Poly-Bernoulli Numbers and Eulerian Numbers

Beáta Bényi
Faculty of Water Sciences
National University of Public Service
Hungary
beata.benyi@gmail.com

Péter Hajnal
Bolyai Institute
University of Szeged
Hungary
and
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Hungary
hajnal@math.u-szeged.hu

April 6, 2018

Abstract

In this note we prove combinatorially some new formulas connecting poly-Bernoulli numbers with negative indices to Eulerian numbers.

1 Introduction

Kaneko [11] introduced Poly-Bernoulli numbers A099594 during his investigations on multiple zeta values. He defined these numbers by their generating
function:
\[ \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{\text{Li}_k(1 - e^{-x})}{1 - e^{-x}}, \]

where
\[ \text{Li}_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k} \]
is the classical polylogarithmic function. As the name indicates, poly-Bernoulli numbers are generalizations of the Bernoulli numbers. For \( k = 1 \) \( B_n^{(1)} \) are the classical Bernoulli numbers with \( B_1 = \frac{1}{2} \). For negative \( k \)-indices poly-Bernoulli numbers are integers (see the values for small \( n, k \) in Table 1) and have interesting combinatorial properties.

| \( n \backslash k \) | 0  | 1  | 2  | 3  | 4  | 5  |
|-------------------|----|----|----|----|----|----|
| 0                 | 1  | 1  | 1  | 1  | 1  | 1  |
| 1                 | 1  | 2  | 4  | 8  | 16 | 32 |
| 2                 | 1  | 4  | 14 | 46 | 146| 454|
| 3                 | 1  | 8  | 46 | 230| 1066|4718|
| 4                 | 1  | 16 | 146| 1066|6906|41506|
| 5                 | 1  | 32 | 454| 4718|41506|329462|

Table 1: The poly-Bernoulli numbers \( B_n^{(-k)} \)

Poly-Bernoulli numbers enumerate several combinatorial objects arisen in different research areas, as for instance lonesum matrices, \( \Gamma \)-free matrices, acyclic orientations of complete bipartite graphs, alternative tableaux with rectangular shape, permutations with restriction on the distance between positions and values, permutations with excedance set \([k]\) etc. In [4, 5, 3] the authors summarize in an actual list the known interpretations, present connecting bijections and give further references.

In this note we are concerned only with poly-Bernoulli numbers with negative indices. For convenience, we denote by \( B_{n,k} \) the poly-Bernoulli numbers \( B_n^{(-k)} \).

Kaneko derived two formulas for the poly-Bernoulli numbers with negative indices: a formula that we call basic formula, and an inclusion-exclusion
type formula. The basic formula is

\[ B_{n,k} = \sum_{m=0}^{\min(n,k)} (m!)^2 \binom{n+1}{m+1} \binom{k+1}{m+1}. \]  

(2)

where \( \binom{n}{k} \) denotes the Stirling numbers of the second kind [A008277] that counts the number of partitions of an \( n \)-element set into \( k \) non-empty blocks [10]. The inclusion-exclusion type formula is

\[ B_{n,k} = \sum_{n=0}^{\infty} (-1)^{n+m} m! \binom{n}{m} (m+1)^k. \]  

(3)

Kaneko’s proofs were algebraic, based on manipulations of generating functions. The first combinatorial investigation of poly-Bernoulli numbers was done by Brewbaker [8]. He defined \( B_{n,k} \) as the number of lonesum matrices of size \( n \times k \). He proved both formulas combinatorially; hence, he proved the equivalence of the algebraic definition and the combinatorial one.

Bayad and Hamahata [2] introduced poly-Bernoulli polynomials by the following generating function:

\[ \sum_{n=0}^{\infty} B^{(k)}(x) \frac{t^n}{n!} = \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt}. \]

For negative indices the polylogarithmic function converges for \( |z| < 1 \) and equals to

\[ \text{Li}_{-k}(z) = \sum_{j=0}^{k} \binom{k}{j} z^{k-j} (1 - z)^{k+1}, \]  

(4)

where \( \binom{k}{j} \) is the Eulerian number [10] [A008282] given for instance by:

\[ \binom{k}{j} = \sum_{i=0}^{j} (-1)^i \binom{k+1}{i} (j - i)^k. \]  

(5)

In [2] the authors used analytical methods to show that for \( k \leq 0 \) it holds

\[ B^{(k)}(x) = \sum_{j=0}^{[k]} \binom{[k]}{j} \sum_{m=0}^{[k]-j} \binom{[k]-j}{m} (-1)^m (x + m - [k] - 1)^n. \]  

(6)

The evaluation of (6) at \( x = 0 \) leads to a new explicit formula of the poly-Bernoulli numbers involving Eulerian numbers.
Theorem 1. \[2\] For all \(k > 0\) and \(n > 0\) it holds

\[
B_{n,k} = \sum_{j=0}^{k} \binom{k}{j} \sum_{m=0}^{k-j} (-1)^m \binom{k-j}{m} (k+1-m)^n.
\]

We see that the Eulerian numbers and the defining generating function of poly-Bernoulli numbers for negative \(k\) are strongly related.

In this note we prove this formula purely combinatorially. Moreover, we show four further new formulas for poly-Bernoulli numbers involving Eulerian numbers.

\section{Main results}

In our proofs a special class of permutations plays the key role. We call this permutation class Callan permutations because Callan introduced this class in \([9]\) as a combinatorial interpretation of the poly-Bernoulli numbers. We use the well-known notation: \([N] := \{1, 2, \ldots, N\}\).

Definition 2. Callan permutation of \([n + k]\) is a permutation such that each substring whose support belongs to \(N = \{1, 2, \ldots, n\}\) or \(K = \{n+1, n+2, \ldots n+k\}\) is increasing.

Let \(C^k_n\) denote the set of Callan permutations of \([n+k]\). We call the elements in \(N\) the left-value elements and the elements in \(K\) the right-value elements. For instance, for \(n = 2, k = 2\) the Callan permutations are (we write the left-value elements in red, right-value elements in blue):

\begin{center}
1234, 1324, 1423, 1342, 2314, 2413, 2341, 3124, 3142, 3241, 3412, 4123, 4132, 4231.
\end{center}

It is easy to see that Callan permutations are enumerated by the poly-Bernoulli numbers, but for the sake of completeness, we recall the sketch of the proof of this theorem.

Theorem 3. \[9\]

\[
|C^k_n| = \sum_{m=0}^{\min(n,k)} (m!)^2 \binom{n+1}{m+1} \binom{k+1}{m+1} = B_{n,k}.
\]
Proof. (Sketch) We extend our universe with 0, a special left-value element and with \( n + k + 1 \), a special right-value element. \( \hat{N} = N \cup \{0\} \) and \( \hat{K} = K \cup (n + k + 1) \). Let \( \pi \in \mathcal{C}_n^k \). Let \( \bar{\pi} = 0\pi(n + k + 1) \). Divide \( \bar{\pi} \) into maximal blocks of consecutive elements in such a way that each block is a subset of \( \hat{N} \) (left blocks) or a subset of \( \hat{K} \) (right blocks). The partition starts with a left block (the block of 0) and ends with a right block (the block of \( n + k + 1 \)). So the left and right blocks alternate, and their number is the same, say \( m + 1 \).

Describing a Callan permutation is equivalent to specifying \( m \), a partition \( \Pi_{\hat{N}} \) of \( \hat{N} \) into \( m + 1 \) classes (one class is the class of 0, the other \( m \) classes are called ordinary classes), a partition \( \Pi_{\hat{K}} \) of \( \hat{K} \) into \( m + 1 \) classes (\( m \) many of them not containing \( n + k + 1 \), these are the ordinary classes), and two orderings of the ordinary classes. After specifying the components, we need to merge the two ordered set of classes (starting with the nonordinary class of \( \hat{N} \) and ending with the nonordinary class of \( \hat{K} \)), and list the elements of classes in increasing order. The classes of our partitions will form the blocks of the Callan permutations. We will refer to the blocks coming from ordinary classes as ordinary blocks.

This proves the claim. \( \square \)

The main results of this note are the next five formulas for the number of Callan permutations and hence, for the poly-Bernoulli numbers. We present elementary combinatorial proofs of the theorems in the next section. Theorem 8 is equivalent to theorem 1; we recall the theorem in the combinatorial setting.

**Theorem 4.** For all \( k > 0 \) and \( n > 0 \) it holds

\[
|\mathcal{C}_n^k| = \sum_{m=0}^{\min(n,k)} \sum_{i=0}^{n} \sum_{j=0}^{k} \binom{n}{i} \binom{k}{j} \binom{n+1-i}{m+1-i} \binom{k+1-j}{m+1-j} = B_{n,k}. \tag{8}
\]

**Theorem 5.** For all \( k > 0 \) and \( n > 0 \) it holds

\[
|\mathcal{C}_n^k| = \sum_{j=0}^{k} \binom{k}{j} \sum_{m=0}^{k+2-j} \binom{k+2-j}{m} (m+j-1)! \binom{n}{m+j-1} = B_{n,k}. \tag{9}
\]

**Theorem 6.** For all \( k > 0 \) and \( n > 0 \) it holds

\[
|\mathcal{C}_n^k| = \sum_{j=0}^{k} \binom{k}{j} \sum_{m=0}^{j-1} (-1)^m \binom{j-1}{m} (k+1-m)^n = B_{n,k}. \tag{10}
\]
Theorem 7. For all $k > 0$ and $n > 0$ it holds

$$|C^k_n| = \sum_{j=0}^{k} \binom{k}{j} \sum_{m=0}^{j+1} \binom{j+1}{m} (m+k-j)! \binom{n}{m+k-j} = B_{n,k}. \quad (11)$$

Theorem 8. For all $k > 0$ and $n > 0$ it holds

$$|C^k_n| = \sum_{j=0}^{k} \binom{k}{j} \sum_{m=0}^{k-j} (-1)^m \binom{k-j}{m} (k+1-m)^n = B_{n,k}. \quad (12)$$

3 Proofs of the main results

Eulerian numbers play the crucial role in these formulas. Though Eulerian numbers are well-known, we think it could be helpful for readers who are not so familiar with this topic to recall some basic combinatorial properties.

3.1 Eulerian numbers

First we need some definitions and notation. Let $\pi = \pi_1 \pi_2 \ldots \pi_n$ be a permutation of $[n]$. We call $i \in [n-1]$ a descent (resp. ascent) of $\pi$ if $\pi_i > \pi_{i+1}$ (resp. $\pi_i < \pi_{i+1}$). Let $D(\pi)$ (resp. $A(\pi)$) denote the set of descents (resp. the set of ascents) of the permutation $\pi$. For instance, $\pi = 361487925$ has 3 descents and $D(\pi) = \{2, 5, 7\}$, while it has 5 ascents and $A(\pi) = \{1, 3, 4, 6, 8\}$.

Eulerian numbers $\langle k \rangle^j$ counts the permutations of $[k]$ with $j - 1$ descents. A permutation $\pi \in S_n$ with $j - 1$ descents is the union of $j$ increasing subsequences of consecutive entries, so called ascending runs. So, in other words $\langle k \rangle^j$ is the number of permutations of $[k]$ with $j$ ascending runs. In our example, $\pi$ is the union of 4 ascending runs: 36, 148, 79, and 25.

There are several identities involving Eulerian numbers, see for instance [7], [10]. We will use a strong connection between the surjections/ordered partitions and Eulerian numbers:

$$r! \binom{k}{r} = \sum_{j=0}^{r} \binom{k}{j} \binom{k-j}{r-j}. \quad (13)$$

Proof. We take all the partitions of $[k]$ into $r$ classes. Order the classes, and list the elements in increasing order. This way we obtain permutations of
[k]. Counting by multiplicity we get \( r! \binom{k}{r} \) permutations. All of these have at most \( r \) ascending runs.

Take a permutation with \( j(\leq r) \) ascending runs. How many times did we list it in the previous paragraph? We split the ascending runs by choosing \( r - j \) places out of the \( k - j \) ascents to obtain all the initial \( r \) blocks. The multiplicity is \( \binom{k-j}{r-j} \). This proves our claim.

Inverting (13) gives immediately,

\[
\langle k \rangle_j = \sum_{j=0}^{\infty} (-1)^j r! \binom{k}{r} \binom{k-r}{j-r}.
\]

In the previous section we mentioned the close relation between Eulerian numbers and the polylogarithmic function \( \text{Li}_k(x) \). Here we recall one possible derivation of the identity (4) following [7].

\[
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i (k+1) \binom{k+1}{j-i} (j-i)^k x^j = \\
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i (k+1) \binom{k+1}{j-i} (j-i)^k x^j = \\
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i}{(j-i)^j} \binom{k+1}{j-i} (j-i)^k x^j = \\
= \sum_{i=0}^{\infty} \frac{(-1)^i}{(1-x)^{j-i}} \binom{k+1}{j-i} x^j = \\
= (1-x)^{k+1} \sum_{i=0}^{\infty} i^k x^i = (1-x)^{k+1} \text{Li}_{k+1}(x).
\]

Plugging (13) for \( \langle k \rangle_j \), exchanging \( i \) to \( j-i \), changing the order of the summation; and finally, applying the binomial theorem we get the result.

### 3.2 Combinatorial proofs of the theorems

Now we turn our attention to the proofs of our theorems. For the sake of convenience, thanks to our color coding (left-value elements are red, and right-value elements are blue), we rewrite the set of right-value elements as \( K = \{1, 2, \ldots, k\} \), and \( \hat{K} = K \cup \{k+1\} \). We can do this without changing
essentially Callan permutations, since we just need the distinction between the elements \( \mathcal{N} \) and \( \mathcal{K} \) and an order in \( \mathcal{N} \) and \( \mathcal{K} \). If we consider separately the left-value elements and right-value elements in the permutation \( \pi \) the elements of \( \mathcal{N} \) form a permutation of \([n]\) and the elements of \( \mathcal{K} \) form a permutation of \([k]\). We let \( \pi^r \) denote the permutation restricted to the right-value elements and we let \( \pi^\ell \) denote the permutation restricted to the left-value elements. For instance, for

\[
\pi = 0231454728183569679,
\]

\[
\pi^r = 145283679, \quad \text{while} \quad \pi^\ell = 0234718569.
\]

**Proof of Theorem 4.** We consider the last entries of the blocks in the restricted permutations \( \pi^\ell \) resp. \( \pi^r \). Some of the blocks end with a descent and some of the blocks not. (The special elements \( 0 \) and \( k + 1 \) are neither descents nor ascents of the permutations.) Let \( i \) be the number of ascending runs and \( j \) the number of ascending runs in \( \pi^r \). Let further \( m \) be the number of ordinary blocks of both types. The \( i - 1 \) descents of \( \pi^\ell \) determine \( i \) blockendings; hence, we are missing \( m - (i - 1) \) further blockendings with ascents. Similarly, the \( j - 1 \) descents of \( \pi^r \) determine \( j - 1 \) blockendings and there are further \( m - (j - 1) \) blockendings with an ascent.

Given a pair \((\pi^\ell, \pi^r)\) with \( |D(\pi^\ell)| = i - 1 \) and \( |D(\pi^r)| = j - 1 \), we can construct a Callan permutations, according to the above arguments. In our running example, \( \pi^\ell = 0234718569 \) and we need to choose \( 3 - 2 = 1 \) blockendings from \( 9 - 2 \) possibilities. In general, we need to choose \( m - (i - 1) \) blockendings from \( n - (i - 1) \) possibilities. And analogously for \( \pi^r \), we need to choose \( m - (j - 1) \) blockendings from \( k - (j - 1) \) possibilities. Hence, for a given a pair \((\pi^\ell, \pi^r)\) with \( |D(\pi^\ell)| = i - 1 \) and \( |D(\pi^r)| = j - 1 \) we have

\[
\binom{n + 1 - i}{m + 1 - i} \binom{k + 1 - j}{m + 1 - j}
\]

different corresponding Callan permutations. Since the number of pairs \((\pi^\ell, \pi^r)\) with \( |D(\pi^\ell)| = i - 1 \) and \( |D(\pi^r)| = j - 1 \) is \( \langle n \rangle \langle k \rangle \rangle \langle i \rangle \langle j \rangle \) Theorem 4 is proven. \( \square \)

Note that (8) is actually a rewriting of the basic combinatorial formula (2) in terms of Eulerian numbers using the relation (13) between the number of ordered partitions and Eulerian numbers.
Now we enumerate Callan permutations according to the number of descents in $\pi^r$. Given a permutation $\pi^r$ with $j - 1$ descents we determine the number of ways to merge $\pi^r$ with left-value elements to obtain a valid Callan permutation. Let $D = \{d_1, d_2, \ldots, d_{j-1}\}$ be the set of descents of $\pi^r$. In our running example, $j = 3$ and $D = \{3, 5\}$. We code the positions of the left-values comparing to the right-values by a word $w$. We let $w_i$ be the number of right-values that are to the left of the left-value $i$. In our example, $w_1 = 5$, since there are 5 right-value elements preceding the left-value 1, $w_2 = 0$, because there are no right-value elements preceding the left-value 2, etc. Hence, $w = 500366356$. Note that the blocks of the left-value elements can be recognized from the word: The positions $i$, for which the values $w_i$ are the same contain the elements of a block. We call a word valid respect to $\pi^r$ if the augmentation of $\pi^r$ according to the word $w$ leads to a valid Callan permutation.

**Observation 9.** A word $w$ is valid respect to a permutation $\pi^r$ if and only if it contains every value $d_i$ of the descent set of $\pi^r$.

**Proof.** In a Callan permutation the substrings restricted to $K$ or $N$ are increasing subsequences. Given $\pi^r$ with descent set $D$, each $d_i \in D$ has to be the last element of a block in the ordered partition of the set $K$, the set of right-value elements. Hence, each right-value with position $d_i$ in $\pi^r$ has to be followed by a left-value element in the Callan permutation. In our word $w$ at the position of this left-value element there is a $d_i$.

For the converse, assume that our word $w$ contains at least one $d_i$, for any $d_i \in D(\pi^r)$. There is at least one left-value element with $d_i$ in $w$ at its position. This implies that if we combine $\pi^r$ and $\pi^\ell$ then in $\pi^r$ the position of the descent will be interrupted by a left-value element. The combined permutation will be a Callan permutation.

**Corollary 10.** The number of valid words respect to $\pi^r$ depends only on the number of descents in $\pi^r$.

We let $w^{j-1}$ denote a word that is valid to a $\pi^r$ with $j - 1$ descents and we let $W(\pi^r)$ denote the set of words $w^{j-1}$. The number of Callan permutations of size $n + k$ is the number of pairs $(\pi^r, w^{j-1})$, where $\pi^r$ is a permutation of $[k]$ with $j - 1$ descents and $w^{j-1} \in W(\pi^r)$. We denote $|W(\pi^r)|$ by $w(j - 1)$. Hence,

$$|C^k_n| = \sum_{j=1}^{k} \binom{k}{j} w(j - 1).$$
The next two proofs are based on two different ways of determining $w(j - 1)$, i.e. enumerating those $w^{j-1}$’s that are valid to a $\pi^r$ with $j - 1$ descents.

**Proof of Theorem 5.** Fix $\pi^r$ and take a $w^{j-1} \in W(\pi^r)$. $w^{j-1}$ corresponds to an ordered partition of $[n]$ into at least $j - 1$ blocks. Let $j - 1 + m$ be the number of the blocks.

First, we take an ordered partition of $\{1, 2, \ldots, n\}$ into $n - (m+1)$ non-empty blocks in $(m + j - 1)!\binom{n}{m+j-1}$ ways. Then we refine the partition of $\pi^\ell$, defined by the descents. For the refinement we need to choose additional places for the $m$ blocks. These places can be before the first element of $\pi^\ell$, or at an ascent. We have $(k+1-\frac{m+2}{m})$ choices. This proves (9).

**Proof of Theorem 6.** Now we calculate $w(j - 1)$ using the inclusion-exclusion principle. The total number of words of length $n$ with entries $\{0, 1, \ldots, k\}$ $(w_i = 0$ if the left-value $i$ is in the first block of the Callan permutation) is $(k+1)^n$. We have to reduce this number with the number of not valid words respect to $\pi^r$, with the words that do not contain at least one of the $d_i \in D$. Let $A_s$ be the set of words that do not contain the value $s$. $|\bigcup_{s \in D} A_s|$ is to be determined. Clearly, $|A_s| = (k+1-1)^n$ and this number does not depend on the choice of $s$; hence, we have $\sum_{s \in D} |A_s| = k^n (j - 1)$. The $|A_s \cap A_t| = (k+1-2)^n$ and $\sum_{s,t \in D} |A_s \cap A_t| = (k+1-1)(j-1)^{\binom{2}{m}}$. Analogously, $|\bigcap_{i=1}^m A_{s_i}| = (k+1-m)^n (j-1)^{\binom{m}{m}}$. The inclusion-exclusion principle gives

$$w(j - 1) = \sum_{m=0}^{j-1} (-1)^m \binom{j - 1}{m} (k+1-m)^n,$$

and this implies (10).

**Proof of Theorem 7. and Theorem 8.** The claims follow by the symmetry of Eulerian numbers. If we reverse a permutation of $[k]$ with $j - 1$ descents we obtain a permutation with $k - (j-1) - 1$ descents. According to our previous arguments a pair $(\pi^r, w^{k-j})$, where $\pi^r$ is a permutation with $k - j$ descents and $w^{k-j}$ is a valid word respect to $\pi^r$ determines a Callan permutation. Hence,

$$|\mathcal{C}_n^k| = \sum_{j=1}^{k} \binom{k}{j} \binom{k}{k-j+1} w(k-j) = \sum_{j=1}^{k} \binom{k}{j} w(k-j).$$
We have two formulas for $w(k - j)$ using the results of the proofs of the previous theorems.

\[
\begin{align*}
    w(k - j) &= \sum_{m=0}^{j+1} \binom{j + 1}{m}(m + k - j)!\left\{ \frac{n}{m + k - j} \right\}, \\
    w(k - j) &= \sum_{m=0}^{k-j} (-1)^m \binom{k - j}{m}(k + 1 - m)^n.
\end{align*}
\]

This implies (11) and (12). \qed

References

[1] T. Arakawa, T. Ibukiyama, M. Kaneko, *Bernoulli numbers and zeta functions*, With an appendix by Don Zagier, Springer Monographs in Mathematics, Springer, Tokyo, 2014.

[2] A. Bayad, Y. Hamahata, Polylogarithms and poly-Bernoulli polynomials, *Kyushu J. Math.* 65 (2011), 15-24.

[3] B. Bényi, *Advances in Bijective Combinatorics*, PhD thesis, (2014), available at \url{http://www.math.u-szeged.hu/phd/dreposit/phdtheses/benyi-beata-d.pdf}.

[4] B. Bényi, P. Hajnal, Combinatorics of poly-Bernoulli numbers, *Studia Sci. Math. Hungarica* 52(4) (2015), 537–558.

[5] B. Bényi and P. Hajnal, Combinatorial properties of poly-Bernoulli relatives, *Integers* 17 (2017), A31.

[6] B. Bényi and G.V. Nagy, Bijective enumerations of $\Gamma$-free 0-1 matrices, *Adv. Appl. Math.* 96 (2018), 195–215.

[7] M. Bóna, *Combinatorics of Permutations*, Discrete Mathematics and its Applications, Chapman Hall/CRC, Boca Raton, 2004.

[8] C. R. Brewbaker, A combinatorial interpretation of the poly-Bernoulli numbers and two Fermat analogues, *Integers* 8 (2008), A02.

[9] Callan, Third comment to [A099594] in [12].
[10] L. Graham, D. Knuth, O. Patashnik, *Concrete mathematics*, Addison-Wesley, 1994.

[11] M. Kaneko, Poly-Bernoulli numbers, *J. Théor. Nombres Bordeaux* 9 (1997), 221–228

[12] N.J.A. Sloane, The on-line encyclopedia of integer sequences, [http://oeis.org](http://oeis.org)

2010 Mathematics Subject Classification: 05A05, 05A15, 05A19, 11B83. Keywords: Combinatorial identities, Eulerian number, poly-Bernoulli number.

(Concerned with sequences: [A008282](http://oeis.org/A008282), [A027641](http://oeis.org/A027641), [A027642](http://oeis.org/A027642), [A008277](http://oeis.org/A008277), [A099594](http://oeis.org/A099594))