Design Algorithms for $B$-splines with certain derivatives of the first and second orders at given points

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Abstract. The article describes the algorithms for creating B-splines in space passing through the given points and with the possibility of setting the values of the first and second derivatives at each of these points. This method of determining the spatial curve provides great opportunities for the lines design and on their basis the surfaces suitable for use in the machine parts, building structures and architectural objects’ design. Software developed by the technology Object ARX of the AutoCAD system. This approach significantly reduces time for software development and makes it possible to use a wide range of AutoCAD system tools, both in development, and by the programs using. The developed programs can be used in computer-aided design systems for various objects independently or as modules.

Introduction
Splines are a powerful modern design tool for spatial lines and surfaces. Many modern automated design systems [1 - 3] have a wide range of tools for working with splines. Most often, such systems implement the functions for working with polynomial splines or B-splines [4 - 5]. But, as it is known, when constructing a polynomial spline, the points through which it passes and the derivatives’ values at the start and end points are set. It is not possible to specify directly the derivatives at intermediate points. This is necessary for some tasks, though. For example, when constructing a reflective surface [6], the first derivative at intermediate points was determined by entering the closely spaced additional points. It is bulky and uncomfortable. To construct B-splines, as a rule, a characteristic polygon is used. Thus, it is also impossible to specify the points through which the B-spline and its derivatives pass. A more productive way to solve this problem is to construct a spline with certain derivatives at given points. For these purposes, B-splines were selected.

The algorithms were developed using ObjectARX technology [7] AutoCAD system [1], in the language C++ [8]. This can significantly reduce software development time and improve its usability, as it gives an opportunity to to use the AutoCAD interface.

Problem Statement
The objective of this study is to develop the software algorithms for creating $B$-splines passing through the given points in space, and the ability to set the values of the first and second derivatives at these points.

Building $B$-splines
Let us consider the standard way to construct a $B$-spline from a given characteristic polygon.
1.1. Building B-splines from a characteristic polygon
The theoretical foundations for constructing B-splines were proposed in [9]. Let a characteristic polygon be given the vertices in space \( B_i, i=1, 2, \ldots, n \). Then the B-spline vector equation has the form:

\[
\vec{r}(t) = \sum_{i=1}^{n} \vec{B}_i N_{i,k}(t).
\]  

In the formula (1) \( t \) – is the spline parameter, varying within the following limits \( t_{\text{min}} \leq t < t_{\text{max}} \), the values \( t_{\text{min}} \) and \( t_{\text{max}} \) depend on the type of nodal vector \( \vec{X} \), which will be discussed below. \( k \) – is the spline order, its value is selected within \( 2 \leq k \leq n \). Value \( k \) determines the spline degree, it is equal to \( k-1 \). \( N_{i,k}(t) \) – are the basis functions. They are calculated by the recursive formulas of Cox - de Boer [10 – 11]

\[
N_{i,1}(t) = \begin{cases} 
1, & t \in [x_i, x_{i+1}); \\
0, & t \notin [x_i, x_{i+1}); 
\end{cases}
\]

\[
N_{i,k}(t) = \frac{(t-x_i)N_{i,k-1}(t)}{x_{i+k} - x_i} + \frac{(x_{i+k} - t)N_{i+k-1}(t)}{x_{i+k} - x_{i+1}}, \quad k \geq 2,
\]  

where \( x_i \) – denotes the components of the nodal vector \( \vec{X} \). The number of the nodal vector components is \( n+k \). The only condition that components should meet is: \( x_i \leq x_{i+1} \). The view of the nodal vector components has a significant effect on the resulting spline. There are many options for defining \( \vec{X} \) components. When implementing this algorithm, an open uniform nodal vector was chosen. The initial and final components \( k \) of such a vector are equal, the values of the others differ by 1. The formula for calculating the components has the form:

\[
x_i = \begin{cases} 
0, & i \leq k; \\
i-k, & k < i \leq n; \\
n-k+1, & i > n.
\end{cases}
\]  

For example, when \( n=5 \) and \( k=3 \), the nodal vector has the following components

\( \vec{X} = \{0 \ 0 \ 0 \ 1 \ 2 \ 3 \ 3 \ \} \).

The minimum value of the spline parameter is \( t_{\text{min}} = x_1 \), the maximum value is \( t_{\text{max}} = x_{n+k} \).

An example of building a B-spline with \( n=7 \), \( k=4 \) is shown in Figure 1. In Figure 1, the constructed spline is denoted by \( s \).

1.2. Building B-splines at given points
Let us consider the construction of a B-spline passing through the given points in space. This task is called the spline “fitting” [5]. Let a point row be given as \( A_i, i=1, 2, \ldots, n \). We construct a B-spline so that it passes through the given points. To do this, find characteristic polygon, corresponding to this spline, i.e. its vertices coordinates \( B_i \). As the spline passes through the given points, then their coordinates should satisfy the equation (1). Obviously, to uniquely determine the spline, the number of vertices of the polygon should be equal to \( n \).

We put the parameter \( t \) in correspondence with each of the given points \( A_i \). To determine the parameters of the given points, we use the linear interpolation method. We assume that the parameter of the corresponding point is proportional to the chords’ lengths sum to this point. By (3) we calculate the components of the nodal vector \( \vec{X} \). Then for the initial and final values of the parameters we have \( t_i = t_{\text{min}} = x_1 = 0, t_n = t_{\text{max}} = x_{n+k} \). The remaining values are calculated by the formula:
\[ t_i = \frac{\sum_{j\geq i} |\vec{A}_j - \vec{A}_{j-1}|}{\sum_{j\geq i} |\vec{A}_j - \vec{A}_{j-1}|} t_n, \quad 1 < i < n. \] (4)

Figure 1. A B-spline construction from a given characteristic polygon

Therefore, to determine the characteristic polygon vertices, it is necessary to solve the following system of linear equations

\[
\begin{align*}
\vec{A}_1 &= \vec{B}_1 N_{1,k}(t_1) + \vec{B}_2 N_{2,k}(t_1) + \cdots + \vec{B}_n N_{n,k}(t_1) ; \\
\vec{A}_2 &= \vec{B}_1 N_{1,k}(t_2) + \vec{B}_2 N_{2,k}(t_2) + \cdots + \vec{B}_n N_{n,k}(t_2) ; \\
&\vdots \\
\vec{A}_n &= \vec{B}_1 N_{1,k}(t_n) + \vec{B}_2 N_{2,k}(t_n) + \cdots + \vec{B}_n N_{n,k}(t_n) ,
\end{align*}
\] (5)

where \( \vec{A}_i \) – is the radius vectors of the given points \( A_i \), \( N_{i,k}(t) \) – determines the basis functions calculated by the formulas (2). Having solved the system of linear equations (5), for example, by the Gauss method, we obtain the coordinates of the characteristic polygon \( B_i \) vertices.

An example of constructing a spline passing through the given points when \( n=6 \) and \( k=4 \), is shown in Figure 2. In Figure 2, the symbol “equiv” indicates the matching points.

Figure 2. A B-spline construction passing through the given points
1.3. Construction of B-splines on the given points with defined first derivatives

To find the first derivative of the spline, we differentiate (1) with respect to the parameter $t$, then

$$ r'(t) = \sum_{i=1}^{n} \tilde{B}_i N'_i(t), $$

where $N'_i(t) = \frac{N_{i,k-1}(t) + (t-x_i)N'_{i,k-1}(t)}{x_{i+k-1}-x_i}$ + \frac{(x_{i+k} - t)N'_{i+1,k-1}(t) - N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}, k \geq 2.$

Thus, if the value of the first derivative is specified at some of the given points $A_i$, then this condition should be added to the system (5) and a vertex added to the characteristic polygon. The system of linear equations will have the following form

$$
\begin{align*}
\bar{A}_1 &= \tilde{B}_1 N_{1,k}(t_1) + \tilde{B}_2 N_{2,k}(t_1) + \cdots + \tilde{B}_m N_{m,k}(t_1), \\
\bar{A}_2 &= \tilde{B}_1 N_{1,k}(t_2) + \tilde{B}_2 N_{2,k}(t_2) + \cdots + \tilde{B}_m N_{m,k}(t_2), \\
\vdots & \quad \vdots \\
\bar{A}_n &= \tilde{B}_1 N_{1,k}(t_n) + \tilde{B}_2 N_{2,k}(t_n) + \cdots + \tilde{B}_m N_{m,k}(t_n); \\
\bar{T}_1 &= \tilde{B}_1 N'_{1,k}(t_1) + \tilde{B}_2 N'_{2,k}(t_1) + \cdots + \tilde{B}_m N'_{m,k}(t_1), \\
\bar{T}_2 &= \tilde{B}_1 N'_{1,k}(t_2) + \tilde{B}_2 N'_{2,k}(t_2) + \cdots + \tilde{B}_m N'_{m,k}(t_2), \\
\vdots & \quad \vdots \\
\bar{T}_p &= \tilde{B}_1 N'_{1,k}(t_p) + \tilde{B}_2 N'_{2,k}(t_p) + \cdots + \tilde{B}_m N'_{m,k}(t_p),
\end{align*}
$$

where $\bar{T}_j = r'(t_j)$ is the given first derivative at the point $A_j$, $m=n+p$, $p$ is the number of the given first derivatives. Having solved the system (7), we obtain a characteristic polygon defining a spline that satisfies the conditions.

An example of calculation is shown in Figure 3, for $n=5$, $k=4$. At the point $A_4$ the first $\bar{T}_4$ derivative is given. For comparison, Figure 3 shows a spline constructed without taking into account the given first derivative. This spline and its characteristic polygon are shown with a dotted line.
I.4. B-splines construction at the given points with defined first and second derivatives

Let us find the second derivative of the spline. We differentiate the equation (1) with respect to the parameter \( t \) two times, then we get:

\[
\dddot{r}(t) = \sum_{i=1}^{n} \dot{B}_i N_{i,k}^{*}(t),
\]

(8)

where \( N_{i,k}^{*}(t) \) – define the second derivatives of the basis function obtained by double differentiation (2) with respect to the parameter \( t \), equal to:

\[
N_{i,k}^{*}(t) = \frac{2N'_{i,k-1}(t) + (t-x_i)N''_{i,k-1}(t) + (x_{i+1} - t)N''_{i+1,k-1}(t) - 2N'_{i+1,k-1}(t)}{x_{i+1} - x_i}, \quad k > 2.
\]

Let at some given points \( A_i \) the second spline derivatives be defined as \( \dddot{S}_i = \dddot{r}(t_i) \), taking into account (8), we add these conditions to the system (7) and obtain:

\[
\begin{align*}
\dddot{A}_1 &= \dddot{B}_1 N_{1,k}(t_1) + \dddot{B}_2 N_{2,k}(t_1) + \ldots + \dddot{B}_m N_{m,k}(t_1); \\
\dddot{A}_2 &= \dddot{B}_1 N_{1,k}(t_2) + \dddot{B}_2 N_{2,k}(t_2) + \ldots + \dddot{B}_m N_{m,k}(t_2); \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \lr...
[4] Zavyalov Yu S, Leus V A, Skorospelov V A 1985 *Splines in engineering geometry* (Engineering, Moscow) 224.

[5] Rogers D, Adams J 2001 *Mathematical Foundations of Machine Graphics* (Mir, Moscow) 604.

[6] Zamyatin A V, Zamyatina E A Formation of reflecting surfaces on the basis of splines-methods *IOP Conference Series: Materials Science and Engineering* 262 (1). Information on http://iopscience.iop.org/article/10.1088/1757-899X/262/1/012107/pdf.

[7] Poleshchuk N N 2015 *Programming for AutoCAD 2013-2015* (DMK Press, Moscow) 462.

[8] Litvinenko N A 1010 C++ programming technology. *Win32 API Applications* (BHV-Petersburg, SPb.) 288.

[9] Schoenberg IJ 1946 Contributions to the problem of approximation of equidistant data by analytic function *Q. Appl. Math.* 4 pp. 45-99, pp. 112-141.

[10] Cox M G 1971 The Numerical evaluation of B-Splines *National Physical Laboratory DNAC 4*.

[11] de Boor C 1972 On Calculation with B-splines *J. Approx. Theory* 6 50-62.