(AK_\omega\) the Fréchet space of \(\omega\)-compatible almost-complex structures. The space \(AK_\omega\) comes naturally equipped with a formal Kähler structure \((\Omega, J)\) (described first by Fujiki in [12]). More precisely, the tangent space \(T_J(AK_\omega)\) at a point \(J\) is identified with the space of \(g\)-symmetric, \(J\)-anti-invariant fields of endomorphisms of \(T(M)\) (where \((\omega, J, g)\) is the corresponding almost-\(K\ddot{a}hler\) metric). Then, for \(A, B \in T_J(AK_\omega)\), the Kähler form \(\Omega\) is given by \(\Omega_J(A, B) = \quad .\)
\[
\int_M \text{tr} (J \circ A \circ B) \frac{\omega^n}{n!} \quad \text{while the } \Omega\text{-compatible (integrable) almost-complex structure } J \text{ is defined by } J_J X = J \circ X.
\]

Let \( \text{Ham}(M, \omega) \) be the group of hamiltonian symplectomorphisms of \((M^{2n}, \omega)\). The Lie algebra of \( \text{Ham}(M, \omega) \) is identified with the space of smooth functions on \( M \) with zero mean value and the Poisson bracket; it is also equipped with an equivariant inner product, given by the \( L^2 \)-norm with respect to \( \frac{\omega^n}{n!} \).

A key observation, made by Fujiki \([12]\) in the integrable case and by Donaldson \([8]\) in the general almost-Kähler case, asserts that the natural action of \( \text{Ham}(M, \omega) \) on \( AK_\omega \) is hamiltonian with momentum given by the hermitian scalar curvature \( s^\nabla \). More precisely, the moment map is

\[
\mu_J(f) = \int_M s^\nabla f \frac{\omega^n}{n!}
\]

where \( s^\nabla \) is the hermitian scalar curvature of the induced almost-Kähler metric \((\omega, J, g)\). The square-norm of the hermitian scalar curvature defines a functional

\[
J \mapsto \int_M (s^\nabla)^2 \frac{\omega^n}{n!}
\]

**Definition 3.1.** The critical points of the functional \((3.4)\) are called *extremal almost-Kähler metrics*.

The functional \((3.4)\) corresponds to the square-norm function of the moment map. This observation was used \([5]\) to deduce that the critical points of \((3.4)\) are precisely the almost-Kähler metrics \( g \) for which the symplectic gradient of their hermitian scalar curvature \( \text{grad}_\omega s^\nabla \), defined as the symplectic dual of \( ds^\nabla \) (i.e. \(-ds^\nabla = \omega(\text{grad}_\omega s^\nabla, \cdot) = g(J\text{grad}_\omega s^\nabla, \cdot)\) and thus \( \text{grad}_\omega s^\nabla = J\text{grad} s^\nabla \)), is a Killing vector field of \( g \); as it is hamiltonian, this is also equivalent to being holomorphic with respect to \( J \). Indeed, it follows from \([11]\) that a point \( x_0 \) is critical for the square norm function of the moment map if and only if the image of \( x_0 \) by the moment map belongs to the Lie subalgebra corresponding to the stabilizer of \( x_0 \) by the action (where the Lie algebra is identified with its dual vector space via the inner product). In our context, this precisely means that \( \text{grad}_\omega s^\nabla \) is a Killing hamiltonian vector field.

**Proposition 3.2.** A metric \( g \) is a critical point of \((3.4)\) if and only if \( \text{grad}_\omega s^\nabla \) is a Killing vector field with respect to \( g \).

**Proof.** We reproduce here a direct verification made by Gauduchon \([16]\) using Mohsen formula \([24]\). The Mohsen formula states that, for a path \( J_t \in AK_\omega \), the first variation of the connection 1-forms \( \alpha_t \) of the hermitian connections on \( K_{J_t}^{-1}(M) \), induced by the canonical hermitian connections \( \nabla_t \) corresponding to \( J_t \), is given by

\[
\frac{d}{dt} \alpha_t = \frac{1}{2} \delta^{\rho^n} J,
\]

where \( \delta^{\rho^n} \) is the codifferential with respect to the metric \( g_t(\cdot, \cdot) = \omega(\cdot, J_t \cdot) \) and \( J = \frac{d}{dt} J_t \). Therefore, by definition, \( \frac{d}{dt} s^\nabla = -\frac{1}{2} \delta^{\rho^n} J \). Hence, by \((3.2)\) and \((2.7)\), we obtain

\[
\frac{d}{dt} s^\nabla = -\Lambda_{\omega} d\delta^{\rho^n} J = \delta^{\rho^n} J = -\delta^{\rho^n} J_t \delta^{\rho^n} J; \quad \text{here } \delta^{\rho^n}_t \text{ is the twisted codifferential with respect to } J_t.
\]

Therefore,

\[
\frac{d}{dt} s^\nabla = -\delta^{\rho^n} J_t \delta^{\rho^n} J.
\]
Using (3.5), we compute the first derivative of (3.4) in the direction of $\dot{J}$

$$\frac{d}{dt} \left( \int_M (s^{-1}) \frac{\omega^n}{n!} \right) = 2 \int_M s^{-1} \left( -\delta \circ J_t \delta \circ \dot{J} \right) \frac{\omega^n}{n!} = 2 \int_M g_t \left( D^{g_t} \dot{J}, s^{-1}, \dot{J} \right) \frac{\omega^n}{n!},$$

where $d_t^c$ is the twisted exterior differential corresponding to $J_t$. However, $\dot{J} = \frac{1}{2} J_t$ is a $g_t$-symmetric, $J_t$-anti-invariant endomorphism of $T(M)$. Hence, $\dot{J}$ is a critical point if and only if the symmetric, $J$-anti-invariant part of $D^{g_t} s^{-1}$ is identically zero. On the other hand, for any vector field $X$ preserving $\omega$, we have

$$0 = (\pounds_X \omega)(JY, Z) = - (\pounds_X g)(Y, Z) + g((\pounds_X J) JY, Z) = - g((D^{g_t} X) Y, Z) - g(Y, (D^{g_t} X) Z) + g((\pounds_X J) JY, Z).$$

It follows that the symmetric part of $D^{g_t} s^{-1}$ is already $J$-anti-invariant. Hence, $\dot{J}$ is a critical point if and only if $D^{g_t} s^{-1}$ is skew-symmetric which means that the symplectic gradient $\nabla_{\omega} \omega$ is a Killing vector field.

**Remark 3.3.** On a complex manifold $(M, J)$, the Calabi problem [7] consists in studying the Calabi functional given by the $L^2$-norm of the scalar curvature of the Kähler metrics whose Kähler form belongs to a fixed Kähler class $\Omega = [\omega]$. It turns out that the critical points of the Calabi functional, called *Calabi extremal Kähler metrics*, are the Kähler metrics for which the symplectic gradient of the scalar curvature is a Killing vector field. The extremal almost-Kähler metric thus appears as a natural extension of the Calabi extremal Kähler metric to the non-integrable case. Indeed, since any two Kähler forms in a fixed Kähler class $\Omega$ are isotopic, the Kähler metrics in $\Omega = [\omega]$ are embedded, via Moser’s lemma [22], in the space of $\omega$-compatible integrable almost-complex structures $K_\omega$.

### 3.2. The extremal vector field

We fix a compact group $G$ in the (infinite dimensional) group $\text{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms of $(M^{2n}, \omega)$. Denote by $g_\omega \subset C^\infty(M)$ the finite dimensional space of smooth functions which are Hamiltonians with zero mean value of elements of $g = \text{Lie}(G)$. It is well-known that $g_\omega$ has a Lie algebra structure given by the Poisson bracket and $g_\omega$ is isomorphic to $g$. Denote by $\Pi_\omega$ the $L^2$-orthogonal projection of a smooth function on $g_\omega$ with respect to the volume form $\frac{\omega^n}{n!}$. Let $AK_\omega^G$ be the space of $\omega$-compatible $G$-invariant almost-complex structures. As $AK_\omega$ is contractible, it is also connected; the same is true for $AK_\omega^G$ by taking the average of a path of metrics in $AK_\omega$ over $G$.

In this context, the following remark, generalizing Lemma 2 in [4], suggests the definition of an *extremal vector field* of $AK_\omega^G$:

**Lemma 3.4.** Let $J_t$ be a smooth family of almost-complex structures compatible with the fixed symplectic form $\omega$, which are invariant under a compact group $G$ of symplectomorphisms acting in a Hamiltonian way on the compact symplectic manifold $(M^{2n}, \omega)$. Then, the $L^2$-orthogonal projection of the hermitian scalar curvature $s^{-1}$ of $(\omega, J_t, g_t)$ on $g_\omega$ is independent of $t$.

**Proof.** By definition, any $f \in g_\omega$ defines a vector field $X = \nabla_{\omega} f$ which is in $g$ and is therefore Killing with respect to any of the metrics $g_t(\cdot, \cdot) = \omega(\cdot, J_t \cdot)$ in $AK_\omega^G$. To prove our claim, we have to show that $\int_M f s^{-1} \frac{\omega^n}{n!}$ is independent of $t$.

Using the relation (3.5), we obtain

$$\frac{d}{dt} \left( \int_M f s^{-1} \frac{\omega^n}{n!} \right) = \int_M (-\delta \circ J_t \delta \circ \dot{J}) f \frac{\omega^n}{n!} = \int_M g_t(\dot{J}, D^{g_t} f) \frac{\omega^n}{n!}.$$
The fact that $X = \text{grad}_\omega f$ is Killing implies that $D^{\text{sym}} d_f^* f$ is an anti-symmetric tensor. The result follows if we recall that $\hat{J}$ is a $g_t$-symmetric endomorphism of $T(M)$. 

**Remark 3.5.** The above lemma can also be viewed as a consequence from the fact that $s^\nabla$ is the momentum map for the action of $\text{Ham}(M,\omega)$. Indeed, consider a Lie subgroup $\mathbb{G} \subset \mathbb{H}$ where $\mathbb{H}$ is a Lie group (equipped with a bi-invariant metric) acting in a hamiltonian way on a symplectic manifold with moment map $\mu$. Let $N$ be a $\mathbb{G}$-invariant connected subspace and denote by $\mu^\mathbb{G}$ the projection (with respect to the inner product) of the image of $\mu^\mathbb{H}$ on the dual of the Lie subalgebra of $\mathbb{G}$. Since $N$ is $\mathbb{G}$-invariant, the differential of $\mu^\mathbb{G}|_N$ (restriction of $\mu^\mathbb{G}$ to $N$) is zero. Therefore, $\mu^\mathbb{G}|_N$ is constant. In our case, $\mathbb{H} = \text{Ham}(M,\omega), G = G, \text{Lie}(G) \cong g_\omega, N = AK^G_\omega$ and $\mu^\mathbb{G}$ is given by (3.3). Lemma 3.4 is equivalent to the fact that $\mu^\mathbb{G}|_N$ is constant.

Given any $J \in AK^G_\omega$, we define $z^G_\omega := \Pi_\omega s^\nabla$, where $s^\nabla$ is the hermitian scalar curvature of $(\omega, J, g)$. This $z^G_\omega$ is independent of $J$ by Lemma 3.4. Let $G = T$ be a maximal torus in $\text{Ham}(M,\omega)$. We obtain the following lemma

**Lemma 3.6.** For any $J \in AK^T_\omega$, the almost-Kähler metric $(\omega, J, g)$ is extremal if and only if

$$s^\nabla = z^T_\omega,$$

where $s^\nabla$ is the integral zero part of the hermitian scalar curvature $s^\nabla$ of $(\omega, J, g)$ given by $s^\nabla = s^\nabla - \frac{\int_M s^\nabla \omega^n}{\int_M \omega^n}$.

**Proof.** Let $g$ be such an extremal metric, then $X = \text{grad}_\omega s^\nabla$ is a Killing field which is invariant by $T$. Denote by $\Xi = \text{span}\{X, t\}$ where $t = \text{Lie}(T)$, then the closure of $\{\exp \Xi\}$ in the (compact) isometry group is a compact torus which contains $T$. By the maximality of the torus, we have $X \in t$. The other direction is obvious. □

**Definition 3.7.** The vector field $Z^T_\omega := \text{grad}_\omega z^T_\omega$ is called the **extremal vector field** relative to $T$.

**Remark 3.8.** The vector field $Z^T_\omega$ is invariant under isotopy of $\omega$: let $\omega_t$ an isotopy of $T$-invariant symplectic forms in $[\omega]$ with $\omega_0 = \omega$, i.e.

$$\omega_t = \omega_0 + d\sigma_t, \quad 0 \leq t \leq 1,$$

where $\sigma_t$ is a $T$-invariant 1-form. The flow $\Phi_t$ of $X_t = -\sigma_t^{\omega_t}$ (which stands for the isomorphism between $T^*(M)$ and $T(M)$ via $\omega_t$) verifies $(\Phi_t)^* \omega_t = \omega_0$ so $(\Phi_t)^*(Z^T_{\omega_0}) = Z^T_\omega$. On the other hand, as the vector field $X_t$ is $T$-invariant, $(\Phi_t)^*(Z^T_{\omega_0}) = Z^T_{\omega_t}$.

It follows that the introduced extremal vector field $Z^T_\omega$ coincides with the one defined by Futaki and Mabuchi in the Kähler case [14]. Indeed, in a fixed $T$-invariant Kähler class, any two Kähler forms $\omega_1$ and $\omega_2$ are isotopic. Therefore, $Z^T_{\omega_1} = Z^T_{\omega_2}$.

Denote by $\mathfrak{M}^T_\omega$ the set all $T$-invariant almost-Kähler metrics induced by $T$-invariant symplectic forms isotopic to $\omega$. The extremal vector field $Z^T_\omega$ is an obstruction to the existence of metrics of constant hermitian scalar curvature in $\mathfrak{M}^T_\omega$. We recall that the space $\mathfrak{M}^T_\omega$ is related (via Moser’s lemma) to the space $AK^T_\omega$. 


Corollary 3.9. If $\mathfrak{M}_ω$ contains a metric with constant hermitian scalar curvature, then $Z^T_ω = 0$. Conversely, If $Z^T_ω = 0$, any extremal metric in $\mathfrak{M}_ω$ is of constant hermitian scalar curvature.

Proof. We readily deduce from the definition of $Z^T_ω$ the first assertion of the corollary. Now, if $Z^T_ω = 0$, then $z^T_ω = 0$ for any $\tilde{ω}$ isotopic to $ω$. In particular, any extremal metric in $\mathfrak{M}_ω$ is of constant hermitian scalar curvature. □

3.3. Periodicity of the extremal vector field. In this section, we show the periodicity of the introduced vector field $Z^T_ω$ relative to a fixed maximal torus $T$ in Ham$(M, ω)$ when $[\frac{ω}{2π}]$ is integral modulo torsion.

Theorem 3.10. Assume that $[\frac{ω}{2π}]$ is integral modulo torsion. Then, there exist a positive integer $ν$ such that $\exp(2πνZ^T_ω) = 1$.

Proof. As in the previous section, we denote by $t_ω ⊂ C^∞(M)$ the finite dimensional space of smooth functions which are hamiltonians with zero mean value of elements of $t = \text{Lie}(T)$.

In the symplectic context, Futaki and Mabuchi defined in [15] a bilinear symmetric form $Φ : t \times t → \mathbb{R}$

$$Φ(X, Y) = \frac{1}{(2π)^2} \int_M f^X f^Y \frac{ω^n}{(2π)^n},$$

where $f^X, f^Y ∈ t_ω$ are momentums of $X, Y$.

By hypothesis, $[\frac{ω}{2π}]$ is an integral cohomology class modulo torsion. Under this condition, Futaki and Mabuchi showed [15] that if $X, Y$ are generators of $S^1$-actions (i.e. $\exp(X) = \exp(Y) = 1$), then $Φ(X, Y) ∈ Q$. To prove our claim, we have to show that $2πΦ(X, Z^T_ω) ∈ Q$ for all $1 ≤ i ≤ k$, where $Z^T_ω$ is the extremal vector field relative to $T$ and $X_1, \cdots, X_k$ are generators of the torus action. This would imply $2πZ^T_ω ∈ \bigoplus_{i=1}^k QX_i$, so $\exp(2πνZ^T_ω) = 1$ for some positive integer $ν$.

To show our claim, we recall Nakagawa’s modified version [25] of Tian’s formula [27]

$$\frac{(n + 1)!2^{n-1}n!}{2π} \int_M f s^\nabla, \frac{ω^n}{(2π)^n} = \sum_{j=0}^n (-1)^j \binom{n}{j} \int_M \left[ \frac{1}{2} Δ^g f + ρ^ν \right] + (N + n - 2j) \left( \frac{f}{2π} + \frac{ω}{2π} \right)^{n+1}

- \left( N + \frac{nμ}{n + 1} \right) 2^n(n + 1)! \int_M \left( \frac{f}{2π} + \frac{ω}{2π} \right)^{n+1},$$

where $f$ is a smooth function on $M$, $s^\nabla$ is the zero integral part of the hermitian scalar curvature $s^ν$ of $(ω, J, g)$, $ρ^ν$ is the hermitian Ricci form, $μ = \int_M ρ^ν ω^{n-1}$, $N$ is an integer and $Δ^g$ is the Riemannian Laplacian with respect to $g$. The formula (3.7) takes in account the normalization (3.1) and is a direct consequence of the fact that $\int_M (Δ^g f) ω^n = 0$ and the identities

$$\sum_{j=0}^n (-1)^j \binom{n}{j} (n - 2j)^k = 0 \text{ if } k < n \text{ or } k = n + 1,$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} (n - 2j)^{n} = 2^n n!.$$
As \( \Phi \) is non-degenerate and \( 2\pi \Phi(X, Z_T^\omega) = \frac{1}{2\pi} \int_M f^X z_T^\omega \omega_n = \frac{1}{2\pi} \int_M f^X s^\omega \omega_n \), we thus reduced the problem to show that the right hand side of (3.7) is rational when \( f = f^X \in T_\omega \) is the momentum with respect to \( \omega \) of a hamiltonian vector field \( X \) generating an \( S^1 \)-action. We have then essentially two integrals in the right hand side of (3.7) which are \( \int_M f^X \omega_n \) and \( \frac{1}{2\pi} \int_M (\frac{1}{2} \Delta g f^X + (N + n - 2j) f^X) \bar{\omega}^n \) with \( \bar{\omega} = \frac{1}{2\pi} + (N + n - 2j) \frac{d}{2\pi} \).

In order to compute the latter integral, we will use the localization formula (2.16). Since \( M \) is compact, we consider an integer \( N \) large enough such that \( \bar{\omega} \) is symplectic. By Lemma (2.6) the vector field \( X \) verifies
\[
2\pi \bar{\omega}(X, \cdot) = -d \left( \frac{1}{2} \Delta g f^X + (N + n - 2j) f^X \right).
\]

On the other hand, Futaki and Mabuchi showed [15] that if \( \int_M f^X \omega_n = 0 \) then \( f^X(p) \in 2\pi \mathbb{Q} \) for any fixed point \( p \in M \) of the \( S^1 \)-action. Moreover, we have \( (\Delta g f^X)(p) = -tr (D^g df^X)_p = tr (JD^g X)_p \), Note that \( (JD^g X)_p \) is a symmetric hermitian endomorphism which is the generator of the induced linear \( S^1 \)-action on \( T_p(M) \) (see e.g. [22]). This implies that the trace \( tr (JD^g X)_p \in 2\pi \mathbb{Z} \) and therefore \( (\Delta g f^X)(p) \in 2\pi \mathbb{Z} \) By (2.16) and using the power series of the exponential function in \( h \), we obtain
\[
\frac{1}{2\pi} \int_M \left( \frac{1}{2} \Delta g f^X + (N + n - 2j) f^X \right) \bar{\omega}^n \in \mathbb{Q}.
\]
This concludes the proof. \( \square \)

3.4. The Hermitian-Einstein condition.

**Definition 3.11.** An almost-Kähler metric \((\omega, J, g)\) is called Hermitian-Einstein if the hermitian Ricci form \( \rho^\nabla \) is a (constant) multiple of the symplectic form \( \omega \), i.e.
\[
\rho^\nabla = \frac{s^\nabla}{2n} \omega,
\]
so the hermitian scalar curvature \( s^\nabla \) is constant.

**Corollary 3.12.** If \( c_1(M, \omega) \) is a multiple of \( [\omega] \) and \( Z_T^\omega = 0 \), then an extremal almost-Kähler metric \((J, g)\) in \( \mathfrak{M}_T^\omega \) is Hermitian-Einstein if and only if \( \rho^\nabla \) is \( J \)-invariant.

**Proof.** Let \((\bar{\omega}, J, g)\) be an extremal almost-Kähler metric in \( \mathfrak{M}_T^\omega \). By Corollary 3.10 and since \( Z_T^\omega = 0 \), we deduce that the hermitian scalar curvature \( s^\nabla \) of \((\bar{\omega}, J, g)\) is constant. To prove our claim, it is enough to show that \( \rho^\nabla \) is co-closed (and therefore harmonic), i.e. \( \delta^g \rho^\nabla = 0 \), where \( \delta^g \) is the codifferential with respect to \( g \). Indeed, as \( \rho^\nabla \) and \( \bar{\omega} \) be two harmonic forms representing the same cohomology class up to a multiple, Hodge theory implies there are equal up to the same multiple.

Denote by \( d^c \) the twisted exterior differential with respect to \( J \) and \( \Lambda_{\bar{\omega}} \) the contraction by \( \bar{\omega} \). By hypothesis \( \rho^\nabla \) is \( J \)-invariant, then \( d^c \rho^\nabla = 0 \). Now, using the relations (2.8) and (3.2), we have \( \delta^g \rho^\nabla = [\Lambda_{\bar{\omega}}, d^c] \rho^\nabla = -d^c \Lambda_{\bar{\omega}} \rho^\nabla = -\frac{i}{2} d^c s^\nabla = 0 \) since \( s^\nabla \) is constant. Therefore, \( \rho^\nabla \) is co-closed. The other direction is obvious. \( \square \)
4. Explicit formula of the hermitian scalar curvature

Let \((M^{2n}, \omega, J, g)\) be an almost-Kähler manifold. By Darboux theorem, there exist coordinates \(\{z_i, t_i\}\) defined on an open set \(U\) such that the symplectic form \(\omega\) has the form \(\omega = \sum_{i=1}^n dz_i \wedge dt_i\) on \(U\); \(\{z_i, t_i\}\) are called the Darboux coordinates. In this section, we give an explicit formula of the hermitian scalar curvature \(s^\nabla\) of \((\omega, J, g)\) in terms of the coordinates \(\{z_i, t_i\}\). On \(U\), the metric \(g\) is of the form,

\[
g = \sum_{i,j=1}^n G_{ij}(z,t) dz_i \otimes dz_j + H_{ij}(z,t) dt_i \otimes dt_j + P_{ij}(z,t) dz_i \otimes dt_j,
\]

where \(G = (G_{ij}), H = (H_{ij})\) are symmetric positive-definite matrix-valued functions which satisfy the compatibility conditions \(GH - P^2 = Id\) and \(HP = tP\) (where \(tP\) denote the transpose of \(P = (P_{ij})\)). We define a local section \(\Phi\) of the anti-canonical bundle \(K_{-1}^J(M)\) by \(\Phi := (K_1^J - \sqrt{-1} JK_2^J) \wedge \cdots \wedge (K_n^J - \sqrt{-1} JK_n^J)\) where \(K_1 = \frac{\sqrt{-1} t g}{\det H}\) and \(t\) stands for the isomorphism between \(T(M)\) and \(T^*(M)\) via \(g\). Let \(\phi\) and \(\psi\) the real \(n\)-forms such that \(\Phi = \phi + \sqrt{-1} \psi\) (in fact \(\psi\) and \(\phi\) are related in the following way \(\psi(JX_1, \ldots, X_n) = \phi(X_1, \ldots, X_n)\)). We still denote by \(\nabla\) the hermitian connection induced on \(K_{-1}^J(M)\) by the canonical hermitian connection. For any vector field \(X\), we have

\[
\nabla_X \phi = a(X)\phi + b(X)\psi
\]

for some real 1-forms \(a, b\) (we can deduce that \(a = \frac{1}{2} d \ln |\phi|^2\)). Moreover, the hermitian Ricci form \(\rho^\nabla\) is given by \(\rho^\nabla = db\). Now, span \(\{K_1, \ldots, K_n\}\) is Lagrangian. Hence, \(\Phi(K_1, \ldots, K_n) = \phi(K_1, \ldots, K_n) = \det g(K_i, K_j) = \det H\) which implies that

\[
(\nabla_X \Phi)(K_1, \ldots, K_n) = (a(X) - ib(X))\Phi(K_1, \ldots, K_n) = (a(X) - ib(X)) \det H.
\]

Let \(\beta(X) := (\nabla_X \Phi)(K_1, \ldots, K_n) / \det H\), then \(\beta = \text{trace} \ (H^{-1} \circ B)\) where \(B(X) = (g(\nabla_X K_i, K_j) + \sqrt{-1} g(\nabla_X K_i, JK_j)) = (2g(\nabla_X K_i^{1,0}, K_j))\). In particular,

\[
(4.1) \quad B(X^{0,1}) = \left(2g(\nabla_X^{0,1} K_i^{1,0}, K_j)\right).
\]

Using Proposition \(2.1\) we can compute, via \(4.1\), the imaginary part of \(B\) and therefore \(\rho^\nabla\) and \(s^\nabla\). If we denote by \(H^{-1} = (H^{ij})\) the inverse of \(H = (H_{ij})\) and
ing Frobenius’ theorem, there exist local coordinates \( \{ \}

\text{is involutive. This condition is automatically satisfied in the K"ahler case . Then, us-

\text{expression } \omega

\text{is generated by a family of hamiltonian vector fileds

\text{fold equipped with an effective hamiltonian action of an

\text{4.1. The toric case. A toric symplectic manifold } \text{is a symplectic mani-

\text{fold equipped with an effective hamiltonian action of an } \text{n-dimensional torus } T. \text{ It is generated by a family of hamiltonian vector fileds } \{ K_1, \cdots, K_n \} \text{ which are linearly independant on a dense open set } \text{and satisfy the condition } \omega(K_i, K_j) = 0 \text{ for all } i, j. \text{ It is well-known that the symplectic form } \omega \text{ has the following local ex-

\text{pression } \omega = \sum_{i=1}^{n} dz_i \wedge dt_i, \text{ where } z_i \text{ is the momentum coordinate and } t_i \text{ is a local coordinate such that } K_i = \frac{\partial}{\partial t_i}. \text{ So, any } \omega\text{-compatible } T\text{-invariant almost-K"ahler}

\text{metric } g \text{ has the local expression

\text{An almost-K"ahler manifold } \text{is called } \text{locally toric if in local coordinates }

\text{the symplectic form is written as } \omega = \sum_{i=1}^{n} dz_i \wedge dt_i \text{ and the almost-K"ahler metric } g \text{ has the form } \text{4.3.}

\text{We deduce from } \text{4.2 } \text{and } \text{4.3 } \text{the expressions of the hermitian Ricci form } \rho^\n
\text{and the hermitian scalar curvature } s^\n
\text{generalizing the formula of } s^\n
\text{given by Abreu } \text{in the integrable case and redis-

\text{covering the expression found by Donaldson } [9].}

\text{Now, we suppose that the (lagrangian) } g\text{-orthogonal distribution to the } T\text{-orbits

\text{is involutive. This condition is automatically satisfied in the K"ahler case. Then, us-

\text{ing Frobenius’ theorem, there exist local coordinates } \{ t_i \} \text{ such that } \{ dt_1, \cdots, dt_n \}

\text{span the annihilator of the orthogonal distribution to the } T\text{-orbits and } \omega = \text{...}
\[ \sum_{i=1}^{n} dz_i \wedge dt_i. \]  
Any such \( \omega \)-compatible \( T \)-invariant almost-Kähler metric \((\omega, g)\) has the local expression
\[ g = \sum_{i,j=1}^{n} \left( G_{ij}(z) dz_i \otimes dz_j + H_{ij}(z) dt_i \otimes dt_j \right) \text{ and } \omega = \sum_{i=1}^{n} dz_i \wedge dt_i. \]

Therefore, the expression \((4.5)\) of \( \rho^\nabla \) further simplifies
\[ \rho^\nabla = -\frac{1}{2} \sum_{i,j,k,l} H_{i,l,i} dz_k \wedge dt_l. \]

**Remark 4.1.** We can deform an \( \omega \)-compatible toric metric of the diagonal form \((4.7)\) by considering \( H' = H + \epsilon U \) and \( G' = (H')^{-1} \) for some non-negative definite matrix-valued function \( U = (U_{ij}) \), say with compact support on an open set and \( \epsilon \) small enough. The fact that the right hand side of \((4.6)\) is under-determined linear differential operator implies that there are infinite dimensional families of \( \omega \)-compatible extremal almost-Kähler metrics around an extremal almost-Kähler metric of the form \((4.7)\). In particular, infinite dimensional families of non-integrable extremal almost-Kähler metrics do exist around any extremal Kähler toric metric.

**Lemma 4.2.** Let \((M^4, \omega, J_0, g_0)\) be an almost-Kähler manifold such that \((\omega, g_0)\) have the form \((4.7)\) on some open set \( V \) of \( M \). Then, there exists an infinite dimensional family of almost-Kähler metrics \((\omega, J_\epsilon, g_\epsilon)\), defined for a sufficiently small \( \epsilon \), such that \( \rho^\nabla = \rho^\nabla_0 \) for all \( \epsilon \); here \( \nabla_\epsilon \) is the canonical hermitian connection corresponding to \( J_\epsilon \). Moreover, if \((\omega, J_0, g_0)\) is Kähler, we obtain an infinite dimensional family of non-integrable \( \omega \)-compatible almost-Kähler metrics.

**Proof.** Let \( H'_{ij}(z) = H_{ij}(z) + \epsilon U_{ij}(z) \), where \( \epsilon \) is a real, \( U_{ij} = f_{ij}(z_1) h_{ij}(z_2) \) with \( f_{ij} = f_{ji}, h_{ij} = h_{ji} \). Now, the condition \( \rho^\nabla - \rho^\nabla_0 = 0 \) gives the following system of O.D.E.'s
\[ \sum_{k=1}^{2} \left( f_{ik}(z_1) h_{ik}(z_2) \right)_{kj} = 0, \]
which reduces to the relations \( f'_{12} = \alpha f_{22}, f'_{11} = \beta f_{12}, h'_{12} = -\beta h_{11}, \) and \( h'_{22} = -\alpha h_{12} \) for some (real) constants \( \alpha, \beta \) which we assume non-zero. Choosing \( f_{11} \) and \( h_{22} \) arbitrary, the above relations determine the remaining functions \( f_{12}, f_{22}, h_{11}, h_{12} \).

In particular, for \( f_{11} \) and \( h_{22} \) with compact support on \( V \), \( U_{ij} \) has a compact support on \( V \). This ensures that for a sufficiently small \( \epsilon \), \( H' = (H'_{ij}) \) is positive-definite.

Letting \( G' = (H')^{-1} \), we obtain the \( \omega \)-compatible metric
\[ g_\epsilon = \left\{ \begin{array}{ll} \sum_{i,j=1}^{2} G'_{ij}(z) dz_i \otimes dz_j + H'_{ij}(z) dt_i \otimes dt_j & \text{on } V, \\ g_0 & \text{elsewhere}. \end{array} \right. \]

One can check directly that for a generic choice of \( f_{11} \) and \( h_{22} \), the corresponding almost-complex structure \( J_\epsilon \) is non-integrable.

**Corollary 4.3.** Let \((M^4, \omega, J, g)\) be a Kähler-Einstein (complex) surface which is locally toric. Then, \( \omega \) admits an infinite dimensional family of non-integrable, \( \omega \)-compatible almost-complex structures inducing Hermitian-Einstein almost-Kähler metrics.

**Examples 4.4.** The corollary applies to the toric Kähler-Einstein surfaces \( \mathbb{CP}^2 \), \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), and \( \mathbb{CP}^1 \# 3 \mathbb{CP}^2 \), but also to the locally symmetric Hermitian-Einstein spaces \( \mathbb{CH}^2 / \Gamma \) and \( (\mathbb{CH}^1 \times \mathbb{CH}^1) / \Gamma \).
4.2. Extremal almost-Kähler metrics saturating LeBrun’s estimates. On an almost-Kähler manifold \((M^4, \omega, J, g)\) of dimension 4, the bundle of 2-forms decomposes as

\[ \Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M), \]

where \(\Lambda^\pm(M)\) correspond to the eigenvalues \((\pm 1)\) under the action of the (Riemannian) Hodge operator \(\ast\) (see e.g. [3]). This decomposition is related to the splitting (2.2) as follows

\[ \Lambda^+(M) = \mathbb{R} \cdot \omega \oplus \Lambda^{J^+}(M) \quad \text{and} \quad \Lambda^-(M) = \Lambda^{J^+}_0(M). \]

The (Riemannian) curvature \(R\), viewed as a (symmetric) linear map of \(\Lambda^2(M)\), decomposes as follows

\[ R = \begin{pmatrix} W^+ + \frac{\omega}{12} \text{Id}_{\Lambda^+(M)} & \frac{\bar{r}_0}{2} \text{Id}_{\Lambda^-(M)} \\ \frac{\bar{r}_0}{2} \text{Id}_{\Lambda^+(M)} & W^- + \frac{\omega}{12} \text{Id}_{\Lambda^-(M)} \end{pmatrix} \]

where \(W^\pm\) are symmetric trace-free endomorphisms of \(\Lambda^\pm(M)\) respectively acting trivially on \(\Lambda^+(M)\) and \(\bar{r}_0\) is the (symmetric) operator defined on \(\Lambda^2(M)\) as \(\bar{r}_0(X \wedge Y) = r_0(X \wedge Y) + X \wedge r_0(Y, \cdot)\); here \(r_0 = r - \frac{\omega^2}{2}\) is the trace-free part of the Ricci tensor \(r\) which is the trace of the Riemannian curvature \(R\) and \(s\) is the Riemannian scalar curvature defined as the trace of the Ricci tensor (for more details see [21]). The tensor \(W^+\) (resp. \(W^-\)) is called the selfdual Weyl tensor (resp. anti-selfdual Weyl tensor).

We recall now a (weak version) of LeBrun’s result in [20, Proposition 2.2].

**Proposition 4.5.** Let \((M^4, \omega)\) be a compact symplectic 4-manifold and \(g \in \text{AK}_\omega\) be a \(\omega\)-compatible almost-Kähler metric. Then

\[ V^\frac{1}{4} \left( \int_M \left( \frac{2}{3} s + 2e \right) \right)^\frac{3}{2} \geq 32\pi^2 |c_1^+|_{L^2}, \]

where \(V = \int_M \omega^2\) is the total volume of \((M^4, \omega)\), \(e(x)\) is the lowest eigevalue of \(W^+\) at \(x\), for any real-valued function \(f\) on \(M\) : \(f_-(x) = \min(f(x), 0)\) and \(c_1^+\) denotes the self dual part of the \(g\)-harmonic 2-form representing \(c_1(M, \omega)\).

Moreover, the equality holds in (4.9) if and only if \(g\) is an extremal almost-Kähler metric with negative constant hermitian scalar curvature \(s^\mathbb{C}\) and at each point \(\omega\) is an eigenform of \(W^+\) corresponding to its lowest eigenvalue.

Corollary 4.3 applied to the (complex) surfaces \(\mathbb{C} \mathbb{H}^2/\Gamma\) and \((\mathbb{C} \mathbb{H}^3 \times \mathbb{C} \mathbb{H}^3)/\Gamma\), provides examples of non-integrable extremal almost-Kähler metrics saturating the inequality (4.12). Indeed, for a Hermitian-Einstein almost-Kähler metric, \(\rho^\mathbb{C}\) is \(J\)-invariant and therefore \(\rho^\ast = R(\omega)\) is (see Sec. 2.3). Since \(R(\omega) = W^+ + \frac{\omega}{12}\), we deduce that \(\omega\) belongs to the eigenspace of \(W^+\) if and only if \(\rho^\mathbb{C}\) is \(J\)-invariant. Recall that in the Kähler case the Riemannian curvature \(R\) and \(\bar{r}_0\) act trivially on \(\Lambda^{J^+}_0(M)\), and the selfdual Weyl tensor \(W^+\) decomposes as

\[ W^+ = \begin{pmatrix} \frac{s}{6} & 0 & 0 \\ 0 & -\frac{s}{12} & 0 \\ 0 & 0 & -\frac{s}{12} \end{pmatrix}. \]

For the Kähler-Einstein metric on \(\mathbb{C} \mathbb{H}^2/\Gamma\) and \((\mathbb{C} \mathbb{H}^3 \times \mathbb{C} \mathbb{H}^3)/\Gamma\), \(s = s^\mathbb{C}\) is negative and thus \(\omega\) belongs to the lowest eigenspace of \(W^+\). Then, by Corollary 4.3 we...
obtain an infinite dimensional family of non-integrable Hermitian-Einstein almost-Kähler metrics whose the almost-Kähler form belongs to the lowest eigenspace of $W^+$ with non-positive constant hermitian scalar curvature $s^\nabla$. Other examples of non-integrable almost-Kähler metrics saturating the inequality (4.9) appear in [26].

5. Further Remarks and Questions

(1) In [9], it is shown that in the toric case the existence of an extremal almost-Kähler metric is closely related (and conjecturally equivalent) to the existence of an extremal Kähler metric; this was further generalized to certain toric bundles in [4]. It will be interesting to establish a similar link in general, assuming that there are integrable complex structures in $AK^T_\omega$.

(2) One expects that there will be infinite families of extremal almost-Kähler metrics (should they exist) in $AK^T_\omega$, thus generalizing the results of [18] and Example 4.4. Indeed, this is suggested from the GIT formal picture in [8], where the existence and uniqueness (modulo the action of $\text{Ham}(M,\omega)$) of the extremal almost-Kähler metrics are expected to hold within a ‘stable’ complexified orbit for the action of $\text{Ham}(M,\omega)$ on $AK_\omega$. However, as $\text{Ham}(M,\omega)$ does not have a nature complexification, the description of the complexified orbits is not obvious and is clearly established only on Kähler manifolds with $H^1(M,\mathbb{R}) = \{0\}$ for the action of $\text{Ham}(M,\omega)$ on the subset $K_\omega$ of integrable, $\omega$-compatible almost-complex structures: In this case, the complexified orbit of $J \in K_\omega$ is identified in [8] (via Moser’s lemma) with the space of Kähler metrics within the Kähler class $\Omega = [\omega]$ on $(M,J)$. In the absence of holomorphic vector fields on $(M,J)$, the stability theorem of [21] and [13] implies that the existence of an extremal Kähler metric in a given complexified orbit is an open condition on the space of such orbits; this was generalized in [4], by fixing a maximal torus $T \subset \text{Auto}_0(M,J) \cap \text{Ham}(M,\omega)$ (where $\text{Auto}_0(M,J)$ is the connected component of the finite dimensional group of complex automorphisms of $(M,J)$) and considering $T$-invariant $\omega$-compatible complex structures. We expect the openness result to generally hold in the non-integrable case, by considering suitable complexified orbits in $AK^T_\omega$.

(3) A well-know result of Calabi states that any extremal Kähler metric $(\omega,g)$ on a compact complex manifold $(M,J)$ is invariant under a maximal connected compact subgroup $G$ of $\text{Auto}(M,J) \cap \text{Ham}(M,\omega)$. It will be desirable to know whether or not such a $G$ (or a maximal torus in it) is also maximal as a compact subgroup of $\text{Ham}(M,\omega)$? More generally, one would like to know whether or not an extremal almost-Kähler metric is necessarily invariant under a maximal torus in $\text{Ham}(M,\omega)$?

(4) It would be interesting to know how does the extremal vector field $Z^T_\omega$, which we introduced as an invariant of a maximal torus $T$ in $\text{Ham}(M,\omega)$, characterize this torus up to conjugacy in $\text{Symp}(M,\omega)$ or $\text{Ham}(M,\omega)$; elaborating on the theory appearing in recent works [2, 17] would be an appealing direction of further investigation.

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