Nonlinear Responses of An Unbalanced Overhung Rotor-Short Journal Bearing System with Some Bifurcation Results

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Abstract
Both autonomous and non-autonomous nonlinear systems display complex responses including the transient and long run responses. An overhung rotor-short journal bearing system (ORSJB) is the excellent paradigm exhibiting nonlinear responses and of practical importance. The unbalanced overhung rotor-journal bearing system is formulated by using modified Laval-Jeffcott rotor model. The nonlinear fluid forces model of the short journal bearing is adopted and rearranged in a suitable form for numerical analysis. This paper investigates numerically a bifurcation point, transient and long run responses of the ORSJB system. The rotational speed is preferable bifurcation parameter herein. The ORSJB system without unbalanced eccentricity for the selected parameters yields the bifurcation value of $\Omega = 3.0091$. Limits cycles in the long-run responses with $e_U = 0.002$ m and without unbalanced eccentricity at long run are presented in the bifurcation regime at the speed of $\Omega = 6.6869$ and $\Omega = 0.01671$ respectively. The transient response with $e_U = 0.002$ m at the speed of $\Omega = 4.7142$ is elucidated. The evident results in this paper give only a partial view of bifurcation behaviours. Bifurcation analysis is a useful tool for design and operation overhung rotor dynamic systems.

1. Introduction
There is a growing body of literature that recognizes some advantages of plain journal bearings, for instance, the ease of manufacturing and lubricant feeding. A considerable amount of literature has been published on nonlinear dynamics of rotor-journal bearing system [1]. Previous research investigated various topics and developed theoretical results in the domain of a plain journal bearing [2, 3, 8]. Short plain journal bearing is the prominent bearing type commonly selected for theoretical study. The chaos of symmetrical in-board rigid rotor interacted with fluid forces in laminar regime from supporting short journal bearings was investigated analytically and experimentally in [2] while the turbulent regime of fluid force dynamics was studied theoretically in [3]. Surveys such as that conducted by Wang et.al [4] explored the analytical model of fluid film force of finite length journal bearing for laminar regime. In-board Laval-Jeffcott rotor with unsymmetrical clearance was analyzed at a fractional speed of whirling as in [5]. Performance of plain journal bearing with effects of viscosity and clearance was analyzed in [6]. Most models used for nonlinear analysis for Laval-Jeffcott rotor journal systems are of an inboard type. To date, most studies have carried out an
overhung rotor for nonlinear motion with the effects of clearance and rubbing [7]. Whilst there have been few empirical investigations into the nonlinear effect of plain journal bearing forces. Overhung rotor dynamic systems with journal bearings have been used wildly in engineering applications such as gas turbine engines, marine propeller shaft systems, to name but a few. This article will formulate governing equations for the proposed overhung Laval-Jeffcott rotor journal system and investigate numerically on effects of nonlinear fluid force with various rotational speeds.

2. An Overhung Rotor-Journal Model and Equations of Motion

Figure 1 presents the 4-DOF model of an overhung rotor journal bearing system.

![Figure 1. An Overhung Rotor-Journal Model](image)

As can be seen from Figure 1 above, the overhung Jeffcott rotor with a journal bearing is proposed. From Figure 1 above we can see that a simple supported bearing is assumed at the left end. It is apparent from this figure that the assumption results has no transverse motion on the left-most plane. While a short plain journal, bearing with clearance is placed at plane 2. The $r_g$ and $r_j$ are radii of bearing and journal respectively. Jeffcott rotor is the massless shaft with point masses at journal and rotor. The relation of length portions can be written as $l = a + b$. The stiffness of the shaft for transverse deflections is $k = \frac{3EJ}{ib^3}$ where $l$ the area moment of inertia of the shaft is. The viscous damping with equally damping coefficients $(d)$ is assumed at the rotor plane 1.

As shown in Figure 1 above, the kinetic energy $(T)$ and potential energy $(U)$ of the system in the can be written in (1) and (2).

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}m_2(\dot{x}_3^2 + \dot{x}_4^2)$$  \hspace{1cm} (1)

$$U = \frac{1}{2}k(x_1 - K_ax_3)^2 + \frac{1}{2}k(x_2 - K_ax_4)^2 + m_1gx_2 + m_2gx_4$$  \hspace{1cm} (2)

where $K_a = l/a$. The Lagrangian is defined as $\mathcal{L} = T - U$. By using $\mathcal{L}_i(\partial \mathcal{L}/\partial \dot{x}_i) - \partial \mathcal{L}/\partial x_i = Q_{xi}$ $i = 1,2,3$ and 4, one can write the governing equations of the above system as in (3)-(5).

$$m_1\ddot{x}_1 + k(x_1 - K_ax_3) = -d\dot{x}_1 + m_1e_\omega \omega^2 \cos(\omega t)$$  \hspace{1cm} (3)

$$m_1\ddot{x}_2 + k(x_2 - K_ax_4) = -d\dot{x}_2 + m_1e_\omega \omega^2 \sin(\omega t) - m_1g$$  \hspace{1cm} (4)

$$m_2\ddot{x}_3 + kK_a(K_ax_3 - x_1) = f_3$$  \hspace{1cm} (5)

$$m_2\ddot{x}_4 + kK_a(K_ax_4 - x_2) = f_4 - m_2g$$  \hspace{1cm} (6)

where the “'” represents the “$d/dt$”. $Q_{xi}$, $i = 1,2,3,4$ are generalized forces in the directions of $x_i$ appearing on the right-hand side of (3)-(6) and $e_\omega$ is the eccentricity of unbalance mass at the rotor. The $f_3$ and $f_4$ are the fluid forces from bearing acting on the journal. These are sources of nonlinear terms dominating behavior of the systems summarizing in the next section.
3. Fluid Film Force of Short Journal Bearing

It can be seen from the data in Figure 1 that a plain short journal bearing was selected. Figure 2 presents description of the quantities used in this study. Regarding a circular journal bearing, $h(\theta, t) = c - e(t)\cos(\theta - \gamma(t))$ is where the bearing clearance is. The eccentricity is $c = r_B - r_j$ and $e$, respectively. We write $ecos(\gamma) = x_3$ and $esin(\gamma) = x_4$, and thus $h(\theta, t) = c - x_3cos\theta - x_4sin\theta$ is provided.

\[
\partial_z \left( \frac{h^3}{12\mu} \partial_z p \right) = \frac{1}{2} \omega \partial_\theta h + \partial_t h \tag{7}
\]

In terms of the Reynolds equation [1], it is specifically shown for a short bearing imposing $\partial_\theta p=0$ with $p(\theta,z)$. It is facing dimensional pressure from the fluid film, and is called a dynamic viscosity of the fluid ($\mu$). The Reynolds equation (7) for a short journal bearing can be rewritten in a non-dimensional form with zero boundary conditions at both ends of the bearing along the axial axis. We use some capital Latin and lowercase Greek letters for non-dimensional quantities, such as $X_i = \frac{x_i}{c}$, $\zeta = z/r_B$, $H = \frac{h}{c}$, $\tau = \omega t$ and $P = \frac{p}{6\mu\omega(r_B/c)}$. After some manipulations, we can obtain the fluid forces in the equation (8)

\[
\left[ \begin{array}{c}
F_3 \\
F_4
\end{array} \right] = \Gamma \Omega \left[ \begin{array}{c}
F_3 \\
F_4
\end{array} \right] \tag{8}
\]

where $F_3$ and $F_4$ are non-dimensional fluid film forces from the bearing acting on the journal. $\Gamma = \frac{\mu r_B^2}{2g h^5c^2 s^5 m_q}$ is bearing’s characteristic number where $m_q = m_2 + m_4 K_a$. $\Omega = \omega \sqrt{c/g}$ is the non-dimensional rotational speed. We have verified and adopted the adorable results from [2] to compute the $F_3$ and $F_4$. We define $" t" = \frac{d}{dt}$, $Q = (1 - X_2^2 - X_4^2)^{1/2}$, $V = X_4cos\alpha - X_3 sin\alpha$ and use $G(X_3, X_4, \alpha) = \frac{2}{Q} \left[ \frac{\pi}{2} + atan \left( \frac{V}{Q} \right) \right]$ with $\alpha = atan \left( \frac{x_4 + 2x_3}{x_3 - 2x_4} \right)$, $\frac{\pi}{2} sign \left( \frac{x_4 + 2x_3}{x_3 - 2x_4} \right) - \frac{\pi}{2} sign \left( X_4 + 2X_3 \right)$. It should be noted that the findings from the experiment provide some support for the $\pi - film$ model of fluid film pressure that is used in this formulation. We then obtain the non-dimensional fluid film forces as:

\[
F_3 = \frac{c\mu}{2} (3X_3 V_\nu - \sin(\alpha) \ G - 2 \cos(\alpha) \ S) \tag{9}
\]

\[
F_4 = \frac{c\mu}{2} (3X_4 V_\nu - \cos(\alpha) \ G - 2 \sin(\alpha) \ S) \tag{10}
\]
where \( C_F = -\sqrt{\frac{(x_3 - 2x_4)^2 + (x_4 + 2x_3')^2}{v^2}} \), \( V'_F = \frac{2 + vG}{v^2} \) and \( S = \frac{x_3 \cos \alpha + x_4 \sin \alpha}{1 - (x_3 \cos \alpha + x_4 \sin \alpha)^2} \).

The fluid film forces in the equations (9) and (10) are sources of nonlinear influencing in the considering rotor-bearing system. The \( F_3 \) and \( F_4 \) represent in dimensionless forms, however, the system equations from (3) to (6) are all in dimensional quantities. Use of the groups of non-dimensional quantities gives us more portrayal of the system behavior. The next section of the study was concerned with the non-dimensional system of (3) to (6).

4. Non-dimensional Equations and State Space Representation

As previously mentioned in Section 3, the non-dimensional fluid film forces are obtained and they are also coupled with the variables \( X_i \) and \( X'_i \). The transformation of all quantities in non-dimensional forms was used to compute the responses of this system. To get all the equations in non-dimensional forms, the systematic method is used to represent in state-space form for the ease of computing the responses.

4.1 Non-dimensional Equations

Starting with the equations (3) to (6), we can normalize the equations and make use of some more dimensionless quantities. The appropriated non-dimensional quantities to be mentioned are \( D = d/m_1 \sqrt{c/g} \), \( K_1 = k c/(m_1 g) \), \( K_2 = k c/(m_2 g) \), and \( E = e_U/c \). The equations (3) to (6) can be written, in non-dimensional forms, as in (11) to (14).

\[
X_1'' + \frac{D}{\Omega} X_1' + \frac{K_1}{\Omega^2} (X_1 - K_a X_3) = E \cos \tau \\
X_2'' + \frac{D}{\Omega} X_2' + \frac{K_1}{\Omega^2} (X_2 - K_a X_4) = E \sin \tau - \frac{1}{\Omega^2} \\
X_3'' + \frac{K_a K_2}{\Omega^2} (K_a X_3 - X_1) = \frac{m_a q}{m_2 \Omega} F_3 \\
X_4'' + \frac{K_a K_2}{\Omega^2} (K_a X_4 - X_2) = \frac{m_a q}{m_3 \Omega} F_4 - \frac{1}{\Omega^2}
\]

4.2 State Space Representation

Representation of the system of equations (11) to (14) in state-space form is preferred for writing computing programs. The following equations (15) to (24) are in the form of ready to use for coding in the Julia Programing Language.

\[
X_1' = X_5 \\
X_2' = X_6 \\
X_3' = X_7 \\
X_4' = X_8 \\
X_5' = -\frac{D}{\Omega} X_5 - \frac{K_1}{\Omega^2} (X_1 - K_a X_3) + E X_9 \\
X_6' = -\frac{D}{\Omega} X_6 - \frac{K_1}{\Omega^2} (X_2 - K_a X_4) + E X_{10} - \frac{1}{\Omega^2} \\
X_7' = -\frac{K_a K_2}{\Omega^2} (K_a X_3 - X_1) + \frac{m_a q}{\Omega m_2} F_3
\]
\[ X_8' = -\frac{K_a \Omega^2}{\Omega^2} (K_a X_4 - X_2) + \frac{m_4 \Gamma}{\Omega m_2} F_4 - \frac{1}{\Omega^2} \] (22)

\[ X_9' = X_9 - X_{10} - (X_9^2 + X_{10}^2) X_9 \] (23)

\[ X_{10}' = X_9 + X_{10} - (X_9^2 + X_{10}^2) X_{10} \] (24)

5. Results and Discussion

The results obtained from the preliminary analysis of numerical bifurcation of the unbalanced rotor-journal bearing system are shown. In general, non-autonomous nonlinear systems can be represented as \( X' = F(\tau, X_i; \Omega) \) where \( \Omega \) is the bifurcation parameter. It is possible for some non-autonomous systems to express as \( X' = F(X_i; \Omega) \) and our system depicted in the equations (15)-(24) is one of the cases. We have chosen the non-dimensional rotational speed \( \Omega \) as the bifurcation parameter in this investigation. In the process of bifurcation investigation, all \( X_i \) are treated as coordinates and the others considered as parameters changing one at a time. The \((X_1, X_2)\) and \((X_3, X_4)\) are of interest because they illustrate the orbits of the rotor and the journal centers. The physical values of the system to yield the parameters are \( a = b = 0.4 \text{ m} \), bearing length \((l_B) = 0.01 \text{ m} \), shaft diameter \((d_s) = 0.03 \text{ m} \), bearing strength \((E_Y) = 200 \times 10^4 \text{ N/m} \), radius \((r_B) = 0.02 \text{ m} \), \( c = 1.0 \times 10^{-4} \text{ m} \), \( \mu = 50 \times 10^{-3} \text{ Pa} \cdot \text{s} \), \( m_1 = 5 \text{ kg} \), \( m_2 = 1 \text{ kg} \), \( g = 9.81 \text{ m/s}^2 \) and \( d = 50 \text{ Ns/m} \). These values yield the parameters of \( D = 0.03192754 \), \( K_1 = 0.37997689 \), \( K_2 = 1.89988446 \), and \( \Gamma = 0.1451 \). In practice, the bearing’s characteristic number can be changed independently since it depends upon the chosen oil viscosity. All numerical results of the orbits are coded in the Julia Programming Language.

We firstly consider the absence of the unbalanced eccentricity \( e_U = 0 = E \). As shown in Figure 3, the equilibrium positions depend on the rotational speeds. It is apparent from this figure that the solid line means stable equilibrium points while the dashed line represents unstable equilibrium points. From this data, we can see that if one starts at some initial conditions at the speeds lower than the critical speed, then the orbits will converge to the equilibrium points at that speed. We now search for the critical speed or the bifurcation point by analyzing the Jacobian matrix of the equations (15) to (22) by setting \( X_9 = X_{10} = 0 \). They have been implemented on MATLAB and the bifurcation value is \( \Omega = 3.0091 \) (\( \omega = 9,100 \text{ rpm} \)). The selected orbits at the speed of \( \Omega = 6.6869 \) (\( \omega = 20,000 \text{ rpm} \)) being in bifurcation regime are shown in Figure 4. The initial conditions used to generate results in this study start with equilibrium points. One should keep running the computation until the orbits reach the limit cycles.

![Figure 3. Equilibrium positions of centers of the rotor and journal at various speeds](image-url)
Figure 4. The bifurcations of the rotor and journal at rotational speed of 20,000 rpm without unbalanced eccentricity

In the presence of unbalanced mass at the rotor with the unbalanced eccentricity $e_U = 0.002 \ m$, the selected results at the rotor were illustrated in Figure 5 and Figure 6. All the selected results herein start to compute from equilibrium points at the corresponding rotational speeds. We have picked, for instance, the orbits at the rotor center to illustrate. In Figure 5, rotational speed is of $\Omega = 0.0167 \ (50 \ rpm)$ below the bifurcation value from the previous case but it reaches into the limit cycle at a very early time ($\tau = 100$). Figure 6 compares the rotational speed is higher than the bifurcation point of the absence of the unbalanced case at the speed of $\Omega = 4.7142 \ (\omega = 14,100 \ rpm)$. The latter case in Figure 6 shows more complex with higher amplitude of the orbit during the transient response. It eventually arrives the limit cycle with bigger amplitudes than the unbalanced case and it is not shown here.

Figure 5. Limit cycle of the rotor with $e_U=0.002 \ m$ at the speed of 50 rpm

Figure 6. Transient response of the rotor with $e_U=0.002 \ m$ at the speed of 14,100 rpm

6. Conclusion

The aim of the present research was to examine an overhung rotor-journal model with nonlinear fluid film forces. His project was undertaken to formulate the governing equation and arrange a non-dimensional form that suited for computation. In this study, the results of the analysis of the numerical bifurcation are also represented. Regarding the absence of an unbalanced eccentricity ($e_U = 0$), it revealed that the bifurcation value is found at $\omega = 9,100 \ rpm$. The limit cycle after the bifurcation point at $\omega = 20,000 \ rpm$ is investigated. In presence of unbalance, we demonstrate both results of higher and lower speeds of bifurcation point. This study has identified the occurrence of the limit cycle at a very low speed of $\omega=50 \ rpm$ well below the bifurcation point of the previous case. At
high speed of $\omega = 14,100 \text{ rpm}$ starting from the equilibrium point, it encounters more complex orbits with large amplitude. The contribution of this study has been to confirm numerical bifurcation simulation is a valuable tool to design and predict a behavior of rotor dynamics. Further research should be carried out to establish the behavior of the overhung journal-rotor system.

References
[1] Kramer E 1993 Dynamics of rotors and foundations (Berlin : Springer-Verlag)
[2] Adiletta G Guido A R and Rossi C 1996 Nonlinear Dynam. 10(3) 251-269 DOI: 10.1007/BF0004
[3] Hashimoto H Wada S and Ito J 1987 J. Tribol.-TASME 109(2) 307-314 DOI: 10.1115/1.3261357
[4] Wang Y L Liu Z S Kang W J and Yan J J 2011 J. Mech. Eng. Sci. 226(5) 1345-1355 DOI: 10.1177/0954406211418302
[5] Childs D W 1982 J. Eng. Power-TASME 104(3) 533-541 DOI: 10.1115/1.3227
[6] Prashad H 1988 Tribol. T, 31(2) 303-309 DOI: 10.1080/10402008808981827
[7] Zilli A Williams R J Ewins D J 2015 J. Eng. for Gas Turb. Power 137(6) 0605001(1-10) DOI: 10.1115/1.4036098
[8] Miura T Inoue T and Kano H 2015 J. Vib. Acoust. 139(3) 031012(12) DOI:10.1115/1.4036098