Learning in Games with Cumulative Prospect Theoretic Preferences

Soham R. Phade and Venkat Anantharam

Abstract

We consider repeated games where players behave according to cumulative prospect theory (CPT). We show that a natural analog for the notion of correlated equilibrium in the CPT case, as defined by Keskin, is not enough to guarantee the convergence of the empirical distribution of action play when players have calibrated strategies and behave according to CPT. We define the notion of a mediated CPT calibrated equilibrium via an extension of the game to a so-called mediated game. We then show, along the lines of Foster and Vohra’s result, that under calibrated learning the empirical distribution of play converges to the set of all mediated CPT correlated equilibria. We also show that, in general, the set of CPT correlated equilibria is not approachable in the Blackwell approachability sense.

1 Introduction

In non-cooperative game theory, a finite $n$-person game models a social system comprised of several decision makers (or players) with possibly different objectives, interacting in a certain type of environment. The notion of equilibrium is central to game theory. The neoclassical economics viewpoint of game theory attempts to explain equilibrium as a self-evident outcome of the optimal behavior of the participating players assuming them to be absolutely rational. Two of the most well-known notions of equilibrium for


A finite $n$-person game are Nash equilibrium [Nash, 1951] and correlated equilibrium [Aumann, 1974]. (See Kreps [1990] for an excellent account of the strengths and weaknesses of these notions.) An alternate approach called learning in games is concerned with dynamic considerations such as repeated games where the players learn from the past play and observations from the environment and adapt accordingly [Aumann et al., 1995, Fudenberg and Levine, 1998, Young, 2004]. In this paper, we will be concerned with this alternate approach.

Since decision makers are an integral part of any social system, their behavioral properties form an important aspect in modeling games. The study of game theory so far has been mainly based on the assumption that the behavior of the players towards their lottery preferences (see section 2 for the definition of a lottery) can be modeled by Von Neumann and Morgenstern [1945] expected utility theory (EUT). EUT has a nice normative appeal to it, in particular when it comes to the independence axiom, which basically says that if lottery $L_1$ is preferred over lottery $L_2$, and $L$ is some other lottery, then, for $0 \leq \alpha \leq 1$, the combined lottery $\alpha L_1 + (1 - \alpha)L$ is preferred over the combined lottery $\alpha L_2 + (1 - \alpha)L$. Even though this seems very intuitive, a systematic deviation from such behavior has been observed in multiple empirical studies (for example, Allais [1953] paradox). This gave rise to the study of non-expected utility theory that does away with the independence axiom. Cumulative prospect theory (CPT) as formulated by Tversky and Kahneman [1992] is one such theory, which accommodates many of the empirically observed behavioral features without losing much tractability [Wakker, 2010]. It is also a generalization of the expected utility theory in the sense that EUT is a special case of CPT.

It becomes even more important to consider non-EUT behavior in the theory of learning in games. For example, in a repeated game, Hart [2005] argues that players tend to use simple procedures like regret minimization. A player $i$ is said to have a small regret if, for each pair of her actions $a_i, \tilde{a}_i$, she does not regret not having played action $\tilde{a}_i$ whenever she played action $a_i$. Such regrets can simply be computed as the difference in the average payoffs received by the player from playing action $\tilde{a}_i$ instead of action $a_i$, assuming the opponents stick to their actions. While evaluating such regrets in the real world, however, the players are prone to systematic deviations. The proposed model based on cumulative prospect theory is an attempt to handle these systematic deviations, arising due to several behavioral features exhibited by the participants, by allowing flexibility over the expected utility

\[1\text{also known as the internal regret or the conditional regret.}\]
model through the probability weighting functions. We pose the following question: How do the predictions of the theory of learning in games change if the players behave according to CPT?

The strategies of the players are said to be in a Nash equilibrium if no player is tempted to deviate from her strategy provided the strategies of the others remain unchanged. Suppose now, before the game is played, there is a mediator who sends each player a private signal to play a certain action. Each player may then choose her action depending on this signal. A correlated equilibrium of the original game is obtained by taking the joint distribution over action profiles of all the players corresponding to a Nash equilibrium of the game with a mediator [Aumann, 1974]. Crawford [1990] studies games where players do not adhere to the independence axiom, and defines an analog for the Nash equilibrium. Keskin [2016] defines analogs for both the notions of equilibrium, Nash and correlated, based on CPT. We call them CPT Nash equilibrium and CPT correlated equilibrium respectively. In section 2 we give a brief review of cumulative prospect theory and Keskin’s definitions for these equilibrium notions. In the absence of the independence axiom, many of the linearities present in the model under EUT are lost. For example, the set of all correlated equilibria is a convex polytope [Aumann, 1987], however, the set of all CPT correlated equilibria need not be convex [Keskin, 2016]. In fact, it can even be disconnected [Phade and Anantharam, 2017].

For a repeated game, Foster and Vohra [1998] describe a procedure based on calibrated learning that guarantees the convergence of the empirical distribution of action play to the set of correlated equilibria, when players behave according to EUT. In section 3 we formulate an analog for their procedure when players behave according to CPT. In example 3.1, we describe a game for which the set of all CPT correlated equilibria is non-convex and we show that the empirical distribution of play does not converge to this set.

We define an extension of the set of CPT correlated equilibria and establish the convergence of the empirical distribution of action play to this extended set. It turns out that this extension has a nice game theoretic interpretation obtained by allowing the mediator to send any private payoff-irrelevant signal (instead of restricting her to send a signal corresponding to some action). We formally define this setup in section 3 and call it a mediated game. Myerson [1986] considers a further generalization in which players first report their type. The mediator collects the reports from all the players and then sends each one of them a private signal. Based on her received signal each player chooses her action. These are called games with communication. Under EUT, the set of all correlated equilibria of a game is characterized by the union over all possible communication systems, of the
sets of joint distributions on the action profiles of all players, arising from all the Nash equilibria for the game with communication (for a detailed exposition see [Myerson, 2013]). This is sometimes referred to as the Bayes-Nash revelation principle or simply the revelation principle. Since mediated games are a specific type of games with communication where players do not report their type, our analysis shows that the revelation principle does not hold under CPT.

Calibrated learning is one way of studying learning in games. Some other approaches originate from Blackwell’s approachability theory and the regret-based framework of online learning (Hart and Mas-Colell [2000], Fudenberg and Levine [1995]). In fact, Foster and Vohra [1998] establish the existence of calibrated learning schemes using such a regret based framework and Blackwell’s approachability theory (See Perchet [2009] for a comparison between these approaches, and also Cesa-Bianchi and Lugosi [2006]). Hannan [1957] introduced the concept of no-regret strategies in the context of repeated matrix games. No-regret learning in games is equivalent to the convergence of the empirical distribution of play to the set of correlated equilibria [Hart and Mas-Colell, 2000, Fudenberg and Levine, 1995]. We establish an analog of this result when players behave according to CPT. We then ask if no-regret learning is possible under CPT.

Blackwell’s approachability theorem prescribes a strategy to steer the average payoff vector of a player towards a given target set, irrespective of the strategies of the other players. The theorem also gives a necessary and sufficient condition for the existence of such a strategy provided the target set is convex and the game environment remains fixed. Here, by game environment, we mean the rule by which the payoff vectors depend on the players’ actions. Under EUT, Hart and Mas-Colell [2000] take these payoff vectors to be the regrets associated to a player and establish no-regret learning by showing that the negative orthant in the space of payoff vectors is approachable. Under CPT, although the target set is convex, the environment is not fixed. It depends on the empirical distribution of play at each step. A similar problem with dynamically evolving environment is considered in Kalathil et al. [2017], where they get around this problem by considering a Stackelberg setting; one player (leader) plays an action first, then, after observing this action, the other player (follower) plays her action. In the absence of a Stackelberg setting, as is in our case, we do not know of any result that characterizes approachability under dynamic environments. However, as far as games with CPT preferences are concerned, we answer this question by giving an example of a game for which a no-regret learning strategy does not exist (example 4.2).
2 Preliminaries

We denote a finite $n$-person normal form game by $\Gamma = (N, (A_i)_{i \in N}, (x_i)_{i \in N})$, where $N = \{1, \ldots, n\}$ is the set of players, $A_i$ is the finite action set of player $i$, and $x_i : A_1 \times \cdots \times A_n \to \mathbb{R}$ is the payoff function for player $i$. Let $A = \prod_{i \in N} A_i$ denote the set of all action profiles $a = (a_i)_{i \in N}$, where $a_i \in A_i$. Let $A_{-i} = \prod_{j \in N \setminus i} A_j$ denote the set of all action profiles $a_{-i} \in A_{-i}$ of all players except player $i$. Let $x_i(a)$ denote the payoff to player $i$ when action profile $a$ is played, and let $x_i(\tilde{a}_i, a_{-i})$ denote the payoff to player $i$ when she chooses action $\tilde{a}_i \in A_i$ while the others stick to $a_{-i}$. For any finite set $S$, let $\Delta(S)$ denote the standard simplex of all probability distributions on the set $S$, i.e.,

$$\Delta(S) = \{(p(s), s \in S) \mid p(s) \geq 0 \forall s \in S, \sum_{s \in S} p(s) = 1\}$$

with the usual topology. Let $e_s$ denote the vector in $\Delta(S)$ with its $s$-th component equal to 1 and the rest 0. Let $\Delta^*(A)$ denote the set of all joint probability distributions that are of product form, i.e.,

$$\Delta^*(A) = \{\mu \in \Delta(A) : \mu(a) = \mu_1(a_1) \times \cdots \times \mu_n(a_n) \forall a \in A\},$$

where $\mu_i(a_i)$ denotes the marginal probability distribution on $a_i$ induced by $\mu$. For a joint distribution $\mu \in \Delta(A)$, let

$$\mu_i(a_i) = \sum_{a_{-i} \in A_{-i}} \mu(a_i, a_{-i})$$

be the marginal distribution of player $i$, and for $a_i$ such that $\mu_i(a_i) > 0$ let

$$\mu_{-i}(a_{-i} | a_i) = \frac{\mu(a_i, a_{-i})}{\mu_i(a_i)}$$

be the conditional distribution on $A_{-i}$.

We now describe the setup for cumulative prospect theory (CPT) (for more details see [Wakker, 2010]). Each person is associated with a reference point $r \in \mathbb{R}$, a corresponding value function $v^r : \mathbb{R} \to \mathbb{R}$, and two probability weighting functions $w^\pm : [0, 1] \to [0, 1]$, $w^+$ for gains and $w^-$ for losses. The function $v^r(x)$ satisfies: (i) it is continuous in $x$; (ii) $v^r(r) = 0$; (iii) it is strictly increasing in $x$. The value function is generally assumed to be convex in the losses frame ($x < r$) and concave in the gains frame ($x > r$), and to be steeper in the losses frame than in the gains frame in the sense that $v^r(r) - v^r(r - z) \geq v^r(r + z) - v^r(r)$ for all $z \geq 0$. However, these assumptions
are not needed for the results in this paper to hold. The probability weighting functions \( w^\pm : [0, 1] \to [0, 1] \) satisfy: (i) they are continuous; (ii) they are strictly increasing; (iii) \( w^\pm(0) = 0 \) and \( w^\pm(1) = 1 \).

Suppose a person faces a lottery (or prospect) \( L = \{ (p_j, z_j) \}_{1 \leq j \leq t} \), where \( z_j \in \mathbb{R}, 1 \leq j \leq t \), denotes an outcome and \( p_j, 1 \leq j \leq t \), is the probability with which outcome \( z_j \) occurs. We assume the lottery to be exhaustive, i.e. \( \sum_{j=1}^{t} p_j = 1 \). (Note that we are allowed to have \( p_j = 0 \) for some values of \( j \).)

Let \( z = (z_j)_{1 \leq j \leq t} \) and \( p = (p_j)_{1 \leq j \leq t} \). We denote \( L \) as \( (p, z) \) and refer to the vector \( z \) as an outcome profile.

Let \( \alpha = (\alpha_1, \ldots, \alpha_t) \) be a permutation of \((1, \ldots, t)\) such that

\[
z_{\alpha_1} \geq z_{\alpha_2} \geq \cdots \geq z_{\alpha_t}.
\]

(2.1)

Let \( 0 \leq j_r \leq t \) be such that \( z_{\alpha_j} \geq r \) for \( 1 \leq j \leq j_r \) and \( z_{\alpha_j} < r \) for \( j_r < j \leq t \). (Here \( j_r = 0 \) when \( z_{\alpha_j} < r \) for all \( 1 \leq j \leq t \).) The CPT value \( V(L) \) of the prospect \( L \) is evaluated using the value function \( v^r(\cdot) \) and the probability weighting functions \( w^\pm(\cdot) \) as follows:

\[
V(L) := \sum_{j=1}^{j_r} \pi_j^+(p, \alpha) v^r(z_{\alpha_j}) + \sum_{j=j_r+1}^{t} \pi_j^-(p, \alpha) v^r(z_{\alpha_j}),
\]

(2.2)

where \( \pi_j^+(p, \alpha), 1 \leq j \leq j_r, \pi_j^-(p, \alpha), j_r < j \leq t \), are decision weights defined via:

\[
\pi_j^+(p, \alpha) = w^+(p_{\alpha_1}),
\]

\[
\pi_j^+(p, \alpha) = w^+(p_{\alpha_1} + \cdots + p_{\alpha_j}) - w^+(p_{\alpha_1} + \cdots + p_{\alpha_j-1}) \quad \text{for } 1 < j \leq t,
\]

\[
\pi_j^-(p, \alpha) = w^-(p_{\alpha_1} + \cdots + p_{\alpha_j}) - w^-(p_{\alpha_1} + \cdots + p_{\alpha_j+1}) \quad \text{for } 1 \leq j < t,
\]

\[
\pi_j^-(p, \alpha) = w^-(p_{\alpha_t}).
\]

Although the expression on the right in equation (2.2) depends on the permutation \( \alpha \), one can check that the formula evaluates to the same value \( V(L) \) as long as the permutation \( \alpha \) satisfies (2.1).

A person is said to have CPT preferences if, given a choice between prospect \( L_1 \) and prospect \( L_2 \), she chooses the one with higher CPT value.

We now describe the notion of correlated equilibrium incorporating CPT preferences, as defined by Keskin [2016]. For each player \( i \), let \( r_i, v_i^r(\cdot) \) and \( w_i^\pm(\cdot) \) be the reference point, the value function, and the probability weighting functions respectively, that player \( i \) uses to evaluate the CPT value \( V_i(L) \) of a lottery \( L \).
Suppose there is a mediator characterized by a joint distribution \( \mu \in \Delta(A) \) who draws an action profile \( a = (a_i)_{i \in N} \) according to the distribution \( \mu \) and sends signal \( a_i \) to each player \( i \). Player \( i \) is signaled only her action \( a_i \) and not the entire action profile \( a = (a_i)_{i \in N} \). We assume that all the players know the distribution \( \mu \). If player \( i \) observes a signal to play \( a_i \), and if she decides to deviate to a strategy \( \tilde{a}_i \in A_i \), then she will face the lottery

\[
L_i(\mu, a_i, \tilde{a}_i) := \{ (\mu_{-i}(a_{-i}|a_i), x_i(\tilde{a}_i, a_{-i})) \}_{a_{-i} \in A_{-i}}.
\]

**Definition 2.1** (Keskin [2016]). A joint probability distribution \( \mu \in \Delta(S) \) is said to be a CPT correlated equilibrium of \( \Gamma \) if it satisfies the following inequalities for all \( i \) and for all \( a_i, \tilde{a}_i \in A_i \) such that \( \mu_i(a_i) > 0 \):

\[
V_i(L_i(\mu, a_i, a_i)) \geq V_i(L_i(\mu, a_i, \tilde{a}_i)).
\] (2.3)

We denote the set of all the CPT correlated equilibria of a game \( \Gamma \) by \( C(\Gamma) \). Note that \( C(\Gamma) \) also depends on the reference points, the value functions and the probability weighting functions of all the players. However we suppress this dependence from the notation.

We now describe the notion of CPT Nash equilibrium as defined by Keskin [2016]. For a mixed strategy \( \mu \in \Delta^*(A) \), if each player \( j \) decides to play \( a_j \), drawn from the distribution \( \mu_j \), then player \( i \) will face the lottery

\[
L_i(\mu_{-i}, a_i) := \{ (\mu_{-i}(a_{-i}|a_i), x_i(a_i, a_{-i})) \}_{a_{-i} \in A_{-i}},
\]

where \( \mu_{-i}(a_{-i}) = \prod_{j \neq i} \mu_j(a_j) \) plays the role of \( \mu_{-i}(a_{-i}|a_i) \), which does not depend on \( a_i \). Suppose player \( i \) decides to deviate and play a mixed strategy \( \tilde{\mu}_i \) while the rest of the players continue to play \( \mu_{-i} \). Then define the average CPT value for player \( i \) by

\[
\mathcal{V}_i(\tilde{\mu}_i, \mu_{-i}) = \sum_{a_i \in A_i} \tilde{\mu}_i(a_i)V_i(L_i(\mu_{-i}, a_i)).
\]

The best response set of player \( i \) to a mixed strategy \( \mu \in \Delta^*(A) \) is defined as

\[
BR_i(\mu) := \{ \mu^*_i \in \Delta(A_i) \mid \forall \tilde{\mu}_i \in \Delta(A_i), \mathcal{V}_i(\mu^*_i, \mu_{-i}) \geq \mathcal{V}_i(\tilde{\mu}_i, \mu_{-i}) \}.
\]

**Definition 2.2** (Keskin [2016]). A mixed strategy \( \mu^* \in \Delta^*(A) \), is a CPT Nash equilibrium iff

\[
\mu^*_i \in BR_i(\mu^*) \text{ for all } i.
\]

\(^2\text{Keskin defines CPT equilibrium assuming } w^+(\cdot) = w^-(\cdot). \text{ However, the definition can be easily extended to our general setting and the proof of existence goes through without difficulty.}\)
Keskin [2016] shows that for every game $\Gamma$, there exists a CPT Nash equilibrium. Further, he also shows that the set of all CPT Nash equilibria of a game $\Gamma$ is equal to $C(\Gamma) \cap \Delta^*(A)$. Thus, as a consequence, we have that the set $C(\Gamma)$ is nonempty. A strategy $\mu \in \Delta^*(A)$ is called a pure strategy if the support of $\mu_i$ is singleton for each $i$. We call $\mu^*$ a pure CPT Nash equilibrium if $\mu^*$ is a pure strategy. Note that every pure CPT Nash equilibrium is a pure Nash equilibrium for the EUT game where each player $i$ computes its value in the action profile $(a_i)_{i \in N}$ as $v^*_i(x_i(a_i, a_{-i}))$.

3 Calibrated learning in games

Let $\Gamma$ be a finite $n$-person game which is played repeatedly at each step (or round), $t \geq 1$. The game $\Gamma$ is called the one shot game (or the stage game) of the repeated game. At every step $t$, each player $i$ draws an action $a_i^t \in A_i$ from a distribution $\sigma_i^t \in \Delta(A_i)$. We assume that the randomizations of the players are independent of each other and of the past randomizations. We assume that after playing her action $a_i^t$, each player observes the actions taken by all the other players and thus at any step $t$ all the players have access to the history of the play $H^{t-1} = (a_1^1, \ldots, a_i^{t-1})$ where $a_i^t = (a_i^t)_{i \in N}$ is the action profile played at step $t$. We also assume that each player knows the one step game $\Gamma$. Let the strategy for player $i$ for the repeated game above be given by $\mathcal{S}_i = (\sigma_i^t, t \geq 1)$ where $\sigma_i^t : H^{t-1} \rightarrow \Delta(A_i)$ for each $t$.

We first describe the result of Foster and Vohra [1997]. Suppose the players follow the following natural strategy: At every step $t$, on the basis of the past history of play, $H^{t-1}$, each player $i$ predicts a joint distribution $\mu_{t-1}^i \in \Delta(A_{-i})$ on the action profile of all the other players. This is player $i$’s assessment of how her opponents might play in the next step. Depending on this assessment, player $i$ chooses a specific action among those that are most preferred by her, called her best reaction. Foster and Vohra [1997] prove that (i) if each player’s assessments are calibrated with respect to the sequence of action profiles of the other players and (ii) if each player plays a best reaction to her assessments, then the limit points of the empirical distribution of action play are correlated equilibria. By action play we mean the sequence of action profiles played by the players. We will give a formal definition of what is meant by calibration shortly. For the moment, roughly, calibration says that the empirical distributions conditioned on assessments converge to the assessments. The best reaction of player $i$ to her assessment

\footnote{Foster and Vohra [1997] refer to it as the best response. In order to avoid confusion with the best response set defined in section 2, we prefer to use the term best reaction.}
\(\mu_{-i}\) of the actions of the other players, as considered by Foster and Vohra [1997], is a specific action \(a^*_i \in A_i\) that maximizes the expected payoff to player \(i\) with respect to her assessment. i.e.,

\[
a^*_i \in \arg \max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \mu_i(a_{-i})x_i(a_i, a_{-i}).
\]

Thus the best reaction is an action in the best response set. Note that it is assumed that each player uses a fixed tie breaking rule if there is more than one action in the best response set.

Suppose now that the players behave according to CPT. Given player \(i\)'s assessment \(\mu_{-i}\) of the play of her opponents, she is faced with the following set of lotteries, one for each of her actions \(a_i \in A_i\):

\[
L_i(\mu_{-i}, a_i) = \{\mu_{-i}(a_{-i}), x_i(a_i, a_{-i})\}_{a_{-i} \in A_{-i}}.
\]

Out of these lotteries, the ones she prefers most are those with the maximum CPT value \(V_i(L_i(\mu_{-i}, a_i))\), evaluated using her reference point \(r_i\), value function \(v^+_i\), and her probability weighting functions \(w^\pm_i\). The choice of the action she makes corresponding to her most preferred lottery (with any arbitrary but fixed tie breaking rule) will be called her best reaction. Thus the best reaction is a specific action in the best response set.

We now ask the following question: Suppose each player’s assessments are calibrated with respect to the sequence of action profiles of the other players and she evaluates her best reaction in accordance with CPT preferences as explained above, then are the limit points of the empirical distribution of play contained in the set of CPT correlated equilibria? Unfortunately, the answer is no (example 3.1). Before seeing why, let us give the promised formal definition of the notion of calibration.

Consider a sequence of outcomes \(y^1, y^2, \ldots\) generated by Nature, belonging to some finite set \(S\). At each step \(t\), the forecaster predicts a distribution \(q^t \in \Delta(S)\). Let \(N(q, t)\) denote the number of times the distribution \(q\) is forecast up to step \(t\), i.e. \(N(q, t) = \sum_{\tau=1}^{t} 1\{q^\tau = q\}\). Let \(\rho(q, y, t)\) be the fraction of the steps on which the forecaster predicts \(q\) for which Nature plays \(y \in S\), i.e.,

\[
\rho(q, y, t) = \begin{cases} 
0 & \text{if } N(q, t) = 0 \\
\frac{\sum_{\tau=1}^{t} 1(q^\tau = q)1(y^\tau = y)}{N(q, t)} & \text{otherwise}.
\end{cases}
\]

The forecast is said to be calibrated with respect to the sequence of plays
Table 1: Payoff matrix for the game in example 3.1. The rows and columns correspond to player 1 and 2’s actions respectively. The first entry in each cell corresponds to player 1’s payoff and second to player 2’s payoff.

|    | I    | II   | III  | IV   |
|----|------|------|------|------|
| 0  | $2\beta, 1$ | $\beta + 1, 1$ | 0, 1 | 1, 1 |
| 1  | 1.99, 0    | 1.99, 0   | 1.99, 0 | 1.99, 0 |

made by Nature if:

$$
\lim_{t \to \infty} \sum_{q \in Q^t} |\rho(q, y, t) - q(y)| \frac{N(q,y,t)}{t} = 0 \text{ for all } y \in S,
$$

where the sum is over the set $Q^t$ of all distributions predicted by the forecaster up to step $t$.

**Example 3.1.** We consider a modification of the 2-player game proposed by Keskin [2016], where it is used to demonstrate that the set of CPT correlated equilibria can be nonconvex. Let the 2-player game be represented by the matrix in table 1 where $\beta = 1/w_+^i(0.5)$. For the probability weighting functions $w_+^i(\cdot)$, we employ the functions of the form suggested by Prelec [1998], which, for $i = 1, 2$, are given by

$$
w_+^i(p) = \exp\{-(-\ln p)^{\gamma_i}\},
$$

where $\gamma_1 = 0.5$ and $\gamma_2 = 1$. We thus have $w_+^1(0.5) = 0.435$ and $\beta = 2.299$. Let the reference points $r_1 = r_2 = 0$. Let $v_i^r(\cdot)$ be identity for $i = 1, 2$. Notice that player 2 is indifferent amongst her actions.

Suppose player 2 plays her actions in a cyclic manner starting with action I at step 1, followed by II, III, IV and then again I and so on. Suppose player 1’s assessment of player 2’s action is $\mu_{\text{odd}} = (0.5, 0, 0, 0, 0.5)$ and $\mu_{\text{even}} = (0, 0.5, 0, 0.5)$ at each odd and even step respectively. Then it is easy to see that player 1’s assessments are calibrated. (Here player 2 plays the role of Nature from the point of view of player 1.) We can evaluate the CPT values of player 1 for the following lotteries as

$$
V_1(L_1(\mu_{\text{odd}}, 0)) = 2\beta w_+^1(0.5) = 2, \quad V_1(L_1(\mu_{\text{odd}}, 1)) = 1.99
$$

$$
V_1(L_1(\mu_{\text{even}}, 0)) = 1 + \beta w_+^1(0.5) = 2, \quad V_1(L_1(\mu_{\text{even}}, 1)) = 1.99
$$

Thus, player 1’s best reaction to both these assessments $\mu_{\text{odd}}$ and $\mu_{\text{even}}$ is action 0. Thus player 1 would play action 0 throughout. The distribution $\mu^*$
represented in table 2 is a limit point of the empirical distribution of action play. However, $\mu^*$ is not a CPT correlated equilibrium because the CPT value

$$ V_1 \left( L_1(\mu^*_{-1}(\cdot|0), 0) \right) = w^+_1(0.75) + \beta w^+_1(0.5) + (\beta - 1) w^+_1(0.25) = 1.985 < 1.99 $$

We have not described player 2's assessments. We would like to show that player 2's assessments are calibrated and her best reactions lead to the cyclic play exhibited. However, to do so we need to modify the game into a 3-person game. Let player 1 have two actions \{0,1\}, and players 2 and 3 both have four actions \{I,II,III,IV\} each. Let the payoffs to all the three players be $-1$ if players 2 and 3 play different actions. If players 2 and 3 play the same action, then let the resulting payoff matrix be as represented in table 1, where the rows correspond to player 1’s actions and the columns correspond to the common actions of players 2 and 3. Player 1 receives the payoff represented by the first entry in each cell and player 2 and 3 each receive the payoff represented by the second entry. Let player 1’s CPT behaviour be characterized by the reference point, value function and probability weighting functions as before in the 2-person game. For players 2 and 3, let them be as for player 2 in that game. Let player 2 and 3 play in the cyclic manner as above, in sync with each other. Let player 1 play action 0 throughout. Let player 2 and player 3’s assessment at step $t$ be the point distribution supported by the action profile $a^t_{-2}$ and $a^t_{-3}$ respectively. This assessment is clearly calibrated. (Here the action pair comprised of the actions of players 1 and 3 play the role of the actions of Nature from the point of view of player 2, and similarly for player 3.) The best reactions of player 2 and 3 are also consistent. Let the assessment of player 1 be $\mu_{odd}$ and $\mu_{even}$ at odd and even steps where now the distribution $\mu_{odd}$ puts 0.5 probability on action profiles (I,I) and (III,III) and $\mu_{even}$ puts 0.5 probability on action profiles (II,II) and (IV,IV). Again player 1’s assessments are calibrated (where now an action pair comprised of the actions of player 2 and player 3 play the role of the action of Nature from the point of view of player 1) and her best reactions are consistent with her play. As in the two player game, the limit point of the empirical distribution is not a CPT correlated equilibrium. Thus we have a game where the assessments of each player are calibrated and each player plays her best reaction, but the
limit empirical distribution exists and does not converge to a CPT correlated equilibrium.

Mediated Games

The non-convexity of the set \( C(\Gamma) \) is essential to the argument in example 3.1. We now describe an extension of the set \( C(\Gamma) \) and establish the convergence of the empirical distribution of action play to this extended set. It turns out that this extended set of equilibria also has a game theoretic interpretation as follows: Suppose we add a signal system \((B_i)_{i \in N}\) to a game \( \Gamma \), where each \( B_i \) is a finite set. Suppose there is a mediator who sends a signal \( b_i \in B_i \) to player \( i \). Let \( B = \prod_{i \in N} B_i \) be the set of all signal profiles \( b = (b_i)_{i \in N} \) and \( B_{-i} = \prod_{j \neq i} B_j \) denote the set of signal profiles \( b_{-i} \) of all players except player \( i \). Let \( \tilde{\Gamma} = (\Gamma, (B_i)_{i \in N}) \) denote such a game with signal system. We call it a mediated game. The mediator is characterized by a distribution \( \psi \in \Delta(B) \) that we call the mediator distribution. Thus, the mediator draws a signal profile \( b = (b_i)_{i \in N} \) from the mediator distribution \( \psi \) and sends signal \( b_i \) to player \( i \). In the definition of a correlated equilibrium the set \( B_i \) is restricted to the set of actions \( A_i \) for each player \( i \).

A randomized strategy for any player \( i \) is given by a function \( \sigma_i : B_i \to \Delta(A_i) \) and a randomized strategy profile \( \sigma = (\sigma_1, \ldots, \sigma_n) \) gives the randomized strategy for all players. We define the best response set of player \( i \) to a randomized strategy profile \( \sigma \) and a mediator distribution \( \psi \) as

\[
BR_i(\psi, \sigma) := \left\{ \sigma_i^* : B_i \to \Delta(A_i) \right\} \quad \text{for all } b_i \in B_i
\]

\[
\text{supp}(\sigma_i^*(b_i)) \subset \arg\max_{a_i \in A_i} \left\{ \bar{\mu}_{-i}(a_{-i}|b_i), x_i(a_i, a_{-i}) \right\}_{a_{-i} \in A_{-i}}
\]

(3.1)

where

\[
\bar{\mu}_{-i}(a_{-i}|b_i) := \sum_{b_{-i} \in B_{-i}} \psi_{-i}(b_{-i}|b_i) \prod_{j \in N \setminus i} \sigma_j(b_j)(a_j).
\]

(3.2)

and \( \text{supp}(\cdot) \) denotes the support of the distribution within the brackets.

Definition 3.2. A randomized strategy profile \( \sigma \) is said to be a mediated CPT Nash equilibrium of a game \( \tilde{\Gamma} \) with respect to a mediator distribution \( \psi \in \Delta(B) \) iff

\[
\sigma_i \in BR_i(\psi, \sigma) \quad \text{for all } i \in N.
\]

Let \( \Sigma(\Gamma, (B_i)_{i \in N}, \psi) \) denote the set of all mediated CPT Nash equilibria of \( \tilde{\Gamma} = (\Gamma, (B_i)_{i \in N}) \) with respect to a mediator distribution \( \psi \in \Delta(B) \).
If all the players choose to ignore the signals sent by the mediator, then the corresponding randomized strategy profile \( \sigma \) consists of constant functions \( \sigma_i(b_i) \equiv \sigma_i^0 \). Further, it follows from definitions 3.3 and 2.1 that \( \sigma \) is a mediated CPT Nash equilibrium of the game \( \Gamma \) with mediator distribution \( \psi \), for any mediator distribution \( \psi \), iff \( \sigma \) is a CPT Nash equilibrium of the game \( \Gamma \). Since, every game \( \Gamma \) has at least one CPT Nash equilibrium, we have that every mediated game \( \tilde{\Gamma} \) with mediator distribution \( \psi \) has at least one mediated CPT Nash equilibrium, for every mediator distribution \( \psi \).

For any mediator distribution \( \psi \in \Delta(B) \), and any randomized strategy profile \( \sigma \), let \( \eta(\psi, \sigma) \in \Delta(A) \) be given by

\[
\eta(\psi, \sigma)(a) := \sum_{b \in B} \psi(b) \prod_{i \in N} \sigma_i(b_i)(a_i).
\]

Thus, \( \eta(\psi, \sigma) \) gives the joint distribution over action profiles of all players corresponding to the randomized strategy \( \sigma \) and mediator distribution \( \psi \).

**Definition 3.3.** A probability distribution \( \mu \in \Delta(A) \) is said to be a mediated CPT correlated equilibrium of a game \( \Gamma \) iff there exist a signal system \( (B_i)_{i \in N} \), a mediator distribution \( \psi \in \Delta(B) \) and a mediated CPT Nash equilibrium \( \sigma \in \Sigma(\Gamma, (B_i)_{i \in N}, \psi) \) such that \( \mu = \eta(\psi, \sigma) \).

Let \( D(\Gamma) \) denote the set of all mediated CPT correlated equilibria of a game \( \Gamma \). This characterizes the union of the sets of equilibria of all mediated games that can be generated from \( \Gamma \). When \( B_i = A_i \) for all \( i \in N \) and \( \sigma_i(b_i)(a_i) = 1 \{ b_i = a_i \} \), \( \sigma \in \Sigma(\Gamma, (B_i)_{i \in N}, \psi) \) if and only if \( \psi \in C(\Gamma) \). Thus, we have \( C(\Gamma) \subset D(\Gamma) \). Under EUT, Aumann [1987] proves that \( D(\Gamma) = C(\Gamma) \). However, under CPT, this property, in general, does not hold true. Lemma 3.4 shows how \( D(\Gamma) \) compares with \( C(\Gamma) \).

For any \( i, a_i, \tilde{a}_i \in A_i \) let \( C(\Gamma, i, a_i, \tilde{a}_i) \) denote the set of all probability vectors \( \pi_{-i} \in \Delta(A_{-i}) \) such that

\[
V_i(\{\pi_{-i}(a_{-i}), x_i(a_i, a_{-i})\}) \geq V_i(\{\pi_{-i}(a_{-i}), x_i(\tilde{a}_i, a_{-i})\}).
\]

It is clear from the definition of CPT correlated equilibrium that, for a joint probability distribution \( \mu \in C(\Gamma) \), provided \( \mu_i(a_i) > 0 \), the probability vector \( \pi_{-i} = \mu(\cdot | a_i) \in \Delta(A_{-i}) \) should belong to \( C(\Gamma, i, a_i, \tilde{a}_i) \) for all \( \tilde{a}_i \in A_i \). Let

\[
C(\Gamma, i, a_i) := \cap_{\tilde{a}_i \in A_i} C(\Gamma, i, a_i, \tilde{a}_i).
\]

Now, for all \( i \), define a subset \( C(\Gamma, i) \subset \Delta(A) \), as follows:

\[
C(\Gamma, i) := \{ \mu \in \Delta(A) | \mu(\cdot | a_i) \in C(\Gamma, i, a_i), \forall a_i \in \text{supp}(\mu_i) \}.
\]

Note that, since \( V_i(\{\pi_{-i}(a_{-i}), x_i(a_i, a_{-i})\}) \) is a continuous function of \( \pi_{-i} \), the sets \( C(\Gamma, i, a_i, \tilde{a}_i), C(\Gamma, i, a_i) \) and \( C(\Gamma, i) \) are all closed.
Lemma 3.4. For any game $\Gamma$, we have

(i) $\overline{\overline{C}(\Gamma, i)} = \{\mu \in \Delta(A) | \mu_{-i}(\cdot | a_i) \in \overline{\overline{C}(\Gamma, i, a_i)} \}, \forall a_i \in \text{supp}(\mu_i)\}$,

(ii) $C(\Gamma) = \cap_{i \in N} C(\Gamma, i)$, and

(iii) $D(\Gamma) = \cap_{i \in N} \overline{\overline{C}(\Gamma, i)}$.

where $\overline{\overline{S}}$ denotes the convex hull of a set $S$.

Proof. Note that, since the sets $C(\Gamma, i)$ and $C(\Gamma, i, a_i)$ are closed, the convex hulls of these sets are closed. Suppose $\mu = \lambda \mu^1 + (1 - \lambda) \mu^2$ where $\mu^1, \mu^2 \in C(\Gamma, i)$ and $0 < \lambda < 1$. If $a_i \in \text{supp}(\mu_i)$, then one of the following three cases holds:

Case 1 $[a_i \in \text{supp}(\mu^1_i), a_i \in \text{supp}(\mu^2_i)]$. Then, $\mu_{-i}(\cdot | a_i) \in C(\Gamma, i, a_i)$ and we have,

$$\mu_{-i}(\cdot | a_i) = \frac{\lambda \mu^1_i(a_i) \mu_{-i}^1(\cdot | a_i) + (1 - \lambda) \mu^2_i(a_i) \mu_{-i}^2(\cdot | a_i)}{\lambda \mu^1_i(a_i) + (1 - \lambda) \mu^2_i(a_i)}.$$ 

Let $\theta = \frac{\lambda \mu^1_i(a_i)}{\lambda \mu^1_i(a_i) + (1 - \lambda) \mu^2_i(a_i)}$. Then $\mu_{-i}(\cdot | a_i) = \theta \mu_{-i}^1(\cdot | a_i) + (1 - \theta) \mu_{-i}^2(\cdot | a_i)$ and hence $\mu_{-i}(\cdot | a_i) \in \overline{\overline{C}(\Gamma, i, a_i)}$.

Case 2 $[a_i \in \text{supp}(\mu^1_i), a_i \notin \text{supp}(\mu^2_i)]$ Here $\mu_{-i}(\cdot | a_i) = \mu_{-i}^1(\cdot | a_i)$. Hence $\mu_{-i}(\cdot | a_i) \in C(\Gamma, i, a_i)$.

Case 3 $[a_i \notin \text{supp}(\mu^1_i), a_i \in \text{supp}(\mu^2_i)]$ can be handled similarly to case 2.

Also, the above argument can be easily extended to when $\mu$ is a convex combination of any finite number of distributions. Since, all our sets are compact subsets of finite dimensional Euclidean spaces, Carathéodory's theorem applies and hence we need to consider only finite convex combinations.

This shows that the set on the LHS is contained in the set on the RHS of the equation in (i).

To prove the other inclusion, let $\mu$ belong to the set on the RHS of the equation in (i). If $a_i \in \text{supp}(\mu_i)$, then $\mu_{-i}(\cdot | a_i)$ is a linear combination of $|A_{-i}|$ joint distributions (allowing repetitions)

$$\zeta_{-i,a_i}^{1}, \ldots, \zeta_{-i,a_i}^{m_i}, \ldots, \zeta_{-i,a_i}^{|A_{-i}|} \in C(\Gamma, i, a_i)$$

with coefficients $\theta_{i,a_i}^{m_i}, 1 \leq m_i \leq |A_{-i}|$ respectively. For each $\zeta_{-i,a_i}^{m_i}$, construct a distribution $\mu_{i,a_i}^{m_i}$ by $\mu_{i,a_i}^{m_i}(b_i, b_{-i}) = 1(b_i = a_i)\zeta_{-i,a_i}^{m_i}(b_i)$. Then $\mu_{i,a_i}^{m_i} \in C(\Gamma, i)$. Let $\lambda_{i,a_i}^{m_i} = \mu_i(a_i)\theta_{i,a_i}^{m_i}$ for all $i, m_i, a_i$ such that $\mu_i(a_i) > 0$. One can now check that $\mu = \sum_{i,m_i,a_i} \lambda_{i,a_i}^{m_i} \mu_{i,a_i}^{m_i}$. Thus $\mu \in \overline{\overline{C}(\Gamma, i)}$. 

14
Statement (ii) follows directly from the definition of CPT correlated equilibrium.

For statement (iii), let $\mu \in \Delta(A)$ be such that $\mu_\ast \in \mathcal{C}(C(\Gamma, i))$ for all $i$. By (i), we have, $\mu_{-i}(\cdot|a_i) \in \mathcal{C}(C(\Gamma, i, a_i))$ for all $a_i$ such that $\mu_i(a_i) > 0$. As above, let $\mu_{-i}(\cdot|a_i)$ be a convex combination of $|A_{-i}|$ joint distributions (allowing repetitions)

$$\zeta_{-i,a_1}, \ldots, \zeta_{-i,a_m}, \ldots, \zeta_{-i,a_{|A_{-i}|}} \in C(\Gamma, i, a_i)$$

with coefficients $\theta_{i,a_i}^{m_i}$, $1 \leq m_i \leq |A_{-i}|$ respectively. For all $i$, let $B_i = A_i \times M_i$ where $M_i = \{1, \ldots, |A_{-i}|\}$. Let the mediator distribution be given by

$$\psi((a_1, m_1), \ldots, (a_n, m_n)) = \frac{\mu(a) \prod_{i=1}^n \{ \theta_{i,a_i}^{m_i} \}}{\sum_{m_i,i \in N} \prod_{i=1}^n \{ \theta_{i,a_i}^{m_i} \}}. \quad (3.5)$$

Let the strategy for each player $i$ be

$$\sigma_i(a_i, m_i) = e_{a_i}. \quad (3.6)$$

From equations (3.3), (3.5) and (3.6) we have

$$\eta(\psi, \sigma)(a) = \sum_{(a_i, m_i) \in B_i, i \in N} \psi((\tilde{a}_1, m_1), \ldots, (\tilde{a}_n, m_n)) \prod_{i \in N} \sigma_i((\tilde{a}_i, m_i))(a_i)$$

$$= \mu(a) \times \sum_{m_i,i \in N} \prod_{i=1}^n \{ \theta_{i,a_i}^{m_i} \} \prod_{i=1}^n \{ \theta_{i,a_i}^{m_i} \}$$

$$= \mu(a).$$

From equations (3.2), (3.5) and (3.6) we have

$$\bar{\mu}_{-i}(a_{-i}|(a_i, m_i)) = \sum_{(\tilde{a}_j, m_j) \in B_j, j \in N \setminus i} \psi_{-i}(((\tilde{a}_j, m_j) \in B_j, j \in N \setminus i)) \prod_{j \in N \setminus i} \sigma_j((\tilde{a}_j, m_j))(a_j)$$

$$\propto \mu(a) \times \frac{\theta_{i,a_i}^{m_i}}{\mu_{-i}(a_{-i}|a_i)}$$

$$= \left\{ \mu_i(a_i) \theta_{i,a_i}^{m_i} \right\} \zeta_{-i,a_i}(a_{-i}).$$
Thus $\tilde{\mu}_i(\cdot | (a_i, m_i)) = \zeta^{m_i}_{i, a_i}$, and we have $\tilde{\mu}_i(\cdot | (a_i, m_i)) \in C(\Gamma, i, a_i)$. Hence $\mu \in D(\Gamma)$ establishing $\cap_{i \in N} \sigma_\Gamma(C(\Gamma, i)) \subset D(\Gamma)$.

To prove the other direction of statement (iii), let $\mu \in D(\Gamma)$. Then there exists a signal system $(B_i)_{i \in N}$, a mediator distribution $\psi \in \Delta(B)$, and a mediated CPT Nash equilibrium $\sigma \in \Sigma(\Gamma, (B_i)_{i \in N}, \psi)$ such that $\mu = \eta(\psi, \sigma)$.

For a $a_i \in \text{supp}(\sigma_i(b_i))$, we have $\tilde{\mu}_i(\cdot | b_i) \in C(\Gamma, i, a_i)$ from equations (3.1) and (3.4). This means, from equations (3.2) and (3.3), that we have $\mu = \eta(\psi, \sigma) \in \cap_{i \in N} \sigma_\Gamma(C(\Gamma, i))$. This completes the proof.

For the 2-person game in example 3.1, we observed that the set $C(\Gamma)$ is non-convex and hence $C(\Gamma) \neq D(\Gamma)$. If $\Gamma$ is a $2 \times 2$ game, i.e., a game with 2 players, each having two actions, and both behaving according to CPT, then Phade and Anantharam [2017] proves that the sets $C(\Gamma, i)$, corresponding to both these players are convex and hence also the set $C(\Gamma)$. From lemma 3.4, we have the following result, having the flavor of the revelation principle:

**Proposition 3.5.** If $\Gamma$ is a $2 \times 2$ game, then the set of all CPT correlated equilibria is equal to the set of all mediated CPT correlated equilibria.

In the context of mediated games, a strategy $\sigma_i$ for player $i$ is said to be pure if $\text{supp}(\sigma_i)$ is singleton and a strategy profile $\sigma = (\sigma_i)_{i \in N}$ is said to be a pure strategy profile if each $\sigma_i$ is a pure strategy.

**Remark 3.6.** From the proof of lemma 3.4, we observe that for any $\mu \in D(\Gamma)$, there exists a signal system $(B_i)_{i \in N}$, a mediator distribution $\psi \in \Delta(B)$, and a mediated CPT Nash equilibrium $\sigma \in \Sigma(\Gamma, (B_i)_{i \in N}, \psi)$ such that $\mu = \eta(\psi, \sigma)$ where $\sigma$ is a pure strategy profile.

**Calibrated learning to mediated CPT correlated equilibrium**

Let $\xi^t$ denote the empirical joint distribution of the action play up to step $t$. Formally,

$$\xi^t = \frac{1}{t} \sum_{\tau=1}^{t} e_{a^\tau},$$

where $e_{a^\tau}$ is an $|A|$-dimensional vector with its $a^\tau$-th component equal to 1 and the rest 0. Let the distance between a vector $x$ and a set $X$ be given by

$$d(x, X) = \inf_{x' \in X} ||x - x'||,$$
where \( \|x\| \) denotes the standard Euclidean norm of \( x \). We say that a sequence 
\[(x^t, t \geq 1) \] converges to a set \( X \) if the following holds:
\[
\lim_{t \to \infty} d(x^t, X) = 0.
\]

**Theorem 3.7.** Assume that all the players use calibrated forecasters to predict assessments and choose the best reaction to these assessments at every step. Then the joint empirical distribution of action play \( \xi^t \) converges to the set of mediated CPT correlated equilibria.

**Proof.** Consider the sequence of empirical distributions \( \xi^t \). Since the simplex \( \Delta(A) \) of all joint distributions over action profiles is a compact set, every sequence has a convergent subsequence. Thus, it is enough to show that the limit of any convergent subsequence of \( \xi^t \) is in \( D(\Gamma) \). Let \( \xi^t_k \) be such a convergent subsequence and denote its limit by \( \hat{\xi} \).

Let \( a_i \) be an action of player \( i \) such that \( \hat{\xi}_i(a_i) > 0 \). Let \( R_i(a_i) \subset \Delta(A_{-i}) \) be the set of all joint distributions \( \mu_{-i} \) for which action \( a_i \) is player \( i \)'s best reaction under CPT preferences. Note that, \( R_i(a_i) \) forms a partition of the simplex \( \Delta(A_{-i}) \). Let \( Q^t_i \) denote the set of assessments made by player \( i \) up to step \( t \). We have,

\[
\xi^t_k(a_{-i}|a_i) = \frac{1}{t_k} \sum_{\tau \leq t_k \text{ s.t. } \mu_{-i}^\tau \in R_i(a_i)} \mathbf{1}\{a_{-i}^\tau = a_{-i}\}
\]

\[
= \frac{1}{t_k} \sum_{q \in R_i(a_i) \cap Q^t_k} \sum_{\tau \leq t_k \text{ s.t. } \mu_{-i}^\tau = q} \mathbf{1}\{a_{-i}^\tau = a_{-i}\}
\]

\[
= \frac{1}{t_k} \sum_{q \in R_i(a_i) \cap Q^t_k} \rho(q, a_{-i}, t_k) N(q, t_k)
\]

\[
= \frac{1}{t_k} \sum_{q \in R_i(a_i) \cap Q^t_k} q(a_{-i}) N(q, t_k)
\]

\[
+ \frac{1}{t_k} \sum_{q \in R_i(a_i) \cap Q^t_k} (\rho(q, a_{-i}, t_k) - q(a_{-i})) N(q, t_k).
\]

Since the forecast being used by player \( i \) is calibrated, the second term in the last expression goes to zero at \( k \to \infty \). Further, we have for all \( k \geq 1 \),

\[
\sum_{q \in R_i(a_i) \cap Q^t_k} q N(q, t_k) / \sum_{q \in R_i(a_i) \cap Q^t_k} N(q, t_k) \in \overline{co}(R_i(a_i)).
\]
Taking the limit as $k \to \infty$ we have, $\hat{\xi}_{-i}(\cdot|a_i) \in \overline{co}(R_i(a_i))$, where $\overline{co}(\cdot)$ denotes the closed convex hull. Note that $R_i(a_i) \subseteq C(\Gamma, i, a_i)$ and $C(\Gamma, i, a_i)$ is closed. Thus $\hat{\xi}_{-i}(\cdot|a_i) \in \overline{co}(C(\Gamma, i, a_i))$ for all $a_i \in A_i$ such that $\hat{\xi}_i(a_i) > 0$. By lemma 3.4, we have $\hat{\xi} \in \overline{co}(C(\Gamma, i))$ and since this is true for all players $i$, we have $\hat{\xi} \in D(\Gamma)$.

**Remark 3.8.** In the proof above, we in fact prove the following stronger statement: if player $i$’s assessments are calibrated and she chooses best reaction to these assessments at every step, then the joint empirical distribution of action play converges to the set $\overline{co}(C(\Gamma, i))$.

This establishes the convergence of the empirical distribution of action play to the set of mediated CPT correlated equilibria under calibrated learning. Oakes [1985] proves that there does not exist a deterministic forecasting scheme that is calibrated for all sequences played by Nature. As noted in [Foster and Vohra, 1997, Theorem 3], however, there exists a randomized forecasting scheme such that no matter what outcome sequence Nature plays, the forecaster is almost surely calibrated. By a randomized forecasting scheme we mean the following: At each step $t$, the forecaster predicts a distribution $q^t \in \Delta(S)$ by drawing one randomly from a distribution over the space of distributions $\Delta(S)$, based on the history $y_1, \ldots, y_{t-1}$. Formally, there exists a randomized forecasting scheme such that the forecaster’s calibration score

$$\sum_{q \in Q^t} |\rho(q, y, t) - q(y)| \frac{N(q, t)}{t} \to 0 \text{ as } t \to \infty \text{ almost surely},$$

irrespective of the outcome sequence played by the Nature$^4$. Combining this result with theorem 3.7 we have,

**Corollary 3.9.** There exist randomized calibrated forecasting schemes for each player, such that if each player predicts her assessments according to her scheme, and plays the best reaction to her assessments, then the empirical distribution of action play converges almost surely to the set of mediated CPT correlated equilibria.

**Proof.** Let player $i$ be the forecaster and let all the opponents together form Nature from the point of view of the player. Thus Nature’s action set is

$^4$Foster and Vohra [1998] prove the existence of a randomized forecasting scheme that makes the forecaster's calibration score tend to zero in probability. However, as noted in Cesa-Bianchi and Lugosi [2006], the same argument proves that the convergence of the calibration score holds, in fact, almost surely.
$S = A_{-i}$. By the Foster and Vohra [1998] result, there exists a randomized forecasting scheme for player $i$ that is calibrated. Let player $i$ use this randomized scheme to predict her assessments. From remark 3.8, it follows that the empirical distribution of play converges to the set $co(C(\Gamma, i))$ almost surely. If each player $i$ follows such a strategy, then we get almost sure convergence to $D(\Gamma)$. \hfill \Box

We now show that, in a certain sense, the set $D(\Gamma)$ is the smallest possible extension of the set $C(\Gamma)$ that guarantees convergence of the empirical distribution of play to this set. In particular, we claim that for all games $\Gamma$ such that the sets $C(\Gamma, i, a_i), i \in N, a_i \in A_i$, do not have any isolated points, if $\mu \in D(\Gamma)$, then there exist calibrated assessments and best reactions for each player such that $\xi^t \to \mu$. Since $\mu \in D(\Gamma)$, as noted in remark 3.6, there exists a signal system $(B_i)_{i \in N}$, a mediator distribution $\psi \in \Delta(B)$, and a mediated CPT Nash equilibrium $\sigma \in \Sigma(\Gamma, (B_i)_{i \in N}, \psi)$ such that $\mu = \eta(\psi, \sigma)$ where $\sigma$ is a pure strategy profile. With abuse of notation, let $\sigma_i(b_i)$ denote the unique element in the support of $\sigma_i(b_i)$. Let $(b^1, b^2, \ldots)$ be a sequence of signal profiles such that the empirical distribution of these signal profiles converges to $\psi$. At step $t$, let player $i$ predict her assessment $\mu_{-i}(\cdot | b_i)$ (as defined in equation (3.2)) and play $\sigma_i(b_i)$. It is easy to see that the assessments of all players are calibrated. Further each of them plays a best response to her assessment and the empirical distribution of play converges to $\mu$. If we can define $\sigma_i(b_i)$ as the best reaction to the assessment $\mu_{-i}(\cdot | b_i)$ for all $b_i$ for all $i$, then the claim is proved. However, if there exist $b_i, \tilde{b}_i$ such that $\mu_{-i}(\cdot | b_i) = \mu_{-i}(\cdot | \tilde{b}_i)$ but $\sigma_i(b_i) \neq \sigma_i(\tilde{b}_i)$, then the above definition of best reaction is not valid.

Let the signal system be as described in the proof of lemma 3.4, where each signal $b_i$ equals $(a_i, m_i)$, $\psi$ is as in equation (3.5) and $\sigma$ as in (3.6). Suppose there exist $b_i = (a_i, m_i), \tilde{b}_i = (\tilde{a}_i, \tilde{m}_i)$ such that $\mu_{-i}(\cdot | b_i) = \mu_{-i}(\cdot | \tilde{b}_i)$ but $\sigma_i(b_i) \neq \sigma_i(\tilde{b}_i)$. In the proof of lemma 3.4 we saw that $\mu_{-i}(\cdot | b_i) = \zeta_{-i,a_i}^m$. Thus, $\pi_{-i} := \zeta_{-i,a_i}^m = \zeta_{-i,\tilde{a}_i}^m$ and $a_i \neq \tilde{a}_i$. Since $C(\Gamma, i, a_i)$ does not have any isolated point, $\pi_{-i}$ is a limit point of the set $C(\Gamma, i, a_i)$. Thus, there exists a sequence $(\tilde{\zeta}_{-i,a_i}^k)_{k \geq 1}$ of distinct distributions in $C(\Gamma, i, a_i)$ such that $\tilde{\zeta}_{-i,a_i}^k \to \zeta_{-i,a_i}^m$ and $(\tilde{\zeta}_{-i,a_i}^k)_{k \geq 1}$ are all distinct from the distributions $(\zeta_{-i,a_i}^m, \forall m_i, a_i)$.

Let player $i$’s assessment be $\tilde{\zeta}_{-i,a_i}^k$ when $b^k = b_i$ for the $k$-th time instead of $\zeta_{-i,a_i}^m$. Since $\tilde{\zeta}_{-i,a_i}^k \to \zeta_{-i,a_i}^m$, player $i$’s assessments are still calibrated and we can also resolve the best reaction to all these assessments without any conflict.
The randomized forecasting scheme proposed in Foster and Vohra [1998] generates a probability distribution on the space of assessments of player $i$. Player $i$ draws her assessment from this distribution and then plays her best reaction. This two-step process gives rise to a randomized strategy for player $i$ at each step. In remark 3.8 we saw that, no matter what actions the opponents play, player $i$ can guarantee that the empirical distribution of action play converges almost surely to the set $\bar{\sigma}(C(\Gamma,i))$.

Under EUT, player $i$ has a strategy that guarantees the almost sure convergence of the empirical distribution of play to the set $C(\Gamma,i)$. This convergence is related to the notion of no-regret learning. We now describe this approach. Suppose at step $t$, player $i$ imagines replacing action $a_i$ by action $\tilde{a}_i$, every timeshe played action $a_i$ in the past. Assuming the actions of the other players did not change, her payoff would become $x_i(\tilde{a}_i,a_{-i})$ for all $\tau \leq t$, such that $a_{\tau}^i = a_i$, instead of $x_i(a_i,a_{-i})$ while for all $\tau \leq t$, such that $a_{\tau}^i \neq a_i$, it will continue to be $x_i(a_i)$. We define the resulting CPT regret of player $i$ for having played action $a_i$ instead of action $\tilde{a}_i$ as

$$K^t_i(a_i,\tilde{a}_i) := \xi^t_i(a_i)\mathcal{R}_i \left\{ \left( \xi^t_{i-1}(a_{-i}|a_i), x_i(\tilde{a}_i,a_{-i}), x_i(a_i,a_{-i}) \right) \right\}_{a_{-i} \in A_{-i}}.$$

where

$$\mathcal{R}_i \left\{ \left( \nu_l, \tilde{z}_l, z_l \right) \right\}_{l=1}^m := V_i \left( \left\{ \nu_l, \tilde{z}_l \right\}_{l=1}^m \right) - V_i \left( \left\{ \nu_l, z_l \right\}_{l=1}^m \right)$$

is the difference in the CPT values of the lotteries $\left\{ \left( \nu_l, \tilde{z}_l \right) \right\}_{l=1}^m$ and $\left\{ \left( \nu_l, z_l \right) \right\}_{l=1}^m$. We associate player $i$ with CPT regrets $\left\{ K^t_i(a_i,\tilde{a}_i), a_i, \tilde{a}_i \in A_i, a_i \neq \tilde{a}_i \right\}$ at each step $t$. Under EUT, this simplifies to

$$K^t_i(a_i,\tilde{a}_i) = \frac{1}{t} \sum_{\tau \leq t, a_{\tau}^i = a_i} \left[ x_i(\tilde{a}_i,a_{\tau-1}^i) - x_i(a^\tau) \right]$$

in agreement with the definition given in Hart and Mas-Colell [2000].

In EUT regrets can be updated recursively without the need to update empirical distributions of action profiles based on the realization of play up to the current time. Based on this observation, Hart [2005] argues that such adaptive heuristics based on regret matching are boundedly rational strategies. In the CPT model we have studied, it would seem that the players need to keep a record of empirical distributions based on the realized sequence of action profiles in order to compute the CPT regrets. This would be in violation of the desired property of bounded rationality. One can counter this
objection by arguing that the computations involved in our model do not necessarily correspond to the computations actually being used by people, i.e. the agents whose behavior is being modeled by CPT. It may well be that the players use some boundedly rational way to recursively update their CPT regrets that does not require them to maintain empirical distributions.

The following proposition shows the connection between regrets and correlated equilibrium.

**Proposition 4.1.** Let \((a^t)_{t=1,2,...}\) be a sequence of action profiles played by the players. Then \(\limsup_{t\to\infty} K^t_i(a_i, \tilde{a}_i) \leq 0\), for every \(i \in N\) and every \(a_i, \tilde{a}_i \in A_i, a_i \neq \tilde{a}_i\), if and only if the sequence of empirical distributions \(\xi^t\) converges to the set \(C(\Gamma)\) of CPT correlated equilibrium.

**Proof.** Since \(\Delta(A)\) is a compact set, \(\xi^t\) converges to the set \(C(\Gamma)\) iff for every convergent subsequence \(\xi^{t_k}\), say converging to \(\tilde{\xi}\), we have \(\tilde{\xi} \in C(\Gamma)\). Let \(\xi^{t_k} \to \tilde{\xi}\) be a convergent subsequence. For each player \(i\), and for every \(a_i, \tilde{a}_i \in A_i, a_i \neq \tilde{a}_i\) such that \(\tilde{\xi}_i(a_i) > 0\), we have,

\[
K^{t_k}_i(a_i, \tilde{a}_i) \to \tilde{\xi}_i(a_i) \mathcal{R}_i \left[ \left\{ \left( \tilde{\xi}_{-i}(a_{-i}|a_i), x_i(a_i, a_{-i}), x_i(a_i, a_{-i}) \right) \right\}_{a_{-i} \in A_{-i}} \right]
\]

by continuity of \(V_i(p, x)\) as a function of the probability vector \(p\) for a fixed outcome profile \(x\). The result is immediate from the definition of CPT correlated equilibrium. \(\Box\)

Player \(i\) is said to have a no-regret learning strategy if her regrets tend to be arbitrarily small almost surely, irrespective of other players’ strategies, i.e.

\[
P \left( \limsup_{t \to \infty} K^t_i(a_i, \tilde{a}_i) \leq 0 \right) = 1, \text{ for every } a_i, \tilde{a}_i \in A_i, a_i \neq \tilde{a}_i.
\]

This is equivalent to asking if the vector of regrets \((K^t_i(a_i, \tilde{a}_i), a_i, \tilde{a}_i \in A_i, a_i \neq \tilde{a}_i)\), converges to the negative orthant almost surely. This is related to the concept of approachability, the setup for which is as follows. Consider a repeated two player game, where now at step \(t\), if the row player and the column player play actions \(a^t_{row}\) and \(a^t_{col}\) respectively, then the row player receives a vector payoff \(\vec{x}(a^t_{row}, a^t_{col})\) instead of a scalar payoff. A subset \(S\) is said to be approachable by the row player, if she has a (randomized) strategy such that no matter how the column player plays,

\[
\lim_{t \to \infty} \mathbb{E} \left( \frac{1}{t} \sum_{\tau=1}^{t} \vec{x}(a^t_{row}, a^t_{col}), S \right) = 0 \text{ almost surely.}
\]
Blackwell’s approachability theorem Blackwell [1956] states that a convex closed set $S$ is approachable if and only if every halfspace $H$, containing $S$, is approachable.

Hart and Mas-Colell [2000] cast the repeated game with stage game $\Gamma$ in the above setup as a two player repeated game where player $i$ is the row player and the opponents together form the column player. Let $\vec{x}(\hat{a}_i, \hat{a}_{-i})$ be the vector payoff when player $i$ plays action $\hat{a}_i$ and the others play $\hat{a}_{-i}$, with components given by

$$
\vec{x}_{a_i, \tilde{a}_i}(\hat{a}_i, \hat{a}_{-i}) = \begin{cases} 
  x_i(\tilde{a}_i, a_{-i}) - x_i(a_i, a_{-i}) & \text{if } \hat{a}_i = \tilde{a}_i \\
  0 & \text{otherwise}
\end{cases}
$$

for all $a_i, \tilde{a}_i \in A_i, a_i \neq \tilde{a}_i$. Under EUT, the average vector payoff of the row player corresponds to the regret of player $i$. Hart and Mas-Colell [2000] show that the negative orthant is approachable for the row player and hence player $i$ has a no-regret learning strategy. Under CPT, if the average vector payoffs were to match with player $i$’s regrets, then the vector payoffs for the row player at step $t$ turn out to depend on the empirical distribution of action play up to step $t$. The following example shows that under CPT, approachability of the nonnegative orthant need not hold. Blackwell’s sufficiency condition [Blackwell, 1956, Section 2] therefore does not hold with such state dependent payoffs. In other words, it can happen under CPT that at least one of the players does not have a no-regret learning strategy.

**Example 4.2.** Consider the 2-player repeated game from example 3.1. Suppose player 1 plays a no-regret learning strategy. Let $\sigma_{odd} = (0.5, 0, 0.5, 0), \sigma_{even} = (0, 0.5, 0, 0.5)$ and $\sigma_{unif} = (0.25, 0.25, 0.25, 0.25)$. Recall that player 1’s action 1 is not a best response to $\sigma_{odd}$ and $\sigma_{even}$ and player 1’s action 0 is not a best response to $\sigma_{unif}$. For an integer $T > 2$, consider the following strategy for player 2:

- play mixed strategy $\sigma_{odd}$ at step 1,
- play mixed strategy $\sigma_{even}$ at step 2,
- play mixed strategy $\sigma_{odd}$ at steps $2T^k < t \leq T^{k+1}$, for $k \geq 0$,
- play mixed strategy $\sigma_{even}$ at steps $T^{k+1} < t \leq 2T^{k+1}$, for $k \geq 0$.

Let $f_{1}^{k+1}$ denote the fraction of times player 2 plays $\sigma_{even}$ up to step $t = T^{k+1}$. We have

$$
f_{1}^{k+1} \leq \frac{2T^k}{T^{k+1}}. \quad (4.1)
$$

22
Let \( f^{k+1}_2 \) denote the fraction of times player 2 plays \( \sigma_{\text{even}} \) up to step \( t = 2T^{k+1} \). We have

\[
f^{k+1}_2 = \frac{T^{k+1} + \frac{T^{k+1} - 1}{T - 1}}{2T^{k+1}} \in \left[ \frac{1}{2}, \frac{1}{2} + \frac{1}{T - 1} \right]. \tag{4.2}\]

We now show that for a suitable choice of \( T, \epsilon > 0 \) and \( \delta > 0 \), there exists an integer \( k_0 \) such that for all \( k \geq k_0 \), we have

\[
P \left( \hat{K}^k > \epsilon \right) > 1 - \delta
\]

where

\[
\hat{K}^k = [K^{T^{k+1}}_1(0, 1)]^+ + [K^{T^{k+1}}_1(1, 0)]^+ + [K^{2T^{k+1}}_1(0, 1)]^+ + [K^{2T^{k+1}}_1(1, 0)]^+,
\]

and \([\cdot]^+ := \max\{, 0\}\).

Consider the subsequence of steps \((t^{l}_{\text{odd}})_{l \geq 1}\) when player 2 played \( \sigma_{\text{odd}} \). Let \( \nu^l_{\text{odd}}(a_1, a_2) \) denote the empirical distribution of the action profile \((a_1, a_2)\), where \( a_1 \in \{0, 1\}, a_2 \in \{I, III\} \), i.e.

\[
\nu^l_{\text{odd}}(a_1, a_2) = \frac{1}{l} \sum_{u=1}^{l} 1\{a^{u}_{\text{odd}} = (a_1, a_2)\}.
\]

We observe that, the sequence \((M^l, l \geq 1)\) is a martingale where \( M^l = l \times (\nu^l_{\text{odd}}(0, I) - \nu^l_{\text{odd}}(0, III)) \). Indeed, we have

\[
\mathbb{E}[M^{l+1} - M^l | M^1, \ldots, M^l] = \mathbb{E}[M^{l+1} - M^l | M^1, \ldots, M^l, a^{l+1}_{\text{odd}} = 0]P(a^{l+1}_{\text{odd}} = 0 | M^1, \ldots, M^l) \\
\quad + \mathbb{E}[M^{l+1} - M^l | M^1, \ldots, M^l, a^{l+1}_{\text{odd}} = 1]P(a^{l+1}_{\text{odd}} = 1 | M^1, \ldots, M^l) \\
\quad = \mathbb{E}[1\{a^{l+1}_{\text{odd}} = (0, I)\} - 1\{a^{l+1}_{\text{odd}} = (0, III)\} | M^1, \ldots, M^l, a^{l+1}_{\text{odd}} = 0]P(a^{l+1}_{\text{odd}} = 0 | M^1, \ldots, M^l) \\
\quad = \frac{1}{2} - \frac{1}{2} = 0,
\]

where the last line follows from the fact that player 2 plays \( \sigma_{\text{odd}} \) at each of the steps \( t^{l}_{\text{odd}} \) independently. Thus, for example by Azuma-Hoeffding inequality, for any \( \delta > 0 \), there exists an integer \( l_{\delta}^1 > 1 \), such that for all \( l \geq l_{\delta}^1 \),

\[
P \left( |\nu^l_{\text{odd}}(0, I)) - \nu^l_{\text{odd}}(0, III)| < \delta \right) > 1 - \delta.
\]

Similarly, we have an integer \( l_{\delta}^2 > 1 \), such that for all \( l \geq l_{\delta}^2 \),

\[
P \left( |\nu^l_{\text{odd}}(1, I)) - \nu^l_{\text{odd}}(1, III)| < \delta \right) > 1 - \delta.
\]
Now consider the sequence of steps \((t^l_{\text{even}})_{l \geq 1}\) when player 2 played \(\sigma_{\text{even}}\). Let \(\nu^l_{\text{even}}(a_1, a_2)\) denote the empirical distribution of the action profile \(a_1 \in \{0, 1\}, a_2 \in \{I, II, IV\}\), i.e.

\[
\nu^l_{\text{even}}(a_1, a_2) = \frac{1}{l} \sum_{u=1}^{l} 1\{a^u_{\text{even}} = (a_1, a_2)\}.
\]

We have an integer \(l^3_\delta > 1\), such that for all \(l \geq l^3_\delta\),

\[
P\left(|\nu^l_{\text{even}}(0, II) - \nu^l_{\text{even}}(0, IV)| < \delta\right) > 1 - \delta,
\]

and an integer \(l^4_\delta > 1\), such that for all \(l \geq l^4_\delta\),

\[
P\left(|\nu^l_{\text{even}}(1, II) - \nu^l_{\text{even}}(1, IV)| < \delta\right) > 1 - \delta.
\]

For a distribution \(\mu \in \Delta(S)\) and \(\epsilon > 0\), let \([\mu]_\epsilon = \{\tilde{\mu} \in \Delta(S) : |\tilde{\mu}(s) - \mu(s)| < \epsilon, \forall s \in S\}\) denote the set of all distributions within \(\epsilon\) error from \(\mu\). Select positive constants \(\epsilon_3, c_3, \epsilon_2, c_2, \epsilon_1, c_1\) as follows:

- Let \(\epsilon_3\) and \(c_3\) be such that the regret \(R_1(\mu, x_1(1, \cdot), x_1(0, \cdot)) > c_3\) for all \(\mu \in [\sigma_{\text{unif}}]_{\epsilon_3}\) (such constants exist because action 0 is not a best response to \(\sigma_{\text{unif}}\)). Let \(\delta_3 = \epsilon_3/2\).

- Let constants \(\epsilon_2 < 0.125\) and \(c_2\) be such that the regret \(R_1(\mu, x_1(0, \cdot), x_1(1, \cdot)) > c_2\) for all \(\mu \in [\sigma_{\text{even}}]_{\epsilon_2}\) (such constants exist because action 1 is not a best response to \(\sigma_{\text{even}}\)). Let \(\delta_2 = \epsilon_2\delta_3\).

- Let constants \(\epsilon_1 < 0.25\) and \(c_1\) be such that the regret \(R_1(\mu, x_1(0, \cdot), x_1(1, \cdot)) > c_1\) for all \(\mu \in [\sigma_{\text{odd}}]_{\epsilon_1}\) (such constants exist because action 1 is not a best response to \(\sigma_{\text{odd}}\)). Let \(\delta_1 = \epsilon_1\delta_2\).

Let \(T > \frac{2}{\delta_1}\) and \(k_0\) be such that \(T^{k_0+1} > \max\{l^1_{\text{odd}}, l^2_{\text{odd}}, l^3_{\text{even}}, l^4_{\text{even}}\}\).

Let \(f^k_{3+1} = \xi^{T_{k+1}}(0)\). For \(k \geq k_0\), from the above observations we have

\[
\xi^{T_{k+1}}(1, \cdot) \in \left[\left(\frac{1 - f^k_{3+1}}{2}, 0, \frac{1 - f^k_{3+1}}{2}, 0\right)\right]_{\delta_1}
\]

with probability at least \(1 - \delta_1\). Indeed, from equation (4.1) and the assumption \(T > \frac{2}{\delta_1}\), we have \(\xi^{T_{k+1}}(1, II) + \xi^{T_{k+1}}(1, III) \leq \frac{2}{T} < \delta_1\), and \(|\xi^{T_{k+1}}(1, I) - \xi^{T_{k+1}}(1, III)| < \delta_1(1 - f^k_{1+1}) < \delta_1\) with probability at least
Putting these together, we get, for the distribution at step $T^k+1$ and $2T^k+1$. We now show that the empirical distribution at step $2T^k+1$ is approximately as shown in table 3 within $\delta_2$ error, i.e. $\xi^{2T^k+1}(0, I) + \xi^{2T^k+1}(0, III) < \delta_2/2$ and hence each term is necessarily less than $\delta_2$. Further, from equation (4.2), we have $\xi^{2T^k+1}(0, I) + \xi^{2T^k+1}(0, III) \in [0.5 - \delta_1 - \delta_2/2, 0.5]$ and since $\xi^{2T^k+1}(0, I) - \xi^{2T^k+1}(0, III) < \delta_1$ with probability at least $1 - \delta_1$, each term lies in the interval $[0.25 - 2\delta_1 - 0.25, 0.25 + \delta_1]$. Since $\epsilon_1 < 0.25$, we have $\xi^{2T^k+1}(0, I), \xi^{2T^k+1}(0, III) \in [0.25 - \delta_2, 0.25 + \delta_2]$. From equation (4.2) and since $|\xi^{2T^k+1}(0, II) - \xi^{2T^k+1}(0, IV)| < \delta_1$ with probability at least $1 - \delta_1$, we have $\xi^{2T^k+1}(0, II), \xi^{2T^k+1}(0, IV) \in [0.25f_4^{k+1} - \delta_1, 0.25f_4^{k+1} + 2\delta_1]$. Similarly, we get $\xi^{2T^k+1}(1, II), \xi^{2T^k+1}(1, IV) \in [0.25(1 - f_4^{k+1}) - \delta_1, 0.25(1 - f_4^{k+1}) + 2\delta_1]$ with probability at least $1 - \delta_1$. Putting these together, we get, $P(\xi^{2T^k+1} \in [\hat{\mu} \delta_2]) \geq 1 - 3\delta_1$.

If $f_3^{k+1} < 1 - \delta_2$, then $\xi^{2T^k+1}(-1) \in [\sigma_{even}]_{\delta_2}$. From the assumption $\delta_2 = \epsilon_2 \delta_3/8$, we have $\frac{\delta_1}{0.25\delta_3 - \delta_2} < \epsilon_2$ and hence the regret $\gamma^{2T^k+1}(1,0) \geq (0.25\delta_3 - \delta_2)c_2$. Thus,

$$P\left(\left[K_1^{2T^k+1}(1,0)\right]^+ \geq (0.25\delta_3 - \delta_2)c_2 1 \left\{ f_3^{k+1} \geq 1 - \delta_2, f_4^{k+1} < 1 - \delta_3 \right\} \right) > 1 - 3\delta_1.$$

If $f_4^{k+1} \geq 1 - \delta_3$ then $\xi^{2T^k+1}(-1) \in [\sigma_{unif}]_{\delta_3}$. From the assumptions $\epsilon_2 < 0.125$ and $\delta_3 = \epsilon_3/3$ we have $\xi^{2T^k+1}(-1) \in [\sigma_{unif}]_{\epsilon_3}$ and thus $\gamma^{2T^k+1}(1,0) \geq (1 - \delta_3)c_3$. Thus,

$$P\left(\left[K_1^{2T^k+1}(1,0)\right]^+ \geq (1 - \delta_3)c_3 1 \left\{ f_3^{k+1} \geq 1 - \delta_2, f_4^{k+1} \geq 1 - \delta_3 \right\} \right) > 1 - 3\delta_1.$$

If we take $\tilde{\epsilon} < \frac{\delta_1}{2} \min\{c_1, c_2, c_3, 1\}$ and $\tilde{\delta} = 8\delta_1$, then we have for all $k \geq k_0$,

$$P\left(K^k > \tilde{\epsilon} \right) > 1 - \tilde{\delta}_1.$$

This contradicts the fact that player 1 has a no-regret learning strategy.
5 Conclusion

We studied how some of the results from the theory of learning in games are affected when the players in the game have cumulative prospect theory preferences. For example, we saw that the notion of mediated games and mediated CPT correlated equilibrium is more appropriate than the notion of CPT correlated equilibrium while studying the convergence of the empirical distribution of play, in particular for calibrated learning schemes. One can ask similar questions with respect to other learning schemes such as follow the perturbed leader [Fudenberg and Levine, 1995], fictitious play [Brown, 1951], etc. We leave this for future work. In general, it seems that the results from learning in games continue to hold under CPT with slight modifications.

References

M. Allais. L’extension des théories de l’équilibre économique général et du rendement social au cas du risque. *Econometrica, Journal of the Econometric Society*, pages 269–290, 1953.

R. Aumann. Correlated equilibrium as an expression of bayesian rationality’, econometrica 55 (1), 1–18. *Google Scholar*, 1987.

R. J. Aumann. Subjectivity and correlation in randomized strategies. *Journal of mathematical Economics*, 1(1):67–96, 1974.

R. J. Aumann, M. Maschler, and R. E. Stearns. *Repeated games with incomplete information*. MIT press, 1995.

D. Blackwell. An analog of the minimax theorem for vector payoffs. *Pacific Journal of Mathematics*, 6(1):1–8, 1956.

G. W. Brown. Iterative solution of games by fictitious play. *Activity analysis of production and allocation*, 13(1):374–376, 1951.
N. Cesa-Bianchi and G. Lugosi. *Prediction, learning, and games*. Cambridge university press, 2006.

V. P. Crawford. Equilibrium without independence. *Journal of Economic Theory*, 50(1):127–154, 1990.

D. P. Foster and R. V. Vohra. Calibrated learning and correlated equilibrium. *Games and Economic Behavior*, 21(1-2):40–55, 1997.

D. P. Foster and R. V. Vohra. Asymptotic calibration. *Biometrika*, 85(2):379–390, 1998.

D. Fudenberg and D. Levine. Learning in games. *European economic review*, 42(3-5):631–639, 1998.

D. Fudenberg and D. K. Levine. Consistency and cautious fictitious play. *Journal of Economic Dynamics and Control*, 19(5-7):1065–1089, 1995.

J. Hannan. Approximation to bayes risk in repeated play. *Contributions to the Theory of Games*, 3:97–139, 1957.

S. Hart. Adaptive heuristics. *Econometrica*, 73(5):1401–1430, 2005.

S. Hart and A. Mas-Colell. A simple adaptive procedure leading to correlated equilibrium. *Econometrica*, 68(5):1127–1150, 2000.

D. Kalathil, V. S. Borkar, and R. Jain. Approachability in stackelberg stochastic games with vector costs. *Dynamic games and applications*, 7(3):422–442, 2017.

K. Keskin. Equilibrium notions for agents with cumulative prospect theory preferences. *Decision Analysis*, 13(3):192–208, 2016.

D. M. Kreps. *Game theory and economic modelling*. Oxford University Press, 1990.

R. B. Myerson. Multistage games with communication. *Econometrica: Journal of the Econometric Society*, pages 323–358, 1986.

R. B. Myerson. *Game theory*. Harvard university press, 2013.

J. Nash. Non-cooperative games. *Annals of mathematics*, pages 286–295, 1951.

D. Oakes. Self-calibrating priors do not exist. *Journal of the American Statistical Association*, 80(390):339–339, 1985.
V. Perchet. Calibration and internal no-regret with random signals. In *International Conference on Algorithmic Learning Theory*, pages 68–82. Springer, 2009.

S. R. Phade and V. Anantharam. On the geometry of nash and correlated equilibria with cumulative prospect theoretic preferences. *arXiv preprint arXiv:1712.00859*, 2017.

D. Prelec. The probability weighting function. *Econometrica*, pages 497–527, 1998.

A. Tversky and D. Kahneman. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and uncertainty*, 5(4):297–323, 1992.

J. Von Neumann and O. Morgenstern. Theory of games and economic behavior. *Bull. Amer. Math. Soc*, 51(7):498–504, 1945.

P. P. Wakker. *Prospect theory: For risk and ambiguity*. Cambridge university press, 2010.

H. P. Young. *Strategic learning and its limits*. OUP Oxford, 2004.