On the Quantizations of the Damped Systems*

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Abstract

Based on a simple observation that a classical second order differential equation may be decomposed into a set of two first order equations, we introduce a Hamiltonian framework to quantize the damped systems. In particular, we analyze the system of a linear damped harmonic oscillator and demonstrate that the time evolution of the Schrödinger equation is unambiguously determined.

* This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contract #DOE91ER40651.B
The search for field theoretic formulations of fundamental anyons$^{1,2}$ has refocused attention on the long outstanding problem of the quantization of (non-Hamiltonian) systems$^{1−7}$ in which only the equation of motion is explicitly known. The common feature of this class of systems is its history dependence. The most notable such system is certainly that of the damped harmonic oscillator. Apart from requiring the knowledge of quantum aspects of this class of systems for practical reasons (for example, the laser system), this problem has its own right to be investigated both for the mathematical interests and exploring the new (perhaps deeper) fundamental features of the quantum theory.

There have been quite a number of attempts to solve this problem, which may be classified into two classes. One is to start with a modified quantization scheme$^{4,5}$ or a bigger system with introduction of additional degrees of freedom$^6$ such that the conventional Schrödinger or Heisenberg dynamical description is valid. The other$^7$ is to formulate the quantum theory in a modified Schrödinger dynamics (a nonlinear one). Clearly, attempts in the second class do not respect the superposition principle as the nonlinearity enters as a consequence of wavefunction dependent potentials. Similarly, the first class of the attempts also has some disadvantages. The introduction of the non-hermitian or time-dependent Hamiltonian in this class often leads to the conclusion of non-existence of the Schrödinger wave description.$^4$

As is usually done in quantum mechanics, the time evolution of the Schrödinger dynamics may be carried out only if one insures the completeness of the Hamiltonian eigenstates. The hermitian Hamiltonian in our standard theory guarantees unique unitary time evolution. For dissipative systems, the time evolution is certainly no longer unitary, yet one
may hope that the Schrödinger quantum mechanics may be still used for unambiguously determining the time evolution. The purpose of this paper is to present an alternative way of modifying the quantization framework in which the Lagrangians are introduced naturally and to show that the Schrödinger dynamics is indeed achieved. Thus the above mentioned puzzle will be resolved.#

We shall begin with a general damped system on the line in an external potential, whose classical motion is govern by the non-linear second order differential equation:

\[ \ddot{x} + k(x)\dot{x} + g(x) = 0, \]  

where the dots denote time derivatives and \( k(x) \) and \( g(x) \) are real functions. We observe that a reduction of Eq. (1) to

\[ \dot{x} + f(x) = 0 \]  

may be defined if \( f \) satisfies

\[ f(f' - k) + g = 0, \]  

where the prime represents an \( x \)-derivative. Note the complex conjugate \( f^* \) of \( f \) will be also a solution of Eq. (3) if \( f \) is a solution. One may verify this reduction by directly differentiating Eq. (2) with respect to time and using of the condition (3). Roughly speaking, this reduction reflects the fact that a second order differential equations may be decomposed into a set of two first order equations. For the obvious reasons, this reduction

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# During the preparation of this paper, we receive a paper by Feshbach and Tikochinsky,\(^5\) in which this puzzle was also resolved in a different way. We thank Roman Jackiw for bringing this reference to our attention.
has at least two advantages. Classically, a first order equation (linear or not) is of course simpler than the second order one,* while quantum mechanically, the first order equations may be regarded as the Hamiltonian equations by properly defining the canonical variables. For later convenience, we denote \( x_i (i = 1, 2) \) as the two (independent) solutions of Eq. (1) or the solutions of the follow set of the first order equations:

\[
\dot{x}_i + f_i(x_i) = 0,
\]

where \( f_i \) are two (independent) solutions of Eq. (3). One may easily show that \( x = x_1 + x_2 \) will be a solution of Eq. (1) only for the linear damped case, namely, \( k=\text{constant} \) and \( g \) is a linear function of \( x \).

The Lagrangians \( L \)

\[
L = \sum_{i=1}^{2} \left[ \dot{x}_i p_i - x_i \dot{p}_i + 2f_i p_i \right],
\]

which may be naturally introduced and give rise to Eq. (3) and

\[
\dot{p}_i - \frac{1}{2} (f'_i(x_i)p_i + p_i f'_i(x_i)) = 0,
\]

where no summation is assumed for the repeat index and the symmetric form is used for the usual reason. Thus the Hamiltonian is obtained in the usual manner (we use the symmetric form to deal with the ambiguous quantum ordering):

\[
H = -\sum_{i} (f_i p_i + p_i f_i).
\]

The symplectic two-form for the system is

\[
\omega = 2 \sum_{i} dp_i \wedge dx_i,
\]

* These ideas may be used to solve the sine-Gordon and other nonlinear equations.8
which defines the Poisson brackets. Therefore, the commutation relations are

\[ [x_i, x_j] = [p_i, p_j] = 0 \]

\[ [x_i, p_j] = \frac{i}{2} \hbar \delta_{ij}. \]  

(9)

Using the rules (9) and the Hamiltonian (7), one may easily verify that the classical equations (4) and (6) are quantum mechanically realized (up to the quantum corrections \( O(\hbar) \)) as the Heisenberg equations:

\[ \dot{x}_j = \frac{1}{i\hbar} [x_j, H], \]

\[ \dot{p}_j = \frac{1}{i\hbar} [p_j, H]. \]  

(10)

Note that \( x_i \) and \( p_i \) may not be considered as hermitian operators and in the case of \( f_1 = f_2^* \) the consistency condition requires the identifications \( x_1 = x_2^* \) and \( p_1 = p_2^* \) classically (in quantum case, they are \( x_1 = x_2^\dagger \) and \( p_1 = p_2^\dagger \)). We will keep this in mind for later discussion.

Now let us analyze the case of a linear damped harmonic oscillator, namely \( k = 2\lambda \geq 0 \) and \( g(x) = \omega^2 x \) (mass=1). Thus we have \( f = -\eta x \) from Eq. (3), where \( \eta_i = -\lambda \pm \sqrt{\lambda^2 - \omega^2} \eta_i(2) \) with the upper (lower) sign] with the convention \( \eta_1 \geq \eta_2 \) (or \( \text{Im} \eta_1 \geq \text{Im} \eta_2 \)), if \( \eta_i \) are real (or complex). Namely, the square root in this paper will be always positive and \( \sqrt{\lambda^2 - \omega^2} = i\sqrt{\omega^2 - \lambda^2} \) for \( \lambda < \omega \). The equations of motion (4) and (6) then become

\[ \dot{x}_i - \eta_i x_i = 0, \]

\[ \dot{p}_i + \eta_i p_i = 0. \]  

(11)

The \( p \)-system is clearly related to the \( x \)-system by a time-reversal transformation \( U_t : t \to -t \), or equivalently \( U_\lambda : \lambda \to -\lambda \). Note that since the complex index \( i \) or the square root is introduced in our reduction, the time reversal transformation has been identified as the complex conjugation or simply shifting \( \sqrt{\quad} \) to \( -\sqrt{\quad} \) in the above. Therefore the combined system \( (x = x_1 + x_2, p = p_1 + p_2) \) may be viewed to be time-reversal invariant.
The Hamiltonian $H$ is from Eq. (7)

$$H = i\hbar(\eta_1 N_1 + \eta_2 N_2), \quad (12)$$

which is indeed invariant under $U_\alpha (\alpha = t, \lambda)$ and $N_i(i = 1, 2)$ are defined as follows

$$N_i = \frac{1}{2}(a_i b_i + b_i a_i) - \frac{1}{2}(A_i B_i + B_i A_i)$$

$$a_i = i A_i, \quad b_i = i B_i, \quad x_i = a_i \exp(\eta_i t), \quad p_i = \frac{1}{2} i \hbar b_i \exp(-\eta_i t). \quad (13)$$

One may easily verify the following commutation relations using the rules

\[ [a_i, a_j] = [b_i, b_j] = 0, \quad [a_i, b_j] = \delta_{ij} \]
\[ [A_i, A_j] = [B_i, B_j] = 0, \quad [B_i, A_j] = \delta_{ij} \]
\[ [N_i, a_j] = -a_i \delta_{ij}, \quad [N_i, b_j] = b_i \delta_{ij} \]
\[ [N_i, A_j] = -A_i \delta_{ij}, \quad [N_i, B_j] = B_i \delta_{ij} \]
\[ [N_1, N_2] = 0. \quad (14) \]

From the relations (14), we immediately have

\[ N_i |n_i^{(\pm)}\rangle = \pm (n_i^{(\pm)} + \frac{1}{2}) |n_i^{(\pm)}\rangle \]
\[ a_i |n_i^{(+)}\rangle = \sqrt{n_i^{(+)} |n_i^{(+)}\rangle - 1} \]
\[ b_i |n_i^{(+)}\rangle = \sqrt{n_i^{(+)} + 1 |n_i^{(+)} + 1\rangle} \]
\[ B_i |n_i^{(-)}\rangle = \sqrt{n_i^{(-)} |n_i^{(-)}\rangle - 1} \]
\[ A_i |n_i^{(-)}\rangle = \sqrt{n_i^{(-)} + 1 |n_i^{(-)} + 1\rangle} \]
\[ \langle n_i^{(\pm)} |n_i^{(\pm)}\rangle = \delta_{n_i n_i'} \delta_{\alpha \alpha'} \quad (\alpha = +, -) \]

\[ (n_i^{(\pm)} = 0, 1, \ldots) \]

(15)

Therefore the Hamiltonian $H$ is exactly of the harmonic oscillator type and has the same wave functions (of course, with the completeness).
Note that since there is no prior reason to identify \( a_i \) (\( b_i \)) as the annihilation (creation) operators, we have used \( A_i = ia_i \) and \( B_i = ib_i \) (these transform the positions into the momenta and vice versa) to obtain the second set of solutions. Namely \( B_i \) (\( A_i \)) are our new annihilation (creation) operators. Alternatively, we may simply identify \( A'_i = -a_i \) as new creation operators. All these are similar to the usual case in which the canonical momentum operator consists of the operations of both annihilation and creation. Furthermore, it is worth pointing out that we have a similar situation to Ref. [5]. An operator, typically \( A = xp + px \), \([x, p] = i\hbar \) (in our case, \( x_i p_i + p_i x_i \)), seems to be hermitian but possess the pure imaginary eigenvalues. We resolve this apparent contradiction in a similar way to that in Ref. [5], namely, giving up the hermitian condition of \( A \). Actually, an alternative method may also be employed, in which one may directly solve \((-2ixd_x - i)\hbar \psi = E\psi \) and assume \( \psi \) to be single-value. Immediately, one gets \( E^\pm_n = \pm i(2n^\pm + 1)\hbar(n^\pm \geq 0, \text{integer}) \). The completeness of the states will be followed as the coherent type weight functions appear in the measure of the inner products.

The above discussion also agrees with the representation theory as below. If one introduces

\[
J_1^i = \frac{1}{4}(a_i^2 + b_i^2) \\
J_2^i = \frac{1}{4}i(a_i^2 - b_i^2) \\
J_3^i = \frac{1}{2}N_i,
\]

one may easily verify that \( J_\beta^i (\beta = 1, 2, 3) \) are generators of the three dimensional Lorentz group \( SO(2, 1) \) or equivalently the two dimensional symplectic group \( Sp(2, R) \):

\[
[J_1^i, J_2^i] = -iJ_3^i, \quad [J_2^i, J_3^i] = iJ_1^i, \quad [J_3^i, J_1^i] = iJ_2^i.
\]
According to Bargmann,\(^9\) all irreducible unitary representations of \(SO(2, 1)\) can be classified as follows: \(D^{(+)}_\mu (\mu < 0)\), \(D^{(-)}_\mu (\mu < 0)\), principal series, supplementary series, and the identity representation, where \(\mu\) is the Casimir parameter, i.e. the Casimir operator 
\[
J_3^2 - J_1^2 - J_2^2 = \mu(\mu + 1).
\]
In terms of the \(A's\) and \(B's\), there will be an additional minus sign in the \(J's\). Because of the sign change of the \(J_i^3\), 
\[
- J_i^3 = \frac{1}{2}(A_i B_i + B_i A_i)
\]
will have the bounded from below spectrum as the square integrable wave functions are chosen. Thus, 
\[
N^{(\pm)}(\pm n + \frac{1}{2}) = \mu = \pm \frac{1}{4} \quad (\mu \text{ is determined by the commutation relations of \(a's\) and \(b's\) or \(A's\) and \(B's\).})
\]
Note that for other values of \(\mu\), one obtains the representations for one dimensional anyons,\(^10\) or for parabosons,\(^11\) depending on one’s viewpoint.

Clearly, the spectrum of \(H\) contains all information of the damped system \(x_i\) and its time reversed partner(growing one) \(p_i\), as one expects. However, the spectrum of \(H\) is lost the usual meaning of energy levels and \(H\) may only be viewed as the dynamical generator.

To be more specific, we focus on the case of the underdamping, \(\lambda < \omega\). In this case, the spectrum of \(H\) can be explicitly displayed as
\[
E^{(+)}_{n_1 n_2} = \hbar \Omega (n_2^{(+)} - n_1^{(+)}) - i \hbar \lambda (n_1^{(+)} + n_2^{(+)} + 1)
\]
\[
E^{(-)}_{n_1 n_2} = \hbar \Omega (n_1^{(-)} - n_2^{(-)}) + i \hbar \lambda (n_1^{(-)} + n_2^{(-)} + 1),
\]
where \(\Omega = \sqrt{\omega^2 - \lambda^2}\) and \(E^{(+)}_{n_1 n_2}\) describe decaying states while \(E^{(-)}_{n_1 n_2}\) describe growing states. One may easily demonstrate that the states with \(E^{(+)}\) is indeed the time reverse of the states with \(E^{(-)}\). Our results (18) agree with those appeared in Ref. \[5\]. Since the well-known harmonic oscillator wave functions are complete, the time-evolution of the Schrödinger equation can be solved without trouble and of course is no longer unitary due to the damping force. For the harmonic oscillator, we see that the states of \(H\) becomes
those of the usual undamped harmonic oscillator in the limit $\lambda \to 0$ if only those states, satisfying the conditions $a_1|\psi> = b_2|\psi> = 0$ are considered. This is similar to neglect the negative energy states which are physically unstable as we interpret $H$ as the energy operator.

Finally, we explicitly present here the Schrödinger time evolution under $H$ for a given initial state $|\psi(0)> = \sum_{n_{1,2}} c_{n_{1,2}}|n_{1,2}>$, where $E_{n_{1,2}}$ are spectrum of $H$. Alternatively, one may easily write down the Schrödinger time evolution of an initial state $|\phi(0)> = \sum_{n_{1,2}} c_{n_{1,2}}|n_{1,2}>$ which decays with time

$$|\phi(t)> = \sum_{n_{1,2}} c_{n_{1,2}} e^{-iE_{n_{1,2}}t}|n_{1,2}>.$$  \hfill (19)

The time dependent operator expectation values can be obtained from either the equations of motion or from the standard Heisenberg representations. For example,

$$\bar{x}_i = <\psi(t)|x_i|\psi(t)>$$

$$= <\psi(0)|x_i(t)|\psi(0)> = <\psi(0)|a_i|\psi(0)> e^{n_i t},$$

where by definition $x_i = \exp(iHt/\hbar)a_i\exp(-iHt/\hbar)$ (from the Heisenberg equation (10)).

Note the definition of the “bra” $<\psi(t)|$ is changed to the time reversed conjugation of $|\psi(t)>$, rather than the usual hermitian conjugation of $|\psi(t)>$. This is natural since the Hilbert space is defined by our time reversal invariant operator $H$. The same definition of the inner products appeared also in Ref. [5]. Likewise, one may easily write down the
correlations

\[ <\psi(0)|x_1(t)x_2(t')|\psi(0)> = <\psi(0)|a_1a_2|\psi(0)> e^{\eta_1t + \eta_2t'}. \] (22)

In conclusion, we have analyzed the linear damped harmonic oscillator using the Hamiltonian framework, which is naturally introduced for describing this system and its time reversed partner. We have shown that the conventional Schrödinger dynamics is explicitly realized. Furthermore, the Schrödinger wave description has the usual interpretation and our method may be easily generalized to quantize three dimensional damped systems. Second quantization may also be carried out so that a field theory for this model is realized.

Since our analysis starts directly by quantizing the classical solutions or phase space, as we have demonstrated, our method may yield a straightforward framework to study general non-Hamiltonian systems. We outline here how this may work. One will need first to identify the asymmetries which are responsible for the non-existence of the Hamiltonian of a system. These asymmetries are usually related to the time-reversal transformations because non-Hamiltonian systems have the common feature of history dependence. Then to find the classical solutions of both the systems will be the next step. Finally, one may find the Hamiltonian in which the original asymmetries will be restored to describe the combined system and the canonical quantization will be carried out directly. One such immediate example is the field theory of anyons. In 2+1 dimensions, the positive and negative spin corresponds to the different models\(^1\,^2\) and they are related by the time reversed symmetry (similar to the magnetic like interactions),\(^2\) one may hope that our method may be used for quantizing anyonic field theories in a similar manner.
Acknowledgements

We thank V. P. Nair and H. C. Ren for fruitful discussions, and M. D. Doyle and V. P. Nair for reading the manuscript.

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