Application of EGSOR Iteration with Nonlocal Arithmetic Discretization Scheme for Solving Burger’s Equation

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Abstract: The purpose of this paper is to investigate the performance of 2-Point Explicit Group Successive Over Relaxation (EGSOR) method with nonlocal arithmetic discretization scheme for solving Burger’s equations. To do this matter, the proposed problems need to be discretized by using second-order implicit scheme to derive the corresponding nonlinear second order implicit finite difference approximate equation. Then, the nonlocal arithmetic discretization scheme is used to transform the corresponding nonlinear implicit approximation equation into a system of linear equations. Furthermore, numerical results of four proposed problems are also included in order to verify the effectiveness of the EGSOR method compared to Gauss-Seidel (GS) and SOR method. Based on numerical results, it can be concluded that the performance of EGSOR method is better than GS and SOR in terms of number of iterations and computational time.

1. Introduction
The Burger’s equation was introduced by Bateman in 1915 and then a steady solution was proposed by Burgers in 1939. This equation is a fundamental of partial differential equations that occurs in various areas of applied mathematics such as fluid mechanics, nonlinear acoustics and gas dynamics [1]. In order to solve Burger’s problem, there are several numerical methods that have been introduced and proven their effectiveness in solving the problems. For example, this problem has been solved by using various numerical methods such as extended modified cubic B-spline differential quadrature method [1] variational iteration method [2], modified trigonometric B-spline [3], compact finite difference method [4], quadratic B-spline collocation finite elements [5] and modified cubic B-splines collocation method [6]. Although the concept of numerical methods is different, all studies have shown good accuracy in their solutions. Hence, this study will consider the implicit finite difference discretization scheme and the nonlocal arithmetic mean scheme as an alternative method to eliminate the nonlinear term for solving Burger’s equation.

Consequently, the purpose of this paper is to transform the corresponding nonlinear implicit approximation equation into the corresponding system of linear equations by using nonlocal arithmetic mean discretization scheme to get the approximate solution of the problem. To investigate the performance of EGSOR method with nonlocal arithmetic mean discretization scheme, numerical results of four examples of proposed problems will be solved numerically via the implementation of block iterations.

To get the numerical solution, let us consider the following one-dimensional Burger’s equation:
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = v \frac{\partial^2 u}{\partial x^2}, \quad a \leq x \leq b, \quad t \in [0, T]
\]

Subject to initial condition:
\[
u(x, 0) = f(x), \quad a \leq x \leq b.
\]

and Dirichlet boundary conditions:
\[
u(a, t) = f_1(t), \quad \nu(b, t) = f_2(t), \quad t > 0.
\]

where \(u = u(x, t)\) is an unknown function, \(v\) is a parameter \((v > 0)\) and \(u \frac{\partial u}{\partial x}\) is the nonlinear term.

For simplifying in discretization process, problem (1) needs to be rewritten by the following equation
\[
\frac{\partial u}{\partial t} + F(x, t, u) \frac{\partial u}{\partial x} = v \frac{\partial^2 u}{\partial x^2}
\]

where \(F(x, t, u) \frac{\partial u}{\partial x}\) is called as a nonlinear term. Before doing the discretization process, we need to construct the finite grid network as shown in Figure 1.

![Figure 1](image)

**Figure 1.** Example of mesh points for the solution domain of the problem.

Based on Figure 1, let solution domain of problem (1) be divided uniformly with grid \(\Delta x\) and \(\Delta t\) in \(x\) and \(t\) directions respectively in which each grid of \(\Delta x\) and \(\Delta t\) can be defined as
\[
\Delta x = \frac{b - a}{m}, \quad \Delta t = \frac{t_n}{n}.
\]

According to Figure 1, the unknown value of \(U(x_i, t_j) = u_{ij}\) can be determined by using an implicit finite difference approximation equation by solving the corresponding linear system.

2. **Formulation of Arithmetic Mean Discretization Scheme**

To formulate the second-order implicit finite difference approximation equation of problem (1), we need
to consider several nonlocal arithmetic mean discretization schemes being given as follows

\[ U_{i,j+1}^2 = U_{i,j+1} U_{i+1,j+1}, \]  
\[ U_{i,j+1}^2 = \left( \frac{U_{i-1,j+1} + U_{i+1,j+1}}{2} \right) U_{i,j+1}, \]  
\[ U_{i,j+1}^2 = \left( \frac{U_{i-1,j+1} + U_{i+1,j+1}}{2} \right) U_{i,j+1}^2. \]

Generally, the second-order implicit finite difference approximation equation for problem (2) can be easily shown as

\[ \frac{U_{i,j+1} - U_{i,j}}{\Delta t} + G_{i,j+1} \left( \frac{U_{i+1,j+1} - U_{i-1,j+1}}{2\Delta x} \right) = \frac{v}{(\Delta x)^2} \left( U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1} \right) \]  

where

\[ G_{i,j+1} = F \left( x_i, t_{j+1}, U_{i,j+1} \right) \]

Since, the expression \( G_{i,j+1} \left( \frac{U_{i+1,j+1} - U_{i-1,j+1}}{2\Delta x} \right) \) shown the nonlinear term of problem (1), the nonlocal arithmetic mean scheme is used to transform the corresponding nonlinear implicit approximation equation into a system of linear equations. To do this, equation (8) can be transformed via the nonlocal arithmetic mean discretization scheme as follows

\[ G_{i,j+1} = F \left( x_i, t_{j+1}, \frac{U_{i+1,j+1} + U_{i-1,j+1}}{2} \right) \]

Referring to equation (7), this implicit approximation equation can be rewritten in the following equation,

\[ -a_i U_{i-1,j+1} + b U_{i,j+1} - c_i U_{i+1,j+1} = F_i \]

where,

\[ a_i = G_{i,j+1} + \frac{v}{(\Delta x)^2}, \quad b = 1 + \frac{2v\Delta t}{(\Delta x)^2}, \quad c_i = \frac{v}{(\Delta x)^2} - G_{i,j+1}, \quad F_i = \frac{U_{i,j}}{\Delta t}. \]

Based on Figure 1 at any time level \((j + 1)\), the implicit approximation equation (10) leads a generated system of implicit approximation equations as follows

\[ A U_{j+1} = F_i \]

where,

\[ A = \begin{bmatrix} b & -c & & & & & & \\ -a_{2} & b & -c & & & & & \\ & & -a_{3} & b & -c_{3} & & & \\ & & & 0 & 0 & 0 & & \\ & & & & & -a_{m-1} & b & -c_{m-1} \\ & & & & & & -a_{m-1} & b \end{bmatrix} \]
\[ U_{j+1} = \begin{bmatrix} U_{i,j+1}, U_{2,j+1}, \ldots, U_{m-2,j+1}, U_{m-1,j+1} \end{bmatrix}^T \]
\[ F_j = \begin{bmatrix} F_{i,j} - a_i U_{i-1,j+1}, F_{2,j} - a_{i+1,j+1} - c_i U_{m,j+1} \end{bmatrix}^T \]

3. Formulation of Explicit Group SOR Iterative Methods

Referring to the linear system in equation (11), it can be seen that its coefficient matrix is large scale. In the effort of getting the numerical solution of the linear system, many studies on various iterative methods had been used for solving any linear system. For an instance, Young [7,8,9] has proposed the SOR iterative method, Saudi and Sulaiman [10,11] has proposed weighted block iterative method using 9-point Laplacian for solving the Laplace’s equation and Evans [12] also proposed four-point block iterative methods via the Explicit Group SOR iterative methods.

Based on the approximation equation in equation (10), the general formulation of the SOR iterative method at time level \((j + 1)\), can be stated as [7,8,9]

\[ U_{i,j+1}^{(k+1)} = (1 - \omega)U_{i,j+1}^{(k)} + \omega \left( F_{i,j} - a_{i-1,j+1}U_{i-1,j+1} - c_i U_{i+1,j+1} \right) \quad (12) \]

for \(i = 1, 2, \ldots, m - 1, j = 0, 1, 2, \ldots, n - 1\) where \(\omega\) represents as a relaxation factor, this parameter, \(\omega\) can be calculated practically by selecting values periodically until the optimum value of \(\omega\) is obtained between the range \(1 \leq \omega < 2\). The parameter \(\omega\) can be used to accelerate the convergence rate and absolute error at any time level can be reduced. The optimal approximate value of \(\omega\) is chosen in which its number of iterations is the smallest. As taking \(\omega = 1\), the SOR method in equation (12) can be reduced as the GS method, which acts as a control method in this study.

As mentioned before, to study the effectiveness of the 2-point EG scheme, let us consider a block

\[
\begin{bmatrix}
  b & -c_i \\
  -a_2 & b
\end{bmatrix}
\begin{bmatrix}
  U_{i,j+1} \\
  U_{i+1,j+1}
\end{bmatrix}
= 
\begin{bmatrix}
  s_1 \\
  s_2
\end{bmatrix}
\quad (13)
\]

of two node points be defined as

where

\[ s_1 = a_i U_{i-1,j+1} + \frac{1}{\Delta t} U_{i,j}, \quad s_2 = c_i U_{i+1,j+1} + \frac{1}{\Delta t} U_{i+1,j}. \]

If the coefficient matrix in equation (13) can be inverted, the general scheme of the 2-point Explicit Group Gauss-Seidel (EGGS) as iterative method can be written as

\[
\begin{bmatrix}
  U_{i,j+1} \\
  U_{i+1,j+1}
\end{bmatrix}
^{(k+1)} = 
\frac{1}{b^2 - (c_i)(a_2)}
\begin{bmatrix}
  b & c_i \\
  -a_2 & b
\end{bmatrix}
\begin{bmatrix}
  s_1 \\
  s_2
\end{bmatrix}
\quad (14)
\]

By referring to equation (12) and adding a weighted parameter, \(\omega\) in equation. (14), the implementation of the 2-Point EGSOR iterative method can be stated as

\[
\begin{bmatrix}
  U_{i,j+1} \\
  U_{i+1,j+1}
\end{bmatrix}
^{(k+1)} = (1 - \omega)
\begin{bmatrix}
  U_{i,j+1} \\
  U_{i+1,j+1}
\end{bmatrix}
^k
\begin{bmatrix}
  b & c_i \\
  -a_2 & b
\end{bmatrix}
\begin{bmatrix}
  s_1 \\
  s_2
\end{bmatrix}
\quad (15)
\]
Hence, based on equation (15), the general algorithm of EGSOR iterative methods to solve the linear system (11) can generally be explained in Algorithm 1.

### Algorithm 1: EGSOR method

1. Initialize $U_{j+1}^{(0)} \leftarrow 0$, $\epsilon \leftarrow 10^{-10}$
2. Assign the optimal value of $\omega$
3. Calculate $U_{j+1}^{(k+1)}$ using
   \[
   \frac{U_{i,j+1}^{(k+1)}}{U_{i+1,j+1}^{(k+1)}} = (1 - \omega) \left[ \frac{U_{i,j+1}^{(k)}}{U_{i+1,j+1}^{(k)}} \right]^{\omega} - \frac{b^2 - (c_1)(a_2)}{a_2} \left[ s_1 \right]
   \]
4. Perform the convergence test, $\left| U_{i,j+1}^{(k+1)} - U_{i,j+1}^{(k)} \right| \leq \epsilon = 10^{-10}$. If yes, proceed to step (v). Otherwise go back to step (iii).
5. Display approximate solutions.

### 4. Numerical Experiments

In this section we proposed four examples of problem (1) in order to verify the effectiveness of the EGSOR iterative method together with nonlocal arithmetic mean scheme as compared to GS and SOR iterative methods. For the comparison purpose, three criteria will be considered such as number of iterations, execution time (second) and maximum absolute error to be recorded in Table 1. All numerical results obtained from the implementation of GS, SOR and EGSOR iterative methods have considered the tolerance error, $\omega = 10^{-10}$ at different grid sizes, $m = 256, 512, 1024, 2048$ and 4096.

The following are four examples that have been used to test the effectiveness of the proposed iterative methods.

#### Example 1 [1]

We consider the following initial value equation:

\[
\frac{x}{1 + \exp\left(\frac{1}{4v}(x^2 - \frac{1}{4})\right)}, \quad \text{for } t > 0.
\]  

with exact solution of problem (16) is given by

\[
u(x,t) = \frac{x}{t^{\frac{1}{2}}} \exp\left(\frac{x^2}{4vt}\right), \quad \text{where } t_o = \exp\left(\frac{1}{8v}\right).
\]  

#### Example 2 [2]

We consider the following initial value equation:

\[
u(x,0) = 2x, \quad \text{for } t > 0.
\]  

with exact solution of problem (18) is given by

\[
u(x,t) = \frac{2x}{1 + 2t}.
\]
Table 2

| Example | M | Number of iterations | Execution time | Maximum absolute error |
|---------|---|----------------------|----------------|------------------------|
|         | GS | SOR | EGSOR | GS | SOR | EGSOR | GS | SOR | EGSOR | GS | SOR | EGSOR |
|         | 256 | 113 | 26 | 19 | 0.14 | 0.08 | 0.05 | 1.603E-04 | 1.603E-04 | 1.603E-04 |
| 1       | 512 | 392 | 48 | 34 | 0.91 | 0.15 | 0.10 | 1.634E-04 | 1.632E-04 | 1.632E-04 |
| 2       | 1024 | 1402 | 89 | 63 | 6.27 | 0.51 | 0.33 | 1.645E-04 | 1.639E-04 | 1.640E-04 |
| 3       | 2048 | 5015 | 163 | 119 | 44.96 | 1.90 | 1.21 | 1.661E-04 | 1.642E-04 | 1.642E-04 |
| 4       | 4096 | 17757 | 315 | 226 | 233.16 | 7.20 | 4.59 | 1.722E-04 | 1.642E-04 | 1.642E-04 |
|         | 256 | 9391 | 317 | 243 | 10.12 | 0.43 | 0.28 | 2.311E-04 | 2.317E-04 | 2.317E-04 |
|         | 512 | 34224 | 633 | 481 | 73.85 | 1.55 | 1.06 | 2.292E-04 | 2.317E-04 | 2.317E-04 |
|         | 1024 | 123648 | 1328 | 945 | 535.63 | 6.38 | 4.05 | 2.217E-04 | 2.317E-04 | 2.317E-04 |
|         | 2048 | 441778 | 2446 | 1910 | 3851.61 | 24.28 | 16.25 | 1.918E-04 | 2.317E-04 | 2.317E-04 |
|         | 4096 | 1556249 | 4913 | 3780 | 30095.34 | 94.12 | 63.90 | 7.776E-05 | 2.317E-04 | 2.317E-04 |
|         | 256 | 1092 | 143 | 86 | 1.20 | 0.28 | 0.13 | 7.396E-04 | 7.399E-04 | 7.399E-04 |
|         | 512 | 3986 | 274 | 167 | 8.72 | 0.68 | 0.39 | 7.372E-04 | 7.387E-04 | 7.387E-04 |
|         | 1024 | 14490 | 530 | 321 | 65.59 | 2.54 | 1.41 | 7.322E-04 | 7.384E-04 | 7.384E-04 |
|         | 2048 | 52197 | 1020 | 614 | 459.38 | 9.72 | 5.28 | 7.133E-04 | 7.383E-04 | 7.384E-04 |
|         | 4096 | 185762 | 2143 | 1177 | 3310.30 | 41.59 | 29.17 | 6.384E-04 | 6.383E-04 | 7.384E-04 |
|         | 256 | 30 | 22 | 13 | 0.11 | 0.08 | 0.06 | 1.044E-08 | 8.013E-10 | 2.065E-10 |
|         | 512 | 89 | 41 | 24 | 0.24 | 0.13 | 0.09 | 4.647E-08 | 4.839E-08 | 1.243E-09 |
|         | 1024 | 304 | 80 | 46 | 1.33 | 0.40 | 0.22 | 1.852E-07 | 2.363E-08 | 1.099E-08 |
|         | 2048 | 1076 | 154 | 90 | 9.24 | 1.46 | 0.82 | 7.601E-07 | 4.470E-08 | 1.668E-08 |
|         | 4096 | 3818 | 300 | 178 | 65.28 | 5.63 | 3.04 | 3.048E-06 | 8.665E-08 | 3.135E-08 |

Example 3 [13]
We consider the following initial value equation:

\[ u(x, 0) = 2v \frac{\pi \sin(\pi x)}{\sigma + \cos(\pi x)}, \quad \text{for} \quad t > 0. \]  \hspace{1cm} (20)

with exact solution of problem (20) is given by

\[ u(x, t) = \frac{2v e^{-\pi \sigma/t} \sin(\pi x)}{\sigma + e^{-\pi \sigma/t} \cos(\pi x)}. \]  \hspace{1cm} (21)

Example 4 [14]
Consider Eq. (1) with initial and boundary condition are taken from the exact solution [5]:

\[ u(x, 0) = \frac{v}{1 + \tan \left( \frac{x}{2 + 2v} \right)}, \quad 0.5 \leq x \leq 1.5, \quad t = 0. \]  \hspace{1cm} (22)
From above four examples, results of numerical experiments obtained from implementation of GS, SOR and EGSOR iterative methods have been summarized in Table 1 with different grid sizes, \( m = 256, 512, 1024, 2048 \) and \( 4096 \). Meanwhile, the reduction percentage of GS, SOR and EGSOR iterative methods are obtained in Table 2. Based on the numerical results obtained in Table 1 and Table 2, it clearly shows that the number of iterations and execution time of the SOR iterative method are smaller than GS iterative method. The number of iterations of SOR iterative method compared to GS iterative method has decreased by approximately 77.0\% – 98.2\%, 96.6\% – 99.7\%, 86.9\% – 98.8\% and 26.7\% – 92.1\% respectively. Meanwhile, the execution time of SOR iterative method also decreased by 42.9\% – 97.8\%, 95.8\% – 99.7\%, 83.3\% – 98.7\% and 27.3\% – 91.4\% as compared with GS iterative method.

In Table 1 also shows that the number of iterations and execution time of EGSOR iterative method are smaller than GS iterative method. The number of iterations of EGSOR iterative method compared to GS iterative method has decreased by approximately 83.2\% – 98.7\%, 97.4\% – 99.8\%, 92.1\% – 99.4\% and 56.7\% – 95.3\% respectively. Meanwhile, the execution time of EGSOR iterative method also decreased by 64.3\% – 98.6\%, 97.2\% – 99.8\%, 89.2\% – 99.4\% and 45.5\% – 95.3\% as compared with GS iterative method. Based on the reduction percentage in Table 2, it can be concluded that the EGSOR is the most efficient method as compared with GS and SOR iterative method.

**Conclusion**

In this paper, it can be observed that problem (1) has been discretized by using second-order implicit scheme to derive the corresponding nonlinear second-order implicit finite difference approximate equation. Then, the nonlocal arithmetic mean discretization scheme is used to transform the corresponding nonlinear implicit approximation equation into a system of linear equations. Based on the approximation equation, the formulation of GS, SOR and EGSOR iterative methods have been presented. Overall, it clearly shows that by applying EGSOR iterative method, the number of iterations and execution time have declined tremendously as compared with GS and SOR iterative methods. In terms of the accuracy of numerical solution obtained, all result of three proposed iterative methods give in a good agreement.

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