Strong CP Problem and the Natural Hierarchy of Yukawa Couplings

Kenzo INOUE *) and Naoki YAMATSU**)  

Department of Physics, Kyushu University, Fukuoka 812-8581, Japan

Abstract

A possible solution to the strong CP problem is presented without using an axion. The model is based on the framework of the supersymmetric vectorlike theory with the spontaneous breakdown of the P-C-T-invariance. It is shown that the characteristic structure of the Yukawa coupling matrices that results from the spontaneous P-C-T-breaking plays the essential role in naturally realizing $\theta = 0$ at the tree level. It is argued that $\theta = 0$ will not be affected by the radiative corrections.

*) E-mail: inoue@phys.kyushu-u.ac.jp  
**) E-mail: yamatsu@higgs.phys.kyushu-u.ac.jp
§1. Introduction

One of the prominent issues of the quantum chromo-dynamics is the strong CP problem.\(^1\) The gauge principle based on the gauge group \(SU_3 \times SU_2 \times U_1\) of the standard model or the minimal supersymmetric standard model (MSSM)\(^2\)–\(^5\) does not forbid the appearance of the gluon \(\theta\)-term that violates the invariance under the space inversion (P) and the time reversal (T):

\[
\mathcal{L}_\theta = \frac{g_c^2}{64\pi^2} \epsilon^{\mu\nu\lambda\sigma} G^a_{\mu\nu} G^a_{\lambda\sigma}.
\]

(1.1)

The upper limit of \(\theta\) is severely constrained by some experiments. The present limit is \(|\theta| \lesssim 10^{-10}.\)\(^6\) The promising candidate for the solution to the problem is an (invisible) axion,\(^7\),\(^8\) which originates from the spontaneous breakdown of the Peccei-Quinn \(U(1)_{\text{PQ}}\) symmetry.\(^9\) For the axion to solve this problem, the \(U(1)_{\text{PQ}}\) symmetry must be an anomalous symmetry that suffers from the gauge anomaly of gluons.

On the other hand, the recent progress of the superstring theories suggests that the various types of the superstring theories are connected in the more fundamental theory. If the theory really admits the chiral as well as the anti-chiral string theories, it might not be unreasonable to imagine that the original theory is the vectorlike theory. The chiral nature of each string theory seems to be realized through the spontaneous breakdown of the P-symmetry. If this is the case, the \(U(1)_{\text{PQ}}\) symmetry, even if exists, may not suffer from the gauge anomalies. This gives the motivation to solve the strong CP problem in an alternative way without using the axion. So far, various attempts have been made for the spontaneous CP violation,\(^10\) where the \(\theta\)-term is sufficiently suppressed but the sizable CP violating phases in the CKM matrix\(^11\) are reserved.

We have examined, in the series of publications,\(^12\)–\(^14\) the vectorlike model\(^15\) that realizes the MSSM as the low-energy effective theory through the spontaneous breakdown of the P-C-T-invariance. In the recent publication,\(^14\) we suggested that the model may solve the strong CP problem. In this paper, we will pursue the extensive analysis of this problem and clarify what is necessary to solve the problem. We omit the lepton sector, which is essentially irrelevant to the subject.

§2. Setup of the model

The model is based on the supersymmetric vectorlike theory with the gauge group \(SU_3 \times SU_2 \times U_1 \times SU(1, 1)\), where the gauge group \(SU(1, 1)\) is a horizontal symmetry\(^16\),\(^17\) governing the generational structures of quarks and leptons. The basic hypothesis is that the model is invariant under the P, C, and T-transformations at the fundamental level. Therefore,
the original value of $\theta$ is $\theta_0 = 0$. The model realizes the MSSM through the spontaneous breakdown of $SU(1,1)$. The nonvanishing vacuum expectation values (VEVs) that break $SU(1,1)$ are shared by the some sets of the finite-dimensional nonunitary $SU(1,1)$ multiplets $\Psi$’s of a type

$$\Psi = \{\psi_{-S}, \psi_{-S+1}, \cdots, \psi_{S-1}, \psi_S\}. \quad (2.1)$$

They are $SU_3 \times SU_2 \times U_1$ singlet and their VEVs are assumed to be roughly on the order of $M \simeq 10^{16}\text{GeV}$ to reproduce the successful MSSM at the low-energy. Although we are not yet able to give the explicit form of their superpotential $W[\text{finite dim.}]$, we make reasonable assumptions on their VEVs based on the intuitive considerations.

The first assumption is that any of the multiplets $\Psi$’s takes its nonvanishing VEV at most at its single component $\langle \psi_M \rangle$ with some $SU(1,1)$ weight $M$. Under the $U(1)_H$ transformation that is a subgroup of $SU(1,1)$, $\psi_M$ transforms as

$$\psi_M \rightarrow e^{iM} e^H \psi_M. \quad (2.2)$$

The second assumption is that any quantity $r_0$ related to the VEVs that has “total weight 0”, such as

$$r_0 = \langle \psi_0 \rangle, \frac{\langle \psi_M \rangle}{\langle \psi_M \rangle}, \frac{\langle \psi_{M} \rangle}{\langle \psi_{M} \rangle}, \frac{\langle \psi_{M} \rangle}{\langle \psi_{M} \rangle}, \text{ etc., } \quad (2.3)$$

has a “natural phase”

$$\frac{r_0}{|r_0|} = e^{ip/q} \quad (2.4)$$

with some set of integers $p$ and $q$, since $W[\text{finite dim.}]$ does not contain any explicitly complex number.

The superpotential for the quark sector of the model is

$$W_{\text{quark}} = (x_Q Q_\alpha Q_{-\alpha} + x_U \bar{U}_\beta U_{-\beta} + x_D \bar{D}_\gamma D_{-\gamma}) \Psi^F + (x_Q' Q_\alpha Q_{-\alpha} + x_U' \bar{U}_\beta U_{-\beta} + x_D' \bar{D}_\gamma D_{-\gamma}) \Psi'^F. \quad (2.5)$$

The multiplet $Q_\alpha$, for example, carries the $SU_3 \times SU_2 \times U_1$ quantum numbers of the quark doublet $q$ and belongs to the infinite-dimensional unitary representation of $SU(1,1)$ with the positive lowest weight $\alpha$, and $\bar{Q}_{-\alpha}$ is its conjugate:

$$Q_\alpha = \{q_\alpha, q_{\alpha+1}, \cdots\}, \quad \bar{Q}_{-\alpha} = \{q_{-\alpha}, q_{-\alpha-1}, \cdots\}. \quad (2.6)$$

The multiplets $\Psi^F$ and $\Psi'^F$ belong to the finite-dimensional nonunitary representations of $SU(1,1)$ with the common highest weight $S^F (\geq 3)$. All coupling constants $x$’s and $x'$’s in
(2.5) are real numbers under the basic hypothesis of the P-C-T-invariance. The nonvanishing VEVs

\[ \langle \bar{\psi}^F \rangle = \langle \psi^F_0 \rangle, \quad \langle \bar{\psi}^D \rangle = \langle \psi^D_0 \rangle \]  

(2.7)

in the superpotential (2.5) generate three generations of the chiral quarks \( q_m, \bar{u}_m, \) and \( \bar{d}_m \) \((m = 0, 1, 2)\) that are embedded in the infinite number of the components of \( Q_\alpha, \bar{U}_\beta, \) and \( \bar{D}_\gamma, \) respectively, in the manner

\[ q_{\alpha+i} = \sum_{m=0}^{2} q_m U^q_{mi} + [\cdots], \quad \bar{u}_{\beta+i} = \sum_{m=0}^{2} \bar{u}_m U^u_{mi} + [\cdots], \quad \bar{d}_{\gamma+i} = \sum_{m=0}^{2} \bar{d}_m U^d_{mi} + [\cdots], \]  

(2.8)

where \([\cdots]\) represents the superheavy massive modes. The mixing coefficient \( U^q_{mi}, \) for example, is derived by the requirement that \( q_m 's \) disappear from the mass operators \( x_Q Q_\alpha \bar{Q}_{-\alpha} \langle \bar{\psi}^F \rangle + x'_Q Q_\alpha \bar{Q}_{-\alpha} \langle \bar{\psi}^F \rangle. \) This gives \(^{14} \)

\[ U^q_{mi} = U^q_m \sum_{r=0}^{\infty} \delta_{i,m+3r} (-\epsilon_q)^r b^q_{mr}(\alpha), \quad m = 0, 1, 2, \]  

(2.9)

with

\[ \epsilon_q = \frac{x'_Q \langle \bar{\psi}^F_0 \rangle}{x_Q \langle \bar{\psi}^F_3 \rangle}. \]  

(2.10)

The function \( b^q_{mr}(\alpha) \) is a real function of the \( SU(1,1) \) Clebsch-Gordan (C-G) coefficients depending on the weights of the multiplets. We note that the mixing parameters \( \epsilon_q, \epsilon_u, \) and \( \epsilon_d \) are in general complex numbers, but they have a common phase. Under the \( U(1)_H \) transformation, they transform as if they have a “weight 3”. Since each component of \( Q_\alpha, \bar{U}_\beta, \) and \( \bar{D}_\gamma \) transforms as

\[ q_{\alpha+i} \rightarrow e^{i(\alpha+i)\varphi_H} q_{\alpha+i}, \quad \bar{u}_{\beta+i} \rightarrow e^{i(\beta+i)\varphi_H} \bar{u}_{\beta+i}, \quad \bar{d}_{\gamma+i} \rightarrow e^{i(\gamma+i)\varphi_H} \bar{d}_{\gamma+i}, \]  

(2.11)

\( q_m, \bar{u}_m, \) and \( \bar{d}_m \) \((m = 0, 1, 2)\) transform under \( U(1)_H \) as

\[ q_m \rightarrow e^{i(\alpha+m)\varphi_H} q_m, \quad \bar{u}_m \rightarrow e^{i(\beta+m)\varphi_H} \bar{u}_m, \quad \bar{d}_m \rightarrow e^{i(\gamma+m)\varphi_H} \bar{d}_m. \]  

(2.12)

This means that the coefficients \( U^q_m, U^u_m, \) and \( U^d_m \) do not receive any transformation. We fix a phase convention of the chiral quarks so that \( U^q_m, U^u_m, \) and \( U^d_m \) are real and positive.

For the higgs sector, it has been shown that the model must have at least double structure for the down-type higgs doublet \( h'. \)\(^{14} \) In this paper, we assume that both of the up-type and the down-type higgses have the double structure, and introduce the \( SU(1,1) \) multiplets \( H_{-\rho}, K_{-\rho-\Delta} (\rho = \alpha + \beta, \rho + \Delta > 0) \) for the up-type higgs doublet \( h, \) and \( H'_{-\sigma}, K'_{-\sigma-\Delta'} (\sigma = \alpha + \gamma, \sigma + \Delta' > 0) \) for the down-type higgs doublet \( h', \) and all of their conjugates. \( \Delta \) and \( \Delta' \) are
there are 4 \times \text{chiral h}

restricted to be an integer (Type-I) or a half-integer (Type-II). The superpotential for the higgs sector is

\[ W_{\text{higgs}} = H \bar{H} \Psi + K K \Phi + K \bar{H} X + H \bar{K} \Omega \]

\[ + H' \bar{H}' \Psi' + K' \bar{K}' \Phi' + K' \bar{H}' X' + H' \bar{K}' \Omega', \] (2.13)

where all real coupling constants have been absorbed into the finite-dimensional nonunitary multiplets \( \Psi \sim \Omega' \). To preserve the vectorlike nature of the model, \( X \) and \( \Omega \), and also \( X' \) and \( \Omega' \) must be assigned to the common \( SU(1,1) \) representations. It has been shown\(^{14}\) that there are \( 4 \times 4 \) cases for the pattern of the VEVs \( \langle \Psi \rangle \sim \langle \Omega' \rangle \) that realize just one set of chiral \( h \) and \( h' \) in \( H, K \) and \( H', K' \) in the manner

\[ h_{-\rho-i} = hU_i + [\cdots], \quad k_{-\rho-\Delta-i} = hV_i + [\cdots], \] (2.14)

\[ h'_{-\sigma-i} = h'U'_i + [\cdots], \quad k'_{-\sigma-\Delta'-i} = h'V'_i + [\cdots]. \] (2.15)

The mixing coefficients \( U_i, V_i, U'_i \), and \( V'_i \) have a general form

\[ U_i = U_0 \epsilon b_i(\rho), \quad V_i = -r_{-\Delta} R_i(\rho) U_i, \] (2.16)

\[ U'_i = U'_0 \epsilon' b'_i(\sigma), \quad V'_i = -r'_{-\Delta'} R'_i(\sigma) U'_i, \] (2.17)

where \( b_i(\rho), b'_i(\sigma), R_i(\rho), \) and \( R'_i(\sigma) \) are the real functions of the C-G coefficients, \( \epsilon \) and \( \epsilon' \) have a weight \(-1\), and \( r_{-\Delta} \) and \( r'_{-\Delta'} \) have a weight \(-\Delta \) and \(-\Delta' \), respectively. For example, a set of VEVs

\[ \langle \Psi \rangle = \langle \psi_1 \rangle, \quad \langle \Phi \rangle = \langle \psi_0 \rangle, \quad \langle X \rangle = \langle \kappa_\Delta \rangle, \quad \langle \Omega \rangle = \langle \omega_{-\Delta} \rangle, \] (2.18)

\[ \langle \Psi' \rangle = \langle \psi'_0 \rangle, \quad \langle \Phi' \rangle = \langle \phi'_0 \rangle, \quad \langle X' \rangle = \langle \kappa'_{\Delta'} \rangle, \quad \langle \Omega' \rangle = \langle \omega'_{-\Delta'} \rangle \] (2.19)

gives

\[ \epsilon = \frac{\langle \kappa_\Delta \rangle \langle \omega_{-\Delta} \rangle}{\langle \psi_1 \rangle \langle \phi_0 \rangle}, \quad \epsilon' = \frac{\langle \phi'_0 \rangle \langle \omega'_{-\Delta'} \rangle}{\langle \chi_{\Delta'} \rangle \langle \omega'_{1-\Delta'} \rangle}, \quad r_{-\Delta} = \frac{\langle \kappa_\Delta \rangle \langle \omega_{-\Delta} \rangle}{\langle \phi_0 \rangle}, \quad r'_{-\Delta'} = \frac{\langle \psi'_0 \rangle \langle \omega'_{-\Delta'} \rangle}{\langle \chi_{\Delta'} \rangle}. \] (2.20)

Notice that \( h \) and \( h' \) transform under the \( U(1)_H \) transformation as

\[ h \to e^{-i \varphi_H} h, \quad h' \to e^{-i \varphi_H} h', \] (2.21)

and \( U_0 \) and \( U'_0 \) do not receive any transformation. For the phase convention of \( h \) and \( h' \), we take \( U_0 \) and \( U'_0 \) to be real and positive.

To reproduce the MSSM in the low-energy, we need the additional sector that generates the \( \mu \)-term \( \mu h h' \). It should be noticed that the coupling of the form \( H H' \Psi \) is forbidden by
the $SU(1, 1)$ symmetry, since both $H$ and $H'$ have negative weights. This fact explains why $\mu$ does not take a huge mass scale $M \approx 10^{16}$GeV. Since the order of $\mu$ should be the supersymmetry breaking scale $m_{\text{SUSY}} \approx 10^{2-3}$GeV, what we have to do in the supersymmetric limit is to generate the effective superpotential $W_\mu$ that is sensitive to the supersymmetry breaking. This means that $W_\mu$ contains the massless particles. The simplest procedure for this sector is to introduce the superpotential that contains the $SU_3 \times SU_2 \times U_1$ singlets $R_{(\rho+\sigma)/2}, R'_{(\rho+\sigma)/2+1}, S_{\rho+\sigma}$, and their conjugates:

$$W_M = (R\bar{R}' + R'\bar{R})\Psi^M + SS\bar{\Psi}^S + \tilde{y}_1(\bar{H}H'S + \bar{H}'H'S) + \tilde{y}_2(RR\bar{S} + \bar{R}\bar{R}S), \quad (2.22)$$

where the coupling constants $\tilde{y}_1, \tilde{y}_2 \simeq O(1)$ are real numbers. Notice that the superpotentials (2.13) and (2.22) respect the Peccei-Quinn $U(1)_PQ$ symmetry with the charges given by

$$Q_{PQ} \begin{array}{cccccc}
H & K & H' & K' & R & R' & S \\
-1 & -1 & -1 & -1 & 1 & 1 & 2
\end{array} | \begin{array}{c}
\Psi^s \\
0
\end{array}. \quad (2.23)$$

All finite-dimensional multiplets $\Psi^s$ are assigned to be $Q_{PQ} = 0$. Thus, each of the conjugate multiplets has opposite $Q_{PQ}$ charge to that of the corresponding multiplets in (2.23). This means that the $U(1)_PQ$ symmetry is a vector symmetry, which is free from any gauge anomalies in the framework of the vectorlike gauge theory. Through the VEV $\langle \psi_0^M \rangle$ in (2.22), the first component $r$ of $R$ and $\bar{r}$ of $\bar{R}$ become massless. The VEV $\langle \psi_0^S \rangle$ gives huge masses to all components of $S$ and $\bar{S}$. Notice that only the first component $\bar{s}$ of $\bar{S}$ couples to $rr$ and the first component $s$ of $S$ couples to $\bar{r}\bar{r}$ and $hh'$. Therefore, the relevant part of the superpotential is

$$\tilde{W}_M = M_s s\bar{s} + y_1 U_0 U'_0 hh' s + y_2 (rr\bar{s} + \bar{r}\bar{s}), \quad (2.24)$$

where

$$M_s = D_0^S \langle \psi_0^S \rangle, \quad y_1 = \tilde{y}_1 C_{0,0}^H, \quad y_2 = \tilde{y}_2 C_{0,0}^R. \quad (2.25)$$

with the real and positive C-G coefficients $D_0^S, C_{0,0}^H$, and $C_{0,0}^R$. The integration of the superheavy $s$ and $\bar{s}$ leads to the effective superpotential

$$W_\mu = -\frac{y_2}{M_s} rr(y_1 U'_0 U'_0 hh' + y_2 \bar{r}\bar{r}). \quad (2.26)$$

Now, let us make the phenomenologically desirable requirements on the characteristics of the supersymmetry breaking. We assume that the supersymmetry breaking occurs in the hidden sector that is singlet under $SU_3 \times SU_2 \times U_1 \times SU(1, 1)$, and does not give the CP-violating phases to the observable sector that consists of $F$’s, $\bar{F}$’s, $\Psi$’s, and the relevant gauge multiplets. Thus, the gauginos have real masses at the energy scale $E \simeq 10^{16}$GeV.
For the Kähler potential, we assume the form

\[ K = \sum_F f_F(z_i, z_i^*) (F^\dagger F + \bar{F}^\dagger \bar{F}) + \sum_A g_A(z_i, z_i^*) K_A(\Psi^A, \Psi^A\dagger) + k_H(z_i, z_i^*) \]  

(2.27)

up to gauge couplings, where \( f_F, g_A, \) and \( k_H \) are real functions of the hidden sector fields \( z_i \). Even if we assume (2.27), we must take into account the tree diagrams connected by the superheavy particles. For example, the superheavy \( \bar{Q} \) inevitably induces the term

\[ \Delta K \propto Q^\dagger (x_Q \Psi^{F^\dagger} + x_Q' \Psi'^{F^\dagger}) (x_Q \Psi^F + x_Q' \Psi'^F) Q, \]  

(2.28)

because \( \Psi^F \)'s couple to \( Q \) and \( \bar{Q} \) in the superpotential (2.5). The chiral modes \( q_m \)'s, however, are realized so that they decouple from the VEVs of \( \Psi^F \)'s. This means that \( q_m \)'s disappear in (2.28) when \( \Psi^F \)'s in (2.28) are replaced by their VEVs. Therefore, as far as the chiral modes are concerned, the Kähler potential (2.27) will be appropriate. Thus, we expect that all chiral modes in the matter multiplets \( F \)'s and \( F \)'s have the generation-independent soft masses \( m_F^2 (= m_F^2) \) at \( E \approx 10^{16} \text{GeV} \). The Kähler potential (2.27) gives the further consequence on the \( A \)-terms. Since the hidden sector fields are \( SU(1, 1) \) singlets, \( A \)-terms should respect the \( SU(1, 1) \) symmetry. Therefore, each of the \( A \)-terms must be exactly proportional, at \( E \approx 10^{16} \text{GeV} \), to the corresponding term in the superpotential.

Under this setup, the supersymmetry breaking terms in the bosonic potential that contain \( r \) and \( \bar{r} \) are given by

\[ V_{s.b.}(r, \bar{r}) = m_r^2 (|r|^2 + |ar{r}|^2) - \left( \frac{y_2 m_0}{M_s} r \bar{r} (y_1 A_1 U_1 U_1' h h' + y_2 A_2 \bar{r} \bar{r}) + \text{h.c.} \right), \]  

(2.29)

where \( m_r, m_0 (= m_{\text{SUSY}}) \) are real and positive, and \( A_1, A_2 (= O(1)) \) are real values. Since \( r \) and \( \bar{r} \) are the \( SU_3 \times SU_2 \times U_1 \) singlets, their tree-level potential takes a form

\[ V(r, \bar{r}) = 4 \frac{y_2^4}{|M_s|^2} |r \bar{r}|^2 (|r|^2 + |ar{r}|^2) + m_r^2 (|r|^2 + |ar{r}|^2) - \left( \frac{y_2^2 A_2 m_0}{M_s} r \bar{r} \bar{r} + \text{h.c.} \right). \]  

(2.30)

This potential has three local minima when \( |A_2| > 2 \sqrt{3} m_r/m_0 \). One is a trivial minimum at the origin. The other two minima are degenerate and sit on the circles specified by

\[ r \eta^* = \pm \bar{r} \eta \varepsilon = \frac{\sqrt{|A_2| + \sqrt{|A_2|^2 - 12(m_r/m_0)^2}}}{2 \sqrt{3} |y_2|} \sqrt{m_0|M_s|} \equiv w_0, \]  

(2.31)

where

\[ |\eta| = 1, \quad |\varepsilon| = 1, \quad \arg[\varepsilon] = \frac{1}{2} \arg\left[ \frac{A_2}{M_s} \right], \]  

(2.32)

and \( \eta \) is a unimodulus constant representing the spontaneous breakdown of the \( U(1)_{\text{PQ}} \) symmetry. Its phase freedom represents the associated Nambu-Goldstone (N-G) boson.
When $|A_2| > 4m_r/m_0$, these minima become deeper than the origin. The VEVs (2.31) generate the $\mu$-term $\mu hh'$ in (2.26) and the $B$-term $\mu Bhh'$ in (2.29) with the desirable magnitudes:

$$
\mu = -2U_0U'_0 y_0 y_2 w^2_0 / M_s, \quad B = m_0 A_1, \tag{2.33}
$$

where we have taken the phase convention $\eta = 1$. Notice that $B$ takes a real value. Expressing $r$ and $\bar{r}$ as

$$
r = w_0 + \tilde{r}, \quad \bar{r} = \varepsilon^*(w_0 + \tilde{r}) \tag{2.34}
$$

and substituting them for (2.26) and (2.29), we find that $\tilde{r}$ and $\bar{r}$ have the mass terms on the order of $m_{\text{SUSY}}$ and the coupling $\tilde{r} hh'$ is suppressed by the factor $\sqrt{m_{\text{SUSY}}/M} \simeq 10^{-7}$. Thus, the effective theory below the energy scale $\sqrt{m_{\text{SUSY}}M}$ is described by the MSSM and the almost decoupled neutral $\tilde{r}$ and $\bar{r}$.

§3. Natural hierarchy of the Yukawa couplings

Let us proceed to the Yukawa couplings of the higgses to the quarks. Owing to the weight constraint of the $SU(1,1)$ symmetry, the superpotential $W_Y$ that describes the Yukawa couplings takes the different form for the integer $\Delta, \Delta'$ (Type-I) and for the half-integer $\Delta, \Delta'$ (Type-II):

$$
\text{Type-I : } W_Y = y_U \bar{U} Q H + y_U^\Delta \bar{U} Q K + y_D \bar{D} Q H' + y_D^\Delta \bar{D} Q K' + \text{[mirror couplings]}, \tag{3.1}
$$

$$
\text{Type-II : } W_Y = y_U \bar{U} Q H + y_D \bar{D} Q H' + \text{[mirror couplings]} \tag{3.2}
$$

Notice that these superpotentials also preserve the $U(1)_{PQ}$ symmetry with the charges

$$
\begin{array}{c|ccc}
Q_{PQ} & Q & \bar{U} & \bar{D} \\
\hline
Q_{PQ} & 1/2 + \delta_B & 1/2 - \delta_B & 1/2 - \delta_B \\
\end{array} \tag{3.3}
$$

in addition to (2.23), where $\delta_B$ represents a contamination of the baryon number charge. We note that both of the $U(1)_{PQ}$ symmetry and the baryon number $U(1)_B$ symmetry are free from gauge anomalies because they are vector symmetries.

Substituting the expressions (2.8), (2.14), and (2.15) for $W_Y$ and extracting the massless modes, we obtain the effective superpotential for the Yukawa couplings of the higgses $h, h'$ to the quarks $q_m, \bar{u}_m, \bar{d}_m$ ($m = 0, 1, 2$):

$$
W_{\text{Yukawa}} = \sum_{m,n=0}^{2} \left( y_u^{mn} \bar{u}_m q_n h + y_d^{mn} \bar{d}_m q_n h' \right). \tag{3.4}
$$
For the contraction of two $SU_2$ indices, we adopt a convention $qh \equiv \varepsilon_{ij}q^i h^j$ with $\varepsilon_{12} = 1$, $\varepsilon_{21} = -1$.

The coupling matrix $y_u^{mn}$ for the Type-I scheme is expressed as

$$y_u^{mn} = y_U \sum_{i,j=0}^{\infty} C_{i,j}^{\alpha,\beta}(0) U_{mi} U_{nj} U_{i+j} + y_U^{\Delta} \sum_{i,j=0}^{\infty} C_{i,j}^{\alpha,\beta}(\Delta) U_{mi} U_{nj} U_{i+j - \Delta},$$

(3.5)

where $C_{i,j}^{\alpha,\beta}(0)$ and $C_{i,j}^{\alpha,\beta}(\Delta)$ are the real C-G coefficients satisfying the symmetry relation

$$C_{i,j}^{\alpha,\beta}(\Delta) = (-1)^{\Delta} C_{j,i}^{\alpha,\beta}(\Delta).$$

(3.6)

The expression of the mixing coefficients (2.9) and (2.16) then gives

$$y_u^{mn} = y_U U_{m} U_{n} U_{0} \varepsilon^{m+n} Y_u^{mn}$$

(3.7)

with

$$Y_u^{mn} = A_u^{mn} - r_U B_u^{mn}, \quad r_U = \frac{y_U^{\Delta}}{y_U} \varepsilon^{-\Delta} r_{-\Delta},$$

(3.8)

where

$$A_u^{mn} = \sum_{r,s=0}^{\infty} (-\varepsilon_U \varepsilon^r)^{s} (-\varepsilon_U \varepsilon^s) \varepsilon^{m+n+3(r+s)} (0) b^\beta_{mr}(\beta) b^\gamma_{ns}(\gamma) b_{m+n+3(r+s)}(\rho),$$

(3.9)

$$B_u^{mn} = \sum_{r,s=0}^{\infty} \theta_{m+n+3(r+s),\Delta} (-\varepsilon_U \varepsilon^r)^{s} (-\varepsilon_U \varepsilon^s) \varepsilon^{m+n+3(r+s)}(\Delta)$$

$$\times b^\beta_{mr}(\beta) b^\gamma_{ns}(\gamma) b_{m+n+3(r+s)-\Delta}(\rho) R_{m+n+3(r+s)-\Delta}(\rho),$$

(3.10)

and $\theta_{m+n+3(r+s),\Delta} = 1$ for $m + n + 3(r + s) \geq \Delta$ and 0 for $m + n + 3(r + s) < \Delta$.

The coupling matrix $y_d^{mn}$ takes a form

$$y_d^{mn} = y_D U_{m} U_{n} U_{0} \varepsilon^{m+n} Y_d^{mn}$$

(3.11)

with

$$Y_d^{mn} = A_d^{mn} - r_D B_d^{mn}, \quad r_D = \frac{y_D^{\Delta'}}{y_D} \varepsilon^{-\Delta'} r_{-\Delta'},$$

(3.12)

where

$$A_d^{mn} = \sum_{r,s=0}^{\infty} (-\varepsilon_D \varepsilon^r)^{s} (-\varepsilon_D \varepsilon^s) \varepsilon^{m+n+3(r+s)} (0) b^\beta_{mr}(\gamma) b^\gamma_{ns}(\alpha) b_{m+n+3(r+s)}(\sigma),$$

(3.13)

$$B_d^{mn} = \sum_{r,s=0}^{\infty} \theta_{m+n+3(r+s),\Delta'} (-\varepsilon_D \varepsilon^r)^{s} (-\varepsilon_D \varepsilon^s) \varepsilon^{m+n+3(r+s)}(\Delta')$$

$$\times b^\beta_{mr}(\gamma) b^\gamma_{ns}(\alpha) b_{m+n+3(r+s)-\Delta'}(\sigma) R_{m+n+3(r+s)-\Delta'}(\sigma),$$

(3.14)
Notice that all of $r_U$, $r_D$, $\epsilon_u e_3$, $\epsilon_q e_3$, $\epsilon_d e_3$, and $\epsilon_q e_3$ have a weight 0.

The coupling matrices $y_u^{mn}$ and $y_d^{mn}$ for the Type-II scheme are obtained by simply setting $r_U = r_D = 0$ in (3.8) and (3.12).

The remarkable aspect of the model is that the coupling matrices $y_u^{mn}$ and $y_d^{mn}$ have the definite transformation property under the $U(1)_H$ transformation:

$$y_u^{mn} \rightarrow e^{-i(m+n)\varphi_H} y_u^{mn}, \quad y_d^{mn} \rightarrow e^{-i(m+n)\varphi_H} y_d^{mn}. \quad (3.15)$$

Although the Yukawa couplings (3.4) are only a subset of the $SU(1, 1)$ invariant Yukawa couplings $W_Y$ given by (3.1) or (3.2), this transformation property, supplemented by (2.12) and (2.21), assures (3.4) to be invariant under the $U(1)_H$ transformation.

§4. Toy model analysis

Before discussing the details of the CP problem of the model, we examine a toy model analysis that will give indispensable information for the later considerations. The toy model is based on the $SU_3 \times SU(1, 1)$ gauge group.

The superpotential of the model is

$$W = (x_P P_\alpha P_{-\alpha} + x_Q Q_\beta Q_{-\beta}) \Psi^F + x_H H_{-\rho} \bar{H}_\rho (\Psi + \Psi^\prime) + y(P_\alpha Q_\beta H_{-\rho} + \bar{P}_{-\alpha} Q_{-\beta} \bar{H}_{\rho}) + W[\text{finite dim.}] . \quad (4.1)$$

The multiplets $P_\alpha$ and $Q_{-\beta}$ are the $SU_3$ triplets, and $H_{-\rho}$ is the singlet:

$$P_\alpha = \{p_\alpha, p_{\alpha+1}, \ldots \}, \quad Q_{-\beta} = \{q_{-\beta}, q_{-\beta-1}, \ldots \}, \quad H_{-\rho} = \{h_{-\rho}, h_{-\rho-1}, \ldots \}, \quad \rho = \alpha + \beta. \quad (4.2)$$

The multiplets $\bar{P}_{-\alpha}$, $\bar{Q}_\beta$, and $\bar{H}_\rho$ are the conjugate representations. The multiplets $\Psi^F$, $\Psi$, and $\Psi'$ have the highest weights $S^F$, $S$, and $S'(= S)$, respectively. The coupling constants $x_P$, $x_Q$, and $y$ in (4.1) are real values.

The superpotential (4.1) respects the global symmetry $U(1)_{PQ}$ with the charges given by

|       | $P_\alpha$ | $Q_{-\beta}$ | $H_{-\rho}$ | $\Psi'$ |
|-------|------------|--------------|-------------|---------|
| $Q_{PQ}$ | 1/2        | 1/2          | -1          | 0       | (4.3) |

in addition to the baryon number $U(1)_B$ symmetry. Both of the $U(1)_{PQ}$ and $U(1)_B$ symmetries may not suffer from the $SU_3$ gauge anomaly because the model is purely vectorlike, which is anomaly-free at any scale. So, if superheavy modes decouple from the low-energy and the other modes generate the anomaly of the $U(1)_{PQ}$ and $U(1)_B$ symmetries, then the anomaly matching condition requires introducing the Wess-Zumino-Witten (WZW)
terms to realize anomaly-free. However, in the following analysis, we deal with all modes including superheavy modes, so we need not consider the WZW terms.

The nonvanishing VEV
\[ \langle \Psi^F \rangle = \langle \psi_{-g}^F \rangle, \]
(4.4)
in the superpotential (4.1) generates \( g \) generations of the chiral quarks
\[ p_m \equiv p_{\alpha+m}, \quad \bar{q}_m \equiv \bar{q}_{\beta+m}, \quad (m = 0, 1, \cdots, g - 1). \]
(4.5)

Other components
\[ p_{I+g} \equiv p_{\alpha+I+g}, \quad \bar{q}_{I+g} \equiv \bar{q}_{\beta+I+g}, \quad \bar{p}_I \equiv \bar{p}_{-\alpha-I}, \quad \bar{q}_I \equiv \bar{q}_{-\beta-I} \quad (I = 0, 1, 2, \cdots) \]
(4.6)
become superheavy. The VEVs
\[ \langle \Psi \rangle = \langle \psi_1 \rangle, \quad \langle \Psi' \rangle = \langle \psi'_0 \rangle \]
(4.7)
generate one chiral higgs \( h \) in \( H_{-\rho} \) in the form
\[ h_{-\rho-i} = U_i h + [\text{massive modes}]. \]
(4.8)

The mixing coefficient \( U_i \) takes a form
\[ U_i = U_0 e^{i b_i(\rho)}, \quad \epsilon = \frac{\langle \psi'_0 \rangle}{\langle \psi_1 \rangle}, \]
(4.9)
where \( b_i(\rho) \) is a real function of the C-G coefficients.

The effective theory below the energy scale \( E \simeq 10^{16} \text{GeV} \) is described by the superpotential
\[ W_{\text{eff}} = \sum_{m,n=0}^{g-1} y C^F_{m,n} U_{m+n} \bar{q}_m p_n h, \]
(4.10)
where \( C^F_{m,n} \) is a real C-G coefficient. The supersymmetry breaking gives the breaking potential
\[ V_{\text{s.b.}} = \sum_{m=0}^{g-1} (m^2_p |p_m|^2 + m^2_q |\bar{q}_m|^2) + m^2_h |h|^2 + \left( \sum_{m,n=0}^{g-1} A m_0 y C^F_{m,n} U_{m+n} \bar{q}_m p_n h + \text{h.c.} \right). \]
(4.11)
The radiative corrections due to the Yukawa couplings pull down \( m^2_h \) to a negative value, and \( h \) eventually acquires the nonvanishing VEV \( \langle h \rangle \). This VEV brings the spontaneous breakdown of the \( U(1)_{\text{PQ}} \) symmetry, and the N-G boson \( G^0 \) appears.

Let us parametrize all VEVs in the form
\[ \langle \psi_{-g}^F \rangle = e^{-i \varphi_H} V^F, \quad \langle \psi_1 \rangle = e^{i \varphi_H} V, \quad \langle \psi'_0 \rangle = V', \quad \langle h \rangle = e^{-i \varphi_H} e^{-i \varphi_{\text{PQ}}} v, \]
(4.12)
where $\varphi_H$ represents the flat direction due to the $SU(1, 1)$ symmetry, and $\varphi_{PQ}$ represents the $U(1)_{PQ}$ symmetry. Due to the $SU(1, 1)$ gauge invariance, the total potential should be flat with respect to $\varphi_H$. The spontaneous breakdown of the $SU(1, 1)$ symmetry will take some value of $\varphi_H$, but any physically observable quantity should be independent of $\varphi_H$.

To evaluate $\theta$, let us write the tree level mass terms of the quarks:

$$m_{\theta} = \sum_{m,n=0}^{\infty} y C_{m,n}^F U_{m+n} \bar{q}_m p_n \langle h \rangle + \sum_{I=0}^{\infty} x_I^P D_I^P p_{I+g} \bar{p}_I \langle \psi^F_{-g} \rangle + \sum_{I=0}^{\infty} x_Q D_I^Q \bar{q}_I \psi^F_{-g}$$

where $D_I^P$ and $D_I^Q$ are real C-G coefficients and

$$\bar{Q} = \{ \bar{q}_I \} \quad Q = \{ p_I \} \quad p_n | p_{K+g} q_L \rangle$$

$$(m,n = 0,1, \cdots, g-1; I,J,K,L = 0,1, \cdots)$$

$$\mathcal{M}_{\text{tree}} = \begin{pmatrix}
  y C_{m,n}^F U_{m+n} \langle h \rangle & y C_{m,K+g}^F U_{m+K+g} \langle h \rangle & 0 \\
  y C_{I+g,n}^F U_{I+g+n} \langle h \rangle & y C_{I+g,K+g}^F U_{I+K+g} \langle h \rangle & x_Q D_I^Q \langle \psi^F_{-g} \rangle \delta_{IK} \\
  0 & x_P D_I^P \langle \psi^F_{-g} \rangle \delta_{JK} & 0
\end{pmatrix}$$

The diagonalization of this mass matrix will generate the $\theta$-term (1.1) with $\theta$ presented by

$$\theta = \arg \det [\mathcal{M}_{\text{tree}}]$$

Since $\mathcal{M}_{\text{tree}}$ is an infinite-dimensional matrix, we do not have a reliable method for the calculation. So, in the following considerations, we take a heuristic approach based on the two alternative assumptions.

**Naive evaluation**

First, we simply write

$$\det [\mathcal{M}_{\text{tree}}] = - \det [y C_{m,n}^F U_{m+n} \langle h \rangle] \times \det \left[ (x_P D_I^P \langle \psi^F_{-g} \rangle \delta_{JK}) \times (x_Q D_I^Q \langle \psi^F_{-g} \rangle \delta_{KL}) \right]$$

$$= - \det [y C_{m,n}^F U_{m+n} \langle h \rangle] \times \det \left[ x_P x_Q D_I^P D_I^Q \langle \psi^F_{-g} \rangle \delta_{JK} \right]$$

following a formula for a finite-dimensional matrix. Then, we have

$$\theta = \pi + \arg \det [y C_{m,n}^F U_{m+n} \langle h \rangle] + \arg \det \left[ x_P x_Q D_I^P D_I^Q \langle \psi^F_{-g} \rangle \delta_{JK} \right]$$
Substituting the expressions (4.9) and (4.12), we find

\[
\arg \det [y C_{m,n}^F U_{m+n} \langle h \rangle] = g(g - 1)(-\varphi_H + \arg [V'/V]) - g\rho\varphi_H - g\varphi_{PQ} + g \arg [v] + \arg \det [y C_{m,n}^F U_0 b_{m+n}(\rho)],
\]

\[
\arg \det \left[ x_P x_Q D_j^P D_Q^Q \langle \psi_{-g}^F \rangle \delta_{JK} \right] = \arg \det [x_P x_Q e^{-i2g\varphi_H} (V^F)^2 \delta_{JK}] + \arg \det \left[ D_j^P D_j^Q \delta_{JK} \right].
\]

(4.19)

Although the C-G coefficients \(D_j^P\) and \(D_j^Q\) have an alternating sign \((-1)^J\), the product \(D_j^P D_j^Q\) is real and positive. So, the second term in (4.20) vanishes. Let us introduce a numerical constant \(c\) by

\[
\arg \det [x_P x_Q e^{-i2g\varphi_H} (V^F)^2 \delta_{JK}] = c \left( -2g\varphi_H + 2 \arg [V^F] + \arg [x_P x_Q] \right).
\]

(4.21)

Formally, \(c\) is a divergent quantity since we have infinite numbers of the superheavy quarks:

\[
c = 1 + 1 + 1 + \cdots.
\]

(4.22)

However, we do not mind it. Summing up all terms in (4.18), we obtain

\[
\theta = \pi - g(g - 1 + \rho + 2c) \varphi_H - g\varphi_{PQ} + g(g - 1) \arg [V'/V] + g \arg [v] + c \left( 2 \arg [V^F] + \arg [x_P x_Q] \right) + \arg \det [y C_{m,n}^F U_0 b_{m+n}(\rho)].
\]

(4.23)

Let us next consider the effect of the radiative corrections induced by the low-energy effective theory to the quark mass matrix \(\mathcal{M}_{\text{tree}}\). The wave function renormalization is irrelevant to \(\theta\).\(^{21}\) This is because, when we perform the wave function renormalization, we must change the path integral measure, and both of the effects sum up to vanish in \(\theta\). Therefore, what we must consider is a vertex correction to \(\mathcal{M}_{\text{tree}}\). Owing to the non-renormalization theorem,\(^{22}\) the origin of the correction is limited to the supersymmetry breaking. This means that the corrections to the masses of the superheavy quarks are suppressed by \(m_{\text{SUSY}}/M\). Thus, the dominant effect is to the mass matrix \(\mathcal{M}_{nn}\) of the \(g\) generations of the chiral quarks. The quark and/or squark loops, however, do not give phases to \(\mathcal{M}_{nn}\) since the loops contain the coupling matrix \(Y_{mn} \equiv C_{m,n}^F U_{m+n}\) in the real form

\[
\text{tr}[Y Y^\dagger Y \cdots Y^\dagger Y Y^\dagger].
\]

(4.24)

The (s)quark line connecting the external quarks contains \(Y\) in the form

\[
(Y Y^\dagger Y \cdots Y^\dagger Y)^{mn}.
\]

(4.25)
This array of matrices preserves the phase structure $U_{m+n}$ of the single $Y^{mn}$. The number of the VEV $\langle h \rangle$ attached to the diagram is larger than that of $\langle h \rangle^*$ just by 1. Thus, the vertex correction does not disturb the phase structure of $M_{\text{tree}}^{mn}$. The effect is limited to the modification of the last term in (4.23). These arguments also apply to the superheavy quark masses. The gluino mass is also real. Therefore, (4.23) holds even after the radiative corrections of the low-energy physics are fully included as long as the determinant in the last term does not change its sign.

It is known that the $SU_3$ instantons give the vacuum energy proportional to $(1 - \cos \theta)$.

Therefore, the $SU(1, 1)$ invariance requires the coefficient of $\varphi_H$ in (4.23) to vanish. Thus, we have, for the consistent value of $c$,

$$c = -\frac{g - 1}{2} - \frac{\rho}{2}, \quad (\rho = \alpha + \beta). \quad (4.26)$$

The expression (4.23) gives further restriction on the value of $c$. Although we assume that the coupling constants $x_P$ and $x_Q$ are real numbers, we always have a freedom to replace them by

$$x_P \rightarrow e^{i2\pi n_P} x_P, \quad x_Q \rightarrow e^{i2\pi n_Q} x_Q, \quad (4.27)$$

with integers $n_P$ and $n_Q$. Since we have no phase transition on the “path” of the extended model space $x_P = e^{i2\pi t} |x_P|$ $(0 \leq t \leq 1)$, this replacement should not shift the value of $\theta$ up to modulus $2\pi$. Thus, $c$ must be an integer. Consequently, the expression (4.26) requires $\rho$ to be an integer. When $\rho$ is an odd integer, $g$ must be an even integer and vice versa.

These results seem to be inconsistent, because we are allowed to have arbitrary number $(g)$ of the chiral quarks, and also allowed to assign the weights $\alpha$ and $\beta$ to arbitrary positive values. Furthermore, this evaluation (4.17) gives the $\varphi_{PQ}$ dependence to $\theta$. From (4.23), we find

$$\theta = -g\varphi_{PQ} + [\varphi_{PQ}-\text{independent terms}]. \quad (4.28)$$

This equation states that, once the $SU(1, 1)$ symmetry is spontaneously broken and the chiral quarks are generated $(g \neq 0)$, the $U(1)_{PQ}$ symmetry becomes an anomalous symmetry although it is originally a vector symmetry. If we accept this result, the N-G boson $G^0$ becomes an axion.

Alternative evaluation

It is reasonable to suspect that the expression (4.17) is incorrect because $M_{\text{tree}}$ is an infinite-dimensional matrix. The $(1,2)$-entry, $(2,1)$-entry, and $(2,2)$-entry sub-matrices of $M_{\text{tree}}$ will contribute to $\theta$ by some amount and modify (4.23) so that the $U(1)_{PQ}$ symmetry is exactly maintained. The potentially complex quantities contained in these sub-matrices are
$\langle h \rangle$ and $\epsilon$ in $U_m$. From the expression (4.12), we find that the only candidate that eliminates the $\varphi_{PQ}$ dependence of $\theta$ in (4.23) is the addition of the term $-g \arg[\langle h \rangle]$. Because $\langle h \rangle$ always appears in $M_{\text{tree}}$ by the combination $yU_0(h)$, what we should add must be $-g \arg[yU_0\langle h \rangle]$. The phase of $\epsilon$ will also modify $\theta$ by some amount. Since we have no way to determine the factor, we add $gc'$ $\arg[\epsilon]$ with some unknown real number $c'$ that will depend on $g$. In principle, we may not rule out a possibility that some function of the product of $\langle h \rangle$, $\epsilon$, and $\langle \psi_f^g \rangle$ that has zero total weight contributes to $\theta$. Since $\langle h \rangle$ moves under the $U(1)_{PQ}$ transformation, it cannot join in this product, and the candidate is limited to the function of $\langle \psi_f^g \rangle/\epsilon^g$. We omit this possibility because it seems to be implausible that $\det[M_{\text{tree}}]$ contains such a term since $M_{\text{tree}}$ is decomposable to the product of the three matrices $M_{\text{tree}} = A(\epsilon)B(\langle h \rangle, \langle \psi_f^g \rangle)A(\epsilon)$. The omission of this term amounts to the assumption that the formula $\det[XY] = \det[X]\det[Y]$ holds for infinite-dimensional matrices. We note that even if we incorporate this term, we will see in §5 that the phase of $\langle \psi_f^g \rangle/\epsilon^g$ must be assigned to $0 \pmod{\pi/4}$ and it does not give a serious problem to a discussion of $\theta$. Therefore, under the above assumption, there is no other way than the addition of the first two terms to modify $\theta$ to recover the $U(1)_{PQ}$ symmetry. This leads to

$$
\theta = \pi - g(g - 1 + c' + 2c) \varphi_H + g(g - 1 + c') \arg[V'/V] + c\left(2\arg[V^F] + \arg[x_Px_Q]\right) + \arg\det[C_{m,n}^F b_{m+n}(\rho)],
$$

(4.29)

from which we find

$$
c = \frac{g - 1 + c'}{2}.
$$

(4.30)

So, we have

$$
\theta = \pi + c \left(-2g \arg[V'/V] + 2\arg[V^F] + \arg[x_Px_Q]\right) + \arg\det[C_{m,n}^F b_{m+n}(\rho)].
$$

(4.31)

Notice that the factor $yU_0(h)$ in the (1,1)-entry sub-matrix of (4.15) is completely eliminated in this expression by the possible contributions from the (1,2), (2,1), and (2,2)-entry sub-matrices, which are related to the superheavy quarks. Also, $V^F$, which represents the masses of the superheavy quarks, appears in (4.31) to cancel the weight of $V'/V$. These results show that the superheavy particles play a significant role in $\theta$. They never decouple from $\theta$.

Although we do not have reliable methods of determining the value of $c$, it must be an integer depending on $g$. In this alternative evaluation, the N-G boson $G^0$ is not an axion. It should be an exactly massless pseudo-scalar particle. The result (4.31) does not impose any constraint on the number of the chiral quarks nor on the values of the weights.
Coupling of $\langle \psi_0^F \rangle$

Let us finally add the coupling $(x'_p P_\alpha \bar{P}_{-\alpha} + x'_{Q} \bar{Q}_{-\beta} Q_{-\beta}) \Psi^F$ with the VEV $\langle \psi_0^F \rangle$ to the superpotential (4.1). Then, the tree level mass terms (4.13) are modified in the form $\bar{M} \mathcal{M} Q$ with

$$\mathcal{M} = \left( \begin{array}{cc} yC_{i+k}^F U_{i+k} \langle h \rangle & xQ D^Q_I \langle \psi_0^F \rangle \delta_{i+g} + x'P D^P_J \langle \psi_0^F \rangle \delta_{j,k} \\ xP D^P_J \langle \psi_0^F \rangle \delta_{j+g,k} + x'P D^P_J \langle \psi_0^F \rangle \delta_{j,k} & 0 \end{array} \right).$$

(4.33)

We deform $\mathcal{M}$ to $\mathcal{M}'$ by eliminating $\langle \psi_0^F \rangle$ by using $\langle \psi_{-g}^F \rangle$ so that $\mathcal{M}$ and $\mathcal{M}'$ have the same determinant. The result is

$$\mathcal{M}' = \left( \begin{array}{cc} y \sum_{r,s=0}^\infty (-\epsilon_q)^r (-\epsilon_p)^s b^Q_r (\beta) b^P_s (\alpha) C^F_{i+gr+k+gs} U_{i+gr+k+gs} \langle h \rangle & xQ D^Q_I \langle \psi_0^F \rangle \delta_{i+g} \\ xP D^P_J \langle \psi_0^F \rangle \delta_{j+g,k} + x'P D^P_J \langle \psi_0^F \rangle \delta_{j,k} & 0 \end{array} \right),$$

(4.34)

where

$$\epsilon_p = \frac{x'_P \langle \psi_0^F \rangle}{x_P \langle \psi_{-g}^F \rangle}, \quad \epsilon_q = \frac{x'_Q \langle \psi_0^F \rangle}{x_Q \langle \psi_{-g}^F \rangle}, \quad b^Q_r (\beta) = \prod_{m=0}^{s-1} \frac{D^P_m}{D^{H}_{k+gm}}, \quad b^P_s (\alpha) = \prod_{m=0}^{r-1} \frac{D^Q_m}{D^{Q}_{i+gm}},$$

(4.35)

This matrix has almost the same structure as (4.15). When we perform the naive evaluation of $\theta$, the difference is only the replacement of the mass matrix of the $g$ generations of chiral quarks, except for the absence of the normalization factors $U_{r,n}(<1)$ and $U_{n,g}(<1)$ of the chiral quarks. Their absence is understandable because the elimination of $\langle \psi_0^F \rangle$ decreases the masses of the superheavy quarks but det $[\mathcal{M}]$ must be unchanged.

Let us evaluate $\theta$ based on the expression (4.34). The naive evaluation gives

$$\theta_{\text{naive}} = \pi - g(g-1+\rho + 2c) \varphi_H - g \varphi_{PQ} + g(g-1) \arg [V'/V] + g \arg [v]$$

$$+ c (2 \arg [V^F] + \arg [x_P x_Q])$$

$$+ \arg \det \left[ y \sum_{r,s=0}^\infty (-\epsilon_q)^r (-\epsilon_p)^s b^Q_r (\beta) b^P_s (\alpha) C^F_{m+gr,n+gs} U_{m+n+g(r+s)} \langle h \rangle \right].$$

(4.36)

To obtain the alternative evaluation of $\theta$, we must examine the detailed structure of the (1,1)-entry matrix of (4.34):

$$\mathcal{M}_{(1,1)}^{ik} = y \sum_{r,s=0}^\infty (-\epsilon_q)^r (-\epsilon_p)^s b^Q_r (\beta) b^P_s (\alpha) C^F_{i+gr,k+gs} U_{i+gr+k+gs} \langle h \rangle$$

16
\[ = y U_0(h) e^{i+k} \sum_{r,s=0}^{\infty} (-\epsilon_q e^g)^r (-\epsilon_p e^g)^s b^q_{ir}(\beta) b^p_{ks}(\alpha) C_{i+gr,k+gs} b_{i+k+g(r+s)}(\rho). \] 

(4.37)

The elimination of the \( \varphi_{PQ} \) dependence from (4.36) is accomplished only by adding a term \(-g \arg[y U_0(h)]\) as we did. Also, because \( e^{i+k} \) is an overall factor, its possible effect is the addition of the term \( g c' \arg[\epsilon] \). The remaining matrix in (4.37) contains potentially complex extra quantities \( \epsilon_q e^g \) and \( \epsilon_p e^g \). Since this matrix is symmetric under the interchange \( (i, \beta, \epsilon_q e^g) \leftrightarrow (k, \alpha, \epsilon_p e^g) \) (4.38) owing to the symmetry relation (3.6) of the C-G coefficient \( C^F_{i,k} \), its possible effect to \( \theta \) should be through the term \( \arg[f_{\beta,\alpha}(\epsilon_q e^g, \epsilon_p e^g)] \), where \( f_{\beta,\alpha}(x,y) = f_{\alpha,\beta}(y,x) \) is some real function of \( x \) and \( y \). Thus, \( \theta \) should take the form

\[
\theta = \pi + c \left(-2 g \arg[\epsilon] + 2 \arg[\psi_{-g}^F] + \arg[x_p x_Q] + \arg[f_{\beta,\alpha}(\epsilon_q e^g, \epsilon_p e^g)] \right) + \arg \det \left[ \sum_{r,s=0}^{\infty} (\epsilon_q e^g)^r (\epsilon_p e^g)^s b^q_{ir} b^p_{ks} C_{i+gr,n+gs} b_{i+k+g(r+s)}(\rho) \right]. \] 

(4.39)

One may doubt about the prescription adopted to eliminate \( \langle \psi^F_0 \rangle \). The unterminated series of \( \langle \psi^F_0 \rangle \) may still contribute to \( \theta \) with a term proportional to \( \arg[\langle \psi^F_0 \rangle] \). However, the coefficient \( c_0 \) of \( \arg[\langle \psi^F_0 \rangle] \) must be an integer, and the limit \( x'_p, x'_Q \to 0 \) must reproduce \( c_0 \to 0 \). Since we have no phase transition in this limit, we should have \( c_0 = 0 \). Notice that (4.39) precisely reproduces (4.31) in this limit.

§5. Tree level analysis of the strong CP problem

Now that we have obtained sufficient knowledge on what happens when the \( SU(1,1) \) symmetry is spontaneously broken, let us return to the main subject. The total superpotential of the model

\[ W = W_{\text{quark}} + W_{\text{higgs}} + W_{M} + W_{Y} + W[\text{finite dim.}] \] 

(5.1)

respects the \( U(1)_{PQ} \) symmetry. The supersymmetry breaking terms also respect this symmetry. This symmetry is spontaneously broken at the energy scale \( \sqrt{m_{\text{SUSY}} M} \) by the VEVs of \( r \) and \( \tilde{r} \). The VEVs of the higgses \( h \) and \( h' \) also break this symmetry at the scale \( m_{\text{SUSY}} \). If we accept the naive evaluation (4.23), the \( U(1)_{PQ} \) symmetry is now an anomalous symmetry because we have three generations of the chiral quarks. This means that the associated N-G boson \( G^0 \) is an invisible axion. Therefore, \( G^0 \) will solve the strong CP problem in the present model. We do not, however, accept this solution, since it seems to lead to some
inconsistency observed in the previous section. We make a trial of solving the problem based on the alternative evaluation of $\theta$ presented in (4.31) and (4.39).

Let us evaluate the value of $\theta$ in the model coming from the diagonalization of the quark mass matrices. Remember that the original value of $\theta$ is $\theta_0 = 0$. The relevant part of the superpotential for the subject is the mass operators of the quarks:

$$W_{\text{mass}} = W_{\text{quark}} + W_Y. \quad (5.1)$$

We introduce a notation

$$Q_\alpha = \begin{pmatrix} U_\alpha^q \\ D_\alpha^q \end{pmatrix}, \quad U_\alpha^q = \{u_\alpha^q, u_{\alpha+1}^q, \ldots\}, \quad D_\alpha^q = \{d_\alpha^q, d_{\alpha+1}^q, \ldots\}. \quad (5.2)$$

We examine the Type-II scheme (3.2) for $W_Y$, which is simpler than the Type-I scheme. The up-type and down-type quark mass operators are expressed as

$$W_{\text{mass}}^u = (x_Q U_\alpha^q \bar{U}_\alpha^q + x_U U_\beta \bar{U}_\beta) \Psi^F + (x'_Q U_\alpha^q \bar{U}_\alpha^q + x'_U U_\beta \bar{U}_\beta) \Psi'^F$$

$$+ y_U (\bar{U}_\alpha U_\beta^2 H_{-\rho} + U_\beta \bar{U}_\alpha H_{-\rho}^2), \quad (5.3)$$

$$W_{\text{mass}}^d = (x_Q D_\alpha^q \bar{D}_\alpha^q + x_D \bar{D}_\gamma D_{-\gamma}) \Psi^F + (x'_Q D_\alpha^q \bar{D}_\alpha^q + x'_D \bar{D}_\gamma D_{-\gamma}) \Psi'^F$$

$$- y_D (\bar{D}_\gamma D_\rho H_{-\rho}^1 + D_{-\rho} \bar{D}_\gamma H_{-\rho}^1), \quad (5.4)$$

with the VEVs

$$\langle \Psi^F \rangle = \langle \psi^F_3 \rangle, \quad \langle \Psi'^F \rangle = \langle \psi'^F_0 \rangle, \quad \langle H_{-\rho}^2 \rangle = \langle h_{-\rho-i}^2 \rangle = \langle h_{-\rho-i}^1 \rangle = \langle h_{-\rho-i}^1 \rangle = \langle h_{-\rho-i}^1 \rangle U_i. \quad (5.5)$$

When we set $x_Q = x_U = x_D = 0$, all quarks become superheavy, but when we set $x'_Q = x'_U = x'_D = 0$, we have three generations of the quarks. This means that there is a critical value $\epsilon_{\text{cr}}$ for the ratios

$$\epsilon_q = \frac{x'_Q \langle \psi'^F_0 \rangle}{x_Q \langle \psi^F_3 \rangle}, \quad \epsilon_u = \frac{x'_U \langle \psi'^F_0 \rangle}{x_U \langle \psi^F_3 \rangle}, \quad \epsilon_d = \frac{x'_D \langle \psi'^F_0 \rangle}{x_D \langle \psi^F_3 \rangle}. \quad (5.6)$$

The critical ratio $\epsilon_{\text{cr}}$ is derived from the normalizable condition $\sum_{i=0}^{\infty} |U_m^f|^2 < \infty$ ($f = q, u, d$) for the mixing coefficients (2.8) of the chiral modes, which requires $\lim_{i \to \infty} |U_m^f| / |U_m^{f+1}| < 1$. This gives $^{13}$

$$\epsilon_{\text{cr}} = \sqrt{\frac{S^F(S^F-1)(S^F-2)}{(S^F+3)(S^F+2)(S^F+1)}}. \quad (5.7)$$

To generate the three generations of $q_m, \bar{q}_m$, and $\bar{d}_m$, all ratios in (5.7) must satisfy

$$|\epsilon_q|, |\epsilon_u|, |\epsilon_d| < \epsilon_{\text{cr}}. \quad (5.8)$$

18
If we set \( x'_Q = x'_U = x'_D = 0 \), the mass operators (5.4) and (5.5) reduce to the duplication of the toy model (4.1). Since the coefficient \( c \) in (4.31) is an integer and it should be independent of the weights \( \alpha, \beta, \) and \( \gamma \), it is sure that the up-type quarks and the down-type quarks take the common value. Thus, the diagonalization of the quark mass matrices will generate \( \theta_{\text{tree}} \) of the amount

\[
\theta_{\text{tree}}^0 = c \left( -6 \arg[\epsilon] + 2 \arg[(\psi^F_3)] + \arg[x_Q x_U] - 6 \arg[\epsilon'] + 2 \arg[(\psi^F_3)] + \arg[x_Q x_D] \right) \\
+ \arg \det[C_{m,n}^{\beta,\alpha} p_{m+n}(\rho)] + \arg \det[C_{m,n}^{\gamma,\alpha} p'_{m+n}(\sigma)]
\]

(5.10)

up to modulus 2\( \pi \).

When we turn on the couplings \( x'_Q, x'_U, \) and \( x'_D \), all chiral quarks are realized as the superpositions of the infinite number of the components of each \( SU(1, 1) \) multiplet. Then, \( \theta_{\text{tree}}^0 \) is modified by \( x'_Q \langle \psi^F_0 \rangle, x'_U \langle \psi^F_0 \rangle, \) and \( x'_D \langle \psi^F_0 \rangle \). From the final result (4.39) of the toy model, we obtain \( \theta_{\text{tree}} \), for the Type-II scheme, in the form

\[
\theta_{\text{tree}} = c \left( - \arg[\epsilon q^{e^3}] - \arg[\epsilon q^{e^3}] - \arg[\epsilon x^{e^3}] - \arg[\epsilon x^{e^3}] + 4 \arg[(\psi^F_0)] + \arg[x'_U x'_D] \right) \\
+ \arg[f^{\beta,\alpha}(\epsilon u^{e^3}, \epsilon q^{e^3})] + \arg[f^{\gamma,\alpha}(\epsilon d^{e^3}, \epsilon q^{e^3})] + \arg \det[A_u^{mn}] + \arg \det[A_d^{mn}].
\]

(5.11)

Let us next examine the Type-I scheme (3.1) for \( W_Y \). When we extend the toy model (4.1) so that it is applicable in this scheme, the matrix (4.37) is replaced by

\[
y U_0 \langle h \rangle \epsilon^{i+k} \left( \sum_{r,s=0}^{\infty} (-\epsilon q^{e^g})^r (-\epsilon p^{e^g})^s b_{tr}^q (\beta) b_{ks}^p (\alpha) C_{i+gr}^{F} (0) b_{i+k+g(r+s)} (\rho) \\
- R_F \sum_{r,s=0}^{\infty} \theta_{i+k+g(r+s)}, \Delta (-\epsilon q^{e^g})^r (-\epsilon p^{e^g})^s \\
\times b_{tr}^q (\beta) b_{ks}^p (\alpha) C_{i+gr}^{F} (\Delta) b_{i+k+g(r+s)} (\rho) \right). \]

(5.12)

Notice that the matrix in the parenthesis is symmetric under the interchange

\[
(i, \beta, \epsilon q^{e^g}, r_F) \leftrightarrow (k, \alpha, \epsilon p^{e^g}, (-1)^{\Delta} r_F)
\]

(5.13)

Therefore, following the same consideration given at the end of the previous section, we obtain the expression of \( \theta_{\text{tree}} \) for the Type-I scheme in the form

\[
\theta_{\text{tree}} = c \left( - \arg[\epsilon q^{e^3}] - \arg[\epsilon q^{e^3}] - \arg[\epsilon x^{e^3}] - \arg[\epsilon x^{e^3}] + 4 \arg[(\psi^F_0)] + \arg[x'_U x'_D] \right) \\
+ \arg[f^{\beta,\alpha}(\epsilon u^{e^3}, \epsilon q^{e^3}, r_U)] + \arg[f^{\gamma,\alpha}(\epsilon d^{e^3}, \epsilon q^{e^3}, r_D)] \\
+ \arg \det[Y_u^{mm}] + \arg \det[Y_d^{mn}],
\]

(5.14)
where \( f^{\beta,\alpha}(x, y, z) \) is a real function of \( x, y, \) and \( z \) satisfying the relation

\[
f^{\beta,\alpha}(x, y, z) = f^{\alpha,\beta}(y, x, (-1)^\Delta z).
\] (5.15)

The expression (5.14) covers both of the Type-I and Type-II \((r_U = r_D = 0)\) schemes. Remember that \( c \) is some real integer, although we cannot determine its value.

Now, we proceed to the main subject of the problem. We must search for a solution to suppress \( \theta_{\text{tree}} \) in (5.14) in a natural manner. What we should attain is \( \theta_{\text{tree}} = 0 \) (mod. \( \pi \)).

We note that the value \( \theta = \pi \) does not bring the strong CP problem. Let us first investigate (5.11) in detail for the Type-II scheme. From their expressions (3.9) and (3.13), we find remarkable facts. Although the matrices \( A_{mn}^u \) and \( A_{mn}^d \) consist of the infinite number of terms, each term contains a potentially complex quantity only by the factor \((-\epsilon_u \epsilon^\prime \epsilon^3)^r (-\epsilon_q \epsilon^3)^s \) or \((-\epsilon_d \epsilon^\prime \epsilon^3)^r (-\epsilon_q \epsilon^3)^s \). If these quantities were complex numbers, it will be hard to expect a miracle occurs so that \( \det[A_{mn}^u] \) and \( \det[A_{mn}^d] \) have the natural phases of realizing \( \theta_{\text{tree}} = 0 \). That is, the phase assignments

\[
\text{arg}[\epsilon_q \epsilon^3] = 0, \quad \text{arg}[\epsilon_u \epsilon^3] = 0, \quad \text{arg}[\epsilon_q \epsilon^3] = 0, \quad \text{arg}[\epsilon_d \epsilon^3] = 0 \quad \text{(mod. } \pi) \] (5.16)

are indispensable. Then, \( \det[A_{mn}^u] \) and \( \det[A_{mn}^d] \) are real. What is surprising is that these phase assignments ensure almost all terms except \( \text{arg}[\langle \psi_0^{\prime F} \rangle] \) in (5.11) to vanish up to modulus \( \pi \). Since \( \epsilon_f \)'s \((f = q, u, d)\) have the common phase up to \( \pi \), the net conditions are

\[
\text{arg}[\epsilon_q \epsilon^3] = 0, \quad \text{arg}[\epsilon_q \epsilon^3] = 0 \quad \text{(mod. } \pi). \quad (5.17)
\]

The remaining subject is the phase of \( \langle \psi_0^{\prime F} \rangle \). The sufficient condition of eliminating this contribution is

\[
\text{arg}[\langle \psi_0^{\prime F} \rangle] = 0 \quad \text{(mod. } \pi/4). \quad (5.18)
\]

Thus, the conditions (5.17) and (5.18) realize

\[
\theta_{\text{tree}} = 0 \quad \text{(mod. } \pi). \quad (5.19)
\]

Notice that the conditions (5.17) fix the relative phase of \( \epsilon \) and \( \epsilon' \) in the way

\[
\text{arg}[\epsilon/\epsilon'] = 0 \quad \text{(mod. } \pi/3). \quad (5.20)
\]

It will be evident that the integer 3 in the denominator of \( \pi/3 \) in (5.20) is a consequence of the fact that we have three generations of the chiral quarks.

Next, we investigate the Type-I scheme. It will be obvious that the conditions (5.17) and (5.18) are also indispensable for realizing \( \theta_{\text{tree}} = 0 \) in a natural way. However, the situation
is somewhat complicated because this scheme contains the additional quantities \(r_U\) and \(r_D\).

The expression of \(\theta_{\text{tree}}\) is now

\[
\theta_{\text{tree}} = \arg \det[Y^u_{mn}] + \arg \det[Y^d_{mn}]
+ \arg[f^{\beta,\alpha}(\epsilon_u \epsilon^3, \epsilon_q \epsilon^3, r_U)] + \arg[f^{\gamma,\alpha}(\epsilon_d \epsilon^3, \epsilon_q \epsilon^3, r_D)] \quad \text{(mod. } \pi),
\]

with real \(\epsilon_f \epsilon^3\) and \(\epsilon_f' \epsilon^3\). Since we cannot expect a delicate cancellation between two nontrivial phases of the first two terms in (5.21), we are forced to require

\[
\arg \det[Y^u_{mn}] = 0, \quad \arg \det[Y^d_{mn}] = 0 \quad \text{(mod. } \pi).
\]

The obvious candidate for meeting these requirements is

**Option-1**: \(\arg[r_U] = 0, \arg[r_D] = 0 \quad \text{(mod. } \pi).\) (5.22)

Then, the third and the fourth terms in (5.21) also vanish, and we have \(\theta_{\text{tree}} = 0 \text{(mod. } \pi)\).

Another candidate, which is rather subtle, is

**Option-2**: \(\arg[r_U] = \frac{\pi}{2}, \arg[r_D] = \frac{\pi}{2} \text{ (mod. } \pi).\) (5.23)

In this case, we need the additional conditions:

\[
\alpha = \beta = \gamma, \quad \frac{x'_Q}{x_Q} = \frac{x'_U}{x_U} = \frac{x'_D}{x_D}, \quad \Delta, \ \Delta' = [\text{odd integer}].
\]

The first two conditions give

\[
b^u_{mr}(\alpha) = b^d_{mr}(\beta) = b^d_{mr}(\gamma), \quad \epsilon_q = \epsilon_u = \epsilon_d.
\]

Then, the third condition implies that the matrices \(A^u_{mn}\) and \(A^d_{mn}\) are symmetric but \(B^u_{mn}\) and \(B^d_{mn}\) are anti-symmetric owing to the symmetry relation (3.6). Consequently, the phase assignment (5.24) realizes the hermitian matrices for \(Y^u_{mn}\) and \(Y^d_{mn}\) whose determinants are real. What is unexpected is that the conditions (5.24) and (5.25) also render the third and the fourth terms in (5.21) to vanish separately. This is because the symmetry relation (5.15) implies, under the odd value of \(\Delta\), that the function \(f^{\alpha,\alpha}(\epsilon_q \epsilon^3, \epsilon_q \epsilon^3, r_U)\) is pure real when \(r_U\) is pure imaginary:

\[
f^{\alpha,\alpha}(\epsilon_q \epsilon^3, \epsilon_q \epsilon^3, r_U) = f^{\alpha,\alpha}(\epsilon_q \epsilon^3, \epsilon_q \epsilon^3, -r_U) = f^{\alpha,\alpha}(\epsilon_q \epsilon^3, \epsilon_q \epsilon^3, r_U)^*.
\]

Of course, we also have the mixed candidates. That is, the up-type quarks take the Option-1 phase assignment and the down-type quarks take the Option-2 assignment and *vice versa.*
The universal quark assignment (5.25) for the Option-2 case seems to invoke the grand unified gauge group \( G \supset SU_3 \times SU_2 \times U_1 \).\(^{23)-26}\) When \( G = SU(5) \), it is not possible to satisfy all requirements in (5.25), and the case is limited to the mixed candidate. One may imagine that \( G = SO(10) \) easily fills (5.25) by simply assigning \( \Psi^F \) and \( \Psi'^F \) to the common representation of \( SO(10) \). This assignment, however, generates the massless right-handed neutrinos. As far as we have examined, it seems to be hard to eliminate the right-handed neutrinos as an illusion\(^{14}\) without disturbing (5.25). The possibility is open for \( G \supset E_6 \).

It is worth noting that, when the up-type and the down-type quarks take the Type-II scheme and/or the Type-I scheme with the Option-1 phase assignment, the phases residing in the Yukawa coupling matrices (3.7) and (3.11) are only in \( \epsilon \) and \( \epsilon' \). This means that the origin of the phases in the CKM matrix is solely the relative phase of \( \epsilon \) and \( \epsilon' \), which has been fixed by (5.20) to be \( \delta \equiv \arg[\epsilon/\epsilon'] = n\pi/3 \) with some integer \( n \). Then, the CKM matrix is restricted to the form

\[
V_{\text{CKM}} = O_u^T P O_d, \quad P = \begin{pmatrix} e^{2i\delta} & 0 & 0 \\ 0 & e^{i\delta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \delta = n\pi/3, (5.28)
\]

where \( O_u \) and \( O_d \) are the real orthogonal matrices, standing on the right when the “reversed” \((m, n = 2, 1, 0)\) coupling matrices \( y_u^{mn} \) and \( y_d^{mn} \) with real and positive \( \epsilon \) and \( \epsilon' \) are diagonalized, respectively.

§6. Consideration on the radiative corrections

In the toy model analysis given in the §4, we found that \( \theta \) does not receive the radiative corrections of the low-energy physics. In the MSSM, we have two higgses \( h \) and \( h' \), each of which has its own Yukawa coupling matrix. Through the supersymmetry breaking terms, \( h^* \) couples to the down-type quarks and \( h'^* \) to the up-type quarks. At the one loop level, this is achieved by the squark-higgsino loop. As a result, the phase structures of the mass matrices \( M_u^{mn} \) and \( M_d^{mn} \) are disturbed. This seems to indicate that \( \theta \) will receive the uncontrollable radiative correction \( \Delta \theta \).

We must, however, remember that the superheavy quarks do not decouple from \( \theta \). We found that the factor \( y_U U_0(h) \) in the tree level mass matrix of the chiral quarks was completely eliminated in the final expression of \( \theta \) presented in (4.39) by the contributions from masses of the superheavy quarks. In the present model, all entries of the mass matrix receive the nontrivial vertex corrections. This means that we must take into account all radiative corrections not only from the chiral particles but also from the superheavy particles. It is not unreasonable to expect that both effects of the radiative corrections exactly cancel each
other when \( \theta = 0 \) is naturally realized at the tree level.

Let us first consider the factors \( y_U \langle h \rangle \) appearing in the mass matrix \( M_{ij}^{\text{tree}} \) of the up-type quarks. Each of them receives a very complicated vertex correction and is modified to the order \( m_{\text{SUSY}} \). However, their \( U(1)_{\text{PQ}} \) transformation property is simply determined, by that of the quarks multiplied to this mass matrix, to be

\[ X_{ij} \rightarrow e^{-i\phi_{\text{PQ}}} X_{ij} \]  

(a coefficient \( c_{ij} \) is an appropriate function of the relevant VEVs that adjust the \( U(1)_H \) and \( U(1)_{\text{PQ}} \) charges). The exact \( U(1)_{\text{PQ}} \) symmetry at the quantum level, therefore, requires that the contribution of \( X_{ij} \) to \( \theta \) must be canceled between the chiral and the superheavy quarks in the same way as we found at the tree level. The radiatively induced new mass operator \( U \tilde{M} U^q \) is suppressed to the negligibly small order \( \tilde{M} \sim m_{\text{SUSY}} (m_{\text{SUSY}}/M)^2 \) and thus it does not disturb this cancellation though \( \tilde{M} \) has \( Q_{\text{PQ}} = 1 \). Because the exact \( U(1)_{\text{PQ}} \) symmetry owes its realization fully to the fact that the model is based on the framework of the vectorlike gauge theory, it will be reasonable to expect that the true origin of this cancellation is the vectorlike feature of the model rather than the exact \( U(1)_{\text{PQ}} \) symmetry. If this consideration is correct, the vertex corrections to the elements of \( M_{ij}^{\text{tree}} \) that contain \( \langle \psi^F_{-3} \rangle \) and \( \langle \psi^F_0 \rangle \) and \( \tilde{M} \) will be also canceled in \( \theta \).

Therefore, the possible source of the modification to \( \theta_{\text{tree}} = 0 \) is the radiative effects to the relative phases of the VEVs of \( \Psi \)'s. If these phases at the tree-level are determined so that they depend continuously on the coupling constants of the model, there will be no reason to expect that these relative phases do not receive radiative corrections, and we will have sizable \( \Delta \theta \). However, when the VEVs \( \langle \Psi \rangle \)'s are determined through their superpotential \( W[\text{finite-dim.}] \) so that they have natural phases given in (2.4) and realize \( \theta_{\text{tree}} = 0 \), it seems to be probable that \( \theta_{\text{tree}} = 0 \) will not be affected by the radiative corrections as we will argue in the following.

As an illustration, let us consider the \( \lambda \phi^4 \) theory with negative mass-square. This theory has two degenerate vacua

\[ \langle \phi \rangle_1 = -\langle \phi \rangle_2 \equiv v_0. \]  

(6.1)

Although the magnitude of \( v_0 \) receives the radiative correction, the relation (6.1) is protected by the symmetry \( \phi \rightarrow -\phi \) of the theory. The directions of the vacua from the origin \( \phi = 0 \) are determined by the tree level analysis. In general, when a theory has some discrete symmetry \( D \), and \( D \) is spontaneously broken, the degenerate vacua should take definite directions in the field-space of the theory specified by the symmetry of the theory.

The present model has the P-C-T-invariance at the fundamental level, and all of these discrete symmetries are spontaneously broken by the VEVs of \( \Psi \)'s. We have realized \( \theta = 0 \) at the tree level in the natural manner, at least for the Type-I scheme with Option-1 phase.
assignment and the Type-II scheme, which depends only on the directions (phases) of the VEVs and does not depend on the detailed values of the coupling constants of the model. Even in the Type-I scheme with Option-2 phase assignment, when the second relation in (5.25) is protected by the gauge symmetry of the grand unified theory, the realization $\theta = 0$ is natural. Remember that $\theta = 0$ is a minimum point of the vacuum energy. Suppose $\theta_{\text{tree}} = 0$ receives the radiative corrections. Then, there will be the direction of the VEVs that realizes $\theta = 0$ in the vicinity of the original direction. However, this direction must depend on the coupling constants of the model to cancel the radiative corrections. It seems to be very unbelievable that the direction of the VEVs, which spontaneously realizes $\theta = 0$, depends on the coupling constants of the model.

One may wonder why $\theta_{\text{tree}} = 0$ that is realized in the models based on the spontaneous breakdown of the CP-invariance$^{10}$ receives the radiative corrections. The essential reason is that the gluon $\theta$-term (1.1) breaks not only the CP-invariance but also the P-invariance. The ordinary models with the spontaneous CP-violation are based on the framework of the chiral theory, which explicitly breaks the P-invariance. Therefore, the emergence of $\theta$ is not a genuine product of the spontaneous symmetry breaking. This means that, even if $\theta_{\text{tree}} = 0$ is achieved in these models, it is not protected against the radiative corrections by the symmetry of the model.

In the present model, all of the P, C, and T symmetries are spontaneously broken. Especially, the spontaneous P breaking generates the chiral particles and the superheavy particles. This means that the original symmetry of the model manifests its characteristics only when all particles, chiral as well as superheavy, are taken into account. It is very hard to show explicitly how the radiative corrections due to the chiral and the superheavy particles cancel each other, because the model inevitably contains the infinite number of particles. In fact, the argument presented in this section does not verify the absence of the radiative correction to $\theta_{\text{tree}} = 0$. However, it seems to be quite tempting and leads us to believe that the present model has a capability of giving $\theta = 0$ to the full order of the quantum effects. We would like to leave the attempt on the rigorous proof to the future study.

§7. Explicit $U(1)_{\text{PQ}}$ breaking

In the previous two sections, we have shown that the strong CP problem will be solvable within the natural phase assignment of the VEVs of the finite-dimensional multiplets $\Psi$’s. We remind that the model is invested with the exact $U(1)_{\text{PQ}}$ symmetry. This symmetry is spontaneously broken at the energy scale $E \simeq \sqrt{m_{\text{SUSY}}M}$. Thus, the model inevitably contains the exactly massless N-G boson $G^0$, which will mediate a long-range force. One
may worry about this long-range force since $G^0$ is shared not only with $r$ and $\bar{r}$ but also with $h$ and $h'$ with a fraction of the order $\langle h \rangle / \langle r \rangle \simeq \sqrt{m_{\text{SUSY}}/M}$. However, $G^0$ will not disturb the Newton’s law of the gravitation, because $G^0$ is a pseudo-scalar particle. It does not give a sizable force between two massive objects. Furthermore, the couplings of $G^0$ to quarks and leptons are suppressed by the factor $\sqrt{m_{\text{SUSY}}/M} \simeq 10^{-7}$, which will be sufficient to clear the present experimental limit.\(^6\)

The $U(1)_{\text{PQ}}$ symmetry has played an indispensable role in deriving the consistent prescription for treating the infinite-dimensional matrix. Nevertheless, we must doubt about this symmetry. This is an accidental symmetry, which happened to emerge in the process of the model building to realize the MSSM at the low-energy by retaining the indispensable couplings of the matter multiplets. It will be probable that there are some other couplings that explicitly break this symmetry even though they are allowed by the $SU(1, 1)$ symmetry. Then, $G^0$ will acquire the mass $m_G$. The magnitude of $m_G$ should depend on how the $U(1)_{\text{PQ}}$ symmetry is explicitly broken. To estimate the values of $m_G$, we must identify the relevant couplings.

We search for the candidates for the couplings. If we limit our considerations to the cubic couplings, the $SU(1, 1)$ invariance gives a rigid restriction on the possible couplings. The couplings $A_{\eta}B_{A}C_{-\zeta}$ and $\bar{A}-\eta \bar{B}_{-\lambda}C_{\zeta}$ are allowed only when $\zeta - \eta - \lambda$ is a nonnegative integer. Let us search for the candidate following the power of $S^{\rho+\sigma}$. The couplings $S^3$ and $\bar{R}S^2$ are trivially forbidden. The possible couplings that contain $S^2$ or $\bar{S}^2$ are

$$\bar{R}'S^2 + R'S^2 \quad \text{with } \rho + \sigma \leq 2/3. \quad (7.1)$$

The candidates containing single $S$ or $\bar{S}$ are

$$RR'S + RR'S \quad \text{with } \rho + \sigma \leq 1, \quad (7.2)$$

$$HKS + H\bar{K}\bar{S} \quad \text{with } \Delta \geq \rho + \sigma, \quad (7.3)$$

$$\bar{H}K'S + H'\bar{K}'\bar{S} \quad \text{with } \Delta' \geq \rho + \sigma. \quad (7.4)$$

The couplings that do not contain $S$ nor $\bar{S}$ are

$$\bar{H}KR + H\bar{K}\bar{R} \quad \text{with } \Delta \geq (\rho + \sigma)/2, \quad (7.5)$$

$$\bar{H}'K'R + H'\bar{K}'\bar{R} \quad \text{with } \Delta' \geq (\rho + \sigma)/2, \quad (7.6)$$

$$HH'R' + \bar{H}H'\bar{R}' \quad \text{with } \rho + \sigma = 2, \quad (7.7)$$

$$RR\bar{R}' + \bar{R}RR' \quad \text{with } \rho + \sigma = 2. \quad (7.8)$$

It will be obvious that, if the supersymmetry is exact, $G^0$ keeps its vanishing mass $m_G = 0$, because we have no mass scale other than $M \simeq 10^{16}\text{GeV}$. This means that the sources that
give the nonvanishing mass to $G^0$ are classified into two types. One is the supersymmetry breaking mass scale $m_0(\simeq m_{\text{SUSY}})$ itself in the $A$-terms. Another one is through the VEVs induced by the supersymmetry breaking. The VEVs relevant to the latter are enumerated as

\begin{align}
\langle H^2 \rangle &= \langle h^2_{-\rho-i} \rangle = \langle h^2 \rangle U_i, \quad \langle K^2 \rangle = \langle k^2_{-\rho-\Delta-i} \rangle = \langle h^2 \rangle V_i, \\
\langle H'^1 \rangle &= \langle h'^1_{-\sigma-i} \rangle = \langle h'^1 \rangle U'_i, \quad \langle K'^1 \rangle = \langle k'^1_{-\sigma-\Delta'-i} \rangle = \langle h'^1 \rangle V'_i, \\
\langle R \rangle &= \langle r_{(\rho+\sigma)/2} \rangle = \langle r \rangle, \quad \langle \bar{R} \rangle = \langle \bar{r}_{-(\rho+\sigma)/2} \rangle = \langle \bar{r} \rangle,
\end{align}

with

\begin{equation}
\langle h^2 \rangle \simeq \langle h'^1 \rangle \simeq m_{\text{SUSY}}, \quad \langle r \rangle \simeq \langle \bar{r} \rangle \simeq \sqrt{m_{\text{SUSY}}M}.
\end{equation}

We note that we must integrate out all of the superheavy particles before substituting these VEVs for the relevant operators.

There are some points to be mentioned for the estimation of $m_G$. First, $G^0$ resides dominantly in $r$ and $\bar{r}$ with the fraction of the order 1. Second, $h^2$ and $h'^1$ contains $G^0$ with the fraction on the order of $\sqrt{m_{\text{SUSY}}/M}$. Thirdly, the $A$-terms corresponding to the couplings (7.1)\textendash(7.8) directly break the $U(1)_{\text{PQ}}$ symmetry. Finally, the supersymmetric $F$-term potential breaks this symmetry only in the cross terms with the different $U(1)_{\text{PQ}}$ charges.

Let us give the results of the analysis of $m_G$. The couplings (7.1)\textendash(7.8) are classified into the three categories:

1. $\rho + \sigma \leq 1$ for any $\Delta, \Delta'$, \hspace{1cm} (7.13)
2. $\rho + \sigma = 2$ for any $\Delta, \Delta'$, \hspace{1cm} (7.14)
3. $1 < \rho + \sigma \neq 2$ and $(\rho + \sigma)/2 \leq \Delta, \Delta'$. \hspace{1cm} (7.15)

The dominant couplings for each category that give the largest $m_G$ are

\begin{align}
1. \quad &R \bar{R}' S + \bar{R} R' \bar{S}; \\
2. \quad &R R \bar{R}' + \bar{R} R R', \\
3. \quad &\bar{H}^{(i)} K^{(i)} R + H^{(i)} \bar{K}^{(i)} \bar{R}.
\end{align}

The category 1 couplings give

\begin{equation}
m_G^2 \sim \frac{m_0}{M^5} \langle r^6 \rangle + \frac{m_0}{M^3} \langle hh' r^2 \rangle = O \left( \frac{m_{\text{SUSY}}^4}{M^2} \right).
\end{equation}

The category 2 and 3 couplings give

\begin{equation}
m_G^2 \sim \frac{m_0}{M^2} \langle hh' r \rangle = O \left( \frac{m_{\text{SUSY}}^{7/2}}{M^{3/2}} \right).
\end{equation}
We must also consider the possibility that the $U(1)_{PQ}$ symmetry is broken in the couplings of the matter multiplets to the superheavy multiplets that we have been discarding. Even in this case, it is confirmed that the largest $m_G$ is limited by (7.20).

These results show that $G_0$ acquires only a tiny mass even if the $U(1)_{PQ}$ symmetry is explicitly broken. Although the coupling of $G_0$ to quarks and leptons are suppressed by the factor $\sqrt{m_{SUSY}/M}$, its effects may be detectable in the future experiments.

§8. Conclusion

First of all, we should state that our trial of solving the strong CP problem has not been completed not only due to the reason that we could not give the proof of the absence of the radiative corrections to $\theta = 0$ but also due to the reason that we are not yet able to give the explicit form of $W[\text{finite dim.}]$ for $\Psi$’s, which gives the desired VEVs through their equations of motions

$$\frac{\partial W[\text{finite dim.}]}{\partial \Psi} = 0. \quad (8.1)$$

The results of the present study bring much information on the structure of $W[\text{finite dim.}]$. The solution of (8.1) must realize all required VEVs satisfying the conditions (5.17), (5.18), and (5.23) or (5.24). They clearly suggest that $\Psi$’s related to the quarks should have the specific couplings to $\Psi$’s related to the higgses. These results will give a valuable hint in the future study of determining the structure of $W[\text{finite dim.}]$. We may have a chance to understand why we have three generations of quarks and leptons in our universe.

Second, the present model predicts, in the reasonable level of probability, the CKM matrix in the form presented in (5.28). Since we have not determined the matrices $O_u$ and $O_d$, this form of the CKM matrix is not able to derive any information from the present experimental observations. However, when we finish the analysis of the mass hierarchies of the quarks and the leptons, we will be able to determine the phase structure of the CKM matrix. If the result reproduces the form (5.28), the present model will obtain a reliable experimental support.

Thirdly, it should be stressed that the supersymmetry plays an indispensable role in realizing the chiral world in the low-energy physics from originally vectorlike theory through the spontaneous breakdown of the P-C-T-invariance by the VEVs of the “chiral $\Psi$’s”. If $\Psi$’s were merely the real scalar fields, their VEVs will not be able to realize the chiral world. In this context, it should be noted that the low-energy physics must be described by the MSSM. Although we simply introduced the superpotential (2.22) to induce the $\mu$-term, we are not allowed to take the essentially different ways. The $SU(1, 1)$ symmetry rejects, for example, the Next-to-MSSM$^{27}$ because the coupling $S^3$ is forbidden by this symmetry. Therefore, the
experimental verification of the MSSM is crucial for the present framework of the model.

The related problem is on the form of the Kähler potential $K$. We assumed (2.27) for its form. Although this form seems to be plausible, it is sure that, if we perform the loop expansion for the quantum effects of the model, we will have the additional terms in $K$. Obviously, we should give a definite answer to the question why the MSSM requires so stringent degeneracy in the soft masses of the squarks and the sleptons to survive in the progress of the experimental study. One simple way of thinking is that the present model is the “effective theory” retaining only the indispensable matter multiplets to reproduce the MSSM at the low-energy, discarding other multiplets $Z_d$, $\bar{Z}_d$, and $\Psi_d$ that the “full theory” contains. It may not be unreasonable to expect the quantum effects of the “full theory” preserve the form of the Kähler potential (2.27). There will, however, be another possibility. The $SU(1,1)$ gauge symmetry itself may play an essential role in this problem within the basic framework of the model as we observed a little bit in (2.28). This will be an exciting and challenging subject.

Finally, we would like to discuss the value of $c$, which has been left undetermined. We presume

$$c = -g \ (= -3). \quad (8.2)$$

Suppose all quarks are superheavy ($g = 0$) and we have some integer $c = c(0)$:

$$c(g = 0) = 1 + 1 + 1 + \cdots = c(0). \quad (8.3)$$

When we have $g$ generations of chiral quarks, we will lose the first $g$ terms of 1’s of this expression, and we will have

$$c(g) = c(0) - g. \quad (8.4)$$

The most plausible value seems to be $c(0) = 0$. In this case, all superheavy quarks (when $g = 0$) decouple from $\theta$. Remember that the superheavy quarks must not decouple only when we have chiral quarks. In fact, (8.2) is desirable for us because we need not mention anything on the superheavy colored multiplets with $g = 0$, which the “full theory” will contain.

We examined the strong CP problem assuming the specific form of the superpotential for quarks and higgses. However, main ingredients of the result of this study will not be modified even if the chiral quarks and higgses are generated through more complicated superpotential as far as they are generated with the definite mixing parameters $\epsilon_f$’s with the weight 3, and $\epsilon$ and $\epsilon'$ with the weight $-1$, which give the characteristic structure of the Yukawa coupling matrices.
Acknowledgments

The authors would like to thank K. Harada, K. Yoshioka, K. Kojima and H. Sawanaka for helpful discussions. This work is supported in part by a grant-in-aid for the scientific research on priority area (♯441) “Progress in elementary particle physics of the 21st century through discoveries of Higgs boson and supersymmetry” (No. 16081209) from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

1) A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, Phys. Lett. B 59 (1975), 85.
   G. ’t Hooft, Phys. Rev. Lett. 37 (1976), 8; Phys. Rev. D 14 (1976), 3432.
   R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976), 172.
   C. G. Callan, R. F. Dashen and D. J. Gross, Phys. Lett. B 63 (1976), 334.
2) P. Fayet, Phys. Lett. B 69 (1977), 489.
3) K. Inoue, A. Kakuto, H. Komatsu and S. Takeshita, Prog. Theor. Phys. 67 (1982), 1889; ibid. 68 (1982), 927.
   L. E. Ibáñez and G. G. Ross, Phys. Lett. B 110 (1982), 215.
4) L. Alvarez-Gaume, J. Polchinski and M. B. Wise, Nucl. Phys. B 221 (1983), 495.
   J. Ellis, J. S. Hagelin, D. V. Nanopoulos and K. Tamvakis, Phys. Lett. B 125 (1983), 275.
5) D. J. H. Chung, L. L. Everett, G. L. Kane, S. F. King, J. D. Lykken and Lian-Tao Wang, Phys. Rep. 407 (2005), 1.
   S. P. Martin, hep-ph/9709356.
6) W. M. Yao et al. (Particle Data Group), J. of Phys. G 33 (2006), 1.
7) S. Weinberg, Phys. Rev. Lett. 40 (1978), 223.
   F. Wilczek, Phys. Rev. Lett. 40 (1978), 279.
8) J. E. Kim, Phys. Rev. Lett. 43 (1979), 103.
   M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 166 (1980), 493.
   A. R. Zhitnitsky, Sov. J. Nucl. Phys. 31 (1980), 260.
   M. Dine, W. Fischler and M. Srednicki, Phys. Lett. B 104 (1981), 199.
   M. Dine and W. Fischler, Phys. Lett. B 120 (1983), 137.
9) R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. 38 (1977), 1440; Phys. Rev. D 16 (1977), 1791.
10) A. Nelson, Phys. Lett. B 136 (1984), 387; ibid. 143 (1984), 165.
S. M. Barr and A. Zee, Phys. Rev. Lett. **55** (1985), 2253; ibid. **65** (1990), 21 [Errata; **65** (1990), 2920].
S. M. Barr, Phys. Rev. D **30** (1984), 1805; ibid. **34** (1986), 1567; ibid. **56** (1997), 1475; Phys. Lett. B **448** (1999), 41.
S. M. Barr and A. Masiero, Phys. Rev. D **38** (1988), 366.
S. M. Barr and E. M. Freire, Phys. Rev. D **41** (1990), 2129.
S. M. Barr and G. Segre, Phys. Rev. D **48** (1993), 302.
A. Masiero and T. Yanagida, hep-ph/9812225.
M. Masip and A. Rasin, Phys. Rev. D **58** (1998), 035007.
R. N. Mohapatra and G. Senjanovic, Phys. Lett. B **79** (1978), 283.
R. N. Mohapatra and A. Rasin, Phys. Rev. Lett. **76** (1996), 3490.
K. S. Babu, B. Dutta and R. N. Mohapatra, Phys. Rev. D **65** (2002), 016005.
11) N. Cabibbo, Phys. Rev. Lett. **10** (1963), 531.
M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49** (1973), 652.
12) K. Inoue, Prog. Theor. Phys. **93** (1995), 403; Prog. Theor. Phys. Suppl. No. 123 (1996), 319.
13) K. Inoue and N. Yamashita, Prog. Theor. Phys. **104** (2000), 677; ibid. **110** (2003), 1087.
14) K. Inoue and N. Yamatsu, Prog. Theor. Phys. **119** (2008), 775.
15) F. A. Wilczek, A. Zee, R. L. Kingsley and S. B. Treiman, Phys. Rev. D **12** (1975), 2768.
A. De. Rújula, H. Georgi and S. L. Glashow, Phys. Rev. D **12** (1975), 3589.
H. Fritzsch, M. Gell-Mann and P. Minkowski, Phys. Lett. B **59** (1975), 256.
K. Inoue, A. Kakuto and Y. Nakano, Prog. Theor. Phys. **58** (1977), 630.
M. Yoshimura, Prog. Theor. Phys. **58** (1977), 972.
16) T. Maehara and T. Yanagida, Prog. Theor. Phys. **61** (1979), 1434.
F. Wilczek and A. Zee, Phys. Rev. Lett. **42** (1979), 421.
17) C. D. Froggatt and H. B. Nielsen, Nucl. Phys. B **147** (1979), 277.
18) J. Ellis, S. Kelley and D. V. Nanopoulos, Phys. Lett. B **260** (1991), 131.
U. Amaldi, W. de Boer and H. Furstenau, Phys. Lett. B **260** (1991), 447.
P. Langacker and M. x. Luo, Phys. Rev. D **44** (1991), 817.
19) G. ’t Hooft, in *Recent Developments in Gauge Theories*, ed. G. ’t Hooft, et al. (Plenum, New York, 1980).
S. Coleman and B. Grossman, Nucl. Phys. B **203** (1982), 205.
20) J. Wess and B. Zumino, Phys. Lett. B **37** (1971), 95.
E. Witten, Nucl. Phys. B **223** (1983), 422.
21) J. Ellis and M. K. Gaillard, Nucl. Phys. B 150 (1979), 141.
   M. Dugan, B. Grinstein and L. Hall, Nucl. Phys. B 255 (1985), 413.
22) K. Fujikawa and W. Lang, Nucl. Phys. B 88 (1975), 61.
   S. Ferrara and O. Piguet, Nucl. Phys. B 93 (1975), 261.
23) H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32 (1974), 438.
24) H. Fritzsch and P. Minkowski, Ann. of Phys. 93 (1975), 193.
   H. Georgi, in *Particles and Fields*, ed. C. E. Carlson (AIP, New York, 1975), p575.
25) F. Gursey, P. Ramond and P. Sikivie, Phys. Lett. B 60 (1975), 177.
26) R. Slansky, Phys. Rep. 79 (1981), 1.
27) P. Fayet, Nucl. Phys. B 90 (1975), 104.