Kernel Estimation of the Instantaneous Frequency

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Abstract

We consider kernel estimators of the instantaneous frequency of a slowly evolving sinusoid in white noise. The expected estimation error consists of two terms. The systematic bias error grows as the kernel halfwidth increases while the random error decreases. For a nonmodulated signal, \( g(t) \), the kernel halfwidth which minimizes the expected error scales as \( h \sim \left( \frac{\sigma^2}{N(\dot{g}(t))^2} \right)^{1/5} \), where \( \sigma^2 \) is the noise variance and \( N \) is the number of measurements per unit time. We show that estimating the instantaneous frequency corresponds to estimating the first derivative of a modulated signal, \( A(t) \exp(i\phi(t)) \). For instantaneous frequency estimation, the halfwidth which minimizes the expected error is larger: \( h_{1,3} \sim \left( \frac{\sigma^2}{A^2N|\dot{\phi}(t)|^2} \right)^{1/7} \). Since the optimal halfwidths depend on derivatives of the unknown function, we initially estimate these derivatives prior to estimating the actual signal.
I. INTRODUCTION

We consider the problem of estimating the instantaneous frequency of one or more slowly evolving sinusoids in white noise. Excellent reviews of the estimation of the instantaneous frequency as well as the general theory of time–frequency distributions can be found in [2,3,10] Cohen & Lee [4] determine an optimal kernel smoother by minimizing the time–frequency spread of the resulting estimate of the instantaneous frequency.

Our approach is based on the theory of kernel smoothers for nonparametric function and derivative estimation [7-9,13-15,17,18,21]. Kernel smoothers are weighted averages of the measured values of a slowly evolving unknown function. We use “kernel smoother” to be consistent with the terminology of nonparametric function estimation. In the electrical engineering literature, the equivalent terminology is “linear transfer function” or “acausal finite impulse response linear filter.” As the kernel halfwidth increases, the random error from the white noise decreases.

Lovell & Williamson (L & W) [12] use a centered difference to estimate the time derivative of the phase, $\phi'(t)$, and then use a kernel smoother to reduce the variance of the estimate of $\phi'(t)$. As in L & W, we treat estimation of the instantaneous frequency as a kernel smoothing problem with circular statistics. However, the kernel smoother has a bias error from systematic evolution of the amplitude and frequency, and this bias error increases with the kernel halfwidth. We calculate the leading order expected estimation error by expanding the unknown function in the ratio of the sampling time to the characteristic time scale on which the unknown signal is evolving. We then determine the optimal kernel halfwidth by minimizing the expected error. In L & W’s pioneering work, the bias error is neglected, and as a result their estimate of the expected error is a monotonically decreasing function of the kernel halfwidth.

A second group of estimators of the instantaneous frequency have been developed which are based on linear regression or linear prediction [6,11,24]. These methods estimate fixed frequencies. Since the frequencies are assumed to be time independent, and these methods neglect the rate of change of the frequencies, and the temporal evolution of the signal frequencies causes a bias error in the estimate. Often, these methods use short subsequences such that on a particular subsequence the bias error (from frequency evolution) is negligible. However, having negligible bias is really a disadvantage because the subsequence length could be increased until the bias error is comparable with the random error. In our approach, we try to minimize the total error by increasing the kernel halfwidth until the rate of increase in the bias error matches the decrease in the variance.

In the next section, we review the theory of nonparametric function and derivative estimation. In Section III, we apply these results to instantaneous frequency estima-
tion. In Section IV, we generalize the analysis to include the correlated errors which are induced by the Hilbert transform. In Section V, we consider multiple signals. In Appendix A, we describe data-adaptive multiple stage kernel estimators which determine a self-consistent optimal halfwidth. In Appendix B, we describe the kernel shapes which minimize the expected error.

II. EXPECTED LOSS OF KERNEL SMOOTHERS

In this section and the appendix, we consider a real digital signal in white noise:

$$y_j = g(t_j) + \tilde{e}_j, \quad j = 1, \ldots, N,$$

where $\tilde{e}_j$ is independently distributed noise with variance $\sigma^2$. Our goal is to estimate the $q$th derivative of $g(t_j)$ with a minimum of expected error. We assume that $g(t)$ has $p$ continuous derivatives and that $g(t)$ varies slowly with respect to the sampling rate. We normalize the measurement times, $t_j$, to be in the closed interval $[0, 1]$.

We consider kernel estimators of $\partial^q g(t)$ of the form:

$$\hat{\partial}^q g(t) = \frac{1}{Nh^{(q+1)}} \sum_{j=1}^{N} K\left(\frac{t - t_j}{h}\right)y_j,$$

where the $\hat{\cdot}$ over $\partial^q g$ denotes the estimate of the $q$th derivative. We define the vector, $\mu(t) = \frac{1}{Nh^{(q+1)}} (K\left(\frac{t - t_u}{h}\right), \ldots, K\left(\frac{t - t_N}{h}\right))^T$. We say a kernel, $\mu$, with halfwidth, $h$, is of order $(q, p)$ if

$$\mu \cdot s^{(m)} = q! \delta_{m,q}, \quad m = 0, \ldots, p - 1,$$

where $s_j^{(m)} \equiv (\frac{t - t_j}{h})^m$. We denote the $p$th moment of a kernel of order $(q, p)$ by $C_{q,p}$:

$$\mu \cdot s^{(p)} = p! C_{q,p}.$$ Kernels of order $(q, p)$ are used to estimate the $q$th derivative of the function to order $O(h^{p-q})$. We normally select $p = q + 2$ and our preferred set of kernels is given in (2.10). For function estimation ($q = 0$), we normally use $p = 2$ and occasionally use $p = 4$. To estimate the instantaneous frequency, we use a kernel smoother of order $(1, 3)$.

The moment conditions (2.3) are also satisfied by the phase difference estimators of Boashash [2]. Boashash’s estimators are chosen to have the shortest possible length: $N = p + 1$. As a result, these phase difference estimators have near minimal bias error and are suitable for high resolution estimates in a noiseless signal. If noise is present, these phase difference estimators will appreciably amplify the noise. In a noisy signal, our kernel estimators reduce the variance by averaging over many more data points than the kernel order, $p$. 

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The variance of the kernel estimator is
\[ \text{Var} \left[ \hat{g}(q)(t, \mu) \right] = \frac{\sigma^2 m_2(\mu)}{Nh^{2q+1}}, \tag{2.4} \]
where \( m_2(\mu) = ||\mu||^2 \times (Nh) \sim \int K(s)^2 ds. \) Expanding \( g(t) \) in a Taylor series about \( g(t) \), the bias of a kernel smoother of order \((q, p)\) is
\[ E \left[ \partial_q^p g(t) \right] - \partial_q^p g(t) = C_{q,p} \partial_q^p g(t) h^{p-q}. \tag{2.5} \]
The leading order total squared error of \( \partial_q^p g(t) \) is
\[ L^2(\partial_q^p g(t); \mu) = C_{q,p}^2 |\partial_q^p g(t_j)|^2 h^{2(p-q)} + \frac{\sigma^2 m_2(\mu)}{Nh^{2q+1}}, \tag{2.6} \]
where the corrections are \( \mathcal{O}(h^{2(p-q)+1}) \). Solving (2.6) for the optimal value of the kernel scale size yields
\[ h_o(\mu) = \left[ \frac{2q+1}{2(p-q)} C_{q,p}^2 \right] \left( \frac{1}{2p+1} \right)^{2p+1} \right] \frac{1}{\sigma^2 m_2(\mu)} \tag{2.7} \]
For this choice of kernel width, \( h_o \), the total squared error of (2.2) is proportional to
\[ L^2(\partial_q^p g(t)) \sim M_{q,p} |\partial_q^p g(t_j)|^{2(2q+1)} \left( \frac{\sigma^2 m_2(\mu)}{N} \right)^{2(p-q)/(2p+1)}, \tag{2.8} \]
where \( M_{q,p} \equiv \left( \frac{2q+1}{2(p-q)} \right)^{2(p-q)/(2p+1)} + \left( \frac{2q+1}{2p+1} \right)^{2(p-q)/(2p+1)}. \) The optimal \( h \) is proportional to \( N^{-1/(2p+1)} \), and the total squared error, \( L^2(\partial_q^p g) \), is proportional to \( N^{-2(p-q)/(2p+1)} \). If \( g(t) \) has \( p \) continuous derivatives, where \( q \leq p \leq p \), the optimal bandwidth scales as \( N^{-1/(2p+1)} \), and the total squared error is proportional to \( N^{-2(p-q)/(2p+1)} \). This convergence rate is optimal for functions with precisely \( p \) continuous derivatives [22].

In [21] and Appendix B, we evaluate the kernel shape (under appropriate constraints) which minimizes the expected error. In the high sampling rate limit, the kernel shapes which minimize the local expected loss (as given by (2.6)) are independent of the kernel halfwidth and can be explicitly evaluated. (See bibliography in [21].) For \( p = q + 2 \), the limiting shape of the optimal kernel is
\[ K(t) = \gamma [P_q(t) - P_{q+2}(t)], \tag{2.9} \]
where \( P_q(t) \) and \( P_{q+2}(t) \) are the Legendre polynomials (or their discrete analog) and \( \gamma \equiv \prod_{k=1}^{q+1} \gamma(\frac{k}{2}). \) In (2.9), the estimation point is at \( t = 0 \) and the kernel support is \([-1, 1]\). For \( q = 0 \), Eq. (2.9) reduces to \( K(t) = \frac{\pi}{4}(1 - t^2) \), and for \( q = 1 \), \( K(t) = \frac{15}{4}(1 - t^3) \). When the domain of the kernel smoother intersects the ends of the dataset,
the kernel requires the more general form: \( K(t, \frac{t-t_j}{h}) \) to continue to be of order \( (q, p) \). The appropriate edge kernels are given in [21].

We have derived the optimal kernel halfwidth assuming that \( g^{(p)}(t) \) is known. In practice, \( g^{(p)}(t) \) is unknown and needs to be estimated. In the appendix, we describe data adaptive methods where we estimate \( \partial_t g(t) \) using a higher order kernel of order \( (p, p+2) \) and substitute \( \hat{\partial}_t g(t) \) into (2.7). More detailed treatments of kernel estimation can be found in [7-9,13-15,17,18,21].

### III. INSTANTANEOUS FREQUENCY ESTIMATES

We now consider models where both the amplitude, \( A(t) \), and the instantaneous frequency, \( \phi'(t) \), are evolving slowly with respect to both the sampling rate and the characteristic oscillation frequency, \( \omega_o \). The measured data satisfies

\[
y_j = A(t_j) \cos(\phi(t_j)) + \tilde{e}_j, \quad j = 1, \ldots N,
\]

where \( \tilde{e}_j \) is i.i.d. noise with variance \( \sigma^2 \). We define the analytic signal, \( z = A[x] = x + iH[x] \), where H is the Hilbert transform. We assume that \( \phi(t) = \omega_o t + \phi_o + \tilde{\phi}(t) \), where the characteristic frequency, \( \omega_o \), is given. In practice, we iterate on \( \omega_o \) with the new value of \( \omega_o \) being the previous estimate of the instantaneous frequency.

We assume that \( A(t) \) and \( \phi(t) \) satisfy the bandwidth conditions [2]: \( \mathcal{F}[A(t)] \) vanishes for \( |\omega| > \omega_b \) and \( \mathcal{F}[\cos(\phi(t))] \) vanishes for \( |\omega| < \omega_b \), where \( \mathcal{F} \) is the Fourier transform and \( \omega_b \) is fixed. In this case, the analytic signal has the phasor representation:

\[
z_j = A(t_j) \exp(i\phi(t_j)) + \epsilon_j, \quad j = 1, \ldots N,
\]

where \( \epsilon \equiv A[e] \). The Hilbert transform couples the \( \epsilon_j \) so that they are not independent. We initially consider the case when the model of (3.2) holds with \( \text{Re}[\epsilon_j] \) and \( \text{Im}[\epsilon_j] \) as independent random variables with variance \( \sigma^2 \). In this case, the exact distribution of the phase is known. Because the distribution of \( \exp(i \arg(z_j)) \) is more nearly Gaussian distributed than is \( \arg(z_j) \) [23], we smooth and differentiate \( \exp(i \arg(z_j)) \) instead of \( \arg(z_j) \).

To apply the optimal kernel smoother theory of Sec. II to (3.2), we require that the sampling rate is fast with respect to the characteristic evolution time. To remove the \( \exp(i\omega_o t) \) modulation from the sampling rate constraint, we demodulate the data about the central frequency. We define \( \tilde{z}_j \sim z_j e^{-i(\omega_o t + \phi_o)} \). By kernel smoothing \( \tilde{z}_j \), we can estimate the real and imaginary parts of \( A(t) e^{i\tilde{\phi}(t)} \). However, for time-frequency representations, we need the modulus, \( A \), and instantaneous frequency, \( \phi'(t) \). We now describe our kernel estimation scheme for the instantaneous frequency.
We begin by computing \( u_j = \exp(i \arg(\tilde{z}_j)) \). Provided that the signal to noise ratio, \( s = \frac{\sigma^2}{\sigma^2 + \sigma^2} \), is high, \( u_j \) is approximately distributed as \( u_j \sim N[\exp(i \tilde{\phi}(t_j)), \sigma^2_\tilde{\phi}] \), where \( \sigma^2_\tilde{\phi} = \frac{\sigma^2}{\sigma^2 + \sigma^2} \). We then estimate \( \exp(i \tilde{\phi}(t)) \) and \( \partial_t \exp(i \tilde{\phi}(t)) \) using kernel smoothers of orders \((0,2)\) and \((1,3)\) respectively. To apply (2.7) \& (2.8), we make the substitutions: \( \exp(i \tilde{\phi}(t)) \rightarrow g(t) \) and \( \sigma^2_\tilde{\phi} = \frac{\sigma^2}{\sigma^2 + \sigma^2} \rightarrow \sigma^2_\phi \).

Using these results, our instantaneous frequency estimate is

\[
\hat{\phi}'(t) = \omega_o + \text{Im} \left[ \frac{\partial_t \exp(i \tilde{\phi}(t))}{\exp(i \tilde{\phi}(t))} \right],
\]

Since our initial guess of the centering frequency, \( \omega_o \), may be inaccurate, we can iterate the local kernel estimates by replacing \( \omega_o \) with \( \omega_o + \hat{\phi}'(t) \). The error estimates for \( \hat{\phi}' \) are dominated by the error in \( \partial_t \exp(i \tilde{\phi}(t)) \). Thus the optimal halfwidth to estimate \( \partial_t \exp(i \tilde{\phi}(t)) \) is

\[
h_o(\mu) = \left( \frac{3}{4} \frac{\sigma^2_\phi m_2(\mu)}{C_{1,3} N |\partial_t \exp(i \tilde{\phi})|^2} \right)^{\frac{1}{7}}.
\]

The resulting error in \( \partial_t \tilde{\phi}(t) \) is

\[
L^2(\partial_t \tilde{\phi}(t_j)) \sim M_{1,3} |C_{1,3} \partial_t^3 \exp(i \tilde{\phi})|^{\frac{6}{7}} \left( \frac{\sigma^2_\phi m_2(\mu)}{N} \right)^{\frac{3}{7}}.
\]

Our work generalizes the kernel estimators of Lovell & Williamson [12] by including bias in estimate of the error and by applying the resulting optimal kernel theory. L & W extract pointwise estimates of \( \phi'(t_j) \) from \( \arg(z_{j+1}) - \arg(z_{j-1}) \) and then smooth these estimates. We have partially reversed the order of the smoothing and nonlinear transformations by working with \( \exp(i \arg(z_j)) \). Our basic algorithm is to choose the kernel smoother/derivative estimator by minimizing the expected loss including the bias error. This algorithm can also be applied to other orderings of the smoothing and nonlinear transformations such as that of L & W. However, the distribution of \( \exp(i \arg(z_j)) \) matches the hypotheses of kernel smoothing better than most other choices [23].

IV. EVOLVING SINUSOIDS IN COLORED NOISE

In Section III, we noted that the Hilbert transformed noise is correlated. To treat this situation, we review kernel estimation with an arbitrary covariance structure [A]: \( \text{Cov}[e_j,e_k] = \sigma^2 R_{j,k} \). In this case, the variance of the kernel estimate of the demodulated data (generalizing (2.4)) is

\[
\text{Var} \left[ \hat{g}(q)(t) \right] = \frac{\sigma^2}{Nh^{2q+1}} \sum_{j,k=1}^N \mu_j R_{j,k} \mu_k \exp(i \omega_o(k-j)).
\]
When the errors are autocorrelated, $R_{j,k} = R(j - k)$, (4.1) can be reformulated in the frequency domain as
\[
\text{Var} [\hat{g}(t)] = \frac{1}{2\pi h^2} \int_{-\pi}^{\pi} S(\omega) |U(\omega - \omega_o)|^2 d\omega, \tag{4.2}
\]
where $U(\omega)$ is the Fourier transform of $\mu$. If $U(\omega)$ is localized near the zero frequency, the variance of the estimate will depend almost exclusively on the spectral density near $\omega_o$, $S(\omega_o)$. Thus $\text{Var}[\hat{g}(t; \mu)] = S(\omega_o) m_2(\mu)/Nh^{2q+1}$.

Returning to the Hilbert transform problem, we use the covariance structure of the Hilbert transform: $\text{Cov}[\epsilon_j, \epsilon_k] = 2\sigma^2 (\delta_{j,k} + i \xi(j - k))$ and $\text{Cov}[\epsilon_j, \epsilon_k] = 0$, where $\xi(j - k) = \frac{2}{\pi(j - k)}$ if $j - k$ is odd and zero otherwise. Thus $E[|\text{Re}(\epsilon_j)|^2] = E[|\text{Im}(\epsilon_j)|^2] = \sigma^2$. Equation (4.2) shows that $\text{Im}(\mathbf{R})$ does not contribute to the variance of the kernel estimate of $A(t)e^{i\phi(t)}$. Thus $\text{Var}[\hat{A}(t)e^{i\phi}] = 2\sigma^2 m_2(\mu)/Nh^{2q+1}$. The factor of two arises because we are estimating both the real and imaginary parts.

In kernel smoothing $\exp(i \arg(z_j))$, the phase error are distributed as
\[
e_{\phi,j} \sim \frac{e^{i(\phi(t_j)+\pi)}}{\sqrt{A^2 + \sigma^2}} \text{Im}[\epsilon_j e^{-i\phi(t_j)}]. \tag{4.3}
\]
Thus $e_{\phi,j}$ has an approximately normal distribution with covariance [12]:
\[
\text{Cov}[e_{\phi,j}, e_{\phi,k}] = \frac{\sigma^2}{A^2 + \sigma^2} \left[ \delta_{j,k} - \xi(j - k)e^{i(\phi(t_j)-\phi(t_k))} \sin (\phi(t_j) - \phi(t_k)) \right]. \tag{4.4}
\]
We substitute (4.4) for (2.4) and apply the resulting kernel halfwidths to (3.3).

V. MULTIPLE EVOLVING SINUSOIDS

We now consider signals which consist of a sum of slowly evolving sinusoids in colored noise:
\[
y_j = \sum_{\ell=1}^{L} A_{\ell}(t_j) \cos(\phi_{\ell}(t_j)) + \tilde{e}_j. \tag{5.1}
\]
We assume that $\phi_{\ell}(t) = \omega_{\ell} t + \phi_{\ell,0} + \tilde{\phi}_{\ell}(t)$, where the frequencies, $\omega_{\ell}$, are given and distinct. We require the bandwidth conditions: $\mathcal{F}[A_{\ell}(t)]$ vanishes for $|\omega| > \omega_b$ and $\mathcal{F}[\cos(\phi_{\ell}(t))]$ vanishes for $|\omega - \omega_{\ell}| < \omega_b$, where $\omega_b$ is fixed. Furthermore, we assume that the supports of $\mathcal{F}[A_{\ell}(t)\cos(\phi_{\ell}(t))], \ell = 1, \ldots, L$, have negligible overlap.

We initially estimate $A_{\ell}\cos(\phi_{\ell})$, $\ell = 1, \ldots, L$, ignoring the bias from the other sinusoids. We then attempt to remove the effect of the other sinusoids. Given estimates, $A_{\ell}\cos(\phi_{\ell})(t)$, of the other sinusoids, we define the $\ell$th corrected data set, $\tilde{y}_j^{(\ell)}$, to be
\[
\tilde{y}_j^{(\ell)} \equiv y_j - \sum_{\ell' = 1, \ell' \neq \ell}^{L} A_{\ell'} \cos(\phi_{\ell'}(t_j)). \tag{5.2}
\]
We estimate the $\ell$th instantaneous frequency using the corrected dataset. The correction and estimation may be iterated.

To determine when we can neglect the bias from the other line frequencies, we compare the relative size of the bias from the time evolution of $A_{\ell}$, with coherent interference from the other line frequencies. We denote $A_{\ell} \cos(\phi_{\ell'}) \equiv A_{\ell'} \cos(\phi_{\ell'}) - A_{\ell'} \cos(\phi_{\ell'})$, which is the error in the kernel estimate of $A_{\ell'} \cos(\phi_{\ell'})$. We assume that the support of the Fourier transform of $A_{\ell'} \cos(\phi_{\ell'})$ is contained in the interval $[-\omega_b, \omega_b]$. We can neglect the interference of the $\ell'$ line in (2.7) if

$$|C_{q,p} \partial_t^p A_{\ell}(t) h^p| \gg \left| \tilde{A}_{\ell'}(t) U(|\omega_{\ell'} - \omega_{\ell'}| - \omega_b) \right|,$$

(5.3)

where $U(\omega)$ is the Fourier transform of $\mu$. The expected size of the error in removing the $\ell'$th sinusoid from the estimate of the $\ell$th sinusoid, $A_{\ell'} \cos(\phi_{\ell'})$, is given by (2.8). The bias from the other line frequencies can be included as a correction to (2.6).

The kernels of (2.9) and Appendix B are designed to minimize the total error under the assumption that the signal is well resolved. These "minimal loss" kernels tend to have larger frequency sidelobes because their design criterion does not explicitly penalize sidelobes. Many other digital filters and differentiators [5] have been designed to have power spectra which decay rapidly away from zero frequency, and thereby reduce the interference terms because $U(\omega_{\ell'} - \omega_{\ell}) \ll 1$.

If interference from sidelobes is significant, we replace the minimal loss kernels with kernel which satisfy $U(\omega_{\ell'} - \omega_{\ell}) \ll 1$ for broad banded bias protection. Our particular choice is to construct a kernel from the first $(p + 1)$ sinusoidal tapers of [19]. The sinusoidal tapers are defined by $v_{n}^{(k)} = \sqrt{2} N + 1 \sin\left(\frac{\pi kn}{N + 1}\right)$ where $k$ is the taper number, $N$ is the length of the kernel and $n = 1 \ldots N$. Imposing (2.3) and the condition that the kernel vanish at the ends of its support gives $p + 1$ conditions and $p + 1$ free parameters.

VI. DISCUSSION

In this article, we have treated time-frequency distributions as an estimation problem for slowly varying sinusoids. This model generalizes the Rife and Boorstyn problem [20] of estimating a pure sinusoid in noise. This approach is valid and appropriate when we know a priori that the signal consists of one or more coherent signals in a noise background. When the signal is incoherent with a slowly varying spectral density, the evolutionary spectrum of Priestley [16] is the appropriate model. In [17], we use a two-dimensional cross-product kernel smoother to estimate the evolutionary spectrum, $S(\omega, t)$.

To our knowledge, none of the previous instantaneous frequency estimators [2] include the effects of bias error from the time variation of the frequency in their
analyses. The multiple stage kernel estimators of the appendix yield the optimal rate of convergence for nonparametric estimation. When the instantaneous frequency is known a priori to have a particular parametric form such as a “chirp”, more accurate estimators are possible. The well-known Cramer-Rao bound of Rife and Boorstin [20] applies when the instantaneous frequency is time independent.

We determine the instantaneous frequency by estimating $e^{i\hat{\phi}(t)}$ and $\partial_t e^{i\hat{\phi}(t)}$ with kernel smoothers. The bias error is proportional to $h^2 \partial_t^3 e^{i\hat{\phi}(t)}$. We demodulate by $e^{-i(\omega_0 t + \varphi_0)}$ to reduce this bias. The variance of the estimate of $\partial_t e^{i\hat{\phi}(t)}$ scales as $\frac{\sigma^2}{A^2 N h_t^3}$. Minimizing the expected error yields the optimal kernel halfwidth, $h_{1,3} \sim \left[ \frac{\sigma^2}{A^2 N |\partial_t^2 e^{i\hat{\phi}(t)}|^2} \right]^{1/3}$ and $L^2(\partial_t \hat{\phi}(t_j)) \sim |\partial_t^2 e^{i\hat{\phi}}| \left( \frac{\sigma^2}{A^2 N} \right)^{2/3}$.

Our optimal kernel smoother approach has two disadvantages. First, it is computationally more intensive than many of the alternative methods. Second, the asymptotic expressions are based on a Taylor series expansion of $e^{i\hat{\phi}(t_j)}$ about $e^{i\phi(t)}$. If $h_t$ is the radius of validity of the third order expansion,

$$e^{i\hat{\phi}(t_j)} \sim e^{i\phi(t_j)} + (t_j - t)\partial_t e^{i\hat{\phi}(t)} + \frac{(t_j - t)^2}{2} \partial_t^2 e^{i\phi(t)} + \frac{(t_j - t)^3}{6} \partial_t^3 e^{i\hat{\phi}(t)} + o((t_j - t)^{3+\delta}) ,$$

our analysis shows that the optimal halfwidth is given by (3.4) if $h_o$ is less than $h_t$. Since the optimal halfwidth scales as the 1/7 root of the signal to noise ratio divided by the number samples per characteristic time, we are often in the limit where $h_o > h_t$. In this case, our analysis shows only that the best kernel halfwidth is greater than or equal to $h_t$. The failure of the Taylor series approximation often corresponds to an order one phase difference between $e^{i\hat{\phi}(t_j)}$ and $e^{i\phi(t)}$. Thus the lower bound on the kernel halfwidth is useful and is often close to optimal value.

APPENDIX A: DATA ADAPTIVE MULTIPLE STAGE KERNEL ESTIMATORS

In this appendix, we construct multiple step kernel estimators which have optimal relative convergence rates. We return to the case of a nonmodulated real signal ($\hat{\phi}(t) \equiv 0$). We consider data adaptive estimators which estimate $g^{(q)}(t)$ in the final stage with a kernel of order $(q, p)$ where the kernel parameters are determined with a kernel pre-estimate of $g^{(p)}(t)$ of order $(p, p + 2)$. The more accurate the estimate of $g^{(p)}(t)$ is, the closer the expected loss of the “plug-in” kernel estimator will be to the optimal value with known $g^{(p)}(t)$. If the estimated value of $\hat{g}^{(p)}(t)$ differs from $g^{(p)}(t)$ by $O(N^{-\alpha})$, then

$$E \left[ |\hat{g}(t|\hat{h}_{0,q}) - g(t)|^2 \right] \sim \left( 1 + o(C_r^2 N^{-2\alpha}) \right) E \left[ |\hat{g}(t|h_{0,q}) - g(t)|^2 \right] , \quad (A1)$$

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where $h_{0,q}$ is given by (2.7) and $\hat{h}_{0,q}$ is its empirical estimate. We say that $C_r N^{-\alpha}$ is the relative convergence rate of the kernel halfwidth estimate. (The convergence is relative to the rate with the known, optimal value of $h$.) If the relative convergence rate tends to zero as $N \to \infty$, then the estimate is asymptotically efficient.

To achieve the optimal rate of convergence, the “pre-estimates” of $\partial_t^p g(t)$ are estimated with a different kernel length than the estimate of $g(t)$. Equation (2.7) shows that the optimal $h$ scales as $N^{-1/5}$ for kernels of order $(0,2)$ and $(2,2)$, and that $h$ scales as $N^{-1/9}$ for kernels of order $(0,4)$ and $(2,4)$. For pre-estimates with kernels of order $(p,p+2)$, the relative convergence rate is $O(N^{-2/(2p+5)})$.

In the first step of any multistep estimation scheme, the kernel halfwidth for the next step needs to be selected. There are three common methods to initialize the kernel smoother: characteristic time scale initialization, parametric fit initialization, and goodness of fit initialization. In the characteristic time scale initialization, the signal is assumed to have a characteristic amplitude, $\tilde{A}$, and to vary on a characteristic time scale, $\tau$, where $\tilde{A}$ and $\tau$ are given a priori. In the initial halfwidth estimate, $\tilde{A}/\tau^p$ is substituted for $g^{(p)}(t)$ in (2.7).

In the parametric fit initialization, $g(t)$ is fit with a prescribed functional form with a small number of free parameters. The parametric fit is then substituted into (2.7) to initialize the kernel estimate.

In the goodness of fit initialization, $h$ is determined by minimizing an expression which includes the residual sum of squared errors but which corrects for the number of degrees of freedom which are used in a kernel smoother [9,14,16,18].

In general, both the characteristic scale initialization and the parametric fit initialization produce order one errors in $g^{(p)}(t)$. The goodness of fit criteria have a slow relative rate of convergence, $O(N^{-1/10})$ [9]. In contrast, the pilot kernel estimator of order $(p,p+2)$ has a relative convergence rate of $O(N^{-2/9})$. Therefore we recommend a multiple stage kernel estimator where $g^{(p)}$ is estimated prior to the estimation of $g(t)$.

For simplicity, we consider a two stage estimate with a characteristic scale initialization. We begin the adaptive estimate by selecting a global halfwidth, $h_{2,4}$, using a characteristic scale initialization. The resulting estimate of $\partial_t^2 g$ achieves the optimal convergence rate of $L^2(\partial_t^2 g) \sim C_2 N^{-4/9}$. Because we use an arbitrary ansatz, $\partial_t^4 g \sim \tau^{-4}\tilde{A}$, in the optimal halfwidth formula, the convergence rate differs from the optimal value by an order one factor.

To estimate $\partial_t g(t)$ (as in instantaneous frequency estimation) we begin with the ansatz that $\partial_t^5 g \sim \tilde{A}/\tau^5$. We insert this ansatz into (2.7) to determine a halfwidth to estimate $\partial_t^3 g(t)$ using a kernel of order $(3,5)$. We then use the estimate of $\partial_t^3 g(t)$ to determine a halfwidth for a kernel of order $(1,3)$.
Our estimate of \( g(t) \) achieves the optimal convergence rate of \( L^2(\hat{g}(t)) \sim CN^{-4/5} \) and the optimal relative convergence rate of \( N^{-4/9} \). A similar, slightly more elaborate, adaptive estimator was proposed by M"uller & Stadtmüller (M-S) \cite{15}. M-S begin by determining a global halfwidth for the (2,4) kernel using the Rice criterion. Goodness of fit initializations improve on the characteristic time scale initialization by selecting an asymptotically efficient global halfwidth for the estimate of \( \partial^2_t g(t) \). The M-S scheme is actually a three stage estimator and the computational effort required on the initial step can be large.

Since \( g(t) \) is \( C^p \), we want \( \hat{g} \) to be smooth as well. The kernel halfwidth of (2.7) using the “plug-in” derivative estimate in an infinite kernel halfwidth. Also, \( \hat{\partial^p_t g} \) is at best is continuous, but need not be smooth. Therefore we convolve \( |\hat{\partial^p_t g}|^2 \) with a regular kernel, \( G(\cdot) \), with \( \int G(s) ds = 1 \). We choose the halfwidth of \( G \) to be \( \tilde{h} \equiv \max_t \{ \hat{h}(t) \} \). To apply our estimate of \( g^{(p)}(t) \) to (2.7), we make the substitution,

\[
|\hat{\partial^p_t g}|^2 \rightarrow \frac{1}{2\tilde{h}} \int_{-\tilde{h}}^{\tilde{h}} G(\frac{s}{\tilde{h}})|\hat{\partial^p_t g}(t-s)|^2 ds . \tag{A2}
\]

This smoothed estimate is asymptotically equal to \( |\partial^p_t g(t)|^2 \), but robustifies the empirical halfwidth for moderate values of \( Nh \). When \( \partial^p_t g(t) \) nearly vanishes, the smoothing in (A2) models the effect of the higher order bias terms. In [15], the robust estimator of \( \partial^2_t g \) in (A2) is replaced by a simpler, but less accurate upper cutoff.

**APPENDIX B: OPTIMAL KERNEL SHAPES**

In (2.6), the expected loss is a quadratic function of the kernel, \( \mu \). For a fixed kernel width, we can minimize the expected loss subject to the constraints that \( \mu \) is of type \((q,p)\). For a given, positive definite, symmetric matrix, \( \bar{R} \), we define the minimal \( \bar{R} \) kernel of order \((q,p)\), \( \mu \) as the minimizer of

\[
\mu^T \bar{R} \mu + \sum_{m=0}^{p-1} \alpha_m (\mu \cdot s_m - \delta_{m,q}) , \tag{B1}
\]

where \( \alpha_m , m = 0 \ldots p - 1 \) are \( p \) Lagrange multipliers and \( T \) denotes “transpose”. When \( \bar{R} \equiv \sigma^2 I \), we call \( \mu \) the minimal variance kernel of type \((q,p)\) and the solution is given in [13]. The approximate expected loss is given by:

\[
L(\mu) \sim \mu^T R \mu + |\hat{\partial^p_t g(t)}|^2 |\mu \cdot s_p|^2 . \tag{B2}
\]

If both \( R \) and \( \hat{\partial^p_t g(t)} \) are given, the minimal loss kernel, \( \mu_L \), corresponds to the choice of \( \bar{R} = R \equiv R + |\hat{\partial^p_t g(t)}|^2 s^{(p)} s^{(p)T} \). Thus the expected loss functional differs from the minimum variance functional by a rank one perturbation.
Thus the minimal $\bar{R}$ norm kernel satisfies

$$\bar{R} \mu = -\sum_{m=1}^{p-1} \alpha_m s^{(m)}, \quad (B3)$$

with the linear constraints of (2.3). We define the $N \times p$ matrix, $S(p) \equiv (s^{(0)}, \ldots, s^{(p-1)})$; i.e. $S(p)$, the matrix of the first $p$ moment vectors, $s^{(m)}$, $0 \leq m < p$. We also define the $p$ vector, $e^{(p)}_q$, to be the unit $p$ vector in the $q$ direction, and $\alpha$ to be the $p$ vector of Lagrange multipliers. The solution of (B2) is

$$S^T \bar{R}^{-1} S \alpha = -e^{(p)}_q, \quad (B4)$$

$$\mu = \bar{R}^{-1} S(S^T \bar{R}^{-1} S)^{-1} e^{(p)}_q. \quad (B5)$$

Thus the minimal loss is $e^{(p)T}_q (S^T \bar{R}^{-1} S)^{-1} e^{(p)}_q$. In [21], the optimal kernel shapes given by (B5) are explicitly evaluated in the large sample limit.

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