SPINS AND CHARGES, THE ALGEBRA AND SUBALGEBRAS OF THE GROUP SO(1,14) AND GRASSMANN SPACE

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ABSTRACT

In a space of $d = 15$ Grassmann coordinates, two types of generators of the Lorentz transformations, one of spinorial and the other of vectorial character, both linear operators in Grassmann space, forming the group $SO(1, 14)$ which contains as subgroups $SO(1, 4)$ and $SO(10) \supset SU(3) \times SU(2) \times U(1)$, define the fundamental and the adjoint representations of the group, respectively. The eigenvalues of the commuting operators can be identified with spins of fermionic and bosonic fields ($SO(1, 4)$), as well as with their Yang-Mills charges ($SU(3)$, $SU(2)$, $U(1)$), offering the unification of not only all Yang-Mills charges but of all the internal degrees of freedom of fermionic and bosonic fields - Yang-Mills charges and spins - and accordingly of all interactions - gauge fields and gravity. The theory suggests that elementary particles are either in the "spinorial" representations with respect to spins and all charges, or they are in the "vectorial" representations with respect to spins and all charges, which indeed is the case with the quarks, the leptons and the gauge bosons. The algebras of the two kinds of generators of Lorentz transformations in Grassmann space were studied and the representations are commented on.

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1. Introduction.

The fact that ordinary space-time is not enough to describe dynamics of particles, was recognized first in 1925 when in addition to the vector space, spanned over the ordinary coordinate space, a space of two vectors, called the internal space of the fermionic spin, was introduced in order to describe the spin ($\frac{1}{2}$) of fermions [1]. In 1928 this internal space was enlarged to four vectors, including the particle-antiparticle degrees of freedom in order to describe relativistic fermions [2]. In 1932 to the internal space of fermionic spins the space of two vectors describing the fermionic isospin, or how it is also called now - the weak charge of fermions, was added [3], and in 1964 to these two spaces the internal space of three vectors describing the colour charge of fermions was added [4]. The unification of electromagnetic and weak interactions makes clear that also the electromagnetic charge origins in the internal space [5, 6, 7].

All symmetries in physics are in theories connected with the appropriate groups. Spins are connected with the Lorentz group $SO(1, 3)$ (in the three dimensional subspace of the four dimensional space time spins are described by the group $SU(2)$), while charges are connected with the group $U(1)$ (the electromagnetic charge), the group $SU(2)$ (the weak charge) and the group $SU(3)$ (the colour charge).

While in ordinary space time the realization of vectorial types of representations for the Lorentz group (integer angular momenta) only are possible, for the internal spaces two types of representations are required: the fundamental and the adjoint. The fundamental representations are used to describe the above mentioned properties of fermions, the adjoint representations are used to describe the corresponding properties of bosons - the gauge vector fields.

To both types of representations two types of singlets have to be added.

Modern theories: the Standard Electroweak Model [7, 6, 5], the Grand Unification Theories [8], the String Theories [9], the Kaluza-Klein Theories [10], the Technicolour Models [11], the Supersymmetric Theories try to unify the internal spaces of charges, but they do not unify the internal spaces of charges with the internal space of spins.

This paper elaborates the idea [13, 14] of unifying all the internal degrees of freedom - spins and charges.

The space of $d$ ordinary commuting and $d$ Grassmann anticommuting coordinates, $d \geq 15$, offers the possibility of describing all degrees of freedom of particles which today are supposed to be elementary. All internal degrees of freedom are in this space described as the dynamics in the Grassmann part of space.

Two kinds of generators of the Lorentz transformations can be defined in Grassmann space: one of spinorial character (spinorial operators,) the other of vectorial character (vectorial operators.) For $d = 15$, contain the representations, if determined by the operators forming the subgroups $SO(1, 3), SU(3), SU(2), U(1)$ of the group $SO(1, 14), all the vectors needed to describe quarks, leptons and all gauge fields. Spinorial representations, including fundamental representations and singlets, describe spins and charges of
fermions, while vectorial representations, including adjoint representations and singlets, describe spins and charges of bosons.

*Spins and charges are in the proposed theory unified.* As the consequence, in this theory elementary particles should either be in the representations, defined by the spinorial operators of the Lorentz group $SO(1,3)$ and all the groups describing charges, or they should be in the representations, defined by the vectorial operators of the Lorentz group and all the groups describing charges.

This is in agreement with the properties of known quarks, leptons and gauge bosons since they are either fermions in the fundamental representations with respect to the Lorentz group and the groups describing charges or they are singlets in the adjoint representations with respect to the Lorentz group and the groups describing charges or they are singlets with respect to these groups.

The Higgs’s boson of the Standard Electroweak Model can in the proposed approach be described as a weak doublet or a constituent field, made of two fermions, one singlet and one doublet with respect to the group $SU(2)$ describing the weak charge.

The supersymmetric partners of weak bosons and gluons, for example, suggested by the minimal supersymmetric extension of the Standard Electroweak Model [15] to be fermions in the adjoint representations with respect to the groups describing charges, can appear in the proposed theory as constituent particles only.

The purpose of this article is to show that generators of the Lorentz transformations in $d = 2n + 1$ dimensional Grassmann space, forming the Lie algebra of the Lorentz group $SO(1,2n)$ with the subalgebra $SO(2n - 4) \times SO(1,4)$, define for the choice $n=7$ the Lorentz subalgebras $SO(1,4)$ and $SO(10)$. Generators of $SO(1,4)$, if having spinorial character, determine spinors or, if having vectorial character, determine scalars and vectors in the four dimensional subspace. Generators of $SO(10)$ subalgebra, being decomposed into generators of $SU(3), SU(2)$ and $U(1)$, accordingly determine colour, weak and electromagnetic charges for spinors or for scalars and vectors.

We study the decompositions of generators of subgroups in terms of generators of the Lorentz transformations of spinorial and vectorial character.

All the generators are differential operators in Grassmann space. We look for Casimir operators of subgroups. Solving the eigenvalue problem for commuting operators, we look for representations of groups and subgroups, defined by both kinds of generators of the Lorentz transformations in Grassmann space, separately, and comment on them.

The theory offers a way of looking for algebras, subalgebras and their representations. It contains a parallel method to Dynkin diagrams and Young tableaux.

2. Coordinate Grassmann space and linear operators.

In this section we briefly repeat a few definitions concerning a $d$-dimensional Grassmann space, linear Grassmann space spanned over the coordinate space, linear operators defined in this space and the Lie algebra of generators of the Lorentz transformations [16, 17].
2.1. Coordinate space with Grassmann character

We define a $d$-dimensional Grassmann space of real anticommuting coordinates \{\(\theta^a\)\}, \(\theta^{a*} = \theta^a\), \(a = 0, 1, 2, 3, 5, 6, \ldots, d\), satisfying the anticommutation relations

\[\theta^a \theta^b + \theta^b \theta^a := \{\theta^a, \theta^b\} = 0, \quad (2.1)\]

called the Grassmann algebra \[14, 16\]. The metric tensor \(\eta_{ab} = \text{diag}(1, -1, -1, -1, \ldots, -1)\) lowers the indices of a vector \(\{\theta^a\} = \{\theta^0, \theta^1, \ldots, \theta^d\}\), \(\theta^a = \eta_{ab} \theta^b\). Linear transformation actions on vectors \((\alpha \theta^a + \beta x^a)\)

\[(\alpha \theta^a + \beta x^a) = L^a_b (\alpha \theta^b + \beta x^b), \quad (2.2)\]

which leave forms

\[(\alpha \theta^a + \beta x^a)(\alpha \theta^b + \beta x^b)\eta_{ab} \quad (2.3)\]

invariant, are called the Lorentz transformations. Here \((\alpha \theta^a + \beta x^a)\) is a vector of $d$ anticommuting components (Eq.(2.1)) and $d$ commuting \((x^a x^b - x^b x^a = 0)\) components, and $\alpha$ and $\beta$ are two complex numbers. The requirement that forms (2.3) are scalars with respect to the linear transformations (2.2), leads to the equations

\[L^a_c L^b_d \eta_{ab} = \eta_{cd}. \quad (2.4)\]

2.2 Linear vector space.

A linear space spanned over a Grassmann coordinate space of $d$ coordinates has the dimension $2^d$. If monomials \(\theta^{a_1} \theta^{a_2} \ldots \theta^{a_m}\) are taken as a set of basic vectors with \(a_m \neq a_j\), half of the vectors have an even (those with an even $m$) and half of the vectors have an odd (those with an odd $n$) Grassmann character. Any vector in this space may be represented as a linear superposition of monomials

\[f(\theta) = \alpha_0 + \sum_{i=1}^{d} \alpha_{a_1 a_2 \ldots a_i} \theta^{a_1} \theta^{a_2} \ldots \theta^{a_i}, \quad a_k < a_{k+1}, \quad (2.5)\]

where constants \(\alpha_0, \alpha_{a_1 a_2 \ldots a_i}\) are complex numbers.

2.3 Linear operators.

In Grassmann space the left derivatives have to be distinguished from the right derivatives, due to the anticommuting nature of the coordinates \[13, 14, 16\]. We shall make use of left derivatives \(\overset{\leftarrow}{\partial}_{a} := \frac{\partial}{\partial \theta^a}\), \(\overset{\leftarrow}{\partial}^a := \eta^{ab} \overset{\leftarrow}{\partial}_{b}\), on vectors of the linear space of monomials \(f(\theta)\), defined as follows:

\[\overset{\leftarrow}{\partial}^b_a \theta^b f(\theta) := \delta^b_a f(\theta) - \theta^b \overset{\leftarrow}{\partial}_{a} f(\theta), \quad (2.6)\]
Here $\alpha$ is a constant of either commuting ($\alpha \theta - \theta \alpha = 0$) or anticommuting ($\alpha \theta + \theta \alpha = 0$) character, and

\[ n_{AB} = \begin{cases} +1, & \text{if } A \text{ and } B \text{ have Grassmann odd character} \\ 0, & \text{otherwise.} \end{cases} \]

We define the following linear operators \cite{13, 14}

\[ p^{\theta} = -i \overrightarrow{\partial^{\theta}} \quad \tilde{a} = i(p^{\theta} - i \theta^a), \quad \tilde{a}^a = -(p^{\theta} + i \theta^a). \] (2.7)

If the inner product is defined as in Sect. (4.1.) it follows

\[ \theta^a + = i\eta^{aa} p^\theta, \quad p^\theta + = i\eta^{aa} \theta^a, \quad \tilde{a} + = \eta^{aa} \tilde{a}^a, \quad \tilde{a}^a + = \eta^{aa} \tilde{a}. \] (2.7a)

We define the generalized commutation relations \cite{13, 14}:

\[ \{ A, B \} := AB - (-1)^{n_{AB}} BA, \] (2.8)

fulfilling the equations

\[ \{ A, B \} = (-1)^{n_{AB}+1} \{ B, A \}, \] (2.9a)

\[ \{ A, BC \} = \{ A, B \} C + (-1)^{n_{AB}} B \{ A, C \}, \] (2.9b)

\[ \{ AB, C \} = A \{ B, C \} + (-1)^{n_{BC}} \{ A, C \} B, \] (2.9c)

\[ (-1)^{n_{AC}} \{ A, \{ B, C \} \} + (-1)^{n_{CB}} \{ C, \{ A, B \} \} + (-1)^{n_{BA}} \{ B, \{ C, A \} \} = 0. \] (2.9d)

We find

\[ \{ p^{\theta a}, p^{\theta b} \} = 0 = \{ \theta^a, \theta^b \}, \] (2.10a)

\[ \{ p^{\theta a}, \theta^b \} = -i \eta^{ab}, \] (2.10b)

\[ \{ \tilde{a}^a, \tilde{a}^b \} = 2 \eta^{ab} = \{ \tilde{a}^a, \tilde{a}^b \}, \] (2.10c)

\[ \{ \tilde{a}^a, \tilde{a}^b \} = 0. \] (2.10d)

We see that $\theta^a$ and $p^{\theta a}$ form a Grassmann odd Heisenberg algebra, while $\tilde{a}^a$ and $\tilde{a}^a$ form the Clifford algebra.
3. Lie algebra of generators of Lorentz transformations.

We define two kinds of operators \[13, 14\]. The first ones are binomials of operators forming the Grassmann odd Heisenberg algebra

\[S_{ab} := (\theta^a p^b - \theta^b p^a). \tag{3.1a}\]

The second kind are binomials of operators forming the Clifford algebra

\[\tilde{S}_{ab} := -\frac{i}{4}[\tilde{a}^a, \tilde{a}^b], \quad \tilde{S}_{ab} := -\frac{i}{4}[\tilde{a}^a, \tilde{a}^b], \tag{3.1b}\]

with \([A, B] := AB - BA\).

Either \(S_{ab}\) or \(\tilde{S}_{ab}\) or \(\tilde{\tilde{S}}_{ab}\) fulfil the Lie algebra of the Lorentz group \(SO(1, d - 1)\) in the \(d\)-dimensional Grassmann space:

\[\{M_{ab}, M_{cd}\} = -i(M_{ad} \eta_{bc} + M_{bc} \eta_{ad} - M_{ac} \eta_{bd} - M_{bd} \eta_{ac}) \tag{3.2}\]

with \(M_{ab}\) equal either to \(S_{ab}\) or to \(\tilde{S}_{ab}\) or to \(\tilde{\tilde{S}}_{ab}\) and \(M_{ab} = -M_{ba}\). There are \(d(d - 1)/2\) operators of each kind in \(d\)-dimensional Grassmann space.

We see that

\[S_{ab} = \tilde{S}_{ab} + \tilde{\tilde{S}}_{ab}, \quad \{\tilde{S}_{ab}, \tilde{S}_{cd}\} = 0 \quad \{\tilde{\tilde{S}}_{ab}, \tilde{\tilde{S}}_{cd}\} = \{\tilde{a}^a, \tilde{\tilde{a}}^b\}. \tag{3.3}\]

The operators \(\tilde{S}_{ab}\), as well as the operators \(\tilde{\tilde{S}}_{ab}\), define what we call the spinorial representations of the Lorentz group \(SO(1, d - 1)\) and of subgroups of this Lorentz group, while \(S_{ab} = \tilde{S}_{ab} + \tilde{\tilde{S}}_{ab}\) define what we call the vectorial representations of the Lorentz group \(SO(1, d - 1)\) and of subgroups of this group.\[13, 14\] The spinorial representations contain what is called the fundamental representations of this Lorentz group and of subgroups and they contain also singlets, while the vectorial representations contain what is called the regular or the adjoint representations of this Lorentz group and of subgroups and they contain also singlets. Group elements are in any of the three cases defined by:

\[U(\omega) = e^{\frac{i}{2} \omega_{ab} M_{ab}}, \tag{3.4}\]

where \(\omega_{ab}\) are the parameters of the group.

Linear transformations, defined in Eq.(2.2), can then be written in terms of group elements as follows

\[\dot{\theta}^a = L^a_b \theta^b = e^{-\frac{i}{2} \omega_{cd} S^{cd}} \theta^a e^{\frac{i}{2} \omega_{cd} S^{cd}}, \tag{3.5a}\]

\[\dot{\tilde{a}}^a = L^a_b \tilde{a}^b = e^{-\frac{i}{2} \omega_{cd} \tilde{S}^{cd}} \tilde{a}^a e^{\frac{i}{2} \omega_{cd} \tilde{S}^{cd}}, \tag{3.5b}\]
\[
\hat{a}^a = L^a_{\hat{b}} \hat{a}^\hat{b} = e^{-\frac{i}{2} \omega_{cd} S^{cd}} \hat{a}^a e^{\frac{i}{2} \omega_{cd} S^{cd}},
\]

(3.5c)

where \( \omega_{cd} \) are the parameters of the transformations. Since \( \{ \tilde{S}^{ab}, \tilde{S}^{cd} \} = 0 \), in Eq.(3.5b) \( S^{cd} \) can be replaced by \( \tilde{S}^{cd} \) and in Eq.(3.5c) \( S^{cd} \) can be replaced by \( \tilde{S}^{cd} \).

By using Eqs.(2.9) and (3.2) it can be proved for any \( d \), that \( M^2 \) is the invariant of the Lorentz group

\[
\{ M^2, M^{cd} \} = 0, \quad M^2 = \frac{1}{2} M_{ab} M_{ab},
\]

(3.6)

and that for \( d=2n \) we can find the additional invariant \( \Gamma \)

\[
\{ \Gamma, M^{cd} \} = 0, \quad \Gamma = \frac{i(-2i)^n}{(2n)!} \epsilon_{a_1 a_2 \ldots a_{2n}} M^{a_1 a_2 \ldots a_{2n-1} a_{2n}},
\]

(3.7)

where \( \epsilon_{a_1 a_2 \ldots a_{2n}} \) is the totally antisymmetric tensor with \( 2n \) indices and with \( \epsilon_{123\ldots 2n} = 1 \). This means that \( M^2 \) and \( \Gamma \) are for \( d = 2n \) the two invariants or Casimir operators of the group \( SO(d) \) (or \( SO(1, d - 1) \), the two algebras differ in the definition of the metric \( \eta^{ab} \)). For \( d = 2n + 1 \) the second invariant cannot be defined. (However, for \( d = 2n + 1 \) one can still define the invariants \( \tilde{\Gamma} = \prod_{a=1,\ldots,d} \sqrt{\eta^{aa}} \tilde{a}^a \) and \( \tilde{\tilde{\Gamma}} = \prod_{a=1,\ldots,d} \sqrt{\eta^{aa}} \tilde{\tilde{a}}^a \), which commute with \( \tilde{S}^{ab} \) and \( \tilde{\tilde{S}}^{ab} \)).

While the invariant \( M^2 \) is trivial in the case when \( M_{ab} \) has spinorial character, since \( (\tilde{S}^{ab})^2 = \frac{1}{4} \eta^{pa} \eta^{pb} = (\tilde{S}^{ab})^2 \) and therefore \( M^2 \) is equal in both cases to the number \( \frac{1}{2} \tilde{S}^{ab} \tilde{S}_{ab} = \frac{1}{2} \tilde{S}^{ab} \tilde{S}_{ab} = d(d - 1) \frac{1}{2} \), it is a nontrivial differential operator in Grassmann space if \( M_{ab} \) have vectorial character \( (M_{ab} = S^{ab}) \). The invariant of Eq.(3.7) is always a nontrivial operator.

We shall discuss the representations of groups in Sect.4.

3.1. Algebras of subgroups of the Lorentz groups

In this section we shall present some subalgebras of the algebras of the groups \( SO(d) \) or \( SO(1, d - 1) \) for a few chosen \( d \). We shall choose subalgebras appropriate for the description of the internal degrees of freedom of fermionic and bosonic fields, that is for spins and known Yang-Mills charges.

We shall present operators forming the desired subalgebras in terms of generators \( M_{ab} \), which have either the spinorial character - \( \tilde{S}^{ab}, \tilde{\tilde{S}}^{ab} \) - and define spinorial representations, or they have the vectorial character - \( S^{ab} \) - and define vectorial representations of the group. We shall analyse subalgebras of \( SO(d) \) or \( SO(1, d - 1) \) for \( d = 4, 5, 6, 8, 15 \). In the case of the group \( SO(d) \) the matrix \( \eta^{ab} \) has all the diagonal elements equal to -1.

It is selfevident that the algebra \( SO(1, d - 1) \) or \( SO(d) \) contains \([13, 14]\) \( N \) subalgebras defined by operators \( \tau^{\alpha i}, \alpha = 1, N, \quad i = 1, N_{\alpha} \), where \( N_{\alpha} \) is the number of elements of each subalgebra, with the properties
\[ [\tau^{\alpha i}, \tau^{\beta j}] = i\delta^{\alpha\beta} f^{\alpha ij k} \tau^{\alpha k}, \]  

(3.8)

If operators \( \tau^{\alpha i} \) can be expressed as linear superpositions of operators \( M^{ab} \)

\[ \tau^{\alpha i} = c^{\alpha i}_{ab} M^{ab}, \quad c^{\alpha i}_{ab} = -c^{\alpha i}_{ba}, \quad \alpha = 1, N, \quad i = 1, N, \quad a, b = 1, d. \]  

(3.8a)

Here \( f^{\alpha ij k} \) are structure constants of the \((\alpha)\) subgroup with \( N \) operators. According to the three kinds of operators \( M^{ab} \), two of spinorial and one of vectorial character, there are three kinds of operators \( \tau^{\alpha i} \) defining subalgebras of spinorial and vectorial character, respectively. All three kinds of operators are, according to Eq.(3.8), defined by the same coefficients \( c^{\alpha i}_{ab} \) and the same structure constants \( f^{\alpha ij k} \). From Eq.(3.8) the following relations among constants \( c^{\alpha i}_{ab} \) follow:

\[ -4c^{\alpha i}_{ab} c^{\beta j b} c^{\gamma k a} - \delta^{\alpha\beta} f^{\alpha ij k} c^{\alpha k}_{ac} = 0. \]  

(3.8b)

In the case when the algebra and the chosen subalgebras are isomorphic, that is if the number of generators of subalgebras is equal to \( \frac{d(d-1)}{2} \), the inverse matrix \( e^{\alpha i}_{ab} \to \) the matrix of coefficients \( c^{\alpha i}_{ab} \) exists \[14\]

\[ M^{ab} = \sum_{\alpha i} e^{\alpha i}_{ab} \tau^{\alpha i}, \]  

(3.8c)

with the properties \( c^{\alpha i}_{ab} e^{\beta j ab} = \delta^{\alpha\beta} \delta^{ij} \), \( c^{\alpha i}_{cd} c^{\alpha i}_{ab} = \delta^{\alpha c} \delta^{db} - \delta^{\beta c} \delta^{ad} \).

When we look for coefficients \( c^{\alpha i}_{ab} \) which express operators \( \tau^{\alpha i} \), forming a subalgebra \( SU(n) \) of an algebra \( SO(2n) \) in terms of \( M^{ab} \), the procedure is rather simple. For spinorial representations we define Grassmann odd operators \( \tilde{b}^i \) and their hermitian conjugate (Eq.(2.7b)) \( \tilde{b}^i + \)

\[ \tilde{b}^i = \frac{1}{2}(\tilde{a}^{(2i-1)} - i\tilde{a}^{(2i)}), \quad \tilde{b}^i + = \frac{1}{2}(\tilde{a}^{(2i-1)} + i\tilde{a}^{(2i)}). \]  

(3.9a)

We take the traceless matrices \((\tilde{\sigma}^{am})_{ij}\) which form the algebra of \( SU(n) \): \( (\tilde{\sigma}^{am})_{ij}(\tilde{\sigma}^{am})_{jk} = \delta^{am} \delta^{ij} \). Here \( \sigma^{amnl} \) are structure constants of the group \( SU(n) \). We then construct operators \( \tilde{\tau}^{am} \) as follows

\[ \tilde{\tau}^{am} = (\tilde{b}^i)^+(\tilde{\sigma}^{am})_{jk} \tilde{b}^k = \]

\[ \frac{1}{4}(\tilde{\sigma}^{am})_{jk}(2\eta_{jk} + \tilde{a}^{+(2j-1)}\tilde{a}^{(2k-1)} + \tilde{a}^{+(2j)}\tilde{a}^{(2k)} + i\tilde{a}^{+(2j)}\tilde{a}^{(2k-1)} - i\tilde{a}^{+(2j-1)}\tilde{a}^{(2k)}), \]

\[ \alpha = \{1, N\}, \quad m = \{1, n^2 - 1\}, \quad \{j, k\} = \{1, n\}. \]  

(3.9b)

Since \( \tilde{a}^{a+} = \eta^{aa} \tilde{a}^a \) (See Eq.(2.7b)) and \( (\tilde{\sigma}^{am})_{jk} \) are traceless matrices, we find for the group \( SO(2n) \):
\[ \tau^{\alpha m} = -\frac{i}{2}(\bar{\sigma}^{\alpha m})_{jk}\{M^{(2j-1)(2k-1)} + M^{(2j)(2k)} + iM^{(2j)(2k-1)} - iM^{(2j-1)(2k)}\}. \quad (3.9c) \]

One can easily prove, if taking into account Eq.(3.2), that operators \( \tau^{\alpha m} \) fulfil the algebra of the group \( SU(n) \) (Eq.(3.8)), for any of three choices for operators \( M^{ab} : S^{ab}, \tilde{S}^{ab}, \tilde{\tilde{S}}^{ab} \). We have generalized expressions for operators \( \tilde{\tau}^{\alpha m} \), which have spinorial character, to all three kinds of operators. Eq. (3.9c) can be found in ref. \[18\]. (Instead of Grassmann odd operators \( \tilde{b}^a \) which are written in terms of Grassmann odd operators \( \tilde{a}^a \), we could define Grassmann even operators in terms of \( \tilde{\gamma}^a \) operators which we shall introduce in the next subsection. Then in a similar way as for spinorial operators \( \tilde{\tau}^{\alpha m} \), the procedure to find \( \tau^{\alpha m} \) for vectorial operators can be found.)

In this section we shall present coefficients \( c^{\alpha i ab} \) for a few cases which seem to be interesting for particle physics. As have we already said, the coefficients are the same for all three kinds of operators, two of spinorial and one of vectorial character. The representations, of course, depend on the operators.

We shall use the notation \( \tilde{\tau}^{\alpha i} \) or \( \tilde{\tilde{\tau}}^{\alpha i} \) for operators \( \tau^{\alpha i} \) (Eq.(3.8a)) when they have spinorial character (in such a case operators \( M^{ab} \) have to be replaced by the corresponding operators of spinorial character \( \tilde{S}^{ab} \) or \( \tilde{\tilde{S}}^{ab} \)). We shall use, however, the same notation as in the general case \( \tau^{\alpha i} \), if operators \( M^{ab} \) describe vectorial character and have therefore to be expressed in terms of \( S^{ab} \).

3.1.1. Subgroups of \( SO(1,3) \)

This problem is discussed for spinorial degrees of freedom in many textbooks \[18, 17, 19, 20\]. We shall follow here ref.\[14\] where the algebra of \( SO(1,3) \) is presented together with the corresponding irreducible representations for operators of spinorial and vectorial character in Grassmann space.

There are six generators \( (d(d-1)/2) \) of the group \( SO(1,3) \) each of three kinds (\( M^{ab} \) is either \( S^{ab} \) or \( \tilde{S}^{ab} \) or \( \tilde{\tilde{S}}^{ab} \)). We find according to Eqs.( 3.6, 3.7) two invariants of the group: \( M^2 := \frac{1}{2}M^{ab}M_{ab} \) and \( \Gamma := -\frac{i}{3!}\epsilon_{abcd}M^{ab}M^{cd} \). For \( M^{ab} = \tilde{S}^{ab} \) or \( M^{ab} = \tilde{\tilde{S}}^{ab} \) the first invariant is the number \( \frac{6}{4}(\tilde{S}^{ab})^2 = \frac{1}{4}\eta^{aa}\eta^{bb} = (\tilde{\tilde{S}}^{ab})^2 \), while for \( M^{ab} = S^{ab} \) it is a nontrivial differential operator in Grassmann space. The second invariant is the nontrivial operator in any of the three cases.

If \( M^{ab} \) is equal to either \( \tilde{S}^{ab} \) or to \( \tilde{\tilde{S}}^{ab} \), the second invariant \( \Gamma \) can be recognized as the operator of chirality, the product of the Dirac \( \gamma^a \) matrices \[14, 20\]. To see this one should write the operator \( \Gamma \) in terms of \( \tilde{a}^a \) or \( \tilde{\tilde{a}}^a \). One finds the product of all operators \( \tilde{\tilde{a}}^a \): \( \tilde{\Gamma} = i\tilde{\tilde{a}}^a\tilde{\tilde{a}}^1\tilde{\tilde{a}}^2\tilde{\tilde{a}}^3 \) or equivalently \( \tilde{\Gamma} = i\tilde{\tilde{a}}^0\tilde{\tilde{a}}^1\tilde{\tilde{a}}^2\tilde{\tilde{a}}^3 \).

Operators \( \tilde{a}^a \) and \( \tilde{\tilde{a}}^a \) are Grassmann odd operators. Operating on spinors they would change spinors to vectors or scalars, changing their Grassmann character from odd to even \[13, 14\]. They cannot, therefore, be recognized as the Dirac \( \gamma^a \) matrices. In the next
subsection we shall find it meaningful to define $-2i\tilde{S}^{5a} = -\tilde{a}^5a^a$ ($-2i\tilde{S}^{5a} = -\tilde{a}^5a^a$) as operators corresponding to the Dirac $\gamma^a$ matrices. Since according to Eq.(2.10c) $(\tilde{a}^5)^2 = -1 = (\tilde{a}^5)^2$, the product of all $\tilde{a}^a$ is equal to the product of the corresponding $\tilde{\gamma}^a$ defining the same invariant $\tilde{\Gamma}$ and equivalently for $\tilde{\Gamma}$.

The corresponding representations will be discussed in Sect.4.

It is easy to find $SO(3)$ as a subgroup of the group $SO(1, 3)$ with three generators $M_{12}, M_{13}, M_{23}$ closing the Lie subalgebra with the properties

$$\tau^{1i} = \frac{1}{2} \epsilon^{ijk} M^{jk}, \quad i, j, k \in \{1, 2, 3\}, \quad (3.10)$$

where $f^{1ijk} = \epsilon^{ijk}$. In this case we see, by comparing Eqs.(3.8a) and (3.10), that

$$c^{1i}_{ab} = \begin{cases} \frac{1}{2} \epsilon_{ab}, & \text{if } a \text{ and } b \in \{1, 2, 3\} \\ 0, & \text{otherwise} \end{cases} \quad (3.10a)$$

The only invariant of the subgroup is $\frac{1}{2} M^{ab} M_{ab}$, which is equal to $\frac{3}{4}$ in the case that $M^{ij} = \tilde{S}^{ij}(\tilde{S}^{ij})$ and is a nontrivial operator if $M^{ij} = S^{ij}$. The two Casimir operators of the group $SO(1, 3)$ are of course the Casimir operators of the subgroup $SO(3)$ as well.

Requiring that two kinds of operators $\tau^{ai}$ exist, $\alpha = 1, 2; \; i = 1, 2$, with the structure constants $f^{aijk} = \epsilon^{ijk}; \; \alpha = 1, 2; \; i, j, k = 1, 2, 3$, one easily finds that

$$c^{ai}_{ab} = \begin{cases} \frac{1}{4} \epsilon_{ab}, & \text{if } a \text{ and } b \in \{1, 2, 3\} \\ (-1)^{\alpha+1} \frac{1}{4}, & \text{if } a = 0 \\ (-1)^{\alpha} \frac{1}{4}, & \text{if } b = 0 \end{cases} \quad (3.10b)$$

Since the two subalgebras are isomorphic to the algebra of $SO(1, 3)$, the matrix $c^{ai}_{ab}$ has the inverse matrix $e^{ai}_{ab}$ (eq.(3.8c))

$$e^{ai}_{ab} = \begin{cases} \epsilon^{ab}, & \text{if } a \text{ and } b \in \{1, 2, 3\} \\ i(-1)^{\alpha}, & \text{if } a = 0 \\ i(-1)^{\alpha+1}, & \text{if } b = 0 \end{cases}, \quad \alpha = 1, 2. \quad (3.10c)$$

We may verify that Eqs.(3.8) are fulfilled and also that the two Casimir operators of the two subgroups are expressible by the two Casimir operators of the group $SO(1, 3)$: $M^2 = \frac{1}{2} M^{ab} M_{ab}$ and $\Gamma = \frac{4\alpha}{3\pi} \epsilon_{abcd} M^{ab} M^{cd}$

$$(\tau^a)^2 = \sum_{i=1}^{3} (\tau^{ai})^2 = \frac{1}{4} \left[ M^2 + (-1)^{\alpha} \frac{3}{2} \Gamma \right], \; \alpha = 1, 2, \quad (3.10d)$$

and oppositely. These two invariant subalgebras are known as the $SU(2) \times SU(2)$ structure of the Lorentz group $SO(1, 3)$. While either generators of spinorial or of vectorial character fulfil the same algebra, they certainly have different representations, which we shall show in Sect.4.
3.1.2. Subgroups of $SO(1,4)$.

The interesting subalgebra of $SO(1,4)$ is $SO(1,3)$ discussed in the previous subsection, since it can be used to describe spins of spinor, vector and scalar fields, that is properties of fields defined in the four dimensional part of the space. If we choose $M^{ab}$ with $a, b \in \{0, 1, 2, 3\}$, to form the subset of generators closing the algebra of $SO(1,3)$, it is meaningful to interpret the remaining generators of the group $SO(1,4)$, that is $M^a_{5a}$, $a \in \{0, 1, 2, 3\}$, which do not form a subgroup, by a special name $\gamma^a = -2iM^a_{5a}$. We showed in ref. [14] that in the case of generators of spinorial character $\tilde{\gamma}^a = -2i\tilde{S}^a_{5a}$ or $\tilde{\tilde{\gamma}}^a = -2i\tilde{\tilde{S}}^a_{5a}$, these generators may be recognized as the Dirac $\gamma^a$ matrices, with all the desired properties.

It is easy to show that the two invariants of the subalgebra $SO(1,3)$, defined in the previous subsection (we shall call them $M^{(4)2}$ and $\Gamma^{(4)}$) can be written as:

$$M^{(4)2} = \frac{1}{2} M^{ab}M_{ab}, \quad \Gamma^{(4)} = \frac{1}{4!} \epsilon_{abcd}M^{ab}M^{cd} = \frac{1}{4!} \epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d,$$

$$a, b, c, d \in \{0, 1, 2, 3\}.$$

This is valid for either of three cases: for $M^{ab} = S^{ab}$ or $M^{ab} = S_{\tilde{5}}^{ab}$ or $M^{ab} = S_{\tilde{\tilde{5}}}^{ab}$. Since $\{M^{ab}, \gamma^d\} \neq 0$, $SO(1,3)$ is not the invariant subalgebra of $SO(1,4)$.

While $\frac{1}{2} M^{ab}M_{ab}$, $ab \in \{0, 1, 2, 3, 5\}$ is the invariant of the group $SO(1,4)$, as well as of the subgroup $SO(1,3)$, the second invariant $\Gamma$ of the group $SO(1,4)$ cannot be defined since $d$ is odd.

3.1.3. Subgroups of $SO(6)$.

The algebra of this group is isomorphic with the algebra of the group $SU(4)$, since both have the same number of generators and the generators of $SU(4)$ may, according to Eqs.(3.8) and Eqs.(3.9), be expressed as superpositions of $M^{ab}$, with the structure constants $f^{ijk}_{1}$ presented in ref. [20].

This group contains as a subgroup the group $SU(3)$. The structure constants of the $SU(3)$ algebra which can be found in several text-books, are presented in Table I, while the coefficients $c^{1i}_{ab}$, defining the generators of the subalgebra in terms of the Lorentz generators $M^{ab}$, are presented in Table II.

| \(i\) | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 6 |
|-----|---|---|---|---|---|---|---|---|---|---|
| \(j\) | 2 | 4 | 5 | 4 | 5 | 4 | 6 | 5 | 7 |
| \(k\) | 3 | 7 | 6 | 6 | 7 | 5 | 7 | 8 | 8 |
| \(f^{1ijk}_{1}\) | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

Table I: Structure constants $f^{1ijk}_{1}$ of the group $SU(3)$. The coefficients $f^{1ijk}_{1}$ are antisymmetric with respect to indices i,j,k.
Table II: Table of coefficients $c^{il}_{ab}$ which determine the generators of the subalgebra $SU(3)$ in terms of the generators $M^{ab}$ forming the Lorentz algebra of $SO(6)$ (Eq. (3.8a)). The coefficients have the property $c^{il}_{ab} = -c^{il}_{ba}$, $i \in \{1, 8\}$. Only nonvanishing coefficients are presented.

We may write, according to Eq.(3.8a) and Table II,

$$
\tau_1 := \frac{1}{2} (M_{14} - M_{23}), \quad \tau_2 := \frac{1}{2} (M_{13} + M_{24}), \quad \tau_3 := \frac{1}{2} (M_{12} - M_{34}),
$$
$$
\tau_4 := \frac{1}{2} (M_{16} - M_{25}), \quad \tau_5 := \frac{1}{2} (M_{15} + M_{26}), \quad \tau_6 := \frac{1}{2} (M_{36} - M_{45}),
$$
$$
\tau_7 := \frac{1}{2} (M_{35} + M_{46}), \quad \tau_8 := \frac{1}{2\sqrt{3}} (M_{12} + M_{34} - 2M_{56}). \quad (3.11)
$$

The two invariants of the group $SO(6)$: $M^2 = \frac{1}{2} M^{ab} M_{ab}$ and $\Gamma = \frac{i(-2i)^3}{6!} \epsilon_{abcdef} M^{ab} M^{cd} M^{ef}$ are at the same time the two invariants of the subgroup $SU(3)$. The two Casimir operators of the group $SU(3)$ [20] $C^1 = \frac{-2i}{3} \sum_{i,j,k} f^{i j k} \tau_i \tau_j \tau_k$ and $C^2 = \sum_{i,j,k} d^{ijk} \tau_i \tau_j \tau_k$ can be expressed in terms of operators of the Lorentz transformations forming the group $SO(6)$. We find for $C^1$

$$
C^1 = \frac{1}{4} \left( \frac{1}{2} M^{ab} M_{ab} + \frac{1}{3} (M_{12} + M_{34} + M_{56})^2 
- 2(M_{12} M_{34} - M_{13} M_{24} + M_{14} M_{23}) - 2(M_{12} M_{56} - M_{15} M_{26} + M_{16} M_{25})
- 2(M_{34} M_{56} - M_{35} M_{46} + M_{36} M_{45}) \right). \quad (3.12a)
$$

The above relations are valid for generators of spinorial ($\tilde{S}^{ab}$, $\tilde{S}^{ab}$) as well as of vectorial ($S^{ab}$) character.

In the case that operators of spinorial character $\tilde{S}^{ab}$ are concerned, eq.(3.12a) simplifies to

$$
\tilde{C}^1 = 1 + \frac{1}{3} \{ \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 \tilde{a}_4 + \tilde{a}_1 \tilde{a}_2 \tilde{a}_5 \tilde{a}_6 + \tilde{a}_3 \tilde{a}_4 \tilde{a}_5 \tilde{a}_6 \}
= 1 + \frac{1}{3} \{ \gamma_1 \gamma_2 \gamma_3 \gamma_4 + \gamma_1 \gamma_2 \gamma_5 \gamma_6 + \gamma_3 \gamma_4 \gamma_5 \gamma_6 \}.
$$

The general expression for the operator $C^2$ is much longer. We shall present it here only for the case of spinorial character, where it simplifies considerably. Using the coefficients $\tilde{d}^{ijk}$ presented in Table III, we find
\[\bar{C}^2 = \frac{5}{6} [\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4 \bar{a}_5 \bar{a}_6 + \frac{1}{3} (\bar{a}_1 \bar{a}_2 + \bar{a}_3 \bar{a}_4 + \bar{a}_5 \bar{a}_6)] \]

\[= \frac{i}{6} [\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6 + \frac{1}{3} (\gamma_1 \gamma_2 + \gamma_3 \gamma_4 + \gamma_5 \gamma_6)]. \quad (3.12b)\]

| \(i\) | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(j\) | 1 | 4 | 5 | 2 | 4 | 5 | 3 | 4 | 5 | 6 | 7 | 4 | 5 |
| \(k\) | 8 | 6 | 7 | 8 | 7 | 6 | 8 | 4 | 5 | 6 | 7 | 8 | 8 |

Table III: The nonvanishing coefficients \(d^{ijk}\), which determine the second Casimir operator of the group \(SU(3)\). The matrix of coefficients \(d^{ijk}\) is totally symmetric.

One obtains equivalent relations for \(\bar{C}^1\) and \(\bar{C}^2\).

3.1.4. Subgroups of \(SO(1,7)\).

We present here as an instructive example the algebra of the group \(SO(1,7)\), which contains as a subgroup \(SO(1,4) \times SU(2)\). The generators \(M^{ab}\), with \(a, b \in \{0, 1, 2, 3, 5\}\), form the subalgebra of the group \(SO(1,4)\) presented in Sect. 3.1.2., while \(M^{ab}\), with \(a, b \in \{6, 7, 8\}\), form the algebra of \(SU(2)\), isomorphic to the algebra of \(SO(3)\), presented in Sect.3.1.1. We shall use these subalgebras in Sect.4 to comment on the representations, the spinorial and the vectorial, describing fields with spins and weak charges. Again \(M^2\) and \(\Gamma\) are the two Casimir operators of the group \(SO(1,7)\).

3.1.5. Subgroups of \(SO(10)\).

We choose this group since it contains as a subgroup the group \(SU(5)\) which contains \(SU(3) \times SU(2) \times U(1)\).

The coefficients \(c^{0i}_{ab}, i \in \{1, 24\}, a, b \in \{1, 10\}\), which according to Eq. (3.8a) determine the operators \(\tau^{0i}\), defining the group \(SU(5)\), are presented in Table VI in such a way that operators \(\tau^{0i}\) demonstrate subalgebras of \(SU(3), SU(2)\) and \(U(1)\). We see that if we define

\[\tau^{1i} := \tau^{0i+15} = c^{0i+15}_{ab} M^{ab}, i \in \{1, 8\}, a, b \in \{1, 10\};\]

\[\tau^{2i} := \tau^{0i+12} = c^{0i+12}_{ab} M^{ab}, i \in \{1, 3\}, a, b \in \{1, 10\};\]

\[\tau^{3i} := \tau^{0i+23} = c^{0i+23}_{ab} M^{ab}, i = 1, a, b \in \{1, 10\},\]

operators \(\tau^{1i}, \tau^{2i}, \tau^{3i}\) close the subalgebras according to the following equations
\[ \{\tau_{1i}, \tau_{1j}\} = i f^{1ijk} \tau_{1k}, \quad \{\tau_{2i}, \tau_{2j}\} = i \epsilon^{ijk} \tau_{2k}, \quad \{\tau_{\alpha i}, \tau_{\beta j}\} = 0, \quad \alpha \neq \beta. \quad (3.13a) \]

Coefficients \(f^{1ijk}\) and \(\epsilon^{ijk}\) are structure constants of the groups \(SU(3)\) and \(SU(2)\), respectively.

| \(i\) | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 7 | 7 | 8 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(a\) | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 3 | 4 | 3 | 4 | 3 | 4 | 3 | 4 |
| \(b\) | 8 | 7 | 7 | 8 | 10 | 9 | 9 | 10 | 8 | 7 | 7 | 8 | 10 | 9 | 9 | 10 |
| \(c^{0}_{ab}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) |

| \(i\) | 9 | 9 | 10 | 10 | 11 | 11 | 12 | 12 | 13 | 13 | 14 | 14 | 15 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(a\) | 5 | 6 | 5 | 6 | 5 | 6 | 7 | 8 | 7 | 8 | 7 | 9 |
| \(b\) | 8 | 7 | 7 | 8 | 10 | 9 | 9 | 10 | 9 | 9 | 10 | 8 | 10 |
| \(c^{0}_{ab}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) |

| \(i\) | 16 | 16 | 17 | 17 | 18 | 18 | 19 | 19 | 20 | 20 | 21 | 21 | 22 | 22 | 23 | 23 | 24 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(a\) | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 4 | 3 | 3 | 4 | 1 | 3 | 5 |
| \(b\) | 4 | 3 | 3 | 4 | 2 | 4 | 6 | 5 | 5 | 6 | 5 | 6 | 2 | 4 | 6 |
| \(c^{0}_{ab}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) | \(-\frac{1}{2}\) |

Table IV: Table of coefficients \(c^{0}_{ab}\), \(c^{0}_{ab} = -c^{0}_{ba}\), \(i \in \{1, 24\}\), \(a, b \in \{1, 10\}\) determining the operators \(\tau^{0i} = c^{0}_{ab} M^{ab}\), which form the algebra of \(SU(5)\). According to Eq. (3.13) operators \(\tau^{0i}\) demonstrate the structure \(SU(3) \times SU(2) \times U(1)\).

From Table IV and Eq.(3.8a) we find the expressions for \(\tau^{1i}\) forming the algebra of \(SU(3)\), they are presented in Eq.(3.11), the expressions

\[ \tau^{21} := \frac{1}{2} (M_{710} - M_{89}), \quad \tau^{22} := \frac{1}{2} (M_{79} + M_{810}), \quad \tau^{23} := \frac{1}{2} (M_{78} - M_{910}), \quad (3.14a) \]

forming the algebra of \(SU(2)\) and the expression

\[ \tau^{31} := \sqrt{\frac{3}{5}} \left[ -\frac{1}{3} (M_{12} + M_{34} + M_{56}) + \frac{1}{2} (M_{78} + M_{910}) \right], \quad (3.14b) \]
forming the algebra $U(1)$.

Operators $\tau^{a\dot{a}}$ define either spinorial (if $M^{ab} = \tilde{S}^{ab}(\tilde{S}^{ab})$) or vectorial (if $M^{ab} = S^{ab}$) representations.

3.1.6. Subgroups of $SO(1,14)$.

The subalgebra of $SO(1,4) \times SO(10)$ of the algebra $SO(1,14)$ may be used to describe spins ($SO(1,4)$) and charges ($SO(10)$) of fields. Both subalgebras were discussed in previous subsections, 3.1.2 and 3.1.5, respectively.

The choice of the group $SO(1,4)$ rather than $SO(1,3)$ enables us to define generators $\gamma^a = -2iM^{5a}$ as we know from Sect.3.1.2. We can then write for one of the invariants of the subgroups $SO(1,3)$ and $SO(10)$, respectively:

$$\Gamma^{(4)} = i\frac{1}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d, \quad a, b, c, d \in 0, 1, 2, 3,$$

and

$$\Gamma^{(10)} = i\frac{1}{10!}\epsilon_{a_1a_2...a_{10}}\gamma^{a_1}\gamma^{a_2}...\gamma^{a_{10}}. \quad \text{This is true for operators of either spinorial or vectorial character.}$$

We shall comment on spinorial and vectorial representations of these subalgebras in Sect.4.

4. Spinorial and vectorial representations of $SO(d)$ or $SO(1, d-1)$ and of subgroups in Grassmann space

In this section we shall present some representations of generators of the Lorentz transformations of spinorial character $\tilde{S}^{ab}$ and representations of operators of vectorial character $S^{ab}$ for the groups discussed in the previous section. The former define representations which include what is called fundamental representations, the latter define representations which include what is called regular or adjoint representations. We shall see that one can find singlets of both types, of spinorial as well as of vectorial character as well. Representations of operators $\tilde{S}^{ab}$ can be found in an equivalent way to those of $\tilde{S}^{ab}$.

The choice of Grassmann space as the coordinate space over which the space of vectors - the basis for the generators of the Lorentz transformations - is spanned, enables the unification of spin and charges, and limits the vectors to those which correspond to spin $\frac{1}{2}$, if operators of spinorial character are concerned and to spin one or zero, if operators of vectorial character are concerned. Spins $\frac{3}{2}$ and 2 follow for tensor fields [13, 14].

According to Eq.(2.5) the vector space, spanned over the d-dimensional Grassmann space, is finite dimensional. It has $2^{d-1}$ vectors of an odd Grassmann character and $2^{d-1}$ vectors of an even Grassmann character.

In this section we shall present operators $M^{ab}$ in the coordinate representation, define the integral over the Grassmann space [15], the inner product of the vectors, as well as irreducible representations of generators $M^{ab}$ of spinorial and vectorial character.

4.1. Integrals on Grassmann space. Inner products.

We assume that differentials of Grassmann coordinates $d\theta^a$ fulfill the Grassmann an-
ticommuting relations \[14, 16\]
\[
\{d\theta^a, d\theta^b\} = 0
\] (4.1)

and we introduce a single integral over the whole interval of \(d\theta^a\)
\[
\int d\theta^a = 0, \quad \int d\theta^a d^a = 1, \quad a = 0, 1, 2, 3, 5, \ldots, d,
\] (4.2)

and the multiple integral over \(d\) coordinates
\[
\int d^d \theta^0 \theta^1 \theta^2 \theta^3 \theta^4 \ldots \theta^d = 1,
\] (4.3)

with
\[
d^d \theta := d\theta^0 \ldots d\theta^d d\theta^1 d\theta^0.
\]

We define \[14\] the inner product of two vectors \(\langle \varphi|\theta \rangle\) and \(\langle \theta|\chi \rangle\), with \(\langle \varphi|\theta \rangle = \langle \theta|\varphi \rangle^*\) as follows:
\[
\langle \varphi|\chi \rangle = \int d^d \theta (\omega \langle \varphi|\theta \rangle) < \theta|\chi > ,
\] (4.4)

with the weight function \(\omega\)
\[
\omega = \prod_{k=0,1,2,3 \ldots, d} (\frac{\partial}{\partial \theta^k} + \theta^k),
\] (4.4a)

which operates on the first function \(\langle \varphi|\theta \rangle\) only, and we define
\[
(\alpha^{a_1}a_2 \ldots a_k \theta^{a_1} \theta^{a_2} \ldots \theta^{a_k})^+ = (\theta^{a_k}) \ldots (\theta^{a_2}) (\theta^{a_1}) (\alpha^{a_1} a_2 \ldots a_k)^*.
\] (4.4b)

Then \(\theta^{a*} = \theta^a\), \(\theta^{a+} = -\eta^{aa} \frac{\partial}{\partial \theta^a}\) and \(\frac{\partial}{\partial \theta^a} \theta^{a+} = \eta^{aa} \theta^{a}\), while \(\tilde{a}^{a+} = \eta^{aa} \tilde{a}^a\) and \(\tilde{a}^{a+} = \eta^{aa} \tilde{a}^a\).

Accordingly the generators of the Lorentz transformations of Eqs.(3.1) are self adjoint \((if \ a \neq 0 \ and \ b \neq 0)\) or anti self adjoint \((if \ a = 0 \ or \ b = 0)\) operators.

Either the volume element \(d^d \theta\) or the weight function \(\omega\) are invariants with respect to the Lorentz transformations \((both \ are \ scalar \ densities \ of \ weight \ - \ 1)\).

4.2. Explicit expressions for operators of the Lorentz transformations in Grassmann space

We express the generators of the Lorentz transformations in terms of coordinates and left derivatives in order to be able to solve the eigenvalue problem in the coordinate representation.

According to Eqs.(2.7) and (3.1) we find
\[
S^{ab} = -i (\theta^a \frac{\partial}{\partial \theta_b} - \theta^b \frac{\partial}{\partial \theta_a}),
\] (4.5a)
\[
\tilde{S}^{ab} = -\frac{i}{2} \left( \frac{\partial}{\partial \theta_a} + \theta^a \right) \left( \frac{\partial}{\partial \theta_b} + \theta^b \right), \quad \text{if} \ a \neq b, \quad (4.5b)
\]
\[
\tilde{S}^{ab} = \frac{i}{2} \left( \frac{\partial}{\partial \theta_a} - \theta^a \right) \left( \frac{\partial}{\partial \theta_b} - \theta^b \right), \quad \text{if} \ a \neq b, \quad (4.5c)
\]
\[
\tilde{a}^a = \left( \frac{\partial}{\partial \theta_a} + \theta^a \right), \quad \tilde{\tilde{a}}^a = i \left( \frac{\partial}{\partial \theta_a} - \theta^a \right). \quad (4.5d)
\]

Accordingly, the superpositions of the operators of the Lorentz transformations, as well as their invariants, can be expressed in terms of \( M^{ab} \) from Eqs.(4.5).

4.3. The eigenvalue problem

To find the irreducible representations of the desired groups or subgroups, we solve the eigenvalue problem for the commuting operators which define the algebra of the group or subgroups

\[
\langle \theta | \tilde{A}_i | \tilde{\varphi} \rangle = \tilde{\alpha}_i < \theta | \tilde{\varphi} >, \quad \langle \theta | A_i | \varphi \rangle = \alpha_i < \theta | \varphi >, \quad i = \{ 1, r \}, \quad (4.6)
\]

where \( \tilde{A}_i \) and \( A_i \) stand for \( r \) commuting operators of spinorial and vectorial character, respectively.

To solve equations (4.6) we express the operators in the coordinate representation (Eqs.(4.5)) and write the eigenvectors as polynomials of \( \theta^a \). We orthonormalize the vectors according to the inner product, defined in Eq.(4.4)

\[
\langle ^a \tilde{\varphi}_i | ^b \tilde{\varphi}_j \rangle = \delta^{ab} \delta_{ij}, \quad \langle ^a \varphi_i | ^b \varphi_j \rangle = \delta^{ab} \delta_{ij}, \quad (4.6a)
\]

where index \( a \) distinguishes between vectors of different irreducible representations and index \( j \) between vectors of the same irreducible representation. Eq.(4.6a) determines the orthonormalization condition for spinorial and vectorial representations, respectively.

4.3.1. Representations of the group \( SO(1, 3) \) and subgroups

\( SO(3) \)

We shall find the representations of the subgroup \( SO(3) \) first. According to Eqs. (3.6) and (3.10), we look for the eigenvectors of the operators \( \tau^{1i} \), choosing \( i = 3 \), and \( \tau^1 \). We find \( \tau^{13} = M^{12} \) and \( \tau^1 = (M^{12})^2 + (-M^{13})^2 + (M^{23})^2 \).

Looking for solutions in the spinorial case we take operators of Eq.(4.5b) and find eight vectors, which can be arranged into two bispinors ( each bispinor has two vectors ) of an odd and two bispinors of an even Grassmann character. We present these eight vectors in Table Va, together with the eigenvalues of the two operators \( \tilde{\tau}^{13} \) and \( \tilde{\tau}^1 \).
Table Va: Eigenvectors of commuting operators of a spinorial character for the group \(SO(3)\), arranged into four irreducible representations, two of an odd and two of an even Grassmann character. Eigenvalues of the operators \(\tilde{\tau}^{13}\) and \((\tilde{\tau}^{1})^{2}\) are also presented. Vectors are orthonormalized according to the inner product defined in Eq. (4.4).

The operators \(\tilde{\tau}_{1}^{\pm} = \tilde{\tau}^{11} \pm i\tilde{\tau}^{12}\) rotate one vector of a bispinor into another vector of the same bispinor. We have, therefore, instead of one, four independent bispinors. To describe fermions in a standard way, the two bispinors of an odd Grassmann character are needed. Bispinors in Table Va are arranged in such a way that the matrix representations of the operators \(\tilde{\tau}_{1}^{i}, i = 1, 2, 3\) are for any bispinor just the Pauli \(\sigma_{i}\) matrices.

To look for the solutions in the vectorial case, we have to take the operators of Eq. (4.5a). We find two scalars and two three vectors. Again, one scalar and one three vector have an odd, another scalar and another three vector have an even Grassmann character. These eight vectors are presented in Table Vb.

Table Vb: Eigenvectors of commuting operators of a vectorial character for the group \(SO(3)\), arranged into two scalars and two three vectors. Half of the vectors have an even and half an odd Grassmann character.

Each of the three vectors are arranged in such a way that the matrix representations of the operators \(\tau^{1i}\) are the usual \(3 \times 3\) matrices \([14]\). The three vectors are in the adjoint
representations with respect to the bispinors of Table Va. To describe vectors and scalars only Grassmann even solutions are taken.

\[ SO(1, 3) \]

To find representations manifesting the \( SU(2) \times SU(2) \) structure of the group \( SO(1, 3) \), one has to solve the eigenvalue problem (4.6) for each of the two types of operators \( \tau^{1i} \) and \( \tau^{2i} \), \( i = 1, 2, 3 \), defined by coefficients (3.10). We gave the solutions in ref. [14] for both types of operators.

In the spinorial case, there are eight bispinors, forming the irreducible representations of the group \( SO(1, 3) \). Half of them have an odd and half an even Grassmann character. They can be further distinguished with respect to the eigenvalue of the operator \( \tilde{\Gamma} \), which is either +1 or -1. We speak about chiral representations. They cannot be arranged into four four spinors unless we define the Grassmann even operators \( \tilde{\gamma}^a \) which connect the two Grassmann odd two bispinors into a four spinor. This can be done [14] within the group \( SO(1, 4) \), which will be discussed in the next subsection.

In the vectorial case we find two scalars and two three vectors of an even Grassmann character and two four vectors of an odd Grassmann character [14]. Three vectors form the adjoint representations with respect to bispinors due to the \( SU(2) \times SU(2) \) structure of the group \( SO(1, 3) \).

4.3.2. Representations of the group \( SO(1, 4) \) and subgroups

\[ SO(1, 3) \]

We see that the two sets of Casimir operators, \( M^2 \) and \( \Gamma \) of Subsect.3.1.1. and \( M^{(4)}^2 \) and \( \Gamma^{(4)} \) of Subsect.3.1.2., are equal since they are both defined in the vector space spanned over the coordinates \( \theta^a, a \in \{0, 1, 2, 3\} \).

We defined in Subsect. 3.1.2. the operators which are not included in the subgroup \( SO(1, 3) \), as the \( \gamma^a \) operators \( \tilde{\gamma}^a = -2i\tilde{S}^5a = -\tilde{a}^5\tilde{a} \) for the spinorial type of operators, and \( \gamma^a = -2iS^5a \) for the vectorial type of operators.

We look for the representations [14] of the subalgebras \( SU(2) \times SU(2) \) (Eq.3.10b) in the space spanned over the five dimensional coordinate space: 16 vectors have an odd and 16 vectors an even Grassmann character. Solving the eigenvalue problem for the operators of spinorial character within the vector space of an odd Grassmann character, we find eight bispinors, which we arrange into four four spinors such that a bispinor which is a singlet with respect to one of the two \( SU(2) \) subgroups and a doublet with respect to the other \( SU(2) \) and a bispinor which is the doublet with respect to the first \( SU(2) \) group and the singlet with respect to the second \( SU(2) \), are transformed into each other by operators \( \tilde{\gamma}^a \). Each bispinor forms the irreducible representation with respect to the generators of the Lorentz transformations in the four dimensional subspace of the five dimensional Grassmann space. We present these vectors, taken from ref. [14], in Table VIa. One can find the matrix representations of the operators in the same reference.
Table VIa: Irreducible representations of the two subgroups $SU(2) \times SU(2)$ embedded into the group $SO(1,4)$ as defined by the generators of spinorial character. There are eight bispinors, two by two bispinors connected into a four spinor by the operators $\tilde{\gamma}^a$ in the way that $\tilde{\gamma}^0$ is diagonal. We present eigenvalues of the commuting operators $\tilde{S}_3 = \tilde{S}^{12}, \tilde{K}_3 = \tilde{S}^{03}, \tilde{\Gamma}^{(4)}, \tilde{\tau}^{13}, (\tilde{\tau}^1)^2, \tilde{\tau}^{23}, (\tilde{\tau}^2)^2$.

Analyzing the space of 16 vectors of an even Grassmann character with respect to the operators of vectorial character, we find two scalars and two three vectors which do not depend on the coordinate $\theta^5$ and two four vectors which do depend on the coordinate $\theta^5$. They are vectors and scalars with respect to the operators of the Lorentz transformations in the four dimensional subspace of the five dimensional space. We present these vectors, taken from ref. [14], in Table VIb. One can find in the above reference also the matrix representation of the operators.
Table VIb. Irreducible representations of the two subgroups $SU(2) \times SU(2)$ embedded into the group $SO(1, 4)$ as defined by the generators of vectorial character. There are two scalars, two three vectors and two four vectors. We present the eigenvalues of commuting operators $S^2 = \frac{1}{2} S^{ab} S_{ab}, S_3 = S^{12}, K_3 = S^{03}, \Gamma, \tau^1, (\tau^1)^2, \tau^2, (\tau^2)^2$.

### 4.3.3. Representations of the group SU(3) embedded in SO(6)

There are two Casimir operators: $C^1$ and $C^2$ (Eq.(3.12)) and two commuting generators: $\tau^3$ and $\tau^8$ (Eq.(3.11)) in this group. We shall be interested in irreducible representations of vectors with an even Grassmann character only, taking into account that the part defined by the group $SO(1, 4)$ determine the Grassmann character of either fermions or bosons.
| $a$ | $i$ | $< \theta | \varphi_i^n >$ | $\tilde{\tau}_3$ | $\tilde{\tau}_3$ |
|-----|-----|-----------------|----------------|----------------|
| 1   | 1   | $\frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{3}}$ |
| 1   | 2   | $\frac{1}{\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 1   | 3   | $-\frac{1}{\sqrt{3}}$ | $0$ | $-\frac{1}{\sqrt{3}}$ |
| 2   | 1   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 2   | 2   | $-\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 2   | 3   | $\frac{1}{2\sqrt{3}}$ | $0$ | $\frac{1}{2\sqrt{3}}$ |
| 3   | 1   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 3   | 2   | $\frac{1}{2\sqrt{3}}$ | $0$ | $\frac{1}{2\sqrt{3}}$ |
| 3   | 3   | $\frac{1}{2\sqrt{3}}$ | $0$ | $\frac{1}{2\sqrt{3}}$ |
| 4   | 1   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 4   | 2   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 4   | 3   | $\frac{1}{2\sqrt{3}}$ | $0$ | $\frac{1}{2\sqrt{3}}$ |
| 5   | 1   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 5   | 2   | $\frac{1}{2\sqrt{3}}$ | $0$ | $\frac{1}{2\sqrt{3}}$ |
| 5   | 3   | $\frac{1}{2\sqrt{3}}$ | $0$ | $\frac{1}{2\sqrt{3}}$ |
| 6   | 1   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 6   | 2   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 6   | 3   | $\frac{1}{2\sqrt{3}}$ | $0$ | $\frac{1}{2\sqrt{3}}$ |
| 7   | 1   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 7   | 2   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 7   | 3   | $\frac{1}{2\sqrt{3}}$ | $0$ | $\frac{1}{2\sqrt{3}}$ |
| 8   | 1   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 8   | 2   | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ |
| 8   | 3   | $\frac{1}{2\sqrt{3}}$ | $0$ | $\frac{1}{2\sqrt{3}}$ |
| 9   | 1   | $\frac{1}{2\sqrt{3}}$ | $0$ | $0$ |
| 10  | 1   | $\frac{1}{2\sqrt{3}}$ | $0$ | $0$ |
| 11  | 1   | $\frac{1}{2\sqrt{3}}$ | $0$ | $0$ |
| 12  | 1   | $\frac{1}{2\sqrt{3}}$ | $0$ | $0$ |
| 13  | 1   | $\frac{1}{2\sqrt{3}}$ | $0$ | $0$ |
| 14  | 1   | $\frac{1}{2\sqrt{3}}$ | $0$ | $0$ |
| 15  | 1   | $\frac{1}{2\sqrt{3}}$ | $0$ | $0$ |
| 16  | 1   | $\frac{1}{2\sqrt{3}}$ | $0$ | $0$ |
Table VIIa: The eight triplets and the eight singlets, the Grassmann even irreducible representations of the generators of spinorial character closing the algebra of $SU(3)$. The generators are expressed by the generators of the Lorentz transformations in 6 dimensional Grassmann space (Eq.(3.11)), forming the group $SO(6)$. Each triplet is arranged in such a way that the matrix representations of the generators of the group $SU(3)$ coincide with the usual matrix representations of triplets [20]. The diagonal matrix elements of $\tilde{\tau}^3$ and $\tilde{\tau}^8$ are also given.

Operators of spinorial character define in the space of $2^5$ vectors of an even Grassmann character eight triplets and eight singlets. We present them in Table VIIa. The triplets are arranged in such a way that the matrix representation of the generators of the group (Eq.(3.11)) agrees for each triplet with the usual matrix representation [20] of the group $SU(3)$ for either triplets ( the first four triplets ) or for anti triplets ( the second four triplets ). We present in Table VIIa only eigenvalues of operators $\tilde{\tau}_3$ and $\tilde{\tau}_8$.

| $a$ | $i$ | $< \theta | \varphi^a_i >$ | $\tau_3$ | $\tau_8$ |
|-----|-----|----------------------|---------|---------|
| 1   | 1   | $(\frac{1}{2})^2(\theta^1 + i\theta^2)(\theta^3 - i\theta^4)(1 + \theta^5\theta^6)$ | $-1$   | $0$     |
| 1   | 2   | $-(\frac{1}{2})^2(\theta^1 - i\theta^2)(\theta^3 + i\theta^4)(1 + \theta^5\theta^6)$ | $1$    | $0$     |
| 1   | 3   | $-\frac{1}{2}(\theta^1\theta^2 - \theta^4\theta^5)(1 + \theta^5\theta^6)$ | $0$    | $0$     |
| 1   | 4   | $-(\frac{1}{2})^2(\theta^1 + i\theta^2)(1 + \theta^3\theta^4)(\theta^5 - i\theta^6)$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ |
| 1   | 5   | $-(\frac{1}{2})^2(\theta^1 - i\theta^2)(1 + \theta^3\theta^4)(\theta^5 + i\theta^6)$ | $\frac{1}{2}$ | $\frac{3}{2}$ |
| 1   | 6   | $(\frac{1}{2})^2(1 + \theta^1\theta^2)(\theta^3 + i\theta^4)(\theta^5 - i\theta^6)$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ |
| 1   | 7   | $-(\frac{1}{2})^2(1 + \theta^1\theta^2)(\theta^3 - i\theta^4)(\theta^5 + i\theta^6)$ | $-\frac{1}{2}$ | $\frac{3}{2}$ |
| 1   | 8   | $\frac{1}{2}(1 + \theta^1\theta^2)(\theta^5\theta^6 - \theta^3\theta^4)$ | $0$    | $0$     |
| 2   | 1   | $(\frac{1}{2})^2(\theta^1 + i\theta^2)(\theta^3 - i\theta^4)(1 - \theta^5\theta^6)$ | $-1$   | $0$     |
| 2   | 2   | $-(\frac{1}{2})^2(\theta^1 - i\theta^2)(\theta^3 + i\theta^4)(1 - \theta^5\theta^6)$ | $1$    | $0$     |
| 2   | 3   | $-\frac{1}{2}(\theta^1\theta^2 - \theta^4\theta^5)(1 - \theta^5\theta^6)$ | $0$    | $0$     |
| 2   | 4   | $-(\frac{1}{2})^2(\theta^1 + i\theta^2)(1 - \theta^3\theta^4)(\theta^5 - i\theta^6)$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ |
| 2   | 5   | $-(\frac{1}{2})^2(\theta^1 - i\theta^2)(1 - \theta^3\theta^4)(\theta^5 + i\theta^6)$ | $\frac{1}{2}$ | $\frac{3}{2}$ |
| 2   | 6   | $(\frac{1}{2})^2(1 - \theta^1\theta^2)(\theta^3 + i\theta^4)(\theta^5 - i\theta^6)$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ |
| 2   | 7   | $-(\frac{1}{2})^2(1 - \theta^1\theta^2)(\theta^3 - i\theta^4)(\theta^5 + i\theta^6)$ | $-\frac{1}{2}$ | $\frac{3}{2}$ |
| 2   | 8   | $-\frac{1}{2}(1 - \theta^1\theta^2)(\theta^5\theta^6 - \theta^3\theta^4)$ | $0$    | $0$     |
| 3   | 1   | $\frac{1}{\sqrt{2}}(1 + \theta^1\theta^2\theta^3\theta^4\theta^5\theta^6)$ | $0$    | $0$     |
| 4   | 1   | $\frac{1}{\sqrt{2}}(1 - \theta^1\theta^2\theta^3\theta^4\theta^5\theta^6)$ | $0$    | $0$     |

Table VIIb. The two singlets and the two octets, forming the Grassmann even irreducible representations of the generators of vectorial character closing the algebra of $SU(3$). The generators are expressed, as in the case of the operators of spinorial character, in terms of the generators of the Lorentz transformations in six dimensional Grassmann space. Eigenvalues of the operators $\tau_3$ and $\tau_8$ are also presented.
Generators of a vectorial character in the vector space of an even Grassmann character define two singlets, two octets and one multiplet of fourteen vectors. We present the singlets and octets in Table VIIb, together with the eigenvalues of the operators $\tau^3$ and $\tau^8$, which are diagonal in these representations. We arrange each of the two octets in such a way that the eight generators $\tau^i$, $i = \{1, 8\}$ from Eq.(3.11), when $M^{ab}$ are replaced by $S^{ab}$, have equal matrix representations for both octets. These matrix representations are in agreement with the matrices defined by the structure constants of the group $SU(3)$ $(T^i)_{jk} = -i f^{1ijk}$, $f^{1ijk}$ taken from Table I, if transformed into the basis in which $T^3$ and $T^8$ are diagonal.

While triplets and singlets, determined by the generators of spinorial character of the group $SU(3)$, can be either identified with fundamental representations of the group (triplets), or can be used to describe fermions without the colour charge (singlets), can octets and singlets, determined by the generators of the vectorial character of the same group, be identified with adjoint representations of the group (octets) or can be used to describe bosons without charges (singlets).

**4.3.4. Representations of the group $SO(1,7)$ in terms of subgroups $SO(1,4) \times SU(2)$**

The representations of the group $SO(1,4) \times SU(2)$, embedded into the Lorentz group $SO(1,7)$ can be deduced from the representations presented in the two Subsect. 4.3.1. and 4.3.2.. In the second subsection we found the irreducible representations of the group $SO(1,4)$, defined in the vector space spanned over the coordinate Grassmann space $\theta^a$, $a = \{0, 1, 2, 3, 5\}$. The vector space of an odd Grassmann character was analyzed with respect to generators of spinorial character. We found eight bispinors arranged into four spinors. The vector space of an even Grassmann character was analyzed with respect to generators of vectorial character. We found two scalars, two three vectors and two four vectors. The fifth dimension was needed to define the $\gamma^a$ matrices.

The vector space spanned over the coordinate space of $\theta^a$, $a = \{6, 7, 8\}$, when analyzed with respect to operators of spinorial character, gives two doublets of an even Grassmann character (Subsect. 4.3.1., if the indices 1,2,3 are replaced by the indices 6,7,8). In the same subspace the operators of vectorial character define one scalar and one three vector of an even Grassmann character.

In the space of eight Grassmann coordinates there exist, with respect to the groups $SO(1,4) \times SU(2)$ embedded into the group $SO(1,7)$ two times eight bispinors (each bispinor has two vectors), which are doublets with respect to the group $SU(2)$ and have an odd Grassmann character, as well as two scalars and two three vectors of an even Grassmann character, which are either triplets or singlets with respect to the group $SU(2)$. Each bispinor (with respect to the group $SO(1,3)$), which is a doublet (with respect to the group $SU(2)$) represents the fundamental representation (with respect to both subgroups), while accordingly a three vector (with respect to the group $SO(1,3)$), which is a triplet (with respect to the group $SU(2)$) represents the adjoint representation with respect to both subgroups. There exists no bispinor which would be a singlet with respect to the
4.3.5. Representations of $SU(3) \times SU(2) \times U(1)$ embedded into the group $SO(10)$

We look for the irreducible representations of generators $\tau^\alpha_i$, $\alpha = 1, 2, 3$, $i = 1, n_\alpha$, expressed in terms of the generators of the Lorentz group $SO(10)$ as defined in Eqs.(3.11) and (3.14). We shall rename the operators $\tau_i, i = 1, 2, \ldots, 8$ from Eq.(3.11) into $\tau^i_1$, $i = 1, 2, \ldots, 8$.

$SU(3)$

The representations of the group $SU(3)$ were already presented in Subsect.4.3.3..

$SU(2)$

The representations of the group $SU(2)$ were presented in Subsect. 4.3.1., since the algebras of $SO(3)$ and $SU(2)$ are isomorphic. However, since the generators of $SU(2)$ are now written in terms of four generators of the Lorentz group rather than in terms of three (see Eq.(3.14a)), we have to solve the eigenvalue problem again.

Solving the eigenvalue problem (Eq.(4.6)) for the operators of spinorial character $\tilde{\tau}^{2i}, i = \{1, 2, 3\}$ (Eq.(3.14a)), expressed in terms of $S^{ab}, a, b = \{7, 8, 9, 10\}$, we find two doublets and four singlets of an even Grassmann character. We present these vectors together with the eigenvalues of the commuting operators in Table VIIIa.

| $a$ | $i$ | $< \theta|\tilde{\tau}^a_i >$ | $\tilde{\tau}^{23}$ | $(\tilde{\tau}^2)^2$ |
|-----|-----|--------------------------|----------------|----------------|
| 1   | 1   | $\frac{1}{2}(1 - i\theta^i \theta^8)(1 + i\theta^9 \theta^{10})$ | $\frac{1}{2}$ | $\frac{2}{3}$ |
| 1   | 2   | $\frac{1}{2}(\theta^i + i\theta^8)(\theta^9 - i\theta^{10})$ | $-\frac{3}{2}$ | $\frac{3}{4}$ |
| 2   | 1   | $\frac{1}{2}(1 + i\theta^i \theta^8)(1 - i\theta^9 \theta^{10})$ | $\frac{1}{2}$ | $\frac{2}{3}$ |
| 2   | 2   | $\frac{1}{2}(\theta^i - i\theta^8)(\theta^9 + i\theta^{10})$ | $\frac{1}{2}$ | $\frac{3}{4}$ |
| 3   | 1   | $\frac{1}{2}(1 + i\theta^i \theta^8)(1 + i\theta^9 \theta^{10})$ | 0 | 0 |
| 4   | 1   | $\frac{1}{2}(\theta^i + i\theta^8)(\theta^9 + i\theta^{10})$ | 0 | 0 |
| 5   | 1   | $\frac{1}{2}(1 - i\theta^i \theta^8)(1 - i\theta^9 \theta^{10})$ | 0 | 0 |
| 6   | 1   | $\frac{1}{2}(\theta^i - i\theta^8)(\theta^9 - i\theta^{10})$ | 0 | 0 |

Table VIIIa: The irreducible representations of the generators of spinorial character $\tilde{\tau}^{2i}, i = \{1, 2, 3\}$, defined in Eqs.(3.14a) with $M^{ab}$ replaced by $S^{ab}$, closing the algebra of $SU(2)$. In the vector space of an even Grassmann character there are two doublets and four singlets. The expectation values of the commuting operators are also added.

Generators of vectorial character $\tau^i_1, i = \{1, 2, 3\}$, expressed in terms of $S^{ab}, a, b = \{7, 8, 9, 10\}$ (Eq.(3.14a)), define one three vector and five singlets of an even Grassmann character. We present the vectors and the eigenvalues of the commuting operators in Table VIIIb.
Table VIIIb. The irreducible representations of the generators of vectorial character \( \tau^{ai} \), \( i = \{1,2,3\} \), defined in Eqs.(3.14a) if \( M^{ab} \) are replaced by \( S^{ab} \), and closing the algebra of \( SU(2) \). In the Grassmann even part of the space there are two doublets and four singlets. The expectation values of the commuting operators are also presented.

\[
U(1)
\]

Solving the eigenvalue problem for the generator \( \tau^{31} \) (Eq.(3.14b)) of the group \( U(1) \) either for operators of spinorial character (when \( M^{ab} = \tilde{S}^{ab} \)) or of vectorial character (when \( M^{ab} = S^{ab} \)), we find that the products of vectors from Table VIIa (VIIb) forming the irreducible representations of \( SU(3) \) with vectors from Table VIIIa (VIIIb), forming the irreducible representations of \( SU(2) \), for either of the two types of operators are the eigenvectors of the operator \( \tau^{31} \) as well. We have, of course, to choose only one set of vectors from each table and make the outer products among them. We always combine the vectors of Table VIIa with the vectors of Table VIIIa, and the vectors of Table VIIb with the vectors of Table VIIIb.

When looking for representations of the operator \( \tilde{\tau}^{31} \) (Eq.(3.14b)) of spinorial character we find that products of triplets from Table VIIa with doublets from Table VIIIa have two different expectation values of the operator \( \tilde{\tau}^{31} \) defining the group \( U(1) \). Products of triplets from Table VIIa with singlets from Table VIIIa have four different eigenvalues of \( \tilde{\tau}^{13} \). Products of singlets from Table VIIa with doublets of Table VIIIa have two different eigenvalues of \( \tilde{\tau}^{13} \), while products of singlets of Table VIIa with singlets of Table VIIIa have three different eigenvalues of \( \tilde{\tau}^{13} \). We present these values in Table IXa.
Table IXa. Eigenvalues of the operator $\tilde{\tau}^{31}$ forming the group of $U(1)$ (Eq.(3.14b)), with $M^{ab} = \tilde{S}^{ab}$. Eigenvectors of $\tilde{\tau}^{31}$ are the outer products of vectors from Table VIIa and Table VIIIa. The first column concerns vectors from Table VIIa, which form either triplets or singlets. The second column concerns vectors from Table VIIIa. They form either doublets or singlets.

When looking for the representations of the operator $\tilde{\tau}^{31}$ (Eq.(3.14b)) of vectorial character, we find that products of octets from Table VIIb with triplets from Table VIIIb and products of singlets from Table VIIb with triplets from Table VIIIb have expectation values of the operator $\tilde{\tau}^{31}$ equal to zero. Products of octets from Table VIIb with singlets from Table VIIIb and products of singlets from Table VIIb with singlets from Table VIIb have expectation values of $\tilde{\tau}^{31}$ zero or $\pm 1$. We present these values in Table IXb.

| $a \in \{1, \ldots, 8\}$ | $b \in \{1, 2\}$ | $\tilde{\tau}^{31}$ |
|---------------------------|------------------|------------------|
| $i \in \{1, 2, 3\}$ | $j \in \{1, 2\}$ | $\sqrt{\frac{3}{5}} \cdot a \in \{1, \ldots, 4\}$ |
| $\in$ triplets | $\in$ doublets | $-\sqrt{\frac{3}{5}} \cdot a \in \{5, \ldots, 8\}$ |

| $a \in \{1, \ldots, 8\}$ | $b \in \{3, \ldots, 6\}$ | $\tilde{\tau}^{31}$ |
|---------------------------|------------------|------------------|
| $i \in \{1, 2, 3\}$ | $j \in \{1\}$ | $-\sqrt{\frac{3}{5}} \cdot a \in \{1, \ldots, 4\}, b \in \{3, 4\}$ |
| $\in$ triplets | $\in$ singlets | $\sqrt{\frac{2}{5}} \cdot a \in \{1, \ldots, 4\}, b \in \{5, 6\}$ |

| $a \in \{9, \ldots, 16\}$ | $b \in \{1, 2\}$ | $\tilde{\tau}^{31}$ |
|---------------------------|------------------|------------------|
| $i \in \{1\}$ | $j \in \{1, 2\}$ | $-\sqrt{\frac{3}{5}} \cdot a \in \{9, \ldots, 12\}$ |
| $\in$ singlets | $\in$ doublets | $\sqrt{\frac{3}{5}} \cdot a \in \{13, \ldots, 16\}$ |

| $a \in \{9, \ldots, 16\}$ | $b \in \{3, \ldots, 6\}$ | $\tilde{\tau}^{31}$ |
|---------------------------|------------------|------------------|
| $i \in \{1\}$ | $j \in \{1\}$ | $-\sqrt{\frac{3}{5}} \cdot a \in \{9, \ldots, 12\}, b \in \{3, 4\}$ |
| $\in$ singlets | $\in$ singlets | $0, a \in \{9, \ldots, 12\}, b \in \{5, 6\}$ |

| $a \in \{9, \ldots, 16\}$ | $b \in \{3, \ldots, 6\}$ | $\tilde{\tau}^{31}$ |
|---------------------------|------------------|------------------|
| $i \in \{1\}$ | $j \in \{1\}$ | $0, a \in \{9, \ldots, 12\}, b \in \{3, 4\}$ |
| $\in$ singlets | $\in$ singlets | $\sqrt{\frac{3}{5}} \cdot a \in \{13, \ldots, 16\}, b \in \{5, 6\}$ |
Table IXb. Eigenvalues of the operator $\tau_{31}$ from Eq.(3.14b), with $M^{ab} = S^{ab}$. The operator $\tau_{31}$ has a vectorial character. We choose representations which are the outer product of vectors from Table VIIb with vectors of Table VIIIb. The first column concerns vectors from Table VIIb, which form either octets or singlets. The second column concerns vectors from Table VIIIb. They form either triplets or singlets.

### 4.3.6. Representations of the group \( SO(1, 14) \) in terms of \( SU(3) \times SU(2) \times U(1) \)

We find the representations of the group \( SO(1, 14) \) as the outer products of the representations of the subgroup \( SO(1, 4) \) and the subgroups \( SU(3), SU(2) \) and \( U(1) \). The representations of these groups are discussed in Subsect. 4.3.2. and 4.3.5.. We look for the spinorial and the vectorial representations in 15 dimensional Grassmann space. The former are defined by the spinorial operators $\tilde{S}^{ab}$, the latter by the vectorial operators $S^{ab}$. In the spinorial case we choose vectors of an odd Grassmann character, while in the vectorial case we choose vectors of an even Grassmann character. The reason for that is the requirement that in the canonical quantization of fields the former quantize to fermions, the later to bosons [14]. Since the character of fields, that is their behaviour as spinors, vectors or scalars, is determined by the behaviour of fields with respect to generators of the Lorentz transformations in the four dimensional subspace of the fifteen dimensional space, we choose vectors to be either of an odd or of an even Grassmann character with respect to the group \( SO(1, 4) \), where the fifth dimension is needed to properly define the Grassmann character of the fields as well as the Dirac $\gamma^a$ matrices. That part of the Grassmann space, which determines charges of fields, is connected with the group \( SO(10) \). It is spanned over coordinate space with indices higher then five. It is chosen to participate to the outer products of vectors by the part of an even Grassmann character only, either for the spinorial or for the vectorial case. The part of an odd Grassmann character will be studied elsewhere[23]. It turns out, for example, that in an outer product...
with representations of the group \( SO(1, 4) \) of an odd Grassmann character, this part offers in the vectorial case three vectors, which are doublets with respect to the group \( SU(2) \).

Representations of the group \( SO(1, 14) \) are the outer products of the representations of the group \( SO(1, 4) \) and of the group \( SO(10) \) discussed in the previous subsections. They all belong either to the spinorial representations or to the vectorial representations. There are spinorial representations which coincide with what is called the fundamental representations with respect to the group \( SO(1, 3) \) and the groups \( SU(3), SU(2) \) and \( U(1) \), but there are also representations which are singlets with respect to any of groups \( SU(3) \) or \( SU(2) \). And there are the vectorial representations which coincide with the adjoint representations with respect to all or some of the groups \( SO(1, 3), SU(3), SU(2) \), some of them are singlets with respect to some or all these groups.

The structure of Grassmann space, with the limited number of vectors limits the possible representations allowed by the group theory.

5. Concluding remarks

In this paper the algebras and subalgebras defined by two kinds of generators of the Lorentz transformations, forming in 15 dimensional Grassmann space the group \( SO(1, 14) \), one of spinorial, the other of vectorial character, both the linear differential operators, were studied and some of their representations were obtained, those, which have in \( SO(1, 4) \) an odd ( when describing fermions) or an even ( when describing bosons) Grassmann character, while in \( SO(10) \) they have only an even Grassmann character. According to two kinds of generators defined in the linear vector space, spanned over Grassmann coordinate space, there are also two kinds of representations: we call them spinorial and vectorial representations, respectively. We find among spinorial representations the representations, which are known as the fundamental representations, and we find also singlets, needed to describe fermions without a particular charge. Among vectorial representations we find accordingly the adjoint or the regular representations, and again we find also singlets, which are needed to describe bosons without a particular charge.

The generators of translations of an odd Grassmann character were used to find the decomposition of the Lorentz group \( SO(2n) \) in terms of the subgroup \( SU(n) \). This decomposition can be found either for operators of spinorial or for operators of vectorial character.

We looked for irreducible representations of the operators closing subalgebras and of the corresponding Casimir operators to find for each of subalgebras two kinds of representations, the spinorial and the vectorial ones in accordance with the two kinds of operators. Since the dimension of the vector space, spanned over Grassmann coordinate space, is finite, all representations are finite dimensional.

Since the group \( SO(1, d - 1) \) contains for \( d = 15 \) as subgroups the groups \( SO(1, 3) \), needed to describe spins of fermions and bosons, as well as \( U(1), SU(2) \) and \( SU(3) \), needed to describe the Yang-Mills charges of fermions and bosons, the spin and the Yang
- Mills charges of either fermions or of bosons are in the presented approach unified. Since spins and charges are described by the representations of the generators of the Lorentz transformations of either fermionic or of bosonic character, it means that fermionic states must belong either to the fundamental representations with respect to the groups, describing charges, or they must be spinorial singlets with respect to those groups, while bosonic states must belong either to the adjoint representations with respect to the groups, describing charges, or they must be vectorial singlets with respect to those groups. One can find [23] for vectorial case besides octets and singlets of SU(3) also triplets and besides triplets and singlets of SU(2) also doublets. But these representations have an odd Grassmann character in SU(10) and are not studied in this article [23].

Among representations of the proposed approach are the ones, needed to describe the quarks, the leptons and the gauge bosons, which appear in the Standard Electroweak Model. We find left handed spinors, SU(3) triplets and SU(2) doublets with U(1) charge $\pm \frac{1}{6}$ and right handed spinors, SU(3) triplets and SU(2) singlets with U(1) charge $\pm \frac{2}{3}$ and $\mp \frac{1}{3}$, which describe quarks. We find left handed spinors, SU(3) singlets and SU(2) doublets with U(1) charge $\mp \frac{1}{2}$ and right handed spinors, SU(3) singlets and SU(2) singlets with U(1) charge $\mp 1$, which describe leptons. We find left and right handed three vectors, SU(3) triplets and SU(2) singlets with U(1) charge 0, describing gluons and left handed SU(3) singlets and SU(2) triplets with U(1) charge 0, describing massless weak bosons and left and right handed SU(3) singlets and SU(2) singlets with U(1) charge 0, describing a U(1) field. The Higgs’s boson of this model appears due to the presented representations as a constituent field, while within above mentioned odd representations it appears [23] as a scalar, which is a SU(3) singlet and SU(2) doublet, with an odd Grassmann character in SO(1, 4) and SU(2) part of the Grassmann space.

The supersymmetric partners of the gauge bosons, required by the supersymmetric extension of the Standard Electroweak Model, can in the proposed theory exist only as constituent particles.

In ref. [13, 14] we presented the action from which it follows that the unification of spins and charges in Grassmann space leads to the unification of all interactions: the gravitational fields in the space of d ordinary and d Grassmann coordinates may manifest under certain conditions in the four dimensional subspace of the d dimensional space as the ordinary gravity and the gauge fields, both depending on ordinary and Grassmann coordinates, if $d \geq 15$. Grassmann coordinates describe all the internal degrees of freedom of fields - their spins and charges. The generators of the Lorentz transformations in Grassmann space, defining the Yang- Mills charges, commute with the generators of the Lorentz transformations in the four dimensional subspace in accordance with the Coleman- Mandula theorem [21] as well as with its extension for the supersymmetric case [22].

The proposed approach also offers representations, not included in the Standard Electroweak Model. These representations, which predict physics beyond the Standard Electroweak Model, need a further study [23].
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References

[1] G.E. Uhlenbeck and S. Goudsmit, Naturwiss. 13, 953 (1925),
[2] P.M. Dirac, Proc. Roy. Soc., A117, 601 (1928), A118, 351 (1928),
[3] W. Heisenberg, Zeitschrift für Phy. 77, 1(1932),
[4] M. Gell-Mann, Phys. Lett. 8, 214 (1964), G. Zweig, CERN preprint 8182/Th. 401 (1964), O.W. Greenberg, Phys. Rev. Lett. bf 13, 598 (1964),
[5] S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967),
[6] A. Salam and J. C. Ward, Phys. lett. 13 168 (1964),
[7] S.L. Glashow, Nucl. Phys. 22, 579 (1961),
[8] H. Georgi and S.L. Glashow, Phys. Rev. Lett.32, 438 (1974)
[9] M. Kaku, Introduction to Superstrings (Springer-Verlag New York Berlin Heidelberg London Paris Tokyo 1988)
[10] M.J. Duff, Nucl. Phys. B 219, 389 (1983), M.J. Duff, B.E.W. Nilsson and C.N. Pope, Phys. Lett. B 139, 154 (1984)
[11] L. Susskin, Phys. Rev. D20, 2613 (1979), S. Weinberg, Phys. Rev. D19, 1279 (1979)
[12] J. Wess, B. Zumino, Nucl. Phys. B 70, 39 (1974)
[13] N. Mankoč Borštnik, Modern Phys. Lett. A 10, 587 (1995), Proceedings of the International Conference Quantum Systems, New Trends and Methods, Minsk, 23-29 May, 1994, p. 312, Ed. by A.O. Barut, I.D. Feranchuk, Ya.M. Shnir, L.M. Tomil’chik, World Scientific, Singapore 1995, Proceedings of the US-Polish Workshop Physics From Plank Scale to Electroweak Scale, Warsaw, 21-24 Sept. 1994, p. 86, Ed. by P. Nath, T. Taylor, S. Pokorski, World Scientific, Singapore 1995, Proceedings of the 7th Adriatic Meetings on High Energy Physics, New Trends in Physics, Brioni, Croatia, 13-22 Sept. 1994, p. 296 Ed. D. Klabučar, I. Picek, D. Tadić, World Scientific, Singapore 1995,
[14] N. Mankoč-Borštnik, *Phys. Lett. B* 292, 25 (1992), *Il Nuovo Cimento A* 105, 1461 (1992), *J. of Math. Phys.* 34, 8 (1993), *Int. J. of Mod. Phys. A* 9, 1731 (1994), *J. of Math. Phys.* 36(4) 1593 (1995),

[15] S. Raby, *Proceedings of the US - Polish Workshop Physics From Plank Scale to the Electroweak Scale, Warsaw, 21-24 Sept., 1994*, p.184, Ed. by P. Nath, T. Taylor, S. Pokorski, World Scientific, Singapore 1995,

[16] F.A. Berezin and M.S. Marinov, *The Methods of Second Quantization*, Pure and Applied Physics (Academic Press, New York, 1966)

[17] D. Lurie, *Particles and Fields*, Interscience Publishers (John Wiley and Sons, New York 1968)

[18] H. Georgi, *Lie Algebra in Particle Physics* (The Benjamin/Cummings Publishing Company, Inc. Advanced Book Program, 1982)

[19] M. Hamermesh, *Group Theory and its Application to Physical Problems* (Dover Publications, Inc. New York, 1989)

[20] W. Greiner, B. Müller, *Quantum Mechanics, Symmetries* (Springer-Verlag New York Berlin Heidelberg London Paris Tokyo 1989)

[21] S. Coleman, J. Mandula, *Phys. Rev.* 159, 1251 (1967)

[22] R. Haag, J.t. Lopuszanski, M.F. Sohnius, *Nucl. Phys.* B88, 257 (1975)

[23] A. Borštnuk, N. Mankoč Borštnik, The Standard Electroweak Model and beyond and Grassmann space, in preparation