Statistical Majorana Bound State Spectroscopy

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Tunnel spectroscopy data for the detection of Majorana bound states (MBS) is often criticized for its proneness to misinterpretation of genuine MBS with low-lying Andreev bound states. Here, we propose a protocol removing this ambiguity by extending single shot measurements to sequences performed at varying system parameters. We demonstrate how such sampling, which we argue requires only moderate effort for current experimental platforms, resolves the statistics of Andreev side lobes, thus providing compelling evidence for the presence or absence of a Majorana center peak.

Introduction.—About a decade after the first proposals for MBS engineering in topological quantum devices [1–4], numerous reports of experimental signatures have been published, see, e.g., Refs. [5–10]. However, opinions remain divided as to whether “Majoranas have been seen” or not. Broadly speaking, experiments aimed at MBS detection can be categorized into two groups: tunnel spectroscopy detecting midgap resonances caused by the assumed presence of an MBS [11–15], and experiments going after unambiguous intrinsic properties of topological states, from unconventional noise correlations [16–24] to full-feathered braiding protocols [25–28].

While the second group remains at the level of theoretical proposals, the former are straightforwardly realizable as a part of the core MBS experiments. However, the downside is that tunnel spectroscopy data can be prone to misinterpretation. Among various other candidates for midgap signatures, pairs of conventional Andreev bound states — which in symmetry class D [29, 30] superconductor environments 1 have a tendency to cluster around zero energy — may leave experimental signatures hard to distinguish from a single MBS [31–38]. At any rate, as witnessed by the current debate on the “topological gap protocol” by the Microsoft Quantum team [10, 39, 40], the community at large does not appear to be ready to take tunnel spectroscopy signatures, even of high quality, as unambiguous evidence for MBS formation.

In this paper, we propose a relatively straightforward upgrade from single shot tunnel spectroscopy measurements to parametric sequences of measurements. Their realization for individual samples neither requires essential new hardware nor measurement protocols beyond what is already available. We argue that the compounded measurement data collected by statistical tunnel spectroscopy does contain compelling evidence for or against MBS formation. Crucially, both the presence and the absence of an MBS will leave unique imprints, provided the required statistical resolution has been met. A second key feature is that disorder or device imperfections, usually considered as unwelcome obstructions to MBS observability [41–46], here assume the role of a resource: our approach works best for significantly disordered systems.

To understand its principle, we need to recall a few signatures of the spectrum of class D superconductors [29–31]. In confined geometries subject to disorder or other sources of “integrability breaking”, the Andreev spectrum is discrete, symmetric around zero energy, and subject to statistical level correlations. Specifically, in the absence of topological midgap states, Andreev bound states exhibit a slight statistical tendency to attraction to zero energy, while they repel amongst themselves. Conversely, if a topological midgap state is present, Andreev states get pushed away from zero energy, and still repel amongst themselves. These signatures find a quantitative representation in the ensemble-averaged spectral density [30, 31],

\[
\langle \rho(\epsilon) \rangle = \frac{1 + c}{2} \frac{\delta(\epsilon)}{\delta_c} + \frac{1}{\delta_c} \left( 1 - c \frac{\sin(2\pi \epsilon / \delta_c)}{2\pi \epsilon / \delta_c} \right),
\]

(1)

where \( \delta_c \) is the average (Andreev bound state) energy-spacing and \( c = +1 (c = -1) \) in the presence (absence) of a MBS. The sinusoidal oscillations in Eq. (1) describe a tendency of the spectrum to “crystallize” into a statistically uniform sequence around zero, with diminishing (\( \sim \epsilon^{-1} \)) rigor. Equation (1) encodes a nonlocal fragmentation of the Hilbert space and is obtained under the idealizing assumption of an infinite ensemble subject to disorder strong enough to couple a large number of levels (random matrix limit [30, 47]).

In experimental reality, there is no mathematical ensemble, disorder may not be quite so strong, and the recorded tunnel conductance data contains wave function fluctuations next to spectral signatures. Further, the MBS peak, if present, will be broadened by spectroscopic resolutions, temperature, and possibly other forms of environmental coupling. However, as we are going to

1 Strictly speaking, the system is either in class B or in class D depending on whether a MBS is present or not. For simplicity, we refer to both cases as “class D.”
argue, and demonstrate by numerical simulations, even relatively small sequences of measurements performed for an engineered ensemble of configurations at limited resolution can reveal the principal signature of the spectral data: a statistical oscillation of period $\delta_\nu$ with opposite sign, depending on the presence or absence of Majorana states. In other words, a positive sign signal assumes the role of a control measurement revealing sufficient resolution for what in the presence of a MBS must flip sign to become a negative sign sequence. These rigidity patterns are deeply non-perturbative signatures of the class D spectrum, which in the $c = 1$ case do require a single midgap state (aka Majorana). On this basis, we reason that a “smoking gun” signature is at hand. While our approach in principle applies to arbitrary Majorana platforms, we illustrate it below for the example of a proximitized topological insulator (TI) slab pierced by a vortex [48–54], where Andreev states correspond to Caroli-de Gennes-Matrion subgap states [55]. In addition, in the SM [56], we comment on alternative implementations in iron-based superconductors [9, 57–64], planar phase-controlled Josephson junctions [65–68], or semiconductor hybrid nanowires [7, 27, 69].

**Statistical spectroscopy principles.**—We propose a protocol where an effectively averaged spectral density $\langle \rho(\epsilon) \rangle$ is obtained by variation of external control parameters. To understand the principle, we note that if integrability is broken by impurities and/or asymmetric system boundaries, the variation of any system parameter will result in new realizations of the chaotic scattering potential [70]. Similar approaches have previously been applied in semiconductor devices [71] and nanowires [72] for generating effective ensemble averages of the tunneling conductance. As concrete example, we here formulate the approach for a TI vortex, cf. Fig. 1(a): An $s$-wave superconductor is deposited on a TI surface except for a circular region of radius $R$. Through this region an integer number, $\nu$, of superconducting flux quanta $\Phi_0 = \pi/e$ is threaded ($\hbar = 1$ throughout). For odd parity of $\nu$, this synthetic vortex binds a zero energy MBS [49].

Variations in the voltage of nearby finger gates, $V_g$, parametrically change the system Hamiltonian. Even in the absence of “intrinsic” disorder, they break integrability and realize an effective ensemble average, provided the perturbation is strong enough to effectively scramble the spectrum of vortex states, cf. Fig.1(b). To estimate the required voltage variations, we make the conservative assumption that the Coulomb interaction across the vortex is strongly screened, and that only local wave functions right under the geometric finger gate surface are susceptible to the perturbation. To first order in perturbation theory, this leads to the estimate $|\langle \Psi \mid \delta V_g \mid \Psi \rangle| \approx \delta V_g \int d^2r |\Psi|^2 \approx \tau_g \delta V_g$ for the distortion of the energy, $\epsilon$, of individual states. Here, the integral extends over the area underneath the gate, we assume approximate statistical uniformity of the wave function modulus, and $0 \leq \tau_g \leq 1$ is the fraction of the gate area relative to that of the vortex. Variations $\delta V_g \gtrapprox \delta_\epsilon/\tau_g$ strong enough that the perturbation exceeds the level spacing, $\delta_\epsilon$, effectively define a new realization of the spectrum, cf. Fig.1(b).

Provided the broadening, $\kappa$, of Andreev states due to disorder exceeds the level spacing $\delta_\epsilon$, we expect level repulsion, and in the consequence the emergence of the spectral density in Eq. (1) upon averaging over an ensemble. Presently, this ensemble average is realized by sampling a large number of configurations distinguished by changes $\delta V_g / \delta_\epsilon = \mathcal{O}(1)$, and subsequently collecting the results in a histogram.

In a concrete experiment where each level spacing is divided into $N_b$ bins and the number of runs is $N_r$, an average number of $n_b = N_r / N_b$ levels will be counted per bin. This number is subject to statistical fluctuations $\delta n_b \sim \mathcal{O}(n_b^{1/2})$. To obtain a reliable result, the relative fluctuation $\delta n_b / n_b$ must be smaller than the relative change $|\rho(\epsilon) - \rho(\infty)|/\rho(\infty)$, computed according to Eq. (1). A straightforward estimate for, say, $\epsilon \approx 2\delta_\epsilon$, leads to the conclusion that $N_r \approx 10^2 N_b$ runs are required to obtain statistical certainty.

**TI vortex.**—In the following, we test the statistical protocol for the TI vortex setup in Fig. 1(a). The single-particle Bogoliubov-de Gennes Hamiltonian describing the proximitized TI surface is given by [49]

$$H_{BdG} = (v\mathbf{p} \cdot \sigma - \mu)\tau_z + \text{Re} \Delta(\mathbf{r}) \tau_x - \text{Im} \Delta(\mathbf{r}) \tau_y,$$  

where $v$ is the surface-state velocity, $\mu$ the chemical potential, and Pauli matrices $\tau_i$ ($\sigma_i$) act in particle-hole (spin) space. In the London gauge, the pair potential is $\Delta(\mathbf{r}) = |\Delta(\mathbf{r})| e^{-i\alpha}$, with polar coordinates $(r, \theta)$ relative to the conclusion that $N_r \approx 10^2 N_b$ runs are required to obtain statistical certainty.

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FIG. 2. Histogram for the positive energy levels of the TI vortex, with energies in units of $E_{\text{cl}} = v/R$. Panels (a)–(d) [(e)–(h)] show numerical results in the presence [absence] of a MBS with increasing disorder strength as obtained by diagonalizing $H_{\text{BdG}}$ for $N_{r} = 5 \times 10^{4}$ disorder realizations. Green, orange and cyan colors refer to the three lowest levels, all others are represented by the grey background. For weak disorder, $\ell \gg R$ [panels (a,e)], the averaged spectral peaks lie isolated, they begin to overlap when $\ell \sim R$ [panels (b,c) and (f,g)], and finally combine to a continuum described by Eq. (1) at $\ell < R$ [panels (d,h)]. The statistics of the lowest level is accurately described by the spacing distribution $P(\epsilon)$ (red curves) discussed in the text.

diffusive. For stronger disorder, the characteristic level spacing shrinks to $\delta_{\epsilon} \sim \ell(R)/(v/R) \equiv E_{\text{Th}}$, i.e., from the inverse of the ballistic time of flight, $E_{\text{cl}}$, to the inverse of the diffusion time across the vortex, $E_{\text{Th}}$, see the SM [56] for details.

Our statistical approach to MBS spectroscopy works for disorder beyond the ballistic/diffusive threshold. To illustrate this point, Fig. 2 shows data histograms obtained from $N_{r} = 5 \times 10^{4}$ disorder realizations and for disorder strengths ranging from an almost perfectly ballistic regime, $\ell = 25R$, to a diffusive one with $\ell = 0.1R$. The columns on the left (right) are for a vortex with (without) MBS, realized here by setting $\nu = 1$ ($\nu = 2$). In the ballistic regime, we observe weakly broadened states with spacings varying strongly at scales $\sim E_{\text{cl}}$. Upon crossing into the diffusive regime, they start to overlap, along with a tendency towards a more uniform spacing — the level crystallization symptomatic for quantum chaotic spectra.

Real experiments have access to the cumulative contribution of all levels, here indicated in grey, where we observe the gradual approach to the profile in Eq. (1), as well as to the distribution of individual levels, cf. the green/orange/cyan histograms for the lowest three positive energy levels. For disorder deep in the diffusive regime, we expect the statistics of these levels to be described by the principles of random matrix theory [30, 47]. Specifically, for class D one expects the probability distribution for the lowest lying level in the case with [without] MBS to be given by $P(\epsilon) \propto \epsilon^{2} \exp(-\epsilon^{2}/2)$ $[P(\epsilon) \propto \exp(-\epsilon^{2}/2)]$ [29, 30, 56]. Figure 2 shows that these distributions, indicated as red curves, are clearly realized by the disordered vortex in the strong disorder regime. However, the most important conclusion is that the presence or absence of a MBS is clearly resolved via the statistics of the cumulative histogram, provided the focus of attention is shifted to the side bands, and the disorder is sufficient to induce inter-level correlations.

**Experimental reality.—** The analysis above assumed arbitrary energy resolution, and averaging over a large number $O(10^{3})$ of realizations. What happens under less ideal conditions? In an experiment, the potential $V(r)$ describing impurities or scattering off device irregularities is fixed and different realizations of the spectrum are generated by variation of externally adjustable parameters. In the TI vortex, the magnetic field strength is likewise fixed, which leaves gate electrodes as the next best choice for generating a parameter set. To generate $N_{r} \sim 10^{2}N_{b}$ samples required for $N_{b}$ bins per level spacing (see above estimate), one may need to work with $f$ finger gates and the resulting $f$-dimensional parameter space. As the gate voltages are meant to mimic “disorder”, it is best to use an asymmetric geometric design as indicated in Fig. 1. Electrodes with large electrode-to-vortex area ratio, $r_{g}$, will generate optimal sensitivity of energy levels, $\sim r_{g}V_{g}$.

The simulations discussed in the following were per-
FIG. 3. Histogram as in Fig. 2 but for a single realization of \( V(r) \) with \( \ell = 0.1R \), using an increasing number of \( N_r \) samples with (left column) and without (right) MBS. The first five extrema are indicated by dashed vertical lines. The green (orange) curves are fits to Eq. (1) for \( c = 1 \) (\( c = -1 \)) within the dark blue regions, i.e., with the \( \delta \)-peak removed, using \( \delta_r \) as single fit parameter. Inferior fits are shown as dashed curves.

formed for \( N_b = 10 \) bins, requiring \( N_r = \mathcal{O}(10^3) \) runs. We worked with \( f = 3 \) electrodes [56], and varying each of their voltages over a range \( \delta V_g \sim \delta_r/r_g \) generated the parameter space for up to \( N_r \approx 10^4 \) statistically independent samples. Finally, we account for the broadening of individual levels due to temperature or environmental coupling by introducing a Lorentzian line width, \( \delta(\epsilon) \sim \Gamma/(\Gamma^2 + \epsilon^2) \), with \( \Gamma = 0.05E_{cl} \). Here, \( \Gamma \lesssim \delta \epsilon \) is required to resolve the oscillatory pattern of the target spectral density, i.e., our method requires the resolvability of individual states.

Given this setup, the minimal goal is a statistically sound distinction between the cases \( c = \pm 1 \) in Eq. (1). Figure 3 illustrates how the two cases are distinguished through a phase shift in the oscillatory spectral density at finite energy. In either case, a midgap peak is present (caused by a broadened MBS for \( c = +1 \), or a statistical accumulation of Andreev states for \( c = -1 \)). While these two peaks are difficult to distinguish, our method focuses on the spectrum away from the center. We note that in either case, the average spectral density contains a sequence of extrema at \( \epsilon \sim \frac{2n + 1}{4} \delta_r \). The difference is that this sequence starts with a maximum (for \( c = 1 \)) or a minimum (for \( c = -1 \)). A more refined signature is obtained by subtracting a constant background and fitting the remaining oscillatory signal for the first, say, five extrema to Eq. (1), using \( \delta_r \) as single fit parameter.

Fig. 3 shows data processed in this way for increasing number of runs \( N_r \), either with (left column) and without (right) MBS. The quality of the data may be assessed, e.g., by calculating the sum of squared distances between the extrema of the fit function and the data. For too low sample number, e.g., for \( N_r = 60 \), no unambiguous pattern of extrema is identifiable. At \( N_r = 600 \) samples, side lobes begin to emerge, but a reliable assignment of extrema is still difficult to ascertain. However, for \( N_r = 6000 \), the extremal energies are evenly spaced, and the squared distance fit accurately determines the correct sign of \( c \). Additional information on system parameters, such as knowledge of the effective broadening \( \Gamma \), may be exploited to develop more informed fitting protocols for the ensemble averaged data. However, we found that such refinements lead only to minor improvements of the results.

Let us briefly comment on the experimental feasibility of the TI vortex setup. Generally speaking the vortex area should be chosen small enough that its quantized levels can be resolved, and large enough that neighboring levels are coupled by disorder and gate variations. With \( \delta_r \approx E_{Th} = v\ell/R^2 \), and given typical values \( v \approx 5 \times 10^3 \text{m/s} \), \( \ell \approx 20 \text{nm} \) [73], with spectral resolution \( \Gamma \approx 30 \mu \text{eV} \), one needs to have \( R \lesssim 3 \mu \text{m} \). Choosing \( R = 300 \text{nm} \) and finger gates of width \( \approx 50 \text{nm} \), we have \( \ell/R \approx 0.1 \). For these values, individual levels can be distinguished and a few finger gate electrodes could be placed over the vortex core. We are thus confident that the requirements for our proposal to work are met by existing setups.

Conclusions.—We have proposed a novel scheme for the detection of MBS in existing device structures which combines tunnel spectroscopy with elements of statistics. The focus of attention is here shifted from the center peak ubiquitous in spectroscopic data — which is notorious for its misinterpretability — to the pattern of side bands. The unavoidable presence of effective disorder becomes a resource in that it induces correlations between levels which, upon averaging over different parametric realizations, lead to the effectively crystalline structure in Eq. (1). The latter originates in a combination of statistics and topology which is unambiguously linked to the presence or absence of a MBS, even if the latter cannot be clearly identified in isolation. Another advantage of the approach is that it includes its own validation: If neither the positive, \( c = 1 \), nor the negative, \( c = -1 \), signal

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2 If no independent information on the characteristic level spacing \( \delta_r \) is available, the latter may be estimated by measuring the spacing between peaks in the averaged spectral density.

3 Another source of uncertainty is due to the fact that tunnel spectroscopy measures point conductance, i.e., a quantity proportional to the product of spectral density and wave function moduli, where the latter remain unknown. However, one may expect that in a large data set, these variations efficiently average out.
can be resolved, the method has not been implemented with sufficient accuracy. The principal conditions for it to work are resolvability of individual levels (where one may argue that this condition must be met anyway for the MBS to become a useful resource), sufficient statistics provided by at least $O(10^3)$ runs, and effective disorder strong enough to cause level correlation. (If the “native” disorder is too weak, one may contemplate lowering the level spacing by increasing the vortex size for diagnostic purposes.) These criteria are realistic for the vortex platform, and we are confident that the same holds for other realizations, such as planar Josephson junctions, leaving sufficient freedom for the placement of gate electrodes. We conclude that this approach has the potential to settle the issue of MBS existence with available measurement protocols and hardware.

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[1] J. Alicea, Rep. Prog. Phys. 75, 076501 (2012).
[2] M. Leijnse and K. Flensberg, Semiconductor Science and Technology 27, 124003 (2012).
[3] C. W. J. Beenakker, Annual Review of Condensed Matter Physics 4, 113 (2013).
[4] S. DasSarma, M. Freedman, and C. Nayak, npj Quantum Inf. 1, 51001 (2015).
[5] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science 336, 1003 (2012).
[6] S. Nadji-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, Science 346, 602 (2014).
[7] R. M. Lutchyn, E. P. A. M. Bakkers, L. P. Kouwenhoven, P. Krogstrup, C. M. Marcus, and Y. Oreg, Nat. Rev. Mat. 3, 52 (2018).
[8] Q. Liu, C. Chen, T. Zhang, R. Peng, Y.-J. Yan, C.-H.-P. Wen, X. Lou, Y.-L. Huang, J.-P. Tian, X.-L. Dong, G.-W. Wang, W.-C. Bao, Q.-H. Wang, Z.-P. Yin, Z.-X. Zhao, and D.-L. Feng, Phys. Rev. X 8, 041056 (2018).
[9] M. Li, G. Li, L. Cao, X. Zhou, X. Wang, C. Jin, C.-K. Chiu, S. J. Pennycook, Z. Wang, and H.-J. Gao, Nature 606, 890 (2022).

MicrosoftQuantum, “Inas-al hybrid devices passing the topological gap protocol.” (2022), arXiv:2207.02472.

[11] K. Sengupta, I. Zutič, H.-J. Kwon, V. M. Yakovenko, and S. Das Sarma, Phys. Rev. B 63, 144531 (2001).
[12] K. T. Law, P. A. Lee, and T. K. Ng, Phys. Rev. Lett. 103, 237001 (2009).
[13] K. Flensberg, Phys. Rev. B 82, 180516 (2010).
[14] H.-H. Sun, K.-W. Zhang, L.-H. Hu, C. Li, G.-Y. Wang, H.-Y. Ma, Z.-A. Xu, C.-L. Gao, D.-D. Guan, Y.-Y. Li, C. Liu, D. Qian, Y. Zhou, L. Fu, S.-C. Li, F.-C. Zhang, and J.-F. Jia, Phys. Rev. Lett. 116, 257003 (2016).
[15] A. Zazunov, R. Egger, and A. Levy Yeyati, Phys. Rev. B 94, 014502 (2016).
[16] C. J. Bolech and E. Demler, Phys. Rev. Lett. 98, 237002 (2007).
[17] J. Nilsson, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. Lett. 101, 120403 (2008).
[18] A. Golub and B. Horovitz, Phys. Rev. B 83, 153415 (2011).
[19] A. Haim, E. Berg, F. von Oppen, and Y. Oreg, Phys. Rev. Lett. 114, 166406 (2015).
[20] D. E. Liu, M. Cheng, and R. M. Lutchyn, Phys. Rev. B 91, 081405 (2015).
[21] K. M. Tripathi, S. Das, and S. Rao, Phys. Rev. Lett. 116, 166401 (2016).
[22] T. Jonckheere, J. Rech, A. Zazunov, R. Egger, and T. Martin, Phys. Rev. B 95, 054514 (2017).
[23] T. Jonckheere, J. Rech, A. Zazunov, R. Egger, A. L. Yeyati, and T. Martin, Phys. Rev. Lett. 122, 097003 (2019).
[24] J. Manousakis, C. Wille, A. Altland, R. Egger, K. Flensberg, and F. Hassler, Phys. Rev. Lett. 124, 096801 (2020).
[25] D. Ansen, M. Hell, R. V. Mishmash, A. Higginbotham, J. Danon, M. Leijnse, T. S. Jespersen, J. A. Folk, C. M. Marcus, K. Flensberg, and J. Alicea, Phys. Rev. X 6, 031016 (2016).
[26] C. W. J. Beenakker, SciPost Phys. Lect. Notes, 15 (2020).
[27] K. Flensberg, F. von Oppen, and A. Stern, Nat. Rev. Mat. 6, 944 (2021).
[28] B. Sibierski, M. Geier, A.-P. Li, M. Brahlek, R. G. Moore, and J. E. Moore, Phys. Rev. B 106, 035413 (2022).
[29] A. Altland and M. R. Zimbauer, Phys. Rev. B 55, 1142 (1997).
[30] C. W. J. Beenakker, Rev. Mod. Phys. 87, 1037 (2015).
[31] D. Bagrets and A. Altland, Phys. Rev. Lett. 109, 227005 (2012).
[32] J. Liu, A. C. Potter, K. T. Law, and P. A. Lee, Phys. Rev. Lett. 109, 267002 (2012).
[33] R. Aguado, Riv. Nuovo Cimento 40, 523 (2017).
[34] C. Moore, T. D. StanesCU, and S. Tewari, Phys. Rev. B 97, 165302 (2018).
[35] A. Vuik, B. Nijholt, A. R. Akhmerov, and M. Wimmer, SciPost Phys. 7, 061 (2019).
[36] E. Prada, P. San-Jose, M. W. A. de Moor, A. Geresdi, E. J. H. Lee, J. Klinovaja, D. Loss, J. Nygård, R. Aguado, and L. P. Kouwenhoven, Nat. Rev. Phys. 2, 225 (2020).
[37] M. Valentini, F. Peñaranda, A. Hofmann, M. Brauns, R. Hauschild, P. Krogstrup, P. San-Jose, E. Prada, R. Aguado, and G. Katsaros, Science 373, 82 (2021).
[38] P. Yu, J. Chen, M. Gomenko, G. Badawy, E. P. A. M. Bakkers, K. Zuo, V. Mourik, and S. Frolov, Nat. Phys. 17, 482 (2021).
[39] S. Frolov and V. Mourik, (2022), majorana fireside pod-cast, https://youtu.be/RnYghkDaHH0.
[40] A. R. Akhmerov, (2022), what can we learn from the reported discovery of Majorana states? Journal Club Condensed Matter, see https://doi.org./10.36471/JCCM
[42] M. Wimmer, A. R. Akhmerov, J. P. Dahlhaus, and C. W. J. Beenakker, New Journal of Physics 13, 053016 (2011).
[43] P. W. Brouwer, M. Duckheim, A. Romito, and F. von Oppen, Phys. Rev. Lett. 107, 196804 (2011).
[44] P. Neven, D. Bagrets, and A. Altland, New Journal of Physics 15, 055019 (2013).
[45] M. Diez, I. C. Fulga, D. I. Pikulin, J. Tworzydło, and C. W. J. Beenakker, New Journal of Physics 16, 063049 (2014).
[46] A. Haim and A. Stern, Phys. Rev. Lett. 122, 126801 (2019).
[47] M. L. Mehta, Random Matrices (Academic Press, 2004).
[48] L. Fu and C. L. Kane, Phys. Rev. Lett. 100, 096407 (2008).
[49] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
[50] P. A. Ioselevich, P. M. Ostrovsky, and M. V. Feigel’man, Phys. Rev. B 86, 035441 (2012).
[51] R. S. Akzyanov, A. V. Rozhkov, A. L. Rakhmanov, and F. Nori, Phys. Rev. B 89, 085409 (2014).
[52] H. S. Reising, R. Ilan, T. Meng, S. H. Simon, and F. Flicker, SciPost Phys. 6, 055 (2019).
[53] A. Ziesen and F. Hassler, Journal of Physics: Condensed Matter 33, 294001 (2021).
[54] B. S. de Mendonça, A. L. R. Manesco, N. Sandler, and L. G. V. D. da Silva, “Can caroli-de gennes-matricon and majorana vortex states be distinguished in the presence of impurities?” (2022), arXiv:2204.05078.
[55] C. Caroli, P. De Gennes, and J. Matricon, Physics Letters 9, 307 (1964).
[56] See the online Supplementary Material (SM), where we provide additional details on peak spacing distributions, on the mean free path, on our numerical simulations, and on other Majorana platforms.
[57] H. Lei, R. Hu, E. S. Choi, J. B. Warren, and C. Petrovíc, Phys. Rev. B 81, 094518 (2010).
[58] G. Xu, B. Lian, P. Tang, X.-L. Qi, and S.-C. Zhang, Phys. Rev. Lett. 117, 047001 (2016).
[59] D. Wang, L. Kong, P. Fan, H. Chen, S. Zhu, W. Liu, L. Cao, Y. Sun, S. Du, J. Schneeloch, R. Zhong, G. Gu, L. Fu, H. Ding, and H.-J. Gao, Science 362, 333 (2018).
[60] K. Jiang, X. Dai, and Z. Wang, Phys. Rev. X 9, 011033 (2019).
[61] M. Hell, M. Leijnse, and K. Flensberg, Phys. Rev. Lett. 118, 107701 (2017).
[62] F. Pientka, A. Keselman, E. Berg, A. Yacoby, A. Stern, and B. I. Halperin, Phys. Rev. X 7, 021032 (2017).
[63] A. Fornieri, A. M. Whiticar, F. Setiawan, E. Portolés, A. C. C. Drachmann, A. Keselman, S. Gronin, C. Thomas, T. Wang, R. Kallober, G. C. Gardner, E. Berg, M. J. Manfra, A. Stern, C. M. Marcus, and F. Nichele, Nature 569, 89 (2019).
[64] H. Ren, F. Pientka, S. Hart, A. T. Pierce, M. Kosowsky, L. Lunczer, R. Schlereth, B. Scharf, E. M. Hankiewicz, L. W. Molenkamp, B. I. Halperin, and A. Yacoby, Nature 569, 93 (2019).
[65] P. Marra, Journal of Applied Physics 132, 231101 (2022).
[66] J. Goldberg, U. Smilansky, M. V. Berry, W. Schweizer, G. Wunner, and G. Zeller, Nonlinearity 4, 1 (1991).
[67] D. M. Zumbühl, J. B. Miller, C. M. Marcus, K. Campman, and A. C. Gossard, Phys. Rev. Lett. 89, 276803 (2002).
[68] L. C. Contamin, L. Jarjat, W. Legrand, A. Cottet, T. Kontos, and M. R. Delbecq, Nature Communications 13, 6188 (2022).
[69] A. A. Taskin, Z. Ren, S. Sasaki, K. Segawa, and Y. Ando, Phys. Rev. Lett. 107, 016801 (2011).
Supplementary Material to “Statistical Majorana Bound State Spectroscopy”

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We here derive the distribution function \( P(\epsilon) \) from random matrix theory (see Sec. I), discuss the mean free path \( \ell \) (see Sec. II), provide details on our numerical calculations in Sec. III, and comment on the application of our approach to alternative Majorana platforms in Sec. IV. Equation or figure numbers containing the index ‘M’ refer to the main text.

I. DISTRIBUTION FUNCTIONS

Analytical results for the distribution function \( P(\epsilon) \) of the lowest finite-energy level \( \epsilon \) can be obtained by random matrix theory arguments [1]. In practice, it is sufficient to calculate \( P(\epsilon) \) for the smallest nontrivial matrix dimension \( d \) for the corresponding class D random matrix ensemble, where \( d = 3 \) (\( d = 2 \)) with (without) a MBS. Using independent real-valued normal-distributed random variables, \( \{a, b, c\} \), the respective random matrices are written as

\[
V_{d=3} = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}, \quad V_{d=2} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}.
\]

The particle-hole symmetric matrix is then given by \( \{0, \pm \lambda_3 = \pm \sqrt{a^2 + b^2 + c^2}\} \) for \( d = 3 \), and by \( \{\pm \lambda_2 = \pm |a|\} \) for \( d = 2 \). Expressing the dependence on \( \{a, b, c\} \) in terms of multivariate \( \chi^2 \)-distributions, we obtain the distributions

\[
P(\lambda_3) = \frac{3}{\pi} \sqrt{\lambda_3^2 e^{-\lambda_3^2/2}}, \quad P(\lambda_2) = \frac{2}{\pi} e^{-\lambda_2^2/2}.
\]

The energy distribution, \( P(\epsilon) \), with \( \epsilon = \lambda_3 \) or \( \epsilon = \lambda_2 \), respectively, now follows as quoted in the main text.

II. MEAN FREE PATH

The mean free path \( \ell \) connects the disorder strength \( \gamma \) to experimentally accessible quantities for the TI vortex platform. Using the standard Born approximation (which applies for our system at finite energy in the presence of weak disorder [2]) and the density of states \( D_2(\epsilon) = |\epsilon|/(2\pi v^2) \) obtained from Eq. (M2), the energy-dependent mean free path is given by

\[
\ell = \frac{v}{2\pi \gamma^2 D_2(\epsilon)} = \frac{\nu^3}{\gamma^2 |\epsilon|}.
\]

For low-energy in-gap states, Eq. (3) is evaluated at \( \epsilon = \delta_\epsilon \), with \( \delta_\epsilon \) the level spacing. Depending on the hierarchy of \( \ell \) and \( R \), \( \delta_\epsilon \) is either given by the ballistic \( \delta_\epsilon \approx E_{\text{Th}} = v\ell/R^2 \) Thouless scale. In the respective limits, we then obtain

\[
\ell \approx \begin{cases} \frac{R^2}{v} \gamma & \gamma \ll \nu, \\ \frac{\nu}{\gamma} R \gamma & \gamma \gg \nu. \end{cases}
\]

Thus, \( \delta_\epsilon \) and \( \ell \) both are suppressed by a factor of \( \nu/\gamma \) in the diffusive regime.

III. NUMERICAL SIMULATION DETAILS

We present details about the numerical simulations employed to verify the gate-based sampling method. The full Bogoliubov-de Gennes (BdG) Hamiltonian in the presence of the Gaussian random potential \( V(r) \) is \( H = H_{\text{BdG}} + V(r)\tau_z \) with \( H_{\text{BdG}} \) in Eq. (M2). For an etched vortex, the radial profile of the pairing gap is well approximated by \( |\Delta(r)| = \Delta \Theta(r - R) \). Moreover, for low-energy in-gap states, we can effectively capture the effects of the superconductor as a boundary condition at \( r = R \) which follows by sending \( \Delta \to \infty \). Below we use the quantum numbers \( \alpha = (n, m) \), with the (integer or half-integer) angular momentum \( m \) and the (integer) radial quantum number \( n \). Introducing also the dimensionless radial variable \( x_\alpha = \epsilon_\alpha r/v \) for finite energy \( \epsilon_\alpha \neq 0 \), the corresponding eigenstates of \( H_{\text{BdG}} \) for \( \mu = \Delta = 0 \) are expressed in terms of Bessel functions,

\[
f_\alpha(r) = N_\alpha e^{i \nu \theta} \begin{pmatrix} i e^{-i(\nu+1)/2} J_{m-\nu+1/2}(x_\alpha) \\ -e^{-i(\nu-1)/2} J_{m-\nu-1/2}(x_\alpha) \\ i \epsilon_\alpha e^{i(\nu-1)/2} J_{m+\nu+1/2}(x_\alpha) \\ e^{i(\nu+1)/2} J_{m+\nu+1/2}(x_\alpha) \end{pmatrix}, \quad (5)
\]

where we use polar coordinates, the four-spinor convention \( (\psi_\uparrow, \psi_\downarrow, \psi_\downarrow^\dagger, -\psi_\uparrow^\dagger) \) in spin and particle-hole space, and a normalization factor \( N_\alpha \). The superconducting boundary condition at \( r = R \) leads to the conditions

\[
\begin{align*}
J_{m-\nu+1/2}(x_\alpha) J_{m+\nu+1/2}(x_\alpha) &= 1, \\
J_{m-\nu+1/2}(x_\alpha) J_{m-\nu+1/2}(x_\alpha) &= 1, \\
c_\alpha &= \frac{J_{m-\nu+1/2}(x_\alpha)}{J_{m+\nu+1/2}(x_\alpha)}.
\end{align*}
\]

Once the energies \( \epsilon_\alpha \) solving the first (implicit) equation in Eq. (6) have been determined, the coefficients \( c_\alpha \) follow from the second equation. In addition, the spectral
FIG. 1. TI vortex setup with six finger gates placed within the vortex core region of radius $R$. The gates cover areas of different sizes, $A_{c,j}$, and are arranged in three pairs indicated by the colors $c \in \{r, b, g\}$. Gates forming a pair are referenced by $j \in \{1, 2\}$.

The problem admits $\nu$ zero-energy solutions, where an even number of such states can hybridize to form topologically trivial Andreev bound states and one finds a single MBS for odd $\nu$. In particular, for $\nu = 1$, the zero-energy state (with $m = n = 0$) is given by

$$f^{(\nu=1)}_{0,0}(r) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ i \\ 0 \end{pmatrix},$$

while for $\nu = 2$, we find zero-energy states with angular momentum $m = \pm 1/2$,

$$f^{(\nu=2)}_{0,-1/2}(r) = \frac{\sqrt{2}}{3\pi} \begin{pmatrix} 0 \\ e^{-i \vartheta_r/r} \\ -i \\ 0 \end{pmatrix},$$
$$f^{(\nu=2)}_{0,1/2}(r) = \frac{\sqrt{2}}{3\pi} \begin{pmatrix} 0 \\ e^{i \vartheta_r/r} \\ i \\ 0 \end{pmatrix},$$

which combine to form an Andreev bound state.

Next, we discuss the matrix elements of the disorder potential in the basis diagonalizing $H_{\text{BDG}}$.

$$V_{\alpha\beta} = \int dr \, f^{\dagger}_\alpha(r) V(r) \tau_z f_\beta(r).$$

In order to avoid the explicit evaluation of the slowly converging and computationally intensive spatial integrals in Eq. (9), we use the fact that, since $V(r)$ is Gaussian distributed, the matrix elements $V_{\alpha\beta}$ have only non-trivial second moments. However, these matrix elements are correlated and therefore the full covariance matrices $\langle V_{\alpha\beta} V_{\gamma\delta}^* \rangle$ and $\langle V_{\alpha\beta} V_{\gamma\delta} \rangle$ are needed, where angular momentum conservation implies

$$\langle V_{\alpha\beta} V_{\gamma\delta}^* \rangle \propto \delta_{m_\alpha - m_\beta, m_\gamma - m_\delta},$$
$$\langle V_{\alpha\beta} V_{\gamma\delta} \rangle \propto \delta_{m_\alpha - m_\beta, -(m_\gamma - m_\delta)}.$$

We note that matrix elements involving zero-energy states also satisfy Eq. (10). With $\Delta m_{\alpha\beta} = m_\alpha - m_\beta$, we next observe that $\langle V_{\alpha\beta} V_{\gamma\delta}^* \rangle \neq 0$ only if $\Delta m_{\alpha\beta} = \Delta m_{\gamma\delta}$.

Similarly, $\langle V_{\alpha\beta} V_{\gamma\delta} \rangle \neq 0$ requires $\Delta m_{\alpha\beta} = -\Delta m_{\gamma\delta}$. Clearly, both covariances can be finite only for a disjoint set of disorder elements unless $\Delta m_{\alpha\beta} = \Delta m_{\gamma\delta} = 0$. It is then convenient to define

$$W_{\alpha\beta} = \begin{cases} V_{\alpha\beta}, & \Delta m_{\alpha\beta} > 0, \\ V_{\alpha\beta}^*, & \Delta m_{\alpha\beta} < 0, \\ V_{\alpha\beta} + i(V_{\alpha\beta}^*), & \Delta m_{\alpha\beta} = 0, \end{cases}$$

with statistically independent auxiliary variables $V'_{\alpha\beta}$ (not entering the Hamiltonian) which are distributed identically as $V_{\alpha\beta}$. The new variables $W_{\alpha\beta}$ fulfill $\langle W_{\alpha\beta} W_{\gamma\delta} \rangle = 0$, and therefore the problem has been reduced to a single Hermitian covariance matrix, $C_{\alpha\beta\gamma\delta} = \langle W_{\alpha\beta} W_{\gamma\delta}^* \rangle$. Using a singular value decomposition, $C = U \Sigma U^\dagger = L L^\dagger$ with $L = U \sqrt{\Sigma}$, we can then map a set of independent normal-distributed complex elements, $X_{\alpha\beta}$, onto correlated matrix elements, $W_{\alpha\beta} = (L X)_{\alpha\beta}$. By using the inverse mapping of Eq. (11), one obtains the matrix elements $V_{\alpha\beta}$ in a numerically efficient manner. We have verified that disorder matrices generated by this mapping obey the same distribution as the spatial integrals in Eq. (9).

We finally describe the simulated gate setup used for generating the data in Fig. M3, which is illustrated in Fig. 1 and contains six finger gates inside the vortex with area $A_{\text{vort}} = \pi R^2$. The gates are paired up into $f = 3$ pairs, where each gate covers the area $A_{c,j}$, with $c \in \{r, b, g\}$ indicating the “color” of a pair and $j \in \{1, 2\}$ numbering both gates forming the pair. The latter are assumed to be oppositely charged by means of oppositely directed gate voltages, $\pm \delta V_c$. All finger gates should be spaced sufficiently far away from each other such that direct tunneling processes between gate electrodes can be ruled out. Discretizing the vortex area $A_{\text{vort}}$ into a grid of infinitesimal area pieces $\delta A$, the perturbation strength due to a gate pair with voltage changes $\pm \delta V_c$ can then be estimated by summing over this spatial grid, where the gate voltage is constant for each finger gate and zero elsewhere. We then arrive at

$$\langle \Psi | \delta V_c | \Psi \rangle \approx \frac{\delta V_c}{A_{\text{vort}}} (A_{c1} - A_{c2}) = r_c \delta V_c.$$

Equation (12) defines the area fraction, $r_g = r_c$, for this gate pair. Independently tuning gate voltages on different gate pairs allows one to sample many disorder realizations, where an uncorrelated new sample is reached by varying the respective gate voltage by $\delta V_c \approx \delta / r_c$.

IV. ALTERNATIVE PLATFORMS

In the main text, we demonstrated our protocol on a numerically simulated proximitized TI vortex. We here
FIG. 2. Possible alternative realizations of statistical spectroscopy. Left: Planar Josephson junction. Right: Semiconductor quantum wire. In either case, a combination of electrostatic side gates and external magnetic fields may realize the ensemble parameter space. The fading red lines indicate the spatial support of a Majorana end state.

discuss some of the most prominent alternative types of MBS hardware and speculate on their suitability for statistical spectroscopic analysis.

Planar Josephson junctions [left panel in Fig. 2] support low-energy bound states along the edge of a two-dimensional electron gas (2DEG) [3–6]. The width, \( W \), of the latter is adjustable and may be increased to allow for the placement of a number of top gates, much as in the TI vortex case. At the same time, the level spacing is proportional to \( 1/W \), meaning that a too wide junction will be in conflict with the required resolvability of levels. We thus have a tradeoff situation which calls for prior numerical optimization. However, generally speaking the planar Josephson junction appears to be a promising candidate for the realization of statistical spectroscopy.

Proximitized nanowires with spin-orbit coupling [right panel in Fig. 2] have been investigated for more than a decade as candidates for the realization of Majorana end states, see Refs. [7, 8] for reviews. We see two principal factors relevant to the applicability of statistical spectroscopy. First, more than one Andreev bound state (next to a prospected center Majorana state) is required to realize the inter-state correlation defining the quasi-oscillatory spectral density profile, i.e., a single-channel limit due to too strong transverse size quantization would prevent our approach from working. The flipside of the same coin is that a large spacing between individual Andreev levels implies their accurate resolvability. In this regard, the nanowire platform appears to be superior to the TI vortex. Second, geometric constrictions may prevent the placement of more than perhaps a single electrode. However, compared to the TI vortex with its rigid flux quantization, there now is larger freedom to apply magnetic fields of different strength and orientation. A combination of gate voltages and field strength parameters may go a long way in realizing a sizeable ensemble. All in all, we consider hybrid nanowires as promising candidates for a statistical extension of the already existing spectroscopic single-shot measurement setups.

Iron-based superconductors [9–12] such as FeSe\(_x\)Te\(_{1-x}\) define another Majorana platform of much current interest. They are expected to host topological surface states [9] where defect-induced vortices in a magnetic field bind Majorana states, similar to the TI vortex case. (The short superconducting coherence length, \( \xi \approx (2 - 12) \) nm [10, 13], here implies a much smaller vortex area.) Noting that the available samples show a large number of surface defects, rather than generating the disorder ensemble by system parameter variations, the averaged spectral density \( \rho(\epsilon) \) could be directly measured by performing tunneling spectroscopy on sufficiently large samples containing many defects. Conductance histograms of this type have already been published [11, 12], and with further refinements may allow to implement our approach.

To summarize, several other currently investigated Majorana hardware platforms appear to be suitable for the statistical spectroscopy approach. In either case, numerical simulations for realistic system parameters are relatively straightforward to implement as trial runs prior to an experimental test.

[1] M. L. Mehta, Random Matrices (Academic Press, 2004).
[2] P. Neven, D. Bagrets, and A. Altland, New Journal of Physics 15, 055019 (2013).
[3] M. Hell, M. Leijnse, and K. Flensberg, Phys. Rev. Lett. 118, 107701 (2017).
[4] F. Pientka, A. Keselman, E. Berg, A. Yacoby, A. Stern, and B. I. Halperin, Phys. Rev. X 7, 021032 (2017).
[5] A. Fornieri, A. M. Whiticar, F. Setiawan, E. Portolés, A. C. C. Drachmann, A. Keselman, S. Gronin, C. Thomas, T. Wang, R. Kallaber, G. C. Gardner, E. Berg, M. J. Manfra, A. Stern, C. M. Marcus, and F. Nichele, Nature 569, 89 (2019).
[6] H. Ren, F. Pientka, S. Hart, A. T. Pierce, M. Kosowsky, L. Lunczer, R. Schlereth, B. Scharf, E. M. Hankiewicz, L. W. Molenkamp, B. I. Halperin, and A. Yacoby, Nature 569, 93 (2019).
[7] R. M. Lutchyn, E. P. A. M. Bakkers, L. P. Kouwenhoven, P. Krogstrup, C. M. Marcus, and Y. Oreg, Nat. Rev. Mat. 3, 52 (2018).
[8] K. Flensberg, F. von Oppen, and A. Stern, Nat. Rev. Mat. 6, 944 (2021).
[9] G. Xu, B. Lian, P. Tang, X.-L. Qi, and S.-C. Zhang, Phys. Rev. Lett. 117, 047001 (2016).
[10] D. Wang, L. Kong, P. Fan, H. Chen, S. Zhu, W. Liu, L. Cao, Y. Sun, S. Du, J. Schneeloch, R. Zhong, G. Gu, L. Fu, H. Ding, and H.-J. Gao, Science 362, 333 (2018).
[11] T. Machida, Y. Sun, S. Pyon, S. Takeda, Y. Kohsaka, T. Hanaguri, T. Sasagawa, and T. Tamegai, Nature Materials 18, 811 (2019).
[12] S. Zhu, L. Kong, L. Cao, H. Chen, M. Papaj, S. Du, Y. Xing, W. Liu, D. Wang, C. Shen, F. Yang, J. Schneeloch, R. Zhong, G. Gu, L. Fu, Y.-Y. Zhang, H. Ding, and H.-J. Gao, Science 367, 189 (2020).
[13] H. Lei, R. Hu, E. S. Choi, J. B. Warren, and C. Petrovic, Phys. Rev. B 81, 094518 (2010).