Radial excitations of scalar and $\eta$, $\eta'$ mesons in a chiral quark model

M.K. Volkov, V.L. Yudichev

Abstract

First radial excitations of the scalar meson nonet and pseudoscalar mesons $\eta$, $\eta'$ are described in a nonlocal chiral quark model of the Nambu–Jona-Lasinio type with 't Hooft interaction. In this model simple form factors are used, which allows us to describe first radial excitations of the mesons and to conserve the gap equations describing spontaneous breaking of chiral symmetry in the standard form. The external parameters of form factors are fixed by masses of excited pseudoscalar mesons and the same form factors are used for predicting the masses of excited scalar mesons. Strong decays of excited scalar mesons and $\eta$, $\eta'$ mesons are described in satisfactory agreement with experiment.

Keywords: quark model, chiral symmetry, scalar mesons, radial excitations, $\eta$ and $\eta'$ mesons.
1. INTRODUCTION

In our previous papers [1–4] a nonlocal chiral quark model of the Nambu–Jona-Lasinio (NJL) type was suggested to describe the first radial excitations of mesons. The nonlocality was introduced into the effective four-quark interaction through form factors represented by first order polynomials of the quark momentum squared \( k^2 \). In [1] it was shown that such form factors can be rewritten in a relativistic form and the internal parameters of these form factors (slope parameters) can be chosen so that the quark condensates and gap equations appearing in the standard NJL model are unchanged. With the form factors thus introduced, all the low energy theorems are fulfilled in the chiral limit (see [1]).

In papers [2–4], this model was used for describing the mass spectrum of excited pions, kaons and of the nonet of vector mesons. The main strong decays were also described therein.

Attempts to construct a model for describing radially excited meson states were made, e. g. in [5] where a model with quasilocal four-quark interaction in the polycritical regime was proposed. Different nonlocal models [6–8] were also suggested.

With the present work, we accomplish the investigation of the excited pseudoscalar meson nonet, considering the excited states of \( \eta \) and \( \eta' \) and studying the first radially excited states of the scalar meson nonet.

For a correct description of the \( \eta, \eta' \) and isoscalar scalar meson masses it is necessary, in addition to the standard four-quark interaction, to introduce six-quark 't Hooft interaction which breaks the chiral symmetry and helps to solve the so called \( U_A(1) \) problem. Contrary to the nonlocal four-quark interaction, 't Hooft vertices are pure local in accordance with their instantaneous origin.

The 't Hooft interaction gives rise to mixing of four pseudoscalar states \( \eta, \eta', \hat{\eta} \) and \( \hat{\eta}' \) (the caret symbol stands for radially excited meson states) and four scalar states \( \sigma, \hat{\sigma}, f_0, \hat{f}_0 \). After diagonalization of the free meson Lagrangians, we get the mass spectrum of these meson states.

Each of the form factors we used has two arbitrary parameters: the slope parameter \( d_\alpha \) and external parameter \( c_\alpha \). There are three slope parameters — \( d_{uu}, d_{us}, \) and \( d_{ss} \). They are uniquely defined by the condition that the excited mesons do not contribute to the quark condensate (tadpoles including one form factor in the vertex equal to zero) and therefore do not contribute to the gap equations. Meanwhile, the constituent quark masses remain constant. The external parameter \( c_\alpha \) influences the interaction of excited states of mesons with quarks or the corresponding four-quark interaction. For the pseudoscalar and vector mesons we define these parameters, using the masses of excited states of mesons. However, for the scalar mesons we use the same form factors as for the pseudoscalar ones. Thus, we can predict the masses of excited scalar meson states. They turn out to be in satisfactory agreement with experiment and allow us to identify the members of scalar meson nonets and tell us which of them are ground and which are radially excited states. This problem is discussed in the Conclusion.

After fitting the parameters \( c_\alpha \) and defining the basic model parameters (constituent quark masses \( m_u \) and \( m_s \) (\( m_u \approx m_d \)), ultraviolet cut-off \( \Lambda \), four-quark coupling constant \( G \) and 't Hooft coupling constant \( K \)), we can describe all the strong coupling constants of mesons and calculate their strong decay widths.

Our paper is organized as follows. In Sec. 2, we introduce the chiral quark Lagrangian
with nonlocal four-quark vertices and local 't Hooft interaction. In Sec. 3, we calculate the effective Lagrangian for isovector and strange mesons in the one-loop approximation. There we renormalize meson fields and transform the free part of the Lagrangian to the diagonal form and obtain meson mass formulae. Section 4 is devoted to isoscalar mesons where we find masses and mixing coefficients. The model parameters are discussed in Sec. 5. In Sec. 6, we calculate the widths of main strong decays of excited states of \( a_0 \), \( \sigma \), \( f_0 \) and \( K_0^* \) mesons. In Sec. 7, we investigate strong decays of the first radial excitations of \( \eta \) and \( \eta' \). In Sec. 8, we analyze our results and compare them with experimental data. Some details of the calculations fulfilled in Sec. 4 and 6 are given in Appendices A and B.

2. \( U(3) \times U(3) \) CHIRAL LAGRANGIAN WITH EXCITED MESON STATES AND 'T HOOFT INTERACTION

We use a nonlocal separable four-quark interaction of a current-current form which admits nonlocal vertices (form factors) in the quark currents, and a pure local six-quark 't Hooft interaction \([9,10]\):

\[
\mathcal{L}(\bar{q}, q) = \int d^4x \, \bar{q}(x)(i\not \! \partial - m^0(x))q(x) + \mathcal{L}^{(4)}_{\text{int}} + \mathcal{L}^{(6)}_{\text{int}},
\]

\[
\mathcal{L}^{(4)}_{\text{int}} = \frac{G}{2} \int d^4x \sum_{a=0}^{8} \sum_{i=1}^{N} [j^a_{S,i}(x)j^a_{S,i}(x) + j^a_{P,i}(x)j^a_{P,i}(x)],
\]

\[
\mathcal{L}^{(6)}_{\text{int}} = -K [\det(\bar{q}(1 + \gamma_5)q) + \det(\bar{q}(1 - \gamma_5)q)].
\]

Here, \( m^0 \) is the current quark mass matrix (\( m^0_u \approx m^0_d \)) and \( j^a_{S(P),i} \) denotes the scalar (pseudoscalar) quark currents

\[
j^a_{S(P),i}(x) = \int d^4x_1 d^4x_2 \, \bar{q}(x_1)F^a_{S(P),i}(x; x_1, x_2)q(x_2)
\]

where \( F^a_{S(P),i}(x; x_1, x_2) \) are the scalar (pseudoscalar) nonlocal quark vertices. To describe the first radial excitations of mesons, we take the form factors in momentum space as follows (see \([12]\)),

\[
F^a_{S,j}(k) = \lambda^a f^a_j, \quad F^a_{P,j} = i\gamma_5 \lambda^a f^a_j
\]

\[
f^a_1 \equiv 1, \quad f^a_2 \equiv f_a(k) = c_a(1 + d_a k^2),
\]

where \( \lambda^a \) are Gell–Mann matrices, \( \lambda^0 = \sqrt{\frac{2}{3}} \mathbf{1} \), with \( \mathbf{1} \) being the unit matrix. Here, we consider the form factors in the rest frame of mesons\(^1\)

\(^1\)The form factors depend on the transversal parts of the relative momentum of quark-antiquark pairs \( k_\perp = k - \frac{2}{3} P \), where \( k \) and \( P \) are the relative and total momenta of a quark-antiquark pair, respectively. Then, in the rest frame of mesons, \( P_{\text{meson}} = 0 \), the transversal momentum is \( k_\perp = (0, \mathbf{k}) \), and we can define the form factors as depending on the 3-dimensional momentum \( \mathbf{k} \) alone.
The part of the Lagrangian (7), describing the ground states and first radial excitations, can be rewritten in the following form (see [9] and [10]):

\[
\mathcal{L} = \int d^4x \left\{ \bar{q}(x)(i\gamma^\mu - m^0)q(x) + \frac{G}{2} \sum_{a=0}^{8} \left[ j_{5,2}^a(x) \right]^2 + \left( j_{P,2}^n(x) \right)^2 + \frac{1}{2} \sum_{a=1}^{9} \left[ G_a^{(-)}(\bar{q}(x)\tau_a q(x))^2 + G_a^{(+)}(\bar{q}(x)i\gamma_5\tau_a q(x))^2 \right] \right\} + G_{us}^{(-)}(\bar{q}(x)\lambda_u q(x))(\bar{q}(x)i\gamma_5\lambda_u q(x)) + G_{us}^{(+)}(\bar{q}(x)i\gamma_5\lambda_u q(x))(\bar{q}(x)i\gamma_5\lambda_u q(x)) \right\},
\]

where

\[
\begin{align*}
\tau_i & = \lambda_i \quad (i = 1, ..., 7), \\
\tau_8 & = \lambda_u = (\sqrt{2}\lambda_0 + \lambda_8)/\sqrt{3}, \\
\tau_9 & = \lambda_s = (-\lambda_0 + \sqrt{2}\lambda_8)/\sqrt{3}, \\
G_{1}^{(+)} & = G_2^{(+)} = G_3^{(+)} = G \pm 4Km_s I_1(m_s), \\
G_{4}^{(+)} & = G_5^{(+)} = G_6^{(+)} = G_7^{(+)} = G \pm 4Km_u I_1(m_u), \\
G_{8}^{(+)} & = G \mp 4Km_u I_1(m_u), \\
G_{9}^{(+)} & = G, \\
G_{10}^{(+)} & = \pm 4\sqrt{2}Km_u I_1(m_u).
\end{align*}
\]

Here \(m_u\) and \(m_s\) are the constituent quark masses and \(I_1(m_q)\) is the integral which for an arbitrary \(n\) is defined as follows

\[
I_n(m_q) = \frac{-iN_c}{(2\pi)^4} \int d^4k \frac{1}{(m_q^2 - k^2)^n}.
\]

The 3-dimensional cut-off \(\Lambda_3\) in (7) is implemented to regularize the divergent integrals.

3. THE MASSES OF ISOVECTOR AND STRANGE MESONS (GROUND AND EXCITED STATES)

After bosonization, the part of Lagrangian (7), describing the isovector and strange mesons, takes the form

\[
\mathcal{L}(a_{0,1}, K_0^{*1}, \pi_1, K_1, a_{0,2}, K_0^{*2}, \pi_2, K_2) = - \frac{a_{0,1}^2}{2G_{a_0}} - \frac{K_0^{*1,2}}{G_{K_0}} - \frac{\pi_1^2}{2G_{\pi}} - \frac{K_2^2}{G_K} - \frac{1}{2G}(a_{0,2}^2 + 2K_0^{*2,2} + \pi_2^2 + 2K_2^2) - iN_c \text{Tr} \ln \left[ 1 + \frac{1}{i\phi} - m \sum_{a=1}^{7} \sum_{j=1}^{2} \lambda_a \left[ \sigma_j^a + i\gamma_5\phi_j^a \right] f_j^a \right],
\]

where \(m = \text{diag}(m_u, m_d, m_s)\) is the matrix of constituent quark masses \((m_u \approx m_d)\), \(\sigma_j^a\) and \(\phi_j^a\) are the scalar and pseudoscalar fields: \(\sum_{a=1}^{3} (\sigma_j^a)^2 \equiv a_{0,j}^2 = (a_{0,j}^0)^2 + 2a_{0,j}^\pm a_{0,j}^-\), \(\sum_{a=1}^{7} (\sigma_j^a)^2 \equiv 2\).
\[ 2K_{0,j}^2 = 2(K_{0,j}^0)^2 \]

\[ + 2(K_{0,j}^+)^2(K_{0,j}^-)^2 \]

\[ \sum_{a=1}^{7}(\varphi_a^2)^2 \equiv \pi_j^2 = (\pi_j^0)^2 + 2\pi_j^+\pi_j^- \]

\[ \sum_{a=4}^{7}(\varphi_a^2)^2 \equiv 2K_j^2 = 2K_{0,j}^0 + 2K_{j}^+K_{j}^- \]

As to the coupling constants \( G_a \), they will be defined later (see Sect. 5 and \( \text{[3]} \)).

The free part of Lagrangian \( \text{[10]} \) has the following form

\[
\mathcal{L}^{(2)}(\sigma, \varphi) = \frac{1}{2} \sum_{i,j=1}^{2} \sum_{a=1}^{7} \left( \sigma_i^a K_{\sigma,ij}^a(P)\sigma_j^a + \varphi_i^a K_{\varphi,ij}^a(P)\varphi_j^a \right) \]

(11)

where the coefficients \( K_{\sigma(\varphi),ij}^a(P) \) are given below,

\[
K_{\sigma(\varphi),ij}^a(P) = -\delta_{ij} \left[ \frac{\delta_{i1}}{G_a^{(\mp)}} + \frac{\delta_{i2}}{G} \right] - iN_c \text{Tr} \left( \frac{d^4k}{(2\pi)^4} \frac{1}{k^+ + P/2 - m_q^a} r_{ij}^{\sigma(\varphi)} f_i^a \frac{1}{k^+ - P/2 - m_q^a} r_{ij}^{\sigma(\varphi)} f_j^a \right) \]

(12)

\[
r_{ij}^{\sigma} = 1, \quad r_{ij}^{\phi} = \gamma_5 \]

(13)

\[
m_q^a = m_a \quad (a = 1, \ldots, 7); \quad m_{q'}^a = m_u \quad (a = 1, \ldots, 3); \quad m_{q''}^a = m_s \quad (a = 4, \ldots, 7), \]

(14)

with \( m_u \) and \( m_s \) being the constituent quark masses and \( f_i^a \) defined in \( \text{[1]} \). Integral \( \text{[12]} \) is evaluated by expanding in the meson field momentum \( P \). To order \( P^2 \), one obtains

\[
K_{\sigma(\varphi),11}^a(P) = Z_{\sigma(\varphi),1}^a(P^2 - (m_q^a \pm m_{q'}^a)^2 - M_{\sigma(\varphi),1}^a), \\
K_{\sigma(\varphi),22}^a(P) = Z_{\sigma(\varphi),2}^a(P^2 - (m_q^a \pm m_{q'}^a)^2 - M_{\sigma(\varphi),2}^a), \\
K_{\sigma(\varphi),12}^a(P) = K_{\sigma(\varphi),21}^a(P) = \gamma_5^a(\varphi)(P^2 - (m_q^a \pm m_{q'}^a)^2), \]

(15)

where

\[
Z_{\sigma,1}^a = 4I_2^a, \quad Z_{\sigma,2}^a = 4I_2^{fa}, \quad \gamma_5^a = 4I_2^{fa}, \]

\[
Z_{\varphi,1}^a = Z^a_{\sigma,1}, \quad Z_{\varphi,2}^a = Z^a_{\sigma,2}, \quad \gamma_5^a = Z^{1/2}\gamma_5^a \]

(16)

(17)

and

\[
M_{\sigma(\varphi),1}^2 = (Z_{\sigma(\varphi),1}^a)^{-1} \left[ \frac{1}{G_a^{(\mp)}} - 4(I_1(m_q^a) + I_1(m_{q'}^a)) \right] \]

(18)

\[
M_{\sigma(\varphi),2}^2 = (Z_{\sigma(\varphi),2}^a)^{-1} \left[ \frac{1}{G} - 4(I_1^{fa}(m_q^a) + I_1^{fa}(m_{q'}^a)) \right] \]

(19)

The factor \( Z \) here appears due to account of \( \pi - a_1 \)-transitions \( \text{[4]} \), \( \text{[3]} \),

\[
Z = 1 - \frac{6m_u^2}{M_{a_1}^2}, \]

(20)

and the integrals \( I_2^{f_a} \) contain form factors:
After the transformations of the meson fields Lagrangians (23) and (24) take the diagonal form:

$$I_2 f_a(m^a_q, m^a_{q'}) = -iN_c(2\pi)^4 \int d^4k \frac{f_a(k) .. f_a(k)}{((m^a_q)^2 - k^2)((m^a_{q'})^2 - k^2)}. \tag{21}$$

Further, we consider only the scalar isovector and strange mesons because the masses of the pseudoscalar mesons have been already described in [2].

After the renormalization of the scalar fields

$$\sigma_i^{ar} = \sqrt{Z_i} \sigma_i^a \tag{22}$$

the part of Lagrangian (11) which describes the scalar mesons takes the form

$$L_{a_0}^{(2)} = \frac{1}{2} \left( P^2 - 4m^2_u - M^2_{a_0,1} \right) a^2_{0,1} + \Gamma_{a_0} \left( P^2 - 4m^2_u \right) a_{0,1}a_{0,2} + \frac{1}{2} \left( P^2 - 4m^2_u - M^2_{a_0,2} \right) a^2_{0,2}; \tag{23}$$

$$L_{K_0}^{(2)} = \frac{1}{2} \left( P^2 - (m_u + m_s)^2 - M^2_{K_0,1} \right) K^*_{0,1} + \Gamma_{K_0} \left( P^2 - (m_u + m_s)^2 \right) K^*_{0,1} K^*_{0,2} + \frac{1}{2} \left( P^2 - (m_u + m_s)^2 - M^2_{K_0,2} \right) K^*_{0,2}; \tag{24}$$

where

$$\Gamma_{\sigma^a} = \frac{I_2 f_a}{\sqrt{I_2 I_2 f_a}}. \tag{25}$$

After the transformations of the meson fields

$$\sigma^a = \cos(\theta_{\sigma,a} - \theta^0_{\sigma,a}) \sigma_1^{ar} - \cos(\theta_{\sigma,a} + \theta^0_{\sigma,a}) \sigma_2^{ar},$$

$$\dot{\sigma}^a = \sin(\theta_{\sigma,a} - \theta^0_{\sigma,a}) \sigma_1^{ar} - \sin(\theta_{\sigma,a} + \theta^0_{\sigma,a}) \sigma_2^{ar}; \tag{26}$$

Lagrangians (23) and (24) take the diagonal form:

$$L_{a_0}^{(2)} = \frac{1}{2} \left( P^2 - M^2_{a_0} \right) a^2_{0} + \frac{1}{2} \left( P^2 - M^2_{a_0} \right) \dot{a}^2_{0}; \tag{27}$$

$$L_{K_0}^{(2)} = \frac{1}{2} \left( P^2 - M^2_{K_0} \right) K^*_{0} + \frac{1}{2} \left( P^2 - M^2_{K_0} \right) \dot{K}^*_{0}. \tag{28}$$

Here we have

$$M^2_{a_0,0} = \frac{1}{2(1 - \Gamma^2_{a_0})} \left[ M^2_{a_0,1} + M^2_{a_0,2} \pm \sqrt{(M^2_{a_0,1} - M^2_{a_0,2})^2 + (2M_{a_0,1}M_{a_0,2} \Gamma_{a_0})^2} \right] + 4m^2_u, \tag{29}$$

$$M^2_{K_0,0} = \frac{1}{2(1 - \Gamma^2_{K_0})} \left[ M^2_{K_0,1} + M^2_{K_0,2} \pm \sqrt{(M^2_{K_0,1} - M^2_{K_0,2})^2 + (2M_{K_0,1}M_{K_0,2} \Gamma_{K_0})^2} \right] + (m_u + m_s)^2, \tag{30}$$
and
\[ \tan 2\theta_{\sigma,a} = \sqrt{\frac{1}{\Gamma_{\sigma a}^2} - 1} \left[ \frac{M_{\sigma a,1}^2 - M_{\sigma a,2}^2}{M_{\sigma a,1}^2 + M_{\sigma a,2}^2} \right], \quad 2\theta_{\sigma,a} = 2\theta_{\sigma,a} + \pi, \] (31)

\[ \sin \theta_{\sigma,a}^0 = \sqrt{\frac{1 + \Gamma_{\sigma a}}{2}}. \] (32)

The caret symbol stands for the first radial excitations of mesons. Transformations (26) express the “physical” fields \( \sigma \) and \( \hat{\sigma} \) through the “bare” ones \( \sigma_{\sigma a}^{pr} \) and for calculations, these equations must be inverted. For practical use, we collect the values of the inverted equations for the scalar and pseudoscalar fields\(^3\) in Table I.

4. THE MASSES OF ISOSCALAR MESONS (THE GROUND AND EXCITED STATES)

The ’t Hooft interaction effectively gives rise to the additional four-quark vertices in the isoscalar part of Lagrangian (7):

\[ \mathcal{L}_{\text{isosc}} = \frac{9}{4} \sum_{a,b=8} \left[ (\bar{q}_a \gamma_5 q_b) T^S_{ab}(\bar{q}_b \gamma_5 q_a) + (\bar{q}_a \gamma_5 \tau_a q_b) T^P_{ab}(\bar{q}_b \gamma_5 \tau_a q_a) \right], \] (33)

where \( T^S(P) \) is a matrix with elements defined as follows (for the definition of \( G_u^{(\pm)}, G_s^{(\pm)} \) and \( G_{us}^{(\pm)} \) see (8))

\[ T^S_{88} = G_u^{(\pm)}/2, \quad T^S_{89} = G_{us}^{(\pm)}/2, \quad T^S_{98} = G_{us}^{(\pm)}/2, \quad T^S_{99} = G_s^{(\pm)}/2. \] (34)

This leads to nondiagonal terms in the free part of the effective Lagrangian for isoscalar scalar and pseudoscalar mesons after bosonization

\[ \mathcal{L}_{\text{isosc}}(\sigma, \varphi) = -\frac{1}{4} \sum_{a,b=8}^9 \left[ \sigma_a^a(T^S_{ab})^{-1}\sigma_b^b + \varphi_a^a(T^P_{ab})^{-1}\varphi_b^b \right] - \frac{1}{2G} \sum_{a=8}^9 \left[ (\sigma_2^a)^2 + (\varphi_2^a)^2 \right] - \frac{1}{iG} \text{Tr} \ln \left\{ 1 + \frac{1}{i\hat{q} - m} \sum_{a=8}^9 \sum_{j=1}^2 \tau^a[\sigma_j^a + i\gamma_5\varphi_j^a] \right\}, \] (35)

where \( (T^S(P))^{-1} \) is the inverse of \( T^S(P) \).

---

\(^3\) Although the formulae for the pseudoscalars are not displayed here (they have been already obtained in [8]) we need the values because we are going to calculate the decay widths of processes where pions and kaons are secondary particles.
\[
\begin{align*}
(T^{SP})^{1}_{88} &= 2G_s^{(\mp)} / D^{(\mp)} , \\
(T^{SP})^{1}_{99} &= 2G_u^{(\mp)} / D^{(\mp)} , \\
D^{(\mp)} &= G_u^{(\mp)} G_s^{(\mp)} - (G_{us}^{(\mp)})^2 .
\end{align*}
\] (36)

From (35), in the one-loop approximation, one obtains the free part of the effective Lagrangian

\[
\mathcal{L}^{(2)}(\sigma, \phi) = \frac{1}{2} \sum_{i,j=1}^{2} \left( \sigma_i^a K_{\sigma,ij}^{[a,b]}(P) \sigma_j^b + \phi_i^a K_{\phi,ij}^{[a,b]}(P) \phi_j^b \right) .
\] (37)

The definition of \( K_{\sigma,ij}^{[a,b]} \) is given in Appendix A.

After the renormalization of both the scalar and pseudoscalar fields, analogous to (22), we come to the Lagrangian which can be represented in a form slightly different from that of (37). It is convenient to introduce 4-vectors of “bare” fields

\[
\Sigma = (\sigma_1^8, \sigma_2^8, \sigma_1^9, \sigma_2^9), \quad \Phi = (\phi_1^8, \phi_2^8, \phi_1^9, \phi_2^9) .
\] (38)

Thus, we have

\[
\mathcal{L}^{(2)}(\Sigma, \Phi) = \frac{1}{2} \sum_{i,j=1}^{4} (\Sigma_i K_{\Sigma,ij}(P) \Sigma_j + \Phi_i K_{\Phi,ij}(P) \Phi_j)
\] (39)

where we introduced new functions \( K_{\Sigma,ij}(P) \) (see Appendix A).

Up to this moment one has four pseudoscalar and four scalar meson states which are the octet and nonet singlets. The mesons of the same parity have the same quantum numbers and, therefore, are expected to be mixed. In our model the mixing is represented by \( 4 \times 4 \) matrices \( R^{\sigma(\phi)} \) which transform the “bare” fields \( \sigma_i^8, \sigma_i^9, \phi_i^8, \phi_i^9 \), entering the 4-vectors \( \Sigma \) and \( \Phi \) to the “physical” ones \( \sigma, \tilde{\sigma}, f_0, \tilde{f}_0, \eta, \eta', \hat{\eta} \) and \( \hat{\eta}' \) represented as components of vectors \( \Sigma_{\text{ph}} \) and \( \Phi_{\text{ph}} \):

\[
\Sigma_{\text{ph}} = (\sigma, \tilde{\sigma}, f_0, \tilde{f}_0), \quad \Phi_{\text{ph}} = (\eta, \hat{\eta}, \eta', \hat{\eta}')
\] (40)

where, let us remind once more, a caret over a meson field stands for the first radial excitation of the meson. The transformation \( R^{\sigma(\phi)} \) is linear and nonorthogonal:

\[
\Sigma_{\text{ph}} = R^\sigma \Sigma, \quad \Phi_{\text{ph}} = R^\phi \Phi .
\] (41)

In terms of “physical” fields the free part of the effective Lagrangian is of the conventional form and the coefficients of matrices \( R^{\sigma(\phi)} \) give the mixing of the \( \bar{u}u \) and \( \bar{s}s \) components, with and without form factors.

Because of the complexity of the procedure of diagonalization for the matrices of dimensions greater than 2, there is no such simple formulae as, e.g., in (26). Hence, we do not implement it analytically but use numerical methods to obtain matrix elements (see Table II).
5. MODEL PARAMETERS AND MESON MASSES

In our model we have five basic parameters: the masses of the constituent \(u(d)\) and \(s\) quarks, \(m_u = m_d\) and \(m_s\), the cut-off parameter \(\Lambda_3\), the four-quark coupling constant \(G\) and the 't Hooft coupling constant \(K\). We have fixed these parameters with the help of input parameters: the pion decay constant \(F\pi\) = 93 MeV, the \(\rho\)-meson decay constant \(g_\rho = 6.14\) MeV (for details of these calculations, see [2,3,10]). Here we give only numerical estimates of these parameters:

\[
m_u = 280\ \text{MeV}, \quad m_s = 405\ \text{MeV}, \quad \Lambda_3 = 1.03\ \text{GeV}, \quad G = 3.14\ \text{GeV}^{-2}, \quad K = 6.1\ \text{GeV}^{-5}.
\]

We also have a set of additional parameters \(c^{\pi,\sigma}_{u \bar{u}}(\varphi)\) in form factors \(f_2^{\pi}\). These parameters are defined by masses of excited pseudoscalar mesons, \(c^{\pi,\sigma}_{u \bar{u}} = 1.44, c^{\pi,\sigma}_{u \bar{u}} = 1.59, c^{\pi,\sigma}_{u \bar{u}} = 1.66\). The slope parameters \(d_{qq}\) are fixed by special conditions satisfying the standard gap equation, \(d_{uu} = -1.78\ \text{GeV}^{-2}, d_{us} = -1.76\ \text{GeV}^{-2}, d_{ss} = -1.73\ \text{GeV}^{-2}\) (see [3]). Using these parameters, we obtain masses of pseudoscalar and scalar mesons which are listed in Table III together with experimental values.

From our calculations we come to the following interpretation of \(f_0(1370), f_J(1710)\) and \(a_0(1470)\) mesons: we consider them as the first radial excitations of the ground states \(f_0(400 - 1200), f_0(980)\) and \(a_0(980)\). Meanwhile, the meson \(f_0(1500)\) is likely a glueball. However, this is just our supposition. Only consideration of a version of the NJL model with glueball states (or dilatons) will allow us to clarify the status of \(f_0(1500)\) and \(f_0(1710)\).

6. STRONG DECAYS OF THE SCALAR MESONS

The ground and excited states of scalar mesons \(f_0, a_0\) decay mostly into pairs of pseudoscalar mesons. In the framework of a quark model and in the leading order of \(1/N_c\) expansion, the processes are described by triangle quark diagrams (see Fig.1). Before we start to calculate the amplitudes, corresponding to these diagrams, we introduce, for convenience, Yukawa coupling constants which naturally appear after the renormalization (22) of meson fields:

\[
\begin{align*}
g_{\sigma u} &\equiv g_{\sigma u}|_{a=1,2,3,8} = [4I_2(m_u)]^{-1/2}, & g_{\kappa 0}^{\pi} &\equiv g_{\sigma 0}|_{a=4,5,6,7} = [4I_2(m_u,m_s)]^{-1/2}, \\
g_{\sigma s} &\equiv g_{\sigma s} = [4I_2(m_s)]^{-1/2}, & g_{\varphi 0}^{\pi} &\equiv Z^{-1/2}g_{\sigma 0}, \\
g_{\pi} &\equiv g_{\varphi 0}|_{a=1,2,3}, & g_{\kappa} &\equiv g_{\varphi 0}|_{a=4,5,6,7}, & g_{\varphi u} &\equiv g_{\varphi s}, & g_{\varphi s} &\equiv g_{\varphi 0}
\end{align*}
\]

4) Here we do not consider vector and axial-vector mesons, however, we have used the relation \(g_\rho = \sqrt{6}g_\sigma\) together with the Goldberger–Treiman relation \(g_\pi = \frac{m_\pi}{F_\pi} = Z^{-1/2}g_\sigma\) to fix the parameters \(m_u\) and \(\Lambda_3\) (see [3]).
\[ \hat{g}_{\sigma^a} \equiv \hat{g}_{\sigma^a} \big|_{a=1,2,3,8} = [4I_2^{ff}(m_a)]^{-1/2}, \quad \hat{g}_{K_0^0} \equiv \hat{g}_{\sigma^a} \big|_{a=4,5,6,7} = [4I_2^{ff}(m_a, m_s)]^{-1/2}, \]
\[ \hat{g}_{\sigma^a} \equiv \hat{g}_{\sigma^a} = [4I_2^{ff}(m_s)]^{-1/2}, \quad \hat{g}_{\varphi^a} = \hat{g}_{\varphi^a} \]
\[ \hat{g}_{\pi} \equiv \hat{g}_{\varphi^a} \big|_{a=1,2,3}, \quad \hat{g}_K \equiv \hat{g}_{\varphi^a} \big|_{a=4,5,6,7}, \quad \hat{g}_{\varphi^a} \equiv \hat{g}_{\varphi^a}, \quad \hat{g}_{\varphi^a} \equiv \hat{g}_{\varphi^a} \]

(44)

They can easily be related to \( Z_{\sigma(\varphi)^i} \), introduced in the beginning of our paper. Thus, the one-loop contribution to the effective Lagrangian can be rewritten in terms of the renormalized fields:

\[ \mathcal{L}_{\text{1-loop}}(\sigma, \varphi) = iN_c \text{Tr} \ln \left[ 1 + \frac{1}{i\hat{g}} - m \sum_{a=1}^{9} \tau_a \left( g_{\sigma^a} \sigma^a_1 + i\gamma_5 g_{\varphi^a} \varphi^a_1 + \right) \right] \left( \hat{g}_{\sigma^a} \sigma^a_2 + i\gamma_5 \hat{g}_{\varphi^a} \varphi^a_2 f_a \right) \]

(45)

All amplitudes that describe processes of the type \( \sigma \to \varphi_1 \varphi_2 \) can be divided into two parts:

\[ T_{\sigma \to \varphi_1 \varphi_2} = C \left( -\frac{iN_c}{(2\pi)^4} \right) \int_{\Lambda_3} d^4k \frac{\text{Tr}[(m + k + p_1)\gamma_5(m + k)\gamma_5(m + k - p_2)]}{(m^2 - k^2)(m^2 - (k + p_1)^2)(m^2 - (k - p_2)^2)} \]
\[ = 4mC \left( -\frac{iN_c}{(2\pi)^4} \right) \int_{\Lambda_3} d^4k \frac{1 - \frac{p_1 \cdot p_2}{m^2 - k^2}}{(m^2 - (k + p_1)^2)(m^2 - (k - p_2)^2)} \]
\[ = 4mC[I_2(m, p_1, p_2) - p_1 \cdot p_2 I_3(m, p_1, p_2)] = T^{(1)} + T^{(2)} \]

(46)

here \( C = 4g_{\sigma}g_{\varphi_1}g_{\varphi_2} \) and \( p_1, p_2 \) are momenta of the pseudoscalar mesons. Using (43) and (14), we rewrite the amplitude \( T_{\sigma \to \varphi_1 \varphi_2} \) in another form

\[ T_{\sigma \to \varphi_1 \varphi_2} \approx 4mZ^{-1/2}g_{\varphi_1} \left[ 1 - p_1 \cdot p_2 \frac{I_3(m)}{I_2(m)} \right], \]

(47)

\[ p_1 \cdot p_2 = \frac{1}{2}(M_{\sigma}^2 - M_{\varphi_1}^2 - M_{\varphi_2}^2). \]

(48)

We assumed here that the ratio of \( I_3 \) to \( I_2 \) slowly changes with momentum in comparison with factor \( p_1 \cdot p_2 \), therefore, we ignore their momentum dependence in (17). With this assumption we are going to obtain just a qualitative picture for decays of the excited scalar mesons.

In eqs. (10) and (17) we omitted the contributions from the diagrams which include form factors in vertices. The whole set of diagrams consists of those containing zero, one, two and three form factors. To obtain the complete amplitude, one must sum up all contributions.

After these general comments, let us consider the decays of \( a_0(1450), f_0(1370) \) and \( f_2(1710) \). First, we estimate the decay width of the process \( \tilde{a}_0 \to \eta\pi \), taking the mixing coefficients from Table \( \| \) and \( \| \) (see Appendix B for the details). The result is

\[ T_{\tilde{a}_0 \to \eta\pi}^{(1)} \approx 0.2 \text{ GeV}, \]

(49)

\[ T_{\tilde{a}_0 \to \eta\pi}^{(2)} \approx 3.5 \text{ GeV}, \]

(50)
\[ \Gamma_{\hat{a}_0 \rightarrow \eta \pi} \approx 160 \text{ MeV}. \quad (51) \]

From this calculation one can see that \( T^{(1)} \ll T^{(2)} \) and the amplitude is dominated by its second part, \( T^{(2)} \), which is momentum dependent. The first part is small because the diagrams with different numbers of form factors cancel each other. As a consequence, in all processes where an excited scalar meson decays into a pair of ground pseudoscalar states, the second part of the amplitude defines the rate of the process.

For the decay \( \hat{a}_0 \rightarrow \pi \eta \) we obtain the amplitudes
\[
T^{(1)}_{\hat{a}_0 \rightarrow \pi \eta} \approx 0.8 \text{ GeV}, \quad (52)
\]
\[
T^{(2)}_{\hat{a}_0 \rightarrow \pi \eta} \approx 3 \text{ GeV}, \quad (53)
\]

and the decay width
\[
\Gamma_{\hat{a}_0 \rightarrow \pi \eta} \approx 36 \text{ MeV}. \quad (54)
\]

The decay of \( \hat{a}_0 \) into kaons is described by the amplitudes \( T_{\hat{a}_0 \rightarrow K^+ K^-} \) and \( T_{\hat{a}_0 \rightarrow \bar{K}^0 K^0} \) which, in accordance with our scheme, can again be divided into two parts: \( T^{(1)} \) and \( T^{(2)} \) (see Appendix B for details):
\[
T^{(1)}_{\hat{a}_0 \rightarrow K^+ K^-} \approx 0.2 \text{ GeV}, \quad (55)
\]
\[
T^{(2)}_{\hat{a}_0 \rightarrow K^+ K^-} \approx 2.1 \text{ GeV}. \quad (56)
\]

and the decay width is
\[
\Gamma_{\hat{a}_0 \rightarrow KK} = \Gamma_{\hat{a}_0 \rightarrow K^+ K^-} + \Gamma_{\hat{a}_0 \rightarrow \bar{K}^0 K^0} \approx 100 \text{ MeV}. \quad (57)
\]

Qualitatively, our results do not contradict the experimental data.
\[
\Gamma^\text{tot}_{\hat{a}_0} = 265 \pm 13 \text{ MeV}, \quad BR(\hat{a}_0 \rightarrow KK) : BR(\hat{a}_0 \rightarrow \pi \eta) = 0.88 \pm 0.23. \quad (58)
\]

The decay widths of radial excitations of scalar isoscalar mesons are estimated in the same way as it was shown above. We obtain:
\[
\Gamma_{\hat{\sigma} \rightarrow \pi \pi} = \begin{cases} 
550 \text{ MeV}(M_{\hat{\sigma}} = 1.3 \text{ GeV}) \\
460 \text{ MeV}(M_{\hat{\sigma}} = 1.25 \text{ GeV}),
\end{cases} \quad (59)
\]
\[
\Gamma_{\hat{\sigma} \rightarrow \eta \eta} = \begin{cases} 
24 \text{ MeV}(M_{\hat{\sigma}} = 1.3 \text{ GeV}) \\
15 \text{ MeV}(M_{\hat{\sigma}} = 1.25 \text{ GeV}),
\end{cases} \quad (60)
\]
\[
\Gamma_{\hat{\sigma} \rightarrow \sigma \sigma} = \begin{cases} 
6 \text{ MeV}(M_{\hat{\sigma}} = 1.3 \text{ GeV}) \\
5 \text{ MeV}(M_{\hat{\sigma}} = 1.25 \text{ GeV}),
\end{cases} \quad (61)
\]
\[
\Gamma_{\sigma \rightarrow KK} \sim 5 \text{ MeV},
\]

\[
\begin{align*}
\Gamma_{f_0(1710) \rightarrow 2\pi} & \approx 3 \text{ MeV}, & \Gamma_{f_0(1500) \rightarrow 2\pi} & \approx 3 \text{ MeV}, \\
\Gamma_{f_0(1710) \rightarrow 2\eta} & \approx 40 \text{ MeV}, & \Gamma_{f_0(1500) \rightarrow 2\eta} & \approx 20 \text{ MeV}, \\
\Gamma_{f_0(1710) \rightarrow \eta\eta'} & \approx 42 \text{ MeV}, & \Gamma_{f_0(1500) \rightarrow \eta\eta'} & \approx 10 \text{ MeV}, \\
\Gamma_{f_0(1710) \rightarrow KK} & \approx 24 \text{ MeV}, & \Gamma_{f_0(1500) \rightarrow KK} & \approx 20 \text{ MeV}.
\end{align*}
\] (63)

The decays of \( f_0(1500) \) and \( f_0(1710) \) to \( \sigma\sigma \) are negligibly small, so we disregard them.

Here we displayed our estimates for both \( f_J(1710) \) and \( f_0(1500) \) resonances. Comparing them will allow us to decide which one to consider as the first radial excitation of \( f_0(980) \) and which a glueball. From the experimental data:

\[
\Gamma^{\text{tot}}_{\sigma'} = 200 - 500 \text{ MeV}, \quad \Gamma^{\text{tot}}_{f_0(1710)} = 133 \pm 14 \text{ MeV}, \quad \Gamma^{\text{tot}}_{f_0(1500)} = 112 \pm 10 \text{ MeV}
\] (64)

we can see that in the case of \( f_0(1500) \) being a \( \bar{q}q \) state there is deficit in the decay widths whereas for \( f_J(1710) \) the result is close to experiment. From this we conclude that the meson \( f_J(1710) \) is a radially excited partner for \( f_0(980) \) and the meson state \( f_0(1370) \) is the first radial excitation of \( f_0(400 - 1200) \). As to the state \( f_0(1500) \), we are inclined to consider it as a glueball which significantly contributes to the decay width.\footnote{Let us emphasize again that it is only our preliminary conclusion. A more careful investigation of this problem will be done in our further works.}

The first radially excited state of the strange scalar \( \hat{K}^*_0 \) decays mostly to \( K\pi \) and is characterized by the width

\[
\Gamma_{\hat{K}^*_0 \rightarrow K\pi} \approx 300 \text{ MeV}.
\] (65)

This value is in agreement with experiment:

\[
\Gamma^{\text{exp}}_{\hat{K}^*_0(1430) \rightarrow K\pi} \approx 287 \pm 23 \text{ MeV}.
\] (66)

The strong decays widths of the ground states of scalar mesons were calculated in paper \cite{10} in the framework of the standard NJL model with 't Hooft interaction where it was shown that a strange scalar meson state with mass about 960 MeV decays into \( K\pi \) with the rate

\[
\Gamma_{\hat{K}^*_0(960) \rightarrow K\pi} = \frac{3}{Z\pi M_{\hat{K}^*_0}} \left( \frac{m_u m_s}{2F_\pi} \right)^2 \times 
\frac{(M_{\hat{K}^*_0}^2 - (M_K - M_\pi)^2)(M_{\hat{K}^*_0}^2 - (M_K + M_\pi)^2)}{M_{\hat{K}^*_0}^4} \approx 360 \text{ MeV}
\] (67)

From comparing this result with the analysis of phase shifts given in \cite{12} where an evidence for existence of a scalar strange meson with the mass equal to 905 ± 50 MeV and decay width 545 ± 170 MeV is shown, we identify the state \( \hat{K}^*_0(960) \) as a member of the ground scalar meson nonet. The state \( \hat{K}^*_0(1430) \) is thereby its first radial excitation.
7. STRONG DECAYS OF $\eta(1295)$ AND $\eta(1440)$.

The mesons $\eta(1295)$ and $\eta(1440)$ have common decay modes: $a_0\pi$, $\eta\pi\pi$, $\eta(\pi\pi)_{S\text{-wave}}$, $K\bar{K}\pi$, moreover, the heavier pseudoscalar $\eta(1440)$ decays also into $KK^*$. For the processes with two secondary particles, the calculations of decay widths are done in the same way as shown in the previous section, by calculating triangle diagrams similar to that in Fig. 1.

Let us consider the decay $\hat{\eta} \rightarrow a_0\pi$. The corresponding amplitude is of the same form as given in (46) for decays of the type $\sigma \rightarrow \varphi_1\varphi_2$. It can also be divided into two parts $T^{(1)}$ and $T^{(2)}$ which in our approximation are constant and momentum-dependent in the sense explained in the previous section (see (47) and the text below):

$$T^{(1)}_{\hat{\eta} \rightarrow a_0\pi} \approx 0.3 \text{ GeV,}$$

$$T^{(2)}_{\hat{\eta} \rightarrow a_0\pi} \approx -1 \text{ GeV.}$$

Therefore, the decay width is

$$\Gamma_{\hat{\eta} \rightarrow a_0\pi} \approx 3 \text{ MeV.}$$

The decay $\hat{\eta} \rightarrow \eta(\pi\pi)_{S\text{-wave}}$ is nothing else than the decay $\hat{\eta} \rightarrow \eta\sigma \rightarrow \eta(\pi\pi)_{S\text{-wave}}$ where we have the $\sigma$-meson in the final state decaying then into pions in the $S$-wave. We simply calculate $\hat{\eta} \rightarrow \eta\sigma$, with $\sigma$ as a decay product.

The calculation of decay widths for the rest of the decay modes with two particles in the final state is similar and the result is given in Table IV.

The decay $\hat{\eta}' \rightarrow KK^*$ differs from the other modes by a strange vector meson among the decay products. In this case we have

$$T^\mu_{\hat{\eta}' \rightarrow KK^*} = 4(p_1 + p_2)^\mu \left( [g_u g_K g_{K^*} I_2(u, s) + \ldots] - \sqrt{2} [g_s g_K g_{K^*} I_2(u, s) + \ldots] \right)$$

where $p_1$ is the momentum of $\hat{\eta}'$; $p_2$, the momentum of $K$; and dots stand for the terms with form factors (not displayed here). These two parts are of the same order of magnitude and differ in sign and therefore cancel each other, which reduces the decay width up to tens of keV:

$$\Gamma_{\hat{\eta}' \rightarrow KK^*} \approx 70 \text{ keV.}$$

When there are three particles in the final state, poles appear in amplitudes, related to intermediate scalar resonances. As it is well known from $\pi\pi$ scattering, these diagrams can play a crucial role in the description of such processes. So, in addition to the "box" diagram we take into account the diagrams with poles provided by $\sigma$, $f_0$, and $a_0$ resonances (see Fig. 2 for the decay $\hat{\eta} \rightarrow \eta\pi\pi$). Here we neglect the momentum dependence in the box diagram, approximating it by a constant. The amplitude is thereby
\[ T_{\hat{\eta} \to \eta \pi \pi} = B + \frac{c_{\sigma \hat{\eta} \eta \pi \pi}}{M_\sigma - s - iM_\sigma \Gamma_\sigma} + \frac{c_{f_0 \hat{\eta} \eta \pi \pi}}{M_{f_0} - s - iM_{f_0} \Gamma_{f_0}} + \frac{c_{a_0 \hat{\eta} \pi \pi \pi}}{M_{a_0} - t - iM_{a_0} \Gamma_{a_0}} + \frac{c_{a_0 \eta \pi \pi \pi}}{M_{a_0} - u - iM_{a_0} \Gamma_{a_0}} + \text{excited}, \tag{74} \]

where \( B \) is given by the "box" diagram:

\[ B = 12 \left( \frac{m_u}{F_\pi} \right)^2 Z^{-1}[R_{11}R_{12} + \ldots] \tag{75} \]

where dots stand for the contribution from diagrams with form factors, and \( R_{ij} \) are taken from Table II (for \( \eta \) and \( \hat{\eta} \)). The coefficients \( c_{\sigma \varphi \varphi} \) represent the amplitudes describing decays of a scalar to a couple of pseudoscalars; the calculation of them is discussed in the previous section. In general, they are momentum-dependent.

The kinematic invariants \( s, t \) and \( u \) are Mandelstam variables:

\[
\begin{align*}
 s &= (p_\pi_1 + p_\pi_2)^2, \\
 t &= (p_\eta + p_\pi_1)^2, \\
 u &= (p_\eta + p_\pi_2)^2
\end{align*}
\]

The "excited" terms are the contributions from excited scalar resonances of a structure similar to that for the ground states. The decay widths of processes \( \hat{\eta} \to \eta \pi \pi \) and \( \hat{\eta}' \to \eta \pi \pi \) are thereby

\( \Gamma_{\hat{\eta} \to \eta \pi \pi} \approx 4 \, \text{MeV}, \quad \Gamma_{\hat{\eta}' \to \eta \pi \pi} \approx 6 \, \text{MeV}. \tag{76} \)

For the processes \( \hat{\eta} \to K \bar{K} \pi \) and \( \hat{\eta}' \to K \bar{K} \pi \) we approximate their decay widths by neglecting the pole-diagram contribution because it turns out that the "box" is dominant here. The result is given in Table [V].

Unfortunately, the branching ratios for different decay modes of \( \eta(1295) \) and \( \eta(1440) \) are not known well from experiment; so one can only find their total decay widths

\[ \Gamma_{\eta(1295)}^{\text{tot}} = 53 \pm 6 \, \text{MeV}, \quad \Gamma_{\eta(1440)}^{\text{tot}} = 50 - 80 \, \text{MeV,} \tag{77} \]

which is in satisfactory agreement with our results.

Strong and electromagnetic decays of the ground states of \( \eta \) and \( \eta' \) mesons were already investigated within the framework of the standard NJL model in [13] and we do not consider them here.

### 8. DISCUSSION AND CONCLUSION

Let us shortly recall some problems concerning the interpretation of experimental data on scalar and \( \eta, \eta' \) mesons. Several years ago, attempts were undertaken to consider the state \( \eta'(1440) \) as a glueball [14]. There is an analogous problem with the interpretation of scalar states \( f_0(1500) \) and \( f_0(1710) \). Moreover, the experimental status of the lightest scalar isoscalar singlet meson was unclear. In some papers, the resonance \( f_0(1370) \) was considered as a member of the ground nonet [15], and it was not until 1998 that the resonance \( f_0(400 - 1200) \) was included into the summary tables of PDG review[16] [11].

---

6) However, in earlier editions of PDG the light \( \sigma \) state still could be found; it was excluded later.
One will find a problem of the same sort in the case of \( K_0^* \). The strange meson \( K_0^*(1430) \) seems too heavy to be the ground state: 1 GeV is more characteristic of the ground meson states (see [12,16]).

From our calculations we conclude that the states \( \eta(1295) \) and \( \eta(1440) \) can be considered as radial excitations of the ground states \( \eta \) and \( \eta' \). The calculation of their strong decay widths also confirms our conclusion. Let us note that these meson states are significantly mixed.

In [14] the authors came to similar conclusions about \( \eta(1295) \) and \( \eta(1440) \) where the radial excitations of the mesons were investigated in the potential \( ^3P_0 \) model.

Our calculations also showed that we can interpret the scalar states \( f_0(1370), a_0(1450), f_0(1710) \) and \( K_0^*(1430) \) as the first radial excitations of \( f_0(400-1200), a_0(980), f_0(980) \) and \( K_0^*(960) \). We estimated their masses and the widths of main decays in the framework of a nonlocal chiral quark model. We would like to emphasize that we did not use additional parameters except those necessary to fix the mass spectrum of pseudoscalar mesons. We used the same form factors both for the scalar and pseudoscalar mesons, which is required by the global chiral symmetry.

We assumed that the state \( f_0(1500) \) is a glueball, and its probable mixing with \( f_0(980), f_0(1370) \) and \( f_J(1710) \) may provide us with a more correct description of the masses of these states\(^7\))(see Table [11] and [7]). We are going to consider this problem in a subsequent publication.

A more complicated situation takes place for the ground state \( a_0(980) \). In the framework of our quark-antiquark model, we have a mass deficit for this meson, 830 MeV instead of 980 MeV. We suspect that this drawback is due to a four-quark component in this state which we did not take into account [18].

In future we are going to consider glueball states [7] and to develop a model with quark confinement [19] to describe the momentum dependence of meson amplitudes.

ACKNOWLEDGMENT

We are very grateful to Prof. S.B. Gerasimov for useful discussion. This work has been supported by RFBR Grant N 98-02-16135.

APPENDIX A: COEFFICIENTS OF THE FREE PART OF EFFECTIVE LAGRANGIAN FOR THE SCALAR ISOSCALAR MESONS.

The functions \( K_{\sigma(\phi),ij}^{[a,a]} \) introduced in Sec. 4 (37) are defined as follows

\[
K_{\sigma(\phi),11}^{[a,a]}(P) = Z_{\sigma(\phi),1}^{a} (P^2 - (m_q^a \pm m_q^{a'})^2 - M_{\sigma(\phi),1}^2),
\]

\[
K_{\sigma(\phi),22}^{[a,a]}(P) = Z_{\sigma(\phi),2}^{a} (P^2 - (m_q^a \pm m_q^{a'})^2 - M_{\sigma(\phi),2}^2),
\]

\(^7\) Our estimates for the masses of \( f_0 \) and \( \hat{f}_0 \): \( M_{f_0} = 1070 \) MeV and \( M_{\hat{f}_0} = 1600 \) MeV are expected to shift to \( M_{f_0} = 980 \) MeV and \( M_{\hat{f}_0} = 1710 \) MeV after mixing with the glueball \( f_0(1500) \).
\[ K^{[a,b]}_{\sigma(\varphi),12}(P) = K^{[a,a]}_{\sigma(\varphi),21}(P) = \gamma_{\sigma(\varphi)}(P^2 - (m_q^a \pm m_q^b)^2), \quad (A1) \]
\[ K^{[8,9]}_{\sigma(\varphi),11}(P) = K^{[9,8]}_{\sigma(\varphi),11}(P) = \frac{1}{2} \left( T^{S(P)} \right)^{-1}_{89}, \]
\[ K^{[8,9]}_{\sigma(\varphi),12}(P) = K^{[9,8]}_{\sigma(\varphi),12}(P) = K^{[8,9]}_{\sigma(\varphi),21}(P) = 0, \]
\[ K^{[9,8]}_{\sigma(\varphi),21}(P) = K^{[8,9]}_{\sigma(\varphi),22}(P) = 0, K^{[9,8]}_{\sigma(\varphi),22}(P) = 0 \]

where the “bare” meson masses are
\[ M^2_{\sigma^8(\varphi^s),1} = (Z^8_{\sigma(\varphi),1})^{-1} \left( \frac{1}{2} (T^{S(P)})^{-1}_{88} - 8I_1(m_u) \right), \]
\[ M^2_{\sigma^9(\varphi^s),1} = (Z^9_{\sigma(\varphi),1})^{-1} \left( \frac{1}{2} (T^{S(P)})^{-1}_{99} - 8I_1(m_s) \right), \]
\[ M^2_{\sigma^8(\varphi^s),2} = (Z^8_{\sigma(\varphi),2})^{-1} \left( \frac{1}{2G} - 8I_1^{ff}(m_u) \right), \]
\[ M^2_{\sigma^9(\varphi^s),2} = (Z^9_{\sigma(\varphi),2})^{-1} \left( \frac{1}{2G} - 8I_1^{ff}(m_s) \right). \]

In the case of isoscalar mesons it is convenient to combine the scalar and pseudoscalar fields into 4-vectors
\[ \Sigma = (\sigma^{8r} \sigma^{9r}, \sigma^{9r} \sigma^{9r}), \quad \Phi = (\varphi^{8r} \varphi^{8r}, \varphi^{9r} \varphi^{9r}), \quad (A3) \]
and introduce 4 \times 4 matrix functions \( K_{\sigma(\varphi),ij} \), instead of old \( K^{[a,b]}_{\sigma(\varphi),ij} \), where indices \( i, j \) run from 1 through 4. This allows us to rewrite the free part of the effective Lagrangian, which then, with the meson fields renormalized, looks as follows
\[ \mathcal{L}^{(2)}(\Sigma, \Phi) = \frac{1}{2} \sum_{i,j=1}^{4} (\Sigma_i K_{\sigma,ij}(P) \Sigma_j + \Phi_i K_{\varphi,ij}(P) \Phi_j). \]

and the functions \( K_{\sigma(\varphi),ij} \) are
\[ K_{\sigma(\varphi),11}(P) = P^2 - (m_u \pm m_u)^2 - M^2_{\sigma^8(\varphi^s),1}, \]
\[ K_{\sigma(\varphi),22}(P) = P^2 - (m_u \pm m_u)^2 - M^2_{\sigma^9(\varphi^s),1}, \]
\[ K_{\sigma(\varphi),33}(P) = P^2 - (m_s \pm m_s)^2 - M^2_{\sigma^9(\varphi^s),1}, \]
\[ K_{\sigma(\varphi),44}(P) = P^2 - (m_s \pm m_s)^2 - M^2_{\sigma^9(\varphi^s),1}, \]
\[ K_{\sigma(\varphi),12}(P) = K_{\sigma(\varphi),21}(P) = \Gamma_{\sigma_u(\eta_u)}(P^2 - (m_u \pm m_u)^2), \]
\[ K_{\sigma(\varphi),34}(P) = K_{\sigma(\varphi),43}(P) = \Gamma_{\sigma_s(\eta_s)}(P^2 - (m_s \pm m_s)^2), \]
\[ K_{\sigma(\varphi),13}(P) = K_{\sigma(\varphi),31}(P) = (Z^8_{\sigma(\varphi),1} Z^9_{\sigma(\varphi),2})^{-1/2} (T^{S(P)})^{-1}_{89}. \]

Now, to transform (A3) to conventional form, one should just diagonalize a 4-dimensional matrix, which is better to do numerically.

**APPENDIX B: THE CALCULATION OF THE AMPLITUDES FOR THE DECAYS OF THE EXCITED SCALAR MESON \( \hat{a}_0 \)**

Here we collect some instructive formulae, which display a part of details of the calculations made in this work. Let us demonstrate how the amplitude of the decay \( \hat{a}_0 \rightarrow \eta \pi \) is
obtained. The mixing coefficients are taken from Table I. Moreover, the diagrams where pion vertices contain form factors are neglected because, as one can see from Table I, their contribution is significantly reduced.

\begin{equation}
T_{\bar{a}_0\rightarrow\eta\pi}^{(1)} = \frac{4m_u^2}{F_\pi} \left\{ 0.82 \cdot 0.71 \cdot Z^{-1/2} \frac{I_2(m_u)}{I_2(m_u)} - \left(1.17 \cdot 0.71 \cdot Z^{-1/2} - 0.82 \cdot 0.11\right) \frac{I_2'(m_u)}{\sqrt{I_2(m_u)I_2'(m_u)}} - 1.17 \cdot 0.11 \cdot \frac{I_2''(m_u)}{I_2'(m_u)} \right\} \approx 0.2 \text{ GeV}, \tag{B1}
\end{equation}

\begin{equation}
T_{\bar{a}_0\rightarrow\eta\pi}^{(2)} = \frac{2m_u^2}{F_\pi} (M_{a_0}^2 - M_{\eta}^2 - M_\pi^2) \left\{ 0.82 \cdot 0.71Z^{-1/2} \frac{I_3(m_u)}{I_2(m_u)} - \left(1.17 \cdot 0.71 \cdot Z^{-1/2} - 0.82 \cdot 0.11\right) \frac{I_3'(m_u)}{\sqrt{I_2(m_u)I_3'(m_u)}} - 1.17 \cdot 0.11 \frac{I_3''(m_u)}{I_3'(m_u)} \right\} \approx 3.5 \text{ GeV}. \tag{B2}
\end{equation}

The decay width thereby is

\begin{equation}
\Gamma_{\bar{a}_0\rightarrow\eta\pi} = \left| \frac{T_{\bar{a}_0\rightarrow\eta\pi}}{16\pi M_{\bar{a}_0}} \right|^2 \sqrt{M_{\bar{a}_0}^2 + M_{\eta}^2 + M_\pi^2 - 2(M_{a_0}^2 M_{\eta}^2 + M_{a_0}^2 M_\pi^2 + M_{\eta}^2 M_\pi^2)} \approx 160 \text{ MeV}. \tag{B3}
\end{equation}

Here \( I_2(m_u) = 0.04, I_2'(m_u) = 0.014c, I_2''(m_u) = 0.015c^2, I_3(m_u) = 0.11 \text{ GeV}^{-2}, I_3'(m_u) = 0.07c \text{ GeV}^{-2}, I_3''(m_u) = 0.06c^2 \text{ GeV}^{-2} \) and \( c \) is the external form factor parameter factored out and cancelled in the ratios of the integrals.

For the decay into strange mesons we obtain (see Fig.1)

\begin{equation}
T_{\bar{a}_0\rightarrow K+K^-} = C_K \left( -\frac{iN_c}{16\pi^2} \right) \int d^4k \frac{\text{Tr}[(m_u + \not{k} + \not{p}_1)\gamma_5(m_s + \not{k})\gamma_5(m_u + \not{k} - \not{p}_2)]}{(m_u^2 - k^2)(m_u^2 - (k - p_1)^2)(m_u^2 - (k - p_2)^2)} \approx 2C_K \left\{ (m_s + m_u)I_2(m_u) - \Delta I_2(m_u, m_s) - [m_s(M_{\bar{a}_0}^2 - 2M_K^2) - 2\Delta^3]I_3(m_u, m_s) \right\}, \tag{B4}
\end{equation}

where \( \Delta = m_s - m_u \) and

\begin{equation}
I_3(m_u, m_s) = -\frac{iN_c}{(2\pi)^4} \int d^4k \frac{d^4k}{\Lambda_3 (m_u^2 - k^2)(m_s^2 - k^2)}. \tag{B5}
\end{equation}

The coefficient \( C_K \) absorbs the Yukawa coupling constants and some structure coefficients. The integral \( I_2(m_u, m_s) \) is defined by \( \Lambda_2 \). This is only the part of the amplitude without form factors. The complete amplitude of this process is a sum of contributions which contain also the integrals \( I_2^{ff} \) and \( I_3^{ff} \) with form factors. Thus, the amplitude is

\begin{equation}
\end{equation}
\[ T_{\hat{a}_0 \rightarrow K^+ K^-} = T^{(1)} + T^{(2)}, \] (B6)
\[ T^{(1)} = \frac{m_u + m_s}{2F_K} \{(m_s + m_u) \cdot 0.13 - \Delta \cdot 0.21\} \approx 0.2 \text{ GeV}, \] (B7)
\[ T^{(2)} = \frac{m_u + m_s}{2F_K} \{[m_s(M_{\hat{a}_0}^2 - 2M_K^2) - 2\Delta^3] \cdot 1 \text{ GeV}^{-2}\} \approx 2.3 \text{ GeV}, \] (B8)
\[ F_K = 1.2F_\pi. \]

The decay width therefore is evaluated to be
\[ \Gamma_{\hat{a}_0 \rightarrow K^+ K^-} = \Gamma_{\hat{a}_0 \rightarrow K^0 K^0} \approx 50 \text{ MeV}. \] (B9)
REFERENCES

1. Volkov, M.K., and Weiss, C., *Phys. Rev. D*, 1997, vol. 56, p. 221.
2. Volkov, M.K., *Phys. At. Nucl.*, 1997, vol. 60, p. 1920; hep-ph/9612456.
3. Volkov, M.K., Ebert, D., and Nagy, M., *Int. J. Mod. Phys. A*, 1998, vol. 13, p. 5443; hep-ph/9705334.
4. Volkov, M.K., Ebert, D., and Yudichev V.L., *JINR Rapid Comm.*, 1998, 6(92)-98, p. 5; hep-ph/9810470.
5. Andrianov, A.A., and Andrianov, V.A., *Nucl. Phys. Proc. Suppl. BC*, 1995, vol. 39, p. 257; Andrianov, A.A., Andrianov, V.A., and Yudichev, V.L., *Theor. Math. Phys.*, 1996, vol. 108, p. 1069.
6. Gerasimov, S.B., and Govorkov A.B., *Z. Phys. C*, 1986, vol. 32, p. 405.
7. Celenza, L.S., Huang, B., and Shakin, C.M., *Phys. Rev. C*, 1991, vol. 59, p. 1041.
8. Burdanov, V.Ja., Efimov, G.V., Nedelko, S.N., and Solunin, S.A., *Phys. Rev. D*, 1996, vol. 54, p. 4483.
9. Vogl, H., and Weise, W., *Progr. Part. Nucl. Phys.*, 1991, vol. 27, p. 195; Klevansky, S.P., *Rev. Mod. Phys.*, 1992, vol. 64, p. 649.
10. Volkov, M.K., Nagy, M., and Yudichev, V.L., *Nuovo Cim. A*, 1999, vol. 112, (in press); JINR Preprint E2-98-101, pp. 8; hep-ph/9804347.
11. Review of Particle Physics, *Europ. Phys. J. C*, 1998, vol. 3, p. 1.
12. Ishida, S., Ishida, M., Ishida, T., Takamatsu, K., and Tsuru, T., *Prog. Theor. Phys.*, 1997, vol. 98, p. 621; Ishida, M.Y., and Ishida, S., *HADRON’97, 4th Int. Conf. on Hadr. Spectr.*, 1997; hep-ph/9712231.
13. Volkov, M.K., *Ann. Phys.*, 1984, vol. 157, p. 282; Volkov, M.K., *Sov. J. Part. Nucl.*, 1986, vol. 17, p. 186.
14. Gerasimov, S.B., and Govorkov, A.B., *Z. Phys. C*, 1985, vol. 29, p. 61; Gerasimov, S.B., and Govorkov, A.B., *Z. Phys. C*, 1987, vol. 36, p. 435.
15. Törnqvist, N., *Phys. Rev. Lett.*, 1982, vol. 49, p. 624; Lanik, J., *Phys. Lett. B*, 1993, vol. 306, p. 139; Dmitrašinović, V., *Phys. Rev. C*, 1996, vol. 53, p. 1383.
16. Scadron, M.D., *Phys. Rev. D*, 1982, vol. 26, p. 239.
17. Kusaka, K., Volkov, M.K., and Weise, W., *Phys. Lett. B*, 1993, vol. 302, p. 145; Jaminon, M., Bossche, B. Van den, *Nucl. Phys. A*, 1997, vol. 619, p. 285.
18. Jaffe, R.L., *Phys. Rev. D*, 1977, vol. 15, p. 267, 281; Achasov, N.N., Devyanin, S.A., and Shestakov, G.N., *Usp. Fiz. Nauk.*, 1984, vol. 142, p. 361.
19. Blaschke, D., Burau, G., Volkov, M.K., and Yudichev, V.L., Preprint Rostock Univ. MPG-VT-UR 178/98; hep-ph/9812503; Volkov, M.K., and Yudichev, V.L., *Phys. At. Nucl.*, 1999, vol. 62, No. 10 (in press).
TABLES

TABLE I. The mixing coefficients for the ground and first radially excited states of the scalar and pseudoscalar isovector and strange mesons. The caret symbol marks the excited states.

|      | $a_0$ | $\tilde{a}_0$ | $K_0^*$ | $\tilde{K}_0^*$ |
|------|-------|---------------|---------|-----------------|
| $a_{0,1}$ | 0.87  | 0.82          | $K_{0,1}^*$ | 0.83           |
| $a_{0,2}$ | 0.22  | -1.17         | $K_{0,2}^*$ | 0.28           | -1.11          |

|      | $\pi$ | $\tilde{\pi}$ | $\bar{K}$ | $\tilde{\bar{K}}$ |
|------|-------|---------------|----------|-------------------|
| $\pi_1$ | 1.00  | 0.54          | $K_1$  | 0.96             |
| $\pi_2$ | 0.01  | -1.14         | $K_2$  | 0.09             | -1.11          |
TABLE II. The mixing coefficients for the isoscalar meson states

|     | $\eta$ | $\eta'$ | $\eta$ | $\eta'$ |
|-----|--------|---------|--------|---------|
| $\varphi_1^8$ | 0.71   | 0.62    | -0.32  | 0.56    |
| $\varphi_2^8$ | 0.11   | -0.87   | -0.48  | -0.54   |
| $\varphi_1^0$ | 0.62   | 0.19    | 0.56   | -0.67   |
| $\varphi_2^0$ | 0.06   | -0.66   | 0.30   | 0.82    |

|     | $\bar{\sigma}$ | $\bar{\sigma}$ | $f_0$ | $f'_{0}$ |
|-----|-----------------|-----------------|-------|---------|
| $\sigma_1^4$ | -0.98           | -0.66           | 0.10  | 0.17    |
| $\sigma_2^4$ | 0.02            | 1.15            | 0.26  | -0.17   |
| $\sigma_1^0$ | 0.27            | -0.09           | 0.82  | 0.71    |
| $\sigma_2^0$ | -0.03           | -0.21           | 0.22  | -1.08   |
TABLE III. The model masses of mesons, MeV

|       | GR | EXC | GR(Exp.)  | EXC(Exp.)  |
|-------|----|-----|-----------|------------|
| $M_\sigma$ | 530 | 1330 | 400 − 1200 | 1200 − 1500 |
| $M_{f_0}$   | 1070 | 1600 | 980 ± 10   | 1712 ± 5   |
| $M_{a_0}$   | 830  | 1500 | 983.4 ± 0.9| 1474 ± 19  |
| $M_{K_0^*}$ | 960  | 1500 | 905 ± 50   | 1429 ± 12  |
| $M_\pi$     | 140  | 1300 | 139.56995 ± 0.00035 | 1300 ± 100 |
| $M_K$       | 490  | 1300 | 497.672 ± 0.031 | 1460(?)    |
| $M_\rho$    | 520  | 1280 | 547.30 ± 0.12 | 1297.8 ± 2.8 |
| $M_{\eta'}$ | 910  | 1470 | 957.78 ± 0.14 | 1440 − 1470 |
TABLE IV. $\eta(1295)$ and $\eta(1440)$ decay modes.

|          | $a_0\pi$ | $\eta\sigma$ | $\eta\pi\pi$ | $KK\pi$ | $KK^*$ | $\Gamma_{\text{tot}}$ |
|----------|----------|---------------|---------------|---------|-------|-------------------|
| $\eta(1295)$ | 3 MeV   | 30 MeV        | 4 MeV         | 5 MeV   | –     | 48 MeV           |
| $\eta(1440)$ | 10 MeV  | 3 MeV         | 6 MeV         | 26 MeV  | 70 keV | 45 MeV           |
FIGURE CAPTIONS

1) Diagrams describing decays of \( a_0 \) to pseudoscalars.

2) Diagrams describing the decay \( \eta' \to \eta \pi \pi \). The black box stands for the sum of “box” diagrams represented by one-loop quark graphs with four meson vertices. The rest of the diagrams is a set of pole graphs with the \( \sigma \), \( f_0 \) and \( a_0 \) scalar resonances. The diagram with \( a_0 \) is to be taken into account for two channels (due to exchange of the pions momenta). There are analogous contributions from radially excited resonances.
FIG. 1.
FIG. 2.