NARROW OPERATORS ON LATTICE-NORMED SPACES

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ABSTRACT. The aim of this article is to extend results of Maslyuchenko O., Mykhaylyuk V. Popov M. about narrow operators on vector lattices. We give a new definition of a narrow operator where a vector lattice as the domain space of a narrow operator is replaced with a lattice-normed space. We prove that every GAM-compact (bo)-norm continuous linear operator from a Banach-Kantorovich space $V$ to a Banach lattice $Y$ is narrow. Then we show that, under some mild conditions, a continuous dominated operator is narrow if and only if its exact dominant is.

1. INTRODUCTION

1.1. Today the theory of narrow operators is a very active area of Functional Analysis [3, 4, 5, 7, 12, 13, 18]. Plichko and Popov were first [20] who systematically studied this class of operators. It is worth remarking, however, that narrow operators have been studied in some particular cases by some others authors before this notion appeared. For example, Ghoussoub and Rosental [8] have considered “norm-signed preserving operators” on $L_1[0,1]$, which are precisely the operators on $L_1[0,1]$ which are not narrow. On the other hand, Enflo and Starbird [6] proved that if $T : L_1(\mu) \to L_1(\nu)$ is $L_1$-complementary singular (i.e. $T$ is invertible on no complemented subspace of $L_1(\mu)$ isomorphic to $L_1(\mu)$) then $T$ is narrow. Johnson, Maurey, Schechtman and Tzafriri [9] proved that every operator $T : L_p(\mu) \to L_p(\nu), 1 < p < 2$ which is $L_p$-complementary singular is narrow. Later Kadets, Shvidkov and Werner had considered narrow operators in a different context [13]. Flores and Ruiz considered narrow operators from a Köthe function space $E$ [7]. Finally, Maslyuchenko, Mykhaylyuk and Popov have considered a general vector-lattice approach to narrow operators [18].

1.2. In this seminal paper [18] the authors gave a new definition of a narrow operator.

Definition 1.1. Let $E$ be an atomless order complete vector lattice, $X$ a Banach space. A map $f : E \to X$ is called narrow if for every $x \in E_+$ and every $\varepsilon > 0$ there exist some $y \in E$ such that $|y| = x$ and $\|f(y)\| < \varepsilon$. We say that $f$ is strictly narrow if for every $x \in E_+$ there exists some $y \in E$ such that $|y| = x$ and $f(y) = 0$.

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In the same paper [18] another definition of a narrow operator for the case when the range space is a vector lattice, was given. Let \( E, F \) be vector lattices with \( E \) atomless. A linear operator \( T : E \to F \) is called \textit{order narrow} if for every \( x \in E_+ \) there exists a net \( (x_\alpha) \) in \( E \) such that \( |x_\alpha| = x \) for each \( \alpha \) and \( Tx_\alpha \to 0 \).

1.3. In this paper we consider narrow operators in the framework of lattice-normed spaces. The notion of a lattice-normed space was introduced by Kantorovich in the first part of 20th century [10]. Later, Kusraev and his school had provided a deep theory. A detailed account the reader can find in [15].

2. Preliminaries

The goal of this section is to introduce some basic definitions and facts. General information on vector lattices, Banach spaces and lattice-normed spaces the reader can find in the books [1, 2, 15, 16, 17, 19].

1.1. Consider a vector space \( V \) and a real archimedean vector lattice \( E \).

A map \( |\cdot| : V \to E \) is a \textit{vector norm} if it satisfies the following axioms:

1) \( |v| \geq 0; \ |v| = 0 \Leftrightarrow v = 0; \ (\forall v \in V) \).

2) \( |v_1 + v_2| \leq |v_1| + |v_2| ; \ (v_1, v_2 \in V) \).

3) \( |\lambda v| = |\lambda| |v| ; \ (\lambda \in \mathbb{R}, v \in V) \).

A vector norm is called \textit{decomposable} if

4) for all \( e_1, e_2 \in E_+ \) and \( x \in V \) from \( |x| = e_1 + e_2 \) it follows that there exist \( x_1, x_2 \in V \) such that \( x = x_1 + x_2 \) and \( |x_k| = e_k, \ (k := 1, 2) \).

A triple \( (V, |\cdot|, E) \) (in brief \( (V, E), (V, |\cdot|) \) or \( V, E \) with default parameters omitted) is a \textit{lattice-normed space} if \( |\cdot| \) is a \( E \)-valued vector norm in the vector space \( V \). If the norm \( |\cdot| \) is decomposable then the space \( V \) itself is called decomposable. We say that a net \( (v_\alpha)_{\alpha \in \Delta} \) \( (bo) \)-converges to an element \( v \in V \) and write \( v = \text{bo-lim} v_\alpha \) if there exists a decreasing net \( (e_\gamma)_{\gamma \in \Gamma} \) in \( E \) such that \( \inf_{\gamma \in \Gamma} (e_\gamma) = 0 \) and for every \( \gamma \in \Gamma \) there is an index \( \alpha(\gamma) \in \Delta \) such that \( |v - v_\alpha(\gamma)| \leq e_\gamma \) for all \( \alpha \geq \alpha(\gamma) \). A net \( (v_\alpha)_{\alpha \in \Delta} \) is called \( (bo) \)-\textit{fundamental} if the net \( (v_\alpha - v_\beta)_{(\alpha, \beta) \in \Delta \times \Delta} \) \( (bo) \)-converges to zero. A lattice-normed space is called \( (bo) \)-\textit{complete} if every \( (bo) \)-fundamental net \( (bo) \)-converges to an element of this space. Let \( e \) be a positive element of a vector lattice \( E \). By \([0, e]\) we denote the set \( \{v \in V : |v| \leq e\} \). A set \( M \subset V \) is called \( (bo) \)-\textit{bounded} if there exists \( e \in E_+ \) such that \( M \subset [0, e] \).

Every decomposable \( (bo) \)-complete lattice-normed space is called a \textit{Banach-Kantorovich space} (a BKS for short).

2.1. Let \( (V, E) \) be a lattice-normed space. A subspace \( V_0 \) of \( V \) is called a \( (bo) \)-ideal of \( V \) if for \( v \in V \) and \( u \in V_0 \), from \( |v| \leq |u| \) it follows that \( v \in V_0 \). A subspace \( V_0 \) of a decomposable lattice-normed space \( V \) is a \( (bo) \)-ideal if and only if \( V_0 = \{v \in V : |v| \in L\} \), where \( L \) is an order ideal in \( E \) [15, 2.1.6.1]. Let \( V \) be a lattice-normed space and \( y, x \in V \). If
\(|x| \perp |y| = 0\) then we call the elements \(x, y\) disjoint and write \(x \perp y\). The equality \(x = \bigcap_{i=1}^{n} x_i\) means that \(x = \sum_{i=1}^{n} x_i \perp x_j\) if \(i \neq j\). An element \(z \in V\) is called a component or a fragment of \(x \in V\) if \(0 \leq |z| \leq |x|\) and \(x \perp (x - z)\). Two fragments \(x_1, x_2\) of \(x\) are called mutually complemented or MC, in short, if \(x = x_1 + x_2\). The notations \(z \subseteq x\) means that \(z\) is a fragment of \(x\). According to [1, p.86] an element \(e > 0\) of a vector lattice \(E\) is called an atom, whenever \(0 \leq f_1 \leq e, 0 \leq f_2 \leq e\) and \(f_1 \perp f_2\) imply that either \(f_1 = 0\) or \(f_2 = 0\). A vector lattice \(E\) is atomless if there is no atom \(e \in E\).

The following object will be often used in different constructions below. Let \(V\) be a lattice-normed space and \(x \in V\). A sequence \((x_n)_{n=1}^{\infty}\) is called a disjoint tree on \(x\) if \(x_1 = x\) and \(x_n = x_{2n} \bigcap x_{2n+1}\) for each \(n \in \mathbb{N}\). It is clear that all \(x_n\) are fragments of \(x\). All lattice-normed spaces below we consider to be decomposable.

2.2. Consider some important examples of lattice-normed spaces. We begin with simple extreme cases, namely vector lattices and normed spaces. If \(V = E\) then the modules of an element can be taken as its lattice norm: \(|v| : = |v| = v \lor (−v); v \in E\). Decomposability of this norm easily follows from the Riesz Decomposition Property holding in every vector lattice. If \(E = \mathbb{R}\) then \(V\) is a normed space.

Let \(Q\) be a compact and let \(X\) be a Banach space. Let \(V := C(Q, X)\) be the space of continuous vector-valued functions from \(Q\) to \(X\). Assign \(E := C(Q, \mathbb{R})\). Given \(f \in V\), we define its lattice norm by the relation \(|f| : t \mapsto \|f(t)\|_X (t \in Q)\). Then \(|\cdot|\) is a decomposable norm [15, lemma 2.3.2].

Let \((\Omega, \Sigma, \mu)\) be a \(\sigma\)-finite measure space, let \(E\) be an order-dense ideal in \(L_0(\Omega)\) and let \(X\) be a Banach space. By \(L_0(\Omega, X)\) we denote the space of \((\text{equivalence classes of})\) Bochner \(\mu\)-measurable vector functions acting from \(\Omega\) to \(X\). As usual, vector-functions are equivalent if they have equal values at almost all points of the set \(\Omega\). If \(\bar{f}\) is the coset of a measurable vector-function \(f : \Omega \to X\) then \(t \mapsto \|f(t)\|_X (t \in \Omega)\) is a scalar measurable function whose coset is denoted by the symbol \(|\bar{f}|\) \(\in L_0(\mu)\). Assign by definition \(E(X) := \{f \in L_0(\mu, X) : |f| \in E\}\).

Then \((E(X), E)\) is a lattice-normed space with a decomposable norm [15, lemma 2.3.7]. If \(E\) is a Banach lattice then the lattice-normed space \(E(X)\) is a Banach space with respect to the norm \(\|\cdot\| := ||f(\cdot)||_X\|_E\).

2.3. Let \(E\) be a Banach lattice and let \((V, E)\) be a lattice-normed space. By definition, \(|x| \in E_+\) for every \(x \in V\), and we can introduce some mixed norm in \(V\) by the formula \(\|\|x\|\| : = \|x\| (\forall x \in V)\).

The normed space \((V, \|\|\cdot\||\)\) is called a space with a mixed norm. In view of the inequality \(|x| - |y| | \leq |x - y|\) and monotonicity of the norm in \(E\),
we have
\[ \| x - y \| \leq \| x - y \| (\forall x, y \in V), \]
so a vector norm is a norm continuous operator from \((V, \| \cdot \|)\) to \(E\). A lattice-normed space \((V, E)\) is called a Banach space with a mixed norm if the normed space \((V, ||\cdot||)\) is complete with respect to the norm convergence.

2.4. Consider lattice-normed spaces \((V, E)\) and \((W, F)\), a linear operator \(T : V \to W\) and a positive operator \(S \in L_+(E, F)\). If the condition
\[ |Tv| \leq S |v| (\forall v \in V) \]
is satisfied then we say that \(S\) dominates or majorizes \(T\) or that \(S\) is dominant or majorant for \(T\). In this case \(T\) is called a dominated or majorizable operator. The set of all dominants of the operator \(T\) is denoted by \(\text{maj}(T)\).

If there is the least element in \(\text{maj}(T)\) with respect to the order induced by \(L_+(E, F)\) then it is called the least or the exact dominant of \(T\) and it is denoted by \(\{T\}\). The set of all dominated operators from \(V\) to \(W\) is denoted by \(M(V, W)\). Denote by \(E_{0+}\) the conic hull of the set \(\{v\} = \{v \in V\}\), i.e., the set of elements of the form \(\sum_{k=1}^{n} v_k\), where \(v_1, \ldots, v_n \in V\), \(n \in \mathbb{N}\).

**Lemma 2.1** ([15], 4.1.2, 4.1.5). Let \((V, E), (W, F)\) be lattice-normed spaces. Suppose \(V\) is decomposable and \(F\) is order complete. Then every dominated operator has the exact dominant \(\{T\}\). The exact dominant of an arbitrary operator \(T \in M(V, W)\) can be calculated by the following formulas:

\[ |T| (e) = \sup \left\{ \sum_{i=1}^{n} |Tv_i| : \sum_{i=1}^{n} |v_i| = e, e \in E_{0+} \right\}; \]

\[ |T| (e) = \sup \{|T| (e_0) : e_0 \in E_{0+}; e_0 \leq e\} (e \in E_+); \]

\[ |T| (e) = |T| (e_+) + |T| (e_-), (e \in E). \]

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3. Definition and some properties of narrow operators

In this section we introduce a new class of operators in lattice-normed spaces and describe some of their properties.

**Definition 3.1.** Let \((V, E)\) be a lattice-normed space, \(X\) a Banach space and suppose that \(E\) is atomless. An operator \(T : V \to X\) is called narrow, if for every \(u \in V\), \(\varepsilon > 0\) there exist two \(MC\) fragments \(u_1, u_2\) of \(u\) such that \(\|T(u_1 - u_2)\| < \varepsilon\). If for every \(u \in V\) there exist two \(MC\) fragments \(u_1, u_2\) of the \(u\) such that \(T(u_1 - u_2) = 0\) for then the operator \(T\) is called strictly narrow.

The set of all narrow operators from a lattice-normed space \((V, E)\) to a Banach space \(X\) we denote by \(N(V, X)\).
Lemma 3.2. If a lattice-normed space \((V, E)\) coincides with \((E, E)\) then definitions \([1.1]\) and \([3.1]\) are equivalent.

Proof. Let \(X\) be a Banach space and let \(T : E \to X\) be a narrow operator in accordance with Definition \([3.1]\). Consider an element \(e \in E_+\) and \(\varepsilon > 0\). Then there exist two \(MC\) fragments \(e_1, e_2\) of \(e\) such that \(\|T(e_1 - e_2)\| < \varepsilon\). Then for \(y = e_1 - e_2\) one has that \(\|Ty\| < \varepsilon\), that is, Definition \([1.1]\) for \(T\) is satisfied.

Now we prove the inverse assertion. Let \(T\) be a narrow operator in accordance with Definition \([1.1]\). Fix any \(x \in E\) and \(\varepsilon > 0\). Then \(x = x_+ - x_-\) and there exist two elements \(x_1'\) and \(x_2'\) such that \(|x_1'| = x_+\), \(\|Tx_1'\| < \frac{\varepsilon}{2}\) and \(|x_2'| = x_-\), \(\|Tx_2'\| < \frac{\varepsilon}{2}\). We consider new elements: \(e_1 := x_1' \vee 0\), \(e_2 := -x_1' \vee 0\) and \(f_1 := x_2' \vee 0\), \(f_2 := -x_2' \vee 0\). So, we have two pairs of \(MC\) fragments \(e_1, e_2\) of \(x_+\) and \(f_1, f_2\) of \(x_-\) such that the following inequalities hold

\[
\|T(e_1 - e_2)\| < \frac{\varepsilon}{2}; \quad \|T(f_1 - f_2)\| < \frac{\varepsilon}{2}.
\]

Then \(x_1 := e_1 - f_2\) and \(x_2 := e_2 - f_1\) are \(MC\) fragments of \(x\), and

\[
\|T(x_1 - x_2)\| = \|T(e_1 - f_2 - e_2 + f_1)\| < \|T(e_1 - e_2)\| + \|T(f_1 - f_2)\| < \varepsilon.
\]

\(\square\)

Lemma 3.3. Let \((V, E)\) be a lattice-normed space and let \((W, F)\) be a Banach space with a mixed norm. An operator \(T : V \to W\) is called order narrow if for every \(u \in V\) there exists a net \((v_\alpha)_{\alpha \in \Lambda}\) where every element \(v_\alpha\) is a difference \(u_\alpha^1 - u_\alpha^2\) of two \(MC\) fragments of \(u\) such that \(T(v_\alpha) \to 0\).

Lemma 3.4. Let \((V, E)\) and \((W, F)\) be the same as in 3.3 and let \(F\) be a Banach lattice with order continuous norm. Then a linear operator \(T : V \to W\) is order narrow if and only if \(T\) is narrow.

Proof. Let \(T\) be an order narrow operator. Then for every \(u \in V\) there exist a net \((v_\alpha)_{\alpha \in \Lambda}\); \(v_\alpha = u_\alpha^1 - u_\alpha^2\), where \(u_\alpha^1\) and \(u_\alpha^2\) are \(MC\) fragments of \(u\) and \(Tv_\alpha \to 0\). Fix any \(\varepsilon > 0\). Using the fact that the norm in \(F\) is order continuous we can find \(\alpha_0 \in \Lambda\) such that \(\|Tv_\alpha\| < \varepsilon\) for every \(\alpha \geq \alpha_0\). In view of Lemma 3.3, the converse is true. \(\square\)
The following lemma will be useful later.

**Lemma 3.5.** Let \((V, E)\) and \((W, F)\) be lattice-normed spaces, \(E\) an atomless vector lattice, \(J\) a \((\text{bo})\)-ideal of \(W\) and \(F_1\) an order ideal of \(F\). If a linear dominated operator \(T : V \rightarrow J\) is order narrow then \(T : V \rightarrow W\) is order narrow as well. Conversely, if a dominated linear operator \(T : V \rightarrow J\) is such that \(T : V \rightarrow W\) is an order narrow then so is \(T : V \rightarrow J\).

**Proof.** The first part is obvious. Let \(T : V \rightarrow J\) be a dominated operator such that \(T : V \rightarrow W\) is order narrow. For any \(v \in V\) we choose a net \((v_\alpha)_{\alpha \in \Delta}\) where every element \(v_\alpha\) is a difference \(u^1_\alpha - u^2_\alpha\) of two \(MC\) fragments of \(u\) such that \(T(v_\alpha) \rightarrow 0\), that is, \(|Tv_\alpha| \leq y_\alpha \downarrow 0\) for some net \((y_\alpha)_{\alpha \in \Delta} \subset F\). Using the fact that the operator \(T\) is dominated and its exact dominant is \([T] : E \rightarrow F_1\), one has that

\[
|Tv_\alpha| \leq |T| |v| = g \in F_1.
\]

Hence, \(|Tv_\alpha| \leq z_\alpha \downarrow 0\) where \(z_\alpha := g \wedge y_\alpha\). So, we have that \((z_\alpha)_{\alpha \in \Delta} \subset F_1\).

Thus, the net \((Tv_\alpha)_{\alpha \in \Delta}\) \((\text{bo})\)-converges to \(0\) in \((J, F_1)\). \(\square\)

4. Narrow and GAM-compact operators

In this section we investigate connections between narrow and GAM-compact operators.

Let \((V, E)\) be a lattice-normed space and let \(Y\) be a Banach space. A linear operator \(T : V \rightarrow Y\) is called \((\text{bo})\)-norm continuous whenever it sends every \((\text{bo})\)-convergent net in \(V\) to a norm convergent net in \(Y\). A linear operator \(T : V \rightarrow Y\) is called generalized \(AM\)-compact or GAM-compact for short if for every \((\text{bo})\)-bounded set \(M \subset V\) its image \(T(M)\) is a relatively compact set in \(Y\). Consider some examples.

**Example 1.** In a particular case when \(V = E\) the sets of GAM-compact and \(AM\)-compact operators from \(V\) to \(Y\) are equal.

**Example 2.** If \(E = \mathbb{R}\) then \(V\) is a normed space and the sets of GAM-compact and compact operators from \(V\) to \(Y\) are equal.

**Example 3.** Let \(X, Y\) be Banach spaces and let \((\Omega, \Sigma, \mu)\) be a finite measure space. The space \(L_1(\mu, X)\) is the space \(\mu\)-Bochner integrable functions on \(\Omega\) with values in \(X\), and \(L_\infty(\mu, X)\) is the space of \(X\)-valued \(\mu\)-Bochner integrable functions on \(\Omega\) that are essentially bounded. A function \(g \in L_\infty(\mu, \mathcal{L}(X, Y))\) is said to have its essential range in the \textit{uniformly compact operators} if there is a compact set \(C\) in \(Y\) such that \(g(\omega)x \in C\) for almost all \(\omega \in \Omega\) and \(x \in X\), \(\|x\| \leq 1\). An operator \(T : L_1(\mu, X) \rightarrow Y\) is called \textit{representable measurable kernel} if there is a bounded measurable function \(g : \Omega \rightarrow \mathcal{L}(X, Y)\) such that

\[
Tf = \int_\Omega fg d\mu \quad \text{for all } f \in L_1(\mu, X).
\]

**Theorem 4.1** ([14], Theorem 2). Let \(X\) be a Banach space such that \(X^*\) has the Radon-Nikodým property. Then there is an isometric isomorphism
between the space of compact operators $K(L_1(\mu, X), Y)$ and the subspace of $L_\infty(\mu, K(X, Y))$ consisting of those functions whose essential range is in the uniformly compact operators. In fact, $T \in K(L_1(\mu, X), Y)$ and $g \in L_\infty(\mu, K(X, Y))$ are in correspondence if and only if

$$T(f) = \int_\Omega g(\omega)f(\omega)\,d\mu(\omega) \text{ for all } f \in L_1(\mu, X).$$

**Lemma 4.2.** Let $X$ be a Banach space such that $X^*$ has the Radon-Nikodým property. Then every representable measurable kernel operator $T : L_1(\mu, X) \to Y$ whose kernel have the essential range in the uniformly compact operators is GAM-compact.

**Proof.** Let $M \subset V$ be a (bo)-bounded set. Then there exists $e \in L_1(\mu)$, $e \geq 0$, such that $\|v\| \leq e$ for every $v \in M$. This implies that the set $M$ is norm bounded in $L_1(\mu, X)$

$$\|v\|_{L_1(\mu, X)} = \int_\Omega |v|\,d\mu = \int_\Omega \|v(\cdot)\|_X\,d\mu \leq \int_\Omega ed\mu = r$$

By Theorem 4.1, the lemma is proved. □

**Lemma 4.3.** Let $(V, E)$ be a Banach space with a mixed norm and let $X$ be a Banach space. Then every GAM-compact operator $T : V \to X$ is norm bounded.

**Proof.** If $T$ were unbounded then we would have a sequence $(v_n) \subset V$ with $\|v_n\| \leq \frac{1}{n}$ and $\|Tv_n\| \to \infty$. The sequence $(v_n)$ is order bounded by the element $e \in E_+ = \sum_{n=1}^\infty |v_n|$. Hence, the sequence $(Tv_n)$ is relatively compact, – a contradiction. □

**Corollary 4.4.** Let $(V, E)$ be a Banach space with a mixed norm, $E$ an order continuous Banach lattice and $X$ a Banach space. Then every GAM-compact operator $T : V \to X$ is (bo)-norm continuous.

The following lemma is well known [11, p.14].

**Lemma 4.5.** Let $(x_i)_{i=1}^n$ be a finite collection of vectors in a finite dimensional normed space $X$ and let $(\lambda_i)_{i=1}^n$ be a collection of reals with $0 \leq \lambda_i \leq 1$ for each $i$. Then there exists a collection $(\theta_i)_{i=1}^n$ of numbers $\theta_i \in \{0, 1\}$ such that

$$\left\| \sum_{i=1}^n (\lambda_i - \theta_i)x_i \right\| \leq \frac{\dim(X)}{2} \max_i \|x_i\|$$

For the rest of the section $(V, E)$ is assumed to be a Banach-Kantorovich space, $E$ is an atomless order complete vector lattice, $X$ is a Banach space and $T : V \to X$ is (bo)-norm continuous linear operator.

**Lemma 4.6.** Let $x \in V$. Then there exist two MC fragments $x_1, x_2$ of $x$ such that $\|Tx_1\| - \|Tx_2\| = 0$. 
Proof. Fix any couple $x_1, x_2$ of $MC$ fragments of $x$. If $\|Tx_1\| - \|Tx_2\| = 0$ then there is nothing to prove. Let $\|Tx_1\| - \|Tx_2\| > 0$. Consider the partially ordered set

$$D := \{y \subseteq x_1 : \|T(x_1 - y)\| - \|T(x_2 + y)\| \geq 0\}$$

where $y_1 \leq y_2$ if and only if $y_1 \subseteq y_2$. If $B \subseteq D$ is a chain then $y^* = \vee B \in D$ by the (bo)-norm continuity of $T$. By the Zorn lemma, there is a maximal element $y_0 \in D$. Now we show $\|T(x_1 - y_0)\| - \|T(x_2 + y_0)\| = 0$. Suppose on the contrary that

$$\alpha = \|T(x_1 - y_0)\| - \|T(x_2 + y_0)\| > 0.$$

Since $E$ is atomless, we can choose a fragment $0 \neq y \subseteq (x_1 - y_0)$ with $\|Ty\| < \frac{\alpha}{3}$. Since $y_0 \downarrow y$, $y_0 + y \subseteq x_1$ and

$$\|T(x_1 - y_0 - y)\| - \|T(x_2 + y_0 + y)\| \geq\]

$$\geq \|T(x_1 - y_0)\| - \|Ty\| - \|T(x_2 + y_0)\| - \|Ty\| > \frac{\alpha}{3},$$

that contradicts the maximality of $y_0$. □

Lemma 4.7. Let $v \in V$ and $(v_n)_{n=1}^{\infty}$ be a disjoint tree on $v$. If $\|Tx_{2n}\| = \|Tx_{2n+1}\|$ for every $n \leq 1$ then

$$\lim_{m \to \infty} \max_{2^{m} \leq i < 2^{m+1}} \|Tv_i\| = 0$$

Proof. Put $\gamma_m = \max_{2^{m} \leq i < 2^{m+1}} \|Tv_i\|$ and $\varepsilon = \limsup_m \gamma_m$. Suppose on the contrary that $\varepsilon > 0$. Then for each $n \subseteq N$ we set

$$\varepsilon_n = \limsup_{m \to \infty} \max_{2^{m} \leq i < 2^{m+1}, v_i \subseteq v_n} \|Tv_i\|.$$ 

Hence, for each $m \subseteq N$ one has

$$\max_{2^{m} \leq i < 2^{m+1}} \varepsilon_i = \varepsilon. \quad (\ast)$$ 

Now we are going to construct a sequence of mutually disjoint elements $(v_{n_j})_{j=1}^{\infty}$ such that $\|Tv_{n_j}\| \leq \frac{\varepsilon}{2}$. At the first step we choose $m_1$ such that

$$\max_{2^{m_1} \leq i < 2^{m_1+1}} \|Tv_i\| \geq \frac{\varepsilon}{2}. \quad (\ast)$$

In accordance with (\ast), we choose $i_1, 2^{m_1} \leq i_1 < 2^{m_1+1}$ so that $\varepsilon_{m_1} = \varepsilon$. Using $\|Tv_{2n_1}\| = \|Tv_{2n_1+1}\|$, we choose $n_1 \neq i_2, 2^{m_1} \leq i < 2^{m_1+1}$ so that $\|Tv_{n_1}\| \leq \frac{\varepsilon}{2}$. At the second step we choose $m_2 > m_1$ so that

$$\max_{2^{m_2} \leq i < 2^{m_2+1}, v_i \subseteq v_{n_1}} \|Tv_i\| \geq \frac{\varepsilon}{2}.$$ 

In accordance with (\ast), we choose $i_2, 2^{m_2} \leq i_2 < 2^{m_2+1}$ so that $\varepsilon_{i_2} = \varepsilon$. Then we choose $m_2 \neq i_2, 2^{m_2} \leq i_2 < 2^{m_2+1}$ so that $\|Tv_{m_2}\| \geq \frac{\varepsilon}{2}$. Proceeding further, we construct the desired sequence. Indeed, $\|Tv_{m_1}\| \geq \frac{\varepsilon}{2}$ by the construction and mutually disjoint for $v_{m_1}, v_{m_j}, j \neq l$ is guaranteed by the condition $m_j \neq i_j$. The elements $v_{m_{j+l}}$ are fragments of $v_i$ which are disjoint to $v_{m_j}$. □
The following lemma indicates that operators with finite dimensional range are also narrow.

**Lemma 4.8.** If $X$ is finite dimensional, then $T$ is narrow.

*Proof.* Fix any $v \in V$, $\varepsilon > 0$ and $\dim(X) = \gamma$. Using Lemma 4.6 we construct a disjoint tree $(v_n)$ on $v$ with $\|Tv_{2n}\| = \|Tv_{2n+1}\|$ for all $n \in \mathbb{N}$. By lemma 4.7 we choose $m$ such that $\gamma \alpha_m < \varepsilon$ where $\alpha_m := \max_{2^m \leq i < 2^{m+1}} \|Tv_i\|$. Then using Lemma 4.5, we choose numbers $(\lambda_i)_{i=1}^n$, $\lambda_i \in \{0, 1\}$ for $i = 2^m, \ldots, 2^{m+1} - 1$ so that

$$\left\| \sum_{i=2^m}^{2^{m+1}-1} \left( \frac{1}{2} - \lambda_i \right)Tv_i \right\| \leq \frac{\gamma}{2} \max_{2^m \leq i < 2^{m+1}} \|Tv_i\| = \frac{\gamma}{2} \alpha_m < \frac{\varepsilon}{2}.$$ 

Now we consider the element $w = 2\left( \sum_{i=2^m}^{2^{m+1}-1} \left( \frac{1}{2} - \lambda_i \right)v_i \right)$. On the other hand, $w = \sum_{i=2^m}^{2^{m+1}-1} u_i$ where $u_i$ are disjoint, and $u_i \in \{ (\pm v_i)_{i=2^m}^{2^{m+1}-1} \}$. Then there exist two $MC$-fragments $v_1$ and $v_2$ of $v$ such that $w = v_1 - v_2$ and $\|Tw = T(v_1 - v_2)\| < \varepsilon$. \qed

Now we are ready to prove the main result of this section.

**Theorem 4.9.** Let $(V, E)$ be a Banach-Kantorovich space, $E$ an atomless order complete vector lattice, $X$ a Banach space. Then every $GAM$-compact (bo)-norm continuous linear operator $T : V \to X$ is narrow.

*Proof.* It is well known that if $H$ is a relatively compact subset of $l_\infty(D)$ for some infinite set $D$ and $\varepsilon > 0$ is an arbitrary positive number then there exists a finite rank operator $S \in l_\infty(D)$ such that $\|x - Sx\| \leq \varepsilon$ for every $x \in H$. So, we may consider $X$ as a subspace of some $l_\infty(D)$ space

$$X \hookrightarrow X^{**} \hookrightarrow l_\infty(BX^\ast) = l_\infty(D) = W.$$ 

By the notation $\hookrightarrow$ we mean isometric embedding. Fix any $v \in V$ and $\varepsilon > 0$. Since $T$ is a $GAM$-compact operator, $K = \{Tu : \|u\| \leq \|v\| \}$ is relatively compact in $X$ and hence, in $W$. Then there exist a finite dimensional operator $S \in L(W)$ such that $\|x - Sx\| \leq \frac{\varepsilon}{2}$ for every $x \in K$. Then $R = S \circ T$ is a (bo)-norm continuous finite dimensional operator. By Lemma 4.8, there exist two $MC$ fragments $v_1, v_2$ of $v$ such that $\|R(v_1 - v_2)\| < \frac{\varepsilon}{2}$. Thus,

$$\|T(v_1 - v_2)\| = \|T(v_1 - v_2) + S(T(v_1 - v_2)) - S(T(v_1 - v_2))\| =$$

$$= \|T(v_1 - v_2) + R(v_1 - v_2) - S(T(v_1 - v_2))\| \leq$$

$$\leq \|R(v_1 - v_2)\| + \|T(v_1 - v_2) - S(T(v_1 - v_2))\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 

\qed
5. Dominated narrow operators

In this section we investigate some properties of the dominated narrow operators. Observe that for every \( x, y \in L_1(\nu) \) the following equality holds

\[
\|x - y\| = \| |x| - |y| \| + \| x \| + \| y \| - \| x + y \|. \quad (\star)
\]

**Theorem 5.1.** Let \( E, F \) be order complete vector lattices such that \( E \) is atomless, \( F \) an ideal of some order continuous Banach lattice and \((V, E)\) a Banach-Kantorovich space. Then every \((bo)\)-continuous dominated linear operator \( T : V \to F \) is order narrow if and only if \(|T|\) is.

**Proof.** First we mention that a Banach lattice \( F \) is a lattice-normed space and the vector norm \( \| \cdot \| \) coincides with the module map \( |\cdot| \). Now we prove the theorem for \( F = L_1(\nu) \). By Lemma 3.4, instead of order narrowness we will consider narrowness. Assume first that \( T \) is narrow. Fix any \( e \in E_+ \) and \( \varepsilon > 0 \). Let \( v \in V \) and \( |v| = e \). Since

\[
\left\{ \sum_{i=1}^{n} |Tv_i| : \sum_{i=1}^{n} v_i = e; \ v_i \perp v_j; \ i \neq j; \ n \in \mathbb{N} \right\}
\]

is an increasing net, using the order continuity of \( L_1(\nu) \), we can choose a finite collection \( \{v_1, \ldots, v_n\} \subset V \) with

\[
e = \bigcup_{i=1}^{n} |v_i|
\]

and

\[
\left\| |T| (e) - \sum_{i=1}^{n} |Tv_i| \right\| < \varepsilon.
\]

Let \( v_i = u_i \bigcup w_i \) be a decomposition into a sum of \( MC \) fragments \( u_i \) and \( w_i \). Then we have

\[
0 \leq |T| (e) - \sum_{i=1}^{n}(|Tu_i| + |Tw_i|) \leq
\]

\[
\leq |T| (e) - \sum_{i=1}^{n} |Tv_i|
\]

Since

\[
e = \sum_{i=1}^{n} |v_i| = \sum_{i=1}^{n} (|u_i| + |w_i|),
\]

we have that

\[
|T| (e) = |T| \left( \sum_{i=1}^{n} (|u_i| + |w_i|) \right) = \sum_{i=1}^{n} (|T| |u_i| + |T| |w_i|).
\]
Since $|T| |u_i| - |Tu_i|$ and $|T| |w_i| - |Tw_i|$ are positive elements of $L_1(\nu)$ for every $i \in \{1, \ldots, n\}$, the sum of their norms equals the norm of their sum. Thus, we obtain
\[
\sum_{i=1}^{n} (|T| |u_i| - |Tu_i| + |T| |w_i| - |Tw_i|) =
\]
\[
= |T| (e) - \sum_{i=1}^{n} (|Tu_i| + |Tw_i|) \leq
\]
\[
\leq |T| (e) - \sum_{i=1}^{n} |Tv_i| < \varepsilon.
\]
For each $i = 1, \ldots, n$ we represent $v_i = u_i \prod w_i$ so that $u_i, w_i \in V$ and $|Tw_i - Tu_i| < \frac{\varepsilon}{n}$. Then putting $u = \prod u_i$, $w = \prod w_i$, $f_1 = |u|$, $f_2 = |w|$ and using inequality $(\ast)$, we obtain
\[
\| |T| f_1 - |T| f_2 \| \leq \sum_{i=1}^{n} (|T| |u_i| - |Tu_i|) \leq
\]
\[
\leq \sum_{i=1}^{n} (|Tu_i| - |Tw_i|) + \sum_{i=1}^{n} (|T| |u_i| - |Tu_i|) \leq
\]
\[
\leq \sum_{i=1}^{n} |Tu_i - Tw_i| + \varepsilon < 2\varepsilon.
\]
Using the arbitrariness of $e \in E_+$ and $\varepsilon > 0$, and the fact that $f_1, f_2$ are two $MC$ fragments of $e$, we deduce that $|T|$ is a narrow operator.

Now let $|T|$ be a narrow operator, $v$ an arbitrary element of $V$, $\varepsilon > 0$, $|v| = e$, $e = \prod_{i=1}^{n} |v_i|$, $v_i \in V$; $\forall i \in \{1, \ldots, n\}$ and again
\[
\| |T| (e) - \sum_{i=1}^{n} |Tv_i| \| < \varepsilon.
\]
For each $i = 1, \ldots, n$ we decompose $v_i = f_i^1 \prod f_i^2$ such that
\[
\| |T| f_i^1 - |T| f_i^2 \| < \frac{\varepsilon}{n}
\]
and let $f^1 = \prod_{i=1}^{n} f_i^1$ and $f^2 = \prod_{i=1}^{n} f_i^2$. Taking into account that the $L_1$-norm of a sum of positive elements equals the sum of their norm, we obtain
\[
\sum_{i=1}^{n} |Tf_i^j| \leq \sum_{i=1}^{n} |Tf_i^j| : j \in \{1, 2\};
\]
\[
\sum_{i=1}^{n} |Tf_i^1| + \sum_{i=1}^{n} |Tf_i^2| = |T| (e);
\]
$$\left\| \sum_{i=1}^{n} |Tf_i^1| + |Tf_i^2| \right\| \leq \| |Tf_1| \|.$$

Then, using again inequality (\ast), we obtain
\[
\|Tf_1 - Tf_2\| \leq \sum_{i=1}^{n} \|Tf_i^1 - Tf_i^2\| = \sum_{i=1}^{n} \|Tf_i^1\| - |Tf_i^2|\| +
+ \sum_{i=1}^{n} \|Tf_i^1\| + |Tf_i^2|\| - \sum_{i=1}^{n} \|Tv_i\| =
= \sum_{i=1}^{n} \|Tf_i^1\| - |Tf_i^2|\| + \|Tf_i^1\| - |Tf_i^2|\| +
+ \sum_{i=1}^{n} \|Tf_i^1\| + |Tf_i^2|\| - \sum_{i=1}^{n} \|Tv_i\| \leq
\leq \sum_{i=1}^{n} \|Tf_i^1\| - |Tf_i^2|\| + \|Tf_i^1\| - |Tf_i^2|\| +
+ \sum_{i=1}^{n} \|Tf_i^1\| - |Tf_i^2|\| + \|Tv_i\| - \sum_{i=1}^{n} \|Tv_i\|.
\]

Finally, using the fact that
\[
\|Tf_1 - Tf_2\| < 3\varepsilon. \quad \text{Since } f_1 \text{ and } f_2 \text{ are } MC \text{ fragments of } v,
\]
we obtain that $\|Tf_1 - Tf_2\| < 3\varepsilon$. Since $f_1$ and $f_2$ are $MC$ fragments of $v$, this proves that $T$ is narrow.

Now we consider the general case. Since $F$ is an ideal of some order continuous Banach lattice $H$, we have by Lemma 3.5 that $T : V \to F$ is order narrow if and only if $T : V \to H$ is.

We consider $T : V \to H$ and $|T| : E \to H$. Fix any $v \in V$. By $E_1$ and $H_1$ we denote the principal bands in $E$ and $H$ generated by $|v|$ and $|T| |v|$ respectively. Using the fact that Boolean algebras of bands $B(V)$ and $B(E)$ are isomorphic [15, 2.1.2.1], we denote $V_1 := \gamma(E_1)$. Here $\gamma : B(E) \to B(V)$ is a boolean isomorphism. Denote by $T_1$ the restriction of $T$ to $V_1$. Operator $|T|$ coincides with the restriction $|T|$ to $E_1$. So $H_1$ is an order continuous Banach lattice with weak unit $|T| |v|$. Then by [17, Theorem 1.b.14] there exists a probability space $(\Omega, \Sigma, \nu)$ and an ideal $H_2$ of $L_1(\nu)$ such that $H_1$ is isomorphic to $H_2$. Let $S : H_1 \to H_2$ be a lattice isomorphism. Then we set $T_2 = S \circ T_1$. Moreover, $|T_2| = S \circ |T_1|$. By our previous consideration, $T_2 : V_1 \to H_2$ is order narrow if and only if $|T_2| : E_1 \to H_2$ is. Thus, we have proved that $T_2$ is order narrow if and only if $|T_2|$ is.

Let $T : V \to H$ be order narrow. Fix an arbitrary $v \in V$. Since $T_1$ is order narrow, so is $T_2$ and hence $|T_2|$. So there exists a net $(v_\alpha)_{\alpha \in \Lambda} \subset V_1$ where every element $v_\alpha$ is a difference $u_\alpha^1 - u_\alpha^2$ two $MC$ fragments of $v$ such that
Let \((V,E),(W,F)\) be lattice-normed spaces and let \(M(V,W)\) be the space of dominated operators from \(V\) to \(W\). The band generated by all lattice homomorphisms from \(E\) to \(F\) we denote \(\mathcal{H}(E,F)\). Then, \(\mathcal{H}(V,W) := \{T \in M(V,W) : |T| \in \mathcal{H}(E,F)\}\).

**Theorem 5.2.** Let \(E,F\) be order complete vector lattices such that \(E\) is atomless, \(F\) an ideal of some order continuous Banach lattice and let \((V,E)\) be a Banach-Kantorovich space. Then every \((bo)\)-continuous dominated linear operator \(T : V \rightarrow F\) is uniquely represented in the form \(T = T_h + T_n\), where \(T_h \in \mathcal{H}(V,F)\) and \(T_n\) is a \((bo)\)-continuous order narrow operator.

**Proof.** By [15, 4.2.1], the set \(M(V,E)\) with the mapping \(p : M(V,E) \rightarrow L_+(E,F)\) is a lattice-normed space. Here \(p(T) = |T|\) for every \(T \in M(V,E)\). By [15, 4.2.6], the vector norm \(p : M(V,F) \rightarrow L_+(E,F)\) is decomposable. This means that for every dominated operator \(T : V \rightarrow F\) and its exact dominant \(|T| : E \rightarrow F\) the following statement hold:

\[
|T| = S_1 + S_2 \Rightarrow \exists T_1, T_2 \in M(V,F);
\]

\[
|T_1| = S_1; \quad |T_2| = S_2; \quad 0 \leq S_1, S_2; \quad S_1 \perp S_2.
\]

Fix an arbitrary \((bo)\)-continuous dominated operator \(T : V \rightarrow F\). By theorem [15, 4.3.2], every dominated operator \(T : V \rightarrow F\) is \((bo)\)-continuous if and only if \(|T| : E \rightarrow F\) is \((o)\)-continuous. Hence, \(|T|\) is \((o)\)-continuous. Then by [18, 11.7] the positive \((o)\)-continuous operator \(|T|\) is uniquely represented as a sum \(|T| = S_D + S_N\) when \(S_D\) is a \((o)\)-continuous operator, \(S_D \in \mathcal{H}(E,F)\) and \(S_N\) is a \((o)\)-continuous narrow operator. Then we obtain

\[
|T| = S_D + S_N \Rightarrow \exists T_1, T_2 \in M(V,F);
\]

\[
|T_1| = S_N; \quad |T_2| = S_D.
\]

Finally, by Theorem 5.1, the proof is completed. \(\square\)

**Theorem 5.3.** Let \(E(\mu)\) be a Köthe function space over probability space \((\Omega, \Sigma, \mu)\), with atomless measure \(\mu\), such that \(E(\mu)\) is order continuous, \(F\) is an order continuous Banach lattice and \(X\) is a Banach space. Then for every \((bo)\)-continuous dominated narrow linear operator \(T : E(X) \rightarrow F\) the inclusion \([0, |T|] \subset \mathcal{N}(E(X),F)\) holds.

**Proof.** By theorem [7, 3.15], for every positive narrow operator \(S : E(\mu) \rightarrow F\), such that \(E(\mu)\) is order continuous, the inclusion \([0,S] \subset \mathcal{N}(E(X),F)\) holds. Now by Theorem 5.1 the proof is completed. \(\square\)
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