Universality in invariant random-matrix models: Existence near the soft edge

E. Kanzeper and V. Freilikher

The Jack and Pearl Resnick Institute of Advanced Technology,
Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel
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We consider two non-Gaussian ensembles of large Hermitian random matrices with strong level confinement and show that near the soft edge of the spectrum both scaled density of states and eigenvalue correlations follow so-called Airy laws inherent in Gaussian unitary ensemble. This suggests that the invariant one-matrix models should display universal eigenvalue correlations in the soft-edge scaling limit.

I. INTRODUCTION

Unitary-invariant random matrix models appear in many physical theories including nuclear physics, string theory, quantum chaos, and mesoscopic physics. They are completely defined by the joint distribution function

$$P[H] = \frac{1}{Z_N} \exp \{-Tr V[H]\}$$

of the entries of the $N \times N$ Hermitian matrix $H$. In Eq. (1) the function $V[H]$ is referred to as “confinement potential”, and it should provide existence of the partition function $Z_N$. Remarkable feature of this random matrix model is that, under certain conditions, a particular form of the confinement potential exerts no influence on the local eigenvalue correlations in the bulk scaling limit.

More precisely, there is a class of strong even confining potentials $V(\varepsilon)$, increasing at least as fast as $|\varepsilon|$ at infinity, for which the two-point kernel in the bulk of the eigenvalue spectrum follows the $\sin\varepsilon$ form in the large-$N$ limit [1,2]:

$$K_{\text{bulk}}(s,s') = \frac{\sin|s-s'|}{\pi|s-s'|}.$$  \hspace{1cm} (2)

This striking property, known as local universality, leads to the conclusion about universality of arbitrary $n$-point correlation functions $R_n(s_1,\ldots,s_n) = \det[K(s_i,s_j)]_{i,j=1,\ldots,n}$ ($n > 1$) in the local regime. In contrast, the global characteristics of eigenspectrum, like density of states or one-point Green’s function, display a great sensitivity to the details of confinement potential [2].

Less is known about eigenvalue correlations near the soft edge which is of special interest in the matrix models of 2D quantum gravity [3]. In the early study [3] the behavior of the density of states near the tail of eigenvalue support has been explored. Authors of Ref. [3] showed that there is a universal crossover from a non-zero density of states to a vanishing density of states which is independent of confining potential in the soft-edge scaling limit. Whereas the universal behavior of the density of states in the soft-edge scaling limit has been proven, the (supposed) universality of $n$-point correlations was not yet considered.

The problem we address in this work is: Whether the eigenvalue correlations in ensembles of large random matrices also possess a universal behavior in the soft-edge scaling limit? To provide an answer to this question we first quote some results for Gaussian unitary ensemble (GUE) that has received the most study near the soft edge, and then we turn to the consideration of eigenspectra of two strongly non-Gaussian ensembles of random matrices associated with quartic and sextic confining potentials.

In the soft-edge scaling limit, GUE is characterized by the Airy two-point kernel [2]

$$K_{\text{GUE}}(s,s') = \frac{\text{Ai}(s)\text{Ai}'(s') - \text{Ai}(s')\text{Ai}'(s)}{s-s'},$$  \hspace{1cm} (3)

whose spectral properties have got the detailed study in Ref. [6]. As a consequence of Eq. (3), the scaled density of states, unlike in the case of bulk scaling limit, cannot already be taken as being approximately constant, and changes in accordance with Airy law:

$$\nu_{\text{GUE}}(s) = \left(\frac{d}{ds} \text{Ai}(s)\right)^2 - s[\text{Ai}(s)]^2$$  \hspace{1cm} (4a)

with asymptotes

$$\nu_{\text{GUE}}(s) = \begin{cases} \frac{\sqrt{|s|}}{\pi} \frac{\cos(4|s|^{3/2}/3)}{4|s|^{3/2}/3}, & s \to -\infty, \\ \frac{1}{s^{3/2}} \exp\left(-4s^{3/2}/3\right), & s \to +\infty. \end{cases}$$  \hspace{1cm} (4b)

Our following treatment of non-Gaussian random matrix ensembles with strong level confinement will be built upon the orthogonal polynomial technique [3] allowing to express the two-point kernel for the random-matrix ensemble defined by Eq. (1) through the polynomials $P_n(\varepsilon)$ orthogonal on the whole real axis with respect to the weight $\exp\{-2V(\varepsilon)\}$. We fix the polynomials $P_n$ satisfying the three-term recurrence formula

$$\varepsilon P_n = a_{n+1} P_{n+1} + a_n P_{n-1}$$  \hspace{1cm} (5)
to be orthonormal,
\[ \int_{-\infty}^{+\infty} d\varepsilon P_n(\varepsilon) P_m(\varepsilon) \exp\left\{-2V(\varepsilon)\right\} = \delta_{nm}. \] (6)

Under these conditions the two-point kernel reads as
\[ K_N(\varepsilon, \varepsilon') = a_N \psi_N(\varepsilon') \psi_{N-1}(\varepsilon) - \psi_N(\varepsilon) \psi_{N-1}(\varepsilon') \over \varepsilon' - \varepsilon, \] (7)
where \[ a_N = k_{N-1}/k_N \] \([k_N \text{ is a leading coefficient of the orthogonal polynomial } P_N(\varepsilon)\)], and the “wavefunctions” \( \psi_N(\varepsilon) = P_N(\varepsilon) \exp\left\{-V(\varepsilon)\right\} \) have been introduced. Inasmuch as our concern is with the matrices of large dimensions, \( N \gg 1 \), the only asymptotics of the “wavefunctions” \( \psi_N(\varepsilon) \) are needed, and also a meaningful scaling limit should be constructed. Quite generally, this can be done by passing from initial energy variable \( \varepsilon \) to a new scaled variable \( s \) that remains finite as \( N \to \infty \): \( \varepsilon = s (N, s) = \varepsilon_s \). Then the scaled two-point kernel is determined by the formula
\[ K(s, s') = \lim_{N\to\infty} K_N(\varepsilon_s, \varepsilon_s') \frac{d\varepsilon_s}{ds}. \] (8)

II. QUARTIC CONFINEMENT POTENTIAL

We choose the quartic confinement potential in the form \( V(\varepsilon) = {1\over 2} \varepsilon^4 \). In this case the differential equation for \( \psi_n(\varepsilon) \) \([\text{index } n \text{ is arbitrary positive integer}]\) can be obtained by the Shohat’s method \([11][12]\):
\[ \frac{d^2}{d\varepsilon^2} \psi_n(\varepsilon) - \left[ \frac{d}{d\varepsilon} \ln \varphi_n(\varepsilon) \right] \frac{d}{d\varepsilon} \psi_n(\varepsilon) + Q_n(\varepsilon) \psi_n(\varepsilon) = 0, \] (9a)
\[ \varphi_n(\varepsilon) = a^2_{n+1} + a^2_n + \varepsilon^2, \] (9b)
\[ Q_n(\varepsilon) = \left( 6\varepsilon^2 - 4\varepsilon^4 - \frac{4\varepsilon^4}{\varphi_n(\varepsilon)} \right) + 4a^2_n \]
\[ \times \left( 4\varphi_n(\varepsilon)\varphi_{n-1}(\varepsilon) + 1 - 4a^2_n\varepsilon^2 - 4\varepsilon^4 - \frac{2\varepsilon^2}{\varphi_n(\varepsilon)} \right). \] (9c)
Here \( a_n \) is the recursion coefficient entering the corresponding three-term recurrence formula for the given set of orthogonal polynomials. Also, the following exact relation takes place:
\[ \psi'_{n-1}(\varepsilon) = \frac{\psi'_n(\varepsilon) + \psi_n(\varepsilon) [V'(\varepsilon) + 4\varepsilon a^2_n]}{4a_n \varphi_n(\varepsilon)}. \] (10)

Thereafter we shall be interested in the behavior of the wavefunction \( \psi_n \) near the soft band edge \( D_n \) in the limit \( n = N \gg 1 \). In this case the endpoint of the spectrum \( D_N = 2a_N \), where \([13]\)
\[ a_N = \left( \frac{N}{12} \right)^{1/4} \left[ 1 + \mathcal{O}(N^{-2}) \right], \] (11)
and,
\[ \varphi_N(\varepsilon) = 2a^2_N + \varepsilon^2 + \mathcal{O}(N^{-1/2}). \] (12)

Let us move the spectrum origin to its endpoint \( D_N \), making replacement \( \varepsilon = D_N + t \), and denote \( \hat{\psi}_N(t) = \psi_N(\varepsilon - D_N) \). It is straightforward to show that this function obeys equation
\[ \frac{d^2}{dt^2} \hat{\psi}_N(t) - 18D^2_N t \cdot \hat{\psi}_N(t) = 0 \] (13)
in the asymptotic limit \( N \gg 1 \). When deriving we supposed the characteristic energy scale \( t_v(N) = \left[ d \ln \hat{\psi}_N(t)/dt \right]^{-1} \) of the variation of \( \hat{\psi}_N(t) \) to be much smaller than the band edge \( D_N \).

Solution to Eq. (13) can be written through the Airy function \( y(x) = \text{Ai}(x) \) satisfying the differential equation
\[ y''(x) - xy(x) = 0; \]
\[ \hat{\psi}_N(t) = \lambda_N \text{Ai} \left( t \cdot (18D^2_N)^{1/3} \right). \] (14)

One can check that the condition \( t_v(N) \sim \mathcal{O}(N^{-5/12}) \ll D_N \) is fulfilled. The coefficient \( \lambda_N \) entering Eq. (14) still remains unknown.

To compute the two-point kernel, Eq. (7), we have to correctly determine the asymptotic behavior of the \( \hat{\psi}_{N-1}(t) \). This can be done by means of the asymptotic analysis of the exact relation Eq. (10), which in the large-

limit comes down to
\[ \hat{\psi}_{N-1}(t) = \hat{\psi}_N(t) + \frac{1}{3D^2_N} \frac{d}{dt} \hat{\psi}_N(t). \] (15)

It is convenient to define the soft-edge scaling limit as
\[ \varepsilon_s = D_N + \frac{s}{(18D^2_N)^{1/3}}. \] (16)

Then the two-point kernel, Eq. (7), and the density of states \( K(\varepsilon_s, \varepsilon_s) \), are given by the formulas
\[ K(\varepsilon_s, \varepsilon_s') = \lambda^2_N \left( \frac{3}{2} D^3_N \right)^{1/3} K_{\text{GUE}}(s, s'), \] (17a)
and
\[ \nu(\varepsilon_s) = \lambda^2_N \left( \frac{3}{2} D^3_N \right)^{1/3} \nu_{\text{GUE}}(s), \] (17b)
respectively. The latter expression provides a possibility to determine the unknown constant $\lambda_N$ by fitting the soft-edge density of states, Eq. (17b), to the bulk density of states [4]

$$
\nu_{bulk}(\varepsilon_s) = \frac{D_N}{\pi} \sqrt{1 - \left( \frac{\varepsilon_s}{D_N} \right)^2} \left[ 1 + 2 \left( \frac{\varepsilon_s}{D_N} \right)^2 \right]^{1/2} (18)
$$
taken near the endpoint of the spectrum, Eq. (16), provided $1 \ll s \ll D_N^{5/3}$. Equations (18), (17b), (16), and (4b) yield the value $\lambda_N^2 = (12D_N)^{1/3}$. Now, making use of the Eqs. (17a), (16), and (8), we arrive at the following expression for the two-point kernel in the soft-edge scaling limit:

$$
K_{soft}(s,s') = K_{GUE}(s,s').
$$

Thus, we conclude that the two-point kernel and the density of states, computed for the random matrix ensemble with quartic confining potential in the soft-edge scaling limit, coincide exactly with those for GUE.

### III. Sextic Confinement Potential

Now we turn to another ensemble of random matrices which is characterized by the confinement potential $V(\varepsilon) = \frac{\varepsilon}{12} \varepsilon^6$. Corresponding wavefunctions $\psi_n$ satisfy the same differential equation, Eq. (9a), but with [4]

$$
Q_n(\varepsilon) = -\frac{1}{4} \varepsilon^4 + \frac{5}{2} \varepsilon^4 - \frac{1}{2} \varepsilon^6 \left[ \frac{d}{d\varepsilon} \ln \varphi_n(\varepsilon) \right]
$$

$$
+ a_n^2 \varphi_n(\varepsilon) \varphi_{n-1}(\varepsilon) - \left( \varepsilon^5 + \pi_n(\varepsilon) - \frac{d}{d\varepsilon} \right) \pi_n(\varepsilon)
$$

$$
- 2\varepsilon \varphi_n(\varepsilon) \varphi_{n-1}(\varepsilon) (2\varepsilon^2 + a_n^2 + a_{n+1}^2),
$$

$$
\pi_n(\varepsilon) = a_n^2 \varepsilon (a_{n-1}^2 + a_n^2 + a_{n+1}^2 + \varepsilon^2),
$$

and

$$
\varphi_n(\varepsilon) = a_{n+1}^2 (a_n^2 + a_{n+1}^2 + a_n^2)
$$

$$
+ a_n^2 (a_{n+1}^2 + a_n^2 + a_{n-1}^2) + \varepsilon^2 (a_{n+1}^2 + a_n^2 + \varepsilon^2).
$$

For $n = N \gg 1$ the recursion coefficient [15]

$$
a_N = \left( \frac{N}{10} \right)^{1/6} \left[ 1 + \mathcal{O}(N^{-2}) \right],
$$

and

$$
\varphi_N(\varepsilon) = 6a_N^4 + \varepsilon^2 (2a_N^2 + \mathcal{O}(N^{-1/2})),
$$

$$
\pi_N(\varepsilon) = a_N^2 \varepsilon (2a_N^2 + \mathcal{O}(N^{-1/3})).
$$

Introducing the shifted energy variable, $\varepsilon = D_N + t$, we are able to rewrite the differential equation (9a) for the function $\psi_N(t) = \psi_N(\varepsilon - D_N)$ in the form

$$
\frac{d^2}{dt^2} \hat{\psi}_N(t) - \frac{225}{128} D_N^9 t \cdot \hat{\psi}_N(t) = 0,
$$

assuming that the characteristic energy scale $t_v(N) = \left[ \frac{d \ln \hat{\psi}_N(t)/d\varepsilon}{d\varepsilon} \right]^{-1}$ of the variation of $\hat{\psi}_N(t) = \psi_N(\varepsilon - D_N)$ is much smaller than the band edge $D_N$.

Solution of Eq. (25) takes the form

$$
\hat{\psi}_N(t) = \lambda_N \psi_N \left( \frac{225D_N^9}{128} t \right),
$$

with coefficient $\lambda_N$ that will be determined later by the same fitting arguments. The assumption $t_v(N) \ll D_N$ is obviously fulfilled.

To get the asymptotic behavior of $\hat{\psi}_{N-1}(t)$ in the large-$N$ limit, we simplify Eq. (21) to

$$
\hat{\psi}_{N-1}(t) = \hat{\psi}_N(t) + \frac{16}{15D_N^3} \frac{d}{dt} \hat{\psi}_N(t).
$$

It is convenient to define the soft-edge scaling limit as

$$
\varepsilon_s = D_N + s \frac{128}{225} D_N^{1/3},
$$

Then the two-point kernel is

$$
K(\varepsilon_s,\varepsilon_s') = \lambda_N^2 D_N^2 \left( \frac{15}{32} \right)^{1/3} K_{GUE}(s,s'),
$$

while the density of states takes the form

$$
\nu(\varepsilon_s) = \lambda_N^2 D_N^2 \left( \frac{15}{32} \right)^{1/3} \nu_{GUE}(s).
$$

The fitting arguments, based on the expansion of the Eq. (29b) and of the bulk density of states [4]

$$
\nu_{bulk}(\varepsilon_s) = \frac{D_N^5}{16\pi} \sqrt{1 - \left( \frac{\varepsilon_s}{D_N} \right)^2},
$$
\[ 3 + 4 \left( \frac{\varepsilon_s}{D_N} \right)^2 + 8 \left( \frac{\varepsilon_s}{D_N} \right)^4 \]  

(30)

near the soft edge \(1 \ll s \ll D_N^3\), yield \(\lambda_N^2 = D_N \left( \frac{15}{4} \right)^{1/3}\). Combining Eqs. (29a), (28), and (8), we end with the following expression for the two-point kernel in the soft-edge scaling limit:

\[ K_{\text{soft}}(s, s') = K_{\text{GUE}}(s, s'). \]  

(31)

This formula demonstrates that in the soft-edge scaling limit the eigenlevel properties for the random matrix ensemble with sextic confinement potential are determined by the same Airy law which is inherent in GUE.

IV. CONCLUDING REMARKS

We have considered for the first time the correlations of the eigenlevels near the soft edge for two strongly non-Gaussian ensembles of large random matrices possessing unitary symmetry, and associated with quartic and sextic confinement potentials. Our treatment has been based on the analysis of the second-order differential equations for the corresponding “wavefunctions” near the soft edge. In both cases it was found that correlations between appropriately scaled eigenvalues are universal, and characterized by the Airy two-point kernel Eq. (3) which previously has been found for GUE.

Together with the fact of universal behavior of the density of states, previously proven in Ref. [3], the consideration presented gives a strong impression that spectral correlations in invariant ensembles of large random matrices with rather strong and monotonous confinement potential are indeed universal near the soft edge.

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