AdS$_3$ backgrounds from 10D effective action of heterotic string theory

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We present a method for calculating solutions and corresponding central charges for backgrounds with AdS$_3$ and $S^k$ factors in $\alpha'$-exact fashion from the full tree-level low energy effective action of heterotic string theory. Three examples are explicitly presented: AdS$_3 \times S^3 \times T^4$, AdS$_3 \times S^2 \times S^1 \times T^4$ and AdS$_3 \times S^3 \times S^3 \times S^1$. Crucial property which enabled our analysis is vanishing of the Riemann tensor calculated from connection with "$\sigma$-model torsion". We show the following: (i) Chern-Simons terms are the only source of $\alpha'$-corrections not only in BPS, but also in non-BPS cases, suggesting a possible extension of general method of Kraus and Larsen, (ii) our results are in agreement with some conjectures on the form of the part of tree-level Lagrangian not connected to mixed Chern-Simons term by supersymmetry (and present in all supersymmetric string theories), (iii) new $\alpha'$-exact result for central charges in AdS$_3 \times S^3 \times S^3 \times S^1$ geometry. As a tool we used our generalization of Sen’s $E$-function formalism to AdS$_p$ with $p > 2$, and paid special attention to proper definition of asymptotic charges.

I. INTRODUCTION

String backgrounds with AdS$_3$ factor play important role in understanding of AdS$_{d+1}$/CFT$_d$ duality conjecture, as they provide examples in which we can perform calculations on both sides of the duality enabling in this way direct comparison. Some of these backgrounds are also connected to asymptotic near-horizon geometries of black strings. A specially important interplay of these two situations appears in microscopic calculations of entropy for extremal black holes. In string theory near-horizon geometries of such black holes typically contain AdS$_2 \times S^1$ factor which happens to be locally isometric to AdS$_3$. By calculating central charges $c_{L,R}$ of dual boundary CFT$_2$ (e.g., by using Cardy formula) one can calculate microcanonical entropies of corresponding extremal black holes in "microscopic" fashion. There are examples in which direct microscopic calculation of black hole entropy (i.e., without using AdS/CFT conjecture) is also possible. Agreement between microscopic and macroscopic (obtained from low-energy effective supergravity action and Wald formula) results for black hole entropies in all known examples (where calculation on both sides is applicable and possible) is one of the big achievements of string theory, as it shows that the theory provides correct statistical interpretation of black hole thermodynamics.

One motivation for studying $\alpha'$-corrections from the macroscopic side (using tree-level effective supergravity actions) is that this allows us to make more precise comparisons with microscopic ("stringy") calculations. In this way one can make precision tests of some of the most important results in string theory, like above mentioned statistical derivation of black hole thermodynamics.

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One can also turn the argument around, and use the (presupposed) agreement between macro- and micro-results to get some knowledge on the structure of the low-energy effective action. We shall use both of these strategies in this paper.

Now, in some cases $\alpha'$-exact results are known on the microscopic side, so it would be desirable to be able to do $\alpha'$-exact calculations also on the macroscopic side. On the first look, this appears as an impossible mission because tree-level low-energy effective actions of string theories are known only partially. In fact, they are known fully only up to $\alpha'^2$- (i.e., 6-derivative) order. Confronted with this obstacle, different strategies were investigated in the literature. Some authors started with dimensionally reduced effective actions truncated to four derivatives ($R^2$-type actions), with the most popular candidates being $R^2$-type supersymmetric actions and (non-supersymmetric) Gauss-Bonnet type actions (for reviews see [1–3]). Surprisingly, in some supersymmetric (BPS) cases these actions produce $\alpha'$-exact results for black hole entropies and/or central charges (with Gauss-Bonnet actions working only in $D = 4$ dimensions). However, it was also shown that this is not the case in non-supersymmetric (non-BPS) cases [1, 4] (the same can be shown for central charges in $D = 5$, by simple extension of the method from [4] to AdS$_3 \times S^2$ geometry), from which follows that these truncated actions are intrinsically incomplete already at 4-derivative, i.e., $\alpha'^1$-, order. Another approach was to analyze 3-dimensional effective actions defined on asymptotic AdS$_3$ geometries. This was immensely fruitful direction, giving along a way a simple method for obtaining $\alpha'$-exact results for central charges. By showing that all relevant information is encoded in Chern-Simons terms, this method also provided explanation for the sufficiency of terms with at most four derivatives in effective actions [5–8]. However, so far all constructions were relying heavily on the presence of supersymmetry in effective 3-dimensional actions.

These developments raise several questions. Two of them, which partly motivated our work presented here, are: (1) What about non-supersymmetric backgrounds, for which the methods from previous paragraph are not applicable? Is the situation really different for them? (2) What is the reason for the mentioned irrelevance of 6- and higher-derivative terms, viewed from the perspective of 10-dimensional string effective actions? In 3-dimensional language combination of supersymmetry and symmetries of AdS$_3$ space guaranty this, but no corresponding argument is known in 10-dimensional language. Well known example are 8-derivative terms multiplied by $\zeta(3)$ number (including famous $\zeta(3)R^{RRRR}$ terms) which are present in 10-dimensional tree-level low-energy actions of all string theories. In many known examples these terms should combine to give vanishing contribution to central charges (and extremal black hole entropies), but no mechanism which enforces this is known (exact structure of these terms is still not known).

Following our previous work [9], we investigate these questions by starting from the full 10-dimensional tree-level low-energy effective action of heterotic string theory, with the idea of taking into account all higher-derivative terms, and finding $\alpha'$-exact solutions (and corresponding central charges) for backgrounds containing factors of AdS$_3$, spheres and tori. We explicitly present three examples - AdS$_3 \times S^3 \times T^3$, AdS$_3 \times S^2 \times S^1 \times T^4$ and AdS$_3 \times S^3 \times S^3 \times S^1$, but method obviously extends to other cases. For first and second example, microscopic results for central charges are known, so we are able to make comparison with our macroscopic calculations. As for the third example, as far as we know the results for central charges are new.
Because the full tree-level action is only partially known, our strategy is to first take into account the part of the effective action which is connected by supersymmetry with (gauge-gravity) mixed Chern-Simons term which we are able to solve directly, without any assumptions. We obtain solutions for all charge-signatures, which include both BPS and non-BPS cases. Comparison of central charges, obtained in this way, with microscopic results (which are known for the first two examples) shows agreement in all cases, BPS and non-BPS. Moreover, all higher-derivative terms except Chern-Simons term happen to be irrelevant in our examples, both for BPS and non-BPS solutions. This provides new insight to question (1) above.

To completely answer question (2), one needs to take into account also the part of the action which is not connected by supersymmetry to mixed Chern-Simons term. This part starts at 8-derivative ($\alpha'^3$) order, and contains above mentioned $\zeta(3)RRRR$ terms. Now, the exact structure of this part is unknown, and only some terms (of 4-point and 5-point type) are completely known. We conjecture that this part of the action gives vanishing contribution in our calculations due to the fact that all our solutions satisfy a property

$$\overline{R}_{MNPQ} = 0 ,$$

where $\overline{R}$ is the "torsional" Riemann tensor in 10-dimensions, which is calculated from modified connection $\overline{\Gamma} = \Gamma - H/2$ ("$\sigma$-model torsion"). Of course, with this we conjecture that this part of the action has a particular form, which in fact was already conjectured previously in the literature ("weak form" of the conjecture was first proposed in [10], and "strong form" in [11]). What is important here is that the most recent calculations [12] show that the known (i.e., 4-point and 5-point) 8-derivative terms indeed are in accord with our conjecture. With this we have answered the question (2).

Here is the outline of the paper. In section II we generalize Sen’s entropy function formalism to AdS$_p$ cases with $p \geq 2$. This formalism, which we call $\mathcal{E}$-function formalism, is for $p = 3$ is equivalent to the so called c-extremization. Section III is the central part of the paper. In subsection III.A we review what is known about the structure of 10-dimensional tree-level low-energy action of heterotic string theory and the strategy of dealing with mixed Chern-Simons term. In subsections III.B-III.D we present explicit solutions and corresponding central charges for three heterotic backgrounds: AdS$_3 \times S^3 \times T^4$, AdS$_3 \times S^2 \times S^1 \times T^4$ and AdS$_3 \times S^3 \times S^3 \times S^1$. Here we paid special attention to the proper definition of asymptotic charges. In subsection III.E we extend the results to corresponding backgrounds in type-II theories. Concluding remarks are left to section IV.

II. SEN’S $\mathcal{E}$-FUNCTION FORMALISM FOR ADS$_p$

A. Backgrounds with AdS$_3$ factor

Our goal is to analyse solutions of higher-derivative gravity theories with geometries given by products of some number of (maximally symmetric) spaces AdS$_p$ and $S^q$. In the special case of AdS$_2 \times S^{D-2}$ geometry, the procedure is developed in [13] and is known as Sen’s entropy function
formalism. As such geometries appear as near-horizon limit of static extremal black holes, Sen’s formalism gained a huge popularity as the simplest method for calculation of entropy of such black holes (for a review including a detailed list of references see [1]). We are interested in generalization of Sen’s formalism to geometries with $\text{AdS}_p$, $p > 2$, factors, in particular $p = 3$ case.$^1$

Let us assume that we have some purely bosonic theory of gravity in $D$-dimensions which is manifestly diffeomorphism covariant, and, if there are gauge symmetries, also manifestly gauge covariant. We want to find solutions with the $\text{AdS}_3 \times S^{D-3}$ geometry which are manifestly symmetric under the full group of isometries, i.e., $SO(2,2) \times SO(D-2)$. In that case the only fields (field strengths for gauge fields) which are not forced to vanish by symmetries are those which can be composed of "elementary tensors", which are metric tensors and volume-forms of AdS$_3$ and $S^{D-3}$ spaces.

For clarity, let us focus on a theories with a local Lagrangian $\mathcal{L}$ and a field content consisting of $D$-dimensional metric $G_{\mu\nu}$, scalars $\phi^{(s)}$, and some number of forms $\chi$ corresponding to gauge (in such cases $\chi$ denotes field strength) and auxiliary fields.$^2$ Then the only potentially non-vanishing fields are metric, scalars and $p$-forms with $p = 3$, $D-3$, or $D$, which are constrained to have the following form:

$$
\begin{align*}
 ds^2 &= G_{\mu\nu}dx^\mu dx^\nu = v_A ds_A^2 + v_S ds_S^2, \quad \phi^{(s)} = u_s \\
 \chi^{(i)} &= h_i \epsilon_A, \quad \chi^{(a)}_{D-3} = h_a \epsilon_S, \quad \chi^{(a)}_D = h_a \epsilon_A \wedge \epsilon_S
\end{align*}
$$

$ds_A^2$ ($ds_S^2$) and $\epsilon_A$ ($\epsilon_S$) denote metric and volume-form of AdS$_3$ ($S^{D-3}$) space with unit radius. This means that $v_A$ ($v_S$) is the squared radius of AdS$_3$ ($S^{D-3}$) appearing in the physical geometry. For convenience (and to make it as close to Sen’s procedure as possible) we choose the coordinates in AdS$_3$ space such that the determinant of AdS$_3$ metric with unit radius is equal to $(-1)$. $v_A$, $v_S$, $u_s$ and $h'$s are constants. If $\chi^{(i)}$ is a gauge field strength, then we denote $h_i = e_i$. If $\chi^{(a)}_{D-3}$ is a gauge field strength, then $p_a = h_a$ is the magnetic charge (in some particular normalization). The rest of $h'$s are variables corresponding to auxiliary fields, which should be determined, together with $v_A$, $v_S$ and $u_s$, by solving the equations of motion.

If we define the function $f$ by

$$
 f(\vec{v}, \vec{u}, \vec{h}; \vec{e}, \vec{p}) = \oint_{S^{D-3}} \sqrt{-G} \mathcal{L},
$$

where we use the AdS$_3 \times S^{D-3}$ Ansatz (2), then solving equations of motion is equivalent to extremization of the function $f$ keeping $\vec{e}$ and $\vec{p}$ fixed, i.e., to solving the algebraic system

$$
 0 = \frac{\partial f}{\partial \vec{v}} = \frac{\partial f}{\partial \vec{u}} = \frac{\partial f}{\partial \vec{h}}
$$

It is more common to express results not in terms of electric fields $\vec{e}$ but in terms of electric charges $\vec{q}$ which are given by

$$
 \vec{q} = \frac{\partial f}{\partial \vec{e}}.
$$

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$^1$ Somewhat different extension of Sen’s entropy function formalism to general AdS$_p$ geometries was developed in [14] (and used in [15, 16]).

$^2$ Bosonic sectors of low energy effective actions of string theories fall in this class.
This transition goes through Legandre transformation, by introducing $\mathcal{E}$-function defined by

$$\mathcal{E}(\vec{v}, \vec{u}, \vec{h}, \vec{e}; \vec{q}, \vec{p}) = 6\pi(\vec{q} \cdot \vec{e} - f).$$

(6)

Extremization of $\mathcal{E}$-function over variables $\vec{v}, \vec{u}, \vec{h}, \vec{e}$

$$0 = \frac{\partial \mathcal{E}}{\partial \vec{v}} = \frac{\partial \mathcal{E}}{\partial \vec{u}} = \frac{\partial \mathcal{E}}{\partial \vec{h}} = \frac{\partial \mathcal{E}}{\partial \vec{e}}$$

(7)

then obviously gives (4) and (5). Solving (7) one gets solutions for variables as functions of electric and magnetic charges $\vec{q}$ and $\vec{p}$. The value of $\mathcal{E}$-function at the extremum gives the central charge $c$ of the dual CFT living on the boundary of AdS$_3$ space

$$c(\vec{q}, \vec{p}) = \mathcal{E}(\vec{v}_0, \vec{u}_0, \vec{h}_0, \vec{e}_0; \vec{q}, \vec{p}), \quad (\vec{v}_0, \vec{u}_0, \vec{h}_0, \vec{e}_0 \text{ satisfy } (7))$$

(8)

It is almost obvious that the above generalization of Sen’s formalism to AdS$_3$ is equivalent to the so-called $c$-extremization method developed by Kraus and Larsen [5, 6] (for a nice review see [17]). To see this, let us first consider a case where there are no electrically charged gauge fields. Then it is easy to check that our $\mathcal{E}$-function is equal to the $c$-function of Kraus and Larsen.

Let us now assume that there are also $n$ electrically charged gauge fields, which 3-form field strengths $F_3^{(i)} = dA_2^{(i)}$ are constrained by (2) to have the form

$$F_3^{(i)} = e_i \epsilon_A, \quad i = 1, \ldots, n$$

(9)

The idea is to pass to the dual magnetic description by making Poincare duality transformation. We introduce $n$ additional $(D - 4)$-form gauge fields $C_{D-4}^{(i)}$ with $(D - 3)$-form gauge field strengths $K_{D-3}^{(i)} = dC_{D-4}^{(i)}$, and define a new Lagrangian density

$$\tilde{\mathcal{L}} = \mathcal{L} - \frac{1}{3!(D-3)!\sqrt{-G}} \sum_{i=1}^{n} \epsilon^{\mu_1 \cdots \mu_D} F_{i\mu_1 \mu_2 \mu_3} K_{i\mu_4 \cdots \mu_D}$$

(10)

where totally antisymmetric tensor density $\epsilon^{\mu_1 \cdots \mu_D}$ by definition receives values $\pm 1, 0$. If we now treat 3-forms $F_3^{(i)}$ as auxiliary fields (instead of gauge field strengths), Lagrangian $\tilde{\mathcal{L}}$ leads to the equations of motion which are equivalent to those obtained from $\mathcal{L}$ (Euler-Lagrange equation for $C_{D-4}^{(i)}$) simply says that $F_3^{(i)}$ is closed, so locally $F_3^{(i)} = dA_2^{(i)}$, i.e., $F_3^{(i)}$ is a gauge field strength).

For AdS$_3 \times S^{D-3}$ solutions $F_3^{(i)}$ and $K_{D-3}^{(i)}$ are constrained by (2) to have the form

$$F_3^{(i)} = e_i \epsilon_A, \quad K_{D-3}^{(i)} = \frac{\vec{p}_i}{\Omega_{D-3}} \epsilon_{D-3}$$

(11)

where $\Omega_{D-3}$ is a volume of the $(D - 3)$-sphere with unit radius, and $\vec{p}_i$ are magnetic charges. We emphasize again that $F_3^{(i)}$ are now auxiliary fields which should be treated as other auxiliary 3-form fields $\chi_3$ in (2) (if such exist). The new Lagrangian $\tilde{\mathcal{L}}$ does not contain any electrically charged gauge fields and so can be used to perform $\mathcal{E}$-function formalism to obtain AdS$_3 \times S^{D-3}$ solutions and central charge of dual CFT. For this we define

$$\tilde{f}(\vec{v}, \vec{u}, \vec{h}, \vec{e}; \vec{p}, \vec{p}) = \int_{S^{D-3}} \sqrt{-G} \tilde{\mathcal{L}},$$

(12)
from which we obtain $\mathcal{E}$-function

$$
\tilde{\mathcal{E}}(\vec{v}, \vec{u}, \vec{h}, \vec{e}, \vec{\tilde{p}}, \vec{p}) = -6\pi \tilde{f} = 6\pi \left( \vec{p} \cdot \vec{e} - f \right).
$$

The second equality follows from (10) and (12). It is now obvious that if we make identification

$$
\vec{p}_i = q_i
$$

then $\mathcal{E}$-function $\tilde{\mathcal{E}}$ (13) is equal to $\mathcal{E}$ from (6), and so it is irrelevant which one is used for finding solutions and central charges. As $\tilde{\mathcal{E}}$ is equivalent to $c$-function of Kraus and Larsen, this completes the proof.

$\mathcal{E}$-function formalism is easily generalised to AdS$_3 \times S^{k_1} \times \ldots \times S^{k_n}$ geometries. Though the generalisation is straightforward, expressions are quite cumbersome and so we shall not write them for general case. Instead, we present in Section III D an explicit example with AdS$_3 \times S^3 \times S^3$ geometry.

### B. Generalisation to theories with Chern-Simons terms

Effective actions of string theories typically contain Chern-Simons terms which are not manifestly gauge or diffeomorphism covariant. This prevents direct application of $\mathcal{E}$-function method. Unfortunately, there is no general recipe for dealing with Chern-Simons terms. Here, we shall restrict ourselves to terms of the type

$$
S_{CS} = \int T^{(D-3)} \wedge \Omega^{(L3)}
$$

where $T^{(D-3)}$ is some manifestly (gauge and diff) covariant $(D-3)$-form, and $\Omega^{(L3)}$ is 3-dimensional gravitational Chern-Simons term defined with

$$
\Omega^{(L3)}_{\mu \nu \rho} = \frac{1}{2} \Gamma^\sigma_{\mu \nu} \partial_\sigma \Gamma^\tau_{\rho \sigma} + \frac{1}{3} \Gamma^\sigma_{\mu \tau} \Gamma^\tau_{\nu \rho} \Gamma^\xi_{\rho \sigma} \text{ (antisym. in } \mu, \nu, \rho) \tag{16}
$$

Applying now AdS$_3 \times S^{k_1} \times \ldots \times S^{k_n}$ Ansatz to $T^{(D-3)}$, one has to properly define the term $\varepsilon^{abc} \Omega^{(L3)}_{abc}$. Obviously, this term can a priori live only on AdS$_3$, $S^3$, or $S^2 \times S^1$, but it is straightforward to check that it gives vanishing contribution for $S^3$ and $S^2 \times S^1$ (after integration over the respective volumes, present in definition of $\mathcal{E}$-function). This leaves us only with AdS$_3$ case, for which we add the following rule to $\mathcal{E}$-function formalism:

- On AdS$_3$ one takes $\varepsilon^{abc} \Omega^{(L3)}_{abc} = \pm 4$, where plus (minus) sign is used for left (right) central charge $c_L$ ($c_R$).

To prove this, we simply refer to $c$-extremization method [17]. From it follows that when we apply $\mathcal{E}$-function formalism by neglecting all terms of the type (15), Eq. (8) is giving us average central charge $(c_L + c_R)/2$. Gravitational Chern-Simons terms introduce diffeomorphism anomaly in dual CFT and generate a difference between $c_L$ and $c_R$

$$
c_L - c_R = 48\pi \beta, \tag{17}
$$
where $\beta$ is (in $\mathcal{E}$-function formalism language) a factor appearing in a contribution of CS term (15) to function $f$ defined in (3). More precisely,

$$f_{CS} = \beta \varepsilon^{abc} \Omega^{(L3)}_{abc}$$

(18)

By using this in (6) and (8), comparison with (17) leads directly to our simple rule, which completes the proof.

In string theory the $D$-dimensional effective theory is frequently obtained by Kaluza-Klein compactification on one or more circles $S^1$. In such cases, sometimes it is more practical to calculate $\mathcal{E}$-function before compactification (in higher-dimensional space). In such cases, it can happen that we need to calculate $\varepsilon^{abc} \Omega^{(L3)}_{abc}$ on $S^2 \times S^1$ space on which metric is not factorized but instead has Kaluza-Klein form

$$ds^2 = g_{ab}(x)dx^a dx^b = \phi(x) \left[ g_{mn}(x)dx^m dx^n + (dy + 2A_m(x)dx^m)^2 \right],$$

(19)

where $1 \leq a, b \leq 3$ and $1 \leq m, n \leq 2$. Following [18, 19] we take

$$\varepsilon^{abc} \Omega^{(L3)}_{abc} = \frac{1}{2} \varepsilon^{mn} \left[ R^{(2)} F_{mn} + 4g^{m'p} g^{q'n} F_{mnm'} F_{pq'q} \right],$$

(20)

where $F_{mn} = \partial_m A_n - \partial_n A_m$ and $R^{(2)}$ is a Ricci scalar calculated from $g_{mn}$. Then (20) gives us the desired manifestly covariant form (in the reduced 2-dimensional space) for the Chern-Simons term. This logic was originally applied in analyses of extremal heterotic black holes (AdS$_2$ case) in [20, 21]. We shall use it here in Sec. III C.

### C. Generalisation to AdS$_{d+1}$

It is natural to contemplate the extension of $\mathcal{E}$-function formalism to the backgrounds with general AdS$_{d+1}$ factor, where $d \geq 1$. It is obvious that we can straightforwardly generalize the formal procedure given in Eqs. (2-8), where now we define $\mathcal{E}_{d}$-function

$$\mathcal{E}_d(\vec{r}, \vec{u}, \vec{h}, \vec{e}, \vec{q}, \vec{p}) = \pi^{[(d+1)/2]} (\vec{q} \cdot \vec{e} - f).$$

(21)

$[x]$ denotes integer part of $x$. Extremal value of $\mathcal{E}_d$-function we denote by $c_d$

$$c_d(\vec{q}, \vec{p}) = \mathcal{E}_d(\vec{v}_0, \vec{u}_0, \vec{h}_0, \vec{e}_0; \vec{q}, \vec{p}) \quad (\vec{v}_0, \vec{u}_0, \vec{h}_0, \vec{e}_0 \text{ satisfy (7)})$$

(22)

The question is what is the meaning of $c_d$? As is known, $2c_1$ is the number of ground states in dual $CFT_1$ (i.e., conformal quantum mechanics) and $6c_2$ is the central charge of dual $CFT_2$. For $d = 2n$ even, one generalization of $d = 2$ case is in fact known - it gives the coefficient in trace anomaly of A-type in the dual $CFT_{2n}$ [22]. More precisely, the trace anomaly is [23]

$$A_{2n}(x) = \frac{c_{2n}}{(4\pi)^n (n!)^2} E_n(x) + \ldots,$$

(23)

where $E_n$ denotes the Euler density in $d = 2n$ dimensions, and dots $\ldots$ denote B-type (conformally invariant) contribution.

From now on we shall deal with $d = 2$ case exclusively.
III. \(\alpha’\)-EXACT SOLUTIONS IN HETEROTIC STRING THEORY

A. Effective action and 10D SUSY

We are interested here in bosonic solutions with AdS factors of the tree-level heterotic low-energy effective action in \(D\)-dimensions, with \(10 - D\) dimensions compactified on a torus \(T^{10-D}\). We restrict ourselves to the most simple case in which torus is flat and all Kaluza-Klein 1-form gauge fields are uncharged (vanishing). In addition, 10-dimensional \((SO(32)\) or \(E_8 \times E_8\)\) Yang-Mills (1-form) field is also taken to vanish. It follows that the only non-vanishing fields present in this sector are metric \(G_{MN}\), Kalb-Ramond 2-form gauge field \(B_{MN}\), and dilaton \(\Phi\). As discussed in detail in [9], effective Lagrangian can be decomposed in the following way

\[
L^{(H)} = L_{01} + \Delta L_{CS} + L_{\text{other}} .
\]

The first term in (24), explicitly written, is

\[
L_{01} = \frac{e^{-2\Phi}}{16\pi G_D} \left[ R + 4(\partial\Phi)^2 - \frac{1}{12} \Pi_{MNP}\Pi^{MNP} \right] ,
\]

where \(G_D\) is \(D\)-dimensional Newton constant. 3-form gauge field strength is not closed, but instead given by

\[
\Pi_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN} - 3\alpha’\Omega_{MNP} ,
\]

where \(\Omega_{MNP}\) is the gravitational Chern-Simons form

\[
\Omega_{MNP} = \frac{1}{2} \Gamma^R_{MQ} \partial_N \Gamma^Q_{PR} + \frac{1}{3} \Gamma^R_{MQ} \Gamma^Q_{NS} \Gamma^S_{PR} \quad (\text{antisym. in } M, N, P)
\]

Bar on the geometric object means that it is calculated using the modified connection

\[
\Gamma^P_{MN} = \Gamma^P_{MN} - \frac{1}{2} \Pi^P_{MN}
\]

in which 3-form \(\Pi\) plays the role of a torsion.

A presence of \(\alpha’\)-correction in \(L_{01}\), induced by Chern-Simons term through (26), breaks the supersymmetry. As shown in [24], one can retrieve supersymmetry (which is \(N = 1\) in \(D = 10\)) by introducing the term \(\Delta L_{CS}\) in Lagrangian (24). As this ”supersymmetrization of Chern-Simons term” is an on-shell construction, it follows that \(\Delta L_{CS}\) contains tower of higher-derivative terms, i.e., (probably infinite) expansion in \(\alpha’\), starting at \(\alpha’1\) (4-derivative) order. In [24] it was shown that there is a field redefinition scheme in which \(\Delta L_{CS}\) can be written by purely using modified Riemman tensor \(\overline{R}_{MNPQ}\) (calculated from modified connection (28))

\[
\overline{R}^M_{NPQ} = R^M_{NPQ} + \nabla_{[P}\overline{R}^M_{Q]N} - \frac{1}{2} \Pi^M_{R[P\overline{R}^R_{Q]N}} ,
\]

\(^3\) In fact, those are not important restrictions. Eventually, we can use \(O(26 - D, 10 - D)\) T-duality of tree-level heterotic theory to obtain from our results central charges in general case.
and the metric tensor (needed just to contract indices). As in [9], we shall use this property to show that $\Delta L_{\text{CS}}$ is giving vanishing contribution to solutions we construct in this paper.

A much less is known about $L_{\text{other}}$ part of the tree-level effective action. It is known that it contains tower of terms starting at 8-derivative ($\alpha'^3$) order, with the notorious $R^4$-type terms multiplied by $\zeta(3)$ transcendental number. By now, only $R^4$, $R^3H^2$ and $RH^2(\nabla H)^2$ terms\(^4\) are fully known, and their structure is consistent with the conjecture that $L_{\text{other}}$ can be written by purely using modified Riemann tensor $\overline{R}_{MNPQ}$ [12]. If this conjecture is true (at least in a weaker form, see section IV), then $L_{\text{other}}$ would also be irrelevant for our results (for the same reason as $\Delta L_{\text{CS}}$). We postpone further discussion to section IV. At the moment we shall simply ignore this term.

We apply the following strategy [9]. First we ignore the terms $\Delta L_{\text{CS}}$ and $\Delta L_{\text{CS}}$ in the tree-level heterotic effective action (24), which leaves us with simplified reduced Lagrangian

$$L_{\text{red}} = L_{01}.$$  

This (non-supersymmetric) action still has nontrivial $\alpha'$-corrections due to presence of gravitational Chern-Simons term in (26). Then we show that all of our exact solutions (obtained from $L_{\text{red}}$) satisfy the condition

$$\overline{R}_{MNPQ} = 0.$$  

Due to the structure of $\Delta L_{\text{CS}}$ term mentioned above, from (31) follows immediately that such solutions are also solutions of the supersymmetric action

$$L_{\text{susyCS}} = L_{01} + \Delta L_{\text{CS}}.$$  

Now, to use $\mathcal{E}$-function method presented in Section II we have to find a way to treat Chern-Simons term. We now show how to do this if it appears in the action as in (15). This can be achieved through the generalization of the method from [9] to general $D$, by the particular Poincare duality transformation (10) in which one takes $F_3 = dB_2$ (where $B_2$ is the 2-form Kalb-Ramond field $B_{MN}$ from (26)). As $\overline{\Pi}_{MNP}$ becomes now an auxiliary field ((26) does not apply), the only appearance of Chern-Simons term is of the form (15).

Let us present this in more detail. The dual, classically equivalent, Lagrangian $\tilde{L}$ is defined by\(^5\)

$$\tilde{L}^{(H)} = \mathcal{L}^{(H)} - \frac{3!}{(24\pi)^2(D-3)!}\zeta^{M_1\cdots M_D}(\overline{\Pi}_{M_1M_2M_3} + 3\alpha'\Omega_{M_1M_2M_3}) K_{M_4\cdots M_D}$$

where it is understood that $(D-3)$-form $K$ is exact, i.e., $K = dC$, and 3-form $\overline{\Pi}$ is treated as an auxiliary field. Using (24) we have

$$\tilde{L}^{(H)} = L_0 + L_{\text{CS}} + \Delta L_{\text{CS}} + L_{\text{other}},$$

\(^4\) Notation $R^kH^m$ denotes all monomials which can be written by multiplying and contracting $k$ Riemann tensors and $m$ 3-form strengths $H$.

\(^5\) Similar dual formulations are known for some time, see, e.g., [25].
where

\[
\bar{\mathcal{L}}_0 = \mathcal{L}_0 - \frac{3!}{(24\pi)^2(D-3)!\sqrt{-G}} \varepsilon^{M_1\cdots M_D} \mathcal{P}_{M_1 M_2 M_3} K_{M_4\cdots M_D} \tag{35}
\]

\[
\bar{\mathcal{L}}_{CS} = -\frac{\alpha'}{32\pi^2(D-3)!\sqrt{-G}} \varepsilon^{M_1\cdots M_D} \mathcal{P}_{M_1 M_2 M_3} K_{M_4\cdots M_D} \tag{36}
\]

In (34) and (35) terms without tilde (\(\mathcal{L}_0\), \(\Delta \mathcal{L}_{CS}\) and \(\mathcal{L}_{other}\)) are the same as before with the important exception that in the dual description \(H_{MNP}\) is now treated as an auxiliary field (so in the dual description we should forget the relation (26)).

Let us pause for a moment to explain the terms in (34). \(\bar{\mathcal{L}}_0\) is the lowest (\(\alpha''_0\)-) order term in \(\alpha'\)-expansion. \(\bar{\mathcal{L}}_{CS}\) is the mixed Chern-Simons term (constructed from torsional connection (28)) and in fact the only term in the whole dualized heterotic effective action containing Chern-Simons term, and so the only term which is not manifestly diffeomorphism-covariant. It is purely of \(\alpha'\)- (4-derivative) order. As mentioned before, \(\Delta \mathcal{L}_{CS}\) represents (probably infinite) tower of terms, starting at \(\alpha'\)-order, which are connected with Chern-Simons term by supersymmetry, and \(\mathcal{L}_{other}\) is ”the rest”, consisting of tower of terms starting at \(\alpha''\)- (8-derivative) order. What is important is that in the dual description, due to the auxiliary nature of 3-form \(\mathcal{P}\), these terms are now free of Chern-Simons term (before dualization they have contained it implicitly because of (26)).

Though we have extracted Chern-Simons term out, we are still not completely satisfied because \(\mathcal{P}_{MNP}\) appearing in (36) is, due to (28), not the normal gravitational Chern-Simons term \(\Omega_{MNP}\) (calculated from ordinary Levi-Civita connection \(\Gamma^P_{MN}\)). This means that (36) is still not of the form (15), which we know how to handle in the \(\mathcal{E}\)-function framework. Now, it can be shown that the difference is given by [26]

\[
\mathcal{P}_{MNP} = \Omega_{MNP} + A_{MNP} \tag{37}
\]

where

\[
A_{MNP} = \frac{1}{4} \partial_M \left( \Gamma^R_{NQ} \mathcal{P}_{RP}^Q \right) + \frac{1}{8} \mathcal{P}_{MQ}^R \nabla_N \mathcal{P}_{RP}^Q - \frac{1}{4} R_{MN}^{QR} \mathcal{P}_{PQR} \\
+ \frac{1}{24} \mathcal{P}_{MQ}^R R_{NR}^{SP} \mathcal{P}_{PS}^Q \tag{38}
\]

First term in (38) gives vanishing contribution to the action obtained from Lagrangian (34) due to \(dK = 0\), and so it can be dropped. It then follows that \(A_{MNP}\) is manifestly diffeomorphism covariant. Using (37) in (36) we have

\[
\bar{\mathcal{L}}_{CS} = \bar{\mathcal{L}}'_1 + \bar{\mathcal{L}}''_1 \tag{39}
\]

where

\[
\bar{\mathcal{L}}'_1 = -\frac{\alpha'}{32\pi^2(D-3)!\sqrt{-G}} \varepsilon^{M_1\cdots M_D} A_{M_1 M_2 M_3} K_{M_4\cdots M_D} \tag{40}
\]

\[
\bar{\mathcal{L}}''_1 = -\frac{\alpha'}{32\pi^2(D-3)!\sqrt{-G}} \varepsilon^{M_1\cdots M_D} \Omega_{M_1 M_2 M_3} K_{M_4\cdots M_D} \tag{41}
\]
The term $\tilde{L}'_1$ is manifestly diffeomorphism covariant, while $\tilde{L}''_1$ contains ordinary (Levi-Civita) Chern-Simons term and is obviously of the form (15). This is what we wanted to achieve, so we can now finally pass to calculations.

Our goal is to calculate $\alpha'$-exact solutions and corresponding central charges for various backgrounds in heterotic string theory which have $\text{AdS}_3$ and $S^k$, $k = 1, 2, 3$ factors. As we shall see a posteriori, all our solutions will satisfy (31). It then follows (see the discussion below Eq. (31)) that $\Delta L_{\text{CS}}$ will not contribute. We conjecture (based on a limited perturbative knowledge) that $L_{\text{other}}$ also should not contribute. It then follows that it is enough to work with the reduced dual Lagrangian given by

$$\tilde{\mathcal{L}}_{\text{red}} = \tilde{\mathcal{L}}_0 + \tilde{\mathcal{L}}_{\text{CS}} = \tilde{\mathcal{L}}_0 + \tilde{\mathcal{L}}'_1 + \tilde{\mathcal{L}}''_1. \quad (42)$$

Note that $\tilde{\mathcal{L}}_{\text{red}}$ has at most 4-derivative terms (i.e., it is $R^2$-type Lagrangian).

We shall use a convention in which $G_D = 2$ and $\alpha' = 16$.

**B. AdS$_3 \times S^3$ backgrounds**

Let us now apply this to $\text{AdS}_3 \times S^3$ solutions in heterotic string theory compactified on $T^4$. Such backgrounds are expected to describe near-horizon geometries of extremal black strings in $D = 6$ dimensions. The non-vanishing fields here are dilaton $\Phi$, 6-dimensional metric $G_{\mu\nu}$, 3-form $\Pi_{\mu
u\rho}$ (treated as auxiliary), and the 2-form gauge field $C_{\mu\nu}$ (with 3-form strength $K_{\mu\nu\rho}$). We now use generalized version of Sen’s $\mathcal{E}$-function formalism presented in Section II, which dictates the following form for the non-vanishing fields

$$ds^2 = v_A ds_A^2 + v_S ds_S^2, \quad e^{-2\Phi} = \frac{u_s}{\pi},$$
$$K = \tilde{e} e_A + \tilde{p} e_S, \quad \Pi = h_A e_A + h_S e_S, \quad (43)$$

where $v_{A,S}$, $u_s$, $h_{A,S}$ are constants, eventually determined from equations of motion (as functions of electric field $\tilde{e}$ and magnetic charge $\tilde{p}$). These 2-charge black string configurations have microscopic interpretation as bound states of some number (connected to electric charge of $\Pi_{\mu\nu\rho}$, which means magnetic charge of $K_{\mu\nu\rho}$) of fundamental strings plus some number (connected to magnetic charge of $\Pi_{\mu\nu\rho}$, which means electric charge of $K_{\mu\nu\rho}$)) of NS5-branes wrapped around the torus $T^4$.

Using (42) we can write function $f$, defined in (3), as

$$f_0 = \frac{1}{8} \left[ u_s (v_A v_S) \frac{3}{2} \left( \frac{3}{v_A} + \frac{3}{v_S} + \frac{h_A^2}{4 v_A^2} - \frac{h_S^2}{4 v_S^2} \right) - h_A \tilde{p} + h_S \tilde{e} \right],$$
$$f'_1 = \frac{h_A^3 \tilde{p}}{4 v_A^2} + \frac{h_S^3 \tilde{e}}{4 v_S^2} - \frac{3}{v_A} h_A \tilde{p} - \frac{3}{v_S} h_S \tilde{p},$$
$$f''_1 = f_{\text{CS}} = \pm 4 \tilde{p}. \quad (45)$$

As for derivation and interpretation of (46), consult section II B. $\mathcal{E}$-function, defined in (6) is now

$$\mathcal{E}(\tilde{v}, u_s, \tilde{h}, \tilde{e}; \tilde{q}, \tilde{p}) = 6\pi (\tilde{e} \tilde{q} - f), \quad (47)$$
where $\tilde{q}$ is an electric charge. Extremization of $E$-function over $v_{1,2}$, $u_s$, $h_{A,S}$, and $\tilde{e}$ gives us then conditions equivalent to equations of motion and central charges, according to (7)-(8). The solution is

$$v_A = v_S = 4(|\tilde{q}| + 4), \quad u_s = \frac{|\tilde{p}|}{4(|\tilde{q}| + 4)}, \quad \tilde{e} = \frac{|\tilde{q}|}{|\tilde{q}| + 4}, \quad h_A = 8 \frac{|\tilde{p}|}{|\tilde{q}|} (|\tilde{q}| + 4), \quad h_S = 8 \frac{|\tilde{q}|}{|\tilde{q}|} (|\tilde{q}| + 4)$$

which is valid for all $\tilde{q}$ and $\tilde{p} \neq 0$. In the special cases when $\tilde{q} = 0$ it is understood that $|0|/0 = \pm 1$, meaning that there are two solutions for fixed choice of $\tilde{p}$.

For central charges we obtain

$$c \equiv \frac{1}{2} (c_L + c_R) = 6 \pi |\tilde{p}| (|\tilde{q}| + 8), \quad c_L - c_R = 48 \pi \tilde{p}$$

(49)

We still have to connect ”canonical” charges $\tilde{q}, \tilde{p}$ with integer-valued charges of microscopic configuration (consisting of fundamental strings and NS5-branes). Normally, one does this by referring to the (if known) lowest-order relations. In the present case, these are well-known, and, in our conventions, are given by

$$w = 4 \pi \tilde{p},$$

(50)

where microscopic charge $w$ is the number of fundamental strings, and

$$m = \frac{\tilde{q}}{4},$$

(51)

where $m$ is the NS5-brane charge. Using (50) and (51) in (48) and (49) we obtain for solution

$$v_A = v_S = 16 (|m| + 1), \quad u_s = \frac{1}{64 \pi (|m| + 1)}, \quad \tilde{e} = \frac{|wm|}{4 \pi m},$$

$$h_A = 32 \frac{w}{|w|} (|m| + 1), \quad h_S = 32 \frac{m}{|m|} (|m| + 1),$$

(52)

and for central charges

$$c \equiv \frac{1}{2} (c_L + c_R) = 6 |w|(|m| + 2), \quad c_L - c_R = 12 w$$

(53)

In the theories with Chern-Simons terms things can get more complicated, as it is known that such terms may introduce shifts between charges defined near the horizon and charges defined in asymptotic infinity (which is standard definition for charges). This effect was previously observed and analyzed in black hole setup in [27, 28]. Now, one way which avoid such issues to express all charges as magnetic charges of some gauge form with closed field strength. For $\tilde{p}$ this is already done, as it is the magnetic charge of a closed 3-form strength $K$, so we do not expect corrections to (50). But, $\tilde{q}$ is electric charge of $K$, so some additional work is necessary.

In [9] it was proposed that one should use 3-form strength $\overline{\mathcal{H}}$. From (26) follows

$$d \overline{\mathcal{H}} = \frac{3}{8} \alpha' \text{tr}(\overline{R} \wedge \overline{R})$$

(54)
and because all our solutions satisfy condition (31) we have $d\overline{\Pi} = 0$. Integer-valued magnetic charge carried by $\overline{\Pi}$ is given by

$$N = \frac{1}{64\pi^2} \oint_{S^3} \overline{\Pi} = \frac{h_S}{32} = \frac{m}{|m|} (|m| + 1). \quad (55)$$

We obtain a shift in the definition of charge. In a new definition of charge (55) doubling of solutions for $m = 0$ becomes natural – from (55) follows that $m = 0$ simply corresponds to two values of $N$, $N = \pm 1$. Using (55) in (52) and (53) we obtain for solution

$$v_A = v_S = 16 |N| , \quad u_s = \frac{1}{64\pi} \left| \frac{w}{N} \right| , \quad \bar{e} = \left| \frac{wN}{4\pi} \right| , \quad h_A = 32 \left| \frac{wN}{w} \right| , \quad h_S = 32 N , \quad (56)$$

and for central charges

$$c \equiv \frac{1}{2} (c_L + c_R) = 6 |w| (|N| + 1) , \quad c_L - c_R = 12 w . \quad (57)$$

The result for central charges $c_{R,L}$ is the same as the one obtained in [29], if we identify $N$ with quantum number $k$ from [29].

However, there is a problem with the above definition of charge, which is visible for more complicated geometries. As a specific example, relevant for us here, let us consider a full black string background for which (52) gives a near-horizon description. Away from the horizon we do not expect (31) to be valid, and so by (54) there is no reason to expect $d\overline{\Pi} = 0$. Without this property, there is no guarantee that "near-horizon charge" (54), which is calculated by taking $S^3$ to be in the near-horizon region where (52) is (approximately) valid, is going to be equal to the standardly defined (asymptotic) charge, which is obtained by taking sphere $S^3$ in (55) to be in an asymptotic infinity.

A simple solution for this problem is that instead of $\overline{\Pi}$, defined in (26), we use a 3-form $H$ defined by

$$H_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN} - 3\alpha' \Omega_{MNP} , \quad (58)$$

which is in fact ”standard” definition for 3-form strength of Kalb-Ramond field in heterotic string theory. The important difference is that $dH$ is given by

$$dH = \frac{3}{8} \alpha' \text{tr}(R \wedge R) , \quad (59)$$

and the right hand side is now topological density which is giving vanishing contribution to the difference between asymptotic and near-horizon charges. From (37) follows that $H$ and $\overline{\Pi}$ are connected by

$$H_{MNP} = \overline{\Pi}_{MNP} + 3\alpha' \mathcal{A}_{MNP} . \quad (60)$$

---

6. The factor of $64\pi^2$, which appears in our conventions, is in fact $2\alpha' \Omega_3$ ($\Omega_3$ is the volume of a unit 3-sphere $S^3$).

7. We thank Ashoke Sen for thorough explanations on this issue and for suggestion that 3-form $H$, instead of $\overline{\Pi}$, should be used to define the proper asymptotic charge.
From (60) and (38) we easily obtain a corresponding magnetic charge

$$Q_5 = \frac{1}{64\pi^2} \oint_{S^3} H = m,$$

which, though it is near-horizon evaluated, is also equal to the standard (asymptotic) charge. We emphasize that $H$ and $\overline{H}$ give the same result for charge calculated in infinity\(^8\), and the difference in near-horizon charge accumulates in the "intermediate region" when one passes from infinity towards horizon.

All in all, we finally obtained that our original definition of electric charge $\tilde{q} = 4m$, which depends on the particular treatment of mixed Chern-Simons term, is correctly describing the magnetic flux, and so $Q_5 = m$ is expected to be identified with the number of NS5-branes.

Comments on our AdS\(_3 \times S^3 \times T^4\) solution:

1. Solution (56) is supersymmetric for all values of charges.

2. It is easy to show that solution (56) satisfies the property (31), which means that we would obtain the same results if we started with more complicated supersymmetric action (32). Moreover, as argued before (and in more detail in section IV), we suggest that (31) also makes $\mathcal{L}_{\text{other}}$ is also irrelevant for our results, which means that we have obtained $\alpha'$-exact solutions and central charges of the full tree-level heterotic effective action (24).

3. Though solution (56) is purely mathematically regular for all $w \neq 0$, from string theory perspective it is meaningful only for $|w/m| \gg 1$ as in this case quantum corrections are expected to be small (effective string coupling $g_s$ satisfies $g_s^2 \sim \exp(2\Phi) \sim |N/w| \ll 1$) and so our purely classical analysis is dominating.

4. Results for central charges $c_{L,R}$ agree with microscopic calculations relying on AdS\(_3 \times \text{CFT}_2\) arguments [29], when we identify our number $N$ (near-horizon charge of 3-form strength $\overline{H}$) with their $k$, which is a level of the affine world-sheet symmetry algebra $\hat{SL}(2)$ in the supersymmetric (right-moving) sector. Also, using $N$ (instead of $m$) we obtain solutions in $\alpha'$-uncorrected form.

5. In view of possible AdS/CFT correspondence, our results for central charges $c_{L,R}$ agree (through the Cardy formula) with results for entropies of 5-dimensional 3-charge extremal black hole entropies calculated in [9], while comparison with calculation which uses $R^2$-type 5-dimensional supersymmetric action shows agreement just for the cases which correspond to BPS black holes (e.g., for $w, m > 0$ it agrees with $c_L$, but not with $c_R$) [4] (for BPS case it exist also a microscopic calculation for the entropy [30]).

\(^8\) The "dangerous" first term in (38), which is not manifestly diff-covariant, is a total derivative and gives vanishing contribution to the integral in (61). The rest of the difference between $H$ and $\overline{H}$ is such that its contribution to the charge calculated in asymptotic infinity vanishes.
C. AdS$_3 \times S^2$ backgrounds

The next example are AdS$_3 \times S^2$ solutions, which should describe near-horizon geometries of extremal black strings in $D = 5$ dimensions. We start from heterotic string theory compactified on $S^1 \times T^4$, taking for the charges coming from Kaluza-Klein fields of $S^1$ reduction to be non-vanishing. The details of this particular Kaluza-Klein reduction are reviewed in [3]. In our notation coordinate radius of $S^1$ is $\sqrt{\alpha'} = 4$. The non-vanishing fields are: string metric $G_{\mu\nu}$, dilaton $\Phi$, modulus $T = (\tilde{G}_{55})^{1/2}$, two Kaluza-Klein gauge fields $A_{(i)\mu}$ $(1 \leq i \leq 2)$, coming from $G_{MN}^{(6)}$ and 2-form potential $C_{MN}^{(6)}$, the 2-form potential $C_{\mu\nu}$ with the strength $K_{\mu\nu\rho}$, one Kaluza-Klein auxiliary two form $\mathbf{D}_{\mu\nu}$ coming from $\mathbf{P}_{(6)}^{(6)}$, and auxiliary 3-form $\mathbf{F}_{\mu\nu\rho}$. $^9$ $\mathcal{E}$-function formalism then dictates the following form for the non-vanishing fields

\[
\begin{align*}
ds^2 &= G_{\mu\nu}dx^\mu dx^\nu = v_A dv_A^2 + v_s dv_s^2, \\
K &= \tilde{\epsilon}_A, \\
F_1 &= \tilde{p}_1 \epsilon_S, \\
F_2 &= \frac{\tilde{p}_2}{16} \epsilon_S, \\
\mathbf{P} &= h_A \epsilon_A, \\
\mathbf{D} &= -\frac{dS}{2} \epsilon_S, \quad (62)
\end{align*}
\]

where now $\epsilon_S$ is a volume-form of unit $S^2$ sphere, $v_{A,S}, u_s, u_t, h_A$ and $dS$ are constants, eventually determined from equations of motion (as functions of electric field $\tilde{\epsilon}$ and magnetic charges $\tilde{p}_{1,2}$). These 3-charge black string configurations have microscopic interpretation as bound states of some number (connected to electric charge of $\mathbf{P}_{\mu\nu\rho}$, which means magnetic charge of $F_{(2)\mu\nu}$) of fundamental strings, some number (connected to magnetic charge of $\mathbf{D}_{\mu\nu}$, which means electric charge of $F_{(1)\mu\nu}$) of NS5-branes wrapped around torus $T^4$, and some number (connected to magnetic charge of $\mathbf{D}_{\mu\nu}$, which means electric charge of $K_{\mu\nu\rho}$) of Kaluza-Klein monopoles with "nut" on $S^1$.

As originally proposed in [20], the most efficient way to calculate the $\mathcal{E}$-function is to lift 5-dimensional background (62) back to 6-dimensions (by using KK reduction relations backwards) and than to perform calculation of $f$ function in 6-dimensions, where the action has much simpler form (presented in section III A). Details of this KK reduction are reviewed in [3]. By using them the background (62) in 6-dimensional language becomes

\[
\begin{align*}
ds^2 &= G_{MN}^{(6)} dx^M dx^N = v_A dv_A^2 + v_s dv_s^2 + u^2 \left(dx^5 - 2 \tilde{p}_1 \cos \theta d\phi \right)^2, \\
K_{012}^{(6)} &= \tilde{\epsilon} \frac{8}{\sqrt{2}} , \\
K_{\theta\phi5}^{(6)} &= -\frac{\tilde{p}_2}{16} \sin \theta, \\
\mathbf{P}_{012}^{(6)} &= h_A, \\
\mathbf{F}_{\theta\phi5}^{(6)} &= d_S \sin \theta. \quad (63)
\end{align*}
\]

Again, using (42) we can write function $f$, defined in (3), as

\[
f(\tilde{v}, \tilde{u}, \tilde{h}, \tilde{\epsilon}, \tilde{p}) = \int_{S^2 \times S^1} \sqrt{-G^{(6)}(\mathbf{L}_{\text{red}} = f_0 + f'_1 + f''_1}, \quad (64)
\]

Using (63) we obtain

\[
f_0 = \frac{1}{4} \left[u_s v_A^{3/2} v_s \left(-\frac{3}{v_A} + \frac{1}{v_s} - \frac{u_t^2 \tilde{p}_1^2}{v_s^2} - \frac{d_S^2}{4 u_t^2 v_s} \right) + h_A \tilde{p}_2 + u_t^2 d_S \tilde{\epsilon} \right] \quad (65)
\]

$^9$ Greek indices are 5-dimensional, i.e., $0 \leq \mu, \nu, \ldots \leq 4$, while capital latin indices are 6-dimensional, i.e., $0 \leq M, N, \ldots \leq 5$. Coordinate on $S^1$ is denoted $x^5$, with $0 \leq x^5 < 8\pi$. 

\[ f'_1 = \frac{6 h_A \tilde{p}_2}{v_A} - \frac{h_A^3 \tilde{p}_2}{2 v_A^3} + \frac{d_S^3 \bar{e}}{2 u_1^2 v_S^3} + \frac{2 u_1^2 d_S \tilde{p}_1^2}{v_S^3} - \frac{2 d_S \bar{e}}{v_S} \]  
\[ f''_1 = \pm 8 \tilde{p}_2 + 4 \bar{e} \left( \frac{u_1^2 \tilde{p}_1}{v_S} - \frac{2 u_1^2 \tilde{p}_1^3}{v_S^3} \right). \]

where for practical purposes we passed to variables \( h_{1,2} \), defined by
\[ h_1 \equiv -\frac{u_s v_2}{2 v_1^{3/2}} h_A, \quad h_2 \equiv \frac{u_s v_1^{3/2}}{2 u_1^2 v_2} d_S, \]
instead of \( h_A \) and \( d_S \). \( E \)-function is given by
\[ E(\bar{v}, \bar{u}, \bar{h}, \bar{c}; \bar{q}, \bar{p}) = 6\pi (\bar{c} \bar{q} - f), \]
where \( \bar{q} \) is electric charge conjugated to \( \bar{c} \). By extremizing \( E \)-function over \( v_{A,S}, u_{t,s}, h_A, d_S \), and \( \bar{c} \) we obtain the solutions. Before writing them down, let us make connection between canonical charges \( \bar{q}, \bar{p}_{1,2} \) and integer-valued microscopic charges. As all U(1) gauge-field strengths are closed, we can safely use lowest-order relations which in our conventions read
\[ \bar{q} = -\frac{W'}{2}, \quad \bar{p}_1 = N', \quad \bar{p}_2 = \frac{w}{8\pi}. \]

In microscopic interpretation (of black string) \( w \) is the number of fundamental strings, \( N' \) is the Kaluza-Klein monopole charge, and \( W' \) is the NS5-brane charge.

Supersymmetric solutions, characterized by \( N'W' \geq 0 \), are given by
\[ v_A = 4 v_S = 16(N'W' + 2), \quad u_s = \frac{1}{8\pi} \frac{|w|}{\sqrt{N'W' + 2}}, \quad u_t = \sqrt{\frac{W'}{N'} \left( 1 + \frac{2}{N'W'} \right)}, \]
\[ \bar{c} = -\frac{|wN'W'|}{\pi W'}, \quad h_A = 32 \frac{w}{|w|} (N'W' + 2), \quad d_S = 4 W' \left( 1 + \frac{2}{N'W'} \right). \]

Central charges in BPS case are
\[ c \equiv \frac{1}{2} (c_L + c_R) = 6 |w|(N'W' + 3), \quad c_L - c_R = 12 w. \]  
For \( w > 0 \) (72) gives
\[ c_L = 6 |w|(N'W' + 4), \quad c_R = 6 |w|(N'W' + 2), \]  
while for \( w < 0 \) one just has to exchange \( c_L \leftrightarrow c_R \).

Non-supersymmetric solutions, characterized by \( N'W' < 0 \), are given by
\[ v_A = 4 v_S = 16 |N'W'|, \quad u_s = \frac{1}{8\pi} \frac{|w|}{\sqrt{|N'W'|}}, \quad u_t = \sqrt{\frac{|W'|}{N'}}, \]
\[ \bar{c} = -\frac{|wN'W'|}{\pi W'}, \quad h_A = 32 \frac{|wN'W'|}{w}, \quad d_S = 4 W'. \]

Central charges in non-BPS case are
\[ c \equiv \frac{1}{2} (c_L + c_R) = 6 |w||N'W'| + 1), \quad c_L - c_R = 12 w. \]
For $w > 0$ (75) gives

$$c_L = 6 |w|(|N'W'| + 2), \quad c_R = 6 |wN'W'|,$$  

(76)

while for $w < 0$ one again just has to exchange $c_L \leftrightarrow c_R$.

It is interesting to find charges calculated from fluxes of 10-dimensional 3-forms $\mathcal{F}$ and $H$. In the case of $\mathcal{F}$, which is defined in (26), the corresponding charge $\mathcal{W}$ is calculated from

$$\mathcal{W} = \frac{1}{128 \pi^2} \oint_{S^2 \times S^1} \mathcal{F} = \frac{h_s}{4}$$  

(77)

In BPS case, using (71), we obtain

$$\mathcal{W} = W' + \frac{2}{N'}$$  

(78)

while in the non-BPS case we obtain a simple uncorrected relation

$$\mathcal{W} = W'.$$  

(79)

Using $\mathcal{W}$ instead of $W'$ puts all solutions (BPS and non-BPS) in $\alpha'$-uncorrected form, and central charges $c_{R,L}$ in the form (75).

However, we argued in section III B that to obtain proper asymptotic charge, instead of $\mathcal{F}$ we should use 3-form strength $H$ defined in (58). The corresponding flux quantum number is now

$$Q_5 \equiv \frac{1}{128 \pi^2} \oint_{S^2 \times S^1} H$$  

(80)

which again can be obtained by using (60) and (38). The result is

$$Q_5 = W' + \frac{1}{N'}$$  

(81)

In the case $N' = 1$ we obtain $Q_5 = W' + 1$. Now, it is known [1, 31] that presence of one Kaluza-Klein monopole adds $(-1)$-unit to NS5-brane charge, so in this case we can again identify of $Q_5$ with the number of NS5-branes.

Comments on our $\text{AdS}_3 \times S^2 \times S^1 \times T^4$ solutions (71) and (74):

1. It is easy to show that both solutions satisfy property (31). Consequences of this are the same as in section III B.

2. Though our solutions are regular for all $|w| \neq 0$, from string theory perspective they are meaningful only for $|w|/\sqrt{|N'W'|} \gg 1$ as in this case quantum corrections are expected to be small (effective string coupling $g_s^2 \sim \exp(2\Phi) = 1/u_s \ll 1$) and so our purely classical analysis is indeed dominant.

3. It is obvious that $\alpha'$-expansion is here effectively $1/|N'W'|$ expansion. So, one would expect problems for $N' = 0$ and/or $W' = 0$. However, we see that (71) is completely regular for $W' = 0$, $N' \neq 0$, though it is singular when $N' = 0$. Now, this is a bit strange because heterotic theory has a particular T-duality on $N' \leftrightarrow W'$ (in which one expects that $T \rightarrow 1/T$ and
\( F_{(1)\mu\nu} \leftarrow D_{\mu\nu} \), which now appears to be broken. This is of course not the case, and the resolution is that in non-trivial \( S^1 \) compactifications higher derivative corrections can change the relations between canonical fields (in our case \( T, F_{(1)\mu\nu} \) and \( D_{\mu\nu} \)) and proper string moduli (in our case \( S^1 \) radius \( R \) and fluxes \( F_{(1)\mu}, D_{\mu\nu} \)), and one needs to find appropriate field redefinitions before making identifications (see, e.g., [32]).

4. Our results for central charges agree with microscopic calculations relying on AdS\(_3\)/CFT\(_2\) arguments [5, 29]. Again, charge \( \hat{W} \) obtained from 3-form \( \hat{H} \) is connected with the total level \( k \) of the world-sheet affine algebra \( \hat{SL}(2) \) in the supersymmetric sector, and the relation is now \( k = N'\hat{W} \). Also, use of \( \hat{W} \) puts solutions in the \( \alpha' \)-uncorrected form.

5. Agreement with microscopic calculations is now also true for \( N' = W' = 0 \), where solutions describe near-horizon geometry of small black (fundamental) string. We mentioned above that in this regime low energy/curvature effective action is not expected to be well defined (as \( \alpha' \)-expansion is not well defined).\(^{10}\) This agreement is in contrast with the case of 6-dimensional black string (analyzed in previous section) where putting \( m = 0 \) gives wrong results for central charges.

6. The same result for \( c \) in BPS case (72) was also obtained from 5-dimensional \( R^2 \)-type action obtained by off-shell supersymmetrization of mixed Chern-Simons term in [36]. However, this action produces wrong result for \( c \) in non-BPS case, deviating from (75) already at \( \alpha'^{1} \)-order.\(^{11}\) This shows that this \( R^2 \)-supersymmetric action is incomplete already at 4-derivative order, a fact already noted in [4, 9].

7. Using AdS/CFT correspondence, our results for central charges \( c_{L,R} \) agree (through the Cardy formula) with results for entropies of 4-dimensional 4-charge extremal black hole entropies calculated in [9]. Comparison with calculation which uses \( R^2 \)-type 4-dimensional supersymmetric action shows agreement just for the cases which correspond to BPS black holes [4].

8. For \( N' = 1 \), 3-charge AdS\(_3 \times S^3 \times T^4 \) solutions become equal to 2-charge AdS\(_3 \times S^2 \times S^1 \times T^4 \) solutions when one identifies the corresponding \( Q_5 \) charges, i.e., for \( m = W' + 1 \). The explanation of this comes from understanding of these backgrounds as near-horizon geometries of black strings. Microscopic explanation of this "charge-shift" was given in [31], while macroscopic explanation, in the framework of \( R^2 \)-type supersymmetric effective action, was given in [27]. In our analysis (and also in [9]), charge-shift appears because Lorentz Chern-Simons term is evaluated on different topologies (\( S^3 \) in one case, and \( S^2 \times S^1 \) in the other).\(^{12}\)

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\(^{10}\) Though there is a proposal that near-horizon properties of small black strings could be effectively described by a simple Lovelock-type action [33]. Analysis from [34] is also giving some wind for this proposal. The extension of small black hole solutions to the whole space-time in the case of Gauss-Bonnet-type action was analyzed in [35].

\(^{11}\) This can be easily shown by simple extension of the method for constructing non-BPS solutions from [4] to AdS\(_3 \times S^2 \) geometries.

\(^{12}\) Note that geometries are locally isomorphic, but not globally.
D. AdS$_3 \times S^3 \times S^3$ backgrounds

Our final example are AdS$_3 \times S^3_+ \times S^3_-$ solutions of the heterotic string theory compactified on $S^1$. Compactification on $S^1$ is trivial (corresponding KK charges are all zero), and $\pm$ subscript on $S^3$ is put just to separate two 3-spheres. Contrary to previous two examples, these backgrounds do not have direct interpretation as near-horizon geometries of some black objects (and so, e.g., are not listed in [37]). Calculations here are similar to those from section III B, with the difference that now we have two 3-spheres and $K$ is a 6-form (because effective space-time is 9-dimensional).

Now, $\mathcal{E}$-function formalism forces the following form for the non-vanishing fields

\[
\begin{align*}
&ds^2 = v_A \, ds^2_A + v_+ \, ds^2_+ + v_- \, ds^2_-, \quad S = u_s, \\
&K = \bar{e}_- \epsilon_A \wedge \epsilon_+ + \bar{e}_+ \epsilon_A \wedge \epsilon_- + \bar{p} \epsilon_+ \wedge \epsilon_-, \quad H = h_A \epsilon_A + h_+ \epsilon_+ + h_- \epsilon_-, \\
&\text{Function } f \text{ is now}
\end{align*}
\]

Using (82) we obtain

\[
\begin{align*}
&f_0 = \frac{\pi^3}{4} \left[ u_s (v_A v_+) \bar{v} \frac{3}{2} \left( \frac{3}{v_+} + \frac{3}{v_-} - \frac{3}{v_A} + \frac{h_A^2}{4 v_A^3} - \frac{h_+^2}{4 v_+^3} - \frac{h_-^2}{4 v_-^3} \right) - h_A \bar{p} + h_+ \bar{e}_+ + h_- \bar{e}_- \right] \\
&f_1' = \frac{\pi^3}{2} \left( \frac{h_A^3 \bar{p}}{v_A^3} + \frac{h_+^3 \bar{e}_+}{v_+^3} + \frac{h_-^3 \bar{e}_-}{v_-^3} \right) - 6\pi^3 \left( \frac{h_A \bar{p}}{v_A} + \frac{h_+ \bar{e}_+}{v_+} + \frac{h_- \bar{e}_-}{v_-} \right) \\
&f_1'' = f_{CS} = \pm 48\pi^4 \bar{p}.
\end{align*}
\]

$\mathcal{E}$-function is now defined by

\[\mathcal{E}(\bar{v}, u_s, \bar{h}, \bar{e}, \bar{q}, \bar{p}) = 6\pi (\bar{e}_+ \bar{q}_+ + \bar{e}_- \bar{q}_- - f).\]

Extremization of $\mathcal{E}$-function over $v_A, u_s, h_A, \bar{e}_+$ and $\bar{e}_-$ gives us the following solution

\[
\begin{align*}
&v_\pm = \frac{2}{\pi^2} |\bar{q}_\pm| + 16, \quad v_A = \frac{v_+ v_-}{v_+ + v_-}, \quad u_s = v_A^{1/2} |\bar{p}| \\
&\bar{e}_\pm = \frac{|\bar{q}_\pm|}{2 |\bar{q}_\pm|} \left( \frac{v_A}{v_\pm} \right)^2, \quad h_A = 2 v_A \frac{\bar{p}}{|\bar{p}|}, \quad h_\pm = 32 \frac{q_\pm}{|q_\pm|} \left( \frac{|q_\pm|}{8\pi^3} + 1 \right)
\end{align*}
\]

From the expression for $h_+$ in (88) we can read that integer-valued fluxes $N^\pm_5$, corresponding to 3-form $\mathcal{H}$, through 3-spheres $S^3_\pm$ are given by

\[N^\pm_5 = \frac{q_\pm}{|q_\pm|} \left( \frac{|q_\pm|}{8\pi^3} + 1 \right).
\]

The remaining integer-valued charge $Q_1$ is given by the well-known lowest-order relation (see, e.g., [38])

\[Q_1 = 8\pi^4 \bar{p}.
\]
Using (89) and (90) in (88) we finally obtain that \( \text{AdS}_3 \times S^3 \times S^3 \) solution is given by

\[
\begin{align*}
 v_\pm &= 16 |N_5^\pm|, \\
v_A &= 16 \frac{|N_5^+ N_5^-|}{|N_5^+| + |N_5^-|}, \\
u_s &= \frac{1}{2(8\pi)^2} \left| \frac{Q_1}{N_5^+ N_5^-} \right| \left( |N_5^+| + |N_5^-| \right)^{-1/2}, \\
\bar{e}_\pm &= \frac{1}{8\pi^4} \frac{|N_5^\pm| Q_1}{N_5^\pm} \left( \frac{v_A}{v_\pm} \right)^2, \\
h_A &= 2 v_A Q_1, \\
h_{\pm} &= 32 N_5^\pm
\end{align*}
\]

We see that by using 3-form \( \Pi \) to define magnetic charges, solutions again have \( \alpha' \)-uncorrected form, and the sole effect of \( \alpha' \)-corrections are charge-shifts (89).

Finally, the central charges are given by

\[
c \equiv \frac{1}{2} (c_L + c_R) = 6 |Q_1| \left( \frac{|N_5^+ N_5^-|}{|N_5^+| + |N_5^-|} + 1 \right), \quad c_L - c_R = 12 Q_1.
\]

For \( Q_1 > 0 \) (92) reads

\[
c_L = 6 |Q_1| \left( \frac{|N_5^+ N_5^-|}{|N_5^+| + |N_5^-|} + 2 \right), \quad c_R = 6 |Q_1| \frac{|N_5^+ N_5^-|}{|N_5^+| + |N_5^-|}, \quad (93)
\]

while for \( Q_1 < 0 \) one just has to exchange \( c_L \leftrightarrow c_R \).

If we instead use 3-form \( H \) to calculate charges \( Q_5^\pm \), we obtain

\[
|Q_5^\pm| = |N_5^\pm| - 1 \quad (94)
\]

Comments on our \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) results:

1. It is easy to show that solution (48) satisfies property (31). Consequences of this are the same as in sections III B and III C.

2. Though solution (48) is regular for all \( Q_1 \neq 0 \), from string theory perspective it is meaningful only for \( |Q_1| \gg |N_5^+ N_5^-|(|N_5^+| + |N_5^-|)^{1/2} \), because then string coupling satisfies \( g_s^2 \sim \exp(2\Phi) \ll 1 \) and our purely classical analysis is valid.

3. Solution is singular for vanishing \( N_5^+ \) or \( N_5^- \). Again, this is not surprising because we see from our solution that \( \alpha' \)-expansion is effectively expansion in \( 1/N_5^+ \), \( 1/N_5^- \), and so it is not well defined when any of the charges vanish.

4. As far as we know, our results for central charges are new. In particular, we are not aware of any \( \alpha' \)-exact microscopic calculation of central charges in this case. In fact, even the microscopic configuration of strings/branes which should lead to such backgrounds is not known. Also, a holographic (CFT2) dual is still not known\(^{14}\), contrary to previous examples analyzed in sections III B and III C.

5. Solution is supersymmetric for all values of charges.

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\(^{13}\) More precisely, \( |N_5^+||N_5^-|^m \) terms will appear at \( \alpha'^m \) order.

\(^{14}\) See [38] for thorough analysis of this issue in type-II string theories.
E. Solutions in type-II superstring theories

All geometries we considered in the paper also appear in NS-NS sector of type-II string theories. We now show that from our analysis directly follows that such type-II solutions will all be α′-uncorrected. The reason is that in type-II theories there are no classical Lorentz Chern-Simons terms (and in particular they are not present in (26)), and the truncated tree-level effective action is given by

\[ \mathcal{L}^{(\text{II})} = \mathcal{L}_0 + \mathcal{L}_{\text{other}}, \]

where \( \mathcal{L}_{\text{other}} \) is the same as in the heterotic case [39]. As the lowest-order solutions (obtained from \( \mathcal{L}_0 \)) all satisfy property (31), we conclude that higher-derivative term \( \mathcal{L}_{\text{other}} \) should be irrelevant in calculations of solutions and corresponding central charges, which stay α′-uncorrected.

This is a well-known fact obtained by other means in the literature. We have shown here how it can be understood as a simple consequence of the form of the 10-dimensional tree-level effective actions of type-II theories.

IV. CONCLUSION

We have shown on several examples of string backgrounds containing AdS\(_3\) factor how one can calculate in α′-exact manner BPS and non-BPS solutions and corresponding conformal central charges from the complete tree-level effective action (by taking into account all higher-derivative terms). Let us here discuss some of the important issues and outcomes of our analysis, which are common to all examples:

1. Though our solutions were obtained from the reduced Lagrangian (30), from the fact that they all satisfy property (31) it follows that they are also solutions of the supersymmetric Lagrangian (32) (obtained in \( D = 10 \) by \( \mathcal{N} = 1 \) supersymmetry completion from Chern-Simons term). Agreement of our ”macroscopic” results for central charges agree with those obtained ”microscopically” shows that Chern-Simons terms are solely responsible for α′-corrections. Now, this is not surprising for supersymmetric (BPS) solutions (56), (71) and (91), because it was shown in effective AdS\(_3\) analyses [7, 8, 17] that this is generally valid if supersymmetry is present in effective 3-dimensional theory. What is new here is that non-supersymmetric example (74) shows that this happens also in cases where 10-dimensional supersymmetry is completely broken in effective 3-dimensional theory. What is new here is that non-supersymmetric example (74) shows that this happens also in cases where 10-dimensional supersymmetry is completely broken in effective 3-dimensional theory. These examples suggest possible extension of the results from [7, 8, 17] to more general (non-supersymmetric) situations.

2. This immediately leads us to the question why the part of the tree-level effective Lagrangian denoted \( \mathcal{L}_{\text{other}} \) in (24) is not giving any contribution to central charges (and, as we are suggesting, neither to the solutions). This part starts at α\(^3\) (8-derivative) order and contains the (in)famous ζ(3)RRRR terms. Contrary to the Δ\( \mathcal{L}_{\text{CS}} \) term, the structure of \( \mathcal{L}_{\text{other}} \) is grossly unknown, with only 4-point sector being completely known [39]. It was shown in [39] that this 4-point sector can be written in simple form ζ(3)\( \overline{R} \overline{R} \overline{R} \overline{R} \), where \( \overline{R} \) stands
for torsional Riemann tensor obtained from the modified ("torsional") connection (28) (and written explicitly in (29)). This has stimulated authors of [11] to conjecture that the whole $\mathcal{L}_{\text{other}}$ can be written in such way (by using $\overline{R}_{MNPQ}$ only). Indeed, the most recent calculations of some 5-point terms ($\zeta(3)RRRH$ and $\zeta(3)R(\Delta H)(\Delta H)HH$) are in accord with this conjecture [12]. What is important for us here is if this conjecture is correct, it would immediately imply that term $\mathcal{L}_{\text{other}}$ does not contribute to our solutions and central charges (a proof is the same as in the case of $\Delta \mathcal{L}_{\text{CS}}$). That would mean that we have found solutions of the full tree-level effective action(s). Of course, we can turn the argument around and claim that agreement of our results (for central charges) with the microscopic calculations argues in favor of the conjecture. However, we should be careful in making strong statements because of the following reasons:

(a) It would be enough for our purposes that every monomial in $\mathcal{L}_{\text{other}}$ is bilinear in $\overline{R}_{MNPQ}$. This weaker version of the conjecture (appearing already in [10]) would also imply that $\mathcal{L}_{\text{other}}$ is irrelevant for our results.

(b) Beside $\overline{R}_{MNPQ} = 0$, our solutions satisfy other common properties, e.g., $R = 0$ (vanishing of 10-dimensional Ricci scalar), $\overline{\mathcal{R}}^2 = 0$, and on top of it all covariant derivatives are zero. So, adding to effective action terms which contain covariant derivative or bilinear in $R$, $\overline{\mathcal{R}}^2$ would not change our results, and so our analysis does not put any constraint on them. To clear this issue, we have to find examples in which we have $\overline{R}_{MNPQ} = 0$ but not these other properties.

3. Reduced Lagrangian $\tilde{\mathcal{L}}_{\text{red}}$ is of 4-derivative type. Our analyses offers direct explanation (in 10-dimensional set-up) why terms with six and more derivatives in 10-dimensional string effective actions are irrelevant for calculations considered here and in [9].

4. All our solutions have the form of $\alpha'$-uncorrected solutions when we used magnetic charges calculated in near-horizon region from 3-form $\overline{\mathcal{H}}$. This is exactly what is obtained in sigma model calculations, despite the fact that those two methods are typically using different field-redefinition schemes (for example, this agreement is not manifestly present in corresponding black hole near-horizon analyses [9, 40]).

5. We have shown in section III E that our strategy can be trivially extended to NS-NS backgrounds in type II string theories. Now, structure of dualities connecting heterotic and type II string theories suggests that the same strategy could be further extended to Ramond-Ramond backgrounds in type II theories. Indeed, it has been proposed [41, 42] that R-R 5-form field couples to the gravity at 8-derivative order exclusively through a relation similar with Eq. 29 (see Eq. (2.10)-(2.11) from [42]). If this proposal is correct, it could be used to argue for the vanishing of corresponding corrections to particular backgrounds, using the same logic which we applied here and in [9].
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