AN APPLICATION OF THE ALMOST PURITY THEOREM TO THE
HOMOLOGICAL CONJECTURES

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Abstract. The aim of this article is to establish the existence of a big Cohen-Macaulay algebra
over a local ring in mixed characteristic \( p > 0 \) in the case where the local ring is finite étale over
a regular local subring after inverting \( p \). The main result follows from the so-called almost purity
theorem proved by Davis and Kedlaya.

1. Introduction

The homological conjectures, which consist of a set of statements for commutative Noetherian
(local) rings, have been of interest in the quest of commutative algebra (see \[12\] and \[18\] for known
results and history). In this article, we would like to consider the following conjecture.

Conjecture 1 (Hochster). A Noetherian local ring \((R, m, k)\) admits an \(R\)-algebra \(B\) such that
\(mB \neq B\) and every system of parameters of \(R\) is a regular sequence on \(B\).

We are interested in the situation where the module-finite map of rings \(R \to S\) for a regular ring
\(R\) is étale after inverting a prime integer \(p > 0\). This comes from the condition imposed on the
statement of the almost purity theorem, which was originally proved by Faltings in \(p\)-adic Hodge
theory, with refinements by Scholze \[19\], Davis and Kedlaya \[6\] more recently. A proof given by
Davis and Kedlaya uses a valuative method and their original motivation was to pin down the class
of rings satisfying the Witt-perfect condition.

A very important aspect of the almost purity theorem is expressed by the following facts:

- If the Frobenius endomorphism on \(R/pR\) is surjective for a \(p\)-torsion free normal ring \(R\)
  and \(R \to S\) is an almost étale extension, then the Frobenius endomorphism on \(S/pS\) is also
  surjective (see Theorem 4.5).
- If \(R\) is almost Cohen-Macaulay and \(R \to S\) is an almost étale extension, then \(S\) is also
  almost Cohen-Macaulay (see Proposition 4.12).

We are going to use the version by Davis and Kedlaya, as their statement assumes only \(p\)-torsion
freeness on rings. Our proof reduces to the case where local rings are complete in the topology
defined by the powers of the maximal ideals, as is commonly done in the search of commutative
algebra. We state our main theorem (see Theorem 4.13).

Main Theorem. Let \(R\) be a regular local ring of mixed characteristic \(p > 0\) and let \(S\) be a torsion
free module-finite \(R\)-algebra such that the localization \(R[\frac{1}{p}] \to S[\frac{1}{p}]\) is finite étale. Then \(S\) has a
balanced big Cohen-Macaulay algebra.

In general, if the field of fractions of a local domain \(R\) has characteristic 0, then a torsion free
module-finite extension \(R \to S\) is generically étale and there is a nonzero element \(f \in R\) such that
\(R[\frac{1}{f}] \to S[\frac{1}{f}]\) is étale. So our theorem treats the special case of finite ring extensions. The almost

Key words and phrases. Almost purity theorem, big Cohen-Macaulay algebra, Frobenius map, Witt vectors.
2000 Mathematics Subject Classification: 13A35, 13B22, 13B40, 13D22, 13K05.
The purity theorem is formulated in the language of almost ring theory. In fact, its idea appeared in Heitmann’s work [9] toward the construction of almost Cohen-Macaulay algebras (see also [2] and [21]), which we employ in the proof of the main theorem. We will then show that almost Cohen-Macaulay algebras map to big Cohen-Macaulay algebras by applying Hochster’s partial algebra modifications. The following corollary, which is regarded as the special case of the Direct Summand Conjecture, is a consequence of the main theorem (see Corollary 4.15):

**Corollary 1.1.** Let $R$ be a Noetherian regular domain and let $S$ be a torsion free module-finite $R$-algebra. Assume that $R$ is $p$-torsion free and the localization $R[\frac{1}{p}] \to S[\frac{1}{p}]$ is finite étale for some prime integer $p > 0$. Then $R \hookrightarrow S$ splits as an $R$-module homomorphism.

We should point out that Bhatt recently proved the corollary in a slightly different setting in [3]. Our corollary is new in that there is no hypothesis on the base ring we start with and we deduce it from the existence of big Cohen-Macaulay algebras, while Bhatt’s result treats a more ramified extension than ours under the assumption that the base ring is étale over a ring that is essentially of finite over a discrete valuation ring. More recently, Bhatt’s result has been generalized in the context of logarithmic algebraic geometry in [8].

2. **Notation and convention**

All rings in this article are commutative with unity. However, we do not always assume rings to be Noetherian. A local ring is a Noetherian ring with a unique maximal ideal. A sequence of elements $(x_1, \ldots, x_n)$ of a local ring $(R, m)$ is a regular sequence on an $R$-module $M$, if

- $(x_1, \ldots, x_n)M \neq M$.
- The multiplication map $M/(x_1, \ldots, x_i)M \xrightarrow{x_{i+1}} M/(x_1, \ldots, x_i)M$ is injective for all $i$.

Here is a definition of (balanced) big Cohen-Macaulay algebra or module.

**Definition 2.1.** Let $(R, m)$ be a Noetherian local ring. Then an $R$-algebra or module $B$ is a big Cohen-Macaulay $R$-algebra or module, if there is a system of parameters of $R$ such that it is a regular sequence on $B$ and $mB \neq B$. Furthermore, $B$ is balanced, if every system of parameters is a regular sequence on $B$.

There are many flavors of Witt vectors in the literature, however the only Witt vectors we consider in this article are the $p$-typical Witt vectors. The basic reference for $p$-typical Witt vectors is Serre’s book [20]. Another good source is an expository paper [16]. We will need to consider the Witt vectors for $p$-torsion free rings to define the Witt-perfect condition. Since the basic part of the Witt vectors is rather involved and we do not require an explicit descriptions, we give only the summary to the extent we need.

Fix a prime number $p > 0$ and a commutative ring $A$ of arbitrary characteristic. Then we denote by $W(A)$ (resp. $W_p^n(A)$) the ring of ($p$-typical) Witt vectors (resp. Witt vectors of length $n$). Then one has a set-theoretic identity: $W_p^n(A) = A^{n+1}$ and a ring-theoretic isomorphism: $W(A) \cong \lim_{\leftarrow n} W_p^n(A)$. On the Witt vectors, there is a well-defined ring homomorphism called the Witt-Frobenius map:

$$F : W_p^n(A) \to W_p^{n-1}(A),$$

which is described as follows. If $A$ is a ring of prime characteristic $p > 0$, then $F(r_1, r_2, \ldots, r_{n+1}) = (r_1^p, r_2^p, \ldots, r_n^p)$ ([6]; Remark 1.5). When $A$ is a general $\mathbb{Z}$-algebra, then the formula is more complicated and found in ([5]; Lemma 1.4). The symbol “Frob” will denote the map $x \mapsto x^p$ for $x \in R$,
where $R$ is a ring of characteristic $p > 0$ to distinguish it from the similar map $F$ as above. We note that the set of maps $F$ with all $n \geq 0$ collate to define a ring homomorphism:

$$F : W(A) \to W(A),$$

which we again call the Witt-Frobenius map. There are also additive Verschiebung maps:

$$V : W_{p^n}(A) \to W_{p^{n+1}}(A)$$

defined by $V(a_0, \ldots, a_n) = (0, a_0, \ldots, a_n)$. For an $R$-module $M$ for an $F_p$-algebra $R$, denote by $R^{(n)} \otimes_R M$ the $n$-th Frobenius-twist of $M$ defined by the rule: $r \otimes m = rs^{p^n} \otimes m$ for $r, s \in R$ and $m \in M$. Here, we identify $R^{(n)}$ with $R$ as rings. For an $R$-algebra $S$, we define the relative Frobenius map

$$R((1)) \otimes_R S \to S$$

by the rule: $r \otimes s \mapsto rs^p$. We use the notation Frac$(A)$ for the total ring of fractions for a ring $A$.

**Definition 2.2.** Let $p > 0$ be a prime number. Say that a ring $A$ has mixed characteristic $p > 0$, or $p$-torsion free, if $p$ is a nonzero divisor in $A$.

**Definition 2.3.** An $F_p$-algebra $B$ is perfect (resp. semiperfect), if the Frobenius map Frob : $B \to B$ is bijective (resp. surjective).

Suppose that $B$ is a perfect $F_p$-domain which is not a field. Then it is seen that $B$ is not Noetherian.

**Definition 2.4 ([1]).** Let $A$ be an integral domain. The absolute integral closure of $A$, denoted by $A^+$, is defined to be the integral closure of $A$ in a fixed algebraic closure of the field of fractions of $A$.

### 3. Preliminaries for the proof of the main theorem

First, we construct a $p$-torsion free big ring and use it to construct an almost Cohen-Macaulay algebra whose basic properties will be discussed in the next section. One requires separate treatments in the unramified and ramified cases. Let $k$ be a perfect field of characteristic $p > 0$.

- Let $(R, m)$ be a complete regular local ring of mixed characteristic $p > 0$ with residue field $k$ such that $p \notin m^2$. Then we can present $R$ as the ring:

$$W(k)[[x_2, \ldots, x_d]].$$

Fix a parameter $\pi$ of the valuation ring $W(k)$. We define

$$R_{p^\infty} := \bigcup_{n>0} R_{p^n},$$

where

$$R_{p^n} := W(k)[[\pi_n]][[x_2^{p^{-n}}, \ldots, x_d^{p^{-n}}]]$$

and $\{\pi_n\}_{n \in \mathbb{N}}$ and $\{x_i^{p^{-n}}\}_{n \in \mathbb{N}}$ are fixed sequences of elements contained in $R^+$ such that $\pi_0 = \pi$ and $\pi_{n+1} = \pi_n$.

- Let $(R, m)$ be a complete regular local ring of mixed characteristic $p > 0$ with residue field $k$ such that $p \in m^2$. Let $T = W(k)[[t_1, \ldots, t_d]]$. Then we can present $R$ as the quotient ring:

$$W(k)[[t_1, \ldots, t_d]]/(p - G)$$
for some element $G \in \mathfrak{m}_T^2/pT$ for the maximal ideal $\mathfrak{m}_T$ of $T$. We define

$$R_{p^\infty} := \bigcup_{n>0} R_{p^n},$$

where

$$R_{p^n} := W(k)[[t_1^{p^n}, \ldots, t_d^{p^n}]]/(p - G).$$

In both unramified and ramified cases, the ring $R_{p^\infty}$ is a non-Noetherian and normal domain with a unique maximal ideal and it is contained in $R^+$ and integral and faithfully flat over $R$, because it is obtained as the filtered colimit of module-finite extensions of regular local rings $R_{p^n}$. Moreover, $\text{Frob} : R_{p^\infty}/pR_{p^\infty} \to R_{p^\infty}/pR_{p^\infty}$ is surjective, but not injective. This is a typical example of Witt-perfect ring, which we will discuss later.

**Remark 3.1.** Assume that $A \to B$ is a finite étale extension. If $A/pA$ is a (semi)perfect $F_p$-algebra, then $B/pB$ is also (semi)perfect. Indeed, this follows by remarking that the map $A/pA \to B/pB$ is finite étale and that the relative Frobenius map

$$(A/pA)^{(1)} \otimes_{A/pA} B/pB \to (B/pB)^{(1)}$$

is an isomorphism. This fact holds more generally for an absolutely flat map $A \to B$ (see [15] for its definition and [7]; Theorem 3.5.13, or [13]; Lemma 3.1.4 for a proof). [13] contains a comprehensive discussion on the tensor category of finite étale algebras over a fixed base ring. The almost purity theorem extends the above fact under almost étale extension of rings.

In order to construct big Cohen-Macaulay algebras from almost Cohen-Macaulay algebras, we need to use partial algebra modifications, which we explain below. The following discussion is taken from [11].

Let $R$ be a ring and let $M$ be an $R$-module. Let $M[X_1, \ldots, X_k] := M \otimes_R R[X_1, \ldots, X_k]$. For an integer $N > 0$, let $M[X_1, \ldots, X_N] \leq N$ be the $R$-submodule spanned by the elements $uX_1^a \cdots X_k^a$ such that $\sum_i a_i \leq N$. Now fix an integer $k \geq 0$ and let $x_{k+1}u_{k+1} = \sum_{i=1}^k x_iu_i$ be a relation for $u_i \in M$ and a fixed system of elements $x_1, \ldots, x_k$ (which we will later take to be part of a system of parameters of a local ring). A partial algebra modification of $M$ is defined as an $R$-module map:

$$M \to M' := \frac{M[X_1, \ldots, X_k] \leq N}{FM[X_1, \ldots, X_k] \leq N-1}$$

with $F := (u_{k+1} - \sum_{i=1}^k x_iX_i)$. This process implies that the map $M \to M'$ trivializes the relation $x_{k+1}u_{k+1} = \sum_{i=1}^k x_iu_i$ and it is expected that it will be useful for constructing big Cohen-Macaulay algebras. Let us specialize this construction. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $x_1, \ldots, x_d$ a system of parameters of $R$ and let $T = M$ be an $R$-algebra. Applying the above construction, we get a sequence of partial algebra modifications:

$$T \to M_1 \to M_2 \to \cdots \to M_r$$

We say that the above sequence is bad, if $\mathfrak{m}M_r = M_r$. Here is a key lemma for the proof of the main theorem.

**Lemma 3.2** ([11]; Lemma 5.1). Let $M$ be a module over a local ring $(R, \mathfrak{m})$, and let $x_1, \ldots, x_d$ be a system of parameters for $R$. Suppose that $T$ is an $R$-algebra, that $c$ is a non-zero divisor of $T$, while there is a $R$-linear map $\alpha : M \to T[c^{-1}]$. Let $M \to M'$ be a partial algebra modification of $M$ with respect to an initial segment of $x_1, \ldots, x_d$, with degree bound $D$. Suppose that for every relation $x_{k+1}t_{k+1} = \sum_{i=1}^k x_it_i$, $t_i \in T$, we have that $ct_{k+1} \in (x_1, \ldots, x_k)T$. Finally, suppose that
\( \alpha(M) \subseteq c^{-N}T \) for some integer \( N > 0 \). Then the map \( \alpha : M \to T[c^{-1}] \) fits into the commutative square:
\[
\begin{array}{ccc}
T[c^{-1}] & \longrightarrow & T[c^{-1}] \\
\alpha \uparrow & & \beta \uparrow \\
M & \longrightarrow & M'
\end{array}
\]
in which \( \beta : M' \to T[c^{-1}] \) is an \( R \)-linear map with image contained in \( c^{-(ND+D+N)}T \).

4. Almost purity theorem and almost Cohen-Macaulay algebras

In this section, we establish the existence of big Cohen-Macaulay algebras in the case that \( R \to S \) is module-finite and \( R[\frac{1}{p}] \to S[\frac{1}{p}] \) is étale and \( R = W(k)[[x_2, \ldots, x_d]] \) or \( R = W(k)[[t_1, \ldots, t_d]]/(p - G) \). We shall recall basic part of almost ring theory, for which we follow the treatment \( [6] \) and then give a definition of almost Cohen-Macaulay algebra based on it. The advantage in working with almost ring theory by Davis and Kedlaya is that their approach is quite flexible and treats a general situation, so that it fits well into the search of big Cohen-Macaulay algebras in mixed characteristic. Some tricks to deal with the ramified case may be found in the monograph \( [8] \). As it is still undergoing revision, we give complete proofs of key facts for the convenience of readers. Note that \( [8] \) contains a detailed proof of some special cases of the Direct Summand Conjecture in the context of logarithmic geometry. We should emphasize here that we consider the trivial \( m \)-adic topology to construct a certain singular scheme, as if it were without any singular points. In general, if \((X, \mathcal{M}_X)\) is a regular log scheme, then the underlying space \( X \) is known to be normal and Cohen-Macaulay (a proof of this fact may be found in \( [8] \)).

**Definition 4.1.** Let \( B \) be a \( p \)-torsion free ring.

(i) A \( p \)-ideal \( I \) of \( B \) is an ideal such that \( I^m \subseteq pB \) for some \( m > 0 \).

(ii) A \( B \)-module \( M \) is almost zero, if \( IM = 0 \) for any \( p \)-ideal \( I \) of \( B \).

(iii) A \( B \)-module \( M \) is almost finite projective, if for any given \( p \)-ideal \( I \subseteq B \), there exists a finite free \( B \)-module \( F \) for which there exists an \( B \)-module map \( M \to F \to M \) given by multiplication by some \( t \in B \) and such that \( I \subseteq tB \).

We need the separatedness of rings or modules with respect to the \( m \)-adic topology to construct a regular sequence out of an almost regular sequence.

**Lemma 4.2.** Assume that \( I \) is an ideal of a ring \( R \) and \( S \) is an \( R \)-algebra. Assume further that \( R \) is a \( p \)-torsion free ring for a fixed prime number \( p > 0 \), \( R \) is \( I \)-adically separated, and \( R \to S \) is a torsion free and almost finite projective extension. Then \( S \) is \( I \)-adically separated.

**Proof.** Fix a \( p \)-ideal \( I = (p) \). Then there exists an element \( t \in R \) together with a finite free \( R \)-module \( F \) and an \( R \)-module homomorphism \( S \to F \to S \) which is multiplication by \( t \). Moreover, \( I \subseteq tR \). Then one can write \( p = at \) for some \( a \in R \). By composing \( S \to F \to S \) with the map \( S \to S \), we may assume that \( S \to F \to S \) is multiplication by \( p \). Consider the composite map:
\[
\bigcap_{n>0} I^n S \to \bigcap_{n>0} I^n F \to \bigcap_{n>0} I^n S
\]
which is multiplication by \( p \). Since the \( R \)-module \( F \) is finite free and \( R \) is \( I \)-adically separated, we have \( \bigcap_{n>0} I^n F = 0 \), which implies that \( p \cdot \bigcap_{n>0} I^n S = 0 \). On the other hand, \( R \to S \) is torsion
free and $R$ is $p$-torsion free by assumption, $S$ is $p$-torsion free. Therefore, $\bigcap_{n>0} I^n S = 0$ and $S$ is $I$-adically separated.

First, we recall the definition of Witt-perfect rings.

**Definition 4.3** (Davis-Kedlaya). Let $A$ be a commutative ring and fix a prime number $p$. Then $A$ is called Witt-perfect (or $p$-Witt-perfect), if the Witt-Frobenius map

$$F : W_p^n (A) \to W_p^{n-1} (A)$$

is surjective for all $n \geq 2$.

Note that the above definition makes sense for any ring $A$, regardless of the characteristic of $A$. Moreover, if $A$ has prime characteristic $p > 0$, the Witt-Frobenius map $F$ in the definition coincides with the lifting of the usual Frobenius $\text{Frob} : A \to A$. A proof of this together with a concrete description of $F$ in the general case is found in ([6]; Lemma 1.4). Since the above definition is not so intuitive, we recall the following more useful criterion.

**Lemma 4.4** (Davis-Kedlaya). Let $B$ be a $p$-torsion free ring. Then the following conditions are equivalent:

(i) $B$ is a Witt-perfect ring.

(ii) $B$ satisfies the following properties:

- The Frobenius map on $B/pB$ is surjective.
- There exist $r, s \in B$ such that $r^p \equiv -p \mod psB$ and $sN \in pB$ for some $N > 0$.

*Proof.* This is ([6]; (ii) $\iff$ (xviii) + (xvii) of Corollary 3.3). □

We state the almost purity theorem due to Davis and Kedlaya ([6]; Theorem 5.2) based on the Witt-perfect condition. This theorem will play a crucial role later in the construction of almost Cohen-Macaulay algebras.

**Theorem 4.5** (Almost purity theorem). Let $B$ be a $p$-torsion free Witt-perfect ring which is integrally closed in $B\{\frac{1}{p}\}$. Let $B \to C$ be a ring homomorphism such that $B\{\frac{1}{p}\} \to C\{\frac{1}{p}\}$ is finite étale and let $C'$ be the integral closure of $B$ in $C\{\frac{1}{p}\}$.

(i) The ring $C'$ is also Witt-perfect.

(ii) $C'$ is an almost finite projective $B$-module.

We shall say that $B \to C$ satisfying the assumption of Theorem 4.5 is an almost étale extension. There is a simple criterion to check when a $p$-torsion free ring is Witt-perfect. We will use this criterion in the following discussion.

**Theorem 4.6** (Davis-Kedlaya). Let $B$ be a $p$-torsion free algebra over a Witt-perfect ring $A$. Then $B$ is Witt-perfect if and only if the Frobenius map on $B/pB$ is surjective.

*Proof.* Since $A$ is Witt-perfect by assumption, this is immediate from Lemma 4.4. □

**Definition 4.7.** A $p$-torsion free ring $B$ is called $p$-big, if $B$ contains sequences of elements $\{\pi_n\}_{n \in \mathbb{N}}$ and $\{u_n\}_{n \in \mathbb{N}}$ such that $\pi_0 = p$, every $u_n$ is a unit of $B$ and $\pi_n u_n = \pi_{n+1}$ for all $n \in \mathbb{N}$.

For later use, we construct a certain valuation ring. Let $k$ be a perfect field of characteristic $p > 0$ and choose a sequence $\{\pi_n\}_{n \in \mathbb{N}}$ as above in $W(k)^+$ and $u_n = 1$ for all $n \in \mathbb{N}$. We define

$$V_{p^\infty} := \bigcup_{n>0} W(k)[\pi_n],$$
In this situation, we may write \( \pi_n \) as \( p^{\pi_n} \). The next lemma is essential for dealing with the unramified case.

**Lemma 4.8.** Let the notation be as above. Then \( V_{p^\infty} \) is a Witt-perfect, non-discrete valuation ring of Krull dimension one.

**Proof.** From the construction, \( V_{p^\infty} \) is the ascending union of discrete valuation rings \( \mathbf{W}(k)[[\pi_n]] \) and its unique maximal ideal is not finitely generated. Thus, \( V_{p^\infty} \) is a non-discrete valuation ring of Krull dimension one. Therefore, it remains to show that it is Witt-perfect. For this, we need to show that the second condition of Lemma 4.4 (ii) holds on \( V_{p^\infty} \). By construction, the Frobenius map:

\[
\text{Frob} : V_{p^\infty}/pV_{p^\infty} \rightarrow V_{p^\infty}/pV_{p^\infty}
\]

is surjective, so it only suffices to verify the second condition of Lemma 4.4 (ii). First, assume \( p > 2 \). Then the equality \((-\pi_1)^p = -p\) satisfies our demand. Next, assume \( p = 2 \). In this case, we choose \( r = \pi_1, s = 2 \) and \( N = 1 \). This finishes the proof that \( V_{p^\infty} \) is Witt-perfect. \( \square \)

The valuation ring \( V_{p^\infty} \) is called deeply ramified in [7] and (non \( p \)-adically complete) perfectoid in [19]. We prove that \( R_{p^\infty} \) has the desired properties.

**Proposition 4.9.** \( R_{p^\infty} \) is a Witt-perfect, normal, and balanced big Cohen-Macaulay \( R \)-algebra.

**Proof.** By the construction, \( R_{p^\infty} \) is a normal domain and faithfully flat over a regular local ring \( R \). Hence \( R_{p^\infty} \) is a balanced big Cohen-Macaulay \( R \)-algebra. Since \( \text{Frob} : R_{p^\infty}/pR_{p^\infty} \rightarrow R_{p^\infty}/pR_{p^\infty} \) is surjective, it suffices to check the second condition of Lemma 4.4 (ii). We need to consider the unramified and ramified cases separately.

First, assume that \( p \notin \mathfrak{m}^2 \). Then \( R_{p^\infty} \) contains \( V_{p^\infty} \) as a subring. Then apply Theorem 4.6 to deduce that \( R_{p^\infty} \) is Witt-perfect.

Next, assume that \( p \in \mathfrak{m}^2 \). We prove that there exist an element \( \pi \in R_{p^\infty} \) and a unit \( u \in R_{p^\infty}^\times \) such that

\[
\pi^p = pu.
\]

Then we may write \( p = \sum_{i=1}^n b_ib'_i \) with \( b_i, b'_i \in \mathfrak{m} \). Let \( \mathfrak{m}_{p^\infty} \) be the unique maximal ideal of \( R_{p^\infty} \). Since the Frobenius map on \( R_{p^\infty}/pR_{p^\infty} \) is surjective, we may find \( d_i, d'_i \in R_{p^\infty} \) and \( c_i, c'_i \in \mathfrak{m}_{p^\infty} \) such that

\[
b_i = c_i^p + pd_i, \quad b'_i = c_i'^p + pd'_i.
\]

Substituting these equations into \( p = \sum_{i=1}^n b_i b'_i \), we obtain \( p(1-h) = \sum_{i=1}^n c_i^p c_i'^p \) with the property that \( 1-h \in R_{p^\infty}^\times \). One can write \( \sum_{i=1}^n c_i^p c_i'^p = f^p + pg \) for some \( f, g \in \mathfrak{m}_{p^\infty} \). Hence we have \( p(1-h) = f^p + pg \) and thus, \( p(1-g-h) = f^p \). So setting \( \pi := f \) and \( u := 1-g-h \) will suffice. Moreover, letting \( \pi \) be as in (4.1), we may find some \( \pi', b \in R_{p^\infty} \) such that \( \pi'^p = \pi + pb \). Set \( u' := 1 + \pi^{-1}pb \). Then \( u' \in R_{p^\infty}^\times \) and

\[
\pi'^p = pu'.
\]

Now to show that \( R_{p^\infty} \) is Witt-perfect, we need to find elements \( r, s \in R_{p^\infty} \) as demanded in the second condition of Lemma 4.4 (ii). We choose \( s = p \) and \( N = 1 \). So it remains to determine \( r \). Since \( p\pi^{-1} \in R_{p^\infty} \), as in the deduction of the equality (4.2), we may find \( z \in R_{p^\infty} \) and \( u'' \in R_{p^\infty}^\times \) such that

\[
z^p = p\pi^{-1}u''.
\]

Set \( t := (\pi'z)^p\pi^{-1} \), where \( \pi' \) is as in (4.2). We claim that \( t \in R_{p^\infty}^\times \). Indeed,

\[
t = \frac{(\pi'z)^p}{p} = \frac{pu'u''}{p} = u'u'' \in R_{p^\infty}^\times ,
\]
hence the claim. We also have $pt = (\pi' z)^p$. By the surjectivity of the Frobenius map on $R_{p^\infty}/pR_{p^\infty}$, we have $w^p \equiv -t^{-1} \pmod p$. Then this transforms into $tw^p \equiv -1 \pmod p$, which gives $ptw^p \equiv -p \pmod {p^2}$ and $(\pi' zw)^p \equiv -p \pmod {p^2}$. Hence it suffices to set $r := \pi' zw$.

**Lemma 4.10.** The algebra $R_{p^\infty}$ is $p$-big; that is, it contains sequences $\{\pi_n\}$ and $\{u_n\}$ such that $\pi_0 = p$, $u_n$ is a unit of $R_{p^\infty}$ and $\pi_n^{p^i} = \pi_n u_n$. In particular, $I_n := (\pi_n)$ is a $p$-ideal.

**Proof.** When $R$ is unramified, this was already shown, as it contains a $p$-big valuation ring $V_{p^\infty}$. When $R$ is ramified, we may put $\pi_0 := p$ and $\pi_1 := \pi'$, where $\pi'$ is as in Proposition 4.9. One may proceed further to define a sequence of elements $\{\pi_n\}$ with the required properties, showing that $R_{p^\infty}$ is $p$-big. \qed

The following definition of almost Cohen-Macaulay algebras is slightly different from the one given in [17]. In fact, the $m$-adic separatedness is not assumed in [17]. Our definition has the advantage that these conditions are sufficient to construct a big Cohen-Macaulay algebra in mixed characteristic.

**Definition 4.11.** Let $B$ be a $p$-torsion free $p$-big algebra over a Noetherian local ring $(R, m)$. Then $B$ is an **almost Cohen-Macaulay $R$-algebra**, if the following conditions are satisfied:

- $B$ is separated in the $m$-adic topology.
- There is a system of parameters $x_1, \ldots, x_d$ of $R$ such that
  \[
  \begin{pmatrix}(x_1, \ldots, x_i) : B & x_{i+1} \\
  (x_1, \ldots, x_i)
  \end{pmatrix}
  \]
  is an almost zero $B$-module for all $i = 0, \ldots, d - 1$ in the above sense.

In particular, $mB \neq B$ from the definition. Once an almost Cohen-Macaulay algebra has been constructed, we apply the partial algebra modifications as discussed previously, to construct a big Cohen-Macaulay algebra.

**Proposition 4.12.** Suppose that $B$ is $p$-torsion free and an almost Cohen-Macaulay algebra over a local ring $(R, m)$. Assume that $C$ is a $B$-algebra such that $B \to C$ is a torsion free and almost finite projective extension. Then the following assertions hold:

(i) $C$ is almost Cohen-Macaulay.

(ii) There exists a $C$-algebra $D$ such that $D$ is a big Cohen-Macaulay $R$-algebra.

**Proof.** (i): The condition on almost Cohen-Macaulay property forces $B$ to be $m$-adically separated. Hence $C$ is $m$-adically separated in view of Lemma 4.2. Moreover, $B$ is $p$-big. Fix a system of parameters $(x_1, \ldots, x_d)$ of $R$ which is an almost regular sequence on $B$. As our $p$-ideal, we choose $I_n = (\pi_n)$ and there exists a finite free $S$-module $F_n$ such that the composite map $C \to F_n \to C$ is multiplication by $t_n \in B$ with the property that $I_n \subset t_n B$ in view of the definition of almost projectivity. We may further take $t_n$ to generate a $p$-ideal by replacing it with some multiple. Since $F_n$ is $B$-free, it is almost Cohen-Macaulay. Applying the functor $(-) \otimes_C C/(x_1, \ldots, x_i)$, we get

\[
C/(x_1, \ldots, x_i) \xrightarrow{f} F_n/(x_1, \ldots, x_i) \xrightarrow{g} C/(x_1, \ldots, x_i).
\]

Then since $\text{Ker}(f) \subset \text{Ker}(g \circ f)$ and $t_n \cdot \text{Ker}(g \circ f) = 0$, we have $t_n \cdot \text{Ker}(f) = 0$. Consider the commutative diagram:

\[
\begin{array}{ccc}
C/(x_1, \ldots, x_i) & \xrightarrow{f} & F_n/(x_1, \ldots, x_i) \\
\downarrow_{x_{i+1}} & & \downarrow_{x_{i+1}} \\
C/(x_1, \ldots, x_i) & \xrightarrow{f} & F_n/(x_1, \ldots, x_i)
\end{array}
\]
Since $F_n$ is almost Cohen-Macaulay, we have

$$t_m \cdot \text{Ker} \left( F_n/(x_1, \ldots, x_i) \to F_n/(x_1, \ldots, x_i) \right) = 0$$

for any $m > 0$. So the equality $t_n \cdot \text{Ker}(f) = 0$, together with the snake lemma implies the following:

$$t_m t_n \cdot \left( (x_1, \ldots, x_i) : C \cdot x_{i+1} \right) \subset (x_1, \ldots, x_i).$$

As both $m$ and $n$ may be taken arbitrarily large and each $t_n$ generates a $p$-ideal, we conclude that $C$ is almost Cohen-Macaulay.

(ii): We shall apply partial algebra modifications. We list the data of $C$ necessary to complete the proof.

- $C$ is $m$-adically separated and $p$-torsion free.
- $(x_1, x_2, \ldots, x_d)$ forms an almost regular sequence on $C$ in the sense defined just above.

Then we may form the diagram consisting of sequences of partial algebra modifications with respect to a fixed system of parameters $(x_1, x_2, \ldots, x_d)$:

$$
\begin{array}{cccccc}
C^{[\frac{1}{p}]} & C^{[\frac{1}{p}]} & \cdots & C^{[\frac{1}{p}]} \\
\uparrow & \uparrow & \cdots & \uparrow \\
C & M_1 & \cdots & M_r
\end{array}
$$

by applying Lemma 3.2. Assume that this sequence is bad. Then we may utilize the properties of $C$ stated as above and argue as in the proof of ([11]; Theorem 5.2) to derive a contradiction. Hence $C$ maps to a big Cohen-Macaulay algebra, as desired. \hfill \Box

We are now ready to prove the main theorem of this article.

**Theorem 4.13.** Let $R$ be a regular local ring of mixed characteristic $p > 0$ and let $S$ be a torsion free module-finite $R$-algebra such that the localization $R^{[\frac{1}{p}]} \to S^{[\frac{1}{p}]}$ is finite étale. Then $S$ has a balanced big Cohen-Macaulay algebra.

**Proof.** First, there exists a flat local extension $(R, m) \to (R', m')$ such that $mR = m'$ and the residue field $k$ of $R'$ is perfect. In particular, $\dim R = \dim R'$ and $R'$ is regular local. Now let $\widehat{R'}$ be the $m'$-adic completion of $R'$. Then $\widehat{R'}$ is isomorphic to $R = W(k)[[x_2, \ldots, x_d]]$ or $R = W(k)[[t_1, \ldots, t_d]]/(p - G)$. By étale base change, $\widehat{R'} \to \widehat{R'} \otimes_R S$ is finite étale after adjoining $\frac{1}{p}$. Since $S \to \widehat{R'} \otimes_R S$ is a faithfully flat map between rings of the same Krull dimension, if $\widehat{R'} \otimes_R S$ maps to a big Cohen-Macaulay algebra $\mathcal{B}(\widehat{R'} \otimes_R S)$ in the sense that a system of parameters of $R$ maps to a regular sequence in $\mathcal{B}(\widehat{R'} \otimes_R S)$, then $\mathcal{B}(\widehat{R'} \otimes_R S)$ is a big Cohen-Macaulay $S$-algebra. Then to prove the theorem, we may replace the original $R \to S$ with the extension:

$$\widehat{R'} \hookrightarrow (\widehat{R'} \otimes_R S).$$

We recall that we have constructed Witt-perfect rings (see Proposition 4.9 for its properties):

$$R_{p^\infty} = \bigcup_{n>0} W(k)[[\tau_n]][[x_2^{p^n}, \ldots, x_d^{p^n}]]$$

and

$$R_{p^\infty} = \bigcup_{n>0} W(k)[[t_1^{p^n}, \ldots, t_d^{p^n}]]/(p - G).$$
Since \( R[\frac{1}{p}] \to S[\frac{1}{p}] \) is étale, \( R_p[\frac{1}{p}] \to (R_p \otimes_R S)[\frac{1}{p}] \) is also finite étale. In particular, \( (R_p \otimes_R S)[\frac{1}{p}] \) is a normal ring. We define

\[
S_{p^\infty} = \text{the integral closure of } R_{p^\infty} \text{ in } (R_p \otimes_R S)[\frac{1}{p}].
\]

Then since \( (R_p \otimes_R S)[\frac{1}{p}] \) is normal, we find that \( S_{p^\infty} \) is the normalization of \( R_{p^\infty} \otimes_R S \) in the total ring of fractions. Therefore, \( S_{p^\infty} \) will be a normal \( S \)-algebra. By applying both Theorem 4.13 and Proposition 4.12, we conclude that \( S_{p^\infty} \) maps to a big Cohen-Macaulay \( S \)-algebra, whose \( m \)-adic completion is a balanced big Cohen-Macaulay \( S \)-algebra ([5]; Corollary 8.5.3), which completes the proof of the theorem.

As a corollary, we have the following.

**Corollary 4.14.** The normal \( S \)-algebra \( S_{p^\infty} \) is almost Cohen-Macaulay.

We deduce another corollary.

**Corollary 4.15.** Let \( R \) be a Noetherian regular domain and let \( S \) be a torsion free module-finite \( R \)-algebra. Assume that \( R \) is \( p \)-torsion free and the localization \( R[\frac{1}{p}] \to S[\frac{1}{p}] \) is finite étale for some prime integer \( p > 0 \). Then \( R \to S \) splits as an \( R \)-module homomorphism.

**Proof.** First, we observe that \( R \to S \) splits if and only if \( R_p \to S_p := R_p \otimes_R S \) splits as an \( R_p \)-homomorphism for all prime ideals \( p \in \text{Spec } R \). If \( p \not\in \mathfrak{p} \), then it follows that \( R_p \to S_p \) is étale. Therefore, this map splits for an obvious reason. So assume that \( p \in \mathfrak{p} \). Then replacing the original extension \( R \to S \) with \( R_p \to S_p \), we may assume that \( R \) is a regular local ring of mixed characteristic \( p \). In particular, \( S \) is a torsion free module-finite \( R \)-algebra and \( S \) is semilocal. Then from Theorem 4.13 it follows that \( S \) maps to a big Cohen-Macaulay algebra. Then by ([10]; Theorem 1), \( R \to S \) splits if and only if we have \((x_1 \ldots x_d)^k \not\in (x_1^{k+1}, \ldots, x_d^{k+1})S\) for a regular system of parameters \( x_1, \ldots, x_d \) of \( R \) and every integer \( k > 0 \). To prove the corollary by contradiction, assume that \( (x_1 \ldots x_d)^k \in (x_1^{k+1}, \ldots, x_d^{k+1})S \). Then mapping this relation to the big Cohen-Macaulay \( S \)-algebra yields a contradiction to the regularity of the sequence \( x_1, \ldots, x_d \). Therefore, \( R \to S \) must split, which completes the proof of the corollary.

Finally, we mention that it is shown in [4] that there is a complete local ring with no small Cohen-Macaulay algebra using Witt-vector cohomology.

5. Remark

Assume that \( R \to S \) is a torsion free module-finite extension of local normal domains of mixed characteristic \( p > 0 \) such that \( R[\frac{1}{p}] \to S[\frac{1}{p}] \) is étale. We define

\[
R \to R^{(p)} \to R^+
\]

to be a unique extension such that \( R[\frac{1}{p}] \to R^{(p)}[\frac{1}{p}] \) is the maximal étale extension. Then we have \( S \subset R^{(p)} \) and \( R^{(p)} \) is a normal domain. Moreover, the main result of this paper asserts that \( R^{(p)} \) maps to a big Cohen-Macaulay algebra. Now we prove that the ring \( R^{(p)} \) is a large integral extension of \( R \). In fact, it is shown that the quotient \( R^{(p)}/pR^{(p)} \) is semiperfect.

**Proposition 5.1.** Let the notation be as above. The Frobenius map on \( R^{(p)}/pR^{(p)} \) is surjective. Moreover, we have \( p^{\frac{1}{p}} \in R^{(p)} \) and the Frobenius endomorphism on \( R^{(p)}/pR^{(p)} \) induces a ring isomorphism:

\[
R^{(p)}/p^R R^{(p)} \cong R^{(p)}/pR^{(p)}.\]
Proof. Since $R$ is a complete local domain, it is henselian and so $R^{(p)}$ is a local domain. We first prove that the Frobenius endomorphism on $R^{(p)}/pR^{(p)}$ is surjective. Let us pick $b \in R^{(p)}$. Consider a polynomial:

$$f(X) := X^{p^2} - pX - b \in R^{(p)}[X].$$

Then we have $f'(X) = p^2X^{p^2-1} - p = p(pX^{p^2-1} - 1)$ and the integral extension

$$R^{(p)} \to R^{(p)}[X]/(f(X))$$

is finite étale after adjoining $\frac{1}{p}$, because of the following fact:

$$pX^{p^2-1} - 1 \in (R^{(p)}[X]/(f(X)))^\times,$$

where the right-hand side is the set of unit elements of a local ring. We can find an element $a \in R^+$ such that $f(a) = 0$. Then there is a commutative diagram:

$$
\begin{array}{ccc}
R^{(p)} & \longrightarrow & R^{(p)}[X]/(f(X)) \\
\| & & \downarrow \\
R^{(p)} & \longrightarrow & R^{(p)}[a] \longrightarrow R^+
\end{array}
$$

and hence, the extension $R \to R^{(p)}[a]$ becomes étale after adjoining $\frac{1}{p}$. Since $R[\frac{1}{p}] \to R^{(p)}[\frac{1}{p}]$ is the maximal étale extension contained in $R^+[\frac{1}{p}]$, it follows that we have $a \in R^{(p)}$. Finally, we have

$$a^{p^2} - b \equiv a^{p^2} - pa - b \equiv 0 \pmod{pR^{(p)}}.$$

Thus, $(a^p)^p \equiv b \pmod{pR^{(p)}}$.

By construction, we have $\frac{1}{p} \in R^{(p)}$. Since $R^{(p)}$ is a normal domain, it is easy to show that the Frobenius map induces an isomorphism

$$R^{(p)}/p^{\frac{1}{p}}R^{(p)} \cong R^{(p)}/pR^{(p)},$$

which completes the proof of the proposition. \qed

The reader will notice that the proof given in the proposition works under the assumption that the ring $R$ is henselian.

Acknowledgement. I would like to thank B. Bhatt, R. Heitmann, and P. Roberts for their careful reading. I also thank M. Asgharzadeh, Alberto F. Boix, and G. Piepmeyer for their comments. My special thanks go to L. E. Miller, who taught the author about Witt vectors.

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