Bohmian mechanics in relativistic quantum mechanics, quantum field theory and string theory

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Abstract. I present a short overview of my recent achievements on the Bohmian interpretation of relativistic quantum mechanics, quantum field theory and string theory. This includes the relativistic-covariant Bohmian equations for particle trajectories, the problem of particle creation and destruction, the Bohmian interpretation of fermionic fields and the intrinsically Bohmian quantization of fields and strings based on the De Donder-Weyl covariant canonical formalism.

1. Introduction – well-established results on the Bohmian interpretation
Let me start with a brief overview of well-established results on the Bohmian interpretation of quantum mechanics and quantum field theory [1, 2, 3, 4, 5]. Consider the Schrödinger equation

\[
\frac{-\hbar^2 \nabla^2}{2m} + V \psi = i\hbar \frac{\partial}{\partial t} \psi.
\]

By writing the wave function in the polar form

\[
\psi(x, t) = R(x, t)e^{iS(x, t)/\hbar},
\]

the complex Schrödinger equation splits up into two real equations, the quantum Hamilton-Jacobi equation

\[
\frac{(\nabla S)^2}{2m} + V + Q = -\frac{\partial}{\partial t} S,
\]

and the conservation equation

\[
\frac{\partial}{\partial t} R^2 + \nabla \left( R^2 \frac{\nabla S}{m} \right) = 0,
\]

where the quantum potential \( Q \) is defined as

\[
Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}.
\]

The conservation equation implies that \( |\psi|^2 \) can be interpreted as a probability density.

The Bohmian interpretation consists in the assumption that the particle has a deterministic trajectory \( x(t) \) satisfying

\[
\frac{dx}{dt} = \frac{\nabla S}{m}.
\]
This equation has the same form as an analogous equation in the classical Hamilton-Jacobi theory (see, however, [6]). When (6) is combined with the quantum Hamilton-Jacobi equation, one obtains the quantum Newton equation

\[ m \frac{d^2 x}{dt^2} = -\nabla (V + Q), \quad (7) \]

which has the same form as the classical Newton equation, except for an additional quantum force generated by the quantum potential \( Q \). Eq. (6) together with the conservation equation (4) provides that the statistical distribution of particle positions always coincides with the quantum distribution \( |\psi|^2 \), provided that this distribution coincides with the quantum distribution at some arbitrary initial time.

To see how the Bohmian interpretation recovers all measurable statistical predictions of standard quantum theory, consider a wave function

\[ \psi(x) = \sum_a c_a \psi_a(x). \quad (8) \]

Interaction with the measuring apparatus induces entanglement resulting in a total wave function of the form

\[ \Psi(x, y) = \sum_a c_a \psi_a(x) \phi_a(y), \quad (9) \]

where \( y \) is the degree of freedom of the measuring apparatus. It is assumed that the wave functions \( \phi_a(y) \) do not overlap, which implies that the \( y \)-particle can only be in one channel \( \phi_a \). Consequently, \( x(t) \) behaves as if the total wave function were \( \psi_a \phi_a \), which explains the effective collapse of the wave function. Thus, in the Bohmian interpretation, there is no need for introducing a true collapse.

The above can be easily generalized to the many-particle case. For a many-particle wave function \( \psi(x_1, \ldots, x_n, t) \) one obtains the corresponding many-particle quantum potential \( Q(x_1, \ldots, x_n, t) \). In general, this quantum potential is nonlocal, which induces a nonlocal instantaneous interaction between particles. Thus, the Bohmian interpretation is a nonlocal hidden-variable interpretation consistent with the Bell theorem.

There are some attempts to show that, in some cases, the predictions of the Bohmian interpretation of nonrelativistic quantum mechanics are not compatible with standard quantum theory. However, such attempts seem to be erroneous (see e.g. [7, 8] and references therein).

All the results above can also be generalized to bosonic fields. Using units \( \hbar = 1 \), quantum field theory for a bosonic field \( \phi \) can be described by a functional Schrödinger equation

\[ H \left[ -i \frac{\delta}{\delta \phi}, \phi \right] \Psi[\phi; t] = i \partial_t \Psi[\phi; t]. \quad (10) \]

In analogy with (6), the Bohmian interpretation assumes that the field has a deterministic dependence on time determined by the equation

\[ \dot{\phi} = \frac{\delta S}{\delta \phi}, \quad (11) \]

which coincides with the analogous classical equation in the Hamilton-Jacobi formulation of classical field theory.

The usual formulation of the Bohmian interpretation has several open questions, such as the following: How to make nonlocality consistent with relativity? How to introduce the Bohmian interpretation for fermionic fields? Should the Bohmian interpretation be applied to particles or fields? Is there an observable consequence of the Bohmian interpretation? Can the Bohmian interpretation be derived (not postulated) from more fundamental principles? What about strings? In the rest of this paper I present an overview of my contributions to the attempts of providing answers to all these questions.
2. Conservation equation and fermionic fields

In general, a wave function $\psi(\vec{\phi}, t)$ (where $\vec{\phi}$ is a many-component continuous degree of freedom) satisfies a Schrödinger equation of the form $\hat{H}\psi = i\partial_t \psi$, where $\rho = \psi^* \psi$ is the probability density. The average velocity can be calculated as

$$\frac{d\langle \vec{\phi} \rangle}{dt} = \int d^n\varphi \rho \vec{u}, \quad (12)$$

where

$$\vec{u} = \text{Re} \frac{i \psi^* [\hat{H}, \vec{\phi}] \psi}{\psi^* \psi}. \quad (13)$$

In general,

$$\partial_t \rho + \vec{\nabla} (\rho \vec{u}) = J \neq 0, \quad (14)$$

where $J$ is some function that, in general, does not need to vanish. On the other hand, for the consistency of the Bohmian interpretation written in the form

$$\frac{d\vec{\phi}}{dt} = \vec{v}, \quad (15)$$

we need a conservation equation of the form

$$\partial_t \rho + \vec{\nabla} (\rho \vec{v}) = 0. \quad (16)$$

To find $\vec{v}$, we use the ansatz

$$\vec{v} = \vec{u} + \rho^{-1} \vec{E}, \quad (17)$$

where $\vec{E}$ is some new function. From the equations above one finds that $\vec{E}$ must satisfy

$$\vec{\nabla} \vec{E} = -J. \quad (18)$$

Thus, for any $J$, one can find a consistent solution $\vec{E}$ [9].

Now let us apply it to fermionic quantum field theory [9]. Any fermionic state can be written as

$$\Psi^F = \sum_n c_n \Psi^F_n, \quad (19)$$

where $\Psi^F_n$ is an $n$-particle state. For each fermionic state $\Psi^F_n$ there is a corresponding bosonic state $\Psi^B_n$ with the same number of particles having the same momenta. This allows us to introduce a map

$$\Psi^F_n \rightarrow \Psi^B_n [\phi]. \quad (20)$$

(The reverse is not true, because there are more bosonic states than fermionic ones.) This provides a bosonic representation of fermionic states. Consequently, the Bohmian interpretation is possible in a similar way as that for bosonic fields, with the conservation equation provided as above.

3. Relativistic quantum mechanics

To make the Bohmian interpretation of particles compatible with the Bohmian interpretation of fields, I propose that both fields and particle positions are fundamental entities [10, 9]. Consider the field operator satisfying the Klein-Gordon equation $(\partial^\mu \partial_\mu + m^2) \hat{\phi}(x) = 0$, where $\hat{\phi}$ is a hermitian (uncharged) field. The $n$-particle wave function is

$$\psi(x_1, \ldots, x_n) = (n!)^{-1/2} S_{\{x_n\}} \langle 0 | \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) | n \rangle, \quad (21)$$
where \( S_{\{x_a\}} \) denotes the symmetric ordering. From this \( n \)-particle wave function one can construct \( n \) conserved particle currents (one for each \( a \)):

\[
j_\mu^a = i \psi^* \frac{\partial}{\partial x_\mu^a} \psi. \tag{22}\]

(For more details on particle currents in quantum field theory, see also [11, 12, 13].) The \( n \)-particle wave function satisfies the \( n \)-particle Klein-Gordon equation, which leads to the relativistic quantum potential

\[
Q = \frac{1}{2m} \sum_a \frac{\partial^\mu \partial_a R}{R}. \tag{23}\]

Here \( Q(x_1, \ldots, x_n) \) is nonlocal, but relativistic invariant! The Bohmian interpretation consists in the assumption that particle trajectories satisfy

\[
\frac{dx_\mu^a}{ds} = -\frac{1}{m} \frac{\partial^\mu}{\partial x_\mu^a} S = \frac{j_\mu^a}{2m\psi^* \psi}. \tag{24}\]

The trajectories in spacetime do not depend on the choice of the auxiliary parameter \( s \), which can be seen by writing (24) as

\[
\frac{dx_\mu^a}{dx_\mu^b} = \frac{j_\mu^a}{j_\mu^b}, \tag{25}\]

which eliminates \( s \). For a more detailed discussion of the covariance of these Bohmian equations of motion, see [14]. Note also that these trajectories may correspond to superluminal motions and motions backwards in time, but that this does not lead to causal paradoxes (see also [15]). To find a measurable consequence of this relativistic Bohmian equation of motion, assume that the interaction with the apparatus that measures particle positions starts at some particular time \( t_1 \). One finds that [16]

\[
\rho(x, t_1) = \begin{cases} 
    j_0(x, t_1) & \text{on } \Sigma', \\
    0 & \text{on } \Sigma^+ \cup \Sigma^-,
\end{cases} \tag{26}\]

where \( \Sigma^- \) is the part of the \( t_1 \)-hypersurface at which \( j_0 < 0 \), \( \Sigma^+ \) is the part of the \( t_1 \)-hypersurface which is connected with \( \Sigma^- \) by Bohmian trajectories for which \( t < t_1 \), and \( \Sigma' \) represents all other points of the \( t_1 \)-hypersurface. We emphasize that this measurable result cannot be obtained without calculating the trajectories. We also emphasize that there is no “standard” prediction for the case \( j_0 < 0 \) [17, 18], while the Bohmian interpretation provides a prediction as above (see also [19]).

4. Particle creation and destruction

What happens with the Bohmian trajectories when particles are created or destructed? A possible answer is that nothing dramatic happens, in the sense that there are no sudden creation or destruction of particle trajectories. Instead, the trajectories exist all the time, but their effectivities (to be defined below) change continuously with time [10, 9]. Consider a QFT state

\[
\Psi = \sum_n \tilde{\Psi}_n, \tag{27}\]

where the tilde on \( \tilde{\Psi}_n \) denotes that the norm of this \( n \)-particle state can be smaller than 1. From (21), we see that an \( n \)-particle wave function is essentially

\[
\psi_n \propto \langle 0| \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) |\Psi\rangle. \tag{28}\]
The important fact is that the Bohmian trajectories do not depend on the norm of $\psi_n$. The effectivity is defined as

$$e_n[\phi; t] = \frac{|\tilde{\Psi}_n[\phi; t]|^2}{\sum_{n'} |\tilde{\Psi}_{n'}[\phi; t]|^2}, \quad (29)$$

which, for a definite Bohmian value of $\phi$, measures the relative contribution of the $n$-particle sector. When $n$ is measured, then the Bohmian evolution of $\phi$ explains the effective collapse $\Psi \rightarrow \Psi_n$. In this case, $e_n = 1$ for this $n$ and $e_{n'} = 0$ for all other $n'$, i.e. only the particles of the $n$-particle sector are effective.

5. Bohmian mechanics from covariant quantization of fields and strings

The Bohmian equation of motion (11) is not relativistic covariant. This is related to the fact that the functional Schrödinger equation (10) is also not covariant. To solve this problem, one possibility is to base the Bohmian interpretation of fields on the many-fingered time Tomonaga-Schwinger equation [20, 21], but the existing results do not seem to be completely satisfying [22], except for theories that satisfy a Hamiltonian constraint [21], such as theories that contain quantum gravity.

Another strategy is to modify the quantization itself to make it manifestly relativistic covariant in a different way [23, 24]. Instead of the noncovariant Hamilton formalism based on the Hamiltonian $\mathcal{H} = \pi \partial_0 \phi - \mathcal{L}$ with the canonical momentum $\pi = \partial \mathcal{L} / \partial (\partial_0 \phi)$, I use De Donder-Weyl covariant Hamilton formalism based on the covariant Hamiltonian

$$\mathcal{H}^{DW} = \pi^\mu \partial_\mu \phi - \mathcal{L}, \quad (30)$$

where the covariant canonical momentum is a vector

$$\pi^\mu = \partial \mathcal{L} / \partial (\partial_\mu \phi). \quad (31)$$

This leads to the covariant Hamilton-Jacobi equation

$$\mathcal{H}^{DW} \left( \frac{\partial S^\alpha}{\partial \phi}, \phi \right) + \partial_\mu S^\mu = 0, \quad (32)$$

and the covariant equation of motion

$$\partial^\mu \phi = \frac{\partial S^\mu}{\partial \phi}. \quad (33)$$

To derive the noncovariant Hamilton-Jacobi equation with $\mathcal{H}^{DW} \rightarrow \mathcal{H}$, it is necessary [23] to use the spatial part $\partial^i \phi = \partial S^i / \partial \phi$ of (33). The covariance then implies that the time part $\partial^0 \phi = \partial S^0 / \partial \phi$ must also be valid. In other words, the determinism of classical mechanics is derived from the requirement of covariance!

To quantize the theory, one modifies the covariant Hamilton-Jacobi equation by adding the quantum potential to the classical Hamiltonian:

$$\mathcal{H}^{DW}_Q = \mathcal{H}^{DW} + Q. \quad (34)$$

Here $Q$ is the same as that for the functional Schrödinger equation, but is written in a covariant form [23]. In the same way as for classical fields, to derive the Schrödinger equation, the spatial part of (33) must be valid. Covariance then implies the time part of (33). This is a derivation of the Bohmian equation of motion from the requirement of covariance!
The idea above can also be applied to strings. This is because strings can be viewed as fields in 2 dimensions. Thus, in the same way, the world-sheet covariance implies the Bohmian equation of motion for strings [25]. In the Bohmian interpretation, the string always has a well-defined shape $X^\mu(\sigma, \tau)$, even when it is not measured. On the other hand, observable properties of strings, such as the mass-spectrum, obey T-duality (see e.g. [26]), which is a symmetry under the change of the compactification radius $R$ with the dual radius $\alpha'/R$ (where $\sqrt{\alpha'}$ is the fundamental length scale of string theory). It is widely believed that T-duality is a fundamental symmetry of string theory that makes the theory nonlocal at distances smaller than $\sqrt{\alpha'}$. However, the Bohmian interpretation breaks T-duality at the fundamental level of hidden variables [27], which makes the role of T-duality less fundamental.

6. Conclusion
Bohmian mechanics is more ambitious (and thus more complicated) than standard quantum theory. The goals of Bohmian mechanics are (i) to recover the predictions of standard quantum theory and (ii) to explain (like classical mechanics and unlike standard quantum theory) what is going on when measurements are not performed. The existing results indicate that it is possible to achieve this for all quantum phenomena.

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