A Provably Robust Multiple Rotation Averaging Scheme for 
$\text{SO}(2)$

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Abstract

We give adversarial robustness results for synchronization on the rotation group over $\mathbb{R}^2$, $\text{SO}(2)$. In particular, we consider an adversarial corruption setting, where an adversary can choose which measurements to corrupt as well as what to corrupt them to. In this setting, we first show that some common nonconvex formulations, which are categorized as “multiple rotation averaging”, may fail. We then discuss a new fast algorithm, called Trimmed Averaging Synchronization, which has exact recovery and linear convergence up to an outlier percentage of $1/4$.

1 Introduction

The synchronization problem involves recovery of an underlying signal of $n$ objects from pairwise measurements between them. It arises, for example, when solving the Structure from Motion (SfM) problem, where one is interested in recovering the three-dimensional orientations and positions of cameras in relation to a scene [Özyeşil et al., 2017]. Here we focus on robust synchronization over $\text{SO}(2)$, the rotation group for $\mathbb{R}^2$. That is, given pairwise rotations in $\text{SO}(2)$, some of which are corrupted, we aim to recover the original signal of $n$ rotations with respect to a fixed frame.

We further clarify the problem. We identify $\text{SO}(2)$ with the complex circle, which we denote by $\mathbb{C}_1$, or equivalently, with the set of angles modulo $2\pi$. We thus also refer to $\text{SO}(2)$ synchronization as angular synchronization. We assume $n$ unknown elements of $\mathbb{C}_1$, which we denote by $z_1^\star, \ldots, z_n^\star$. We form a graph $G([n], E)$, where $[n] := \{1, \ldots, n\}$ indexes the $n$ unknown elements and $E$ designates the edges for which measurements of relative rotations are taken. For each $jk \in E$ one is provided a measurement of

$$
    z_{jk}^\star = z_j^\star \overline{z_k}^\star,
$$

where $\overline{\cdot}$ denotes the complex conjugate. We think of $z_i^\star$ as the orientation of point $i$ with respect to a fixed planar coordinate system. We can think of $z_{jk}^\star$ in the following way: If we are oriented in the coordinate system with respect to node $k$, where the real axis is identified with $\text{Sp}(z_k^\star)$, then $z_{jk}^\star$ rotates our coordinate system into the coordinate system we would see if we were sitting at node $j$. This synchronization formulation extends to any given group, where one needs to recover $\{g_i\}_{i=1}^n$, a subset of the group, given measurements of the group ratios $\{g_ig_j^{-1}\}_{i,j=1}^n$.

In reality, we cannot hope to measure exactly the pairwise rotations in (1). In fact, in many real systems, noise and corrupted measurements frequently occur: we focus here on corrupted measurements. The corruption model is assumed to be fully adversarial, where an adversary is allowed to choose which measurements to corrupt, as well as what value these corruptions take. A related model is considered in [Lerman and Shi (2019)], where the authors focus on corrupted cycles rather than corrupted edges. More specifically, our model is:

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We observe corrupted (or “bad”) edges $E_b \subset E$, where all edges in $E_b$ have a corresponding arbitrary corruption. The adversary is allowed to choose $E_b$ (and thus may to some small degree influence the connectivity of $E \setminus E_b$) as well as the corrupted values $z_{jk}$.

The rest of the observed edges are uncorrupted (or “good”) edges $E_g = E \setminus E_b$, where each edge in $E_g$ has an associated measurement given by (1).

Theoretically guaranteed methods for robust synchronization are still lacking, especially in adversarial and nonconvex settings. The development of these methods is quite important because in practice measurements are usually quite corrupted, and, as was mentioned by [Ozyesil et al., 2017], results on robustness of existing methods are lacking in the SfM pipeline.

We establish here what we refer to as exact recovery. That is, given a set of corrupted measurements, we wish to exactly recover $z_1^*, \ldots, z_n^*$ up to a global rotation. We will show that this is possible for a nonconvex method even in the presence of significant amount of arbitrary corruption.

Our theory focuses on the class of multiple rotation averaging algorithms [Govindu, 2004, Martinec and Pajdla, 2007, Hartley et al., 2013], which are highly efficient and popular but attempt to solve a nonconvex program. Thus, guarantees for these methods are lacking. It is imperative to develop a theoretical understanding of these methods and their robust counterparts [Hartley et al., 2011, Chatterjee and Govindu, 2017]. Moreover, as we discuss later, there are only few robustness guarantees for group synchronization with adversarial corruption, and we thus make a significant contribution to this area. This work is also of general appeal to the nonconvex optimization community, since we are able to prove convergence results in the complex nonconvex landscape of robust multiple rotation averaging. As we discuss later, the landscapes of this problem exhibit many local minima and spurious fixed points, which we are able to avoid with our new method.

1.1 Contributions of This Work

We are now ready to briefly state the main contributions of this work.

1. We give a new algorithm, termed Trimmed Averaging Synchronization (TAS), which exactly recovers an underlying signal in the presence of a significant amount of adversarial outliers in fully connected graphs. This is the first algorithm for multiple rotation averaging with a guarantee of robustness to adversarial corruption. We also give an extension of this algorithm to cases where $G$ is not fully connected but satisfies an additional “well-connectedness condition”.

2. We discuss issues arising for current multiple rotation averaging algorithms. In particular, we demonstrate a “poorly-tempered landscape”, where multiple rotation averaging schemes based on a robust energy may fail.

While we carefully review related work later in Section 2, we emphasize here our contributions in terms of the most relevant works. The only published robustness results for SO($d$) synchronization are in [Wang and Singer, 2013]. However, the partially adversarial model in this work is very restrictive (see details in Section 2) and the proposed method is slow for large $n$. Very recently, [Lerman and Shi, 2019] have proposed a general method for group synchronization that is guaranteed to be robust to adversarial corruption. However, multiple rotation averaging schemes are faster by an order of $n$: the former method uses information from 3-cycles, that is, triangles in the graph, and so the speedup is the ratio between the number of triangles and the number of edges in the graph. On the other hand, our work considers extraction of only pairwise information and avoids higher-order cycles. We bound the ratio of corrupted edges, whereas [Lerman and Shi, 2019] bounds the ratio of corrupted triangles. However, we also require a well-connectedness condition of the full graph. We remark though that somewhat similar conditions appear, sometimes implicitly, in works minimizing similar energy functions [Wang and Singer, 2013, Hand et al., 2018, Lerman et al., 2018, Huang et al., 2017].
At last we mention that the absolute deviation energy function considered here, over $SO(2)^n$, is similar to the one of Maunu et al. [2019] over the Grassmannian for the different problem of robust subspace recovery (RSR) [Lerman and Maunu, 2018]. However, the latter energy is “well-tempered” under a general generic condition, which implies general recovery guarantee for an adversarial setting for RSR [Maunu and Lerman, 2019]. On the other hand, there are significant issues that arise in trying to prove such well-tempered results in a synchronization framework. In this work, we point out some possible difficulties that arise in establishing a well-tempered landscape for the absolute deviation energy over $SO(2)^n$, and therefore we resort to a novel robust procedure to avoid local minima. Finally, although there are theoretical issues in proving convergence for least absolute deviations methods, it is hard to find examples where previous methods actually converge to spurious minima. This fact points to some deeper phenomenon at play in practical settings.

1.2 Structure of the Rest of the Paper

First, we review related work in Section 2. Then, we review preliminary notions to understand our main results in Section 3. Following this, we give a new algorithm, called Trimmed Averaging Synchronization (TAS) in Section 4, along with its theoretical guarantees of robustness and convergence. Following this, we discuss why a similar analysis for previous multiple rotation averaging algorithms does not work in Section 5. Finally, Section 6 concludes this work with open directions. Supplemental proofs are provided in the appendix.

2 Related Work

Interest in the synchronization problem has grown in recent years due to applications in computer vision and image processing, such as SfM [Govindu, 2004, Martinec and Pajdla, 2007, Arie-Nachimson et al., 2012] Hartley et al., [2013] Tron and Vidal [2009] Ozyesil et al. [2015] Boumal [2016], cryo-electron microscopy [Wang and Singer, 2013] and Simultaneous Localization And Mapping (SLAM) [Rosen et al., 2019].

The most common formulation for solving angular and other group synchronization problems involve a non-convex least squares formulation that can be addressed by spectral methods [Singer, 2011] or semidefinite relaxation [Bandeira et al., 2017]. On the other hand, the work of Wang and Singer [2013] uses a semidefinite relaxation of a least absolute deviations formulation to obtain a robust estimate for $SO(d)$ synchronization. They prove recovery for the pure optimizer of this convex problem in a restricted setting. In this setting the full graph is complete, every edge is corrupted with a certain probability $p$ (in the case of $SO(2)$, they require that $p \leq 0.543$) and the corrupted group ratios are distributed uniformly on $SO(2)$. In practice, they advocate using an alternating direction augmented Lagrangian to solve their optimization problem. One may also use methods like the Burer-Monteiro formulation [Boumal et al., 2018], although current guarantees require the rank of the semidefinite program to be at least $O(\sqrt{n})$, which results in storing iterates much larger than the underlying signal that is a vector of size $n$ [Waldspurger and Waters, 2018].

For a survey of robust rotation synchronization, see [Tron et al., 2016]. Some early works on rotation synchronization include Govindu [2001, 2006], Martinec and Pajdla [2007], with later works by [Hartley et al., 2013], Chatterjee and Madhav Govindu [2013], Chatterjee and Govindu [2017]. The later works discuss some least absolute deviations based approaches to multiple rotation averaging which we will discuss later. Most past works focused on SO(3), while for the sake of complete guarantees we focus on SO(2).

For example, [Hartley et al., 2011] and [Hartley et al., 2013] study a least absolute deviations formulation over SO(3) using successive averaging with a Weiszfeld algorithm and a gradient based algorithm. The authors also give a counterexample that shows that local minima exist and thus the global minimum of their problem may be hard to find in general. However, the authors give no guarantee of convergence or recovery in any setting.

Another recent work tries to leverage a low-rank plus sparse decomposition for robust synchronization [Arrigoni et al., 2018]. However, this work does not contain robustness guarantees.
Bandeira et al. [2017] study maximum likelihood estimation of the angular synchronization problem and show that the associated semidefinite relaxation is tight. More recently, message passing algorithms have been used for maximum likelihood estimation in the Gaussian setting [Perry et al., 2018]. A different message passing procedure that incorporates consistent information from cycles was proposed and guaranteed to be robust for the adversarial setting in Lerman and Shi [2019]. This work applies to any compact group and its adversarial setting is very general. However, it requires a bound on the ratio of corrupted cycles per edge and not on the corrupted edges. Furthermore, the use of cycles results in more computationally intensive algorithm than the one in this work that only uses pairwise information. Other recent results leverage multiple phases to obtain better results in noisy settings [Gao and Zhao, 2019].

This problem is tied to optimization on the manifold $SO(2)$, and more generally optimization on the manifold $SO(d)$ [Taylor and Kriegman, 1994, Arora, 2009]. Indeed, to extend our results to higher dimensional settings, one would need to leverage the associated manifold structures.

Guarantees for exact recovery with adversarial, or partially adversarial, corruption appear in few other synchronization problems. The adversarial corruption in $Z_2$ synchronization is very special, since there is a single choice to corrupt a group ratio. Under a special probabilistic model, Bandeira [2018] established asymptotic and probabilistic exact recovery for the SDP relaxation of the least squares energy function of $Z_2$ synchronization. The model assumes that $G([n], E)$ is an Erdős-Rényi graph with probability $p$ of connection, edges are randomly corrupted with probability $q$ and $p(1 - 2q)^2 \leq 0.5$. Huang et al. [2017] analyzed a solution for regularized least absolute deviations formulation for one-dimensional translation synchronization. Their generic condition is rather complicated and in order to interpret it they assume that $G([n], E)$ is complete. They also restrict the maximal degree of the random graph $G([n], E_b)$. Nevertheless, their interpretive model is not adversarial, but noisy. Hand et al. [2018] and Lerman et al. [2018] established asymptotic exact recovery under a probabilistic model for solutions of the different problem of location recovery from pairwise orientations. In this problem ratios of the Euclidean group are normalized to the sphere. They assume an i.i.d. Gaussian generative probabilistic model for the ground truth locations and an Erdős-Rényi model for the graph $G([n], E)$ and further bounded the ratio of maximal degree of $G([n], E_b)$ over $n$. In both works these bounds approach zero as $n$ approaches infinity, unlike the constant bound of this work.

Another related problem is the synchronization of Kuramoto oscillators. In particular, a main question is the minimal graph connectivity requirement ensuring that the energy landscape is nice. The weakest known requirement is that every vertex is connected to at least $0.7889n$ other vertices [Lu and Steinerberger, 2019]. The conjectured bound is $0.75n$, which is reminiscent of the bound we require for the local recovery of our method with adversarial corruption.

Our work also fits in with the growing body of work analyzing nonconvex energy landscapes and procedures [Dauphin et al., 2014, Hardt, 2014, Jain et al., 2014, Netrapalli et al., 2014, Yi et al., 2016, Ge et al., 2015, Lee et al., 2015, Arora et al., 2015, Mei et al., 2018, Ge et al., 2016, Sun et al., 2015b, a, Lerman and Maunu, 2017, Cherapanamjeri et al., 2017, Ma et al., 2018, Maunu et al., 2019].

3 Preliminaries for Robust Synchronization over $SO(2)$

In this section, we give the necessary preliminary ideas to understand multiple rotation averaging methods. First, Section 3.1 gives definitions of some geometrical objects on $SO(2)$, which we identify with $C_1$ for mathematical convenience. Then, Section 3.2 defines the notion of robustness we consider for the synchronization problem.

3.1 The Geometry of $C_1$

Borrowing terminology and ideas from Riemannian geometry, we define a few structures related to the manifold $C_1$. First, the tangent space is given by $\mathbb{R}$. Let $v \in T_z C_1$ be a unit direction in the tangent space at $z_j$ (i.e., $v = \pm 1$). The geodesic originating at $z_j$ in the direction $v$ is given by $\gamma(t) = e^{itv}z_j$, $t \in [0, \pi/|v|]$. The
exponential map and inverse exponential map (log map) on this 1-dimensional manifold are given by
\[ \exp_z(\theta) = e^{i\theta}z, \theta \in (-\pi, \pi], \quad \log_y(z) = \arg(y\overline{z}). \] (2)

Finally, the cut-locus of a point \( z \in \mathbb{C}_1 \) is defined as the set of points for which there is not a unique geodesic from \( z \). It is not hard to see that this is given by \( C_l(z) = \{-z\} \).

Recall that we seek an underlying signal \( z^* \in \mathbb{C}_1^\ast \). Notice that its elements, \( z^*_j \in \mathbb{C}_1 \) for \( j \in [n] \), can be parameterized by angles, \( z^*_j = e^{i\theta_j} \). This angle is also known as the argument of the complex number, and so we write \( \arg(e^{i\theta}) = \theta \), where \( \theta \in (-\pi, \pi] \). The angular, or geodesic, distance between \( z_1 \) and \( z_2 \in \mathbb{C}_1 \) is
\[ d_\angle(z_1, z_2) = |\arg(z_1\overline{z_2})|. \] (3)

For later reference, we plot the extended angular distance function in Figure 1.

![Figure 1: The angular distance function \( d_\angle(e^{i\theta}, 1) \).](image)

Recall that if \( jk \in E_g \), then the edge measurement is correct, that is, \( z_{jk} = z^*_jz^*_k \), where \( z^*_j \) is defined in (1). For \( jk \in E_b \), the measurement \( z_{jk} \) is assumed to be an arbitrary element of \( \mathbb{C}_1 \). Within the measurements \( z_{jk}^* \), \( jk \in E_g \), \( z^* \) is only identified up to a global rotation, since one can sandwich a factor \( z\overline{z} = 1 \) in the middle of (1). To deal with this ambiguity, the following function will be used to demonstrate convergence of a sequence to \( z^* \). Define the deviation \( \delta \) of a point \( z \) with respect to \( z^* \) as
\[ \delta(z) = \max_{jk \in E} d_\angle(z_{jk}z^*_j, z_{jk}z^*_k) = \max_{jk \in E} |\arg(z_{jk}\overline{z^*_jz^*_k})|. \] (4)

Notice that \( \delta(z) = 0 \iff z = z^* \cdot y \) for some rotation \( y \in \mathbb{C}_1 \). Therefore, convergence of \( \delta(z) \) to zero indicates convergence of \( z \) to \( z^* \), and an algorithm exactly recovers \( z^* \) iff \( \delta(z) \rightarrow 0 \).

### 3.2 Notions of Robustness

In order to quantify the robustness we introduce the following terminology. We refer to \( E_g \) and \( E_b \) as inlier and outlier edge sets, respectively. For any \( j \in [n] \), we define the following local notions of inlier and outlier edge sets
\[ E^j_g = \{k \in [n] : jk \in E_g\}, \quad E^j_b = \{k \in [n] : jk \in E_b\}. \] (5)

We will denote by \( \alpha_0 \) the maximum percentage of outliers per node. That is, \( \alpha_0 \) is the maximum of \( \#(E^j_b)/d_j \) over all \( j \in [n] \), where throughout the rest of the paper \( \#(\cdot) \) denotes the number of points in a set and \( d_j \) is the degree of node \( j \). The following notion of recovery threshold is analogous to the notion of breakdown point in robust statistics. However, the compactness of \( \text{SO}(2) \) requires a modified version, which is similar to the notion of RSR breakdown point given in Section 1.1 of [Maunu and Lerman [2019]].
Definition 1 (Recovery Threshold). The recovery threshold of an algorithm is the largest value of \( \alpha_0 \) such that the algorithm outputs \( \hat{z} \sim z^\star \).

The simplest information-theoretic bound for the recovery threshold is \( \alpha_0 \leq 1/2 \). Indeed, if \( \alpha_0 > 1/2 \), then one could easily choose \( E_b \) to have a subgraph identical to \( E_g \). Then, as long as one forms a consistent measurement system on \( E_b \), it would be impossible to know what the true signal is. Further, when \( \alpha_0 > 1/2 \), one can actually disconnect the measurement graph \( E_g \), making exact recovery impossible. In either case, the signal \( z^\star \) is not recoverable.

For many nonconvex methods, good initialization is quite important. As we will see in the theory we develop, we only obtain recovery after initializing in a neighborhood of \( z^\star \). For reference, we state the initialization assumption here.

Assumption 2. If \( z^0 \) is the initial iterate of a given multiple rotation averaging algorithm, then we assume \( \delta(z^0) < \pi \).

We can guarantee a good initialization in the following scenario, where we assume some prior knowledge of \( z^\star \). Suppose that \( |\arg(z^\star_j z^\star_k)| < \pi/2 \) for all \( jk \). Then, it is not hard to see that \( \delta(1) < \pi \), where \( 1 \) is the vector of all ones. In general, such an assumption would be based on the assumption that the orientations of \( z^\star \) are somewhat close to each other.

4 Trimmed Averaging Synchronization

In order to yield a provable algorithm, we resort to trimmed means.\(^1\) This is the first instance of such methods being used for the synchronization problem.

We will show in the following that using a trimmed rotation averaging scheme converges to the underlying solution when the percentage of outliers is at most \( \alpha_0 \leq 1/4 \) when \( G \) is fully connected. In the case where \( G \) is not fully connected, we give a condition on the connectedness of \( G \) that ensures recovery with the bound \( \alpha_0 \leq 1/4 \).

We note that this fraction is similar to the one given in Lerman and Shi \[2019\], although there the bound is formulated for corrupted triangles in the graph. Beyond exact recovery, trimmed means are also appealing due to the fact that they are more efficient than the median for Cauchy noise distributions \[Bloch\ 1966\], although it is unclear if such results translate to the compact set \( C_1 \). The algorithm we propose is also quite computationally efficient, and converges linearly when \( \alpha_0 \leq 1/4 \).

First, in Section 4.1, we explain this new algorithm, which we call Trimmed Averaging Synchronization. Then, Section 4.2 gives the adversarial recovery guarantee when \( G \) is assumed to be fully connected, as well as an extension to the setting where \( G \) is not fully connected.

4.1 Trimmed Averaging Algorithm

The Trimmed Averaging Synchronization (TAS) algorithm follows. First, for a discrete \( A \subset \mathbb{R} \) and a fraction \( 0 < p < 1 \), define the \( p \)th quantile of \( A \) by \( A_p \). With this, we define the following trimming operator

\[
T_{0.25}A = \left\{ a \in A : A_{0.25} < a < A_{0.75} \right\}.
\]

We also denote the average of a dataset \( X \subset \mathbb{R} \) by \( \text{ave}(X) \). That is, \( \text{ave}(X) = \left( \sum_{x \in X} x \right) / \#(X) \).

To motivate the algorithm, \( \{z_{jk} z'_k : k \in [n] \setminus j\} \) is a set of estimates of the orientation \( z_j \) based on the current guesses of \( z_k \). To update our guess of \( z_j \), it makes sense to consider some form of average of these measurements. This is precisely the motivation behind multiple rotation averaging schemes. With this in mind, we must actually do the averaging over the manifold \( C_1 \). In our procedure, instead of calculating the mean on the manifold \( C_1 \), we instead find the mean in the tangent space and then project back to the

\(^1\)A strategy based on Windsorized means would also work because the underlying principle would be the same.
manifold. Consider \( \{ \theta_k^t = \log_{z^t_j}(z_{jk}z_k^t) : k \neq j \} \), which contains the angles of the estimates of \( z_j \) with respect to \( z_j^t \). Our proposal is to take a trimmed average of these angles, and then use the exponential map to project this average back to \( \mathbb{C}_1 \). An illustration of one trimmed averaging step is given in Figure 3.

Figure 2: Illustration of the TAS algorithm at a fixed step and a fixed node \( i \). The measurement is \( z_j = z = 1 \). After projecting into the tangent space, the outermost points in red are filtered, and the green points are averaged.

The indices \( j \) are updated in a cyclic fashion, and we call one pass over all of indices an epoch. We give the TAS algorithm in Algorithm 1. To allow for damping of the updates, we include the parameter \( \eta \in (0, 1] \). When \( \eta < 1 \), we refer to the algorithm as \( \eta \)-Damped TAS, or \( \eta \)-DTAS for short, and when \( \eta = 1 \) the algorithm is simply referred to as TAS.

Input: \( z^0 \), number of epochs \( T \), damping parameter \( \eta \in (0, 1] \)
for \( t = 1, \ldots, T \) do
\[ z^{t+1} \leftarrow z^t \]
for \( j = 1, \ldots, n \) do
\[ z_j^{t+1} \leftarrow \exp_{z_j^t} \left( \eta \cdot \text{ave} \left( T_{0.25} \left\{ \log_{z_j^{t+1}} (z_{jk}z_k^{t+1}) : k \neq j \right\} \right) \right) \]
end
end
return \( z^T \)

Algorithm 1: \( (\eta \text{-Damped}) \) Trimmed Averaging Synchronization

4.2 Recovery Guarantees for TAS With Fully Connected Graphs

We have the following recovery result for the TAS algorithm. While the algorithm converges linearly, the rate we derive depends on \( n \) and is worst-case. In the few simulations we have run, the algorithms seems to converge at a faster rate. The proof of this theorem is given in Appendix A.1.

Theorem 3. Suppose that \( \alpha_0 \leq 1/4 \), \( G \) is fully connected, Assumption 2 holds, and \( \{ z^t \}_{t \in \mathbb{N}} \) is the sequence generated by \( \eta \)-DTAS. Then, the sequence converges linearly: \( \delta(z^t) \leq \left( \frac{(n - 1 - \eta)}{(n - 1)} \right)^{T-1} \delta(z^0) \).

We can extend the analysis of the \( \eta \)-DTAS algorithm to the case where \( G \) is not fully connected provided that it satisfies an additional condition, which we call well-connectedness. We now assume that every node \( j \) has degree \( 0 < d_j \leq n - 1 \), and we define \( E_j = \{ k : jk \in E \} \).

Assumption 4 (Well-connectedness condition). For \( J \subset [n] \) such that \( \#(J) \leq n/2 \), there exists an index \( j \in J \) such that
\[ \# \left[ E^j \cap ([n] \setminus J) \right] > \# \left[ E^j \cap J \right]. \]
In words, this nodes \( j \) is connected to more nodes outside of \( J \) than inside of \( J \).
We believe that this assumption is nontrivial, and we give some examples of simple graphs satisfying this condition in Appendix B. However, further examination of this condition is left to future work.

The following theorem gives the main recovery result for the $\eta$-DTAS algorithm. Its proof is left to Appendix A.2.

**Theorem 5.** Suppose that $\alpha_0 \leq 1/4$, Assumption 2 holds, $G$ satisfies the well-connectedness condition of Assumption 4, and $(z^t)_{t \in N}$ is the sequence generated by $\eta$-DTAS, for $\eta \in (0, 1)$. Then, $\delta(z^t) \to 0$, and the algorithm exactly recovers $z^\star$.

## 5 $L_1$ Multiple Rotation Averaging

We conclude with a consideration of a previous scheme for robust multiple rotation averaging which utilizes an $L_1$ energy [Dai et al., 2009; Hartley et al., 2011; Chatterjee and Govindu, 2017]. Past works on this method offer no guarantees of robustness, although they show strong empirical performance. Many $L_1$ Multiple Rotation Averaging algorithms ($L_1$-MRA) boil down to doing coordinate descent, while others use IRLS on individual coordinates [Hartley et al., 2011] or on the whole set of coordinates [Chatterjee and Govindu, 2017]. These previous cases also primarily focused on optimization over $SO(3)$. Here, we consider optimization over $SO(2)$ since the theoretical analysis is simpler.

In this section, we review the existing approaches to synchronization based on the least absolute deviations energy. First, Section 5.1 reviews the $L_1$ energy minimization functional proposed for synchronization. After this, in Section 5.2 we discuss perhaps the simplest variant of $L_1$-MRA, which uses coordinate descent. After this, Section 5.3 discusses some issues arising in the analysis of this algorithm.

### 5.1 Robust Synchronization by Energy Minimization

[Dai et al., 2009], [Hartley et al., 2011], [Wang and Singer, 2013] and [Chatterjee and Govindu, 2017] consider the following least absolute deviations formulation

$$\min_{z \in \mathbb{C}^n_{\mathbb{R}}} F_\angle(z) := \sum_{j < k} d_\angle(z_j, z_k^j).$$

(7)

In the same way that the median is a robust estimator of the center of a dataset, the least absolute deviations estimator is expected to give a robust estimate of the underlying signal in synchronization.

Next, we define the coordinate energy function

$$F^j_\angle(y; z) = \sum_{k \in [n] \setminus j} d_{\angle}(z_j, z_{jk}^k).$$

(8)

It will be useful to calculate directional derivatives of $F^j_\angle$. For ease of notation, we define some auxiliary datasets (which multisets). Notice that the energy (7) minimizes the deviations between $z_j$ and estimates of $z_j$ given by $z_{jk}^k$, for $k \in [n] \setminus j$. For a given $z$, we break up the set of these estimates into parts. Define the multisets

$$C_+(j; z) := \left\{ y = z_{jk}^k : \log_{z_j}(y) \in (0, \pi), k \in [n] \setminus j \right\},$$

(9)

$$C_-(j; z) := \left\{ y = z_{jk}^k : \log_{z_j}(y) \in (-\pi, 0), k \in [n] \setminus j \right\}.$$  

(10)

The set $C_+(j; z)$ represents all estimates $z_{jk}^k$ such that they have positive value in the tangent space at $z_j$, and similarly the set $C_-(j; z)$ have negative value in the tangent space at $z_j$. We also define the multisets

$$C_0(j; z) := \left\{ y = z_{jk}^k = z_j \right\},$$

$$C_1(j; z) := \left\{ y = z_{jk}^k = -z_j \right\}.$$  

(11)
The set $C_l(j; z)$ represents points that lie at the cut-locus with respect to $z_j$, while $C_0(j; z)$ represent estimates $z_{jk}z_k$ that exactly equal $z_j$. We emphasize that all of these datasets are multisets, since we allow for repeated values from the measurements.

Let $v \in T_z \mathcal{C}_1$ be a unit direction and $\gamma(t) = e^{ivt}z_j$ a geodesic. After looking at Figure 1 it is not hard to see that the directional derivative of $F^j_\mathcal{L}(\cdot; z)$ at $z_j$ is given by

$$
\frac{d}{dt} F^j_\mathcal{L}(\gamma(t); z) \bigg|_{t=0} = \lim_{h \to 0} \frac{F^j_\mathcal{L}(\gamma(h); z) - F^j_\mathcal{L}(\gamma(0); z)}{h} = \frac{\sum_{k \in [n]} \sum_{j} d_\mathcal{L}(e^{ivt}z_j, z_{jk}z_k) - \sum_{k \in [n]} d_\mathcal{L}(z_j, z_{jk}z_k)}{h} = \text{sign}(v)(-\#C_+(j; z) + \#C_-(j; z)) - \#C_l(j; z).
$$

### 5.2 Variants of the $L_1$-MRA Algorithm

We present a variant of coordinate descent based $L_1$-MRA [Dai et al., 2009, Hartley et al., 2011]. At epoch $t$, we update all indices $j$ in a cyclic fashion. Of course, this is just for ease of analysis, and one could consider other strategies, such as randomly ordering within each epoch, or by just randomly sampling $j$ for every update.

The coordinate descent algorithm follows. Suppose that we wish to update index $j$ within epoch $t$, where at the beginning of the epoch we set $z^t \leftarrow z^{t-1}$. Then, to make the most improvement in minimizing (7), we would like to solve $z^t_j \in \text{argmin}_{z \in \mathbb{C}} F^j_\mathcal{L}(z; z^t)$. The easiest way to theoretically control the iterates is coordinate descent. In this algorithm, we search for a local minimum of $F^j_\mathcal{L}(z; z^t)$ after initializing at $z^t_j$, which is done using a line-search in a direction of decrease from $z^t_j$. This direction can be found by choosing $v = \pm 1$ such that $\frac{d}{dt} F^j_\mathcal{L}(z_j; z) \bigg|_{t=0} < 0$. Along the geodesic given by $e^{ivt}z_j$, the directional derivatives $\frac{d}{dt} F^j_\mathcal{L}(e^{ivt}z_j; z)$ are constant until $e^{ivt}z_j = \pm z_{jk}z_k$, for some $k \in [n]$. In this case, the value of $\frac{d}{dt} F^j_\mathcal{L}(e^{ivt}z_j; z)$ either increases or decreases by $\# \{ k : e^{ivt}z_j = \pm z_{jk}z_k \}$. Therefore, the algorithm searches in the direction $v$ until the first point $e^{ivt}z_j$ is found such that

$$
e^{ivt}z_j \in \left\{ \pm z_{jk}z_k : k \in [n] \right\}, \quad \frac{d}{dt} F^j_\mathcal{L}(e^{ivt}z_j; z) \bigg|_{t=0} \geq 0.
$$

This variant of $L_1$-MRA is termed Gradient Descent $L_1$-MRA (GD-$L_1$-MRA), and we give the full algorithm in Algorithm 2.

**Algorithm 2: GD-$L_1$ Multiple Rotation Averaging** [Dai et al., 2009, Hartley et al., 2011]
It is easy to see that the only fixed points for this algorithm occur when every point is a local minimum with respect to the coordinate directions. We state this as follows. If \( z \) is a fixed point of the GD-L1-MRA algorithm, then \( \partial_v F^L_2(z_j; z) \geq 0 \) for \( v \in \{ \pm 1 \} \) and for all \( j \). Therefore, each coordinate sits at its own local minimum, which is just to say that moving each coordinate alone does not decrease the cost. However, this does not preclude the existence of a geodesic in \( C^n_1 \) along which the full cost \( F(z) \) decreases. This issue is discussed further in Section 5.3.

We finish by giving two simple lemmas on the properties of using GD-L1-MRA to solve (7). The first lemma implies, with proper initialization, the iterates of L1-MRA must remain in the local neighborhood \( \delta(z^t) < \pi \), as long as the ratio of bad edges per node is not too large. Notice here that there are no restrictions on the measurement graph other than it must be connected. The proof is given in Appendix A.3.1.

**Lemma 6.** Suppose Assumption 2 holds, \( G \) is connected, and that \( \alpha_0 < 1/2 \). Then, if \( (z^t)_{t \in \mathbb{N}} \) is the sequence generated by GD-L1-MRA, \( \delta(z^{t+1}) \leq \delta(z^t) \).

The second lemma shows that GD-L1-MRA is strictly monotonic, and its proof is given in Appendix A.3.2.

**Lemma 7.** If \( (z^t)_{t \in \mathbb{N}} \) is the sequence generated by GD-L1-MRA, then the sequence of function values \( (F_L(z^t))_{t \in \mathbb{N}} \) is strictly monotonic.

These results are insufficient to give convergence of L1-MRA to \( z^* \) due to two issues we discuss in Section 5.3. These issues are: 1) the coordinate descent mapping defined by GD-L1-MRA is not necessarily closed in the adversarial setting, and 2) there may be spurious fixed points. Indeed, it has been pointed out before that the general proof of convergence for such methods is hard, as is mentioned previously in Section 7.4 of [Hartley et al., 2013].

### 5.3 Issues Arising in the Analysis of L1-MRA

We briefly state two results arising in the analysis of the coordinate descent mapping. First, we show that the mapping defined by this algorithm may not be closed. Then, we discuss spurious fixed points for the GD-L1-MRA procedure.

First, the coordinate mapping given by Algorithm 2 may not be closed.

**Example 8** (The map \( z^t \rightarrow z^{t+1} \) is not closed). As a simple example, suppose we have a signal with 6 components, and the GD-L1-MRA procedure is initialized at \( z^0 = (1, e^{ic}, e^{ic}, -e^{-ic}, -1, -1) \) for \( c < \pi \), and \( z_{jk} = 1 \) for all \( jk \in E \). Suppose further that we wish to update \( z^1_0 \) at iteration 1. Then, it is obvious that \( \partial_{z^0} F^L_2(z^0) \) at iteration 1. Then, it is obvious that \( \partial_{z^0} F^L_2(1; z^0) < 0 \) since this direction moves the signal closer to the coordinates \( z_k \) for \( k = 2, 3, 4, 5 \). Therefore, we find that \( z^1 = (e^{ic}, e^{ic}, e^{-ic}, -e^{-ic}, -1, -1) \). However, it is quite easy to come up with a sequence that takes steps in the opposite direction. Indeed, consider the sequence \( z^0_l = (1, e^{ic}, e^{ic}, e^{-ic}, -e^{-il}, -e^{-il}) \) for \( l \in \mathbb{N} \). We see that \( z^0_l \rightarrow z^0 \). However, updating the first coordinate of \( z^0_l \) yields \( z^1_l = (e^{ic}, e^{ic}, e^{ic}, e^{-ic}, -e^{-il}, -e^{-il}) \). As \( l \rightarrow \infty \), \( z^1_l \not\rightarrow z^1 \). Therefore, GD-L1-MRA mapping is not closed in general.

As the example illustrates, all issues with the map not being closed have to do with the points located at the cut locus of \( z^t_j \). This property prevents application of Zangwill’s global convergence theorem [Zangwill, 1969].

A second issue is the existence of spurious fixed points. These even exist in the case where \( G \) is fully connected and there is one outlier per edge. The proof of this lemma is given in Appendix A.3.3.

**Lemma 9** (No Outliers). Suppose that \( G \) is a fully connected graph, and for each \( j \), \( E_b^j \) only contains one element. Then, there exists adversarial corruptions on \( E_b \) such that there are spurious fixed points \( \hat{z} \) where \( 0 < \delta(\hat{z}) < \pi \).

In a certain sense, this states that the subgraph of good edges cannot be too well-connected to achieve convergence of GD-L1-MRA. Further, for any point \( \hat{z} \) satisfying the implications of Lemma 2, the energy
decreases along geodesic between $\hat{z}$ and $z^*$. However, in order to find this decrease, we see that one must move blocks of coordinates together, which GD-$L_1$-MRA cannot do.

We currently do not have easy fixes for these problems. Some ideas to deal with the issues discussed here involve regularization, either by smoothing the energy function or by using some sort of random subsampling. Both of these ideas would be interesting directions for future work.

6 Conclusion

The rigorous development of a condition for SO($d$) is left to future work. We are further curious what the limits of this analysis are, as well as if tighter analyses can yield larger recovery thresholds. At least for the case of SO(2), the trimmed average type estimator discussed in this paper has significant recovery thresholds, while also being computationally tractable. In particular, it is most important to extend these ideas to $d = 3$, although it is not immediately clear how to do this.

Two more concrete directions for future work would be to see if the well-connectedness condition in Assumption 4 can be relaxed at all, and further what its implications are and when it actually holds. Another direction is to explain the phenomenon observed in practice that $L_1$-MRA does not actually seem to converge to spurious fixed points. One could also include noise in these analyses.

In this paper we focus on robustness to adversarial outliers. However, it would be interesting to see what sorts of results are achievable in the non-adversarial setting. For example, in the problem of robust subspace recovery, one can take the percentage of corruption to 1 and still obtain exact recovery [Maunu et al., 2019]. It would be interesting to see if nonconvex methods for synchronization enjoy similar guarantees.

Finally, perhaps the most important direction for future work is to give theoretically justified algorithms for a larger range of algorithms employed for SfM [Ozyesil et al., 2017, Bianco et al., 2018]. Indeed, such theoretical work can lead to new and improved algorithms and help to push state-of-the-art forward.

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A Supplementary Proofs

A.1 Proof of Theorem 3

The first lemma in the proof of this theorem does not need the assumption that $G$ is fully connected. In the following lemmas, we denote the degree of node $j$ by $d_j$. The assumption of fully connectedness is only needed in Lemma 11, where it is used to ensure the existence of no other fixed points but $z^*$. We define the residual distance function $\Delta_{jk}(z) := d_{\angle}(z_j, z_k z_k z_k)$, so that $F_{\angle}(z) = \sum_{j<k} \Delta_{jk}(z)$.

\textbf{Lemma 10.} Suppose that $\alpha_0 \leq 1/4$ and that $E$ is connected. Then, any $z \sim z^*$ is a fixed point of the $\eta$-DTAS algorithm.

\textit{Proof.} First, since $\alpha_0 \leq 1/4$, we have

$$\left\{ k : \Delta_{jk}(z^*) = 0 \right\} \geq 3d_j/4, \forall j. \quad (14)$$

Therefore, in the trimming step, any outliers such that $\Delta_{jk}(z^*) \neq 0$ are trimmed. This implies that $T_{0.25}\{\log_{z_j}(z_k z_k z_k) : k \neq j\} = \{0, \ldots, 0\}$. Therefore, $z^*$ is a fixed point.

With these lemmas in hand, the proof of Theorem 3 follows from the following lemma.
Lemma 11. Suppose that Assumption\textsuperscript{2} holds, $\alpha_0 \leq 1/4$, $G$ is fully connected, and $\eta < (n - 1)/(n + 1)$. Then, the sequence generated by $\eta$-DTAS is strictly monotonic with respect to $\delta(\cdot)$ over epochs, with bound

$$\delta(z^t) < \left(\frac{n - 1 - \eta}{n - 1}\right) \delta(z^{t-1}),$$

unless $\delta(z^t) = 0$, in which case $z^t$ is a fixed point and $z \sim z^*$. 

Proof. We begin epoch $t$ by setting $z^t \leftarrow z^{t-1}$. In this proof, we assume without loss of generality that $z_j^t, z_k^t \in B(1, \delta(z^t)/2)$ for all $j$, where for $x \in \mathbb{C}_1$, we denote an open ball of radius $r$ by $B(x, r) = \{y \in \mathbb{C}_1 : d_{\mathbb{C}}(x, y) < r\}$ and $\overline{A}$ of a set $A$ is the closure of $A$, while $\tau$ of $z \in \mathbb{C}$ is the complex conjugate of $z$. In this proof, we will write $\delta = \delta(z^t)$ as a shorthand.

Notice that, for all $j \in [n]$, we can write the estimates with respect to $z_j^t$,

$$\log_{z_j^t}(z_j^t z_k^t) = \begin{cases} \arg(z_j^t z_k^t z_j^t z_k^t) = e^{i(-\theta_j^t + \theta_j^t + (\theta_k^t - \theta_k^t))}, & jk \in E_g, \\ \arg(z_j^t z_k^t z_j^t z_k^t), & jk \in E_b. \end{cases}$$

Since $\alpha_0 \leq 1/4$, we know that $j$ must have good connections with more than $n/4$ indices in $I_-(t)$, since $I_-(t)$ contains at least $n/2$ nodes. For all such $k$,

$$\arg(z_j^t z_k^t z_j^t z_k^t) \in (-\theta_j^t + \theta_j^t, -\theta_j^t + \theta_j^t - \delta(z^t)/2).$$

Therefore, the trimmed set of angles must contain at least one point in this set.

Within these sets, denote

$$\delta_j = \theta_j^t - \theta_j^t \in [-\delta(z^t)/2, \delta(z^t)/2].$$

These are the normalized angular coordinates of our points $z_1^t, \ldots, z_n^t$. Also, define the sets

$$I_+(t) = \{ k : \arg(z_j^t z_k^t) > 0 \}, \quad I_-(t) = \{ k : \arg(z_j^t z_k^t) \leq 0 \}.$$ 

Notice that we must have

$$\min(#I_+(t), I_-(t)) \leq n/2,$$

unless $I_+(t) = I_-(t) = [n]$, in which case $z^t \sim z^*$. 

First, for the update with respect to index $j$, all good pairwise measurements must lie in $-\delta_j + [-\delta_j, \delta_j]$. Since there are at least $3n/4$ good measurements per index, all trimmed points must lie in this interval as well. Therefore, for all $j \in I_-(t)$, after updating we have

$$z_j^t z_k^t \in \exp\left[i[-\delta_j, \eta \delta_j/2]\right].$$

Using this fact, we will now show that the indices in $I_+(t)$ must move inwards. Indeed, since $\#I_+(t) \leq n/2$, we must have $\#(E_g \cap I_-(t)) \leq 1$. Therefore, for each trimmed mean for $j \in I_+(t)$, we have

$$\text{ave} \left(\mathcal{T}_{0.25} \left( \left\{ \log_{z_j^t}(z_j^t z_k^t) : k \neq j \right\} \right) \right) \leq \frac{2}{n - 1} \left[ \eta \frac{\delta}{2} - \delta_j + \frac{2}{n - 1} \left( \frac{n - 1}{2} - 1 \right) \frac{\delta}{2} - \delta_j \right]$$

$$\leq \left( \frac{n - 3}{n - 1} - \frac{2}{n - 1} \eta \right) \cdot \frac{\delta}{2} - \delta_j.$$ 

Thus,

$$\eta \text{ave} \left(\mathcal{T}_{0.25} \left( \left\{ \log_{z_j^t}(z_j^t z_k^t) : k \neq j \right\} \right) \right) + \delta_j \leq \eta \left( \frac{n - 3 - 2\eta}{n - 1} \right) \cdot \frac{\delta}{2} + (1 - \eta)\delta_j$$

$$\leq \eta \left( \frac{n - 3}{n - 1} \right) \cdot \frac{\delta}{2} + \frac{1 - \eta}{2} \delta_j$$

$$= \frac{\delta}{2} \left( 1 - \eta \left( \frac{2}{n - 1} \right) \right).$$
Therefore, after the coordinate update, we have that for all \( j \in I_+(t) \),
\[
z_j^{t+1} \exp \left[ i \left( -\frac{\delta}{2} \left( 1 - \eta \left( \frac{2}{n-1} \right) \frac{\delta}{2} \right) \right) \right].
\] (22)

After repeating this argument for all \( j \) over the course of the epoch, this yields that
\[
z_j^{t+1} \exp \left[ i \left( -\frac{\delta}{2} \left( 1 - \eta \left( \frac{2}{n-1} \right) \frac{\delta}{2} \right) \right) \right],
\]
as long as \( \eta < (n-1)/(n+1) \). The width of this interval is \((n-1-\eta)\delta(z^t)/(n-1)\), which yields the desired result.

The proof of Theorem 3 follows from these three lemmas. Since the algorithm is strictly monotonic and closed, it converges to a fixed point \( \hat{z} \) such that \( \delta(\hat{z}) < \pi \). Since the only fixed points are equivalent to \( z^* \), the result follows.

### A.2 Proof of Theorem 5

First, we prove that the mapping defined by \( \eta \)-DTAS for \( \eta \in [0,1] \) is closed.

**Lemma 12.** Suppose that we wish to apply the coordinate mapping defined by Algorithm 7 to a point \( z \) such that \( \delta(z) < \pi \). If we assume that \( \alpha_0 < 1/4 \), then, for any \( j \), this mapping is closed.

**Proof.** Suppose that \((z^t)_{t \in \mathbb{N}}\) is a sequence converging to \( z \). Then, it is obvious that \( z_j^t z_k^t \to z_j z_k \), which also implies that \( \log_{z_j}(z_j^t z_k^t) \to \log_{z_j}(z_j z_k) \) unless \( z_j z_k = -z_j \) (i.e., if \( z_j z_k \) is in the cut-locus of \( z_j \)).

Therefore, the only potential issue for non-convergence of the trimmed means occurs when points lie at the boundary: i.e., when \( z_j z_k = -z_j \). In this case, the direction from which \( z_j^t \) approaches \( z_j \) would contribute very different values to the resulting sample mean. However, due to the assumption \( \delta(z) < \pi \) and \( \alpha_0 < 1/4 \), less than \( 3d_j/4 \) points \( jk \) can satisfy this property, where \( d_j \) is the degree of \( j \). Indeed, since \( \delta(z) < \pi \), no point \( z_j^* z_k^* \) can lie in the cut-locus of \( z_j \). Since \( \alpha_0 < 1/4 \), there are more than \( 3d_j/4 \) points that do not lie in the cut-locus. Therefore, all cut-locus values must be filtered, and the map is closed.

One can also show that the sequence remains bounded.

**Lemma 13.** Suppose that we wish to apply the coordinate mapping defined by Algorithm 7 to a point \( z \) such that \( \delta(z) < \pi \). Denote the output of this map by \( z^+ \). If we assume that \( \alpha_0 < 1/4 \), then \( \delta(z^+) \leq \delta(z) \).

**Proof.** Suppose that we update index \( j \) with our coordinate descent step. Then, since \( \delta(z) < \pi \), we know that all \( \log_{z_j}(z_j z_k), jk \in E_g \), lie in some interval \( I \) such that \( \text{length}(I) < \pi \) and \( I \subset (-\pi, \pi) \). Since \( I \) contains more than \( 3d_j/4 \) points, the trimmed set of angles must be contained within \( I \), and therefore the trimmed average must lie within \( I \). It is then easy to see that this implies, by the same reasoning as the proof of Lemma 6, that \( \delta(z^+) \leq \delta(z) \).

To finish the proof of this theorem, we just need the following lemma that demonstrates strict monotonicity after sufficiently many epochs.

**Lemma 14.** Suppose that Assumption 2 holds, \( \alpha_0 < 1/4 \), and \( E \) satisfies Assumption 4. Then, the sequence generated by \( \eta \)-DTAS is strictly monotonic with respect to \( \delta(\cdot) \) after sufficiently many epochs, unless \( \delta(z^t) = 0 \), in which case \( z^t \) is a fixed point and \( z \sim z^* \).

**Proof.** The proof is similar to that of Lemma 11, with some slight changes. We begin epoch \( t \) by setting \( z^t \leftarrow z^{t-1} \). As before, in this proof we assume without loss of generality that \( z_j^t z_k^t \in B(1, \delta(z^t)/2) \) for all \( j \).
Define the sets
\[ M(t) = \{ j : z_j T \delta(z^t) / 2 \}, \quad m(t) = \{ j : z_j \delta(z^t) / 2 \}. \] (23)

Notice that we must have
\[ \min(#M(t), m(t)) \leq n/2, \]
unless \( M(t), m(t) = [n] \), in which case \( z^t \sim z^* \). Without loss of generality, assume that \( #M(t) \leq n/2 \).

Notice that
\[ J_{0.25} \left( \log_{z_{j+1}} \left( \frac{z_j z_{j+1}^t}{z_k} : k \neq j \right) \right) \leq 0, \forall j \in M(t), \] (24)
by the reasoning of Lemma. For any \( j \in M(t) \), define
\[ N(j, t) := \left\{ k \in E^j_g : \Delta_{jk}(z^t) = 0 \right\}. \] (25)

By Assumption there is at least one index \( j \) such that
\[ #\left[ E^j \cap ([n] \setminus M(t)) \right] \geq \frac{d_j}{2}. \] (26)

Further, since \( \alpha_0 < 1/4 \), we must have
\[ #\left[ E^j_g \cap ([n] \setminus M(t)) \right] > \frac{d_j}{4}. \] (27)

Since \( \log_z (z_j z_k) < 0 \) for all \( k \in E^j_g \cap [n] \setminus M(t) \), we must have that \( z_{j+1}^t \in B(1, \delta(z^t) / 2) \), that is, it moves to the interior.

On the other hand, due to the damping of steps, it is easy to see that no points \( z_j^t \in B(1, \delta(z^t) / 2) \) can have an update that puts them on the boundary. This is due to the fact that the trimmed averages must always lie in
\[ \log_{z_j}(\left[ e^{-i\delta(z^t) / 2}, e^{i\delta(z^t) / 2} \right]), \] (28)
following the reasoning in Lemma.

Therefore, after a finite number of iterations, all elements of \( M(t) \) must move into the interior of \( B(1, \delta(z^t) / 2) \), and the sequence is strictly monotonic with respect to \( \delta \).

\[ \square \]

### A.3 Proof of Lemmas for \( L_1 \)-MRA

#### A.3.1 Proof of Lemma

Within epoch \( t \), we will show that the coordinate descent mapping satisfies the bounded iteration property.

Suppose we are updating the \( j \)th coordinate within epoch \( t \). Then, the updated coordinate \( z_{j+1}^t \) is found by running gradient descent initialized at \( z_j^t \) until a local minimum is found for the energy function \( F^j_{z_k}(z; z^t) \).

The next iterate minimizes a sum of absolute deviations with \( z_j z_k^t \), where the inlier terms are given by \( z_j \delta(z^t) \).

Notice that, for all \( j, k \in E_g \),
\[ \max_{l,k} d_c(z_j z_l^t, z_j z_k^t) = \max_{l,k} d_c(z_j^t, z_l z_j z_k^t) = \max_{l,k} d_c(z_j^t, z_k z_k^t) \leq \delta(z^t). \] (29)

This implies that all the points \( z_j^t z_k^t \) lie within a circular interval of width less than \( \pi \), which we write as \( \epsilon^{[a, b]} \). Notice that, for sufficiently small \( \epsilon \), \( \partial_{z^t} F^j_k(z; z^t) < 0 \) for \( z \in \epsilon^{(a-\epsilon, a)} \) and \( \partial_{-1} F^j_k(z; z^t) < 0 \) for \( z \in \epsilon^{(b, b+\epsilon)} \) since more than \( n/2 \) points lie in \( \epsilon^{[a, b]} \). Therefore, the function \( F^j_k(z; z^t) \) must have at least one local minimum in \( \epsilon^{[a, b]} \) that is found by \( L_1 \)-MRA. It then follows that
\[ \delta(z^{t+1}) \leq \delta(z^t). \] (30)
A.3.2 Proof of Lemma 7

We can break up the energy function $F$ in the following way:

$$F(z) = \sum_{j<k} d_\angle(z_j, z_k z_k)$$

(31)

$$= \sum_{k \in [n] \setminus j} d_\angle(z_j, z_k z_k) + \sum_{l<k \in [n] \setminus j} d_\angle(z_j, z_k z_k).$$

All dependence on $z_j$ is in the first sum. Notice that our iteration finds a local minimum for the first sum with respect to $z_j$, and further that it must find a local minimum that has less energy than $z^*_j$ since the line-search procedure starts at $z^*_j$. If $z^*_j$ is not already a local minimum for its coordinate function, then one of the directional derivatives is strictly negative. This implies that the $L_1$-MRA procedure is strictly monotonic unless $z^*_j$ is fixed.

A.3.3 Proof of Lemma 9

Divide $[n]$ into two sets, both of size $n/2$ if $n$ is even, and if $n$ is odd, one of size $(n - 1)/2$ and one of size $(n + 1)/2$. Call these sets $J$ and $K$. Choose $\hat{z}$ to be such that $\hat{z}_j z_j^\star = \hat{z}_j z_j^\star$ for all $j, j' \in J$, and $\hat{z}_k z_k^\star = \hat{z}_k z_k^\star$ for all $k, k' \in K$, and such that $0 < \delta(\hat{z}) < \pi$.

In the case of even $n$, we see that each $j \in J$ is connected to $n/2 - 1$ points in $J$ and $n/2$ points in $K$. The same holds for all $k \in K$. Therefore, as long as one chooses the adversarial edge for each $j$ to point in the opposite direction of the points $\hat{z}_k z_k^\star$, for $k \in K$, the energy $F_j(\hat{z}; \hat{z})$ has a local minimum for $\hat{z}_j$. The same argument holds for all $k \in K$ since these adversarial edges give a conjugate direction for their derivatives.

This argument extends to the odd case as well by a similar argument.

B Graphs Satisfying Well-Connectedness Condition

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Figure 3: Graphs that satisfy the well-connectedness condition. For all subsets $J$ of size at most $n/2$, there exists a node $j \in J$ such that $\#(E_j \cap ([n] \setminus J)) > \#(E_j \cap J)$. 