Momentum exchange between an electromagnetic wave and a dispersive medium

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We present an elementary discussion of the momentum transferred by an electromagnetic wave propagating in a dispersive medium. Our analysis is based on Minkowski’s electromagnetic momentum density which have been recently seen to be consistent with a fully covariant expression of the energy-momentum tensor of the electromagnetic field in a dispersive medium and with all the experimental evidence. We show that the medium may be either pulled or pushed as the electromagnetic pulse enters in it depending on the value of the frequency.

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The description of momentum exchange between an electromagnetic wave and a dielectric media is an intriguing issue which has been dominated historically by Abraham-Minkowski’s controversy. Initially the controversy was focused on which, between Minkowski’s energy density and Abraham density was defined above. In this process, it became necessary to re-interpret even such a familiar concept as Poynting’s energy density. Poynting\textsuperscript{3} proposed his energy conservation equation for homogeneous isotropic materials with time-independent linear susceptibility. Later on, it was recognized the necessity of identifying the electromagnetic contributions to the matter energy-momentum tensor and the importance of a better understanding of the criteria which allow to distinguish between matter and field. In the process, we have shown that (3) does not represent the field energy alone but corresponds to the energy of the whole electromagnetic wave, understood as a mixed entity with contributions of the field and the polarizations. We first identify the force density on matter to be given by,

\[ f_{d}^{\mu} = \frac{1}{2} D_{\alpha \beta} \partial_{\mu} F^{\alpha \beta} , \]

with \( D_{\alpha \beta} \) the space-time dipolar density, whose spatial part is the magnetization density \( D_{ij} = \epsilon_{ijk} M_{k} \) and whose temporal part is the polarization \( D_{0k} = -D_{k0} = P_{k} \). Then we show that the energy momentum tensor of the electromagnetic field compatible with this force is given by

\[ T_{\mu \nu}^{\text{em}} = -\frac{1}{16\pi} F_{\alpha \beta} F^{\alpha \beta} \eta_{\mu \nu} + \frac{1}{4\pi} F_{\alpha}^{\mu} H^{\nu \alpha} , \]

where \( H^{\mu \nu} = F^{\mu \nu} - 4\pi D^{\mu \nu} \) and \( \eta_{\mu \nu} \) is the metric tensor with the signature \((-,+,+,+\)). For this tensor, which is valid in particular for a dispersive polarizable medium, the energy density is

\[ u_{F} = T_{F}^{00} = \frac{1}{8\pi} (E^{2} + B^{2}) + E \cdot P , \]

the energy current density \( cT_{F}^{0i} = S = cE \times H/4\pi \) is Poynting’s vector and the momentum density \( c^{-1}T_{F}^{0i} = g_{F} = D \times B/4\pi c \) coincides with Minkowski’s expression\textsuperscript{1}. Maxwell’s stress tensor is given by,

\[ T_{F}^{ij} = \frac{1}{8\pi} (E^{2} + B^{2}) \delta_{ij} - B \cdot M \delta_{ij} - \frac{1}{4\pi} (E_{i} D_{j} + H_{i} B_{j}) . \]

The difference between the energy density above and the energy density of the vacuum, \( u_{F} - u_{\text{vac}} = P \cdot E \), is the
negative of the electrostatic energy density of polarization which should be considered part of the energy of matter. This makes physical sense because it contributes to the inertia of matter in exactly the same way as nuclear interaction energy contributes to nuclei mass. Note that there is no similar magnetic term since no potential magnetic energy exists. The energy density of matter is given by,

\[ u_M(P, M, E) = u_b(P, M) - P \cdot E \]  

(10)

where \( u_b \) is the bare energy density that does not depend on the electromagnetic field. It may be splitted in a term which does not depend on \( P \) and \( M \) and one that does,

\[ u_b(P, M) = u_{b0} + u_{PM}(P, M) \]  

(11)

Similarly the momentum density of matter is shown to have the structure,

\[ g_M(P, M, E, B) = g_b(P, M, E) - \frac{1}{c} P \times B \]  

(12)

When matter is immersed in an electromagnetic field the energy density of matter changes. This energy difference can be calculated integrating the power expression obtained from \( E \). For time-independent linear polarizabilities the work done on matter does not depend on the way the fields change in time, it depends only on the final values of the fields

\[ \Delta u_M = u_M(E, B) - u_M(0, 0) = \int dw_d \]

(13)

\[ = -\frac{1}{2} (E \cdot P + B \cdot M) \]

Since it includes the electrical potential energy density \(-E \cdot P\) we can write

\[ \Delta u_M = -E \cdot P + \frac{1}{2} (E \cdot P - B \cdot M) \]  

(14)

Poynting’s energy density turns out to be

\[ u_P = u_{Pd} + \Delta u_M = \frac{1}{8\pi} (E^2 + B^2) + \frac{1}{2} (E \cdot P - B \cdot M) \]  

(15)

supporting the interpretation mentioned above. In \([5]\), we discuss how this approach is consistent with all the experimental evidence.

When these ideas are applied to the elementary example of an electromagnetic wave propagating in a linear homogeneous polarizable medium interacting with a conducting sheet \([6]\) we found that conservation of momentum is achieved when Minkowski’s expression, which is consistent with our approach, is used but not when Abraham’s expression is considered.

In this letter we are interested in momentum exchange between an electromagnetic pulse and a dispersive medium in the simplest case, when only a resonant frequency appears in the dispersion relation. As the electromagnetic wave passes through the medium it induces a dipole momentum in the atoms but in first approximation, it does not produce any appreciable changes in its positions. In a simple textbook approach we can take the medium as a collection of damped harmonic oscillators satisfying,

\[ m\ddot{r} = -kr - \alpha m\dot{r} - eE \]  

(16)

with \( m, k \) and \( \alpha \) phenomenological constants. The dipolar momentum of the atom is \( p = -e\dot{r} \) and satisfies,

\[ \dot{p} = -\omega_0^2 p - \alpha \dot{p} + \frac{e^2}{m} E \]  

(17)

In an homogeneous material the polarization \( P \) is linear in the individual moments and we can write

\[ \dot{P} = -\omega_0^2 P - \alpha \dot{P} + \chi_0 \omega_0^2 E \]  

(18)

with \( \chi_0 \) a constant with susceptibility units. Independently of the simple model from which it has been deduced we take Eq.\([18]\) as the one which is characteristic of a dispersive medium with a single resonance. Note that the same equation is obtained up to second order in the quantum mechanically perturbative treatment.

For a plane wave in the usual complex notation we have the fields

\[ \dot{E} = E e^{-i\omega t} \]  

\[ \dot{P} = P e^{-i\omega t} \]  

(19)

related by the susceptibility \( \chi(\omega) \) by,

\[ P = \chi(\omega) E \]  

(20)

Substituting in the equation we have the usual single resonant expression for the susceptibility \([\overline{7}]\)

\[ \chi(\omega) = \frac{\chi_0 \omega_0^2}{\omega_0^2 - \omega^2 - i\omega} \]  

(21)

Let us first discuss energy exchange. Multiplying the equation above by \( \dot{P} \) we have,

\[ \frac{1}{2} \frac{d}{dt} \dot{P}^2 + \frac{\omega_0^2}{2} \frac{d}{dt} P^2 - \chi_0 \omega_0^2 \frac{d}{dt} (E \cdot P) + \chi_0 \omega_0^2 \dot{E} \cdot P = -\alpha \dot{P}^2 \]  

(22)

This is expressed more conveniently as,

\[ \frac{d}{dt} \left( \frac{1}{2\chi_0 \omega_0^2} \dot{P}^2 + \frac{1}{2\chi_0} P^2 - E \cdot P \right) = -\frac{\alpha}{\chi_0 \omega_0^2} \dot{P}^2 - \dot{E} \cdot P \]  

(23)

The first term in the right hand side corresponds to the energy loses of our model due to damping and goes to increase \( u_{b0} \). According to the relativistic expression of the force density \([\overline{6}]\), the second term is the power which the field gives to matter

\[ c_f^0 = -P \cdot \frac{\partial E}{\partial t} - M \cdot \frac{\partial B}{\partial t} \]  

(24)
Energy equilibrium is achieved by taking

$$u_{PM}(P, M) = \frac{1}{2\chi_0 \omega_0^2} \dot{P}^2 + \frac{1}{2\chi_0} P^2$$

(25)

in our model. The first term in (23) is naturally interpreted as the kinetic energy and the second term as the elastic term. We insist that \(-E \cdot P\) is the electrostatic polarization energy.

So, according to (10) and (8) we have the expressions,

$$u_M = u_{b0} + \frac{1}{2\chi_0 \omega_0^2} \dot{P}^2 + \frac{1}{2\chi_0} P^2 - E \cdot P$$

(26)

$$u_F = \frac{1}{8\pi} (E^2 + B^2) + E \cdot P$$

(27)

for the matter and field energy densities. Here \(u_{b0}\) is the non electromagnetic energy density of matter. The energy of the full electromagnetic wave is,

$$u_W = u_F + u_M - u_{b0}$$

$$= \frac{1}{8\pi} (E^2 + B^2) + \frac{1}{2\chi_0 \omega_0^2} \dot{P}^2 + \frac{1}{2\chi_0} P^2$$

(28)

This generalizes Poynting’s formula which is recovered in the static case.

In order to test this result in a specific situation we consider a dilute gas of polar atoms. We use the simplified textbook description of the phenomenon, assuming that there is no appreciable interaction between the atoms that constitute the gas and that the losses are negligible, \(\alpha \approx 0\). We suppose that the gas is confined in some region but that the boundary is diffuse so that there is not a reflected ray when the light hits the gas. We consider a wave packet coming from vacuum with a transversal section \(A\) and a length \(cT\) with \(T\) some time interval. For definiteness we take the electromagnetic pulse traveling in the direction of \(\hat{e}_1\). It has an energy density,

$$u = \frac{1}{8\pi} (E^2 + B^2) = \frac{1}{4\pi} (E^2)$$

(29)

and a time averaged energy density,

$$\langle u \rangle = \frac{1}{8\pi} (E_0^2)$$

(30)

which results in a total energy in vacuum given by

$$U_0 = cT A E_0 \frac{E_0}{8\pi}$$

(31)

Here \(E_0\) is the amplitude of the electric field in vacuum. In the medium the wave takes the form [7],

$$E = E_1 e^{i(kx_1 - \omega t)} \hat{e}_2, \quad B = B_1 e^{i(kx_1 - \omega t)} \hat{e}_3$$

(32)

where using \(\epsilon = 1 + 4\pi \chi(\omega)\), we have

$$k = \frac{\omega}{c} \sqrt{\epsilon} = \frac{\omega}{c} \sqrt{1 + 4\pi \chi(\omega)}, \quad B_1 = \sqrt{\epsilon} E_1$$

(33)

It has a phase velocity given by

$$v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1 + 4\pi \chi(\omega)}}$$

(34)

When the denominator is negative the medium is opaque. The group velocity is computed from

$$v_g = \frac{d\omega}{dk} = \frac{1}{\frac{d\omega}{d\omega}}$$

(35)

Doing the algebra one gets,

$$\frac{dk}{d\omega} = \frac{1}{v_{ph}(1 + 4\pi \chi(\omega))} [1 + 4\pi \chi(\omega) - \frac{\omega_0^2}{\omega_0^2 - \omega^2}]$$

(36)

$$v_g = \frac{v_{ph}(1 + 4\pi \chi(\omega))}{1 + 4\pi \chi(\omega) \frac{\omega_0^2}{\omega_0^2 - \omega^2}} = c \sqrt{\frac{(1 + 4\pi \chi(\omega))}{1 + 4\pi \chi(\omega) \frac{\omega_0^2}{\omega_0^2 - \omega^2}}}$$

(37)

The group velocity is always smaller than \(c\). In Figure 1 we show the dependency of the group velocity with the frequency. At low frequencies \(v_g \rightarrow v_{ph}\).

After the pulse has completely entered the medium it has a length \(v_g T\) and we will suppose that the energy loses are negligible. The energy of the wave in the gas should be computed using (28). For the time average we have to take care of the phase difference between \(\mathbf{P}\) and \(\mathbf{P}\). Denoting by \(E_1\) and \(P_1\) the magnitudes of \(\mathbf{E}\) and \(\mathbf{P}\) in the medium, we have \(P_1 = \chi(\omega) E_1\) and we get,

$$\langle \frac{1}{2\chi_0 \omega_0^2} \dot{P}^2 + \frac{1}{2\chi_0} P^2 \rangle = \frac{P_1^2 (\omega_0^2 + \omega^2)}{4\chi_0 \omega_0^2}$$

(38)

Then,

$$\langle u_W \rangle = \frac{1}{16\pi} (E_1^2 + \frac{c^2}{\epsilon_{ph}} E_1^2) + \frac{\chi^2(\omega)E_1^2 (\omega_0^2 + \omega^2)}{4\chi_0 \omega_0^2}$$

$$= \frac{E_1^2}{8\pi} [1 + 4\pi \chi(\omega) \frac{\omega_0^2}{\omega_0^2 - \omega^2}]$$

(39)

Multiplying by the volume we get the total energy,

$$U_1 = \langle u_W \rangle AvgT = \frac{E_1^2}{8\pi} AcT \sqrt{1 + 4\pi \chi(\omega)}$$

(40)
Since we are supposing that energy is conserved $U_0 = U_1$, we obtain the relation between the amplitudes,

$$E_1^2 = \frac{E_0^2}{\sqrt{1 + 4\pi \chi(\omega)}} = \frac{E_0^2}{\sqrt{\epsilon}}$$  \hspace{1cm} (41)

The momentum density in vacuum is

$$g_0 = \frac{1}{4\pi c} E \times B = \frac{1}{4\pi c} E_0^2 \hat{e}_1$$  \hspace{1cm} (42)

with the time average,

$$\langle g_0 \rangle = \frac{1}{8\pi c} E_0^2 \hat{e}_1.$$  \hspace{1cm} (43)

The total momentum in vacuum is,

$$P_0 = \langle g_0 \rangle A c T = \frac{1}{8\pi} E_0^2 A T \hat{e}_1$$  \hspace{1cm} (44)

In the medium, using Minkowski’s density

$$g_0 = \frac{1}{4\pi c} D \times B = \frac{1}{4\pi c} D B \hat{e}_1$$  \hspace{1cm} (45)

and

$$\langle g_1 \rangle = \frac{1}{8\pi c} D_1 B_1 \hat{e}_1 = \frac{(1 + 4\pi \chi(\omega))}{8\pi v_{ph}} E_1^2 \hat{e}_1.$$  \hspace{1cm} (46)

$$\langle g_1 \rangle = \frac{\sqrt{(1 + 4\pi \chi(\omega))^3}}{8\pi c} E_1^2 \hat{e}_1 = \frac{\epsilon E_0^2}{8\pi c} \hat{e}_1.$$  \hspace{1cm} (47)

Finally,

$$P_1 = \langle g_1 \rangle A v_0 T = \frac{\sqrt{(1 + 4\pi \chi(\omega))^3}}{[1 + 4\pi \chi(\omega) \frac{\omega_0^2}{\omega_0^2 - \omega^2}]} P_0$$ \hspace{1cm} (48)

$$P_1 = \frac{\epsilon v_0^2}{c} P_0$$  \hspace{1cm} (49)

Figure 2 displays the behavior of the transmitted electromagnetic momentum. Below the resonance frequency it is larger than the incoming momentum. Conservation of momentum implies that the gas acquires mechanical momentum in the direction opposite to the propagation of the wave. As the frequency increases the excess of transmitted electromagnetic momentum diminishes. At some value the transmission is recoilless. Finally near the resonant frequency the transmitted electromagnetic momentum is only a fraction of the incoming momentum and the gas acquires mechanical momentum in the direction of propagation of the wave. For frequencies above the opaque region the gas also acquires mechanical momentum in the direction of propagation of the wave.

The simplicity of the picture presented in this paper suggests that it may be also useful to describe the main features of momentum transmission in other more complicate situations. In an example discussed recently [3], the anisotropic expansion of a Bose-Einstein condensate when one of the light beams of the confining traps is switched off was portrayed as a negative effective mass effect. Instead, it can be view as the result of momentum transfer by the transmitted beam. We note again that the main features of our computation translate to the perturbative quantum mechanical treatment. As another more familiar situation where the work presented in this paper may be of relevance, we may point to the experiments discussed in [9, 10]. There, it was shown that light either entering or leaving a steady liquid exerts a net outward force at the liquid surface. Nevertheless for the discussion of this experiments one should include the reflected wave and the possibility of multiple resonant frequencies. A related calculation was presented in [4] where we showed, using the solution of Maxwell’s equations, that an incoming wave which enters a region filled by an homogeneous dielectric medium exerts on it a force which is opposite to the direction in which it propagates.