BRAIDED HOPF ALGEBRAS FROM TWISTING.

ARKADIUSZ BOCHNIAK AND ANDRZEJ SITARZ

Abstract. We show that a class of braided Hopf algebras, which includes the braided $SU_q(2)$ of [7], is obtained by twisting. We show further examples and demonstrate that twisting of bicovariant differential calculi gives braided bicovariant differential calculi.

1. Introduction

Hopf algebras are a natural generalization of symmetries in mathematics, providing an unifying picture of discrete groups, Lie groups and Lie algebras. Apart from their role as symmetries of hypothetical quantum space-time in models of quantum gravity, they appear in the description of renormalization and arise in integrable models and around solutions of Yang-Baxter equation. Some interesting generalizations of Hopf algebras are, however, slightly different, as their coproduct is an algebra homomorphism only if one passes to the braided category. Such objects are natural candidates to generalize symmetries of quantum physical systems with nontrivial statistics, like anyons, but they appear frequently as models of noncommutative spaces and noncommutative geometries [3]. So far the existence of braided symmetries there has been not studied there.

In [7] the authors introduced and studied a family of deformations of the algebra of functions on the $SU(2)$ group, which are braided Hopf algebras. Though the deformation and the description in the above paper was in the $C^*$-algebraic language it is interesting to notice that the structure is easily translated for the Hopf algebra of the quantum $SU(2)$ group.

Our original motivation was to construct braided differential calculi for the braided $SU_q(2)$. In this note we show that the general framework of twisting provides the answer, that twisting applied to bicovariant differential calculi over the Hopf algebra (as classified by Woronowicz in [13]) gives braided bicovariant calculus.

We provide also a further class of examples that include braided $SU_q(n)$ and braided quantum double torus.

Throughout the paper we work with braided Hopf algebras, which are Hopf algebras in a braided tensor category, with a given braiding $\Psi$ as defined in [19].

2. Twisting Hopf Algebras

Let $H$ be a Hopf algebra with the coproduct $\Delta$, the counit $\epsilon$ and the antipode $S$. Let $\chi : H \to \mathbb{C}[z, z^{-1}]$ be a Hopf algebra homomorphism, where $\mathbb{C}[z, z^{-1}]$ is the Hopf algebra of the group algebra of $\mathbb{Z}$.

Lemma 2.1. The following maps define left and right coactions of $\mathbb{C}[z, z^{-1}]$ on $H$,

$$\Delta_L = (\chi \otimes \text{id})\Delta, \quad \Delta_R = (\text{id} \otimes \chi)\Delta,$$
which are equivariant under the coproduct in \( H \).

**Proof.** We check only \( \Delta_L \). Using the fact that \( \chi \) is Hopf algebra morphism:
\[
(id \otimes \Delta_L) \Delta_L(x) = (id \otimes \chi \otimes id)(id \otimes \Delta)(\chi \otimes id)\Delta(x) = \chi(x_{(1)}) \otimes \chi(x_{(2)}) \otimes x_{(3)}
\]
\[
= (\Delta_{\mathbb{C}[z,z^{-1}]} \otimes id)(\chi(x_{(1)}) \otimes x_{(2)}) = (\Delta_{\mathbb{C}[z,z^{-1}]} \otimes id)\Delta_L(x),
\]
and
\[
(\epsilon_{\mathbb{C}[z,z^{-1}]} \otimes id)\Delta_L = (\epsilon_{\mathbb{C}[z,z^{-1}]} \otimes id)(\chi \otimes id)\Delta = (\epsilon \otimes id)\Delta = id.
\]
The equivariance under the coproduct in \( H \) means:
\[
(id \otimes \Delta)\Delta_L(x) = (id \otimes \Delta)(\chi(x_{(1)}) \otimes x_{(2)}) = \chi(x_{(1)}) \otimes x_{(2)} \otimes x_{(3)}
\]
\[
= \Delta_L(x_{(1)}) \otimes x_{(2)} = (\Delta_L \otimes id)\Delta(x).
\]
\[\square\]

**Definition 2.2.** We say that \( x \in H \) is a homogeneous element of degrees \( \mu(x), \nu(x) \), where \( \mu(x), \nu(x) \in \mathbb{Z} \) if:
\[
\Delta_L(x) = z^{\mu(x)} \otimes x, \quad \Delta_R(x) = x \otimes z^{\nu(x)}.
\]
We also define the index
\[
\delta(x) = \mu(x) - \nu(x).
\]
We shall study now properties of the maps \( \mu \) and \( \nu \).

**Lemma 2.3.** The degrees are additive, that is:
\[
\mu(xy) = \mu(x) + \mu(y), \quad \nu(xy) = \nu(x) + \nu(y).
\]
This follows directly from the definition of degrees and the fact that \( \mathbb{C}[z,z^{-1}] \) is a \( \mathbb{Z} \)-graded algebra.

From now on we assume that the Hopf algebra \( H \) has a countable basis of homogeneous elements, so that any element of \( H \) could be presented as a finite sum of elements with fixed degrees \( \mu \) and \( \nu \).

**Lemma 2.4.** We have the following identities for any \( x \in H \) and \( \Delta x = \sum_i x_{i\,(1)}^i \otimes x_{i\,(2)}^i \), where \( x_{i\,(1)}^i \) and \( x_{i\,(2)}^i \) are homogeneous:
\[
\mu(x) = \mu(x_{i\,(1)}^i), \quad \nu(x) = \nu(x_{i\,(2)}^i), \quad \text{for any } i,
\]
and
\[
\mu(x_{i\,(2)}^i) = \nu(x_{i\,(1)}^i), \quad \text{for any } i.
\]

**Proof.** From the identity:
\[
(id \otimes \Delta)\Delta_L = (\Delta_L \otimes id)\Delta,
\]
we see that for a homogeneous \( x \) of degree \( \mu(x) \):
\[
(id \otimes \Delta)\Delta_L(x) = z^{\mu(x)} \otimes \Delta(x) = \sum_i z^{\mu(x)} \otimes x_{i\,(1)}^i \otimes x_{i\,(2)}^i,
\]
but on the other hand:
\[
(\Delta_L \otimes id)\Delta(x) = \sum_i \Delta_L(x_{i\,(1)}^i) \otimes x_{i\,(2)}^i
\]
By comparing these two expressions and taking into account that each \(x^{i}_{(1)}\) in the sum are homogeneous and are the basis of the algebra, we obtain

\[
\mu(x) = \mu(x^{i}_{(1)}), \quad \text{for every } i.
\]

Similarly we prove the remaining identities. \(\square\)

In particular, as a consequence we have:

**Corollary 2.5.** If \(x = \sum_{i} x^{i}_{(1)} \otimes x^{i}_{(2)}\) then, for any \(i\):

\[
\delta(x) = \delta(x^{i}_{(1)}) + \delta(x^{i}_{(2)}).
\]

**Lemma 2.6.** For each homogeneous \(x\) we have:

\[
\mu(S(x)) = -\nu(x), \quad \nu(S(x)) = -\mu(x).
\]

**Proof.** Since for each element \(x\) we have:

\[
\Delta(S(x)) = S(x_{(2)}) \otimes S(x_{(1)}),
\]

composing it with \(\chi\), which is a Hopf algebra morphism, so \(\chi(S(x)) = S_{\mathbb{C}[z, z^{-1}]}(\chi(x))\) and using the form of the antipode on \(\mathbb{C}[z, z^{-1}]\) algebra we obtain that for a homogeneous \(x\) with degrees \(\mu(x), \nu(x)\) we have:

\[
\Delta_{L}(S(x)) = S_{\mathbb{C}[z, z^{-1}]}(z^{\mu(x)}) \otimes S(x) = z^{-\nu(x)} \otimes S(x),
\]

\[
\Delta_{R}(S(x)) = S(x) \otimes S_{\mathbb{C}[z, z^{-1}]}(z^{\mu(x)}) = S(x) \otimes z^{-\mu(x)}.
\]

\(\square\)

One of the consequences of the compatibility of coactions with the coproduct and the antipode is the following fact:

**Lemma 2.7.** For any homogeneous element \(x\), \(\epsilon(x) = 0\) if \(\delta(x) \neq 0\).

**Proof.** Using the property that \(\Delta, \epsilon, \chi\) are morphisms we have:

\[
m \circ (\chi \otimes \epsilon)\Delta(x) = \chi(x_{(1)})\epsilon(x_{(2)}) = \chi(x).
\]

However, for a homogeneous \(x\):

\[
m(\chi \otimes \epsilon)\Delta(x) = z^{\mu(x)}\epsilon(x).
\]

On the other hand:

\[
m(\epsilon \otimes \chi)\Delta(x) = \epsilon(x_{(1)})\chi(x_{(2)}) = \chi(x),
\]

and

\[
m(\epsilon \otimes \chi)\Delta(x) = \epsilon(x)z^{\nu(x)}.
\]

Therefore, for any homogeneous \(x\) we have:

\[
z^{\delta(x)}\epsilon(x) = \epsilon(x).
\]

\(\square\)
2.1. The twisted product.

**Lemma 2.8.** The following map defined on homogeneous elements \( x, y \in H \) for any \( \phi \in \mathbb{R} \):
\[
\Phi(x, y) = e^{i\phi(\mu(x)\nu(y) - \nu(x)\mu(y))}
\]
extends to a two-cocycle on \( H \), that is a map \( \Phi : H \otimes H \to U(1) \), which satisfies for all \( x, y, z \in H \):
\[
\Phi(x, yz)\Phi(y, z) = \Phi(x, y)\Phi(xy, z).
\]

**Proof.** Since for homogeneous elements \( x, y \in A \) we have Lemma 2.3, we further compute,
\[
\Phi(x, yz)\Phi(y, z) = e^{i\phi(\mu(x)\nu(yz) - \nu(x)\mu(yz))} e^{i\phi(\mu(y)\nu(z) - \nu(z)\mu(y))}
\]
\[
= e^{i\phi(\mu(x)(\nu(y)+\nu(z))-\nu(x)(\mu(y)+\mu(z)))} e^{i\phi(\mu(y)\nu(z) - \nu(z)\mu(y))}
\]
\[
= e^{i\phi(\mu(x)\nu(y)-\nu(x)\mu(y))} e^{i\phi(\mu(xy)\nu(z)-\nu(z)\mu(xy))}
\]
\[
= \Phi(x, y)\Phi(xy, z).
\]

\( \square \)

As a natural consequence we have,

**Lemma 2.9.** The following product defined on homogeneous elements of \( H \) extends linearly to an associative product on \( H \):
\[
x \ast y = \Phi(x, y) x \cdot y.
\]

Moreover, note that since \( \mu(1) = \nu(1) = 0 \) the algebra we obtain is unital (with the same unit 1 as in \( H \)). We will denote this algebra by \( H_{\phi} \).

2.2. The braiding. We introduce the following prebraiding on \( H \otimes H \).

**Lemma 2.10.** The map defined on homogeneous elements \( x, y \in H_{\phi} \):
\[
(2.3) \quad \Psi(x \otimes y) = e^{2i\phi(x)\delta(y)} y \otimes x,
\]
defines a prebraiding on \( H \otimes H \).

**Proof.** We can introduce degrees of tensor products of homogeneous elements, in the following way. Taking \( x = x_1 \otimes x_2 \otimes \cdots \otimes x_n \), we define:
\[
\delta(x) = \sum_{i=1}^{n} \delta(x_i).
\]
then, for \( x \) as above and \( y = y_1 \otimes \cdots \otimes y_k \) we define the braiding on tensors as:
\[
\Psi(x \otimes y) = e^{2i\phi(x)\delta(y)} (y \otimes x).
\]

In this way we obtain a prebraided monoidal structure. Indeed one can easily check that
\[
\Psi((x \otimes y) \otimes z) = (\Psi \otimes \text{id})(\text{id} \otimes \Psi)(x \otimes y \otimes z),
\]
\[
\Psi(x \otimes (y \otimes z)) = (\text{id} \otimes \Psi)(\Psi \otimes \text{id})(x \otimes y \otimes z),
\]
if \( \delta \) satisfies \( \delta(x \otimes y) = \delta(x) + \delta(y) \). \( \square \)
From this prebraided structure defined above we obtain a braided monoidal category of vector spaces, if as morphisms we take maps which commute with $\Psi$. For $F : H \to H$ it is sufficient to check that $\delta \circ F = \delta$. This follows from the fact that in this case the naturality conditions:

$$\Psi(F \otimes \text{id}) = (\text{id} \otimes F)\Psi, \quad \Psi(\text{id} \otimes F) = (F \otimes \text{id})\Psi,$$

are satisfied. We shall denote this category by $\mathcal{H}_\phi^\otimes \equiv (\mathcal{H}_\phi^\otimes, \text{Mor} (\mathcal{H}_\phi^\otimes), \Psi)$. As a result we have that $\Psi$ is a Yang-Baxter operator, which can be also checked directly:

$$(\text{id} \otimes \Psi)(\Psi \otimes \text{id})(\text{id} \otimes \Psi)(x \otimes y \otimes z) =$$

$$= e^{2i\phi(\delta(x)\delta(y) + \delta(x)\delta(z) + \delta(y)\delta(z))} (z \otimes y \otimes x)$$

$$= (\Psi \otimes \text{id})(\Psi \otimes \text{id})(x \otimes y \otimes z),$$

This follows again from the properties of $\delta$.

2.3. The coproduct.

**Theorem 2.11.** Using a similar the one used to define the braiding we construct the coproduct on $H_\phi$:

$$\Delta_\phi(x) = \sum_j e^{i\phi(\delta(x_1)\delta(x_2))} x_{(1)}^j \otimes x_{(2)}^j,$$

**Proof.** For simplicity we skip here the sum and the summation indices, one should remember that we are using the basis of homogeneous elements and the coproduct $\Delta(x) = x_{(1)} \otimes x_{(2)}$ is in fact a well-defined unique expression and it is meant to be $\sum_j x_{(1)}^j \otimes x_{(2)}^j$ where all terms are homogeneous elements of the basis.

Let us verify that the coproduct (2.4) is coassociative:

$$(\text{id} \otimes \Delta_\phi)\Delta_\phi(x) = e^{i\phi(\delta(x_1)\delta(x_2))} e^{i\phi(\delta(x_{(2,1)}))} x_{(1)} \otimes x_{(2,1)} \otimes x_{(2,2)},$$

but

$$(\Delta_\phi \otimes \text{id})\Delta_\phi(x) = e^{i\phi(\delta(x_1))} e^{i\phi(\delta(x_{(1,1)}))} x_{(1,1)} \otimes x_{(1,2)} \otimes x_{(2)},$$

Then, both sides are equal to each other if Lemma 2.4 is satisfied. \qed

**Theorem 2.12.** The coproduct $\Delta_\phi$ (2.4) satisfies

$$\Delta_\phi(x \ast y) = \Delta_\phi(x) \ast \Delta_\phi(y),$$

where $(a \otimes b) \ast (c \otimes d) := a \Psi(b \otimes c)d$.

**Proof.** Let us take two homogeneous elements $x, y \in H$ with their coproducts:

$$\Delta x = x_{(1)} \otimes x_{(2)}, \quad \Delta y = y_{(1)} \otimes y_{(2)}.$$

We compute first:

$$\Delta_\phi(x \ast y) = e^{i\phi(\mu(x)\nu(y) - \nu(x)\mu(y))} \Delta_\phi(xy)$$

$$= e^{i\phi(\mu(x)\nu(y) - \nu(x)\mu(y))} e^{i\phi(\delta(x_1) + \delta(y_1))} (x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}).$$
On the other hand:
\[
\Delta_\phi(x) \ast \Delta_\phi(y) = e^{i \phi(x(1)) \phi(x(2))} e^{i \phi(y(1)) \phi(y(2))} \left( x(1) \otimes x(2) \right) \ast \left( y(1) \otimes y(2) \right)
\]
\[
= e^{i \phi(x(1)) \phi(x(2))} e^{i \phi(y(1)) \phi(y(2))} e^{2i \phi(x(2)) \phi(y(1))}.
\]

A simple computation, using Lemma 2.4 allows to demonstrate that both scalar factors are identical, hence the map \(\Delta_\phi\) satisfies \(\Delta_\phi(x \ast y) = \Delta_\phi(x) \ast \Delta_\phi(y)\).

**Theorem 2.13.** \(H_\phi\) is a braided bialgebra in braided category with the braiding given by \([2.3]\).

**Proof.** Let us denote \(m_\ast(x \otimes y) = x \ast y\). From Lemma 2.9 we know that \((H_\phi, m_\ast, 1)\) is an algebra. Theorem 2.11 implies coassociativity of \(\Delta_\phi\), properties of counit follows from Lemma 2.7. It means that \((H_\phi, \Delta_\phi, \epsilon)\) forms coalgebra.

Now, we check that the structures \(m_\ast, \Delta_\phi, \epsilon\) commute with \(\Psi\), i.e.

\[
\begin{align*}
(2.5) & \quad \Psi(id \otimes m_\ast) = (m_\ast \otimes id)(id \otimes \Psi)(\Psi \otimes id) \\
(2.6) & \quad \Psi(id \otimes m_\ast)(\Psi \otimes id)(id \otimes \Psi) \\
(2.7) & \quad (id \otimes \Delta_\phi)\Psi = (\Psi \otimes id)(id \otimes \Psi)(\Delta_\phi \otimes id) \\
(2.8) & \quad (\Delta_\phi \otimes id)\Psi = (id \otimes \Psi)(\Psi \otimes id)(id \otimes \Delta_\phi)
\end{align*}
\]

Indeed, one can easily compute that
\[
\Psi(id \otimes m_\ast)(x \otimes y \otimes z) = e^{i \phi(\mu(y) \mu(z) - \mu(z) \mu(y))} e^{2i \phi(x) \phi(y)} y z \otimes x
\]
and
\[
(m_\ast \otimes id)(id \otimes \Psi)(\Psi \otimes id) = e^{i \phi(\mu(y) \mu(z) - \mu(z) \mu(y))} e^{2i \phi(x) \phi(y)} e^{2i \phi(x) \phi(y)} y z \otimes x.
\]

Hence we obtain (2.5) and similarly (2.6). To prove (2.7), (2.8) we do analogous computation but in the last step we use (2.1).

Moreover, by Theorem 2.12 we know that \(\Delta_\phi\) is a morphism of algebras. To finish the proof we need to show that \(\epsilon\) is an algebra map:
\[
\epsilon(x \ast y) = \epsilon(x) \epsilon(y).
\]

By definition we have:
\[
\epsilon(x \ast y) = e^{i \phi(\mu(x) \mu(y) - \nu(x) \mu(y))} \epsilon(x) \epsilon(y).
\]

The only nontrivial case is when \(\epsilon(x) \neq 0\) and \(\epsilon(y) \neq 0\). But by Lemma 2.7 then \(\mu(x) = \nu(x)\) and \(\mu(x) = \nu(y)\) hence the above equality holds. \(\square\)

### 2.4. The antipode.

Finally, we extend an antipode \(S\) defining \(S_\phi : H_\phi \rightarrow H_\phi\) in the following way.

**Theorem 2.14.** The antipode, defined on homogeneous elements as:
\[
S_\phi(x) = e^{i \phi(x)^2} S(x),
\]

makes \(H_\phi\) is a braided Hopf algebra in \(H_\phi^\otimes\).
Proof. To show that it is also a Hopf algebra we need to check that $S_\phi \in \mathcal{M}or\left(\mathcal{H}_\phi^\otimes\right)$. It is enough to check that $\delta \circ S = \delta$. But it simply follows from lemma 2.6
\begin{equation}
(\delta \circ S)(x) = \mu(S(x)) - \nu(S(x)) = -\nu(x) - (-\mu(x)) = \mu(x) - \nu(x) = \delta(x),
\end{equation}
i.e. $\delta \circ S = \delta$. Hence we have $S_\phi \in \mathcal{M}or\left(\mathcal{H}_\phi^\otimes\right)$.
Next, we verify:
\[m_*(S_\phi \otimes \text{id})\Delta_\phi(x) = e^{i\phi(x(1))\delta(x(2))}S_\phi(x(1)) \ast x(2)\]
\[= e^{i\phi(x(1))\delta(x(2))} e^{i\phi(x(1)) \nu(x(2)) + \mu(x(2))} S(x(1)) \ast x(2)\]
\[= e^{i\phi(x(1))\delta(x(2))} e^{i\phi(x(1)) \nu(x(2)) + \mu(x(2))} S(x(1)) \ast x(2)\]
\[= e^{i\phi(x(1))\delta(x)} \epsilon(x) = \epsilon(x).
\]
The last step follows from the fact that $\delta(x)$ does not vanish only on kernel of $\epsilon$, therefore once $\epsilon(x) \neq 0$ then the scalar factor in front $\epsilon(x)$ is 1.

We finish the section by observing that if $H$ is Hopf*-algebra and $\chi$ is a *-morphism then $H_\phi$ is a braided Hopf *-algebra with
\[(x \otimes y)^* = \psi(x, y)x^* \otimes y^*,\]
where $\psi(x, y)$ is the phase factor in the braiding, i.e. $\Psi(x \otimes y) = \psi(x, y)y \otimes x$.

Remark 2.15. Observe, that due to the theorem of Schauenburg: [11], for a Hopf algebra in the prebraided monoidal category $(\mathcal{C}, \Psi)$, we have:
\[\Psi = (m_* \otimes m_*)(S_\phi \otimes (\Delta_\phi \circ m_*) \otimes S_\phi)(\Delta_\phi \otimes \Delta_\phi).
\]
We verify, that for two homogeneous elements $x, y \in H$:
\[\Psi(x, y) = S_\phi(x(1)) \ast (x(2) \ast y(1))_{(1)} \otimes (x(2) \ast y(1))_{(2)} \ast S_\phi(y(2))\]
\[= e^{i\phi(x, y)} y \otimes x,
\]
where the phase $\zeta(x, y)$ reads, using shortcut $z = x(2)y(1)$,
\[\zeta(x, y) = \delta(x(1))\delta(x(2)) + \delta(y(1))\delta(y(2)) + \delta(x(1))^2 + \delta(y(2))^2\]
\[+ \left(\mu(x(2))\nu(y(1)) - \nu(x(2))\mu(y(1))\right) + \delta(z(1))\delta(z(2))\]
\[+ \left(\mu(Sx(1))\nu(z(1)) - \mu(z(1))\nu(Sx(1))\right) + \left(\mu(z(2))\nu(Sy(2)) - \mu(Sy(2))\nu(z(2))\right)
\]
Of course, the phase factor enters only when the relevant terms are non-zero, which happens only if the counit does not vanish, and that is exactly when:
\[\epsilon(x(1))\epsilon((x(2))(1)) \neq 0, \quad \epsilon((y(1))(2))\epsilon(y(2)) \neq 0\]
Let's take elements with $x$ only:
\[\delta(x(1))\delta(x(2)) + \delta(x(1))^2 + \delta((x(2))(1))\delta((x(2))(2)) + \left(-\nu(x(1))\nu((x(2))(1)) + \mu(x(1))\mu((x(2))(1))\right)
\]
2.5. Twisting and untwisting. Let us note that the twisting of the product, coproduct and braiding depends solely on the bigrading $\mu, \nu$ that satisfied the properties from Lemmas 2.3, 2.4, 2.6 and Lemma 2.7. It is easy to see that neither of this properties is changed in any way if we pass to the algebra $H_\phi$ with the braiding (2.3). Both $H$ and $H_\phi$ share the same basis of elements with fixed grading $\mu, \nu$ and since all products and coproducts are deformed in a homogeneous way, neither of above condition is changed. Therefore, $H_\phi$ can be further twisted using the above procedure and after a twist by an angle $\eta$ we obtain, in fact a Hopf algebra, which is a twist of the original Hopf algebra $H$ by $\phi + \eta$.

3. Examples

3.1. The algebra $\mathbb{C}[z, z^{-1}]$. Consider the $H = \mathbb{C}[z, z^{-1}]$ with $\chi$ being the identity map. Since the algebra is cocommutative then $\mu(x) \equiv \nu(x)$ for every homogeneous $x$ and therefore $H_\phi = H$ in this case.

3.2. The algebra $C_\lambda(a, b)$. Consider a unital Hopf algebra generated by $a, a^{-1}, b$ with relation:

$$ab = \lambda ba,$$

and coproduct,

$$\Delta(a) = a \otimes a, \quad \Delta(b) = a \otimes b + b \otimes 1,$$

counit and antipode:

$$\epsilon(a) = 1, \quad \epsilon(b) = 0, \quad S(a) = a, \quad S(b) = -a^{-1}b.$$  

If, additionally $\lambda$ is real then it is Hopf $*$-algebra with $b = b^*$ and $a^* = a^{-1}$.

We take the map $\chi : C_\lambda(a, b) \to \mathbb{C}[z, z^{-1}]$ as:

$$\chi(a) = z, \quad \chi(b) = 0,$$

which is a morphism of Hopf algebras (and of Hopf $*$-algebras if we consider the $*$ structure). we easily see that:

$$\mu(a) = \nu(a) = \mu(b) = 1, \quad \nu(b) = 0,$$

so that $\delta(a) = 0$ and $\delta(b) = 1$.

The twisted braided Hopf algebra $C_\lambda(a, b)_\phi$ is then generated by $a, b$ which obey the relation:

$$a * b = e^{-2i\phi} \lambda b * a,$$

with the same coproduct $\Delta_\phi = \Delta$ and the antipode:

$$S_\phi(a) = S(a), \quad S_\phi(b) = e^{i\phi} S(b),$$

with the only nontrivial braiding:

$$\Psi(b, b) = e^{2i\phi} b \otimes b.$$
3.3. The Hopf algebra $\mathcal{A}(SU_q(n))$.

The $\mathcal{A}(SU_q(n))$ algebra is generated by a unitary matrix elements $u_{ij}, 1 \leq i, j \leq n$, with relations:

\[
\begin{align*}
&u_{ik}u_{jk} = qu_{jk}u_{ik}, (i < j), & u_{ki}u_{kj} = qu_{kj}u_{ki}, (i < j), \\
u_{il}u_{jk} = u_{jk}u_{il}, (i < j; k < l), & u_{ik}u_{jl} - u_{il}u_{jk} = (q - q^{-1})u_{jk}u_{il}, (i < j; k < l), \\
\sum_{\sigma} (-q)^{|\sigma|} u_{1\sigma(1)} \cdots u_{n\sigma(n)} = 1,
\end{align*}
\]

where $|\sigma|$ is a number of inversions in permutation $\sigma \in S_n$.

The Hopf algebra structure on $\mathcal{A}(SU_q(n))$ is given by

\[
\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \epsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = (u_{ji})^*,
\]

where the $\ast$-structure is given by

\[
(u_{ij})^* = (-q)^{j-i} \sum_{\sigma} (-q)^{|\sigma|} u_{k_1\sigma(1)} \cdots u_{k_{n-1}\sigma(l_{n-1})},
\]

where

\[
(k_1, \ldots, k_{n-1}) = (1, \ldots, n) \setminus \{i\}, \quad (l_1, \ldots, l_{n-1}) = (1, \ldots, n) \setminus \{j\}
\]
treated as ordered sets.

Let $p : \{1, 2, \ldots, n\} \to \mathbb{Z}$ be a function such that $\sum_{k=1}^n p(k) = 0$.

**Lemma 3.1.** The following map:

\[
\chi_p : u_{ij} \mapsto z^{p(i)} \delta_{ij},
\]

is a $\ast$-algebra homomorphism from $\mathcal{A}(SU_q(n))$ to $\mathbb{C}[z, z^{-1}]$.

**Lemma 3.2.** $\chi_p$ is a coalgebra morphism.

**Proof.** We calculate

\[
(\chi_p \otimes \chi_p)\Delta(u_{ij}) = (\chi_p \otimes \chi_p) \left( \sum_k u_{ik} \otimes u_{kj} \right) = \sum_k z^{p(i)} \delta_{ik} \otimes z^{p(j)} \delta_{jk} = z^{p(i)} \otimes z^{p(j)} \delta_{ij}.
\]

On the other hand we have

\[
\Delta_{\mathbb{C}[z, z^{-1}]}(\chi_p(u_{ij})) = \Delta_{\mathbb{C}[z, z^{-1}]}(z^{p(i)} \delta_{ij}) = z^{p(i)} \otimes z^{p(j)} \delta_{ij}.
\]

Moreover,

\[
(\chi_p \otimes \chi_p)\Delta(1) = 1 \otimes 1 = \Delta_{\mathbb{C}[z, z^{-1}]}(\chi_p(1)).
\]

**Lemma 3.3.** $\chi_p$ is a Hopf algebra morphism.

**Proof.** Simple calculation, using definition of $\ast$-structure, gives us $\chi_p(u_{ij}^*) = z^{-p(i)} \delta_{ij}$. Hence

\[
(\chi_p \circ S)(u_{ij}) = \chi_p(u_{ji}^*) = z^{-p(j)} \delta_{ij}
\]

but on the other hand we have

\[
(S_{\mathbb{C}[z, z^{-1}]} \circ \chi_p)(u_{ij}) = S_{\mathbb{C}[z, z^{-1}]}(z^{p(i)} \delta_{ij}) = z^{-p(i)} \delta_{ij}.
\]
Let us now calculate degrees of generators.
\[ \Delta_L(u_{ij}) = (\chi_p \otimes \text{id}) \left( \sum_k u_{ik} \otimes u_{kj} \right) = \sum_k \delta_{ik} z^{p(i)} \otimes u_{kj} = z^{p(i)} \otimes u_{ij}. \]

Similarly \( \Delta_R(u_{ij}) = u_{ij} \otimes z^{p(j)}. \)

Hence \( \mu(u_{ij}) = p(i), \nu(u_{ij}) = p(j) \) and
\[ \delta(u_{ij}) = p(i) - p(j). \]

Therefore, by the general construction we obtain

**Corollary 3.4.** The algebra \( A_{\phi}(SU_q(n)) \) is a braided Hopf algebra with the product:
\[ u_{ij} \ast u_{kl} = e^{i\phi(p(i)p(l) - p(k)p(j))} u_{ij} \cdot u_{kl}, \]
the coproduct:
\[ \Delta_{\phi}(u_{ij}) = \sum_k e^{i\phi(p(i)p(k) - p(j)p(l))} u_{ik} \otimes u_{kj}, \]
the braiding:
\[ \Psi(u_{ij} \otimes u_{kl}) = e^{2i\phi(p(i) - p(j))} u_{kl} \otimes u_{ij}, \]
the same counit and with the antipode:
\[ S_{\phi}(u_{ij}) = e^{i\phi(p(i) - p(j))} S(u_{ij}). \]

### 3.4. The \( A(SU_q(2)) \) Hopf algebra.
Here we discuss in details the case of the braided Hopf algebra \( A(SU_q(2)) \) and its explicit presentation. The results could be directly compared with the presentation in [7].

Let \( q \) be a real number with \( 0 < q < 1 \), and let \( A = A(SU_q(2)) \) be the \( \ast \)-algebra generated by \( a \) and \( b \), subject to the following commutation rules:

\begin{align*}
(3.2) \quad ab &= qba, \quad ab^* = q^*b^*a, \quad bb^* = b^*b, \\
(3.3) \quad a^*a + b^*b &= 1, \quad aa^* + q^2b^*b &= 1.
\end{align*}

As a consequence, \( a^*b = q^{-1}ba^* \) and \( a^*b^* = q^{-1}b^*a^* \). This becomes a Hopf \( \ast \)-algebra under the coproduct
\[ \Delta a := a \otimes a - q b^* \otimes b, \]
\[ \Delta b := b \otimes a + a^* \otimes b, \]
counit \( \epsilon(a) = 1, \epsilon(b) = 0 \), and the antipode
\[ Sa = a^*, \quad Sb = -qb, \quad Sb^* = -q^{-1}b^*, \quad Sa^* = a. \]

Let \( \chi : A \to C(S^1) \) be the following map:
\[ \chi(a) = z, \quad \chi(b) = 0. \]

It is an Hopf algebra homomorphism from \( \chi \) to \( \mathbb{C}[z, z^{-1}] \), so we have:

**Lemma 3.5.** The following establishes the grading of homogeneous elements as defined in the previous section:
\[ \mu(a) = 1 = \nu(a), \quad \mu(b) = -1 = -\nu(b). \]
**Theorem 3.6.** Let $0 \leq \phi < 2\pi$. The deformed $*$-algebra $A_\phi(SU_q(2))$ is isomorphic to the algebra with the following generators and relations:

\begin{align*}
(3.4) & \quad \alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma^*\gamma = 1, \\
(3.5) & \quad \gamma\gamma^* = \gamma^*\gamma \\
(3.6) & \quad \alpha\gamma = qe^{i\phi}\gamma\alpha, \quad \alpha^*\gamma = qe^{-i\phi}\gamma^*\alpha.
\end{align*}

**Proof.** We compute:

\begin{align*}
a * a^* &= aa^*, \quad a^* * a = a^*a, \\
b * b^* &= bb^*, \quad b^* * b = b^*b, \\
a * b &= e^{2i\phi}ab, \quad a^* b^* = e^{-2i\phi}b^* * a, \\
b * a &= e^{-2i\phi}ba, \quad b^* * a = e^{2i\phi}a * b^*.
\end{align*}

If $\rho$ is the identity map $\rho : H \to H_\phi$, setting $\rho(a) = \alpha$ and $\rho(b) = \gamma$ we obtain from the relations (3.3) the relations of $A_\phi(SU_q(2))$. 

For $A(SU_q(2))$ we have:

\[ \delta(a) = 0 = \delta(a^*), \quad \delta(b^*) = 2 = -\delta(b). \]

Next, we can see how the coproduct is changed for $A_\phi(SU_q(2))$:

\begin{align*}
(3.7) & \quad \Delta_\phi(\alpha) = \alpha \otimes \alpha - qe^{-4i\phi}\gamma^* \otimes \gamma \\
(3.8) & \quad \Delta_\phi(\alpha^*) = \alpha^* \otimes \alpha^* - qe^{-4i\phi}\gamma \otimes \gamma^* \\
(3.9) & \quad \Delta_\phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma \\
(3.10) & \quad \Delta_\phi(\gamma^*) = \gamma^* \otimes \alpha^* + \alpha \otimes \gamma^*
\end{align*}

In the $A(SU_q(2))$ case the only nontrivial braiding phase factors $\psi$ between the generators $\alpha, \gamma$ are:

\[ \psi(\gamma, \gamma) = e^{8i\phi} = \psi(\gamma^*, \gamma^*), \quad \psi(\gamma^*, \gamma) = e^{-8i\phi} = \psi(\gamma, \gamma^*). \]

We compute the antipode on the the generators of $A(SU_q(2))$:

\begin{align*}
(3.11) & \quad S_\phi(\alpha) = \alpha^*, \quad S_\phi(\alpha^*) = \alpha \\
(3.12) & \quad S_\phi(\gamma) = -qe^{4i\phi}\gamma, \quad S_\phi(\gamma^*) = -q^{-1}e^{4i\phi}\gamma^*.
\end{align*}

### 3.5. The quantum double torus.

Though the noncommutative torus is not a Hopf algebra, there exists a Hopf algebra structure over a direct sum of the commutative and noncommutative torus described by Hajac and Masuda [6].

Let $A = C(T^2) \oplus C(T^2_\theta)$ with generators $u, v$ and $U, V$ of $C(T^2)$ and $C(T^2_\theta)$, respectively. With the following coproduct:

\begin{align*}
\Delta(u) &= u \otimes u + V \otimes U, \quad \Delta(v) = v \otimes v + U \otimes V, \\
\Delta(U) &= U \otimes u + v \otimes U, \quad \Delta(V) = V \otimes v + u \otimes V,
\end{align*}

counit

\[ \epsilon(u) = \epsilon(v) = 1, \quad \epsilon(U) = \epsilon(V) = 0 \]
and the antipode:

\[ S(u) = u^*, \quad S(v) = v^*, \quad S(U) = V^*, \quad S(V) = U^* , \]

\( \mathcal{A} \) is a Hopf algebra called a quantum double torus.

**Lemma 3.7.** Let us define \( \chi : \mathcal{A} \to \mathbb{C}[z, z^{-1}] \) by

\[ \chi(u) = z, \quad \chi(v) = z^{-1}, \quad \chi(U) = \chi(V) = 0. \]

Then \( \chi \) is a Hopf algebra map.

Using the general construction we obtain a braided Hopf algebra \( \mathcal{A}_\phi \) with the following properties:

1. as algebras \( \mathcal{A} \cong \mathcal{A}_\phi \),
2. the coproduct on \( \mathcal{A}_\phi \) is as follows:
   
   \begin{align}
   \Delta_\phi(x) &= \Delta(x), \quad x \in C(\mathbb{T}_q^2), \\
   \Delta_\phi(u) &= u \otimes u + \epsilon^{-4i\phi} V \otimes U, \\
   \Delta_\phi(v) &= v \otimes v + \epsilon^{-4i\phi} U \otimes V,
   \end{align}

3. the only non-trivial braiding is on the noncommutative torus alone, using the shorthand \( \Psi(x \otimes y) = \psi(x, y)y \otimes x \):

   \begin{align*}
   \psi(U, U) &= \psi(V, V) = \psi(U^*, U^*) = \psi(V^*, V^*) = \epsilon^{8i\phi}, \\
   \psi(U, V) &= \psi(V, U) = \psi(U^*, U) = \psi(V, V^*) = \psi(U^*, V) = \epsilon^{-8i\phi}, \\
   \psi(U, V^*) &= \psi(V^*, U) = \psi(U^*, V) = \psi(V, U^*) = \epsilon^{8i\phi}.
   \end{align*}

So we can see that the same algebra has a family of coproducts each in a different braided category, which make it a braided Hopf algebra.

### 4. Bicovariant differential calculi

Recall that a left action of a (braided) Hopf algebra \( H \) on vector space \( \Gamma \) is a linear map \( \triangleright : H \otimes \Gamma \to \Gamma \) s.th. \( (hg) \triangleright x = h \triangleright (g \triangleright x) \) and \( 1 \triangleright x = x \) for all \( g, h \in H, x \in \Gamma \). Similarly we define right action. Then we say that \( \Gamma \) is a left (resp. right) \( H \)-module. A left coaction of \( H \) on \( \Gamma \) is a linear map \( \delta_L : \Gamma \to H \otimes \Gamma \) satisfying

\[ (\Delta \otimes \text{id}) \circ \delta_L = (\text{id} \otimes \delta_L) \circ \delta_L, \]

and

\[ (\epsilon \otimes \text{id}) \delta_L = \text{id}. \]

Analogously we define the right one. We shall say \( \Gamma \) is a left (resp. right) comodule and very often we shall use Sweedler’s notation:

\[ \delta_L(x) = x_{(-1)} \otimes x_{(0)}, \quad \delta_R(x) = x_{(0)} \otimes x_{(1)}. \]

If \( \Gamma \) is both left and right \( H \)-module, with the actions that are commutative with each other:

\[ (h \triangleright x) \lhd g = h \triangleright (x \lhd g), \]

then we call \( H \)-bimodule. We define \( H \)-bicomodule in a similar way.
Definition 4.1. We say that $\Gamma$ is a left covariant bimodule if it is a left $H$-bimodule which is a left comodule over $H$ with coaction $\delta_L$ such that for any $a, b \in H$, $\rho \in \Gamma$ the following are satisfied
\[
\delta_L(a \rho) = \Delta(a)\delta_L(\rho), \quad \delta_L(\rho a) = \delta_L(\rho)\Delta(a).
\]
In the above relations we used shorter notation: $x < \rho = x\rho$ and $\rho \triangleright x = \rho x$. The multiplication $(x \otimes y)(z \otimes \rho)$ for $x, y, z \in H$, $\rho \in \Gamma$ is defined by:
\[
(x \otimes y)(z \otimes \rho) = (m \otimes \triangleright)(x \otimes \tilde{\Psi}(y \otimes z) \otimes \rho),
\]
where $\tilde{\Psi}$ is the braiding of the braided Hopf algebra $H$. Similarly for the right module structure:
\[
(z \otimes \rho)(x \otimes y) = (m \otimes \triangleleft)(z \otimes \hat{\Psi}(\rho \otimes x) \otimes y),
\]
where $\hat{\Psi} : \Gamma \otimes H \to H \otimes \Gamma$ is braiding between $H$ and $\Gamma$.
Similarly we define right covariant bimodule with compatibility conditions
\[
\delta_R(a \rho) = \Delta(a)\delta_R(\rho), \quad \delta_R(\rho a) = \delta_R(\rho)\Delta(a).
\]
Note, that the above conditions require that $\Gamma \otimes H$ and $H \otimes \Gamma$ have a bimodule structure over $H \otimes H$. This is obvious in the case $H$ is not braided, however, if $H$ is a braided Hopf algebra it requires an introduction of the braiding between $\Gamma$ and $H$.

Definition 4.2. We say that $\Gamma$ is a bicovariant bimodule or Hopf bimodule over $H$ if it is both $H$-bicomodule, left and right covariant bimodule.

Let us recall the definition of left and right covariant differential calculus for a braided Hopf algebra. Let $H$ be a braided Hopf algebra and $(\Gamma, d)$ be the first order differential calculus (FODC), that is $\Gamma$ is a Hopf bimodule over $H$ and $d : H \to \Gamma$ satisfies the Leibniz rule and is surjective in the sense that every element of $\Gamma$ is of the form $\sum a_k \triangleright db_k$ for some $a_k, b_k \in H$.

Definition 4.3. We say that the first order differential calculus $(\Gamma, d)$ over a braided Hopf algebra $H$ is braided left covariant if $\Gamma$ is a Hopf bimodule over $H$ and
\[
\delta_L(dh) = (id \otimes d)\Delta(h), \quad \forall h \in H.
\]
In a similar way we define the braided right covariant calculus. We say that the calculus is braided bicovariant if it is left and right covariant.

4.1. The universal differential calculus. Let $H$ be a braided Hopf algebra with the braiding $\Psi$ and $\Gamma_u = \ker m \subset H \otimes H$ be the Hopf bimodule of the universal first order differential calculus, i.e. with $da = a \otimes 1 - 1 \otimes a$.

Lemma 4.4. The following maps,
\[
\delta_L : \Gamma_u \to H \otimes \Gamma_u, \quad \delta_L(x \otimes y) = (m \otimes id \otimes id)(x(1) \otimes \Psi(x(2) \otimes y(1)) \otimes y(2)),
\]
\[
\delta_R : \Gamma_u \to \Gamma_u \otimes H, \quad \delta_R(x \otimes y) = (id \otimes id \otimes m)(x(1) \otimes \Psi(x(2) \otimes y(1)) \otimes y(2)),
\]
make $(\Gamma, D)$ a bicovariant FODC.

Proof. Using fact that if on one leg we have unity then $\Psi$ acts on such tensor product as usual flip, we obtain the following
\[
\delta_L(a \otimes 1) = a(1) \otimes a(2) \otimes 1, \quad \delta_L(1 \otimes a) = a(1) \otimes 1 \otimes a(2),
\]
\[
\delta_R(a \otimes 1) = a(1) \otimes 1 \otimes a(2), \quad \delta_R(1 \otimes a) = 1 \otimes a(1) \otimes a(2).
\]
Hence we obtain
\[
\delta_L(da) = a_{(1)} \otimes (a_{(2)} \otimes 1 - 1 \otimes a_{(2)}) = (\text{id} \otimes d)\Delta(a),
\]
\[
\delta_R(da) = (a_{(1)} \otimes 1 - 1 \otimes a_{(1)}) \otimes a_{(2)} = (d \otimes \text{id})\Delta(a).
\]

\[\square\]

4.2. Braided bicovariant calculi over twisted braided Hopf algebras. Let us assume now that we have a braided Hopf algebra \( H_\phi \) constructed from a Hopf algebra \( H \) (as stated in Theorem 2.13) and let \((\Gamma, d)\) be a bicovariant first order differential calculus over \( H \).

Then, \( \Gamma \) has a left and right coaction of \( C[z, z^{-1}] \) defined in a similar way as on \( H \):

\[
\hat{\delta}_L(\omega) = (\chi \otimes \text{id})\delta_L(\omega), \quad \hat{\delta}_R(\omega) = (\text{id} \otimes \chi)\delta_R(\omega).
\]

We say that an element \( \omega \) is homogeneous of degrees \( \hat{\mu}, \hat{\nu} \) iff:

\[
\hat{\delta}_L(\omega) = z^{\hat{\mu}(\omega)} \otimes \omega, \quad \hat{\delta}_R(\omega) = \omega \otimes z^{\hat{\nu}(\omega)}.
\]

We have:

\[
\hat{\mu}(dh) = \mu(h), \quad \hat{\nu}(dh) = \nu(h),
\]

\[
\hat{\mu}(x \omega) = \mu(x) + \hat{\mu}(\omega), \quad \hat{\nu}(x \omega) = \nu(x) + \hat{\nu}(\omega),
\]

\[
\hat{\mu}(\omega y) = \hat{\mu}(\omega) + \mu(y), \quad \hat{\nu}(\omega y) = \hat{\nu}(\omega) + \nu(y).
\]

Indeed, using fact that \( \chi \) is Hopf algebra morphisms we have

\[
(\Delta \otimes \text{id})\hat{\delta}_L = (\Delta \circ \chi \otimes \text{id})\delta_L = (\chi \otimes \chi \otimes \text{id})(\Delta \otimes \text{id})\delta_L = (\text{id} \otimes (\chi \otimes \text{id}) \otimes \delta_L)(\chi \otimes \text{id})\delta_L = (\text{id} \otimes \hat{\delta}_L)(\chi \otimes \text{id})\delta_L.
\]

Similarly for the right one.

Next, suppose that \( h \) has degrees \( \mu(h), \nu(h) \), i.e.

\[
(\chi \otimes \text{id})\Delta(h) = z^{\mu(h)} \otimes h \quad \text{and} \quad (\text{id} \otimes \chi)\Delta(h) = h \otimes z^{\nu(h)}.
\]

Then

\[
\hat{\delta}_L(dh) = (\chi \otimes \text{id})\delta_L(dh) = (\chi \otimes \text{id})(\text{id} \otimes d)\Delta(h) = (\text{id} \otimes d)(\chi \otimes \text{id})\Delta(h) = (\text{id} \otimes d)(z^{\mu(h)} \otimes h) = z^{\mu(h)} \otimes h.
\]

Hence \( \hat{\mu}(dh) = \mu(h) \) and similarly for the right degree, \( \hat{\nu}(dh) = \nu(h) \).

Now, left covariance of the Hopf bimodule \( \Gamma \) means \( \delta_L(x \omega) = \Delta(x)\delta_L(\omega) \). It implies the following

\[
\hat{\delta}_L(x \omega) = (m \circ \triangleright)(\chi \otimes \tau \otimes \text{id})(\Delta \otimes (\chi \otimes \text{id}) \delta_L)(x \otimes \omega) = (m \otimes \triangleright)(\text{id} \otimes \tau \otimes \text{id})((\chi \otimes \text{id})\Delta \otimes (\chi \otimes \text{id})\delta_L)(x \otimes \omega) = z^{\mu(x) + \hat{\mu}(\omega)} \otimes x \omega.
\]

Analogously we can prove other equalities.

Lemma 4.5. Taking \( \Gamma_\phi = \Gamma \) as a vector space, the following defines a left and right module structure on \( \Gamma_\phi \) over \( H_\phi \):

\[
x \ast \omega = e^{i\phi(\mu(x)\hat{\nu}(\omega) - \nu(x)\hat{\mu}(\omega))} x \omega, \quad \omega \ast x = e^{i\phi(\hat{\mu}(\omega)\nu(x) - \hat{\nu}(\omega)\mu(x))} \omega x.
\]
Proof. Using the properties of $\hat{\mu}, \mu$ and $\hat{\nu}, \nu$, we obtain analogous cocycle conditions as for $\mu, \nu$ alone in Lemma 2.8

\begin{align*}
\mu(x)\hat{\nu}(y) - \nu(x)\hat{\mu}(y, \omega) + \mu(y)\hat{\nu}(\omega) - \nu(y)\hat{\mu}(\omega) &= \\
= \mu(xy)\hat{\nu}(\omega) - \nu(xy)\hat{\mu}(\omega) + \nu(x)\hat{\nu}(y) - \nu(x)\hat{\mu}(y), \\
\hat{\mu}(\omega)\nu(xy) - \hat{\nu}(\omega)\mu(xy) + \mu(x)\nu(y) - \nu(x)\mu(y) &= \\
= \hat{\mu}(\omega x)\nu(y) - \hat{\nu}(\omega x)\mu(y) + \hat{\mu}(\omega)\nu(x) - \hat{\nu}(\omega)\mu(x). 
\end{align*}

That implies that we obtain left and right module structure. \qed

**Lemma 4.6.** The following defines the left (right) coaction of $H_{\phi}$ on $\Gamma_{\phi}$ on homogeneous elements as:

\[ \delta_{L\phi}(\omega) = e^{i\phi x(1)\delta(\omega)} x(0) \otimes \omega, \]

and

\[ \delta_{R\phi}(\omega) = e^{i\phi x(2)\delta(\omega)} x(1) \otimes \omega(1), \]

with the braiding between the bimodule $\Gamma_{\phi}$ and $H_{\phi}$ defined in the same way:

\[ \Psi(\omega \otimes x) = e^{2i\phi \delta(x)\delta(\omega)} x \otimes \omega, \]

\[ \Psi(x \otimes \omega) = e^{2i\phi \delta(x)\delta(\omega)} \omega \otimes x. \]

**Proof.** The proof follows exactly the one for the the braided Hopf algebra $H_{\phi}$. \qed

**Theorem 4.7.** With the above definitions $(\Gamma_{\phi}, d)$ is a braided bicovariant differential calculus.

**Proof.** We have already demonstrated that $\Gamma_{\phi}$ is a Hopf bimodule, so it remains to show that the external derivative $d$ is compatible with the covariance. This is, however, an immediate consequence of Lemma 4.6. Indeed, if we have a homogeneous $x$ then:

\[ (1d \otimes d)\Delta_{\phi}(x) = \mu(x(1)\delta(\omega(2))) x(0) \otimes dx(2). \]

On the other hand using Lemma 4.6

\[ \delta_{L\phi}(dx) = \mu(x(1)\delta(dx(2))) x(0) \otimes dx(0), \]

and combining it with Lemma 4.5 and the fact that $(\Gamma, d)$ is a bicovariant first order differential calculus we obtain that the same holds for the braided $(\Gamma_{\phi}, d)$. \qed

5. Outlook

In this note we have presented a construction of class of braided Hopf algebras, which are obtained by twisting. It is interesting to observe that this includes the braided $SU_q(2)$ algebra and therefore, in particular, a family of so-called theta deformations of the sphere $S^3$, introduced by Matsumoto [10]. These objects have been so far considered only as algebras corresponding to noncommutative homogeneous spaces on which some (unbraided) Hopf algebras act and coact.

The existence of braided Hopf algebra symmetries is a new element that could possibly have some implications for the geometries of theta-deformed three spheres. In particular, it would be interesting to verify that the Connes’ spectral geometries and spectral triples for the theta-spheres [3] are compatible with the structure of the braided Hopf algebras.

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Institute of Physics, Jagiellonian University, prof. Stanisława Łojasiewicza 11, 30-348 Kraków, Poland
Copernicus Center for Interdisciplinary Studies, Szczepańska 1/5, 31-011 Kraków, Poland

Institute of Physics, Jagiellonian University, prof. Stanisława Łojasiewicza 11, 30-348 Kraków, Poland.
Institute of Mathematics of the Polish Academy of Sciences, Sniadeckich 8, 00-656 Warszawa, Poland.

E-mail address: andrzej.sitarz@uj.edu.pl