Comparison of the $\mu$-invariants of an abelian variety and its dual abelian variety

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Abstract

In this note, we compare the dual Selmer groups of an abelian variety with that of its dual over certain large Galois field. We give a formula which relates the generalized Iwasawa $\mu$-invariants associated with their dual Selmer groups under the natural isogeny.

1 Introduction

Let $F$ be a number field and $\bar{F}$ a fixed algebraic closure of $F$. Let $A$ be an abelian variety defined over $F$ of dimension $d$ with no complex multiplication over $\bar{F}$. Let $p$ be an odd prime number. By $A_{p^\infty}$, we denote the set of all $p$-th power torsion points of $A$ and by $F_{\infty}$ we denote the field of definition $F(A_{p^\infty})$ of $A_{p^\infty}$. Let $G_{\infty} = \text{Gal}(F_{\infty}/F)$. By a well known result of Serre, if $A$ does not admit complex multiplication over $\bar{F}$, then $G_{\infty}$ embeds into $GL_{2d}(\mathbb{Z}_p)$, whose the cohomological dimension is the same as its manifold dimension as a $p$-adic Lie group and hence is equal to $2d$. By $F_n$ we denote the field extension of $F$ obtained by adjoining $p^{n+1}$-torsion points of $A$ to $F$. Let $F^{cyc}$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$. From the Weil pairing it is clear that $F^{cyc}$ is contained in $F_{\infty}$. Let $\Gamma = \text{Gal}(F^{cyc}/F)$.

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Let $S$ be a finite set of primes of $F$ containing the all primes dividing $p$ and primes at which $A$ has bad reduction. Let $F_S$ denotes the maximal extension of $F$ which is unramified at all the places not contained in $S$ and at Archimedean primes. Let $G_S = \text{Gal}(F_S/F)$. Then clearly $F_\infty$ is contained in $F_S$. So we have the following tower of extensions of $F$.

$$F_S \supseteq F_\infty \supseteq G_\infty \supseteq F^{\text{cyc}} \supseteq \Gamma \supseteq F$$  

(1)

For any extension $L$ of $F$ contained in $F_S$, let $G_S(L)$ denote the Galois group $\text{Gal}(F_S/L)$. Then for any extension $L$ of $F$ contained in $F_\infty$ the $p^\infty$-Selmer group of $A$ over $L$ denoted by $\text{Sel}_{p^\infty}(A/L)$ is defined by the exact sequence

$$0 \longrightarrow \text{Sel}_{p^\infty}(A/L) \longrightarrow \text{H}^1(G_S(L), A_{p^\infty}) \xrightarrow{\lambda_L} \bigoplus_{v \in S} J_v(A/L)$$  

(2)

where $J_v(A/L) = \lim_{\longrightarrow} \oplus \text{H}^1(K_w, A)(p)$, $K$ runs over the finite extension of $F$ contained in $L$ and the direct limit is taken with respect to the restriction maps.

For any profinite group $G$, let $\Lambda(G) := \lim_{\leftarrow} \mathbb{Z}_p[G/U]$, $(U$ being open normal subgroup of $G$) be the Iwasawa algebra of $G$ which is a complete ring. The Galois group $G_\infty$ acts on $\text{Sel}_{p^\infty}(A/F_\infty)$ continuously and the action is extended continuously so that it becomes a discrete module over the Iwasawa algebra $\Lambda(G_\infty)$. We consider its Pontrjagin dual $\hat{\mathfrak{X}}(A/F_\infty) = \text{Sel}_{p^\infty}(A/F_\infty)$ which is a compact $\Lambda(G_\infty)$-module. Similarly $\hat{\mathfrak{X}}(A/F^{\text{cyc}})$ is a $\Lambda(\Gamma)$-module. It is known (11) that $\hat{\mathfrak{X}}(A/F^{\text{cyc}})$ is a finitely generated $\Lambda(\Gamma)$-module and that $\hat{\mathfrak{X}}(A/F_\infty)$ is a finitely generated $\Lambda(G_\infty)$-module.

The following is conjectured.

**Conjecture 1:** Let $F^{\text{cyc}} \subset L \subset F_S$. Then
i) the map $\lambda_L$ in the exact sequence (2) is surjective.
ii) $\text{H}^2(G_S(L), A_{p^\infty}) = 0$. 



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It is already known (Theorem 2.10 in [2]) that $H^2(G_\infty(F_\infty), A_{p,\infty}) = 0$ but this is still only a conjecture for an arbitrary $L$.

**Conjecture 2:** $\mathfrak{x}(A/F_\infty)$ is a torsion $\Lambda(G_\infty)$-module.

We assume both these conjectures. In fact conjecture 1 is equivalent to conjecture 2 if $A$ has potentially ordinary reduction at all the places dividing $p$. If $G_\infty$ is pro-$p$ and if $M$ is a finitely generated $\Lambda(G_\infty)$-module, then ([4]) we have

$$\mu(M) = \mu_{G_\infty}(M) = \log_p(\chi(G_\infty, \widehat{M}(p))).$$

With the above notation, we state the formula for the $\mu$-invariants of isogenous abelian varieties which has been proved by Susan Howson ([4]) in the case of elliptic curves.

**Proposition 1** Let $A_1$ and $A_2$ be non-CM abelian varieties defined over $F$ and let $f : A_1 \to A_2$ be an isogeny defined over $F$. We assume that $F_\infty = F(A_{1,p,\infty}) = F(A_{2,p,\infty})$ be a pro-$p$-extension of $F$. Let $C = (\ker f)(p)$ and $\tilde{C}_v$ be the image of $C$ under the reduction map at $v : A_1 \to \tilde{A}_1$. Then as $\Lambda(G_\infty)$-modules

$$\mu(\mathfrak{x}(A_1/F_\infty)) - \mu(\mathfrak{x}(A_2/F_\infty))$$

$$= \sum_{v|\infty} \log_p(\#C(F_v)) - [F : \mathbb{Q}] \log_p(\#C) + \sum_{v|p} \log_p(|\#\tilde{C}_v|_v)$$

where $|\cdot|_v$ is the $v$-adic norm.

Proof. The argument is exactly the same as that in the $GL_2$ case ([4]) in the case of an isogeny between two abelian varieties with $F_\infty = F(A_{i,p,\infty})$. Hence we omit it.

We remark that Proposition [1] has been proved for the cyclotomic extension by Perrin-Riou ([7] and P. Schneider ([8]).
2 Comparing the abelian variety with its dual

In this section, we prove that the \( \mu \)-invariants of an abelian variety and that of its dual abelian variety are the same.

Suppose \( A_1 = A \) and \( A_2 = A^t \), the dual abelian variety. We always have (6) an isogeny \( f : A_1 \to A_2 \).

The main ingredients of the proof are the following lemmas. For a finite module \( M \), we define \( M^D := \text{Hom}(M, \mu_{p^\infty}) \). Recall that \( C \) is the kernel of the isogeny \( f : A \to A^t \). For a prime \( v \), the kernel of the reduction of \( C \) modulo \( v \) is denoted by \( C_v \).

**Lemma 2** Let \( v \) be a prime of \( F \) dividing \( p \). Maintaining the above notation and hypotheses on the variety \( A \), we have

\[ \#C_v = \#\tilde{C}_v = \#\tilde{C}_v^t = \#C^t. \]

**Proof.** Let \( A_v \) be the kernel of the reduction of \( A \) modulo \( v \). Then we have the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \to & C & \to & A_{v,p^\infty} & \to & A_{v,\infty}^t & \to & 0 \\
0 & \to & C & \to & A_{p^\infty} & \to & A_{p,\infty}^t & \to & 0 \\
0 & \to & \tilde{C}_v & \to & \tilde{A}_{v,p^\infty} & \to & \tilde{A}_{v,\infty}^t & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

By Milne [6] we know that \( C^t = \text{Hom}(C, \mu_{p^\infty}) \) which gives perfect Weil pairing \( C \times C^t \to \mu_{p^\infty} \). Further the exact annihilator of \( A_{p^n} \) is \( A_{p^n}^t \) and therefore we see that \( A_{p^n} \) is Pontrjagin dual to \( A_{p^n}^t \). Using this in the pairing \( C \times C^t \to \mu_{p^\infty} \),
we see that the Pontrjagin dual of $C_v$ is $\tilde{C}_v(-1)$. This implies that $\# C = \tilde{C}_v$. Similarly, $\# C' = \tilde{C}_v$. We now show that $\# C_v = \# \tilde{C}_v$.

The snake lemma induced by multiplication by $p^n$ for the top row in the above diagram gives an exact sequence

$$0 \to C_{v,p^n} \to A_{v,p^n} \to A_{v,p^n}' \to \frac{C_v}{p^nC_v} \to 0$$

since $A_{v,p^n}$ is divisible. Taking inverse limit over $n$, we get an exact sequence

$$0 \to T_pA_v \to T_pA_v' \to C_v \to 0$$

since $C_v$ is a finite $p$-primary group and hence $\lim \leftarrow C_{v,p^n} = 0$.

Next recall that $A_{v,p^n}$ is Pontrjagin dual to $\tilde{A}_{v,p^n}'(-1)$. This implies,

$$0 \to \tilde{A}_{v,p^n}'(-1) \to \tilde{A}_{v,p^n}'(-1) \to C_v \to 0.$$ 

Taking the Pontrjagin dual along this exact sequence, we have

$$0 \to \tilde{C}_v \to \tilde{A}_{v,p^n}'(-1) \to \tilde{A}_{v,p^n}'(-1) \to 0.$$ 

But by the bottom row of diagram (2), $\# \tilde{C}_v = \# \tilde{C}_v(-1)$ and hence $\# C_v = \# \tilde{C}_v$. Hence the lemma.

\[\square\]

**Lemma 3** If $v \nmid \infty$ and $G_v = \text{Gal}(\bar{F}_v/F_v)$, then $\# C(F_v) = \chi(F_v, C)$ and $\# C'(F_v) = \chi(F_v, C')$.

Proof. First let $v$ be a complex prime so that $F_v = \mathbb{C}$. Then $G_v = \{ e \}$ is the trivial group. This implies $H^1(G_v, C) = H^2(G_v, C) = 0$ and hence $\chi(G_v, C) = \# H^0(G_v, C) = \# C$. Similarly, $\chi(G_v, C') = \# H^0(G_v, C') = \# C'$.

Now let $v$ be a real prime so that $F_v = \mathbb{R}$ and $\bar{F}_v = \mathbb{C}$ with $G_v = \mathbb{Z}/2\mathbb{Z}$. As $G_v$ is a finite cyclic group, its Herbrand quotient is 1. Therefore $\chi(G_v, C) = \# H^0(G_v, C) = \# C(F_v)$. Similar argument for the dual abelian variety gives $\chi(G_v, C') = \# C'(F_v)$.

\[\square\]
Proposition 4 \ With same notation and assumptions as above, \[ \mu(\mathfrak{X}(A/F_{\infty})) = \mu(\mathfrak{X}(A^t/F_{\infty})). \]

Proof. By Proposition 1, we have the formula \[ \mu(\mathfrak{X}(A/F_{\infty})) - \mu(\mathfrak{X}(A^t/F_{\infty})) = \sum_{\nu|\infty} \log_p(\#C(F_\nu)) - [F : \mathbb{Q}] \log_p(\#C) \]
\[ + \sum_{\nu|p} \log_p(\#\tilde{C}_v|_\nu); \]
and similarly by interchanging the roles of $A$ and $A^t$ we have another formula \[ \mu(\mathfrak{X}(A^t/F_{\infty})) - \mu(\mathfrak{X}(A/F_{\infty})) = \sum_{\nu|\infty} \log_p(\#C^t(F_\nu)) - [F : \mathbb{Q}] \log_p(\#C^t) \]
\[ + \sum_{\nu|p} \log_p(\#\tilde{C}^t_v|_\nu). \]

By Lemma 2 \[ \#\tilde{C}_v^t = \tilde{C}_v \] for primes lying above $p$ and by Lemma 3 we have equality \[ \prod_{\nu|\infty} \#C(F_\nu) = \prod_{\nu|\infty} \#C^t(F_\nu). \] These two together imply that the left hand side expressions of the above two formulae are equal. But clearly, they are negatives of each other. Hence left hand side expression is zero. That is \[ \mu(\mathfrak{X}(A/F_{\infty})) = \mu(\mathfrak{X}(A^t/F_{\infty})). \] Hence the proposition.

\[ \square \]

In the cyclotomic case, Proposition 4 has been proved by K. Matsuno ([5]). Our methods give a different proof of his result which carries over to the $GL_2$-case. It is not a priori clear how to generalize Matsuno’s proof.

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