Simple Baryon-Meson Mass Relations
From A Logarithmic Interquark Potential

Tom D. Imbo*

Department of Physics
University of Illinois at Chicago
845 W. Taylor St.
Chicago, IL 60607-7059

I consider the quantity \( \delta(m_1 m_2 m_3) \equiv M_{q_1 q_2 q_3} - (M_{q_1 \bar{q}_2} + M_{q_2 \bar{q}_3} + M_{q_1 \bar{q}_3})/2 \), where the \( M \)'s represent the ground state spin-averaged hadron masses with the indicated quark content and the \( m \)'s the corresponding constituent quark masses. I assume a logarithmic interquark potential, the validity of a nonrelativistic approach, and various standard potential model inputs. Simple scaling arguments then imply that the quantity \( R(x) \equiv \delta(m m m_3)/\delta(m_0 m_0 m_0) \) depends only on the ratio \( x = m/m_3 \), and is independent of \( m_0 \) as well as any parameters appearing in the potential. A simple and accurate analytic determination of \( \delta(m m m_3) \), and hence \( R(x) \), is given using the \( 1/D \) expansion where \( D \) is the number of spatial dimensions. When applicable, this estimate of \( R(x) \) compares very well to experiment — even for hadrons containing light quarks. A prediction of the above result which is likely to be tested in the near future is \( M_{\Sigma^*}^b/2 + (M_{\Lambda_b} + M_{\Sigma^b})/4 = 5774 \pm 4 \text{ MeV}/c^2 \).

* imbo@uic.edu
In 1983, Nussinov \([1]\) suggested that

\[
M_{q_1 q_2 q_3} > (M_{q_1 \bar{q}_2} + M_{q_2 \bar{q}_3} + M_{q_1 \bar{q}_3})/2
\]

is a rigorous inequality in QCD. Here \(M_{q_1 q_2 q_3}\) represents the mass of any spin state of the ground state baryon with the indicated quark content, and \(M_{q_i \bar{q}_j}\) the mass of the ground state meson in the spin state imposed by the chosen baryon. He was able to demonstrate this assuming a separation of the three-body Hamiltonian for the quarks in the baryon into a sum of two-body pieces — a result which holds at both weak and strong coupling in QCD. Although this does not constitute a complete proof, the inequality (1) is indeed satisfied by the observed ground state baryons and mesons. The arguments in \([1]\) apply not only to specific spin states as noted above, but also to spin-averaged states \([2]\). That is, we can take the \(M\)’s in (1) to represent the spin-averaged ground state baryon and meson masses. This view of the Nussinov inequality, which we adopt below, is also supported by experiment.

Is it possible to further understanding the relationship between ground state baryon and meson masses? More precisely, if we write

\[
M_{q_1 q_2 q_3} = (M_{q_1 \bar{q}_2} + M_{q_2 \bar{q}_3} + M_{q_1 \bar{q}_3})/2 + \delta(m_1 m_2 m_3),
\]

(2)
can we say anything about the positive quantity \(\delta(m_1 m_2 m_3)\)? How does \(\delta\) depend on the quark masses \(m_1, m_2\) and \(m_3\)? Does anything special happen when one or more of these masses get very large, or when one or more are equal? To begin to answer these questions clearly requires more specific dynamical input than that used in arguing for the inequality (1). A first principles analytic computation from QCD is clearly beyond our current theoretical resources. Numerical results from lattice QCD have not yet reached the level of accuracy required to address these questions seriously. So, in order to proceed at all, a more phenomenological approach is called for. For instance, we can attempt to study \(\delta\) in the context of potential models. That is, we can postulate a specific form for the interaction between the quarks in mesons and baryons, and then solve the appropriate two and

\footnote{This inequality was also discussed independently in the context of nonrelativistic potential models by Richard \([3]\). The special case of equal quark masses was first treated by Ader, Richard and Taxil \([3]\). Similar inequalities were also derived at this time by Weingarten \([4]\) and Witten \([5]\). Various “improved” versions of (1) have also been discussed over the years. See, for example, \([6]\) and references therein.}
three body problems to obtain the desired spectrum. Of course, there are necessarily free parameters in this approach which must be fit to a subset of the experimental data. This is because we do not know the exact form of the interquark potential, nor do we know the exact constituent quark masses that should appear in the above Hamiltonians. Moreover, three body equations are notoriously difficult to solve or approximate analytically, so that one must in general resort to sophisticated numerical methods. So even this very natural idea of potential models, although having been applied very successfully for over 30 years, is found to lack a certain simplicity and elegance.

We will find, however, that a simple, accurate, and completely analytic approximation to $\delta$ can be obtained if one assumes a logarithmic interquark potential and nonrelativistic dynamics governed by the Schrödinger equation — at least when two of the quark masses are equal. The log potential approach to the hadron spectrum has been around for two decades [7]. It was originally motivated by the observation that the $\psi(2S)$-$\psi(1S)$ and $\Upsilon(2S)$-$\Upsilon(1S)$ mass splittings are approximately equal, as this splitting is quark mass independent in the log potential. After many years, and the discovery of many new states, fits to the mass spectrum of mesons containing $b$, $c$ and $s$ quarks still support the quasi-logarithmic nature of the interquark potential [8]. There is also evidence supporting this picture in the low-lying baryon spectrum, although computations are more difficult here and experimental results more sparse [8][9]. Of course, it is clear from QCD that this description must eventually break down at very short and very large distance scales. However, more realistic QCD-inspired potentials — such as the famous “Coulomb-plus-linear” potential [10] — are all quasi-logarithmic over the distance scales relevant for the low-lying spectrum of the observed heavy hadrons [11][12]. In short, the log potential and its relatives have been very successful.

The approach used here is to approximate $\delta$ to leading order in the $1/D$ expansion, where $D$ is the number of spatial dimensions. That is, the relevant two and three body Hamiltonians are first generalized from 3 to $D$ spatial dimensions, and then we assume that $D$ is large. It is well known that a systematic $1/D$ expansion can be developed [13][14]. There is also strong evidence that the leading order term is already very accurate for quasi-logarithmic potentials, in both the two and three body cases, even for $D = 3$. The two

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2 Other physical quantities such as production and decay rates, which are sensitive to the wave function at the origin, depend on physics at shorter distance scales. Here there are substantial differences between the predictions of quasi-logarithmic potentials and more realistic ones, even though they have the same “shape” near the RMS radii of the states. See, for example, [12].
body results in the $1/D$ expansion also have the nice property that they are completely analytic for all power-law potentials $V(r) = Ar^b$. Since $\ell n(r) = \lim_{b \to 0}(r^b - 1)/b$, the results are also analytic for the log potential. In the three body case this property no longer holds for general power-law potentials except in the case of three equal masses. For instance, when only two masses are equal, the leading order large-$D$ result requires the numerical solution of a transcendental algebraic equation \[15\]. However, for the log limit I will show that a completely analytic solution can be found. Moreover, when put together with the two body result, this leads to an extremely simple and accurate form for $\delta$. (For three unequal masses the situation is more complicated \[15\], and will not be treated here. Related applications of large-$D$ results to the ground state meson and baryon spectra can be found in \[15\].)

Let's begin with the specific form of the interquark forces. I will assume that the potential between a quark and an antiquark in a color singlet meson is $V_{q\bar{q}}(r) = A\ell n(r/r_0)$, where $r$ is the interquark separation and $A, r_0 > 0$ are parameters. I make the further standard assumption \[3\] that the quark-quark potential in a baryon (that is, two quarks in an overall color antitriplet) is $1/2$ of the color singlet quark-antiquark potential at equal separation: $V_{qq}(r) = V_{q\bar{q}}(r)/2$. (I ignore the effects of the direct three body interactions that arise in QCD, as these have been shown on many occasions to be negligible for the questions that we are considering. See, for example, \[3\].) I also assume that the dynamics are governed by the appropriate nonrelativistic Schrödinger equation. If $H_2\psi = E_{m_1m_2}\psi$ and $H_3\phi = E_{m_1m_2m_3}\phi$ are the two and three body Schrödinger equations (for the ground state) respectively, then we have $\delta(m_1m_2m_3) = E_{m_1m_2m_3} - (E_{m_1m_2} + E_{m_2m_3} + E_{m_1m_3})/2$. Throughout, the effects of spin are neglected and the results are interpreted as representing the masses and binding energies of spin-averaged meson and baryon states. The simple rescaling $r_i \to h r_i/\sqrt{m_i A}$ of the quark position vectors in $H_2$ and $H_3$ shows that

$$E_{m_1m_2} = (A/2)[\ell n(h^2/\mu Ar_0^2) + C],$$
$$E_{m_1m_2m_3} = (3A/4)[\ell n(h^2/m_1 Ar_0^2) + f(w, x)],$$

where $w = m_1/m_2$, $x = m_1/m_3$, and $\mu = m_i m_j/(m_i + m_j)$ is the two body reduced mass. Both the constant $C$ and the function $f$ are independent of $h, A$ and $r_0$. These scaling relations further imply that

$$\delta(m_1m_2m_3) = Ag(w, x),$$

(3)
where again $g$ is independent of any dimensionful parameter in the problem other than the constituent quark masses, and even this dependence is only through the two dimensionless ratios $w$ and $x$.

Note that an immediate consequence of (11) is that in the case of equal quark masses $m_1 = m_2 = m_3 \equiv m$, $\delta$ is independent of $m$. Assuming that the constituent masses of the $u$ and $d$ quarks are identical and equal to some value $m_n$, where here and below the symbol $n$ denotes either $u$ or $d$, we may then determine $\delta(m_n m_n m_n)$ from experiment. More precisely, $\delta(m_n m_n m_n) = M_{nnn} - 3(M_{n\bar{n}})/2$. Assuming further the standard technique for constructing the spin-averaged masses, we have $M_{nnn} = (M_N + M_\Delta)/2$ and $M_{n\bar{n}} = (M_\pi + 3M_\rho)/4$. Putting this all together with the measured masses of the $N$, $\Delta$, $\pi$ and $\rho$ states gives $\delta(m_n m_n m_n) = 171 \pm 2$ MeV/$c^2$. Thus, we should expect $\delta(m_s m_s m_s)$ to also be in this range, where $m_s$ is the mass of the strange quark. Unfortunately, this cannot be tested directly since the spin 1/2 ground state baryon with three $s$ quarks does not exist. There is only the spin 3/2 $\Omega^-$. (This is because the spin wave function of the identical $s$ quarks must be totally symmetric in the ground state.) There is a similar problem in the meson sector where only the spin 1 $\phi$ can be interpreted as a nearly pure $s\bar{s}$ ground state. (The spin 0 combination mixes strongly with $u\bar{u}$ and $d\bar{d}$ states.) Thus, some additional theoretical input is needed to obtain the spin-averaged baryon and meson masses. However, any reasonable model of the spin-spin interactions between quarks, when applied to the $\Omega^-$ and $\phi$ in order to reconstruct the spin-averaged states, will yield a value of $\delta(m_s m_s m_s)$ around the desired range. A similar test of $\delta(mmm)$ for charm and bottom quarks cannot be made since the heavy $ccc$ and $bbb$ baryons have not yet been observed.

It is interesting to note that a numerical evaluation of $\delta$ in the case of equal quark masses yields $\delta(mmm) \simeq (0.225)A$. (That is, $g(1,1) \simeq 0.225$.) If we then use this result to “fit” $A$ to the observed $\delta$ for $u$ and $d$ quarks, we obtain $A = 760 \pm 10$ MeV. This is not too far from the value $A = 733$ MeV obtained from a fit of the log potential spectrum to meson states containing only heavy quarks! Not only do we see a striking consistency between the meson and baryon sectors, but we also gain some faith in the application of our nonrelativistic model to hadrons containing “light” quarks. This should not be too surprising since nonrelativistic models have often been successful on such semirelativistic quark systems in the past. The standard philosophy regarding this is that many relativistic

\footnote{Unless otherwise stated, all experimental results are taken from [18]. I also average over the electromagnetic splittings within a given isospin multiplet.}
corrections can be incorporated into a nonrelativistic model by a redefinition of the constituent quark masses and simple modifications of the potential such as the addition of an overall constant, at least for low-lying states \[19\]. (Note that both of these modifications do not affect the result for \(\delta(mmm)\) in the log potential.) Moreover, in our case there may be some cancellations of additional relativistic corrections occurring between the two body and three body contributions to \(\delta\). I will maintain this viewpoint in what follows and continue to apply the results of the above model to systems containing light and/or heavy quarks.

What about the case when only two of the quark masses are equal? Here, \(\delta(mmm_3) = Ag(1, x)\), where \(x = m/m_3\). Instead of resorting to a time-consuming numerical determination of \(g\) as a function of \(x\), let’s consider \(\delta\) to leading order in the \(1/D\) expansion. To this end, let \(H^{(D)}_2 \psi = E^{(D)}_{m_1 m_j} \psi\) and \(H^{(D)}_3 \phi = E^{(D)}_{m_1 m_2 m_3} \phi\) be the generalizations of the two and three body Schrödinger equations to an arbitrary number of spatial dimensions. That is, we keep the interquark potentials the same and simply replace each three-dimensional Laplacian appearing in the kinetic energy terms by its \(D\)-dimensional counterpart. We can still write \(E^{(D)}_{m_1 m_j}\) and \(E^{(D)}_{m_1 m_2 m_3}\) as in (3), only now the constant \(C\) and the function \(f(w, x)\) depend on \(D\). These binding energies can be obtained in the leading order of the \(1/D\) expansion by taking the appropriate limit of the corresponding results for power-law potentials. For the two body case, this has already been done \[20\]. The result is

\[
E^{(D)}_{m_1 m_j} = (A/2)\{\ln(e\hbar^2 D^2/4\mu A r_0^2) + O(1/D)\}.
\]

(5)

The \(D \to \infty\) limit is a classical limit (distinct from the \(\hbar \to 0\) limit), and the above result can be interpreted as \(V_{\text{eff}}(R_0)\) where \(V_{\text{eff}}\) is the large-\(D\) effective potential. Here \(R_0\) is the interquark distance at which this effective potential is minimized. In other words, \(R_0\) is the “size” of the meson. For the above log potential we have \(R_0 = \sqrt{\hbar^2/4\mu A}\).

In the three body case, the leading order large-\(D\) result for power-law potentials is not completely analytic, as noted previously. However, the log limit can still be recovered in closed form when two of the masses are equal. A careful analysis yields

\[
E^{(D)}_{mmm_3} = (3A/4)\{\ln(e\hbar^2 D^2(1 + 2x)(2a)^{1/3}/2mA r_0^2(2 - a)^2) + O(1/D)\},
\]

(6)

where \(x = m/m_3\) and \(a = 8/[\sqrt{(2x + 1)(2x + 25) + 2x + 5}]\). This corresponds to a minimum energy configuration of the large-\(D\) effective potential in which the distance between \(q_3\) and either of the other two quarks is \(\sqrt{\hbar^2(1 + 2x)/2mA(2 - a)^2}\), and the
angle $\theta$ between the two associated displacement vectors satisfies $\cos(\theta) = 1 - a$. After some messy but straightforward algebra, one obtains the following expression for $\delta(mmm_3) = E_{mmm_3}^{(D)} - E_{mm_3}^{(D)} - E_{mm}^{(D)}/2$:

\[
\delta(mmm_3) = (A/4)[5\ln(\sqrt{(2x+1)(2x+25)} + 2x + 5) + 3\ln(2x + 1) - 2\ln(x + 1) + O(1/D)].
\] (7)

Note that $\delta(mmm_3)$ is independent of $D$ as $D \to \infty$. When $x = 1$, that is $m_3 = m$, we have $\delta(mmm) = (3A/4)\ln(4/3) \simeq (0.216)A$, which compares well with the numerical result of $(0.225)A$. (It is interesting to note that the large-$D$ result for $\delta(mmm)$ is exactly the same as that obtained in [21] by seemingly independent means.) It seems reasonable to believe that at least some of this accuracy is due to a partial cancellation of the $1/D$ corrections between the two and three body cases. We can also easily extract the results for $x = 0$ and $x \to \infty$ from (7). We have $\delta(x = 0) = (A/4)\ln(3125/1458) \simeq (0.191)A$ and $\delta(x \to \infty) = (A/4)\ln(2) \simeq (0.173)A$. Scaling arguments imply that the latter result is actually exact! Unfortunately, in this limit we do not expect the log potential to provide an accurate description of quark dynamics. This is because we can think of the $x \to \infty$ limit as either $m$ getting very large or $m_3$ getting very small. In the former case, the two very heavy quarks of mass $m$ in the baryon are clearly in the perturbative “Coulombic” and not the logarithmic regime, while in the latter case a nonrelativistic treatment of the very light quark is not justified. (Similar problems do not occur in the $x \to 0$ limit viewed as $m_3$ getting very large.) As mentioned earlier, however, we do believe that the nonrelativistic log potential is applicable to systems containing $u, d, s, c$ and $b$ quarks. Here, $x$ can get as large as $m_b/m_n \simeq 15$.

It is often a bit more useful to consider the ratio $R(x) \equiv \delta(mmm_3)/\delta(m_0m_0m_0)$. This quantity has the advantage of being independent of the parameters appearing in the log potential. A plot of $R(x)$ in the large-$D$ limit is shown in Fig. 1. It possesses a single critical point (a global maximum) in the range $0 \leq x < \infty$. This occurs at $x = 1$ where $R(1) = 1$. (Note that the large-$D$ result for $R(x)$ is forced to be exact at $x = 1$, so that one might expect it to be somewhat more accurate in the region of physical interest than the corresponding result for $\delta$, which already seems to be within a few percent of the exact number for all $x$. More on this below. Of course, $R(x)$ is no longer exact as $x \to \infty$. But this is a trade-off that we will gladly accept since, as noted above, our model is not applicable at very large $x$.) The asymptotic values $R(0) \simeq 0.883$ and $R(\infty) \simeq 0.803$ represent completely parameter independent relations for the log potential, at least in the
large-$D$ limit. Note also that $R(x)$ only changes by about 20% over an infinite range of $x$ values, and only by about 10% in the physical region $m_n/m_b \leq x \leq m_b/m_n$. Higher order $1/D$ corrections to the two and three body energies, and hence to $\delta$ and $R$, can in principle be computed. Although the resulting formulas will be more complicated, it is nice to know that there is a systematic analytic procedure for improving our results. For instance, using the results of [22] one obtains

$$
\delta(mm) = (3A/4)[\ln(n)/3 + (2\sqrt{3} - \sqrt{2} - 2)/D + O(1/D^2)]
\equiv [3\ln(n)/4A/4\{1 + \alpha/D + O(1/D^2)}].
$$

(8)

For $D = 3$ this yields $\delta(mm) \approx 0.228$ which is 99% accurate, to be compared to the 96% accuracy of the leading term alone. Of course, the above $1/D$ correction has no effect on $R(1)$ which is already exact at leading order. However, for $x \neq 1$ our result for $R(x)$ will have $1/D$ corrections. As an example, consider $R(0)$. A lengthy computation using the results of [13] yields

$$
\delta(x = 0) = (A/4)[\ln(n)/4 + (2\sqrt{3} + \sqrt{2} + 2\sqrt{6} - 5\sqrt{2} - 5)/D + O(1/D^2)]
\equiv [\ln(n)/4A/4\{1 + \beta/D + O(1/D^2)}].
$$

(9)

For $D = 3$ this gives $\delta(x = 0) \approx 0.202$ which is 99% accurate, compared to the leading order result of approximately $0.191A$. Putting (9) together with (8) we obtain

$$
R(0) = [\ln(n)/4A/4\{1 + (\beta - \alpha)/D + O(1/D^2)}].
$$

(10)

For $D = 3$ we have $(\beta - \alpha)/D \approx 0.0016$, so that the change in $R(0)$ due to $1/D$ corrections is about 1/6 of 1%. $(R(0)$ changes from about 0.8833 to approximately 0.8848.) This is quite remarkable convergence. So although not exact as for $R(1)$, the leading order result for $R(0)$ should be extremely accurate. I believe the above examples to be illustrations of a quite general pattern: the leading order large-$D$ result for $\delta(mm)_{3}$ is accurate to within a few percent, while the corresponding result for $R(x)$ is accurate to within a fraction of a percent — at least over a large range of $x$. To be safe, one can compute the $1/D$ corrections to $R(x)$ for a general $x$ using the results of [23]. However, it seems clear that the leading order results are accurate enough in general that the added complexity introduced by adding higher order corrections is not justified.
How do the above results compare with observation? Let’s first consider $x = m_n/m_s$ and $x = m_s/m_n$ (we have already discussed the situation for $x = 1$). In the former case we have

$$
\delta(m_n m_n m_n) = \frac{(M_\Lambda + M_\Sigma)}{4} + \frac{M_{\Sigma^*}}{2} - \frac{(M_\pi + 3M_\rho)}{8} - \frac{(M_K + 3M_{K^*})}{4}
$$

(11)

Recalling that $\delta(m_n m_n m_n) = 171 \pm 2$ MeV/c$^2$ as well, this leads to the experimental value $R_{\exp}(m_n/m_s) = 1.00 \pm 0.02$. However, we cannot obtain an unambiguous theoretical prediction for $R(m_n/m_s)$ from (11) since $m_n$ and $m_s$ have not been determined in our model. One can use, among other things, the ratio of the $K-K^*$ and $\pi-\rho$ mass splittings as a naive estimate of $m_n/m_s$. This gives $m_n/m_s \approx 0.63$. (For comparison, the ratio of the $\Sigma-\Sigma^*$ to the $N-\Delta$ splitting yields $m_n/m_s \approx 0.65$.) But luckily, $R$ does not change too much within the entire range of “reasonable” values of $m_n/m_s$. For $0.5 \leq m_n/m_s \leq 0.8$, $R(m_n/m_s)$ stays between 0.985 and 1.00, in good agreement with the experimental number. Using the corresponding range $1.25 \leq m_s/m_n \leq 2.0$, the log model prediction for $R(m_s/m_n)$ also varies between 0.985 and 1.00. Unfortunately, we cannot obtain a purely experimental number to compare this with since certain states needed to find the required spin-averaged
masses $M_{s\bar{s}}$ and $M_{s\bar{n}}$ do not exist. However, estimating the masses of these missing states from other spin-splittings of hadrons containing $u$, $d$ and $s$ quarks yields an “experimental” value of $R(m_s/m_n)$ in the desired range.

We now turn to hadrons containing one heavy quark — that is, one $c$ or $b$ quark. Taking the ratio of the $D$-$D^*$ and $\pi$-$\rho$ mass splittings yields the estimate $m_n/m_c \simeq 0.22$. (Using the recent measurement $M_{\Sigma^*_c} = 2519 \pm 2$ MeV/c$^2$ made by the CLEO collaboration [24], we can also estimate $m_n/m_c$ from the ratio of the $\Sigma_c - \Sigma^*_c$ and $N - \Delta$ splittings. This again gives $m_n/m_c \simeq 0.22$.) From (7) we then obtain the prediction $R \simeq 0.95$. As before, the deviation from this value is not too large over the entire range of reasonable values of $m_n/m_c$. More precisely, for $0.15 \leq m_n/m_c \leq 0.3$, the value of $R$ varies from approximately 0.935 to 0.965. The experimental formula for $\delta(m_n m_n m_c)$ can be obtained from (11) by replacing $K$ and $K^*$ by $D$ and $D^*$, as well as substituting $\Lambda$, $\Sigma$ and $\Sigma^*$ for $\Lambda$, $\Sigma$ and $\Sigma^*$.

Similarly, the ratio of the $B$-$B^*$ and $\pi$-$\rho$ splittings yields $m_n/m_b \simeq 0.07$, which when plugged into (7) gives $R \simeq 0.91$. Moreover, for the range $0.05 \leq m_n/m_b \leq 0.08$, the value of $R$ stays between 0.905 and 0.915. The experimental value can again be obtained from (11), this time by substituting $B$ and $B^*$ for $K$ and $K^*$, and replacing $\Lambda$, $\Sigma$ and $\Sigma^*$ by $\Lambda$, $\Sigma_b$ and $\Sigma^*_b$. But since the experimental status of some of the bottom baryons is uncertain (see below), I instead use this formula in conjunction with the above result for $R$ to obtain a rather precise prediction for the $bnn$ center of gravity — namely, $M_{bnn} = 5774 \pm 4$ MeV/c$^2$. To what extent can this result be checked? In other words, what is our current state of knowledge about the $bnn$ baryons? As of a year ago, the $\Lambda_b$ had been seen but its mass was not known very accurately, the world average being $5641 \pm 50$ MeV/c$^2$. Since then, there have been three additional measurements. The most accurate is that of the CDF collaboration [25]. They found a $\Lambda_b$ mass of $5621 \pm 5$ MeV/c$^2$. The other two results are from the ALEPH [26] and DELPHI [27] collaborations. The ALEPH number is $5614 \pm 21$ MeV/c$^2$, while DELPHI found $5668 \pm 18$ MeV/c$^2$. Taking the weighted average of these new results and the previous world average gives a new world average for the $\Lambda_b$ mass of $5624 \pm 5$ MeV/c$^2$, which we will use below. Recently, the DELPHI collaboration has also reported the first evidence for the $\Sigma_b$ and $\Sigma^*_b$ baryons [28]. Their preliminary mass values are $M_{\Sigma_b} - M_{\Lambda_b} = 173 \pm 14$ MeV/c$^2$ and $M_{\Sigma^*_b} - M_{\Lambda_b} = 229 \pm 9$ MeV/c$^2$. Noting that

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4 I thank C. Kreuter for informing me of the new (larger) error for $M_{\Sigma_b} - M_{\Lambda_b}$.
\[ M_{bnn} = M_{\Lambda_b} + (M_{\Sigma_b^*} - M_{\Lambda_b})/2 + (M_{\Sigma_b} - M_{\Lambda_b})/4, \]

we can plug in the above measurements and obtain the experimental value \( M_{bnn}^{\text{exp}} = 5782 \pm 8 \text{ MeV}/c^2 \). Though consistent with the above prediction, it would clearly be desirable to have a more precise measurement of all the \( bnn \) baryon masses in order to make a better comparison. At this point it is also worth noting one strange feature of the measured values of the \( \Sigma_b \) and \( \Sigma_b^* \) masses — namely, their difference.\(^5\) We obtain from the DELPHI numbers that \( M_{\Sigma_b^*} - M_{\Sigma_b} = 56 \pm 17 \text{ MeV}/c^2 \). An estimate of this difference from the spin-splitting pattern in the meson sector would be around 20 MeV/\( c^2 \). (By contrast, recall the amazing consistency of the meson and baryon hyperfine splittings in the \( snn \) and \( cnn \) cases.) Various other simple estimates yield a similar number \([29]\). Given this discrepancy, I would not be surprised to see these preliminary mass values change somewhat in the near future and come into better agreement with both the mesonic spin-splittings and the predictions of the log model.

It is worth stressing again that there is every reason to believe the above predictions to be extremely trustworthy. As noted earlier, the nonrelativistic log model should provide an accurate description of the ground state hadrons for some values of the parameters, even for light quarks. However, all of the parameter dependence has cancelled out in \( R(x) \) except for a relatively weak dependence on the quark mass ratio \( x \), and we have incorporated our ignorance about these ratios into the overall uncertainty. Finally, \( 1/D \) corrections also seem to be completely under control — the additional uncertainty in our results due to the neglect of such corrections is swamped by the quark mass uncertainties just mentioned. The appeal of the above approach is in its simplicity, accuracy and parameter-independence, and not in any rigorous connection to more fundamental ideas.

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