Bappaditya Bhowmik and Nilanjan Das

Bohr radius and its asymptotic value for holomorphic functions in higher dimensions

Volume 359, issue 7 (2021), p. 911-918

<https://doi.org/10.5802/crmath.237>
Bohr radius and its asymptotic value for holomorphic functions in higher dimensions

Bappaditya Bhowmik* and Nilanjan Das

* Corresponding author.

1. Introduction and the main results

Let $X$ be a complex Banach space and $G \subset X$, $\Omega \subset \mathbb{C}$ be two domains. For any holomorphic mapping $f : G \to \Omega$, let $D^k f(x)$ denote the $k^{th}$ Fréchet derivative ($k \in \mathbb{N}$) of $f$ at $x \in G$, which is a bounded symmetric $k$-linear mapping from $\prod_{i=1}^{k} X$ to $\mathbb{C}$. Any such $f$ can be expanded into the series

\[ f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(x_0) \left( (x-x_0)^k \right) \]

in a neighborhood of any given $x_0 \in G$. It is understood that $D^0 f(x_0)(x^0) = f(x_0)$ and

\[ D^k f(x_0) \left( x^k \right) = D^k f(x_0)(x,x,\ldots,x) \]
k-times
for \(k \geq 1\). The reader is referred to [14] for a more general and detailed discussion in this area. Now, let us denote by \(K^n_X(\Omega)\) the supremum of all \(r \in [0, 1]\) such that the inequality

\[
\sum_{k=1}^{\infty} \frac{1}{k!} D^k f(0) \left( x^k \right) \leq d \left( f(0), \partial \Omega \right)
\]

holds for all \(x \in r \mathbb{G}\) and for all holomorphic mappings \(f\) from a bounded balanced domain \(G \subset X\) to \(\Omega \subset \mathbb{C}\) with an expansion (1) in a neighborhood of \(x_0 = 0\). We clarify that \(G\) is balanced if \(u \mathbb{G} \subset G\) for all \(u \in \overline{D}\), and \(d(f(0), \partial \Omega)\) is the Euclidean distance between \(f(0)\) and the boundary \(\partial \Omega\) of the domain \(\Omega\). The sharp version of the famous theorem of Harald Bohr [8] states that \(K^n_G(\mathbb{D}) = 1/3\). After this theorem found an application to the characterization problem of Banach algebras satisfying the von Neumann inequality (cf. [12]), problems of similar type started being studied extensively in different settings (see for example [1–7, 10, 11, 15–21] and the references therein), and gained popularity by the name Bohr phenomenon. It is worth mentioning here that Bohr inequalities of type (2) have been considered in [2, 15, 18]. Of particular interest to us is [1, Theorem 8], which shows that for any balanced domain \(\mathbb{G} \subset X\) to \(\Omega \subset \mathbb{C}\) with an expansion (1) in a neighborhood of \(x_0 = 0\). We clarify that \(G\) is balanced if \(u \mathbb{G} \subset G\) for all \(u \in \overline{D}\), and \(d(f(0), \partial \Omega)\) is the Euclidean distance between \(f(0)\) and the boundary \(\partial \Omega\) of the domain \(\Omega\). The sharp version of the famous theorem of Harald Bohr [8] states that \(K^n_G(\mathbb{D}) = 1/3\). After this theorem found an application to the characterization problem of Banach algebras satisfying the von Neumann inequality (cf. [12]), problems of similar type started being studied extensively in different settings (see for example [1–7, 10, 11, 15–21] and the references therein), and gained popularity by the name Bohr phenomenon. It is worth mentioning here that Bohr inequalities of type (2) have been considered in [2, 15, 18]. Of particular interest to us is [1, Theorem 8], which shows that for any balanced domain \(G\) centered at 0 in \(\mathbb{C}^n\), \(K^n_G(\mathbb{D}) \geq 1/3\), and assuming \(G\) convex it was shown that \(K^n_G(\mathbb{D}) = 1/3\). As a consequence of a more general theorem, it was further proved in [15, Corollary 3.2] that \(K^n_G(\mathbb{D}) = 1/3\) for any bounded balanced domain \(G \subset X\), \(X\) being a complex Banach space. In the following theorem, we replace \(\mathbb{D}\) with more general domains \(\Omega\) and establish sharp Bohr phenomena. To this end, we define the following two quantities for any given complex Banach space \(X\) and a bounded balanced domain \(G \subset X\):

\[
\tilde{K}^G_X = \inf \{K^n_X(\Omega) : \Omega \subset \mathbb{C} \text{ is simply connected} \},
\]

and

\[
\tilde{K}^G_X = \inf \{K^n_X(\Omega) : \Omega \subset \mathbb{C} \text{ is convex} \}.
\]

**Theorem 1.** \(\tilde{K}^G_X = 3 - 2\sqrt{2}\) and \(\tilde{K}^G_X = 1/3\).

For \(X = \mathbb{C}\) and \(G = \mathbb{D}\), the above theorem gives [18, Theorem 1 and Remark 1] back. Also, in some sense, the second part of our Theorem 1 generalizes [15, Corollary 3.2].

Before we proceed further, we need to introduce some concepts. Let \(D^n = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n : \|z\|_\infty := \max_{1 \leq k \leq n} |z_k| < 1\}\) be the open unit polydisk in the \(n\)-dimensional complex plane \(\mathbb{C}^n\). Any holomorphic \(f : D^n \rightarrow \mathbb{C}\) can be expanded in the power series

\[
f(z) = c_0 + \sum_{|\alpha| \in \mathbb{N}} c_\alpha z^\alpha, z \in \mathbb{D}^n.
\]

Here and hereafter, we use the standard multi-index notation: \(\alpha\) means an \(n\)-tuple \((\alpha_1, \alpha_2, \cdots, \alpha_n)\) of nonnegative integers, \(|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n\), \(z\) denotes an \(n\)-tuple \((z_1, z_2, \cdots, z_n)\) of complex numbers, and \(z^\alpha\) is the product \(z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}\). It is evident that in our previous discussion, if the Banach space \(X\) is chosen to be \(\mathbb{C}^n\) and \(G = \mathbb{D}^n\), then for any \(k \in \mathbb{N}\) and for any \(f\) as in (3) we have

\[
\frac{1}{k!} D^k f(0) \left( z^k \right) = \sum_{|\alpha| = k} c_\alpha z^\alpha, z \in \mathbb{D}^n.
\]

Hence, we are motivated to consider a “stronger” Bohr phenomenon in this case. To be more specific, we denote by \(K^n_r(\Omega)\) the supremum of all \(r \in [0, 1]\) such that

\[
\sum_{k=1}^{\infty} \left( \sum_{|\alpha| = k} |c_\alpha z^\alpha| \right) \leq d \left( f(0), \partial \Omega \right)
\]

for all \(z \in \mathbb{D}^n\) satisfying \(\|z\|_\infty \leq r\) and for all holomorphic \(f : D^n \rightarrow \Omega\) with an expansion (3). Lower and upper bounds for \(K^n_r(\mathbb{D})\) were obtained in [7, 10] and the recent article [4] has improved over...
previously known lower bounds. Although for any \( n > 1 \) the exact value for \( K_n(\mathbb{D}) \) is yet unknown, it is known from [3] that \( K_n(\mathbb{D}) \) behaves asymptotically as \( \sqrt{\log n}/\sqrt{n} \). Let us define

\[
\tilde{K}_n := \inf \{ K_n(\Omega) : \Omega \subset \mathbb{C} \text{ is simply connected} \}.
\]

In the next theorem, we show that \( \tilde{K}_n \) has the same asymptotic behaviour as \( K_n(\mathbb{D}) \).

**Theorem 2.** \( \lim_{n \to \infty} \tilde{K}_n \sqrt{n}/(\log n) = 1 \).

The aim of the penultimate theorem of this article is to give lower bounds on \( K_n(\Omega) \) for simply connected and convex domains \( \Omega \subset \mathbb{C} \). For this purpose, we need to be familiar with the quantity \( S(k, n) \), known as the Sidon constant for the index set \( \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) : |\alpha| = k\} \). \( S(k, n) \) is defined as the smallest constant \( C \) such that

\[
\sum_{|\alpha| = k} |a_\alpha| \leq C \sup_{z \in \partial \mathbb{D}^n} \left| \sum_{|\alpha| = k} a_\alpha z^\alpha \right|
\]

for any \( k \)-homogeneous polynomial \( P(z) = \sum_{|\alpha| = k} a_\alpha z^\alpha \) in \( n \)-complex variables (see f.i. [11]).

**Theorem 3.** Suppose \( \Omega \subset \mathbb{C} \) is a simply connected domain and \( f : \mathbb{D}^n \to \Omega \) is holomorphic with an expansion (3). Then \( K_n(\Omega) \geq r_0, r_0 \) being the only root in \((0, 1)\) of the equation

\[
y + \sum_{k=2}^{\infty} kS(k, n) y^k = \frac{1}{4}.
\]

In addition, if \( \Omega \) is assumed to be convex, then \( K_n(\Omega) \geq r_1, r_1 \) being the only root in \((0, 1)\) of the equation

\[
y + \sum_{k=2}^{\infty} S(k, n) y^k = \frac{1}{2}.
\]

We remark here that the lower bound \( r_1 \) of \( K_n(\Omega) \) in the second part of the above Theorem 3 was obtained for \( \Omega = \mathbb{D} \) in [4, Theorem 3.3]. Finally, we concentrate on the functions mapping into \( \mathbb{D} \). In particular, very recently a refined version of [1, Theorem 8] has appeared in [17, Theorem 2.1] along with several other Bohr or Bohr-type inequalities (see also [20, Theorem 2] for one variable versions of [17, Theorem 2.1]). Like [1, Theorem 8], [17, Theorem 2.1] also deals with the holomorphic functions defined on a balanced domain \( G \subset \mathbb{C}^n \), and the result was shown to be sharp for \( G \) convex. Motivated by [15, Corollary 3.2], we show in the last theorem of this article that [17, Theorem 2.1] extends for the holomorphic functions defined on a balanced bounded domain \( G \) in any complex Banach space \( X \).

**Theorem 4.** Suppose \( X \) is a complex Banach space, \( G \subset X \) is a bounded balanced domain and \( f : G \to \mathbb{D} \) is a holomorphic function with an expansion (1) in a neighborhood of \( z_0 = 0 \). Then

\[
a^p + \sum_{k=1}^{\infty} \left| \frac{1}{k!} D^k f(0) \left( x^k \right) \right|^2 \leq 1
\]

for \( x \in r_p(a)G \) and \( r \leq r_p(a) \), where \( r_p(a) = (1 - a^p)/(2 - a^2 - a^p) \), \( a = |f(0)| \) and \( p > 0 \). The number \( r_p(a) \) and the factor \( 1/(1 + a) \) in (7) cannot be improved.

It may be noted that the other Bohr-like inequalities, i.e. [17, the Theorems 2.3, 2.6, 2.7 and the Corollary 2.8] can also be proved in sharp form for the holomorphic functions defined on a bounded balanced domain of a complex Banach space \( X \) in a similar manner as in Theorem 4.
2. Proofs of the theorems

**Proof of Theorem 1.** We start with any arbitrary simply connected domain $\Omega \subset \mathbb{C}$. For any holomorphic $f : G \to \Omega$ with an expansion (1) around $x_0 = 0$, we observe that the holomorphic function

$$f_1(u) := f(u\beta) = f(0) + \sum_{k=1}^{\infty} \left( \frac{1}{k!} D^k f(0) (\beta^k) \right) u^k, \quad u \in \mathbb{D}$$

maps $\mathbb{D}$ into the same domain $\Omega$ for any fixed $\beta \in \mathbb{G}$. Therefore, $f_1$ is subordinate to $g$ in $\mathbb{D}$, where $g$ is the univalent Riemann mapping from $\mathbb{D}$ onto $\Omega$, satisfying $g(0) = f_1(0) = f(0)$. Now, using the well-known theorem of de Branges (cf. [9, Theorem 1.1]), we have

$$\left| \frac{1}{k!} D^k f(0) (\beta^k) \right| \leq k |f'_1(0)|$$

for all $k \geq 1$. Since $\beta$ is arbitrary, [18, Lemma 1] gives

$$\sum_{k=1}^{\infty} \left| \frac{1}{k!} D^k f(0) (\beta^k) \right| \leq 4kd (f(0), \partial \Omega)$$

for any $x \in G$. Hence, given any $r \in (0, 1)$,

$$\sum_{k=1}^{\infty} \left| \frac{1}{k!} D^k f(0) (\beta^k) \right| \leq \frac{4r}{(1-r)^2} d (f(0), \partial \Omega)$$

for all $y \in rG$. This, inequality (2) is satisfied whenever $4r/(1-r)^2 \leq 1$, i.e. if $r \leq 3 - 2\sqrt{2}$, which implies that $K_x^G(\Omega) \geq 3 - 2\sqrt{2}$, and therefore $K_x^G \geq 3 - 2\sqrt{2}$. To show that $K_x^G$ is actually equal to $3 - 2\sqrt{2}$, we adopt the approach of [15]. For any $\hat{r} \in (3 - 2\sqrt{2}, 1)$, there exists $c \in (0, 1)$ and $V \in \partial G$ such that $c\hat{r} > 3 - 2\sqrt{2}$ and $c\sup_{x \in \partial G} \|x\| < \|V\|$. Now, we consider the Koebe function $K(u) = u/(1-u)^2, u \in \mathbb{D}$ and define the holomorphic function $f$ on $G$ by $f(x) = K(c\phi_V(x)/\|V\|)$, where $\phi_V$ is a bounded linear functional on $X$ with $\phi_V(V) = \|V\|$ and $\|\phi_V\| = 1$. It is easy to see that $f$ maps inside a simply connected domain $\Omega$, which is, in this case, the range of $K$, i.e. the whole plane $\mathbb{C}$ minus the part of the negative real axis from $-1/4$ to infinity. Thus, for $x = \hat{r}V$,

$$\sum_{k=1}^{\infty} \left| \frac{1}{k!} D^k f(0) (\beta^k) \right| = \frac{c\hat{r}}{(1-c\hat{r})^2} > \frac{1}{4} = d (f(0), \partial \Omega),$$

showing that $K_x^G$ cannot be bigger than $3 - 2\sqrt{2}$. Similar argument can be used for completing the proof of the case $K_x^G = 1/3$. We only need to note that the right hand side of the inequality (9) will be replaced by $2d(f(0), \partial \Omega)$ (see [18, Lemmas 2, 3]), and for the proof of the sharpness of the constant $1/3$, we have to use $L(u) = u/(1-u), u \in \mathbb{D}$ instead of $K(u)$, and observe that $L$ maps $\mathbb{D}$ onto the half-plane $c(w) > -1/2$.

**Proof of Theorem 2.** We follow the ideas of [3] in this proof. Given any $k$-homogeneous ($k \geq 1$) complex polynomial $P(z) = \sum_{|a| = k} a_m z^a$ defined in $\mathbb{C}^n$ and for any pre-assigned $\epsilon > 0$, there exists $\mu > 0$ such that

$$\left( \sum_{|a| = k} |a_m|^{2k} \right)^{1/2k} \leq \mu (1 + \epsilon)^k \sup_{\|z\|_\infty = 1} \left| \sum_{|a| = k} a_m z^a \right|$$

(see [3, Theorem 1.1]). Now, for any holomorphic function $f$ which maps $\mathbb{D}^n$ into a simply connected domain $\Omega \subset \mathbb{C}$ and has an expansion (3), it is immediate from (9) that

$$\left( \sum_{|a| = k} |a_m|^{2k} \right)^{1/2k} \leq 4k(1 + \epsilon)^k d (f(0), \partial \Omega).$$

Hence, using the Hölder’s inequality and the estimate

$$\binom{n + k - 1}{k} \leq \frac{(n + k - 1)^k}{k!} < \left( \frac{e}{k} \right)^k (n + k - 1)^k < e^k \left( 1 + \frac{n}{k} \right)^k,$$
we get, by setting \( r = (1 - 2\epsilon) \sqrt{(\log n)/n} \)

\[
\sum_{k=1}^{\infty} r^k \sum_{|a| = k} |c_a| \leq \sum_{k=1}^{\infty} r^k \left( \sum_{|a| = k} |c_a| e^{2k} \right)^{\frac{k+1}{2k}} \left( \binom{n + k - 1}{k} \right)^{\frac{k-1}{2k}}
\]

\[
< 4\mu d (f(0), \partial \Omega) \sum_{k=1}^{\infty} k(r \varpi(1 + \epsilon))^k \left(1 + \frac{n}{k}\right)^{\frac{k+1}{2k}}
\]

\[
= 4\mu d (f(0), \partial \Omega) \sum_{k=1}^{\infty} k \left( \sqrt{\log n/n} \sqrt{\varpi(1 - 2\epsilon)(1 + \epsilon)} \right)^k \left(1 + \frac{n}{k}\right)^{\frac{k-1}{2k}}.
\]

For \( n \) large enough,

\[
t_n := \frac{\sqrt{\log n}}{n^{1/4}} \sqrt{2e(1 - 2\epsilon)(1 + \epsilon)} < 1,
\]

and for \( k > \sqrt{n} \), observe that

\[
\left(1 + \frac{n}{k}\right)^{\frac{k-1}{2k}} < (2\sqrt{n})^k.
\]

Using both the above facts,

\[
\sum_{k > \sqrt{n}} k \left( \sqrt{\log n/n} \sqrt{\varpi(1 - 2\epsilon)(1 + \epsilon)} \right)^k \left(1 + \frac{n}{k}\right)^{\frac{k-1}{2k}} \leq \sum_{k > \sqrt{n}} k \left( \sqrt{\log n/n} \sqrt{2e(1 - 2\epsilon)(1 + \epsilon)} \right)^k \leq \frac{t_n}{(1 - t_n)^2},
\]

which goes to 0 as \( n \to \infty \). For \( k \leq \sqrt{n} \), we start by making \( n \) sufficiently large such that \( 2 < k_0 \leq \log n \) can be chosen for which the inequalities

\[
k_0^{\frac{1}{k_0}} \leq 1 + \frac{\epsilon}{2}, \quad \sum_{k_0 \leq k \leq \sqrt{n}} k \left(1 - 2\epsilon\right)(1 + \epsilon)^{3/2})^k \leq \frac{1}{8\mu} \quad \text{and} \quad \left(1 + \frac{k_0}{n}\right)^{\frac{k_0-1}{2(k_0-1)}} \leq \frac{\epsilon}{2}
\]

are satisfied. Observing that \( x^{1/(x-1)} \) is decreasing and \( (x - 2)/2(x - 1) \) is increasing in \((1, \infty)\), we obtain, for \( k \geq k_0 \):

\[
\left( k^\frac{k}{(k_0)^{1/k}} \left( \frac{1}{n} + \frac{k}{k_0} \right) \right)^{\frac{k-1}{k}} \leq \left( \left( \frac{1}{n} \right)^{\frac{k_0-1}{2(k_0-1)}} + k^\frac{1}{k_0} \right)^{\frac{k-1}{k}}
\]

\[
\leq \left( \left( \frac{1}{n} \right)^{\frac{k_0-1}{2(k_0-1)}} + k^\frac{1}{k_0} \right)^{\frac{k-1}{k}} \leq (1 + \epsilon)^{\frac{k-1}{k}} \leq 1 + \epsilon,
\]

which, after a little simplification, gives

\[
\left(1 + \frac{n}{k}\right)^{\frac{k-1}{2k}} \leq (1 + \epsilon)^{\frac{k-1}{k}} \frac{n^{\frac{k}{2}}}{k^{\frac{k}{2}}}.
\]

Therefore, observing that \( x \to n^{1/x} \) is decreasing up to \( x = \log n \) and increasing thereafter, we get

\[
\sum_{k_0 \leq k \leq \sqrt{n}} k \left( \sqrt{\log n/n} \sqrt{\varpi(1 - 2\epsilon)(1 + \epsilon)} \right)^k \left(1 + \frac{n}{k}\right)^{\frac{k-1}{2k}}
\]

\[
\leq \sum_{k_0 \leq k \leq \sqrt{n}} k \left( \sqrt{\log n(1 - 2\epsilon)(1 + \epsilon)^{3/2}} \sqrt{1/n^{1/k}} \right)^k
\]

\[
\leq \sum_{k_0 \leq k \leq \sqrt{n}} k \left(1 - 2\epsilon\right)(1 + \epsilon)^{3/2})^k \leq \frac{1}{8\mu}.
\]
It remains to analyze the case $1 \leq k \leq k_0$. In this case, we observe that for $n$ large enough,

$$\frac{k}{n} + 1 \leq \frac{k_0}{n} + 1 \leq \epsilon + 1,$$

and hence

$$\left(1 + \frac{n}{k}\right) \frac{k}{n} \leq \left(1 + \epsilon\right) k \left(\frac{n}{k}\right)^{\epsilon}.$$

Making use of the above inequality and the fact that $x \to n^{1/x} x$ is decreasing in $[1, k_0]$, it is easily seen that

$$\sum_{k=1}^{k_0} k \left(\sqrt{\frac{\log n}{n}} \sqrt{\epsilon(1-2\epsilon)(1+\epsilon)}\right)^k \left(1 + \frac{n}{k}\right) \frac{k}{n} \leq \sum_{k=1}^{k_0} k \left(\sqrt{\epsilon n(1-2\epsilon)(1+\epsilon)^{3/2}} \frac{k^{1/(2k)}}{k_0^{1/2} n^{1/(2k_0)}}\right)^k,$$

which tends to 0 as $n \to \infty$. Combining all the above three estimates, we have

$$\sum_{k=1}^{\infty} r^k \sum_{|a|=k} |c_a| \leq 4 \mu d(f(0), \partial \Omega) \left(\frac{1}{\delta \mu} + o(1)\right) \leq d(f(0), \partial \Omega)$$

for $n$ large enough. Therefore, $K_n(\Omega) \geq (1-2\epsilon) \sqrt{\log n}/n$, provided $n$ is sufficiently large. On the other hand, it is known from [7] that $\lim_{n \to \infty} K_n(\Delta) / \sqrt{n/(\log n)} \leq 1$. Since $\tilde{K}_n \leq K_n(D)$, our proof is complete.

**Proof of Theorem 3.** First suppose that $\Omega$ is simply connected. Hence, using the definition of Sidon constant, it follows from (9) that

$$\sum_{|a|=k} |c_a| \leq 4 k S(k, n) d(f(0), \partial \Omega)$$

for all $k \geq 1$. Assume that $\|z\|_\infty \leq r$. Then applying the above inequality and noting that $S(1, n) = 1$, it is clear that (4) is satisfied if

$$4 \sum_{k=1}^{\infty} k S(k, n) r^k = 4 \left(r + \sum_{k=2}^{\infty} k S(k, n) r^k\right) \leq 1. \quad (11)$$

It is easily seen that

$$R(y) := y + \sum_{k=2}^{\infty} k S(k, n) y^k - 1/4$$

is increasing in $(0, 1)$, $R(0) = -1/4 < 0$, $R(1/2) = 1/4 + \sum_{k=2}^{\infty} k S(k, n)(1/2)^k > 0$, and therefore $R$ has exactly one root $r_0$ in $(0, 1)$. As a consequence, the inequality (11) holds if $\|z\|_\infty \leq r_0$, where $r_0$ is the only root in $(0, 1)$ of the equation (5), i.e. $K_n(\Omega) \geq r_0$. For convex $\Omega$, we only have to start from the inequality

$$\sum_{|a|=k} |c_a| \leq 2 S(k, n) d(f(0), \partial \Omega)$$

for all $k \geq 1$, and argue exactly as above. \qed

**Proof of Theorem 4.** We construct $f_1$ as in the proof of Theorem 1, which then becomes a holomorphic self mapping of $D$ with an expansion (8). Since $\beta \in G$ is arbitrary, [17, Theorem A(b)] asserts the validity of (7) under the conditions $x \in r_p(a) G$ and $r \leq r_p(a)$, $\bar{r}_p(a)$ as defined in the statement of Theorem 4. To prove the sharpness part, we again need to use arguments similar to that of the article [15]. For the sake of completeness, it is included here. Given any $a \in [0, 1)$, we begin by considering the function $F(u) = (a-u)/(1-av)$, $u \in D$. For any $\bar{\tau} \in (r_p(a), 1)$, there exists $c \in (0, 1)$ and $V \in \partial G$ such that $c \bar{\tau} > r_p(a)$ and $c \sup_{x \in \partial G} \|x\| < \|V\|$. Now we define the holomorphic function $f$ on $G$ by $f(x) = F(c\phi_V(x)/\|V\|)$, where $\phi_V$ is a bounded linear functional.
on $X$ with $\phi_V(V) = \|V\|$ and $\|\phi_V\| = 1$. Hence, for $x = \bar{r}V$ and $r = \bar{r}$, the left hand side of the inequality (7) reduces to

$$\eta(a, \bar{r}) := a^p + \frac{c\bar{r}(1 - a^2)}{1 - ac\bar{r}} + \frac{1 + a\bar{r}}{(1 + a)(1 - \bar{r})} \left(1 - a^2\right)^2 \left(c\bar{r}^2\right) > a^p + \left(1 - a^2\right) \frac{c\bar{r}}{1 - c\bar{r}} > 1.$$ 

On the other hand, let us assume that the quantity $1/(1 + a)$ in (7) can be replaced by a bigger number $A$ and the resulting inequality is still valid for all $x \in r_p(a)G$ and $r = r_p(a)$. We use the same $f$ and $F$ as already defined, but instead of fixing some $\bar{r}$, we will work with $r_p(a)$ itself; and for any $c \in (0, 1)$, we get a $V \in \partial G$ as above. Now for $x = cr_p(a)V$ and $r = r_p(a)$, the left hand side of the modified inequality (7) is bigger than $\eta(a, cr_p(a))$, which is again bigger than $a^p + (1 - a^2)(c^2 r_p(a)) / (1 - c^2 r_p(a))$. It is evident that the last quantity approaches to 1 as $c \to 1 -$, and at the same time the modified inequality (7) is satisfied for this particular $x$ and $r$ as well. Therefore,

$$\lim_{c \to 1^-} \left(a^p + \frac{c^2 r_p(a)(1 - a^2)}{1 - ac^2 r_p(a)} + \left(A + \frac{r_p(a)}{1 - r_p(a)}\right) \frac{(1 - a^2)^2 \left(c^2 r_p(a)\right)^2}{1 - (ac^2 r_p(a))^2}\right) \right.$$ 

$$= a^p + \frac{r_p(a)(1 - a^2)}{1 - ar_p(a)} + \left(A + \frac{r_p(a)}{1 - r_p(a)}\right) \frac{(1 - a^2)^2 \left(r_p(a)\right)^2}{1 - (ar_p(a))^2}$$ 

$$= 1 = a^p + (1 - a^2) \frac{r_p(a)}{1 - r_p(a)},$$

i.e. $A = 1/(1 + a)$. Summarizing the above discussion, we conclude that neither the number $r_p(a)$ nor the factor $1/(1 + a)$ could be improved. $\square$

References

[1] L. Aizenberg, "Multidimensional analogues of Bohr's theorem on power series", Proc. Am. Math. Soc. 128 (2000), no. 4, p. 1147-1155.
[2] ———, "Generalization of results about the Bohr radius for power series", Stud. Math. 180 (2007), no. 2, p. 161-168.
[3] F. Bayart, D. Pellegrino, I. B. Seoane-Sepúlveda, "The Bohr radius of the $n$-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$", Adv. Math. 264 (2014), p. 726-746.
[4] L. Bernal-González, H. J. Cabana, D. García, M. Maestre, G. A. Muñoz-Fernández, J. B. Seoane-Sepúlveda, "A new approach towards estimating the $n$-dimensional Bohr radius", Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM 115 (2021), no. 2, article no. 44 (10 pages).
[5] B. Bhowmik, N. Das, "Bohr phenomenon for operator-valued functions", Proc. Edinb. Math. Soc. 64 (2021), no. 1, p. 72-86.
[6] ———, "A characterization of Banach spaces with nonzero Bohr radius", Arch. Math. 116 (2021), no. 5, p. 551-558.
[7] H. P. Boas, D. Khavinson, "Bohr's power series theorem in several variables", Proc. Am. Math. Soc. 125 (1997), no. 10, p. 2975-2979.
[8] H. Bohr, "A theorem concerning power series", Proc. Lond. Math. Soc. 13 (1914), p. 1-5.
[9] L. de Branges, "A proof of the Bieberbach conjecture", Acta Math. 154 (1985), no. 1-2, p. 137-152.
[10] A. Defant, L. Frerick, "A logarithmic lower bound for multi-dimensional Bohr radii", Isr. J. Math. 152 (2006), p. 17-28.
[11] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes, K. Seip, "The Bohnenblust–Hille inequality for homogeneous polynomials is hypercontractive", Ann. Math. 174 (2011), no. 1, p. 485-497.
[12] P. G. Dixon, "Banach algebras satisfying the non-unital von Neumann inequality", Bull. Lond. Math. Soc. 27 (1995), no. 4, p. 359-362.
[13] P. L. Duren, Univalent functions, Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, 1983.
[14] I. Graham, G. Kohr, Geometric function theory in one and higher dimensions, Pure and Applied Mathematics, Marcel Dekker, vol. 255, Marcel Dekker, 2003.
[15] H. Hamada, T. Honda, G. Kohr, "Bohr's theorem for holomorphic mappings with values in homogeneous balls", Isr. J. Math. 173 (2009), p. 177-187.
[16] H. Hamada, T. Honda, Y. Mizuta, "Bohr phenomenon on the unit ball of a complex Banach space", Math. Inequal. Appl. 23 (2020), no. 4, p. 1325-1341.
[17] M.-S. Liu, S. Ponnusamy, “Multidimensional analogues of refined Bohr’s inequality”, Proc. Am. Math. Soc. 149 (2021), no. 5, p. 2133-2146.
[18] Y. A. Muhanna, “Bohr’s phenomenon in subordination and bounded harmonic classes”, Complex Var. Elliptic Equ. 55 (2010), no. 11, p. 1071-1078.
[19] V.I. Paulsen, G. Popescu, D. Singh, “On Bohr’s inequality”, Proc. Lond. Math. Soc. 85 (2002), no. 2, p. 493-512.
[20] S. Ponnusamy, R. Vijayakumar, K.-J. Wirths, “New inequalities for the coefficients of unimodular bounded functions”, Results Math. 75 (2020), no. 3, article no. 107 (11 pages).
[21] G. Popescu, “Bohr inequalities for free holomorphic functions on polyballs”, Adv. Math. 347 (2019), p. 1002-1053.