ON DIVERGENCE FORM SPDES WITH VMO
COEFFICIENTS IN A HALF SPACE

N.V. KRYLOV

Abstract. We extend several known results on solvability in the Sobolev
spaces $W^1_p$, $p \in [2, \infty)$, of SPDEs in divergence form in $\mathbb{R}^d_+$ to equations
having coefficients which are discontinuous in the space variable.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with an increasing filtration
$\{\mathcal{F}_t, t \geq 0\}$ of complete with respect to $(\mathcal{F}, P)$ $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$. Denote
by $\mathcal{P}$ the predictable $\sigma$-field in $\Omega \times (0, \infty)$ associated with $\{\mathcal{F}_t\}$. Let $w^k_t$, $k = 1, 2, \ldots$, be independent one-dimensional Wiener processes with respect
$\{\mathcal{F}_t\}$.

We fix a stopping time $\tau$ and for $t \leq \tau$ in
$\mathbb{R}^d_+ = \{x = (x^1, x') : x^1 > 0, x' = (x^2, \ldots, x^d) \in \mathbb{R}^{d-1}\}$, $d \geq 2$,
$\mathbb{R}^1_+ = \mathbb{R}_+ = (0, \infty)$
consider the following equation

$$
du_t = (L_t u_t + D_i f^i_t + f^0_t) dt + (\Lambda^k_t u_t + g^k_t) dw^k_t, \quad (1.1)
$$



where $u_t = u_t(\omega, x)$ is an unknown function,

$L_t \psi(x) = D_j \left(a^{ij}_t(x)D_i \psi(x) + a^i_t(x)\psi(x) \right) + b^i_t(x)D_i \psi(x) + c_t(x)\psi(x),$

$\Lambda^k_t \psi(x) = \sigma^{ik}_t(x)D_i \psi(x) + \nu^k_t(x)\psi(x),$

the summation convention with respect to $i, j = 1, \ldots, d$ and $k = 1, 2, \ldots$ is
enforced and detailed assumptions on the coefficients and the free terms will
be given later. Equation (1.1) is supplemented with zero initial data and
zero boundary condition on $x^1 = 0$. Other initial conditions can also be con-
sidered by a standard method of continuing them for $t > 0$ and subtracting
the result of continuation from $u$. However, for simplicity of presentation
we confine ourselves to the simplest case of zero initial condition.

One of possible approaches to equation (1.1) is to rewrite it in the nondi-
vergence form assuming that the coefficients $a^{ij}_t$ and $a^i_t$ are differentiable in
$x$ and then one could apply the results from [2] to obtain the solvability in
It turns out that the differentiability of $a_{ij}^t$ and $a_t^i$ is not needed for the corresponding counterparts of the results in [2] to be true, which is shown in [1], where the coefficients $a$ and $\sigma$ are just continuous in $x$. Recent development in the theory of parabolic PDEs allows one to further reduce the regularity assumption on $a$ (but not $\sigma$) and require that $a$ be in VMO with respect to the space variable and showing this is the main purpose of this article.

The main guidelines we follow are quite common: getting a priori estimates and using the method of continuity. The method of continuity requires a starting point, which in our case is the solvability of the equation

$$
\begin{align*}
\frac{du}{dt} &= (\Delta u_t + D_if_t^i + f_0^i) \, dt + g_t^k \, dw_t^k
\end{align*}
$$

for sufficiently large class of $f^j, g^k$, say, smooth with compact support. By the way, introducing a new unknown function $v_t = u_t - \int_0^t g_s^k \, dw_s^k$ reduces (1.2) to the heat equation with random free term, which makes proving the solvability of (1.2) quite elementary. Here is the only point where we rely on the theory of SPDEs with constant coefficients.

Our methods of obtaining a priori estimates are quite different from the methods of [1] and do not require developing first the theory of SPDEs in $\mathbb{R}^d_+$ or in $\mathbb{R}^d$ with coefficients independent of $x$ (but depending on $t$ and $\omega$). In our case this theory does not help because the usual method of freezing the coefficients does not lead to small perturbations due to the fact that, generally, $a$ is not continuous in $x$.

Instead, we use new interior estimates of independent interest for SPDEs in $\mathbb{R}^d$ (Theorem 3.3) which we then apply to get an a priori estimate for equations in $\mathbb{R}^d_+$ of the highest norm of the solution in terms of its lowest norm (Theorem 4.1 and Corollary 4.2). Then in Section 4 we develop a new method of estimating the lowest norm of the solution again avoiding considering equations with coefficients independent of $x$.

We work in Sobolev spaces with weights which is unavoidable if the stochastic terms in the equation do not vanish on $\partial \mathbb{R}^d_+$. It is interesting that, even if they vanish identically, our results are new. By the way, in that deterministic case the restriction $p \geq 2$ can be relaxed to $p \in (1, \infty)$ by using a standard duality argument. Also in a standard way our results can be extended to cover SPDEs with VMO coefficients in $C^1$ domains. The interested reader is referred to [1] for necessary techniques to do that.

Our results cover the classical case that $p = 2$ when no continuity hypotheses is needed and even in this case the results are new in what concerns weights. In the case when $p \neq 2$ and $a$ is only measurable in $x$ the best results can be found in [3], where $\sigma \equiv 0$ and $p \geq 2$ is sufficiently close to 2.
2. Main results

Throughout the article the coefficients $a_{ij}^t$, $a_i^t$, $b_i^t$, $\sigma_t^k$, $c_t$, and $\nu_t^k$ are assumed to be measurable with respect to $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel $\sigma$-field on $\mathbb{R}^d$. We understand equation (1.1) in the sense of generalized functions. To be more specific we introduce appropriate Banach spaces.

Fix some numbers $p \geq 2$, $\theta \in (d - 1, d - 1 + p)$, and denote $L_p = L_p(\mathbb{R}^d_+)$

$$L_{p,\theta} = \{ f : M^{(\theta - d)/p} f \in L_p \}, \quad \| f \|_{L_{p,\theta}} = \| M^{(\theta - d)/p} f \|_{L_p},$$

where $M$ is the operator of multiplying by $x^1$, so that $(M^{(\theta - d)/p} f)(x) = (x^1)^{(\theta - d)/p} f(x)$. We use the same notation $L_p$ and $L_{p,\theta}$ for vector- and matrix-valued or else $\ell_2$-valued functions such as $g_i = (g_i^k)$ in (1.1). For instance, if $u(x) = (u^1(x), u^2(x), ...)$ is an $\ell_2$-valued measurable function on $\mathbb{R}^d$, then

$$\| u \|_{L_p}^p = \int_{\mathbb{R}^d_+} |u(x)|^p \, dx = \int_{\mathbb{R}^d_+} \left( \sum_{k=1}^{\infty} |u^k(x)|^2 \right)^{p/2} \, dx.$$

Denote $D_i = \frac{\partial}{\partial x^i}$, $i = 1, ..., d$, $\Delta = D_1^2 + ... + D_d^2$.

By $Du$ we mean the gradient with respect to $x$ of a function $u$ on $\mathbb{R}^d_+$. By $W^1_{p,\theta}$ we mean the space of functions such that $u, MDu \in L_{p,\theta}$. The norm in this space is introduced in an obvious way. As is easy to see

$$\| M^{-1} u \|_{W^1_{p,\theta}} \sim \| M^{-1} u \|_{L_{p,\theta}} + \| Du \|_{L_{p,\theta}}. \quad (2.1)$$

Recall that $\tau$ is a fixed stopping time and set

$$L_{p,\theta}(\tau) = L_p((0, \tau], \mathcal{P}, L_{p,\theta}), \quad W^1_{p,\theta}(\tau) = L_p((0, \tau], \mathcal{P}, W^1_{p,\theta}).$$

We also need the space $\mathfrak{M}^1_{p,\theta}(\tau)$, which is the space of functions $u_t = u_t(\omega, \cdot)$ on $\{(\omega, t) : 0 \leq t \leq \tau, t < \infty\}$ with values in the space of generalized functions on $\mathbb{R}^d_+$ and having the following properties:

(i) We have $M^{-1} u_0 \in L_p(\Omega, \mathcal{F}_0, L_{p,\theta})$;

(ii) We have $M^{-1} u \in W^1_{p,\theta}(\tau)$;

(iii) There exist real valued $f^0 \in M^{-1} L_{p,\theta}(\tau)$, $f^1, ..., f^d \in L_{p,\theta}(\tau)$, and an $\ell_2$-valued $g = (g^k, k = 1, 2, ...)$ in $L_{p,\theta}(\tau)$ such that for any $\varphi \in C^\infty_0(\mathbb{R}^d_+)$ with probability 1 for all $t \in [0, \infty)$ we have

$$u_{t \wedge \tau, \varphi} = (u_0, \varphi) + \sum_{k=1}^{\infty} \int_0^t I_{s \leq \tau}(g^k, \varphi) \, dw^k_s$$

$$+ \int_0^t I_{s \leq \tau}(- (f^i_s, D_i \varphi) + (f^0_s, \varphi)) \, ds. \quad (2.2)$$
In particular, for any \( \phi \in C_0^\infty(\mathbb{R}_+^d) \), the process \((u_{t\wedge \tau}, \phi)\) is \( \mathcal{F}_t \)-adapted and continuous.

In case that property \( \text{(2.2)} \) holds, we write

\[
d u_t = (D_i f^i_t + f^0_t) \, dt + g^k_t \, dw^k_t
\]

(2.3)

for \( t \leq \tau \) and this explains the sense in which equation \( \text{(1.1)} \) is understood. Of course, we still need to specify appropriate assumptions on the coefficients and the free terms in \( \text{(1.1)} \).

For \( u \in \mathcal{M}^1_{p,0}(\tau) \) we write \( u \in \mathcal{M}^1_{p,0,0}(\tau) \) if \( u_0 = 0 \).

**Remark 2.1.** It is worth noting that, if \( u \in \mathcal{M}^1_{p,0,0}(\tau) \), then for any \( \phi \in C_0^\infty(\mathbb{R}_+^d) \) the function \( u\phi \in \mathcal{W}^1_{p,0}(\tau) \) (we remind the definition of \( \mathcal{W}^1_{p,0}(\tau) \) later) and as any element of \( \mathcal{W}^1_{p,0}(\tau) \) is indistinguishable from an \( L^p \)-valued \( \mathcal{F}_t \)-adapted continuous process (see, for instance, [7]).

In the following assumption we use a parameter \( K \geq 0 \), which will be specified later as a small constant.

**Assumption 2.1.** For all values of indices and arguments we have

\[
|Ma^i_t| + |Mb^i_t| + |M^2c_i| + |Mu_t|_{\ell_2} \leq K, \quad c_t \leq 0.
\]

**Remark 2.2.** Assumption \( \text{2.1} \) shows that \( a^i_t, b^i_t, c_t, \) and \( \nu_t \) go to zero as \( x^1 \to \infty \). Actually, in applications to SPDEs in bounded domain this is irrelevant because far from the boundary everything is taken care of by the theory in the whole space. On the other hand, \( a^i_t, b^i_t, c_t, \) and \( \nu_t \) can blow up to infinity for \( x^1 \) approaching zero.

**Assumption 2.2.** For a constant \( \delta \in (0, 1] \) for all values of the arguments and \( \xi \in \mathbb{R}^d \) we have

\[
\alpha^{ij}_t \xi^i \xi^j \leq \delta^{-1}|\xi|^2, \quad (a^{ij}_t - \alpha^{ij}_t) \xi^i \xi^j \geq \delta|\xi|^2,
\]

(2.4)

where

\[
\alpha^{ij}_t = (\sigma^i_t, \sigma^j_t)_{\ell_2}.
\]

Notice that we do not assume that the matrix \((a^{ij}_t)\) is symmetric.

**Remark 2.3.** Observe that if \( M^{-1}u \in \mathcal{W}^1_{p,0}(\tau) \), then

\[
M^{-1}u \in \mathbb{L}_{p,0}(\tau), \quad Du \in \mathbb{L}_{p,0}(\tau),
\]

and all

\[
a^{ij}D_i u, \quad a^i u, \quad Mb^iD_i u, \quad Mcu, \quad \sigma^i D_i u, \quad \nu u
\]

belong to \( \mathbb{L}_{p,0}(\tau) \), so that the right-hand side of \( \text{(1.1)} \) has the form of the right-hand side of \( \text{(2.3)} \) with some \( f^j \) and \( g^k \) there and \( \text{(1.1)} \) makes perfect sense for any \( u \in \mathcal{M}^1_{p,0}(\tau) \).
For functions \( h_t(x) \) on \( \mathbb{R}^{d+1} \) and balls \( B \) in \( \mathbb{R}^d \) introduce

\[
h_t(B) = \frac{1}{|B|} \int_B h_t(x) \, dx,
\]
where \( |B| \) is the volume of \( B \).

If \( \rho > 0 \), set \( B_\rho = \{ x : |x| < \rho \} \) and for locally integrable \( h_t(x) \) and continuous \( \mathbb{R}^d \)-valued function \( x_t, t \geq 0 \), introduce the integral oscillation of \( h \) relative to \( B \) and \( x \) by

\[
osc_\rho (h, x_t) = \sup_{s \geq 0} \frac{1}{\rho^2} \int_s^{s+\rho^2} (|h_t - h_t(B+x_t)|)(B+x_t) \, dr,
\]

where \( B = B_\rho \). Also for \( y \in \mathbb{R}^d \) set

\[
Osc_\rho (h, y) = \sup_{|x| \leq \rho} \sup_{r \leq \rho} osc_r (h, y + x),
\]

where \( |x|_C \) is the sup norm of \( |x| \). Observe that \( osc_\rho (h, x) = 0 \) if \( h_t(x) \) is independent of \( x \).

Denote by \( \beta_0 \) one third of the constant \( \beta_0(d, p, \delta) > 0 \) from Lemma 5.1 of [8].

**Assumption 2.3.** There exist a constant \( \varepsilon \in (0, 1] \) such that for any \( y \in \mathbb{R}_+^d \) (and \( \omega \)) we have

\[
Osc_{\varepsilon y^1} (a^{ij}, y) \leq \beta_0, \quad \forall i, j.
\]  

Furthermore,

\[
(a^{jk}_t(x) - a^{jk}_t(y))\xi^j \xi^k \geq \delta |\xi|^2
\]

for all \( t, \xi, \) and \( x \) satisfying \( |x - y| \leq \varepsilon y^1 \).

**Remark 2.4.** This assumption is quite substantially weaker than similar assumptions known in the literature (see, for instance, [1] and the references therein), where the oscillation of \( a^{ij} \) in (2.5) is understood as

\[
\sup_{t \geq 0} \sup_{|x-y| \leq (x^1 \wedge y^1)} |a^{ij}_t(x) - a^{ij}_t(y)|.
\]

It is easy to see that if, for an \( \varepsilon \in (0, 1] \), (2.6) is less than a \( \beta > 0 \), then the left hand-side of (2.5) is also less than \( \beta \) if we replace there \( \varepsilon \) with \( \varepsilon/4 \).)

With such substitution \( a^{ij}_t(x) \) will have jumps at each point \( x \in \mathbb{R}_+^d \) not larger than \( \beta_0 \), which is a small constant.

On the other hand, if \( a^{ij}_t(x) \) is independent of \( t \), then, for \( 0 < y^1 \leq 2, \) (2.5) is satisfied if \( a \in \text{VMO} \), which is the class of functions with vanishing mean oscillation and which for \( d = 2 \) contains, for instance, the function \( 2 + \sin f(x) \), where \( f(x) = \ln^{1/3}(|x - e| \wedge 1) \) and \( e \) is the first basis vector in \( \mathbb{R}^d \). The usual oscillation of this function at \( e \) is 2.

**Remark 2.5.** It follows from our proofs that if \( \sigma \equiv 0 \), then we can relax condition (2.5) by using the modified integral oscillations which are defined by taking \( x_t \equiv 0 \).
Let $\beta_1 = \beta_1(d, p, \delta, \varepsilon) > 0$ be the constant from Lemma 5.2 of [8].

**Assumption 2.4.** There exists a constant $\varepsilon_1 > 0$ such that for any $t \geq 0$ we have

$$|\sigma_i^t(x) - \sigma_i^t(y)|_{L^2} \leq \beta_1,$$

whenever $x, y \in \mathbb{R}^d, \ |x - y| \leq \varepsilon_1(x^1 \vee y^1), \ i = 1, ..., d.$

Our first main result is the following.

**Theorem 2.6.** Let $\overline{\delta} > 0$ be a constant such that for any $\xi \in \mathbb{R}^d$ and all values of arguments we have

$$\overline{\delta} \left( \sum_i a^{i1}_t \xi^i \right)^2 \leq (a^{ij}_t - \alpha^{ij}_t) \xi^i \xi^j.$$  \hspace{1cm} (2.7)

Let Assumptions 2.1 through 2.4 be satisfied with a (small) constant $K = K(d, p, \delta, \overline{\delta}, \theta, \varepsilon, \varepsilon_1) > 0$, an estimate from below for which can be obtained from the proof. Set

$$\gamma = \theta - d - p + 1 \ (< 0)$$

and assume that

$$|\gamma| (\overline{\delta} \delta)^{1/2}(p - 1) > p|\gamma + 1|,$$

which holds, for instance, if $\theta = d + p - 2$ when $\gamma + 1 = 0$. Then for any $f^0, ..., f^d,$ and $g = (g^k) \in L^p, \theta$ satisfying

$$M f^0, f^i, g = (g^k) \in L^p, \theta(\tau), \ i = 1, ..., d$$

there exists a unique $u \in W^1_{p, \theta}(\tau)$ satisfying (1.1) in $\mathbb{R}^d_+$. Furthermore, for this solution

$$\|M^{-1} u\|_{W^1_{p, \theta}(\tau)} \leq N \left( \|M f^0\|_{L^p, \theta(\tau)} + \sum_{i=1}^d \|f^i\|_{L^p, \theta(\tau)} + \|g\|_{L^p, \theta(\tau)} \right), \hspace{1cm} (2.8)$$

where $N$ depends only on $d, p, \delta, \theta, \overline{\delta}, \varepsilon, \varepsilon_1.$

**Remark 2.7.** As it follows from the proof of Theorem 2.6 if $p = 2$, Assumptions 2.3 and 2.4 are not needed. Thus we obtain the classical result on Hilbert space solvability of SPDEs in half spaces with one improvement that we can allow spaces with weights. By the way, observe that the proof of Theorem 2.6 does not use the Hilbert space theory of SPDEs.

To state our second result we need an additional assumption.

**Assumption 2.5.** (i) There exists a constant $\tilde{\delta} \in (0, 1]$ such that for all $\xi \in \mathbb{R}^d$ and all arguments we have

$$\tilde{\delta} \left( \sum_j \hat{a}^{1j}_t \xi^j \right)^2 \leq a^{11}_t (a^{ij}_t - \alpha^{ij}_t) \xi^i \xi^j,$$  \hspace{1cm} (2.9)

where

$$\hat{a}^{ij}_t = (1/2)(a^{ij}_t + a^{ji}_t).$$
(ii) It holds that
\[ d - 1 + p \left[ 1 - \frac{1}{p(1 - \delta) + \delta^2} \right] < \theta < d - 1 + p. \] (2.10)

(iii) For a constant \( \beta_2 > 0 \), if \( x, y \in \mathbb{R}^d_+ \) are such that \( |x - y| \leq x^1 \land y^1 \), then for all \( i = 1, \ldots, d \) and \( t > 0 \)
\[ |\hat{a}_{1i}^1(x) - \hat{a}_{1i}^1(y)| \leq \beta_2. \] (2.11)

**Remark 2.8.** In previous works on a similar subject (see, for instance, [1] or [9]) a condition stronger than (2.9) used to be assumed:
\[ \tilde{\delta} a_{ij}^1 \xi^i \xi^j \leq (a_{ij}^0 - \hat{a}_{ij}^1) \xi^i \xi^j. \] (2.12)

That (2.12) is stronger than (2.9) follows from the fact that for the positive definite matrix \((\hat{a}_{ij}^1)\) and \( \eta = (1, 0, \ldots, 0) \) it holds that
\[ (\sum_j \hat{a}_{ij}^1 \xi^j)^2 = (\sum_j \tilde{\delta}a_{ij}^1 \eta^j \xi^j)^2 \leq (\hat{a}_{ij}^1 \eta^i \eta^j) \hat{a}_{ij}^1 \xi^i \xi^j = a_{ij}^1 \hat{a}_{ij}^1 \xi^i \xi^j. \]

Also observe that sometimes (2.9) holds with \( \tilde{\delta} = 1 \) and (2.12) does not. This happens, for instance, if \( \alpha_{ij}^0 \equiv 0 \) for all \( j \) and \( \hat{a}_{ij}^1 \equiv 0 \) for \( j \neq 1 \). Finally, in the case when \( \sigma \equiv 0 \) condition (2.9) is satisfied with \( \tilde{\delta} = 1 \) and then condition (2.10) becomes \( d - 1 < \theta < d - 1 + p \) which is the widest range possible for \( \theta \) even in the deterministic case for the heat equation.

**Remark 2.9.** Condition (2.11) is imposed only on \( \hat{a}_{ij}^1 \). As is discussed in [1] (also see references therein), this condition allows rather sharp oscillations of \( \hat{a}_{ij}^1(x) \) near \( \partial \mathbb{R}^d_+ \). The other entries of \((\hat{a}_{ij}^1(x))\) are still allowed to be discontinuous in \( x \) but yet kind of belong to VMO (cf. Remark 2.4).

**Theorem 2.10.** There exist (small) constants \( K > 0 \) and \( \beta_2 > 0 \), depending only on \( d, p, \tilde{\delta}, \beta, \theta, \varepsilon_1 \), \( \varepsilon_1 \) and estimates from below which can be obtained from the proof, such that if Assumptions (2.4) through (2.8) are satisfied with these constants, then the assertion of Theorem (2.6) holds true again with \( \tilde{\delta} \) in place of \( \hat{\delta} \) in the arguments of \( N \).

We prove Theorems (2.6) and (2.10) in Section 5 after preparing necessary tools in Section 3, where we treat equations in \( \mathbb{R}^d \), and in Section 4 containing auxiliary results for equations in \( \mathbb{R}^d_+ \).

3. Auxiliary results for equations in \( \mathbb{R}^d \)

The assumptions in this section are somewhat different from the assumptions of Section 2 apart from the assumption about the measurability of the coefficients.

To investigate the equations in \( \mathbb{R}^d_+ \) we need a few results about equations in \( \mathbb{R}^d \). To state them we remind the reader the definition of spaces \( \mathcal{W}^1_p(\tau) \) and \( \mathcal{V}^1_p(\tau) \) introduced in [7] (which is somewhat different from \( \mathcal{H}^1_p(\tau) \) in [1] or [5], see the discussion of the differences in [8]).
As usual, 

\[ W^1_p = \{u \in L_p(\mathbb{R}^d) : Du \in L_p(\mathbb{R}^d)\}, \quad \|u\|_{W^1_p} = \|u\|_{L_p(\mathbb{R}^d)} + \|Du\|_{L_p(\mathbb{R}^d)}. \]

Recall that \( \tau \) is a stopping time and set

\[ L_p(\tau) := L_p((0, \tau], \mathcal{F}_0, L_p(\mathbb{R}^d)), \quad W^1_p(\tau) := L_p((0, \tau], \mathcal{F}_0, W^1_p). \]

The space \( W^1_p(\tau) \) is introduced as the space of functions \( u_t = u_t(\omega, \cdot) \) on \( \{(\omega, t) : 0 \leq t \leq \tau, t < \infty\} \) with values in the space of generalized functions on \( \mathbb{R}^d \) and having the following properties:

(i) We have \( u_0 \in L_p(\Omega, \mathcal{F}_0, L_p(\mathbb{R}^d)) \);
(ii) We have \( u \in W^1_p(\tau) \);
(iii) There exist \( f^i \in L_p(\tau), i = 0, \ldots, d, \) and \( g = (g^1, g^2, \ldots) \in L_p(\tau) \) such that for any \( \varphi \in C_0^\infty(\mathbb{R}^d) \) with probability 1 for all \( t \in [0, \infty) \) we have

\[
(u_{t \wedge \tau}, \varphi) = (u_0, \varphi) + \sum_{k=1}^{\infty} \int_0^t I_{s \leq t} (g^k_s, \varphi) dw^k_s \\
+ \int_0^t I_{s \leq t} ((f^0_s, \varphi) - (f^i_s, D_i \varphi)) ds. \tag{3.1}
\]

In particular, for any \( \phi \in C_0^\infty \), the process \( (u_{t \wedge \tau}, \phi) \) is \( \mathcal{F}_t \)-adapted and continuous.

The following result is a somewhat weakened version of Corollary 5.5 in \[8\].

**Lemma 3.1.** Let \( G \subset \mathbb{R}^d \) be a domain (perhaps, \( G = \mathbb{R}^d \)) and let \( K \geq 0, \varepsilon > 0, \) and \( \varepsilon_1 \in (0, \varepsilon/4] \) be some constants.

(i) Let \( f^j, g \in L_p(\tau), \) and \( u \in W^1_{p,0}(\tau) \) satisfy (1.1) in \( \mathbb{R}^d \) for \( t \leq \tau \) and be such that \( u_t(x) = 0 \) if \( x \not\in G \).

(ii) Suppose that Assumption 2.2 is satisfied and suppose that for \( y \in G \) and all values of indices and other arguments

\[
|a^i_j(y)| + |b^i_j(y)| + |c^i_j(y)| + |\nu^i(y)|_{\ell_2} \leq K, \quad c^i_j(y) \leq 0.
\]

(iii) Assume that, for any \( x_0, \) such that \( \text{dist}(x_0, G) \leq \varepsilon_1, \) we have

\[
\text{Qsc}_x(a^{ij}, x_0) \leq \beta_0, \quad \forall i, j, \tag{3.2}
\]

where \( \beta_0 \) is the one third of \( \beta_0(d, p, \delta) > 0 \) from Lemma 5.1 of \[8\], and

\[
|\sigma^i_j(x) - \sigma^i_j(x_0)|_{\ell_2} \leq \beta_1, \quad (a^i_j(y) - a^i_j(x_0))\xi^i_j \xi^k \geq \delta |\xi|^2 \tag{3.3}
\]

for all values of indices and arguments such that \( |x - x_0| \leq \varepsilon_1 \) and \( |y - x_0| \leq \varepsilon, \) where \( \beta_1 = \beta_1(d, \delta, p, \varepsilon/2) > 0 \) is the constant from Lemma 5.2 of \[8\].

Then there exist a constant \( N \) depending only on \( d, p, K, \delta, \varepsilon, \) and \( \varepsilon_1 \) such that

\[
\|Du\|_{L_p(\tau)} \leq N(\sum_{i=1}^{d} \|f^i\|_{L_p(\tau)} + \|g\|_{L_p(\tau)} + \|f^0\|_{L_p(\tau)}^{1/2} \|u\|_{L_p(\tau)}^{1/2} + \|u\|_{L_p(\tau)}).
\]
Next we give a version of Lemma 3.1 for some particular domains $G$ the most important of which will be $\{|x|^2 \leq R\}$. We state it in a slightly more general setting suitable for investigating interior smoothness of solutions in $\mathbb{R}^d$ or in $\mathbb{R}^d_+$.

We fix an integer $d_1 \in [1,d]$ and for $x \in \mathbb{R}^d$ introduce

$$|x|^r = \left( \sum_{i=1}^{d_1} (x^i)^2 \right)^{1/2}, \quad B'_R = \{ x \in \mathbb{R}^d : |x|^r < R \}.$$

**Theorem 3.2.** Take some $\epsilon > 0$, $\epsilon_1 \in (0, \epsilon/4]$, $K \geq 0$, and $R > 0$.

(i) Let $f^j, g \in L_p(\tau)$, and $u \in W^1_{p,0}(\tau)$ satisfy (1.1) in $\mathbb{R}^d$ for $t \leq \tau$.

(ii) Assume that $u_t(x) = 0$ if $x \notin B'_R$.

(iii) Suppose that Assumption 2.2 is satisfied and for $y \in B'_R$ and all values of the indices and other arguments

$$R|a^i_t(y)| + R|b^i_t(y)| + R|\nu_t(y)|_{\ell_0} + R^2|c_t(y)| \leq K, \quad c_t(y) \leq 0.$$

(iv) Assume that (3.2) with $\varepsilon R$ in place of $\varepsilon$ and (3.3) hold for any $x_0$, such that $|x_0|^r \leq (1 + \varepsilon)R$, and $x, y$ such that $|x - x_0| \leq \varepsilon_1 R$, $|y - x_0| \leq \varepsilon R$, and all values of indices and other arguments.

Then there exists a constant $N = N(d,p, \delta, K, \varepsilon, \varepsilon_1)$ such that

$$\|Du\|_{L_p(\tau)} \leq N \left( \sum_{i=1}^d \|f^i\|_{L_p(\tau)} + \|g\|_{L_p(\tau)} + \|f^0\|_{L_p(\tau)}^{1/2} \|u\|_{L_p(\tau)}^{1/2} + R^{-1}\|u\|_{L_p(\tau)} \right).$$

(3.4) Proof. If $R = 1$, the result follows directly from Lemma 3.1. The case of general $R$ we reduce to the particular one by using dilations. Introduce

$$\hat{\mathcal{F}}_t = \mathcal{F}_{Rt}, \quad \hat{\tau} = R^{-2}\tau, \quad \hat{u}^k_t = R^{-1}u^k_{Rt}.$$

$$(\hat{a}^i_t, \hat{a}^i_0, \hat{b}^i_t, \hat{c}_t, \hat{\sigma}_t, \hat{\tau}_t)(x) = (a^i_{Rt}, Ra^i_{Rt}, Rb_{Rt}, R^2c_{Rt}, \sigma_{Rt}, R\nu_{Rt})(Rx),$$

$$\hat{u}_t(x) = u_{Rt}(Rx), \quad \hat{f}^i_t(x) = Rf^i_{Rt}(Rx), \quad i = 1, \ldots, d,$$

$$\hat{\nu}_t(x) = R^2\nu_{Rt}(Rx).$$

Also introduce the operators $\hat{L}_t$ and $\hat{\Lambda}^k_t$ constructing them from the above introduced coefficients. It is easily seen that $\hat{u}^k_t$ are independent $\hat{\mathcal{F}}_t$-Wiener processes, $\hat{\tau}$ is an $\hat{\mathcal{F}}_t$-stopping time, all the above processes with hats are predictable with respect to the filtration $\{\hat{\mathcal{F}}_t\}$, and $\hat{u} \in W^1_{p,0}(\hat{\tau})$, $\hat{f}, \hat{\nu} \in \hat{L}_p(\hat{\tau})$, where the spaces with hats are defined on the basis of $\{\hat{\mathcal{F}}_t\}$.

Observe that for $t < \hat{\tau}$

$$\hat{L}_t \hat{u}_t(x) = (D^i_j(\hat{a}^{ij}_t(x)D^i_t \hat{u}_t(x) + \hat{a}^{ji}_t(x)\hat{u}_t(x))) + \hat{b}^i_t(x)D^i_t \hat{u}_t(x) + \hat{c}_t(x)\hat{u}_t(x)$$

$$= R^2(D^i_j(a^i_{Rt}D^i_t u_{Rt} + a^j_{Rt} u_{Rt}) + a^i_{Rt}D^i_t u_{Rt} + c_{Rt}u_{Rt})(Rx)$$

$$= R^2L_{Rt}u_{Rt}(Rx), \quad D^i_t \hat{f}^i_t(x) = R^2(D^i_t f^i_{Rt})(Rx),$$

$$\int_0^{t \land \hat{\tau}} [\hat{L}_s \hat{u}_s(x) + D^i_t \hat{f}^i_s(x) + \hat{f}^0_s(x)] ds$$
\[ = \int_0^{(R^2)\wedge \tau} [L_s u_s(Rx) + D_t f_s^0(Rx)] \, ds. \]

Of course, we understand this equality in the sense of distributions:
\[ \int_0^{(R^2)\wedge \tau} (\hat{L}_s \hat{u}_s + D_t \hat{f}_s^i + \hat{f}_s^0, \phi) \, ds = \int_0^{(R^2)\wedge \tau} ([L_s u_s + D_t f_s^i + f_s^0](R \cdot), \phi) \, ds \]
for any \( \phi \in C^\infty_0(\mathbb{R}^d) \). One also knows that if \( \hat{h}_t \) is an \( \mathcal{F}_t \)-predictable process satisfying a natural integrability condition with respect to \( t \), then
\[ \int_0^t \hat{h}_s \, d\hat{w}_s^k = R^{-1} \int_0^{R^2 t} \hat{h}_{R^{-2} s} \, dw_s^k \quad \text{(a.s.)} \]

Therefore, (a.s.)
\[ \int_0^{(R^2)\wedge \tau} [\hat{L}_s^k \hat{u}_s + \hat{g}_s^k](x) \, d\hat{w}_s^k = R \int_0^{(R^2)\wedge \tau} [\Lambda_{R^2 s}^k u_{R^2 t} + g_{R^2 s}^k](Rx) \, d\hat{w}_s^k \]
\[ = \int_0^{(R^2)\wedge \tau} [\Lambda_{s}^k u_s + g_s^k](Rx) \, du_s^k. \]

It follows that (a.s.)
\[ \int_0^{(R^2)\wedge \tau} [\hat{L}_s \hat{u}_s(x) + D_t \hat{f}_s^i(x) + \hat{f}_s^0(x)] \, ds \]
\[ + \int_0^{(R^2)\wedge \tau} [\Lambda_{s}^k \hat{u}_s(x) + \hat{g}_s^k(x)] \, d\hat{w}_s^k = u((R^2)\wedge \tau)(Rx) = \hat{u}_{(R^2)\wedge \tau}(x), \]
so that \( \hat{u} \) satisfies equation (1.1) with new operators and free terms. It is also easy to see that our objects with hats satisfy the assumptions of the theorem with \( R = 1 \). Therefore, by the result for \( R = 1 \)
\[ \|D\hat{u}\|_{L_p(\hat{\tau})} \leq N \left( \sum_{j=1}^d \|\hat{f}_j\|_{L_p(\hat{\tau})} + \|\hat{g}\|_{L_p(\hat{\tau})} + \|\hat{f}_0\|_{L_{p/(2)}(\hat{\tau})}^{1/2} \|\hat{u}\|_{L_{p/(2)}(\hat{\tau})}^{1/2} + \|\hat{u}\|_{L_p(\hat{\tau})} \right). \]

Now it only remains to notice that changing variables shows that this inequality is precisely (3.4). The theorem is proved.

Here is an interior estimate for equations in \( \mathbb{R}^d \). In its spirit it is similar to Theorem 2.3 of [6].

**Theorem 3.3.** Let assumptions (i), (iii), and (iv) of Theorem 3.2 be satisfied. Then, for any \( r \in (0, R) \), we have
\[ \|I_{B_r^d} D u\|_{L_p(\tau)} \leq N \left( \|I_{B_r^d} f^0\|_{L_{p/(2)}(\tau)}^{1/2} \|I_{B_r^d} u\|_{L_{p(\tau)}}^{1/2} \right. \]
\[ + \sum_{i=1}^d \|I_{B_r^d} f_i\|_{L_p(\tau)} + \|I_{B_r^d} g\|_{L_{p(\tau)}} + N(R - r)^{-1} \|u\|_{L_{p(\tau)}} \right), \quad (3.5) \]
where \( N = N(d, p, \delta, K, \varepsilon, \varepsilon_1) \).
Proof. We follow a usual procedure taken from the theory of PDEs. Let \( \chi(s) \) be an infinitely differentiable function on \( \mathbb{R} \) such that \( \chi(s) = 1 \) for \( s \leq 0 \) and \( \chi(s) = 0 \) for \( s \geq 1 \). For \( m = 0, 1, 2, \ldots \) introduce, \( (r_0 = r) \)

\[
r_m = r + (R - r) \sum_{j=1}^{m} 2^{-j}, \quad \zeta_m(x) = \chi(2^{m+1}(R - r)^{-1}(|x|^2 - r_m)).
\]

As is easy to check, for

\[
Q(m) = B'_{r_m},
\]

it holds that

\[
\zeta_m = 1 \text{ on } Q(m), \quad \zeta_m = 0 \text{ outside } Q(m + 1).
\]

Also (observe that \( N2^{m+1} = N_12^m \) with \( N_1 = 2N \))

\[
|D\zeta_m| \leq N2^m(R - r)^{-1}.
\]

Next, the function \( \zeta_m u_t \) is in \( \mathcal{W}^{1,p}_{t,0}(\tau) \) and satisfies

\[
d(\zeta_m u_t) = (L_t(\zeta_m u_t) + D_j f_{mt} + f_{0mt}^0) dt + (A_t^k(\zeta_m u_t) + g_{mt}^k) du_t^k, \quad (3.6)
\]

where

\[
f_{mt}^j = -a_t^{ij} u_t D_i \zeta_m + \zeta_m f_{i}^j, \quad j = 1, \ldots, d,
\]

\[
f_{0mt}^0 = -a_t^{ij} (D_t u_t) D_j \zeta_m - u_t a_t^{ij} D_j \zeta_m - u_t b_t^i D_t \zeta_m + \zeta_m f_{0t}^i - f_{i}^i D_t \zeta_m.
\]

\[
g_{mt}^k = \zeta_m g_t^k - u_t \sigma_t^{ik} D_t \zeta_m.
\]

Notice that

\[
|f_{mt}^j| \leq N2^m(R - r)^{-1}|\zeta_{m+1} u_t| + \zeta_m |f_{i}^j|, \quad j = 1, \ldots, d,
\]

\[
|f_{0mt}^0| \leq N2^m(R - r)^{-1}|\zeta_{m+1} u_t| + N2^m(R - r)^{-2}|\zeta_{m+1} u_t|
\]

\[
+N \zeta_m |f_{0t}^i| + N2^m(R - r)^{-1} \zeta_{m+1} \sum_{j=1}^{d} |f_{j}^j|
\]

\[
\leq N2^m(R - r)^{-1}|D(\zeta_{m+1} u_t)| + N2^m(R - r)^{-2}|\zeta_{m+1} u_t|
\]

\[
+N \zeta_m |f_{0t}^i| + N2^m(R - r)^{-1} \zeta_{m+1} \sum_{j=1}^{d} |f_{j}^j|
\]

\[
|g_{mt}|_{L^2} \leq \zeta_m |g_t|_{L^2} + N2^m(R - r)^{-1}|\zeta_{m+1} u_t|.
\]

Since \( \zeta_m u_t(x) = 0 \) for \( x \not\in B'_{r_m} \), by Theorem 3.2 and Young’s inequality we have

\[
D_m := \| D(\zeta_m u) \|_{L^p(\tau)} \leq NF + N2^m(R - r)^{-1} U_{m+1}
\]

\[
+ N2^{m/2}(R - r)^{-1/2} D_{m+1}^{1/2} U_{m+1}^{1/2}
\]

\[
\leq NF + N2^m(R - r)^{-1} U_{m+1} + 2^{-2} D_{m+1},
\]

where

\[
U_m := \| \zeta_m u \|_{L^p(\tau)}, \quad F := \sum_{i=1}^{d} \| I_B^i f_i^i \|_{L^p(\tau)} + \| I_B^i g_i \|_{L^p(\tau)}
\]
\[ + \|I_{B_R} f_0\|_{L^p}^{1/2} \|I_{B_R} u\|_{L^p}^{1/2}. \]

It follows that
\[ D_0 + \sum_{m=1}^{\infty} 2^{-2m} D_m \leq NF + N(R - r)^{-1} \|u I_{B_R'} f_0\|_{L^p} + \sum_{m=1}^{\infty} 2^{-2m} D_m. \]

By canceling like terms we estimate \( D_0 \) by the right-hand side of (3.5). Its left-hand side is certainly smaller than \( D_0 \). This would yield (3.5) provided that what we canceled is finite.

Obviously,
\[ D_m \leq N \|Du\|_{L^p} + N2^m (R - r)^{-1} \|u\|_{L^p} \]
and the terms in question are finite since \( u \in W^1_p(\tau) \). The theorem is proved.

4. Auxiliary results for equations in \( \mathbb{R}^d_+ \)

In this section we are investigating the local regularity of solutions in \( \mathbb{R}^d_+ \) and give preliminary a priori estimates.

For \( r > 0 \) denote
\[ G_r = \{ x \in \mathbb{R}^d : 0 < x^1 < r \}. \]

Here is the divergence form counterpart of Theorem 4.3 of [6].

**Theorem 4.1.** Take an \( R \in (0, \infty) \) and suppose the following.

(i) Assumptions 2.1 through 2.4 are satisfied.

(ii) We have a function \( u \) such that \( \phi u \in W^1_{p,0}(\tau) \) for any \( \phi \in \mathcal{C}_{0}^{\infty}(G_R) \) and \( u \) satisfies (1.1) in \( \mathbb{R}^d_+ \) for \( t \leq \tau \) with some \( f^j, g \in L^p(\tau), i = 1, \ldots, d \).

Then, for any \( r \in (0, R/4) \),
\[ \|I_{G_r} Du\|_{L^p,\theta}(\tau) \leq N \|I_{G_R} Mf_0\|_{L^p,\theta}(\tau)^{1/2} \|I_{G_R} M^{-1} u\|_{L^p,\theta}(\tau)^{1/2} \]
\[ + N \sum_{i=1}^{d} \|I_{G_R} f^i\|_{L^p,\theta}(\tau) + N \|I_{G_R} g\|_{L^p,\theta}(\tau) + N \|I_{G_R} M^{-1} u\|_{L^p,\theta}(\tau), \] (4.1)

where \( N = N(d, p, \delta, \varepsilon, \varepsilon_1, K) \).

**Proof.** We are going to apply Theorem 3.3 to shifted \( B'_R \) when \( d_1 = 1 \).

For \( n = -1, 0, 1, \ldots \), set \( r_n = 2^{-n/3} R \). Observe that if \( n \geq 0 \), then the half width of \( G_{r_n-1} \setminus G_{r_{n+2}} \) equals \( \rho_n := r_{n+2}/2 \) and
\[ r_{n-1} + \rho_n \leq 2r_{n-1} < 4r < R, \quad r_{n+2} - \rho_n = \rho_n. \]

Let \( c_n = (r_{n-1} + r_{n+2})/2 \) and observe that for \( x_0 \in \mathbb{R}^d_+ \) such that \( |x_0^1| \leq (1 + \varepsilon)\rho_n \) we have \( \rho_n \leq x_0^1 \) because \( \varepsilon \leq 1 \). It follows that
\[ \text{Osc}_{\varepsilon\rho_n} (a^i j, x_0) \leq \beta_0. \]
Also, for \( y \in G_{r_{n-1}} \setminus G_{r_{n+2}} \) we have \( \rho_n \leq y^1 \) and
\[ \rho_n |a_1^i (y)| + \rho_n |b_1^i (y)| + \rho_n |c_1^i (y)| \leq \rho_n^2 |c_1^i (y)| \leq K, \quad c_1^i (y) \leq 0. \]
Next, if \( |x_0^1 - c_n| \leq (1 + \varepsilon)\rho_n \) and \( |y - x_0| \leq \varepsilon \rho_n \), then \( |y - x_0| \leq \varepsilon x_0^1 \) and

\[
(a^j_1(y) - a^j_1(x_0))\xi^j \xi^k \geq \delta |\xi|^2.
\]

Finally, define \( \gamma \in (0, \varepsilon/4] \) by

\[
\frac{\gamma}{1 - \gamma} = \varepsilon_1 \wedge \frac{\varepsilon}{4}
\]

and observe that if \( |x_0^1 - c_n| \leq (1 + \varepsilon)\rho_n \) and \( |x - x_0| \leq \gamma \rho_n \), then

\[
|x - x_0| \leq \gamma x_0^1 \leq \gamma(x_0^1 \wedge x^1) \leq \varepsilon_1(x_0^1 \wedge x^1) \tag{4.2}
\]

if \( x^1 \geq x_0^1 \) and, if \( x^1 < x_0^1 \), then \( x_0^1 - x^1 \leq \gamma x_0^1, x_0^1 \leq (1 - \gamma)^{-1}(x_0^1 \wedge x^1) \) and the inequality between the extreme terms in (4.2) holds again. In that case

\[
|\sigma^j_1(x) - \sigma^j_1(x_0)| \varepsilon_2 \leq \beta_1.
\]

This means that the assumptions of Theorem 3.3 about the coefficients are satisfied if we shift \( c_n \) into the origin.

Furthermore, if \( n \geq 0, \zeta \in C^\infty(0, R) \), and \( \zeta(z) = 1 \) for \( r_{n+2} \leq z \leq r_{n-1} \), then \( \zeta u \) satisfies (1.1) in \( \mathbb{R}^d \) with certain \( f \) and \( g \) which on \( G_{r_{n-1}} \setminus G_{r_{n+2}} \) coincide with the original ones. Finally, if \( n \geq 0 \), then the distance between the boundaries of \( G_{r_n} \setminus G_{r_{n+1}} \) and \( G_{r_{n-1}} \setminus G_{r_{n+2}} \) is \((2^{1/3} - 1)r_{n+2}\).

It follows by Theorem 3.3 that for \( n \geq 0 \)

\[
\|I_{G_{r_n} \setminus G_{r_{n+1}}}Du\|_{L^p(\tau)}^p \leq N\left(\|I_{G_{r_n} \setminus G_{r_{n+2}}}f^0\|_{L^p(\tau)}^{p/2}\|I_{G_{r_{n-1}} \setminus G_{r_{n+2}}}u\|_{L^p(\tau)}^{p/2}\right) + \sum_{i=1}^d \|I_{G_{r_n} \setminus G_{r_{n+2}}}f^i\|_{L^p(\tau)}^p + \|I_{G_{r_n} \setminus G_{r_{n+2}}}g\|_{L^p(\tau)}^p + Nr_{n+2}^{-p}\|I_{G_{r_{n-1}} \setminus G_{r_{n+2}}}u\|_{L^p(\tau)}^p.
\]

Young’s inequality yields that for any constant \( \chi > 0 \)

\[
\|I_{G_{r_n} \setminus G_{r_{n+1}}}Du\|_{L^p(\tau)}^p \leq Nr_{n+2}^{-p}(1 + \chi)\|I_{G_{r_n} \setminus G_{r_{n+2}}}u\|_{L^p(\tau)}^p + N\left(\|I_{G_{r_n} \setminus G_{r_{n+2}}}f\|_{L^p(\tau)}^{p} + \sum_{i=1}^d \|I_{G_{r_n} \setminus G_{r_{n+2}}}f^i\|_{L^p(\tau)}^p + \|I_{G_{r_n} \setminus G_{r_{n+2}}}g\|_{L^p(\tau)}^p\right).
\]

We multiply both parts by \( r_{n+2}^q \) and use the facts that \( r_{n-1} = 2r_{n+2} \) and on \( G_{r_n} \setminus G_{r_{n+2}} \) the ratio \( x^1/r_{n+2} \) satisfies

\[
1 \leq x^1/r_{n+2} \leq 2.
\]

Then we obtain

\[
\|I_{G_{r_n} \setminus G_{r_{n+1}}}Du\|_{L^p, (\tau)}^p \leq N(1 + \chi)\|I_{G_{r_n} \setminus G_{r_{n+2}}}M^{-1}u\|_{L^p, (\tau)}^p + N(\chi^{-1}\|I_{G_{r_n} \setminus G_{r_{n+2}}}Mf\|_{L^p, (\tau)}^p + \sum_{i=1}^d \|I_{G_{r_n} \setminus G_{r_{n+2}}}f^i\|_{L^p, (\tau)}^p + \|I_{G_{r_n} \setminus G_{r_{n+2}}}g\|_{L^p, (\tau)}^p).
\]
Upon summing up these inequalities over \( n \geq 0 \) we conclude
\[
\| I_{G_r} Du \|_{L^p_{p, \theta} (\tau)}^p \leq N (1 + \chi) \| I_{G_r - 1} M^{-1} u \|_{L^p_{p, \theta} (\tau)}^p
\]

\[
+ N \left( \chi^{-1} \| I_{G_{r-1}} M f^0 \|_{L^p_{p, \theta} (\tau)}^p + \sum_{i=1}^d \| I_{G_{r-1}} f^i \|_{L^p_{p, \theta} (\tau)}^p + \| I_{G_{r-1}} g \|_{L^p_{p, \theta} (\tau)}^p \right);
\]

which after minimizing with respect to \( \chi > 0 \) leads to a result which is even somewhat sharper than (4.1). The theorem is proved.

By letting \( r \to \infty \) in (4.3) we get the following.

**Corollary 4.2.** If the assumptions of Theorem 4.1 hold with \( R = \infty \), then
\[
\| Du \|_{L^p_{p, \theta} (\tau)} \leq N \| M f^0 \|_{L^p_{p, \theta} (\tau)}^{1/2} \| M^{-1} u \|_{L^p_{p, \theta} (\tau)}^{1/2}
\]

\[
+ N \sum_{i=1}^d \| f^i \|_{L^p_{p, \theta} (\tau)} + N \| g \|_{L^p_{p, \theta} (\tau)} + N \| M^{-1} u \|_{L^p_{p, \theta} (\tau)},
\]

where \( N = N (d, p, \delta, \varepsilon, \varepsilon_1, K) \). In particular, if \( \| M^{-1} u \|_{L^p_{p, \theta} (\tau)} < \infty \), then \( u \in W^1_{p, \theta, 0} (\tau) \).

Corollary 4.2 reduces obtaining an estimate for \( \| M^{-1} u \|_{W^1_{p, \theta, 0} (\tau)} \) to estimating \( \| M^{-1} u \|_{L^p_{p, \theta} (\tau)} \). Estimating the latter will be done by using the following “energy” estimate. Recall that
\[
\gamma = \theta - d - p + 1 \quad (< 0).
\]

**Lemma 4.3.** Let \( u \in W^1_{p, \theta, 0} (\tau), \ M f^0 \in L^p_{p, \theta} (\tau), \ f^i \in L^p_{p, \theta} (\tau), \ i = 1, \ldots, d, \ g = (g^k) \in L^p_{p, \theta} (\tau) \) and assume that (2.3) holds for \( t \leq \tau \) in the sense of generalized functions on \( \mathbb{R}_+^d \). Then
\[
E \int_0^\tau \left( \int_{\mathbb{R}_+^d} \left[ p M^{\gamma + 1} |u_t|^{p-2} u_t f^0_t - p(p-1) M^{\gamma + 1} |u_t|^{p-2} f^1_t D u_t \right. \right.
\]

\[
- p(\gamma + 1) M^{\gamma} |u_t|^{p-2} u_t f^1_t + \left( 1/2 \right) p(p-1) M^{\gamma + 1} |u_t|^{p-2} |g_{t(\cdot)}|_{2}^2 \right) dx \ dt \]

\[
\geq EI_{t < \infty} \int_{\mathbb{R}_+^d} M^{\gamma + 1} |u_{\tau}|^p \ dx \quad (4.3)
\]

with an equality in place of the inequality if \( \tau \) is bounded.

**Proof.** First of all observe that the concavity of the function \( \log t \) implies that
\[
a_1^\alpha_1 \cdots a_n^\alpha_n \leq \alpha_1 a_1 + \cdots + \alpha_n a_n
\]

if \( a_i, \alpha_i \geq 0 \) and \( \alpha_1 + \cdots + \alpha_n = 1 \). It follows that for any \( \kappa > 0 \)
\[
M^{\gamma + 1} |u_t|^{p-1} |f^0_t| \leq \kappa M^{\theta - d} |M^{-1} u_t|^p + \kappa M^{\theta - d} |M f^0_t|^p,
\]

\[
M^{\gamma + 1} |u_t|^{p-2} |f^1_t D u_t| \leq \kappa M^{\theta - d} (|M^{-1} u_t|^p + |D u_t|^p) + \sum_{i=1}^d M^{\theta - d} |f^i_t|^p,
\]

\[
M^\gamma |u_t|^{p-1} |f^1_t| \leq \kappa M^{\theta - d} |M^{-1} u_t|^p + \kappa M^{\theta - d} |f^1_t|^p,
\]
\[ M^{\gamma + 1}|u_t|^p |g_t|_2^2 \leq \kappa M^{\theta - d} |M^{-1}u_t|^p + NM^{\theta - d} |g_t|^p, \quad (4.4) \]

where the constants \( N \) depend only on \( \kappa \) and \( p \). The right-hand sides in these estimates are summable over \( [0, \tau] \times \mathbb{R}^d \), implying that the expectation in (4.3) makes perfect sense.

Next take a nonnegative function \( \phi \) of one variable \( x \) of class \( C_0^\infty(\mathbb{R}_+) \) and notice that
\[
d(M^{(\gamma + 1)/p}u_t \phi) = (M^{(\gamma + 1)/p} f_t^0 \phi - M^{(\gamma + 1)/p} f_t^1 \phi')
-(\gamma + 1)p^{-1} M^{(\gamma + 1)/p - 1} \phi f_t^1 + D_t(M^{(\gamma + 1)/p} f_t^1 \phi)) dt + M^{(\gamma + 1)/p} g_t^k \phi dw_t^k.
\]

This equation holds in \( \mathbb{R}^d \) rather than only in \( \mathbb{R}_+^d \). Hence, by Corollary 2.2 of [7]
\[
E \int_0^\tau \left( \int_{\mathbb{R}^d} M^{\gamma + 1} \left\{ p|u_t|^{p-2} u_t \phi^{p-1} |f_t^1 - p f_t^1 \phi' - (\gamma + 1) M^{-1} \phi f_t^1 \right\} 
-p(p-1)|u_t|^{p-2} \phi^{p-1} f_t^1 \phi D_t u_t + (1/2)p(p-1) \phi^p |u_t|^{p-2} |g_t|^2 \right) dx \right) dt
\geq EI_{\tau < \infty} \int_{\mathbb{R}^d} M^{\gamma + 1}|u_t|^p \phi^p dx \quad (4.5)
\]

with an equality in place of the inequality if \( \tau \) is bounded.

By recalling what was said in the beginning of the proof and having in mind the dominated convergence theorem and Fatou’s lemma we easily see that, to prove inequality (4.3), now it suffices to find a sequence of \( \phi_n \in C_0^\infty(\mathbb{R}_+) \) such that \( 0 \leq \phi_n \leq 1, \phi_n \to 1 \), and
\[
E \int_0^\tau \int_{\mathbb{R}^d} M^{\gamma + 1}|u_t|^{p-1} |f_t^1 \phi_n'| dx dt \to 0.
\]

Furthermore, since estimates (4.4) imply that
\[
E \int_0^\tau \int_{\mathbb{R}^d} M^{\gamma}|u_t|^{p-1} |f_t^1| dx dt < \infty,
\]

the dominated convergence theorem shows that it suffices to find a sequence of \( \phi_n \in C_0^\infty(\mathbb{R}_+) \) such that \( 0 \leq \phi_n \leq 1, \phi_n \to 1, M \phi_n' \) are uniformly bounded, and \( M \phi_n' \to 0 \) in \( \mathbb{R}_+ \).

To construct such a sequence, take some nonnegative \( \eta, \zeta \in C_0^\infty(\mathbb{R}) \) such that \( \eta = 0 \) near the origin, \( \eta(x) = 1 \) for \( x \geq 1, \zeta = 1 \) near the origin, and \( \eta, \zeta \leq 1 \). Then define \( \phi_n(x) = \eta(nx) \zeta(x/n) \). The reader will easily check that the required properties are satisfied.

To prove that (4.3) holds with the equality sign if \( \tau \) is bounded, we write (4.5) with the equality sign and pass to the limit by the dominated convergence theorem knowing already that the right-hand side of (4.3) is finite. The lemma is proved.

**Corollary 4.4.** Let Assumptions 2.1 and 2.2 be satisfied. Let \( u \in \mathcal{M}_{p,\theta,0}(\tau) \), \( M f^0 \in L_{p,\theta}(\tau), f^i \in L_{p,\theta}(\tau), i = 1, \ldots, d, g = (g^k) \in L_{p,\theta}(\tau) \) and assume
that \( u \) satisfies \((1.1)\) for \( t \leq \tau \). Then for any constant \( \chi > 0 \) there exist constants \( N^* = N^*(d, p, \delta) \) and \( N = N(\chi, d, p, \delta) \) such that

\[
 p(p - 1)E \int_0^\tau \int_{\mathbb{R}^d} M^{\gamma + 1} u_t |u_t|^{p - 2} (a_{ij} \tau - \alpha_t) (D_i u_t) D_j u_t \, dx \, dt \\
+ p(\gamma + 1) E \int_0^\tau \int_{\mathbb{R}^d} M^\gamma |u_t|^{p - 2} u_t a_{ij}^1 D_i u_t \, dx \, dt \\
\leq N(\|M f^0\|_{\mathbb{L}^p_{p, \delta}(\tau)} + \sum_{i=1}^d \|f_i^p\|_{\mathbb{L}^p_{p, \delta}(\tau)} + (1 + K^p) \|g\|_{\mathbb{L}^p_{p, \delta}(\tau)})^p \\
\quad + [N^* K (1 + K) + \chi] I, \quad (4.6)
\]

where

\[
 I = \int_0^\tau \int_{\mathbb{R}^d} (M^{\gamma - 1} |u_t|^p + M^\gamma |u_t|^{p - 2} |Du_t|^2) \, dx \, dt \leq N^* \|M^{-1} u\|_{W^1_{p, \delta}(\tau)},
\]

To derive this result observe that by Lemma 4.3

\[
 E \int_0^\tau \left( \int_{\mathbb{R}^d} [p M^{\gamma + 1} |u_t|^{p - 2} u_t (b_i^1 D_i u_t + c_i u_t + f_i^0) \\
- p(p - 1) M^{\gamma + 1} |u_t|^{p - 2} (a_{ij} \tau - \alpha_t) (D_i u_t) D_j u_t \\
- p(\gamma + 1) M^\gamma |u_t|^{p - 2} u_t (a_{ij}^1 D_i u_t + a_{ij}^1 u_t + f_i^1) \\
+ (1/2) p(p - 1) M^{\gamma + 1} |u_t|^{p - 2} |\sigma_t^i D_i u_t + \nu_t u_t + g_i^2| dx \right) \, dt \geq 0,
\]

which after rearranging the terms becomes

\[
p(p - 1) E \int_0^\tau \int_{\mathbb{R}^d} M^{\gamma + 1} u_t |u_t|^{p - 2} (a_{ij} \tau - \alpha_t) (D_i u_t) D_j u_t \, dx \, dt \\
+ p(\gamma + 1) E \int_0^\tau \int_{\mathbb{R}^d} M^\gamma |u_t|^{p - 2} u_t a_{ij}^1 D_i u_t \, dx \, dt \\
\leq E \int_0^\tau \int_{\mathbb{R}^d} [M^\gamma |u_t|^{p - 2} u_t A_i^1 D_i u_t + M^{\gamma - 1} |u_t|^p B_t] \, dx \, dt \\
+ E \int_0^\tau \int_{\mathbb{R}^d} [M^\gamma |u_t|^{p - 2} u_t F_t + M^{\gamma + 1} |u_t|^{p - 2} G_t + M^{\gamma + 1} |u_t|^{p - 2} H_i^1 D_i u_t] \, dx \, dt,
\]

where

\[
 A_i^1 = p M b_i^1 - p(p - 1) M a_i^1 + p(p - 1) (\sigma_t^i \nu_t)_{\ell_2}, \\
 B_t = p M^2 c_t - p(\gamma + 1) M a_i^1 + (1/2) p(p - 1) M^2 |\nu_t|_{\ell_2}^2, \\
 F_t = p M f_t^0 - p(\gamma + 1) f_t^1 + p(p - 1) (M \nu_t, g_t)_{\ell_2}, \\
 G_t = (1/2) p(p - 1) |g_t|_{\ell_2}^2, \\
 H_i^1 = p(p - 1) (\sigma_t^i, g_t)_{\ell_2} - p(p - 1) f_i^1.
\]

To estimate the first expectation on the right, one uses the following simple estimates

\[
 |A_i^1| \leq N^* K, \quad |B_t| \leq N^* K (1 + K), \\
 M^\gamma |u_t|^{p - 1} |Du_t| = (M^{(\gamma - 1)/2} |u_t|^{p/2}) (|Du_t| M^{(\gamma + 1)/2} |u_t|^{(p - 2)/2}) \\
\leq M^{\gamma - 1} |u_t|^p + M^{\gamma + 1} |u_t|^{p - 2} |Du_t|^2 = M^{p - d} |M^{-1} u_t|^p.
\]
The second expectation is estimated by using inequalities like (4.4). For instance,

\[ M^{γ+1}|u_t|^p |Du_t|^2 |H_t| \]
\[ = (M^{γ(1-\frac{p-2}{p})}|u_t|^{p-2}/2)(M^{γ(1-\frac{p-2}{p})}|Du_t|)(M^{θ-δ/p}|H_t|) \]
\[ ≤ χ(M^{γ-1}|u_t|^p + M^{γ+1}|u_t|^{p-2}/2|Du_t|^2) + NM^{θ-δ}|H_t|^p. \]

Now we prepare to estimate from below the left-hand side of (4.6) in terms of a quantity equivalent to \( ||M^{-1}u||_{l^{p,θ}(γ)} \). The following two results will not be used in the proof of Theorem 2.6.

**Lemma 4.5.** Let \( β, ε ∈ (0, ∞) \) be some constants and let \( a \) be a measurable bounded \( R^d \)-valued function on \( R^d_+ \) such that

\[ |a(x) - a(y)| ≤ β \]

whenever \( x, y ∈ R^d_+ \) and \( |x - y| ≤ ε(x^1 ∧ y^1) \). Then for any \( u ∈ MW_{p,θ}^1 \) we have

\[ |I + p^{-1}γ∫_{R^d_+} a_iM^{γ-1}|u|^p dx| ≤ Nβ||M^{-1}u||_{W_{p,θ}^1}, \]

where \( N = N(d, p, θ, ε) \) and

\[ I := ∫_{R^d} M^{γ}|u|^{p-2}u a_i D_i u dx. \]

Proof. Since \( C^∞_0(R^d_+) \) is dense in \( MW_{p,θ}^1 \) we may assume that \( u ∈ C^∞_0(R^d_+) \). Take a nonnegative \( ζ ∈ C^∞_0(R^d_+) \) with unit integral and such that \( ζ(x) = 0 \) if \( x^1 ∉ (1, 1 + ε/2) \) or \( |x'| ≥ ε/2 \). For \( y ∈ R^d_+ \) define

\[ ζ^y(x) = (x^1)^{γ+1}ζ(y^1 x^1, y^1(y^1 - x^1))(y^1)^{d-1}. \]

Observe that for \( x ∈ R^d_+ \)

\[ ∫_{R^d_+} ζ^y(x) dy = (x^1)^{γ} ∫_{R^d_+} ζ(y) dy = (x^1)^{γ}. \]

It follows that

\[ I = ∫_{R^d_+} I(y) dy, \]

where \( I(y) = I_1(y) + I_2(y), \)

\[ I_1(y) = ∫_{R^d_+} ζ^y|u|^{p-2}u a_i(y)D_i u dx, \quad y = ((y^1)^{-1}, y^1), \]

\[ I_2(y) = ∫_{R^d_+} ζ^y|u|^{p-2}u[a_i(x) - a_i(y)]D_i u dx. \]

By the choice of \( ζ \) we have that if \( ζ^y(x) ≠ 0 \), then \( 1 < y^1 x^1 < 1 + ε/2 \) and \( y^1 |y^1 - x^1| < ε/2 \) implying that

\[ y^1 x^1 < (1 + ε/2)y^1, \quad |y^1 - x^1| < y^1 ε/2 = (ε/2)(x^1 ∧ y^1), \]
\[ 0 < x^1 - \bar{y}^1 < \bar{y}^1 \varepsilon/2, \quad |x^1 - \bar{y}^1| < (\varepsilon/2)(x^1 \wedge \bar{y}^1), \]
\[ |x - \bar{y}| < \varepsilon (x^1 \wedge \bar{y}^1), \quad |\bar{a}(x) - \bar{a}(\bar{y})| \leq \beta. \]

Hence,
\[ |I_2(y)| \leq \beta \int_{\mathbb{R}^d_+} \zeta |u|^{p-1} |Du| \, dx, \]
\[ \int_{\mathbb{R}^d_+} |I_2(y)| \, dy \leq \beta \int_{\mathbb{R}^d_+} M^\gamma |u|^{p-1} |Du| \, dx \leq N \beta \|M^{-1}u\|_{W^{1,p}_p}^p, \]
\[ |I - \int_{\mathbb{R}^d_+} I_1(y) \, dy| \leq N \beta \|M^{-1}u\|_{W^{1,p}_p}^p. \tag{4.10} \]

To deal with \( I_1(y) \) we integrate by parts observing that
\[ |u|^{p-2} u D_i u = p^{-1} D_i (|u|^p). \]

Then we find
\[ I_1(y) = -p^{-1} \int_{\mathbb{R}^d_+} (D_i \zeta^y) \bar{a}^i(\bar{y}) |u|^p \, dx = -p^{-1} J_1(y) - p^{-1} J_2(y), \]
where
\[ J_1(y) = \int_{\mathbb{R}^d_+} (D_i \zeta^y) [\bar{a}^i(\bar{y}) - \bar{a}^i(x)] |u|^p \, dx, \]
\[ J_2(y) = \int_{\mathbb{R}^d_+} (D_i \zeta^y) \bar{a}^i |u|^p \, dx. \]

As is easy to see
\[ \int_{\mathbb{R}^d_+} D_i \zeta^y \, dy = D_i ((x^1) \gamma) = \delta^{i1} \gamma (x^1)^{\gamma - 1}, \]
\[ \int_{\mathbb{R}^d_+} J_2(y) \, dy = \gamma \int_{\mathbb{R}^d_+} \bar{a}^1 M^{\gamma - 1} |u|^p \, dx \]
and by \[4.10\]
\[ |I + p^{-1} \gamma \int_{\mathbb{R}^d_+} \bar{a}^1 M^{\gamma - 1} |u|^p \, dx| \leq N \beta \|M^{-1}u\|_{W^{1,p}_p}^p + p^{-1} \int_{\mathbb{R}^d_+} |J_1(y)| \, dy. \tag{4.11} \]

Furthermore,
\[ |J_1(y)| \leq \beta \int_{\mathbb{R}^d_+} |D \zeta^y| |u|^p \, dx. \]

Here
\[ |D \zeta^y(x)| \leq |\gamma + 1|(x^1)^{-1} \zeta^y(x) + \zeta^y(x)y^1, \]
\[ \zeta^y(x) := (x^1)^{\gamma - 1} |D \zeta|(y^1 x^1, y^1 (y^1 - x^1))(y^1)^{d-1}, \]
\[ \int_{\mathbb{R}^d_+} |D \zeta^y(x)| \, dy \leq |\gamma + 1|(x^1)^{\gamma - 1} + (x^1)^{\gamma - 1} \int_{\mathbb{R}^d_+} |D \zeta(y)| y^1 \, dy = N(x^1)^{\gamma - 1}. \]
It follows that
\[ \int_{\mathbb{R}^d_+} |J_1(y)| \, dy \leq N\beta \int_{\mathbb{R}^d_+} M^{\gamma - 1} |u|^p \, dx = N\beta \|M^{-1}u\|_{L^p,\theta}, \]
which after being combined with (4.11) leads to (4.8) and proves the lemma.

The following lemma is a simple consequence of Lemma 6.6 of [6], where
the estimate is stronger. The proof of Lemma 6.6 of [6] follows the same
lines as that of Lemma 4.5. Lemma 4.6 will be used for \( \bar{a}^{ij} = (a_t^{11})^{-1} \hat{a}_t^{11} \).

**Lemma 4.6.** Let \( \beta, \epsilon \in (0, \infty) \) be some constants and let \( \bar{a}(x) \) be a measurable function given on \( \mathbb{R}^d_+ \) with values in the set of symmetric nonnegative
matrices and such that \( |\bar{a}^{ij}| \leq \delta \) and
\[ |\bar{a}^{ij}(x) - \bar{a}^{ij}(y)| \leq \beta \] (4.12)
whenever \( x, y \in \mathbb{R}^d_+ \) and \( |x - y| \leq \epsilon (x^1 \wedge y^1) \). Then for any \( u \in MW_{p,\theta}^1 \) and \( \chi > 0 \) and \( \kappa \in (0,1] \) we have
\[ \int_{\mathbb{R}^d_+} M^{\gamma + 1} |u|^{p - 2} \bar{a}^{ij}(D_i u) D_j u \, dx \]
\[ \geq (1 - \kappa)\gamma^2 p - 2 \int_{\mathbb{R}^d_+} M^{\gamma - 1} \bar{a}^{11} |u|^p \, dx \]
\[ - N \left( (\epsilon^{-1} R + 1)\beta + \kappa^{-1} \chi \right) \|M^{-1}u\|_{L^p,\theta}^{p-1}, \]
where \( N = N(d, p, \delta, \theta) \) and \( \ln R = N(d, p)\chi^{-1/2} \).

5. **Proof of Theorems 2.6 and 2.10**

With start with a theorem that says that to prove the solvability of (1.1)
we only need to have an a priori estimate of the lowest norm of \( u \).

**Theorem 5.1.** Let Assumptions 2.1 through 2.4 be satisfied. Assume that
there is a constant \( N_0 < \infty \) such that for any \( \lambda \in [0,1] \), \( u \in MW_{p,\theta,0}^1(\tau) \), and
\( f^0, ..., f^d \) and \( g = (g^k) \), satisfying
\[ Mf^0, f^i, g = (g^k) \in L_{p,\theta}(\tau), \quad i = 1, ..., d, \]
we have the a priori estimate
\[ \|M^{-1}u\|_{L^p,\theta}(\tau) \leq N_0 \left( \|Mf^0\|_{L^p,\theta}(\tau) + \sum_{i=1}^d \|f^i\|_{L^p,\theta}(\tau) + \|g\|_{L^p,\theta}(\tau) \right) \]
provided that
\[ du_t = (\lambda \Lambda^k_t u_t + g^k_t) \, dw_t^k \]
\[ + [(\lambda L_t + (1 - \lambda)\Delta)u_t + f^0_t + D_i f^i_t] \, dt, \quad t \leq \tau, \]
in \( \mathbb{R}^d_+ \) (estimate (5.2) is not supposed to hold if there is no solution \( u \in MW_{p,\theta,0}^1(\tau) \) of (5.3)).
Then for any \( f^0, \ldots, f^d \), and \( g = (g^k) \) satisfying (5.1) there exists a unique \( u \in W^1_{p,\theta,0}(\tau) \) satisfying (1.1) in \( \mathbb{R}^d_+ \) for \( t \leq \tau \). Furthermore, for this solution \( u \),

\[
\|Du\|_{L_{p,\theta}(\tau)} \leq N \left( \|Mf^0\|_{L_{p,\theta}(\tau)} + \sum_{i=1}^d \|f^i\|_{L_{p,\theta}(\tau)} + \|g\|_{L_{p,\theta}(\tau)} \right), \tag{5.4}
\]

where \( N \) depends only on \( d, p, \delta, K, \varepsilon, \varepsilon_1, \) and \( N_0 \).

Proof. We call a \( \lambda \in [0, 1] \) “good” if for any for any \( f^0, \ldots, f^d \), and \( g = (g^k) \) satisfying (5.1) there exists a unique \( u \in W^1_{p,\theta,0}(\tau) \) satisfying (5.3) in \( \mathbb{R}^d_+ \). By Corollary 1.2 and assumption (5.2) estimate (5.1) holds for solutions of (5.3) if \( \lambda \) is a “good” point. It follows that to prove the theorem it suffices to prove that all points of \([0, 1]\) are “good”.

We are going to use the method of continuity observing that the fact that the point \( 0 \) is “good” is known from [9] (or is easily obtained as suggested after (1.2)). We will achieve our goal if we show that there exists a constant \( \mu > 0 \) such that if \( \lambda_0 \) is a “good” point, then all points in the interval \([\lambda_0 - \mu, \lambda_0 + \mu] \cap [0, 1]\) are “good”. So fix a “good” point \( \lambda_0 \) and fix some \( f^0, \ldots, f^d \), and \( g = (g^k) \) satisfying (5.1).

For any \( v \in W^1_{p,\theta}(\tau) \) consider the equation

\[
du_t = [(\lambda_0 L_t + (1 - \lambda_0)\Delta)u_t + (\lambda - \lambda_0)(L_t - \Delta)v_t + D_i f^i_t + f^0_t) dt \\
+ (\lambda_0 \Lambda^k_i u_t + (\lambda - \lambda_0)\Lambda^k v_t + g^k_t) dv^k_t. \tag{5.5}
\]

Observe that

\[
(L_t - \Delta)v_t = D_j ((a^{ij} - \delta^{ij})D_i v_t + a^i_j v_t) + b_i D_i v_t + cv_t,
\]

where by assumption

\[
|(a^{ij} - \delta^{ij})D_i v_t| \leq N |Dv_t|, \quad |a^i_j v_t| \leq NM^{-1} |v_t|, \quad M |b_i D_i v_t| \leq N |Dv_t|,
\]

and the right-hand sides in these estimates are in \( L_{p,\theta}(\tau) \). Hence by the assumption that \( \lambda_0 \) is “good”, equation (5.5) has a unique solution \( u \in W^1_{p,\theta,0}(\tau) \). (\( \subset W^1_{p,\theta,0}(\tau) \)).

In this way, for \( f^j \) and \( g \) being fixed, we define a mapping \( v \rightarrow u \) in the space \( W^1_{p,\theta,0}(\tau) \). It is important to keep in mind that the image \( u \) of \( v \in W^1_{p,\theta,0}(\tau) \) is always in \( W^1_{p,\theta,0}(\tau) \). Take \( v', v'' \in W^1_{p,\theta}(\tau) \) and let \( u', u'' \) be their corresponding images. Then \( u := u' - u'' \) satisfies

\[
dt u = [(\lambda_0 L_t + (1 - \lambda_0)\Delta)u_t + (\lambda - \lambda_0)(L_t - \Delta)v_t) dt \\
+ (\lambda_0 \Lambda^k_i u_t + (\lambda - \lambda_0)\Lambda^k v_t) dv^k_t,
\]

where \( v = v' - v'' \). It follows by (5.2) and (5.4) that

\[
\|M^{-1} u\|_{W^1_{p,\theta}(\tau)} \leq N |\lambda - \lambda_0| \|M^{-1} v\|_{W^1_{p,\theta}(\tau)}
\]

with a constant \( N \) independent of \( f, g, v', v'', \lambda_0, \) and \( \lambda \). For \( \lambda \) sufficiently close to \( \lambda_0 \), our mapping is a contraction and, since \( W^1_{p,\theta}(\tau) \) is a Banach
space, the mapping has a fixed point. This fixed point is in $W^{1,1}_p(\tau)$ and, obviously, satisfies (5.3). As is explained above, this proves the theorem.

**Proof of Theorem 2.6** According to Theorem 5.1 it suffices to find $K = K(d, p, \delta, \theta, \varepsilon, \epsilon_1) > 0$ such that Assumptions 2.1 through 2.4 would imply that (5.2) holds for any solution $u \in W^{1,1}_p(\tau)$ of (1.1) for $t \leq \tau$ and $N_0$ depends only on $d, p, \delta, \theta, \varepsilon, \varepsilon_1$. From the start we will only consider $K \leq 1$. This assumption allows us to eliminate $K$ from the lists of what $N$'s depend on in Theorem 5.1 and Corollary 4.2.

By Hölder’s inequality
\[
I := \left| E \int_0^\tau \int_{\mathbb{R}^d} M^\gamma |u_t|^{p-2} u_t a_{ij}^t D_i u_t \, dx \, dt \right| \leq I_1^{1/2} I_2^{1/2},
\]
where
\[
I_1 = E \int_0^\tau \int_{\mathbb{R}^d} M^{\gamma+1} |u_t|^{p-2} \left( \sum_{i} a_{ij}^t D_i u_t \right)^2 \, dx \, dt,
\]
and
\[
I_2 = \|M^{-1} u\|_{L^p_d(\tau)}^p.
\]
By assumption (2.7)
\[
I_1 \leq \delta^{-1} E \int_0^\tau \int_{\mathbb{R}^d} M^{\gamma+1} |u_t|^{p-2} (a_{ij}^t - \alpha_{ij}^t) (D_j u_t) D_i u_t \, dx \, dt =: \delta^{-1} I_3.
\]
By Lemma 6.1 of [4] (Hardy’s inequality) and Assumption 2.2 we have
\[
\gamma^2 I_2 \leq p^2 E \int_0^\tau \int_{\mathbb{R}^d} M^{\gamma+1} |u_t|^{p-2} |D u_t|^2 \, dx \, dt \leq p^2 \delta^{-1} I_3.
\]
It follows that
\[
I \leq \delta^{-1/2} \delta^{-1/2} p |\gamma|^{-1} I_3,
\]
so that the left hand side of (1.16) dominates
\[
p(p - 1) I_3 - p |\gamma| + 1 |\delta^{-1/2} |\gamma|^{-1} p \delta^{-1/2} I_3.
\]
By assumption the sum of the coefficients of $I_3$ is strictly positive. Therefore, a strictly positive factor of $I_3$ admits an estimate in terms of the right-hand side of (1.16). Estimate (1.6) shows that the same is true for $I_2$. In other words,
\[
\|M^{-1} u\|_{L^p_d(\tau)}^p \leq NJ + \left[ N^* K (1 + K) + \chi \right] \|M^{-1} u\|_{W^{1,1}_p_d(\tau)}^p,
\]
where $\chi > 0$ is arbitrary, $N = N(\chi, d, p, \delta, \theta, \varepsilon)$, $N^* = N^*(d, p, \delta, \theta, \varepsilon)$ and
\[
J = \|M f^0\|_{L^p_d(\tau)} + \sum_{i=1}^d \|f^i\|_{L^p_d(\tau)} + \|g\|_{L^p_d(\tau)}.
\]
Upon combining this with Corollary 4.2 we find
\[
\|M^{-1} u\|_{W^{1,1}_p_d(\tau)}^p \leq NJ + \left[ N^* K (1 + K) + \chi \right] \|M^{-1} u\|_{W^{1,1}_p_d(\tau)}^p,
\]
where $N = N(\chi, d, p, \delta, \theta, \varepsilon, \varepsilon_1)$ and $N^* = N^*(d, p, \delta, \theta, \varepsilon, \varepsilon_1)$. 

SPDES WITH VMO COEFFICIENTS 21
Now it is clear how to find $\chi > 0$ and $K > 0$, depending only on $d, p, \delta, \theta, \varepsilon$, and $\varepsilon_1$, so that the last estimate would imply that the estimate

$$
\|M^{-1}u\|_{L_p,\theta(\tau)}^p \leq \|M^{-1}u\|_{W^{1,p}_{p,\theta}(\tau)}^p \leq N_0 J,
$$

implying (5.2), holds for any solution $u \in W^d_{p,\theta,0}(\tau)$ of (1.1) with $N_0$ depending only on $d, p, \delta, \theta, \varepsilon$, and $\varepsilon_1$. The theorem is proved.

**Proof of Theorem 2.10** As in the above proof, given $d, p, \delta, \theta, \varepsilon$, and $\varepsilon_1$, it suffices to show how to find $K > 0$ and $\beta_2 > 0$ in such a way that Assumptions 2.1 through 2.5 would allow us to derive (5.2). Again without loss of generality we assume that $K, \beta_2 \leq 1$. By Lemma 4.5

$$
p(\gamma + 1)E \int_0^T \int_{\Omega_d} M^{\gamma} |u_t|^{p-2} u_t a^{ij}_t D_i u_t dx dt
\geq -\gamma(\gamma + 1)E \int_0^T \int_{\Omega_d} a^{ij}_t M^{\gamma-1} |u_t|^p dx dt - N \beta_2 \|M^{-1}u\|_{W^{1,p}_{p,\theta}(\tau)}^p,
$$

where $N = N(d, p, \theta)$. By Assumption 2.5 and Lemma 4.6 for $\tilde{a}^{ij} = (a^{ij}_1)^{-1} a^{ij}_0 a^{ij}_2$ we have

$$
p(p - 1)E \int_0^T \int_{\Omega_d} M^{\gamma+1} |u_t|^{p-2} (a^{ij}_0 - a^{ij}_t)(D_i u_t) D_j u_t dx dt
\geq p(p - 1)\tilde{\delta} E \int_0^T \int_{\Omega_d} M^{\gamma+1} |u_t|^{p-2} a^{ij}_t (D_i u_t) D_j u_t dx dt
\geq p^{-1}(p - 1)\tilde{\delta}(1 - \kappa)\gamma^2 E \int_0^T \int_{\Omega_d} a^{ij}_1 M^{\gamma-1} |u_t|^p dx dt
-N ((R + 1)\beta_2 + \kappa^{-1} \chi) \|M^{-1}u\|_{W^{1,p}_{p,\theta}(\tau)},
$$

where $N = N(d, p, \delta, \theta, \varepsilon)$, $\ln R = N(d, p)\chi^{-1/2}$, and $\kappa \in (0, 1]$ and $\chi > 0$ are arbitrary.

Observe that, as $\kappa \downarrow 0$,

$$
-\gamma(\gamma + 1) + p^{-1}(p - 1)\tilde{\delta}(1 - \kappa)\gamma^2 \rightarrow -\gamma[\gamma + 1 - p^{-1}(p - 1)\tilde{\delta}\gamma].
$$

The latter is a strictly positive constant since $\gamma < 0$ and

$$
\gamma + 1 + p^{-1}(p - 1)\tilde{\delta}\gamma = \frac{p - \tilde{\delta}p + \tilde{\delta}}{p \theta + \frac{p}{p - \tilde{\delta}p + \tilde{\delta}} - d - p + 1} > 0
$$

by Assumption 2.5. It follows by (4.6) that after fixing $\kappa = \kappa(d, p, \theta, \tilde{\delta}) \in (0, 1]$ appropriately we can find an $N = N(d, p, \theta, \tilde{\delta}, \delta)$ such that for any $\chi > 0$

$$
\|M^{-1}u\|_{L_p,\theta(\tau)}^p \leq N ((R + 1)\beta_2 + K + \chi) \|M^{-1}u\|_{W^{1,p}_{p,\theta}(\tau)}^p
+ N^*(\|M f^0\|_{L_p,\theta(\tau)}^p + \sum_{i=1}^d \|f_i\|_{L_p,\theta(\tau)}^p + (1 + \beta^p) \|g\|_{L_p,\theta(\tau)}^p),
$$

(5.7)

where $N^* = N^*(d, p, \theta, \tilde{\delta}, \delta, \chi)$. 
By (5.7) and Corollary [4.2] for any \( \chi > 0 \),

\[
\|M^{-1}u\|_{W_{1}^{p,\theta}(\tau)}^{p} \leq N_1((R + 1)\beta_2 + K + \chi)\|M^{-1}u\|_{W_{1}^{p,\theta}(\tau)}^{p}
\]

\[
+ N_2(\|Mf^0\|_{L_{p,\theta}(\tau)}^{p} + \sum_{i=1}^{d} \|f_i\|_{L_{p,\theta}(\tau)}^{p} + \|g\|_{L_{p,\theta}(\tau)}^{p}),
\tag{5.8}
\]

where (recall that \( K \leq 1 \))

\[
N_1 = N_1(d, p, \theta, \tilde{\delta}, \delta, \varepsilon, \varepsilon_1), \quad N_2 = N_2(d, p, \theta, \tilde{\delta}, \delta, \varepsilon, \varepsilon_1, \chi).
\]

Now we fix a \( \chi = \chi(d, p, \theta, \tilde{\delta}, \delta, \varepsilon, \varepsilon_1) > 0 \) so that \( N_1\chi \leq 1/4 \) and then find a \( \beta_2 = \beta_2(d, p, \theta, \tilde{\delta}, \delta, \varepsilon, \varepsilon_1) \) such that \( N_1 R \beta_2 \leq 1/4 \). Then estimate (5.8) will implies (5.2) which along with Theorem 5.1 brings the proof of Theorem 2.10 to an end.

**References**

[1] Kyeong-Hun Kim, *On \( L_p \)-theory of stochastic partial differential equations of divergence form in \( C^1 \) domains*, Probab. Theory Related Fields, Vol. 130 (2004), No. 4, 473-492.

[2] Kyeong-Hun Kim, *On stochastic partial differential equations with variable coefficients in \( C^1 \) domains*, Stochastic Process. Appl., Vol. 112 (2004), No. 2, 261-283.

[3] Kyeong-Hun Kim, *\( L_p \) estimates for SPDE with discontinuous coefficients in domains*, Electron. J. Probab., Vol. 10 (2005), No. 1, 1-20.

[4] N.V. Krylov, *Weighted Sobolev spaces and Laplace’s and the heat equations in a half space*, Communications PDE, Vol. 24 (1999), No. 9-10, 1611-1653.

[5] N.V. Krylov, *An analytic approach to SPDEs*, pp. 185-242 in Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs, Vol. 64, AMS, Providence, RI, 1999.

[6] N.V. Krylov, *On parabolic PDEs and SPDEs in Sobolev spaces \( W_2^2 \) without and with weights*, pp 151-198 in Topics in Stochastic Analysis and Nonparametric Estimation, P.-L. Chow, B. Mordukhovich, G. Yin eds., IMA Volumes in Mathematics and its Applications, Vol. 145 (2008), Springer, New York.

[7] N.V. Krylov, *Itô’s formula for the \( L_p \)-norm of stochastic \( W_1^1 \)-valued processes*, http://arxiv.org/abs/0806.1557

[8] N.V. Krylov, *On divergence form SPDEs with VMO coefficients*, http://arxiv.org/abs/0806.1925

[9] N.V. Krylov and S.V. Lototsky, *A Sobolev space theory of SPDEs with constant coefficients in a half space*, SIAM J. Math. Anal., Vol. 31 (1999), No. 1, 19-33.

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455

E-mail address: krylov@math.umn.edu