Compilation of relations for the antisymmetric tensors defined by the Lie algebra cocycles of $\text{su}(n)$

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Abstract

This paper attempts to provide a comprehensive compilation of results, many new here, involving the invariant totally antisymmetric tensors (Omega tensors) which define the Lie algebra cohomology cocycles of $\text{su}(n)$, and that play an essential role in the optimal definition of Racah-Casimir operators of $\text{su}(n)$. Since the Omega tensors occur naturally within the algebra of totally antisymmetrised products of $\lambda$-matrices of $\text{su}(n)$, relations within this algebra are studied in detail, and then employed to provide a powerful means of deriving important Omega tensor/cocycle identities. The results include formulas for the squares of all the Omega tensors of $\text{su}(n)$. Various key derivations are given to illustrate the methods employed.
1 Introduction

A recent paper [1] gives a systematic account of the invariant symmetric and skewsymmetric primitive tensors that may be constructed on a compact simple Lie algebra \( \mathfrak{g} \) of rank \( l \). The new family of symmetric tensors introduced in [1] allows the direct construction of the \( l \) primitive Racah-Casimir operators for \( \mathfrak{g} \); the antisymmetric tensors determine the \( l \) primitive Lie algebra cohomology cocycles of \( \mathfrak{g} \). We refer e.g. to [1, 2] and references therein for the definitions and explanation of the significance of the invariant skewsymmetric tensors

\[
\Omega^{(2m-1)}_{i_1i_2\ldots i_{2m-1}} \equiv \Omega_{i_1i_2\ldots i_{2m-1}} \quad (i_1, i_2 \ldots \in \{1,2,\ldots, \dim \mathfrak{g}\})
\]

that are associated with the primitive cocycles of \( \mathfrak{g} \), of \( l \) different orders \( q = 2m - 1 \), where \( m \) is the order of the associated Racah-Casimir operators. The \( l \) allowed values of \( q \) for the different Lie algebra cohomology groups of each \( \mathfrak{g} \) are well-known (see e.g. [1, 2] for tables and further references). For \( su(n) = A_l \), \( l = n - 1 \), \( m = 2,3,\ldots,n \), the cocycles \( \Omega^{(2m-1)} \) have orders \( q = 3,5,\ldots,2m - 1 \) and indices \( i_1,\ldots,i_{(2m-1)} = 1,\ldots,n^2 - 1 \). Each
tensor (1) is used in [1] to define exactly one member of a family of maximally traceless (Sec.2) fully symmetric tensors $t$. These have a very favourable status within the set of totally symmetric tensors, because each of them can be used to define one primitive Racah-Casimir operator for $\mathfrak{g}$, $l$ in all, and no more. For example, if one specialises to $su(3)$ the general $su(n)$ definition of an $m$-th order tensor $t^{(m)}$ of our favoured family of symmetric tensors, one finds that the definition collapses to zero for any $m > 3$, in keeping with the fact that $su(3)$ has no primitive Racah-Casimir operators of any order greater than three.

These matters are fully discussed and illustrated in [1, 3], which pay especial attention to the case of the $SU(n)$ group. However, if one wants to make application of tensors like (1), for example, in the construction of higher supercharges [4] and in the quantum mechanics of particles with $SU(n)$ colour, or in the definition of BRST-like operators and higher order Hodge analysis [5], one finds the need for more identities involving them than are to be found in [1] or elsewhere. There are also other areas in which good control over the properties of the Omega-tensors/cocycles (1) is valuable. One of these is in the discussion of multi-bracket generalisations of Lie algebras [2] and higher order linear Poisson structures of the type introduced in [4]. Another is in the construction of Wess-Zumino terms for effective actions in space-times of various dimensions (see [4, 8, 9] and references therein) and, in general, in the group theory factors that may appear in particle physics. A recent study of this last subject is [10].

The aim in this paper is to collect all that we currently know regarding the properties of and the identities involving the $su(n)$ algebra skewsymmetric Omega tensors (see (9)–(12) below). Our approach divides itself into three stages. The first presents a discussion of the Omega tensors of $su(n)$ that sets out from their definitions, and utilises only the properties of the $f$- and $d$-tensors [11, 1] to deliver its output, which is then employed in the second stage. This stage is based on the fact that the Omega tensors play a central role in the discussion of the algebra of totally antisymmetrised products of an even or odd number of lambda-matrices of $su(n)$

$$\lambda_{[ijk...s]} = \lambda_{i}^{j} \lambda_{j}^{k} \lambda_{k}^{s} \ldots \lambda_{s}^{i}, \quad i, j, \ldots, s = 1, \ldots, n^{2} - 1,$$

and accordingly we provide an extensive discussion of results within this algebra. The completeness properties and trace identities for such products thus obtained then give rise to the powerful approach contained in the third stage of our programme, which allows us to derive, amongst other results, one of special importance. Defining the fully contracted scalar

$$(\Omega^{(2s+1)})^{2} = \Omega_{i_{1}j_{1}...i_{s}j_{s}k} \Omega_{i_{1}j_{1}...i_{s}j_{s}k} ,$$

we derive the recursion relation (195)

$$(\Omega^{(2s+1)})^{2} = \frac{4}{2s(2s-1)} (n^{2} - s^{2}) (\Omega^{(2s-1)})^{2} ,$$

and its explicit solution (196).

The content of this paper is organised as follows. Sec. 2 gives the basic definitions of Omega tensors. Then Sec. 3 gives results, within the first stage of our study, classified roughly according to types, e.g. Jacobi or $Ad$-invariance results, contractions including those with the structure constants $f_{ijk}$ of $su(n)$ (that define the three-cocycle $\Omega^{(3)}$), recursive relations, duality relations, product relations, uniqueness questions for antisymmetric tensors, etc. Some derivations (of formulas not derived elsewhere) are given in Sec. 4 in order to illustrate methods employed within stage one of our work. In Sec. 5, we turn
to the development of the algebra of the quantities (2), using results for Omega tensors deduced independently of it. This then enables the attack in Sec. 6 on the identities (3) and (4), using completeness and trace properties of the products (2) found in Sec. 5. Within this approach some auxiliary results are merely quoted in Sec. 6, and proved in Sec. 7. Some critical questions raised within the discussion in Sec. 3.7 of the uniqueness of Omega tensors are answered in Sec. 8, using lambda-matrix methods.

There have been in the literature very many discussions of the invaritensors, symmetric and skewsymmetric, associated with a simple Lie algebra \( g \) of rank \( l \). A significant recent one is [10]; lists of references are given in [1, 2]. However such studies often do not consider the full set of invariant tensors for \( g \), neglecting the \((l - 1)\) higher order Lie algebra cocycles \( i.e., \) the invariant higher order antisymmetric Omega tensors. Our paper [3] emphasises the crucial role these Omega tensors play not only in the method we advocate to define Racah-Casimir operators, but also in our discussion of their eigenvalues and the corresponding generalised Dynkin indices. One additional motivation for the present paper is to make readily available a comprehensive listing of results involving Omega tensors that are needed for that programme.

## 2 Definitions of \( su(n) \) \( d, \Omega \) and \( t \) tensors

We start with a family of symmetric invariant tensors, the \( d \)-family. It is easily defined recursively [12] starting from the standard Gell-Mann totally symmetric tensor \( d_{ijk} \) (see eq. (110)). First, one constructs

\[
d^{(r+1)}_{i_1 \ldots i_{r+1}} = d^{(r)}_{i_1 \ldots i_{r-1} j} d^{(3)}_{ji i_{r+1}} , \quad r = 3, 4, \ldots .
\]

For \( r \geq 3 \), eq. (3) does not define totally symmetric tensors. The \( d \)-family of symmetric tensors is obtained by symmetrising over all free indices in (3) and hence is defined by

\[
d^{(r)}_{(i_1 \ldots i_r)} ,
\]

where the round brackets indicate symmetrisation with unit weight over the set of indices enclosed. This should be done as economically as possible, \( e.g., \)

\[
d^{(4)}_{ijkl} = \frac{1}{3}(d_{ijkt}d_{kilt} + d_{jktl}d_{kitl} + d_{kitl}d_{jkl}) .
\]

The lowest order symmetric tensor, the Cartan-Killing metric (since \( g \) is compact and the generators hermitian it will be taken as the unity) may be viewed as the order two member of the \( d \)-family (3), \( d_{ij} \equiv \delta_{ij} \). Since the iteration process (3),(4) can go on indefinitely, it is clear that not all tensors of the \( d \)-family are primitive, since for a simple algebra \( g \) of rank \( l \) there are only \( l \) invariant primitive symmetric tensors (or, equivalently, \( l \) primitive Racah-Casimir operators).

We now turn to the totally antisymmetric Omega tensors (1), referring to [1, 2] for an explanation of their cohomological origin. Thus we define

\[
\Omega^{(3)}_{ijk} \equiv f_{ijk} = f^{a}_{ij} d_{ak} ;
\]

\[
\Omega^{(5)}_{ijklm} \equiv \Omega_{ijklm} = f^{a}_{[ij} f^{b}_{kl]} d_{abm} ;
\]

\[
\Omega^{(7)}_{ijklmpq} \equiv \Omega_{ijklmpq} = f^{a}_{[ij} f^{b}_{kl} f^{c}_{mp} d^{(4)}_{(abcq)} ;
\]

\[
\Omega^{(9)}_{ijklmpqrs} \equiv \Omega_{ijklmpqrs} = f^{a}_{[ij} f^{b}_{kl} f^{c}_{mp} f^{d}_{qr} d^{(5)}_{(abcds)} ;
\]

\[
\Omega^{(11)}_{ijklmpqrsuv} \equiv \Omega_{ijklmpqrsuv} = f^{a}_{[ij} f^{b}_{kl} f^{c}_{mp} f^{d}_{qr} f^{e}_{su} d^{(6)}_{(abcdef)} .
\]
and so on. Here, the square brackets imply total unit weight antisymmetrisation over the set of indices which they enclose. The raising of indices is trivial from a metric point of view, and is usually used in this paper in order to exempt certain indices from the antisymmetrisation (or symmetrisation) effect of the square (round) brackets. We note that the $\Omega^{(2m-1)}$ tensor is fully skewsymmetric in its $(2m-1)$ indices in spite of the fact that only $(2m-2)$ indices are explicitly antisymmetrised in the r.h.s. above \[1\]. Some further discussion of the properties of the $d$-tensors and their role in the definitions of the Omega tensors is given below as Sec. 2.1.

Next we use the Omega tensors to define a second family of invariant fully symmetric tensors, the $t$-tensors, as follows \[1\]

$$\Omega_{ijm} f_{ija} = t^{(2)}_{am} ,$$  
$$\Omega_{ijklm} f_{ija} f_{klb} = t^{(3)}_{abm} ,$$  
$$\Omega_{ijklmpq} f_{ija} f_{klb} f_{mpc} = t^{(4)}_{abcq} ,$$  
$$\Omega_{ijklmpqrs} f_{ija} f_{klb} f_{mpc} f_{qrd} = t^{(5)}_{abcds} ,$$  

etc.; they are fully symmetric on account of the skewsymmetry of the $\Omega$'s. The tensors on the right hand side of (14)–(16) have been evaluated before \[1\]. We give below the expression of the lower order $su(n)$ $t$ tensors in terms of members of the $d$-family,

$$t^{(2)}_{ij} = n \delta_{ij} ,$$  
$$t^{(3)}_{ijk} = \frac{1}{3} n^2 d_{ijk} ,$$  
$$t^{(4)}_{ijkl} = \frac{1}{15} (n(n^2 + 1) d^{(4)}_{ijkl} - 2(n^2 - 4) \delta_{ij} \delta_{kl}) ,$$  
$$t^{(5)}_{ijklm} = \lambda(n) \left( n(n^2 + 5) d^{(5)}_{ijklm} - 2(3n^2 - 20) d_{ij} \delta_{kl} \delta_{im} \right) ,$$

where the function $\lambda(n)$, not determined in \[1\], turns out from the work of Sec. 6 to be given by

$$\lambda(n) = \frac{n^2}{105} .$$

The tensors (18) and higher collapse to zero when their order $m$ is larger than $n$. While Eq. (20) can be indeed be used as it stands (and will be to avoid circularity of argumentation below), much of the information we need will be seen to follow from the definitions (14)–(16) and the properties of Omega tensors.

The $t$-tensors are totally symmetric and, unlike the higher $(m > 3)$ order $d$-tensors, they are orthogonal to all other $t$-tensors of different order (Lemma 3.3 of \[1\]). For instance, for $t^{(4)}$

$$t^{(4)}_{ijkl} \delta_{ij} = 0 , \quad t^{(4)}_{ijkl} t^{(3)}_{ijk} = 0 .$$

In contrast, since trace formulas for $d$-tensors easily give

$$d^{(4)}_{ijkl} d_{ijm} = \frac{2}{3} (n^2 - 8) d_{klm} ,$$

the contraction of only two indices gives

$$t^{(4)}_{ijkl} t^{(3)}_{ijm} = \frac{1}{3} n^2 t^{(4)}_{ijkl} d_{ijm} = \frac{2}{45} n^2 (n^2 - 9) t^{(3)}_{klm} .$$

For $t^{(5)}$ we have

$$t^{(5)}_{ijklm} \delta_{ij} = 0 , \quad t^{(5)}_{ijklm} t^{(3)}_{ijk} = 0 , \quad t^{(5)}_{ijklm} t^{(4)}_{ijkl} = 0$$

\[1\]Eqs. (18) and (19) above correct the overall factors of (6.13) and (6.14) in \[1\]; the $i$ difference in \[1\] is due to the fact that here we take the generators of $g$ hermitian, see (128).
or, in other words, the maximal contraction of the indices of two \( t \)-tensors of different order is zero.

Another issue concerns the claims made above that \( t^{(4)} \) vanishes identically for \( n = 3 \), and that \( t^{(5)} \) vanishes identically for \( n = 3 \) and \( n = 4 \). This is a point one will see clearly illustrated in many places, and is the result of factors becoming zero or of relations expressing the \( d^{(m)} \) tensors in terms of primitive ones when \( m > n \). The necessary identities special to \( su(3) \) and \( su(4) \) are noted in [1]. Indeed almost all we need here in the way of identities involving the \( d \)- and \( f \)-tensors of \( su(n) \) is presented in [1], especially in the appendix. See also [11] and [13].

### 2.1 More about \( d \)-tensors

Below we shall need the identities that express the \( Ad \)-invariance of the \( d \)-tensors, namely,

\[
d_{t(ij)f}^{(4)}_{st} = 0 , \tag{26}
\]
\[
d_{t(ijk)f}^{(4)}_{st} = 0 , \tag{27}
\]
\[
d_{t(ijkt)f}^{(5)}_{st} = 0 , \tag{28}
\]

and so on.

One use of (26) is as follows. Referring first to (9), we note that the symmetry properties of the \( SU(n) \)-invariant \( d \)- and \( f \)-tensors permit the left-hand square bracket to be moved, without altering the definition, one place to the right, so as to enclose only \( j, k \) and \( l \). Then application of (26) allows one to show that the right side of (9) is antisymmetric under the interchange of \( i \) and \( m \), and hence, as mentioned, indeed defines a totally antisymmetric quantity.

Referring next to (10), we note that it may be simplified in either of two ways but not simultaneously both: one of these is the move of the left hand square bracket one place to the right (or alternatively the right hand one to the left), the other employs the result

\[
d_{t(abq)}^{(4)} = d_{t(abq)}^{(4)} , \tag{29}
\]

and thereby allows \( d_{t(abq)}^{(4)} \) to be replaced in (11) by one of the terms of \( d_{t(abq)}^{(4)} \), e.g. \( d_{x}d_{y}d_{z}d_{q} \). It is the use of (29) that deserves close attention. It displays a simplifying feature of the \( d^{(4)} \) situation that does not generalise systematically to \( d^{(r)} \) for \( r > 4 \). For \( r = 5 \), we have

\[
d_{t(abcdq)}^{(5)} = \frac{1}{5} d_{t(abq)}^{(5)} d_{t(cde)}^{(5)} + \frac{4}{5} d_{t(abcdq)}^{(5)} , \tag{30}
\]

where

\[
d_{t(abq)}^{(5)} d_{t(cde)}^{(5)} = d_{t(abq)}^{(5)} d_{t(cde)}^{(5)} , \tag{31}
\]
\[
d_{t(abcdq)}^{(5)} = d_{t(abq)}^{(5)} d_{t(cde)}^{(5)} d_{t(dq)}^{(5)} . \tag{32}
\]

Inspection of the evident tree-diagram representation of the tensors occurring here makes clear the fractions seen in (31).

In (30) we meet an obstacle to extending, to (12) and beyond, the simple proof that allowed the right hand round bracket in (10) to be moved one place to the left. It is nevertheless a generally allowed step, providing a valuable simplification of the definitions of \( \Omega^{2s+1} \) for \( s \geq 4 \). However, to obtain a convenient proof of this, we need to have recourse to lambda matrix methods, and so, will return to the matter in Sec.5.2.

Similarly

\[
d_{t(abcdq)}^{(6)} = \frac{1}{3} d_{t(abq)}^{(6)} d_{t(cde)}^{(6)} + \frac{2}{3} d_{t(abcdq)}^{(6)} , \tag{33}
\]
The fractions in the RHS of (33) arise because the tree diagrams in use are trees with four equivalent end twigs and two equivalent non-end twigs.

An additional complication enters for symmetric tensors of order six, one that has already been observed in [1]. The tensor \(d^{(6)}\) that enters the definition (11) of \(\Omega^{(11)}\) is the \(r = 6\) member of the family (6). But it is not the only primitive symmetric sixth order tensor that can be defined. One also has \(d^{(6)'}\) given by

\[
d^{(6)'}_{abcdef} = d_{ab}^{x}d_{cd}^{y}d_{ef}^{z}d_{xyz} .
\]  

(34)

The tensors \(d^{(6)}_{abcdef}\) and \(d^{(6)'}_{abcdef}\) are for our purposes equivalent. It is shown in [1] (below eq. (A.21) there) that they differ by non-primitive terms which are symmetrised products of lower order \(d\)-tensors. The claimed equivalence follows from the fact that such non-primitive terms cannot contribute to the definition (11) of \(\Omega^{(11)}\) because of Jacobi identities.

Inspection of the relevant tree diagram shows that

\[
d^{(6)'}_{abcdeq} = d^{(6)'}_{abcde} .
\]  

(35)

3 Identities involving the Omega tensors

These are mostly displayed for Omega tensors of lower order for obvious reasons. But one can often see patterns that would guide an attack on higher order analogues that may now seem out of reasonable reach, or perhaps just until the need of a specific application provides the necessary motivation. The trace methods for products of the hermitian \(D\)- and \(F\)-matrices [14], where \((D)_{jk} = d_{ijk}\) and \((F)_{jk} = -if_{ijk}\), such as are seen in use in the derivations presented in Sec. 4, become discouraging when one cannot avoid doing a trace that is more than of fourth order, unless one can harness computational skills like those of [10]. Also, finding a viable path through increasing complication becomes progressively more taxing. The necessity for going on in later sections to develop an alternative approach – that which makes systematic use of lambda matrices – come into evidence in this way.

3.1 Jacobi identities

First, we consider the Jacobi identities which express the \(Ad\)-invariance of the Omega tensors. The \(Ad\)-invariance of \(\Omega^{(3)}_{ijk} = f_{ijk}\) is expressed by the Jacobi identity,

\[
f_{t[ij}f_{k]lt} = f_{t[ij}f_{k]ls}\delta_{st} = 0 .
\]  

(36)

For higher \(\Omega\)’s \(Ad\)-invariance gives

\[
\Omega_{t[ijkl}f_{pq]st} = 0 ,
\]

(37)

\[
\Omega_{t[ijklpq}f_{rs]st} = 0 ,
\]

(38)

\[
\Omega_{t[ijklpqrs}f_{uv]st} = 0 .
\]  

(39)

In analogy with the second way of writing the Jacobi identity (36), we may usefully expand (37)–(39) in terms of the higher members of the \(d\)-family (6) getting

\[
f^{a}_{[ij}f^{b}_{kl}f^{c}_{pq}d_{abc} = 0 ,
\]  

(40)

\[
f^{a}_{[ij}f^{b}_{kl}f^{c}_{pq}f^{d}_{rs]}d^{(4)}_{abcd} = 0 .
\]  

(41)
In the last two results, obviously the right hand square bracket can be moved one place to the right. In the case of (41) this allows the round symmetrising brackets to be taken off $d^{(4)}$ since $f^{a}_{[ij}f^{b}_{kl}f^{c}_{pq}f^{d}_{rs]}$ is fully symmetric in $abcd$.

Also, using (38), eq. (41) can be rearranged to read

$$\Omega_{t[ijkl}\Omega_{pqrs]}_{st} = 0,$$

(42)

an expression that may be understood as the generalised Jacobi identity (GJI) for a higher (fourth) order multibracket algebra. In the general case, the GJI reads [2, 6]

$$\Omega_{t[i_1...i_{2m-2}}\Omega_{i_{2m-1}...i_{4m-3}}i_{4m-4}t = 0,}$$

(43)

This is an identity that follows directly if one takes the coordinates of $\Omega^{(2m-1)}$ as the generalised structure constants (with $(2m - 1)$ antisymmetric indices) of a multi-bracket Lie algebra of order $(2m - 2)$. As in the standard case (for $m = 2$, eq. (13) reduces to the JI, eq. (36)), only the associativity of the $(2m - 2)$ entries in the fully skewsymmetric multibracket is required to obtain eq. (13).

The importance of the Jacobi identities can hardly be overemphasised: they are essential to the simplification of other identities in all classes, as will be seen below. Certain other results which bear a close resemblance to (37)–(39) are also valid, namely

$$\Omega_{t[i_{1}...i_{r}}\Omega_{i_{r+1}...i_{r+s-1}}i_{r+s}t = 0 \quad (r \text{ and } s \text{ even}),}$$

(47)

and constitute consistency relations that must be satisfied by the generalised structure constants, and hence have the same origin as the GJI (see [14]). Here we have followed the converse path, showing that these relations follow from the definition of the $\Omega$ tensors.

It is also worth noting that many of the results of this section can be identified as cases of the general result Lemma 3.1 of [2]:

$$f^{p_1}_{i_{1}j_1}\cdots f^{p_s}_{i_{s}j_s}k_{(p_1\cdots p_s)} = 0,$$

(48)

where $k_{(p_1\cdots p_s)}$ is any $Ad$-invariant totally symmetric tensor of order $s$.

### 3.2 On the definition of the $\Omega$ tensors

Since we have introduced the Omega tensors using the recursively defined $d$-tensors (3), and then used the Omega tensors to obtain the preferred family of $t$-tensors (eqs.(13)–(16), (17)–(20) etc.), one might well ask why we did not need the latter in order to start the process off in the first place. The answer is that the non-primitive product terms that appear as the tails of the $t$-tensors cannot contribute to the Omega tensors at all in virtue of Jacobi identities of the type given in Sec. 3.1. In fact, non-primitive invariant symmetric tensors do not contribute to the $\Omega$ tensors, making the definitions (1)–(12)
unique (see [1], Cor. 3.1). For example (cf. (10)), there is no need to contemplate a contribution to $\Omega^{(7)}$ proportional to

$$f^a_{[ij} f^b_{kl} f^c_{mp} \delta_{(ab \delta_{eq})},$$

since it vanishes by eq. (36). To see that a similar state of affairs applies to a putative contribution to $\Omega^{(9)}$ like

$$f^a_{[ij} f^b_{kl} f^c_{mp} f^d_{qr} d_{(abc \delta_{ds})},$$

requires the use of both (36) and (40), depending upon where $s$ occurs in the five terms of the expansion of $d_{(abc \delta_{ds})}$. Considerations like those described often employ steps like $[ijklmp...] = [ijklm[p...]$; unit weighted brackets are convenient for such use. Thus apart from overall normalisation, replacing $d$-tensors (6) by $f$-tensors (see eq. (18)) in the definitions of the Omega tensors has of no effect since, by virtue of (48), the non-primitive parts in which they differ do not contribute.

### 3.3 Recursive identities

We note the important results relating $\Omega^{(5)}$ to $\Omega^{(7)}$ and $\Omega^{(7)}$ to $\Omega^{(9)}$ respectively

$$\Omega^{(7)}_{ijklmpq} = \Omega^{(5)}_{t[ijkl} f^s_{mp} d_{s]qt},$$

$$\Omega^{(9)}_{ijklmpqrs} = \Omega^{(7)}_{t[ijklmp} f^u_{qr} d_{u]st},$$

which are the two lowest versions of a general result ([1], eq. (7.6)). Having written these results, one sees that the definition of $\Omega^{(5)}$ provides the first member of the series, the identification

$$\Omega^{(3)}_{ijk} = f_{ijk},$$

having been noted already in (8).

Each of the two results just displayed can usefully be presented in a different form

$$\Omega^{(7)}_{ijklmpq} = f_{x[ij} \Omega^{(5)}_{klmp} y_{d_q]st},$$

$$\Omega^{(9)}_{ijklmpqrs} = \Omega^{(7)}_{x[ijklmp} f^u_{qr} y_{d_s]ut},$$

evident generalisations may be expected to hold. It is easy also to use Jacobi identities to show if one replaces $d$-tensors by $f$-tensors on the right sides of (54) and (55), one gets the answer zero.

### 3.4 Contraction of higher order Omega tensors with lower order ones

In view of the antisymmetry of the Omega tensors, it is clear that these are amongst the most important contractions to be considered. We find

$$f_{ijk} f_{ijl} = n \delta_{kl} = t^{(2)}_{kl},$$

$$\Omega_{tijkl} f_{ij} = \frac{1}{2} n f_{u[kl} d_{u]as},$$

$$\Omega_{tijkl} f_{ij} f_{klv} = \frac{1}{3} n^2 d_{tuv} = t^{(3)}_{tuv},$$

$$\Omega_{tijklpq} f_{ij} f_{klt} f_{pqw} = \frac{1}{15} (n(n^2 + 1) d^{(4)}_{(uv wt)} - 2(n^2 - 4) \delta_{(uv \delta_{wt})}) = t^{(4)}_{tuvw}.$$
The last two of course are just the definitions of \( t^{(3)} \) and \( t^{(4)} \) seen from a different viewpoint. We may contract (57) further, obtaining

\[
\Omega_{ijkl} f_{ijk} = 0
\]  

(60)

It is sometimes relevant to observe the absence of the display of formulas that might naively be guessed as, e.g., for \( \Omega_{ijklpq} f_{ij} \), cf. (57). This will not reduce simply to a multiple of \( \Omega_{ijklpq} d_{ijus} \), since in this case there are other quantities with the required symmetries available to complicate matters. Although a useful reduction can be achieved, the result is not clean enough to be displayed.

Families of more complicated but still useful contractions include the following

\[
\begin{align*}
\Omega^{(5)}_{ijkl} \Omega^{(7)}_{ijklpq} &= \frac{2}{15} (n^2 - 9) f_{ulpq} d_{rus} \\
\Omega^{(7)}_{ijklpq} \Omega^{(9)}_{ijklpqst} &= \frac{2}{105} (n^2 - 9)(n^2 - 16) f_{vst} d_{urs} \\
\Omega^{(5)}_{ijkl} \Omega^{(7)}_{ijklpq} f_{pq} &= \frac{2}{15} n^2 (n^2 - 9) d_{rst} \\
\Omega^{(7)}_{ijklpq} \Omega^{(9)}_{ijklpqst} f_{stv} &= \frac{4}{45} n^2 (n^2 - 9)(n^2 - 16) d_{uv}
\end{align*}
\]

where factors \((n^2 - 9)\) and \((n^2 - 16)\) reflect the respective facts that \( \Omega^{(7)} \) is absent for \( n = 3 \), and \( \Omega^{(9)} \) is absent for \( n = 4 \). Eq. (61) can be rewritten also as

\[
\Omega^{(5)}_{ijkl} \Omega^{(7)}_{ijklpq} = 2 \cdot \frac{1}{15} (n^2 - 9) f_{ij} \Omega^{(5)}_{ijklpq}
\]  

(65)

which is a recursion relation of sorts, one that be generalised along obvious lines, as indeed (61) itself has been in the production of (62). Eq. (63) is an easy consequence of (61), which also implies

\[
\Omega^{(5)}_{ijklpq} \Omega^{(7)}_{ijklpq} = 0
\]  

(66)

The same applies to (64) and (62), so that also

\[
\Omega^{(7)}_{ijklpqst} \Omega^{(9)}_{ijklpqst} = 0
\]  

(67)

Results such as (60), (66) and (67) are evident enough, since there is no \( SU(n) \)-invariant totally antisymmetric tensor of order two. They point strongly towards an analogue to the orthogonality result for the \( t \)-tensors, eqs. (14)–(16), and suggest that the maximal contraction of two Omega tensors of different order is zero. We do not have a general proof, but as questions regarding it arise it will be shown that this is indeed the case. Thus one might ask about the claim

\[
\Omega^{(7)}_{ijklpq} f_{pq} = 0
\]  

(68)

Using the methods of this section it is indeed possible, but not easy, to verify this by direct calculation. Alternatively, we may have recourse to the assertion that there exist no \( SU(n) \)-invariant totally antisymmetric tensors of order four. Similarly, eq. (62) below indicates the absence of any \( SU(n) \)-invariant totally antisymmetric tensors of rank six. It follows that we may write

\[
\Omega^{(9)}_{ijklpqrst} f_{rst} = 0
\]  

(69)

and similarly

\[
\Omega^{(11)}_{ijklpqrstuv} \Omega^{(5)}_{ijklp} = 0
\]  

(70)

Such arguments fail for the proof of

\[
\Omega^{(11)}_{ijklpqstuv} f_{tuv} = 0
\]  

(71)
because for \(su(n)\) with \(n > 5\) (so that \(\Omega^{(11)}\) exists) there is a (non-primitive) totally antisymmetric \(SU(n)\)-invariant tensor of order eight. Such matters are discussed systematically in Sec 3.7, the eighth-order tensor being there displayed as \(\Omega^{(11)}\). Eq. (71) is nevertheless true, although we need the methods of later sections, lambda-matrix methods, to obtain a convenient approach (see Sec. 8) to its proof. Thus, as is known to be true for the \(t\)-tensors, so also it seems for the Omega tensors that the only invariants that can be built out of them are their fully contracted squares. We begin the task of evaluating these in the next paragraph.

### 3.5 Product identities

We begin with the results

\[
\begin{align*}
f_{ijs}f_{ijt} &= \phi_3(n)\delta_{st}, \\
\Omega_{ijkl}\Omega_{ijkl} &= \phi_5(n)\delta_{st}, \\
\Omega_{ijklpq}\Omega_{ijklpq} &= \phi_7(n)\delta_{st},
\end{align*}
\]

that define a family of quantities of which the first few members are

\[
\begin{align*}
\phi_3(n) &= n, \\
\phi_5(n) &= \frac{1}{3}n(n^2 - 4), \\
\phi_7(n) &= \frac{2}{15}(n^2 - 9)\phi_5(n).
\end{align*}
\]

These imply the consequences

\[
\begin{align*}
f_{ijs}f_{ijs} &= \psi_3(n), \\
\Omega_{ijkl}\Omega_{ijkl} &= \psi_5(n), \\
\Omega_{ijklpq}\Omega_{ijklpq} &= \psi_7(n),
\end{align*}
\]

where

\[
\begin{align*}
\psi_3(n) &= n(n^2 - 1), \\
\psi_5(n) &= \frac{1}{3}n(n^2 - 1)(n^2 - 4), \\
\psi_7(n) &= \frac{2}{15}n(n^2 - 1)(n^2 - 4)(n^2 - 9).
\end{align*}
\]

One can speculate with confidence on the extension of these results to higher Omega tensors. As is discussed in related contexts in \[\text{[1]}\], results collapse to zero equal zero for low values of \(n\), for which the relevant primitive cocycles do not exist. For example \(su(2)\) has no cocycle of order higher than three and \(su(3)\) none higher than five. The right hand sides of \((82)\) and \((83)\) have explicit factors zero for the corresponding \(n\)-values. The proofs of \((82)\) and \((83)\) are given in Sec. 4.

One often needs results more general than those so far mentioned. We have

\[
\begin{align*}
\Omega_{ijklpq}\Omega_{ijklrs} &= \frac{1}{6}((n^2 - 6)f_{pqf_{rst}} + n(\delta_{ps}\delta_{qs} - \delta_{ps}\delta_{qr})) , \\
\Omega_{ijklpq}\Omega_{ijklpq} &= \frac{2}{135}(n^2 - 9)((n^2 - 8)f_{qr}f_{stu} + 2n(\delta_{qs}\delta_{rt} - \delta_{qt}\delta_{rs})),
\end{align*}
\]

which imply \((3)\) and \((74)\), as they should. The right hand sides of \((84)\) and \((85)\) involve linear combination of the only two fourth order tensors with the correct symmetries (so that these are a basis in the vector space in question). The \(\Omega^{(7)}\) result collapses for \(n = 3\) (\(su(3)\) has no seven-cocycle) because of the explicit factor \(n^2 - 9\). Although there is no factor \((n^2 - 4)\) in \((84)\) causing it to collapse for \(su(2)\), the right side of \((84)\) nevertheless vanishes because, for \(su(2)\), we have \(f_{ijk} = \epsilon_{ijk}\), and \(\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kp} - \delta_{jq}\delta_{kp}\).
3.6 Duality results

We give results here mainly for $su(3)$ although in principle the analysis could be extended to higher $su(n)$.

We mention first results involving the totally antisymmetric eighth order epsilon tensor:

$$
\epsilon_{ijklmnpq} \epsilon_{ijklmnpq} = 8! , \\
\epsilon_{ijklmnpq} \epsilon_{ijklmnpq} = 7! \delta_{gt} , \\
\epsilon_{ijklmnpq} \epsilon_{ijklmnpq} = 6! (\delta_{ps} \delta_{qt} - \delta_{pt} \delta_{qs}) , \\
\epsilon_{ijklmnpq} \epsilon_{ijklmnpq} = 5! 3! (\delta_{p[} \delta_{r]} \delta_{q]}).
$$  \tag{86}

We note here that the factor 3! is present because the square brackets imply antisymmetrisation with unit weight. Next, from [1], we note

$$
12 \sqrt{3} \Omega_{ijklm} = \epsilon_{ijklmpqr} f_{pqr} , \tag{87}
20 \sqrt{3} f_{stu} = \epsilon_{ijklmstu} \Omega_{ijklm} . \tag{88}
$$

Again results more general than the above are often called for, as in [5]. From [5] we quote

$$
\epsilon_{ijklrstu} \Omega_{ijklm} = 16 \sqrt{3} \delta_{m[r} \Omega_{ijkl]} , \tag{89}
\epsilon_{ijklmpqr} f_{qrs} = 24 \sqrt{3} \delta_{s[p} \Omega_{ijklm]} . \tag{90}
$$

One may check that \eqref{87} implies \eqref{89}, and that \eqref{88} implies \eqref{87}. However to prove \eqref{88} one must insert \eqref{87} and use an identity from the family \eqref{86}. Similarly insertion of \eqref{88} allows proof of \eqref{87}. Further, one may use \eqref{89} to show that \eqref{88} follows from \eqref{87}.

For an $su(4)$ result, see Sec. 8 of [1]. Recent work of the authors [4] actually uses duality to obtain information about $\Omega^{(5)}$ for $su(5)$, having used MAPLE programs for data about the lower Omega tensors.

3.7 Non-primitive antisymmetric tensors

In Sec. 2, we defined for $su(n)$ its Omega tensors, which are a set of $l = (n - 1)$ primitive antisymmetric tensors of orders

$$
3, 5, 7, \ldots , (2n - 1) . \tag{91}
$$

We have described the fundamental role they play in the discussion of primitive Racah-Casimir operators of $su(n)$ (see also [4]) but they are not the only antisymmetric tensors that can be defined. One can form non-primitive, tilded tensors $\tilde{\Omega}$, as totally antisymmetrised products of primitive tensors $\Omega$, e.g.

$$
\tilde{\Omega}^{(8)}_{ijklpqrs} = \Omega^{(3)}_{[ijkl} \Omega^{(5)}_{pqrs]} , \tag{92}
$$

which for $su(3)$ is a multiple of the eigth order $\epsilon$-tensor. In terms of forms, eq. \eqref{92} determines a non-primitive de Rham cocycle on the $SU(n)$ group manifold (e.g., the volume form on $SU(3)$, ignoring factors). A more interesting example arises for $su(8)$, $l = 7$, which has seven Omega tensors of orders 3, 5, 7, 9, 11, 13, 15. In this case, one can form a second, tilded antisymmetric tensor of order 15

$$
\tilde{\Omega}^{(15)}_{i_1 \ldots i_{15}} = \Omega^{(3)}_{[i_1 i_2 i_3} \tilde{\Omega}^{(5)}_{i_4 \ldots i_8} \Omega^{(7)}_{i_9 \ldots i_{15}]} , \tag{93}
$$
which is non-primitive and not maximal on the \((n^2 - 1) = 63\)-dimensional space (manifold in the case of forms on \(SU(n)\)). The discussion of Sec. 5.1 suggests that it should have zero full contraction with \(\Omega^{(15)}_{i_1 \cdots i_{15}}\), in virtue of results like (68). This example illustrates a significant restriction on the construction of non-trivial non-primitive antisymmetric tensors: all the primitive Omega tensors used must have different orders. To see this, notice that we may write, \(e.g.\),

\[
\Omega^{(3)}_i = \Omega^{(3)}_i \omega^i \wedge \omega^j \wedge \omega^k, \quad (94)
\]

where the \(\omega^i\) are the left-invariant (LI) Maurer-Cartan one-forms on the \(SU(n)\) groups manifold, dual to the LI \(su(n)\) generators, so that \(\Omega^{(3)}\) is the invariant de Rham three-cocycle of coordinates \(\Omega^{(3)}_{ijk}\). Obviously, \(\Omega^{(3)} \wedge \Omega^{(3)} = 0\) and hence

\[
\Omega^{(3)}_{[ijk} \Omega^{(3)}_{pqr]} = f^{(3)}_{ijk} f^{(3)}_{pqr} = 0. \quad (95)
\]

In general, the skewsymmetrisation of two copies of the same Omega tensor is zero since this corresponds to taking the wedge product of a primitive \(SU(n)\) de Rham cocycle by itself, which is zero because all these cocyles are represented by odd, \((2m - 1)\)-forms on the \(SU(n)\) group manifold.

### 3.8 9-cocycle results

We used the ninth order Omega tensor to define the fifth order \(t\)-tensor \(t^{(5)}\), quoting the result \([1]\) for it as (20). This enables us to calculate

\[
\Omega^{(9)}_{ijklmnpqrt} \Omega^{(9)}_{ijklmnpqrt}, \quad (96)
\]

to within an overall normalisation constant. We may use (16) to derive

\[
\Omega_{ijklmnpqrt} \Omega_{ijklmnpqrt} = \Omega_{ijklmnpqrt} f^{(a)}_{ij} f^{(b)}_{kl} f^{(c)}_{pq} f^{(d)}_{rs} d^{(5)}_{(abcdm)} \]

\[
= t^{(5)}_{abcdm} d^{(5)}_{(abcdm)} \]

\[
= \lambda(n) \prod_{r=1}^{4} (n^2 - r^2). \quad (97)
\]

Here the first line uses the definition (12) and the second one uses (13). The last line may be evaluated from the second one using (20) and the two following results, the first of which is the last equation of the appendix in \([1]\), and the other is much more easily obtained:

\[
d^{(5)}_{(abcdm)} d^{(5)}_{(abcdm)} = \frac{1}{15n^4} (n^2 - 1)(n^2 - 4)(5n^4 - 96n^2 + 480), \quad (98)
\]

\[
d^{(5)}_{(abcdm)} \delta_{(abdec)} = \frac{1}{5n^2} (n^2 - 1)(n^2 - 4)(3n^2 - 20). \quad (99)
\]

Equation (97) already displays the essential factors anticipated in Sec. 3.5. We confirm its correctness in Sec. 6, using lambda-matrix techniques which allow the factor \(\lambda(n)\) to be determined, the result having been given above as (21).

### 4 Selected Derivations
4.1 Equations (84), (73) and (79)

The first result involving $\Omega^{(5)}$ that is not straightforward to derive from definitions is (84). We develop

$$
\Omega_{ijklm} \Omega_{ijkpq} = \Omega_{mijkl} \Omega_{qijkp} = f^a_m [i f^b_j k] d_{abl} f^x_q [i f^y_j k] d_{xyp} \quad .
$$

(100)

The second set of square brackets can now be dropped, leaving behind the sum of three sixth order products of $d$- and $f$-tensors to be reduced by trace methods using formulas from the appendix of [1]. Two of the three terms coincide, and nothing more than the evaluation of four-fold traces needs to be done. A rough graphical representation of any term helps (here and elsewhere) to see the best way to use trace formulas. In this vertices correspond in evident manner to $d$- and $f$-tensors, while closed loops indicate the traces.

The result comes out initially in terms of products of $\delta \delta$ and $dd$ terms. But there is an identity valid for all $su(n)$ ([11], eq. (2.10)) which allows the latter to be given in terms of $\delta \delta$ and $ff$ terms as displayed.

It is not difficult to reduce (84) to confirm the correctness of (73)–(77) and hence of (79)–(83). However, a direct attack on either of the latter along the lines just indicated is a good way to get up to speed on methods useful in the current study.

4.2 Equations (85), (74) and (80)

Eq. (85) perhaps discourages such a direct approach as Sec. 4.1 uses, so one adopts a different approach. This requires, as a preliminary, the knowledge of (80). Hence we first develop

$$
\Omega_{ijklmpq} \Omega_{ijklmpq} = \Omega_{ijklmpq} f^a_{[ij} f^b_{kl} f^c_{mp]} d^{(4)}(abcq) = t^{(4)}_{abcq} d^{(4)}(abcq) = t^{(4)}_{abcq} d_{ctd} d_{ctd} = \frac{2}{45} n^2 (n^2 - 9) d_{ctd} d_{ctd} \quad .
$$

(101)

The first line uses the definition (12), from which the square brackets can be dropped, so that (15) can be used in the second line. Next the symmetry properties of the $t$-tensors allow the replacement of $d^{(4)}$ by one of its terms, whereupon (24) may be used. Since

$$
d_{abc} d_{abd} = \frac{(n^2 - 4)}{n} \delta_{cd} \quad ,
$$

(102)

eq. (80) follows.

Returning now to (85), we use that the two terms on the right side of (84) are a basis for the vector space of tensors with the the required symmetry properties. Hence we seek a result of the form

$$
\Omega_{ijklpq} \Omega_{ijklpq} = b(n) f_{qru} f_{stu} + a(n) (\delta_{qs} \delta_{rt} - \delta_{qt} \delta_{rs}) \quad .
$$

(103)

To determine the coefficients, we must perform contractions with $\delta_{qs}$ and with $f_{stu}$. As we show below, this gives equations

$$
(n^2 - 2) a(n) + b(n) n = \phi_7(n) \quad ,
$$

$$
2a(n) + b(n) n = \frac{1}{3} \phi_7(n) \quad ,
$$

(104)
which can be solved to complete the derivation of (85). It is easy to get all the terms here except the one on the right side of the second equation. A viable starting point is elusive. Consider therefore

\[ \Omega_{ijklpst} f_{gst} f^a_{[ij} f^b_{kl} f^c_{p]} d^{(4)}_{(abcr)} . \]  

(105)

The square brackets can again be dropped. This lead us to

\[ f_{cpqd} d^{(4)}_{(abcr)} t^{(4)}_{abgp} \]  

(106)

using (15). Now, we need to check

\[ d^{(4)}_{(abcr)} \delta_{(abgp)} = \frac{2}{g} \left( \frac{n^2-4}{n} \right) \delta_{gp} \delta_{cr} \]  

(107)

use (23), and complete the computation

\[ f_{cpqd} d^{(4)}_{(abcr)} d_{age} d_{bpe} = \frac{1}{n^2} (n^2 - 4)(n^2 - 8) f_{qrg} . \]  

(108)

This last one requires use of the four-fold \( d \)-tensor trace of (A.10) of [1], some simpler results also found there and some patience. Then all the pieces of the calculation have to be put together to complete the derivation of (85). One is guided through something of a morass to an answer one knows is right by the fact that both the unknowns \( a(n) \) and \( b(n) \) must contain a factor \( (n^2 - 9) \) that vanishes at \( n = 3 \).

\section{Omega tensors and \( su(n) \) lambda-matrices}

\subsection{Antisymmetrised products of \( su(n) \) lambda-matrices}

We use the lambda-matrices of ref. [1] which are subject to

\[ \text{Tr} \lambda_i = 0 \quad , \quad \text{Tr} \lambda_i \lambda_j = 2 \delta_{ij} \quad , \quad \lambda^\dagger_i = \lambda_i \quad , \]  

(109)

\[ \lambda_i \lambda_j = \frac{2}{n} \delta_{ij} + (d + if)_{ijk} \lambda_k \quad , \quad d_{ijk} \delta_{ij} = 0 \quad , \]  

(110)

and define totally antisymmetrised products of unit weight of lambda-matrices

\[ \lambda_{[ijk...s]} = \lambda_{[i} \lambda_j \lambda_k \ldots \lambda_s] \quad . \]  

(111)

Simple computations using (3) and (33) lead directly to

\[ \lambda_{[ijk]} = \frac{2}{n} i (f_{ijk} + \Omega_{ijkab} f_{abq} \lambda_q) = \frac{2}{n} i f_{ijk} + i f_{[ijd_k]sq} \lambda_q \quad , \]  

(112)

\[ \lambda_{[ijkl]} = - \Omega_{ijkl} \lambda_t \quad . \]  

(113)

These imply the trace results

\[ \text{Tr} \lambda_{[ijkl]} = 0 \quad , \]  

(114)

\[ \text{Tr} \lambda_{[ijk]} = 4 \Omega_{ijkab} f_{abt} \quad , \]  

(115)

which may be contrasted with

\[ \text{Tr} \lambda_{[ijk]} = \text{Tr} \lambda_{[ijk]} = 2 i f_{ijk} \quad , \]  

(116)

\[ \text{Tr} \lambda_{[ijkpq]} = \text{Tr} \lambda_{[ijkpq]} = - 2 \Omega_{ijkpq} \quad , \]  

(117)
where the first two equalities use the cyclic nature of the trace. For odd traces as (116), terms related by cyclicity add up, whereas for even ones they cancel pairwise and indeed completely:

$$\text{Tr} \lambda_{[i_1i_2\ldots i_{2s}]} = 0 \ .$$  \hfill (118)

For the five-fold case one finds using $\lambda_{[ij kpq]} = \lambda_{[[ij kp]q]}$ and (38) that

$$\lambda_{[ij kpq]} = -\frac{2}{n} \Omega_{ij kpq} - \Omega_{[ijk]p}d_{rq} \lambda_t \ ,$$  \hfill (119)

from which (117) can be recovered. For the six-fold antisymmetrised product of $\lambda$'s we use $\lambda_{[ijkpqr]} = \lambda_{[[ijkp]qr]}$ to deduce

$$\lambda_{[ijkpqr]} = -i \Omega_{[ijk]p}f_{qr} \lambda_s \lambda_t$$  \hfill (120)

$$= -i \Omega_{ijkpqr} \lambda_s \ ,$$  \hfill (121)

and

$$\text{Tr} \lambda_{[ijkp]q} = -2 \Omega_{[ijk]p}d_{qrst} \ .$$  \hfill (122)

Here, to derive (121), we used (110) for $\lambda_s \lambda_t$, Ad-invariance (eq. (37)) to discard the first term, and the following steps to discard another term,

$$\Omega_{s[ij kp f_{qr}]t} = f_{[ij}f_{kp}f_{qr]}d_{syt} \lambda_t$$  \hfill (123)

$$= f_{[ij}f_{kp}f_{qr]}d_{s(xyt)m} = 0 \ ,$$

upon using (26). Similar steps, using (7) and (10), show how the $d$-term of (110) features in the production of (121).

For the seven-fold product we find

$$\lambda_{[ijklpqr]} = -\frac{2}{n} i \Omega^{(7)}_{ijklpqr} - i \Omega^{(7)}_{s[ijklpq]d_{rst}} \lambda_t \ ,$$  \hfill (124)

with the aid of (38) and

$$\text{Tr} \lambda_{[ijklpqr]} = -2 i \Omega^{(7)}_{ijklpqr} \ .$$  \hfill (125)

Eq. (124) is a natural generalisation of the odd traces (112) and (119), and we may infer the result for the odd case

$$\lambda_{[i_1i_2\ldots i_{2s}]} = 2 \ i^s \Omega^{(2s+1)}_{i_1i_2\ldots i_{2s}k} + i^s \Omega^{(2s+1)}_{p[i_1i_2\ldots i_{2s}d_{k]pq]} \lambda_q} \ .$$  \hfill (126)

Also we may use (125) and

$$\text{Tr} \lambda_{[ijklpqr]} = \text{Tr} \lambda_{[ijklpqr]} = \text{Tr} \lambda_{[[ijkl]pq]} \ ,$$  \hfill (127)

to check our work by reproducing the recursive identity (51).

Writing the elementary result

$$\lambda_{[ij]} = \frac{1}{2} [\lambda_i , \lambda_j] = if_{ijk} \lambda_k \ ,$$  \hfill (128)

and comparing it also with the even case (113), and (121), one gets for the antisymmetrised product of an even number of $\lambda$'s the result

$$\lambda_{[i_1i_2\ldots i_{2s}]} = i^s \Omega^{(2s+1)}_{i_1i_2\ldots i_{2s}k} \lambda_k \ ,$$  \hfill (129)

which implies (118) and

$$\text{Tr} \lambda_{[i_1i_2\ldots i_{2s}]} = 2 i^s \Omega^{(2s+1)}_{i_1i_2\ldots i_{2s}k} \ .$$  \hfill (130)

We note that (129), and in particular (113) and (121), provide an explicit realisation of the $(2m - 2)$-bracket Lie algebras \cite{4} for $su(n)$. As mentioned, the coordinates of the $\Omega^{(2m-1)}$ determine the associated higher order structure constants (above, $s = m - 1$), and satisfy the GJI (12).
5.2 General proofs of results for the $\lambda_{[i_1\cdots i_s]}$

The above results are general, due to the nature of the $\Omega^{(2m-1)}$ tensors as generalised structure constants (for instance, eq. (128) may be looked at as a consequence of Th. 3.1 in [2]). However, the above eqs., and in particular (126), were presented on the basis of inspection of a modest number of low value cases. It is thus necessary to show their general validity, particularly since, as defined above by eqs. (10)–(11), $\Omega^{(5)}$ and $\Omega^{(7)}$ involve $d$-tensors with simple properties that do not generalise straightforwardly to the $d$-tensors involved in the definition of higher $\Omega$-tensors.

Let us look first at (129). We write

$$\lambda_{[i_1j_1\cdots i_sj_s]} = i^s f^{p_1}_{[i_1j_1} \cdots f^{p_s}_{i_sj_s]} \lambda_{(p_1\cdots p_s)}$$  \hspace{1cm} (131)$$

If we apply (110) repeatedly, making full use of the symmetry properties that are implied by the round brackets, we can establish a result of the form

$$\lambda_{(p_1\cdots p_s)} = \tilde{k}_{(p_1\cdots p_s)t} \lambda_t + k_{(p_1\cdots p_s t)}$$  \hspace{1cm} (132)$$

where the $k$-tensors are Ad-invariant tensors with the indicated symmetries. Eq. (18) tells us that $k$ does not contribute to (131). Also $\tilde{k}_{(p_1\cdots p_s)t}$ differs from $d^{(s+1)}(p_1\cdots p_s)t$ only by some linear combination of non-primitive terms, which, also by (18), do not contribute to (131). Further

$$\tilde{k}_{(p_1\cdots p_s)t} = \frac{1}{2} \mbox{Tr} \lambda_{(p_1\cdots p_s)t} = \frac{1}{2} \mbox{Tr} \lambda_{(p_1\cdots p_s t)} = \tilde{k}_{(p_1\cdots p_s t)}$$  \hspace{1cm} (133)$$

all of which allows us to replace the lambda-matrix factor of (134) by $d^{(s+1)}(p_1\cdots p_s)\lambda_t$, so that also

$$\lambda_{[i_1j_1\cdots i_sj_s]} = \Omega^{(2s+1)}_{i_1 j_1 \cdots i_s j_s t} \lambda_t$$  \hspace{1cm} (134)$$

The trace of (134) now confirms (118).

Eq. (130) also follows easily. To obtain (126), we multiply (134) by $\lambda_k$ and use (110). Then (126) follows directly, after the use of the Ad-invariance of $\Omega^{(2s+1)}$ to drop the contribution of the $f$-term of (110).

Inspection of (133) shows that it is tantamount to the statement that, in the definition (11) of $\Omega^{(9)}$, e.g., one is, after all, allowed to move the right hand round bracket one place to the left. It is of interest to see this explicitly, because, amongst other things, a further class of identities for $d$-tensors emerges as a by-product. We illustrate this for $s = 4$. One evaluation of the trace involved leads to

$$\lambda_{(abcd)} = \frac{4}{n^3} \delta_{(ab} \delta_{cd)} + \frac{2}{n} d^{(4)}_{(abcd)} + \frac{2}{n} d_{(ab} \lambda_{cd)} + \frac{2}{n} \delta_{(ab} d_{cd)} y_{\lambda_{y}} + d^{(5)}_{(abcd)y} \lambda_y$$  \hspace{1cm} (135)$$

The key trace result (cf. (133))

$$\mbox{Tr} \lambda_{(abcd)e} = \mbox{Tr} \lambda_{(abcde)}$$  \hspace{1cm} (136)$$

now leads to

$$d^{(5)}_{(abcd)e} = d^{(5)}_{(abcd)e} + \frac{1}{n} \delta_{(ab} d_{cde)} - \frac{1}{n} \delta_{(ab} d_{cd)e}$$  \hspace{1cm} (137)$$

The difference between the two $d^{(5)}$ tensors here, and as in the general discussion above, makes no contribution to the evaluation of $\lambda_{[i_1j_1\cdots i_4j_4]}$. It follows then that in the definition (11), we can replace $d^{(5)}_{(abcd)e}$ by $d^{(5)}_{(abcd)e}$, which of course has fewer terms.
Another question arises here: how does (136) relate to (30)? To answer, we note that a different way of evaluating the trace gives rise to
\[
\lambda_{abcd} = \frac{4}{n} \delta_{(ab} \delta_{cd)} + \frac{2}{n} d^{(4)}_{(abcd)} + \frac{4}{n} \delta_{(ab} d_{cd)y} \lambda_{y} + d^{(5)}_{(ab} \gamma_{cd) \lambda_{y}} ,
\]
and hence
\[
d^{(5)}_{(abcede)} = d^{(5)}_{(abe \, cde)} - \frac{4}{n} \delta_{(ab} d_{cde)} + \frac{4}{n} \delta_{(ab} d_{cde)x} .
\]
Now (30) follows obviously from (135) and (139).

There is another instructive way to make the point that the three \(d^{(5)}\) tensors can be used interchangeably in the definition (11) of \(\Omega^{(9)}\). It follows by comparison of
\[
\lambda_{ijklpqrs} = \lambda_{[[ijklpq][rs]]} = \Omega_{x[ijklpqf_{rs}] y} \lambda_{x} \lambda_{y} = f^{a}_{[ij} f^{b}_{kl} f^{c}_{pq} f^{d}_{rs]} d^{(5)}_{(abcd) \lambda_{t}} \lambda_{t} ,
\]
and
\[
\lambda_{ijklpqrs} = \lambda_{[[ijk][pqrs]]} = f^{a}_{[ij} f^{b}_{kl} f^{c}_{pq} f^{d}_{rs]} d^{(5)}_{(abc \gamma) \lambda_{t}} \lambda_{t} .
\]
The discussion just given for \(\Omega^{(9)}\) generalises naturally for higher Omega tensors.

### 5.3 Use of the completeness relation for the \(su(n)\) lambda-matrices

We set out from the result well-known for \(su(n)\) (13)
\[
\lambda_{i ab} \lambda_{i cd} = 2 \delta_{ad} \delta_{cb} - \frac{2}{n} \delta_{ab} \delta_{cd} ,
\]
and note also its consequences
\[
- i f_{ij k} \lambda_{j ab} \lambda_{k cd} = \lambda_{iad} \delta_{cb} - \lambda_{icb} \delta_{ad} \quad (143)
\]
\[
d_{ijk} \lambda_{j ab} \lambda_{k cd} = \lambda_{iad} \delta_{bc} + \lambda_{icb} \delta_{ad} - \frac{2}{n} \left( \lambda_{i ab} \delta_{cd} + \lambda_{i cd} \delta_{ab} \right) .
\]
From (142) we may compute
\[
\lambda_{[ij]} ab \lambda_{[ij]} cd = - n \lambda_{i ab} \lambda_{i cd} \quad (145)
\]
\[
\lambda_{[ijk]} ab \lambda_{[ijk]} cd = - \frac{2}{3} (n^2 - 4) \lambda_{i ab} \lambda_{i cd} - \frac{4}{n} (n^2 - 1) \delta_{ab} \delta_{cd} \quad (146)
\]
\[
\lambda_{[ijkl]} ab \lambda_{[ijkl]} cd = \frac{2}{3} (n^2 - 4) \lambda_{i ab} \lambda_{i cd} \quad (147)
\]
\[
\lambda_{[ijklm]} ab \lambda_{[ijklm]} cd = \frac{2}{15} (n^2 - 4) (n^2 - 6) \lambda_{i ab} \lambda_{i cd} + \frac{4}{3n} (n^2 - 1) (n^2 - 4) \delta_{ab} \delta_{cd} ,
\]
and so on. One can make checks on these results by putting \(b = c\) to reach
\[
\lambda_{i} \lambda_{i} = \frac{2}{n} (n^2 - 1) I
\]
\[
\lambda_{[ij]} \lambda_{[ij]} = - 2 (n^2 - 1) I
\]
\[
\lambda_{[ijk]} \lambda_{[ijk]} = - \frac{4}{3n} (n^2 - 1)^2 I
\]
\[
\lambda_{[ijkl]} \lambda_{[ijkl]} = \frac{2}{3} (n^2 - 4) (n^2 - 4) I
\]
\[
\lambda_{[ijklm]} \lambda_{[ijklm]} = \frac{4}{15n} (n^2 - 1)^2 (n^2 - 4) I .
\]
These results are of use themselves and may be verified by other means. It is tempting to speculate on the nature of results beyond (148), but it gets increasingly hard to compute directly the \(n\)-dependences. Use of \(\text{Tr} \, I = n\) yields obvious trace formulas.
For the purpose, central to the aims of this paper, of computing the quantities \((\Omega^{(2n+1)})^2\) explicitly in closed form, it is enough to analyse the traced analogues of (147)–(148), obtained by putting \(b = c\) and \(d = a\). For this analysis, one of the approaches available employs another set of lemmas that follow from (142). For any \(n\)-dimensional matrix \(M\), eq. (142) gives

\[
(\lambda_i M \lambda_i)_{ab} = 2\delta_{ab} \text{Tr} M - \frac{2}{n} M_{ab} \quad .
\]

This provides us with a method for obtaining the results

\[
\lambda_i \lambda_j \lambda_i = -\frac{2}{n} \lambda_j \quad , \quad (155)
\]

\[
\lambda_i \lambda_{[jk]} \lambda_i = -\frac{2}{n} \lambda_{[jk]} \quad , \quad (156)
\]

\[
\lambda_i \lambda_{[jkl]} \lambda_i = 4i f_{jkl} - \frac{2}{n} \lambda_{[jkl]} \quad , \quad (157)
\]

\[
\lambda_i \lambda_{[jklpq]} \lambda_i = -4\Omega_{jklpq} - \frac{2}{n} \lambda_{[jklpq]} \quad , \quad (158)
\]

\[
\lambda_i \lambda_{[i_1 j_1 \ldots i_m j_m k]} \lambda_i = 4i m \Omega^{(2m+1)}_{i_1 j_1 \ldots i_m j_m k} - \frac{2}{n} \lambda_{[i_1 j_1 \ldots i_m j_m k]} \quad , \quad (159)
\]

\[
\lambda_i \lambda_{[i_1 j_1 \ldots i_m j_m]} \lambda_i = -\frac{2}{n} \lambda_{[i_1 j_1 \ldots i_m j_m]} \quad . \quad (160)
\]

The last result follows from (154) because of (118).

We note here further simple results that may help streamline larger tasks, for example one approach to the proof of (149)–(153):

\[
\lambda_i \lambda_{[ij]} = n \lambda_j \quad . \quad (161)
\]

\[
\lambda_i \lambda_{[ijk]} = \frac{2}{3} \frac{n^2 - 1}{n} \lambda_{[jk]} \quad , \quad (162)
\]

\[
\lambda_{[ij]} \lambda_{[ijk]} = -\frac{2}{3} (n^2 - 1) \lambda_k \quad , \quad (163)
\]

\[
\lambda_i \lambda_{[i j k]} = \frac{5}{3} \lambda_{[jkl]} - if_{jkl} \quad , \quad (164)
\]

\[
\lambda_{[ij]} \lambda_{[ijkl]} = -\frac{1}{3} (n^2 - 4) \lambda_{[kl]} \quad , \quad (165)
\]

\[
\lambda_{[ijkl]} \lambda_{[ijkl]} = -\frac{7}{6} (n^2 - 4) \lambda_i \quad , \quad (166)
\]

\[
\lambda_{[ijkl]} \lambda_{[ijklm]} = \frac{2}{3n} (n^2 - 1) \lambda_{[jklm]} - \frac{4}{3} if_{jklm} \lambda_{[lm]} \quad , \quad (167)
\]

\[
\lambda_{[ij]} \lambda_{[ijklm]} = -\frac{1}{5} (n^2 - 4) \lambda_{[klm]} \quad , \quad (168)
\]

\[
\lambda_{[ijk]} \lambda_{[ijklm]} = -\frac{2}{15n} (n^2 - 1)(n^2 - 4) \lambda_{[lm]} \quad , \quad (169)
\]

\[
\lambda_{[ijkl]} \lambda_{[ijklm]} = \frac{2}{15} (n^2 - 1)(n^2 - 4) \lambda_m \quad . \quad (170)
\]

Hermitian conjugation gives results such as

\[
\lambda_{[ji]} \lambda_i = n \lambda_j \quad . \quad (171)
\]

Inspection of (161), (162), (164) and (167) suggests the general result

\[
\text{Tr} \lambda_i \lambda_{[i_2 \ldots i_s]} = 0 \quad (s \text{ even or odd}) \quad , \quad (172)
\]

which is easily proved using the results of Sec. 5.2. If \(s\) is even we find, using (129),

\[
\text{Tr}(\lambda_{ij} \lambda_{[i_2 \ldots i_s]}) \sim \text{Tr}(\lambda_{ij} \Omega_{i_2 \ldots i_s k} \lambda_k) = 2\Omega_{i_2 \ldots i_s i} = 0 \quad .
\]

If \(s\) is odd, \(\text{Tr}(\lambda_{ij} \lambda_{[i_2 \ldots i_s]}) = \text{Tr}(\lambda_{ij} \lambda_{[i_2 \ldots i_s]}) \sim \text{Tr}(\lambda_{ij} \lambda_{[i]}) \Omega_{i_2 \ldots i_s k} = 0 \quad .
\]

Some of the principal results to be derived below require as input more trace results. First, and in agreement with results displayed above, we expect

\[
\text{Tr} (\lambda_{ij} \lambda_{[ij_1 \ldots i_{2s}]}) = 0 \quad . \quad (173)
\]

A typical proof here, using the methods of Sec. 5.2, is

\[
\text{Tr}(\lambda_{ij} \lambda_{[ijklpq]}) = \text{Tr}(\lambda_{ij} (-i\Omega_{ijklpq} \lambda_r)) = 2\Omega_{ijklpq} f_{ijr} = 0 \quad , \quad (174)
\]
upon use of (68). The analogues to (173) for $s = 4$ and $s = 5$ of (173) however depend on (69) and (71), results which remain unproved until the methods of Sec. 8 can be called upon. Second

$$\text{Tr} (\lambda_{[ijk]}\lambda_{[ijki_{4...i_{2}u}]} = 0 . \quad (175)$$

A typical proof, here for $s = 3$, is

$$\text{Tr} (\lambda_{[ijk]}\lambda_{[ijklpq]} = \text{Tr} (\lambda_{[ijkl]}(-i\Omega_{ijklpq}\lambda_{r})) = \Omega_{ijklpq} f_{nijklpq} \Omega_{ijklpq} \lambda_{r} = 2\Omega_{ijklpq} f_{ijklpq} = 0 \ , \quad (176)$$

where (121), (115) and (77) have been used. The last equality follows from the total symmetry of $d^{(3)}$ and the antisymmetry of $\Omega^{(7)}$.

### 5.4 Further trace results

We are interested here in trace results of the type

$$\text{Tr} (\lambda_{[ij...s]}\lambda_{[ij...s]} \ ) . \quad (177)$$

Even traces of this sort are of primary interest in virtue of their relationship to $(\Omega^{(2s+1)})^2$. For example, for $s = 2$,

$$4(\Omega^{(5)})^2 = \text{Tr} \lambda_{[ijkpq]} \text{Tr} \lambda_{[ijkpq]} = \text{Tr} \lambda_{[ijkpq]} \text{Tr} \lambda_{[jkpq]} = 2\text{Tr} \lambda_{[jkpq]} = \quad (178)$$

where (142) and (114) have been used. However, proceeding recursively for higher $s$ brings the odd traces into the picture. We have two approaches to either even or odd traces, and one works better for the odd and the other for the even traces. We begin with the even traces for which we have a nice general result. We note a generalisation of results embedded in the previous subsections:

$$\text{Tr} (\lambda_{[i_{1}...i_{2}s...i_{s}]}\lambda_{[i_{1}...i_{2}s...i_{s}]} \ ) = \frac{2}{(2s+1)} \frac{(n^2-1)}{n} \text{Tr} (\lambda_{[i_{1}...i_{2}s...i_{s}]}\lambda_{[i_{1}...i_{2}s...i_{s}]} \ ) . \quad (179)$$

A brief look at the case $s = 3$ will indicate clearly that this result is valid in general. Thus we write

$$\text{Tr} (\lambda_{[ijklpqr]}\lambda_{[i_{1}...i_{2}s...i_{s}]}\lambda_{[i_{1}...i_{2}s...i_{s}]} \lambda_{[ijklpq]} = \frac{1}{7} \text{Tr} (\lambda_{[ijklpq]}\lambda_{[ijklpq]} \ )$$

Now we use the cyclic property of the trace to justify the use of results of the type (159) and (160) in the first six terms, and of (149) to the seventh. One can see from (118) the Omega tensor terms of (153) do not contribute (which is why this approach is better for the odd traces than for the even ones), and then it is easy to see that, after taking due care of the signs of the first six terms, everything cancels except the contribution of the seventh term of (180), which gives the right side of (179) at $s = 3$.

Reduction of the right side of (179) is much harder because the same approach brings in the Omega tensor pieces of (157), (159) etc., non-trivially. This caused us to adopt a related but distinct approach to such traces in the next section, although the approach just followed does work, but rather less well. To say enough to allow a comparison of
methods to be made, let us refer back to (178). We may drop the second set of square brackets and reinsert others judiciously in suitable places whenever this is allowed by existing antisymmetries. Then the development of the first set of square brackets yields

\[ 8(\Omega^{(5)})^2 = \text{Tr}(\lambda_j \lambda_{[kpq]} \lambda_j \lambda_{[kpq]}) - \text{Tr}(\lambda_k \lambda_{[pqj]} \lambda_j \lambda_{[kpq]}) + \text{Tr}(\lambda_p \lambda_{[qjk]} \lambda_j \lambda_{[kpq]} \lambda_q) - \text{Tr}(\lambda_q \lambda_{[jkp]} \lambda_j \lambda_{[kpq]} \lambda_q) \] . \tag{181} 

The cyclic property of the trace now allows use of (157), (156), (155) and (149), in that order so that after cancellations, we obtain

\[ 8(\Omega^{(5)})^2 = \text{Tr}(\lambda_{[kpq]}[4i f_{kpq} + \left(\frac{2}{n} + \frac{2}{n^2} - 1\right)] \text{Tr}(\lambda_{[kpq]} \lambda_k \lambda_p \lambda_q)) \] . \tag{182} 

Now use of (116) and (151) leads directly to the answer obtained before: (79) with (82). The higher order even traces get successively harder in this approach, but we will see a comparable increase in the price associated with passing to higher \(s\) is present also in our favoured method of Sec. 6.

6 The recursion relations for the \( (\Omega^{(2m-1)})^2 \)

We illustrate the general approach by reference to the case \(m = 5\). Since, by eq. (130),

\[ 2\Omega_{ijklpqrs} = \text{Tr}(\lambda_{[ijklpqrs]} t) \] , \tag{183} 

we may write

\[ 4(\Omega^{(9)})^2 = 4\Omega_{ijklpqrs} \Omega_{ijklpqrs} = \text{Tr}(\lambda_{[ijklpqrs]} \lambda_t) \text{Tr}(\lambda_{[ijklpqrs]} \lambda_t) = 2\text{Tr}(\lambda_{[ijklpqrs]} \lambda_{[ijklpqrs]}) \] . \tag{184} 

Here we have used (142) and the trace result (118). The key steps now follow. We can remove the first set of square brackets completely and then reinsert them round the indices \(jklpqrs\). Then we expand the second set of square brackets to expose, in each of the eight terms that thereby arise, the matrix \(\lambda_i\):

\[ 16(\Omega^{(9)})^2 = \lambda_{i ab} \lambda_{[ijklpqrs]} \lambda_{[ijklpqrs]} \lambda_{[ijklpqrs]} = \delta_{cd} \lambda_i \lambda_{[ijklpqrs]} \delta_{ea} \lambda_{[ijklpqrs]} + \lambda_{[rs]} \lambda_{[ijklpqrs]} \delta_{ea} \lambda_{[ijklpqrs]} - \lambda_{[grs]} \lambda_{[ijklpqrs]} \delta_{ea} \lambda_{[ijklpqrs]} + \lambda_{[pqrs]} \lambda_{[ijklpqrs]} \delta_{ea} \lambda_{[ijklpqrs]} - \lambda_{[klpqrs]} \lambda_{[ijklpqrs]} \delta_{ea} \lambda_{[ijklpqrs]} + \lambda_{[klpqrs]} \lambda_{[ijklpqrs]} \delta_{ea} \lambda_{[ijklpqrs]} - \lambda_{[ijklpqrs]} \delta_{ea} \lambda_{[ijklpqrs]} \lambda_{[ijklpqrs]} - \lambda_{[ijklpqrs]} \delta_{ea} \lambda_{[ijklpqrs]} \lambda_{[ijklpqrs]} \] . \tag{185} 

where \(a, b, \ldots, c = 1, \ldots, n\) are matrix element indices, \(\lambda_{i ab} \equiv (\lambda_i)_{ab}\). Now we may use (142) once more. The second term of (142) gives zero contribution, or rather its contributions to the eight terms of (185) cancel pairwise. Turning next to the contributions that come from the first term of (142), we see the second, fourth, sixth and seventh terms of (185) vanish because of trace results such as (118). The first term gives

\[ 2\text{Tr}(\lambda_{[ijklpqrs]} \lambda_{[ijklpqrs]}) = -8(\Omega^{(7)})^2 \] , \tag{186} 

where
by steps like those that yielded (184). The eighth term gives

\[-2\text{Tr} \left( \lambda_{ijklpqrs} \lambda_{ijklpqrs} \right) \text{Tr} I_n = \left(-2\right)^2 \left(\frac{n^2 - 1}{n}\right) \text{Tr} \left( \lambda_{ijklpq} \lambda_{ijklpq} \right) n = -\frac{8}{9} (n^2 - 1)(\Omega^7)^2, \tag{187}\]

where the result (179) has been used. There thus remains to be treated a set of two terms, one each from the third and fifth lines of (188). We next display the two terms in question together with the results of evaluating them

\[2\text{Tr}(\lambda_{ijklpqrs} \lambda_{[rs])}\text{Tr}(\lambda_{ijklpq}) = -8 \cdot \frac{7}{9} (\Omega^7)^2, \tag{188}\]

\[2\text{Tr}(\lambda_{ijklpqrs} \lambda_{[pqrs]}\text{Tr}(\lambda_{ijkl}) = -8 \cdot \frac{3}{9} (\Omega^7)^2. \tag{189}\]

Proofs of (188) and (189) are given in below in Sec. 7. We may now collect the contributions (186)-(189) to produce the final answer

\[(\Omega^9)^2 = \frac{1}{14} (n^2 - 16)(\Omega^7)^2. \tag{190}\]

We have presented this calculation in detail because every aspect of it works for higher cases in almost exactly the same fashion. The main difference for the case \(m = 5\) of \((\Omega^{11})^2\) is that there are now three terms in the set of terms that arise in the same way as did (188) and (189), namely

\[2\text{Tr}(\lambda_{ijklpqrs} \lambda_{[rs]}\text{Tr}(\lambda_{ijklpq}) = -8 \cdot \frac{7}{9} (\Omega^9)^2, \tag{191}\]

\[2\text{Tr}(\lambda_{ijklpqrs} \lambda_{[pqrs]}\text{Tr}(\lambda_{ijkl}) = -8 \cdot \frac{3}{9} (\Omega^9)^2. \tag{192}\]

\[2\text{Tr}(\lambda_{ijklpqrs} \lambda_{[pqrs]}\text{Tr}(\lambda_{ijkl}) = -8 \cdot \frac{3}{9} (\Omega^9)^2. \tag{193}\]

With the aid of these results, proved below in Sec 7.1, we reach the \(m = 5\) analogue of (190)

\[(\Omega^{11})^2 = \frac{2}{45} (n^2 - 25)(\Omega^7)^2. \tag{194}\]

Indeed it is possible to infer the general result relating the squares of the \((2s - 1)\)- and \((2s + 1)\)-cocycles associated with the Racah-Casimir operators of order \(s\) and \((s + 1)\):

\[(\Omega^{2s+1})^2 = \frac{4}{2s(2s-1)}(n^2 - s^2)(\Omega^{2s-1})^2, \tag{195}\]

and hence

\[(\Omega^{2m-1})^2 = \frac{2^{2m-3} n}{(2m-2)!} \prod_{r=1}^{m-1} (n^2 - r^2). \tag{196}\]

The last two results show in full the expected factors that force the absence of the \(su(n)\)-algebra cocycle/Omega tensor \(\Omega^{2m-1}\) whenever \(m > n\). Indeed, the last factor in (196) is \((n^2 - (m-1)^2)\) and hence \((\Omega^{2m-1})^2 = 0\) whenever \(n < m\). These results are also crucial in the discussion [3] of Racah-Casimir operators, their eigenvalues and of generalised Dynkin indices for \(su(n)\).

7 Proof of results like (188)-(193)

We begin with the simplest result (188) for which we develop

\[2\text{Tr}(\lambda_{ijklpqrs} \lambda_{[rs]}\text{Tr}(\lambda_{ijklpq}) = -4i\text{Tr} \left[ \left(-\frac{2}{n} i\Omega_{ijklpqrs} - i\Omega_{ijklpq} d_s_{tx} \lambda_x \right) i\Omega_{ijklpq} \right] = \tag{197}\]

\[22\]
where we used (124) and (119) in the first line. Next, by opening up the square brackets, we find seven terms of which two vanish upon use of (66), while the remaining five are seen to be equal after relabelling. This accounts for the fraction that appears in the third line. In the fourth line we have used the definition of \( \Omega_{jklpq} \), which sets the scene for using the definition (10) of \( \Omega_{tklpqrs} \) to reach the last line. It may be noted that it is the first Omega tensor which allows the required symmetries to be implied for the remaining factors in order to build the second Omega tensor.

It should suffice to illustrate things fully to sketch the proof of the most complicated member of the set of results (188)–(193). This requires the space

\[
\text{Tr}(\lambda_t\lambda_{(abc)}) = \text{Tr} \lambda_{(abc)} = \frac{4}{n} \delta_t(a\delta_{bc}) + 2d_{st(a\delta_{bc})s} \quad ,
\]

which also follows from (133). Then, putting in some brackets, we get

\[
2\text{Tr}(\lambda_{[jklpqrsuv]}\lambda_{][pq][rs][uv]}) \lambda_{[jkl]} = 4\text{Tr} \Omega_{w[jklpqrsuv]d_v[u]t\lambda_t\lambda_{f_pqa\lambda_\lambda_{f_rsb\lambda_{f_uvc\lambda_{a\delta_{bc}}\lambda_{f_jkl}}}} = 8\Omega_{w[jklpqrsuv]d_v[u]t\lambda_t\lambda_{f_pqa\lambda_\lambda_{f_rsb\lambda_{f_uvc\lambda_{a\delta_{bc}}\lambda_{f_jkl}}}} = 8\frac{3}{5} \Omega_{wklpqrsuvf_{jkl}f_{pqa\lambda_\lambda_{f_rsb\lambda_{f_uvc\lambda_{a\delta_{bc}}\lambda_{f_jkl}}}} = \Omega^{(9)}_{wjklpqrsuvf_{jkl}} = 0 \quad ,
\]

and the remaining three are equal. The first term of (198) fails to contribute to line two of (199) because of Jacobi identities that also rely on the antisymmetry properties provided by the first Omega tensor.

It can be seen that as one goes, notionally, to higher \( m \) all the same patterns persist. Although this may require results beyond those explicitly provided here, no problems should be encountered in finding these, the generalisations of (133) being given in Sec.8. We remark also that the coefficients that appear on the right side of (188) and (189), and on the right side of (191)–(193) also conform to a rather obvious pattern, which affords a check on the work, and is instrumental in producing the crucial \( (n^2 - s^2) \) factors of the recursion relations (195).

### 8 Proof of (71)

We first prove here that

\[
\Omega^{(11)}_{ijklpqrsuv}f_{tuv} = 0 \quad .
\]
This is a critical case because it is the simplest one of the type discussed in Sec. 3.4 in which there is a non-trivial invariant totally antisymmetric tensor of the same order as the right side of the identity to be proved, namely the tensor $\tilde{\Omega}^{(8)}$ of (12). In this case, moreover, the methods of Sec. 3.4 do not offer a viable approach. The method of proof to be given for (201) extends straightforwardly to its analogue for $\Omega^{(13)}$. But then we meet a further critical case

$$\Omega^{(15)}_{ijklpqrsuvwxyz} f_{xyz} = 0 \quad .$$

(202)

This case is critical because it is the simplest one in which there is a non-trivial invariant totally antisymmetric tensor of the same order as $\Omega^{(15)}$ itself, the tensor $\tilde{\Omega}^{(15)}$ of (13). However there is no obstacle to extending to this case the method of proof to be given for (201).

To prove (201), we start with

$$4\Omega^{(11)}_{ijklpqrsstu} f_{tuv} = \text{Tr} \lambda_{ijklpqrstuv} \text{Tr} \lambda_{[tuv]} = \text{Tr} \lambda_{ijklpqrstuv} \text{Tr} \lambda_{[tuv]} = 2 \text{Tr} \lambda_{ijklpqrstuv} \lambda_{[tu]} \quad ,$$

(203)

using now familiar steps.

To make progress with computing the right side of (203), we drop the second set of square brackets and open out the other set to expose $\lambda_u$ in each of its ten terms. This serves to enable a second use of (142):

$$\begin{align*}
\left( \lambda_{ijklpqrst} ab \delta_{dc} - \lambda_{ijklpqrst} ab \delta_{ij} dc \\
+ \lambda_{[klpqrst} ab \lambda_{ij]} dc - \lambda_{[lpqrst} ab \lambda_{ijk]} dc \\
+ \delta_{qrst ab} \lambda_{ijklpq} dc - \lambda_{[stab]} \lambda_{ijklpq} dc \\
+ \lambda_{[tub] \lambda_{ijklpqrs} dc} - \delta_{ab} \lambda_{ijklpqrs} dc \right) \frac{1}{5} \lambda_{u bd} \lambda_{t ce} \lambda_{u ea} ,
\end{align*}$$

(204)

where the labels $a, \ldots, e = 1, \ldots, n$ are matrix element indices, hence unaffected by antisymmetrisation.

It is easy to check that all ten contributions from the second term of (142) cancel pairwise. The ten contributions from the first term of (142) then are $\frac{2}{5}$ times

$$\begin{align*}
\text{Tr} \lambda_{ijklpqrs} \text{Tr} \lambda_t & - \text{Tr} \lambda_{ijklpqrs} \text{Tr} \lambda_i t \\
+ \text{Tr} \lambda_{[klpqrs} \text{Tr} \lambda_{ij]} t & - \text{Tr} \lambda_{[pqrs] \text{Tr} \lambda_{ijk]} t} \\
+ \text{Tr} \lambda_{[prst} \text{Tr} \lambda_{ijklp]} t & - \text{Tr} \lambda_{[qrst]} \text{Tr} \lambda_{ijklp]} t \\
+ \text{Tr} \lambda_{[rst} \text{Tr} \lambda_{ijklpq]} t & - \text{Tr} \lambda_{[st]} \text{Tr} \lambda_{ijklpq]} t \\
+ \text{Tr} \lambda_{[t] \text{Tr} \lambda_{ijklpqrs]} t & - n \text{Tr} \left( \lambda_{ijklpqrs} \lambda_t \right) 
\end{align*}$$

(205)

Terms 1 and 9 here are zero trivially, terms 2, 4, 6, 8 are zero using (118). Also term 10 is zero by (172). This leaves terms 3, 5 and 7. Terms 3, 5 and 7 are, to within a common factor, given by

$$\begin{align*}
\Omega^{(7)}_{[klpqrs} f_{ij]} t \\
\Omega^{(5)}_{[pqrs] \Omega^{(5)}_{ijkl]} t} \\
f_{rst} \Omega^{(7)}_{ijklpq]} t \quad .
\end{align*}$$

(206) - (208)

The terms (206), (208) are zero since they are the result of extending the antisymmetrisation of expressions that are already zero by $Ad$-invariance, cf. (38). Similarly the term (207) is zero by (12).
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References

[1] J.A. de Azcárraga, A.J. Macfarlane, A.J. Mountain and J.C. Pérez Bueno, *Invariant tensors for simple Lie groups*, Nucl. Phys. **B510**, 657-687 (1998), [physics/9706006](http://arxiv.org/abs/physics/9706006).

[2] J.A. de Azcárraga and J.C. Pérez Bueno, *Higher order simple Lie algebras*, Commun. Math. Phys. **184**, 669-681 (1997), [hep-th/9605213](http://arxiv.org/abs/hep-th/9605213).

[3] J.A. de Azcárraga and A.J. Macfarlane, *Optimally defined Racah-Casimir operators for su(n) and their eigenvalues for various classes of representations*, J. Math. Phys. **42**, 419-433 (2001), [math-ph/0006013](http://arxiv.org/abs/math-ph/0006013).

[4] J.A. de Azcárraga and A.J. Macfarlane, *Fermionic realisations of Lie algebras and their group invariant fermionic operators*, Nucl. Phys. **B581**, 743-760 (2000), [hep-th/0003111](http://arxiv.org/abs/hep-th/0003111).

[5] C. Chryssomalakos, J.A. de Azcárraga, A.J. Macfarlane and J.C. Perez Bueno, *Higher order BRST and anti-BRST operators and cohomology for compact Lie algebras*, J. Math. Phys. **40**, 6009-6033 (1999), [hep-th/9810212](http://arxiv.org/abs/hep-th/9810212).

[6] J.A. de Azcárraga, A. Perelomov and J.C. Pérez Bueno, *The Schouten-Nijenhuis bracket, cohomology and generalised Poisson structures*, J. Phys. **A29**, 7993-8009 (1996), [hep-th/9605067](http://arxiv.org/abs/hep-th/9605067).

[7] E. d’Hoker and S. Weinberg, *General effective actions*, Phys. Rev. **D50**, R6050-R6053 (1994), [hep-ph/9409402](http://arxiv.org/abs/hep-ph/9409402).

[8] E. d’Hoker, *Invariant effective actions, cohomology of homogeneous spaces and anomalies*, Nucl. Phys. **B451**, 725-748 (1995), [ep-th/9502162](http://arxiv.org/abs/ep-th/9502162).

[9] J.A. de Azcárraga, A.J. Macfarlane and J.C. Pérez Bueno, *Effective actions, relative cohomology and Chern Simons forms*, Phys. Lett. **B419B**, 186-194 (1998), [hep-th/9711064](http://arxiv.org/abs/hep-th/9711064); J.A. de Azcárraga and J.C. Pérez Bueno, *On the general structure of gauged Wess-Zumino terms*, Nucl. Phys. **B534**, 653-674 (1998), [hep-th/9802192](http://arxiv.org/abs/hep-th/9802192).

[10] T. van Ritbergen, A.N. Schellekens and J.A.M Vermaseren, *Group theory factors for Feynman diagrams*, Int. J. Mod. Phys. **14A**, 41-96 (1999), [hep-ph/9802376](http://arxiv.org/abs/hep-ph/9802376).

[11] A.J. Macfarlane, A. Sudbery and P.H. Weisz, *On Gell-Mann’s λ-matrices, d- and f-tensors, octets and parametrizations of SU(3)*, Commun. Math. Phys. **11**, 77-90 (1968).

[12] A. Sudbery, PhD thesis, Cambridge University, (1970); *Computer friendly d-tensor identities for SU(n)*, J. Phys. **A23**, L705-L709 (1990).
[13] A.J. Macfarlane and Hendryk Pfeiffer, *On characteristic equations, trace identities and Casimir operators for simple Lie algebras*, J. Math. Phys. 41, 3192-3225 (2000).

[14] L.M. Kaplan and M. Resnikoff, *Matrix products and the explicit 3,6,9 and 12-j coefficients of the regular representation of SU(n)*, J. Math. Phys. 8, 2194-2205 (1967).

[15] J.A. de Azcárraga, J.M. Izquierdo and J.C. Pérez Bueno, *An introduction to some novel applications of Lie algebra cohomology in mathematics and physics*, to appear in the Proc. of the VI Fall Workshop on Geometry and Physics (Salamanca, 1997), physics/9803046.