Quantum entanglement driven by electron-nanomechanical coupling

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Abstract — We study the effects of an electron-electron effective interaction on the formation of entangled states in a two-qubit system, driven by the coupling of electronic states with vibrational modes. The system is composed by four quantum dots separated in pairs, each pair with one excess electron, which is able to tunnel between the dots. Also, the dots from each pair are coupled with different vibrational modes. The combined action of both, this effective interaction and the electronic tunneling explains not only features on the spectrum and the eigenstates of the Hamiltonian, but also the formation of electronic Bell states by exploiting the quantum dynamics of the system.

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I. INTRODUCTION

For many years, the single-molecule electronics has being an outstanding issue due to its potentiality for future implementations of a cheaper and faster single-electron transistor[11,12]. There are already many successful examples of single-molecule devices that operate in the Coulomb-blockade regime, presenting transistor-like behavior[3]. To increase further the functionality of a single-electron transistor, it is possible to couple nanomechanical and electronic degrees of freedom[13]. It is well known that this kind of interaction plays a significant role, bringing a wealth of interesting effects, such as quantum-shuttles in quantum dots (QDs) system[14,15], local cooling[16], phonon-assisted transport in molecular quantum dot junctions[17,18], and Franck-Condon blockade[19].

Concerning this single electron systems, there is an emerging interest in the interplay between the electronic degrees of freedom, confined inside nanoparticles, and discrete vibrational modes. In particular, this problem has being analyzed considering two different contexts: carbon nanotubes quantum dots and vibrational modes of a cavity coupled with quantum dots.

Carbon nanotubes (CNT) become one of the most successful new materials in front of their wide set of direct applications[20]. These include the implementation of ultrahigh tunable frequency resonators[21,22], nanoradios[23], and ultrasensitive mass sensors[24,25]. Moreover, when operated as mechanical resonators, nanotubes show high quality factor[26,27] being possible, for instance, to excite, detect and control specific vibrational modes of a CNT with a current being injected from a scanning tunneling microscopy (STM) tip into a CNT[28]. Additionally, it have been reported strong coupling regimes between single-electron tunneling and nanomechanical motion on a suspended nanotube, tuned via electrical gates[29]. Regarding applications in micro and nanoelectronics, carbon nanotubes presented ballistic conduction[24] and Coulomb blockade effect in single and double nanotube based quantum dot devices[25]. Particularly, it was proposed a mechanically induced “two-qubit” quantum gate and the generation of entanglement between electronic spin states in CNT[26]. Also, phonons on CNTs showed its potentiality as “flying” qubits for electron spin communications over long distances[27].

Alternatively, regarding the second experimental context, the coupling between electronic degrees of freedom and vibrational modes can also be accomplished in piezoelectric phononic cavities, based on Bragg mirrors that confine a vibrational modes[29]. This coupling was experimentally demonstrated in transport measurements as a phonon assisted tunneling in a double quantum dot structure embedded in the cavity. The electron-vibrational mode coupling factor in this quantum dots system is ten times bigger if compared with couplings found on cavity quantum electrodynamics (CQED) and can be enhanced, on demand, by a factor of 20-500 of its regular value[30].

In front of the advantages cited above, both systems have a great potentiality as a solid state based quantum information processing. In the present work, we explore the properties of a Hamiltonian which describes two electrons on four quantum dots, considering a general model of coupling between electrons and vibrational modes, which applies to both experimental setups. Our proposal is based on charge-qubits, instead of spin, where the states of two qubits are defined depending of the occupation of the quantum dots. We are interested on the interplay between the electron-vibrational mode coupling and the tunneling of electrons between adjacent quantum dots.

In Sec. [I] by using the unitary transformation of Lang-Firsov, we demonstrate that both couplings are responsible for the apparition of an effective electron-electron interaction. Section [II] is devoted to the exploration of the signatures of this effective interaction and correlated phenomena on the spectrum and eigenstates of the model. Finally, in Sec. [III], using our previous experience on quantum dynamics on coupled quantum dots[26,29], we study the formation of Bell states under specific conditions. Section [IV] contains our final remarks.
FIG. 1. A possible experimental setup of the system of interest: quantum dots 1 and 3 (2 and 4) are coupled with the vibrational (bosonic) mode 1 (2) on carbon nanotubes. Also, the dots 1 (3) and 2 (4) are coupled to each other by tunneling, encoding a qubit.

II. MODEL

Consider a multipartite system with two main parts: the electronic subspace $D$, consisting on four quantum dots, and the bosonic subspace $V$, with two devices containing vibrational modes. Vibrational mode 1 couples with electronic levels of the dots 1 and 3, while vibrational mode 2 couples with dots 2 and 4. Tunneling is allowed between dots 1 (3) and 2 (4) so the pair 1 − 2 (3−4) can be described as a qubit. A possible experimental setup is illustrated in Fig. 1, where the bosonic devices are carbon nanotubes. Electronic levels on the quantum dots can be populated by using sources and drains, with additional gates to control the process of tunneling between the dots. The electron-vibrational interaction does not change the electronic population, although it creates or destroys vibrational excitations in both devices. Although electrons in different qubits could interact, for instance, by Coulomb interaction, we assume this interaction is weak enough to be ignored.

The Hamiltonian that describes the two charged qubits and vibrational modes is given by

$$H = H_D \otimes I_V + I_D \otimes H_V + V_{DV}.$$  

Here $H_D$ and $H_V$ are the free Hamiltonians of the quantum dots and vibrational modes subspaces, respectively, $V_{DV}$ is the dots-vibrational modes coupling and $I_D(V)$ is the identity matrix for quantum dots ($D$) or the vibrational modes ($V$) subspaces. While each pair of dots is a 2-dimensional subspace, the vibrational mode spans on an infinite dimension subspace $N_v$, where $N_v \to \infty$ for $v$-th subspace ($v = 1, 2$). The elements of the computational basis have the general form $|n_1 n_2 n_3 n_4 \rangle_D \otimes |N_1 N_2 \rangle_V = |n_1 n_2 n_3 n_4; N_1 N_2 \rangle$ with the first four indexes indicating the occupation of the specific dot (0-empty and 1-occupied) and the last two being the population of the vibrational modes. One example is the state $|1010, 00 \rangle$ which represents the situation where dots 1 and 3 contain a single electron each and there is no excitations in the vibrational modes.

We proceed to write each term of the Hamiltonian. The free Hamiltonian for the four quantum dots subsystem is written as ($\hbar = 1$):

$$H_D = \sum_{i=1,2} \varepsilon_i N_i^D + \Delta_{12} \left( S_1^i S_2^i + S_2^i S_1^i \right) \otimes I^\otimes 2 + I^\otimes 2 \otimes \sum_{j=3,4} \varepsilon_j N_j^D + \Delta_{34} \left( S_3^j S_4^j + S_4^j S_3^j \right)$$

where $S_{i(j)}^\dagger \left[ S_{i(j)} \right]$ are the creation (annihilation) operators for the $i(j)$-th quantum dot and $N_{i(j)}^D = S_{i(j)}^\dagger S_{i(j)}$ ($i = 1, 2$ while $j = 3, 4$). The parameters $\varepsilon_{i(j)}$ are the electronic levels for each dot while $\Delta_{12(34)}$ describes the tunneling coupling. If we consider a single vibrational mode per bosonic subspace, the free Hamiltonian $H_V$ becomes

$$H_V = \omega_1 B_1^\dagger B_1 \otimes I^\otimes N_2 + I^\otimes N_1 \otimes \omega_2 B_2^\dagger B_2,$$

where $\omega_{1(2)}$ is the energy of the corresponding vibrational mode. Here $B_{1(2)}^\dagger \left( B_{1(2)} \right)$ creates (annihilates) an excitation in a $v$-th vibrational mode subspace.

Now we focus on the term $V_{DV}$, which provides the electron-vibrational mode coupling. This coupling happens when a single electron on dot $i$ interacts with the vibrational subsystem thus creating or annihilating one excitation in the corresponding vibrational mode. We consider that the electron-vibrational mode coupling is the same for both bosonic modes, with a strength given by $g_v$. The term $V_{DV}$ is written as

$$V_{DV} = g_1 \left( N_1^D \otimes I^\otimes 2 + I^\otimes 2 \otimes N_2^D \right) \otimes \left( B_1^\dagger + B_1 \right) \otimes I^\otimes N_2 + g_2 \left( N_2^D \otimes I^\otimes 2 + I^\otimes 2 \otimes N_1^D \right) \otimes I^\otimes N_1 \otimes \left( B_2^\dagger + B_2 \right).$$

As pointed out by Sowa et. al., there can be a phase difference in the coupling parameters $g_v$ given by where $\phi_v = k_v \cdot d_v$, with $k_v$ being the wavevector of the $v$ vibrational mode and $d_v$ the distance between dots coupled with this specific mode. Here, we assume that the distance between dots inside a molecule is smaller than the vibrational wavelength so the phase difference can be ignored and $g_1 = g_2 = g$.

In order to analyze the action of electron-vibrational mode and tunneling couplings, we apply the Lang-Firsov unitary transformation over the Hamiltonian in Eq. (1) calculating $\tilde{H} = e^S \tilde{H} e^{-S}$, where

$$S = \alpha_1 \left( N_1^D \otimes I^\otimes 2 + I^\otimes 2 \otimes N_2^D \right) \otimes \left( B_1^\dagger - B_1 \right) \otimes I^\otimes N_2 + \alpha_2 \left( N_2^D \otimes I^\otimes 2 + I^\otimes 2 \otimes N_1^D \right) \otimes I^\otimes N_1 \otimes \left( B_2^\dagger - B_2 \right),$$

where $\alpha_1$ and $\alpha_2$ are the coupling strengths.
with $\alpha_v = \frac{e^2}{\varepsilon}$. This calculation results on a new form for the Hamiltonian written as

$$\tilde{H} = (\tilde{H}_D + V_{\text{eff}}^D) \otimes I_V + I_D \otimes H_V + \Delta_{T \nu}^J,$$

where

$$\tilde{H}_D = \sum_{i=1,2} \tilde{e}_{i} N_i^D \otimes I^{\otimes 2} + I^{\otimes 2} \otimes \sum_{j=3,4} \tilde{e}_{j} N_j^D,$$

is the transformed Hamiltonian for the dots with $\tilde{e}_{i}(j)$ being an energy level shifted due to the action of the electron-vibrational mode coupling. Specifically, $\tilde{e}_{i}(3) = \varepsilon_{i}(3) - \alpha_1^2 \omega_1$ while $\tilde{e}_{i}(4) = \varepsilon_{i}(4) - \alpha_2^2 \omega_2$. Apart from this dressed uncoupled electronic Hamiltonian, we want to highlight two new terms on Eq. (6). The first one can be seen as an effective electron-electron interaction

$$V_{\text{eff}}^D = -2\alpha_1^2 \omega_1 N_1^D \otimes N_2^D - 2\alpha_2^2 \omega_2 N_2^D \otimes N_1^D,$$

which cooperates with tunneling in order to generate maximally entangled states. The last term

$$\Delta_{T \nu}^J = \left[ \left( \Delta_{12} S_1^1 S_2^1 \otimes I^{\otimes 2} + I^{\otimes 2} \otimes \left( \Delta_{34} S_3^1 S_4^1 \right) \right) \otimes X_{12} \right. + \left. \left[ \left( \Delta_{12} S_1^2 S_2^1 \otimes I^{\otimes 2} + I^{\otimes 2} \otimes \left( \Delta_{34} S_3^2 S_4^1 \right) \right) \otimes X_{12} \right],$$

(9)

describes an effective interaction between the two vibrational modes considered on our problem. Here $X_{12} = e^{-\alpha_1 (B_1 - B_1^2)} \otimes e^{-\alpha_2 (B_2 - B_2^2)} = D_1(\alpha_1) \otimes D_2(\alpha_2)$, being a tensorial product of displacement operators for the quantum harmonic oscillator.

The new transformed Hamiltonian, Eq. (6) and its terms Eqs. (7)-(9), highlights important effects of the couplings considered on this particular physical system. The first is a shift on the value of the electronic levels which depends on both, the coupling parameter $g$ and $\omega_v$. The second is the effective electron-electron interaction which couples the electrons from different qubits, which is mediated by the electron-vibrational mode coupling. As we discuss below, the effective electron-electron interaction, together with electronic tunneling, is behind the apparition of entangled eigenstates of the full Hamiltonian and the subsequent possibility of generation, by quantum dynamics, of Bell states.

**III. SPECTRAL ANALYSIS**

We proceed to explore the characteristics of energy spectrum and eigenstates of the Hamiltonian in Eq. (1). Along with the study of energy spectrum, we are interested on the entanglement properties of the eigenstates. It is well known that Coulomb interaction is behind the formation of entangled states in coupled quantum dots molecule\[^{29,34}\], once this interaction couples two single electrons, making viable the encoding of two qubits. In the present problem, we expected the apparition of signatures of the effective electron-electron interaction on the entanglement degree of the eigenstates.

To perform our numerical analysis, both basis associated with the vibrational modes are truncated at $N_1 = N_2 = 13$, although both basis for vibrational modes have infinite dimension. This number of computational states is enough to guarantee the accuracy of the calculation of lower energies and eigenstates. To analyze the formation of entangled states, we first build up a density matrix for each eigenstate in the complete basis $\hat{\rho}_i = |\psi_i\rangle \langle \psi_i|$, where $|\psi_i\rangle$ is the ith eigenstate of Hamiltonian (1). The second step is the calculation of the reduced 4 x 4 density matrix for the two qubits, by tracing out the degrees of freedom of the vibrational modes so $\hat{\rho}_{D,l} = \text{Tr}_V[\hat{\rho}_l]$. Then, the concurrence, defined by Wootter\[^{35}\], is used as a measure of entanglement.

FIG. 2. The first 24 eigenvalues of Hamiltonian (1) varying the detuning $\delta = \varepsilon_1 - \varepsilon_2$ considering $\varepsilon_3 = \varepsilon_4 = 0$ and $g = 0.5 \omega$ and $\Delta_{12} = \Delta_{34} = 5 \times 10^{-2} \omega$. The energy increases from panel (c) to panel (a) showing the first four states (black solid lines and squares), the next eight states (red dashed lines and circles), and the following twelve states (blue dotted lines and triangles).

An auxiliary Hermitian operator\[^{36}\] $R_l$ is defined as $R_l = \sqrt{\hat{\rho}_{D,l} \hat{\rho}_{D,l} \sqrt{\hat{\rho}_{D,l}}}$, where $\hat{\rho}_{D,l} = (\sigma_y \otimes \sigma_y) \hat{\rho}_{D,l} (\sigma_y \otimes \sigma_y)$, is the spin-flipped matrix with $\hat{\rho}_{D,l}$.
being the complex conjugate of $\hat{\rho}_{D,1}$. The concurrence is obtained once $C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$ where $\lambda_k$ ($k = 1..4$) are the eigenvalues of the operator $R_l$ in decreasing order.

Figures 2(a) to (c) show the behavior of the first 24 eigenvalues of the Hamiltonian as a function of detuning $\delta = \varepsilon_1 - \varepsilon_2$, considering $\varepsilon_3 = \varepsilon_4 = 0$, a resonance condition for the dots 3 and 4. We consider that both vibrational modes have the same frequency value so $\omega_1 = \omega_2 = \omega$. The coupling parameters are defined in terms of the frequency $\omega$ being $g = \omega/2$ the electron-vibrational mode coupling and $\Delta_{12} = \Delta_{34} = \Delta/2 = \omega/20$ the tunneling rates.

The energy spectrum shows the emergence of branches, spanned on an energy interval $\Delta E = \omega$, each with an increasing number of inner states as energy increases: while the first branch (solid black lines) has the four states shown in Fig. 2 (c) including the ground level, the second branch (dashed red lines) has eight states, as seen in Fig. 2 (b). The subsequent branch (dotted blue lines) contains twelve eigenstates, shown in Fig. 2 (a). The branches share some common features. The first is the appearance of anticrossings at $\delta = \pm 0.5$, due to first order transitions that switch only one electron per time, e.g., $|1001\rangle_D \leftrightarrow |1010\rangle_D$. The second is a little anticrossing arising at $\delta = 0$, related to higher order transition processes. For instance, two electrons can start at dots 1 and 4 (state $|1001\rangle_D$) ending at dots 2 and 3 (state $|0110\rangle_D$).

For each branch, the inner states have interesting properties concerning entanglement as can be seen from Fig. 3 (a) to (c), where we show the behavior of concurrence as a function of detuning $\delta$. Comparing our findings with results on a previous work, some similarities let us to conclude that the effective electron-electron coupling is behind the apparition of dressed Bell states as eigenstates. A signature of this fact is the increasing value of maximally entangled states at $\delta = 0$, as shown by the scattered plots: there is one on the black branch [filled squares on Fig. 3 (a) and Fig. 2 (c)], two inside the red branch [filled and open circles on Fig. 3 (b) and Fig. 2 (b)] and three on the blue branch [filled and open triangles point up and down on Fig. 3 (c) and Fig. 2 (a)]. Additionally, some satellite peaks at $\delta = \pm 0.5$ are observed, having a lower value of concurrence. While those secondary peaks are related to first order processes, the sharp peak at zero energy arises from second and higher order transitions.

Those features of the eigenstates can be explored theoretically through a straightforward calculation (see Appendix A for details) which consists on perform a basis transformation going from the electronic computational four-dimensional basis given by $\{|1010\rangle_D, |0101\rangle_D, |1001\rangle_D, |0110\rangle_D\}$ to an electronic Bell basis ordered as $\{|\Psi_-\rangle_D, |\Phi_-\rangle_D, |\Psi_+\rangle_D, |\Phi_+\rangle_D\}_D$, where $|\Psi_\pm\rangle_D = \frac{1}{\sqrt{2}}(|1001\rangle_D \pm |0110\rangle_D)$ and $|\Phi_\pm\rangle_D = \frac{1}{\sqrt{2}}(|1010\rangle_D \pm |0101\rangle_D)$. For the specific case of equal tunneling rates and $\delta = 0$, the calculation shows that the terms of the Hamiltonian regarding $|\Psi_-\rangle_D$ can be written as a tensorial product given by

$$
H_{\text{with} |\Psi_-\rangle} = (|\Psi_\pm\rangle_D \otimes \sum \alpha |N'_1 \rangle_{\text{V}} |N'_2 \rangle_{\text{V}} \langle N_1 N_2|) \langle \Psi_+ | \Psi_- \rangle |\Psi_\pm\rangle_D \otimes |\Phi_\pm\rangle_D,$$

Other terms on Hamiltonian cannot be written as a tensorial product of the form $|\psi\rangle_D \otimes \sum \alpha |N'_1 \rangle_{\text{V}} |N'_2 \rangle_{\text{V}} \langle N_1 N_2|$ terms with $|\Psi_+\rangle$ are coupled with $|\Phi_\pm\rangle$ by electron-vibrational mode interaction, while elements $|\Phi_\pm\rangle$ and $|\Psi_\pm\rangle$ are also coupled to each other by tunneling (see Appendix A).

In the Eq. 10 we use bold type and the square brackets, $[,]$, to emphasize the new dressed basis $\{|\psi_{\text{Bell}, N_1 N_2}\rangle\}$. The number of eigenstates per branch and the number of maximally entangled molecular states at $\delta = 0$ are linked with the dimension of original subspaces with the same value of the sum $N_1 + N_2$, as can be seen from Eq. (A1). Although these subspaces are coupled with each other, each branch can be seen as dressed Bell states, with an energy increasing as $N = N_1 + N_2$ grows.

IV. DYNAMICAL GENERATION OF ELECTRONIC BELL STATES

After studying the properties of the eigenstates of the model, it is worthwhile to explore the preparation of electronic entangled states by quantum dynamics. We again use a numerical approach, considering the general Hamiltonian $[\hat{H}]$, to simulate the quantum dynamics through the evolution of the density matrix $\rho(t) = |\psi(t)\rangle \langle \psi(t)|$. Tracing out the vibrational degrees of freedom, it results on the electronic reduced density matrix

$$
\rho_D(t) = \text{Tr}_V[\rho(t)],
$$

used to explore the behavior of the electronic part of the system through the calculation of physical properties as populations, fidelity, and concurrence. Initially, we perform a test to check the precision of our calculation together with our findings about the special character of
the electronic Bell state \(|\Psi_-\rangle_D\) as discussed on Sec. \(\text{III}\). We consider \(\Delta_{12} = \Delta_{34}\) (equal tunneling couplings), detuning \(\delta = 0\) and the initial state given by

\[
\rho(0) = (|\Psi_-\rangle \langle \Psi_-|)_D \otimes |00\rangle_V \langle 00|.
\] (12)

After the calculation of \(\rho(t)\) and the \(\rho_D(t)\), we obtain the fidelity of the evolved electronic state with this initial state being \(\mathcal{F}(t) = \text{Tr}_D[\rho_D(t)\rho_D(0)]\). Our results shown that the fidelity remains constant with value \(\mathcal{F}(t) = 1\), showing that, from the electronic point of view, any initial state with \((|\Psi_-\rangle \langle \Psi_-|)_D\) as the electronic part, acts as a stationary state.

In order to explore the generation of electronic entangled states, we now assume the system is being prepared as

\[
\rho_0 = |1001\rangle_D \langle 1001| \otimes |00\rangle_V \langle 00|,
\] (13)

which is an experimentally feasible initial state, once experimental setups include a set of sources and drains attached to the quantum dots, that can inject electrons. We consider the condition of \(\delta = 0\) and the same choices of physical parameters used in Figs. [2][3]. In Fig. [4](a), we show the evolution of the populations of electronic states \(|1001\rangle, |0110\rangle, |0101\rangle, |1010\rangle\). At initial times, the values is given by \(P_{1001} = 1\) (black line), \(P_{0110} = 0\) (red line), and \(P_{0101} = P_{1010} = 0\) (green line) in agreement with the initial condition. As time evolves, \(P_{1001}\) and \(P_{0110}\) oscillates out of phase between zero (non-occupied) and 1 (occupied), while \(P_{1010}\) and \(P_{0101}\) remain oscillating close to zero. This is a promising sign toward the formation of a state similar with \(|\Psi_+\rangle\) states on Bell basis, once the population of states \(|1001\rangle\) and \(|0110\rangle\) is 0.5 at \(\omega t = 300\). We also plot the concurrence, solid blue line in Fig. [4](a), which reaches the value \(C \approx 1\) when \(\rho_{1001} = \rho_{0110}\), indicating the actual formation of a maximally entangled electronic state. A similar effect was originally reported in Ref. [25] in the context of two-qubits coupled via Coulomb interaction. In the present case, this evolution shows that the effective electron-electron interaction mediated by the vibrational modes turns out to be responsible for the coupling between the qubits.

In order to give a closer look at the entangled states created by quantum dynamics, we compute the fidelity compared with a pre-defined target Bell state, \(\mathcal{F}(t) = \text{Tr}_D[\rho_D(t)\rho_{\text{tar}}^D]\). For this calculation, we use as target state the electronic density matrix \(\rho_{\text{tar}}^D = |\Psi(\varphi)\rangle_D \langle \Psi(\varphi)|\), where

\[
|\Psi(\varphi)\rangle_D = \frac{1}{\sqrt{2}}[|1001\rangle_D + e^{i\varphi}|0110\rangle_D],
\] (14)

where \(\varphi\) is a relative phase between the state. Note that if \(\varphi = 0\) (\(\varphi = \pi\)) then \(|\Psi(0)\rangle \equiv |\Psi_+\rangle\) (\(|\Psi(\pi)\rangle \equiv |\Psi_-\rangle\)). In Fig. [4](b) we show how \(\mathcal{F}\) evolves with time for different values of \(\varphi\). For \(\varphi = 0\) the fidelity remains stationary, with a value of 0.5, while for \(\varphi = \pi\), the fidelity shows small oscillations around those value. By setting the \(\varphi = \pm \pi/2\), we fulfill our goal of find the correct relative phase of the dynamically created entangled state, once the fidelity for both cases oscillates out of phase between 0 and 1. The concurrence maximum values being 1 let us to conclude that the electronic entangled state alternate between \(|\Psi(\pi/2)\rangle_D\), at \(\omega t = 300\) and \(|\Psi(-\pi/2)\rangle_D\), at \(\omega t = 600\). This means that the dynamics shows not only the ability of create Bell states related with \(|\Psi_+\rangle_D\) but also an additional ingredient, which is the imprint of a relative phase \(\varphi\).

Finally in Fig. [5] the effect of the electron-vibrational mode coupling parameter, \(g\), on the formation of the electronic entangled states, going into a weak coupling regime. We first notice that, by decreasing the value of \(g\), we still obtain values of \(C \approx 1\) at some evolved time.

FIG. 3. Behavior of the concurrence, C, for the electronic part of the first 24 eigenstates. We use the same physical parameters and the same color, lines and symbol conventions that in Fig. 2. Note that the condition \(\delta = 0\) is related with the apparition of electronic maximally entangled states, with \(C = 1\).
FIG. 4. Dynamics of populations, concurrence, and fidelity of electrons in quantum dots considering $\delta = \varepsilon_1 - \varepsilon_2 = 0$, $\varepsilon_3 = \varepsilon_4 = 0$, $g = 0.5\omega$ and $\Delta_{12} = \Delta_{34} = 5 \times 10^{-2}\omega$. (a) Populations $P_{1001}$ (black line), $P_{0110}$ (red line) and $P_{1010} = P_{0101}$ (green line), and concurrence (blue line) as functions of $\omega t$. (b) Dynamics of the fidelity between the evolved state and the target state given by Eq. (14), for different values of relative phase $\varphi$: $\varphi = -\pi/2$ (blue line), $\varphi = 0$ (green line), $\varphi = +\pi/2$ (black line), and $\varphi = \pi$ (red line).

Second feature is that it becomes clear that the period of oscillations of the formation of the electronic Bell states is governed by this coupling parameter. This shows that the formation of electronic maximally entangled states is a quite robust effect, which can become accessible in a wide range of experimental devices, even for weak values of electron-vibrational mode coupling.

V. CONCLUSION

In this work, we analyze a general model which describes a system where two electrons inside a set of four quantum dots interact with vibrational modes, shown in Fig. 1. The goal is to explore the interplay between tunneling, detuning of the electronic levels inside the quantum dots, and the electron-vibrational mode coupling. The model can describe several experimental scenarios, including electrons inside carbon nanotubes quantum dots or the coupling between electrons and an acoustic cavity.

Our findings include the study of the characteristics of the spectrum, the eigenstates, and the quantum dynamics of the system, focusing on the search of electronic maximally entangled states. We find that the electron-vibrational mode coupling is responsible for the apparition of dressed electronic Bell states. Concerning the dynamics, the generation of those entangled states is possible for a wide range of physical parameters.

VI. ACKNOWLEDGMENTS

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Appendix A: The matrix representation of hamiltonian on the dressed Bell basis

The entanglement behavior of the system of interest can be explored by writing down its Hamiltonian in terms of the electronic Bell states. Let us calculate the representation of the original Hamiltonian $\hat{H}$ as a matrix written in the dressed basis $|\psi_{\text{Bell}}, N_1 N_2\rangle$, where the electronic part is ordered following

$$|\psi_{\text{Bell}}\rangle_D = \{|\Psi_-\rangle, |\Phi_-\rangle, |\Psi_+\rangle, |\Phi_+\rangle\}_D,$$
where $|\Psi_+\rangle_D = \frac{1}{\sqrt{2}}(|1001\rangle \pm |0110\rangle)$ and $|\Phi_+\rangle_D = \frac{1}{\sqrt{2}}(|1010\rangle \pm |0101\rangle)$. We choose to keep together the states with the same number of total excitations $N = N_1 + N_2$. This choice remarks the fact that the basis for the whole system has an internal structure of coupled subspaces. For each value of $N$, there is an associated family of subspaces $S_{B,(N_1N_2)}$: $N = 0$ has only the subspace $S_{B,(00)}$ with four inner states, $N = 1$ has two subspaces being $S_{B,(10)}$ and $S_{B,(01)}$ (eight inner states), $N = 3$ has twelve inner states associated with $S_{B,(11)}$, $S_{B,(20)}$, and $S_{B,(02)}$, and so on.

Let us write the matrix representation of the Hamiltonian for this first six 4D subspaces $S_{B,(N_1N_2)}$, ordered as $\{S_{B,(00)}, S_{B,(01)}, S_{B,(10)}, S_{B,(11)}, S_{B,(02)}, S_{B,(20)}\}$:

$$H = \begin{pmatrix}
G_2 & 0 & 0 & 0 & 0 & 0 \\
G_1 & G_0 & 0 & 0 & 0 & 0 \\
0 & G_1 & G_2 & 0 & 0 & 0 \\
0 & 0 & G_0 & G_2 & 0 & 0 \\
0 & 0 & 0 & G_0 & G_2 & 0 \\
0 & 0 & 0 & 0 & G_0 & G_2
\end{pmatrix}. \quad (A1)$$

using the order $|\Psi_-, N_1N_2\rangle$, $|\Phi-, N_1N_2\rangle$, $|\Psi+, N_1N_2\rangle$, and $|\Phi+, N_1N_2\rangle$, the 4D matrices $B_{N_1N_2}$ and $G_v$ are defined as

$$B_{N_1N_2} = \begin{pmatrix}
E_{N_1N_2} & \Delta_+ & \delta_-/2 & 0 & 0 & 0 \\
\Delta_+ & E_{N_1N_2} & 0 & \delta_+/2 & 0 & 0 \\
\delta_-/2 & 0 & E_{N_1N_2} & 0 & \Delta_+ & 0 \\
0 & \delta_+/2 & 0 & E_{N_1N_2} & 0 & \Delta_+
\end{pmatrix}. \quad (A2)$$

and

$$G_v = \begin{pmatrix}
g_v & 0 & 0 & 0 & 0 & 0 \\
0 & g_v/2 & 0 & (-1)^{(v-1)}g_v/2 & 0 & 0 \\
0 & 0 & g_v & 0 & 0 & 0 \\
0 & (-1)^{(v-1)}g_v/2 & 0 & g_v/2 & 0 & 0
\end{pmatrix}. \quad (A3)$$

where $E_{N_1N_2} = \sum_{i=1,2} \sum_{j=3,4} \sum_{\alpha=1,2} (\varepsilon_i + \varepsilon_j + \omega_\alpha)$ are the energy of the state $|\psi_{B_\alpha, N_1N_2}\rangle$, the tunneling couplings are defined as $\Delta_\pm = \Delta_{34} \pm \Delta_{12}$ and $\delta_\pm = \delta_{12} \pm \delta_{34}$, with $\delta_{lm} = \varepsilon_l - \varepsilon_m$ ($l, m = 1, 3$ and $m = 2, 4$).

The first matrix resembles the rotated matrix on Bell basis, whose properties discussed in details on Ref. 28 and the matrices $G_v$ depends on $g_v$ and carry on the effect of electron-vibrational mode coupling, where the factor $\sqrt{N_2}$ appears on the specific elements of the matrix (A1) which depends on the values of $N_2$ of the coupled subspaces. If we consider a full resonance condition between electronic levels $\delta_\pm = 0$, the matrix $B_{N_1N_2}$ becomes

$$B_{N_1N_2} = \begin{pmatrix}
E_{N_1N_2} & 0 & 0 & 0 \\
0 & E_{N_1N_2} & 0 & 0 \\
0 & 0 & E_{N_1N_2} & \Delta_+ \\
0 & \Delta_+ & E_{N_1N_2}
\end{pmatrix}. \quad (A4)$$

At first sight, it seems that the states with electronic part being $|\Psi_-\rangle_D$ and $|\Phi_-\rangle_D$ are decoupled, at the same time that $|\Psi_+\rangle_D$ and $|\Phi_+\rangle_D$ are not, in the same way that in Ref. 28. Nevertheless, if elements for the first two lines on matrix (A1), associated with $|\Psi-, 00\rangle$ and $|\Psi-, 00\rangle$ respectively, are written using the notation $|\rangle \langle |$, we obtain:

$$H = |\Psi-, 00\rangle \left(E_{00} |\Psi-, 00\rangle + g_2 |\Psi-, 01\rangle + g_1 |\Psi-, 10\rangle \right)$$

$$+ |\Phi-, 00\rangle \left(E_{00} |\Phi-, 00\rangle + g_2/2 |\Phi-, 01\rangle - g_2/2 |\Phi-, 10\rangle \right)$$

$$+ g_1/2 |\Phi-, 10\rangle + g_1/2 |\Phi-, 10\rangle + \ldots \right) \quad (A5)$$

The term with $|\Psi_-\rangle_D$ can be written as $\left(|\Psi_-\rangle \langle \Psi_-|\right)_D \otimes (E_{00} |00\rangle \langle 00| + g_2 |00\rangle \langle 01| + g_1 |00\rangle \langle 10|)$, while the others do not permit the same. Continuing with the calculation, we realize that only the terms on Hamiltonian associated with $|\Psi_-\rangle_D$ are decoupled, at least from the electronic point of view, from the rest of the Bell basis. In this way, there is a Bell state, dressed by vibrational modes, becoming an eigenstate of the Hamiltonian (A1) for the specific condition of equal tunneling couplings, $\Delta_{34} = \Delta_{12}$ and full resonance between the electronic levels.

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Although not shown here, the second-order time-dependent perturbation theory can be used to show this transition.