Chern-Simons-Matter Theory

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December 12, 2013

Abstract

In this paper we will deform a ABJ theory in $\mathcal{N} = 3$ harmonic superspace without breaking any supersymmetry. We will analyse this ABJ theory and show that it retains the full $\mathcal{N} = 6$ supersymmetry. We will then analyse the gauge fixing and ghost terms for this model in various gauges. We will also analyse the corresponding BRST and anti-BRST symmetries of this model.

1 Introduction

Chern-Simons theories are also important in condensed matter physics due to their relevance in fractional quantum Hall effect [1]-[4]. In fractional quantum Hall effect the electrons are described as bosons in combined external and statistical magnetic fields. At special values of the filling fraction the statistical field cancels the external field, in the mean field sense. The system at these values of the filling fraction is described as a gas of bosons feeling no net magnetic field. These bosons condense into a homogeneous ground state. Thus, by coupling Chern-Simons theory to the fermions in two dimensions, fermions can be described as charged bosons carrying an odd integer number of flux quanta. Recently, supersymmetric generalisation of fractional quantum Hall effect have been investigated [5]-[14]. Furthermore, a relation between fractional quantum Hall effect and noncommutative field theories has also been investigated [15]-[18]. Thus, the noncommutative field theories have interesting condensed matter applications. Thus, it will be interesting to analyse what effect the graviphoton deformation can have on the properties of supersymmetric quantum Hall systems. In fact, it will also be interesting to analyse the theory dual to these deformations by using $AdS/CMT$ correspondence [19]-[20]. This is another motivation for studying Chern-Simons theories. It may be noted that the holography of two dimensional conformal field theories is special because in all higher dimensional examples, the propagating modes of a bulk gauge field are dual to a symmetry current in the boundary theory. However, in two dimensional case, the boundary currents are captured by topological terms in the bulk. It will also be interesting to analyse the gravity dual to the deformed ABJ theory.

In this paper we will analyse the ABJ theory in harmonic superspace. The harmonic superspace variable parameterize the coset $SU(2)/U(1)$ and are well
suited for analysing theories with high amount of supersymmetry. Thus, the harmonic superspace has been used for studying theories with \( N = 2 \) supersymmetry in four dimensions \([21]-[22]\). They have also been used for studying theories with \( N = 3 \) supersymmetry in three dimensions \([23]-[25]\). The ABJM theory has also been analysed in harmonic superspace \([26]\). ABJM theory is a superconformal Chern-Simons-matter theory with manifest \( N = 6 \) supersymmetry which is expected to get enhanced to \( N = 8 \) supersymmetry \([37]-[38]\). In this theory gauge fields are governed by the Chern-Simons action and the matter fields live in the bifundamental representation of the gauge group \( U(N) \times U(N) \) \([27]-[32]\). It is thought to be a low energy description of \( N \) M2-branes on \( C^4/Z_k \) orbifold because it coincides with the BLG theory for the only known example of a Lie 3-algebra \([33]-[36]\). A generalization of the ABJM theory is called the ABJ theory \([39]-[42]\). In this theory the matter fields live in the bifundamental representation of gauge group \( U(M) \times U(N) \) with \( M \neq N \). This theory also has \( \mathcal{N} = 6 \) supersymmetry, but unlike the ABJM theory, non-planar corrections to the two-loop dilatation generator of ABJ theory mix states with positive and negative parity, and this mixing is proportional to \( M - N \) \([43]\). The ABJ theory reduces to the ABJM theory when \( M = N \), and there is clearly no mixing for the ABJM theory.

In string theory the NS backgrounds cause a noncommutative deformation of the spacetime \([46]-[49]\) and the RR backgrounds causes a non-anticommutative deformation of the Grassmann coordinates which in-turn partially break the supersymmetry of the theory \([55]-[60]\). However, gravitino backgrounds cause a noncommutative deformation between the spacetime and Grassmann coordinates \([51]-[54]\). The non-anticommutative deformation of harmonic superspace has already been analysed \([62]-[65]\). As M-theory is dual to type II string theory a deformation of the string theory side will also generate a deformation on the M-theory side. So, in this paper we will thus analyse a deformation of the ABJ theory caused by a non-vanishing commutator between the spacetime and Grassman coordinates. It will also be interesting to analyse the gravity dual of the deformed ABJ theory. This motivates the study of deformation of the ABJ theory.

We will analyse the BRST and the anti-BRST symmetries of this theory in various gauges. The BRST and the anti-BRST symmetries occur for theories with a gauge degrees of freedom \([66]\). In Landau and Curci-Ferrari gauges the BRST and the anti-BRST transformations along with few other transformations generate a algebra known as the Nakanishi-Ojima algebra \([68]-[71]\). In fact, this Nakanishi-Ojima algebra is mass-deformed in massive Curci-Ferrari gauge \([72]\). The BRST symmetry for the ordinary Chern-Simons theory \([67]-[73]\) and \( N = 1 \) Chern-Simons theory \([74]-[75]\) has been already studied. The BRST symmetry of noncommutative pure Chern-Simons theory has also been analysed \([76]-[77]\). The BRST symmetry for the deformed ABJ theory has already been studied \([78]\). In this paper we will generalize this work to include the anti-BRST symmetry. Thus, we will analyse the BRST and the anti-BRST symmetries of the deformed ABJ theory in harmonic superspace.
2 Harmonic superspace

The $\mathcal{N} = 3$ harmonic superspace is constructed using the following derivatives

\[
\begin{align*}
D^{++} &= \partial^{++} + 2i\theta^{++a}\theta^{a}_b \partial^A_{ab} + \theta^{++a} \frac{\partial}{\partial \theta^{0a}} + 2\theta^{0a} \frac{\partial}{\partial \theta^{--a}}, \\
D^{--} &= \partial^{--} - 2i\theta^{--a}\theta^{a}_b \partial^A_{ab} + \theta^{--a} \frac{\partial}{\partial \theta^{0a}} + 2\theta^{0a} \frac{\partial}{\partial \theta^{++a}}, \\
D^0 &= \partial^0 + 2\theta^{++a} \frac{\partial}{\partial \theta^{0a}} - 2\theta^{--a} \frac{\partial}{\partial \theta^{0a}},
\end{align*}
\]

and

\[
\begin{align*}
D^a_{--} &= \frac{\partial}{\partial \theta^{++a}} + 2i\theta^{--b}\theta^{b}_a \partial^A_{ab}, \\
D^a^{++} &= \frac{\partial}{\partial \theta^{--a}},
\end{align*}
\]

where

\[
\begin{align*}
\partial^{++} &= u^+_i \frac{\partial}{\partial u_i}, \\
\partial^{--} &= u^-_i \frac{\partial}{\partial u_i}, \\
\partial^0 &= u^+_i \frac{\partial}{\partial u_i} - u^-_i \frac{\partial}{\partial u_i}.
\end{align*}
\]

Here the harmonic variables $u^\pm_i$ are subjected to the constraints

\[
u^+_i u^-_i = 1, \quad u^+_i u^+_i = u^-_i u^-_i = 0.
\]

These derivatives are satisfy

\[
\begin{align*}
\{D^a_{++}, D^b_{--}\} &= 2i\theta^{ab}_A, \\
\{D^a_0, D^b_0\} &= -i\theta^{ab}_A, \\
[D^{++}, D^{++}] &= 2D^0, \\
[D^{--}, D^{--}] &= 2\partial^0, \\
[D^0, D^0] &= \pm 2D^{\pm\pm}, \\
\partial^0 &= [\partial^{++}, \partial^{--}], \\
[D^{++}, D^{--}] &= D^0.
\end{align*}
\]

The full harmonic superspace is parameterized by

\[
\begin{align*}
z &= (x^{ab}_A, \theta^{++}_a, \theta^{--}_a, \theta^0_a, u^\pm_i),
\end{align*}
\]

and the analytic superspace is parametrized by

\[
\begin{align*}
\zeta_A &= (x^{ab}_A, \theta^{++}_a, \theta^0_a, u^\pm_i),
\end{align*}
\]

where

\[
x^{ab}_A = (\gamma_m)^{ab} x^m_A = x^{ab} + i(\theta^{++a}\theta^{--b} + \theta^{++b}\theta^{--a}).
\]

This is because the analytic superspace is defined to be independent of the $\theta^{--}_a$,

\[
D^a_{++}\Phi_A = 0 \iff \Phi_A = \Phi_A(\zeta_A).
\]

In this superspace the generators of the supersymmetry are denoted by

\[
\begin{align*}
Q^a_{++} &= u^+_i u^+_j Q^i_a, \\
Q^a_{--} &= u^-_i u^-_j Q^i_a, \\
Q^a_0 &= u^+_i u^-_j Q^i_a.
\end{align*}
\]
where
\[ Q_{ij}^a = \frac{\partial}{\partial \theta^a_{ij}} - \theta^a_{ij} \partial_{ab}, \]  
and the superspace measures are denoted by
\[ d^9z = -\frac{1}{16} d^3x (D^+)^2 (D^-)^2 (D^0)^2, \]
\[ d\zeta^{(-4)} = \frac{1}{4} d^3x_A du (D^-)^2 (D^0)^2. \]  

A conjugation in this superspace is defined by
\[
\begin{align*}
\tilde{u}^\pm_i &= u^\pm_i, \\
\tilde{x}_A^m &= x_A^m, \\
\tilde{\theta}_a^{\pm \pm} &= \theta_a^{\pm \pm}, \\
\tilde{\theta}_a^0 &= \theta_a^0.
\end{align*}
\]  

Thus, the analytic superspace measure is real and the full superspace measure is imaginary.

Now we can analyse the deformation of this superspace, caused by a gravitino background. This deformation gives rise to the following commutator
\[ [\theta^{++\mu}, x^\mu] = A^{a\mu}, \]  
and so this deformation does not break any supersymmetry. This deformation induces the following star product in this superspace,
\[
V^{++}(z) \star V^{++}(z) = \exp \left( -\frac{1}{2} \left( A^{a\mu} (\partial_a^2 \partial_\mu^2 + \partial_\mu^2 \partial_a^2) \right) \right)
\times V^{++}(z_1)V^{++}(z_2) \Big|_{z_1 = z_2 = z}.
\]  

Now we can construct the action for ABJ theory in this deformed superspace using \( V^{++}_L \) and \( V^{++}_R \), which are defined by
\[
\begin{align*}
V^{++}_L &= u^+_i u^+_j V^{ij}_L, \\
V^{++}_R &= u^+_i u^+_j V^{ij}_R,
\end{align*}
\]  

where \( V^{ij}_L \) and \( V^{ij}_R \) are fields transforming under the gauge group \( U(M) \) and \( U(N) \), respectively. We also define matter fields \( q^+ \) and \( \bar{q}^+ \), which transform under the bifundamental representation of the group \( U(N) \times U(M) \). Now the action for this deformed ABJ theory, which is invariant under the gauge group \( U(N) \times U(M) \), can be written as
\[
S = S_{CS,k}[V^{++}_L], + S_{CS,-k}[V^{++}_R], + S_M[q^+, \bar{q}^+],
\]  
where
\[
\begin{align*}
S_{CS,k}[V^{++}_L] &= \frac{ik}{4\pi} \text{tr} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \int d^3x d^6\theta du_1 \ldots du_n H^{++}_L, \\
S_{CS,-k}[V^{++}_R] &= -\frac{ik}{4\pi} \text{tr} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \int d^3x d^6\theta du_1 \ldots du_n H^{++}_R, \\
S_M[q^+, \bar{q}^+] &= \text{tr} \int d\zeta^{(-4)} \bar{q}^+ \ast \nabla^{++} \ast q^+,
\end{align*}
\]
and

\[ H_{L}^{++} = \frac{V^{++}(z,u_1)L \star V^{++}(z,u_2)L \cdots V^{++}(z,u_n)L}{(u_1^L u_2^L) \cdots (u_n^L u_1^L)}, \]
\[ H_{R}^{++} = \frac{V^{++}(z,u_1)_R \star V^{++}(z,u_2)_R \cdots V^{++}(z,u_n)_R}{(u_1^R u_2^R) \cdots (u_n^R u_1^R)}, \]
\[ \nabla^{++}q^+ = D^{++}q^+ + V^{++}_L \star q^+ - q^+ \star V^{++}_R, \]
\[ \nabla^{++}\bar{q}^+ = D^{++}\bar{q}^+ - \bar{q}^+ \star V^{++}_L + V^{++}_R \star \bar{q}^+. \] (19)

The covariant derivatives for the matter fields in the deformed ABJ theory are given by

\[ \nabla^{++}q^+ = D^{++}q^+ + V^{++}_L \star q^+ - q^+ \star V^{++}_R, \]
\[ \nabla^{++}\bar{q}^+ = D^{++}\bar{q}^+ - \bar{q}^+ \star V^{++}_L + V^{++}_R \star \bar{q}^+, \] (20)

It is useful to define \( V_{L}^{-} \) and \( V_{R}^{-} \) as

\[ V_{L}^{-} = \sum_{n=1}^{\infty} (-1)^n \int du_1 \cdots du_n E^{++}_L, \]
\[ V_{R}^{-} = \sum_{n=1}^{\infty} (-1)^n \int du_1 \cdots du_n E^{++}_R, \] (21)

where

\[ E^{++}_L = \frac{V^{++}_L(z,u_1) \star V^{++}_L(z,u_2) \cdots V^{++}_L(z,u_n)}{(u^L u_1^L)(u_1^L u_2^L) \cdots (u_n^L u_1^L)}, \]
\[ E^{++}_R = \frac{V^{++}_R(z,u_1) \star V^{++}_R(z,u_2) \cdots V^{++}_R(z,u_n)}{(u^R u_1^R)(u_1^R u_2^R) \cdots (u_n^R u_1^R)}. \] (22)

It is also useful to define \( W_{L}^{++} \) and \( W_{R}^{++} \) as

\[ W_{L}^{++} = -\frac{1}{4} D^{+++} D^+_a V_L^{-}, \]
\[ W_{R}^{++} = -\frac{1}{4} D^{+++} D^+_a V_R^{-}. \] (23)

This ABJ theory is invariant under the following infinitesimal gauge transformations

\[ \delta q^+ = \Lambda_L \star q^+ - q^+ \star \Lambda_R, \]
\[ \delta \bar{q}^+ = \Lambda_R \star \bar{q}^+ - \bar{q}^+ \star \Lambda_L, \]
\[ \delta V^{++}_L = \nabla^{++} \Lambda_L, \]
\[ \delta V^{++}_R = \nabla^{++} \Lambda_R. \] (24)

where

\[ \nabla^{++} \Lambda_L = -D^{++} \Lambda_L - [V^{++}_L, \Lambda_L]_\star, \]
\[ \nabla^{++} \Lambda_R = -D^{++} \Lambda_R - [V^{++}_R, \Lambda_R]_\star. \] (25)
and the following $\mathcal{N} = 3$ supersymmetric transformations

$$
\begin{align*}
\delta q^+ &= i\epsilon^a \hat{\nabla}_a^0 \star q^+, \\
\delta \bar{q}^+ &= i\epsilon^a \hat{\nabla}_a^0 \star \bar{q}^+, \\
\delta V_L^{++} &= \frac{8\pi}{k} \epsilon^a \theta_0^a \star q^+ \star \bar{q}^+, \\
\delta V_R^{++} &= \frac{8\pi}{k} \epsilon^a \theta_0^a \star \bar{q}^+ \star q^+,
\end{align*}
$$

(26)

where

$$
\begin{align*}
\hat{\nabla}_a^0 \star q^+ &= \nabla_a^0 \star q^+ + \theta_a^- (W_L^{++} \star q^+ - q^+ \star W_R^{++}), \\
\nabla_a^0 \star q^+ &= D_a^0 q^+ + V_L^0 a \star q^+ - q^+ \star V_R^0 a, \\
V_{L,R}^0 &= \frac{1}{2} D^{++} V_{L,R}^{--}.
\end{align*}
$$

(27)

Thus, apart from the original manifest $\mathcal{N} = 3$ supersymmetry, this model has additional $\mathcal{N} = 3$ supersymmetry. So, this ABJM theory has $\mathcal{N} = 6$ supersymmetry.

### 3 Linear Gauge

As the deformed ABJ theory is invariant under gauge transformations given by Eq. (24), we cannot quantize it without fixing a gauge. Thus, we choose the gauge fixing conditions,

$$
D^{++} \star V_{L,R}^{++} = 0, \quad D^{++} \star V_{R,L}^{++} = 0.
$$

(28)

To incorporate these gauge fixing conditions at a quantum level, we add the following gauge fixing term to the original Lagrangian density,

$$
\mathcal{L}_{gf} = \int d\zeta \left( -\frac{4}{2} \text{tr} \left[ b_L \star (D^{++} V_{L}^{++}) + \frac{\alpha}{2} b_L \star b_L \\
- b_R \star (D^{++} V_{R}^{++}) + \frac{\alpha}{2} b_R \star b_R \right] \right).
$$

(29)

In order to ensure unitarity of the model we also add the following ghost term to the original Lagrangian density,

$$
\mathcal{L}_{gh} = \text{tr} \int d\zeta \left[ \bar{c}_L \star \tau^{++} \star c_L - \bar{c}_R \star \tau^{++} \star c_R \right].
$$

(30)

The sum of the gauge fixing term and the ghost term, $\mathcal{L}_g = \mathcal{L}_{gf} + \mathcal{L}_{gh}$, is a total BRST of $\Phi$, and a total anti-BRST variation of $\Phi$, 

$$
\mathcal{L}_g = \int d\zeta (-s) \text{tr} [\Phi]\]
$$

(31)
where

\[ \Phi = c_L \left( D^{++} V_L^{++} - \frac{i \alpha}{2} b_L \right) - c_R \left( D^{++} V_R^{++} - \frac{i \alpha}{2} b_R \right), \]
\[ \bar{\Phi} = \bar{\tau}_L \left( D^{++} V_L^{++} - \frac{\alpha}{2} b_L \right) - \bar{\tau}_R \left( D^{++} V_R^{++} - \frac{\alpha}{2} b_R \right). \]  

(32)

Here the BRST transformations are given by

\[ s V_L^{++} = \nabla^{++} c_L, \quad s V_R^{++} = \nabla^{++} c_R, \]
\[ s c_L = \left[ c_L, \tau_L \right], \quad s \tau_R = -b_R - 2 [\tau_R, c_R], \]
\[ s \tau_L = b_L, \quad s c_R = -[c_R, \tau_R], \]
\[ s b_L = 0, \quad s b_R = -[b_R, \tau_R], \]
\[ s q^+ = c_L \ast q^+ - \bar{q}^+ \ast c_L, \quad s \bar{q}^+ = c_R \ast \bar{q}^+ - \bar{q}^+ \ast c_L. \]

(33)

and the anti-BRST transformations are given by

\[ \bar{s} V^{++} = \nabla^{++} \bar{\tau}_L, \quad \bar{s} V^{++}_R = \nabla^{++} \bar{\tau}_R, \]
\[ \bar{s} c_L = -b_L - 2 [\bar{\tau}_L, \tau_L], \quad \bar{s} \tau_R = b_R, \]
\[ \bar{s} \tau_L = -[\tau_L, \bar{\tau}_L], \quad \bar{s} c_R = -[c_R, \bar{\tau}_R], \]
\[ \bar{s} b_L = b_L, \quad \bar{s} b_R = 0, \]
\[ \bar{s} q^+ = \bar{\tau}_L \ast q^+ - q^+ \ast \bar{\tau}_L, \quad \bar{s} \bar{q}^+ = \bar{\tau}_R \ast \bar{q}^+ - \bar{q}^+ \ast \bar{\tau}_L. \]

(34)

Both these transformations are nilpotent, \( s^2 = \bar{s}^2 = 0 \). In fact, they also satisfy, \( s \bar{s} + \bar{s} s = 0 \). Now as the sum of the ghost term and the gauge fixing term is expressed as a total BRST or a total anti-BRST variation, it invariant under them. This is because of the nilpotency of these transformations. The BRST or the anti-BRST variation of the original classical Lagrangian density is its gauge variations with the gauge parameter replaced by the ghosts or the anti-ghosts. As the original classical Lagrangian density was gauge invariant, so it is also invariant under these BRST and anti-BRST transformations. Thus, the effective Lagrangian density, which is defined to be a sum of the original classical Lagrangian density, the gauge fixing term and the ghost term, is also invariant under these BRST and anti-BRST transformations. The sum of the gauge fixing term and ghost term takes a simple form in the Landau gauge, \( \alpha = 0 \),

\[ \mathcal{L}_g = \int d\zeta (-4) s \text{tr} \left[ \tau_L \ast (D^{++} V_L^{++}) - \tau_R \ast (D^{++} V_R^{++}) \right] \]
\[ = \int d\zeta (-4) \bar{s} \text{tr} \left[ c_L \ast (D^{++} V_L^{++}) - c_R \ast (D^{++} V_R^{++}) \right]. \]

(35)

In fact, in Landau gauge this can be expressed as combination of a total BRST and a total anti-BRST variation. Thus, in Landau gauge sum of the gauge fixing term and the ghost term is given by

\[ \mathcal{L}_g = -\frac{1}{2} \int d\zeta (-4) s \bar{s} \text{tr} [Z] \]
\[ = \frac{1}{2} \int d\zeta (-4) s \bar{s} \text{tr} [Z], \]

(36)

where

\[ Z = V_L^{++} \ast V_L^{++} - V_R^{++} \ast V_R^{++}. \]

(37)
4 Non-Linear Gauges

For gauge theories sum of the gauge fixing term and the ghost term can also be expressed as a combination of the total BRST and the total anti-BRST variation, for any value of $\alpha$ in Curci-Ferrari gauge [68]-[71]. Here we will show that this also holds for a deformed ABJ theory in $\mathcal{N} = 3$ harmonic superspace formalism. The non-linear BRST transformations for the deformed ABJ theory in $\mathcal{N} = 3$ harmonic superspace are given by

$$ sV_L^{++} = \nabla^{++} \ast c_L, \quad s\bar{V}_R^{++} = \nabla^{++} \ast \bar{c}_R, \quad s\bar{V}_L^{++} = \nabla^{++} \ast \bar{c}_L, \quad s\bar{V}_R^{++} = \nabla^{++} \ast \bar{c}_R, $$

$$ s\alpha = -[c_L, c_L], \quad s\bar{b}_L = -[\bar{b}_L, \bar{c}_L], \quad s\bar{c}_L = -[c_L, c_L], \quad s\bar{c}_R = -[c_R, c_R], $$

and the non-linear anti-BRST transformations for the deformed ABJM theory in $\mathcal{N} = 3$ harmonic superspace are given by

$$ \bar{s}V_L^{++} = \bar{nabla}^{++} \ast \bar{c}_L, \quad \bar{s}b_L = -[b_L, c_L], \quad \bar{s}\bar{c}_L = -[\bar{c}_L, c_L], \quad \bar{s}\bar{c}_R = -[\bar{c}_R, c_R], $$

$$ \bar{s}\bar{q}^+ = c_L \ast \bar{q}^+ - \bar{q}^+ \ast c_L, \quad \bar{\bar{s}}\bar{q}^+ = \bar{c}_R \ast \bar{\bar{q}}^+ - \bar{\bar{q}}^+ \ast \bar{c}_L. $$

(38)

These transformations also satisfy $s^2 = \bar{s}^2 = s\bar{s} + \bar{s}s = 0$. Thus, both these transformations are also nilpotent. We can now write sum of the gauge fixing term and the ghost term for this deformed ABJ theory as a combination of a total BRST and a total anti-BRST variation

$$ \mathcal{L}_g = \frac{1}{2} \int d\zeta (-4) s\bar{s} tr [\mathcal{Z} + \mathcal{Y}], $$

$$ = -\frac{1}{2} \int d\zeta (-4) \bar{s}s tr [\mathcal{Z} + \mathcal{Y}], $$

(40)

where

$$ \mathcal{Y} = \alpha \bar{c}_R \ast c_R - \alpha \bar{c}_L \ast c_L. $$

(41)

In gauge theories [68]-[71], the addition of a bare mass term breaks the nilpotency of the BRST and the anti-BRST transformations. Here we will show that this also occurs for a deformed ABJ theory in $\mathcal{N} = 3$ harmonic superspace formalism. The bare mass term is added to Curci-Ferrari model to obtain a massive Curci-Ferrari model as follows

$$ \mathcal{L}_g = -\frac{1}{2} \int d\zeta (-4) [s\bar{s} + im^2] tr [\mathcal{Z} + \mathcal{Y}], $$

$$ = \frac{1}{2} \int d\zeta (-4) [s\bar{s} - im^2] tr [\mathcal{Z} + \mathcal{Y}]. $$

(42)

Now the BRST transformations get modified as follows

$$ sV_L^{++} = \nabla^{++} \ast c_L, \quad s\bar{b}_L = im^2 c_L - [b_L, c_L], \quad s\bar{c}_L = -[c_L, c_L], $$

$$ s\bar{c}_R = -[c_R, c_R], \quad s\bar{q}^+ = c_L \ast \bar{q}^+ - \bar{q}^+ \ast c_L, $$

$$ \bar{s}\bar{q}^+ = \bar{c}_R \ast \bar{\bar{q}}^+ - \bar{\bar{q}}^+ \ast \bar{c}_L. $$

(38)
\[
\begin{align*}
s_{CL} &= -[cL, cl]s, & s\bar{v}_L &= b_L - [\bar{v}_L, cl]s, \\
s_D^{++} &= \nabla^{++} + c_R, & s b_R &= im^2 e_R - [b_R, c_R]s - [\bar{v}_R, [c_R, c_R]]s, \\
s_{cR} &= -[c_R, c_R]s, & s\bar{v}_R &= b_R - [\bar{v}_R, c_R]s, \\
\bar{\pi} q^+ &= \bar{v}_L \ast q^+ - q^+ \ast \bar{v}_R, & \bar{\pi} \tilde{q}^+ &= \bar{v}_R \ast \tilde{q}^+ - \tilde{q}^+ \ast \bar{v}_L, 
\end{align*}
\]

and the anti-BRST transformations get modified as
\[
\begin{align*}
\bar{\pi} V_L^{++} &= \nabla^{++} + \bar{v}_L, & \bar{\pi} b_L &= im^2 \bar{v}_L - [b_L, \bar{v}_L]s + [cL, [\bar{v}_L, \bar{v}_L]]s, \\
\bar{\pi} c_L &= -[cL, \bar{v}_L]s, & s\bar{v}_L &= -b_L - \bar{v}_L[cL]s, \\
\bar{\pi} V_R^{++} &= \nabla^{++} + \bar{v}_R, & \bar{\pi} b_R &= im^2 \bar{v}_R - [b_R, \bar{v}_R]s + [cR, [\bar{v}_R, \bar{v}_R]]s, \\
\bar{\pi} c_R &= -[cR, \bar{v}_R]s, & s\bar{v}_R &= -b_R - \bar{v}_R[cR]s, \\
\bar{\pi} q^+ &= \bar{v}_L \ast q^+ - q^+ \ast \bar{v}_R, & \bar{\pi} \tilde{q}^+ &= \bar{v}_R \ast \tilde{q}^+ - \tilde{q}^+ \ast \bar{v}_L.
\end{align*}
\]

These modified BRST and anti-BRST transformations now satisfy
\[
s^2 = \bar{s}^2 \sim 2im^2.
\]

Thus, the addition of the bare mass term breaks the nilpotency of the BRST and the anti-BRST transformations. However, in the zero mass limit, the nilpotency of the BRST and the anti-BRST transformations is restored.

5 Nakanishi-Ojima Algebra

When ever the sum of the gauge fixing term and the ghost term can be written as a combination of the total BRST and the total anti-BRST variation, the total Lagrangian density is invariant under a set of symmetry transformations which obey a \(SL(2, R)\) algebra called Nakanishi-Ojima algebra. We will show that this algebra also hold for the ABJ theory in \(N = 3\) harmonic superspace. To do so we first define the following transformations for the deformed ABJ theory,
\[
\begin{align*}
\delta_1 b_L &= [c_L, c_L]s, & \delta_1 b_R &= [c_R, c_R]s, & \delta_1 c_L &= 0, \\
\delta_1 c_R &= 0, & \delta_1 \bar{v}_L &= c_L, & \delta_1 \bar{v}_R &= c_R, \\
\delta_1 V_L^{++} &= 0, & \delta_1 V_R^{++} &= 0, & \delta_1 q^+ &= 0, \\
\delta_2 c_L &= \bar{v}_L, & \delta_2 c_R &= \bar{v}_R, & \delta_2\bar{v}_L &= 0, \\
\delta_2 \bar{v}_R &= 0, & \delta_2 q^+ &= 0 & \delta_2 \tilde{q}^+ &= 0.
\end{align*}
\]

Now we can see that in Landau and Curci-Ferrari gauges these transformations, the BRST transformation and the anti-BRST transformation along with the \(FP\)-conjugation form the Nakanishi-Ojima \(SL(2, R)\) algebra,
\[
\begin{align*}
[s, s]_s &= 0, & [\bar{\pi}, \bar{\pi}]_s &= 0, \\
[s, \bar{\pi}]_s &= 0, & [\delta_1, \delta_2]_s &= -2\delta_{FP}, \\
[\delta_1, \delta_{FP}]_s &= -4\delta_1, & [\delta_2, \delta_{FP}]_s &= 4\delta_2, \\
[s, \delta_{FP}]_s &= -2s, & [\bar{\pi}, \delta_{FP}]_s &= 2\bar{\pi}, \\
[s, \delta_1]_s &= 0, & [\bar{\pi}, \delta_1]_s &= -2\bar{\pi}, \\
[s, \delta_2]_s &= 2\bar{\pi}, & [\bar{\pi}, \delta_2]_s &= 0.
\end{align*}
\]
A bare mass term breaks the nilpotency of the BRST and the anti-BRST trans- 
formations in the massive Curci-Ferrari gauge. However, the FP-conjugation is 
not broken in the massive Curci-Ferrari gauge. Thus, we are able to con- 
struct a mass-deformed version of this Nakanishi-Ojima in massive Curci-Ferrari gauge, 

\[
\begin{align*}
[s, s]_* &= -2im^2\delta_1, \\
[s, s]_* &= 2im^2\delta_{FP}, \\
[\delta_1, \delta_{FP}]_* &= -2\delta_1, \\
[\delta_2, \delta_{FP}]_* &= 4\delta_2, \\
[s, \delta_{FP}]_* &= -2s, \\
[\delta_1, \delta_{FP}]_* &= 2\pi, \\
[s, \delta_1]_* &= 0, \\
[\delta_2, \delta_{FP}]_* &= 0.
\end{align*}
\]

(48)

6 Physical States

It is now possible to construct current corresponding to the BRST and the anti-BRST symmetries of this theory. The current associated with the BRST symmetry is given by

\[
2J^\mu_{(B)} = \int d\zeta(-4) \text{tr} \left[ \frac{\partial L_{eff}}{\partial \partial_\mu V_L^{++}} * s V_L^{++} + \frac{\partial L_{eff}}{\partial \partial_\mu c_L} * s c_L + \frac{\partial L_{eff}}{\partial \partial_\mu \bar{c}_L} * s \bar{c}_L \\
+ \frac{\partial L_{eff}}{\partial \partial_\mu b_L} * s b_L + \frac{\partial L_{eff}}{\partial \partial_\mu V_R^{++}} * s V_R^{++} + \frac{\partial L_{eff}}{\partial \partial_\mu c_R} * s c_R \\
+ \frac{\partial L_{eff}}{\partial \partial_\mu \bar{c}_R} * s \bar{c}_R + \frac{\partial L_{eff}}{\partial \partial_\mu b_R} * s b_R + \frac{\partial L_{eff}}{\partial \partial_\mu \bar{q}^+} * s \bar{q}^+ \\
+ \frac{\partial L_{eff}}{\partial \partial_\mu \bar{q}^+} * s \bar{q}^+ \right],
\]

(49)

and current associated with the anti-BRST symmetry is given by

\[
2\overline{J}^\mu_{(B)} = \int d\zeta(-4) \text{tr} \left[ \frac{\partial L_{eff}}{\partial \partial_\mu V_L^{++}} * \bar{s} V_L^{++} + \frac{\partial L_{eff}}{\partial \partial_\mu c_L} * \bar{s} c_L + \frac{\partial L_{eff}}{\partial \partial_\mu \bar{c}_L} * \bar{s} \bar{c}_L \\
+ \frac{\partial L_{eff}}{\partial \partial_\mu b_L} * \bar{s} b_L + \frac{\partial L_{eff}}{\partial \partial_\mu V_R^{++}} * \bar{s} V_R^{++} + \frac{\partial L_{eff}}{\partial \partial_\mu c_R} * \bar{s} c_R \\
+ \frac{\partial L_{eff}}{\partial \partial_\mu \bar{c}_R} * \bar{s} \bar{c}_R + \frac{\partial L_{eff}}{\partial \partial_\mu b_R} * \bar{s} b_R + \frac{\partial L_{eff}}{\partial \partial_\mu \bar{q}^+} * \bar{s} \bar{q}^+ \\
+ \frac{\partial L_{eff}}{\partial \partial_\mu \bar{q}^+} * \bar{s} \bar{q}^+ \right],
\]

(50)

where

\[
\int d\zeta(-4) |L_{eff}| = \mathcal{L}_c + \mathcal{L}_{gh} + \mathcal{L}_{gf}.
\]

(51)

If we restrict the deformations to space-like deformations, then the charges corresponding to the BRST and the anti-BRST symmetries are given by

\[
Q_B = \int d^3y \, J^0_{(B)}, \\
\overline{Q}_B = \int d^3y \, \overline{J}^0_{(B)},
\]

(52)
So, from now on we shall restrict the deformations to space-like deformations. Now these charge associated with the BRST and the anti-BRST symmetries transformation commutes with the Hamiltonian and are conserved. These charges are nilpotent for all gauges except the massive Curci-Ferrari gauge,

\[ Q^2_B = ar{Q}^2_B = 0. \]  

(53)

However, for massive Curci-Ferrari gauge these charges are not nilpotent

\[ Q^2_B \neq 0, \]
\[ \bar{Q}^2_B \neq 0. \]  

(54)

Now we define physical states as states that are annihilated by \( Q_B \)

\[ Q_B |\phi_p\rangle = 0. \]  

(55)

We can equivalently define the physical states as states that are annihilated by \( \bar{Q}_B \)

\[ \bar{Q}_B |\phi_p\rangle = 0. \]  

(56)

The inner product of those physical states, which are obtained from unphysical states by the action of these charges, vanishes with all other physical states. This is because if \( |\phi_{up}\rangle \) is a unphysical state, then

\[ \langle \phi_p | Q_B |\phi_{up}\rangle = 0, \]
\[ \langle \phi_p | \bar{Q}_B |\phi_{up}\rangle = 0. \]  

(57)

So, all the relevant physical information lies in the physical states which are not obtained by the action of these charges on unphysical states. Now if the asymptotic physical states are given by

\[ |\phi_{pa,out}\rangle = |\phi_{pa}, t \rightarrow \infty\rangle, \]
\[ |\phi_{pb,in}\rangle = |\phi_{pb}, t \rightarrow -\infty\rangle, \]  

(58)

then a typical \( S \)-matrix element can be written as

\[ \langle |\phi_{pa,out}| S|\phi_{pb}\rangle = \langle |\phi_{pa}| S^\dagger S|\phi_{pb}\rangle. \]  

(59)

As these charges commute with the Hamiltonian, the time evolution of any physical state will also also be annihilated by them. This implies that the states \( S|\phi_{pb}\rangle \) must be a linear combination of states physical states. So we can write

\[ \langle |\phi_{pa}| S^\dagger S|\phi_{pb}\rangle = \sum_i \langle |\phi_{pa}| S^\dagger |\phi_{0,i}\rangle \langle \phi_{0,i}| S|\phi_{pb}\rangle. \]  

(60)

Since the full \( S \)-matrix is unitary this relation implies that the \( S \)-matrix restricted to physical sub-space is also unitarity. It may be noted that the nilpotency of these charges was essential for proving the unitarity of the resultant theory. However, in massive Curci-Ferrari gauge these charges are not nilpotent,

\[ Q^2_B |\phi\rangle \neq 0, \]
\[ \bar{Q}^2_B |\phi\rangle \neq 0, \]  

(61)
so the $S$-matrix does not factorize in the massive Curci-Ferrari gauge,

$$
\langle \phi_{pa} | S^I S | \phi_{pb} \rangle \neq \sum_i \langle \phi_{pa} | S^I | \phi_{0,i} \rangle \langle \phi_{0,i} | S | \phi_{pb} \rangle,
$$

and thus the resultant theory is not unitarity. However, as the nilpotency is restored in the zero mass limit, the unitarity is also restored in the zero mass limit.

7 Conclusion

In this paper we have analysed a deformed ABJ theory in $\mathcal{N} = 3$ harmonic superspace. The classical Lagrangian density was represented by the difference of the two Chern-Simons sectors for the left and right gauge groups plus the Lagrangian density of the matter fields which was minimally coupled to the gauge superfields. No explicit superfield potential was needed in the action. We analysed the BRST and the anti-BRST symmetries of this model in various gauges. The sum of the ghost term and the gauge fixing term was expressed as a combination of a total BRST and a total anti-BRST variation, in Landau and Curci-Ferrari gauges.

It will also be interesting to generalized this work to include boundaries. The existence of boundaries can have a lot of applications in condensed matter physics, due to the existence of edge currents [79]-[84]. There is a well-known connection between Chern-Simons theories on a three dimensional manifold and the two dimensional conformal field theories on its boundaries [85]. For pure Chern-Simons theory with suitable boundary conditions, a component of the gauge field, say $A_0$, appears linearly in the action and so can be integrated out, imposing the constraint $F_{12} = 0$ [86]-[87]. This constraint can be solved explicitly resulting in a two dimensional WZW model. Even though the ABJM is not topological, it is still conformal. So, in presence of a boundary the ABJM action also gets modified. In this new modified ABJM action the Chern-Simons gauge potential is coupled to a boundary WZW model. This boundary action reproduces the pure WZW action when starting from a pure Chern-Simon action. Thus, it is gauge invariant even in presence of a boundary. This has been done for the ABJM theory in $\mathcal{N} = 1$ superspace [88]. This has also been done for the BLG theory in $\mathcal{N} = 1$ superspace [89].

Just like strings can end on D-branes in string theory, M2-branes can end on M5-branes, M9-branes or gravitational waves in M-theory [90]. So, M2-branes in M-theory are analogous to string in string theory. Furthermore, just like various background fluxes can cause various deformations in string theory, the presence of a background flux can also cause deformation in the M-theory [91]. Thus, the open M2-brane action can be studied to learn about M5-brane action. The BLG model has been used to motivate a novel quantum geometry on the M5-brane world-volume, by analysing a system of multiple M2-branes ending on a M5-brane with a constant $C$-field [92]. The the BLG action with Nambu-Poisson 3-bracket has also been identified with a M5-brane action, with a large worldvolume $C$-field [93]. Furthermore, by analysing the action for a single open M2-branes, a non-commutative deformation of string theory on the M5-brane worldvolume has been studied [94]-[97]. It will also be interesting to analyse these results for the ABJ theory in harmonic superspace.
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