Determining the Number of Samples Required to Estimate Entropy in Natural Sequences

Andrew D. Back, Member, IEEE, Daniel Angus, Member, IEEE and Janet Wiles, Member, IEEE
School of ITEE, The University of Queensland, Brisbane, QLD, 4072 Australia.

Abstract—Calculating the Shannon entropy for symbolic sequences has been widely considered in many fields. For descriptive statistical problems such as estimating the N-gram entropy of English language text, a common approach is to use as much data as possible to obtain progressively more accurate estimates. However in some instances, only short sequences may be available. This gives rise to the question of how many samples are needed to compute entropy. In this paper, we examine this problem and propose a method for estimating the number of samples required to compute Shannon entropy for a set of ranked symbolic “natural” events. The result is developed using a modified Zipf-Mandelbrot law and the Dvoretzky-Kiefer-Wolfowitz inequality, and we propose an algorithm which yields estimates for the minimum number of samples required to obtain an estimate of entropy with a given confidence level and degree of accuracy.

I. INTRODUCTION

Consider a sequence of symbolic information given by \( Y_i = [v_0, \ldots, v_{n-1}]^T \), where \( v_j \in \Sigma^* \), and \( \Sigma^* \) is an alphabet or finite nonempty set with symbolic members. Suppose we are interested in the information content in such a message sequence. One way to approach this problem is by measuring what is new or novel in a given sequence. If a string consists of symbols \( \text{aaa, bbb, aaa} \) - beyond the first few words, there is little novelty or ‘surprise’ about the message. On the other hand, if a string consists of symbols \( \text{acb, bde, zxy, eqa} \) - then it is evident that the message has a higher degree of novelty. The idea that the randomness of a message can give a measure of the information it conveys formed the basis of Shannon’s entropy\(^1\) theory which gives a means of assigning a value to the information carried within a message \( [1,2] \). The way in which Shannon formulated this principle is that, given a single random variable \( x \) which may take \( M \) distinct values, and is in this sense symbolic, where each value occurs independently with probability \( p(x_i), i \in [1,M] \), then the single symbol Shannon entropy is defined as:

\[
H_1(X) = - \sum_{i=1}^{M} p(x_i) \log_2(p(x_i))
\]

(1)

This extends to the case where the probabilities of multiple symbols occurring together are taken into account. Hence, \( H_2 \) indicates the entropy from the probabilities of two symbols occurring consecutively. The general N-gram entropy, which is a measure of the information due to the statistical probability of \( N \) adjacent symbols occurring consecutively, can be derived as

\[
H_N(X|B) = - \sum_{i,j} p(b_i, x_j) \log_2(p(x_j|b_i))
\]

(2)

where \( b_i \in \sum_{N-1} \) is a block of \( N - 1 \) symbols, \( x_j \) is an arbitrary symbol following \( b_i \), \( p(b_i, x_j) \) is the probability of the N-gram \( (b_i, x_j) \), \( p(x_j|b_i) \) is the conditional probability of \( x_j \) occurring after \( b_i \) and is given by \( p(b_i, x_j)/p(b_i) \).

Shannon demonstrated the concept of entropy by applying it to English text, obtaining estimates of entropy by using a list of 1027 words which were sampled from 100,000 words of English text \( [3] \). Entropy has since been applied to a diverse range of problems including word entropy estimation \( [4] \), statistical keyword detection \( [5] \), phylogenetic diversity measurement \( [6] \), population biology \( [7] \), language assessment of Pictish symbols \( [8] \), facial recognition \( [9] \), and interpreting gene expression data in functional genomics for drug discovery \( [10] \).

One of the limitations of computing entropy accurately is the dependence on large amounts of data, even more so when computing N-gram entropy. As a concrete example, in language analytics, estimates of entropy based on letter, word and N-gram statistics have often relied on large data sets \( [11,12] \). The reliance on long data sequences to estimate the probability distributions used to calculate entropy and attempts to overcome this in coding schemes is discussed in \( [13] \) where they provide an estimate of letter entropy extrapolated for infinite text lengths.

Various approaches to estimating entropy over finite sample sizes have been considered. A method of computing the entropy of dynamical systems which corrects for statistical fluctuations of the sample data over finite sample sizes has been proposed in \( [14] \). Estimation techniques using small datasets have been proposed in \( [15] \), and a novel approach for calculating entropy using the idea of estimating probabilities from a quadratic function of the inverse number of symbol coincidences was proposed in \( [16] \). An online approach for estimating entropy in limited resource environments was proposed in \( [17] \). Entropy estimation over short symbolic sequences was considered in the context of dynamical time series models based on logistic maps and correlated Markov chains, where an effective shortened sequence length was proposed which accounted for the correlation effect \( [18] \).

A question which naturally arises then is how many samples are required in order to obtain an accurate estimate of entropy.
II. Estimating Samples Required for Entropy

A. Shannon Entropy Algorithm

The Shannon entropy of a discrete random variable $X$ with discrete probability distribution $p_s(x, t)$ is defined as:

$$H_s(X, t) = - \sum_{x \in S_t} p_s(x, t) \log_2 (p_s(x, t))$$

where $H_s(X, t)$ is computed on $p_s(x)$ for $\pi_p(t) = [x(t_0), \ldots, x(t_{p-1})]^T$ over $p$ consecutive samples$^3$ where $S_t \subset X$. The usual approach to calculating entropy is by estimating $p_s(x)$. Assuming some theoretical, true values for the probabilities $p_s(x, t)$, the accuracy of $H_s(X, t)$ is determined by the accuracy with which $\{p_s(x)\}$ is estimated.

For small values of $p_s(x, t)$, $|\log_2 p_s(x)|$ becomes large and hence small $p_s(x, t)$ may contribute significantly to $H_s(X, t)$. Now, given a finite set of samples, the accuracy with which $H_s(X, t)$ can be computed will depend on the accuracy with which $p_0(x)$ can be computed, where the probability of the most infrequent event occurrence is defined$^2$ as

$$p_0(x) = \inf \{p(x, r)|x \in X, r \in [1, M]\}$$

where the $M$ probabilities (the alphabet size) are computed from $N$ observations and the empirical probability is defined as

$$N_0 = \frac{n}{p_0(x)}$$

This raises questions of how large $N_0$ should be and is there a relationship between $M$ and the number of samples $N_0$ required to obtain a specified degree of accuracy with some level of confidence? Intuitively, one would expect that the larger the alphabet size, then the greater the number of observations required. In the next sections, we develop a method for determining the number of samples required to estimate the entropy of natural sequences derived from a given alphabet.

B. Dvoretzky-Kiefer-Wolfowitz Inequality

Given a finite set of randomly sampled iid (independent and identically distributed) observations $X_1, \ldots, X_n$ for which

$$F(\lambda) = P\{X_j \leq \lambda\} \quad (6)$$

and an empirical distribution function$^3$ is available, defined by

$$\hat{F}_n(x) = \frac{1}{n} \sum_{j=1}^{n} I\{X_j \leq x\} \quad (7)$$

where $I\{X_j \leq x\}$ is the indicator function defined as

$$I\{X_j \leq x\} = \begin{cases} 1 & \text{if } X_j \leq x \\ 0 & \text{otherwise} \end{cases}$$

The Dvoretzky-Kiefer-Wolfowitz (DKW) inequality$^1$ extends earlier asymptotic results by Kolmogorov and Smirnov$^2$ and provides a probabilistic bound on the difference between the empirical and true distributions. A tighter probabilistic bound was obtained by Masaart$^3$, which allows the DKW inequality to be expressed as

$$P\left(\sup_{x \in \mathbb{R}} |\hat{F}_n(x) - F(x)| > \epsilon \right) \leq 2e^{-2n\epsilon^2} \quad (9)$$

Hence, using this inequality, for every $\epsilon > 0$, and $(1 - \zeta) > 0$, it is possible to calculate $N(\epsilon, \zeta)$ such that if $n \geq N(\epsilon, \zeta)$, then

$$P\left(\sup_{x \in \mathbb{R}} |\hat{F}_n(x) - F(x)| > \epsilon \right) \leq 1 - \zeta. \quad (10)$$

A novel application of the DKW inequality to determining a probabilistic upper bound on the entropy of an unknown distribution based on a sample from that distribution was given recently by Learned-Miller & DeStefano$^4$. In the work presented here, we consider the number of samples required to obtain a specified degree of accuracy with a given confidence level. Hence, suppose we wish to determine $n$, such that with some degree of confidence $\zeta$, the maximum difference between $\hat{F}_n(x)$ and $F(x)$ is $\epsilon$, then it follows that a solution many be found as

$$P\left(\sup_{x \in \mathbb{R}} |\hat{F}_n(x) - F(x)| < \epsilon \right) \leq \zeta. \quad (11)$$

Now, the DKW inequality specified a bound for the difference in distribution functions $\hat{F}_n(x)$ and $F(x)$. For discrete random variables with discrete probabilities, we have

$$\hat{F}_n(r) = P(X \leq r) \quad (12)$$

$$= \sum_{k=0}^{r} \hat{p}(k) \quad (13)$$

Hence, we obtain

$$P\left(\sup_{x \in \mathbb{R}} \left|\sum_{k=0}^{r} (\hat{p}(k) - p(k)) \right| < \epsilon_0 \right) \leq \zeta \quad (14)$$

where,

$$1 - \zeta = 2e^{-2n\epsilon_0^2} \quad (15)$$

$^2$ As a notational convention, we designate $t_0 = t - p$, where $x(t-1)$ indicates the value of a variable $x$ one sample before $x(t)$ which in some cases is indicated as $x_t$. Also note that while Shannon entropy is commonly specified using base 2, it is also possible to formulate entropy using base $e$ or base 10.

$^3$ We expect there to be a greatest lower bound on the set of probabilities, but a precise minimum does not necessarily exist.

$^4$ Following historical convention, we will at times use notation where $P(x)$ refers to probabilities, typically distribution probabilities associated with an event, and $p_s(x)$ refers also to probabilities, where $p_s(x) = P(X = x)$. In this section we introduce the notation $\hat{p}(x)$ to represent empirical probability.
and rearranging, this becomes
\[ n = \frac{1}{2\epsilon^2} \ln \left( \frac{2}{1 - \zeta} \right). \] (16)

Now, the confidence level is specified by \( \zeta \), and through consideration of \( H_t(X, t) \) it is clear that \( \epsilon \) can be specified by the number of samples required to discriminate between the two closest probabilities used in the entropy calculation through the DKW inequality as
\[ P \left\{ \sup_{x \in X} \left| \hat{p}_n(r) - \hat{p}(r) \right| < \epsilon_r \right\} \leq \zeta \] (17)

Hence, given a finite set of randomly sampled iid observations \( X_1, \ldots, X_n \) for which it is assumed that there exists a corresponding set of monotonic probabilities for a set of \( M \) possible events, i.e. \( \hat{p}(x_1), \ldots, \hat{p}(x_M) \), then define
\[ \Delta_0 = \inf \left\{ \left| \hat{p}(x_i) - \hat{p}(x_{i+1}) \right| \right\} \quad i = 1, \ldots, M - 1 \] (18)

Now, it follows that \( \epsilon_r < \Delta_0 \), moreover, it can be observed that if \( \epsilon_r = \Delta_0/2 \), then at worst, we cannot still reliably discriminate between \( p_i(r) \) and \( p_j(r) \). Hence, for this case, \( \sup \{ \hat{p}_n(r) \} = \inf \{ \hat{p}_n(r) \} \), where
\[ \hat{p}_e(r) \in \mathbb{R} : \hat{p}(r) - \epsilon_r \leq \hat{p}_e(r) \leq \hat{p}(r) + \epsilon_r \] (19)

Hence, consider a rule for determining the entropy probabilities as a function of rank, alphabet size and some parametrization, i.e. \( \hat{p}(r) = f(r, M; \theta) \). From (14), then for a number of observations \( N_0 \) and expected number of events \( n_0 \), with the empirical probability defined in (5), an expression is required for
\[ \epsilon_r = f (\hat{p}(r)) , \quad r \in [1, M] \] (20)
\[ = f (r, M) \] (21)
which will be considered in the next section.

C. Probabilistic Event Model

For various natural sequences, the probability of information events can generally be ranked into monotonically decreasing order. This phenomena has been examined extensively, in particular, it was demonstrated by Zipf’s early work that the frequency of ranked words in a text occur in such a way that they can be described by a power law [24]. For natural language, it has been shown that Zipf’s law approximates the distribution of probabilities of letter or words across a corpus of sufficient size for the larger probabilities [25]. The universality of Zipf’s law has been challenged and in particular, it has been shown to arise as a result of the choice of rank as an independent variable [26, 27]. Nevertheless, Zipfian laws have been proven to be useful as a tool enabling the transformation between the ranking of symbolic events and an estimate of their expected probabilities. Hence, it is useful in forming a model of symbolic information transmission which is organized on the basis of sentences made by words in interaction with each other (this may be considered in a general sense of natural sequences, not just human language) [30]. Thus, Zipfian based models can be useful as a tool for describing the probabilistic rankings of the symbols employed in natural sequences.

For the calculation of entropy, the accuracy will depend on the accuracy of calculating the set of probabilities. It follows that we might expect the accuracy of the probability calculations will be determined by the most infrequent event occurrences. Therefore, the number of samples required to estimate the probability of the least frequent event \( p_0(r) \) determines the number of samples required to estimate the entropy for the corresponding set of data.

We proceed by imposing a probabilistic model of the symbolic events for a given sequence. Since we are dealing with natural sequences of symbolic data, we consider a Zipfian model approach. The most basic form of Zipf’s law models the frequency rank \( r \) of a word, i.e. the \( r \)-th most frequent word, by a simple inverse power law, such that the frequency of a word \( f(r) \) scales according to an equation which is approximately
\[ f(r) \propto \frac{1}{r^\alpha} \] (22)
where a constant of proportionality dependent on the particular corpus may be introduced, [26] and where typically \( \alpha \approx 1.1 \). Thus, if \( p_i(x) \) follows a Zipfian law, then \( p_0(x) \propto 1/M \) and \( p_i(x) = \varphi f(r) \). Numerous other variations of this general law have been proposed to provide more accurate representations, including the Bradford Law, Lotka Law [31, 32]. A more well known approach is the Zipf-Mandelbrot law [33]. Given symbols \( x \in \Sigma^{M+1} \) from an alphabet of size \( M + 1 \) which includes a blank space \( w_s \), then for any random word of length \( L \), given by \( v_k(L) = \{ w_s, x_1, \ldots, x_L, w_S \} \), \( k = 1, \ldots, M^L \) the frequency of occurrence is determined as
\[ p_i(L) = \frac{\lambda}{(M + 1)^L} \] (23)
then Li showed that \( \lambda \) can be determined via the summation of all probabilities of such words [26], hence
\[ \sum_{L=1}^{\infty} M^L p_i(L) = 1 \] (24)
\[ \sum_{L=1}^{\infty} M^L \frac{\lambda}{(M + 1)^L} = \frac{\lambda M}{(M + 1)^{L+2}} \] (25)
\[ \lambda = \frac{(M + 1)^2}{M} \] (26)
which leads to
\[ p_i(L) = \frac{1}{M(M + 1)^L} \] (27)

Now, defining the rank of a given word \( v_k(L) \) as \( r(L) \), then after performing an exponential transformation from the word

A word or N-gram is not necessarily referring to human language, but indicates a specific set of sequentially occurring symbols.
length to word rank, the probability of occurrence of a given word in terms of rank can be defined as [28], [34]:

\[ p(r) = \frac{\gamma}{(r + \beta)^\alpha} \]  

(28)

where, for iid samples, Li showed the constants can be computed as [26]:

\[ \alpha = \log_2(M + 1)^\beta = \frac{M}{M + 1}, \gamma = \frac{M^{\alpha - 1}}{(M - 1)^\alpha} \]  

(29)

We introduce a normalization step as follows. Since

\[ \sum_{i=1}^{M} p(i) = 1, \sum_{i=1}^{\infty} \frac{\gamma}{(r + \beta)^\alpha} = \kappa \]  

we introduce

\[ \gamma' = \frac{\gamma}{\kappa} \]  

(31)

which leads to

\[ p(r) = \frac{\gamma'}{(r + \beta)^\alpha} \]  

(32)

Now we have

\[ p(r) = \frac{M^{\alpha - 1}}{([M - 1](r + \beta)]^\alpha} \]  

(33)

We seek to determine

\[ \varphi_0(x, r) = \inf_r \{|p(x, r - 1) - p(x, r)|, x \in X, r \in [1, M]\} \]  

(34)

Hence we define,

\[ \theta_j = d_j - d_{j+1}, \]  

(35)

where

\[ d_j = \frac{p(r - j) - p(r - j + 1)}{\gamma'} \]  

(36)

\[ = \frac{1}{(r - j + \beta)^\alpha} - \frac{1}{(r - j + 1 + \beta)^\alpha} \]  

(37)

then for \( M \geq 1, \alpha > 1 \), and hence \( \tilde{\theta}_j > \theta_j \), where we seek to establish whether \( \tilde{\theta}_j > 0 \) (it is not). For notational convenience, let \( \phi_j = r - j + \beta \), then we define

\[ \tilde{d}_j = \frac{1}{\phi_j} - \frac{1}{\phi_{j+1}} \]  

(38)

and

\[ \tilde{\theta}_j = \left( \frac{1}{\phi_j} - \frac{1}{\phi_{j+1}} \right) - \left( \frac{1}{\phi_{j+1}} - \frac{1}{\phi_j} \right) \]  

(39)

\[ = \frac{\phi_j - \phi_{j+1}}{\phi_j\phi_{j+1}} - \frac{\phi_j - \phi_{j+1}}{\phi_j\phi_{j+1}} \]  

(40)

\[ \tilde{\theta}_j = \phi_{j+1}(\phi_j - \phi_{j+1}) - \phi_j(\phi_j - \phi_{j+1}) \]  

(41)

\[ \phi_{j+1}(\phi_j - \phi_{j+1}) - \phi_j(\phi_j - \phi_{j+1}) \]  

(42)

\[ = (\phi_{j+1} - \phi_{j+1}) \]  

(43)

as \( \phi_{j+1} - \phi_j = 1 \), and \( \phi_j - \phi_{j+1} = 1 \). Therefore,

\[ \tilde{d}_j = (r - (j + 1) + \beta) - (r - (j - 1) + \beta) \]  

(44)

\[ = -2 \]  

(45)

Hence, \( \tilde{\theta}_j < 0 \) and since \( \tilde{\theta}_j > \theta_j \), therefore \( \theta_j < 0 \). Hence it follows that \( \tilde{d}_j < \tilde{d}_{j+1} \), and

\[ p(r - j) - p(r - j + 1) < p(r - (j + 1)) - p(r - j) \]  

(46)

By induction, we have

\[ \varphi_0(x, r) = p(M - 1) - p(M) \]  

(47)

which provides a result which we use in the following to determine an expression for \( N_0 \). Hence, using empirical probabilities we can solve (18) as

\[ \Delta_0 = \hat{p}(M - 1) - \hat{p}(M) \]  

(48)

Now, since

\[ \sup \left\{ \hat{p}(r) + \frac{\Delta_0}{2} \right\} = \inf \left\{ \hat{p}(r - 1) - \frac{\Delta_0}{2} \right\}_{r=M}, \]  

(49)

then we have

\[ \epsilon_r \geq \frac{\Delta_0}{4} \]  

(50)

Hence, the value of \( n_0 \) can be calculated using the DKW inequality and the result in (10) as follows.

\[ n_0 = \frac{1}{2\epsilon^2} \ln \left( \frac{2}{1 - \zeta} \right) \]  

(51)

\[ \leq \frac{1}{2 \left( \frac{\Delta_0}{4} \right)^2} \ln \left( \frac{2}{1 - \zeta} \right) \]  

(52)

\[ \leq \frac{8}{\Delta_0} \ln \left( \frac{2}{1 - \zeta} \right) \]  

(53)

Hence, from (5), the minimum number of observations required to estimate the entropy is

\[ N_0 = \frac{n_0}{\hat{p}_0(r)} \]  

(54)

where,

\[ \hat{p}_0(r) = \hat{p}(M) \]  

(55)

\[ = \frac{M^{\alpha - 1}}{[(M - 1)(M + \beta)]^\alpha} \]  

(56)

Now, this gives the number of observations required to estimate the entropy within a specified degree of confidence and within specified bounds by considering the smallest probabilities used. This algorithm is suitable for estimating the minimum number of samples required to compute the \( N \)-gram entropy of a sequence.

(41)

D. Samples required for coarse entropy classification

Suppose we wish to detect major differences between entropies \( H_i, H_{i+1} \) due to changes in the most frequent symbol probabilities. What then is the minimum number of samples \( N_0 \) required? In this case, to find \( N_0 \) for \( \sup(\epsilon_0) \), implies detecting changes due to \( p(r), p(r + 1), \ldots \) where \( r \ll M \), e.g. \( r = 1, 2 \). Hence define a new alphabet size \( M_c < M \), then since

\[ \sum_{i=1}^{M_c} \hat{p}(i) \approx \sum_{i=1}^{M} \hat{p}(i) \]  

(57)
we may omit \{\hat{p}(i)\} ∀i > M_c (i.e. the most infrequently occurring symbols), and thus we select

\[ \Delta_0 = \hat{p}(M_c - 1) - \hat{p}(M_c) \] (58)

where the final number of samples may be approximated directly from (53) and (54) substituting \( M_c \) for \( M \). Similarly, this admits other related approaches to varying \( \Delta_0 \) for example, the top \( q\% \) of the probabilities, e.g.

\[ r = \left\lceil \frac{M}{4} \right\rceil \] (59)

An example of using this approach is given in the next section.

### III. Example results

1) First order entropy of English text : Consider the following example of calculating the number of samples required to determine the entropy. Let the alphabet size be \( M = 26 \), and confidence level of the calculation be 95%, i.e. \( \zeta' = 0.95 \). Hence,

\[ n = \frac{8}{\Delta^2_{0}} \ln \left( \frac{2}{1 - 0.95} \right) \] (60)

where the probabilities are computed from (32) as

\[ \hat{p}(26) = \frac{\gamma'}{(26 + \beta)^{\alpha}} \] (61)

with the parameters computed as:

\[ \alpha = \frac{\log_2(26 + 1)}{\log_2(26)} \approx 1.012, \quad \beta = \frac{26}{26 + 1} \approx 0.963, \] (62)

\[ \gamma' = \frac{\kappa 26^{\alpha - 1}}{(26 - 1)^{\alpha}} \approx 0.351 \] (63)

Hence, it can be found that \( \Delta_0 = 4.88 \times 10^{-4} \), and therefore \( n = 1.24 \times 10^8 \). Now, this implies the total number of observations required is then

\[ \hat{N}_0 = \frac{1.24 \times 10^8}{\hat{p}(26)} = 1.06 \times 10^9 \] (64)

Thereby, this gives an estimate of the number of observation samples which may be required in order to obtain an estimate of the entropy which takes into account the smallest contributing probabilities, i.e. the most infrequent symbols. The value of approximately 1 billion samples appears to be consistent with reports in the literature, and provides a useful indication of the upper bound required to compute entropy in this case.

2) Coarse entropy classification for small alphabet size : Consider the following example of calculating the number of samples required to detect the difference between entropies by finding the least number of samples required to detect major changes in entropy due to changes in the most frequent symbol probabilities. Let the alphabet size of interest be \( M = 3 \), and confidence level of the calculation be 75%, i.e. \( \zeta' = 0.75 \). Hence,

\[ n = \frac{8}{\Delta^2_{0}} \ln \left( \frac{2}{1 - 0.75} \right) \] (65)

It can be found that \( \Delta_0 = 0.223 \) and therefore \( n = 334 \). Now, this implies the total number of observations required is then

\[ \hat{N}_0 = \frac{334}{\hat{p}(2)} = 1151 \] (66)

Hence, this gives an estimate of the number of samples required in order to distinguish two different natural sequences using the most frequently occurring symbols.

This result is demonstrated by simulation in Figs. 1 and 2. In this case, a small alphabet size of \( M = 3 \) was selected which admits a set of probabilities following a Zipfian law. Using ensemble averaging, a set of data giving rise to empirical probabilities and hence entropies was generated for a series of differing sample values \( (N) \). The mean square error (mse) between the true and estimated entropies was obtained as a function of the number of samples. The efficacy of the method can then be readily observed particularly in the linear scale plot (Fig 2).

### IV. Conclusions

Shannon entropy is a well known method of measuring the information content in a sequence of probabilistic symbolic events. In this paper, we have proposed a method of estimating the number of samples required to estimate the Shannon Entropy for natural sequences. Using a modified Zipf-Mandelbrot-Li law and the Dvoretzky-Kiefer-Wolfowitz
inequality, we propose a model which yields an estimate for the minimum number of samples required to obtain an estimate of entropy with a given confidence level and degree of accuracy. Examples have been given which show the efficacy of the proposed methodology. It would be of interest to apply this method to various real world applications to compare the theoretical results against experimentally obtained results. In terms of information theoretic analytical tools, it may be of interest to consider just how few samples may be required in order to obtain useful results. Future improvements may possibly by re-considering some of the assumptions, such as iid samples for the parametrization of the Zipf-Mandelbrot-Li model.

REFERENCES

[1] C. E. Shannon, “A mathematical theory of communication (parts I and II),” Bell System Technical Journal, vol. XXVII, pp. 379–423, 1948.
[2] ———, “A mathematical theory of communication (part III),” Bell System Technical Journal, vol. XXVII, pp. 623–656, 1948.
[3] ———, “Prediction and entropy of printed English,” Bell System Technical Journal, pp. 50–64, 1951.
[4] G. Barnard, “Statistical calculation of word entropies for four western languages,” IRE Transactions Inf. Theory, pp. 49–53, March 1955.
[5] J. Herrera and P. Purv, “Statistical keyword detection in literary corpora,” Eur. Phys. J. B, vol. 63, pp. 135–146, 05 2008.
[6] B. Allen, M. Kon, and Y. Bar-Yam, “A new phylogenetic diversity measure generalizing the shannon index and its application to phyllostomid bats,” American Naturalist, vol. 174, no. 2, pp. 236–243, 2009.
[7] C. Rao, “Diversity and dissimilarity coefficients: a unified approach,” Theoretical Population Biology, vol. 21, pp. 24–43, 1982.
[8] R. Lee, P. Jonathan, and P. Ziman, “Fictitious symbols revealed as a written language through application of shannon entropy,” Proc. R. Soc. A, vol. 466, pp. 2545–2560, 2010.
[9] R. A. Khan, A. Meyer, H. Konik, and S. Bouakaz, “Facial Expression Recognition using Entropy and Brightness Features,” in 11th Int. Conf. on Intell. Sys. Design and Appl.(ISDA), IEEE, Ed., Dec 2011.
[10] S. Fuhrman, M. J. Cunningham, X. Wen, G. Zweiger, J. J. Seilheimer, and R. Somogyi, “The application of Shannon entropy in the identification of putative drug targets,” Biosystems (A5E), vol. 55, no. 1–3, pp. 5–14, 2000.
[11] W. Ebeling and T. Pöschel, “Entropy and long-range correlations in literary English,” Europhysics Letters, vol. 26, no. 4, p. 241, 1994.
[12] I. Moreno-Sánchez, F. Font-Clos, and Á. Corral, “Large-scale analysis of Zipf’s law in English texts,” PLOS ONE, vol. 11, no. 1, 01 2016.