Research Article

Construction for the Sequences of Q-Borderenergetic Graphs

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This research intends to construct a signless Laplacian spectrum of the complement of any \( k \)-regular graph \( G \) with order \( n \). Through application of the join of two arbitrary graphs, a new class of \( Q \)-borderenergetic graphs is determined with proof. As indicated in the research, with a regular \( Q \)-borderenergetic graph, sequences of regular \( Q \)-borderenergetic graphs can be constructed. The procedures for such a construction are determined and demonstrated. Significantly, all the possible regular \( Q \)-borderenergetic graphs of order \( 7 < n \leq 10 \) are determined.

1. Introduction

All graphs considered in this paper are simple, unweighted, and undirected. Let \( G \) be a graph of order \( n = |V (G)| \), where \( V (G) \) is the vertex set of \( G \). The complement of \( G \) is denoted by \( \overline{G} \). The complete graph of order \( n \) is denoted by \( K_n \). Denote the average vertex degree of \( G \) by \( \overline{d} \). The join of two graphs \( H_1 \) and \( H_2 \) is the graph \( H_1 \sqcup H_2 \) with the vertex set \( V (H_1) \sqcup V (H_2) \) and the edge set consisting of all the edges of \( H_1 \) and \( H_2 \) together with the edges joining each vertex of \( H_1 \) with every vertex of \( H_2 \). For details on graph theory and spectral graph theory; see [1–4].

Let \( A(G) \) and \( D(G) \) be the adjacency matrix and the diagonal matrix of the vertex degrees of \( G \), respectively. Then, \( L(G) = D(G) − A(G) \) and \( Q(G) = D(G) + A(G) \) are called the Laplacian matrix and the signless Laplacian matrix of \( G \), respectively. In particular, the signless Laplacian spectra of join of two regular graphs are already determined [5].

The energy \( E(G) \) of \( G \) is defined as the sum of the absolute value of the eigenvalues of its adjacency matrix \( A(G) \) [6, 7]. Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) be the eigenvalues of \( A \). Then,

\[
E(G) = \sum_{i=1}^{n} |\lambda_i|.
\]

For additional information on graph energy and its applications in chemistry, we refer to [8–10]. The eigenvalues of the Laplacian matrix \( L(G) \) of graph \( G \) are denoted by \( \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n = 0 \). The Laplacian energy [11] of \( G \) is defined as

\[
LE(G) = \sum_{i=1}^{n} |\xi_i - \overline{d}|.
\]

The eigenvalues of the signless Laplacian matrix \( Q \) of graph \( G \) are denoted by \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n \) which forms the signless Laplacian spectrum \( \text{Spec}_Q(G) \). The signless Laplacian energy of \( G \) [12] is defined as \( QE(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}| \).

In 2015, Gong et al. [13] proposed the concept of borderenergetic graphs, namely graphs of order \( n \) satisfying \( E(G) = 2(n−1) \). The corresponding results on borderenergetic graphs can be seen in [14–17]. For the Laplacian energy of a graph \( G \), Tura [18] proposed the concept of L-borderenergetic graphs, that is, a graph \( G \) of order \( n \) is
L-borderenergetic if $LE(G) = LE(K_n) = 2(n-1)$. More results on L-borderenergetic graphs, we can refer to [18–22].

Recently, Tao and Hou [23] extended this concept to the signless Laplacian energy of a graph. If a graph has the same signless Laplacian energy as the complete graph $K_n$, i.e., $QE(G) = QE(K_n) = 2(n-1)$, then it is called Q-borderenergetic. In [23, 24], several classes of Q-borderenergetic graphs are constructed.

Moreover, in this paper, through using the joint of two graphs, we construct a new class of Q-borderenergetic graphs and present a procedure to find sequences of regular Q-borderenergetic graphs. Especially, all regular Q-borderenergetic graphs of order $7 < n \leq 10$ are presented. In addition, we obtain the signless Laplacian spectrum of the complement of any $k$-regular graph $G$ of order $n$.

2. Construction on Q-Borderenergetic Graphs

At first, the signless Laplacian spectrum of the complement of any $k$-regular graph $G$ with order $n$ is given in Lemma 1. Denote the signless Laplacian matrix of $G$ by $\overline{Q}$.

**Lemma 1.** Let $G$ be a $k$-regular connected graph of order $n$. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the eigenvalues of $Q(G)$, then the eigenvalues of $Q(\overline{G})$ are as follows:

$$2(n-1) - \mu_1 = 2(n-1-k) \geq n - 2 - \mu_2 \geq n - 2 - \mu_3 \geq \cdots \geq n - 2 - \mu_n.$$  \hspace{1cm} (3)

**Proof.** Note that the signless Laplacian matrix of the complement graph $G'$ of $G$ is written as

$$Q(\overline{G}) = [(n-1)I - D(G)] + (J - A(G) - I).$$

$$= (n-2)I + J - Q(G),$$  \hspace{1cm} (4)

where $I$ is an identity matrix and $J$ is the matrix with each of whose entries is equal to 1. Since $G$ is $k$-regular, we have that $\mu_i = 2k$ with corresponding eigenvector $e = (1, 1, \ldots, 1)^T$. Let $x_1, \ldots, x_{n-1}, x_n$ be the eigenvectors of $Q(G)$ corresponding to the eigenvalues $\mu_2, \ldots, \mu_{n-1}, \mu_n$, respectively. Thus, we have $Q(\overline{G})x_i = \mu_ix_i, i = 2, \ldots, n$. Since $Q(G)$ is symmetric, all the eigenvectors $e, x_2, \ldots, x_{n-1}, x_n$ are orthogonal to each other. Thus, we obtain $e^T x_i = 0, i = 2, \ldots, n$. As $J$ can be presented as

$$J = \begin{pmatrix} e^T \\ e^T \\ \vdots \\ e^T \end{pmatrix},$$  \hspace{1cm} (5)

it arrives at $Jx_i = 0, i = 2, \ldots, n$. Therefore,

$$Q(\overline{G})x_i = ((n-2)I + J - Q(G))x_i,$$

$$= (n-2)x_i + Jx_i - Q(G)x_i,$$

$$= (n-2 - \mu_i)x_i, i = 2, \ldots, n.$$  \hspace{1cm} (6)

Thus, $n - 2 - \mu_i$ is an eigenvalue with corresponding eigenvector $x_i$ of $Q(\overline{G})$, where $i = 2, \ldots, n$. As $\overline{G}$ is $(n-1-k)$-regular, $2(n-1-k)$ is an eigenvalue with corresponding eigenvector $e = (1, 1, 1)^T$.

Using Lemma 1, we obtain the signless Laplacian spectrum of the join of two special graphs in the following theorem.

**Theorem 1.** Let $G_1$ be a $k$-regular graph on $n$ vertices and $G_2$ be an empty graph on $n-k$ vertices. If $2k = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the signless Laplacian eigenvalues of $G_1$, then the signless Laplacian eigenvalues of $G_1 \cup G_2$ are

$$n - k + \mu_3, n - k + \mu_3, \ldots, n - k + \mu_n, n(n-k-1), k, 2n.$$  \hspace{1cm} (7)

**Proof.** Note that the join of $G_1$ and $G_2$ can also be expressed with

$$G_1 \cup G_2 = G_1 \cup \overline{G_2}.$$  \hspace{1cm} (8)

Since $2k = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $0^{n-k}$ are the signless Laplacian eigenvalues of $G_1$ and $G_2$, respectively, by Lemma 1, we have that the signless Laplacian spectra of $\overline{G_1}$ and $\overline{G_2}$ are as follows:

$$\{n - 2 - \mu_1, n - 2 - \mu_2, \ldots, n - 2 - \mu_n\},$$

$$\{n - k - 2(n-k-1), k, 2n\}.$$  \hspace{1cm} (9)

Thus, the set of the signless Laplacian eigenvalues of $\overline{G_1} \cup \overline{G_2}$ is composed of the above two sets. Using Lemma 1, we obtain the signless Laplacian eigenvalues of $G_1 \cup G_2$ as follows:

$$n - k + \mu_3, n - k + \mu_3, \ldots, n - k + \mu_n, n(n-k-1), k, 2n.$$  \hspace{1cm} (10)

Using Theorem 1, from any $k$-regular Q-borderenergetic graph, we can construct a new class of Q-borderenergetic graphs in the following theorem.

**Theorem 2.** Let $G$ be a $k$-regular Q-borderenergetic graph with $n$ vertices. Then $GVK_{-k,n}$ is Q-borderenergetic.

**Proof.** Let $2k = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the signless Laplacian eigenvalues of $G$. Since $G$ is Q-borderenergetic, then we have

$$\sum_{i=1}^{n} |\mu_i - k| = 2n - 2.$$  \hspace{1cm} (11)

Let $p = n - k$. By Theorem 1, the Q-spectrum of $GVK_{p}$ is

$$\text{Spec}_{Q}(GVK_{p}) = \{p + \mu_3, p + \mu_3, \ldots, p + \mu_n, n, n, n, n, k, 2n\}.$$  \hspace{1cm} (12)

Since $p = n - k$, the average degree $\overline{d}$ of graph $GVK_{-p}$ is

$$\overline{d} = \frac{nk + 2np}{n + p} = \frac{nk + 2n(n-k)}{n + n - k} = n.$$  \hspace{1cm} (13)
By the definition of signless Laplacian energy of a graph with (11), we have

\[
QE(G\vee K_P) = \sum_{i=1}^{n} |\mu_i - k| + |n - n| (n - k - 1) \\
+ |k - n| + |2n - n| \\
= \sum_{i=1}^{n} |\mu_i - k| - |\mu_i - k| + n - k + n \\
= (2n - 2) - (2k - k) + n - k + n = 2(2n - k - 1).
\]

Since \(|V(G\vee K_P)| = 2n - k\), from the above result, we conclude that \(G\vee K_P\) is \(Q\) borderenergetic. □

### 3. Sequences of \(Q\)-Borderenergetic Graphs

In this section, by using Theorem 2 repeatedly, an infinite sequence of \(Q\)-borderenergetic graphs is constructed. Let \(G^{(0)}\) be any \(k\)-regular \(Q\)-borderenergetic graph with \(n\) vertices. Consider an infinite sequence \(H\) of graphs, i.e., \(H = \{G^{(0)}, G^{(1)}, \ldots, G^{(s)}, \ldots\}\) such that

\[
G^{(1)} = G^{(0)} \vee K_{n-k}, G^{(2)} = G^{(1)} \vee K_{n-k}, \ldots, G^{(s)} = G^{(s-1)} \vee K_{n-k}, \ldots
\]

One can easily see that graph \(G^{(s)}(s = 1, 2, \ldots)\) is of orders \(n + s(n - k)\) and \(n + (s - 1)(n - k)\)-regular. And the signless Laplacian spectrum of \(G^{(0)}\) is given in the following lemma.

**Lemma 2.** Let \(G^{(0)}\) be a \(k\)-regular \(Q\)-borderenergetic graph of order \(n\) with signless Laplacian eigenvalues \(2k = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n\). Then for any \(G^{(s)} \in H\) \((s \geq 1)\), the signless Laplacian spectrum of \(G^{(s)}\) is the following:

\[
\operatorname{Spec}_Q(G^{(s)}) = \left( \begin{array}{c}
\mu_2 + s(n - k), \mu_3 + s(n - k), \ldots, \mu_n + s(n - k), n + (s - 1)(n - k), \ldots, n + (s - 1)(n - k), \\
n + (s - 2)(n - k), 2n + 2(s - 1)(n - k) \end{array} \right)
\]

Proof. We prove this lemma by mathematical induction on \(s\). For \(s = 1\), by Theorem 2, (16) holds. We now assume that the result holds for \(s = t\). Then we have

\[
\operatorname{Spec}_Q(G^{(t)}) = \left( \begin{array}{c}
\mu_2 + t(n - k), \mu_3 + t(n - k), \ldots, \mu_n + t(n - k), n + (t - 1)(n - k), \ldots, n + (t - 1)(n - k), \\
n + (t - 2)(n - k), 2n + 2(t - 1)(n - k) \end{array} \right)
\]

Now, we have \(G^{(t+1)} = G^t \vee K_{n-k}\). By Theorem 1, we obtain

\[
\operatorname{Spec}_Q(G^{(t+1)}) = \left( \begin{array}{c}
\mu_2 + (t + 1)(n - k), \mu_3 + (t + 1)(n - k), \ldots, \mu_n + (t + 1)(n - k), n + t(n - k), \ldots, n + t(n - k), \\
n + (t - 1)(n - k), 2n + 2t(n - k) \end{array} \right)
\]
Proof. Obviously, the average degree of $G$ is $k$. The former equality holds by Lemma 4. Moreover,
\[
\text{QE}(G) = \sum_{i=1}^{n} |\mu_i - k| = \sum_{i=1}^{n} |k + \lambda_i - k| = \text{E}(G).
\]
(20)

This completes the proof of the theorem.

For a $k$-regular graph of order $n$, if $G$ is borderenergetic, then $G$ is $Q$-borderenergetic and $L$-borderenergetic. In [13], Gong et al. found all the borderenergetic graphs with order $7 \leq n \leq 9$. Bearing in mind that there are no noncomplete borderenergetic graphs with order $n < 7$. Furthermore, Li et al. [17] searched for the borderenergetic graphs of order 10. Thus, we can find all the regular $Q$ or $L$-borderenergetic graph of order $n$, $7 \leq n \leq 10$ (Figure 1). Denote the $i$-th $k$-regular $Q$-borderenergetic graph of order $n$ by $G_{n,k}$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{Q-borderenergetic_graphs.png}
\caption{The regular $Q$-borderenergetic graphs (a) $G_{8,5}^{1}$, (b) $G_{6,4}^{2}$, (c) $G_{9,4}^{3}$, (d) $G_{10,7}^{4}$, (e) $G_{10,6}^{5}$, and (f) $G_{10,6}^{6}$.}
\end{figure}

\section*{Data Availability}

The data, cited from the paper [17], used to support the findings of this study are included within the article.

Proof. Since the graph $G^{(i)}$ is $n + (s - 1)(n - k)$-regular with order $n + s(n - k)$, by Lemma 2 and the definition of signless Laplacian energy, we have

\[
\text{QE}(G^{(i)}) = \sum_{i=1}^{n} |\mu_i + s(n - k) - n - (s - 1)(n - k)| + s(n - k - 1)[n + (s - 1)(n - k) - n - (s - 1)(n - k)]
+ s[n + (s - 2)(n - k) - n - (s - 1)(n - k)] + 2(n + (s - 1)(n - k)) - n - (s - 1)(n - k)
\]
\[
= 2n - 2 - |\mu_1 - k| + |s(n - k) + (n + (s - 1)(n - k))| = 2n - 2 - |\mu_1 - k| + n + (2s - 1)(n - k)
\]
(19)

\section*{Conflicts of Interest}

The authors declare that they have no conflicts of interest.

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