Constructing Point Form Mass Operators from Vertex Interactions

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Abstract

A relativistic few-body theory is formulated using point form quantum mechanics, in which all of the interactions are in the four-momentum operator and Lorentz transformations are kinematic. The four-momentum operator is written as a product of mass and four-velocity operators, where the mass operator is the sum of free and interacting mass operators. Interacting mass operators are constructed from vertices, products of local field operators, evaluated at the space-time point zero. Matrix elements of such mass operators, evaluated on four-velocity eigenstates of a truncated Fock space, which is the space of the few-body theory, are shown to behave like relativistic potentials. Examples for a simple vertex are given.

1 Introduction

One of the goals of few-body nonrelativistic nuclear physics has been to use phenomenological potentials determined by two-body systems (for example the deuteron and nucleon-nucleon scattering) to predict the behavior of three or more body systems. Similarly, current operators determined by one and two body systems are used to predict form factors for three or more body

\[1\] P.A.C.S. 21.30.Fe, 24.10.Jv, 25.10.+s, 25.80.Dj
systems. In the energy domain where nonrelativistic quantum mechanics is valid this program has been quite successful [1].

To carry out a similar program at higher energies requires introducing some form of relativistic quantum mechanics, not only because of the higher energies involved, but also because particle (for example pi meson) production becomes important. One of the ways to generate a Poincaré covariant theory is via the so-called Bakamjian-Thomas procedure [2][3], where interactions for the few body system are put into a mass operator, which takes the place of the Hamiltonian. While this procedure is indeed relativistic, it generally violates cluster separability properties, wherein if two clusters of interacting particles are widely separated, there should be no interactions between the separated clusters. It has been shown that cluster separability properties can be satisfied by introducing appropriate unitary operators [4], but this procedure is unwieldy, and in any event it is not entirely clear how to extend it to deal with particle production.

Quantum field theory on the other hand provides a setting in which particle production occurs naturally, as does cluster separability. However when the Fock space on which the quantum field theory is formulated is truncated to a finite number of degrees of freedom, relativity is lost. The goal of this paper is to construct mass operators acting on a truncated Fock space for an n-body system. Up to a given truncation the mass operator should determine all the possible reactions that were allowed by the field theory. Further the many-body theory should satisfy the following properties:

- For a given truncation of Fock space the dynamics should arise from a many-body mass operator.
- Generators built out of the mass operator (and other kinematic operators) should satisfy the Poincaré commutation relations.
- Forces come from vertex interactions, which generate the mass operator, yet should look like relativistic ”potentials”.
- The theory should be Lorentz covariant.
- Bound state problems are solved as eigenvalue-eigenvector problems, from which renormalized masses can be calculated.

The theory to be formulated here is intermediate between a relativistic quantum mechanics for a fixed number of particles, and quantum field theory, as a theory with infinite numbers of degrees of freedom, and incorporating full particle production and vacuum structure. It is reminiscent of earlier models such as the Lee model [5], with its mass renormalization and limited particle production. Though the original Lee model was not relativistically covariant,
Fuda [3] has shown, using the front form, how to make it properly relativistic. The theory developed here differs from this generalized Lee model in that it allows for an arbitrary number of produced particles, up to a prespecified limit, given by the original truncation. As will be discussed in the conclusion there are still open questions about the cluster properties for such a theory.

The procedure for obtaining a mass operator will be carried out in the context of point form relativistic quantum mechanics [4], in which all of the interactions are put in the four-momentum operator and the Lorentz generators are kinematic. The four-momentum operator is written as the product of a mass operator times a four-velocity operator, which is the Bakamjian-Thomas construction in the point form. The interacting part of the mass operator is obtained from matrix elements of a vertex, products of local field operators evaluated at the space-time point zero, using velocity states, eigenstates of the four-velocity operator, in which the matrix elements are diagonal in the four-velocity.

In section 2 the elements of point form quantum mechanics are reviewed and both free and interacting mass operators are constructed, using velocity states, diagonal in the overall four-velocity of an n particle state. In section 3 the techniques discussed in section 2 are applied to the simple example of a scalar ”nucleon”, scalar ”pion” vertex.

2 Point Form Relativistic Quantum Mechanics and Mass Operators

In point form relativistic quantum mechanics all interactions are put in the four-momentum operator. The Lorentz generators contain no interactions, so the point form is manifestly Lorentz covariant. In fact the Lorentz generators will almost never be used; instead global Lorentz transformations will define the transformation properties of operators and states. If $U_{\Lambda}$ is a unitary operator representing the Lorentz transformation $\Lambda$, then the four-momentum operator must satisfy the following point form equations:

$$[P_\mu, P_\nu] = 0 \quad (1)$$

$$U_{\Lambda}P_\mu U^{-1}_{\Lambda} = (\Lambda^{-1})^\nu_\mu P_\nu. \quad (2)$$

These equations are simply one way of writing the Poincaré commutation relations in which the relations of the four-momentum operators are empha-
sized. The mass operator is defined to be $M := \sqrt{P \cdot P}$ and must have a nonnegative spectrum.

Since the four-momentum operators are the generators of space-time translations, they can be used to define the relativistic generalization of the time dependent Schrödinger equation,

$$i \frac{\partial \Psi_x}{\partial x^\mu} = P_\mu \Psi_x,$$

(3)

where $\Psi_x$ is an element of the Hilbert space. $x = x_\mu$ is a space-time point and acts as four parameters, playing the same role as the time parameter in the nonrelativistic time dependent Schrödinger equation. From Eq.3 it follows that $\Psi_x$ satisfies a generalized Klein-Gordon equation,

$$\left( \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} + M^2 \right) \Psi_x = 0,$$

(4)

where $M$ is the mass operator. It is Eqs.3,4 that take the place of equations of motion for operators. If $P_\mu$ has no explicit space-time dependence then Eq.3 can be written as an eigenvalue equation for the four-momentum operator:

$$P_\mu \Psi = p_\mu \Psi.$$

(5)

The simplest way in which the point form equations can be satisfied is with single particle irreducible representations of the Poincaré group. Then the Hilbert space is $L^2(R^3) \otimes V^j$, where $V^j$ is the $2j+1$ dimensional spin space for a particle of spin $j$, and the state $|p, \sigma>$ transforms under Poincaré transformations as

$$P_\mu |p, \sigma> = p_\mu |p, \sigma>,$$

(6)

$$U_\Lambda |p, \sigma> = \sum |\Lambda p, \sigma'> D^{j}_{\sigma' \sigma} (R_W),$$

(7)

$$R_W : = B^{-1}(\Lambda v)\Lambda B(v).$$

(8)

Here the four-momentum $p$ satisfies the mass constraint $p \cdot p = m^2$ and the four-velocity $v$ is defined by $v = p/m$. $B(v)$ is a boost (see reference [8]), a Lorentz transformation that takes the rest four-momentum, $p^{rest} = (m, 0, 0, 0)$ to the four-momentum $p$: $B(v)p^{rest} = p$. $R_W$ is a Wigner rotation and $D^{j}(\cdot)$ is a matrix element of the rotation group. It is clear that the operators Eqs. 6,7 satisfy, by construction, the point form equations.
Many-particle states with the same transformation properties as the single particle ones are conveniently obtained by introducing creation and annihilation operators. Let \( a^\dagger(p, \sigma) \) be the operator that creates the state \( |p, \sigma> \) from the vacuum. If \( a(p, \sigma) \) is its adjoint, these operators must satisfy the following relations:

\[
[a(p, \sigma), a^\dagger(p', \sigma')]_\pm = E \delta^3(p - p') \delta_{\sigma, \sigma'} \\
U_a a^\dagger(p, \sigma) U_a^{-1} = e^{ip \cdot a}(p, \sigma),
\]

\( P_\mu(fr) = \sum \int \frac{d^3p}{E} p_\mu a^\dagger(p, \sigma) a(p, \sigma), \)

\( U_\Lambda a^\dagger(p, \sigma) U_\Lambda^{-1} = \sum a^\dagger(\Lambda p, \sigma') D^i_{\sigma', \sigma}(R_W). \)

Here \( P_\mu(fr) \) is the free four-momentum operator and plays a role analogous to the free Hamiltonian in nonrelativistic quantum mechanics. Again it is straightforward to show that \( P_\mu \) satisfies the point form equations. \( U_a \) in Eq.10 is the unitary operator representing the four-translation \( a \).

To prepare for the construction of interacting mass operators, it is convenient to introduce velocity states, states with simple Lorentz transformation properties. If a Lorentz transformation is applied to a many-particle state, \( |p_1, \sigma_1...p_n, \sigma_n> = a^\dagger(p_1, \sigma_1)...a^\dagger(p_n, \sigma_n)|0> \), then it is not possible to couple all the momenta and spins together to form spin or orbital angular momentum states, because the Wigner rotations for each momentum state are different. However, velocity states, defined as \( n \)-particle states in their overall rest frame boosted to a four-velocity \( v \) will have the desired Lorentz transformation properties:

\[
|v, \vec{k}_i, \mu_i>: = U_{B(v)}|k_1, \mu_1...k_n, \mu_n> \\
= \sum |p_1, \sigma_1...p_n, \sigma_n> \prod D^i_{\sigma_i, \mu_i}(R_W_i). \\
U_\Lambda|v, \vec{k}_i, \mu_i> = U_\Lambda U_{B(v)}|k_1, \mu_1...k_n, \mu_n> \\
= U_{B(\Lambda v)} U_{R_W} |k_1, \mu_1...k_n, \mu_n> \\
= \sum |\Lambda v, R_W \vec{k}_i, \mu_i'> \prod D^i_{\mu_i', \mu_i}(R_W). \]

Unlike the Lorentz transformation of an \( n \)-particle state, where all the Wigner rotations of the D functions are different, in Eq.15 it is seen that the Wigner rotations in the D functions are all the same and given by Eq.8. Moreover the same Wigner rotation also multiplies the internal momentum vectors, which means that for velocity states, spin and orbital angular momentum can be
coupled together exactly as is done nonrelativistically (for more details see reference [8]). The relationship between single particle and internal momenta is given by

\[ p_i = B(v)k_i, \sum k_i = 0; \]

Eq. 14 provides the link between velocity and many-particle states, where \( R_{W_i} \) is given by replacing \( p \) by \( k_i \) and \( \Lambda \) by \( B(v) \) in Eq. 8. From the definition of velocity states it then follows that

\[ V_\mu|v, \vec{k}_i, \mu_i> = v_\mu|v, \vec{k}_i, \mu_i>, \]

\[ M(fr)|v, \vec{k}_i, \mu_i> = m_n|v, \vec{k}_i, \mu_i>, \]

\[ P_\mu(fr)|v, \vec{k}_i, \mu_i> = m_n v^\mu|v, \vec{k}_i, \mu_i>, \]

(16) (17) (18)

with \( m_n = \sum \sqrt{m_i^2 + k_i^2} \) the 'mass' of the \( n \)-particle velocity state and \( P_\mu(fr) = M(fr)V_\mu \). On velocity states the free four-momentum operator has been written as the product of the four-velocity operator times the free mass operator, which is the Bakamjian-Thomas construction for the point form. More generally four-momentum operators are written as \( P_\mu = MV_\mu \), where the four-velocity operator is defined by \( V_\mu := \frac{P_\mu(fr)}{M_{fr}} \); \( V_\mu \) is diagonal on velocity states, as seen in Eq. 16. The mass operator is the sum of free and interacting mass operators, \( M = M_{fr} + M_I \); if the mass operator commutes with the four-velocity operator and Lorentz transformations, then the point form Eqs. 1, 2 will be satisfied.

The goal now is to construct an interacting mass operator on a truncated Fock space, where the Fock space is formed from creation and annihilation operators. The truncated Fock space is the space of the many-body system, a system with a finite number of degrees of freedom. For example if the creation and annihilation operators are those for (bare) pions and nucleons, the truncated Fock space could be chosen as the direct sum of two nucleon spaces with zero to \( n \) pions, appropriate for the scattering of two nucleons as well as the production of mesons.

The same creation and annihilation operators can also be used to form local fields; the general procedure for this construction is given in reference [9] and is reviewed in detail in one of the papers in this series, dealing with constructing current operators for arbitrary spin particles [10]. Moreover reference [9] shows how to take products of such arbitrary spin fields and couple them together to form Lorentz scalar densities, which, following this reference will be denoted by \( \mathcal{H}(x) \). Examples of such scalar densities of products of local fields are the familiar \( \bar{\Psi}\gamma_5\Psi\phi \) for pion-nucleon vertices and \( \bar{\Psi}\gamma_\mu\Psi A^\mu \) for electron-photon vertices.
The fact that $\mathcal{H}(x)$ is a Lorentz scalar density means that $U_\Lambda \mathcal{H}(x) U_\Lambda^{-1} = \mathcal{H}(\Lambda x)$. Therefore at the space-time point $x = 0$, $U_\Lambda \mathcal{H}(0) U_\Lambda^{-1} = \mathcal{H}(0)$, so that $\mathcal{H}(0)$ is a Lorentz scalar. The velocity state matrix element of $\mathcal{H}(0)$ on the truncated Fock space, $<v'\vec{k}',\mu'_i|\mathcal{H}(0)|v\vec{k},\mu_i >|_{v' = v}$, where the initial and final four-velocities are the same, can be viewed as the kernel of an interacting mass operator on the truncated Fock space, because it commutes with the four-velocity operator (since it is diagonal in the four-velocity, by construction) and commutes with Lorentz transformations.

However, since $\mathcal{H}(0)$ is constructed from products of fields, its velocity state matrix elements may not give kernels of well defined operators on the truncated Fock space. In order that the interacting mass operator be well defined on the truncated Fock space, matrix elements of $\mathcal{H}(0)$ are multiplied by "form factors", functions of $(p' - p)^2 = (m'v' - mv)^2$, where $m$ and $m'$ are the free masses of the many-particle state, given in Eq.17. But since the initial and final four-velocities are the same, the form factor will be a function of the magnitude of mass differences only, namely $\Delta m := |m' - m|$. The definition of the interacting many-body mass operator is then given by

$$M_I : = <v',\vec{k}',\mu'_i|\mathcal{H}(0)|v,\vec{k},\mu_i >|_{v' = v} f(\Delta m),$$

and by virtue of the form factor $f(\Delta m)$ is a well defined operator on the truncated Fock space. It commutes with Lorentz transformations and the four-velocity operator, so that $P_{\mu} = (M_{fr} + M_I) V_{\mu}$ satisfies the point form equations, Eq.1,2 and provides the starting point for a relativistic many-body theory.

It should be noted that $M_I$ actually is independent of the four-velocity:

$$M_I : = <v',\vec{k}',\mu'_i|\mathcal{H}(0)|v,\vec{k},\mu_i >|_{v' = v} f(\Delta m)$$
$$= <k'_1\mu'_1...|U_{B(v)}^{-1}\mathcal{H}(0)U_{B(v)}|k_1\mu_1... >|_{v' = v} f(\Delta m)$$
$$= <k'_1\mu'_1...|\mathcal{H}(0)|k_1\mu_1... >|_{v' = v} f(\Delta m),$$

where the $k_i$ and $k'_i$ are the internal momenta defined in Eq.13 and use has been made of the fact that $\mathcal{H}(0)$ is a Lorentz scalar. Also, since under Lorentz transformations the internal momenta are rotated by the Wigner rotation of Eq.15 the mass operator is rotationally invariant and satisfies all the properties of a ”potential”. Finally, since $M_I$ arises from products of fields, it always has only off-diagonal matrix elements in the truncated Hilbert space.
3 An Example: The Scalar Nucleon-Pion Vertex

To show how mass operators and generalized Lippmann-Schwinger equations result from vertices, this section works out the example of a scalar “nucleon” interacting with a scalar “pion”. The vertex is of the form $\mathcal{H}(x) = g \Psi^\dagger(x)\Psi(x)\phi(x)$ where $g$ is a coupling constant. The interacting mass operator can be written from Eq.19 as

$$M_I = gf(\Delta m) < v, \vec{k}, \mu_i | \Psi^\dagger(0)\Psi(0)\phi(0) | v, \vec{k}', \mu_i' >,$$

and various baryon number sectors evaluated.

As a first truncation consider the baryon number one sector, with the Hilbert space the direct sum of bare nucleon plus bare nucleon and bare pion. Then the total mass operator can be written as a two by two matrix operator in the direct sum space:

$$M = \begin{bmatrix} m_N(0) & gK^\dagger \\ gK & D_2(0) \end{bmatrix}$$ (21)

where the internal momentum variables are defined by

$p_N = B(v)(\omega_N(0), \vec{k})$ and $p_\pi = B(v)(\omega_\pi, -\vec{k})$ with $\omega_N(0) = \sqrt{m_N^2(0) + \vec{k}^2}$

and $\omega_\pi = \sqrt{m_\pi^2 + \vec{k}^2}$. $m_N(0)$ is the bare nucleon mass and $D_2(0) := \sqrt{m_N^2(0) + \vec{k}^2} + \sqrt{m_\pi^2 + \vec{k}^2} = \omega_N(0) + \omega_\pi$, a diagonal operator in the two particle space, is the relativistic two particle mass. The interacting mass operator that connects the two particle space to the one nucleon space is the kernel

$$< \vec{k} | K | > = f(\Delta m) = f(\omega_N(0) + \omega_\pi - m_N(0)).$$ (22)

On the direct sum space the problem to be solved is the mass operator eigenvalue-eigenvector problem, $M\phi = m\phi$, where, for the bound state problem, $m = m_N$ is the physical nucleon mass, while for the scattering problem, $m = \sqrt{s}$ is the invariant relativistic energy for pion-nucleon scattering. Designating the components of the physical one nucleon state as $(\phi_1^N, \phi_2^N)$, the mass operator eigenvalue equation becomes

$$m_N(0)\phi_1^N + g(K, \phi_2^N) = m_N\phi_1^N$$ (23)
\begin{align}
g K \phi_1^N + D_2(0) \phi_2^N &= m_N \phi_2^N \quad (24) \\
\phi_2^N &= g (m_N - D_2(0))^{-1} K \phi_1^N \quad (25) \\
g^2 (K, (m_N - D_2(0))^{-1} K) &= m_N - m_N(0), \quad (26)
\end{align}

where in the last equation, \( \phi_1^N \) has been cancelled from both sides of Eq.26 because it is a number (to be determined by normalization requirements).

For a given truncation, Eq.26 provides the relationship between the bare and physical nucleon mass. If \( m_N \) is taken to be the physical nucleon mass, then, for a given form factor, the bare nucleon mass, \( m_N(0) \), is fixed. Written out more explicitly, Eq.26 becomes

\[ g^2 \int \frac{d^3k}{\omega_N(0)\omega_\pi} \frac{|f(\omega_N(0) + \omega_\pi - m_N(0))|^2}{m_N - \omega_N(0) - \omega_\pi} = m_N - m_N(0), \quad (27) \]

and \( f \) is used to regularize the large momentum components in the integral. The nucleon wave function is normalized to one; that is, \(|\phi_1^N|^2 + (\phi_2^N, \phi_2^N) = 1\), where \( \phi_2^N \) is given in Eq.25.

To formulate the pion-nucleon scattering problem, the mass operator, Eq.21, is written as the sum of the free mass operator, \( M_0 \), and the interacting mass operator, \( M_I \), \( M = M_0 + M_I \), where

\begin{align}
M_0 &= \begin{bmatrix} m_N(0) & 0 \\
0 & D_2(0) \end{bmatrix} \quad (28) \\
M_I &= \begin{bmatrix} 0 & gK^\dagger \\
gK & 0 \end{bmatrix} \quad (29)
\end{align}

The relativistic Lippmann-Schwinger equation can then be written as an integral equation on the direct sum space with scattering wave function \( \Psi = (\Psi_1, \Psi_2) \):

\begin{align}
\Psi &= \Phi + (\sqrt{s} - M_0 + i\epsilon)^{-1} M_I \Psi \quad (30) \\
\Psi_1 &= \frac{g}{\sqrt{s} - m_N(0)} (K, \Psi_2) \quad (31) \\
\Psi_2 &= \phi_2 + \frac{g^2}{\sqrt{s} - m_N(0)} (\sqrt{s} - D_2(0))^{-1} K (K, \Psi_2), \quad (32)
\end{align}

where the second term in Eq.32 looks like a separable potential with a coupling constant \( g^2/(\sqrt{s} - m_N(0)) \). That is, in this simple truncation the
Lippmann-Schwinger equation can be solved exactly to compute the pion-nucleon scattering amplitude. \( \Phi = (0, \phi_2) \), where \( \phi_2 \) is the free pion-nucleon momentum state, and all \( \sqrt{s} \) factors in Eqs.30,31 should be replaced by \( \sqrt{s} + i \epsilon \).

The function \( f(\Delta m) \) used to relate bare and physical mass, Eq.27 and generate the separable potential, Eq.32, can also be used in other baryon number sectors. For baryon number two, consider the truncated space consisting of two bare nucleons plus two bare nucleons and bare pion, a direct sum of two particle and three particle spaces. In this case the mass operator has the form

\[
M = \begin{pmatrix}
D_2(0) & gK^+ \\
gK & D_3(0)
\end{pmatrix}
\]

where the internal momenta are related to single particle momenta by the following relations: \( p_1' = B(v)(\omega, \vec{k}), p_2' = B(v)(\omega, -\vec{k}) \), with \( \omega = \sqrt{m^2_N(0) + \vec{k}^2} \); \( p_1 = B(v)(\omega_1, \vec{k}_1), p_2 = B(v)(\omega_2, \vec{k}_2), p_\pi = B(v)(\omega_\pi, \vec{k}_\pi) \), with \( \omega_i = \sqrt{m^2_N(0) + \vec{k}^2_i} \) and \( \vec{k}_1 + \vec{k}_2 + \vec{k}_\pi = 0 \). The diagonal operator \( D_2(0) = 2|K| \) is the bare two particle relativistic mass, while the diagonal operator \( D_3(0) = \omega_1 + \omega_2 + \omega_\pi \) is the bare three particle mass. The two particle to three particle transition kernel is given by

\[
< \vec{k}_1 \vec{k}_2 | K | \vec{k} > = f(\omega_1 + \omega_2 + \omega_\pi - 2\omega) \]

\[
(\delta^3(\vec{k} + \vec{k}_1) + \delta^3(\vec{k} - \vec{k}_1) + \delta(\vec{k} + \vec{k}_2) + \delta(\vec{k} - \vec{k}_2))
\]

where the argument in \( f \) is (the magnitude of) the mass difference between the three and two particle states. The four delta functions are a consequence of the two nucleons being identical particles.

With the mass operator given in Eq.33 a "deuteron" bound state can be calculated as a two component mass operator eigenvector, corresponding to the deuteron mass \( m_D \):

\[
M \phi_D^D = m_D \phi_D^D, \quad (35)
\]

\[
D_2(0) \phi_2^D + gK^+ \phi_3^D = m_D \phi_2^D \quad (36)
\]

\[
gK \phi_2^D + D_3(0) \phi_3^D = m_D \phi_3^D \quad (37)
\]

\[
\phi_3^D = g(m_D - D_3(0))^{-1} K \phi_2^D \quad (38)
\]

\[
D_2(0) \phi_2^D + g^2K^+(m_D - D_3(0))^{-1} K \phi_2^D = m_D \phi_2^D. \quad (39)
\]
Though the first term in Eq.39 is the bare relativistic kinetic energy, the second term, which looks like a potential energy term, contains the unknown deuteron mass eigenvalue; because the component of the deuteron wavefunction in the three particle sector has been eliminated, Eq.39 is not of the usual form of an eigenvalue equation, but is known in coupled channel problems. The deuteron wave function is normalized by writing $||\phi_2^D||^2 + ||\phi_3^D||^2 = 1$.

The mass operator, Eq.33 can be used to calculate scattering in the nucleon-nucleon and nucleon-nucleon-pion channels by again writing the mass operator as a sum of the free mass operator plus interacting mass operator, $M = M_0 + M_I$:

$$M_0 = \begin{bmatrix} D_2(0) & 0 \\ 0 & D_3(0) \end{bmatrix}$$

$$M_I = \begin{bmatrix} 0 & gK^\dagger \\ gK & 0 \end{bmatrix}$$

$$\Psi = \Phi + (\sqrt{s} - M_0 + i\epsilon)^{-1}M_I\Psi;$$

$$\Psi_2 = \phi_2 + (\sqrt{s} - D_2(0))^{-1}gK^\dagger\Psi_3;$$

$$\Psi_3 = (\sqrt{s} - D_3(0))^{-1}gK\Psi_2;$$

$$\Psi_2 = \phi_2 + g^2(\sqrt{s} - D_2(0))^{-1}K^\dagger(\sqrt{s} - D_3(0))^{-1}K\Psi_2;$$

where $\Phi = (\phi_2, 0)$, and the driving term is the two free nucleon momentum state, $\phi_2 = \delta^3(\vec{k} - \vec{k}_{\text{initial}})$. Again Eq.45 has the form of a driving term plus "potential", from which it is possible to calculate scattering amplitudes for the two processes mentioned above.

Finally, the baryon zero sector gives the ground state and physical pion mass. However in the truncation where there are no nucleon-antinucleon pairs, and only the direct sum of Fock vacuum, and one or more bare pion states, there is no pion mass renormalization and the physical vacuum is the Fock vacuum. If however a nucleon-antinucleon two particle space is added to the Fock vacuum along with a nucleon-antinucleon-pion space and a one and two pion space, the mass operator will have off-diagonal terms linking the Fock vacuum to the nucleon-antinucleon-pion Hilbert space, and pion mass renormalization will occur, as well as pion-pion scattering.

More generally for a given truncation the mass operator will be a quint-diagonal matrix operator for which bound and scattering states can be calculated. If the function $f$ occurring in the off-diagonal kernal is fixed, then for the given truncation all channels will be determined by $f$. 

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4 Conclusion

This paper has shown how to compute interacting mass operators from a given vertex, by multiplying velocity state matrix elements of the vertex operator by a form factor that makes the mass operator a well defined operator on the truncated Fock space appropriate for a few-body system. The total mass operator, the sum of free and interacting mass operators, has a discrete spectrum that gives the bound states, and continuous spectrum the scattering states. And just as the total momentum and spin projection are extracted from an isolated many-body nonrelativistic system, so in the point form the overall four-velocity and spin projection are extracted from an isolated many-body relativistic system. Since the interacting mass operator comes from a vertex operator, the interactions are all due to the exchange of particles. Yet, as shown in section 3 the coupled channel problem can be rewritten so that the relativistic Lippmann-Schwinger equations contain relativistic "potentials".

There are a number of applications of the formalism discussed here. The most obvious is a more realistic treatment of the pion-nucleon system, in which the spins are correctly taken into account. Then the interacting mass operator will contain spinors, but, as can be seen from the way in which fields for arbitrary spin particles can be coupled together to form Lorentz scalar densities [9], the coupling can also include the spin 3/2 delta resonance. Lee and coworkers [11], as well as Fuda [12] have written T matrix equations for pion-nucleon scattering, using basically the instant form of relativistic quantum mechanics, in which vertices are used to generate interactions in the Hamiltonian. A similar procedure should be possible in the point form.

A second application is to the study of the meson and baryon spectrum, arising as bound states of quarks. While the work of Plessas et al [13] can be interpreted as a point form mass operator, the hyperfine part of the mass operator only indirectly comes from a vertex. Closer in spirit to what is developed here is the dissertation of Krassnigg [14] who studies the vector mesons as bound states of quark-antiquark pairs. The truncated Hilbert space is the direct sum of quark-antiquark and quark-antiquark, pseudoscalar meson spaces, and the hyperfine interaction is given from a vertex of the form $\mathcal{H}(x) = \Psi(x)\gamma_5\Psi(x)\phi(x)$, where $\Psi(x)$ is the quark field and $\phi(x)$ the pseudoscalar meson field. The main difference with the mass operator discussed in this work is that the free mass operator is replaced by a harmonic oscillator mass operator to simulate quark confinement. But otherwise the
coupled channel equations are very similar to those of section 3, and result, for example, in excited states having widths and shifted masses.

For a given truncation there is a definite mass renormalization, which however will change if the truncation changes. Already at this point violations of cluster properties will occur, for if a set of bare particles are moved far away from another set, the mass renormalization for the separated set will not be the same as with all the particles. But for a given truncation and mass renormalization, it should be possible to construct unitary operators (packing operators, see reference [4]) that give the desired cluster properties for the many-body system. Further there is reason to believe that the packing operators for vertex interactions should be relatively straightforward to calculate. The desirable cluster properties come about because $f$ has as an argument the difference of the relativistic masses of the initial and final sets of particles. The mass operator is the product of this function multiplied by velocity state matrix elements of the vertex operator. For a given vertex only three particles interact while the momentum of the others are unchanged, resulting in the cancellation of energies in the argument of $f$. Then the argument of $f$ depends only on the momentum of the interacting particles and not the others. Because of this peculiar feature of the argument of $f$ in the point form, the cluster properties may be relatively simple. But this remains to be investigated.

More generally as pointed out in the introduction, the coupled channel mass operators that result from vertex interactions allow for a limited amount of particle production, depending on the truncation chosen. Such theories are intermediate between relativistic direct interaction theories for a fixed number of particles and quantum field theories with infinite degrees of freedom. Moreover, quantum field theories satisfy cluster separability properties (see for example, [9]), while relativistic direct interaction theories for a fixed number of particles require packing operators to satisfy cluster properties [4]. While the relativistic coupled channel theory discussed in this paper has limited particle production and requires mass renormalization, the way mass renormalization depends on cluster properties remains an open question. This issue is particularly relevant in light of a theorem by Aks [15] that says in a four dimensional quantum field theory particle production and vacuum polarization necessarily go together.

The procedures given in this paper for constructing strongly interacting mass operators from vertices can also be applied to electromagnetic vertices, where $H(0)$ is of the form $J_\mu(0)A^\nu(0)$; in the following papers current oper-
ators are constructed for particles with arbitrary spin and form factors [10], as well as analyzing the Gupta-Bleuler formalism appropriate for the above coupling [16]. Finally the last paper in this series [17] shows how the free current operator is modified from its one-body form to generate the appropriate many-body current operator in the presence of interactions. The goal then, in this series of papers is to see whether the nonrelativistic methods that have been so successful in low energy nuclear physics, where the Hamiltonian is the sum of matter, photon, and electromagnetic coupling Hamiltonians, can be generalized to higher energies where relativity is required and the mass operator is similarly the sum of matter, photon and electromagnetic coupling mass operators.

5 Appendix

In this appendix the interacting mass operator is obtained from $\mathcal{H}(x)$ by integrating over the forward hyperboloid to get the interacting four-momentum operator. Let $\mathcal{H}(x)$ be a polynomial in free fields which is a Lorentz scalar density, $U_\Lambda \mathcal{H}(x) U^{-1}_\Lambda = \mathcal{H}(\Lambda x)$. Then an interacting four-momentum operator which satisfies the point form equations can be defined by integrating $\mathcal{H}(x)$ over the forward hyperboloid:

$$P_\mu(I) : = \int d^4x \delta(x \cdot x - \tau^2) x_\mu \theta(x_0) \mathcal{H}(x); \quad (46)$$

$$[P_\mu(I), P_\nu(I)] = 0. \quad (47)$$

The components of the interacting four-momentum operator commute with one another since the commutator of $\mathcal{H}(x)$ with $\mathcal{H}(y)$ is zero if the space-time points $x$ and $y$ are space-like separated. But both these points lie on the same forward hyperboloid specified by $\tau$ and hence are space-like separated. It also follows from Eq.46 that $P_\mu(I)$ transforms as a Lorentz four-vector, since $\mathcal{H}(x)$ transforms as a Lorentz scalar density. Hence the interacting four-momentum operator satisfies the point form equations, Eq.1,2.

The total hadronic four-momentum operator is the sum of free and interacting four-momentum operators, $P_\mu(h) = P_\mu(fr) + P_\mu(I)$. Again it can be shown that the total four-momentum operator satisfies the point form equations. The Lorentz transformation part follows from the fact that both terms transform as four-vectors. To show that the components of the total four-momentum operator commute, it suffices to note that the components
of the free part commute among themselves (from Eq. 11), as do the components of the interacting part (Eq. 47). Thus what must be shown is that the sum of the cross term commutators give zero. Now from the definition of \( \mathcal{H}(x) \) as a polynomial in free fields, it follows that \( U_a \mathcal{H}(x) U_a^{-1} = \mathcal{H}(x + a) \). But \( U_a \) is the exponential of \( P(fr) \cdot a \), so if \( a \) is made infinitesimal, then

\[
[P_\nu(fr), P_\mu(I)] = \int d^4 x \delta(x \cdot x - \tau^2) \theta(x_0) x_\mu \frac{\partial}{\partial x^\nu} \mathcal{H}(x); \quad (48)
\]

\[
[P_\mu(fr), P_\nu(I)]
+ [P_\mu(I), P_\nu(fr)] = [P_\mu(fr), P_\nu(I)] - [P_\nu(fr), P_\mu(I)]
= \int d^4 x \delta(x \cdot x - \tau^2) \theta(x_0) (x_\nu \frac{\partial}{\partial x^\mu} - x_\mu \frac{\partial}{\partial x^\nu}) \mathcal{H}(x)
= 0, \quad (49)
\]

since \( \mathcal{H}(x) \) is a Lorentz scalar density. Thus the total four-momentum operator satisfies the point form equations.

Given the total four-momentum operator, the sum of Eqs. 11, 46, bound and scattering states could in principle be found by solving the eigenvalue equation, Eq. 5. But the interacting four-momentum operator defined in Eq. 46 is not a well defined operator because it involves the product of local field operators at the same space-time point. Moreover, if a restriction to a finite degree of freedom system is made, the theory is no longer properly relativistic because Eq. 47 is not valid on a truncated space. Finally, the interacting four-momentum operator given in Eq. 47 is never of Bakamjian-Thomas form; that is, it can never be written as the product of a mass operator times a four-velocity operator. Nevertheless, a Bakamjian-Thomas interacting mass operator can be constructed from Eq. 46 by examining matrix elements of the interacting four-momentum operator, where the states are velocity states with the same initial and final four-velocities:

\[
<v, \vec{k}_i, \mu_i | P_\mu(I) | v, \vec{k}'_i, \mu'_i> = <v, | \vec{k}_i, \mu_i | \mathcal{H}(0) | v, \vec{k}'_i, \mu'_i>
= \int d^4 x \delta(x \cdot x - \tau^2) x_\mu \theta(x_0) e^{i(\Delta m)v \cdot x}, \quad (50)
\]

where \( \Delta m \) is \( m - m' \) and \( m \) (respectively \( m' \)) is the mass of the velocity state, as given in Eq. 17. \((\Delta m)\nu \) is always a timelike four-vector, although the sign of the time component may be positive or negative, depending on the sign of \( \Delta m \).
The integral in Eq. 50 is a special function and is evaluated in the following paragraph. The relevant point is that, if the initial and final four-velocities are arbitrary, the integral is not diagonal in the four-velocity, which is another way of saying that an interacting four-momentum operator in quantum field theory is never of Bakamjian-Thomas form.

But Eq. 50 can be used to construct an interacting mass operator, generated from vertex interactions, by defining

\[ M_I = \langle v \vec{k}_i \mu_i | H(0) | v \vec{k}_i' \mu_i' \rangle f(\Delta m), \quad (51) \]

where \( f \) is the function defined in Eq. 50 and evaluated in the following paragraph. If it is replaced with an arbitrary form factor, the mass operator is the same as that given in Eq. 50, which was the starting point for a many-body theory.

The integral in Eq. 50 can be computed as follows. Define the Lorentz invariant function \( I^\pm_\tau(p) \) as

\[ I^\pm_\tau(p) = \int d^4x \delta(x \cdot x - \tau^2) \theta(x_0) e^{\pm i p \cdot x} \quad (52) \]

\[ I^\pm_\tau(\Lambda p) = I^\pm_\tau(p), \quad (53) \]

\[ I^\pm_\tau(sp) = s^{-2} I^\pm_\tau(p) \quad (54) \]

From reference [18] the function \( I^\pm_\tau(p) \) is given by

\[ I^+_\tau(p) = \pi^2 \tau (p \cdot p)^{-1/2} \left[ N_1(\tau \sqrt{p \cdot p}) - i \epsilon(p_0) J_1(\tau \sqrt{p \cdot p}) \right] \quad (55) \]

\[ I^-_\tau(p) = (I^+_\tau(p))^*, \quad (56) \]

for \( p \) timelike.

Then for \( \Delta m > 0 \),

\[ \int d^4x \delta(x \cdot x - \tau^2) x_\mu \theta(x_0) e^{i(\Delta m)v \cdot x} = \frac{(i \Delta m)^{-1}}{\partial \partial v^\mu I^+_\tau(\Delta mv)} = -i (\Delta m)^{-3} \partial \partial v^\mu I^+_\tau(\Delta mv) = v^\mu f_\tau(\Delta m), \quad (57) \]

where \( f_\tau(\Delta m) = -\pi^2 \tau (\Delta m)^{-2} \alpha^{-1} \partial \partial \alpha [\alpha^{-1} J_1(\Delta m \tau \alpha) + i \alpha^{-1} N_1(\Delta m \tau \alpha)]_{\alpha=1} \) with \( \alpha := \sqrt{v \cdot v} \). For \( \Delta m < 0 \), the desired integral is

\[ \int d^4x \delta(x \cdot x - \tau^2) x_\mu \theta(x_0) e^{-i(\Delta m)v \cdot x} = i |\Delta m|^{-3} \partial \partial v^\mu I^-_\tau(\Delta mv) = v^\mu f_\tau^*(|\Delta m|). \quad (58) \]
The integrals in Eqs.57,58 are complex conjugates of one another, with both depending on the magnitude of the difference between final and initial masses. $J_1$ and $N_1$ are various forms of Bessel functions, as given in reference [8].

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