Geometric Solutions to Non-linear Differential Equations

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A general formalism to solve nonlinear differential equations is given. Solutions are found and reduced to those of second order nonlinear differential equations in one variable. The approach is uniformized in the geometry and solves generic nonlinear systems. Further properties characterized by the topology and geometry of the associated manifolds may define global properties of the solutions.
1 Introduction

The systematic solution to non-linear partial differential equations has prohibited many advances in mathematics and physics. These equations, contrary to standard theory and linear equations, appear disparate and unsolvable in the general case. In the past the solutions to these systems of equations have involved many equation dependent techniques that are different in various regimes. A systematic formalism to the solutions of these problems is required.

In this presentation a foundation is given based on geometric methods allowing for a uniform treatment of the solutions to these sets of non-linear partial differential systems. The treatment here is to give a solution set to these equations and to provide an optimum calculus for future improvements.

The general non-linear system is solved for by finding the geodesics on an associated manifold. Even in the generic multi-dimensional system, the solution space is found by solving only second order non-linear equations in one proper time variable (e.g. the geodesics). The reduction in dimensionality is useful for computations and for manifesting global properties of the solutions, such as existence and behavior in various regimes.

2 Setup

The work begins with the algebraic solution to systems of polynomial equations as generated in [1], involving toric and geodesic properties as found for example in [2],[3], [4]. The sets of equations are those of the form,

\[ P_c(z_i; x_j) = \sum_{j=1}^{m} a_{\sigma(j)} \prod_{l=1}^{n_{\sigma(j,l)}} z_{\sigma(j,l)}^{\sigma(j,l)}, \quad (2.1) \]

with \( c \) labeling the equation, and \( \sigma_c(j,l) \) labeling the \( l \)th term of the \( c \)th equation. The permutations of the terms labeled by \( \sigma \). An example set is,

\[ z_1^n + 2z_2^m z_4^n + 3z_3^n = 0 \quad (2.2) \]

\[ 3z_2^m + 2z_3^n z_2^m + z_4^m = 0 \quad (2.3) \]
The equations in (2.1) are general and contain, for example, the well known case pertaining to Fermat’s last theorem,

\[ P(z_i) = z_1^n + z_2^n - z_3^n \]  \hspace{1cm} (2.4)

Equations with \( m = \infty \) are also contained in the set.

The individual terms are accorded to the function(s) solved for together with the derivatives. For example, in the Fermat’s set of equations, one would have \( z_1 = u(w,t) \), \( z_2 = u_x(w,t) \), and \( z_3 = u_t(w,t) \); this pertains to a quantum mechanics problem in the two-dimensional space of \((w,t)\).

The interest here, however, is in modeling general sets of (non-linear) differential equations in a uniform manner. In the interpretation of (2.1), the dynamics of the system is governed by a set of equations with initial conditions spanning a (potentially singular) manifold. To set a basic groundwork, the equations in (2.1) parameterize a holomorphically complex manifold as in [1].

The spaces considered are (toric) Calabi-Yau, although more general manifolds are required. These manifolds have varieties defined by the equation set in (2.1), and may potentially be singular. The beginning setup is then, to every non-linear differential equation, there is an equation describing a fibration of a Calabi-Yau manifold \( M_Z \) over the possibly curved spacetime \( M_X \). The fibration is general and an example is,

\[ P(z; x) = z_1^n + R(x)z_2^m + z_3^n = 0 \]  \hspace{1cm} (2.5)

with \( R(x) \) a generic function describing the \( x \) dependence of the non-linear differential equation. The other coordinates are \( z_1 = u(x,t) \), \( z_2 = u_x(x,t) \), and \( z_3 = u_t(x,t) \). Hence, the fibration is obvious in this example, as is the differential equation. Most general forms are described by,

\[ P_c(z_i; x_j) = \sum_{j=1}^{m} a_{\sigma(j)} n_{\sigma(j,l)} \prod_{l=1}^{Q_{(j,l)}} Q(z_{\sigma(j,l)}) \]  \hspace{1cm} (2.6)

These equations are modeled by the flow in a ‘product’ of \( Z \otimes X \), and are analyzed in this work.

One example in particular is the Navier-Stokes equations,
\[
\frac{\partial}{\partial t} u_i + \sum_{i=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x, t),
\]

which has a manifold interpretation in terms of \( v_i = z_{(i,0)} \), \( v_{(i,j)} = \frac{\partial u_i}{\partial x_j} \) with \( j = 0, 1, \ldots \), and \( p(x) \) and \( f_i \) general functions in the space \( X \). The remaining equations is contained in the divergence condition,

\[
\sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0.
\]

These equations are solvable in the approach here.

The fibration \( Z \otimes X \) is important to consider first. First, the initial conditions are defined by a point in this space.

The solutions are found via flows in the total space \( X \otimes Z \). There are integrability conditions in the direction of the flow given the derivatives, i.e. \( u(x) = Z \) and \( u_x(x,t) = \partial_x Z \). As an example, consider a manifold \( M_Z \) spanned by the variables,

\[
u, \quad u_x, \quad u_{xx}, \quad u_{xxt},
\]

which are functions of the base space variables \( x \) and \( t \). Then the dynamics is described by the differential equation \( P(u, u_x, u_{xx}, u_{xxt}) \), with \( u(x,t) \) a function of the variables on the base \( R(x,t) \) (e.g. flat space \( R^2 \)). Label the functions as,

\[
z_1 = u, \quad z_2 = \frac{\partial u}{\partial x} = \frac{\partial z_1}{\partial x}, \quad z_3 = \frac{\partial z_2}{\partial x}, \quad z_4 = \frac{\partial z_3}{\partial t}.
\]

The initial conditions \( u^o, u_x^o, u_{xx}^o, \) and \( u_{xxt}^o \) at \( (x^o, t^o) \) generate a point on the fibration \( M_Z \otimes X \). (These initial values must satisfy the differential equations.)

In order to find the values of \( u \) (and \( u_x, u_{xx}, u_{xxt} \)) at a different point \( (x,t) \), the former variables follow a flow from the point \( (x^o, t^o) \) to \( (x, t) \) while preserving the conditions in \( (2.10) \). The path in \( M_Z \) generated through the equations in \( (2.10) \) are integrability conditions on the coordinates \( (x^o, t^o) \) to \( (x, t) \).

3 The case of \( P(z_i) = 0 \).

There are two cases of interest, when \( P(z_i; x_j) = P(z_i) \) and when there is explicit base dependence. The former case is analyzed first. The \( z_i \) coordinates are the functions \( u, \)
The values from the initial point $z_i$ to the final point $z_i$ are constrained on the manifold $P(z_i)$ and are generated via a geodesic flow. Then the coordinates $x_j$ are determined via the integrability conditions in (2.10); the inclusion of an auxiliary complex space $\bar{P}(\bar{z}_i)$ is generally required.

The first step, as in [1], is to model a space via the polynomial equation $P(z_i) = 0$ (with possibly the non-holomorphic modification of $\bar{P}(\bar{z}_i)$). The geodesics on these space will require solutions.

Formally the spaces used in the work are toric Calabi-Yau, which may have singularities; the general manifolds are holomorphically toric, that is, have two polynomials specifying the both the holomorphic and nonholomorphic sides. A modified $\mathcal{N} = 2$ D-term specification can be used to find these metrics.

These polynomials give information about the existence of solutions, further from the locus of points in (2.1). In the Kähler examples, label the space pertinent to (2.1) as $M_P(z_i)$ and its Riemannian metric as $g_{\mu\nu}$. Its Kähler so that both $g_{\mu\nu} = \partial_\mu \partial_\nu \ln \phi(z_i, \bar{z}_i)$ (in terms of $z$ and $\bar{z}$, $g = g_{ij}$) and $\Gamma_{\rho,\mu\nu} = 1/2 \partial_\rho \partial_\mu \partial_\nu \ln \phi(z_i, \bar{z}_i)$ hold.

The polynomial sets of finite degree are modeled by a finite dimensional Calabi-Yau; the equation sets of infinite degree (i.e. transcendental) are described by an infinite dimensional manifold.

The geodesic equation is, with the coordinates $x = (z_i, \bar{z}_i)$,

$$\frac{d^2 x^\rho}{d\tau^2} + \Gamma^{\rho,\mu\nu} \frac{dx_\mu}{d\tau} \frac{dx_\nu}{d\tau} = 0 \tag{3.1}$$

The Kähler example admits a complex form with the connection components,

$$\Gamma^{\rho,\mu\nu} = 1/2 \partial^\rho \partial^\mu \partial^\nu \ln \phi(x_\mu, \bar{x}_\nu) \tag{3.2}$$

The coordinates $x$ contain both the holomorphic and anti-holomorphic pieces describing the geometry. Its complex Kähler form is

$$\frac{dz_i}{d\tau} + \frac{dz_j}{d\tau} \partial^i \partial^j \ln(\phi) = 0 \tag{3.3}$$

or

$$\frac{dz_i}{d\tau} + \frac{dz_j}{d\tau} g^{\jmath,i} = 0 \tag{3.4}$$
These equations are second order in derivatives, but non-linear because of the Kähler potential. Solving these equations gives the $z_i$ coordinates as a function of the 'proper time' $\tau$; the $2n$ initial and final coordinates $z_i^0$ and $z_i^f$ determine the unknowns.

Once the path from $z_i^0$ to $z_i$ is found, the coordinates in the base are determined. As the $z_i$ coordinates are explicit functions of the initial conditions and $\tau$, the integrability conditions in (2.10) are first order in derivatives and the coordinates $x_j$ requires solutions in $\tau$. Take the example of the equation,

$$u^5 + u_{xx}^4 + u_x = 0.$$  \hfill (3.5)

The $z_i$ are $z_1 = u(x)$, $z_2 = u_x(x)$ and $z_3 = u_{xx}(x)$. The integrability conditions in (2.10) requires,
The function $x$ is found via,

$$
\frac{dx}{d\tau} = \frac{1}{z_2(\tau)} \frac{\partial z_1}{\partial \tau} .
$$

(3.7)

The coordinate in (3.7) can be solved by an integral

$$
dx = \frac{z_1}{z_2} \frac{d\tau}{d\tau} \quad (3.8)
$$

and gives its $\tau$ dependence. Further equations need to be solved, and these put conditions on the solutions to $z_i$. For example, the equation $z_3 = dz_2/dx$ is consistent only if,

$$
\frac{z_1}{z_2} = \frac{z_2}{z_3} \rightarrow \frac{1}{2} \frac{\partial}{\partial \tau} \frac{z_2}{z_3} = \frac{\partial z_1}{\partial \tau} .
$$

(3.9)

The equation in (3.9) is satisfied only if the final coordinates $z_1^f$, $z_2^f$, and $z_3^f$ (and the initial $z_1^0$, $z_2^0$, and $z_3^0$ at the point $x$) are chosen well. These further requirements are algebraic in nature, and are hand in hand with the existence of solutions.

\section{Integrability conditions}

In determining the coordinates $x_j$ from the conditions in (2.10) there are complications due to the number of equations and the number of free variables $x_i$. These complications are removed by enlarging the toric space to include a non-holomorphic side $\bar{P}(\bar{z}_i) = 0$, which may be of a large degree. The manifold in principle is no longer Kähler, and holomorphically separated. The solution to the non-linear pde’s is solved by the geodesic flow represented in the holomorphic coordinates $z_i$, while the constraints are satisfied via the influence on $z_i$ via the flow in the non-holomorphic side $\bar{z}_i$. This is analyzed in this section.

Consider a differential equation containing,

$$
z_1 = u \quad z_2 = u_{x_1} \quad z_3 = u_{x_2} \quad z_4 = u_{x_1 x_1} \quad z_5 = u_{x_2 x_2} \quad z_6 = u_{x_1 x_2} .
$$

(4.1)
There are fewer coordinates $x_i$ than integrability conditions in (4.1). After solving for the coordinates $x_1(\tau)$ and $x_2(\tau)$ with,

$$\frac{dz_1}{dx_1} = z_2 \quad \frac{dz_1}{dx_2} = z_3 \ ,$$

there are four integrability conditions,

$$\frac{dz_2}{dx_1} = z_4 \quad \frac{dz_2}{dx_2} = z_6 \quad \frac{dz_3}{dx_1} = z_6 \quad \frac{dz_3}{dx_2} = z_5 \ ,$$

and two relations between them,

$$\frac{dz_2}{z_4} = \frac{dz_3}{z_6} \quad \frac{dz_2}{z_6} = \frac{dz_3}{z_5} .$$

The integrability conditions, e.g.,

$$\frac{dz_2}{d\tau} (\frac{dx_1}{d\tau})^{-1} = z_4$$

determine four constants, i.e. four of the $z_i^f$ in solving the geodesic motion. However, if the solutions are not possible, which is very probable, then the final points are taken to be arbitrary and a different route (or geodesic) is required; this different route changes the functional form of the solution $z_i$ so as to make these integrability conditions satisfied. The change in the route is made possible by altering the manifold on the non-holomorphic side $\bar{P}(\bar{z})$, while preserving the holomorphic polynomial $P(z_i)$ which models the differential equation.

In a separate issue, perturbations may be added to make the integrability conditions linear in derivatives; this makes the analysis easier in solving these equations. If the differential equation does not contain the first derivative in the coordinate, then either: (1) the first non-vanishing term (e.g. $u_{xx}$ or higher order) must be used to define the spatial coordinate $x_j$, or (2) a perturbation of the form $u_x$ has to be added to the differential to define $x_j$ as in the previous. The first option requires a second order differential equation to solved, as opposed to a first order one, $z_2 = d^2 z_1/dx^2,$

$$z_2 = -\left(\frac{\partial x}{\partial \tau}\right)^{-3} \frac{d^2 x}{d\tau^2} z_{1,\tau} + \left(\frac{\partial x}{\partial \tau}\right)^{-2} z_{1,\tau\tau} .$$

Equations of these types (to be solved for $x$) are more complicated, and first order linearity is preferred. The perturbations $\alpha_i u_{x_i}$ are simpler to add, and post the analysis the coefficients $\alpha_i$ are taken to zero. Of course, if the solutions are singular then the previous method should be used.
The example given in (3.5) describes how the general non-linear differential equation is solved for \( z_i(\tau; z^o_i, x^o_j) \) and \( x_j(\tau; z^o_i, x^o_j) \). The general solution requires two non-trivial steps: (1) the solution of a non-linear second-order differential equation describing the geodesic condition, and (2) the solution to a set of algebraic conditions describing the integrability relations. These steps are uniform to all of the non-linear partial differential equations that satisfy \( P(z_i) = 0 \), i.e. without any \( x_i \) dependence.

The integrability conditions, if not satisfied via the solution to the direct equations of motion via the toric variety with complex conjugates \( P(z_i) = 0 \), and \( P(\bar{z}_i) = 0 \), then a procedure is developed based on a different manifold is used. Consider a general space defined by the constraint,

\[
P_c(z_i) = \sum_{\sigma(i)} a_{\sigma(i)} \prod_i z^{\sigma(i)},
\]

(4.7)

and

\[
\bar{P}_c(\bar{z}_i) = \sum_{\bar{\sigma}(i)} \bar{a}_{\bar{\sigma}(i)} \prod_i \bar{z}^{\bar{\sigma}(i)},
\]

(4.8)

with possible logarithmic modifications \( \ln(z) \) and \( \ln(\bar{z}) \). In general, the Laurent expansion with positive and negative powers of the coordinates span all of the non-analytic terms. There are in principle an infinite number of constants in the general non-holomorphic side represented in (4.8). These constants are used to satisfy the integrability conditions.

The point of introducing the coefficients \( a_{\sigma} \) and \( \bar{a}_{\bar{\sigma}} \) is to allow functional dependence in the components of the Christoffel connection, so as to allow the integrability conditions (e.g. (2.10)) to be satisfied. In general the metric is a function of these coefficients \( a \) and \( \bar{a} \). The holomorphic piece \( P(z_i) = 0 \) is used to describe the analyzed non-linear differential equation. The non-holomorphic constraint \( \bar{P}(\bar{z}_i) \), with the coefficients \( \bar{a} \) have enough terms (an infinite defining a non-analytic function) so as to allow the constraints to be satisfied.

As the holomorphic side of the manifold is parameterized by \( P(z_i) \), the final solution for \( Z^\mu_f = Z^\mu(\tau_f) \) will be obtained by a geodesic condition in the total manifold spanned by \( P(z_i) \) and \( \bar{P}(\bar{z}_i) \), in such a manner that the consistency conditions are satisfied. The latter are maintained by choosing the \( \bar{P}(\bar{z}_i) \) and the initial and final non-holomorphic coordinates, \( \bar{z}_i \) and \( \bar{z}_f \), appropriately to repeat their solution via the geodesic motion in the total space. The solution to the consistency conditions could be complicated due to the possibly complicated solutions to the coordinates \( z_i \), although
in only one variable. The geodesic equations are non-holomorphic in the sense that both \( z^i \) and \( \bar{z}^i \) are required (and the mixed indexed \( \Gamma \)) in the individual flows; however, the solution to the non-linear partial differential equations are obtained from only the holomorphic piece.

In the procedure described, the general metric on a non-complex (holomorphically toric) manifold is required together with the solution to the geodesics on the space. However, the holomorphic side is described by the original function \( P(z_i) \) describing the non-linear partial differential equation. The reproduction of the integrability conditions, given for the one example in (2.10), requires \textit{algebraic} solutions to the parameters \( \bar{a}_{\sigma(i)} \) (the solutions may require transcendental algebra).

The primary difficulty in the solution is solving for the geodesics in the Calabi-Yau manifold and also solving the algebraic equations. However, the treatment of all the non-linearities is reduced to only second-order partial differential equations in one variable (an analysis of algebraic equations is presented in [1], requiring also second order partial differential equation solutions).

The existence of solutions and the singularities are described by the steps in the previous example, and are generally given by the behavior of the metric on the associated manifold. It is relevant that the non-linearities and the types of solutions may be described by the geodesics and properties of the underlying geometry. The consistency conditions describe a submanifold connected to the initial conditions \( z^0_i \) and \( x^0_j \).

The solutions via this method generate the \( \tau \) dependence in the coordinates \( x_j \) and the functions \( z_i \), i.e. \( x_j(\tau; x_i^0, z_i^0) \) and \( z_i(\tau; x_j^0, z_k^0) \). The \( \tau \) dependence may be solved for as a function of \( x_j \) and substituted in the solution to find \( u \) \((u_x, \ldots)\), or the \( z_i \), in order to manifest the solution’s \( x_i \) dependence.

The interpretation of the auxiliary non-holomorphic components \( z_i \) may be put in the form of integrability constants, perhaps with the the non-holomorphic polynomial \( \bar{P}(\bar{z}) \) having a group theory interpretation.

Spatial dependence of the initial conditions should be commented on. The initial conditions of \( u(x, t) \), \( u_x(x, t) \), \ldots, correspond to a point in the manifold parameterized by \( P_c(z_i) \). A spatially dependent ‘wave-packet’ at time \( t = 0 \) as an initial condition may be constructed by choosing an appropriate point \( Q \) in the manifold and finding the values in a localized region of space \( \vec{x} \in M_i \) at time \( t = 0 \) in which \( u(\vec{x}, 0) \) is given, via the transports from the point \( Q \) to the ‘final’ points in this region. The solutions at these ‘final’ points are evolved into the required regions \( t \neq 0 \) and the \( \vec{x} \).
5 The case of \( P(z_i; x_j) = 0 \).

The primary difference in this case is the inclusion of explicit coordinate dependence in the set of coupled non-linear partial differential equations. For example, there might be a general term \( p(x) \) as in the Navier-Stokes equations,

\[
\frac{\partial}{\partial t} u_i + \sum_{i=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p(x_i)}{\partial x_i} + f_i(x, t). \tag{5.1}
\]

More general equations could have the \( x \) coordinates spanning a curved spacetime, with the (coupled set of) non-linear equations taking values on it and there being explicit metric dependence.

The general \( x \) dependence in the equations described by \( P_a(z_i; x_j) \) is incorporated by enlarging the flow in the \( z_i \) coordinates to include \( x_j \). In this manner, the constraints \( P_a(z_i; x_j) \) are always preserved. The non-holomorphic polynomial(s) \( \bar{P}_a(z_i) \) describe the remaining side of the manifold and are used in the geodesic flows as in the previous case of no explicit \( x \)-dependence.

6 Concluding remarks

The general set of coupled non-linear differential equations is described and reduced in terms of flows in manifolds, and the solution of algebraic equations. The solution to the differential equations is described by geodesic flow, that is they reduce to second order non-linear differential equations in one variable only. The multi-variable aspect is further transferred to the required solving of a set of algebraic equations (of potentially infinite degree) in an auxiliary set of variables \( \bar{a} \).

The second order differentiability in one variable bears a resemblance to a Lagrangian particle description. A Hamiltonian first order description would be useful.

The nonlinear partial differential equations describes a manifold; due to the type of manifold, the solutions inherit modular properties and potentially further ones akin to a mirror symmetry on toric varieties. Furthermore, the manifolds’ topological properties potentially may be used to classify solutions.

The geodesic flows, and the sub-manifolds they parameterize, describe chaotic versus periodic solutions and characterize the solutions according to initial conditions. For example, a closed geodesic obviously generates non-chaotic behavior; these
topological properties characterize the integrability, and ergodicity, of the solutions. An ideal situation entails cohomological calculations to show integrability properties.

The obvious use of the proper-time solutions to the non-linear partial differential systems, and their form, merits further investigation. Also, the connection to the geometry of the manifolds perhaps requires input from its topological properties for characterizations of the solutions. The explicit form of the geodesics, and the metrics required in order to find them, are not presented in this work.

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