Quantizing SU(N) gauge theories without gauge fixing

G B Tupper and F G Scholtz

Institute of Theoretical Physics
University of Stellenbosch
7600 Stellenbosch
South Africa

(January 17, 1995)

Abstract

We generalize and extend the quantization procedure of [1] which is designed to quantize SU(N) gauge theories in the continuum without fixing the gauge and thereby avoid the Gribov problem. In particular we discuss the BRS symmetry underlying the effective action. We proceed to use this BRS symmetry to discuss the perturbative renormalization of the theory and show that perturbatively the procedure is equivalent to Landau gauge fixing. This generalizes the result of [1] to the non-abelian case and confirms the widely held believe that the Gribov problem manifests itself on the non-perturbative level, while not affecting the perturbative results. A relation between the gluon mass and gluon condensate in QCD is obtained which yields a gluon mass consistent with other estimates for values of the gluon condensate obtained from QCD sum rules.

PACS numbers 11.15.–q, 11.10.Gh
I. INTRODUCTION

A major obstacle in the quantization of non-abelian gauge theories is the Gribov problem [2], especially as formulated by Singer [3]: consider a compact, semisimple non-abelian gauge theory in Euclidean space-time with boundary conditions at infinity implying the identification of space-time with $S^4$, then due to a topological obstruction no global continuous gauge fixing is possible. This excludes a very general class of gauge fixing conditions and in particular all the practically implementable ones.

It seems to be generally accepted that the Gribov problem is not important in the perturbative domain, but that it may play an important role in understanding the non-perturbative aspects of a gauge theory [4]. Our present results confirm that the Gribov problem is perturbatively unimportant. However, the situation regarding the non-perturbative aspects is much less clear. Indeed, Fujikawa [5] argued that non-perturbatively the usual BRS symmetry [6] is spontaneously broken when a Gribov problem is present, invalidating the associated Slavnov-Taylor identities. Thus, while perturbatively innocuous, the Gribov problem casts doubt on e.g. the program of solving the Schwinger-Dyson equations non-perturbatively via the gauge technique [7]. It is therefore of paramount importance that the Gribov problem should be brought under control before reliable investigations into the non-perturbative aspects of gauge theories can be launched.

Within the continuum limit there has been two recent proposals for quantizing non-abelian gauge theories without gauge fixing and thereby avoiding the Gribov problem. The first of these, 'soft gauge fixing', due to Zwanziger [8] and Jona-Lasinio [9] amounts to an implementation of Popov’s suggestion [10] that the Faddeev-Popov trick should be generalized to $1 = \int [dU] F[U, G] / I[G]$ with $I[G] = \int [dU] F[U, G]$ and $F$ a non-gauge invariant function of the gauge
field $G$ such that $I[G]$ exists. For a particular choice of $F[G]$ this method has been shown by Fachin \cite{11} to reproduce the usual perturbative result for the renormalized Landau-gauge propagator in a suitable limit.

The second approach is due to the present authors \cite{1}. Here the point of departure is a supersymmetry-like resolution of the identity in terms of bosonic and fermionic auxiliary fields combined with a U-gauge transformation to give the gauge field a mass. Since the starting identity has a BRS symmetry one also expects this to be the case for the effective action resulting from this quantization scheme. This is indeed the case as is shown below.

There are, however, several outstanding problems connected to the procedure of \cite{1}. Firstly it has not yet been demonstrated how the usual perturbative results can be recovered from this procedure for a non-abelian theory. Indeed one may appreciate that since the program invokes a massive U-gauge like non-abelian gauge field the ordinary perturbation theory and renormalization analysis are problematic. Secondly it can be shown (see section 6) that the original proposition of breaking the gauge symmetry at the tree level is perturbatively incompatible with the BRS invariance as it would imply the spontaneous breaking of the BRS symmetry. Thirdly the sequential scheme originally proposed to deal with SU($N>2$) is clumsy and obscures the BRS symmetry underlying the quantization procedure.

In this paper we will show how these problems are overcome. We begin in section 2 by reformulating our procedure so as to effectively deal with any SU($N>2$) theory without resorting to the sequential scheme of \cite{1}. This is done by imbedding in a U($N$) gauge theory with the identity realized in terms of local U($N$)$_L \otimes$ global U($N$)$_R$ fields. In section 3 we discuss the BRS symmetry, which becomes very transparent in the present formulation, for a SU($N$) theory. In the remaining sections the BRS symmetry and the pinching technique of Cornwall \cite{12} are used to perform the perturbative renormal-
ization, which becomes much more tractable in the present formulation, of
the effective theory. This culminates in our main result, namely, that in the
perturbative domain the present quantization procedure corresponds to Lan-
dau gauge fixing for non-abelian theories. The normal perturbative results
are therefore recovered, showing that the Gribov problem has no effect in the
perturbative domain. Some technical results are collected in two appendices.

II. THE U(N) FORMALISM

In [1] the auxiliary bosonic and fermionic fields were taken as vector valued
in the fundamental representation of SU(N). To integrate the gauge degrees
of freedom out in this setting one has to resort to a sequential procedure in
which the gauge symmetry is broken down according to SU(N) ⊃ SU(N-1) ....
This procedure obscures many aspects of the theory and leads to technical
complications.

Here we avoid this sequential procedure by taking the auxiliary bosonic
and fermionic fields to be matrix valued in the fundamental representation of
U(N). We exploit, as a matter of convenience, the fact that SU(N) ⊂ U(N). Our
procedure is as follows: consider a pure Yang–Mills type theory (for notation
see appendix A)

\[ L_{YM} = -\frac{1}{2} \text{tr}(G_{\mu\nu}^2) \]  

\[ G_{\mu\nu} = \partial_{\mu}G_{\nu} - \partial_{\nu}G_{\mu} + ig[G_{\mu}, G_{\nu}], \quad G_{\mu} = G_{\mu}^{a}t_{a} \]

This lagrangian is invariant under the gauge transformations

\[ G_{\mu} \rightarrow G'_{\mu} = -\frac{i}{g}U D_{\mu}U^\dagger, \quad U \in \text{U}(N) \]

where

\[ D_{\mu} = \partial_{\mu} + igG_{\mu} \]
is the gauge covariant derivative in the fundamental representation. Now for some set of gauge invariant functionals \( O[G] = O[G'] \) define

\[
\langle O \rangle = Z^{-1} \int [dG] O[G] \exp \left( i \int d^4x L_{YM} \right),
\]

\[
Z = \int [dG] \exp \left( i \int d^4x L_{YM} \right)
\]

(4)

For the restricted class of functionals, \( O \), which depend on the SU(N) gauge field

\[
G_\mu = G_\mu^a t_a
\]

(5)

only, it follows from \( U(N) \cong SU(N) \otimes U(1) \) that

\[
\langle O \rangle = Z^{-1} \int [dG] O[G] \exp \left( i \int d^4x L_{YM} \right),
\]

\[
Z = \int [dG] \exp \left( i \int d^4x L_{YM} \right)
\]

(6)

which is canonical. Thus quantization of an SU(N) gauge theory proceeds from (4) applied to \( O \).

While (4) and (6) are well defined on the lattice, in the continuum the measure

\[
[dG] \propto \Pi_{x,\mu,a} dG_\mu^a(x)
\]

(7)

is itself gauge invariant, \( [dG'] = [dG] \), so \( \langle O \rangle \sim \infty/\infty \) which is ill defined.

To factor the volume of the gauge group while eschewing the Faddeev–Popov ansatz together with its associated Gribov problem consider the (minimal) identity

\[
1 = \int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger] \exp \left( i \int d^4x L_{aux} \right),
\]

\[
L_{aux} = \text{tr}(\Phi^\dagger D_\mu^\dagger D^\mu \Phi + \zeta^\dagger D_\mu^\dagger D^\mu \zeta).
\]

(8)

Herein the auxiliary fields are matrix valued in the fundamental representation of U(N)
\[ \Phi = \Phi_a t_a \, , \quad \Phi^\dagger = \Phi_a^\dagger t_a \quad \text{and} \quad \zeta = \zeta_a t_a \, , \quad \zeta^\dagger = \zeta_a^\dagger t_a \]  \hspace{1cm} (9)

representing \( N^2 \) complex (\( 2N^2 \) real) scalar degrees of freedom and \( N^2 \) complex Grassmann valued scalar degrees of freedom (ghosts), respectively. Furthermore \( D_\mu^\dagger \) acts to the left

\[ D_\mu^\dagger = \partial_\mu - igG_\mu \]  \hspace{1cm} (10)

The measure on the auxiliary fields is

\[ [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger] = \Pi_{x,a} \frac{d\Phi_a(x)d\Phi_a^\dagger(x)}{2\pi i} d\zeta_a(x)d\zeta_a^\dagger(x) \]  \hspace{1cm} (11)

By construction \( L_{\text{aux}} \) has local \( U(N)_L \) invariance, under the transformations

\[ G_\mu \to -\frac{i}{g} U_L D_\mu U_L^\dagger \, , \quad \Phi \to U_L \Phi \, , \quad \Phi^\dagger \to \Phi^\dagger U_L^\dagger \]  \hspace{1cm} (12)

\[ \zeta \to U_L \zeta \, , \quad \zeta^\dagger \to \zeta^\dagger U_L^\dagger \, , \quad U_L = \exp (i\theta^L_a t_a) \in U(N)_L \]

as well as an independent global \( U(N)_R \) symmetry:

\[ G_\mu \to G_\mu \, , \quad \Phi \to \Phi U_R^\dagger \, , \quad \Phi^\dagger = U_R \Phi^\dagger \]  \hspace{1cm} (13)

\[ \zeta \to \zeta U_R^\dagger \, , \quad \zeta^\dagger \to U_R \zeta^\dagger \, , \quad U_R = \exp (i\theta^R_a t_a) \in U(N)_R \]

For infinitesimal transformations

\[ \delta_L \Phi_a = 2i \text{tr}(t_a t_c t_b) \theta^L_c \Phi_b = i\theta^L_c (T_c)_{ab} \Phi_b \]  \hspace{1cm} (14)

whereas

\[ \delta_R \Phi_a = i \theta^R_c (-T^*_a)_{ab} \Phi_b \]  \hspace{1cm} (15)

which, together with (14), identifies \( T_a \) and \(-T^*_a \) as generators of the adjoint representation of \( U(N)_L \) and \( U(N)_R \) respectively; our auxiliary fields transform as the basis for the adjoint representation of chiral \( U(N) \). Note that this is in contrast to [1] where the auxiliary fields transform as the fundamental representation. Writing
\[ \int d^4 x L_{\text{aux}} = \int d^4 x \int d^4 y \left( \frac{1}{2} \Phi_b^\dagger(x) M_{ab}(x, y) \Phi_b(y) + \frac{1}{2} \zeta_a^\dagger(x) M_{ab}(x, y) \zeta_b(y) \right), \]  

(16)

\[
M_{ab}(x, y) = -D_{\mu}^{ae} D_{\ell b}^{\mu} \delta^{(4)}(x - y)
\]

with

\[
D_{\mu}^{ab} = \delta_{ab} \partial_{\mu} + igG_{\mu}^{se}(T_e)_{ab}
\]

(17)

the gauge covariant derivative in the adjoint representation of U(N)_L, the proof of the identity is immediate:

\[
\int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger] \exp \left( i \int d^4 x L_{\text{aux}} \right) = \frac{\det (M)}{\det (M)} = 1.
\]

(18)

Injecting (8) into (4)

\[
\langle Q \rangle = Z^{-1} \int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger][dG][Q][G] \exp \left( i \int d^4 x (L_{\text{YM}} + L_{\text{aux}}) \right),
\]

(19)

\[ Z = \int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger][dG] \exp \left( i \int d^4 x (L_{\text{YM}} + L_{\text{aux}}) \right) \]

we make the change of variables

\[
\Phi = -U(\pi)\phi, \quad \Phi^\dagger = -\phi U^\dagger(\pi)
\]

(20)

\[ U(\pi) = \exp \left( i \pi_a t_a \right) \]

followed by

\[
G_{\mu} \rightarrow G_{\mu}' = \frac{i}{g} U(\pi) D_{\mu} U^\dagger(\pi), \quad \zeta \rightarrow \zeta' = U(\pi)\zeta, \quad \zeta^\dagger \rightarrow \zeta'^\dagger = \zeta^\dagger U^\dagger(\pi)
\]

(21)

which is the form of a (unitary) gauge transformation. Using the result (39)

\[
\langle Q \rangle = Z^{-1} \int [dU][\det (J(\phi))][d\phi][d\zeta][d\zeta^\dagger][dG][Q][G] \exp \left( i \int d^4 x L_{\text{eff}} \right)
\]

(22)

\[ Z = \int [dU][\det (J(\phi))][d\phi][d\zeta][d\zeta^\dagger][dG] \exp \left( i \int d^4 x L_{\text{eff}} \right) \]

where
\[ L_{\text{eff}} = L_{\text{YM}} + \text{tr}(\phi D^\dagger_\mu D^\mu \phi + \zeta^\dagger D^\dagger_\mu D^\mu \zeta) \] (23)

and

\[ J_{ab}(\phi) = D_{abc} \phi_c \] (24)

The integrand being independent of \( U(\pi) \) the volume of the gauge group, \( \int [dU] \), factors and cancels in the normalization, leaving

\[ <Q> = Z^{-1} \int [\det (J(\phi))d\phi][d\zeta][d\zeta^\dagger][dG] Q[G] \exp (i \int d^4 x L_{\text{eff}}) \] (25)

\[ Z = \int [\det (J(\phi))d\phi][d\zeta][d\zeta^\dagger][dG] \exp (i \int d^4 x L_{\text{eff}}) \]

Comparing our formulation to that of [8,9] one sees that the latter constitutes a non-linear realization of the chiral symmetry with the ghosts represented by pseudofermions.

**III. BRS SYMMETRY**

The essential content of the identity (8) is that \( G_\mu \) appears as an external source field, expanding in powers of \( G_\mu^a \) one observes that for each closed \( \Phi \)--loop there is a closed \( \zeta \)--loop with a relative minus sign from ghost statistics so they exactly cancel – including vacuum graphs – much as in a supersymmetry. In turn this suggests that our procedure possesses a type of BRS symmetry, and such is indeed the case as we now proceed to show. For the representation of the auxiliary fields used here the relevant (anti) BRS transformations generated by \((\bar{S})S\) are
\( S\Phi^\dagger = \zeta^\dagger \quad \bar{S}\Phi^\dagger = 0 \)

\( S\Phi = 0 \quad \bar{S}\Phi = -\zeta \)

\( S\zeta^\dagger = 0 \quad \bar{S}\zeta^\dagger = \Phi^\dagger \quad \) (26)

\( S\zeta = \Phi \quad \bar{S}\zeta = 0 \)

\( SG_\mu = 0 \quad \bar{S}G_\mu = 0 \)

Clearly \( S \) and \( \bar{S} \) are nilpotent but do not anticommute, rather

\[
(SS + \bar{S}S)(\Phi^\dagger, \Phi, \zeta^\dagger, \zeta) = (\Phi^\dagger, -\Phi, \zeta^\dagger, -\zeta)
\] (27)

Using

\[ S(XY) = (SX)Y \pm X(SY) \quad \bar{S}(XY) = (\bar{S}X)Y \pm X(\bar{S}Y) \] (28)

where the \(+\) \((-\) sign applies if \(X\) is ghost even (odd) one sees that

\[ L_{\text{aux}} = SW = \bar{S}W^* \] (29)

with

\[
W = \text{tr}(\Phi^\dagger D_\mu^\dagger D^\mu \zeta) \quad W^* = \text{tr}(\zeta^\dagger D_\mu^\dagger D^\mu \Phi).
\] (30)

Thus from nilpotency \( L_{\text{aux}} \) is (anti) BRS invariant and moreover a simple calculation shows the BRS invariance of the measure in (8). Then, for \( F \) any functional of the auxiliary fields and \( G_\mu \)

\[
\int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger] F \exp (i \int d^4x L_{\text{aux}}) = \\
\int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger] (F + \chi SF) \exp (i \int d^4x L_{\text{aux}})
\] (31)

where we have performed a BRS transformation with \( \chi \) a global Grassmann variable. It follows that

\[
0 = \int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger] (SF) \exp (i \int d^4x L_{\text{aux}})
\] (32)
From (32) we have an alternative proof of (3): let

\[ I[G] = \int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger] \exp \left( i \int d^4 x L_{\text{aux}} \right). \]  

(33)

By (29), differentiation of \( I[G] \) with respect to \( G_\mu \) produces a quantity of the form (32), i.e. \( \delta I[G]/\delta G_\mu = 0 \) which is to say \( I[G] \) is a constant that may be normalized to unity. Note from (23) that \( L_{\text{aux}} \) corresponds to a topological field theory of the Witten type [13].

Next we need to establish what becomes of this BRS symmetry under the change of variables (20), (21). We begin by defining the ‘BRS current’.

\[ C = U^\dagger(\pi)SU(\pi) \]  

(34)

\[ SU(\pi) = U(\pi)C, \quad SU^\dagger(\pi) = -CU^\dagger(\pi) \]

Then

\[ S(-\phi U^\dagger(\pi)) = -(S\phi)U^\dagger(\pi) + \phi CU^\dagger(\pi) = \zeta U^\dagger(\pi) \]  

(35)

or

\[ S\phi = -\zeta^\dagger + \phi C \]  

(36)

while

\[ S(U(\pi)\phi) = U(\pi)C\phi + U(\pi)(S\phi) \]  

(37)

so

\[ S\phi = -C\phi. \]  

(38)

Similarly

\[ S(\zeta^\dagger U^\dagger(\pi)) = (S\zeta^\dagger)U^\dagger(\pi) + \zeta^\dagger CU^\dagger(\pi)) = 0 \]

\[ S(U(\pi)\zeta) = U(\pi)C\zeta + U(\pi)(S\zeta)) = -U(\pi)\phi \]
yielding

\[ S\zeta^\dagger = -\zeta^\dagger C \]  

\[ S\zeta = -\phi - C\zeta \]  

Finally (note \(SG' = 0\) now)

\[ SG_\mu = S \left( -\frac{i}{g} U^\dagger(\pi)(\partial_\mu + igG'_\mu)U(\pi) \right) \]

\[ = -\frac{i}{g} (-CU^\dagger(\pi))(\partial_\mu + igG'_\mu)U(\pi) - \frac{i}{g} U^\dagger(\pi)(\partial_\mu + igG'_\mu)(U(\pi)C) \]

\[ = -CG_\mu + G_\mu C - \frac{i}{g} \partial_\mu C \]

so

\[ SG_\mu = -\frac{i}{g} [D_\mu, C] \]  

What is most crucial is that consistency between (36) and (38) requires

\[ \zeta^\dagger = \{\phi, C\} \]  

or in component form

\[ \zeta^\dagger_a = D_{abe} \phi e C_b = J_{ab}(\phi)C_b \]

where \(J\) is the same matrix whose determinant appears in the measure in (22). Thus, changing variables from \(\zeta^\dagger\) to \(C\)

\[ <Q> = Z^{-1} \int [dG][d\phi][d\zeta][dC][Q][G] \exp \left( i \int d^4x L_{\text{eff}} \right) \]

\[ Z = \int [dG][d\phi][d\zeta][dC] \exp \left( i \int d^4x L_{\text{eff}} \right) \]

\[ L_{\text{eff}} = L_{\text{YM}} + \text{tr}(\phi D_{\mu}^\dagger D_\mu \phi + \{\phi, C\} D_{\mu}^\dagger D^\mu \zeta) \]

The BRS transformations are now

\[ S\phi = -C\phi \]

\[ SC = -CC \]  

\[ S\zeta = -\phi - C\zeta \]  

\[ SG_\mu = -\frac{i}{g} [D_\mu, C] \]
where that for $C$ follows directly from the definition (34) and nilpotency of $S$ – it is a straightforward exercise to show the nilpotency of the BRS transformations (45) and that

$$L_{\text{eff}} = L_{YM} + \text{Str}(-\phi D_{\mu}^\dagger D^\mu \zeta).$$ 

As $SG_\mu$ has the form of a gauge transformation BRS invariance of $L_{\text{eff}}$ is immediate. Moreover, again a simple calculation demonstrates that the measure in (44) is BRS invariant; notable in this regard is that the $G_\mu$ and $C$, $\phi$ and $\zeta$ contributions to the superdeterminant cancel pairwise. In turn this is understood in that the change of variables (20), (21) absorbs the auxiliary field $\pi$ into the gauge field.

A parallel development may be made for the fate of the anti BRS symmetry. With $U^\dagger(\pi)\bar{SU}(\pi) \equiv C^\dagger$:

$$\bar{S}\phi = \phi C^\dagger = \zeta - C^\dagger\phi$$
$$\bar{S}C^\dagger = -C^\dagger C^\dagger$$
$$\bar{S}\zeta^\dagger = -\phi - \zeta^\dagger C^\dagger$$
$$\bar{S}\zeta = -C^\dagger \zeta$$
$$\bar{S}G_\mu = -\frac{i}{g}[D_\mu, C^\dagger]$$

Substituting $(\zeta^\dagger)$ $\zeta$ in terms of $\phi$ and $(C) C^\dagger$ in the corresponding (anti) BRS transformations one obtains

$$\{\phi, SC^\dagger\} + \{C, C^\dagger\} \phi = -\phi = \{\phi, \bar{S}C\} + \phi \{C, C^\dagger\}$$

so

$$\{\phi, SC^\dagger + \bar{S}C + CC^\dagger + C^\dagger C\} = -2\phi$$

with solution
\[ SC^\dagger + \bar{S}C + \{C, C^\dagger\} = -I \quad (51) \]

or in component form

\[ SC_a^\dagger + \bar{S}C_a + i f_{abc} C_b C_c^\dagger = -\sqrt{2N} \delta_{a0} . \quad (52) \]

It is then easy to verify that the BRS and anti BRS transformations now do satisfy both nilpotency and anti commutation

\[ S^2 = SS + \bar{S}S = \bar{S}^2 = 0 \quad (53) \]

as well as that

\[ L_{\text{eff}} = L_{YM} + \bar{S} \text{tr}(\phi D_\mu^\dagger D^\mu \phi) . \quad (54) \]

The essential difference between our BRS algebra and the usual one \[ \text{[6]} \] lies in the nonvanishing right hand side of \[ (51) \]

Transformation from the \( \zeta \) to \( C^\dagger \) ghost in \[ (44) \] leads to the occurrence of \( 1/\det(J(\phi)) \) in the measure while exacerbating the problem of higher than quartic vertices in \( L_{\text{eff}} \), however, and thus we implement only the BRS symmetry in what follows.

**IV. EXTENSIONS**

The identity \[ \text{[8]} \] is not the most general such we could write down; any \( L_{aux} \) possessing the chiral symmetry \[ \text{[12], [13]} \] and the BRS symmetry \[ \text{[26]} \] will do. If on the other hand, we impose the condition of perturbative renormalizability (in the sense of orthodox gauge fixing) and require that \( L_{aux} \) retain the scale invariance of \( L_{YM} \) then the most general form is

\[ 1 = \int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger] \exp \left( i \int d^4x L_{aux} \right) , \]

\[ L_{aux} = \text{tr} \left( \Phi^\dagger D_\mu^\dagger D^\mu \Phi + \zeta^\dagger D_\mu^\dagger D^\mu \zeta - \frac{\lambda_1}{N}(\Phi^\dagger \Phi + \zeta^\dagger \zeta)^2 \right) - \frac{\lambda_2}{N^2}(\text{tr}(\Phi^\dagger \Phi + \zeta^\dagger \zeta))^2 \quad (55) \]
This may be proven by uncompleting the square: let

\[ I[G] = N^{-1} \int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger][d\Sigma] \exp \left( i \int d^4x (\tilde{L}_{\text{aux}} + L_{\Sigma}) \right), \]

\[ N = \int [d\Sigma] \exp \left( i \int d^4x L_{\Sigma} \right), \]

\[ \tilde{L}_{\text{aux}} = \text{tr}(\Phi^\dagger D^\mu \Phi + \zeta^\dagger D^\mu \zeta - (\Phi^\dagger \Phi + \zeta^\dagger \zeta) \Sigma), \quad \Sigma = \Sigma_a t_a, \]

\[ L_{\Sigma} = \frac{N}{4\lambda_1} \text{tr}(\Sigma^2) - \frac{1}{4\lambda_1} \frac{\lambda_2}{\lambda_1 + 2} (\text{tr} \Sigma)^2 \]

Integrating out the \( \Sigma \)-field gives

\[ I[G] = \int [d\Phi][d\Phi^\dagger][d\zeta][d\zeta^\dagger] \exp \left( i \int d^4x L_{\text{aux}} \right) \]

with \( L_{\text{aux}} \) as in (55). Conversely, integrating out the auxiliary fields \( \Phi, \Phi^\dagger, \zeta \) and \( \zeta^\dagger \) first, with now

\[ M_{ab}(x, y) = -(D^\mu D_{\mu})_{ab} + \Sigma_a (T^a_{\mu})_{\mu} \delta^{(4)}(x - y) \]

yields

\[ I[G] = N^{-1} \int [d\Sigma] \frac{\text{det}(M)}{\text{det}(M)} \exp \left( i \int d^4x L_{\Sigma} \right) = 1 \]

Alternately one may observe that

\[ L_{\text{aux}} = SW, \]

\[ W = \text{tr} \left( \Phi^\dagger D^\mu \Phi - \frac{\lambda_1}{N} \text{tr}(\Phi^\dagger \Phi S(\Phi^\dagger \Phi)) \right) - \frac{\lambda_2}{N^2} \text{tr}(\Phi^\dagger \zeta) \text{Str}(\Phi^\dagger \zeta) \]

so (32) holds here also: differentiation of \( I[G] \) by \( G^a_{\mu} \), \( \lambda_1 \) and \( \lambda_2 \) yields a vanishing quantity, \( I[G] \) thereby being a constant which may be normalized to unity. Further, clearly, one may again follow the steps leading to (44), but now with

\[ L_{\text{eff}} = L_{\text{YM}} + SW, \]

\[ W = \text{tr} \left( -\phi D^\mu \zeta - \frac{\lambda_1}{N} \phi \zeta S(\phi \zeta) \right) - \frac{\lambda_2}{N^2} \text{tr}(\phi \zeta) \text{Str}(\phi \zeta) \]
Due to the BRS invariance of $L_{\text{eff}}$ and the measure in (44) we have analogous to (32)

$$0 = \int [dG][d\phi][d\zeta][dC](SF) \exp \left( i \int d^4x L_{\text{eff}} \right)$$ (62)

for $F$ any functional of $G_{\mu}$ and the auxiliary fields $\phi$, $\zeta$ and $C$. From (61) and (62)

$$\frac{\partial}{\partial \lambda_1} Z = \frac{\partial}{\partial \lambda_2} Z = 0$$ (63)

One should take care to note that the independence of the ‘partition function’ $Z$ and $<O>$ from $\lambda_1$ and $\lambda_2$ does not mean that we should blindly set $\lambda_1 = \lambda_2 = 0$. Indeed, a cursory examination of the system $L_{YM} + L_{\text{aux}}$ treated with perturbative gauge fixing demonstrates the generation of chiral invariant quartic interactions among the auxiliary fields which require the additional terms in (55) for their renormalization.

Thus far we have restricted ourselves to pure Yang–Mill theory, but for applications to, in particular QCD, and also for the purpose of discussions to follow we need the extension of our formalism to SU($N$) gauge fields coupled to matter. Consider therefore

$$L = L_{YM} + \bar{\psi}(i\gamma^\mu D^\mu - m)\psi$$ (64)

where $L_{YM}$ is as in (1), and

$$D^\mu = \partial^\mu + igG^\mu$$ (65)

with $m$ the (bare) fermion mass or, for $N_f$ flavors, the (diagonal) mass matrix.

This lagrangian is invariant under (2) together with

$$\psi \rightarrow U\psi, \quad \bar{\psi} \rightarrow \bar{\psi}U^\dagger, \quad U \in \text{U}(N)/\text{U}(1)$$ (66)

The gauge invariant observables to be considered are $O = O[G, \bar{\psi}, \psi]$,
\[ < Q > = Z^{-1} \int [d\psi][d\bar{\psi}][dG][\mathcal{O}[G], \bar{\psi}, \psi] \exp (i \int d^4x L), \]
\[ Z = \int [d\psi][d\bar{\psi}][dG] \exp (i \int d^4x L) \]

Supplementing the BRS transformations (26) by
\[ S\psi = S\bar{\psi} = 0 \]
and the change of variables (21) with
\[ \psi \rightarrow \psi' = U(\pi)\psi, \quad \bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}U^\dagger(\pi), \]
\[ U(\pi) = \exp (i\pi_{\mathbf{a}}t_{\mathbf{a}}) \]

one straightforwardly arrives at
\[ < Q > = Z^{-1} \int [d\psi][d\bar{\psi}][d\phi][d\zeta][dC][\mathcal{O}[G], \bar{\psi}, \psi] \exp (i \int d^4x L_{\text{eff}}), \]
\[ Z = \int [d\psi][d\bar{\psi}][d\phi][d\zeta][dC] \exp (i \int d^4x L_{\text{eff}}) \]

where
\[ L_{\text{eff}} = L + SW \]

and
\[ W = \text{tr} \left( -\phi D^\mu D_\mu \zeta - \frac{\lambda_1}{N} \phi \zeta S(\phi \zeta) \right) - \frac{\lambda_2}{N^2} \text{tr}(\phi \zeta)S\text{tr}(\phi \zeta) \]

while the BRS transformations are
\[ S\phi = -C\phi \]
\[ SC = -CC \]
\[ S\zeta = -\phi - C\zeta \]
\[ SG_\mu = -\frac{i}{g} [D_\mu, C] \]
\[ S\psi = -C\psi, \quad \bar{C} = C_{\mathbf{a}}t_{\mathbf{a}} \]
\[ S\bar{\psi} = \bar{\psi}\bar{C}. \]
(note the matter fields are ghost even). By BRS invariance of $L_{\text{eff}}$ and the
measure in (70)

$$0 = \int [d\psi][d\bar{\psi}][dG][d\phi][d\zeta][dC](SF) \exp \left(i \int d^4x L_{\text{eff}}\right)$$

(74)

for any functional $F$ of $\psi, \bar{\psi}, G_\mu$ and the auxiliary fields $\phi, \zeta$ and $C$.

V. RENORMALIZATION

The intrinsic BRS symmetry of our quantization prescription imposes
some important constraints on the renormalizations which will be needed
below to deal with infinities. In particular, preservation of (26) for the renormalized fields requires a common wavefunction renormalization constant:

$$(\Phi_B^\dagger, \Phi_B, \zeta_B^\dagger, \zeta_B) = (\tilde{Z})^{1/2}(\Phi_R^\dagger, \Phi_R, \zeta_R^\dagger, \zeta_R)$$

(75)

wherein $B(R)$ denotes bare (renormalized). Also, in view of the remarks in section 4, we write for the renormalization of the quartic couplings

$$\lambda_B^i = \delta \lambda_i / (\tilde{Z})^2,$$

(76)

i.e., we hold the renormalized coupling constants $\lambda_R^i$ to vanish.

For the gauge field and its coupling we take

$$G_B^{\mu} = (Z_3)^{1/2} G_R^{\mu}$$

(77)

$$g_B = Z g_R$$

Carrying out the transformations (21), (22) with $U_B(\pi) = U_R(\pi) = U(\pi)$
we have

$$\phi_B = (\tilde{Z})^{1/2} \phi_R$$

(78)

$$C_B = C_R = C$$
Note the non-renormalization of $C$, as follows from its definition (34) as a symmetry current. Defining the renormalized covariant derivative

$$D_{R\mu} = \partial_\mu + iZ_g\sqrt{Z_3}g_RG_{R\mu}$$

one immediately obtains the renormalized BRS transformations

$$S\phi_R = -C\phi_R$$
$$SC = -CC$$

$$S\zeta_R = -\phi_R - C\zeta_R$$
$$SG_{R\mu} = -\frac{i}{Z_g\sqrt{Z_3}}g_R[D_{R\mu},C]$$

That (80) are indeed the BRS symmetry transformations of

$$L_B^{\text{eff}} = -\frac{Z_3}{2}\text{tr}(G_{R\mu\nu}^2) + \tilde{Z}\text{tr}(\phi_RD_{R\mu}^\dagger D_{R\mu}\phi_R + \{\phi_R, C\}D_{R\mu}^\dagger D_{R\mu}\zeta_R) - \frac{\delta\lambda_1}{N}\text{tr}((\phi_R^2 + \{\phi_R, C\}\zeta_R)^2) - \frac{\delta\lambda_2}{N^2}(\text{tr}(\phi_R^2 + \{\phi_R, C\}\zeta_R))^2 ,$$

for the pure gauge theory is by inspection.

Including fermions, we need the additional field and mass renormalizations

$$(\bar{\psi}_B, \psi_B) = (Z_2)^{1/2}(\bar{\psi}_R, \psi_R)$$

$$m_B = m_RZ_m/Z_2$$

and then

$$L_B^{\text{eff}} = \bar{\psi}_R(Z_2i\gamma^\mu D_{R\mu} - m_RZ_m)\psi_R - \frac{Z_3}{2}\text{tr}(G_{R\mu\nu}^2) + SW_B ,$$

$$W_B = \tilde{Z}\text{tr}(-\phi_RD_{R\mu}^\dagger D_{R\mu}\zeta_R) - \frac{\delta\lambda_1}{N}\text{tr}(\phi_R\zeta_RS(\phi_R\zeta_R)) - \frac{\delta\lambda_2}{N^2}\text{tr}(\phi_R\zeta_R)\text{Str}(\phi_R\zeta_R)$$

is invariant under the BRS transformations (80) together with

$$S\psi_R = -C\psi_R, \quad S\bar{\psi}_R = \bar{\psi}_RC.$$
Per say, $Z_3$ and $Z_g$ are independent, however, it emerges below that the peculiarities of perturbation theory in our approach imply

$$Z_g \sqrt{Z_3} = 1 \quad (85)$$

as occurs in the background field gauge [14]. With this, dropping the subscript $R$, the effective lagrangian becomes

$$L_{\text{eff}}^B = L_{\text{eff}} + \delta L_{\text{eff}} \quad (86)$$

wherein

$$L_{\text{eff}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2} \text{tr}(G_{\mu\nu}^2) + \text{tr}(\phi D_\mu^\dagger D^\mu\phi + \{\phi, C\} D_\mu^\dagger D^\mu\zeta) \quad (87)$$

$$\delta L_{\text{eff}} = \bar{\psi}((Z_2 - 1)i\gamma^\mu D_\mu - (Z_m - 1)m)\psi - \frac{(Z_3 - 1)}{2} \text{tr}(G_{\mu\nu}^2) +$$

$$(\tilde{Z} - 1)\text{tr}(\phi D_\mu^\dagger D^\mu\phi + \{\phi, C\} D_\mu^\dagger D^\mu\zeta) -$$

$$\frac{\delta \lambda_1}{N} \text{tr}(\phi^2 + \{\phi, C\}\zeta)^2 - \frac{\delta \lambda_2}{N^2} (\text{tr}(\phi^2 + \{\phi, C\}\zeta))^2 \quad (88)$$

while the BRS transformations remain those of (73).

VI. PERTURBATION THEORY

Albeit we have succeeded in our objective of factoring the volume of the gauge group from $< O >$, we would also appear to have painted ourselves into the proverbial corner in that an inspection of $L_{\text{eff}}$ shows that (i) the term quadratic in the gauge field involves the transverse projection operator $(g_{\mu\nu} - \Box^{-1}\partial_\mu\partial_\nu)$ so the corresponding propagator does not exist, and (ii) there is no kinetic term for the ghosts while $L_{\text{eff}}$ is a polynomial higher than quartic in the fields. Actually the latter is an unfortunate byproduct of our transformation from $\zeta^\dagger$ to $C$ and can be avoided by retaining $\zeta^\dagger$ and $\zeta$ as the ghosts.

In contrast, the problem with the gauge field is profound; to redress this situation suppose
Decomposing φ as
\[ \phi = \phi + vI = \phi_a t_a + vI \] (90)
and denoting
\[ m_G = gv \] (91)
one has
\[
L_{\text{eff}} = L + m_G^2 \text{tr}(G_\mu G^\mu) + \text{tr}(\phi D_\mu^\dagger D^\mu \phi) + 2gm_G \text{tr}(G_\mu G^\mu \phi) + 2v \text{tr}(C D_\mu^\dagger D^\mu \zeta) + \text{tr}((\{\phi, C\} D_\mu^\dagger D^\mu \zeta) .
\] (92)

Now the gauge field propagator exists as the (unitary gauge) propagator of a massive vector field
\[
i D_{\mu \nu}^{ab}(k) = \frac{i \delta_{ab}}{k^2 - m_G^2} \left[ -g_{\mu \nu} + \frac{k_\mu k_\nu}{m_G^2} \right]
\] (93)
and we have an identifiable, if unconventional, kinetic term for the ghosts.

Since for a BRS invariant vacuum \(0 = \langle SF \rangle = S \langle F \rangle\), (89) requires \(Sv = 0\). Then \(L_{\text{eff}}\) and the measure remain invariant under the amended BRS transformations
\[
Sv = 0 \quad S\phi = -\nu C - C\phi \\
S C = -CC \quad S\zeta = -\phi - vI - C\zeta \quad (94)
\]
\[
SG_\mu = -\frac{i}{g}[D_\mu, C] \\
S\bar{\psi} = -C\psi , \quad S\bar{\psi} = \bar{\psi}C
\]

That we may assign a vacuum expectation value to the scalar field while leaving (a version of) the BRS invariance intact should come as no surprise
since a similar situation exists in supersymmetric theories. Crucial here is that from the BRS identity \(0 = \text{Str} < (v - \phi)\zeta >\) one obtains

\[ Nv^2 = \text{tr} < \phi^2 + \{\phi, C\} \zeta > \tag{95} \]

as the right hand side is of order \(\hbar\) the tree–level breaking of the gauge symmetry is ruled out.

Of course this does not deny the possibility of a dynamical mechanism for generating \(v\), one such being that of Coleman and Weinberg \([15]\): injecting in (88) leads to

\[
\delta L_{\text{eff}} = (\tilde{Z} - 1)m_G^2 \text{tr}(G \mu G^\mu) + (\delta \lambda_1 + \delta \lambda_2) v^4 - \\
\left(6 \frac{\delta \lambda_1}{N} v^2 + 2 \frac{\delta \lambda_2}{N} v^2 \right) \text{tr} \phi^2 - \left(4 \frac{\delta \lambda_2}{N} v^2 \right) \frac{1}{N} (\text{tr} \phi)^2 - \\
\left(2 \frac{\delta \lambda_1}{N} v^2 + 2 \frac{\delta \lambda_2}{N} v^2 \right) \text{tr}(2vC\zeta) + \ldots \tag{96} \]

Using \(n(= 4 - 2\epsilon)\)–dimensional regularization, finiteness of the one–loop scalar self energy, Figure 1a, at zero momentum gives the modified minimal subtraction constants

\[ \delta \lambda_1 = \delta \lambda_2 = 3 N^2 g^4 \frac{1}{128\pi^2} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right) \tag{97} \]

which also yield a finite zero momentum ghost self energy, Figure 1b, and a finite effective potential for \(v\)

\[ V_{\text{eff}}(v) = \frac{3N^2 g^4}{64\pi^2} v^4 \left[\ln \left(\frac{g^2 v^2}{\mu^2}\right) - \frac{5}{6}\right], \tag{98} \]

\(\mu\) being the dimensional regularization scale. This effective potential has a minimum for \(\ln(m_G^2/\mu^2) = \frac{1}{3}\) which gives the scalar mass matrix

\[ (m^2_\phi)_{ab} = \frac{3Ng^2}{32\pi^2} m_G^2 (\delta_{ab} + \delta_{a0}\delta_{b0}) \tag{99} \]

while the ghost mass matrix vanishes. We observe that the \(\delta \lambda_i\) in (97) coincide with the values obtained by applying Landau gauge field perturbation theory to \(L_{\text{YM}} + L_{\text{aux}}\). By matching the vacuum energy density
\[ \epsilon_{\text{vac}} = -\frac{3N^2}{128\pi^2}m_G^4 \]  

(100)

to the canonical pure gauge expression \[\tag{101}\]

\[ \epsilon_{\text{vac}} = \left\langle \frac{\beta(g)}{8g}G \cdot G \right\rangle \]

and using the one-loop $\beta$–function the resulting gluon mass in QCD can be estimated as

\[ m_G^4 \approx \frac{44\pi^2}{9N}\left\langle \frac{\alpha_s}{\pi} G \cdot G \right\rangle \]  

(102)

With $\left\langle \frac{\alpha_s}{\pi} G \cdot G \right\rangle \approx (330 \text{ MeV})^4$ from QCD sum rules \[\tag{107}\] one finds $m_G \approx 660$ MeV. Of course, since the corresponding renormalization scale is $\mu = m_G e^{-1/6} \approx 560 \text{MeV}$ we should take this perturbative calculation with a large grain of salt, yet it is intriguing to note that a relation almost identical to (102) has been obtained by Lavelle \[\tag{108}\] from the operator product expansion while a value $m_G = 660 \pm 80 \text{MeV}$ for the effective gluon mass has been extracted by Consoli and Field \[\tag{109}\] from a study of charmonium decay.

VII. RENORMALIZED GAUGE FIELD PROPOGATOR

Next we face the problems posed by the gauge field propagation \[\tag{93}\] which is order 1 by power counting and through its troublesome longitudinal part mixes orders in $g$. Thus, for example, the gauge field contributions to the gauge field self energy, Figure 2, are

\[ i\Pi_{G\mu
u}^{ab}(q) = \frac{Nq^2}{2} \delta_{ab} \mu^2 \int \frac{d^n \ell}{(2\pi)^n} \left\{ \left( n - 1 - \frac{q^2}{m_G^2} \right) \left[ \frac{(2\ell + q)_\mu (2\ell + q)_\nu}{(\ell^2 - m_G^2)((\ell + q)^2 - m_G^2)} - \frac{2g_{\mu\nu}}{\ell^2 - m_G^2} \right] + 2\left( 4 - \frac{q^2}{m^2} \right) \frac{(q^2 g_{\mu\nu} - q_{\mu} q_{\nu})}{(\ell^2 - m_G^2)((\ell + q)^2 - m_G^2)} + \frac{(\ell_{\mu} q_{\nu}^2 - q_{\mu} \ell \cdot q)(\ell_{\nu} q_{\mu}^2 - q_{\nu} \ell \cdot q)}{m_G^4(\ell^2 - m_G^2)((\ell + q)^2 - m_G^2)} \right\} \]  

(103)

While transverse, the pole part...
\[ \Pi_{G}^{ab}(q) = \frac{Nq^2 \delta_{ab}}{32\pi^2} \epsilon(q^2 g_{\mu\nu} - q_{\mu}q_{\nu}) \left[ 7 - \frac{7}{6} \frac{q^2}{m_G^2} - \frac{1}{12} \left( \frac{q^2}{m_G^2} \right)^2 \right] + \text{finite} \]  

(104)
cannot be cancelled by a local counterterm.

All is not lost, however; the gauge field propagation does not belong to the class \( \mathcal{Q} \). Consider instead the correlator of fermionic scalar currents

\[ G(x,y) = \langle \bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y) \rangle \]  

(105)
which does belong to \( \mathcal{Q} \). To order \( g^2 \) the diagrams contributing to \( G \) are given in Figure 3 and it is straightforward to show that the contributions due to the longitudinal part of \( \Pi^{ab} \) cancel – indeed one may there replace the unitary gauge propagator by

\[ iD_{T,\mu\nu}^{ab}(k) = \frac{i\delta_{ab}}{k^2 - m_G^2} \left[ -g_{\mu\nu} + \frac{k_{\mu}k_{\nu}}{k^2} \right]. \]  

(106)

Now define

\[ G_{\infty}(x,y) = \lim_{x_0 \to +\infty, y_0 \to -\infty} G(x,y) \]  

(107)
which clearly also belongs to \( \mathcal{Q} \). Using

\[ iS_F(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{E} \sum_{\lambda} \left[ \theta(x_0 - y_0)u(p,\lambda)\bar{u}(p,\lambda)e^{-ip(x-y)} - \theta(y_0 - x_0)v(p,\lambda)\bar{v}(p,\lambda)e^{-ip(x-y)} \right] \]  

(108)
one sees that in \( G_{\infty}(x,y) \) the fermion lines adjacent to \( x \) and \( y \) are driven on–shell, i.e., \( G_{\infty}(x,y) \) is the form of a convolution with an S-matrix element.

The order \( g^2 \) graphs contributing to \( G_{\infty}(x,y) \) are those of Figure 4 for which the replacement \( \Pi^{ab} \) holds by current conservation. Indeed the fermion self energy parts can be exactly cancelled by the fermion mass counterterm, leaving the first or exchange diagram of Figure 4 with \( \Pi^{ab} \) representing the effective gauge field propagation.

Of course the above example is somewhat trivial in that to order \( g^2 \) \( G(x,y) \) and \( G_{\infty}(x,y) \) are abelian in character and as is well known an abelian gauge
field mass is harmless. The important point is that the analysis extends to order $g^4$ where nonabelian aspects appear. The order $g^4$ exchange diagrams contributing to $G_\infty(x,y)$ are those of Figure 5. Note that Figure 5(d) contains Figure 2 as a subgraph. When the dangerous longitudinal part of the unitary gauge propagator (93) is contracted at a gauge–fermion vertex in Figure 5(a–c) it triggers the simple Ward identity

$$\gamma^\mu k_\mu = S^{-1}_F(k+p) - S^{-1}_F(p)$$  \hspace{1cm} (109)$$

to cancel an internal fermion propagator. The triple gauge vertex has longitudinal parts which do the same thing; this is the essence of the ‘pinch technique’ of Cornwall [12], and the pinch diagrams corresponding to Figure 5(a–c) are given in Figure 6 (a–c). Omitting trivial external factors one finds for the pinch part,

\begin{align*}
6 \text{(a)} : & \frac{N}{2} g^2 \delta_{ab} \mu^\lambda \frac{d^n \ell}{(2\pi)^n} \frac{1}{(\ell^2 - m^2_G) [((\ell + q)^2 - m^2_G) - 2g_{\mu\nu} \ell^2]} \left\{ \frac{\ell_\nu}{m^4_G} g_\lambda^\mu - \frac{2g_{\mu\nu}}{m^2_G} \right\} \\
6 \text{(b)} : & \frac{N}{2} g^2 \delta_{ab} \mu^\lambda \frac{d^n \ell}{(2\pi)^n} \left\{ \frac{\ell_\mu (q^2 \ell_\alpha - q_\alpha \ell \cdot q)}{m^4_G (\ell^2 - m^2_G) [((\ell + q)^2 - m^2_G)]} - \frac{\ell_\mu (2\ell + q)_\alpha + 2q^2 g_{\mu\alpha}}{m^2_G (\ell^2 - m^2_G) [((\ell + q)^2 - m^2_G)]} + \frac{g_{\mu\alpha}}{m^2_G (\ell^2 - m^2_G)} \right\} D^\alpha_G(q)
\end{align*}

while that of Figure 6(c) obtains from 6(b) by $\mu \leftrightarrow \nu$. Appealing to current conservation we arrive at the transverse ‘pinch contribution’ to the gauge field self energy:

\begin{align*}
i \Pi^{th}_{F\mu\nu}(q) = & \frac{N g^2}{2} \delta_{ab} \mu^\lambda \frac{d^n \ell}{(2\pi)^n} \left\{ \frac{q^2}{m^2_G} - 1 \right\} \left[ \frac{(2\ell + q)_\mu (2\ell + q)_\nu}{((\ell^2 - m^2_G) [((\ell + q)^2 - m^2_G)]} - \frac{2g_{\mu\nu}}{\ell^2 - m^2_G} \right] + \\
& 2 \left( \frac{q^2}{m^2_G} - \frac{m_G^2}{q^2} \right) \frac{(q^2 g_{\mu\nu} - q_\mu q_\nu)}{(\ell^2 - m^2_G) [((\ell + q)^2 - m^2_G)]} + \\
& \left\{ \frac{1}{(q^2)^2} - \frac{1}{m^4_G} \right\} \frac{((\ell + q)^2 - m^2_G)}{(\ell^2 - m^2_G) [((\ell + q)^2 - m^2_G)]} \right\} D^\alpha_G(q)
\end{align*}

(111)

The sum of (108) and (111) is
\[ i\Pi^{ab}_{G+P\mu\nu}(q) = \frac{N g^2}{2} \delta_{ab} \mu^{2\epsilon} \int \frac{d^n \ell}{(2\pi)^n} \left\{ (n - 2) \left[ \frac{(2\ell + q)_\mu(2\ell + q)_\nu}{(\ell^2 - m_G^2)((\ell + q)^2 - m_G^2)} - \frac{2g_{\mu\nu}}{\ell^2 - m_G^2} \right] + 2 \left( 4 - \frac{m_G^2}{q^2} \right) \frac{(q^2 g_{\mu\nu} - q_\mu q_\nu)}{(\ell^2 - m_G^2)((\ell + q)^2 - m_G^2)} + \frac{(\ell_\mu q^2 - q_\mu \ell \cdot q)(\ell_\nu q^2 - q_\nu \ell \cdot q)}{(q^2)^2(\ell^2 - m_G^2)((\ell + q)^2 - m_G^2)} \right\} \] (112)

Still to be included are the ghost and scalar contributions, Figure 7. The ghost part of the gauge field self-energy is wholly transverse

\[ i\Pi^{ab}_{\xi\mu\nu}(q) = -\frac{N g^2}{2} \delta_{ab} \mu^{2\epsilon} \int \frac{d^n \ell}{(2\pi)^n} \left\{ \frac{\ell_\mu \ell_\nu - m_G^2 g_{\mu\nu}}{(\ell + q)^2(\ell^2 - m_G^2)} - \frac{1}{8} \frac{(2\ell + q)_\mu(2\ell + q)_\nu}{\ell^2(\ell + q)^2} \right\} \] (113)

but the scalar part is not

\[ i\Pi^{ab}_{\delta\mu\nu}(q) = N g^2 \delta_{ab} \mu^{2\epsilon} \int \frac{d^n \ell}{(2\pi)^n} \left\{ \frac{\ell_\mu \ell_\nu - m_G^2 g_{\mu\nu}}{(\ell + q)^2(\ell^2 - m_G^2)} - \frac{1}{8} \frac{(2\ell + q)_\mu(2\ell + q)_\nu}{\ell^2(\ell + q)^2} \right\} \] (114)

On the other hand owing to current conservation at the gauge–fermion vertex we only need the transverse projection

\[ i\Pi^{ab}_{ST\mu\nu}(q) = N g^2 \delta_{ab} \mu^{2\epsilon} \int \frac{d^n \ell}{(2\pi)^n} \left\{ \frac{-m_G^2}{q^2} \frac{(q^2 g_{\mu\nu} - q_\mu q_\nu)}{(\ell + q)^2(\ell^2 - m_G^2)} + \frac{(\ell_\mu q^2 - q_\mu \ell \cdot q)(\ell_\nu q^2 - q_\nu \ell \cdot q)}{(q^2)^2(\ell + q)^2(\ell^2 - m_G^2)} + \frac{1}{8} \frac{(2\ell + q)_\mu(2\ell + q)_\nu}{\ell^2(\ell + q)^2} \right\} . \] (115)

The sum of (112), (113) and (115) gives the effective gauge field self-energy

\[ \Pi^{ab}_{\mu\nu}(q) = \Pi^{ab}_{G+P\mu\nu}(q) + \Pi^{ab}_{\xi\mu\nu}(q) + \Pi^{ab}_{ST\mu\nu}(q) = \delta_{ab}(q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2) \] (116)

and one finds

\[ \Pi(q^2) = \frac{N g^2}{16\pi^2} \left( \frac{11}{3} - \frac{3 m_G^2}{2 q^2} \right) \left( \frac{1}{\epsilon - \gamma + \ln 4\pi} \right) + \text{finite} . \] (117)

Finally there are the field and gauge mass counterterms implied by (88) and (96) respectively – for the latter we again need the transverse projection so

\[ \delta\Pi(q^2) = (\tilde{Z} - 1) \frac{m_G^2}{q^2} - (Z_3 - 1) \] (118)
Thus we see that

$$\Pi_R(q^2) = \Pi(q^2) + \delta \Pi(q^2)$$  \hspace{1cm} (119)

is rendered finite for

$$\check{Z} = 1 + \frac{3Ng^2}{32\pi^2} \left( \frac{1}{\epsilon} - \gamma + \ln 4\pi \right), \hspace{1cm} (120)$$
$$Z_3 = 1 + \frac{11Ng^2}{48\pi^2} \left( \frac{1}{\epsilon} - \gamma + \ln \pi \right), \hspace{1cm} (121)$$

and so is the effective one-loop renormalized gauge field propagator

$$iD_{\mu\nu}^{ab}(k) = \frac{i\delta^{ab}}{k^2[1 + \pi_R(k^2)] - m_G^2} \left[ -g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right]; \hspace{1cm} (122)$$

this is our principle perturbative result. We note that as for $\delta \lambda_1$ and $\delta \lambda_2$, $\check{Z}$ agrees with the value obtained from perturbative Landau gauge fixing applied to $L_{YM} + L_{aux}$.

Further, as promised, via (85) $Z_3$ of (121) contains the full information regarding the one-loop gauge coupling renormalization. This feature may be traced to the role of the longitudinal pieces of the triple gauge vertex in the pinch program; an inspection of the calculation in [12] shows that in gauge fixed perturbation theory the gauge invariant proper vertices laboriously constructed from the pinch technique coincide with what is obtained through the background field method of Abbott [14].

Indeed, from a strict perturbative viewpoint – i.e. leaving aside possible dynamical mechanisms for generating $\nu \neq 0$ – one could proceed from the minimal identity of section 2, introducing $\nu$ as in (90) through (94) with the meaning of a ‘gauge parameter’ to define the intermediate steps with the limit $\nu \to 0$ (after pinching to remove the pieces singular as $\nu \to 0$) understood. Taking $m_G \to 0$ in (116)

$$\Pi(q^2) \longrightarrow -i \frac{Ng^2}{2} \left[ 8 - \left( \frac{n - 2}{n - 1} \right) \right] \mu^{2\epsilon} \int \frac{d^d \ell}{(2\pi)^n} \frac{1}{\ell^2(\ell + q)^2} \hspace{1cm} (123)$$
\[ \Pi_R(q^2) \to \frac{N g^2}{16 \pi^2} \left[ -\frac{11}{3} \ln \left( \frac{-q^2}{\mu^2} \right) + \frac{67}{9} \right] \]  

which are canonical to the background field gauge.

Another perspective is gained by observing that because the amended BRS symmetry (94) is non-intrinsic \( S \) and \( \frac{\partial}{\partial v} \) do not commute, but rather

\[ \frac{\partial}{\partial v} SW = S \frac{\partial}{\partial v} W - 2i g \text{tr}(G^\mu \partial_\mu \phi) \]

there follows, integrating by parts,

\[ \frac{\partial}{\partial v} \ln Z = - < g \int d^4 x \phi_a \partial \cdot G^a > , \]

\[ \frac{\partial}{\partial v} < O > = < O > < g \int d^4 x \phi_a \partial \cdot G^a > - < O > g \int d^4 x \phi_a \partial \cdot G^a > \]

and in the perturbative regime where we can apply ordinary gauge fixing in the form \( \partial \cdot G = 0 \) both \( \partial \ln Z/\partial v \) and \( \partial < O >/\partial v \) vanish. Moreover, even when mass is dynamically generated through the Coleman-Weinberg mechanism as discussed in the preceding section \( \partial \ln Z/\partial v \) must vanish since \( \epsilon_{\text{vac}} \) is an extremum; one readily verifies that the diagrams of figure 8 give a null result for the right hand side of (126).

\section*{VIII. DISCUSSION AND CONCLUSIONS}

We have shown how a SU(N) gauge theory can be quantized in the continuum without fixing the gauge. In this way the Gribov problem is circumvented. The underlying BRS symmetry has been identified and the associated Slavnov-Taylor identities can be derived in the usual way. We used this BRS symmetry to perform the perturbative renormalization of the effective theory and showed that in the perturbative regime the procedure is equivalent to Landua gauge fixing. All the usual perturbative results can therefore be recovered.
IX. ACKNOWLEDGEMENTS

This work was supported by a grant from the Foundation of Research Development.
APPENDIX A:

Herein we collect some notation and useful relations for the groups $\text{U}(N)$ and $\text{SU}(N) \subset \text{U}(N)$. In the fundamental representation the $\text{U}(N)$ generators are denoted

$$t_0 = 1/\sqrt{2N}, \quad t_a = \lambda_a/2$$

where (non) underlined indices run from (0) 1 to $N^2 - 1$; the $\lambda_a$’s are the $\text{SU}(N)$ generalizations of the Gell–Mann matrices, $\text{tr}(\lambda_a \lambda_b) = 2 \delta_{ab}$, so the $t_a$’s are normalized to

$$\text{tr}(t_a t_b) = \frac{1}{2} \delta_{ab}$$

and generate the $\text{U}(N)$ algebra

$$[t_a, t_b] = i F_{abc} t_c, \quad F_{ab0} = 0$$

$$(A1)$$

$$\{t_a, t_b\} = D_{abc} t_c, \quad D_{ab0} = \sqrt{\frac{2}{N}} \delta_{ab}$$

with $(F_{abc}) D_{abc}$ the totally (anti) symmetric structure constants. Clearly the $t_a$ generate the $\text{SU}(N)$ subalgebra.

$$[t_a, t_b] = i f_{abc} t_c \quad , \quad f_{abc} = F_{abc}$$

$$(A3)$$

$$\{t_a, t_b\} = \frac{\delta_{ab}}{N} + d_{abc} t_c \quad , \quad d_{abc} = D_{abc}$$

By completeness

$$(t_a)_{ij} (t_a)_{k\ell} = \frac{1}{2} \delta_{ij} \delta_{k\ell}, \quad i, j = 1, \ldots, N;$$

$$(A5)$$

hence

$$t_a t_a = \frac{N}{2}, \quad t_a t_b t_a = \frac{1}{2} \sqrt{\frac{N}{2}} \delta_{b0}$$

$$(A6)$$

while for the $\text{SU}(N)$ subgroup

$$t_a t_a = C_F, \quad C_F = \frac{N^2 - 1}{2N}$$

$$(A7)$$

$$t_a t_b t_a = (C_F - C_A/2) t_b, \quad C_A = N$$
Then

\[ F_{acd}F_{bcd} = -2\text{tr}(\{t_a, t_c\}|t_b, t_c\}) \]

\[ = 2\text{tr}(\{t_a, t_b\}t_c t_c - 2t_a t_c t_b t_c) \]

\[ = N(\delta_{ab} - \delta_{a0}\delta_{b0}) \equiv N\delta_{ab} \quad (A8) \]

and similarly

\[ D_{acd}D_{bcd} = 2\text{tr}(\{t_a, t_c\}\{t_b, t_c\}) = N(\delta_{ab} + \delta_{a0}\delta_{b0}) \quad (A9) \]

Now, define the \( N^2 \times N^2 \) matrices \( T_a \) as

\[ (T_a)_{bc} \equiv \frac{1}{2}(D_{abc} - iF_{abc}) \quad (A10) \]

so from (A3)

\[ t_a t_b = (T_b)_{ac} t_c. \quad (A11) \]

There follows

\[ \text{tr}(t_a t_b t_c) = \frac{1}{2}(T_b)_{ac} \quad (A12) \]

and

\[ \text{tr}(t_a t_b t_c t_d) = \frac{1}{2}(T_b)_{ac}(T_c)_{ed} \]

\[ = \frac{1}{2}(T_a)_{de}(T_b)_{ec} \]

\[ = iF_{bec}\frac{1}{2}(T_c)_{ad} + \frac{1}{2}(T_c)_{ae}(T_b)_{ed} \quad (A13) \]

from which

\[ [T_a, T_b] = iF_{abc}T_c, \]

\[ [T_a, -T_b^*] = 0, \quad (A14) \]

\[ [-T_a^*, -T_b^*] = iF_{abc}(-T_c^*), \]

i.e., the \( T_a \) and \(-T_a^*\) generate the chiral algebra \( \text{U}(N)_L \otimes \text{U}(N)_R \). They are normalized by
\[ \text{tr}(T_a T_b) = \frac{N}{2} \delta_{ab} = \text{tr}(T_a^* T_b^*), \tag{A15} \]

as a consequence of (A8) and (A9). The diagonal subalgebra generated by

\[ (T_a - T_a^*)_bc = -iF_{abc} \tag{A16} \]

is isomorphic to the adjoint representation of SU(N).
Consider the change of variables (20); we define

\[ \ell_a(\pi) \equiv U^\dagger(\pi) \frac{\partial}{\partial \pi_a} U(\pi) \]

\[ = \int_0^1 d\lambda U^\dagger(\lambda \pi) i t_a U(\lambda \pi) \]  

(B1)

and note by repeated use of (A11)

\[ t_a U(\pi) = U_{ab}(\pi) t_b \]  

(B2)

where \( U(\pi) \) is the \( N^2 \times N^2 \) matrix

\[ U(\pi) = \exp (i \pi_a T_a) . \]  

(B3)

Then

\[ \ell_a(\pi) = L_{ab}(\pi) i t_b = i t_b L_{ba}(\pi) , \]  

(B4)

\[ L(\pi) = \int_0^1 d\lambda \exp (i \lambda \pi_a (T_a - T_a^*)) \]

and since

\[ \text{tr}(\partial_\mu U^\dagger(\pi) \partial^\mu U(\pi)) = -\text{tr}(\ell_a \ell_b) \partial_\mu \pi_a \partial^\mu \pi_b \]

\[ = \frac{1}{2} (L(\pi) L(-\pi))_{ab} \partial_\mu \pi_a \partial^\mu \pi_b \]

\[ = \frac{1}{2} g_{ab}(\pi) \partial_\mu \pi_a \partial^\mu \pi_b \]  

(B5)

the invariant measure on the U(N) group space is

\[ dU = \sqrt{\det (g(\pi))} \Pi_a d\pi_a = \det (L(\pi)) \Pi_a d\pi_a . \]  

(B6)

Now

\[ \frac{\partial}{\partial \pi_a} U(\pi) = U(\pi) \ell_a(\pi) , \quad \frac{\partial}{\partial \pi_a} U^\dagger(\pi) = -\ell_a(\pi) U^\dagger(\pi) \]  

(B7)

so
\[ \frac{\partial \Phi_a}{\partial \pi_b} = \frac{2\text{tr}(-t_a U(\pi)\ell_b(\pi)\phi)}{2\pi} = (-U(\pi)i\phi e^T L(-\pi))_{ab}, \]

\[ \frac{\partial \Phi_a}{\partial \phi_b} = \frac{2\text{tr}(-t_a U(\pi)t_b)}{2\pi} = -U_{ab}(\pi), \]

\[ \frac{\partial \Phi^\dagger_a}{\partial \pi_b} = \frac{2\text{tr}(\phi\ell_b(\pi)U^\dagger(\pi)t_a)}{2\pi} = (U^*(\pi)i\phi e^T e^T L(-\pi))_{ab}, \]

\[ \frac{\partial \Phi^\dagger_a}{\partial \phi_b} = \frac{2\text{tr}(-t_b U^\dagger(\pi)t_a)}{2\pi} = -U^*_{ab}(\pi) \]  

and the Jacobian factorizes:

\[ \Pi_a \frac{d\Phi_a d\Phi_a^\dagger}{2\pi i} = \det \left( \begin{array}{c|c} U(\pi) & 0 \\ \hline & \end{array} \right) \det \left( \begin{array}{c|c} -\phi e^T L(-\pi) & -I \\ \hline & \end{array} \right) \Pi_a \frac{d\phi_a d\pi_a}{2\pi} \]

\[ = \det(U(\pi)U^T(\pi)) \det(\phi e^T + \phi e) \det(L(\pi))\Pi_a \frac{d\phi_a d\pi_a}{2\pi} \]

\[ = dU \det(J(\phi)) \Pi_a \frac{d\phi_a d\pi_a}{2\pi} \]  

\[ \text{with} \]

\[ J_{ab}(\phi) = D_{abc}\phi_c \]  

(B10)
REFERENCES

[1] F. G. Scholtz and G. B. Tupper, Phys. Rev. D48, 1792 (1993).

[2] V. N. Gribov, Nucl. Phys. B139, 1 (1978).

[3] I. M. Singer, Commun. Math. Phys. 60, 7 (1978).

[4] Hue Sun Chan and M.B. Halpern, Phys. Rev. D33, 540 (1986).

   D. Zwanziger, Nucl. Phys. B209, 336 (1982); B323, 513 (1989); B321, 591 (1989); B345, 461 (1990).

   S.V. Shabanov, Phys. Lett. B255, 398 (1991).

   L.J. Carson, Nucl. Phys. B266, 357 (1986).

   K. Fujikawa, Prog. Theor. Phys. 61, 627 (1979).

   M. B. Halpern and J. Koplik, Nucl. Phys. B132, 239 (1978).

   R.E. Cutkosky, Phys. Rev. D30, 447 (1984).

   C.M. Bender, T. Eguchi and H. Pagels, Phys. Rev. D17, 1086 (1978).

   R.D. Peccei, Phys. Rev. D17, 1097 (1978).

   R. Jackiw, I. Muzinich and C. Rebbi, Phys. Rev. D17, 1576 (1978).

   H. Neuberger, Phys. Lett. B183, 337 (1987).

   G. Siopsis, Phys. Lett. B136, 175 (1984); B187, 351 (1987).

   T. Maskawa and H. Nakajima, Prog. Theor. Phys. 63, 641 (1980).

[5] K. Fujikawa, Nucl. Phys. B223, 218 (1983).

[6] C. Becchi, A. Rouet and R. Stora, Ann. Phys. 98, 287 (1976).

[7] M. Baker, J. S. Ball and F. Zachariasen, Nucl. Phys. B186, 531, 560 (1981).

   N. Brown and M. R. Pennington, Phys. Rev. D38, 2266 (1988), D39, 2723 (1989).

[8] D. Zwanziger, Nucl. Phys. B345, 461 (1990).
[9] C. Parrinello and G. Jona-Lasinio, Phys. Lett. B251, 175 (1990).

[10] V. N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983), p.44.

[11] S. P. Fachin, Phys. Rev. D47, 3487 (1993).

[12] J. M. Cornwall and J. Papavassiliou, Phys. Rev. D40, 3474 (1989).

[13] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209, 129 (1991).

[14] L. F. Abbot, Nucl. Phys. 185, 189 (1981).

[15] S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973).

[16] J. C. Colling, A. Duncan and S.D. Joglekar, Phys. Rev. D16, 438 (1977).

K. N. Nielsen, Nucl. Phys. B210, 212 (1977).

[17] L. J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rep. 127, 1 (1985).

[18] M. Lavelle, Phys. Rev. D44, R26 (1991).

[19] M. Consoli and J. H. Field, Phys. Rev. D49, 1293 (1994).
FIGURES

FIG. 1. One-loop diagrams contributing to (a) the scalar and (b) the ghost self-energy; wavey lines denote the gauge field.

FIG. 2. Gauge field contributions to the gauge field self-energy.

FIG. 3. Order $g^2$ (two-loop) contributions to $\mathcal{G}(x,y)$.

FIG. 4. Order $g^2$ graphs contributing to $\mathcal{G}_\infty(x,y)$.

FIG. 5. Order $g^4$ graphs contributing to the exchange part of $\mathcal{G}_\infty(x,y)$.

FIG. 6. Pinch parts of the exchange diagrams contributing to $\mathcal{G}_\infty(x,y)$ in order $g^4$.

FIG. 7. (a) Ghost and (b) scalar contributions to the exchange part of $\mathcal{G}_\infty(x,y)$ in order $g^4$.

FIG. 8. Two-loop diagrams contributing to $\partial \ln Z / \partial v$. The square denotes the operator insertion of (126).