Operator estimates for the Neumann sieve problem

Andrii Khrabustovskyi

Received: 4 October 2022 / Accepted: 23 January 2023 / Published online: 10 February 2023

Abstract

Let $\Omega$ be a domain in $\mathbb{R}^n$, $\Gamma$ be a hyperplane intersecting $\Omega$, $\varepsilon > 0$ be a small parameter, and $D_{k,\varepsilon}$, $k = 1, 2, 3 \ldots$ be a family of small “holes” in $\Gamma \cap \Omega$; when $\varepsilon \to 0$, the number of holes tends to infinity, while their diameters tend to zero. Let $\mathcal{A}_\varepsilon$ be the Neumann Laplacian in the perforated domain $\Omega_{\varepsilon} = \Omega \setminus \Gamma_{\varepsilon}$, where $\Gamma_{\varepsilon} = \Gamma \setminus (\bigcup_k D_{k,\varepsilon})$ (“sieve”). It is well-known that if the sizes of holes are carefully chosen, $\mathcal{A}_\varepsilon$ converges in the strong resolvent sense to the Laplacian on $\Omega \setminus \Gamma$ subject to the so-called $\delta'$-conditions on $\Gamma \cap \Omega$. In the current work we improve this result: under rather general assumptions on the shapes and locations of the holes we derive estimates on the rate of convergence in terms of $L^2 \to L^2$ and $L^2 \to H^1$ operator norms; in the latter case a special corrector is required.

Keywords Homogenization · Perforated domain · Neumann sieve · Resolvent convergence · Operator estimates · Spectrum

Mathematics Subject Classification 35B27 · 35B40 · 35P05 · 47A55

1 Introduction

One of the main directions in homogenization theory concerns boundary value problems in perforated domains of the form $\Omega_{\varepsilon} = \Omega \setminus \Gamma_{\varepsilon}$, where $\Omega \subset \mathbb{R}^n$ is fixed, while the set $\Gamma_{\varepsilon}$ depends on a small parameter $\varepsilon > 0$ and its geometry is getting more and more complicated as $\varepsilon \to 0$. Homogenization theory is aimed to construct an effective (homogenized) boundary value problem in the unperturbed domain $\Omega$ such that its solutions approximate solutions of the initial problem in $\Omega_{\varepsilon}$ as $\varepsilon \to 0$.

The typical example of such a domain $\Omega_{\varepsilon}$ is the one with a lot of tiny holes, that is $\Gamma_{\varepsilon} = \bigcup_k D_{k,\varepsilon}$, where $D_{k,\varepsilon} \subset \Omega$ are pairwise disjoint compact sets whose diameters tends to
zero as $\varepsilon \to 0$, while their number (per finite volume) tends to infinity. The holes $D_{k,\varepsilon}$ may be distributed in the entire volume or along some hypersurface.

In the current work we consider the perforation $\Gamma_{\varepsilon}$ of another type—the one having the form of a "sieve", and revisit the so-called Neumann sieve problem studying Neumann Laplacian in such a perforated domain $\Omega_{\varepsilon} \setminus \Gamma_{\varepsilon}$. In the next subsection we recall the problem setting and known results.

1.1 The Neumann sieve problem

Let $\Omega$ be a domain in $\mathbb{R}^n$, and $\Gamma$ be a hyperplane intersecting $\Omega$ and dividing it on two subsets $\Omega^+$ and $\Omega^-$. Further, let $\varepsilon > 0$ be a small parameter, and $\{D_{k,\varepsilon}, k = 1, 2, 3 \ldots\}$ be a family of small subsets of $\Gamma \cap \Omega$ ("holes"). We set

$$
\Omega_{\varepsilon} = \Omega^+ \cup \Omega^- \cup \left( \bigcup_{k \in \mathbb{N}} D_{k,\varepsilon} \right) = \Omega \setminus \Gamma_{\varepsilon}, \quad \text{where } \Gamma_{\varepsilon} = \Gamma \setminus \left( \bigcup_{k} D_{k,\varepsilon} \right) \text{ ("sieve").}
$$

Since the set $\Gamma_{\varepsilon}$ has Lebesgue measure zero, the Hilbert spaces $L^2(\Omega_{\varepsilon})$ and $L^2(\Omega)$ coincide. In the following, we will use the same notation $\mathcal{H}$ for both these spaces.

Let $f \in \mathcal{H}$. We consider the following problem in $\Omega_{\varepsilon}$:

Find $u_{\varepsilon} \in H^1(\Omega_{\varepsilon}) : (u_{\varepsilon}, v)_{H^1(\Omega_{\varepsilon})} = (f, v)_{\mathcal{H}}, \forall v \in H^1(\Omega_{\varepsilon}).$ (1.1)

It is well-known that this problem has the unique solution $u_{\varepsilon}$, and the mapping $\mathcal{R}_{\varepsilon} : f \mapsto u_{\varepsilon}$ defines a bounded self-adjoint invertible operator in $\mathcal{H}$. The operator $\mathcal{A}_{\varepsilon} = \mathcal{R}_{\varepsilon}^{-1} - I$ is called the Neumann Laplacian in $\Omega_{\varepsilon}$ (hereinafter, the notation I stands either for the identity operator or the identity matrix).

Homogenization theory is aimed to describe the asymptotic behavior of the solution $u_{\varepsilon}$ as $\varepsilon \to 0$, when the number of holes tends to infinity, while their diameters tend to zero.

Assume for simplicity that the holes $D_{k,\varepsilon}$ are identical (up to a rigid motion) and distributed periodically along the $\varepsilon$-periodic lattice on $\Gamma$. It is well-known [4, 5, 21, 22, 32, 34, 36, 38–40] (for more details see the overview of existing literature in Sect. 1.3) that if the size of the holes is carefully chosen, namely, one has

$$
\exists \lim_{\varepsilon \to 0} \frac{C_{\varepsilon}}{4\varepsilon^{n-1}} = \gamma \geq 0, \quad \text{where } C_{\varepsilon} \text{ is the capacity of } D_{k,\varepsilon},
$$

then

$$
\|u_{\varepsilon} - u\|_{\mathcal{H}} \to 0 \text{ as } \varepsilon \to 0,
$$

and the limiting function $u \in H^1(\Omega \setminus \Gamma)$ is the unique solution to the following problem:

Find $u \in H^1(\Omega \setminus \Gamma) : (u, v)_{H^1(\Omega \setminus \Gamma)} + (\gamma[u], [v])_{L^2(\Gamma \cap \Omega)} = (f, v)_{\mathcal{H}}, \forall v \in H^1(\Omega \setminus \Gamma).$ (1.4)
Here \([\cdot]\) stands for the jump of enclosed function across \(\Gamma\). It is easy to see that \(u\) is a weak solution to the boundary value problem

\[
\begin{align*}
-\Delta u^+ + u^+ &= f^+ \text{ on } \Omega^+,
-\Delta u^- + u^- &= f^- \text{ on } \Omega^-,
-\frac{\partial u^+}{\partial \nu^+} &= \frac{\partial u^-}{\partial \nu^-} = \gamma(u^+ - u^-) \text{ on } \Gamma \cap \Omega,
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

where \(u^\pm = u \restriction_{\Omega^\pm}\), \(f^\pm = f \restriction_{\Omega^\pm}\), \(\frac{\partial}{\partial \nu^\pm}\) is the derivative along the outward (with respect to \(\Omega^\pm\)) normal to \(\Gamma\), \(\partial \nu\) is the derivative along the unit normal to \(\partial \Omega\). We define the homogenized operator \(\mathcal{A}\) by \(\mathcal{A} = \mathcal{R}^{-1} - I\), where the bounded invertable operator \(\mathcal{R}\) in \(\mathcal{H}\) is given by \(\mathcal{R} : f \mapsto u\). The operator is self-adjoint and non-negative. Note that in some recent works (see, e.g., [7, 8]) the interface conditions \(-\frac{\partial u^+}{\partial \nu^+} = \frac{\partial u^-}{\partial \nu^-} = \gamma(u^+ - u^-)\) are called \(\delta^\prime\)-interaction of the strength \(\gamma\).

Similar result holds if \(\Gamma\) is a closed smooth surface: under suitable assumptions on the holes \(D_{k, \varepsilon} \subset \Gamma\), which resemble (1.2), the homogenized problem has the form (1.4) with some non-negative and, in general, non-constant function \(\gamma\).

### 1.2 Main results

In the aforementioned papers the convergence (1.3) was established for a fixed \(f \in \mathcal{H}\). In the language of operator theory this means that \(\mathcal{A}_\varepsilon\) converges to the operator \(\mathcal{A}\) as \(\varepsilon \to 0\) in the strong resolvent sense. In the current work we are aimed to upgrade (1.3) to the norm resolvent convergence and derive estimates on its rate. In the homogenization literature (see the overview below) they are usually called operator estimates.

Our first result (see Theorem 2.2) is the estimate

\[
\|u_\varepsilon - u\|_{\mathcal{H}} \leq \mu_\varepsilon \|f\|_{\mathcal{H}}
\]

with some \(\mu_\varepsilon \to 0\) as \(\varepsilon \to 0\) (see (2.17)). An immediate consequence of this result is a convergence of spectra of the underlying operators in the Hausdorff metrics (see Corollary 2.3). Secondly, besides the convergence in the \(\mathcal{H} \to \mathcal{H}\) operator norm, we also establish the convergence in the \(\mathcal{H} \to H^1(\Omega \setminus \Gamma)\) operator norm. In this case, to get a reasonable result one needs a special corrector; the corresponding estimate reads (see Theorem 2.4):

\[
\|u_\varepsilon - u - w_\varepsilon^\pm\|_{H^1(\Omega^\pm)} \leq \mu_\varepsilon \|f\|_{\mathcal{H}}.
\]

The correctors \(w_\varepsilon^\pm \in H^1(\Omega^\pm)\) are supported in a small neighborhood of \(\Gamma\) and satisfy

\[
\|w_\varepsilon^\pm\|_{L^2(\Omega^\pm)} = o(\mu_\varepsilon) \|f\|_{\mathcal{H}}, \quad \|\nabla w_\varepsilon^\pm\|_{L^2(\Omega^\pm)}^2 = \frac{1}{2} \|\gamma^{1/2}[u]\|_{L^2(\Gamma)}^2 + O(\kappa_\varepsilon) \|f\|_{\mathcal{H}}^2,
\]

with some \(\kappa_\varepsilon \to 0\) as \(\varepsilon \to 0\) (see (2.11)).

The assumptions we impose on the shapes and locations of the holes are rather general (i.e., beyond periodic setting). In the last section we present two examples for which these assumptions are satisfied.

We restrict ourselves to the case when the domain \(\Omega\) is unbounded and the hyperplane \(\Gamma\) belongs to \(\Omega\), moreover \(\text{dist}(\Gamma, \partial \Omega) > 0\). For example, \(\Omega\) may coincide with the whole \(\mathbb{R}^n\) or \(\Omega\) may be an unbounded waveguide-like domain. The case when \(\Gamma\) intersects \(\partial \Omega\)
is not considered in this work (except for the case, when the holes are located within a subset $\Gamma' \subset \Gamma \cap \Omega$ such that $\text{dist}(\Gamma', \partial \Omega) > 0$–we briefly comment on it in Remark 2.5). The technical difficulties arising when the holes approaches $\partial \Omega \cap \Gamma$ are explained in Remark 3.16. We postpone the investigation of this case, as well as the case of $\Gamma$ being a closed surface, to a subsequent work.

Our proof is based on the abstract result from [2], which is a part of the toolbox developed in [41, 42] for studying resolvent and spectral convergence of operators in varying Hilbert spaces. Originally this toolbox was developed to handle convergence of the Laplace-Beltrami operator on manifolds which shrink to a graph. It also has shown to be effective for homogenization problems in perforated domains, see [2, 30, 31]. The main ingredient of the proof is a construction of a suitable operator from $H^1(\Omega \setminus \Gamma)$ to $H^1(\Omega_\varepsilon)$ (these spaces are the domains of the sesquilinear forms associated with the operators $\mathcal{A}_\varepsilon$ and $\mathcal{A}$).

1.3 Literature

1.3.1 The sieve problem

The Neumann sieve problem was treated first in [36]. In this article $\Gamma$ is a closed surface, and the holes $D_{k,\varepsilon} \subset \Gamma$ obey quite general assumptions (though, resembling (1.2)). It was proven that the Green’s function associated with the operator $\mathcal{A}_\varepsilon$ converges pointwise to the Green’s function associated with the operator $\mathcal{A}$ as $\varepsilon \to 0$. The proof in [36] is based on methods of potential theory. Later, in [32] the convergence of the resolvents of $\mathcal{A}_\varepsilon$ has been proven by using variational methods; in this work the author considered sieves with even more general geometry. Both articles [32, 36] deal with $n = 3$. For more details we refer to the monograph [34, Chapter 3].

In the mid 80’s the problem was revisited in [4, 5, 22, 38, 40]. In [4, 22, 38, 40] the sieve $\Gamma_\varepsilon$ is flat (a subset of a hyperplane) and the holes are distributed periodically, while in [5] $\Gamma$ is a closed surface, and the holes $D_{k,\varepsilon} \subset \Gamma$ satisfy some general assumptions. The proofs in the above papers are based on variational methods. We also mention later articles [21, 39], where the periodic Neumann sieve problem was studied via unfolding method; in particular, in [21] the authors treated not only Laplacians, but also more general elliptic operators.

Finally, we refer to the articles [3, 23] dealing with a sieve having non-zero thickness, i.e. $\Gamma_\varepsilon$ is thin layer perforated by a lot of narrow passages connecting the upper and lower faces of this layer.

1.3.2 Operator estimates

Derivation of operator estimates is a rather young topic in homogenization theory originating from the pioneer works [9, 10, 25, 26, 44–46] (see also the overview [47]) concerning homogenization of periodic elliptic operators with rapidly oscillating coefficients.

In the last decade there appeared a plenty of works where operator estimates were established for homogenization problems in domains with a lot of holes. The case when holes are distributed in the entire domain was addressed in [43, 46, 47] (Neumann conditions on the boundary of holes), [2, 31] (Dirichlet conditions, also [2] treats Neumann holes), [30] (linear Robin conditions), [11, 17] (Dirichlet and nonlinear Robin conditions). In [30, 43, 44] the holes are distributed periodically (in [31]–locally periodically), and are identical, while in [2, 11, 17] quite general assumptions on sizes and location of holes are imposed. The surface distribution of holes was treated in [15, 18, 19, 24].
We also mention the closely related works [12–14, 20] and [16], where operator estimates were deduced, respectively, for elliptic operators with frequently alternating boundary conditions and for boundary value problems in domains with fast oscillating boundary.

2 Setting of the problem and main results

Let \( n \in \mathbb{N}, n \geq 2 \). In the following, \( x' = (x^1, \ldots, x^{n-1}) \) and \( x = (x', x^n) \) stand for the Cartesian coordinates in \( \mathbb{R}^{n-1} \) and \( \mathbb{R}^n \), respectively. Also, by \( B(r, z) \) we denote the open ball in \( \mathbb{R}^n \) of radius \( r > 0 \) and center \( z \in \mathbb{R}^n \). By \( C, C_1, C_2, \ldots \) we denote generic positive constants being independent of \( \varepsilon \); note that these constants may vary from line to line.

Let \( \Gamma_1 \) be a hyperplane given by
\[
\Gamma_1 := \{ x = (x', x^n) \in \mathbb{R}^n : x^n = 0 \},
\]
and \( \Omega_1 \) be an unbounded Lipschitz domain satisfying
\[
\Gamma_1 \subset \Omega_1, \quad \text{dist}(\Gamma_1, \partial \Omega_1) > 0 \tag{2.1}
\]
We set
\[
\Omega_1^{\pm} := \Omega_1 \cup \{ x = (x', x^n) \in \mathbb{R}^n : \pm x^n > 0 \}.
\]
We connect \( \Omega_1^+ \) and \( \Omega_1^- \) by making a lot of holes in \( \Gamma_1 \).

Let \( \varepsilon \in (0, \varepsilon_0] \) be a small parameter.

Let \( \{ D_{k, \varepsilon} \subset \Gamma_1, k \in \mathbb{N} \} \) be a family of relatively open in \( \Gamma_1 \) connected sets; below we will make further assumptions on their locations and sizes. We define the “sieve” \( \Gamma_\varepsilon \) as follows (see Fig. 2):
\[
\Gamma_\varepsilon := \Gamma \setminus \left( \bigcup_{k \in \mathbb{N}} D_{k, \varepsilon} \right).
\]
The resulting domain \( \Omega_\varepsilon \) (see Fig. 3) is given by
\[
\Omega_\varepsilon := \Omega^+ \cup \Omega^- \cup \left( \bigcup_{k \in \mathbb{N}} D_{k, \varepsilon} \right) = \Omega \setminus \Gamma_\varepsilon.
\]

Next we introduce the Neumann Laplacian \( A_\varepsilon \) in \( \Omega_\varepsilon \). Recall that we use the notation \( \mathcal{H} \) for both spaces \( L^2(\Omega_\varepsilon) \) and \( L^2(\Omega) \), which coincide since the set \( \Gamma_\varepsilon \) has measure zero. We define the sesquilinear form \( a_\varepsilon \) in \( \mathcal{H} \) by
\[
a_\varepsilon[u, v] = (\nabla u, \nabla v)_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \sum_{i=1}^{n} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^i} \, dx, \quad \text{dom}(a_\varepsilon) = H^1(\Omega_\varepsilon). \tag{2.2}
\]
This form is symmetric, densely defined, closed, and positive. By the first representation theorem [28, Chapter 6, Theorem 2.1], there exists the unique self-adjoint and positive operator \( \mathcal{A}_\varepsilon \) associated with \( a_\varepsilon \), i.e. \( \text{dom}(\mathcal{A}_\varepsilon) \subset \text{dom}(a_\varepsilon) \) and
\[
\forall u \in \text{dom}(\mathcal{A}_\varepsilon), \forall v \in \text{dom}(a_\varepsilon) : \mathcal{A}_\varepsilon u, v)_{\mathcal{H}} = a_\varepsilon[u, v].
\]
Note that \( u_\varepsilon = (\mathcal{A}_\varepsilon + I)^{-1} f \) is the solution to the problem (1.1).

Our goal is to describe the behaviour of the resolvent of \( \mathcal{A}_\varepsilon \) as \( \varepsilon \to 0 \) under the assumptions on the distribution and sizes of the holes \( D_{k, \varepsilon} \) stated below (see (2.3)–(2.5) and (2.11)).
Fig. 2 The sieve $\Gamma_\varepsilon$. The larger discs with dotted boundary correspond to the sets $B_{k,\varepsilon} \cap \Gamma$; due to (2.3) these discs are pairwise disjoint. The smaller discs with dotted boundary correspond to the sets $\mathcal{B}(d_{k,\varepsilon}, x_{k,\varepsilon}) \cap \Gamma$. The figure respects the assumption (2.8)

Let $d_{k,\varepsilon}$ and $x_{k,\varepsilon}$ be the radius and the center of the smallest ball $\mathcal{B}(d_{k,\varepsilon}, x_{k,\varepsilon})$ containing $D_{k,\varepsilon}$; evidently, $x_{k,\varepsilon} \in \Gamma$. We make the following assumptions: there exist a sequence $\{\varrho_k, \varepsilon, k \in \mathbb{N}\}$ of positive numbers such that

$$B_{k,\varepsilon} \cap B_{l,\varepsilon} = \emptyset \quad \text{for} \quad k \neq l,$$

where $B_{k,\varepsilon} := \mathcal{B}(\varrho_k, x_{k,\varepsilon})$, (2.3)

$$\sup_{k \in \mathbb{N}} \varrho_k, \varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$ (2.4)

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \sup_{k \in \mathbb{N}} \gamma_k, \varepsilon < \infty,$$ (2.5)

where the numbers $\gamma_k, \varepsilon$ are defined by

$$\gamma_k, \varepsilon := \begin{cases} d_{k,\varepsilon}^{n-2} \varrho_k^{1-n}, & n \geq 3, \\ |\ln d_{k,\varepsilon}|^{-1} \varrho_k^{-1}, & n = 2. \end{cases}$$

It follows easily from (2.4) and (2.5) that

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \sup_{k \in \mathbb{N}} \varrho_k, \varepsilon < 1,$$ (2.6)

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \sup_{k \in \mathbb{N}} \gamma_k, \varepsilon < \frac{\text{dist}(\Gamma, \partial \Omega)}{2},$$ (2.7)

$$\forall \varepsilon \in (0, \varepsilon_0), \forall k \in \mathbb{N}: \quad d_{k,\varepsilon} \leq \frac{\varrho_k, \varepsilon}{8}$$ (2.8)

for small enough $\varepsilon_0$; in particular, (2.3) and (2.8) imply $D_{k,\varepsilon} \cap D_{l,\varepsilon} = \emptyset$ for $k \neq l$. In the following we assume that $\varepsilon_0$ is small enough so that the conditions (2.6)–(2.8) are satisfied.

To formulate our last assumption, we introduce the capacity-type quantity $\mathcal{C}(D_{k,\varepsilon})$ by

$$\mathcal{C}(D_{k,\varepsilon}) = \inf_{U \in H_0^1(B_{k,\varepsilon})} \{U |_{\mathcal{B}_{k,\varepsilon}} = 1 \| \nabla U \|_{L^2(B_{k,\varepsilon})}^2 \}.$$ (2.9)

Note that $\mathcal{C}(D_{k,\varepsilon}) > 0$ despite $D_{k,\varepsilon}$ has Lebesgue measure zero. We also introduce the set

$$S_{k,\varepsilon} := B_{k,\varepsilon} \cap \Gamma,$$ (2.10)

and denote by $\langle v \rangle_{S_{k,\varepsilon}}$ the mean value of a function $v : \Gamma \rightarrow \mathbb{C}$ in $S_{k,\varepsilon}$, i.e.

$$\langle v \rangle_{S_{k,\varepsilon}} := \frac{1}{\text{area}(S_{k,\varepsilon})} \int_{S_{k,\varepsilon}} v(x) \, dx = (x', 0) \in S_{k,\varepsilon}.$$

\copyright Springer
Then our last assumption reads as follows: there exists $\gamma \in C^1(\Gamma) \cap W^{1,\infty}(\Gamma)$ such that

$$
\left| \frac{1}{4} \sum_{k \in \mathbb{N}} \mathcal{G}(D_{k,\varepsilon})(g) S_{k,x} \langle \overline{h} \rangle S_{k,x} - (\gamma g, h)_{L^2(\Gamma)} \right| \leq \kappa \varepsilon \| g \|_{H^{3/2}\nabla^1(\Gamma)} \| h \|_{H^{1/2}(\Gamma)} \text{ with } \kappa \varepsilon \to 0 \quad (2.11)
$$

for any $g \in H^{3/2}(\Gamma)$ and $h \in H^{1/2}(\Gamma).$ In Sect. 4 we present two examples on the distribution and the sizes of the holes for which the condition (2.11) is fulfilled.

Now, we introduce the limiting operator $A.$ In the following, for $f \in \mathcal{H},$ we denote by $f^+$ and $f^-$ the restrictions of $f$ on $\Omega^+$ and $\Omega^-$, respectively. The same notations $f^\pm$ will be used for the traces of $f^\pm$ on $\Gamma$ provided $f \in H^1(\Omega \setminus \Gamma)$ (in this case the traces $f^\pm$ are indeed well-defined and belong to $H^{1/2}(\Gamma)$). In $\mathcal{H}$ we define the sesquilinear form $a$ via

$$
a[f, g] = (\nabla f^+, \nabla g^+)_L^2(\Omega^+) + (\nabla f^-, \nabla g^-)_L^2(\Omega^-) + (\gamma [f], [g])_{L^2(\Gamma)}, \quad \text{dom}(a) = H^1(\Omega \setminus \Gamma), \quad (2.12)
$$

where $[\cdot]$ stands for the jump of the enclosed function across $\Gamma$:

$$
[h] := (h^+ - h^-) |_{\Gamma}, \quad h \in H^1(\Omega \setminus \Gamma).
$$

This form is symmetric, densely defined, closed, and non-negative [7, Proposition 3.1]). We denote by $\mathcal{A}$ the associated self-adjoint operator. One has [7, Theorem 3.3]:

$$
f \in \text{dom}(\mathcal{A}) \iff \begin{cases}
f^\pm \in H^1(\Omega^\pm), \\
\Delta f^\pm \in L^2(\Omega^\pm), \\
\frac{\partial f^\pm}{\partial n^\pm} \in L^2(\partial\Omega^\pm), \quad \frac{\partial f^\pm}{\partial n^\pm} |_{\Gamma^\pm} = \mp \gamma [f], \quad \frac{\partial f^\pm}{\partial n^\pm} |_{\partial\Omega^\pm \setminus \Gamma} = 0,
\end{cases} \quad (2.13)
$$

where $\frac{\partial}{\partial n^\pm}$ is the derivative along the outward (with respect to $\Omega^\pm$) pointing normal to $\partial\Omega^\pm.$

The operator $\mathcal{A}$ acts as follows:

$$
(\mathcal{A} f)^\pm = -\Delta f^\pm.
$$

In fact, the functions from dom($\mathcal{A}$) possesses even more regularity, but we postpone the corresponding statement until the proof of main results, see Lemma 3.7.
Remark 2.1 It is easy to show that if (2.3)–(2.5) hold and, moreover, the left-hand-side of (2.11) converge to zero for any fixed \( g, h \in C_0^\infty (\Gamma) \), then

\[
\frac{1}{4} \sum_{k \in \mathbb{N}} \mathcal{G}(D_{k,\varepsilon}) \delta_{x_k,\varepsilon} \rightharpoonup \gamma \text{ in } \mathcal{D}'(\Gamma) \text{ as } \varepsilon \to 0, \tag{2.14}
\]

where \( \delta_{x_k,\varepsilon} \) is the Dirac delta function supported at \( x_k, \varepsilon \). Furthermore, if the domain \( \Omega \) is bounded, one can prove that \( \mathcal{A}_\varepsilon \) converges to \( \mathcal{A} \) is the strong resolvent sense (i.e., (1.3) holds) provided the assumptions (2.3)–(2.5) and (2.14) are satisfied. However, to derive operator estimates we require stronger assumption (2.11) instead of (2.14).

We are now in position to formulate the main results of this work. For \( k \in \mathbb{N} \) we set

\[
\eta_{k,\varepsilon} := \begin{cases} \frac{d_{k,\varepsilon}}{\rho_{k,\varepsilon}}, & n \geq 5, \\ \frac{d_{k,\varepsilon}}{\rho_{k,\varepsilon}} \ln \frac{d_{k,\varepsilon}}{\rho_{k,\varepsilon}}, & n = 4, \\ \left( \frac{d_{k,\varepsilon}}{\rho_{k,\varepsilon}} \right)^{1/2}, & n = 3, \\ \left( \frac{\ln d_{k,\varepsilon}}{\rho_{k,\varepsilon}} \right)^{-1/2}, & n = 2. \end{cases} \tag{2.15}
\]

It follows from (2.4), (2.5) that \( \sup_{k \in \mathbb{N}} \eta_{k,\varepsilon} \to 0 \) as \( \varepsilon \to 0 \).

Theorem 2.2 One has

\[
\forall f \in \mathcal{H} : \quad \| (\mathcal{A}_\varepsilon + 1)^{-1} f - (\mathcal{A} + 1)^{-1} f \|_{\mathcal{H}} \leq C_{\mu,\varepsilon} \| f \|_{\mathcal{H}}, \tag{2.16}
\]

where \( \mu_{\varepsilon} \) is defined by

\[
\mu_{\varepsilon} := \begin{cases} \max_{k \in \mathbb{N}} \{ \kappa_{k,\varepsilon} ; \sup_{k \in \mathbb{N}} \eta_{k,\varepsilon} \}, & n \geq 3, \\ \max_{k \in \mathbb{N}} \{ \kappa_{k,\varepsilon} ; \sup_{k \in \mathbb{N}} \left( \frac{1}{\rho_{k,\varepsilon}} \ln \rho_{k,\varepsilon} \right) \}, & n = 2. \end{cases} \tag{2.17}
\]

An immediate consequence of Theorem 2.2 is a convergence of spectra. Recall that for closed sets \( X, Y \subset \mathbb{R} \) the Hausdorff distance \( d_H(X, Y) \) is given by

\[
d_H(X, Y) := \max \left\{ \sup_{x \in X} \inf_{y \in Y} |x - y| ; \sup_{y \in Y} \inf_{x \in X} |y - x| \right\}.
\]

Corollary 2.3 One has:

(A) For any \( \lambda \in \sigma(\mathcal{A}) \) there exists a family \( (\lambda_{k,\varepsilon})_{\varepsilon>0} \) with \( \lambda_{k,\varepsilon} \in \sigma(\mathcal{A}_\varepsilon) \) such that \( \lambda_{k,\varepsilon} \to \lambda \) as \( \varepsilon \to 0 \);

(B) For any \( \lambda \in \mathbb{R} \setminus \sigma(\mathcal{A}) \) there exist \( \delta > 0 \) such that \( \sigma(\mathcal{A}_\varepsilon) \cap (\lambda - \delta, \lambda + \delta) = \emptyset \) for sufficiently small \( \varepsilon \).

Moreover, one has the estimate

\[
d_H(\sigma((\mathcal{A}_\varepsilon + 1)^{-1}), \sigma((\mathcal{A} + 1)^{-1}) \leq C_{\mu,\varepsilon}. \tag{2.18}
\]

Proof For two normal bounded operators \( R_1 \) and \( R_2 \) in the Hilbert space \( \mathcal{H} \) one has the following inequality (see, e.g., [27, Lemma A.1]):

\[
d_H(\sigma(R_1), \sigma(R_2)) \leq \| R_1 - R_2 \|_{\mathcal{H}}.
\]

Applying it for \( R_1 := (\mathcal{A}_\varepsilon + 1)^{-1} \) and \( R_2 := (\mathcal{A} + 1)^{-1} \) and taking into account (2.16), we arrive at the estimate (2.18). By virtue of [27, Lemma A.2] the statements (A) and (B) follow immediately from the fact \( d_H(\sigma((\mathcal{A}_\varepsilon + 1)^{-1}), \sigma((\mathcal{A} + 1)^{-1}) \to 0 \) as \( \varepsilon \to 0 \). \( \square \)
The second main theorem establishes the closeness of resolvents \((\mathcal{A}_\varepsilon + I)^{-1}\) and \((\mathcal{A} + I)^{-1}\) in the \((L^2 \to H^1)\) operator norm. Here, to get a reasonable result, one needs to employ a special corrector. We denote

\[
B_{k,\varepsilon}^\pm := B_{k,\varepsilon} \cap \Omega^\pm.
\]

Let \(U_{k,\varepsilon} \in H^1_0(B_{k,\varepsilon})\) be the solution to the problem

\[
\begin{aligned}
\Delta U &= 0, \quad x \in B_{k,\varepsilon} \setminus \overline{D_{k,\varepsilon}}, \\
U &= 1, \quad x \in \partial D_{k,\varepsilon} = \overline{D_{k,\varepsilon}}, \\
U &= 0, \quad x \in \partial B_{k,\varepsilon}.
\end{aligned}
\]

It is well-known that

\[
\|\mathcal{C}(D_{k,\varepsilon})\|_{L^2(B_{k,\varepsilon})}^2
\]

We introduce the operator \(\mathcal{K}_\varepsilon : \mathcal{H} \to H^1(\Omega \setminus \Gamma)\) via

\[
(\mathcal{K}_\varepsilon f)(x) = \begin{cases} 
-\frac{1}{\varepsilon} \langle [g] \rangle_{S_{k,\varepsilon}} U_{k,\varepsilon}^+(x), & x \in B_{k,\varepsilon}^+, \\
\frac{1}{\varepsilon} \langle [g] \rangle_{S_{k,\varepsilon}} U_{k,\varepsilon}^-(x), & x \in B_{k,\varepsilon}^-, \\
0, & x \in \Omega \setminus (\cup_{k \in \mathbb{N}} B_{k,\varepsilon}),
\end{cases}
\]

where \(g = (\mathcal{A} + I)^{-1} f\).

Here \(U_{k,\varepsilon}^\pm := U_{k,\varepsilon} \upharpoonright_{B_{k,\varepsilon}^\pm}\); recall that \([g] = (g^+ - g^-) \mid \Gamma\), and \(\langle [g] \rangle_{S_{k,\varepsilon}}\) stands for the mean value of \([g]\) in \(S_{k,\varepsilon}\).

**Theorem 2.4** One has

\[
\forall f \in \mathcal{H} : \|((\mathcal{A}_\varepsilon + I)^{-1} f - (\mathcal{A} + I)^{-1} f - \mathcal{K}_\varepsilon f\|_{H^1(\Omega \setminus \Gamma)} \leq C \varepsilon \|f\|_{\mathcal{H}}.
\]

The correcting term \(\mathcal{K}_\varepsilon f\) obeys the following properties:

\[
\|(\mathcal{K}_\varepsilon f)^\pm\|_{L^2(\Omega^\pm)} \leq C \sup_{k \in \mathbb{N}} \epsilon \frac{1}{\sqrt{k}} \|f\|_{\mathcal{H}},
\]

\[
\|\nabla(\mathcal{K}_\varepsilon f)^\pm\|_{L^2(\Omega^\pm)} \leq \frac{1}{2} \|\gamma^{1/2} [g]\|_{L^2(\Gamma)}^2 + O(\epsilon) \|f\|_{\mathcal{H}}^2, \quad \text{where } g = (\mathcal{A} + I)^{-1} f.
\]

**Remark 2.5** Besides \(\Omega\) satisfying (2.1) one can also address the case when \(\Gamma\) intersects \(\Omega\). However, here we face technical difficulties that so far we cannot overcome—see Remark 3.16 for details. The only case, which we still are able to handle is when the holes are located non-trivial interface conditions. The proof is similar to the case (2.1).

We assume that \(\{D_{k,\varepsilon} \subset \Gamma\}, \quad k \in \mathbb{N}_0 \subseteq \mathbb{N}\) are relatively open connected sets satisfying the assumptions (2.3)–(2.5) (with \(k \in \mathbb{N}\) being replaced by \(k \in \mathbb{M}_0\)), and there is a function \(\gamma \in C^1(\Gamma) \cap W^{1,\infty}(\Gamma)\) with \(\text{supp}(\gamma) \subset \Gamma\) such that the assumption (2.11) is fulfilled. In this case Theorems 2.2 and 2.4 remain valid with the homogenized operator \(\mathcal{A}_\varepsilon\) being defined in a similar way. The assumption \(\text{supp}(\gamma) \subset \Gamma\) means that we get non-trivial interface conditions on \(\Gamma\), while the set \((\Gamma \cap \Omega) \setminus \Gamma\) is impenetrable—from both sizes we have the Neumann conditions. The proof is similar to the case (2.1).

**Remark 2.6** Instead of Laplacians one can also consider Schrödinger operators. Namely, let \(V \in L^\infty(\Omega)\) be real, then the estimates (2.16), (2.23) hold true with \(\mathcal{A}_\varepsilon + I\) and \(\mathcal{A} + I\) being replaced by \(\mathcal{A}_\varepsilon + V - \Lambda I\) and \(\mathcal{A} + V - \Lambda I\), respectively; \(\Lambda \in \mathbb{R}\) is small enough in order to be in the resolvent sets of \(\mathcal{A}_\varepsilon + V\) and \(\mathcal{A} + V\) (for example, one can choose \(\Lambda = -\|V\|_{L^\infty(\Omega)} - 1\)).
Remark 2.7 The choice of the boundary conditions on $\partial \Omega$ is inessential: the above results remain valid if we impose Dirichlet, Robin, mixed or any other $\varepsilon$-independent conditions on $\partial \Omega$ (of course, the homogenized operator then must be changed accordingly).

Remark 2.8 Interestingly, the form of $\eta_{k, \varepsilon}$ changes drastically when passing through $n = 4$. The reason for this peculiarity is the Sobolev-type inequality (3.19) playing the key role in the proof of the estimate (3.66)—the only estimate where $\eta_{k, \varepsilon}$ pops up. To achieve a better convergence rate in (3.66) we utilize (3.19) with the largest $p$ for which it is fulfilled. For $n = 2, 3$ this largest $p$ is $p = \infty$, for $n \geq 5$ it is $p = \frac{2n}{n-4}$, while for $n = 4$ the largest $p$ does not exist (for $n = 4$ one has the Sobolev embedding $H^2 \hookrightarrow L^p$ for $p < \infty$, but not for $p = \infty$), and in order to get the best convergence rate in the case $n = 4$, we choose the special $\varepsilon$-dependent $p$ (see (3.70)) leading to the appearance of a logarithm at the expression for $\eta_{k, \varepsilon}$. Note that the same convergence rate $\eta_{k, \varepsilon}$ appears also in the operator estimates for homogenization of the Robin Laplacian in a domain with a lot of tiny holes [30].

The remaining part of the work is organized as follows. In the next section we present the proof of Theorems 2.2 and 2.4. In Sect. 3.1 we collect several useful estimates which will be widely used further. In Sect. 3.2 we recall the abstract result from [2] for studying convergence of operators in varying Hilbert spaces. In Sect. 3.3 we verify the conditions of this abstract result in our concrete setting. The proofs of are completed in Sect. 3.4. Finally, in Sect. 4 we present two examples for which the condition (2.11) is fulfilled.

3 Proof of main results

3.1 Auxiliary estimates

In the following, by $\langle v \rangle_B$ we denote the mean value of the function $v$ in the measurable bounded set $B$ with $\text{vol}(B) \neq 0$ (hereinafter, $\text{vol}(B)$ stands for the volume of $B$), i.e.,

$$\langle v \rangle_B := \frac{1}{\text{vol}(B)} \int_B v(x) \, dx.$$  

The same notation will be use for the mean value of a function defined on a subset $S$ of the hyperplane $\Gamma$ with area$(S) \neq 0$:

$$\langle v \rangle_S := \frac{1}{\text{area}(S)} \int_S v(x') \, dx', \quad x = (x', 0).$$

Recall that $S_{k, \varepsilon}^\pm$ and $B_{k, \varepsilon}^\pm$ are defined by (2.10) and (2.19), respectively.

Lemma 3.1 One has

$$\forall v \in H^1(B_{k, \varepsilon}^\pm) : \|v - \langle v \rangle_{B_{k, \varepsilon}^\pm} \|_{L^2(B_{k, \varepsilon}^\pm)}^2 \leq C \varrho_{k, \varepsilon}^2 \|\nabla v\|_{L^2(B_{k, \varepsilon}^\pm)}^2,$$  

(3.1)

$$\forall v \in H^1_0(B_{k, \varepsilon}) : \|v\|_{L^2(B_{k, \varepsilon})}^2 \leq C \varrho_{k, \varepsilon}^2 \|\nabla v\|_{L^2(B_{k, \varepsilon})}^2,$$  

(3.2)

$$\forall v \in H^1(B_{k, \varepsilon}^\pm) : \|v\|_{L^2(B_{k, \varepsilon}^\pm)}^2 \leq C \left( \varrho_{k, \varepsilon} \|v\|_{L^2(S_{k, \varepsilon})}^2 + \varrho_{k, \varepsilon}^2 \|\nabla v\|_{L^2(B_{k, \varepsilon}^\pm)}^2 \right),$$  

(3.3)

$$\forall v \in H^1(B_{k, \varepsilon}^\pm) : |\langle v \rangle_{S_{k, \varepsilon}} - \langle v \rangle_{B_{k, \varepsilon}^\pm}|^2 \leq C \varrho_{k, \varepsilon}^{2-n} \|\nabla v\|_{L^2(B_{k, \varepsilon}^\pm)}^2.$$  

(3.4)
Proof We denote $B := B(1, 0)$, $B^± := B \cap \{x ∈ ℝ^n : ±x^n > 0\}$, $S := B \cap Γ$. We have the following standard Poincaré-type inequalities:

$$∀ v ∈ H^1(B^±) : \|v − ⋅ B^±\|_{L^2(B^±)}^2 \leq \Lambda_N^{-1} \|∇v\|_{L^2(B^±)}^2,$$

(3.5)

$$∀ v ∈ H^1_0(B) : \|v\|_{L^2(B)}^2 \leq \Lambda_D^{-1} \|∇v\|_{L^2(B)}^2,$$

(3.6)

$$∀ v ∈ H^1(B^±) : \|v\|_{L^2(B^±)}^2 \leq \Lambda_R^{-1} \left(\|v\|_{L^2(S)}^2 + \|∇v\|_{L^2(B^±)}^2\right),$$

(3.7)

where $Λ_N > 0$ is the smallest non-zero eigenvalue of the Neumann Laplacian on $B^±$, $Λ_D > 0$ is the smallest eigenvalue of the Dirichlet Laplacian on $B$, and $Λ_R > 0$ is the smallest eigenvalue of the Laplacian on $B^±$ subject to the Robin boundary conditions $\frac{∂v}{∂ν} + v = 0$ ($ν$ is the unit normal pointed outward of $B^±$) on $S$ and the Neumann boundary conditions on $∂B^± \setminus S$. Then, via the coordinate transformation

$$y = q_{k,ε}^{-1}(x − x_{k,ε}),$$

(3.8)

we reduce (3.5)–(3.7) to (3.1)–(3.3). Furthermore, one has also the trace estimate

$$∀ v ∈ H^1(B^±) : \|v\|_{L^2(S)}^2 \leq C \|v\|_{H^1(B^±)}^2,$$

which, after the coordinate transformation (3.8), reduces to

$$∀ v ∈ H^1(B^±_{k,ε}) : \|v\|_{L^2(S_{k,ε})}^2 \leq C \left(q_{k,ε}^{-1} \|v\|_{L^2(B^±_{k,ε})}^2 + q_{k,ε} \|∇v\|_{L^2(B^±_{k,ε})}^2\right).$$

(3.9)

Using the Cauchy-Schwarz inequality, (3.1) and (3.9), we get the last estimate (3.4):

$$\left|\langle v \rangle_{S_{k,ε}} − \langle v \rangle_{B^±_{k,ε}}\right|^2 \leq \frac{1}{\text{area}(S_{k,ε})} \|v − \langle v \rangle_{B^±_{k,ε}}\|_{L^2(S_{k,ε})}^2 \leq C q_{k,ε}^{-1} \|v − \langle v \rangle_{B^±_{k,ε}}\|_{L^2(B^±_{k,ε})}^2 + q_{k,ε} \|∇v\|_{L^2(B^±_{k,ε})}^2 \leq C q_{k,ε}^{2−n} \|∇v\|_{L^2(B^±_{k,ε})}^2.$$  

The lemma is proven. □

We denote

$$\tilde{d}_{k,ε} := \begin{cases} 2d_{k,ε}, & n ≥ 3, \\ (q_{k,ε}d_{k,ε})^{1/2}, & n = 2, \end{cases}$$

(3.10)

$$\tilde{D}^±_{k,ε} := B(\tilde{d}_{k,ε}, x_{k,ε}) \cap \{x ∈ ℝ^n : ±x^n > 0\}. $$

(3.11)

It follows (2.8) that

$$∀ ε ∈ (0, ε_0), ∀ k ∈ ℕ : \tilde{d}_{k,ε} < \frac{q_{k,ε}}{2}. $$

(3.12)

whence, in particular, $\bar{B}^±_{k,ε} \subset B^±_{k,ε} ∪ S_{k,ε}$.

Lemma 3.2 One has

$$∀ v ∈ H^1(B^±_{k,ε}) : \|v\|_{L^2(\tilde{D}^±_{k,ε})}^2 \leq \begin{cases} C \left(\frac{d_{k,ε}}{q_{k,ε}}\right)^n \|v\|_{L^2(B^±_{k,ε})}^2 + d_{k,ε}^2 \|∇v\|_{L^2(B^±_{k,ε})}^2, & n ≥ 3, \\ C \left(\frac{d_{k,ε}}{q_{k,ε}}\right)^2 \|v\|_{L^2(B^±_{k,ε})}^2 + d_{k,ε} q_{k,ε} |\ln(d_{k,ε} q_{k,ε})| \cdot \|∇v\|_{L^2(B^±_{k,ε})}^2, & n = 2. \end{cases} $$

(3.13)
Proof Evidently, it is enough to prove (3.13) for \( v \in C^\infty(B^-_\varepsilon) \).

We denote \( C^\pm_{k,\varepsilon} := B^\pm_{k,\varepsilon} \setminus \Gamma \). Similarly to (3.3), we obtain the estimate
\[
\|v\|^2_{L^2(B^\pm_{k,\varepsilon})} \leq C \left( \|\tilde{d}_{k,\varepsilon} v\|^2_{L^2(C^\pm_{k,\varepsilon})} + (\tilde{d}_{k,\varepsilon})^2 \|\nabla v\|^2_{L^2(\tilde{D}^\pm_{k,\varepsilon})} \right).
\] (3.14)

We introduce the spherical coordinate \((r, \theta)\) in \(B^\pm_{k,\varepsilon} \setminus \tilde{D}^\pm_{k,\varepsilon}\). Here \( r \in [\tilde{d}_{k,\varepsilon}, \varrho_{k,\varepsilon}] \) stands for the distance to \(x_{k,\varepsilon}, \theta = (\theta_1, \ldots, \theta_{n-1})\) are the angular coordinates \((\theta_k \in [0, \pi])\). One has
\[
\nu(\tilde{d}_{k,\varepsilon}, \theta) = \nu(r, \theta) - \int_{\tilde{d}_{k,\varepsilon}}^r \frac{\partial \nu(\tau, \theta)}{\partial \tau} \mathrm{d}\tau,
\]
whence
\[
|\nu(\tilde{d}_{k,\varepsilon}, \theta)|^2 \leq 2|\nu(r, \theta)|^2 + 2 \left| \int_{\tilde{d}_{k,\varepsilon}}^r \frac{\partial \nu(\tau, \theta)}{\partial \tau} \mathrm{d}\tau \right|^2 
\leq 2|\nu(r, \theta)|^2 + 2M_r \int_{\tilde{d}_{k,\varepsilon}}^\varrho_{k,\varepsilon} \left| \frac{\partial \nu(\tau, \theta)}{\partial \tau} \right|^2 \tau^{n-1} \mathrm{d}\tau
\]
and then integrating over \( r \in [\tilde{d}_{k,\varepsilon}, \varrho_{k,\varepsilon}], \theta_j \in (0, \pi), j = 1, \ldots, n-1 \), we get
\[
\|v\|^2_{L^2(C^\pm_{k,\varepsilon})} \leq 2(\tilde{d}_{k,\varepsilon})^{n-1} \left( (N_{\varepsilon})^{n-1} \left\| v \right\|^2_{L^2(B^\pm_{k,\varepsilon} \setminus \tilde{D}^\pm_{k,\varepsilon})} + M_{\varepsilon} \left\| \nabla v \right\|^2_{L^2(B^\pm_{k,\varepsilon} \setminus \tilde{D}^\pm_{k,\varepsilon})} \right).
\] (3.15)

It is easy to see that
\[
M_{\varepsilon} \leq \begin{cases} C(\tilde{d}_{k,\varepsilon})^{2-n}, & n \geq 3, \\ C|\ln \tilde{d}_{k,\varepsilon}|, & n = 2 \end{cases}
\] (3.16)

(in the case \( n = 2 \) we took into account that, due to (2.6), (3.12), \( \ln \tilde{d}_{k,\varepsilon} < \ln \varrho_{k,\varepsilon} < 0 \)).

Furthermore, due to (3.12), one has
\[
(N_{\varepsilon})^{n-1} \leq C \varrho_{k,\varepsilon}^{n-1}.
\] (3.17)

The required estimate (3.13) follows from (3.10), (3.14)–(3.17) (in the case \( n = 2 \) we also have take into account that, by virtue of (2.6), (3.12), \( |\ln \tilde{d}_{k,\varepsilon}| \geq C > 0 \)). The lemma is proven. \( \square \)

Let \( \square \subset \mathbb{R}^n \) be a cube, and \( \square_\varepsilon \equiv \varepsilon \square \). One has the following estimate [31, Lemma 4.3]:
\[
\forall v \in H^2(\square_\varepsilon) \text{ with } \int_{\square_\varepsilon} v(x) \, \mathrm{d}x = 0 : \ |v|_{L^p(\square_\varepsilon)} \leq C_{n,p} \varepsilon^{n/p+(2-n)/2} |v|_{H^2(\square_\varepsilon)},
\]
provided \( p \) satisfies
\[
1 \leq p \leq \frac{2n}{n-4} \text{ as } n \geq 5, \quad 1 \leq p < \infty \text{ as } n = 4, \quad 1 \leq p \leq \infty \text{ as } n = 2, 3
\] (3.18)
are not able to choose the largest constant \( C \) came from the Sobolev embedding theorem \([1, \text{Theorem 5.4 and Remark 5.5(6)}]\): if \( D \subset \mathbb{R}^n \) is a bounded Lipschitz domain, then the space \( H^2(D) \) is embedded continuously into the space \( L^p(D) \) provided \( p \) satisfies (3.18). Repeating verbatim the proof of \([31, \text{Lemma 4.3}]\) for \( B_{k,\varepsilon}^\pm \cong \partial_{k,\varepsilon} B^\pm \) one arrives on the estimate below.

**Lemma 3.3** One has
\[
\forall v \in H^2(B_{k,\varepsilon}^\pm) \text{ with } \int_{B_{k,\varepsilon}^\pm} v(x) \, dx = 0 : \|v\|_{L^p(B_{k,\varepsilon}^\pm)} \leq C_{n, p} \cdot \varepsilon_{k, \varepsilon}^{n/p+(2-n)/2} \|v\|_{H^2(B_{k,\varepsilon}^\pm)} \tag{3.19}
\]
provided \( p \) satisfies (3.18). The constant \( C_{n, p} \) depends on \( n \) and \( p \) only.

**Remark 3.4** Further we will apply (3.19) for the largest \( p \) satisfying (3.18). For \( n = 4 \) we are not able to choose the largest \( p \) (in the dimension 4 the embedding \( H^2 \leftrightarrow L^p \) holds for any \( p < \infty \), but not for \( p = \infty \)), and in this case we need the following estimate on the constant \( C_{4, p} \) in the right-hand-side of (3.19) \([30, \text{Lemma 4.4}]\):
\[
C_{4, p} \leq Cp, \tag{3.20}
\]
where the constant \( C > 0 \) is independent of \( p \).

### 3.2 Abstract scheme

Let \( \mathcal{H} \) be a Hilbert space, \( \mathcal{A}_\varepsilon \) and \( \mathcal{A} \) be non-negative, self-adjoint, unbounded operators in \( \mathcal{H} \), and \( \alpha_\varepsilon \) and \( \alpha \) be the associated sesquilinear forms. Further, we introduce the energetic spaces \( \mathcal{H}^1_\varepsilon \) and \( \mathcal{H}^1 \) associated with the forms \( \alpha_\varepsilon \) and \( \alpha \), respectively:
\[
\begin{align*}
\mathcal{H}^1_\varepsilon &= \text{dom}(\alpha_\varepsilon), \quad \|u\|_{\mathcal{H}^1_\varepsilon}^2 = \alpha_\varepsilon[u, u] + \|u\|_{\mathcal{H}}^2, \\
\mathcal{H}^1 &= \text{dom}(\alpha), \quad \|f\|_{\mathcal{H}^1}^2 = \alpha[f, f] + \|f\|_{\mathcal{H}}^2.
\end{align*}
\tag{3.21}
\]

We also introduce the Hilbert space \( \mathcal{H}^2 \) via
\[
\mathcal{H}^2 := \text{dom}(\mathcal{A}), \quad \|f\|_{\mathcal{H}^2} := \|\mathcal{A} f\|_{\mathcal{H}}.
\tag{3.22}
\]

The proof of Theorems 2.2 and 2.4 is based on the following abstract result from \([2]\).

**Proposition 3.5** \([2, \text{Proposition 2.5}]\) Let \( \mathcal{J}_\varepsilon : \mathcal{H}^1 \to \mathcal{H}^1_\varepsilon \), \( \mathcal{J}' : \mathcal{H}^1 \to \mathcal{H}^1 \) be linear operators satisfying
\[
\begin{align*}
\|\mathcal{J}_\varepsilon f - f\|_{\mathcal{H}^1_\varepsilon} &\leq \delta_\varepsilon \|f\|_{\mathcal{H}^1_\varepsilon}, & \forall f \in \mathcal{H}^1_\varepsilon, \\
\|\mathcal{J}' u - u\|_{\mathcal{H}^1} &\leq \delta_\varepsilon \|u\|_{\mathcal{H}^1}, & \forall u \in \mathcal{H}^1_\varepsilon, \\
|\alpha_\varepsilon[\mathcal{J}_\varepsilon f, u] - \alpha[f, \mathcal{J}' u]| &\leq \delta_\varepsilon \|f\|_{\mathcal{H}^2} \|u\|_{\mathcal{H}^1_\varepsilon}, & \forall f \in \mathcal{H}^2, \ u \in \mathcal{H}^1_\varepsilon
\end{align*}
\tag{3.23-3.25}
\]
with some \( \delta_\varepsilon \geq 0 \). Then one has the estimate
\[
\forall f \in \mathcal{H} : \|\mathcal{A}_\varepsilon + I\| f - \mathcal{J}_\varepsilon (\mathcal{A} + I)\| f\|_{\mathcal{H}^1_\varepsilon} \leq C\delta_\varepsilon \|f\|_{\mathcal{H}} ,
\]
with some absolute constant \( C > 0 \).

**Remark 3.6** In fact, Proposition 2.5 from \([2]\) covers even more general case, when the operators \( \mathcal{A}_\varepsilon \) and \( \mathcal{A} \) act in distinct Hilbert spaces \( \mathcal{H}_\varepsilon \) and \( \mathcal{H} \); in this case one requires also certain identification operators between \( \mathcal{H}_\varepsilon \) and \( \mathcal{H} \). However, if these spaces coincide and these identification operators are chosen to be identity, Proposition 2.5 from \([2]\) reduces to Proposition 3.5.
3.3 Utilization of the abstract scheme

We apply Proposition 3.5 for $\mathcal{H} = L^2(\Omega) = L^2(\Omega_\epsilon)$, and the self-adjoint non-negative operators $\mathcal{A}_\epsilon$ and $\mathcal{A}$ being associated with the sesquilinear forms $a_\epsilon$ (2.2) and $a$ (2.12), respectively. We introduce the spaces $\mathcal{H}_\epsilon^1, \mathcal{H}^1, \mathcal{H}^2$ as in (3.21) and (3.22), i.e.

$$
\mathcal{H}_\epsilon^1 = H^1(\Omega_\epsilon), \quad \|u\|_{\mathcal{H}_\epsilon^1}^2 = \|u\|_{H^1(\Omega_\epsilon)}^2,
\mathcal{H}^1 = H^1(\Omega \setminus \Gamma), \quad \|f\|_{\mathcal{H}^1}^2 = \|\nabla f^+\|_{L^2(\Omega^+)}^2 + \|\nabla f^-\|_{L^2(\Omega^-)}^2 + \|y^{1/2}[f]\|_{L^2(\Gamma)}^2 + \|f\|_{\mathcal{H}}^2,
\mathcal{H}^2 = \text{dom}(\mathcal{A}), \quad \|f\|_{\mathcal{H}^2}^2 = -\Delta f^+ + f^+\|_{L^2(\Omega^+)}^2 + \|\Delta f^- + f^-\|_{L^2(\Omega^-)}^2
$$

(recall that $f^\pm = f|_{\Omega^\pm}$, and $[f] = (f^+ - f^-)|_{\Gamma}$). Note that

$$
\|f\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}_\epsilon^1} \leq \|f\|_{\mathcal{H}^1} \quad (3.26)
$$

Since $\text{dom}(a_\epsilon) \subset \text{dom}(a)$, we define $\mathcal{J}_\epsilon^1 : \mathcal{H}_\epsilon^1 \rightarrow \mathcal{H}^1$ be equal to the identity operator:

$$
\mathcal{J}_\epsilon^1 u = u, \quad u \in H^1(\Omega_\epsilon).
$$

With such a choice of $\mathcal{J}_\epsilon^1$, the condition (3.24) is fulfilled with $\delta_\epsilon = 0$. (3.27)

To define an appropriate operator $\mathcal{J}_\epsilon : \mathcal{H}^1 \rightarrow \mathcal{H}_\epsilon^1$, we first need to introduce two auxiliary functions $\phi_{k,\epsilon} \in C(\mathbb{R}^n)$ and $\psi_{k,\epsilon} \in C^\infty(\mathbb{R}^n)$ via

$$
\phi_{k,\epsilon}(x) = \begin{cases} 1, & |x - x_{k,\epsilon}| \leq d_{k,\epsilon}, \\
\frac{G(|x - x_{k,\epsilon}|) - G(\tilde{d}_{k,\epsilon})}{G(d_{k,\epsilon}) - G(\tilde{d}_{k,\epsilon})}, & d_{k,\epsilon} < |x - x_{k,\epsilon}| < \tilde{d}_{k,\epsilon}, \\
0, & \tilde{d}_{k,\epsilon} \leq |x - x_{k,\epsilon}|,
\end{cases}
$$

$$
\psi_{k,\epsilon}(x) := \psi \left( \frac{4|x - x_{k,\epsilon}|}{q_{k,\epsilon}} \right) \quad (3.28)
$$

Here $\tilde{d}_{k,\epsilon}$ is defined by (3.10), the function $G : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$
G(t) := \begin{cases} t^{2-n}, & n \geq 3, \\
-t^n, & n = 2,
\end{cases} \quad (3.29)
$$

and $\psi \in C^\infty([0, \infty))$ is a fixed cut-off function satisfying

$$
0 \leq \psi(t) \leq 1, \quad \psi(t) = 1 \quad \text{as} \quad t \leq 1, \quad \psi(t) = 0 \quad \text{as} \quad t \geq 2.
$$

Let $f \in H^1(\Omega \setminus \Gamma)$. We define

$$
(\mathcal{J}_\epsilon f)(x) := \begin{cases} f^+(x) + \sum_{k \in \mathbb{N}} \left( (f^+)_{B_{k,\epsilon}} - f^+(x) \right) \phi_{k,\epsilon}^+(x) - \frac{1}{2} f_{k,\epsilon} U_{k,\epsilon}^+ (x) \psi_{k,\epsilon}^+(x), & x \in \Omega^+ \\
-\sum_{k \in \mathbb{N}} \left( (f^-)_{B_{k,\epsilon}} - f^-(x) \right) \phi_{k,\epsilon}^-(x) + \frac{1}{2} f_{k,\epsilon} U_{k,\epsilon}^- (x) \psi_{k,\epsilon}^-(x), & x \in \Omega^-.
\end{cases}
$$

Here $\phi_{k,\epsilon}^\pm := \phi_{k,\epsilon}|_{\Omega^\pm}$, $\psi_{k,\epsilon}^\pm := \psi_{k,\epsilon}|_{\Omega^\pm}$, $U_{k,\epsilon}^\pm := U_{k,\epsilon}|_{\Omega^\pm}$, where $U_{k,\epsilon}(x)$ is the solution to (2.20) extended by 0 to $\mathbb{R}^n \setminus B_{k,\epsilon}$, and $f_{k,\epsilon} := (f^+)_{B_{k,\epsilon}} - (f^-)_{B_{k,\epsilon}}$.
One has $\text{supp}(\phi_{k,\varepsilon}) \subset B_{k,\varepsilon}$, $\phi_{k,\varepsilon} = 1$ in $\mathcal{B}(d_{k,\varepsilon}, x_{k,\varepsilon})$; the same hold for $\psi_{k,\varepsilon}$ (cf. (2.8)). Using these properties we conclude that $(\mathcal{J}_f \varphi)^\pm \in H^1(\Omega^\pm)$, furthermore, the traces of $(\mathcal{J}_f \varphi)^+$ and $(\mathcal{J}_f \varphi)^-$ coincide on $D_{k,\varepsilon}$ (with the constant $\frac{1}{2}(\langle f^+ \rangle_{B_{k,\varepsilon}} + \langle f^- \rangle_{B_{k,\varepsilon}})$). Thus, $\mathcal{J}_f \varphi \in H^1(\Omega^\pm)$, and we have a well-defined linear operator $\mathcal{J}_f : \mathcal{H}_1 \to \mathcal{H}_1^\varepsilon$.

Our goal is to show that the above introduced operators $\mathcal{J}_f$ and $\mathcal{J}_f^\varepsilon$ satisfy the conditions (3.23) and (3.25) with some $\delta_{\varepsilon} \to 0$.

First, we establish some properties of the functions from the domain of the limiting operator $\mathcal{J}$, Recall that we assume $\text{dist}(\Gamma, \partial \Omega) > 0$ (cf. (2.1)). We introduce the domains

$$O^\pm := \left\{ x = (x', x^n) : |x^n| < \frac{\text{dist}(\Gamma, \partial \Omega)}{2}, \pm x^n > 0 \right\}.$$

Lemma 3.7 Let $f \in \mathcal{H}^2$. Then $f^\pm \in H^2(O^\pm)$. Furthermore, the following estimate holds true:

$$\forall f \in \mathcal{H}^2 : \| f^\pm \|_{H^2(O^\pm)} \leq C \| f \|_{\mathcal{H}^2}.$$  \hspace{1cm} (3.30)

Proof For $i = (i^1, \ldots, i^{n-1}) \in \mathbb{Z}^{n-1}$ we introduce the sets

$$\square_i := \left\{ x = (x^1, \ldots, x^n, 0) \in \Gamma : |x^j - i^j| < \frac{1}{2}, \frac{1}{j} = 1, \ldots, n-1 \right\},$$

$$\widehat{\square}_i := \left\{ x = (x^1, \ldots, x^n, 0) \in \Gamma : |x^j - i^j| < 1, j = 1, \ldots, n-1 \right\},$$

$$O_i^\pm := \left\{ x = (x', x^n) \in \mathbb{R}^n : x' \in \square_i, |x^n| < \frac{1}{2} \text{dist}(\Gamma, \partial \Omega), \pm x^n > 0 \right\},$$

$$\widehat{O}_i^\pm := \left\{ x = (x', x^n) \in \mathbb{R}^n : x' \in \widehat{\square}_i, |x'n| < \text{dist}(\Gamma, \partial \Omega), \pm x^n > 0 \right\}.$$

Note that

$$\Gamma = \bigcup_{i \in \mathbb{Z}^n} \square_i, \quad \overline{O_i^\pm} = \bigcup_{i \in \mathbb{Z}^n} \overline{O_i^\pm}, \quad \square_i \cap \square_j = \emptyset, \quad O_i^\pm \cap O_j^\pm = \emptyset \text{ if } i \neq j. \hspace{1cm} (3.31)$$

One has (cf. (2.13))

$$f^\pm \in H^1(\widehat{O}_i^\pm), \quad \Delta f^\pm \in L^2(\widehat{O}_i^\pm). \hspace{1cm} (3.32)$$

Since $f^\pm \in H^{1/2}(\widehat{\square}_i)$ and $\gamma \in C^1(\Gamma)$, we have $\gamma f^\pm \in H^{1/2}(\widehat{\square}_i)$ (see, e.g., [37, Theorem 3.20]), whence

$$\frac{\partial f^\pm}{\partial \nu^\pm} |_{\square_i} = \mp \gamma[f] |_{\square_i} \in H^{1/2}(\widehat{\square}_i). \hspace{1cm} (3.33)$$

By virtue of [37, Theorem 4.18(ii)] the properties (3.32)–(3.33) imply $f \in H^2(O_i^\pm)$, moreover, the following estimate holds true:

$$\| f^\pm \|^2_{H^2(O_i^\pm)} \leq C \left( \| f^\pm \|^2_{H^1(\widehat{O}_i^\pm)} + \| \gamma[f] \|^2_{H^{1/2}(\widehat{\square}_i)} + \| - \Delta f^\pm + f^\pm \|^2_{L^2(\widehat{O}_i^\pm)} \right) \hspace{1cm} (3.34)$$

(evidently, the constant $C$ in (3.34) is independent of $i$). One has:

$$\sum_{i \in \mathbb{Z}^n} \| - \Delta f^\pm + f^\pm \|^2_{L^2(\widehat{O}_i^\pm)} \leq 2^{n-1} \| - \Delta f^\pm + f^\pm \|^2_{L^2(\bigcup_{i \in \mathbb{Z}^n} \widehat{O}_i^\pm)} \leq 2^{n-1} \| - \Delta f^\pm + f^\pm \|^2_{L^2(\Omega^\pm)} \leq 2^{n-1} \| f \|^2_{\mathcal{H}^2}, \hspace{1cm} (3.35)$$

\text{Springer}
Lemma 3.8

Combining (3.31), (3.34)–(3.37) and taking into account (3.26), we arrive at (3.30).

\[ \sum_{i \in \mathbb{Z}^n} \| f^\pm \|_{H^1(\overline{\Omega}_i)}^2 \leq 2^n - 1 \| f \|_{\mathcal{H}^1}^2 \quad (3.36) \]

The factor $2^n$ comes from the fact that each $x \in \bigcup_{i \in \mathbb{Z}^n} \overline{\Omega}_i$ belongs to $2^n$ sets $\overline{\Omega}_i^\pm$. Furthermore, introducing the function $\tilde{\gamma} \in C^1(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ via $\tilde{\gamma}(x', x^n) := \gamma(x')$, we get

\[ \sum_{i \in \mathbb{Z}^n} \| \gamma [f] \|_{H^{1/2}(\Omega_i)}^2 \leq 2 \sum_{i \in \mathbb{Z}^n} \left( \| \gamma f^+ \|_{H^{1/2}(\Omega_i)}^2 + \| \gamma f^- \|_{H^{1/2}(\Omega_i)}^2 \right) \]

\[ \leq C \sum_{i \in \mathbb{Z}^n} \left( \| \tilde{\gamma} f^+ \|_{H^1(\overline{\Omega}_i)}^2 + \| \tilde{\gamma} f^- \|_{H^1(\overline{\Omega}_i)}^2 \right) \]

\[ \leq 2^n C \left( \| \tilde{\gamma} f \|_{H^1(\Omega_1^+) \cup \Omega_2^+)}^2 + \| \tilde{\gamma} f \|_{H^1(\Omega_1^- \cup \Omega_2^+)}^2 \right) \leq C \| f \|_{H^1(\Omega \setminus \Gamma)}^2 \leq C \| f \|_{\mathcal{H}^1}^2 \quad (3.37) \]

Combining (3.31), (3.34)–(3.37) and taking into account (3.26), we arrive at (3.30).

\[ \forall f \in \mathcal{H}^1 : \| \mathcal{J}_\varepsilon f - f \|_{\mathcal{H}^1} \leq C \mu_\varepsilon \| f \|_{\mathcal{H}^1} \quad (3.38) \]

\[ \mathcal{C}(D_{k,\varepsilon}) \leq \| \nabla \phi_{k,\varepsilon} \|_{L^2(B_{k,\varepsilon})}^2 \]

(recall that $\phi_{k,\varepsilon}$ is given in (3.28)). Straightforward computations yield $\| \nabla \phi_{k,\varepsilon} \|_{L^2(B_{k,\varepsilon})}^2 \leq C (G(d_{k,\varepsilon}))^{-1}$, whence, taking into account (2.5), one arrives at the estimate

\[ \mathcal{C}(D_{k,\varepsilon}) \leq C \phi_{k,\varepsilon}^{n-1} Y_{k,\varepsilon} \quad (3.39) \]

Let $f \in \mathcal{H}$. Recall that the sets $\tilde{B}_{k,\varepsilon}^{\pm}, \tilde{D}_{k,\varepsilon}^{\pm}$ are defined in (2.19), (3.11); note that, due to (2.7) and (3.12), $\tilde{D}_{k,\varepsilon}^{\pm} \subset B_{k,\varepsilon}^{\pm} \subset \Omega^{\pm}$. One has $|\phi_{k,\varepsilon}^\pm| \leq 1, |\psi_{k,\varepsilon}^\pm| \leq 1$, furthermore, we have

\[ \text{supp}(\phi_{k,\varepsilon}^\pm) \subset \tilde{D}_{k,\varepsilon}^{\pm} \quad (3.40) \]

\[ \text{supp}(\psi_{k,\varepsilon}^\pm) \subset B_{k,\varepsilon}^{\pm} \quad (3.41) \]

Using these properties and (2.3), we obtain

\[ \| \mathcal{J}_\varepsilon f - f \|_{L^2(\Omega^{\pm})}^2 = \sum_{k \in \mathbb{N}} \left( (f^+)_{B_{k,\varepsilon}^+} - f^+ \right) \phi_{k,\varepsilon}^+ + \frac{1}{2} U_{k,\varepsilon} \left( (f^-)_{B_{k,\varepsilon}^-} - (f^+)_{B_{k,\varepsilon}^+} \right) \psi_{k,\varepsilon}^+ \| \|_{L^2(\tilde{B}_{k,\varepsilon}^+)}^2 \]

\[ \leq \sum_{k \in \mathbb{N}} \left( 2 \| (f^+)_{B_{k,\varepsilon}^+} - f^+ \|_{L^2(\tilde{D}_{k,\varepsilon}^+)}^2 + \| (f^-)_{B_{k,\varepsilon}^-} - (f^+)_{B_{k,\varepsilon}^+} \|_{L^2(\tilde{B}_{k,\varepsilon}^+)}^2 \right) \| U_{k,\varepsilon} \|_{L^2(B_{k,\varepsilon}^+)}^2 \quad (3.42) \]

Using the estimates (3.1) and (3.13), we get

\[ n \geq 3 : \sum_{k \in \mathbb{N}} \| f^+ - (f^+)_{B_{k,\varepsilon}^+} \|_{L^2(\tilde{D}_{k,\varepsilon}^+)}^2 \]

\[ \leq C \sum_{k \in \mathbb{N}} \left( \left( \frac{d_{k,\varepsilon}}{\psi_{k,\varepsilon}} \right)^n \| f^+ - (f^+)_{B_{k,\varepsilon}^+} \|_{L^2(B_{k,\varepsilon}^+)}^2 + d_{k,\varepsilon}^2 \| \nabla f^+ \|_{L^2(B_{k,\varepsilon}^+)}^2 \right) \]

\[ \square \text{ Springer} \]
\[
\leq C_1 \sum_{k \in \mathbb{N}} \left( \frac{d_{k,e}^n}{n-2} + d_{k,e}^2 \right) \| \nabla f^+ \|_{L^2(B^+_{k,e})}^2 \\
\leq C_2 \sum_{k \in \mathbb{N}} d_{k,e}^2 \| \nabla f^+ \|_{L^2(B^+_{k,e})}^2 
\leq C_2 \sup_{k \in \mathbb{N}} d_{k,e}^2 \| f \|_{H^1}^2
\] (3.43)

(in the penultimate estimate we also use \( \varrho_{k,e}^{2-n} < d_{k,e}^{2-n} \)). Similarly, one has

\[
n = 2 : \sum_{k \in \mathbb{N}} \| f^+ - (f^+)_{B^+_{k,e}} \|_{L^2(B^+_{k,e})}^2 \leq C \sup_{k \in \mathbb{N}} (d_{k,e} \varrho_{k,e} | \ln(d_{k,e} \varrho_{k,e})|) \| f \|_{H^1}.
\] (3.44)

Further, the estimate (3.2) together with (3.39), (2.21) yield

\[
\| U^+_{k,e} \|_{L^2(B^+_{k,e})} \leq \| U_{k,e} \|_{L^2(B_{k,e})} \leq C \varrho_{k,e} \| \nabla U_{k,e} \|_{L^2(B_{k,e})} = C \varrho_{k,e} \| D_{k,e} \| \leq C_1 \varrho_{k,e}^n \eta_{k,e}.
\] (3.45)

Using the Cauchy-Schwarz inequality, (3.3), (2.4) and (2.6), we get

\[
| (f^+)_{B^+_{k,e}} |^2 \leq C \varrho_{k,e}^{-n} \| f^+ \|^2_{L^2(B_{k,e})} \leq C \varrho_{k,e}^{-n} \left( \varrho_{k,e} \| f^+ \|^2_{L^2(S_{k,e})} + \varrho_{k,e} \| \nabla f^+ \|^2_{L^2(B_{k,e})} \right) \leq C \varrho_{k,e}^{-n} \left( \| f^+ \|^2_{L^2(S_{k,e})} + \| \nabla f^+ \|^2_{L^2(B_{k,e})} \right).
\] (3.46)

Combining (3.45), (3.46) and the trace estimate

\[
\| f^\pm \|_{L^2(\Gamma)} \leq C \| f^\pm \|_{H^1(\Omega^\pm)}
\] (3.47)

we arrive at

\[
\sum_{k \in \mathbb{N}} \left( | (f^-)_{B^-_{k,e}} |^2 + | (f^+)_{B^+_{k,e}} |^2 \right) \| U^+_{k,e} \|_{L^2(B^+_{k,e})} \leq C \sum_{k \in \mathbb{N}} \varrho_{k,e}^2 \eta_{k,e} \left( \| f^- \|^2_{L^2(S_{k,e})} + \| \nabla f^+ \|^2_{L^2(B_{k,e})} \right) + \| f^+ \|^2_{L^2(S_{k,e})} + \| \nabla f^+ \|^2_{L^2(B_{k,e})} \leq C \sup_{k \in \mathbb{N}} (\varrho_{k,e}^2 \eta_{k,e}) \| f \|_{H^1}.
\] (3.48)

It follows from (3.42), (3.43), (3.48) that

\[
\| \mathcal{J}_e f - f \|_{L^2(\Omega^+)} \leq C \xi_e \| f \|_{H^1},
\] (3.49)

where \( \xi_e \) is given by

\[
\xi_e := \begin{cases} 
\left( \sup_{k \in \mathbb{N}} \frac{d_{k,e}^2}{n-2} + \varrho_{k,e}^2 \eta_{k,e} \right)^{1/2}, & n \geq 3, \\
\left( \sup_{k \in \mathbb{N}} (d_{k,e} \varrho_{k,e} | \ln(d_{k,e} \varrho_{k,e})|) + \varrho_{k,e}^2 \eta_{k,e} \right)^{1/2}, & n = 2.
\end{cases}
\]

Repeating verbatim the above arguments we get similar estimate for \( \Omega^- \):

\[
\| \mathcal{J}_e f - f \|_{L^2(\Omega^-)} \leq C \xi_e \| f \|_{H^1}.
\] (3.50)
It is easy to show, using (2.5), (2.6), (2.8), that
\[
\zeta_\varepsilon \leq C \sup_{k \in \mathbb{N}} \eta_{k, \varepsilon} \leq C \mu_\varepsilon.
\] (3.51)

The estimate (3.38) follows immediately from (3.50)–(3.51). The lemma is proven. \(\square\)

Now, we start the estimation of the difference \(a_\varepsilon[\mathcal{J}_\varepsilon f, u] - a[f, \mathcal{J}_\varepsilon' u]\), where \(f \in \text{dom}(\mathcal{A})\), \(u \in \text{dom}(a_\varepsilon)\). Owing to (3.40)–(3.41) we can represent it as follows:
\[
a_\varepsilon[\mathcal{J}_\varepsilon f, u] - a[f, \mathcal{J}_\varepsilon' u] = I_k^1 + I_k^2 + I_k^3.
\] (3.52)

Here
\[
I_k^1 := \sum_{k \in \mathbb{N}} \left( \nabla((f^+)_{B_k, \varepsilon}^+ - f_{B_k, \varepsilon}^+) \nabla u^+ \right)_{L^2(D_k, \varepsilon)},
\]
\[
I_k^2 := \frac{1}{2} \sum_{k \in \mathbb{N}} f_k \left( \nabla(U_{k, \varepsilon}^+ \psi_{k, \varepsilon}^+), \nabla u^+ \right)_{L^2(B_k, \varepsilon)},
\]
\[
I_k^3 := - (\gamma(f), [u])_{L^2(\Gamma)}
\]
(as usual, \(f^\pm = f |_{\Omega^\pm}, u^\pm = u |_{\Omega^\pm}, \gamma = (f^+ - f^-) |_{\Gamma}, [u] = (u^+ - u^-) |_{\Gamma}\).

To proceed further we require several properties of the function \(U_{k, \varepsilon}\).

**Lemma 3.9** One has:
\[
\frac{\partial U_{k, \varepsilon}^\pm}{\partial v^\pm} = 0 \text{ on } S_{k, \varepsilon} \setminus D_{k, \varepsilon},
\] (3.53)
\[
\frac{\partial U_{k, \varepsilon}^+}{\partial v^+} = \frac{\partial U_{k, \varepsilon}^-}{\partial v^-} \text{ on } D_{k, \varepsilon},
\] (3.54)
\[
\mathcal{C}(D_{k, \varepsilon}) = 2 \int_{D_{k, \varepsilon}} \frac{\partial U_{k, \varepsilon}^\pm}{\partial v^\pm} \, dx'.
\] (3.55)

where \(v^\pm = \mp(0, \ldots, 0, 1)\) is the normal to \(\Gamma\) pointed outward of \(\Omega^\pm\). Furthermore, the following pointwise estimates hold true:
\[
0 \leq U_{k, \varepsilon}(x) \leq \tilde{\Psi}_{k, \varepsilon}(x) := \frac{G(|x - x_{k, \varepsilon}|) - G(\partial_{k, \varepsilon})}{G(d_{k, \varepsilon}) - G(\partial_{k, \varepsilon})}, \quad x \in B_{k, \varepsilon} \setminus \partial(d_{k, \varepsilon}, x_{k, \varepsilon}),
\] (3.56)
\[
\forall j \in \{1, \ldots, n\}, \quad \left| \frac{\partial U_{k, \varepsilon}}{\partial x_j}(x) \right| \leq C \frac{e^{-1/4} x_{k, \varepsilon}}{G(d_{k, \varepsilon}) - G(\partial_{k, \varepsilon})}, \quad x \in B_{k, \varepsilon} \setminus \partial(d_{k, \varepsilon}, x_{k, \varepsilon}).
\] (3.57)

where the function \(G(t)\) is defined by (3.29).

**Proof** Standard regularity theory for elliptic PDEs yields
\[
U_{k, \varepsilon} \in C^\infty(B_{k, \varepsilon} \setminus D_{k, \varepsilon}).
\] (3.58)

Evidently, \(U_{k, \varepsilon}\) is symmetric with respect to the hyperplane \(\Gamma\), i.e.
\[
U_{k, \varepsilon}^+(x', \tau) = U_{k, \varepsilon}^-(x', -\tau), \quad \tau > 0.
\] (3.59)

\footnote{In fact, one has even the property \(\zeta_\varepsilon = o(\sup_{k \in \mathbb{N}} \eta_{k, \varepsilon})\) as \(\varepsilon \to 0\) (which is stronger than (3.51)). However, the knowledge of this fact gives us no profits, since the convergence rate \(\sup_{k \in \mathbb{N}} \eta_{k, \varepsilon}\) appears later in the estimate for \(a_\varepsilon[\mathcal{J}_\varepsilon f, u] - a[f, \mathcal{J}_\varepsilon' u]\).}
Then (3.53)–(3.54) follow immediately from (3.58) and (3.59). Furthermore, since \( U_{k,\varepsilon} \) is the solution to the problem (2.20), one has the following Green’s identity:

\[
\int_{D_{k,\varepsilon}} \left( \frac{\partial U_{k,\varepsilon}^+}{\partial \nu^+} + \frac{\partial U_{k,\varepsilon}^-}{\partial \nu^-} \right) \, dx' = \| \nabla U_{k,\varepsilon} \|_{L^2(B_{k,\varepsilon})}^2.
\]

(3.60)

From (2.21), (3.54) and (3.60), we infer (3.55).

Now, we proceed to the proof of (3.56). By the maximum principle

\[
0 \leq U_{k,\varepsilon} \leq 1.
\]

(3.61)

Furthermore, the function \( \tilde{\psi}_{k,\varepsilon} \) is harmonic in \( B_{k,\varepsilon} \setminus \mathcal{B}(d_{k,\varepsilon}, x_{k,\varepsilon}) \), it is equal to 0 on \( \partial B_{k,\varepsilon} \), and it is equal to 1 on \( \partial \mathcal{B}(d_{k,\varepsilon}, x_{k,\varepsilon}) \). Hence, the function \( \psi_{k,\varepsilon} - U_{k,\varepsilon} \) is harmonic in \( B_{k,\varepsilon} \setminus \mathcal{B}(d_{k,\varepsilon}, x_{k,\varepsilon}) \), equal to 0 on \( \partial B_{k,\varepsilon} \) and is non-negative on \( \partial \mathcal{B}(d_{k,\varepsilon}, x_{k,\varepsilon}) \). Again applying the maximum principle, we conclude that

\[
\tilde{\psi}_{k,\varepsilon} - U_{k,\varepsilon} \geq 0 \text{ in } B_{k,\varepsilon} \setminus \mathcal{B}(d_{k,\varepsilon}, x_{k,\varepsilon}).
\]

(3.62)

The estimate (3.56) follows from (3.61) and (3.62).

Finally, we prove (3.57). Let \( x \in B_{k,\varepsilon} \setminus \mathcal{B}(\frac{\partial B}{4}, x_{k,\varepsilon}) \). We denote \( \tau_x := \frac{1}{2} (|x - x_{k,\varepsilon}| - x_{k,\varepsilon}) \) (that is \( \tau_x \in (1/4, 1) \)), and \( \ell_x := \min\{\frac{1}{2}, 1 - \tau_x\} \). Due to (2.8) one has

\[
\mathcal{B}(\ell_x Q_{k,\varepsilon}, x) \subset B_{k,\varepsilon} \setminus \mathcal{B}(d_{k,\varepsilon}, x_{k,\varepsilon}).
\]

Since \( U_{k,\varepsilon} \) is a harmonic function, its partial derivatives are harmonic functions too. Using the mean value theorem for harmonic functions and then integrating by parts, we get

\[
\frac{\partial U_{k,\varepsilon}}{\partial x^j}(x) = \frac{1}{\text{vol}(\mathcal{B}(\ell_x Q_{k,\varepsilon}, x))} \int_{\mathcal{B}(\ell_x Q_{k,\varepsilon}, x)} \frac{\partial U_{k,\varepsilon}}{\partial x^j}(y) \, dy
\]

\[
= \frac{1}{\text{vol}(\mathcal{B}(\ell_x Q_{k,\varepsilon}, x))} \int_{\partial \mathcal{B}(\ell_x Q_{k,\varepsilon}, x)} u(y) v_j(y) \, ds_y,
\]

(3.63)

where \( ds_y \) is a surface element on the sphere \( \partial \mathcal{B}(\ell_x Q_{k,\varepsilon}, x) \), \( v_j(y) \) is the \( j \)-th component of the outward pointing unit normal on \( \partial \mathcal{B}(\ell_x Q_{k,\varepsilon}, x) \) (thus \( |v_j(y)| \leq 1 \)). Using (3.57) we infer from (3.63):

\[
\left| \frac{\partial U_{k,\varepsilon}}{\partial x^j}(x) \right| \leq C(\ell_x Q_{k,\varepsilon})^{-1}(G(d_{k,\varepsilon}) - G(Q_{k,\varepsilon}))^{-1} \max_{y \in \partial \mathcal{B}(\ell_x, x)} (G(|y - x_{k,\varepsilon}|) - G(Q_{k,\varepsilon}))
\]

\[
= CQ_{k,\varepsilon}^{-1-n}(G(d_{k,\varepsilon}) - G(Q_{k,\varepsilon}))^{-1} \theta_x,
\]

where

\[
\theta_x := \begin{cases} ((\tau_x - \ell_x)^{2-n} - 1) \ell_x^{-1}, & n \geq 3, \\ - \ln(\tau_x - \ell_x)\ell_x^{-1}, & n = 2. \end{cases}
\]

(here we use the fact that \( \max_{y \in \partial \mathcal{B}(\ell_x, x)} G(|y - x_{k,\varepsilon}|) \) is attained at the point \( y_0 \in \partial \mathcal{B}(\ell_x Q_{k,\varepsilon}, x) \) lying on the interval between \( x \) and \( x_{k,\varepsilon} \), that is \( |y_0 - x_{k,\varepsilon}| = (\tau_x - \ell_x)Q_{k,\varepsilon} \). It is easy to show that \( \theta_x \) is uniformly bounded on \( \{(\tau_x, \ell_x) : \tau_x \in (1/4, 1), \ell_x = \min\{\frac{1}{8}, 1 - \tau_x\}\} \). The estimate (3.57) is proven. \( \square \)

**Remark 3.10** Alternative proofs of (3.56)–(3.57)-type estimates can be found in [34, Lemma 2.2] and [35, Lemma 2.4]. They are based on Green’s third identity and known pointwise estimates for Green’s function for the Dirichlet Laplacian on a ball.
Now, we can further transform the terms \( I_{k,\varepsilon}^{2,\pm} \). Integrating by parts twice and taking into account (3.53), (3.55) and the properties 

\[ \psi_{k,\varepsilon} = 1 \text{ in a neighborhood of } D_{k,\varepsilon}, \quad \psi_{k,\varepsilon} = 0 \text{ in a neighborhood of } \partial B_{k,\varepsilon}, \quad \frac{\partial \psi_{k,\varepsilon}}{\partial \nu} = 0 \text{ on } \Gamma, \]

we get:

\[
I_{\varepsilon}^{2,\pm} = \pm \frac{1}{2} \sum_{k \in \mathbb{N}} f_{k,\varepsilon}(\Delta(U_{k,\varepsilon}^{\pm}, \psi_{k,\varepsilon}^{\pm}), u^{\pm})_{L^2(B_{k,\varepsilon}^{\pm})} + Q_{\varepsilon}^{1,\pm}
\]

\[
= \pm \frac{1}{2} \sum_{k \in \mathbb{N}} f_{k,\varepsilon}(\Delta(U_{k,\varepsilon}^{\pm}, \psi_{k,\varepsilon}^{\pm}), (u^{\pm})_{B_{k,\varepsilon}^{\pm}})_{L^2(B_{k,\varepsilon}^{\pm})} + Q_{\varepsilon}^{1,\pm} + Q_{\varepsilon}^{2,\pm}
\]

\[
= \pm \frac{1}{2} \sum_{k \in \mathbb{N}} f_{k,\varepsilon}(u^{\pm})_{B_{k,\varepsilon}^{\pm}} \left( \int_{D_{k,\varepsilon}} \frac{\partial U_{k,\varepsilon}^{\pm}}{\partial \nu} \right) + Q_{\varepsilon}^{1,\pm} + Q_{\varepsilon}^{2,\pm}
\]

\[
= \pm \frac{1}{4} \sum_{k \in \mathbb{N}} f_{k,\varepsilon}(u^{\pm})_{B_{k,\varepsilon}^{\pm}} C(\psi_{k,\varepsilon}^{\pm}) + Q_{\varepsilon}^{1,\pm} + Q_{\varepsilon}^{2,\pm}
\]

\[
= Q_{\varepsilon}^{1,\pm} + Q_{\varepsilon}^{2,\pm} + Q_{\varepsilon}^{3,\pm} + Q_{\varepsilon}^{4,\pm},
\]

where

\[
Q_{\varepsilon}^{1,\pm} := \pm \frac{1}{2} \sum_{k \in \mathbb{N}} f_{k,\varepsilon} \int_{D_{k,\varepsilon}} \frac{\partial U_{k,\varepsilon}^{\pm}}{\partial \nu} u^{\pm} \, dx',
\]

\[
Q_{\varepsilon}^{2,\pm} := \pm \frac{1}{2} \sum_{k \in \mathbb{N}} f_{k,\varepsilon}(\Delta(U_{k,\varepsilon}^{\pm}, \psi_{k,\varepsilon}^{\pm}), (u^{\pm})_{B_{k,\varepsilon}^{\pm}})_{L^2(B_{k,\varepsilon}^{\pm})},
\]

\[
Q_{\varepsilon}^{3,\pm} := \pm \frac{1}{4} \sum_{k \in \mathbb{N}} (f_{k,\varepsilon}(u^{\pm})_{B_{k,\varepsilon}^{\pm}} - \langle [f] \rangle_{B_{k,\varepsilon}^{\pm}} (u^{\pm})_{S_{k,\varepsilon}}) C(D_{k,\varepsilon}),
\]

\[
Q_{\varepsilon}^{4,\pm} := \pm \frac{1}{4} \sum_{k \in \mathbb{N}} ([f]_{S_{k,\varepsilon}} (u^{\pm})_{S_{k,\varepsilon}} C(D_{k,\varepsilon})
\]

Denoting

\[
I_{\varepsilon}^{4} := Q_{\varepsilon}^{4,+} + Q_{\varepsilon}^{4,-} + I_{\varepsilon}^{3},
\]

we can rewrite (3.52) as follows:

\[
a_{\varepsilon} \left[ J_{\varepsilon} f, u \right] - a[f, J'_{\varepsilon} u] = I_{\varepsilon}^{1} + \sum_{k=1}^{3} \left( Q_{\varepsilon}^{k,+} + Q_{\varepsilon}^{k,-} \right) + I_{\varepsilon}^{4}.
\]

(3.64)

By virtue of (3.54) and the fact that \( u \) is continuous across \( D_{k,\varepsilon} \), we immediately get

\[
Q_{\varepsilon}^{1,+} + Q_{\varepsilon}^{1,-} = 0.
\]

(3.65)

In Lemmata 3.11–3.14 below we estimate the other terms in the right-hand-side of (3.64). Recall that \( \eta_{k,\varepsilon} \) is defined by (2.15).

**Lemma 3.11** One has

\[
|I_{\varepsilon}^{1,\pm}| \leq C \sup_{k \in \mathbb{N}} \eta_{k,\varepsilon} \| f \|_{\mathcal{H}^2} \| u \|_{\mathcal{H}^1}.
\]

(3.66)
Proof Using $|\phi_{k,e}^\pm| \leq 1$ and the Cauchy-Schwarz inequality, we get

$$|I_{e}^{1,\pm}| = \left| \sum_{k \in \mathbb{N}} \left( -(\phi_{k,e}^\pm \nabla f^\pm, \nabla u^\pm)_{L^2(\tilde{D}_{k,e}^\pm)} + ((f^\pm)_{B_{k,e}^\pm} - f^\pm)\nabla \phi_{k,e}^\pm, \nabla u^\pm \right)_{L^2(\tilde{D}_{k,e}^\pm)} \right|$$

$$\leq \left\{ \left( \sum_{k \in \mathbb{N}} \|\nabla f^\pm\|_{L^2(\tilde{D}_{k,e}^\pm)}^2 \right)^{1/2} + \left( \sum_{k \in \mathbb{N}} \|(f^\pm - (f^\pm)_{B_{k,e}^\pm})\nabla \phi_{k,e}^\pm\|_{L^2(\tilde{D}_{k,e}^\pm)}^2 \right)^{1/2} \right\} \|u\|_{\mathcal{H}_1^e}. \tag{3.67}$$

Using Lemma 3.2, we obtain

$$\|\nabla f\|_{L^2(\tilde{D}_{k,e}^\pm)} \leq C\tilde{\eta}_{k,e} \|f^\pm\|_{\mathcal{H}^2(B_{k,e}^\pm)}, \tag{3.68}$$

where $\tilde{\eta}_{k,e}$ is defined by

$$\tilde{\eta}_{k,e} := \begin{cases} \max \left\{ d_{n,\pm}^{n/2} - n/2 ; d_{k,e} \right\}, & n \geq 3, \\ \max \left\{ d_{k,e}^{1/2} - 1/2 ; (d_{k,e} \ln(d_{k,e} \ln d_{k,e})) \right\}^{1/2}, & n = 2. \end{cases}$$

It is easy to see that $\tilde{\eta}_{k,e} \leq \eta_{k,e}$, whence, using (3.68), (2.3), $B_{k,e}^\pm \subset O^\pm$ (this follows from (2.7)) and Lemma 3.7, we get

$$\left( \sum_{k \in \mathbb{N}} \|\nabla f^\pm\|_{L^2(\tilde{D}_{k,e}^\pm)}^2 \right)^{1/2} \leq C \sup_{k \in \mathbb{N}} \tilde{\eta}_{k,e} \|f\|_{\mathcal{H}^2(O^\pm)} \leq C_1 \sup_{k \in \mathbb{N}} \eta_{k,e} \|f\|_{\mathcal{H}^2}. \tag{3.69}$$

To estimate the second term in the right-hand-side of (3.67), we define the numbers $p, q$ via

$$p = \frac{2n}{n - 4} \quad \text{if} \quad n \geq 5, \quad p = 2 |\ln d_{k,e} - \ln q_{k,e}| \quad \text{if} \quad n = 4, \quad p = \infty \quad \text{if} \quad n = 2, 3, \tag{3.70}$$

$$q = \frac{n}{2} \quad \text{if} \quad n \geq 5, \quad q = \frac{2}{1 - |\ln d_{k,e} - \ln q_{k,e}|^{-1}} \quad \text{if} \quad n = 4, \quad q = 2 \quad \text{if} \quad n = 2, 3. \tag{3.71}$$

Note that, due to (2.8), we have $|\ln d_{k,e} - \ln q_{k,e}| \leq \ln 8 > 1$. It is easy to see that $p, q \in [2, \infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then, by virtue of the Hölder inequality, we get

$$\|(f^\pm - (f^\pm)_{B_{k,e}^\pm}) \nabla \phi_{k,e}^\pm\|_{L^2(\tilde{D}_{k,e}^\pm)} \leq \|(f^\pm - (f^\pm)_{B_{k,e}^\pm})\nabla \phi_{k,e}^\pm\|_{L^2(\tilde{D}_{k,e}^\pm)}^2$$

$$\leq \|f^\pm - (f^\pm)_{B_{k,e}^\pm}\|_{L^p(B_{k,e}^\pm)} \|\nabla \phi_{k,e}^\pm\|_{L^q(B_{k,e}^\pm)}^2. \tag{3.72}$$

Using Lemma 3.3 for $v := f^\pm - (f^\pm)_{B_{k,e}^\pm}$ and $p$ as in (3.70) and taking into account (3.20), we obtain the estimate

$$\|f^\pm - (f^\pm)_{B_{k,e}^\pm}\|_{L^p(B_{k,e}^\pm)} \leq C \|f^\pm\|_{\mathcal{H}^2(B_{k,e}^\pm)} \begin{cases} q_{k,e}^{-1} |\ln d_{k,e} - \ln q_{k,e}|^{-1} & n \geq 5, \\ \ln d_{k,e} - \ln q_{k,e} \cdot q_{k,e}^{-1} & n = 4, \\ q_{k,e}^{-1/2} & n = 3, \\ 1 & n = 2. \end{cases} \tag{3.73}$$
For \(q\) as in (3.71) we get via straightforward calculations:

\[
\| \nabla \phi_{k,e}^+ \|_{L^q(B_{k,e}^+)} \leq C \begin{cases} 
  d_{k,e}, & n \geq 5, \\
  \frac{d_{k,e}^{1-2} |\ln d_{k,e} - \ln \kappa_{k,e}|^{-1}}{\kappa_{k,e}}, & n = 4, \\
  \frac{d_{k,e}^{1/2}}{\kappa_{k,e}}, & n = 3, \\
  |\ln d_{k,e} - \ln \kappa_{k,e}|^{-1/2}, & n = 2.
\end{cases}
\] (3.74)

Combining (3.72)–(3.74) and taking into account that \(d_{k,e}^{1-2} |\ln d_{k,e} - \ln \kappa_{k,e}|^{-1} \exp(2)\), we get

\[
\|(f^+ - (f^+)_{B_{k,e}^+}) \nabla \phi_{k,e}^+ \|_{L^2(D_{k,e}^+)} \leq C \eta_{k,e} \| f^+ \|_{H^2(B_{k,e}^+)},
\] (3.75)

where \(\eta_{k,e}\) is given by (2.15). Taking into account (3.30), we deduce from (3.75):

\[
\left( \sum_{k \in \mathbb{N}} \| (f^+ - (f^+)_{B_{k,e}^+}) \nabla \phi_{k,e}^+ \|_{L^2(D_{k,e}^+)}^2 \right)^{1/2} \leq C \sup_{k \in \mathbb{N}} \eta_{k,e} \| f \|_{H^2}.
\] (3.76)

Combining (3.67), (3.69), (3.76), we arrive at the required estimate (3.66). The lemma is proven.

\[\square\]

**Lemma 3.12** One has

\[
|Q_{k,e}^{2,+}| \leq C \sup_{k \in \mathbb{N}} \eta_{k,e}^{1/2} \| f \|_{H^1} \| u \|_{H^1},
\] (3.77)

where

\[\eta_{k,e} := \begin{cases} 
  1, & n \geq 3, \\
  |\ln \kappa_{k,e}|, & n = 2.
\end{cases}\]

**Proof** Using the Cauchy-Schwarz inequality, we get

\[
|Q_{k,e}^{2,+}| \leq \left( \sum_{k \in \mathbb{N}} \eta_{k,e} \eta_{k,e}^{-2} \eta_{k,e}^{-2} |f_{k,e}|^2 \| \Delta(U_{k,e}^+ \psi_{k,e}^+) \|_{L^2(B_{k,e}^+)}^2 \right)^{1/2} \times \left( \sum_{k \in \mathbb{N}} \eta_{k,e}^{-1} \eta_{k,e}^{-2} \| u^+ - \langle u^+ \rangle_{B_{k,e}^+} \|_{L^2(B_{k,e}^+)}^2 \right)^{1/2}.
\] (3.78)

Since \(\Delta U_{k,e} = 0\), we have

\[
\Delta(U_{k,e}^+ \psi_{k,e}^+) = 2(\nabla U_{k,e}^+ \psi_{k,e}^+, \nabla \psi_{k,e}^+) + U_{k,e}^+ \Delta \psi_{k,e}^+.
\] (3.79)

Furthermore, one has

\[
\text{supp}(\nabla \psi_{k,e}^+) \cup \text{supp}(\Delta \psi_{k,e}^+) \subset B_{k,e}^+ \setminus \mathcal{B}(\Omega_{k,e}^+, x_{k,e}).
\] (3.80)

\[
|\psi_{k,e}^+(x)| \leq 1, \quad |\nabla \psi_{k,e}^+(x)| \leq C \eta_{k,e}^{-1}, \quad |\Delta \psi_{k,e}^+(x)| \leq C \eta_{k,e}^{-2}.
\] (3.81)

Then, using (3.56)–(3.57) and (3.79)–(3.81), we conclude

\[
\| \Delta(U_{k,e}^+ \psi_{k,e}^+) \|_{L^2(B_{k,e}^+)}^2 \leq C \eta_{k,e}^2 \| \nabla U_{k,e}^+ \psi_{k,e}^+ \|_{L^2(B_{k,e}^+ \setminus \mathcal{B}(\Omega_{k,e}^+, x_{k,e}))}^2 + C \eta_{k,e}^4 \| U_{k,e}^+ \psi_{k,e}^+ \|_{L^2(B_{k,e}^+ \setminus \mathcal{B}(\Omega_{k,e}^+, x_{k,e}))}^2 \leq C \eta_{k,e}^4 \left( \eta_{k,e}^{-2} \| \nabla U_{k,e}^+ \|_{L^2(B_{k,e}^+ \setminus \mathcal{B}(\Omega_{k,e}^+, x_{k,e}))}^2 + \eta_{k,e}^{-2} \| U_{k,e}^+ \|_{L^2(B_{k,e}^+ \setminus \mathcal{B}(\Omega_{k,e}^+, x_{k,e}))}^2 \right).
\]
Combining (3.46), (3.47), (3.82) we arrive at

\[
\sum_{k \in \mathbb{N}} q_{k,e} y_{k,e}^{-2} x_{k,e}^{-2} |f_{k,e}|^2 \|\Delta(U_{k,e}^+ \psi_{k,e})\|_{L^2(B_{k,e}^+)}^2 
\leq 2 \sum_{k \in \mathbb{N}} q_{k,e} y_{k,e}^{-2} x_{k,e}^{-2} \left( |(f^-)_{B_{k,e}^-}|^2 + |(f^+)_{B_{k,e}^+}|^2 \right) \|\Delta(U_{k,e}^+ \psi_{k,e})\|_{L^2(B_{k,e}^+)}^2 
\leq C \sum_{k \in \mathbb{N}} \left( \|f^+\|_{L^2(S_{k,e})}^2 + \|\nabla f^+\|_{L^2(B_{k,e}^+)}^2 + \|f^-\|_{L^2(S_{k,e})}^2 + \|\nabla f^-\|_{L^2(B_{k,e}^-)}^2 \right) \leq C_1 \|f\|_{\mathcal{H}^1}^2. 
\]

(3.83)

Finally, by using (3.1), we estimate the second factor in (3.78):

\[
\sum_{k \in \mathbb{N}} q_{k,e}^{-1} y_{k,e}^2 x_{k,e}^2 \|u - (u)_{B_{k,e}^-}\|_{L^2(B_{k,e}^+)}^2 \leq C \sum_{k \in \mathbb{N}} q_{k,e} y_{k,e}^2 x_{k,e} \|\nabla u\|_{L^2(B_{k,e}^+)}^2 
\leq C \left( \sup_{k \in \mathbb{N}} q_{k,e}^{-1/2} y_{k,e} x_{k,e} \right)^2 \|u\|_{\mathcal{H}^1}^2. 
\]

(3.84)

The required estimate (3.77) for \(Q_{\varepsilon}^{2,+}\) follows from (3.78), (3.83), (3.84). For \(Q_{\varepsilon}^{2,-}\) the proof is similar. The lemma is proven. \(\square\)

**Lemma 3.13** One has

\[
|Q_{\varepsilon}^{3,\pm}| \leq C \sup_{k \in \mathbb{N}} (q_{k,e}^{-1/2} y_{k,e} x_{k,e}) \|f\|_{\mathcal{H}^1} \|u\|_{\mathcal{H}^1}. 
\]

(3.85)

**Proof** One can represent \(Q_{\varepsilon}^{3,\pm}\) is the form \(Q_{\varepsilon}^{3,1} + Q_{\varepsilon}^{3,2}\), where

\[
Q_{\varepsilon}^{3,1} = \frac{1}{4} \sum_{k \in \mathbb{N}} \left( (f^+)_{B_{k,e}^+} - (f^-)_{S_{k,e}} \right) \left( (f^-)_{B_{k,e}^-} - (f^+)_{S_{k,e}} \right) \langle u^+ \rangle_{B_{k,e}^+} \langle u^- \rangle_{S_{k,e}} \mathcal{C}(D_{k,e}) 
\]

\[
Q_{\varepsilon}^{3,2} = \frac{1}{4} \sum_{k \in \mathbb{N}} \left( (f^+)_{S_{k,e}} - (f^-)_{S_{k,e}} \right) \left( \langle u^+ \rangle_{B_{k,e}^+} - \langle u^- \rangle_{S_{k,e}} \right) \mathcal{C}(D_{k,e}). 
\]

Using (3.4) (applied for \(f^\pm\)), (3.46) and (3.47) (applied for \(u^\pm\)), (3.39), we get

\[
|Q_{\varepsilon}^{3,1}| \leq \frac{1}{4} \left( 2 \sum_{k \in \mathbb{N}} \langle \mathcal{C}(D_{k,e}) \rangle^2 q_{k,e}^{-n} \left( |(f^+)_{B_{k,e}^+} - (f^-)_{S_{k,e}}|^2 + |(f^-)_{B_{k,e}^-} - (f^+)_{S_{k,e}}|^2 \right) \right)^{1/2} 
\]

\[
\times \left( \sum_{k \in \mathbb{N}} q_{k,e}^{-1} \langle u^+ \rangle_{B_{k,e}^+}^2 \right)^{1/2} \leq C \left( \sum_{k \in \mathbb{N}} q_{k,e} y_{k,e}^2 \left( \|\nabla f^+\|_{L^2(B_{k,e}^+)}^2 + \|\nabla f^-\|_{L^2(B_{k,e}^-)}^2 \right) \right)^{1/2} 
\]

\[
\times \left( \sum_{k \in \mathbb{N}} \|u^+\|_{L^2(B_{k,e}^+)}^2 + \|u^-\|_{L^2(S_{k,e})}^2 \right)^{1/2} \leq C \sup_{k \in \mathbb{N}} (q_{k,e}^{-1/2} y_{k,e} x_{k,e}) \|f\|_{\mathcal{H}^1} \|u\|_{\mathcal{H}^1}. 
\]

(3.86)

Similarly, applying the Cauchy-Schwarz inequality (for \(f^\pm\), (3.4) (for \(u^+\)), and (3.39), we estimate \(Q_{\varepsilon}^{3,2}\):

\[
|Q_{\varepsilon}^{3,2}| \leq \frac{1}{4} \left( 2 \sum_{k \in \mathbb{N}} \langle \mathcal{C}(D_{k,e}) \rangle^2 q_{k,e}^{-n} \left( |(f^+)_{S_{k,e}}|^2 + |(f^-)_{S_{k,e}}|^2 \right) \right)^{1/2} 
\]

\[
\times \left( \sum_{k \in \mathbb{N}} q_{k,e}^{-2} \langle u^+ \rangle_{B_{k,e}^+}^2 - \langle u^- \rangle_{S_{k,e}}^2 \right)^{1/2} \]

\[\leq C \sup_{k \in \mathbb{N}} (q_{k,e}^{-1/2} y_{k,e} x_{k,e}) \|f\|_{\mathcal{H}^1} \|u\|_{\mathcal{H}^1}. \]
Combining (3.86) and (3.87) we arrive at the required estimate (3.85) for $Q^3_{\varepsilon}$. For $Q^3_{\varepsilon}$ the proof is similar. The lemma is proven.

**Lemma 3.14** One has

$$ |I^4_{\varepsilon}| \leq C \kappa_{\varepsilon} \| f \|_{\mathcal{H}^2} \| u \|_{\mathcal{H}^1},$$

where $\kappa_{\varepsilon} \to 0$ is given in (2.11).

**Proof** Since $f^\pm \in \mathcal{H}^2(O^\pm)$ (see Lemma 3.7) and $u^\pm \in \mathcal{H}^1(\Omega^\pm)$, the trace theorem yields $f^\pm \in \mathcal{H}^{3/2}(\Gamma)$ and $u^\pm \in \mathcal{H}^{1/2}(\Gamma)$, moreover

$$ \| f^\pm \|_{\mathcal{H}^{3/2}(\Gamma)} \leq C \| f^\pm \|_{\mathcal{H}^2(O^\pm)}, \quad \| u^\pm \|_{\mathcal{H}^{1/2}(\Gamma)} \leq C \| u^\pm \|_{\mathcal{H}^1(O^\pm)}. $$

Then, using (2.11), (3.89) and (3.30), we obtain

$$ |I^4_{\varepsilon}| \leq \left| \frac{1}{4} \sum_{k \in \mathbb{N}} \| \langle f \rangle \|_{S_{k,\varepsilon}} \| \langle u \rangle \|_{S_{k,\varepsilon}} \langle D_{k,\varepsilon} \rangle - \langle \gamma[f], \langle u \rangle \rangle_{L^2(\Gamma)} \right| \leq \kappa_{\varepsilon} \| f \|_{\mathcal{H}^{3/2}(\Gamma)} \| u \|_{\mathcal{H}^{1/2}(\Gamma)}$$

$$ \leq \kappa_{\varepsilon} \left( \| f^+ \|_{\mathcal{H}^{3/2}(\Gamma)} + \| f^- \|_{\mathcal{H}^{3/2}(\Gamma)} \right) \left( \| u^+ \|_{\mathcal{H}^{1/2}(\Gamma)} + \| u^- \|_{\mathcal{H}^{1/2}(\Gamma)} \right)$$

$$ \leq C \kappa_{\varepsilon} \left( \| f^+ \|_{\mathcal{H}^1(O^+)} + \| f^- \|_{\mathcal{H}^1(O^-)} \right) \left( \| u^+ \|_{\mathcal{H}^1(O^+)} + \| u^- \|_{\mathcal{H}^1(O^-)} \right) \leq C_{1,\varepsilon} \| f \|_{\mathcal{H}^2} \| u \|_{\mathcal{H}^1}. $$

The lemma is proven.

Using the definitions of $\eta_{k,\varepsilon}$ and $\gamma_{k,\varepsilon}$ one can easily get

$$ q^{1/2}_{k,\varepsilon} \gamma_{k,\varepsilon} = \eta_{k,\varepsilon} q^{1/2}_{k,\varepsilon} A_{k,\varepsilon}, $$

where

$$ A_{k,\varepsilon} := \begin{cases} \frac{(d_{k,\varepsilon})^{n-4}}{\bar{d}_{k,\varepsilon}}, & n \geq 5, \\ \ln \frac{d_{k,\varepsilon}}{\bar{d}_{k,\varepsilon}} \bigg|^{-1}, & n = 4, \\ 1, & n = 3, \\ 1 + \gamma_{k,\varepsilon} q_{k,\varepsilon} \ln q_{k,\varepsilon} \bigg|^{1/2}, & n = 2. \end{cases} $$

Note that

$$ A_{k,\varepsilon} \leq C $$

(actually, we even have $\sup_{k \in \mathbb{N}} A_{k,\varepsilon} \to 0$ as $\varepsilon \to 0$ for $n \geq 4$). The equality (3.90) shows that the estimate (3.66) provides worse (or the same) convergence rate than the estimate (3.85) and, if $n \geq 3$, than the estimate (3.77). In the case $n = 2$ one has a “competition” between the converge rates $\sup_{k \in \mathbb{N}} \eta_{k,\varepsilon}$ coming from (3.66) and $\sup_{k \in \mathbb{N}} (q^{1/2}_{k,\varepsilon} \gamma_{k,\varepsilon} \ln q_{k,\varepsilon})$ coming from (3.77): for example, if $\gamma_{k,\varepsilon} \geq C_1 > 0$, then the second one dominates, but for very small $d_{k,\varepsilon}$ (namely, if $\sup_{k \in \mathbb{N}} (q^{1/2}_{k,\varepsilon} \ln q_{k,\varepsilon}) \to 0$ as $\varepsilon \to 0$) the first one gives worse convergence rate. Taking into account the above observations, (3.26) and the definition (2.17) of $\mu_{\varepsilon}$, we conclude from (3.64)–(3.66), (3.77), (3.85), (3.88), the final estimate for $a_{\varepsilon}([\mathcal{J}_{\varepsilon} f, u] - a[f, \mathcal{J}_{\varepsilon} u])$.

**Lemma 3.15** One has

$$ \forall f \in \mathcal{H}^2, \ u \in \mathcal{H}^1 : |a_{\varepsilon}([\mathcal{J}_{\varepsilon} f, u] - a[f, \mathcal{J}_{\varepsilon} u])| \leq C \mu_{\varepsilon} \| f \|_{\mathcal{H}^2} \| u \|_{\mathcal{H}^1}. $$

\(\square\) Springer
3.4 End of proofs of Theorems 2.2 and 2.4

By virtue of Proposition 3.5, the properties (3.27), (3.38), (3.92) imply the estimate
\[
\forall f \in \mathcal{H}: \quad \| (\mathcal{A} + I)^{-1} f - \mathcal{J}_e (\mathcal{A} + I)^{-1} f \|_{\mathcal{H}} \leq \| (\mathcal{A}_e + I)^{-1} f - \mathcal{J}_e (\mathcal{A} + I)^{-1} f \|_{\mathcal{H}} \leq C \mu_e \| f \|_{\mathcal{H}} \tag{3.93}
\]
(the first inequality above follows trivially from the definition of \( \| \cdot \|_{\mathcal{H}} \)). Then, using (3.26), (3.38), (3.93) we get
\[
\| (\mathcal{A}_e + I)^{-1} f - (\mathcal{A} + I)^{-1} f \|_{\mathcal{H}} \leq \| (\mathcal{A}_e + I)^{-1} f - \mathcal{J}_e (\mathcal{A} + I)^{-1} f \|_{\mathcal{H}} + \| (\mathcal{J}_e - I) (\mathcal{A} + I)^{-1} f \|_{\mathcal{H}} \leq C \mu_e \| f \|_{\mathcal{H}} + \| (\mathcal{A} + I)^{-1} f \|_{\mathcal{H}} \leq C \mu_e \| f \|_{\mathcal{H}},
\]
and Theorem 2.2 is proven.

Now, we proceed to the proof of Theorem 2.4. Let \( f \in \mathcal{H} \) and \( g = (\mathcal{A} + I)^{-1} f \). One has
\[
(\mathcal{A}_e + I)^{-1} f - (\mathcal{A} + I)^{-1} f - \mathcal{J}_e f = (\mathcal{A}_e + I)^{-1} f - \mathcal{J}_e (\mathcal{A} + I)^{-1} f + (\mathcal{Y}_e + \mathcal{W}_e) g. \tag{3.94}
\]
Here \( \mathcal{H}_e \) is defined by (2.22), and
\[
(\mathcal{Y}_e g) \pm (x) := \sum_{k \in \mathbb{N}} \left( (g^\pm)_{B_{k,e}^\pm} - g^\pm (x) \right) \phi_{k,e}^\pm (x), \quad x \in \Omega^\pm,
\]
\[
(\mathcal{W}_e g) \pm (x) := \pm \frac{1}{2} \sum_{k \in \mathbb{N}} \left( \{g\} \right)_{k,e} U_{k,e}^\pm (x) \mp \frac{1}{2} \sum_{k \in \mathbb{N}} g_{k,e} U_{k,e}^\pm (x) \phi_{k,e}^\pm (x), \quad x \in \Omega^\pm,
\]
where \( g_{k,e} := (g^+)_{B_{k,e}^+} - (g^-)_{B_{k,e}^-} \). Due to (3.93) we have
\[
\| (\mathcal{A}_e + I)^{-1} f - \mathcal{J}_e (\mathcal{A} + I)^{-1} f \|_{H^1(\Omega; \Gamma)} = \| (\mathcal{A}_e + I)^{-1} f - \mathcal{J}_e (\mathcal{A} + I)^{-1} f \|_{\mathcal{H}} \leq C_1 \| f \|_{\mathcal{H}}. \tag{3.95}
\]
Using the estimates (3.43), (3.44), (3.69), (3.76) (applied for \( f^\pm \) instead of \( f^\mp \)) and (3.51), we get
\[
\| (\mathcal{Y}_e g)^\pm \|_{L^2(\Omega^\pm)} \leq \sum_{k \in \mathbb{N}} \left\| g^\pm - (g^\pm)_{B_{k,e}^\pm} \right\|^2_{L^2(\overline{B}_{k,e}^\pm)} \leq \| \zeta_e \| g \|_{\mathcal{H}} \leq C_1 \mu_e \| g \|_{\mathcal{H}} \leq C_1 \mu_e \| g \|_{\mathcal{H}}, \tag{3.96}
\]
\[
\| \nabla (\mathcal{Y}_e g)^\pm \|_{L^2(\Omega^\pm)} \leq \sum_{k \in \mathbb{N}} \left[ \| \nabla g^\pm \|_{L^2(\overline{B}_{k,e}^\pm)} \right]^2 + \left( \sum_{k \in \mathbb{N}} \| (g^\pm - (g^\pm)_{B_{k,e}^\pm}) \nabla \phi_{k,e}^\pm \|_{L^2(\overline{B}_{k,e}^\pm)} \right)^2 \leq C \sup_{k \in \mathbb{N}} \eta_{k,e} \| g \|_{\mathcal{H}} \| g \|_{\mathcal{H}} \leq C \mu_e \| g \|_{\mathcal{H}}. \tag{3.97}
\]
Let us now estimate \( (\mathcal{W}_e g)^\pm \). One has
\[(W_{\epsilon} g)^+ = \frac{1}{2} \sum_{k \in \mathbb{N}} (g^+_{S_k, \epsilon} - g^+_{B_k, \epsilon}) U^+_{k, \epsilon} - \frac{1}{2} \sum_{k \in \mathbb{N}} (g^-_{S_k, \epsilon} - g^-_{B_k, \epsilon}) U^+_{k, \epsilon} \]
\[+ \frac{1}{2} \sum_{k \in \mathbb{N}} (g^+_{B_k, \epsilon} - g^-_{B_k, \epsilon}) U^+_{k, \epsilon} (1 - \psi^+_{k, \epsilon}).\]

whence, taking into account (2.3), we infer

\[\| (W_{\epsilon} g)^+ \|_{H^1(\Omega^+)} \leq \frac{1}{2} \left( \sum_{k \in \mathbb{N}} \| (g^+_{S_k, \epsilon} - g^+_{B_k, \epsilon}) U^+_{k, \epsilon} \|^2_{H^1(B_k^+, \epsilon)} \right)^{1/2} \]
\[+ \frac{1}{2} \left( \sum_{k \in \mathbb{N}} \| (g^-_{S_k, \epsilon} - g^-_{B_k, \epsilon}) U^+_{k, \epsilon} \|^2_{H^1(B_k^+, \epsilon)} \right)^{1/2} \]
\[+ \frac{1}{2} \left( 2 \sum_{k \in \mathbb{N}} \| (g^+_{B_k, \epsilon})^2 + (g^-_{B_k, \epsilon})^2 \|_{H^1(B_k^+, \epsilon)} \right)^{1/2} (1 - \psi^+_{k, \epsilon}) \|^2_{H^1(B_k^+, \epsilon)} \right)^{1/2} .\]

From (3.45) we get

\[\| U^+_{k, \epsilon} \|^2_{H^1(B_k^+, \epsilon)} = \| \nabla U^+_{k, \epsilon} \|^2_{L^2(B_k^+, \epsilon)} + \| U^+_{k, \epsilon} \|^2_{L^2(B_k^+, \epsilon)} \leq C \xi_{k, \epsilon} \| \nabla g^\pm \|_{H^1(\Omega^+)} \times (3.99)\]

Also, using the properties

\[0 \leq \psi^+_{k, \epsilon} \leq 1, \quad |\nabla \psi^+_{k, \epsilon}| \leq C \xi_{k, \epsilon}^{-1}, \quad \text{supp}(1 - \psi^+_{k, \epsilon}) \cap B^+_k \subset B^+_k \setminus \mathcal{B} (\frac{\xi_{k, \epsilon}}{4}, \epsilon)\]

and (3.56)–(3.57), we deduce easily the estimate

\[\| U^+_{k, \epsilon} (1 - \psi^+_{k, \epsilon}) \|^2_{H^1(B_k^+, \epsilon)} = \| \nabla (U^+_{k, \epsilon} (1 - \psi^+_{k, \epsilon})) \|^2_{L^2(B_k^+, \epsilon)} + \| U^+_{k, \epsilon} \|^2_{L^2(B_k^+, \epsilon)} \leq C \xi_{k, \epsilon}^{-1} \| \nabla \psi^+_{k, \epsilon} \|^2_{L^2(B_k^+, \epsilon)} \times (3.100)\]

By Lemma 3.7 \[g^\pm \in H^2(O^\pm), \quad \text{whence } \nabla g^\pm \in H^1(O^\pm).\] Using the estimate (3.3) (applied for \[\nabla g^\pm, (2.4)\]) and the trace estimate \[\| \nabla g^\pm \|^2_{L^2(\Gamma)} \leq C \| g^\pm \|^2_{H^1(O^\pm)}, \quad \text{we obtain}\]

\[\sum_{k \in \mathbb{N}} \xi_{k, \epsilon} \gamma_{k, \epsilon} \| \nabla g^\pm \|^2_{L^2(B_k^+, \epsilon)} \leq C \sum_{k \in \mathbb{N}} \xi_{k, \epsilon} \gamma_{k, \epsilon} \| g^\pm \|^2_{L^2(S_k^+, \epsilon)} + \xi_{k, \epsilon} \sum_{i, j=1}^n \left\| \frac{\partial g^\pm}{\partial x^i} \frac{\partial g^\pm}{\partial x^j} \right\|^2_{L^2(B_k^+, \epsilon)} \]
\[\leq C \sup_{k \in \mathbb{N}} \xi_{k, \epsilon} \gamma_{k, \epsilon} \left( \| \nabla g^\pm \|^2_{L^2(\Gamma)} + \sum_{i, j=1}^n \left\| \frac{\partial g^\pm}{\partial x^i} \frac{\partial g^\pm}{\partial x^j} \right\|^2_{L^2(O^\pm)} \right) \]
\[\leq C \sup_{k \in \mathbb{N}} \xi_{k, \epsilon} \gamma_{k, \epsilon} \| g^\pm \|^2_{H^2(O^\pm)}.\]
Using (3.4), (3.46), (3.47), (3.98)–(3.101) and taking into account (3.26), (3.90), (3.91), and
\( \varrho_{k,e} \gamma^{1/2} = \eta_{k,e} \varrho^{1/2} A_{k,e} \), where \( A_{k,e} \) is given in (3.90) \( \tag{3.102} \)
(the above equality follows directly from (3.90)), we get
\[
\| (\mathcal{A}_{\varepsilon} g)^+ \|_{\mathcal{H}^1(\Omega^+)} \leq C \left( \sum_{k \in \mathbb{N}} \varrho_{k,e} \gamma_{k,e} \| \nabla g^+ \|^2_{L^2(B_{k,e}^+)} \right)^{1/2} + C \left( \sum_{k \in \mathbb{N}} \varrho_{k,e} \gamma_{k,e} \| \nabla g^- \|^2_{L^2(B_{k,e}^-)} \right)^{1/2}
\]
\[+ C \left( \sum_{k \in \mathbb{N}} \varrho_{k,e} \gamma_{k,e}^2 \left( \| g^+ \|^2_{L^2(B_{k,e}^+)} + \| g^- \|^2_{L^2(B_{k,e}^-)} \right) \right)^{1/2}
\]
\[\leq C \sup_{k \in \mathbb{N}} (\varrho_{k,e} \gamma_{k,e}) \left( \| g^+ \|^2_{H^2(O^+)} + \| g^- \|^2_{H^2(O^-)} \right)
\]
\[+ C \sup_{k \in \mathbb{N}} (\varrho_{k,e} \gamma_{k,e}) \left( \| g^+ \|^2_{\mathcal{H}^1(\Omega^+)} + \| g^- \|^2_{\mathcal{H}^1(\Omega^-)} \right) \leq C_1 \mu \varepsilon \| g \|_{\mathcal{H}^2}.
\] \( \tag{3.103} \)

Similarly,
\[
\| (\mathcal{A}_{\varepsilon} g)^- \|_{\mathcal{H}^1(\Omega^-)} \leq C \mu \varepsilon \| g \|_{\mathcal{H}^2}.
\] \( \tag{3.104} \)

It follows from (3.94)–(3.97), (3.103), (3.104) and \( \| g \|_{\mathcal{H}^1} \leq \| g \|_{\mathcal{H}^2} = \| f \|_{\mathcal{H}} \) that
\[
\| (\mathcal{A}_{\varepsilon} + I)^{-1} f - (I + \mathcal{A}_{\varepsilon}) (\mathcal{A}_{\varepsilon} + I)^{-1} f \|_{\mathcal{H}^1(\Omega^1)} \leq C \mu \varepsilon \| f \|_{\mathcal{H}}.
\]

It remains to prove (2.24)–(2.25). Let \( f \in \mathcal{H} \) and \( g = (\mathcal{A}_{\varepsilon} + I)^{-1} f \). Using (2.3), (3.45)–(3.47) and the Cauchy-Schwarz inequality, we get
\[
\| (\mathcal{A}_{\varepsilon} f)^\pm \|_{L^2(\Omega^\pm)}^2 = \frac{1}{4} \sum_{k \in \mathbb{N}} \| \{ g \} s_{k,e} \|_{L^2(\Omega^\pm)}^2
\]
\[\leq C \sum_{k \in \mathbb{N}} \| u_{\varepsilon} \|^2_{L^2(B_{k,e}^\pm)} \varrho_{k,e} \gamma_{k,e} \left( \| g^+ \|^2_{L^2(B_{k,e}^\pm)} + \| g^- \|^2_{L^2(B_{k,e}^\pm)} \right)
\]
\[\leq C \sup_{k \in \mathbb{N}} (\varrho_{k,e} \gamma_{k,e}) \| g \|^2_{\mathcal{H}^1} \leq C \sup_{k \in \mathbb{N}} (\varrho_{k,e} \gamma_{k,e}) \| g \|^2_{\mathcal{H}^2}
\]
\[= C \sup_{k \in \mathbb{N}} (\varrho_{k,e} \gamma_{k,e}) \| f \|^2_{\mathcal{H}} \leq C_2 \sup_{k \in \mathbb{N}} (\varrho_{k,e} \gamma_{k,e}) \| f \|^2_{\mathcal{H}}
\]
(on the last step we use (3.102) and (3.91)). Furthermore, using (2.21) and (3.59), we obtain
\[
\| \nabla (\mathcal{A}_{\varepsilon} f)^\pm \|_{L^2(\Omega^\pm)}^2 = \frac{1}{4} \sum_{k \in \mathbb{N}} \| \nabla u_{\varepsilon} \|^2_{L^2(B_{k,e}^\pm)} \| \{ g \} s_{k,e} \|^2 = \frac{1}{2} \left( \| \gamma^{1/2} [g] \|^2_{L^2(\Gamma)} + L_\varepsilon \right)
\]
where \( L_\varepsilon := \frac{1}{2} \sum_{k \in \mathbb{N}} \mathcal{G}(D_{k,e}) \| \{ g \} s_{k,e} \|^2 - \| \gamma^{1/2} [g] \|^2_{L^2(\Gamma)} \). Using (2.11), the trace theorem and Lemma 3.7, we get the required estimate on the remainder \( L_\varepsilon \):
\[
| L_\varepsilon | \leq k \varepsilon \| g \|^2_{H^2(\Gamma)} \| g \|_{H^1(\mathcal{H})} \leq C \kappa \varepsilon \left( \| g \|^2_{H^2(O^+)} + \| g \|^2_{H^2(O^-)} \right) \left( \| g \|^2_{H^1(\Omega^+)} + \| g \|^2_{H^1(\Omega^-)} \right)
\]
\[\leq C_1 \kappa \varepsilon \| g \|^2_{\mathcal{H}^2} \| g \|^2_{\mathcal{H}^1} \leq C_1 \kappa \varepsilon \| g \|^2_{\mathcal{H}^2} \| f \|^2_{\mathcal{H}}.
\]

Theorem 2.4 is proven.

**Remark 3.16** Lemma 3.7 is the only place where we use the special form of the domain \( \Omega \), see (2.1). If \( \Gamma \) intersects \( \Omega \), we face a problem–\( H^2 \)-regularity fails in a neighborhood of \( \partial \Omega \cap \Gamma \), where \( \delta' \)-interface conditions “meet” with the Neumann boundary conditions. This
difficulty can be overcome in the case when the holes belong to a subset $\Gamma' \subset \Gamma \cap \Omega$ with $\text{dist}(\Gamma', \partial \Omega) > 0$ and $\text{supp}(\gamma) \subset \Gamma'$ (cf. Remark 2.5). In this case the functions from $\text{dom}(\gamma')$ belong to $H^2(\partial \Omega')$, where $O^\pm := \{x = (x', x^n) : x' \in \Gamma', |x^n| < \text{dist}(\Gamma', \partial \Omega)/2, \pm x^n > 0\}$, and the estimate (3.30) holds with $O^\pm$ being replaced by $O^\pm$—these properties allows to carry out the proof in the same was as it is done above for $\Omega$ satisfying (2.1).

### 4 Examples

In this section we present two examples for which the assumption (2.11) holds true.

#### 4.1 Very small holes

The first example deals with the case when the holes $D_{k,\varepsilon}$ are so small that the limiting function $\gamma$ is zero.

Assume that the sets $D_{k,\varepsilon} \subset \Gamma$ satisfies the conditions (2.3) and (2.4), while (2.5) is complemented by the stronger assumption

$$\lim_{\varepsilon \to 0} \sup_{k \in \mathbb{N}} \gamma_{k,\varepsilon} \to 0.$$ 

Let $g \in H^{3/2}(\Gamma)$ and $h \in H^{1/2}(\Gamma)$. Using the Cauchy-Schwarz inequality, the estimate (3.39) and taking into account that $\text{area}(S_{k,\varepsilon}) = C \|g\|^{-1} = C \|D_{k,\varepsilon}\gamma_{k,\varepsilon}^{-1}$, we obtain

$$\left| \sum_{k \in \mathbb{N}} \mathcal{G}(D_{k,\varepsilon})(g)S_{k,\varepsilon}(\bar{h})S_{k,\varepsilon} \right| \leq \left( \sum_{k \in \mathbb{N}} \mathcal{G}(D_{k,\varepsilon})\|g\|_{S_{k,\varepsilon}}^2 \right)^{1/2} \left( \sum_{k \in \mathbb{N}} \mathcal{G}(D_{k,\varepsilon})\|\bar{h}\|_{S_{k,\varepsilon}}^2 \right)^{1/2} \leq C \sup_{k \in \mathbb{N}} \gamma_{k,\varepsilon} \|g\|_{L^2(\Gamma)} \|h\|_{L^2(S_{k,\varepsilon})} \leq C \sup_{k \in \mathbb{N}} \gamma_{k,\varepsilon} \|g\|_{H^{1/2}(\Gamma)} \|h\|_{H^{1/2}(\Gamma)}.$$ 

Thus (2.11) holds with $\gamma = 0$ and $\kappa_{\varepsilon} = C \sup_{k \in \mathbb{N}} \gamma_{k,\varepsilon}$.

#### 4.2 Regularly distributed holes

In the second case we consider the holes whose location is subordinate to some partition of $\Gamma$ and show that their sizes can be chosen such a way that (2.11) holds with any predefined $\gamma$.

For simplicity, we assume that $n \geq 3$. Furthermore, we assume that the holes $D_{k,\varepsilon} \subset \Gamma$ are balls, i.e. $D_{k,\varepsilon} = B(d_{k,\varepsilon}, r_{k,\varepsilon}) \cap \Gamma$. As before, we assume that there exist a sequence $\{q_{k,\varepsilon}, k \in \mathbb{N}\}$ of positive numbers such that (2.3), (2.4) hold. Additionally, we assume that there exists a family $\{\Gamma_{k,\varepsilon}, k \in \mathbb{N}\}$ of relatively open subsets of $\Gamma$ such that

1. $B_{k,\varepsilon} \cap \Gamma \subset \Gamma_{k,\varepsilon}$ (recall that $B_{k,\varepsilon} = B(q_{k,\varepsilon}, s_{k,\varepsilon})$),
2. $\Gamma_{k,\varepsilon} \cap \Gamma_{l,\varepsilon} = \emptyset$ if $k \neq l$,
3. $\bigcup_{k \in \mathbb{N}} \Gamma_{k,\varepsilon} = \Gamma$,
4. $\exists m \in \mathbb{N} \forall x \in \Gamma : \#(\{k \in \mathbb{N} : x \in \text{conv}(\Gamma_{k,\varepsilon})\}) \leq m$,
5. $r_{k,\varepsilon} \leq C q_{k,\varepsilon}$.
where \( r_{k,\varepsilon} := \text{diam}(\Gamma_{k,\varepsilon}) \), by \( \text{conv}(\Gamma_{k,\varepsilon}) \) we denote the convex hull of \( \Gamma_{k,\varepsilon} \), and \(#(\ldots)\) stands for the cardinality of the enclosed set (that is, if all \( \Gamma_{k,\varepsilon} \) are convex, then (4.4) holds with \( m = 1 \)). The above conditions are illustrated on Fig. 4.

We specify the numbers \( d_{k,\varepsilon} \). Let \( \gamma \in C^1(\Gamma) \cap W^{1,\infty}(\Gamma) \) be arbitrary positive function.

Set

\[
d_{k,\varepsilon} = (4\alpha - 1) \gamma(x_k,\varepsilon) \text{area}(\Gamma_{k,\varepsilon})^{1/(n-2)},
\]

where the constant \( \alpha \) is the Newtonian capacity of the set \( \mathcal{B}(1,0) \cap \Gamma \), i.e.

\[
\alpha = \inf_U \| \nabla U \|^2_{L^2(\mathbb{R}^n)},
\]

where the infimum is taken over \( U \in C_0^\infty(\mathbb{R}^n) \) being equal to 1 on a neighbourhood of \( \mathcal{B}(1,0) \cap \Gamma \) (with the neighbourhood depending upon \( U \)). Due to (4.5) (which implies \( \text{area}(\Gamma_{k,\varepsilon}) \leq C \varrho_k^{-1} \)) this choice of \( d_{k,\varepsilon} \) is indeed admissible, i.e. the condition (2.5) is fulfilled.

We are in position to formulate the main result of this subsection.

**Proposition 4.1** Let the assumptions (2.3), (2.4), (4.1)–(4.6) hold true. Then the condition (2.11) is fulfilled with \( \kappa_{\varepsilon} = C \sup_{k \in \mathbb{N}} \varrho_k^{1/2} \).

**Proof** First, we introduce several new sets:

- \( \hat{\Gamma}_{k,\varepsilon} := \text{conv}(\Gamma_{k,\varepsilon}) \),
- \( \hat{P}_{k,\varepsilon} := \{ x = (x',x^n) \in \mathbb{R}^n : x' \in \Gamma_{k,\varepsilon}, 0 < |x^n| < \varrho_{k,\varepsilon}, x^n > 0 \} \),
- \( \hat{\tilde{P}}_{k,\varepsilon} := \{ x = (x',x^n) \in \mathbb{R}^n : x' \in \hat{\Gamma}_{k,\varepsilon}, 0 < |x^n| < \varrho_{k,\varepsilon}, x^n > 0 \} \).

Due to (2.7), one has \( \hat{\tilde{P}}_{k,\varepsilon} \subset O^+ \). Furthermore, by virtue of (4.4), one has the estimate

\[
\forall f \in L^2(O^+) : \sum_{k \in \mathbb{N}} \| f \|_{L^2(\hat{\tilde{P}}_{k,\varepsilon})}^2 \leq m \| f \|_{L^2(\bigcup_{k \in \mathbb{N}} \hat{\tilde{P}}_{k,\varepsilon})}^2 \leq m \| f \|_{L^2(O^+)}^2.
\]

**Lemma 4.2** One has

\[
\forall v \in H^1(\hat{\tilde{P}}_{k,\varepsilon}) : \| v \|_{L^2(\hat{\tilde{P}}_{k,\varepsilon})} \leq 2 \varrho_{k,\varepsilon}^{-1} \| v \|_{L^2(\hat{\tilde{P}}_{k,\varepsilon})}^2 + 2 \varrho_{k,\varepsilon} \| \nabla v \|_{L^2(\hat{\tilde{P}}_{k,\varepsilon})}^2,
\]

\[
\forall v \in H^1(\hat{\tilde{P}}_{k,\varepsilon}) : \| v - (v)_{\hat{\tilde{P}}_{k,\varepsilon}} \|_{L^2(\hat{\tilde{P}}_{k,\varepsilon})} \leq C \varrho_{k,\varepsilon}^2 \| \nabla v \|_{L^2(\hat{\tilde{P}}_{k,\varepsilon})}^2.
\]
Proof It is enough to proof (4.8) for $v \in C^\infty(\hat{P}_{k,e})$. Let $(x', 0) \in \hat{\Gamma}_{k,e}$ and $(x', z) \in \hat{P}_{k,e}$ with $z \in (0, \varrho_{k,e})$. One has

$$v(x', 0) = v(x', z) - \int_0^z \frac{\partial v}{\partial x^n}(x', \tau) \, d\tau,$$

whence, using the Cauchy-Schwarz inequality, we get

$$|v(x', 0)|^2 \leq 2|v(x', z)|^2 + 2\varrho_{k,e} \int_0^{\varrho_{k,e}} \left|\frac{\partial v}{\partial x^n}(x', \tau)\right|^2 \, d\tau \leq 2|v(x', z)|^2 + 2\varrho_{k,e} \int_0^{\varrho_{k,e}} \left|\nabla v(x', \tau)\right|^2 \, d\tau.$$

Integrating this inequality over $z \in (0, \varrho_{k,e})$ and over $x' \in \hat{\Gamma}_{k,e}$, then dividing by $\varrho_{k,e}$, we arrive at the estimate (4.8).

To prove the estimate (4.9) we need the following result [6, Theorem 3.2]: let $D$ be a convex domain, then

$$\|v - \langle v \rangle_D\|_{L^2(D)} \leq \frac{\text{diam}(D)}{\pi} \|\nabla v\|_{L^2(D)}, \ \forall v \in H^1(D).$$

Applying it for $D = \hat{P}_{k,e}$ (evidently, this set is complex) and taking into account (cf. (4.5)) that $\text{diam}(\hat{P}_{k,e}) \leq C\varrho_{k,e}$ (here we use $\text{diam}(\hat{\Gamma}_{k,e}) = \text{diam}(\Gamma_{k,e}) \leq C\varrho_{k,e}$), we immediately get the required estimate (4.9). The lemma is proven. \hfill \Box

Lemma 4.3 One has

$$\forall v \in H^1(\hat{P}_{k,e}) : \left|\langle v \rangle_{\Gamma_{k,e}} - \langle v \rangle_{\hat{P}_{k,e}}\right|^2 \leq C_1 \varrho_{k,e}^2 \|\nabla v\|_{L^2(\hat{P}_{k,e})}^2,$$

(4.10) and

$$\forall v \in H^1(\hat{P}_{k,e}) : \left|\langle v \rangle_{\Gamma_{k,e}} - \langle v \rangle_{\partial \hat{P}_{k,e}}\right|^2 \leq C_2 \varrho_{k,e}^2 \|\nabla v\|_{L^2(\hat{P}_{k,e})}^2,$$

(4.11) Proof Using the Cauchy-Schwarz inequality, the estimates (4.8)–(4.9), and taking into account that $\text{area}(\Gamma_{k,e}) \geq C\varrho_{k,e}^{n-1}$, we get

$$\left|\langle v \rangle_{\Gamma_{k,e}} - \langle v \rangle_{\hat{P}_{k,e}}\right|^2 = \left|\langle v - \langle v \rangle_{\hat{P}_{k,e}}\rangle_{\Gamma_{k,e}}\right|^2 \leq \frac{1}{\text{area}(\Gamma_{k,e})} \left\|v - \langle v \rangle_{\hat{P}_{k,e}}\right\|_{L^2(\Gamma_{k,e})}^2 \leq \frac{1}{\text{area}(\Gamma_{k,e})} \left\|v - \langle v \rangle_{\hat{P}_{k,e}}\right\|_{L^2(\Gamma_{k,e})}^2 \leq \frac{C}{\text{area}(\Gamma_{k,e})} \left(\varrho_{k,e}^{-1} \left|v - \langle v \rangle_{\hat{P}_{k,e}}\right|_{L^2(\hat{P}_{k,e})}^2 + \varrho_{k,e} \|\nabla v\|_{L^2(\hat{P}_{k,e})}^2\right) \leq C_1 \varrho_{k,e}^2 \|\nabla v\|_{L^2(\hat{P}_{k,e})}^2.$$

(4.12)

To proceed further we need an auxiliary result from [29]: let $D \subset \mathbb{R}^n$ be a bounded convex domain, $D_1, D_2$ be arbitrary measurable subsets of $D$ with $\text{vol}(D_1) \neq 0, \text{vol}(D_2) \neq 0$, then

$$\forall v \in H^1(D) : \left|\langle v \rangle_{D_1} - \langle v \rangle_{D_2}\right|^2 \leq \frac{(\text{diam}(D))^{n+2}}{\text{vol}(D_1) \cdot \text{vol}(D_2)} \|\nabla v\|_{L^2(D)}^2,$$

where the constant $C > 0$ depends only on $n$. Applying this estimate for $D := \hat{P}_{k,e}, D_1 := \hat{P}_{k,e}$ and $D_2 := B^+_{k,e}$ and taking into account that $\text{diam}(\hat{P}_{k,e}) \leq C\varrho_{k,e}$ and $\text{vol}(\hat{P}_{k,e}) \geq \text{vol}(B^+_{k,e}) = C\varrho_{k,e}^n$, we obtain

$$\forall v \in H^1(\hat{P}_{k,e}) : \left|\langle v \rangle_{\hat{P}_{k,e}} - \langle v \rangle_{B^+_{k,e}}\right|^2 \leq C \varrho_{k,e}^2 \|\nabla v\|_{L^2(\hat{P}_{k,e})}^2.$$

(4.13)
Combining (4.12), (4.13) and (3.4), we arrive at the estimate (4.11). The lemma is proven. □

Lemma 4.4 One has:

\[
\alpha d_{k,e}^{n-2} \leq \mathcal{C}(D_{k,e}) \leq \alpha d_{k,e}^{n-2}(1 + C_{D_{k,e}}).
\] (4.14)

Proof Let \( \alpha \) be the Newtonian capacity of the set \( D_{k,e} = B(d_{k,e}, x_{k,e}) \cap \Gamma \), i.e.

\[
\alpha = \inf_U \| \nabla U \|^2_{L^2(\mathbb{R}^n)},
\] (4.15)

where the infimum is taken over \( U \in C_0^\infty(\mathbb{R}^n) \) being equal to 1 on a neighbourhood of \( D_{k,e} \).

Simple re-scaling arguments yield

\[
\alpha = \alpha d_{k,e}^{n-2}.
\] (4.16)

Let \( U_{k,e} \) be the solution to the problem (2.20) extended by zero to the whole \( \mathbb{R}^n \). We wish to use \( U_{k,e} \) as a test-function in (4.15), but since \( U_{k,e} \) is not smooth we have to modify it slightly. Namely, for each \( \delta > 0 \) there exists \( U_{\delta k,e} \in C_0^\infty(\mathbb{R}^n) \) such that

\[
\| \nabla U_{\delta k,e} - \nabla U_{k,e} \|_{L^2(\mathbb{R}^n)} \leq \delta \quad \text{and} \quad U_{\delta k,e} = 1 \text{ on a neighborhood of } D_{k,e}.
\]

Using (4.15) and (2.21), we get

\[
\forall \delta > 0 : \quad \alpha_{\delta}^{1/2} \leq \| \nabla U_{k,e} \|_{L^2(\mathbb{R}^n)} \leq \| \nabla U_{k,e} \|_{L^2(\mathbb{R}^n)} + \delta = \sqrt{\mathcal{C}(D_{k,e})} + \delta,
\]

whence, sending \( \delta \to 0 \), we obtain

\[
\alpha \leq \mathcal{C}(D_{k,e}).
\] (4.17)

Combining (4.16) and (4.17) we get the first estimate in (4.14).

To prove the second estimate we introduce the function \( V_{k,e} \)–the solution to the problem

\[
\begin{aligned}
\Delta V &= 0, \quad x \in \mathbb{R}^n \setminus D_{k,e}, \\
V &= 1, \quad x \in \partial D_{k,e} = D_{k,e}, \\
V &\to 0, \quad |x| \to \infty.
\end{aligned}
\]

It is well-known that

\[
\alpha = \| \nabla V_{k,e} \|^2_{L^2(\mathbb{R}^n)}.
\]

Recall that the function \( \psi_{k,e} \) is defined in (3.28). Using (2.9) and taking into account that \( V_{k,e} \psi_{k,e} \in H_0^1(B_{k,e}) \) and \( (V_{k,e} \psi_{k,e}) |_{D_{k,e}} = 1 \), we get

\[
\mathcal{C}(D_{k,e}) \leq \| \nabla (V_{k,e} \psi_{k,e}) \|_{L^2(B_{k,e})}^2 = \| \nabla (V_{k,e} \psi_{k,e}) \|_{L^2(\mathbb{R}^n)}^2
\]

\[
= \| \nabla V_{k,e} \|^2_{L^2(\mathbb{R}^n)} + R_e = \alpha + R_e,
\] (4.18)

where \( R_e = R_1^e + R_2^e \) with

\[
R_1^e := \| \nabla (V_{k,e} \psi_{k,e} - 1) \|^2_{L^2(\mathbb{R}^n)}, \quad R_2^e := 2(\nabla V_{k,e}, \nabla (V_{k,e} \psi_{k,e} - 1))_{L^2(\mathbb{R}^n)}.
\]

The function \( V_{k,e} \) obeys the following estimates:

\[
\forall x \in \mathbb{R}^n \setminus B(d_{k,e}, x_{k,e}) : \quad |V_{k,e}(x)| \leq C d_{k,e}^{n-2} |x - x_{k,e}|^{2-n}.
\]
The desired estimate follows from (4.16), (4.18), (4.22). The lemma is proven.

where

Then, using the Cauchy-Schwarz inequality, (4.11), (4.7) and the fact (cf. (3.39)) that (the proof is similar to the proof of (3.56)–(3.57), an alternative proof can be found in [35, Lemma 2.1]). Using (4.19) and the properties

we obtain

Similarly,

Combining (4.20)–(4.21) and taking into account that $d_{k,e}^{n-2} \leq C \varrho_{k,e}^{n-1}$, we finally get

The desired estimate follows from (4.16), (4.18), (4.22). The lemma is proven.

Now, we proceed to the proof of Proposition 4.1. Let $g \in H^{3/2}(\Gamma)$ and $h \in H^{1/2}(\Gamma)$. Taking into account (4.2)–(4.3) one can represent the left-hand-side of (2.11) as follows:

where

It is well-know (see, e.g., [33, Theorem 3.1]) that there exists a bounded linear operator $\mathcal{E} : H^{1/2}(\Gamma) \to H^1(O^+)\,$, which is a right inverse of the trace operator $H^1(O^+) \to H^1(\Gamma)$. Then, using the Cauchy-Schwarz inequality, (4.11), (4.7) and the fact (cf. (3.39)) that

$$
\mathcal{E}(D_{k,e}) \leq C \varrho_{k,e}^{n-1}, \quad \text{area}(S_{k,e}) = C_1 \varrho_{k,e}^{n-1}, \quad \text{area}(\Gamma_{k,e}) \geq C \varrho_{k,e}^{n-1},
$$

(4.23)
we get

\[
|P^1_\epsilon| \leq \frac{1}{4} \left( \sum_{k \in \mathbb{N}} \mathcal{C}(D_k, \epsilon) (g) S_{k, \epsilon} \right)^2 \left( \sum_{k \in \mathbb{N}} \mathcal{C}(D_k, \epsilon) (\frac{\Delta}{\Gamma_1} S_{k, \epsilon} - \frac{\Delta}{\Gamma_1} \Gamma_{k, \epsilon} \right)^2
\]

\[
\leq C \left( \sum_{k \in \mathbb{N}} g \|g\|_{L^2(S_{k, \epsilon})} \right)^{1/2} \left( \sum_{k \in \mathbb{N}} \partial_{k, \epsilon} \|\nabla (\partial h)\|_{L^2(\partial_k \epsilon)} \right)^{1/2}
\]

\[
\leq C_1 \sup_{k \in \mathbb{N}} \partial_{k, \epsilon}^{1/2} \|g\|_{L^2(\Gamma)} \|\nabla (\partial h)\|_{L^2(\Omega^+)} \leq C_2 \sup_{k \in \mathbb{N}} \partial_{k, \epsilon}^{1/2} \|g\|_{L^2(\Gamma)} \|h\|_{H^{1/2}(\Gamma)},
\]

(4.24)

and, similarly,

\[
|P^2_\epsilon| \leq C \sup_{k \in \mathbb{N}} \partial_{k, \epsilon}^{1/2} \|g\|_{H^{1/2}(\Gamma)} \|h\|_{L^2(\Gamma)}.
\]

(4.25)

Further, by virtue of (4.6), (4.14) and (2.5) one has

\[
\left| \frac{1}{4} \mathcal{C}(D_k, \epsilon) - \gamma(x_{k, \epsilon}) \text{area}(\Gamma_{k, \epsilon}) \right| = \frac{1}{4} \left| \mathcal{C}(D_k, \epsilon) - \alpha d_{k, \epsilon}^{n-2} \right|
\]

\[
\leq C d_{k, \epsilon}^{n-2} \partial_{k, \epsilon} \leq C_1 \partial_{k, \epsilon}^{n},
\]

whence, taking into account that area(\Gamma_{k, \epsilon}) \geq C \partial_{k, \epsilon}^{n-1}, we get

\[
|P^3_\epsilon| \leq C \left( \sum_{k \in \mathbb{N}} \partial_{k, \epsilon} (g) \Gamma_{k, \epsilon} \right)^{1/2} \left( \sum_{k \in \mathbb{N}} \partial_{k, \epsilon} (h) \Gamma_{k, \epsilon} \right)^{1/2} \leq C \sup_{k \in \mathbb{N}} \partial_{k, \epsilon} \|g\|_{L^2(\Gamma)} \|h\|_{L^2(\Gamma)}.
\]

(4.26)

Using (4.7)–(4.10), we obtain

\[
|P^4_\epsilon| \leq C \left( \sum_{k \in \mathbb{N}} g \|g\|_{L^2(\Gamma_{k, \epsilon})} \right)^{1/2} \left( \sum_{k \in \mathbb{N}} \|h - \langle h \rangle \Gamma_{k, \epsilon}\|_{L^2(\Gamma_{k, \epsilon})} \right)^{1/2}
\]

\[
\leq C \|g\|_{L^2(\Gamma)} \left( \sum_{k \in \mathbb{N}} \partial_{k, \epsilon}^{-1} \|\partial h - \langle \partial h \rangle \Gamma_{k, \epsilon}\|_{L^2(\partial_k \epsilon)} + \partial_{k, \epsilon} \|\nabla \partial h\|_{L^2(\partial_k \epsilon)} \right)^{1/2}
\]

\[
\leq C \|g\|_{L^2(\Gamma)} \left( \sum_{k \in \mathbb{N}} \partial_{k, \epsilon}^{-1} \|\partial h - \langle \partial h \rangle \partial_k \epsilon\|_{L^2(\partial_k \epsilon)} + \partial_{k, \epsilon} \|\nabla \partial h\|_{L^2(\partial_k \epsilon)} \right)^{1/2}
\]

\[
\leq C \sup_{k \in \mathbb{N}} \partial_{k, \epsilon}^{1/2} \|g\|_{L^2(\Gamma)} \|\nabla \partial h\|_{L^2(\Omega^+)}
\]

(4.27)
Finally, since \(|\gamma(x) - \gamma(x_k,\varepsilon)| \leq C|x - x_k,\varepsilon|\) (recall that \(\gamma \in C^1(\Gamma) \cap W^{1,\infty}(\Gamma)\)), we infer

\[
|P^5_\varepsilon| \leq C \sup_{k \in \mathbb{N}} \rho_k \varepsilon \sum_{k \in \mathbb{N}} \|g\|_{L^2(\Gamma)} \|h\|_{L^2(\Gamma)}.
\]  

(4.28)

The statement of the proposition follows immediately from (4.23)–(4.28).

Acknowledgements The author gratefully acknowledges financial support by the Czech Science Foundation (GAČR) through the project 22-18739S and by the research program “Mathematical Physics and Differential Geometry” of the Faculty of Science of the University of Hradec Králové. The author thanks Jussi Behrndt and VladimirLotoreichik for useful suggestions.

References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Anné, C., Post, O.: Wildly perturbed manifolds: norm resolvent and spectral convergence. J. Spectr. Theory 11(1), 229–279 (2021). https://doi.org/10.4171/JST/340
3. Ansini, N.: The nonlinear sieve problem and applications to thin films. Asymptotic Anal. 39(2), 113–145 (2004). https://content.iospress.com/articles/asymptotic-analysis/asy636
4. Attouch, H.: Variational Convergence for Functions and Operators. Pitman Advanced Pub, Boston (1984)
5. Attouch, H., Picard, C.: Comportement limité de problèmes de transmission unilateraux à travers des grilles de forme quelconque. Rend. Semin. Mat., Torino, 45(1):71–85 (1987)
6. Bebendorf, M.: A note on the Poincaré inequality for convex domains. Z. Anal. Anwend. 22(4), 751–756 (2003). https://doi.org/10.4171/ZAA/1170
7. Behrndt, J., Exner, P., Lotoreichik, V.: Schrödinger operators with \(\delta\) and \(\delta'\)-interactions on Lipschitz surfaces and chromatic numbers of associated partitions. Rev. Math. Phys. 26(8), 1450015 (2014). https://doi.org/10.1142/S0129055X14500159
8. Behrndt, J., Langer, M., Lotoreichik, V.: Schrödinger operators with \(\delta\) and \(\delta'\)-potentials supported on hypersurfaces. Ann. Henri Poincaré 14(2), 385–423 (2013). https://doi.org/10.1007/s00023-012-0189-5
9. Birman, M.S., Suslina, T.A.: Second order periodic differential operators. Threshold properties and homogenization. St. Petersb. Math. J. 15(5), 639–714 (2004). https://doi.org/10.1090/S1061-0022-04-00827-1
10. Birman, M.S., Suslina, T.A.: Homogenization with corrector term for periodic elliptic differential operators. St. Petersbg. Math. J. 17(6), 897–973 (2006). https://doi.org/10.1090/S1061-0022-06-00935-6
11. Borisov, D.: Operator estimates for non-periodically perforated domains with Dirichlet and nonlinear Robin conditions: strange term. J. Math. Anal. Appl. 497(2), 1346–1383 (2021). https://doi.org/10.1016/j.jmaa.2021.124748
12. Borisov, D., Bunoiu, R., Cardone, G.: On a waveguide with frequently alternating boundary conditions: uniform homogenized Neumann condition. Ann. Henri Poincaré 11(8), 1591–1627 (2010). https://doi.org/10.1007/s00023-010-0065-0
13. Borisov, D., Bunoiu, R., Cardone, G.: Waveguide with non-periodically alternating Dirichlet and Robin conditions: homogenization and asymptotics. Z. Angew. Math. Phys. 64(3), 439–472 (2013). https://doi.org/10.1007/s00033-012-0264-2
14. Borisov, D., Cardone, G.: Waveguide of the planar waveguide with frequently alternating boundary conditions. J. Phys. A Math. Theor. 42(36), 365205 (2009). https://doi.org/10.1088/1751-8113/42/36/365205
15. Borisov, D., Cardone, G., Durante, T.: Homogenization and norm-resolvent convergence for elliptic operators in a strip perforated along a curve. Proc. R. Soc. Edinb. Sect. A Math. 146(6):1115–1158, (2016). https://doi.org/10.1017/S0308210516000019
16. Borisov, D., Cardone, G., Faella, L., Perugia, C.: Uniform resolvent convergence for strip with fast oscillating boundary. J. Differ. Equations 255(12), 4378–4402 (2013). https://doi.org/10.1016/j.jde.2013.08.005
17. Borisov, D.I., Križ, J.: Operator estimates for non-periodically perforated domains with Dirichlet and nonlinear Robin conditions: vanishing limit. Anal. Math. Phys. (2023). https://doi.org/10.1007/s13324-022-00765-8
18. Borisov, D.I., Mukhametzhanov, A.I.: Uniform convergence and asymptotics for problems in domains finely perforated along a prescribed manifold in the case of the homogenized Dirichlet condition. Sb. Math. 212(8), 1068–1121 (2021). https://doi.org/10.1070/SMM435

Springer
19. Borisov, D.I., Mukhametrakhimova, A.I.: Norm convergence for problems with perforation along a given manifold with nonlinear Robin condition on boundaries of cavities. arXiv:2202.10767 [math.AP], (2022). https://doi.org/10.48550/arXiv.2202.10767
20. Chechkina, A.G., D’Apice, C., De Maio, U.: Operator estimates for elliptic problem with rapidly alternating Steklov boundary condition. J. Comput. Appl. Math. 376, 112802 (2020). https://doi.org/10.1016/j.cam.2020.112802
21. Cioranescu, D., Damlamian, A., Griso, G., Onofrei, D.: The periodic unfolding method for perforated domains and Neumann sieve models. J. Math. Pures Appl. 89(3), 248–277 (2008). https://doi.org/10.1016/j.matpur.2007.12.008
22. Damlamian, A.: Le problème de la passoire de Neumann. Rend. Semin. Mat. Torino 43:427–450, (1985)
23. Del Vecchio, T.: The thick Neumann’s sieve. Ann. Mat. Pura Appl. IV. Ser. 147:363–402, (1987). https://doi.org/10.1007/BF01762424
24. Gómez, D., Pérez, M. E., Shaposhnikova, T. A.: Spectral boundary homogenization problems in perforated domains with Robin boundary conditions and large parameters. In Integral methods in science and engineering. Progress in numerical and analytic techniques. Proceedings of the conference, IMSE, Bento Gonçalves, Rio Grande do Sul, Brazil, July 23–27, (2012), pp. 155–174. New York: Birkhäuser/Springer, (2013). https://doi.org/10.1007/978-1-4614-7828-7_11
25. Griso, G.: Error estimate and unfolding for periodic homogenization. Asymptot. Anal. 40(3–4), 269–286 (2004)
26. Griso, G.: Interior error estimate for periodic homogenization. Anal. Appl. 4(1), 61–79 (2006). https://doi.org/10.1142/S021953050600070X
27. Herbst, I., Nakamura, S.: Schrödinger operators with strong magnetic fields: Quasi-periodicity of spectral orbits and topology. Differential operators and spectral theory. M. Sh. Birman’s 70th anniversary collection. Providence, RI: American Mathematical Society. Transl. Ser. Am. Math. Soc. 189(41), 105-123 (1999)
28. Kato, T.: Perturbation Theory for Linear Operators. Springer, Berlin (1966)
29. Khrabustovskyi, A.: Homogenization of eigenvalue problem for Laplace-Beltrami operator on Riemannian manifold with complicated bubble-like microstructure. Math. Meth. Appl. Sci. 32(16), 2123–2137 (2009). https://doi.org/10.1002/mma.1128
30. Khrabustovskyi, A., Plum, M.: Operator estimates for homogenization of the Robin Laplacian in a perforated domain. J. Differ. Equations 338, 474–517 (2022). https://doi.org/10.1016/j.jde.2022.08.005
31. Khrabustovskyi, A., Post, O.: Operator estimates for the crushed ice problem. Asymptotic Anal. 110(3–4), 137–161 (2018). https://doi.org/10.3233/ASY-181480
32. Khruslov, E.Y.: On the Neumann boundary problem in a domain with composite boundary. Math. USSR Sb. 12(4), 553–571 (1971)
33. Lamberti, P.D., Provenzano, L.: On trace theorems for Sobolev spaces. Matematiche 75(1), 137–165 (2020)
34. Marchenko, V.A., Khruslov, E.Y.: Boundary value problems in domains with a fine-grained boundary. Naukova Dumka, Kiev (1974)
35. Marchenko, V.A., Khruslov, E.Y.: Homogenization of Partial Differential Equations. Birkhäuser, Boston (2006)
36. Marchenko, V.A., Suzikov, G.V.: The second boundary value problem in regions with a complex boundary. Mat. Sb. (N.S.), 69(1):35–60 (1966)
37. McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge (2000)
38. Murat. F.: The Neumann sieve. Nonlinear variational problems. Int. Workshop, Elba/Italy 1983, Res. Notes Math. 127, 24–32 (1985)
39. Onofrei, D.: The unfolding operator near a hyperplane and its application to the Neumann sieve model. Adv. Math. Sci. Appl. 16(1), 239–258 (2006)
40. Picard, C.: Analyse limite d’équations variationnelles dans un domaine contenant une grille. Modélisation Math. Anal. Numér. 21(2), 293–326 (1987). https://doi.org/10.1051/m2an/1987210202931
41. Post, O.: Spectral convergence of quasi-one-dimensional spaces. Ann. Henri Poincaré 7(5), 933–973 (2006). https://doi.org/10.1007/s00023-006-0272-x
42. Post, O.: Spectral Analysis on Graph-Like Spaces. Springer, Berlin (2012)
43. Suslina, T.A.: Spectral approach to homogenization of elliptic operators in a perforated space. Rev. Math. Phys. 30(8), 1840016 (2018). https://doi.org/10.1142/S0129055X18400160
44. Zhikov, V.V.: On operator estimates in homogenization theory. Dokl. Math. 72(1), 534–538 (2005)
45. Zhikov, V.V.: Spectral method in homogenization theory. Proc. Steklov Inst. Math. 250, 85–94 (2005)
46. Zhikov, V.V., Pastukhova, S.E.: On operator estimates for some problems in homogenization theory. Russ. J. Math. Phys. 12(4), 515–524 (2005)
47. Zhikov, V.V., Pastukhova, S.E.: Operator estimates in homogenization theory. Russ. Math. Surv. 71(3), 417–511 (2016). https://doi.org/10.1070/RM9710

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.