Relativistically speaking: Let’s walk or run through the rain?

Armando V.D.B. Assis*

armando.assis@pgfsc.ufsc.br

December 3, 2010

Abstract

We analyse under a simple approach the problem one must decide the best strategy to minimize the contact with rain when moving between two points through the rain. The available strategies: walk (low speed boost \(< < c\)) or run (relativistic speed boost \(\approx c\)).

1 On the General Aspects of the Problem

We will restrict ourselves to the domain of relativistic kinematics. Consider an inertial cartesian reference frame 0\(xyz\). Put the velocity field \(\vec{u}(\vec{r}; t) \equiv \vec{u}(x, y, z; t)\) of the raindrops observed in the inertial reference frame 0\(xyz\). Let a closed surface \(S\) move with constant velocity \(\vec{v}\) in 0\(xyz\). Now, let 0\(x'\)y'z'\ be the inertial reference frame encrusted to the proper surface \(S'\) and \(\vec{u}'(\vec{r}'; t') \equiv \vec{u}'(x', y', z'; t')\) the velocity field of the raindrops observed in 0\(x'\)y'z'. Suppose \(S'\) as being transparent to the field \(\vec{u}'(\vec{r}'; t')\), i.e., an imaginary proper control surface. Therefore, the flux \(\phi'\) of the velocity field \(\vec{u}'(x', y', z'; t')\) through \(S'\) is given by:

\[
\phi' = \oint_{S'} \vec{u}'(x', y', z'; t') \cdot \hat{n} dS',
\]

where \(\hat{n} dS'\) is the elemental surface vector normal to \(S'\) at a point \(P(x', y', z'; t') \in S' \times \{t'\}\). Obviously, a real body moving through the rain will be opaque to the field \(\vec{u}'(\vec{r}'; t')\). In cases in which there exists a shadow region regarding the surfaces where the field \(\vec{u}'(\vec{r}'; t')\) is null, the amount of rain that would penetrate through the surfaces of the opaque body in a case of transparency is given by eq. (1):

\[
\phi' = \oint_{S'} \vec{u}'(x', y', z'; t') \cdot \hat{n} dS' = \oint_{S'_r} \vec{u}'_r(x', y', z'; t') \cdot \hat{n} dS' \quad \text{and} \quad \oint_{S'_{wall}} \vec{0} \cdot \hat{n} dS' = 
\]

\[
\phi' = \phi'_r = \oint_{S'_r} \vec{u}'_r(x', y', z'; t') \cdot \hat{n} dS',
\]

where \(\phi'_r\) is the flux that would penetrate (in a case of transparency) through the region of \(S'\) touching raindrops, namely \(S'_r\). Hence, eq. (2) gives the amount of water touching \(S'\) through \(S'_r \subseteq S'\). \(S'_{wall}\) is the part of \(S'\) at shadow region where \(\vec{u}' = \vec{0} \forall P(x', y', z'; t') \in S'_{wall} \times \{t'\}\).

2 A Simple Approach

Let’s do some simplifications in analyzing the problem. The surface \(S'\) will be of a proper (there is not Lorentz contraction regarding \(S'\)) rectangular prism with square base in 0\(x'\)y'z', as drawn on the left. This schematic drawing shows the mentioned reference frames, 0\(xyz\) and 0\(x'\)y'z', and the boost vector \(\vec{v}\). We also note the surface \(DAFGBCD\), taken as the surface \(S'_r\) touching the raindrops (these ones are not depicted yet).

In the schematic drawing below, we have a lateral vision of the problem as observed in 0\(xyz\), with the terminal velocity field of the raindrops \(\vec{u}(x, y, z; t)\), the position vectors \(\vec{r}\) and \(\vec{r}'\), of an arbitrary rain drop \(P\), the displacement vector of the origin \(0'\) in relation to the origin 0, vector \(\hat{n}\) and other trivially denoted elements inherent to the simplification we are from now on considering.

* Dedicated to YHVH.

1 Each point belonging to \(S\) is moving with the same constant velocity \(\vec{v}\) in 0\(xyz\).
Let’s consider the two reference frames in the canonical configuration, i.e., coincident origins at \( t = t' = 0 \) keeping the spacelike parallelism of the axes \( x \equiv x', y \equiv y' \) and \( z \equiv z' \).

The Lorentz transformations of the 3-velocity field of the raindrops gives \( \vec{u} = \frac{d\vec{r}}{dt} \) in the \( 0x'y'z' \) reference frame. Since \((u_x, u_y, u_z) = (0, 0, -u_{\infty})\) in \( 0xyz \), where \( u_{\infty} \) is the terminal velocity of the raindrops as measured in \( 0xyz \), we have:

\[
\begin{align*}
  u_{x'} &= \frac{u_x - v}{1 - v u_x / c^2} = -v; \\
  u_{y'} &= \frac{u_y}{\gamma (1 - v u_x / c^2)} = 0; \\
  u_{z'} &= \frac{u_z}{\gamma (1 - v u_x / c^2)} = -\frac{u_{\infty}}{\gamma},
\end{align*}
\]

where \( \gamma = 1/\sqrt{1 - v^2/c^2} \), \( c \) is the speed of light. We tacitly started supposing the uniform field of the terminal raindrops as a simplification. As mentioned above, we are concerned to the kinematics, in which case such a constraint is sufficient to the flux calculation, regardless the physics of drag upon the raindrops and so on. We must discretize the field, since we should not associate a vector to every point of space \( 0xyz \) in virtue of the fact we are not dealing with continuous current lines of fluid. Putting the above results in eq. (2), asseverating discreteness by the label \( d \), we have:

\[
\phi_r' = \int_{DABCD} \left( -v \vec{e}_{x'} - \frac{u_{\infty}}{\gamma} \vec{e}_{z'} \right) \cdot \hat{n} dS',
\]

where \( \vec{e}_{x'} \) and \( \vec{e}_{z'} \) are the unitary vectors along the axes \( 0'x' \) and \( 0'z' \) respectively.

We discretize the raindrops field at the points of the proper surfaces DABCD and AFGBA in the \( 0'x'y'z' \) reference frame via the Dirac delta functions at the points of these surfaces that simultaneously cross the discrete rain rays. Obviously, these non-proper surface points in \( 0xyz \) should not associate a vector to every point of space \( 0xyz \) in virtue of the fact we are not dealing with continuous current lines of fluid. Putting the above results in eq. (2), asseverating discreteness by the label \( d \), we have:

\[
\phi_r' = \int_{DABCD} \left( -v \vec{e}_{x'} - \frac{u_{\infty}}{\gamma} \vec{e}_{z'} \right) \cdot \hat{n} dS',
\]

where \( \vec{e}_{x'} \) and \( \vec{e}_{z'} \) are the unitary vectors along the axes \( 0'x' \) and \( 0'z' \) respectively.

We discretize the raindrops field at the points of the proper surfaces DABCD and AFGBA in the \( 0'x'y'z' \) reference frame via the Dirac delta functions at the points of these surfaces that simultaneously cross the discrete rain rays. Obviously, these non-proper surface points in \( 0xyz \) cross the discrete rain rays non-simultaneously in \( 0xyz \). It does not matter at all, since we are analyzing the problem under \( 0'x'y'z' \) point of view. In fact, the rain rays appears to be rotated when compared to the non-relativistic case in which the angle \( \theta \) between the rain drops ray and the \( 0'x' \) axe is given by \( \tan \theta = u_{\infty}/v \), instead of \( \tan \theta = u_{\infty}/(\gamma v) \). Hence, simultaneous raindrops under non-relativistic case are shifted through the surfaces, becoming non-simultaneous ones. The non-null raindrops field at the surfaces DABCD and AFGBA at an instant \( t' \) in \( 0'x'y'z' \) world description reads:

\[
\begin{align*}
( -v \vec{e}_{x'} - \frac{u_{\infty}}{\gamma} \vec{e}_{z'} )_{DABCD} &= \left( -v \vec{e}_{x'} - \frac{u_{\infty}}{\gamma} \vec{e}_{z'} \right) \sum_{i=1}^{N_x(t')} \sum_{j=1}^{N_y(t')} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta[x' - x'_i(t')] \delta[y' - y'_j(t')] \; dx' dy'; \\
( -v \vec{e}_{x'} - \frac{u_{\infty}}{\gamma} \vec{e}_{z'} )_{ABGFA} &= \left( -v \vec{e}_{x'} - \frac{u_{\infty}}{\gamma} \vec{e}_{z'} \right) \sum_{j=1}^{N_y(t')} \sum_{k=1}^{N_z(t')} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta[y' - y'_j(t')] \delta[z' - z'_k(t')] \; dy' dz',
\end{align*}
\]

with \( N_x(t') \geq 1 \forall j \in \{1, 2, 3\} \), where \( N_x(t')N_y(t') = N_{DABCD}(t') \) is the number of raindrops simultaneously touching the proper surface DABCD at the instant \( t' \) and \( N_y(t')N_z(t') = N_{ABGFA}(t') \) is the number of raindrops simultaneously touching the proper surface ABGFA at the instant \( t' \), both measured in \( 0'x'y'z' \).

By the schematic drawing below, we depict the situation under \( 0'x'y'z' \) point of view. We denote \( u \equiv u_{\infty}/\gamma \). From the depicted geometry, we infer the following relations:

\[
SA = AS \cong \frac{DA}{N_{x'}}, \quad AT \cong \frac{AF}{N_{y'}}, \quad \frac{AT}{AS} = \frac{AF}{DA} \frac{N_{y'}}{N_{x'}}, \quad \therefore
\]

\[
\frac{AT}{AS} = \frac{AF}{DA} \frac{N_{y'}N_{y'AB}}{N_{x'}N_{y'AB}}.
\]

\( N_{x'}N_{y'} = N_{DABCD} \) is the number of raindrops touching the proper surface DABCD, from now denoted by \( N_s \), and \( N_{y'}N_{z'} = N_{ABGFA} \) the number of raindrops touching the proper surface ABGFA, from now denoted by \( N_f \).

\footnote{\( s \) denotes \textit{superior} and \( f \) denotes \textit{frontal}.}
Where the function \( I \) reads:

\[
\begin{align*}
\tau(t')/T_{x_j} &= \sum_{k=0}^{\lfloor \tau(t')/T_{x_j} \rfloor} V_{x_j}(t') N_{x_j} \delta(t' - kT_{x_j}); \\
N_{y'}(t')N_{z'}(t') a_{ij}^{f}(t')v &= \sum_{l=0}^{\lfloor \tau(t')/T_{x_j} \rfloor} V_{y'}^{f}(t') N_{y'} \delta(t' - lT_{x_j}); \\
N_{x'}(t')N_{y'}(t') &= N_{x}(t'); \quad N_{y'}(t')N_{z'}(t') = N_{f}(t'),
\end{align*}
\]

where the function \( I(\tau'/T_{x_j}^') : \tau'/T_{x_j}^' \in \mathbb{R}^+ \to I(\tau'/T_{x_j}^') \in \mathbb{Z}^+ \) returns the integer part of \( \tau/T_{x_j}^' \), the ratio between the total elapsed time \( \tau' \) in \( 0'x'y'z' \) world and the period \( T_{x_j}^' \) of a plane orthogonal to \( 0'x_j' \) containing simultaneous raindrops in \( 0'x'y'z' \) world; \( \delta \) is the Dirac delta function; \( N_{x_j}^' \) was properly defined through the march that led to eqs. (6), (7) and (8) above. Furthermore, since \( \delta \) is a distribution, the inherent context regarding it will become clear in the time integration we will perform at the final of our calculation. \( N_{x}(t') \) and \( N_{f}(t') \) are, respectively, the \( t' \) instantaneous number of raindrops touching the top and the frontal proper surfaces of the prism in \( 0'x'y'z' \) world. \( a_{ij}^{f}(t') \) is the \( 0'z' \) orthogonally projected area of a single raindrop \( t' \) instantaneously passing through the superior (top) proper adjacent surface of the prism at some \( (i,j) \)-position, \( V_{y'}^{f}(t') \) is the \( t' \) instantaneous raindrop volume passing through \( a_{ij}^{f}(t') \), both observed in \( 0'x'y'z' \) world; \( a_{jm}^{f}(t') \) and \( V_{jm}^{f}(t') \) have analogous definitions related to the frontal proper surface of the prism.

We asseverate that the identity of raindrops touching the frontal proper surface ABGFA, \( N_{f} \), and the identity of raindrops touching de top surface ABCDA, \( N_{s} \), must be the same in both reference frames, but not simultaneous in both systems of reference. These observers will not agree about the identities of the raindrops simultaneously touching the top and frontal surfaces. This lack of reciprocal simultaneity, an inherent character of Einstein’s theory of relativity, tell us we must avoid a measure of the number of raindrops based on identities of raindrops simultaneously touching the surfaces in both frames of reference, as naturally one would do in a case of galilean relativity. Straightforwardly, imagine (at \( 0xyz! \)) a set of horizontal planes equally spaced \( (d_{0z}) \) (each containing horizontal lattice of raindrops \( 0z \times 0y \) spaced) falling vertically at constant terminal velocity \( v \). The prism touches a set of vertical planes periodically with period \( d_{0z}/v \). At each instant, the vertical displacement between consecutive raindrops touching the frontal surface of the prism is a constant, say \( d_{0z} \). The distance \( AT \) is \( d_{0z} \) as easily verified in the above schematic instantaneous. Indeed, the vertical raindrops touching the frontal surface do that simultaneously in each frame of reference, since two subsequent raindrops in a vertical plane parallel to \( yz \) simultaneously touch the frontal surface as observed in \( 0xyz \), say at \( t_1 = t_2 \), with
\[ t'_2 - t'_1 = \gamma \left( t_2 - \frac{v_{y2}}{c^2} \right) - \gamma \left( t_1 - \frac{v_{y1}}{c^2} \right) = 0; \]  
(12)

since \( z'_2 - z'_1 = z_2 - z_1 \) for any pair of events under canonical Lorentz transformation, \( AT = d_{0z} \). Unfortunately, the simultaneity of raindrops touching the top area in \( 0xyz \) does not hold in \( 0'x'y'z' \), since \( x_2 \neq x_1 \) for any pair of raindrops contained in the \( xy \) plane simultaneously touching the top area as observed in \( 0xyz \), although \( t_2 = t_1 \). Hence, since we are considering simultaneity in \( 0'x'y'z' \) in the situation depicted in the above schematic drawing, the proper top plane events in \( 0'x'y'z' \) like A, S etc., at an instant \( t' \) must be under the spacetime constraint condition in \( 0xyz \) (spacetime set of points correlated to \( t' \)):

\[ t' = constant = \gamma \left( t - \frac{v_x}{c^2} \right), \]  
(13)

where a particular value of the constant \( t' \) defines a particular set of spacetime points coordinated in \( 0xyz \) simultaneous in \( 0'x'y'z' \) at \( t' \) as measured by synchronous \( 0'x'y'z' \) clocks. These spacetime points in \( 0xyz \times t \) must have different heights in \( 0z \). Two of such points are \( z \) shifted at \( t < t_2 \):

\[ \gamma \left( t_2 - \frac{v_{y2}}{c^2} \right) = \gamma \left( t_1 - \frac{v_{y1}}{c^2} \right) \Rightarrow t_2 - t_1 = \frac{v}{c^2} (x_2 - x_1), \quad \frac{v}{c^2} n_x d_{0x} = \frac{n_z d_{0z}}{u_\infty}, \]

(14)

\( n_x, n_z \) integers characterizing the \( 0x \) and \( 0z \) shifts of these pair of arbitrary points belonging to some of the hypersurface families given by eq. (13). At the instant \( t' \) depicted in the above drawing, a pair of points like A, S etc., simultaneously touching the top surface in \( 0'x'y'z' \) reference frame, are \( x' \) simultaneous spaced by an amount given by the Lorentz transformation of the correlated two points in \( 0xyz \) spaced (in \( 0xyz \) reference frame) by an amount given by eq. (14) Hence:

\[ x'_2 - x'_1 = \gamma (x_2 - vt_2) - \gamma (x_1 - vt_1) = \gamma [(x_2 - x_1) - v(t_2 - t_1)] = \gamma \left( 1 - \frac{v^2}{c^2} \right) n_x d_{0x} = \gamma^{-1} n_z d_{0z} \]

(15)

Hence, consecutive points simultaneously touching the proper top surface of the prism in \( 0'x'y'z' \) reference frame are spaced apart by the amount given by \( n_x = 1 \) in eq. (15) as:

\[ AS = \gamma^{-1} d_{0z}, \]

(16)

as one would intuitively expect as being the Lorentz contraction of the proper displacement \( d_{0z} \). Unfortunately, one should take some care in defining simultaneous horizontal raindrops in \( 0'x'y'z' \) when seeking the instantaneous flux calculation in relativistic (Einstein’s) discrete situations. Strictly speaking, in virtue of eq. (14), such simultaneity occurs in \( 0'x'y'z' \) only if:

\[ n_z = \frac{u_\infty v d_{0z}}{c^2 d_{0x}}, \]

(17)

for consecutive points, with \( n_z \in \mathbb{Z} \). In a galilean case, \( uv << c^2 \Rightarrow n_z d_{0x} \approx 0 \), and there would not exist the necessity of a \( z \) shift condition to guarantee a top proper plane of simultaneity in \( 0'x'y'z' \). Physically, the vertical velocity of the raindrops \( u_\infty \) is the cause of the absence of a general simetry regarding the usage of simultaneity for purposes of instantaneous flux calculation. In a case in wich one was able to adhocally consider continuity, one would straightforwardly use simultaneous reasoning in \( 0'x'y'z' \) ab initio, since such would imply:

\[ n_x d_{0z} \rightarrow dx, \quad n_z d_{0z} \rightarrow dz, \]

(18)

but this is not, until now, the case being analysed here. One shall infer that simultaneous points in \( 0'x'y'z' \) belonging to the respective \( x'y' \) plane parallel to the top surface of the prism must move diagonally in \( 0'x'y'z' \), must move downwards with velocity \( u_\infty \gamma \) and backwards with velocity \( -v \), since these are the velocity components of these points in \( 0'x'y'z' \). Considering one of such horizontal plane of simultaneous (if possible) raindrops in \( 0'x'y'z' \), one also shall infer from eq. (14) that the respective raindrops in \( 0xyz \) must be diagonally located through successive horizontal \( xy \) planes of simultaneity in \( 0xyz \), being \( \alpha = \arctan \frac{u_\infty v}{c^2} \) the angle between this diagonal and \( 0x \). These diagonals move downwards with velocity \( u_\infty \) while the prism moves forward with velocity \( v \) in \( 0xyz \) world. Each diagonal corresponds to a \( x'y' \) plane of raindrops simultaneously touching the proper top plane of the prism in \( 0'x'y'z' \) world. The interpretation is simple: since raindrops touch the top surface of the prism simultaneously in \( 0xyz \) world, they do not simultaneously touch in \( 0'x'y'z' \) world. This lack of simultaneity is asseverated by the necessity \( n_x << 1 \) as stated by eq. (17).

From now on, we model our raindrops distribution from the perspective of an \( 0xyz \) observer, as one would naturally do, with the following characteristics:

\footnote{See the meaning of \( AS \) in eqs. (6), (7) and (8).}
• The average proper displacement of raindrops along 0x is \(d_{0x}\);
• The average proper displacement of raindrops along 0y is \(d_{0y}\);
• The average proper displacement of raindrops along 0y is \(d_{0y}\);
• We will neglect relativistic (Einstein’s) effects of raindrops in 0xyz;
• The raindrops’ average behavior in 0xyz will be translated to a tridimensional raindrops \(d_{0x} \times d_{0y} \times d_{0z}\), orthogonally spaced infinite lattice falling at terminal velocity \(u_\infty\), being the sites’ basis vectors given by \(\{\vec{d}_{0x} = d_{0x}e_x, \vec{d}_{0y} = d_{0y}e_y, \vec{d}_{0z} = d_{0z}e_z\}\), where \(\{e_x, e_y, e_z\}\) is the canonical spacelike 3D euclidian orthonormal basis of 0xyz.

3 Solving the Problem

Hence, the contact with the top plane of the prism is simultaneous in the 0xyz world, implying non-simultaneity of these raindrops in the 0’x’y’z’ world. The distribution of these raindrops must have, instantaneously at \(t’\) in 0’x’y’z’ world, the following characteristics:

• The displacement between two consecutive raindrops correlated to the respective simultaneous ones in 0xyz, these latter displaced by the proper distance \(x_{i+1} - x_i\) along 0x and belonging to the falling xy plane in 0xyz, is given by:

\[
x'_{i+1}(t') - x'_i(t') = \gamma^{-1} (x_{i+1} - x_i) = \gamma^{-1} d_{0x}.
\]

(19)

• The displacement between two consecutive raindrops correlated to the respective simultaneous ones in 0xyz, these latter displaced by the proper distance \(z_{i+1} - z_i = 0\) along 0z and belonging to the falling xy plane in 0xyz, is given by:

\[
z'_{i}(t') - z'_{i+1}(t') = \frac{u_\infty v d_{0x}}{c^2}.
\]

(20)

• The distance between consecutive raindrop planes \(\Pi_{i+1}\) and \(\Pi_i, \forall i, d_{0z}\). The raindrop planes are inclined in relation to the x’y’ plane by the angle:

\[
\alpha = \pi - \arctan\left(\frac{\gamma u_\infty v}{c^2}\right).
\]

(21)

Indeed, let’s derive these facts. Firstly, instantaneously at \(t\) in 0xyz, two consecutive raindrops \(0x\) along, are time delayed in 0’x’y’z’ × \(\{t’\}\) world by the amount:

\[
t'_{i+1} - t'_i = \gamma \left( t - \frac{v}{c^2} x_{i+1}\right) - \gamma \left( t - \frac{v}{c^2} x_i\right) = -\gamma \frac{v}{c^2} (x_{i+1} - x_i) = -\gamma \frac{v}{c^2} d_{0x},
\]

(22)

and the \(i\)-raindrop is late in relation to the \((i + 1)\)-raindrop. Hence, backwarding the \(t'_i\) clocks down to the the \(t'_{i+1}\) instant, the \(i\)-raindrop must move the amounts: \(\delta z’\) upwards and \(\delta x’\) to the right, being these amounts given by:

\[
\delta z' = \left(-\frac{u_\infty}{\gamma}\right) \times \left(-\gamma \frac{v}{c^2} d_{0x}\right) = \frac{u_\infty v d_{0x}}{c^2}; \quad \delta x' = (-v) \times \left(-\gamma \frac{v}{c^2} d_{0x}\right) = \frac{v^2 \gamma d_{0x}}{c^2},
\]

(23)

since \((-v e_{x’} - (u_\infty/\gamma)e_{z’}\)) is the velocity of raindrops in 0’x’y’z’. But, at \(t\), the \(i\)-raindrop and the \((i + 1)\)-raindrop have got the same \(z\) coordinate, since they are in a xy plane, and, since the \(z \rightarrow z’\) Lorentz map is identity, these raindrops must have the same \(z’\) coordinate at their respective transformed instants. Hence, backwarding \(t'_i\) clocks down to the the \(t'_{i+1}\) instant, one concludes that the \(\delta z’\) in eq. (21) is the instantaneous, at same \(t’, \) height shift between consecutive raindrops that simultaneously touches the top plane of the prism in 0xyz. The \(x \rightarrow x’\) Lorentz map is not identity, implying one must calculate the \(x'_{i+1} - x'_i\) shift at the 0xyz instantaneous \(t\):

\[
x'_{i+1}(t) - x'_i(t) = \gamma (x_{i+1} - vt) - \gamma (x_i - vt) = \gamma (x_{i+1} - x_i) = \gamma d_{0x}.
\]

(24)

Hence, backwarding \(t'_i\) clocks down to the the \(t'_{i+1}\) instant, this amount is reduced by the amount \(\delta x’\) given by eq. (22):

\[
x'_{i+1}(t') - x'_i(t') = \gamma d_{0x} - \gamma d_{0x} \frac{v^2}{c^2} = \gamma d_{0x} \left(1 - \frac{v^2}{c^2}\right) = \gamma^{-1} d_{0x}.
\]

(25)

From eqs. (23) and (24) we reach the eqs. (19) and (20). One shall infer that the non-instantaneous displacement (non-instantaneous in 0’x’y’z’) given by eq. (24) is the distance between two successive
non-instantaneous raindrops marks assigned upon the proper top plane of the prism in \( 0' x' y' z' \). This fact is easy to understand, as these instantaneously assigned marks (instantaneous in \( 0 x y z \)) become splayed in \( 0' x' y' z' \), since the prism turns out to be Lorentz contracted in \( 0 x y z \). Also, one shall infer that eq. (25) gives the \( t' \) instantaneous displacement of falling raindrops along \( 0' x' \). The reason why the distance between consecutive raindrops marks \( \gamma d_{0z} \) is bigger than the contracted distance \( \gamma^{-1} d_{0z} \) of the two consecutive falling raindrops is explained by the non-simultaneity between these raindrops when touching the proper top plane of the prism in the \( 0' x' y' z' \) world, straightforwardly seem by the inclination between the raindrop plane containing these two consecutives raindrops and the proper top plane of the prism; i.e., when the first raindrop touches the prism at the proper top plane assigning the first mark, the second travels an amount \( \delta x' \) to the left given by eq. (23) before touching the proper top plane of the prism, assigning the second mark. A \( xy \) instantaneous falling plane containing raindrops in \( 0 x y z \) world becomes an inclined instantaneous falling plane in \( 0' x' y' z' \) world, being the inclination, eq. (21), easily derived from eqs. (23) and (25):

\[
\tan (\pi - \alpha) = \frac{\delta z'(t')} {x'_{i+1}(t') - x'_i(t')} = \frac{\gamma u_{\infty} v} {c^2},
\]

giving the eq. (21).

The \( t' \) instantaneous distance, \( D_c \) along \( 0' x' \), between two consecutive inclined planes of raindrops crossing the proper top plane of the prism is given by

\[
\tan (\pi - \alpha) = \frac{d_{0z}} {D_c} \Rightarrow D_c = \frac{d_{0z} c^2} {\gamma u_{\infty} v}.
\]

Hence, the \( t' \) instantaneous fractional number of planes \( n^x(t') \) crossing the proper top plane of the prism read:

\[
n^x(t') = \frac{AD} {D_c} = \frac{\gamma (AD) u_{\infty} v} {d_{0z} c^2}.
\]

Raindrops belonging to one of these planes periodically cross, one by one, where:

\[
T_{x'} = \frac{\delta z'} {\left( u_{\infty} / \gamma \right)} = \frac{\gamma v d_{0z}} {c^2}.
\]

is the period. Hence, \( n^x(t') \) is the \( t' \) instantaneous number of raindrops periodically crossing the proper top surface of the prism with period \( T_{x'} \) at an arbitrary instant \( t' \) in a time interval \((t'_0; t'_0 + \delta t^x_{z'})\) such that:

\[
\delta t^x_{z'} = \frac{d_{0z}} {\left( u_{\infty} / \gamma \right)} = \frac{\gamma d_{0z}} {u_{\infty}},
\]

is the amount of time two vertically successive raindrops in two inclined raining planes crosses the proper top surface of the prism as observed in \( 0' x' y' z' \) world; \( t'_0 \) is the instant the first among these two successive raindrops touches the proper top surface of the prism as observed in \( 0' x' y' z' \) world.

Inserting the results of eqs. (4) and (5) in eq (3), one obtains:

\[
\phi'_e = \int_{ABCD} (ABCD) \left(-v \hat{e}_{x'} - \frac{u_{\infty}} {\gamma} \hat{e}_{z'} \right) \cdot \hat{n} dS' + \int_{ABGFA} \left(-v \hat{e}_{x'} - \frac{u_{\infty}} {\gamma} \hat{e}_{z'} \right) \cdot \hat{n} dS' = \int_0^{AB} \left(-v \hat{e}_{x'} - \frac{u_{\infty}} {\gamma} \hat{e}_{z'} \right) \sum_{i=1}^{N_{x'}} \sum_{j=1}^{N_{y'}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta |x' - x'_i(t')| \delta |y' - y'_j(t)| \ dx' \ dy' \ \hat{e}_{z'} \ dx' \ dy' + \int_0^{FG} \int_0^{GB} \left(-v \hat{e}_{x'} - \frac{u_{\infty}} {\gamma} \hat{e}_{z'} \right) \sum_{k=1}^{N_{x'}} \sum_{j=1}^{N_{y'}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta |y' - y'_j(t')| \delta |z' - z'_k(t')| \ dy' \ dz' \ \hat{e}_{x'} \ dy' \ dz' = \sum_{i=1}^{N_{x'}} \sum_{j=1}^{N_{y'}} -\frac{u_{\infty}} {\gamma} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_0^{DA} \int_0^{AB} \ dx' \ dy' \ \delta |x' - x'_i(t')| \delta |y' - y'_j(t')| \ dx' \ dy' + \int_0^{FG} \int_0^{GB} \ dy' \ dz' \ \delta |y' - y'_j(t')| \delta |z' - z'_k(t')| \ dy' \ dz'.
\]

One should infer that the double integrals between brackets, in the first and in the second integrals of

---

\(^4\)The vertical distance, along \( 0' z' \), between two consecutive planes of raindrops in \( 0' x' y' z' \) is \( d_{0z} \), since two vertically located raindrops (0z consecutively along), \( t' \) instantaneously in \( 0 x y z \), have same \( t \), same \( x \) and \( \delta z = d_{0z} \). This implies the same \( x' \), the same \( t' \) and the same \( \delta z' = d_{0z} \) Lorentz transformation for both in \( 0' x' y' z' \).

\(^5\)See the drawing at page 1.
the above expression, are the functions passing through the respective Dirac \( \delta \) filters, returning, each, respectively, the projected areas of the raindrops passing through the respective surfaces of contact at the points \((x'_i(t'), y'_i(t'))\) and \((y'_j(t'), z'_k(t'))\). Denoting these projected areas by \( a^p_{ij} \) and \( a^f_{jk} \), one has:

\[
\phi'_r = \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} \frac{-u}{\gamma} a^p_{ij} + \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} -v a^f_{jk}.
\]

Writing down the contact flux \( \phi'_r \), the instantaneous rate of variation of the raining water volume touching the prism while the prism runs through the rain to the finish line:

\[
\frac{dV}{dt'} = \phi'_r = \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} \frac{-u}{\gamma} a^p_{ij} + \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} -v a^f_{jk}.
\]

The variation of the proper volume \( dV_p \) quantity entering the proper contact area of the prism in \( 0'x'y'z' \) world is related to the eq. (32) simply by:

\[
\gamma^{-1} \frac{dV_p}{dt'} = \phi'_r = \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} \frac{-u}{\gamma} a^p_{ij} + \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} -v a^f_{jk} \Rightarrow
\]

\[
\frac{dV_p}{dt'} = \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} \frac{-u}{\gamma} \left( \gamma a^p_{ij} \right) + \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} -v \left( \gamma a^f_{jk} \right),
\]

and one easily verify the proper orthogonally projected areas, \( a^p_{ij} \) and \( a^f_{jk} \), of the raindrops through the proper top and frontal contact surfaces of the prism, respectively, as being given by:

\[
a^p_{ij} = \pi R_0^2 = \gamma a_{ij} = \gamma^{-1} \pi R_0^2;
\]

\[
a^f_{jk} = \pi R_0^2 = \gamma a_{jk} = \gamma^{-1} \pi R_0^2,
\]

where we denoted by \( R_0 \) the proper radius of the supposed spherical proper raindrops. Putting these last results in eq. (32), one has:

\[
\frac{dV}{dt'} = -\pi R_0^2 \left[ \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} \gamma^{-2} u + \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} \gamma^{-1} v \right].
\]

The minus sign is provided by our initial convention regarding the flux as being positive when exiting from a control surface. Hence the flux is negatively exiting the prism, i.e., entering through the prism. Summing up eq. (36):

\[
\frac{dV}{dt'} = -\pi R_0^2 \left[ \gamma^{-2} u \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} N_{x'} N_{y'} - \gamma^{-1} v \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} N_{x'} N_{z'} \right],
\]

we must consider de eqs. (9) and (10).

Firstly, the period \( T_{x'} \) is given by eq. (29). The period \( T_{x'} \), the amount of time the prism spends to touch two consecutively spaced vertical planes of raindrops parallel to \( y'z' \) in \( 0'x'y'z' \) world, reads:

\[
T_{x'} = \frac{\gamma^{-1} d_{0x}}{v},
\]

since \( \gamma^{-1} d_{0x} \) is the \( t' \) instantaneous \( 0'x' \) along distance between two consecutive raindrops contained in a same inclined raindrop plane in \( 0'x'y'z' \) world, hence the distance between two consecutively spaced vertical (parallel to \( y'z' \) planes in \( 0'x'y'z' \) world, \(-v \) is the \( 0'x' \) (retrograd) velocity component of these planes in \( 0'x'y'z' \) world. The time duration blocks to be used in which these periodic touchs hold is \( \delta t_{x'} \), given by eq. (30). The summation limits in eqs. (9) and (10) are given by:

\[
I \left( \tau' / T_{x'} \right) = \int_{\tau' / T_{x'}}^\infty I \left( \frac{c^2}{u_{\infty} v} \, d_{0x} \right);
\]
In virtue of eq. (37), we easily obtain from the eqs. (9) and (10):

\[
\gamma^{-2} \pi R_0^3 u_\infty N_x(t') N_y(t') = \frac{4}{3} \pi \gamma^{-1} R_0^3 \sum_{k=0}^{l} N_{x'} N_{y'} \delta \left( t' - k \frac{\gamma v \Delta x}{c^2} \right); 
\]

\[
\gamma^{-1} \pi R_0^3 u_\infty N_y(t') N_y(t') = \frac{4}{3} \pi \gamma^{-1} R_0^3 \sum_{l=0}^{\infty} N_{y'} N_{z'} \delta \left( t' - l \gamma^{-1} \frac{\Delta x}{v} \right). 
\]

The spent time, in \( t' \gamma' \gamma' \), to cross the finish line is given by the instant in which the origin \( t' \) crosses the proper distance \( L \) measured in \( \text{xyz} \):

\[
t'_{L} = 0 = \gamma \left( \frac{L}{v} - \frac{v}{c^2} L \right) = \gamma \left( \frac{L}{v} v \right) = \gamma^{-1} \frac{L}{v},
\]

being the Lorentz contraction of the spent time, \( L/v \), in \( \text{xyz} \). From eqs. (37), (41), (42) and (43), the total non-proper volume \( V_L \), passing through a transparently adjacent proper contact surfaces of the prism is given by:

\[
V_L = \frac{4}{3} \pi \gamma^{-1} R_0^3 \int_{0}^{\gamma^{-1} L/v} \left[ \sum_{k=0}^{l} N_{x'} N_{y'} \delta \left( t' - k \frac{\gamma v \Delta x}{c^2} \right) + \sum_{l=0}^{\infty} N_{y'} N_{z'} \delta \left( t' - l \gamma^{-1} \frac{\Delta x}{v} \right) \right] dt',
\]

where the time duration blocks are now summed up to \( \tau'_L = \gamma^{-1} L/v \). Since the Dirac delta filters raining drops in the time domain related to the proper length \( L \), one extends de integration domain with the condition: \( N_{x'} N_{y'} \) and \( N_{y'} N_{z'} \) are void in the interval \( t' \notin [0; \gamma^{-1} L/v] \). Hence, the eq. (44) reads:

\[
V_L = \frac{4}{3} \pi \gamma^{-1} R_0^3 \left[ \sum_{k=1}^{\infty} \int_{0}^{\infty} N_{x'} N_{y'} \delta \left( t' - k \frac{\gamma v \Delta x}{c^2} \right) dt' + \sum_{l=1}^{\infty} \int_{0}^{\infty} N_{y'} N_{z'} \delta \left( t' - l \gamma^{-1} \frac{\Delta x}{v} \right) dt' \right].
\]

In virtue of eq. (11), one writes:

\[
V_L = \frac{4}{3} \pi \gamma^{-1} R_0^3 \left[ \sum_{k=1}^{\infty} N_s \left( k \frac{\gamma v \Delta x}{c^2} \right) + \sum_{l=1}^{\infty} N_f \left( l \gamma^{-1} \frac{\Delta x}{v} \right) \right] \Rightarrow
\]

\[
V_L = \frac{4}{3} \pi \gamma^{-1} R_0^3 \left[ \frac{1}{v} I \left( \gamma^{-2} \frac{c^2}{v^2} \frac{L}{\Delta x} \right) \gamma \frac{v^2}{c^2} A_s + \frac{1}{u_\infty} I \left( \frac{L}{\Delta x} \right) \right].
\]

From eq. (8) and (27) the eq. (46) becomes:

\[
V_L = \frac{4}{3} \pi \gamma^{-1} R_0^3 u_\infty N_f \left[ \frac{1}{v} I \left( \gamma^{-2} \frac{c^2}{v^2} \frac{L}{\Delta x} \right) \gamma \frac{v^2}{c^2} A_s + \frac{1}{u_\infty} I \left( \frac{L}{\Delta x} \right) \right].
\]

Hence, the proper volume accumulating upon the proper prism surface is given by:

\[
V^p_L = \frac{4}{3} \pi R_0^3 u_\infty N_f \left[ \frac{1}{v} I \left( \gamma^{-2} \frac{c^2}{v^2} \frac{L}{\Delta x} \right) \gamma \frac{v^2}{c^2} A_s + \frac{1}{u_\infty} I \left( \frac{L}{\Delta x} \right) \right],
\]

or the number of raindrops, \( V^p_L/(4\pi R_0^3/3) \):

---

\(^{8}\)When one extends the domain, it should be asseverated that the Dirac delta is under a distribution context. The raindrops filter process is strongly concentrated, but not instantaneous. Hence, the total of raindrops being filtered by the integration over the Dirac delta summation must be properly treated in the left hand side of eq. (44) regarding the volume integration. If one states that none raindrop is being filtered by the delta at \( t' = 0 \) and that at \( t = \gamma^{-1} L/v \) all raindrops are properly filtered, one shall perform the delta summation starting from \( k = 1 \) and, \( V_L \), the volume variation, straightforwardly gives the totality of filtered raindrops. If raindrops start to be filtered at \( t' = 0 \), as we wrote down, the volume integration must be rigorously understood as (e.g., the first filtering process) \( V_L |_{t'=0} = \lim_{x \to 0} \left( \int_{0}^{L/v} \frac{dV}{\Delta x} + \int_{0}^{\gamma^{-1} L/v} \frac{dV}{\Delta x} \right) = (4/3) \pi \gamma^{-1} R_0^3 + \int_{0}^{\gamma^{-1} L/v} \frac{dV}{\Delta x} = (4/3) \pi \gamma^{-1} R_0^3 + V_L \). This process leads one to conclude the \( k \) summation index must starts from \( l = 1 \) in any case.

\(^{9}\)Remembering: \( AT = \Delta x \) and \( AS = D_c \).
Let’s analyze the classical limit. Taking the limit $c \to \infty$, one obtains the classical (galilean) non-relativistic number of raindrops $N_L^c$: 

$$N_L^c = N_f I \left( \frac{u_\infty}{v} \left( \frac{\gamma c^2}{v^2 d_{0z}} \right) \frac{A_s}{c^2 A_f} + I \left( \frac{L}{d_{0z}} \right) \right),$$

being the asymptotically minimum contact, $N_f L/d_{0z}$, at $v \to \infty$. Relativistically (Einstein’s), the same minimum amount of contact is reached at $v = c$, but not asymptotically. One stays more dry by running faster, but in a Lorentzian world one always stays less wet.