Doubly Heavy Baryons Expanded in $1/m_Q$

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Starting from the semirelativistic Hamiltonian for a doubly heavy baryon system ($QQq$) with Coulomb and linear confining scalar potentials, and operating the naive Foldy-Wouthuysen-Tani transformation on two heavy quarks, I construct a formulation how to calculate mass spectra and wave functions of doubly heavy baryons. Based on our formulation, their masses and wave functions are expanded in $1/m_Q$ with a heavy quark mass $m_Q$ and the $\lambda$ mode lowest-order equation is examined carefully to obtain a complete set of eigenfunctions. I claim that the $\rho$ mode wave function should be given by other nonrelativistic methods, e.g., the Godfrey-Isgur model to give a complete wave function for a doubly heavy baryon.

KEYWORDS: heavy baryon, heavy quark symmetry, spectrum

1. Introduction

Heavy quark symmetry can be respected in hadrons in which at least one heavy quark is included and other quarks are light quarks. The simplest case is a heavy-light meson and the next will be a doubly heavy baryon $QQq$ because compared with $Qqq$ baryon, there is only one light quark involved. Treating two heavy quark is rather easier compared with two light quarks. The heavy-light meson has been extensively studied by us in Ref. [1] and also by many groups (see Ref. [2] for a review).

In this article, we will describe how to treat doubly heavy baryons using the method adopted in the former paper [1] where heavy-light mesons have been studied. Starting from the semirelativistic Hamiltonian for a doubly heavy baryon system ($QQq$) with Coulomb and linear confining scalar potentials, and operating the naive Foldy-Wouthuysen-Tani transformation on the heavy quarks, I expand all the physical quantities, Hamiltonian $H$, wave function $\psi$, and energy eigenvalue $E$, in $1/m_Q$ with heavy quark mass $m_Q$. However, as for the so-called $\rho$ mode wave function, we need to use other methods to calculate. Godfrey-Isgur relativised potential model is one choice. This is because by using the FWT transformation, we cannot obtain matrix structure for the $\rho$ mode Hamiltonian so that we cannot distinguish a couple of spin states of a heavy diquark.

2. Formulation

Using heaviness of $c$ or $b$ quarks compared with light quarks, $u, d,$ and $s$, we apply the method developed in Ref. [1] to a doubly heavy baryon. In Ref. [1], we have used the Foldy-Wouthuysen-Tani (FWT) transformation to expand the system in $1/m_Q$. Since there are two heavy quarks in a doubly heavy baryon, we need to operate two kinds of the FWT transformation on $QQq$.

Instead of using the Cartesian coordinates $\vec{r}_1, \vec{r}_2,$ and $\vec{r}_3$, we use the Jacobi relative coordinates $\vec{\lambda}$ and $\rho$. If we regard quarks 1 and 2 are heavy with mass $m_1 = m_2 = m_Q$ and quark 3 is light with...
\[ \rho = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2), \quad \lambda = \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3). \]  

(1)

After expansion, we obtain the lowest order eigenvalue equation for the \( \lambda \) mode wave function as,

\[ \tilde{H}_0 = -\vec{p}_\lambda \cdot \vec{a}_3 + m_q \beta_3 + 2V(\lambda') + 2\beta_3 S(\lambda'), \quad \tilde{H}_0 \psi_0 = E_0 \psi_0, \]

(2)

where \( \lambda' = \lambda/\sqrt{6} \), one-gluon exchange potential \( V(r) = -2\alpha_s/(3r) \), and confining linear potential \( S(r) = r/a^2 + b \). This equation expresses nothing but the one for interaction between a light quark and doubly heavy diquark, which is automatically derived from our formulation.

An angular part of a solution to Eq. (2) is explicitly given as follows. We define the following eigenfunctions for Eq. (2),

\[ y^k_{jm}(\Omega) = \frac{1}{\sqrt{2(j + 1)}} \left( \begin{array}{c} \sqrt{j + 1 - m} Y^m_{jm+1/2} \\ -\sqrt{j + 1 + m} Y^m_{jm+1/2} \end{array} \right), \]

\[ y^{-k}_{jm}(\Omega) = \frac{1}{\sqrt{2j}} \left( \begin{array}{c} \sqrt{j + 1 - m} Y^{-m}_{jm+1/2} \\ -\sqrt{j + 1 + m} Y^{-m}_{jm+1/2} \end{array} \right) = (\vec{\sigma} \cdot \vec{n}) y^k_{jm}(\Omega), \]

(3)

where \( Y^m_j \) are spherical harmonics, and \( k = j + 1/2 \). Here is a relation between \( k, l, \) and \( j \) as

\[ \text{when } l = k \pm \frac{1}{2}, \text{ then } k = \pm \left( j + \frac{1}{2} \right), \]

(5)

and \( k \) is an eigenvalue of an operator \( K = -\beta_q \left( \vec{S}_q \cdot \vec{L}_\lambda + 1 \right) \). Then a general solution to Eq. (2) is given by

\[ \psi^k_{jm} = \frac{1}{r} \left( \begin{array}{c} u_k(r) \\ -iv_k(r) \end{array} \right) y^k_{jm}(\Omega), \]

where functions \( u_k(r) \) and \( v_k(r) \) satisfy the following eigenvalue equation:

\[ \left( \begin{array}{cc} m_q + 2S + 2V & -\partial_r + \frac{k}{r} \\ -m_q - 2S + 2V & \partial_r + \frac{\lambda}{r} \end{array} \right) \left( \begin{array}{c} u_k(r) \\ v_k(r) \end{array} \right) = E_0^k \left( \begin{array}{c} u_k(r) \\ v_k(r) \end{array} \right), \]

(7)

which can be solved like in Ref. [1] using the variational method.

3. Physics related to heavy-light systems

1) Relation among \( L_\rho, s_\rho, \) and state symbols
Because the total wave function of a diquark should be antisymmetric for two heavy quarks, there is a relation between $L_\rho$ and $s_\rho$. First, I should mention that a diquark with the same two heavy quarks, $c$ or $b$, is flavor symmetric and color antisymmetric. As for a combination of two heavy quarks, if a diquark has spin $s_\rho = 0$, i.e., two heavy quarks have opposite spin directions, a spin wave function is antisymmetric. On the other hand, if a diquark has spin $s_\rho = 1$, a spin wave function is symmetric. When we denote spin as $s_\rho$, angular momentum as $L_\rho$, and parity as $P_\rho$ for a $\rho$ mode diquark, we have the following combinations,

\[
\begin{align*}
(s_\rho = 0, \quad L_\rho = 1, 3, \ldots, \quad P_\rho = -) \\
(s_\rho = 1, \quad L_\rho = 0, 2, \ldots, \quad P_\rho = +)
\end{align*}
\]  

(8) (9)

As you can see from these combinations, when the value of $L_\rho$ is given, the value of $s_\rho$ becomes unique. Together with these quantum numbers, we need to consider quantum numbers coming from the $D$ meson and quark interaction is given by the following combinations,

\[
\begin{align*}
\rho \text{ are equivalent but Ref. [3] lacks information on } 2) \text{ Threshold behaviors}
\end{align*}
\]

\[
\begin{align*}
\text{Ref.[2]} : \quad (N_\rho L_\rho n_\lambda l_\lambda) J^P, \\
\text{Ref.[3]} : \quad (N_\rho L_\rho n_\lambda l_\lambda) J^P, \\
\text{Ours} : \quad (N_\rho L_\rho n_\lambda l_\lambda k) J^P.
\end{align*}
\]

(10) (11) (12)

where $P = (-)^{L_\rho + L_\lambda}, k = \pm(j_\rho + 1)$, and the $\rho$ mode principal quantum number $N_\rho$. Ref. [4] and ours are equivalent but Ref. [3] lacks information on $\rho$ mode quantum numbers.

2) Threshold behaviors

Let us consider the similarity of doubly heavy baryons to heavy-light mesons, especially to $D_s(0^+, 1^+)$, which have very narrow widths. This is described as follows: Assume that $SU(3)$ light meson and quark interaction is given by

\[
\mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2} f_\pi} \bar{q}_i \gamma_\mu \gamma_5 \partial^\mu \phi_j q^i,
\]

(13)

\[
(\phi)_{ij} = \frac{\sqrt{2}}{\sqrt{1 + \epsilon^2}} \begin{pmatrix}
\phi_3 + \frac{1}{\sqrt{6}} \phi_8 & \pi^+ \\
\pi^- & \phi_3 + \frac{1}{\sqrt{6}} \phi_8 \\
K^- & K^0
\end{pmatrix},
\]

(14)

\[
\begin{pmatrix}
\pi^0 \\
\eta
\end{pmatrix} = \frac{1}{\sqrt{1 + \epsilon^2}} \begin{pmatrix}
1 & \epsilon \\
-\epsilon & 1
\end{pmatrix} \begin{pmatrix}
\phi_3 \\
\phi_8
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
\phi_3 \\
\phi_8
\end{pmatrix} = \frac{1}{\sqrt{1 + \epsilon^2}} \begin{pmatrix}
1 & -\epsilon \\
\epsilon & 1
\end{pmatrix} \begin{pmatrix}
\pi^0 \\
\eta
\end{pmatrix}.
\]

(15)

Then, since $s$ quark ($i, j = 3$) inside of $D_s$ couples to $\phi_8$ which is mixed with pion ($\pi^0$), heavy-light meson can decay into another heavy-light meson + $\pi^0$ with a small mixing parameter $\epsilon = 1.0 \times 10^{-2}$.

If $M(D_s(0^+)) > M(D(0^-)) + M(K)$, then we would expect a broad decay width of the $D_s(0^+)$. However, what we have found is the opposite situation so that the decay width of this state becomes very narrow because the allowed decay channel is $D_s(0^+) \rightarrow D_\lambda(0^-) + \pi$ which occurs through very small $\pi^0 - \eta$ coupling.

A similar process to this in doubly heavy baryons is given by, if $M(\Omega_{cc}(3/2^+)) < M(\Xi_{cc}(1/2^+)) + M(K),$

\[
\Omega_{cc}(3/2^+)^+(ccs) \rightarrow \Omega_{cc}(1/2^-)^+(ccs) + \pi^0.
\]

(16)

If $M(\Omega_{cc}(3/2^+)) > M(\Xi_{cc}(1/2^+)) + M(K)$, we expect a broad decay width due to the existence of the process,

\[
\Omega_{cc}(3/2^+)^+(ccs) \rightarrow \Xi_{cc}(1/2^-)^+(ccu) + K^-.
\]

(17)
where \( K \) has mass of 494–498 MeV, the quark content of \( \Xi_{cc} (1/2^+) \) is \( ccd \), while that of \( \Xi_{cc} (1/2^+)^++ \) is \( ccu \). Doubly heavy baryons have excitations in \( \rho \) mode, i.e., the first excitation is given by \( L_\rho L_\lambda = P_\rho S_\lambda \) with a total angular momentum \( J^P = 3/2^+ \) and the ground states are given by \( S_\rho S_\lambda \) with \( J^P = 1/2^- \).

2) Mixing angles in heavy-quark symmetry

In principle, there could be mixing between states with the same quantum number \( J^P \). For instance, there may be mixing between states with \( J^P = 1/2^- \), e.g., \((1S1p)1/2^-(j_\lambda = 1/2), (1S1p)1/2^-(j_\lambda = 3/2)\) and \((1P1s)1/2^-(j_\lambda = 1/2)\). We have checked whether there is mixing, e.g., between \((1S1p)1/2^-(j_\lambda = 1/2)\) and \((1P1s)1/2^-(j_\lambda = 3/2)\). It turns out that as long as we consider matrix elements with interactions of spin-spin and \( LS \) coupling, we obtain no mixing between them, i.e., there is mixing between the states with the same \( L_\lambda \) and different \( L_\rho \) but no mixing between \( S_\rho P_\lambda \) and \( P_\rho S_\lambda \). Because we use the heavy quark symmetry, we obtain mixing angles between certain states à la heavy-light mesons [5]. One example is given by,

\[
\begin{pmatrix}
(1S1p)1/2^- \\
(1S1p)1/2^-
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
(1S1p)1/2^- (j_\lambda = 1/2) \\
(1S1p)1/2^- (j_\lambda = 3/2)
\end{pmatrix},
\]

with \( \theta = \arctan(1/\sqrt{2}) = 35.3^\circ \) in the heavy quark limit.

4. Summary

In this article, I have described how one can obtain the \( \lambda \) mode wave function with an explicit angular part. Other physical quantities, good quantum number \( K \), state symbols, possible threshold behaviors, and mixing angles among states with the same quantum numbers, have been discussed.

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References

[1] T. Matsuki and T. Morii, Phys. Rev. D 56, 5646 (1997) doi:10.1103/PhysRevD.56.5646 [hep-ph/9702366]; T. Matsuki, T. Morii and K. Sudoh, Eur. Phys. J. A 31, 701 (2007) doi:10.1140/epja/i2006-10287-1 [hep-ph/0610186]; T. Matsuki, T. Morii and K. Sudoh, Prog. Theor. Phys. 117, 1077 (2007) doi:10.1143/PTP.117.1077 [hep-ph/0605019].

[2] E. S. Swanson, Phys. Rept. 429, 243 (2006) doi:10.1016/j.physrep.2006.04.003 [hep-ph/0601110].

[3] D. Ebert, R. N. Faustov, V. O. Galkin and A. P. Martynenko, Phys. Rev. D 66, 014008 (2002) doi:10.1103/PhysRevD.66.014008 [hep-ph/0201217].

[4] Q. F. Lü, K. L. Wang, L. Y. Xiao and X. H. Zhong, Phys. Rev. D 96, no. 11, 114006 (2017) doi:10.1103/PhysRevD.96.114006 [arXiv:1708.04468 [hep-ph]].

[5] T. Matsuki, T. Morii and K. Seo, Prog. Theor. Phys. 124, 285 (2010) doi:10.1143/PTP.124.285 [arXiv:1001.4248 [hep-ph]].