A Note on Deformed Ladder Operators for Noncommutative Morse Oscillator

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Abstract

Morse oscillator is one of the known solvable potentials which attracts many applications in quantum mechanics especially in quantum chemistry. One of the interesting results of this study is the generation of ladder operators for Morse potential. The operators are a representation of the shifting energy levels of the states exhibited by the wavefunction. From this result, we manipulate and deform the operators in such a way that it gives a noncommutative property to promote noncommutative quantum mechanics (NCQM). The resultant NC feature can be shown in the spatial coordinates and finally the Hamiltonian. In this study, we consider two-dimensional Morse potential where the ladder operators are in the form of the corresponding 2D Morse.

Keywords: Non-commutative Quantum Mechanics, Morse oscillator, Operator Method, Ladder Operators

1 Introduction

To date, noncommutative quantum mechanics has been widely studied \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]\) and for one of many reasons, its importance is seen to be related with quantum potentials. The apparent element to differentiate noncommutative quantum mechanics from the standard quantum mechanics is the association of additional parameters to the commutation relations for position and momentum operators. The ordinary canonical commutation relations for these operators

\begin{align}
[x_i, x_j] &= 0, \\
[p_i, p_j] &= 0, \\
[x_i, p_j] &= i\hbar\delta_{ij}
\end{align}

are deformed to become

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\[ [\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\zeta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}. \]

where \( \theta_{ij} \) and \( \zeta_{ij} \) are anti-symmetric real matrices. Here, the NC operators are defined in the same Hilbert space as the commutative ones [14]. The idea of translating the mathematical means of classical mechanics to quantum mechanics is realisable through the transformation of the commutative algebra of classical observables to that of quantum mechanical observables which are non-commutative. The obvious difference of these two is that in order to show the non-commutative property of the algebra, it has to be at least in two dimensions. (1.2) shows the non-commutative algebra of both spatial and momentum operators with \( \theta_{ij} \) and \( \zeta_{ij} \) being the non-zero non-commutative parameters while \( \delta_{ij} \) is the standard Kronecker delta for \( i, j = 1, 2 \).

In this paper, we shall see this exact representation for our case study associated to two dimensional Morse oscillator. That being said, the purpose of this paper is to highlight the idea of non-commutativity of a diatomic quantum potential through its ladder operators. In this case, we study Morse potential and present the obtained ladder operators from previous work by [15, 18]. From the algebra of the these operators, we deform them to show the NC quantum mechanics property which later modifies the Hamiltonian.

This paper is organised sectionally where in Section 2, the ordinary two dimensional ladder operators are introduced. In Section 3, the deformed operators with respect to NC features are considered, which later lead to the alteration of the energy spectrum of the potential. The Casimir operator associated to the algebra is also reviewed. Concluding remarks are featured in the final section.

## 2 The two-dimensional Morse oscillator

We consider a two-dimensional Morse oscillator model obtained by superposition of two one-dimensional isotropic Morse oscillators. The Hamiltonian of the Morse potential is in the form [18]

\[ H_{M_i} = \frac{p_i^2}{2\mu} + V_0(e^{-2\alpha x_i} - 2e^{-\alpha x_i}), \quad \text{for} \quad i = 1, 2 \]

where \( V_0 \) represents the potential depth, \( \alpha \) a constant related to the range of the potential, such that \( \alpha \) is inversely proportional to the width of the potential, \( x \) the (relative) position, \( \mu \) the reduced mass, and \( p \) the momentum operator. The Hamiltonian of two dimensional Morse potential is simply [18]

\[ H_M = H_{M_1} + H_{M_2}. \]

The operator algebra for the two-dimensional Morse oscillator model is [19]

\[ K_{-i} = qI - \frac{y_i}{2} + \frac{ip_{y_i}}{\hbar \alpha}, \]

\[ K_{+i} = qI - \frac{y_i}{2} - \frac{ip_{y_i}}{\hbar \alpha}. \]
where \( q \) is a real quantity related to the number operator and \( I \) is a unit operator with the corresponding generators

\[
[K_{-i}, K_{+i}] = 2K_{0i}, \quad [K_{0i}, K_{\mp i}] = \mp K_{\mp i},
\]

such that \( K_{0i} = qI \). Here in (2.3), we can see that \( K_{-i} \) and \( K_{+i} \) are in terms of \( K_{0i} \). \( K_{0i} \) is also related to the number operator in the following form

\[
K_{0i} = \left(n_i - \frac{\nu - 1}{2}\right),
\]

for \( n_i \in \mathbb{N}, \nu = \sqrt{\frac{8n_iV_0}{\alpha^2\hbar^2}} \) and \( s_i = \sqrt{\frac{-2mE_0}{\alpha^2\hbar^2}} \) related by

\[
2s_i + 1 - \nu = -2n_i \quad \text{for } i = 1, 2.
\]

\( K_{-i}, K_{+i} \) and \( K_{0i} \) are ladder and number operators respectively which are written in terms of coordinate \( y_i \) and momentum \( p_{yi} \) operators. We define

\[
y_i \equiv ve^{-\alpha x_i}, \quad p_{yi} \equiv -i\hbar \frac{\partial}{\partial x_i}
\]

Now, we impose the non-commutativity feature to the commutator of the spatial coordinates by using the Baker-Campbell-Hausdorff formula to show

\[
[y_i, y_j] = ve^{-\alpha(x_i+x_j)}(e^{\theta/2} - e^{-\theta/2}), \quad [y_i, y_i] = [y_j, y_j] = 0,
\]

for \( i, j = 1, 2 \), in the case of non-commutative two dimensional quantum mechanics where \( \theta \) is a nonzero non-commutative parameter. From equation (2.3), coordinate \( y_i \) and momentum \( p_{yi} \) operators thus can be expressed in terms of ladder operators

\[
y_i = 2K_{0i} - (K_{+i} + K_{-i}), \quad p_{yi} = \frac{i\hbar \alpha}{2}(K_{+i} - K_{-i}),
\]

where the commutator of the operators is simply

\[
[y_i, p_{yi}] = -i\hbar \alpha (4K_{0i} - y_i),
\]

given that \( [y_i, y_j] = [p_{yi}, p_{yj}] = 0 \). The Hamiltonian of equation (2.1) can then be rewritten as

\[
H_{Mi} = \frac{\hbar^2 \alpha^2}{2\mu} (K_{+i}K_{-i} - K_{0i}^2).
\]

With (2.10), we can then compute the commutator of \( H_{Mi} \) with \( K_{-i} \) and \( K_{+i} \),

\[
[H_{Mi}, K_{-i}] = \frac{\hbar^2 \alpha^2}{2\mu} (-3K_{0i}K_{-i} - K_{-i}K_{0i}),
\]

\[
[H_{Mi}, K_{+i}] = \frac{\hbar^2 \alpha^2}{2\mu} (3K_{+i}K_{0i} + K_{0i}K_{+i}).
\]

From the relations above, we can tell that the commutators of Hamiltonian and lowering operator and creating operator each shows annihilation and creation in the energy eigenstate respectively.
3 Deformed Morse Operators

For one-dimensional ordinary ladder operators, 

\[ K_- = (qI + n) - \frac{y}{2} + \frac{ip_y}{\hbar \alpha} \]
\[ K_+ = (qI + n) - \frac{y}{2} - \frac{ip_y}{\hbar \alpha} \]

the generators are calculated to be

\[ K_- \phi_n = C_n \phi_{n-1}, \]
\[ K_+ \phi_n = C_{n+1} \phi_{n+1}, \]

with \( K_- \) analogous to annihilation operator, \( K_+ \) analogous to creation operator and \( n \) as the principle quantum number. When acting on the wavefunction \( \phi_n \), \( K_- \) and \( K_+ \) each lowers and raises the state by 1 respectively, where \( C_n = \sqrt{n(n + 2q - 1)} \). For two-dimensional deformed ladder operators which are indicated explicitly as follows

\[ g_{11} K_- \phi_{n,m} + g_{12} K_- \phi_{n-1,m}, \]
\[ g_{21} K_- \phi_{n,m} + g_{22} K_- \phi_{n-1,m}, \]
\[ g_{11} K_+ \phi_{n,m} + g_{12} K_+ \phi_{n-1,m}, \]
\[ g_{21} K_+ \phi_{n,m} + g_{22} K_+ \phi_{n-1,m}, \]

are in terms of the ordinary ladder operators having being operated with some matrix \( g \) with elements \( g \in GL(2, \mathbb{C}) \) such that \( g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \). This \( g \) with specific form will give an explicit example of NCQM in problems as discussed in [20]. For

\[ g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

the case of NCQM becomes the case of ordinary QM such that

\[ K_-^g \phi_{n,m} = K_- \phi_{n,m}, \]
\[ K_+^g \phi_{n,m} = K_+ \phi_{n,m}, \]

for two-dimensional problems.

The deformed form of these operators is indicated by \( g \) to differentiate it from the ordinary form. The corresponding two-dimensional deformed generators are thus

\[ g_{11} K_- \phi_{n,m} + g_{12} K_- \phi_{n-1,m}, \]
\[ g_{21} K_- \phi_{n,m} + g_{22} K_- \phi_{n-1,m}, \]
\[ g_{11} K_+ \phi_{n,m} + g_{12} K_+ \phi_{n-1,m}, \]
\[ g_{21} K_+ \phi_{n,m} + g_{22} K_+ \phi_{n-1,m}, \]
for \( C_n = \sqrt{n(n+2q-1)} \) and \( C_m = \sqrt{m(m+2q-1)} \) which correspond to the energy spectra of the potential. Here, (3.3) shows linear combination of ordinary ladder operators \( K_{\pm i} \). It is also worth to mention that the deformed number operator generates the ordinary number operator such that

\[
(3.8) \quad K_{01}^g \phi_n^g = K_0 \phi_n^g.
\]

From the operators constructed and (2.4), we can compute the commutation relations of the deformed ladder operators which are in terms of the ordinary form. The only survived commutation relations are

\[
[K_{\mp i}^g, K_{\pm j}^g] = \pm 2[g_{11} K_{0i} + g_{21} K_{0j}],
\]

\[
[K_{\mp i}^g, K_{\pm j}^g] = \pm 2[g_{11} g_{12} K_{0i} + g_{21} g_{22} K_{0j}],
\]

\[
[K_{01}^g, K_{\mp i}^g] = \mp g_{11} K_{\pm 1}, \quad [K_{01}^g, K_{\mp 2}^g] = \mp g_{11} K_{\mp 2}
\]

\[
[K_{02}^g, K_{\pm i}^g] = \mp g_{21} K_{\pm 2}, \quad [K_{02}^g, K_{\mp 2}^g] = \mp g_{21} g_{22} K_{\pm 2}
\]

The deformed Casimir operator is thus

\[
\tilde{C}^2 = (K_{01}^g)^2 + K_{01}^g - K_{\pm 1}^g K_{\mp 1}^g = K_{0i}^g \mp K_0^g - K_{\pm 1}^g K_{\pm 1}^g,
\]

for \( i = 1, 2 \). Explicitly, one can have

\[
(3.9) \quad \tilde{C}^2 = K_{01}^g \mp K_{0i}^g - K_{\pm 1}^g K_{\mp 1}^g,
\]

\[
= K_{01}^g \mp K_{01}^g - g_{11} K_{\pm 1}^g K_{\pm 1}^g - g_{11} g_{12} (K_{\pm 1} + K_{\mp 1}^g) - g_{11} g_{12} K_{\pm 2}^g K_{\mp 1}^g,
\]

for \( i = 1 \) and

\[
(3.10) \quad \tilde{C}^2 = K_{02}^g \mp K_{02}^g - K_{\pm 2}^g K_{\mp 2}^g,
\]

\[
= K_{02}^g \mp K_{02}^g - g_{21} K_{\pm 1}^g K_{\pm 1}^g - g_{21} g_{22} (K_{\pm 1}^g + K_{\pm 2}^g) - g_{21} g_{22} K_{\pm 2}^g K_{\mp 1}^g,
\]

for \( i = 2 \). The purpose of this relation is to show the original Casimir and additional terms associated to the deformed operators.

Interestingly, these results can be used to obtain deformed coordinate \( q_{ji}^g \) and momentum \( p_{ji}^g \) operators where the commutator is \([q_{ji}^g, p_{ij}^g] = i\hbar \left( g_{12} \left[ K_{j+} + K_{j-} \right] - 2 \left[ g_{11} g_{12} K_{0i} + g_{21} g_{22} K_{0j} \right] \right)\) for \( i, j = 1, 2 \). In addition to that, we now can write the deformed Hamiltonian

\[
(3.11) \quad H_M^g = H_{M1}^g + H_{M2}^g,
\]

\[
= \frac{\hbar^2 \alpha^2}{2\mu} \left[ K_{\pm 1}^g K_{\mp 1}^g - (K_{31}^g)^2 \right] + \frac{\hbar^2 \alpha^2}{2\mu} \left[ K_{\pm 2}^g K_{\mp 2}^g - (K_{32}^g)^2 \right],
\]

\[
= \frac{\hbar^2 \alpha^2}{2\mu} \left[ (g_{11} K_{1+} + g_{12} K_{1-}) (g_{11} K_{1-} + g_{12} K_{1+}) - K_{01}^2 \right]
\]

\[
+ \frac{\hbar^2 \alpha^2}{2\mu} \left[ (g_{21} K_{1+} + g_{22} K_{1-}) (g_{21} K_{1-} + g_{22} K_{1+}) - K_{02}^2 \right],
\]

\[
= \frac{\hbar^2 \alpha^2}{2\mu} \left[ (g_{11}^2 + g_{21}^2) K_{1+} K_{1-} + (g_{11} g_{12} + g_{12} g_{22}) (K_{1+} K_{1-} + K_{1+} K_{1-}) + (g_{12}^2 + g_{22}^2) K_{1+} K_{1-}
\]

\[
- K_{01}^2 - K_{02}^2 \right].
\]
where non-commutative quantum mechanics in two dimensions is considered. The deformed Hamiltonian can be transformed to the ordinary Hamiltonian, taking \(\text{(3.5)}\) so it becomes

\[
H_M = \frac{\hbar^2 \alpha^2}{2\mu} [K_{+1}K_{-1} - K_{01}^2] + \frac{\hbar^2 \alpha^2}{2\mu} [K_{+2}K_{-2} - K_{02}^2].
\]

Here, we show that by operating a specific matrix \(g\), that is, a unit operator, the deformed Hamiltonian is converted back to its ordinary form as in \(\text{(2.2)}\).

4 Conclusion Remarks

In this work, we present the algebra of the ladder operators of Morse oscillator and introduce the two-dimensional form of the operators corresponding to the 2D Morse. The energy spectrum of the function can be shown from the action of the annihilation and creation operators on the associated wavefunction. The Hamiltonian can then be rewritten in a way that it is a summation of the individual Hamiltonian of each dimension \([18]\). The commutation relations of the ladder operators are seen to be comparable to the SU(1,1) algebra, which later develops the Casimir operator associated to the algebra from the commutation relations of the operators. We then propose a method of obtaining the noncommutativity of a quantum potential by quantisation of its ladder operators. In other words, the ordinary ladder operators are being deformed for some parameterised matrix \([20]\) which in results show the noncommutativity property. Many interesting results could be obtained from this work, which particularly includes the commutation relation between the spatial coordinates and between the spatial and momentum operators. Here, the NC property is not shown for the commutation relation of the momentum operators and thus is let such that they, the momentum operators commute. From the computation we could arrive to our final result, which is deformed Hamiltonian in two dimensions which directly link to the originally ordinary ladder operators of the potential. We also prove that for a specific 2 by 2 matrix, we could have the deformed Hamiltonian reverted to its ordinary form, particularly in 2D.

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7