SYMMETRIC RANDOM WALKS ON CERTAIN AMALGAMATED FREE PRODUCT GROUPS

KEN DYKEMA

ABSTRACT. We consider nearest–neighbor random walks on free products of finitely many copies of the integers with amalgamation over nontrivial subgroups. When all the subgroups have index two, we find the Green function of the random walks in terms of complete elliptic integrals. Our technique is to apply Voiculescu’s operator–valued R–transform.

1. INTRODUCTION AND DESCRIPTION OF RESULTS

Symmetric random walks on groups have been much studied since Kesten’s classic paper [5] on the subject. See, for example, [13] and [14] and references therein.

Given a group $G$ with a finite, symmetric generating set $S$, we consider the random walk on the associated Cayley graph of the group which starts at the identity element $e$ and at each step moves to any nearest neighbor with equal probability. Let $p_n$ be the probability of return to $e$ on the $n$th step. The Green function of the random walk is

$$G(z) = \sum_{n=0}^{\infty} p_n z^n$$

and the spectral radius of the random walk is the reciprocal of the radius of convergence of this power series, i.e.

$$r = \limsup_{n \to \infty} p_n^{1/n}.$$ 

We will consider groups of the form $G = \mathbb{Z} \ast_H \mathbb{Z}$ or $G = (\ast_H)^N \mathbb{Z}$, which are amalgamated free products of copies of the integers over a subgroup $H \cong \mathbb{Z}$, where $H$ is embedded in the $j$th copy of $\mathbb{Z}$ as the subgroup of index $m_j$. We will denote this group by $G = G_{m_1, m_2, \ldots, m_N}$. Our generating sets will be $S = S_{m_1, \ldots, m_N} = \{a_1, a_1^{-1}, \ldots, a_N, a_N^{-1}\}$, where $a_j$ is a generator of the $j$th copy of $\mathbb{Z}$. For convenience, we will call the corresponding random walk the standard random walk on the group $G_{m_1, \ldots, m_N}$. When $m_j = 2$ for all $j$, we will write the Green function of the random walk in terms of Legendre’s complete elliptic integrals. In other cases, the Green function is equal to the integral of an algebraic function, and arbitrarily many terms of its power series expansion can be easily found.
It is known that the spectral radius of the standard random walk on $G = \langle G_1, \ldots, G_N \rangle$ is an algebraic number. Indeed, $H$ is a normal (in fact, central) subgroup of $G$ that is amenable. By [5, Cor. 2] and [6], the spectral radius of the standard random walk on $G$ equals the spectral radius of the resulting random walk on the quotient group $G/H$. But

$$G/H \cong \langle \mathbb{Z}/m_1 \mathbb{Z} \rangle * \langle \mathbb{Z}/m_2 \mathbb{Z} \rangle * \cdots * \langle \mathbb{Z}/m_N \mathbb{Z} \rangle$$

with the generating set $S_{m_1, \ldots, m_N}$ mapping to a union of generating sets for the cyclic groups $\mathbb{Z}/m_j \mathbb{Z}$. The algebraicity of the spectral radius of this random walk is well known and has been proved by different authors; see [12], [3], [2]. See [3] and [7] for some results about random walks on other amalgamated free product groups.

Our techniques rely on Voiculescu’s operator–valued free probability theory, and in particular on the operator–valued R–transform; this is reviewed below, but see [10], [11] or [8] for more.

The paper is organized as follows. In §2 we review the elements of operator–valued free probability theory that we will need, principally Voiculescu’s R–transform. In §3 we will show that the $B$–valued Cauchy transform of the standard random walk on $G_{m_1, \ldots, m_N}$ is an algebraic function, where $B = C^*_r(H)$ is the reduced group $C^*$–algebra of the group over which we amalgamate. We will also find the Green function when $m_1 = \cdots = m_N = 2$.

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2. Preliminaries

Let $A$ be a unital $C^*$–algebra and let $B$ be a unital $C^*$–subalgebra of $A$ having a conditional expectation $E : A \to B$. The pair $(A, E)$ forms what is called a $B$–valued noncommutative probability space. Given $T \in A$, the $B$–valued Cauchy transform of $T$ is the function

$$C_T^{(B)}(b) = \sum_{n=0}^{\infty} E((bT)^n b) = E((1 - bT)^{-1} b)$$

from a neighborhood of 0 in $B$ into $B$.

We now review the means, devised by Voiculescu [10], of finding the $B$–valued $R$–transform $R_T^{(B)}$ of $T$ from the $B$–valued Cauchy transform. The function $C_T^{(B)}$ has an inverse $K_T^{(B)} = (C_T^{(B)})^{-1}$ with respect to composition. Moreover, (see [11] Prop. 2.3)), $K_T^{(B)}$ maps some neighborhood of 0 in $B$ bijectively onto a neighborhood of 0 in $B$ and maps invertible elements in this neighborhood to invertible elements. Then the $R$–transform of $T$ is

$$R_T^{(B)}(b) = K_T^{(B)}(b)^{-1} - b^{-1}.$$ 

This is defined for $b$ invertible and of small norm. However, given that $C_T^{(B)}$ is a power–series like sum of multilinear maps, both $K_T^{(B)}$ and $R_T^{(B)}$ are seen to have a similar structure; see also the combinatorial description in [8]. The above definition of $R_T^{(B)}$ on invertible elements actually determines it on all of a neighborhood of 0 in $B$. 

Suppose for every $i$ in an index set $I$, $A_i \subseteq A$ is a $C^*$-subalgebra of $A$ with $B \subseteq A_i$. The family $(A_i)_{i \in I}$ is said to be free if $E(a_1 \ldots a_n) = 0$ whenever $a_j \in A_{i_j} \cap \ker E$ and $i_1 \neq i_2$, $i_2 \neq i_3$, \ldots, $i_{n-1} \neq i_n$. A family $(T_i)_{i \in I}$ of elements of $A$ is said to be free if the family $(C^*(B \cup \{T_i\}))_{i \in I}$ of $C^*$-subalgebras is free.

**Theorem 2.1** (Voiculescu [10]). In a $B$-valued noncommutative probability space $(A, E)$, suppose $T_i \in A$ are such that $(T_i)_{i=1}^N$ is a free family. Then

$$R_{T_1+\ldots+T_N}^{\mathcal{B}} = R_{T_1}^{\mathcal{B}} + \ldots + R_{T_N}^{\mathcal{B}}.$$

We will now consider how these results from free probability theory may be applied to the study of random walks on amalgamated free products of groups. Given a group $G$, let $\lambda = \lambda(G)$ be the left regular representation of $G$ as unitary operators on $\ell^2(G)$, extended linearly to a $*$-representation of the complex group algebra $C[G]$. The reduced group $C^*$-algebra is

$$\text{span} \{ \lambda(g) \mid g \in G \}.$$

(We will sometimes write $\lambda_g$ instead of $\lambda(g)$.) Note that the canonical trace on $C[G]$, which extracts the coefficient of the identity element, extends to the tracial state $\tau = \tau_G$ on $C^*_r(G)$ given by $\tau(x) = \langle x\delta_e, \delta_e \rangle$. Thus, $(C^*_r(G), \tau)$ is a $C^*$-valued noncommutative probability space. If $H$ is a subgroup of $G$, then $\lambda(G)(C[H])$ is isomorphic to $C^*_r(H)$, and will be denoted as such. The projection $\ell^2(G) \to \ell^2(H)$ implements the canonical conditional expectation $E : C^*_r(G) \to C^*_r(H)$, which satisfies

$$E(\lambda(g)) = \begin{cases} 
\lambda(g), & g \in H \\
0, & \text{otherwise.}
\end{cases}$$

Thus, $(C^*_r(G), E)$ is a $C^*_r(H)$-valued noncommutative probability space. Note that $E$ is $\tau$-preserving.

Suppose $G = (H_i)_{i=1}^N G_i$ is a free product of groups $G_i$ with amalgamation over subgroups $H \subseteq G_i$. For ease of writing, we will take $N = 2$ and write $G = G_1 \ast_H G_2$, though similar considerations apply for general $N < \infty$. The group inclusions $G_i \subseteq G$ give rise to $C^*$-subalgebras $C^*_r(G_i) \subseteq C^*_r(G)$, and the pair $(C^*_r(G_1), C^*_r(G_2))$ is free with respect to the canonical conditional expectation $E : C^*_r(G) \to C^*_r(H)$. Suppose $S_i$ is a finite, symmetric generating set for $G_i$. Then $S = S_1 \cup S_2$ a finite, symmetric generating set for $G$. Let $T_i = \sum_{a \in S_i} \lambda(a) \in C^*_r(G_i) \subseteq C^*_r(G)$ and let $T = T_1 + T_2$. The adjacency operator $T$ is related to the random walk associated to $S$ by $\tau(T^n) = |S|^np_n$ and the $C^*$-valued Cauchy transform

$$C_T^{(C)}(\zeta) = \sum_{n=0}^\infty \tau(T^n)\zeta^{n+1}$$

is related to the Green function $G$ of this random walk [11] by

$$C_T^{(C)}(\zeta) = \zeta G(|S|\zeta).$$

(2)

Let $B = C^*_r(H)$. We have that $C_T^{(C)}$ is $\tau \circ C_T^{(B)}$ restricted to scalar multiples of the identity operator.
What follows, then, is a strategy for finding the Green function of this random walk on \( G_1 \ast_H G_2 \). If the \( B \)-valued Cauchy transforms \( C^{(B)}_{T_1} \) and \( C^{(B)}_{T_2} \) are known, then the \( B \)-valued R–transforms of \( T_1 \) and \( T_2 \) can be found and used, with Voiculescu’s Theorem 2.1 to compute the \( B \)-valued R–transform of \( T \), from with the \( B \)-valued Cauchy transform of \( T \) can be found. Composing with \( \tau \) then yields the \( C \)-valued Cauchy transform of \( T \).

### 3. Green functions

In this section, we consider standard random walks on the amalgamated free product groups \( G_{m_1,...,m_N} \). We find the Green function \( G_{m_1,...,m_N} \) of this random walk when \( m_j = 2 \) for all \( j \), and we show how to derive information about the Green function in other cases.

**Proposition 3.1.** Let \( n \in \{2,3,4,\ldots\} \) and consider the subgroup \( nZ = H \subseteq G = Z \). Let \( E \) be the canonical conditional expectation from \( A = C^*_{r}(G) \) onto the subalgebra \( B = C^*_{r}(H) \). Consider the adjacency operator \( T = \lambda_1 + \lambda_{-1} \in A \). Then the \( B \)-valued Cauchy transform of \( T \) is

\[
C^{(B)}_{T}(b) = \frac{bp(b)}{q(b) - bp(\lambda_n + \lambda_{-n})},
\]

where \( p \) and \( q \) are polynomials with integer coefficients, each with constant term equal to 1 and with \( \deg(p) \leq n - 1 \) and \( \deg(q) \leq n \).

**Proof.** Using the Fourier transform, \( A = C^*_{r}(Z) \) is seen to be isomorphic to \( C(T) \), the algebra of all continuous functions on the circle, and we henceforth make this identification of \( A \) with \( C(T) \). Thus \( \lambda_k \in A \) is identified with the function that is the map \( T \ni z \mapsto z^k \). The subalgebra \( B = C^*_{r}(nZ) \), is identified with the set of functions invariant under rotation of the domain \( T \) by angle \( 2\pi/n \), and for every \( f \in A \),

\[
( Ef)(z) = \frac{1}{n} \sum_{k=0}^{n-1} f(\omega^k z),
\]

where \( \omega = \omega_n = \exp(2\pi i/n) \). For \( b \in B \) of sufficiently small norm,

\[
C^{(B)}_{T}(b) = E(b(1 - Tb)^{-1}) = \frac{b}{n} \sum_{k=0}^{n-1} \frac{1}{1 - (\omega^k \lambda_1 + \omega^{-k} \lambda_{-1})b}
\]

\[
= \left( \frac{b}{n} \sum_{k=0}^{n-1} \prod_{j=0}^{n-1} (1 - (\omega^j \lambda_1 + \omega^{-j} \lambda_{-1})b) \right) \left/ \prod_{j=0}^{n-1} (1 - (\omega^j \lambda_1 + \omega^{-j} \lambda_{-1})b) \right).
\]

Consider the ring

\[
R = \left\{ \sum_{k \in \mathbb{Z}} a_k \lambda_k \mid a_k \in \mathbb{Z}[\omega], \text{ all but finitely many } a_k = 0 \right\},
\]
which is the group ring of \( \mathbb{Z} \) with coefficients from \( \mathbb{Z}[\omega] \). Consider the denominator \( Q \) of (4) as a polynomial in variable \( b \) with coefficients from \( R \). Then \( Q \) is of degree \( \leq n \) and has constant term equal to 1. The coefficient of \( b^k \), \( 1 \leq k \leq n \), is of the form

\[
a_{-k} \lambda_{-k} + a_{-k+1} \lambda_{-k+1} + \cdots + a_k \lambda_k
\]

(5)

with all \( a_j \in \mathbb{Z}[\omega] \). We see from (4) that \( Q \) is invariant under the automorphism of \( R \) given by \( \lambda_j \mapsto \omega^j \lambda_j \). So only \( a_0 \) can be nonzero in (5) if \( 1 \leq k \leq n - 1 \), while with \( k = n \), we find that the coefficient of \( b^n \) in \( Q \) is of the form

\[
a_{-n} \lambda_{-n} + a_0 + a_n \lambda_n.
\]

(6)

But since the coefficient of \( b_n \) equals

\[
(-1)^n \prod_{j=0}^{n-1} (\omega^{-j} \lambda_{-1} + \omega_j \lambda_1),
\]

we see

\[
a_n = (-1)^n \prod_{j=0}^{n-1} \omega^j = (-1)^n \omega^{(n-1)n/2} = -1
\]

and \( a_n = a_n^{-1} = -1 \). Finally, \( Q \) is invariant under the transformation \( \omega \mapsto \omega^d \) whenever \( d \) is relatively prime to \( n \). So by the fundamental theorem of Galois theory, we get \( a_0 \in \mathbb{Z} \) in (3) and the coefficient of \( b^k \) is an integer for \( 1 \leq k \leq n - 1 \). Let \( q \) be the polynomial given by \( Q(b) = q(b) - b^n(\lambda_n + \lambda_n) \).

By the same reasoning as for \( Q \), we see that the numerator, \( P \), of (4) is equal to \( \frac{1}{n} b \tilde{P}(b) \), where \( \tilde{P} \) is a polynomial of degree \( \leq n - 1 \) having integer coefficients, and where the constant coefficient of \( \tilde{P} \) is equal to \( n \). We need only show that the coefficients of \( \frac{1}{n} \tilde{P} \) are all integers. Equating two descriptions of \( C_{T}^{(B)}(b) \), we get

\[
\sum_{k=0}^{\infty} E(b(Tb)^k) = \frac{b \tilde{P}(b)}{1 - (\tilde{q}(b) + b^n(\lambda_n + \lambda_n))} = \frac{b}{n} \tilde{P}(b) \sum_{k=0}^{\infty} (\tilde{q}(b) + b^n(\lambda_n + \lambda_n))^k,
\]

where \( \tilde{q}(b) = 1 - q(b) \). Since \( E(b(Tb)^k) \) can be written as a linear combination of \( \{\lambda_j \mid j \in n\mathbb{Z}, |j| \leq k\} \) with coefficients from \( \mathbb{Z}[b] \) and since \( \tilde{q} \) has integer coefficients and zero constant coefficient, we conclude that \( \frac{1}{n} \tilde{P} \) has integer coefficients. □

Examples 3.2. Let \( C_n \) denote the \( B \)-valued Cauchy transform considered in equation (3) of Proposition 3.1. Using the formula (4), we find

\[
C_2(b) = \frac{b}{1 - 2b^2 - b^2(\lambda_{-2} + \lambda_2)} \quad C_3(b) = \frac{b - b^3}{1 - 3b^2 - b^3(\lambda_{-3} + \lambda_3)}
\]

\[
C_4(b) = \frac{b - 2b^3}{1 - 4b^2 + 2b^4 - b^4(\lambda_{-4} + \lambda_4)} \quad C_5(b) = \frac{b - 3b^3 + b^5}{1 - 5b^2 + 5b^4 - b^5(\lambda_{-5} + \lambda_5)}
\]

From Proposition 3.1 and the procedure for obtaining the \( R \)-transform as described in (2) we see that the \( C_n(H) \)-valued \( R \)-transform of the operator \( T \) considered above is an algebraic function. A precise formulation is below.
Proposition 3.3. Let $n \in \{2, 3, 4, \ldots \}$, let $B = C_r^*(n\mathbb{Z}) \subseteq A = C_r^*(\mathbb{Z})$, $E : A \to B$ and $T$ be as in Proposition 3.1. Let $R = R_T^{(B)}$ be the $B$–valued $R$–transform of $T$. Then there is an irreducible polynomial $Q_n$ in three variables and with integer coefficients such that
\[ Q_n(R(b), b, \xi) = 0, \]
where $\xi = \lambda_p + \lambda_p$, for $p$ a generator of $H$.

Examples 3.4. Below are listed some of the irreducible polynomials $Q_n = Q_n(R(b), b, \xi)$ from Proposition 3.3:
\begin{align*}
Q_2 &= bR^2 + R - b(2 + \xi) \\
Q_3 &= b^2R^2 + 2bR^2 + (1 - 3b^2)R - b(2 + b\xi) \\
Q_4 &= b^3R^4 + 3b^2R^3 + b(3 - 4b^2)R^2 + (1 - 6b^2)R - b(2 - 2b^2 + b^2\xi) \\
Q_5 &= b^4R^5 + 6b^3R^4 + (6b^2 - 5b^4)R^3 + (4b - 12b^3)R^2 + (1 - 9b^2 + 5b^4)R - b(2 - 4b^2 + b^3\xi)
\end{align*}

Proposition 3.5. Let $G = G_{m_1, \ldots, m_N}$ with generating set $S = S_{m_1, \ldots, m_N}$ be as described in the introduction. Let $B = C_r^*(H) \subseteq C_r^*(G) = A$ equipped with the canonical conditional expectation $E : A \to B$. Let $T = \sum_{a \in S} \lambda_a \in A$ be the adjacency operator for the standard random walk. Let $C = C_T^{(B)}$ and $R = R_T^{(B)}$ be the $B$–valued Cauchy transform and $R$–transform of $T$, respectively. Then there is are irreducible polynomials $P_{m_1, \ldots, m_N}$ and $Q_{m_1, \ldots, m_N}$, each in three variables and with integer coefficients, such that
\[ P_{m_1, \ldots, m_N}(C(b), b, \xi) = 0, \quad Q_{m_1, \ldots, m_N}(R(b), b, \xi) = 0, \]
where $\xi = \lambda_{-p} + \lambda_p$, for $p$ a generator of $H$.

Proof. We have $S = \{ a_1^{-1}, a_1, \ldots, a_N^{-1}, a_N \}$ where $a_j$ is a generator of the $j$th copy of $\mathbb{Z}$ in the amalgamated free product $G = (\ast_{j=1}^N \mathbb{Z})$, and $T = T_1 + \cdots + T_N$, where $T_j = \lambda_{a_j^{-1}} + \lambda_{a_j}$. By additivity of the $B$–valued $R$–transform (Theorem 3.3),
\[ R_T^{(B)} = R_{T_1}^{(B)} + \cdots + R_{T_N}^{(B)}. \]
By Proposition 3.3, each $R_T^{(B)}(b)$ is an algebraic function of $b$ and $\xi$, being the root of the polynomial with integer coefficients, so the same is true for $R_T^{(B)}$. Now the procedure for finding $C_T^{(B)}$ from $R_T^{(B)}$ yields the polynomial $P_{m_1, \ldots, m_N}$ from $Q_{m_1, \ldots, m_N}$.

The polynomials $Q_{m_1, \ldots, m_N}$ and $P_{m_1, \ldots, m_N}$ are easily found, as is illustrated in the following three examples. In the first two of these examples, we are able to write explicitly the Green functions of the random walks.

Example 3.6. Consider the case $G = G_{2, 2}$. Note that $G$ is an amenable group. Then from $Q_2$ of Examples 3.4, we get immediately
\[ Q_{2,2}(R(b), b, \xi) = bR^2 + 2R - 4b(2 + \xi). \]
and

\[ P_{2,2}(C, b, \xi) = (1 - 8b^2 - 4b^2\xi)C^2 - b^2. \]  

(7)

Letting \( C^{(B)}_{2,2} \) denote the \( B \)-valued Cauchy transform of the adjacency operator \( T \), namely the quantity \( C \) in (7) above, and using the asymptotic behavior \( C^{(B)}_{2,2}(b) = b + O(\|b\|^2) \) as \( \|b\| \to 0 \) to choose the branch of the square root, we get

\[ C^{(B)}_{2,2}(b) = \frac{b}{\sqrt{1 - 4b^2(2 + \xi)}}. \]

Letting \( C_{2,2} \) denote the scalar–valued Cauchy transform of \( T \), we have for \( \zeta \in \mathbb{C} \),

\[ C_{2,2}(\zeta) = \tau \circ C^{(B)}_{2,2}(\zeta), \]

where \( \tau \) is the canonical trace on \( C^*_r(H) \). Thus, taking \( |\zeta| \) small,

\[
\begin{align*}
C_{2,2}(\zeta) &= \frac{\zeta}{2\pi i} \int_{|z|=1} \frac{dz}{z \sqrt{1 - 4\zeta^2(z^{-1} + z + 2)}} \\
&= \frac{\zeta}{2\pi i} \int_{|z|=1} \frac{dz}{\sqrt{z(-4\zeta^2z^2 + (1 - 8\zeta^2)z - 4\zeta^2)}} \\
&= \frac{1}{4\pi i} \int_{|z|=1} \frac{dz}{\sqrt{z(z - z_1)(z_2 - z)}}, \quad (8)
\end{align*}
\]

where

\[
\begin{align*}
z_1 &= z_1(\zeta) = \frac{1 - 8\zeta^2 - \sqrt{1 - 16\zeta^2}}{8\zeta^2} = 4\zeta^2 + O(|\zeta|^4) \\
z_2 &= z_2(\zeta) = \frac{1 - 8\zeta^2 + \sqrt{1 - 16\zeta^2}}{8\zeta^2} = \frac{1}{4\zeta^2} + O(1)
\end{align*}
\]

with the indicated asymptotics as \( \zeta \to 0 \) and where in \( \mathbb{R} \), the branch of \( \sqrt{z_2 - z} \) close to \( 1/2\zeta \) is chosen. Replacing the contour \( |z| = 1 \) by the contour drawn in Figure 1 and letting the circles shrink to the point of disappearing, we get

![Figure 1. Contour used in evaluating the integral](image)
Solving for $C$ and using analogous notation, we find the integers with amalgamation over their index–two subgroups. As in the previous example and integrating yields the Green function of the standard random walk on $G$

\[ \zeta = 4 \frac{2 - z^2 - 2\sqrt{1 - z^2}}{2 - z^2 + 2\sqrt{1 - z^2}} \]

(9)

where we have made the change of variables $t = \sin^2 \phi$, where $F_1$ is Legendre’s complete elliptic integral of the first kind (see [4]) and where we take the branch of $\sqrt{z_2(\zeta)}$ that is close to $1/(2\zeta)$; it is not necessary to specify the branch of $\sqrt{z_1(\zeta)/z_2(\zeta)}$. Therefore, the Green function of the standard random walk on on $G_{2,2}$ is

\[ G_{2,2}(z) = \frac{4}{\pi\sqrt{2 - z^2 + 2\sqrt{1 - z^2}}} F_1 \left( \sqrt{\frac{2 - z^2 - 2\sqrt{1 - z^2}}{2 - z^2 + 2\sqrt{1 - z^2}}} \right). \]

Example 3.7. Let $G = G_{2,...,2} = (\mathbb{Z})^N\mathbb{Z}$ be the free product of $N \geq 3$ copies of the integers with amalgamation over their index–two subgroups. As in the previous example and using analogous notation, we find

\[ Q_{2,...,2} = bR^2 + NR - bN^2(2 + \xi) \]

\[ P_{2,...,2} = (1 - b^2N^2(2 + \xi))C^2 + b(N - 2)C - b^2(N - 1). \]

Solving for $C$ gives

\[ C_{2,...,2}^{(B)}(b) = \left( \frac{2 - N + N\sqrt{1 - 4(N - 1)b^2(2 + \xi)}}{2(1 - N^2b^2(2 + \xi))} \right) b, \]

and integrating yields the $\mathbf{C}$–valued Cauchy transform

\[ C_{2,...,2}(\zeta) = \frac{\zeta}{2\pi i} \int_{|z|=1} \frac{2 - N + N\sqrt{1 - 4(N - 1)b^2(2 + \zeta - 1 + z)}}{2z(1 - N^2b^2(2 + \zeta - 1 + z))} dz \]

\[ = \frac{(2 - N)\zeta}{4\pi i} \int_{|z|=1} \frac{dz}{-N^2\zeta^2z^2 + (1 - 2N^2\zeta^2)z - N^2\zeta^2} \]

\[ + \frac{N\zeta}{4\pi i} \int_{|z|=1} \frac{\sqrt{1 - 4(N - 1)b^2(2 + \zeta - 1 + z)}}{-N^2\zeta^2z^2 + (1 - 2N^2\zeta^2)z - N^2\zeta^2} dz, \]

(11)

for $\zeta$ sufficiently small. The denominator in the integrals (11) and (12) has roots

\[ z_3 = z_3(\zeta) = \frac{1 - 2N^2\zeta^2 - \sqrt{1 - 4N^2\zeta^2}}{2N^2\zeta^2} = N^2\zeta^2 + O(|\zeta|^4) \]

(13)

\[ z_4 = z_4(\zeta) = \frac{1 - 2N^2\zeta^2 + \sqrt{1 - 4N^2\zeta^2}}{2N^2\zeta^2} = \frac{1}{N^2\zeta^2} + O(1). \]

(14)
The value of the first term \((11)\) is, thus,
\[
\frac{(2 - N)\zeta}{4\pi i} \int_{|z| = 1} \frac{dz}{N^2\zeta^2(z - z_3)(z_4 - z)} = \frac{(2 - N)\zeta}{2N^2\zeta^2(z_4 - z_3)} = \frac{(2 - N)\zeta}{2\sqrt{1 - 4N^2\zeta^2}}.
\] (15)

The second term \((12)\) equals
\[
\frac{N\zeta}{4\pi i} \int_{|z| = 1} \sqrt{\frac{1}{z} (z - 4(N - 1)\zeta^2(2z + 1 + z^2))} \frac{dz}{N^2\zeta^2(z - z_3)(z_1 - z)} = \frac{1}{4N\zeta i} \int_{|z| = 1} \frac{p(z)}{(z - z_3)(z_4 - z)} \sqrt{z p(z)} \, dz
\] (16)
where
\[
p(z) = z - 4(N - 1)\zeta^2(2z + 1 + z^2) = -4(N - 1)\zeta^2z^2 + (1 - 8(N - 1)\zeta^2)z - 4(N - 1)\zeta^2.
\]
The roots of \(p(z)\) are
\[
z_5 = z_5(\zeta) = \frac{1 - 8(N - 1)\zeta^2 - \sqrt{1 - 16(N - 1)\zeta^2}}{8(N - 1)\zeta^2} = 4(N - 1)\zeta^2 + O(|\zeta|^4)
\]
(17)
\[
z_6 = z_6(\zeta) = \frac{1 - 8(N - 1)\zeta^2 + \sqrt{1 - 16(N - 1)\zeta^2}}{8(N - 1)\zeta^2} = \frac{1}{4(N - 1)\zeta^2} + O(1)
\]
(18)
and the quantity \((16)\) equals
\[
\frac{1}{8N\sqrt{N - 1}\zeta^2i} \int_{|z| = 1} \frac{p(z)}{(z - z_3)(z_4 - z)} \sqrt{z p(z)} \, dz.
\] (19)
But we have
\[
\frac{p(z)}{(z - z_3)(z_4 - z)} = 4(N - 1)\zeta^2 + \frac{a_1}{z - z_3} + \frac{a_2}{z_4 - z},
\]
where
\[
a_1 = \frac{(N - 2)\zeta^2z_3}{\sqrt{1 - 4N^2\zeta^2}}, \quad a_2 = \frac{(N - 2)\zeta^2z_4}{\sqrt{1 - 4N^2\zeta^2}}.
\]
Thus, \((19)\) becomes
\[
\sqrt{\frac{N - 1}{2N\pi i}} \int_{|z| = 1} \frac{dz}{\sqrt{z(z - z_5)(z_6 - z)}}
\] (20)
\[
+ \frac{(N - 2)^2z_3}{8N\pi i\sqrt{N - 1}\sqrt{1 - 4N^2\zeta^2}} \int_{|z| = 1} \frac{dz}{(z - z_3)\sqrt{z(z - z_5)(z_6 - z)}}
\] (21)
\[
+ \frac{(N - 2)^2z_4}{8N\pi i\sqrt{N - 1}\sqrt{1 - 4N^2\zeta^2}} \int_{|z| = 1} \frac{dz}{(z_4 - z)\sqrt{z(z - z_5)(z_6 - z)}}.
\] (22)
Using the contour in Figure 1 but with \( z_5 \) replacing \( z_1 \), we see that the term (20) equals
\[
2\sqrt{N - 1} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \frac{2z_5}{z_6}\sin^2\phi}} = \frac{2\sqrt{N - 1}}{N\pi\sqrt{z_6(\zeta)}} F_1(\sqrt{\frac{z_5}{z_6}(\zeta)}),
\]
while the term (22) equals
\[
\frac{(N - 2)^2}{2N\pi \sqrt{N - 1}\sqrt{1 - 4N^2\zeta^2}\sqrt{z_6}} \int_0^{\pi/2} \frac{d\phi}{(1 - \frac{2z_5}{z_4}\sin^2\phi)\sqrt{1 - \frac{2z_5}{z_6}\sin^2\phi}} = \frac{(N - 2)^2}{2N\pi \sqrt{N - 1}\sqrt{1 - 4N^2\zeta^2}\sqrt{z_6}(\zeta)} \Pi_1(\frac{z_5(\zeta)}{z_4(\zeta)}, \sqrt{\frac{z_5}{z_6}(\zeta)}),
\]
where \( \Pi_1 \) is Legendre’s complete elliptic integral of the third kind and where in (23) and (24), we take the branch of \( \sqrt{z_6(\zeta)} \) that is close to \( \frac{1}{2\sqrt{N - 1}\zeta} \). We now consider the term (21). We see from the asymptotics (13), (17) and (18) that for \( \zeta \) sufficiently small we have \( |z_5| < |z_3| < 1 \). Hence, picking up the residue at \( z_3 \), we see that the term (21) equals
\[
\frac{(N - 2)^2 z_3}{4N\sqrt{N - 1}\sqrt{1 - 4N^2\zeta^2}\sqrt{z_3(z_3 - z_5)(z_6 - z_3)}} - \frac{(N - 2)^2}{2N\pi \sqrt{N - 1}\sqrt{1 - 4N^2\zeta^2}\sqrt{z_6}} \int_0^{\pi/2} \frac{d\phi}{(1 - \frac{2z_5}{z_3}\sin^2\phi)\sqrt{1 - \frac{2z_5}{z_6}\sin^2\phi}}
\]
where we choose the branch of
\[
\sqrt{z_3(\zeta)(z_3(\zeta) - z_5(\zeta))(z_6(\zeta) - z_3(\zeta))} = \sqrt{\frac{N^2(N - 2)^2}{4(N - 1)^2}} \zeta^2 + O(\zeta^4)
\]
that is close to \( \frac{N(N - 2)}{2\sqrt{N - 1}} \zeta \) and, again, the branch of \( \sqrt{z_6(\zeta)} \) that is close to \( \frac{1}{2\sqrt{N - 1}\zeta} \). However,
\[
(z_3 - z_5)(z_6 - z_3) = \frac{(N - 2)^2}{4N^2(N - 1)\zeta^2} z_3
\]
and the residue (25) equals
\[
\frac{(N - 2)\zeta}{2\sqrt{1 - 4N^2\zeta^2}},
\]
which exactly cancels (15). Collecting all terms, we have
\[
C_{2,\ldots,2}(\zeta) = \frac{2}{N\pi \sqrt{N - 1}\sqrt{z_6(\zeta)}} \left( (N - 1) F_1(\sqrt{\frac{z_5}{z_6}(\zeta)}) + \frac{(N - 2)^2}{4\sqrt{1 - 4N^2\zeta^2}} \left( \Pi_1(\frac{z_5(\zeta)}{z_3(\zeta)}, \sqrt{\frac{z_5}{z_6}(\zeta)}) - \Pi_1(\frac{z_5(\zeta)}{z_4(\zeta)}, \sqrt{\frac{z_5}{z_6}(\zeta)}) \right) \right).
\]
which yields for the Green function of the standard random walk on $G_{2,\ldots,2}$ the expression

$$
G_{2,\ldots,2}(z) = \frac{8}{N\pi\sqrt{w_6(z)}} (N-1) F_1 \left( \frac{w_5(z)}{w_6(z)} \right) + \frac{(N-2)^2}{2\sqrt{4 - N^2 z^2}} \left( \Pi_1 \left( \frac{N^2 w_5(z)}{(N-1)w_4(z)}, \frac{w_5(z)}{w_6(z)} \right) - \Pi_1 \left( \frac{N^2 w_5(z)}{(N-1)w_3(z)}, \frac{w_5(z)}{w_6(z)} \right) \right),
$$

where

$$
w_3(z) = 8 - N^2 z^2 - 4\sqrt{4 - N^2 z^2},
$$

$$
w_4(z) = 8 - N^2 z^2 + 4\sqrt{4 - N^2 z^2},
$$

$$
w_5(z) = 2 - (N-1) z^2 - 2\sqrt{1 - (N-1)z^2},
$$

$$
w_6(z) = 2 - (N-1) z^2 + 2\sqrt{1 - (N-1)z^2}.
$$

**Example 3.8.** Consider the case $G = G_{2,3}$. Using Maple to find a Groebner basis, one quickly computes $Q_{2,3}$ from $Q_2$ and $Q_3$. (We checked this result in Mathematica by back–substitution and elimination.) We found that $Q_{2,3} = Q_{2,3}(R, b, \xi)$ from Proposition 3.5 is

$$
Q_{2,3} = b^7 R^6 + 7b^4 R^5 + b^3(19 - 12b^2 - 3b^2\xi)R^4 + b^2(25 - 56b^2 - 14b^2\xi - 2b^3\xi)R^3 + b(16 - 93b^2 + 21b^4 - 23b^2\xi + 7b^3\xi + 12b^4\xi + 3b^4\xi^2)R^2 + (4 - 65b^2 + 49b^4 + 26b^2\xi - 9b^3\xi + 28b^4\xi - 6b^5\xi + 7b^4\xi^2 - 6b^5\xi^2)R - b(16 - 28b^2 + 2b^4 + 4b^2\xi - 17b^2\xi + 7b^3\xi - 3b^4\xi - 4b^2\xi^2 + 7b^3\xi^2 - b^4\xi^2 + b^4\xi^3).
$$

From this, we find $P_{2,3} = P_{2,3}(C, b, \xi)$ from Proposition 3.5 is

$$
P_{2,3} = (1 - 12b^2 + 21b^4 - 2b^6 - 3b^2\xi - 2b^3\xi + 12b^4\xi - 6b^5\xi + 3b^6\xi + 3b^4\xi^2 - 6b^5\xi^2 + b^6\xi^2 - b^6\xi^3)C^3 + (b - 8b^3 + 7b^5 - 2b^2\xi - b^4\xi + 4b^5\xi - b^6\xi + b^5\xi^2 - b^6\xi^2)C^2 - (b^2 - b^4\xi + b^5\xi - b^6\xi)C - b^3 + b^5.
$$

(26)

The $B$–valued Cauchy transform, $C_{2,3}^{(B)}$, of the adjacency operator $T$ is an algebraic function of degree 3. To compute explicitly the $C$–valued Cauchy transform of $T$ by integrating as was performed in Examples 3.6 and 3.7 seems, thus, to be difficult. However, starting from $C_{2,3}^{(B)}(b) = b + O(||b||^2)$, further terms of the power series expansion for $C_{2,3}^{(B)}$ can be computed from the polynomial $P_{2,3}$. We obtain, for instance,
\[ C_{2,3}^{(B)}(b) = b + (4 + \xi)b^3 + \xi b^4 + (26 + 12\xi + \xi^2)b^5 + 5\xi(3 + \xi)b^6 \\
+ (194 + 132\xi + 25\xi^2 + \xi^3)b^7 + 7\xi(5 + \xi)(5 + 2\xi)b^8 \\
+ (1542 + 1392\xi + 432\xi^2 + 52\xi^3 + \xi^4)b^9 \\
+ \xi(1887 + 1593\xi + 406\xi^2 + 30\xi^3)b^{10} \\
+ (12714 + 14320\xi + 6275\xi^2 + 1350\xi^3 + 125\xi^4 + \xi^5)b^{11} + O(||b||^{12}). \quad (27) \]

Taking \(\tau\) of (27) and using

\[
\tau(\xi^n) = \begin{cases} 
\frac{n}{2} & n \text{ even} \\
0 & n \text{ odd}, 
\end{cases}
\]

we get the expansion for the \(C\)-valued Cauchy transform of \(T\), which gives the following expression for the first several terms of the Green function for the standard random walk on \(G_{2,3}:\)

\[
G_{2,3}(z) = 1 + 4 \left(\frac{\xi}{4}\right)^2 + 28 \left(\frac{\xi}{4}\right)^4 + 10 \left(\frac{\xi}{4}\right)^5 + 244 \left(\frac{\xi}{4}\right)^6 + 210 \left(\frac{\xi}{4}\right)^7 \\
+ 2412 \left(\frac{\xi}{4}\right)^8 + 3366 \left(\frac{\xi}{4}\right)^9 + 26014 \left(\frac{\xi}{4}\right)^{10} + O(|z|^{11}).
\]

Example 3.9. Here is the case \(G = G_{2,4}\), using notation as in the previous example:

\[
P_{2,4} = (1 - 16\xi^2 + 60\xi^4 - 32\xi^6 + 4\xi^8 - 4b^2\xi + 30b^4\xi - 40b^6\xi + 4b^8\xi + 6b^4\xi^2 \\
- 28b^6\xi^2 - 3b^8\xi^2 + 4b^6\xi^3 - 2b^8\xi^3 + b^8\xi^4)G^4 \\
+ (2b - 24b^3 + 60b^5 - 16b^7 - 6b^5\xi + 30b^5\xi - 20b^7\xi + 6b^5\xi^2 \\
- 14b^7\xi^2 - 2b^7\xi^3)G^3 \\
+ (-2b^4 + 10b^6 + b^6\xi - 2b^8\xi - 3b^8\xi^2)G^2 \\
+ (-2b^3 + 8b^5 - 2b^7 + 2b^5\xi - b^7\xi)G - b^4 + 2b^6
\]

\[
C_{2,4}^{(B)}(b) = b + (4 + \xi)b^3 + (26 + 13\xi + \xi^2)b^5 + (196 + 150\xi + 30\xi^2 + \xi^3)b^7 \\
+ (1588 + 1644\xi + 545\xi^2 + 60\xi^3 + \xi^4)b^9 \\
+ (13424 + 17540\xi + 8160\xi^2 + 1585\xi^3 + 110\xi^4 + \xi^5)b^{11} + O(||b||^{13})
\]

\[
G_{2,4}(z) = 1 + 4 \left(\frac{\xi}{4}\right)^2 + 28 \left(\frac{\xi}{4}\right)^4 + 256 \left(\frac{\xi}{4}\right)^6 + 2684 \left(\frac{\xi}{4}\right)^8 + 30404 \left(\frac{\xi}{4}\right)^{10} + O(|z|^{12}).
\]

Example 3.10. Here is the case \(G = G_{2,5}\).

\[
P_{2,5} = (1 - 20b^2 + 115b^4 - 180b^6 + 45b^8 - 2b^{10} - 5b^2\xi + 60b^4\xi - 2b^5\xi - 145b^6\xi \\
- 30b^7\xi + 20b^8\xi + 10b^9\xi - 5b^{10}\xi + 10b^4\xi^2 - 60b^6\xi^2 - 20b^7\xi^2 + 25b^8\xi^2 \\
- 10b^9\xi^2 + b^{10}\xi^2 - 10b^6\xi^3 + 20b^8\xi^3 - 10b^9\xi^3 - 5b^{10}\xi^3 + 5b^8\xi^4 - b^{10}\xi^5)G^5 \\
+ (3b - 48b^3 + 207b^5 - 216b^7 + 27b^9 - 12b^3\xi + 108b^5\xi - 3b^6\xi - 174b^7\xi \\
- 27b^8\xi + 12b^9\xi + 3b^{10}\xi + 18b^6\xi^2 - 72b^7\xi^2 - 18b^8\xi^2 + 15b^9\xi^2 - 3b^{10}\xi^2 \\
- 12b^7\xi^3 + 12b^8\xi^3 + 3b^{10}\xi^3 + 3b^9\xi^4)G^4
\]
Random walks on amalgamated free products

\[ + (2b^2 - 27b^4 + 97b^6 - 75b^8 + 2b^{10} - 6b^4\xi + 45b^6\xi - 6b^7\xi - 61b^8\xi \\
- 7b^9\xi - b^{10}\xi + 6b^6\xi^2 - 21b^8\xi^2 - 8b^9\xi^2 + 46b^{10}\xi^2 - 2b^{10}\xi^3 + 3b^{10}\xi^3)G^3 \\
+ (-2b^3 + 13b^5 - 7b^7 - 5b^9 + 4b^5\xi - 6b^7\xi - 4b^8\xi - 8b^9\xi \\
- 2b^7\xi^2 - 3b^9\xi^2 - b^{10}\xi^2)G^2 \\
+ (-3b^4 + 15b^6 - 11b^8 + 3b^8\xi - 3b^8\xi - b^{10}\xi - b^{10}\xi)G - b^5 + 3b^7 - b^9 \]

\[ C_{2,5}^{(B)}(b) = b + (4 + \xi)b^3 + (26 + 12\xi + \xi^2)b^5 + \xi b^6 + (196 + 132\xi + 24\xi^2 + \xi^3)b^7 \\
+ (21\xi + 7\xi^2)b^8 + (1590 + 1408\xi + 400\xi^2 + 40\xi^3 + \xi^4)b^9 \\
+ (306\xi + 189\xi^2 + 27\xi^3)b^{10} \\
+ (13482 + 14800\xi + 5741\xi^2 + 940\xi^3 + 60\xi^4 + \xi^5)b^{11} \\
+ (3861\xi + 3388\xi^2 + 924\xi^3 + 77\xi^4)b^{12} + O(\|b\|^{13}) \]

\[ G_{2,5}(z) = 1 + 4 \left( \frac{\xi}{4} \right)^2 + 28 \left( \frac{\xi}{4} \right)^4 + 244 \left( \frac{\xi}{4} \right)^6 + 14 \left( \frac{\xi}{4} \right)^7 + 2396 \left( \frac{\xi}{4} \right)^8 \\
+ 378 \left( \frac{\xi}{4} \right)^9 + 25324 \left( \frac{\xi}{4} \right)^{10} + 7238 \left( \frac{\xi}{4} \right)^{11} + O(|z|^{12}). \]

References

[1] L. Aagaard, A Banach algebra approach to amalgamated R– and S–transforms, preprint.
[2] K. Aomoto and Y. Kato, Green functions and spectra on free products of cyclic groups, Ann. Inst. Fourier, Grenoble 38 (1988), 59-85.
[3] D.I. Cartwright and P.M. Soardi, Random walks on free products, quotients and amalgams, Nagoya Math. J. 102 (1986), 163-180.
[4] A. Cayley, Elementary Treatise on Elliptic Functions, Bell & Sons, 1876.
[5] H. Kesten, Symmetric random walks on groups, Trans. Amer. Math. Soc. 92 (1959), 336-354.
[6] H. Kesten, Full Banach mean values on countable groups, Math. Scand. 7 (1959), 146-156.
[7] M. Picardello and W. Woess, Random walks on amalgams, Mh. Math. 100 (1985), 21-33.
[8] R. Speicher, Combinatorial theory of the free product with amalgamation and operator–valued free probability theory, Mem. Amer. Math. Soc. 132 (1998), no. 627.
[9] D.V. Voiculescu, Symmetries of some reduced free product C*–algebras, Operator Algebras and Their Connections with Topology and Ergodic Theory, H. Araki, C.C. Moore, S. Strătilă and D. Voiculescu, (Eds.), Lecture Notes in Mathematics 1132, Springer-Verlag, 1985, pp. 556–588.
[10] D.V. Voiculescu, Operations on certain non-commutative operator–valued random variables, Recent advances in operator algebras (Orléans, 1992), Astérisque No. 232 (1995), pp. 243-275.
[11] D.V. Voiculescu, K.J. Dykema and A. Nica, Free Random Variables, CRM Monograph Series 1, American Mathematical Society, 1992.
[12] W. Woess, Nearest neighbour random walks on free products of discrete groups, Boll. Un. Mat. Ital. B (6) 5 (1986), 961-982.
[13] W. Woess, Random walks on infinite graphs and groups — a survey on selected topics, Bull. London Math. Soc. 26 (1994), 1-60.
[14] W. Woess, Random walks on infinite graphs and groups, Cambridge University Press, 2000.

Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, 48149 Münster, Germany

Permanent Address: Department of Mathematics, Texas A&M University, College Station TX 77843–3368, USA

E-mail address: kdykema@math.tamu.edu