PRIMENESS RESULTS FOR VON NEUMANN ALGEBRAS ASSOCIATED WITH SURFACE BRAID GROUPS

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Abstract. In this paper we introduce a new class $P$ of non-amenable groups which give rise to prime von Neumann algebras. This means that for every $\Gamma \in P$ its group von Neumann algebra $L(\Gamma)$ cannot be decomposed as a tensor product of diffuse von Neumann algebras. We show that $P$ is fairly large as it contains many examples of groups intensively studied in various areas of mathematics, notably: all infinite central quotients of pure surface braid groups—in particular, most pure braid groups on punctured surfaces of genus at least 1; all mapping class groups of (punctured) surfaces of genus $0, 1, 2$; most Torelli groups and Johnson kernels of (punctured) surfaces of genus $0, 1, 2$; and, all groups hyperbolic relative to finite families of residually finite, exact, infinite, proper subgroups.

1. Introduction

In pioneering work [Po81] Sorin Popa discovered that the (non-separable) factors $L(\mathcal{F})$ arising from uncountably generated free groups $\mathcal{F}$ are prime, i.e., $L(\mathcal{F})$ cannot be decomposed as a tensor product of diffuse factors. Much later, using Voiculescu’s influential free probability theory [VDN92, V94, V96], Liming Ge was able to show primeness for all factors associated with countably generated, non-abelian free groups as well, [G98]. In the context of free probability other examples of prime factors were subsequently unveiled [Shl00, Shl04, Ju07].

By developing a different perspective, largely based on $C^*$-algebraic methods, Narutaka Ozawa obtained a far-reaching generalization of these results by showing that that all factors $L(\Gamma)$ associated with non-elementary hyperbolic groups $\Gamma$ are in fact solid, i.e., for every diffuse, amenable subalgebra $\Lambda \subset L(\Gamma)$ its relative commutant $\Lambda' \cap L(\Gamma)$ is again amenable; in particular, it follows that $L(\Gamma)$ is prime. Notice that Ozawa’s solidity result holds for all factors associated with bi-exact groups [Oz03, Oz05, BO08].

In the early 2000’s Popa introduced a completely new conceptual framework to study von Neumann algebras, now termed Popa’s deformation/rigidity theory. This novel approach has generated over the last decade a spectacular progress, leading to complete solutions to many longstanding open problems in the classification of von Neumann algebras and equivalence relations arising from group actions, [Po01, Po03, Po04, IPP05, Po06, Po06b]. The theory develops a powerful technical apparatus designed to incorporate meaningful cohomological, geometric, and algebraic information of a group and its actions in the analytic context of von Neumann algebras. Overtime, these methods
began more and more precise and sophisticated and revealed unprecedented connections with cohomological, geometric, and dynamical aspects in group theory.

These techniques are very suitable to study the primeness phenomenon as well. Indeed, using his free malleable deformations in combination with a novel spectral gap argument, Popa was able to find a new, elementary proof for Ozawa’s solidity result for the non-amenable free group factors, [Po06]. His approach laid out the foundations for many important subsequent developments regarding the algebraic structure of factors. For instance, it provided the correct insight which later allowed Ozawa and Popa to show in a remarkable work [OP07] that all non-amenable free group factors $L(F)$ are in fact strongly solid, i.e., for every diffuse amenable subalgebra $A \subset L(F)$, its normalizing group $N_{L(F)}(A) = \{u \in \mathbb{U}(L(F)) : uAu^* = A\}$ generates an amenable von Neumann subalgebra in $L(F)$—a result of great influence for the entire subsequent development on the classification of normalizers of algebras in many classes of factors.

Exploiting a new viewpoint which originates in his recent ingenious study of unbounded derivations on von Neumann algebras, Jesse Peterson further showed that every non-amenable, icc group with positive first $\ell^2$-Betti number gives rise to a prime factor, [Pe06].

These results along with Ozawa’s earlier solidity results have spawned a rich activity in the classification of von Neumann algebras. Numerous technical outgrowth of these methods by several authors have led overtime to the discovery to many striking structural results including primeness, (strong) solidity, uniqueness of Cartan subalgebra, and beyond for large classes of von Neumann algebras, [Po08, OP07, OP08, CH08, CI08, Pe09, PV09, FV10, Io10, IPV10, HPV10, CP10, Si10, Va10, Fi11, CS11, CSU11, Io11, PV11, HV12, PV12, Io12, Bo12, BHR12, Is12, BV13, Va13, Is14, CIK13, VV14, BC14]

1.1. **Statements of main results.** In this paper we introduce new families of groups which give rise to prime von Neumann algebras. Many of these groups are intensively studied in various areas of Mathematics, especially topology and geometric group theory. Over time, via deep topological and geometric methods, many strong classification results emerged regarding the structure of these groups in both discrete and measurable setting. However, momentarily, little is known about the structure of the von Neumann algebras associated with these groups and this paper initiates a study in this direction. Formally, we define our class of groups as follows:

**Definition 1.1.** A group $\Gamma$ belongs to class $\mathcal{P}$ if the following two conditions are satisfied:

- $\textbf{NC}_1$: $\Gamma$ is non-amenable and admits an unbounded quasi-cocycle valued into one of its mixing, weakly-$\ell^2$, orthogonal representations (see Section 4 for relevant definitions);
- $\textbf{Quot}(\mathcal{C}_{\text{rss}})$: $\Gamma$ is a finite-step extension of groups belonging to $\mathcal{C}_{\text{rss}}$—the collection of all non-elementary hyperbolic groups and non-amenable, non-trivial free products of exact groups (see Sections 2.3 and 3 for relevant definitions).

While at a first look this definition may seem a little restrictive and not entirely illuminating, we will show however that $\mathcal{P}$ is fairly large, containing many important examples of groups, such as:
a) Any infinite, central quotient of the pure braid group \( PB_\infty(S_{g,k}) \) of \( n \) strands on a connected, compact and orientable surface \( S_{g,k} \) of genus \( g \) with \( k \) boundary components—in particular, all surface pure braid groups \( PB_\infty(S_{g,k}) \), for \( n \geq 1 \) and either \( g = 1 \) and \( k \geq 1 \) or \( g \geq 2 \) and \( k \geq 0 \);

b) Any mapping class group \( \text{Mod}(S_{g,k}) \), for \( 0 \leq g \leq 2 \) and \( 2g + k \geq 4 \);

c) Any Torelli group \( \mathcal{I}(S_{g,k}) \) and Johnson kernel \( \mathcal{K}(S_{g,k}) \), for \( g = 1, 2 \) and \( 2g + k \geq 4 \);

d) Any group that is hyperbolic relative to a finite family of exact, residually finite, infinite, proper subgroups.

For the proofs of these results as well as other basic properties of \( \mathcal{P} \) we refer the reader to Sections 3 and 4 in the sequel.

The central result of the paper shows that all groups in \( \mathcal{P} \) give rise to prime von Neumann algebras; in particular, for any \( \Gamma \in \mathcal{P} \) its von Neumann algebra cannot be decomposed as \( L(\Gamma) = L(\Omega \times \Sigma) \), for any infinite groups \( \Omega \) and \( \Sigma \). In fact, we are able to show a slightly more general structural theorem regarding the commuting diffuse subalgebras inside the von Neumann algebra of any group that is commensurable up to finite kernel with a group in \( \mathcal{P} \). (Notice that from Theorem 4.6, Proposition 3.2, and [VV14, Proposition 4.7] it follows that \( \mathcal{P} \) is closed under commensurability).

**Theorem A.** Let \( \Gamma \) be a group commensurable up to finite kernel with a group in \( \mathcal{P} \) and let \( L(\Gamma) \) be its the corresponding von Neumann algebra. If \( \mathcal{P} \in L(\Gamma) \) is a nonzero projection, then any two diffuse, commuting subalgebras \( B, C \subseteq pL(\Gamma)p \) generate together a von Neumann subalgebra \( B \vee C \) which has infinite Pimsner-Popa index in \( pL(\Gamma)p \). In particular, \( L(\Gamma) \) is prime and hence \( L(\Gamma) \not\cong L(\Omega \times \Sigma) \), for any infinite groups \( \Omega \) and \( \Sigma \).

In this form the result is sharp, as in general there are groups \( \Gamma \) in \( \mathcal{P} \) whose algebras \( L(\Gamma) \) do contain commuting, non-amenable, diffuse subalgebras which, together generate subalgebras in \( L(\Gamma) \) of infinite index. To see some basic examples, consider \( B_n \) to be the braid group on \( n \geq 6 \) strands and denote by \( Z \) its center. By [FM11, Section 9.2], the quotient \( \Gamma = B_n/Z \) can be realized as a subgroup of index \( n \) inside the mapping class group of a \( (n+1) \)-punctured surface of genus zero and hence Theorem 4.6 and Examples 4.8 c) further imply that \( \Gamma \in \text{NC}_1 \). Moreover, using Birman short exact sequence, Corollary 3.5 shows that \( \Gamma \) is a \( (n-2) \)-step extension of non-abelian free groups and hence \( \Gamma \in \mathcal{P} \). On the other hand, notice that \( B_n \) contains mixed braid subgroups of the form \( B_p \times B_q < B_n \), where \( p + q = n \) with \( p, q \geq 3 \). Then one can check that the quotients \( \Gamma_1 = B_p/Z \) and \( \Gamma_2 = B_q/Z \) are commuting, non-amenable subgroups of \( \Gamma \) which together generate a subgroup \( \langle \Gamma_1, \Gamma_2 \rangle < \Gamma \) of infinite index. This canonically implies that \( L(\Gamma_1) \) and \( L(\Gamma_2) \) are commuting, non-amenable, diffuse subalgebras of \( L(\Gamma) \) which together generate a von Neumann subalgebra \( L(\langle \Gamma_1, \Gamma_2 \rangle) \subset L(\Gamma) \) of infinite index. Notice that, all such groups in \( \mathcal{P} \) will give rise to prime von Neumann algebras which are not solid in the sense of Ozawa, [Oz03].

We believe that Theorem A can be further improved by showing that the algebra \( B \vee C \) is actually never co-amenable inside \( pL(\Gamma)p \), [Po86]. Notice that this will follow verbatim from our current proofs if one will be able to show an analogue of Proposition 2.4 in the context of co-amenable inclusions rather than finite index inclusions.

The proof of our result is based on Popa’s deformation/rigidity theory and is obtained by induction on \( n \) where \( \Gamma \in \text{NC}_1 \cap \text{Quot}_n(\mathcal{C}_{rss}) \). For the induction step we use
in an essential way recent, powerful results due to Popa and Vaes [PV11, PV12] and to Ioana [Io12] regarding the classification of normalizers of subalgebras in von Neumann algebras arising from actions by non-elementary hyperbolic groups and by free product groups, respectively. Assuming by contradiction that \( B \vee C \subseteq pL(\Gamma)p \) has finite index then condition \( \Gamma \in \text{Quot}_{n}(\mathfrak{c}_{rss}) \) enables us to employ these structural results, via the methods developed in [CIK13], to intertwine \( B \) (or \( C \)) onto a subalgebra \( B_0 \subseteq qL(\Sigma)q \subset qL(\Gamma)q \), where \( \Sigma \triangleleft \Gamma \) is a normal subgroup satisfying \( \Sigma \in \text{Quot}_{n-1}(\mathfrak{c}_{rss}) \) and \( q \in B_0 \) is a nonzero sub-projection of \( p \). Moreover there exists a subalgebra \( C_0 \subseteq qL(\Sigma)q \) commuting to \( B_0 \) such that \( B_0 \vee C_0 \subseteq qL(\Sigma)q \) has finite index. Since \( \Gamma \in \text{NC}_1 \) then by [CSU13, Theorem 2.1] we have \( \Sigma \in \text{NC}_1 \cap \text{Quot}_{n-1}(\mathfrak{c}_{rss}) \) and by the induction hypothesis one can find a nonzero corner \( rB_0r \) of finite index in \( rL(\Sigma)r \). On the other hand, since \( \Gamma \in \text{NC}_1 \), then a spectral gap argument shows that the corresponding weak deformations \( V_t \) on \( L(\Gamma) \) arising from an unbounded quasi-cocycle on \( \Gamma \) [CS11] will converge uniformly to the identity on the unit ball \( (B_1) \). From this, developing new aspects in the infinitesimal analysis of \( V_t \) (Section 6) we further show that \( V_t \) converges uniformly to the identity on the unit ball \( (rB_0r)_1 \) and by the finite index assumption it follows that a \( V_t \) has a uniform decay on the unit ball \( (rL(\Sigma)r)_1 \). Hence the quasi-cocycle is bounded on \( \Sigma \) and by [CSU13, Theorem 2.1] it is bounded on \( \Gamma \), which is a contradiction; thus \( B \vee C \) must have infinite index in \( pL(\Gamma)p \).

Notice that, a spectral gap argument [CS11, Proposition 1.7 (3)] shows that for any group \( \Gamma \in \text{NC}_1 \), any two commuting, infinite subgroups \( \Gamma_1, \Gamma_2 < \Gamma \) generate an infinite index subgroup \( \langle \Gamma_1, \Gamma_2 \rangle \) of \( \Gamma \) (in other words, \( \Gamma \) is not presentable by products). This should be seen as evidence supporting the far-reaching conjecture that Theorem A actually holds for all groups \( \Gamma \) satisfying only condition \( \text{NC}_1 \) (even, without the mixing assumption on the representation). However, from a technical point of view, a successful implementation of this argument in the von Neumann algebra setting seems out of reach momentarily and depending heavily on investigating new aspects of the infinitesimal analysis of the weak deformations arising from quasi-cocycles. Unlike in the case of 1-cocycles, the weak deformations arising from quasi-cocycles of groups seem to lack good averaging and uniform bimodularity properties which makes their analysis quite difficult. In our situation some of these difficulties could be by-passed through the knowledge that \( \Gamma \) admits a “finite resolution” by groups in \( \mathfrak{c}_{rss} \). Hence our result can be viewed as a first instance when knowing a little bit more information about the group (in addition of being in \( \text{NC}_1 \)) could decisively enhance the analysis on the weak deformations to conclude primeness results for many such group von Neumann algebras. It is then conceivable that there is actually an entire spectrum of such properties and a thorough investigation may reveal interesting results in this direction.

As a byproduct of these methods, we obtain new applications of deformation/rigidity techniques to the algebraic structure of groups. Indeed, outgrowths of our methods in combination with the techniques developed in [CSU13] allows us to deduce the following strengthening of [CSU13, Theorem 3.5] in the case of groups satisfying condition \( \text{NC}_1 \) above.

**Theorem B.** For any group \( \Gamma \) satisfying \( \text{NC}_1 \) there exists a short exact sequence \( 1 \to F \to \Gamma \to \Gamma_0 \to 1 \) such that \( F \) is finite and \( \Gamma_0 \) is a non-inner amenable, icc group satisfying \( \text{NC}_1 \).
In particular, this provides a more quasi-cohomological explanation for some recent results on non-inner amenability of acylindrically hyperbolic groups by Dahmani, Guirardel, and Osin [DGO11], Osin [Os13], and Minasyan and Osin [MO13]. This result also implies that every groups in \( P \) gives von Neumann algebra with finite dimensional center, result that is implicitly used in the proof of Theorem A.

1.2. Notations. In this section we establish some notions that we will be used throughout the paper.

A *tracial von Neumann algebra* \((M, \tau)\) is a pair that consists of a von Neumann algebra \(M\) and a faithful normal tracial state \(\tau\). We denote by \(M_+\) the set of all positive elements \(x \in M\) and by \(Z(M)\) the center of \(M\). For \(x \in M\), we denote by \(\|x\|\) the operator norm of \(x\), and by \(\|x\|_2 = \sqrt{\tau(x^*x)}\) the 2-norm of \(x\). Throughout the paper, we denote by \(L^2(M)\) the Hilbert space obtained by completing \(M\) with respect to \(\|\cdot\|_2\), and consider the standard representation \(M \subset B(L^2(M))\).

If \(M\) is a von Neumann algebra together with a subset \(S \subset M\), then a state \(\phi : M \to \mathbb{C}\) is called \(S\)-central if \(\phi(xT) = \phi(Tx)\), for all \(T \in M\) and \(x \in S\). A tracial von Neumann algebra \((M, \tau)\) is called *amenable* if there exists an \(M\)-central state \(\phi : B(L^2(M)) \to \mathbb{C}\) such that \(\phi(x) = \tau(x)\), for all \(x \in M\). By a well-known theorem of A. Connes [Co75], \((M, \tau)\) is amenable if and only if it is approximately finite dimensional.

All inclusions of von Neumann algebras that appear in the paper are assumed to be unital unless specified otherwise. Let \((M, \tau)\) be a tracial von Neumann algebra and \(P \subset M\) a von Neumann subalgebra. *Jones's basic construction* \(\langle M, e_P \rangle \subset B(L^2(M))\) is the von Neumann algebra generated by \(M\) and the orthogonal projection \(e_P : L^2(M) \to L^2(P)\). It is endowed with a faithful semifinite trace \(\text{Tr}\) given by \(\text{Tr}(xe_Py) = \tau(xy)\), for all \(x, y \in M\). Also, we note that \(E_P := e_{P|M} : M \to P\) is the unique \(\tau\)-preserving conditional expectation onto \(P\).

We say that \(P \subset M\) is a *masa* if it is a maximal abelian \(*\)-subalgebra. The *normalizer of \(P\) inside \(M\)*, denoted by \(N_M(P)\), is the set of all unitaries \(u \in M\) such that \(uPu^* = P\). We say that \(P\) is *regular in \(M\)* if \(N_M(P)^\prime\prime = M\).

Often if \(M\) is a von Neumann algebra we denote by \(M^h\) its hermitian part and by \(M_+\) its positive part. If \(S \subseteq M\) is a subset and \(c > 0\) we denote by \((S)_c\) the set of all elements in \(S\) whose operatorial norm does not exceed \(c\).

Also if \(B, C \subseteq M\) are subalgebras then we will denote by \(B \vee C\) the von Neumann algebra generated by \(B \cup C\) in \(M\).

Finally, whenever \(\Gamma\) is a group and \(K, H \subseteq \Gamma\) are subsets we will be denoting by \(KH = \{kh : k \in K, h \in H\}\) and by \(\langle K \rangle\) the subgroup generated by \(K\) in \(\Gamma\).

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2. Some Preliminaries on Intertwining Results

2.1. Popa's intertwining techniques. Over a decade ago Popa developed a powerful technology for conjugating subalgebras of tracial von Neumann algebras, now termed the *intertwining-by-bimodules techniques*, [Po03, Theorem 2.1 and Corollary 2.3]. For further reference we recall the following theorem.
Theorem 2.1 (Popa, [Po03]). Let \((M, \tau)\) be a separable tracial von Neumann algebra and \(P, Q\) be two (not necessarily unital) von Neumann subalgebras of \(M\).

Then the following are equivalent:

1. There exist non-zero projections \(p \in P, q \in Q\), a \(*\)-homomorphism \(\theta : pPp \to qQq\) and a non-zero partial isometry \(v \in qQqPp\) such that \(\theta(x)v = vx\), for all \(x \in pPp\).
2. There is no sequence \(u_n \in \mathcal{U}(P)\) satisfying \(\| E_Q(xu_ny) \|_2 \to 0\), for all \(x, y \in M\).

If one of the two equivalent conditions in the theorem above holds then we say that a corner of \(P\) embeds into \(Q\) inside \(M\), and write \(P \preceq_M Q\). If in addition we have that \(Pp' \preceq_M Q\), for any non-zero projection \(p' \in P' \cap 1_P M 1_P\), then we write \(P \preceq^s_M Q\).

2.2. Finite index inclusions of tracial von Neumann algebras. If \(P \subseteq M\) are II\(_1\) factors, then the Jones index of the inclusion \(P \subseteq M\), denoted \([M : P]\), is the dimension of \(L^2(M)\) as a left \(P\)-module. M. Pimsner and S. Popa showed that the number \([M : P]\) can be interpreted as the best constant appearing in several inequalities involving the conditional expectation \(E_P\) [PP86, Theorem 2.2]. It also follows from their work that these constants can be used to define a “probabilistic” index of any inclusion of tracial von Neumann algebras [PP86, Remark 2.4].

Definition 2.2 (Pimsner & Popa, [PP86]). Let \((M, \tau)\) be a tracial von Neumann algebra with a von Neumann subalgebra \(P\). Let

\[
\lambda = \inf \{ \| E_P(x) \|^2/\| x \|^2 : x \in M_+ \}.
\]

The index of the inclusion \(P \subseteq M\) is defined as \([M : P] = \lambda^{-1}\), under the convention that \(\frac{1}{\infty} = \infty\).

For further use we note the following basic facts:

Lemma 2.3. [PP86, Lemma 2.3] Let \((M, \tau)\) be a tracial von Neumann algebra and \(P \subseteq M\) be a von Neumann subalgebra of finite index such that \([M : P] < \infty\). Then the following hold:

1. For every projection \(p \in P\) we have \([pM : pP] < \infty\);
2. \(M \preceq^s_M P\).

As explained in [CIK13], it turns out that this precise notion of index is a well suited technical tool to study global decomposition properties for von Neumann algebras, up to intertwining. For instance, it enables one to show the following version of [CIK13, Proposition 3.6] involving commuting subalgebras rather than masa’s. Its proof is similar to the one presented in [CIK13] but we include all details for reader’s convenience.

Proposition 2.4. Let \((M, \tau)\) be a tracial von Neumann algebra and let \(z \in M\) be a non-zero projection. Assume that \(P \subseteq zMz\) and \(N \subseteq M\) are von Neumann subalgebras such that \(P \vee (N' \cap zMz) \subseteq zMz\) has finite index and that \(P \preceq_M N\).

Then one can find a scalar \(s > 0\), non-zero projections \(r \in N, p \in P\), a subalgebra \(P_0 \subseteq rNr\), and a \(*\)-isomorphism \(\theta : pPp \to P_0\) such that the following properties are satisfied:

1. \(P_0 \vee (P_0' \cap rNr) \subseteq rNr\) has finite index;
2. there exist a non-zero partial isometry \(v \in M\) such that \(rE_N(v^*v) = E_N(v^*v)r \geq sr\) and \(\theta(pPp)v = P_0v = rvPpPp\);
3. \(E_N(v(pPpP \cap pM)p^*)^\perp \subseteq P_0' \cap rNr\).
Proof. Since \( P \preceq_{\operatorname{M}} N \), one can find nonzero projections \( p \in P \) and \( q \in N \), a nonzero partial isometry \( v \in M_{z} \), and a \( * \)-homomorphism \( \theta : pPp \to qNq \) such that \( \theta(x)v = vx \), for all \( x \in pPp \). Notice that \( v^*v \in pPp' \cap pMp \) and \( q' := vv^* \in \theta(pPp') \cap qMq \). Moreover, without any loss of generality, we can assume that the support projection of \( E_N(q') \) equals \( q \). Observe that for every \( x \in pPp \) and \( y \in pPp' \cap pMp \) and we have
\[
\theta(x)vv^* = \theta(x)vv^*vv^* = vv^*vv^* = vv^*vxv^* = \theta(x)vvv^* = vvv^*\theta(x).
\]
Thus we have \( v(pPp' \cap pMp)v^* \subseteq \theta(pPp') \cap qMq \) and hence
\[
E_N(v(pPp' \cap pMp)v^*) \subseteq \theta(pPp') \cap qNq.
\]
Since the inclusion \( P \vee (P' \cap zM_{z}) \subseteq zM_{z} \) has finite index then also \( pPp \vee (pPp' \cap pMp) = p(P \vee (P' \cap M)p) \subseteq pMp \) has finite index and hence \( v(pPp \vee (pPp' \cap pMp))v^* \subseteq \nu pMpv^* = q'Mq' \) has finite index too.

For every \( s > 0 \) we denote by \( q_s = 1_{s,\infty}(E_N(q')) \) and notice that \( ||q_s - q|| \to 0 \), as \( s \to 0 \) and \( rE_N(q') = E_N(q')r \geq sr \). This further implies that \( ||q_s v - v|| \to 0 \), as \( s \to 0 \); in particular, we can pick \( s > 0 \) such that \( q_s v \neq 0 \). Applying [CIK13, Lemma 2.3] (see also [Io11, Lemma 1.6(1)]) it follows that the inclusion
\[
E_n(v(pPp \vee (pPp' \cap pMp))v^*)''q_s \subseteq q_s Nq_s
\]
has finite index. Since \( v^*v \in pPp' \cap pMp \) then \( v(pPp \vee (pPp' \cap pMp))v^* \subseteq v(pPp)v^* \vee v(pPp' \cap pMp)v^* = \theta(pPp)v^* \vee v(pPp' \cap pMp)v^* \). This further implies
\[
E_n(v(pPp \vee (pPp' \cap pMp))v^*)'' \subseteq E_n(\theta(pPp)v^* \vee v(pPp' \cap pMp)v^*)'' \subseteq \theta(pPp) \vee E_n(v(pPp' \cap pMp)v^*)''.
\]
This last containment together with relation (2.2) give that the inclusion \( \theta(pPp)q_s \vee q_s E_n(v(pPp' \cap pMp)v^*)''q_s \subseteq q_s Nq_s \), has finite index. Then the statement follows from this and (2.1) by letting \( r := q_s , \) and \( P_0 := \theta(pPp)q_s \subseteq q_s Nq_s \).

2.3. Dichotomy for normalizers inside crossed products. Recently, Sorin Popa and Stefaan Vaes obtained a series of ground breaking results regarding the classification of normalizers of algebras inside crossed products arising from large families of groups including free groups [PV11, Theorem 1.6] or hyperbolic groups [PV12, Theorem 1.4]. Motivated by these results and by the remarkable subsequent developments in the realm of amalgamated free products due to Adrian Ioana [Io12] (see also [Va13]) we will introduce a new class of groups. However to be able to state it properly we need one more definition.

Definition 2.5. [OP07, Section 2.2] Let \(( M, \tau )\) be a tracial von Neumann algebra, \( p \in M \) a projection, and \( P \subseteq pMp, Q \subseteq M \) von Neumann subalgebras. We say that \( P \) is amenable relative to \( Q \) inside \( M \) if there exists a \( P \)-central state \( \phi : p\langle M, e_Q \rangle p \to \mathbb{C} \) such that \( \phi(x) = \tau(x) \), for all \( x \in pMp \).

Definition 2.6. A group \( \Gamma \) belongs to class \( \mathcal{C}_{\text{rel}}(\Sigma) \) if it is exact [KW99] and there exists a malnormal, proper subgroup \( \Sigma < \Gamma \) for which the following dichotomy property holds:
Assume $\Gamma \curvearrowleft B$ is any trace preserving action on a tracial von Neumann algebra $(B, \tau)$ and denote by $M = B \rtimes \Gamma$. Let $p \in M$ be a projection and $A \subset \text{pM}_p$ a von Neumann subalgebra that is amenable relative to $B \rtimes \Sigma$ inside $M$. Then either $A \lessapprox_M B \rtimes \Sigma$ or $p := \text{NpM}_p(A)^{''}$ is amenable relative to $B \rtimes \Sigma$ inside $M$.

Summarizing the results described above, the following classes of groups belong to $\mathcal{C}_{rss}(\Sigma)$:

1. [PV11, Theorem 3.1, Lemma 4.1, and Theorem 7.1] Any weakly amenable group with positive first $\ell^2$-Betti number—here $\Sigma = \langle e \rangle$;
2. [PV12, Theorem 3.1] Any weakly amenable, non-amen-able, bi-exact group—here $\Sigma = \langle e \rangle$;
3. [Io12, Theorem 1.6], [Va13, Theorem A] Any non-amen-able, nontrivial free product $\Gamma = \Gamma_1 \ast \Lambda \Gamma_2$, where $\Gamma_i$ are exact and $\Lambda < \Gamma_i$ is assumed malnormal—here $\Sigma = \Lambda$.

From now on, whenever $\Sigma = \langle e \rangle$ then the class $\mathcal{C}_{rss}(\Sigma)$ will be simply denoted by $\mathcal{C}_{rss}$. In the same spirit as in [VV14] one can establish that the class $\mathcal{C}_{rss}$ is closed under commensurability.

3. Class Quot($\mathcal{C}$)

Let $\mathcal{C}$ be a class of groups. We define $\text{Quot}_1(\mathcal{C}) = \mathcal{C}$. Given an integer $n \geq 2$, we say that a group $\Gamma$ belongs to the class $\text{Quot}_n(\mathcal{C})$ if there exist:

1. a collection of groups $\Gamma_k$, $1 \leq k \leq n$, such that $\Gamma$ and $\Gamma_n$ are commensurable and $\Gamma_1 \in \mathcal{C}$, and
2. a collection of surjective homomorphisms $\pi_k : \Gamma_k \rightarrow \Gamma_{k-1}$ such that $\ker(\pi_k) \in \mathcal{C}$, for all $2 \leq k \leq n$.

**Definition 3.1.** We denote by $\text{Quot}(\mathcal{C}) := \bigcup_{n \in \mathbb{N}} \text{Quot}_n(\mathcal{C})$ and any group $\Gamma \in \text{Quot}(\mathcal{C})$ is called a finite-step extension of groups in $\mathcal{C}$.

Below we list some useful basic algebraic properties of this family of groups. We will omit most of the proofs as they are either straightforward or already contained in [CIK13, Lemmas 2.9-2.10].

**Proposition 3.2.** The following properties hold:

1. If $\Gamma \in \text{Quot}_1(\mathcal{C})$ and $p : \Lambda \rightarrow \Gamma$ is a surjective homomorphism such that $\ker(p) \in \mathcal{C}$ then $\Lambda \in \text{Quot}_{n+1}(\mathcal{C})$.
2. If $\Gamma_i \in \text{Quot}_{n_i}(\mathcal{C})$ for all $1 \leq i \leq k$ then $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k \in \text{Quot}_{n_1+n_2+\cdots+n_k}(\mathcal{C})$.
3. If $\mathcal{C}$ is closed under commensurability then $\text{Quot}(\mathcal{C})$ is also closed under commensurability.
4. Let $\Gamma \in \text{Quot}_n(\mathcal{C})$ and let $\pi_k : \Gamma_k \rightarrow \Gamma_{k-1}$ be the surjective homomorphisms satisfying the previous definition and let $p_n = \pi_2 \circ \pi_3 \circ \cdots \circ \pi_n : \Gamma_n \rightarrow \Gamma_1$. Then the following hold:
   a. If $\Lambda < \Gamma$ is a subgroup such that $p_n(\Lambda) \in \mathcal{C}$ then $\Lambda \in \text{Quot}_n(\mathcal{C})$;
   b. $\ker(p_n) \in \text{Quot}_{n-1}(\mathcal{C})$; moreover, if $\mathcal{C}$ is closed under commensurability and $\Lambda < \Gamma$ is a subgroup such that $p_n(\Lambda)$ is finite then $\Lambda \in \text{Quot}_{n-1}(\mathcal{C})$. 
(5) If \( C \) is closed under commensurability up to finite kernel then Quot\(_n(C)\) is also closed under commensurability up to finite kernel.

(6) If all the groups in \( C \) are exact then so are all the groups in Quot\(_n(C)\), [DL14].

Proof. We will show only (5). Assume that \( \Lambda_1 \) is commensurable up to finite kernel with \( \Lambda_2 \) and \( \Lambda_1 \in \text{Quot}_n(C) \). Thus one can find finite index subgroups \( \Sigma_i \triangleleft \Lambda_i, \) a group \( H, \) and surjective homomorphisms \( \phi_i : \Sigma_i \to H \) with finite kernels \( \Omega_i := \ker(\phi_i) \). Denote by \( \hat{\phi}_i : \Sigma_i/\Omega_i \to H \) the induced isomorphisms. Since \( \Lambda_1 \in \text{Quot}_n(C) \), \( \Sigma_1 \triangleleft \Lambda_1 \) is finite index, and \( C \) is closed under taking finite index subgroups, it follows that \( \Sigma_1 \in \text{Quot}_n(C) \). Thus there exist surjections \( \pi_k : \Gamma_k \to \Gamma_{k-1} \) such that \( \ker(\pi_k) \in C \) for every \( 2 \leq k \leq n \), \( \Gamma_1 \in C \), and \( \Sigma_1 = \Gamma_n \). Let \( \Theta_n := \Omega_1 \) and then for every \( 2 \leq k \leq n \) define recursively \( \Theta_k := \pi_k(\Theta_{k-1}) \subset \Gamma_{k-1} \); since \( \pi_k \)'s are surjections then \( \Theta_{k-1} \) is a finite normal subgroup of \( \Gamma_{k-1} \). Moreover, for each \( 2 \leq k \leq n \) the map \( \tilde{\pi}_k : \Gamma_k/\Theta_k \to \Gamma_{k-1}/\Theta_{k-1} \) given by \( \tilde{\pi}_k(x\Theta_k) = \pi_k(x)\Theta_{k-1} \) is a surjective homomorphism and \( \ker(\tilde{\pi}_k) = \ker(\pi_k)/\Theta_k \). By the isomorphism theorem we have \( \ker(\tilde{\pi}_k) \cong \ker(\pi_k)/(\ker(\pi_k) \cap \Theta_k) \) and since \( \Theta_k \) is finite and \( C \) is closed under commensurability up to finite kernel it follows that \( \ker(\tilde{\pi}_k) \in C \), for all \( 2 \leq k \leq n \) and \( \Gamma_1/\Theta_1 \in C \). If \( p : \Sigma_2 \to \Sigma_2/\Omega_2 \) denotes the canonical projection then the formulas \( \tilde{\pi}_k' := \pi_n \circ \hat{\phi}_2^{-1} \circ \hat{\phi}_1 \circ p : \Sigma_2 \to \Gamma_{n-1}/\Theta_{n-1} \) and \( \tilde{\pi}_k' := \tilde{\pi}_k : \Gamma_k/\Theta_k \to \Gamma_{k-1}/\Theta_{k-1} \) for every \( 2 \leq k \leq n \) define surjective homomorphisms. Moreover, since the kernel \( \ker(\tilde{\pi}_n') = p^{-1}(\hat{\phi}_2^{-1}(\ker(\pi_n))) \) satisfies \( \ker(\tilde{\pi}_n')/\Omega_2 \cong \ker(\pi_n) \) and \( \Omega_2 \) is finite we have \( \ker(\tilde{\pi}_n') \in C \). By above we have \( \ker(\tilde{\pi}_n') \in C \), for all \( 2 \leq k \leq n-1 \), and \( \Gamma_1/\Theta_1 \in C \) which further imply that \( \Sigma_2 \in \text{Quot}_n(C) \). Finally, since from construction \( \Sigma_2 \) is a finite index subgroup of \( \Lambda_2 \) we conclude that \( \Lambda_2 \in \text{Quot}_n(C) \).

3.1. Relative hyperbolic groups. Let \( \Gamma \) be a group that is hyperbolic relative to \( \mathcal{P} = \{P_i : i \in I\} \) a finite family of infinite, proper, residually finite subgroups. Using very deep methods in geometric group theory Denis Osin was able to prove a powerful algebraic Dehn filling analog for this class of groups. As a consequence, it was shown in [Os06, Theorem 1.1] that there exist a non-elementary hyperbolic group \( H \), a surjective homomorphism \( \psi : \Gamma \to H \), and finite index, normal subgroups \( N_i \triangleleft P_i \) such that \( \ker(\psi) \) is the normal closure of \( \cup_i N_i \) in \( \Gamma \), i.e., \( \ker(\psi) = \langle \cup_i N_i \rangle^\Gamma \). More recently, using the concept of very rotating family of subgroups, Dahmani, Guirardel, and Osin [DGO11, Theorem 7.9] were able to describe more concretely the structure of \( \ker(\psi) \), as a nontrivial free product. Precisely, they showed that there exist a family of nonempty subsets \( T_i \subset \Gamma \) such that \( \ker(\psi) = \ast_{i \in I}(\ast_{y \in T_i} N_i^y) \), where \( N_i^y = \gamma N_i \gamma^{-1} \), (see also Osin’s argument in [CIK13, Corollary 5.1]). Summarizing we have the following:

Theorem 3.3 ([Os06, DGO11]). Let \( \mathcal{H} \) be the class consisting of all non-elementary hyperbolic groups and all non-amenable, non-trivial free product of exact groups. Then for any group \( \Gamma \) that is hyperbolic relative to a finite family of residually finite, exact, infinite, proper subgroups we have \( \Gamma \in \text{Quot}_2(\mathcal{H}) \).

3.2. Mapping class groups. Let \( S_{g,k} \) be a connected, compact and orientable surface of genus \( g \) with \( k \) boundary components. Throughout the paper, we assume that a surface satisfies these conditions unless otherwise mentioned. Denote by \( \text{Mod}(S_{g,k}) \) the mapping class group of \( S_{g,k} \), i.e., the group of isotopy classes of orientation-preserving
homeomorphisms from $S_{g,k}$ onto itself, where isotopy may move points of the boundary of $S_{g,k}$. Using the Birman short exact sequence in combination with the earlier results of Birman and Hilden [BH73] it was shown in [CIK13, Section 4.3] the following result.

**Theorem 3.4.** If $\mathcal{F}$ denotes the class of all groups commensurable with non-abelian free groups then we have the following:

(i) If $g = 0$, $k \geq 4$ then $\text{Mod}(S_{g,k}) \in \text{Quot}_{k-3}(\mathcal{F})$;

(ii) If $g = 1$, $k \geq 1$ then $\text{Mod}(S_{g,k}) \in \text{Quot}_k(\mathcal{F})$;

(iii) If $g = 2$, $k \geq 0$ then $\text{Mod}(S_{g,k}) \in \text{Quot}_{k+3}(\mathcal{F})$.

It follows from [FM11] that, for every positive integer $k$, the central quotient of the braid group with $k$ strands $B_k$ can be canonically identified with a finite index subgroup of the mapping class group $\text{Mod}(S_{g,k+1})$. Moreover, notice that the central quotient of the pure braid group with $k$ strands $P_k$ is a normal subgroup of index $k!$ in $B_k$. Combining with the above theorem we have

**Corollary 3.5.** For every $k \geq 3$, we have that $\tilde{B}_k, \tilde{P}_k \in \text{Quot}_{k-2}(\mathcal{F})$.

### 3.3. Surface braid groups

Throughout this subsection, we let $M = S_{g,b}$ be a surface. Consider $\text{PMod}(M)$ the pure mapping class group of $M$, i.e., the subgroup of $\text{Mod}(M)$ that consists of isotopy classes of orientation-preserving homeomorphisms from $M$ onto itself which also preserve each component of $\partial M$ as a set. Then $\text{PMod}(M)$ is a subgroup in $\text{Mod}(M)$ of index $b!$.

Fix $k$ a positive integer. Denote by $F_k(M)$ the space of ordered $k$ distinct points of $M$ and let $\text{PB}_k(M)$ be the fundamental group of $F_k(M)$. The group $\text{PB}_k(M)$ is called the pure braid group of $k$ strands on $M$. From definitions we have the equality $\text{PB}_1(M) = \pi_1(M)$. For the basic properties of the group $\text{PB}_k(M)$ we refer to [B69a, B69b, B74, PR99].

We fix $x_1, \ldots, x_k$, mutually distinct points of $M$. By [FaNe62, Theorem 3], the map from $F_{k+1}(M)$ into $F_k(M)$ sending a point $(t_1, \ldots, t_{k+1})$ of $F_{k+1}(M)$ to $(t_1, \ldots, t_k)$ is a locally trivial fibration which has the fiber $F_1(M \setminus \{x_1, \ldots, x_k\})$. Thus, we have the following exact sequence

$$\cdots \to \pi_2(F_{k+1}(M)) \to \pi_2(F_k(M)) \to \pi_1(M \setminus \{x_1, \ldots, x_k\}) \to \text{PB}_{k+1}(M) \to \text{PB}_k(M) \to 1.$$  

If $(g, b) \neq (0, 0)$, then by [FaNe62, Corollary 2.2] we have $\pi_2(F_1(M)) = 0$, for any positive integer $l$, and hence (3.1) gives the following short exact sequence

$$1 \to \pi_1(M \setminus \{x_1, \ldots, x_k\}) \to \text{PB}_{k+1}(M) \to \text{PB}_k(M) \to 1.$$  

Following the terminology from [PR99], we say that $M$ is large if the group $\pi_1(M)$ is non-elementarily hyperbolic. This is the case if and only if $(g, b) \neq (0, 0), (0, 1), (0, 2), (1, 0)$. Let $S = S_{g,b+k}$ be a surface and choose $k$ components of $\partial S$. We suppose that $M$ is obtained by filling a disk to each of these $k$ components of $\partial S$. Thus we get the following Birman exact sequence

$$\text{PB}_k(M) \xrightarrow{\psi} \text{PMod}(S) \to \text{PMod}(M) \to 1.$$  

corresponding to the canonical embedding of $S$ into $M$ ([B69b, Theorem 1], [FM11, Theorem 9.1] and [Iv02, Theorem 2.8.C]). Denote by $Z_k(M)$ the center of $\text{PB}_k(M)$ and notice that by [B69b, Corollary 1.2] we have $\ker j < Z_k(M)$. Denote by $\text{PB}_k(M) := \text{PB}_k(M)/Z_k(M)$. When $M$ is large then [PR99, Proposition 1.6] implies that the center $Z_k(M)$ is trivial and hence $j$ is an injective homomorphism and also $\text{PB}_k(M) = \text{PB}_k(M)$.

**Remark 3.6.** For any $M$ non-large, we will argue below that $\ker j = Z_k(M)$; in particular, the group $\tilde{\text{PB}}_k(M)$ can be identified with a normal subgroup of $\text{PMod}(S)$ through $j$.

To justify our claim notice that when $g = 0$ and $b \leq 2$ the group $\text{PMod}(M)$ is trivial. Then using the exact sequence (3.3) we have that $\text{PB}_k(M)/\ker j \simeq \text{PMod}(S)$. Since by [FM11, Section 3.4] the center of $\text{PMod}(S)$ is trivial, this further implies that $\ker j = Z_k(M)$; in this case we also get that $\tilde{\text{PB}}_k(M) \simeq \text{PMod}(S)$.

Now assume that $g = 1$ and $b = 0$. In [B69b, Corollary 1.3], two generators of $\ker j$ were described through the presentation of $\text{PB}_k(M)$ in [B69a, Theorem 5], and it was shown that $\ker j$ is isomorphic to $\mathbb{Z}^2$. Combining this with [PR99, Proposition 4.2], we obtain the equality $\ker j = Z_k(M)$.

**Theorem 3.7.** For every positive integer $k$, the following assertions hold:

(i) If $M$ is large and $b \geq 1$, then $\text{PB}_k(M) \in \text{Quot}_k(\mathcal{F})$;

(ii) If $M$ is large and $b = 0$, then $\text{PB}_k(M) \in \text{Quot}_k(\mathcal{F})$;

(iii) If $g = 0, b \leq 2$ and $b + k \geq 4$, then $\text{PB}_k(M) \in \text{Quot}_{b+k−3}(\mathcal{F})$;

(iv) If $g = 1, b = 0$ and $k \geq 2$, then $\text{PB}_k(M) \in \text{Quot}_{k−1}(\mathcal{F})$.

**Proof.** Assertions (i) and (ii) follow inductively from the short exact sequence (3.2) and the equality $\text{PB}_1(M) = \pi_1(M)$.

If $g = 0, b \leq 2$, and $b + k \geq 4$ then using the Remark 3.6 above we have that $\tilde{\text{PB}}_k(M) \simeq \text{PMod}(S)$. Thus, assertion (iii) follows from Theorem 3.4 (i).

Suppose that $g = 1, b = 0$ and $k \geq 2$. Let $\rho: \text{PB}_{k+1}(M) \to \text{PB}_k(M)$ be the surjection from the exact sequence (3.2). As mentioned in Remark 3.6, two generators of $Z_k(M)$ are described in [B69b, Corollary 1.3] and [PR99, Proposition 4.2]. The homomorphism $\rho$ is induced by the map from $F_{k+1}(M)$ into $F_k(M)$ which sends a point $(t_1, \ldots, t_{k+1})$ of $F_{k+1}(M)$ to $(t_1, \ldots, t_k)$. It follows from the description of generators of $Z_k(M)$ that $\rho(Z_{k+1}(M)) = Z_k(M)$ which further gives rise to a surjection $\tilde{\rho}: \tilde{\text{PB}}_{k+1}(M) \to \tilde{\text{PB}}_k(M)$.

Since $\pi_1(M \setminus \{x_1, \ldots, x_k\})$ is isomorphic to the free group of rank $k+1$ and its center is trivial, then (3.2) induces in a canonical way a short exact sequence

\[
(3.4) \quad 1 \to \pi_1(M \setminus \{x_1, \ldots, x_k\}) \to \tilde{\text{PB}}_{k+1}(M) \to \tilde{\text{PB}}_k(M) \to 1.
\]

Since $\text{PB}_1(M) = \pi_1(M) \simeq \mathbb{Z}^2$ then the group $\tilde{\text{PB}}_1(M)$ is trivial. Hence, assertion (iv) follows from (3.4), by induction on $k$. \[\square\]

**Remark 3.8.** Next we briefly discuss the exceptional cases for $\text{PB}_k(M)$ not covered by Theorem 3.7. If $(g, b) = (0, 0)$ and $k \leq 3$, then the first theorem in [FV62, Section VI.2] implies that $\text{PB}_k(M)$ is finite. If $(g, b) = (0, 1)$, then $\text{PB}_1(M)$ is trivial, and $\text{PB}_2(M)$ is the Artin pure braid group of two strands and thus is isomorphic to $\mathbb{Z}$ ([FM11, Section 9.3]). If $(g, b) = (0, 2)$, then $\text{PB}_2(M) \simeq \mathbb{Z}$. If $(g, b) = (1, 0)$, then $\text{PB}_1(M) \simeq \mathbb{Z}^2$. 


3.4. The Torelli group and the Johnson kernel. Let \( S = S_{g,k} \) be a surface. A simple closed curve in \( S \) is called *essential* in \( S \) if it is neither homotopic to a single point of \( S \) nor isotopic to a component of \( \partial S \). When there is no risk of confusion, by a curve in \( S \) we mean either an essential simple closed curve in \( S \) or its isotopy class. A curve \( \alpha \) in \( S \) is called *separating* in \( S \) if \( S \setminus \alpha \) is not connected. Otherwise \( \alpha \) is called *non-separating* in \( S \). Whether \( \alpha \) is separating in \( S \) or not depends only on the isotopy class of \( \alpha \). A pair of non-separating curves in \( S \), \( \{ \beta, \gamma \} \), is called a *bounding pair* (BP) in \( S \) if \( \beta \) and \( \gamma \) are disjoint and non-isotopic and \( S \setminus \{ \beta \cup \gamma \} \) is not connected. This condition depends only on the isotopy classes of \( \beta \) and \( \gamma \). Given a curve \( \alpha \) in \( S \), we denote by \( t_\alpha \in \text{PMod}(S) \) the *Dehn twist* about \( \alpha \).

We define the *Torelli group* \( \mathcal{J}(S) \) to be the group generated by all elements of the form \( t_\alpha \) and \( t_\beta t_\gamma^{-1} \) with \( \alpha \) a separating curve in \( S \) and \( \{ \beta, \gamma \} \) a BP in \( S \). We define the *Johnson kernel* \( \mathcal{K}(S) \) as the group generated by all \( t_\alpha \) with \( \alpha \) a separating curve in \( S \). We refer the reader to [FM11, Chapter 6] for more background of these groups.

When \( g \geq 2 \) and \( k \leq 1 \), the Torelli group of \( S \) is originally defined as the group of elements of \( \text{Mod}(S) \) acting on \( H_1(S, \mathbb{Z}) \) trivially. Using the results in [Jo79] and [Pow78], this original group turns out to be equal to the group \( \mathcal{J}(S) \) defined above. When \( g = 0 \), any curve in \( S \) is separating in \( S \), and therefore we have \( \mathcal{J}(S) = \mathcal{K}(S) = \text{PMod}(S) \).

**Theorem 3.9.** The following assertions hold:

(i) If \( g = 1 \) and \( k \geq 2 \), then \( \mathcal{J}(S), \mathcal{K}(S) \in \text{Quot}_{k-1}(\mathcal{F}) \);

(ii) If \( g = 2 \) and \( k \geq 0 \), then \( \mathcal{J}(S), \mathcal{K}(S) \in \text{Quot}_{k+1}(\mathcal{F}) \).

**Proof.** If \( g = 2 \) and \( k = 0 \), then the equality \( \mathcal{J}(S) = \mathcal{K}(S) \) holds, and it is further isomorphic to the free group of infinite rank by results in [Me92, BBM10]. The theorem for this case follows.

Suppose that either \( g = 1 \) and \( k \geq 2 \) or \( g = 2 \) and \( k \geq 1 \). Choose a component of \( \partial S \), and denote by \( R \) the surface obtained by filling up a disk in this chosen component of \( \partial S \). Then by [FM11, Theorem 4.6] we have the Birman exact sequence

\[
1 \rightarrow \pi_1(R) \rightarrow \mathcal{J}(S) \rightarrow \mathcal{K}(S) \rightarrow \text{PMod}(R) \rightarrow 1
\]

(3.5)

corresponding to the canonical embedding of \( S \) into \( R \). By [Iv02, Lemma 4.1.I], through the injection \( j \), each of standard generators of \( \pi_1(R) \) induces either the element \( t_\alpha \) with \( \alpha \) a separating curve in \( S \), its inverse, or the element \( t_\beta t_\gamma^{-1} \) with \( \{ \beta, \gamma \} \) a BP in \( S \). It follows that \( j(\pi_1(R)) < \mathcal{J}(S) \). Also, by the definition of \( \mathcal{J}(S) \) we have \( q(\mathcal{J}(S)) = \mathcal{J}(R) \). Combining these with (3.5) we obtain the following short exact sequence

\[
1 \rightarrow \pi_1(R) \rightarrow \mathcal{J}(S) \rightarrow \mathcal{K}(S) \rightarrow \mathcal{K}(R) \rightarrow 1.
\]

(3.6)

If \( g = 1 \) and \( k = 2 \), then \( \mathcal{J}(R) \) is trivial because there exist neither a separating curve in \( R \) nor a BP in \( R \). Hence, when \( g = 1 \) and \( k \geq 2 \) we get \( \mathcal{J}(S) \in \text{Quot}_{k-1}(\mathcal{F}) \), by using (3.6) and induction on \( k \). Similarly, when \( g = 2 \) and \( k \geq 0 \) we have \( \mathcal{J}(S) \in \text{Quot}_{k+1}(\mathcal{F}) \), by induction on \( k \), starting from the result in the first paragraph of the proof.

Suppose again that either \( g = 1 \) and \( k \geq 2 \) or \( g = 2 \) and \( k \geq 1 \). Restricting the the short exact sequence (3.5), we obtain the following short exact sequence

\[
1 \rightarrow j(\pi_1(R)) \cap \mathcal{K}(S) \rightarrow \mathcal{K}(S) \rightarrow \mathcal{K}(R) \rightarrow 1.
\]

(3.7)
Denote by \( N = j(\pi_1(\mathbb{R})) \cap \mathcal{K}(S) \). Through the injection \( j \), any simple loop in \( \mathbb{R} \) surrounding exactly one component of \( \partial R \) and cutting a cylinder from \( \mathbb{R} \) induces either the element \( t_\alpha \) with \( \alpha \) a separating curve in \( S \) or its inverse; in particular, it follows that \( N \) is infinite. Also, relying on [Jo80] it was proved in [Ki09, Proposition 2.2] that for any free \( \mathbb{R} \} \) in \( S \), no non-zero power of \( t_\beta t_\gamma^{-1} \) lies in \( \mathcal{K}(S) \). This further implies that \( N \) has infinite index in \( j(\pi_1(\mathbb{R})) \) and hence, by [DD89, Theorem V.12.5], \( N \) is a free group of infinite rank. Finally, we obtain the conclusion of the theorem about \( \mathcal{K}(S) \), by using (3.7) and induction on \( k \).

\[ \square \]

4. Class NC\(_1\)

Let \( \Gamma \) be a countable discrete group and let \( \pi : \Gamma \to O(\mathcal{H}) \) be an orthogonal representation. A map \( q : \Gamma \to \mathcal{H} \) is called a quasi-cocycle for \( \pi \) if there exists a constant \( C \geq 0 \) such that

\[
(4.1) \quad \sup_{\gamma, \lambda \in \Gamma} \| q(\gamma \lambda) - q(\gamma) - \pi_\gamma(q(\lambda)) \| \leq C.
\]

The infimum of all constants \( C \) satisfying equation (4.1) is called the defect of quasi-cocycle \( q \) and is denoted by \( D(q) \). When the defect vanishes the quasi-cocycle \( q \) is actually a 1-cocycle with coefficients in \( \pi \), [BHV05]. Throughout the paper we will assume, without any loss of generality, that any quasi-cocycle is anti-symmetric, i.e., \( q(\gamma) = -\pi_\gamma(q(\gamma)) \), for all \( \gamma \in \Gamma \). We can make this assumption because in fact every quasi-cocycle is within a bounded distance from an anti-symmetric quasi-cocycle, [Tho09]. The set of all unbounded, anti-symmetric quasi-cocycles for \( \pi \) will be denoted by \( \mathcal{Q}^1_{\pi}(\Gamma, \pi) \).

If \( \mathcal{B}_{as}(\Gamma, \pi) \) denotes the set of all bounded, anti-symmetric maps \( b : \Gamma \to \mathcal{H} \) then we obviously have that \( \mathcal{Q}^1_{\pi}(\Gamma, \pi) + \mathcal{B}_{as}(\Gamma, \pi) = \mathcal{Q}^1_{\pi}(\Gamma, \pi) \) and \( (\mathbb{R} \setminus \{0\}) \cdot \mathcal{Q}^1_{\pi}(\Gamma, \pi) = \mathcal{Q}^1_{\pi}(\Gamma, \pi) \).

**Definition 4.1.** A orthogonal representation \( \pi : \Gamma \to O(\mathcal{H}) \) is called mixing if for every \( \xi, \eta \in \mathcal{H} \) we have \( \lim_{\gamma \to \infty} \langle \pi_\gamma(\xi), \eta \rangle = 0 \).

Basic examples are any multiple of the (real) left regular representation, \( \oplus \ell^2(\Gamma) \). The mixing property is preserved under many basic operations on representations including: taking sub-representations, restrictions to infinite subgroups, direct sums, tensor products, and inductions to finite index supra-groups.

**Definition 4.2.** A representation \( \pi : \Gamma \to O(\mathcal{H}) \) is called weakly-\( \ell^2 \) if it is weakly contained in the left regular representation \( \ell^2(\Gamma) \).

It follows from the definitions that any restriction of a weakly-\( \ell^2 \) representation to any of its subgroups is again weakly-\( \ell^2 \). Moreover, the weakly-\( \ell^2 \) property is preserved under direct sum and under induction to supragroups (this follows from a similar proof with [BHV05, Theorem F.3.5]). Thus, if \( \Gamma \) is a group and \( \{ \Sigma_i \} \) is a countable family of amenable subgroups then by above it follows that the multiple \( \oplus_i \ell^2(\Gamma/\Sigma_i) \) is weakly-\( \ell^2 \).

**Definition 4.3** (Notation 0.1 in [CSU13]). We say that a group \( \Gamma \) belongs to class NC\(_1\) if it is non-amenable and there exists a weakly-\( \ell^2 \), mixing, orthogonal representation \( \pi : \Gamma \to O(\mathcal{H}) \) such that \( \mathcal{Q}^1_{\pi}(\Gamma, \pi) \neq \emptyset \).
Notice that this class includes the class $D_{\text{reg}}$ introduced by Thom in [Tho09]. In the remaining part of this subsection we underline some basic properties of the class NC$_1$. Some of these have been already discussed in [CSU13, Section 1] but we will include them here to make the text more self contained. As we will see this class is quite rich, containing large families of groups which are intensively studied in various areas of mathematics, especially topology, geometric group theory, or logic. In addition, we study permanence properties of NC$_1$ under various canonical constructions in group theory. Many of these properties are either folklore or already appeared in the literature so many of their proofs will be skipped.

**Proposition 4.4.** The following properties hold:

a) If $\Sigma < \Gamma_1, \Gamma_2$ are groups with $\Sigma$ finite and $[\Gamma_1 : \Sigma] \geq 2, [\Gamma_2 : \Sigma] \geq 3$ then $\Gamma_1 \ast_{\Sigma} \Gamma_2 \in \text{NC}_1, [\text{PV09, CP10}];$

b) Given $\Sigma < \Gamma$ groups with $\Sigma$ nontrivial finite, $\Gamma$ infinite and $\Theta : \Sigma \to \Gamma$ is a monomorphism denote by $\text{HNN}(\Gamma, \Sigma, \Theta)$ the corresponding HNN-extension; then $\text{HNN}(\Gamma, \Sigma, \Theta) \in \text{NC}_1, [\text{FV10, CP10}];$

c) If a non-amenable group $\Gamma$ acts on a tree with finite stabilizers on edges then $\Gamma \in \text{NC}_1;$

d) The class NC$_1$ is closed under taking non-amenable normal subgroups;

e) If $\Gamma$ is either a direct product of infinite groups or admits a infinite normal amenable subgroup then $\Gamma \not\in \text{NC}_1, [\text{MS04, Po08, Pe06, CS11}].$

**Proof.** We will only very briefly justify d). Let $\Gamma \in \text{NC}_1$ and let $\Sigma \triangleleft \Gamma$ be any non-amenable, normal subgroup. Then there exists a weakly-$\ell^2$, mixing, orthogonal representation $\pi : \Gamma \to O(\mathcal{H})$ and $q \in \mathcal{O}\mathcal{H}_1(\Gamma, \pi)$. From the previous observations it follows that the restriction $\pi_{\Sigma} : \Sigma \to O(\mathcal{H})$ is again weakly-$\ell^2$ and mixing. Moreover, since $\Sigma$ is normal in $\Gamma$ then [CSU13, Theorem 2.1] implies that the restriction $q_{|\Sigma} \in \mathcal{O}\mathcal{H}_1(\Sigma, \pi_{\Sigma})$, and hence $\Sigma \in \text{NC}_1$. $\square$

**Lemma 4.5.** The class NC$_1$ is closed under taking supragroups of finite index.

**Proof.** Let $\Omega \in \text{NC}_1$ and let $\Omega < \Gamma$ be finite index supragroup. By passing to a finite index subgroup of $\Omega$ we can assume without any loss of generality that $\Omega$ is normal in $\Gamma$. Since $\Gamma$ satisfies NC$_1$ there exists a mixing, weakly-$\ell^2$, orthogonal representation $\pi : \Gamma \to O(\mathcal{H})$ and an unbounded quasi-cocycle $q : \Gamma \to \mathcal{H}$. To get our conclusion we will use a similar construction as in the proof of Kaloujnine-Krasner embedding theorem [KK50]. Denote by $\hat{\Gamma} = \Gamma/\Omega = \{\hat{\lambda} = \Omega \lambda : \lambda \in \Gamma\}$ the (finite) quotient group. Fix $\{t_{\lambda} : \lambda \in \Gamma\} \subset \Gamma$ a complete set of representatives for the cosets $\Omega$ in $\Gamma$ and notice that $t_{\lambda} = \hat{\lambda}$, for every $\lambda \in \Gamma$; hence for every $\gamma, \lambda \in \Gamma$, we have $f_{t_{\gamma}}(\hat{\lambda}) := t_{\gamma}^{-1}t_{\lambda}^{-1} \in \Omega$.

Now we define $\tilde{\pi} : \Gamma \to O(\oplus_{\Gamma}\mathcal{H})$ by letting $\tilde{\pi}_{\gamma}(\oplus_{\lambda}\xi_{\lambda}) = \oplus_{\lambda}(\pi_{t_{\gamma}(\hat{\lambda})}(\xi_{\lambda}))$, for every $\gamma \in \Gamma$ and $\oplus_{\lambda}\xi_{\lambda} \in \oplus_{\Gamma}\mathcal{H}$. It is a straightforward exercise to check that $\tilde{\pi}$ is an orthogonal representation.

Next we show that $\tilde{\pi}$ is mixing. To see this fix $\tilde{\xi} = \oplus_{\lambda}\xi_{\lambda}, \tilde{\eta} = \oplus_{\lambda}\eta_{\lambda} \in \oplus_{\Gamma}\mathcal{H}$ and let $\{\gamma_n : n \in \mathbb{N}\} \subseteq \Gamma$ be an infinite sequence. Using the definition of $\tilde{\pi}$ we see that
\[
\|\langle \pi y_n (\xi), \eta \rangle \| = \left| \sum_{\lambda} \langle \pi y_n (\lambda) (\xi_{\lambda y_n}), \eta_{\lambda} \rangle \right| \\
\leq \sum_{\lambda} \left| \langle \pi y_n (\lambda) (\xi_{\lambda y_n}), \eta_{\lambda} \rangle \right| \\
= \sum_{\lambda} \left| \langle \pi y_n (\pi_{-1} (\xi_{\lambda y_n})), \pi_{-1} (\eta_{\lambda}) \rangle \right|
\]

(4.2)

Since \( \{ \pi_{-1} (\xi_{\lambda y_n}) : n \in \mathbb{N}, \lambda \in \Gamma \} \) and \( \{ \pi_{-1} (\eta_{\lambda}) : \lambda \in \Gamma \} \) are finite subsets of \( \mathcal{H} \) and \( \pi \) is mixing it follows that the last quantity in (4.2) converges to 0, as \( n \to \infty \); hence \( \|\langle \pi y_n (\xi), \eta \rangle \| \to 0 \), as \( n \to \infty \), showing that \( \hat{\pi} \) is mixing.

Now we briefly argue that \( \hat{\pi} \) is weakly-\( \ell^2 \). Since \( \pi \) is weakly-\( \ell^2 \) then \( \pi \) can be approximated in Fell’s topology by the left regular representation \( \pi' : \Gamma \to \mathcal{O}(\ell^2_R (\Gamma)) \). Moreover, since \( \hat{\pi} \) is a finite set, then using the definition of \( \hat{\pi} \) one can show that \( \hat{\pi} \) can be approximated in Fell’s topology by the orthogonal representation \( \hat{\lambda} : \Gamma \to \mathcal{O}(\oplus \ell^2_R (\Gamma)) \) given by \( \hat{\lambda}_\gamma (\oplus \xi_{\gamma \lambda}) = \oplus \lambda' (\xi_{\gamma \lambda'}) \), for every \( \gamma \in \Gamma \) and \( \oplus \xi_{\gamma \lambda} \in \oplus \ell^2_R (\Gamma) \). Thus, to derive our conclusion it suffices to show that \( \hat{\lambda} \) is equivalent to a multiple of \( \ell^2_R (\Gamma) \). To see this, for every \( \lambda, \gamma \in \Gamma \), consider the vector \( \xi_{\gamma \lambda} \in \oplus \ell^2_R (\Gamma) \) given by the formula \( (\xi_{\gamma \lambda})_{\mu} = 0 \), if \( \mu \neq \lambda \), and \( (\xi_{\gamma \lambda})_{\lambda} = \Delta_{\gamma} \), if \( \mu = \lambda \); here \( \Delta_{\gamma} \in \ell^2_R (\Gamma) \) is the Dirac function supported on \( \gamma \). Notice that \( \{ \xi_{\gamma \lambda} : \gamma, \lambda \in \Gamma \} \) is an orthonormal basis of \( \oplus \ell^2_R (\Gamma) \), invariant under \( \hat{\pi} \). Also note that \( \bar{\pi}_{\lambda} (\xi_{\gamma \lambda}) = \xi_{\gamma \lambda} \) if and only if \( x = e \). Using these facts one can show that there exist elements \( \gamma, \lambda \in \Gamma \) such that we can decompose the orthogonal representation \( \oplus \ell^2_R (\Gamma) = \oplus \mathcal{H}_i \), with \( \mathcal{H}_i = \text{span}_{\mathbb{C}}(\bar{\pi}_{\lambda} (\xi_{\gamma \lambda}) : x \in \Gamma \} \equiv \ell^2_R (\Gamma) \), which proves our claim.

In the remaining part we show that the map \( \bar{q} : \Gamma \to \oplus \mathcal{H} \) defined by \( \bar{q} (\gamma) = \oplus \xi q (f_\gamma (\hat{\lambda})) \), for every \( \gamma \in \Gamma \), is an unbounded quasi-cocycle, which will conclude the proof. To see this, fix \( \gamma, \delta \in \Gamma \) and using the definitions together with the quasi-cocycle inequality for \( q \) we get

\[
\| \bar{q} (\gamma \delta) - \bar{q} (\gamma) \| = \sum_{\lambda} \| q (f_\gamma (\phi (\hat{\lambda})) - \pi_{\gamma \lambda} (\xi_{\lambda y_n}) q (f_\delta (\lambda y_n))) - q (f_\gamma (\hat{\lambda})) \| \\
= \sum_{\lambda} \| q (f_\gamma (\phi (\lambda y_n)) - \pi_{\gamma \lambda} (\xi_{\lambda y_n}) q (f_\delta (\lambda y_n))) - q (f_\gamma (\hat{\lambda})) \|^2 \\
\leq \sum_{\lambda} D^2 (q) = |\hat{\pi}| D^2 (q).
\]

This computation shows that \( \bar{q} \) is a quasi-cocycle satisfying \( D (\bar{q}) \leq |\hat{\pi}|^{1/2} D (q) \).

Finally, we argue that \( \bar{q} \) is unbounded. First notice that the quasi-cocycle inequality together with the triangle inequality and the antisymmetry, gives that \( \| q (xyz^{-1}) \| \geq \| q (y) \| - \| q (x) \| - \| q (z) \| - 2 D (q) \), for every \( x, y, z \in \Gamma \). Using this in combination with
the definitions and the Cauchy-Schwarz inequality, for every \( \gamma \in \Gamma \), we have

\[
\| \bar{q}(\gamma) \|^2 = \sum_{\lambda} \| q(f_\gamma(t_\lambda)) \|^2 \\
\geq \frac{1}{|\hat{H}|} (\sum_{\lambda} \| q(t_\lambda \gamma t_\lambda^{-1}) \|)^2 \\
\geq \frac{1}{|\hat{H}|} (\sum_{\lambda} \| q(\gamma) \| - \| q(t_\lambda) \| - \| q(t_\lambda^{-1}) \| - 2D(q))^2 \\
\geq |\hat{H}| (\| q(\gamma) \| - 2\max_{\lambda} \| q(t_\lambda) \| - 2D(q))^2.
\]

Since \( q \) is unbounded it follows that \( \bar{q} \) is unbounded too.

Two groups \( \Gamma_1 \) and \( \Gamma_2 \) are called \textit{commensurable up to finite kernel} if there exist finite index subgroups \( \Lambda_i \leq \Gamma_i \), a group \( H \), and surjections \( \psi_i : \Lambda_i \to H \) such that \( \ker(\psi_i) \) are finite. One can easily check that this is the smallest equivalence relation whose classes are closed under taking both finite index subgroups and quotients under normal finite subgroups. The following result parallels similar results for the class of acylindrically hyperbolic groups, [MO13, Lemma 3.8].

\textbf{Theorem 4.6.} The class \( \text{NC}_1 \) is closed under commensurability up to finite kernel.

\textit{Proof.} Assuming \( \Gamma_i \) are groups such that \( \Gamma_1 \in \text{NC}_1 \) and \( \Gamma_2 \) is commensurable up to finite kernel with \( \Gamma_2 \) we need to show that \( \Gamma_2 \in \text{NC}_1 \). From the definition there exist finite index subgroups \( \Lambda_i \leq \Gamma_i \), a group \( H \), and surjections \( \psi_i : \Lambda_i \to H \) such that \( \ker(\psi_i) \) are finite. Since \( \Gamma_1 \in \text{NC}_1 \) and \( \Lambda_1 \leq \Gamma_1 \) is finite index then \( \Lambda_1 \in \text{NC}_1 \) and hence there exist \( \pi : \Lambda_1 \to \mathcal{O}(\mathcal{H}_1) \) a weakly-\( \ell^2 \), mixing, orthogonal representation and \( q \in \Omega \mathcal{H}_1(\Lambda_1, \pi_1) \) an unbounded quasi-cocycle. We see that for every \( \gamma \in H, \xi \in \mathcal{H}_1 \) the formula \( \bar{\pi}_\gamma(\xi) = \omega^{-1} \sum_{\delta \in \psi_1^{-1}(\gamma)} \pi_\delta(\xi) \) where \( \omega := |\ker(\psi_1)| \), defines an orthogonal representation \( \bar{\pi} : H \to \mathcal{O}(\mathcal{H}) \). The reader may verify that since \( \pi \) is mixing and weakly-\( \ell^2 \) then it follows that \( \bar{\pi} \) is also mixing and weakly-\( \ell^2 \). Next, we check that the map \( \bar{q} : H \to \mathcal{H}_1 \) given by the formula \( \bar{q}(\gamma) = \omega^{-1} \sum_{\delta \in \psi_1^{-1}(\gamma)} q(\delta) \), for every \( \gamma \in H \), defines an unbounded quasi-cocycle for \( \bar{\pi} \).

To see this fix \( \gamma_1, \gamma_2 \in H \) and \( \delta_1 \in \psi_1^{-1}(\gamma_1), \delta_2 \in \psi_1^{-1}(\gamma_2) \). Since \( |\psi_1^{-1}(\gamma)| = |\ker(\psi_1)| = \omega \), for all \( \gamma \in H \), then from definitions we have

\[
\| \bar{q}(\gamma_1 \gamma_2) - \bar{\pi}_{\gamma_1}(\bar{q}(\gamma_2)) - \bar{q}(\gamma_1) \| = \| \omega^{-1} \sum_{s \in \psi_1^{-1}(\gamma_1 \gamma_2)} q(s) - \omega^{-2} \sum_{s \in \psi_1^{-1}(\gamma_1), t \in \psi_1^{-1}(\gamma_2)} \pi_s(q(t)) \\
- \omega^{-1} \sum_{s \in \psi_1^{-1}(\gamma_1)} q(s) \|.
\]
Using the relations $\psi^{-1}_t(\gamma_1) = \ker(\psi_1)\delta_t$, the last term above equals to
\[
= \|\omega^{-1} \sum_{s \in \ker(\psi_1)} q(s\delta_1\delta_2) - \omega^{-2} \sum_{s,t \in \ker(\psi_1)} \pi_{s\delta_1}(q(t\delta_2)) - \omega^{-1} \sum_{s \in \ker(\psi_1)} q(s\delta_1)\|
= \omega^{-2} \| \sum_{s,t \in \ker(\psi_1)} (q(s\delta_1\delta_2) - \pi_{s\delta_1}(q(t\delta_2)) - q(s\delta_1))\|
= \omega^{-2} \| \sum_{s,t \in \ker(\psi_1)} (q(s\delta_1\delta_2) - \pi_{s\delta_1}(q(\delta_2\delta_2^{-1}t\delta_2)) - q(s\delta_1))\|.
\]

Finally, since $\pi$ is an orthogonal representation then using the quasi-cocycle relation for $q$ and the normality of $\ker(\psi_1)$ in $\Lambda_1$ we see that the last term above is smaller than
\[
\leq 3D(q) + \omega^{-2} \| \sum_{s,t \in \ker(\psi_1)} \pi_{s}(q(\delta_1\delta_2)) - \pi_{s\delta_1}(q(\delta_2)) - \pi_{s\delta_1\delta_2}(q(\delta_2^{-1}t\delta_2)) - \pi_{s}(q(\delta_1))\|
\leq 3D(q) + \omega^{-1} \| \sum_{t \in \ker(\psi_1)} q(\delta_1\delta_2) - \pi_{s\delta_1}(q(\delta_2)) - q(\delta_1) - \pi_{s\delta_1\delta_2}(q(\delta_2^{-1}t\delta_2))\|
\leq 4D(q) + \max_{t' \in \ker(\psi_1)} \|q(t')\|.
\]

Altogether, these inequalities show that $\tilde{q}$ is a quasi-cocycle whose defect satisfies $D(\tilde{q}) \leq 4D(q) + \max_{t' \in \ker(\psi_1)} \|q(t')\|$. Moreover, one can check from the definition that since $q$ is unbounded then it follows that $\tilde{q}$ is also unbounded. Notice that for every $\gamma \in \Lambda_2$ and $\xi \in \mathcal{H}_1$ the formula $\tilde{\pi}^t_\gamma(\xi) = \tilde{\pi}_{\psi_2(\gamma)}(\xi)$ defines an orthogonal representation $\tilde{\pi}^t : \Lambda_2 \to \mathcal{O}(\mathcal{H})$. Since $\tilde{\pi}$ is mixing and weakly-$t^2$ and $\ker(\psi_2)$ it follows that $\tilde{\pi}^t$ is also mixing and weakly-$t^2$. Also it is a straightforward exercise to check that the map $\tilde{q}^t : \Lambda_2 \to \mathcal{H}_1$ given by $\tilde{q}^t(\gamma) = \tilde{q}(\psi_2(\gamma))$ for every $\gamma \in \Lambda_2$ defines an unbounded quasi-cocycle for $\tilde{\pi}^t$ whose defect satisfies $D(\tilde{q}^t) \leq D(\tilde{q})$. Thus $\Lambda_2 \in NC_1$ and since $\Lambda_2$ has finite index in $\Gamma_2$ if follows from Lemma 4.5 above that $\Gamma_2 \in NC_1$.

The following result will be used in the sequel.

**Lemma 4.7.** Let $\Gamma$ be a group and let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be a weakly-$t^2$, mixing orthogonal representation such that $\mathcal{O}(\mathcal{H})^\Gamma \neq \emptyset$. If we fix $q \in \mathcal{O}(\mathcal{H})^\Gamma$ then for every infinite subgroup $\Lambda \subset \Gamma$ we have that the centralizer $\mathcal{C}_\Gamma(\Lambda)$ is amenable or there exists $C \geq 0$ such that $\langle \Lambda, C(\Lambda) \rangle \subseteq B_C^2 = \{ \lambda \in \Gamma : \|q(\lambda)\| \leq C \}$. Here, for every subset $H \subseteq \Gamma$, we have denoted by $C(\mathcal{H}) = \{ \gamma \in \Gamma : \gamma h = h \gamma, \text{ for all } h \in H \}$.

**Proof.** Assume that the centralizer $\mathcal{C}_\Gamma(\Lambda)$ is non-amenable. Then proceeding as in the first part of the proof of [CSU13, Proposition 2.6] one can find $C_1 \geq 0$ such that $\Lambda \subseteq B_{C_1}^2$. Finally since every element of $\mathcal{C}_\Gamma(\Lambda)$ normalizes the subgroup $\Lambda$, $\pi$ is mixing, and $\Lambda$ is infinite, then by the first part in [CSU13, Theorem 2.1] one can find $C \geq 0$ such that $\langle \Lambda, C(\Lambda) \rangle \subseteq B_C^2$.

In the end of this subsection we describe some recent important progress in building quasi-cocycles through innovative methods in geometric group theory. Some of the first results emerged from the seminal work of Mineyev [Mi01] and Mineyev, Monod, and Shalom [MMS03], who showed that every Gromov hyperbolic group $\Gamma$ admits an
unbounded (even proper) quasi-cocycle into a finite multiple of its left-regular representation and hence it is in $\mathcal{D}_{\text{reg}}$. This was generalized by Mineyev and Yaman to relative hyperbolic groups, [MY09]. Hamenstädt [Ha] showed that all weakly acylindrical groups, in particular, non-elementary mapping class groups and $\text{Out}(\mathbb{F}_n)$, $n \geq 2$, belong to the class $\mathcal{D}_{\text{reg}}$. More recently, Hull and Osin [HO11] and independently Bestvina, Bromberg and Fujiwara [BBF13] were able to find a unified approach to these results by showing that for every group which admits a non-degenerate, hyperbolically embedded subgroup belongs to the class $\mathcal{D}_{\text{reg}}$. Their key results are some beautiful extension theorems on quasi-cohomology. In fact, by very recent work of Osin [Os13] the weak curvature conditions used in both papers, as well as Hamenstädt’s weak acylindricity condition, are equivalent to the notion of acylindrical hyperbolicity formulated by Bowditch, cf. [op. cit.]. Collecting these results together, the following families of groups are known to be acylindrically hyperbolic. In particular, they belong to the class $\mathcal{D}_{\text{reg}}$ and thus will be contained in class $\mathcal{NC}_1$:

**Examples 4.8.**

a. Gromov hyperbolic groups [Mi01, MMS03];

b. Groups which are hyperbolic relative to a family of subgroups as in [MY09, HO11];

c. The mapping class group $\text{Mod}(S_{g,k})$ of a surface $S_{g,k}$ with $3g + k - 4 \geq 0$, [Ha];

d. $\text{Out}(\mathbb{F}_n)$, $n \geq 2$ [Ha].

**Proposition 4.9.** Let $M = S_{g,b}$ be a surface and let $k$ be a positive integer. Then, the following assertions hold:

(i) If $M$ is large, then $\text{PB}_k(M) \in \mathcal{NC}_1$.

(ii) If either $g = 0$, $b \leq 2$ and $b + k \geq 4$ or $g = 1$, $b = 0$ and $k \geq 2$, then $\tilde{\text{PB}}_k(M) \in \mathcal{NC}_1$.

**Proof.** Let $S = S_{g,b+k}$ be the surface in the exact sequence (3.3). Suppose that $M$ is large. As mentioned right before Remark 3.6, $\text{PB}_k(M)$ is isomorphic to a normal subgroup of $\text{PMod}(S)$. By Proposition 4.4 d) and Examples 4.8 c., we have $\text{PB}_k(M) \in \mathcal{NC}_1$ which gives assertion (i).

Next, we suppose the conditions in assertion (ii) are satisfied. By Remark 3.6, we have that $\tilde{\text{PB}}_k(M)$ is isomorphic to a normal subgroup of $\text{PMod}(S)$. Then Proposition 4.4 d) and Examples 4.8 c. again imply that $\tilde{\text{PB}}_k(M) \in \mathcal{NC}_1$, which gives assertion (ii). □

5. The class $\mathcal{P}$

Since all our main structural results are applicable to von Neumann algebras arising from groups belonging to $\mathcal{P} = \mathcal{NC}_1 \cap \text{Quot}(\mathcal{C}_{\text{rss}})$, it would be interesting a thorough investigation of this class of groups. While a a complete understanding of this class of groups remains an open problem for future study, following the previous two subsections, we know it includes all groups that are commensurable with the following concrete groups:

(1) Any infinite, central quotient of the pure braid group $\text{PB}_n(S_{g,k})$ of $n$ strands on a surface $S_{g,k}$—in particular, all surface pure braid groups $\text{PB}_n(S_{g,k})$, for $n \geq 1$ and either $g = 1$ and $k \geq 1$ or $g \geq 2$ and $k \geq 0$;

(2) Any mapping class group $\text{Mod}(S_{g,k})$, for $0 \leq g \leq 2$ and $2g + k \geq 4$;
(3) Any Torelli group $\mathcal{J}(S_{g,k})$ and Johnson kernel $\mathcal{K}(S_{g,k})$, for $g = 1, 2$ and $2g + k \geq 4$;

(4) Any group that is hyperbolic relative to a finite family of exact, residually finite, infinite, proper subgroups.

6. Gaussian Deformations Arising From Quasi-cocycles on Groups

Throughout this section we will assume that $\Gamma$ is a countable group and $\pi : \Gamma \to \mathcal{O} (\mathcal{H})$ is an orthogonal representation such that $\Omega \mathcal{H}^1_{\mathrm{as}} (\Gamma, \pi) \neq \emptyset$. Following [PS09, Si10] to the orthogonal representation $\pi$ one can associate, via the Gaussian construction, a probability measure space $(Y_{\pi}, \mu_{\pi})$ and a family $\{\omega (\xi) : \xi \in \mathcal{H}\}$ of unitaries in $L^\infty (Y_{\pi}, \mu_{\pi})$ such that $L^\infty (Y_{\pi}, \mu_{\pi})$ is generated as a von Neumann algebra by the $\omega (\xi)$’s and the following relations hold:

1. $\omega (0) = 1$, $\omega (\xi_1 + \xi_2) = \omega (\xi_1) \omega (\xi_2)$, $\omega (\xi)^* = \omega (-\xi)$, for all $\xi_1, \xi_2 \in \mathcal{H}$;
2. $\tau (\omega (\xi)) = \exp (-\|\xi\|^2)$, where $\tau$ is the trace on $L^\infty (Y_{\pi})$ given by integration.

Furthermore, there is a p.m.p. action $\Gamma \curvearrowright \mathcal{H}$ $(Y_{\pi}, \mu_{\pi})$ called the Gaussian action associated to $\pi$ which in turn induces a trace preserving action $\Gamma \curvearrowright L^\infty (Y_{\pi}, \mu_{\pi})$ that satisfies $\mathcal{H}_{\gamma} (\omega (\xi)) = \omega (\pi_{\gamma} (\xi))$, for all $\gamma \in \Gamma$ and $\xi \in \mathcal{H}$.

Assume that $(N, \tau)$ is a finite von Neumann algebra endowed with a trace $\tau$, $\Gamma \curvearrowright^\sigma (N, \tau)$ is a trace preserving action and denote by $M = N \rtimes_{\sigma, \tau} \Gamma$ the corresponding crossed product von Neumann algebra. Then consider the Gaussian dilation of $M$ is defined as the crossed product algebra $\tilde{M} = (N \bar{\otimes} L^\infty (Y_{\pi}, \mu_{\pi})) \rtimes_{\sigma \otimes \pi} \Gamma$.

Fix $q \in \Omega \mathcal{H}^1_{\mathrm{as}} (\Gamma, \pi)$ an unbounded quasi-cocycle. Following [Si10] (see also [CS11]) we construct a deformation arising from $q$ through a canonical exponentiation procedure—throughout the text this will be referred to as the Gaussian deformation associated with $q$. For every $t \in \mathbb{R}$ consider the unitary $V_t \in \mathcal{U} (L^2 (N) \bar{\otimes} L^2 (Y_{\pi}, \mu_{\pi}) \bar{\otimes} L^2 (\Gamma))$ defined by the formula

$$V_t (x \otimes y \otimes \delta_\gamma) = x \otimes \omega (t(q(\gamma))) y \otimes \delta_\gamma,$$

for every $x \in L^2 (N)$, $y \in L^2 (Y_{\pi}, \mu_{\pi})$, and $\gamma \in \Gamma$. In [CS11] it was proved that $V_t$ is a strongly continuous one parameter group of unitaries also satisfying the following transversality property, [Po08]:

**Proposition 6.1.** [CS11, Lemma 2.8] Under the previous assumptions, for each $t$ and any $\xi \in L^2 (M)$, we have

$$2 \| e_M \cdot V_t (\xi) \|^2_2 \geq \| \xi - V_t (\xi) \|^2_2 \geq \| e_M \cdot V_t (\xi) \|^2_2,$$

where $e_M$ denotes the orthogonal projection of $L^2 (\tilde{M})$ onto $L^2 (M)$ and $e_M^2 = 1 - e_M$.

Notice also that the deformation $V_t$ satisfies an “asymptotic bimodularity” property, a key notion to incorporate the “bounded equivariance” of quasi-cocycles into von Neumann algebra context.

**Theorem 6.2.** [CS11, Lemma 2.6] For every $x, y \in N \rtimes_{\sigma, \tau} \Gamma$ (the reduced crossed product) we have that

$$\lim_{t \to 0} \left( \sup_{\xi \in L^2 (M)} \| x V_t (\xi) y - V_t (x \xi y) \|_2 \right) = 0.$$
Lemma 6.3. There exists a function $f : \mathbb{R} \to [0, 2^{1/2}]$ satisfying $\lim_{t \to 0} f(t) = 0$ and such that for every $x, y \in M$ and $z \in N \times_{\Gamma_{alg}} \Gamma$ we have the following inequality:

\[\max(\|V_t(xy) - xV_t(y)\|_2, \|V_t(yx) - V_t(y)x\|_2) \leq 2\|x\|_\infty \|y - z\|_2 + \|z\|_\infty \sup(z)^{1/2} (\|V_{2^{1/2}t}(x) - x\|_2 + f(t)\|x\|_2).\]

Here we denoted by $\sup(z)$ the cardinality of the support of $z$ in $\Gamma$.

Proof. Using the triangle inequality and the fact that $V_t$ is a unitary we have that

\[\|V_t(xy) - xV_t(y)\|_2 \leq \|V_t(x(y - z))\|_2 + \|xV_t(y - z)\|_2 + \|V_t(xz) - xV_t(z)\|_2 \leq 2\|x\|_\infty \|y - z\|_2 + \|V_t(xz) - xV_t(z)\|_2.\]

Next, let $z = \sum_\mu z_\mu u_\mu \in N \times_{\Gamma_{alg}} \Gamma$, with $z_\mu \in N$, and let $x = \sum_\lambda x_\lambda u_\lambda$, with $x_\lambda \in N$, be the Fourier decompositions of $z$ and $x$ respectively. Thus, using the Cauchy-Schwarz inequality together with the formula for $V_t$, we see that

\[\|V_t(xz) - xV_t(z)\|_2 = \bigg| \sum_\mu V_t(xz_\mu u_\mu) - xV_t(z_\mu u_\mu) \bigg|_2 \leq \sup(z)^{1/2} \bigg( \sum_\mu \|V_t(xz_\mu u_\mu) - xV_t(z_\mu u_\mu)\|_2^{1/2} \bigg)^{1/2} \leq \sup(z)^{1/2} \bigg( \sum_\mu \|x_\lambda \sigma_\lambda(z_\mu) \otimes (\omega(tq(\lambda u)) - \omega(t\tau_\lambda(q(\mu))))\|_2 \bigg)^{1/2} \leq \sup(z)^{1/2} \bigg( \sum_{\mu, \lambda} (2 - 2e^{-t^2\|q(\lambda u) - \tau_\lambda(q(\mu))\|^2}) \|x_\lambda \sigma_\lambda(z_\mu)\|_2 \bigg)^{1/2} \]

Furthermore, using successively the basic inequalities $\|x_\lambda \sigma_\lambda(z_\mu)\|_2 = \|z_\mu\|_\infty \|x_\lambda\|_2 \leq \|z\|_\infty \|x_\lambda\|_2$, $\|q(\lambda u) - \tau_\lambda(q(\mu))\|^2 \leq 2\|q(\lambda)\|^2 + 2D(q)^2$, and $e^{-2t^2\|q(\lambda)\|^2} \leq 1$ we see that the last term above is smaller than
Thus, applying the triangle inequality and then using the first inequality in (6.11) in the
previous lemma together with (6.12), (6.27), and (6.8), we have

\[\| V_{t} (xy) - xy\| \leq \| V_{t} (xy) - xV_{t} (y)\| + \| V_{t} (y) - y\|\]
\[\leq 2\| y - y\| + \| y\| \| \sup (y)\|^{1/2} (\| V_{2t/2} \| - x\| + f(t)) + \| V_{t} (y) - y\|
\leq \varepsilon / 3 + \varepsilon / 6 + \varepsilon / 6 + \varepsilon / 3 = \varepsilon ,\]

which finishes the first part of the proof. The remaining part of the statement follows
from Proposition 6.1.
Corollary 6.5. Let $X \subseteq (M)_1$ be a subset and for each $i = 1, 2$ let $a_i \in M_+$ be positive elements, $f_i \in M$ be projections, and $\lambda_i > 0$ be scalars satisfying $f_i a_i = a_i f_i \geq \lambda_i f_i$. If $e_{M}^{1} \cdot V_t \to 0$ uniformly on $a_1 X a_2$ then $e_{M}^{1} \cdot V_t \to 0$ uniformly on $f_1 X f_2$.

Proof. Since $f_i a_i = a_i f_i \geq \lambda_i f_i$ then one can find $x_i \in M$ such that $a_i x_i = f_i$. The statement follows then from the transversality property (Proposition 6.1) and Proposition 6.4.

Remark 6.6. More generally, instead of being a quasi-cocycle assume that $q$ is an s-array on $\Gamma$, i.e., there exists a linear function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|q(\lambda) - \tau_\lambda(q(\gamma))\| \leq \psi(\|q(\lambda)\|)$, for all $\lambda, \gamma \in \Gamma$. Then, one can easily check that a version of Lemma 6.3 still holds, with some constants in the inequality (6.11) and the formula for $f$ there slightly modified. Consequently, Proposition 6.4 and Corollary 6.5 will also hold in this case.

Lemma 6.7. Let $\Sigma < \Gamma$ be a subgroup and let $a \in N \rtimes \Sigma =: \Pi$ be an element satisfying $0 \leq a \leq 1$. Then for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $|t| \leq t_\varepsilon$ and all $x \in (\Pi)_1$ we have

$$
\|e_{M}^{1} \cdot V_t (a' x)\|_2^2 \leq \|e_{M}^{1} \cdot V_t (ax)\|_2^2 + \varepsilon,
$$

where $a' = E_\Pi (a)$.

Proof. As before consider the Gaussian dilations $\tilde{M} = (N \otimes L^\infty(X_{\pi}, \mu_{\pi})) \rtimes \Gamma$ and by $\tilde{P} = (N \otimes L^\infty(X_{\pi}, \mu_{\pi})) \rtimes \Sigma$ and we notice the following commuting square condition $E_\Pi \circ E_M = E_{M} \circ E_{\Pi} = E_{P}$.

Fix $\varepsilon > 0$. By Kaplansky density theorem there exists $a_\varepsilon \in N \rtimes \text{alg} \Gamma$ with $\|a_\varepsilon\|_\infty \leq 1$ such that $\|a - a_\varepsilon\|_2 \leq \varepsilon/4$. Also, using Theorem 6.2, there exists $t_\varepsilon > 0$ such that for all $x \in (\Pi)_1$ and all $0 \leq |t| \leq t_\varepsilon$ we have

$$
\|e_{M}^{1} \cdot V_t (a_\varepsilon x) - a_\varepsilon e_{M}^{1} \cdot V_t (x)\|_2 \leq \varepsilon/4,
$$

and

$$
\|e_{M}^{1} \cdot V_t (a' x) - a' \varepsilon e_{M}^{1} \cdot V_t (x)\|_2 \leq \varepsilon/4,
$$

where we have denoted by $a' = E_\Pi (a_\varepsilon)$. Thus, inequalities (6.10) in combination with relations $e_{\Pi} \cdot e_{M}^{1} \cdot V_t (x) = e_{M}^{1} \cdot V_t (x)$, for all $x \in \Pi$, $e_{\Pi} a_{\varepsilon} e_{P} = E_\Pi (a_\varepsilon e_{\Pi})$, and the basic inequality $\|e_{\Pi} (\xi)\|_2 \leq \|\xi\|_2$, for all $\xi \in L^2(\tilde{M})$, show that for every $|t| \leq t_\varepsilon$ we have

$$
\|e_{M}^{1} \cdot V_t (ax)\|_2 \geq \|e_{M}^{1} \cdot V_t (a_\varepsilon x)\|_2 - \varepsilon/4
$$

$$
\geq \|a_\varepsilon e_{M}^{1} \cdot V_t (x)\|_2 - \varepsilon/2
$$

$$
= \|a_\varepsilon \cdot e_{\Pi} \cdot e_{M}^{1} \cdot V_t (x)\|_2 - \varepsilon/2
$$

$$
\geq \|e_{\Pi} (a_\varepsilon) e_{M}^{1} \cdot V_t (x)\|_2 - \varepsilon/2
$$

$$
\geq \|e_{M}^{1} \cdot V_t (E_\Pi (a_\varepsilon) x)\|_2 - 3\varepsilon/4
$$

$$
\geq \|e_{M}^{1} \cdot V_t (E_\Pi (a) x)\|_2 - \varepsilon.
$$

Since $\varepsilon > 0$ was arbitrary then inequality (6.9) follows by squaring (6.11).

It can be easily seen that Proposition 6.4 implies that whenever $P \subseteq M$ is a subfactor such that if $V_t \to \text{Id}$ uniformly on $(\Pi)_1$ then for every intermediate subfactor $P \subseteq Q \subseteq M$
such that $P \subseteq Q$ has finite index we have that $V_t \to \text{Id}$ uniformly on $(Q)_1$. Essentially, this follows from the existence of a finite Pimsner-Popa basis for the inclusion $P \subseteq Q$, [PP86]. Below we will show that similar statements hold for finite index inclusions of arbitrary von Neumann algebras, [PP86].

**Lemma 6.8.** Let $\Sigma < \Gamma$ be a subgroup and let $p \in N \rtimes \Sigma =: P$ be a nonzero projection. Assume that $Q \subseteq pPp$ is a diffuse subalgebra of finite index such that $e^+_M \cdot V_t \to 0$ uniformly on $(Q)_1$. Then there exists $0 < A \leq 1$ such that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $|t| \leq t_\varepsilon$ and all $x \in (pPp)^h$ we have

\[(6.12) \quad \|e^+_M \cdot V_t(x)\|_2^2 \leq 2(1 - A^2)||x||_2^2 + \varepsilon.\]

**Proof.** Since $Q \subseteq pPp$ has finite index then one can find a constant $0 < A \leq 1$ such that for all $x \in (pPp)^+_+$ we have

\[(6.13) \quad \|E_Q(x)\|_2 \geq A||x||_2.\]

Next we claim that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $|t| \leq t_\varepsilon$ and all $x \in (pPp)^+_+$ we have

\[(6.14) \quad \|e^+_M \cdot V_t(x)\|_2 \leq (1 - A^2)^{1/2}||x||_2 + \varepsilon||x||_\infty.\]

Fix $\varepsilon > 0$. Since by assumption $e^+_M \cdot V_t \to 0$ uniformly on $(Q)_1$ there exists $t_\varepsilon > 0$ such that for all $|t| \leq t_\varepsilon$ and all $x \in Q_+$ we have that

\[(6.15) \quad \|e^+_M \cdot V_t(E_Q(x))\|_2 \leq \varepsilon||E_Q(x)||_\infty \leq \varepsilon||x||_\infty.\]

Thus using the triangle inequality together with (6.13) and (6.15) we see that

\[
\|e^+_M \cdot V_t(x)\|_2 = \|e^+_M \cdot V_t(x - E_Q(x) + E_Q(x))\|_2 \\
\leq \|x - E_Q(x)\|_2 + \|e^+_M \cdot V_t(E_Q(x))\|_2 \\
\leq (||x||_2^2 - ||E_Q(x)||_2^2)^{1/2} + \varepsilon||x||_\infty \\
\leq (1 - A^2)^{1/2}||x||_2 + \varepsilon||x||_\infty,
\]

which shows (6.14).

Next we show that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $|t| \leq t_\varepsilon$ and all $x \in (pPp)^h$ we have

\[(6.16) \quad \|e^+_M \cdot V_t(x)\|_2 \leq (2(1 - A^2))^{1/2}||x||_2 + \varepsilon||x||_\infty.\]

Again fix $\varepsilon > 0$ and let $x \in (pPp)^h$. First one can write $x = x_+ - x_-$ for $x_+, x_- \in (pPp)^+_+$ satisfying $x_+ x_- = 0$. Thus applying (6.15) for $x_+ \geq 0$, $x_- \geq 0$, and $\varepsilon/2 > 0$ we see that

\[
\|e^+_M \cdot V_t(x)\|_2 \leq \|e^+_M \cdot V_t(x_+)\|_2 + \|e^+_M \cdot V_t(x_-)\|_2 \\
\leq (1 - A^2)^{1/2}||x_+||_2 + ||x_-||_2 + \varepsilon/2 (||x_+||_\infty + ||x_-||_\infty) \\
\leq (2(1 - A^2))^{1/2}||x||_2 + \varepsilon||x||_\infty,
\]

as desired.

Finally, taking the square in (6.16) and shrinking $\varepsilon > 0$ if necessary we get (6.12). □
**Lemma 6.9.** Assume the previous notations. Let \( p \in \mathbb{N} \times \Sigma =: \mathcal{P} \) be a nonzero projection and assume there is a constant \( 0 < A < 1 \) such that for every \( \varepsilon > 0 \) there exists \( t_\varepsilon > 0 \) such that for all \( |t| \leq t_\varepsilon \) and all \( x \in (p\mathcal{P})_1^h \) we have

\[
\|e_M^1 \cdot V_t(x)\|_2^2 \leq 2(1 - A^2)\|x\|_2^2 + \varepsilon.
\]

(6.17)

Then for every \( \varepsilon > 0 \) there exists \( t_\varepsilon > 0 \) such that for all \( |t| \leq t_\varepsilon \) and all \( y \in (p\mathcal{P})_1 \) we have

\[
\|e_M^1 \cdot V_t(y)\|_2^2 \leq 2(1 - A^2)\|y\|_2^2 + \varepsilon.
\]

(6.18)

**Proof.** Denote by \( J : L^2(\hat{M}) \to L^2(\hat{M}) \) Tomita’s conjugation operator. Throughout the proof, for a vector \( \xi \in L^2(\hat{M}) \) we will denote by \( \text{Re}(\xi) = (\xi + J(\xi))/2 \) and by \( \text{Im}(\xi) = (\xi - J(\xi))/(2i) \). Also, since the quasi-cocycle \( q \) is antisymmetric then from [CS11, Proposition 2.5] it follows that \( V_t J(\xi) = J V_t(\xi) \) for all \( t \in \mathbb{R} \) and \( \xi \in L^2(M) \). This further implies that for every vector \( \eta \in L^2(M) \) we have that \( e_M^1 \cdot V_t(\text{Re}(\eta)) = e_M^1 \cdot V_t(\text{Im}(\eta)) = (e_M^1 \cdot V_t(\text{Re}(\eta))) = (e_M^1 \cdot V_t(\text{Im}(\eta))) / 2 = e_M^1 \cdot V_t(\text{Re}(\eta)) + J(e_M^1 \cdot V_t(\text{Im}(\eta))). \) Similarly we have \( e_M^1 \cdot V_t(\text{Im}(\eta)) = \text{Im}(e_M^1 \cdot V_t(\text{Im}(\eta))). \) Using these relations together with some basic computations we see that for every \( x \in (P)_1 \) and every \( t \in \mathbb{R} \) we have

\[
\|e_M^1 \cdot V_t(x)\|_2^2 = \|\text{Re} (e_M^1 \cdot V_t(x))\|_2^2 + \|\text{Im} (e_M^1 \cdot V_t(x))\|_2^2
\]

(6.19)

Fix \( \varepsilon > 0 \). From (6.17) there exists \( t_\varepsilon > 0 \) such that for all \( |t| \leq t_\varepsilon \) and all \( x \in (P)_1 \) we have

\[
\|e_M^1 \cdot V_t(\text{Re}(x)p)\|_2^2 \leq 2(1 - A^2)\|\text{Re}(x)p\|_2^2 + \varepsilon/2, \text{ and}
\]

\[
\|e_M^1 \cdot V_t(\text{Im}(x)p)\|_2^2 \leq 2(1 - A^2)\|\text{Im}(x)p\|_2^2 + \varepsilon/2.
\]

Using these inequalities together with (6.19) we see that for all \( |t| \leq t_\varepsilon \) and all \( x \in (P)_1 \) we have

\[
\|e_M^1 \cdot V_t(x)\|_2^2 \leq 2(1 - A^2)(\|\text{Re}(x)p\|_2^2 + \|\text{Im}(x)p\|_2^2) + \varepsilon/4
\]

\[
= 2(1 - A^2)\|xp\|_2^2 + \varepsilon,
\]

thus giving (6.18). \( \square \)

**Lemma 6.10.** Assume the previous notations. Let \( p \in \mathbb{N} \times \Sigma =: \mathcal{P} \) be a nonzero projection and assume there is a constant \( 0 < A < 1 \) such that for every \( \varepsilon > 0 \) there exists \( t_\varepsilon > 0 \) such that for all \( |t| \leq t_\varepsilon \) and all \( x \in (p\mathcal{P})_1 \) we have

\[
\|e_M^1 \cdot V_t(x)\|_2^2 \leq 2(1 - A^2)\|x\|_2^2 + \varepsilon.
\]

(6.20)

Then there exists a nonzero element \( r \in \mathbb{Z}(\mathcal{P}) \) with \( 0 < r \leq 1 \) such that for every \( \varepsilon > 0 \) one can find \( t_\varepsilon > 0 \) such that for all \( |t| \leq t_\varepsilon \) and all \( y \in (P)_1 \) we have

\[
\|e_M^1 \cdot V_t(yr)\|_2^2 \leq 2(1 - A^2)\|yr\|_2^2 + \varepsilon.
\]

(6.21)
Proof. First we claim that there exists \( r' \in Z(P) \) a projection such that \( r'p \neq 0 \) and for every \( \varepsilon_1 > 0 \) there exists \( t_{\varepsilon_1} > 0 \) such that for all \( |t| \leq t_{\varepsilon_1} \), \( y \in (P)_1 \) we have

\[
\|e_M^r \cdot V_t(yr'p)\|_2^2 \leq 2(1 - A^2)\|yr'p\|_2^2 + \varepsilon_1. \tag{6.22}
\]

To see this we use a standard convexity argument [CP10], and [Va10]. Notice that the set \( S := \{n\nu_\gamma : n \in U(N), \gamma \in \Sigma \} \) forms a dense subgroup of \( P \). Consider the closed convex hull \( K(p) = \text{conv}_F\{zpz^* : z \in S\} \) and denote by \( q \in K(p) \) the unique element of minimal \( \|\cdot\|_2 \). Notice that since \( \|zqz^*\|_2 = \|q\|_2 \) and \( zqz^* \in K(p) \) for every \( z \in S \) then by uniqueness we have that \( zqz^* = q \) for every \( z \in S \). Thus \( q \in P \cap G' = Z(P) \) and since \( \text{ctr}(K(p)) = \text{ctr}_p(p) \) we conclude that \( q = \text{ctr}_p(p) \in K(p) \).

Fix \( \delta A^2 \geq \varepsilon > 0 \). Thus from the definition of \( K(p) \) one can find a finite subset \( F \subset S \) and \( 0 < c_s, s \in F \) with \( \sum s \in F, c_s = 1 \) such that

\[
\|q - \sum_{s \in F} c_s sps^*\|_2 \leq \varepsilon/8. \tag{6.23}
\]

Moreover, by Proposition 5.1 and Theorem 5.2 above there exists \( t^1_\varepsilon > 0 \) such that for all \( |t| \leq t^1_\varepsilon \), \( s \in F \), and \( x \in (P)_1 \) we have

\[
\|se_M^r \cdot V_t(ps^*xp) - e_M^r \cdot V_t(sps^*xp)\|_2 \leq \varepsilon/8. \tag{6.24}
\]

Also from (6.20) there exists \( t^2_\varepsilon > 0 \) such that for all \( |t| \leq t^2_\varepsilon \), \( s \in F \), and \( x \in (P)_1 \) we have

\[
\|e_M^r \cdot V_t(x)\|_2^2 \leq 2(1 - A^2)\|x\|_2^2 + \varepsilon/8. \tag{6.25}
\]

Using the triangle inequality together with \( \|V_t(\xi)\|_2 \leq \|\xi\|_2 \), for \( \xi \in L^2(M) \), (6.23), (6.24), (6.25) and Cauchy-Schwarz inequality for every \( x \in (P)_1 \) and \( |t| \leq \min(t^1_\varepsilon, t^2_\varepsilon) \) we have

\[
\|e_M^r \cdot V_t(qxp)\|_2^2 \leq \left(\|e_M^r \cdot V_t((q - \sum_{s \in F} c_s sps^*)xq)\|_2 + \sum_{s \in F} c_s\|e_M^r \cdot V_t(sps^*xq)\|_2\right)^2
\]

\[
\leq \left(\|e_M^r \cdot V_t((q - \sum_{s \in F} c_s sps^*)xq)\|_2 + \sum_{s \in F} c_s\|e_M^r \cdot V_t(sps^*xp) - se_M^r \cdot V_t(ps^*xp)\|_2 + \sum_{s \in F} c_s\|e_M^r \cdot V_t(ps^*xp)\|_2\right)^2
\]

\[
\leq \left(\varepsilon/8 + \varepsilon/8 + \sum_{s \in F} c_s\|e_M^r \cdot V_t(ps^*xp)\|_2\right)^2
\]

\[
\leq \varepsilon^2/16 + \varepsilon/2 + \sum_{s \in F} c_s\|e_M^r \cdot V_t(ps^*xp)\|_2^2
\]

\[
\leq \varepsilon^2/16 + \varepsilon/2 + (1 - A^2)\varepsilon/4 + 2(1 - A^2)\sum_{s \in F} c_s\|ps^*xp\|_2^2.
\]
Proof. Applying the assumption for γ \| such that for all q

(6.27)

\| (\tau(x^* q x p) + \|q - \sum_{s \in F} c_s q s^*\|_2) \|

\leq \varepsilon^2 / 16 + \varepsilon / 2 + (1 - \Lambda^2) \varepsilon / 4 + 2(1 - \Lambda^2) \tau (x^* (\sum_{s \in F} c_s q s^*) x p)

\leq \varepsilon^2 / 16 + \varepsilon / 2 + (1 - \Lambda^2) \varepsilon / 2 + 2(1 - \Lambda^2) \|q x p\|_2^2

\leq \varepsilon + 2(1 - \Lambda^2) \|q x p\|_2^2.

Altogether, we have obtained that for every \varepsilon > 0 there exists \varepsilon_1 > 0 such that for all |t| \leq \varepsilon_1 and all x \in (P)_1 we have

(6.26)

\|e_M^1 \cdot V_\varepsilon(q x p)\|_2^2 \leq \varepsilon + 2(1 - \Lambda^2) \|q x p\|_2^2.

For every \mu > 0 we denote by q_{\mu} the spectral projection of q corresponding to the interval [\mu, \infty) and notice that \mu \nrightarrow r := \text{supp}(q) increasingly in SO- topology, as \mu \searrow 0. Thus there exists \delta > 0 such that q_{\delta} q \neq 0 and since q = \text{ctr}_P(p) it follows that q_{\delta} p \neq 0. Moreover, since q_{\delta} q \geq \delta q_{\delta} there exists an element x_{\delta} \in \mathcal{Z}(P) such that q q_{\delta} x_{\delta} = q_{\delta} and \|x_{\delta}\|_\infty \leq \delta^{-1}.

Fix \varepsilon_1 > 0. From (6.26) there exists \varepsilon_1 such that for all |t| \leq \varepsilon_1 and all x \in (P)_1 we have

\|e_M^1 \cdot V_\varepsilon(q x p)\|_2^2 \leq \varepsilon_1 \delta^2 + 2(1 - \Lambda^2) \|q x p\|_2^2.

If in this inequality we let x = \delta q_{\delta} x_{\delta} y for arbitrary y \in (P)_1 then we get our claim for r' = q_{\delta}.

Finally, we notice that our claim together with same averaging argument as used in its proof further implies (6.21), where r = \text{ctr}_P(r' p). In fact, the arguments presented above verbatim and we leave the details to the reader. \square

Lemma 6.11. Assume the previous notations and let N \times \Sigma =: P. Assume that there exists a finite subgroup \Omega < \Sigma such that \mathcal{Z}(P) \subseteq N \times \Omega. Also suppose there is a constant 0 < \Lambda \leq 1 and a nonzero element 0 \leq r \leq 1 with r \in \mathcal{Z}(P) such that for every \varepsilon > 0 there exists \varepsilon_1 > 0 such that for all |t| \leq \varepsilon_1 and all x \in (P)_1 we have

(6.27)

\|e_M^1 \cdot V_\varepsilon(x r)\|_2^2 \leq 2(1 - \Lambda^2) \|x r\|_2^2 + \varepsilon.

Then the quasi-cocycle q is bounded on \Sigma.

Proof. Applying the assumption for \varepsilon := \Lambda^2 \|r\|_2^2 it follows that there exists t > 0 such that for all \gamma \in \Sigma we have

(6.28)

\|e_M^1 \cdot V_\varepsilon(u_r t)\|_2^2 \leq 2(1 - \Lambda^2) \|u_r t\|_2^2 + \Lambda^2 \|r\|_2^2 = 2(1 - \Lambda^2 / 2) \|r\|_2^2.
Consider \( r = \sum_{\omega \in \Omega} r_{\omega} u_{\omega} \) be its Fourier decomposition. Thus using the definition of \( V_{t} \) we see that (6.28) is equivalent to
\[
\sum_{\omega \in \Omega} (2 - 2e^{-t^2\|q(\gamma \omega)\|^2}) ||r_{\omega}||^2_2 \leq 2(1 - \Lambda^2/2) \sum_{\omega \in \Omega} ||r_{\omega}||^2_2.
\]
In particular, this inequality implies that for every \( \gamma \in \Sigma \) there exists \( \omega \in \Omega \) such that \( 2 - 2e^{-t^2\|q(\gamma \omega)\|^2} \leq 2(1 - \Lambda^2/2) \) or equivalently \( \|q(\gamma \omega)\| \leq (\ln(2/\Lambda^2)^{1/2})/t. \) Using the quasi-cocycle relation this, further entails that \( \|q(\gamma)\| \leq D(q) + \|q(\omega)\| + (\ln(2/\Lambda^2)^{1/2})/t. \) Altogether, we have \( \|q(\gamma)\| \leq D(q) + \sup_{\omega \in \Omega} \|q(\omega)\| + (\ln(2/\Lambda^2)^{1/2})/t, \) for all \( \gamma \in \Sigma, \) and since \( \Omega \) is finite it follows that \( q \) is bounded on \( \Sigma. \)

**Remark 6.12.** Finally, we leave to the reader to check that all the previous Lemmas 6.7-6.11 still hold if instead of being a quasi-cocycle one assumes that \( q \) is just an (anti)symmetric [CS11] s-array on \( \Gamma, \) as defined in Remark 6.6. Essentially, all the proofs will follow in the same way, except for the proof of Lemma 6.9 in the case of symmetric s-array where the technical form of the decompositions used in equation (6.19) will be slightly different.

### 7. Applications to Group Theory

The presence of a non-trivial quasi-cocycle on a group taking values in its left regular representation restricts significantly the internal structure of the group; for instance, it excludes most relations of “order one” like (asymptotic) commutation, etc. In the same spirit, we will show that any such group has at most finitely many finite conjugacy classes. More precisely, appealing to the representation theory techniques which steam mainly from [CSU13] we show the following more general statement.

**Theorem 7.1.** Let \( \Gamma \) be a non-amenable group together with \( \Sigma \triangleleft \Gamma, \) a normal non-amenable subgroup. If \( \Gamma \in NC \) then there are only finitely many finite orbits for the action of \( \Sigma \) on \( \Gamma \) by conjugation.

**Proof.** Since \( \Gamma \in NC \) there exists a weakly-\( \ell^2 \), mixing orthogonal representation \( \pi : \Gamma \to 0(\mathcal{H}_1) \) and \( q \in \Omega \mathcal{H}_1 \) of \( \Gamma. \)

Let \( \{0_n : n \in \mathbb{N}\} \) be an enumeration of all the (disjoint) finite orbits for the action of \( \Sigma \) on \( \Gamma \) by conjugation. Notice that \( \cup_{n \in \mathbb{N}} 0_n = \{\gamma \in \Gamma : [\Sigma : C_{\Sigma}(\gamma)] < \infty\} =: \Lambda, \) where \( C_{\Gamma}(\gamma) \) is the centralizer of \( \gamma \) in \( \Gamma. \) Since \( \Sigma \) is normal in \( \Gamma, \) it is a straightforward exercise to show that \( \Lambda \) is a normal subgroup of \( \Gamma. \)

Fixing a finite subset \( F \subset \Lambda \) we denote by \( \langle F \rangle \) the subgroup generated by \( F \) and we see from the definitions that \( C_{\Sigma}(\langle F \rangle) = C_{\Sigma}(F) = \cap_{\lambda \in \Lambda} C_{\Sigma}(\lambda) \) is a finite index subgroup of \( \Sigma. \) Since \( \Sigma \) is non-amenable then so is \( C_{\Sigma}(\langle F \rangle). \) If we would assume that the group \( \langle F \rangle \) is not finite then Lemma 4.7 would further imply that the quasicocycle \( q \) is bounded on \( C_{\Sigma}(\langle F \rangle). \) Thus, by [CSU13, Theorem 2.1], \( q \) would be bounded on \( \Sigma, \) by the finite index assumption, and hence on \( \Gamma \) by the normality assumption. This however is a contradiction, and hence we have shown that every finite subset \( F \subset \Lambda \) generates a finite subgroup \( \langle F \rangle \) in \( \Gamma. \)
For every \( n \in \mathbb{N} \) denote by \( \Lambda_n := \langle \cup_{i=1}^{n} O_i \rangle \) which, by the previous observation, is a finite subgroup of \( \Lambda \). Since the orbits \( O_i \) are invariant under the action of \( \Sigma \) by conjugation then one can check that \( \Sigma \) is contained in \( N_\Gamma(\Lambda_n) \), the normalizing group of \( \Lambda_n \) in \( \Gamma \). Moreover, by construction, \( \{ \Lambda_n : n \in \mathbb{N} \} \) forms an ascending sequence of subgroups such that \( \cup_n \Lambda_n = \Lambda \). In particular, it follows that \( \Lambda \) is a torsion, amenable group. To conclude our proof, it suffices to show that \( \Lambda \) is finite which we do next.

The proof relies heavily on the techniques used in [CSU13, Theorems 3.1 and 3.5] so we will only include a brief sketch on how to fit together these results. Denote by \( M = L(\Gamma) \) the corresponding group von Neumann algebra and let \( \xi_n \), with \( n \geq 1 \), be the canonical group unitaries. For every \( n \in \mathbb{N} \) denote by \( \xi_n = |\Lambda_n|^{1/2} \sum_{a \in \Lambda_n} u_a \in M \subset L^2(\tilde{M}) \). Then a basic calculation shows that for every \( \gamma \in \Sigma \) and \( n \in \mathbb{N} \) we have

\[
\xi_n = \xi_n \gamma_n.
\]

Denote by \( \tilde{M} = L^\infty(\Sigma) \times \Gamma \) be the Gaussian dilation associated with \( \pi \). Let \( V_t : L^2(\tilde{M}) \to L^2(\tilde{M}) \), with \( t \in \mathbb{R} \), be the Gaussian deformation corresponding to \( q \) as defined in Section 6. Since \( \Sigma \) is non-amenable and \( \pi \) is weakly-\( \ell^2 \) one can find a finite subset \( E \subset \Sigma \) and \( K \geq 0 \) such that for every \( \xi \in L^2(\tilde{M}) \) we have that

\[
\sum_{\gamma \in E} \| u_\gamma \xi - \xi u_\gamma \|_2 \geq K \| \xi \|_2.
\]

Then Proposition 6.2 above combined with the same spectral gap argument from the beginning of the proof of theorem [CSU13, Theorem 3.1] show that

\[
\lim_{t \to 0} \left( \sup_n \| e_{\tilde{M}}^t \cdot V_t(\xi_n) \|_2 \right) = 0.
\]

Thus the transversality property (Proposition 6.1) will further imply that

\[
\lim_{t \to 0} \left( \sup_n \| \xi_n - V_t(\xi_n) \|_2 \right) = 0.
\]

Then a simple calculation shows that for every \( \epsilon > 0 \) there exists \( C \geq 0 \) such that

\[
\sup_n \| \xi_n - P_{B_\gamma}(\xi_n) \|_2 \leq \epsilon.
\]

As before, we have denoted by \( P_{B_\gamma} \) the orthogonal projection from \( \ell^2(\Gamma) \) onto the Hilbert subspace \( \ell^2(B_\gamma) \) with \( B_\gamma = \{ \lambda \in \Gamma : \| q(\lambda) \| \leq C, \lambda \neq e \} \) being the ball of radius \( C \) centered and pierced at the identity element \( e \).

Then the same argument as in the proof of [CSU13, Theorem 3.5, pages 15-16] shows that for every \( \epsilon > 0 \) and every \( \gamma \in \Sigma \) there exists \( C \geq 0 \) such that

\[
\limsup_n \| P_{A_\gamma}(\xi_n) \|_2^2 \geq 1 - 6\epsilon^2,
\]

where \( A_\gamma = \gamma B_{\gamma} \gamma^{-1} \cap B_\gamma \). Since \( \Sigma \) is normal in \( \Gamma \) then by [CSU13, Theorem 2.1] the quasi-cocycle is unbounded on \( \Sigma \). Thus one can pick \( \gamma \in \Sigma \setminus B_{2\mathbb{C}+2D(q)} \) and notice that by [CSU13, Theorem 2.1] again if follows that \( A_\gamma \) is finite. Moreover, using the definition of \( \xi_n \) we see that \( \| P_{A_\gamma}(\xi_n) \|_2^2 = |A_\gamma \cap \Lambda_n| |\Lambda_n|^{-1} \), for every \( n \). This together
with (7.2) imply that there exists an integer \( n_0 \) such that \( \Lambda_k = \Lambda_{n_0} \), for every \( k \geq n_0 \); hence \( \Lambda = \Lambda_{n_0} \) is finite.

If we let \( \Sigma = \Gamma \) in the previous theorem we notice the following immediate corollary.

**Corollary 7.2.** Let \( \Gamma \in \text{NC}_1 \). Then \( \Gamma \) has only finitely many finite conjugacy classes. Hence there exists a short exact sequence of groups \( 1 \to F \to \Gamma \to \Gamma_0 \to 1 \), where \( F \) is finite and \( \Gamma_0 \) is infinite conjugacy class. In particular, if \( \Gamma \) is assumed torsion free then \( \Gamma_0 \) is infinite conjugacy class.

When the previous corollary is combined with the results in [CSU13] we obtain the following more complete form of [CSU13, Theorem 3.5].

**Corollary 7.3.** Let \( \Gamma \in \text{NC}_1 \). Then there exists a short exact sequence of groups \( 1 \to F \to \Gamma \to \Gamma_0 \to 1 \) such that \( F \) is finite and \( \Gamma_0 \in \text{NC}_1 \) and is infinite conjugacy class. In particular, \( \Gamma_0 \) is non inner-amenable and infinite conjugacy class.

**Proof.** From the previous theorem there exists a short exact sequence \( 1 \to F \to \Gamma \to \Gamma_0 \to 1 \), where \( F \) is finite and \( \Gamma_0 \) is infinite conjugacy class. Thus \( \Gamma \) is commensurable up to finite kernel with \( \Gamma_0 \) and from Theorem 4.6 we have that \( \Gamma_0 \in \text{NC}_1 \). The remaining part of the statement follows from [CSU13, Theorem 3.5].

The following corollary is a straightforward consequence of the previous results.

**Corollary 7.4.** Let \( \Gamma \in \text{NC}_1 \) be a non-amenable group together with \( \Sigma \triangleleft \Gamma \) a non-amenable normal subgroup. Assume that \( \Gamma \not\sim N \) is a trace preserving action on a finite von Neumann algebra \( N \). If \( N \rtimes \Gamma \) denotes the corresponding crossed product von Neumann algebra then there exists \( \Lambda \triangleleft \Gamma \) a finite normal subgroup such that \( \Sigma(N \rtimes \Sigma) \subseteq (N \rtimes \Sigma)' \cap (N \rtimes \Gamma) \subseteq (N \rtimes \Lambda) \).

### 8. Primeness Results for von Neumann Algebras of Groups in \( \mathcal{P} \)

In this section we will use the technical results from the previous sections to derive the proof of Theorem A. Our arguments are similar in essence with the ones used in [CIK13] but they have slightly different technical forms. For the sake of completeness we include all details. We start start by proving some technical lemmas which are essential in the main proofs.

**Lemma 8.1.** Let \( \Gamma \) be a group together with \( \Sigma, \Omega \triangleleft \Gamma \) normal subgroups such that \( \Omega \) is finite. Denote by \( \Gamma' := \Gamma/\Omega, \Sigma' := (\Omega\Sigma)/\Omega \), fix a section \( s : \Gamma' \to \Gamma \), and let \( M = L(\Gamma') \). Consider the \( * \)-homomorphism \( \psi : M \to M \bar{\otimes} L(\Gamma) \) given by \( \psi(\upsilon\lambda) = u_{\lambda} \otimes (|\Omega|^{-1} \sum_{\omega \in \Omega} \upsilon_{\upsilon s(\lambda)}) \), for every \( \lambda \) in \( \Gamma' \). If \( Q \) is any finite von Neumann algebra, \( p \in M \) is a nonzero projection, and \( B \subseteq Q \bar{\otimes} pMp \) is a subalgebra such that \( (1 \otimes \psi)(B) \subseteq Q \bar{\otimes} M \bar{\otimes} L(\Sigma) \) then \( B \subseteq Q \bar{\otimes} M \bar{\otimes} L(\Sigma') \).

**Proof.** Since \( (1 \otimes \psi)(B) \subseteq Q \bar{\otimes} M \bar{\otimes} L(\Gamma) \) \( Q \bar{\otimes} M \bar{\otimes} L(\Sigma) \) then using the Kaplansky density theorem and Popa’s intertwining techniques one can find \( K > 0 \) and a finite set \( x_1, x_2, \ldots, x_n \in (L(\Gamma))_1 \) such that for every \( b \in \mathcal{U}(B) \) we have

\[
\sum_{i=1}^{n} \|E_{B \bar{\otimes} M \bar{\otimes} L(\Sigma)}((1 \otimes 1 \otimes x_i)(1 \otimes \psi)(b))\|_2^2 \geq K.
\]
We claim that for every $1 > \varepsilon > 0$ and every $x \in (L(\Gamma))_1$ there exists a finite set $F_{x,\varepsilon} \subset \Gamma'$ such that for every $b \in (B_1)$ we have

\[
\|E_{B \hat{\otimes} M \hat{\otimes} L(\Sigma)}((1 \otimes 1 \otimes x) \langle 1 \otimes \psi \rangle(b))\|^2 \leq 3|\Omega|\|P_{\Sigma'F_{x,\varepsilon}}(b)\|^2 + \varepsilon,
\]

where $P_{\Sigma'F_{x,\varepsilon}}$ is the orthogonal projection onto the closure of the linear span $\{x \otimes v_\lambda : x \in Q, \lambda \in \Sigma'F_{x,\varepsilon}\}$.

Notice that inequalities (8.1) and (8.2) will then imply that $\|P_{\Sigma'F}(b)\|^2 \geq K/(6|\Omega|n)$, for every $b \in U(B)$, where $F = U_{\ell_1}F_{x,\varepsilon}$ is a finite set, and hence, by Theorem 2.1, we get $B \aleq_{Q,\hat{\otimes} L(\Sigma')}$. Thus, to complete our proof it only remains to show our claim. For this fix $1 > \varepsilon > 0$, take the Fourier expansion $b = \sum \lambda b_\lambda u_\lambda$ and $\sqrt{\varepsilon}/2$-approximate in $\|\cdot\|_2$ by finite sum $x \cong \sum_{\gamma \in F} b_\gamma v_\gamma$. By definitions, we have

\[
\|E_{Q,\hat{\otimes} M \hat{\otimes} L(\Sigma)}((1 \otimes 1 \otimes x) \langle 1 \otimes \psi \rangle(b))\|^2 = 2|\Omega|^{-1} \sum_{\lambda \in \Gamma'} |b_\lambda|^2 \sum_{\gamma \in F, \omega \in \Omega} |b_\lambda|^2 \|\mu_\gamma E_{L(\Sigma)}(v_{\omega s(\gamma)})\|^2 + \varepsilon
\]

\[
= 2|\Omega|^{-1} \sum_{\lambda \in \Gamma'} |b_\lambda|^2 \sum_{\gamma \in F, \omega \in \Omega, \omega s(\gamma) \in \Sigma} |b_\lambda|^2 \|\mu_\gamma v_{\omega s(\gamma)}\|^2 + \varepsilon.
\]

The condition $\gamma ws(\lambda) \in \Sigma$ implies that $s(\lambda) \in (\Omega \Sigma)^{-1}$ and hence $\lambda \in \Sigma'F_{x,\varepsilon}$ where $F_{x,\varepsilon}$ is the set of the disjoint cosets $F^{-1}/\Omega$. Also notice that $\|\sum_{\gamma \in F, \omega \in \Omega, \omega s(\gamma) \in \Sigma} |b_\lambda|^2 \|\mu_\gamma v_{\omega s(\gamma)}\|^2 \leq |\Omega|^2 (\sum_{\gamma} |b_\lambda|^2) \leq |\Omega|^2 (\varepsilon + 2) < 3|\Omega|^2$. Altogether, these show that the last term in identity (8.3) does not exceed $3|\Omega| \sum_{\omega \in \Omega, \omega s(\gamma) \in \Sigma} |b_\lambda|^2 + \varepsilon$, which proves our claim.  

Lemma 8.2. Let $\Lambda, \Gamma$ be groups together with $\Omega \triangleleft \Gamma$ a finite normal subgroup and $\theta : \Lambda \rightarrow \Gamma/\Omega$ a surjective homomorphism. Denote by $\Gamma' := \Gamma/\Omega$, and fix a section $s : \Gamma' \rightarrow \Gamma$. Denote by $M = L(\Lambda)$ and consider the following $*$-homomorphisms:

1. $\tilde{\psi} : M \hat{\otimes} L(\Gamma') \rightarrow L(\Gamma') \hat{\otimes} L(\Gamma)$, given by $\psi(u_\lambda) = u_\lambda \otimes v_{\theta(\lambda)s}$ for every $\lambda \in \Lambda$;

2. $\psi : L(\Gamma') \rightarrow L(\Gamma') \hat{\otimes} L(\Gamma)$, given by $\psi(u_\gamma) = u_\gamma \otimes (|\Omega|^{-1} \sum_{\omega \in \Omega} v_{s(\gamma)}), \text{ for every } \gamma \in \Gamma'$.

Let $\tilde{M} := M \hat{\otimes} L(\Gamma') \hat{\otimes} L(\Gamma)$ and $\Delta := (1 \otimes \psi) \circ \tilde{\psi} : M \rightarrow \tilde{M}$. If there exists a non-zero projection $q \in \tilde{M}$ such that $q\Delta(M)p$ is amenable relative to $M \hat{\otimes} L(\Gamma')$ inside $\tilde{M}$ then $\Gamma'$ is amenable.

Proof. First notice that $r = |\Omega|^{-1} \sum_{\omega \in \Omega} v_\omega$ is a central projection of $L(\Gamma)$ and $\tilde{s}_\gamma = |\Omega|^{-1} \sum_{\omega \in \Omega} v_{s(\gamma)} \in rL(\Gamma)r$ satisfy $\tilde{s}_\gamma \tilde{s}_\delta = \tilde{s}_{\gamma \delta}$, for every $\gamma, \delta \in \Gamma'$. Hence, $q\Delta(M)p \subseteq r\tilde{M}r$ and by [Io11, Remark 2.2] we can assume that $q \in \Delta(M)' \cap r\tilde{M}r$. In particular, there exists a $\Delta(M)$-equivalence class $\Psi : q(\tilde{M}, e_{M \hat{\otimes} L(\Gamma')})q \rightarrow C$ such that $\Psi_{|q\tilde{M}q} = \tau$. Consider the canonical $*$-isomorphism $\tau : M \hat{\otimes} L(\Gamma') \hat{\otimes} B(\ell_2 \Gamma) \rightarrow \langle \tilde{M}, e_{M \hat{\otimes} L(\Gamma')} \rangle$ and define $\Phi : B(\ell_2 \Gamma) \rightarrow C$ by letting $\Phi(T) = \Psi(q\pi(1 \otimes 1 \otimes T)q)$. If $\mu$ denotes the left regular representation on $\ell_2 \Gamma$ denote by $x_\mu$ the left regular representation on $\ell_2 \Gamma$ denote by $x_\mu' = |\Omega|^{-1} \sum_{\omega \in \Omega} \mu_{s(\gamma)}$ for every $\gamma \in \Gamma$ and notice that $x_\mu' \in L(\Gamma) \subseteq B(\ell_2 \Gamma)$ is a non-zero projection. Since for every $\lambda \in \Lambda$ and $T \in B(\ell_2 \Gamma)$ we have $\pi(1 \otimes 1 \otimes (x_\mu T x_\mu \lambda)) = (u_\lambda \otimes v_{\theta(\lambda)} \otimes x_{\theta(\lambda)})(1 \otimes 1 \otimes T)(u_\lambda \otimes v_{\theta(\lambda)} \otimes x_{\theta(\lambda)})^*$ and $(u_\lambda \otimes v_{\theta(\lambda)} \otimes x_{\theta(\lambda)})q \in \theta(M)q$ and $\Psi$ is $\theta(M)q$-central we get that
Moreover, the case (2) above further implies, by the same dichotomy theorem, that either
\[ \ker = \text{a collection of surjective homomorphisms} \]
Throughout the proof we will denote by \( \theta : \Lambda \rightarrow \Lambda/\Lambda = \Gamma_1 \) the canonical projection. Consider the \( * \)-homomorphism \( \tilde{\theta} : M \rightarrow M \hat{\otimes} L(\Gamma_1) \) given by \( \tilde{\theta}(u_\lambda) = u_\lambda \otimes \nu_{\theta(\Lambda)}, \) for all \( \lambda \in \Lambda. \) Assume by contradiction that \( B, C \subseteq pMp \) are two commuting, diffuse subalgebras such that the inclusion \( B \lor C \subseteq pMp \) has finite index. Hence \( \tilde{\theta}(B), \tilde{\theta}(C) \) are commuting, diffuse subalgebras of \( \tilde{\theta}(p) \) \( \text{and} \) \( \tilde{\theta}(p) \) is amenable relatively to \( M \otimes 1 \) inside \( M \hat{\otimes} L(\Gamma_1). \) Thus by the dichotomy property we either have that
\[ \begin{align*}
\Phi(\chi_{\theta(\lambda)}^{\prime}) \& T(\chi_{\theta(\lambda)}^{\prime})* = \\
= \Psi((u_\lambda \otimes \nu_{\theta(\lambda)} \otimes \tilde{x}_{\theta(\lambda)})) \tau(1 \otimes 1 \otimes T)((u_\lambda \otimes \nu_{\theta(\lambda)} \otimes \tilde{x}_{\theta(\lambda)}) q)^*) \\
= \Phi(T).
\end{align*} \]

Since \( \Phi(1) = \tau(q) > 0 \) then (8.4) implies that \( \tau(q)^{-1} \Phi : r'B(t^2 \Gamma) \rightarrow C \) is a \( \{ \chi_{\theta(\lambda)}^{\prime}\}_{\lambda \in \Lambda} \)
-central state. Since \( r'B(t^2 \Gamma) \rightarrow C \) can be canonically identified with \( B(\mathcal{H}) \) where \( \mathcal{H} = r'(t^2 \Gamma) \) then one can find a sequence of unit vectors \( \xi_n \in \mathcal{H} \otimes \mathcal{K} \) such that \( \| (\chi_{\theta(\lambda)}^{\prime} \otimes 1) \xi_n - \xi_n (1 \otimes x_{\theta(\lambda)}') \|_{\mathcal{H} \otimes \mathcal{K}} \rightarrow 0 \) for all \( \lambda \in \Lambda, \) as \( n \rightarrow \infty. \) In particular, the representation \( \pi' : \Gamma' \rightarrow \mathcal{U}(\mathcal{H} \otimes \mathcal{K}) \) defined by \( \pi'_n(\xi) = (x_{\theta(\lambda)}' \otimes 1) \xi (1 \otimes x_{\theta(\lambda)}') \) weakly contains the trivial representation \( 1_{\Gamma'}. \) Denoting by \( B = \{ x'_\gamma : \gamma \in \Gamma' \}'' \subset r'L(\Gamma) \) and using a left and respectively a right Pimsner-Popa basis \( [PP86] \) one can decompose \( \mathcal{H} = \oplus_i B_{a_i} \) and \( \mathcal{H} = \oplus_j b_j B, \) as \( B \)-bimodules. Hence, the \( B \)-bimodule \( \mathcal{H} \otimes \mathcal{K} \) is isomorphic to a multiple of the coarse \( B \)-bimodule and since \( \pi' \) weakly contains the trivial representation, it follows that \( \Gamma' \) is amenable. \( \square \)

Now we proceed to the main proofs and to simplify the writing we introduce a notation:

**Notation 8.3.** Fixing a group \( \Gamma_n \in \text{Quot}_n(\mathcal{C}_{rss}), \) there exist groups \( \Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1} \) and a collection of surjective homomorphisms \( \tau_k : \Gamma_k \rightarrow \Gamma_{k-1} \) such that \( \Gamma_1 \in \mathcal{C}_{rss}, \) and \( \ker(\tau_k) \in \mathcal{C}_{rss}, \) for all \( 2 \leq k \leq n. \) Then we define \( \theta_n = \tau_2 \circ \cdots \circ \tau_n : \Gamma_n \rightarrow \Gamma_1 \) and notice that by Proposition 3.2 we have that \( \ker(\theta_n) \in \text{Quot}_{n-1}(\mathcal{C}_{rss}). \)

**Proof of theorem A.** Let \( \Lambda \) be commensurable up to finite kernel with a group \( \Gamma_n \in \text{NC}_1 \cap \text{Quot}_n(\mathcal{C}_{rss}). \) Since both \( \text{NC}_1 \) and \( \text{Quot}_n(\mathcal{C}_{rss}) \) are closed under commensurability then it will be sufficient to treat each of the following two cases when either

a) there exists a finite, normal subgroup \( \Omega < \Lambda \) such that \( \Lambda/\Omega = \Gamma_1, \) or

b) there exists a finite, normal subgroup \( \Omega < \Gamma \) such that \( \Lambda = \Gamma_n/\Omega. \)

Throughout the proof we will denote by \( \{ u_\lambda, \lambda \in \Lambda \} \subset L(\Lambda) =: M \) and \( \{ v_\gamma, \gamma \in \Gamma_1 \} \subset L(\Gamma_1) \) the canonical unitaries. In each of the cases a) and b) above we will prove our statement by induction on \( n. \)

First we argue for \( n = 1. \) Assuming case a) we have \( \Lambda/\Omega = \Gamma_1 \in \mathcal{C}_{rss} \) and denote by \( \theta : \Lambda \rightarrow \Lambda/\Omega = \Gamma_1 \) the canonical projection. Consider the \( * \)-homomorphism \( \tilde{\theta} : M \rightarrow M \hat{\otimes} L(\Gamma_1) \) given by \( \tilde{\theta}(u_\lambda) = u_\lambda \otimes \nu_{\theta(\Lambda)}, \) for all \( \lambda \in \Lambda. \) Assume by contradiction that \( B, C \subseteq pMp \) are two commuting, diffuse subalgebras such that the inclusion \( B \lor C \subseteq pMp \) has finite index. Hence \( \tilde{\theta}(B), \tilde{\theta}(C) \) are commuting, diffuse subalgebras of \( \tilde{\theta}(p) \) \( \text{and} \) \( \tilde{\theta}(p) \) is amenable relatively to \( M \otimes 1 \) inside \( M \hat{\otimes} L(\Gamma_1). \) Thus by the dichotomy property we either have that

1. \( \tilde{\theta}(P) \cong M \hat{\otimes} L(\Gamma_1) M \otimes 1, \) or
2. \( \tilde{\theta}(C) \) is amenable relative to \( M \otimes 1 \) inside \( M \hat{\otimes} L(\Gamma_1). \)

Moreover, the case (2) above further implies, by the same dichotomy theorem, that either
(3) \( \tilde{\theta}(C) \preceq \bar{M} \bar{\otimes} \bar{L}(\Gamma_1) \) \( M \otimes 1 \), or
(4) \( \tilde{\theta}(B \vee C) \) is amenable relative to \( M \otimes 1 \) inside \( \bar{M} \bar{\otimes} \bar{L}(\Gamma_1) \).

As in the proof of [CIK13, Theorem 3.1] we show that case (4) above will lead to a contradiction. Indeed since \( B \vee C \subseteq pMp \) has finite index then \( pMp \preceq pMp \) \( B \vee C \) and hence \( pMp \) is amenable relative to \( B \vee C \) inside \( pMp \). This implies that \( \tilde{\theta}(pMp) \) is amenable relative to \( \theta(B \vee C) \) inside \( \bar{M} \bar{\otimes} \bar{L}(\Gamma_1) \) and by [OP07, Proposition 2.4] it follows that \( \tilde{\theta}(pMp) \) is amenable relative to \( M \otimes 1 \) inside \( \bar{M} \bar{\otimes} \bar{L}(\Gamma_1) \). Finally, [CIK13, Proposition 3.5] further implies that \( \tilde{\theta}(\Lambda) = \Gamma_1 \) is amenable which is a contradiction.

In conclusion, for every \( P \subseteq B \) be an arbitrary diffuse, amenable subalgebra we have either (1) or (3) above. But by [BO08, Theorem] this further implies that either \( \tilde{\theta}(B) \preceq M \otimes 1 \) or \( \tilde{\theta}(C) \preceq M \otimes 1 \). Due to the symmetry, we can assume without any loss of generality that \( \tilde{\theta}(B) \preceq M \otimes 1 \). By [CIK13, Proposition 3.4] this further implies that \( B \preceq M \bar{L}(\Omega) \) and since \( \Omega \) is finite it follows that \( B \) is not diffuse which contradicts the assumptions; this settles case a).

Next assume we have case b) so \( \Lambda = \Lambda_1/\Omega \) with \( \Gamma_1 \in \mathfrak{e}_{rss} \). Fix a section \( s : \Gamma_1/\Omega \to \Gamma_1 \) such that \( s(1) = 1 \) and consider the *-homomorphism \( \psi : M \to \bar{M} \bar{\otimes} \bar{L}(\Gamma_1) \) given by \( \psi(u_\lambda) = u_\lambda \otimes (|\Omega|^{-1} \sum_{\omega \in \Omega} v_{\omega s(\lambda)}) \), for every \( \lambda \in \Lambda \). Again, as before, assume by contradiction that \( B, C \subseteq pMp \) are two commuting, diffuse non-amenable subalgebras such that the inclusion \( B \vee C \subseteq pMp \) has finite index. Hence \( \psi(B), \psi(C) \) are commuting, diffuse subalgebras of \( \psi(p) (\bar{M} \bar{\otimes} \bar{L}(\Gamma_1)) \psi(p) \). Thus the same arguments as before show that one of the following must hold:

(5) \( \psi(B) \preceq M \otimes 1 \);
(6) \( \psi(C) \preceq M \otimes 1 \);
(7) \( \psi(B \vee C) \) is amenable relative to \( M \otimes 1 \) inside \( \bar{M} \bar{\otimes} \bar{L}(\Gamma_1) \).

Assuming (5), then Lemma 8.1 for \( Q = C1 \) implies that \( B \preceq M C1 \) and hence \( B \) is not diffuse which contradicts the assumptions. Proceeding in a similar manner, the case (6) above leads to a contradiction too.

Assuming (7) and proceeding as in the case (4) above we obtain that \( \psi(pMp) \) is amenable relative to \( M \otimes 1 \) inside \( \bar{M} \bar{\otimes} \bar{L}(\Gamma_1) \) and by Lemma 8.2 when \( \theta \) is the identity we obtain that \( \Gamma_1/\Omega \) is amenable and hence \( \Gamma_1 \) is amenable which is a contradiction; this settles case b).

Next we show the inductive step. First we treat case a), i.e., \( \Lambda/\Omega = \Gamma_n \in \mathfrak{e}_{rss} \). Notice there exists a surjection \( \theta' = \theta_n \circ \theta : \Lambda \to \Gamma_n \), where \( \theta_n \) is the homomorphism from Notation 8.3. This allows us to define a *-homomorphism \( \tilde{\theta}' : M \to \bar{M} \bar{\otimes} \bar{L}(\Gamma_1) \) by letting \( \tilde{\theta}'(u_\lambda) = u_\lambda \otimes v_{\theta_n(\lambda)} \), for all \( \lambda \in \Lambda \). Assume by contradiction that \( B, C \subseteq pMp \) are two commuting, diffuse subalgebras such that the inclusion \( B \vee C \subseteq pMp \) has finite index. Thus \( \tilde{\theta}'(B), \tilde{\theta}'(C) \) are two commuting diffuse subalgebras of \( \tilde{\theta}'(p) (\bar{M} \bar{\otimes} \bar{L}(\Gamma_1)) \tilde{\theta}'(p) \). Proceeding as in \( n = 1 \) case a) one of the following must hold:

(8) \( \tilde{\theta}'(B) \preceq M \otimes 1 \);
(9) \( \tilde{\theta}'(C) \preceq M \otimes 1 \);
(10) \( \tilde{\theta}'(B \vee C) \) is amenable relative to \( M \otimes 1 \) inside \( \bar{M} \bar{\otimes} \bar{L}(\Gamma_1) \).

As in that proof, case (10) implies that \( \theta'(\Lambda) = \Gamma_1 \) is amenable which is a contradiction and cases (8) and (9) implies that \( B \preceq M (\ker(\theta')) \) and \( C \preceq M (\ker(\theta')) \), respectively.
Also, notice that if $B$ is amenable (and hence $C$ non-amenable!) we automatically have that $B \preceq_M L(\ker(\theta'))$. Thus, by the previous discussion, it suffices to treat only the case $B \preceq_M L(\Sigma)$, where $\Sigma = \ker(\theta')$. By Proposition 2.4 one can find $s > 0$, non-zero projections $r \in N, q \in B$, a subalgebra $B_0 \subseteq rNr$, and a $*$-isomorphism $\theta : qBq \to B_0$ such that the following properties are satisfied:

(8.5) $B_0 \vee (B'_0 \cap rL(\Sigma)r) \subset rL(\Sigma)r$ has finite index;

(8.6) there exist a non-zero partial isometry $v \in M$ such that

$$rE_N(vv^*) = E_N(vv^*)r \geq sr \text{ and } \theta(qBq)v = B_0v = rvqBq.$$  

By Theorem 4.6 and Proposition 3.2, we have $\Sigma/(\Sigma \cap \Omega) = \ker(\theta')/(\ker(\theta') \cap \Omega) = \ker(\theta_n)$ in $NC_1 \cap \text{Quot}_{n-1}(C_{rss})$. Thus, by the induction hypothesis, it follows from the non-zero projections $r \in N, q \in B$, a subalgebra $B_0 \subseteq rNr$, and a $*$-isomorphism $\theta : qBq \to B_0$ such that the inclusion

$$B_o \vee (B'_o \cap rL(\Sigma)r) \subset rL(\Sigma)r$$

has finite index and from Lemma 2.3 it follows that the inclusion $B_o \subseteq rL(\Sigma)r$ has finite index and from Lemma 2.3 it follows that the inclusion $B_0 \subseteq rL(\Sigma)r$ has finite index, too.

Let $\varphi \in QM^1(\Gamma, \pi)$ be an unbounded quasi-cocycle and let $V_\tilde{t} : L^2(M) \to L^2(\tilde{M})$ be corresponding Gaussian deformation as defined in the Section 6, where $M \subseteq \tilde{M}$ is the Gaussian dilation of $M$. Denote by $e_M$ the orthogonal projection on $L^2(\tilde{M})$ onto $L^2(M)$. Since $C$ can always be assumed non-amenable then using the same à la Popa spectral gap argument (see for instance [CS11, Theorem 3.2]) we have that $e_M^1 \cdot V_\tilde{t} \to 0$ uniformly on $(B)_t$, as $t \to 0$. Using Propostion 6.4 this further implies that $e_M^1 \cdot V_\tilde{t} \to 0$ uniformly on $rv(\theta'(qBq))_t$, as $t \to 0$. Using (8.6) and Proposition 6.4 again we get that $e_M^1 \cdot V_\tilde{t} \to 0$ uniformly on $(B_0)_t$,$rvv^*$, as $t \to 0$. Moreover, Lemma 6.7 further gives that $e_M^1 \cdot V_\tilde{t} \to 0$ uniformly on $(B_0)_t$,$rE_N(vv^*)$, as $t \to 0$. Hence, by (8.6) and Corollary 6.5 we obtain that $e_M^1 \cdot V_\tilde{t} \to 0$ uniformly on $(B_0)_t$,$r$, as $t \to 0$ and by Proposition 6.4 again we conclude that $e_M^1 \cdot V_\tilde{t} \to 0$ uniformly on $(B_0)_t$, as $t \to 0$. Since $B_0 \subseteq pL(\Sigma)p_0$ has finite index then Lemmas 6.8-6.11 and Corollary 7.4 imply that the quasi-cocycle $\varphi$ is bounded on $\Sigma$. Moreover, since $\Sigma$ is normal in $\Gamma$ then [CSU13, Theorem 2.1] further implies that $\varphi$ is bounded on $\Gamma$ and we have reached a contradiction; this settles a).

Assume be have case b), i.e., $\Lambda = \Gamma_n/\Omega$ with $\Gamma_n \in C_{rss}$. Let $\theta_n : \Gamma_n \to \Gamma_1$ be the surjection from Notation 8.3 and denote by $\pi : \Gamma_n/\Omega \to \Gamma_1/\Omega'$ the induced surjection, where $\Omega' = \theta_n(\Omega)$. Fix a section $s : \Gamma_1/\Omega' \to \Gamma_1$ with $s(1) = 1$. As before, consider the $*$-homomorphism $\hat{\pi} : M \to M\hat{\otimes}L(\Gamma_1/\Omega')$, defined by $\hat{\pi}(u_\lambda) = u_\lambda \otimes v_{\gamma}(\lambda)$, and the $*$-homomorphism $\psi : L(\Gamma_1/\Omega') \to L(\Gamma_1/\Omega')\hat{\otimes}L(\Gamma_1)$ defined by $\psi(v_{\gamma})(\lambda) = v_{\gamma} \otimes (|\Omega'|^{-1} \sum_{\omega} x_{w(l)}(\lambda))$. Then define the $*$-homomorphism $\Delta = (I \otimes \psi) \circ \hat{\pi} : M \to M \otimes L(\Gamma_1/\Omega')\hat{\otimes}L(\Gamma_1) = : \tilde{M}$.

Assume by contradiction that $B, C \subseteq pMp$ are two commuting, diffuse subalgebras such that the inclusion $B \vee C \subseteq pMp$ has finite index. Thus $\Delta(B), \Delta(C)$ are two commuting diffuse subalgebras of $\Delta(p) (M\hat{\otimes}L(\Gamma_1/\Omega')\hat{\otimes}L(\Gamma_1)) \Delta(p)$. Proceeding as in $n = 1$ case b) we obtain that

(11) $\Delta(B) \preceq_{\tilde{M}} M\hat{\otimes}L(\Gamma_1/\Omega')$;

(12) $\Delta(C) \preceq_{\tilde{M}} M\hat{\otimes}L(\Gamma_1/\Omega')$;
Using Lemma 8.1 we see that (11) and (12) above imply that \( \sim \) and \( \hat{\sim} \) inside \( \tilde{M} \).

Also notice that \( \ker(\pi) \cap \Omega \) is finite and \( \ker(\theta_n) \cap \Omega \) where \( \ker(\theta_n) \cap \Omega \) is finite and \( \ker(\theta_n) \in NC_1 \cap Quot_{n-1}(E_{rss}) \), by Theorem 4.6 and Proposition 3.2 (4). Thus, by the induction assumption, a corner of \( B_o \) satisfies the following properties: \( B_o \cap \ker(\theta_n) \cap \Omega \cap \ker(\pi) \cap \Omega \) has finite index and there exist a non-zero partial isometry \( v \in M \) such that \( \sigma_1(E_\pi(\nu^s)) = E_\pi(\nu^s) \sigma_1 \) and \( \theta(qBq)v = B_o v = rvqBq \).

To finish the proof it remains to analyze the case (13) above. First we observe that \( (\omega) \) is amenabe which is a contradiction.

\[ \Delta(\mathbb{B} \cup C) \] is amenable relative to \( M \otimes L(\Gamma_1 / \Omega') \) inside \( \tilde{M} \).

\[ \Delta(\mathbb{B} \cup C) \] is amenable relative to \( M \otimes L(\Gamma_1 / \Omega') \) inside \( \tilde{M} \). Using Lemma 8.1 we see that (11) and (12) above imply that \( \pi(\mathbb{B}) \preceq M \otimes 1 \) and \( \pi(\mathbb{C}) \preceq M \otimes 1 \), respectively. Due to symmetry again, we treat only one case so assume \( \pi(\mathbb{B}) \preceq M \otimes 1 \). By [CIK13, Proposition 3.3] this implies that \( B \preceq M \preceq \mathbb{L}(\ker(\pi)) \). Furthermore, using Proposition 2.4, this further entails that one can find a scalar \( s > 0 \), non-zero projections \( r \in N, q \in B \), a subalgebra \( B_o \subseteq rN_1 \), and a \( * \)-isomorphism \( \theta : qBq \to B_o \) satisfying the following properties: \( B_o \cap \ker(\theta_n) \cap \Omega \cap \ker(\pi) \cap \Omega \) has finite index and there exist a non-zero partial isometry \( v \in M \) such that \( \sigma_1(E_\pi(\nu^s)) = E_\pi(\nu^s) \sigma_1 \) and \( \theta(qBq)v = B_o v = rvqBq \).

Also notice that \( \ker(\pi) = \ker(\theta_n) \cap \Omega \) and \( \ker(\pi) / \Omega = \ker(\theta_n) / (\ker(\theta_n) \cap \Omega) \) where \( \ker(\theta_n) \cap \Omega \) is finite and \( \ker(\theta_n) \in NC_1 \cap Quot_{n-1}(E_{rss}) \), by Theorem 4.6 and Proposition 3.2 (4). Thus, by the induction assumption, a corner of \( B_o \cap \ker(\theta_n) \cap \Omega \) is completely atomic and then proceeding exactly as in the proof of the inductive step case a) above one obtains a contradiction.

To finish the proof it remains to analyze the case (13) above. First we observe that proceeding as in the proof of [CIK13, Theorem 3.1] this further implies that \( \Delta(pMp) \) is amenable relative to \( M \otimes L(\Gamma_1 / \Omega) \) inside \( \tilde{M} \). Thus by Lemma 8.2 we have that \( \Gamma_1 / \Omega \) is amenable and hence \( \varGamma_1 \) is amenable which is a contradiction. \( \square \)

8.1. Further results. Remark that our techniques can be further developed to show that the groups in our class \( \mathcal{P} \) also verify the unique prime decomposition phenomenon discovered by Ozawa and Popa in [OP03] for bi-exact groups. Hence, our examples will add to the subsequent examples found in [Pe06, CS11, SW12, IS14].

\[ \text{Theorem 8.4. For every } 1 \leq i \leq n \text{ let } \varGamma_i \in \mathcal{P} \text{ and for every } 1 \leq j \leq m \text{ let } P_j \text{ be a II}_1 \text{ factor. If we assume that } L(\varGamma_1) \otimes L(\varGamma_2) \otimes \cdots \otimes L(\varGamma_n) \cong P_1 \otimes P_2 \otimes \cdots \otimes P_m \text{ then } n \geq m. \text{ If we assume in addition that } P_j = L(\varGamma'_j) \text{ for some } \varGamma'_j \in \mathcal{P} \text{ then we have } n = m \text{ and moreover there exist } \sigma \text{ a permutation of the set } \{1, \ldots, n\} \text{ and positive scalars } t_i \text{ with } t_1 t_2 \cdots t_n = 1 \text{ such that } L(\varGamma_i)^{t_i} \cong P_{\sigma(i)}, \text{ for all } 1 \leq i \leq n. \]

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