ON A CONJECTURE OF R. M. MURTY AND V. K. MURTY II

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Abstract. Let $\omega^*(n)$ be the number of primes $p$ such that $p - 1$ divides $n$. In 1955, Prachar proved that $\sum_{n \leq x} \omega^*(n)^2 = O(x(\log x)^2)$. Recently, Murty and Murty improved this to

$$x(\log \log x)^3 \ll \sum_{n \leq x} \omega^*(n)^2 \ll x \log x.$$  

They further conjectured that there is some positive constant $C$ such that

$$\sum_{n \leq x} \omega^*(n)^2 \sim Cx \log x$$  

as $x \to \infty$. In a former note, the author gave the correct order of it by showing that

$$\sum_{n \leq x} \omega^*(n)^2 \approx x \log x.$$

In this subsequent article, we provide a conditional proof of their conjecture.

The investigations of the normal order of certain arithmetic functions start from the paper of Hardy and Ramanujan [6]. Let $\omega(n)$ be the number of distinct prime divisors of $n$. Hardy and Ramanujan found that all most all integers $n$ satisfy $\omega(n) \sim \log \log n$ as $x \to \infty$. Later, Turán [11] simplified the proof significantly. After Turán, the theorem of Hardy and Ramanujan follows from the following two asymptotic formulae:

$$\sum_{n \leq x} \omega(n) = x \log \log x + Bx + O(x/ \log x)$$

and

$$\sum_{n \leq x} \omega(n)^2 = x(\log \log x)^2 + O(x \log \log x),$$

where $B$ is a constant.

In 1955, Prachar [10] considered a variant arithmetic function of $\omega$. Let $\omega^*(n)$ be the number of primes $p$ such that $p - 1$ divides $n$. Prachar proved that

$$\sum_{n \leq x} \omega^*(n) = x \log \log x + Bx + O(x/ \log x)$$

and

$$\sum_{n \leq x} \omega^*(n)^2 = O(x(\log x)^2).$$

Also, Prachar proved that

$$\omega^*(n) > \exp \left( a \log n/(\log \log n)^2 \right)$$
for infinitely many integers \( n \), where \( a \) is an absolute constant. Adleman, Pomerance and Rumely [1] improved this to

\[
\omega^*(n) > \exp\left(a \log \frac{n}{\log \log n}\right).
\]

Motivated by Prachar’s work, Erdős and Prachar [5] proved that the number of pairs of primes \( p \) and \( q \) so that the least common multiple \( [p - 1, q - 1] \leq x \) is bounded by \( O(x \log \log x) \). Following a remark of Erdős and Prachar, Murty and Murty [8] improved this to \( O(x) \). By this improvement, they reached the nice bounds

\[
x(\log \log x)^3 \ll \sum_{n \leq x} \omega^*(n)^2 \ll x \log x.
\]

As remarked by Murty and Murty, the above lower bound means that \( \omega^*(n) \) does not have a normal order. With these in hands, Murty and Murty conjectured that there is some some positive constant \( C \) such that

\[
\sum_{n \leq x} \omega^*(n)^2 \sim Cx \log x
\]
as \( x \to \infty \), or equivalently,

\[
\sum_{\lcm[p - 1, q - 1] \leq x} \frac{1}{[p - 1, q - 1]} \sim C \log x.
\]

In a former note, the author [3] gave a slight improvement of the result of Murty and Murty by showing that there are two absolute constants \( a_1 \) and \( a_2 \) such that

\[
a_1 x \log x \leq \sum_{n \leq x} \omega^*(n)^2 \leq a_2 x \log x.
\]

In this subsequent article, we shall confirm their conjecture under the following remarkable assumption which is well–believed to be true.

**Elliott–Halberstam Conjecture.** [4] Let \( x \geq 2 \) be a number. For any \( A > 0 \), the following estimates

\[
\sum_{d \leq x^\theta} \max_{y \leq x} \left| \pi(y; d, 1) - \frac{\text{liy}}{\phi(d)} \right| \ll_{\theta, A} \frac{x}{(\log x)^A}
\]

holds for all \( \theta < 1 \), where \( \pi(y; d, 1) \) is the number of primes \( p \equiv 1 \pmod{d} \) up to \( y \) and

\[
\text{li}_y = \int_2^y \frac{1}{\log t} \, dt.
\]

In our proof, we shall use frequently the following classical inequality.

**Brun–Titchmarsh inequality.** [9] Let \( x \) be a positive real number, and let \( k, a \) be relatively prime positive integers. Then

\[
\pi(x; k, a) \leq \frac{2x}{\phi(k) \log(x/k)},
\]

provided that \( x > k \).
Theorem 1. Assuming the Elliott–Halberstam Conjecture, there is a constant $C$ such that

$$\sum_{n \leq x} \omega^*(n)^2 \sim Cx \log x$$

as $x \to \infty$.

Proof. Throughout our proof, the number $x$ is sufficiently large. The implied constants are absolute unless otherwise indicated. From the paper of Murty and Murty [8, equation (4.10)], we have

$$\sum_{n \leq x} \omega^*(n)^2 = x \sum_{m \leq x} \varphi(m) \left( \sum_{\substack{p \leq x \mod m \equiv 1 \mod m}} \frac{1}{p} \right)^2 + O(x). \tag{1}$$

Note that

$$\sum_{\substack{p \leq x \mod m \equiv 1 \mod m}} \frac{1}{p(p-1)} < 2 \sum_{n \equiv 1 \mod m} \frac{1}{n^2} \ll \frac{1}{m^2}, \tag{2}$$

it follows from equations (1) and (2) that

$$\sum_{n \leq x} \omega^*(n)^2 = x \sum_{m \leq x} \varphi(m) \left( \sum_{\substack{p \leq x \mod m \equiv 1 \mod m}} \frac{1}{p} + O\left(\frac{1}{m^2}\right) \right)^2 + O(x)$$

$$= x \sum_{m \leq x} \varphi(m) \left( \sum_{\substack{p \leq x \mod m \equiv 1 \mod m}} \frac{1}{p} \right)^2 + O(x), \tag{3}$$

where we have used the trivial estimate

$$\sum_{\substack{p \leq x \mod m \equiv 1 \mod m}} \frac{1}{p} \leq \sum_{n \leq x \mod m \equiv 1 \mod m} \frac{1}{n} \ll \frac{\log x}{m}.$$ Integrating by parts, we get

$$\sum_{\substack{p \leq x \mod m \equiv 1 \mod m}} \frac{1}{p} = \frac{\pi(x; m, 1)}{x} + \int_2^x \frac{\pi(t; m, 1)}{t^2} dt.$$

This together with equation (3) lead to

$$\sum_{n \leq x} \omega^*(n)^2 = S_1(x) + S_2(x) + S_3(x) + O(x), \tag{4}$$

where

$$S_1(x) = x \sum_{m \leq x} \varphi(m) \left( \int_2^x \frac{\pi(t; m, 1)}{t^2} dt \right)^2,$$

$$S_2(x) = 2 \sum_{m \leq x} \varphi(m) \pi(x; m, 1) \int_2^x \frac{\pi(t; m, 1)}{t^2} dt.$$
and
\[ S_3(x) = \frac{1}{x} \sum_{m \leq x} \varphi(m) \pi(x; m, 1)^2. \]

It turns out that \( S_2(x) \) and \( S_3(x) \) offer the error terms. The sum \( S_3(x) \) is easy to bound. By the Brun–Titchmarsh inequality, we have
\[
\frac{1}{x} \sum_{m \leq x} \varphi(m) \pi(x; m, 1)^2 \ll x \sum_{m \leq x} \frac{1}{\varphi(m)} \frac{1}{\log^2(x/m)} \\
\ll x \sum_{m \leq x} \frac{\log \log m}{m} \frac{1}{\log^2(x/m)} \\
\ll x \log \log x.
\]

Thus, by trivial estimate we obtain
\[
S_1(x) = \frac{1}{x} \sum_{x/\log x < m \leq x} \varphi(d) \pi(x; m, 1)^2 + O(x \log \log x) \\
\ll x \sum_{x/\log x < m \leq x} \frac{\varphi(m)}{m^2} + x \log x \log x \\
\ll x \log \log x. \tag{5}
\]

Next, we bound the sum \( S_2(x) \). Since \( \pi(t; m, 1) = 0 \) for \( t \leq m \), we have
\[
\int_{x/2}^{x} \frac{\pi(t; m, 1)}{t^2} \, dt = \int_{m}^{2m} \frac{\pi(t; m, 1)}{t^2} \, dt + \int_{2m}^{x/2} \frac{\pi(t; m, 1)}{t^2} \, dt + \int_{x/2}^{x} \frac{\pi(t; m, 1)}{t^2} \, dt \\
\ll \int_{m}^{2m} \frac{dt}{mt} + \int_{2m}^{x/2} \frac{dt}{\varphi(m) t \log(t/m)} + \int_{x/2}^{x} \frac{dt}{mt} \\
\ll \frac{1}{m} + \frac{\log \log x - \log \log m}{\varphi(m)}. \tag{6}
\]

via the trivial estimate and the Brun–Titchmarsh inequality, provided that \( 8 < m \leq x/8 \). From which we deduce that
\[
\sum_{8 < m \leq x/8} \varphi(m) \pi(x; m, 1) \int_{x/2}^{x} \frac{\pi(t; m, 1)}{t^2} \, dt \ll x \sum_{8 < m \leq x/8} \frac{\varphi(m)}{m \log(x/m)} \frac{\log \log x}{\varphi(m)} \\
\ll x(\log \log x)^2.
\]

For \( x/8 < m \leq x \), by trivial estimate we have
\[
\sum_{x/8 < m \leq x} \varphi(m) \pi(x; m, 1) \int_{x/2}^{x} \frac{\pi(t; m, 1)}{t^2} \, dt \ll x \sum_{x/8 < m \leq x} \frac{\varphi(m)}{m} \int_{m}^{x} \frac{1}{mt} \, dt \\
\ll x \sum_{x/8 < m \leq x} \frac{\log x - \log m}{m} \\
\ll x.
\]
And from equation (6), it is clear that
\[ \sum_{m \leq 8} \varphi(m) \pi(x; m, 1) \int_2^x \frac{\pi(t; m, 1)}{t^2} dt \ll x \log \log x. \]
Thus, we have proved that \( S_2(x) \ll x(\log \log x)^2 \). We are left over to compute \( S_1(x) \).

Let \( \varepsilon > 0 \) be an arbitrarily small number. The sum \( S_1(x) \) can be split into the following three shorter sums:

\[
S_1(x) = x \left( \sum_{8 < m \leq x} + \sum_{x < m \leq x^{1-\varepsilon}} + \sum_{x^{1-\varepsilon} < m \leq x} \right) \varphi(m) \left( \int_2^x \frac{\pi(t; m, 1)}{t^2} dt \right)^2 + O(x(\log \log x)^2)
\]
\[ = S_{11}(x) + S_{12}(x) + S_{13}(x) + O(x(\log \log x)^2), \]

where the error term comes from estimate of \( m \leq 8 \) via equation (6). Again by equation (6), we have

\[
S_{11}(x) \ll x \sum_{m \leq x^{\varepsilon}} \varphi(m) \left( \frac{1}{m} + \frac{\log \log x - \log \log m}{\varphi(m)} \right)^2
\]
\[ \ll \varepsilon x \log x + x \sum_{8 < m \leq x^{\varepsilon}} \frac{\log \log x - \log \log m}{m} + x \sum_{8 < m \leq x^{\varepsilon}} \frac{(\log \log x - \log \log m)^2}{\varphi(m)}
\]
and

\[
S_{13}(x) \ll x \sum_{x^{1-\varepsilon} < m \leq x} \varphi(m) \left( \frac{1}{m} + \frac{\log \log x - \log \log m}{\varphi(m)} \right)^2
\]
\[ \ll \varepsilon x \log x + x \sum_{x^{1-\varepsilon} < m \leq x} \frac{\log \log x - \log \log m}{m} + x \sum_{x^{1-\varepsilon} < m \leq x} \frac{(\log \log x - \log \log m)^2}{\varphi(m)}.
\]

It is plain that
\[
\sum_{8 < m \leq x^{\varepsilon}} \frac{\log \log x - \log \log m}{m} \ll \int_8^{x^{\varepsilon}} \frac{\log \log x - \log \log t}{t} dt \ll \varepsilon \log x + 1
\]
and
\[
\sum_{x^{1-\varepsilon} < m \leq x} \frac{\log \log x - \log \log m}{m} \ll \int_{x^{1-\varepsilon}}^{x} \frac{\log \log x - \log \log t}{t} dt \ll \varepsilon \log x + 1.
\]

We are in a position to introduce the following well-known results (see for example [7]) due to Landau

\[
\sum_{m \leq y} \frac{1}{\varphi(m)} = A \log y + O(1) \quad \text{and} \quad \sum_{m \leq y} \frac{m}{\varphi(m)} = Ay + O(\log y),
\]
where \( A \) is an absolute constant. (We shall show later that the constant \( C \) in our theorem is actually equal to \( 2A \).) Integrating by parts and then using equation (8), we
have
\[
\sum_{8 < m \leq x^\varepsilon} \frac{(\log \log x - \log \log m)^2}{\varphi(m)} \ll \int_{8}^{x^\varepsilon} \frac{(\log \log x - \log \log t)^2}{t} d \sum_{8 < m \leq t} \frac{m}{\varphi(m)} \ll \varepsilon \log x + (\log \log x)^2.
\]

The same argument yields
\[
\sum_{x^{1-\varepsilon} < m \leq x} \frac{(\log \log x - \log \log m)^2}{\varphi(m)} \ll \varepsilon \log x + (\log \log x)^2.
\]

Hence, we get
\[
S_{11}(x) + S_{13}(x) \ll \varepsilon x \log x + x(\log \log x)^2.
\]

The rest of the proof will be devoted to the manipulations of \(S_{12}(x)\). For any \(0 \leq j \leq \left\lfloor \frac{\log x}{(1-2\varepsilon) \log 2} \right\rfloor\), let \(Q_j = 2^j x^\varepsilon\). Then \(x^\varepsilon \leq Q_j \leq x^{1-\varepsilon}\) for all \(0 \leq j \leq \left\lfloor \frac{\log x}{(1-2\varepsilon) \log 2} \right\rfloor\). By the Elliott–Halberstam Conjecture, the estimate
\[
\sum_{Q_j < m \leq 2Q_j} \max_{y \leq z} \left| \pi(y; m, 1) - \frac{\li y}{\varphi(m)} \right| \ll \varepsilon \frac{z}{(\log z)^5}
\]
holds for any \(0 \leq j \leq \left\lfloor \frac{\log x}{(1-2\varepsilon) \log 2} \right\rfloor\) and \(Q_j^{1+\varepsilon} < z \leq x\). It follows from equation (10) that for any \(Q_j^{1+\varepsilon} < z \leq x\) the estimate
\[
\max_{y \leq z} \left| \pi(y; m, 1) - \frac{\li y}{\varphi(m)} \right| < \frac{\li y}{\varphi(m) \log z}
\]
holds for all \(Q_j < m \leq 2Q_j\) with at most \(O(Q_j/(\log Q_j)^2)\) exceptions. This immediately leads to the fact that
\[
\pi(y; m, 1) = \frac{\li y}{\varphi(m)} + O\left(\frac{\li y}{\varphi(m) \log y}\right) \quad (z/2 < y \leq z)
\]
is valid for all \(Q_j < m \leq 2Q_j\) but at most \(O(Q_j/(\log Q_j)^2)\) exceptions. Considering the dichotomy of \(z\) between the interval \((Q_j^{1+\varepsilon}, x]\), we have
\[
\pi(y; m, 1) = \frac{\li y}{\varphi(m)} + O\left(\frac{\li y}{\varphi(m) \log y}\right) \quad (\forall Q_j^{1+\varepsilon} < y \leq x)
\]
for all \(Q_j < m \leq 2Q_j\) but at most
\[
\frac{\log x}{\log 2} - Q_j/(\log Q_j)^2 \ll \varepsilon Q_j / \log x
\]
exceptions. Let \(\mathcal{J}_j\) be the set of all exceptions in the interval \(Q_j < m \leq 2Q_j\) for \(0 \leq j \leq \left\lfloor \frac{\log x}{(1-2\varepsilon) \log 2} \right\rfloor\). Then \(|\mathcal{J}_j| \ll \varepsilon Q_j / \log x\). Thus, we conclude that for all
\[
m \in \left((x^\varepsilon, x^{1-\varepsilon}] \setminus \bigcup_{j=0}^{\left\lfloor \frac{\log x}{(1-2\varepsilon) \log 2} \right\rfloor} \mathcal{J}_j\right) := \mathcal{R}, \quad \text{say},
\]
we have
\[
\int_{m^{1+\varepsilon}}^{x} \frac{\pi(t; m, 1)}{t^2} dt = \int_{m^{1+\varepsilon}}^{x} \frac{1}{\varphi(m)t \log t} dt + O \left( \int_{m^{1+\varepsilon}}^{x} \frac{1}{\varphi(m)t \log^2 t} dt \right) \\
= \log \log x - \log \log m \frac{1}{\varphi(m)} + O \left( \frac{1}{\varphi(m) \log m} + \frac{\varepsilon}{\varphi(m)} \right)
\]
from equation (11). Note that
\[
\int_{m^{1+\varepsilon}}^{x} \frac{\pi(t; m, 1)}{t^2} dt \ll \int_{m}^{m^{1+\varepsilon}} \frac{1}{mt} dt \ll \frac{1}{m \log m}
\]
from the trivial estimate, hence, for integers \( m \) located in (12) we have
\[
\int_{2}^{x} \frac{\pi(t; m, 1)}{t^2} dt = \log \log x - \log \log m \frac{1}{\varphi(m)} + O \left( \frac{1}{\varphi(m) \log m} + \frac{\varepsilon}{\varphi(m)} \right)
\]
from equation (13). We now turn back to calculate \( S_{12}(x) \). By definition of \( S_{12}(x) \),
\[
S_{12}(x) = x \left( \sum_{j=0}^{[\log x/(1-2\varepsilon) \log 2]} \sum_{m \in \mathcal{S}_j} \varphi(m) \left( \int_{2}^{x} \frac{\pi(t; m, 1)}{t^2} dt \right)^2 \right) \]
\[
+ O \left( x \sum_{\frac{1}{2} x^{1-\varepsilon} \leq m \leq x^{1-\varepsilon}} \varphi(m) \left( \int_{2}^{x} \frac{\pi(t; m, 1)}{t^2} dt \right)^2 \right),
\]
where the error term comes from the possible overlaps between the sets. From equation (6), the error term can be bounded as
\[
x \sum_{x^{1-\varepsilon} / 2 \leq m \leq x^{1-\varepsilon}} \varphi(m) \left( \int_{2}^{x} \frac{\pi(t; m, 1)}{t^2} dt \right)^2 \ll x (\log \log x)^2.
\]
Employing equation (6) again, we obtain
\[
x \left( \sum_{j=0}^{[\log x/(1-2\varepsilon) \log 2]} \sum_{m \in \mathcal{S}_j} \varphi(m) \left( \int_{2}^{x} \frac{\pi(t; m, 1)}{t^2} dt \right)^2 \right) \ll x \sum_{j=0}^{[\log x/(1-2\varepsilon) \log 2]} \sum_{m \in \mathcal{S}_j} \frac{(\log \log x)^2}{\varphi(m)}
\]
\[
\ll \varepsilon x (\log \log x)^2 \sum_{j=0}^{[\log x/(1-2\varepsilon) \log 2]} \frac{1}{\log x}
\]
\[
\ll \varepsilon x (\log \log x)^2,
\]
where we have used the facts \(|\mathcal{S}_j| \ll \varepsilon Q_j / \log x\) and (hence)
\[
\sum_{m \in \mathcal{S}_j} \frac{1}{\varphi(m)} \ll \varepsilon \frac{1}{\log x}.
\]
For \( m \in \mathcal{R} \), from equation (14) we have
\[
x \sum_{m \in \mathcal{R}} \varphi(m) \left( \int_{2}^{x} \frac{\pi(t; m, 1)}{t^2} dt \right)^2 \]
\[
= x \sum_{m \in \mathcal{R}} \varphi(m) \left( \frac{\log \log x - \log \log m}{\varphi(m)} + O \left( \frac{1}{\varphi(m) \log m} + \frac{\varepsilon}{\varphi(m)} \right) \right)^2 \]
\[
= x \sum_{m \in \mathcal{R}} \frac{(\log \log x - \log \log m)^2}{\varphi(m)} + O \left( x(\log \log x)^2 + \varepsilon x \log x \right). \]

Noting that
\[
\sum_{m \in \mathcal{R}} \frac{(\log \log x - \log \log m)^2}{\varphi(m)} \]
\[
= \sum_{x^\varepsilon < m \leq x^{1-\varepsilon}} \frac{(\log \log x - \log \log m)^2}{\varphi(m)} + O_\varepsilon \left( (\log \log x)^2 \right) \]
in view of equation (16) and
\[
\sum_{x^\varepsilon < m \leq x^{1-\varepsilon}} \frac{(\log \log x - \log \log m)^2}{\varphi(m)} = \int_{x^\varepsilon}^{x^{1-\varepsilon}} \frac{(\log \log x - \log \log t)^2}{t} dt \sum_{x^\varepsilon < m \leq t} \frac{m}{\varphi(m)} = A f(\varepsilon) \log x + O((\log \log x)^2) \]
via equation (8), we can conclude that
\[
x \sum_{m \in \mathcal{R}} \varphi(m) \left( \int_{2}^{x} \frac{\pi(t; m, 1)}{t^2} dt \right)^2 = A f(\varepsilon) x \log x + O(x(\log \log x)^2 + \varepsilon x \log x), \quad (17) \]
where
\[
f(\varepsilon) = 2(1 - \varepsilon) + (1 - \varepsilon) \log^2(1 - \varepsilon) - \varepsilon \log^2 \varepsilon + 2(1 - \varepsilon) \log \frac{1}{1 - \varepsilon} + 2 \varepsilon \log \varepsilon. \]
Thus, we finally get
\[
S_{12}(x) = A f(\varepsilon) x \log x + O_\varepsilon(x(\log \log x)^2) + O(\varepsilon x \log x). \]

Looking back the estimates from equations (4) to (17), we have established
\[
\sum_{n \leq x} \omega^*(n)^2 = A f(\varepsilon) x \log x + O_\varepsilon(x(\log \log x)^2) + O(\varepsilon x \log x) \quad (18) \]
for any \( \varepsilon > 0 \). From above equation (18), we have
\[
Af(\varepsilon) + O(\varepsilon) \leq \liminf_{x \rightarrow \infty} \frac{\sum_{n \leq x} \omega^*(n)^2}{x \log x} \leq \limsup_{x \rightarrow \infty} \frac{\sum_{n \leq x} \omega^*(n)^2}{x \log x} \leq Af(\varepsilon) + O(\varepsilon). \]
Making \( \varepsilon \rightarrow 0 \), we have
\[
2A \leq \liminf_{x \rightarrow \infty} \frac{\sum_{n \leq x} \omega^*(n)^2}{x \log x} \leq \limsup_{x \rightarrow \infty} \frac{\sum_{n \leq x} \omega^*(n)^2}{x \log x} \leq 2A \]
since \( f(\varepsilon) \rightarrow 2 \) for \( \varepsilon \rightarrow 0 \). Therefore,
\[
\sum_{n \leq x} \omega^*(n)^2 \sim 2Ax \log x, \quad (as \ x \rightarrow \infty). \]
Remark. It is clear that we have the following corollary
\[
\sum_{p,q \leq x} \frac{1}{|p-1, q-1|} \sim C \log x
\]
as \(x \to \infty\) under the Elliott–Halberstam Conjecture due to (see \([8, \text{page 6, last line}]\))
\[
\sum_{n \leq x} \omega^*(n)^2 = \sum_{p,q \leq x} x \frac{x}{|p-1, q-1|} + O(x).
\]

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