Slope equality of non-hyperelliptic Eisenbud–Harris special fibrations of genus 4

Makoto Enokizono

Makoto Enokizono, Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki Noda, Chiba, 278-8510, Japan
E-mail: enokizono_makoto@ma.noda.tus.ac.jp

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Abstract
The Horikawa index and the local signature are introduced for relatively minimal fibered surfaces whose general fiber is a non-hyperelliptic curve of genus 4 with unique trigonal structure.

1. Introduction
Let $S$ (resp. $B$) be a non-singular projective surface (resp. curve) defined over $\mathbb{C}$ and $f : S \to B$ a relatively minimal fibration whose general fiber $F$ is a non-hyperelliptic curve of genus 4. According to [2], we say that $f$ is Eisenbud–Harris special or E-H special for short (resp. Eisenbud–Harris general) if $F$ has a unique $g_1^1$ (resp. two distinct $g_1^1$’s), or equivalently, the canonical image of $F$ lies on a quadric surface of rank 3 (resp. rank 4) in $\mathbb{P}^3$.

For E-H general fibrations of genus 4, two important local invariants, the local signature and the Horikawa index, are introduced in the appendix in [2]. The purpose of this short note is to show that an analogous result also holds for E-H special fibrations of genus 4, that is, to show the following:

**Theorem 1.1.** Let $A$ be the set of fiber germs of relatively minimal E-H special fibrations of genus 4. Then, the Horikawa index $\text{Ind} : A \to \mathbb{Q}_{\geq 0}$ and the local signature $\sigma : A \to \mathbb{Q}$ are defined so that for any relatively minimal E-H special fibration $f : S \to B$ of genus 4, the slope equality

$$K_f^2 = \frac{24}{7} \chi_f + \sum_{p \in B} \text{Ind}(f^{-1}(p)),$$

and the localization of the signature

$$\text{Sign}(S) = \sum_{p \in B} \sigma(f^{-1}(p)),$$

hold.

Note that the above slope equality was established in [7] under the assumption that the multiplicative map $\text{Sym}^2 f_* \omega_f \to f_* \omega_f^{\otimes 2}$ is surjective, and that for non-hyperelliptic fibrations of genus 4, the slope inequality

$$K_f^2 \geq \frac{24}{7} \chi_f,$$

was shown independently in [3] and [6].
2. Proof of theorem

In this section, we prove Theorem 1.1. Let \( f : S \to B \) be a relatively minimal E-H special fibration of genus 4. Since the general fiber \( F \) of \( f \) is non-hyperelliptic, the multiplicative map \( \text{Sym}^2 f_*\omega_f \to f_*\omega_f^{\otimes 2} \) is generically surjective from Noether’s theorem. Thus, we have the following exact sequences of sheaves of \( \mathcal{O}_B \)-modules:

\[
0 \to \mathcal{L} \to \text{Sym}^2 f_*\omega_f \to f_*\omega_f^{\otimes 2} \to T \to 0,
\]

(2.1)

where the kernel \( \mathcal{L} \) is a line bundle on \( B \) and the cokernel \( T \) is a torsion sheaf on \( B \). Then, the first injection defines a section \( q \in H^0(B, \text{Sym}^2 f_*\omega_f \otimes \mathcal{L}^{-1}) = H^0(\mathbb{P}(f_*\omega_f), 2T - \pi^*\mathcal{L}) \), where \( \pi : \mathbb{P}(f_*\omega_f) \to B \) is the projection and \( T = \mathcal{O}_{\mathbb{P}(f_*\omega_f)}(1) \) is the tautological line bundle on \( \mathbb{P}(f_*\omega_f) \). The section \( q \) can be regarded as a relative quadratic form \( q : (f_*\omega_f)^\tau \to f_*\omega_f \otimes \mathcal{L}^{-1} \), which defines the determinant \( \det(q) : \det(f_*\omega_f)^{\tau^{-1}} \to \det(f_*\omega_f) \otimes \mathcal{L}^{-3} \). Note that for a non-hyperelliptic fibration \( f \) of genus 4, \( \det(q) = 0 \) if and only if \( f \) is E-H special. On the other hand, \( Q = (q) \in |2T - \pi^*\mathcal{L}| \) is regarded as the unique relative quadric on \( \mathbb{P}(f_*\omega_f) \) containing the image of the relative canonical map \( \Phi_f : S \dashrightarrow \mathbb{P}(f_*\omega_f) \). Since \( f \) is E-H special, the general fiber of \( \pi |_Q : Q \to B \) is a quadric of rank 3 on \( \mathbb{P}(H^0(F, K_f)) = \mathbb{P}^3 \). The closure of the set of vertexes of general fibers of \( \pi |_Q \) defines a section \( v : B \to Q \), which corresponds to some quotient line bundle \( \mathcal{F} \) of \( f_*\omega_f \). Let \( E \) be the kernel of the surjection \( f_*\omega_f \to \mathcal{F} \) and put \( P = \mathbb{P}(f_*\omega_f) \) and \( P = \mathbb{P}(E) \). Let \( \tau : \tilde{P} \to P \) be the blow-up of \( P \) along the section \( v(B) \). Then, the relative projection \( P \dashrightarrow P' \) from the section \( v(B) \) extends to the morphism \( \tau' : \tilde{P} \to P' \) with

\[
\tau'^{*}T = \tau^{*}T - E,
\]

where \( T' = \mathcal{O}_{\mathbb{P}(E)}(1) \) is the relative tautological line bundle of \( \mathbb{P}(E) \) and \( E \) is the exceptional divisor of \( \tau \). Let \( \tilde{Q} \) denote the proper transform of \( Q \) on \( \tilde{P} \). It follows that in \( \text{Pic}(\tilde{P}) \),

\[
\tilde{Q} = \tau^{*}Q - 2E = \tau'^{*}(2T' - \pi'^{*}\mathcal{L}),
\]

where \( \pi' : P' \to B \) is the projection. Let \( Q' = \tau'(\tilde{Q}) \) be the image of \( \tilde{Q} \) via \( \tau' \). It follows that \( Q' \subseteq |2T' - \pi'^{*}\mathcal{L}| \) and \( \tilde{Q} = \pi'^{*}Q' \). The general fiber of \( \pi |_{Q'} : Q' \to B \) is a conic on \( \mathbb{P}(H^0(F, E|_P)) = \mathbb{P}^2 \) of rank 3, which is isomorphic to \( \mathbb{P}^1 \). Note that the composite \( \tau' \circ \Phi_f : S \dashrightarrow Q' \subseteq P' \) of the relative canonical map \( \Phi_f : S \dashrightarrow P \) and the projection \( \tau' : P \dashrightarrow P' \) determines the unique trigonal structure of the general fiber \( F \) of \( f \). Let \( q' \in H^0(P', 2T' - \pi'^{*}\mathcal{L}) = H^0(B, \text{Sym}^2 E \otimes \mathcal{L}^{-1}) \) be a section which defines \( Q' = (q') \).

Then \( q' \) can be regarded as a relative quadratic form \( q' : E' \to E \otimes \mathcal{L}^{-1} \), which has non-zero determinant \( \det(q') : \det(E)^{-1} \to \det(E) \otimes \mathcal{L}^{-3} \) since \( Q' \) is of rank 3. Thus, \( \det(q') \in H^0(B, \det(E)^{\otimes 2} \otimes \mathcal{L}^{-3}) \) defines an effective divisor \( \Delta_{q'} = (\det(q')) \) on \( B \). The degree of \( \Delta_{q'} \) is

\[
\deg \Delta_{q'} = 2\deg E - 3\deg \mathcal{L}.
\]

(2.2)

Let \( \rho : \tilde{S} \to S \) be the minimal desingularization of the rational map \( \tau^{-1} \circ \Phi_f : S \dashrightarrow \tilde{P} \) and \( \tilde{\Phi} : \tilde{S} \to \tilde{P} \) the induced morphism. Put \( \Phi = \tau \circ \tilde{\Phi} : \tilde{S} \to P, \Phi' = \tau' \circ \tilde{\Phi} : \tilde{S} \to P', M = \Phi^*T \) and \( M' = \Phi'^*T' \). Then we can write \( \rho^*K_f = M + Z \) for some effective vertical divisor \( Z \) on \( \tilde{S} \). Since \( M' = M - \tilde{\Phi}^*E \), we can also write \( \rho^*K_f = M' + Z' \), where \( Z' = Z + \tilde{\Phi}^*E \) is also an effective vertical divisor on \( \tilde{S} \). Since \( \Phi' \) is of degree 3 onto the image \( Q' \), we have \( \Phi'_*\tilde{S} = 3Q' \) as cycles. It follows that

\[
M^2 = \Phi'^{*}T' \tilde{S} = \Phi'^{*}\Phi'_*\tilde{S} = 3T'^2Q' = 3T'^2(2T' - \pi'^{*}\mathcal{L}) = 6\deg E - 3\deg \mathcal{L},
\]

while we have

\[
M^2 = (\rho^*K_f - Z)^2 = K_f^2 - (\rho^*K_f + M')Z'.
\]

Hence, we get

\[
K_f^2 = 6\deg E - 3\deg \mathcal{L} + (\rho^*K_f + M')Z'.
\]

(2.3)
From (2.2) and (2.3), we can delete the term \( \deg \mathcal{L} \) and then we have
\[
\deg \mathcal{L} = \frac{1}{6} K_j^2 - \frac{1}{6}(\rho^* K_r + M')Z' - \frac{1}{2} \deg \Delta_f.
\] (2.4)

On the other hand, taking the degree of (2.1), we get
\[
K_j^2 = 4 \chi_f - \deg \mathcal{L} + \text{length} \mathcal{T}.
\] (2.5)

Substituting (2.4) in the equation (2.5), we get
\[
K_j^2 = \frac{24}{7} \chi_f + \frac{1}{7}(\rho^* K_r + M')Z' + \frac{3}{7} \deg \Delta_f + \frac{6}{7} \text{length} \mathcal{T}.
\]

For a fiber germ \( f^{-1}(p) \), we define \( \text{Ind}(f^{-1}(p)) \) by
\[
\text{Ind}(f^{-1}(p)) = \frac{1}{7}(\rho^* K_r + M')Z_p' + \frac{3}{7} \text{mult}_p \Delta_f + \frac{6}{7} \text{length}_p \mathcal{T},
\]
where \( Z = \sum_{p \in B} Z_p \) is the natural decomposition with \( (f \circ \rho)(Z_p) = \{p\} \) for any \( p \in B \). For the definitions of \( M', Z', \) etc., we do not use the completeness of the base \( B \). Thus, we can modify the definition of \( \text{Ind} \) for any fiber germ of relatively minimal E-H special fibrations of genus 4 which is invariant under holomorphically equivalence. Thus, we can define the Horikawa index \( \text{Ind} : \mathcal{A} \rightarrow \mathbb{Q}_{\geq 0} \) such that
\[
K_j^2 = \frac{24}{7} \chi_f + \sum_{p \in B} \text{Ind}(f^{-1}(p)).
\]

The non-negativity of \( \text{Ind}(f^{-1}(p)) \) is as follows. From the nefness of \( K_r \), we have \( \rho^* K_r Z_p' \geq 0 \). For a sufficiently ample divisor \( a \) on \( B \), the linear system \( |M' + (f \circ \rho)^* a| \) is free from base points. Thus, by Bertini’s theorem, there is a smooth horizontal member \( C \in |M' + (f \circ \rho)^* a| \) and then \( M' Z_p' = (M' + (f \circ \rho)^* a)Z_p' = CZ_p' \geq 0 \).

Once the Horikawa index is introduced, we can define the local signature since \( \text{Sign}(S) = K_j^2 - 8 \chi_f \) and \( e_f = 12 \chi_f - K_j^2 \) is localized by using the topological Euler numbers of the singular fibers (cf. [1, Section 2]). Indeed, we put
\[
\sigma(f^{-1}(p)) = \frac{7}{15} \text{Ind}(f^{-1}(p)) - \frac{8}{15} e_f(f^{-1}(p)) + 6
\]
for \( p \in B \). Then we have \( \text{Sign}(S) = \sum_{p \in B} \sigma(f^{-1}(p)) \).

**Remark 2.1.** In [5], we define a Horikawa index \( \text{Ind}_{g,n} \) for fibered surfaces of genus \( g \) admitting a cyclic covering of degree \( n \) over a ruled surface (called primitive cyclic covering fibrations of type \((g, 0, n)\)). For \( g = 4 \) and \( n = 3 \), these fibrations are non-hyperelliptic E-H special fibrations of genus 4. One can check the Horikawa index \( \text{Ind}_{4,3}(f^{-1}(p)) \) in [5, (4.5)] and \( \text{Ind}(f^{-1}(p)) \) in Theorem 1.1 are coincide by using the technique of [4, Appendix] which we left to the reader.

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