NOTES ON THE CODIMENSION ONE CONJECTURE IN THE OPERATOR CORONA THEOREM

MARIA F. GAMAL’

ABSTRACT. Answering on the question of S.R.Treil [23], for every δ, 0 < δ < 1, examples of contractions are constructed such that their characteristic functions \( F \in H^\infty(\mathcal{E} \to \mathcal{E}_s) \) satisfy the conditions

\[ \|F(z)x\| \geq \delta \|x\| \text{ and } \dim \mathcal{E}_s \cap F(z)\mathcal{E} = 1 \text{ for every } z \in \mathbb{D}, \ x \in \mathcal{E}, \]

but \( F \) are not left invertible. Also, it is shown that the condition

\[ \sup_{x \in \mathbb{D}} \|I - F(z)^* F(z)\|_{\mathcal{E}_s} < \infty, \]

where \( \mathcal{G}_1 \) is the trace class of operators, which is sufficient for the left invertibility of the operator-valued function \( F \) satisfying the estimate

\[ \|F(z)x\| \geq \delta \|x\| \text{ for every } z \in \mathbb{D}, \ x \in \mathcal{E}, \] with some \( \delta > 0 \) (S.R.Treil, [22]), is necessary for the left invertibility of an inner function \( F \) such that \( \dim \mathcal{E}_s \cap F(z)\mathcal{E} < \infty \) for some \( z \in \mathbb{D} \).

1. INTRODUCTION

Let \( \mathbb{D} \) be the open unit disk, let \( \mathbb{T} \) be the unit circle, and let \( \mathcal{E} \) and \( \mathcal{E}_s \) be separable Hilbert spaces. The space \( H^\infty(\mathcal{E} \to \mathcal{E}_s) \) is the space of bounded analytic functions on \( \mathbb{D} \) whose values are (linear, bounded) operators acting from \( \mathcal{E} \) to \( \mathcal{E}_s \). If \( F \in H^\infty(\mathcal{E} \to \mathcal{E}_s) \), then \( F \) has nontangential boundary values \( F(\zeta) \) for a.e. \( \zeta \in \mathbb{T} \) with respect to the Lebesgue measure \( m \) on \( \mathbb{T} \). Every function \( F \in H^\infty(\mathcal{E} \to \mathcal{E}_s) \) has the inner-outer factorization, that is, there exist an auxiliary Hilbert space \( \mathcal{D} \) and two functions \( \Theta \in H^\infty(\mathcal{E} \to \mathcal{D}) \) and \( \Omega \in H^\infty(\mathcal{D} \to \mathcal{E}_s) \) such that \( F = \Theta \Omega \), \( \Theta \) is inner, that is, \( \Theta(\zeta) \) is an isometry for a.e. \( \zeta \in \mathbb{T} \), and \( \Omega \) is outer. The definition of outer function is not recalled here, but it need to mentioned that for an outer function \( \Omega \), \( \text{clo} \Omega(z)D = \mathcal{E}_s \) for all \( z \in \mathbb{D} \) and for a.e. \( z \in \mathbb{T} \). Recall that a function \( F \in H^\infty(\mathcal{E} \to \mathcal{E}_s) \) is called +-inner (+-outer), if the function \( F_\ast \in H^\infty(\mathcal{E}_s \to \mathcal{E}) \), \( F_\ast(z) = F^*(\zeta), z \in \mathbb{D} \), is inner (outer) (see [16, Ch.V], also [14, §A.3.11.5]).

Every analytic operator-valued function \( F \in H^\infty(\mathcal{E} \to \mathcal{E}_s) \) such that \( \|F\| \leq 1 \) can be represented as an orthogonal sum of a unitary constant and a purely contractive function \( F_0 \), that is, there exist representations of Hilbert spaces \( \mathcal{E} = \mathcal{E} \oplus \mathcal{E}_0 \) and \( \mathcal{E}_s = \mathcal{E}_s \oplus \mathcal{E}_{s0} \) and a unitary operator \( W : \mathcal{E}_s \to \mathcal{E}_s \) such that \( F(z)\mathcal{E}_s \subset \mathcal{E}_s, F(z)|\mathcal{E}_s = W, F(z)\mathcal{E}_0 \subset \mathcal{E}_{s0}, F_0(z) = F(z)|\mathcal{E}_0 \) for every \( z \in \mathbb{D} \), and \( \|F_0(0)x\| < \|x\| \) for every \( x \in \mathcal{E}_0, x \neq 0 \) ([16, Proposition V.2.1]). In all questions considered in this note it can be supposed that \( \|F\| \leq 1 \) and \( F \) is purely contractive.

2010 Mathematics Subject Classification. Primary 30H80; Secondary 47A45, 47B20.

Key words and phrases. Operator corona theorem, contraction, similarity to an isometry.

 Partially supported by RFBR grant No. 14-01-00748-a.
The Operator Corona Problem is to find necessary and sufficient condition for a function \(F \in H^\infty(\mathcal{E} \to \mathcal{E}_s)\) to be left invertible, that is, to exist a function \(G \in H^\infty(\mathcal{E}_* \to \mathcal{E}_s)\) such that \(G(z)F(z) = I_{\mathcal{E}}\) for all \(z \in \mathbb{D}\). If \(F \in H^\infty(\mathcal{E} \to \mathcal{E}_s)\) is left invertible, then there exists \(\delta > 0\) such that
\[
\|F(z)x\| \geq \delta \|x\| \quad \text{for all } x \in \mathcal{E}, \ z \in \mathbb{D}.
\]
It is easy to see that if \(F\) satisfies (1.1) and \(F = \Theta\Omega\) is the inner-outer factorization of \(F\), then the outer function \(\Omega\) is invertible, and the inner function \(\Theta\) satisfies (1.1) (may be with another \(\delta\)).

The condition (1.1) is sufficient for left invertibility, if \(\dim \mathcal{E} < \infty\) [19], but is not sufficient in general [20], [21]. Also, (1.1) is not sufficient under additional assumption
\[
\dim \mathcal{E}_* \ominus F(z)\mathcal{E} = 1 \quad \text{for all } z \in \mathbb{D}.
\]
In [23], for every \(\delta, 0 < \delta < 1/3\), two functions \(F_1, F_2 \in H^\infty(\mathcal{E} \to \mathcal{E}_s)\) are constructed such that
\[
\|x\| \geq \|F_k(z)x\| \geq \delta \|x\| \quad \text{for all } x \in \mathcal{E}, \ z \in \mathbb{D},
\]
(1.2) is fulfilled for \(F_k, k = 1, 2\), \(F_1(\zeta)\mathcal{E} = \mathcal{E}_s\) for a.e. \(\zeta \in \mathbb{T}\), \(\dim \mathcal{E}_* \ominus F_2(\zeta)\mathcal{E} = 1\) for a.e. \(\zeta \in \mathbb{T}\), but \(F_1\) and \(F_2\) are not left invertible. It is mentioned in [23] that the method from [20], [21] gives examples of such functions for \(\delta < 1/\sqrt{2}\), and a question was posed if for every \(\delta, 0 < \delta < 1\), there exists \(F \in H^\infty(\mathcal{E} \to \mathcal{E}_s)\) such that \(F\) satisfies (1.2) and (1.3), and \(F\) is not left invertible. In this note, it is shown that such function \(F\) exists for every \(0 < \delta < 1\), and any from the following cases can be realized:
\[
F(\zeta)\mathcal{E} = \mathcal{E}_s \quad \text{for a.e. } \zeta \in \mathbb{T},
\]
(1.4) or
\[
\dim \mathcal{E}_* \ominus F(\zeta)\mathcal{E} = 1 \quad \text{for a.e. } \zeta \in \mathbb{T},
\]
(1.5) or
\[
\dim \mathcal{E}_* \ominus F(\zeta)\mathcal{E} = 1 \quad \text{for a.e. } \zeta \in \mathbb{E}, \text{ and } F(\zeta)\mathcal{E} = \mathcal{E}_s \quad \text{for a.e. } \zeta \in \mathbb{T} \setminus \mathbb{E},
\]
(1.6) where \(E \subset \mathbb{T}\) is a closed set satisfying the Carleson condition with \(0 < m(E) < 1\) (see Sec. 5 of this note where the definition is recalled).

Actually, not operator-valued functions, but contractions are constructed, and the required functions are the characteristic functions of these contractions [16, Ch. VI], see Sec. 3 of this note.

In [22], some sufficient conditions are given, which imply the left invertibility of functions, in particular, it is proved in [22], that if \(F \in H^\infty(\mathcal{E} \to \mathcal{E}_s)\) satisfies to (1.1) and
\[
\sup_{z \in \mathbb{D}} \|I_{\mathcal{E}} - F(z)^*F(z)\|_{\mathcal{E}_1} < \infty,
\]
(1.7) where \(\mathcal{E}_1\) is the trace class of operators, then \(F\) is left invertible. In this note, it is shown that the condition (1.7) is necessary for left invertibility of \(F\), if \(F\) is inner and \(\dim \mathcal{E}_* \ominus F(z)\mathcal{E} < \infty\) for some \(z \in \mathbb{D}\). Actually, an appropriate fact is proved for contractions with such characteristic functions, and the statement on function follows from the fact for contractions.
The function $F \in H^\infty(\mathcal{E} \to \mathcal{E}_s)$ has a left scalar multiple if there exist $G \in H^\infty(\mathcal{E}_s \to \mathcal{E})$ and a function $\rho \in H^\infty$, where $H^\infty$ is the algebra of all bounded analytic functions on $\mathbb{D}$, such that $\rho(z)I_\mathcal{E} = G(z)F(z)$ for all $z \in \mathbb{D}$. The left invertibility of $F$ means that $F$ has a scalar multiple, which is invertible in $H^\infty$. Functions $F_1$ and $F_2$ from [23] mentioned above do not have scalar multiple, the proof is actually the same as the proof that $F_1$ and $F_2$ are not left invertible. In this note, it is shown that the existence of the left scalar multiple of $F$ with (1.1) is not sufficient for the left invertibility of $F$, even if $F$ is inner and $F$ satisfies (1.2). In this case, $I_\mathcal{E} - F(z)^*F(z) \in \mathcal{S}_1$ for every $z \in \mathbb{D}$, but $\sup_{z \in \mathbb{D}} \|I_\mathcal{E} - F(z)^*F(z)\|_{\mathcal{S}_1} = \infty$.

Again, an appropriate contraction is constructed, and $F$ is the characteristic function of this contraction.

We shall use the following notation: $\mathbb{D}$ is the open unit disk, $\mathbb{T}$ is the unit circle, $m$ is the normalized Lebesgue measure on $\mathbb{T}$, and $H^2$ is the Hardy space in $\mathbb{D}$. For a positive integer $n$, $1 \leq n < \infty$, $H^2_n$ and $L^2_n$ are orthogonal sums of $n$ copies of spaces $H^2$ and $L^2 = L^2(\mathbb{T}, m)$, respectively. The unilateral shift $S_n$ and the bilateral shift $U_n$ of multiplicity $n$ are the operators of multiplication by the independent variable in spaces $H^2_n$ and $L^2_n$, respectively. For a Borel set $\sigma \subset \mathbb{T}$, by $U(\sigma)$ we denote the operator of multiplication by the independent variable on the space $L^2(\sigma, m)$ of functions from $L^2$ that are equal to zero a.e. on $\mathbb{T} \setminus \sigma$.

For a Hilbert space $H$, by $I_H$ and $\mathbb{O}_H$ the identity and the zero operators acting on $H$ are denoted, respectively.

Let $T$ and $R$ be operators on spaces $H$ and $K$, respectively, and let $X : H \to K$ be a (linear, bounded) operator which intertwines $T$ and $R$: $XT = RX$. If $X$ is unitary, then $T$ and $R$ are called unitarily equivalent, in notation: $T \cong R$. If $X$ is invertible (the inverse $X^{-1}$ is bounded), then $T$ and $R$ are called similar, in notation: $T \approx R$. If $X$ a quasiaffinity, that is, $\ker X = \{0\}$ and $\text{clos} X H = K$, then $T$ is called a quasiaffine transform of $R$, in notation: $T \prec R$. If $T \subset R$ and $R \subset T$, then $T$ and $R$ are called quasisimilar, in notation: $T \sim R$.

Let $\mathcal{H}$ be a Hilbert space, and let $T : \mathcal{H} \to \mathcal{H}$ be a (linear, bounded) operator. $T$ is called a contraction, if $\|T\| \leq 1$. Let $T$ be a contraction on a space $\mathcal{H}$. $T$ is of class $C_1$, $(T \in C_1)$, if $\lim_{n \to \infty} \|T^n x\| > 0$ for each $x \in \mathcal{H}$, $x \neq 0$. $T$ is of class $C_0$, $(T \in C_0)$, if $\lim_{n \to \infty} \|T^n x\| = 0$ for each $x \in \mathcal{H}$, and $T$ is of class $C_\alpha$, $a = 0, 1$, if $T^a$ is of class $C_\alpha$.

It is easy to see that if a contraction $T$ is a quasiaffine transform of an isometry, then $T$ is of class $C_1$, and if $T$ is a quasiaffine transform of a unilateral shift, then $T$ is of class $C_{10}$.

The paper is organized as follows. In Sec. 2 and 3, the known facts about contractions, their relations to isometries, and their characteristic functions are collected. In Sec. 4, the necessity of (1.7) to the left invertibility of some operator-valued functions is proved. Sec. 5 is the main section of this paper, where for any $\delta$, $0 < \delta < 1$, examples of subnormal contractions are constructed such that their characteristic functions satisfy (1.2), and (1.3) with $\delta$, and are not left invertible.
2. ISOMETRIC AND UNITARY ASYMPTOTES OF CONTRACTIONS

For a contraction $T$ the isometric asymptote $(X_{T^+}, T^{(a)}_{+})$, and the unitary asymptote $(X_{T}, T^{(a)})$ was defined, see, for example, [16, Ch. IX.1]. There are some ways to construct the isometric asymptote of the contraction, for our purpose, it is convenient to use the following, see [11].

Let $(\cdot, \cdot)$ be the inner product on the Hilbert space $H$, and let $T: H \to H$ be a contraction. Define a new semi-inner product on $H$ by the formula $\langle x, y \rangle = \lim_{n \to \infty} \langle T^n x, T^n y \rangle$, where $x, y \in H$. Set

$$H_0 = H_{T,0} = \{ x \in H : \langle x, x \rangle = 0 \}.$$  

Then the factor space $H/H_0$ with the inner product $\langle x+H_0, y+H_0 \rangle = \langle x, y \rangle$ will be an inner product space. Let $H_+^{(a)}$ denote the resulting Hilbert space obtained by completion, and let $X_{T^+}: H \to H_+^{(a)}$ be the natural imbedding, $X_{T^+}x = x + H_0$. Clearly, $X_{T^+}$ is a (linear, bounded) operator, and $\|X_{T^+}\| \leq 1$. Clearly, $(Tx, Ty) = \langle x, y \rangle$ for every $x, y \in H$. Therefore, $T_1 : x + H_0 \mapsto Tx + H_0$ is a well-defined isometry on $H/H_0$. Denote by $T_{+}^{(a)}$ the continuous extension of $T_1$ to the space $H_+^{(a)}$. Clearly, $X_{T^+}T = T_{+}^{(a)}X_{T^+}$. The pair $(X_{T^+}, T_{+}^{(a)})$ is called the isometric asymptote of a contraction $T$. The operator $X_{T^+}$ is called the canonical intertwining mapping.

A contraction $T$ is similar to an isometry $V$ if and only if $X_{T^+}$ is an invertible operator, that is, ker $X_{T^+} = \{0\}$ and $X_{T^+}H = H_+^{(a)}$, and in this case, $V \cong T_{+}^{(a)}$ (see [11, Theorem 1]). In particular, if $T$ is a contraction of class $C_{10}$, and $T_{+}^{(a)}$ is a unitary operator, then $T$ is not similar to an isometry.

Denote by $T^{(a)}$ the minimal unitary extension of $T_{+}^{(a)}$, by $H^{(a)} \supset H_+^{(a)}$ the space on which $T^{(a)}$ acts, and by $X_T$ the imbedding of $H$ into $H^{(a)}$. Clearly, $X_T T = T^{(a)}X_T$ and $X_T x = X_T x$ for every $x \in H$. The pair $(X_T, T^{(a)})$ is called the unitary asymptote $(X, T^{(a)})$ of a contraction $T$.

3. CONTRACTIONS AND THEIR CHARACTERISTIC FUNCTIONS

All statement of this section are well-known and can be found in [16, Ch. VI], see also [14, Ch. C.1].

Let $H$ be a separable Hilbert space, and let $T: H \to H$ be a contraction. A contraction $T$ is called completely nonunitary, if $T$ has no invariant subspace such that the restriction of $T$ on this subspace is unitary. For a contraction $T$ put $D_T = \text{clos}(I_H - T^*T)H$. It is easy to see that

$$\text{if } x \in H \ominus D_T, \text{ then } x = T^*Tx \text{ and } \|Tx\| = \|x\|. \leqno{(3.1)}$$

Also, $TD_T \subset D_{T^*}$ and $T(H \ominus D_T) = H \ominus D_{T^*}$ (see [16, Ch. I.3.1]), therefore,

$$\dim D_{T^*} \ominus TD_T = \dim H \ominus TH. \leqno{(3.2)}$$

Since $I_H - T^*T = (I_{D_T} - T^*T|_{D_T}) \oplus \mathcal{H} \ominus D_T$,

$$\|I_H - T^*T\|_{\mathcal{S}_1} = \|I_{D_T} - T^*T|_{D_T}\|_{\mathcal{S}_1}. \leqno{(3.3)}$$

Lemma 3.1. Let $T: H \to H$ be a contraction, and let $0 < \delta \leq 1$. Then $\|Tx\| \geq \delta\|x\|$ for every $x \in H$ if and only if $\| Tx \| \geq \delta \|x\|$ for every $x \in D_T$. 

Proof. Indeed, it need to prove the “if” part only. Let \( x \in \mathcal{D}_T \), and let \( y \in \mathcal{H} \otimes \mathcal{D}_T \). Then, by (3.1), \((Tx,Ty) = (x,T^*Ty) = (x,y) = 0 \) and \(|Ty| = |y| \geq \delta |y| \). Therefore, \(|T(x+y)|^2 = |Tx|^2 + |Ty|^2 \geq \delta^2 |x|^2 + \delta^2 |y|^2 = \delta^2 |x+y|^2\). \( \square \)

The characteristic function \( \Theta_T \) of the contraction \( T \) is the analytic operator-valued function acting by the formula

\[
(3.4) \quad \Theta_T(z) = (-T+z(I-TT^*)^{1/2}(I-zT^*)^{-1}(I-T^*T)^{1/2})|_{\mathcal{D}_T}, \quad z \in \mathbb{D}.
\]

For every \( z \in \mathbb{D} \) the inclusion \( \Theta_T(z)\mathcal{D}_T \subset \mathcal{D}_{T^*} \) holds, the mapping \( z \mapsto \Theta_T(z) \) is an analytic function from \( \mathbb{D} \) to the space of all (linear, bounded) operators from \( \mathcal{D}_T \) to \( \mathcal{D}_{T^*} \), and \( \|\Theta_T(z)\| \leq 1 \) for every \( z \in \mathbb{D} \). That is, \( \Theta_T \in H^\infty(\mathcal{D}_T \rightarrow \mathcal{D}_{T^*}) \), and \( \|\Theta_T\| \leq 1 \). It is easy to see that \( \Theta_T \) is purely contractive. Conversely, for every analytic operator-valued function \( F \in H^\infty(\mathcal{E} \rightarrow \mathcal{E}_1) \) such that \( \|F\| \leq 1 \) and \( F \) is purely contractive there exists a contraction \( T \) such that \( F = \Theta_T \) [16, Ch. VI].

The following theorem was proved in [15], see also [14, C.1.5.5].

**Theorem A.** [15] The contraction \( T \) is similar to an isometry if and only if \( \Theta_T \) is left invertible.

Let \( T \) be a completely nonunitary contraction. Then \( T \) is of class \( C_1 \), if and only if \( \Theta_T \) is \(*\)-outer, and \( T \) is of class \( C_0 \), if and only if \( \Theta_T \) is inner [16, VI.3.5].

Recall that the *multiplicity* of an operator is the minimum dimension of its reproducing subspaces. An operator is called *cyclic* if its multiplicity is equal to 1.

The following theorem was proved in [7], [10], [17], [24], [25].

**Theorem B.** Let \( T \) be a contraction, and let \( 1 \leq n < \infty \). The following are equivalent:

1. \( T \prec S_n \);
2. \( T \) is of class \( C_{10} \), \( \dim \ker T^* = n \), and \( I - T^*T \in \mathcal{S}_1 \);
3. \( \Theta_T \) is an inner \(*\)-outer function, \( \Theta_T \) has a left scalar multiple, and \( \dim \mathcal{D}_T \cap \Theta_T(\lambda)\mathcal{D}_T = n \) for some \( \lambda \in \mathbb{D} \).

Moreover, if \( T \) is a contraction such that \( T \prec S_n \), \( 1 \leq n < \infty \), then the following are equivalent:

4. \( T \sim S_n \);
5. multiplicity of \( T \) is equal to \( n \);
6. \( \Theta_T \) has an outer left scalar multiple.

**Remark.** If \( T \) is a contraction and \( T \prec S_n \), \( 1 \leq n < \infty \), then \( b_\lambda(T) \) is a contraction and \( b_\lambda(T) < b_\lambda(S_n) \cong S_n \). Therefore, \( I - b_\lambda(T)^*b_\lambda(T) \in \mathcal{S}_1 \) for every \( \lambda \in \mathbb{D} \). Here \( b_\lambda(T) = (T - \lambda)(I - \lambda T)^{-1} \).

Let \( T \) be a completely nonunitary contraction. Put

\[
(3.5) \quad \Delta_\omega(\zeta) = (I_{\mathcal{D}_T} - \Theta_T(\zeta)\Theta_T(\zeta^*)^{1/2}, \quad \zeta \in \mathbb{T}, \quad \omega_T = \{ \zeta \in \mathbb{T} : \Delta_\omega(\zeta) \neq \emptyset \}.
\]

Then the unitary asymptote \( T^{(a)} \) of a completely nonunitary contraction \( T \) is unitarily equivalent to the operator of multiplication by the independent
variable $\zeta$ on $\text{clos} \Delta_s L^2(D_{T^*})$. In particular, $T^{(a)}$ is cyclic if and only if
\begin{equation}
\dim \Delta_s(\zeta)D_{T^*} \leq 1 \quad \text{for a.e. } \zeta \in \mathbb{T},
\end{equation}
and in this case $T^{(a)} \cong U(\omega_T)$ (see [16, Ch. IX.2]). Also, if the function $\Theta_T$ is inner and (3.6) holds, then
\begin{equation}
\omega_T = \{ \zeta \in \mathbb{T} : \dim D_{T^*} \ominus \Theta(\zeta)D_T = 1 \}
\end{equation}
and $T \setminus \omega_T = \{ \zeta \in \mathbb{T} : \Theta(\zeta)D_T = D_{T^*} \}$.

For $\lambda \in \mathbb{D}$ put $b_\lambda(z) = \frac{z - \lambda}{1 - \lambda z}$, $z \in \mathbb{D}$. Then $b_\lambda(T) = (T - \lambda)(I - \overline{\lambda}T)^{-1}$ is a contraction. For every $\lambda \in \mathbb{D}$ there exists unitary operators $V_\lambda : D_T \to D_{b_\lambda(T)}$ and $V^*_\lambda : D_{b_\lambda(T)^*} \to D_{T^*}$ such that
\begin{equation}
V_\lambda \Theta b_\lambda(T)(z)V^*_\lambda = \Theta_T(b_\lambda(z)).
\end{equation}
Setting $z = 0$ in (3.4) and (3.8), we conclude that
\begin{equation}
\Theta_T(\lambda) = -V_\lambda^* b_\lambda(T)V_\lambda \quad \text{for every } \lambda \in \mathbb{D}
\end{equation}
([16, Ch. VI.1.3]).

The following lemma is a straightforward consequence of (3.2), (3.3), (3.9), and Lemma 3.1.

**Lemma 3.2.** Suppose $T : \mathcal{H} \to \mathcal{H}$ is a completely nonunitary contraction.

(i) Let $0 < \delta \leq 1$. Then $\|\Theta_T(\lambda)x\| \geq \delta \|x\|$ for every $x \in D_T$, $\lambda \in \mathbb{D}$, if and only if $\|b_\lambda(T)x\| \geq \delta \|x\|$ for every $x \in \mathcal{H}$, $\lambda \in \mathbb{D}$.
(ii) $\dim D_{T^*} \ominus \Theta_T(\lambda)D_T = \dim \mathcal{H} \ominus b_\lambda(T)\mathcal{H}$ for every $\lambda \in \mathbb{D}$.
(iii) $\|I_{D_T} - \Theta_T^*(\lambda)\Theta_T(\lambda)\|_{\mathcal{E}_1} = \|I_{\mathcal{H}} - b_\lambda(T)^*b_\lambda(T)\|_{\mathcal{E}_1}$ for every $\lambda \in \mathbb{D}$.

**4. On contractions similar to an isometry**

The following theorem is actually proved in [7, Theorem 2.1] (see also enlarged version on arXiv).

**Theorem 4.1.** [7] Suppose $T$ is a contraction with finite multiplicity, and $T$ is similar to an isometry. Then
\begin{equation}
\sup_{\lambda \in \mathbb{D}} \|I - b_\lambda(T)^*b_\lambda(T)\|_{\mathcal{E}_1} < \infty.
\end{equation}

**Corollary 4.2.** Suppose $\mathcal{E}$, $\mathcal{E}_s$ are Hilbert spaces, $F \in \mathcal{H}^\infty(\mathcal{E} \to \mathcal{E}_s)$ is an inner function, and $\dim \mathcal{E}_s \ominus F(\lambda)\mathcal{E} < \infty$ for some $\lambda \in \mathbb{D}$. If $F$ is left invertible, then $F$ satisfies (1.7).

**Proof.** Let $\mathcal{H}$ be a Hilbert space, and let $T : \mathcal{H} \to \mathcal{H}$ be a contraction such that $\Theta_T = F$, where $\Theta_T$ is the characteristic function of $T$ (see [16, Ch. VI.3]). Since $F$ is inner, $T \in C_0$, see [16, VI.3.5]. By Lemma 3.2(ii),
\begin{equation}
\dim \mathcal{H} \ominus b_\lambda(T)\mathcal{H} = \dim \mathcal{E}_s \ominus F(\lambda)\mathcal{E} < \infty.
\end{equation}
Now suppose that $F$ is left invertible. Then, by Theorem A, there exist a Hilbert space $\mathcal{K}$ and an isometry $V : \mathcal{K} \to \mathcal{K}$ such that $T \cong V$. Since $T \in C_0$, $V$ is a unilateral shift, and the multiplicity of $V$ is equal to
\begin{equation}
\dim \mathcal{K} \ominus b_\lambda(V)\mathcal{K} = \dim \mathcal{H} \ominus b_\lambda(T)\mathcal{H} < \infty.
\end{equation}
By Theorem 4.1, $T$ satisfies (4.1). By Lemma 3.2(iii), $F$ satisfies (1.7).
5. Subnormal contractions

Operators that are considered in this sections are subnormal ones, and are studied by many authors, the reader can consult with the book [6].

Let \( \nu \) be a positive finite Borel measure on the closed unit disk \( \mathbb{D} \). Denote by \( P^2(\nu) \) the closure of analytic polynomials in \( L^2(\nu) \), and by \( S_\nu \) the operator of multiplication by the independent variable in \( P^2(\nu) \), i.e.

\[
S_\nu : P^2(\nu) \to P^2(\nu),
\]

\[
(S_\nu f)(z) = zf(z) \text{ for a.e. } z \in \mathbb{D} \text{ with respect to } \nu, \quad f \in P^2(\nu).
\]

Clearly, \( S_\nu \) is a contraction.

Recall that \( m \) is the Lebesgue measure on \( \mathbb{T} \). If \( \nu = m \), then \( S_\nu \) is the unilaterial shift of multiplicity 1, it is denoted by \( S \) in this section.

The following lemma is a straightforward consequence of the construction of the isometric asymptote of a contraction from [11], see Sec. 2 of this paper, so its proof is omitted.

**Lemma 5.1.** Suppose \( \nu \) is a positive finite Borel measure on \( \mathbb{D} \), \( \mathcal{H} = P^2(\nu) \), and \( T = S_\nu \). Then \( \mathcal{H}_{0,T} = \{ f \in P^2(\nu) : f = 0 \text{ a.e. on } \mathbb{T} \text{ with respect to } \nu \} \), \( \mathcal{H}^{(a)} = P^2(\nu|_\mathbb{T}) \),

\[
X_{T,+} : P^2(\nu) \to P^2(\nu|_\mathbb{T}), \quad X_{T,+} f = f|_\mathbb{T}, \quad f \in P^2(\nu),
\]

is the natural imbedding, and \( T^{(a)} = S|_\mathbb{T} \).

The proof of the following lemma is obvious and omitted.

**Lemma 5.2.** Suppose \( \nu \) is a positive finite Borel measure on \( \mathbb{D} \), and \( f \in P^2(\nu) \). Then there exists \( \lambda \in \mathbb{D} \) such that \( \|b_\lambda f\| = \|f\| \) if and only if \( f(z) = 0 \) for a.e. \( z \in \mathbb{D} \) with respect to \( \nu \).

**Corollary 5.3.** Suppose \( \nu \) is a positive finite Borel measure on \( \mathbb{D} \), and \( \lambda \in \mathbb{D} \). Then

\[
P^2(\nu) \ominus D_{b_\lambda(S_\nu)} \subset \{ f \in P^2(\nu) : f(z) = 0 \text{ for a.e. } z \in \mathbb{D} \text{ with respect to } \nu \}
\]

and

\[
P^2(\nu) \ominus D_{b_\lambda(S_\nu)^*} \subset \{ f \in P^2(\nu) : f(z) = 0 \text{ for a.e. } z \in \mathbb{D} \text{ with respect to } \nu \}.
\]

**Proof.** Let \( P_+ : L^2(\nu) \to P^2(\nu) \) be the orthogonal projection. It is easy to see that \( (b_\lambda(S_\nu)f)(z) = b_\lambda(z)f(z) \) for a.e. \( z \in \mathbb{D} \) with respect to \( \nu \), and \( b_\lambda(S_\nu)^*f = P_+(\overline{b_\lambda f}), \quad f \in P^2(\nu) \). If \( f \in P^2(\nu) \ominus D_{b_\lambda(S_\nu)} \), then, by (3.1), \( \|f\| = \|b_\lambda f\| \), and, by Lemma 5.2, \( f(z) = 0 \) for a.e. \( z \in \mathbb{D} \) with respect to \( \nu \).

If \( f \in P^2(\nu) \ominus D_{b_\lambda(S_\nu)^*} \), then, by (3.1), \( \|f\| = \|P_+(\overline{b_\lambda f})\| \leq \|b_\lambda f\| = \|b_\lambda f\| \), and, by Lemma 5.2, \( f(z) = 0 \) for a.e. \( z \in \mathbb{D} \) with respect to \( \nu \). \( \square \)

Denote by \( m_\alpha \) the normalized Lebesgue measure on the unit disk \( \mathbb{D} \), for \(-1 < \alpha < \infty \) put \( dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dm_2(z) \). It is well known that the Bergman space \( P^2(A_\alpha) \) has the following properties: \( f \in P^2(A_\alpha) \) if and only if \( f \) is an analytic function in \( \mathbb{D} \) and \( f \in L^2(A_\alpha) \), the functional \( f \mapsto f(z) \) is bounded on \( P^2(A_\alpha) \) for every \( z \in \mathbb{D} \), and there exists a constant \( C_\alpha > 0 \) (which depends on \( \alpha \)) such that

\[
|f(z)| \leq C_\alpha \frac{\|f\|_{P^2(A_\alpha)}}{(1 - |z|^2)^{1+\alpha/2}}, \quad z \in \mathbb{D},
\]
(see, for example, [9, Sec. 1.1 and 1.2]). It is easy to see that \( S_{A_0} \in C_{00} \).

**Lemma 5.4.** [3, Lemma 4.2] Let \(-1 < \alpha < \infty\). Then for every \( f \in P^2(A_0) \) and \( \lambda \in \mathbb{D} \)
\[
\int_\mathbb{D} |b_\lambda f|^2 dA_0 \geq \frac{1}{\alpha + 2} \int_\mathbb{D} |f|^2 dA_0.
\]

**Corollary 5.5.** Let \(-1 < \alpha < \infty\), and let \( \mu \) be a positive finite Borel measure on \( \mathbb{T} \). Then for every \( f \in P^2(A_0 + \mu) \) and \( \lambda \in \mathbb{D} \)
\[
\int_\mathbb{D} |b_\lambda f|^2 d(A_0 + \mu) \geq \frac{1}{\alpha + 2} \int_\mathbb{D} |f|^2 d(A_0 + \mu).
\]

**Proof.** Clearly, \( P^2(A_0 + \mu) \subset P^2(A_0) \). Let \( f \in P^2(A_0 + \mu) \), and let \( \lambda \in \mathbb{D} \).
We have
\[
\int_\mathbb{D} |b_\lambda f|^2 d(A_0 + \mu) = \int_\mathbb{D} |b_\lambda f|^2 dA_0 + \int_\mathbb{D} |b_\lambda f|^2 d\mu 
\geq \frac{1}{\alpha + 2} \int_\mathbb{D} |f|^2 dA_0 + \int_\mathbb{T} |f|^2 d\mu \geq \frac{1}{\alpha + 2} \int_\mathbb{D} |f|^2 d(A_0 + \mu),
\]
because of \( |b_\lambda| = 1 \) on \( \mathbb{T} \) and \( 1 > 1/(\alpha + 2) \) for \(-1 < \alpha < \infty\). \( \square \)

Recall the following definition.

**Definition.** Let \( E \) be a closed subset of \( \mathbb{T} \), and let \( \{J_k\}_k \) be the collection of open arcs of \( \mathbb{T} \) such that \( J_k \cap J_\ell = \emptyset \) for \( k \neq \ell \) and \( E = \bigcup_k J_k \). The set \( E \) satisfies the Carleson condition if \( \sum_k m(J_k) \log m(J_k) > -\infty \).

Let \( w \in L^1(\mathbb{T}, m) \), \( w \geq 0 \) a.e. on \( \mathbb{T} \). Then \( P^2(wm) = L^2(wm) \) if and only if \( \log w \notin L^1(\mathbb{T}, m) \), and then \( S_{wm} \cong U(\sigma) \), where \( \sigma \subset \mathbb{T} \) is a measurable set such that \( wm \) and \( m|_\sigma \) are mutually absolutely continuous. If \( \log w \in L^1(\mathbb{T}, m) \), then there exists an outer function \( \psi \in H^2 \) such that \( |\psi|^2 = w \) a.e. on \( \mathbb{T} \). Then
\[
(5.2) \quad P^2(wm) = H^2_{\psi} = \left\{ \frac{h}{\psi} : h \in H^2 \right\}, \quad \left\| \frac{h}{\psi} \right\|_{P^2(wm)} = \|h\|_{H^2}, \quad h \in H^2,
\]
and \( S_{wm} \cong S \) (see, for example, [6, Ch. III.12] or [14, A.4.1.5]).

In Theorems 5.6 and 5.7, we consider nontangential boundary values of functions from \( P^2(\mu) \) for some measures \( \mu \). Nontangential boundary values of functions from \( P^t(\mu) \) with \( 1 \leq t < \infty \) are considered in [4] in relation to another questions, see also references therein, especially [1], [12], [13], [18], and [2]. In Theorems 5.6 and 5.7 we formulate particular cases of these results in the form convenient to our purpose.

Theorems 5.6 and the main part of Theorem 5.7 were proved in [8, Sec. 2] for \( \alpha = 0 \), but the proofs are the same in the case of \(-1 < \alpha \leq 0 \) (because the estimate (5.1) involves the estimate \( |f(z)| \leq C_\alpha \frac{|f||P^2(A_0)\|}{\|f\|_{\mathbb{D}}} \) for \(-1 < \alpha \leq 0 \), which is used in [8, Sec. 2]), therefore, the proofs of Theorem 5.6 and of the main part of Theorem 5.7 are omitted. In addition, to prove Theorems 5.6 and 5.7, one needs to apply the notion of isometric asymptote (see Sec. 2 of this paper and references therein).
Theorem 5.6. [8] Let $-1 < \alpha \leq 0$, and let $E \subset \mathbb{T}$ be a closed set such that $0 < m(E) < 1$ and $E$ satisfies the Carleson condition. Then the functional $f \mapsto f(z)$ is bounded on $P^2(A_{\alpha} + m| E)$ for every $z \in \mathbb{D}$. Furthermore, for $f \in P^2(A_{\alpha} + m| E)$ the restriction $f |_E$ is analytic on $\mathbb{D}$, $f |_E$ has nontangential boundary values a.e. on $E$ with respect to $m$, which coincide with $f |E$. Therefore, $S_{A_{\alpha} + m| E} \in C_{10}$. Also, $I - S_{A_{\alpha} + m| E}^* S_{A_{\alpha} + m| E}$ is compact, and $(S_{A_{\alpha} + m| E})^n = U(E)$. Thus, $S_{A_{\alpha} + m| E}$ is not similar to an isometry.

Theorem 5.7. [8] Let $-1 < \alpha \leq 0$, and let $w \in L^1(\mathbb{T}, m)$. Suppose that for every closed arc $J \subset \mathbb{T} \setminus \{1\}$ there exist two constants $0 < c_J < C_J < \infty$ such that $c_J \leq w \leq C_J$ a.e. on $J$ (with respect to $m$). Then the functional $f \mapsto f(z)$ is bounded on $P^2(A_{\alpha} + w| E)$ for every $z \in \mathbb{D}$. Furthermore, for $f \in P^2(A_{\alpha} + w| E)$ the restriction $f |_E$ is analytic on $\mathbb{D}$, $f |_E$ has nontangential boundary values a.e. on $\mathbb{T}$ with respect to $m$, which coincide with $f |\mathbb{D}$. Therefore, $S_{A_{\alpha} + w| E} \in C_{10}$. Also, $I - S_{A_{\alpha} + w| E}^* S_{A_{\alpha} + w| E}$ is compact, $(S_{A_{\alpha} + w| E})^n = S_{w| E}$, and the canonical mapping which intertwines $S_{A_{\alpha} + w| E}$ with $S_{w| E}$ is the natural imbedding

$$P^2(A_{\alpha} + w| E) \rightarrow P^2(w| E), \quad f \mapsto f |\mathbb{D}, \quad f \in P^2(A_{\alpha} + w| E).$$

Therefore,

(i) $S_{A_{\alpha} + w| E} \sim S$ if and only if $\log w \in L^1(\mathbb{T}, m)$;

(ii) $S_{A_{\alpha} + w| E}$ is similar to an isometry if and only if $S_{A_{\alpha} + w| E} \approx S$;

(iii) $S_{A_{\alpha} + w| E} \approx S$ if and only if $\log w \in L^1(\mathbb{T}, m)$ and for every $h \in H^2$ there exists $f \in P^2(A_{\alpha} + w| E)$ such that $f |\mathbb{D} = h/\psi \, \text{a.e. on } \mathbb{T}$ (with respect to $m$), where $\psi \in H^2$ is an outer function such that $|\psi|^2 = w$ a.e. on $\mathbb{T}$.

Remark. In the conditions of Theorem 5.7, let $h, \psi \in H^2$, $\psi(z) \neq 0$ for every $z \in \mathbb{D}$, $f \in P^2(A_{\alpha} + w| E)$, and $f |\mathbb{D} = h/\psi \, \text{a.e. on } \mathbb{T}$ (with respect to $m$). Then $f(z) = h(z)/\psi(z)$ for every $z \in \mathbb{D}$. Indeed, set $g(z) = h(z)/\psi(z)$, $z \in \mathbb{D}$. Then $g$ is a function analytic on $\mathbb{D}$, and $g$ has nontangential boundary values $h(\zeta)/\psi(\zeta)$ for a.e. $\zeta \in \mathbb{T}$. Then $f - g$ is a function analytic on $\mathbb{D}$, and $f - g$ has zero nontangential boundary values a.e. on $\mathbb{T}$. By Privalov’s theorem (see, for example, [5, Theorem 8.1]), $f(z) = g(z)$ for every $z \in \mathbb{D}$. Therefore, if the conditions (iii) of Theorem 5.7 are fulfilled, then $P^2(A_{\alpha} + w| E) = H^2/\psi$ as the set, and the norms on these spaces are equivalent.

Proof of Theorem 5.7. The main part of Theorem 5.7 is proved in [8, Sec. 2]. Let $X$ be the imbedding

$$X: P^2(A_{\alpha} + w| E) \rightarrow P^2(w| E), \quad Xf = f |\mathbb{D}, \quad f \in P^2(A_{\alpha} + w| E).$$

Then $XS_{A_{\alpha} + w| E} = S_{w| E} X$. Since $S_{A_{\alpha} + w| E} \in C_{10}$, ker $X = \{0\}$, therefore, $X$ is a quasiisometry which realizes the relation $S_{A_{\alpha} + w| E} \approx S_{w| E}$. If $\log w \in L^1(\mathbb{T}, m)$, then $S_{w| E} \approx S$, therefore, $S_{A_{\alpha} + w| E} \approx S$. Since $S_{A_{\alpha} + w| E}$ is a cyclic contraction, $S_{A_{\alpha} + w| E} \sim S$ by Theorem B(5). The “if” part of (i) is proved.

The assumptions of the “if” part of (iii) mean that $XP^2(A_{\alpha} + w| E) = P^2(w| E)$, see (5.2). Thus, $X$ realizes the relation $S_{A_{\alpha} + w| E} \approx S_{w| E}$, and, since $S_{w| E} \approx S$, the relation $S_{A_{\alpha} + w| E} \approx S$ is proved.

Now suppose that $S_{A_{\alpha} + w| E} \approx V$, where $V$ is an isometry. Then, by [11, Theorem 1] (see Sec. 2 of this paper), $XP^2(A_{\alpha} + w| E) = P^2(w| E)$ and $V \approx S_{w| E}$. Since $S_{A_{\alpha} + w| E} \in C_{10}$, $S_{w| E} \in C_{10}$. By [14, A.4.1.5] (see
the description of $S_{wm}$ before (5.2) in this paper), $\log w \in L^1(\mathbb{T}, m)$ and $S_{wm} \cong S$. The parts (ii) and (iii) are proved.

Now suppose that $S_{A_n + wm} \sim S$. By [11, Theorem 1], see also [16, Ch. IX.1], there exists an operator $Y$: $P^2(wm) \to H^2$ such that $YS_{wm} = SY$ and $\text{clos} YP^2(wm) = H^2$. If $\log w \notin L^1(\mathbb{T}, m)$, then $S_{wm}$ is unitary, and from the relations $Y^*S^* = S^*w^*Y^*$ and $\text{ker} Y^* = \{0\}$ we conclude that $S^* \in C_1$, a contradiction. Therefore, $\log w \in L^1(\mathbb{T}, m)$. The “only if” part of (i) is proved.

The following lemma is a variant of [9, Theorem 1.7].

**Lemma 5.8.** Let $-1 < \alpha < \infty$, and let $\beta \in \mathbb{R}$. Put $\varphi_\beta(z) = 1/(1 - z)^\beta$, $z \in \mathbb{D}$. Then $\varphi_\beta \in H^2$ if and only if $\beta < 1/2$, and $\varphi_\beta \in P^2(A_\alpha)$ if and only if $\beta < 1 + \alpha/2$.

**Proof.** Put $v_n = 2(\alpha + 1) \int_0^1 (1 - r^2)^\alpha dr$, $n \geq 0$. Then $v_n = n!\Gamma(\alpha + 2)$, where $\Gamma$ is the Gamma function, and for every function $f$ analytic on $\mathbb{D}$

$$\int_{\mathbb{D}} |f|^2 dA_\alpha = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 v_n.$$ 

If $\beta \leq 0$, then $\varphi_\beta \in P^2(A_\alpha)$ for every $\alpha$, $-1 < \alpha < \infty$. Suppose $\beta > 0$. Since $\hat{\varphi}_\beta(n) = \frac{\Gamma(\beta + n)}{n!\Gamma(\beta)}$, $n \geq 0$, we have that $\varphi_\beta \in P^2(A_\alpha)$ if and only if the series $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + n)^2}{n!\Gamma(\alpha + n + 2)}$ converges. By Stirling’s formula,

$$\frac{\Gamma(\beta + n)^2}{n!\Gamma(\alpha + n + 2)} \sim (n + 1)^{2\beta - \alpha - 3} \quad \text{as } n \to \infty.$$ 

Therefore, the series $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + n)^2}{n!\Gamma(\alpha + n + 2)}$ converges if and only if $\beta < 1 + \alpha/2$.

The first statement of the lemma can be proved similarly.

**Lemma 5.9.** Let $-1 < \alpha < 0$, and let $\beta < -1 - \alpha$. Then $S_{A_\alpha + |\varphi_\beta|m} \sim S$, but $S_{A_\alpha + |\varphi_\beta|m}$ is not similar to an isometry.

**Proof.** By Theorem 5.7(i), $S_{A_\alpha + |\varphi_\beta|m} \sim S$. Put $\psi = \varphi_\beta/2$. Clearly, $|\psi|^2 = |\varphi_\beta|$. By (5.2), $P^2(|\varphi_\beta|m) = H^2/\psi$. By Theorem 5.7(iii), if $S_{A_\alpha + |\varphi_\beta|m}$ is similar to an isometry, then $H^2/\psi = P^2(A_\alpha + |\varphi_\beta|m) \subset P^2(A_\alpha)$. Take $\gamma = 1 + \alpha/2 + \beta/2 \leq \gamma < 1/2$. Put $h = \varphi_\gamma$. Then $h \in H^2$, and $h/\psi = \varphi_{\gamma-\beta/2} \notin P^2(A_\alpha)$ by Lemma 5.8. Therefore, $S_{A_\alpha + |\varphi_\beta|m}$ is not similar to an isometry.

**Corollary 5.10.** Let $0 < \delta < 1$, and let $E \subset \mathbb{T}$ be a closed set satisfying the Carleson condition and such that $0 < m(E) < 1$. Then there exist operator-valued inner functions $F_k$, such that $F_k$ satisfy (1.2), and (1.3) with $\delta$, and $F_k$ are not left invertible, $k = 1, 2, 3$. Also, $F_1$, $F_2$, $F_3$ satisfy (1.4), (1.6), (1.5), respectively, $F_3$ has an outer left scalar multiple, and $I - F_3(z)^*F_3(z) \in \mathcal{S}_1$ for every $z \in \mathbb{D}$.

**Remark.** Since $F_3$ is not left invertible, $F_3$ does not satisfy (1.7), see [22].

**Proof of Corollary 5.10.** Put $\alpha = 1/\max(\delta^2, 1/2) - 2$, then $-1 < \alpha \leq 0$. Take $\beta < -1 - \alpha$. Put

$$H_1 = P^2(A_\alpha), \quad H_2 = P^2(A_\alpha + m|_E), \quad H_3 = P^2(A_\alpha + |\varphi_\beta|m),$$
Clearly, $T_k$ are cyclic contractions, $k = 1, 2, 3$, $T_1 \in C_{00}$, and $T_k \in C_{10}$ by Theorems 5.6 and 5.7, $k = 2, 3$. By Corollary 5.5, 

\[(5.3) \quad \|b_\lambda(T_k)f\|^2 \geq \delta^2\|f\|^2 \quad \text{for every } \lambda \in \mathbb{D}, \ f \in \mathcal{H}_k, \ k = 1, 2, 3.\]

By Theorems 5.6 and 5.7, the functionals 

\[f \mapsto f(z), \ \mathcal{H}_k \to \mathbb{C},\]

are bounded for every $z \in \mathbb{D}$, $k = 1, 2, 3$. Therefore, 

\[(5.4) \quad \dim \mathcal{H}_k \ominus b_\lambda(T_k)\mathcal{H}_k = 1 \quad \text{for every } \lambda \in \mathbb{D}, \ k = 1, 2, 3.\]

Indeed, let $k$ be fixed, and let $\lambda \in \mathbb{D}$. Then there exists $g_\lambda \in \mathcal{H}_k$ such that $f(\lambda) = (f, g_\lambda)$ for every $f \in \mathcal{H}_k$. Since $(b_\lambda(T_k)f)(z) = b_\lambda(z)f(z)$ for every $z \in \mathbb{D}$, $f \in \mathcal{H}_k$, it is clear that $g_\lambda \in \mathcal{H}_k \ominus b_\lambda(T_k)\mathcal{H}_k$. Since $T_k$ is cyclic, $T_k - \lambda I$ is cyclic, too, therefore, $\dim \mathcal{H}_k \ominus b_\lambda(T_k)\mathcal{H}_k = \dim \mathcal{H}_k \ominus (T_k - \lambda I)\mathcal{H}_k \leq 1$. The equality (5.4) is proved.

Now find the unitary asymptotes of $T_k$, $k = 1, 2, 3$, see Sec. 2 of this paper and references therein. Since $T_1 \in C_{00}$, $T_1^{(a)} = \emptyset$. By Theorem 5.6, $T_2^{(a)} = U(E)$. By Lemma 5.9, $T_3 \sim S$, therefore, $T_3^{(a)} = U(S)$, the bilateral shift of multiplicity 1. Since $T^{(a)} \cong U(\omega_T)$ for every cyclic completely nonunitary contraction $T$, where $\omega_T$ is defined in (3.5), we conclude that 

\[(5.5) \quad \omega_{T_1} = \emptyset, \ \omega_{T_2} = E, \ \omega_{T_3} = T.\]

Also, $T_1$ is not similar to an isometry, because $T_1 \in C_{00}$, and $T_2$ and $T_3$ are not similar to an isometry by Theorem 5.6 and Lemma 5.9, respectively.

Now put $F_k = \Theta_{T_k}$, that is, $F_k$ is the characteristic function of the contraction $T_k$, $k = 1, 2, 3$, see Sec. 3 of this paper and references therein. Since $T_k \in C_{00}$, $F_k$ are inner. By (5.3) and Lemma 3.2(i), $F_k$ satisfy (1.3) with $\delta$. By (5.4) and Lemma 3.2(ii), $F_k$ satisfy (1.2). $F_k$ are not left invertible, because of $T_k$ are not similar to an isometry, see Theorem A. $F_1$, $F_2$, $F_3$ satisfy (1.4), (1.6), (1.5), respectively, because of (3.7) and (5.5). Since $T_3 \sim S$ (by Lemma 5.9), $F_3$ has an outer left scalar multiple by Theorem B(6), and $I - F_3(z)^*F_3(z) \in \mathcal{S}_1$ for every $z \in \mathbb{D}$ by Lemma 3.2(iii) and Theorem B(2).

\[\square\]

References

[1] J. R. Akeroyd, Another look at some index theorems for the shift, Indiana Univ. Mat. J. 50 (2001), 705–718.

[2] J. R. Akeroyd, A note on harmonic measure, Comput. Methods Funct. Theory 7 (2007), no. 1, 91–104.

[3] A. Aleman, S. Richter, and C. Sundberg, Invariant subspaces for the backward shift on Hilbert spaces of analytic functions with regular norm, in: Bergman spaces and related topics in complex analysis. Contemp. Math. 404 (2006) AMS, Providence, RI, 1–25.

[4] A. Aleman, S. Richter, and C. Sundberg, Nontangential limits in $\mathcal{P}^k(\mu)$-spaces and the index of invariant subspaces, Ann. of Math. (2) 169 (2009), 449–490.
NOTES ON THE CODIMENSION ONE CONJECTURE

[5] E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge Univ. Press, 1966.

[6] J. B. Conway, *The theory of subnormal operators*, Math. Surveys and Monographs, Amer. Math. Soc., v. 36, 1991.

[7] M. F. Gamal’, *On contractions that are quasiaffine transforms of unilateral shifts*, Acta Sci. Math. (Szeged) **74** (2008), 755–765.

[8] M. F. Gamal’, *On contractions with compact defects*, Zap. Nauchn. Sem. POMI **366** (2009), 13–41 (Russian); translation in J. Math. Sci. (N. Y.) **165** (2010), no. 4, 435–448.

[9] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*. Graduate Texts in Mathematics, **199**, New York, 2000.

[10] V. V. Kapustin and A. V. Lipin, *Operator algebras and lattices of invariant subspaces, I, II*, Zap. Nauchn. Semin. LOMI, **178** (1989), 23–56; **190** (1991), 110–147 (Russian); English translation in: J. Soviet Math., **61** (1992), 1963–1981, J. Math. Sci., **71** (1994), 2240–2262.

[11] L. Kérchy, *Isometric asymptotes of power bounded operators*. – Indiana Univ. Math. J. **38** (1989), 173–188.

[12] T. L. Miller and R. C. Smith, *Nontangential limits of functions in some $P^2(\mu)$ spaces*, Indiana Univ. Math. J. **39** (1990), no.1, 19–26.

[13] T. L. Miller, W. Smith, and L. Yang, *Bounded point evaluations for certain $P^d(\mu)$ spaces*, Illinois J. Math. **43** (1999), no.1, 131–150.

[14] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Volume I: Hardy, Hankel, and Toeplitz, Volume II: Model operators and systems*, Math. Surveys and Monographs **92**, AMS, 2002.

[15] B. Sz.-Nagy and C. Foias, *On the structure of intertwining operators*, Acta Sci. Math. (Szeged) **35** (1973), 225–254.

[16] B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy, *Harmonic analysis of operators on Hilbert space*, Springer, New York, 2010.

[17] K. Takahashi, *On quasiaffine transforms of unilateral shifts*, Proc. Amer. Math. Soc. **100** (1987), 683–687.

[18] J. E. Thomson and L. Yang, *Invariant subspaces with the codimension one property in $L^p(\mu)$*, Indiana Univ. Math. J. **44** (1995), no.4, 1163–1173.

[19] V. A. Telokkonikov, *Estimates in the Carleson corona theorem, ideals of the algebra $H^\infty$, a problem of Sz.-Nagy*, Zap. Nauchn. Semin. LOMI **113** (1981), 178–198 (Russian); English translation in: J. Soviet Math. **22** (1983), 1814–1828.

[20] S. R. Treil’, *Angles between co-invariant subspaces, and the operator corona problem. The Szőkefalvi-Nagy problem*, Dokl. Akad. Nauk SSSR,
302 (1988), no. 5, 1063–1068 (Russian); English translation in Soviet Math. Dokl., 38 (1989), no. 2, 394–399.

[21] S. Treil, *Geometric methods in spectral theory of vector-valued functions: Some recent results*, in: Toeplitz operators and spectral function theory, Oper. Theory, Adv. Appl. 42 (1989), 209-280.

[22] S. Treil, *An operator corona theorem*, Indiana Univ. Math. J. 53 (2004), 1763–1781.

[23] S. Treil, *Lower bounds in the matrix corona theorem and the codimension one conjecture*, Geom. Funct. Anal. 14 (2004), 1118–1133.

[24] M. Uchiyama, *Contractions and unilateral shifts*, Acta Sci. Math. (Szeged) 46 (1983), 345–356.

[25] M. Uchiyama, *Curvatures and similarity of operators with holomorphic eigenvectors*, Trans. Amer. Math. Soc., 319 (1990), 405–415.

St. Petersburg Branch, V. A. Steklov Institute of Mathematics, Russian Academy of Sciences, Fontanka 27, St. Petersburg, 191023, Russia

E-mail address: gamali@pdmi.ras.ru