Some properties of the equation of fast diffusion and its multidimensional exact solutions

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Abstract

The invariance for the equation of fast diffusion in the 2D coordinate space has been proved, and its reduction to the 1D (with respect to the spatial variable) analog is demonstrated. On the basis of these results, new exact multi-dimensional solutions, which are dependent on arbitrary harmonic functions, are constructed. As a result, new exact solutions of the well-known Liouville equation – the steady-state analog for the fast diffusion equation with the linear source – have been obtained. Some generalizations for the systems of quasi-linear parabolic equations, as well as systems of elliptic equations with Poisson interaction, which are applied in the theory of semiconductors, are considered.

Introduction

The paper investigates the equation of fast diffusion

\[ u_t = \Delta \ln u, \quad u = u(x, y, t), \]

in the 2D coordinate space, which is characteristic of many applied problems, for example, in the description of spreading of superfine monomolecular layers of liquid under the influence of Van der Waals forces [1]. It arises in modeling of diffusion phenomena in semiconductors, polymers, etc. It is known [2] that eq. (1) is special from the viewpoint of the group theory since it assumes finite-dimensional algebra of point symmetries. This means that eq. (1) in the 2D coordinate space possesses an infinite stock of invariant solutions [3]. In the literature, the relation (1) is sometimes called the Ricci equation.

This paper considers some properties of (1). On the basis of these properties some new – exact and multidimensional – solutions of this equation, which are dependent on arbitrary harmonic functions, are constructed.

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1 Invariance of the equation of fast diffusion

Let invariance be understood as nonvariability of the form of this equation under the effect of any transformation.

Definition[4]. Harmonic functions $\xi(x,y)$ and $\eta(x,y)$ are called conjugate in the simple connected domain $D$ if the function $F(z) = \xi(x,y) + i\eta(x,y)$ is some analytical function of the argument $z = x + iy$ in the domain $D$.

Conjugate harmonic functions are related by Cauchy-Riemann equations

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y}, \quad \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x},$$

define one another everywhere in $D$ with the to an additive constant and, consequently, possess the following properties

$$(\nabla \xi, \nabla \eta) = 0, \quad |\nabla \xi|^2 = |\nabla \eta|^2. \quad (2)$$

The harmonic polynomials

$$\xi(x,y) = (x^2 + y^2)^{\frac{n}{2}} \cos(n\varphi), \quad \eta(x,y) = (x^2 + y^2)^{\frac{n}{2}} \sin(n\varphi),$$

where

$$\varphi = \arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right), \quad \text{or} \quad \varphi = \arcsin \left( \frac{y}{\sqrt{x^2 + y^2}} \right), \quad n \in \mathbb{N},$$

are the simplest examples of the conjugate harmonic functions.

Lemma 1. If the function $\eta(x,y)$ is harmonic then the function $\ln |\eta_x^2 + \eta_y^2|$ is also harmonic.

Proof. For the purpose of comfort let us introduce the denotation $\psi(x,y) = \eta_x^2 + \eta_y^2$.

$$\Delta \ln \psi = \left[ \psi (\psi_{xx} + \psi_{yy}) - (\psi_x^2 + \psi_y^2) \right] \psi^{-2}. \quad (3)$$

By consecutive computing necessary partial derivatives, we obtain

$$\psi_{xx} + \psi_{yy} = 2 \left( \eta_{xx}^2 + \eta_{yy}^2 \right) + 4\eta_{xy}^2 + 2\eta_{x} (\eta_{xx} + \eta_{yy})_x + 2\eta_{y} (\eta_{xx} + \eta_{yy})_y,$$

$$\psi_x^2 + \psi_y^2 = 8\eta_x \eta_y (\eta_{xx} + \eta_{yy}) + 4\eta_x^2 (\eta_{xx} + \eta_{xy}^2) + 4\eta_y^2 (\eta_{yy}^2 + \eta_{xy}^2).$$

Since due to the harmonic character of $\eta_{xx} = -\eta_{yy}$, the latter relations will, respectively, assume the following form:

$$\psi_{xx} + \psi_{yy} = 4 \left( \eta_{xx}^2 + \eta_{xy}^2 \right),$$

$$\psi_x^2 + \psi_y^2 = 4 \left( \eta_{xx}^2 + \eta_{xy}^2 \right) \left( \eta_{xx}^2 + \eta_{xy}^2 \right).$$
Hence, expression (3) turns identically zero. \(\Box\)

Now let us formulate one of the main results of the work.

**Theorem 1.** If \(\xi = \xi(x,y), \eta = \eta(x,y)\) are conjugate harmonic functions then the equation of fast diffusion (1) is invariant with respect to the transformation

\[
u(x,y,t) = \rho(x,y)v(\xi,\eta,t),
\]

where \(\rho(x,y) = |\nabla \eta|^2\), i.e. under the effect of (4) it transforms into itself

\[
v_t = \Delta v, \ln v.
\] (5)

**Proof.** After substitution of relation (4) into eq. (1) and simple computations, on account of the properties of conjugate harmonic functions (2), we obtain

\[
\rho v_t = \Delta_{xy} \ln \rho + \rho \Delta_{\xi\eta} \ln v.
\]

Hence, due to Lemma 1, relation (5) immediately follows. \(\Box\)

Therefore, expression (4) is a form of autotransformation for the equation of fast diffusion (1) and, consequently, it is the formula of branching solutions.

**Example 1.** Consider an exact solution of eq. (5) of the form [5, 6]

\[
v(\xi,\eta,t) = 2 \text{th}(t) \xi + \eta \text{th}^2(t).
\]

By applying transformation (4) with arbitrarily chosen harmonic functions to this solution let us construct a few new exact nonautomodel explicit solutions of eq. (1), which are anisotropic with respect to spatial variables

\[
\begin{align*}
u(x,y,t) &= 2 \frac{\text{th}(t)}{\xi^2 + \eta^2 \text{th}^2(t)}, \\
u(x,y,t) &= 2 \frac{\text{th}(t)}{1 + \text{tg}^2(x) \text{tg}^2(y) \text{th}^2(t)}, \\
u(x,y,t) &= 18 \frac{\text{tg}^2(x^3 - 3xy^2) + \text{th}^2(3x^2y - y^3) (x^2 + y^2) \text{th}(t)}{1 + \text{tg}^2(x^3 - 3xy^2) \text{th}^2(3x^2y - y^3) \text{th}^2(t)}, \\
u(x,y,t) &= 18 \frac{\text{tg}^2(\exp(3x)a(y)) + \text{th}^2(\exp(3x)b(y)) \exp(x)\text{th}(t)}{1 + \text{tg}^2(\exp(3x)a(y)) \text{th}^2(\exp(3x)b(y)) \text{th}^2(t)},
\end{align*}
\]

where the following denotations are used

\[
a(y) = \cos^3(y) - 3 \cos(y) \sin^2(y), \quad b(y) = 3 \cos^2(y) \sin(y) - \sin^3(y).
\]
The solutions obtained are significant in virtue of the fact that for \( t \to +\infty \) they are stabilized in the form of steady-state solutions.

A result similar to that of Theorem 1 can be extended onto some class of systems of quasilinear parabolic equations. Hence, the following theorem is valid.

**Theorem 2.** Let \( \xi = \xi(x,y) \), \( \eta = \eta(x,y) \) are conjugate harmonic functions, and \( f_i(u_1, u_2, \ldots, u_m), i = 1, 2, \ldots, n, m \leq n \) are homogeneous, the degree of homogeneity being one, i.e.

\[
f_i(\lambda u_1, \lambda u_2, \ldots, \lambda u_m) = \lambda f_i(u_1, u_2, \ldots, u_m). \tag{6}
\]

The n the system of n equations

\[
\frac{\partial u_i}{\partial t} = \Delta_{xy} \ln u_i + f_i(u_1, u_2, \ldots, u_m) \tag{7}
\]

is invariant with respect to the transformations

\[
u_i(x, y, t) = \rho(x, y)v_i(\xi, \eta, t), \tag{8}
\]

where \( \rho = |\nabla \eta|^2 \), i.e. it transforms into itself

\[
\frac{\partial v_i}{\partial t} = \Delta_{\xi\eta} \ln v_i + f_i(v_1, v_2, \ldots, v_m). \tag{9}
\]

**Proof.** By substituting functions (8) into eqs. (7) and taking account of the conditions of homogeneity (6), we obtain the system

\[
\rho \frac{\partial v_i}{\partial t} = \Delta_{xy} (\ln \rho + \ln v_i) + \rho f_i(v_1, v_2, \ldots, v_m), \tag{10}
\]

furthermore, when using the properties of the conjugate harmonic functions (2), it can readily be shown that

\[
\Delta_{xy} \ln v_i = \rho \Delta_{\xi\eta} \ln v_i.
\]

Therefore, from (10), due to Lemma 1, it is possible to obtain the system of equations (9). \( \square \)

2 Reduction of the equation of fast diffusion to its one-dimensional (with respect to the spatial variable) analog

**Theorem 3.** Let \( \eta(x,y) \) be an arbitrary harmonic function, which is different from the constant one. Then with the aid of the transformation

\[
u(x, y, t) = \left(\eta_x^2 + \eta_y^2\right) v(\eta, t), \tag{11}
\]
the equation of fast diffusion (1) can be reduced to its one dimensional (with respect to the spatial variable $\eta$) analog

$$\frac{\partial v}{\partial t} = \frac{\partial^2 \ln v}{\partial \eta^2}. \quad (12)$$

**Proof.** After substituting expression (11) into eq. (1), we obtain the equality

$$\psi (x, y) \frac{\partial v}{\partial t} = \Delta_{xy} (\ln \psi (x, y) + \ln v), \quad (13)$$

furthermore, it can easily be shown that

$$\Delta_{xy} \ln v = \psi (x, y) \frac{\partial^2 \ln v}{\partial \eta^2},$$

where $\psi (x, y) = \eta_x^2 + \eta_y^2$. Therefore, from equality (13), due to Lemma 1, we obtain eq. (12). $\square$

Consequently, by integrating eq. (12) and using formula (11), it is possible to construct a class of exact solutions of fast diffusion equations (1), which depend on an arbitrary harmonic function.

**Example 2.** Let $\eta (x, y)$ be a 4th degree harmonic polynomial, i.e. $\eta (x, y) = x^4 - 6x^2y^2 + y^4$, or $\eta (x, y) = 4(x^3y - xy^3)$. Hence $\eta_x^2 + \eta_y^2 = 16(x^2 + y^2)^3$. In this case, the transformation

$$u (x, y, t) = 16(x^2 + y^2)^3 v (\eta, t),$$

gives an anisotropic (with respect to spatial variables $x$ and $y$) solution of eq. (1), furthermore, the function $v (\eta, t)$ can be determined from the relation (12). Since with the aid of the substitution $v = w^{-1}$ eq. (12) can be reduced to the equation with quadratic nonlinearities

$$w_t = w w_{\eta\eta} - (w_\eta)^2,$$

it is possible, in addition, to construct some exact solutions of eq. (12) on the basis of the approach described in [7].

$$v (\eta, t) = \frac{1}{\lambda} \cdot \frac{\sqrt{k_1^2 + k_2^2 \sh (\lambda t)}}{k_1 \cos (\eta) + k_2 \sin (\eta) + \sqrt{k_1^2 + k_2^2 \ch (\lambda t)}},$$

$$v (\eta, t) = \frac{1}{\lambda} \cdot \frac{\sqrt{k_1^2 + k_2^2 \cos (\lambda t)}}{k_1 \cos (\eta) + k_2 \sin (\eta) - \sqrt{k_1^2 + k_2^2 \sin (\lambda t)}},$$

$$v (\eta, t) = \frac{1}{\lambda} \cdot \frac{\sqrt{k_1^2 - k_2^2 \cos (\lambda t)}}{k_1 \ch (\eta) + k_2 \sh (\eta) + \sqrt{k_1^2 - k_2^2 \sin (\lambda t)}}.$$
\[v(\eta, t) = \frac{1}{\lambda} \cdot \frac{\sqrt{k_1^2 - k_2^2 \text{sh}(\lambda t)}}{k_1 \text{ch}(\eta) + k_2 \text{sh}(\eta) - \sqrt{k_1^2 - k_2^2 \text{ch}(\lambda t)}}.\]

Here \(\lambda \neq 0, k_1 > k_2\) are arbitrary numerical parameters.

**Remark.** The proof of an analog of Theorem 3 for the system of quasilinear parabolic equations (7) has been described in the author’s paper [8].

It is known rather well [9, .84] that the Laplace equation in the 2D coordinate space is invariant with respect to the conformal transformation of independent variables

\[
\frac{\partial x}{\partial q_2} = \frac{\partial y}{\partial q_1}, \quad \frac{\partial x}{\partial q_1} = -\frac{\partial y}{\partial q_2},
\]

furthermore, \(\frac{\partial x}{\partial q_1}, \frac{\partial y}{\partial q_2}\) do not simultaneously turn zero. The system of parabolic coordinates

\[x = q_2^2 - q_1^2, \quad y = 2q_1q_2.\]

is an example of such transformation. Now, let us apply the transformation (14) to eq. (1) under scrutiny. The latter may be rewritten as follows:

\[u_t = f(q_1, q_2) \Delta_{q_1q_2} \ln u, \quad u = u(q_1, q_2, t).\]

Here the denotation \(f(q_1, q_2) = x_{q_1}^2 + x_{q_2}^2\) is used. Since due to (14) the function \(x = x(q_1, q_2)\) is harmonic, Lemma 1 implies that

\[\Delta_{q_1q_2} \ln f(q_1, q_2) = 0.\]

Consequently, the following theorem is valid.

**Theorem 4.** If relation (16) holds then with the aid of the transformation

\[u(q_1, q_2, t) = \frac{1}{f(q_1, q_2)} \left[ f_{q_1}^2 + f_{q_2}^2 \right] v(\eta, t), \quad \eta = \ln f(q_1, q_2),\]

the nonhomogeneous equation of fast diffusion (15) can be reduced to the homogenous and one-dimensional (with respect to the spatial variable \(\eta\)) equation (12).

The proof of this theorem is similar to that of Theorem 3. To the end of proving it is necessary to put \(\eta(q_1, q_2) = \ln f(q_1, q_2)\) in (11) written in terms of variables \(q_1\) and \(q_2\).

**Example 3.** The equation of nonlinear diffusion

\[u_t = \exp(x^2 - y^2) \Delta \ln u\]

or \(w_t = \exp(x^2 - y^2 - w) \Delta w,\)

where \(w(x, y, t) = \ln v(x, y, t)\), has the following exact solution, which is asymmetric with respect to the spatial variables

\[u(x, y, t) = 4(x^2 + y^2) \exp(x^2 - y^2)v(\eta, t), \quad \eta = x^2 - y^2.\]

Furthermore, the function \(v(\eta, t)\) satisfies eq. (12), some exact solutions of which have been given in Example 2.


3 Exact solutions of Liouville equation in the 2D coordinate space

It can easily be shown that principal results of the sections 1 and 2 can be directly generalized onto the equation of fast diffusion with a linear source (sink)

\[ u_t = \Delta \ln u - \lambda u, \]  
where \( \lambda \in \mathbb{R} \setminus \{0\} \). Since formulas (4), (11), (17) are obviously independent of time, Theorems 1, 3 and 4 remain valid for the steady-state (elliptical) eq. (18)

\[ \Delta \ln u = \lambda u, \]

which by the substitution \( u = \exp(\lambda w) \) may be reduced to the well-known Liouville equation:

\[ \Delta w = \exp(\lambda w). \]  

Proposition 1. The Liouville equation (19) is invariant with respect to the transformation

\[ w(x, y) = v(\xi, \eta) + \frac{1}{\lambda} \ln |\rho(x, y)|, \]  

where \( \rho = |\nabla \eta|^2 \), and \( \xi(x, y), \eta(x, y) \) are conjugate harmonic functions.

By applying transformation (11) to eq. (19) it is possible to obtain an ordinary differential equation (ODE), which can be easily integrated, and make sure that the following proposition is valid.

Proposition 2. The Liouville equation (19) in the 2D coordinate space has exact solutions of the form

\[ w(x, y) = \frac{1}{\lambda} \ln \left| \frac{2A^2}{\lambda} \left( \eta_x^2 + \eta_y^2 \right) \sec^2(A\eta) \right|, \]

\[ w(x, y) = \frac{1}{\lambda} \ln \left| \frac{2A^2}{\lambda} \left( \eta_x^2 + \eta_y^2 \right) \sech^2(A\eta) \right|, \]

where \( \eta(x, y) \) is an arbitrary harmonic function, which is different from the constant one, \( A, \lambda \in \mathbb{R} \setminus \{0\} \).

Similarly, from Theorem 4 it follows that for a steady-state nonhomogeneous equation (15) having a linear adjunct the following proposition is valid.

Proposition 3. The nonhomogeneous Liouville equation

\[ \Delta w = \eta(x, y) \exp(\lambda w), \]  

where \( \eta(x,y) \) is an arbitrary harmonic function different from the constant one, has the exact solution
\[
w(x,y) = \frac{1}{\lambda} \ln \left| \frac{3 \eta^2_x + \eta^2_y}{\eta^3} \right|.
\]

Note that exact solutions of the nonhomogeneous Liouville equation (21), when \( \eta(x,y) \) is a holomorphic function of special form, can be found in [10].

As a case of generalization of the results obtained in this section onto some different systems of equations consider the steady-state mathematical model of charge transfer, which implies Poisson interactions, known from the theory of semiconductors.

**Example 4.** The elliptical system of equations of the form
\[
\Delta u = \exp(u) + A|\nabla|^2, \quad \Delta v = \exp(v) - B|\nabla|^2, \\
\Delta = \exp(v) - \exp(u),
\]
has an exact solution of the form
\[
u(x,y) = \psi(\eta) + \ln \left| \eta^2_x + \eta^2_y \right|, \\
u(x,y) = \varphi(\eta),
\]
where \( \eta(x,y) \) is an arbitrary harmonic function, which is different from the constant one; \( A, B \in \mathbb{R}\setminus\{0\} \). Furthermore, the functions \( f, \psi, \varphi \) can be defined from the system of ODE
\[
f'' = \exp(f) + A\varphi^2, \quad \psi'' = \exp(\psi) - B\varphi^2, \quad \varphi'' = \exp(\psi) - \exp(f).
\]
In the case when \( A = -B \), system (22) assumes a particular exact solution \( u(x,y) = v(x,y) = f(\eta) + \ln \left| \eta^2_x + \eta^2_y \right|, (x,y) = \eta(x,y) \), furthermore, the function \( f(\eta) \) satisfies the linear 2nd order ODE
\[
f'' = f + A.
\]

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