Stewart-Lyth inverse problem

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Abstract

In this paper the Stewart-Lyth inverse problem is introduced. It consists of solving two nonlinear differential equations for the first slow-roll parameter and finding the inflaton potential. The equations are derived from the Stewart-Lyth equations for the scalar and tensorial perturbations produced during the inflationary period. The geometry of the phase planes transverse to the trajectories is analyzed, and conclusions about the possible behavior for general solutions are drawn.

I. INTRODUCTION

The analysis of observations of the cosmic microwave background radiation, recently reported by the experiments Boomerang and Maxima-1\textsuperscript{[1,2]}, confirmed that, at present-day, inflation\textsuperscript{[3,4]} still remains as the favourite model for the origin of structure in the Universe. The main reason is that the currently obtained observational data on structures in the Universe are interpreted in a natural way within the framework of the Gaussian adiabatic and nearly scale-invariant density perturbations that the usual models produce. On the other hand, it is one of the simplest paradigms within which rigorous theoretical predictions can be achieved.

However, the inflationary paradigm is quite a broad one, and there are several equally satisfactory implementations of the inflationary idea. The simplest scenario arises when the dynamics of inflation (both classical and quantum) are dominated by a single scalar field (inflaton) evolving in a nearly flat potential. Even in this case, there are a large number of candidates for the potential. For this scenario it is well established\textsuperscript{[5,6]} that, to a good approximation, the scalar and tensor perturbations will take on a power-law form, with the tensor ones giving a subdominant (and almost negligible) contribution. A way

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of discriminating between candidates for inflationary potential is the comparison between observed and predicted perturbations spectra. However, the present level of accuracy of the observations is well below the accuracy of the predictions for most of the models, allowing a large number of them to remain as candidates.

Another way of accomplishing the task of choosing a proper potential is to reconstruct it from the observed spectra, but again the low accuracy of measurements as well as the fact that the reconstruction is based on a Taylor expansion of the potential (with higher order derivatives given in dependence of the amount and quality of observational data), implies a large uncertainty in the outcomes of the standard reconstruction procedure, from the point of view of the potential uniqueness. Anticipating the near-future launch of satellites capable of measuring microwave background anisotropies to an accuracy of a few percent or better, across a wide range of angular scales, an assessment of the accuracy of predictions of the anisotropies for given cosmological models has been stressed. Grivell and Liddle have confirmed that for most models of inflation, the Stewart and Lyth analytic calculations should give extremely accurate predictions.

Lyth and Stewart began with the precise calculation for power-law inflation and then they went on to use this exact result to analytically compute the next-order slow-roll correction to the standard formula. At this level of approximation, the Stewart and Lyth equations for the spectral indices can be rewritten as nonlinear differential equations in terms of the first slow-roll parameter. Hence, a third option for the determination of the inflationary potential, that we shall discuss in this paper, is to use observational information about density perturbations and gravitational waves spectra as input in these differential equations, to solve for the corresponding inflationary potential. We call this procedure the Stewart-Lyth inverse problem.

In this work, after introducing the Stewart-Lyth inverse problem, we test the feasibility of the method by finding the proper potential for the well-known scenario of power-law inflation. Nevertheless, the involved system of differential equations composed by a second order and a first order equations, have proved to be very difficult to solve for general functional forms of the perturbations spectra. Taking this into account, in this paper we restrict our analysis to a qualitative study of the dynamics described by these equations. To draw conclusions about the behavior of the general solutions, the equations are studied using the spectral indices as parameters. For the corresponding reduced equations, we analyze the phase space for the second order equation which presents a singularity that strongly determines the flow geometry. Not cyclical orbits are found. The reduced first order equation is solved for any value of the tensorial spectral index and the possible behaviors are analyzed with all of the solutions being monotonic. Except in the case of null tensorial index, the singularity is also observed. In general, for solutions of the Stewart-Lyth inverse problem with smoothly and slowly changing spectral indices, periodic, quasiperiodic, or chaotic general solutions should not be expected. The theoretical analysis as well as observations suggest that power-law solutions are still valid.

In the next section we briefly describe the theoretical frame for the Stewart-Lyth calculations and present their originally algebraic equations in terms of the spectral indices. In Sec. we introduce the Stewart-Lyth inverse problem. Section is devoted to the qualitative analysis of the phase-spaces of the reduced second order equation for the first slow-roll parameter. The analysis of the solutions for the reduced first order equation is
presented in Sec. V. We summarize the main results obtained in Sec. VI.

II. THE STEWART-LYTH EQUATIONS

A. The single scalar field scenario

The theoretical frame for the Stewart-Lyth calculation is the flat Friedmann-Robertson-Walker universe containing a single scalar field equivalent to a perfect fluid with equations of motions given by

\[ H^2 = \frac{\kappa}{3} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right), \]

\[ \ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \]

where \( \phi \) is the inflaton, \( V(\phi) \) is the inflationary potential, \( H = \dot{a}/a \) is the Hubble parameter, \( a \) is the scale factor, dot and prime stand for derivatives with respect to cosmic time and \( \phi \) respectively, \( \kappa = 8\pi/m_{Pl}^2 \) is the Einstein constant and \( m_{Pl} \) is the Planck mass.

In this framework, the first three slow-roll parameters were respectively defined in Ref. [15], we shall write them here as in Ref. [7]:

\[ \epsilon(\phi) \equiv 3\frac{\dot{\phi}^2}{2} \left[ \frac{\dot{\phi}^2}{2} + V(\phi) \right]^{-1} = \frac{2}{\kappa} \left[ \frac{H'}{H} \right]^2, \]

\[ \eta(\phi) \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon - \frac{\epsilon'}{\sqrt{2\kappa\epsilon}}, \]

\[ \xi(\phi) \equiv \left[ \epsilon\eta - \left( \frac{2}{\kappa} \right)^{\frac{1}{2}} \eta' \right]^{\frac{1}{2}}. \]

Up to a constant, the first slow-roll parameter (3) is a measure of the relative contribution of the kinetic energy to the total energy of the field. By definition \( \epsilon \geq 0 \), and because inflation can be defined as \( \ddot{a} > 0 \), for inflation to proceed: \( \epsilon < 1 \).

B. Power-law inflation

Few models of inflation allow to exactly calculate scalar and tensorial perturbations. One of them is the power-law model [12], a particular scenario of inflation, where

\[ a(t) \propto t^p, \]

\[ H(\phi) \propto \exp \left( -\sqrt{\frac{\kappa}{2p}} \phi \right), \]

\[ V(\phi) \propto \exp \left( -\sqrt{\frac{2\kappa}{p}} \phi \right), \]

with \( p \) being a positive constant. It follows from Eqs. (3), (4), and (5) that in this case the slow-roll parameters are constant and equal each other.
C. The indices equations

Exact expressions for the asymptotic scalar (density perturbations) and tensorial (gravitational waves) power spectra for the case of power-law inflation was correspondingly derived by Lyth and Stewart \[11\] and Abbot and Wise \[16\].

Assuming that the deviation of the higher slow-roll parameters from $\epsilon$ is small (power-law approximation) and that $\epsilon$ is small with respect to unity (slow-roll approximation), Stewart and Lyth \[10\] derived next-to-leading order expressions for both spectra. In terms of the spectral indices, these expressions are

\[ 1 - n_s(k) \simeq 4\epsilon - 2\eta + 8(C + 1)\epsilon^2 - (10C + 6)\epsilon\eta + 2C\xi^2, \]
\[ n_T(k) \simeq -2\epsilon \left[ 1 + (2C + 3)\epsilon - 2(C + 1)\eta \right], \]

where the notation is that of \[7\]; $n_s(k)$ and $n_T(k)$ are the scalar and tensorial spectral indices respectively, $k$ is the wave number of the comoving scale, and $C \approx -0.73$ is a constant related to the Euler constant originated in the expansion of Gamma function. The symbol $\simeq$ is used to indicate that these equations were obtained using the power-law and slow-roll approximations. Hereafter we shall use the $=$ sign in our calculations, but the meaning of approximation should be added whenever it applies.

For a given expression of the scale factor, the Hubble parameter and the potential are determined, and then by substituting definitions (3), (4), and (5) in the Stewart-Lyth equations (9) and (10), the scale-dependent spectral indices are obtained. For instance, giving Eq. (7) and substituting in Eqs. (3), (4), and (5), one obtains $\epsilon = \eta = \xi = 1/p$, which in turn substituted in Eqs. (4) and (10) yield $n_s - 1 = n_T = \text{const.}$ This is an alternative definition for power-law inflation. We should note that the spectral indices are directly related with the power spectra (for details see Ref. \[7\]).

III. FORMULATION OF THE STEWART-LYTH INVERSE PROBLEM

In this section we shall formulate a method for finding the inflaton potential using observational information on the spectral indices. Denoting $T \equiv \dot{\phi}^2/2$ and using definitions (3), (4), (5), together with Eqs. (1) and (2) then, in a straightforward manner, we obtain

\[ \epsilon = 3T \frac{T}{T + V} = \frac{\kappa T}{H^2}, \]
\[ \eta = \frac{\kappa}{dH^2} \frac{dT}{dH^2}, \]
\[ \xi^2 = \kappa \epsilon \frac{dT}{dH^2} + 2\kappa \epsilon H^2 \frac{d^2T}{d(H^2)^2}. \]

Defining $\tau \equiv \ln H^2$, $\delta(k) \equiv n_T(k)/2$ and $\Delta(k) \equiv [n_s(k) - 1]/2$, and substituting Eqs. (11), (12), and (13) in Eqs. (9) and (10), the indices equations in terms of the first slow-roll parameter $\epsilon$ and its derivatives with respect to $\tau$ ($\dot{\epsilon} \equiv d\epsilon/d\tau$ and $\ddot{\epsilon} \equiv d^2\epsilon/d\tau^2$) become

\[ 2C\epsilon\ddot{\epsilon} - (2C + 3)\epsilon\dot{\epsilon} + \epsilon^2 + \epsilon + \Delta = 0, \]
\[ 2(C + 1)\epsilon\dot{\epsilon} - \epsilon^2 - \epsilon - \delta = 0. \]
The second order nonlinear differential equation (14) was first introduced in Ref. [13], while the first order equation (15) was derived first in Ref. [14]. Notice that Eqs. (14) and (15) are just alternative representations of the Stewart-Lyth Eqs. (9) and (10). The approach of Ref. [13] for the potential reconstruction is incomplete because is restricted to the case $\Delta(k) = \text{const}$, and, on the other hand, the equation for the tensorial spectral index (15) is considered only in the trivial case of power-law inflation [14]. As we shall see, Eq. (15) imposes rigorous constraints upon the set of solutions of Eq. (14).

By the Stewart-Lyth inverse problem we mean the problem consisting in solving Eqs. (14) and (15) for $\epsilon$, given expressions for the spectral indices, and finding the corresponding inflaton potential using the definitions of the first slow-roll parameter (3) and (11). We prefer this denomination instead of “reconstruction” because we will use explicit functional forms of the spectra as input in the problem and correspondingly we will obtain functional forms for the potentials rather than pieces of information about them.

As was already mentioned, having an expression for $\epsilon(\tau)$ the corresponding potential as a function of $\tau$ can be obtained from Eq. (11):

$$V(\tau) = \frac{1}{\kappa} [3 - \epsilon(\tau)] \exp(\tau). \quad (16)$$

On the other hand, taking into account that $\dot{\phi} = \dot{\tau} \dot{\phi}$ and using Eq. (3), the scalar field as a function of $\tau$ is given by

$$\phi(\tau) = -\frac{1}{\sqrt{2\kappa}} \int \frac{d\tau}{\sqrt{\epsilon(\tau)}} + \phi_0, \quad (17)$$

where $\phi_0$ is an integration constant.

Finally, the inflationary potential as parametric function of the inflaton can be given:

$$V(\phi) = \begin{cases} 
\phi(\tau), \\
V(\tau).
\end{cases} \quad (18)$$

The above expressions are similar to those used in Ref. [13] but we would like to stress that the functional form for $\epsilon$ in expressions (16) and (17) must be solution of both Eqs. (14) and (15).

As a simple example of a resolution of the Stewart-Lyth inverse problem, let us analyze the case of power-law inflation, where the spectral indices are constants. Substituting Eq. (15) in Eq. (14) with constant $\Delta$ and $\delta$, an algebraic equation for $\epsilon$ is obtained. Since the coefficients of this equation are scale-independent, all the solutions for this algebraic equation are just of the form: $\epsilon = 1/p$ with $p = \text{const}$. Substituting this expression for $\epsilon$ in Eq. (17) and after some algebra, we obtain for the Hubble parameter

$$H(\phi) = \exp \left( -\sqrt{\frac{\kappa}{2p}} (\phi - \phi_0) \right). \quad (19)$$

We can see that, up to a constant, Eq. (19) is equivalent to (7). Now, substituting $\epsilon = 1/p$ and $H^2$ in Eq. (16), we obtain,

$$V(\phi) = \frac{3p - 1}{\kappa p} \exp \left( -\sqrt{\frac{2\kappa}{p}} (\phi - \phi_0) \right), \quad (20)$$
in complete correspondence with Eq. (8).

On the relevance of taking into account the first-order equation, we want to remark that, although Eq. (14) has a large number of solutions for a constant scalar index, once Eq. (15) with $\delta = \text{const}$ was used as a first integral of Eq. (14), the unique remaining solution is just the potential given by Eq. (20).

Even for a scale-independent scalar spectrum, Eq. (14), when it is taken with no regards to Eq. (15), has proved to be very difficult to be analytically solved; solutions have been found only for a few fixed values of $\Delta$ (see Refs. [14,17] and references therein). This obstacle is even harder to overcome when an scale-dependent scalar spectrum is assumed; even for the first order Eq. (15) it is hard to find an explicit solution for $\epsilon$ for any value of $\delta = \text{const}$, and even more difficult for $\delta = \delta[\tau(k)]$.

In the present work, we will focus in the dynamical aspect of the Stewart-Lyth inverse problem, i.e., in the dynamics of the first slow-roll parameter determined by Eqs. (14) and (15). The problem of integrating for the inflationary potential is left to be done in a near future.

IV. PHASE SPACES ANALYSIS OF THE REDUCED SECOND ORDER EQUATION

For an understanding of the dynamics behind Eqs. (14) and (15), the best approach seems to be a qualitative analysis of the corresponding phase-spaces. Not yet having an explicit expression for $\Delta(\tau)$ and $\delta(\tau)$, we use the following approach in order to draw conclusions about the dynamics described by Eqs. (14) and (15): if we consider $\Delta(\tau)$ and $\delta(\tau)$ as the forcing element in these equations, then we can assume the dynamics to be characterized by one more dimension. The $(\epsilon, \dot{\epsilon})$ planes [$\left(\tau, \epsilon\right)$ for the first order equation] corresponding to the different values of this new coordinate are transverse to the trajectories given by these equations. Having the phase-portraits on the planes $(\epsilon, \dot{\epsilon})$ for any value of $\Delta = \text{const}$ in Eq. (14) and the solutions $(\tau, \epsilon)$ for any $\delta = \text{const}$ in Eq. (15), the geometry of the surfaces along which the real trajectories spread out could be outlined, assuming slow variation for $\Delta(\tau)$ and $\delta(\tau)$.

Hereafter, we will refer to Eqs. (14) and (15) with constant $\Delta$ and $\delta$ as the reduced equations of the Stewart-Lyth inverse problem, and the space for solutions depending on $\tau$ as the extended phase-space.

Solutions for the reduced first order equation (15) will be studied in the next section. Let us now proceed with the analysis of the dynamics given by Eq. (14). This equation can be rewritten as

$$\ddot{\epsilon} - \frac{1}{2C} \left(2C + 3 + \frac{1}{\epsilon}\right) \dot{\epsilon} + \frac{1}{2C} \left(\epsilon + 1 + \frac{\Delta}{\epsilon}\right) = 0. \tag{21}$$

With the change of variables

$$x_1 = \epsilon,$$
$$x_2 = \dot{\epsilon}, \tag{22}$$

we obtain the system
\[ \dot{x}_1 \equiv F(x_1, x_2) = x_2, \]
\[ \dot{x}_2 \equiv G(x_1, x_2) = -\frac{1}{2C} \left( x_1 + 1 + \frac{\Delta}{x_1} \right) + \frac{1}{2C} \left( 2C + 3 + \frac{1}{x_1} \right) x_2. \]  
(23)

For \( \Delta = \text{const} \), the condition for existence of solution \([15]\), i.e., the continuity condition for the vector field, holds at every point \((x_1, x_2) \in \mathcal{P} \equiv \mathbb{R}^2 \setminus \{(0, x_2), \forall x_2 \} \). On the other hand,
\[
\left| G(x_1, x_2^{(1)}) - G(x_1, x_2^{(2)}) \right| = \frac{1}{2|C|} \left( 2C + 3 + \frac{1}{x_1} \right) \left| x_2^{(1)} - x_2^{(2)} \right|, \tag{24}
\]
where the upper indices denote any two different values of \( x_2 \). Then, the uniqueness condition, i.e., Lipschitz condition, holds for the same set \( \mathcal{P} \). Therefore, unique solution for the equations system (23) certainly exists at any point in \( \mathcal{P} \). In a similar fashion, the differentiability of solution with respect to initial conditions and parameters of the system is also satisfied. Hence, we can use the results of qualitative theory of dynamical systems in the plane for the study of system (23) \([19]\).

The set of singular points of system (23) is the union of the set of fixed points \((F = G = 0)\) and the \( x_2 \) axis. We should note that around point \( p_s = (0, \Delta) \) the field is well defined along directions for which \( x_2 \to \Delta \) faster than \( x_1 \to 0 \). This case will be analyzed in detail in Sec. IV C.

### A. Fixed points

The fixed points of this system are the set \( \{(x_1, x_2)\} \) such that
\[
F(x_1, x_2) = 0, \quad G(x_1, x_2) = 0.
\]
Therefore \( x_2 = 0 \). From \( G(x_1, x_2) = 0 \) and for \( x_1 \neq 0 \), we have
\[
x_1^2 + x_1 + \Delta = 0. \tag{25}
\]
The roots of this quadratic polynomial are
\[
\alpha_\pm = -\frac{1}{2} \pm \frac{\sqrt{1 - 4\Delta}}{2}, \tag{26}
\]
where the symbol \( \alpha_\pm \) stands for the two different roots of Eq. (25). These roots are placed symmetrically with respect to \( x_1 = -0.5 \). From now on we shall use the symbol \( \alpha \) without lower indices while referring to any of these roots.

According to the Grobman-Hartman theorem \([20]\), the local behavior near the equilibrium position depends on the linear part (if exists) of the vector field, in those cases where the eigenvalues of the Jacobian matrix of the vector field associated to the system have nonzero real part. Hence, in order to develop local analysis of the equilibrium positions, we calculate
\[
J(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2C} \left( -1 + \frac{\Delta - x_2}{x_1^2} \right) & \frac{1}{2C} \left( 2C + 3 + \frac{1}{x_1} \right) \end{pmatrix}. \tag{27}
\]
Because the equilibrium positions are of type \((\alpha, 0)\) the Jacobian matrix in these points is written as

\[
J(\alpha, 0) = \begin{pmatrix} 0 & 1 \\ g_1 & g_2 \end{pmatrix},
\]

where

\[
g_1 = \frac{1}{2C} \left( \frac{\Delta}{\alpha^2} - 1 \right),
\]

\[
g_2 = \frac{1}{2C} \left( 2C + 3 + \frac{1}{\alpha} \right).
\]

The characteristic polynomial, corresponding to the eigenvalue problem of Eq. (28), is

\[
p(\lambda) = \lambda^2 - g_2 \lambda - g_1,
\]

with roots

\[
\lambda_{\pm} = \frac{1}{2} g_2 \pm \frac{1}{2} \sqrt{g_2^2 + 4g_1}.
\]

We study the possible behavior near the equilibrium positions depending on the values of the polynomial roots (32). For a complete discussion, definitions and main results, see Refs. [18–20]. In general, one is concerned about whether the eigenvalues are real or complex, and if their real parts are greater, equal or less than zero.

The roots (32) depend on the value of discriminant \(D = g_2^2 + 4g_1\); they will be real if \(D \geq 0\). Hence, the following inequality holds:

\[
D = 1 + \frac{1}{C} + \left( \frac{3}{2C} \right)^2 + \left( \frac{1}{C} + \frac{3}{2C^2} \right) \frac{1}{\alpha} + \left( \frac{1}{4C^2} + \frac{2\Delta}{C} \right) \frac{1}{\alpha^2} \geq 0.
\]

This inequality can be written as

\[
\left[ (2C + 3)^2 - 8C \right] \alpha^2 + 2(2C + 3)\alpha + 8C\Delta + 1 \geq 0.
\]

The polynomial in Eq. (34) has roots

\[
\alpha_{+, -} = \frac{-(2C + 3) \pm 2\sqrt{2C (1 - [(2C + 3)^2 - 8C] \Delta)}}{(2C + 3)^2 - 8C}.
\]

We shall see soon that the fixed point \((\alpha, 0)\) is always a saddle point. Hence, the current analysis has sense only for \(\alpha_+\). Comparing the expressions for this root given by Eqs. (26) and (35), solving for \(\Delta\) and substituting the value of \(C \approx -0.73\), we find out that roots (32) are real if and only if \(\Delta \leq 0.124273\).

Now we are able to classify the fixed points. For \(\Delta > 0.25\), case (a), there are not fixed points and the flow is deformed from a laminar one only near the point \(p_s\) (see Fig. 4, case (a) and Sec. IV C). Hereafter, in the figures we represent the flow direction by means of arrows.
FIG. 1. Reduced phase portraits for $\Delta = 0.3$, case (a), and for $\Delta = 0.25$, case (b).

For $\Delta = 0.25$, case (b), one of the eigenvalues has zero real part and hence the stability of the unique and degenerated fixed point $(-0.5, 0)$ cannot be determined by linearization. The remaining eigenvalue has positive real part, then the local behavior transverse to the center manifold is controlled by the exponentially expanding flows in the local unstable manifold. According to Center Manifold theorem for flows [19], the center manifold (it could be not unique) is tangent at $(-0.5, 0)$ to the center subspace that, in this case, coincides with the $x_1$ axis. On the other hand, the unstable manifold is tangent to the unstable subspace, i.e., the line $x_2 = g_2 x_1$ with $g_2$, for this case, being $(2C + 1)/2C \approx 0.315068$ [Fig. 1, case (b)]. We recall that each subspace is spanned by the eigenvectors corresponding to eigenvalues (32). Further, for $\Delta < 0.25$ and $\Delta \neq 0$ we have two nondegenerate fixed points over the $x_1$ axis symmetrically distributed with respect to $x_1 = -0.5$.

FIG. 2. Reduced phase portrait for $\Delta = 0.227694$, case (c), and for $\Delta = 0.124273$, case (d).

Let us proceed with the fixed points classification paying attention to the leftmost one.
FIG. 3. Reduced phase portrait for \( \Delta = 0 \), case (e), and for \( \Delta = -0.2 \), case (f).

Obviously, the value of \( \alpha_- \) for this point is always negative for any \( \Delta \) and hence \( g_1 > 0 \). This way, the leftmost fixed point is always a saddle. The rate \( |\lambda_+| / |\lambda_-| \) is equal to 1 for \( \Delta = -0.25 + 2(C+1)/(2C+3) \approx 0.227694 \), value of \( \Delta \) that corresponds to \( \alpha_+ = -0.64935 \). For \( \Delta \) greater (less) than 0.227694 the trajectories converge to the saddle point faster (slower) than they diverge from it.

For the rightmost fixed point the situation is richer. First, from Eq. (30) one can see that \( g_2 > 0 \) for any \( \Delta < 0.25 \) and \( \Delta \neq 0 \). For \( 0 < \Delta < 0.25 \) we have \( g_1 < 0 \), then in the interval \( 0.124273 < \Delta < 0.25 \), case (c), the rightmost fixed point is an unstable focus (Fig. 2). As can be observed in the figure, one of the stable separatrices of the saddle point seems to be directly spiraling out from the focus. For \( 0 < \Delta \leq 0.124273 \), case (d), the rightmost fixed point is an unstable node (Fig. 2). In the case \( \Delta = 0 \), case (e), \( g_2 < 0 \) and we have a single saddle located at \((-1, 0)\) (see Fig. 3). For \( \Delta < 0 \), case (f), the rightmost fixed point becomes a saddle (Fig. 3) located on the right half-plane.

B. Closed orbits

Note that the sign of the vector field divergence of system (23) does not depend on \( \Delta \), thus the value of the parameter is of no importance while applying the Bendixson criterion [19], which states that no closed path can be found in plane regions where the vector field divergence is null or not undergoes sign changes. This criterion, applied to system (23), divides the plane in four regions without divergence sign changes:

\[
\mathcal{P}_1 \equiv \{(x_1, x_2); x_1 > 0, \forall x_2\}, \quad \text{with diverging vector field,}
\]
\[
\mathcal{P}_2 \equiv \{(x_1, x_2); -0.64935 < x_1 < 0, \forall x_2\}, \quad \text{with a converging vector field,}
\]
\[
\mathcal{P}_3 \equiv \{(-0.64935, x_2); \forall x_2\}, \quad \text{with null divergence, and}
\]
\[
\mathcal{P}_4 \equiv \{(x_1, x_2); x_1 < -0.64935, \forall x_2\}, \quad \text{with positive divergence.}
\]

It is worthy to note that, even if both half-planes \( x_1 < 0, x_1 > 0 \) are connected through point \( p_s \), this connection should take place in one and only one direction. This way, no
closed orbit could be found in regions $\mathcal{P}_1$ and $\mathcal{P}_3$ or entirely lying in $\mathcal{P}_2$ or $\mathcal{P}_4$. Any closed trajectory should belong to $\mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$. Then, three possible scenarios with closed path could take place. First, a homoclinic orbit (i.e., a saddle connection) enclosing the second fixed point (focus or node, we will refer to it as $\alpha_+$). The second possibility is a limit cycle to which converges all of the trajectories starting from $\alpha_+$ and from which spiral out the orbits. Finally, it could be possible to find a homoclinic orbit with an embedded limit cycle around $\alpha_+$. Each of these scenarios has two possible realizations depending on whether both fixed points lie in $\mathcal{P}_2$ or the saddle lies in $\mathcal{P}_4$ and $\alpha_+$ lies in $\mathcal{P}_2$. The realizations for the more general scenario (a homoclinic orbit with an embedded limit cycle) are schematically represented in Fig. 4.

![FIG. 4. Scheme of a homoclinic orbit with an embedded limit cycle. The dashed curve represents the limit cycle and the dot-dashed line, the trajectory given by Eq. (36). The dotted line is $x_1 = -0.64935$.](image)

One conclude from the left scheme in Fig. 4 that not closed trajectory will be found with both fixed points lying in $\mathcal{P}_2$ because sign changes in the derivative for the vector field along $x_1$, at each side of axis $x_2 = 0$, are not allowed by the first equation of system (23). With regards to the realization plotted on the right part of Fig. 4, we shall see in Sec. IV C that for this range of $x_1$ there is a trajectory (represented by the dot-dashed line on the figure) that can be fairly approximated by

$$x_2 = \left(1 - \frac{x_1}{\alpha_+}\right) \Delta,$$

then no closed orbit can neither exists here.

**C. Flow near point $p_s$**

As we have already mentioned, a sharp feature of these phase spaces is the collisions of the trajectories with the border $x_1 = 0$. Our system possess here a vector field discontinuity
that strongly determines the behavior of the trajectories near this border. Such a piecewise
continuous map is a more general system than those piecewise differentiable maps previously
reported and analyzed in literature (see, for example, Refs. [21,22]).

A particularly interesting situation is exhibited by the trajectories near the point $p_s = (0, \Delta)$. First of all, let us note that no statement can be done about the existence or
uniqueness of solutions of system (23) at this point. To analyze the behavior of the flow in
the neighborhood of this point we rewrite the system as

$$\hat{x}_1 = x_2, \quad (37)$$

$$\hat{x}_2 = \frac{1}{2C} \left\{ (3 + 2C)\Delta - x_1 - 1 + [(3 + 2C)x_1 + 1] \frac{(x_2 - \Delta)}{x_1} \right\}. \quad (38)$$

We shall look for a solution $x_2 = \Delta + f(x_1)$ for the phase curves in the neighborhood of
the above mentioned point. Differentiating this expression with respect to $x_1$, taking into
account that $\hat{x}_2 = \hat{x}_1 dx_2/dx_1 = x_2 df/dx_1$, substituting in Eq. (37) for the approximation
$x_1 \ll 1$ and $f(x_1) \ll 1 (x_2 \approx \Delta)$, we obtain the following nonhomogeneous linear equation
for $f(x_1)$:

$$\frac{df}{dx_1} - \frac{1}{2C\Delta x_1} f = \frac{(3 + 2C)\Delta - 1}{2C\Delta}. \quad (39)$$

For $\Delta \neq 1/2C$ the solution of Eq. (39) is

$$f(x_1) = \left(1 - \frac{3\Delta}{1 - 2C\Delta}\right) x_1 + K x_1^{1/2C\Delta}, \quad (40)$$

while for $\Delta = 1/2C$,

$$f(x_1) = \frac{3}{2C} x_1 \ln x_1 + K x_1, \quad (41)$$

where $K$ is an integration constant depending on the initial conditions. Hence, the behavior
around the point $p_s$ is described by the following family of curves

$$x_2 = \begin{cases} \Delta + \left(1 - \frac{3\Delta}{1 - 2C\Delta}\right) x_1 + K x_1^{1/2C\Delta}, & \text{if } \Delta \neq 1/2C, \\ \frac{1}{2C} + \frac{3}{2C} x_1 \ln x_1 + K x_1, & \text{if } \Delta = 1/2C. \end{cases} \quad (42)$$

Concerning the qualitative behavior of the trajectories near the point $p_s$ there are four
interesting intervals of $\Delta$. First, for $|\Delta| \gg 1$ the flow seems to ignore the existence of the
special singular point $(0, \Delta)$. All the trajectories flows to (or from) the $x_2$ axis along parallel
lines with slope $1 + 3/2C \simeq -1.054794$. From these lines only that intercepting the $x_2$
axis at $p_s$ could arrive to or depart from this point. For positive (but not too large) $\Delta$ the
trajectories have the distribution observed in the left part of Fig. [3]. In this figure we present
the case $\Delta = 1/(3 + 2C) \simeq 0.64935$ where the line with $K = 0$ coincides with the stationary
solution for $x_1$. Then, for $\Delta > 0$ one and only one trajectory could arrive to singular point
$p_s$ and one and only one trajectory could leave it. It is precisely this trajectory which can
be represented by Eq. (36) for $-0.64935 < \alpha_+ < 0$. For $\Delta = 0$, in the left half-plane, all
of the trajectories between the unstable separatrices of the saddle point and the ordinates
axis move away from $p_s$ diverging from the line $x_2 = x_1$. In the right half-plane, all of the trajectories converge asymptotically to the solution that behaves as $x_2 = x_1$ in the
neighborhood of $p_s$ [recall Fig. 3, case (e)]. All of the trajectories converge to $p_s$ or diverge from it in the same direction. Finally, in the case of negative $\Delta$ (but again with not too
large absolute value), the trajectories converge to $p_s$ or diverge from it tangent to the lines
$K = 0$ with slope given by Eq. (42) (Fig. 5, right part).

V. THE REDUCED FIRST ORDER EQUATION

We have already stressed the importance of Eq. (13) for constraining the possible solutions of Eq. (14). Let us consider now the equation

$$2(C + 1)\dot{\epsilon} + \epsilon^2 - \epsilon - \delta = 0,$$

with constant $\delta$.

The solutions of this equation also depend on the parameter value. For $\delta > 0.25$, we have

$$\ln \left[ \frac{|\epsilon^2 + \epsilon + \delta|}{B} \right] - \frac{1}{\sqrt{\delta - 1/4}} \arctan \left( \frac{\epsilon + 1/2}{\sqrt{\delta - 1/4}} \right) - \frac{\tau}{C + 1} = 0,$$

where $B$ is the integration constant.

For $\delta = 0.25$, the solution of Eq. (13) is that of the algebraic equation

$$(\epsilon + 1/2)^2 - B \exp \left( \frac{\tau}{C + 1} - \frac{1}{\epsilon + 1/2} \right) = 0.$$

Now, for $\delta < 0.25$, and $-0.5 - 0.5\sqrt{1 - 4\delta} < \epsilon(\tau) < -0.5 + 0.5\sqrt{1 - 4\delta}$, the solution is obtained from
\[ |\epsilon^2 + \epsilon + \delta| - B \left( \frac{\sqrt{1 - 4\delta} - 2\epsilon - 1}{\sqrt{1 - 4\delta} + 2\epsilon + 1} \right) \frac{\tau}{C + 1} \exp \left( \frac{\tau}{C + 1} \right) = 0, \quad (46) \]

and, for \( \delta < 0.25 \) but \( \epsilon(\tau) < -0.5 - 0.5\sqrt{1 - 4\delta} \) and \( \epsilon(\tau) > -0.5 + 0.5\sqrt{1 - 4\delta} \), the solution results from

\[ |\epsilon^2 + \epsilon + \delta| - B \left( \frac{2\epsilon + 1 - \sqrt{1 - 4\delta}}{2\epsilon + 1 + \sqrt{1 - 4\delta}} \right) \frac{\tau}{C + 1} \exp \left( \frac{\tau}{C + 1} \right) = 0. \quad (47) \]

Note that the solution for \( \delta = 0 \) is a special case of (47), i.e.,

\[ \epsilon(\tau) = (\epsilon_0 + 1) \exp \left( \frac{\tau}{2(C + 1)} \right) - 1, \quad (48) \]

where \( \epsilon_0 \) is the initial condition.

Possible behaviors of solutions of Eq. (43) for different values of \( \delta \) are summarized in Fig. 6. The five interesting intervals are represented by vertical bands in the above plot.

**FIG. 6.** Solutions of the reduced first order equation for different values of \( \delta \). In each of the five regions divided by vertical lines typical solutions for \( \delta \) in the corresponding interval are presented. In every region each curve branch that goes from left to right is a solution with different initial values.

In each band the curves branch spreading from left to right is a solution starting from a different initial condition. We can see that all of the solutions are monotonic. In some cases they evolve unbounded, in some other cases bounded by the stationary solutions given by the roots of

\[ \epsilon^2 + \epsilon + \delta = 0. \]
VI. CONCLUSIONS

We introduced the Stewart-Lyth inverse problem as the determination of the inflaton potential through expressions for the first slow-roll parameter obtained as a solution of differential equations. These equations were derived from the Stewart-Lyth equations for the spectral indices. We tested the feasibility of the method by solving the problem with constant spectral indices as input, corresponding with power-law inflation.

To draw conclusions about the behavior of general $\tau$-depending solutions we analyze reduced equations using the spectral indices as parameters. The phase space for the reduced second order equation is richer than the generic ones, particularly due to the singularity $\epsilon = 0$ and to the existence of the special singular point $[0, 0.5(n_s - 1)]$ near which the flow is deformed even when there are not fixed points. Do not exist cyclical orbits given by the reduced second order equation for any value of $n_s$. The condition for the existence of stationary solutions with positive first slow-roll parameter is $n_s < 1$, and there is only one such a solution for every constant value of the scalar spectral index.

The reduced first order equation was solved for any value of the tensorial spectral index. Five possible behaviors were found for the solutions: stationary, asymptotically increasing, asymptotically decreasing, boundlessly increasing, and boundlessly decreasing. The singularity $\epsilon = 0$ was also observed, except in the case $n_T = 0$. The condition for the existence of stationary solutions with positive first slow-roll parameter is $n_T < 0$, and there is only one such a solution for every constant value of the tensorial spectral index.

In general, for solutions of the Stewart-Lyth inverse problem with smoothly and slowly changing spectral indices in the expected range of values, we shall find that the trajectories would be confined in one of the sectors of the extended phase space divided by the axis $\epsilon = 0$. The exception could be those system with the tensorial index crossing through value $n_T = 0$ while the scalar index is greater than 1. Due to the lack of periodic solutions for the reduced equations, periodic, quasiperiodic, or chaotic extended solutions should not be expected. Suitable (and unique in each case) power-law solutions will exist if and only if $n_s < 1$ and (consistently with the definition of power-law inflation) $n_T < 0$. Taking this last result into account, estimations of the scalar index based in the recent observations by the collaborations Boomerang and Maxima-1 indicate that, despite its simplicity, power-law scenario is still a good candidate for the inflationary stage of the early universe.

A great inside about general properties of scale-dependent solutions for smoothly and slowly changing spectral indices was gained in this work. Further efforts will be focus on obtaining solutions for the Stewart-Lyth inverse problem, i.e., explicit functional forms of the inflationary potential.

ACKNOWLEDGMENTS

This research was supported in part by the CONACyT grant No. 32138-E and the Sistema Nacional de Investigadores (SNI). The work of one of the authors (RM) was partially supported by project DGAPA IN122498. We want to thank Andrew Liddle and Eckehard Mielke for helpful discussions.
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