Warped Compactification with a Four-Brane

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Warped compactification of a six-dimensional bulk with a negative cosmological constant is realized using a 4-brane along with an abelian gauge theory. We find that no fine tuning of couplings is needed to obtain a vanishing cosmological constant in four dimensions, just as in the case of a bulk with a positive cosmological constant.

§1. Introduction

The higher-dimensional approach to the cosmological constant problem employed by Rubakov and Shaposhnikov is based on the idea that a vanishing cosmological constant in four dimensions might be realized through warped compactification, even in the presence of a nonvanishing higher-dimensional cosmological constant. This might be possible, since four-dimensional effective flatness is not necessarily in contradiction with large curvature of spacetime in a higher-dimensional space.

In a previous paper, we investigated the case of a six-dimensional bulk with a positive cosmological constant and realized a vanishing four-dimensional cosmological constant without metric singularity by introducing an abelian gauge field. In this paper, we consider the case of a negative bulk cosmological constant to obtain regular and compact extra dimensions. We are led to introduce a 4-brane in the bulk along with an abelian gauge field, again the case of a positive cosmological constant.

§2. The model

Let us consider an $SO(2)$-symmetric warped metric as a six-dimensional background:

$$ds^2 = g_{MN}dx^M dx^N = \sigma(r)\bar{g}_{\mu\nu}dx^\mu dx^\nu - dr^2 - \rho(r)d\theta^2.$$  \hspace{1cm} (1)

Here, $\bar{g}_{\mu\nu}$ denotes the four-dimensional metric, independent of $(r, \theta)$, with $0 \leq \theta < 2\pi$. The action of this metric with an abelian gauge field and a 4-brane at $r = r_1$ is given by

$$S = \int d^6x\sqrt{-g} \left( \frac{1}{2} R - \frac{1}{4} F_{MN} F^{MN} - \Lambda \right) - \int d^5x\sqrt{-g_5} \lambda \delta(r - r_1),$$  \hspace{1cm} (2)

where $\Lambda < 0$ and $g_5$ is the determinant of the induced metric on the 4-brane.
The gauge field equations of motion are obtained as
\[ \partial_M(\sqrt{-g}F^{MN}) = 0, \]  
and the background configuration with the SO(2) symmetry is given by
\[ A_\mu = A_r = 0, \quad A_\theta = a(r). \]
These yield the field strength
\[ F^{r\theta} = \frac{B}{\sigma^2 \sqrt{\rho}}, \]
where \( B \) is an integration constant. With this gauge field configuration, the Einstein equations lead to
\[
\begin{align*}
\frac{3}{2} \frac{\sigma''}{\sigma} + \frac{3}{4} \frac{\sigma'}{\rho} - \frac{1}{4} \rho'' + \frac{1}{4} \rho' - \frac{A_4}{\sigma} = -\frac{B^2}{2\sigma^4} - \Lambda - \lambda \delta(r - r_1), \\
\frac{3}{2} \frac{\sigma'^2}{\sigma^2} + \frac{\sigma'}{\rho} - \frac{2A_4}{\sigma} = \frac{B^2}{2\sigma^4} - \Lambda,
\end{align*}
\]
where the prime denotes differentiation with respect to \( r \) and we have used the four-dimensional Einstein equations for the metric \( \bar{g}_{\mu\nu} \) with the cosmological constant \( A_4 \), which comes out as an integration constant.

We obtain the junction conditions across the 4-brane from the equations of motion as
\[
\begin{align*}
\sigma'(r_1 + 0) - \sigma'(r_1 - 0) &= -\frac{1}{2} \lambda \sigma(r_1), \\
\rho'(r_1 + 0) - \rho'(r_1 - 0) &= -\frac{1}{2} \lambda \rho(r_1).
\end{align*}
\]
In the regions \( r < r_1 \) and \( r > r_1 \), the bulk Einstein equations are reduced to the equation of motion
\[
\begin{align*}
\frac{z''}{z} &= -\frac{\partial V(z)}{\partial z}; \\
V(z) &= \frac{25}{96} B^2 z^{-6/5} + \frac{5}{16} A_4 z^2 - \frac{25}{24} A_4 z^{6/5}, \\
\sigma &= z^{4/5}, \\
\rho &= C_\pm z^{2} z^{-6/5},
\end{align*}
\]
where \( C_- \) and \( C_+ \) are integration constants for the two regions. Note that this equation describes the motion of a particle with position \( z \) at time \( r \) subject to a potential \( V(z) \).

§3. The solutions

Now we seek regular metric solutions in the case of interest, \( A_4 = 0 \). The junction conditions Eqs. (9) and (10) can be expressed in terms of \( z \) with the condition \( \lambda \neq 0 \)
\[ z'_+ - z'_- = -\frac{5}{8}\lambda z_1, \quad (13) \]
\[ z'_+ z'_- = -\frac{5}{16}B^2z_1^{-6/5} + \frac{5}{8}A\bar{z}_1^2, \quad (14) \]

where \( z'_\pm = z'(r_1 \pm 0) \) and \( z_1 = z(r_1) \) (which may be normalized to 1 without loss of generality). Equations (13) and (14) can be solved for \( z'_\pm \), provided \( 5\lambda^2 + 32A \geq 0 \), with \( z'_- > 0 \) and \( z'_+ < 0 \). This change of the velocity \( z' \) from positive to negative is crucial for compactification of the extra dimensions, which is achieved by means of the 4-brane with \( \lambda > 0 \).

The equation of motion (11) implies \( z' = 0 \) at \( r = r_0, \bar{r} \), with \( z' \neq 0 \), for \( r_0 < r < r_1 \) and \( r_1 < r < \bar{r} \). The conservation of ‘energy’ as \( z \) changes in each of the ‘time’ regions \( r_0 \leq r < r_1 \) and \( r_1 < r \leq \bar{r} \) implies

\[ \frac{25}{96}B^2z_0^{-6/5} + \frac{5}{16}A\bar{z}_0^2 = \frac{1}{2}z'_- + \frac{25}{96}B^2z_1^{-6/5} + \frac{5}{16}A\bar{z}_1^2, \quad (15) \]
\[ \frac{25}{96}B^2\bar{z}^{-6/5} + \frac{5}{16}A\bar{z}_0^2 = \frac{1}{2}z'_+ + \frac{25}{96}B^2z_1^{-6/5} + \frac{5}{16}A\bar{z}_1^2, \quad (16) \]

respectively. Here we have defined \( z_0 = z(r_0) \) and \( \bar{z} = z(\bar{r}) \), whose values are determined by these equations.

The continuity of \( \rho \) across the 4-brane imposes the condition

\[ C_-^{-2}z_-^2 = C_+^{-2}z_+^2 \quad (17) \]

from Eq. (12). For regularity of the higher-dimensional metric, we have the condition

\[ (\sqrt{\rho})'(r_0) = -(\sqrt{\rho})'(\bar{r}) = 1, \quad (18) \]

which gives

\[ C_-^{-3/5}z_0^{-11/5} = \frac{5}{16}B^2z_0^{-11/5} - \frac{5}{8}A\bar{z}_0, \quad (19) \]
\[ C_+^{-3/5}\bar{z}^{-11/5} = \frac{5}{16}B^2\bar{z}^{-11/5} - \frac{5}{8}A\bar{z}. \quad (20) \]

The above three conditions can be solved for the three integration constants \( B \) and \( C_\pm \). This yields a reflection-symmetric metric with respect to the 4-brane. In other words, the effective four-dimensional cosmological constant \( \Lambda_4 \) can vanish for generic values of \( \Lambda < 0 \) and \( \lambda > 0 \) with the extra dimensions compactified.

We note that, in addition to the 4-brane at \( r = r_1 \), 3-branes may be introduced at \( r = r_0 \) and/or \( r = \bar{r} \) straightforwardly. For example, if we put a 3-brane with tension \( \lambda_0 \) at \( r = r_0 \), the regularity condition Eq. (19) is replaced by \(^2\)

\[ C_- \left( 1 - \frac{\lambda_0}{2\pi} \right) z_0^{-3/5} = \frac{5}{16}B^2z_0^{-11/5} - \frac{5}{8}A\bar{z}_0, \quad (21) \]

which also allows the full equations of motion to be satisfied without a singularity.
§4. Conclusion

We have obtained a regular metric with compact extra dimensions in the case of a negative bulk cosmological constant in six dimensions. We found that the four-dimensional cosmological constant $\Lambda_4$ can vanish even without the fine tuning of the Lagrangian parameters $\Lambda < 0$ and $\lambda > 0$ (or the tensions of the 3-branes, if such exist).

We note that regular and compact metrics also exist for $\Lambda_4 \neq 0$ in the case of a negative bulk cosmological constant, just as in the case of a positive bulk cosmological constant. It is thus seen that $\Lambda_4$ is simply an integration constant to be determined by the boundary conditions, which might be supplied by some other sector to be added to the model.

References

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