ON THE VECTOR BUNDLES ASSOCIATED TO THE IRREDUCIBLE REPRESENTATIONS OF COCOMPACT LATTICES OF \( \text{SL}(2, \mathbb{C}) \)

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ABSTRACT. In this continuation of [BM], we prove the following: Let \( \Gamma \subset \text{SL}(2, \mathbb{C}) \) be a cocompact lattice, and let \( \rho : \Gamma \to \text{GL}(r, \mathbb{C}) \) be an irreducible representation. Then the holomorphic vector bundle \( E_\rho \to \text{SL}(2, \mathbb{C})/\Gamma \) associated to \( \rho \) is polystable. The compact complex manifold \( \text{SL}(2, \mathbb{C})/\Gamma \) has natural Hermitian structures; the polystability of \( E_\rho \) is with respect to these natural Hermitian structures. In [BM] it was shown that if \( \rho(\Gamma) \subset \text{U}(r) \), then \( E_\rho \) equipped with the Hermitian structure given by \( \rho \), and \( \text{SL}(2, \mathbb{C})/\Gamma \) equipped with a natural Hermitian structure, together produce a solution of the Strominger system of equations. A polystable vector bundle also has a natural Hermitian structure, which is known as the Hermitian–Yang–Mills structure. It would be interesting to find similar applications of the Hermitian–Yang–Mills structure on the above polystable vector bundle \( E_\rho \). We show that the polystable vector bundle \( E_\rho \) is not stable in general.

1. Introduction

We first recall the set–up, and some results, of [BM]. Let

\[
\Gamma \subset \text{SL}(2, \mathbb{C})
\]

be a discrete cocompact subgroup. Fixing a \( \text{SU}(2) \)-invariant Hermitian form on the Lie algebra \( sl(2, \mathbb{C}) \), we get a Hermitian structure \( h \) on the compact complex manifold \( M := \text{SL}(2, \mathbb{C})/\Gamma \). The \((1,1)\)-form \( \omega_h \) on \( M \) associated to \( h \) satisfies the identity \( d\omega_h^2 = 0 \). Take any homomorphism

\[
\rho : \Gamma \to \text{GL}(r, \mathbb{C}).
\]

This \( \rho \) produces a holomorphic vector bundle \( E_\rho \) of rank \( r \) on \( M \) equipped with a flat holomorphic connection \( \nabla^\rho \). The homomorphism \( \rho \) is called irreducible if \( \rho(\Gamma) \) is not contained in some proper parabolic subgroup of \( \text{GL}(r, \mathbb{C}) \).

If \( \rho(\Gamma) \subset \text{U}(r) \), then \( E_\rho \) is equipped with a Hermitian structure \( H^\rho \) such that the associated Chern connection is \( \nabla^\rho \).

If

\[
\begin{align*}
\bullet \ & \rho(\Gamma) \subset \text{U}(r) \quad \text{and} \\
\bullet \ & \rho(\Gamma) \text{ is irreducible,}
\end{align*}
\]

2000 Mathematics Subject Classification. 81T30, 14D21, 53C07.

Key words and phrases. Strominger system, polystability, cocompact lattice, irreducible representation.
then the quadruple \((M, h, E_\rho, H^\rho)\) satisfies the Strominger system of equations \([BM, \text{Theorem } 4.6]\). In particular, the vector bundle \(E_\rho\) is stable \([BM, \text{Proposition } 4.5]\).

Now assume that \(\rho\) is irreducible, but do not assume that \(\rho(\Gamma) \subset U(r)\). Our aim here is to prove the following (see Theorem 2.2):

The holomorphic vector bundle \(E_\rho\) is polystable with respect to the Hermitian structure \(h\) on \(M\).

It is known that under some minor condition, the group \(\Gamma\) admits some free groups of more than one generators as quotients \([La, \text{p. 3393, Theorem } 2.1]\). Therefore, there are many examples of pairs \((\Gamma, \rho)\) of the above type satisfying the irreducibility condition.

Since \(E_\rho\) is polystable, the holomorphic vector bundle \(E_\rho\) has an Hermitian–Yang–Mills structure \(H^\rho\) \([LY]\) (see also \([Bu]\)). It may be worthwhile to investigate this Hermitian structure \(H^\rho\). We should clarify that \(H^\rho\) need not be flat. An Hermitian–Yang–Mills structure on a polystable vector bundle with vanishing Chern classes over a compact Kähler manifold is flat, but \(M\) is not Kähler.

It is natural to ask whether the polystable vector bundle \(E_\rho\) is stable. If we take \(\rho\) to be the inclusion of \(\Gamma\) in \(SL(2, \mathbb{C})\), then \(\rho\) is irreducible, but the associated holomorphic vector bundle \(E_\rho\) is holomorphically trivial, in particular, \(E_\rho\) is not stable (see Lemma 2.3 for the details).

Infinitesimal deformations of the complex structure of \(M\) are investigated in \([Ra]\).

2. Polystability of associated vector bundle

The Lie algebra of \(SL(2, \mathbb{C})\), which will be denoted by \(sl(2, \mathbb{C})\), is the space of complex \(2 \times 2\) matrices of trace zero. Consider the adjoint action of \(SU(2)\) on \(sl(2, \mathbb{C})\). Fix an inner product \(h_0\) on \(sl(2, \mathbb{C})\) preserved by this action; for example, we may take the Hermitian form \((A, B) \mapsto \text{trace}(AB^*)\) on \(sl(2, \mathbb{C})\). Let \(h_1\) be the Hermitian structure on \(SL(2, \mathbb{C})\) obtained by right–translating the Hermitian form \(h_0\) on \(T_{\text{Id}}SL(2, \mathbb{C}) = sl(2, \mathbb{C})\).

Let \(\Gamma\) be a cocompact lattice in \(SL(2, \mathbb{C})\). So \(\Gamma\) is a discrete subgroup of \(SL(2, \mathbb{C})\) such that the quotient

\[
M := SL(2, \mathbb{C})/\Gamma
\]

is compact. This \(M\) is a compact complex manifold of complex dimension three. The left–translation action of \(SL(2, \mathbb{C})\) on itself descends to an action of \(SL(2, \mathbb{C})\) on \(M\). We will call this action of \(SL(2, \mathbb{C})\) on \(M\) the left–translation action. The Hermitian structure \(h_1\) on \(SL(2, \mathbb{C})\) descends to an Hermitian structure on \(M\). This descended Hermitian structure on \(M\) will be denoted by \(h\). Let \(\omega_h\) be the \(C^\infty(1, 1)\)–form on \(M\) associated to \(h\). Then

\[
d\omega_h^2 = 0
\]

\([BM, \text{Corollary } 4.1]\).
For a torsionfree nonzero coherent analytic sheaf $F$ on $M$, define
\[
\text{degree}(F) := \int_M c_1(F) \wedge \omega_h^2 \in \mathbb{R} \quad \text{and} \quad \mu(F) := \frac{\text{degree}(F)}{\text{rank}(F)} \in \mathbb{R}.
\]
A torsionfree nonzero coherent analytic sheaf $F$ on $M$ is called \textit{stable} (respectively, \textit{semistable}) if for every coherent analytic subsheaf $V \subset F$ such the $\text{rank}(V) \in [1, \text{rank}(F) - 1]$ and the quotient $F/V$ is torsionfree, the inequality
\[
\mu(V) < \mu(F) \quad \text{(respectively,} \quad \mu(V) \leq \mu(F))
\]
holds (see [KO, Ch. V, § 7]). A torsionfree nonzero coherent analytic sheaf $F$ on $M$ is called \textit{polystable} if it is semistable and is isomorphic to a direct sum of stable sheaves.

\textbf{Remark 2.1.} Since a polystable coherent analytic sheaf $F$ is semistable, if $F = \bigoplus_{i=1}^{\ell} F_i$, then $\mu(F_i) = \mu(F)$ for all $i$.

Take any homomorphism (2.2)\[
\rho : \Gamma \longrightarrow \text{GL}(r, \mathbb{C}).
\]
Let $(E_{\rho}, \nabla^\rho)$ be the flat holomorphic vector bundle of rank $r$ over $M$ associated to the homomorphism $\rho$. We recall that the total space of $E_{\rho}$ is the quotient of $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ where two points
\[
(z_1, v_1), (z_2, v_2) \in \text{SL}(2, \mathbb{C}) \times \mathbb{C}^r
\]
are identified if there is an element $\gamma \in \Gamma$ such that $z_2 = z_1\gamma$ and $v_2 = \rho(\gamma^{-1})(v_1)$.

The trivial connection on the trivial vector bundle $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r \longrightarrow \text{SL}(2, \mathbb{C})$ of rank $r$ descends to the connection $\nabla^\rho$. The left–translation action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C})$ and the trivial action of $\text{SL}(2, \mathbb{C})$ on $\mathbb{C}^r$ together define an action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$. This action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ descends to an action (2.3)
\[
\tau : \text{SL}(2, \mathbb{C}) \times E_{\rho} \longrightarrow E_{\rho}
\]
of $\text{SL}(2, \mathbb{C})$ on the vector bundle $E_{\rho}$. The action $\tau$ in (2.3) is clearly a lift of the left–translation action of $\text{SL}(2, \mathbb{C})$ on $M$.

The homomorphism $\rho$ in (2.2) is called \textit{reducible} if there a nonzero linear subspace $S \subset \mathbb{C}^r$ such that $\rho(\Gamma)(S) = S$. The homomorphism $\rho$ is called \textit{irreducible} if it is not reducible.

\textbf{Theorem 2.2.} Assume that the homomorphism $\rho$ in (2.2) is irreducible. Then the corresponding holomorphic vector bundle $E_{\rho}$ is polystable.

\textit{Proof.} Since $E_{\rho}$ has a flat connection, the Chern class $c_1(\det E_{\rho}) = c_1(E_{\rho}) \in H^2(M, \mathbb{R})$ vanishes. Hence we have $\text{degree}(E_{\rho}) = 0$ (see [BM, Lemma 4.2]).

We will first show that $E_{\rho}$ is semistable. Assume that $E_{\rho}$ is not semistable. Let (2.4)
\[
0 \subset W_1 \subset \cdots \subset W_{\ell-1} \subset W_\ell = E_{\rho}
\]
be the Harder–Narasimhan filtration $E_{\rho}$; see [Br] for the construction of the Harder–Narasimhan filtration of vector bundles on compact complex manifolds. Since $E_{\rho}$ is not semistable, we have $\ell \geq 2$ and $W_1 \neq 0$.

Consider the action $\tau$ of $\text{SL}(2, \mathbb{C})$ on $E_{\rho}$ constructed in (2.3). From the uniqueness of the Harder–Narasimhan filtration it follows immediately that $\tau(\{g\} \times W_1) = W_1$ for every $g \in \text{SL}(2, \mathbb{C})$. Therefore, we have

(2.5) \[ \tau(\text{SL}(2, \mathbb{C}) \times W_1) = W_1. \]

Let $C(W_1) \subset M$ be the closed subset over which $W_1$ fails to be locally free. Since $\tau$ is a lift of the left–translation action of $\text{SL}(2, \mathbb{C})$ on $M$, from (2.5) we conclude that $C(W_1)$ is preserved by the left–translation action of $\text{SL}(2, \mathbb{C})$ on $M$. As the left–translation action of $\text{SL}(2, \mathbb{C})$ on $M$ is transitive, it follows that $C(W_1)$ is the empty set. Therefore, $W_1$ is a holomorphic vector bundle on $M$. Similarly, the closed proper subset of $M$ over which $W_1$ fails to be a subbundle of $E_{\rho}$ is preserved the left–translation action of $\text{SL}(2, \mathbb{C})$ on $M$. Hence this subset is empty, and $W_1$ is a holomorphic subbundle of $E_{\rho}$.

We will show that the flat connection $\nabla^\rho$ on $E_{\rho}$ preserves the subbundle $W_1$ in (2.4).

To show that $\nabla^\rho$ preserves $W_1$, first note that the flat sections of the trivial connection on the trivial vector bundle $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r \to \text{SL}(2, \mathbb{C})$ are of the form

\[ \text{SL}(2, \mathbb{C}) \to \text{SL}(2, \mathbb{C}) \times \mathbb{C}^r, \quad g \mapsto (g, v_0), \]

where $v_0 \in \mathbb{C}^r$ is independent of $g$. On the other hand, the image of such a section is an orbit for the action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$; recall that the action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ is the diagonal one for the left–translation action of $\text{SL}(2, \mathbb{C})$ on itself and the trivial action of $\text{SL}(2, \mathbb{C})$ on $\mathbb{C}^r$ (see the construction of $\tau$ in (2.3)). Also, recall that the connection $\nabla^\rho$ on $E_{\rho}$ is the descent of the trivial connection on the trivial vector bundle $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r \to \text{SL}(2, \mathbb{C})$. Combining these, from (2.5) we conclude that $\nabla^\rho$ preserves $W_1$.

The homomorphism $\rho$ is given to be irreducible. Therefore, the only holomorphic subbundles of $E_{\rho}$ that are preserved by the associated connection $\nabla^\rho$ are 0 and $E_{\rho}$ itself. But $\ell \geq 2$ and $W_1 \neq 0$ in (2.4). So $W_1$ neither 0 nor $E_{\rho}$.

In view of the above contradiction, we conclude that the holomorphic vector bundle $E_{\rho}$ is semistable.

We will now prove that $E_{\rho}$ is polystable.

Consider all nonzero coherent analytic subsheaves $V$ of $E_{\rho}$ such that

- $V$ is polystable, and
- $\text{degree}(V) = 0$.

Let

(2.6) \[ \mathcal{F} \subset E_{\rho} \]
be the coherent analytic subsheaf generated by all $V$ satisfying the above two conditions. It is know that $\mathcal{F}$ is polystable with $\mu(\mathcal{F}) = \mu(E_\rho) = 0$ (see [HL, page 23, Lemma 1.5.5]). Therefore, the subsheaf $\mathcal{F}$ is uniquely characterized as follows: the subsheaf $\mathcal{F}$ is the unique maximal coherent analytic subsheaf of $E_\rho$ such that

- $\mathcal{F}$ is polystable, and
- $\text{degree}(\mathcal{F}) = 0$.

Note that the quotient $E_\rho/\mathcal{F}$ is torsionfree, because if $T \subset E_\rho/\mathcal{F}$ is the torsion part, then $\varphi^{-1}(T) \subset E_\rho$, where

$$\varphi : E_\rho \longrightarrow E_\rho/\mathcal{F}$$

is the quotient map, also satisfies the above two conditions, while $\mathcal{F} \subsetneq \varphi^{-1}(T)$ if $T \neq 0$.

Consider the action $\tau$ of $\text{SL}(2, \mathbb{C})$ on $E_\rho$ constructed in (2.3). From the above characterization of the subsheaf $\mathcal{F}$ in (2.6) it follows immediately that

$$\tau(\text{SL}(2, \mathbb{C}) \times \mathcal{F}) = \mathcal{F}.$$  

As it was done for $W_1$, from (2.7) we conclude that $\mathcal{F}$ is a holomorphic subbundle of $E_\rho$.

As it was done for $W_1$, from (2.7) it follows that the flat connection $\nabla^\rho$ on $E_\rho$ preserves the subbundle $\mathcal{F}$ in (2.6). Since $\rho$ is irreducible, either $\mathcal{F} = 0$ or $\mathcal{F} = E_\rho$. The rank of $\mathcal{F}$ is at least one because the semistable vector bundle $E_\rho$ of degree zero has a nonzero stable subsheaf of degree zero. Therefore, we conclude that $\mathcal{F} = E_\rho$. Consequently, $E_\rho$ is polystable. \hfill \Box

We may now ask whether the polystable vector bundle $E_\rho$ in Theorem 2.2 is stable. The following lemma shows that $E_\rho$ is not stable in general.

Let

$$\delta : \Gamma \hookrightarrow \text{SL}(2, \mathbb{C})$$

be the inclusion map. This homomorphism $\delta$ is clearly irreducible. Let $(E_\delta, \nabla^\delta)$ be the corresponding flat holomorphic vector bundle on $M$.

**Lemma 2.3.** The above holomorphic vector bundle $E_\delta$ is holomorphically trivial.

**Proof.** Recall that the vector bundle $E_\delta$ is a quotient of $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^2$. Consider the holomorphic map

$$\text{SL}(2, \mathbb{C}) \times \mathbb{C}^2 \longrightarrow \text{SL}(2, \mathbb{C}) \times \mathbb{C}^2$$

defined by $(g, v) \longmapsto (g, g(v))$. This map descends to a holomorphic isomorphism of vector bundles

$$E_\delta \longrightarrow M \times \mathbb{C}^2$$

over $M$. Therefore, this descended homomorphism provides a holomorphic trivialization of $E_\delta$. \hfill \Box
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