Muonium annihilation into $\nu_e \bar{\nu}_\mu$ and $\nu_e \bar{\nu}_\mu \gamma$

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Abstract

We calculate in detail the annihilation of Muonium ($Mu$) into $\nu_e \bar{\nu}_\mu$ and $\nu_e \bar{\nu}_\mu \gamma$ states. For $Mu \rightarrow \nu_e \bar{\nu}_\mu$ we obtain the branching ratio $Br = 6.6 \cdot 10^{-12}$ and for $Mu \rightarrow \nu_e \bar{\nu}_\mu \gamma$ in the limit of high energy photon $Br = 4.3 \cdot 10^{-11}$. 
Introduction

In the light of new possibilities of experimental verification of the Standard Model one needs to reconcile the different theoretical predictions. The difference has place, e.g., in the case of Muonium (Mu) system decay. The dominant decay channel occurs by the muon beta decay: \( \mu^+ \rightarrow e^+ \nu_e \overline{\nu}_\mu \), but there is another possible channel (Muonium annihilation): \( Mu \rightarrow \nu_e \overline{\nu}_\mu \). The corresponding decay rate is rather small but detectable in the planned experiments \([1]\) as an invisible decay of Muonium. The branching ratio of this process was calculated in some detail and was found to be \( \sim 10^{-12} \) in the works \([2]\), \([3]\) and \( \sim 10^{-10} \) in the work \([4]\). To recheck these results we provide here the full calculations and also estimate the full decay width and a photon energy spectrum of the reaction \( Mu \rightarrow \nu_e \overline{\nu}_\mu \gamma \).

1. Mu \( \rightarrow \nu_e \overline{\nu}_\mu \) annihilation

\( \mu^+e^- \rightarrow \nu_e \overline{\nu}_\mu \) process

We start with a calculation of the amplitude of \( \mu^+e^- \rightarrow \nu_e \overline{\nu}_\mu \) process assuming both \( \mu^+ \) and \( e^- \) to be free states. In our case particles have small energies in comparison with the masses of the weak bosons, so the amplitude has the form

\[
M_0 = \frac{G_F}{\sqrt{2}} j^\mu_1 j^\mu_2,
\]

where

\[
j^\mu_1 = \overline{\nu}_\mu(k')O^\mu \nu_e(p'),
\]
\[
j^\mu_2 = \overline{\nu}_e(p)O^\mu \nu_\mu(k).
\]

Here \( u \) and \( v \) are the solutions to the Dirac equation with positive and negative frequencies respectively, \( p, p', k, k' \) are 4-momenta of \( e, \mu, \nu_e \) and \( \nu_\mu \) respectively, \( O^\mu = \gamma^\mu (\gamma^5 + 1) \). For the square of this matrix element neglecting the neutrino masses we have,

\[
|M_0|^2 = 128G_F^2(k \cdot p')(p \cdot k').
\]

Since electron and muon are supposed to be non-relativistic and both neutrinos are ultra relativistic then

\[
p = (m_e, \overrightarrow{p}), \quad k = (m_\mu/2, m_\mu/2), \quad (1)
\]
\[
p' = (m_\mu, -\overrightarrow{p}), \quad k' = (m_\mu/2, -m_\mu/2), \quad (2)
\]

so, the final expression for the matrix element squared is

\[
|M_0|^2 \approx 32G_F^2 \left( m_\mu^2 m_e + m_e^2 |\overrightarrow{p}| \right) \cos \theta \left( m_\mu + |\overrightarrow{p}| \cos \theta \right) \cos \theta,
\]

where \( \theta \) is the angle between muon \( \mu \) and electron neutrino \( \nu_e \). Hereafter we will use the leading part of this expression assuming that \( |\overrightarrow{p}| \approx 0 \) and \( m_e \ll m_\mu \), so, the accuracy of our calculations has order \( m_e/m_\mu \).
Bound states

The amplitude associated with the bound state expresses in terms of the amplitude of the free process as

\[ M = \sqrt{2m} \int \frac{d^3q}{(2\pi)^3} \psi^*(\vec{q}) \frac{1}{\sqrt{2m_\mu}} \frac{1}{\sqrt{2m_e}} M_0, \]

where \( m \) is the mass of the bound state which in our case equals to \( m_\mu \) within our accuracy, \( \psi(\vec{q}) \) is a Fourier transform of the Schroedinger wave function of the bound state which we set to be equal to the S ground state. Hence, the full angular momentum of the system is determined by the summary spin of \( \mu^+ \) and \( e^- \). Multipliers \( (1/\sqrt{2m_e}) \), \( (1/\sqrt{2m_\mu}) \) provide the integral normalizing to unity and the factor \( \sqrt{2m} \) in front of the integral needs for accordance with the cross section formula [5]. Since \( M_0 \) does not contain any dependency on \( \vec{q} \), this expression simplifies, so for its square we have,

\[ |M|^2 = \frac{1}{2m_e} |M_0|^2 \cdot |\psi(0)|^2. \]

The wave function of the ground state of Muonium is the same as for hydrogen:

\[ \psi(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \]

where \( a \) is the Muonium Bohr radius \( a^{-1} = |\vec{p}| = \alpha m_{rd} \), and \( m_{rd} = m_em_\mu/(m_e+m_\mu) \approx m_e \) with our precision. Therefore

\[ |\psi(0)|^2 = \frac{m_e^3 \alpha^3}{\pi}. \quad (3) \]

Decay width

Decay width of Muonium is written in a usual way (again \( m \approx m_\mu \)):

\[ \Gamma = \frac{1}{2m_\mu} \int |M|^2 d\Pi_2. \]

A two-particle phase volume \( d\Pi_2 \) for \( 2 \to 2 \) reactions, which are symmetrical with respect to the collision axis, can be reduced to the form

\[ d\Pi_2 = \frac{1}{16\pi} d\cos \theta \frac{2|\vec{k}|}{E_{CM}}, \]

where \( E_{CM} \approx m_\mu \) is the energy of particles in the center-mass system and \( |\vec{k}| \) is the momentum of the outgoing particles, \( |\vec{k}| \approx m_\mu/2 \). The fact that the dependence on the polar angle \( \theta \) contains only in \( |M_0|^2 \) allows us to write,

\[ \Gamma = \frac{1}{4m_em_\mu} |\psi(0)|^2 \int |M_0|^2 d\Pi_2. \quad (4) \]

Integration over the phase volume and substitution the explicit expression for \(|\psi(0)|^2 \) [3] leads to the expression:

\[ \Gamma = \frac{G_F^2 \alpha^3 m_e^3 m_\mu^2}{\pi^2}. \]
Averaging over the initial polarizations gives,

$$\frac{G_F^2 \alpha^3 m_e^3 m_\mu^2}{4 \pi^2} = 48\pi \left( \frac{\alpha m_e}{m_\mu} \right)^3 \Gamma_{\mu \to e\nu_\mu\nu_e},$$

so the branching ratio is

$$Br = \Gamma_\tau = 6.6 \cdot 10^{-12}. \quad (5)$$

2. $Mu \to \nu_e \nu_\mu \gamma$ annihilation

To the leading order in $\alpha$ there are three diagrams which give contribution to the $Mu \to \nu_e \nu_\mu \gamma$ process. They are presented in fig. 1.

![Figure 1: The leading order diagrams for free $\mu^+ e^- \to \nu_e \nu_\mu \gamma$ process.](image)

The first of them contains two virtual W-boson propagators and hence its amplitude is suppressed significantly. Naively, the leading order part of the matrix element squared of $\mu^+ e^- \to \nu_e \nu_\mu \gamma$ process involving the term from this diagram is proportional to

$$|M_0|^2 \sim \frac{e^2 G_F^2}{m_W^2} \frac{m_\mu^7}{m_e^2},$$

and for its contribution to the full decay width of $Mu$ one can obtain:

$$\Gamma \sim \frac{\alpha^4 G_F^2 m_\mu^7}{m_W^4} \left( 1 + O \left( \frac{m_e}{m_\mu} \right) \right).$$

The corresponding branching ratio has an order $Br \sim 10^{-15}$ and turns out to be small even in comparison with the next order loop corrections. The second diagram is also suppressed by a factor $\sim m_e/m_\mu$ and we will not consider it in accordance with the discussion above. The last diagram describes the emission of a photon from the external electron line. Its amplitude is equal to

$$M_{0\gamma} = \frac{G_F}{\sqrt{2}} \frac{i}{-2p \cdot q} \left( \bar{\nu}_e(k)\gamma_\mu(1 + \gamma_5)(\gamma_\mu(p^\rho - q^\rho) - m_e)\gamma_\sigma \epsilon^{\sigma\tau}(q)u_\nu(p) \right) \times \left( \bar{\nu}_\mu(p')\gamma^\mu(1 + \gamma_5)u_\nu^\prime(k') \right).$$

Here $q$ is the photon momentum. Conjugation, production and averaging over the initial polarizations lead to the following expression,

$$|M_{0\gamma}|^2 = \frac{8e^2 G_F^2}{(p \cdot q)^2} \left[ 2(p \cdot l)(k' \cdot l)(k \cdot p') - l^2(k' \cdot p)(k \cdot p') \right], \quad (6)$$

where $l = p - q$. For 4-momenta of the particles we have,

$$p = (m_e, \vec{p}), \quad p' = (m_\mu, -\vec{p}), \quad k = (\omega_1, \vec{k}_1), \quad k' = (\omega_2, \vec{k}_2), \quad q = (\omega, \vec{k}).$$
For the differential decay width one can write similar to (4),

\[ \Gamma_\gamma = \frac{1}{2m_\mu} \int |M_\gamma|^2 d\Pi_3 = \frac{\langle \psi(0) \rangle^2}{4m_e m_\mu} \int |M_{0\gamma}|^2 d\Pi_3, \quad (7) \]

where \( d\Pi_3 \) is now a differential three-particle phase volume

\[ d\Pi_3 = (2\pi)^4 \delta(\omega_1 + \omega_2 + \omega_\gamma - m_\mu) \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_\gamma) \frac{d^3 \vec{k}_1}{2\omega_1} \frac{d^3 \vec{k}_2}{2\omega_2} \frac{d^3 \vec{k}_\gamma}{2\omega_\gamma}. \]

One can integrate over the 3-momenta \( \vec{k}_1 \) and \( \vec{k}_2 \) using the equality \[6\],

\[ \int k_\alpha k'_\beta \frac{d^3 \vec{k}_1}{\omega_1} \frac{d^3 \vec{k}_2}{\omega_2} \delta^4(k + k' - Q) = \frac{\pi}{6} \left( Q^2 g_{\alpha\beta} + 2Q_\alpha Q_\beta \right), \quad (8) \]

where \( Q = p + p' - q \) in our case. Noting that \( d^3 \vec{k}_\gamma = 4\pi \omega_\gamma^2 d\omega_\gamma \) one can reduce the differential decay width to the form

\[ \frac{d\Gamma_\gamma}{d\omega_\gamma} = \frac{G_F^2 \alpha^4}{12\pi^3} m_e m_\mu^4 F(x), \quad (9) \]

where \( F(x) = x(3 - 4x) \) is a photon spectrum function, \( x = \omega_\gamma/m_\mu, \ x \leq 0.5 \). In accordance with our assumption this expression is valid for \( \omega_\gamma \gg m_\mu \), i.e. for \( x \gg 5 \cdot 10^{-3} \). Integration over \( \omega_\gamma \) leads to the final expression,

\[ \Gamma_\gamma = \frac{5G_F^2 \alpha^4 m_e m_\mu^4}{288\pi^3}. \quad (10) \]

The appropriate branching ratio is \( Br = \Gamma_\gamma \tau = 4.3 \cdot 10^{-11} \).

**Results**

The branching ratio \[5\] for the reaction \( Mu \to \nu_e \bar{\nu}_\mu \) is in agreement with the works \[2\] and \[3\] but differs from the one obtained in the work \[4\]. Thus, its value lies in the detectable range of an experiment proposed in \[1\]. The value \[10\] of the full decay width in \( Mu \to \nu_e \bar{\nu}_\mu \gamma \) reaction differs slightly from the result in the work \[3\]. The differential decay width \[9\] is valid for the range of photon energies \( \omega_\gamma \sim 5 - 50 MeV \), as we discussed above, and can be also verified in the new experiments.

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References

[1] S. N. Gninenko, N. V. Krasnikov and V. A. Matveev, Phys. Rev. D 87, 015016 (2013) [arXiv:1209.0060 [hep-ph]].

[2] A. Czarnecki, G. P. Lepage and W. J. Marciano, Phys. Rev. D 61, 073001 (2000) [hep-ph/9908439].

[3] P.-J. Li, Z.-Q. Tan and C.-E. Wu, J. Phys. G 14, 525 (1988) [J. Phys. G G 14, 525 (1988)].

[4] L. Chatterjee, A. Chakrabarty, G. Das and S. Mondal, Phys. Rev. D 46, 5200 (1992).

[5] Michael E. Peskin and Dan V. Schroeder. An Introduction To Quantum Field Theory (Frontiers in Physics). Westview Press, October 1995.

[6] Okun, L.B. Leptons and Quarks. Elsevier Science, 1985.