Blow-up of weak solutions to a chemotaxis system under influence of an external chemoattractant

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Abstract
We study nonnegative radially symmetric solutions of the parabolic–elliptic Keller–Segel whole space system

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^n, t > 0, \\
0 &= \Delta v + u + f(x), & x \in \mathbb{R}^n, t > 0, \\
u(x, 0) &= u_0(x), & x \in \mathbb{R}^n,
\end{aligned}
\]

with prototypical external signal production

\[
f(x) := \begin{cases}
\int_0^R |x|^{-\alpha}, & \text{if } |x| \leq R - \rho, \\
0, & \text{if } |x| \geq R + \rho,
\end{cases}
\]

for \( R \in (0, 1) \) and \( \rho \in \left(0, \frac{R}{2}\right) \), which is still integrable but not of class \( L^{2+\delta_0}(\mathbb{R}^n) \) for some \( \delta_0 \in (0, 1) \). For corresponding parabolic-parabolic Neumann-type boundary-value problems in bounded domains \( \Omega \), where \( f \in L^{2+\delta_0}(\Omega) \cap C^\alpha(\Omega) \) for some \( \delta_0 \in (0, 1) \) and \( \alpha \in (0, 1) \), it is known that the system does not emit blow-up solutions if the quantities \( \|u_0\|_{L^{2+\delta_0}(\Omega)}, \|f\|_{L^{2+\delta_0}(\Omega)} \) and \( \|v_0\|_{L^\theta(\Omega)} \) for some \( \theta > n \), are all bounded by some \( \varepsilon > 0 \) small enough.

We will show that whenever \( f_0 > \frac{2\alpha}{n}(n-2)(n-\alpha) \) and \( u_0 \equiv c_0 > 0 \) in \( B_1(0) \), a measure-valued global-in-time weak solution to the system above can be constructed which blows up immediately. Since these conditions are independent of \( R \in (0, 1) \) and \( c_0 > 0 \), we obtain a strong indication that in fact \( \delta_0 = 0 \) is critical for the existence of global bounded solutions under a smallness conditions as described above.
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1. Introduction

Chemotaxis is a biological mechanism whereby the movement of cells is influenced by a chemical substance. This mechanism appears in multiple biological processes, e.g. aggregation of bacteria or the inflammatory response of leukocytes. One of the first PDE systems modelling these processes dates back to the pioneering works [10] and [11] by Keller and Segel. Variants of the original Keller–Segel model have also been incorporated in more complex biological processes ranging from pattern formation ([1]) to angiogenesis in early stages of cancer ([13]). For a broader spectrum of applications and an overview of known results we refer to the survey articles [2, 7] and [8].

The Keller–Segel system which is the basis this article has the form

\[
\begin{align*}
  u_t &= \Delta u - \nabla \cdot (u \nabla v), \quad x \in \Omega, \; t > 0, \\
  v_t &= \Delta v - v + u, \quad x \in \Omega, \; t > 0, \\
  \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \; t > 0, \\
  (u(x,0), v(x,0)) &= (u_0(x), v_0(x)), \quad x \in \Omega, \\
\end{align*}
\]

(KS)

wherein \( u(x, t) \) represents the density of the moving cells and \( v(x, t) \) denotes the concentration of an attracting chemical substance influencing said movement at place \( x \) in the bounded domain \( \Omega \) and at time \( t \). In the mathematical study of chemotaxis, blow-up solutions, i.e. the existence of some \( T \in (0, \infty) \) such that \( \lim \sup_{T \to T} \|u\|_{L^\infty(\Omega)} = \infty \), are of utmost importance. The existence of such solutions is identified with the occurrence of self-organizing patterns within the cell population. As such, the formulation of conditions which allow for blow-up to happen, or conditions negating blow-up completely are widely sought after.

For (KS) conditions negating blow-up are well known. In particular it was shown for suitable \( \Omega \subset \mathbb{R}^n \) and nonnegative initial values \( u_0 \in C^0(\bar{\Omega}), \; v_0 \in C^0(\bar{\Omega}) \), that the corresponding maximally extended classical solution \((u, v)\) of (KS) fulfills:

If \( n = 2 \): If \( \int_\Omega u_0 \, dx < 4\pi \) (or \( 8\pi \) in the radial symmetric setting), then \((u, v)\) is global and bounded with regard to the \( L^\infty(\Omega) \)-norm ([12, 5]).

If \( n \geq 3 \): It was proven in [17] that there exists a bound for \( u_0 \in L^q(\Omega) \) and for \( \nabla v_0 \in L^p(\Omega) \), with \( q > \frac{n}{2} \) and \( p > n \) such that the solution \((u, v)\) is global in time and bounded. This result has further been extended to the critical case \( q = \frac{n}{2} \) and \( p = n \) ([4]).

In our previous work ([3]) we considered an extension of the (KS) model by introducing an external signal production to (KS). External signal productions are motivated by the fact that frequently in biological experiments artificial gradients of chemoattractants are introduced to observe the migration of cells, see for instance [15] where the migration of hematopoietic progenitor cells in response to attracting stroma cells introduced to the surroundings was studied. Going further than just observing the response to external stimulants one could ask if and how a population of cells can be influenced in a desired way through these external signal, which could in particular be interesting for tumor treatment. Mathematically this would translate to an optimal control problem which, after a given time, nets a desired distribution of cells through adjustment of the external signal. Since the occurrence of self-organizing patterns is
closely linked to blow-up solutions, the first step before thinking of an appropriate optimal control formulation, is to verify for what class of external signal production functions blow-up may occur.

As a basic prototype for a chemotaxis model with external source we studied in [3] the system

\[
\begin{aligned}
&u_t = \Delta u - \nabla \cdot (u \nabla v), \quad x \in \Omega, t > 0, \\
&\tau v_t = \Delta v - \nabla u + f(x, t), \quad x \in \Omega, t > 0, \\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
&u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega
\end{aligned} \tag{KS_\delta}
\]

in a bounded and smooth domain \( \Omega \subseteq \mathbb{R}^n \) with \( n \geq 2, \tau > 0, u_0 \in C^0(\Omega), v_0 \in W^{1,\theta}(\Omega) \) for some \( \theta > n \) and \( f \in L^\infty([0, \infty); L^{\frac{n+2}{n-2}+\theta}(\Omega) \cap C^0(\Omega \times (0, \infty)) \) with \( \delta_0 \in (0, 1) \) and \( \alpha > 0 \). Obviously for \( f \equiv 0 \), this coincides with (KS). Applying the usual fixed point arguments, as illustrated in [16] for the system (KS), we were able to verify the existence of classical solutions to (KS_\delta). Furthermore, we were able to expand most of the known boundedness results for (KS) to the setting \( f \not\equiv 0 \). Firstly, we obtained a critical mass result for \( n = 2 \) and constant signal production \( f \in L^{\frac{n+2}{n-2}+\theta}(\Omega) \cap C^0(\Omega) \) similar to the one by [12] and [5]. Secondly, and important for the context of our current work, we were able to prove a result similar to the one above by [17]. We cite the theorem here in a short version without including the statements regarding asymptotic behavior of the solution. For the full version see [3, theorem 1.3].

**Theorem 1.1.** Let \( 0 < \delta_0 < 1, \ 0 < \alpha, \ n > \theta < \frac{n^2 + 2n\delta_0}{n - 2\delta_0} \) and \( 1 < r \). Then there exist constants \( \varepsilon_0 > 0 \) and \( C > 0 \) with the following property: If \( u_0 \in C^0(\Omega), \ v_0 \in W^{1,\theta}(\Omega) \) and \( f \in L^\infty([0, \infty); L^{\frac{n+2}{n-2}+\theta}(\Omega) \cap C^0(\Omega \times (0, \infty)) \) are nonnegative with

\[
\|u_0\|_{L^{\frac{n+2}{n-2}+\theta}(\Omega)} \leq \varepsilon, \ \|\nabla v_0\|_{L^\infty(\Omega)} \leq \varepsilon \text{ and } \|f\|_{L^\infty([0, \infty); L^{\frac{n+2}{n-2}+\theta}(\Omega))} \leq \varepsilon
\]

for some \( \varepsilon < \varepsilon_0 \), then there exists a global classical solution \((u, v)\) of (KS_\delta) with \( \|u\|_{L^\infty(\Omega)} \) and \( \|v\|_{W^{1,\theta}(\Omega)} \) remaining bounded for all times.

Unfortunately, the methods applied to prove the result above do not yield any information whether \( \delta_0 = 0 \) is the critical boundary for the existence of such small-data solutions, that is blow-up occurs for \( f \) not of class \( L^{\frac{n+2}{n-2}+\theta}(\Omega) \) for some \( \delta_0 \in [0, 1) \). Since proving blow-up for parabolic-parabolic chemotaxis systems is a challenging and mostly unsolved task, there are some commonly made simplifications making at least some blow-up results attainable. The most important of these simplifications assumes that the chemical molecules diffuse much faster than the cells and as such \( v \) approximately satisfies an elliptic equation \( (\tau = 0) \) rather than a parabolic one \( (\tau > 0) \). These results for the simplified parabolic–elliptic system are then in turn taken as an indication for the long term behavior of the solutions to the corresponding fully parabolic system. That \( \delta_0 = 0 \) could be the critical boundary for the existence of small-data solutions in (KS_\delta) seems to be strongly suggested by the results from [14], where a simplified parabolic–elliptic version of (KS_\delta) in the radially symmetric setting on the whole space \( \mathbb{R}^n \) for \( n = 2 \) without degrading chemical was considered, that is

\[
\begin{aligned}
&u_t = \Delta u - \nabla \cdot (u \nabla v), \quad x \in \mathbb{R}^n, t > 0, \\
&0 = \Delta v + u + f(x), \quad x \in \mathbb{R}^n, t > 0, \\
&u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n
\end{aligned} \tag{KS_\delta^0}
\]
with a Dirac-distributed signal production \( f(x) = f_0 \delta(x) \). It was shown in a radially symmetric setting that for any choice of \( f_0 > 0 \) certain generalized solutions, so called radial weak solutions, blow up immediately and depending on the size of the initial mass \( \mu := \int_{\mathbb{R}^n} u_0(x) \, dx < \infty \), compared to the critical mass \( 8\pi - 2f_0 \), form a Dirac singularity.

It is the purpose of the present work to examine whether there is any indication that \( \delta_0 = 0 \) is also a critical boundary for the existence of such small-data solutions in higher dimensions. To this end we will study the behavior of solutions in dimensions \( n \geq 3 \) for constant-in-time functions \( f \), that are not of class \( L^{2+\delta_0}(\mathbb{R}^n) \) for some \( \delta_0 \in [0, 1) \) but still integrable. Following the approach of [14], we will consider \((KS)^0_f\) in the radially symmetric setting on the whole space \( \mathbb{R}^n \) for \( n \geq 3 \), with given radially symmetric and nonnegative \( u_0 \). Furthermore we assume, that \( u_0 \not\equiv 0 \) has finite mass \( \mu \) and that \( f \) is nonnegative and radially symmetric as well.

Using a transformation introduced in [9] and employed in [14], we will prove that generalized global-in-time measure-valued solutions of \((KS)^0_f\) blow up immediately for prototypical signal production functions \( f \) satisfying

\[
f(x) := \begin{cases} 
    f_0 \, |x|^{-\alpha}, & \text{if } |x| \leq R - \rho, \\
    0, & \text{if } |x| \geq R + \rho
\end{cases}
\]

and smooth in between with some \( 1 > R > 0, \alpha > 2, \rho \in (0, \frac{R}{2}) \) and \( f_0 > \frac{2n}{\alpha}(n-2)(n-\alpha) \).

Our main theorem reads as follows:

**Theorem 1.2 (Immediate blow-up of radial weak solutions).** For \( 1 > R > 0, \rho \in (0, \frac{R}{2}), n > \alpha > 2 \) and \( f_0 > 0 \) satisfying \( f_0 > \frac{2n}{\alpha}(n-2)(n-\alpha) \), let \( f(r) \) be defined as in (2.2). Furthermore, assume the initial data satisfy \( u_0 \equiv c_0 \) in \( \overline{B}(0) \) for some positive constant \( c_0 > 0 \). Then for all \( t_0 \geq 0 \) there exists a globally defined radial weak solution \( u \) of \((KS)^0_f\), in the sense of definition 2.1 below, such that this solution satisfies

\[
\|u\|_{L^{\infty}(\mathbb{R}^n \times [t_0, t_0 + \eta])} = \infty \quad \text{for all } \eta > 0.
\]

The theorem above states a sufficient condition for the occurrence of immediate blow-up in \((KS)^0_f\), with external production of the form described in (2.2). The only restriction on \( u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) involves \( c_0 > 0 \), which can be arbitrary small. Obviously, one can thereby find smooth initial values with arbitrary small \( L^p \) norms for which the theorem is still applicable and since the assumption imposed on \( f_0 \) is independent of the parameter \( R \), even signal production functions with small norm may lead to blow-up—in the case where \( f \) is not of class \( L^{2+\delta_0}(\mathbb{R}^n) \) (for some \( 0 \leq \delta_0 < 1 \)). Though we have to leave open the question if blow-up may also occur for scaling factors \( f_0 \) smaller than \( \frac{2n}{\alpha}(n-2)(n-\alpha) \), these results for the simplified parabolic–elliptic setting may be still be taken as a strong indication that the case \( \delta_0 = 0 \) is indeed a critical one for the existence of small-data solutions to \((KS)^0_f\) even in higher dimensions.

We should further remark that blow-up only occurs in the origin, which is indicated by the measure valued transformation identity (2.9) below. However, it is not clear whether the solution forms a Dirac singularity at the origin or some milder singularity, that is the solution remains bounded in some \( L^p \) space for all times. The two dimensional results by [14] suggest that the formation of a Dirac singularity at the origin could be dependent on the size of \( f_0 \), but further analysis is necessary to verify if this is indeed the case here.
In the subsequent sections, if not stated otherwise, $n$ will always denote the space dimension and $\mu := \int_{\mathbb{R}^n} u_0(x) dx < \infty$ the initial mass.

2. Blow-up for less regular signal production

We would like to consider prototypical signal production functions of the form

$$f(x) := \begin{cases} f_0 |x|^{-\alpha}, & \text{if } |x| < R, \\ 0, & \text{if } |x| \geq R \end{cases}$$

with some $f_0 > 0$ and $0 < R < 1$, which for $\alpha \in (2, n)$ are still integrable but not of class $L^{2+\epsilon}(\mathbb{R}^n)$ for any $\epsilon > 0$. However, in order to take advantage of well-known regularity results for partial differential equations we will instead work with a smoother version of (2.1).

More precisely, for $\rho \in (0, \frac{R}{2})$ we consider radially symmetric functions $f$ satisfying

$$f(r) := \begin{cases} f_0 r^{-\alpha}, & \text{if } r \leq R - \rho, \\ 0, & \text{if } r \geq R + \rho, \end{cases}$$

such that $r \mapsto f(r)$ is smooth and monotonically decreasing on $(0, \infty)$. Consequently the function $F(s) := \int_0^s f(r) r^{n-1} dr$ is monotonically increasing and smooth on $(0, \infty)$ with

$$F(s) = \int_0^s f(r) r^{n-1} dr = \begin{cases} \frac{f_0}{n-\alpha} \frac{s^{n-\alpha}}{n-\alpha}, & \text{if } 0 \leq s \leq R - \rho, \\ \leq f_0 (R + \rho)^{\frac{n-\alpha}{n-\alpha}}, & \text{if } s \geq R + \rho. \end{cases}$$

In addition the first derivative is monotonically decreasing and satisfies

$$F'(s) = \frac{1}{n} f(s^{\frac{1}{\alpha}}) = \begin{cases} \frac{f_0}{n} \frac{s^{\frac{\alpha}{n}}}{\alpha}, & \text{if } 0 < s \leq R - \rho, \\ 0, & \text{if } s \geq R + \rho. \end{cases}$$

Both of these functions will play an important role in the transformation of $(KS^0_f)$ introduced in the next section, which is an adjustment to higher space dimensions of the transformation applied in [14].

2.1. Radial weak solutions

Following the approach employed in [14] and [9], we will first make use of spherical coordinates to transform $(KS^0_f)$ into the related degenerate parabolic initial-boundary value problem (2.8). The solutions of these PDE problems are connected by the notion of radial weak solutions stated in definition 2.1. Our objective is then to prove the immediate blow-up of $W_s$ and in turn, by the identity (2.9), the blow-up of the radial weak solution $u$.

The transformation in question is defined by

$$W(s, t) := \frac{n}{[S_{n-1}]} \int_{B(0, s^\frac{1}{n})} u(x, t) dx, \quad s \geq 0, t \geq 0,$$
with $|S_{n-1}|$ representing the surface area of the unit sphere in $n$ dimensions and $B(0, r)$ denoting the ball around the origin with radius $r$. For radially symmetric $u = u(r, t)$, by using spherical coordinates, this can also be expressed as

$$W(s, t) = \frac{n}{|S_{n-1}|} \int_{S_{n-1}} \int_0^1 u(r, t) r^{n-1} dr dS = n \int_0^1 u(r, t) r^{n-1} dr, \ s \geq 0, t \geq 0.$$  

Formal calculation, without regarding the regularity of $u$ for now, shows

$$W(s, t) = nu(s^{1/2}, t)(s^{1/2})^{n-1} \frac{1}{s^{1/2}} = u(s^{1/2}, t) \text{ for } s > 0, t \geq 0$$  

(2.6)

and thus

$$W_0(s, t) = \frac{1}{n} s^{-\frac{n-1}{n}} u_r(s^{1/2}, t) \text{ for } s > 0, t \geq 0.$$  

Considering these expressions and the first equation of $(KS_0)$ we thereby see that $W(s, t)$ formally fulfills

$$W(s, t) = n^2 s^{\frac{2n-2}{n}} W_0(s, t) + W(s, t) W_0(s, t) + nF(s)W_0(s, t)$$  

with $F(s)$ as in (2.2). Let us briefly recall further statements from [14] regarding the transformed problem. Letting $W_0 := W(s, 0) = n \int_0^1 u_0(r) r^{n-1} dr$ for $s \geq 0$, we observe that if $u_0$ is nonnegative and bounded fulfilling $\mu := \int u_0(x) dx < \infty$, then $W_0$ satisfies

$$W_0 \in W^{1, \infty}(0, \infty),$$  

$$W_0 \geq 0 \text{ in } (0, \infty),$$  

$$W_0(s) \rightarrow \frac{n\mu}{|S_{n-1}|} \text{ as } s \rightarrow \infty.$$  

(2.7)

If $u$ satisfies the mass-conservation property $\int u(x, t) dx = \mu$ for all $t \geq 0$, then for each $t \geq 0$ there holds $W(s, t) \rightarrow \frac{n\mu}{|S_{n-1}|}$ as $s \rightarrow \infty$. Thereby, this formally leads to the following degenerate parabolic initial-boundary value problem:

$$\begin{cases}
W_t = n^2 s^{\frac{2n-2}{n}} W_0 + W W + nFW, & s > 0, t > 0, \\
W(0, t) = 0, \lim_{s \rightarrow \infty} W(s, t) = \frac{n\mu}{|S_{n-1}|}, & t > 0, \\
W(s, 0) = W_0(s) & s > 0.
\end{cases}$$  

(2.8)
By the definition of the transformation in (2.5) and the nonnegativity of \( u \), it is obvious that for each \( t \) the function \( W(\cdot, t) \) must be nondecreasing. In particular, \( W \) is bounded and since the PDE in (2.8) is uniformly parabolic with smooth coefficients in each cylinder \((s_0, \infty) \times (0, \infty)\) with \( s_0 > 0 \), standard theory implies the smoothness of \( W \) in \((0, \infty) \times (0, \infty)\). Thus, for \( s > 0 \) we expect the identity \( W(s, t) = u(s^{\frac{1}{n}}, t) \) suggested by (2.6) to hold. Consequently, discontinuities for \( W \) can only occur at \( s = 0 \), and given a solution \( W \) of (2.8) we can reconstruct \( u \) by taking into account the jump size \( W(t, 0) \) at the origin, in terms of the measure-valued identity

\[
u(x, t) := W([x], t) + \frac{|S_{n-1}|}{n} W(0^{+}, t) \delta(x) \quad (2.9)
\]

for \( t > 0 \), where \( \delta(x) \) denotes the Dirac delta function in \( n \) dimensions. In [14] it was therefore suggested to act in the framework of radially symmetric Radon measures \( M_{rad}(\mathbb{R}^n) \), that is the space of all functionals \( \psi \), radially symmetric about \( x = 0 \), defined on the space \( C_{0}^{0}(\mathbb{R}^n) \) of compactly supported continuous functions over \( \mathbb{R}^n \). To be more precise

\[
M_{rad}(\mathbb{R}^n) := \{ \psi : C_{0}^{0}(\mathbb{R}^n) \to \mathbb{R} \, | \, \psi(\zeta \circ \nu) = \psi(\zeta) \text{ for all test functions } \zeta \in C_{0}^{\infty}(\mathbb{R}^n) \text{ and all rotations } \nu \in SO(n) \},
\]

where \( SO(n) \) denotes the special orthogonal group in \( n \) dimensions. This way, we translate the notion of radial weak solutions given in [14, definition 1.1] to our equation in the following way:

**Definition 2.1.** Assume \( u_0 \in L^{\infty}(\mathbb{R}^n) \) is nonnegative and \( \mu := \int_{\mathbb{R}^n} u_0(x)dx \) is finite. Then we call

\[
\{ \nu, u \} \in C^{0}(0, \infty) ; M_{rad}(\mathbb{R}^n)
\]

a radial weak solution of \((KS_f)\) in \( \mathbb{R}^n \times (0, \infty) \) if the function \( W : [0, \infty) \times [0, \infty) \) defined by (2.5) satisfies

\[
W(s, t) \to \frac{n\mu}{|S_{n-1}|} \text{ as } s \to \infty \text{ for all } t > 0
\]

and

\[
- \int_0^\infty \int_0^\infty \zeta W - \int_0^\infty \zeta(0) W_0 = n^2 \int_0^\infty \int_0^\infty (s^{\frac{2n-2}{n}} \zeta_0) W - \frac{1}{2} \int_0^\infty \int_0^\infty \zeta W^2 - n \int_0^\infty \int_0^\infty (F \zeta)_s W
\]

for all \( \zeta \in C_{0}^{\infty}(0, \infty) \times [0, \infty) \), where \( W_0(s) := \frac{n}{|S_{n-1}|} \int_{B(0, \sqrt{s})} u_0(x) dx \) for \( s \geq 0 \).
\[ \chi^{(c)} \equiv 0 \text{ on } [0, \varepsilon/2], \quad \chi^{(c)} \equiv 1 \text{ on } [\varepsilon, \infty) \quad \text{and} \quad \chi^{(c)}_s \geq 0 \text{ on } [0, \infty). \] (2.10)

Additionally, \( \chi^{(c)} \) satisfies the inequalities

\[ |\chi^{(c)}_s| \leq \frac{c_\chi}{\varepsilon} \quad \text{and} \quad |\chi^{(c)}| \leq \frac{c_\chi}{\varepsilon}, \] (2.11)

with \( c_\chi := \|\chi^{(c)}_t\|_{L^\infty(0,\varepsilon)} + \|\chi^{(c)}_s\|_{L^\infty(0,\varepsilon)} \). Moreover, \( \chi^{(c)}(s) \not\rightarrow 1 \) as \( \varepsilon \rightarrow 0 \) holds for all \( s > 0 \).

The cut-off function at hand, we now introduce the approximate problem for (2.8):

\[
\begin{aligned}
&W_t(s,t) = n^2s^{-1}W^{(c)} + \chi^{(c)}W_{ss}^{(c)} + nF^{(c)}W_s^{(c)}, & s > 0, t > 0, \\
&W^{(c)}(0,t) = 0, \quad \lim_{s \rightarrow \infty} W^{(c)}(s,t) = \frac{n\mu}{|S_{n-1}|}, & t > 0, \\
&W^{(c)}(s,0) = W_0(s), & s > 0.
\end{aligned}
\] (2.12)

Although we are more interested in the behavior of solutions, we cannot completely skip examining solvability and other important properties. Let us therefore briefly state some results whose proofs we omit since these results can be proven by using the same arguments as shown in [14, lemmas 1.2–1.5].

**Lemma 2.2.** For \( 1 > R > 0, \rho \in (0, \frac{R}{2}), n > \alpha > 2 \) and \( f_0 > 0 \) let \( F \) be defined as in (2.2).

Assume \( W \) and \( W \) belong to \( C^{0}([0, \infty) \times [0, \varepsilon)) \cap C^{2,1}((0, \varepsilon) \times (0, \varepsilon)) \) and satisfy

\[ W \geq n^2s^{-1}W + \chi^{(c)}W + n\chi^{(c)}F, \]

and

\[ W \leq n^2s^{-1}W + \chi^{(c)}W + n\chi^{(c)}F, \]

for all \( s > 0 \) and \( t > 0 \). Additionally, suppose \( W_0 := \lim_{s \rightarrow \infty} W(s,0) \) and \( W_0 := \lim_{s \rightarrow \infty} W(s,0) \) satisfy (2.7) with positive numbers \( \mu_1 \) and \( \mu_2 \) respectively and that \( W_0 \) holds on \( (0, \varepsilon) \). Moreover, assume \( W(0,t) \geq W(0,t) \) for all \( t > 0 \) and \( \lim_{s \rightarrow \infty} W(s,t) \geq \lim_{s \rightarrow \infty} W(s,t) \) for all \( t > 0 \). Then \( W \geq W \) in \( [0, \varepsilon) \times [0, \varepsilon) \).

**Remark 2.3.** The comparison principle above also holds for bounded space-time cylinders \( [0, s_0] \times [0, t_0] \), if we assume that \( W \) and \( W \) belong to \( C^{0}([0, s_0] \times [0, t_0]) \cap C^{2,1}((0, s_0) \times (0, t_0)) \) and fulfill the inequalities \( W(0,t) \leq W(0,t) \) for \( t \in [0, t_0] \), \( W(s_0,t) \leq W(s_0,t) \) for \( t \in [0, t_0] \) and \( W_0 \in W_0 \) in \( [0, s_0] \) instead of the inequalities stated in the lemma above.

In addition to allowing for this comparison principle, the approximate problem is also uniquely solvable in the classical sense.

**Lemma 2.4.** For \( 1 > R > 0, \rho \in (0, \frac{R}{2}), n > \alpha > 2 \) and \( f_0 > 0 \) let \( F(s) \) be defined as in (2.2).

Assume \( W_0 \) satisfies (2.7) with some \( \mu > 0 \). Then for each \( \varepsilon \in (0, 1) \) there exists a unique function \( W^{(c)} \in C^{0}([0, \varepsilon) \times (0, \varepsilon)) \cap C^{2,1}((0, \varepsilon) \times (0, \varepsilon)) \) which satisfies (2.12) in the classical sense.

Taking the limit \( \varepsilon \downarrow 0 \) to obtain \( W^{(c)} \) in \( (0, \varepsilon) \times (0, \varepsilon) \), which—adopting the notion of [14] and [6]—we will call the proper solution of (2.8). This limit procedure combined with the backwards transformation in (2.9) then results in a desired radial weak solution in the sense of definition 2.1.
Lemma 2.5. For $1 > R > 0$, $\rho \in (0, \frac{R}{2})$, $n > \alpha > 2$ and $f_0 > 0$ let $F(s)$ be defined as in (2.2). Assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is a radially symmetric function satisfying $\mu := \int_{\mathbb{R}^n} u_0 dx < \infty$. Then there exists at least one radial weak solution $u$ of $(KS')$ in the sense of definition 2.1. Such a solution can be obtained by applying the backwards transformation (2.9) to the proper solution of (2.8).

3. Immediate blow-up of radial weak solutions

We require two further preparatory results not connected to the solution. The first is a variation of Gronwall’s lemma and can also be found in [14, lemma 2.1], whereto we refer for proof once again.

Lemma 3.1. Suppose that $\Phi \in W^{1,\infty}_l(\mathbb{R})$ is non-decreasing, and that for some $t_1, t_2 \in \mathbb{R}$ and $c \in \mathbb{R}$ we are given two functions $y \in C^1([t_1, t_2])$ and $z \in C^1([t_1, t_2])$ such that

$$\int_{t_1}^{t_2} \Phi(y(t)) dt = \Phi(y_1) - \Phi(y_2)$$

and

$$\frac{dy}{dt} = \Phi(y(t)) \quad \text{for all } t \in (t_1, t_1 + T).$$

Then

$$y(t) > z(t) \quad \text{for all } t \in (t_1, t_1 + T).$$

The next lemma will provide us with functions fulfilling the role of test functions later in the proof of our main theorem. This lemma is an adjusted version of the corresponding construction from [14, lemma 2.2].

Lemma 3.2. For $1 > R > 0$, $\rho \in (0, \frac{R}{2})$, $n > \alpha > 2$ and $f_0 > 0$ satisfying

$$f_0 > \frac{2n}{\alpha} (n - 2)(n - \alpha),$$

let $F(s)$ be defined as in (2.2). Furthermore, we define $h(n, \alpha, f_0) := (n - \alpha)(3n - 4) - f_0$ and fix $\xi \in (4 - \frac{4}{n}, 4)$ as well as $\delta \in (0, 1)$ fulfilling the condition

$$\delta > \max\left\{ \frac{n - \alpha}{n}, \frac{h(n, \alpha, f_0) + \sqrt{h(n, \alpha, f_0)^2 + 4f_0(n - \alpha)^2}}{2n(n - \alpha)} \right\}.$$

Then there exist positive constants $a$, $b$, $k_0$ and $K_0$ depending only on $n, f_0, \alpha$ and $\xi$, such that for any $\gamma > \frac{4}{R - \rho}$ the function $\varphi^{(\gamma)} := \varphi : (0, \infty) \to \mathbb{R}$ defined by

$$\varphi(s) := \begin{cases} \frac{a}{\gamma} s^{-\frac{\gamma}{n}} - b & \text{if } s < \frac{\xi}{\gamma} \\
 e^{-\gamma s} & \text{if } s \geq \frac{\xi}{\gamma} \end{cases}$$

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is positive, belongs to \( W^{2,\infty}_0((0, \infty)) \) and satisfies
\[
n^2 s^{n-2} \varphi_{ss} + 4(n^2 - n) s^{n-2} \varphi_s - n F \varphi_s - n F \varphi \geq k \alpha^2 \varphi \quad \text{a.e in } (0, \infty),
\]
(3.4)
as well as
\[
\int_0^\infty \frac{\varphi^2(s)}{\varphi(s)} \, ds \leq \frac{K_0}{\gamma^2}.
\]
(3.5)

**Proof.** Let us first verify that the assumed property (3.1) for \( f_0 \) ensures that \( \delta \in (0, 1) \) can be chosen to satisfy (3.2). Clearly \( \frac{n-\alpha}{n} < 1 \), thus we only have to verify that (3.1) implies
\[
h(n, \alpha, f_0) + \sqrt{n(n, \alpha, f_0)^2 + 4f_0(n - \alpha)^2} < 2n(n - \alpha).
\]

Multiplication of (3.1) with \( \frac{4\alpha}{n-\alpha} \) implies
\[
8n(n-2) < \frac{4\alpha}{n-\alpha} f_0.
\]
Adding \( n^2 + \frac{f_0}{(n-\alpha)^2} + \frac{8 - 4\alpha - 2n}{n-\alpha} f_0 - 8n + 16 \) to both sides we obtain
\[
9n^2 + \frac{f_0}{(n-\alpha)^2} - 24n + \frac{8 - 4\alpha - 2n}{n-\alpha} f_0 + 16 < n^2 + \frac{f_0}{(n-\alpha)^2} - 8n + \frac{8 - 2n}{n-\alpha} f_0 + 16,
\]
which may also be expressed as
\[
\left( 3n - 4 - \frac{f_0}{n-\alpha} \right)^2 + 4f_0 < \left( 4 - n + \frac{f_0}{n-\alpha} \right)^2.
\]
This implies
\[
\sqrt{\left( 3n - 4 - \frac{f_0}{n-\alpha} \right)^2 + 4f_0} < 4 - n + \frac{f_0}{n-\alpha}
\]
and thus, multiplying with \( n - \alpha \), recalling the definition of \( h(n, \alpha, f_0) \) and reordering the terms appropriately, we obtain
\[
h(n, \alpha, f_0) + \sqrt{n(n, \alpha, f_0)^2 + 4f_0(n - \alpha)^2} < 2n(n - \alpha).
\]

Next, we observe that the fact \( \gamma > \frac{4}{R-\rho} \) implies that \( \xi \) satisfies the inequalities \( (R-\rho)\gamma > \xi > 4 - \frac{4}{n} > \delta \). Setting \( a := \frac{\xi+1}{\delta} e^{-\xi} \) and \( b := \left( \frac{\xi}{\delta} - 1 \right) e^{-\xi} \) we have \( a > 0 \) and \( b > 0 \). With the constants \( a, b \) defined this way, the function \( \varphi(s) \) in (3.3) fulfills
\[
\lim_{s \downarrow 0^+} \varphi(s) = \frac{a}{\gamma} \alpha^\frac{\gamma}{\delta} e^{-\xi} - b = \frac{\xi}{\delta} e^{-\xi} - \left( \frac{\xi}{\delta} - 1 \right) e^{-\xi} = e^{-\xi} = \varphi \left( \frac{\xi}{\gamma} \right).
\]
and thus, $\varphi$ is continuous in $0, (\infty)$. Similarly, simple calculation shows

$$
\varphi(s) = \begin{cases} 
\frac{-a_b}{\gamma^b}s^{-(\delta+1)} & \text{if } s < \frac{\xi}{\gamma}, \\
-\gamma e^{-\gamma s} & \text{if } s > \frac{\xi}{\gamma}.
\end{cases}
$$

Therefore, $\varphi$ is monotonically decreasing in $0, (\infty)$, which by the definition of $\varphi$ immediately implies $\varphi > 0$ in $0, (\infty)$. Moreover, it holds that

$$
\lim_{s \to \frac{\xi}{\gamma}} \varphi(s) = \frac{-a_b}{\gamma^b} \frac{\xi^{\delta+1}}{\xi^{\delta+1}} = -\gamma e^{-\xi} = \varphi\left(\frac{\xi}{\gamma}\right)
$$

and hence $\varphi$ is continuous in $0, (\infty)$ as well. One further differentiation shows

$$
\varphi^\prime(s) = \begin{cases} 
\frac{a_b(\delta + 1)}{\gamma^\delta}s^{-(\delta+2)} & \text{if } s < \frac{\xi}{\gamma}, \\
\gamma^2 e^{-\gamma s} & \text{if } s > \frac{\xi}{\gamma},
\end{cases}
$$

therefore clearly $\varphi \in W^{2,\infty}_{loc}(0, \infty)$. Utilizing the expressions above, we see that for $s > \frac{\xi}{\gamma}$ holds:

$$
L_\varphi(s) := n^2 \frac{2n-2}{n-\alpha} \varphi^\prime_\gamma(s) + 4(n^2 - n)\frac{\alpha-2}{n-\alpha} \varphi(s) - nF(s)\varphi^\prime(s) - nF(s)\varphi(s)
= \left(n^2 \frac{2n-2}{n-\alpha} \gamma^2 - 4(n^2 - n)\frac{\alpha-2}{n-\alpha}\gamma\right) e^{-\gamma s} + (n\gamma F(s) - nF(s)) e^{-\gamma s}
$$

(3.6)

To estimate $n\gamma F(s) - nF(s)$ from below we distinguish the cases $s > R - \rho$ and $R - \rho > s > \frac{\xi}{\gamma}$.

In the first case we have

$$
n\gamma F(s) - nF(s) \geq n\gamma F(R - \rho) - nF(R - \rho) = \left(\frac{n\gamma}{n-\alpha}(R - \rho) - 1\right)f_0 (R - \rho)^{\frac{n}{\alpha}},
$$

by the monotonicity of $F$ and $F_\gamma$ and their explicit values at $s = R - \rho$ shown in (2.2) and (2.4), respectively. Because of $n > \alpha$ and $(R - \rho)\gamma > 4 - \alpha > 1$, we have $\frac{n\gamma(R - \rho)}{n-\alpha} > 1$ and since $f_0 > 0$ and $R - \rho > 0$ also hold, this shows

$$
n\gamma F(s) - nF(s) \geq 0,
$$

In the case $R - \rho > s > \frac{\xi}{\gamma}$ we make use of $\frac{n-\alpha}{n} > \frac{s}{\gamma} - \frac{\xi}{\gamma}$, to obtain

$$
n\gamma F(s) - nF(s) = \left(\frac{n\gamma}{n-\alpha} \frac{n-\alpha}{n} - s \frac{\alpha}{\pi}\right)f_0 \geq \left(\frac{n\xi}{n-\alpha} - 1\right)f_0 \frac{s}{\gamma}^{\frac{n-\alpha}{n}},
$$

This term is nonnegative as well, since $\frac{n-\alpha}{n} < 1 < \xi$ implies $\frac{n\xi}{n-\alpha} > 1$. Thus we may drop the last term in (3.6) and acquire
\[ L\varphi(s) \geq \left( \frac{2n-2}{n^2} - 4(n^2-n)\gamma s^{-\frac{n-2}{n}} \right) s^{-\gamma s} \]

\[ = \left( \frac{2n-2}{n^2} - 4(n^2-n)\gamma s^{-\frac{n-2}{n}} \right) \varphi(s) \quad \text{for all } s > R - \rho. \]

Using \( s > \frac{\xi}{\gamma} \) again, we estimate \( s^{-\frac{n-2}{n}} \geq \frac{\xi}{\gamma} \), which leads to

\[ L\varphi(s) \geq (n^2\xi - 4(n^2-n))\gamma s^{-\frac{n-2}{n}} \varphi(s) \quad \text{for all } s > R - \rho. \]

Here the choice of \( \xi > 4 - \frac{4}{n} \) implies that the foremost factor is nonnegative and thus, using \( s > \frac{\xi}{\gamma} \) once more, we obtain

\[ L\varphi(s) \geq (n^2\xi - 4(n^2-n))\gamma s^{-\frac{n-2}{n}} \varphi(s) =: c_1 \gamma \varphi(s) \quad \text{for all } s > R - \rho. \]

On the other hand for \( 0 < s < \frac{\xi}{\gamma} < R - \rho < 1 \):

\[ L\varphi(s) = n^2 \frac{a\delta(\delta + 1)}{\gamma^\delta} s^{-\delta} - \frac{2}{\pi} \int_{s^\delta} - 4(n^2-n) \frac{a\delta}{\gamma^\delta} s^{-\delta} - 2 + \frac{n \delta}{n - \alpha} \frac{a\delta}{\gamma^\delta} s^{-\delta} - \frac{\alpha}{\gamma^\delta} - f_0 s^{-\frac{\alpha}{\gamma^\delta} - b} \]

\[ \geq \left( n^2\delta(\delta + 1) - 4\delta(n^2-n)\gamma s^{-\frac{n-2}{n}} + \left( \frac{\delta n}{n - \alpha} - 1 \right) f_0 s^{-\frac{\alpha}{\gamma^\delta}} \right) \frac{a}{\gamma^\delta} s^{-\delta}. \]

Recalling the first argument of the maximum in (3.2), we have \( \frac{\delta n}{n - \alpha} - 1 > 0 \) and since \( 1 > s \) and \( \alpha > 2 \) imply \( s^{-\frac{\alpha}{\gamma^\delta}} > s^{-\frac{n-2}{n}} \), we obtain

\[ L\varphi(s) \geq \left( n^2\delta(\delta + 1) - 4\delta(n^2-n)\gamma s^{-\frac{n-2}{n}} + \left( \frac{\delta n}{n - \alpha} - 1 \right) f_0 s^{-\frac{\alpha}{\gamma^\delta}} \right) \frac{a}{\gamma^\delta} s^{-\delta} \]

\[ = \left( n^2\delta^2 + \left( \frac{nf_0}{n - \alpha} - 3n^2 + 4n \right) \delta - f_0 \right) s^{-\frac{\alpha}{\gamma^\delta}} \quad \text{for all } 0 < s < \frac{\xi}{\gamma}. \]

Observing that the larger root of the equation \( n^2\delta^2 + \left( \frac{nf_0}{n - \alpha} - 3n^2 + 4n \right) \delta - f_0 = 0 \) for \( \delta \) equals the second argument of the maximum in (3.2), the choice of \( \delta \) implies that \( n^2\delta^2 + \left( \frac{nf_0}{n - \alpha} - 3n^2 + 4n \right) \delta - f_0 > 0 \) holds. Thus the factor in front is positive and we may estimate

\[ L\varphi(s) \geq \left( n^2\delta^2 + \left( \frac{nf_0}{n - \alpha} - 3n^2 + 4n \right) \delta - f_0 \right) s^{-\frac{2}{\pi} \gamma \varphi} \quad \text{for all } 0 < s < \frac{\xi}{\gamma}. \]

Now we can use \( s < \frac{\xi}{\gamma} \) to obtain

\[ L\varphi(s) \geq \left( n^2\delta^2 + \left( \frac{nf_0}{n - \alpha} - 3n^2 + 4n \right) \delta - f_0 \right) s^{-\frac{2}{\gamma} \varphi} =: c_2 \gamma \varphi \quad \text{for all } 0 < s < \frac{\xi}{\gamma}. \]
Choosing \( k_0 := \min\{c_1, c_2\} \), the asserted inequality (3.4) holds a.e in \((0, \infty)\). To show (3.5) we use \( b > 0 \) again to calculate:

\[
\int_0^\infty \frac{\xi^2}{|\varphi|} \, ds \leq \int_0^\infty \frac{a^2 s^{-2\delta}}{\delta \gamma s^{-\delta} - 1} \, ds = \int_0^\infty \frac{a \xi^{2-\delta}}{\delta (2-\delta) \gamma^2} \, ds
\]

and

\[
\int_0^\infty \frac{\xi^2}{|\varphi|} \, ds = \int_0^\infty \frac{e^{-\gamma \xi}}{\gamma e^{-\gamma \xi}} \, ds = \frac{e^{-\xi}}{\gamma^2}.
\]

Hence the asserted statement in (3.5) holds for \( K_0 := \left( \frac{a \xi^{2-\delta}}{2-\delta} + e^{-\xi} \right) \), which completes the proof.

Before we begin with the proof of our main result let us fix some parameters.

**Lemma 3.3.** Assume that the conditions of lemma 3.2 hold and \( W_0 \) satisfies (2.7) for some \( \mu > 0 \), as well as \( W_0 \in C^0([0, 1]) \cap C^2((0, 1)) \), \( W_0(s) \geq c_0 s \) for some \( c_0 > 0 \) and all \( s \in [0, 1] \), \( W_0(s) \geq 0 \) for all \( s \in (0, 1) \) and \( W_0(s) \geq 0 \) for all \( s \in [0, 1] \). Denote by \( W \) the corresponding proper solution of (2.8). Furthermore, we set \( \zeta = 4 \) and let \( a, b, k_0, K_0 \) be the positive constants according to lemma 3.2. Then for any \( t_0 \geq 0 \) and \( \eta > 0 \) we can find \( \kappa > 0 \) fulfilling

\[
\kappa \leq \frac{k_0 \eta}{8} \quad (3.7)
\]

and \( \gamma > \frac{4}{k - p} \) such that

\[
1 + \frac{2 k_0 K_0 e^{-\gamma \eta}}{W(\kappa \gamma)^{\frac{2 \gamma \eta}{\kappa - p}, t_0 + \frac{\eta}{2} \gamma^{\eta - \gamma \eta}} \leq e^{2 \gamma \eta}\quad (3.8)
\]

as well as

\[
\gamma > \left( \frac{\zeta}{\kappa} \right)^{\frac{n}{2}} \quad (3.9)
\]

hold.

**Proof.** Because of \( \eta > 0 \) and \( k_0 > 0 \), we may choose a sufficiently small \( \kappa > 0 \) which satisfies the inequality (3.7). For \( \varepsilon \in (0, 1) \) let \( W^{(\varepsilon)} \) denote the solution to (2.12). We want to estimate \( W^{(\varepsilon)} \) from below in \([0, 1] \times [0, t_0 + \frac{\eta}{2}] \) by a suitable subsolution. To this end we observe that we have \( W_0(1) > 0 \), since \( W_0(s) \geq c_0 s \) for every \( s \in [0, 1] \). Thus, by the strong maximum principle applied to (2.12), \( W^{(\varepsilon)} \) is positive in \((\frac{1}{4}, \infty) \times (0, \infty) \). Hence, the number

\[
c_1 := \inf_{\tau \in [0, t_0 + \frac{\eta}{2}]} \left\{ \left( \frac{1}{W^{(\varepsilon)}(t_0 + \tau)}, \frac{W^{(\varepsilon)}(t_0 + \tau)}{W_0(1)} \right) \right\}
\]
is positive and well-defined. In fact, setting $W(s, t) := c_2 s^2 W_0(s)$ we see that

$$W(0, t) = 0 = W^{(e)}(0, t) \quad \text{for all } t \in \left[0, t_0 + \frac{\eta}{2}\right],$$

$$W(1, t) = c_1 W_0(1) \leq W^{(e)}(1, t) \leq W^{(e)}(1, t) \quad \text{for all } t \in \left[0, t_0 + \frac{\eta}{2}\right],$$

and

$$W(s, 0) = c_2 s^2 W_0(s) \leq W_0(s), \quad \text{for } s \in [0, 1].$$

Furthermore we have

$$n^2 s^{n-2} W_{ss} + \chi^{(c)} W_s W_s + n \chi^{(c)} F_{ss} = n^2 s^{n-2} (2 c_1 W_0 + c_3 s^2 W_{0ss} + 4 s \gamma W_0)$$

$$\geq 0 = W_t,$$

in $(0, 1) \times (0, t_0 + \frac{\eta}{2})$, since $W_0(s) \geq 0$ and $W_{0ss}(s) \geq 0$ for all $s \in (0, 1)$. Thus, we may use the comparison principle, see remark 2.3, to deduce

$$W^{(e)}(s, t) \geq W(s, t) \text{ in } [0, 1] \times \left[0, t_0 + \frac{\eta}{2}\right].$$

In particular, since $W^{(e)} / W$ as $\varepsilon \searrow 0$, we have

$$W \left( s, t_0 + \frac{\eta}{2} \right) \geq W^{(e)}(s, t_0 + \frac{\eta}{2}) \geq c_3 s^2 W_0(s) \geq c_1 c_0 s^3 =: p(s) \quad \text{for } s \in [0, 1] \quad (3.10)$$

since $W_0(s) \geq c_0 s$ in $[0, 1]$. This inequality at hand we can now verify that by choosing $\gamma$ sufficiently large (3.8) is indeed fulfilled. To this end, let $s_0 \in (0, 1)$ be so small, such that the inequalities

$$\frac{k_0 K_0}{\kappa} \leq c_0 c_1 s_0^3 \sinh \left( \kappa \left( \frac{\kappa}{s_0} \right)^{\frac{3}{n-2}} \right) \quad (3.11)$$

and

$$s_0 < \left( \frac{2 \kappa \frac{n-2}{3(n-2)}}{s_0^2} \right)^{\frac{n-2}{n}} \quad (3.12)$$

hold. Rearranging (3.12) we see that

$$\frac{3}{2} s_0^2 < \frac{\kappa^{\frac{n-2}{n}}}{n-2} s_0^{\frac{2n-6}{n-2}}$$
Hence, taking $s_0 = 0$ and $s = 0$, we obtain
\[ (3.9) \] as well as
\[ (3.10) \]
for all $s \in (0, 0)$. Hence, taking $\gamma > 1$, such that $\beta > \eta > 0$.

Proposition 3.4. Suppose that the conditions of lemma 3.2 hold. Then for any $\beta > 1$ and each
\[ W(\kappa, a, b, c) = \infty \] for all $\kappa > 0$.

In particular, we have
\[ \sup_{\kappa > 0} W(\kappa, a, b, c) = \infty \] for all $\kappa > 0$.

(3.14)

(3.15)

Inspired by the methods of [14, lemma 2.3], we can use functions of the type described
in lemma 3.2 to prove that the spatial derivative of the proper solution $W_{\kappa}(t)$ blows up
immediately:
\[ \frac{\partial W_{\kappa}(t)}{\partial t} = -\frac{\kappa^2}{2} \frac{\partial W_{\kappa}(t)}{\partial \kappa} \]
\[ -\frac{\kappa^2}{2} \frac{\partial W_{\kappa}(t)}{\partial \kappa} \]
\[ \beta > \eta > 0 \quad \text{and} \quad \gamma > 1. \]

We work along the lines of a contradiction argument and assume to this end that there
exist $\beta > 1, \eta > 0$, and $c > 0$ such that
\[ W(\kappa, a, b, c) = \infty \] for all $\kappa > 0$.

Utilizing $s_0 < 1$, this yields
\[ p(s) > \kappa \sinh \left( \frac{s}{\kappa} \right) \] for all $s \in (0, 0)$. Making use of this monotonicity and (3.11) we obtain
\[ (3.13) \]
for all $s \in (0, 0)$. This implies that $g(s) = \kappa \sinh \left( \frac{s}{\kappa} \right)$ is monotonically decreasing since
holds for all $s \in (0, 0)$. Hence,
\[ \frac{\partial g(s)}{\partial s} < 0 \] for all $s \in (0, 0)$. This implies that $g(s) = \kappa \sinh \left( \frac{s}{\kappa} \right)$ is monotonically decreasing since
holds for all $s \in (0, 0)$. Hence,
\[ \frac{\partial g(s)}{\partial s} < 0 \] for all $s \in (0, 0)$. This implies that $g(s) = \kappa \sinh \left( \frac{s}{\kappa} \right)$ is monotonically decreasing since
holds for all $s \in (0, 0)$. Hence,
\[ \frac{\partial g(s)}{\partial s} < 0 \] for all $s \in (0, 0)$. This implies that $g(s) = \kappa \sinh \left( \frac{s}{\kappa} \right)$ is monotonically decreasing since
holds for all $s \in (0, 0)$. Hence,
\( W(s, t) \leq c s^\beta \) for all \( s \geq 0 \) and \( t \in [t_0, t_0 + \eta] \). (3.16)

Since \( \beta \geq 1 \), we can fix \( \delta \in (0, 1) \) satisfying \( \beta > \delta \) as well as the property (3.2). Furthermore set \( \xi = 4 \) and let \( a, b, k_0 \) and \( K_0 \) be the positive constants defined in lemma 3.2. Corresponding to these parameters let \( \kappa, \gamma \) be the positive constants given by lemma 3.3.

With these parameters we define \( \varphi := \varphi^{(\xi)} \) as in (3.3) of lemma 3.2. In particular, \( \varphi \) fulfills the differential inequality (3.4) and the integral inequality (3.5). Both will be required later in this proof. Recalling the cut-off functions \( \chi^{(\epsilon)} \) for \( \epsilon \in (0, 1) \), mentioned in (2.10) and (2.11), we multiply the approximation problem (2.12) by \( \chi^{(\epsilon)} \varphi \). Integration by parts over \( ss, 0 < s < \infty \), where \( s_0 \) is an arbitrary number satisfying \( s_0 > max \{ \frac{\xi}{\gamma}, \frac{2n-2}{\gamma}, \frac{\xi}{2}, \epsilon \} \), results in

\[
\frac{d}{ds} \int_0^{s_0} \chi^{(\epsilon)} \varphi \, W^{(\epsilon)} \, ds = \int_0^{s_0} \chi^{(\epsilon)} \varphi \left[ n_s^2 \frac{2n-2}{n} W^{(\epsilon)} + \frac{1}{2} \chi^{(\epsilon)} ((W^{(\epsilon)})^2)_s + n \chi^{(\epsilon)} F W^{(\epsilon)}_s \right] \, ds
\]

\[
= n^2 \int_0^{s_0} \left( s^{2n-2} \chi^{(\epsilon)} \varphi \right)_s \, W^{(\epsilon)} \, ds - \frac{1}{2} \int_0^{s_0} (\chi^{(\epsilon)} F \varphi)_s \, W^{(\epsilon)} \, ds
\]

\[
- n \int_0^{s_0} (\chi^{(\epsilon)} F^2) \varphi W^{(\epsilon)} \, ds + B(t) \text{ for } t > 0,
\]

where \( B(t) \) are the collected boundary terms, that is

\[
B(t) := \left[ n_s^2 \frac{2n-2}{n} \chi^{(\epsilon)} \varphi \, W^{(\epsilon)}_s - n^2 \left( s^{2n-2} \chi^{(\epsilon)} \varphi \right)_s \, W^{(\epsilon)} + \frac{1}{2} (\chi^{(\epsilon)} F \varphi)^2 \varphi + n \chi^{(\epsilon)} W^{(\epsilon)} F \right]^{s_0}_0.
\]

Calculating the mixed derivative term

\[
n^2 \left( s^{2n-2} \varphi \chi^{(\epsilon)} \right)_s = n^2 \frac{2n-2}{n} \varphi \chi^{(\epsilon)} + n^2 \frac{2n-2}{n} \varphi \chi^{(\epsilon)} + n(2n-2)s^{n-2} \varphi \chi^{(\epsilon)},
\]

as well as its companions

\[
\frac{1}{2} \varphi \chi^{(\epsilon)} = \frac{1}{2} \chi^{(\epsilon)} \varphi \chi^{(\epsilon)},
\]

\[
n \varphi F = n \chi^{(\epsilon)} \varphi F + 2n \chi^{(\epsilon)} \varphi F + n \chi^{(\epsilon)} \varphi F,
\]

and

\[
n^2 \left( s^{2n-2} \varphi \chi^{(\epsilon)} \right)_s = n^2 \frac{2n-2}{n} \varphi \chi^{(\epsilon)} + 2n^2 \frac{2n-2}{n} \varphi \chi^{(\epsilon)} + 4n(n-1)s^{n-2} \varphi \chi^{(\epsilon)}
\]

\[
+ n^3 \frac{2n-2}{n} \varphi \chi^{(\epsilon)} + 4n(n-1)s^{n-2} \varphi \chi^{(\epsilon)} + (2n^2 - 6n + 4)s^2 \varphi \chi^{(\epsilon)},
\]

we can make use of the facts \( \chi^{(\epsilon)} = 0 \) on \( [0, \frac{\xi}{\gamma}] \), \( \chi^{(\epsilon)} = 1 \) on \( [\epsilon, \infty) \), \( \varphi(s) = e^{-\gamma s} \) on \( [\frac{\xi}{\gamma}, \infty) \) and \( s_0 > max \{ \frac{1}{n\gamma}, \frac{\xi}{\gamma}, \epsilon \} \), to express (3.18) as
\[ B(t) = n^2 s_0^{2n-2} e^{-\gamma s_0} W_s^{(c)}(s_0, t) + \left( n^2 s_0^{2n-2} \gamma e^{-\gamma s_0} - n(2n - 2)s_0^{n-2} e^{-\gamma s_0} \right) W_s^{(c)}(s_0, t) + \frac{1}{2} (W_s^{(c)}(s_0))_2 e^{-\gamma s_0} + n W_s^{(c)}(s_0, t) e^{-\gamma s_0} F(s_0) \] for all \( t > 0 \).

Recalling that \( W_s^{(c)} \geq 0 \) and \( F \geq 0 \) we can drop multiple nonnegative terms to obtain
\[ B(t) \geq (n^2 s_0^{2n-2} \gamma e^{-\gamma s_0} - n(2n - 2)s_0^{n-2} e^{-\gamma s_0} \right) W_s^{(c)}(s_0, t) \] for all \( t > 0 \),

which by choice of \( s_0 \) then implies \( B(t) \geq 0 \) for all \( t > 0 \). Thus, inserting (3.20)–(3.22) into (3.17), we get
\[
\frac{d}{dt} \int_0^t \varphi(s) W_s^{(c)}(s) \, ds \geq \int_0^t \varphi(s) W_s^{(c)} \left[ n^2 s_0^{2n-2} \gamma s_0^{n-2} - n(2n - 2)s_0^{n-2} e^{-\gamma s_0} \right] \varphi(s) \, ds + \frac{1}{2} \int_0^t \varphi(s) W_s^{(c)}(s) + I_1(t) + I_2(t) \quad \text{for all } t > 0,
\]

where we set \( I_1(t) := \int_0^t \varphi(s) W_s^{(c)} \left( 2n^2 s_0^{2n-2} \gamma s_0^{n-2} \varphi(s) + 4n(2n - 2)s_0^{n-2} e^{-\gamma s_0} \right) \, ds \) and \( I_2(t) := \int_0^t \varphi(s) W_s^{(c)}(s) \). Using the properties of the cut-off function \( \varphi(s) \) we can estimate both \( I_1 \) and \( I_2 \) from below. For that, we first recall that \( \varphi(s) \leq \frac{\gamma_s}{\gamma} \), \( \varphi(s) \leq 1 \), \( \varphi(s) \equiv 0 \) on \( (\varepsilon, \infty) \) and \( W_s^{(c)} \leq W \leq \frac{n^2}{S_{n-2}} \) for all \( s \geq 0 \) and \( t > 0 \). Thus, using \( F(s) \leq F(R + \rho) \) for all \( s > 0 \), we can estimate
\[
I_1(t) \geq \int_0^t \left( 2n^2 s_0^{2n-2} \gamma s_0^{n-2} \varphi - \varphi \chi^{(c)} W_s^{(c)} - 2n \chi^{(c)} \varphi F \right) \chi^{(c)} W_s^{(c)} \, ds \geq - \int_0^t \left( 2n^2 s_0^{2n-2} \varphi \chi^{(c)} W_s^{(c)} + \varphi W + 2n \varphi F(R + \rho) \right) \chi^{(c)} W_s^{(c)} \, ds.
\]

Next, as long as \( \varepsilon < \frac{\rho}{\gamma} \), we make use of the definitions of \( \varphi, F \) and our contradiction assumption \( W \leq \varepsilon \), for all \( t \in (t_0, t_0 + \eta) \) and \( s \geq 0 \) in (3.16), to obtain
\[
I_1(t) \geq \int_0^t \left( 2n^2 s_0^{2n-2} \gamma s_0^{n-2} \varphi - \varphi \chi^{(c)} W_s^{(c)} - 2n \chi^{(c)} \varphi F \right) \chi^{(c)} W_s^{(c)} \, ds \geq - \int_0^t \left( 2n^2 s_0^{2n-2} \varphi \chi^{(c)} W_s^{(c)} + \varphi W + 2n \varphi F(R + \rho) \right) \chi^{(c)} W_s^{(c)} \, ds.
\]

Since \( \varepsilon < 1 \) and \( (R + \rho)^{\frac{n-\alpha}{\alpha}} < 2 \), one further estimation shows
\[
I_1(t) \geq - \int_0^t \left( 2n^2 s_0^{2n-2} \gamma s_0^{n-2} \varphi - \varphi \chi^{(c)} W_s^{(c)} - 2n \chi^{(c)} \varphi F \right) \chi^{(c)} W_s^{(c)} \, ds \quad \text{for all } t \in (t_0, t_0 + \eta)
\]

as long as \( \varepsilon < \frac{\rho}{\gamma} \). Similarly, recalling \( |\chi^{(c)}| \leq \frac{\gamma_s}{\gamma} \), we have
\[ |J_2(t)| \leq n^2 c(c) a \int_0^\varepsilon \frac{s^{n+2-\varepsilon}}{s^{n^2-n+4}s^2} \frac{\varepsilon^2}{\gamma^2} \int_{t_0}^t \frac{s-\varepsilon^2}{s^{n^2+n+4}} ds \]

\[
\leq n^2 c(c) a \int_0^\varepsilon s^{3-\varepsilon} + 1 \int_{t_0}^t s^{3-\varepsilon} ds \quad \text{for all } t \in (t_0, t_0 + \eta) \tag{3.25}
\]

as long as \( \varepsilon < \frac{\gamma}{2} \). By the choice of \( \delta < \beta \), the combination of (3.24) and (3.25) yields

\[(l_1 + l_2)(t) > -c_2\varepsilon^{\beta-\delta} \text{ in } (t_0, t_0 + \eta) \text{ for some } c_2 > 0 \text{ independent of } \varepsilon, \]

as long as \( \varepsilon < \frac{\gamma}{5} \), so that (3.23) reduces to

\[
\frac{d}{dt} \int_0^\varepsilon \varphi \chi(c) W(c) ds
\]

\[
\geq \int_0^\varepsilon \chi(c) W(c) \left[ \frac{n \chi(c)}{\varepsilon^{n+2-\varepsilon}} \varphi_\varepsilon + 4n(n-1)s^{n-2}\varphi - s^{n^2-n+4}s^2 \varphi \right. \\
- n\chi(c)\varphi F - n\chi(c)\varphi F_s \right] ds - \frac{1}{2} \int_0^\varepsilon \varphi_\varepsilon (\chi(c))^2 (W(c))^2 ds - c_2\varepsilon^{\beta-\delta} \\
\geq \int_0^\varepsilon \chi(c) W(c) \left[ \frac{n \chi(c)}{\varepsilon^{n+2-\varepsilon}} \varphi_\varepsilon + 4n(n-1)s^{n-2}\varphi - n\varphi_\varepsilon \chi(c) - nF\varphi_\varepsilon \chi(c) - nF_s\varphi_\varepsilon \chi(c) \right] ds \\
- \frac{1}{2} \int_0^\varepsilon \varphi_\varepsilon (\chi(c))^2 (W(c))^2 ds - c_2\varepsilon^{\beta-\delta} \quad \text{for all } t \in (t_0, t_0 + \eta),
\]

as long as \( \varepsilon < \frac{\gamma}{2} \). Now (3.4) of lemma 3.2 implies

\[ n^{n-2} \varphi_\varepsilon + 4n(n-1)s^{n-2}\varphi - (nF\varphi_\varepsilon + nF_s\varphi) \chi(c) \geq k_0\gamma^{n-2} \varphi + (1 - \chi(c))(nF\varphi_\varepsilon + nF_s\varphi) \]

a.e in \((0, \infty)\) and thus we obtain

\[
\frac{d}{dt} \int_0^\varepsilon \varphi \chi(c) W(c) ds
\]

\[
\geq \int_0^\varepsilon \chi(c) W(c) \left[ \frac{n \chi(c)}{\varepsilon^{n+2-\varepsilon}} \varphi_\varepsilon + 4n(n-1)s^{n-2}\varphi - (nF\varphi_\varepsilon + nF_s\varphi) \chi(c) \right] ds + \int_0^\varepsilon n\chi(c)F\varphi_\varepsilon \chi(c) W(c) ds \\
- \frac{1}{2} \int_0^\varepsilon \varphi_\varepsilon (\chi(c))^2 (W(c))^2 ds - c_2\varepsilon^{\beta-\delta} \quad \text{for all } t \in (t_0, t_0 + \eta),
\]

for \( \varepsilon < \frac{\gamma}{2} \). Setting \( y^{(c),\eta}(t) := \int_0^\varepsilon \varphi(s) \chi(c)(s) W(c)(s,t) ds \) and integrating over \((t_0 + \eta, t) =: (t, t)\), the inequality above takes the form

\[
y^{(c),\eta}(t) \geq y^{(c),\eta}(t_0) + k_0\gamma^{n-2} \int_{t_0}^t \int_0^\varepsilon \varphi \chi(c) W(c) ds dt - \frac{1}{2} \int_{t_0}^t \int_0^\varepsilon \varphi_\varepsilon (\chi(c))^2 (W(c))^2 ds dt \\
+ n \int_{t_0}^t \int_0^\varepsilon F\varphi (1 - \chi(c)) \chi(c) W(c) ds dt + n \int_{t_0}^t \int_0^\varepsilon F\varphi (1 - \chi(c)) \chi(c) W(c) ds dt \\
- c_2\varepsilon^{\beta-\delta} (t - t_0) \quad \text{for all } t \in (t_0, t_0 + \eta) \text{ and } 0 < \varepsilon < \min \left\{ 1, \frac{\gamma}{2} \right\}. \]
Observing that \[ \left| \int_0^{\infty} \varphi \chi^{(c)W^{(c)}} ds \right| \leq \left( \frac{a c^{(c)}}{s} + \frac{\varepsilon}{\gamma} \right) \frac{1}{|b|} \text{ holds for all } s \in (0, \infty) \] we may use the monotone convergence theorem to take \( s_0 \to \infty \) and obtain
\[
\lim_{s_0 \to \infty} \int_0^{s_0} \varphi \chi^{(c)W^{(c)}} dsdr = \int_0^t \int_0^{\infty} \varphi \chi^{(c)W^{(c)}} dsdr < \infty \quad \text{for all } t \in (t_0, t_0 + \eta). \]

In a similar fashion—using not only the exponential decay of \( \varphi \) but also of \( \varphi_2 \)—we can apply the monotone convergence theorem for
\[
\lim_{s_0 \to \infty} \int_0^{s_0} \varphi_2 \chi^{(c)W^{(c)}} dsdr = \int_0^t \int_0^{\infty} \varphi_2 \chi^{(c)W^{(c)}} dsdr
\]
and
\[
\lim_{s_0 \to \infty} \int_0^{s_0} F_2 \varphi (1 - \chi^{(c)}) \chi^{(c)W^{(c)}} dsdr
\]
to see that the function \( y^{(c)}(t) := \lim_{s_0 \to \infty} y^{(c,s_0)}(t) = \int_0^{\infty} \varphi(s) \chi^{(c)}(s) W^{(c)}(s, t) ds \) satisfies the inequality
\[
y^{(c)}(t) \geq y^{(c)}(t_1) + k_0 \varepsilon^2 \int_0^t \int_0^{\infty} \varphi \chi^{(c)W^{(c)}} dsdr - \frac{1}{2} \int_0^t \int_0^{\infty} \varphi_2 \chi^{(c)W^{(c)}} dsdr
\]
\[
+ n \int_0^t \int_0^{\infty} F_2 \varphi (1 - \chi^{(c)}) \chi^{(c)W^{(c)}} dsdr + n \int_0^t \int_0^{\infty} F_2 \varphi (1 - \chi^{(c)}) \chi^{(c)W^{(c)}} dsdr
\]
\[
- c_2 \varepsilon^{\beta - \delta} (t - t_1) \quad \text{for all } t \in (t_0, t_0 + \eta) \text{ and } 0 < \varepsilon < \min \left\{ \frac{1}{\gamma}, \frac{\xi}{\gamma} \right\}. \quad (3.26)
\]

In order to take \( \varepsilon \to 0 \) we recall the definition of \( \varphi \) in (3.3) to see that
\[
|\varphi_2(s)| \leq \begin{cases} \frac{a d^\beta}{\gamma^\delta s^{(\delta + 1)}} & \text{if } s < \frac{\varepsilon}{\gamma} \\ \gamma e^{-\gamma s} & \text{if } s > \frac{\varepsilon}{\gamma} \end{cases} \leq c_3 (1 + s^{-(\delta + 1)}) e^{-\gamma s},
\]
for some \( c_3 > 0 \) and \( s > 0 \). And similarly
\[
|\varphi(s)| \leq c_4 (1 + s^{\beta}) e^{-\gamma s},
\]
for some \( c_4 > 0 \) in \( (0, \infty) \). Combining these inequalities with our assumption (3.16) and the definitions of \( F \) and \( F_2 \) in (2.2) and (2.4), respectively, we observe that
\[
\left| F(s) \varphi_2(s)(1 - \chi^{(c)}(s)) \chi^{(c)}(s) W^{(c)}(s, t) \right| \leq \left| F(R + \rho) \varphi_2(s) W(s, t) \right| \leq c_5 (s^\beta + s^{\beta - 1}) e^{-\gamma s}
\]
for some \( c_5 > 0 \) a.e. in \((0, \infty)\), as well as

\[
[F(s) \varphi(s)(1 - \chi^{(e)}(s))\chi^{(e)}(s)W^{(e)}(s, t)] \leq [F(s) \varphi(s)W(s, t)] \leq c_6(s^{3-\frac{2}{\beta}} + s^{3-\frac{\beta}{n}-\delta})e^{-\gamma s}
\]

for some \( c_6 > 0 \) and all \( s > 0 \), holds independent of \( e \). Because of \( \beta > \delta \) and \( n > \alpha \) the integrals

\[
\int_0^t \int_0^\infty (s^{3-\frac{\beta}{n}-\delta})e^{-\gamma s}dsdt \quad \text{and} \quad \int_0^t \int_0^\infty (1 + s^{3-\frac{\beta}{n}-\delta})e^{-\gamma s}dsdt
\]

converge, so that an additional application of the dominated convergence theorem shows

\[
\lim_{\epsilon \searrow 0} \int_0^t \int_0^\infty F\varphi(1 - \chi^{(e)}(s))\chi^{(e)}W^{(e)}dsdt = \int_0^t \int_0^\infty \lim_{\epsilon \searrow 0} F\varphi(1 - \chi^{(e)}(s))\chi^{(e)}W^{(e)}dsdt = 0
\]

and

\[
\lim_{\epsilon \searrow 0} \int_0^t \int_0^\infty F\varphi(1 - \chi^{(e)}(s))\chi^{(e)}W^{(e)}dsdt = 0
\]

for all \( t \in (t_1, t_0 + \eta) \). For the two remaining integral terms in (3.26) we make use of the monotonicity of \( \chi^{(e)} \) and \( W^{(e)} \) with respect to \( s \) and apply the monotone convergence theorem to conclude that \( y(t) := \int_0^\infty \varphi Wds \) satisfies

\[
y(t) \geq y(t_1) + k_0 \gamma^2 \int_0^t \int_0^\infty \varphi Wds + \frac{1}{2} \int_0^t \int_0^\infty \varphi^2 W^2dsdt - \frac{1}{2} \int_0^t \int_0^\infty \varphi W^2dsdt
\]

\[
= y(t_1) + k_0 \gamma^2 \int_0^t \int_0^\infty \varphi Wds + \frac{1}{2} \int_0^t \int_0^\infty \varphi |W|^2dsdt \quad \text{for all } t \in (t_1, t_0 + \eta).
\]

Using Hölder’s inequality and the definition of \( y(t) \) we see that

\[
y^2(t) = \left( \int_0^\infty \varphi Wds \right)^2 \left( \int_0^\infty \varphi^2 W^2ds \right) \left( \int_0^\infty |\varphi|^2 |W|^2ds \right) \quad \text{for all } t > 0,
\]

which in turn by lemma 3.2 (3.5) implies

\[
y^2(t) \leq \frac{K_0}{\gamma^2} \int_0^\infty |\varphi|^2 |W|^2ds \quad \text{for all } t > 0.
\]

Combination with (3.27) therefore yields

\[
y(t) \geq y(t_1) + k_0 \gamma^2 \int_0^t y(\tau)d\tau + \frac{\gamma^2}{2K_0} \int_0^t y^2(\tau)d\tau \quad \text{for all } t \in (t_1, t_0 + \eta).
\]

In view of lemma 3.1 \( y(t) \) thus satisfies the inequality \( y(t) \geq z(t) \) for all \( t \in (t_1, t_0 + \eta) \), where \( z \) is the solution of

\[
\begin{cases}
z' = Az + Bz^2, & t > t_1, \\
z(t_1) = y(t_1)
\end{cases}
\]
with \( A = k_0 \gamma \pi^2 \) and \( B = \gamma^2 / 2k_0 \). For \( \gamma(t_1) > 0 \) this Bernoulli-type initial-value problem has the explicit solution

\[
z(t) = \frac{1}{(1/\gamma(t_1) + \theta / \gamma) e^{-\gamma(t-t_1)}} - \theta / \gamma, \quad t \in (t_2, t_1 + T),
\]

with maximal existence time determined by \( T = 1 / A \log \left( 1 + \frac{A}{B\gamma(t_1)} \right) \). Next, we utilize \( \kappa \gamma^{3/2} > \xi \) from (3.9) and the fact \( W_0 \) is \( \gamma \) from below with

\[
y(t_1) = \int_0^\infty \varphi(s)W(s, t_1)ds \geq c_\gamma \int_{\kappa \gamma^{3/2}}^\infty \varphi(s)ds \geq c_\gamma \int_{\kappa \gamma^{3/2}}^\infty e^{-\gamma s}ds = \frac{c_\gamma}{\gamma} e^{-\kappa \gamma^{3/2}},
\]

where we set \( c_\gamma := W(\kappa \gamma^{3/2}, t_1) \). Accordingly, recalling inequality (3.8) from lemma 3.3 we have

\[
T \leq \frac{1}{k_0 \gamma \pi^2} \log \left( 1 + \frac{k_0 \gamma \pi^2}{\kappa \gamma^{3/2} e^{-\kappa \gamma^{3/2}}} \right) = \frac{1}{k_0 \gamma \pi^2} \log \left( 1 + \frac{2k_0 k_0}{c_\gamma \gamma^{3/2} e^{-\kappa \gamma^{3/2}}} \right) \\
\leq \frac{1}{k_0 \gamma \pi^2} \log \left( e^{2\gamma \pi^2} \right) = \frac{2\kappa}{k_0}.
\]

by definitions of \( A \) and \( B \). But this means, see (3.7), that \( T \leq \frac{\eta}{4} \). Thus \( y \) blows up before or at \( t = t_1 + \eta / 4 = t_0 + \eta / 4 \), which—since \( \varphi \) is integrable because of \( \delta < 1 \)—is a contradiction to

\[
y(t) = \int_0^\infty \varphi(s)W(s, t)ds \leq \frac{\eta}{|S_{\kappa \eta}|} \int_0^\infty \varphi(s)ds \leq c_\theta \forall t > 0.
\]

Hence our assumption in (3.16) must have been false, which completes the proof of (3.14). To verify (3.15) we take \( \beta = 1 \) and conclude that \( W(s, t) \) cannot be Lipschitz continuous on \( (0, \infty) \times (t_0, t_0 + \eta) \) for each \( t_0 \geq 0 \) and every \( \eta > 0 \), which then implies the asserted unboundedness of \( W_s \).

Having this result for the derivative of \( W \) at hand, we can now show our main result, which corresponds in part to [14, theorem 0.1].

**Proof of theorem 1.2.** The assumption \( u_0 \equiv c_0 \) on \( B(0) \) for some \( c_0 > 0 \), implies \( W_0 \in C^0([0, 1]) \cap C^2(0, 1) \), \( W_0(s) = c_0 \) on \([0, 1], W_0 = c_0 > 0 \) on \((0, 1) \) and \( W_{0\theta} = 0 \) on \((0, 1) \). This allows us to choose one of our generalized test functions \( \varphi \) fulfilling the important inequalities (3.4) and (3.5) of lemma 3.2, with parameters chosen as in lemma 3.3. Using this test function as show in by proposition 3.4, we obtain \( \|W\|_{L^\infty((0, \infty) \times (t_0, t_0 + \eta))} = \infty \) for all \( \eta > 0 \) (see (3.15)). Now we recall the measure-valued reconstruction identity \( u(x, t) = W([x]^{\eta}, \eta) + \frac{|S_{\kappa \eta}|}{\eta} W(0+, t) \delta(x) \) (shown in (2.9)) to verify that the radial weak solution of \( (KS_0^\eta) \), in the sense of definition 2.1, blows up immediately at \( x = 0 \).
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