Three results on representations of Mackey Lie algebras

Alexandru Chirvasitu

Abstract

I. Penkov and V. Serganova have recently introduced, for any non-degenerate pairing \( W \otimes V \to \mathbb{C} \) of vector spaces, the Lie algebra \( \mathfrak{gl}^M = \mathfrak{gl}^M(V,W) \) consisting of endomorphisms of \( V \) whose duals preserve \( W \subseteq V^* \). In their work, the category \( \mathcal{T}_{\mathfrak{g}^M} \) of \( \mathfrak{g}^M \)-modules which are finite length subquotients of the tensor algebra \( T(W \otimes V) \) is singled out and studied. In this note we solve three problems posed by these authors concerning the categories \( \mathcal{T}_{\mathfrak{g}^M} \). Denoting by \( \mathcal{T}_V \otimes W \) the category with the same objects as \( \mathcal{T}_{\mathfrak{g}^M} \) but regarded as \( V \otimes W \)-modules, we first show that when \( W \) and \( V \) are paired by dual bases, the functor \( \mathcal{T}_{\mathfrak{g}^M} \to \mathcal{T}_V \otimes W \) taking a module to its largest weight submodule with respect to a sufficiently nice Cartan subalgebra of \( V \otimes W \) is a tensor equivalence. Secondly, we prove that when \( W \) and \( V \) are countable-dimensional, the objects of \( \mathcal{T}_{\End(V)} \) have finite length as \( \mathfrak{g}^M \)-modules. Finally, under the same hypotheses, we compute the socle filtration of a simple object in \( \mathcal{T}_{\End(V)} \) as a \( \mathfrak{g}^M \)-module.

Keywords: Mackey Lie algebra, finite length module, large annihilator, weight module, socle filtration

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Introduction

In the recent paper [4], the authors study various categories of representations for Lie algebras associated to pairs of complex vector spaces \( V,W \) endowed with a non-degenerate bilinear form \( W \otimes V \to \mathbb{C} \). This datum realizes \( W \) as a subspace of the full dual \( V^* \), and vice versa. The associated Mackey Lie algebra \( \mathfrak{g}^M = \mathfrak{g}^M(V,W) \) is then simply the set of all endomorphisms of \( V \) whose duals leave \( W \subseteq V^* \) invariant. It can be shown that the definition is symmetric in the sense that reversing the roles of \( V \) and \( W \) produces canonically isomorphic Lie algebras. When \( W = V^* \), the resulting Lie algebra is simply \( \End(V) \).

Categories \( \mathcal{T}_{\mathfrak{g}^M} \) of \( \mathfrak{g}^M \)-representations are then introduced. They consist of modules for which all elements have appropriately large annihilators; see [4, § 7.3]. One remarkable result is that all these categories, for all possible non-degenerate pairs \( (V,W) \), are in fact equivalent as tensor categories (i.e. symmetric monoidal abelian categories). Moreover, they are also equivalent to the
categories $\mathbb{T}_{\mathfrak{sl}(V, W)}$ from [4, § 3.5] and $\mathbb{T} = \mathbb{T}_{\mathfrak{sl}(\infty)}$ introduced and studied earlier in [1]; all of this follows from [4, Theorems 5.1 and 7.9].

In view of the abstract equivalence between $\mathbb{T}_{\text{End}(V)} \simeq \mathbb{T}_{\mathfrak{sl}(V, W)}$ noted above, it is a natural problem to try to find as explicit and natural a functor as possible that implements this equivalence. In order to do this, we henceforth specialize to the case when $W = V_*$ is a vector space whose pairing with $V$ is given by a pair of dual bases $v_\gamma \in V$, $v^*_\gamma \in V_*$ for $\gamma$ ranging over some (possibly uncountable) set $I$. This assumption ensures the existence of a so-called local Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{sl}(V, V_*)$ [4, 1.4].

Denote $\mathfrak{g} = \mathfrak{sl}(V, V_*)$. In our setting, for a local Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, let $\Gamma_{\mathfrak{h}}^{\text{wt}}$ be the functor from $\text{End}(V)$-modules to $\mathfrak{g}$-modules which picks out the $\mathfrak{h}$-weight part of a representation. Similarly, denote by $\Gamma^{\text{wt}}$ the functor $\bigcap_{\mathfrak{h}} \Gamma_{\mathfrak{h}}^{\text{wt}}$, where the intersection ranges over all local Cartan subalgebras of $\mathfrak{g}$. We will abuse notation and denote by these same symbols the restrictions of $\Gamma_{\mathfrak{h}}^{\text{wt}}$ and $\Gamma^{\text{wt}}$ to various categories of $\text{End}(V)$-modules.

With these preparations (and keeping the notations we’ve been using), the following seems reasonable ([4, 8.4]).

**Conjecture 0.1** The functor $\Gamma^{\text{wt}}$ implements an equivalence from $\mathbb{T}_{\text{End}(V)}$ onto $\mathbb{T}_{\mathfrak{g}}$.

One of the main results of this note is a proof of this conjecture. The outline of the note is as follows:

In the next section we prove 0.1, making use of the results in [6] on a certain universality property for the category $\mathbb{T}_{\mathfrak{g}}$.

In Section 2 we specialize to a pairing $V_* \otimes V \to \mathbb{C}$ of countable-dimensional vector spaces $V$, $V_*$. In this case, noting that $V^*/V_*$ is a simple $\mathfrak{g}^M = \mathfrak{sl}^M(V, V_*)$-module, the authors of [4] ask whether all objects of $\mathbb{T}_{\text{End}(V)}$ are finite-length $\mathfrak{g}^M$-modules. We show that this is indeed the case in Theorem 2.1.

Finally, Theorem 3.5 in Section 3 contains the description of the socle filtration as a $\mathfrak{g}^M$-module of a simple object in $\mathbb{T}_{\text{End}(V)}$. This solves a third problem posed in the cited paper.

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## 1 Explicit equivalence between $\mathbb{T}_{\text{End}(V)}$ and $\mathbb{T}_{\mathfrak{g}}$

We will actually prove a slightly strengthened version of 0.1. Before formulating it, recall our setting: We are considering a pairing between $V$ and $V_*$ determined by dual bases $v_\gamma \in V$ and $v^*_\gamma \in V_*$, and $\mathfrak{g}$ stands for $\mathfrak{sl}(V, V_*)$. By $\mathfrak{h} \subseteq \mathfrak{g}$ we denote the local Cartan subalgebra spanned linearly by the elements $v_\gamma \otimes v^*_\gamma - v_\gamma \otimes v^*_\gamma \in \mathfrak{g} \subseteq V \otimes V_*$ for $\gamma \neq \tilde{\gamma} \in I$.

Throughout, ‘tensor category’ means symmetric monoidal, $\mathbb{C}$-linear and abelian. Similarly, tensor functors are symmetric monoidal and $\mathbb{C}$-linear, and tensor natural transformations are symmetric monoidal.

Our main result in the present section reads as follows.

**Theorem 1.1** The functor $\Gamma_{\mathfrak{h}}^{\text{wt}}$ implements a tensor equivalence from $\mathbb{T}_{\text{End}(V)}$ onto $\mathbb{T}_{\mathfrak{g}}$. 
Before embarking on the proof, note that as claimed above, the theorem implies the conjecture.

**Corollary 1.2** 0.1 is true.

**Proof** On the one hand, the functor $\Gamma^{\text{wt}}$ from the statement of the conjecture is a subfunctor of $\Gamma^\#:$. On the other hand though, the theorem says that $\Gamma^\#$ already lands inside the category $\mathcal{T}_\vartheta$ which consists of weight modules for any local Cartan subalgebra of $\mathfrak{g}$ (because it consists of modules embeddable in finite direct sums of copies of the tensor algebra $T(V \oplus V_\eta)$; see e.g. [4, 7.9]). In conclusion, we must have $\Gamma^{\text{wt}} = \Gamma^\#$, and we are done. ■

We will make use of the following simple observation.

**Lemma 1.3** Let $H$ be a cocommutative Hopf algebra over an arbitrary field $\mathbb{F}$, and $\text{Fin} : H{-}\text{mod} \to H{-}\text{mod}$ the functor sending an $H$-module $M$ to the largest $H$-submodule of $M$ which is a union of finite-dimensional $H$-modules. Then, $\text{Fin}$ is a tensor functor.

**Proof** Let $S$ be the antipode of $H$. For an $H$-module $V$, denote by $V^*$ the algebraic dual of $V$ made into an $H$-module via $(hf)(v) = f(S(h)v)$ for $h \in H$, $v \in V$ and $f \in V^*$. Then the usual evaluation $V^* \otimes V \to \mathbb{F}$ is an $H$-module map if $V^* \otimes V$ is a module via the tensor category structure of $H{-}\text{mod}.

Now let $M, N$ be $H$-modules, and $V \subseteq \text{Fin}(M \otimes N)$ be a finite-dimensional $H$-submodule. We need to show that $V$ is in fact a submodule of $\text{Fin}(M) \otimes \text{Fin}(N)$.

Denote by $N_V \subseteq N$ the image of the $H$-module morphism $M^* \otimes V \subseteq M^* \otimes M \otimes N \rightarrow N$, where the last arrow is evaluation on the first two tensorands. Similarly, denote by $M_V \subseteq M$ the image of $V \otimes N^* \subseteq M \otimes N \otimes N^* \rightarrow M$. Then $M_V$ and $N_V$ are $H$-submodules of $M$ and $N$ respectively, being images of module maps. It is now easily seen from their definition that $M_V$ and $N_V$ are finite-dimensional, and that the inclusion $V \subseteq M \otimes N$ factors through $M_V \otimes N_V \subseteq M \otimes N$. ■

**Remark 1.4** The cocommutativity of $H$ is used in the proof to conclude that the category $H{-}\text{mod}$ is symmetric monoidal, and hence $N \otimes N^* \rightarrow \mathbb{F}$ is an $H$-module map because its domain is isomorphic to $N^* \otimes N$.

Although we do not need this in the sequel, as Lemma 1.3 will only be applied to universal envelopes of Lie algebras, the above proof can be generalized to show that the functor $\text{Fin}$, defined in the obvious fashion, is monoidal for any Hopf algebra with bijective antipode. In the definition of $V_N$ one would need to use the evaluation map $N \otimes ^*N \rightarrow \mathbb{F}$ instead, where $^*N$ is the full dual of $N$ made into an $H$-module using the inverse of the antipode instead of the antipode. ♦

We now need a characterization of the category $\mathcal{T}_{\text{End}(V)} \simeq \mathcal{T}_\vartheta$ in terms of a universality property which defines it uniquely up to tensor equivalence. The following result is Theorem 3.4.2 from [6], where the category $\mathcal{T}_\vartheta$ is denoted by $\text{Rep}(\text{GL})$.

**Theorem 1.5** For any tensor category $\mathcal{C}$ with monoidal unit $1$ and any morphism $b : x \otimes y \rightarrow 1$ in $\mathcal{C}$, there is a left exact tensor functor $F : \mathcal{T}_{\text{End}(V)} \rightarrow \mathcal{C}$ sending the pairing $V^* \otimes V \rightarrow \mathbb{C}$ in $\mathcal{T}_{\text{End}(V)}$ to $b$. Moreover, $F$ is unique up to tensor natural isomorphism. ■

As an immediate consequence we have:

**Corollary 1.6** A left exact tensor functor $\mathcal{T}_{\text{End}(V)} \rightarrow \mathcal{T}_\vartheta$ turning the pairing $V^* \otimes V \rightarrow \mathbb{C}$ into the pairing $V_\eta \otimes V \rightarrow \mathbb{C}$, is a tensor equivalence.
Proof The abstract tensor equivalence $\mathcal{T}_{\text{End}(V)} \simeq \mathcal{T}_g$ established in [4, 5.1, 7.9] identifies the two bilinear pairings in the statement. The conclusion then follows from Theorem 1.5 in the usual manner (a universality property implies uniqueness up to equivalence).

The proof of Theorem 1.1 makes use of the following auxiliary result.

Lemma 1.7 Let $\mathfrak{h}$ be a complex abelian Lie algebra. For any functional $\varphi \in \mathfrak{h}^*$, let $M^\varphi \in \mathfrak{h} \mod$ be an $\mathfrak{h}$-module all of whose elements are vectors of weight $\varphi$. Then, using the notation $\text{FIN}$ from Lemma 1.3, we have

$$\text{FIN} \left( \prod_{\varphi \in \mathfrak{h}^*} M^\varphi \right) = \bigoplus_{\varphi \in \mathfrak{h}^*} M^\varphi.$$

Proof We denote the direct product $\prod_{\varphi} M^\varphi$ by $M$. Let $x \in M$ be an element contained in some $d$-dimensional $\mathfrak{h}$-submodule $N$ of $M$.

Assume there are $d + 1$ distinct functionals $\varphi_0$ up to $\varphi_d$ such that the components $x_i$, $0 \leq i \leq d$ of $x$ in $M^{\varphi_i}$ are all non-zero. Because $\varphi_i \in \mathfrak{h}^*$ are distinct, we can find some element $h \in \mathfrak{h}$ such that the scalars $t_i = \varphi_i(h)$ are distinct (as $h$ cannot be the union of the kernels of $\varphi_i - \varphi_j$, $0 \leq i \neq j \leq d$; here we use the fact that we are working over $\mathbb{C}$, or more generally, over an infinite field). The claim now is that $x, hx, \ldots, h^d x$ are linearly independent, contradicting the assumption $\dim N = d$.

To prove the claim, consider the images of the vectors $h^i x$, $0 \leq i \leq d$ through the projection $M \to \prod_{\varphi} M^{\varphi}$. They are linear combinations of the $x_i$’s, and their coefficients form the columns of the $(d + 1) \times (d + 1)$ non-singular Vandermonde matrix

$$\begin{pmatrix}
1 & t_0 & \cdots & t^d_0 \\
1 & t_1 & \cdots & t^d_1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & t_d & \cdots & t^d_d
\end{pmatrix}$$

This finishes the proof.

We are now ready to prove our first main result.

Proof of Theorem 1.1 First, recall from [1, §4] that $\mathcal{T}_{\text{End}(V)}$ has enough injectives, that the tensor products $(V^*)^{\otimes m} \otimes V^{\otimes n}$ contain all indecomposable injectives as summands, and also [5] that all morphisms between such tensor products are built out of the pairing $V^* \otimes V \to \mathbb{C}$ by taking tensor products, permutations, and linear combinations. Therefore, if we show that $\Gamma = \Gamma^\text{wt} : \mathcal{T}_{\text{End}(V)} \to g \mod$ sends $V^* \otimes V \to \mathbb{C}$ to $V_* \otimes V \to \mathbb{C}$ and is a left exact tensor functor, its image will automatically lie in $\mathcal{T}_g$. We can then apply Corollary 1.6 to conclude that the resulting functor from $\mathcal{T}_{\text{End}(V)} \to \mathcal{T}_g$ is a tensor equivalence.

The functor $\Gamma$, regarded as a functor from $g \mod$ to $\mathfrak{h}$-weight $\mathfrak{g}$-modules, is the right adjoint of the exact inclusion functor going in the opposite direction; it’s thus clear that it is left exact.

Since the pairing $V_* \otimes V \to \mathbb{C}$ is simply the restriction of the full pairing $V^* \otimes V \to \mathbb{C}$ and $\Gamma$ is compatible with inclusions, we will be done as soon as we prove that it is a tensor functor and it sends $V^*$ to $V_*$. We prove tensoriality first. In fact, since compatibility with the symmetry is clear, it is enough to prove monoidality. That is, that the inclusion $\Gamma(M) \otimes \Gamma(N) \subseteq \Gamma(M \otimes N)$ is actually an isomorphism.
for any $M, N \in \mathbb{T}_{\text{End}(V)}$. To see that this is indeed the case, note that every object of $\mathbb{T}_{\text{End}(V)}$, being embedded in some finite direct sum of tensor products $(V^*)^m \otimes V^n$, is certainly a submodule of a direct product of $\mathfrak{h}$-weight spaces. Lemma 1.7 now shows that every finite-dimensional $\mathfrak{h}$-submodule of an object in $\mathbb{T}_{\text{End}(V)}$ is automatically an $\mathfrak{h}$-weight module. Conversely, $\mathfrak{h}$-weight modules are unions of finite-dimensional $\mathfrak{h}$-modules. It follows that $\Gamma$ coincides with the functor $\text{Fin}$ considered in Lemma 1.3 for the Hopf algebra $H = U(\mathfrak{h})$ (i.e. the universal enveloping algebra of $\mathfrak{h}$); the lemma finishes the job of proving monoidality.

Finally, it is almost immediate that $\Gamma(V^*) = V^*_\ast$; simply note that the $\mathfrak{h}$-weight subspaces of $V^*$ are the lines spanned by the basis elements $v^*_i$.

2 Restrictions from $\mathbb{T}_{\text{End}(V)}$ to $\mathfrak{g}^M(V, V_\ast)$ have finite length

In what follows $V_\ast \otimes V \to \mathbb{C}$ will be a non-degenerate pairing between countable-dimensional vector spaces. In this case, it is shown in [3] that we can find dual bases $v_i, v_i^\ast$, $i \in \mathbb{N} = \{0, 1, \ldots\}$ for $V$ and $V_\ast$ respectively, in the sense that $v_i^\ast(v_j) = \delta_{ij}$. Denote $\mathfrak{g} = \mathfrak{sl}(V, V_\ast)$ and $\mathfrak{g}^M = \mathfrak{sl}^M(V, V_\ast)$. In general, for a vector space $W$ and a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0)$, denote by $W_\lambda$ the image of the Schur functor corresponding to $\lambda$ applied to $W$.

We think of elements of $V^*$ as row vectors indexed by $\mathbb{N}$, on which the Lie algebra $\text{End}(V)$ of $\mathbb{N} \times \mathbb{N}$ matrices with finite columns acts on the left via – right multiplication. The subspace $V_\ast \subseteq V^*$ consists of row vectors with only finitely many non-zero entries. We will often think of elements of $V^*/V_\ast$ as row vectors as well, keeping in mind that changing finitely many entries does not alter the element. For a subset $I \subseteq \mathbb{N}$ and a vector $x \in V^*$, the restriction $x|_I$ is the vector obtained by keeping the entries of $x$ indexed by $I$ intact, and turning all other entries to zero. The same terminology applies to $x \in V^*/V_\ast$.

Let $I \subseteq \mathbb{N}$ be a subset. An element of $V^*$ (respectively $V^*/V_\ast$) is $I$-concentrated, or concentrated in $I$ if all of its non-zero entries (respectively all but finitely many of its non-zero entries) belong to $I$. Similarly, a matrix in $\text{End}(V)$ is $I$-concentrated if all of its non-zero entries are in $I \times I$.

Our result is:

Theorem 2.1 The objects of $\mathbb{T}_{\text{End}(V)}$ have finite length when regarded as $\mathfrak{g}^M$-modules.

Remark 2.2 In fact, the proof of the theorem could be adapted to the more general case covered in the previous section: $V_\ast$ and $V$ could be allowed to be uncountable-dimensional, so long as they are still paired by means of dual bases.

We need some preparations. The following result is very likely well known.

Lemma 2.3 Let $\mathfrak{G}$ be a Lie algebra over some field $\mathbb{F}$, and $I \subseteq \mathfrak{G}$ an ideal. Let $U$ be a simple $\mathfrak{G}/I$-module, and $W$ an $\mathfrak{G}$-module on which $I$ acts densely and irreducibly, and such that $\text{End}_I(W) = \mathbb{F}$. Then, $U \otimes W$ is a simple $\mathfrak{G}$-module.

Proof Let $x = \sum_{i=1}^n u_i \otimes w_i$ be a non-zero element of $U \otimes W$, with the tensor product decomposition chosen such that the $u_i$ are linearly independent and all $w_i$ are non-zero. We have to show that $x$ generates $U \otimes W$ as an $\mathfrak{G}$-module.

Because $I$ annihilates $U$, it acts on $\bigoplus_{i=1}^n \mathbb{F}u_i \otimes W \cong W^\otimes n$. By the simplicity of $W$ over $I$, there are vectors $w'_i \in W$, $i \in \{1, \ldots, n\}$, with $w'_1 = w_1$ and such that the projection $W^\otimes n \to W$ onto the first component maps the $I$-submodule of $W^\otimes n$ generated by $\sum u_i \otimes w'_i$ isomorphically onto $W$.  

5
Now note that $u_i' \mapsto w_i'$, $i > 1$ extend to $I$-module automorphisms of $W$. By the condition $\text{End}_I(W) = \mathbb{F}$, these automorphisms are scalar: $u_i' = t_i w_1'$ for all $i > 1$; for simplicity, set $t_1 = 1$ so that this identity holds for all $i$. Now, substituting $\sum_i t_i u_i$ for $u_1$ and denoting $u_1 = u$, we may assume that a non-zero simple tensor $u \otimes w \in U \otimes W$ belongs to the $I$-span of $x$.

Starting with a simple tensor $u \otimes w$ as above, note first that the enveloping algebra $U(I)$ can act so as to obtain any other tensor of the form $u \otimes w'$ (because $I$ annihilates $U$ and acts irreducibly on $W$).

On the other hand, for $h \in \mathfrak{g}$, we have $h(u \otimes w) = h u \otimes w + u \otimes h w$. Since $I$ acts densely on $W$, we can find $k \in I$ such that $k w = h w$. In this case we have $(h - k)(u \otimes w) = hu \otimes w$; since $U$ is simple over $\mathfrak{g}$, all simple tensors of the form $u' \otimes w$ are in the $\mathfrak{g}$-span of $u \otimes w$. Combining this with the previous paragraph, we get the desired conclusion. 

We now need the following infinite-dimensional analogue of Schur-Weyl duality.

**Proposition 2.4** For any partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_m \geq 0)$, the $\mathfrak{g}^M$-module $(V^*/V_*)_\lambda$ is simple.

**Proof** Let $k = |\lambda| = \sum_{i=1}^m \lambda_i$. Choose an arbitrary non-zero $x \in (V^*/V_*)^\otimes k$, thought of as a sum $\sum_{\ell} x^\ell$ for $x^\ell = x^{\ell}_1 \otimes \ldots \otimes x^{\ell}_k$.

We denote the symmetric group on $k$ letters by $S_k$. Partition $\mathbb{N}$ into $k$ infinite subsets $I_1, \ldots, I_k$ such that the element of $(V^*/V_*)^\otimes k$ defined by

$$x_{\text{RES}} = \sum_{\ell} \sum_{\sigma \in S_k} (x^\ell_{I_1(1)}) \otimes \ldots \otimes (x^\ell_{I_k(1)})$$

is non-zero; we leave it to the reader to show that this is possible. Now choose $k$ complex numbers $t_j$, $j \in \{1, \ldots, k\}$ such the sums $\sum_j m_j t_j$ for non-negative integers $\sum_j m_j = k$ are distinct for different choices of tuples $m_1, \ldots, m_k$ (e.g. $t_j$ could equal $(k + 1)^j$), and let $h \in \mathfrak{g}^M$ be the diagonal matrix whose $I_j$-indexed entries are equal to $t_j$. By breaking everything up into $h$-eigenspaces, we see that the $\mathfrak{g}^M$-module generated by $x$ contains $x_{\text{RES}}$. In order to keep notation simple, we substitute $x_{\text{RES}}$ for $x$ and assume that the individual tensorands $x^{\ell}_j$ of each summand $x^\ell$ of $x$ are concentrated in distinct $I_j$'s.

Now consider the subspace $W_1$ of $V^*/V_*$ generated by all $I_1$-concentrated $x^{\ell}_j$'s, and let $p_1, \ldots, p_s$ be rank one idempotent $I_1$-concentrated matrices in $\mathfrak{g}^M$ such that $\sum_i p_i$ acts as the identity on $W_1$. Since $x = \sum_i p_i x$, some $p_i x$ must be non-zero. Substitute it for $x$, and repeat the process with $I_2$ in place of $I_1$, etc. The resulting non-zero element, again denoted by $x$, will now be a linear combination of simple tensors $x^\ell$ as before, with tensorands $x^{\ell}_j$ concentrated in distinct $I_j$'s for each $\ell$, and such that all $x^{\ell}_j$'s concentrated in $I_j$ (for all $\ell$) are equal. Denoting by $x_j$ this common $I_j$-concentrated vector, our element $x$ is a linear combination of permutations of $x_1 \otimes \ldots \otimes x_k$.

Note that the entire procedure we have just described is $S_k$-equivariant: If the vector we started out with was in $(V^*/V_*)_\lambda \subseteq (V^*/V_*)^\otimes k$, then so is the output of the process. We assume this to be the case for the rest of the proof.

Because $\mathfrak{g}^M$ acts transitively on $V^*/V_*$ (in the sense that any non-zero element can be transformed into any other element by acting on it with some matrix in $\mathfrak{g}^M$), we can find $k^2$ elements $a_{ij}$ of $\mathfrak{g}^M$ such that $a_{pq} x_r = \delta_{qr} x_p$. The elements $a_{ij}$ generate a Lie algebra isomorphic to $\mathfrak{gl}(k)$, and by ordinary Schur-Weyl duality we conclude that the $\mathfrak{g}^M$-module generated by $x$ contains $c_\lambda(W^\otimes k)$, where $W$ is the linear space spanned by the $x_j$, and $c_\lambda$ is the Young symmetrizer corresponding to $\lambda$. 


Since for each \( j \) the vector \( x_j \) can be transformed into any other \( I_j \)-concentrated vector by acting on it with some \( I_j \)-concentrated matrix, the conclusion from the previous paragraph applies to any choice of \( x_j \)’s. The desired result follows from the fact that every element of \( c_\lambda(V^*/V_*)^{\otimes k} \) is a sum of elements from \( c_\lambda(W^{\otimes k}) \) for various \( W \) spanned by various tuples \( \{ x_j \} \).

As a consequence of Lemma 2.3 and Proposition 2.4 we get:

**Corollary 2.5** Let \( W \) be a simple object in \( T_{g,M} \), and \( \lambda \) be a partition. Then, the \( g^M \)-module \((V^*/V_*)_\lambda \otimes W \) is simple.

**Proof** We apply Lemma 2.3 to the Lie algebra \( \mathfrak{g} = g^M \), the ideal \( I = g \), and the modules \( U = (V^*/V_*)_\lambda \) and \( W \). We already know that \( W \) is simple over \( g \) and is acted upon densely by the latter Lie algebra ([4, Corollary 7.6]), and the remaining condition \( \text{End}_g(W) = C \) follows for example from the fact that all simple modules in \( T_g = T_{g,M} \) are highest weight modules with respect to a certain Borel subalgebra of \( g \).

We can now turn to Theorem 2.1.

**Proof of Theorem 2.1** Since all objects of \( T_{\text{End}(V)} \) are isomorphic to subquotients of finite direct sums of tensor products \((V^*)^{\otimes m} \otimes V^{\otimes n}\), it suffices to prove the conclusion for these tensor products. In turn, when regarded as \( g^M \)-modules, these tensor products have filtrations by finite direct sums of objects of the form \((V^*/V_*)^{\otimes m_1} \otimes V^{\otimes n_2} \otimes V^{\otimes n_3} \). The leftmost tensorand \((V^*/V_*)^{\otimes m_1}\) breaks up as a direct sum of images \((V^*/V_*)_\lambda \) of Schur functors, while \( V^{\otimes n_2} \otimes V^{\otimes n_3} \) has a finite filtration by simple modules from the category \( T_{g,M} \). The conclusion now follows from Corollary 2.5 above.

3 **Socle filtrations of \( T_{\text{End}(V)} \)-objects over \( g^M(V, V_*) \)**

We now tackle the problem of finding the socle filtrations of simples in \( T_{\text{End}(V)} \) as \( g^M \)-modules. We start with a definition.

**Definition 3.1** A filtration \( M^0 \subseteq M^1 \subseteq \ldots \subseteq M^n = M \) of an object \( M \) in an abelian category is essential if for every \( p < q < r \), the module \( M^q/M^p \) is essential in \( M^r/M^p \), i.e. intersects every non-zero submodule of \( M^r/M^q \) non-trivially.

**Remark 3.2** It can be shown by induction on \( r - p \) that the condition in Definition 3.1 is equivalent to \( M^{p+1}/M^p \) being essential in \( M^{p+2}/M^p \) for all \( p \).

For dealing with tensor products of copies of \( V^* \) and \( V_* \) we use the following notation: For a binary word \( r = (r_1, \ldots, r_k) \), \( r_i \in \{0, 1\} \), let \( V^r \) be the tensor product \( \bigotimes_{i=1}^k V^{r_i} \), where \( V^0 = V_* \) and \( V^1 = V^* \). We denote \( \sum_i r_i \) by \( |r| \).

Now consider the following (ascending) filtration of \( W = (V^*)^{\otimes m} \otimes V^{\otimes n} \):
\[
W^k = \sum_{|r| \leq k} V^r \otimes V^{\otimes n}, \quad \text{for every } 0 \leq k \leq m. \tag{1}
\]

**Proposition 3.3** The filtration (1) of \( W = (V^*)^{\otimes m} \otimes V^{\otimes n} \) is essential in \( g^M \) – mod.

**Proof** By Remark 3.2, it suffices to show that for any \( k \geq 0 \), the \( g^M \)-module generated by any element \( x \in W^{k+2} - W^{k+1} \) intersects \( W^{k+1} - W^k \) (the minus signs stand for set difference).
Moreover, it is enough to assume that $x$ is a sum of simple tensors $y = y_1 \otimes \ldots \otimes y_m \otimes x_1 \otimes \ldots \otimes x_n$ for $y_i \in V^*$ and $x_j \in V$ such that exactly $k + 2$ of the $y_i$'s are in $V_*$.

Acting on a term $y$ as above with an element $q$ of $g$ which annihilates all $y_i \in V_*$ and all $x_j$ will produce an element of $W^{k+1}$, which belongs to $W^{k+1} - W^k$ provided it is non-zero; this element can be written as a sum of simple tensors, each of which has the tensorands $y_i \in V_*$ and $x_j$ in common with the original term $y$. Hence, focusing on the action of $g$ on only those tensorands $y_i$ which do not belong to $V_*$, it is enough to prove the following claim (which we apply to $s = m - (k + 2)$):

The annihilator in $g$ of an element $z \in (V^*)^\otimes s$ whose image in $(V^*/V_*)^\otimes s$ is non-zero does not contain a finite corank subalgebra (as defined in [4, § 3.5]).

Fixing $p \in \mathbb{N}$, we have to prove that some matrix $a \in g$ concentrated in $\mathbb{N}_{\geq p} = \{p, p + 1, \ldots \}$ does not annihilate $z$. In fact, it is enough to prove this for $a \in g^M$. Indeed, it would then follow that for sufficiently large $q > p$, the vector $(az)_{\leq q}$ obtained by annihilating all coordinates with index larger than $q$ is non-zero. But we can find some large $r$ such that $(az)_{\leq q}$ equals $(a_{\leq r}z)_{\leq r}$, where $a_{\leq r}$ is the $\{p, \ldots, r\}$-concentrated truncation of $a$. We would then conclude that $a_{\leq r}z$ is non-zero, and the proof would be complete.

Finally, to show that some $\mathbb{N}_{\geq p}$-concentrated $a \in g^M$ does not annihilate $z$, it suffices to pass to the quotient by $V_*$, and regard $z$ as a non-zero element of $(V^*/V_*)^\otimes s$. Since $g^M$ acts on $V^*/V_*$ via its quotient $g^M/\mathfrak{g}$, being $\mathbb{N}_{\geq p}$-concentrated no longer matters: any element of $g^M$ can be brought into $\mathbb{N}_{\geq p}$-concentrated form by adding an element of $\mathfrak{g}$. In conclusion, the desired result is now simply that no non-zero element of $(V^*/V_*)^\otimes s$ is annihilated by $g^M$; this follows immediately from Proposition 2.4, for example.

The proof is easily applicable to traceless tensors in $(V^*)^\otimes m \otimes V^\otimes n$, i.e. the intersection of the kernels of all $mn$ evaluation maps

$$(V^*)^\otimes m \otimes V^\otimes n \to \bigoplus_{i,j} V^i \otimes V^j \otimes n.$$

In other words:

**Corollary 3.4** Let $W \subseteq (V^*)^\otimes m \otimes V^\otimes n$ be the space of traceless tensors, and set

$$W^k = W \cap \bigoplus_{|r| \leq k} V^r \otimes V^\otimes n, \quad \text{for every } 0 \leq k \leq m. \quad (2)$$

The filtration $\{W^k\}$ of $W$ is essential over $g^M$.

We can push this even further, making use of the $S_m \times S_n$-equivariance of the corollary. Recall that the irreducible objects in $\mathcal{T}_{\text{End}(V)}$ are precisely the modules $W_{\lambda,\mu}$ of traceless tensors in $(V^*)^\otimes |\lambda| \otimes V^\otimes |\mu|$, for partitions $\lambda, \mu$ (see e.g. [4, Theorem 4.1] and discussion preceding it). For any pair of partitions, the intersection of (2) (for $m = |\lambda|$ and $n = |\mu|$) with $W_{\lambda,\mu}$ is a filtration of $W_{\lambda,\mu}$ by $g^M$-modules. It turns out that it is precisely what we are looking for:

**Theorem 3.5** For any two partitions $\lambda, \mu$, the intersection of (2) with $W_{\lambda,\mu}$ is the socle filtration of this latter module over $g^M$.

**Proof** Immediate by the proof of Proposition 3.3: simply work with $W_{\lambda,\mu}$ instead of $(V^*)^\otimes |\lambda| \otimes V^\otimes n$.
We now rephrase the theorem slightly, to give a more concrete description of the quotients in the socle filtration. To this end, recall that the ring SYM of symmetric functions is a Hopf algebra over \( \mathbb{Z} \) (e.g. §2 of [2]), with comultiplication \( \Delta \), say. We regard partitions as elements of SYM by identifying them with the corresponding Schur functions, and we always think of \( \Delta(\lambda) \) as a \( \mathbb{Z} \)-linear combination of tensor products \( \mu \otimes \nu \) of partitions.

For a partition \( \lambda \) we denote \( \Delta(\lambda) \) by \( \lambda^{(1)} \otimes \lambda^{(2)} \). Note that this is is a slight notational abuse, as \( \Delta(\lambda) \) is not a simple tensor but rather a sum of tensors; we are suppressing the summation symbol to streamline the notation. The summation suppression extends to Schur functors: The expression \( M_{\lambda^{(1)}} \otimes M_{\lambda^{(2)}} \), for instance, denotes a direct sum over all summands \( \mu \otimes \nu \) of \( \Delta(\lambda) \).

Finally, one last piece of notation: For an element \( \nu \in \text{SYM} \) and \( k \in \mathbb{N} \), we denote by \( \nu^k \in \text{SYM} \) the degree-\( k \) homogeneous component of \( \nu \) with respect to the usual grading of \( \text{SYM} \).

We can now state the following consequence of Theorem 3.5, whose proof we leave to the reader (it consists simply of running through the definition of the coproduct of SYM). Recall that we denote simple modules in \( T_{\text{End}(V)} \) by \( W_{\lambda,\mu} \); similarly, simple modules in \( g^M \) are denoted by \( V_{\mu,\nu} \).

**Corollary 3.6** Let \( \lambda, \mu \) be two partitions. The semisimple subquotient \( W^k/W^{k-1} \), \( k \geq 0 \) of the socle filtration \( 0 = W^{-1} \subseteq W^0 \subseteq W^1 \ldots \) of \( W_{\lambda,\nu} \) in \( g^M \) \( \mod \) is isomorphic to

\[
(V^*/V_*)_{\lambda^{(1)}}^k \otimes V_{\lambda^{(2)}}^{(1)} \otimes \mu^{(2)}.
\]

Finally, it seems likely that as a category of \( g^M \)-modules, \( T_{\text{End}(V)} \) has a universal property of its own, reminiscent of Theorem 1.5. Denoting by \( T\text{res} \) the full (tensor) subcategory of \( g^M \) \( \mod \) on the objects of \( T_{\text{End}(V)} \), the following seems sensible.

**Conjecture 3.7** For any tensor category \( C \) with monoidal unit \( 1 \), any morphism \( b : x \otimes y \rightarrow 1 \) in \( C \), and any subobject \( x' \subseteq x \), there is a left exact tensor functor \( F : T\text{res} \rightarrow C \) sending the pairing \( V^* \otimes V \rightarrow C \) to \( b \) and turning the inclusion \( V_+ \subseteq V^* \) into \( x' \subseteq x \). Moreover, \( F \) is unique up to tensor natural isomorphism.

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