ON THE BEREZIN DESCRIPTION OF KÄHLER QUOTIENTS

Iliana Carrillo-Ibarra
Departamento de Matemáticas
Centro de Investigación y de Estudios Avanzados del IPN
Apdo. Postal 14-740, 07000, México D.F., México
iliana@math.cinvestav.mx

Hugo García-Compeán
Departamento de Física
Centro de Investigación y de Estudios Avanzados del IPN
Apdo. Postal 14-740, 07000, México D.F., México
compean@fis.cinvestav.mx

Abstract We survey geometric prequantization of finite dimensional affine Kähler manifolds. This prequantization and the Berezin’s deformation quantization formulation, as proposed by Cahen et al., is used to quantize their corresponding Kähler quotients. Equivariant formalism greatly facilitates the description.

Keywords: Berezin’s Deformation Quantization, Kähler quotients, Equivariant Formalism

1. Motivation

Chern-Simons (CS) gauge theory in 2+1 dimensions is a very interesting quantum field theory which has been very useful to describe diverse sorts of physical and mathematical systems. On the physical side, we have the fractional statistics of quasi-particles in the fractional quantum Hall effect [1], Einstein gravity in 2+1 dimensions with nonzero cosmological constant [2]. On the mathematical side it is related to beautiful mathematics like knot and link invariants [3] and to quantum groups [4]. There is also a nice relation with conformal field theory in two dimensions [3, 5]. From the quantization of this theory we also have learned
a lot of things like a very non-trivial generalization of the original Jones representations of the braid group [6].

Canonical Quantization

In 2+1 dimensions CS gauge theory is based in the Lagrangian

\[ L_{CS} = \frac{k}{4\pi} \int_M Tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \]  

(1)

where \( A \) is a Lie algebra \( (\text{Lie}(G)) \) valued gauge connection on the \( G \)-bundle \( E \) over \( M \) and where \( G \) is a compact and simple finite dimensional Lie group, e.g. \( SU(2) \). Thus \( A = \sum_{a=1}^{\text{dim} G} A^a_I dx^I \) (with \( I, J = 0, 1, 2 \)) where \( T_a \) are the generators of \( \text{Lie}(G) \) and \( k \in \mathbb{Z} \) is the level of the theory.

In the classical theory the equations of motion are given by the “flat connection” condition

\[ F(A) = dA + A \wedge A = 0 \iff F^a_{IJ} = 0. \]

(2)

While the constraints are given by the Gauss law:

\[ \frac{\delta L}{\delta A^a_0} = \epsilon_{I,J} F^a_{IJ} = 0. \]

(3)

Canonical quantization on \( M = \Sigma \times \mathbb{R} \) consists in the construction of a Hilbert space \( \mathcal{H}_\Sigma \) associated to the two-dimensional surface \( \Sigma \). The construction of \( \mathcal{H}_\Sigma \) is as follows. First, the decomposition of \( M \) allows to fix the temporal gauge \( A_0 = 0 \). In this gauge, the Poisson bracket is given by

\[ \{ A^a_I(x), \pi^b_J(y) \} = \frac{4\pi}{k} \epsilon_{I,J} \delta^{ab} \delta^2(x - y), \]

(4)

where \( \pi^a_J(x) = \frac{\partial L}{\partial \dot{A}^a_J} = A^a_J(x) \).

The phase space \( \mathcal{A} \) consist of the solutions of Eq. (2). That means all flat connections on \( \Sigma \). The incorporation of the constraints (3) leads to the moduli space of certain families of vector bundles over \( \Sigma \). This can be identified with the reduced phase space \( \mathcal{M} = \mathcal{A}/\mathcal{G} \). Pick a complex structure \( J \) on \( \Sigma \) leads to consider to \( \mathcal{M} \) as a compact Kähler manifold \( \mathcal{M}_J \) of finite dimension. This space is precisely the moduli space of certain family of holomorphic vector bundles over \( \Sigma \). The quantization of the manifold \( \mathcal{M}_J \) leads to the quantization of the Chern-Simons theory given by the Lagrangian (1). This can be constructed with the help of the Quillen determinant line bundle \( \mathcal{L} \). This is a line bundle over \( \mathcal{M}_J \) whose first Chern class \( c_1(\mathcal{L}) \) coincides with the simplectic form \( \omega_0 \).
Berezin's Quantization of Kähler Quotients

which determines the Poisson brackets \( \{ A^a(x), A^b(y) \} = \omega_0^{-1}(dA, dA) \). For the arbitrary level \( k \) the symplectic form \( \omega = k\omega_0 \) is given by the first Chern class of the \( k \)-th power of the line bundle \( \mathcal{L}^{\otimes k} \). Finally the Hilbert space \( \mathcal{H} \) is constructed from the space of \( L^2 \)-completion of holomorphic sections \( H^0_{L^2}(\mathcal{M}_J, \mathcal{L}^{\otimes k}) \) of the determinant line bundle \( \mathcal{L}^{\otimes k} \).

The quantization of the CS theory is given by the geometric quantization of the physical reduced phase space \( \mathcal{M}_J \). This was given by Axelrod, Della Pietra and Witten [6].

In the present note we use the Berezin’s deformation quantization formalism [7, 8, 9, 10, 11] to quantize the reduced phase space \((\mathcal{M}_J, \tilde{\omega})\) which can be regarded as a finite dimensional Kähler quotient space. This is a preliminary step to apply the Berezin’s formalism to quantize the more involved infinite dimensional case of Chern-Simons gauge theory in three dimensions with compact groups. One possible guide to address the infinite dimensional case would be the case of the quantization of \( \mathbb{C}P^\infty \) [12]. This will be reported in a forthcoming paper [13]. Recent interesting applications of Berezin’s quantization are found in [14, 15].

2. Geometric Quantization of Kähler Quotients

Up-stairs Geometric Quantization

We first consider the finite dimensional affine symplectic manifold \((\mathcal{A}, \omega)\) with a chosen complex structure \( J \) on \( \mathcal{A} \), invariant under affine translations [6]. This induces a Kähler structure on \( \mathcal{A} \). In order to quantize the affine Kähler space \( \mathcal{A}_J \) we consider the prequantum line bundle \( \mathcal{L} \) over \( \mathcal{A} \) with connection \( \nabla \). This connection has curvature \([\nabla, \nabla] = -i\omega \) since symplectic connection \( \omega \) is of the \((1,1)\) type and type \((0,2)\) component of the curvature vanishes. This induces a holomorphic structure on \((\mathcal{L}, \nabla)\). Consider the \( L^2 \)-completion of the subset of holomorphic sections \( \mathcal{H}_\mathcal{Q}|_J = H^0_{L^2}(\mathcal{A}_J, \mathcal{L}) \) of the prequantum Hilbert space \( H^0_{L^2}(\mathcal{A}, \mathcal{L}) \). This constitutes the Hilbert space of the quantization of \((\mathcal{A}_J, \omega)\). The Kähler quantization depends on the choice of \( J \). In the theory of geometric quantization it is impossible to choose in a natural way a Kähler polarization. That is, to choose a complex structure \( J \) for \( \omega \) has the properties of a Kähler form. Thus the Kähler polarization is not unique. In the procedure of quantization we should make sure that the final result will be independent on this complex structure \( J \) and it depends only on the underlying symplectic geometry. Thus the idea is to find a canonical identification of the \( \mathcal{H}_\mathcal{Q}|_J \) as \( J \) varies (for further details see [6]).
Down-stairs Geometric Quantization

Now we will consider symplectic quotients of finite-dimensional affine symplectic spaces. The idea is geometric quantize the reduced phase space \((M_J, \tilde{\omega})\) where \(M_J = A_J/\mathcal{G}\) is the Marsden-Weinstein quotient.

We start from \((A_J, \omega)\) with the action of the group \(\mathcal{G}\) acting on \(A_J\) by symplectic diffeomorphisms. Let \(g\) be the Lie algebra of \(\mathcal{G}\) and consider the map \(T : g \to \text{Vect}(A_J)\). Since \(\mathcal{G}\) preserves the symplectic form, the image of \(T\) is a subspace of \(\text{Vect}(A_J)\) consisting in the symplectic vector fields on \(A_J\). The co-moment map is given by \(F : g \to \text{Ham}(A_J)\) where \(\text{Ham}(A_J)\) is the space of Hamiltonian functions on \(A_J\). Take a basis of \(g\) to be \(\{L_a\}\) and we have \(T_a = T(L_a)\). \(T\) is a Lie algebra representation since \([T_a, T_b] = f_{ab}^c T_c\) with \(a, b, c = 1, \ldots, \dim \mathcal{G}\).

For each \(x \in A_J\), \(F_a(x)\) are the components of a vector in the dual space \(g^V\). That means that there is a mapping \(F : A_J \to g^V\) called the moment map. \(F^{-1}(0)\) is \(\mathcal{G}\)-invariant so one can define the symplectic quotient of \(A_J\) and \(\mathcal{G}\) as \(M_J = F^{-1}(0)/\mathcal{G}\). Thus one have \(\pi : F^{-1}(0) \to F^{-1}(0)/\mathcal{G} \equiv M_J\) where \(x \mapsto \tilde{x}\). \(M_J\) also have structure of a symplectic manifold whose symplectic structure \(\tilde{\omega}\) is given by \(\tilde{\omega}_{\tilde{x}}(\tilde{V}, \tilde{W}) = \omega_x(V, W)\) for \(\tilde{V}, \tilde{W} \in T_{\tilde{x}}A_J\).

We consider on \(A_J\) only \(\mathcal{G}\)-invariant quantities so that when restricted to \(F^{-1}(0) \subset A_J\) they are pushed-down to the corresponding objects in \(M_J\).

The prequantum line bundle can be pushed-down as follows. The symplectic action of \(\mathcal{G}\) on \(A_J\) can be lifted to \(L\) in such a way that it preserves the connection and Hermitian structure on \(L\). One may define the push-down bundle \(\tilde{L}\) by stating that its sections \(\Gamma(M_J, \tilde{L}) \equiv \Gamma(F^{-1}(0), L)^\mathcal{G}\) constitutes a \(\mathcal{G}\)-invariant subspace of the space of sections \(\Gamma(A_J, L)\). The connection also can be pushed-down and it satisfies \(\tilde{\nabla}_{\tilde{V}} \psi = \nabla_V \psi\). Meanwhile the curvature of the connection \(\tilde{\nabla}\) is \(-i\tilde{\omega}\). Thus the prequantization is given by \((\tilde{L}, \tilde{\nabla}, \langle \cdot | \cdot \rangle_{\tilde{L}})\). Here \(\langle \cdot | \cdot \rangle_{\tilde{L}}\) is the \(\mathcal{G}\)-invariant inner product \(\langle \cdot | \cdot \rangle_{\tilde{L}}^\mathcal{G}\).

3. Berezin Quantization of Kähler Quotients

The main goal of this section is to describe the Berezin’s quantization of the Kähler manifold \((M_J, \omega)\) where \(M_J\) is the Marsden-Weinstein quotient. That means we find an associative and noncommutative family of algebras \((\tilde{S}_B, \tilde{\ast}_B)\) with \(\tilde{S}_B \subset C^\infty(M_J)\) which is indexed with a real and positive parameter \(\bar{\hbar}\) which helps to recover the classic limit when \(\hbar \to 0\). Here we set \(\bar{\hbar} = 1\).
In order to do that we follow the same strategy that for the geometric quantization case of the previous section. We first Berezin quantize $(\mathcal{A}_J, \omega)$, i.e. we find an associative and noncommutative family of algebras $(\mathcal{S}_B, \ast_B)$ with $\mathcal{S}_B \subset C^\infty(\mathcal{A}_J)$ (see [8, 9]). After that we project out all relevant quantities to be $G$-invariant i.e. $\tilde{\mathcal{S}}_B \subset C^\infty(\mathcal{A}_J)^G \equiv C^\infty(F^{-1}(0)/G)$ with $F^{-1}(0) \subset \mathcal{A}_J$.

Up-Stairs Berezin’s Quantization

Let $(\mathcal{L}, \nabla, \langle \cdot | \cdot \rangle_{\mathcal{L}})$ be a prequantization of the affine Kähler manifold $\mathcal{A}_J$. The inner product $\langle \cdot | \cdot \rangle_{\mathcal{L}}$ is compatible with the connection $\nabla$ and it is defined as

$$\langle \chi | \psi \rangle_{\mathcal{L}} \equiv \int_{\mathcal{A}_J} \langle \chi | \psi \rangle \omega^n \frac{1}{n!},$$

for all $\chi, \psi \in H^0_{L^2}(\mathcal{A}_J, \mathcal{L})$ where $\langle \chi | \psi \rangle = \exp(-\Phi)\overline{\chi}\psi$ and $\Phi$ is the Kähler potential $\Phi(Z, \overline{Z}) = \sum_i Z_i \overline{Z}_i^*$. The norm of an element $\psi$ of $H^0_{L^2}(\mathcal{A}_J, \mathcal{L})$ is given by $\langle \psi | \psi \rangle_{\mathcal{L}} \equiv ||\psi||_{\mathcal{L}}^2$. As $\mathcal{A}_J$ is topologically trivial, the line bundle $\mathcal{L}$ can be identified with the trivial holomorphic line bundle whose holomorphic sections are holomorphic functions $\psi$ with Hermitian structure i.e. $\langle \psi | \psi \rangle_{\mathcal{L}} = \exp(-\Phi)|\psi|^2$. The curvature of the connection compatible with the Hermitian structure is given by $\bar{\partial}(-\Phi) = \sum_i dZ_i \wedge d\overline{Z}_i = -i\omega$. Of course the existence of a prequantization bundle implies that $\frac{i}{2\pi} \in H^2(\mathcal{A}_J, \mathbb{Z})$.

Take $Q \in \mathcal{L}_0$ and $\pi[Q] = x \in \mathcal{A}_J$ with local complex coordinates $\{Z^i, \overline{Z}^j\}$. Here $\mathcal{L}_0$ is the line bundle $\mathcal{L}$ without the zero section. Now consider $\psi \in H^0_{L^2}(\mathcal{A}_J, \mathcal{L})$ a holomorphic section such that $\psi(x) = \psi[\pi(Q)] = L_Q(\psi)Q$, where $L_Q(\psi)$ is a linear continuous functional of $\psi$.

By the Riesz theorem there is a section $e_Q \in H^0_{L^2}(\mathcal{A}_J, \mathcal{L})$ such that

$$L_Q(\psi) = \langle e_Q | \psi \rangle_{\mathcal{L}},$$

with $Q \in \mathcal{L}_0$. $e_Q$ is known in the literature as a generalized coherent state.

Now consider a bounded operator $\hat{O} : H^0_{L^2}(\mathcal{A}_J, \mathcal{L}) \rightarrow H^0_{L^2}(\mathcal{A}_J, \mathcal{L})$. Define the covariant symbol of this operator as

$$O_B(x) = \frac{\langle e_Q | \hat{O} | e_Q \rangle_{\mathcal{L}}}{||e_Q||_{\mathcal{L}}^2},$$

where $Q \in \mathcal{L}_0$ and $\pi(Q) = x$. Define the space of covariant symbols $\mathcal{S}_B = \{O_B(x), \text{ which are covariant symbols of operators } \hat{O} \}$. Each covariant symbol can be analytically continued to the open dense subset of $\mathcal{A}_J \times \mathcal{A}_J$.
in such a way \( \langle e_{Q}|e_{Q'}\rangle_{\mathcal{L}} \neq 0 \) with \( \pi(Q) = x \) and \( \pi(Q') = y \) (with local coordinates \( \{W', \overline{W}'\} \)). The obtained symbol is holomorphic in the first entry and anti-holomorphic in the second entry. This analytic continuation is reflected in the covariant symbol in the form

\[
O_{B}(Z, \overline{W}) = \frac{\langle e_{Q}|\hat{\mathcal{O}}|e_{Q'}\rangle_{\mathcal{L}}}{\langle e_{Q}|e_{Q'}\rangle_{\mathcal{L}}}. \tag{8}
\]

The operator \( \hat{\mathcal{O}} \) can be recovered from its symbol in the form

\[
\hat{\mathcal{O}} \psi(Z) = \langle e_{Q}|\hat{\mathcal{O}}|\psi\rangle_{\mathcal{L}} Q. \tag{9}
\]

The consideration of the completeness condition \( 1 = \int_{A_{J}} |e_{Q}| \langle e_{Q}| \exp (-\Phi(Z, \overline{Z})) \frac{\omega^{n}}{n!}(Z, \overline{Z}) \rangle Q \) leads to

\[
\hat{\mathcal{O}} \psi(Z) = \int_{A_{J}} O_{B}(Z, \overline{W}) B_{Q}(Z, \overline{W}) \psi(W) \exp (-\Phi(W, \overline{W})) \frac{\omega^{n}}{n!}(W, \overline{W}) Q, \tag{10}
\]

where \( \psi(W) = \langle e_{Q'}|\psi\rangle_{\mathcal{L}} \), \( \pi(Q) = x \), and \( B_{Q}(Z, \overline{W}) \equiv \langle e_{Q}|e_{Q'}\rangle_{\mathcal{L}} \). \( B_{Q}(Z, \overline{W}) \) is the generalized Bergman kernel.

In order to connect this global description with the (local) standard Berezin formalism, it is important to set a dense open subset \( U_{J} \) of the affine space \( A_{J} \). Then there is a holomorphic section \( \psi_{0} : U_{J} \to \mathcal{L}_{0} \) and a holomorphic function \( \phi : U_{J} \to \mathbf{C} \) such that \( \psi(x) = \phi(x) \psi_{0}(x) \) with \( x \in U_{J} \subset A_{J} \). Define the map \( U_{J} \to H_{L^{2}}^{0}(U_{J}, \mathcal{L}) \) such that \( x \mapsto \phi_{x} \). Let \( \phi_{x} \) and \( \phi'_{x} \) be two elements of \( H_{L^{2}}^{0}(U_{J}, \mathcal{L}) \) then

\[
\phi'(x) = \langle \phi'|\phi_{x}\rangle_{U_{J}} = \int_{U_{J}} \phi'(y) \overline{\phi_{x}}(y) |\psi_{0}|^{2}(y) \exp (-\Phi(y)) \frac{\omega^{n}}{n!}(y), \tag{11}
\]

where \( \langle \cdot | \cdot \rangle_{U_{J}} \) is the inner product in \( H_{L^{2}}^{0}(U_{J}, \mathcal{L}) \).

Let \( \hat{\mathcal{O}} \) be a bounded operator on \( H_{L^{2}}^{0}(A_{J}, \mathcal{L}) \). Define the corresponding operator \( \hat{\mathcal{O}}_{0} \) acting on \( H_{L^{2}}^{0}(U_{J}, \mathcal{L}) \) by \( \hat{\mathcal{O}} \psi = [\hat{\mathcal{O}}_{0} \phi] \psi_{0} \) where \( \psi \in H_{L^{2}}^{0}(A_{J}, \mathcal{L}) \) and \( \psi = \phi \psi_{0} \) on \( U_{J} \).

The analytic continuation of the covariant symbol when restricted to \( U_{J} \times U_{J} \) is given by

\[
O_{B(0)}(Z, \overline{W}) = \frac{\langle \phi_{x} \psi_{0}|\hat{\mathcal{O}}_{0}|\phi_{y} \psi_{0}\rangle_{U_{J}}}{\langle \phi_{x} \psi_{0}|\phi_{y} \psi_{0}\rangle_{U_{J}}} = \frac{\langle \phi_{x}|\hat{\mathcal{O}}_{0}|\phi_{y}\rangle_{U_{J}}}{\langle \phi_{x}|\phi_{y}\rangle_{U_{J}}}. \tag{12}
\]

The function \( O_{B(0)}(Z, \overline{Z}) \in C^\infty(U_{J}) \) is called the **covariant symbol** of the operator \( \hat{\mathcal{O}}_{0} \). Now if \( O_{B(0)}(Z, \overline{W}) \) and \( O'_{B(0)}(Z, \overline{W}) \) are two covariant
symbols of $\hat{O}_0$ and $\hat{O}'_0$, respectively, then the covariant symbol of $\hat{O}_0\hat{O}'_0$ is given by the Berezin-Wick star product $O_{B(0)} * B O'_{B(0)}$ given by

$$(O_{B(0)} * B O'_{B(0)})(Z, \overline{Z}) = \int_{U_J} O_{B(0)}(Z, \overline{W}) O'_{B(0)}(W, \overline{Z}) \frac{B(Z, \overline{W})B(W, \overline{Z})}{B(Z, \overline{Z})} \exp \{-\Phi(W, \overline{W})\} \frac{\omega^n}{n!}(W, \overline{W})$$

where $\Phi(Z, \overline{Z}) := \Phi(Z, \overline{W}) + \Phi(W, \overline{Z}) - \Phi(Z, \overline{Z}) - \Phi(W, \overline{W})$ is called the Calabi diastatic function and $B(Z, \overline{Z})$ is the usual Bergman kernel.

Thus we have find the pair $(S_B, *_B)$ which constitutes the Berezin’s quantization of $(A_J, \omega)$.

**Down-Stairs Berezin’s Quantization**

Finally we are in position to get the desired Berezin quantization of the Kähler quotient $(M_J, \tilde{\omega})$. That means to find the family of algebras $(\tilde{S}_B, *_{B})$. Having the Berezin’s quantization $(\tilde{S}_B, *_{B})$ of $(A_J, \omega)$ and following the description of the pushed-down prequantization bundle, we restrict ourselves to $F^{-1}(0) \subset A_J$ and consider only $G$-invariant quantities.

Consider $(\tilde{L}, \tilde{\nabla}, \langle \cdot | \cdot \rangle_{\tilde{L}})$ the pushed-down prequantization with $\tilde{L} = \mathcal{L}^0_G$ being the $G$-complex line bundle over $M_J$. The inner product $\langle \cdot | \cdot \rangle_{\tilde{L}}$ is the $G$-invariant product $\langle \cdot | \cdot \rangle_{\tilde{L}}$ given by

$$\langle \tilde{\chi} | \tilde{\psi} \rangle_{\tilde{L}} = \langle \chi | \psi \rangle^G_{\tilde{L}} = \int_{M_J} \langle \tilde{\chi} | \tilde{\psi} \rangle \frac{\tilde{\omega}}{n!} = \langle \chi | \psi \rangle_{\tilde{L}}$$

for all $\tilde{\chi}, \tilde{\psi} \in H^0_{L^2}(M_J, \tilde{L}) = H^0_{L^2}(F^{-1}(0), \mathcal{L}^0_J)$ where $\langle \tilde{\chi} | \tilde{\psi} \rangle = \langle \chi | \psi \rangle$ and $\tilde{\omega}$ is preserved by the action of $G$, i.e. $\omega$ is $G$-invariant. The norm of an element $\tilde{\psi}$ of $H^0_{L^2}(F^{-1}(0), \mathcal{L}^0_J)$ is given by $\langle \tilde{\psi} | \tilde{\psi} \rangle_{\tilde{L}} = \| \tilde{\psi} \|^2_{\tilde{L}}$.

Now take $\tilde{Q} \in \tilde{L}_0$, $\pi[\tilde{Q}] = \tilde{x} \in M_J$ with local complex coordinates $\{z^i, \overline{z}^j\}$ and $\pi[\overline{\tilde{Q}}] = \overline{\tilde{y}} \in M_J$ with local complex coordinates $\{w^i, \overline{w}^j\}$. Here $\tilde{L}_0$ is the line bundle $\tilde{L}$ without the zero section. Now consider $\tilde{\psi}(\tilde{x}) = \tilde{\psi}[\pi(\tilde{Q})] = \tilde{L}_0[\tilde{\psi}]$ with $\tilde{L}_0[\tilde{\psi}]$ being a linear functional of $\tilde{\psi}$. The group $G$ acts on $H^0_{L^2}(F^{-1}(0), \mathcal{L}^0_J)$ in the form

$$(\tilde{\Gamma}\tilde{\psi})(\tilde{x}) = \tilde{\Gamma}\tilde{\psi}(\tilde{x})$$

where $\tilde{\Gamma} \in G$, $\tilde{x} \in M_J$ and $\tilde{\psi} \in H^0_{L^2}(F^{-1}(0), \mathcal{L}^0_J)$. 

**Berezin’s Quantization of Kähler Quotients**

7
Again the Riesz theorem ensures the existence of a section \( \tilde{e}_Q \in H^{0}_{L^2}(F^{-1}(0), \mathcal{L}_j^G) \) such that \( L_Q[\tilde{\psi}] = \langle \tilde{e}_Q | \tilde{\psi} \rangle \) with \( \tilde{Q} \in \tilde{L}_0 \). \( \tilde{e}_Q \) is the push-down of the generalized coherent state \( e_Q \).

Let \( \tilde{\mathcal{O}}^G : H^{0}_{L^2}(F^{-1}(0), \mathcal{L}_j^G) \to H^{0}_{L^2}(F^{-1}(0), \mathcal{L}_j^G) \) be a bounded operator. The covariant symbol of this operator is defined as

\[
\mathcal{O}_{B}^{G}(\tilde{x}) = \frac{\langle e_Q | \tilde{\mathcal{O}}^G | e_Q \rangle_{\tilde{L}}}{||e_Q||^2_{\tilde{L}}} = \frac{\langle e_Q | \tilde{\mathcal{O}}^G | e_Q \rangle_{\tilde{L}}}{||e_Q||^2_{\tilde{L}}},
\]

where \( \tilde{Q} \in \tilde{L}_0 \) and \( \pi(\tilde{Q}) = \tilde{x} \). Now the space of covariant symbols \( \tilde{S}_B \) is defined as the pushing-down of \( S_B \), i.e. \( \tilde{S}_B \).

Similarly to the case of the quantization of \( (\mathcal{A}_j, \omega) \), each covariant symbol can be analytically continued to the open dense subset of \( \mathcal{M}_j \times \mathcal{M}_j \) in such a way \( \langle \tilde{e}_Q | \tilde{e}_{Q'} \rangle_{\tilde{L}} \neq 0 \) with \( \pi(\tilde{Q}) = \tilde{x} \) and \( \pi(\tilde{Q}') = \tilde{y} \) which is holomorphic in the first entry and anti-holomorphic in the second entry. This analytic continuation is written as

\[
\mathcal{O}_{B}^{G}(z, \bar{w}) = \frac{\langle e_Q | \tilde{\mathcal{O}}^G | e_Q \rangle_{\tilde{L}}}{\langle e_Q | e_Q \rangle_{\tilde{L}}},
\]

Similar considerations apply to other formulas. But an essential difference with respect to the quantization of \( (\mathcal{A}_j, \omega) \) is that, in the present case, the Kähler quotient is topologically nontrivial and therefore the line bundle \( \tilde{\mathcal{L}} \) is non-trivial. It is only locally trivial i.e. \( \tilde{\mathcal{L}}_{(j)} = \mathcal{W}^{(j)} \times \mathbb{C} \) for each dense open subset \( \mathcal{W}_j^{(j)} \subset \mathcal{M}_j \) with \( j = 1, 2, \ldots, N \). Analogous global formulas found on \( \mathcal{L} \), can be applied only on each local trivialization of \( \tilde{\mathcal{L}} \). Of course, transition functions on \( \mathcal{W}_j^{(i)} \cap \mathcal{W}_j^{(j)} \) with \( i \neq j \) are very important and sections and other relevant quantities like the Bergman kernel, Kähler potential, covariant symbols, etc., transform nicely under the change of the open set (see [12]). Thus in a particular trivialization \( \tilde{\mathcal{L}}_{(j)} \) and in the local description, the function \( \mathcal{O}_{B(0)}^{(j)}(z, \bar{z}) \in C^\infty(\mathcal{W}_j^{(j)}) \) is called the covariant symbol of the operator \( \tilde{\mathcal{O}}_0^{(j)} \). Now if \( \mathcal{O}_{B(0)}^{(j)}(z, \bar{z}) \) and \( \mathcal{O}_0^{(j)}(z, \bar{z}) \) are two covariant symbols of \( \tilde{\mathcal{O}}_0^{(j)} \) and \( \tilde{\mathcal{O}}_0^{(j)} \), respectively, then the covariant symbol of \( \tilde{\mathcal{O}}_0^{(j)} \tilde{\mathcal{O}}_0^{(j)} \) is given by the Berezin-Wick star product \( \mathcal{O}_{B(0)}^{(j)} \tilde{\mathcal{O}}_0^{(j)} \)

\[
\langle \mathcal{O}_{B(0)}^{(j)} \tilde{\mathcal{O}}_0^{(j)} \rangle (z, \bar{z}) = \int_{\mathcal{W}_j^{(j)}} \mathcal{O}_{B(0)}^{(j)}(z, \bar{w}) \mathcal{O}_0^{(j)}(w, \bar{z}) \frac{\mathcal{B}^{(j)}(z, \bar{w}) \mathcal{B}^{(j)}(w, \bar{z})}{\mathcal{B}^{(j)}(z, \bar{z})} \exp \left\{ -\Phi^{(j)}(w, \bar{w}) \right\} \frac{\bar{\omega}}{n!}(w, \bar{w})
\]
Where \( K^{(j)}(z, z | w, w) := \Phi^{(j)}(z, w) + \Phi^{(j)}(w, z) - \Phi^{(j)}(z, z) - \Phi^{(j)}(w, w) \) is called the Calabi diastatic function on \( W^{(j)} \). This construction is valid for all local prequantization \((\tilde{\mathcal{L}}^{(j)}, \tilde{\nabla}^{(j)}, \langle \cdot | \cdot \rangle_{\tilde{\mathcal{L}}^{(j)}})\). Finally, this structure leads to the pair \((\tilde{\mathcal{S}}_B, \tilde{\ast}_B)\) which constitutes the Berezin quantization of \((\mathcal{M}_J, \tilde{\omega})\).

Acknowledgments

H. G.-C. wish to thank the organizers of the First Mexican Meeting on Mathematical and Experimental Physics, for invitation. H. G.-C. wish also to thank M. Przanowski and F. Turrubiates for very useful discussions. I. C.-I. is supported by a CONACyT graduate fellowship. This work was partially supported by the CONACyT grant No. 33951E.

References

[1] F. Wilczek (ed.), Fractional Statistics and Anyon Superconductivity, Singapore, (World Scientific, 1990).
[2] A. Achúcarro and P.K. Townsend, Phys. Lett. B 180 (1986) 89; E. Witten, Nucl. Phys. B 311 (1988) 46.
[3] E. Witten, Commun. Math. Phys. 121 (1989) 351.
[4] E. Witten, Nucl. Phys. B 330 (1990) 285.
[5] G. Moore and N. Seiberg, Phys. Lett. B 220 (1989) 422; S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. B 326 (1989) 108.
[6] S. Axelrod, S. Della Pietra and E. Witten, J. Diff. Geom. 33 (1991) 787.
[7] E.A. Berezin, Math. USSR-Izv. 6 (1972), 1117; Soviet Math. Dokl. 14 (1973) 1209; Math. USSR-Izv. 8 (1974) 1109; Math. USSR-Izv. 9 (1975) 341; Commun. Math. Phys. 40 (1975) 153.
[8] M. Cahen, S. Gutt and J. Rawnsley, J. Geom. Phys. 7 (1990) 45.
[9] M. Cahen, S. Gutt and J. Rawnsley, Trans. Amer. Math. Soc. 337 (1993) 73.
[10] N. Reshetikhin and L.A. Takhtajan, “Deformation Quantization of Kähler Manifolds”, math.QA/9907171.
[11] M. Schlichenmaier, “Deformation Quantization of Compact Kähler Manifolds by Berezin-Toeplitz Quantization”, math.QA/9910137.
[12] H. García-Compeán, J.F. Plebański, M. Przanowski and F.J. Turrubiates, “Deformation Quantization of Geometric Quantum Mechanics”, hep-th/0112049.
[13] H. García-Compeán, “Berezin’s Quantization of Chern-Simons Gauge Theory”, to appear (2002).
[14] M. Spradlin and A. Volovich, “Noncommutative Solitons on Kähler Manifolds”, hep-th/0106180.
[15] J.M. Isidro, “Darboux’s Theorem and Quantization”, quant-ph/0112032.