The Pinch Technique Approach To
Gauge-Independent $n$-Point Functions

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Abstract
The pinch technique (PT) is an algorithm for the rearrangement of contributions to
conventional, gauge-dependent $n$-point functions in gauge theories to obtain a formul-
ation of one-loop perturbation theory in terms of “effective” $n$-point functions which,
principle among many other desirable properties, are individually entirely independent of
the particular gauge-fixing procedure used. The aim of this talk is to give an introduc-
tory account of (i) the phenomenological motivations for the PT approach, (ii) the PT
algorithm, including the principle properties of the PT $n$-point functions, (iii) an example
application, and (iv) recent progress in extending the PT beyond the one-loop level.

1. Motivation
In order to quantize a gauge theory, it is necessary to remove the redundant degrees of
freedom resulting from the gauge symmetry. The standard procedure thus involves adding to
the classical lagrangian a gauge-fixing term, together with an associated Fadeev-Popov ghost
term:

$$\mathcal{L}\text{(classical)} \rightarrow \mathcal{L}\text{(classical)} + \mathcal{L}\text{(gauge-fixing)} + \mathcal{L}\text{(ghost)}.$$  

The gauge symmetry of the classical lagrangian is thus replaced by the BRST symmetry
of the gauge-fixed lagrangian. This procedure has two immediate consequences. First, the
individual $n$-point (Green’s) functions in general depend explicitly and non-trivially on the
particular choice of the gauge-fixing term in (1). Second, for a non-abelian theory, the
ensemble of $n$-point functions satisfy complicated Slavnov-Taylor identities involving ghost
fields and associated with the BRST invariance of the gauge-fixed lagrangian, rather than
the gauge invariance of the original classical lagrangian.

As long as one is interested in computing strictly order-by-order in perturbation theory
$S$-matrix elements for the scattering of on-shell fields, neither of the above two facts presents
a problem. In particular, order-by-order the $S$-matrix element (assumed to be defined) for
a given process is guaranteed by general proofs to be independent of the given gauge-fixing
procedure. Thus, one is at liberty to choose the gauge (assumed to be renormalizable etc.) in
which the calculations are simplest, safe in the knowledge that the final result, to any order
in perturbation theory, must be independent of this choice.

However, there are (at least) two classes of application for which the conventional approach
to perturbation theory just outlined is not sufficient:

1. **Partial summations of perturbation theory.** There are various situations in which one
wishes to sum up some infinite subset of contributions to a given process, i.e. go beyond
order-by-order computation. The simplest example of this is the Dyson summation of
the self-energy of a given field. Such a summation occurs in two closely-related cases:

   - **Unstable particles.** In order to regulate the pole which occurs in the tree level
     propagator for such particles, one is obliged to Dyson-sum the self-energy so that
     it appears in the denominator of the radiatively-corrected propagator. The imaginary
     part of the self-energy then specifies the decay width of the given particle.
• **Effective charges.** In order to provide a well-defined basis for renormalon analyses, and also to improve predictions in phenomenology, one would like to be able to extend directly at the diagrammatic level the Gell-Mann–Low concept of an effective charge from QED to non-abelian gauge theories.

In both cases, the fundamental problem is the ambiguity of the self-energies involved due to their gauge dependence: in general, the self-energies are gauge-dependent at all $q^2$ away from the propagator pole position. Furthermore, even if this gauge dependence is “mild” (usually a meaningless concept), the naive Dyson summation in general leads to potentially catastrophic violations of gauge invariance (i.e. current conservation) when embedded in a matrix element.

2. **Explicitly off-shell processes.** The second class of applications occurs when one wishes to consider amplitudes for processes in which some or all of the external fields are off-shell. Such cases include:

• **Form factors.** A popular way of comparing theory with experiment is via the use of form factors to parameterize loop corrections to vertices involving off-shell fields, e.g. the $\gamma W^+W^-$ and $ZW^+W^-$ vertices measured at LEP. Similarly, electroweak “oblique” i.e. self-energy corrections are often parameterized in terms of $S, T, U$.

• **Matching of full and effective gauge theories.** The description of the low-energy physics of a theory involving light and heavy fields via an effective lagrangian requires a matching at low energies of the off-shell $n$-point functions of the full and effective theories.

• **Schwinger-Dyson analyses.** While in principle able to give exact, non-perturbative information, in practice the Schwinger-Dyson equations for gauge theories require some truncation in order to be tractable, together with the use of perturbative $n$-point functions as building blocks.

Again, in all of these examples, a fundamental problem is the ambiguity of the $n$-point functions involved due to their gauge dependence. At best, in the above cases, this ambiguity necessitates very careful consideration when comparing theory with experiment (e.g. when looking for signals of “new physics”); at worst, it renders analyses useless.

The PT provides a framework for perturbation theory in which these problems are tackled at source. In particular, the $n$-point functions obtained in the PT approach are gauge-independent, i.e. they are entirely independent of the particular gauge-fixing procedure used (independent of $\xi, n_\mu$ etc.), and also gauge-invariant, i.e. they satisfy simple tree-level-like Ward identities associated with the gauge symmetry of the classical lagrangian of the theory. Note that, throughout, we distinguish between gauge independence and gauge invariance; the former does not follow from the latter. It is important to emphasize that, in addition to being both gauge-independent and gauge-invariant, the PT $n$-point functions display a wide variety of further desirable theoretical properties.

2. **The Pinch Technique Algorithm**

The PT is based on the observation that one-loop Feynman diagrams that naively contribute to a given $n$-point function in fact in general contain components identical in structure to diagrams which contribute to $m$-point functions, $m < n$. In order to see this, consider the now-canonical case of quark-quark scattering at one loop in QCD. The corrections to the
the conventional one-loop gluon self-energy (2-point function) is given by

\[ \Pi^{(1)}(q^2, \xi) = i(q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi^{(1)}(q^2, \xi) , \] (2)

with, for \( n_f \) flavours of massless quark, in \( d = 4 - 2\epsilon \) dimensions,

\[ \Pi^{(1)}(q^2, \xi) = \frac{g^2}{16\pi^2} \left\{ \left[ \frac{3\xi - 13}{6} C_A + \frac{4}{3} T_F n_f \right] - C_{UV} + \ln \left( \frac{-q^2}{\mu^2} \right) \right\} + \frac{9\xi^2 + 18\xi + 97}{9} C_A - \frac{20}{9} T_F n_f \right\} - (Z_3 - 1)^{(1)} , \] (3)

where \( \mu \) is the 't Hooft mass scale and \( C_{UV} = \epsilon^{-1} + \ln(4\pi) - \gamma_E \). However, the vertex and box diagrams also have components—the so-called “pinch parts”—which have exactly the same structure as the gluon self-energy correction. The identification of these pinch parts involves two steps: first, the isolation in the \textit{integrands} for the diagrams of all factors of longitudinal four-momentum associated with the gauge fields propagating in the loops; and second, the systematic use of the elementary Ward identities obeyed by the tree level vertices when contracted with these longitudinal factors.

The contribution of this diagram is given in standard notation by

\[
\text{Fig. 1a} = \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} i g \gamma_\mu T^m i D^{\mu\nu}(q) g f^{mrst} \Gamma_{\mu\rho\sigma}(q, k_1, k_2) i D^{\rho\sigma'}(k_1) i D^{\sigma\rho'}(k_2) i g \gamma_\sigma T^s i S(p - k_1) i g \gamma_\rho T^r ,
\] (4)

where \( S(p - k_1) = (\not{p} - \not{k_1} - m + i\epsilon)^{-1} \) is the internal fermion propagator (here \( m = 0 \)). In Eq. (4), the sources of longitudinal internal gauge field four-momentum \( k_{1\rho}, k_{2\sigma} \) are the two internal gauge propagators and the triple gauge vertex. For the latter, one writes

\[
\Gamma_{\mu\rho\sigma}(q, k_1, k_2) = (k_1 - k_2)_\mu g_{\sigma\rho} - 2q_\rho g_{\sigma\mu} + 2q_\sigma g_{\rho\mu} - k_{1\rho} g_{\sigma\mu} + k_{2\sigma} g_{\rho\mu} .
\] (5)

The component \( \Gamma^{F}_{\mu\rho\sigma}(q; k_1, k_2) \) contributes no factors of longitudinal internal four-momentum \( k_{1\rho}, k_{2\sigma} \) and obeys a Ward identity involving the difference of inverse Feynman gauge propagators:

\[
g^{\mu} \Gamma^{F}_{\mu\rho\sigma}(q; k_1, k_2) = D^{F}_{\rho\sigma}(k_1)^{-1} - D^{F}_{\rho\sigma}(k_2)^{-1} .
\] (6)
When the longitudinal factors from the internal gauge propagators and the triple gauge vertex in Eq. (4) are contracted with the corresponding Dirac matrices occurring in the tree level gauge-fermion vertices, they trigger the elementary Ward identities

\[ k_1 = S^{-1}(p) - S^{-1}(p - k_1), \quad k_2 = S^{-1}(p - k_1) - S^{-1}(p + q). \]  

(7)

The inverse fermion propagators \( S^{-1}(p - k_1) \) cancel ("pinch") the internal fermion propagator in Eq. (4), leading to the pinch part of the diagram shown schematically in Fig. 1b. Using \( f_{mrs}T^sT^r = -\frac{1}{2}i T^m \), we see that the pinch part of the vertex (3-point) diagram couples to the pair of external fermion lines in exactly the same way as the conventional gluon self-energy (2-point) diagram. The terms in Eq. (4) involving no longitudinal factors, i.e. the Feynman gauge component \( D_F \) of the gauge propagators and the component \( \Gamma^F \) of the triple gauge vertex, give the “non-pinched” or “genuine” vertex contribution shown in Fig. 2c. The terms proportional to \( S^{-1}(p) \) and \( S^{-1}(p + q) \) in the Ward identities Eqs. (7) give the contribution denoted by the ellipsis in Fig. 1. This last contribution vanishes when the external quarks are on-shell.

A similar procedure may be carried out for the remaining one-loop vertex and box diagrams for the process in order to isolate their self-energy-like pinch parts. Adding these pinch parts to the conventional self-energy, one obtains the PT gluon “effective” 2-point function \( \hat{\Pi}^{(1)}(q^2) \):

\[ \hat{\Pi}^{(1)}(q^2) = \frac{g^2}{16\pi^2} \left\{ \beta_0 \left[ -C_{UV} + \ln \left( -\frac{q^2}{\mu^2} \right) \right] + \frac{67}{9} C_A - \frac{20}{9} T_n f \right\} - (Z_3 - 1)^{(1)}, \]  

(8)

where \( \beta_0 = -\frac{11}{3} C_A + \frac{4}{3} T_n f \) is the first coefficient of the \( \beta \) function. It is seen from the above expression that the PT self-energy is entirely gauge-independent, with the coefficient of the logarithm given by the \( \beta \) function, just like the vacuum polarization in QED.

The PT “effective” \( n \)-point functions, \( n = 3, 4, \ldots \), i.e. vertices, boxes etc., may be similarly constructed \([3-8]\). For example, the PT one-loop triple gauge vertex (3-point) function \( \hat{\Gamma}^{(1)}_{\alpha\beta\gamma}(q_1, q_2, q_3) \) may be obtained from the consideration of diagrams with three external quark lines. It is found \([3]\) that this function is (i) gauge-independent, and (ii) obeys a simple tree-level-like Ward identity associated with the gauge invariance of the classical lagrangian:

\[ q_1^{\alpha \hat{\Gamma}^{(1)}\alpha\beta\gamma}(q_1, q_2, q_3) = \hat{\Pi}^{(1)}_{\beta\gamma}(q_2) - \hat{\Pi}^{(1)}_{\beta\gamma}(q_3), \]  

(9)

where \( \hat{\Pi}^{(1)}_{\beta\gamma}(q) = (q^2 g_{\beta\gamma} - q_{\beta} q_{\gamma}) \hat{\Pi}^{(1)}(q^2) \). It is emphasized that, in the conventional approach in ordinary covariant gauges, the above Ward identity holds for the fermionic contributions but not for the bosonic contributions.

The above cases are the simplest examples of the implementation of the PT algorithm. In general, the PT is a systematic rearrangement of contributions to conventional one-loop \( n \)-point functions according simply to the structure of their couplings to external fields. This rearrangement takes place at the level of the Feynman integrands, and results in the set of PT one-loop “effective” \( n \)-point functions for the given theory, be it unbroken, e.g. QCD, or spontaneously broken, e.g. the electroweak standard model. The principle properties of the PT one-loop \( n \)-point functions may be summarised as follows:

- **Gauge independence.** The PT one-loop \( n \)-point functions are entirely independent of the particular gauge-fixing procedure used, be it covariant or non-covariant, linear or non-linear \([1-13]\).
• **Gauge invariance.** The PT one-loop $n$-point functions satisfy simple tree-level-like Ward identities associated with the gauge invariance of the classical lagrangian [1–8].

• **Renormalizability.** The PT one-loop $n$-point functions are multiplicatively renormalizable by local counterterms, regardless of the renormalizability of the initial gauge [1–8].

• **Universality.** The PT one-loop $n$-point functions are independent of the quantum numbers of the external fields in the particular process from which they are obtained [9, 11].

• **Analyticity and unitarity.** The PT one-loop $n$-point functions display properties of analyticity and unitarity exactly analogous to those of scalar field theories [10, 12].

• **Relation to observables.** The PT one-loop $n$-point functions are in principle directly related to physical observables via dispersion relations [12].

• **Feynman rules.** The PT one-loop $n$-point functions for any given gauge theory may in fact be obtained directly using a well-defined set of Feynman rules [17–19].

In addition to these general properties, specific PT $n$-point functions display further important properties. In particular, it has been shown that the PT 2-point functions may be summed in Dyson series [10, 11], and that they do not shift the position of the propagator pole [10]. It is the above collection of properties which has led to the PT being advocated as the appropriate theoretical framework for the wide range of applications described in the introduction for which the conventional perturbative approach is inadequate [10–13, 16].

### 3. Relation to the Background Field Method

It was observed some time ago [17–19] that the PT one-loop $n$-point functions coincide with the background field $n$-point functions obtained in the background field method (BFM) in the Feynman-like quantum gauge $\xi_Q = 1$. This correspondence holds for the cases both of unbroken and spontaneously broken gauge symmetry. To obtain an heuristic explanation of this correspondence [17–20], consider the simpler unbroken case, e.g. QCD again.

The basic idea of the BFM [21] is first to make a shift of the gauge field variable in the classical lagrangian: $A \rightarrow \tilde{A} + A$, where $\tilde{A}$ is the background gauge field and $A$ is the quantum gauge field. The quantum field is the functional integration variable. The gauge-fixing term for the background gauge fields may then be chosen entirely independently. Demanding that the shift be made such that the quantum gauge field has vanishing vacuum expectation value, the background gauge fields are the classical fields which do not propagate in loops (since the path integral is only over the quantum fields), while the quantum gauge fields do not appear as external fields (since $\langle A^a_\mu \rangle = 0$).

The gauge-fixing term for the quantum field is then chosen so as to retain invariance of the generating functional under gauge transformations of the background field:

$$\mathcal{L}(\text{quantum gauge-fixing}) = -\frac{1}{2\xi_Q} \left[ D^a_{\mu}(\tilde{A}) A_b^{b\mu} \right]^2,$$  \hspace{1cm} (10)

where $D^a_{\mu}(\tilde{A})$ is the background covariant derivative. The gauge-fixing term for the background gauge field may then be chosen entirely independently. Demanding that the shift be made such that the quantum gauge field has vanishing vacuum expectation value, the background gauge fields are the classical fields which do not propagate in loops (since the path integral is only over the quantum fields), while the quantum gauge fields do not appear as external fields (since $\langle A^a_\mu \rangle = 0$).

As a result of the explicitly-retained background gauge invariance, the background field $n$-point functions are gauge-invariant, i.e. they satisfy to all orders simple tree-level-like Ward identities associated with the gauge invariance of the classical lagrangian. However, the background field $n$-point functions remain gauge-dependent, i.e. they still depend explicitly on the gauge-fixing parameter $\xi_Q$. 
At the level of Feynman rules, the effect of the non-linear gauge-fixing Eq. (10) is to alter not only the quadratic quantum gauge field terms $AA$ from the classical lagrangian, but also the cubic and quartic terms $\tilde{A}AA$ and $\tilde{A}AA$ [21]. In particular, the Feynman rule for the $\tilde{A}AA$ vertex is given by (cf. Eq. (5))

$$\tilde{A}_\mu^m(q)A^r_{\nu}(k_1)A^s_{\sigma}(k_2) : g^F_{mrs} \left( \Gamma^F_{\mu\nu\sigma}(q; k_1, k_2) + (1 - \xi_Q^{-1}) \Gamma^P_{\mu\nu\sigma}(q; k_1, k_2) \right). \quad (11)$$

Thus, choosing the Feynman-like quantum gauge $\xi_Q = 1$, neither the quantum gauge field propagator nor the background-quantum-quantum triple gauge vertex supplies factors of longitudinal gauge field four-momentum. Thus, in this particular gauge, at the one-loop level, there are no longitudinal factors to trigger the PT rearrangement (there is “no pinching”). In this way, one understands at an heuristic level how the Feynman rules of the BFM with $\xi_Q = 1$ give the PT one-loop $n$-point functions directly.

The relation between the PT and the BFM may be summarised as follows:

1. For $\xi_Q = 1$ in the BFM, for both unbroken and spontaneously broken theories, at the one-loop level there are no factors of longitudinal internal gauge field four-momentum to trigger the PT rearrangement. The BFM at $\xi_Q = 1$ thus provides the gauge in which one obtains the PT one-loop $n$-point functions directly [17,19].

2. For $\xi_Q \neq 1$, the PT algorithm can be implemented in the BFM formalism to obtain, once again, the PT gauge-independent one-loop $n$-point functions for a given theory, unbroken or broken [20].

3. For $\xi_Q \neq 1$, the BFM $n$-point functions display properties (e.g. imaginary parts with unphysical thresholds [10,12]) that preclude their use in the cases described in the introduction. The PT is thus not just a special case of the BFM, as claimed in [17].

4. Beyond one loop, the correspondence between the PT $n$-point functions and those obtained in the BFM with $\xi_Q = 1$ does not persist [14,15]. The correspondence at one loop is thus simply a convenient fortuity.

4. Relation to Physical Observables

It is well known that in QED with fermions $f = e, \mu, \ldots$, the imaginary part of, e.g., the muon contribution to the one-loop vacuum polarization is directly related to the tree level cross section for the process $e^+e^- \to \mu^+\mu^-$. This relation is a result purely of the unitarity of the $S$-matrix $S = 1 + iT$, expressed in the optical theorem for the particular case of forward scattering in the process $e^+e^- \to e^+e^-$:

$$\text{Im} \langle e^+e^- | T | e^+e^- \rangle = \frac{1}{2} \sum_i \int d\Gamma_i |\langle i | T | e^+e^- \rangle|^2. \quad (12)$$

In Eq. (12), the sum on the r.h.s. is over all on-shell physical states $|i\rangle$ with the quantum numbers of $|e^+e^-\rangle$; in each case, the integral is over the available phase space $\Gamma_i$.

Consider now the case of the standard electroweak model, in particular the tree level contribution of the gauge boson pair $|i\rangle = |W^+W^-\rangle$ to the r.h.s. of Eq. (12), i.e. the process $e^+e^- \to W^+W^-$. The $T$-matrix element for this process is given by

$$\langle W^+W^- | T | e^+e^- \rangle = \epsilon^\mu_\nu(p_1)\epsilon^\nu_\sigma(p_2) \Theta(k_2)T^{\mu\nu}u(k_1), \quad (13)$$
three self-energy-like components are proportional at high energy to the \( \beta \) \( d \) the components 

after the systematic and exhaustive implementation of this Ward identity. It is found [12] that 

individually diverge at high energy, and (ii) specifies the non-vanishing contributions, of 

purely non-abelian origin, which remain after this cancellation. The components 

the PT decomposition are then defined by the 

\( i, j \) 

\( s \), and so violate unitarity. Nevertheless, their sum is well-behaved. However, 

the high-energy behaviour of the \( d\sigma/d\Omega \) precludes the use of the optical theorem [12] 

to interpret individually the components \{\( \sigma_{\gamma\gamma}, \sigma_{\gamma Z}, \sigma_{ZZ} \)\}, \{\( \sigma_{\nu\nu}, \sigma_{Z\nu} \)\} and \{\( \sigma_{\nu\nu} \)\} of the tree 

level cross section for \( e^+e^- \to W^+W^- \) in terms of the imaginary parts of renormalizable 

one-loop self-energy-, vertex- and box-like \( W^+W^- \) contributions, respectively, to the process 

\( e^+e^- \to e^+e^- \).

In the PT decomposition [12], one first contracts into \( (\overline{T}_{\mu\nu}u), (\overline{v}T_{\mu\nu}^\ast u) \ast \) the longitudinal 

correlation length appearing in the \( W^\pm \) polarization sums in Eq. (15), triggering the elementary 

Ward identity obeyed by \( T_{\mu\nu} \). This Ward identity (i) enforces the exact cancellation among 

contributions from diagrams with distinct s- and t-channel propagator structure and which 

individually diverge at high energy, and (ii) specifies the non-vanishing contributions, of 

purely non-abelian origin, which remain after this cancellation. The components \( d\sigma_{ij}/d\Omega \) in 

the PT decomposition are then defined by the \( i, j = \gamma, Z, \nu \) propagator structure obtained 

after the systematic and exhaustive implementation of this Ward identity. It is found [12] that 

the components \( d\sigma_{ij}/d\Omega \) are individually well-behaved at high energy. In particular, the 

three self-energy-like components are proportional at high energy to the \( W \) contribution to 

the corresponding electroweak \( \beta \) functions (and so are negative).
Fig. 3. The schematic representation of the LEP2 process $e^+e^- \rightarrow W^+W^- \rightarrow 4$ fermions.

One may then define, by direct analogy with QED, the imaginary parts of one-loop $W^+W^-$ (including associated would-be Goldstone boson and ghost) contributions to self-energy-like functions $\hat{\Sigma}_{ij}$ directly in terms of the corresponding self-energy-like components of the tree level $e^+e^- \rightarrow W^+W^-$ cross section:

$$ e^2 \frac{1}{s^2} \text{Im} \hat{\Sigma}_{\gamma\gamma}(s) = \hat{\sigma}_{\gamma\gamma}(e^+e^- \rightarrow W^+W^-) $$

$$ 2e^2 \frac{a}{s_wc_w} \frac{1}{s(s-M_W^2)} \text{Im} \hat{\Sigma}_{\gamma Z}(s) = \hat{\sigma}_{\gamma Z}(e^+e^- \rightarrow W^+W^-) $$

$$ e^2 \left( \frac{a^2 + b^2}{s_w^2c_w^2} \right) \frac{1}{(s-M_Z^2)^2} \text{Im} \hat{\Sigma}_{ZZ}(s) = \hat{\sigma}_{ZZ}(e^+e^- \rightarrow W^+W^-) , $$

where $s_w$ ($c_w$) is the sine (cosine) of the weak mixing angle, $a = \frac{1}{2} - s_w^2$, $b = \frac{1}{4}$. Given the imaginary parts of the self-energy functions, the full renormalized functions may then be constructed from twice-subtracted dispersion relations. It is found that the resulting functions are precisely those obtained at one loop in the PT [12].

The significance of this result is that the individual differential cross section components $d\hat{\sigma}_{ij}/d\Omega$ can in principle be projected out from the full differential cross section [12]. In this way, the PT self-energy functions are directly related to physical observables.

5. Example Application: Unstable Particles

Perhaps the most important application of the PT is to the physics of unstable particles [10, 13]. For example, consider the LEP2 process $e^+e^- \rightarrow W^+W^- \rightarrow 4$ fermions involving intermediate W pair production, as shown in Fig. 3. At tree level, the W propagators are singular at $p_i^2 = M_W^2$, $i = 1, 2$. Clearly, in order to evaluate amplitudes for such processes at arbitrary values of the kinematic parameters, it is essential to regulate such singularities. This involves the introduction in some way of a finite decay width.

Originally, two approaches were popular. In the “fixed width” approach, one makes the systematic replacement $(q^2-M^2)^{-1} \rightarrow (q^2-M^2+iM\Gamma)^{-1}$ for the propagators, where $\Gamma$ is the physical width of the particle of mass $M$. However, the introduction of a non-zero width for space-like $q^2$ is clearly unphysical, and leads to violations of unitarity. In the “running width” approach, the introduction of a $q^2$-dependent width $\Gamma(q^2)$ enables unitarity to be restored. In general, however, both of these approaches lead to violations of gauge invariance. Although in favourable cases, such as at LEP1, the resulting effects can be shown to be small, in other cases, such as those involving longitudinal gauge bosons at high energies, the loss of gauge invariance can lead to catastrophically large effects, and hence useless results.

The fundamental problem is that the decay width of an unstable particle arises from an infinite subset of diagrams, i.e. from the Dyson summation of the 1PI self-energy $\Sigma(q^2)$. This leads to the replacement

$$ \frac{1}{q^2 - M^2} \rightarrow \frac{1}{q^2 - M^2 + \text{Re} \Sigma(q^2) + i\text{Im} \Sigma(q^2)} . $$

(20)
At \( q^2 = M_p^2 \), where \( M_p \) is the (complex) pole mass, defined from \( M_p^2 - M^2 + \Sigma(M_p^2) = 0 \), the function \( \Sigma(q^2) \) is gauge-independent. In particular, at the one-loop level, the lowest order width \( \Gamma^{(1)} \) is given by \( M^{\Gamma^{(1)}} = \Im \Sigma^{(1)}(M^2) \). At \( q^2 \neq M_p^2 \), however, the conventional self-energy \( \Sigma(q^2) \) is in general gauge-dependent. For a given process, summing the infinite subset of contributions involved in the Dyson series while including the remaining contributions only to some finite order then in general leads to two problems (see e.g. [23]). First, the delicate order-by-order gauge cancellation mechanism is distorted, so that gauge independence is lost. And second, the Ward identities are violated, so that gauge invariance is also lost.

The most satisfactory approach to these problems is to attempt to identify the infinite subset of contributions which encode the physics of the unstable particles. In particular, the observation that, at lowest order, the gauge bosons of the standard model decay exclusively into fermions has led to the “fermion loop scheme” [24]. In this approach, for a given tree level process involving unstable gauge bosons, e.g. Fig. 3, one includes all possible fermionic one loop corrections, with those contributing to the self-energy summed in Dyson series. This infinite subset of corrections is both gauge-independent, since the individual fermion loop corrections are gauge-independent, and gauge-invariant, since the ensemble of fermion loop corrections, including the Dyson-summed self-energies, exactly satisfy tree-level-like Ward identities.

However, it is known that bosonic corrections too can produce large effects, especially at high energies. In order to account for bosonic effects, the BFM has been advocated as an appropriate framework [25]. However, as already stated, although the ensemble of BFM \( n \)-point functions are gauge-invariant, they remain individually gauge-\( \xi_Q \)-dependent. In particular, for \( \xi_Q \neq 1 \) their imaginary parts have unphysical thresholds [10, 12]. Thus, the naive Dyson summation of the BFM self-energies leads to gauge-dependent matrix elements.

By contrast, the PT provides a general framework for the treatment of unstable particles at the one-loop level in which not only gauge invariance and gauge independence are maintained, but also a range of further requisite field-theoretic properties are satisfied. Detailed discussions of these issues are given in [10] for gauge bosons and [13] for the Higgs boson.

6. Progress Beyond One Loop

Most recently, work on the PT has concentrated on the attempt to extend the algorithm beyond one loop [14, 15]. From a theoretical point of view, this extension is clearly essential if it is to be shown that the PT is more than just an artefact of the one-loop approximation. From a phenomenological point of view, many of the applications of the PT increasingly demand accuracy beyond the one-loop level. There are two basic problems:

1. \textit{How to deal consistently with triple gauge vertices all three legs of which are associated with gauge fields propagating in loops?} In the PT at the one-loop level, the factors of longitudinal four-momentum associated with gauge fields propagating in loops originate from tree level gauge field propagators and triple gauge vertices. In particular, the triple gauge vertices which occur in one-loop diagrams always have one external leg \( A^\rho_\mu(q) \) and two internal legs \( A^\rho_\mu(k_1), A^\rho_\mu(k_2) \). It is then possible to decompose such vertices so as to isolate unambiguously the longitudinal factors \( k_1\mu, k_2\rho \) associated with the internal gauge fields. Beyond one loop, however, there occur triple gauge vertices for which all three legs are internal. It is thus unclear how to decompose such vertices in order to identify the associated longitudinal factors which then trigger the PT rearrangement.

2. \textit{How to deal consistently with the “induced” factors of longitudinal internal gauge field four-momentum which originate from internal loop corrections?} Beyond the one-loop
level, in addition to the factors of longitudinal internal gauge field four-momentum from tree level gauge field propagators and triple gauge vertices, there occur further such factors originating from the invariant tensor structure of internal loop corrections. A simple example are the longitudinal factors occurring in the transverse structure of the gluon self-energy in QCD. It is thus unclear whether or not such “induced” factors should also be used to trigger the PT rearrangement; and, if so, how this may be done consistently.

At the two-loop level, the two problems are uncoupled and so can be investigated separately and successively: one first of all has to solve (1) in order to identify the one-loop internal corrections needed before one can address (2).

The approach to problem (1) taken in [14] is based on the assumption that there exists a skeleton expansion for QCD [26] analogous to that of QED [27]. In the case of the quark self-energy, such a diagrammatic representation is easy to construct, and involves the concept of a Bethe-Salpeter scattering kernel. This representation follows from the Schwinger-Dyson equations for QCD [28]. Expanded out at the two-loop level, this representation is as shown in Fig. 4, with the lowest order kernel as shown in Fig. 5. In Figs. 4 and 5, the one-loop internal corrections and the tree level propagators are the conventional gauge-dependent functions.

It was shown in [14] that, demanding that the renormalized one-loop internal corrections in Figs. 4b–4e be the PT gauge-independent one-loop functions, the corresponding PT kernel term which automatically results from the necessary rearrangement is as shown in Fig. 6.

Thus, the PT kernel is identical to the conventional kernel in the Feynman gauge, except that the triple gauge vertex in Fig. 6c involves only the component $\Gamma^F$ (cf. Eq. (5)), with orientation as indicated. Crucially, the PT kernel thus supplies no longitudinal factors associated with the external gluon legs, and so, when embedded in Fig. 4f, cannot trigger any further PT rearrangement. In this way, the two-loop PT rearrangement for the quark self-energy is self-consistent.

It was explicitly shown [14] that this PT two-loop quark self-energy is gauge-independent at all momenta, does not shift the position of the propagator pole, and is multiplicatively renomalizable by local counterterms. Furthermore, it was shown by this example that the general correspondence between the PT $n$-point functions and those obtained in the BFM at $\xi_Q = 1$ does not persist beyond one loop.

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**Fig. 4.** The skeleton expansion representation of the renormalized two-loop quark self-energy. The box in (f) represents the lowest order contribution to the Bethe-Salpeter quark-gluon scattering kernel.

**Fig. 5.** The contributions to the lowest order conventional Bethe-Salpeter quark-gluon scattering kernel.

**Fig. 6.** The contributions to the PT kernel. The propagator and vertex in (c) are $D^F$ and $\Gamma^F$ (line indicates orientation of $\Gamma^F$).
The quark self-energy is the first non-trivial \( n \)-point function to be tackled beyond one loop in the PT approach. Despite the progress which has been made, the general solution of the two problems described above appears still to represent a formidable challenge in field theory. For a discussion of problem (2), see [15].

7. Conclusions
The PT provides an approach to perturbation theory at the one-loop level in which the one-loop \( n \)-point functions are both gauge-independent, i.e. they are individually entirely independent of the particular gauge-fixing procedure used, and gauge-invariant, i.e. they satisfy simple tree-level-like Ward identities associated with the gauge symmetry of the classical lagrangian. In addition to these two basic properties, the PT \( n \)-point functions display a wide range of further desirable field-theoretic properties. The PT thus provides a general perturbative framework which enables one to deal with a broad range of applications in which one is forced to go beyond the strictly order-by-order computation of \( S \)-matrix elements, or to consider amplitudes for explicitly off-shell processes.

Despite the appeal of the PT approach, there remain, however, two basic criticisms which need to be answered. First, it must now be shown in general how the PT can be extended beyond the one loop approximation. And second, it is now highly desirable to have a more formal understanding of the PT, beyond diagrammatics. Thus, there remains much work to be done if the PT approach is to fulfill its promise.

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