Statistical Lorentzian geometry
and the closeness of Lorentzian manifolds

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Abstract
I introduce a family of closeness functions between causal Lorentzian geometries of finite volume and arbitrary underlying topology. When points are randomly scattered in a Lorentzian manifold, with uniform density according to the volume element, some information on the topology and metric is encoded in the partial order that the causal structure induces among those points; one can then define closeness between Lorentzian geometries by comparing the sets of probabilities they give for obtaining the same posets. If the density of points is finite, one gets a pseudo-distance, which only compares the manifolds down to a finite volume scale, as illustrated here by a fully worked out example of two 2-dimensional manifolds of different topology; if the density is allowed to become infinite, a true distance can be defined on the space of all Lorentzian geometries. The introductory and concluding sections include some remarks on the motivation for this definition and its applications to quantum gravity.

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I. Introduction

The purpose of this paper is to propose a definition of closeness between Lorentzian geometries, where by Lorentzian geometry I mean a diffeomorphism equivalence class $G = \{(M, g)\}$ of manifolds with Lorentzian metrics. More specifically, I will first define a pseudo-distance function $d_n(G, G')$ of two geometries $G = \{(M, g)\}$ and $G' = \{(M', g')\}$ with finite volumes $V_M$ and $V_{M'}$, depending on an integer $n$, such that whenever $d_n(G, G')$ is small, the two geometries are close at large volume scales compared to $V_M/n$ and $V_{M'}/n$, up to a global scale transformation; most of the paper is devoted to this pseudo-distance and its properties, but I will also extend the definition to a distance function $d_\ell(G, G')$ depending on a length parameter $\ell$. Notice that the geometries in question can be based on two entirely different manifolds $M$ and $M'$.

There are various contexts in which such a definition is useful, but the ones that motivated this work are mostly related to quantum gravity. There is a growing amount of evidence, initially suggested by analogies with other theories and simple consistency arguments but increasingly supported by more rigorous results, that the structure of spacetime at the smallest scales (of the order of or smaller than the Planck volume $\ell_P^4 = (G\hbar/c^3)^2$ —just for this equation, $G$ stands for Newton’s gravitational constant) differs significantly from that of the four-dimensional, topologically flat differentiable manifold we use as a model in ordinary physics [1, 2, 3]. Very many different proposals exist for what to replace this manifold with; I will mention only a few of them here, as examples of situations in which one needs to talk about the closeness of Lorentzian geometries.

If one assumes that large quantum fluctuations of the metric on small scales will be associated with fluctuations in the topology itself, but differentiable manifolds are still valid models for the geometry, one is led to the notion of *spacetime foam* [4, 5, 6], a bubbling topological magma in which topological entities like geons and wormholes fluctuate into and out of existence. Spacetime is a quantum superposition of differentiable manifolds of different topology, and the ones that contribute most to the classical spacetime we see are such that each topological fluctuation occupies on the average one Planck volume; at larger scales they are all thought to be close to each other, and essentially indistinguishable from a topologically flat manifold.

On the other hand, there are hints that the very notion of manifolds and continuity may have to be abandoned for models that describe spacetime at Planck scales. In the *causal set* proposal, spacetime is considered as a locally finite partially ordered set [7, 8, 9, 10, 11, 12]; if the elements are thought of as events, occupying on the average one Planck volume each, the partial order is interpreted as giving the causal relations between them. In *spin foam* type proposals, the basic structure is also that of a graph, but with extra variables attached to the edges and vertices [13, 14, 15, 16, 17, 18]. In either case, the continuum and the rest of the Lorentzian manifold structure we see at large scales emerge as a thermodynamic limit, much like the description of a gas by thermodynamic quantities such as pressure and temperature emerges at large scales. Part of the reason why this limit exists is that, even though there are infinitely many Lorentzian manifolds which can
smoothly interpolate between the elements of a given discrete set, they are all supposedly indistinguishable at scales larger than $\ell_4$.

A definition of closeness between Lorentzian metrics on the same manifold $M$, in the form of a scale dependent function $d_\lambda(g, g')$, has already been given in Ref 19 (the main idea can also be found in Ref 20). However, that definition is not diffeomorphism invariant, in the sense that, if $\phi$ is a diffeomorphism of $M$, in general $d_\lambda(g, g') \neq d_\lambda(g, \phi^*g')$. In principle, given such a $d_\lambda$ one can construct an invariant one [19], but in this case $d_\lambda$ is difficult to work with; and, more importantly, it is not defined for metrics on different manifolds. My goal here is to set up a definition that is applicable to any two Lorentzian manifolds, analogously to the one given for Riemannian geometries by Gromov using geometrical concepts [21] or by Seriu using spectral techniques [22]. Unfortunately, the ideas behind those distances rely heavily on the positive-definite nature of the metrics; the one I use here comes instead from causal set theory: $G$ and $G'$ are close if, when we distribute the same number of points at random with uniform density in (one representative of) each of them, the probability of obtaining any given induced partial order among those points is about the same in the two cases. A few of the ideas that led to this work appeared earlier in a different form in Ref 23.

The use of uniform distributions of points is what makes the definition diffeomorphism-invariant, by not requiring us to identify points in the two manifolds; we are comparing instead the two geometries by independently sampling them, Montecarlo style, which brings a probabilistic aspect into the definition. Therefore, I begin in section II by briefly reviewing the definition and some properties of a uniform random distribution of points in a manifold, with respect to a given volume element. For simplicity, I will assume that all manifolds $(M, g)$ have a finite total volume $V_M = \int_M d^Dx \sqrt{-g}$, where $D$ is the dimension of $M$. If the manifolds have no closed timelike curves, each $n$-point sprinkling is endowed with a partial order by the causal structure on the manifold, and defines an element of the set $\mathcal{C}_n$ of all partially ordered sets (posets) on $n$ points. The idea then is to define $d_n(G, G')$ by comparing the two probabilities on $\mathcal{C}_n$ corresponding to $G$ and $G'$. The rest of the section is devoted to constructing a procedure for calculating those probabilities. Section III contains the definition of the family of pseudo-distances and a derivation of some of its properties, and section IV an example in which calculations are carried out in detail. In section V, I introduce a family of distances, which uses sprinklings of arbitrarily high numbers of points. The discussion is kept at a general level, independently of any applications, but the concluding section VI contains additional remarks on applications of this work as well as open issues.

Finally, a few words concerning notation and terminology. Poset elements are denoted by $p, q, ...$; manifold points by $x, y, ...$; by the past or future of a point $x$ in a Lorentzian manifold, I will mean its chronological past or future $I^\pm(x)$ (this convention is adopted mainly for the sake of definiteness, since most of our considerations will depend just on the volume of those sets or of their intersections and unions, which for well-behaved geometries would be the same if I had used instead causal pasts/futures or their closures); and the
relationship $\mathbf{y} \in I^+(\mathbf{x})$, or $\mathbf{x} \in I^-(\mathbf{y})$, will be indicated by $\mathbf{x} < \mathbf{y}$. Finally, $P$’s will stand for probabilities, $\tilde{P}$’s for probability densities, $C$’s for posets, $C$’s for sets of posets, $V_R$ or $V(R)$ for the volume of the region $R \subseteq M$, and $R \setminus R' = \{ \mathbf{x} \mid \mathbf{x} \in R, \mathbf{x} \notin R' \}$ for the difference between sets.

II. Random Point Distributions and Partially Ordered Sets

This section contains the elements that will go into the definition of the closeness measures. I begin with a summary of the few notions we will need regarding uniform distributions of points in a manifold, and then discuss how to obtain probabilities for different resulting partial orders.

Given any manifold $M$ with a volume element, in particular one with a metric (which at this point could be Riemannian or Lorentzian, possibly even degenerate—but not everywhere, lest we get $V_M = 0$!), such that the total volume $V_M$ is finite, we can define a random process of sprinkling points uniformly by stating that, each time a point is chosen in $M$, the probability density that a particular $\mathbf{x}$ be picked is

$$\tilde{P}_M(\mathbf{x}|\sqrt{-g}) = \frac{1}{V_M} \sqrt{-g(\mathbf{x})},$$

in any coordinate system; equivalently, the probability that $\mathbf{x}$ fall in any given measurable region $R \subseteq M$ (such as any interval or any finite union or intersection of such sets [24]) is

$$P_M(\mathbf{x} \in R) = \int_R \tilde{P}_M(\mathbf{x}|\sqrt{-g}) d^D\mathbf{x} = \frac{V_R}{V_M}.$$ (2)

If the process is repeated $n$ times, we get a uniform, random sprinkling of points with density $\rho := n/V_M$, or, if we forget the order they came in, an unlabelled $n$-point distribution; these are the events we are interested in, and for which we will calculate probabilities.

One of the probabilities one uses most often in such cases is the one for exactly $k$ points out of $n$ to fall inside $R$ (without specifying which ones). This probability follows a binomial distribution,

$$P(k, R | n, M) = \binom{n}{k} \prod_{i=1}^{k} P(\mathbf{x}_i \in R) \prod_{j=k+1}^{n} P(\mathbf{x}_j \in M \setminus R)$$

$$= \binom{n}{k} \left( \frac{V_R}{V_M} \right)^k \left( 1 - \frac{V_R}{V_M} \right)^{n-k},$$ (3)

which, as $V_M$ and $n$ become very large, with $\rho = \text{const}$, approaches a Poisson distribution,

$$P(k, R | n, M) \approx \frac{e^{-\rho V_R}}{k!} (\rho V_R)^k.$$ (4)

This last equation justifies the name Poisson distribution that is often used for the sets of points used in this paper, and corresponds to the infinite volume situation. The fact that in that case $P(k, R | n, M)$ can be written in the (exact) form (4), where only $\rho$ appears and not $n$ or $V_M$, indicates that it may be possible to generalize the definitions and results
of this paper to infinite volumes, although in that case we do not have the probability density (1) available, which is what we would use to carry out an actual sprinkling, e.g., in a computer simulation.

When we randomly sprinkle $n$ points in $M$, the volume element $\sqrt{-g}$ determines statistically where they will fall; given their positions, the causal structure $\hat{g}$ determines then the causal relations between them. From now on, all metrics will have Lorentzian signature and satisfy the past and future distinguishing condition (see, e.g., Refs 25 and 26). In particular, this implies the causality condition (no closed causal curves), which guarantees that a partial order is induced on each sprinkling, defining an $n$-element poset $C \in \mathcal{C}_n$; the slightly stronger distinguishing condition implies that “there are no almost closed causal curves,” in a specific sense which gives some additional benefits, as I will discuss below. Different geometries $G = \{(M, g)\}$ and $G' = \{(M', g')\}$ will then in general give different probabilities $P_n(C|G)$ and $P_n(C|G')$ of obtaining each $C \in \mathcal{C}_n$, which we may compare as a way to determine how close the geometries themselves are. It is therefore important to have a general procedure available for calculating, in principle at least, the probabilities $P_n(C|G)$.

Let us start by fixing our notation. While $\mathcal{C}_n$ is the set of unlabelled posets $C$ on $n$ elements, $\overline{\mathcal{C}}_n$ will denote the set of labelled $n$-element posets $\overline{C}$, and $\Sigma_n(M)$ the set of $n$-point sprinklings $\sigma = (x_1, \ldots, x_n)$ in $M$. (One may argue that the labelling of the points should not be important; I am considering sprinklings to be ordered $n$-tuples of points here for convenience.) As already stated, our random events are $n$-point sprinklings $\sigma$ obtained as a result of a random process with uniform density. The volume element $\sqrt{-g}$ on $M$ induces a probability density on $\Sigma_n(M)$; since the points are independently sprinkled, this can be obtained from products of single point probability densities (1) [27],

$$\tilde{P}_M(x_1, \ldots, x_n | \sqrt{-g}) = \prod_{i=1}^n \tilde{P}_M(x_i | \sqrt{-g}) = \frac{1}{V_M^n} \prod_{i=1}^n \sqrt{-g(x_i)} . \quad (5)$$

If the spacetime $(M, g)$ has no closed timelike curves, i.e., satisfies the chronology condition, the relation $x_1 < x_2$ induced by the conformal structure $\hat{g}$ on $M$ is a partial order, so the sprinkling $\sigma$ becomes a labelled poset $\overline{C} := \{p_i \mid p_i < p_j \text{ iff } x_i < x_j \text{ in } \sigma\}$, i.e., we get a map $\Phi_{\hat{g}} : \Sigma_n(M) \to \overline{\mathcal{C}}_n$ given by $\sigma \mapsto \overline{C}$. This map is many-to-one, and the inverse image of any $\overline{C}$ is the set $S = \Phi_{\hat{g}}^{-1}(\overline{C}) \subset \Sigma_n(M)$ of all sprinklings with the same induced labelled partial order. This set has non-zero measure in $\Sigma_n(M)$; in fact, its probability is

$$P_S(S | \sqrt{-g}) = \int_S \tilde{P}_M(x_1, \ldots, x_n | \sqrt{-g}) \, d^p x_1 \ldots d^p x_n , \quad (6)$$

where $S$ is specified by conditions on the relations between the sprinkled points giving, for each $x_i$, a region $M_i \subset M$ it can fall into in order to have the right relations with the previously sprinkled $x_j$ with $j < i$, according to $\overline{C}$. Thus, the probability (6) is of the form

$$P_S(S | \sqrt{-g}) = \frac{1}{V_M^n} \prod_{i=1}^n \int_{M_i(x_1, \ldots, x_{i-1}; \overline{C})} \sqrt{-g(x_i)} \, d^p x_i . \quad (7)$$
This expression gives the probability that the sprinkling give rise to a labelled poset \( \mathcal{C} \); we will see below how to specify the \( M_i \) explicitly.

What we really want to find is the probability that the sprinkling give rise to an unlabelled poset \( \mathcal{C} \). Each \( \mathcal{C} \) can be labelled in \( n! \) ways, but in general some of these labellings are indistinguishable in terms of the order relation; more specifically, the number of permutations of elements of \( \mathcal{C} \) that give the same \( \mathcal{C} \) is the number of automorphisms of \( \mathcal{C} \), \( |\text{Aut}(\mathcal{C})| \) (this number is a property of \( \mathcal{C} \), independent of the specific \( \mathcal{C} \) chosen), and we get that each \( \mathcal{C} \in \mathcal{C}_n \) can be obtained from \( n!/|\text{Aut}(\mathcal{C})| \) different labelled \( \mathcal{C} \)'s, so the probability we are looking for is

\[
P_n(C|G) := \frac{n!}{|\text{Aut}(\mathcal{C})|} \frac{1}{V_M^n} \int_{M_i(x_1, \ldots, x_{i-1}; \mathcal{C})} \sqrt{-g(x_i)} \, d^n x_i,
\]

(8)

where \( \mathcal{C} \) is an arbitrary labelling of \( \mathcal{C} \).

Suppose a given labelling \( \mathcal{C} = \{p_i\} \) of \( \mathcal{C} \) has been chosen to carry out the sprinkling. This means that, in order for the \( \{x_i\} \) to be a realization of \( \mathcal{C} \), each \( x_i \) must be in the future of the \( x_j \)'s such that \( p_i > p_j \), among the previously sprinkled ones, in the past of the ones such that \( p_i < p_j \), and spacelike related to the remaining ones. In other words, while \( x_1 \) can be anywhere, \( M_1(\mathcal{C}) = M \), points \( x_i \) with \( i > 1 \) must fall in the regions

\[
M_i(x_1, \ldots, x_{i-1}; \mathcal{C}) = \bigcap_{j<i} M_{ij}(x_j; \mathcal{C}), \quad M_{ij}(x_j; \mathcal{C}) = \begin{cases} I_j^+ & \text{if } p_i > p_j \\ I_j^- & \text{if } p_i < p_j \\ M \setminus I_j & \text{otherwise} \end{cases}
\]

(9)

where for futures and pasts I use the abbreviations \( I_i^\pm := I^\pm(x_i) \) and \( I_i := I_i^- \cup I_i^+ \). The most convenient labelling \( \mathcal{C} \) to use in each case may vary. It is often a good choice to pick one compatible with the partial order on \( \mathcal{C} \), in the sense that if \( p_i < p_j \) then \( i < j \), which can always be done (in fact, it just means “start labelling from the bottom and work your way up,” and the choice is almost never unique); this has the advantage that, to reproduce the partial order on \( \mathcal{C} \), no \( x_i \) needs to be in the past of any of the previously sprinkled \( x_j \)'s with \( j < i \), which eliminates the second case in (9).

This completes the prescription for calculating the probabilities to be used in the closeness function. In practice, the dependence of each \( M_i \) on the points \( x_1, \ldots, x_{i-1} \) makes the probability very difficult to calculate analytically, and one would normally use other means such as computer methods, except for very simple situations like the one in section IV.
III. The Pseudo-Distance

In this section, I will take the point of view that geometries $G$ can be identified with the sets of probabilities $\{P_n(C|G) \mid C \in \mathcal{C}_n\}$, with a degree of approximation that improves as $n$ increases. The task of defining a pseudo-distance between $G$ and $G'$ is then reduced to that of defining a distance between their respective sets of probabilities. I will do so, and then consider some properties of the resulting pseudo-distance.

Various functions can be used as distances between sets of numbers; some simple ones to handle would be the $\ell^1$-type distance $d_n^{(1)}(G, G') = \frac{1}{2} \sum_{C \in \mathcal{C}_n} |P_n(C|G) - P_n(C|G')|$, the Euclidean distance, or simply the “sup” distance, but in view of the interpretation of the numbers as probabilities, I will use instead the statistical distance introduced by Wootters [28] in the context of rays in Hilbert space, which is proportional to the number of statistically distinguishable, in an appropriate sense, intermediate probability sets between the two sets being compared. Let us then define, for any two geometries,

$$d_n(G, G') := \frac{2}{\pi} \arccos \left[ \sum_{C \in \mathcal{C}_n} \sqrt{P_n(C|G)} \sqrt{P_n(C|G')} \right].$$

(10)

Geometrically, the fact that $\sum_{C \in \mathcal{C}_n} P_n(C|G) = 1$ means that $\sqrt{P_n(C|G)}$ can be interpreted as the coordinates of a point on the unit sphere, identifying a direction in probability space $\mathbb{R}^{\mathcal{C}_n}$, and $d_n(G, G')$ is then proportional to the angle defined by the two corresponding directions; notice that, because all coordinates are non-negative, that angle is at most $\pi/2$, so with this definition $d_n(G, G')$ is at most equal to 1.

Clearly, $d_n(G, G')$ is not positive-definite. For each $n$, the number $|\mathcal{C}_n|$ of posets that can be made out of the $n$ sprinkled points, although very large, is finite; thus, the value of $d_n(G, G')$ depends on a finite number of parameters, and cannot capture all of the information contained in the geometries. This means that $d_n$ cannot be an actual distance function in the infinite-dimensional space of Lorentzian geometries. One possibility would be to take the limit $n \to \infty$; this may indeed give a distance, but it may be a trivial one, as I discuss below, and we shall consider a better alternative in section V. However, even for finite $n$, two geometries for which $d_n(G, G') = 0$ are close when probed at scales larger than the mean point spacing, and this is what we really need in some applications.

Let us consider the other extreme situation, $d_n(G, G') = 1$. For finite $n$, this can happen only for highly degenerate geometries, since it requires that the argument of the arccos function in (10) be zero, in other words that there be no $C \in \mathcal{C}_n$ for which both $P_n(C|G)$ and $P_n(C|G')$ are non-vanishing, i.e., which can be embedded in both geometries. One of the possible $C$’s is always the totally ordered $n$-element poset (a chain), so one of the geometries (say, $G$) must assign zero probability to pairs of timelike related points; in $G$, the light cones of all points must have degenerated away to lines. Another possible poset is the totally disconnected one (an antichain), so one of the geometries (necessarily the other one, $G'$) must assign zero probability to pairs of spacelike related points; $G'$ has the wide open light cones of the infinite speed of light limit, or is a one-dimensional timelike line. No poset $C \in \mathcal{C}_n$ other than those two can be embedded in either geometry. We
conclude that the inequality $d_n(G, G') \leq 1$ cannot be saturated other than as a degenerate limit of sequences of geometries of the type just described.

In the limit $n \to \infty$, however, the situation may change. We know from continuum results that the topology, differentiable, and conformal structures of a past and future distinguishing Lorentzian geometry can be recovered just from the knowledge of the causal relations between all pairs of points \[29,30\] and that, if one considers instead pairs of points in a sequence of uniform sprinklings of increasing density, the same is true in the limit $n \to \infty$, with the added bonus that the volume element can be recovered as well, up to a global factor \[31,20\]; thus, in that limit, sequences of posets $\{C_n\}$, where each $C_n$ has $n$ elements and is a subposet of the next one, $C_n \subset C_{n+1}$, can be embedded at most in a single geometry $G$. This means that

$$\forall G \neq G', \quad \lim_{n \to \infty} P_n(C_n|G) \cdot P_n(C_n|G') = 0 \quad \forall \{C_n\}. \quad (11)$$

In fact, it is also true that each individual probability $P_n(C_n|G)$ or $P_n(C_n|G')$ tends to zero as $n \to \infty$. But the number of terms in the summation in (10) grows very fast with $n$ (faster than exponentially \[32\]), and the limiting value of $d_n(G, G')$ for $G \neq G'$ depends on the rate of approach to zero of these probabilities. It is possible that $d_\infty(G, G') = 1$ for all $G \neq G'$ (in terms of the discussion above, many posets may be embeddable in both $G$ and $G'$, but the limit is 1 because all products of probabilities in (11) go to zero fast enough); in this case $d_\infty$ would be a distance, but a trivial, not very useful one.

We will see what $d_\infty(G, G')$ can be replaced by later in the paper, and return now to examining properties of $d_n$ with finite $n$. In addition to its much greater ease of computation, the function $d_n(G, G')$ is also made interesting by the following reasonable conjectures:

(i) In a sense, for large enough $n$, it is “almost a distance,” or “positive-definite up to differences on small scales;” (ii) For any subset of geometries labelled by a finite number $N$ of parameters (analogous to the “minisuperspaces” used for spatial geometries), there is a finite $n$ such that $d_n$ is a true distance function on this set, and (iii) For any two arbitrary (distinguishing, finite-volume) different geometries $G$ and $G'$ there is a finite $n$ such that $d_n(G, G') > 0$, with $d_n(G, G') \to 1$ as $n \to \infty$.

To start with, I will prove the intuitively obvious, and nice property of the closeness measure that it is a monotonically increasing function of $n$:

$$\forall G, G' \quad d_n(G, G') \leq d_{n+1}(G, G'). \quad (12)$$

To prove this inequality, consider the process of sprinkling $n + 1$ points in a geometry $G$ as an $n$-point sprinkling, followed by the choice of one more point. Then, the probability of the first $n$ points yielding any given $C \in C_n$ is a sum over probabilities for different $C' \in C_{n+1}$ obtained when the extra point is added,

$$P_n(C|G) = \sum_{C' \in C_{n+1}} f_{C,C'} \cdot P_{n+1}(C'|G), \quad f_{C,C'} := \frac{1}{n+1} \binom{C'}{C}, \quad (13)$$
where \( f_{C,C'} \) is the fraction of \( n \)-element subsets of \( C' \) that are isomorphic to \( C \), which can be expressed in terms of the number \( \binom{C'}{C} \) of ways of picking an \( n \)-element subset of \( C' \) that is isomorphic to \( C \) (this number may be called “\( C' \) choose \( C \),” and I will use the convention that it vanishes if \( C \) is not a subposet of \( C' \)); notice that it is clear from the definition that \( \sum_{C \in C_n} f_{C,C'} = 1 \), for any \( C' \). Then, we can write

\[
d_n(G, G') = \frac{2}{\pi} \arccos \left[ \sum_{C} \sqrt{ \sum_{C'} f_{C,C'} P_{n+1}(C'|G) \sqrt{ \sum_{C''} f_{C,C''} P_{n+1}(C''|G') } } \right],
\]

where it is understood that \( C \in C_n \) and \( C', C'' \in C_{n+1} \). For each \( C \), the corresponding term in the summation in (14) is of the form \( \sqrt{(\sum_i a_i)(\sum_j b_j)} \) where all \( a_i \) and \( b_j \) are non-negative, for which the general inequality

\[
\sqrt{(\sum_i a_i)(\sum_j b_j)} \geq \sum_i \sqrt{a_i b_i}
\]

holds. To prove this inequality, we can square the two sides, which gives \( \sum_i \sum_j a_i b_j \) and \( \sum_i \sum_j \sqrt{a_i b_i} \sqrt{a_j b_j} \), respectively; the terms with \( i = j \) are equal; separate the other ones in pairs, \( a_i b_j + a_j b_i \) and \( 2 \sqrt{a_i b_i} \sqrt{a_j b_j} \), respectively, and square them; since we always have \( a_i^2 b_j^2 + 2 a_i a_j b_i b_j + a_j^2 b_i^2 \geq 4 a_i a_j b_i b_j \), (15) follows. Applying this to (14) gives

\[
d_n(G, G') \leq \frac{2}{\pi} \arccos \left[ \sum_{C} \sum_{C'} \sqrt{(f_{C,C'})^2 P_{n+1}(C'|G) P_{n+1}(C''|G')} \right]
= \frac{2}{\pi} \arccos \left[ \sum_{C'} \left( \sum_{C} f_{C,C'} \right) \sqrt{P_{n+1}(C'|G) P_{n+1}(C''|G')} \right]
= d_{n+1}(G, G').
\]

As a consequence of the proof, we also see that

\[
d_n(C, C') = d_{n+1}(C, C') \quad \text{iff} \quad d_{n+1}(C, C') = 0,
\]

since the inequality in (16) can only be saturated if (15) is, and this will happen only if for all \( i \) and \( j \), \( a_i b_j = a_j b_i \), which in terms of our probabilities reduces to \( P_{n+1}(C'|G) = P_{n+1}(C''|G') \). As a byproduct, we also obtain the equality (13), which may be useful for calculating \( P_n(C|G) \), or one of the \( P_{n+1}(C'|G) \)'s if the others are known.
IV. A Simple Example

As an illustration of the definition of $d_n(G,G')$ and the procedure for calculating $P_n(C|G)$ introduced in section II, we consider a very simple example, which already involves two 1-parameter families of geometries with different underlying manifolds: a finite-size rectangular portion of 2-dimensional Minkowski space, with line element $ds^2 = -dt^2 + dx^2$ and topology $M \simeq \mathbb{R}^2$, $G_\gamma = \{(M,\eta)\}$, and a similar one obtained after a spatial identification, with the same line element and topology $M' \simeq \mathbb{R} \times S^1$, $G_\delta = \{(M',\eta)\}$. I will first introduce each geometry and calculate the simplest probabilities, $P_2(C|G_\gamma)$ and $P_2(C|G_\delta')$, then use these to find $d_2(G_\gamma,G_\delta')$; the results will give us an indication of features and limitations of $d_2$, and we will then see how to overcome these limitations by calculating the $P_3$’s and using $d_3(G_\gamma,G_\delta')$.

The geometry $G_\gamma$ is the rectangle $M := \{x \mid 0 \leq x \leq a, 0 \leq t \leq b\}$ in two-dimensional Minkowski space. Since the probabilities we are looking for are invariant under a global rescaling, they cannot depend on the volume $V_M = ab$, but only on the aspect ratio $\gamma := b/a$. For $n = 2$, $G_\gamma$ has two elements, the connected two-element poset $\mathbb{I}$ and the disconnected one $••$; we must calculate the integrals in (8) for these two posets.

To get the connected poset $\mathbb{I}$ in a two-point sprinkling with the “bottom-up” labelling, we need $x_2$ to fall in the future of $x_1$, or $M_2(x_1,\overline{C}) = M_{21}(x_1,\overline{C}) = I^+_1$, and (8) becomes

$$P_2(\mathbb{I} | G_\gamma) = \frac{2!}{1 \cdot V_M^2} \int_M d^2x_1 \int_{I^+_1} d^2x_2 = \frac{2}{(ab)^2} \int_0^a dx \int_0^b dt \ V(I^+(x,t)), \quad (18)$$

where $x_1 = (x,t)$, and the volume $V(I^+(x,t))$ is $(b-t)^2$, with correction terms that are needed for some values of $(x,t)$ (see Fig. 1),

$$V(I^+_1) = (b-t)^2 - \frac{(b-t-x)^2}{2} \theta(x < b \text{ and } t < b-x) - \frac{(b-t-a+x)^2}{2} \theta(x > a-b \text{ and } t < b-a+x). \quad (19)$$

(The step function $\theta$ equals 1 if its argument is true, 0 otherwise.) If we assume that $\gamma \geq 1$, so that $x < b$ and $x > a-b$ are always satisfied, (18) gives

$$P_2(\mathbb{I} | G_\gamma) = \frac{1 - 4\gamma + 6\gamma^2}{6\gamma^2}, \quad (20)$$

To get the disconnected poset $••$ we need the points to be causally unrelated, $M_2(x_1,\overline{C}) = M \setminus I_1$, so (8) becomes

$$P_2(•• | G_\gamma) = \frac{2!}{2 \cdot V_M^2} \int_M d^2x_1 \int_{M \setminus I_1} d^2x_2 = 1 - P_2(\mathbb{I} | G) = \frac{4\gamma - 1}{6\gamma^2}, \quad (21)$$

where instead of doing another integral I have used $P_2(\mathbb{I} | G_\gamma) + P_2(•• | G_\gamma) = 1$.

For the case $b < a$, we can now either integrate (19) again, or use simple symmetry considerations. If we flip the rectangle by exchanging $a \leftrightarrow b$, the manifold transforms
according to $G_2 \leftrightarrow G_{1/\gamma}$; if we take the two sprinkled points $x_1$ and $x_2$ along, the posets are also turned into each other, $I \leftrightarrow \mathbf{\bullet \bullet}$. Thus,

$$P_2(\mathbf{\bullet \bullet} \mid G_\gamma) = P_2(\mathbf{\bullet \bullet} \mid G_{1/\gamma}) = \frac{4\gamma - \gamma^2}{6}$$

$$P_2(\mathbf{\bullet \bullet} \mid G_\gamma) = P_2(\mathbf{\bullet \bullet} \mid G_{1/\gamma}) = \frac{\gamma^2 - 4\gamma + 6}{6}. \tag{22}$$

The geometry $G'_\gamma$ is the cylinder $M'$ one obtains applying the spatial identification $(t, 0) \sim (t, a)$ to the rectangle in $G_\gamma$, with the same line element; again, the probabilities only depend on the aspect ratio $\gamma := b/a$. Similar calculations to the ones leading to (20) and (21), but now integrating

$$V(\Pi_1^+) = (b - t)^2 - (b - t - \frac{1}{2} a)^2 \theta(t < b - \frac{1}{2} a) \tag{23}$$

(see Fig. 2) over $M'$, give

$$P_2(\mathbf{\bullet \bullet} \mid G'_\gamma) = \frac{1 - 6 \gamma + 12 \gamma^2}{12 \gamma^2}, \quad P_2(\mathbf{\bullet \bullet} \mid G'_\gamma) = \frac{6 \gamma - 1}{12 \gamma^2}, \tag{24}$$

for $\gamma \geq \frac{1}{2}$, and

$$P_2(\mathbf{\bullet \bullet} \mid G'_\gamma) = \frac{2}{3} \gamma, \quad P_2(\mathbf{\bullet \bullet} \mid G'_\gamma) = 1 - \frac{2}{3} \gamma, \tag{25}$$

for $\gamma \leq \frac{1}{2}$, when $V(\Pi_1^+)$ is just $(b - t)^2$; we cannot use a trick like the one in (22) here, but this probability is the easiest one to calculate anyway.

If we now use the definition (10) to calculate

$$d_2(G_\gamma, G'_\gamma) = \frac{2}{\pi} \arccos \left[ \sqrt{P_2(\mathbf{\bullet \bullet} \mid G_\gamma) P_2(\mathbf{\bullet \bullet} \mid G'_\gamma) + P_2(\mathbf{\bullet \bullet} \mid G_\gamma) P_2(\mathbf{\bullet \bullet} \mid G'_\gamma)} \right], \tag{26}$$

where the two geometries are characterized by the same parameter value $\gamma$, we get the function plotted in Fig. 3, which goes to zero as the aspect ratio $\gamma \to 0$ or $\infty$, and the difference between the two manifolds becomes immaterial because all pairs $(x_1, x_2)$ are spacelike or timelike related, respectively, in both geometries; but $d_2(G_\gamma, G'_\gamma)$ is not zero for any non-degenerate cases. However, if we use $d_2$ for geometries with different parameter values, we find that, for example,

$$P_2(\mathbf{\bullet \bullet} \mid G_\gamma) = P_2(\mathbf{\bullet \bullet} \mid G'_\delta) \quad \text{when} \quad \delta = \gamma - \frac{1}{4} \gamma^2, \tag{27}$$

i.e., for each $\gamma$ there is a $\delta$ such that $d_2(G_\gamma, G'_\delta)$ vanishes; with two sprinkled points, the single available parameter $P_2(\mathbf{\bullet \bullet} \mid G)$ cannot distinguish all geometries in this example. We will now see that this can be done using three points.

What we need to show is that, for all values of the parameters, among the elements of $\mathcal{C}_3 = \{\mathbf{\bullet \bullet \bullet}, \mathbf{\bullet \bullet I}, \mathbf{\bullet Y}, \mathbf{\bullet \Lambda}, \mathbf{\bullet I}\}$, there is at least one which is embeddable in $G$ and $G'$ with different probabilities; because of property (12), we actually only need to do this for the parameter values for which $d_2(G_\gamma, G'_\delta) = 0$. Let us consider the three-element poset for which the probabilities are easiest to calculate, the linear order $\mathbf{\bullet \bullet I}$ (linear orders are always the easiest ones, because $M_{ij}(x_j, \overline{c})$ reduces just to $I_j^+$ and $M_i(x_1, ..., x_{i-1}; \overline{c})$ to $I_{i-1}^+$ in
(9), if \( C \) is the “bottom-up” labelling). The calculations again proceed along the lines of those for (20) but are somewhat longer, since we now have to evaluate

\[
P_3(\{ | G) = \frac{3!}{2 \cdot V_M^3} \int_M d^2x_1 \int_{I_1^+} d^2x_2 \int_{I_1^+} d^2x_3 .
\]  

(28)

I will restrict myself to the case \( \gamma \leq \frac{1}{2} \), where by explicitly integrating (28) one gets

\[
P_3(\{ | G_\gamma) = \frac{1}{5} \gamma^2 - \frac{1}{12} \gamma^3, \quad P_3(\{ | G_\delta' = \frac{1}{5} \gamma^2 .
\]  

(29)

When \( \delta = \gamma - \frac{1}{4} \gamma^2 \), a simple calculation gives

\[
P_3(\{ | G_\gamma) - P_3(\{ | G_\delta') = \frac{1}{20} \gamma^3 (\frac{1}{3} - \frac{1}{4} \gamma),
\]  

(30)

which does not vanish for \( \gamma \leq \frac{1}{2} \), so \( d_3(G_\gamma, G_\delta') \neq 0 \) as expected. The two families of geometries are different enough that the induced order on a random three-element subset will pick out the difference. The other three-point probabilities \( P_3(C|G) \) can be found without too much additional effort. I will now show how to do this, since the information those probabilities capture about the geometries is interesting in its own right, and because this will illustrate the use of some of the general relationships introduced above, as well as one new relationship and possible symmetries.

Suppose we want to calculate the values of the probabilities \( P_3(\bullet \bullet | G) \), \( P_3(\bullet | G) \), \( P_3(\bigvee | G) \), \( P_3(\bigwedge | G) \), and \( P_3(\{ | G) \) for some geometry \( G \), for which we already know the values of \( P_2(\bullet | G) \) and \( P_2(\{ | G) \); this includes having calculated \( V(I_1^+) \) for every \( x_i \in M \), as in (19), or some other similar integral. It would be to our advantage to use as many relationships as possible among the \( P_3(C|G) \)'s. One is always given by the identity

\[
\sum_{C \in C_3} P_3(C|G) = 1,
\]  

(31)

and two more are always given by

\[
\frac{1}{3} P_3(\bullet | G) + \frac{2}{3} P_3(\bigvee | G) + \frac{2}{3} P_3(\bigwedge | G) + P_3(\{ | G) = P_2(\{ | G),
\]  

(32)

\[
P_3(\bullet \bullet | G) + \frac{2}{3} P_3(\bullet | G) + \frac{1}{3} P_3(\bigvee | G) + \frac{1}{3} P_3(\bigwedge | G) = P_2(\bullet \bullet | G),
\]  

(33)

arising from (13). Notice however that only two of the three relationships (31)–(33) are independent, since (13) already implies \( \sum_{C \in C_{n+1}} P_n(C'|G) = 1 \) if one uses valid \( P_n \)'s and \( f_{C,C'} \)'s, satisfying \( \sum_{C \in C_n} P_n(C|G) = 1 \) and \( \sum_{C \in C_n} f_{C,C'} = 1 \). An additional relationship, not as simple as (32)–(33) but still useful, can be found by considering probabilities for subsets of \( C_3 \) rather than just single \( P_3(C|G) \)'s. Since the outcomes \( \bigvee \) and \( \{ \) are mutually exclusive,

\[
P_3(\bigvee \) or \( \{ | G) = P_3(\bigvee | G) + P_3(\{ | G),
\]  

(34)

which means that

\[
P_3(\bigvee | G) = P_3(\bigvee \) or \( \{ | G) - P_3(\{ | G)
\]  

\[
= \frac{3!}{2 \cdot V_M^3} \int_M d^2x_1 \int_{I_1^+} d^2x_2 \int_{I_1^+} d^2x_3 - P_3(\{ | G)
\]  

\[
= \frac{3}{V_M^3} \int_M d^2x (V(I^+(x,t)))^2 - P_3(\{ | G),
\]  

(35)
where I have used the fact that, in a “bottom-up” labelling, the condition for obtaining \( \mathbf{Y} \) or \( \mathbf{I} \) is simply that both \( \mathbf{x}_2 \) and \( \mathbf{x}_3 \) be in the future of \( \mathbf{x}_1 \). Finally, if \( G \) was time reversal invariant (the \( G_\gamma \) and \( G'_\delta \) in our example both are), we would get an additional relationship,

\[
P_3(\mathbf{A} | G) = P_3(\mathbf{Y} | G).
\]

We have found four relationships among the \( P_3(C | G) \)'s; if they hold, only one probability needs to be calculated by direct application of (8). Also, in specific cases, it may be possible to use other symmetries of \( G \) to derive relationships of other types; for example, the one I used in the trick of (22) involving different parameter values.

In our example, we start by calculating the integral in (35) using (19); if \( \gamma < \frac{1}{2} \),

\[
\int_M d^2x \left( V(I^+(x,t)) \right)^2 = \int_0^a dx \int_0^b dt \left( b - t \right)^4
+ \int_0^b dx \int_0^{b-x} dt \left[ \frac{(b-t-x)^4}{4} - (b-t)^2(b-t-x)^2 \right]
+ \int_{a-b}^a dx \int_0^{b-a+x} dt \left[ \frac{(b-t-a+x)^4}{4} - (b-t)^2(b-t-a+x)^2 \right]
= \frac{1}{5} ab^5 - \frac{17}{180} b^6,
\]

where one of the cross terms in the square of (19) does not contribute because in this case the intervals \( x < b \) and \( x > a - b \) don’t overlap; then, using \( P_3(\mathbf{I} | G) \) from (29), we find \( P_3(\mathbf{Y} | G) \), and from (36) we find \( P_3(\mathbf{A} | G) \); with these, (32) gives \( P_3(\mathbf{I} | G) \), and (33) gives \( P_3(\mathbf{I} \mathbf{I} | G) \). Analogous calculations can be done for \( G' \), starting with

\[
\int_{M'} d^2x_1 \left( V(I_1^+(x_1)) \right)^2 = \int_0^a dx \int_0^b dt \left( b - t \right)^4 = \frac{1}{5} ab^5.
\]

To conclude, the full sets of probabilities we have calculated for \( \gamma < \frac{1}{2} \), are

\[
\begin{align*}
P_2(\mathbf{I} | G_\gamma) &= \frac{2}{3} \gamma - \frac{1}{6} \gamma^2 & P_2(\mathbf{I} | G'_\gamma) &= \frac{2}{3} \gamma \\
P_2(\mathbf{I} \mathbf{I} | G_\gamma) &= 1 - \frac{2}{3} \gamma + \frac{1}{6} \gamma^2 & P_2(\mathbf{I} \mathbf{I} | G'_\gamma) &= 1 - \frac{2}{3} \gamma
\end{align*}
\]

for two-point sprinklings, and

\[
\begin{align*}
P_3(\mathbf{I} | G_\gamma) &= \frac{1}{5} \gamma^2 - \frac{1}{12} \gamma^3 & P_3(\mathbf{I} | G'_\gamma) &= \frac{1}{5} \gamma^2 \\
P_3(\mathbf{I} \mathbf{I} | G_\gamma) &= \frac{2}{5} \gamma^2 - \frac{1}{5} \gamma^3 & P_3(\mathbf{I} \mathbf{I} | G'_\gamma) &= \frac{2}{5} \gamma^2 \\
P_3(\mathbf{I} \mathbf{I} \mathbf{I} | G_\gamma) &= \frac{2}{5} \gamma^2 - \frac{1}{5} \gamma^3 & P_3(\mathbf{I} \mathbf{I} \mathbf{I} | G'_\gamma) &= \frac{2}{5} \gamma^2 \\
P_3(\mathbf{I} | G_\gamma) &= 2 \gamma - \frac{27}{10} \gamma^2 + \frac{21}{20} \gamma^3 & P_3(\mathbf{I} | G'_\gamma) &= 2 \gamma - \frac{11}{5} \gamma^2 \\
P_3(\mathbf{I} \mathbf{I} | G_\gamma) &= 1 - 2 \gamma + \frac{17}{10} \gamma^2 - \frac{17}{30} \gamma^3 & P_3(\mathbf{I} \mathbf{I} | G'_\gamma) &= 1 - 2 \gamma + \frac{6}{5} \gamma^2
\end{align*}
\]

for three-point sprinklings. The corresponding ones for \( \gamma > \frac{1}{2} \) can be similarly calculated with the formalism described above.
V. The Distance Function

We can now extend the definition of the closeness function to a distance. Only the definition and a few comments will be given here; a more extensive study of its properties is left for future work. From the previous discussion, it should be clear that in this case we need to let the number of points sprinkled in each manifold go to infinity, so that we probe its structure at arbitrarily small scales. Also, it is not sufficient to let \( n \to \infty \) in \( d_n(G, G') \), both because the resulting distance may be trivial, and because, like all \( d_n \)'s, it would not distinguish between different values of the total volumes \( V_M \) and \( V_{M'} \). Each random event considered in previous sections was the choice of \( n \) points in each manifold; since \( n \) was the same for both manifolds, the outcomes gave us no information on their total volumes. We can overcome this limitation in the distance by letting each random event consist in the choice of both \( n \) and the location of the points, and this will give me an opportunity to mention one feature of the closeness functions that had not explicitly come up until now.

Consider two geometries \( G = \{(M, g)\} \) and \( G' = \{(M', g')\} \), as before. We will draw the number of points \( n \) to be sprinkled in each manifold from Poisson distributions, whose means will be proportional to the respective volumes, and will thus in general be different, the relationship between the two means being given by the point densities they correspond to. However, the two manifolds may be of different dimensions, \( D \) and \( D' \) respectively, in which case it would be meaningless to require the two volume densities of points to be equal; what we can require is equality of the “mean point spacings”, i.e., that the volume densities \( \rho \) and \( \rho' \) satisfy \( (\rho)^{1/D} = (\rho')^{1/D'} = \ell^{-1} \). (For example, in quantum gravity applications, we can think of \( \ell \) as being the Planck length \( \ell_P \), and the issue of different dimensionalities is relevant for higher-dimensional theories such as the Kaluza-Klein ones—for a recent review, see Ref. 33—where we might want to compare a macroscopic four-dimensional manifold \( 4M \) to a \( D \)-dimensional fundamental one which is, at least locally, considered to be a product of the type \( D M \simeq 4M \times D^{-4}M \), with \( D^{-4}M \) of volume \( \ell_P^{D-4} \).)

To define the distance function, choose a positive mean point spacing \( \ell \) around which most of the contribution to the distance will come from; sprinkle points in \( G \) and \( G' \) by first choosing, each time, the number of points according to Poisson distributions

\[
P_\mu(n) = \frac{e^{-\mu} \mu^n}{n!}, \quad P_\mu'(n) = \frac{e^{-\mu'} \mu'^n}{n!},
\]

respectively, where \( \mu := V_M / \ell^D \) and \( \mu' := V_{M'} / \ell^{D'} \), and distribute in \( (M, g) \) and \( (M', g') \) the chosen numbers of points uniformly at random; the probabilities of obtaining any given poset \( C \in C_n \) as a result are now, respectively,

\[
P_\ell(n, C | G) = P_\mu(n) P_n(C | G), \quad P_\ell(n, C | G') = P_\mu'(n) P_n(C | G');
\]

(39)
finally, compare these probabilities by extending (10) to

\[ d_\ell(G, G') := \frac{2}{\pi} \arccos \left[ \sum_{n=0}^{\infty} \sum_{C \in \mathcal{C}_n} \sqrt{P_\ell(n, C|G)} \sqrt{P_\ell(n, C|G')} \right] = \frac{2}{\pi} \arccos \left[ \sum_{n=0}^{\infty} \left( \sum_{C \in \mathcal{C}_n} \sqrt{P_\mu(n) P_{\mu'}(n)} \right) \sum_{C \in \mathcal{C}_n} \sqrt{P_n(C|G) P_n(C|G')} \right]. \] (41)

Here, I am assuming we have defined the probabilities \( P_n(C|G) \) for the sets of one-element posets, \( \mathcal{C}_1 = \{ \bullet \} \), and zero-element posets, \( \mathcal{C}_0 = \{ \emptyset \} \); if we set \( P_\ell(\bullet|G) = 1 \), consistently with the general definition, and adopt the convention that \( P_0(\emptyset|G) = 1 \), the argument of the \( \arccos \) function can be written as

\[ \sqrt{e^{-\mu} e^{-\mu'}} + \sqrt{\mu \mu' e^{-\mu} e^{-\mu'}} + \sum_{n=2}^{\infty} \left( \sum_{C \in \mathcal{C}_n} \sqrt{P_\mu(n) P_{\mu'}(n)} \right) \sum_{C \in \mathcal{C}_n} \sqrt{P_n(C|G) P_n(C|G')} \]. (42)

The expressions (41) and (42) are clearly well-defined; the rapid decrease of \( P_\mu(n) \) and \( P_{\mu'}(n) \) for large \( n \) makes them finite, and the fact that they are probabilities implies that (41) actually gives a number between 0 and 1, for the same reason as the pseudo-distance (10) did. To examine these two extreme situations, consider (42). This expression vanishes only in the large \( \mu \) or \( \mu' \) limit (\( \ell \to 0 \), or at least one of the volumes \( \to \infty \)), so that the first two terms vanish, and if the contribution from all \( n \geq 2 \) vanishes; this may imply that the conformal structures have the degeneracies described earlier in section III for \( d_\ell(G, G') \), at least if the manifolds have equal volumes, but in any case we can already see that \( d_\ell(G, G) = 1 \) only in situations obtained as limits of ones of the type under consideration.

The more interesting situation is when \( d_\ell(G, G') = 0 \). We can see from (41) that this implies \( P_n(C|G) = P_n(C|G') \) for all \( n \) and \( C \), since for all \( n \) the summation over \( C \in \mathcal{C}_n \) must equal 1, and \( P_\mu(n) = P_{\mu'}(n) \) for all \( n \), since the sum over \( n \) also must give 1. The latter conditions obviously mean that \( V_M = V_{M'} \), and the former set implies \( G = G' \); the sketch of a proof goes as follows. When we sprinkle an increasing number of points in a manifold, we build a sequence of posets \( \{ C_n \}_{n \in \mathbb{N}} \), with \( C_n \subset C_{n+1} \) for all \( n \). In the limit \( n \to \infty \), we obtain the direct limit of this sequence, \( C_\omega := \bigcup_{n=1}^{\infty} C_n \), and the equality of all \( P_n(C|G) = P_n(C|G') \) turns into the equality of appropriately defined probabilities \( P_\infty(C|G) \) and \( P_\infty(C|G') \) on \( \mathcal{C}_\infty \). Now, if we apply a completion procedure to \( C_\omega \), analogous to the Dedekind cut construction of real numbers from rational ones, we get the points of the original manifold together with their causal relations and the conformal factor, i.e., we get back the geometry \( G \) [20,31]. But if the two probabilities \( P_\infty(C|G) \) and \( P_\infty(C|G') \) are equal, and the completion of infinite posets drawn from them gives respectively \( G \) and \( G' \), with probability one, it must be that \( G = G' \), and thus \( d_\ell \) is positive-definite.

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VI. Concluding Remarks

To measure the closeness of Lorentzian geometries, I have introduced a family of pseudo-distances $d_n(G,G')$ and a family of distances $d_\ell(G,G')$ on the space of all past and future distinguishing Lorentzian geometries of finite volume. The main idea was to sprinkle points uniformly at random in $G$ and $G'$, and use the resulting probabilities $P_n(C|G)$ as the basic ingredients for the functions; in this paper, those probabilities were combined using one specific distance between probability measures, but others are known and may be more suitable in some applications. The closeness functions presented here, together with other possible such functions based on the same probabilities, are the only non-trivial diffeomorphism-invariant ones on this space that I am aware of.

A number of interesting questions arise about the statistical approach to Lorentzian geometry discussed here. Even before their use in the closeness functions, the probabilities $P_n(C|G)$ are interesting in themselves, as a complete set of invariants (together with the volume) of finite volume, distinguishing Lorentzian geometries. It would be worth while to study the type of information about the manifold that those different invariants contain; for example, how they encode dimensionality, how they are affected by conformal transformations as opposed to changes in the conformal structure, or how one can tell “localized” changes from “global” changes in a manifold from their effect on the $P_n(C|G)$. Possible starting points in answering these questions may consist in examining examples along the lines of the one in section IV but in which different parameters are varied, e.g., comparing a two-dimensional and a three-dimensional manifold, or modifying one by a conformal transformation; and studying analytically the effect of small variations $g \mapsto g + \delta g$.

The answer to questions of the above type may then allow us to word in a more precise way statements like “geometries for which $d_n(G,G')$ is small are close down to the scale $V_M/n$;” understand how the topology induced by the $d_n$’s and $d_\ell$’s on the space of Lorentzian geometries relates to previously studied ones [34]; and place bounds on the value of $d_n(G,G')$ when the actual value cannot be calculated, for example through bounds on the probabilities $P_n$ due to the non-embeddability of some $C$’s in a geometry, such as $C$’s that require higher dimensions. The infinite density limit and the properties of $d_\ell$ need to be understood better than what is sketched in section V, and the limit in which the “regulator” $\ell$ is taken to zero is a potentially useful one. It would also be useful to extend the present work to a definition of closeness that applies to infinite volume manifolds, as mentioned in section II; in that case, one may need to introduce a quasi-local element in the definition, and use finite size subsets of sprinklings of density $\rho$.

On the physical side, this work may be related to definitions of approximate solutions of Einstein’s equation [35, 36, 37], and spacetimes with approximate symmetries [38], which have been considered for various reasons, including their relevance to the issue of gravitational entropy and the smoothing problem in cosmology [39]. These problems, in addition to the motivation coming from quantum gravity, make it an interesting issue to study properties of $P_n(C|G)$, by analytical methods or numerical simulations.
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References

1. P.E. Gibbs, “The small-scale structure of spacetime: A bibliographical review,” [hep-th/9506171].
2. A. Kempf, “On the structure of spacetime at the Planck scale,” [hep-th/9810215].
3. A. Ashtekar, “Quantum mechanics of geometry,” [gr-qc/9901023].
4. J.A. Wheeler, “On the nature of quantum geometrodynamics,” Ann. Phys. 2, 604-614 (1957);
   ——Geometrodynamics (Academic Press 1962), pp 71–83.
5. S.W. Hawking, “Space-time foam” Nucl. Phys. B144, 349-362 (1978).
6. L. Crane and L. Smolin, “Spacetime foam as a universal regulator” Gen. Rel. Grav. 17, 1209 (1985).
7. J. Myrheim, “Statistical geometry,” 1978 preprint, Ref. TH 2538-CERN, unpublished.
8. G. ’t Hooft, “Quantum gravity: a fundamental problem and some radical ideas,” in Recent Developments in Gravitation. Cargèse 1978, M. Lévy and S. Deser, eds. (Plenum 1979).
9. L. Bombelli, J. Lee, D.A. Meyer and R.D. Sorkin, “Spacetime as a causal set,” Phys. Rev. Lett. 59, 521-524 (1987); see also C. Moore, “Comment to ‘Spacetime as a causal set’” (and reply), Phys. Rev. Lett. 60, 655-656 (1988).
10. L. Bombelli, Spacetime as a Causal Set, PhD thesis, Syracuse University 1987.
11. R.D. Sorkin, “A specimen of theory construction from quantum gravity,” in The Creation of Ideas in Physics, J Leplin, ed (Kluwer 1995), and gr-qc/9511063.
12. G. Brightwell and R. Gregory, “Structure of random discrete spacetime,” Phys. Rev. Lett. 66, 260-263 (1991).
13. M.P. Reisenberger and C. Rovelli, “Sum over surfaces form of loop quantum gravity,” Phys. Rev. D 56, 3490-3508 (1997), and gr-qc/9612033.
14. J.C. Baez, “An introduction to spin foam models of quantum gravity and BF theory,” gr-qc/9905087.
15. R. De Pietri, “Canonical loop quantum gravity and spin foam models,” gr-qc/9903076.
16. J. Iwasaki, “A surface theoretic model of quantum gravity,” gr-qc/9903112.
17. F. Markopoulou and L. Smolin, “Quantum geometry with intrinsic local causality,” Phys. Rev. D 58: 084032 (1998), and gr-qc/9712067.
18. S. Gupta, “Causality in spin foam models,” Phys. Rev. D 61 (2000) 064014, and gr-qc/9908018.
19. L. Bombelli and R.D. Sorkin, unpublished work.
20. L. Bombelli and D.A. Meyer, “The origin of Lorentzian geometry,” *Phys. Lett.* **141A**, 226-228 (1989).
21. M. Gromov, *Structures métriques pour les variétés riemanniennes* (Paris: Cedic/F Nathan 1981).
22. M. Seriu, “The spectral representation of the spacetime structure: The ‘distance’ between universes with different topologies,” *Phys. Rev.* D **53**, 6902-6920 (1996), and gr-qc/9603002.
   ———“Space of spaces as a metric space,” *Commun. Math. Phys.* **209**, 393-405 (2000), and gr-qc/9908078.
23. L. Bombelli, “Causal sets and the closeness of Lorentzian manifolds,” in *Relativity in General* (Spanish Relativity Meeting 1993), J. Díaz Alonso and M. Lorente Páramo, eds. (Editions Frontière 1994).
24. L.B. Szabados, “Causal measurability in chronological spaces,” *Gen. Rel. Grav.* **19**, 1091-1100 (1987).
25. S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press 1973).
26. R.M. Wald, *General Relativity* (University of Chicago Press 1984).
27. D.A. Meyer, *The Dimension of Causal Sets*, PhD thesis, MIT 1988.
28. W.K. Wootters, “Statistical distances in Hilbert spaces,” *Phys. Rev.* D **23**, 357-362 (1981).
29. S.W. Hawking, A.R. King and P.J. McCarthy, “A new topology for curved space-time which incorporates the causal, differential and conformal structures,” *J. Math. Phys.* **17**, 174-181 (1976).
30. D. Malament, “The class of continuous timelike curves determines the topology of spacetime,” *J. Math. Phys.* **18**, 1399-1404 (1977).
31. D.A. Meyer and R.D. Sorkin, unpublished work.
32. D.J. Kleitman and B.L. Rothschild, “Asymptotic enumeration of partial orders on a finite set,” *Trans. Am. Math. Soc.* **205**, 205-220 (1975).
33. J.M. Overduin and P.S. Wesson, “Kaluza-Klein gravity,” *Phys. Rep.* **283**, 303-380 (1997), and gr-qc/9805018.
34. J.K. Beem and P.E. Ehrlich, *Global Lorentzian Geometry* (Dekker 1981).
35. L. Bel, “The quality factor of approximate solutions of Einstein’s equations,” *Gen. Rel. Grav.* **19**, 1127-1130 (1987).
36. J. Garriga and E. Verdaguer, “Testing the quality factor with some approximate solutions to Einstein’s eqs,” *Gen. Rel. Grav.* **20**, 1249-1262 (1988).
37. S. Detweiler and L.H. Brown, “The post Minkowskii expansion of general relativity,” *Phys. Rev.* D **56**, 826-841 (1997), and gr-qc/9609010.
38. R. Zalaletdinov, “Approximate symmetries in general relativity,” gr-qc/9912021.
39. M. Seriu, “Spectral representation and the averaging problem in cosmology,” gr-qc/0001014, to appear in *Gen. Rel. Grav.*
Figure Captions

**Figure 1:** The geometry $G_{\gamma}$. The drawing shows the case $b < a$, or $\gamma < 1$, with a sprinkled point $x_1$ and its future light cone. For this particular point, $V(I^+(x_1)) = (b - t)^2 - \frac{1}{2} [(b - t) - (a + x)]^2$, where the area of the small triangle with dashed edges in the upper right hand corner must be subtracted, since $a - x < b - t$.

**Figure 2:** The geometry $G'_{\gamma}$. The drawing shows the case $b > a/2$, or $\gamma > \frac{1}{2}$, with a sprinkled point $x_1$ and its future light cone. For this particular point, $V(I^+(x_1)) = (b - t)^2 - (b - t - a/2)^2$, where the area of the two small triangles with dashed edges must be subtracted, since $b - t > a/2$. Any two points whose $x$ coordinates differ by $a$ are identified; in particular, the two outer vertical dashed lines are to be identified with each other.

**Figure 3:** Plot of $d_2(G_{\gamma}, G'_{\gamma})$ as a function of $\gamma$, for $0 < \gamma < 5$; for $\gamma > 5$, the function decreases monotonically, and approaches zero as $\gamma \to \infty$. 


