UNDECIDABLY SEMILOCALIZABLE METRIC MEASURES SPACES

THIERRY DE PAUW

Abstract. We characterize measure spaces such that the canonical map \( L_\infty \rightarrow L_1^* \) is surjective. In case of \( d \) dimensional Hausdorff measure of a complete separable metric space \( X \) we give two equivalent conditions. One is in terms of the order completeness of a quotient Boolean algebra associated with measurable sets and with locally null sets. Another one is in terms of the possibility to decompose space in a certain way into sets of nonzero finite measure. We give examples of \( X \) and \( d \) so that whether these conditions are met is undecidable in ZFC, including one with \( d \) equals the Hausdorff dimension of \( X \).

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1. Foreword

Some questions pertaining to the calculus of variations would benefit from a useful description of the dual of the Banach space \( BV(\mathbb{R}^n) \) of functions of bounded variation in the sense of E. De Giorgi. The question occurs as Problem 7.4 is [1]. Measures belonging to this dual space have been characterized by N.G. MEYERS and W.P. ZIEMER in [25]. A description of the other members was obtained (in a slightly different context) by F.J. ALMGREN in [5] under the Continuum Hypothesis and the particular description was proved to be independent of Zermelo-Fraenkel axioms by the present author in [9]. Recently, following former work of R.D. MAULDIN, N. FUSCO and D. SPECTOR have given a more precise description under the Continuum Hypothesis, [20].

In [9] the problem is shown to be related to describing the dual of the Banach space \( L_1(\mathbb{R}^n, \mathcal{H}^{n-1}) \) where \( \mathcal{H}^{n-1} \) denotes Hausdorff \( n - 1 \) dimensional measure in \( \mathbb{R}^n \). Here we will restrict to the case when \( n = 2 \) and we shall aim for results in ZFC. The notation \( L_1(\mathbb{R}^n, \mathcal{H}^{n-1}) \) however is misleading as it assumes the problem to be independent of the underlying \( \sigma \)-algebra. As we shall see, this is not the case.
Let $(X, \mathcal{A}, \mu)$ be a measure space. There is a natural linear retraction
\[ T : L_\infty(X, \mathcal{A}, \mu) \to L_1(X, \mathcal{A}, \mu)^* \] (1)
which sends $g$ to $f \mapsto \int_X g f \, d\mu$ where $g$ and $f$ represent $g$ and $f$ respectively. In general $T$ does not need to be injective or surjective. It has been understood for a long time that it is surjective or not, the situation is the following.

(1) If $X$ is a complete separable metric space and $0 < d < \infty$ then the measure space $(X, \mathcal{B}(X), \mathcal{H}^d)$ is semifinite. Here $\mathcal{B}(X)$ denotes the $\sigma$-algebra of Borel subsets of $X$ and $\mathcal{H}^d$ is the $d$ dimensional Hausdorff measure on $X$. In case $X = \mathbb{R}^n$ this was proved by R.O. Davies, \[8\] and in general by J. Howroyd, \[22\].

(2) According to D.H. Fremlin, \[17\] 439H] the measure space $(\mathbb{R}^2, \mathcal{A}_{ge^1}, \mathcal{H}^1)$ is not semifinite, where $\mathcal{A}_{ge^1}$ denotes the $\sigma$-algebra consisting of $\mathcal{H}^1$ measurable subsets of $\mathbb{R}^2$. This is based on the existence of «large» universally null subsets of $[0, 1]$ established by E. Grzegorek, \[21\]. See also the article of O. Zindulka \[29\].

Nonetheless, recalling our work \[9\] it is the surjectivity of $T$ that is relevant for the existence of a certain integral representation of members of the dual of $BV(\mathbb{R}^2)$. Injectivity pertains to its uniqueness.

Under the assumption that $(X, \mathcal{A}, \mu)$ is semifinite, a necessary and sufficient condition for the surjectivity of $T$ has been known for a long time. It asks for the quotient Boolean algebra $\mathcal{A} / \mathcal{N}_\mu$ to be order complete, where $\mathcal{N}_\mu = \mathcal{A} \cap \{ N : \mu(N) = 0 \}$ is the $\sigma$-ideal of $\mu$ null sets. Semifinite measure spaces with this property are sometimes called Maharam, \[14\] 211G]. A stronger condition sometimes called decomposable, generalizes the idea of $\sigma$-finiteness to possibly uncountable decomposition into sets of finite measure, together with a new condition called locally determined (that measurability be determined by sets of finite measure), see \[6, 1\] for the definition of locally determined and \[14\] 211E] for the definition of decomposable. If the quotient $\sigma$-algebra $\mathcal{A} / \mathcal{N}_\mu$ is not too big then decomposability implies Maharam according to E.J. McShane, \[24\] but not in general according to D.H. Fremlin, \[14\] 216E].

If $X$ is a Polish space and $\mathcal{B}(X)$ denotes the $\sigma$-algebra consisting of its Borel subsets, and if the measure space $(X, \mathcal{B}(X), \mu)$ is decomposable, then it is $\sigma$-finite. I learned the «counting argument» to prove this from D.H. Fremlin, see \[5, 5\]. In view of (1) above it shows that $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathcal{H}^1)$ is not decomposable. Since decomposability is stronger in general than the surjectivity of $T$, we need to argue a bit more to show that $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathcal{H}^1)$ is not Maharam, see below. This observation calls for developing a criterion for the surjectivity of $T$ without assuming that $(X, \mathcal{A}, \mu)$ be semifinite in the first place. We do this in Section 4. Thus regarding the question whether
\[ T : L_\infty(\mathbb{R}^2, \mathcal{A}, \mathcal{H}^1) \to L_1(\mathbb{R}^2, \mathcal{A}, \mathcal{H}^1)^* \]
is surjective or not, the situation is the following.

(3) If $\mathcal{A} = \mathcal{B}(\mathbb{R}^2)$ then $T$ is not surjective. Since $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathcal{H}^1)$ is semifinite according to (1), and not $\sigma$-finite, it is not decomposable, \[5, 5\]. The argument of E.J. McShane, \[5, 5\] does not show $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathcal{H}^1)$ is not Maharam (the reason being that its completion is not locally determined). However we give below a simple argument to the extent that it is not Maharam, based on Fubini’s Theorem.

(4) If $\mathcal{A} = \mathcal{A}_{ge^1}$ then the surjectivity of $T$ is undecidable in $ZFC$. The consistency of its surjectivity is a consequence of the Continuum Hypothesis, \[5, 3\] 5.3 and 4.6. The consistency of it not being surjective was first noted in \[9\] although in a slightly different disguise. The idea is explained below.
The present paper grew out of the attempt to adapt the techniques used to prove (3) and (4) to the case where \( \mathbb{R}^2 \) is replaced with a small compact subset \( X \subseteq \mathbb{R}^2 \) – as small as it can possibly be, i.e. of Hausdorff dimension 1 (of course not of \( \sigma \)-finite \( \mathcal{H}^1 \) measure, for in that case \( (X, \mathcal{B}(X), \mathcal{H}^1) \) and \( (X, \mathcal{A}, \mathcal{H}^1) \) are both Maharam and \( \mathcal{Y} \) is surjective, [4.4].

Why however would the answer depend on the \( \sigma \)-algebra under consideration? In order to understand this, let us try to prove that \( \mathcal{Y} \) is surjective.

We know from the classical Riesz’ Theorem that \( \mathcal{Y} \) is surjective whenever \( (X, \mathcal{A}, \mu) \) is a finite measure space. This suggests to consider \( \mathcal{A}_\mu^f = \mathcal{A} \cap \{ A : \mu(A) < \infty \} \) and for each \( A \in \mathcal{A}_\mu^f \) the map

\[ \mathcal{T}^A : L_\infty(A, \mathcal{A}_\mu, \mu_A) \rightarrow L_1(A, \mathcal{A}_\mu, \mu_A)^* \]

where \( (A, \mathcal{A}_\mu, \mu_A) \) is the obvious measure subspace. Thus \( \mathcal{T}^A \) is an isometric linear isomorphism and given \( \sigma \in L_1(X, \mathcal{A}, \mu) \) there exist \( g_A \in gA \in L_\infty(A, \mathcal{A}_\mu, \mu_A) \) such that

\[ (\sigma \circ \iota_A)(f) = \int_A g_A f d\mu_A \]

whenever \( f \in f \in L_1(A, \mathcal{A}_\mu, \mu_A) \), where \( \iota_A : L_1(A, \mathcal{A}_\mu, \mu_A) \rightarrow L_1(X, \mathcal{A}, \mu) \) is the obvious embedding. From the \( \mu_A \) almost everywhere uniqueness of the Radon-Nikodým derivative \( g_A \) we infer that if \( A, A' \in \mathcal{A}_\mu^f \) then \( \mu(A \cap A' \cap \{ g_A \neq g_A' \}) = 0 \). Thus \( (g_A)_{A \in \mathcal{A}_\mu^f} \) is what we call, from now on a compatible family of locally defined measurable functions and the question is whether it corresponds to a globally defined measurable function, i.e. whether there exists an \( \mathcal{A} \)-measurable \( g : X \rightarrow \mathbb{R} \) such that \( \mu(A \cap \{ g \neq g_A \}) = 0 \) for every \( A \in \mathcal{A}_\mu^f \).

If such \( g \) exists let us call it a gluing of the compatible family \( (g_A)_{A \in \mathcal{A}_\mu^f} \). This is reminiscent of, and not entirely unrelated to the sheaf property of the functor \( U \mapsto C(U) \) where \( U \) is a subset of a topological space, see [3,19]Q3).

It turns out to be rather useful to notice that the question whether a gluing exists or not can be asked in a slightly more general setting since it depends on the measure \( \mu \) only insofar as its \( \mu \) null sets are involved. Thus a measureable space with negligibles \( (X, \mathcal{A}, \mathcal{N}) \) consists of a measurable space \( (X, \mathcal{A}) \) and a \( \sigma \)-ideal \( \mathcal{N} \subseteq \mathcal{A} \). Given any \( \mathcal{E} \subseteq \mathcal{A} \) one can readily define the notion of a compatible family \( (g_E)_{E \in \mathcal{E}} \) by asking that \( E \cap E' \cap \{ g_E \neq g_E' \} \in \mathcal{N} \) whenever \( E, E' \in \mathcal{E} \), and by saying that an \( \mathcal{A} \)-measurable function \( g : X \rightarrow \mathbb{R} \) is a gluing of \( (g_E)_{E \in \mathcal{E}} \) provided \( E \cap \{ g \neq g_E \} \in \mathcal{N} \) for all \( E \in \mathcal{E} \). One then shows, [3,13] that each compatible family admits a gluing if and only if each \( \mathcal{E} \subseteq \mathcal{A} \) admits an \( \mathcal{N} \) essential supremum \( A \in \mathcal{A} \). This means that

(i) For every \( E \subseteq \mathcal{E} \) one has \( E \setminus A \in \mathcal{N} \);

(ii) For every \( B \in \mathcal{A} \), if \( E \setminus B \in \mathcal{N} \) whenever \( E \in \mathcal{E} \), then \( A \setminus B \in \mathcal{N} \).

We say that a measurable space with negligibles is localizable if it has this property.

In this paper we characterize those measure spaces such that \( \mathcal{Y} \) is surjective, [4,6]. To state this we first define

\[ \mathcal{N}_\mu \mathcal{A}_\mu = \mathcal{A} \cap \{ N : \mu(A \cap N) = 0 \text{ for all } A \in \mathcal{A}_\mu^f \}. \]

It is a \( \sigma \)-ideal, whose members one is tempted to call locally \( \mu \) null.

**Theorem.** — For any measure space \( (X, \mathcal{A}, \mu) \), the map \( \mathcal{Y} \) (recall [1]) is surjective if and only if the measurable space with negligibles \( (X, \mathcal{A}, \mathcal{N}_\mu \mathcal{A}_\mu^f) \) is localizable.

We call a measure space semilocalizable if it has this property – thus no semifiniteness is assumed. We study the connection with the notion of *almost decomposable* measure space introduced in [9,5,3] and [6,5] thereby generalizing to non semifinite measure spaces the classical theory briefly alluded above. We call a measure space \( (X, \mathcal{A}, \mu) \) *almost decomposable* if there exists a disjointed family \( \mathcal{B} \subseteq \mathcal{A}_\mu^f \) such that

\[ \forall A \in \mathcal{P}(X) : (\forall G \in \mathcal{B} : A \cap G \in \mathcal{A}) \Rightarrow A \in \mathcal{A}, \]

where
and
\[ \forall A \in \mathcal{A} : \mu(A) < \infty \Rightarrow \mu(A) = \sum_{G \in \mathcal{G}} \mu(A \cap G). \]

Using an idea of E.J. McShane [24] and the fact that there are not too many equivalence classes of measurable sets with respect to a Borel regular outer measure on a Polish space, we prove the following.

**Theorem.** — Let \( X \) be a complete separable metric space and \( 0 < d < 1 \). For the measure space \((X, \mathcal{A}, \mathcal{H}_d)\) the following are equivalent.

1. The canonical map \( \mathcal{T} \) is surjective;
2. \((X, \mathcal{A}_d, \mathcal{H}_d)\) is semilocalizable;
3. \((X, \mathcal{A}_d, \mathcal{H}_d)\) is almost decomposable.

Let us now consider the measure space \((\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathcal{H}_1)\) in view of the notion of semilocalizability. We know it is not semilocalizable, (3) above, but we promised to show how this is a consequence of Fubini’s Theorem. Define the vertical sections \( V_s = \{ s \} \times \mathbb{R}, \ s \in \mathbb{R} \), and the horizontal sections \( H_t = \mathbb{R} \times \{ t \}, \ t \in \mathbb{R} \). Assume if possible that \( A \in \mathcal{B}(\mathbb{R}^2) \) is an \( \mathcal{N}_d \left[ \mathcal{B}(\mathbb{R}^2)^f, \mathcal{H}_1 \right] \) essential supremum of the family \((V_s)_s \in \mathbb{R}\). It would then readily follow that

- \( \mathcal{H}_1(V_s \setminus A) = 0 \) for every \( s \in \mathbb{R} \);
- \( \mathcal{H}_1(H_t \cap A) = 0 \) for every \( t \in \mathbb{R} \).

Indeed upon noticing that \( V_s \) and \( H_t \) have \( \sigma \)-finite \( \mathcal{H}_1 \) measure, (a) is a rephrasing of (i) above and (b) follows from (ii) applied with \( B = A \setminus H_t \). Applying Fubini’s Theorem twice would yield

\[ \mathcal{L}^2(\mathbb{R}^2 \setminus A) = \int_{\mathbb{R}} \mathcal{H}_1(V_s \setminus A) d\mathcal{L}_1(s) = 0 \]

according to (a), and

\[ \mathcal{L}^2(\mathbb{R}^2 \cap A) = \int_{\mathbb{R}} \mathcal{H}_1(H_t \cap A) d\mathcal{L}_1(t) = 0 \]

according to (b). In turn \( \mathcal{L}^2(\mathbb{R}^2) = 0 \), a contradiction. Clearly the same argument applies with \( \mathbb{R}^2 \) replaced by any Borel set \( X \subseteq \mathbb{R}^2 \) such that \( \mathcal{L}^2(X) > 0 \), to showing that \((X, \mathcal{B}(X), \mathcal{H}_1)\) is not semilocalizable.

There are two cases when the above argument is not conclusive:
- \( (\alpha) \) when \( A \) is not \( \mathcal{L}^2 \) measurable (because Fubini’s Theorem does not apply);
- \( (\beta) \) when \( \mathcal{L}^2(X) = 0 \) (because no contradiction ensues).

With regard to case \( (\alpha) \) indeed, when we replace the \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^2) \) by the larger \( \mathcal{A}_d \), then \((\mathbb{R}^2, \mathcal{A}_d, \mathcal{H}_1)\) is consistently semilocalizable. This is a consequence of the Continuum Hypothesis and a much more general statement holds\[ \[ \text{5.4} \] \]
As noticed in \[ \[ \text{9} \] \] it turns out however that \((\mathbb{R}^2, \mathcal{A}_d, \mathcal{H}_1)\) is also consistently not semilocalizable. Here is the reason why. We assume that \( A \in \mathcal{A}_d \) is an \( \mathcal{N}_d \left[ \mathcal{A}_d^f, \mathcal{H}_1 \right] \) essential supremum of the family \((V_s)_s \in \mathbb{R}\). For each \( s \in \mathbb{R} \) we define \( T_s = \mathbb{R} \cap \{ t : (s, t) \in V_s \setminus A \} \), thus \( \mathcal{L}_1(T_s) = 0 \) according to (i). Now choose \( E \subseteq \mathbb{R} \) such that \( \mathcal{L}_1(E) > 0 \) and \( E \) has least cardinal among all sets with nonzero Lebesgue measure, and let \( \text{non}(\mathcal{N}_d) \) denote this cardinal. Assume that there exists \( t \in \mathbb{R} \setminus \bigcup_{s \in E} T_s \). Then for each \( s \in E, \ t \notin T_s \), i.e. \((s, t) \in H_t \cap A \). Therefore \( \mathcal{L}_1(E) = 0 \) according to (ii), a contradiction. Of course we can reach this contradiction only if \( \mathbb{R} \not= \bigcup_{s \in E} T_s \), which depends upon how big \( E \) is. We denote as \( \text{cov}(\mathcal{N}_d) \) the least cardinal of a covering of \( \mathbb{R} \) by \( \mathcal{L}_1 \) negligible sets. Thus if \( \text{card} E = \text{non}(\mathcal{N}_d) < \text{cov}(\mathcal{N}_d) \) then the argument goes through. It turns out that this strict inequality of cardinals (appearing in the so-called Cichoń diagram) is consistent.*

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*I learned it from [12, 2.5.10]. Unfortunately the presentation there does not allow for putting emphasis on the role played by the choice of a particular \( \sigma \)-algebra.
with ZFC, [6] Chapter 7) or [12, 552H and 552G]. We will refer to this idea below as the «vertical-horizontal method». This argument is from [9]; I learned it from D.H. Fremlin.

With regard to case (β) above we observe again that the Continuum Hypothesis implies that \( (X, \mathcal{A}_{\mathcal{H}^1}, \mathcal{H}^1) \) is semilocalizable for any compact set \( X \subseteq \mathbb{R}^2 \) regardless whether it has zero \( \mathcal{L}^2 \) measure or not.\(^5\) The question is therefore whether \( (X, \mathcal{A}_{\mathcal{H}^1}, \mathcal{H}^1) \) is semilocalizable in ZFC or consistently not semilocalizable. The latter occurs when the «vertical-horizontal method» generalizes from \( X = \mathbb{R}^2 \) to \( X \). For instance it clearly generalizes to \( X = [a, b] \times [c, d] \) but it is not instantly obvious how to proceed if \( \mathcal{L}^2(X) = 0 \).

In connection with this question we now mention a result due to G. Alberti, M. Csörnyei and D. Preiss. Given \( X \subseteq \mathbb{R}^2 \), say compact, we call \( \tau : X \rightarrow G(\mathbb{R}^2, 1) \) a weak tangent field to \( X \) if for every \( C^1 \) curve \( \Gamma \subseteq \mathbb{R}^2 \) one has \( \text{tan}(\Gamma, x) = \tau(x) \) at \( \mathcal{H}^1 \) almost every \( x \in \Gamma \cap X \). The above argument using Fubini’s Theorem shows that if \( \mathcal{L}^2(X) > 0 \) then \( X \) does not admit an \( \mathcal{L}^2 \) measurable weak tangent field. On the other hand under the Continuum Hypothesis any compact \( X \subseteq \mathbb{R}^2 \) admits an \( \mathcal{H}^1 \) measurable weak tangent field. One considers indeed the \( \mathcal{N}_{\mathcal{H}^1} \) compatible family of line fields \( \tau_x : \Gamma \cap X \rightarrow G(\mathbb{R}^2, 1) \) for all \( x \in \Gamma \cap X \). It is indeed compatible since \( \text{tan}(\Gamma_x, x) = \text{tan}(\Gamma_y, x) \) for \( \mathcal{H}^1 \) almost every \( x \in \Gamma_x \cap \Gamma_y \). Hence, the existence of such gluing ensues from the semilocalizability of \( (X, \mathcal{A}_{\mathcal{H}^1}, \mathcal{H}^1) \), which holds under the Continuum Hypothesis, \([5, 3]\) and \([5, 3]\). However G. Alberti, M. Csörnyei and D. Preiss proved the following striking result in ZFC, \([2]\) and \([1]\). If \( \mathcal{L}^2(X) = 0 \) then \( X \) admits a Borel measurable weak tangent field. One may wonder if this is a consequence, in ZFC of the localizability of \( (X, \mathcal{H}(X), \mathcal{N}) \) for some \( \sigma \)-ideal \( \mathcal{N} \), for example \( \mathcal{N}_{\mathcal{H}^1} \) the \( \sigma \)-ideal consisting of purely \( (\mathcal{H}^1, 1) \) unrectifiable subsets of \( \mathbb{R}^2 \). This however is not the case: We give in \([11, 1]\) below an example of a «purely rectifiable» \( \mathcal{L}^2 \) negligible compact set \( X \subseteq \mathbb{R}^2 \) such that for all \( \sigma \)-algebra \( \mathcal{H}(X) \subseteq \mathcal{A} \subseteq \mathcal{P}(X) \) and a large collection of \( \sigma \)-ideals – including \( \mathcal{N}_{\mathcal{H}^1} \) and \( \mathcal{N}_{\mathcal{H}^d} \) – the measure space with negligibles \( (X, \mathcal{A}, \mathcal{N}) \) is consistently not localizable. Furthermore \( X \) is nearly as small as it can be for this to happen: \( \mathcal{H}^1 \sqcap X \) is not \( \sigma \)-finite but \( X \) has Hausdorff dimension \( \frac{3}{2} \). See the last stated Theorem of this introduction.

Thus we ought to explain how the «vertical-horizontal method» described above, showing that if \( \text{non}(\mathcal{N}_{\mathcal{H}^1}) < \text{cov}(\mathcal{N}_{\mathcal{H}^1}) \) then \( (\mathbb{R}^2, \mathcal{A}_{\mathcal{H}^1}, \mathcal{H}^1) \) is not semilocalizable, can be adapted to the case where \( \mathbb{R}^2 \) is replaced with some suitable subset \( X \subseteq \mathbb{R}^2 \). We give a rather general version below, \([8, 3]\). First of all we make the useful observation that if \( (S, \mathfrak{P}(S), \sigma) \) is a probability space, \( S \) is Polish and \( \sigma \) is diffuse, then \( \text{non}(\mathcal{A}) = \text{non}(\mathcal{N}_{\mathcal{A}}) \) and \( \text{cov}(\mathcal{A}) = \text{cov}(\mathcal{N}_{\mathcal{A}}) \). This ensues from the Kuratowski Isomorphism Theorem, \([8, 2]\). A careful inspection of the argument leads to the following, \([8, 5]\).

**Theorem.** — Let \( 0 < d < 1 \) and let \( C_d \subseteq [0, 1] \) be the standard self-similar Cantor set of Hausdorff dimension \( 0 < d < 1 \). Whether the measure space \( (C_d \times C_d, \mathcal{A}_{\mathcal{H}^d}, \mathcal{H}^d) \) is semilocalizable is undecidable in ZFC.

Incidentally, constructing a certain isomorphism in the category of measurable spaces with negligibles we are able to infer the following, \([9, 8]\).

**Theorem.** — Whether the measure space \( (\{0, 1\}, \mathcal{A}_{\mathcal{H}^{1/2}}, \mathcal{H}^{1/2}) \) is semilocalizable is undecidable in ZFC.

Here the exponent \( 1/2 \) reflects the nature of the argument, viewing the space \( X \) as a product of a kind, where «vertical» sets \( V_x \) and «horizontal» sets \( H_y \) of the same size

\(^†\)Recall that \( N \subseteq \mathbb{R}^2 \) is purely \( (\mathcal{H}^1, 1) \) unrectifiable if and only if \( \mathcal{H}^1(N \cap \Gamma) = 0 \) for all \( C^1 \) curves \( \Gamma \), and notice that a weak tangent field to \( X \) is well defined \( \mathcal{N}_{\mathcal{H}^1} \) almost everywhere.

\(^\ddagger\)Of course for each Borel set \( Y \subseteq \mathbb{R}^2 \) such that \( \mathcal{H}^1 \sqcap Y \) is \( \sigma \)-finite the measure space \((Y, \mathcal{A}(Y), \mathcal{H}^1)\) is semilocalizable, \([3, 8]\) and \( Y \) admits the obvious Borel weak tangent field defined on its rectifiable part
make sense, their intersections behaving according to some technical assumptions (see the statement of [8.3]). The sets \( X = C_d \times C_d \) are purely \((\mathcal{H}_1, 1)\) unrectifiable, and therefore irrelevant to the question whether the existence of a weak tangent field for \( X \) follows from a localizability property: Any map \( X \to G(\mathbb{R}^2, 1) \) is a weak tangent field if \( X \) is purely \((\mathcal{H}_1, 1)\) unrectifiable, let alone the fact that \( d = 1 \) is omitted above.

Therefore we next seek to apply the «vertical-horizontal method» to an \( \mathcal{L}^2 \) negligible compact set \( X \subseteq \mathbb{R}^2 \) which is not purely \((\mathcal{H}_1, 1)\) unrectifiable and prove that \((X, \mathcal{A}, \mathcal{P}_1, \mathcal{H}_1)\) is not semilocicalizable. Let us choose \( X \) as small as possible, i.e. of Hausdorff dimension 1, say \( X = C \times [0, 1] \) where \( C \subseteq [0, 1] \) is a Cantor set of Hausdorff dimension 0. It is of course clear that \( V_s = \{ s \} \times [0, 1], \; s \in C \), can be chosen as our vertical sets, yet the choice \( H_t = C \times \{ t \}, \; t \in [0, 1] \), will be of no use since \( \mathcal{H}_1(H_t) = 0 \) and therefore no contradiction can ensue when implementing the «vertical-horizontal method». Instead we proceed as follows to define \( H_t \). Let \( \mu \) be a diffuse probability measure on \( C \) and let \( f(t) = \mu([0, t]) \), \( t \in [0, 1] \), be its distribution function (this is a version of the Cantor-Vitali devil staircase for our 0 dimensional set \( C \)). Consider the graph \( G \) of the function \( \frac{1}{2} f \); thus \( G \) is a rectifiable curve, and intersects non \( \mathcal{H}_1 \) trivially the set \( X \). We then define \( H_t = G + t \cdot e_2, \; t \in [0, 1/2], \) where \( e_2 = (0, 1) \). It turns out that these will successfully play the role of horizontal sets, the details are in section [10]. The following subsumes [10.7] and [11.1].

**Theorem.** — Assume that

1. \( C \subseteq [0, 1] \) is some Cantor set of Hausdorff dimension 0;
2. \( X = C \times [0, 1] \);
3. \( \mathcal{A} \) is a \( \sigma \)-algebra and \( \mathcal{B}(X) \subseteq \mathcal{A} \subseteq \mathcal{P}(X) \);
4. \( \mathcal{N} = \mathcal{N}_{\mathcal{B},1} \) or \( \mathcal{N} = \mathcal{N}_{\mu} \).

It follows that the measurable space with negligibles \((X, \mathcal{A}, \mathcal{N})\) is consistently not localizable.

Of course this particular set \( X \) admits an obvious constant weak tangent field: \( \tau(x) = \text{span}\{e_2\}, \; x \in X \). Notice in particular that for \( \mathcal{H}_1 \) almost every \( x \in G \cap X \) the tangent line to \( G \) at \( x \) is vertical. This illustrates that localizability is indeed much stronger than the existence of a weak tangent field.

My thanks are due to David H. Fremlin, not only for his inspiring treatise «Measure Theory» [15][14][16][17][18][19] but also for many helpful conversations. I am also indebted to Francis Borceux for useful conversations regarding [3.19].

## 2. Preliminaries

### 2.1. — A measurable space consists of a pair \((X, \mathcal{A})\) where \( X \) is a set and \( \mathcal{A} \) is a \( \sigma \)-algebra of subsets of \( X \). Whenever \((X, \mathcal{A})\) and \((Y, \mathcal{B})\) are measurable spaces and \( f : X \to Y \) we say that \( f \) is \((\mathcal{A}, \mathcal{B}) \) measurable provided \( f^{-1}(B) \in \mathcal{A} \) for all \( B \in \mathcal{B} \). In the particular case when \( Y = \mathbb{R} \) it is always understood that \( \mathcal{B} = \mathcal{B}(\mathbb{R}) \) is the \( \sigma \)-algebra consisting of Borel subsets of \( \mathbb{R} \) and we say that \( f \) is \( \mathcal{A} \) measurable instead of \((\mathcal{A}, \mathcal{B}(\mathbb{R})) \) measurable.

We let \( L_0(X, \mathcal{A}) \) denote the collection of \( \mathcal{A} \) measurable functions \( X \to \mathbb{R} \). It is an algebra and a Riesz space (under the pointwise operations and partial order).

### 2.2. — As usual a measure space \((X, \mathcal{A}, \mu)\) consists in a measurable space \((X, \mathcal{A})\) and a measure \( \mu \) defined on the \( \sigma \)-algebra. We let \( L_1(X, \mathcal{A}, \mu) \) denote the subspace of \( L_0(X, \mathcal{A}) \) consisting of those \( f \) such that \( |f| \) is \( \mu \)-summable. The corresponding space of equivalence classes with respect to equality \( \mu \) almost everywhere is denoted \( L_1(X, \mathcal{A}, \mu) \).

If \( f \in L_1(X, \mathcal{A}, \mu) \) we let \( f^* \) denote its equivalence class in \( L_1(X, \mathcal{A}, \mu) \). Thus \( f \in f^* \in L_1(X, \mathcal{A}, \mu) \) means that \( f \) is an actual function representing the equivalence class \( f^* \), i.e. \( f^* = f \).
2.3. — If $(X, \mathcal{A}, \mu)$ is a measure space and $\mathcal{B} \subseteq \mathcal{A}$ is a $\sigma$-algebra, we let $\mu|_{\mathcal{B}}$ denote the restriction of $\mu$ to $\mathcal{B}$. If $A \in \mathcal{A}$ we let $(A, \mathcal{A}_A, \mu_A)$ denote the measure space where $\mathcal{A}_A = \mathcal{B} \cap \{B : B \subseteq A\}$ and $\mu_A = \mu|_{\mathcal{A}_A}$.

2.4. — With a measure space $(X, \mathcal{A}, \mu)$ we associate an outer measure $\bar{\mu}$ on $X$ by the usual formula

$$\bar{\mu}(S) = \inf \{\mu(A) : A \supseteq S\}.$$

2.5. — If $X$ is a set and $\phi$ an outer measure on $X$ we let $\mathcal{A}_{\phi}$ denote the $\sigma$-algebra consisting of those subsets of $X$ which are $\phi$ measurable in the sense of Caratheory.

2.6. — If $(X, \mathcal{A}, \mu)$ is a measure space and $\bar{\mu}$ is associated with it as in (2.4) then

1. $\mathcal{A} \subseteq \mathcal{A}_{\mu}$;
2. For every $A \in \mathcal{A}$ one has $\bar{\mu}(A) = \mu(A)$;
3. For every $S \subseteq X$ there exists $A \supseteq S$ such that $\bar{\mu}(S) = \mu(A)$.

Let $A \in \mathcal{A}$ and $S \subseteq X$. We ought to show that $\bar{\mu}(S) \geq \mu(S \cap A) + \bar{\mu}(S \setminus A)$. Let $S \supseteq B \supseteq A$ and notice that $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$. Since $B$ is arbitrary the proof of (1) is complete. Given $A \in \mathcal{A}$ and $\mathcal{A} \supseteq B \supseteq A$ we clearly have $\mu(A) \leq \mu(B)$ and, since $B$ is arbitrary $\mu(A) \leq \bar{\mu}(A)$. Letting $B = A$ proves the equality of conclusion (2) and we now turn to establishing (3). If $\bar{\mu}(S) = \infty$ then take $A = X$. If not choose $\mathcal{A} \ni A' \supseteq S$ such that $\mu(A'_n) \leq n^{-1} + \bar{\mu}(S)$, $n \in \mathbb{N}^*$. Let $A_n = \cap_{m \geq n} A'_m$, $A = \cap_{n \geq 2} A_n$ and notice that $\mathcal{A} \ni A \supseteq S$ and $\bar{\mu}(S) \leq \mu(A) = \lim_n \mu(A_n) = \lim_n \mu(A'_n) = \bar{\mu}(S)$.

2.7. — If $X$ is a Polish space and $\phi$ an outer measure on $X$ we say that $\phi$ is Borel regular if

1. $\mathcal{B}(X) \subseteq \mathcal{A}_{\phi}$, i.e. each Borel subset of $X$ is $\phi$ measurable;
2. For every $A \subseteq X$ there exists a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\phi(A) = \phi(B)$.

When $B$ is associated with $A$ as in (2) we call it a Borel hull of $A$. In this case one readily checks that $\phi(B') \leq \phi(B')$ whenever $B' \in \mathcal{B}(X)$ and $A \subseteq B'$.

2.8. — If $X$ is a Polish space, $\phi$ is a Borel regular outer measure on $X$, and $\mu = \phi|_{\mathcal{B}(X)}$, then $\bar{\mu} = \phi$.

This is a particular case of (2.6)(2).

2.9. — If $(X, \mathcal{B}(X), \mu)$ is a measure space where $X$ is Polish then $\bar{\mu}$ is Borel regular and $\bar{\mu}|_{\mathcal{B}(X)} = \mu$.

This is a particular case of (2.6).

2.10. — Let $X$ be a metric space and $0 < d < \infty$. Given $0 < \delta \leq \infty$ and $A \subseteq X$ we define

$$\mathcal{H}_{\delta}^d(A) = \inf \left\{ \sum_{i \in I} (\text{diam } A_i)^d : A \subseteq \bigcup_{i \in I} A_i, I \text{ is at most countable, and } \text{diam } A_i \leq \delta \right\}.$$

We further let

$$\mathcal{H}^d(A) = \lim_{\delta \to 0^+} \mathcal{H}_{\delta}^d(A) = \inf \left\{ \mathcal{H}_{\delta}^d(A) : 0 < \delta \leq \infty \right\}.$$

Thus $\mathcal{H}^d$ is a Borel regular outer measure on $X$. Notice that our definition differs from that of [2.10.2] by a constant multiplicative factor. This does not affect the results stated in the present paper, except for the specific constants in [9.6] which are of no relevance otherwise to our concerns.

3. Measurable Spaces with Negligibles

Most of the material in this Section is either known, or folklore or both, with the possible exception of the Definitions and Facts in [3.14] and [3.15] needed in the next Section. I learned about the concept of measurable space with negligibles in D.H. Fremlin’s treatise on Measure Theory. Here I call localizable a measurable space with negligibles whose
A measurable space with negligibles consists of a triple \((X, \mathcal{A}, \mathcal{N})\) where \((X, \mathcal{A})\) is a measurable space and \(\mathcal{N} \subseteq \mathcal{A}\) is a \(\sigma\)-ideal of \(\mathcal{A}\). The latter means that:

1. \(\emptyset \in \mathcal{N}\);
2. If \(A \in \mathcal{A}, B \in \mathcal{N}\) and \(A \subseteq B\) then \(A \in \mathcal{N}\);
3. If \((A_n)_{n \in \mathbb{N}}\) is a sequence in \(\mathcal{N}\) then \(\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{N}\).

Given a measure space \((X, \mathcal{A}, \mu)\) we define

\[\mathcal{N}_\mu = \mathcal{A} \cap \{ N : \mu(N) = 0 \}\]

so that clearly \((X, \mathcal{A}, \mathcal{N}_\mu)\) is a measurable space with negligibles. Even though it seems the natural measurable space with negligibles associated with \((X, \mathcal{A}, \mu)\), it is by no means the only one that will matter in this paper, see [14, 213N]. Similarly if \(\phi\) is an outer measure on a set \(X\) then

\[\mathcal{N}_\phi = \mathcal{P}(X) \cap \{ N : \phi(N) = 0 \}\]

is a \(\sigma\)-ideal of \(\mathcal{P}(X)\).

3.2. — Given a measurable space with negligibles \((X, \mathcal{A}, \mathcal{N})\) and \(g \in L_0(X, \mathcal{A})\) we define

\[\|g\|_\mathcal{N} = \inf \{ t : X \cap \{ x : |g(x)| > t \} \in \mathcal{N} \} \in [0, \infty] \]

and we say that \(g\) is \(\mathcal{N}\) essentially bounded if \(\|g\|_\mathcal{N} < \infty\). Letting \(L_0(X, \mathcal{A}, \mathcal{N})\) denote the collection of such functions and be equipped with the operations and partial order inherited from \(L_0(X, \mathcal{A})\) one checks it is an algebra and a Riesz space. Furthermore \(\| \cdot \|_\mathcal{N}\) is a seminorm defined on \(L_0(X, \mathcal{A}, \mathcal{N})\). One classically shows that \(\|g\|_\mathcal{N} = 0\) if and only if \(X \cap \{ g \neq 0 \} \in \mathcal{N}\) and we let \(L_\infty(X, \mathcal{A}, \mathcal{N})\) be the corresponding quotient space equipped with the corresponding norm. The following is established in exactly the same way as in the case of measure spaces.

3.3. — Given a measurable space with negligibles \((X, \mathcal{A}, \mathcal{N})\), \(L_\infty(X, \mathcal{A}, \mathcal{N})\) is both a Banach space and a Banach lattice.

3.4. — Let \((X, \mathcal{A}, \mathcal{N})\) be a measurable space with negligibles. Forgetting about the stability of \(\mathcal{A}\) and \(\mathcal{N}\) under countable (rather than finite) operations we view \(\mathcal{A}\) as a Boolean algebra and \(\mathcal{N}\) as an ideal of \(\mathcal{A}\). As such the quotient \(\mathcal{A}/\mathcal{N}\) is a Boolean algebra as well.

Given an arbitrary Boolean algebra \(B\) we recall that its Stone representation \(\text{Spec}(B)\) is a totally disconnected compact Hausdorff topological space of which the Boolean algebra of clopen sets is isomorphic to \(B\), see e.g. [16, 311E and 311I]. By a totally disconnected topological space we mean one whose connected subsets are all singletons; if the space is assumed to be compact Hausdorff this is equivalent to the existence of a basis for the topology consisting of clopen (closed and open) subsets.

3.5. Proposition. — Given a measurable space with negligibles \((X, \mathcal{A}, \mathcal{N})\), the Banach spaces \(L_\infty(X, \mathcal{A}, \mathcal{N})\) and \(C(\text{Spec}(\mathcal{A}/\mathcal{N}))\) are isometrically isomorphic.

Proof. Letting \(L_\infty(X, \mathcal{A}, \mathcal{N})\) denote the linear subspace of \(L_\infty(X, \mathcal{A}, \mathcal{N})\) corresponding to those simple functions \(g \in L_\infty(X, \mathcal{A}, \mathcal{N})\), i.e. those having finite range, we define
\( \Xi : L_{\infty, \lambda}(X, \mathcal{A}, \mathcal{N}) \to C(\text{Spec}(\mathcal{A}, \mathcal{N})) \) by the formula

\[
\Xi(a^*) = \sum_{y \in \text{St}(X)} y^* \mathbb{1}_{\text{St}^{-1}(\{y\})}
\]

where \( \text{St} : \mathcal{A}, \mathcal{N} \to \text{Spec}(\mathcal{A}, \mathcal{N}) \) is the Stone isomorphism and the superscript bullet denotes the equivalence class. Since each \( \mathbb{1}_{\text{St}^{-1}(\{y\})} \), \( a^* \in \mathcal{A} \), is continuous, \( \Xi \) is well defined. It is easy to check that \( \Xi \) is a linear isometry onto its image. The basic Approximation Lemma of measurable functions by simple functions implies that \( L_{\infty, \lambda}(X, \mathcal{A}, \mathcal{N}) \) is dense in \( L_{\infty}(X, \mathcal{A}, \mathcal{N}) \), therefore \( \Xi \) uniquely extends to a linear isometry \( \hat{\Xi} : L_{\infty}(X, \mathcal{A}, \mathcal{N}) \to C(\text{Spec}(\mathcal{A}, \mathcal{N})) \). Upon noticing that \( \text{im} \; \Xi \) is a subalgebra of \( C(\text{Spec}(\mathcal{A}, \mathcal{N})) \) that contains the constant functions and that separates points we infer from the Stone-Weierstrass Theorem that \( \text{im} \; \Xi \) is dense and in turn that \( \hat{\Xi} \) is surjective. \( \square \)

3.6. — Let \((X, \mathcal{A}, \mathcal{N})\) be a measurable space with negligibles and \( \mathcal{E} \subseteq \mathcal{A} \). We say that \( A \in \mathcal{A} \) is an \( \mathcal{N} \) essential supremum of \( \mathcal{E} \) whenever the following holds:

1. For every \( E \in \mathcal{E} \) one has \( E \setminus A \in \mathcal{N} \);
2. If \( B \in \mathcal{A} \) is such that \( E \setminus B \subseteq A \) for every \( E \in \mathcal{E} \), then \( A \setminus B \in \mathcal{N} \).

In particular if \( A, A' \in \mathcal{A} \) are both \( \mathcal{N} \) essential suprema of \( \mathcal{E} \) it follows that \( A \cap A' \in \mathcal{N} \) where \( \cap \) denotes the symmetric difference of two sets. If \( A \) verifies condition (1) but necessarily condition (2) we call it an \( \mathcal{N} \) essential upper bound of \( \mathcal{E} \).

3.7. — We say that a measurable space with negligibles \((X, \mathcal{A}, \mathcal{N})\) is localizable whenever each family \( \mathcal{E} \subseteq \mathcal{A} \) admits an \( \mathcal{N} \)-essential supremum. Given a measurable space with negligibles \((X, \mathcal{A}, \mathcal{N})\) the following conditions are equivalent:

1. \((X, \mathcal{A}, \mathcal{N})\) is localizable;
2. The Boolean algebra \( \mathcal{A}, \mathcal{N} \) is order complete;
3. The Stone space \( \text{Spec}(\mathcal{A}, \mathcal{N}) \) is extremally disconnected;
4. The Banach lattice \( C(\text{Spec}(\mathcal{A}, \mathcal{N})) \) is order complete;
5. The Banach space \( C(\text{Spec}(\mathcal{A}, \mathcal{N})) \) is isometrically injective.

That (1) be equivalent to (2) is routine verification. If \( \mathcal{B} \) is a Boolean algebra then \( \mathcal{B} \) is order complete if and only if \( \text{Spec}(\mathcal{B}) \) is extremally disconnected, see e.g. [16, 314S]. We recall that a compact Hausdorff topological space \( K \) is called extremally disconnected if the closure of any open set is open. Furthermore a compact Hausdorff space \( K \) is extremally disconnected if and only if \( C(K) \) is order complete, see e.g. [3] Problems 4.5 and 4.6. This shows the equivalence between (3) and (4). The equivalence between (4) and (5) is a consequence of the Goodner-Nachbin Theorem, see [3, 4.3.6].

An important class of examples of localizable spaces with negligibles is given below, with a proof for the reader’s convenience.

3.8. PROPOSITION. — If \((X, \mathcal{A}, \mu)\) is a \( \sigma \)-finite measure space then \((X, \mathcal{A}, N_\mu)\) is localizable (recall [3, 7]).

Proof. If \((X, \mathcal{A}, \mu)\) is not finite choose a partition \((X_n)_{n \in \mathbb{N}}\) of \( X \) into members of \( \mathcal{A} \) such that \( 0 < \mu(X_n) < \infty \) for every \( n \in \mathbb{N} \) and define a measure \( \nu \) on \( \mathcal{A} \) by the formula

\[
\nu(A) = \sum_{n \in \mathbb{N}} 2^{-n} \mu(X_n)^{-1} \mu(X_n \cap A),
\]

\( A \in \mathcal{A} \). Observing that \( N_\mu = N_\mu \) and that \( \nu(X) = 2 \) we conclude that the proposition follows from its special case when \((X, \mathcal{A}, \mu)\) is finite.

We henceforth assume that \( \mu(X) < \infty \). Let \( \mathcal{E} \subseteq \mathcal{A} \) and define

\[
\mathcal{F} = \mathcal{A} \setminus \{ F : \mu(F \cap E) = 0 \text{ for every } E \in \mathcal{E} \}.
\]

Notice that \( \mathcal{F} \) is a \( \sigma \)-ideal. Put \( \tau = \sup \{ \mu(F) : F \in \mathcal{F} \} < \infty \). There exists a nondecreasing sequence \((F_n)_{n \in \mathbb{N}}\) in \( \mathcal{F} \) such that \( \mu(F_n) \geq \tau - (n + 1)^{-1} \) for every \( n \in \mathbb{N} \). Thus \( F := \bigcup_{n \in \mathbb{N}} F_n \in \mathcal{F} \) and \( \mu(F) = \tau \). In particular \( \mu(G \setminus F) = 0 \) for every \( G \in \mathcal{F} \) (for otherwise
μ(F ∪ (G \ F)) > μ(F) and F ∪ (G \ F) ∈ ℙ, a contradiction). We now claim that A = X \ F is an $\mathcal{A}_\mu$-essential supremum of $\mathcal{E}$. Indeed:

1. Given $E ∈ \mathcal{E}$, $μ(E \setminus A) = μ(E \cap F) = 0$ since $F ∈ ℙ$;
2. If $B ∈ \mathcal{A}$ is such that $μ(E \setminus B) = 0$ for every $E ∈ \mathcal{E}$ then $G = X \setminus B ∈ ℙ$ and hence $0 = μ(G \setminus F) = μ(A \setminus B)$.

\[\Box\]

3.9. — The first obstacle that comes to mind for a measurable space with negligibles $(X, \mathcal{A}, \mathcal{N})$ to be localizable is that $\mathcal{A}$ is not required to be stable under arbitrary unions. If it were, then condition (1) of 3.6 would be obviously satisfied with $A = \cup \mathcal{E} ∈ \mathcal{A}$. This is not the end of the story however as condition (2) may well fail for such choice of $A$. In fact we give an example below [11.2] (the paragraph before (Q7)) of a measurable space with negligibles of the type $(X, ℙ(X), \mathcal{N})$ which is consistently not localizable. Worse yet we exhibit a proper $\sigma$-algebra $\mathcal{B} ⊆ ℙ(X)$ such that $(X, \mathcal{A}, \mathcal{A} \cap \mathcal{N})$ is consistently non localizable whenever $\mathcal{B} ⊆ \mathcal{A} ⊆ ℙ(X)$ is a $\sigma$-algebra. This ruins the hope that with each measurable space with negligibles one can associate a localizable version of it by «adding enough measurable sets» to the given $\sigma$-algebra. Accepting to enlarge the base set $X$, see 3.19(4) for resurrecting some weak hope.

3.10. — Assume $(X, \mathcal{A}, \mathcal{N})$ and $(Y, \mathcal{B}, \mathcal{M})$ are measurable spaces with negligibles and $f : X → Y$ is a bijection such that

$$\mathcal{A} = ℙ(X) \cap \{f^{-1}(B) : B ∈ \mathcal{B}\}$$

and

$$\mathcal{N} = ℙ(X) \cap \{f^{-1}(M) : M ∈ \mathcal{M}\}.$$  

It follows that $(X, \mathcal{A}, \mathcal{N})$ is localizable if and only if $(Y, \mathcal{B}, \mathcal{M})$ is localizable.

This can be checked directly by routine verifications from the definition of essential supremum or by observing that the quotient Boolean algebras $\mathcal{A}/\mathcal{N}$ and $\mathcal{B}/\mathcal{M}$ are isomorphic and referring to 3.7. Such $f$ is an instance of an isomorphism in the category to be discussed in 3.19. We will use this result in 9.8 below.

3.11. — If $(X, \mathcal{A})$ is a measurable space and $E ∈ \mathcal{A}$ we associate with it its subspace $(E, \mathcal{A}_E)$ where $\mathcal{A}_E = ℙ(E) \cap \{E \cap A : A ∈ \mathcal{A}\}$.

3.12. — Let $(X, \mathcal{A}, \mathcal{N})$ be a measurable space with negligibles and let $\mathcal{E} ⊆ \mathcal{A}$. A family subordinated to $\mathcal{E}$ is a family $(g_E)_{E ∈ \mathcal{E}}$ such that

1. $g_E : E → R$ is $\mathcal{A}_E$-measurable for every $E ∈ \mathcal{E}$.

We further say that $(g_E)_{E ∈ \mathcal{E}}$ is compatible if also

2. For every pair $E_1, E_2 ∈ \mathcal{E}$ one has $E_1 \cap E_2 ∩ \{g_{E_1} ≠ g_{E_2}\} ∈ \mathcal{N}$.

A gluing of a compatible family $(g_E)_{E ∈ \mathcal{E}}$ subordinated to $\mathcal{E}$ is a function $g : X → R$ such that

1. $g$ is $\mathcal{A}$-measurable;
2. $E ∩ \{g ≠ g_E\} ∈ \mathcal{N}$ for every $E ∈ \mathcal{E}$.

3.13. Proposition. — Let $(X, \mathcal{A}, \mathcal{N})$ be a measurable space with negligibles. The following are equivalent.

1. $(X, \mathcal{A}, \mathcal{N})$ is localizable.
2. For every $\mathcal{E} ⊆ \mathcal{A}$, every compatible family subordinated to $\mathcal{E}$ admits a gluing.

Proof. (1) ⇒ (2) Let $\mathcal{E} ⊆ \mathcal{A}$ and let $(g_E)_{E ∈ \mathcal{E}}$ be a compatible family subordinated to $\mathcal{E}$. With each $q ∈ Q$ and $E ∈ \mathcal{E}$ we associate $E_q = E ∩ \{g_E ≥ q\} ∈ \mathcal{A}$. Thus given $q ∈ Q$ the family $\{E_q : E ∈ \mathcal{E}\}$ admits an $\mathcal{N}$ essential supremum which we denote as $A_q ∈ \mathcal{A}$. Define $\tilde{g} : X → [−∞, +∞]$ by the formula $\tilde{g}(x) = \sup\{q : x ∈ A_q\}$, $x ∈ X$, where as usual $\inf \emptyset = −∞$. Notice that if $q ∈ Q$ then $\{\tilde{g} > q\} = \cup_{r > q} A_r ∈ \mathcal{A}$, thus $\tilde{g}$ is $\mathcal{A}$-measurable.
Given $E \in \mathcal{E}$ we shall now establish that
\[ E \cap \{g_E \neq \bar{g}\} \in \mathcal{N}. \tag{2} \]
If $x \in E \cap \{g_E < \bar{g}\}$ then there exists $q \in \mathbb{Q}$ such that $g_E(x) < q$ and $x \in A_q$. Accordingly,
\[ E \cap \{g_E < \bar{g}\} \subseteq \bigcup_{q \in \mathbb{Q}} E \cap (A_q \setminus E_q). \tag{3} \]
Now if $q \in \mathbb{Q}$ and $E' \in \mathcal{E}$ then
\[ E_q' \setminus (E' \cup E_q) = E \cap \{g_E < q\} \cap E' \cap \{g_E' \geq q\} \subseteq E \cap E' \cap \{g_E \neq g_E'\} \in \mathcal{N}. \]
Since $E' \in \mathcal{E}$ is arbitrary we infer that
\[ \mathcal{N} \ni A_q \setminus (E' \cup E_q) = E \cap (A_q \setminus E_q) \]
and it therefore ensues from \((3)\) that
\[ E \cap \{g_E > \bar{g}\} \in \mathcal{N}. \tag{4} \]
Next if $x \in E \cap \{g_E > \bar{g}\}$ then there exists $q \in \mathbb{Q}$ such that $g_E(x) > q$ and $x \notin A_q$. Consequently,
\[ E \cap \{g_E > \bar{g}\} \subseteq \bigcup_{q \in \mathbb{Q}} E \cap (E_q \setminus A_q) \in \mathcal{N}. \tag{5} \]
It now follows from \((4)\) and \((5)\) that \(\mathcal{G}\) holds.

Finally we let $A = \{\bar{g} \in \mathbb{R}\} \in \mathcal{A}$ and $g = \bar{g} \cdot 1_A$ (with the usual convention that $(-\infty,0) = 0$). Thus $g$ is $\mathcal{A}$-measurable and, for each $E \in \mathcal{E}$, $E \cap \{g \neq g_E\} \subseteq E \cap \{\bar{g} \neq g_E\} \in \mathcal{N}$ since $g_E$ is $\mathbb{R}$ valued. Whence $g$ is a gluing of $(g_E)_{E \in \mathcal{E}}$.

(2) $\Rightarrow$ (1) Let $\mathcal{E} \subseteq \mathcal{A}$ and define $\mathcal{E}^* = \mathcal{E} \setminus \mathcal{N}$ as well as
\[ \mathcal{F} = \mathcal{A} \cap \{F : F \cap E \in \mathcal{N} \text{ for every } E \in \mathcal{E}^*\}. \]

Notice that $\mathcal{F} \cap \mathcal{E}^* = \emptyset$. Put $\mathcal{G} = \mathcal{E}^* \cup \mathcal{F}$ and define a family $(g_E)_{E \in \mathcal{G}}$ subordinated to $\mathcal{G}$ as follows. If $E \in \mathcal{E}^*$ then $g_E = 1_E$, and if $F \in \mathcal{F}$ then $g_F = 0 \cdot 1_F$. One easily checks that $(g_E)_{E \in \mathcal{G}}$ is a compatible family, thus it admits a gluing $g$ by assumption. Let $A = \{g = 1\} \in \mathcal{A}$. We ought to show that $A$ is an $\mathcal{N}$ essential supremum of $\mathcal{E}$.

First let $E \in \mathcal{E}$. If $E \in \mathcal{N}$ then clearly $E \setminus A \in \mathcal{N}$. Otherwise $E \in \mathcal{E}^*$ and hence $E \setminus A = E \cap \{g \neq 1\} = E \cap \{g \neq g_E\} \in \mathcal{N}$. Suppose now that $B \in \mathcal{E}$ is such that $E \setminus B \in \mathcal{N}$ for every $E \in \mathcal{E}$. Let $F = A \setminus B \in \mathcal{A}$. Given $E \in \mathcal{E}$ we observe that $F \cap E = E \cap (A \setminus B) \subseteq A \setminus B \in \mathcal{N}$. Therefore $F \in \mathcal{F}$. It follows that $A \setminus B = F \cap A = F \cap \{g = 1\} \subseteq F \cap \{g \neq g_F\} \in \mathcal{N}$ and the proof is complete. \(\square\)

3.14. — Given a measurable space with negligibles $(X,\mathcal{A},\mathcal{N})$ and $\mathcal{F} \subseteq \mathcal{A}$ an arbitrary family, we define
\[ \mathcal{N}[\mathcal{F}] = \mathcal{A} \cap \{A : A \cap F \in \mathcal{N} \text{ for every } F \in \mathcal{F}\}. \]

The following are immediate consequences of the definition.

(1) $\mathcal{N}[\mathcal{F}]$ is a $\sigma$-ideal in $\mathcal{A}$.
(2) $\mathcal{N} \subseteq \mathcal{N}[\mathcal{F}]$.
(3) If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ then $\mathcal{N}[\mathcal{F}_1] \supseteq \mathcal{N}[\mathcal{F}_2]$.
(4) If $g$ and $g'$ are both gluings of a compatible family $(g_E)_{E \in \mathcal{E}}$ subordinated to $\mathcal{E}$ then $\{g \neq g'\} \in \mathcal{N}[\mathcal{E}]$.

3.15. — Let $(X,\mathcal{A},\mathcal{N})$ be a measurable space with negligibles. We say that $\mathcal{I} \subseteq \mathcal{A}$ is an ideal in $\mathcal{A}$ whenever the following holds:

(1) $\emptyset \in \mathcal{I}$;
(2) If $A \in \mathcal{I}$, $B \in \mathcal{I}$ and $A \subseteq B$ then $A \in \mathcal{I}$;
(3) If $A_1,\ldots,A_N \in \mathcal{I}$ then $\bigcup_{n=1}^N A_n \in \mathcal{I}$. 


One observes that each \( \mathcal{E} \subseteq \mathcal{A} \) is contained in a smallest ideal which we denote as ideal(\( \mathcal{E} \)). The reader will easily check that

\[
\text{ideal}(\mathcal{E}) = \mathcal{A} \cap \{ A : \text{ there exist } E_1, \ldots, E_n \in \mathcal{E} \text{ such that } A \subseteq \bigcup_{i=1}^{N} E_n \}.
\]

Therefore

(4) \( \mathcal{N}[\mathcal{E}] = \mathcal{N}[\text{ideal}(\mathcal{E})] \).

(5) \( A \in \mathcal{A} \) is an \( \mathcal{N} \) essential supremum of \( \mathcal{E} \) if and only if \( A \) is an \( \mathcal{N} \) essential supremum of ideal(\( \mathcal{E} \)).

There is no difficulty in showing that the latter is a consequence of the definition of essential supremum and of the following claim: If \( C \in \mathcal{A} \) is such that \( E \setminus C \in \mathcal{N} \) for every \( E \in \mathcal{E} \) then also \( F \setminus C \in \mathcal{N} \) for every \( F \in \text{ideal}(\mathcal{E}) \).

3.16 (Partition of unity). — Let \( (X, \mathcal{A}, \mathcal{N}) \) be a measurable space with negligibles, and \( \mathcal{I} \subseteq \mathcal{A} \) an ideal. A partition of unity relative to \( \mathcal{I} \) is a collection \( \mathcal{E} \subseteq \mathcal{I} \) such that

(1) \( \mathcal{E} \cap \mathcal{N} = \emptyset \);

(2) For every \( E_1, E_2 \in \mathcal{E} \), if \( E_1 \neq E_2 \) then \( E_1 \cap E_2 \in \mathcal{N} \);

(3) For every \( A \in \mathcal{I} \setminus \mathcal{N} \) there exists \( E \in \mathcal{E} \) such that \( A \cap E \notin \mathcal{N} \).

3.17. Lemma. — Let \( (X, \mathcal{A}, \mathcal{N}) \) be a measurable space with negligibles, and \( \mathcal{I} \subseteq \mathcal{A} \) an ideal. There exists a partition of unity \( \mathcal{E} \) relative to \( \mathcal{I} \). Furthermore \( \mathcal{E} \neq \emptyset \) in case \( \mathcal{I} \subseteq \mathcal{N} \).

Proof. This is a routine application of Zorn’s Lemma. \( \square \)

3.18 (Magnitude). — Let \( (X, \mathcal{A}, \mathcal{N}) \) be a measurable space with negligibles, \( \mathcal{I} \) an ideal in \( \mathcal{A} \), and \( \kappa \) a cardinal. We say that \( (X, \mathcal{A}, \mathcal{N}) \) has magnitude less than \( \kappa \) relative to \( \mathcal{I} \) whenever for every \( \mathcal{E} \subseteq \mathcal{I} \) with the following properties:

(1) \( \mathcal{E} \cap \mathcal{N} = \emptyset \);

(2) For every \( E_1, E_2 \in \mathcal{E} \), if \( E_1 \neq E_2 \) then \( E_1 \cap E_2 \in \mathcal{N} \);

one has \( \text{card } \mathcal{E} \leq \kappa \).

3.19 (Functorial point of view). — Here we refer to [2] for the vocabulary of category theory, and we offer some questions stated in this language. We start by defining a category MSN. Its objects are the measurable spaces with negligibles. A morphism from an object \( (X, \mathcal{A}, \mathcal{N}) \) to an object \( (Y, \mathcal{B}, \mathcal{M}) \) consists in a map \( f : X \to Y \) which is \( (\mathcal{A}, \mathcal{B}) \) measurable and such that \( f^{-1}(M) \in \mathcal{N} \) for every \( M \in \mathcal{M} \). For instance if \( (X, \mathcal{A}, \mu) \) is a measure space, \( (Y, \mathcal{B}) \) a measurable space, and \( f : X \to Y \) an \( (\mathcal{A}, \mathcal{B}) \) measurable map, we recall that the measure \( \nu = f_* \mu \) is defined on \( \mathcal{B} \) by the formula \( (f_* \mu)(B) = \mu(f^{-1}(B)) \), \( B \in \mathcal{B} \), and we readily check that \( f \) defines a morphism between \( (X, \mathcal{A}, \mathcal{N}_\mu) \) and \( (Y, \mathcal{B}, \mathcal{N}_\nu) \). On then infers the following from the Kuratowski Isomorphism Theorem, [28] 3.4.23. If \( X \) is an uncountable Polish space and \( \mu \) is a diffuse Borel probability measure on \( X \) then \( (X, \mathcal{B}(X), \mathcal{N}_\mu) \) and \( ((0, 1], \mathcal{B}([0, 1]), \mathcal{N}_{\mathcal{L}^1}) \) are isomorphic objects of MSN, where \( \mathcal{L}^1 \) is the Lebesgue measure on the unit interval \([0, 1] \). Without a separability assumption of the base space, Maharam’s Theorem [16] 332B gives a classification of probability spaces at the level of measure algebras but not as strong it seems as to describe isomorphism classes in the category MSN. Examples of isomorphisms in the category MSN that are not obtained via the Kuratowski Isomorphism Theorem are used below in the proof of [18].

Next we define a full subcategory LOC of MSN by letting its objects be those localizable measurable spaces with negligibles and we consider the corresponding forgetful functor \( F : \text{LOC} \to \text{MSN} \).

(Q1) It would be desirable to study the categorical properties of MSN and LOC. We claim for instance that they both admit equalizers and coproducts, preserved by \( F \). We also claim that MSN admits products (but the corresponding \( \sigma \)-ideal in the
product is not of «Fubini type»), yet LOC may not. Finally we claim that MSN admits coequalizers, yet LOC may not.

We now consider the category $\text{Bool}$ whose objects are Boolean algebras and whose morphisms are the homomorphisms of Boolean algebras. We also consider $\text{CBool}$ the full subcategory of $\text{Bool}$ whose objects are the order complete Boolean algebras, together with the corresponding forgetful functor $\text{CBool} \to \text{Bool}$. One then defines a contravariant functor $A : \text{MSN} \to \text{Bool}$ in the following way. The image by $A$ of an object $(X, \mathcal{A}, \mathcal{N})$ is the quotient Boolean algebra $\mathcal{N} / \mathcal{A}$ considered already in [3,4]. Given a morphism $f$ between $(X, \mathcal{A}, \mathcal{N})$ and $(Y, \mathcal{B}, \mathcal{M})$ we let $A(f)$ be the homomorphism of Boolean algebras $\mathcal{M} / \mathcal{B} \to \mathcal{N} / \mathcal{A}$ that maps $B^*$ to $f^{-1}(B)^*$, $B \in \mathcal{B}$, and one checks it is well defined. We notice that $A(f)$ is in fact more than a homomorphism of Boolean algebras: It is sequentially order continuous. It is even order continuous when for instance the so-called countable chain condition, which is the case when $\mathcal{A}$ is in fact more than a homomorphism of Boolean algebras: It is sequentially order continuous. It is even order continuous when for instance $\mathcal{A}$ is assumed to have the so-called countable chain condition, which is the case when $(X, \mathcal{A}, \mathcal{N})$ has magnitude less than $\aleph_0$ relative to $\mathcal{A}$ (an example being $(X, \mathcal{A}, \mathcal{N})$ with $(X, \mathcal{A}, \mu)$ a $\sigma$-finite measure space). The functor $A$ lifts to a contravariant functor $A_c : \text{LOC} \to \text{CBool}$, leading to the following commutative diagram where vertical arrows denote the forgetful functors.

\[
\begin{array}{ccc}
\text{LOC} & \xrightarrow{A_c} & \text{CBool} \\
\downarrow & & \downarrow \\
\text{MSN} & \xrightarrow{A} & \text{Bool}
\end{array}
\]

(Q2) As mentioned in the first paragraph of this number, Maharam’s Theorem pertains to a classification of some objects in $\text{CBool}$ arising as measure algebras of probability spaces, but not necessarily a classification in $\text{LOC}$. This raises the question of «realization» in $\text{LOC}$ of certain morphisms in $\text{CBool}$. Specifically if $u : \mathcal{P} \to \mathcal{A}$ is a sequentially order continuous homomorphism of Boolean algebras, under what conditions on $(X, \mathcal{A}, \mathcal{N})$ and $(Y, \mathcal{B}, \mathcal{M})$ does there exist a morphism $f$ between these two objects in $\text{MSN}$ such that $A(f) = u$? Inspired by the special case [13] it may be that the key ingredients are the existence of a «lifting» in $(Y, \mathcal{B}, \mathcal{M})$ and the existence of a compact class in $\mathcal{P}(X)$ that «generates» $(X, \mathcal{A}, \mathcal{N})$ in an appropriate sense. See also [16] Chapter 34. It further raises the question of identifying classes of measurable spaces with negligibles that admit a «lifting», beyond the case of measure spaces (where the situation is well understood).

Next we let $\text{Comp}_{\text{tot}}$ (resp. $\text{Comp}_{\text{extr}}$) denote the category of totally disconnected (resp. extremely disconnected) Hausdorff compact topological spaces and their continuous maps. We complete the diagram above by adding the central horizontal arrows which are (contravariant) equivalences of categories.

\[
\begin{array}{ccccccc}
\text{LOC} & \xrightarrow{A_c} & \text{CBool} & \longrightarrow & \text{Comp}_{\text{extr}} & \longrightarrow & \text{Ban}_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{MSN} & \xrightarrow{A} & \text{Bool} & \longrightarrow & \text{Comp}_{\text{tot}} & \longrightarrow & \text{Ban}_1
\end{array}
\]

The objects of the category $\text{Ban}_1$ are the Banach spaces and its morphisms are the linear contractions. The horizontal contravariant functors in the right hand row map an object $K$ to $C(K)$, and $K$ is extremally disconnected if and only if $C(K)$ is an injective object of $\text{Ban}_1$. The composition of the horizontal functors in the diagram associates contravariantly with $(X, \mathcal{A}, \mathcal{N})$ the Banach space $C(\text{Spec}(\mathcal{A}))$. The isometric isomorphism described in [3,5] is functorial: There exists a natural transformation between the composition of horizontal functors in the above diagram and the functors $L_\infty : \text{MSN} \to \text{Ban}_1$ (resp. $L_{\infty,C} : \text{MSN} \to \text{Ban}_1$) defined as follows. An object $(X, \mathcal{A}, \mathcal{N})$ is mapped to $L_\infty(X, \mathcal{A}, \mathcal{N})$. Given a
morphism \( f \) between the objects \((X, \mathcal{A}, \mathcal{N})\) and \((Y, \mathcal{B}, \mathcal{M})\) we let \( L_\infty(f) : L_\infty(Y, \mathcal{B}, \mathcal{M}) \to L_\infty(X, \mathcal{A}, \mathcal{N}) \) be the linear contraction (well) defined by \( L_\infty(f)(u^*) = (u \circ f)^* \).

(Q3) There is a subtlety in the definition of localizability that is worth pointing out, related with its sheaf-like quality. Even though \( U \mapsto C(U) \) is a sheaf, and despite not all measurable spaces with negligibles are localizable. This can be likely expressed in terms of properties of the functor \( A \) related to «subspaces».

Somewhat related to the search for a «localizable version» of an arbitrary measurable space with negligibles (recall (4.9)) is the fact that the vertical forgetful functor on the right hand row of (6) has no left adjoint (see [27] for a proof of the Gaifman-Hales Theorem to the extent that there exists no free, order complete Boolean algebra generated by an infinite set), thus this forgetful functor does not satisfy the «small set condition» in Freyd’s adjoint functor Theorem [7, 3.3.3]. Yet one can consider the following process.

(Q4) Given a measurable space with negligibles \((X, \mathcal{A}, \mathcal{N})\) we seek to produce a new one, localizable, in a canonical (and functorial) way. We start with \( \mathcal{A}\mathcal{N} \) and we consider its Dedekind completion, say \( A \), see e.g. [16, 314T]. Since \( A \) is order \( \sigma\)-complete we can next associate its Loomis-Sikorski realization (see [16, 314M]) or [26, §29]), a measurable space with negligibles \((\hat{X}, \hat{\mathcal{A}}, \hat{\mathcal{N}})\) such that \( A \) and \( \mathcal{A}\mathcal{N} \) are isomorphic as Boolean algebras. What are the properties of the functor thus defined? Can one identify and understand this localizable version for instance in cases described in [17,2, Q7]?

4. Semifinite and Semilocalizable Measure Spaces

For a measure space \((X, \mathcal{A}, \mu)\) the canonical map from \( L_\infty(X, \mathcal{A}, \mu) \) to \( L_1(X, \mathcal{A}, \mu)^* \) is in general neither injective nor surjective. It is known that injectivity is equivalent to semifiniteness of \((X, \mathcal{A}, \mu)\). We identify here a condition equivalent to surjectivity, which we call semilocalizability.

4.1. — Let \((X, \mathcal{A}, \mu)\) be a measure space. We consider the map

\[
\Gamma : L_\infty(X, \mathcal{A}, \mu) \to L_1(X, \mathcal{A}, \mu)^*
\]

defined in the following way. Given \( g \in L_\infty(X, \mathcal{A}, \mu) \) and \( f \in L_1(X, \mathcal{A}, \mu) \) we let \( \Gamma(g)(f) = \int_X g f \, d\mu \) for a choice of \( g \in g \) and \( f \in f \). In general \( \Gamma \) is neither injective nor surjective. In this Section we state a necessary and sufficient condition for \( \Gamma \) to be injective (namely that the measure space be semifinite) and a necessary and sufficient condition for \( \Gamma \) to be surjective (namely that the measure space be semilocalizable).

4.2. — We say that a measure space \((X, \mathcal{A}, \mu)\) is **semifinite** whenever the following holds: For every \( A \in \mathcal{A} \) such that \( \mu(A) = \infty \) there exists \( B \in \mathcal{A} \) such that \( B \subseteq A \) and \( 0 < \mu(B) < \infty \). Clearly all \( \sigma\)-finite measure spaces are semifinite. We recall that \( \Gamma \) is injective if and only if \((X, \mathcal{A}, \mu)\) is semifinite and in that case \( \Gamma \) is an isometry, [14, 243G(a)]. Furthermore if \((X, \mathcal{A}, \mu)\) is semifinite then \( \Gamma \) is bijective if and only if \((X, \mathcal{A}, \mathcal{N}_\mu)\) is localizable, [14, 243G(b)]. Below we give a necessary and sufficient condition for \( \Gamma \) to be surjective (not assuming that it be injective in the first place). This seems to be new.

4.3. — Let \((X, \mathcal{A}, \mu)\) be a measure space. We say that \( A \in \mathcal{A} \) is **purely infinite** if for every \( B \in \mathcal{A} \) such that \( B \subseteq A \) one has \( \mu(B) = 0 \) or \( \mu(B) = \infty \). Thus \((X, \mathcal{A}, \mu)\) is semifinite if and only if there exists no purely infinite \( A \in \mathcal{A} \). We define

\[
\mathcal{A}\mu = \mathcal{A} \cap \{ A : \mu(A) = 0 \}
\]

\[
\mathcal{A}\mu^f = \mathcal{A} \cap \{ A : \mu(A) < \infty \}
\]

\[
\mathcal{A}\mu^m = \mathcal{A} \cap \{ A : A \text{ is purely infinite} \}
\]

D.H. Fremlin calls localizable a measure space \((X, \mathcal{A}, \mu)\) which is semifinite and such that \((X, \mathcal{A}, \mathcal{N}_\mu)\) is (in the vocabulary introduced in the present paper) a localizable measurable space with negligibles.
and we abbreviate \( \mathcal{A}^I = \mathcal{A}_I^I \) and \( \mathcal{A}^{PI} = \mathcal{A}_I^{PI} \) when no confusion occurs, which is almost always. Clearly \( \mathcal{A}^I \) is an ideal, whereas \( \mathcal{N}_\mu \) and \( \mathcal{N}_\mu \cup \mathcal{A}^{PI} \) are \( \sigma \)-ideals. In fact one easily checks that \( \mathcal{N}_\mu^{[\mathcal{A}^I]} = \mathcal{N}_\mu \cup \mathcal{A}^I \) and also that \( \mathcal{N}_\mu \cup \mathcal{A}^{PI} = \mathcal{N}_\mu^{[\mathcal{A}^I]} \) where \((X, \mathcal{A}, \mu_\mathcal{A})\) is the semifinite version of \((X, \mathcal{A}, \mu)\), see \(\text{[13]}\) 213x(c) and also \(4.4\) below.

Referring to \(3.14\) we consider the \(\sigma\)-ideal \(\mathcal{N}_\mu^{[\mathcal{A}^I]}\) which will play the major role in the present Section. When we need to refer to its members we call these \textbf{locally \(\mu\) null}.

4.4. Lemma. — Let \((X, \mathcal{A}, \mu)\) be a measure space. The following are equivalent:

1. \((X, \mathcal{A}, \mu)\) is semifinite;
2. \(\mathcal{N}_\mu^{[\mathcal{A}^I]} \subseteq \mathcal{N}_\mu^{[\mathcal{A}^I]}\).

\textbf{Proof.} It follows from \(\text{[3.14]}\) (1) that (2) is equivalent to \(\mathcal{N}_\mu^{[\mathcal{A}^I]} \subseteq \mathcal{N}_\mu^{[\mathcal{A}^I]}\). Therefore \(\neg\) (2) is equivalent to \(\mathcal{N}_\mu^{[\mathcal{A}^I]} \setminus \mathcal{N}_\mu^{[\mathcal{A}^I]} \neq \emptyset\). One easily observes that \(\mathcal{N}_\mu^{[\mathcal{A}^I]} \setminus \mathcal{N}_\mu^{[\mathcal{A}^I]} = \mathcal{A}^{PI}\). Since \((X, \mathcal{A}, \mu)\) is semifinite if and only if \(\mathcal{A}^{PI} = \emptyset\) the proof is complete. \(\square\)

4.5. — A measure space \((X, \mathcal{A}, \mu)\) is called \textbf{semilocalizable} if the measurable space with negligibles \((X, \mathcal{A}, \mathcal{N}_\mu^{[\mathcal{A}^I]})\) is localizable.

4.6. Proposition. — Let \((X, \mathcal{A}, \mu)\) be a measure space. The following are equivalent:

1. \((X, \mathcal{A}, \mu)\) is semilocalizable.
2. \(\mathcal{T}\) is surjective.

\textbf{Proof.} (1) \(\Rightarrow\) (2) To each \(E \in \mathcal{A}^I\) we associate the linear isometry \(\beta_E : L_1(E, \mathcal{A}_E, \mu_E) \rightarrow L_1(X, \mathcal{A}, \mu)\) defined in the obvious way (extending \(f \in L_1(E, \mathcal{A}_E, \mu_E)\) by zero outside of \(E\), as well as the linear map \(\rho_E : L_1(X, \mathcal{A}, \mu) \rightarrow L_1(E, \mathcal{A}_E, \mu_E)\) (restricting \(f \in L_1(X, \mathcal{A}, \mu)\) to \(E\)). Thus \((\rho_E \circ \beta_E)(f) = f\) for every \(f \in L_1(E, \mathcal{A}_E, \mu_E)\) and \((\beta_E \circ \rho_E)(f) = f\) for every \(f \in L_1(X, \mathcal{A}, \mu)\) such that \(E^c \cap \{f \neq 0\} \in \mathcal{N}_\mu, f \in \mathcal{A}^I\). Given \(\alpha \in L_1(X, \mathcal{A}, \mu)\) and \(E \in \mathcal{A}^I\) it follows that \(\alpha \circ \beta_E \in L_1(X, \mathcal{A}_E, \mu_E)\). Since \((E, \mathcal{A}_E, \mu_E)\) is a finite measure space the classical Riesz Representation Theorem yields an \(\mathcal{A}^E\)-measurable function \(g_E : E \rightarrow \mathbb{R}\) such that \((\alpha \circ \beta_E)(f) = \int_E g_E f d\mu_E\) for every \(f \in L_1(E, \mathcal{A}_E, \mu_E)\) and \(\sup |g_E| \leq \|\alpha\|\), where we shall now observe that the family \((g_E)_{E \in \mathcal{A}^I}\) is compatible. Let \(E_1, E_2 \in \mathcal{A}^I, n \in \mathbb{N}^+\) and define \(Z_n = E_1 \cap E_2 \cap \{g_{E_1} \leq n^{-1} + g_{E_2}\}\). Thus \(f_n = 1_{Z_n} \in L_1(X, \mathcal{A}, \mu)\) and

\[
\alpha(f_n) = (\alpha \circ \beta_{E_1} \circ \rho_{E_2})(f_n) = \int_{E_1} g_{E_1} f_n d\mu_{E_1} + \int_{E_2} g_{E_2} f_n d\mu = n^{-1} \mu(Z_n) + \int_{Z_n} g_{E_1} d\mu + \int_{E_2} g_{E_2} f_n d\mu = (\alpha \circ \beta_{E_1} \circ \rho_{E_2})(f_n) = \alpha(f_n).
\]

Therefore \(\mu(Z_n) = 0\), thus also \(E_1 \cap E_2 \cap \{g_{E_1} \leq g_{E_2}\} = \bigcup_{n \in \mathbb{N}} Z_n \in \mathcal{N}_\mu\), and in turn \(E_1 \cap E_2 \cap \{g_{E_1} \neq g_{E_2}\} \in \mathcal{N}_\mu \subseteq \mathcal{N}_\mu^{[\mathcal{A}^I]}\). Since \((X, \mathcal{A}, \mathcal{N}_\mu^{[\mathcal{A}^I]})\) is localizable it follows from \(\text{[3.13]}\) that \((g_E)_{E \in \mathcal{A}^I}\) admits a gluing \(\tilde{g} : X \rightarrow \mathbb{R}\). We let \(Z = X \cap \{|g| > \|\alpha\|\} \subseteq \mathcal{A}^I\). It ensues from our choice of a special representative \(g_E = g\) that \(E \cap Z \subseteq E \cap \{g \neq g_E\} \in \mathcal{N}_\mu^{[\mathcal{A}^I]}\) for every \(E \in \mathcal{A}^I\). Hence the function \(g = \tilde{g} \upharpoonright Z\) is also a gluing of \((g_E)_{E \in \mathcal{A}^I}\), and furthermore \(\sup |g| \leq \|\alpha\| < \infty\). Therefore \(g \in L_{\infty}(X, \mathcal{A}, \mu)\) and it remains to establish that \(T(g) = \alpha\).

Let \(f \in f \in L_1(X, \mathcal{A}, \mu)\), define \(A = \{f \neq 0\}\), and \(A_n = \{|f| > n^{-1}\}, n \in \mathbb{N}^+\). Thus \(A = \bigcup_{n \in \mathbb{N}} A_n\) and \(A_n \in \mathcal{A}^I\) for each \(n \in \mathbb{N}^+\). Letting \(f_n = f : A_n\) we notice that \((f_n)_{n \in \mathbb{N}^+}\) converges to \(f \in L_1(X, \mathcal{A}, \mu)\), whence \(\lim_{n} \alpha(f_n) = \alpha(f)\). We also notice that \(g f_n \rightarrow g f\) as \(n \rightarrow \infty\), everywhere, and that \(|g f_n| \leq |g f| \in L_1(X, \mathcal{A}, \mu)\), so that the Dominated
Convergence Theorem applies to \((g f_n)_{n \in \mathbb{N}}\). Accordingly,
\[
\lim_n \alpha(f_n) = \lim_n (\alpha \circ \beta_{A_n} \circ \rho_{A_n})(f_n)
\]
\[
= \lim_n \int_{A_n} g_{A_n} f_n d\mu_{A_n}
\]
\[
= \lim_n \int_X g f_n d\mu \quad \text{(because } A_n \cap \{g \neq g_{A_n}\} \in \mathcal{M} \text{ according to } 3.4\)
\]
\[
= \lim_n \int_X g f d\mu
\]
\[
= \int_X g f d\mu
\]
\[
= \Upsilon(g)(f).
\]

(2) \Rightarrow (1) Let \(\mathcal{E} \subseteq \mathcal{A}\). We ought to show that \(\mathcal{E}\) admits an \(\mathcal{M}[\mathcal{A}/f]\)-essential supremum in \(\mathcal{A}\). According to 3.15(5) there is no restriction to assume that \(\mathcal{E}\) is an ideal. We will define some \(\alpha \in L_1(X, \mathcal{A}, \mu)^*\) associated with \(\mathcal{E}\). We start by defining \(\alpha(f) \in \mathbb{R}_+\) associated with \(f \in L_1(X, \mathcal{A}, \mu), f \geq 0\), by the following formula:
\[
\alpha(f) = \sup_{E \in \mathcal{E}} \int_E f d\mu.
\]
We claim that the following hold:

(a) For every \(f \in L_1(X, \mathcal{A}, \mu)^+\) one has \(0 \leq \alpha(f) \leq \int_X f d\mu < \infty\);
(b) For every \(f_1, f_2 \in L_1(X, \mathcal{A}, \mu)^+\) one has \(\alpha(f_1 + f_2) = \alpha(f_1) + \alpha(f_2)\);
(c) For every \(f \in L_1(X, \mathcal{A}, \mu)^+\) and every \(t \geq 0\) one has \(\alpha(t f) = t \alpha(f)\);
(d) For every \(f_1, f_2, f_1', f_2' \in L_1(X, \mathcal{A}, \mu)^+\) if \(f_1 - f_2 = f_1' - f_2' \geq 0\) then \(\alpha(f_1) - \alpha(f_2) = \alpha(f_1') - \alpha(f_2')\).

Claims (a) and (c) are obvious. For proving (d) we notice that \(\alpha(f_1) + \alpha(f_2') = \alpha(f_1 + f_2') = \alpha(f_1') + \alpha(f_2)\), according to (b). Regarding (b) we first notice that \(\alpha(f_1 + f_2) \leq \alpha(f_1) + \alpha(f_2)\). Furthermore given \(\varepsilon > 0\) there are \(E_j \in \mathcal{E}, j = 1, 2\), such that \(\alpha(f_j) < \varepsilon + \int_{E_j} f_j d\mu\). Therefore
\[
\alpha(f_1) + \alpha(f_2) < 2\varepsilon + \int_{E_1} f_1 d\mu + \int_{E_2} f_2 d\mu < 2\varepsilon + \int_{E_1 \cup E_2} f_1 d\mu + \int_{E_2 \setminus E_1} f_2 d\mu
\]
\[
< 2\varepsilon + \int_{E_1 \cup E_2} (f_1 + f_2) d\mu < 2\varepsilon + \alpha(f_1 + f_2)
\]
because \(E_1 \cup E_2 \in \mathcal{E}\). Since \(\varepsilon > 0\) is arbitrary, claim (b) follows.

Now if \(f \in L_1(X, \mathcal{A}, \mu)\) we define \(\alpha(f) = \alpha(f^+) - \alpha(f^-) \in \mathbb{R}\) — a definition compatible with the previous one when \(f \geq 0\). It easily follows from (d) that \(\alpha\) is additive. Observing that \(\alpha(-f) = -\alpha(f)\) when \(f \geq 0\), it follows from (c) that \(\alpha\) is homogeneous of degree 1.

In other words \(\alpha\) is linear. Furthermore (a) implies that \(|\alpha(f)| \leq \int_X |f| d\mu\). It is now clear that \(\alpha(f) = \alpha(f), f \in L_1(X, \mathcal{A}, \mu)^*\) is well defined and that \(\alpha \in L_1(X, \mathcal{A}, \mu)^*\).

It ensues from the hypothesis that there exists \(g \in \mathfrak{g} \in L_1(X, \mathcal{A}, \mu)^+\) such that
\[
\int_X g f d\mu = \alpha(f) = \alpha(f) = \sup_{E \in \mathcal{E}} \int_E f d\mu
\]
for all \(0 \leq f \in L_1(X, \mathcal{A}, \mu)^+\). We define \(A = \{g \neq 0\} \in \mathcal{A}\) and we will next check that \(A\) is an \(\mathcal{M}[\mathcal{A}/f]\)-essential supremum of \(\mathcal{E}\).

Let \(E \in \mathcal{E}\). Define \(Z = E \setminus A = E \cap \{g = 0\}\). Given \(F \in \mathcal{A}/f\) let \(1_{F \cap Z} \in L_1(X, \mathcal{A}, \mu)^+\). Thus
\[
0 = \int_X 1_{F \cap Z} d\mu = \alpha(1_{F \cap Z}) \geq \int_E 1_{F \cap Z} d\mu = \mu(F \cap (E \setminus A)).
\]
Since \(F \in \mathcal{A}/f\) is arbitrary it follows that \(E \setminus A \in \mathcal{M}[\mathcal{A}/f]\).
We next claim that if \( F \in \mathcal{A}^f \) then \( F \cap \{ g < 0 \} \in \mathcal{N}_\mu \). Letting \( Z_n = \{ g \leq -n^{-1} \}, \ n \in \mathbb{N}^* \), we notice that \( \mathbb{I}_{F \cap Z_n} \in L_1(X, \mathcal{A}, \mu)^* \) whence:

\[
-n^{-1} \mu(F \cap Z_n) \geq \int_X g \mathbb{I}_{F \cap Z_n} \, d\mu = \alpha(\mathbb{I}_{F \cap Z_n}) \geq 0.
\]

Thus clearly \( \mu(F \cap Z_n) = 0 \) and, since \( n \in \mathbb{N}^* \) is arbitrary \( \mu(F \cap \{ g < 0 \}) = 0 \).

Finally we assume that \( B \in \mathcal{A}^f \) is such that \( \mu(F \cap (E \setminus B)) = 0 \) for every \( F \in \mathcal{A}^f \). We define \( Z = A \setminus B \in \mathcal{A} \). Let \( F \in \mathcal{A}^f \) and notice once again that \( \mathbb{I}_{F \cap Z} \in L_1(X, \mathcal{A}, \mu)^* \), therefore:

\[
\int_X g \mathbb{I}_{F \cap Z} \, d\mu = \alpha(\mathbb{I}_{F \cap Z}) = \sup_{E \in \mathcal{E}} \int_E \mathbb{I}_{F \cap Z} \, d\mu.
\]

Now given \( E \in \mathcal{E} \) we observe that \( E \cap F \cap Z = F \cap (E \cap (A \setminus B)) \subseteq F \cap (E \setminus B) \in \mathcal{N}_\mu \) by our assumption about \( B \). Since \( E \in \mathcal{E} \) is arbitrary we infer that:

\[
\int_X g \mathbb{I}_{F \cap Z} \, d\mu = 0.
\]

It follows from the previous paragraph and the definition of \( Z \) that \( g > 0, \mu \) almost everywhere on \( Z \cap F \). Consequently \( \mu(F \cap Z) = 0 \) and the proof is complete. □

4.7. — Let \( (X, \mathcal{A}, \mathcal{N}) \) be a measurable space with negligibles. Here we recall that \( \mathcal{L}_{\mathcal{A}^f}(X, \mathcal{A}, \mathcal{N}) \) is isometrically isomorphic to a dual Banach space then \( (X, \mathcal{A}, \mathcal{N}) \) is localizable – indeed in this case \( C(\text{Spec}(\mathcal{A}^f)) \) is isometrically isomorphic to a dual Banach space according to [3, 5] whence it is isometrically injective [4, 4.3.8(i)], and it remains to recall [3, 7]. However if \( (X, \mathcal{A}, \mathcal{N}) \) is localizable then \( \mathcal{L}_{\mathcal{A}^f}(X, \mathcal{A}, \mathcal{N}) \) does not need to be isometrically isomorphic to a dual Banach space in general (see [4] Problems 4.8 and 4.9) for an example due to R. Dixmier), yet below we show the conditions are equivalent in the class of measurable spaces with negligibles of the type \( (X, \mathcal{A}, \mathcal{N}_\mu[\mathcal{A}^f]) \) for some measure space \( (X, \mathcal{A}, \mu) \).

4.8. PROPOSITION. — Let \( (X, \mathcal{A}, \mu) \) be a measure space. The following are equivalent.

1. \( (X, \mathcal{A}, \mathcal{N}_\mu[\mathcal{A}^f]) \) is localizable;
2. \( \mathcal{L}_{\mathcal{A}^f}(X, \mathcal{A}, \mathcal{N}_\mu[\mathcal{A}^f]) \) is isometrically isomorphic to a dual Banach space.

In this case \( \mathcal{L}_{\mathcal{A}^f}(X, \mathcal{A}, \mathcal{N}_\mu[\mathcal{A}^f]) \) is isometrically isomorphic to \( L_1(X, \mathcal{A}, \mu)^* \).

Proof. That (2) \( \Rightarrow \) (1) follows from the general argument in [4, 7]. We henceforth assume that \( (X, \mathcal{A}, \mathcal{N}_\mu[\mathcal{A}^f]) \) is localizable and we let \( \mathcal{L}_{\mathcal{A}^f}(X, \mathcal{A}, \mu) \) denote the linear space consisting of those \textit{bounded}, \( \mathcal{A} \) measurable functions \( g : X \to \mathbb{R} \). In the exact same way as in [4, 7] we define a linear map:

\[
\hat{T} : \mathcal{L}_{\mathcal{A}^f}(X, \mathcal{A}, \mu) \to \mathcal{L}_1(X, \mathcal{A}, \mu)^*.
\]

We claim that \( \ker \hat{T} \) consists of those \( g \in \mathcal{L}_{\mathcal{A}^f}(X, \mathcal{A}, \mu) \) such that \( S_g = \{ g \neq 0 \} \notin \mathcal{N}_\mu[\mathcal{A}^f] \). If \( g \) has this property and \( f \in L_1(X, \mathcal{A}, \mu) \) then \( \{ f \neq 0 \} = \bigcup_{n \in \mathbb{N}} \{ n \leq |f| \} \) and since each \( \{ n \leq |f| \} \in \mathcal{A}^f \) it follows that \( \{ g f \neq 0 \} = \{ g \neq 0 \} \cap \{ f \neq 0 \} \in \mathcal{N}_\mu \) and in turn \( \hat{T}(g)(f) = \int_X g f \, d\mu = 0 \). The other way around we let \( g \in \ker \hat{T} \) and we define \( S^g = \{ g > 0 \} \) so that \( g \notin \mathcal{N}_\mu[\mathcal{A}^f] \). Given \( A \in \mathcal{A}^f \) and letting \( f = \mathbb{I}_A \cap S^g \) we infer that \( 0 = \hat{T}(g)(f) = \int_A g^* \, d\mu \) thus \( S^g \cap A \in \mathcal{N}_\mu \). Thus \( S^g \in \mathcal{N}_\mu[\mathcal{A}^f] \), and similarly \( \mathcal{N}_\mu[\mathcal{A}^f] \).

Since \( \mathcal{L}_1(X, \mathcal{A}, \mu)^* \) is linearly isomorphic to \( \mathcal{L}_{\mathcal{A}^f}(X, \mathcal{A}, \mu)/\ker \hat{T} \), the claim being established we now easily infer that \( \mathcal{L}_1(X, \mathcal{A}, \mu)^* \) is linearly isomorphic to \( \mathcal{L}_{\mathcal{A}^f}(X, \mathcal{A}, \mathcal{N}_\mu[\mathcal{A}^f]) \). It remains to show that the corresponding linear isomorphism associated with \( \hat{T} \) is an isometry. We leave the details to the reader. □
5. Almost Decomposable Measure Spaces

In this Section we state basic facts on the notion of almost decomposable measure space introduced in [9]. It is an appropriate generalization to non semifinite measure spaces of the notion of decomposable measure space (also called strictly localizable measure space). I learned [5,4] from [12] 2.5.10 (in a different language than here). I learned the idea in [5,5] from D.H. Fremlin.

5.1. — Let \((X, \mathcal{A}, \mu)\) be a measure space. An almost decomposition of \((X, \mathcal{A}, \mu)\) is a family \(\mathcal{A} \subseteq \mathcal{A} \) with the following properties:

1. \(\forall G \in \mathcal{A} : \mu(G) < \infty;\)
2. \(\mathcal{A}\) is disjointed;
3. \(\forall A \in \mathcal{A}(X) : (\forall G \in \mathcal{A} : A \cap G \in \mathcal{A} ) \Rightarrow A \in \mathcal{A};\)
4. \(\forall A \in \mathcal{A}: \mu(A) < \infty \Rightarrow \mu(A) = \sum_{G \in \mathcal{A}} \mu(A \cap G).\)

We say that \((X, \mathcal{A}, \mu)\) is almost decomposable if it admits an almost decomposition.

5.2. — Almost decomposable measure spaces generalize \(\sigma\)-finite measure spaces. In fact, assuming that \((X, \mathcal{A}, \mu)\) is semifinite, if \(\mathcal{A}\) is an almost decomposition of \((X, \mathcal{A}, \mu)\) and \(\mathcal{A}\) is (at most) countable then \(\mu\) is \(\sigma\)-finite. Indeed \(S = \cup_{G \in \mathcal{A}} G\) (either because \(\mathcal{A}\) is countable or according to 5.1(3)), and we ought to show that \(\mu(X \setminus S) = 0.\) If \(A \in \mathcal{A}, \mu(A) < \infty\) and \(A \subseteq X \setminus S\) then \(\mu(A) = 0\) according to 5.1(4). Since \(\mu\) is semifinite this implies \(\mu(Z \setminus S) = 0.\)

5.3. Proposition. — If a measure space admits an almost decomposition then it is semifinalizable.

Proof. Let \(\mathcal{A}\) be an almost decomposition of the measure space \((X, \mathcal{A}, \mu)\) and let \(\mathcal{E} \subseteq \mathcal{A}\) be an arbitrary family. With each \(G \in \mathcal{A}\) we associate \(\mathcal{E}_G = \{ G \cap E : E \in \mathcal{E}\} \subseteq \mathcal{A}_G.\) Since \((G, \mathcal{A}_G, \mu_G)\) is a finite measure space, \((G, \mathcal{A}_G, \mathcal{N}_{\mu_G})\) is localizable according to 3.8 and we let \(A_G \in \mathcal{A}_G \subseteq \mathcal{A}\) be an \(\mathcal{N}_{\mu_G}\) essential supremum of \(\mathcal{E}_G.\) We now define a subset of \(X\)

\[ A = \cup_{G \in \mathcal{A}} A_G. \]

Since \(A \cap G = A_G \in \mathcal{A}\) for every \(G \in \mathcal{A}\) it follows from condition (3) of the definition of an almost decomposition that \(A \in \mathcal{A}.\)

We shall now show that \(E \setminus A \in \mathcal{N}_{\mu}[\mathcal{A}]\) for every \(E \in \mathcal{E}.\) Let \(E \in \mathcal{E}\) and \(F \in \mathcal{A}.\) It follows that

\[ \mu[F \cap (E \setminus A)] = \sum_{G \in \mathcal{A}} \mu[F \cap G \cap (E \setminus A)] \quad \text{(by 5.1(4))} \]
\[ \leq \sum_{G \in \mathcal{A}} \mu[(E \cap G) \setminus A_G] \]
\[ = 0. \]

Finally we assume that \(B \in \mathcal{A}\) is such that \(E \setminus B \in \mathcal{N}_{\mu}[\mathcal{A}]\) for all \(E \in \mathcal{E}\) and we ought to show that \(A \setminus B \in \mathcal{N}_{\mu}[\mathcal{A}].\) Let \(F \in \mathcal{A}.\) We must show that \(\mu[F \cap (A \setminus B)] = 0.\) Notice that

\[ \mu[F \cap (A \setminus B)] = \sum_{G \in \mathcal{A}} \mu[F \cap G \cap (A \setminus B)] \quad \text{(by 5.1(4))} \]
\[ \leq \sum_{G \in \mathcal{A}} \mu[G \cap (A \setminus B)]. \]

Thus it suffices to establish that \(\mu[G \cap (A \setminus B)] = 0\) for each \(G \in \mathcal{A}.\) Fix \(G \in \mathcal{A}\) and let \(B_G = B \cap G.\) Thus \(\mu[(E \cap G) \setminus B_G] = \mu[B_G \setminus B] = 0\) for every \(E \in \mathcal{E}.\) Therefore \(\mu(A_G \setminus B_G) = 0.\) Since \(A_G \setminus B_G = (A \cap G) \setminus (B \cap G) = G \cap (A \setminus B)\) the proof is complete. □
5.4. Proposition (ZFC + CH). — Let X be a Polish space and let φ be a Borel regular outer measure on X (recall [2.7]). Assuming the Continuum Hypothesis, the measure space \((X, \mathcal{A}, \phi)\) admits an almost decomposition.

Proof. In case \(X\) is finite the conclusion clearly holds. We henceforth assume \(X\) is infinite. In that case \(\text{card } \mathcal{P}(X) = 2^{\aleph_0}\) (the upper bound follows from the fact that Borel sets are Suslin, and Suslin sets are continuous images of closed subsets of a particular Polish space, the Baire space, see e.g. [28, 3.3.18]). We abbreviate \(\mathcal{B} = \mathcal{P}(X)\) and as usual \(\mathcal{B}^f = \mathcal{B} \cap \{B : \phi(B) < \infty\}\). It now follows from the Continuum Hypothesis that \(\mathcal{B}^f\) admits a well-ordering \(\leq\) such that every initial segment \(\mathcal{B}^f_c = \mathcal{B}^f \cap \{C : C \leq B \text{ and } C \neq B\}, B \in \mathcal{B}^f\), is at most countable and therefore \(\bigcup \mathcal{B}^f_c \in \mathcal{B}\). With each \(B \in \mathcal{B}^f\) we associate the Borel set \(G_B = B \setminus \bigcup \mathcal{B}^f_c\). We claim that \(\mathcal{G} = \{G_B : B \in \mathcal{B}^f\}\) is an almost decomposition of \((X, \mathcal{A}, \phi)\). Conditions (1) and (2) of 5.1 are readily satisfied.

In order to check that 5.1(3) holds, we let \(A \subseteq X\) be such that \(A \cap G\) is \(\phi\)-measurable for each \(G \in \mathcal{G}\) and we ought to show that \(A\) is \(\phi\)-measurable. Let \(S \subseteq X\) be arbitrary. We must establish that

\[\phi(S) \geq \phi(S \cap A) + \phi(S \setminus A)\]

Clearly we may assume that \(\phi(S) < \infty\). We choose \(\mathcal{B}^f \ni B \supseteq S\) with \(\phi(S) = \phi(B)\) and we number \(G_0, G_1, G_2, \ldots\), the sets \(G_C\) corresponding to \(C \in \mathcal{B}^f\) with \(C \leq B\) and \(C \neq B\). Thus \((G_n)_{n \in \mathbb{N}}\) is a disjointed sequence of Borel sets whose union is \(B\). In turn \(B \cap A = \bigcup_{n \in \mathbb{N}} (G_n \cap A)\) is \(\phi\)-measurable according to our hypothesis about \(A\). Therefore \(B \setminus A = B \cap (B^c \cup A^c) = B \cap (B \cap A)^c\) is also \(\phi\)-measurable, whence

\[\phi(S) = \phi(B) = \phi(B \cap A) + \phi(B \setminus A) \geq \phi(S \cap A) + \phi(S \setminus A)\]

and the proof of (3) is complete.

We turn to proving that condition [5.14] holds. Let \(A \subseteq X\) be \(\phi\)-measurable and such that \(\phi(A) < \infty\). Owing to the Borel regularity of \(\phi\) there exists \(\mathcal{B}^f \ni B \supseteq A\) such that \(\phi(A) = \phi(B)\). Associate \((G_n)_{n \in \mathbb{N}}\) with \(B\) as above. It follows that \(A = \bigcup_{n \in \mathbb{N}} A \cap G_n\) and of course each \(A \cap G_n\) is \(\phi\)-measurable. Therefore

\[\phi(A) = \sum_{n \in \mathbb{N}} \phi(A \cap G_n)\]

Furthermore if \(C \in \mathcal{B}^f, C \neq B\) and \(B \leq C\) then \(A \cap G_C \subseteq B \cap G_C = \emptyset\) and therefore \(\phi(A \cap G_C) = 0\). Consequently

\[\phi(A) = \sum_{G \in \mathcal{G}} \phi(A \cap G)\]

\[\square\]

5.5. Proposition. — Assume X is a Polish space and \(\mu\) a Borel measure in \(X\). If the measure space \((X, \mathcal{H}(X), \mu)\) is semifinite and almost decomposable then it is \(\sigma\)-finite.

Proof. Assume if possible that \((X, \mathcal{H}(X), \mu)\) is semifinite and almost decomposable but not \(\sigma\)-finite. Let \(\mathcal{G}\) be an almost decomposition of \(\mathcal{G}\) it would ensue from [5.2] that \(\mathcal{G}\) is uncountable. Let \(\kappa = \text{card } \mathcal{G}\). It follows from the axiom of choice that there exists \(A \subseteq X\) such that \(A \cap G\) is a singleton for each \(G \in \mathcal{G}\). Thus card \(A = \kappa\), and \(A \in \mathcal{H}(X)\) according to [5.13]. Now \(A\) being an uncountable Suslin subset of a Polish space, \(\kappa = \text{card } A = \kappa\), see e.g. [28, 4.3.5]. Furthermore if \(B \in \mathcal{H}(A)\) then for each \(G \in \mathcal{G}\) the set \(B \cap G\) is either empty or a singleton, therefore \(B \in \mathcal{H}(X)\) as follows from [5.13]. Consequently \(2^\kappa = 2^\kappa = \text{card } \mathcal{H}(A) \leq \text{card } \mathcal{H}(X) = \kappa\) (where the last equality was already recalled at the beginning of the proof of [5.4]), in contradiction with G. Cantor’s Theorem that \(\kappa < 2^\kappa\). \[\square\]

5.6. Corollary. — Let \(X\) be an uncountable separable complete metric space, and \(0 < d < \infty\). It follows that either the measure space \((X, \mathcal{H}(X), \mathcal{H}^d)\) is \(\sigma\)-finite or it is not almost decomposable.
Indeed 5.5 applies because $(X, \mathcal{B}(X), \mathcal{H}^d)$ is semifinite according to J. Howroyd’s Theorem [23] or [17, 471S].

6. Locally Determined Measure Spaces of Magnitude less than Continuum

Almost decomposable measure spaces are semilocalizable, but the converse does not hold. A classical counter-example (in case of semifinite measure spaces) is due to D.H. Fremlin [14, 216E]. However if the corresponding quotient Boolean algebra is “not too large” the converse holds. In case of semifinite measure spaces this is due to E. J. McShane [24]. Here we deal with the non semifinite case, 6.5.

6.1. — A measure space $(X, \mathcal{A}, \mu)$ is called locally determined whenever the following holds:

$$\forall A \in \mathcal{P}(X) : \left[ \forall F \in \mathcal{A}^f : A \cap F \in \mathcal{A} \right] \Rightarrow A \in \mathcal{A}$$

where as usual

$$\mathcal{A}^f = \mathcal{A} \cap \{ A : \mu(A) < \infty \}.$$ 

6.2. Lemma. — Let $\phi$ be an outer measure on a set $X$ and assume that $\phi$ has measurable hulls, i.e.

$$(\forall S \in \mathcal{P}(X)) \left( \exists A \in \mathcal{A}_\phi : S \subseteq A \text{ and } \phi(S) = \phi(A) \right).$$

It follows that $(X, \mathcal{A}_\phi, \phi)$ is locally determined.

Proof. Let $A \in \mathcal{P}(X)$ and assume that $A \cap F$ is $\phi$ measurable whenever $F$ is $\phi$ measurable and $\phi(F) < \infty$. We ought to show that $A$ is $\phi$ measurable. It suffices to establish that

$$\phi(S) \geq \phi(S \cap A) + \phi(S \setminus A)$$

whenever $S \in \mathcal{P}(X)$ and $\phi(S) < \infty$. Let $B \in \mathcal{A}_\phi$ be a $\phi$ measurable hull of $S$. Thus $B \in \mathcal{A}^f$ so that $A \cap B \in \mathcal{A}_\phi$ by assumption, and hence also $B \setminus A = B \setminus (A \cap B) \in \mathcal{A}_\phi$. Therefore

$$\phi(S) = \phi(B) = \phi(B \cap A) + \phi(B \setminus A) \geq \phi(S \cap A) + \phi(S \setminus A)$$

and the proof is complete. □

6.3. Proposition. — Assume that $X$ is a Polish space and that $\phi$ is a Borel regular outer measure on $X$ (recall 2.7). It follows that $(X, \mathcal{A}_\phi, \phi)$ has magnitude (recall 3.18) less than $\epsilon$ (the power of continuum).

Proof. Let $\mathcal{E} \subseteq \mathcal{A}^f$ be as in 3.18. With each $E \in \mathcal{E}$ we associate a Borel hull $B_E \in \mathcal{B}(X)$ such that $E \subseteq B_E$ and $\phi(E) = \phi(B_E)$. Since $\phi(B_E) < \infty$ and both $E$ and $B_E$ are $\phi$ measurable, we infer that $\phi(B_E \setminus E) = 0$. We now claim that if $E_1, E_2 \in \mathcal{E}$ and $E_1 \neq E_2$ then $B_{E_1} \neq B_{E_2}$. Indeed, assuming $E_1 \neq E_2$, we see that

$$B_{E_1} \cap B_{E_2} \subseteq (B_{E_1} \setminus E_1) \cup (E_1 \cap B_{E_2}) \subseteq (B_{E_1} \setminus E_1) \cup (E_1 \cap E_2) \cup (B_{E_2} \setminus E_2)$$

is $\phi$ negligible. Assuming if possible that $B_{E_1} = B_{E_2}$, it would ensue that $B_{E_1}$ is $\phi$ negligible, whence also $E_1$, a contradiction. In other words the map $\mathcal{E} \rightarrow \mathcal{B}(X) : E \mapsto B_E$ is injective. Since card $\mathcal{B}(X) \leq \epsilon$ the proof is complete. □

6.4. Proposition. — Let $(X, \mathcal{A}, \mu)$ be a measure space which is complete and locally determined, and let $\mathcal{E}$ be such that

1. $\mathcal{E} \subseteq \mathcal{A}^f$;
2. $\mathcal{E}$ is disjoint;
3. $(\forall A \in \mathcal{A}^f \setminus \mathcal{N}_\mu)(\exists E \in \mathcal{E}) : A \cap E \notin \mathcal{N}_\mu$.

It follows that $\mathcal{E}$ is an almost decomposition of $(X, \mathcal{A}, \mu)$. 

6.5. Proposition. — According to 3.17 applied with \( F \) each \( A \) at most countable, the first term in the union of the right hand side above belongs to \( F \).

We first claim that \( \mathcal{E}_A \) is at most countable. Indeed if \( \mathcal{F} \subseteq \mathcal{E}_A \) is finite then

\[
\sum_{E \in \mathcal{F}} \mu(E \cap A) = \mu((\cup \mathcal{F}) \cap A) \leq \mu(A)
\]

because \( \mathcal{F} \) is disjointed. Letting \( \mathcal{E}_{A,n} = \mathcal{E}_A \cap \{ E : \mu(E \cap A) \geq n^{-1} \} \) we infer that \( \mathcal{E}_A = \bigcup_{n=1}^{\infty} \mathcal{E}_{A,n} \) and \( \text{card } \mathcal{F} \leq n \mu(A) < \infty \). This completes the proof of the claim.

Define

\[ B = \bigcup_{E \in \mathcal{E}_A} E \cap A \]

and notice that \( B \in \mathcal{A} \) because \( \mathcal{E}_A \) is at most countable. Let \( C = A \setminus B \). Assume if possible that \( \mu(C) > 0 \). Since \( \mu(C) \leq \mu(A) < \infty \) it follows from hypothesis (3) that the exists \( E \in \mathcal{E} \) such that \( \mu(E \cap C) > 0 \). Thus \( C \in \mathcal{E}_A \) and in turn \( E \cap C \subseteq B \) so that \( E \cap C = E \cap (C \cap B) = \emptyset \) by the definition of \( B \), a contradiction. Thus indeed \( \mu(C) = 0 \). Finally

\[
\mu(A) = \mu(B) = \sum_{E \in \mathcal{E}_A} \mu(E \cap A) = \sum_{E \in \mathcal{E}} \mu(E \cap A).
\]

It remains to establish that condition (3) of [5.1] holds. Since \( (X, \mathcal{A}, \mu) \) is locally determined it suffices to show the following: If \( A \in \mathcal{P}(X) \) and \( A \cap E \in \mathcal{A} \) for every \( E \in \mathcal{E} \), then \( A \cap F \) for every \( F \in \mathcal{A}^f \). Fix \( A \in \mathcal{P}(X) \) that meets this condition. Let \( F \in \mathcal{A}^f \). We apply the preceding paragraph to \( F \):

\[ F = \left( \bigcup_{E \in \mathcal{E}_F} E \cap F \right) \cup N \]

for some \( N \in \mathcal{N}_\mu \). Therefore

\[ A \cap F = \left( \bigcup_{E \in \mathcal{E}_F} ((A \cap E) \cap F) \right) \cup (A \cap N). \]

Now each \( A \cap E \in \mathcal{A} \) by assumption, thus also \( A \cap E \cap F \in \mathcal{A} \), \( E \in \mathcal{E} \). Since \( \mathcal{E}_F \) is at most countable, the first term in the union of the right hand side above belongs to \( \mathcal{A} \). Furthermore \( A \cap N \in \mathcal{N}_\mu \subseteq \mathcal{A} \) because \( (X, \mathcal{A}, \mu) \) is complete. Therefore \( A \cap F \in \mathcal{A} \). Since \( F \in \mathcal{A}^f \) is arbitrary we are done. \( \Box \)

6.5. Proposition. — Assume that a measure space \( (X, \mathcal{A}, \mu) \):

(1) is complete;
(2) is locally determined;
(3) has magnitude less than \( \epsilon \);
(4) is semilocalizable.

It follows that it is almost decomposable.

Proof. According to [3.1] applied with \( \mathcal{A} = \mathcal{A}^f \) and \( \mathcal{N} = \mathcal{N}_\mu \), there exists \( \mathcal{E} \subseteq \mathcal{A} \) such that

(a) \( \mathcal{E} \cap \mathcal{N}_\mu = \emptyset \);
(b) For every \( E_1, E_2 \in \mathcal{E} \), if \( E_1 \neq E_2 \) then \( E_1 \cap E_2 \in \mathcal{N}_\mu \);
(c) For every \( A \in \mathcal{A}^f \setminus \mathcal{N}_\mu \) there exists \( E \in \mathcal{E} \) such that \( A \cap E \notin \mathcal{N}_\mu \).

By hypothesis (3), \( \text{card } \mathcal{E} \leq \epsilon \) thus there exists an injective map \( u : \mathcal{E} \to [0,1] \). With each \( E \in \mathcal{E} \) we associate \( g_E = u(E)1_E \) which is \( \mathcal{A} \) measurable, thus \( \{g_E\}_{E \in \mathcal{E}} \) is a family subordinated to \( \mathcal{E} \). It is compatible relative to \( (X, \mathcal{A}, \mathcal{N}_\mu[\mathcal{A}^f]) \) because if \( E_1, E_2 \in \mathcal{E} \) and \( E_1 \cap E_2 \in \mathcal{N}_\mu \), then

\[ E_1 \cap E_2 \cap \{g_{E_1} \neq g_{E_2}\} \subseteq E_1 \cap E_2 \subseteq \mathcal{N}_\mu \subseteq \mathcal{N}_\mu[\mathcal{A}^f]. \]

By hypothesis (4) it therefore admits a gluing \( g \), i.e. an \( \mathcal{A} \) measurable function \( g : X \to \mathbb{R} \) such that for every \( E \in \mathcal{E} \),

\[ E \cap \{g \neq g_E\} \in \mathcal{N}_\mu[\mathcal{A}^f]. \]
Now for $E \in \mathcal{E}$ we define

$$G_E = E \cap g^{-1}(u(E)) \in \mathcal{A}.$$ 

If $E_1 \neq E_2$ then $G_{E_1} \cap G_{E_2} \subseteq g^{-1}(u(E_1)) \cap g^{-1}(u(E_2)) = \emptyset$ because $u$ is injective. Accordingly, $\mathcal{G} = \{G_E : E \in \mathcal{E}\}$ is disjointed. Also, $\mu(G_E) \leq \mu(E) < \infty$ when $E \in \mathcal{E}$, thus $\mathcal{G} \subseteq \mathcal{A}$.

In order to establish that $\mathcal{G}$ verifies condition (3) of 6.4, assume to the contrary that there exists $A \in \mathcal{A} \setminus \mathcal{N}_\mu$ such that $A \cap G_E \in \mathcal{N}_\mu$ for all $E \in \mathcal{E}$. Given $E \in \mathcal{E}$ we shall now show that $A \cap E \in \mathcal{N}_\mu$, in contradiction with (c) above, thereby completing the proof. Since $A \cap E = (A \cap G_E) \cup (A \cap (E \setminus G_E))$ and $A \cap G_E \in \mathcal{N}_\mu$, it suffices to show that $A \cap (E \setminus G_E) \in \mathcal{N}_\mu$. Recall that $E \setminus G_E = E \setminus \{g \neq g_E \} \in \mathcal{N}[(\mathcal{A})]$.

Since $A \in \mathcal{A}$ we infer indeed that $A \cap (E \setminus G_E) \in \mathcal{N}_\mu$.

\[\square\]

6.6. Corollary. — Assume that $(X, \mathcal{A}, \mu)$ is a complete, locally determined measure space and has magnitude less than $\epsilon$. The following are equivalent.

1. $\mathcal{T}$ is surjective;
2. $(X, \mathcal{A}, \mu)$ is semilocalizable;
3. $(X, \mathcal{A}, \mu)$ admits an almost decomposition.

7. Hausdorff Measures in Complete Separable Metric Spaces

7.1. Theorem. — Let $X$ be a complete separable metric space and $0 < d < \infty$. For the measure space $(X, \mathcal{A}_{\mathcal{F}d}, \mathcal{H}^d)$ the following are equivalent:

1. The canonical map $\mathcal{T} : L_0(X, \mathcal{A}_{\mathcal{F}d}, \mathcal{H}^d) \to L_1(X, \mathcal{A}_{\mathcal{F}d}, \mathcal{H}^d)$ is surjective;
2. $(X, \mathcal{A}_{\mathcal{F}d}, \mathcal{H}^d)$ is semilocalizable;
3. $(X, \mathcal{A}_{\mathcal{F}d}, \mathcal{H}^d)$ is almost decomposable.

Proof. That (1) and (2) be equivalent for any measure space was established in 4.6 and that (3) $\Rightarrow$ (2) for any measure space was proved in 5.3. It remains to observe that (2) $\Rightarrow$ (3) under the present assumptions is a consequence of 6.5. The measure space $(X, \mathcal{A}_{\mathcal{F}d}, \mathcal{H}^d)$ is clearly complete. The outer measure $\mathcal{H}^d$ being Borel regular, it follows from 6.2 that $(X, \mathcal{A}_{\mathcal{F}d}, \mathcal{H}^d)$ is locally determined, and from 6.3 and the separability assumption of $X$ that it has magnitude less than $\epsilon$.

\[\square\]

8. An Abstract Condition for the Consistency of not being Semilocalizable

8.1 (Cardinals related to $\sigma$-ideals). — Let $X$ be a set and let $\mathcal{N} \subseteq \mathcal{P}(X)$ be a $\sigma$-ideal in $\mathcal{P}(X)$. We recall the following cardinals associated with $\mathcal{N}$:

$$\non(\mathcal{N}) = \min\{\text{card } S : S \subseteq X \text{ and } S \notin \mathcal{N}\} \quad \text{and} \quad \cov(\mathcal{N}) = \min\{\text{card } C : C \subseteq \mathcal{N} \text{ and } X = \cup C\}.$$ 

Letting $\mathcal{L}$ denote the restriction of the Lebesgue outer measure to the interval $[0, 1]$ we consider the corresponding cardinals $\non(\mathcal{N}_{\mathcal{L}})$ and $\cov(\mathcal{N}_{\mathcal{L}})$. These are part of the so-called Cichoń diagram, see [18, 522]. Below we will use the fact that the strict inequality $\non(\mathcal{N}_{\mathcal{L}}) < \cov(\mathcal{N}_{\mathcal{L}})$ is consistent with ZFC, see [6, Chapter 7] or [19, 552H and 552G].

8.2. Lemma. — Let $X$ be a Polish space and let $\mu$ be a diffuse probability measure defined on $\mathcal{B}(X)$. It follows that:

1. $\non(\mathcal{N}_\mu) = \non(\mathcal{N}_{\mathcal{L}})$;
2. $\cov(\mathcal{N}_\mu) = \cov(\mathcal{N}_{\mathcal{L}})$.

Proof. Since $\mu$ is diffuse and nonzero, $X$ is uncountable and therefore the Kuratowski Isomorphism Theorem applies. [28, 3.4.23]: There exists a bijection $f : X \to [0, 1]$ such that both $f$ and $f^{-1}$ are Borel measurable, and $f_\mu = \lambda$ where $\lambda = \mathcal{L}_{\mathcal{L}[0,1]}$. We claim that:

For every $S \subseteq X : \mu(S) = 0$ if and only if $\mathcal{L}(f(S)) = 0$. 

Assume that \( \mu(S) = 0 \). Since \( \mu \) is Borel regular, \[ \exists \mathcal{A} \ni B \supseteq S \text{ such that } \mu(B) = 0. \] As \( f(B) \) is Borel one has \( \mathcal{L}^1(f(B)) = \lambda(f(B)) = (f,\mu)(f(B)) = \mu(B) = 0 \) and therefore \( \mathcal{L}^1(f(S)) = 0 \) because \( f(S) \subseteq f(B) \). The other way round one argues similarly, referring to the Borel regularity of \( \mathcal{L}^1 \).

We now prove (1). Assume \( S \subseteq [0,1] \) and \( S \notin \mathcal{N}_f \), i.e. \( \mathcal{L}^1(S) > 0 \). It follows from the claim above claim that \( f^{-1}(S) \notin \mathcal{N}_f \). Since card \( S = \text{card} \ f^{-1}(S) \) we infer that \( \text{non}(\mathcal{N}_f) \leq \text{non}(\mathcal{N}_f) \). The reverse inequality is proved in a similar fashion.

Let \((N_i)_{i \in I} \subseteq \mathcal{N}_f \) be such that \([0,1] = \cup_{i \in I} N_i \). It follows from the claim above that \((f^{-1}(N_i))_{i \in I} \subseteq \mathcal{N}_f \), and clearly \( X = \cup_{i \in I} f^{-1}(N_i) \), thus \( \text{cov}(\mathcal{N}_f) \leq \text{cov}(\mathcal{N}_f) \). The reverse inequality follows similarly and the proof of (2) is complete.

8.3. THEOREM (ZFC + \( \text{non} (\mathcal{N}_f) < \text{cov}(\mathcal{N}_f) \)). Assume that \((X, \mathcal{A}, \mathcal{N}) \) is a measurable space with negligibles and that \((S, \mathcal{B}(S), \sigma), (T, \mathcal{B}(T), \tau) \) are probability spaces with \( S \) and \( T \) being Polish spaces. Furthermore assume the existence of maps \( S \to \mathcal{A} : s \mapsto V_s \) and \( T \to \mathcal{A} : t \mapsto H_t \) with the following properties.

(1) For every \( s \in S \) and every \( t \in T \) one has \( \emptyset \neq V_s \cap H_t \in \mathcal{N} \);
(2) For every \( Z \in \mathcal{A} \) one has:
(a) For every \( s \in S \) if \( V_s \cap Z \in \mathcal{N} \) then
\[ \tau(T \cap \{ t : H_t \cap V_s \cap Z \neq \emptyset \}) = 0; \]
(b) For every \( t \in T \) if \( H_t \cap Z \in \mathcal{N} \) then
\[ \tau(S \cap \{ s : V_s \cap H_t \cap Z \neq \emptyset \}) = 0. \]

Under the consistent assumption that \( \text{non} (\mathcal{N}_f) < \text{cov}(\mathcal{N}_f) \) it follows that \((X, \mathcal{A}, \mathcal{N}) \) is not localizable.

Proof. Proceeding toward a contradiction, assume if possible that \((X, \mathcal{A}, \mathcal{N}) \) is localizable. The family \( \mathcal{E} = \{ V_s : s \in S \} \subseteq \mathcal{A} \) would then admit an \( \mathcal{N} \) essential supremum, say \( A \in \mathcal{A} \). Thus,

(A) For every \( s \in S : V_s \setminus A \in \mathcal{N} \);
(B) For every \( t \in T : H_t \cap A \in \mathcal{N} \).

Condition (A) readily follows from the definition of an essential supremum whereas condition (B) is established in the following manner. Fix \( t \in T \) and consider the set \( B = A \setminus H_t \) and observe that for every \( s \in S \) one has \( V_s \setminus B = (V_s \setminus A) \cup (V_s \cap H_t) \in \mathcal{N} \) since the first term is a member of \( \mathcal{N} \) by (A), and the second by hypothesis (1). As \( s \) is arbitrary, \( B \) is an \( \mathcal{N} \) essential upper bound of \( \mathcal{E} \). Therefore \( A \setminus B = A \cap H_t \).

With each \( s \in S \) we now associate the set
\[ T_s = T \cap \{ t : H_t \cap V_s \setminus A^c \neq \emptyset \}. \]
Since \( V_s \setminus A^c \in \mathcal{N} \) by (A), our hypothesis (2)(a) implies that \( \tau(T_s) = 0 \). Now let \( E \subseteq S \) be such that \( \tau(E) > 0 \) and
\[ \text{card} \ E = \text{non} (\mathcal{N}_f) = \text{non} (\mathcal{N}_f) < \text{cov}(\mathcal{N}_f) = \text{cov}(\mathcal{N}_f), \]
where the second and last equalities follow from (2) and the strict inequality is our consistent assumption. We next define \( F = \cup_{s \in E} T_s \subseteq T \). Since each \( T_s \in \mathcal{N}_f \) and card \( E < \text{cov}(\mathcal{N}_f) \) it ensues that \( F \neq T \). Pick \( t \in T \setminus F \). Thus for each \( s \in E \) one has \( t \notin T_s \), i.e. \( H_t \cap V_s \cap A^c = \emptyset \) and in turn (since \( V_s \cap H_t \neq \emptyset \) by assumption (1)) \( H_t \cap V_s \neq \emptyset \). Accordingly,
\[ E \subseteq S \cap \{ s : V_s \cap H_t \cap A \neq \emptyset \}. \]
Yet \( H_t \cap A \in \mathcal{N} \) for every \( t \in T \) by (B), thus \( \tau(E) = 0 \) by assumption (2)(b), a contradiction.
8.4. — One checks (in a similar way as in 8.3 below) that 8.3 applies to the measurable space with negligibles $(X, \mathcal{A}, \mathcal{N})$ where $X = [0, 1] \times [0, 1]$. $\mathcal{A}$ is a $\sigma$-algebra of subsets of $X$ containing the Borel subsets of $X$, and $\mathcal{N}$ consists of those members of $\mathcal{A}$ that are $\mathcal{H}^d$-negligible. The hypotheses are met with both $(S, \mathcal{B}(S), \sigma)$ and $(T, \mathcal{B}(T), \tau)$ being $([0, 1], \mathcal{B}([0, 1]), \mathcal{L}^d|_{\mathcal{B}([0, 1])})$ and $V_s = \{s\} \times [0, 1]$, $H_s = [0, 1] \times \{t\}$, $s, t \in [0, 1]$. Thus $V_s \cap H_s = \{(s, t)\}$ and the condition $V_s \cap H_s \cap Z \neq \emptyset$ is equivalent to $(s, t) \in Z$, and therefore (2)(a) (resp. (2)(b)) of 8.3 holds since the corresponding slice of $Z$ is assumed to be $\mathcal{H}^1$-negligible, and the projection on the second (resp. first) axis contracts $\mathcal{H}^1$ measure. The case when $\mathcal{A}$ consists of those $\mathcal{H}^1$ measurable subsets of $X$ was proved in [9]. The notion of a measurable space with negligibles makes it a possibility to dispense altogether with a measure being defined on $\mathcal{A}$ and, consequently allows for our slightly more general statement here. The point being that the nature of the statement does not involve a measure. See also [11.2(5)]

8.5. THEOREM. — Let $0 < d < 1$ and let $C_d \subseteq [0, 1]$ be a self-similar Cantor set of Hausdorff dimension $d$ described in [23] 4.10. The measure space $(C_d \times C_d, \mathcal{A}_{|[d_0]}$, $\mathcal{H}^d)$ is consistently not semilocalizable.

Proof. We let $X = C_d \times C_d$, we let $\phi$ be the Hausdorff $\mathcal{H}^d$ measure restricted to $X$ and $\mathcal{A} = \mathcal{A}_{|[d_0]}$. We also put $\mathcal{N} = N_{d_0}[\mathcal{A}_{|[d_0]}]$ and we aim at checking that 8.3 applies to the measurable space with negligibles $(X, \mathcal{A}, \mathcal{N})$. To this end we consider the probability spaces $(S, \mathcal{B}(S), \sigma)$ and $(T, \mathcal{B}(T), \tau)$ both equal to $(C_d, \mathcal{A}_{|[d_0]} \subseteq C_d, \mathcal{H}^d \subseteq C_d)$. We further define $V_s = \{s\} \times C$ and $H_s = C \times \{s\}$, $s, t \in C$. These belong to $\mathcal{A}$ because they are Borel and $\phi$ is Borel regular. For each $s, t \in C$, $\emptyset \neq V_s \cap H_t = \{(s, t)\} \in \mathcal{A}_\emptyset \subseteq \mathcal{N}$ thus hypothesis (1) of 8.3 is verified. Now let $Z \subseteq X_d$ be $\mathcal{H}^d$ measurable, $s \in C$, and assume that $V_s \cap Z \in \mathcal{N} = N_{d_0}[\mathcal{A}_{|[d_0]}]$. Since $\mathcal{H}^d(V_s) = \mathcal{H}^d(C) < \infty$ and $V_s \cap Z = V_s \cap (V_s \cap Z)$ we instantly infer that $\mathcal{H}^d(V_s \cap Z) = 0$. Furthermore,

$$C \cap \{t : H_t \cap V_s \cap Z \neq \emptyset\} = C \cap \{t : (s, t) \in V_s \cap Z\} = \pi_1(V_s \cap Z)$$

and in turn

$$\mathcal{H}^d(C \cap \{t : H_t \cap V_s \cap Z \neq \emptyset\}) \leq \text{Lip}_1 \mathcal{H}^d(V_s \cap Z) = 0.$$

This proves that condition (2)(b) of 8.3 is satisfied in the present case. Part (b) is checked in a similar fashion.

9. PURELY UNRECTIFIABLE EXAMPLE

9.1 (The purely unrectifiable set $X$). — We are given a sequence $(\lambda_k)_{k \in \mathbb{N}}$ of positive real numbers such that $\lambda_0 = 1$ and $0 < \lambda_k < \lambda_{k-1}/2$ for every $k \geq 1$. We will define inductively a sequence $(\mathcal{X}_k)_{k \in \mathbb{N}}$ of sets of squares in $\mathbb{R}^2$. We start with $X_{0,0} = [0, 1] \times [0, 1]$ and $\mathcal{X}_0 = \{X_{0,0}\}$. We let $\mathcal{X}_k$ consists of $4^k$ closed squares: It contains four subsquares of each $S \in \mathcal{X}_{k-1}$, each having a vertex in common with $S$ and sidelength $\lambda_k$. We let

$$X = \bigcap_{k=1}^\infty \cup \mathcal{X}_k.$$

It clearly follows from the definitions that $\cup \mathcal{X}_k \subseteq \cup \mathcal{X}_{k-1}$ and that $\mathcal{X}_k$ consists of $4^k$ pairwise disjoint nonempty compact sets. Consequently $X$ is (topologically) a Cantor space.

One next defines $C \subseteq [0, 1]$ as $C = \bigcap_{k=0}^\infty \cup \mathcal{C}_k$, where $(\mathcal{C}_k)_{k \in \mathbb{N}}$ is defined inductively as follows. $\mathcal{C}_0 = \{[0, 1]\}$. We let $\mathcal{C}_k$ consists of $2^k$ closed intervals: It contains two subintervals of each $I \in \mathcal{C}_{k-1}$, each having an endpoint in common with $I$ and length $\lambda_k$.

(1) $X = C \times C$ and if $\mathcal{L}^1(C) = 0$ then $X$ is purely $(\mathcal{H}^1, 1)$ unrectifiable.

The first assertion follows from the observation that $S \in \mathcal{X}_k$ if and only if $S = I \times J$ for some $I, J \in \mathcal{C}_k$. The second assertion follows from [23] 18.10(4).
9.2 (Numbering of $\mathcal{A}_k$ and $\mathcal{J}_k$). — We observe that each $S \in \mathcal{A}_k$ is contained in a unique $T \in \mathcal{A}_k$. It will be convenient to number $\mathcal{A}_k = \{X_{k,j} : j = 0, \ldots, 4^k - 1\}$ in such a way that $X_{k,j} \subseteq X_{k-1, \lfloor j/4 \rfloor}, k \in \mathbb{N}$, $j = 0, \ldots, 4^k - 1$. This is readily feasible.

We next consider the sequence $\{\mathcal{J}_k\}_{k \in \mathbb{N}}$ of subsets of $[0,1]$ defined as follows. We put $I_{0,0} = [0,1]$ and $\mathcal{J}_0 = \{I_{0,0}\}$, and we let $\mathcal{J}_k$ consist of $4^k$ nonoverlapping compact subintervals of $[0,1]$, each of length $4^{-k}$, such that $[0,1] = \cup \mathcal{J}_k$. We notice that each $I \in \mathcal{J}_k$ is contained in a unique $J \in \mathcal{A}_k$. We choose a numbering of $\mathcal{J}_k = \{I_{k,\ell} : \ell = 0, \ldots, 4^k - 1\}$ in such a way that $I_{k,\ell} \subseteq I_{k-1, \lfloor \ell/4 \rfloor}, k \in \mathbb{N}$, $\ell = 0, \ldots, 4^k - 1$.

Given two integers $j, j' \in \mathbb{N}$ we say that $j'$ is a daughter of $j$ if $j = \lfloor j'/4 \rfloor$. We say that a sequence $(j_k)_{k \in \mathbb{N}}$ of nonnegative integers is a lineage if $j_{k-1}$ is a daughter of $j_k$ for every $k \geq 1$. The following now follows from our choice of numbering.

(1) Let $(j_k)_{k \in \mathbb{N}}$ be a sequence of nonnegative integers. The sequence $(X_{k,j_k})_{k \in \mathbb{N}}$ (resp. $(I_{k,j_k})_{k \in \mathbb{N}}$) is decreasing if and only if $(j_k)_{k \in \mathbb{N}}$ is a lineage.

9.3 (Coding). — Here we will define functions $j : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\ell : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $Y \subseteq [0,1]$ is to be described momentarily. Given $x \in X$ and $k \in \mathbb{N}$, there exists a unique $j(k,x) \in \{0, \ldots, 4^k - 1\}$ such that $x \in X_{k,j(k,x)}$. This is because the family $\mathcal{A}_k$ is disjointed. Furthermore, $(X_{k,j(k,x)})_{k \in \mathbb{N}}$ is decreasing, i.e. $(j(k,x))_{k \in \mathbb{N}}$ is a lineage.

If $y \in [0,1]$ and $k \in \mathbb{N}$ there does not necessarily exist a unique $\ell \in \{0, \ldots, 4^k - 1\}$ such that $y \in I_{k,\ell}$. Then:

(1) For every $y \in [0,1]$ and every $k \in \mathbb{N}$ there exists a unique $\ell \in \{0, \ldots, 4^k - 1\}$ such that $y \in I_{k,\ell}$ if and only if $y \notin D_k$ where $D_k = \{j.4^{-k} : j = 1, \ldots, 4^k - 1\}$. If instead $y \in \{j.4^{-k} : j = 1, \ldots, 4^k - 1\}$ then there are exactly two such $\ell$'s.

(2) Assume $y \in [0,1]$. There are at most two lineages $(\ell_k)_{k \in \mathbb{N}}$ such that $y \in I_{k,\ell_k}$ for every $k \in \mathbb{N}$.

In order to prove this, assume $(\ell_k)_{k \in \mathbb{N}}$, $(\ell'_k)_{k \in \mathbb{N}}$ and $(\ell''_k)_{k \in \mathbb{N}}$ are three lineages, at least two of which are distinct, such that $y \in I_{k,\ell_k} \cap I_{k,\ell'_k} \cap I_{k,\ell''_k}$ for every $k \in \mathbb{N}$. Let $k_0$ be the least integer such that $(\ell_{k_0}, \ell'_{k_0}, \ell''_{k_0})$ is not a singleton. Renaming the sequences if necessary we may assume $\ell_{k_0} \neq \ell'_{k_0}$. Since any three distinct members of $\mathcal{A}_{k_0}$ have empty intersection it follows that either $\ell''_{k_0} = \ell_{k_0}$ or $\ell''_{k_0} = \ell'_{k_0}$. Renaming again the sequences if necessary we may assume the first case occurs. It remains to observe, by induction on $m$ that $\ell''_{k_0+m} = \ell_{k_0+m}$, $m \in \mathbb{N}$. This is because of the two members of $\mathcal{A}_{k_0+m}$ that contain $y$, only one is contained in $I_{k_0+m}$.

To close this number we define $D = \cup_{k \in \mathbb{N}} D_k$ and $Y = [0,1] \setminus D$. Thus for every $y \in Y$ and every $k \in \mathbb{N}$ there exists a unique $(\ell(k,y)) \in \{0, \ldots, 4^k - 1\}$ such that $y \in I_{k,\ell(k,y)}$. It follows that $(I_{k,\ell(k,y)})_{k \in \mathbb{N}}$ is decreasing, hence $(\ell(k,y))_{k \in \mathbb{N}}$ is a lineage.

9.4. Proposition. — There exists a Borel isomorphism $f : X \to [0,1]$ and a countable set $E \subseteq X$ with the following properties.

(1) For every $k \in \mathbb{N}$ and every $S \in \mathcal{A}_k$ there exists $I \in \mathcal{J}_k$ such that $f(S \setminus E) \subseteq I$;
(2) For every $k \in \mathbb{N}$ and every $I \in \mathcal{J}_k$ there exists $S \in \mathcal{A}_k$ such that $f^{-1}(I \setminus D) \subseteq S$;
(3) $f(E) = D$.

Proof. We start by defining a map $g : X \to [0,1]$. Given $x \in X$ we consider the lineage $(j_k(x))_{k \in \mathbb{N}}$ defined in (2.2) it follows from (2.4) that $(I_{k,j(k,x)})_{k \in \mathbb{N}}$ is a decreasing sequence of compact intervals, whose $k^\text{th}$ term has length $4^{-k}$. Accordingly there exists $g(x) \in [0,1]$ such that

$$\{g(x)\} = \cap_{k \in \mathbb{N}} I_{k,j(k,x)}.$$
Now for each \( k \in \mathbb{N} \) and \( j \in \{0, \ldots, 4^k - 1\} \) we pick \( y_{k,j} \in I_{k,j} \) arbitrarily, and we observe that

\[
g(x) = \lim_{k} \sum_{j=0}^{4^k-1} y_{k,j} \mathbb{1}_{X_{k,j}}(x),
\]

\( x \in X \). This shows that \( g \) is Borel measurable.

Letting \( D \subseteq [0, 1] \) be defined as in \([9,3]\) and \( E = g^{-1}(D) \) we infer that \( g|_{X\setminus E} \) is injective. Suppose indeed that \( x, x' \in X \) are such that \( g(x) = g(x') \notin D \). It follows from \([9,3]\) and the definition of \( g \) that \( j(k, x) = j(k, x') \), hence \( \|x - x'\| \leq \text{diam} X_{k(k(x))} = 4^{-k} \sqrt{2} \), for all \( k \in \mathbb{N} \), thus \( x = x' \). We further claim that \( g(X \setminus E) = [0, 1] \setminus D \). If indeed \( y \in [0, 1] \setminus D \) we consider the sequence \( (\ell(k, y))_{k \in \mathbb{N}} \) defined in \([9,3]\) so that \( (X_{k,\ell(k,y)})_{k \in \mathbb{N}} \) is a decreasing sequence according to \([9.2]\) and hence there exists \( h(y) \in X \) such that

\[
\{h(y)\} = \cap_{k \in \mathbb{N}} X_{k,\ell(k,y)}.
\]

Upon observing that \( j(k, g(y)) = \ell(k, y) \) it follows from the definition of \( g \) that \( g(h(y)) = y \).

By definition of \( E \), \( h(y) \in X \setminus E \). In other words \( h \) is the inverse of the bijection \( X \setminus E \to [0, 1] \setminus D : x \mapsto g(x) \). Picking \( x_{k,j} \in X_{k,j} \) arbitrarily, \( k \in \mathbb{N}, j = 0, \ldots, 4^k - 1 \), we note that

\[
h(y) = \lim_{k} \sum_{j=0}^{4^k-1} x_{k,j} \mathbb{1}_{I_{k,j}}(y),
\]

\( y \in [0, 1] \setminus D \), thereby showing that \( h \) is Borel measurable.

Next we infer from \([9.3]\) and the definition of \( g \) that \( g^{-1}\{y\} \) contains at most two members, \( y \in D \). Since \( D \) is countable it follows that so is \( E = g^{-1}(D) \). Choose arbitrarily a bijection \( \varphi : E \to D \) and define \( f : X \to [0,1] \) by

\[
f(x) = \begin{cases} g(x) & \text{if } x \notin E \\ \varphi(x) & \text{if } x \in E. \end{cases}
\]

It is now obvious that \( f \) is a bijection, and that both \( f|_{X\setminus E} \) and \( f|_{E} \) are Borel isomorphisms. Thus \( f \) itself is a Borel isomorphism.

Let \( k_0 \in \mathbb{N} \) and \( S \in \mathcal{I}_{k_0} \). It readily follows from the definition of \( f \) that \( f(S \setminus E) \subseteq g(S) \).

Let \( j_0 \) be such that \( S = X_{k_0,j_0} \). If \( x \in S \) then \( j(k_0, x) = j_0 \) so that, by definition of \( f \), \( g(x) \in I_{k_0,j_0} \). This proves (1). Similarly let \( I = I_{k_0,j_0} \in \mathcal{I}_{k_0} \). Clearly \( f^{-1}(I \setminus D) = h(I \setminus D) \).

If \( y \in I \setminus D \) then \( \ell(k_0, y) = j_0 \) whence \( h(y) \in X_{k_0,j_0} \). This proves (2). \( \Box \)

9.5 (Choice of a Cantor set \( C_d \) and corresponding \( X_d = C_d \times C_d \). — Recall the construction of \( X \) in \([9.1]\). For the remaining part of this section \( C \) will be a self-similar Cantor set of Hausdorff dimension \( 0 < d \leq \frac{\log 2}{\log 4} \). Thus we let the family \( \mathcal{C}_d \) consists of \( 2^k \) members of length \( \lambda_k = \lambda^k \) where \( \lambda = \frac{\log 2}{\log 4} \), i.e. \( 0 < \lambda \leq \frac{1}{4} \) (see e.g. \([23, 4.10]\) or \([12, 2.10.28]\)). We choose our notation to reminisce about this choice by letting \( C_d \) denote the corresponding Cantor set, and \( X_d = C_d \times C_d \).

9.6. Proposition. — For every \( Z \subseteq X_d \) one has

\[
\left( \frac{1}{2} \right)^{1/2} \mathcal{H}^d(Z) \leq \mathcal{H}^{1/2}(f(Z)) \leq \left( \frac{2}{3} \right)^d \mathcal{H}^d(Z).
\]

In particular \( \mathcal{H}^d(Z) = 0 \) if and only if \( \mathcal{H}^{1/2}(f(Z)) = 0 \).

Proof. We start by observing that for every \( k \in \mathbb{N} \) and every \( S \in \mathcal{I}_k \), \( I \in \mathcal{I}_k \) one has

\[
\sqrt{\text{diam} I} = \sqrt{4^{-k}} = 2^{-k} = \lambda^{kd} \geq \left( \frac{1}{\sqrt{2}} \text{diam} S \right)^d
\]

because \( 2^k(\lambda^k)^d = 1 \) by our choice of \( \lambda \) and \( d \), \([9.5]\).
Let $Z \subseteq X_d$ and let $\varepsilon > 0$. In order to prove the right hand inequality there is no restriction to assume that $\mathcal{H}^d(Z) < \infty$. There exists a finite or countable covering $(U_i)_{i \in I}$ of $Z$ in $X$ such that $\sum_i (\text{diam } U_i)^d < \varepsilon + \mathcal{H}^d(Z)$ and $\text{diam } U_i < \varepsilon$ for every $i \in I$. We may assume that each $U_i$ is open and nonempty. Abbreviate $\delta_i = \text{diam } U_i$, $i \in I$. Choosing $x_i \in U_i$ and letting $U_i'' = U(x_i, \delta_i) \cap X$ we see that $\text{diam } U_i'' \leq 2\delta_i$, $i \in I$. Choose $k_i \in \mathbb{N}$ such that $\lambda^{-k_i+1} \leq \text{diam } U_i'' < \lambda^{-k_i}$. Thus $U_i''$ intersects some $X_{k_i,j_i}$, $j_i \in \{0, \ldots, 4k_i - 1\}$, and since $\lambda < \frac{1}{d}$ it intersects only one of them. Therefore $U_i'' \subseteq X_{k_i,j_i}$.

Notice that $\text{diam } X_{k_i,j_i} = \lambda^{k_i} \sqrt{2} < 2\sqrt{2} \lambda^{-1} \delta_i$. Now $Z \subseteq \cup_{i \in I} \subseteq \cup_{i \in I} X_{k_i,j_i}$, thus also $Z \setminus E \subseteq \cup_{i \in I} (X_{k_i,j_i} \setminus E)$ and it follows from (9.4(1)) that $f(Z \setminus E) \subseteq \cup_{i \in I} I_{k_i,j_i}$ for some integers $j_i \in \mathbb{N}, i \in I$. In turn (7) implies that

$$
\mathcal{H}^d_{\eta}(f(Z \setminus E)) \leq \sum_{i \in I} \sqrt{\text{diam } I_{k_i,j_i}} \leq \left( \frac{2}{\lambda} \right)^d \sum_{i \in I} (\text{diam } U_i)^d \leq \left( \frac{2}{\lambda} \right)^d \left( \varepsilon + \mathcal{H}^d(Z) \right),
$$

where $\eta_d = 2^{3d-1} \lambda^{-2d} \varepsilon^{2d}$. Since $\varepsilon > 0$ is arbitrary we infer that $\mathcal{H}^d(f(Z \setminus E)) \leq 2^d \lambda^{-d} \mathcal{H}^d(Z)$. As $f(Z) \subseteq f(Z \setminus E) \cup f(E)$ and $f(E) = D$ is countable, $\mathcal{H}^d(f(E)) = 0$ and the right hand inequality follows.

Let $A \subseteq [0, 1]$ and let $\varepsilon > 0$. In order to prove the left hand inequality we may of course suppose that $\mathcal{H}^d(A) < \infty$. There exists a covering $(U_i)_{i \in I}$ of $A$ in $[0, 1]$ such that $\sum_i \sqrt{\text{diam } U_i} < \varepsilon + \mathcal{H}^d(A)$ and $\text{diam } U_i < \varepsilon$ for every $i \in I$. There is no restriction to assume that each $U_i$ is a nondegenerate interval. We abbreviate $\delta_i = \text{diam } U_i$ and we let $k_i \in \mathbb{N}$ be such that $4\lambda^{-k_i-1} \leq \delta_i < 4\lambda^{-k_i}$. Notice that $U_i$ intersects at most two members of $\mathcal{K}_k$, i.e. there exist $j_{i,1}, j_{i,2} \in \mathcal{K}_k$ such that $U_i \subseteq J_{i,1} \cup J_{i,2}$. Furthermore $\text{diam } J_{i,q} < 4\text{diam } U_i$, $i \in I$, $q = 1, 2$. Now observe that

$$
f^{-1}(A \setminus D) \subseteq \cup_{i \in I} \left( f^{-1}(J_{i,1} \setminus D) \cup f^{-1}(J_{i,2} \setminus D) \right) \subseteq \cup_{i \in I} (X_{k_i,j_{i,1}} \cup X_{k_i,j_{i,1}})
$$

for some integers $j_{i,1}, j_{i,2} \in \{0, \ldots, 4k_i - 1\}$, according to (9.4(2)). Therefore it follows from (7) that

$$
\mathcal{H}^d_{\eta}(f^{-1}(A \setminus D)) \leq \sum_{i \in I} \left( (\text{diam } X_{k_i,j_{i,1}})^d + (\text{diam } X_{k_i,j_{i,2}})^d \right)
\leq 2^{1+\frac{d}{2}} \sum_{i \in I} \sqrt{\text{diam } U_i} \leq 2^{1+\frac{d}{2}} \left( \varepsilon + \mathcal{H}^d(A) \right),
$$

where $\eta_d = 2^{3/4}(4\varepsilon)^{1/2}$. Since $\varepsilon > 0$ is arbitrary, $\mathcal{H}^d(f^{-1}(A \setminus D)) = 0$. Finally $\mathcal{H}^d(f^{-1}(A)) \leq 2^{1+\frac{d}{2}} \mathcal{H}^d(A)$ because $f^{-1}(A) \subseteq f^{-1}(A \setminus D) \cup f^{-1}(D)$ and $f^{-1}(D) = E$ is countable whence $\mathcal{H}^d(f^{-1}(A)) = 0$.

*9.7. —* We say that a measure space $(X, \mathcal{A}, \mu)$ is *undecidably semilocalizable* if the proposition «$(X, \mathcal{A}, \mu)$ is semilocalizable» is undecidable in ZFC. In case $X$ is a complete separable metric space and $0 < d < \infty$, the measure space $(X, \mathcal{A}_{\geq d}, \mathcal{H}^d)$ is undecidably semilocalizable if and only if the proposition «$(X, \mathcal{A}_{\geq d}, \mathcal{H}^d)$ is almost decomposable» is undecidable in ZFC. This is a consequence of (7.1).

*9.8. Theorem. —* The measure space $\left([0, 1], \mathcal{A}_{\geq 1/2}, \mathcal{H}^{1/2}\right)$ is undecidably semilocalizable.

*Proof. —* It is consistently semilocalizable according to (5.4). Fix $0 < d \leq \frac{\log 2}{\log 3}$ arbitrarily. Let $f : X_d \to [0, 1]$ be as before and define a $\sigma$-algebra of subsets of $X_d$ by the formula

$$
\mathcal{A} = \mathcal{P}(X_d) \cap \left\{ f^{-1}(A) : A \in \mathcal{A}_{\geq 1} \right\}.
$$
We claim that $\mathcal{B}(X_d) \subseteq \mathcal{A}$. Indeed if $B \subseteq X_d$ is Borel then so is $f(B)$, according to \textbf{9.4}. Thus $f(B) \in \mathcal{A}_{\mathcal{G}^{1/2}}$ and in turn $B = f^{-1}(f(B)) \in \mathcal{A}$. We also define

$$N = \mathcal{P}(X_d) \cap \left\{ f^{-1}(N) : N \in \mathcal{A}_{\mathcal{G}^{1/2}} \right\}.$$ 

It ensues from their construction that the measurable spaces with negligibles $(X_d, \mathcal{A}, N)$ and $\left( [0,1], \mathcal{A}_{\mathcal{G}^{1/2}}, N, \mathcal{A}_{\mathcal{G}^{1/2}} \left\{ f^{-1}(N) \right\} \right)$ are isomorphic in the category MSN. According to \textbf{8.5} one is localizable if and only if the other one is. Reasoning as in the proof of \textbf{8.5} we will now show that the former is localizable. First we notice that $V_s = \{ s \} \times C_d$ and $H_t = C_d \times \{ t \}$, $s, t \in C_d$, indeed belong to $\mathcal{A}$ because $\mathcal{B}(X_d) \subseteq \mathcal{A}$. Let $Z \in \mathcal{A}$ and $s \in C_d$ be such that $V_s \cap Z \in N$. This means that $f(V_s \cap Z) \in \mathcal{A}_{\mathcal{G}^{1/2}} \left\{ f^{-1}(N) \right\}$.

Since $f(V_s \cap Z) = f(V_s) \cap f(Z)$ and $f(V_s) \in \mathcal{A}_{\mathcal{G}^{1/2}}$ according to \textbf{9.6} we infer that $\mathcal{H}^d(f(V_s \cap Z)) = 0$ and in turn $\mathcal{H}^d - V_s \cap Z = 0$ again thanks to \textbf{9.6}. Reasoning as in \textbf{8.5} we conclude that $\mathcal{H}^d(C \cap \{ t : H_t \cap V_s \cap Z \neq \emptyset \}) = 0$. Similarly the condition (2)(b) of \textbf{8.5} holds as well and the proof is complete. $\square$

10. **Purely Rectifiable Example**

10.1 (A Cantor set). — We let $\{ 0, 1 \}^{\mathbb{N}}$ be the Cantor space equipped with its usual topology and its usual Borel, probability, product measure $\lambda$. For each $j \in \mathbb{N}^+$ let $\mathcal{J}_j$ be a collection of disjoint, compact subintervals of $[0,1]$, and we let $(\ell_j)_{j \in \mathbb{N}^+}$ be a sequence in $(0,1)$, with the following properties:

1. card $\mathcal{J}_j = 2^j$;
2. For every $T \in \mathcal{J}_j$ one has card $\mathcal{J}_{j+1} \cap \{ S : S \subseteq T \} = 2$;
3. For every $S \in \mathcal{J}_j$ one has $\mathcal{L}^1(S) = \ell_j$.

We then define $C = \cap_{j \in \mathbb{N}^+} \cup \mathcal{J}_j$. This way we can realize a set $C$ of any Hausdorff dimension $0 \leq d < 1$, see e.g. \textbf{12.4} and \textbf{11.1}. We will be mostly interested in the case $d = 0$. In any case we will henceforth assume that $\mathcal{L}^1(C) = 0$.

For each $j \in \mathbb{N}^+$ we number the members of $\mathcal{J}_j$ as $S_{j,0}, \ldots, S_{j,2^j-1}$ in such a way that max $S_{j,k} < \min S_{j+1,k}$, $k = 0, \ldots, 2^j - 1$. Thus $S_{j+1,2k} \cup S_{j+1,2k+1} \subseteq S_{j,k}$ for all $j \in \mathbb{N}^+$ and all $k = 0, \ldots, 2^j - 1$. Now given $\xi \in \{ 0,1 \}^{\mathbb{N}^+}$ we define inductively $(k_\xi(j))_{j \in \mathbb{N}^+}$ as follows: $k_\xi(1) = \xi(1)$ and $k_\xi(j+1) = 2k_\xi(j) + \xi(j+1)$. In turn we define the usual coding of $C$,

$$\varphi : \{ 0,1 \}^{\mathbb{N}^+} \rightarrow C$$

by letting $\varphi(\xi)$ be the only point of $[0,1]$ such that

$$\{ \varphi(\xi) \} = \cap_{j \in \mathbb{N}^+} S_{j,k_\xi(j)}.$$

Thus $\varphi$ is a homeomorphism.

10.2 (The measures $\mu$ and $\mu_j$). — We define a Borel probability $\mu$ measure on $[0,1]$ by the formula $\mu(B) = \lambda(\varphi^{-1}(B \cap C))$, $B \in \mathcal{B}([0,1])$. For each $j \in \mathbb{N}^+$ we define a Borel probability measure $\mu_j$ on $[0,1]$ by the formula

$$\mu_j = \left( \frac{1}{2j} \ell_j \right) \mathcal{L}^1 (\cup \mathcal{J}_j).$$

10.3. Lemma. — The sequence $(\mu_j)_{j \in \mathbb{N}^+}$ converges weakly* to $\mu$.

Proof. First we let $S \in \mathcal{J}_j$ for some $j \in \mathbb{N}^+$. Observe that $\mu(S) = 2^{-j}$. If $k \geq j$ then $\mu_k(S) = 2^{-k} \mathcal{L}^1(S \cap \mathcal{J}_k) = 2^{-k} \mathcal{L}^1(2^{k-j} S_k) = 2^{-j}$. In particular $\lim_k \mu_k(S) = \mu(S)$.

Next we let $U \subseteq [0,1]$ be relatively open. There exists a disjointed sequence $(S_n)_{n \in \mathbb{N}}$ of members of $\cup_{j \in \mathbb{N}^+} \mathcal{J}_j$ such that each $S_n \subseteq U$ and

$$C \cap U = C \cap (\cup_{n \in \mathbb{N}} S_n).$$
It suffices indeed to let \((S_n)_{n \in \mathbb{N}}\) be a numbering of \(\mathcal{F} = \bigcup_{j \in \mathbb{N}} \mathcal{T}_j\) where \((\mathcal{T}_j)_{j \in \mathbb{N}}\) is defined inductively as follows: \(\mathcal{T}_1 = \mathcal{T} \cap \{ S : S \subseteq U \}\) and \(\mathcal{T}_{j+1} = \mathcal{T}_{j+1} \cap \{ S : S \subseteq U \} \text{ and } S \cap \bigcup_{k=1}^{j} \mathcal{T}_k = \emptyset \). Now given \(\varepsilon > 0\) there exists \(N \in \mathbb{N}\) such that
\[
\sum_{n \in \mathbb{N}} \mu(S_n) \leq \varepsilon + \sum_{n=0}^{N} \mu(S_n).
\]
Furthermore,
\[
\mu(U) = \sum_{n \in \mathbb{N}} \mu(S_n) \leq \varepsilon + \sum_{n=0}^{N} \mu(S_n) = \varepsilon + \sum_{n=0}^{N} \lim_k \mu_k(S_n)
= \varepsilon + \lim_k \mu_k \left( \bigcup_{n=0}^{N} S_n \right) \leq \varepsilon + \liminf_k \mu_k(U).
\]
Since \(\varepsilon > 0\) is arbitrary it follows that \(\mu(U) \leq \lim\inf_k \mu_k(U)\).

Recalling that \(\mu([0,1]) \neq \mu([0,1] \setminus K)\), for all \(k \in \mathbb{N}\) we infer that for every compact \(K \subseteq [0,1]\),
\[
\mu(K) = \mu([0,1]) - \mu([0,1] \setminus K) \geq \mu_k([0,1]) - \liminf_k \mu_k([0,1] \setminus K) = \limsup_k \mu_k(K).
\]
The conclusion follows from Portmanteau’s Theorem. \(\square\)

10.4 (The mappings \(F\) and \(F_j\)). — We associate with \(\mu\) its distribution function
\[
f : [0,1] \rightarrow [0,1] : t \mapsto \mu([0,t])
\]
and we observe that \(f\) is continuous (because \(\mu\) is diffuse) and nondecreasing. We also define
\[
F : [0,1] \rightarrow \mathbb{R}^2 : t \mapsto (t, f(t))
\]
and we observe that the set \(\Gamma = \text{graph}(f) = F([0,1])\) is 1-rectifiable and \(\mathcal{H}^1(\Gamma) < \infty\). This most easily follows from the «bow-tie lemma» (see e.g. \cite{10} 4.8.3) applied with \(n = m + 1 = 2, S = \Gamma, r = 3, \sigma = \sin(\pi/4)\) and \(W = \text{span}\{e_1 + e_2\}\).

We will approximate \(f\) by the functions
\[
f_j : [0,1] \rightarrow [0,1] : t \mapsto \mu_j([0,t])
\]
which are nondecreasing and Lipschitz. Given \(t \in [0,1]\) we notice that \(\text{Bdry}[0,t] = \{0,t\}\) is \(\mu\)-null, whence
\[
f(t) = \mu([0,t]) = \lim_j \mu_j([0,t]) = f_j(t),
\]
according to \cite[10.3]{10} and \cite[1.9 Theorem 1]{11}. Thus the sequence \((f_j)_{j \in \mathbb{N}}\) converges pointwise to \(f\).

We next record that each \(f_j\) is differentiable \(\mathcal{L}^1\) almost everywhere. In fact upon defining
\[
\sigma_j = \frac{1}{2\ell_j}
\]
one has
\[
f_j'(t) = \begin{cases} 0 & \text{if } t \notin \bigcup \mathcal{T}_j \\ \sigma_j & \text{if } t \in \text{Int } \bigcup \mathcal{T}_j. \end{cases}
\]
We finally define
\[
F_j : [0,1] \rightarrow \mathbb{R}^2 : t \mapsto (t, f_j(t))
\]
and related to the Jacobian of \(F_j\) we define
\[
e_j = 2^{\ell_j} \sqrt{1 + \sigma_j^2} = \sigma_j^{-1} \sqrt{1 + \sigma_j^2}.
\]
Since \( L^1(C) = 0 \) we infer that \( \lim \sup_j \sigma_j = \infty \) (for otherwise \( f \) would be Lipschitz) and in turn
\[
\lim_j c_j = \lim_j \sigma_j^{-1} \sqrt{1 + \sigma_j^2} = 1.
\]

10.5. Lemma. — For every \( j \in \mathbb{N}^* \) and every Borel set \( B \subseteq [0, 1] \) one has
\[
\mathcal{H}^1(F_j(B)) \geq c_j \mu_j(B).
\]

Proof. Let \( B \subseteq [0, 1] \) be Borel and define \( B' = B \cap (\cup \mathcal{J}_j) \). It follows from the definition of \( \mu_j \) that
\[
\mu_j(B) = \mu_j(B') = \sigma_j L^1(B').
\]
Recalling [10.4] it follows from the «area formula» in this simple case (see e.g. [11, §3.3 Theorem 1] for the general case)
\[
\mathcal{H}^1(F_j(B)) \geq \mathcal{H}^1(F_j(B')) = \int_{B'} \sqrt{1 + f_j'(t)^2} \, d L^1(t) = \sqrt{1 + \sigma_j^2} L^1(B').
\]

10.6. Lemma. — Let \( S \subseteq [0, 1] \) be any set. It follows that
\[
\mathcal{H}^1(F(S)) \geq \frac{\bar{\mu}(S)}{\sqrt{2}}.
\]

Proof. We start with the case when \( S = [a, b] \subseteq [0, 1] \) is a closed interval. Since \( F(S) \subseteq F([0, 1]) \) is 1-rectifiable (and compact) the following «integral geometric inequality» follows for instance from [12, 3.2.27] (\( \pi_1 \) and \( \pi_2 \) denote resp. the projection from \( \mathbb{R}^2 \) onto its first and second axis):
\[
\mathcal{H}^1(F(S)) \geq \sqrt{a_1^2 + a_2^2}
\]
where
\[
a_1 = \int_{\mathbb{R}} \text{card}(F(S) \cap \pi_1^{-1}(x)) \, d L^1(x) = L^1(S) = b - a,
\]
and
\[
a_2 = \int_{\mathbb{R}} \text{card}(F(S) \cap \pi_2^{-1}(y)) \, d L^1(y) = L^1(f(S)) = f(b) - f(a).
\]
Similarly the other inequality from [12, 3.2.27] applies to \( F_j(S) \):
\[
a_{1,j} + a_{2,j} \geq \mathcal{H}^1(F_j(S))
\]
where
\[
a_{1,j} = \int_{\mathbb{R}} \text{card}(F_j(S) \cap \pi_1^{-1}(x)) \, d L^1(x) = L^1(S) = b - a,
\]
and
\[
a_{2,j} = \int_{\mathbb{R}} \text{card}(F_j(S) \cap \pi_2^{-1}(y)) \, d L^1(y) = L^1(f_j(S)) = f_j(b) - f_j(a).
\]
Accordingly,
\[
\mathcal{H}^1(F(S)) \geq \sqrt{(b-a)^2 + (f(b) - f(a))^2}
\]
\[
= \lim_{j} \sqrt{(b-a)^2 + (f_j(b) - f_j(a))^2} \quad \text{(by 10.4)}
\]
\[
\geq \frac{1}{\sqrt{2}} \lim_{j} ((b-a) + (f_j(b) - f_j(a)))
\]
\[
\geq \frac{1}{\sqrt{2}} \lim_{j} (a_{t_j} + a_{z_j})
\]
\[
\geq \frac{1}{\sqrt{2}} \lim \sup \mathcal{H}^1(F(S))
\]
\[
\geq \frac{1}{\sqrt{2}} \lim \sup \epsilon_j \mu_j(S) \quad \text{(by 10.5)}
\]
\[
= \frac{\mu(S)}{\sqrt{2}} \quad \text{(according to 10.3 since} \mu(\text{Bdry } S) = 0)
\]

This completes the proof in case \(S\) is a closed interval.

We now turn to the case when \(S \subseteq [0, 1]\) is Borel. To this end we define \(\nu(B) = \sqrt{2}\mathcal{H}^1(F(B)), B \in \mathcal{B}([0, 1])\). Since \(F(B) \in \mathcal{B}(\mathbb{R}^2)\) whenever \(B \in \mathcal{B}([0, 1])\) (because \(F\) is continuous, hence Borel measurable, and injective, see [28, 5.4.5]) and since \(B_1 \cap B_2 = \emptyset\) implies that \(F(B_1) \cap F(B_2) = \emptyset\) it follows that \(\nu\) is a measure on \(\mathcal{B}([0, 1])\). Thus \(\mu\) and \(\nu\) are two finite Borel measures on \([0, 1]\) such that \(\mu(I) = \nu(I)\) whenever \(I \subseteq [0, 1]\) is a closed interval. Since \(\nu\) is also clearly diffused we infer that \(\mu(I) = \nu(I)\) whenever \(I = (m2^{-n}, (m + 1)2^{-n})\), for some \(n \in \mathbb{N}^+\) and \(m = 0, \ldots, 2^n - 1\). Since each relatively open set \(U \subseteq (0, 1)\) is the union of a disjointed sequence of such dyadic semi-intervals it follows that \(\mu(U) = \nu(U)\). Finally the outer regularity of \(\nu\) yields \(\mu(B) = \nu(B)\) for all \(B \in \mathcal{B}([0, 1])\).

We come to the case when \(S \subseteq [0, 1]\) is arbitrary. We choose a Borel set \(B_1 \subseteq [0, 1]\) such that \(S \subseteq B_1\) and \(\mu(S) = \mu(B_1)\), we choose a Borel set \(B_2 \subseteq \mathbb{R}^2\) such that \(F(S) \subseteq B_2\) and \(\mathcal{H}^1(F(S)) = \mathcal{H}^1(B_2)\), we let \(B_3 = F^{-1}(B_2) \subseteq [0, 1]\) which is Borel as well, and finally we define \(B = B_1 \cap B_3\). Since \(F\) is injective and \(F(S) \subseteq B_2\) we see that \(S = F^{-1}(F(S)) \subseteq F^{-1}(B_2) = B_3\), thus \(S \subseteq B_1 \cap B_3 = B\). Therefore \(\mu(S) = \mu(B) = \mu(B_1) = \mu(B_3)\) and we conclude that \(\nu(S) = \mu(B)\). Similarly, from \(S \subseteq B \subseteq B_3\) and the definition of \(B_2\) we infer that \(\mathcal{H}^1(F(S)) = \mathcal{H}^1(F(B)) \subseteq B_2\) and in turn \(\mathcal{H}^1(F(S)) \subseteq \mathcal{H}^1(F(B)) \subseteq \mathcal{H}^1(B_2) = \mathcal{H}^1(F(S))\) so that \(\mathcal{H}^1(F(S)) = \mathcal{H}^1(F(B))\). Finally it follows from the previous paragraph that
\[
\mathcal{H}^1(F(S)) = \mathcal{H}^1(F(B)) \geq \frac{\mu(B)}{\sqrt{2}} = \frac{\mu(S)}{\sqrt{2}}.
\]

\[\square\]

10.7. Theorem (ZFC + \(\text{non} (\mathcal{N}_{2^1}) < \text{cov} (\mathcal{N}_{2^1})\)). — Assume that

(1) \(C \subseteq [0, 1]\) is a Cantor set such as in [10.7] and \(X = C \times [0, 2] \subseteq \mathbb{R}^2\);
(2) \(\mathcal{N}\) is a \(\sigma\)-algebra of subsets of \(X\) such that \(\mathcal{B}(X) \subseteq \mathcal{N} \subseteq \mathcal{P}(X)\);
(3) \(\mathcal{N} \subseteq \mathcal{A}\) is a \(\sigma\)-ideal with the following property:
(a) \(\{x\} \in \mathcal{N}\) for every \(x \in X\);
(b) For every \(A \in \mathcal{N}\) and every \(1\)-rectifiable set \(M \subseteq \mathbb{R}^2\) if \(A \cap M \in \mathcal{N}\) then
\(\mathcal{H}^1(A \cap M) = 0\);  
(4) \(\text{non} (\mathcal{N}_{2^1}) < \text{cov} (\mathcal{N}_{2^1})\).

It follows that \((X, \mathcal{A}, \mathcal{N})\) is not localizable.

Proof. In this proof \(e_1, e_2\) denotes the canonical basis of \(\mathbb{R}^2\) and \(\pi_1, \pi_2\) the canonical projections of \(\mathbb{R}^2\) on its first and second axis respectively. The result will be obtained as a
consequence of \(8.3\) applied to \((X,\mathcal{A},\mathcal{N})\) as in the statement, \((S,\mathcal{B}(S),\sigma) = (C,\mathcal{B}(C),\mu)\), \((T,\mathcal{B}(T),\tau) = ([0,1],\mathcal{B}([0,1]),\mathcal{L}^1)\),

\[
V_s = \{s\} \times [0,2] \in \mathcal{B}(X) \subseteq \mathcal{A},
\]

\(s \in C\), and

\[
H_t = (\Gamma + t.e_2) \cap X \in \mathcal{B}(X) \subseteq \mathcal{A},
\]

\(t \in [0,1]\), where \(\Gamma = F([0,1])\).

We now check that condition (1) of \(8.3\) is satisfied. Let \(s \in C\) and \(t \in [0,1]\). Since \(H_t\) is contained in the graph of a function and \(V_s\) is contained in a vertical line, \(V_s \cap H_t\) is either empty or a singleton, therefore a member of \(\mathcal{N}\) according to our current hypothesis \((3)(a)\). It is easy to see that \(p_{s,t} = (s,f(s) + t) \in V_s \cap H_t\), so that \(V_s \cap H_t \neq \emptyset\).

We next verify that condition \((2)(a)\) of \(8.3\) is satisfied. Fix \(s \in C\) and \(Z \in \mathcal{A}\) such that \(V_s \cap Z \in \mathcal{N}\). Observe that

\[
[0,2] \cap \{t : H_t \cap V_s \cap Z \neq \emptyset\} = [0,1] \cap \{t : p_{s,t} \in V_s \cap Z\} = [0,2] \cap \{t : t \in \pi_2(V_s \cap Z) - f(s)\},
\]

and therefore

\[
\mathcal{L}^1([0,2] \cap \{t : H_t \cap V_s \cap Z \neq \emptyset\}) \subseteq \mathcal{H}^1(V_s \cap Z) = 0
\]

where the last equality follows from our assumption \((3)(b)\) because \(V_s\) is 1 rectifiable.

Finally we ought to show that condition \((2)(b)\) of \(8.3\) is satisfied. Let \(t \in [0,1]\) and \(Z \in \mathcal{A}\) be such that \(H_t \cap Z \in \mathcal{N}\). Observe that

\[
C \cap \{s : V_s \cap H_t \cap Z = \emptyset\} = C \cap \{s : p_{s,t} \in H_t \cap Z\} = \pi_1(H_t \cap Z).
\]

Since \(H_t \cap Z = (\Gamma + t.e_2) \cap Z\) and \(\Gamma + t.e_2\) is 1 rectifiable, our hypothesis \((3)(b)\) implies that \(\mathcal{H}^1(H_t \cap Z) = 0\). Abbreviating \(E = \pi_1(H_t \cap Z) \subseteq C\) it ensues from \(10.6\) that

\[
0 = \mathcal{H}^1(H_t \cap Z) = \mathcal{H}^1(F(E) + t.e_2) = \mathcal{H}^1(F(E)) \geq \frac{\mu(E)}{\sqrt{2}},
\]

and the proof is complete. \(\Box\)

10.8. COROLLARY. — Let \(C \subseteq [0,1]\) be a Cantor set as in \(10.1\) and \(X = C \times [0,2]\). It follows that \((X,\mathcal{A}_{\mathcal{H}^1},\mathcal{H}^1)\) is undecidably semilocalizable. \(\Box\)

Proof. It is consistently semilocalizable according to \(8.4\) and it is consistently not semilocalizable according to \(10.7\) applied with \(\mathcal{A} = \mathcal{A}_{\mathcal{H}^1}\) and \(\mathcal{N} = \mathcal{A}_{\mathcal{H}^1}\). \(\Box\)

11. CONCLUDING REMARKS AND OPEN QUESTIONS

11.1. — One may apply \(10.7\) to other \(\sigma\)-ideals than \(\mathcal{A}_{\mathcal{H}^1}\). For example let

\[
\mathcal{A}_{pu} = \mathcal{P}(\mathbb{R}^2) \cap \{S : S\text{ is purely } (\mathcal{H}^1,1) \text{ unrectifiable}\}.
\]

Recall that a set \(S \subseteq \mathbb{R}^2\) is called **purely** \((\mathcal{H}^1,1)\) **unrectifiable** whenever \(\mathcal{H}^1(S \cap M) = 0\) for every 1 rectifiable \(M \subseteq \mathbb{R}^2\). It then follows from \(10.7\) that for any \(\sigma\)-algebra \(\mathcal{B}(X) \subseteq \mathcal{A} \subseteq \mathcal{P}(X)\) the measurable space with negligibles \((X,\mathcal{A},\mathcal{A} \cap \mathcal{N}_{\mathcal{A}_{pu}})\) is consistently not localizable.

11.2. — We turn back to \(10.7\) applied with \(\mathcal{A}_{\mathcal{H}^1}\). It follows that \((X,\mathcal{B}(X),\mathcal{B}(X) \cap \mathcal{A}_{\mathcal{H}^1})\) is consistently not semilocalizable. It further follows in ZFC from \(5.6\) that \((X,\mathcal{B}(X),\mathcal{B}(X) \cap \mathcal{N}_{\mathcal{A}_{pu}})\) is not almost decomposable.

(Q5) I do not know whether, in ZFC, \((X,\mathcal{B}(X),\mathcal{B}(X) \cap \mathcal{A}_{\mathcal{H}^1})\) is not semilocalizable. Notice that, under \(\text{CH}\), semilocalizability does not follow from \(5.4\). A more general version of this question is the following.

(Q6) I do not know whether in \(5.3\) the word **almost decomposable** might be replaced by the word **semilocalizable** without affecting the validity of the statement.
Notice that [6.5] does not seem to apply in this situation, for the following reason. If $(X, \mathcal{B}(X), \mu)$ is such that $X$ is Polish and $\mu$ is semifinite, then I do not see a reason that the completion of $(X, \mathcal{B}(X), \mu)$ be locally determined.

Another consequence of [10.7] is that there does not exist, in ZFC, a «localizable version» of $(X, \mathcal{B}(X), \mathcal{B}(X) \cap \mathcal{N}_{\mathcal{P}^1})$ obtained by simply «enlarging» the given $\sigma$-algebra $\mathcal{B}(X)$ to another one $\mathcal{B}(X) \subseteq \mathcal{A} \subseteq \mathcal{P}(X)$. Instead it seems necessary to enlarge $X$ first. As stated in [3.19] Q4 it would be interesting to investigate the following.

(Q7) Assuming one has defined a specific left adjoint functor to the forgetful functor \( \text{LOC} \to \text{MSN} \), describe its effect on \((X, \mathcal{B}(X), \mathcal{B}(X) \cap \mathcal{N}_{\mathcal{P}^1})\). It would be particularly interesting to give a geometric interpretation of the process.

11.3. — Here we ask whether the behavior exhibited by the specific set $X$ of section [10] is shared by other compact subsets of $\mathbb{R}^2$ of Hausdorff dimension 1 but non $\sigma$-finite $\mathcal{H}^1$ measure.

(Q8) Let $X \subseteq \mathbb{R}^2$ be a compact set of Hausdorff dimension 1 and such that $\mathcal{H}^1(X \cap U) = \infty$ for every open set $U \subseteq \mathbb{R}^2$ with $X \cap U \neq \emptyset$. Is $(X, \mathcal{A}_{\mathcal{P}^1}, \mathcal{H}^1)$ undeniably semilocalizable?

11.4. — In regard to (Q8), of particular interest would be an example of such $X$ which is purely $(\mathcal{H}^1, 1)$ unrectifiable. Let us for instance consider the following set $X$, using the notations of [9.1] Choosing $\lambda_k = k^{-\frac{1}{4}}$ one checks that $X$ has Hausdorff dimension 1 and $\mathcal{H}^1(X \cap U) = \infty$ whenever $U \subseteq \mathbb{R}^2$ is open and $X \cap U \neq \emptyset$. Indeed $\mathcal{H}^d(X) = 0$ when $1 < d$, is a consequence of the definition of Hausdorff measure, and if $S \in \mathcal{H}_k$ for some $k \in \mathbb{N}^*$ then $\mathcal{H}^1(S \cap K) = \infty$ according to [12] 2.10.27 because $X \cap S = K \times K$ for some $K \subseteq [0, 1]$ with $\mathcal{H}^1(K) = \infty$ according to [12] 2.10.28. Also observe, as in [9.1] 1 that $X$ is purely $(\mathcal{H}^1, 1)$ unrectifiable.

(Q9) With the set $X$ described here, is $(X, \mathcal{A}_{\mathcal{P}^1}, \mathcal{H}^1)$ undeniably semilocalizable?

Viewing $X$ as a product as in section [9] does not seem to be immediately helpful since it gives information about a Hausdorff measure essentially of dimension 1/2. Trying to use the graphs of distribution functions as in section [10] is no more successful since these graphs are rectifiable and $X$ is purely unrectifiable; their intersection will always be $\mathcal{H}^1$ null. One may also attempt to produce families $V_\xi$ and $H_\xi$ needed in [8.3] as random Cantor subsets of $X$: the $V_\xi$ using more often a specific set of three subsquares at each generation, and the $H_\xi$ using more often a distinct specific set of three subsquares at each generation. However random sets constructed this way tend to intersect too often, making it hard to guarantee condition (2) of [8.3].

11.5. — In the notation of section [9] and with the same restriction on $d$ as in the proof of [9.8]

(Q10) I do not know whether the measurable spaces with negligibles $(X_d, \mathcal{A}_{\mathcal{P}^1 d}, \mathcal{N}_{\mathcal{P}^1 d})$ and $([0, 1], \mathcal{A}_{\mathcal{P}^1 [0, 1]}, \mathcal{N}_{\mathcal{P}^1 [0, 1]})$ are isomorphic in the category MSN.

This boils down to deciding whether the $\sigma$-algebra $\mathcal{A}$ defined in the proof of [9.8] coincides with the $\sigma$-algebra $\mathcal{A}_{\mathcal{P}^1 d}$. The fact the answer to this question is not known turned out to be no obstacle thanks to the freedom allowed in [8.3] regarding the $\sigma$-algebra $\mathcal{A}$.

11.6. — Our last question here concerns [9.8] The proof, based on [8.3] seems to require a product structure that forces the dimension to be $1/2$.

(Q11) Let $0 < d < 1$ and $d \neq \frac{1}{2}$. Is the measure space $([0, 1], \mathcal{A}_{\mathcal{P}^1 d}, \mathcal{H}^d)$ consistently not semilocalizable?

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School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200062, P.R. of China, and NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai, 3663 Zhongshan Road North, Shanghai 200062, China

On leave of absence from: Université Paris Diderot, Sorbonne Université, CNRS, Institut de Mathématiques de Jussieu – Paris Rive Gauche, IMJ-PRG, F-75013, Paris, France

E-mail address: thdepauw@math.ecnu.edu.cn, thierry.de-pauw@imj-prg.fr