Dispersion management in optical fiber links: 
Integrability in leading nonlinear order

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Abstract We show that an integro-differential equation model for pulse propagation in optical transmission lines with dispersion management, is integrable at the leading nonlinear order. This equation can be transformed into the nonlinear Schrödinger equation by a near-identity canonical transformation for the case of weak dispersion. We also derive the next order (nonintegrable) correction.

I. INTRODUCTION AND MAIN IDEAS

The impressive progress in high-bit-rate optical fiber communications achieved during the last few years was primarily due to the invention of the dispersion management (DM) technique [1], which has become the key technological component of fiber telecommunications. A recent example of just how successful this technique is the 1.02 Tbit/s data transmission achieved in standard optical fiber with dispersion management over 1000km [3].

The main factor that limits the bit-rate is pulse-broadening due to the chromatic dispersion of optical fiber. This broadening is characterized by the dispersion length, $Z_{\text{dis}} \sim (d \times (BR)^2)^{-1}$. Here, $d$ is the fiber chromatic dispersion and $BR$ is the bit-rate. The dispersion length $Z_{\text{dis}}$ is the distance at which the pulse-width approximately doubles due to dispersive broadening. This distance decreases as a quadratic function of the bit-rate.

The basic idea of the DM technique is to compensate the fiber chromatic dispersion by periodically incorporating an additional element, such as a dispersion compensating fiber or fiber chirped gratings, whose sign of the chromatic dispersion coefficient is opposite to that of the main optical fiber. (See Fig.).

Modern optical transmission systems must satisfy very strict requirements for bit-error-rate ($\text{BER} \approx 10^{-12}$ to $10^{-15}$). Therefore, the pulse amplitude should be large enough so that it can be effectively detectable. As a consequence, the Kerr nonlinearity of the fiber refractive index $n = n_0 + \alpha I$ ($n_0$ – linear part of the refractive index, $I$ – pulse intensity, and $\alpha$ – coefficient of Kerr nonlinearity) should be taken into account. The spectrum of an optical pulse with characteristic power $P_0$ will experience noticeable nonlinear distortion at distances greater than the characteristic nonlinear length $Z_{\text{nl}} = (\alpha P_0)^{-1}$. A natural idea is to control the nonlinear spectral distortion by properly choosing the value of the residual dispersion over the period of the fiber link.

The dispersion management technique is currently the focus of intensive experimental, numerical and theoretical research [2-3] because of its excellent practical performance. Optical pulses in DM fiber links exhibit unexpected soliton-like properties, such as high stability and elastic interaction, despite the fact that the governing equation does not belong any known class of integrable equations.

In this paper, we demonstrate that the leading-order equation describing the slow dynamics of optical pulses in such systems is close to integrable for the case of...
weak dispersion management. We also present the next order (nonintegrable) correction. This correction is small for weak dispersion management.

The problem of weak dispersion management was considered in [12] for the first time as a theoretical example of periodic variation of the dispersion coefficient in the model of transmission link. The power of Lie transform technique proposed by authors earlier in [13,14] was also shown in [12]. This paper was published long before dispersion compensation became popular and before its strength was proven experimentally. The main result of this paper was that pulse propagation in fiber links with weak variation of dispersion can be described in leading order by an unperturbed nonlinear Schroedinger equation. Using a formalism which is common in weak turbulence theory [15], we derived similar results and investigated its validity by calculating the higher order corrections.

II. BASIC EQUATIONS

The relevant equation for the electric field envelope \( E(z, t) \) is the Nonlinear Schroedinger equation (NLS) with periodically varying dispersion and an external force representing fiber losses and amplification:

\[
iE_z + \frac{1}{2} \frac{Z_{nl}}{Z_{dis}} d(z) E_{tt} + |E|^2 E = iR'(z)E
\]

\[
R'(z)E = Z_{nl} \left[-\gamma + r \sum_{k=1}^{\infty} \delta(z - z_k)\right].
\]

Here, \( \gamma \) describes the fiber losses. The amplifiers that compensate the fiber losses are placed periodically at points \( z_k \) and separated by distances \( Z_n \approx 1/\gamma \). The coefficient of amplification \( r \) is chosen to exactly compensate energy losses after passing the amplification distance \( Z_n \).

The function \( d(z) \) is defined by

\[
d(z) = \begin{cases} 
1, & \text{in the transmission fiber,} \\
-d_{\text{comp}}/d_{\text{trans}}, & \text{in the compensating fiber}
\end{cases}
\]

Without loss of generality, we consider the case when the period of amplification is equal to the period of compensation. The theory can be trivially extended to the general case.

In practice, \( Z_{nl}/Z_{dis} \gg 1 \) and \( \gamma Z_{nl} \gg 1 \). Therefore, on the scale of the amplification distance \( Z_n \), the pulse dynamics is practically linear. On the other hand, the linear evolution of the signal preserves the Fourier spectrum of the pulse. One can consider optical pulse propagation over one period of a fiber link to be a mapping of the input pulse into the output pulse. To preserve the bit pattern, this mapping must be both stationary and stable. Stationarity can be achieved by a proper choice of the amplification coefficient of the amplification and by complete compensation of dispersion. In this case, stationarity will be achieved for any pulse shape. Stability, on the other hand, is more complicated, and can be investigated through the slow nonlinear pulse dynamics.

In order to consider slow nonlinear pulse dynamics it is therefore natural to transform the equations into the Fourier representation, and study the slow dependence of the spectrum on the coordinate \( z \). One can see the separation of scales in the equation (1): \( E_z \) is dominated by the RHS and dispersion term. Let us remove the external force (RHS) from (1) by the transformation \( E = q(z, t) \exp(R(z)) \), and we obtain the equation

\[
iq_z + \frac{1}{2} \frac{Z_{nl}}{Z_{dis}} d(z) q_{tt} + c(z)|q|^2 q = 0,
\]

where \( c(z) = \exp(2R(z)) \).

Let us decompose \( d(z) \) into the sum

\[
d(z) = \langle d(z) \rangle + \hat{d}(z), \quad \langle \hat{d}(z) \rangle = 0,
\]

where

\[
\langle f(z) \rangle = \frac{1}{Z_n} \int_0^z f(z) dz.
\]

Following [4], we represent \( q(t, z) \) as

\[
q(z, t) = \int_{-\infty}^{\infty} d\omega a_{\omega}(z) \exp(i\omega t + i\omega z/2) \frac{Z_{nl}}{Z_{dis}} \int_0^z \hat{d}(z) dz,
\]

and compute the equation for the slow dynamics of the pulse’s Fourier spectrum in Hamiltonian form

\[
\frac{\partial a_\omega}{\partial z} + \frac{\delta H}{\partial a_\omega^*} = 0.
\]

with the Hamiltonian

\[
H = \int \frac{1}{2} \frac{Z_{nl}}{Z_{dis}} < d(z) > \omega^2 |a_\omega|^2 d\omega + \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \times F_{\omega_1 \omega_2}^{\omega_3 \omega_4} \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) a_{\omega_1}^* a_{\omega_2}^* a_{\omega_3} a_{\omega_4},
\]

where

\[
F_{\omega_1 \omega_2}^{\omega_3 \omega_4} = c(z) \exp(i(\omega_1^2 + \omega_2^2 - \omega_3^2 - \omega_4^2) Z_{nl} \int_0^z \hat{d}(\xi) d\xi).
\]
Note that the kernel $F_{\omega_1 \omega_2}$ depends only on the composed variable $g = (\omega_1^2 + \omega_2^2 - \omega_3^2 - \omega_4^2)$, which, as we will see later, leads to important consequences. The leading-order approximation for the slow nonlinear dynamics can be obtained by averaging (6) over $Z_a$. The resulting equation is valid if the pulse spectrum does not change appreciably over the amplification distance. The next order correction to the averaged equation was calculated in [16] by using a Lie transform [13,14]. An alternative method for calculating these corrections was proposed in [17].

In the simple case of communication through one frequency channel, with the additional assumption that pulses do not interact with each other, one can use the quasi self-similar character of the single-pulse behavior [19], which was proposed for the first time by Talanov (lens transformation) [20]. As a result, the slow dynamics of a single pulse is described by the Nonlinear Schroedinger equation with a quadratic potential. This approach was extensively discussed in [11],[19],[21], and allows one to explicitly take into account the evolution of the single pulse spectra within the system period. Another advantage of the quasi self-similarity is that it also holds for very small values of the residual dispersion. It was shown in [11], [22] that stable single pulse propagation is possible for zero or even negative residual dispersion for special configurations of the dispersion map.

However, Talanov’s method cannot be applied to the case of several interacting pulses within one frequency channel, nor can it describe the evolution of an optical pulse propagating in a system that utilizes the wavelength division multiplexing (WDM) technique. In this case, the WDM technique exploits several frequency channels for data transmission. Optical pulse streams in systems that use this technique propagate in every channel with their own velocities and interact through the common refractive-index matrix.

In the fiber links with dispersion management, the nonlinear effects are smaller than the dispersive effects, therefore the Fourier spectrum of a pulse does not change much between amplifiers. Hence, equation (3) can be replaced by the equation averaged between the amplifiers. In the corresponding averaged Hamiltonian, $F_{\omega_1 \omega_2}$ is replaced by $T_{\omega_1 \omega_2}$, so that this Hamiltonian becomes

$$H = \int \frac{1}{2} \frac{Z_{\text{nl}}}{Z_{\text{dis}}} < d(z) > \omega^2 |a_\omega|^2 d\omega + \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \times T_{\omega_1 \omega_2} \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) a_{\omega_1}^* a_{\omega_2}^* a_{\omega_3} a_{\omega_4},$$

here

$$T_{\omega_1 \omega_2} \equiv \frac{1}{2a} \int_{0}^{2a} dz c(z)$$

$$\times \exp \left( i (\omega_1^2 + \omega_2^2 - \omega_3^2 - \omega_4^2) \frac{1}{2} \frac{Z_{\text{nl}}}{Z_{\text{dis}}} \int_{0}^{z} d(\xi) d\xi \right),$$

This averaging is equivalent to assuming that higher order corrections in the Lie expansion are small compared to the leading order.

III. INTEGRABILITY AT THE LEADING ORDER

In our further analysis we will utilize ideas that are well developed in the theory of weak turbulence [15]. We rewrite the Hamiltonian (3) as

$$H = \int Q_\omega a_\omega a_\omega^* d\omega + \int T_{\omega_1 \omega_2} a_\omega^* a_{\omega_3} a_{\omega_4} \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) d\omega_1 d\omega_2 d\omega_3 d\omega_4,$$

where we define the leading order quadratic dispersion relation as

$$Q_\omega = \beta < d(z) > \omega^2, \quad \beta = \frac{1}{2} \frac{Z_{\text{nl}}}{Z_{\text{dis}}},$$

Then $a_\omega$ satisfies the canonical equation of motion:

$$\frac{\partial a_\omega}{\partial z} + i \frac{\delta H}{\delta a_\omega} = 0.$$ (10)

If $T_{\omega_1 \omega_2}$ is a constant independent of the $\omega$’s, then equations (8) [10] correspond to the usual NLS equation. With $T_{\omega_1 \omega_2}$ given by (3), these equations describe the averaged, slow-time evolution of an optical pulse propagating in a fiber-optical link with periodic amplifiers, damping, and dispersion compensators.

To find the leading-order approximation to (3), we follow [15,23,24] and perform the transformation

$$a_\omega = b_\omega + \int B_{\omega_2 \omega_3} b_{\omega_1}^* b_{\omega_2} b_{\omega_3} \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) + \text{higher order interactions.}$$ (11)

The transformation (11) is canonical up to and including the cubic terms if the coefficients $B_{\omega_1 \omega_2 \omega_3}$ satisfy the symmetry condition [15]

$$B_{\omega_2 \omega_3} = B_{\omega_3 \omega_2} = B_{\omega_1 \omega_2} = -(B_{\omega_1 \omega_2})^*.$$ (12)

Substitution of (11) into the Hamiltonian (8) transforms (8) into...
Now we choose the canonical transformation kernel as

$$g \equiv \text{leading nonlinear order.}$$

and quadratic dispersion relation (9) is the NLS, and ear approximation the equation (10) with Hamiltonian fold (13), (14) is equal to a constant independent of $\omega_i$. The transformed Hamiltonian acquires the form

$$H = \int Q_\omega b_\omega b_\omega^* d\omega + \int T_0 b_\omega^* b_\omega b_\omega b_\omega d\omega_1 d\omega_2 d\omega_3 d\omega_4 + \text{higher order interactions},$$

which is the Hamiltonian of the NLS equation to leading order. Thus equation (10) with Hamiltonian (1) and quadratic dispersion relation (1) is integrable at the leading nonlinear order.

The condition of quasi-identity of transformation (11) reads as

$$B_{\omega_1 \omega_2} \ll 1.$$  

Let us estimate the $B_{\omega_1 \omega_2}$ in (16). For the case of a loss-

less piece-wise constant dispersion map with alternating pieces of fibers of length $l_1$ with dispersion $< d > + d_1$ and of length $l_2$ with dispersion $< d > + d_2$, ($l_1 d_1 + l_2 d_2 = 0$), $T_{\omega_1 \omega_2}$ acquires the form

$$T_{\omega_1 \omega_2} = \frac{\sin(\delta g)}{\delta g},$$

where $\delta$ is the strength of a dispersion map. Thus condition (19) becomes

$$\delta^2 \ll \langle d \rangle,$$

so that the transformation (11) is quasi-identical for the so-called weak dispersion maps, i.e. for the cases of small variation of the dispersion on top of big dispersion.

In this paper, we only study the sixth-order interaction in the averaged equations. V.E. Zakharov [12] recently pointed out that a canonical transformation similar to (11) can also be used to bring the original nonaveraged problem (3) into a form analogous to that developed here for the averaged equation, namely, the Nonlinear Schrödinger Equation with higher-order corrections. We believe that the present work is an important stepping stone in the proper development of a more sophisticated transformation, which will give us sufficient understanding to attack this much harder problem.
IV. SIX-WAVE INTERACTIONS ON A RESONANT SURFACE

In this section, we calculate the sixth-order interaction matrix element for the Hamiltonian (15). The Hamiltonian (15) is the averaged Hamiltonian, and, including sixth-order terms, has the form:

\[ H = \int d\omega Q_\omega |b_\omega|^2 + \int d\omega d\omega' d\omega'' d\omega''' b_\omega^* b_\omega' b_\omega'' b_\omega''' \delta_{\omega + \omega' + \omega'' + \omega'''} \]

The analytical expression corresponding to this diagram is

\[ T_\omega = \frac{\int \frac{d\omega}{Q_\omega} + \frac{d\omega'}{Q_\omega'} - \frac{d\omega''}{Q_\omega''} - \frac{d\omega'''}{Q_\omega'''} = \sum_{i=1}^{9} \frac{T(g_i)^2}{g_i}, \]

where the nine variables \( g_i \), \( i = 1, \ldots, 9 \) are given by the following expressions

\[ g_i = \{-2(\omega_4 - \omega_1)(\omega_4 - \omega_2), -2(\omega_4 - \omega_1)(\omega_4 - \omega_3), -2(\omega_4 - \omega_1)(\omega_5 - \omega_2), -2(\omega_5 - \omega_1)(\omega_5 - \omega_3), -2(\omega_5 - \omega_1)(\omega_6 - \omega_2), -2(\omega_6 - \omega_1)(\omega_6 - \omega_3), -2(\omega_6 - \omega_2)(\omega_6 - \omega_3)\}. \]

The resulting expression can be viewed as the measure of the “distance” of the perturbed system from its integrable analog. Indeed direct calculations show that for the case of NLS, when \( T_\omega = \text{const} \), this sixth order matrix element is identically equal to zero on the resonant manifold

\[ Q_\omega + Q_{\omega_1} + Q_{\omega_2} = Q_{\omega_3} + Q_{\omega_4} + Q_{\omega_5}, \quad \omega + \omega_1 + \omega_2 = \omega_3 + \omega_4 + \omega_5. \]

In order to bring equation (15) as close to the integrable limit as possible, we must minimize \( T_{\omega_1,\omega_2} \) over all the tunable system parameters.

For the case of a lossless piece-wise constant dispersion map, when the fourth order matrix element is given by (24), we find (using Mathematica 3.0), that the six wave interaction matrix element

\[ T_{\omega_1,\omega_2} \propto \delta^2/(d), \]

on the 6-wave resonant manifold (24). Therefore the above perturbation technique is valid for the case of a weak dispersion map. In leading order, the slow pulse evolution in optical fiber is governed by the integrable NLS equation.
V. CONCLUSION

We showed that the leading-order equation describing pulse dynamics in optical fiber links with a weak dispersion map is close to integrable. Using canonical transformations, we found a nonintegrable higher order correction to the integrable part of the model equation describing the slow evolution of optical pulses.

VI. ACKNOWLEDGMENT

We are very grateful to Roberto Camassa, Gary Doolen, Gregor Kovacic, Victor Lvov and Vladimir Zakharov for helpful discussions. This work was supported by DOE contract W-7-405-ENG-36 and the DOE Program Applied Mathematical Sciences KJ-01-01.

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