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A sundial with hour lines portraying the Earth

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The study of sundials is an age-old discipline of science. The greatest developments were achieved in the Antiquity and in the Middle Ages, but modern physicists also contributed their part. In 1956, a former student of Wolfgang Pauli, Heinz Schilt, constructed an underwater sundial in his garden pond, using refraction to reshape the sundial’s hour lines. His approach is developed further in this paper and combined with Thales’ geometric concept of gnomonic projection. Together, this leads to a beautiful result: Snell’s law can be used to build a sundial whose hour lines portray the Earth and its meridians. A prototype has been built to demonstrate the idea. © 2019 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).

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I. INTRODUCTION

Since Babylonian times sundials have been an integral part of science and civilization.1 Hardly any other measuring instrument has had such a long and profound impact on public life and personal habits, reaching deep into everyday life.2 The Antiquity sundials had been displaying hours of unequal length3 until someone in the Middle Ages thought of using the hour angle of the Sun to measure time. Who this was remains unknown.4–6 The idea represented a great step forward and solved a problem of great importance at its time: the synchronization of mechanical clocks.4 This was a result of the hour angle being a far better time signal than those previously in use. With the new definition, all hours of the year were of nearly equal duration, with fluctuations in the length of less than 1.3 s. Also, the hour angle was available everywhere and always remains in phase with daylight. Solar time, as the hour angle signal is now called, remained the time standard for several hundred years until it was replaced by the mean solar time in the 19th century.7 Later, increasingly accurate time signals became available from quartz and atomic clocks.1,8–9

Most current sundials use a gnomonic projection10 to display solar time on a plane surface. On such sundials, the hours are represented by straight lines that converge in a single point: the projection of the celestial pole. For this paper, such sundials shall be referred to as classical sundials. Their hour lines are obtained from a gnomonic projection of a system of reference lines aligned with the Earth’s poles. When classical sundials were invented is unclear. They rapidly became popular in the 15th century6,11 and were easy to construct and easy to read. They have been considered the perfect, final form of a sundial, with nothing left to invent.12 The geometrical concept of gnomonic projection was discovered much earlier than the classical sundial, likely by Thales (580 B.C.).10

For every point on Earth except the poles, solar time $t$ is defined as the Sun’s hour angle $h$ plus half a turn (or half a day), such that $t = h + 12 \pm \frac{1}{2}$ h.15,14 The hour angle is the difference between the longitude of a celestial object (the Sun in this case) and the observer’s longitude and measured in twenty fourths of a full turn. The terrestrial counterpart of the hour angle is the geographical longitude, measured with respect to Greenwich. Although the hour angle and the geographical longitude serve to measure different things (the position of a celestial object for the first and the position of a terrestrial object for the second), they only differ by an arbitrary additive constant. This is why chronometers can be used to accurately determine the longitude.15 Thus, the hour lines of a sundial (lines of equal hour angle) always represent the Earth and its meridians (lines of equal longitude). Yet, on classical sundials, the Earth does not appear in the shape of the hour lines, not in an obvious way at least. Neither the straight hour lines nor the conic declination lines are readily recognizable as longitude and latitude lines of the Earth (Fig. 1). A particular challenge, therefore, is to design a sundial that naturally displays the relationship between hour lines and terrestrial meridians. A geometrical solution to this problem has been known for a long time: it consists in abandoning the flat dial surface in favor of a spherical dial surface, either solid16 or hollow17,18 (Fig. 2). In this way, the sundial and its hour lines represent a (nearly) congruent scale model of the Earth and its meridians.

There is, interestingly, another solution to this problem, maintaining the flat dial surface. The starting point is a sundial that was built in 1956 by Heinz Schilt, a student of Wolfgang Pauli at ETH Zürich.19 His sundial had its hour lines underwater and its gnomon just touching the surface of the water from above.20 It decorates the bottom of a pond in his former garden. In contrast to the hour lines of classical sundials, the hour lines of his sundial are curved.

II. SHAPING HOUR LINES BY SNELL’S LAW

As demonstrated by Schilt’s sundial, the refractive properties of water provide a means to reshape the hour lines. The first step is to adjust the definition of the sundial “A sundial consists of an object casting a shadow on a dial surface carrying a scale”6,21 to include “with a refracting medium in between the two.” Using a medium other than air gives the sundial maker an extra degree of freedom that enables him or her to redirect the rays of sunlight within the possibilities offered by Snell’s law. This can be used in various ways; for example, to recreate the Miracle of Ahaz22,23 or, as done here, to reshape hour lines to represent the Earth’s longitude lines. A number of sundials using the principle of refraction have been built20 or proposed in patent applications,24 but none of these aim at using the hour lines to portray the Earth.

Our analysis is based on a result of geometrical optics (Fig. 3): the ensemble of all rays of light traveling in a plane $h$, incident at $O$ on a plane-parallel plate with refractive index $n$, forms an elliptical cone within the plate. A proof of...
this assertion is given in the Appendix. The proof is devised in simple terms and can be used as a classroom exercise in optics or astronomy classes. If \( h \) is an hour plane (a plane of constant hour angle \( h \)), an hour line is formed at the intersection of the refracted conical surface associated with \( h \) and the base of the plane-parallel plate (i.e., if a refracted ray of sunlight passing through \( O \) intersects that hour line, then \( t = h + 12 \)). If the procedure is repeated for the other 23 hour planes of the celestial sphere, the hour lines of the sundial are obtained.

In purely geometrical terms, the setup shown in Fig. 3 is a simple modification of Thales’ gnomonic projection, in which the projection lines deflect at \( O \) according to Snell’s law. If the refractive index is set to 1, a conventional gnomonic projection is obtained. Since light obeys Snell’s law, a sundial can be built on that principle by only letting rays of light enter the plane-parallel plate through a small transparent area at \( O \), while the rest of the surface is treated to scatter sunlight. Of course, sundials can be realized only with an adequate material (aerogels, transparent polymers, glasses, crystals, diamond, etc.). Each material will lead to other hour lines, depending on the refractive index.

The equations for calculating the hour lines are included in the Appendix. The wave properties of light, the parallax of the Sun, and the not entirely predictable motions of the Earth’s poles, which have tiny effects on the positions of hour lines, have been neglected. The result is that the hour lines always appear as arcs of ellipses for \( n > 1 \). The ellipses have a common center and semi-major axes of equal length (see the Appendix). Figure 4 shows hour lines and lines of equal declination for a range of refractive indices. For \( n > 1 \), the diameter of the image is always finite. It decreases in inverse proportion to \( n \), but this can be compensated by increasing the thickness of the plane-parallel plate. In order to make the connection with the Earth more obvious, geographical features such as the shorelines of continents have been inserted by associating hour lines with the longitude.

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Fig. 1. General structure of the classical sundial. (a) Hour lines of a historic specimen from Schaffhausen (Switzerland), facing south. (b) Ensemble of straight hour lines and conic declination lines of a modern specimen configured for New York facing south-west, including declinations inaccessible to the Sun (grey areas). Earth does not appear in the lines of such sundials.

Fig. 2. Spherical sundials. Time is indicated by the position of the point of normal incidence (of sunlight onto the rounded dial surface) with respect to the hour lines. (a) Solid sphere sundial. (b) Hollow sphere sundial from the Roman period found at Windisch in Switzerland, with a reconstruction of the antique hour lines (hours of non-equal duration). It is dated between 20 B.C. and 75 A.D (Ref. 17). The gnomon was not preserved on the find. Photos A: Erich Baumann; B: Béla A. Polyvás (Ref. 29).

Fig. 3. Refraction of an hour plane by a plane-parallel plate through a fixed point \( O \). The broken line \( S-O-T \) shows an example path of light. The points A and C mark the grazing incidence for light travelling along the \( x \)-axis shown in the figure and B marks the steepest incidence.
and declination with the latitude. The ellipticity of the hour lines, which is the key property that provides the appearance of the Earth, is clearly visible in Fig. 4. In contrast, the shapes of the declination lines are quite fancy, especially for \( n \) close to 1.

### III. DISCUSSION

As previously mentioned, the lines for solar time and declination converge towards the lines of a classical sundial as \( n \to 1 \). The convergence is best visible in the central part of the maps in Fig. 4. The classical sundial is thus one limiting case of our arrangement.

As the refractive index is increased beyond 1, the hour lines are forced into a circular envelope enclosing a distorted image of a terrestrial globe (Fig. 4). With \( n \) close to 1, the globe is strongly distorted, but as \( n \) increases, the distortion rapidly decreases. From \( n \geq 1.5 \), the image of the globe appears more and more familiar. In the limiting case as \( n \) approaches infinity, the distortions disappear and the image converges to an orthographic projection of the Earth or, in terms of sundials, to an orthographic projection of a spherical sundial. Thus, the spherical sundial, or more precisely its orthographic projection, represents a second limiting case for \( n \to \infty \) of our sundial. The mathematical framework in the Appendix thus unites two distinct sundial types: the classical sundials and the spherical sundials.

When Heinz Schilt devised his underwater sundial, he did not realize that he could have imaged the Earth on the bottom of his pond (or if he did, he did not mention it). However, its image would have appeared mirror-inverted. For \( n = 1.33 \), the refractive index of water, it would also have looked somewhat flattened in the central area (Fig. 4(d)).

### IV. APPLICATION

To demonstrate the idea, a prototype has been built. It was designed to stand on the sill of a sunny window. Its body consists of a thick plate of PMMA glass. In contrast to Fig. 3, the plane plate is oriented vertically. It could have been oriented horizontally or at any tilt. The Sun-facing side is translucent, except for a small polished area around \( O \), which is transparent. The opposing side is configured as a matt screen. As the Sun moves across the sky, the sunbeams are refracted at \( O \) as previously discussed. In addition to showing solar time, the hour lines for modern-day standard time have been added (this avoids to convert solar time to standard time at each reading). The hour lines of standard time are easily constructed from the hour lines of solar time by using the equation of time and assuming the orbital
parameters of the Earth to be constant. They form the characteristic octal shape of the analemmatic correction.27,28

Figure 5 shows a comparison of the dials for a classical sundial (n = 1) and our device (for n = 1.5). The image corresponds to the matt screen being viewed from the side opposite to the Sun. The hour lines in the lower half of the dial are fully functional, indicating time during the sunshine. They correspond to a position of the Sun above the mathematical horizon. The hour lines in the upper half of the dial circle correspond to a position of the Sun below the horizon. Thus, this portion of the hour scale is not functional as a sundial. It is included to portray the Earth-Sun system as a celestial-mechanical clock which visually shows the relationship among the time of day, the seasons of the year, and the relative position of the Sun, the Earth, and the sundial in a practically self-explanatory way, very much like a planetarium does, but with no moving parts except for the astronomical bodies.

Fig. 5. Shapes of hour lines for solar time (dotted lines) and for standard time (grey/black figure eight-shaped loops) for a plane plate sundial facing south in Zürich. (a) Dial resulting from a refractive index of 1.0. (b) Dial resulting from a refractive index of 1.5. The black half-loops mark the standard hour lines between winter solstice and summer solstice and the grey half-loops those between summer solstice and winter solstice.

Figure 6 shows a comparison of the dials for a classical sundial (n = 1) and our device (for n = 1.5). The image corresponds to the matt screen being viewed from the side opposite to the Sun. The hour lines in the lower half of the dial are fully functional, indicating time during the sunshine. They correspond to a position of the Sun above the mathematical horizon. The hour lines in the upper half of the dial circle correspond to a position of the Sun below the horizon. Thus, this portion of the hour scale is not functional as a sundial. It is included to portray the Earth-Sun system as a celestial-mechanical clock which visually shows the relationship among the time of day, the seasons of the year, and the relative position of the Sun, the Earth, and the sundial in a practically self-explanatory way, very much like a planetarium does, but with no moving parts except for the astronomical bodies.

The prototype of the new sundial is shown in Fig. 6. A plane-parallel plate with a refractive index of 1.49 was used for its construction. Continental features are not shown on this model: the Earth and its position in space appear solely through the shape and orientation of the hour lines. The sundial shows solar time with an accuracy of ±1 min in the central area and somewhat less near the edges. This is the accuracy expected for a small sundial.

V. CONCLUSION

A novel sundial has been presented. The novelty includes the realization of elliptical hour lines portraying the Earth and its network of longitude lines, as part of a miniature cosmic model in two dimensions. The underlying framework is based on a modification of Thales’ gnomonic projection that uses Snell’s law of refraction to redirect the projection lines. The appearance of the hour lines depends on the refractive index n of a plane-parallel plate inserted between the device casting the shadow and the dial surface. For n → 1, the hour lines converge to those of the most popular sundial of modern times: the classical sundial. For n → ∞, they converge to an image of the most popular sundial of Antiquity: the spherical sundial. Between these limits, there is a plethora of possibilities to configure a sundial for various values of n. The larger the n value is, the more the lines of the sundial resemble a familiar orthographic projection of Earth. The smaller the n value is, the more they resemble a gnomonic projection of Earth. With its attractive and intuitive hour lines, the proposed sundial is likely to become a worthy new member of the family of sundials.
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APPENDIX: REFRACTION OF AN HOUR PLANE

The ensemble of all rays of light traveling in a plane \( h \), incident at point \( O \) on a plane-parallel plate with refractive index \( n \), forms an elliptical cone within the plate. In the following, this assertion is proven. Light is assumed to travel according to the laws of geometrical optics. Without loss of generality, the plane-parallel plate can be considered of unit thickness.

The assertion is proven in two steps. We begin by constructing a sphere \( s \) of unit radius centered at \( O \) and consider a ray of light that travels from \( S \), the Sun, to \( O \), the entry point to the plane-parallel plate (see Figs. 3 and 7) The angle between \( SO \) and the \( z \)-axis perpendicular to the plane-parallel plate is \( \theta \). The \( x \)-axis is laid on the intersection of \( h \) and the front face of the plane plate (Fig. 3). In the absence of refraction, the ray would continue from \( O \) to a point \( P \) on \( s \) without changing the direction. With refraction, the ray of light deflects and travels from \( O \) to the point \( R \) on \( s \) and, from there, continues straight to \( T \) on the base \( p \) of the plane-parallel plate (Fig. 7(a)). On the proposed sundial, this point \( T \) is a point on the hour line corresponding to the hour plane \( h \).

The projections of \( O \), \( P \), and \( R \) onto the base of the plane-parallel plate are denoted by the same respective symbols with a prime (Fig. 7(b)). The projection of the intersection of the sphere \( s \) and the plane \( h \) is an ellipse \( L_0 \) with the semi-major axis of length 1 and the semi-minor axis of length \( \sin \theta_0 \), where \( \theta_0 \) is the angle of steepest incidence for a ray in the plane \( h \). The right half of \( L_0 \) is the locus of all points \( P' \) which results from arbitrarily varying the position of \( S \) in \( h \). Knowing \( L_0 \), we can construct the locus \( L_1 \) of all points \( R' \) derived from \( P' \). Since \( R' \in O'P' \) and \( u := |O'R'| = \sin \theta' = (1/n) \sin \theta \) by virtue of Snell’s law, \( L_1 \) is an ellipse congruent to \( L_0 \), down scaled by \( 1/n \) with respect to the latter. However, \( L_1 \) is the hour line of \( h \) on the surface \( s \), not on the surface \( p \), which we require. The hour line on \( p \) is the set of all points \( T \) derived from \( R' \). We denote it \( L_2 \). By analysis of panel A, we observe that \( T \in O'P' \) and \( v := |O'T| = \tan \theta' \).

In the second step, we prove that \( L_2 \) is, like \( L_1 \), an ellipse and that it is not congruent to \( L_0 \). To show this, we note that in polar coordinates \((r, \phi)\), the equation of an ellipse aligned with the coordinate system and centered on its origin takes the following form:

\[
\frac{1}{r^2} = \frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2}.
\]

Since \( L_1 \) is an ellipse satisfying these conditions (for reasons of symmetry, the center of \( L_1 \) coincides with \( O' \) and the \( y \)-axis is a symmetry axis), we have

\[
\frac{1}{u^2} = \frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2},
\]

where \( a = 1/n \) and \( b = (\sin \theta_0)/n \) denote the lengths of the respective half-axes in the \( x \) and \( y \)-directions. In addition, we know from Snell’s law of refraction and from trigonometry that for \( 0 \leq \theta' \leq \pi/2 \),

\[
u = \sin \theta' = \frac{1}{n} \sin \theta; \quad \sin \theta = n \sin \theta';
\]

\[
v = \tan \theta' = \frac{u}{\sqrt{1 - u^2}}. \quad \text{(A4)}
\]

Using these two expressions, \( 1/v^2 \) can be written as

\[
\frac{1}{v^2} = \frac{1 - u^2}{u^2} = 1 - \frac{1}{u^2} = 1 - \frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2} - 1. \quad \text{(A5)}
\]

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**Fig. 7.** Views of the path of a ray of sunlight incident at \( O \) on a transparent plane-parallel plate with refractive index \( n \). (a) Side view in the plane defined by the incident ray (from \( S \) to \( O \)) and the refracted ray (from \( O \) to \( T \)). (b) Top-view looking down from the \( z \)-axis onto the base \( p \) of the plane-parallel plate.
The last expression is indeed one of an ellipse with half-axes $a'$ and $b'$ provided that $n > 1$. Thus, $L_2$ is an ellipse. Its size scales proportionally to the plate thickness (Fig. 7(a)). This proves the assertion.