INTEGRABLE EXTENSIONS OF N=2 SUPERSYMMETRIC KdV HIERARCHY ASSOCIATED WITH THE NONUNIQUENESS OF THE ROOTS OF THE LAX OPERATOR

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Abstract.

We present a new supersymmetric integrable extensions of the a=4,N=2 KdV hierarchy. The root of the supersymmetric Lax operator of the KdV equation is generalized, by including additional fields. This generalized root generate new hierarchy of integrable equations, for which we investigate the hamiltonian structure. In special case our system describes the interaction of the KdV equation with the two MKdV equations.

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1 Introduction

The Korteweg -de Vries (KdV) equation, which has been extensively studied by mathematicians as well as physicists [1-3] in the last 30 years, is probably the most popular soliton equation. On the other side, various different generalizations of the soliton equation have recently been proposed as the Kadomtsev - Petviashvili and Gelfand - Dickey hierarchies and supersymmetrization [4-11]. The motivation for studying these are diverse. In the supersymmetric generalization, one expects that, in the so called bosonic sector of the supersymmetry (SUSY) a new class of the integrable models may appear.

From the soliton point of view we can distinguish two important classes of the supersymmetric equations: the non-extended ($N = 1$) and extended ($N > 1$) cases. Considerations of the extended case may imply new bosonic equations whose properties need further investigation. This may be viewed as a bonus, but this extended case is in no way more fundamental than the non-extended one.

In order to get a SUSY theory we have to add to a system of $k$ bosonic equations $kN$ fermions and $k(N - 1)$ boson fields ($k = 1, 2...N = 1, 2..$) in such a way that the final theory becomes SUSY invariant. Interestingly enough, it appeared that during the supersymmetrizations, some typical SUSY effects (compared with the classical theory) occurred. For example, the Lax operator for the one of the extended ($N = 2$) supersymmetric extension of the KdV equation ($a=4$) possesses nonunique roots [12]. The supersymmetric Boussinesq [13] equation does not reduce to the classical Boussinesq equation. The $N=1$ supersymmetric KdV equation possesses the non-local conservation laws [14]. These effects rely strongly on the descriptions of the generalized systems of equations which we would like to supersymmetrize.

Until now supersymmetric KdV equation have been constructed for $N = 1, 2, 3$ and 4 [4,8,15-18] based on their relation to the superconformal algebras. For extended $N = 2$ supersymetric case it appeared that it is possible to construct three different integrable extensions of the KdV equation [8,19,20]. All these extensions have Lax pair representations. On the other side a new variety of generalizations of all different supersymmetric extensions of the KdV equation have been proposed [10,21-23] recently. In order to obtain such extensions the Lax operator is modified by including, in a proper way, the additional fields. These Lax operators produce new integrable hierarchy of equations. However such procedure is technically complicated.

For the supersymmetric $N = 2, a = 4$ KdV equation we have additional possibility of the generalizations of this equation. Indeed, we can generalize one of the root of the Lax operator, by including to it additional fields. As the result the Lax operator is also generalized. It simplify the process of generalization of the Lax operator. In this paper we describe such construction in more details. As a result we obtain new extensions of the $N = 2, a = 4$ supersymmetric KdV equation. This extensions constitute the hamiltoninan system. We investigate also the reduction of our generalized supersymmetric equation. Finally let us mention that our generalization is different then this proposed by Ivanov and Krivons [21].
2 Notation

The basic objects in the supersymmetric analysis are superfields and the supersymmetric derivatives. The Taylor expansion of the superfield with respect to the $\theta$ is

$$\phi(x, \theta_1, \theta_2) = w(x) + \theta_1 \xi_1 + \theta_2 \xi_2 \theta_2 u,$$

where the fields $w, u$ are to be interpreted as the boson (fermion) fields for the superboson (superfermion) field while $\xi_1, \xi_2$, as the fermion (boson) for the superboson (superfermions) respectively. The superderivatives are defined as

$$D_1 = \partial_{\theta_1} + \theta_1 \partial,$$

$$D_2 = \partial_{\theta_2} + \theta_2 \partial,$$

and satisfy $D_1^2 = D_2^2 = \partial$ and $D_1 D_2 + D_2 D_1 = 0$.

Below we shall use the following notation: $(D_i F)$ denotes the outcome of the action of the superderivative on the superfield $F$, while $D_i F$ denotes the action itself of the superderivative on the superfield $F$.

3 Supersymmetric N=2 KdV equation

This equation could be written down in the following one-parameter family of super-Hamiltonian evolution equations

$$\phi_t = -\phi_{xxx} + 3(\phi D_1 D_2 \phi)_x + \frac{1}{2} (a - 1)(D_1 D_2 \phi^2)_x + 3a \phi^2 \phi_x$$

$$= (D_1 D_2 \partial + 2 \partial \phi + 2 \phi \partial - D_1 \phi D_1 - D_2 \phi D_2) \frac{\delta}{\delta \phi} \int \frac{1}{2} (\phi D_1 D_2 \phi + \frac{a}{3} \phi^3) dX$$

where $dX = dx \theta_1 \theta_2$ and $a$ is an arbitrary parameter and $\phi$ is a superboson.

The operator

$$P_2(\phi) := D_1 D_2 \partial + 2 \partial \phi + 2 \phi \partial - D_1 \phi D_1 - D_2 \phi D_2$$

in (5) is the hamiltonian operator stemming from the $N = 2$ extension of the Virasoro algebra.

Although for arbitrary values of $a$ a Miura transformation for (4) had been given in [8], only for three cases ($a = -2, 4, 1$) Lax formulations had been found [8,20]. They are given by

$$a = 1 : \quad L := \partial + \partial^{-1} D_1 D_2 \phi$$

$$a = -2 : \quad L := \partial^2 + D_1 \phi D_2 - D_2 \phi D_1$$

$$a = 4 : \quad L := \partial^2 - (D_1 D_2 \phi) - \phi^2 + (D_2 \phi) D_1 - (D_1 \phi) D_2 - 2 \phi D_1 D_2$$

For $a = -2, 4$ the equation (4) is equivalent to [8]

$$\frac{d}{dt} L = -4 [(L^2)_+ , L]$$
where + denotes the (super) differential part of the operator.

For $a = 1$ the equation (4) is equivalent to [20]

$$\frac{d}{dt} L = \left[ (L^3)_{\geq 1}, L \right]$$

where $\geq 1$ denotes purely (super) differential part of the operator.

The fractional powers of $L$ can be obtained from the square roots of the Lax operators defined by the following expansion into (inverse) powers of the differential operator

$$L^\frac{1}{2} = \partial + \sum_{k=1}^{\infty} (a_k + b_k D_1 + c_k D_2 + d_k D_1 D_2) \partial^{-k};$$

where $a_k, d_k$ are superbosons while $b_k, c_k$ are superfermions.

However such square root can be defined also in a different manner

$$L^\frac{1}{2} = i \left(D_1 D_2 + a_0 + \sum_{k=1}^{\infty} (aa_k + bb_k D_1 + cc_k D_2 + dd_k D_1 D_2) \partial^{-k} \right).$$

where $aa_k, dd_k$ are superbosons while $bb_k, cc_k$ are superfermions.

Indeed the Lax operator (9) for the $N = 2, a = 4$ supersymmetric KdV equation can be written down as

$$\hat{L} = D_1 D_2 + \phi.$$ (14)

Due to it also Lax equations of the type

$$\frac{d}{dt} \hat{L} = \left[ (\hat{L}^n \hat{L}^\frac{1}{2})_+, \hat{L} \right]$$

can be considered in addition to (10). Furthermore from the Lax operator $\hat{L}$ additional conserved quantities for the equation of motion are given by the residues of the pseudo-differential operators $L^{k+\frac{1}{2}} \hat{L}$. These correspond to the additional Hamiltonian functions $H_{2k}$ which reduce to 0 when passing to the $N = 1$ case.

4 The generalization of the supersymmetric a=4 KdV equation

The formula (13) suggests to consider more general form of the root of the Lax operator as (14). Therefore we consider two different generalizations of the operator (14):

$$L_b = D_1 D_2 + \phi + h \partial^{-1} D_1 D_2 g - g \partial^{-1} D_1 D_2 h$$

$$L_f = D_1 D_2 + \phi + h \partial^{-1} D_1 D_2 g + g \partial^{-1} D_1 D_2 h$$

where $h, g$ are the superbosons for $L_b$ while they are superfermions for $L_f$.

Now it is rather technical problem to construct the Lax operator as the second power of $L_f$ or $L_b$. Next we can easily compute second root of such constructed Lax operator, using the formula (12) and finally to construct the whole hierarchy of integrable equations utilizing the formula (15). Explicitely the second flow for the superbosonic case is
\[ \phi_t = \left( -D_1D_2\phi - 2\phi^2 - 2(D_2gD_2h) - 2(D_1gD_1h) + 2gh_x - 2g_x h \right) / 2, \]  
\[ h_t = -\left( 2D_1D_2h_x + (D_1\phi D_1h) + (D_2\phi D_2h) + \phi_x h + 4(D_2hD_1h)g \\
\quad + 2(D_1gD_2h)h - 2(D_2gD_1h)h + 2(D_1D_2g)h^2 - 4(D_1D_2h)gh + 4\phi h_x \right) / 2, \]  
\[ g_t = -\left( 2D_1D_2g_x + (D_1\phi D_1g) + (D_2\phi D_2g) + \phi_x g - 4(D_2gD_1g)h \\
\quad - 2(D_1gD_2g)h + 2(D_2gD_1h)g + 2(D_1D_2g)gh - 2(D_1D_2h)g^2 + 4\phi g_x \right) / 2. \]

while for the superfermionic case is

\[ \phi_t = \left( -(D_1D_2\phi) - 2\phi^2 + 2g_x h - 2gh_x - 2(D_1gD_1h) - 2(D_2gD_2h) \right) / 2, \]  
\[ h_t = \left( -2D_1D_2h_x - 4h_x \phi - (D_1\phi D_1h) - (D_2\phi D_2h) - 2g(D_1D_2h)h \\
\quad - h\phi_x + 2h(D_1gD_2h) - 2h(D_2gD_1h) \right) / 2, \]  
\[ g_t = \left( -2D_1D_2g_x - 4g_x \phi - (D_1\phi D_1g) - (D_2\phi D_2g) - 2(D_1D_2g)gh \\
\quad - g\phi_x - 2g(D_1gD_2h) + 2g(D_2gD_1h) \right) / 2. \]

The conserved charges for our systems are given by the residues of the pseudo-differential operators \( L^{k+\frac{1}{2}} \). Explicitly the first three charges are:

for the superbosonic case

\[ H_1 = \int dX \phi, \]  
\[ H_2 = \int dX \left( 4g_x h + \phi^2 \right), \]  
\[ H_3 = \int dX \left( \phi(D_1D_2\phi) + \frac{4}{3} \phi^3 + 6h(D_1D_2g_x) + 3(D_1g)(D_1h)\phi + \\
\quad 3(D_2g)(D_1g)\phi + 3g_x h\phi - 3gh_x \phi - 3(D_1g)(D_2h)gh - \\
\quad 3(D_2g)(D_1g)h^2 + 3(D_2g)(D_1h)h g - 3(D_2h)(D_1h)g^2 \right) / 12. \]

while for the superfermionic case

\[ H_1 = \int dX \phi, \]  
\[ H_2 = \int dX \left( -4g_x h + \phi^2 \right), \]  
\[ H_3 = \int dX \left( \phi(D_1D_2\phi) + \frac{4}{3} \phi^3 - 6(D_1D_2g_x)h - 3g_x h\phi + 3gh_x \phi \\
\quad - 3gh(D_1g)(D_2h) + 3gh(D_2g)(D_1h) + 3\phi(D_1g)(D_1h) + \\
\quad 3\phi(D_2g)(D_2h) \right) / 12. \]

The systems of equations (18-20) and (21-23) are hamiltonians systems with the following Hamiltonian operator:
for the superbosonic case

\[ P = \begin{pmatrix} \partial & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \] (30)

and for the superfermionic case

\[ P = \begin{pmatrix} \partial & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \] (31)

Using this Hamiltonian operator our equations could be written down in the hamiltonian form as

\[ \frac{d}{dt}(\phi, g, h) = P(\frac{\delta H_3}{\delta \phi}, \frac{\delta H_3}{\delta g}, \frac{\delta H_3}{\delta h})^t, \] (32)

where \( t \) denotes transposition.

5 Reduction

Using formula (15) it is possible to construct a whole hierarchy of integrable equations. In the next we consider the case in which \( g = h \). Then the superbosonic Lax operator \( L_b \) reduces to the usual Lax operator for the supersymmetric \( N = 2, a = 4 \) Korteweg - de Vries equation. For the superfermionic case we obtain interesting possibilities in which

\[ L_f = D_1D_2 + \phi + 2h\partial^{-1}D_1D_2h \] (33)

This Lax operator produces the new hierarchy of equations also. The second flow could be obtained from the formulas (21 - 23) and reads as

\[ \phi_t = \left(- (D_1D_2\phi - 2\phi^2 + 4h_xh - 2(D_1h)^2 - 2(D_2h)^2)\right)_x/2 \] (34)

\[ h_t = \left(-(2(D_1D_2h_x - 4h_x\phi - (D_1\phi)(D_2h) - (D_2\phi)(D_2h) - h\phi_x)\right)/4. \] (35)

The third flow produces us the generalization of the supersymmetric \( N = 2, a = 4 \) KdV equation in the form

\[ \phi_t = \left( - \phi_{xx} + 6\phi(D_1D_2\phi) - 3(D_2\phi)(D_1\phi) + 4\phi^3 + 12h(D_1D_2h_x) - \\
12h_x(D_1D_2h) - 24h_xh\phi + 12\phi(D_1h)^2 + 12\phi(D_2h)^2\right)_x/4, \] (36)

\[ h_t = \left(- 4h_{xxx} + 12\phi(D_1D_2h_x) + 6\phi_x(D_1D_2h) - 3(D_2\phi_x)(D_1h) - \\
6(D_2\phi)(D_1h_x) + 3(D_1\phi_x)(D_2h) + 6(D_1\phi)(D_2h_x) + \\
12h_x((D_1h)^2 + (D_2h)^2 + \phi^2) + 6\phi((D_1\phi)(D_1h) + (D_2\phi)(D_2h) + \phi_xh) - \\
6h_x(D_1D_2\phi) + 3h(D_1D_2\phi_x)\right)/2. \] (37)

These equations are hamiltonian equations with the following Hamiltonian operator

\[ P = \begin{pmatrix} \partial & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \] (38)
The Hamiltonian which produce the equations (34-35) is

\[ H_3 = \int dX \left( \phi(D_1D_2\phi) + \frac{3}{4}\phi^3 - 6(D_1D_2h)xh - 6hxh\phi + 
+ 3\phi(D_1h)^2 + 3\phi(D_2h)^2 \right) / 12. \] (39)

while this which give us equations (36-37) is

\[ H_5 = \int dX \left( \frac{1}{2}\phi_x^2 + \frac{3}{2}\phi^2(D_1D_2\phi) + \phi^4 - 2ff_{xxx} + 3f f_x((D_1f)^2 + (D_2f)^2) 
+ \phi(12f(D_1D_2f) - 12f_x(D_1D_2f) - 12\phi f + 6\phi(D_1f)^2 + 6\phi(D_2f)^2) \right). \] (40)

Now let us investigate the bosonic limit of the equations (34-37). Assuming that

\[ \phi = \phi_o + \theta_2\theta_1\phi, \] (41)
\[ h = \theta_1h_o + \theta_2h. \] (42)

the bosonic sector of the equations 34-35 has the following form

\[ \begin{align*}
\phi_{ot} &= \left( -\phi - 2\phi_o^2 - 2h_o^2 - 2h_1^2 \right) / 2, \\
\phi_{1t} &= \left( \phi_{ot} - 4\phi_1\phi_o + 8h_o h_1 - 8h_1 h_0 \right) / 2, \\
h_{ot} &= \left( -2h_{1xx} - 2\phi_{ox} h_o - 4\phi_o h_1 - \phi_1 h_1 \right) / 2, \\
h_{1t} &= (2h_{ox} - 2\phi_{ox} h_1 - 4\phi_o h_1 + \phi_1 h_0) / 2. 
\end{align*} \] (43)

The bosonic sector of the equations (36-37) is

\[ \begin{align*}
\phi_{ot} &= \left( -\phi_{ot} + 6\phi_1\phi_o + 4\phi_o^3 + 12\phi_o(h_1^2 + h_0^2) \right) / 4, \\
\phi_{1t} &= \left( -\phi_{1t} - 3\phi_{ox}^2 - 6\phi_{ox}h_o + 12\phi_1\phi_o^2 + 3\phi_1^2 \\
&\quad + 12\phi_1(h_1^2 + h_2^2) + 12h_{ox}h_o - 12h_0^2 - 48\phi_o h_1 h_o + \\
&\quad 12h_{1xx} h_1 - 12h_1^2 + 48\phi_o h_1 h_0 \right) / 4, \\
h_{ot} &= \left( -4\phi_{ox} + 12h_{ox} h_2^2 + 12h_{ox} h_1^2 + 12\phi_{ax} h_1 h_1 + 3\phi_{ox}h_1 + \\
&\quad 12\phi_o h_{xx} + 12\phi_o h_{ox} + 12\phi_o h_1 h_o + 6\phi_1 h_1 \right) / 2, \\
h_{1t} &= \left( -4h_{ox} + 12h_{1xx} h_1^2 + 12h_{1xx} h_o^2 - 12\phi_{ax} h_2 - 3\phi_{ox} h_o - \\
&\quad 12\phi_o h_{ox} + 12\phi_o h_1 h_1 + 12\phi_o h_1 h_o - 6\phi_1 h_0 \right) / 2. 
\end{align*} \] (44-47)

The Hamiltonian operator for these systems could be extracted also from the supersymmetric Hamiltonian operator (38) and reads as

\[ P = \begin{pmatrix}
0 & \partial & 0 & 0 \\
\partial & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}. \] (48)

Similarly we can computed the corresponding hamiltonians using the formulas (39-40).

Interestingly the system of equations (44-47) allows us to make additional reduction in which \( \phi_o = 0 \). In this limit these equations reduces to
\[ \phi_{tt} = \left( -\phi_{1xx} + 3\phi_1^2 + 12\phi_1(h_0^2 + h_1^2) + 12h_{oxx}h_o - 12h_o^2 + 12h_{1xx}h_1 - 12h_1^2 \right)/4, \]  
(49)

\[ h_{ot} = \left( -2h_{oxxx} + 6h_{ox}h_o^2 + 6h_{ox}h_1^2 \right), \]  
(50)

\[ h_{tt} = \left( -2h_{1xxx} + 6h_{1x}h_1^2 + 6h_{1x}h_o^2 \right). \]  
(51)

and describes the interactions of Korteweg - de Vries field with two Modified Korteweg - de Vries fields.

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