Measurement-induced nonlocality based on the trace norm

Ming-Liang Hu and Heng Fan

1 School of Science, Xi’an University of Posts and Telecommunications, Xi’an 710121, China
2 Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
3 Author to whom any correspondence should be addressed.
E-mail: mingliang0301@163.com and hfan@iphy.ac.cn

Keywords: Measurement-induced nonlocality, Trace norm, Quantum correlations

Abstract
Nonlocality is one unique property of quantum mechanics that differs from the classical world. One of its quantifications can be properly described as the maximum global effect caused by locally invariant measurements, known as measurement-induced nonlocality (MIN) (2011 Phys. Rev. Lett. 106 120401). Here, we propose quantifying the MIN by the trace norm. We show explicitly that this measure is monotonically decreasing under the action of a completely positive trace-preserving map, which is the general local quantum operation, on the unmeasured party for the bipartite state. This property avoids the undesirable characteristic appearing in the known measure of MIN defined by the Hilbert–Schmidt norm which may be increased or decreased by trivial local reversible operations on the unmeasured party. We obtain analytical formulas of the trace-norm MIN for any \(2 \times n\)-dimensional pure state, two-qubit state, and certain high-dimensional states. As with other quantum correlation measures, the newly defined MIN can be directly applied to various models for physical interpretations.

1. Introduction
Quantum physics differs in many aspects from our conventional intuition. One such intriguing difference is the celebrated notion of nonlocality, which arises from debate among scientists in the early 20th century. One line of the debate originated with Einstein, Podolsky, and Rosen, who proposed the thought experiment known as the EPR paradox and the so-called ‘spooky-action-at-a-distance’ [1]. Their predictions of quantum mechanics are, in sharp contrast to the conventional view, that physical processes should obey the principle of locality.

Nonlocality of quantum physics has been studied from different points of view, such as by Bell inequality and entanglement. The Bell-type inequalities [2–4] are derived by the local hidden variable theory, and may be violated by realizable experimentally quantum measurement outcomes [5]. The violation of Bell inequalities implies the existence of entanglement in a system [3], however, this is not always true for the opposite case, as there exist mixed states which are entangled but do not violate any Bell-type inequalities [6]. Other studies demonstrate that there is also nonlocality without entanglement [7], or nonlocality without quantum correlations other than entanglement [8]. So it seems that nonlocality can be quantified reasonably by different measures.

Nowadays, we realize that nonlocality is not only a central concept of quantum mechanics, but may also be used to improve the efficiency of many quantum information processing (QIP) tasks [9, 10]. Meanwhile, it is also interrelated with other foundational theories of quantum mechanics such as the uncertainty principle [11]. These delicate and intriguing features of nonlocality prompted a huge surge of interest from the quantum physics community, with notable progress being achieved in the past few years [12–15].

Apart from the traditional line for quantum nonlocality related to entanglement or Bell inequalities, it is also important to investigate it from other perspectives. Recently, Luo and Fu [16] presented a new measure of nonlocality which they termed measurement-induced nonlocality (MIN), motivated by the definition of quantum discord [17, 18]. As the name itself indicates, MIN characterizes nonlocality from a measurement...
perspective, and thus is different from other nonlocality measures. It is a manifestation of the global disturbance to the overall state of a system caused by the locally non-disturbing measurement on one subsystem, and can also be considered as one kind of nonclassical correlation measure that is different from entanglement and quantum discord.

2. MIN quantified by the Hilbert–Schmidt norm

The MIN has been a research focus for years [19–27]. However, its quantification based on the Hilbert–Schmidt norm (we call it conventional MIN for brevity), while intuitively appealing and conceptually significant, has certain discouraging properties. To see this explicitly, we recall its definition, which reads [16]

\[ N_2(\rho_{AB}) = \max_{\Pi} \left\| \rho_{AB} - \Pi^A(\rho_{AB}) \right\|_2, \]

for a bipartite state \( \rho_{AB} \) in \( \mathcal{H}_{AB} \). Here, \( \|X\| = \sqrt{\text{Tr} (X^\dagger X)} \) denotes the Hilbert–Schmidt norm, and the maximum is taken over the full set of local projective measurements \( \Pi^A = \{ \Pi_k^A \} \) that keep the reduced state \( \rho^B = \text{Tr} \_B(\rho_{AB}) \) invariant, namely, \( \sum_k \Pi_k^A \rho_{AB} \Pi_k^A = \rho^B \). An analytical formula of the conventional MIN for any \((2 \times n)\)-dimensional state \( \rho_{AB} \) can be obtained [16]. Here, we argue that the conventional MIN in equation (1), despite being favored for its convenience of calculation, may have certain undesirable properties. More specifically, we will show that it can increase or decrease under trivial local reversible operations on the unmeasured subsystem \( B \) of \( \rho_{AB} \). Consider, for instance, a channel \( \Gamma_B \) acting as \( \Gamma_B(\rho_{AB}) = \rho_{AB} \otimes \rho_C \) (i.e., it introduces a local ancilla to \( B \)), then by making use of the multiplicativity of the Schatten \( p \)-norm (which reduces to the Hilbert–Schmidt norm when \( p = 2 \)) under tensor products, we obtain

\[ N_2(\rho_{ABC}) = N_2(\rho_{AB}) \text{Tr} \rho_C^2. \]

This equality means that \( N_2(\rho_{ABC}) \leq N_2(\rho_{AB}) \) as the purity of a state is no larger than one, \( \text{Tr} \rho_C^2 \leq 1 \). Particularly, if \( \rho_C = I_n/n \) with \( I_n \) being the \( n \)-dimensional identity operator, we obtain \( N_2(\rho_{ABC}) = N_2(\rho_{AB})/n \). Then \( N_2(\rho_{ABC}) \) will approach zero when \( n \) takes the limit of infinity. This behavior differs completely from our intuition that the nonlocal properties of a system should not be affected by trivially adding or removing an uncorrelated local ancillary state.

We remark here that the above perplexity is reminiscent of the phenomena encountered for the geometric measure of quantum discord (GQD) [28–31]. In this case, several well-defined measures of GQD have been introduced to remedy the problem [32–36].

3. MIN based on the trace norm

Motivated by the proposition for modifying GQD via the trace norm [34], we propose defining the MIN for a bipartite state \( \rho_{AB} \) as

\[ N_1(\rho_{AB}) = \max_{\Pi} \left\| \rho_{AB} - \Pi^A(\rho_{AB}) \right\|_1, \]

where \( \|X\| = \text{Tr} \sqrt{X^\dagger X} \), and \( \Pi^A \) still denotes the projective measurements that satisfy \( \Pi^A = \{ \Pi_k^A \} \). We call \( N_1(\rho_{AB}) \) the trace MIN hereafter. The physical interpretation of this new nonlocality measure can still be presented as the maximal global effect, or more explicitly, the maximal trace distance that the postmeasurement state \( \Pi^A(\rho_{AB}) \) departs from its premeasurement state \( \rho_{AB} \) caused by locally invariant measurements.

The nonlocality measure defined above can circumvent the problem incurred for the conventional MIN as implied by equation (2). Explicitly, let us repeat the analysis by adding an uncorrelated ancilla when the new definition is used, then \( N_1(\rho_{ABC}) = N_1(\rho_{AB}) \) due to the normalization condition \( \text{Tr} \rho_C = 1 \). Therefore, \( N_1(\rho_{AB}) \) does not increase under the action of \( \Gamma_B \), namely, it is unaffected by adding or removing a factorized local ancilla on the unmeasured party. Here, we further show a more general and powerful result related to the trace MIN in equation (3).

**Theorem 1.** The trace MIN \( N_1(\rho_{AB}) \) defined in equation (3) is nonincreasing under the action of any completely positive trace-preserving (CPTP) channel \( \mathcal{E}_B \) on the unmeasured party \( B \), i.e., we always have

\[ N_1(\rho_{AB}) \geq N_1(\mathcal{E}_B(\rho_{AB})). \]

**Proof.** Let \( \mathcal{E}_B \) be an arbitrary CPTP channel acting on party \( B \) of \( \rho_{AB} \), and \( \Pi^A_k \) be the optimal projection-valued measurement on party \( A \) that maximizes the trace norm on the right-hand side of equation (3) for \( N_1(\mathcal{E}_B(\rho_{AB})) \), namely,
then, by noting that any local channel on party B and the measurement made on party A commute, we obtain
\[ \mathcal{P}^A \{ \mathcal{E}_B(\rho_{AB}) \} = \mathcal{E}_B [ \mathcal{P}^A (\rho_{AB}) ] , \]
and therefore, by denoting \( \mathcal{P}^A \) (note that \( \mathcal{P}^A \neq \mathcal{P}^A \) in general) the optimal measurement for obtaining \( N_1(\rho_{AB}) \), we have
\[ N_1(\rho_{AB}) = \| \rho_{AB} - \mathcal{P}^A (\rho_{AB}) \| \]
\[ \geq \| \rho_{AB} - \mathcal{P}^A (\rho_{AB}) \| \]
\[ \geq \| \mathcal{E}_B (\rho_{AB}) - \mathcal{E}_B [ \mathcal{P}^A (\rho_{AB}) ] \| \]
\[ = N_1(\mathcal{E}_B (\rho_{AB})) , \]
where the first inequality comes from the fact that \( \mathcal{P}^A (\rho_{AB}) \) is not necessarily the optimal state to \( \rho_{AB} \), and the second inequality is due to the contractivity of the trace norm under CPTP map (Theorem 9.2 of [37]). This completes the proof.

The above theorem means that no physical process on B can increase the maximum trace distance between a state \( \rho_{AB} \) and its postmeasurement state \( \mathcal{P}^A (\rho_{AB}) \) obtained after the locally invariant measurements on party A, and therefore it successfully circumvents the problem in the conventional MIN.

We now list some other basic properties of the trace MIN. (i) \( N_1(\rho_{AB}) = 0 \) for all the product states \( \rho_{AB} = \rho_A \otimes \rho_B \), and the classical–quantum states \( \rho_{AB} = \sum p_k \Pi_k^A \otimes \rho_k^B \) with nondegenerate \( \rho_k = \sum p_k \Pi_k^A \). (ii) \( N_1(\rho_{AB}) \) is invariant under locally unitary operation \( U = V_A \otimes W_B \) on \( \rho_{AB} \), namely, \( N_1(U \rho_{AB} U^\dagger) = N_1(\rho_{AB}) \), which is obvious as the trace norm is preserved under unitary transformations [37].

The proposed trace MIN can be used to detect the effect of a locally invariant measurement on the overall state of a system, and the zero trace MIN implies that the state of the system cannot be disturbed by any locally invariant measurement, namely, the measurement of one subsystem cannot determine the corresponding result of a measurement of the other, and therefore this system obeys the principle of locality. Moreover, from the basic properties listed above, one can note that while any entangled or discordant state possesses nonvanishing trace MIN, there also exist states with nonvanishing trace MIN but which do not produce correlations of entanglement or discord. Therefore, the trace MIN is an important complement to, but different from, entanglement and quantum discord.

4. Analytical formulas of the trace MIN

The maximization in equation (3) over the full set of locally invariant measurements on party A can be obtained for a certain family of states, and in turn the trace MIN can be evaluated analytically. We present them via the following theorems.

**Theorem 2.** For any \( (2 \times n) \)-dimensional pure state \( |\psi\rangle \) with the Schmidt decomposition
\[ |\psi\rangle = \sum_{i=1}^{n} \sqrt{\lambda_i} \left( |\phi_i^A\rangle \otimes |\phi_i^B\rangle \right), \]
the trace MIN is given by
\[ N_1(|\psi\rangle \langle \psi |) = 2 \sqrt{\lambda_1 \lambda_2} . \]

**Proof.** We denote \( \rho^w = |\psi\rangle \langle \psi | \), and \( \rho^w = \text{Tr}_{AB} \rho^w \) for simplicity, then if \( \rho^w \) is nondegenerate, the optimal measurement \( \mathcal{P}^A = |\phi^A\rangle \langle \phi^A | \), and we have
\[ \mathcal{P}^A (\rho^w) = \sum_{k=1}^{n} \left| \phi_k^A \right\rangle \left\langle \phi_k^A \right| \otimes \left| \phi_k^B \right\rangle \left\langle \phi_k^B \right| , \]
therefore
\[ \rho^w - \mathcal{P}^A (\rho^w) = \sqrt{\lambda_1 \lambda_2} \left( \left| \phi_1^A \right\rangle \left\langle \phi_1^A \right| \otimes \left| \phi_2^B \right\rangle \left\langle \phi_2^B \right| \right) + \left| \phi_2^A \right\rangle \left\langle \phi_2^A \right| \otimes \left| \phi_1^B \right\rangle \left\langle \phi_1^B \right| , \]

the singular values of which can be obtained as \( \epsilon_{1,2} = \sqrt{\lambda_1 \lambda_2} \), and thus
\[
N_i(\rho^\Psi) = 2 \sqrt{\lambda_1 \lambda_2}.
\] (10)

If \( \rho^\Psi \) is degenerate (i.e., \( \lambda_{1,2} = 1/2 \)), assuming the optimal locally invariant measurement to be \( \tilde{\Pi}_k^A = |k\rangle \langle k| \), with
\[
|k\rangle = a_k^A |\phi_1^A\rangle + a_k^B |\phi_2^A\rangle,
\] (11)
then as one can always find a unitary operator \( U_A \) such that \( U_A |\phi_k^A\rangle = |\tilde{k}\rangle \), and as \( N_i(\rho^\Psi) \) is locally unitary invariant, we obtain \( N_i(\rho^\Psi) = 1 \) after a similar analysis to that performed for the nondegenerate case, and this completes our proof. \( \square \)

As the entanglement of formation (EoF) for \( |\Psi\rangle \) was given by \( E_f = -\sum_i \lambda_i \log_2 \lambda_i \) [38], the above theorem implies that \( N_i(|\Psi\rangle \langle \Psi|) \) constitutes an entanglement monotone. But this is not the case for general states. Moreover, one can derive \( N_i(|\Psi\rangle \langle \Psi|) = 2N_i(2|\Psi\rangle \langle \Psi|) \), which means that for this special case, both \( N_i(|\Psi\rangle \langle \Psi|) \) and \( N_2(|\Psi\rangle \langle \Psi|) \) give qualitatively the same characterizations of nonlocality.

For a general \( (m \times n) \)-dimensional pure state in the Schmidt expression \( |\Psi\rangle = \sum_{i=1}^d \sqrt{\lambda_i} |\phi_i^A\rangle \otimes |\phi_i^B\rangle \) with \( d = \min \{m, n\} \), and \( \rho^\Psi = |\Psi\rangle \langle \Psi| \), we have
\[
\rho^\Psi - \tilde{\Pi}^A(\rho^\Psi) = \sum_{i,j} \sqrt{\lambda_i \lambda_j} |\phi_i^A\rangle \langle \phi_j^A| \otimes |\phi_i^B\rangle \langle \phi_j^B|,
\] (12)
when \( \rho_A = \text{Tr}_B \rho^\Psi \) is nondegenerate, but a closed form of its singular values cannot be derived for \( m \geq 3 \), and in turn it is difficult to obtain an analytical formula of \( N_i(\rho^\Psi) \). For the degenerate \( \rho_A^\Psi \), an analysis similar to that for \( m = 2 \) yields \( N_i(\rho^\Psi) = 2(m - 1)/m \).

Now we calculate the trace MIN for a general two-qubit state \( \tau_{AB} \), which has been proven to be locally unitary equivalent to \( \rho_{AB} \) of the following form [39]
\[
\rho_{AB} = \frac{1}{4} \left( \mathbb{1}_2 \otimes \mathbb{1}_2 + \vec{x} \cdot \vec{\sigma} \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \vec{y} \cdot \vec{\sigma} + \sum_{i=1}^3 \epsilon_i \sigma_i \otimes \sigma_i \right),
\] (13)
where the vectors \( \vec{x} = (x_1, x_2, x_3), \vec{y} = (y_1, y_2, y_3) \), and \( x_i = \text{Tr} \rho_{AB} (\sigma_i \otimes \mathbb{1}_2), y_i = \text{Tr} \rho_{AB} (\mathbb{1}_2 \otimes \sigma_i), \epsilon_i = \text{Tr} (\sigma_i \otimes \sigma_i), \) and \( \sigma_{1,2,3} \) are the three Pauli operators.

The local unitary invariance of the trace MIN enables \( N_i(\tau_{AB}) = N_i(\rho_{AB}) \), and therefore it suffices to consider the representative family of states \( \rho_{AB} \) in equation (13), for which \( N_i(\rho_{AB}) \) can be evaluated analytically.

**Theorem 3.** For any two-qubit state of the form of equation (13), the trace MIN can be obtained as
\[
N_i(\rho_{AB}) = \begin{cases} 
\frac{\sqrt{\chi_+} + \sqrt{\chi_-}}{2 ||\vec{x}||_2} & \text{if } \vec{x} \neq 0, \\
\max \{ |c_1|, |c_2|, |c_3| \} & \text{if } \vec{x} = 0,
\end{cases}
\] (14)
where \( \chi_\pm = \alpha \pm 2 \sqrt{\beta} ||\vec{x}||_2, \) with \( \alpha = ||\vec{x}||_2^2 - \sum i \epsilon_i^2 x_i^2 \), \( \vec{x} = (c_1, c_2, c_3), \beta = \sum (i|\vec{k}) x_i^2 c_i^2 \), and the summation runs over all the cyclic permutations of \( \{1, 2, 3\} \).

**Proof.** If \( \rho_A = (\mathbb{1}_2 + \vec{x} \cdot \vec{\sigma})/2 \) is nondegenerate, that is, if \( \vec{x} \neq 0 \), then the unique projective measurement leaving \( \rho_A \) invariant is induced by its spectral resolutions
\[
\tilde{\Pi}_{x,\vec{\sigma}}^A = \frac{1}{2} \left( \mathbb{1}_2 \pm \frac{\vec{x} \cdot \vec{\sigma}}{||\vec{x}||_2} \right).
\] (15)
Thus we have [14]
\[
\Pi^A(\rho_{AB}) = \frac{1}{4} \left( \mathbb{1}_2 \otimes \mathbb{1}_2 + \vec{x} \cdot \vec{\sigma} \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \vec{y} \cdot \vec{\sigma} + \frac{\vec{x} \cdot \vec{\sigma}}{||\vec{x}||_2} \otimes \sum_{i=1}^3 \epsilon_i \sigma_i \right).
\] (16)
and therefore
\[
\rho_{AB} - \tilde{\Pi}^A(\rho_{AB}) = \frac{1}{4} \left( \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i - \frac{\bar{x} \cdot \bar{\sigma}}{\|\bar{\sigma}\|^2} \otimes \sum_{i=1}^{3} c_i \sigma_i \right). \tag{17}
\]

After a straightforward algebra, one can obtain the singular values of \(\rho_{AB} - \tilde{\Pi}^A(\rho_{AB})\) as
\[
\epsilon_{1,2} = \frac{\sqrt{\bar{x} \cdot \bar{\sigma}}}{4 \|\bar{\sigma}\|}, \quad \epsilon_{3,4} = \frac{\sqrt{\bar{x} \cdot \bar{\sigma}}}{4 \|\bar{\sigma}\|}, \tag{18}
\]
with \(\chi\) being given below equation (14), and therefore
\[
N_1(\rho_{AB}) = \frac{\sqrt{\bar{x} \cdot \bar{\sigma}} + \sqrt{\bar{x} \cdot \bar{\sigma}}}{2 \|\bar{\sigma}\|}. \tag{19}
\]
If \(\rho_A\) is degenerate, i.e., \(\bar{x} = 0\), then by adopting a similar line to \([35]\), one can obtain
\[
N_1(\rho_{AB}) = \frac{1}{2} \sqrt{2 \max \hat{\epsilon}}, \tag{20}
\]
with \(\hat{\epsilon} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\) being a unit vector in \(\mathbb{R}^3\). By further ordering the singular values of the correlation tensor \(\mathcal{R} = \text{diag} \{c_1, c_2, c_3\}\) as \(c_+ \geq c_0 \geq c_-\), we have \([35]\)
\[
h(\hat{\epsilon}) = Q + \sqrt{H}, \tag{21}
\]
where \(Q\) and \(H\) are given by
\[
Q = c_+^2 + c_0^2 - \sin^2 \theta \left[ c_0^2 - c_-^2 + \cos^2 \phi \left( c_+^2 - c_0^2 \right) \right],
\]
\[
H = A(\theta) \sin^4 \phi + B(\theta) \sin^2 \phi + C(\theta), \tag{22}
\]
with the \(\theta\)-dependent functions \(A(\theta), B(\theta),\) and \(C(\theta)\) (note that there is a misprint in \([35]\), \(\gamma_2^2 + \gamma_3^2\) in the expression for \(C(\theta)\) should be \(\gamma_2^2 - \gamma_3^2\)) being given by
\[
A(\theta) = \sin^4 \theta \left( c_+^2 - c_0^2 \right)^2,
\]
\[
B(\theta) = 2 \left( c_+^2 - c_0^2 \right) \left[ \sin^2 \theta \left( c_+^2 + c_0^2 - 2c_+^2 \right) - \sin^4 \theta \left( c_+^2 - c_-^2 \right) \right],
\]
\[
C(\theta) = \left[ c_+^2 - c_0^2 - \sin^2 \theta \left( c_+^2 - c_-^2 \right) \right]^2. \tag{23}
\]
By combining equations (22) and (23), one can see that both \(Q\) and \(H\) reach their maxima when \(\phi = \pi/2\), for which
\[
Q_{\max} = c_+^2 + c_0^2 - \sin^2 \theta \left( c_0^2 - c_-^2 \right),
\]
\[
H_{\max} = \left[ c_+^2 - c_0^2 + \sin^2 \theta \left( c_0^2 - c_-^2 \right) \right]^2. \tag{24}
\]
Therefore, we have \(\max h(\hat{\epsilon}) = 2c_+^2\), and thus
\[
N_1(\rho_{AB}) = c_+. \tag{25}
\]
This completes our proof. \(\square\)

We would like to point out here that for the two-qubit \(\rho_{AB}\) with nondegenerate \(\rho_A\), the calculation of the trace MIN can also be performed in a similar manner to that for the degenerate case. But now the expression for \(h(\hat{\epsilon})\) in equation (21) is more complicated (see equation [31] in \([35]\) for more detail), and therefore we adopted the procedure listed above as the optimal \(\tilde{\Pi}_{1,2}\), which can be written explicitly for this case.

In figure 1(a), we presented an exemplified plot of the level surfaces of \(N_1(\rho_{BD}) = 0.45\) for the Bell-diagonal states \(\rho_{BD}\) (i.e., \(\bar{x} = 0\) and \(\bar{y} = 0\) in equation (13)). As physical \((c_1, c_2, c_3)\) belongs to a tetrahedron \(T\) (see figure 1), and \(N_1(\rho_{BD}) = \max [c_1, c_2, c_3]\), the surfaces of constant trace MIN correspond to the cross
sections of the six surfaces of a cube $C$ of side length $N_1(\rho^{\text{BD}})$ with $T$. When $N_1(\rho^{\text{BD}}) \leq 1/3$, the surfaces of $C$ are also the surfaces of constant trace MIN, while for $N_1(\rho^{\text{BD}}) > 1/3$, part of them is cut by the four surfaces of $T$.

By denoting $c_+, c_0$ and $c_-$ the maximum, intermediate, and minimum values of $\{ |c_1|, |c_2|, |c_3| \}$, respectively, one can derive a relation between $N_1(\rho^{\text{BD}})$ and $N_2(\rho^{\text{BD}}) = (c_+^2 + c_-^2)/4$ for the Bell-diagonal states $\rho^{\text{BD}}$ as
\[
N_1(\rho^{\text{BD}}) = \sqrt{4N_2(\rho^{\text{BD}})} - c_+.
\] (26)

This implies that the two different MIN measures may impose different orderings of nonlocality, as when $c_+$ keeps unchanged, $N_1(\rho^{\text{BD}})$ also remains unchanged, while $N_2(\rho^{\text{BD}})$ increases (decreases) with the increasing (decreasing) value of $c_0$ in the region of $c_- \leq c_0 \leq c_+$. Thus there is no one-to-one correspondence between the well-defined trace MIN and the conventional MIN in general, and we hope this simple example may provide some intuition about the subtle issue concerning the appropriateness of using the Hilbert–Schmidt norm as a distance for quantifying nonlocality, just like the appropriateness of using it for defining GQD [30].

Moreover, it is also worthwhile to point out that when $\rho^{\text{BD}}$ is subject to the $S^{(i)}$ channel (with $i = 1, 2, 3$ representing respectively, the bit flip, bit-phase flip, and phase flip channels), we have $c_i(t) = c_i(0)$, and $c_{jk}(t) = c_{jk}(0) p(t)(i \neq j \neq k)$, where $p(t) = e^{-\gamma t}$ for the one-sided channel $S^{(i)} \otimes I$, or $I \otimes S^{(i)}$, and $p(t) = e^{-2\gamma t}$ for the two-sided channel $S^{(i)} \otimes S^{(j)}$, with $\gamma$ being the decay rate. As a consequence, if $|c_i(0)| = \max \{ |c_1(0)|, |c_2(0)|, |c_3(0)| \}$ at the initial time, we obtain $N_1(S^{(i)}(\rho^{\text{BD}})) = |c_i(0)|$ by equation (14), which is not destroyed by the $S^{(i)}$ noise during the whole time period. This is in sharp contrast to other nonclassical correlation measures which remain constant only for a finite time interval [40]. This unique and novel characteristic of the trace MIN is not only conceptually significant, but is also appealing for potential quantum algorithms relying on it.

Figure 1(b) plots the valid regions of $(c_1, c_2, c_3)$ for which $N_1(\rho^{\text{BD}})$ can evade the detrimental effects of the phase flip channel. They belong to two hexahedra with vertices $(0, 0, 0), (\pm 1, \mp 1, 0), (\pm 1, \pm 1/3, \pm 1/3, 0)$, and $(0, 0, 0), (\pm 1, \pm 1, -1), (\pm 1, \mp 1, -1/3, -1/3)$, respectively. The results for the bit (bit-phase) flip channel are similar, with $c_3$ replacing $c_i(0)$.

So far we have obtained analytical formulas of the trace MIN for any $(2 \times n)$-dimensional pure state and a general two-qubit state, and discussed several interesting implications. We now turn to consider two high-dimensional states with symmetry. The analytical expressions of some quantum correlation measures (see [41] for a review) for them have already been obtained [29, 33, 42, 43].

Consider first the celebrated Werner state on $\mathbb{C}^d \otimes \mathbb{C}^d$ [6], which can be written as
\[
\rho^W = \frac{d - x}{d^2 - d} I_d^2 + \frac{dx}{d^2 - d} \sum_{ij} |ij\rangle\langle ji|, \quad x \in [-1, 1],
\] (27)
which admits the local unitary invariance, i.e., $\rho^W = (U \otimes U)\rho^W(U^\dagger \otimes U^\dagger)$ for any local unitary operation $U$ and therefore one can choose the optimal measurement basis to be $\Pi_i^A = |i\rangle\langle i|$, which yields
\[
\rho^W - \Pi_i^A(\rho^W) = \frac{dx}{d^2 - d} \sum_{ij} |ij\rangle\langle ji|.
\] (28)
As $\sum_{i,j}|ij\rangle\langle ji|$ constitutes a permutation matrix (a binary matrix with exactly one entry 1 in each row and each column and zeros elsewhere), the singular values of $\rho^W - \tilde{H}^A(\rho^W)$ can be evaluated directly as $|dx - 1|/(d^2 - d)$ with multiplicity $d$ ($d - 1$). Then, by the definition (3) we obtain

$$N_1(\rho^W) = \frac{|dx - 1|}{d + 1},$$

(29)

therefore $N_1(\rho^W)$ vanishes only when $x = 1/d$, and it implies that for the present case, the trace MIN disappears only when $\rho^W$ reduces to the maximally mixed one. Meanwhile, the conventional MIN for $\rho^W$ had also been derived analytically [14], from which we obtain

$$N_1(\rho^W) = \sqrt{d(d - 1)N_2(\rho^W)}.$$  

(30)

This means that both $N_1$ and $N_2$ give qualitatively the same descriptions of nonlocality for $\rho^W$ with finite $d$. But it should be noted that their asymptotic behaviors are different because $\lim_{d \to \infty} N_1(\rho^W) = |x|$, and $\lim_{d \to \infty} N_2(\rho^W) = 0$.

The second high-dimensional state we want to consider is the $(d \times d)$-dimensional isotropic state expressed as follows

$$\rho^I = \frac{1}{2} - \frac{x}{d^2 + 1} 1_{d^2} + \frac{d^2x - 1}{d^2 + 1} |\Phi\rangle\langle \Phi|, \quad x \in [0,1],$$

(31)

with $|\Phi\rangle = \frac{1}{d} \sum |ii\rangle$, and $|i\rangle$ denotes the computational basis on $\mathbb{C}^d$. For this state, let $\tilde{H}^A = |\tilde{k}\rangle\langle \tilde{k}|$ be the measurement basis that maximizes the trace norm in equation (3), then due to the symmetry of $|\Phi\rangle$, one can always find local unitary operation $U$ such that $(U \otimes U)|\Phi\rangle = |\tilde{k}\rangle\langle \tilde{k}|$, and the local unitary invariance of $N_1$ enables $N_1(\rho^I) = N_1([U \otimes U] \rho^I (U^+ \otimes U^+))$. Therefore, by denoting $\rho^I_U = (U \otimes U) \rho^I (U^+ \otimes U^+)$, we obtain

$$\rho^I_U - \tilde{H}^A(\rho^I_U) = \frac{d^2x - 1}{d(d^2 + 1)} \sum_{k\neq \tilde{k}} |\tilde{k}\rangle\langle \tilde{k}|,$$

(32)

de the singular values of which can be evaluated analytically as $|dx - 1|/(d^2 + d)$ with multiplicity 1 and $|d^2x - 1|/(d^2 - d)$ with multiplicity $d - 1$, and this yields

$$N_1(\rho^I) = \frac{2}{d} \frac{|d^2x - 1|}{d(d + 1)}.$$  

(33)

Here, the trace MIN $N_1(\rho^I) = 0$ only when $x = 1/d^2$, namely, when $\rho^I$ comes to be maximally mixed. Moreover, the analytical expression for the conventional MIN can be obtained from [14], through the combination of which we obtain

$$N_1(\rho^I) = \frac{2}{d} \sqrt{(d - 1)N_2(\rho^I)}.$$  

(34)

Their asymptotic values are given by $\lim_{d \to \infty} N_1(\rho^I) = 2x$, and $\lim_{d \to \infty} N_2(\rho^I) = x^2$, respectively. It implies that the two MIN measures still give qualitatively the same characterizations of nonlocality for the isotropic state.

Moreover, it is remarkable that for the special case $x = 1$, i.e., when $\rho^I$ reduces to the maximally entangled state, we have $N_1(\rho^I) = 2(d - 1)/d$, which is just twice that of the conventional MIN.

5. Summary and discussion

To summarize, we have introduced a well-defined measure of nonlocality by making use of the trace norm. It can remedy the undesirable property of the conventional MIN which can be changed arbitrarily and reversibly by trivial local action on the subsystem. We proved explicitly that the proposed trace MIN is nonincreasing under the action of general CPTP quantum channels on the unmeasured subsystem. This property by itself has a conceptual significance, as it has already been proven that the Schatten 1-norm (trace norm) is the only p-norm that can be used to give a well-defined quantum correlation measure [34]. Here, the fascinating properties of the trace MIN again show the ubiquitousness and intrinsic significance of the Schatten 1-norm for defining MIN. We hope this may shed some new light on the issue concerning the characterization and quantification of nonlocality from a measurement perspective.

We have also presented analytical formulas of the trace MIN for any $(2 \times n)$-dimensional pure state, two-qubit state, as well as the Werner states and the isotropic states on $\mathbb{C}^d \otimes \mathbb{C}^d$ which possess high symmetry. We
revealed through these results that the trace MIN captures the nonlocal property of a system more intrinsically than that of the conventional MIN. Moreover, we revealed a unique and appealing characteristic of this new proposed nonlocality measure, namely, it can evade the detrimental effects of certain noisy channels during the whole time period for elaborately designed initial states. This may have potential applications in QIP owing to its coherence protecting property.

We remark that the entropic measure of MIN based on the von Neumann entropy [19], or its equivalent form based on the relative entropy [20], is also monotonically decreasing due to the monotonicity of the quantum mutual information under channels on $B$ (see [30] for a detailed proof). Moreover, it has already been pointed out that one can remedy the MIN via the square root of the considered density matrix [33]. Here, we mention that it is also natural to define the MIN as

$$N_B(\rho_{AB}) = 2 \max_{\Pi^A} \left\{ 1 - \sqrt{F(\rho_{AB}, \Pi^A(\rho_{AB}))} \right\},$$

(35)

via the Bures distance [36], with $\Pi^A$ being the locally invariant measurement and $F(\rho, \sigma) = [\text{Tr} (\sqrt{\rho} \sigma \sqrt{\rho})^{1/2}]^2$ denoting the Uhlmann fidelity. By using the monotonicity of the Bures distance [37] and after a similar analysis to that for proving Theorem 1, one can show directly that $N_B$ is also nonincreasing under general CPTP channels. But its evaluation may be intractable and further investigation is needed. Thus the measure presented in this paper may be widely used for its concise and simple form.

The significance for this computable measure of the trace MIN in QIP can be studied parallel to that of quantum entanglement and quantum discord, see reviews [9, 41]. Additionally, it may be applied as a new technique to various models to study physical phenomena such as quantum phase transitions and topologies of those systems, similar to the applications of entanglement, see for example [45, 46, 44]. Moreover, as nonlocality is quantitatively related with Heisenberg’s uncertainty principle [11], which provides the basis for the security of quantum cryptography [47], the obvious physical significance of the trace MIN and its convenience of calculation is also hoped to play a role in these related issues.

Acknowledgments

This work was supported by NSFC (11205121, 11175248), NSF of Shaanxi Province (2014JM1008), and SRP of the Education Department of Shaanxi Province (12JK0986).

References

[1] Einstein A, Podolsky B and Rosen N 1935 Phys. Rev. 47 777
[2] Bell J S 1964 Physics 1 195
[3] Clauser J F, Horne M A, Shimony A and Holt R A 1969 Phys. Rev. Lett. 23 880
[4] Augusiak R, Cavalcanti D, Prettico G and Acín A 2010 Phys. Rev. Lett. 104 230401
[5] Genovese M 2005 Phys. Rep. 413 319
[6] Werner R F 1989 Phys. Rev. A 40 4277
[7] Bennett C H et al 1999 Phys. Rev. A 59 1070
[8] Horodecki R, Horodecki M and Horodecki P 1999 Phys. Rev. A 60 4144
[9] Niset J and Cerf N J 2006 Phys. Rev. A 74 052103
[10] Walgate J and Hardy L 2002 Phys. Rev. Lett. 89 147901
[11] Mor T 2006 Int. J. Quantum Inf. 4 161
[12] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Rev. Mod. Phys. 81 865
[13] Bennett C H and Wiesner S J 1992 Phys. Rev. Lett. 69 2881
[14] Bennett C H et al 1993 Phys. Rev. Lett. 70 1895
[15] Bennett C H et al 2001 Phys. Rev. Lett. 87 077902
[16] Oppenheim J and Wehner S 2010 Science 330 1072
[17] Fu L B 2006 Europhys. Lett. 75 1
[18] Gharibian S, Kampermann H and Bruß D 2009 Quantum Inf. Comput. 9 1013
[19] Luo S and Fu S 2010 Europhys. Lett. 92 20004
[20] Luo S and Luo S 2011 Int. J. Quantum Inf. 9 1587
[21] Luo S and Fu S 2011 Phys. Rev. Lett. 106 120401
[22] Ollivier H and Zurek W H 2001 Phys. Rev. Lett. 88 017901
[23] Luo S 2008 Phys. Rev. A 77 042303
[24] Hu M, Liu and Fan H 2012 Ann. Phys. 327 2343
[25] Li Y 2012 Phys. Rev. A 85 042325
[26] Zhang G F, Fan H, Ji A L and Liu W M 2012 Eur. Phys. J. D 66 34
[27] Czekmak B, Karpat G and Gedik Z 2012 Phys. Lett. A 376 2982
[28] Mirafla Z, Sargolzahi I, Ahanj A, Javidan K and Sarbishei M 2013 Quantum Inf. Comput. 13 0479
[29] Guo Y and Hou J 2013 J. Phys. A: Math. Theor. 46 032301
[30] Guo Y 2013 Int. J. Mod. Phys. B 27 1350067
[31] Tian Z and Jing J 2013 Ann. Phys. 333 76
[25] Sen A, Sarkar D and Bhar A 2012 J. Phys. A: Math. Theor. 45 405306
Sen A, Sarkar D and Bhar A 2013 Quantum Inf. Process. 12 3007
Rana S and Parashar P 2013 Quantum Inf. Process. 12 2523
Ramzan M 2013 Quantum Inf. Process. 12 2721
[26] Mohamed A B A 2013 Optik 124 5369
[27] Wu S X, Zhang J, Yu C S and Song H S 2014 Phys. Lett. A 378 344
[28] Dakić B, Vedral V and Brukner Č 2010 Phys. Rev. Lett. 105 190502
[29] Luo S and Fu S 2010 Phys. Rev. A 82 034302
[30] Piani M 2012 Phys. Rev. A 86 034101
Hu X, Fan H, Zhou D L and Liu W M 2013 Phys. Rev. A 87 032340
[31] Dakić B et al 2012 Nat. Phys. 8 666
[32] Girolami D, Tufarelli T and Adesso G 2013 Phys. Rev. Lett. 110 240402
[33] Chang L and Luo S 2013 Phys. Rev. A 87 062303
[34] Rana S and Parashar P 2013 Phys. Rev. A 87 016301
Montealegre J D, Paula F M, Saguia A and Sarandy M S 2013 Phys. Rev. A 87 042115
Paula F M, de Oliveira T R and Sarandy M S 2013 Phys. Rev. A 87 064101
Aaronson B, Franco R L, Compagno G and Adesso G 2013 New J. Phys. 15 093022
[35] Ciccarello F, Tufarelli T and Giovannetti V 2014 New J. Phys. 16 013038
[36] Spehner D and Orszag M 2013 New J. Phys. 15 103001
Spehner D and Orszag M 2014 J. Phys. A: Math. Theor. 47 035302
[37] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[38] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Phys. Rev. A 54 3824
[39] Horodecki R and Horodecki M 1996 Phys. Rev. A 54 1838
[40] Mazzola L, Piló J and Maniscalco S 2010 Phys. Rev. Lett. 104 200401
[41] Modi K, Brodutch A, Cable H, Paterek Z and Vedral V 2012 Rev. Mod. Phys. 84 1655
[42] Luo S 2008 Phys. Rev. A 77 022301
[43] Chitambar E 2012 Phys. Rev. A 86 032110
[44] Amico L, Fazio R, Osterloh A and Vedral V 2008 Rev. Mod. Phys. 80 517
[45] Cui J, Gu M, Kwek L C, Santos M F, Fan H and Vedral V 2012 Nat. Commun. 3 812
[46] Wang D, Liu Z, Cao J P and Fan H 2013 Phys. Rev. Lett. 111 186804
[47] Berta M, Christandl M, Colbeck R, Renes J M and Renner R 2010 Nat. Phys. 6 659