Unveiling transient to steady effects in reduced order models of thermomechanical plates via global dynamics

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Unveiling transient to steady effects in reduced order models of thermomechanical plates via global dynamics

Valeria Settimi, Giuseppe Rega and Eduardo Saetta

Abstract A reduced model of third-order shear deformable plate with cubic temperature is used to investigate the system nonlinear dynamic response in a full thermomechanical coupling framework. Numerical investigations of local and global dynamics allow to highlight distinct response features as occurring under different (constant or dome-shaped) prescribed spatial temperatures on the plate surfaces. In both cases, the important role played by global analysis for unveiling meaningful transient to steady effects in the system dynamics clearly comes out.

1 Introduction

Reduced order modeling and nonlinear dynamics of composite plates under different excitation conditions in a thermomechanical environment have been the subject of recent papers aimed at highlighting the role of multiphysics coupling and the main local and global features of the nonlinear response [1–5].

In the framework of a unified formulation of the thermomechanical problem based on the Tonti approach to physical theories [6], two different 2D models of laminated plates with von Kármán nonlinearities have been proposed, by either neglecting [1] or considering [2] shear deformability and by consistently assuming a corresponding linear or cubic variation of the unknown thermal field along the plate

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thickness. For symmetric cross-ply laminates, proper and controllable dimension reduction accomplished via Galerkin approximations has allowed in both cases to attain a minimal model (with one mechanical and two thermal equations/unknowns) still exhibiting the fundamental features of geometrical nonlinearity and thermomechanical coupling embedded in the underlying, yet more complicated, continuum models.

The simpler (shear indeformable with linear temperature) reduced model –labelled CTC for being based on the Classical theory with Thermomechanical Coupling –has been employed for extended investigations of the plate nonlinear dynamic response under both passive [5] and active [3, 4] thermal conditions. The former refer to a situation in which thermal phenomena are dragged into the system overall response by the distributed transverse mechanical excitation, as a result of the full coupling; the latter accounts also for the presence of a thermal source (of variable nature) entailing direct activation of the plate temperature field, in addition to mechanical excitation. Compressive in-plane forces distributed along the plate edges have been considered, too (see Fig. 1(a)).

Local and global nonlinear dynamics have been investigated, highlighting the transition to mechanically- or thermally-induced buckled responses and also focusing on the different role played by coupling effects in different excitation conditions, with the ensuing possibility to consider simplified or partially coupled models [3]. Global dynamics has shown to be of major importance mostly in active thermal conditions [3, 4], where it turns out to be decisive for reliably catching the non-trivial influence of the slow transient thermal dynamics on the steady outcome of the faster mechanical response.

In contrast, no parallel nonlinear dynamic analyses have been conducted yet with the richer (shear deformable with cubic temperature) reduced model, labelled TTC for being based on the Third-order theory with Thermomechanical Coupling. Yet, its major richness, inherently embedded in the description of the thermal field [2], allows to consider a remarkably larger set of thermal boundary conditions with respect to the CTC model.

The present paper is a first step in this direction, and aims at further highlighting how proper consideration of system global dynamics turns out to be essential to reliably unveil the transient to steady effects due to thermomechanical coupling. Parametric investigation of the response under two different conditions of prescribed temperature on the external surfaces (of interest in a variety of multiphysics structural applications) is accomplished by means of local bifurcation diagrams, phase portraits and planar cross sections of the four-dimensional basins of attraction. This allows us to highlight some ensuing meaningful qualitative changes and to get an overall confirmation of the role played by global analysis for attaining a comprehensive understanding of system dynamics.


2 Thermomechanically coupled models

The mathematical model describing the motion of the thermomechanical plate under analysis is derived in the framework of a unified 2D formulation presented in [2], to refer to for all details, in which von Kármán nonlinearities, third-order shear deformability and a cubic temperature distribution along the thickness are considered (TTC model). Moving from seven (five mechanical and two thermal) generalized 2D variables, and under the assumption of symmetric cross-ply laminates, kinematic condensation of the in-plane displacements and shear angles is performed at the continuum and discretized level, respectively; then, a minimal dimension reduction via a Galerkin procedure with dome-shape functions assumed for the remaining (two) thermal and (one) mechanical variables is developed. Thanks to the richness and flexibility of the underlying continuum formulation, the model allows to account for a variety of thermomechanical assumptions, excitations and boundary conditions. Thus, it represents a substantial improvement of the CTC thermomechanical model with shear deformability and linear temperature variation along the thickness, previously investigated by the authors in the nonlinear dynamics regime [3–5], in which the sole thermal boundary condition of free heat exchange between plate and environment can be taken into account. Under this condition, anyway, the governing thermomechanical equations of the TTC and CTC models are formally equal, of course with different expressions of the coefficients. Moreover, results not reported here have shown that for the thin (i.e. ratio $1/100$ between length and thickness) orthotropic single-layered epoxy/carbon fibre composite plate with simply supported, movable and isothermal edges [4], herein considered (Fig. 1(a)), the outcomes furnished by the two models are practically coincident in terms of both local and global dynamics analysis, as somewhat expected at least from the mechanical viewpoint. For this reason, the following sections are devoted to the description of the TTC model response under two different thermal boundary conditions while, when needing to bring up the results relevant to the free heat exchange case, in a comparison perspective, reference will be made to the corresponding outcomes presented for the CTC model in [4].

![Composite plate subjected to mechanical loads (a); Spatially constant (b) and dome-shaped (c) temperature distributions on the external surfaces.](image-url)
2.1 TTC model with constant prescribed temperature on the external surfaces

The first thermal boundary condition considered is associated with the condition of prescribed temperature on the external, upper and lower, surfaces of the plate, with the temperature distribution which is assumed to be constant on each of the surfaces (TTC\(_C\) model, Fig. 1(b)). Referring to Eqs. 72 of [2], nondimensionalization with respect to time and plate thickness allows to obtain the following set of governing equations

\[
\ddot{W} + a_{12} \dot{W} + a_{13} W + a_{14} W^3 + a_{15} T_{R1} + a_{16} W \cdot T_{R0} + a_{17} \cos(t) = 0, \quad (1)
\]
\[
\dot{T}_{R0} + a_{22} T_{R0} + a_{23} \alpha_1 (T_{up} + T_{down}) + a_{24} W \cdot W + a_{25} e_0(t) = 0, \quad (2)
\]
\[
\dot{T}_{R1} + a_{32} T_{R1} + a_{33} \dot{W} + a_{34} e_1(t) + a_{35} \alpha_1 (T_{up} - T_{down}) = 0, \quad (3)
\]

in terms of the unknown 0D configuration nondimensional reduced variables \(W\) (deflection of the center of the plate), \(T_{R0}\) (membrane temperature), \(T_{R1}\) (bending temperature). Note that, for the sake of generality, besides the non-vanishing harmonic transversal mechanical excitation in Eq. (1), body thermal membrane \((e_0)\) and bending \((e_1)\) excitations also appear in Eqs. (2)-(3); however, they are given here zero values for the interest being in evaluating the effects of the sole boundary conditions. The parameters representing the thermal boundary conditions are \(T_{up}\) and \(T_{down}\), corresponding to the dimensional (in Kelvin) prescribed constant variations of the temperature on the upper and lower external surface, respectively, with respect to the reference value \(T_{ref}\). The expressions of the \(a_{ij}\) coefficients are not reported here for the sake of brevity. However, the comparison with those obtained in Eq. 2.1 of [4], for the case of plate with free heat exchange, points out the general increase of the coefficients values in case of the TTC\(_C\) model, due to the different physical process activated by the two types of thermal boundary conditions. The choice of a different boundary condition causes the replacement of the \(T_{\infty}\) term (expressing the difference between plate and environment temperatures) in the membrane equation (2) by a substantially equivalent thermal term and, more important, the addition, into the bending temperature equation (3), of a new term related to the difference between the upper and lower temperatures on the plate surfaces. As a consequence, when the two faces have different temperatures, both the membrane and the bending thermal variables are activated. This differs from the free heat exchange condition which, considering a unique temperature for the external environment, determines the triggering of the sole membrane thermal variable.

The different effect of the thermal boundary condition on the system equations of motion reflects also in the dynamical response of the system, as highlighted by the bifurcation diagram of Fig. 2 as a function of the temperature on the lower surface \(T_{down}\). In fact, the symmetry characterizing the dynamics of the buckled responses in the system with free heat exchange (see Fig. 3a of [4]) is here broken, due to the contemporary activation of the (along the thickness) symmetric membrane and anti-symmetric bending thermal variables. In particular, the results of Fig. 2 are
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Fig. 2 Bifurcation diagram of the transversal displacement $W$ of the TTC$_C$ model as a function of the temperature on the lower surface $T_{down}$, for $p = 2.51$ and $T_{up} = 100$. Circle: saddle-node bifurcation; diamond: period-doubling bifurcation.

obtained for a pre-buckling in-plane mechanical precompression value $p = 2.51$ and fixed temperature on the upper surface $T_{up} = 100$ able to bring up the two high-amplitude buckled responses $P_{1III}$ and $P_{1IV}$. Starting from this scenario, the varying temperature on the lower plate surface drastically modify the mechanical system response. For $T_{down} > 0$ the buckled scenario is strengthened by the arise of the two low-amplitude $P_{1I}$ and $P_{1II}$ solutions which, as the other ones, differ from each other by amplitude as well as by region of occurrence. When the lower surface is cooled (i.e. for $T_{down} < 0$), conversely, buckling is reduced up to the disappearance of multistability. However, for $-633 < T_{down} < -356$, the TTC$_C$ system displays a peculiar behavior, not present in the model with free heat exchange, marked out by the presence of chaotic responses representing the sole stable solutions for the system. Their characterization in terms of phase portraits is reported in Fig. 3, with the first Lyapunov exponent being $+0.05$. Finally, for lower $T_{down}$ values, the cross-well pre-buckling $P_1$ solution regains stability.

To complete the description of the dynamical response for the TTC$_C$ model, the global dynamics is investigated by realizing planar cross sections, with fixed ther-

Fig. 3 For $p = 2.51$, $T_{up} = 100$ and $T_{down} = -400$, phase portraits of the chaotic solution for the TTC$_C$ model in the mechanical (a), membrane temperature (b) and bending temperature (c) planes.
mal initial conditions (i.c.), of the four-dimensional basins of attractions. As a sample case, the general situation of different prescribed temperatures on the external surfaces (i.e. $T_{up} = 100$, $T_{down} = 400$) is reported in Fig. 4. When considering trivial thermal initial conditions, representing the most natural configuration from a physical viewpoint, the outcomes furnished by the global dynamics analysis, reported in Fig. 4(a), are evidently different from those obtained by the bifurcation diagram of Fig. 2. In fact, despite the contemporary presence of the five main 1-period solutions detected by the bifurcation diagram, only two basins are identified by the global analysis, corresponding to the P1 and P1$^{IV}$ solutions. This apparent discrepancy of results between local and global dynamics analyses has to be attributed to the effect of the thermal transient dynamics, as already pointed out in [4] for the free heat exchange case. As confirmation, basins of Fig. 4(a) are compared with those obtained by the relevant uncoupled system and reported in Fig. 4(c). The latter model is described by the sole mechanical equation (1) in which the thermal boundary condition is taken into account by substituting the thermal variables $T_{R0}$ and $T_{R1}$ with the relevant steady mean values achieved at the end of their temporal evolution, thus neglecting the thermal transient. The outcomes display a strongly different scenario, with the evident presence of the buckled, positive and negative, wells and the identification of all five 1-period basins. This is coherent with what detected with the local dynamics analysis, with also the addition of a basin related to a 2-period response, in pink, not reported in the bifurcation diagram for the sake of readability of that figure. The crucial role played by the thermal transient emphasizes the importance of the choice of the thermal i.c. when analyzing the TTC$_C$ model, since their selection is fundamental in determining the lasting of the thermal evolution. In fact, if the thermal transient is neglected also in the coupled model, i.e. the thermal i.c. are assumed equal to the relevant steady state values as in Fig. 4(b), the obtained response turns out to be coincident with that of the uncoupled system (Fig. 4(c)); this underlines how the latter is able to describe a specific dynamical scenario of the system achievable only under selected, and physically barely realizable, thermal

![Fig. 4](image_url)

**Fig. 4** For $p = 2.51$, $T_{up} = 100$ and $T_{down} = 400$, cross sections of the basins of attraction of the TTC$_C$ model in the $(W, W)$ plane and thermal initial conditions $T_{R0} = 0.0$, $T_{R1} = 0.0$ (a), $T_{R0} = 2.30275$, $T_{R1} = 2.77037$ (b), basins of attraction of the purely mechanical model (c). Red basin: P1$^{II}$ solution; Blue basin: P1$^{IV}$ solution; Cyan basin: P1$^{II}$ solution; Gray basin: P1 solution; Orange basin: P1$^{I}$ solution; Pink basin: P2 solution.
conditions. As a final remark, the non-symmetric behavior of the TTC\(_C\) model highlighted by the bifurcation diagram is confirmed also by the basins analysis, which organize inside the two wells in a clearly different way.

2.2 TTC model with dome-shape prescribed temperature on the external surfaces

In order to ensure consistency between modeling of internal and boundary temperatures, the thermal condition of prescribed temperature on the external surfaces of the plate is here considered by referring to a dome-shape profile in both the upper and lower faces (TTC\(_{DS}\) model, Fig. 1(c)). The relevant governing equations read

\[
\ddot{W} + a_{12} \dot{W} + a_{13} W + a_{14} W^3 + a_{15} T_{R1} + a_{16} W \cdot T_{R0} + a_{17} \cos(t) + a_{18} (T_{up} + T_{down}) W + a_{19} (T_{up} - T_{down}) = 0, \tag{4}
\]

\[
\dot{T}_{R0} + a_{22} T_{R0} + a_{23} \alpha_1 (T_{up} + T_{down}) + a_{24} W \cdot W + a_{25} \epsilon_0(t) = 0, \tag{5}
\]

\[
\dot{T}_{R1} + a_{32} T_{R1} + a_{33} W + a_{34} \epsilon_1(t) + a_{35} \alpha_1 (T_{up} - T_{down}) = 0. \tag{6}
\]

Comparing Eq. (4) with Eq. (1), the presence of two new terms related to the \(T_{up}\) and \(T_{down}\) parameters into the mechanical equation is pointed out, modifying the linear mechanical stiffness and adding a constant external excitation. However, looking at the numerical values of the relevant coefficients it can be observed that \(a_{18}\) and \(a_{19}\) parameters are two orders of magnitude lower than the others, so that their effect on the transversal displacement can be grasped only if temperatures on the surfaces have great sum or difference. As a general observation, it can be noted that the numerical coefficients of the TTC\(_{DS}\) model are equal to those of the TTC\(_C\) model, with exception of those related to \(T_{up}\) and \(T_{down}\) which are higher in the latter case.

![Fig. 5 Bifurcation diagrams of the transversal displacement \(W\) as a function of the temperature on the lower surface \(T_{down}\), for \(p = 2.51\) and \(T_{up} = 100\); comparison between TTC\(_C\) (black) and TTC\(_{DS}\) (red) models. Circle: saddle-node bifurcation; diamond: period-doubling bifurcation.](image-url)
For $p = 2.51$, $T_{up} = 100$ and $T_{down} = -400$, phase portraits of the P1$^{\text{III}}$ solution for the TTC$_{DS}$ model in the mechanical (a), membrane temperature (b) and bending temperature (c) planes.

The local dynamics analysis is here performed by realizing again bifurcation diagrams as a function of the $T_{down}$ parameter, with $p = 2.51$ and $T_{up} = 100$, and comparing in Fig. 5 the results (in red) with those already presented in Fig. 2 (here reported in black). Looking at positive values of $T_{down}$, the main differences between the responses of the two models can be detected inside the negative buckled well, corresponding to the plate bending towards the upper surface, which for $T_{down} > 100$ (representing most of the range here considered) is the colder side of the plate. The differences pertain to amplitude as well as existence region of the main periodic solutions. Conversely, responses around the positive buckled configuration are almost coincident in the two models. Diverse behavior between TTC$_{C}$ and TTC$_{DS}$ model can be observed also for $T_{down} < 0$, where the chaotic region is substituted, in the TTC$_{DS}$ model, by the low-amplitude buckled P1$^{\text{III}}$ response represented in terms of phase portraits in Fig. 6, which remains stable in the whole negative range analyzed.

Moving to the analysis of the basins of attraction of the TTC$_{DS}$ model, the results presented in Fig. 7 allow to confirm the importance of properly describing the thermal transient via the coupled model in order to determine the steady state response.

For $p = 2.51$, $T_{up} = 100$ and $T_{down} = 400$, cross sections of the basins of attraction of the TTC$_{DS}$ model in the ($W, W$) plane and thermal initial conditions $T_{R_0} = 0.0$, $T_{R_1} = 0.0$ (a), $T_{R_0} = 1.41818$, $T_{R_1} = 1.70726$ (b), basins of attraction of the purely mechanical model (c). Red basin: P1$^{\text{II}}$ solution; Blue basin: P1$^{\text{IV}}$ solution; Cyan basin: P1$^{\text{III}}$ solution; Gray basin: P1 solution; Pink basin: P2 solution.
of the system. This proves to be a general characteristic of the thermomechanical model under analysis, irrespective of the thermal boundary condition considered. In fact, the behavior of the coupled TTC\textsubscript{DS} model with trivial thermal i.c. reported in Fig. 7(a) is clearly different from the outcomes obtained by the relevant uncoupled mechanical system with prescribed thermal steady values (Fig. 7(c)), which however can be perfectly reproduced setting the i.c. to the relevant steady values (Fig. 7(b)). Moreover, comparing Fig. 4(a) and Fig. 7(a), it can be observed that the shape chosen for modeling the prescribed temperature on the external surfaces is able to influence the steady dynamics of the system, by modifying the role of the P1 (gray) and P1\textsuperscript{IV} (blue) basins, the former dominating the response of the TTC\textsubscript{C} model, the latter becoming the main basin for the TTC\textsubscript{DS} system, as shown also in Fig. 8(a). Here, apart from highlighting the different mechanical response achieved by the two models, the effect of the boundary conditions on the thermal variable evolution can be observed. As a general comment, the time histories of Fig. 8 clearly stress the length of the thermal transients with respect to the mechanical one, even if they are shorter than those relevant to the case of plate with free heat exchange (see, e.g., Fig. 7(ii) of [4]). Furthermore, the steady values reached by the thermal variables of the TTC\textsubscript{DS} model are lower than those of the TTC\textsubscript{C} model, a behavior which is confirmed by the bifurcation diagrams with respect to the thermal variables, not reported here, thus showing to be robust with respect to possible variations of the thermal boundary parameters. Finally, also when neglecting the thermal transient, e.g., in Fig. 7(b) and 4(b), TTC\textsubscript{DS} and TTC\textsubscript{C} models display differences in the basins organization, mostly localized in the negative well, as already deduced by the bifurcation diagram of Fig. 5.

3 Conclusions

A reduced model of third-order shear deformable laminated plate with spatially assumed cubic variation of the unknown thermal field along the thickness has been used for the first time to investigate the nonlinear dynamic response of an orthotropic single-layered epoxy/carbon fibre composite plate with simply supported, movable and isothermal edges. The analysis has been conducted in a full (i.e., two-way)
thermomechanical coupling framework. Besides mechanical (transverse harmonically varying and in-plane constant) excitations and body thermal excitations, the third-order model allows to consider a remarkable variety of thermal boundary conditions to be possibly prescribed on the plate upper and lower surfaces. A condition of prescribed temperature entailing direct activation of the plate temperature field has been considered in the numerical investigation; yet, two relevant spatial shapes, either constant or dome-shaped, have been considered, the latter being more consistent with the modeling assumption about the spatial distribution of the unknown thermal field made in the Galerkin modal reduction. Local and global nonlinear dynamics have been investigated, highlighting the transition to thermally-induced buckled responses, however with meaningfully distinct response features occurring when considering either one of the two different prescribed shapes. Nonetheless, the remarkable influence of the slow transient thermal dynamics on the steady outcome of the faster mechanical response clearly emerges in both cases. It can be suitably caught only through the construction and comparison of proper planar cross sections of the system actual four-dimensional basins of attraction, getting an overall confirmation of the role played by global analysis for attaining a comprehensive understanding of system dynamics. Owing to its considerable richness and flexibility, the third-order model is currently being used to perform systematic investigations of the effects of a variety of physically meaningful thermal boundary conditions on the nonlinear dynamic response, by also looking at the results reliability in terms of consistency of assumptions made in both the modeling and the analysis stage.

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