Minimal non-ergodic $C^1$-diffeomorphisms of the circle

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Abstract. We construct, for each irrational number $\alpha$, a minimal $C^1$-diffeomorphism of the circle with rotation number $\alpha$ which is not ergodic with respect to the Lebesgue measure.

1. Introduction

A diffeomorphism $f$ of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is ergodic with respect to the Lebesgue measure if there is no $f$-invariant Borel set with Lebesgue measure strictly between zero and one. In [1], Denjoy proved that a $C^1$-diffeomorphism with bounded variation derivative is ergodic. In the other direction, Denjoy constructed examples of $C^1$-diffeomorphism in any rotation class with invariant Cantor sets of positive measure. These examples, having wandering intervals, are not minimal. In this paper we construct, for each irrational number $\alpha \in (0, 1)$, an orientation preserving $C^1$-diffeomorphism of the circle which is minimal, i.e. has every orbit dense, but is not ergodic.

Given an orientation preserving $C^1$-diffeomorphism $f : \mathbb{T} \to \mathbb{T}$, to define its rotation number, $\rho(f)$, lift $f$ to $\tilde{f} : \mathbb{R} \to \mathbb{R}$ and take

$$\rho(f) := \lim_{n \to \infty} \frac{\tilde{f}^n(x) - x}{n}$$

for $x \in \mathbb{R}$, where $\tilde{f}^n, n \in \mathbb{Z}$, denotes the iterates of $\tilde{f}$. An equivalent, more enlightening, combinatorial definition can be found in de Melo and van Strien [4, p. 33]. If $\rho(f) = \alpha$ and $f$ is minimal then $f$ is conjugated to the rotation $R_\alpha : x \in \mathbb{T} \mapsto x + \alpha \in \mathbb{T}$. This means that there is an orientation-preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ R_\alpha = f \circ h$.

To construct $f$, we go the other way around; we construct an homeomorphism $h$ such that $f := h \circ R_\alpha \circ h^{-1}$ is a $C^1$-diffeomorphism with the required properties. We define $h$...
first as an order-preserving map on $\mathcal{O}$, the non-negative orbit of $0=1$ under $R_\alpha$, in such a way that $h(\mathcal{O})$ is dense in $\mathbb{T}$. Then, it is immediate, $h$ extends to $\mathbb{T}$ as an homeomorphism.

To define $h$ on $\mathcal{O}$ we need to understand the relationship between the dynamical and linear orders on $\mathcal{O}$, an information which is encoded in the continued fraction expansion of $\alpha$. We recall this classical formalism in a way that fits our needs in §2. In §3 we collect the lemmas we will need to prove, in §4, the following theorem.

**Theorem 1.** Given an irrational number $\alpha$, $0 < \alpha < 1$, there is a minimal orientation-preserving $C^1$-diffeomorphism, $f : \mathbb{T} \rightarrow \mathbb{T}$, which is not ergodic with respect to Lebesgue measure and such that $\rho(f) = \alpha$.

2. Continued fractions and towers

Fix $\alpha \in (0, 1)$ as irrational and let $a_1, a_2, a_3, \ldots$ be the sequence of its partial quotients and $p_n/q_n$ be the sequence of its approximants. Let $(T)$, $T = T(aq_{n+1} + q_n)$, be the set of partitions of $\mathbb{T}$ by intervals with extremes

$$\{R^i_\alpha(0)\}_{i=0}^{a_{n+1} + q_n - 1},$$

for $n \geq 1$ and $1 \leq a \leq a_{n+2}$, well ordered by the relation of refinement.

For easy reference it is convenient to stack the intervals of $T$ into a pair of towers so that we get from $T$ to its next refinement, $\tilde{T}$, by the process of ‘cutting and stacking’. This procedure is explained in detail in [3, Appendix 1] (actually, Katznelson and Ornstein work with ‘towers with balconies’ separating balconies from towers give our towers). Figure 1 shows, $T(q_{n+1} + q_n)$ for $n$ even. In this figure the integer $k$ at the side of a level stands for $R^k_\alpha(0)$. In general, $T(aq_{n+1} + q_n)$ has a left tower made of left intervals or $L$-intervals

$$\{R^i_\alpha[R^j_\alpha(a_{n+1} + q_n, 0)]\}_{i=0}^{a_{n+1} - 1}$$
and a right tower made of \( R \)-intervals
\[
\{ R^j_a[0, R^{a+1}_a(0)](a^{-1})_q = 0, q \}
\]
for \( n \) odd; for \( n \) even change left for right.

The idea for constructing \( h \), and therefore \( f \), is very simple: at each stage in the process of forming towers, points of \( T \) fall either in a \( L \) or \( R \)-interval. Suppose we choose a side, say right, for a fixed increasing subsequence \( T_k \) of the above sequence of towers, which, it should be emphasized, is entirely determined by (the continued fraction expansion of) \( \alpha \).

The set of points \( S \) which are not in the orbit of zero but which are in an interval of the right side of \( T_k \) for \( k \) arbitrarily large is clearly \( R^\alpha \)-invariant. We are going to construct \( h \) in such a way that
\[
1 > \mu(h(S)) > 0 \quad \text{where} \quad \mu \text{ is the Lebesgue measure.}
\]
This is easily done. We just have to imitate the construction of the Cantor map and conveniently distort the intervals in the right side as they appear in the process of refinement. If we do that, then \( f := h \circ R^\alpha \circ h^{-1} \) is a homeomorphism but, of course, not, in general, a \( C^1 \)-diffeomorphism. Our task is then to show that we can accomplish this change of lengths and also get a diffeomorphism. To control those distortions we will need the following lemma.

**Lemma 2.** Let \( T \) be a tower of \( R \). Denote the \( L \)-intervals of \( T \) by \( L \), and the \( R \)-intervals by \( R \). Cut and stack \( T \) to get a new tower \( \tilde{T} \). Denote the intervals in the left tower of \( \tilde{T} \) by \( l \) and similarly define the intervals \( r \). These smaller intervals decompose each left interval of \( T \) into \( n_l \) and \( m_l \) intervals, respectively. Analogously they decompose the right intervals of \( T \) into \( n_r \) and \( m_r \) intervals, respectively. Then, cutting and stacking, we get infinitely often pairs of towers \( \tilde{T} \) such that
\[
\frac{1}{5} < \frac{m_r}{n_l} < \rho := \frac{(m_l + m_r)r}{(n_l + n_r)r} < \frac{m_l}{n_r} < 5.
\]

**Proof.** Since our thesis is invariant under scaling, we can consider the rescaled first return map to the bottom of the towers and assume the initial towers with height one. Cut and stack \( T \) to get \( \tilde{T} \) with \( n_l \), \( n_r \), \( m_l \) and \( m_r \). Then \( n_r + n_l \) is the height of the right tower of \( \tilde{T} \) and \( n_l + n_r \) the height of its left tower. We can take these quantities as large as we please since, say, \( n_r / n_l + n_r \) goes to \( \alpha \) as we cut and stack, by the unique ergodicity of \( R^\alpha \).

We have
\[
\begin{pmatrix} L \\ R \end{pmatrix} = \begin{pmatrix} n_l & m_l \\ n_r & m_r \end{pmatrix} \begin{pmatrix} 1 \\ r \end{pmatrix}
\]
and \( n_l m_r - n_r m_l = 1 \) since this matrix is a product of the matrices
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

We have \( m_r n_l / n_l < m_r n_l / n_r \). We will show that by further cutting and stacking of \( \tilde{T} \) we can get \( T \) such that
\[
\frac{\delta^-}{\delta^+} = \frac{\rho - m_r n_l}{n_r (n_r + n_l)} \quad \text{and} \quad \frac{\delta^-}{\delta^+} = \frac{m_l n_r}{n_l (n_r + n_l)}
\]
where
\[
\delta^- = \rho - \frac{m_r}{n_r} \quad \text{and} \quad \delta^+ = \frac{m_l}{n_l} - \rho.
\]
which will prove the lemma. Consider two cases: either \( a_n = 1 \) for \( n \) large enough or \( a_n > 1 \) for \( n \) arbitrarily large. Since our thesis is invariant by flipping sides, left and right, we can suppose, without loss of generality, that \( r > l \) and \( n' + n' < m' + m' \). If the second possibility holds, cut and stack \( T \) to get \( \tilde{T} \) critical immediately preceding the appearance of a partial quotient \( a_n = \lfloor r/l \rfloor > 1 \). Cut and stack \( \tilde{T} \) a \( := [a_n/2] \geq 1 \) times to get \( \tilde{\tilde{T}} \). We have

\[
L = n'l + m'l = (n' + m'a)l + m'(r - al)
\]
\[
R = n'r + m'r = (n' + m'a)l + m'(r - al)
\]

and therefore

\[
\rho = \frac{(m' + m')(r - al)}{(m' + m')(m' + m') + a} = \frac{a_n + f - a}{(m' + m')(m' + m') + a}
\]

where \( f \) is the fractional part of \( r/l \). Thus

\[
\frac{a_n - a}{1 + a} \leq \rho \leq \frac{a_n - a + 1}{a}
\]

and, using the definition of \( a \), we have

\[
\frac{1}{2} \leq \frac{a}{1 + a} \leq \rho \leq \frac{a + 3}{a} \leq 4
\]

as required. For \( \delta^+ \) we have

\[
\delta^+ = \frac{1}{(n' + m')(r - al)} \frac{r - al}{l} \leq \frac{4}{m' + m'} \leq \frac{1}{4}
\]

and similar estimates hold for \( \delta^- \).

Now suppose \( a_n = 1 \) for \( n \) large enough. Cut and stack \( T \) to get \( \tilde{T} \) as the partial quotients \( a_n \) converge to 1. Cut and stack \( \tilde{T} \) twice to get \( \tilde{\tilde{T}} \). We have

\[
L = (n' + m')(2l - r) + (2m' + n')(r - l)
\]
\[
R = (n' + m')(2l - r) + (2m' + n')(r - l)
\]

and therefore

\[
\rho = \frac{(2m' + n' + 2m' + n')(r - l)}{(n' + m' + n' + m')(2l - r)} = \frac{(r - l)}{(1 + \frac{m' + m'}{n' + m' + n' + m'}) (2l - r)}
\]

which shows that \( 1 \leq \rho \leq 4 \) as required, since \( 0 \leq r - l < l \), \( 0 \leq l - (r - l) = 2l - r < r - l \) and \( 0 \leq (r - l) - (2l - r) < 2l - r \). As for, say, \( \delta^+ \) we have, using the above inequalities, that

\[
\delta^+ = \frac{1}{(n' + m')(n' + m' + n' + m')} \frac{r - l}{2l - r} < \frac{1}{4}
\]
3. **Pl towers**

Now take a pair of towers $T$ of $R_{\alpha}$ and change the lengths of their intervals except for the top and bottom ones. This change is subject only to the condition that the sum of the lengths of all intervals is one. The new pair of towers, $\tilde{T}$, with its scheme of mappings and identifications, defines, in the obvious way, a unique (modulo rotations) piecewise (pl) homeomorphism of the circle, $f$, and a pl conjugacy, $h$, between $f$ and $R_{\alpha}$. We just have to take every map in sight as orientation preserving and affine. More precisely we consider towers, which we shall call pl towers, such that:

1. the combinatorics of the towers are the combinatorics of some $R_{\alpha}$ tower;
2. the maps one floor up are affine;
3. the top and bottom levels are the same as in the corresponding $R_{\alpha}$ tower and therefore the top to bottom maps are the same isometries as for $R_{\alpha}$.

Since an orientation-preserving affine homeomorphism between intervals is unique, any diagram of such maps commutes and we have a pl conjugacy, $h$, between $f$ and $R_{\alpha}$ on the mid levels. As the top and bottom levels were unchanged the same holds there and the conjugacy thus defined ensures that $\rho(f) = \alpha$ and the minimality of $f$. Therefore $h$ is affine on all levels and, in fact, the identity on the top and bottom levels, the iterates of 0 under $f$ coincide with the corresponding ones of $R_{\alpha}$ on the top and bottom levels and are an affine image of them on the other levels.

Observe that if we have one of these towers and we cut and stack its intervals (as prescribed by the pl homeomorphism $f$ it defines) any number of times and change their lengths in the allowed way, we still get a tower of the same sort which defines a pl homeomorphism $f$ with rotation number $\alpha$. Also, if we take one of these towers and start to cut and stack, the maximum length of the intervals in the towers goes down monotonically to zero since the cutting orbit is dense in $\mathbb{T}$.

The derivative of a pl homeomorphism defines a positive step map $\varphi$ which, by definition, is a real-valued map defined on $\mathbb{T}$ such that there are distinct points $p_0, p_1, \ldots, p_k = p_0 \in \mathbb{T}$, indexed in the counterclockwise sense, such that $\varphi$ is constant in each interval $[p_i, p_{i+1})$. If $p$ is a point in $\mathbb{T}$ we define the jump of $\varphi$ at $p$ as

$$J(\varphi, p) = \left| \lim_{x \to p^+} \varphi(x) - \lim_{x \to p^-} \varphi(x) \right|.$$ 

If $\varphi_n$ is a sequence of step maps, satisfying:

1. $\sum_{n} \|\varphi_{n+1} - \varphi_n\| < \infty$ where $\|\varphi\|$ is the supremum norm of $\varphi$;
2. $\max_{p \in \mathbb{T}} J(\varphi_n, p) \to 0$ as $n \to \infty$;

then clearly $\varphi_n$ converges uniformly to a continuous map $\varphi$. Integrating we get the following lemma.

**Lemma 3.** Let $f_n$ be a sequence of pl homeomorphisms of $\mathbb{T}$ converging pointwise to a pl homeomorphism $f$. Suppose the sequence of derivatives $\varphi_n = Df_n$ satisfies (i) and (ii) above and is uniformly bounded away from zero. Then $f$ is a $C^1$-diffeomorphism and $Df = \varphi$ where $\varphi = \lim \varphi_n$.

Let $I_1$ and $I_2$ be two intervals, and $f : I_1 \to I_2$ be the affine orientation-preserving homeomorphism between them. Let $T_1 = \{p_j\}_{j=0}^{k}$ be a partition of $I_1$ and
\[ T_2 = f(T_1) = \{ p_{2j} \}. \] Fix two indices \( 1 < k_1 < k_2 < k \) and move the points of \( T_1 \) without changing their relative order to form a new partition \( \tilde{T}_1 = \{ \tilde{p}_{ij} \} \) of \( I_1 \) and do the same for \( T_2 \) thus getting \( \tilde{T}_2 = \{ \tilde{p}_{2j} \}. \) We are no longer assuming that \( \tilde{T}_2 = f(\tilde{T}_1). \) Consider \( \tilde{f} : I_1 \to I_2 \) as the orientation-preserving \( \text{pl} \) homeomorphism defined by the partitions \( \tilde{T}_1 \) and \( \tilde{T}_2. \) Suppose the change is such that:

1. \( \tilde{p}_{ik_1} = p_{ik_1} \) and \( \tilde{p}_{ik_2} = p_{ik_2}, \) \( i = 1, 2, \) i.e. the marked points are unchanged;
2. \( \tilde{f} \) is affine in the intervals \( [\tilde{p}_{11}, \tilde{p}_{1k_1}] \) and \([\tilde{p}_{1k_2}, \tilde{p}_{1k_1-1}]; \)
3. the intervals of \( T_1 \) (respectively \( T_2 \)) in \([\tilde{p}_{1k_1}, \tilde{p}_{1k_2}] \) (respectively \([\tilde{p}_{2k_1}, \tilde{p}_{2k_2}] \)) are partitioned in three groups of intervals \( C_1, \tilde{C}_1 \) and \( Q_1 \) (their image by \( \tilde{f} \) being respectively \( C_2, \tilde{C}_2 \) and \( Q_2 \)). Denoting the corresponding intervals of \( T = \tilde{T}_1 \) (respectively \( \tilde{T}_2 \)) by adding \( \tilde{\cdot} \) we assume that the intervals of \( \tilde{C}_1 \) (respectively \( \tilde{C}_2 \)) are contracted by a factor of \( \lambda_1, \) \( 0 < \lambda_1 < 1 \) (respectively \( \lambda_2, \) \( 0 < \lambda_2 < 1 \)). The intervals of \( \tilde{Q}_1 \) (respectively \( \tilde{Q}_2 \)) remain with length equal to their counterparts in \( T \) (respectively \( \tilde{T} \)). The remaining intervals in \( \tilde{E}_1 \) (respectively \( \tilde{E}_2 \)) are, consequently, expanded. We assume this expansion is uniform.

**Lemma 4.** Under the above hypothesis, defining \( c = \mu(\cup C_1), e = \mu(\cup E_1), s = \mu(\cup Q_1), \) \( \kappa = \max\{c/e, e/c\} \) and

\[
r = \max \left\{ \frac{\tilde{p}_{1i} - p_{0i}}{\tilde{p}_{ik} - p_{0k}}, \frac{\tilde{p}_{ik} - \tilde{p}_{ik-1}}{\tilde{p}_{ik} - p_{ik}} \right\}_{i = 1, 2},
\]

and assuming \( r < 1/2, \) we have

\[
\| D\tilde{f} - Df \| \leq \begin{cases} \kappa |\lambda_2/\lambda_1 - 1| \| Df \|, & \text{for intervals in } [\tilde{p}_{1k_1}, \tilde{p}_{1k_2}] \\ 4r \| Df \|, & \text{for intervals in } [\tilde{p}_{11}, \tilde{p}_{1k_1}] \text{ or } [\tilde{p}_{1k_2}, \tilde{p}_{1k_1-1}]. \end{cases}
\]

**Proof.** For an interval \([\tilde{p}_{1j-1}, \tilde{p}_{1j}]\) in \( \tilde{C}_1 \) we have

\[
\| D\tilde{f} - Df \| = \left| \frac{\lambda_2}{\lambda_1} - 1 \right| \| Df \| 
\]

If \([\tilde{p}_{1j-1}, \tilde{p}_{1j}]\) is in \( \tilde{E}_1 \) we have in the same way

\[
\| D\tilde{f} - Df \| \leq \frac{v_2}{v_1} - 1 \| Df \|
\]

where \( v_i \) is the expansion undergone by the intervals in \( \tilde{E}_i, \) \( i = 1, 2. \) Now \( c + e + s = p_{ik_2} - p_{ik_1} = \lambda_1 c + v_1 e + s \) and \( a(p_{ik_2} - p_{ik_1}) = p_{2k_2} - p_{2k_1} = \lambda_2 a c + v_2 ae + as, \) where \( a = |I_2|/|I_1|. \) Then \( c + e = \lambda_1 c + v_1 e = \lambda_2 c + v_2 e \) from which we get

\[
\frac{|v_2|}{v_1} - 1 = \left| \frac{(c + e)/c - \lambda_2}{(c + e)/c - \lambda_1} - 1 \right| = \frac{\lambda_1 - \lambda_2}{(c + e)/c - \lambda_1} 
\]

\[
\leq \frac{c}{e} \frac{\lambda_2}{\lambda_1} - 1 \leq \kappa \frac{\lambda_2}{\lambda_1} - 1.
\]
For intervals in $[\tilde{p}_{11}, \tilde{p}_{1k_1}]$ (or $[\tilde{p}_{1k_2}, \tilde{p}_{1k_1}^{-1}]$) we have

$$\|D\tilde{f} - Df\| = \left| \frac{p_{2k_1} - \tilde{p}_{21}}{p_{1k_1} - p_{11}} - \frac{p_{2k_1} - p_{20}}{p_{1k_1} - p_{10}} \right|$$

$$= \left| \frac{p_{2k_1} - p_{20} - (\tilde{p}_{21} - p_{20})}{p_{1k_1} - p_{10} - (\tilde{p}_{11} - p_{10})} - \frac{p_{2k_1} - p_{20}}{p_{1k_1} - p_{10}} \right|$$

$$\leq 2 \left( \frac{p_{2k_1} - p_{20}}{p_{2k_1} - p_{20} + p_{11} - p_{10}} + \frac{p_{2k_1} - p_{20}}{p_{1k_1} - p_{10}} \right)^2 \frac{p_{2k_1} - p_{20}}{p_{1k_1} - p_{10}} \leq 4r\|Df\|.$$

Using that $Df$ is a step map and the triangle inequality we have

$$|JD\tilde{f}(\tilde{p})| \leq \begin{cases} 2\kappa|\lambda_2/\lambda_1 - 1||Df||, & \text{for } \tilde{p} \in (\tilde{p}_{1k_1}, \tilde{p}_{1k_2}), \\ (|\lambda_2/\lambda_1 - 1| + 4r)||Df||, & \text{for } \tilde{p} = \tilde{p}_{1k_1} \text{ or } \tilde{p} = \tilde{p}_{1k_2}, \\ |D\tilde{f}(p_{10}) - Df(p_{10})| + 4r\|Df\||, & \text{for } \tilde{p} = p_{11}, \\ |D\tilde{f}(p_{1k_1}) - Df(p_{1k_1})| + 4r\|Df\||, & \text{for } \tilde{p} = \tilde{p}_{1k_1}^{-1}. \end{cases}$$

Since the hypothesis is symmetric we have similar estimates for $D\tilde{f}^{-1}$ and $JD\tilde{f}^{-1}$, changing $\lambda_2/\lambda_1$ to $\lambda_1/\lambda_2$.

Let $T = T(aq_{n+1} + q_n)$ be a pl tower defined by a pl homeomorphism $f$, $\rho(f) = \alpha$. Denote its intervals by $[L_i]^h_{i=0}$ and $[R_i]^h_{i=0}$, where the heights are, say, $h^i = aq_{n+1} + q_n$ and $h^r = q_{n+1}$, indexed from bottom to top, and define $S = \cup R_0$. Fix positive integers $0 < h < \min\{h^i, h^r\}$, $\tilde{h}$ and $q' \in R_0$, $q^i \in L_0$ so close to zero that $R_{q'}^{-1}(0) \in f^{h^i-1}[q^i, f^h(q^i)] \cup f^{h^r-1}[q^r, f^h(q^r)]$ but not in the $R_q$-orbit of zero.

Cut and stack $T$ to a tower $\tilde{T} = \tilde{T}(\tilde{a}a_{n+1} + \tilde{q}_n)$ and define l- and r-intervals, $l^i$, $m^i$, $n^i$ and $m^r$ as in Lemma 2. We suppose $0 < \tilde{h} < \tilde{h}/2$ and $\tilde{h}/2$, the heights of $\tilde{T}$, where we have started to use the convention of adding $\tilde{q}_n$ to the names of objects referring to $\tilde{T}$. Define $X = X(h)$ as the set of intervals on the first or top h levels of $T$ and $\tilde{D} = \tilde{D}(q', q^r)$ as the complement of the set of intervals in $\tilde{T}$ whose projections on the bottom of $T$ are contained in $[q^r, f^h(q^r)] \cup [f^h(q^r), q^r]$. Define also $\tilde{F} = X \cup \tilde{D}$ and $\tilde{P} = \tilde{F}^c$. See Figure 2. Denote by $\tilde{l}^r_{ij}$, a generic l-interval that decomposes $L_i$, where the superscript refers to the side, left, of the interval, $L_i$. A missing l or r refers to either l or r. Define similarly $\tilde{l}^r_{ij}$, $\tilde{l}^l_{ij}$ and $\tilde{r}^l_{ij}$. Since the towers are affine the l-intervals that enter into the decomposition of a fixed interval, say $L_i$, all have the same length which we denote by $\tilde{l}^l_{ij}$ (respectively for $\tilde{l}^r_{ij}$, $\tilde{l}^l_{ij}$ and $\tilde{r}^l_{ij}$). Since $T$ is affine we have

$$\frac{n^l_{ij}}{L_i} = n^l_{ij}/L_i = n^l_{ij}/L_0,$$

which shows that this quantity (respectively for $m^l_{ij}/L_i$, $n^l_{ij}/R_i$ and $m^r_{ij}/R_i$) depends only on the corresponding $R_{a_i}$-tower. Thus using Lemma 2, we can assume from now on that

$$\frac{1}{5} \leq \frac{m^l_{ij}}{n^l_{ij}} < \frac{m^r_{ij}}{n^r_{ij}} < 5. \quad (1)$$
FIGURE 2.

Let \( \{p_{ij}\}_{j=0}^{k'} \), where \( k' = n' + m' \), be the extreme points of the partition of \( L_i \) by the \( l \)- and \( r \)-intervals. In a similar way define \( p'_{ij} \) partitioning \( R_i \). Now take \( k'_r \) such that 
\[
\max_{0 \leq j \leq k'} p'_{0k'_r} < q < \min_{0 \leq j \leq k'} p'_{0k'_{r+1}}.
\]
Since \( \| \tilde{T} \| \to 0 \), as we cut and stack, on account of \( f \) being minimal, we see that 
\[
k'_r \to \infty \quad \text{and} \quad p'_{0k'_r} \to q .
\] 
In a similar way take positive integers \( k'_l, k'_1 \) and \( k'_2 \) such that 
\[
\max_{0 \leq j \leq k'_l} p'_{0k'_l} < f_{h'}(q) \leq \min_{0 \leq j \leq k'_l} p'_{0k'_l} + 1 ,
\]
\[
\max_{0 \leq j \leq k'_2} p'_{0k'_2} - 1 < q \leq \min_{0 \leq j \leq k'_2} p'_{0k'_2} ,
\]
respectively. Fix \( 0 < \xi < 1 \). We are going to change \( \tilde{T} \) to a pl tower, 
\[
\tilde{T} = (h, \tilde{h}, q', q', \xi),
\]
defining a pl homeomorphism \( \tilde{f} \) by contracting the intervals \( l'_{ij} \) and \( r'_{ij} \) inside the intervals of \( T \). We change the lengths of the \( l \)- and \( r \)-intervals by moving slightly the point \( p_{ij} \) to a point \( \tilde{p}_{ij} \). Start by changing the position of the points \( f^i(p'_{0i}) \), for \( i = 0, 1, \ldots, h' + h' - 1 \) (which are equal to \( p'_{0i} \) or \( p'_{1-h'1} \)), taking \( \tilde{p}_{0i} = p'_{0i} \) and moving the point \( f^i(p'_{0i}) \), \( i > 0 \), to the point at distance \( (p'_{0i} - p'_{00})g'_{00}g'_{10} \cdots g'_{i-10} \) from the nearest extreme (respectively \( p'_{00} \) or \( p'_{1-h'0} \)) where \( g'_{i0} = \sqrt{D^+f(p'_{00})D^-f(p'_{00})} \) is the geometric mean of the lateral derivatives at the extreme points of the intervals. This changes the points next to the lower extremes of the \( L \)- and \( R \)-intervals. The points next to the upper extremes \( f^i(p'_{0k'_l-1}) \), for 
\( i = 0, 1, \ldots, h' + h' - 1 \), are similarly moved. Since we have kept the extremes of the large intervals fixed and changed the length of the small intervals next to them in order that both lateral derivatives are now equal to the geometric mean of the previous lateral derivatives, it follows that \( \tilde{f} \) agrees with \( f \) and is smooth at the extremes of the intervals of \( T \). To move the remaining points \( p'_{ij} \) and to define the one-floor-up maps in \( S \), we use the previous lemma \( h' - 1 \) times taking \( I_1 = R_i, I_2 = R_{i+1}, k_1 = k'_r, k_2 = k'_2, Q = \tilde{X} = \tilde{X}(h) \) and \( C \) the remaining \( l'_{ij} \) intervals. The contraction \( \lambda'_{ij} \) will change as we move in the levels of the
right tower as follows:

\[ \lambda_i' = \begin{cases} 
\xi^i, & \text{for } i = 0, 1, \ldots, h = 1, \\
\xi^{h' - i - 1}, & \text{for } i = h' - h, h' - h + 1, \ldots, h' - 1, \\
\xi^h, & \text{otherwise.} 
\end{cases} \]

To move the points in the left tower of \( \tilde{T} \) we proceed in the same way, only now contracting the intervals \( r_i \) not in \( \tilde{X} \). Note that \( \lambda_i + 1/\lambda_i \) is either 1 or \( \xi^{\pm 1} \) and \( 0 < 1 - \xi \leq \xi^{-1} - 1 \).

Using (1) we see that we can take \( \kappa = 6 \) in Lemma 4. In fact, take \( \tilde{T} \) further down in the sequence of pairs of towers, if necessary, in order to keep the ratios \( \mu(\cup \tilde{C})/\mu(\cup \tilde{E}) \) in (1/6, 6) \( \supseteq (1/5, 5) \) which is possible since

\[ \mu(\cup \tilde{X}) \leq 4\tilde{h} \| \tilde{T} \| 
\]

and \( \| \tilde{T} \| \to 0 \).

To define the top-to-bottom maps of \( \tilde{T} \) we use Lemma 4 again on the intervals \( I_1 = L_{h' - 1}, I_2 = f(L_{h' - 1}) \) (respectively \( I_1 = R_{h' - 1}, I_2 = f(R_{h' - 1}) \)) with the \( k \)'s as before and the partitions given by the points \( p_{ij} \) where we omit the points 0 = \( p_0^i \), which has no pre-image, and \( f^{\tilde{a}_q q_i + q_i - 1}(0) \), which has no image in the stretch of trajectory from 0 to \( \tilde{a}_q q_i + q_i - 1 \) which we are considering. The sets \( C \) and \( Q \) will not, in fact, matter since we are going to take, as we must, \( \lambda_1 = \lambda_2 = 1 \) because we need \( \tilde{T} \) to be an isometry near \( R^i_\alpha(0) \) in order to keep the rotation number the same. Recall that in Lemma 4 the points \( p_{k_1} \) and \( p_{k_2} \) are unchanged and from the definition of the \( q \)'s and \( k_1 \) we have

\[ R^i_\alpha(0) \in f^{h' - 1}[p_{0 k_1}^i, p_{0 k_2}^i] \cup f^{h' - 1}[p_{0 k_1}^i, p_{0 k_2}^i]. \]

The definition of \( \tilde{T} \) is now complete and we have the following lemma.

**Lemma 5.** Given \( T, f, \) such that \( 1/2 < \| Df \| < 2 \), with \( h, \tilde{h}, q' \) and \( q'' \) as above, there are towers \( \tilde{T} \), infinitely often in the sequence of towers, which deformed to \( \tilde{T} = \tilde{T}(h, \tilde{h}, q', q'', \xi) \) as described satisfy:

(a) \( \| D\tilde{f} - Df \| \leq \max \left\{ \tilde{h} \| J Df \|, 4r \| Df \|, 6(\xi^{-1} - 1) \| Df \| \right\} \);

(b) \( \| J D\tilde{f} \| \leq \max \left\{ \tilde{h} \| J Df \| + 4r \| Df \|, (6(\xi^{-1} - 1) + 4r) \| Df \| \right\} \);

(c) \( \mu(\tilde{S} - S) \leq \mu(\cup \tilde{F}) + \tilde{h} \mu(S) + \mu(\cup \tilde{X}) \);

(d) \( \mu(\tilde{S}) \geq (1 - \tilde{h}) \mu(S) - \mu(\cup \tilde{F}) - \mu(\cup \tilde{X}) \);

(e) \( \rho(\tilde{f}) = \alpha \) and \( \tilde{f}'(0) = f'(0), \ i = 0, \ldots, aq_{i+1} + q_i - 1 \);

where

\[ r = \max \left\{ \frac{p_{k+1} - p_{k0}}{p_{k1} - p_{k0}}, \frac{p_{k0} - p_{k-1}}{p_{k1} - p_{k0}} \right\}, \]

with the maximum taken over all intervals in \( T \). (a) and (b) hold for \( D\tilde{f}^{-1} \).

**Proof.** (e) is obvious from the choice of the points \( q \) and from the definition of the new top to bottom maps. (a) and (b) follow easily from Lemma 4 and its following remark plus the easy fact that the geometric mean, \( \sqrt{\rho \lambda} \), of two numbers \( \rho \) and \( \lambda \) in the interval (1/2, 2)
satisfy \( \sqrt{\rho - \rho} < 2/\lambda \) and \( \lambda - \sqrt{\rho - \rho} < 2/\lambda \). (c) follows from

\[
\mu(S - S) = \sum_{i,j} T_{ij}^{l} \leq \mu(S) + \sum_{i,j} T_{ij}^{r} \leq \mu(S) + \xi^{b} \mu(S) + \mu(\cup \tilde{X})
\]

and (d) from

\[
\mu(\tilde{S}) = \sum_{i,j} T_{ij}^{l} + \sum_{i,j} T_{ij}^{r} \geq \sum_{i,j} T_{ij}^{l} = \mu(S) - \sum_{i,j} T_{ij}^{r}
\]

\[
\geq \mu(S) - \mu(\cup F) = \sum_{i,j} T_{ij}^{l} \geq \mu(S) - \mu(\cup F) - \xi^{b} \mu(S) - \mu(\cup \tilde{X}).
\]

4. Proof of Theorem 1

We use the notation of Lemma 5. Fix \( \nu, 0 < \nu < 1/2 \), and make \( \sigma = 0 \) and \( \sigma_{n} = \sum_{i=1}^{\nu} \nu^{i} \).

Take \( a \) and \( b \), \( 0 < a < 1/6 \) and \( 5/6 < b < 1 \), and fix a positive integer \( n_{0} \) such that

\( a + \nu_{n_{0}} < 1/6, b + \nu_{n_{0}} < 1 \) and \( 2\nu_{n_{0}} < \nu \). Take a sequence \( x_{n}, n = 0, 1, 2, \ldots \), such that

\( 0 < x_{n+1} < x_{n} \) and \( 3/4 x_{n} < x_{n+1}, \forall n \) and \( \sum x_{n} = 1 \). Define positive integers

\[
h_{n} = 1 + \left[ \frac{\ln(v_{n+1}^{n_{0}}(v - 2\nu_{n_{0}})) - \ln(100 + x_{n+1}/2)}{\ln(100 + x_{n+1}/2) - \ln(100 + x_{n}/2)} \right], \quad n = 0, 1, 2, \ldots
\]

Then there is a sequence \( f_{n}, n = 0, 1, 2, \ldots \), of \( \alpha \) homeomorphisms defined by a sequence of pairs of towers \( T_{n} \) such that, denoting by \( S_{n} \) the union of the intervals in the right tower of \( T_{n} \), we have:

1. \( \rho(f_{n}) = \alpha, \forall n \geq 0 \);
2. \( f_{n} = f_{n-1} \) on the extremes of \( T_{n-1}, \forall n \geq 1 \);
3. \( \|DF_{n} - DF_{n-1}\| \) and \( \|DF_{n}^{-1} - DF_{n-1}^{-1}\| \leq x_{n}, \forall n \geq 1 \);
4. \( \|JD_{f_{n}}\| \) and \( \|JD_{f_{n}}^{-1}\| \leq x_{n}, \forall n \geq 1 \);
5. \( a + \nu^{n+n_{0}} \leq \mu(S_{n}) \leq b + \nu^{n_{0}} \sigma, \forall n \geq 0 \);
6. \( \mu(S_{n} - S_{n-1}) \leq \nu^{n+n_{0}}, \forall n \geq 1 \);
7. \( h_{n}^{l} \) and \( h_{n}^{r} > 2h_{n}, \forall n \geq 0 \), where \( h_{n}^{l} \) and \( h_{n}^{r} \) denote the heights of the left and right towers of \( T_{n} \), respectively;
8. \( \mu(\cup X(h_{n})) \leq \frac{1}{2} v_{n+2n_{0}}, \forall n \geq 0 \);
9. \( \|T_{n}\| \leq x_{n}, \forall n \geq 0 \).

We construct the sequences \( f_{n} \) and \( T_{n} \) by induction on \( n = 0, 1, \ldots \).

For \( n = 0 \) we take a tower \( T_{0} \) of \( R_{n} \). More precisely, according to Lemma 2, constructing the towers of \( R_{n} \), we obtain \( 1/5 < (1 - \mu(S_{0})/\mu(S_{0})) < 5/6 \) \( \mu(S_{0}) < 5/6 \) infinitely often, where \( S_{0} \) is the union of the intervals in the right tower of \( T_{0} \). Fix one of these towers that is high and thin enough to satisfy (7), (8) and (9) as \( T_{0} \).

Now suppose we have constructed towers \( T_{0}, T_{1}, \ldots, T_{n} \) defining \( f_{0}, f_{1}, \ldots, f_{n} \), respectively, \( \alpha \) homeomorphisms satisfying (1) to (9). (3) implies

\[
\|DF_{n}\| \text{ and } \|DF_{n}^{-1}\| \leq 1 + \sum_{i=1}^{n} x_{i} < 2.
\]

Define \( T_{n+1} = T_{n} \) given by the previous lemma where we take \( T = T_{n}, h = h_{n}, \)

\( \tilde{h} = h_{n+1}, \xi = (100 + x_{n+1}/2)/(100 + x_{n+1}) \) (then \( \xi^{-1} - 1 < x_{n+1}/100 \) and \( \xi^{h_{n}} < v^{n+n_{0}}(v - 2\nu_{n_{0}}) \)), \( q^{l} \) and \( q^{r} \) so close to zero that
(i) 
\[ \mu(\cup D(q^l, q^r)) < \frac{1}{2} \nu^{n+1+2n_0} \quad \text{and} \quad |q| < \frac{x_{n+1}}{2h_n + h_{n_0}}, \]

and \( \widetilde{T} \) so high in the sequence of towers that

(ii) 
\[ \| \widetilde{T} \| < \min \left\{ x_{n+1}, \frac{\nu^{n+1+2n_0}}{8h_{n+1}} \right\}, \]

(iii) \( \widehat{h'} \) and \( \widehat{h} > 2h_{n+1} \).

(iv) 
\[ \max \left\{ \frac{p_{i_1} - p_{i_0}}{p_{i_1} - p_{i_0}}, \frac{p_{i_2} - p_{i_1}}{p_{i_2} - p_{i_1}} \right\} < \frac{x_{n+1}}{100}, \]

for both right and left \( p \)'s. This is possible since, for instance, \( p_{i_1}^l \to p_{i_0} \) and \( p_{i_1}^l \to f^q_{h_n+1} \) as we move up in the sequence of towers:

(v) \( \mu(\cup X(h_{n+1})) \leq \frac{1}{4} \nu^{n+1+2n_0}. \)

Using (iv) and the definition of the points \( \bar{p}_{i,j} \) next to the extremes we see that

\[ r = \max \left\{ \frac{\bar{p}_{i_1} - p_{i_0}}{p_{i_1} - p_{i_0}}, \frac{p_{i_2} - \bar{p}_{i_2}}{p_{i_2} - p_{i_1}} \right\} < \frac{x_{n+1}}{100}, \]

since, for instance,

\[ \frac{\bar{p}_{i_1}^l - p_{i_0}^l}{p_{i_1}^l - p_{i_0}^l} = \frac{(p_{i_1}^l - p_{i_0}^l)g_{i_0}g_{i_1} \cdots g_{i_{10}}}{p_{i_1}^l - p_{i_0}^l} \]

\[ = \frac{p_{i_1}^l - p_{i_0}^l}{p_{i_1}^l - p_{i_0}^l} \sqrt{\frac{D^{-f^i(p_{i_0}^l)}}{D^+\nu^{i}f^i(p_{i_0}^l)}} \leq \frac{x_{n+1}}{100} < x_{n+1}. \]

We check that the assertions (1)–(9) hold. (1) and (2) follow from (e). (3) is obtained from Lemma 5(a) since

\[ \frac{2}{3} \| JDf_n \| \leq \frac{2}{3} x_n < x_{n+1} \]

\[ 4r \| JDf_n \| \leq 8r \leq x_{n+1} \]

\[ 6(\xi^{-1} - 1) \| JDf_n \| \leq 12(\xi^{-1} - 1) < x_{n+1}. \]

(4) follows from Lemma 5(b) and the definitions. The same estimates hold for \( Df_{n+1}^{-1} \) and \( JDf_{n+1}^{-1} \).

Checking (6) we have

\[ \mu(S_{n+1} - S_n) \leq \mu(\cup X_n) + \xi h_n \mu(S_n) + \mu(\cup X_{n+1}) \]

\[ \leq \mu(\cup D) + \mu(\cup X_n) + \xi h_n + \mu(\cup X_{n+1}) \]

\[ \leq \frac{1}{2} \nu^{n+1+2n_0} + \frac{1}{2} \nu^{n+2n_0} + \nu^{n+2n_0} (\nu - 2\nu^{2n_0}) + \frac{1}{2} \nu^{n+1+2n_0} < \nu^{n+1+n_0}. \]

The second inequality of (5) is easy to prove:

\[ \mu(S_{n+1}) \leq \mu(S_n) + \mu(S_n - S_n) \leq b + \nu^{n_0} \sigma_n + \nu^{n+1+n_0} = b + \nu^{n_0} \sigma_{n+1}. \]
The first is a bit longer:

\[
\mu(S_{n+1}) \geq \mu(S_n) - \mu(\cup\mathcal{X}_n) - \xi_n \mu(S_n) - \mu(\cup X_{n+1}) \geq a + v^{n+n_0} - (\frac{1}{2}v^{n+1+2n_0} + \frac{1}{2}v^{n+2n_0} + v^{n+2n_0} + 2v^{n_0} + \frac{1}{2}v^{n+1+2n_0}) \\
\geq a + v^{n+1+n_0}.
\]

Condition (7) is (iii) and (8) is (v).

(9) is clearly true for the \(l\)-or \(r\)-intervals outside the intervals \([\tilde{p}_{ik_1}, \tilde{p}_{ik_2}]\) since, for example,

\[
\tilde{p}_{ik_1} - \tilde{p}_{i0} < f'_i(q') - p'_0 = Df'_n(p_00)(q' - p_00) < 2\left|q'\right|
\]

which is less than \(x_{n+1}\) by (i), and also for the intervals there which are contracted or remain with the same length on account of (ii). If the interval is expanded, say an interval \(r'_{ij}\) in \(R_i\), denoting this expansion by \(\nu_i\) we have

\[
M\nu_i r'_{ij} < R_i \leq \|T\| \leq x_n, \text{ where } M \text{ is the number of intervals expanded, but then } \nu_i r'_{ij} \leq x_n/M < x_n/2 < x_{n+1}, \text{ completing the induction.}
\]

Now (2) implies that \(h_n\) converges pointwise on \(O\) to an order-preserving map \(h\) and, similarly, \(f_n\) converges on \(h(O)\), a set dense in \(T^1\), to an order-preserving map \(f\). Using Herman’s Lemma 3.3 [2, p. 140], these maps extend to homeomorphisms of \(T^1\), also denoted by \(h\) and \(f\), respectively, such that \(h \circ R_{\alpha} = f \circ h\). That \(f\) is \(C^1\) follows from Lemma 3 and (3) and (4). It remains to see that \(f\) is not ergodic. Denote by \(T^0_n\) the pair of towers of \(R_{\alpha}\) corresponding to \(T_n\) and, similarly, write \(S^0_n = h^{-1}_n(S_n) = h^{-1}(S_n)\) as its right tower. The set \(S = \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} S^0_k\) is \(R_{\alpha}\) invariant and therefore \(h(S) = \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} S_k\) is \(f\) invariant. But

\[
\mu(h(S)) = \limsup_{n \to \infty} \mu(S_n) + \mu(S_{n+1} - S_n) + \cdots \\
\leq \limsup_{n \to \infty} \mu(S_n) + v^{n+1+n_0} + \cdots \\
\leq \limsup_{n \to \infty} \mu(S_n) \leq \limsup_{n \to \infty} b + v^{n_0}\sigma_n < b + v^{n_0} < 1.
\]

On the other hand,

\[
\mu(h(S)) = \lim_{n \to \infty} \mu\left(\bigcup_{k=n}^{\infty} S_k\right) \geq \liminf_{n \to \infty} \mu(S_n) \geq \liminf_{n \to \infty} a + v^{n+n_0} = a > 0.
\]

REFERENCES

[1] A. Denjoy. Sur les courbes définies par les équations différentielles à la surface du tore. J. Math. Pures Appl. 9(11) (1932), 333–375.
[2] M. R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. Publ. Math. IHES 49 (1979), 5–234.
[3] Y. Katznelson and D. Ornstein. The differentiability of the conjugation of certain diffeomorphisms of the circle. Ergod. Th. & Dynam. Sys. 9 (1989), 643–680.
[4] W. de Melo and S. van Strien. One Dimensional Dynamics. Springer, New York, 1993.