Tachyon Effective Actions
In Open String Theory

David Kutasov and Vasilis Niarchos

Enrico Fermi Inst. and Dept. of Physics, University of Chicago
5640 S. Ellis Ave., Chicago, IL 60637-1433, USA

We argue that the Dirac-Born-Infeld (DBI) action coupled to a tachyon, that is known to reproduce some aspects of open string dynamics, can be obtained from open string theory in a certain limit, which generalizes the limit leading to the usual DBI action. This helps clarify which aspects of the full open string theory are captured by this action.
1. Introduction

Recent work on the dynamics of unstable D-branes in string theory has led to an effective action for the open string tachyon \( T \) and massless open string modes, \( A_\mu \) (the gauge field on the D-brane) and \( Y^I \) (the scalar fields parametrizing the location of the D-brane in the transverse directions) \([1]-[18]\). This action has the form

\[
S = \int d^{p+1}L, \\
L = -V(T)\sqrt{-\det G},
\]

with \( V(T) \) the tachyon potential (see below), and

\[
G_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu T \partial_\nu T + \partial_\mu Y^I \partial_\nu Y^I + F_{\mu\nu}.
\]

The action (1.1) is known to reproduce several non-trivial aspects of open string dynamics, such as the following:

(1) Choosing \([19,20]\)

\[
V(T) = \frac{1}{\cosh \alpha T^2},
\]

with \( \alpha = 1 \) for the bosonic string, and \( \alpha = \sqrt{2} \) for the non-BPS D-brane in the superstring, one finds from (1.1) the correct stress-tensor \( T_{\mu\nu} \) in homogenous tachyon condensation (the rolling tachyon solution which starts at the top of the potential at \( x^0 \to -\infty \)) \([7,21,20]\).

(2) With the potential (1.3), one finds that the theory contains static solitonic solutions corresponding to lower dimensional D-branes, with the correct tension.

(3) For the case of an unstable D-brane in type II string theory, one can construct a codimension one BPS D-brane as a solitonic solution of (1.1). Small excitations of the soliton correspond to massless fields, similar to \( Y^I \) and \( A_\mu \) in (1.2), and by using (1.1) one finds \([22]\) that the effective action for these excitations of the soliton is precisely the DBI action\(^2\).

(4) Inhomogenous solutions of the equations of motion which follow from the action (1.1) encode non-trivial information about the decay of higher dimensional branes into lower dimensional ones; in particular, they contain information about the relative velocities of the lower dimensional branes created in the process of tachyon condensation \([24,25]\).

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1 We use the conventions \( \alpha' = 1, \eta_{\mu\nu} = (-1, +1, \cdots, +1) \).

2 Given by (1.1), (1.2) with the tachyon \( T \) set to zero; see [23] for a review.
(5) For non-BPS D-branes in type II string theory, the potential (1.3) leads to the correct value of the mass of the tachyon on the D-brane (for the bosonic string, this is not the case) [20].

These and other successes lead one to believe that the action (1.1) captures some class of phenomena in the full classical open string theory. This action should presumably be thought of as a generalization of the DBI action describing the gauge field $A_\mu$ and scalars $Y^I$ on the brane. The DBI action is valid in the full open string theory, in situations where $F_{\mu\nu}$ and $\partial_\mu Y^I$ are arbitrary (i.e. not necessarily small), but slowly varying [23]. The question we would like to address in this note is whether there exists a similar regime, in which the action (1.1) describes the interactions of the tachyon in the full open string theory. We will argue that the answer is affirmative, and identify such a regime.

2. An effective action for tachyons

At first sight it seems difficult to incorporate the tachyon in an effective action such as the DBI action, since its mass is of order the string scale. Solutions of the equations of motion, $T(x^\mu)$, vary rapidly in spacetime, and in general one cannot decouple the tachyon from other (non-tachyonic) modes with string scale masses.

To proceed, one can use the following fact. Consider a homogenous tachyon $T(x^0)$ in the open bosonic string. The general solution of the linearized equation of motion for the tachyon is

$$T(x^0) = T_+ e^{x_0} + T_- e^{-x_0}.$$  \hspace{1cm} (2.1)

It is known that (2.1) is an exact solution of the full open string equations of motion.\footnote{We will discuss the generalization to non-BPS branes in the superstring later.} Thus, on-shell homogenous tachyons do not in fact couple to higher mass open string modes. It is natural to expand around the exact solution (2.1) and study tachyon profiles of the form

$$T(x^\mu) = T_+(x^\mu) e^{x_0} + T_-(x^\mu) e^{-x_0},$$  \hspace{1cm} (2.2)

where $T_\pm(x^\mu)$ are slowly varying on the string scale. What is the effective action describing the dynamics of such slowly varying perturbations? This action should have the property that arbitrary constant values of $T_+$ and $T_-$ correspond to a solution of the equations

\footnote{To be precise, this is known to be the case in the Euclidean theory obtained by taking $x^0 \to ix$ [26,27,28], and is believed to be the case in the Minkowski theory as well.}
of motion. It should describe the leading interactions in an expansion in derivatives of $T_{\pm}(x^\mu)$. Such an action would be non-perturbative in $T$, $\partial_\mu T$, since it would be valid for generic $T$ of the form (2.1). We will argue below that the action in question coincides (after a certain field redefinition) with (1.1) - (1.3).

Actually, if both $T_+$ and $T_-$ in (2.1) are non-vanishing, it is not obvious that an effective action of the sort we want exists. The reason is that in this case, the system is very far from the perturbative open string vacuum both at very early ($x^0 \to -\infty$), and very late ($x^0 \to \infty$) time. Since the natural observables in string theory are S-matrix elements of perturbative string modes, and the background one gets as $T \to \pm \infty$ is not believed to contain any physical open string excitations, it is not obvious that in this case one can make sense of the S-matrix, and therefore of the action. In the case that either $T_+$ or $T_-$ vanishes, the situation is better. Consider, say, the case

$$T(x^0) = T_+ e^{x^0}. \quad (2.3)$$

At early times, the tachyon goes to zero and the system approaches the perturbative open string vacuum. Thus, one can define and study observables analogous to an S-matrix as follows. The tachyon vertex operator in the bosonic string is

$$T_{\vec{k}} = e^{i\vec{k} \cdot \vec{x} - wx^0}; \quad \vec{k}^2 + w^2 = 1. \quad (2.4)$$

For $|\vec{k}| \ll 1$ one finds two solutions,

$$w_\pm = \pm(1 - \frac{1}{2}k^2) + O(k^4) \quad (2.5)$$

and thus the vertex operator (2.4) takes the form

$$T_{\vec{k}}^{(\pm)} = e^{i\vec{k} \cdot \vec{x} - w_x^0}. \quad (2.6)$$

Now, consider the correlation functions

$$\langle T_{\vec{k}_1}^{(+)} \cdots T_{\vec{k}_n}^{(+)} T_{\vec{p}_1}^{(-)} \cdots T_{\vec{p}_m}^{(-)} \rangle_{T_+}. \quad (2.7)$$

This case was studied in [29].

For now we restrict to amplitudes involving only tachyons. We will comment on including massless fields below.
The subscript $T_+$ means that we are computing these correlation functions in a background with a non-zero tachyon condensate (2.3). The correlation functions (2.7) vanish when $\vec{k}_1 = \cdots = \vec{k}_n = \vec{p}_1 = \cdots = \vec{p}_m = 0$. From the spacetime point of view, this is due to the fact that (2.1) is an exact solution of the full classical open string equations of motion. On the worldsheet, the vanishing of these amplitudes is directly related to the fact that the boundary perturbation $\lambda \int d\tau \cos x(\tau)$ is truly marginal in the Euclidean case ($x^0 \to ix$).

The effective action we are after is the action that reproduces the correlation functions (2.7) to leading order in $\vec{k}_i$, $\vec{p}_j$. In the next section we will compute this effective action.

We conclude this section with a few comments.

(a) To calculate (2.7) perturbatively in $T_+$, it is convenient to continue to Euclidean space, $x^0 \to ix$. Then, the correlation functions (2.7) involve momentum modes, whose momentum vectors are almost aligned with a particular, arbitrarily chosen, axis in space.

(b) It might seem that the action we have introduced is not Poincare invariant, since it involves a choice of preferred direction in spacetime. This is not the case, since as $x^0 \to -\infty$ the background approaches a Poincare invariant one (the open string vacuum), and the action should be valid for arbitrary perturbations away from this vacuum obtained from (2.3) by a Poincare transformation. In other words, the apparent breaking of Poincare symmetry in (2.3) is spontaneous.

(c) One can think of the effective action we have introduced as a special case of a more general construction, which includes the usual DBI action as another special case. In the $\sigma$-model approach to string theory (see, for example, [30,31]), one thinks of the configuration space of the theory as the space of (in general non-conformal) worldsheet theories, with conformal theories corresponding to solutions of the spacetime equations of motion. The coupling $T_+$ in (2.3) parametrizes a line of fixed points of the worldsheet renormalization group (RG), or classical solutions of the spacetime action. The effective action we have introduced describes infinitesimal deviations from this line of fixed points. Since turning on a large constant $T_+$ in (2.2) clearly does not take us away from this line, the size of the deviation from the line of fixed points is governed by the size of the derivatives of $T_+(x^\mu)$, and by $T_-$. This is analogous to the situation in the DBI case, where there is a surface of fixed points of the worldsheet

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7 Solutions with different $T_+ \neq 0$ are related by time translation and are thus equivalent.
RG corresponding to constant $F_{\mu\nu}, \partial_\mu Y^I$. The DBI action governs small fluctuations away from this surface of fixed points.

(d) It is easy to generalize the considerations of this section to the case of non-BPS branes in the superstring. The homogenous tachyon takes in this case the form

$$T(x^0) = T_+ \exp\left(\frac{1}{\sqrt{2}} x^0\right) + T_- \exp\left(-\frac{1}{\sqrt{2}} x^0\right).$$  \hspace{1cm} (2.8)

This is also the vertex operator in the $-1$ picture. Eq. (2.8) corresponds again to an exactly marginal perturbation of the worldsheet theory, and thus to an exact solution of the spacetime equations of motion. One can study small fluctuations around the solution with $T_+ \neq 0, T_- = 0$, as in (2.2), and define an effective action for these fluctuations. This effective action should reproduce the leading small $\vec{k}$ behavior in correlation functions of the analogs of the vertex operators (2.6). In the fermionic case one has $(-1)$-picture vertex operators

$$T_{\vec{k}}^{(\pm)} = e^{i \vec{k} \cdot \vec{x} - w \pm x^0},$$  \hspace{1cm} (2.9)

with

$$w_\pm = \pm \frac{1}{\sqrt{2}} (1 - \vec{k}^2) + O(\vec{k}^4).$$  \hspace{1cm} (2.10)

The corresponding 0-picture vertex operators are

$$T_{\vec{k}}^{(\pm)} = i(\vec{k} \cdot \vec{\psi} - w \pm \psi^0)e^{i \vec{k} \cdot \vec{x} - w \pm x^0}$$  \hspace{1cm} (2.11)

and the correlation function (2.7) contains $2n - 2$ 0-picture vertex operators and two $(-1)$-picture ones.

(e) There are some well known ambiguities associated with going from on-shell S-matrix elements to off-shell actions. One is the freedom to perform field redefinitions $T \rightarrow f(T, \partial_\mu T, \partial_\mu \partial_\nu T, \cdots)$. Another is the freedom to use the equations of motion at lower order in $T$ to change higher order terms in the Lagrangian. For example, for a field $T$ of unit mass, a cubic vertex $g_3 T^3$ gives the same on-shell three point function as the derivative interaction $g_3 (\partial_\mu \partial^\mu T)T^2$. We will arrive at a specific form of the action by fixing all these ambiguities in a particular way, but of course one can write the action in a different form by using them.
3. Computing the effective action

In this section we will compute the effective Lagrangian for the tachyon discussed above. We will require the Lagrangian to be symmetric under $T \rightarrow -T$. In the fermionic case (the non-BPS brane), this is due to the standard $Z_2$ symmetry of the theory, $\psi^\mu \rightarrow -\psi^\mu$, under which the open string tachyon is odd. In the bosonic case, one should in principle start from a Lagrangian without such a symmetry but, as we mention below, imposing one of the conditions that $\mathcal{L}$ should satisfy leads to a Lagrangian even under $T \rightarrow -T$. Thus, we impose this symmetry from the outset in the bosonic case as well.

Since the action is designed to reproduce only the leading terms in the S-matrix elements (2.7) as $\vec{k}_i, \vec{p}_j \rightarrow 0$, we can furthermore take the Lagrangian $\mathcal{L}$ to depend on $T$ and $\partial_\mu T$ only,

$$\mathcal{L} = \mathcal{L}(T, \partial_\mu T) \ .$$

(3.1)

This is essentially the statement that Lagrangians of the form (3.1) have a sufficient number of free parameters to match the leading terms in (2.7) for all $n, m$. We will return to this point and will make it more precise in section 4. For now, we will take the fact that we can bring $\mathcal{L}$ to the form (3.1) for granted, and proceed.

Note that (3.1) partially fixes the field redefinition ambiguity mentioned at the end of section 2, but it leaves a residual freedom of taking

$$T \rightarrow T f(T^2) \ ,$$

(3.2)

where $f(x) = 1 + C_1 x + C_2 x^2 + \cdots$.

To summarize, one expects the Lagrangian to take the form

$$\mathcal{L} = \sum_{n=0}^{\infty} \mathcal{L}_{2n}(T, \partial_\mu T) \ ,$$

(3.3)

where $\mathcal{L}_{2n}$ includes all the terms that go like $T^{2n}$,

$$\mathcal{L}_{2n} = \sum_{l=0}^{n} a_l^{(n)} (\partial_\mu T \partial^\mu T)^l T^{2(n-l)} \ .$$

(3.4)

It is important to emphasize that in the preceding discussion we have assumed that the effective Lagrangian (3.1) is analytic around $T = 0$. This is in fact not guaranteed, and we will see that this assumption fails in some cases.
Under the assumptions outlined above, the problem of determining the Lagrangian reduces to computing the constants \( a^{(n)}_l \). A non-trivial constraint is that the equations of motion that follow from the Lagrangian (3.3), (3.4) should allow the solution (2.1). Since (2.1) should be a solution for arbitrary (constant) \( T_\pm \), the equations of motion that follow from (3.4) should allow this solution for each \( n \) separately.

Varying \( \mathcal{L}_{2n} \), one finds the equation of motion

\[
\sum_{l=1}^{n} la^{(n)}_l \partial^\mu \left[ (\partial_\lambda T \partial^\lambda T)^l - 1 (\partial_\mu T)^2 \right] = \sum_{l=0}^{n} (n-l) a^{(n)}_l T^{2(n-l)-1} (\partial_\mu T \partial^\mu T)^l .
\]  

(3.5)

Plugging (2.1) in (3.5) leads to a recursion relation for the \( a^{(n)}_l \),

\[
a^{(n)}_{l+1} = \frac{(n-l)(2l-1)}{(2l+1)(l+1)} a^{(n)}_l ,
\]

(3.6)

with the solution

\[
a^{(n)}_l = \frac{(n-1)!}{(n-l)!l!(2l-1)} a^{(n)}_1 .
\]

(3.7)

We see that all couplings in (3.4) are fixed in terms of any one of them by the requirement that (2.1) be a solution of the equations of motion (3.5). It should be mentioned that if one starts with a Lagrangian without the symmetry \( T \to -T \), the requirement that (2.1) is a solution implies that all terms odd under this symmetry must vanish.

To fix \( \mathcal{L} \), we have to compute the remaining unknown coefficients, \( a^{(n)}_1 \). Note that the requirement that (2.1) be a solution fixes the field redefinition freedom (3.2), and we expect to find a unique solution for the Lagrangian (3.3).

In order to compute the couplings \( a^{(n)}_1 \) (3.7), one can proceed as follows. On general grounds, one expects that the on-shell spacetime action should be equal to the disk partition sum \([30,32,31]\). Both the spacetime action and the disk partition sum involve an integral over \( x^0 \), and in both of them there is a natural object that can be defined by stripping off the integral over \( x^0 \). In the case of the spacetime action, the resulting object is the on-shell Lagrangian (3.3). In the case of the worldsheet partition sum, it is the disk path integral over the non-zero modes of \( x^0 \), with the zero mode unintegrated, \( Z'(x^0) \). It is natural to conjecture that these two objects are equal to each other,

\[
\mathcal{L}_{\text{on-shell}}(x^0) = Z'(x^0) .
\]

(3.8)

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\(^8\) See \([6,14]\) for a related discussion.
This assumption was used successfully in [21], and we will use it here as well. Using (3.8), one can fix all the \( a_1^{(n)} \) as follows. Substitute the solution (2.3) into the Lagrangian (3.3), (3.4), and compare the resulting function of \( x^0 \) to that obtained by evaluating \( Z'(x^0) \) [21]

\[
Z'(x^0) = \frac{1}{1 + T e^{x^0}}. \tag{3.9}
\]

In doing that, one encounters a surprise. The on-shell Lagrangian (3.3), (3.4), (2.3) only involves terms that go like \( \exp(2nx^0) \), \( n = 0, 1, 2, 3, \ldots \), while \( Z' \) (3.9) also has in its expansion odd powers of \( \exp(x^0) \). Thus, it seems that one of the assumptions that went into the analysis above must be incorrect. We will soon see that the problematic assumption is that of analyticity of \( L(T, \partial \mu T) \) near \( T = 0 \), but for now let us set this problem aside and turn to the fermionic string (the non-BPS brane in type II), which as we will see is easier to understand.

First note that the analysis leading to (3.7) is slightly modified in this case, since the solution we want is not (2.1) but (2.8). It is easy to see that the correct form of (3.7) for the fermionic case is

\[
a_{i}^{(n)} = \frac{(n - 1)!2^i - 1}{(n - l)!!(2l - 1)} a_{1}^{(n)} . \tag{3.10}
\]

We can now attempt to fix the coefficients \( a_{1}^{(n)} \) by computing the effective Lagrangian (3.3), (3.4) for the solution \( T(x^0) = T_+ \exp(x^0/\sqrt{2}) \) (see (2.8)), and comparing it to the disk partition sum \( Z'(x^0) \). The latter was computed in [21]:

\[
Z' = \frac{1}{1 + \frac{1}{2} T_+^2 e^{x^0}}. \tag{3.11}
\]

We see that in this case the expansion of \( Z' \) involves only even powers of \( T(x^0) \), and we can use (3.8) to determine \( a_{1}^{(n)} \). One finds the following result:

\[
a_{i}^{(n)} = -\frac{(-1)^n 2^i - 2^{n-1} n!!}{i!(n - i)!(2i - 1)} , \tag{3.12}
\]

where we have used the identity

\[
\sum_{s=0}^{n} \frac{(-1)^s}{s!(n - s)!(2s - 1)} = \frac{2^n}{(2n - 1)!!} . \tag{3.13}
\]
As a check, (3.12) can be easily verified to satisfy (3.10). Plugging (3.12) into (3.3), (3.4) one finds

\begin{equation}
\mathcal{L} = -\frac{1}{1 + \frac{1}{2}T^2} \sqrt{1 + \frac{1}{2}T^2 + \partial_\mu T \partial^\mu T}.
\end{equation}

(3.14)

Finally, making the redefinition

\begin{equation}
\frac{T}{\sqrt{2}} = \sinh \frac{\tilde{T}}{\sqrt{2}},
\end{equation}

(3.15)

which is a transformation of the form (3.2), one finds

\begin{equation}
\mathcal{L} = -\frac{1}{\cosh \frac{T}{\sqrt{2}}} \sqrt{1 + \partial_\mu \tilde{T} \partial^\mu \tilde{T}}.
\end{equation}

(3.16)

This is exactly the tachyonic part of the Lagrangian (1.1) - (1.3).

Having understood the string theory origin of the (tachyon part of the) Lagrangian (1.1) for the non-BPS D-brane in type II string theory, we return to the bosonic case. Let us assume that the Lagrangian (1.1), (1.2) with the potential (1.3) is still correct in this case, i.e. that in some definition of the tachyon field, \( \tilde{T} \), the Lagrangian is

\begin{equation}
\mathcal{L} = -\frac{1}{\cosh \frac{T}{\sqrt{2}}} \sqrt{1 + \partial_\mu \tilde{T} \partial^\mu \tilde{T}}.
\end{equation}

(3.17)

The equation of motion of (3.17) has a solution

\begin{equation}
\sinh \frac{\tilde{T}}{2} = \exp \left( \frac{1}{2}x^0 \right)
\end{equation}

(3.18)

whose energy density is the same as that of the original D-brane. It is easy to see that this solution corresponds in the parametrization (2.1) to (2.3). Thus, the parametrizations (3.18), (2.3) are related by a map of the form

\begin{equation}
T = C \sinh^2 \frac{\tilde{T}}{2},
\end{equation}

(3.19)

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9 With the help of another identity,

\[ \sum_{s=0}^{n-l} 2^{-s} \frac{(2(l+s)-3)!!}{s!} = \frac{2^{l-n}(2n-1)!!}{(n-l)!(2l-1)}. \]
with $C$ some constant. In particular, the map is non-analytic near $T = 0$. One has:

$$T = a\tilde{T}^2 + b\tilde{T}^4 + \cdots .$$  \hspace{1cm} (3.20)

The Lagrangian (3.17), which is analytic in $\tilde{T}$, corresponds in terms of $T$ to a non-analytic Lagrangian

$$\mathcal{L} = \text{const} - \frac{1}{2}(\partial_\mu \tilde{T})^2 + \frac{1}{8}\tilde{T}^2 + \cdots = \text{const} - \frac{1}{8a} \left(\frac{\partial_\mu T}{T}\right)^2 + \frac{1}{8a} T + \cdots .$$  \hspace{1cm} (3.21)

We see that, as suggested by the expansion of the disk partition sum (3.9), the on-shell Lagrangian does contain odd powers of $T \simeq \exp(x^0)$, and these are due to the non-analytic structure of the effective Lagrangian for $T$ near $T = 0$. One can in fact check that plugging in the solution (3.18) into the Lagrangian (3.17) leads to a result which agrees with the disk partition sum (3.9), as one would expect from (3.8).

4. Discussion

In the previous sections we have argued that the action (1.1) - (1.3) arises from string theory in a particular limit, in which one considers slowly varying $T_\pm(x^\mu)$ in (2.2), (2.8) (in the bosonic and fermionic cases, respectively), expanded around a solution with $T_+ \neq 0$, $T_- = 0$. In this section we will discuss in more detail the regime of validity of the action (1.1) - (1.3), comment on possible extensions of our results, and discuss some consequences of our analysis for the issues mentioned in the introduction.

What is the precise string theory question, the answer to which is the action (1.1) - (1.3)? Consider, for example, the scattering amplitude of tachyons on a non-BPS brane, (2.9) - (2.11),

$$\langle T_{\vec{k}_1}^{(+)\cdots T_{\vec{k}_n}^{(+)} T_{\vec{p}_1}^{(-)} \cdots T_{\vec{p}_n}^{(-)} \rangle$$  \hspace{1cm} (4.1)

in the limit $|\vec{k}_i|, |\vec{p}_j| \ll 1$. At $\vec{k}_i = \vec{p}_j = 0$, this amplitude vanishes, since (2.8) is an exact solution of the full classical open string equations of motion. Thus, the leading non-vanishing terms in (4.1) scale like momentum (various combinations of $\vec{k}_i$, $\vec{p}_j$) squared. What we have shown in sections 2 and 3 is that (1.1) - (1.3) is the action that reproduces these momentum squared terms in the $2n$-point function (4.1). This way of thinking about it also explains why one can choose the Lagrangian to depend only on $T$ and $\partial_\mu T$, as we have in (3.1).
Indeed, as mentioned in section 3, Lagrangians of the form (3.1) have enough free parameters to incorporate the fact that (4.1) vanishes when all $\vec{k}_i = \vec{p}_j = 0$, and to parametrize the most general possible $O((\vec{k}, \vec{p})^2)$ terms in (4.1). The vanishing of the amplitude at zero momentum corresponds to the requirement that (2.8) is an exact solution of the equations of motion, and leads to the determination of all terms at order $T^{2n}$ in terms of one constant (see (3.10)). The fact that the remaining single free parameter at each order, $a_i^{(n)}$, is sufficient to parametrize the most general momentum squared terms in (4.1) can be seen in the following way.

The $2n$-point amplitude (4.1) has two different types of contributions: vertices with less than $2n$ external legs connected by propagators, and one-particle-irreducible vertices with $2n$ external legs. To determine the form of the effective action, we are only interested in the 1PI contributions. These are local, i.e. polynomial, in the momenta $\vec{k}_i$ and $\vec{p}_j$ and their form is severely constrained by symmetries, as follows. The most general local quadratic polynomial in $\vec{k}_i$, $\vec{p}_j$, compatible with rotation invariance can be written in the form

$$
\sum_{i,j} a_{ij} \vec{k}_i \cdot \vec{k}_j + \sum_{i,j} b_{ij} \vec{p}_i \cdot \vec{p}_j + \sum_{i,j} c_{ij} \vec{k}_i \cdot \vec{p}_j
$$

(4.2)

where $a_{ij}$, $b_{ij}$, $c_{ij}$ are free parameters that have to satisfy additional symmetry constraints. First, since the amplitude (4.1) is by definition symmetric under interchange of any of the $T(+)’s$ and separately under interchange of any of the $T(−)$’s, (i.e. it is completely symmetric under interchange of the $\{\vec{k}_i\}$, and separately symmetric under interchange of the $\{\vec{p}_j\}$) it can be written as

$$
a_1 \sum_i |\vec{k}_i|^2 + a_2 \sum_{i \neq j} \vec{k}_i \cdot \vec{k}_j + b_1 \sum_i |\vec{p}_i|^2 + b_2 \sum_{i \neq j} \vec{p}_i \cdot \vec{p}_j + c \sum_{i,j} \vec{k}_i \cdot \vec{p}_j
$$

(4.3)

Since $\sum_i \vec{k}_i + \sum_j \vec{p}_j = 0$ by momentum conservation, the last term (proportional to $c$) is not independent of the other terms, and we can set $c = 0$. Furthermore, the symmetry $(\vec{x}, t) \to (−\vec{x}, −t)$ implies that the amplitude should also be invariant under interchange of the $T(+)’s$ with the $T(−)$’s, i.e. it should be invariant under the interchange of $\{\vec{k}_i\}$ with $\{\vec{p}_j\}$. This implies that $a_1 = b_1$ and $a_2 = b_2$ and we are left with only two independent constants, $a_1$ and $a_2$. The fact that (4.1) is an amplitude in a Lorentz invariant theory, and thus we can write it in terms of Mandelstam invariants, implies that $a_1$ and $a_2$ are also related. Altogether, we conclude that to quadratic order in momenta the 1PI part of the $2n$-point function (4.1) is unique up to one free coefficient. The action that was
constructed in sections 2 and 3 also has one free coefficient at each order, and the argument leading to (3.12) determines it.

Since the action (1.1) - (1.3) describes the leading terms in the $2n$-point function (4.1), one can also use it to describe scattering amplitudes in the presence of a non-zero $T_+$, as in (2.7), perturbatively in $T_+$. Note that one cannot use it when both $T_+$ and $T_-$ in (2.8) are non-zero, since then the spectrum of scaling dimensions changes (it is no longer true that $\Delta(e^{w x_0}) = w^2$). This is the worldsheet manifestation of the fact (mentioned in section 2) that in this case the system is far from the open string vacuum both at early and at late times.

We now move on to possible extensions of our work. One natural extension is to add the gauge field on the D-brane $A_\mu$, and scalars $Y^I$, to derive the full action (1.1) - (1.3). This should be possible along the lines of comment (c) at the end of section 2. To construct the tachyon action (3.16) we utilized the existence of exact classical solutions (or, equivalently, manifolds of fixed points of the worldsheet boundary RG) labeled by $T_+$. We also noted that constant $F_{\mu \nu}, \partial_\mu Y^I$, correspond to surfaces of solutions as well, and studying the vicinity of these surfaces leads to the DBI action.

One can combine the two observations and study surfaces of solutions labeled by constant $F_{\mu \nu}, \partial_\mu Y^I$, and $T_+$. This is particularly simple when $F_{\mu 0} = \partial_0 Y^I = 0$, since then the solution for the tachyon (2.1), (2.8) is not modified by the expectation values of the massless fields. More generally, one has to replace (2.1), (2.8) by a solution of the tachyon equations of motion in the open string metric, but one still expects to have a surface of solutions labeled by constant expectation values of $F_{\mu \nu}, \partial_\mu Y^I$, and $T_+$. The full action describing slowly varying $F_{\mu \nu}, \partial_\mu Y^I, T_+ \text{and small and slowly varying } T_- \text{ is very likely given by (1.1) - (1.3). This leads to a uniform treatment of the tachyon and massless fields: the action (1.1) - (1.3) describes the physics in the vicinity of the surface of exact solutions corresponding to constant } F_{\mu \nu}, \partial_\mu Y^I, T_+.$

Another generalization is to couple the action (1.1) - (1.3) to massless closed strings. For (NS,NS) sector closed strings, this is expected to lead to a structure similar to the usual DBI action. The tachyon couplings to massless (R,R) sector fields are an interesting open problem.

In the introduction we mentioned that the action (1.1) - (1.3) seems to reproduce some aspects of the dynamics of the full open string theory like properties of rolling tachyon and other solutions. Our understanding of the role of this action in string theory should help clarify which aspects of the full problem should be captured by this action, and which
should not. We next briefly comment on this issue in the context of the points mentioned in section 1.

The fact that the action (1.1) - (1.3) should reproduce the correct stress-tensor in homogenous tachyon condensation should be clear from the point of view of our analysis. This is the statement that the one point function of a zero momentum graviton in the full string theory is the same as the expectation value of the stress-tensor corresponding to (1.1) on the solution (2.1), (2.8). Clearly, this one point function probes an infinitesimal deviation from the solution that the action (1.1) is designed to describe, and thus the result obtained from (1.1) must agree with that obtained in the full string theory. Indeed, this is known to be the case for the solution (2.3).

For the general solutions (2.1), (2.8), with both $T_+$ and $T_-$ non-zero, it is known that the stress tensor computed in the field theory (1.1) does not precisely reproduce that computed in the full string theory [33]. For example, in the case of non-BPS branes in the superstring, when the energy density $E$ is smaller than the D-brane tension, the tachyon effective action (3.14) gives [20] the stress-energy tensor

$$
T_{00} = E, \quad T_{ij} = -\frac{1}{E} \frac{1}{1 + \frac{u^2}{2} \cosh^2 \left( \frac{x_0}{\sqrt{2}} \right)} \delta_{ij} = -\delta_{ij} \left( \frac{1}{1 + ce^{\sqrt{2}x_0}} + \frac{1}{1 + ce^{-\sqrt{2}x_0}} - 1 \right),
$$

(4.4)

where $T_+ = T_- = \frac{u}{2}$ in (2.8), $c = \frac{b-1}{b+1}$, and $b^2 = 1 + \frac{u^2}{2} = \frac{1}{E^2}$. The exact open string calculation gives [8]

$$
T_{00} = \frac{1}{2} (1 + \cos u), \quad T_{ij} = -f(x^0) \delta_{ij},
$$

(4.5)

with

$$
f(x^0) = \frac{1}{1 + \frac{1}{2} \sin^2 \left( \frac{u}{2} \right) e^{\sqrt{2}x_0}} + \frac{1}{1 + \frac{1}{2} \sin^2 \left( \frac{u}{2} \right) e^{-\sqrt{2}x_0}} - 1.
$$

(4.6)

The two results (4.4) and (4.3) have the same rough form, and agree to leading order in an expansion in $u$, but the detailed dependence on the strength of the tachyon field is not the same. This is consistent with our discussion in section 2. Note that as pointed out in [33], even in this case the effective field theory (1.1) - (1.3) is still valid at late times. This is very natural from the point of view of the discussion in section 2, since at late times one can replace the interaction (2.1) by (2.3), with a renormalized value of the coupling $T_+$ (this renormalization is the origin of the periodic dependence on the tachyon in (4.6), from this point of view).

---

10 We have set the D-brane tension to one.
We mentioned in section 1 that the action (1.1) describes correctly lower dimensional D-branes, which correspond to solitons of this action. This is easy to understand from the point of view of our analysis. As explained in [34], one way to describe codimension one D-branes in the full string theory is to turn on a tachyon with the profile

\[ T(x) = e^{wx^0} \cos kx. \]  

(4.7)

(in [34], \(\exp(wx^0)\) is replaced by a scale dependent coupling, but this is an inessential difference). In the bosonic string, one finds at late times codimension one D-branes located at the minima of (4.7), while in the fermionic case one finds codimension one BPS D-branes located at the zeroes of (4.7).

As mentioned in [34], taking \(k \rightarrow 0\) in (4.7) allows one to focus on a single codimension one D-brane, with the other branes sent to infinity in the limit. Perturbations of the form (4.7) with \(k \rightarrow 0\) (which correspond to \(T_+(x) \propto \cos kx\) in (2.2), (2.8)) are precisely what the action (1.1) is designed to describe. \(T^+(x)\) is slowly varying and the perturbation never takes one far from the surface of exact solutions corresponding to constant \(T^+\). Therefore, one expects (1.1) to give the correct tension of the codimension one brane, and (for the non-BPS brane in type II) to give the Dirac-Born-Infeld action as the action describing small excitations of the soliton\(^{11}\). In the full string theory, the approximation that gives rise to (1.1) on the original non-BPS brane is exactly the same as the approximation that gives rise to the DBI action on the BPS brane. One goes smoothly into the other under the tachyon condensation (4.7) with \(k \ll 1\).

What happens if we turn on a perturbation (4.7) with \(k \simeq 1\)? Since \(T_+(x)\) is no longer slowly varying, the action (1.1) need not describe the full open string dynamics in this case. However, as discussed in [34], at late times one actually expects the worldsheet operator \(\cos kx\) to approach the identity operator, and the full boundary CFT is thus expected to approach the surface of solutions \(T^+ = \text{constant}\), on which the action (1.1) is reliable. Thus, in situations like this, one expects the dynamics to be well approximated by (1.1) at late times, when the tachyon is large. This agrees with the discussion in [33].

General inhomogenous solutions of the equation of motion of the tachyon action (1.1) in 1 + 1 dimensions were analyzed in [24]. It was found that generic solutions develop caustics at a finite time. It was then argued in [33] that the effective action (1.1) breaks down in these situations, since the tachyon develops large gradients. In [25] it was pointed

\(^{11}\) In the bosonic case, one expects to get an action of the form (1.1) on the soliton.
out that the presence/absence of caustics in the solutions is directly related to the dynamics of the $D0$-branes that the $D1$-brane decomposes into via tachyon condensation. If these $D0$-branes are at rest relative to each other, the late time solution is smooth. If they have non-zero relative velocities, caustics appear and (1.1) breaks down. From the point of view of our analysis here, this is very reasonable. The situation with vanishing relative velocities can be Lorentz transformed\footnote{This is true for equidistant $D0$-branes at late times. Non-equidistant $D0$-branes at rest are described by a simple generalization of (4.7).} to (4.7), which as we already explained is well described by (1.1). When the relative velocities are non-zero, it is easy to see that the system is getting farther and farther away from the surface of solutions $T_+ = \text{constant}$ at late times. Thus, the action (1.1) is not expected to be reliable in this case, and the caustics presumably signal its breakdown.

Finally, we mentioned in section 1 that the action (1.1) - (1.3) reproduces the correct mass of the tachyon on the non-BPS D-brane in type II, while it does not give the correct tachyon mass in the bosonic string. From the point of view of our analysis, the fact that the mass is reproduced in the type II case is true by construction, while the disagreement in the bosonic string is understood to be due to the non-analytic relation (3.19) between the open string tachyon $T$ and the field that appears in (1.1), which is really $\tilde{T}$. Taking the map (3.19) into account, one in fact finds the correct tachyon mass (again, by construction).

The action (1.1) - (1.3) is quite different from the actions that were found in boundary string field theory (BSFT)\footnote{This is true for equidistant $D0$-branes at late times. Non-equidistant $D0$-branes at rest are described by a simple generalization of (4.7).}. It is sometimes said that these actions might be related by field redefinitions, which involve derivatives of $T$. From the point of view presented here, it is clear why these actions look so different. The actions computed in BSFT are valid far off the mass-shell of $T$, e.g. for $T \simeq a + ux^2$ in the bosonic string. The action (1.1) - (1.3) on the other hand is valid for approximately on-shell configurations, such as (2.4), (2.5). The two actions have rather different regimes of validity, and it is not surprising that they look different.

The general point of view on effective actions presented here might be useful for thinking about other related problems as well. For example, one can ask whether it is possible to generalize the analysis to the study of closed string tachyons. In order to do that, one needs to identify an analog of the exact solution (2.1) for this case. As a naive attempt, one may try to study the worldsheet theory in the presence of a perturbation

\[ \delta L_{ws} = \lambda \cosh 2x^0. \] (4.8)
Unfortunately, unlike its open string analog, this perturbation is not expected to be truly marginal. Analytically continuing $x^0 \to ix$, one finds the Sine-Gordon interaction, which is marginally relevant. The central charge goes to zero in the IR (the worldsheet field $x$ becomes massive). In the string theory context, this means that turning on $\lambda$ leads to a large backreaction of the metric and dilaton at late times, and one needs to understand it before proceeding. It is possible that, as was suggested recently in [38], the perturbation

$$\delta \mathcal{L}_{ws} = \lambda \exp(2x^0)$$

(4.9)
is exactly marginal, in which case one could use it as a closed string analog of (2.3), although it is not clear what suppresses the backreaction of the metric and dilaton to the stress-tensor of the tachyon. Another natural arena in which effective actions such as (1.1) might be useful is localized closed string tachyon condensation [39-43], where the backreaction is milder, and in particular the central charge does not change.

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