Non-Perturbative Schwinger-Dyson Equations for 3d $\mathcal{N} = 4$ Gauge Theories

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Abstract: We analyze symmetries corresponding to separated topological sectors of 3d $N = 4$ gauge theories with Higgs vacua, compactified on a circle. The symmetries are encoded in Schwinger-Dyson identities satisfied by correlation functions of a certain gauge-invariant operator, the “vortex character.” Such a character observable is realized as the vortex partition function of the 3d gauge theory, in the presence of a 1/2-BPS Wilson line defect. The character enjoys a double refinement, interpreted as a deformation of the usual characters of finite-dimensional representations of quantum affine algebras. We derive and interpret the Schwinger-Dyson identities for the 3d theory from various physical perspectives: in the 3d gauge theory itself, in a 1d gauged quantum mechanics, in 2d $q$-Toda theory, and in 6d little string theory. We establish the dictionary between all approaches. Lastly, we comment on the transformation properties of the vortex character under the action of three-dimensional Seiberg duality.
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1 Introduction

Since its inception, supersymmetry has been a formidable tool to understand the dynamics of gauge theories in various dimensions. More recently, the success of localization methods [1] has brought about a flourish of new results, often shedding light on hidden structures and symmetries in the strongly coupled regime. Among those, novel symmetries of gauge theories in four dimensions were uncovered by exhibiting non-perturbative Schwinger-Dyson type equations satisfied by certain correlators of the quantum field theory [2, 3].

Concretely, a correlator is defined via a path integral in quantum field theory. Schwinger-Dyson equations can be understood as constraints that must be satisfied by such a correlator. This comes about from demanding that the path integral be invariant under an infinitesimal shift of the contour (under the condition that the integral measure is left invariant by such a shift). The particular question asked in [2] was to determine what type of constraints must be satisfied by correlators in Yang-Mills theory, when a contour gets shifted from a given topological sector to another distinct topological sector of the theory, related to the former by a large gauge transformation. Recall that the connected components of the space of gauge fields are labeled by an integer called the instanton charge: \( \frac{1}{8\pi^2} \int \text{Tr} F \wedge F \), where \( F \) is the field strength and the domain of integration is the spacetime. Then, the problem can be recast in a simple way: what symmetries of the gauge theory are made manifest when the instanton charge varies?

The question was answered in the context of supersymmetric \( \mathcal{N} = 2 \) Yang-Mills, on a regularized spacetime called the \( \Omega \)-background [4–6] on \( \mathbb{C}^2 \). On this background, the instanton number can be changed by adding and removing point-like instantons in a controlled way, and the shift of contour in the definition of the path integral turns into the discrete operation of adding and removing boxes in a Young tableau [7].

To mediate the change in instanton number of the theory, it is convenient to construct a local 1/2-BPS codimension-4 “Y-operator,” as a function of an auxiliary complex parameter \( M \in \mathbb{C} \). Then, Schwinger-Dyson equations are understood as regularity conditions for the vev \( \langle Y(M) \rangle \) in the parameter \( M \). Put differently, the correlator typically has poles in the fugacity \( M \), but the Schwinger-Dyson equations tell us that there exists a precise sum of Y-operator vevs which is pole-free in \( M \), nicknamed the \( \text{qq} \)-character observable. Here, each “q” stands for one of the two parameters of the \( \Omega \)-background on \( \mathbb{C}^2 \), and the term “character” is used because the observable is a (deformed) character of a finite dimensional representation of a Yangian algebra.

The above construction can be generalized in many ways, for example by considering additional defects in the background [8, 9], by studying different gauge groups [10, 11], or by going away from four dimensions: the case of a five-dimensional gauge theory compactified on a circle has been an particularly fruitful area of research [12–21], where the \( \text{qq} \)-character observable arises not as an object defined in the representation theory of Yangians, but instead in the representation theory of quantum affine algebras. Likewise, in the case of a six-dimensional gauge theory compactified on a 2-torus [22, 23], the \( \text{qq} \)-character observable
becomes an object in the representation theory of quantum elliptic algebras. Remarkably, equivariant localization on the Ω-background can be performed to yield exact expressions for the \( qq \)-character observables in all of the above cases.

Meanwhile, supersymmetric gauge theories in codimension-2 lower dimensions share many common features with their higher-dimensional counterparts, but have not yet been studied in any systematic way. Most notably, there exist once again distinct topological sectors of the theory, this time labeled by an integer called the vortex charge:

\[
\frac{1}{2\pi} \int \text{Tr} F,
\]

where \( F \) is the field strength and the integration is over the two real dimensions transverse to the vortex. By the logic we reviewed above, one should then expect non-perturbative Schwinger-Dyson equations to exist also in dimensions 2, 3 (compactified on a circle) and 4 (compactified on a 2-torus). This time around, invariance under a slight shift of contour in the path integral should translate to a change in vortex number. To mediate such a shift, one could hope to construct as before a 1/2-BPS \( Y \)-operator, this time around of codimension-2, as a function of (at least) one auxiliary parameter \( M \). Then, Schwinger-Dyson equations would again be understood as regularity conditions that the vev \( \langle Y(M) \rangle \) needs to satisfy in the parameter \( M \).

Indeed, the existence of such non-perturbative equations has been anticipated for two-dimensional gauged linear sigma models with \( \mathcal{N} = (4, 4) \) supersymmetry: to exhibit the equations and their associated symmetries, a new vortex \( qq \)-character observable can be defined \([8, 18]\), with the same Yangian symmetry as its four-dimensional counterpart, but involving different twists.

The aim of this paper is to give a first-principles construction of low-dimensional non-perturbative Schwinger-Dyson equations, and interpret them from various physical perspectives. We find it convenient to work in a K-theoretic framework, i.e. we study three-dimensional \( \mathcal{N} = 4 \) gauge theories \( G^{3d} \) on the manifold \( \mathbb{C} \times S^1 \). Results for two-dimensional gauged linear sigma model with \( \mathcal{N} = (4, 4) \) supersymmetry can be obtained by reducing the 3d theory on the circle \( S^1 \). The theories \( G^{3d} \) we focus on will be of quiver-type, labeled by an \( ADE \) Lie algebra, with unitary gauge groups and fundamental flavors. We require the amount of flavors to be large enough in order for \( G^{3d} \) to be Higgsable, and introduce non-abelian versions of NielsenOlesen vortex solutions at the Higgs vacua. Our investigations leads us to define of a vortex character observable with quantum affine symmetry\(^1\).

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\(^1\)The literature regarding the representation theory of quantum affine algebras is rich. As a short guide, there are two popular presentations of finite-dimensional representations, one due to Jimbo [24], and another due to Drinfeld [25]. In our physical context, it is the latter presentation that is relevant; see also [26, 27]. Characters of finite-dimensional representations of quantum affine algebras, dubbed “\( q \)-characters,” were first constructed by Frenkel and Reshetikhin in the 90’s [28]. They were later rediscovered in a physical context when discussing the quantum geometry of 5d supersymmetric quiver gauge theories [29, 30]. A deformed character depending on two parameters was introduced in [31–33] (for related work on \( t \)-analogues of \( q \)-characters, see also [34]). This “\( qq \)-character” was again rediscovered in the study of 5d supersymmetric gauge theories [2]. In this paper, we study how it furthermore arises in the study of 3d supersymmetric quiver gauge theories.
In this three-dimensional setting, the codimension-2 operator mediating the change in vortex number is a 1/2-BPS Wilson loop wrapping the circle $S^1$. Recall that a Wilson loop is formulated as the trace of a holonomy matrix, where a quark is parallel transported along the circle, and the trace is evaluated in some representation of the gauge group. We are able to give four definitions of the vortex $qq$-character observable:

- The vortex character is the Witten index of a one-dimensional $\mathcal{N} = 2$ gauged quantum mechanics living on the vortices of $G^{3d}$, interacting with the quark in the Wilson loop.

- The vortex character is the half-index of the 3d $\mathcal{N} = 4$ gauge theory $G^{3d}$ in the presence of a defect, the Wilson loop.

- The vortex character is a deformed $W_{q,t}$-algebra correlator on an infinite cylinder, with stress tensor and higher spin current insertions, including a distinguished set of “fundamental” vertex operators.

- The vortex character is the partition function of the six-dimensional $(2,0)$ little string theory compactified on the cylinder, in the presence of various codimension-4 defects. The defects are all realized as D3 branes of type IIB wrapping 2-cycles of a resolved $ADE$ singularity.

We will analyze each perspective in detail, and prove that all four definitions are in fact equivalent. Let us briefly comment on them. The most obvious perspective is perhaps the one-dimensional one. There, we describe microscopically the gauged quantum mechanics on the vortices in some Higgs vacuum of $G^{3d}$. In three dimensions, both the vortices and the quark of the Wilson loop are particles wrapping the circle $S^1$. In particular, the magnetically charged vortex will experience a Lorentz force in the presence of the electrically charged quark, and the quantum mechanics captures the corresponding dynamics. We count vortices in this background by computing the Witten index of the theory, with appropriate chemical potentials turned on. We show that this index is a deformed character of a finite-dimensional representation of a quantum affine algebra.

This Witten index can be reinterpreted directly from the perspective of $G^{3d}$ itself, as a half-index, or holomorphic block [35], and where the Wilson loop is treated as a codimension-2 line defect. In this picture, the half-index of the 3d theory is computed via Coulomb branch localization, and the line defect is coupled to the bulk theory via gauging of its flavor symmetries. We show that this coupled 3d/1d index is, up to overall normalization, precisely the vortex $qq$-character constructed from the vortex quantum mechanics.

The 3d perspective is useful to make contact with certain vertex operator algebras called $W(g)$-algebras. These are labeled by a simple Lie algebra $g$, which in this work will be simply-laced, and the choice of a nilpotent orbit, which in this work will be the maximal
one. They realize the symmetry of Toda theory, here defined on an infinite cylinder. The particular case $\mathfrak{g} = A_1$ is known as Liouville theory, which enjoys Virasoro symmetry. When $\mathfrak{g} \neq A_1$, the Virasoro stress tensor remains, but there are also higher spin currents. In the 90’s, Frenkel and Reshetikhin introduced a two-parameter deformation of the $\mathcal{W}$-algebras, denoted as $\mathcal{W}_{q,t}(\mathfrak{g})$ [33], and sometimes referred to as deformed $\mathcal{W}$-algebras. Crucially, while an ordinary $\mathcal{W}$-algebra has conformal symmetry, its deformation does not: instead, it is the symmetry of the so-called $q$-Toda theory on the cylinder. Correlators are defined in the free field formalism, as integrals over the positions of some deformed screening currents on the cylinder. We show that the vortex $qq$-character of $G^{3d}$ is such a correlator: the 3d gauge content is realized as screening current insertions, the 3d flavor symmetry is realized as fundamental vertex operator insertions, and the Wilson loop is realized as the insertion of a generating current operator. This latter type of operator includes the deformed stress tensor, but also “higher spin” currents of the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra; they are all constructed in free field formalism as the commutant of the screening currents. There are $\text{rank}(\mathfrak{g})$ independent generators constructed this way, with spin $s$ in the range $2 \leq s \leq \text{rank}(\mathfrak{g}) + 1$. The vortex Schwinger-Dyson equations of the gauge theory $G^{3d}$ are now interpreted as Ward identities satisfied by the correlator in the $\mathcal{W}_{q,t}(\mathfrak{g})$-algebra.

Finally, the various operators appearing in the $\mathcal{W}(\mathfrak{g})$-algebra construction have a natural interpretation in $(2,0)$ little string theory compactified on the cylinder: they are all D3 brane defects at points on this cylinder. Some D3 branes realize the screening charges, other D3 branes realize the flavor vertex operators, and a last set of D3 branes realizes the stress tensor and higher spin currents of the $\mathcal{W}$-algebra.

In hindsight, some of the relations are not too surprising: for instance, given a three-dimensional supersymmetric gauge theory defined on a $S^1$-bundle over a 2-manifold, the partition function (with adequate twists) is expected to contain information about the vortex sector of the theory. This makes plausible the relation between the 3d half-index with line defect and the Witten index of the vortex quantum mechanics. Furthermore, the relation from gauge theory supersymmetric indices to $\mathcal{W}$-algebra correlators is an illustration of the so-called BPS/CFT correspondence [5, 36]. Lastly, it is known that the effective theory on D3 branes in $(2,0)$ little string is precisely the 3d $\mathcal{N} = 4$ theory under study [37]. The goal of this paper is to make use of 1/2-BPS Wilson loops to flesh out these ideas in detail, and exhibit new non-perturbative physics in the process.

As an application, we briefly analyze the action of 3d Seiberg duality on the vortex $qq$-character observable. This duality relates different 3d gauge theories as defined in the UV, but which flow to the same theory in the IR [38]. Here, we construct Seiberg-dual characters directly from the vortex quantum mechanics, where the duality manifests itself as a wall-crossing phenomenon in the Witten index [39]. This perspective gives us complete control over the action of the duality in three dimensions.

The paper is organized as follows: in Section 2, we construct the vortex quantum
mechanics of $G^{3d}$ in the presence of a Wilson loop, and show its Witten index is a vortex character. We comment on how to interpret our result as a set of non-perturbative Schwinger-Dyson identities. In section 3, we re-derive the vortex character directly from the 3d perspective, coupled to a loop defect. In Section 4, we make contact with Ward identities in the deformed $\mathcal{W}(\mathfrak{g})$-algebra picture. In Section 5, we define the vortex character straight from little string theory in the presence of codimension-4 D-brane defects. In section 6, we discuss Seiberg duality and future directions. In section 7, we showcase in full detail all the results of the paper for the case of 3d $\mathcal{N}=4$ SQCD.

2 Schwinger-Dyson Equations: the Vortex Quantum Mechanics Perspective

We start with a lightning review on three-dimensional gauge theories with 8 supercharges, along with the various 1/2-BPS objects that enter our story.

2.1 3d $\mathcal{N}=4$ Gauge Theory, Vortices and Wilson Loops

We consider a 3d $\mathcal{N}=4$ quiver gauge theory $G^{3d}$ on the manifold $\mathbb{C} \times S^1(\hat{R})$, where the quiver is labeled by a simply-laced Lie algebra $\mathfrak{g}$ of rank $n$, of shape the Dynkin diagram of $A_n$, $D_n$ or $E_n$. The radius of the circle is denoted by $\hat{R}$. For concreteness, the Lagrangian gauge group is a product of $n$ unitary groups,

$$G = \prod_{a=1}^{n} U(N^{(a)}),$$

(2.1)

We introduce flavor symmetry through the gauge group

$$G_F = \prod_{a=1}^{n} U(N_F^{(a)}),$$

(2.2)

where the associated gauge fields of $U(N_F^{(a)})$ are frozen. This produces $N_F^{(a)}$ hypermultiplets on node $a$, in the bifundamental representation $(N^{(a)}, \overline{N}_F^{(a)})$ of the group $U(N^{(a)}) \times U(N_F^{(a)})$.

Finally, we have hypermultiplets in the bifundamental representation $\oplus_{b>a} \Delta^{ab} (N^{(a)}, \overline{N}_F^{(b)})$ of the group $\prod_{a,b} U(N^{(a)}) \times U(N^{(b)})$, where $\Delta^{ab}$ is the incidence matrix of $\mathfrak{g}$: $\Delta^{ab}$ is equal to 1 if there is a link connecting nodes $a$ and $b$ in the Dynkin diagram of $\mathfrak{g}$, and is 0 otherwise. Then, $G^{3d}$ contains a total of $n-1$ such bifundamental hypermultiplets.

The $a$-th gauge group in the quiver contains an abelian factor $U(1) \in U(N^{(a)})$, from which one can define a conserved current $j^{(a)} = \frac{1}{2\pi} * \text{Tr} F^{(a)}$; the associated global symmetry
makes up the so-called topological symmetry of $G^{3d}$. Coupling this current to a $U(1)$ factor from the gauge group results in a Fayet-Iliopoulos (FI) term for the $a$-th node of the quiver.

The theory $G^{3d}$ has a moduli space of vacua with Coulomb and Higgs branches, and a corresponding $SU(2)_C \times SU(2)_H$ R-symmetry, where each $SU(2)$ acts on the two branches separately. In particular, each of the $n$ FI parameters is a triplet under $SU(2)_H$; we decompose each such triplet into a real FI parameter and a complex one. Under the R-symmetry, the 3d $\mathcal{N} = 4$ Poincaré supercharges transform in the representation $(2,2)$. They obey the anticommutator relation

$$\{Q^\alpha_{a,a'}, Q^\beta_{b,b'}\} = \epsilon_{ab} \epsilon_{a'b'} (\gamma_\mu C)^{\alpha\beta} P^\mu,$$

where we introduced $SO(2,1)$ $\gamma$-matrices, the charge conjugation matrix $C$, and the three-momentum $P^\mu$. The upper index $\alpha$ is a spinor index for $SO(2,1)$, while the lower indices $a, a'$ are indices for $SU(2)_C$ and $SU(2)_H$, respectively. Additionally, the above supercharges obey a reality condition, which we omit writing explicitly here.

The aim of this work is to exhibit certain symmetries associated to finite energy configurations of BPS vortices, which sit at Higgs vacua of $G^{3d}$. Therefore, from now on, we require that all theories under study possess a Higgs branch, and moreover that all vacua we study be Higgs vacua. In other words, the flavor symmetry group $G_F$ should have a large enough rank. The vortices then arise as semi-local non-abelian versions of Nielsen-Olson solutions; they are codimension-2 particles, transverse to the C-line and wrapping $S^1(\hat{R})$.

Then we tune the moduli to sit at such a Higgs vacuum, and the gauge group $G$ breaks to its $U(1)$ centers. We furthermore turn on the $n$ real FI parameters. The complex FI parameters are set to zero throughout this paper. The R-symmetry is broken to $SU(2)_C \times U(1)_H$, and 1/2-BPS vortices solutions appear in the moduli space. They can be described as a one-dimensional $\mathcal{N} = 4$ supersymmetric quantum mechanics\footnote{Here, we mean by 1d $\mathcal{N} = 4$ supersymmetry the reduction of 2d $\mathcal{N} = (2,2)$ supersymmetry to one dimension.}, preserving the supercharges $Q^1_{a,1}$ and $Q^2_{b,2}$ of the 3d theory. Those four supercharges anticommute to the generator of translations along the vortices, which we denote as $H$:

$$\{Q^1_{a,1}, Q^2_{b,2}\} = \epsilon_{ab} H$$

Independently of vortices, it is also possible to introduce 1/2-BPS Wilson lines for $G^{3d}$. Specifically, a Wilson line operator is labeled by a choice of path and a representation $R$ of the gauge group. In this work, we choose a loop wrapping the circle $S^1(\hat{R})$ and sitting at the origin of $C$. The Wilson loop operator vev on node $a$ reads:

$$\langle W^a_R \rangle = \text{Tr}_R P \exp \oint d\tau i \left[ A^a_\mu \dot{x}^\mu + \Phi^a(\sqrt{-\dot{x}^2}) \right]$$

\[ (2.5) \]
The second term in the exponent is required by supersymmetry, and $\Phi^{(a)}$ is the scalar belonging in the 3d $\mathcal{N} = 2$ $a$-th vector multiplet inside the $\mathcal{N} = 4$ vector multiplet. This operator clearly breaks the $SU(2)_C \times SU(2)_H$ R-symmetry to $U(1)_C \times SU(2)_H$, since $SU(2)_C$ used to act on the triplet of $\mathcal{N} = 4$ vector multiplet scalars.

One way to realize such a supersymmetric Wilson loop operator is to couple the 3d bulk to a one-dimensional $\mathcal{N} = 4$ supersymmetric quantum mechanics. The Wilson loop is then the theory of a 1d $\mathcal{N} = 4$ Fermi multiplet, meaning a complex chiral fermion. This can be achieved in a supersymmetric way by gauging the flavor symmetry of the Fermi multiplet, meaning we couple it to a 1d $\mathcal{N} = 4$ vector multiplet embedded inside the 3d $\mathcal{N} = 4$ vector multiplet. Then, integrating out the 1d Fermi multiplet has the effect of inserting the Wilson loop (2.5) in the path integral of the bulk theory $G^{3d}$.

The 1d $\mathcal{N} = 4$ theory on the Wilson loop preserves the supercharges $Q_{1,a'}^1$ and $Q_{2,b'}^2$ of the 3d theory. Those four supercharges anticommute to the generator of translations along the loop, which we denote as $H$:

$$\{Q_{1,a'}^1, Q_{2,b'}^2\} = \epsilon_{a'b'} H \tag{2.6}$$

The supersymmetric vortices and Wilson loops we described above both preserve 4 supercharges, but only $Q_{1,1}^1$ and $Q_{2,2}^2$ are preserved at the same time. Therefore, in the presence of a Wilson loop, the vortex quantum mechanics only has 1d $\mathcal{N} = 2$ supersymmetry.

### 2.2 The Vortex Quantum Mechanics

Let us first consider $G^{3d}$ without any Wilson loop. In that case, the 1d $\mathcal{N} = 4$ quantum mechanics on its vortices is well-known [40–43]. Just like the bulk theory, it is a $g$-type quiver theory of rank $n$ which we call $T^{1d}_{\text{pure}}$, where the subscript “pure” emphasizes the absence of defect Wilson loop for now. The Higgs branch of this quiver theory is the moduli space of $(k^{(1)}, k^{(2)}, \ldots, k^{(n)})$ vortices of $G^{3d}$, where $k^{(a)}$ is a positive integer denoting the rank of the $a$-th gauge group in the quantum mechanics. Concretely, the gauge group of $T^{1d}_{\text{pure}}$ is

$$\tilde{G} = \prod_{a=1}^{n} U(k^{(a)}) \tag{2.7}$$

For a 3d gauge group $U(N^{(a)})$ with field strength $F^{(a)}$, each 1d rank above is identified as the nontrivial first Chern class $k^{(a)} = -\frac{1}{2\pi} \int \text{Tr} F^{(a)}$, where the integral is taken over the $\mathbb{C}$-line transverse to the vortex.

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3Here, we mean by 1d $\mathcal{N} = 4$ supersymmetry the reduction of 2d $\mathcal{N} = (0, 4)$ supersymmetry to one dimension. Note that this is different from the 1d $\mathcal{N} = 4$ supersymmetry we described for the vortices, which was a reduction of 2d $\mathcal{N} = (2, 2)$ supersymmetry.

4The description of the moduli space of vortices relies on a D-brane construction in the work [40]. However, some of the dynamical degrees of freedom considered there turn out to be non-normalizable zero modes when performing a careful field-theoretic analysis [41–43]; as a result, the Kähler potentials on the vortex moduli space are in general different in the brane and field theory approaches, with agreement only on certain BPS solutions [44, 45]. In our context, the index of the vortex quantum mechanics we compute is insensitive to these discrepancies.
There are chiral multiplets in the bifundamental representation \( \oplus_{b>a} \Delta^{ab} (k^{(a)}, k^{(b)}) \) and in the bifundamental representation \( \oplus_{b>a} \Delta^{ab} (k^{(a)}, k^{(b)}) \) of \( \prod_{a,b} (U(k^{(a)}) \times U(k^{(b)})) \), where \( \Delta^{ab} \) is again denoting the incidence matrix of \( g \): \( \Delta^{ab} \) is equal to 1 if there is a link connecting nodes \( a \) and \( b \) in the Dynkin diagram of \( g \), and is 0 otherwise.

Additionally, there is fundamental and antifundamental chiral matter which manifests itself as additional “teeth” in the 1d quiver. The precise determination of such matter requires specifying the gauge group \( G = \prod_{a=1}^{n} U(N^{(a)}) \) and flavor group \( G_F = \prod_{a=1}^{n} U(N_F^{(a)}) \) of the 3d bulk theory. We denote this flavor symmetry by \( \hat{G}_F \). When the rank of \( G_F \) is large enough, fully Higgsing the 3d quiver theory is always possible, for any \( ADE \) Lie algebra. The resulting 1d theory is then a generic handsaw quiver variety \([46]\), with 1d chiral matter on all \( n \) nodes. Namely, on the \( a \)-th node, there are \( P^{(a)} \) chirals in the representation \( (k^{(a)}, P^{(a)}) \) of \( U(k^{(a)}) \times U(P^{(a)}) \), and \( Q^{(a)} \) chirals in the representation \( (k^{(a)}, Q^{(a)}) \) of \( U(k^{(a)}) \times U(Q^{(a)}) \).

As we reviewed in the previous section, the R-symmetry group of \( T_{1d}^{\text{pure}} \) is \( SU(2)_C \times U(1)_H \), and the R-charge assignment of the various fields is constrained by the superpotential, readable from the “closed loops” in the quiver diagram.

The 3d FI parameter \( c_{3d}^{(a)} \) of the gauge group \( U(N^{(a)}) \) sets the BPS tension of the vortex on node \( a \). It is related to the 1d gauge coupling \( \xi_{1d}^{(a)} \) of the gauge group \( U(k^{(a)}) \), according to \( \xi_{1d}^{(a)} \propto 1/(\zeta_{3d}^{(a)})^2 \). Meanwhile, the 3d gauge coupling \( \xi_{3d}^{(a)} \) of the gauge group \( U(N^{(a)}) \) is related to the 1d FI parameter \( c_{1d}^{(a)} \) of the gauge group \( U(k^{(a)}) \), according to \( s_{1d}^{(a)} \propto 1/(s_{3d}^{(a)})^2 \).

![Figure 1](image_url)

**Figure 1**: Example of the \( G^{3d} \) theory \( T_p[SU(N^{(n+1)})] \), and its vortex quantum mechanics \( T_{1d}^{\text{pure}} \). Note we use a 3d \( \mathcal{N} = 4 \) notation for the quiver on top, and a 1d \( \mathcal{N} = 4 \) notation for the quiver on the bottom.

Let us pause and look at an example, the \( g = A_n \) case, showcased in Figure 1.
The particular quiver theory $G^{3d}$ on top is sometimes called $T_ρ[SU(N^{(n+1)})]$, labeled by a partition $ρ = [N^{(1)}, N^{(2)} - N^{(1)}, \ldots, N^{(n+1)} - N^{(n)}]$ [47]. The $n$ circles label the gauge group $G = \prod_{a=1}^{n} U(N^{(a)})$, the box on the right labels a flavor symmetry group $G_F = U(N_F^{(n)}) \equiv U(N^{(n+1)})$. An arrow between two circles labels a $N = 4$ bifundamental hypermultiplet, while the arrow between the $n$-th circle and the box labels $N^{(n+1)}$ hypermultiplets in the fundamental representation of $U(N^{(n)})$. The corresponding 1d $N = 4 \ (k^{(1)}, k^{(2)}, \ldots, k^{(n)})$ vortex world-line theory $T_{pure}^{1d}$ is shown on the bottom. The circles label the gauge group $G = \prod_{a=1}^{n} U(k^{(a)})$, the looping arrows label adjoint chiral multiplets, and the straight arrows label fundamental/antifundamental chiral multiplets, which makes up the flavor symmetry $\hat{G}_F = \prod_{a=1}^{n+1} U(N^{(a)} - N^{(a-1)})$. Specifically, in our previous notation, the number of fundamental chiral multiplets at node $a$ is $P^{(a)} = N^{(a)} - N^{(a-1)}$, while the number of antifundamental chiral multiplets at node $a$ is $Q^{(a)} = N^{(a+1)} - N^{(a)}$. There are two types of cubic contributions to the superpotential: the first type of terms is due to the bifundamental/adjoint chiral multiplets, while the second type is due to the bifundamental/fundamental/antifundamental chiral multiplets, meaning the flavor teeth. These superpotential terms can simply be read off the various triplets of arrows making closed loops in the quiver diagram. Mathematically, the theory $T_{pure}^{1d}$ in this example is known as a handsaw quiver, isomorphic to a parabolic Laumon space. This is the moduli space of (based) quasi-maps from $\mathbb{P}^1$ into the flag variety [46, 48, 49].

We now come to the new physics, and consider the vortex quantum when a 1/2-BPS Wilson loop wraps $S^1(\hat{R})$ in $G^{3d}$; we call the resulting theory $T^{1d}$. Introducing such a Wilson loop for the 3d gauge group $U(N^{(a)}) \ (a \in \{1, \ldots, n\})$ can be done with the use of a new (nondynamical) defect group $U(L^{(a)})$ for a one-dimensional complex fermion field $\chi^{(a)}$ localized on the $S^1(\hat{R})$, transforming in the representation of $(N^{(a)}, \overline{L}^{(a)})$ of $U(N^{(a)}) \times U(L^{(a)})$. We thereby refer to the defect group for the entire quiver as:

$$\hat{G}_{decat} = \prod_{a=1}^{n} U(L^{(a)}) \ . \quad (2.8)$$

The fermions make up the dimensional reduction of a 2d $N = (0, 4)$ Fermi multiplet to 1d, coupled to the 3d fields in the bulk as [50, 51]:

$$S^{3d/1d} = \int dt \chi^{(a)}_i \left( δ_{ρσ}(δ_{ij} i \partial_t + A^{3d,(a)}_{t,ij} + Φ^{3d,(a)}_{ij}) - δ_{ij} \tilde{A}^{(a)}_{t,ρσ} \right) \chi^{(a)}_j \ . \quad (2.9)$$

Above, $A^{3d,(a)}_t$ and $Φ^{3d,(a)}_{t,ij}$ are the pullback of the 3d gauge field and the adjoint scalar of the $U(N^{(a)})$-vector multiplet, respectively. $\tilde{A}^{(a)}_t$ is the background $U(L^{(a)})$ gauge field the 1d fermions couple to. $i$ and $j$ are indices for the fundamental representation of $U(N^{(a)})$, while $ρ$ and $σ$ are indices for the fundamental representation of $U(L^{(a)})$. The variable $t$ is periodic, with period $\hat{R}/(2\pi)$. In the rest of this paper, an important role will be played by the eigenvalues $\{M^{(a)}_ρ\}$ of the background gauge field $\tilde{A}^{(a)}_t$, which are (large) masses for the
fermions and set the energy scale for their excitation.

One can integrate out the fermions exactly, in which case the path integral organizes itself as a generating function of Wilson loops in the $L^{(a)}$-fold tensor product of the fundamental representation of $SU(N^{(a)})$, with expansion parameters the (exponentiated) defect fermions $\{M_ρ^{(a)}\}$ [50]. For example, if the defect group is $U(L^{(a)}) = U(1)$, the path integral is a series in the parameter $\exp(\hat{R}M^{(a)})$ comprised of $N^{(a)} + 1$ terms, where each coefficient is a Wilson loop $\langle \cdot \rangle$ valued in one of fundamental representations of $SU(N^{(a)})$ (including the trivial one).

An important point is that because we study $G^{3d}$ on its Higgs branch, the 3d theory is massive, the gauge group $G$ is broken and the Coulomb moduli are frozen to flavor masses. It follows that the Wilson loop we consider technically becomes a collection of “flavor” loops at that locus of the moduli space, determined from the Higgsing pattern. After turning on the FI parameters, the Wilson loop fermions make up the degrees of freedom of 1d $\mathcal{N} = 2$ Fermi multiplets.

What are the details of the quantum mechanics $T^{1d}$? First recall from the last section that only two supercharges are preserved by the vortex quantum mechanics after adding in the Wilson loop. In particular, the $\mathcal{N} = 4$ multiplets that previously defined $T^{1d}_{\text{pure}}$ are still present, but should now be understood as a collection of $\mathcal{N} = 2$ vector, chiral, and Fermi multiplets in $T^{1d}$. Moreover, the $SU(2)_C \subset SU(2)_C \times U(1)_H$ part of the $T^{1d}_{\text{pure}}$ R-symmetry is now broken to $U(1)_C$. From the paragraph above, it is furthermore clear that the Wilson loop contributes extra $\mathcal{N} = 2$ Fermi multiplets, coupling the defect fermions to the various flavors. The interactions are encoded in the superpotential, now written in terms of holomorphic functions called E and J-terms for the Fermi multiplets, which constrains the R-charge of the various fields.

More nontrivially, we claim there are additional multiplets present due to the coupling of the Wilson loop to the vortex. These are $\mathcal{N} = (0, 4)$ twisted hypermultiplets and Fermi multiplets in the bifundamental representation of $U(k^{(a)}) \times U(L^{(a)})$; see Figure 2 for an illustration in our previous $T^{1d}_{\mu}[SU(N^{(n+1)})]$ example. Those multiplets decompose into $\mathcal{N} = 2$ chiral and Fermi multiplets in $T^{1d}$. Heuristically, the existence of these extra multiplets can be justified as follows: note that on the manifold $\mathbb{C} \times S^1(\hat{R})$, the Wilson loop is the world-line of an electrically charged quark, while the vortex is a magnetically charged particle. Then, when a vortex moves in the presence of a quark, it experiences a Lorentz force\footnote{An analogous Lorentz force was identified in a five-dimensional context, where the Wilson loop quark is moving instead in a nontrivial instanton background [12].}, and correspondingly the Higgs branch of $T^{1d}$ should be understood as describing a generalized vortex moduli space. We leave the precise mathematical characterization of this modified moduli space to future work\footnote{This program was recently carried out successfully in instanton physics in five dimensions [2, 10, 11, 13, 18, 21]: there, the problem of counting instantons in the presence of a Wilson loop is not solved by localizing the loop on usual ADHM solutions [52], but by defining instead a more general “crossed instanton” moduli space from the onset. The fact that a similar problem arises in our context is not too surprising, since}. The major takeaway is that it is not enough to...
simply localize the Wilson loop (2.5) at the solutions of usual BPS vortex equations in the absence of the loop. This would just result in the Fermi multiplet contributions (2.9). Such contributions should be understood as “classical,” or topologically trivial, in the sense that they exist already at zero vortex charge $k = 0$, but they will undergo an infinite number of corrections due to the other sectors $k > 0$. These corrections are represented by the double green line in Figure 2, and they will be crucial in making the non-perturbative symmetries of $G^{3d}$ manifest.

Figure 2: On the top, a Wilson loop defect is placed in the 3d theory $T_{\rho}[SU(N^{(n+1)})]$, transforming in the $L^{(2)}$-fold tensor product of the fundamental representation of $SU(N^{(2)})$. We denoted the loop by a green cross. On the bottom, the vortex quantum mechanics $T^{1d}$ is displayed. Black links and arrows denote 1d $\mathcal{N} = 4$ multiplets obtained by reduction of 2d $\mathcal{N} = (2, 2)$ supersymmetry, as before. Meanwhile, The double green link labels both a 1d $\mathcal{N} = 4$ twisted hypermultiplet and Fermi multiplet, obtained by reduction of 2d $\mathcal{N} = (0, 4)$ supersymmetry. The green dashed arrows represent 1d $\mathcal{N} = 2$ Fermi multiplets, as reduced from 2d $\mathcal{N} = (0, 2)$, which is the topologically trivial contribution of flavor Wilson loops. Ultimately, all multiplets in the picture should be decomposed into appropriate 1d $\mathcal{N} = 2$ vector, chiral and Fermi multiplets, as this is the supersymmetry of $T^{1d}$. We refrained from doing so in the bottom quiver not to clutter the figure.

For now, we justify the existence of these extra multiplets inside $T^{1d}$ a fortiori, by showing that they correctly (and uniquely) account for the symmetries of $G^{3d}$ under a shift of vortex number; in other words, they describe the physics of a non-perturbative Schwinger-Dyson identity. We will give a direct proof of the existence of these multiplets in section 5, where we analyze an underlying string picture. Having described the quantum vortices are ultimately related to instantons on a codimension-2 locus.
mechanics $T^{1d}$, we turn to the definition of its Witten index. As we will soon see, this index has remarkable properties in our context.

2.3 The Index of the Quantum Mechanics

Recall that the gauge theory $G^{3d}$ is defined on $\mathbb{C} \times S^1(\hat{R})$. Let us denote by $U(1)_\omega$ the symmetry associated with rotating the $C$-line. Then, the global symmetry of the vortex theory $T^{1d}$ is $(\hat{G}_F/U(1)^n) \times \hat{G}_{\text{defect}} \times U(1)_C \times U(1)_H \times U(1)_\omega$, with $\hat{G}_F$ the chiral matter producing the teeth of the handsaw quiver, and all other groups as introduced previously. The diagonal combination of $U(1)_H \times U(1)_\omega$ commutes with the supersymmetry and acts as a flavor symmetry; we call it $U(1)_{\epsilon_1}$, with generator $J_-$. We further define $r$ as the generator of $U(1)_H$, and $J_3$ as the generator of $U(1)_C$.

Then, the refined Witten index of the $\mathcal{N} = 2$ gauged quantum mechanics $T^{1d}$ has the form [53–55]:

$$[\chi]^{(L_1^{(1)}, \ldots, L_n^{(n)})}_{1d} = \text{Tr} \left[ (-1)^F e^{-\hat{R}(Q, \overline{Q})} e^{\hat{R} \epsilon_2 (2J_3 - r)} e^{2\hat{R} \epsilon_1 J_-} \prod_{a=1}^n e^{\hat{R} \zeta^{(a)}_3} e^{\hat{R} \Sigma_d \omega^{(a)}_d \Pi^{(a)}_d} e^{\hat{R} \Sigma_\rho \Lambda^{(a)}_\rho} \right].$$  \hspace{1cm} (2.10)

This index has a path integral interpretation as a twisted partition function on $S^1(\hat{R})$. The trace is over the Hilbert space of the theory, and the index counts states in $Q$-cohomology, where we have redefined the supercharges as $Q \equiv Q_{1,1}$ and $\overline{Q} \equiv Q_{2,2}^3$ in the notation of section 2.1. $F$ is the fermion number. $\{\Pi^{(a)}_i\}$ and $\{\Lambda^{(a)}_\rho\}$ are Cartan generators of the flavor group $\hat{G}_F$ and the Wilson line defect group $\hat{G}_{\text{defect}}$, respectively. We have also defined conjugate variables for these generators: the fundamental/antifundamental chiral multiplet masses $\{m^{(a)}_d\}$, and the Wilson loop fermion masses $\{M^{(a)}_\rho\}$. Furthermore, the integer $k^{(a)} = \frac{-1}{2\pi} \int \text{Tr} F^{(a)}$ is the topological $U(1)$ charge for the $a$-th gauge group, conjugate to the vortex counting fugacity $\zeta^{(a)}_3$, which as we reviewed is the 3d FI parameter\footnote{Recall that throughout our analysis, we set the complex FI parameter to zero. $\zeta^{(a)}_3$ is the real FI parameter, but because the 3d theory is compactified on $S^1(\hat{R})$, the parameter is in fact complexified by the holonomy of the corresponding background gauge field around the circle.}. Finally, we have introduced the variables $\epsilon_1$ and $\epsilon_2$, respectively conjugate to $J_-$ and $2J_3 - r$.

The fugacity $e^{\hat{R} \epsilon_1}$ is well-known in the context of the 3d gauge theory on $\mathbb{C} \times S^1(\hat{R})$, where it is called the $\Omega$-background [4–7]. We will analyze it in detail when discussing the 3d gauge theory perspective. In the rest of this paper, the following redefined fugacities will come in handy:

$$\epsilon_+ \equiv \frac{\epsilon_1 + \epsilon_2}{2}, \quad \epsilon_- \equiv \frac{\epsilon_1 - \epsilon_2}{2}.$$ \hspace{1cm} (2.11)

The index is the grand canonical ensemble of vortex BPS states. The natural grading by the integers $k^{(a)}$ means that the index can be organized as a sum over vortex sectors $(k^{(1)}, k^{(2)}, \ldots, k^{(n)})$. 

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By standard arguments \[56, 57\], the Witten index does not depend on the circle scale \(\hat{R}\). In particular, we can work in the limit \(\hat{R} \to 0\), where it reduces to Gaussian integrals around saddle points. These saddle points are parameterized by \(\phi^{(a)} = \hat{R} \varphi_{1d}^{(a)} + i \hat{R} A_{t, 1d}^{(a)}\)

with \(A_{t, 1d}^{(a)}\) the gauge field and \(\varphi_{1d}^{(a)}\) the scalar in the \(a\)-th vector multiplet of the quantum mechanics. The (complexified) eigenvalues of \(\phi^{(a)}\) are denoted as \(\phi_1^{(a)}, \ldots, \phi_{k^{(a)}}^{(a)}\). Performing the Gaussian integrals over massive fluctuations, the index reduces to a zero mode integral of various 1-loop determinants, which we write schematically as:

\[
[X]_{1d}^{(L^{(1)}, \ldots, L^{(n)})} = \sum_{k^{(1)}, \ldots, k^{(n)} = 0}^{\infty} \prod_{a=1}^{n} \frac{e^{\hat{R} \varphi_{1d}^{(a)} k^{(a)}}}{k^{(a)}!} \prod_{\rho=1}^{L^{(a)}} Z_{\text{defect}, \varnothing}^{(a)} \\
\times \oint \frac{d\phi^{(a)}}{2\pi i} Z_{\text{pure}, \text{vec}}^{(a)} \cdot Z_{\text{pure}, \text{adj}}^{(a)} \cdot Z_{\text{pure, teeth}}^{(a)} \cdot \prod_{b > a}^{n} Z_{\text{pure, bif}}^{(a, b)} \cdot Z_{\text{defect}, k}^{(a)} ,
\]

where

\[
Z_{\text{pure, vec}}^{(a)} = \prod_{I \neq J, I, J = 1}^{k^{(a)}} \frac{\text{sh} (\phi_I^{(a)} - \phi_J^{(a)})}{\text{sh} (\phi_I^{(a)} - \phi_J^{(a)} + \epsilon_2)}
\]

\[
Z_{\text{pure, adj}}^{(a)} = \prod_{I, J = 1}^{k^{(a)}} \frac{\text{sh} (\phi_I^{(a)} - \phi_J^{(a)} + \epsilon_1 + \epsilon_2)}{\text{sh} (\phi_I^{(a)} - \phi_J^{(a)} + \epsilon_1)}
\]

\[
Z_{\text{pure, teeth}}^{(a)} = \prod_{I = 1}^{k^{(a)}} \prod_{i = 1}^{p^{(a)}} \frac{\text{sh} (\phi_I^{(a)} - \mu_i^{(a)} + (\epsilon_1 - \epsilon_2)/2 + \epsilon_2)}{\text{sh} (\phi_I^{(a)} - \mu_i^{(a)} + (\epsilon_1 - \epsilon_2)/2)} \prod_{j = 1}^{Q^{(a)}} \frac{\text{sh} (-\phi_I^{(a)} + \mu_j^{(a)} + (\epsilon_1 + \epsilon_2)/2 + \epsilon_2)}{\text{sh} (-\phi_I^{(a)} + \mu_j^{(a)} + (\epsilon_1 + \epsilon_2)/2)}
\]

\[
Z_{\text{pure, bif}}^{(a, b)} = \left[ \prod_{I = 1}^{k^{(a)}} \frac{\text{sh} (\phi_I^{(a)} - \phi_I^{(b)} + \epsilon_2)}{\text{sh} (\phi_I^{(a)} - \phi_I^{(b)} + \epsilon_2)} \prod_{J = 1}^{k^{(b)}} \frac{\text{sh} (-\phi_J^{(b)} + \phi_J^{(a)} - \epsilon_1)}{\text{sh} (-\phi_J^{(b)} + \phi_J^{(a)} - \epsilon_1)} \right] \Delta^{ab}
\]

\[
Z_{\text{defect}, \varnothing}^{(a)} = \prod_{U(N^{(a)}) \to \prod_{i = 1}^{b} U(P^{(b)})} \prod_{i = 1}^{b} \text{sh} (\mu_i^{(b)} - M^{(b)}_\rho + \epsilon_2 - \#_{i}^{(b)} (\epsilon_1 + \epsilon_2)/2)
\]

\[
Z_{\text{defect}, k}^{(a)} = \prod_{I = 1}^{k^{(a)}} \frac{\text{sh} (\phi_I^{(a)} - M^{(a)}_\rho - (\epsilon_1 - \epsilon_2)/2)}{\text{sh} (\phi_I^{(a)} - M^{(a)}_\rho - (\epsilon_1 - \epsilon_2)/2)} \prod_{J = 1}^{k^{(b)}} \frac{\text{sh} (-\phi_J^{(b)} + M^{(b)}_\rho - (\epsilon_1 + \epsilon_2)/2)}{\text{sh} (-\phi_J^{(b)} + M^{(b)}_\rho - (\epsilon_1 + \epsilon_2)/2)}.
\]

Some comments are in order:

We use the convenient notation \(\text{sh}(x) = 2\sinh(\hat{R}x/2)\). \(\mathcal{M}_k\) is the set of poles enclosed by the contours, which we will characterize below. The prefactor \(\prod_{a=1}^{n} 1/k^{(a)}!\) is the Weyl group order of the 1d gauge group \(\hat{G} = \prod_{a=1}^{n} U(k^{(a)})\).

The factor \(Z_{\text{pure, vec}}^{(a)}(\{\phi_I^{(a)}\}, \epsilon_2)\) is the contribution of the (reduction from 2d \(\mathcal{N} = (2, 2)\) to 1d \(\mathcal{N} = 4\) vector multiplet on node \(a\), decomposed into \(\mathcal{N} = 2\) Fermi and chiral multiplets.
The factor $Z_{\text{pure,adj}}^{(a)}(\{\phi_I^{(a)}\}, \epsilon_1, \epsilon_2)$ is the contribution of the (reduction from 2d $\mathcal{N} = (2, 2)$ to) 1d $\mathcal{N} = 4$ adjoint chiral multiplet on node $a$, decomposed into $\mathcal{N} = 2$ Fermi and chiral multiplets.

The factor $Z_{\text{pure,teeth}}^{(a)}(\{\phi_I^{(a)}\}, \{\mu_i^{(a)}\}, \{\bar{\mu}_i^{(a)}\}, \epsilon_1, \epsilon_2)$ is the contribution of the (reduction from 2d $\mathcal{N} = (2, 2)$ to) 1d $\mathcal{N} = 4$ flavors on node $a$. Note that the numbers $P^{(a)}$ of fundamental chirals (with corresponding masses $\{\mu_i^{(a)}\}$) and $Q^{(a)}$ of antifundamental chirals (with corresponding masses $\{\bar{\mu}_i^{(a)}\}$) are fully determined in terms of the ranks of the 3d gauge group $G$, the 3d flavor group $G_F$, and the choice of the 3d vacuum\(^8\). For instance, consider the 3d theory $T_{\rho}[SU(N^{(n+1)})]$, where the fundamental matter ranks are $N_F^{(a)} = 0$ for $a = 1, \ldots, n - 1$, and $N_F^{(n)} = N^{(n+1)}$ on the last node, with corresponding masses $\{m_d^{(n)}\}_{d=1,\ldots,N^{(n+1)}}$. Then, in the quantum mechanics, $P^{(a)} = N^{(a)} - N^{(a-1)}$, while $Q^{(a)} = N^{(a+1)} - N^{(a)}$, and the matter factor becomes:

$$Z_{\text{pure,teeth}}^{(a)} = \prod_{I=1}^{k^{(a)}} \prod_{i=N^{(a)-1}}^{N^{(a)}} \frac{\text{sh} \left( \phi_I^{(a)} - m_i^{(n)} + \epsilon_- + \epsilon_2 \right)}{\text{sh} \left( \phi_I^{(a)} - m_i^{(n)} + \epsilon_- \right)} \prod_{j=N^{(a)}}^{N^{(a+1)}} \frac{\text{sh} \left( -\phi_I^{(a)} + m_j^{(n)} + \epsilon_+ + \epsilon_2 \right)}{\text{sh} \left( -\phi_I^{(a)} + m_j^{(n)} + \epsilon_+ \right)}.$$  

(2.13)

In particular, the chiral multiplet masses $\{\mu_i^{(a)}\}$ and $\{\bar{\mu}_i^{(a)}\}$ are now written exclusively in terms of the $N^{(n+1)}$ 3d masses $\{m_i^{(n)}\}$, as they should. The generalization to $D_n$ and $E_n$ algebras is straightforward, even though the Higgsing pattern is more intricate to write down explicitly\(^5\).

The factor $Z_{\text{pure,bif}}^{(a,b)}(\{\phi_I^{(a)}\}, \{\phi_I^{(b)}\}, \epsilon_1, \epsilon_2)$ is the contribution of the (reduction from 2d $\mathcal{N} = (2, 2)$ to) 1d $\mathcal{N} = 4$ bifundamental matter between nodes $a$ and $b$. It is only nontrivial when the incidence matrix $\Delta^{ab}$ is as well. Recall that the matrix $\Delta^{ab}$ equals 1 if there is a link connecting nodes $a$ and $b$, and equals 0 otherwise.

The above factors account for all the multiplets present in the vortex quantum mechanics $T^\text{pure}_1$ of a 3d $\mathcal{N} = 4$ gauge theory in the absence of a Wilson loop. Now, recall that the loop is characterized by the defect group $\hat{G}_{\text{defect}} = \prod_{\rho=1}^n U(L^{(\rho)})$ for additional 1d fermions. The superscript notation we use for the index, $[\chi_1^{(L^{(1)})},\ldots,L^{(n)}]$, makes the dependence on this defect group explicit. Those fermions are responsible for two universal contributions to the index.

First, the factor $Z_{\text{defect,}\varnothing}^{(a)}(\{M_\rho^{(a)}\}, \{\mu_i^{(a)}\}, \epsilon_1, \epsilon_2)$ is the contributions of (the reduction from 2d $\mathcal{N} = (0, 2)$ to) 1d $\mathcal{N} = 2$ Fermi multiplets. We previously called this factor “classical,” in the sense that it exists even in the zero-vortex sector. Hence, it sits outside of the 1-loop determinant integrals and is denoted by the subscript $\varnothing$. The symbols $\#_i^{(b)}$ stand for positive integers, which are uniquely fixed by $R$-symmetry once the 3d Higgs vacuum is specified. As is the case for the factor $Z_{\text{pure,teeth}}^{(a)}$, each mass $\mu_i^{(a)}$ is equal to one of the 3d

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\(^8\)This is the data that determines the Higgsing of the 3d theory, which is to say the fundamental mass each Coulomb modulus is frozen to. In our 1d notation, the masses and $\{\mu_i^{(a)}\}$ and $\{\bar{\mu}_i^{(a)}\}$ should eventually be expressed in terms of the 3d masses $\{m_a^{(a)}\}$. 

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masses \( \{m_d^{(b)}\} \), determined by the choice of the vacuum. The product is over all \( b \) such that \( U(N^{(a)}) \to \prod_{b} U(P^{(b)}) \); this is to indicate the breaking of gauge Wilson loop to a product of flavor Wilson loops on the Higgs branch.

Second, there is the interaction between the defect fermions and the vortices: we denote it as the factor \( Z_{\text{defect},k}^{(a)}(\{\phi_{I}^{(a)}\}, \{M_{\rho}^{(a)}\}, \epsilon_{1}, \epsilon_{2}) \), which is the contributions of (the reduction from 2d \( \mathcal{N} = (0, 4) \) to) 1d \( \mathcal{N} = 4 \) twisted hypermultiplets and Fermi multiplets, decomposed into \( \mathcal{N} = 2 \) chiral and Fermi multiplets.

Because the theory is valued on a circle of radius \( \hat{R} \), it is useful in what follows to introduce K-theoretic fugacities for each of the equivariant parameters:

\[
\tilde{\gamma}^{(a)} = e^{\hat{R} \hat{\xi}^{(a)}},
\]

\[
q = e^{\hat{R} \epsilon_{1}}, \quad t = e^{-\hat{R} \epsilon_{2}}, \quad v = e^{\hat{R} \epsilon_{+}} = \sqrt{q/t}, \quad u = e^{\hat{R} \epsilon_{-}} = \sqrt{q/t},
\]

\[
f_{d}^{(a)} = e^{-\hat{R} \nu_{d}^{(a)}}, \quad \tilde{f}_{d}^{(a)} = e^{-\hat{R} \tilde{\nu}_{d}^{(a)}}, \quad \gamma_{d}^{(a)} = e^{-\hat{R} \mu_{d}^{(a)}}.
\]

Crucially, the Witten index also depends implicitly on additional continuous parameters in a piecewise constant manner: the \( n \) FI parameters \( \hat{\xi}_{id}^{(a)} \), which are themselves \( k^{(a)} \)-vectors, one for each abelian factor in \( \hat{G} \). Indeed, when such a parameter changes sign and crosses the value \( \hat{\xi}_{id}^{(a)} = 0 \), a non-compact Coulomb branch opens up, and some vacua may appear or disappear, resulting in wall crossing and a jump in the index. This dependence on the 1d FI parameters is in one-to-one correspondence with the choice of the index integration contours, which we now turn to.

We adopt the so-called Jeffrey-Kirwan (JK) residue prescription [59]. It was first popularized in our context in a two-dimensional setup [60], and used in our quantum mechanical context in [53–55]. Let us briefly review its main features. First note that each \( Z^{(a)} \)-factor in the integrand has the following general form:

\[
\frac{\prod_{i=1}^{n_{1}} \text{sh}(\hat{\rho}_{i} \phi_{i} + \ldots)}{\prod_{j=1}^{n_{2}} \text{sh}(\hat{\rho}_{j} \phi_{j} + \ldots)},
\]

where \( \hat{\rho} \) is a \( k \)-tuple vector, with \( k = \sum_{a=1}^{n} k^{(a)} \). The entries of \( \hat{\rho} \) are all in the set \( \{0, \pm 1\} \), and \( n_{1}^{(a)} \) and \( n_{2}^{(a)} \) are positive integers specified by the details of the vortex quantum mechanics. The dots “…” stand for a linear function of the spacetime fugacities \( \epsilon_{1}, \epsilon_{2} \), as well as all the other 1d flavor fugacities. Since \( \sinh(0) = \sinh(i\pi) = 0 \), there can be many poles in (2.15). We denote a pole locus as \( \hat{\phi} = \hat{\phi}_{+} \).

Now, we assemble the \( n \) FI parameters \( \xi_{id}^{(a)} \) into a vector \( \xi_{id} \) of size \( k = \sum_{a=1}^{n} k^{(a)} \). As we pointed out, the Witten index depends on the choice of a chamber for \( \xi_{id} \). Apart from the FI parameter vector \( \xi_{id} \), the JK prescription instructs us to define yet another auxiliary \( k \)-vector \( \eta \), though the index ultimately does not depend on \( \eta \). We are a priori free to work with any \( k \)-vector \( \eta \) we want to carry out the JK residue prescription, but there exists a particularly convenient choice \( \eta = \xi_{id} \). Indeed, on general grounds, the index
The integral \((2.12)\) has \(\phi\)-poles at \(\pm\infty\) with nonzero residues; one can show that the choice \(\eta = \zeta_{id}\), meaning \(\eta\) generic but chosen in the same chamber as \(\zeta_{id}\), guarantees that the contributions of \(\phi\)-residues at \(\pm\infty\) vanish. Unless specified otherwise, in this paper we work in a chamber where all components of \(\zeta_{id}\) are positive. We will work in different chambers when discussing 3d Seiberg duality later on. Having defined \(\eta\), we are to choose \(k\) hyperplanes from the arguments of \(\sinh\) functions in the denominator of \((2.15)\). Those hyperplanes will take the following form:

\[
\vec{\rho}_j \cdot \vec{\phi}_j + \ldots = 0, \quad \text{where } j = 1, \ldots, k.
\]

(2.16)

The contours of the index are then chosen to enclose poles which are solutions of this linear system of equation, but only if the vector \(\eta\) also happens to lie in the cone spanned by the vectors \(\vec{\rho}_j\). A practical way to test this condition is to construct a \(k \times k\) matrix \(Q\) where \(Q_{ji} = (\rho_j)_i\), and test if all the components of \(\eta Q^{-1}\) are positive. We collect the poles \(\vec{\phi}_s\) satisfying the condition in a set \(M_k\).

Summing over all the poles in \(M_k\), the Witten index takes the form

\[
[\chi]_{1d}^{(L^{(1)}, \ldots, L^{(n)})} = \sum_{k(1), \ldots, k(n)=0}^{\infty} \prod_{a=1}^{n} \left( q^{(a)}_{k(a)} \right)^{k(a)} \sum_{\vec{\phi}_s} \text{JK-res}_{\vec{\phi}_s}(Q_s, \eta) Z^{(a)}_{\text{integrand}},
\]

(2.17)

where \(Z^{(a)}_{\text{integrand}}\) is the integrand of \((2.12)\), and the JK-residue is defined as

\[
\text{JK-res}_{\vec{\phi}_s}(Q_s, \eta) = \begin{cases} 
\frac{1}{|\det(Q)|} & \text{if } \eta \in \text{cone } (Q) \\
0 & \text{otherwise}
\end{cases}
\]

(2.18)

The condition \(\eta \in \text{cone } (Q)\) means that the vector \(\eta\) should lie in the cone spanned by the rows of the matrix \(Q\). It can happen that a solution of the system of equations \((2.16)\) yields additional zeroes in the denominator of \((2.15)\). This typically results in degenerate poles, which can be dealt with using a constructive definition of the JK residue and the so-called flag method [60, 61]. This is an involved procedure to implement analytically, and we will refrain from doing so in this paper, treating potential degenerate poles on a case-by-case basis instead.

We now come to our main object of study, the derivation of non-perturbative Schwinger-Dyson equations for the gauge theory \(G^{3d}\). As we now show, they arise as a regularity condition of the quantum mechanics index \([\chi]_{1d}^{(L^{(1)}, \ldots, L^{(n)})}\) on the defect fermion masses \(\{M^{(a)}_\mu\}\).

### 2.4 The Index is a Vortex \(qq\)-character

We evaluate the index, using the JK residue prescription above to define the contours. As a warmup, let us practice with the index of \(T_{1d}^{\text{pure}}\) which is the vortex quantum mechanics of the “pure” 3d \(\mathcal{N} = 4\) theory, in the absence of Wilson loop defects. We call the corresponding
index:
\[
[\chi]^{(0,\ldots,0)}_{1d} = \sum_{k^{(1)},\ldots,k^{(n)}=0}^{\infty} \prod_{a=1}^{n} \left( \frac{\tilde{q}^{(a)} k^{(a)}}{k^{(a)}!} \right) \prod_{b>a}^{n} Z_{\text{pure},b/a}^{i(a,b)}
\]

\[
\times \oint_{\mathcal{M}_k^{\text{pure}}} \left[ \frac{d\phi_f^{(a)}}{2\pi i} \right] Z_{\text{pure,vec}}^{(a)} Z_{\text{pure,adj}}^{(a)} Z_{\text{pure,teeth}}^{(a)} \prod_{b>a}^{n} Z_{\text{pure,bif}}^{i(a,b)} .
\]

Working in the \( \zeta_{1d} > 0 \) chamber, the poles that end up contributing to the \( T_{1d}^{\text{pure}} \) index make up the set \( \mathcal{M}_k^{\text{pure}} \). The elements of this set satisfy:

\[
\phi_f^{(a)} = \phi_f^{(a)}(I) - \epsilon_1 ,
\]

\[
\phi_f^{(a)} = \phi_f^{(a)}(J) - \epsilon_2 ,
\]

\[
\phi_f^{(a)} = \mu_i^{(a)} - \epsilon_-, \text{ for some } i \in \{1, \ldots, P^{(a)}\} ,
\]

\[
\phi_J^{(a)} = \phi_f^{(a)} , \text{ if there is a link between nodes } a \text{ and } b > a ,
\]

\[
\phi_J^{(a)} = \phi_f^{(a)} + 2\epsilon_+ , \text{ if there is a link between nodes } a \text{ and } b < a .
\]

The poles (2.20) arise from the \( N = 4 \) adjoint chiral factor,

\[
Z_{\text{pure,adj}}^{(a)} = \prod_{I,J=1}^{k^{(a)}} \frac{\text{sh} \left( \phi_f^{(a)}(I) - \phi_f^{(a)}(J) + \epsilon_1 + \epsilon_2 \right)}{\text{sh} \left( \phi_f^{(a)}(I) - \phi_f^{(a)}(J) + \epsilon_1 \right)}. \tag{2.25}
\]

The poles (2.21) arise from the \( N = 4 \) vector multiplet,

\[
Z_{\text{pure,vec}}^{(a)} = \frac{\prod_{I,J=1}^{k^{(a)}} \text{sh} \left( \phi_f^{(a)}(I) - \phi_f^{(a)}(J) \right)}{\prod_{I,J=1}^{k^{(a)}} \text{sh} \left( \phi_f^{(a)}(I) - \phi_f^{(a)}(J) + \epsilon_2 \right)}. \tag{2.26}
\]

The poles (2.22) arise from the \( N = 4 \) flavor factor,

\[
Z_{\text{pure,teeth}}^{(a)} = \prod_{I=1}^{k^{(a)}} \prod_{i=1}^{P^{(a)}} \frac{\text{sh} \left( \phi_f^{(a)}(I) - \mu_i^{(a)} + \epsilon_- + \epsilon_2 \right)}{\text{sh} \left( \phi_f^{(a)}(I) - \mu_i^{(a)} + \epsilon_- \right)} \prod_{j=1}^{Q^{(a)}} \frac{\text{sh} \left( \phi_f^{(a)}(I) + \tilde{\mu}_j^{(a)} + \epsilon_- + \epsilon_2 \right)}{\text{sh} \left( \phi_f^{(a)}(I) + \tilde{\mu}_j^{(a)} + \epsilon_- \right)}. \tag{2.27}
\]

Specifically, the JK contours enclose poles coming from the fundamental chirals only (the \( P^{(a)} \)-product), and none of the antifundamental chirals. We wrote the poles in terms of 1d flavor fugacities \( \{\mu_i^{(a)}\} \) as a shorthand notation, which are really placeholders for the rank(\( GF \)) 3d fundamental masses \( \{m_d^{(b)}\} \). The poles (2.23) and (2.24) are due to the
bifundamental contributions,

\[ Z_{\text{pure, bif}}^{(a,b)} = \left[ \prod_{l=1}^{k_{(a)}} \prod_{j=1}^{k_{(b)}} \frac{\sinh(\phi_{l}^{(b)} - \phi_{j}^{(a)} + \epsilon_{2})}{\sinh(\phi_{l}^{(b)} - \phi_{j}^{(a)})} \frac{\sinh(-\phi_{l}^{(b)} + \phi_{j}^{(a)} - \epsilon_{1})}{\sinh(-\phi_{l}^{(b)} + \phi_{j}^{(a)} - \epsilon_{1} - \epsilon_{2})} \right]^{\Delta_{ab}}. \] (2.28)

Two important remarks are in order. First, even though the contours enclose the JK-poles (2.21), the resulting residues are always trivial, because the numerators in \( Z_{\text{pure, teeth}}^{(a,b)} \) create a zero at this locus. Second, because of the bifundamental factor \( Z_{\text{pure, bif}}^{(a,b)} \), some of the enclosed poles are non-simple for generic rank \( k^{(a)} \). However, a careful application of the flag method to construct the JK-residue shows that the poles we enclose above make up an exhaustive list; this was checked numerically in [39]. Putting it all together, and writing the 1d fundamental chiral masses \( \{\mu_{i}^{(a)}\} \) in terms of the 3d masses \( \{m_{i}^{(b)}\} \), the various poles which end up contributing with nonzero residue are of the form

\[ \phi_{l}^{(a)} = m_{i}^{(b)} - \epsilon_{-} - (s_{i} - 1)\epsilon_{1} + 2 \#^{(ab)} \epsilon_{+}, \quad \text{with } s_{i} \in \{1, \ldots, k_{i}^{(a)}\}, \ i \in \{1, \ldots, N^{(a)}\}, \] (2.29)

for some mass index \( b \in \{1, \ldots, n\} \), and where \( (k_{1}^{(a)}, \ldots, k_{N^{(a)}}^{(a)}) \) is a partition of the vortex charge \( k^{(a)} \) into \( N^{(a)} \) non-negative integers. The pair of integers \( (i, s_{i}) \) is assigned to one of the integers \( I \in \{1, \ldots, k^{(a)}\} \) exactly once, and \( \#^{(ab)} \) is a non-negative integer equal to the number of links between nodes \( a \) and \( b < a \) in the Dynkin diagram of \( g \) (and \( \#^{(ab)} = 0 \) if \( b > a \)).

As an explicit example, consider the \( A_{n} \) theory \( G^{3d} = T_{\mu}[SU(N^{(n+1)})] \) from figure 1. Then, the poles with nonzero residue are all of the form

\[ \phi_{l}^{(a)} = m_{i}^{(n)} - \epsilon_{-} - (s_{i} - 1)\epsilon_{1}, \quad \text{with } s_{i} \in \{1, \ldots, k_{i}^{(a)}\}, \ i \in \{1, \ldots, N^{(a)}\}. \] (2.30)

and \( (k_{1}^{(a)}, \ldots, k_{N^{(a)}}^{(a)}) \) is a partition of \( k^{(a)} \) into \( N^{(a)} \) non-negative integers, and the pair of integers \( (i, s_{i}) \) is assigned to one of the integers \( I \in \{1, \ldots, k^{(a)}\} \) exactly once. Performing the residue integral, one finds the following closed-form expression, which is well-known [39]:

\[
[\chi]^{(0, \ldots, 0)}_{1d} = \sum_{k^{(1)}, \ldots, k^{(n)} = 0}^{\infty} \prod_{a=1}^{n} (\hat{g}^{(a)})^{k^{(a)}} \sum_{\sum_{i,j=1}^{N^{(a)}} k^{(a)} = k^{(a)}} \left[ \prod_{i,j=1}^{N^{(a)}-1} \prod_{s=1}^{k^{(a)}_{i,j}} \frac{\sinh(m_{i} - m_{j} + \epsilon_{2} - (s-1)\epsilon_{1})}{\sinh(m_{i} - m_{j} - (s-1)\epsilon_{1})} \right]^{\prod_{i,j=1}^{N^{(a)}-1} k^{(a)}_{i,j}} \times \left[ \prod_{i=1}^{N^{(a+1)}-1} \prod_{j=1}^{N^{(a)}} \prod_{p=1}^{k^{(a+1)}_{i,j}} \frac{\sinh(m_{i} - m_{j} + \epsilon_{2} + p\epsilon_{1})}{\sinh(m_{i} - m_{j} + p\epsilon_{1})} \right].
\] (2.31)
We now consider the vortex quantum mechanics of $G^{3d}$ in the presence of the Wilson loop, that is to say the index of $T^{1d}$ (2.12). For a given vortex number $k = \sum_{a=1}^{n} k^{(a)}$, the set of poles to be enclosed is denoted as $\mathcal{M}_k$. This set contains the set $\mathcal{M}_k^{pure}$ of poles we just reviewed for the theory $T^{1d}$ (the index in the absence of defect). There are also additional poles depending on the fermion masses $M^{(a)}$, which belong in the set $\mathcal{M}_k \setminus \mathcal{M}_k^{pure}$. Specifically, the new poles are of the form:

\[
\phi_{I}^{(a)} = M^{(a)}_{p} + \epsilon + , \quad \phi_{J}^{(b)} = \phi_{I}^{(a)} , \quad \text{if there is a link between nodes } a \text{ and } b > a , \quad \phi_{J}^{(b)} = \phi_{I}^{(a)} + 2\epsilon + , \quad \text{if there is a link between nodes } a \text{ and } b < a . \quad (2.32)
\]

The poles (2.32) arise because of the interactions between the vortices and the Wilson loop fermions,

\[
Z^{(a)}_{defect,k} = \prod_{i=1}^{k^{(a)}} \frac{\sh(\phi_{I}^{(a)} - M^{(a)}_{p} - \epsilon_{-}) \sh(-\phi_{I}^{(a)} + M^{(a)}_{p} - \epsilon_{-})}{\sh(\phi_{I}^{(a)} - M^{(a)}_{p} - \epsilon_{+}) \sh(-\phi_{I}^{(a)} + M^{(a)}_{p} - \epsilon_{+})} . \quad (2.35)
\]

The remaining poles (2.33) and (2.34) are again due to the bifundamental contributions,

\[
Z^{(a,b)}_{pure,bif} = \left[ \prod_{i=1}^{k^{(a)}} \prod_{j=1}^{k^{(b)}} \frac{\sh(\phi_{I}^{(b)} - \phi_{J}^{(a)} + \epsilon_{2}) \sh(-\phi_{I}^{(b)} + \phi_{J}^{(a)} - \epsilon_{1})}{\sh(\phi_{I}^{(b)} - \phi_{J}^{(a)}) \sh(-\phi_{I}^{(b)} + \phi_{J}^{(a)} - \epsilon_{1} - \epsilon_{2})} \right]^{\Delta_{ab}} . \quad (2.36)
\]

For a given vortex number $k$, we now argue that the content of the set $\mathcal{M}_k \setminus \mathcal{M}_k^{pure}$ makes it possible to reinterpret the index as the character of a finite-dimensional representation of a quantum affine algebra. In order to prove this, we define a new quantity, the vacuum expectation value of a loop defect operator on node $a$, with corresponding fermion mass $q^{(a)}_{p} \equiv z$:

\[
\left\langle Y^{(a)}_{1d}(z) \right\rangle \equiv \sum_{k^{(1)}, \ldots, k^{(n)} = 0}^{\infty} \prod_{b=1}^{n} \frac{(q^{(b)}_{p})^{k^{(b)}}}{k^{(b)!}} \times \oint_{\mathcal{M}_k^{pure}} \left[ \frac{d\phi_{I}^{(b)}}{2\pi i} \right] Z^{(b)}_{pure,vec} \cdot Z^{(b)}_{pure,adj} \cdot Z^{(b)}_{pure,teeth} \cdot \prod_{c>b} Z^{(b,c)}_{pure,bif} \cdot \left[ Z^{(a)}_{defect,k}(z) \right]^{\pm 1} . \quad (2.37)
\]

Even though the defect factor $Z^{(a)}_{defect,k}(z)$ is present inside the integrand, the contour integral is defined to only enclose poles in the set $\mathcal{M}_k^{pure}$, the same poles as in the pure index (2.19).

Remarkably, the index of $T^{1d}$ can be written as a finite Laurent series in such $Y$-operator vevs. In order to be as concise as possible, we find it convenient to normalize the index by the classical Wilson loop contribution and the index of the vortex quantum mechanics
As usual, \( T_{pure}^{1d} \), in the absence of Wilson loop:

\[
\langle \chi \rangle_{1d}(L^{(1)},...,L^{(n)}) (\{z^a_{\rho} \}) \equiv \frac{\prod_{a=1}^n L^{(a)}(z^a_{\rho})}{\prod_{b=1}^L \chi_{1d}(0,...,0)}
\]

As a function of the fermion masses \( \{z^a_{\rho} \} \), our first main result is that the normalized index can be written as:

\[
\langle \chi \rangle_{1d}(L^{(1)},...,L^{(n)}) (\{z^a_{\rho} \}) = \frac{1}{\langle \chi \rangle_{1d}(0,...,0)} \sum_{\omega \in V(\lambda)} \prod_{b=1}^n (q^{(b)})^{d^b_{\omega}} c_{d^b_{\omega}}(q,t) \left( Q_{d^b_{\omega}}^{(b)} (\{z^a_{\rho} \}) \right) \left[ Y_{1d}(\{z^a_{\rho} \}) \right]_\omega.
\]

(2.39)

We will prove this statement momentarily. For now, let us unpack the notation.

\( \{z^a_{\rho} \} \) denotes collectively the \( \sum_{a=1}^n L^{(a)} \) fermion masses \( z^a_{\rho} \equiv e^{-RM^a_{\rho}} \). The sum runs over all the weights \( \omega \) of the finite-dimensional irreducible representation \( V(\lambda) \) of the quantum affine algebra \( U_q(\hat{g}) \), with highest weight \( \lambda = \sum_{a=1}^n L^{(a)} \lambda_a \). Here, \( g \) is the simply-laced Lie algebra denoting the 3d quiver gauge theory (as well as its vortex quantum mechanics), and \( \lambda_a \) the \( a \)-th fundamental weight of \( g \). The label \( d^b_{\omega} \) is a positive integer that is determined by solving

\[
\omega = \lambda - \sum_{b=1}^n d^b_{\omega} \alpha_b.
\]

(2.40)

Namely, a given weight \( \omega \) is reached by lowering the highest weight \( \lambda \) a finite number of times, using the positive simple roots \( \{\alpha_b\}_{b=1,...,n} \). This procedure is referred to as building the weight \( \omega \) out of \( sl_2 \) strings. The equivariant parameter \( \tilde{q}^{(b)} \) is the 3d FI parameter for the \( b \)-th gauge group.

The factors \( c_{d^b_{\omega}}(q,t) \) are coefficients depending only on \( q \) and \( t \).

The function \( Q_{d^b_{\omega}}^{(b)} (\{z^a_{\rho} \}) \) is the residue of \( Z_{pure, teeth} \) evaluated at the poles (2.32), (2.33), and (2.34). The function is therefore made up of fundamental and antifundamental chiral multiplet contributions, such as:

\[
\prod_{a=1}^n P^{(b)} \prod_{b=1}^L \prod_{i=1}^L \prod_{j=1}^L \left( \frac{1 - v \#_{i,j}^{(b)} f_i^{(b)} / z_{\rho}^{(a)}}{1 - v \#_{i,j}^{(b)} f_i^{(b)} / z_{\rho}^{(a)}} \right) \prod_{j=1}^L \left( \frac{1 - t v \#_{j}^{(b)} f_j^{(b)} / z_{\rho}^{(a)}}{1 - t v \#_{j}^{(b)} f_j^{(b)} / z_{\rho}^{(a)}} \right)
\]

(2.41)

As usual, \( P^{(b)} \) stands for the number of fundamental chiral at node \( b \), with masses \( \{f_i^{(b)} \} \), while \( Q^{(b)} \) stands for the number of antifundamental chiral at node \( b \), with masses \( \{f_j^{(b)} \} \).

The symbols \( \#_{i,j}^{(a)} \) and \( \#_{j}^{(a)} \) stand for other non-negative integers, which are fixed by the choice of the 3d Higgs vacuum.

Finally, the operator \( Y_{1d}(\{z^a_{\rho} \}) \), for a given weight \( \omega \), is the expectation value of
a rational function of $Y$-operators $\left\langle \Pi_\alpha \left[ Y_{id}^{(a)} \right]^{\pm 1} \right\rangle$ and derivatives thereof$^9$, where each operator $\left[ Y_{id}^{(a)} \right]^{\pm 1}$ is a function of a fermion mass $z_\rho^{(a)}$. The arguments of each factor is shifted by powers of $q$ and $t$, determined uniquely from (2.40).

All in all, the index is a twisted$^{10}$ character of a finite dimensional irreducible representation $V(\lambda)$ of $U_q(\mathfrak{g})$, with highest weight $\lambda = \sum_{a=1}^{n} L^{(a)} \lambda_a$. Starting with the highest weight $\lambda$, each term in the character can then be obtained by successive “vortex-Weyl” reflections, which generalize the usual Weyl group action of the Lie algebra $\mathfrak{g}$. Because of the dependence on the two fugacities $q = e^{Rt_1}$ and $t = e^{-R_{t_2}}$, the vortex character is a $qq$-character, in the denomination of [2].

Two remarks are in order. First, similar $qq$-characters have been constructed in the related context of counting instantons in the presence of a 1/2-BPS Wilson loop on the manifold $\mathbb{C}^2 \times S^1(\hat{R})$. There, the functional form of the character, meaning its dependence on the $Y$-operators $\left[ Y_{id}\{z_\rho^{(a)}\}\right]_\omega$ and on the weights $\omega$, is identical to what we found here in the context of vortex counting. This is because the $Y$-operator dependence is entirely fixed by the choice of the algebra $\mathfrak{g}$ and the representation $(L^{(1)}, \ldots, L^{(n)})$ in which the Wilson loop transforms. In particular, the functional form of the character does not depend on whether we study instanton or vortex counting, nor does it depend on the dimension of the manifold. Of course, there are still notable differences according to which gauge theory setup we study: this is encoded in the expressions for the vevs $\langle \ldots \rangle$, and the functions $Q^{(b)}(\{z_\rho^{(a)}\})$ in (2.39). For instance, in the context of an instanton quantum mechanics, these functions are contributions of $\mathcal{N} = 2$ Fermi multiplets exclusively, while in the vortex context, we found here that the functions are made of both $\mathcal{N} = 2$ Fermi and chiral multiplets.

Second, one can consider the limit where we shrink the circle size to zero. There are a priori many ways to take this limit, so we should be specific: here, we require that all flavor fugacities of the quantum mechanics remain fixed as we take $\hat{R} \to 0$. In practice, all the trigonometric functions present in the 1-loop determinants of the quantum mechanics index will become rational functions of their arguments instead. The 3d gauge theory $G^{3d}$ turns into a 2d $\mathcal{N} = (4, 4)$ gauged sigma model on $\mathbb{C}^2$, and the Wilson loop wrapping $S^1(\hat{R})$ becomes a 1/2-BPS point defect at the origin. Correspondingly, the index we computed becomes a vortex $qq$-character of the 2d theory, whose general form was first conjectured

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$^9$An example where such a derivative term can appear is the index of the $D_4$ theory with a fundamental Wilson loop insertion on node 2, $\chi_D^{(1.0.0)}(z_2^{(2)})$. The partition function organizes itself as a Laurent series of 29 $Y$-operator terms, one of which involves derivatives of $Y_{id}^{(a)}$-operators. Note that the second fundamental representation of $D_4$ is only 28-dimensional. However, finite dimensional irreducible representations of quantum affine algebras are notoriously bigger than their non-affine counterpart. Indeed, the second fundamental representation $V(\lambda_2)$ of $U_q(D_4)$ decomposes into irreducible representations of $U_q(D_4)$ as $V(\lambda_2) = 28 \oplus 1$. Put differently, one necessarily has to add the trivial representation 1, an extra null weight, to the 28 in order to obtain an irreducible representation of $U_q(D_4)$.

$^{10}$The character is twisted because of the presence of the 3d FI parameters $\hat{q}^{(b)}$ and the flavor matter factors $Q^{(b)}$. 
in [8] (section 7). Our work in this section can be seen as a microscopic derivation of the expression presented there.

It remains to prove that the index of $T^{1d}$ is indeed equal to the character expression (2.39). Since a fully explicit proof would require the knowledge of a specific Higgs vacuum for the 3d theory, we find it more worthwhile to outline the universal features of the proof here in the general case, and showcase it in detail later when discussing an example in section 7. The proof consists of two parts. First, recall that the contours of the index enclose the poles $\mathcal{M}_k$, for a given vortex number $k$. In contrast, the contours of the $Y$-operator (2.37) only enclose the poles $\mathcal{M}_k^{\text{pure}}$ of the index in the absence of Wilson loop. Because the set $\mathcal{M}_k \setminus \mathcal{M}_k^{\text{pure}}$ is non-empty for every $k$, it follows that the index has following expansion:

$$
[X]_{1d}^{(L^1,\ldots,L^n)}(\{\rho(a)\}) = \prod_{a=1}^{n} \prod_{\rho=1}^{L(a)} Z_{\text{defect},\phi}(z_\rho) \left( \prod_{a=1}^{n} \prod_{\rho=1}^{L(a)} Y_{1d}(z_\rho) \right) + \ldots ,
$$

(2.42)

where each term in the dots “...” stands for a residue of $[X]_{1d}^{(L^1,\ldots,L^n)}$ at one of the poles (2.32), (2.33), or (2.34), in $\mathcal{M}_k \setminus \mathcal{M}_k^{\text{pure}}$. These extra poles making up the dotted terms need to be included, as dictated by the JK-prescription, and our first observation is that there is only a finite number of them. This last point is highly nontrivial, and is derived from the explicit form of the integrand (2.39). As an example, suppose that there exists a pole of the first kind (2.32), at the locus $\phi_I^{(a)} - M_{\rho,\ast}^{(a)} - \epsilon_+ = 0$, for some $I \in \{1,\ldots,k^{(a)}\}$. Then, there exist no pole at the locus $\phi_J^{(a)} - M_{\rho,\ast}^{(a)} - \epsilon_+ = 0$ for any $J \neq I$. This is because there is a zero at the locus $\phi_I^{(a)} - \phi_J^{(a)} = 0$, due to the numerator in $Z_{\text{pure,vec}}^{(a)}$. Similarly, the JK-residue prescription predicts a pole at the locus $\phi_I^{(a)} - \phi_I^{(a)} + \epsilon_1 = 0$, due to the denominator of $Z_{\text{pure,adj}}^{(a)}$, and a pole at the locus $\phi_J^{(a)} - \phi_I^{(a)} + \epsilon_2 = 0$, due to the denominator of $Z_{\text{pure,vec}}^{(a)}$. However, there is a zero at both loci, due to the zeros at the numerators of $Z_{\text{defect},k}^{(a)}$. All in all, the locus (2.32) contributes a single new $M_{\rho,\ast}^{(a)}$-pole to the index. One can similarly show that the only other $M_{\rho,\ast}^{(a)}$-poles are exclusively due to the loci (2.33), (2.34), and that this list of poles is bounded above for all $k$. Namely, for all vortex number $k$, the size of the set $\mathcal{M}_k \setminus \mathcal{M}_k^{\text{pure}}$ is always smaller or equal to some fixed integer $k'$. By carrying out the JK-residue procedure explicitly, we can determine $k'$ exactly: one finds that $k' + 1$ is equal to the dimension of the finite-dimensional irreducible representation of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ with highest weight $(L^1,\ldots,L^n)$. From this perspective, the first term in (2.42) before the dots is nothing but the highest weight of the representation. This ends the first part of the proof.

It remains to show that each of the $k'$ terms in the dotted expansion (2.42) is precisely of the form (2.39). This follows from a remarkable fact, which again can be proved by direct computation: for a given vortex number $k$, each contour enclosing $j'$ of the $k'$ poles in $\mathcal{M}_k \setminus \mathcal{M}_k^{\text{pure}}$ can be traded for an integration contour which encloses $k - j'$ poles only, where $j' = 1,\ldots,k'$. The price to pay for such a trade of contours is the introduction in
the integrand of extra $Y$-operator insertions, along with the residue at the $j'$ poles of the chiral matter factors $Z^{(a)}_{\text{pure, teeth}}$. Performing this change of contours for all $k'$ dotted terms, and normalizing by the classical Wilson loop contribution $\prod_{\alpha=1}^{n} \prod_{\rho=1}^{L^{(a)}_{\text{defect, } \phi}} Z^{(a)}_{\text{defect, } \phi}(z^{(a)}_{\rho})$, we arrive at the advertised expression for the vortex $qq$-character.

Figure 3: The black crosses denote poles in the set $M^{\text{pure}}_k$, from the pure index, while the red dot denotes a pole in the set $M_k \setminus M^{\text{pure}}_k$. Such a pole is due to the factor $Z^{(a)}_{\text{defect, } \phi}$ in the integrand. On the left, we show a possible contour for the computation of the index at $k = 1$. Note that by the JK prescription, we must in particular enclose the new pole in red. Remarkably, it is equivalent to trade this contour for the one on the right, which now only encloses the poles in the set $M^{\text{pure}}_1$, but with a modified integrand; in the latter contour, the integrand will now contain insertions of additional $Y$-operators, with a vortex charge shift of one unit to account for the missing pole.

We emphasize that at no point in the discussion did we need to know the content of the set $M^{\text{pure}}_k$, that is to say the poles of the quantum mechanics index in the absence of Wilson loop. What is instead relevant here to derive the $qq$-character is the set of poles $M_k \setminus M^{\text{pure}}_k$, due entirely to the insertion of the defect. We now explain why this vortex character can be understood as a non-perturbative Schwinger-Dyson equation for the 3d theory $G^{3d}$.

2.5 Physics of the Schwinger-Dyson Equations

Let us focus on the case of a fundamental Wilson loop on the $a$-th node of $G^{3d}$: $L^{(a)} = 1$ for some $a \in \{1, \ldots, n\}$ and $L^{(b)} = 0$ for $b \neq a$. Correspondingly, the defect group is $\hat{G}_{\text{defect}} = U(1)$ in that case, parameterized by the fermion mass $z^{(a)}_{1} \equiv z$. 

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The $Y$-operator we constructed mediates the change in vortex charge $k$ of the theory $G^{3d}$. More precisely, the vev $\langle Y^{(a)}_{1d}(z) \rangle$ represents the insertion of a Wilson loop with fermion mass $z$ on node $a$, enabling vortex particles to appear and disappear out of the bulk. This changes the topological sector of $G^{3d}$, and the $qq$-character (2.39) encodes a corresponding quantum affine symmetry of the theory. Put differently, the Schwinger-Dyson equation for the $Y$-operator vev is the statement that even though $\langle Y^{(a)}_{1d}(z) \rangle$ is singular in $z$, as is obvious from the explicit expression (2.37), a particular Laurent series of $Y$-operator vevs, the $qq$-character, has regularity properties in $z$. The precise statement of the Schwinger-Dyson equations is as follows:

$$[\chi^{(0,...,0,1,0,...,0)}_{1d}] (z) \text{ is regular in } z \text{ except for the poles of the function } Q^{(b)}_{d_b}(z) \text{ in (2.39).}$$

To prove this statement, one has to show that the residues at the various poles of the index do not develop singularities in the variable $z = e^{-M}$, other than at the denominators of $Q^{(b)}_{d_b}(z)$. A singularity in $z$ will arise if two poles of the integrand pinch one of the contours. It turns out that most potential singularities are canceled in a subtle manner by zeroes in the integrand, resulting in an almost regular structure in $z$. Indeed, a finite set of $z$-singularities is produced after integration, and makes up the various poles of the function $Q^{(b)}_{d_b}(z)$.

For instance, let $a \in \{1,\ldots,n\}$ and $I \in \{1,\ldots,k^{(a)}\}$, and consider the following two loci of poles in the integrand:

$$\phi^{(a)}_I - M - \epsilon_+ = 0 \quad \text{and} \quad \phi^{(a)}_I - \tilde{\mu}^{(a)}_j - \epsilon_+ = 0 \text{ for some mass } \{\tilde{\mu}^{(a)}_j\} . \quad (2.43)$$

The first pole locus is due to the denominator of $Z^{(a)}_{k^{defect,k}}$, while the second pole locus is due to the denominator of $Z^{(a)}_{k^{pure,teeth}}$, the antifundamental chiral matter contribution. By the JK-residue prescription, the first locus is inside the integration contour, as we saw in (2.32). Furthermore, the JK-residue instructs us to only enclose the $\{\mu^{(a)}_j\}$-poles coming from fundamental chiral multiplets (2.22), and none of the $\{\tilde{\mu}^{(a)}_j\}$-poles coming from antifundamental chiral multiplets. It follows that the second locus is outside the integration contour. Then, the poles can freely coalesce and pinch the contour, resulting in the singular locus:

$$M = \tilde{\mu}^{(a)}_j . \quad (2.44)$$

This singularity manifests itself as a simple pole in the function $Q^{(b)}_{d_b}(z)$.

Given a generic theory, providing the comprehensive list $z$-singularities in the index is a tedious exercise, though it presents no technical difficulties; one simply proceeds as above, analyzing the various sets of poles which can potentially pinch the contours. We will carry out this procedure in detail when presenting an example in section 7.

As a last remark, note that this discussion straightforwardly generalizes to a Wilson loop in an arbitrary irreducible representation of the 3d gauge group $G$. In that case, the Schwinger-Dyson equations are still regularity conditions on the associated $qq$-character,
but involving correlation functions of a higher number of $Y$-operators. Typically, the index 
$[\chi_{\lambda_1}^{(L_1^{(a)}, \ldots, L_n^{(a)})}(\{z^{(a)}_\rho\})]$ will develop even more singularities in the defect fermion masses $\{z^{(a)}_\rho\}$.

We now give an alternate derivation of the non-perturbative Schwinger-Dyson equations obeyed by 3d $\mathcal{N} = 4$ gauge theories, directly from three dimensions and without resorting to its vortex quantum mechanics.

3 Schwinger-Dyson Equations: the Three-Dimensional Perspective

Consider a 3d supersymmetric gauge theory on a 3-manifold. There is by now overwhelming evidence that the partition function on such a space, with adequate twists, contains information about the vortex sector of the theory [35, 62–72]$.^{11}$ For instance, in the case where the 3-manifold is a $\mathbb{C}$-bundle over $S^1$, and the theory is the $\mathcal{N} = 4$ quiver $G^{3d}$ we previously considered, the vortex part of the 3d partition function is precisely the index of the quantum mechanics $T^{1d}_{\text{pure}}$ from last section [39].

In the spirit of the above body of works, in this section we propose a half-index for $G^{3d}$ on $\mathbb{C} \times S^1(\tilde{R})$ in the presence of a 1/2-BPS Wilson loop wrapping $S^1(\tilde{R})$, and derive the vortex $qq$-character from it. The definition of the index is quite nontrivial in the 3d picture, but we will argue that it is sensible since it correctly reproduces the results we derived in the vortex quantum mechanics picture for $T^{1d}$.

3.1 A Half-Index Presentation

Let us first review how to define a half-index for $G^{3d}$ on $\mathbb{C} \times S^1(\tilde{R})$ in the absence of Wilson loop. This index is also referred to as a holomorphic block [35]$^{12}$. We first consider the 3-manifold in the $\Omega$-background, to regularize the non-compactness of $\mathbb{C}$; namely, if we let $z$ be a complex coordinate on the complex line, we can view the 3-manifold as a $\mathbb{C}$-bundle over $S^1(\tilde{R})$, where as we go around the circle, we make the identification

$$ z \sim z e^{\tilde{R}\epsilon_1}, \quad \epsilon_1 \in \mathbb{R}. \quad (3.1) $$

$^{11}$Similar results exist for 2d gauge theories on 2-manifolds, see for instance [73, 74]

$^{12}$Formally, a holomorphic block is defined in the IR of the 3d $\mathcal{N} = 4$ theory: on the manifold $\mathbb{C} \times S^1(\tilde{R})$, one has to specify boundary conditions at infinity on $\mathbb{C}$ by choosing an IR vacuum. Alternatively, one can consider a “half-index” on $D \times S^1(\tilde{R})$, where $D$ is a finite disk, with boundary conditions defined in the UV on the edge of the disk. The boundary conditions will flow to some boundary condition in the IR which may or may not agree with the one defined in the holomorphic block formalism. It would be important to explore these subtleties in our context.
From now on, we denote the \( \mathbb{C} \)-line in this background as \( \mathbb{C}_q \), with \( q = e^{\hat{R} \epsilon_1} \). Then, the partition function of \( G^{3d} \) is defined via the following half-index:

\[
\left[ \tilde{\chi}_{3d}^{(0, \ldots, 0)} \right] = \text{Tr} \left[ (-1)^F e^{-\hat{R} \{ Q, \overline{Q} \}} q^{S_1 - S_R} t^{S_2 + S_R} \prod_{a=1}^{n} (\tilde{q}^{(a)})^{k(a)} \prod_{d=1}^{N_d} (\alpha_d^{(a)} \Pi_d^{(a)}) \right]. \tag{3.2}
\]

The trace is taken over the Hilbert space of states on \( \mathbb{C}_q \). The index counts states in \( Q \)-cohomology, where \( Q = Q_{1,1} \) and \( \overline{Q} = Q_{2,2} \) were defined in section 2.1. \( F \) is the fermion number. \( S_1 \) is a rotation generator for \( \mathbb{C}_q \), while \( S_2 \) is a R-symmetry generator; indeed, \( S_2 \) generates a \( U(1)_C \) symmetry which is a subgroup of the \( SU(2)_C \) R-symmetry acting on the vector multiplet scalars. Meanwhile, \( S_H \) generates a \( U(1)_H \) symmetry which is a subgroup of the \( SU(2)_H \) R-symmetry acting on the hypermultiplet scalars. \( \{ \Pi_d^{(a)} \} \) are Cartan generators for the flavor group \( G_F \), with conjugate fundamental masses \( \{ m_d^{(a)} \} \). The integer \( k(a) = -\frac{1}{2\pi} \int \text{Tr} F(a) \) is the topological \( U(1) \) charge for the \( a \)-th gauge group, and the conjugate fugacity \( \tilde{q}^{(a)} \) is the real FI parameter on node \( a \), complexified by the holonomy of the corresponding background gauge field around \( S^1(\hat{R}) \). The field configurations which preserve the supersymmetries of the index are solutions to the vortex equations on \( \mathbb{C} \); the integers \( k(a) \) then provide a natural grading on the moduli space of vortices.

So far we have not talked about the gauge symmetry group \( G = \prod_{a=1}^{n} U(N^{(a)}) \). We start by treating it as a global symmetry, which we make abelian by breaking it to its maximal torus. The associated equivariant parameters are denoted collectively as “\( y \)”. We then gauge the symmetry by projecting to \( G \)-invariant states, which amounts to integrating over those parameters. Namely,

\[
\oint d_{\text{Haar}} y = \oint \prod_{a=1}^{n} \prod_{i=1}^{N(a)} \frac{dy_i^{(a)}}{y_i^{(a)}}. \tag{3.3}
\]

Above, the contour is chosen to project to states neutral under the \( G \)-symmetry. Because the parameters \( \{ y_i^{(a)} \} \) parameterize part of the Coulomb branch of \( G^{3d} \), this presentation of the index is referred to as Coulomb branch localization:

\[
\left[ \tilde{\chi}_{3d}^{(0, \ldots, 0)} \right] = \oint_{\mathcal{M}^{\text{bulk}}} dy \left[ I_{3d}^{\text{bulk}}(y) \right]. \tag{3.4}
\]

The choice of contours \( \mathcal{M}^{\text{bulk}} \) determines a vacuum for \( G^{3d} \). In this three-dimensional setup, the contours are once again fixed by the JK-residue prescription, where we choose to work with the auxiliary vector \( \eta = (1, \ldots, 1) \), as we did before. The integrand \( I_{3d}^{\text{bulk}}(y) \) stands for the contribution of all the various multiplets to the index. These can be read off directly from the 3d \( \mathcal{N} = 4 \) quiver description of the theory. This bulk contribution has the form
The factor
\[ I_{\text{vec}}(y) = \prod_{a=1}^{n} \prod_{i=1}^{N(a)} \left( y_i^{(a)} / y_j^{(a)} \right) \prod_{1 \leq i < j \leq N(a)} \left( t y_i^{(a)} / y_j^{(a)} ; q \right) \theta \left( t y_i^{(a)} / y_j^{(a)} ; q \right) \theta \left( y_i^{(a)} / y_j^{(a)} ; q \right) \]

(3.7)

stands for the contribution of a $\mathcal{N} = 4$ vector multiplet for the gauge group $U(N(a))$. Above, we use the following definitions of the $q$-Pochhammer symbol,
\[ (x ; q)_{\infty} \equiv \prod_{l=0}^{\infty} \left( 1 - q^l x \right), \]

(3.8)

and of the theta function,
\[ \Theta (x ; q) \equiv (x ; q)_{\infty} (q/x ; q)_{\infty}. \]

(3.9)

In particular, decomposing the $\mathcal{N} = 4$ vector multiplet as a $\mathcal{N} = 2$ vector multiplet and $\mathcal{N} = 2$ adjoint chiral multiplet, the numerator factor $\left( y_i^{(a)} / y_j^{(a)} ; q \right)_{\infty}$ is the contribution of the W-bosons in the $\mathcal{N} = 2$ vector multiplet, while the denominator factor $\left( t y_i^{(a)} / y_j^{(a)} ; q \right)_{\infty}$ is the contribution of the $\mathcal{N} = 2$ adjoint chiral multiplet\textsuperscript{13}.

The factor
\[ I_{\text{bif}}^{(a,b)}(y) = \prod_{1 \leq i \leq D(a)} \prod_{1 \leq j \leq D(b)} \left[ \left( t y_i^{(a)} / y_j^{(b)} ; q \right)_{\infty} \right]^{\Delta_{ab}} \]

(3.10)
is the contribution of $\mathcal{N} = 4$ bifundamental hypermultiplets. We use the same notations as

\textsuperscript{13}The ratio of theta functions has a natural interpretation when the manifold is thought of as $D \times S^1(\hat{R})$, with $D$ the disk. The boundary of that manifold is $S^1 \times S^1 = T^2$, and one needs to specify the 2d theory on this torus $T^2$. In principle, any choice of 2d $\mathcal{N} = (0,2)$ boundary conditions will do, as long as the theory is anomaly-free. In our context, one should specify the 3d chiral multiplet boundary conditions, which are either Dirichlet or Neumann. The gauge fields have Neumann boundary conditions, and the appearance of theta functions in the 3d vector multiplet is understood as the contribution of the 2d elliptic genus on the boundary torus. For details, see [71, 76, 77], and the related discussion in [75].
introduced previously: $\Delta^{ab}$ is the incidence matrix of the Lie algebra $\mathfrak{g}$, and $v = \sqrt{q/t}$.

The factor

$$I^{(a)}_{\text{flavor}}(y, \{x_d^{(a)}\}) = \prod_{d=1}^{N_F} \prod_{i=1}^{N^{(a)}} \frac{(v x_d^{(a)}/y_i^{(a)}; q)_{\infty}}{(v x_d^{(a)}/y_i^{(a)}; q)_{\infty}} \quad (3.11)$$

stands for the contribution of $\mathcal{N} = 4$ hypermultiplets in the fundamental representation of the $a$-th gauge group. The $\mathcal{N} = 4$ supersymmetry fixes the R-charge assignments of the various fugacities in the arguments of the $q$-Pochhammer symbols. In particular, note the presence of a cubic superpotential term due to the bifundamental/adjoint chiral multiplets in the $\mathcal{N} = 2$ language.

Let us briefly discuss the contours: there are three distinct sets of poles in the set $\mathcal{M}^{\text{bulk}}$, following the JK-residue prescription: the first is due to the denominators \(\left(v x_d^{(a)}/y_i^{(a)}; q\right)_{\infty}\) from the fundamental matter contribution (3.11), resulting in:

$$y_i^{(a)} = v x_d^{(a)} q^s \quad , \quad s = 0, 1, 2, \ldots , \quad d \in \{1, \ldots , N^{(a)}\} . \quad (3.12)$$

Second, there are the denominators $\left(v y_j^{(a)}/y_j^{(b)}; q\right)_{\infty}$ from the bifundamentals (3.10), resulting in:

$$y_i^{(b)} = v y_j^{(a)} q^s \quad , \quad s = 0, 1, 2, \ldots , \text{if there is a link between nodes } a \text{ and } b > a . \quad (3.13)$$

Third, there are the denominators $\left(t \ y_i^{(a)}/y_j^{(a)}; q\right)_{\infty}$ from the vector multiplets (3.7). However, such $t$-dependent poles turn out to have vanishing residue, as can be easily checked$^{14}$. The above pole structure makes explicit the grading over the vortex charge, so we naturally denote the set of poles as $\mathcal{M}^{\text{bulk}}_k$, summed over all $k = \sum_{a=1}^{n} k^{(a)}$.

We now want to introduce a 1/2-BPS Wilson loop wrapping $S^1(\hat{R})$, which is a codimension-2 defect from the point of view of $G^{3d}$. As we reviewed, the fact that this is possible in the first place is because such a loop preserves the supersymmetries $Q$ and $\bar{Q}$. We couple the one-dimensional $\mathcal{N} = 4$ theory on the loop to the bulk three-dimensional theory by considering the flavor symmetries of the 1d theory and gauging them with 3d $\mathcal{N} = 4$ vector multiplets. From the point of view of the index, this translates into gauging the 1d masses, turning them into the scalars of the corresponding 3d $\mathcal{N} = 4$ vector multiplets. When the vector multiplet is dynamical, the scalar becomes an eigenvalue $y$ to be integrated over, while in the case of a background vector multiplet, the scalar becomes a mass from the 3d point of view. To achieve this, we start by defining a defect $Y$-operator vacuum

---

$^{14}$The fact that poles of the third kind give a vanishing residue is characteristic of indices for 3d theories with $\mathcal{N} = 4$ supersymmetry. In particular, our argument does not apply to 3d theories with less supersymmetry. A similar phenomenon was noted in the 1d vortex quantum mechanics index.
expectation value, written as an integral over the Coulomb moduli of the 3d theory:

\[
\left\langle \left[ \tilde{Y}_{3d}^{(a)}(z) \right]^{\pm 1} \right\rangle \equiv \oint_{\mathcal{M}_{\text{bulk}}} dy \left[ I_{3d}^{\text{bulk}}(y) \cdot \left[ \tilde{Y}_{\text{defect}}^{(a)}(y, z) \right]^{\pm 1} \right]. \tag{3.14}
\]

For now, let us simply state that the integrand factor is defined as

\[
\tilde{Y}_{\text{defect}}^{(a)}(y, z) = \prod_{i=1}^{N(a)} \frac{1 - t y_i^{(a)}/z}{1 - y_i^{(a)}/z}. \tag{3.15}
\]

There is another piece of the defect $Y$-operator which is not integrated over, as it couples the loop to the flavor symmetry of $G^{3d}$; this contribution has the generic form

\[
\tilde{Y}_{\text{flavor}}^{(a)}(\{x_d^{(b)}\}, z) = \prod_{b=1}^{N_F} \prod_{d=1}^{n} \frac{1 - t v^{(ab)}/z}{1 - t v^{(ab)}/z}, \tag{3.16}
\]

where $\#^{(ab)}$ is a non-negative integer equal to the number of links between nodes $a$ and $b$ in the Dynkin diagram of $g$.

Note that the contour definition for the above $Y$-operator vev (3.14) is the same as the contour definition for the 3d index (3.4) in the absence of defect. In particular, the contours are defined not to enclose the potential $z$-poles from the factor $\prod_{a=1}^{n} \tilde{Y}_{\text{defect}}^{(a)}$.

Then, we define the (normalized) half-index of $G^{3d}$ in the presence of a Wilson loop, or 3d/1d index for short, as the expansion:

\[
\left[ Y_{3d}^{(L^{(1)}, \ldots, L^{(n)})}(\{z^{(a)}_p\}) \right]_\omega = \frac{1}{[\lambda]_{3d}} \sum_{\omega \in V(\lambda)} \prod_{b=1}^{n} \left( \tilde{Q}_{d_b}^{(b)}(\{z^{(a)}_p\}) \right) \left[ Y_{3d}(\{z^{(a)}_p\}) \right]_\omega. \tag{3.17}
\]

In the above, the factor $[Y_{3d}(\{z^{(a)}_p\})]_\omega$ is defined as the vev of a rational function of the $Y$-operators $\left\langle \prod_a \left[ Y_{3d}^{(a)} \right]^{\pm 1} \right\rangle$ (3.14), and possible derivatives thereof. All the other functions and notations appearing above are the same as were introduced in section 2.4\textsuperscript{15}. This implies that the index is once again a twisted $qq$-character of a finite dimensional irreducible representation $V(\lambda)$ of the quantum affine algebra $U_q(\widehat{g})$, with highest weight $\lambda = \sum_{a=1}^{m} L^{(a)} \lambda_a$.

The above definition of the 3d/1d half-index seems ad-hoc from the three-dimensional

\textsuperscript{15} There is one subtle difference: in the quantum mechanics $T^{1d}$, the function $Q_{d_b}^{(b)}(z^{(a)}_p)$ was the residue of $Z_{\text{bare, teeth}}^{(b)}$ at the various poles of $\mathcal{M}_k \setminus \mathcal{M}_k^{\text{bare}}$. In the 3d setup used here, we have defined a function $\tilde{Q}_{d_b}^{(b)}(z^{(a)}_p)$, which is still written as contributions of 1d $\mathcal{N} = 4$ chirals, but not quite the same expressions as in the quantum mechanics. Giving exact formulas would require specifying the theory $G^{3d}$, see section 7 for a detailed example.
perspective, but we will see later that it is in fact very natural in the light of the BPS/CFT correspondence. We end this section by exhibiting the relation between the 3d/1d index (3.17) and the Witten index (2.39) of the 1d quantum mechanics: up to normalization, they turn out to be one and the same!

### 3.2 Relation between the 3d and 1d Expressions for the \( qq \)-character

As we reviewed, the choice of contour for the 3d half-index fixes a vacuum for \( G^{3d} \). Let \( T^\text{1d}_{\text{pure}} \) be the vortex quantum mechanics defined on that vacuum, and let \( T^\text{1d} \) be the vortex quantum mechanics in the presence of a Wilson loop. We now prove that the index of \( T^\text{1d} \) is, up to a constant factor, the 3d/1d half-index introduced above. The proof rests on establishing a relation between the Wilson loop \( Y \)-operator vev \( \langle \tilde{Y}_{3d}^{(a)}(z) \rangle \) and its quantum mechanical counterpart \( \langle Y_{1d}^{(a)}(z) \rangle \). First recall that in defining the 1d \( Y \)-operator vev (2.37), one sums over the poles (2.29) in the set \( \mathcal{M}_k^{\text{pure}} \) for each vortex charge \( k = \sum_a k^{(a)} \),

\[
\phi_{\gamma}^{(a)} = \mu^{(b)}_i - \epsilon_+ - (s_i - 1)\epsilon_1 + 2 \#(ab)\epsilon_+, \quad \text{with } s_i \in \{1, \ldots, k^{(a)}_i\}, \quad i \in \{1, \ldots, N^{(a)}\},
\]

(3.18)

for some mass index \( b \in \{1, \ldots, n\} \). Recall that in the notation above, \( (k^{(a)}_1, \ldots, k^{(a)}_{N^{(a)}}) \) is a partition of the vortex charge \( k^{(a)} \) into \( N^{(a)} \) non-negative integers, and \( \#(ab) \) is a non-negative integer equal to the number of links between nodes \( a \) and \( b \) in the Dynkin diagram of \( g \). We write the collection of all such \( \phi \)-loci as \( \tilde{\phi}_s \). After performing the residue sum, the \( Y \)-operator vev can be schematically written as:

\[
\langle Y_{1d}^{(a)}(M) \rangle = \sum_{k=0}^{\infty} \sum_{\tilde{\phi}_s \in \mathcal{M}_k^{\text{pure}}} Z^\text{1d}_{\text{pure}}(\tilde{\phi}_s) \cdot Z^\text{1d}_{\text{defect},k}(\tilde{\phi}_s, M), \quad (3.19)
\]

where we collected all the contributions independent of the defect inside a factor \( Z^\text{1d}_{\text{pure}} \), while the remaining factor is due to the interaction (2.35) between the loop and the vortices, rewritten here for convenience:

\[
Z^\text{1d}_{\text{defect},k}(\phi_{\gamma}^{(a)}, M) = \prod_{l=1}^{k} \frac{\text{sh} \left( \phi_{\gamma}^{(a)} - M^{(a)}_l - \epsilon_- \right) \text{sh} \left( -\phi_{\gamma}^{(a)} + M^{(a)}_l - \epsilon_+ \right)}{\text{sh} \left( \phi_{\gamma}^{(a)} - M^{(a)}_l + \epsilon_- \right) \text{sh} \left( -\phi_{\gamma}^{(a)} + M^{(a)}_l - \epsilon_+ \right)}. \quad (3.20)
\]

Meanwhile, in the three-dimensional setup, we sum over the poles \( \{\tilde{y}_s\} \in \mathcal{M}_k^{\text{bulk}} \), and the \( Y \)-operator vev (3.14) becomes:

\[
\langle \tilde{Y}_{3d}^{(a)}(z) \rangle \equiv \tilde{Y}_{\text{flavor}}^{(a)}(\{x_d\}, z) \sum_{k=0}^{\infty} \sum_{\{\tilde{y}_s\} \in \mathcal{M}_k^{\text{bulk}}} f^\text{3d}_{\text{bulk}}(\tilde{y}_s) \cdot \tilde{Y}_{\text{defect}}^{(a)}(\tilde{y}_s, z). \quad (3.21)
\]

It is well known that in the absence of loop defect, the 3d index \( \sum_{k=0}^{\infty} \sum_{\{\tilde{y}_s\} \in \mathcal{M}_k^{\text{bulk}}} f^\text{3d}_{\text{bulk}}(\tilde{y}_s) = [\hat{X}]_{3d}^{(0,\ldots,0)} \) is in fact equal to the quantum mechanical index \( \sum_{k=0}^{\infty} \sum_{\{\tilde{\phi}_s\} \in \mathcal{M}_k^{\text{pure}}} Z^\text{1d}_{\text{pure}}(\tilde{\phi}_s) = \).
We further renormalize the masses to make contact with their 3d definitions: We switch to K-theoretic variables \( z \).

We recognize the first product as the 3d/1d contribution

After a finite number of telescopic cancellations, the 1d \( Y \)-operator at the locus (3.18) becomes:

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where the leftover factor have been collected in the expression \( c_{3d/1d}(\{x_d^{(b)}\}) \); this factor can be determined exactly by simply comparing the above result to the contribution (3.16), if one wishes. Putting it all together, we have shown that the \( Y \)-operator vevs are proportional to each other:

Now, the 3d/1d half-index is a Laurent series in \( Y \)-operator vevs, with the same functional form and number of terms as the index of \( T^{1d} \). This does not yet guarantee the indices are the same, since \( c_{3d/1d}^{(a)} \) appears as a relative factor between the various terms of the character. Remarkably, one can show after computing each term in the character that they all share the same proportionality factor \( c_{3d/1d}^{(a)} \); so it can be factored out entirely. Considering a general Wilson loop in the \((L^{(1)}, \ldots, L^{(n)})\) representation of \( G \), the normalized indices of

\[ \tilde{\chi}_{1d}^{(0, \ldots, 0)} \], up to overall normalization. We refer the reader to the references [30, 35, 39, 75] for details. Because the contents of the sets \( \mathcal{M}_k^{bulk} \) and \( \mathcal{M}_k^{pure} \) are in one-to-one correspondence for all \( k \), this implies in particular that the summands \( I_{bulk}^{3d}(\tilde{y}_s) \) and \( Z_{pure}^{1d}(\tilde{\phi}_s) \) are the same, up to normalization.

We therefore need to simply investigate the remaining factor \( Z_{defect,k}^{(a)}(\tilde{\phi}_s, M) \) in (3.19). We switch to K-theoretic variables \( z = e^{-M} \), \( f_i^{(a)} = e^{-\mu_i^{(a)}} \), and let

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G^{3d}$ and its quantum mechanics $T^{1d}$ therefore satisfy:
\[
[\tilde{\chi}]^{(L^{(1)},...,L^{(n)})}_{3d} (\{z^{(a)}_{\rho}\}) = c_{3d/1d} \cdot [\tilde{\chi}]^{(L^{(1)},...,L^{(n)})}_{1d} (\{z^{(a)}_{\rho}\}) .
\] (3.28)

The proportionality constant $c_{3d/1d}$ is simply:
\[
c_{3d/1d} = \prod_{a=1}^{n} \prod_{\rho=1}^{L^{(a)}} c^{(a)}_{3d/1d}(\{x^{(b)}_{d}\},z^{(a)}_{\rho}) .
\] (3.29)

4 Schwinger-Dyson Equations: the $\mathcal{W}_{q,t}(\mathfrak{g})$-Algebra Perspective

The BPS/CFT correspondence predicts that Schwinger-Dyson equations for the gauge theory $G^{3d}$ should have a counterpart as a set of Ward identities for a conformal field theory, or a deformation thereof, on a Riemann surface. In this paper, we show that the vortex $qq$-character is a certain (chiral) correlator a deformed $\mathcal{W}(\mathfrak{g})$-algebra on the cylinder.

4.1 The Deformed $\mathcal{W}_{q,t}(\mathfrak{g})$-Algebra

Let $\mathfrak{g}$ be a simply-laced Lie algebra. In the work [33], a deformation of $\mathcal{W}(\mathfrak{g})$-algebras was proposed, in free field formalism, based on a certain canonical deformation of the screening currents; this is the symmetry algebra of the so-called $\mathfrak{g}$-type $q$-Toda theory on a cylinder. See also [31] for the special case $\mathfrak{g} = A_1$, and [32, 78] for the case $\mathfrak{g} = A_n$. The starting point is to define a $(q,t)$-deformed Cartan matrix$^{16}$:
\[
C_{ab}(q, t) = (qt^{-1} + qt) \delta_{ab} - [\Delta_{ab}]_q .
\] (4.1)

Let us explain the notation: a number in square brackets is called a quantum number, defined as
\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} ,
\] (4.2)
and the incidence matrix is $\Delta_{ab} = 2 \delta_{ab} - C_{ab}$. Later, we will also need the inverse of the Cartan matrix:
\[
M(q, t) = C(q, t)^{-1} .
\] (4.3)

$^{16}$In what follows, we follow the conventions of [33]. Other definitions of the deformed Cartan matrix are possible, by introducing explicit “bifundamental masses” in the off-diagonal entries [14].
On then constructs a deformed Heisenberg algebra, generated by $n$ positive simple roots, satisfying:

$$[\alpha_a[k], \alpha_b[m]] = \frac{1}{k} (q^k - q^{-k} )(t^k - t^{-k} )C_{ab}(q^k, t^k) \delta_{k,-m} .$$

(4.4)

In the above, it is understood that the zero-th generator commutes with all others: $[\alpha_a[k], \alpha_b[0]] = 0$, for $k$ an arbitrary integer. The Fock space representation of this algebra is given by acting on the vacuum state $|\psi\rangle$:

$$\alpha_a[0]|\psi\rangle = \langle \psi, \alpha_a |\psi\rangle|\psi\rangle$$

$$\alpha_a[k]|\psi\rangle = 0, \quad \text{for } k > 0 .$$

(4.5)

Then, we define deformed screening operators as\(^{17}\):

$$S^{(a)}(y) = y^{-\alpha_a[0]} : \exp \left( \sum_{k\neq 0} \frac{\alpha_a[k]}{q^k - q^{-k} } y^k \right) : .$$

(4.6)

Note all operators in this section are written up to a center of mass zero mode, whose effect is simply to shift the momentum of the vacuum. Up to redefinition of the vacuum $|\psi\rangle$, we safely ignore such factors.

The $W_{q,t}(g)$-algebra is defined as the associative algebra whose generators are Fourier modes of the operators commuting with the screening charges,

$$Q^{(a)} = \int \, dy \, S^{(a)}(y) .$$

(4.7)

We denote the generating currents as $W^{(s)}(z)$, labeled by their “spin” $s$. We therefore have\(^{18}\):

$$[W^{(s)}(z), Q^{(a)}] = 0 , \quad \text{for all } a = 1, \ldots, n, \text{ and } s = 2, \ldots, n+1 .$$

(4.8)

In this way, one finds that every generating current can be written as a Laurent polynomial in certain vertex operators, which we call $Y$-operators for reasons that will soon be clear:

$$Y^{(a)}(z) = e^{\lambda_a[0]} : \exp \left( - \sum_{k\neq 0} w_a[k] t^{-k/2} z^k \right) : .$$

(4.9)

---

\(^{17}\)The screening operators we write down are called “magnetic” in [33], and are defined with respect to the parameter $q$. Another set of “electric” screening and vertex operators can be constructed using the parameter $t$ instead, but these will not enter our discussion. From the point of view of the 3d gauge theory, this amounts to having $G^{3d}$ defined on the manifold $\mathbb{C}_t \times S^1(\hat{R})$ instead of the manifold $\mathbb{C}_q \times S^1(\hat{R})$. We made the choice of working on the latter manifold in this paper, hence why we only make use of magnetic screenings here.

\(^{18}\)Note that the generating currents must also commute with the set of electric screening charges we mentioned in the previous footnote.
Degenerate vertex operators are constructed out of $n$ fundamental weight generators,

$$ [\alpha_a[k], w_b[m]] = \frac{1}{k} (q^k - q^{-k})(t^k - t^{-k}) \delta_{ab} \delta_{k,-m}. \tag{4.10} $$

These are dual to the operators $\alpha_a[k]$. Put differently,

$$ \alpha_a[k] = \sum_{b=1}^{n} C_{ab}(q^k, t^k) w_b[k]. \tag{4.11} $$

For completeness, we also write the commutator of two coweight generators,

$$ [w_a[k], w_b[m]] = \frac{1}{k} (q^k - q^{-k})(t^k - t^{-k}) M_{ab}(q^k, t^k) \delta_{k,-m}. \tag{4.12} $$

Among the vertex operators of the theory, a distinguished class which will enter our story is the set of so-called fundamental vertex operators [33]:

$$ V^{(a)}(x) = x^{w_a[0]} : \exp \left( - \sum_{k \neq 0} \frac{w_a[k]}{q^k - q^{-k}} x^k \right) :. \tag{4.13} $$

Before we proceed, let us briefly comment on an important limit: if we rescale $q = \exp(\hat{R} \epsilon_1)$, $t = \exp(-\hat{R} \epsilon_2)$ and take $\hat{R} \to 0$, the deformed $W_{q,t}(\mathfrak{g})$-algebra becomes the standard $W(\mathfrak{g})$-algebra, which is the symmetry algebra of $\mathfrak{g}$-type Toda conformal field theory, for an extensive review, see [86]. In particular, if we set $b = -\epsilon_2/\epsilon_1$, the central charge of the theory is $c = m + 12 \langle Q, Q \rangle$, where $Q = \rho (b + 1/b)$ is the background charge, $\rho$ the Weyl vector of $\mathfrak{g}$, and the bracket is the Cartan-Killing form. For the Heisenberg algebra (4.4) to keep making sense, the root (and weight) generators should also be rescaled by $\hat{R}$ and $\epsilon_1$ in this limit. The deformed screenings currents (4.6) become

$$ S^{(a)}(y) = :e^{\langle \alpha_a, \varphi(y) \rangle/b} :, \tag{4.14} $$

with $\alpha_a$ the $a$-th simple root of $\mathfrak{g}$, and $\varphi$ a $n$-dimensional boson. The deformed fundamental vertex operators (4.13) become vertex operators of unit momentum,

$$ V^{(a)}(x) = :e^{\langle w_a, \varphi(x) \rangle/b} :, \tag{4.15} $$

with $w_a$ the $a$-th fundamental weight of $\mathfrak{g}$. Furthermore, in the limit $\hat{R} \to 0$, the deformed generators $W^{(s)}(z)$ become the stress tensor and higher spin currents of the $W(\mathfrak{g})$-algebra. The special case $\mathfrak{g} = A_1$ is called the Liouville CFT, and $W(A_1)$ is more commonly called...
the Virasoro algebra, generated by the spin 2 stress energy tensor \( W^{(2)}(z) \). When \( \mathfrak{g} \) is a higher rank algebra, the stress tensor \( W^{(2)}(z) \) is still present, but there are also more currents \( W^{(s)}(z) \) of higher spin \( s > 2 \).

As a concrete example, consider the deformed stress tensor of \( \mathcal{W}_{q,t}(A_1) \). It is a Laurent polynomial in the \( Y \) operators (4.9):

\[
W^{(2)}(z) = Y(z) + \left[ Y(v^{-2}z) \right]^{-1}.
\] (4.16)

This can be checked explicitly by computing the commutator \([W^{(s)}(z),Q^{(a)}]\), and finding out that it indeed vanishes. In the limit \( q = \exp(R\epsilon_1), \ t = \exp(-R\epsilon_2) \), and the further rescaling of the Heisenberg algebra generators, one finds

\[
W^{(2)}(z) \rightarrow -\frac{1}{2} : (\partial_z \phi(z))^2 : + Q : \partial^2_z \phi(z) :.
\] (4.17)

The reader will recognize the Liouville stress energy tensor of the \( \mathcal{W}(A_1) \)-algebra.

### 4.2 The Vortex \( qq \)-character is a Deformed \( \mathcal{W}_{q,t}(g) \)-Algebra Correlator

We are interested in evaluating the following correlator:

\[
\left< \psi' \left| \prod_{a=1}^{n} \prod_{d=1}^{N^{(a)}} V^{(a)}(x_d^{(a)}) \left( Q^{(a)} \right)^{N^{(a)}} \prod_{s=2}^{n+1} \prod_{\rho=1}^{L^{(s-1)}} W^{(s)}(z^{(s-1)}_{\rho}) \right| \psi \right>. \] (4.18)

In what follows, we use the shorthand notation \( \langle \ldots \rangle \) for a vacuum expectation value. The incoming and outgoing states are written as \( |\psi\rangle \) and \( |\psi'\rangle \) respectively, instead of the trivial vacuum \( |0\rangle \). Because the theory is defined in the free-field formalism, the above correlator can be evaluated using straightforward Wick contractions, as an integral over the positions \( y \) of the \( N^{(a)} \) screening currents. Namely, after taking into account the normal ordering of the various operators, the correlator (4.18) becomes the integral

\[
\int d_{\text{Haar}} y \ I_{\text{Toda}}(y), \] (4.19)

where the Haar measure is given by

\[
d_{\text{Haar}} y = \prod_{a=1}^{n} \prod_{i=1}^{N^{(a)}} \frac{dy_i^{(a)}}{y_{i}^{(a)}}. \] (4.20)

The integrand \( I_{\text{Toda}}(y) \) is made up of various factors. First, we have

\[
\prod_{a=1}^{n} \prod_{i=1}^{N^{(a)}} \left( y_{i}^{(a)} \right)^{(\psi,\alpha_a)}, \] (4.21)
We recognize the vector multiplet contribution \((3.7)\) to the index \(\tilde{\chi}^{3d}\).

There are also various two-point functions: for a given \(a \in \{1, \ldots, n\}\), we find by direct computation:

\[
\prod_{1 \leq i < j \leq N(a)} \left\langle S^{(a)}(y_i^{(a)}) S^{(a)}(y_j^{(a)}) \right\rangle = \prod_{1 \leq i \neq j \leq N(a)} \frac{(y_i^{(a)}/y_j^{(a)}; q)_\infty}{(t y_j^{(a)}/y_i^{(a)}; q)_\infty} \prod_{1 \leq i < j \leq N(a)} \frac{\Theta(t y_j^{(a)}/y_i^{(a)}; q)}{\Theta(y_j^{(a)}/y_i^{(a)}; q)}.
\]

We recognize the bifundamental contribution \((3.10)\) to the index \(\tilde{\chi}^{3d}\).

The two-point of a fundamental vertex operator with a screening current equals:

\[
\prod_{1 \leq i \leq N(a)} \prod_{1 \leq j \leq N(b)} \left\langle S^{(a)}(y_i^{(a)}) S^{(b)}(y_j^{(b)}) \right\rangle = \prod_{1 \leq i \leq N(a)} \prod_{1 \leq j \leq N(b)} \left[ (t v y_i^{(a)}/y_j^{(b)}; q)_\infty \right]^{\delta_{ab}}.
\]

We recognize the flavor contributions \((3.11)\) to the index \(\tilde{\chi}^{3d}\).

We come to the two-point function of a screening current with a \(\mathcal{Y}\)-operator, which evaluates to:

\[
\prod_{i=1}^{N(b)} \left\langle S^{(b)}(y_i^{(b)}) \mathcal{Y}^{(a)}(z) \right\rangle = \prod_{i=1}^{N(b)} \left[ \frac{1 - t y_i^{(b)}/z}{1 - y_i^{(b)/z}} \right]^{\delta_{ab}}.
\]

We recognize part of the Wilson loop contribution \((3.15)\) to the index \(\tilde{\chi}^{3d}\). Note the zero mode of the \(\mathcal{Y}\)-operator \((4.9)\) acts nontrivially on the vacuum \(|\psi\rangle\). As a result, the two-point of a screening with a \(\mathcal{Y}\)-operator generates a relative shift of one unit of 3d FI parameter between the various terms of the generating current \(W^{(s)}(z)\).

The last missing ingredient is the two-point of a fundamental vertex operator with a \(\mathcal{Y}\)-operator, which at first sight takes a far less elegant form:

\[
\left\langle V^{(b)}(x_d^{(b)\ Y}^{(a)}(z) \right\rangle = \exp \left( \sum_{k>0} M_{ba}(q^k, t^k) \frac{t^k - 1}{k} \left( x_d^{(a)}/z \right)^k \right).
\]

Recall \(M_{ba}\) is the inverse of the deformed Cartan matrix. Fortunately, this two-point can
be rewritten as:

$$\left\langle V^{(b)}(x^{(b)}_d) \mathcal{Y}^{(a)}(z) \right\rangle = B(x^{(b)}_d, z) \frac{1 - v^{#(ab) + 1}}{1 - t v^{#(ab) + 1}} \frac{x^{(b)}_d}{z}.$$  \tag{4.27}$$

where $#^{(ab)}$ is a non-negative integer equal to the number of links between nodes $a$ and $b$ in the Dynkin diagram of $g$, and $B(x^{(b)}_d, z)$ is defined by the above two equations: it is literally the exponential in (4.26) divided by the ratio on the right-hand side of (4.27).

It may seem like we have not gained much by trading the exponential (4.26) for a prefactor of the form (4.27), with the same prefactor $B(x^{(b)}_d, z)$ for each term. This implies that the prefactor $B(x^{(b)}_d, z)$ can be factorized out of the correlator integral altogether. Note that a related factorization had also been noticed in the gauge theory picture, see the discussion under (3.27).

To fully specify the correlator integral, we also need to make a choice of contour. Here, the contours are simply chosen to be the ones we used in defining the 3d/1d half-index, enclosing the poles in $\mathcal{M}^{bulk}$. In particular, the contours will avoid all poles depending on the generating current fugacity $z$.

For a general correlator, we can now claim:

$$\left\langle \psi' \left[ \prod_{a=1}^{n} \prod_{d=1}^{N(a)} V^{(a)}(x^{(a)}_d) (Q^{(a)})^{N(a)} \mathcal{Y}^{(a)} \prod_{s=2}^{n+1} \prod_{\rho=1}^{L^{(s-1)}} W^{(s)}(z^{(s-1)}_{\rho}) \right] \psi \right\rangle \tag{4.28}$$

where the overall prefactor is

$$B \left( \{x^{(b)}_d\}, \{z^{(s-1)}_{\rho}\} \right) = \prod_{s=2}^{n+1} \prod_{\rho=1}^{L^{(s-1)}} \prod_{b=1}^{n} \prod_{d=1}^{N(b)} B \left( x^{(b)}_d, z^{(s-1)}_{\rho} \right).$$  \tag{4.29}$$

Naturally, $B(\{x_d\}, \{z^{(s)}_{\rho}\})$ stands outside the correlator integrals, since it does not depend on the $y$-integration variables.

In the end, we find that the 3d/1d index of the gauge theory $G^{3d}$ with Wilson loop is a deformed $W_{q,t}(g)$-algebra correlator, up to the constant $B(\{x_d\}, \{z^{(s)}_{\rho}\})$. Recall that this prefactor contains an exponential; we mention in passing that this “phase” has a
natural interpretation when $g = A_n$, in the general framework of Ding-Iohara-Miki (DIM) algebras [87, 88]; for a detailed study, see [16]. Roughly speaking, in the DIM formalism, a vertex operator $V^{(a)}(x^{(a)})$ is built using intertwiners, as the product of a $V^{(a)}(x^{(a)})$ vertex operator from the $W_{q,t}(A_n)$-algebra, and another vertex operator coming from an additional Heisenberg algebra. This extra Heisenberg algebra comes with its own Fock space, and contributes to the correlator in a way to precisely cancel the prefactor $B([x_d], [z^{(s)}])$.

5 Schwinger-Dyson Equations: the Little String Perspective

We have showcased three different physical frameworks where we can make sense of a vortex character for the 3d $\mathcal{N} = 4$ gauge theory $G^{3d}$. Moreover, we saw explicitly how the regularity properties of this character implies a non-perturbative type of Schwinger-Dyson identities for the theory. Ultimately, all the various perspectives are unified in a string theory picture. In the process of describing it, we will learn about the dynamics of new defects in $(2,0)$ little string theory. The literature on BPS-defects of the little string has been steadily growing in the last few years, with rich physical and mathematical implications: among them, we find codimension-2 defects [58, 89–91], codimension-4 defects [75, 92], point and codimension-2 defects [18], and in this present work, new codimension-4 defects.

5.1 Little String Basics

We consider ten-dimensional type IIB string theory compactified on an $ADE$ surface $X$ times a circle, meaning type IIB on $X \times M_6$. The six-manifold $M_6$ is the product of an infinite cylinder $C = \mathbb{R} \times S^1(R)$ of radius $R$ and two complex lines, which we distinguish using the subscript notation $C_q$ and $C_t$, so that $M_6 = C \times C_q \times C_t$. $X$ is a resolution of a $\mathbb{C}^2/\Gamma$ singularity, where $\Gamma$ is one of the discrete subgroups of $SU(2)$. By the McKay correspondence, such a discrete subgroup is labeled by one of the simply-laced Lie algebras $g = A, D, E$; we call $n$ the rank of $g$. Explicitly, the singularity is resolved by blow-up: the exceptional divisor is a collection of 2-spheres $S_a$, $a = 1, \ldots, n$, which organize themselves in the shape of the Dynkin diagram of $g$.

We focus our attention on a sector of the theory which has far less degrees of freedom than are present in the full IIB string. That is, we decouple gravity and focus only on the degrees of freedom supported near the origin of $X$ by sending the string coupling to $g_s \to 0$. In this limit, the type IIB string on $X$ becomes a six-dimensional string theory on $M_6$, known as the $(2,0)$ little string of type $g = A, D, E$ [93–95]. It is not a local QFT [96]. It is instead a theory of strings proper (inherited from the ten-dimensional IIB strings),

\[^{21}\text{Equivalently, these are all T-dual defects in the (1,1) little string theory.}\]
with finite tension $m^2_s$, the square of the string mass. There are a few good reviews in the literature, most notably [97, 98].

The moduli space of the $(2, 0)$ little string is

$$(\mathbb{R}^4 \times S^1)^n / W(g) ,$$

(5.1)

where $W(g)$ is the Weyl group of $g$. The moduli come from periods of various 2-forms along the 2-cycles $S_a$ of the surface $X$: the $S^1$ modulus is the R-R 2-form $C^{(2)}$ of the ten-dimensional type IIB string theory integrated over $S_a$. Meanwhile, the $\mathbb{R}^4$ moduli come from the NS-NS B-field $B^{(2)}$, and a triplet of self-dual 2-forms $\omega_{IJK}$, which exist because $X$ is a hyperkähler manifold. To get the correct R-R and NS-NS normalizations, one needs to recall the low energy action of the type IIB superstring. In particular, the R-R field is not accompanied by any power of $g_s$. Moreover, the mass dimension of a scalar in a theory of 2-forms should be 2. Then, in canonical normalization, we obtain:

$$m^4_s \int_{S_a} \omega_{IJK}, \quad m^2_s \int_{S_a} B^{(2)}, \quad m^2_s \int_{S_a} C^{(2)} .$$

(5.2)

The above periods remain fixed in the limit $g_s \to 0$.

As is, this background preserves 16 supercharges. Ultimately, we want to make contact with three-dimensional physics and produce nontrivial dynamics. We can achieve both goals at once by introducing various supersymmetric branes. Since our construction originates in type IIB, we naturally consider adding certain D-branes, whose tension should remain finite in the $g_s \to 0$ limit. As we will argue, the relevant branes to consider here are D3 branes wrapping 2-cycles of the surface $X$, which we now turn to.

### 5.2 The Effective Theory on D3-Branes

To be more quantitative, we introduce some notations: According to the McKay correspondence, the second homology group $H_2(X, \mathbb{Z})$ of $X$ is identified with the root lattice $\Lambda$ of $g$. Then, $H_2(X, \mathbb{Z})$ is spanned by $n$ vanishing 2-cycles $S_a$, which we identify as the positive simple roots $\alpha_a$. The intersection pairing in homology is further identified with the Cartan Killing metric of $g$; explicitly,

$$\#(S_a \cap S_b) = -C_{ab} ,$$

(5.3)

where $C_{ab}$ is the Cartan matrix of $g$.

We also consider the second relative homology group $H_2(X, \partial X, \mathbb{Z})$. This group is spanned by non-compact 2-cycles $S^*_a$, $a = 1, \ldots, n$, where each $S^*_a$ is constructed as the fiber of the cotangent bundle $T^*S_a$ over a generic point on $S_a$. The group $H_2(X, \partial X, \mathbb{Z})$ is identified with the weight lattice $\Lambda_\ast$ of $g$; correspondingly, the 2-cycle $S^*_a$ is identified with the $a$-th fundamental weight $\lambda_a$ of $g$. In particular, the following orthonormality relation holds in homology:

$$\#(S_a \cap S^*_b) = \delta_{ab} .$$

(5.4)
Note that $H_2(X, Z) \subset H_2(X, \partial X, Z)$, since compact 2-cycles can be understood as elements of $H_2(X, \partial X, Z)$ with trivial boundary at infinity. This is just the homological version of the familiar statement that the root lattice of $\mathfrak{g}$ is a sublattice of the weight lattice, $\Lambda \subset \Lambda_\ast$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{vanishing_cycle.png}
\caption{A vanishing 2-cycle of an $A_n$ singularity, labeled by $S_a$ (the black 2-sphere), and the dual non-compact 2-cycle $S_a^\ast$ (the black cigar).}
\end{figure}

Consider a total of $N$ D3\textsubscript{gauge} branes wrapping the compact 2-cycles of $X$ and one of the complex lines $C_q$ in $M_6$, while sitting at the origin of the transverse complex line $C_t$. This results in a net non-zero D3\textsubscript{gauge} brane charge, measured by a class $[S] \in H_2(X, Z)$. We expand $[S]$ in terms of positive simple roots as

$$[S] = \sum_{a=1}^{n} N^{(a)} \alpha_a \in \Lambda , \quad (5.5)$$

with $N^{(a)}$ non-negative integers. The $N$ D3\textsubscript{gauge} branes are points on the cylinder $C$, with coordinates $\{y^{(a)}_i\}$.

Next, we consider a total of $N_f$ D3\textsubscript{flavor} branes wrapping non-compact 2-cycles in $X$, along with the same complex line $C_q$ in $M_6$, while also sitting at the origin of the transverse complex line $C_t$. The charge for these branes is measured by a class $[S^\ast] \in H_2(X, \partial X, Z)$. We expand $[S^\ast]$ in terms of fundamental weights as

$$[S^\ast] = -\sum_{a=1}^{n} N_{f}^{(a)} \lambda_a \in \Lambda_\ast , \quad (5.6)$$

where $N_{f}^{(a)}$ are non-negative integers commonly called Dynkin labels. The $N_f$ D3\textsubscript{flavor}
branes are points on the cylinder $\mathcal{C}$, with coordinates $\{x_{d}^{(a)}\}$.

Lastly, we introduce $L \text{D}_3^\text{defect}$ branes wrapping the non-compact 2-cycles in $X$ and the transverse complex line $\mathbb{C}_t$, while sitting at the origin of $\mathbb{C}_q$. The charge for these $\text{D}_3^\text{defect}$ branes is measured by a class $[S^*_\text{defect}] \in H_2(X, \partial X, \mathbb{Z})$, expanded in terms of fundamental weights as:

$$[S^*_\text{defect}] = -\sum_{a=1}^{n} L^{(a)} \lambda_a \in \Lambda,$$

where $L^{(a)}$ are non-negative integers, once again Dynkin labels. The $L \text{D}_3^\text{defect}$ branes are points on the cylinder $\mathcal{C}$, with coordinates $\{z_{\rho}^{(a)}\}$.

Let us first ignore the $\text{D}_3^\text{defect}$ branes. At energies $E$ well below the string scale, $E/m_s \ll 1$, the effective theory on the $\text{D}_3^\text{gauge}$ branes is a three-dimensional gauge theory with $N = 4$ supersymmetry\textsuperscript{22}, on the manifold $\mathbb{C}_q \times S^1(R)$. At first sight, our brane configuration in type IIB: there are $N \text{D}_3^\text{gauge}$ branes wrapping compact 2-cycles $S_a$ and $\mathbb{C}_q$ (yellow), $N_f \text{D}_3^\text{flavor}$ branes wrapping non-compact 2-cycles $S^*_a$’s and $\mathbb{C}_q$ (red). There are also $L \text{D}_3^\text{defect}$ branes wrapping the non-compact 2-cycles $S^*_a$’s and $\mathbb{C}_q$ (green). All branes are points on the cylinder $\mathcal{C}$. Later, we will also consider the quantum mechanics of $k \text{D}_1^\text{vortex}$ branes (not pictured) wrapping the compact 2-cycles $S_a$’s.

\textsuperscript{22}Recall we have defined the periods of a triplet $\vec{\omega} = (\omega_1, \omega_2, \omega_N)$ of self-dual 2-forms (5.2). The $\text{D}_3^\text{flavor}$ branes wrapping the non-compact 2-cycles preserve the same supersymmetry only if the vectors $\int_{\mathbb{C}_q} \hat{\omega}$ all point in the same direction, for all $a = 1, \ldots, n$. Having made such a choice, we then have to worry about the supersymmetry preserved by the $\text{D}_3^\text{gauge}$ branes wrapping the compact 2-cycles. This is determined by
setup may suggest that the gauge theory we obtain should only be two-dimensional, with \( \mathcal{N} = (4, 4) \) supersymmetry. However, it is not the case: the D3\(_{\text{gauge}} \) branes are points on the cylinder \( \mathcal{C} = \mathbb{R} \times S^1(R) \), and strings wrap around the circle, resulting in a tower of Kaluza-Klein states on the T-dual circle of radius \( \hat{R} = 1/(m_s^2 R) \), which modifies the low-energy physics. Put differently, the (2,0) little string compactified on \( S^1(R) \) enjoys T-duality (inherited from type IIB), under which it becomes the (1,1) little string theory compactified on \( S^1(\hat{R}) \). Then, the D3\(_{\text{gauge}} \) branes at points on the cylinder \( \mathcal{C} = \mathbb{R} \times S^1(R) \) in the (2,0) little string are exactly the same as D4\(_{\text{gauge}} \) branes wrapping the circle of the T-dual cylinder \( \mathcal{C}' = \mathbb{R} \times S^1(\hat{R}) \), in the (1,1) little string. It is clear in the second description that the low energy theory really is three-dimensional, on \( \mathbb{C}_q \times S^1(\hat{R}) \). We call this gauge theory \( G^{3d} \). The choice of this denomination is not innocent, since we will now argue that the low energy theory on the branes is precisely the 3d theory we have studied in the rest of this paper.

The precise characterization of \( G^{3d} \) was determined by Douglas and Moore \cite{99}: it is a quiver gauge theory of shape the Dynkin diagram of \( g = ADE^{23} \). The gauge group is

\[
G = \prod_{a=1}^{n} U(N^{(a)}) ,
\]

(5.8)

where the ranks \( N^{(a)} \) were defined in (5.5) as the number of D3\(_{\text{gauge}} \) branes wrapping the compact 2-cycle \( S_a \),

\[
[S] = \sum_{a=1}^{n} N^{(a)} \alpha_a \in \Lambda .
\]

The flavor symmetry is the gauge group

\[
G_F = \prod_{a=1}^{n} U(N_f^{(a)}) ,
\]

(5.9)

where the ranks \( N_f^{(a)} \) were defined in (5.6) as the number of D3\(_{\text{flavor}} \) branes wrapping the non-compact 2-cycle \( S_a^* \),

\[
[S^*] = - \sum_{a=1}^{n} N_f^{(a)} \lambda_a \in \Lambda^* .
\]

Note that because the 2-cycle \( S_a^* \) are non-compact, the associated gauge fields of \( U(N_f^{(a)}) \) are frozen. This produces \( N_f^{(a)} \) hypermultiplets on node \( a \), in the bifundamental representation.

the periods of the 2-forms through the 2-cycles \( S_a \), which is the choice of a metric on \( X \). Then, it is always possible to choose D3 branes wrapping the compact and non-compact 2-cycles and which break the same supersymmetry.

\(^{23}\)The original analysis of \cite{99} was carried out in the full type IIB background, and the quiver gauge theory there was labeled by an affine Dynkin diagram. Here, we are working in the little string limit \( g_s \to 0 \), and the affine node is decoupled.
Such multiplets come about from quantizing the strings between the D3 branes wrapping the compact 2-cycle $S_a$ and the non-compact 2-cycle $S_a^*$. Finally, for $a \neq b$, we have hypermultiplets coming from the intersection of 2-cycles $S_a$ and $S_b$ at a point. The intersection pairing $\#(S_a \cap S_b)$ in homology is identified with the incidence matrix of $g$. Open strings with one end on the $a$-th D3 gauge brane and the other end on the $b$-th D3 gauge brane results in a hypermultiplet in the bifundamental representation $(N^{(a)}, N^{(b)})$ of $U(N^{(a)}) \times U(N^{(b)})$.

So far, the above stringy construction applies a priori to any configuration of D3 gauge and D3 flavor branes. The resulting effective 3d $\mathcal{N} = 4$ gauge theory then inherits an arbitrary gauge and flavor content. The aim of this work is to exhibit certain symmetries associated to BPS vortices, which sit at Higgs vacua. Therefore, from now on, we require the number $N_F$ of D3 flavor branes to be large enough, so that $G^{3d}$ possesses a Higgs branch and that its vacua be Higgs vacua.

Consider now adding the D3 defect branes in the background. Recall that those branes wrap the non-compact 2-cycles of the geometry. As such, they are not dynamical. They are point defects on the $\mathbb{C}_q$-line, and break supersymmetry further by half. The number of such D3 defect branes is $L = \sum_{a=1}^{n} L^{(a)}$. Correspondingly, the defects carry a background gauge group of their own,

$$\hat{G}_{\text{defect}} = \prod_{a=1}^{n} U(L^{(a)}) \, . \quad (5.10)$$

Figure 6: T-duality tells us that the D3 branes at points on the cylinder $\mathcal{C}$ in the (2,0) little string are the same as D4 branes wrapping the T-dual cylinder $\mathcal{C}'$ in the (1,1) little string.
The D3\textsubscript{defect} branes are points on C, or equivalently, following the same line of reasoning as before, they are D4\textsubscript{defect} branes wrapping the circle \(S^1(\hat{R})\) of the T-dual cylinder \(C'\). Thus, from the point of view of the gauge theory \(G^{3d}\), the D3\textsubscript{defect} branes really are 1/2-BPS defects wrapping the \(S^1(\hat{R})\), and at the origin of \(C\); in other words, they make up a Wilson loop of \(G^{3d}\) \footnote{The realization of supersymmetric Wilson loops in string theory was first proposed in [100, 101], in the context of holography; namely, a loop in the first fundamental representation of \(SU(N)\) is described as a fundamental string whose worldsheet ends at the loop, located at the boundary of \(AdS\). Later, a description of the loops was given in terms of D-branes instead [50, 102, 103], allowing for more general representations. This D-brane perspective is the one relevant to us here; in particular, the D3\textsubscript{defect} branes we study are T-dual to the D5' branes appearing in the work [51].}.

Translating the geometry to gauge theory data, the periods (5.2) of the \((2, 0)\) little string become parameters of \(G^{3d}\). Namely, the modulus coming from the NS-NS \(B^{(2)}\)-field through the 2-cycle \(S^2_a\) is identified with the gauge coupling on node \(a\) of the quiver gauge theory. The triplet of self-dual two-forms \(\omega_{I,J,K}\) are the FI parameters. The positions of the \(N\) D3\textsubscript{gauge} branes on the cylinder \(C\) are (part of the) Coulomb moduli of \(G^{3d}\). The positions of the \(N_f\) D3\textsubscript{flavor} branes on \(C\) are mass parameters for the fundamental hypermultiplets of \(G^{3d}\). Finally, the positions of the \(L\) D3\textsubscript{defect} branes on \(C\) are the fermion masses (2.9) for the Wilson loop. All of the above moduli and parameters are complexified, due to the presence of the circle \(S^1(\hat{R})\) the 3d theory and the Wilson loop live on. This is precisely the gauge theory setup we studied throughout this paper.

We end this section with comments on the limit \(m_s \to \infty\). In that regime, we lose the one scale of the theory and flow to a \((2, 0)\) SCFT on \(M_6\), labeled by the same simply-laced Lie algebra \(g\) as the \((2, 0)\) little string\footnote{As usual, there are a priori many ways to take this limit. The limit described here is the same one that turned deformed \(W_{q,t}(g)\)-algebras into the usual \(W(g)\)-algebras.}. The various moduli of the little string are kept fixed in the limit, and become moduli of the SCFT. We further insist on keeping the Riemann surface we compactified the theory on fixed, along with the position of the various D3 branes on it; that is, the cylinder \(C = \mathbb{R} \times S^1(\hat{R})\) remains fixed. Recall however that \(G^{3d}\) is naturally defined on the T-dual cylinder \(C' = \mathbb{R} \times S^1(\hat{R})\), where \(\hat{R} = 1/(m_s^2 R)\). Therefore, if \(R\) is kept fixed, the dual radius \(\hat{R}\) vanishes in the SCFT limit and the theory on the branes becomes effectively two-dimensional, with \(\mathcal{N} = (4, 4)\) supersymmetry. The gauge coupling on the D3 branes becomes infinite, meaning the theory cannot be described as a gauge theory anymore. Meanwhile, the Wilson loop wrapping \(\hat{R}\) becomes a 1/2-BPS point defect, and supersymmetry is broken to \(\mathcal{N} = (0, 4)\). In the rest of this paper, we keep \(m_s\) finite.

5.3 The Index of the Little String is a \(qq\)-character

Our goal is to compute the partition function of the \((2, 0)\) little string theory on \(M_6 = \mathcal{C} \times \mathbb{C}_q \times \mathbb{C}_t\). Note this is fully equivalent to computing the partition function of the \((1, 1)\)
little string on $M'_6 = C' \times C_q \times C_t$, where $C'$ is the T-dual cylinder of radius $\hat{R}$. In the latter setup, and in the absence of D-branes, the partition function is naturally expressed as a supersymmetric index:

$$\text{Tr} \left[ (-1)^F q^{S_1-S_R} t^{-S_2+S_R} \right] .$$  

(5.11)

The trace is defined over the circle $\hat{R}$, and $q^{S_1-S_R} t^{-S_2+S_R}$ turns the manifold $M'_6$ into a twisted product. As we go around $S^1(\hat{R})$, the line $C_q$ is rotated by $q$, and the line $C_t$ is rotated by $t^{-1}$, describing an $\Omega$-background. $S_1$ is the generator of the $C_q$-rotations, while $S_2$ is the generator of the $C_t$-rotations. $S_R$ is the generator of a $U(1) \subset SU(2)$ subgroup of the R-symmetry of the 6d theory. Without any branes, the index is trivial by pairwise cancellations of bosons and fermions, since the 6d theory has too much supersymmetry.

Working in the T-dual setup, we first add the D4$_{\text{gauge}}$ and D4$_{\text{flavor}}$ branes in the background. By a supersymmetric localization argument, the partition function of the bulk little string becomes the partition function on the defects. Indeed, supersymmetry is only broken near the locus of the defect branes, while the supersymmetries of the full $(1,1)$ little string are preserved away from the defect. It follows that the partition function on the D4 branes is precisely the half-index (3.2) of $G^{3d}$. In particular, the 3d FI parameter contribution (3.6) comes from turning on the periods of the self-dual two-form $\omega_I$. The $\mathcal{N} = 4$ vector multiplet contribution on node $a$ (3.7) comes about from quantizing the D4$_{\text{gauge}}$/D4$_{\text{gauge}}$ strings, with the D4 branes wrapping the $a$-th compact 2-cycle. The $\mathcal{N} = 4$ bifundamental hypermultiplets contribution between nodes $a$ and $b$ (3.10) comes about from quantizing the D4$_{\text{gauge}}$/D4$_{\text{gauge}}$ strings, with one set of D4 branes wrapping the $a$-th compact 2-cycle, and the other set of D4 branes wrapping the $b$-th compact 2-cycle. The $\mathcal{N} = 4$ fundamental hypermultiplet contribution on node $a$ (3.11) comes about from quantizing the D4$_{\text{gauge}}$/D4$_{\text{flavor}}$ strings, with the D4$_{\text{gauge}}$ branes wrapping the $a$-th compact 2-cycle, and the D4$_{\text{flavor}}$ branes wrapping the dual non-compact 2-cycle.

We now introduce the D4$_{\text{defect}}$ branes. These branes are nondynamical as they do not wrap $C_q$, but they nonetheless modify the index. We conjecture here that the new string sectors realize the $Y$-operator defect in the gauge theory. Namely, (3.15) is the contribution of D4$_{\text{gauge}}$/D4$_{\text{defect}}$ strings at node $a$, while (3.16) is the contribution of D4$_{\text{flavor}}$/D4$_{\text{defect}}$ strings at node $a$. All in all, this implies that the index of the little string in the presence of all three types of branes localizes to the vortex $qq$-character observable. We rewrite here for convenience, normalized by the index in the absence of D4$_{\text{defect}}$ branes:

$$[\tilde{\chi}]_{D4_g/D4_f/D4_d}^{(L^{(1)}, \ldots, L^{(n)})} (\{z^{(a)}_p\}) = \frac{1}{[\tilde{\chi}]_{D4_g/D4_f}^{(0, \ldots, 0)}} \sum_{\omega \in \nu(\lambda)} \prod_{b=1}^{n} \left( \tilde{q}^{(b)} \right)^{d_{\omega}^{(b)}} c_{d_{\omega}^{(b)}} (q, t) \left( \tilde{Q}_{d_{\omega}^{(b)}}^{(b)} (\{z^{(a)}_p\}) \right) \left[ \tilde{Y}_{3d} (\{z^{(a)}_p\}) \right]_\omega .$$

(5.12)
The superscripts in the index designate the $D_{4\text{ defect}}$ charge, while the subscripts indicate which types of branes are present ($D_{4g}$ for $D_{4\text{ gauge}}$ and so on).

This above identification makes the dictionary to deformed $\mathcal{W}$-algebras explicit: the $N$ screening charges (4.7) are the $N$ $D_{4\text{ gauge}}$ branes, the $N_F$ fundamental vertex operators (4.13) are the $N_F$ $D_{4\text{ flavor}}$ branes, and the $L$ generating currents (4.8) are the $L$ $D_{4\text{ defect}}$ branes. The little string index can therefore be recast as the $q$-Toda correlator (4.28).

To make contact with the vortex quantum mechanics $T^{1d}$, we need to do a little more work. Namely, we freeze the moduli of the $D_{4\text{ gauge}}$ branes to be equal to the moduli of the $D_{4\text{ flavor}}$ branes. This describes the root of the Higgs branch for $G^{3d}$. Geometrically, this means we can recombine the $D_{4\text{ gauge}}$ branes with the $D_{4\text{ flavor}}$ branes so that they exclusively make up a collection of $N_F$ $D_{4'}\text{ flavor}$ branes wrapping the non-compact 2-cycles of $X$, and the theory is effectively massive.

![Figure 7](image_url): Illustration of the Higgsing procedure in the string theory picture. On the right side, the gauge and flavor branes have recombined exclusively into flavor branes.

Now, we would like to introduce vortices for $G^{3d}$. First note that a generic collection of vortices is BPS if the 3d FI parameters are aligned in the same direction. For each $a = 1, \ldots, n$, the triplet of FI terms $\int_{S_n} \omega_{I,J,K}$ transforms as a vector under the R-symmetry group $SU(2)_R$ rotating the hypermultiplet scalars. We identify $SU(2)_R$ as the $SU(2)$ R-symmetry of the little string. We then turn on the periods $\int_{S_n} \omega_I > 0$, while setting the other periods to zero, $\int_{S_n} \omega_{J,K} = 0$, for all $a$. Correspondingly, this turns on a real FI parameter on each node $a$ (complexified as usual due to the presence of the cylinder $C'$), while the complex FI parameters are set to zero. This describes a generic point on the Higgs branch of $G^{3d}$, and $SU(2)_R$ is broken to $U(1)_R$, which acts by rotating the periods of...
\(\omega_J\) and \(\omega_K\). This background indeed allows for 1/2-BPS vortex solutions: they are D2\(_{\text{vortex}}\) branes wrapping the compact 2-cycles of \(X\) and the circle of the cylinder \(C'\) in the (1, 1) little string. Alternatively, they are D1\(_{\text{vortex}}\) branes wrapping the compact 2-cycles of \(X\) at a point on \(C\) in the (2, 0) little string. With all branes present, only two supercharges are preserved in the background.

The effective theory on \(k\) D2\(_{\text{vortex}}\) branes is precisely the quantum mechanics \(T^{1d}\). It follows that the index of the (1, 1) little string in the presence of the D4\(_f\)\_\text{flavor}\) branes, the D4\(_d\)\_\text{defect}\) branes, and the new D2\(_{\text{vortex}}\) branes, is the 1d \(N = 2\) Witten index (2.10). In its integral representation (2.12), the index is comprised of 1-loop determinants, all of which can be attributed to the various strings stretching between the branes. Let us explain the factors resulting in the Wilson loop physics: the classical Wilson loop contribution \(Z^{(a)}_{\text{defect,}}\) is attributed to D4\(_f\)\_\text{flavor}\)/D4\(_d\)\_\text{defect}\) strings, which provide exclusively fermions; for details, see the T-dual setup of D0/D8 branes studied in [104], as well as [51]. Meanwhile, the interaction \(Z^{(a)}_{\text{defect,k}}\) of the vortices with the Wilson loop is attributed to D2\(_{\text{vortex}}\)/D4\(_d\)\_\text{defect}\) strings on node \(a\). It provides the degrees of freedom for (the reduction from 2d \(N = (0, 4)\) to) 1d \(N = 4\) twisted hypermultiplets and Fermi multiplets; these 1-loop determinants were also worked out in T-dual setups, see [13, 18, 105].

All in all, the Witten index of \(T^{1d}\) is a little string index, which can be naturally expressed once again as a vortex \(qq\)-character:

\[
\left[\tilde{\chi}(L^{(1)},\ldots,L^{(n)})\right]_{\text{D2v/D4f/D4d}}(\{z^{(a)}_{\rho}\}) = \frac{1}{\left[\chi\right]_{\text{D2v/D4f/D4d}}} \sum_{\omega \in V(\lambda)} \prod_{b=1}^{n} \left(\tilde{q}^{(b)}\right)^{d_{\omega}^{(b)}} c_{d_{\omega}^{(b)}}(g,t) \left(Q^{(b)}_{d_{\omega}^{(b)}}(\{z^{(a)}_{\rho}\})\right) \left[Y_{1d}(\{z^{(a)}_{\rho}\})\right]_{\omega}.
\]

(5.13)

Again, the superscripts indicate the D4\(_d\)\_\text{defect} charge, while the subscripts indicate which types of branes are present in the background.

Recall that this index depends on a choice of sign for the 1d FI parameter. In the little string context, this is the sign of the period for the NS-NS \(B\)-field \(\int_{S^2} B^{(2)}\). In particular, as we will argue next, changing the sign of this period gives a little string realization of 3d Seiberg duality.

6 Discussion

6.1 Seiberg Duality of the Vortex Character

Our focus in this paper was on the UV physics in three dimensions, and in particular we have not paid attention to what the IR description of \(G^{3d}\) may look like. Our only requirement was that the theory had Higgs vacua, and the number of hypermultiplets in that range allows for many distinct behaviors in the IR [47, 106]. It can happen that two
distinct UV theories flow to the same IR point, a phenomenon called Seiberg duality \cite{38}. We would like to ask what the action of Seiberg duality (if any) is on the $qq$-character observable we constructed.

Luckily, we have the one-dimensional quantum mechanics description at our disposal, where Seiberg duality is understood microscopically as a change of sign in the 1d FI parameter $\zeta_{1d}$ \cite{39}; in the absence of a Wilson loop, 3d $\mathcal{N} = 4$ gauge theories which are Seiberg-dual in the UV typically look very different from each other, but turn out to have identical partition functions. In the 1d quantum mechanics picture, Seiberg-dual theories happen to have one and the same gauge theory description $T_{\text{pure}}^{1d}$. What happens in the presence of a Wilson loop? For starters, the quantum mechanics $T^{1d}$ is no longer invariant: this is because the Wilson loop transforms in some representation of the 3d gauge group $G$, not the 3d flavor group $G_F$, a distinction which breaks a symmetry of Seiberg-duality. In particular, the dual Wilson loop is expected to map to a flavor Wilson loop, already in the topologically trivial sector. In our presentation of the Witten index, this is the contribution of the Fermi multiplets $Z_{\text{defect,}\emptyset}^{(a)}$ present at $k = 0$. We also want to understand the duality in a background with arbitrary vortex charge $k$. A detailed proof of the construction of the Seiberg-dual character will be given in the case $\mathfrak{g} = A_1$ in section 7. Here, we only sketch the main points.

From the quantum mechanics picture, the $qq$-character of a Seiberg-dual theory can easily be obtained after changing the sign of some of the 1d FI parameters $\zeta_{1d}^{(a)}$ from positive to negative in the Witten index (2.12). In particular, a different set of poles from the one considered so far in this paper will be enclosed by the contours of the dual theory. This modification of the contours is perfectly tractable, and we can readily compute the index by the JK-residue. As our end result, we find that up to normalization by the “classical” contribution $Z_{\text{defect,}\emptyset}^{(a)}$, the vortex character of a Seiberg-dual theory is obtained by switching the signs of the various flavor masses, defect fermion masses and 3d FI parameters in the quantum mechanics. There is one caveat, however: the FI parameters are continuous, so when changing their sign, wall crossing can happen at the value $\zeta_{1d}^{(a)} = 0$. This typically results in new states from the Coulomb branch contributing to the Witten index, and these should be identified carefully by considering the residues at $\phi \to \pm\infty$.

Another important subtlety is that the Seiberg-dual theory we identify in this formalism is only correct on the Higgs branch, where the vortex solutions are defined. In particular, at a more general point on the moduli space, the 3d Seiberg-dual theories we identify can happen to disagree globally. Put differently, the duality we identify is only strictly true at a special point in the moduli space. For physical considerations on this point, see \cite{107–111}. For mathematical considerations, see \cite{112–114}.

### 6.2 Future Directions

Starting from our construction of the vortex character, there are many important questions to investigate. Let us list a few pressing open problems:
One important question would be to understand what the D3\textit{defect} branes of the little string mean in geometry, most notably in the language of quantum K-theory of Nakajima quiver varieties [37, 75].

Defining the vortex characters should be possible for classical gauge groups using orientifold arguments [10], and for more general quiver theories than the ADE ones. For instance, so-called “fractional” quiver theories, which include the non simply-laced BCFG algebras, should be obtained by folding [15, 18, 75]. Affine quivers theories could also be studied, modifying some of our arguments in a straightforward way.

The vortex characters we constructed are naturally defined for 3d theories on the manifold $\mathbb{C} \times S^1(\hat{R})$. As we mentioned in the text, reducing the theory on the circle produces vortex characters for 2d $\mathcal{N} = (4,4)$ gauged sigma models [8]. It should likewise be straightforward to study the uplift to 4d $\mathcal{N} = 2$ theories on the manifold $\mathbb{C} \times T^2$; such a lift is expected to produce elliptic vortex characters [22].

Recently, a vortex $qq$-character was defined for certain 3d $\mathcal{N} = 2$ gauge theories of handsaw-type [18] obtained from the Higgsing a five-dimensional theory, using a similar construction to the one we presented for $\mathcal{N} = 4$ theories. In the $\mathcal{W}_{q,t}(g)$-algebra formalism, the characters arise as correlators similar to those studied here, but with the insertion of deformed “primary” vertex operators at points on the cylinder rather than the fundamental vertex operators (4.13) compatible with $\mathcal{N} = 4$ supersymmetry. It would be important to construct the characters for more generic 3d $\mathcal{N} = 2$ theories, and understand their realization in the language of $\mathcal{W}(g)$-algebras and string theory, as well as the action of Seiberg-like duality [115, 116].

Mirror symmetry is known to exchange Wilson loops with so-called vortex loops in 3d $\mathcal{N} = 4$ gauge theories [51, 117]. By Wilson loop, we mean here the classical contribution $Z_{\text{defect},\emptyset}^{(\alpha)}$. It our work, we crucially made use of the idea that the presence of such a loop should generalize the vortex moduli space altogether and introduce new multiplets in the quantum mechanics (as opposed to localizing the loop on the vortex solutions in the absence of a loop). It would be interesting to study the action of mirror symmetry in our setup, and notably how the symmetry acts on the vortex character.

7 A Case Study: 3d $\mathcal{N} = 4$ SQCD

In this section, we illustrate in detail all the statements made in the paper for the Lie algebra $\mathfrak{g} = A_1$. Namely, consider the 3d $\mathcal{N} = 4$ gauge theory $G^{3d}$ with gauge group $G = U(N)$.
and flavor group $G_F = U(N_F)$, on the manifold $\mathbb{C} \times S^1(\hat{R})$, with a 1/2-BPS Wilson loop at the origin of $\mathbb{C}$ and wrapping $S^1(\hat{R})$.

——- The 1d Quantum Mechanics ——-

Let us first describe the theory in the absence of Wilson loop. We freeze each equivariant parameter of $G$ to be one of the equivariant parameters of $G_F = U(N_F)$, describing the root of the Higgs branch of $G^{3d}$. We turn on the real FI parameter $\zeta_{3d} > 0$, complexified because of the circle, and consider the moduli space of $k$ vortices. Let $T_{\text{pure}}^{1d}$ be the quantum mechanics on the vortices. It is a theory with (the reduction from 2d $\mathcal{N} = (2,2)$ to) 1d $\mathcal{N} = 4$ supersymmetry on $S^1(\hat{R})$, with gauge group $\hat{G} = U(k)$. The flavor symmetry is $\hat{G}_F = U(N) \times U(N_F - N)$, where the first group is the symmetry of $N$ fundamental chiral multiplets, while the second group is the symmetry of $N_F - N$ antifundamental chiral multiplets.

**Figure 8**: The 3d gauge theory $G^{3d}$ and the vortex quantum mechanics $T_{\text{pure}}^{1d}$.

The Witten index (2.10) of the quantum mechanics is expressed as the following integral:

$$
\begin{align*}
\chi^{(L)}_{1d} &= \sum_{k=0}^{\infty} \frac{e^{\zeta_{3d} k}}{k!} \int_{\mathcal{M}_{k}^{\text{pure}}} \left[ \frac{d\phi_I}{2\pi i} \right] Z_{\text{pure,vec}} \cdot Z_{\text{pure,adj}} \cdot Z_{\text{pure,teeth}}, \\
Z_{\text{pure,vec}} &= \prod_{I,J=1}^{k} \sh (\phi_I - \phi_J) \\
&= \frac{\prod_{I,J=1}^{k} \sh (\phi_I - \phi_J + \epsilon_2)}{\prod_{I,J=1}^{k} \sh (\phi_I - \phi_J + \epsilon_2)}
\end{align*}
$$
\[ Z_{\text{pure,adj}} = \prod_{I,J=1}^{k} \frac{\text{sh}(\phi_I - \phi_J + \epsilon_1 + \epsilon_2)}{\text{sh}(\phi_I - \phi_J + \epsilon_1)} \]

\[ Z_{\text{pure,teeth}} = \prod_{I \in \{1, \ldots, k\}} \prod_{i=1}^{N} \prod_{j=1}^{N-I} \frac{\text{sh}(\phi_I - m_i + \epsilon_+ + \epsilon_2)}{\text{sh}(\phi_I - m_i + \epsilon_+)} \prod_{j=1}^{N-I-N} \frac{\text{sh}(-\phi_I + m_j + \epsilon_+ + \epsilon_2)}{\text{sh}(-\phi_I + m_j + \epsilon_+)} . \]

We made use of the notations \( \text{sh}(x) = 2 \sinh(\tilde{R}x/2) \), \( \epsilon_+ = (\epsilon_1 + \epsilon_2)/2 \) and \( \epsilon_- = (\epsilon_1 - \epsilon_2)/2 \). Crucially, the index depends on the sign of the 1d FI parameter, which we take here to be \( \zeta_{1d} > 0 \). After applying the JK-residue prescription in that FI-chamber, the poles that end up contributing at vortex charge \( k \) to the \( T_{1d}^{\text{pure}} \) index make up a set \( \mathcal{M}_k^{\text{pure}} \). The elements of this set satisfy:

\[ \phi_I = \phi_J - \epsilon_1, \quad (7.2) \]

\[ \phi_I = \phi_J - \epsilon_2, \quad (7.3) \]

\[ \phi_I = m_i - \epsilon_-, \quad i \in \{1, \ldots, N\} . \quad (7.4) \]

The poles (7.2) arise from the adjoint chiral factor \( Z_{\text{pure,adj}} \), the poles (7.3) arise from the vector multiplet \( Z_{\text{pure,vec}} \), and the poles (7.4) arise from flavor factor \( Z_{\text{pure,teeth}} \). Most notably, the last set of contours only encloses poles originating from the fundamental chiral multiplets, and none of the antifundamental chiral multiplets. Furthermore, the residues at the locus (2.21) are all zero, thanks to the numerator of \( Z_{\text{pure,teeth}} \). Putting it all together, the various poles which end up contributing with nonzero residue are of the form

\[ \phi_I = m_i - \epsilon_- - (s_i - 1)\epsilon_1, \quad \text{with } s_i \in \{1, \ldots, k\}, \quad i \in \{1, \ldots, N\} . \quad (7.5) \]

In this notation, \( (k_1, \ldots, k_N) \) is a partition of \( k \) into \( N \) non-negative integers, and the pair of integers \( (i, s_i) \) is assigned to one of the integers \( I \in \{1, \ldots, k\} \) exactly once.

Performing the residue integral, one finds the following well-known expression [118]:

\[ [\chi]^{(0)}_{1d} = \sum_{k=0}^{\infty} e^{C_{1d}k} \sum_{\sum_i k_i = k, k_i \geq 0} \left[ \prod_{i,j=1}^{N} \prod_{s=1}^{k_i} \frac{\text{sh}(m_i - m_j + \epsilon_2 - (s - k_j - 1)\epsilon_1)}{\text{sh}(m_i - m_j - (s - k_j - 1)\epsilon_1)} \right] \times \left[ \prod_{i=N+1}^{N_f} \prod_{j=1}^{N} \prod_{p=1}^{k_j} \frac{\text{sh}(m_i - m_j + \epsilon_2 + p\epsilon_1)}{\text{sh}(m_i - m_j + p\epsilon_1)} \right] . \quad (7.6) \]

Let us now consider the inclusion of the Wilson loop, transforming in the fundamental representation of \( G = U(N) \). This loop is a \( 1/2 \)-BPS codimension-2 defect from the point of view of \( G^{3d} \); we introduce a defect group \( \hat{G}_{\text{defect}} = U(1) \) for the 1d fermions in the loop, with associated mass fugacity \( M \). The inclusion of the loop modifies the vortex quantum mechanics, which we now call \( T_{1d} \).
Figure 9: The 3d gauge theory $G^{3d}$ with a Wilson loop defect and the vortex quantum mechanics $T^{1d}$. The notations are as in Figure 2.

Its Witten index is given by:

$$[\chi]^{(1)}_{1d} = \sum_{k=0}^{\infty} \frac{e^{\zeta_{1d} k}}{k!} Z_{\text{defect},\varnothing} \oint_{M_k} \left[ \frac{d\phi_I}{2\pi i} \right] Z_{\text{pure,vec}} \cdot Z_{\text{pure,adj}} \cdot Z_{\text{pure,teeth}} \cdot Z_{\text{defect},k},$$

(7.7)

$$Z_{\text{defect},\varnothing} = \prod_{i=1}^{N} \text{sh} (m_i - M + \epsilon_2)$$

$$Z_{\text{defect},k} = \prod_{I=1}^{k} \frac{\text{sh} (\phi_I - M - \epsilon_-) \text{sh} (-\phi_I + M - \epsilon_-)}{\text{sh} (\phi_I - M - \epsilon_+) \text{sh} (-\phi_I + M - \epsilon_+)}.$$

We once again work in the FI-chamber $\zeta_{1d} > 0$. For a given vortex charge $k$, the set of poles to be enclosed by the contours is called $\mathcal{M}_k$. This set contains the set $\mathcal{M}_k^{\text{pure}}$ we specified in (7.5) for the theory $T^{1d}_{\text{pure}}$, and an extra locus depending on the defect fermion mass $M$:

$$\phi_I = M + \epsilon_+,$$

for some $I \in \{1, \ldots, k\}$.

(7.8)

No other pole depending on $M$ exists, by the following argument: consider the pole at $\phi_I = M + \epsilon_+$ for $I$ fixed. The JK-prescription indicates there is a pole at the locus $\phi_J - M - \epsilon_+ = 0$, with $J \neq I$. But the residue there is zero, because of the factor $\text{sh}(\phi_I - \phi_J)$ in the numerator of $Z_{\text{pure,vec}}$. Similarly, the JK prescription requires us to include the hyperplanes $\phi_I - \phi_J + \epsilon_1 = 0$ and $\phi_I - \phi_J + \epsilon_2 = 0$. But the numerators of $Z_{\text{defect},k}$ guarantee a zero residue at this loci, so we have succeeded in showing that there is
exactly one $M$-dependent pole in $\mathcal{M}_k$. Put differently, $|\mathcal{M}_k| = |\mathcal{M}^{\text{pure}}_k| + 1$, for all $k$. We conclude that for a given vortex charge $k$, we choose one of the contours to pick up the unique $M$-dependent pole in $\mathcal{M}_k$, namely (7.8), and the $k−1$ other poles are to be chosen in the set $\mathcal{M}^{\text{pure}}_{k−1}$, according to (7.5).

We introduce a (renormalized) $1/2$-BPS codimension-2 defect operator for the loop, with associated vev:

$$\langle [Y_1d(M)]^{±1} \rangle = \sum_{k=0}^{\infty} \frac{e^{\xi_0 k}}{k!} \oint_{\mathcal{M}^{\text{pure}}_k} \frac{d\phi_I}{2\pi i} Z_{\text{pure,vec}} \cdot Z_{\text{pure,adj}} \cdot Z_{\text{pure,teeth}} \cdot [Z_{\text{defect},k}(M)]^{±1} .$$

(7.9)

Note the contour is defined to exclusively enclose poles in the set $\mathcal{M}^{\text{pure}}_k$, thereby avoiding the pole at $\phi_I = M + \epsilon_+$. In what follows, we will freely make use of K-theoretic notations,

$$\tilde{q} = e^{\xi_0}, \quad q = e^{\epsilon_1}, \quad t = e^{-\epsilon_2}, \quad v = e^{\epsilon_+} = \sqrt{qt}, \quad u = e^{\epsilon_-} = \sqrt{qt},$$

$$f_d = e^{-m_d}, \quad z = e^{-M} .$$

Furthermore, for ease of presentation, we renormalize the index by the classical Wilson loop contribution and the index of the vortex quantum mechanics $T^{1d}_{\text{pure}}$ in the absence of Wilson loop:

$$[\tilde{\chi}]^{(1)}_{1d}(z) \equiv \frac{[\chi]^{(1)}_{1d}(z)}{Z_{\text{defect},0}(z) \cdot [\chi]^{(0)}_{1d}} .$$

(7.11)

Then, we find that the normalized index can be expressed in terms of the $Y$-operators, as a sum of exactly two terms:

$$[\tilde{\chi}]^{(1)}_{1d}(z) = \frac{1}{[\chi]^{(0)}_{1d}} \left[ \langle Y_1d(z) \rangle + \tilde{q} \prod_{i=1}^{N} \frac{1 - q t^{-1} f_i / z}{1 - q f_i / z} \prod_{j=N+1}^{N_F} \frac{1 - t f_j / z}{1 - f_j / z} \langle Y_1d(z v^{-2}) \rangle \right] .$$

(7.12)

This is a twisted $qq$-character of the fundamental representation of the quantum affine algebra $U_q(\hat{A}_1)$. The meaning of the above character is as follows: The first term on the right-hand side encloses almost all the “correct” poles in the index integrand, but it is missing exactly one: the extra pole at $\phi_I = M + \epsilon_+ = 0$. The second term on the right-hand side makes up for this missing pole, and relies on a key observation: we can trade a contour enclosing this extra pole for a contour which does not enclose it, at the expense of inserting the operator $Y(z v^{-2})^{-1}$ inside the vev. This result is derived at once from the integral expression (7.7), and the $Y$-operator definition (7.9). Finally, note the presence of the 3d FI parameter $\tilde{q}$ in the second term; it counts exactly one vortex, to make up for the missing $M$-pole, consistent with the fact that $|\mathcal{M}_k| = |\mathcal{M}^{\text{pure}}_k| + 1$. 

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It is instructive to recast the above result in terms of the general expression for the index presented in the main text:

\[
[\chi^{(1)}_{1d}(z)] = \frac{1}{|\chi^{(0)}_{1d}|} \sum_{\omega \in \mathcal{V}^{(1)}} (q)^{d} c_{d}(q,t) Q_{d}(z) [Y_{1d}(z)]_{\omega}.
\] (7.13)

In that notation, the sum is over two weights exactly, the highest weight \(\omega_{1} = [1]\) of the spin-1/2 representation of \(A_{1}\), and the \(\omega_{2} = [-1]\), obtained by lowering \(\omega_{1}\) by the positive simple root \(\alpha\) of \(A_{1}\): \(1 - [2] = [-1]\). The coefficients \(c_{d}(q,t)\) are all 1, while

\[
Q_{d}(z) = 1,
\]

\[
Q_{d-1}(z) = \prod_{i=1}^{N} \frac{1 - q t^{-1} f_i/z}{1 - q f_i/z} \prod_{j=N+1}^{N_{F}} \frac{1 - t f_j/z}{1 - f_j/z},
\] (7.14)

and

\[
[Y_{1d}(z)]_{1} = \langle Y_{1d}(z) \rangle,
\]

\[
[Y_{1d}(z)]_{-1} = \left\langle \frac{1}{Y_{1d}(z v^{-2})} \right\rangle.
\] (7.15)

Non-perturbative Schwinger-Dyson identities for \(G^{3d}\) follow from the regularity conditions of the vortex character in the defect fugacity \(z = e^{-M}\). Namely, a \(z\)-singularity will arise if two poles of the integrand pinch one of the contours. Let us list such pairs of poles, where the poles on the left are inside the contours and the poles on the right are outside the contours, by the JK-prescription. We let \(I \in \{1, \ldots, k\}\) and find:

\[
\phi_{I} - M - \epsilon_{+} = 0 \quad \text{and} \quad \phi_{I} - m_{j} - \epsilon_{+} = 0 \quad (j \in \{N + 1, \ldots, N_{F}\})
\] (7.16)

\[
\phi_{I} - M - \epsilon_{+} = 0 \quad \text{and} \quad \phi_{I} - \phi_{J} - \epsilon_{1} = 0 \quad (J \neq I)
\] (7.17)

\[
\phi_{I} - M - \epsilon_{+} = 0 \quad \text{and} \quad \phi_{I} - \phi_{J} - \epsilon_{2} = 0 \quad (J \neq I)
\] (7.18)

\[
\phi_{I} - m_{i} + \epsilon_{-} = 0 \quad \text{and} \quad \phi_{I} - M + \epsilon_{+} = 0 \quad (i \in \{1, \ldots, N\})
\] (7.19)

\[
\phi_{I} - \phi_{J} + \epsilon_{1} = 0 \quad \text{and} \quad \phi_{I} - M + \epsilon_{+} = 0 \quad (J \neq I)
\] (7.20)

\[
\phi_{I} - \phi_{J} + \epsilon_{2} = 0 \quad \text{and} \quad \phi_{I} - M + \epsilon_{+} = 0 \quad (J \neq I)
\] (7.21)

The sets of poles (7.17), (7.18), (7.20) and (7.21) pinch the contour, but the singularity is canceled by a zero in the integrand. For instance, the set (7.20) implies a singularity at the locus \(\phi_{J} - M - \epsilon_{-} = 0\), but there is a zero there due to the numerator of \(Z_{\text{defect}, k}\). The sets of poles (7.16) and (7.19) genuinely pinch the contours, and result in singularities

\[
M = m_{j} \quad (j \in \{N + 1, \ldots, N_{F}\}),
\] (7.22)

\[
M = m_{i} + \epsilon_{2} \quad (i \in \{1, \ldots, N\}),
\] (7.23)

The singularity (7.23) is formally canceled by the Fermi multiplets numerators of \(Z_{\text{defect}, 0}\) in the index (7.7), but we normalized by this classical contribution when defining the vortex character. The singularity is absent if we decide not to normalize the index by \(Z_{\text{defect}, 0}\), at the cost of slightly modifying the twist of the \(qq\)-character. Our choice to normalize by this
classical contribution was purely cosmetic, and ultimately does not affect any result. On the other hand, the singularity \((7.22)\) is an unavoidable feature of 3d \(\mathcal{N}=4\) theories. Removing this singularity can be done by inserting an additional flavor Wilson loop in the 3d theory, transforming in the fundamental representation of the flavor subgroup \(U(N_f - N) \subset U(N_F)\); this would result in extra Fermi multiplet contributions to the index, of the form \(Z_{\text{defect},a} (7.7)\). We decided against adding unnecessary Wilson loops in this work. Ultimately, our presentation of the index simply means that we have to live with these flavor singularities. The Schwinger-Dyson identities are the statement that these are the only singularities present.

——— The 3d Gauge Theory Perspective ———

In the absence of Wilson loop, the half-index of the 3d theory reads

\[
[\hat{\chi}]_{3d}^{(0)} = \oint_{\mathcal{M}^{\text{bulk}}} dy \left[ I_{3d}^{\text{bulk}}(y) \right].
\]

(7.24)

where the bulk contribution reads

\[
I_{3d}^{\text{bulk}}(y) = \prod_{i=1}^N y_i^{(\zeta_{3d}-1)} I_{\text{vec}}(y) \cdot I_{\text{flavor}}(y, \{x_d\}).
\]

(7.25)

The factor

\[
\prod_{i=1}^N y_i^{(\zeta_{3d})}
\]

(7.26)

is the contribution of the 3d FI parameter.

The factor

\[
I_{\text{vec}}(y) = \prod_{1 \leq i \neq j \leq N} \frac{(y_i/y_j; q)_{\infty}}{(t y_i/y_j; q)_{\infty}} \prod_{1 \leq i < j \leq N} \frac{\Theta (t y_i/y_j; q)}{\Theta (y_i/y_j; q)}
\]

(7.27)

stands for the contribution of the \(\mathcal{N}=4\) vector multiplet for the gauge group \(G = U(N)\).

The factor

\[
I_{\text{flavor}}(y, \{x_d\}) = \prod_{d=1}^{N_F} \prod_{i=1}^N \frac{(t v x_d/y_i; q)_{\infty}}{(v x_d/y_i; q)_{\infty}}
\]

(7.28)

stands for the contribution of the \(\mathcal{N}=4\) hypermultiplets in the fundamental representation of the \(a\)-th gauge group. The set of poles to be enclosed by the contours is denoted as \(\mathcal{M}^{\text{bulk}}\). Following the JK-residue prescription, the poles which contribute with nonzero
residue are located at
\[
y_i = v_x d q_s, \quad s = 0, 1, 2, \ldots, \quad d \in \{1, \ldots, N_F\},
\]
(7.29)
where each integer \(\{i\}\) gets mapped uniquely to one of the integers \(\{d\}\). Note that the \(t\)-dependent poles coming from the vector multiplet denominators \((t y_i^{(a)}/y_j^{(a)}; q)\) are allowed by the JK-prescription, but they contribute with zero residue due to the fundamental hypermultiplet numerators \((t v x_d/y_i; q)\). Having identified the poles, one can perform the residue integral to recover the Witten index (7.6) of the quantum mechanics \(T_{\text{pure}}^{1d}\) (up to normalization by irrelevant infinite quantum dilogarithm factors).

We now introduce the 1/2-BPS Wilson loop wrapping \(S^1(R)\) via gauging its 1d degrees of freedom, as explained in the main text. The corresponding defect \(Y\)-operator vev is written as an integral over the Coulomb moduli of the 3d theory:
\[
\langle \tilde{Y}_{3d}(z) \rangle^{\pm 1} = \oint_{\mathcal{M}^{\text{bulk}}} dy \left[ f^{3d}_{\text{bulk}}(y) \cdot \tilde{Y}_{\text{defect}}(y, z) \right]^{\pm 1},
\]
(7.30)
with
\[
\tilde{Y}_{\text{defect}}(y, z) = \prod_{i=1}^{N_F} \frac{1 - t y_i/z}{1 - y_i/z}.
\]
(7.31)
There is also the flavor part of the defect which we have to include,
\[
\tilde{Y}_{\text{flavor}}(\{x_d\}, z) = \prod_{d=1}^{N_F} \frac{1 - v x_d/z}{1 - t v x_d/z}.
\]
(7.32)
Then, the (normalized) index of \(G^{3d}\) in the presence of a Wilson loop is defined as the following vortex character:
\[
[\tilde{\chi}]^{(1)}_{3d}(z) = \left[ \frac{1}{[\chi]^{(0)}_{3d}} \tilde{Y}_{\text{flavor}}(\{x_d\}, z) \langle \tilde{Y}_{3d}(z) \rangle + \left\langle \frac{1}{\tilde{Y}_{3d}(z v^{-2})} \right\rangle \right].
\]
(7.33)
This is once more a twisted \(qq\)-character of the fundamental representation of \(U_q(\hat{A}_1)\). It is again instructive to recast the above result in terms of the general expression for the index presented in the main text:
\[
[\tilde{\chi}]^{(1)}_{3d}(z) = \frac{1}{[\chi]^{(0)}_{3d}} \sum_{\omega \in V(\lambda)} (\tilde{q})^{d\omega} c_{d\omega}(q, t) \tilde{Q}_{d\omega}(z) \langle Y_{3d}(z) \rangle_{\omega}.
\]
(7.34)
Just as in the case of the quantum mechanics presentation, the sum is over two weights exactly: the highest weight \(\omega_1 = [1]\) of the spin-1/2 representation of \(A_1\), and the \(\omega_2 = [-1]\), obtained by lowering \(\omega_1\) by the positive simple root \(\alpha\) of \(A_1\): \([1] - [2] = [-1]\). The coefficients
$c_d(q,t)$ are all 1, while

$$\tilde{Q}_{d^1}(z) = \tilde{Y}_{\text{flavor}}(\{x_d\}, z), \quad \tilde{Q}_{d^{-1}}(z) = 1,$$

and

$$[\tilde{Y}_{3d}(z)]_1 = \langle \tilde{Y}_{3d}(z) \rangle, \quad [\tilde{Y}_{3d}(z)]_{-1} = \frac{1}{\langle Y_{3d}(zv^{-2}) \rangle}.$$

The correctness of this definition can be checked by comparing it to the Witten index of the vortex theory; recall that we previously derived:

$$[\tilde{\chi}]^{(1)}_{1d}(z) = \frac{1}{[\chi]^{(0)}_{1d}} \left[ \langle Y_{1d}(z) \rangle + \tilde{q} \prod_{i=1}^{N} \frac{1 - q t^{-1} f_i/z}{1 - q f_i/z} \prod_{j=N+1}^{N_F} \frac{1 - t f_j/z}{1 - f_j/z} \langle Y_{1d}(z v^{-2}) \rangle \right].$$

(7.37)

We can perform the residue integral over the poles (7.5) explicitly, and define “3d variables” as:

$$y_{i,*} = f_i q^{k_i+1}, \quad i \in \{1, \ldots, N\},$$

(7.38)

along with the rescaling of the 1d masses to define them in terms of the 3d masses,

$$f_i = v x_i, \quad i = 1, \ldots, N_F.$$

(7.39)

The defect residue in the quantum mechanics becomes:

$$Z_{\text{defect}, k}(y_{i,*}, z) = \prod_{i=1}^{N} \frac{1 - t v x_i/z}{1 - y_{i,*}/z} \prod_{i=1}^{N} \frac{1 - f_i/z}{1 - f_i/z} \prod_{j=N+1}^{N_F} \frac{1 - v x_i/z}{1 - v x_i/z} \cdot \prod_{i=N+1}^{N_F} \frac{1 - t v x_i/z}{1 - v x_i/z}.$$

(7.40)

In other terms, we find that the $Y$-operator vev as defined in the quantum mechanics can be written in terms of the $Y$-operator vev as defined in the 3d theory.
Let us now look at the second term in the 1d vortex character:

\[
\prod_{i=1}^{N} \frac{1 - q t^{-1} f_i / z}{1 - q f_i / z} \prod_{j=N+1}^{N_F} \frac{1 - t f_j / z}{1 - f_j / z} \frac{1}{Z_{\text{defect},k}(y_i, z \nu)}
\]

\[
= \prod_{i=4}^{N} \frac{1 - q t^{-1} f_i / z}{1 - q f_i / z} \prod_{j=N+1}^{N_F} \frac{1 - t f_j / z}{1 - f_j / z} \prod_{i=1}^{N} \frac{1 - v^2 y_i / z}{1 - t v^2 y_i / z} \prod_{j=N+1}^{N} \frac{1 - q f_j / z}{1 - t^{-1} f_j / z}
\]

\[
= \prod_{i=1}^{N} \frac{1 - v^2 y_i / z}{1 - t v^2 y_i / z} \prod_{j=N+1}^{N_F} \frac{1 - t v x_j / z}{1 - v x_j / z}
\]

\[
= \frac{1}{\left[\tilde{Y}_{\text{defect}}(y_*, z \nu^-)\right]} \prod_{i=N+1}^{N_F} \frac{1 - t v x_i / z}{1 - v x_i / z}
\]  

(7.41)

After the above astonishing cancellations, we find that the characters are in fact proportional to each other! Denoting the proportionality factor as

\[
c_{3d/1d} = \prod_{i=N+1}^{N_F} \frac{1 - t v x_i / z}{1 - v x_i / z},
\]

(7.42)

we proved that

\[
[\tilde{\chi}]_{3d}^{(1)} (z) = c_{3d/1d} \cdot [\tilde{\chi}]_{1d}^{(1)} (z).
\]

(7.43)

--- The \(W_{q,t}(A_1)\)-Algebra ---

\(q\)-Liouville theory on the cylinder \(C\) enjoys a \(W_{q,t}(A_1)\)-algebra symmetry, which is generated by the deformed stress tensor \(W^{(2)}(z)\). This generating current is constructed as the commutant of the screening charge. We find:

\[
W^{(2)}(z) = [\mathcal{Y}(z) : + : (\mathcal{Y}(v^- z)^{-1}) :]
\]

(7.44)

where \(\mathcal{Y}\) is the operator defined in (4.9). We consider the correlator:

\[
\left\langle \psi' \left| \prod_{d=1}^{N_f} V(x_d) Q^N W^{(2)}(z) \right| \psi \right\rangle.
\]

(7.45)

The contours are specified as to not enclose any pole in the \(z\) variable. The state \(|\psi\rangle\) is
defined such that:

\[\alpha[0]|\psi\rangle = \langle \psi, \alpha |\psi\rangle\]
\[\alpha[k]|\psi\rangle = 0, \quad \text{for } k > 0,\]

where the \(\alpha[k]\) generate a \(q\)-deformed Heisenberg algebra:

\[[\alpha[k], \alpha[m]] = \frac{1}{k}(q^k - q^{-k})(t^k - t^{-k})(v^k + v^{-k})\delta_{k,-m}.\]

We compute the various two-points making up the correlator; first, the bulk contributions

\[
\prod_{1 \leq i \neq j \leq N} \langle S(y_i) S(y_j) \rangle = \prod_{1 \leq i \neq j \leq N} \frac{(y_i/y_j; q)^\infty}{(t y_i/y_j; q)^\infty} \prod_{1 \leq i < j \leq N} \Theta(y_j/y_i; q) \Theta(y_j/z; q) \prod_{i=1}^{N} (t v x_d/y_i; q)^\infty \prod_{i=1}^{N} (v x_d/y_i; q)^\infty.
\]

The two-point of fundamental vertex operators with themselves will drop out after normalization, so we omit writing it.

We now come to the contributions involving the Wilson loop. First, the two-point of the stress tensor with the screening currents

\[
\prod_{i=1}^{N} \langle S(y_i) W^{(2)}(z) \rangle = \prod_{i=1}^{N} \frac{1 - t y_i/z}{1 - y_i/z} + \prod_{i=1}^{N} \frac{1 - v^2 y_i/z}{1 - t v^2 y_i/z}.
\]

Note that in the actual correlator, the vacuum is labeled by \(|\psi\rangle\) instead of \(|0\rangle\), resulting in a relative shift of \(\tilde{q}\) between the two terms. A more involved computation is the two-point of the fundamental vertex operator with the stress tensor:

\[
\langle V(x_d) W^{(2)}(z) \rangle = \exp \left( \sum_{k>0} \frac{1}{k} \frac{t^k - 1}{v^k + v^{-k}} \left( \frac{x_d}{z} \right)^k \right)
+ \exp \left( - \sum_{k>0} \frac{1}{k} \frac{t^k - 1}{v^k + v^{-k}} \left( \frac{v^2 x_d}{z} \right)^k \right)
= \exp \left( - \sum_{k>0} \frac{1}{k} \frac{t^k - 1}{v^k + v^{-k}} \left( \frac{v^2 x_d}{z} \right)^k \right) \cdot \left( \frac{1 - v x_d/z}{1 - t v x_d/z} + 1 \right)
= B(x_d, z) \cdot \left( \frac{1 - v x_d/z}{1 - t v x_d/z} + 1 \right).
\]

In the first line, we used the commutator

\[[w[k], w[n]] = \frac{1}{k}(q^k - q^{-k})(t^k - t^{-k}) \frac{1}{v^k + v^{-k}} \delta_{k,-n} ,\]

\[\text{(7.52)}\]
which is dual to the relation (7.47). In the second line we used the identity \( \exp(-\sum_{k>0}^z v^k) = (1-x) \). In the third line, we gave a name to the overall exponential factor,

\[
B(x_d, z) \equiv \exp \left( -\sum_{k>0}^z \frac{t^k - 1}{k} \left( \frac{v^2 x_d}{z} \right)^k \right).
\]

(7.53)

All in all, the normalized \( W_{q,t}(A_1) \) correlator comes out to be proportional to the 3d vortex character,

\[
\frac{\langle \psi' \prod_{d=1}^{N_f} V(x_d) QD W^{(2)}(z) \mid \psi \rangle}{\langle \psi' \prod_{d=1}^{N_f} V(x_d) QD \mid \psi \rangle} = B(\{x_d\}, z) [\chi]_{3d}^{(1)}(z).
\]

(7.54)

with

\[
B(\{x_d\}, z) \equiv \prod_{d=1}^{N_F} \exp \left( -\sum_{k>0}^z \frac{t^k - 1}{k} \left( \frac{v^2 x_d}{z} \right)^k \right),
\]

(7.55)

As explained in the main text, this factor can be naturally canceled out in the Ding-Iohara-Miki formalism, where it arises as an extra \( U(1) \) due to an auxiliary Heisenberg algebra [16].

The non-perturbative Schwinger-Dyson equation for \( G^{3d} \) manifests itself here as a Ward identity. It is interpreted as a statement about the regularity in the fugacity \( z \) of the correlator \( \langle [\ldots] W^{(2)}(z) \rangle \).

It is straightforward to generalize this discussion to the case of a Wilson loop in a higher spin representation, by considering a defect group \( \hat{G}_{\text{defect}} = U(L) \). This corresponds to considering a Wilson loop valued in the representation \( N \otimes \ldots \otimes N \) of \( SU(N) \), where the fundamental representation \( N \) is tensored \( L \) times with itself. This is a flavor Wilson loop after Higgsing. The JK-residue prescription dictates that for each \( \rho \in \{1, 2, \ldots, L\} \), the contours of the quantum mechanics should enclose a pole at

\[
\phi_I - M_\rho - \epsilon_+ = 0.
\]

(7.56)

Once again, the partition function can be expressed as a \( qq \)-character of \( U_q(A_1) \), with highest weight \( \{L\} \) (the spin \( L/2 \) representation). In the \( q \)-Liouville picture, one would simply consider a deformed \( W_{q,t}(A_1) \)-algebra correlator with \( L \) insertions of the deformed stress tensor:

\[
\frac{\langle \psi' \prod_{d=1}^{N_f} V(x_d) QD \prod_{\rho=1}^L W^{(2)}(z_\rho) \mid \psi \rangle}{\langle \psi' \prod_{d=1}^{N_f} V(x_d) QD \mid \psi \rangle} = B(\{x_d\}, \{z_\rho\}) [\chi]_{3d}^{(L)}(\{z_\rho\}).
\]

(7.57)
Let $X$ be a resolved $A_1$ singularity, and consider type IIB string theory on $X \times \mathbb{C} \times \mathbb{C}_q \times \mathbb{C}_t$, with $\mathcal{C} = \mathbb{R} \times S^1(R)$ an infinite cylinder of radius $R$, and $\mathbb{C}_q$ and $\mathbb{C}_t$ two complex lines. We introduce $N$ D3$_{\text{gauge}}$ branes wrapping the compact 2-cycle $S$ of $X$ and $\mathbb{C}_q$. We further introduce $N_f$ D3$_{\text{flavor}}$ branes wrapping the dual non-compact 2-cycle $S^*$ and $\mathbb{C}_q$. We also add to this background $L$ D3$_{\text{defect}}$ branes wrapping that same 2-cycle $S^*$ and $\mathbb{C}_t$. This background preserves 4 supercharges. We send the string coupling to $g_s \to 0$; the tensions of the various D3 branes survive in the limit. Then, this amounts to studying the $(2,0)$ $A_1$ little string on $\mathcal{C} \times \mathbb{C}_q \times \mathbb{C}_t$ in the presence of various codimension-4 defects. At energies below the string scale, the dynamics are fully captured by the theory on the D3$_{\text{gauge}}$ branes: the effective theory on the branes is the 3d gauge theory $G^{3d}$, with gauge group $G = U(N)$, defined on the manifold $\mathbb{C}_q \times S^1(\hat{R})$. Note that this is the T-dual circle to the original circle $S^1(R)$ of the cylinder, meaning $\hat{R} = 1/(m_s^2 R)$. The D3$_{\text{flavor}}$ branes realize the fundamental matter content $G_F = U(N_F)$. From the 3d gauge theory point of view, the $L$ D3$_{\text{defect}}$ branes make up a 1/2-BPS Wilson loop wrapping $S^1(\hat{R})$ and sitting at the origin of $\mathbb{C}_q$. Let us focus on the case $L = 1$, which fixes the Wilson loop representation to be the fundamental one.

The index of the $(2,0)$ little string in this background localizes to the 3d/1d half-index (7.33) of the 3d gauge theory:

$$[\tilde{\chi}]^{(1)}_{\text{D3}_g, \text{D3}_f, \text{D3}_d}(z) = \frac{1}{[\chi]^{(0)}_{\text{D3}_g, \text{D3}_f}} \prod_{d=1}^{N_F} \frac{1 - v x_d/z}{1 - t v x_d/z} \left( \langle \tilde{Y}^{3d}(z) \rangle + \langle \frac{1}{\tilde{Y}^{3d}(z v^{-2})} \rangle \right). \tag{7.58}$$

Up to an overall normalization, this also happens to be the $q$-Liouville correlator (7.54) on the cylinder, see Figure 10.

**Figure 10:** Example of a correlator in $q$-Liouville, along with the corresponding D-branes at points on the cylinder. The specific correlator pictured here is $\langle \psi' | \prod_{d=1}^{2} V(x_d) (Q)^3 T(z) | \psi \rangle$. 

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**Defects of the $A_1$ $(2,0)$ Little String**
We break $G$ by freezing the D3$_{gauge}$ moduli, and reorganize the branes to make up a set of D3'$_{flavor}$ branes exclusively. We turn on the period $\int_S \omega_i > 0$, which is the 3d FI parameter, and study the vortex solutions on the Higgs branch of $G^{3d}$. These are D1$_{vortex}$ branes wrapping the 2-cycle $S$ in the (2, 0) string. The partition function localizes to the Witten index (7.12) of the quantum mechanics $T_1^{1d}$ on the D1$_{vortex}$ branes. Up to an overall normalization, this is again the same vortex character:

$$\left[\tilde{\chi}^{(1)}_{D3',D3_d,D1_v}(z)\right] = \frac{1}{\left[\chi^{(0)}_{D3',D1_v}\right]} \left[\chi(t_0) + \bar{q} \prod_{i=1}^N \frac{1-q t^{-1} f_i/z}{1-q f_i/z} \prod_{j=N+1}^{N_F} \frac{1-t f_j/z}{1-f_j/z} \left\langle \frac{1}{\Lambda_v(z v^{-2})} \right\rangle \right].$$

(7.59)

——— Seiberg Duality of the Vortex Character ———

Let us first go back to the index of the quantum mechanics $T_1^{1d}$ in the absence of the Wilson loop, which we rewrite here for convenience:

$$[\chi]^{(L)}_{1d} = \sum_{k=0}^{\infty} \frac{e^{k \zeta_1 d}}{k!} \int_{\mathcal{M}_k^{\text{pure}}} \left[\frac{d\phi_I}{2\pi i}\right] Z_{\text{pure,vec}} \cdot Z_{\text{pure,adj}} \cdot Z_{\text{pure,teeth}},$$

(7.60)

$$Z_{\text{pure,vec}} = \prod_{I,J=1}^k \frac{\text{sh} (\phi_I - \phi_J)}{\text{sh} (\phi_I - \phi_J + \epsilon_2)} \prod_{I,J=1}^k \frac{\text{sh} (\phi_I - \phi_J + \epsilon_1 + \epsilon_2)}{\text{sh} (\phi_I - \phi_J + \epsilon_1)} \prod_{j=1}^{N_F-N} \frac{\text{sh} (-\phi_I + m_j - \epsilon_+ + \epsilon_2)}{\text{sh} (-\phi_I + m_j - \epsilon_+)}.$$

Once again, we used the notations $\epsilon_+ = (\epsilon_1 + \epsilon_2)/2$ and $\epsilon_- = (\epsilon_1 - \epsilon_2)/2$. We now study the Witten index in a chamber with a negative 1d FI parameter, $\zeta_{1d} < 0$. We then apply the JK-residue prescription in that FI-chamber\textsuperscript{27}. For each vortex charge $k$, the poles that end up contributing make up the set $\mathcal{M}_k^{\text{pure}}$. The elements of this set satisfy:

$$\phi_I = \phi_J + \epsilon_1,$$

$$\phi_I = \phi_J + \epsilon_2,$$

$$\phi_I = m_j + \epsilon_+ , \quad j \in \{N+1, \ldots, N_F\}.$$

\textsuperscript{27} Just as before, the JK-residue requires us to define an auxiliary vector $\eta$ of size $k$, and we once again choose $\eta = \zeta_{1d}$ to remove contributions from $\phi$-poles at $\pm \infty$. We find the choice $\eta = (-1, \ldots, -1)$ convenient here.
The poles (7.61) arise from the adjoint chiral factor $Z_{\text{pure,adj}}$, the poles (7.62) arise from the vector multiplet factor $Z_{\text{pure,vec}}$, and the poles (7.63) arise from flavor factor $Z_{\text{pure,teeth}}$. The last set of contours now encloses poles originating from the antifundamental chiral multiplets, and none of the fundamental chiral multiplets. Furthermore, the residue at the locus (7.62) is zero. Putting it all together, the various poles which end up contributing with nonzero residue are of the form:

$$\phi_I = m_j + \epsilon_+ + (s_i - 1)\epsilon_1, \quad \text{with} \quad s_i \in \{1, \ldots, k_i\}, \quad i \in \{N + 1, \ldots, N_F\}. \quad (7.64)$$

In this notation, $(k_1, \ldots, k_N)$ is a partition of $k$ into $N_F - N$ non-negative integers, and the pair of integers $(i, s_i)$ is assigned to one of the integers $I \in \{1, \ldots, k\}$ exactly once. Performing the residue integral, we get the closed-form expression:

$$[\chi]^{(0)}_{1d, \zeta_{1d} < 0} = \sum_{k=0}^{\infty} (-e^{\zeta_{1d}})^k \sum_{k_i \geq 0} \left[ \prod_{i,j=N+1}^{N_F} \prod_{s=1}^{k_i} \frac{\sinh (m_i - m_j + \epsilon_2 - (s - k_j - 1)\epsilon_1 \sinh (m_i - m_j - (s - k_j - 1)\epsilon_1)}{\sinh (m_i - m_j + \epsilon_2 + p\epsilon_1)} \right] \prod_{i=N+1}^{N_F} \prod_{j=1}^{k_j} \prod_{p=1}^{N} \frac{\sinh (m_i - m_j + \epsilon_2 + p\epsilon_1)}{\sinh (m_i - m_j + p\epsilon_1)}.$$  \quad (7.65)

After flipping the signs of the $N_F$ masses $\{m_d\} \to \{-m_d - \epsilon_2\}$ (the shift by $-\epsilon_2$ is inconsequential at this stage, but will matter later) and the sign of the 3d FI parameter $\tilde{q} \to -\tilde{q}$, we recognize the index of a 3d $U(N_F - N)$ gauge theory with $N_F$ fundamental flavors. As predicted, changing the sign of the 1d FI parameter in the quantum mechanics realizes 3d Seiberg duality [39]. For comparison, we rewrite the index of the $U(N)$ gauge theory with $N_F$ fundamental flavors we previously derived in the chamber $\zeta_{1d} > 0$:

$$[\chi]^{(0)}_{1d, \zeta_{1d} > 0} = \sum_{k=0}^{\infty} e^{\zeta_{1d}k} \sum_{k_i = k} \left[ \prod_{i,j=1}^{N} \prod_{s=1}^{k_i} \frac{\sinh (m_i - m_j + \epsilon_2 - (s - k_j - 1)\epsilon_1 \sinh (m_i - m_j - (s - k_j - 1)\epsilon_1)}{\sinh (m_i - m_j + \epsilon_2 + p\epsilon_1)} \right] \prod_{i=N+1}^{N_F} \prod_{j=1}^{k_j} \prod_{p=1}^{N} \frac{\sinh (m_i - m_j + \epsilon_2 + p\epsilon_1)}{\sinh (m_i - m_j + p\epsilon_1)}.$$  \quad (7.66)

Having reviewed the pure case, let us now introduce the Wilson loop. Recall that the Witten index of the quantum mechanics $T^{1d}$ now reads:

$$[\chi]_{1d} = \sum_{k=0}^{\infty} \frac{e^{\zeta_{1d}k}}{k!} Z_{\text{defect,}\emptyset} \times \oint_{\mathcal{M}_k} \frac{d\phi_I}{2\pi i} Z_{\text{pure,vec}} \cdot Z_{\text{pure,adj}} \cdot Z_{\text{pure,teeth}} \cdot Z_{\text{defect,k}}, \quad (7.67)$$
\[ Z_{\text{defect,}\varnothing} = \prod_{i=1}^{N} \text{sh} \left( m_i - M + \epsilon_2 \right), \]
\[ Z_{\text{defect},k} = \prod_{l=1}^{k} \frac{\text{sh} \left( \phi_I - M - \epsilon_- \right) \text{sh} \left( -\phi_I + M - \epsilon_- \right)}{\text{sh} \left( \phi_I - M - \epsilon_+ \right) \text{sh} \left( -\phi_I + M - \epsilon_+ \right)}. \]

In the FI-chamber \( \zeta_{1d} < 0 \), we have a new pole at the locus \( \phi_I = M - \epsilon_+ \), for some \( I \in \{1, \ldots, k\} \).

(7.68)

No other pole depending on \( M \) exists, by the same arguments invoked in the case \( \zeta_{1d} > 0 \). For each vortex charge \( k \), the set of poles \( \mathcal{M}_k \) is therefore the set \( \mathcal{M}_k^{\text{pure}} \), augmented by the pole (7.68). The (renormalized) codimension-2 \( Y \)-operator is defined as before, avoiding this new pole:

\[ \langle [Y_{1d}(M)]^{\pm 1} \rangle = \sum_{k=0}^{\infty} \frac{e^{\zeta_{1d} k}}{k!} \int_{\mathcal{M}_k^{\text{pure}}} \left[ \frac{d\phi_I}{2\pi i} \right] Z_{\text{pure,vec}} \cdot Z_{\text{pure,adj}} \cdot Z_{\text{pure,teeth}} \cdot [Z_{\text{defect},k}(M)]^{\pm 1}. \]

We renormalize the index by the classical Wilson loop contribution and the index of the vortex quantum mechanics \( T_{1d}^{\text{pure}} \) in the absence of Wilson loop:

\[ [\bar{\chi}]_{1d, \zeta_{1d} < 0}(z) \equiv \frac{[\chi]_{1d}^{(1)}(z)}{Z_{\text{defect,}\varnothing}(z) \cdot [\chi]_{1d, \zeta_{1d} < 0}^{(0)}}, \]

(7.69)

and derive at once the vortex character in the FI-chamber \( \zeta_{1d} < 0 \), written in K-theoretic notation as:

\[ [\bar{\chi}]_{1d, \zeta_{1d} < 0}(z) = \frac{1}{[\chi]_{1d, \zeta_{1d} < 0}^{(0)}} \left[ \langle Y_{1d}(z) \rangle - \tilde{q} \prod_{i=1}^{N} \frac{1 - f_i/z}{1 - t f_i/z} \prod_{j=N+1}^{N_F} \frac{1 - t^2 q^{-1} f_j/z}{1 - t q^{-1} f_j/z} \left( \frac{1}{Y_{1d}(z v^2)} \right) \right]. \]

(7.70)

If we flip the sign of the \( N_F \) masses \( \{m_d\} \rightarrow \{-m_d - \epsilon_2\} \) (or \( \{f_d\} \rightarrow \{t^{-1} f_d^{-1}\} \) in the new variables), flip the defect fermion mass as \( M \rightarrow -M \) (or \( z \rightarrow z^{-1} \) in the new variables), and flip the 3d FI parameter as \( \tilde{q} \rightarrow -\tilde{q} \), we recognize the vortex character of a 3d \( U(N_F - N) \) gauge theory with \( N_F \) fundamental flavors. Note the nontrivial rescaling of the \( N_F \) flavor masses by \( t^{-1} \).

For comparison, we also rewrite the vortex character of the 3d \( U(N) \) gauge theory with \( N_F \) fundamental flavors (7.12):

\[ [\bar{\chi}]_{1d, \zeta_{1d} > 0}(z) = \frac{1}{[\chi]_{1d, \zeta_{1d} > 0}^{(0)}} \left[ \langle Y_{1d}(z) \rangle + \tilde{q} \prod_{i=1}^{N} \frac{1 - q t^{-1} f_i/z}{1 - q f_i/z} \prod_{j=N+1}^{N_F} \frac{1 - t f_j/z}{1 - f_j/z} \left( \frac{1}{Y_{1d}(z v^{-2})} \right) \right]. \]

(7.71)
As a last remark, note that the index $[\chi]_{1d, \zeta_{1d}>0}^{(0)}$ (7.66) in the positive FI chamber is not equal to the index $[\chi]_{1d, \zeta_{1d}<0}^{(0)}$ (7.65) in the negative FI chamber. This is because new states appear and contribute to the index at $\zeta_{1d} = 0$, due to the opening of the Coulomb branch there. The vortex mechanics $T_{pure}^{1d}$ experiences wall-crossing, and the BPS index of the extra states can be computed explicitly by identifying the residues at asymptotic infinity, enclosing the $\phi$-poles of the integrand (7.60) at $\pm \infty$. A quick computation shows that such residues can be summed up exactly to give the contribution of $2N - N_F$ decoupled twisted hypermultiplets, which do exist on the Coulomb branch of $G^{3d}$ [39, 47, 110, 111, 118]. Explicitly, the wall-crossing contribution can be written as a plethystic exponential:

$$\text{PE}\left[\frac{\text{sh}(2\epsilon_+) \text{sh}((2N - N_F)\epsilon_2)}{\text{sh}(\epsilon_1) \text{sh}(\epsilon_2)} q^\beta\right]$$

(7.72)

Note that such a contribution vanishes when $N_F = 2N$, in which case the indices agree: $[\chi]_{1d, \zeta_{1d}<0}^{(0)} = [\chi]_{1d, \zeta_{1d}>0}^{(0)}$.

We can carry out the same computation for the index $[\chi]_{1d}^{(1)}$ in the presence of the Wilson loop (7.67), to find that the extra contributions due to $\phi$-poles at $\pm \infty$ are the same as above: there are $2N - N_F$ extra decoupled twisted hypermultiplets, resulting in a decoupled factor (7.72). Because the vortex character observable is the index $[\chi]_{1d}^{(1)}$ normalized by the pure index $[\chi]_{1d}^{(0)}$, the twisted hypermultiplets contributions cancel out at any rate in our context.

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A Some Examples of Vortex Characters

We write some explicit expressions for the vortex $qq$-character observables of some 3d gauge theories, in the 3d/1d half-index formalism. It is straightforward to write the observables in the quantum mechanics or $q$-Toda variables instead, if one wishes. All characters should be normalized by the pure index $[\chi]_{3d}^{(0,...,0)}$, which we omitted here not to overburden the expressions.
Figure 11: The $T_p[SU(N^{(n+1)})]$ theory, with a Wilson loop defect producing the first fundamental vortex character (top), and a loop defect producing the $n$-th fundamental vortex character (bottom).

For the $T_p[SU(N^{(n+1)})]$ theory on top of Figure 11, we compute:

\[
[\tilde{\chi}]^{(1),0,...,0}_{3d}(z) = \prod_{d=1}^{N^{(n+1)}} \frac{1 - v^n x_d/z}{1 - t v^n x_d/z} \langle \tilde{Y}^{(1)}_{3d}(z) \rangle + \tilde{q}^{(1)} \prod_{d=1}^{N^{(n+1)}} \frac{1 - v^n x_d/z}{1 - t v^n x_d/z} \langle \tilde{Y}^{(2)}_{3d}(z v^{-1}) \rangle + \tilde{q}^{(1)} \tilde{q}^{(2)} \prod_{d=1}^{N^{(n+1)}} \frac{1 - v^n x_d/z}{1 - t v^n x_d/z} \langle \tilde{Y}^{(3)}_{3d}(z v^{-2}) \rangle + \ldots + \prod_{a=1}^{n} \tilde{q}^{(n)} \langle \tilde{Y}^{(n)}_{3d}(z v^{-n+1}) \rangle .
\]

For the $T_p[SU(N^{(n+1)})]$ theory on the bottom of Figure 11, we compute:

\[
[\tilde{\chi}]^{0,...,0,1}_{3d}(z) = \prod_{d=1}^{N^{(n+1)}} \frac{1 - v x_d/z}{1 - t v x_d/z} \langle \tilde{Y}^{(n)}_{3d}(z) \rangle + \tilde{q}^{(n)} \langle \tilde{Y}^{(n-1)}_{3d}(z v^{-1}) \rangle + \tilde{q}^{(n)} \tilde{q}^{(n-1)} \langle \tilde{Y}^{(n-2)}_{3d}(z v^{-2}) \rangle + \ldots + \prod_{a=1}^{n} \tilde{q}^{(n)} \langle \tilde{Y}^{(1)}_{3d}(z v^{-n+1}) \rangle .
\]
Figure 12: A $D_4$ theory with fundamental matter on node 3, with a Wilson loop defect producing the first fundamental vortex character.

For the $D_4$ theory in Figure 12, we compute:

$$\chi^{(1,0,0)}_{3d}(z) = \prod_{d=1}^{N_F^{(3)}} \frac{1 - v^3 x_d/z}{1 - t v^3 x_d/z} \left\langle \tilde{Y}_{3d}^{(1)}(z) \right\rangle$$

$$+ \bar{q}^{(1)} \prod_{d=1}^{N_F^{(3)}} \frac{1 - v^3 x_d/z}{1 - t v^3 x_d/z} \left\langle \frac{\tilde{Y}_{3d}^{(2)}(z v^{-1})}{\tilde{Y}_{3d}^{(1)}(z v^{-2})} \right\rangle$$

$$+ \bar{q}^{(1)} \bar{q}^{(2)} \prod_{d=1}^{N_F^{(3)}} \frac{1 - v^3 x_d/z}{1 - t v^3 x_d/z} \left\langle \frac{\tilde{Y}_{3d}^{(3)}(z v^{-2}) \tilde{Y}_{3d}^{(4)}(z v^{-2})}{\tilde{Y}_{3d}^{(2)}(z v^{-3})} \right\rangle$$

$$+ \bar{q}^{(1)} \bar{q}^{(2)} \bar{q}^{(3)} \left\langle \frac{\tilde{Y}_{3d}^{(4)}(z v^{-2})}{\tilde{Y}_{3d}^{(3)}(z v^{-4})} \right\rangle$$

$$+ \bar{q}^{(1)} \bar{q}^{(2)} \bar{q}^{(3)} \bar{q}^{(4)} \prod_{d=1}^{N_F^{(3)}} \frac{1 - v^3 x_d/z}{1 - t v^3 x_d/z} \left\langle \frac{\tilde{Y}_{3d}^{(3)}(z v^{-2})}{\tilde{Y}_{3d}^{(4)}(z v^{-4})} \right\rangle$$

$$+ \bar{q}^{(1)} \bar{q}^{(2)} \bar{q}^{(3)} \bar{q}^{(4)} \left\langle \frac{\tilde{Y}_{3d}^{(2)}(z v^{-3})}{\tilde{Y}_{3d}^{(3)}(z v^{-4}) \tilde{Y}_{3d}^{(4)}(z v^{-4})} \right\rangle$$

$$+ \bar{q}^{(1)} \left[ \bar{q}^{(2)} \bar{q}^{(3)} \bar{q}^{(4)} \right]^2 \left\langle \frac{\tilde{Y}_{3d}^{(1)}(z v^{-4})}{\tilde{Y}_{3d}^{(2)}(z v^{-5})} \right\rangle$$

$$+ \left[ \bar{q}^{(1)} \right]^2 \left[ \bar{q}^{(2)} \right]^2 \left[ \bar{q}^{(3)} \right]^2 \left\langle \frac{1}{\tilde{Y}_{3d}^{(1)}(z v^{-6})} \right\rangle. \quad (A.3)$$
References

[1] V. Pestun et al., Localization techniques in quantum field theories, *J. Phys. A* **50** (2017) 440301, [1608.02952].

[2] N. Nekrasov, BPS/CFT correspondence: non-perturbative Dyson-Schwinger equations and qq-characters, *JHEP* **03** (2016) 181, [1512.05388].

[3] N. Nekrasov, BPS/CFT correspondence II: Instantons at crossroads, moduli and compactness theorem, *Adv. Theor. Math. Phys.* **21** (2017) 503–583, [1608.07272].

[4] N. A. Nekrasov, Seiberg-Witten prepotential from instanton counting, *Adv. Theor. Math. Phys.* **7** (2003) 831–864, [hep-th/0206161].

[5] A. S. Losev, A. Marshakov and N. A. Nekrasov, Small instantons, little strings and free fermions, [hep-th/0302191].

[6] N. Nekrasov and E. Witten, The Omega Deformation, Branes, Integrability, and Liouville Theory, *JHEP* **09** (2010) 092, [1002.0888].

[7] N. Nekrasov and A. Okounkov, Seiberg-Witten theory and random partitions, *Prog. Math.* **244** (2006) 525–596, [hep-th/0306238].

[8] N. Nekrasov, BPS/CFT correspondence IV: sigma models and defects in gauge theory, *Lett. Math. Phys.* **109** (2019) 579–622, [1711.11011].

[9] S. Jeong and N. Nekrasov, Opers, surface defects, and Yang-Yang functional, [1806.08270].

[10] N. Haouzi and J. Oh, On the Quantization of Seiberg-Witten Geometry, [2004.00654].

[11] N. Haouzi, Quantum Geometry and θ-Angle in Five-Dimensional Super Yang-Mills, [2005.13565].

[12] D. Tong and K. Wong, Instantons, Wilson lines, and D-branes, *Phys. Rev.* **D91** (2015) 026007, [1410.8523].

[13] H.-C. Kim, Line defects and 5d instanton partition functions, *JHEP* **03** (2016) 199, [1601.06841].

[14] T. Kimura and V. Pestun, Quiver W-algebras, [1512.08533].

[15] T. Kimura and V. Pestun, Fractional quiver W-algebras, [1705.04410].

[16] A. Mironov, A. Morozov and Y. Zenkevich, DingIoharaMiki symmetry of network matrix models, *Phys. Lett.* **B762** (2016) 196–208, [1603.05467].

[17] B. Assel and A. Sciarappa, Wilson loops in 5d $\mathcal{N} = 1$ theories and S-duality, *JHEP* **10** (2018) 082, [1806.09636].

[18] N. Haouzi and C. Kozaz, Supersymmetric Wilson Loops, Instantons, and Deformed W-Algebras, [1907.03838].

[19] J.-E. Bourgine, M. Fukuda, K. Harada, Y. Matsuo and R.-D. Zhu, $(p, q)$-webs of DIM representations, 5d $\mathcal{N} = 1$ instanton partition functions and qq-characters, *JHEP* **11** (2017) 034, [1703.10759].

[20] J.-E. Bourgine and S. Jeong, New quantum toroidal algebras from 5D $\mathcal{N} = 1$ instantons on orbifolds, *JHEP* **05** (2020) 127, [1906.01625].
[21] C.-M. Chang, O. Ganor and J. Oh, An index for ray operators in 5d \( E_n \) SCFTs, *JHEP* 02 (2017) 018, [1608.06284].

[22] T. Kimura and V. Pestun, Quiver elliptic \( W \)-algebras, *Lett. Math. Phys.* 108 (2018) 1383–1405, [1608.04651].

[23] P. Agarwal, J. Kim, S. Kim and A. Sciarappa, Wilson surfaces in \( M5 \)-branes, *JHEP* 08 (2018) 119, [1804.09932].

[24] M. Jimbo, A \( q \)-analogue of \( U(g[(N+1)]) \), Hecke algebra, and the Yang-Baxter equation, *Letters in Mathematical Physics* 11 (1986) 247.

[25] V. G. Drinfeld, A New realization of Yangians and quantized affine algebras, *Sov. Math. Dokl.* 36 (1988) 212–216.

[26] V. Chari and A. Pressley, Quantum affine algebras and their representations, *hep-th/9411145*.

[27] V. Chari, Minimal affinizations of representations of quantum groups: The \( U_q(g) \) module structure, *hep-th/9411144*.

[28] E. Frenkel and N. Reshetikhin, The \( q \)-characters of representations of quantum affine algebras and deformations of \( \mathfrak{w} \)-algebras, in *Contemporary Math* 248 (2000) , [math/9810055].

[29] N. Nekrasov, V. Pestun and S. Shatashvili, Quantum geometry and quiver gauge theories, *Commun. Math. Phys.* 357 (2018) 519–567, [1312.6689].

[30] M. Bullimore, H.-C. Kim and P. Koroteev, Defects and Quantum Seiberg-Witten Geometry, *JHEP* 05 (2015) 095, [1412.6081].

[31] J. Shiraishi, H. Kubo, H. Awata and S. Odake, A Quantum deformation of the Virasoro algebra and the Macdonald symmetric functions, *Lett. Math. Phys.* 38 (1996) 33–51, [q-alg/9507034].

[32] H. Awata, H. Kubo, S. Odake and J. Shiraishi, Quantum \( W(N) \) algebras and Macdonald polynomials, *Commun. Math. Phys.* 179 (1996) 401–416, [q-alg/9508011].

[33] E. Frenkel and N. Reshetikhin, Deformations of \( \mathfrak{w} \)-algebras associated to simple Lie algebras, *Comm. Math. Phys.* 197 (1998) 1–32, [q-alg/9708006].

[34] H. Nakajima, \( t \)-analogue of the \( q \)-characters of finite dimensional representations of quantum affine algebras, *Physics and Combinatorics* (2000) 196–219, [math/0009231].

[35] C. Beem, T. Dimofte and S. Pasquetti, Holomorphic Blocks in Three Dimensions, *JHEP* 12 (2014) 177, [1211.1986].

[36] L. F. Alday, D. Gaiotto and Y. Tachikawa, Liouville Correlation Functions from Four-dimensional Gauge Theories, *Lett. Math. Phys.* 91 (2010) 167–197, [0906.3219].

[37] M. Aganagic and A. Okounkov, Quasimap counts and Bethe eigenfunctions, *Moscow Math. J.* 17 (2017) 565–600, [1704.08746].

[38] N. Seiberg, Electric - magnetic duality in supersymmetric nonAbelian gauge theories, *Nucl. Phys. B* 435 (1995) 129–146, [hep-th/9411149].

[39] C. Hwang, P. Yi and Y. Yoshida, Fundamental Vortices, Wall-Crossing, and Particle-Vortex Duality, *JHEP* 05 (2017) 099, [1703.00213].
[40] A. Hanany and D. Tong, *Vortices, instantons and branes*, *JHEP* **07** (2003) 037, [hep-th/0306150].

[41] M. Shifman and A. Yung, *Non-Abelian semilocal strings in N=2 supersymmetric QCD*, *Phys. Rev. D* **73** (2006) 125012, [hep-th/0603134].

[42] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Manifestly supersymmetric effective Lagrangians on BPS solitons*, *Phys. Rev. D* **73** (2006) 125008, [hep-th/0602289].

[43] M. Eto, J. Evslin, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi et al., *On the moduli space of semilocal strings and lumps*, *Phys. Rev. D* **76** (2007) 105002, [0704.2218].

[44] M. Shifman, W. Vinci and A. Yung, *Effective World-Sheet Theory for Non-Abelian Semilocal Strings in N = 2 Supersymmetric QCD*, *Phys. Rev. D* **83** (2011) 125017, [1104.2077].

[45] M. Shifman, W. Vinci and A. Yung, *Quantum Dynamics of Low-Energy Theory on Semilocal Non-Abelian Strings*, *Phys. Rev. D* **84** (2011) 065018, [1107.3779].

[46] H. Nakajima, *Handsaw quiver varieties and finite W-algebras*, *Moscow Math. J.* **12** (2012) 633, [1107.5073].

[47] D. Gaiotto and E. Witten, *S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory*, *Adv. Theor. Math. Phys.* **13** (2009) 721–896, [0807.3720].

[48] M. Aganagic, N. Haouzi and S. Shakirov, *Aν-Triality*, 1403.3657.

[49] J. Gomis and F. Passerini, *Holographic Wilson Loops*, *JHEP* **08** (2006) 074, [hep-th/0604007].

[50] B. Assel and J. Gomis, *Mirror Symmetry And Loop Operators*, *JHEP* **11** (2015) 055, [1506.01718].

[51] M. Atiyah, N. J. Hitchin, V. Drinfeld and Y. Manin, *Construction of Instantons*, *Phys. Lett. A* **65** (1978) 185–187.

[52] K. Hori, H. Kim and P. Yi, *Witten Index and Wall Crossing*, *JHEP* **01** (2015) 124, [1407.2567].

[53] C. Cordova and S.-H. Shao, *An Index Formula for Supersymmetric Quantum Mechanics*, 1406.7853.

[54] C. Hwang, J. Kim, S. Kim and J. Park, *General instanton counting and 5d SCFT*, *JHEP* **07** (2015) 063, [1406.6793].

[55] E. Witten, *Constraints on Supersymmetry Breaking*, *Nucl. Phys. B* **202** (1982) 253.

[56] L. Alvarez-Gaume, *SUPERSYMMETRY AND INDEX THEORY*, in *1984 NATO ASI on Supersymmetry*, pp. 1–44, 9, 1986.

[57] M. Aganagic and N. Haouzi, *ADE Little String Theory on a Riemann Surface (and Triality)*, 1506.04183.

[58] E. Witten, *Localization for nonabelian group actions*, 9307001.

[59] L. Jeffreys and F. Kirwan, *Elliptic Genera of 2d \N = 2 Gauge Theories*, *Commun. Math. Phys.* **333** (2015) 1241–1286, [1308.4896].
[61] A. Szenes and M. Vergne, *Toric reduction and a conjecture of Batyrev and Materov*, *Inventiones Mathematicae* **158** (June, 2004) 453–495, [math/0306311].

[62] S. Pasquetti, *Factorisation of N = 2 Theories on the Squashed 3-Sphere*, *JHEP* **04** (2012) 120, [1111.6905].

[63] N. A. Nekrasov, *Instanton partition functions and M-theory*, in *15th International Seminar on High Energy Physics*, 2008.

[64] C. Krattenthaler, V. Spiridonov and G. Vartanov, *Superconformal indices of three-dimensional theories related by mirror symmetry*, *JHEP* **06** (2011) 008, [1103.4075].

[65] T. Dimofte, D. Gaiotto and S. Gukov, *Gauge Theories Labelled by Three-Manifolds*, *Commun. Math. Phys.* **325** (2014) 367–419, [1108.4389].

[66] M. Taki, *Holomorphic Blocks for 3d Non-abelian Partition Functions*, 1303.5915.

[67] S. Cecotti, D. Gaiotto and C. Vafa, *tt* geometry in 3 and 4 dimensions, *JHEP* **05** (2014) 055, [1312.1008].

[68] M. Fujitsu, M. Honda and Y. Yoshida, *Higgs branch localization of 3d \( \mathcal{N} = 2 \) theories*, *PTEP* **2014** (2014) 123B02, [1312.3627].

[69] F. Benini and W. Peelaers, *Higgs branch localization in three dimensions*, *JHEP* **05** (2014) 030, [1312.6078].

[70] C. Hwang, H.-C. Kim and J. Park, *Factorization of the 3d superconformal index*, *JHEP* **08** (2014) 018, [1211.6023].

[71] Y. Yoshida and K. Sugiyama, *Localization of 3d \( \mathcal{N} = 2 \) Supersymmetric Theories on \( S^1 \times D^2 \)*, 1409.6713.

[72] F. Benini and A. Zaffaroni, *A topologically twisted index for three-dimensional supersymmetric theories*, *JHEP* **07** (2015) 127, [1504.03698].

[73] N. Doroud, J. Gomis, B. Le Floch and S. Lee, *Exact Results in D=2 Supersymmetric Gauge Theories*, *JHEP* **05** (2013) 093, [1206.2606].

[74] F. Benini and S. Cremonesi, *Partition Functions of \( \mathcal{N} = (2,2) \) Gauge Theories on \( S^2 \) and Vortices*, *Commun. Math. Phys.* **334** (2015) 1483–1527, [1206.2356].

[75] M. Aganagic, E. Frenkel and A. Okounkov, *Quantum q-Langlands Correspondence*, 1701.03146.

[76] A. Gadde, S. Gukov and P. Putrov, *Walls, Lines, and Spectral Dualities in 3d Gauge Theories*, *JHEP* **05** (2014) 047, [1302.0015].

[77] T. Dimofte, D. Gaiotto and N. M. Paquette, *Dual boundary conditions in 3d SCFTs*, *JHEP* **05** (2018) 060, [1712.07654].

[78] B. Feigin and E. Frenkel, *Quantum W algebras and elliptic algebras*, *Commun. Math. Phys.* **178** (1996) 653–678, [q-alg/9508009].

[79] F. Nieri and Y. Zenkevich, *Quiver \( W_{\epsilon_1,\epsilon_2} \) algebras of 4d \( \mathcal{N} = 2 \) gauge theories*, *J. Phys. A* **53** (2020) 275401, [1912.09969].

[80] V. S. Dotsenko and V. A. Fateev, *Conformal Algebra and Multipoint Correlation Functions in Two-Dimensional Statistical Models*, *Nucl. Phys.* **B240** (1984) 312.
[81] R. Dijkgraaf and C. Vafa, *Toda Theories, Matrix Models, Topological Strings, and N=2 Gauge Systems*, 0909.2453.
[82] H. Itoyama, K. Maruyoshi and T. Oota, *The Quiver Matrix Model and 2d-4d Conformal Connection*, Prog. Theor. Phys. 123 (2010) 957–987, [0911.4244].
[83] A. Mironov, A. Morozov and S. Shakirov, *Conformal blocks as Dotsenko-Fateev Integral Discriminants*, Int. J. Mod. Phys. A25 (2010) 3173–3207, [1001.0563].
[84] A. Morozov and S. Shakirov, *The matrix model version of AGT conjecture and CIV-DV prepotential*, JHEP 08 (2010) 066, [1004.2917].
[85] K. Maruyoshi, β-Deformed Matrix Models and 2d/4d Correspondence, 1412.7124.
[86] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, Phys. Rept. 223 (1993) 183–276, [hep-th/9210010].
[87] J. Ding and K. Iohara, *Generalization and Deformation of Drinfeld quantum affine algebras*, q-alg/9608002.
[88] K. Miki, *A (q,γ) analog of the w_{1+∞} algebra*, J. Math. Phys. 48 (2007).
[89] N. Haouzi and C. Schmid, *Little String Origin of Surface Defects*, JHEP 05 (2017) 082, [1608.07279].
[90] N. Haouzi and C. Schmid, *Little String Defects and Bala-Carter Theory*, 1612.02008.
[91] N. Haouzi and C. Kozaz, *The abcdefg of little strings*, 1711.11065.
[92] M. Berkooz, M. Rozali and N. Seiberg, *Matrix description of M theory on T**4 and T**5*, Phys. Lett. B 408 (1997) 105–110, [hep-th/9704089].
[93] A. Losev, G. W. Moore and S. L. Shatashvili, *M & m’s*, Nucl. Phys. B522 (1998) 105–124, [hep-th/9707250].
[94] M. Berkooz, M. Rozali and N. Seiberg, *Matrix description of M theory on T**4 and T**5*, Phys. Lett. B 408 (1997) 105–110, [hep-th/9704089].
[95] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity*, Eur. Phys. J. C22 (2001) 379–394, [hep-th/9803001].
[96] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, JHEP 02 (2005) 010, [hep-th/0501109].
[103] S. Yamaguchi, Wilson loops of anti-symmetric representation and D5-branes, JHEP 05 (2006) 037, [hep-th/0603208].

[104] T. Banks, N. Seiberg and E. Silverstein, Zero and one-dimensional probes with N=8 supersymmetry, Phys. Lett. B401 (1997) 30–37, [hep-th/9703052].

[105] N. Nekrasov and N. S. Prabhakar, Spiked Instantons from Intersecting D-branes, Nucl. Phys. B914 (2017) 257–300, [1611.03478].

[106] A. Kapustin, B. Willett and I. Yaakov, Tests of Seiberg-like dualities in three dimensions, JHEP 08 (2020) 114, [1012.4021].

[107] B. Assel and S. Cremonesi, The Infrared Physics of Bad Theories, SciPost Phys. 3 (2017) 024, [1707.03403].

[108] A. Dey and P. Koroteev, Good IR Duals of Bad Quiver Theories, JHEP 05 (2018) 114, [1712.06068].

[109] D. Bashkirov, Relations between supersymmetric structures in UV and IR for N = 4 bad theories, JHEP 07 (2013) 121, [1304.3952].

[110] D. Gaiotto and P. Koroteev, On Three Dimensional Quiver Gauge Theories and Integrability, JHEP 05 (2013) 126, [1304.0779].

[111] I. Yaakov, Redeeming Bad Theories, JHEP 11 (2013) 189, [1303.2769].

[112] M. Bullimore, T. Dimofte and D. Gaiotto, The Coulomb Branch of 3d N = 4 Theories, Commun. Math. Phys. 354 (2017) 671–751, [1503.04817].

[113] H. Nakajima, Towards a mathematical definition of Coulomb branches of 3-dimensional N = 4 gauge theories, I, Adv. Theor. Math. Phys. 20 (2016) 595–669, [1503.03676].

[114] A. Braverman, M. Finkelberg and H. Nakajima, Towards a mathematical definition of Coulomb branches of 3-dimensional N = 4 gauge theories, II, Adv. Theor. Math. Phys. 22 (2018) 1071–1147, [1601.03586].

[115] O. Aharony, IR duality in d = 3 N=2 supersymmetric USp(2N(c)) and U(N(c)) gauge theories, Phys. Lett. B 404 (1997) 71–76, [hep-th/9703215].

[116] F. Benini, C. Closet and S. Cremonesi, Comments on 3d Seiberg-like dualities, JHEP 10 (2011) 075, [1108.5373].

[117] T. Dimofte, N. Garner, M. Geracie and J. Hilburn, Mirror symmetry and line operators, JHEP 02 (2020) 075, [1908.00013].

[118] H.-C. Kim, J. Kim, S. Kim and K. Lee, Vortices and 3 dimensional dualities, 1204.3895.