Distributed medium access control for energy efficient transmission in cognitive radios

Oshri Naparstek1, Amir Leshem1 (Senior Member, IEEE) and Eduard Jorswieck2 (Senior Member, IEEE)

Abstract—In the last few years the issue of energy-efficient network design has gained increasingly more importance [1], [2], both in academia and in industry. Information and communication technologies (ICT) represent about 2% of the entire world’s energy consumption, and the situation is likely to reach a point where ICT equipments in large cities will require more energy than is actually available [3]. In this paper we develop and investigate methods for energy efficiency optimization for wireless networks with cognitive units. We show that the problem can be solved without explicit message passing using a modified distributed auction algorithm. Then we introduce a fast converging algorithm that approximates the solution to the maximal energy efficiency problem. We state conditions under which the fast algorithm is asymptotically optimal in terms of the number of users. Finally, we provide simulated examples of the methods developed in this article.

I. INTRODUCTION

While for decades communication networks have been designed to optimize performance measures such as bit-error-rate, latency, data-rate, etc., in the last few years the issue of energy-efficient network design has gained increasingly more importance [1], [2], both in academia and in industry. Information and communication technologies (ICT) represent about 2% of the entire world’s energy consumption, and the situation is likely to reach a point where ICT equipment in large cities will require more energy than is actually available [3]. As for data networks, contrary to intuition more energy is consumed in access networks than in core networks, because the number of devices in access networks (i.e. mobile terminals, base stations, and data modems installed on customers’ premises), is orders of magnitude larger than the number of communication devices (routers, multiplexers, etc.) in the core network. For this reason, research in the field of wireless networks has focused on the problem of optimizing the Energy Efficiency (EE) at the physical layer, by maximising the ratio of a SINR-based function over the consumed power.

The first and most widely used definition of EE is the ratio between the throughput and the transmit power [4], [5], [6], [7], [8], [9], [10], and references therein. Another proposed metric uses the goodput in place of the throughput [11]. In all of the above works, as far as the computation of the consumed power is concerned, only the transmit power is considered, whereas the power that is dissipated in the electronic circuitry of each terminal in order to keep the terminal active is neglected. This assumption was relaxed in [12], by defining the consumed power as the sum of the transmit power plus a constant term, independent of the transmit power, which models the circuit power needed to operate the terminal. Following [12], in [13], [14], [15] the consumed power is also defined as the sum of the transmitted power and the circuit power. Moreover, in these papers the throughput is replaced by the achievable rate in the definition of the energy efficiency. In [14], [15] a single user channel is considered and the optimization is carried out through transmit power control. By contrast, in [13], multiuser interference channels are considered and, as well as in references [4], [5], [6], [7], [8], [9], [10], [11], a competitive scenario in which users selfishly aim at individual EE maximization is addressed.

Each transmitter $k$ is not only interested in maximizing its own performance in terms of achieved SINR $\gamma_k$, but also in saving as much battery energy as possible. This trade-off is well modeled by defining the EE of a given terminal $k$, as the ratio between the so-called efficiency function which measures the SINR-based performance of user $k$ and the power consumed to attain this performance level [13], [12], [14], [15]: namely

$$EE_k = \frac{f(\gamma_k|p_k)}{p_k + P_{c,k}}.$$  \hfill (1)

In (1), $P_{c,k}$ is the power that is required by the transmitter electronic circuitry to operate the device, and which is dissipated even during non-transmission periods. For further details on the circuitry power term, we refer the reader to [16] and references therein, where several power consumption models for wireless networks are developed. As for $f(\gamma)$, in principle it can be a generic increasing function of the $k$-th user’s SINR, with $f(0) = 0$ and such that $\frac{f}{\gamma} | \gamma \rightarrow 0$ tends to zero for growing $p_k$.

Two widely used efficiency functions are

1) $f(\gamma_k) = R(1 - e^{-\gamma_k})$, where $R$ is the communication rate and $(1 - e^{-\gamma_k})$ an approximation of the probability of correct symbol reception. A similar approximation was used in [5], [6]. Thus, $f$ is the number of bits that are correctly demodulated at receiver $k$ per unit of time.

2) $f(\gamma_k) = W \log(1 + \gamma_k)$, where $W$ is the communication bandwidth. For strictly static channels $f$ represents the $k$-th user’s achievable rate. For quasi-static channels, the use of $f$ for resource allocation purposes is still well-motivated in view of the assumption that the channel coefficients remain constant for longer than the resource allocation phase.

Variations of option 1) are also available in the literature in the form of $f(\gamma_k) = R(1 - e^{-\gamma_k})^M$ and $f(\gamma_k) = R(1 - e^{-\gamma_k/2})^M$, and in this case the function $f(\gamma_k)$ is an approximation of

\[^1\text{Faculty of Engineering, Bar-Ilan university, Ramat-Gan, 52900, Israel.} \]
\[^2\text{School of Engineering Sciences, TU Dresden, 01062 Dresden, Germany.} \]
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the probability of error-free reception of a data packet of \( M \) symbols. An EE that considers both the case of \( M > 1 \) and the circuit power \( P_{c,k} \) was considered in [12] for a single-hop system. There, it was shown that an equilibrium for the power allocation algorithm existed, but the convergence could not be proved. The techniques developed in this paper could be used to extend the results of [12] to the relay-assisted scenario, as well. However, in the following we choose to focus on the equally well-motivated case of \( M = 1 \). Thus, for any \( M \), the resulting EE \( (1) \) is a measure of the number of bits that are correctly decoded at the receiver, per unit of time and per Joule of energy drained from the battery of the transmitter. Moreover, all the efficiency functions that we consider result in an EE \( (1) \) which is measured in bits over Joules, thus representing a natural measure of the efficiency with which each Joule of energy drained from the battery is being used.

Two pertinent social welfare performance metrics are the average EE and the system global EE (GEE), respectively defined as \[6\], \[8\], \[7\], \[10\]

\[
\text{EE}^{av} = \frac{1}{K} \sum_{k=1}^{K} \frac{f(\gamma_k)}{P_{c,k} + p_k}
\]

and

\[
\text{GEE} = \frac{\sum_{k=1}^{K} f(\gamma_k)}{\sum_{k=1}^{K} P_{c,k} + p_k}.
\]

Customarily, the GEE is used to describe the overall system EE. Centralised and decentralised resource allocation in multi-hop networks for energy-efficiency maximization was studied in [17], [18].

Unlike cellular networks where the energy efficiency can be optimized in a centralized manner, in many cases centralized optimization is not possible. In this case distributed protocols are needed to maximize the energy efficiency of the network. Several approaches had recently been suggested for optimal distributed allocation. These approaches utilize an opportunistic version of carrier sensing to determine an orthogonal allocation which is optimal or near optimal in the sense of exploiting channel diversity. The first example is the use of the well known Gale-Shapley stable marriage theorem [19] to allocate spectra in a multichannel setup [20], [21]. In that work, Leshem et al. demonstrated how a stable channel allocation can be obtained without any explicit transmission using carrier sensing as a mechanism to prioritize channel. Analysis of this technique for Rayleigh fading channels appeared in [27]. Other approaches to the distributed channel allocation problem include game theoretic bargaining solutions [22], [23], [24] and distributed allocation using multichannel ALOHA [25], [26]. While the stable allocation which turns out to be the greedy assignment is almost optimal for Rayleigh fading channels, it is desirable to obtain the optimal allocation. It is well known that this allocation can be computed by solving a linear programming problem [28]. However, in order to compute the optimal distributed solution [29], [30] revised the auction technique of Bertsekas [31] which requires a shared memory or price exchange between the bidders and the auctioneer. Instead of knowledge of the highest price this technique only requires knowledge of local prices. Based on the local prices an algorithm which can be implemented using multichannel opportunistic CSMA is presented and its optimality is proved.

In this paper we present fully distributed MAC protocols to maximize the energy efficiency of wireless networks. The protocols does not require any explicit communication between the wireless devices and are asymptotically optimal. The paper is organized as follows: in Section II we define the maximal energy efficiency problem. In Section III we present the algorithm and show how it can be implemented in a fully distributed manner using opportunistic CSMA. In Section IV we present a relaxation to the maximal energy efficiency problem and show that the algorithm for the relaxed problem terminates within \( O(N \log(N)) \) iterations with high probability. We then show that the algorithm can be used to yield an asymptotically optimal solution to the maximal energy efficiency problem. In Section VI we discuss simulated results for the proposed algorithm.

II. PROBLEM FORMULATION

We consider the case of \( N \) cognitive transceiver pairs sharing a time slotted unlicensed frequency band divided into \( K \) sub bands. We assume \( K \geq N \). Each user \( n \) is preassigned a target rate \( R_n \). The transmission power \( P(n, k) \) is defined as the minimal power needed by the \( n \)’th user to achieve his target rate in the \( k \)’th channel. We assume all the users have continuous sensing over all channels. We also assume that only one user can transmit on each channel in each time slot and consider out of cell interferences as noise. We seek a fully distributed method to maximize the overall energy efficiency of the system. We assume that the coherence time of the channel is long enough for the algorithm to converge. Under this assumption, \( P(n, k) \) is the solution to the following equation

\[
R_n = \log_2 \left( 1 + \frac{|H(n, k)|^2 P(n, k)}{\sigma_n^2} \right)
\]

where \( H \) is the channel matrix and \( \sigma_n^2 \) is the noise variance of the \( n \)’th receiver. The solution to the above equation is

\[
P(n, k) = \frac{(2^{R_n} - 1) \sigma_n^2}{|H(n, k)|^2}.
\]

\( P(n, k) \) can also be chosen according to a goodput requirement. Alternatively, instead of a rate requirement \( R_n \) per user we can choose \( P(n, k) \) to fulfill a goodput requirement

\[
T_n = R(1 - e^{-|H(n, k)|^2 P(n, k) \sigma_n^2})
\]

for fixed \( R > 0 \) and \( T_n \). Solve for \( P(n, k) \) leads to

\[
P(n, k) = \frac{\log \left( 1 - \frac{T_n}{2R} \right) \sigma_n^2}{|H(n, k)|^2}
\]

using the formulation of equation [5]. The utility for the \( n \)’th user in the \( k \)’th channel in the computation of the \( \text{EE}^{av} \) criterion is given by:

\[
U_{av}(n, k) = \frac{R_n}{|H(n, k)|^2} + c(n)
\]
where \( c(n) \) is the minimal amount of energy the \( n \)'th device requires to operate at any given time. Under the assumption of preassigned target rates the GEE simplifies into a power minimization problem under rate constraints. We further assume that the instantaneous transmission power is limited. We denote the maximum instantaneous transmission power per user as \( P_{\text{max}} \). For simplicity, we formulate the utility of the GEE as a maximization problem. We define the utility matrix \( U_{\text{GEE}}(n,k) \) for the GEE criterion as

\[
U_{\text{GEE}}(n,k) = \begin{cases} P_{\text{max}} - P(n,k), & P(n,k) \leq P_{\text{max}} \\ 0, & P(n,k) > P_{\text{max}}. \end{cases}
\]  

(9)

The maximum energy efficiency problem can be formulated as an integer programming problem

\[
\max \sum_{n=1}^{N} \sum_{k=1}^{K} U(n,k) \eta(n,k)
\]

s.t.

\[
\sum_{k} \eta(n,k) = 1, \quad \forall n = 1, 2, ..., N \\
\sum_{n} \eta(n,k) = 1, \quad \forall k = 1, 2, ..., K \\
\eta(n,k) \in \{0,1\}, \quad \forall n, k
\]  

(10)

The constraint matrix of this problem is totally unimodular. This means that the solution to the relaxed problem where we replace the integer constraint by \( 0 \leq \eta(n,k) \leq 1 \) is also the solution to the original problem. The relaxed problem is a linear programming (LP) problem and can be solved efficiently in a centralized manner by LP solutions methods such as the Hungarian Method \[38\]. In the next section we review distributed algorithms for the channel assignment problem.

### III. DISTRIBUTED PROTOCOL FOR MAXIMAL ENERGY EFFICIENCY

In this section we propose the use of a fully distributed channel assignment algorithm that does not require any explicit message passing or a shared memory between the users. The algorithm relies on the auction algorithm \[31\] and the distributed algorithm suggested in \[30\] for sum-rate maximization. The distributed protocol consists of a bidding stage and an assignment stage. The description of the algorithm is as follows. The utility matrix \( U \) is an \( N \times K \) matrix of energy efficiency indexes and \( C \) is an \( N \times K \) cost matrix. The cost of a channel \( C(n,k) \) is a unitless number that merely represents how much user \( n \) wants channel \( k \) in comparison to the other users. We define the profit of user \( n \) from channel \( k \) as the reward (i.e. energy efficiency index) minus the price of the channel \( U(n,k) - C(n,k) \). In the initialization stage each user sets the cost for all of the channels to be 0; i.e., \( C(n,k) = 0, \forall n,k \), select \( \epsilon > 0 \) and sets his state to unassigned. \( \tilde{k}_n \) is defined as the most profitable channel of the \( n \)'th user. The distributed protocol proceeds in iterations. In each iteration two stages are sequentially performed, a bidding stage where users raise the price on their most profitable channel and an assignment stage where channels are assigned to the users who proposed the highest prices. In the bidding stage, each unassigned user \( n \) finds his most profitable channel \( \tilde{k}_n \) and the profits from that channel \( \gamma_n \) and his second most profitable channel \( \omega_n \)

\[
\tilde{k}_n = \arg \max_k (U(n,k) - C(n,k)) \\
\gamma_n = U(n, \tilde{k}_n) - C(n, \tilde{k}_n) \\
\omega_n = \max_{k \neq \tilde{k}_n} (U(n,k) - C(n,k))
\]  

(11)

Each unassigned user raises the price on his most profitable channel by

\[
C(n, \tilde{k}_n) = C(n, \tilde{k}_n) + \gamma_n - \omega_n + \epsilon
\]  

(12)

where \( \epsilon \) is a predetermined constant that can be seen as the price of participating in the auction. After the unassigned users update their prices, all the users bid on their most profitable channels. If user \( n \) gets assigned to channel \( k \) he continues to bid on that channel without raising his bid. If a user \( n \) is unassigned he bids on \( \tilde{k}_n \) with the new bid \( C(n, \tilde{k}_n) \). In the assignment stage each channel is assigned to the highest bidding user. A channel without bids stays unassigned and users who were not assigned to channels become unassigned. The bidding and assignment stages proceed in iterations until all the users are assigned to channels. Once all of the users are assigned to channels, no one raises his bid and as a result the assignment becomes static. When all the users are assigned we say that the algorithm has converged. The distributed auction algorithm appears in Table \[I\]. It was proven in \[30\] that the distributed auction algorithm converges in finite time to a solution within \( N \epsilon \) from the optimal solution. The distributed auction algorithm can be applied using an opportunistic CSMA protocol.

One can implement the distributed protocol without the use of explicit message passing using opportunistic CSMA. Opportunistic CSMA \[32\] is a distributed transmission protocol suggested for wireless sensor networks. Opportunistic CSMA is composed of carrier sensing and a waiting strategy. Continuous sensing of all channels by all users is assumed. Each user in the network calculates a fitness measure \( \psi_n \) and maps it into a waiting time \( \tau_n \) based on a predetermined common decreasing function \( f(\psi_n) \). An example of a mapping function from fitness to waiting times is shown in Figure \[I\]. Each user waits until his waiting time ends and if no one transmitted on his most wanted channel then he is allowed

| TABLE I | DISTRIBUTED AUCTION ALGORITHM |
|--------|-------------------------------|
| Select \( \epsilon > 0 \), set all the users as unassigned and set \( C(n,k) = 0, \forall n,k \) |
| **Repeat** |
| 1. Each unassigned user \( n \) calculates his own maximum profit: \( \gamma_n = \max_k (U(n,k) - C(n,k)) \) |
| 2. Each unassigned user \( n \) calculates his second maximum profit: \( \omega_n = \max_{k \neq \tilde{k}_n} (U(n,k) - C(n,k)) \) |
| 3. Each unassigned user \( n \) updates the price of his best channel \( \tilde{k}_n \) to be \( C(n, \tilde{k}_n) = C(n, \tilde{k}_n) + \gamma_n - \omega_n + \epsilon \) |
| 4. All the users bid. The unassigned users bid on their new best channel with the updated bid. The assigned users bid on the last channel they bid on and with the same price. |
| 5. Assign channel to the highest bidder (channels with no bids stay unassigned) |
| **Until** all users are assigned |
The price expected number of iterations. is asymptotically optimal and can be solved with next sections we suggest a relaxation to the maximal energy $O$ is described in Table II. It was shown in \[30\] that the number times must converge in a finite number of iterations as in the auction algorithm. The distributed auction algorithm for cognitive radio systems using opportunistic CSMA can be used as an auctioneer. The opportunistic CSMA protocol does not require explicit message passing between users and hence it is suitable for the implementation of the fully distributed solution on the maximum energy efficiency problem. As seen above, the resulting assignment is within $N\epsilon$ from the optimal solution as in the auction algorithm. We define the reward that each user $n$ gets from channel $k$ to be the energy efficiency of that channel $U(n,k)$. Using the opportunistic CSMA scheme, each user $n$ tries to access his best profit channel defined by

$$\tilde{k}_n = \arg\max_k (U(n,k) - C(n,k))$$

with a backoff time of

$$\tau_n = f(\rho_{n,\tilde{k}_n})$$

where $f(x)$ is a positive monotonically decreasing function. The price $C(n,k)$ is determined and updated if necessary as described in Table II. The prices and their corresponding waiting times must converge in a finite number of iterations as in the distributed auction algorithm. The distributed auction algorithm for cognitive radio systems using opportunistic CSMA is described in Table II. It was shown in \[30\] that the number of iterations needed until the distributed auction algorithm is converged is bounded by $O\left(N^3\right)$ with respect to $N$. In the next sections we suggest a relaxation to the maximal energy efficiency problem. We show that the suggested relaxation is asymptotically optimal and can be solved with $N\log(N)$ expected number of iterations.

IV. FAST MATCHING ALGORITHM

In Section II we formulated the maximum energy efficiency channel assignment problem for arbitrary values. However, if $U \in \{0,1\}^{N \times K}$ the problem is reduced to finding matchings on bipartite graphs. To formulate the maximum energy efficiency assignment problem as a matching problem on bipartite graphs we use the following definitions \[33\]:

**Definition 1:** Let $G = (V,E)$ be a graph with a vertex set $V$ and an edge set $E$. The neighborhood of vertex $v \in V$ is given by

$$n_v = \{u \in V : (u,v) \in E\}.$$  \hspace{2cm} (15)

**Definition 2:** Let $G = (\tilde{V},E)$ be a graph with a vertex set $\tilde{V}$ and an edge set $E$. If $\tilde{V}$ can be divided into two subsets $\tilde{U},\tilde{V}$ such that

$$n_u \cap \tilde{U} = \emptyset, \quad \forall u \in \tilde{U}$$

$$n_v \cap \tilde{V} = \emptyset, \quad \forall v \in \tilde{V}.$$ we say that $G$ is a bipartite graph and we denote it by $G(\tilde{U},\tilde{V},E)$.

**Definition 3:** Let $G = (U,V,E)$ be a bipartite graph with a vertex sets $|U| = |V| = N$ and an edge set $E$. Let $M \subseteq E$ and let $\tilde{G} = (U,V,M)$ be a bipartite subgraph of $G$ with vertex sets $|\tilde{U}|,|\tilde{V}| = N$ and an edge set $M$. $M$ is a matching on $G$ if

$$\max_{n \in \tilde{U} \cup \tilde{V}} |n_v| = 1.$$  \hspace{2cm} (16)

**Definition 4:** Let $G = (U,V,E)$ be a bipartite graph with vertex sets $|U| = |V| = N$ and an edge set $E$. $M$ is a perfect matching if $M$ is a matching and $|M| = N$.

**Definition 5:** Let $G = (U,V,E)$ be a bipartite graph with vertex sets $|U| = |V| = N$ and an edge set $E$. Let $M \subseteq E$ and let $\tilde{G} = (U,V,M)$ be a bipartite subgraph of $G$ with vertex sets $|\tilde{U}|,|\tilde{V}| = N$ and an edge set $M$. A vertex $v \in U \cup V$ is free if $|n_v| = 0$ otherwise we say it is not free.

Using the above definitions we define the maximum cardinality matching problem as follows: Let $G = (U,V,E)$ be bipartite graph with vertex sets $|U| = |V| = N$ and an edge set $E$. Find a matching $M$ such that $|M|$ is maximal.

The MCM problem can also be formulated as a max-energy efficiency problem \[10\] where the reward matrix is a binary matrix with 0, 1 values. We now present an algorithm that finds a maximum cardinality matching on bipartite graphs which can be implemented in a fully distributed manner.

### Table II

**DISTRIBUTED AUCTION ALGORITHM USING OPPORTUNISTIC CARRIER SENSING**

| Step | Description |
|------|-------------|
| 1. | Each unassigned user $n$ calculates its own maximum profit: $\gamma_n = \max_k (U(n,k) - C(n,k))$ |
| 2. | Each unassigned user $n$ calculates its own second maximum profit: $\tilde{k}_n = \arg\max_k (U(n,k) - C(n,k))$ |
| 3. | Each unassigned user $n$ updates the price of his best channel $\tilde{k}_n$ to be $C(n,\tilde{k}_n) = C(n,\tilde{k}_n) + \gamma_n - \omega_n + \epsilon$ |
| 4. | Each unassigned user $n$ maps the new best profit into backoff time $\tau_n = f(C(n,\tilde{k}_n))$ |
| 5. | Each user waits for $\tau_n$ milliseconds. If the user detects that $\tilde{k}_n$ is free when its backoff time has expired, he transmits his packet on $\tilde{k}_n$ and sets his state to assigned. |

**Select** $\epsilon > 0$, each user sets himself as unassigned and set $C(n,k) = 0, \forall k$.

**Repeat**

1. Each unassigned user $n$ calculates its own maximum profit:

$$\gamma_n = \max_k (U(n,k) - C(n,k))$$

2. Each unassigned user $n$ calculates its own second maximum profit:

$$\tilde{k}_n = \arg\max_k (U(n,k) - C(n,k))$$

$$\omega_n = \max_{k \neq \tilde{k}_n} (U(n,k) - C(n,k))$$

3. Each unassigned user $n$ updates the price of his best channel $\tilde{k}_n$ to be $C(n,\tilde{k}_n) = C(n,\tilde{k}_n) + \gamma_n - \omega_n + \epsilon$

4. Each unassigned user $n$ maps the new best profit into backoff time $\tau_n = f(C(n,\tilde{k}_n))$

5. Each user waits for $\tau_n$ milliseconds. If the user detects that $\tilde{k}_n$ is free when its backoff time has expired, he transmits his packet on $\tilde{k}_n$ and sets his state to assigned.

**Until** all users are assigned
TABLE III
ALGORITHM FOR MAXIMAL MATCHING

1) Initialize $h_0 = 0, \forall v \in V$ and set $M = \emptyset$
2) While $|M| \leq N$ do
   a) Choose $u \in U_{\text{free}}$
   b) $j = \arg \min_{v \in \mathcal{G}_{\text{old}}} h_v$
   c) $M = M \cup (u,j)$
   d) $u_{\text{old}} = \{u \in U : (u,j) \in M\}$
   e) $M = M \setminus (u_{\text{old}},j)$
   f) $U_{\text{free}} = U_{\text{free}} \cup (u_{\text{old}},j)$
   g) $h_j = h_j + 1$
3) Return

We propose an iterative algorithm that assigns an unassigned user to a channel according to the following scheme: Each channel $k \in K$ is assigned a value $h_k$ that represents how many times the channel was reassigned to different users. Let $h^{(i)} = [h_1^{(i)}, h_2^{(i)}, ..., h_K^{(i)}]$ be the vector of the values of the channels on the $i$'th iteration. At the beginning of the algorithm all the values of the channels are initialized to 0; i.e.,

$$h_k^{(0)} = 0, \quad \forall k = 1, 2, ..., K.$$  

Let $U_{\text{free}}$ be the set of all free users. On each iteration, an unassigned user $u \in U_{\text{free}}$ is chosen. $u$ is then assigned to the channel with a minimal value he can access and raises its value by 1. The algorithm is depicted in Table III.

A. Expected number of iterations of the matching algorithm

In this section we analyze the expected number of iterations until the algorithm converges for random bipartite graphs. A random bipartite graph is defined as follows:

**Definition 6:** $G = (U, V, E)$ is called a bipartite random graph if $G$ is a bipartite graph and the edges in $E$ are independently chosen with probability $p$; i.e.,

$$\Pr ((u,v) \in E) = p, \quad \forall u \in U, \quad \forall v \in V.$$  

Denote the set of all random bipartite graphs with vertex sets $|U| = |V| = N$ and probability $p$ for an edge by $B(N,p)$. The following known result on perfect matching in random bipartite graphs was proven by Erdős and Rényi in [34].

**Theorem 1:** Let $p = \frac{(1+\epsilon) \log(N)}{N}$ and $G \in B(N,p)$ then

$$\lim_{N \to \infty} \Pr (G \text{ contains a perfect matching}) = e^{-2N^{-\epsilon}} = 0.$$  

(17)

The next theorem proven in [36] shows that for random bipartite graphs with $p \geq \frac{(1+\epsilon) \log(N)}{N}$ the number of iterations until the convergence of the algorithm is less than $\frac{cN \log(N) \log(Np)}{\log(Np)}$ with high probability and $c > 0$ is some constant.

**Theorem 2:** Let $G = (U, V, E)$ be a random bipartite graph with $|U| = |V| = N$ and $p \geq \frac{(1+\epsilon) \log(N)}{N}$. Let $T$ be the number of iterations until the algorithm converges then

$$\lim_{N \to \infty} \Pr \left( T \leq \frac{cN \log(N) \log(Np)}{\log(Np)} \right) = 1.$$  

(18)

Theorem 2 ensures that the fast matching finds a perfect matching with a probability that approaches 1 in $O(N \log(N))$ iterations. In the next section we show how the fast matching algorithm can be implemented to find asymptotically optimal solutions to the energy efficiency maximization problem with a small number of iterations.

V. FAST ENERGY EFFICIENCY MAXIMIZATION ALGORITHM

In this section we show how the algorithm described in Section IV can be used to give asymptotically optimal solutions to obtain maximally energy efficient channel assignment. The main idea is to transform $U$ into a binary 0, 1 matrix $\tilde{U}$ and apply the matching algorithm on $\tilde{U}$. The transformation from $U$ to $\tilde{U}$ is done by applying a threshold $a_n^{\text{thresh}} \geq 0$ for each row.

$$\tilde{U}(n,k) = \begin{cases} 1, & U(n,k) \geq a_n^{\text{thresh}} \\ 0, & U(n,k) < a_n^{\text{thresh}}. \end{cases}$$  

(19)

To ensure asymptotically optimal solutions to the max-energy efficiency problem, $a_n^{\text{thresh}}$ must satisfy the following requirements:

1) Only the best channels of each user should be above $a_n^{\text{thresh}}$.

2) $\tilde{U}$ should contain a perfect matching with high probability.

The first condition is to ensure that the solution to $\tilde{U}$ will provide a good solution to the max-sum problem and the second condition is to ensure that with high probability all of the users will get assigned by the algorithm. Assume that each row $n$ of $U$ consists of i.i.d random variables and assume that the cumulative distribution function (CDF) of each entry of $U(n,k)$ is given by $F_n$. A proper choice of $a_n^{\text{thresh}}$ would be

$$a_n^{\text{thresh}} = F_n^{-1} \left( 1 - \frac{\alpha \log(N)}{N} \right),$$  

(20)

where $\alpha > 1$. The choice of the threshold satisfies the first condition since only $\alpha \log(N)$ best channels of each user are above the threshold. Theorem 1 ensures that with high probability $\tilde{U}$ contains a perfect matching if $a_n^{\text{thresh}} \geq 0$ for all $n = 1, 2, ..., N$. Hence, the target rates should be chosen such that with high probability $a_n^{\text{thresh}} \geq 0$ for all $n = 1, 2, ..., N$.

A. Target rates for Rayleigh channels

In this section we analyze the fast matching algorithm for Rayleigh fading. We model the channels of each user as Rayleigh fading channels; i.e., the channel attenuation $|H(n,k)|^2$ is an exponential random variable given by:

$$|H(n,k)|^2 = G_n \cdot F_n \cdot \frac{1}{r_n^\alpha}$$  

(21)

where $G_n$ is a global normalizing factor, $F_n$ is an exponentially distributed gain (due to the Rayleigh fading channel with a multipath effect), $r_n$ is the distance between the $n$'th transmitter from its receiver and $\alpha$ is the path loss exponent; therefore:

$$f_{|H(n,k)|^2}(x) = \lambda_n e^{-\lambda_n x}$$  

(22)

where

$$\lambda_n = \frac{r_n^\alpha}{G_n}$$  

(23)
The rate of user \( n \) in channel \( k \) is given by [35]:

\[
R_n(k) = \log_2 \left( 1 + \frac{\|H(n,k)\|^2 P(n,k)}{\sigma_n^2} \right)
\]  

(24)

We now derive sufficient conditions on \( R_n \) such that the fast algorithm will converge within \( O(N \log(N)) \) expected number of iterations.

**Theorem 3:** The fast matching algorithm will converge within \( O(N \log(N)) \) expected number of iterations in Rayleigh fading if the following requirement is satisfied for all \( n \):

\[
R_n \leq \log_2 \left( 1 + \frac{P_{\max} \log \left( \frac{N}{\alpha \log(N)} \right)}{\lambda_n \sigma_n^2} \right).
\]  

(25)

**Proof:** As shown in equation (5) the minimal power needed to achieve rate \( R_n \) is given by:

\[
P(n,k) = \frac{(2R_n - 1) \sigma_n^2}{\|H(n,k)\|^2}.
\]

For Rayleigh channels \( \|H(n,k)\|^2 \) is exponentially distributed with CDF

\[
F_{\|H(n,k)\|^2}(x) = 1 - e^{-\lambda_n x}.
\]  

(26)

As a result, the CDF of \( P(n,k) \) is given by:

\[
F_{P(n,k)}(x) = e^{-\lambda_n \sigma_n^2 (2R_n - 1)/x}.
\]  

(27)

The expected number of iterations will be \( O(N \log(N)) \) only if there exists a perfect matching in the graph with a probability of at least \( 1 - \frac{2}{N^{\alpha \log(N)}} \). From Theorem 2 there exists a perfect matching with a probability of at least \( 1 - \frac{2}{N^{\alpha \log(N)}} \) only if the expected number of edges connected to each vertex is at least \( \alpha \log(N) \) for \( \alpha > 1 \). Hence, in order to fulfill this requirement the expected number of channels in which each user is able to transmit without violating his power constraint is at least \( \alpha \log(N) \). A user can transmit on a channel only if the power needed on the channel to achieve the target rate is less than \( P_{\max} \).

\[
F_{P(n,k)}(P_{\max}) \geq \frac{\alpha \log(N)}{N}.
\]  

(28)

Hence, the target rates for each user must satisfy:

\[
F_{P_{\max}^{-1}} \left( \frac{\alpha \log(N)}{N} \right) \leq P_{\max}.
\]  

(29)

The inverse CDF of \( F_{P(n,k)}(x) \) is given by:

\[
F_{P_{\max}^{-1}}(x) = \frac{\lambda_n \sigma_n^2 (2R_n - 1)}{\log(x)}.
\]  

(30)

Hence, the target rates must satisfy

\[
F_{P_{\max}^{-1}} \left( \frac{\alpha \log(N)}{N} \right) = \frac{\lambda_n \sigma_n^2 (2R_n - 1)}{\log(N)} \leq P_{\max}
\]  

(31)

which implies that if all \( R_n \) satisfy:

\[
R_n \leq \log_2 \left( 1 + \frac{P_{\max} \log \left( \frac{N}{\alpha \log(N)} \right)}{\lambda_n \sigma_n^2} \right),
\]  

(32)

then the fast algorithm converges with an expected \( O(N \log(N)) \) number of iterations.

### B. Asymptotical optimality

In this section we shall show that the fast assignment is asymptotically optimal for Rayleigh fading with properly chosen target rates as shown in [V-A]. For the analysis we use some known results from order statistics [37].

**Definition 7:** Let \( A \) be a random variable with CDF \( F_A(r) \) and let \( A_1:N < A_2:N < \ldots < A_{N:N} \) be random variables obtained by taking \( N \) samples from \( A \) and ordering the samples in an increasing order. \( A_{k:k} \) is called the \( k \) th order statistic of \( A \) with \( N \) samples.

**Definition 8:** Let \( k_N \) be a function of \( N \) such that \( k_N \to \infty \) as \( N \to \infty \) and \( \lim_{N \to \infty} k_N/N = 0 \) then \( A_{N-k_N+1:N} \) and \( A_{k_k:N} \) are called intermediate order statistics.

**Definition 9:** Let \( F_A(x) \) be a differentiable, absolutely continuous distribution function. If

\[
\lim_{x \to F^{-1}(1)} \frac{d}{dx} \left( \frac{1 - F(x)}{f(x)} \right) = 0
\]  

(33)

then the third Von Mises condition is satisfied.

An important result on intermediate order statistics was found by Falk [38].

**Theorem 4:** (Falk 1989). Let \( F \) be an absolutely continuous CDF satisfying one of the Von Mises conditions. Suppose \( k_N \to \infty \) as \( N \to \infty \) and \( \lim_{N \to \infty} k_N/N = 0 \). Then there exist norming constants \( \alpha_N \) and \( \beta_N > 0 \) such that

\[
\frac{A_{N-k_N+1:N} - \alpha_N}{\beta_N} \to N(0,1).
\]  

(34)

where \( \alpha_N = F^{-1}(1 - k_N/N) \) and \( \beta_N = \sqrt{\frac{k_N}{f(\alpha_N)N}} \).

We now state the main result of this section:

**Theorem 5:** Let \( A_{\text{opt}}^{\text{GEE}} \) be the optimal solution to the max-energy efficiency problem for the GEE and let \( A_{\text{opt}}^{\text{GEE}} \) be the solution obtained by the fast assignment algorithm. We assume that the rates are properly chosen and a perfect matching exists. Then for Rayleigh channels

\[
\lim_{N \to \infty} \frac{E \left( A_{\text{opt}}^{\text{GEE}} \right)}{E \left( A_{\text{opt}}^{\text{fast}} \right)} = 1
\]  

(35)

**Proof:** We start by formulating the probability distribution and quantile function of \( U(n,k) \) given that the power is lower than \( P_{\max} \).

\[
F_{U_{\text{GEE}}(n,k)}(x) = 1 - e^{-a_n \frac{x}{P_{\max}}} e^{-\frac{\sigma_n^2}{2P_{\max}}} \}
\]  

(36)

where \( a_n = \lambda_n \sigma_n^2 (2R_n - 1) \) and

\[
F_{U_{\text{GEE}}(n,k)}^{-1}(\rho) = P_{\max} + \frac{\alpha_n}{\log(1 - \rho) - \frac{a_n}{P_{\max}}}.
\]  

(37)

We now observe that this probability distribution satisfies the third Von Mises condition and as a result

\[
\lim_{N \to \infty} E \left( U_{N-\alpha \log(N)+1:N} \right) = F^{-1}(1 - \frac{\alpha \log(N)}{N}) = \frac{P_{\max} + \frac{a_n}{\log(N)}}{\log(\log(N)) - \log(N) + \frac{a_n}{P_{\max}}}.
\]  

(38)
We can now obtain simple bounds on $E (A_{\text{fast}}^{GEE})$ and $E (A_{\text{opt}})$.

\[ E (A_{\text{opt}}) \leq \sum_{n=1}^{N} E (A_{N:N}) = \]
\[ = N P_{\text{max}} - \sum_{n=1}^{N} \frac{a_n}{\frac{\alpha_n}{P_{\text{max}}}} \log(N) \]
and
\[ E (A_{\text{fast}}^{GEE}) \leq \sum_{n=1}^{N} E (A_{N:N}) = \]
\[ = N P_{\text{max}} - \sum_{n=1}^{N} \frac{a_n}{\frac{\alpha_n}{P_{\text{max}}}} \log(N) \]
it is now easy to see
\[ \frac{N P_{\text{max}} - \sum_{n=1}^{N} \frac{a_n}{\frac{\alpha_n}{P_{\text{max}}}} \log(N)}{N P_{\text{max}} - \sum_{n=1}^{N} \frac{a_n}{\frac{\alpha_n}{P_{\text{max}}}} \log(N)} \leq \]
\[ \leq \frac{E (A_{\text{fast}}^{GEE})}{E (A_{\text{opt}})} \leq 1 \]
However,
\[ \lim_{N \to \infty} \frac{N P_{\text{max}} - \sum_{n=1}^{N} \frac{a_n}{\frac{\alpha_n}{P_{\text{max}}}} \log(N)}{N P_{\text{max}} - \sum_{n=1}^{N} \frac{a_n}{\frac{\alpha_n}{P_{\text{max}}}} \log(N)} = 1. \]
and as a result
\[ \lim_{N \to \infty} \frac{E (A_{\text{fast}}^{GEE})}{E (A_{\text{opt}})} \to 1. \]

**Theorem 6:** Let $A_{\text{opt}}^{\alpha}$ be the optimal solution to the max-energy efficiency problem for the EE$^{AV}$ and let $A_{\text{fast}}^{GEE}$ be the solution obtained by the fast assignment algorithm. We assume that the rates are properly chosen and a perfect matching exists. Then for Rayleigh fading
\[ \lim_{N \to \infty} \frac{E (A_{\text{fast}}^{GEE})}{E (A_{\text{opt}}^{\alpha})} = 1. \]

**Proof:** The proof is identical to the proof of Theorem 5. The only difference is that the CDF $F_{U_{\pi}(n,k)}(x)$ is given by:
\[ F_{U_{\pi}(n,k)}(x) = 1 - e^{-\frac{\lambda_{\pi}(x^{\pi_{n,k}}) \log(2)}{\sigma^2}} \] (45)
It is easy to verify that all arguments applied to the CDF in Theorem 5 also apply to this CDF.

The fully distributed matching algorithm for cognitive radio systems using prioritized CSMA is depicted in Table IV.

### VI. Simulations

In this section we present simulated results for the methods described in the paper. The simulated results are consistent with the theory. In the first set of simulations we compared the distributed auction algorithm and the fast matching algorithm. We generated $N \times N$ random energy efficiency matrices for independent Rayleigh channels with $N$ varying from $2^5$ to $2^9$. The target rates were chosen according to Section V-A and the threshold was chosen to be $a_n^{\text{thresh}} = F^{-1}(1 - \frac{2 \log(N)}{N})$. Figure 2 shows the ratio between the expected transmitted power achieved by the fast matching protocol and the expected transmitted power achieved by the distributed auction algorithm. It is easy to see that the ratio converges to 1 with the number of users. This coincides with Theorem 5 which states that this ratio should converge to 1 as the number of users and channels goes to infinity. In the next set of simulations we investigated the expected number of iterations achieved by both algorithms. First, we verified Theorem 18 which states that the number of iterations in the fast matching algorithm exceeds $c N \log(N)$ with a probability of less than $\frac{1}{N}$. In Figure 3 we plotted the

**TABLE IV**

**MATCHING ALGORITHM USING OPPORTUNISTIC CARRIER SENSING FOR THE n’TH USER**

Initialize $h_k = 0, \forall k \in K$, set assigned=false
set $b_n$ to be the indices of $\alpha \log(N)$ best channels of the n’th user, set $N_{\text{iter}} = 0$

**Repeat**
1. If assigned=false then
   1.1 $N_{\text{iter}} = N_{\text{iter}} + 1$
   1.2 Find the channel with minimal value $j = \arg \min_{m \in b_n} h_m$
   1.3 Choose random backoff time $\tau_n$
   1.4 If channel $m$ was accessed before $\tau_n$ then
      1.4.1 No transmission attempt by the n’th user in the current time slot
      1.4.2 $h_m = h_m + 1$
   1.5 Else
      1.5.1 Access the j’th channel.
      1.5.2 $h_j = h_j + 1$
      1.5.3 Set assigned=true
   1.6 End if
2. Else
   2.1 If channel $m$ was accessed before $\tau_{\text{max}}$ then
      set $h_m = h_m + 1$
   2.2 $N_{\text{iter}} = N_{\text{iter}} + 1$
   2.3 If $m = j$
      2.3.1 No transmission attempt by the n’th user in the current time slot
      2.3.2 Set assigned=false
   2.4 Else access the j’th channel.
3. End If
**Until** all users are assigned or $N_{\text{iter}} = (N - 1)^2$
run the distributed auction algorithm in Table I

**End If**

![Fig. 2. Ratio of the average optimal EE and the sum rate obtained by the matching algorithm.](image)
empirical probability that the fast algorithm exceeds \( N \log(N) \) iterations against the theoretical bound of \( \frac{1}{N} \) for \( N = 10 \ldots 10^4 \). Again, the simulations support the theoretical results.

We then presented a fast algorithm to solve the problem. The algorithm was shown to converge within \( O(N \log(N)) \) iterations with high probability. We showed that under mild assumptions, the fast algorithm can produce asymptotically optimal solutions to the max energy efficiency problem. We then showed how to implement the matching algorithm in a fully distributed manner using prioritized CSMA.

**Appendix**

Here we present an outline with the lemmas and theorems used in the proof without the proof of the lemmas. The full proof is given in [39]. We start by noting the following trivial property of the fast matching algorithm:

**Lemma 7:** Let \( T \) be the number of iterations until the algorithm terminates and let \( h_v \) be the value of vertex \( v \) at termination, then

\[
T = \sum_{v=1}^{N} h_v
\]

**Proof:** The proof is trivial since on every iteration the value of exactly one vertex is increased by 1.

Now, the theorem would be proven if we can prove that with high probability the maximal value of \( h_v \) is less than \( \alpha \log(N) \). This is done by proving the following claims:

**Lemma 8:** Let \( G = (U, V, E) \) be a bipartite graph with vertex sets \( |U| = |V| = N \) and an edge set \( E \). Let \( M(i) \subseteq E \) be a non maximal matching obtained by the algorithm in the \( i \)’th iteration. Let \( h_v(i) \) be the value of vertex \( v \) in the \( i \)’th iteration of the algorithm. Let \( D_l(i) \) be a subset of \( V \) defined by:

\[
D_l(i) = \{ v \in V : h_v(i) \geq l \}.
\]

If \( v_0 \in D_l(i) \) and \( (u, v_0) \in M(i) \) then

\[
n_u \subseteq D_{l-1}(i)
\]

**Lemma 9:** Let \( G = (U, V, E) \) be a bipartite graph with vertex sets \( |U| = |V| = N \) and an edge set \( E \). Let \( M(i) \) be a non maximal matching obtained by the algorithm in the \( i \)’th iteration. Let \( u_0 \in U \) be a free vertex such that in the \( i \)’th iteration of the algorithm \( n_{u_0} \subseteq D_l(i) \). Let \( u_1 \in U \) be the end point of an alternating path \( P \) with \( |P| = 2 \) starting from \( u_0 \) then

\[
v \notin D_{l-2}(i) \setminus D_{l-1}(i) \quad \forall v \in n_{u_1}
\]

**Lemma 10:** Let \( G = (U, V, E) \) be a bipartite graph with vertex sets \( |U| = |V| = N \) and an edge set \( E \). Let \( M(i) \) be a non maximal matching obtained by the algorithm in the \( i \)’th iteration and let \( u_0 \in U \) be a free vertex such that in the \( i \)’th iteration of the algorithm \( n_{u_0} \subseteq D_l(i) \); then every augmenting path of \( G \) on \( M(i) \) starting from \( u_0 \) is at least of length \( 2l + 1 \).

A well-known theorem by Berge [39] states that if a bipartite graph \( G \) contains a perfect matching there exists an augmenting path in \( G \) with respect to any non maximal matching \( M \).

**Theorem 11:** [39] Let \( G = (U, V, E) \) be a bipartite graph with vertex sets \( |U| = |V| = N \) and an edge set \( E \). If \( G \)
Definition 11: $G \in \bar{B}(N, p)$ if $G \in B(N, p)$ and for any non-maximal matching $M$ there exists an augmenting path for $M$ of length at most $2L+1$ where $L = \frac{c \log(N)}{\log(Np)}$ and $c > 0$ is some constant.

Lemma 15: Let $G \in B(N, p)$ and let and let $T$ be the number of iterations until the algorithm terminates then

$$T \leq N(L+1).$$

where $L = \frac{c \log(N)}{\log(Np)}$.

The following theorem was proven in [40]:

Theorem 16: Let $G \in B(N, p)$ where $p \geq \frac{(1+\epsilon) \log(N)}{N}$ then for every $\gamma > 0$ there exists $N_\gamma$ such that for every $N \geq N_\gamma$

$$\Pr(G \in \bar{B}(N, p)) \geq 1 - N^{-\gamma}.$$ (53)

The following theorem was proven in [41]:

Theorem 17: Let $G = (U, V, E) \in B(N, p)$ and $p = \frac{c \log(N)}{N}, c > 2$ then

$$\lim_{N \to \infty} \Pr(G \text{ contains a perfect matching}) = e^{-2N^{1-c}}.$$ (54)

Using the above theorems and lemmas Theorem [18] is proven

Theorem 18: Let $G \in B(N, p)$ be a random bipartite graph with $N > N_\gamma$ vertices on each side and $p = \frac{c \log(N)}{N}, c > 2$.

Then the expected number of iterations until the algorithm terminates is

$$E(T) \leq O\left(\frac{N \log(N)}{\log(Np)}\right)$$ (55)

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