Abstract. We investigate full Lipschitzian and full Hölderian stability for a class of control problems governed by semilinear elliptic partial differential equations, where all the cost functional, the state equation, and the admissible control set of the control problems undergo perturbations. We establish explicit characterizations of both Lipschitzian and Hölderian full stability for the class of control problems. We show that for this class of control problems the two full stability properties are equivalent. In particular, the two properties are always equivalent in general when the admissible control set is an arbitrary fixed nonempty, closed, and convex set.

Key words. Perturbed control problem, semilinear elliptic partial differential equations, full Lipschitzian stability, full Hölderian stability, coderivative, combined second-order sub-differential.

AMS subject classifications. 49K20, 49K30, 35J61.

1 Introduction

The notion of full Lipschitzian stability was introduced and studied by Levy, Poliquin, and Rockafellar [14] in the finite dimensional setting. This property then was investigated in the infinite dimensional setting by Mordukhovich and Nghia [18]. Moreover, in [18], the authors also introduced and studied the notion of full Hölderian stability. The full Lipschitzian and Hölderian stability is defined on the basis of two types of parameters: Basic parameters and tilt ones, where the last type is related to the notion of tilt stability introduced by Poliquin and Rockafellar [21].

According to [18, Example 4.4] (see also [25] for the original example), the two notions of full Lipschitzian and Hölderian stability are really different, where the example shows that the full Hölderian stability is strictly weaker than the Lipschitzian one. However, it is interesting to know which classes of optimization models the Lipschitzian and Hölderian full stability properties are equivalent.

In the paper [18], the authors provided various characterizations of both Lipschitzian and Hölderian full stability in general optimization models. Then they applied these results to characterize (only) the full Lipschitzian stability of optimal control problems governed by
semilinear elliptic partial differential equations, where the basic parameters appear only in the cost functional while the state equation and the admissible control set of the problems are fixed.

In this paper, we study both Lipschitzian and Hölderian full stability for a broader class of control problems governed by semilinear elliptic partial differential equations. Moreover, all the cost functional, the state equation, and the admissible control set of the control problems undergo perturbations of basic parameters. We will provide explicit characterizations of both Lipschitzian and Hölderian full stability for the class of optimal control problems on the basis of the second-order subdifferential characterizations of the latter stability properties for parametric optimization problems obtained in [18]. In particular, we show that for this class of optimal control problems the two full stability properties are equivalent.

Related to our class of control problems with the cost functional not involving the usual quadratic term for the control and the admissible control set being fixed, we refer the reader to [24] for the Hölderian stability of bang-bang optimal controls in $L^1$ that is not deduced from our results in this paper; see also [5, 7, 22, 23] for more details on second-order optimality conditions, solution stability, and numerical methods for bang-bang controls.

It is worth mentioning another approach to the local Lipschitzian stability of solutions to parametric optimal control problems for nonlinear systems; see, e.g., [9, 15, 16] and the references therein. The main tools in stability analysis for such problems are Robinson’s implicit function theorem for generalized equations given in [26] and an extension of Robinson’s theorem obtained in [8]. However, Robinson’s implicit function theorem does not provide us with any information on the gap between sufficient conditions and necessary conditions of Lipschitzian stability while the extension of Robinson’s theorem allows to establish necessary and sufficient conditions of Lipschitzian stability provided the sufficiently strong dependence of data on the parameters. For instance, in [16] the authors established a second-order sufficient condition for Lipschitzian stability of solutions to elliptic optimal control problems under nonlinear perturbations in general. The latter condition is also a second-order necessary condition for Lipschitzian stability only if the parameter of the problems is a linear perturbed parameter. Namely, in the terminology of full stability, there is only tilt parameter in the problems while the basic parameter disappears. In contrast to this approach, our results obtained in this paper are necessary and sufficient conditions for Lipschitzian and Hölderian stability for both basic and tilt perturbations.

The rest of the paper is organized as follows. A class of optimal control problems together with assumptions posed on the initial data of these problems are stated in Section 2. The section also recalls some notions and facts from variational analysis and basic results for optimal control problems of semilinear elliptic partial equations.

In Section 3, we show that the two properties of Lipschitzian and Hölderian full stability for a class of optimization problems, where the cost functional and the state equation of the problems are perturbed while the admissible control set of the problems is fixed, are equivalent. This result is applied to deduce that the two properties of full stability are always equivalent for our class of optimal control problems when the admissible control set does not undergo perturbations. In addition, explicit characterizations of the full stability properties for the class of optimal control problems in this case are also provided.

Section 4 is devoted to investigate full stability for the class of optimal control problems, where all the cost functional, the state equation, and the admissible control set of the problems are perturbed. We establish explicit characterizations of both Lipschitzian and Hölderian full stability properties for the class of optimal control problems. Interestingly, the two full stability properties are also equivalent in this setting. Note that the equivalence of the two properties of full stability is due to special structures of perturbed admissible control sets, the properties of full stability are not always equivalent in general for perturbed

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admissible control sets in contrast to the setting considered in the previous section.

Some concluding remarks and further investigations on the full stability are provided in the last section.

2 Problem statement and preliminaries

Consider the optimal control problem

\[
\begin{aligned}
\text{Minimize} & \quad J(u) = \int_{\Omega} L(x, y_u(x))dx + \frac{1}{2} \int_{\Omega} \zeta(x)u(x)^2dx \\
\text{subject to} & \quad \alpha(x) \leq u(x) \leq \beta(x) \quad \text{for a.e. } x \in \Omega,
\end{aligned}
\]

(2.1)

where \( \zeta \in L^2(\Omega) \) satisfies \( \zeta(x) \geq \zeta_0 > 0 \) for a.e. \( x \in \Omega \), and \( y_u \) is the weak solution associated to the control \( u \) of the Dirichlet problem

\[
\begin{aligned}
Ay + f(x, y) &= u \quad \text{in } \Omega \\
y &= 0 \quad \text{on } \Gamma,
\end{aligned}
\]

(2.2)

where \( A \) denotes the second-order differential elliptic operator of the form

\[
Ay(x) = -\sum_{i,j=1}^N \partial_{x_j}(a_{ij}(x)\partial_{x_i}y(x)).
\]

(2.3)

From now on, we denote the set of admissible controls by

\[
U_{ad} = \{ u \in L^2(\Omega) \mid \alpha(x) \leq u(x) \leq \beta(x) \text{ for a.e. } x \in \Omega \}.
\]

(2.4)

Observe that if \( \alpha, \beta \in L^\infty(\Omega) \) with \( \alpha(x) \leq \beta(x) \) for a.e. \( x \in \Omega \), then \( U_{ad} \) is nonempty, closed, bounded, and convex in \( L^2(\Omega) \).

In this paper, we will study solution stability of problem (2.1), where the state equation (2.2), the admissible control set (2.4), and the cost functional \( J(\cdot) \) given in (2.1) undergo perturbations. We are interested in the perturbed control problem as follows

\[
\begin{aligned}
\text{Minimize} & \quad J(u, e) = J(u + e_y) + (e_J, y_u + e_y)_{L^2(\Omega)} \\
\text{subject to} & \quad u \in U_{ad}(e),
\end{aligned}
\]

(2.5)

where \( y_{u+e_y} \) is the weak solution associated to \( u + e_y \) of the perturbed Dirichlet problem

\[
\begin{aligned}
Ay + f(x, y) &= u + e_y \quad \text{in } \Omega \\
y &= 0 \quad \text{on } \Gamma,
\end{aligned}
\]

(2.6)

the perturbed admissible control set \( U_{ad}(e) \) is given by

\[
U_{ad}(e) = \{ u \in L^2(\Omega) \mid \alpha(x) + e_\alpha(x) \leq u(x) \leq \beta(x) + e_\beta(x) \text{ for a.e. } x \in \Omega \},
\]

(2.7)

and \( e_J, e_y \in L^2(\Omega), e_\alpha, e_\beta \in L^\infty(\Omega) \) are parameters. We denote the space of parameters by \( E = L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega) \) with the norm \( \| \cdot \|_E \) of \( e = (e_y, e_J, e_\alpha, e_\beta) \in E \) defined by

\[
\| e \|_E = \| e_y \|_{L^2(\Omega)} + \| e_J \|_{L^2(\Omega)} + \| e_\alpha \|_{L^\infty(\Omega)} + \| e_\beta \|_{L^\infty(\Omega)}.
\]

Let us fix any parameter \( \bar{e} = (\bar{e}_J, \bar{e}_y, \bar{e}_\alpha, \bar{e}_\beta) \in E \) and consider the following problem

\[
\begin{aligned}
\text{Minimize} & \quad J(u, e) = J(u + e_y) + (e_J, y_u + e_y)_{L^2(\Omega)} \\
\text{subject to} & \quad u \in U_{ad}(\bar{e}),
\end{aligned}
\]

(2.8)
where \( y_{u+\varepsilon y} \) is the weak solution associated to \( u + \varepsilon y \) of the Dirichlet problem (2.6). The corresponding perturbed problem of problem (2.8) is defined as follows

\[
P(u^*, e): \text{Minimize } J(u, e) - \langle u^*, u \rangle \text{ subject to } u \in U_{ad}(e),
\]

where \( J : L^2(\Omega) \times E \to IR \) is defined in (2.5) and \((u^*, e) \in L^2(\Omega) \times E\) are the parameters. We interpret \( J \) as the basic parameter perturbations and \( u^* \in L^2(\Omega) \) as the tilt ones. Note that problem (2.8) reduces to problem (2.1) when \( \varepsilon = (0,0,0,0) \in E \).

The main contributions of this paper are the following:

(a) We show that the two properties of full Lipschitzian and full Hölderian stability for a class of optimization problems, where the objective function is of class \( C^2 \) and the constraint set of the problems is fixed, are always equivalent in general. We will apply this result to deduce that the full Lipschitzian and full Hölderian stability properties for problem (2.9) are equivalent when the admissible control set \( U_{ad} \) of the control problem does not undergo perturbation, i.e., the basic parameter is always in the form \( e = (\varepsilon_0, \varepsilon_1, 0, 0) \). Moreover, we provide explicit characterizations of the two properties for problem (2.9) in this setting.

(b) We establish explicit characterizations of both Lipschitzian and Hölderian full stability for problem (2.9), where all the cost functional, the state equation, and the admissible control set of the problem undergo perturbations. In comparison between the characterizations of the full stability properties obtained we show that the two full stability properties are also equivalent for this setting.

(c) Our results show that the equivalence of the Lipschitzian and Hölderian full stability properties depend on the structure of perturbed admissible control sets. Namely, the full stability properties are not always equivalent when the admissible control sets undergo perturbations.

Given \((\tilde{u}^*, \varepsilon) \in L^2(\Omega) \times E\), \( \tilde{u} \in U_{ad}(\varepsilon) \), \((u^*, e) \in L^2(\Omega) \times E\), and \( \gamma > 0 \), associated with these data we define

\[
\begin{align*}
m_\gamma(u^*, e) &= \inf_{u \in U_{ad}(e), \|u-u^*\|_{L^2(\Omega)} \leq \gamma} \left\{ J(u, e) - \langle u^*, u \rangle \right\}, \\
M_\gamma(u^*, e) &= \arg\min_{u \in U_{ad}(e), \|u-u^*\|_{L^2(\Omega)} \leq \gamma} \left\{ J(u, e) - \langle u^*, u \rangle \right\}.
\end{align*}
\]

(2.10)

Following [14] and [18], we recall the concepts of Lipschitzian and Hölderian full stability of the problem \( P(\tilde{u}^*, \varepsilon) \) in (2.9) as follows:

- The control \( \tilde{u} \) is said to be a Lipschitzian fully stable local minimizer of the problem \( P(\tilde{u}^*, \varepsilon) \) if there exists a number \( \gamma > 0 \) such that the mapping \((u^*, e) \mapsto M_\gamma(u^*, e) \) in (2.10) is single-valued and locally Lipschitz continuous with \( M_\gamma(\tilde{u}^*, \varepsilon) = \tilde{u} \) and the function \((u^*, e) \mapsto m_\gamma(u^*, e) \) is also Lipschitz continuous around \((\tilde{u}^*, \varepsilon) \).

- We say that the control \( \tilde{u} \) is a Hölderian fully stable local minimizer of the problem \( P(\tilde{u}^*, \varepsilon) \) if there are \( \gamma, \kappa > 0 \) such that the mapping \((u^*, e) \mapsto M_\gamma(u^*, e) \) from (2.10) is single-valued around \((\tilde{u}^*, \varepsilon) \) with \( M_\gamma(\tilde{u}^*, \varepsilon) = \tilde{u} \) and the Hölder property

\[
\| M_\gamma(u^*, e) - M_\gamma(\tilde{u}^*, \varepsilon) \|_{L^2(\Omega)} \leq \kappa \left( \| u^* - \tilde{u}^* \|_{L^2(\Omega)} + \| e - \varepsilon \|_{L^2(\Omega)} \right)^{1/2}
\]

(2.11)

holds for any pairs \((u^*, e), (\tilde{u}^*, \varepsilon)\) in a neighborhood \( U^* \times V \) of \((\tilde{u}^*, \varepsilon)\), and that the function \((u^*, e) \mapsto m_\gamma(u^*, e) \) is Lipschitz continuous on \( U^* \times V \).
We now assume that \( \Omega \subset \mathbb{R}^N \) with \( N \in \{1, 2, 3\} \) and \( \alpha, \beta \in L^2(\Omega) \) with \( \alpha(x) < \beta(x) \) for a.e. \( x \in \Omega \). Moreover, the function \( L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are Carathéodory functions of class \( C^2 \) with respect to the second and third variables satisfying the following assumptions:

(A1) The function \( f \) is of class \( C^2 \) with respect to the second variable, and

\[
 f(\cdot, 0) \in L^2(\Omega) \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) \geq 0 \quad \text{for a.e. } x \in \Omega,
\]

and for all \( M > 0 \) there exists a constant \( C_{f,M} > 0 \) such that

\[
 \left| \frac{\partial f}{\partial y}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) \right| \leq C_{f,M},
\]

and

\[
 \left| \frac{\partial^2 f}{\partial y^2}(x, y) - \frac{\partial^2 f}{\partial y^2}(x, y_1) \right| \leq C_{f,M} |y_2 - y_1|,
\]

for a.e. \( x \in \Omega \) and \( |y|, |y_1|, |y_2| \leq M \).

(A2) The function \( L(\cdot, 0) \in L^1(\Omega) \) and for all \( M > 0 \) there are a constant \( C_{L,M} > 0 \) and a function \( \psi_M \in L^2(\Omega) \) such that

\[
 \left| \frac{\partial L}{\partial y}(x, y) \right| \leq \psi_M(x), \quad \left| \frac{\partial^2 L}{\partial y^2}(x, y) \right| \leq C_{L,M},
\]

and

\[
 \left| \frac{\partial^2 L}{\partial y^2}(x, y_2) - \frac{\partial^2 L}{\partial y^2}(x, y_1) \right| \leq C_{L,M} |y_2 - y_1|,
\]

for a.e. \( x \in \Omega \) and \( |y|, |y_1|, |y_2| \leq M \).

(A3) The set \( \Omega \) is an open and bounded domain in \( \mathbb{R}^N \) with Lipschitz boundary \( \Gamma \), the coefficients \( a_{ij} \in L^\infty(\Omega) \) of the second-order differential elliptic operator \( A \) defined by (2.3) satisfy the condition

\[
 \lambda_A \|\xi\|^2_{\mathbb{R}^N} \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j, \quad \forall \xi \in \mathbb{R}^N, \quad \text{for a.e. } x \in \Omega,
\]

for some constant \( \lambda_A > 0 \).

**Theorem 2.1** (See [6, Theorem 2.1]) Suppose that (A1) holds. Then, for every \( u \in L^2(\Omega) \), the state equation (2.2) has a unique solution \( y_u \in H_0^1(\Omega) \cap C(\bar{\Omega}) \). In addition, there exists a constant \( M_{\alpha, \beta} \) such that

\[
 \|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar{\Omega})} \leq M_{\alpha, \beta}, \quad \forall u \in \mathcal{U}_{ad}.
\]

Furthermore, if \( u_n \rightharpoonup u \) weakly in \( L^2(\Omega) \), then \( y_{u_n} \to y_u \) strongly in \( H_0^1(\Omega) \cap C(\bar{\Omega}) \).

**Theorem 2.2** (See [6, Theorem 2.4]) If assumption (A1) holds, then the control-to-state mapping \( G : L^2(\Omega) \to H_0^1(\Omega) \cap C(\bar{\Omega}) \), defined by \( G(u) = y_u \), is of class \( C^2 \). Moreover, for every \( u, v \in L^2(\Omega) \), \( z_{u,v} = G'(u)v \) is the unique weak solution of

\[
 \begin{cases}
 Az + \frac{\partial f}{\partial y}(x, y)z = v & \text{in } \Omega \\
 z = 0 & \text{on } \Gamma.
\end{cases}
\]
Finally, for every \( v_1, v_2 \in L^2(\Omega) \), \( z_{v_1v_2} = G''(u)(v_1, v_2) \) is the unique weak solution of

\[
\begin{aligned}
Az + \frac{\partial f}{\partial y}(x, y)z + \frac{\partial^2 f}{\partial y^2}(x, y)z_{u,v_1}z_{u,v_2} &= 0 & \text{in } \Omega \\
&\quad \text{in } \Omega \\
&z = 0 & \text{on } \Gamma;
\end{aligned}
\]

where \( y = G(u) \) and \( z_{u,v_i} = G'(u)v_i \) for \( i = 1, 2 \).

Related to the results on the solution of the state equation (2.12) we refer the reader to Chapter 4 for more details. We introduce the space \( Y = H^1_0(\Omega) \cap C(\bar{\Omega}) \) endowed with the norm

\[
\|y\|_Y = \|y\|_{H^1_0(\Omega)} + \|y\|_{L^\infty(\Omega)}.
\]

**Theorem 2.3** (See [6, Theorem 2.6 and Remark 2.8]) Suppose that (A1) and (A2) hold. The cost functional \( J : L^2(\Omega) \to \mathbb{R} \) is of class \( C^2 \). Moreover, for every \( u, v, v_1, v_2 \in L^2(\Omega) \), the first and second derivatives of \( J(\cdot) \) are given by

\[
J'(u)v = \int_{\Omega} (\zeta u + \varphi_u)vdx,
\]

and

\[
J''(u)(v_1, v_2) = \int_{\Omega} \left( \frac{\partial^2 L}{\partial y^2}(x, y_u)z_{u,v_1}z_{u,v_2} + \zeta_1 v_1 v_2 - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u)z_{u,v_1}z_{u,v_2} \right)dx,
\]

where \( y_u = G(u) \), \( z_{u,v_i} = G'(u)v_i \) for \( i = 1, 2 \), and \( \varphi_u \in W^{2,p}(\Omega) \) is the adjoint state of \( y_u \) defined as the unique weak solution of

\[
\begin{aligned}
A^* \varphi + \frac{\partial f}{\partial y}(x, y_u)\varphi &= \frac{\partial L}{\partial y}(x, y_u) & \text{in } \Omega \\
\varphi &= 0 & \text{on } \Gamma
\end{aligned}
\]

with \( A^* \) being the adjoint operator of \( A \).

For any \( p \in [1, \infty] \), we denote \( \bar{B}_p^\varepsilon(\bar{u}) \) the closed ball in the space \( L^p(\Omega) \) with the center at \( \bar{u} \in L^p(\Omega) \) and the radius \( \varepsilon > 0 \), i.e.,

\[
\bar{B}_p^\varepsilon(\bar{u}) = \{ v \in L^p(\Omega) \mid \| v - \bar{u} \|_{L^p(\Omega)} \leq \varepsilon \}.
\]

An element \( \bar{u} \in \mathcal{U}_{ad} \) is said to be a solution/global minimum of problem (2.1) if \( J(\bar{u}) \leq J(u) \) for all \( u \in \mathcal{U}_{ad} \). We will say that \( \bar{u} \) is a local solution/local minimum of problem (2.1) in the sense of \( L^p(\Omega) \) if there exists a closed ball \( \bar{B}_p^\varepsilon(\bar{u}) \) such that \( J(\bar{u}) \leq J(u) \) for all \( u \in \mathcal{U}_{ad} \cap \bar{B}_p^\varepsilon(\bar{u}) \). The local solution \( \bar{u} \) is called strict if \( J(\bar{u}) < J(u) \) holds for all \( u \in \mathcal{U}_{ad} \cap \bar{B}_p^\varepsilon(\bar{u}) \) with \( u \neq \bar{u} \). Under the above assumptions (A1)-(A3), solutions of problem (2.1) exist.

**Theorem 2.4** (See [6, Theorem 2.2]) For the assumptions (A1)-(A3), the control problem (2.1) has at least one solution.

Let us recall concepts and facts of variational analysis and generalized differentiation taken from [17]. Unless otherwise stated, every reference norm in a product normed space is the sum norm. Given a point \( u \) in a Banach space \( X \) and \( \rho > 0 \), we denote \( B_{\rho}(u) \) the open ball of center \( u \) and radius \( \rho \) in \( X \), and \( \overline{B}_{\rho}(u) \) is the corresponding closed ball. Let \( F : X \rightrightarrows W \) be a multifunction between Banach spaces. The graph of \( F \), denoted by \( \text{gph} F \), is the set \( \{(u, v) \in X \times W \mid v \in F(u)\} \). We say that \( F \) is locally closed around the point
\( \tilde{\omega} = (\tilde{u}, \tilde{v}) \in \text{gph} F \) if \( \text{gph} F \) is locally closed around \( \tilde{\omega} \), i.e., there exists a closed ball \( \tilde{B}_r(\tilde{\omega}) \) such that \( \tilde{B}_r(\tilde{\omega}) \cap \text{gph} F \) is closed in \( X \times W \). For a multifunction \( \Phi : X \rightrightarrows X^* \), the sequential Painlevé-Kuratowski upper limit of \( \Phi \) as \( u \to \tilde{u} \) is defined by

\[
\limsup_{u \to \tilde{u}} \Phi(u) = \left\{ u^* \in X^* \mid \text{there exist } u_n \to \tilde{u} \text{ and } u_n^* \xrightarrow{w^*} u^* \text{ with } u_n^* \in \Phi(u_n) \text{ for every } k \in \mathbb{N} = \{1, 2, \ldots \} \right\}.
\]  

(2.17)

Let \( \phi : X \to \overline{\mathbb{R}} \) be a proper extended-real-valued function on an Asplund space \( X \) (see [1] for more details on Asplund spaces). Assume that \( \phi \) is lower semicontinuous (lsc) around \( \tilde{u} \) from the domain \( \text{dom} \phi = \{ u \in X \mid \phi(u) < \infty \} \). The regular subdifferential of \( \phi \) at \( \tilde{u} \in \text{dom} \phi \) is

\[
\partial \phi(\tilde{u}) = \left\{ u^* \in X^* \mid \liminf_{u \to \tilde{u}} \frac{\phi(u) - \phi(\tilde{u}) - \langle u^*, u - \tilde{u} \rangle}{\| u - \tilde{u} \|} \geq 0 \right\},
\]

(2.18)

while the limiting subdifferential (known also as Mordukhovich subdifferential) of \( \phi \) at \( \tilde{u} \) is defined via the sequential outer limit \( 2.17 \) by

\[
\partial \phi(\tilde{u}) = \limsup_{u \to \tilde{u}} \widehat{\partial} \phi(u),
\]

(2.19)

where the notation \( u \xrightarrow{\phi} \tilde{u} \) means that \( u \to \tilde{u} \) with \( \phi(u) \to \phi(\tilde{u}) \).

Given a nonempty set \( \Theta \subset X \) locally closed around \( \tilde{u} \in \Omega \), the regular and limiting normal cones to \( \Theta \) at \( \tilde{u} \in \Theta \) are respectively defined by

\[
\widehat{N}(\tilde{u}; \Theta) = \widehat{\partial} \delta(\tilde{u}; \Theta) \quad \text{and} \quad N(\tilde{u}; \Theta) = \partial \delta(\tilde{u}; \Theta),
\]

(2.20)

where \( \delta(\cdot; \Theta) \) is the indicator function of \( \Theta \) defined by \( \delta(u; \Theta) = 0 \) for \( u \in \Theta \) and \( \delta(u; \Theta) = \infty \) otherwise. The regular and Mordukhovich coderivatives of the multifunction \( F : X \rightrightarrows W \) at the point \( (\tilde{u}, \tilde{v}) \in \text{gph} F \) are respectively the multifunction \( \hat{D}^* F(\tilde{u}, \tilde{v}) : W^* \rightrightarrows X^* \) defined by

\[
\hat{D}^* F(\tilde{u}, \tilde{v})(v^*) = \left\{ u^* \in X^* \mid (u^*, -v^*) \in \widehat{N}( (\tilde{u}, \tilde{v}); \text{gph} F ) \right\}, \quad \forall v^* \in W^*.
\]

and the multifunction \( D^* F(\tilde{u}, \tilde{v}) : W^* \rightrightarrows X^* \) given by

\[
D^* F(\tilde{u}, \tilde{v})(v^*) = \left\{ u^* \in X^* \mid (u^*, -v^*) \in N( (\tilde{u}, \tilde{v}); \text{gph} F ) \right\}, \quad \forall v^* \in W^*.
\]

Given any \( \tilde{u}^* \in \partial \phi(\tilde{u}) \), the combined second-order subdifferential of \( \phi \) at \( \tilde{u} \) relative to \( \tilde{u}^* \) is the multifunction \( \partial^2 \phi(\tilde{u}, \tilde{u}^*) : X^{**} \rightrightarrows X^* \) with the values

\[
\partial^2 \phi(\tilde{u}, \tilde{u}^*)(u) = \left( \hat{D}^* \partial \phi(\tilde{u}, \tilde{u}^*) \right)(u), \quad \forall u \in X^{**}.
\]

(2.21)

Note that for \( \phi \in C^2 \) around \( \tilde{u} \) with \( \tilde{u}^* = \nabla \phi(\tilde{u}) \) we have \( \partial^2 \phi(\tilde{u}, \tilde{u}^*)(u) = \{ \nabla^2 \phi(\tilde{u})u \} \) for all \( u \in X^{**} \) via the symmetric Hessian operator \( \nabla^2 \phi(\tilde{u}) \).

We say that the multifunction \( F : X \rightrightarrows W \) is locally Lipschitz-like, or \( F \) has the Aubin property [10], around a point \( (\tilde{u}, \tilde{v}) \in \text{gph} F \) if there exist \( \ell > 0 \) and neighborhoods \( U \) of \( \tilde{u} \), \( V \) of \( \tilde{v} \) such that

\[
F(u_1) \cap V \subset F(u_2) + \ell \| u_1 - u_2 \| B_W, \quad \forall u_1, u_2 \in U,
\]

where \( B_W \) denotes the closed unit ball in \( W \). Characterization of this property via the mixed Mordukhovich coderivative of \( F \) can be found in [17] Theorem 4.10.

Let us recall the concepts of prox-regularity and subdifferential continuity of extended-real-valued functions from [18]. Given a function \( \psi : X \times E \to \overline{\mathbb{R}} \) finite at \( (\tilde{u}, \tilde{e}) \) and given
a partial limiting subgradient \( \bar{u}^* \in \partial_u \psi(\bar{u}, \bar{e}) \) of \( \psi(\cdot, \bar{e}) \) at \( \bar{u} \), we say that \( \psi \) is prox-regular in \( u \) for \( \bar{u}^* \) with compatible parameterization by \( e \) at \( \bar{e} \) if there are neighborhoods \( U \) of \( \bar{u} \), \( U^* \) of \( \bar{u}^* \), and \( V \) of \( \bar{e} \) along with numbers \( \varepsilon > 0 \) and \( r > 0 \) such that

\[
\psi(u, e) \geq \psi(v, e) + \langle u^*, u - v \rangle - \frac{r}{2} \| u - v \|^2, \forall u \in U,
\]

whenever

\[
u^* \in \partial_u \psi(v, e) \cap U^*; \quad (v, e) \in U \times V, \quad \psi(v, e) \leq \psi(\bar{u}, \bar{e}) + \varepsilon.
\]

The function \( \psi \) is subdifferentially continuous in \( u \) at \( \bar{u}^* \) with compatible parameterization by \( e \) at \( \bar{e} \) if the mapping \( (u, e, u^*) \mapsto \psi(u, e) \) is continuous relative to \( \text{gph} \partial_u \psi \) at \((\bar{u}, \bar{e}, \bar{u}^*)\). When \( \psi \) is prox-regular and subdifferentially continuous in \( u \) at \( \bar{u}^* \) with compatible parameterization by \( e \) at \( \bar{e} \), \( \psi \) is said to be parametrically continuously prox-regular at \((\bar{u}, \bar{e})\) for \( \bar{u}^* \). These concepts are comprehensively studied in finite dimensional settings; see [14]. The nonparametric versions of these concepts can be found in [20], [27]. We say that the basic constraint qualification (BCQ) holds at a point \((\bar{u}, \bar{e})\) if the epigraphical mapping \( F : E \rightrightarrows X \times \mathbb{R} \) defined by \( F(e) = \text{epi} \psi(\cdot, e) \) is locally Lipschitz-like around \((\bar{e}, \bar{u}, \psi(\bar{u}, \bar{e})) \in \text{gph} F \), where the set

\[
\text{epi} \psi(\cdot, e) := \{(u, \tau) \in X \times \mathbb{R} | \tau \geq \psi(u, e)\}
\]

is the epigraph of the function \( \psi(\cdot, e) \).

### 3 Fixed admissible control set and full stability

Let \( X \) be a Hilbert space and let \( E \) be an Aslund space. Fix any \( \bar{e} \in E \) and consider the problem

\[
\text{Minimize } \psi(u, \bar{e}) \text{ over } u \in X,
\]

where \( \psi : X \times E \rightarrow \mathbb{R} \) is an extended-real-valued function. The corresponding perturbed problem of (3.22) is as follows

\[
\mathcal{P}_{\text{Abs}}(u^*, \bar{e}) : \text{Minimize } \psi(u, e) - \langle u^*, u \rangle \text{ over } u \in X.
\]

Here, the subscript “Abs” refers to the abstract nature of this optimization problem. We recall [18] Theorem 4.9] on characterization for full Lipschitzian stability via a perturbed positive definiteness of the regular coderivative of the function \( \partial_u \psi(\cdot, \cdot) \).

**Theorem 3.1** (See [18] Theorem 4.9]) Assume that the BCQ holds at \((\bar{u}, \bar{e})\) and that \( \psi \) is parametrically continuously prox-regular at \((\bar{u}, \bar{e})\) for \( \bar{u}^* \in \partial_u \psi(\bar{u}, \bar{e}) \). Then, the following are equivalent:

(i) The point \( \bar{u} \) is a Lipschitzian fully stable local minimizer of \( \mathcal{P}_{\text{Abs}}(\bar{u}^*, \bar{e}) \) in (3.23).

(ii) The graphical mapping \( e \mapsto \text{gph} \partial_u \psi(\cdot, e) \) is locally Lipschitz-like around \((\bar{e}, \bar{u}, \bar{u}^*)\) and there are \( \eta, \delta > 0 \) such that for all \((u, e, u^*) \in \text{gph} \partial_u \psi \cap B_{\eta}(\bar{u}, \bar{e}, \bar{u}^*) \) we have

\[
\langle v^*, v \rangle \geq \delta \| v \|^2_{L^2(\Omega)} \quad \text{whenever } (v^*, e^*) \in \hat{D}^*(\partial_u \psi)(u, e, u^*)(v) \quad \text{for } v \in X.
\]

And, we also recall [18] Theorem 4.7] on characterization for full Hölderian stability via a perturbed positive definiteness of the combined second-order subdifferential of the function \( \psi_e(\cdot) = \psi(\cdot, e) \).
Theorem 3.2 (See [18, Theorem 4.7]) Assume that the BCQ holds at $(\bar{u}, \bar{e}) \in \text{dom } \psi$ and that $\psi$ is parametrically continuously prox-regular at $(\bar{u}, \bar{e})$ for $\bar{u}^* \in \partial_u \psi(\bar{u}, \bar{e})$. Then, the following are equivalent:

(i) The point $\bar{u}$ is a Hölderian fully stable local minimizer of $\mathcal{P}_{\text{Abs}}(\bar{u}^*, \bar{e})$ in (3.23).

(ii) There are $\eta, \delta > 0$ such that for all $(u, e, u^*) \in \text{gph } \partial_u \psi \cap \bar{B}_\eta(\bar{u}, \bar{e}, \bar{u}^*)$ we have

$$\langle v^*, v \rangle \geq \delta \|v\|^2_{L^2(\Omega)} \text{ whenever } v^* \in \partial^2 \psi_e(u, u^*)(v) \text{ for } v \in X. \quad (3.25)$$

We now consider the case that

$$\psi(u, e) = \phi(u, e) + \delta(u; K),$$

where $\phi : X \times E \to \overline{I}$ is $C^2$ around the point $(\bar{u}, \bar{e}) \in \text{dom } \psi$ and $K$ is a closed and convex subset of $X$. This means that problem [(5.22)] can be rewritten as follows

Minimize $\phi(u, \bar{e})$ subject to $u \in K$.

In this case, we prove that the properties of full Lipschitzian stability and full Hölderian stability are equivalent.

Theorem 3.3 Let $(\bar{u}, \bar{e}) \in \text{dom } \psi$ and $\bar{u}^* \in \partial_u \psi(\bar{u}, \bar{e})$ be given. The following statements are equivalent:

(i) The point $\bar{u}$ is a Lipschitzian fully stable local minimizer of $\mathcal{P}_{\text{Abs}}(\bar{u}^*, \bar{e})$ in (3.23).

(ii) The point $\bar{u}$ is a Hölderian fully stable local minimizer of $\mathcal{P}_{\text{Abs}}(\bar{u}^*, \bar{e})$ in (3.23).

(iii) There are $\eta, \delta > 0$ such that for all $(u, e, u^*) \in \text{gph } \partial_u \psi \cap \bar{B}_\eta(\bar{u}, \bar{e}, \bar{u}^*)$ we have

$$\langle v^*, v \rangle \geq \delta \|v\|^2_{L^2(\Omega)} \text{ whenever } (v^*, e^*) \in \hat{D}^*(\partial_u \psi)(u, e, u^*)(v) \text{ for } v \in X.$$

(iv) There are $\eta, \delta > 0$ such that for all $(u, e, u^*) \in \text{gph } \partial_u \psi \cap \bar{B}_\eta(\bar{u}, \bar{e}, \bar{u}^*)$ we have

$$\langle v^*, v \rangle \geq \delta \|v\|^2_{L^2(\Omega)} \text{ whenever } v^* \in \partial^2 \psi_e(u, u^*)(v) \text{ for } v \in X.$$

Proof. According to the proof of [18, Theorem 6.3], the BCQ holds at $(\bar{u}, \bar{e}) \in \text{dom } \psi$ and $\psi$ is parametrically continuously prox-regular at $(\bar{u}, \bar{e})$ for $\bar{u}^* \in \partial_u \psi(\bar{u}, \bar{e})$. We now verify that the graphical mapping $F(e) := \text{gph } \partial_u \psi(\cdot, e)$ is locally Lipschitz-like around $(\bar{e}, \bar{u}, \bar{u}^*)$. Take any $e_1, e_2 \in \bar{B}_\delta(\bar{e})$ and $(u, u^*) \in F(e_1) \cap \bar{B}_\delta(\bar{e}, \bar{u}, \bar{u}^*)$ for $\delta$ small enough. Then, by setting $\hat{u}^* := u^* + \phi'_u(u, e_2) - \phi'_u(u, e_1)$, we have $(u, \hat{u}^*) \in F(e_2)$. Hence, we obtain

$$(u, u^*) \in (u, \hat{u}^*) + \ell \|e_1 - e_2\|_{E \bar{B}_{L^2(\Omega)} \times L^2(\Omega)},$$

where $\ell > 0$ is a Lipschitz constant of $\phi'_u(\cdot, \cdot)$ around $(\bar{u}, \bar{e})$. This shows that

$$F(e_1) \cap V \subset F(e_2) + \ell \|e_1 - e_2\|_{E \bar{B}_{L^2(\Omega)} \times L^2(\Omega)}, \quad \forall e_1, e_2 \in U,$$

where $U := \bar{B}_\delta(\bar{e})$ and $V := \bar{B}_\delta(\bar{e}, \bar{u}, \bar{u}^*)$. This means that $F$ is locally Lipschitz-like around $(\bar{e}, \bar{u}, \bar{u}^*) \in \text{gph } F$. 

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In addition, for each \((u, e, u^*)\) \(\in \text{gph} \partial_t \psi \cap \tilde{B}_0(\bar{u}, \bar{e}, \bar{u}^*)\) and for every \(v \in X\), it holds from [17] Theorem 1.62] that
\[
\tilde{D}^*(\partial_t \psi)(u, e, u^*)(v) = \tilde{D}^*(\phi''_u(u, e)v, \phi''_{ue}(u, e)v) = (\phi''_u(u, e)v, \phi''_{ue}(u, e)v) + \tilde{D}^* N(\cdot; K)(u, u^* - \phi'_u(u, e))v(v) \times \{0_E\}
\]
\[
= (\phi''_u(u, e)v + \tilde{D}^* N(\cdot; K)(u, u^* - \phi'_e(u, e))v(v) \times \{\phi''_{ue}(u, e)v\}
\]
\[
= \tilde{D}^*((\phi'_e)(\cdot) + N(\cdot; K))(u, u^*)(v) \times \{\phi''_{ue}(u, e)v\}
\]
\[
= \tilde{D}^2\psi_e(u, u^*)(v) \times \{\phi''_{ue}(u, e)v\}.
\]
Hence, we obtain
\[
(v^*, e^*) \in \tilde{D}^*(\partial_t \psi)(u, e, u^*)(v) \iff \begin{cases} v^* \in \tilde{D}^2\psi_e(u, u^*)(v) \\ e^* = \phi''_{ue}(u, e)v. \end{cases}
\]
This implies that (3.24) is equivalent to (3.25). Applying Theorems 3.1 and 3.2 we get the assertion of the theorem.

We are going to apply Theorem 3.3 to provide explicit characterizations of full stability properties for problem (2.8), where the admissible control set of the problem is fixed. In this case, problem (2.8) is rewritten as follows

Minimize \(J(u, e) = J(u + e_y) + (e_J, y_{ue + e_y})_{L^2(\Omega)}\) subject to \(u \in \mathcal{U}_{ad}\), (3.26)

and the corresponding perturbed problem of problem (3.26) is defined by

\(P_0(u^*, e) : \text{Minimize } J(u, e) - \langle u^*, u \rangle \text{ subject to } u \in \mathcal{U}_{ad}\), (3.27)

where the basic parameter \(e \in E\) always appears in the form \(e = (e_y, e_J, 0, 0)\).

Following [11], we say that a closed and convex subset \(K\) of a Banach space \(X\) is polyhedral at \(\bar{u} \in K\) for \(\bar{u}^* \in N(\bar{u}; K)\) if we have the representation

\(T_K(\bar{u}) \cap \{\bar{u}^*\} = \text{cl}(\text{cone}(K - \bar{u})) \cap \{\bar{u}^*\}/\), (3.28)

where \(\text{cone}(K - \bar{u}) = \bigcup_{t>0} t^{-1}(K - \bar{u})\) is the tangent cone to \(K\) at \(\bar{u}\). The set \(K\) is said to be polyhedral if \(K\) is polyhedral at every \(u \in K\) for any \(u^* \in N(u; K)\). The polyhedral property of a set is first introduced in [11] and then applied extensively in optimal control; see, e.g., [1], [4], [12] and the references therein.

**Theorem 3.4** (See [18] Theorem 6.2]) For any \(\bar{u} \in K\) and \(\bar{u}^* \in N(\bar{u}; K)\), we have

\(\text{dom } \tilde{D}^2\delta(\cdot; K)(\bar{u}, \bar{u}^*) \subset - (T_K(\bar{u}) \cap \{\bar{u}^*\})^/\). (3.29)

If, in addition, \(K\) is polyhedral at \(\bar{u} \in K\) for \(\bar{u}^*\), then the equality

\(\tilde{D}^2\delta(\cdot; K)(\bar{u}, \bar{u}^*)(u) = (T_K(\bar{u}) \cap \{\bar{u}^*\})^/\) (3.30)

holds for all \(u \in - (T_K(\bar{u}) \cap \{\bar{u}^*\})^/\).

**Remark 3.5** According to [2] Lemma 4.13] (see also [13] Lemma 2.4]), the set of admissible controls \(\mathcal{U}_{ad}\) is polyhedral at every \(u \in \mathcal{U}_{ad}\) for any \(u^* \in N(u; \mathcal{U}_{ad})\), and thus \(\mathcal{U}_{ad}\) is polyhedral. Note that this property also holds for the perturbed admissible control set \(\mathcal{U}_{ad}(e)\) in (2.7) for \(e \in E\). This result is a particular instance of the more general result [4] Theorem 3.58] which holds true in general Banach lattices.
From now on, for every pair \((u, u^*)\) with \(u^* \in N(u; U_{ad})\), we define the critical cone
\[
C_0(u, u^*) = T_{U_{ad}}(u) \cap \{u^*\}^\perp. \tag{3.31}
\]
Since \(U_{ad}\) is polyhedral at every \(u \in U_{ad}\) for any \(u^* \in N(u; U_{ad})\), from (3.30) and (3.31) we deduce that
\[
\partial^2 \delta(\cdot; U_{ad})(u, u^*)(v) = C_0(u, u^*)^*, \quad \forall v \in -C_0(u, u^*). \tag{3.32}
\]
Let us define the normal cone mapping \(N_0 : L^2(\Omega) \to \mathbb{R}^n\) by setting
\[
N_0(u) = N(u; U_{ad}), \quad \forall u \in L^2(\Omega). \tag{3.33}
\]
Applying Theorem 3.3 and [18, Theorem 6.3], we obtain the second-order characterization of Lipschitzian and Hölderian full stability for the problem \(P_0(\bar{u}, \bar{e})\) in the following theorem.

**Theorem 3.6** Assume that the assumptions (A1)-(A3) hold. Given \((\bar{u}, \bar{e}) \in U_{ad} \times E\) with \(\bar{e} = (\bar{e}_y, \bar{e}_f, 0, 0)\), let \(\bar{u}^* \in J_u'(\bar{u}, \bar{e}) + N_0(\bar{u})\) and define \(u^* = \bar{u}^* - J_u'(\bar{u}, \bar{e}) \in N_0(\bar{u})\). Then, the following are equivalent:

(i) The control \(\bar{u}\) is a Lipschitzian fully stable local minimizer for \(P_0(\bar{u}, \bar{e})\) in (3.27).

(ii) The control \(\bar{u}\) is a Hölderian fully stable local minimizer for \(P_0(\bar{u}, \bar{e})\) in (3.27).

(iii) There exist \(\eta > 0\) and \(\delta > 0\) such that for \((u, u^*) \in \text{gph} \ N_0 \cap B_\eta(\bar{u}, u^*)\) and \(e \in B_\eta(\bar{e})\), we have
\[
J''_{uu}(u, e)v^2 \geq \delta \|v\|^2_{L^2(\Omega)}, \quad \forall v \in C_0(u, u^*), \tag{3.34}
\]
where \(C_0(u, u^*)\) is defined by (3.31).

**Proof.** We see that \((\bar{u}, \bar{e}) \in \text{dom} \ \psi\) and \(\bar{u}^* \in \partial_u \psi(\bar{u}, \bar{e})\), where \(\psi(u, e) = J(u, e) + \delta(u; U_{ad})\) with \(J(\cdot, \cdot)\) being \(C^2\) around \((\bar{u}, \bar{e})\). Since the set \(U_{ad}\) is polyhedral by Remark 3.5 according to [18, Theorem 6.3] the control \(\bar{u}\) is a Lipschitzian fully stable local minimizer for \(P_0(\bar{u}, \bar{e})\) in (3.27) if and only if (3.34) holds. Therefore, applying Theorem 3.3 we obtain the assertion of the theorem.

This theorem shows that a certain condition on the positive definiteness of \(J''_{uu}\) near \((\bar{u}, \bar{e})\) is necessary and sufficient for stability. In the remainder of this section, we will derive an equivalent condition, which is posed only on \(J''_{uu}(\bar{u}, \bar{e})\).

We now analyze Theorem 3.6 to derive an explicit characterization for Lipschitzian and Hölderian full stability of the problem \(P_0(\bar{u}, \bar{e})\). We denote \(C^0_w(\bar{u}, \bar{u}^*)\) the sequential outer limit of critical cones \(C_0(u, u^*)\) in the weak* topology of \(L^2(\Omega)\), i.e.,
\[
C^0_w(\bar{u}, \bar{u}^*) = \limsup_{(u, u^*) \in \text{gph} \ N_0(\bar{u}, \bar{u}^*)} C_0(u, u^*) \tag{3.35}
\]
and \(C^0_s(\bar{u}, \bar{u}^*)\) the sequential outer limit of \(C_0(u, u^*)\) in the strong topology of \(L^2(\Omega)\), i.e.,
\[
C^0_s(\bar{u}, \bar{u}^*) = \left\{ v \in L^2(\Omega) \mid \exists (u_n, u^*_n) \xrightarrow{\text{gph} \ N_0} (\bar{u}, \bar{u}^*), v_n \in C_0(u_n, u^*_n), v_n \rightharpoonup v \right\}. \tag{3.36}
\]

**Lemma 3.7** Let \(\bar{u} \in U_{ad}\) and \(\bar{u}^* \in N_0(\bar{u})\). Then, both \(C^0_w(\bar{u}, \bar{u}^*)\) and \(C^0_s(\bar{u}, \bar{u}^*)\) are computed by the formula
\[
C^0_w(\bar{u}, \bar{u}^*) = C^0_s(\bar{u}, \bar{u}^*) = \left\{ v \in L^2(\Omega) \mid v(x)\bar{u}^*(x) = 0 \text{ for a.e. } x \in \Omega \right\}, \tag{3.37}
\]
where \(C^0_w(\bar{u}, \bar{u}^*)\) and \(C^0_s(\bar{u}, \bar{u}^*)\) are respectively given by (3.35) and (3.36).
Proof. The set of admissible controls $\mathcal{U}_{ad}$ is convex and polyhedric due to Remark \ref{r:3.5}. In addition, by \cite[Lemma 4.11]{2}, we have the representation of $T_{\mathcal{U}_{ad}}(\bar{u})$ as follows

$$T_{\mathcal{U}_{ad}}(\bar{u}) = \left\{ v \in L^2(\Omega) \left| \begin{array}{ll}
v(x) \geq 0 & \text{for } x \in \{ \bar{u} = \alpha \} \\
v(x) \leq 0 & \text{for } x \in \{ \bar{u} = \beta \}
\end{array} \right. \right\},$$

(3.38)

where

$$\{ \bar{u} = \alpha \} = \{ x \in \Omega | \bar{u}(x) = \alpha(x) \} \quad \text{and} \quad \{ \bar{u} = \beta \} = \{ x \in \Omega | \bar{u}(x) = \beta(x) \}.$$

Using the convexity and polyhedricity of $\mathcal{U}_{ad}$ and the representation of $T_{\mathcal{U}_{ad}}(\bar{u})$ in (3.38), by arguing similarly as in the proof of \cite[Proposition 7.3]{18} we obtain (3.37).

A quadratic form $Q : H \to \mathbb{R}$ on a Hilbert space $H$ is said to be a Legendre form if $Q$ is sequentially weakly lower semicontinuous and that if $h_n$ converges weakly to $h$ in $H$ and $Q(h_n) \to Q(h)$ then $h_n$ converges strongly to $h$ in $H$.

Lemma 3.8 For any $(\bar{u}, \bar{e}_y) \in \mathcal{U}_{ad} \times L^2(\Omega)$, define the quadratic form $Q : L^2(\Omega) \to \mathbb{R}$ by

$$Q(h) := \mathcal{J}''_{u\bar{u}}(\bar{u}, \bar{e}_y)h^2, \quad \forall h \in L^2(\Omega),$$

(3.39)

where $\mathcal{J}(:,:,\cdot)$ is given in (2.5). Then, $Q$ is a Legendre form on $L^2(\Omega)$.

Proof. We first show that the quadratic form $R(h) := J''(\bar{u} + \bar{e}_y)h^2$ defined on $L^2(\Omega)$, where $J(\cdot)$ is given in (2.1), is a Legendre form on $L^2(\Omega)$. Due to (2.16), we have

$$R(h) = \int_{\Omega} \left( \frac{\partial^2 L}{\partial \bar{y}^2}(x, \bar{u} + \bar{e}_y) \frac{\partial^2}{\partial \bar{y}^2} f(x, \bar{u} + \bar{e}_y) \right) dx = R_1(h) + R_2(h),$$

where

$$R_1(h) := \int_{\Omega} \left( \frac{\partial^2 L}{\partial \bar{y}^2}(x, \bar{u} + \bar{e}_y) \frac{\partial^2}{\partial \bar{y}^2} f(x, \bar{u} + \bar{e}_y) \right) \bar{z}_{\bar{u} + \bar{e}_y}^2 dx$$

and

$$R_2(h) := \int_{\Omega} \zeta h^2 dx.$$ 

Since $G'(\bar{u} + \bar{e}_y) : h \mapsto \bar{z}_{\bar{u} + \bar{e}_y}h$ from $L^2(\Omega)$ into $L^2(\Omega)$ is compact, $R_1(h)$ is a weakly continuous quadratic form on $L^2(\Omega)$. In addition, we observe that $R_1(th) = t^2 R_1(h)$ for all $h \in L^2(\Omega)$ and $t > 0$, i.e., $R_1$ is positively homogeneous of degree 2. On the other hand, $R_2$ is an elliptic quadratic form on $L^2(\Omega)$ because $R_2$ is continuous and

$$R_2(h) = \int_{\Omega} \zeta h^2 dx \geq \zeta_0 \| h \|_{L^2(\Omega)}, \quad \forall h \in L^2(\Omega),$$

where $\zeta(x) \geq \zeta_0 > 0$ for a.e. $x \in \Omega$. Due to \cite[Proposition 3.76]{4}, $R_2$ is a Legendre form on $L^2(\Omega)$, and thus $R = R_1 + R_2$ is also a Legendre form on $L^2(\Omega)$.

We now verify that $Q$ is a Legendre form on $L^2(\Omega)$. Indeed, we see that $Q$ is sequentially weakly lower semicontinuous. Moreover, we have

$$Q(h) = \mathcal{J}''_{uu}(\bar{u}, \bar{e})h^2 = J''(\bar{u} + \bar{e}_y)h^2 + \left( \bar{e}_J, G''(\bar{u} + \bar{e}_y)h^2 \right)_{L^2(\Omega)}$$

$$= R(h) + \left( \bar{e}_J, G''(\bar{u} + \bar{e}_y)h^2 \right)_{L^2(\Omega)},$$

where $\mathcal{J}(:,:,\cdot)$ is a Legendre form on $L^2(\Omega)$. Then, $Q(h) = R(h) + \left( \bar{e}_J, G''(\bar{u} + \bar{e}_y)h^2 \right)_{L^2(\Omega)}$.
where $R(h) = J''(\bar{u} + \bar{e}_y)h^2$ is a Legendre form on $L^2(\Omega)$. Suppose that $h_n$ converges weakly to $h$ in $L^2(\Omega)$ and $Q(h_n) \to Q(h)$. Then, $z_{a+\bar{e}_y,h_n}$ converges strongly to $z_{a+\bar{e}_y,h}$ in $L^2(\Omega)$, and thus $G''(\bar{u} + \bar{e}_y)h_n^2$ converges strongly to $G''(\bar{u} + \bar{e}_y)h^2$ in $L^2(\Omega)$. Consequently, we have
\[
(\bar{e}_J, G''(\bar{u} + \bar{e}_y)h_n^2)_{L^2(\Omega)} \to (\bar{e}_J, G''(\bar{u} + \bar{e}_y)h^2)_{L^2(\Omega)}.
\]
Combining this with $R(h_n) = Q(h_n) - (\bar{e}_J, G''(\bar{u} + \bar{e}_y)h_n^2)_{L^2(\Omega)}$ we deduce that
\[
R(h_n) \to R(h) - (\bar{e}_J, G''(\bar{u} + \bar{e}_y)h^2)_{L^2(\Omega)} = R(h).
\]
This implies that $h_n$ converges strongly to $h$ since $R$ is a Legendre form. Therefore, we have shown that $Q$ is a Legendre form.

\[\square\]

**Lemma 3.9** Consider $\mathcal{J}(\cdot, \cdot)$ in (2.5). The map $\mathcal{J}''_{uu} : L^2(\Omega) \times E \to L^2(\Omega) \times L^2(\Omega)$ defined by setting
\[
(u, e) \mapsto \mathcal{J}''_{uu}(u, e), \ \forall (u, e) \in L^2(\Omega) \times E,
\]
(3.40)
is continuous on $L^2(\Omega) \times E$.

**Proof.** Let any sequence $(u, e)$ converge to $(\bar{u}, \bar{e})$ in $L^2(\Omega) \times E$, where $e = (e_y, e_J, e_\alpha, e_\beta)$ and $\bar{e} = (\bar{e}_y, \bar{e}_J, \bar{e}_\alpha, \bar{e}_\beta)$. According to (2.5), we have
\[
\mathcal{J}(u, e) = J(u + e_y) + (e_J, y_{u+e_y})_{L^2(\Omega)} = J(u + e_y) + (e_J, G(u + e_y))_{L^2(\Omega)}.
\]
By Theorems 2.2 and 2.3, $J(\cdot)$ and $G(\cdot)$ are of class $C^2$, thus we have
\[
\mathcal{J}''_{uu}(u, e)(h_1, h_2) = J''(u + e_y)(h_1, h_2) + (e_J, G''(u + e_y)(h_1, h_2))_{L^2(\Omega)}.
\]
In addition, when $(u, e) \to (\bar{u}, \bar{e})$, we have $u + e_y \to \bar{u} + \bar{e}_y$ and $e_J \to \bar{e}_J$, and it follows that
\[
J''(u + e_y) \to J''(\bar{u} + \bar{e}_y) \quad \text{and} \quad G''(u + e_y) \to G''(\bar{u} + \bar{e}_y)
\]
(3.41)
strongly in $L^2(\Omega) \times L^2(\Omega)$ and $\mathcal{L}(L^2(\Omega) \times L^2(\Omega), L^2(\Omega))$, respectively. Moreover, we have
\[
\| (e_J, G''(u + e_y)(\cdot, \cdot))_{L^2(\Omega)} - (\bar{e}_J, G''(\bar{u} + \bar{e}_y)(\cdot, \cdot))_{L^2(\Omega)} \|_{L^2(\Omega) \times L^2(\Omega)}
\leq \| (e_J, G''(u + e_y)(\cdot, \cdot) - G''(u + e_y)(\cdot, \cdot))_{L^2(\Omega)} \|_{L^2(\Omega) \times L^2(\Omega)}
\]
\[
+ \| (e_J - \bar{e}_J, G''(\bar{u} + \bar{e}_y)(\cdot, \cdot))_{L^2(\Omega)} \|_{L^2(\Omega) \times L^2(\Omega)}
\]
\[
= \sup_{\|h_1\| = \|h_2\| = 1} \left| (e_J, [G''(u + e_y) - G''(u + e_y)](h_1, h_2))_{L^2(\Omega)} \right|
\]
\[
+ \sup_{\|h_1\| = \|h_2\| = 1} \left| (e_J - \bar{e}_J, G''(\bar{u} + \bar{e}_y)(h_1, h_2))_{L^2(\Omega)} \right|
\]
(3.42)
\[
\leq \| e_J \|_{L^2(\Omega)} \sup_{\|h_1\| = \|h_2\| = 1} \| [G''(u + e_y) - G''(u + e_y)](h_1, h_2) \|_{L^2(\Omega)}
\]
\[
+ \| e_J - \bar{e}_J \|_{L^2(\Omega)} \sup_{\|h_1\| = \|h_2\| = 1} \| G''(\bar{u} + \bar{e}_y)(h_1, h_2) \|_{L^2(\Omega)}
\]
\[
\leq \| e_J \|_{L^2(\Omega)} \| G''(u + e_y) - G''(\bar{u} + \bar{e}_y) \|_{\mathcal{L}(L^2(\Omega) \times L^2(\Omega), L^2(\Omega))}
\]
\[
+ \| e_J - \bar{e}_J \|_{L^2(\Omega)} \| G''(\bar{u} + \bar{e}_y) \|_{\mathcal{L}(L^2(\Omega) \times L^2(\Omega), L^2(\Omega))}
\]
\[
\to 0.
\]
From (3.41) and (3.42) we deduce that
\[
\| \mathcal{J}''_{uu}(u, e) - \mathcal{J}''_{uu}(\bar{u}, \bar{e}) \|_{L^2(\Omega) \times L^2(\Omega)} \to 0.
\]
This shows that the map $(u, e) \mapsto \mathcal{J}''_{uu}(u, e)$ is continuous on $L^2(\Omega) \times E$. \[\square\]
Theorem 3.10 Assume that the assumptions (A1)-(A3) hold. Given \((\bar{u}, \bar{e}) \in \mathcal{U}_{ad} \times E\) with \(\bar{e} = (\bar{e}_y, \bar{e}_J, 0, 0)\), let

\[
\bar{u}^* \in \zeta(\bar{u} + \bar{e}_y) + \varphi_{\bar{u} + \bar{e}_y} + G'(\bar{u} + \bar{e}_y)^* \bar{e}_J + \mathcal{N}_0(\bar{u})
\]

and define

\[
\bar{u}^* = \bar{u}^* - \zeta(\bar{u} + \bar{e}_y) - \varphi_{\bar{u} + \bar{e}_y} - G'(\bar{u} + \bar{e}_y)^* \bar{e}_J.
\]

Then, the following statements are equivalent:

(i) The control \(\bar{u}\) is a Lipschitzian fully stable local minimizer for \(\mathcal{P}_0(\bar{u}^*, \bar{e})\) in (3.27).

(ii) The control \(\bar{u}\) is a Hölderian fully stable local minimizer for \(\mathcal{P}_0(\bar{u}^*, \bar{e})\) in (3.27).

(iii) The following condition holds that

\[
\mathcal{J}_u''(\bar{u}, \bar{e})v^2 \geq 0, \quad \forall v \neq 0 \text{ with } v(x)\bar{u}^*(x) = 0 \text{ for a.e. } x \in \Omega,
\]

where \(\bar{e} = (\bar{e}_y, \bar{e}_J, 0, 0)\).

Proof. According to (2.15), we have

\[
\mathcal{J}_u'(\bar{u}, \bar{e})v = J'(\bar{u} + \bar{e}_y)v + (\bar{e}_J, G'(\bar{u} + \bar{e}_y)v)_{L^2(\Omega)} = \int_{\Omega} (\zeta(\bar{u} + \bar{e}_y) + \varphi_{\bar{u} + \bar{e}_y}) vdx + \int_{\Omega} G'(\bar{u} + \bar{e}_y)^* \bar{e}_J vdx
\]

\[
= \int_{\Omega} (\zeta(\bar{u} + \bar{e}_y) + \varphi_{\bar{u} + \bar{e}_y} + G'(\bar{u} + \bar{e}_y)^* \bar{e}_J) vdx,
\]

(3.44)

From (3.44) it follows that

\[
\mathcal{J}_u'(\bar{u}, \bar{e}) = \zeta(\bar{u} + \bar{e}_y) + \varphi_{\bar{u} + \bar{e}_y} + G'(\bar{u} + \bar{e}_y)^* \bar{e}_J \in L^2(\Omega).
\]

Thus, we obtain the inclusions \(\bar{u}^* \in \mathcal{J}_u'(\bar{u}, \bar{e}) + \mathcal{N}_0(\bar{u})\) and \(\hat{u}^* \in \mathcal{J}_u'(\bar{u}, \bar{e})\).

By Theorem 3.6, (i) is equivalent to (ii).

We now verify that (i) is equivalent to (iii). Assume that the control \(\bar{u}\) is a Lipschitzian fully stable local minimizer for the problem \(\mathcal{P}_0(\bar{u}^*, \bar{e})\). By Theorem 3.6 there exist \(\eta > 0\) and \(\delta > 0\) such that for each \((u, u^*) \in \text{gph}\mathcal{N}_0 \cap \bar{B}_\eta(\bar{u}, \bar{u}^*)\) and \(e \in \bar{B}_\eta(\bar{e})\), we have (3.34). Fix any \(v \in L^2(\Omega)\) with \(v \neq 0\) and \(v(x)\bar{u}^*(x) = 0\) for a.e. \(x \in \Omega\). It follows that \(v \in \mathcal{C}_\eta^0(\bar{u}, \bar{u}^*)\) due to (3.36). From (3.36) one can find sequences \((u_n, u_n^*) \rightarrow (\bar{u}, \bar{u}^*)\) with \((u_n, u_n^*) \in \text{gph}\mathcal{N}_0\), \(v_n \in \mathcal{C}_\eta(0, u_n^*)\) such that \(v_n \rightarrow v\) as \(n \rightarrow \infty\). Since \(v_n \in \mathcal{C}_\eta(0, u_n^*)\), from (3.34) we deduce that

\[
\mathcal{J}_u''(u_n, e_n)v_n^2 \geq \delta \|v_n\|^2_{L^2(\Omega)}, \quad \forall n \in \mathbb{N},
\]

(3.45)

where \(e_n = \bar{e}\) for every \(n \in \mathbb{N}\). By passing (3.45) to the limit as \(n \rightarrow \infty\), we get (3.43).

Conversely, suppose that (3.43) holds. In order to apply Theorem 3.6 to deduce that \(\bar{u}\) is a Lipschitzian fully stable local minimizer for the problem \(\mathcal{P}_0(\bar{u}^*, \bar{e})\), we have to prove that (3.34) holds. Suppose to the contrary that (3.34) does not hold. Then, one can find sequences \((u_n, u_n^*) \rightarrow (\bar{u}, \bar{u}^*)\) with \((u_n, u_n^*) \in \text{gph}\mathcal{N}_0\), \(e_n \rightarrow \bar{e}\), and \(v_n \in \mathcal{C}_\eta(0, u_n^*)\) such that

\[
\mathcal{J}_u''(u_n, e_n)v_n^2 < \frac{1}{n} \|v_n\|^2_{L^2(\Omega)}, \quad \forall n \in \mathbb{N},
\]

(3.46)

where \(e_n = (e_{yn}, e_{Jn}, 0, 0)\) for every \(n \in \mathbb{N}\). We may assume that \(\|v_n\|_{L^2(\Omega)} = 1\) for every \(n \in \mathbb{N}\). Without loss of generality we may assume that \(v_n \rightarrow v\) weakly (and also weakly...
star) in $L^2(\Omega)$. From (3.35) we deduce that $v \in C^0_\text{ad}(\bar{u}, \hat{u}^*)$, which implies $v(x)\hat{u}^*(x) = 0$ for a.e. $x \in \Omega$ due to Lemma 3.7. Now, on one hand, we have

$$
\mathcal{J}''_{uu}(u_n, e_n)v_n^2 = \int_\Omega \left( \frac{\partial^2 L}{\partial y^2}(x, y_{u_n+e_n}) - \varphi_{u_n+e_n} \frac{\partial^2 f}{\partial y^2}(x, y_{u_n+e_n}) \right) v_n^2 \, dx + \int_\Omega \zeta v_n^2 \, dx + \left( e_{J_n}, G''(u_n + e_n)v_n^2 \right)_{L^2(\Omega)}.
$$

(3.47)

By our assumptions posed on the functions $L$ and $f$, since $R_2(h) = \int_\Omega \zeta h^2 \, dx$ is sequentially weakly lower semicontinuous and $G(\cdot)$ is of class $C^2$, from (3.47) we deduce that

$$
\liminf_{n \to \infty} \mathcal{J}''_{uu}(u_n, e_n)v_n^2 \geq \int_\Omega \left( \frac{\partial^2 L}{\partial y^2}(x, y_{\bar{u}+\bar{e}_y}) - \varphi_{\bar{u}+\bar{e}_y} \frac{\partial^2 f}{\partial y^2}(x, y_{\bar{u}+\bar{e}_y}) \right) \bar{v}^2_{\bar{u}+\bar{e}_y} \, dx + \int_\Omega \zeta \bar{v}^2 \, dx + \left( e_{J_n}, G''(\bar{u} + \bar{e}_y)v^2 \right)_{L^2(\Omega)}
$$

$$
= \mathcal{J}''_{uu}(\bar{u}, \bar{e})v^2.
$$

From this and (3.43) we get $\mathcal{J}''_{uu}(\bar{u}, \bar{e})v^2 \leq 0$. This yields $v = 0$ due to (3.43). It follows that

$$
\lim_{n \to \infty} \mathcal{J}''_{uu}(u_n, e_n)v_n^2 = \mathcal{J}''_{uu}(\bar{u}, \bar{e})v^2 = 0.
$$

(3.48)

On the other hand, we have

$$
\mathcal{J}''_{uu}(u_n, e_n)v_n^2 - \mathcal{J}''_{uu}(\bar{u}, \bar{e})v^2 = \mathcal{J}''_{uu}(u_n, e_n)v_n^2 - \mathcal{J}''_{uu}(\bar{u}, \bar{e})v_n^2 + \mathcal{J}''_{uu}(\bar{u}, \bar{e})v^2 - \mathcal{J}''_{uu}(\bar{u}, \bar{e})v^2
$$

$$
= (\mathcal{J}''_{uu}(u_n, e_n) - \mathcal{J}''_{uu}(\bar{u}, \bar{e}))v_n^2 + Q(v_n) - Q(v).
$$

(3.49)

where $Q$ is the Legendre form defined by (3.39). From (3.48), (3.49) and by Lemma 3.9 we infer that

$$
Q(v_n) - Q(v) = \mathcal{J}''_{uu}(u_n, e_n)v_n^2 - \mathcal{J}''_{uu}(\bar{u}, \bar{e})v^2 - (\mathcal{J}''_{uu}(u_n, e_n) - \mathcal{J}''_{uu}(\bar{u}, \bar{e}))v_n^2 \to 0,
$$

when $n \to \infty$. Since $Q$ is a Legendre form on $L^2(\Omega)$, $v_n$ converges strongly to $v$ (with $v = 0$) in $L^2(\Omega)$. We arrive at a contradiction due to $\|v_n\|_{L^2(\Omega)} = 1$ for every $n \in \mathbb{N}$.

\[\square\]

\textbf{Remark 3.11} In [16], it is proved that a coercivity condition (called (AC) in [16]) equivalent to (3.43) is sufficient for the local Lipschitzian stability of solutions to parametric optimal control problems for nonlinear systems. In addition, this condition was proved to be necessary for local Lipschitzian stability of solutions with respect to only tilt parameter. The results in [16] are established on the basis of Robinson’s implicit function theorem [26] and its extension obtained in [8]. See also [15] for the local Lipschitzian stability of solutions to parametric optimal control problems for parabolic equations.

\section{Perturbed admissible control set and full stability}

In this section, we will establish explicit characterizations of the full stability properties for problem (2.8) with the corresponding perturbed problem (2.9), where the perturbed admissible control set $\mathcal{U}_{ad}(e)$ is given by (2.7) with respect to $e \in E = L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$.
Let us consider the function $\psi : L^2(\Omega) \times E \to \mathbb{R}$ defined by
\[
\psi(u,e) = J(u,e) + \delta(u;U_{ad}(e)),
\] and the normal cone mapping $N : L^2(\Omega) \times E \rightrightarrows L^2(\Omega)$ given by
\[
N(u,e) = N(u;U_{ad}(e)),
\] for every $(u,e) \in L^2(\Omega) \times E$. For each $e \in E$, we put $J_e(u) = J(u,e)$, $\delta_e(u) = \delta(u;U_{ad}(e))$, $N_e(u) = N(u,e)$, and $\psi_e(u) = J_e(u) + \delta_e(u)$ for all $u \in L^2(\Omega)$.

For every $(u,e,u^*)$ with $u^* \in N(u,e)$, we define the critical cone
\[
C(u,e,u^*) = T_{U_{ad}(e)}(u) \cap \{u^*\}^\perp.
\] For each $e \in E$, using (3.29) with $K = U_{ad}(e)$ we obtain
\[
\text{dom } \partial \delta_e(u,u^*) = \text{dom } \partial \delta_e(\cdot;U_{ad}(e))(u,u^*) \subset -(T_{U_{ad}(e)}(u) \cap \{u^*\}^\perp),
\] and thus combining this with (1.52) yields
\[
\text{dom } \partial \delta_e(u,u^*) \subset -C(u,e,u^*). \tag{4.53}
\] Moreover, since $U_{ad}(e)$ is polyhedral at every $u \in U_{ad}(e)$ for any $u^* \in N(u,e)$, from (4.50) and (4.52) we deduce that
\[
\partial \delta_e(u,u^*)(v) = \partial \delta(\cdot;U_{ad}(e))(u,u^*)(v) = (C(u,e,u^*))^*, \forall v \in -C(u,e,u^*). \tag{4.54}
\]

For every pair $(u,e) \in L^2(\Omega) \times E$ satisfying $u \in U_{ad}(e)$, we define the three subsets of $\Omega$ as follows
\[
\Omega_1(u,e) = \left\{ x \in \Omega \mid u(x) = \alpha(x) + e_\alpha(x) \right\},
\]
\[
\Omega_2(u,e) = \left\{ x \in \Omega \mid u(x) \in (\alpha(x) + e_\alpha(x), \beta(x) + e_\beta(x)) \right\},
\]
\[
\Omega_3(u,e) = \left\{ x \in \Omega \mid u(x) = \beta(x) + e_\beta(x) \right\}. \tag{4.55}
\] Then, $N(u,e) = N(u;U_{ad}(e))$ is computed by
\[
N(u,e) = \left\{ u^* \in L^2(\Omega) \mid \begin{array}{l}
u^*(x) \leq 0 \quad \text{if } x \in \Omega_1(u,e), \vspace{1mm} \\
u^*(x) = 0 \quad \text{if } x \in \Omega_2(u,e), \vspace{1mm} \\
u^*(x) \geq 0 \quad \text{if } x \in \Omega_3(u,e)
\end{array} \right\} \tag{4.56}
\]
for a.e. $x \in \Omega$.

Again, we obtain the equivalence of the conditions of the abstract stability Theorems 3.2 and 3.3.

**Theorem 4.1** Assume that the assumptions (A1)-(A3) hold. Let $(\bar{u},\bar{e}) \in \text{dom } \psi$ be such that there exists $\sigma > 0$ satisfying
\[
\alpha(x) + \bar{e}_\alpha(x) + \sigma \leq \beta(x) + \bar{e}_\beta(x) \text{ for a.e. } x \in \Omega, \tag{4.57}
\] where $\bar{e} = (\bar{e}_y, \bar{e}_f, \bar{e}_\alpha, \bar{e}_\beta)$. Let any $\bar{u}^* \in \partial \psi(\bar{u},\bar{e})$ be given. Then, the following statements are equivalent:

(i) The control $\bar{u}$ is a Lipschitzian fully stable local minimizer of $P(\bar{u}^*,\bar{e})$ in (2.9).

(ii) The control $\bar{u}$ is a Hölderian fully stable local minimizer of $P(\bar{u}^*,\bar{e})$ in (2.9).
(iii) There are \( \eta, \delta > 0 \) such that for all \((u, e, u^*) \in \text{gph} \partial \psi \cap B_\delta(\bar{u}, \bar{\varepsilon}, \bar{u}^*)\) we have
\[
\langle v^*, v \rangle \geq \delta \|v\|_{L^2(\Omega)}^2 \quad \text{whenever} \quad v^* \in \partial^2 \psi(u, u^*)(v) \text{ for } v \in L^2(\Omega).
\] (4.58)

**Proof.** For the function \( \psi(u, e) \) in (4.50), we have \( \partial_e \psi(u, e) = J'_\psi(u, e) + N'(u, e) \) whenever \((u, e) \in U_{\text{ad}}(e) \times E \). Due to (4.57), by choosing \( \delta \in (0, \sigma/2) \), we have \( U_{\text{ad}}(e) \neq \emptyset \) for every \( e \in B_\delta(\bar{e}) \). We first verify that the BCQ holds at \((\bar{u}, \bar{\varepsilon}) \in \text{dom} \psi\), i.e., the epigraphical mapping \( \mathcal{E}(e) := \text{epi} \psi(\cdot, e) \) is locally Lipschitz-like around the point \((\bar{e}, \bar{u}, \psi(\bar{e}, \bar{u})) \in \text{gph} \mathcal{E}\).

Select any \( e_1 = (e^1_y, e^1_f, e^1_\alpha, e^1_\beta) \in B_\delta(\bar{e}) \) and \( e_2 = (e^2_y, e^2_f, e^2_\alpha, e^2_\beta) \in B_\delta(\bar{e}) \) with \( e_1 \neq e_2 \) (because there is nothing to do if \( e_1 = e_2 \)). For any \((u_1, \tau_1) \in \mathcal{E}(e_1) \cap B_\delta(\bar{u}, \psi(\bar{u}, \bar{\varepsilon}))\), we construct \( u_2 \in U_{\text{ad}}(e_2) \) as follows. Since \( u_1 \in U_{\text{ad}}(e_1) \) with
\[
U_{\text{ad}}(e_1) = \{ u \in L^2(\Omega) | \alpha(x) + e^1_\alpha(x) \leq u(x) \leq \beta(x) + e^1_\beta(x) \text{ for a.e. } x \in \Omega \},
\]
we have \( u_1 = \lambda(\alpha + e^1_\alpha) + (1 - \lambda)(\beta + e^1_\beta) \) with some \( \lambda(x) \in [0, 1] \) for a.e. \( x \in \Omega \). By setting
\[
u_2 := \lambda(\alpha + e^2_\alpha) + (1 - \lambda)(\beta + e^2_\beta),
\](4.59)
we obtain \( u_2 \in U_{\text{ad}}(e_2) \) and
\[
|u_1 - u_2| \leq \lambda|e^1_\alpha - e^2_\alpha| + (1 - \lambda)|e^1_\beta - e^2_\beta| \leq |e^1_\alpha - e^2_\alpha| + |e^1_\beta - e^2_\beta|
\]
for a.e. \( x \in \Omega \). This yields
\[
\|u_1 - u_2\|_{L^\infty(\Omega)} \leq \|e^1_\alpha - e^2_\alpha\|_{L^\infty(\Omega)} + \|e^1_\beta - e^2_\beta\|_{L^\infty(\Omega)} \leq \|e_1 - e_2\|_E.
\]
It follows that
\[
\|u_1 - u_2\|_{L^2(\Omega)} \leq |\Omega|^{1/2} \|e_1 - e_2\|_E.
\]

By choosing \( \tau_2 = \tau_1 + \ell_\psi\|(u_1, e_1) - (u_2, e_2)\|_{L^2(\Omega) \times E} \geq \psi(u_2, e_2) \), where \( \ell_\psi > 0 \) is a Lipschitz constant of \( \psi \) around \((\bar{u}, \bar{\varepsilon})\), we have \( (u_2, \tau_2) \in \mathcal{E}(e_2) \) and
\[
\|(u_1, \tau_1) - (u_2, \tau_2)\|_{L^2(\Omega) \times R} = \|u_1 - u_2\|_{L^2(\Omega)} + |\tau_1 - \tau_2| = \|u_1 - u_2\|_{L^2(\Omega)} + \ell_\psi\|(u_1, e_1) - (u_2, e_2)\|_{L^2(\Omega) \times E} = (\ell_\psi + 1)\|u_1 - u_2\|_{L^2(\Omega)} + \ell_\psi\|e_1 - e_2\|_E \leq ((\ell_\psi + 1)|\Omega|^{1/2} + \ell_\psi)\|e_1 - e_2\|_E.
\]
Therefore, setting \( \ell_\mathcal{E} = (\ell_\psi + 1)|\Omega|^{1/2} + \ell_\psi > 0 \), we obtain
\[
(u_1, \tau_1) \in (u_2, \tau_2) + \ell_\mathcal{E}\|e_1 - e_2\|_E B_{\ell_\mathcal{E}}(\Omega) \times R.
\]
We have shown that
\[
\mathcal{E}(e_1) \cap V \subset \mathcal{E}(e_2) + \ell_\mathcal{E}\|e_1 - e_2\|_E B_{\ell_\mathcal{E}}(\Omega) \times R, \quad \forall e_1, e_2 \in U,
\]
where \( U := B_\delta(\bar{e}) \) and \( V := B_\delta(\bar{u}, \psi(\bar{u}, \bar{\varepsilon})) \). Thus, \( \mathcal{E} \) is locally Lipschitz-like around the point \((\bar{e}, \bar{u}, \psi(\bar{e}, \bar{\varepsilon})) \in \text{gph} \mathcal{E}\).

We now check that \( \psi \) is parametrically continuously prox-regular at the point \((\bar{u}, \bar{\varepsilon}) \) for \( \bar{u}^* \in \partial_\psi \psi(\bar{u}, \bar{\varepsilon}) \). Let \( r > 0 \) satisfy \( \|J'_\psi(u, e)\|_{L^2(\Omega) \times L^2(\Omega)} \leq r \) for every \((u, e) \in U_{\text{ad}}(e) \times B_\delta(\bar{e}) \). Fix an arbitrary \( \varepsilon > 0 \) and take any \( u^* \in \partial_\psi \psi(v, e) \cap B_\varepsilon(\bar{u}) \) with \((v, e) \in B_\varepsilon(\bar{u}) \times B_\delta(\bar{e}) \) and \( \psi(v, e) \leq \psi(\bar{u}, \bar{\varepsilon}) + \varepsilon \). From this we have \( v \in U_{\text{ad}}(e) \). For every \( u \in B_\varepsilon(\bar{u}) \), if \( u \notin U_{\text{ad}}(e) \), then we get \( \psi(u, e) = +\infty \), and thus
\[
\psi(u, e) \geq \psi(v, e) + \langle u^*, u - v \rangle - \frac{r}{2}\|u - v\|_{L^2(\Omega)}^2.
\] (4.60)
Consider the case that $u \in \mathcal{U}_{ad}(e)$. Note that since $u^* \in \partial_u \psi(v,e)$, we have $u^* = \mathcal{J}_u'(v,e) + u_v^*$ for some $u_v^* \in \mathcal{N}(v,e)$. Using a Taylor expansion, we have
\[
\mathcal{J}_u'(v,e)(u-v) = \mathcal{J}(u,e) - \mathcal{J}(v,e) - \frac{1}{2} \mathcal{J}''(\bar{u},e)(u-v)^2
\]
\[
\leq \mathcal{J}(u,e) - \mathcal{J}(v,e) + \frac{r}{2} \|u - v\|_{L^2(\Omega)}^2, \tag{4.61}
\]
where $\bar{u} = v + \theta(u-v)$ with $\theta \in (0,1)$. Since $u^* = \mathcal{J}_u'(v,e) + u_v^*$ with $u_v^* \in \mathcal{N}(v,e)$, from (4.61) we get
\[
\langle u^*, u-v \rangle \leq \mathcal{J}(u,e) - \mathcal{J}(v,e) + \frac{r}{2} \|u - v\|_{L^2(\Omega)}^2.
\]
This yields that (4.60) holds. In addition, the mapping $(u,e,u^*) \mapsto \psi(u,e)$ is also continuous relative to $\text{gph } \partial_u \psi$ at $(\bar{u}, \bar{e}, \bar{u}^*)$. We have shown that $\psi$ is parametrically continuously prox-regular at $(\bar{u}, \bar{e})$ for $u^* \in \partial_u \psi(\bar{u}, \bar{e})$.

Finally, we show that the graphical mapping $F(e) := \text{gph } \partial_u \psi(\cdot, e)$ is locally Lipschitz-like around $(\bar{e}, \bar{u}, \bar{u}^*) \in \text{gph } F$. For every $e_1 = (e_1^1, e_1^2, e_1^3), e_2 = (e_2^1, e_2^2, e_2^3) \in B_3(e)$, pick any $(u_1, u_1^*) \in F(e_1) \cap B_3(\bar{u}, \bar{u}^*)$. Then, we have $u_1^* = \mathcal{J}_u'(u_1, e_1) + \hat{u}^*$ for some $\hat{u}^* \in \mathcal{N}(u_1, e_1)$ due to $\partial_u \psi(u,e) = \mathcal{J}_u'(u,e) + \mathcal{N}(u,e)$. By constructing $u_2$ the same as in (4.59), we obtain $u_2 \in \mathcal{U}_{ad}(e_2)$ and
\[
\| u_1 - u_2 \|_{L^2(\Omega)} \leq \| \Omega \|^{1/2} \| e_1 - e_2 \|_E. \tag{4.62}
\]

According to (4.59), from the definition of $u_2$ we get $\Omega(u_1, e_1) = \Omega(u_2, e_2)$ for all $i = 1, 2, 3$. By (4.56), it follows that $\mathcal{N}(u_1, e_1) = \mathcal{N}(u_2, e_2)$. Thus, by setting
\[
u_2^* := \mathcal{J}_u'(u_2, e_2) + \hat{u}^* \in \mathcal{J}_u'(u_2, e_2) + \mathcal{N}(u_2, e_2) = \partial_u \psi(u_2, e_2),
\]
we have $(u_2, u_2^*) \in F(e_2)$ and by (4.62) we deduce that
\[
\| (u_1, u_1^*) - (u_2, u_2^*) \|_{L^2(\Omega) \times L^2(\Omega)} = \| u_1 - u_2 \|_{L^2(\Omega)} + \| u_1^* - u_2^* \|_{L^2(\Omega)}
\]
\[
= \| u_1 - u_2 \|_{L^2(\Omega)} + \| \mathcal{J}_u'(u_1, e_1) - \mathcal{J}_u'(u_2, e_2) \|_{L^2(\Omega)}
\]
\[
\leq \| u_1 - u_2 \|_{L^2(\Omega)} + \mathcal{J}_u''(u_1, e_1, e_2) + \mathcal{J}_u''(u_2, e_2, e_2) + \mathcal{J}_u''(u_2, e_2, e_2)
\]
\[
\leq (\mathcal{J}_u'' + 1) \| u_1 - u_2 \|_{L^2(\Omega)} + \mathcal{J}_u'' \| e_1 - e_2 \|_E
\]
\[
\leq (\mathcal{J}_u'' + 1) \| \Omega \|^{1/2} \| e_1 - e_2 \|_E + \mathcal{J}_u'' \| e_1 - e_2 \|_E
\]
\[
= (\mathcal{J}_u'' + 1) \| \Omega \|^{1/2} + \mathcal{J}_u'' \| e_1 - e_2 \|_E,
\]
where $\mathcal{J}_u'' > 0$ is a Lipschitz constant of $\mathcal{J}_u''(\cdot, \cdot)$ around the point $(\bar{u}, \bar{e})$. Therefore, by setting $\ell_F := (\mathcal{J}_u'' + 1) \| \Omega \|^{1/2} + \mathcal{J}_u'' > 0$, we obtain
\[
(u_1, u_1^*) \in (u_2, u_2^*) + \ell_F \| e_1 - e_2 \|_E \bar{B} L^2(\Omega) \times L^2(\Omega).
\]
This implies that
\[
F(e_1) \cap V_0 \subset F(e_2) + \ell_F \| e_1 - e_2 \|_E \bar{B} L^2(\Omega) \times L^2(\Omega), \quad \forall e_1, e_2 \in U_0,
\]
where $U_0 := \bar{B} \beta(\bar{e})$ and $V_0 := \bar{B} \beta(\bar{u}, \bar{u}^*)$. This means that $F$ is locally Lipschitz-like around the point $(\bar{e}, \bar{u}, \bar{u}^*) \in \text{gph } F$.

By Theorem 8.2 and Corollary 4.8, the assertion of the theorem is fulfill. \hfill \Box

**Remark 4.2** Note that when the parametric space $E = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ is replaced with $E = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ in Theorem 4.1, the full stability properties cannot hold. Indeed, the set $\mathcal{U}_{ad}(e)$ has no interior points in $L^2(\Omega)$, hence there are perturbations $e$ arbitrarily close to $\bar{e}$ in $E$ for which such $\mathcal{U}_{ad}(e)$ can be empty. This is the
main reason why the full stability properties fail to hold for this setting. Therefore, we can investigate the full stability for problem \((2.8)\) in the corresponding perturbed problem \((2.9)\) with respect to the parametric metric space \(\tilde{E}_0 = \{e \in \tilde{E} | Uad(e) \neq \emptyset\}\) instead of \(\tilde{E}\). Note further that in our setting of Theorem 4.1 where the parametric space \(E\) is considered, condition \((4.57)\) ensures that \(Uad(e) \neq \emptyset\) for \(e\) near \(\tilde{e}\) enough.

**Theorem 4.3** Assume that the assumptions (A1)-(A3) hold. Let \((\bar{u}, \bar{e}) \in Uad(\bar{e}) \times E\) satisfy condition \((4.57)\) for some \(\sigma > 0\), where \(\bar{e} = (e_y, \bar{e}_J, \bar{e}_\alpha, \bar{e}_\beta)\). Let \(\tilde{u} \in J'_\eta(\bar{u}, \bar{e}) + \mathcal{N}(\bar{u}, \bar{e})\) and define \(\tilde{u}^* = \tilde{u} - J'_\eta(\bar{u}, \bar{e}) \in \mathcal{N}(\bar{u}, \bar{e})\). Then, the following statements are equivalent:

(i) The control \(\bar{u}\) is a Lipschitzian fully stable local minimizer of \(P(\bar{u}^*, \bar{e})\) in \((2.9)\).

(ii) The control \(\bar{u}\) is a Hölderian fully stable local minimizer of \(P(\bar{u}^*, \bar{e})\) in \((2.9)\).

(iii) There exist \(\eta > 0\), \(\delta > 0\) such that for each \((u, e, u^*) \in gph \mathcal{N} \cap \tilde{B}_\eta(\bar{u}, \bar{e}, \tilde{u}^*)\), we have
\[
J''_{\eta u}(u, e)v^2 \geq \delta \|v\|^2_{L^2(\Omega)}, \quad \forall v \in C(u, e, u^*),
\]
where \(C(u, e, u^*)\) is given by \((4.52)\).

**Proof.** We observe that our assumptions satisfy all the assumptions of Theorem 4.1 hence by Theorem 4.1 we deduce that (i) is equivalent to (ii).

We now verify that (ii) is equivalent to (iii). Suppose that the control \(\bar{u}\) is a Hölderian fully stable local minimizer of the problem \(P(\bar{u}^*, \bar{e})\) in \((2.9)\). According to Theorem 4.1 condition \((4.58)\) holds for some \(\eta_1, \delta_1 > 0\) and for all \((u, e, u^*) \in gph \partial_e \psi \cap \tilde{B}_{\eta_1}(\bar{u}, \bar{e}, \tilde{u}^*)\).

Let us select any \(v \in C(u, e, u^*)\) for \((u, e, u^*) \in gph \mathcal{N} \cap \tilde{B}_{\eta_1}(\bar{u}, \bar{e}, \tilde{u}^*)\) with some constant \(\eta > 0\) satisfying \(\eta_1/3 \geq \eta(\ell + 1) > 0\), where \(\ell > 0\) is a Lipschitz constant of \(J'_\eta(\cdot, \cdot)\) around the point \((\bar{u}, \bar{e})\). Since \(\partial_e \psi(u, e) = J'_\eta(u, e) + \mathcal{N}(u, e) = (J_e)'(u) + \mathcal{N}_e(u) = \partial \psi_e(u)\), we have \((J_e)'(u) + u^* = J'_\eta(u, e) + u^* \in \partial \psi_e(u, e)\). In addition, we have
\[
\left\|\left(u, e, (J_e)'(u) + u^*\right) - (\bar{u}, \bar{e}, \tilde{u}^*)\right\|_{L^2(\Omega) \times L^2(\Omega)}
\leq \eta_1/3 + \eta_1/3 + \|((J_e)'(u) + u^*) - \tilde{u}^*\|_{L^2(\Omega)}
\leq 2\eta_1/3 + \|((J_e)'(u) - (J_e)'(\bar{u}))\|_{L^2(\Omega)} + \|u^* - \tilde{u}^*\|_{L^2(\Omega)}
\leq 2\eta_1/3 + \ell \eta + \eta \leq \eta_1.
\]
This implies that \((u, e, (J_e)'(u) + u^*) \in gph \partial_e \psi \cap \tilde{B}_{\eta_1}(\bar{u}, \bar{e}, \tilde{u}^*)\). It follows from \((4.54)\) with noting \(u^* \in C(u, e, u^*)\) that
\[
-(J_e)''(u)v + u^* \in (J_e)''(u)(-v) + \partial \delta_e(u, u^*)(-v)
= ((J_e)''(u) + \partial \delta_e(u, u^*))(v)
\]
\[
= ((J_e)''(u) + \partial \delta_e(u, u^*))(-v).
\]
According to \([17]\) Theorem 1.62, we have
\[
\partial \delta_e(u, (J_e)'(u) + u^*)(-v) = \partial \delta_e(u, (J_e)'(u) + u^*)(-v)
= \partial \delta_e((J_e)' + \mathcal{N}_e)(u, (J_e)'(u) + u^*)(-v)
= ((J_e)''(u) + \partial \delta_e(u, u^*))(-v).
\]
From \((4.64)\) and \((4.65)\) we get
\[
u^* - (J_e)''(u)v \in \partial \delta_e(u, (J_e)'(u) + u^*)(-v).
\]
Since $v \in C(u, e, u^*)$, using condition (4.58) it follows from (4.66) that
\[(J_e)''(u)v^2 = \langle u^* - (J_e)''(u)v, -v \rangle \geq \delta_1 \|v\|^2_{L^2(\Omega)},\]
or, equivalently, as follows
\[J_{uu}(u, e)v^2 \geq \delta_1 \|v\|^2_{L^2(\Omega)}.
By choosing $\delta = \delta_1$, we obtain (4.63) in (iii).

Conversely, suppose that (iii) holds. To prove (ii) it suffices to verify that Theorem 4.1(iii) holds for the positive
Lemma 4.4
where $\overline{\psi} \in C(u, e, u^*)$. Let any $v \in L^2(\Omega)$, pick any $v^* \in \partial^2 \psi_e(u, u^*)(v)$ with
\[\langle \psi, v - u \rangle = \langle \psi, v \rangle + \langle u^* - (J_e)'(u), -v \rangle \geq 0. \]
We have $(u, v, e, u^*) \in \partial \psi_e \cap B_m(\nu, \bar{e}, \bar{u}^*)$ with $\eta/3 \geq \eta_1(1 + \ell) > 0$. According to [17] Theorem 1.62, we have
\[\partial^2 \psi_e(u, u^*)(v) = \tilde{D}^* (\partial \psi_e)(u, u^*) (v)\]
where $\partial \psi_e = (J_e)'(u) + \tilde{N}_e (u, u^*) - (J_e)'(u)(v)$. From (4.53) we deduce that
\[v \in \text{dom} \partial^2 \psi_e (u, u^* - (J_e)'(u)) \subset -C(u, e, u^* - (J_e)'(u)). \quad (4.67)\]
Since $v^* - (J_e)'(u)v \in \partial^2 \psi_e(u, u^* - (J_e)'(u))(v) = \tilde{D}^* \tilde{N}_e (u, u^* - (J_e)'(u))(v)$ and $\tilde{N}_e$ is a maximal monotone operator, applying [19] Lemma A.2 we get $\langle v^* - (J_e)'(u)v, v \rangle \geq 0$. We see that
\[\|v\|^2_{L^2(\Omega)} = \|v - \tilde{u}\|^2_{L^2(\Omega)} + \|e - \tilde{e}\|^2_{E} + \|u^* - (J_e)'(u) - \tilde{u}^*\|^2_{L^2(\Omega)} \leq \eta/3 + \eta/3 + \|u^* - \tilde{u}^*\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \leq 2\eta/3 + \eta_1 + \ell \eta_1 \leq \eta.
Combining the latter with (4.67) and using condition (4.63) in (iii) of the theorem we deduce that
\[(J_e)''(u)v^2 = J_{uu}(u, e)((-v, -v) \geq \delta \|v\|^2_{L^2(\Omega)} = \delta \|v\|^2_{L^2(\Omega)}.
From this and the fact that $\langle v^* - (J_e)'(u)v, v \rangle \geq 0$ we obtain the estimate
\[\langle v^*, v \rangle = \langle v^* - (J_e)'(u)v, v \rangle + \langle J_e''(u)v^2 \geq \delta_1 \|v\|^2_{L^2(\Omega)},\]
where $\delta_1 := \delta$. We have shown that Theorem 4.1(iii) holds for the positive $\eta_1, \delta_1$. \hfill \Box

In the sequel, we will equivalently characterize condition (4.63) by an assumption on the reference point $(\bar{u}, \bar{e})$ only.

We now define the sequential outer limits of the critical cones $C(u, e, u^*)$ respectively via (2.17) in the weak* topology of $L^2(\Omega)$ by
\[C_{w^*}(\bar{u}, \bar{e}, \bar{u}^*) = \limsup_{(u, e, u^*) \overset{\text{gph} \tilde{N}}{\longrightarrow} (\bar{u}, \bar{e}, \bar{u}^*)} C(u, e, u^*), \quad (4.68)\]
and in the strong topology of $L^2(\Omega)$ as follows
\[C_s(\bar{u}, \bar{e}, \bar{u}^*) = \left\{ v \in L^2(\Omega) \left| \exists (u_n, e_n, u^*_n) \overset{\text{gph} \tilde{N}}{\longrightarrow} (\bar{u}, \bar{e}, \bar{u}^*), v_n \in C(u_n, e_n, u^*_n), v_n \to v \right\}. \quad (4.69)\]

**Lemma 4.4** Let any $(\bar{u}, \bar{e}) \in \mathcal{U}_{ad}(\bar{e}) \times E$ satisfy (4.57) for some $\sigma > 0$ and let $\bar{u}^* \in \tilde{N}(\bar{u}, \bar{e})$, where $\bar{e} = (\bar{e}_y, \bar{e}_f, \bar{e}_a, \bar{e}_\beta)$. Then, $C_{w^*}(\bar{u}, \bar{e}, \bar{u}^*)$ and $C_s(\bar{u}, \bar{e}, \bar{u}^*)$ are computed by the formula
\[C_{w^*}(\bar{u}, \bar{e}, \bar{u}^*) = C_s(\bar{u}, \bar{e}, \bar{u}^*) = \left\{ v \in L^2(\Omega) \left| v(x)\bar{u}^*(x) = 0 \text{ for a.e. } x \in \Omega \right\}, \quad (4.70)\]
where $C_{w^*}(\bar{u}, \bar{e}, \bar{u}^*)$ and $C_s(\bar{u}, \bar{e}, \bar{u}^*)$ are respectively given by (4.68) and (4.69).
Proof. Under our assumptions, for each \((u, e) \in \mathcal{U}_{ad}(e) \times E\) near \((\tilde{u}, \tilde{e})\) enough, the perturbed admissible control set \(\mathcal{U}_{ad}(e)\) is convex and polyhedric by Remark 3.5. Moreover, due to [2, Lemma 4.11], \(T_{\mathcal{U}_{ad}(e)}(u)\) is computed by

\[
T_{\mathcal{U}_{ad}(e)}(u) = \left\{ v \in L^2(\Omega) \mid \begin{array}{ll} v(x) \geq 0 & \text{for } x \in \Omega_1(u, e) \\ v(x) \leq 0 & \text{for } x \in \Omega_3(u, e) \end{array} \right\},
\]

(4.71)

where \(\Omega_1(u, e)\) and \(\Omega_3(u, e)\) are defined by (4.55). In addition, for any \(u^* \in \mathcal{N}(u, e)\), we have

\[
C(u, e, u^*) = T_{\mathcal{U}_{ad}(e)}(u) \cap \{u^*\}^\perp \quad = \{v \in T_{\mathcal{U}_{ad}(e)}(u) \mid v(x)u^*(x) = 0\text{ for a.e. } x \in \Omega\}. \quad (4.72)
\]

In order to prove (4.70), we first verify the following inclusion

\[
C_u^*(\tilde{u}, \tilde{e}, \hat{u}^*) \subset \{v \in L^2(\Omega) \mid v(x)\hat{u}^*(x) = 0\text{ for a.e. } x \in \Omega\}. \quad (4.73)
\]

Pick any \(v \in C_u^*(\tilde{u}, \tilde{e}, \hat{u}^*)\). Due to (4.68), we can find sequences \((u_n, e_n, u_n^*) \rightarrow (\tilde{u}, \tilde{e}, \hat{u}^*)\) with \((u_n, e_n, u_n^*) \in \text{gph}\mathcal{N}\) and \(v_n \rightharpoonup v\) with \(v_n \in C(u_n, e_n, u_n^*)\) for every \(n \in \mathbb{N}\). This implies by (4.72) that \(v_n \in T_{\mathcal{U}_{ad}(e_n)}(u_n)\) and \(v_n(x)u_n^*(x) = 0\) for a.e. \(x \in \Omega\) for every \(n \in \mathbb{N}\). This yields for any measurable set \(\Theta \subset \Omega\) that

\[
\int_\Theta v(x)\hat{u}^*(x)dx = 0,
\]

that is \(v(x)\hat{u}^*(x) = 0\) for a.e. \(x \in \Omega\). Hence, (4.73) has been verified.

Since \(C_s(\tilde{u}, \tilde{e}, \hat{u}^*) \subset C_u^*(\tilde{u}, \tilde{e}, \hat{u}^*)\), to obtain (4.70) it suffices to prove that

\[
\{v \in L^2(\Omega) \mid v(x)\hat{u}^*(x) = 0\text{ for a.e. } x \in \Omega\} \subset C_s(\tilde{u}, \tilde{e}, \hat{u}^*). \quad (4.74)
\]

Select any \(v\) from the set on the left-hand side of (4.74). We define functions \(v_1\) and \(v_2\) by

\[
v_1(x) = \begin{cases} \max\{0, -v(x)\}, & \text{for } x \in \Omega_1(\tilde{u}, \tilde{e}) \\ 0, & \text{for } x \in \Omega_2(\tilde{u}, \tilde{e}) \\ \min\{0, -v(x)\}, & \text{for } x \in \Omega_3(\tilde{u}, \tilde{e}), \end{cases}
\]

and \(v_2 = v + v_1\). According to (4.71), we have \(v_1, v_2 \in T_{\mathcal{U}_{ad}(e)}(\tilde{u})\). In addition, we also have \(v_1, v_2 \in \{\hat{u}^*\}^\perp\) due to \(v(x)\hat{u}^*(x) = 0\) for a.e. \(x \in \Omega\). Thus, we get \(v_1, v_2 \in C(\tilde{u}, \tilde{e}, \hat{u}^*)\). Since \(\mathcal{U}_{ad}(e)\) is polyhedric, by (3.28) we find sequences \(v_{1,n} \rightarrow v_1, v_{2,n} \rightarrow v_2, t_{1,n} \downarrow 0, t_{2,n} \downarrow 0\) such that \(\tilde{u} + t_{1,n}v_{1,n} \in \mathcal{U}_{ad}(\tilde{e}), \tilde{u} + t_{2,n}v_{2,n} \in \mathcal{U}_{ad}(\hat{e})\), and \(v_{1,n}, v_{2,n} \in \{\hat{u}^*\}^\perp\). Since \(\mathcal{U}_{ad}(\hat{e})\) is convex, by setting \(t_n = \min\{t_{1,n}, t_{2,n}\}\) we have

\[
\begin{cases} u_n := \tilde{u} + t_nv_{1,n} \in \mathcal{U}_{ad}(\hat{e}) \\ \bar{u}_n := \tilde{u} + t_nv_{2,n} \in \mathcal{U}_{ad}(\hat{e}). \end{cases}
\]

Now, by choosing \(e_n = \tilde{e}, u_n^* = \hat{u}^*,\) and \(v_n = v_{2,n} - v_{1,n}\), we have \((u_n, e_n, u_n^*) \rightarrow (\tilde{u}, \tilde{e}, \hat{u}^*)\) with \((u_n, e_n, u_n^*) \in \text{gph}\mathcal{N}\) and \(v_n \rightharpoonup v\) with

\[
v_n = v_{2,n} - v_{1,n} = \frac{\bar{u}_n - u_n}{t_n} \in \text{cone}(\mathcal{U}_{ad}(e_n) - u_n) \cap \{u_n^*\}^\perp \subset C(u_n, e_n, u_n^*).
\]

This yields \(v \in C_s(\tilde{u}, \tilde{e}, \hat{u}^*)\), and therefore (4.74) holds. \(\Box\)

From Theorem 4.3 and Lemma 4.4 we obtain the following result.
Theorem 4.5 Assume that the assumptions (A1)-(A3) hold. Let \((\tilde{u}, \tilde{e}) \in U_{ad}(\tilde{e}) \times E\) satisfy condition (4.57) for some \(\sigma > 0\), where \(\tilde{e} = (\tilde{e}_y, \tilde{e}_J, \tilde{e}_\alpha, \tilde{e}_\beta)\). Let

\[
\tilde{u}^* \in \varsigma (\tilde{u} + \tilde{e}_y) + \varphi_{\tilde{u} + \tilde{e}_y} + G'(\tilde{u} + \tilde{e}_y)^* \tilde{e}_J + \mathcal{N}(\tilde{u}, \tilde{e}),
\]

and define

\[
\hat{u}^* = \tilde{u}^* - \varsigma (\tilde{u} + \tilde{e}_y) - \varphi_{\tilde{u} + \tilde{e}_y} - G'(\tilde{u} + \tilde{e}_y)^* \tilde{e}_J.
\]

Then, the following statements are equivalent:

(i) The control \(\tilde{u}\) is a Lipschitzian fully stable local minimizer for \(\mathcal{P}(\tilde{u}^*, \tilde{e})\) in (2.9).

(ii) The control \(\tilde{u}\) is a Hölderian fully stable local minimizer for \(\mathcal{P}(\tilde{u}^*, \tilde{e})\) in (2.9).

(iii) The following condition holds that

\[
\mathcal{J}'_{uu}(\tilde{u}, \tilde{e}) v^2 > 0, \quad \forall v \neq 0 \text{ with } v(x) \hat{u}^*(x) = 0 \text{ for a.e. } x \in \Omega,
\]

where \(\tilde{e} = (\tilde{e}_y, \tilde{e}_J, \tilde{e}_\alpha, \tilde{e}_\beta)\).

Proof. As it has been seen in the proof of Theorem 3.10, we have

\[
\mathcal{J}'_{u}(\tilde{u}, \tilde{e}) = \varsigma (\tilde{u} + \tilde{e}_y) + \varphi_{\tilde{u} + \tilde{e}_y} + G'(\tilde{u} + \tilde{e}_y)^* \tilde{e}_J \in L^2(\Omega).
\]

Therefore, the inclusions \(\tilde{u}^* \in \mathcal{J}'_{u}(\tilde{u}, \tilde{e}) + \mathcal{N}(\tilde{u}, \tilde{e})\) and \(\hat{u}^* = \tilde{u}^* - \mathcal{J}'_{u}(\tilde{u}, \tilde{e}) \in \mathcal{N}(\tilde{u}, \tilde{e})\) follow. By Theorem 4.3 (i) is equivalent to (ii). In order to verify the equivalence of (ii) and (iii), it suffices to prove that (iii) is equivalent to (ii). Assume that Theorem 4.3(iii) holds. Let any \(v \in L^2(\Omega)\) satisfy \(v \neq 0\) and \(v(x) \hat{u}^*(x) = 0\) for a.e. \(x \in \Omega\). By \((4.72)\), we have \(v \in \mathcal{C}_s(\tilde{u}, \tilde{e}, \hat{u}^*)\). From this and \((4.69)\) there exist sequences \((u_n, e_n, u_n^*) \to (\tilde{u}, \tilde{e}, \hat{u}^*)\) with \((u_n, e_n, u_n^*) \in \text{gph N}^\alpha\), \(v_n \in C(u_n, e_n, u_n^*)\) such that \(v_n \to v\) when \(n \to \infty\). By \(v_n \in C(u_n, e_n, u_n^*)\) and by \((4.68)\) we get

\[
\mathcal{J}_{uu}(u_n, e_n)v_n^2 \geq \delta \|v_n\|^2_{L^2(\Omega)},
\]

Passing \((4.76)\) to the limit as \(n \to \infty\), we obtain \(\mathcal{J}_{uu}(\tilde{u}, \tilde{e}) v^2 > 0\).

Conversely, assume that (iii) holds. Suppose to the contrary that Theorem 4.3(iii) does not hold. Then, there exist sequences \((u_n, e_n, u_n^*) \to (\tilde{u}, \tilde{e}, \hat{u}^*)\) with \((u_n, e_n, u_n^*) \in \text{gph N}^\alpha\) and \(v_n \in C(u_n, e_n, u_n^*)\) such that

\[
\mathcal{J}_{uu}(u_n, e_n)v_n^2 < \frac{1}{n} \|v_n\|^2_{L^2(\Omega)}, \quad \forall n \in \mathcal{N}.
\]

We may assume that \(\|v_n\|_{L^2(\Omega)} = 1\) for every \(n \in \mathcal{N}\) and may also assume that \(v_n \to v\) in \(L^2(\Omega)\). This implies that \(v \in \mathcal{C}_{w^*}(\tilde{u}, \tilde{e}, \hat{u}^*)\) due to \((4.68)\). Using \((4.72)\) and arguing similarly as in the proof of Theorem 3.10, we obtain a contradiction.

Remark 4.6 We observe that under linear perturbations of the admissible control set of the control problem, we obtain condition \((4.76)\) with the same structure as condition \((3.13)\). In other words, when the admissible control sets undergo linear perturbations, condition \((3.13)\), which is a characterization of both Lipschitzian and Hölderian full stability, is still “stable” provided that the assumption \((4.57)\) holds. Our result considerably extends the results of [15] and [16] because we consider basic perturbations of all the cost functional, the state equation, and the admissible control set of the control problem while the admissible control sets considered in [15] and [16] are fixed.

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5 Concluding remarks

Explicit characterizations of both Lipschitzian and Hölderian full stability for a class of optimal control problems are established in this paper. When the admissible control set of the problems is fixed, we show that the two full stability properties are always equivalent. This fact relies on a general stability result for optimization problems. In the perturbed admissible controls setting, the two full stability properties are also equivalent for this class of control problems. From our results we see that the equivalence of the two full stability properties in this case is due to the special structures of the perturbed admissible control sets, these full stability properties are not always equivalent in general. We think that the main reason for the equivalence of the full Lipschitzian stability and the full Hölderian one in our paper is that all the cost functional, the state equation, and especially the admissible control set of the control problems undergo linear perturbations. Motivated by this remark, in order to understand deeply about phenomena related to the full stability in optimal control, in future research we intend to study the full stability for classes of control problems under nonlinear perturbations.

References

[1] E. Asplund, *Fréchet differentiability of convex functions*, Acta Math., 121 (1968), pp. 31–47.

[2] T. Bayen, J. F. Bonnans, F. J. Silva, *Characterization of local quadratic growth for strong minima in the optimal control of semi-linear elliptic equations*, Trans. Amer. Math. Soc., 366 (2014), pp. 2063–2087.

[3] J. F. Bonnans, *Second-order analysis for control constrained optimal control problems of semilinear elliptic systems*, Appl. Math. Optim., 38 (1998), pp. 303–325.

[4] J. F. Bonnans, A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer-Verlag, New York, 2000.

[5] E. Casas, *Second order analysis for bang-bang control problems of PDEs*, SIAM J. Control Optim., 50 (2012), pp. 2355–2372.

[6] E. Casas, J. C. de los Reyes, F. Tröltzsch, *Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints*, SIAM J. Optim., 19 (2008), pp. 616–643.

[7] E. Casas, D. Wachsmuth, G. Wachsmuth, *Sufficient second-order conditions for bang-bang control problems*, Preprint, (2017), pp. 1–25.

[8] A. L. Dontchev, *Characterizations of Lipschitz stability in optimization*, Recent Developments in Well-Posed Variational Problems, pp. 95–115, Math. Appl., 331, Kluwer Acad. Publ., Dordrecht, 1995.

[9] A. L. Dontchev, K. Malanowski, *A characterization of Lipschitzian stability in optimal control*, Calculus of Variations and Optimal Control, (Haifa, 1998), pp. 62–76, Chapman and Hall, Boca Raton, 2000.

[10] A. L. Dontchev and R. T. Rockafellar, *Implicit Functions and Solution Mappings*, Springer, Dordrecht, 2009.
[11] A. Haraux, How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities, J. Math. Soc. Japan, 29 (1977), pp. 615–631.
[12] K. Ito, K. Kunisch, Lagrange multiplier approach to variational problems and applications, SIAM, Philadelphia, 2008.
[13] B. T. Kien, V. H. Nhu, N. H. Son, Second-order optimality conditions for a semi-linear elliptic optimal control problem with mixed pointwise constraints, Set-Valued Var. Anal., 25 (2017), pp. 177–210.
[14] A. B. Levy, R. A. Poliquin, R. T. Rockafellar, Stability of locally optimal solutions, SIAM J. Optim., 10 (2000), pp. 580–604.
[15] K. Malanowski, F. Tröltzsch, Lipschitz stability of solutions to parametric optimal control problems for parabolic equations, Z. Anal. Anwendungen, 18 (1999), pp. 469–489.
[16] K. Malanowski, F. Tröltzsch, Lipschitz stability of solutions to parametric optimal control for elliptic equations, Control Cybernet., 29 (2000), pp. 237–256.
[17] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation, I: Basic Theory, Springer-Verlag, Berlin, 2006.
[18] B. S. Mordukhovich, T. T. A. Nghia, Full Lipschitzian and Hölderian stability in optimization with applications to mathematical programming and optimal control, SIAM J. Optim., 24 (2014), pp. 1344–1381.
[19] B. S. Mordukhovich, T. T. A. Nghia, Second-order variational analysis and characterizations of tilt-stable optimal solutions in infinite-dimensional spaces, Nonlinear Anal., 86 (2013), pp. 159–180.
[20] R. A. Poliquin, R. T. Rockafellar, Prox-regular functions in variational analysis, Trans. Amer. Math. Soc., 348 (1996), pp. 1805–1838.
[21] R. A. Poliquin, R. T. Rockafellar, Tilt stability of a local minimum, SIAM J. Optim., 8 (1998), pp. 287–299.
[22] F. Pörner, D. Wachsmuth. An iterative Bregman regularization method for optimal control problems with inequality constraints, Optimization, 65 (2016), pp. 2195–2215.
[23] F. Pörner, D. Wachsmuth. Tikhonov regularization of optimal control problems governed by semi-linear partial differential equations, Preprint, (2017), pp. 1–25.
[24] N. T. Qui, D. Wachsmuth, Stability for bang-bang control problems of partial differential equations, Preprint, (2017), pp. 1–20.
[25] S. M. Robinson, Generalized equations and their solutions. II. Applications to non-linear programming. Optimality and stability in mathematical programming, Math. Programming Stud., 19 (1982), pp. 200–221.
[26] S. M. Robinson, Strongly regular generalized equations, Math. Oper. Res., 5 (1980), pp. 43–62.
[27] R. T. Rockafellar, R. J.-B. Wets, Variational Analysis, Springer-Verlag, Berlin, 1998.
[28] F. Tröltzsch, Optimal Control of Partial Differential Equations. Theory, Methods and Applications, American Mathematical Society, Providence, RI, 2010.