Minimal see-saw model predicting best fit lepton mixing angles

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A R T I C L E   I N F O

Article history:
Received 23 May 2013
Accepted 5 June 2013
Available online 7 June 2013
Editor: M. Cvetić

A B S T R A C T

We discuss a minimal predictive see-saw model in which the right-handed neutrino mainly responsible for the atmospheric neutrino mass has couplings to \((\nu_e, \nu_\mu, \nu_\tau)\) proportional to \((0, 1, 1)\) and the right-handed neutrino mainly responsible for the solar neutrino mass has couplings to \((\nu_e, \nu_\mu, \nu_\tau)\) proportional to \((1, 4, 2)\), with a relative phase \(\eta = -2\pi/5\). We show how these patterns of couplings could arise from an \(A_4\) family symmetry model of leptons, together with \(Z_3\) and \(Z_2\) symmetries which fix \(\eta = -2\pi/5\) up to a discrete phase choice. The PMNS matrix is then completely determined by one remaining parameter which is used to fix the neutrino mass ratio \(m_2/m_3\). The model predicts the lepton mixing angles \(\theta_{12} \approx 34^\circ, \theta_{23} \approx 41^\circ, \theta_{13} \approx 9.5^\circ\), which exactly coincide with the current best fit values for a normal neutrino mass hierarchy, together with the distinctive prediction for the CP violating oscillation phase \(\delta \approx 106^\circ\).

1. Introduction

Daya Bay [1] and RENO [2] have measured a non-zero reactor angle \(\theta_{13} \approx 0.15\) which excludes Tri-Bimaximal (TB) mixing [3]. Recent global fits also hint at deviations of the atmospheric and solar angles from their TB values [for a recent review see e.g. [4]]. Such deviations may be expressed in terms of the deviation parameters \((s, a\) and \(r\)) from TB mixing [5] (for a related parametrisation see [6]):

\[
\sin \theta_{12} = \frac{1}{\sqrt{3}} (1 + s), \quad \sin \theta_{23} = \frac{1}{\sqrt{2}} (1 + a), \quad \sin \theta_{13} = \frac{r}{\sqrt{2}}. \tag{1}
\]

With zero solar and atmospheric deviations from TB mixing, \(s = a = 0\), and Cabibbo-like reactor mixing described by \(r = \lambda\), with \(\lambda = 0.225\) being the Wolfenstein parameter, one is led to Tri-Bimaximal Cabibbo (TBC) mixing [7]. However, as mentioned above, current global fits prefer non-zero solar and atmospheric TB deviation parameters,

\[
s = -\lambda^2/2, \quad a = -\lambda/3, \quad r = \lambda, \tag{2}
\]

corresponding to the angles,

\[
\theta_{12} = 34.2^\circ, \quad \theta_{23} = 40.8^\circ, \quad \theta_{13} = 9.15^\circ. \tag{3}
\]

These angles are close to the best fit values for all three global fits in the case of a normal neutrino mass ordering [4]. Assuming a normal neutrino mass hierarchy with \(m_1 = 0\), one is led to [13],

\[
\frac{m_2}{m_3} = \frac{3}{4} \lambda, \tag{4}
\]

corresponding to \(m_2/m_3 \approx 0.17\), close to the best fit value [4]. The deviation parameters in Eq. (2) have the feature that the atmospheric mixing angle is in the first octant and the solar mixing angle is somewhat less than its tri-maximal value, in agreement with the latest global fits for the case of a normal neutrino mass ordering. In particular it reproduces the best fit values of angles of all three global fits [4] to within one standard deviation.

There have been many attempts to describe the lepton mixing angles based on the type I see-saw model [8] combined with sequential dominance (SD) [9] in which the right-handed neutrinos contribute with sequential strength. Constrained sequential dominance (CSD) [10] involves the right-handed neutrino mainly responsible for the atmospheric neutrino mass having couplings to \((\nu_e, \nu_\mu, \nu_\tau)\) proportional to \((0, 1, 1)\) and the right-handed neutrino mainly responsible for the solar neutrino mass having couplings to \((\nu_e, \nu_\mu, \nu_\tau)\) proportional to \((1, 4, 2)\) and it led to TB mixing. CSD2 [11] proposed a new non-zero reactor angle and is based on the same atmospheric alignment but with right-handed neutrino mainly responsible for the solar neutrino mass having couplings to \((\nu_e, \nu_\mu, \nu_\tau)\) proportional to \((1, 0, -2)\) or \((1, 2, 0)\) yielding a reactor angle \(\theta_{13} \approx 6^\circ\) which unfortunately is too small, although the situation can be rescued by invoking charged lepton corrections [12]. The CSD3 model in [13] involves the right-handed neutrino mainly responsible for the solar neutrino mass having...
couplings to \((v_e, \mu, \tau)\) proportional to \((1, 3, 1)\) or \((1, 1, 3)\) with a relative phase \(\mp \pi/3\) yielding a reactor angle \(\theta_{13} \approx 8.5^\circ\) close to the observed value. However CSD3 predicts approximate TBC mixing with an almost maximal atmospheric mixing angle disfavoured by the latest global fits, and so it may soon be challenged.

In this Letter we shall propose a model based on a new possibility called CSD4 which predicts the above best fit angles in Eq. (3) of the PMNS lepton mixing matrix and also makes predictions for the physical CP violating phases. Similar to all SD models, the CSD4 model involves effectively two right-handed neutrinos and a normal neutrino mass hierarchy, leading to \(m_1 = 0\). As in CSD2 and CSD3, the CSD4 model only requires one input parameter, namely the ratio of neutrino masses which is selected to be \(m_2/m_3 \approx 3/4\), which is a natural value that one would expect in such models. Also as in CSD2 and CSD3, once this value is chosen, the entire PMNS mixing matrix is then fixed by the theory (up to a discrete choice of phases) with no remaining free parameters. In the CSD4 model, the right-handed neutrino mainly responsible for the atmospheric neutrino mass has couplings to \((v_e, \nu_\mu, \nu_\tau)\) proportional to \((0, 1, 1)\) and the right-handed neutrino mainly responsible for the solar neutrino mass has couplings to \((v_e, \nu_{\mu}, \nu_\tau)\) proportional to \((1, 4, 2)\), with a relative phase \(\eta = -2\pi/5\). These couplings and phase relation were first discovered in [13] and shown to lead to lepton mixing angles in good agreement with the latest global fits, but no model has been proposed based on CSD4. The goal of this Letter is to show how CSD4 can arise from an \(A_4\) family symmetry, together with additional discrete \(Z_5\) and \(Z_3\) symmetries, and to present the first model of leptons along these lines. This is necessary since it is far from clear whether alignments such as \((1, 4, 2)\) are possible to achieve within a realistic model. The CSD4 model presented here predicts the best fit PMNS angles in Eq. (3) with the distinctive prediction for the oscillation phase \(\delta \approx 106^\circ\).

### 2. A minimal predictive \(A_4\) model of leptons

In this section we outline a supersymmetric (SUSY) \(A_4\) model of leptons with CSD4 along the lines of the \(A_4\) models of leptons discussed in [11,14]. The basic idea is that the three families of lepton doublets \(L\) form a triplet of \(A_4\) while the right-handed charged leptons \(e^c, \mu^c, \tau^c\), right-handed neutrinos \(N_{\text{atm}}, N_{\text{solar}}\) and the two Higgs doublets \(H_1, H_2\) required by SUSY are all singlets of \(A_4\). In addition the model employs an additional \(Z_5^\text{sol}\) family symmetry in order to account for the charged lepton mass hierarchy.

The vacuum alignment that is required for the model discussed in Appendix A is shown in Table 1. As in models of this kind [11,14], the RH neutrinos have no mass terms at the renormalisable level, but they become massive after some \(A_4\) singlet flavons \(\xi_{\text{atm}}\) and \(\xi_{\text{solar}}\) develop their vevs due to the renormalisable superpotential:

\[
\mathcal{W}_\text{Yuk}^{\nu} \sim \frac{1}{\Lambda} H_2(\phi_{\text{atm}} \cdot L)N_{\text{atm}} + \frac{1}{\Lambda} H_2(\phi_{\text{solar}} \cdot L)N_{\text{solar}}.
\]

Again the higher order corrections are completely negligible. The neutrino sector of the model exploits the \(A_4\) triplet flavons \(\phi_{\text{atm}} \equiv \psi_{\text{atm}}\) and \(\phi_{\text{solar}} \equiv \psi_{\text{solar}}\) whose alignment is discussed in Appendix A.

As is typical in models of this kind [11,14], the RH neutrinos have no mass terms at the renormalisable level, but they become massive after some \(A_4\) singlet flavons \(\xi_{\text{atm}}\) and \(\xi_{\text{solar}}\) develop their vevs due to the renormalisable superpotential.
require a special phase relation see-saw mechanism naturally leads to the neutrino mass matrix example by taking physical neutrino masses as discussed in [12], and we shall follow the strategy of predicting the best fit values of the lepton mixing angles.

To ensure that the mixed terms are absent at renormalisable order we have imposed a right-handed neutrino flavour symmetry $Z_5^{\text{atm}}$ under which $N_{\text{atm}}$ and $\xi_{\text{atm}}$ transform as $\rho^2$ and $\rho$ (where $\rho = e^{i\pi/5}$) while $\phi_{\text{atm}}$ transforms as $\rho^2$ with all other fields being singlets. We have also imposed a similar symmetry $Z_5^{\text{sol}}$ under which the "solar" fields transform in an analogous way. We remark that these charge assignments are consistent with the flavon superpotential in Eq. (29), where we identify $\phi_{\text{atm}} \equiv \psi_{\nu_3}$, and $\phi_{\text{sol}} \equiv \psi_{\nu_4}$, with suitable charges assigned to the driving fields.

With all masses, couplings and messenger scales set approximately equal, the driving flavon superpotentials would predict $|\langle \xi_{\text{atm}} \rangle| \approx |\langle \xi_{\text{sol}} \rangle|$ and hence approximately equal right-handed neutrino masses $M_A \approx M_B$. Similarly, from Appendix A with $\phi_{\text{atm}} \equiv \psi_{\nu_3}$ and $\phi_{\text{sol}} \equiv \psi_{\nu_4}$, we see that $|\langle \phi_{\text{atm}} \rangle| \approx |\langle \phi_{\text{sol}} \rangle|$ would lead to

$$\langle \phi_{\text{atm}} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \nu_{\text{atm}}, \quad \langle \phi_{\text{sol}} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \nu_{\text{sol}},$$

where $\nu_{\text{atm}} \approx \nu_{\text{sol}}$.

The above charge assignments allow higher order non-renormalisable mixed terms such as

$$\Delta W_R \sim \frac{1}{\Lambda} (\phi_{\text{atm}} \cdot \phi_{\text{sol}}) N_{\text{atm}} N_{\text{sol}},$$

which contribute off-diagonal terms to the right-handed neutrino mass matrix of a magnitude which depends on the absolute scale of the flavon vevs $\langle \phi_{\text{atm}} \rangle$ and $\langle \phi_{\text{sol}} \rangle$ compared to $\langle \xi_{\text{atm}} \rangle$ and $\langle \xi_{\text{sol}} \rangle$. If all flavon vevs and messenger scales are set equal then these terms are suppressed by $e^{-\phi_{\text{atm}}}$ according to the estimate below Eq. (6), however they may be even more suppressed. We shall ignore the contribution of such off-diagonal mass terms in the following.

Implementing the see-saw mechanism, the effective neutrino mass matrix has the form

$$m^{\nu} \sim \frac{v^2}{\Lambda^2} \left( \frac{\langle \phi_{\text{atm}} \rangle}{\langle \xi_{\text{atm}} \rangle} \frac{\langle \phi_{\text{atm}} \rangle^T}{\langle \xi_{\text{atm}} \rangle} + \frac{\langle \phi_{\text{sol}} \rangle}{\langle \xi_{\text{atm}} \rangle} \frac{\langle \phi_{\text{sol}} \rangle^T}{\langle \xi_{\text{sol}} \rangle} \right),$$

where $v = (H_2)$. Hence it can be parameterised, up to an overall irrelevant phase, as

$$m^{\nu} = m_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_1 e^{i2\eta \epsilon_{\nu}} \begin{pmatrix} 1 & 4 & 2 \\ 4 & 16 & 8 \\ 2 & 8 & 1 \end{pmatrix},$$

where $m_0$ and $\epsilon_{\nu}$ are real mass parameters which determine the physical neutrino masses $m_1$ and $m_2$ and we have written the relative phase difference between the two terms as $2\eta$. Using Eq. (10) the see-saw mechanism naturally leads to the neutrino mass matrix in Eq. (13) with $\epsilon_{\nu} \approx 2/21 \approx 0.1$. Hence the desired value of $\epsilon_{\nu} \approx 0.06$ assumed below is not unreasonable, and may be achieved for example by taking $M_B \approx 2M_A$. As discussed in [12], we shall also require a special phase relation $\eta = -2\pi/5$ in order to achieve our goal of predicting the best fit values of the lepton mixing angles.

The phase difference $\eta = -2\pi/5$ between flavon vevs can be obtained in the context of spontaneous CP violation from discrete symmetries as discussed in [12], and we shall follow the strategy outlined there. The basic idea is to impose CP conservation on the theory so that all couplings and masses are real. Note that the $A_4$ assignments in Table 1 do not involve the complex singlets ’1’, ’1’ or any complex Clebsch–Gordan coefficients so that the definition of $\epsilon$ is straightforward in this model and hence $\epsilon$ may be defined in different ways which are equivalent for our purposes (see [12] for a discussion of this point). The CP symmetry is broken in a discrete way by the form of the superpotential terms. We shall follow [12] and suppose that the flavon vevs $\langle \phi_{\text{atm}} \rangle$ and $\langle \phi_{\text{sol}} \rangle$ to be real with the phase $\eta$ in Eq. (13) originating from the solar right-handed neutrino mass matrix due to the flavon vev $\langle \xi_{\text{atm}} \rangle \sim M_B e^{i\pi/5}$ having a complex phase of $4\pi/5$, while the flavon vev $\langle \xi_{\text{sol}} \rangle \sim M_A$ is real and positive. This can be arranged if the right-handed neutrino flavon vevs arise from the superpotential,

$$W_{\text{flavon}, R} = g P \left( \frac{\xi_{\text{atm}}^5}{A^2} - M^2 \right) + g' P' \left( \frac{\xi_{\text{sol}}^5}{A'^2} - M'^2 \right).$$

where, as in [12], the driving singlet fields $P, P'$ denote linear combinations of identical singlets and all couplings and masses are real due to CP conservation. The F-term conditions from Eq. (14) are,

$$\frac{\langle \xi_{\text{atm}} \rangle^5}{A^2} - M^2 \quad \frac{\langle \xi_{\text{atm}} \rangle^5}{A'^2} - M'^2 = 0.$$
diagonalised by the TM$_1$ mixing matrix [20] where the first column is proportional to \( s \), and the second column is proportional to \( c \). Note that these predictions assume \( \eta \approx -2\pi /5 \). The predictions are obtained numerically using the Mixing Parameter Tools package based on [15]. The leading order analytic results are not reliable as discussed in Appendix B.

Exact TM$_1$ mixing angle and phase relations are obtained by equating moduli of PMNS elements to those of the first column of the TB mixing matrix (see also [20]):

\[
\begin{align*}
C_{12}C_{13} &= \sqrt{\frac{7}{3}}, \\
|C_{23}S_{12} + S_{13}S_{23}c_{12}e^{i\delta}| &= \frac{1}{\sqrt{6}}, \\
|S_{23}S_{12} - S_{13}S_{23}c_{12}e^{i\delta}| &= \frac{1}{\sqrt{6}}.
\end{align*}
\]

From Eq. (16) we see that TM$_1$ mixing approximately preserves the successful TB mixing for the solar mixing angle \( \theta_{12} \approx 35^\circ \) as the correction due to a non-zero but relatively small reactor angle is of second order. While general TM$_1$ mixing involves an undetermined reactor angle \( \theta_3 \), we emphasise that CSD4 fixes this reactor angle. For \( \eta \approx -2\pi /5 \) the reactor angle is in the correct range as shown in Table 2.

In an approximate linear form, the relations in Eqs. (16)-(18) imply the atmospheric sum rule relation \( a = r \cos \delta \) [11], hence,

\[ \theta_{23} \approx 45^\circ + \sqrt{2}\theta_3 \cos \delta. \]

For \( \eta = -2\pi /5 \) the predictions shown in Table 2 for the small deviations of the atmospheric angle from maximality are well described by the sum rule in Eq. (19). In the present model this sum rule is satisfied by particular predicted values of angles and CP phase which only depend on the neutrino mass ratio \( m_3/m_3 \). Over the successful range of \( m_2/m_3 \) we predict CP violation with \( \delta \approx 106^\circ \) and \( \theta_{23} \approx 41^\circ \) which satisfy the sum rule. Note that according to this sum rule, non-maximal atmospheric mixing is linked to non-maximal CP violation.

### 3. Conclusions

There is long history of attempts to explain the neutrino mixing angles starting from the type I seesaw mechanism and using SD, first using CSD to account for TB mixing, then using CSD2 to obtain a small reactor angle before going to CSD3 where the correct reactor angle can be reproduced along with maximal atmospheric mixing. We have discussed a minimal predictive seesaw model based on CSD4 in which the right-handed neutrino mainly responsible for the atmospheric neutrino mass has couplings to \((\nu_e, \nu_\mu, \nu_\tau)\) proportional to \((0, 1, 1)\) and the right-handed neutrino mainly responsible for the solar neutrino mass has couplings to \((\nu_e, \nu_\mu, \nu_\tau)\) proportional to \((1, 4, 2)\), with a relative phase \( \eta = -2\pi /5 \). We have shown how these patterns of couplings and phase could arise from an \( A_4 \) family symmetry model of leptons. We remark that the type of model presented here is referred to as “indirect” according to the classification scheme of models in [4], meaning that the family symmetry is completely broken by flavons and its only purpose is to generate the desired vacuum alignments. By contrast, the “direct” models where the symmetries of the neutrino and charged lepton mass matrices is identified as a subgroup of the family symmetry, requires rather large family symmetry groups in order to account for the reactor angle [21].

It is possible to have “semi-direct” models, either at leading order or emerging due to higher order corrections [4], but these are inherently less predictive. In the light of the observed reactor angle, “indirect models” therefore offer the prospect of full predictivity at the leading order from a small family symmetry group. Spontaneous CP violation seems to be an important ingredient in the “indirect” approach since a particular phase relation between flavons a crucial requirement.

The particular indirect model presented here, in which CSD4 emerges from an \( A_4 \) family symmetry, offers a highly predictive framework involving only one free parameter which is used to fix the neutrino mass ratio \( m_2/m_3 \), together with an overall neutrino mass scale which is used to fix the atmospheric neutrino mass \( m_3 \). Remarkably, the model then predicts the PMNS angles \( \theta_{12} \approx 34^\circ \), \( \theta_{13} \approx 41^\circ \), \( \theta_{23} \approx 9.5^\circ \), which exactly coincide with the current best fit values for a normal neutrino mass hierarchy, together with the distinctive prediction for the CP violating oscillation phase \( \delta \approx 106^\circ \). These predictions will surely be tested by current and planned high precision neutrino oscillation experiments.

### Acknowledgements

S.F.K. would like to thank A. Merle for help with MPT, Christoph Luhn and Stefan Antusch for discussions and A. Kusenko and T. Yanagida and the IPMU for hospitality and support. S.F.K. also acknowledges partial support from the STFC Consolidated ST/J000396/1 and EU ITN grants UNILHC 237920 and INVISIBLES 289442.

### Appendix A. Vacuum alignment

In this appendix we shall discuss how to achieve the following vacuum alignment,

\[
\frac{\langle \phi_{\text{atm}} \rangle}{\Lambda} \propto \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{\langle \phi_{\text{sol}} \rangle}{\Lambda} \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
\]

which we refer to as CSD4. The vacuum alignments associated with TB mixing have been very well studied. Here we shall focus on the family symmetry \( A_4 \) as it is the smallest non-Abelian finite group with an irreducible triplet representation. The generators of the \( A_4 \) group, can be written as \( S \) and \( T \) with \( S^2 = T^3 = (ST)^3 = 1 \). \( A_4 \) has four irreducible representations, three singlets \( 1, 1' \) and \( 1'' \), and one triplet. The products of singlets are:

\[
\begin{array}{c}
1 \otimes 1 = 1, \\
1' \otimes 1'' = 1, \\
1' \otimes 1' = 1''.
\end{array}
\]

We work in the basis [19],

\[
S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

In this basis one has the following Clebsch rules for the multiplication of two triplets,

\[
(ab)_{1} = a_{1}b_{1} + a_{2}b_{2} + a_{3}b_{3},
\]
(ab)_{11} = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3,
(ab)_{12} = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3,
(ab)_{31} = (a_3 b_1, a_3 b_1, a_1 b_2),
(ab)_{32} = (a_3 b_2, a_1 b_1, a_2 b_1),
\quad (23)
where \omega^3 = 1, \quad a = (a_1, a_2, a_3) \quad \text{and} \quad b = (b_1, b_2, b_3).

Following the methods of [14] it is straightforward to obtain the vacuum alignments for charged lepton flavon alignments suitable for a diagonal charged lepton mass matrix. The charged lepton flavon alignments used to generate a diagonal charged lepton mass matrix are obtained from the renormalisable superpotential [14].

\[ W_{A4}^{\text{flavon},i} \sim A_{\tau} \psi_e \psi_e + A_{\tau} \psi_\mu \psi_\mu + A_{\tau} \psi_\tau \psi_\tau + O_{\epsilon \mu} \psi_e \psi_\mu + O_{\epsilon \tau} \psi_e \psi_\tau + O_{\mu \mu} \psi_e \psi_e. \]
\quad (24)
The triplet driving fields \( A_{\epsilon,\mu,\tau} \) give rise to flavon alignments \( \langle \psi_{\epsilon,\mu,\tau} \rangle \) with two zero components, and the singlet driving fields \( O_{ij} \) require orthogonality among the three flavon vevs so that we arrive at the vacuum structure [14].

\[ \langle \psi_e \rangle = v_e \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \psi_\mu \rangle = v_\mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \]
\[ \langle \psi_\tau \rangle = v_\tau \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]
\quad (25)
Of more interest to us in this Letter are the new neutrino flavon alignments. The starting point for the discussion is the usual standard TB neutrino flavon alignments proportional to the respective columns of the TB mixing matrix,

\[ \langle \psi_{\nu_1} \rangle = v_{\nu_1} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \langle \psi_{\nu_2} \rangle = v_{\nu_2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \]
\[ \langle \psi_{\nu_3} \rangle = v_{\nu_3} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \]
\quad (26)
We will also employ the alternative TB alignments which are related by phase redefinitions,

\[ \langle \psi_{\nu_1}' \rangle = v_{\nu_1}' \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \langle \psi_{\nu_2}' \rangle = v_{\nu_2}' \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \]
\[ \langle \psi_{\nu_3}' \rangle = v_{\nu_3}' \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \]
\quad (27)
In the remainder of this subsection we shall show how to obtain the neutrino flavon alignments including the new alignment,

\[ \langle \psi_{\nu_4} \rangle = v_{\nu_4} \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \]
\quad (28)
which corresponds to the CSD4 solar flavon alignment in Eq. (20). We shall identify \( \phi_{3m} \equiv \psi_{\nu_3} \) and \( \phi_{3a} \equiv \psi_{\nu_4} \). The renormalisable superpotential involving the driving fields necessary for aligning the neutrino-type flavons is given as

\[ W_{A4}^{\text{flavon},4} = A_{\nu_1} (g_1 \psi_{\nu_1} \psi_{\nu_2} + g_2 \psi_{\nu_2} \psi_{\nu_1}) + A_{\nu_2} (g_1' \psi_{\nu_2} \psi_{\nu_1} + g_2' \psi_{\nu_1} \psi_{\nu_2}) + O_{\nu_1 \nu_3} g_3 \psi_{\nu_1} \psi_{\nu_3} + O_{\nu_1 \nu_2} g_4 \psi_{\nu_1} \psi_{\nu_2} + O_{\nu_1 \nu_3} g_5 \psi_{\nu_1} \psi_{\nu_3} + O_{\nu_1 \nu_2} g_6 \psi_{\nu_1} \psi_{\nu_2}, \]
\quad (29)
where \( A_{\nu_1} \) is a triplet driving field and \( O_{ij} \) are singlet driving fields whose F-terms lead to orthogonality relations between the accompanying flavon fields. Here \( g_i \) are dimensionless coupling constants. The first line of Eq. (29) produces the vacuum alignment \( \langle \psi_{\nu_3} \rangle \propto (1, 1, -1)^T \) of Eq. (26) and \( \langle \psi_{\nu_4} \rangle \propto (1, -1, 1)^T \) of Eq. (27) as can be seen from the F-term conditions

\[ 2g_1 \begin{pmatrix} \langle \psi_{\nu_2} \rangle_1 \\ \langle \psi_{\nu_2} \rangle_2 \\ \langle \psi_{\nu_2} \rangle_3 \end{pmatrix} + 2g_2 \begin{pmatrix} \langle \psi_{\nu_1} \rangle_1 \\ \langle \psi_{\nu_1} \rangle_2 \\ \langle \psi_{\nu_1} \rangle_3 \end{pmatrix} = 0 \quad \text{and} \quad 2g_1 \begin{pmatrix} \langle \psi_{\nu_2} \rangle_1 \\ \langle \psi_{\nu_2} \rangle_2 \\ \langle \psi_{\nu_2} \rangle_3 \end{pmatrix} = 0 \]
\quad (30)
plus similar conditions involving the primed flavons. The first two terms in the second line of Eq. (29) give rise to orthogonality conditions which uniquely fix the alignment \( \langle \psi_{\nu_3} \rangle \propto (0, 1, 1)^T \) of Eq. (26),

\[ \langle \psi_e \rangle^T \cdot \langle \psi_{\nu_3} \rangle = \langle \psi_{\nu_2} \rangle^T \cdot \langle \psi_{\nu_3} \rangle = 0 \quad \text{and} \quad \langle \psi_{\nu_3} \rangle \propto \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \]
\quad (31)
The last two terms in the second line of Eq. (29) give rise to orthogonality conditions which uniquely fix the alignment \( \langle \psi_{\nu_3} \rangle \propto (2, -1, 1)^T \) of Eq. (26),

\[ \langle \psi_{\nu_2} \rangle^T \cdot \langle \psi_{\nu_3} \rangle = \langle \psi_{\nu_2} \rangle^T \cdot \langle \psi_{\nu_3} \rangle = 0 \quad \text{and} \quad \langle \psi_{\nu_3} \rangle \propto \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}. \]
\quad (32)
Similarly the terms in the third line of Eq. (29) give rise to orthogonality conditions which fix the alternative TB alignments in Eq. (27) corresponding to a different choice of phases.

The terms in the fourth line of Eq. (29) give rise to orthogonality conditions which fix the alignments of the auxiliary flavon fields \( \psi_{\nu_3} \) and \( \psi_{\nu_4} \),

\[ \langle \psi_{\nu_4} \rangle^T \cdot \langle \psi_{\nu_3} \rangle = \langle \psi_{\nu_4} \rangle^T \cdot \langle \psi_{\nu_3} \rangle = 0 \quad \text{and} \quad \langle \psi_{\nu_3} \rangle \propto \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}. \]
\quad (33)
The neutrino-type flavon of interest labelled as \( \psi_{\nu_4} \) gets aligned by the remaining terms in the fifth line of Eq. (29), leading to the desired alignment in Eq. (28),

\[ \langle \psi_{\nu_3} \rangle^T \cdot \langle \psi_{\nu_4} \rangle = \langle \psi_{\nu_3} \rangle^T \cdot \langle \psi_{\nu_4} \rangle = 0 \quad \text{and} \quad \langle \psi_{\nu_4} \rangle \propto \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}. \]
\quad (35)
So far we have only shown how to align the flavon vevs and have not enforced them to be non-zero. In order to do this we shall introduce the additional non-renormalisable superpotential terms which include,

\[ \text{We remark that the general alignment derived from these F-term conditions is} \quad \langle \psi_{\nu_3} \rangle \propto (\pm 1, \pm 1, \pm 1)^T. \] One can, however, show that all of them are equivalent up to phase redefinitions. Note that \((1, 1, 1)\) is related to permutations of the minus sign as well as to \((-1, -1, -1)\) by \( A_4 \) transformations. The other four choices can be obtained from these by simply multiplying an overall phase (which would also change the sign of the \( \xi_{\nu_4} \) vev).
Appendix B. Leading order analytic results

For the case of atmospheric alignments of the form \((0, z_1, 1)\) and solar alignments of the form \((1, z_2, z_3)\), the leading order analytic results in [9,13] give:

\[
\tan \theta_{23} \approx |z_1|, \tag{38}
\]

\[
\cot \theta_{12} \approx c_{23} z_2 \cos \left( \eta_2 - \frac{\beta}{2} \right) - s_{23} z_3 \cos \left( \eta_3 - \frac{\beta}{2} \right), \tag{39}
\]

\[
\theta_{13} \approx m_2^2 s_{12} c_{23} |z_3| + |z_2| \tan \theta_{23} e^{i(\eta_2 - \eta_3)}, \tag{40}
\]

where \(\eta_2 = \arg(z_2/z_1)\) and \(\eta_3 = \arg(z_3)\), while \(\beta\) is a Majorana phase. With \(z_1 = 1\) and arbitrarily assuming \(\beta = 0\) and real phases \(\pm 1\) associated with \(\eta_2\) and \(\eta_3\) one finds the relations:

\[
\tan \theta_{23} \approx 1, \tag{41}
\]

\[
\cot \theta_{12} \approx 1 \sqrt{2} \left| z_2 - z_3 \right|, \tag{42}
\]

\[
\theta_{13} \approx \frac{m_2}{m_3} \frac{1}{3} \sqrt{2} \left| z_2 + z_3 \right|. \tag{43}
\]

We should say immediately that the assumption \(\beta = 0\) is not justified so these results can be at best suggestive. With this caveat, we note that approximately tri-maximal solar mixing \(\cot \theta_{12} \approx 1/\sqrt{2}\) results from the general condition \(|z_2 - z_1| = 2\) which is satisfied by all the proposed forms of CSD.\(^2\) Moreover, CSD with \(z_2 = 1, z_3 = -1\) leads to \(\theta_{13} \approx 0,\) CSD2 with \(z_2 = 2, z_3 = 0\) leads to \(\theta_{13} \approx \frac{m_2}{m_3} 1/\sqrt{2},\) CSD3 with \(z_2 = 3, z_3 = 1\) leads to \(\theta_{13} \approx \frac{m_2}{m_3} 4/\sqrt{3},\) CSD4 with \(z_2 = 4, z_3 = 2\) leads to \(\theta_{13} \approx \frac{m_2}{m_3} \sqrt{2}.\)

\(^2\) I would like to thank Stefan Antusch (private communications) for emphasizing the condition \(|z_2 - z_1| = 2.\)

Although the above leading order results provide a qualitative understanding of the results obtained for CSD, CSD2, CSD3 and CSD4, they have large corrections of order \(m_2/m_3,\) much larger than the errors in the global fits and so do not give reliable predictions. In addition there is a strong dependence on the phase difference between the solar and atmospheric alignments which these results ignore. Moreover, the phase \(\eta\) does not appear in the leading order formula for \(\theta_{13},\) but in practise the reactor angle depends strongly on \(\eta,\) as discussed in [13]. On the other hand, while the phases do appear in the solar angle formula, we have arbitrarily and incorrectly assumed \(\beta = 0.\)

In summary, the leading order results, while providing a qualitative understanding, are quantitatively unreliable and cannot be used to estimate the mixing angles to the required accuracy. The general analysis as performed in [13] did not rely on the leading order results in any way and was not inspired by them. Starting from an exact master formula, the analysis [13] determined from first principles not only the moduli \(|z|\) but also the phases \(\eta\), which are required for a proper definition of any new type of CSD. For example CSD4 with solar alignment \((1, 4, 2)\) is only properly defined once the phases \(\eta_2 = \eta_3 = -2\pi /5\) are specified. In retrospect, it is rather fortuitous that the condition \(|z_2 - z_3| = 2\) is satisfied for all the proposed forms of CSD. Indeed more other complicated but equally successful examples were found that violated the condition \(|z_2 - z_3| = 2\) and these were also tabulated in [13].

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