ON MINIMAL PARABOLIC FUNCTIONS
AND TIME-HOMOGENEOUS PARABOLIC $h$-TRANSFORMS

Krzysztof Burdzy
Thomas S. Salisbury
University of Washington
York University and the Fields Institute

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Abstract. Does a minimal harmonic function $h$ remain minimal when it is viewed as a parabolic function? The question is answered for a class of long thin semi-infinite tubes $D \subset \mathbb{R}^d$ of variable width and minimal harmonic functions $h$ corresponding to the boundary point of $D$ “at infinity.” Suppose $f(u)$ is the width of the tube $u$ units away from its endpoint and $f$ is a Lipschitz function. The answer to the question is affirmative if and only if $\int_{\infty}^\infty f^3(u)du = \infty$. If the test fails, there exist parabolic $h$-transforms of space-time Brownian motion in $D$ with infinite lifetime which are not time-homogenous.

1. Introduction and main results. We want to compare the parabolic Martin boundary of a domain in $\mathbb{R}^d$ with its Martin boundary, both topologically and probabilistically. In many cases, the two boundaries are related in a very simple way. This provides a complete description of the parabolic Martin boundary in those cases (quite many) when the Martin boundary is known. We plan to present a detailed discussion of this general problem in a separate publication. This paper is devoted to a narrower aspect of the relationship between the two boundaries. We will start with a very informal discussion of a special case which motivated our study. The concepts of the usual and parabolic Martin boundary will be reviewed in a rigorous way later in the introduction. The basic ideas of the classical potential theory and Brownian motion may be found in Doob (1984).

Consider a strip $D = \{(x^1, x^2) \in \mathbb{R}^2 : |x^2| < 1\}$. Let $X_t$ be a Brownian motion starting from $(0, 0)$. Then $\dot{X}_t = (X_t, -t)$ is a space-time Brownian motion starting from $(0, 0, 0)$. First fix some $s > 0$, a point $z \in \partial D$ and a sequence of points $\{z_k\}$ in $D$ converging to $z$ as $k \to \infty$. Condition $\dot{X}$ to be at $(z_k, -s)$ at time $s$ and to not leave $D \times \mathbb{R}$ before time $s$. Then let $k$ go to infinity. The conditioned processes...

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converge in distribution to a process whose first coordinate is a Brownian motion conditioned to exit $D$ through $z$ at time $s$. The lifetime of this process is finite. This conditioned space-time Brownian motion is not time-homogeneous, i.e., its transition probabilities $P(\dot{X}_u \in (dy, -du) \mid \dot{X}_t \in (dx, -dt))$ depend not only on $u - t$, but on the values of $t$ and $u$ as well.

Next suppose that $c > 0$ is a constant and consider $\dot{X}$ conditioned to be at $(ck, 0, -k)$ at time $k$ and to not leave $D \times \mathbb{R}$ before time $k$. In the limit, as $k \to \infty$, we obtain a process whose spatial component escapes “to $+\infty$” within $D$ at rate $c$. The first coordinate of the space process is a one-dimensional Brownian motion with drift $c$. This conditioned space-time Brownian motion is time-homogeneous and its lifetime is infinite.

The domain in our example, a strip, seems to be typical and we would expect that many domains have the property stated in the following problem.

(1.1) Problem. Find necessary and sufficient conditions, of a geometric nature in $D$, such that for every minimal parabolic function $h$ in $\hat{D}$, the corresponding $h$-transform of the space-time Brownian motion is time homogeneous if and only if its lifetime is a.s. infinite.

Another source of motivation may be explained in purely analytic language. Recall the domain of our first example, $D = \{(x^1, x^2) \in \mathbb{R}^2 : |x^2| < 1\}$. Consider a minimal positive harmonic function $h(x), x \in D$. Let $g(x, t) = h(x)$ for all $x \in D$ and $t \in \mathbb{R}$. Evidently, $g$ is a parabolic function, and we may therefore identify every harmonic function with a parabolic function. Since $h$ is minimal harmonic, it corresponds to a minimal Martin boundary point $y$ of $D$. Suppose that $y$ is also a Euclidean boundary point, say, $y = (1, 1)$. Then $g$ is not minimal as a parabolic function, i.e., it is a mixture of different parabolic functions. An easy probabilistic justification can be based on the fact that Brownian motion conditioned by $h$ has a random lifetime. Thus the space-time Brownian motion conditioned by $g$ is a mixture of processes conditioned to exit $D$ through $y$ at different times $s$, i.e., a mixture of $g_s$-transforms for different parabolic functions $g_s$. However, if $y$ is the point at “$+\infty$” then $g$ is minimal in the space of parabolic functions. While not completely obvious, this is simple to show directly, and also follows from our main result, Theorem 1.3 below. Our informal discussion suggests that in many domains, a minimal harmonic function is also minimal in the space of parabolic functions if and only if it corresponds to a “point at infinity.” We propose the following problem.

(1.2) Problem. Determine which minimal harmonic functions are minimal in the space of parabolic functions.

We are not able to give a complete answer to either of the two problems but we hope that our main result, Theorem 1.3 below, will shed light on both.

We proceed with a rigorous presentation of our results. We start with a review of basic definitions and facts concerning Martin boundaries and conditioned Brownian motion. Let $D$ be a Euclidean domain, that is, an open connected subset of $\mathbb{R}^d$
for some $d \geq 2$. We will consider the domain $\hat{D} \equiv D \times (-\infty, 0) \subset \mathbb{R}^{d+1}$. Let $G(x, y) = G_D(x, y)$ and $\hat{G}(u, v) = \hat{G}_D(u, v)$ be the Green functions for $(1/2)\Delta$ on $D$ and for the heat operator $(1/2)\Delta - \partial/\partial t$ on $\hat{D}$ where $\Delta$ is the Laplace operator (see Doob (1984) 1.VII.1 and 1.XVII.4). Thus $G : D \times D \rightarrow [0, \infty]$ and $\hat{G} : \hat{D} \times \hat{D} \rightarrow [0, \infty]$. For $u = (x, s) \in \hat{D}$ and $v = (y, s - t) \in \hat{D}$ we have that

$$\hat{G}(u, v) = \begin{cases} p_t(x, y), & \text{for } t > 0 \\
0, & \text{for } s < t \leq 0 \end{cases},$$

where $p_t = p_t^D$ is the heat kernel on $D$ (that is, the transition function for Brownian motion killed upon leaving $D$). Note that this formula can also be used to define $\hat{G}((x, s), v)$ when $s = 0$. A function $h : D \rightarrow [0, \infty)$ is harmonic if $\Delta h = 0$ on $D$. A function $g : \hat{D} \rightarrow [0, \infty)$ is parabolic if it solves the heat equation

$$\frac{\partial g}{\partial t} = \frac{1}{2} \Delta_x g$$

in $\hat{D}$. In this case, it is superparabolic as well. That is,

$$g(x, s) \geq \int g(y, s - t)p_t(x, y)dy$$

for every $(x, s) \in \hat{D}$ and $t > 0$. We may extend $g$ by letting

$$g(x, 0) \equiv \lim_{t \downarrow 0} \int g(y, -t)p_t(x, y)dy$$

(the limit is easily seen to be monotone). We say that $g$ is admissible if $g(x_0, 0) < \infty$.

Now recall the definitions of the Martin boundary in the elliptic and parabolic contexts (Doob (1984) 1.XII.3 and 1.XIX.3). Fix some $x_0 \in D$ and let

$$K(x, y) \equiv \frac{G(x, y)}{G(x_0, y)}$$

for $x, y \in D$. Then, up to homeomorphism there is a unique metrizable compactification $D^M$ of $D$ such that

(i) the function $K(\cdot, \cdot)$ may be extended continuously to $D \times (D^M \setminus \{x_0\})$;

(ii) $K(\cdot, x) \equiv K(\cdot, y)$ if and only if $x = y$.

The set $\partial^M D \equiv D^M \setminus D$ is called the Martin boundary of $D$. For $z \in \partial^M D$ and $y_k \in D$, we have $y_k \rightarrow z$ if and only if $K(x, y_k) \rightarrow K(x, z)$ for every $x \in D$. A harmonic function $h > 0$ is said to be minimal if, whenever $h' > 0$ is harmonic, and $h' \leq h$, it follows that $h' = ch$ for some constant $c$. A point $z \in \partial^M D$ is said
to be minimal if $K(\cdot, z)$ is minimal. For every $h > 0$ harmonic, there is a unique measure $\mu$, concentrated on the set $\partial_0^M D$ of minimal points of $\partial^M D$, such that

$$h(x) = \int_{\partial_0^M D} K(x, z) \mu(dz),$$

for every $x \in D$ (See Doob (1984) 1.XII.9).

Now define $\dot{K}$ on $\dot{D} \times \dot{D}$ by

$$\dot{K}((x, s), (y, t)) \overset{\text{df}}{=} \frac{\dot{G}((x, s), (y, t))}{G((x_0, 0), (y, t))} = \begin{cases} p_{s-t}(x, y)/p_{s-t}(x_0, y), & t < s < 0 \\ 0, & s \leq t < 0. \end{cases}$$

Then up to homeomorphism, there is a unique metrizable compactification $\dot{D}^M$ of $\dot{D}$ with the following properties:

(i) the function $\dot{K}$ has an extension to $\dot{D} \times \dot{D}^M$ such that for each $(x, s) \in \dot{D}$, the function $\dot{K}((x, s), \cdot)$ is finite valued and continuous on $\dot{D}^M \setminus \{(x, s)\};$
(ii) $\dot{K}(\cdot, u) = \dot{K}(\cdot, v)$ if and only if $u = v$.

We call $u$ the pole of $\dot{K}(\cdot, u)$. We write $\partial^M \dot{D} \overset{\text{df}}{=} \dot{D}^M \setminus \dot{D}$ and call it the Martin boundary of $\dot{D}$ (or the parabolic Martin boundary of $D$). We have again that, for $z \in \partial^M \dot{D}$ and $(y_k, t_k) \in \dot{D}$, $(y_k, t_k) \to z$ if and only if $\dot{K}((x, t), (y_k, t_k)) \to \dot{K}((x, t), z)$ for every $(x, t) \in \dot{D}$. Every $\dot{K}(\cdot, z)$ is admissible (see 1.XIX.3.1 of Doob (1984)).

We denote by $\hat{0}$ the unique point of $\partial^M \dot{D}$ for which $\dot{K}(\cdot, \hat{0}) \equiv 0$. It is unique by (ii) and exists as the limit of some subsequence of $(x_0, 1/n)$. A point $z \in \partial^M \dot{D}$ is minimal if $\dot{K}(\cdot, z)$ is minimal as a parabolic function, and $\dot{K}((x_0, 0), z) = 1$. The set of minimal points is denoted $\partial_0^M \dot{D}$. The integral representation of admissible parabolic functions as

$$g(x, t) = \int_{\partial_0^M D} \dot{K}((x, t), z) \mu(dz)$$

is entirely analogous to that of the harmonic setting (See Doob (1984) 1.XIX.7).

Let $(\Omega, \mathcal{F})$ be a measurable space with $X : \Omega \times [0, \infty) \to \mathbb{R}^d \cup \{\delta\}$ a stochastic process. We use the notation $X_t$ and $X(t)$ interchangeably. $P^x$ is a probability measure under which $X$ is a standard $d$-dimensional Brownian motion started from $x$, and killed upon leaving $D$. We write $E^x$ for the corresponding expectation. In particular, $\delta$ is a cemetery point adjoined to $D$, $X$ is continuous on a random time interval $[0, \zeta)$, and $X_t = \delta$ for $t \geq \zeta$.

Let $\tau_t = \tau_0 - t$ be a process measuring absolute time, and write $\dot{X}_t = (X_t, \tau_t)$. By enlarging $\Omega$ if necessary, we may suppose that for each $s \leq 0$, there are probability
measures $P^{x,s}$ under which $X$ has the same law as under $P^x$, and $\tau_0 = s$. That is, \{\dot{X}_t, t \geq 0\} is a space-time Brownian motion starting from $(x, s)$.

If $h : D \to (0, \infty]$ is a superharmonic function then

$$P^h_t(x, y) \overset{df}{=} \frac{h(y)p_t(x, y)}{h(x)}$$

is the transition function of a Markov process $X^h$, called an $h$-transform, or conditioned Brownian motion. We write $P^x_h$ and $E^x_h$ for the corresponding probability measure, and its expectations. By convention, $h$ is taken to vanish at $\delta$. If $x \in D^M$, $x \neq x_0$ then we write $X^x$ for $X^{K(\cdot, x)}$. If $h = \int_{\partial^0 M} K(\cdot, z) \mu(dz)$ is harmonic, then

$$P^h_x = \frac{1}{h(x)} \int_{\partial^0 M} K(x, z) P^x_z \mu(dz).$$

The paths of $X^h$ converge a.s. to points of the minimal Martin boundary, at their lifetimes (see Doob (1984) 3.III.1, or section 7.2 of Pinsky (1995)).

Similarly, if $g : \dot{D} \to [0, \infty]$ is a superparabolic function, then

$$P^g_t((x, s), (y, s - t)) \overset{df}{=} \frac{g(y, s - t)p_t(x, y)}{g(x, s)}$$

is the transition function for a Markov process $X^g$ taking values in $\dot{D} \cup \{\delta\}$ (actually in $\{\delta\} \cup \{u \in \dot{D}; g(u) > 0\}$) that we call a conditioned space-time Brownian motion. We will use $P^{x,s}_g$ to denote a probability measure under which $X^g$ has this transition function and starts from $(x, s)$. We write $X^g$ for the spatial component of $X^g$ (with $X^g_t = \delta$ for $t \geq \zeta$), and note that

$$\dot{X}^g_t = \begin{cases} (X^g_t, \tau_t) \in \dot{D}, & \text{for } t < \zeta \\ \delta & \text{for } t \geq \zeta. \end{cases}$$

We will also refer to $X^g$ as an $g$-transform. This abuse should cause no confusion, as it is easy to check that if $h$ is superharmonic and we define a superparabolic function $g$ by $g(x, t) = h(x)$ then $X^h = X^g$. If $u \in \dot{D}^M$ then we write $\dot{X}^u$, $X^u$, $P^{x,s}_u$ instead of $\dot{X}^{K(\cdot, u)}$, etc. Strictly speaking, the above formulae hold under $P^{x,s}_g$ only for $s < 0$, but by taking $X^g_0 = x$ under $P^{x,0}_g$, we obtain extensions valid for $s = 0$ as well, provided $g$ is admissible. If $g$ is actually parabolic, then each $g$-process approaches the one-point boundary of $\dot{D}$ at its lifetime $\zeta$ (Doob (1984) 2.X.12), in other words, it eventually leaves every compact subset of $\dot{D}$. In the Martin topology, the paths of $\dot{X}$ converge at their lifetimes, to points of the minimal parabolic Martin boundary, and the measures $P^{x,s}_g$ can be represented in terms of the $P^{x,s}_u$, for $u \in \partial^0 M \dot{D}$, just as in the harmonic setting.
For \((x^1, x^2, \ldots, x^d) \in \mathbb{R}^d\) let \(\tilde{x} = (x^1, x^2, \ldots, x^{d-1})\). We will restrict our attention to “tubes” with variable width. For a non-negative function \(f : \mathbb{R} \to \mathbb{R}\), let

\[
D_f \overset{\text{df}}{=} \{ x \in \mathbb{R}^d : |\tilde{x}| < f(x^d) \}.
\]

We will always assume that \(f\) is strictly positive on \((a, b)\) for some \(-\infty \leq a < b \leq \infty\) and equal to 0 on \((-\infty, a] \cup [b, \infty)\). We will focus on domains \(D_f\) corresponding to functions \(f\) which are Lipschitz on \((a, b)\) (the function may have a jump at \(a\) or \(b\)). If \(f\) is Lipschitz and \(b = \infty\), then each sequence \(x_k\) of points in \(D_f\) such that \(x_k^d \to \infty\) converges in the Martin topology to a point (the same for all such sequences) which we will denote as \(\infty\). The proof of this claim is easy — it may be based on the boundary Harnack principle. The same result should be true for all functions \(f\) (not necessarily Lipschitz) but we do not see an obvious argument. An analogous remark applies to \(-\infty\).

Any positive harmonic function \(h\) corresponding to \(\infty \in \partial M D_f\) vanishes on \(\{ x \in \partial D_f : x^d < b \}\) and, moreover, \(h(x) \to 0\) when \(x^d \to -\infty\).

Let \(\Lambda_s = \{ x \in D_f : x^d = s \}\). The stopping time \(\inf\{ t > 0 : X_t \in A \}\) will be denoted \(T(A)\). We write \(\tau(A)\) for the absolute time \(\tau_{T(A)} = \tau_0 - T(A)\).

Recall that a harmonic function \(h\) is identified with a parabolic function by letting \(h(x, t) = h(x)\).

**Theorem.** Suppose that \(b = \infty\) and \(f\) is a function which is Lipschitz on \((a, b)\) and such that

\[
\limsup_{v \to \infty} f(v) < \infty
\]

and

\[
\int_u^\infty f(v)dv = \infty
\]

for all \(u < \infty\). Let \(h\) be the minimal harmonic function corresponding to \(\infty \in \partial^M D_f\). Fix some \(x_0 \in D_f\).

(i) Suppose that either

(a) \(\int_u^\infty f^3(v)dv < \infty\) or

(b) the Lipschitz constant of \(f\) is sufficiently small (it will suffice to assume that it is less than the \(\lambda\) in (iv) of Theorem 1.6) and \(\int_u^\infty f^3(v)dv < \infty\) for some \(u < \infty\).

Each one of assumptions (a) or (b) implies (A)-(D) below.

(A) For some function \(g : (a, \infty) \to (-\infty, 0]\) with \(\lim_{u \to \infty} g(u) = -\infty\), we have the following. For each \(s \in \mathbb{R}\) there is a minimal point \(z_s \in \partial^M D_f\), which is the limit of all sequences \((x_k, (g(x_k^d) - s_k) \wedge 0)\) with \(x_k^d \to \infty\) and \(s_k \to s\).

(B) If \(s_1 \neq s_2\) then \(z_{s_1} \neq z_{s_2}\).
(C) Let \( h_s \) denote a minimal parabolic function with pole at \( z_s \). Then \( h = \int_R h_s \mu(ds) \) for some measure \( \mu \) which charges all non-degenerate intervals. In particular, \( h \) is not minimal in the space of parabolic functions on \( D_f \).

(D) Let \( s \in \mathbb{R} \) and \((x,t) \in \mathcal{D} \). The process \( X \) is not time-homogeneous under \( P_{x,t}^{x,t} \). In fact, \( g(u) - \tau(\Lambda_u) \to s \) as \( u \to \infty \) \( P_{x,t}^{x,t}\)-a.s. Hence, \( \lim_{u \to \infty} (T(\Lambda_u) + g(u)) \) exists \( P_{h}^{x,t} \)-a.s.

(ii) If \( \int_u^\infty f^3(v)dv = \infty \) for all \( u < \infty \) then \( h \) is minimal in the space of parabolic functions on \( \mathcal{D}_f \).

(1.5) Remarks.

The lifetime of Brownian motion conditioned by \( h \) is infinite if and only if \( \int_u^\infty f(v)dv = \infty \) for all \( u < \infty \), according to Theorem 1.6 below. If this condition is not satisfied, the function \( h \) is not minimal as a parabolic function (see the discussion preceding Problem 1.2).

The proof of Theorem 1.3 hinges on estimates of the variance of \( h \)-path lifetimes. Since the estimates may have some independent interest, we state them as Theorem 1.6 below.

Several authors have addressed the problem of when, given a domain \( D \subset \mathbb{R}^d \), there is a constant \( c = c(D) < \infty \) such that for any \( x \in D \) and any positive harmonic function \( h \) in \( D \) we have \( \text{E}_h^x \zeta < c \). The pioneering work was done by Cranston and McConnell (1983) and Cranston (1985). The existence of the finite upper bound \( c \) is known for a wide class of domains; see, e.g., Bañuelos and Davis (1992) or Bass and Burdzy (1992) and references therein. Higher moments of \( h \)-path lifetimes have been studied by Davis (1988), Davis and Zhang (1994) and Zhang (1996).

Chris Rogers has pointed out to us that a related equivalence, between non-minimality and the variance of hitting times, has been established in the context of one-dimensional diffusions. There, the speed measure and coupling can be used to give a simple proof. See Rogers (1988), which synthesizes earlier work of Fristedt and Orey (1978), Küchler and Lunze (1980), and Rösler (1979).

Recall that we are concerned with functions \( f \) which are strictly positive and Lipschitz on \((a,b)\) and equal to 0 on \((-\infty,a) \cup [b,\infty)\). Our next result holds for all functions \( f \) which are Lipschitz on \((a,b)\). However, in order to simplify the notation we will prove it only in the case when \( f \) is Lipschitz with the constant equal to 1, i.e., from now on we will assume that \( |f(u) - f(v)| \leq |u - v| \) for \( u,v \in (a,b) \). Fix some \( s_0 \in (a,b) \) and define \( s_k \) inductively by \( s_{k+1} = s_k + f(s_k)/2 \) for \( k \geq 0 \) and \( s_{k-1} = s_k - f(s_k)/2 \) for \( k \leq 0 \). If \( s_k \geq b \) for some \( k \) then we redefine \( s_j \) for \( j \geq k \) and we let \( s_j = b \) for all \( j \geq k \). A similar remark applies to the case when \( s_k \leq a \). Note that it may happen that \( s_k < b \) for all \( k > 0 \) and/or \( s_k > a \) for all \( k < 0 \). However, we always have \( \lim_{k \to \infty} s_k = b \) and \( \lim_{k \to -\infty} s_k = a \). Let \( k_f = \inf\{k : s_k = b\} \) and recall that \( \Lambda_{s_k} = \{x \in D_f : x^d = s_k\} \). Let \( D_j \) be the component of \( D_f \setminus \Lambda_{s_j} \) which contains points \( x \) with \( x^d < s_j \).

(1.6) Theorem. Let \( h \) be a positive harmonic function in \( D_f \) which vanishes on \( \{x \in \partial D : x^d < b\} \). If \( b = \infty \) then \( h \) corresponds to \( \infty \in \partial_0 M D_f \). In the
following statements, $x$ ranges over the elements of $D_f$ with $x^d < b - f(b-) \ (\text{here } \infty - \infty = \infty)$.

(i) For some $c_1, c_2 \in (0, \infty)$,

\begin{equation}
(1.7) \quad c_1 \int_{x^d}^{b} f(v)dv \leq E_{h}^x \zeta \leq c_2 \int_{x^d}^{b} f(v)dv.
\end{equation}

(ii) If $\int_{x^d}^{b} f(v)dv = \infty$ then $\zeta = \infty$ $P_h^x$-a.s.

(iii) If $\zeta < \infty$ $P_h^x$-a.s. then for some $c_3, c_4 \in (0, \infty)$,

\begin{equation}
(1.8) \quad c_3 \int_{x^d}^{b} f^3(v)dv \leq \text{Var}_{h}^x \zeta \leq c_4 \int_{x^d}^{b} f^3(v)dv.
\end{equation}

(iv) There exists $\lambda > 0$ such that if the Lipschitz constant of $f$ is less than $\lambda$ then

\begin{equation}
(1.9) \quad \text{Var}_{h}^x \zeta \leq c_5 \int_{x^d}^{b} f^3(v)dv.
\end{equation}

(v) If $\int_{x^d}^{b} f^3(v)dv = \infty$ then for each $c_6 < \infty$ and $c_7 > 0$ there is a $k_0 < \infty$ such that for all $k > k_0$ and $u \in \mathbb{R}$,

\[ P_h^x(T(\Lambda_{s_k}) \in (u, u + c_6)) < c_7. \]

(1.10) Remarks.

(i) The constants $c_j$ in Theorem 1.6 depend only on the dimension $d$ and the Lipschitz constant of $f$. However, the proof will be given only in the case when the Lipschitz constant of $f$ is equal to 1 so all the constants in Section 2 will depend only on the dimension $d$.

(ii) The bound (1.9) holds for $d \geq 4$ without any assumptions on the value of the Lipschitz constant of $f$ but it does not hold without such an assumption for $d < 4$. We are not going to prove the latter. It essentially follows from a theorem of Davis and Zhang (1994).

(iii) We can give a meaning to (1.8) and (1.9) even if $\zeta = \infty$ $P_h^x$-a.s. Note that in such a case we necessarily have $b = \infty$ (see (1.7)). For all $k < \infty$ and $x \in D_f$ such that $x^d < s_k$, $\text{Var}_{h}^x T(\Lambda_{s_k}) < c_4 \int_{a}^{b} f^3(v)dv$ with the same constant $c_4$ as in (1.8). This and the analogous modification of (1.9) can be proved by applying the theorem to the function $\tilde{f}(v) \overset{df}{=} f(v) \mathbf{1}_{(-\infty, s_k)}(v)$.

(iv) In the two-dimensional case, part (i) of Theorem 1.6 is due to Xu (1990). This was generalized in Bañuelos and Davis (1992).
(v) Suppose that \( d = 2 \), the Lipschitz constant of \( f \) is small and let \( \rho \) be the supremum of areas of discs contained in \( D_f \). Then (1.7) and (1.9) imply that \( \text{Var}_{h}^x \zeta \leq c_1 \rho E_{h}^x \zeta \). Davis (1988) discovered this inequality and proved that it holds for all simply connected planar domains \( D \) provided \( h \) is a minimal positive harmonic function or a Green function.

We would like to thank Rodrigo Bañuelos, Rich Bass and Burgess Davis for some very useful discussions of \( h \)-path lifetimes.

2. Moments of \( h \)-transform lifetimes. This section contains the proof of Theorem 1.6. We start with a short review of some useful facts about \( h \)-processes. The proofs may be found in Doob (1984) and Meyer, Smythe and Walsh (1972).

Let \( D \subset \mathbb{R}^d \) be a Greenian domain and \( h \) be a positive superharmonic function in \( D \). Suppose that \( M \) is a closed subset of \( D \) and let \( L = \sup \{ t < \zeta : X_t \in M \} \) be the last exit time from \( M \). Let

\[
Y_1(t) = X(t), \quad t \in (0, T(M)),
\]

\[
Y_2(t) = X(T(M) + t), \quad t \in (0, \zeta - T(M)),
\]

\[
Y_3(t) = X(t), \quad t \in (0, L),
\]

\[
Y_4(t) = X(L + t), \quad t \in (0, \zeta - L),
\]

\[
Y_5(t) = X(\zeta - t), \quad t \in (0, \zeta).
\]

Under \( P^x_h \), each process \( Y_k \) is an \( h_k \)-transform in a domain \( D_k \), where \( D_1 = D_4 = D \setminus M \) and \( D_2 = D_3 = D_5 = D \). Moreover, \( h_1 = h_2 = h \). The function \( h_3 \) is a potential supported by \( \partial M \). The function \( h_4 \) is harmonic and has the boundary values 0 on \( \partial M \) and the same boundary values as \( h \) on \( \partial D \setminus \partial M \). The function \( h_5 \) is the Green function \( G_D(x, \cdot) \) if \( x \in D \) or a harmonic function with a pole at \( x \) if \( x \in \partial D \).

If \( \mu(dy) \) is the \( P^x \)-distribution of \( X(T(M)) \) then the \( P^x_h \)-distribution of this random variable is \( \mu(dy)h(y)/h(x) \).

(2.1) Lemma. (Brownian scaling) Suppose \( h \) is a positive superharmonic function in a domain \( D \subset \mathbb{R}^d \) and \( x \in D^M \). For a fixed \( a \in (0, \infty) \) let

\[
D_a \overset{\text{def}}{=} \{ y \in \mathbb{R}^d : y/a \in D \},
\]

\[
h_a(y) \overset{\text{def}}{=} h(y/a) \quad \text{for} \quad y \in D_a,
\]

\[
x_a \overset{\text{def}}{=} ax,
\]

\[
X^a_t \overset{\text{def}}{=} aX_{t/a^2} \quad \text{for} \quad t \geq 0.
\]

If \( X \) has the distribution \( P^x_h \), then \( X^a \) has the distribution \( P^{x/a}_h \).

Proof. The lemma follows immediately from the scaling properties of Brownian motion and superharmonic functions. \( \square \)

A domain \( D \subset \mathbb{R}^d, d \geq 2 \), is called a Lipschitz domain if for every \( x \in \partial D \) there is a neighborhood \( U_x \) of \( x \), an orthonormal coordinate system \( CS_x \) and a Lipschitz function \( f_x : \mathbb{R}^{d-1} \to \mathbb{R} \) with constant \( \lambda \) (independent of \( x \)) such that \( \partial D \cap U_x \) is a
part of the graph of \( f_x \) in \( CS_x \). Note also that the index on any constant \( c_1, c_2, \ldots \) is local in nature. That is, new results or sections of proofs will start numbering their constants with \( c_1 \) as well.

**Lemma.** *(Boundary Harnack principle)*

(a) Suppose \( f : \mathbb{R}^{d-1} \to \mathbb{R} \) is a Lipschitz function with constant \( \lambda > 0 \), \( |f(x)| \leq 1 \) for all \( x \in \mathbb{R}^{d-1} \), and let

\[
D = \{ x \in \mathbb{R}^d : |\tilde{x}| < 1, f(\tilde{x}) < x^d < 2 \},
\]

\[
D_1 = \{ x \in D : |\tilde{x}| < 1/2, x^d < 3/2 \}.
\]

There exists \( c_1 > 0 \) which depends on \( \lambda \) but otherwise does not depend on \( f \) such that for all \( x, y \in D_1 \) and all positive harmonic functions \( g, h \) in \( D \) which vanish continuously on \( \{ z \in \partial D : z^d = f(\tilde{z}) \} \) we have

\[
\frac{g(x)}{g(y)} \geq c_1 \frac{h(x)}{h(y)}.
\]

(b) Suppose \( D \) is a Lipschitz domain, \( Q \) is a compact set and \( A \) is an open set such that \( Q \cap \overline{D} \subseteq A \). There exists \( c_2 > 0 \) such that for all \( x, y \in Q \cap D \) and all positive harmonic functions \( g, h \) in \( D \) which vanish continuously on \( \partial D \cap A \) we have

\[
\frac{g(x)}{g(y)} \geq c_2 \frac{h(x)}{h(y)}.
\]

□

For the first proofs of the boundary Harnack principle, see Ancona (1978), Dahlberg (1977) and Wu (1978). Stronger versions of the result may be found in Bass and Burdzy (1991) or Bañuelos, Bass and Burdzy (1991).

Part (a) of Lemma 2.2 holds (with the same \( c_1 \)) in domains which may be obtained from \( D \) by scaling.

When applying the boundary Harnack principle we will sometimes leave it to the reader to find the right choice of \( D \) and \( D_1 \) or \( D \), \( A \) and \( Q \).

**Lemma.** Suppose \( D \) is a domain, \( D_1 \) is a Lipschitz subdomain of \( D \), \( Q \) is a compact set, \( A \) is an open set such that \( Q \cap \overline{D} \subseteq A \), \( A \cap D \subseteq D_1 \), and \( M \) is a Borel subset of \( D \setminus A \). Assume that \( h \) is a positive superharmonic function in \( D \) which vanishes on \( \partial D \cap A \) and is harmonic in \( D_1 \). Then

\[
P^x_h(T(M) < \infty) \leq c_1 P^y_h(T(M) < \infty)
\]

for all \( x, y \in Q \cap D \). The constant \( c_1 \) depends only on \( D_1, Q \) and \( A \).

**Proof.** The function

\[
x \to E^x[T(M) < T(\partial D), h(X(T(M)))]
\]
is positive and harmonic in \( A \cap D \) and the same is true for \( x \to h(x) \). Let \( D_2 \) be a Lipschitz subdomain of \( A \cap D \) which contains \( Q \). By the boundary Harnack principle (2.2)(b), applied in \( D_2 \),

\[
P^x_h(T(M) < \infty) = \frac{1}{h(x)} E^x[T(M) < T(\partial D), h(X(T(M)))] \\
\leq c_2 \frac{1}{h(y)} E^y[T(M) < T(\partial D), h(X(T(M)))] \\
= c_2 P^y_h(T(M) < \infty). \quad \Box
\]

(2.4) **Lemma.** Suppose \( D \) is a domain and for each \( k = 1, 2 \),

(i) \( D_k \) is a subdomain of \( D \),

(ii) \( A_k \overset{\text{def}}{=} \partial D_k \cap D \),

(iii) \( V_k \) is an open set and \( Q_k \) is a compact set such that \( Q_k \cap \overline{D} \subset V_k \) and \( \overline{V_k} \cap D \subset D_k \),

(iv) \( (D_1 \cup V_1) \cap (D_2 \cup V_2) = \emptyset \),

(v) there is a \( c_k > 0 \) such that for all \( x, y \in Q_k \cap D \) and all positive harmonic functions \( f, g \) in \( D_k \) which vanish on \( V_k \cap \partial D \) we have

\[
\frac{f(x)}{f(y)} \geq c_k \frac{g(x)}{g(y)}.
\]

Assume that \( x_1, x_2 \in Q_1 \cap D \) and \( h_1, h_2 \) are positive superharmonic functions in \( D \) which vanish continuously on \( \partial D \setminus V_2 \) and are harmonic in \( D \setminus Q_2 \). Let \( T_1 \overset{\text{def}}{=} T(A_1) \) and let \( T_2 \) be the last exit time from \( A_2 \). The distributions of \( \{X_t, t \in [T_1, T_2]\} \) under \( P^x_{h_1} \) and \( P^x_{h_2} \) are mutually absolutely continuous and their Radon-Nikodym derivative is bounded below by \( c_1 c_2 \).

**Proof.** We will consider only the case when \( x_k \in Q_1 \cap D \) and \( h_k(\cdot) = G_D(\cdot, y_k) \) for some \( y_k \in Q_2 \cap D \). Other points \( x_k \) and functions \( h_k \) may be treated analogously.

Under \( P^x_{y_k} \), the process \( \{X_t, t \in [T_1, \zeta]\} \) is an \( G_D(\cdot, y_k) \)-process with the initial distribution

\[
\mu_k(\cdot) \overset{\text{def}}{=} P^x_{y_k}(X(T_1) \in \cdot) = P^{x_k}(T_1 < T(D^c), X(T_1) \in \cdot) G_D(\cdot, y_k)/G_D(x_k, y_k),
\]

supported on \( A_1 \). For a fixed \( z \in A_1 \), the process \( Y_t \overset{\text{def}}{=} X_{t-} \) under \( P^z_{y_k} \) has the distribution \( P^z_{y_k} \). If \( T_3 = \inf\{t : Y_t \in A_2\} \) then \( T_3 = \zeta - T_2 \). The process \( \{Y_t, t \in [T_3, \zeta]\} \) under \( P^z_{y_k} \) is a \( G_D(\cdot, z) \)-process with the initial distribution

\[
\nu_k(\cdot) \overset{\text{def}}{=} P^y_{y_k}(T(A_2) < T(D^c), X(T(A_2)) \in \cdot) G_D(\cdot, z)/G_D(y_k, z).
\]
For a fixed $v \in A_2$, the function $y \mapsto P^y(T(A_2) < T(D^c), X(T(A_2)) \in dv)$ is positive and harmonic in $D_2$ and vanishes on $V_2 \cap \partial D$ and the same is true for $z \mapsto G_D(v, z)$. By (2.5),

$$\frac{dv_k}{dv_{3-k}}(v) = \frac{P^{y_k}(T(A_2) < T(D^c), X(T(A_2)) \in dv)G_D(v, z)G_D(y_{3-k}, z)}{G_D(y_k, z)P^{y_{3-k}}(T(A_2) < T(D^c), X(T(A_2)) \in dv)G_D(v, z)} \geq c_2.$$  

After reversing time again, we see that the distributions of $X(T_2)$ under $P_{y_1}^z$ and $P_{y_2}^z$ have Radon-Nikodym derivative bounded below by $c_2$. The process \{X_t, t \in [T_1, T_2]\} under $P_{y_1}^z$ is a mixture of $h$-transforms converging to $w$ with the mixing measure $P_{y_1}^z(X(T_2) \in dw)$ and the same remark applies to $P_{y_2}^z$. Hence, the distributions of \{X_t, t \in [T_1, T_2]\} under $P_{y_1}^z$ and $P_{y_2}^z$ have a Radon-Nikodym derivative bounded below by $c_2$.

We can prove in a similar way that $d\mu_k(\cdot)/d\mu_{3-k}(\cdot) \geq c_1$. The distributions of \{X_t, t \in [T_1, T_2]\} under $P_{y_1}^{x_1}$ and $P_{y_2}^{x_2}$ have the Radon-Nikodym derivative bounded below by $c_1c_2$ because $P_{y_k}^{x_k}$ is a mixture of the measures $P_{y_k}^z$ with the mixing measure $\mu_k$. □

(2.6) **Lemma.** Suppose that $f : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz with constant $\lambda$ and assume that $|f(x)| \leq 1$ for all $x$. Let

$$D = \{x \in \mathbb{R}^d : |\bar{x}| < 1, f(\bar{x}) < x^d < 2\}.$$  

There exists $c < \infty$ (which may depend on $\lambda$ but does not otherwise depend on $f$) such that for every $x \in D$ and every positive harmonic function $h$ in $D$

$$(2.7) \quad E_h^x \zeta < c.$$  

**Proof.** The result is essentially due to Cranston (1985) but we refer the reader to the paper by Bass and Burdzy (1992). Our domain $D$ is a special case of a “twisted Hölder domain” and (2.7) follows from Theorem 1.1 (i) (a) (C) of Bass and Burdzy (1992). A direct inspection of its proof shows that $c$ depends only on the volume and diameter of $D$ (under the assumption that $f$ is Lipschitz with constant $\lambda$) and these quantities may be bounded independently of the particular form of $f$. □

(2.8) **Remark.** It is not necessary to assume in Lemma 2.6 that $f$ is Lipschitz. It is enough to suppose that $f$ is upper semicontinuous and $f(x)$ is bounded in the $L^p$-norm for a suitable $p = p(d)$. This version of the result uses Theorem 1.1 (i) (a) (A) of Bass and Burdzy (1992) which has a considerably more complicated proof than Theorem 1.1 (i) (a) (C). We feel it would not be fair to ask the reader to go through the former proof in order to check that the constants may be chosen independently of $f$.  

12
Lemma. Suppose that $D \subset \mathbb{R}^d$ is a domain, $x, y \in \overline{D}$, and for each $v = x, y$ there exist an orthonormal coordinate system $CS_v$, a point $z_v \in D$, a Lipschitz function $f_v$ with constant $\lambda$ and a constant $c_v > 0$ such that $|f_v| \leq c_v$,

$$D_v \overset{df}{=} \{ z \in D : |\tilde{z}| < c_v, -c_v < z^d < 2c_v \text{ in } CS_v \} = \{ z \in \mathbb{R}^d : |\tilde{z}| < c_v, f_v(\tilde{z}) < z^d < 2c_v \text{ in } CS_v \},$$

$$z_v = (0, 0, \ldots, 0, 3c_v/2) \text{ in } CS_v,$$

$$|\tilde{v}| \leq c_v/2 \text{ and } v^d \leq 3c_v/2 \text{ in } CS_v,$$

$$D_x \cap D_y = \emptyset.$$

If $E^{z_x}_y \zeta = c_1$ then

$$E^{x}_y \zeta \leq c_2 c_1 + c_3 (c_x^2 + c_y^2)$$

where $c_2$ and $c_3$ depend only on the dimension $d$ and the Lipschitz constant $\lambda$.

Proof. For $v = x, y$ let

$$D^1_v = \{ z \in D_v : |\tilde{z}| < 3c_v/4, z^d < 7c_v/4 \text{ in } CS_v \},$$

$$A_v = \partial D^1_v \cap D,$$

$$Q_v = \{ z \in \overline{D}_v : |\tilde{z}| \leq c_v/2, z^d \leq 3c_v/2 \text{ in } CS_v \},$$

$$V_v = \{ z \in \mathbb{R}^d : \text{dist}(z, Q_v) < c_v/8 \}.$$

By the boundary Harnack principle (2.2)(a), applied in $D_v$, assumption (2.5) of Lemma 2.4 holds. Let $T_1$ be the first hitting time of $A_x$ and $T_2$ be the last exit time from $A_y$. By Lemma 2.4,

$$(2.10) \quad E^{x}_y (T_2 - T_1) \leq c_4 E^{z_x}_y (T_2 - T_1) \leq c_4 E^{z_x}_y \zeta.$$

Lemma 2.6 and Brownian scaling (2.1) imply that

$$(2.11) \quad E^{x}_y T_1 \leq c_5 c_x^2.$$

The same lemma and time-reversal show that

$$(2.12) \quad E^{x}_y (\zeta - T_2) \leq c_5 c_y^2.$$

The lemma follows from (2.10)-(2.12). □

We now return to the specific domains, hypotheses, and notation of Theorem 1.6.
(2.13) Lemma. Assume that $a < s_{j-1} < s_j < b$. There exists $c_1 > 0$ such that for every positive harmonic function $h$ in $D_j$ which vanishes on $\partial D_j \setminus \Lambda_{s_j}$ and every $x \in \Lambda_{s_{j-1}}$,

$$E^x_\zeta \geq c_1 f^2(s_{j-1}).$$

Moreover, there is a non-negative, non-constant and bounded random variable $Y$ such that for every $j$ and $x \in \Lambda_{s_{j-1}}$, the distribution of $\zeta$ under $P^x_\zeta$ is stochastically larger than that of $f^2(s_{j-1})Y$.

Proof. Let $B(y, r)$ denote the ball with center $y$ and radius $r$. Let $c_2$ be the expected lifetime of conditioned Brownian motion in $B(0, 1)$ starting from 0 and converging to $x \in \partial B(0, 1)$. The constant $c_2$ is strictly positive and does not depend on $x$ by symmetry. For any harmonic function $g$ in $B(0, 1)$, the $g$-process starting from 0 is a mixture of processes conditioned to go to some point of $\partial B(0, 1)$ so its expected lifetime is also equal to $c_2$. By scaling, the expected lifetime of any Brownian motion conditioned by a harmonic function in $B(y, r)$ and starting from $y$ is equal to $c_2 r^2$.

Let

$$B_0 = B([0, \ldots, 0, s_{j-1} + f(s_{j-1})/4, f(s_{j-1})/8),$$

$$T_1 = \inf \{ t > T(B_0) : |X_t - X(T(B_0))| = f(s_{j-1})/16 \}.$$ 

Note that $B_0 \subset D_j$. By the strong Markov property applied at $T(B_0)$,

$$E^x_\zeta \geq E^x_\zeta[(T_1 - T(B_0))1_{\{T(B_0) < \infty\}}] = c_2 (f(s_{j-1})/16)^2 P^x_\zeta(T(B_0) < \infty).$$

(2.14) 

Let $x_0 = (0, \ldots, 0, s_{j-1})$. By Lemma 2.3, for all $x \in \Lambda_{s_{j-1}}$,

$$P^x_\zeta(T(B_0) < \infty) \geq c_3 P^{x_0}_\zeta(T(B_0) < \infty).$$

(2.15) 

It is not hard to see that the constant $c_3$ may be chosen independently of the particular form of $f$. The probability $P^{x_0}_\zeta(T(B_0) < \infty)$ is not less than

$$P^{x_0}(T(B_0) < T(\partial D_j)) \inf_{y \in B_0} h(y)/h(x_0).$$

It is elementary to see that $P^{x_0}(T(B_0) < T(\partial D_j))$ is bounded below and the usual Harnack principle shows that the same is true for $\inf_{y \in B_0} h(y)/h(x_0)$. Hence, $P^{x_0}(T(B_0) < \infty)$ is bounded below by $c_4 > 0$ which together with (2.14) and (2.15) implies

$$E^x_\zeta \geq c_2 (f(s_{j-1})/16)^2 c_3 c_4.$$ 

It is clear from our proof that $Y$ can be chosen as follows. Let $\tilde{\zeta}$ be the hitting time of $\partial B(0, 1/16)$ by a Brownian motion starting from 0 and let $W$ be an independent random variable with $P(W = 1) = 1 - P(W = 0) = c_3 c_4$. Then let $Y = WY'$, where $Y' = c_2 \min(\tilde{\zeta}, 1)$. □
(2.16) **Lemma.** Suppose that $s_j < s_n$. Let $T_j^1 = T(\Lambda_{s_j})$ and

$$S_j^k = \inf\{t > T_j^k : X_t \in \Lambda_{s_{j-1}} \cup \Lambda_{s_{j+1}}\}, \quad k \geq 1,$$

$$T_j^k = \inf\{t > S_j^{k-1} : X_t \in \Lambda_{s_j}\}, \quad k > 1.$$ 

There exist $c_1 < \infty$ and $p < 1$ such that for all $k$ and for every positive harmonic function $h$ in $D_n$ which vanishes on $\partial D_n \setminus \Lambda_{s_n}$ and every $x \in D_n$

$$P_h^x(T_j^k < \infty) < c_1p^k.$$ 

Moreover, if $i \geq 0$, $j + i < n$ and $x \in \Lambda_{s_{j+i}}$, then

$$P_h^x(T_j^k < \infty) < c_1p^{k+i}.$$ 

**Proof.** Suppose $s_k < s_{k+1} \leq s_n$. We have

$$h(x) = \int_{\Lambda_{s_{k+1}}} h(y)P_x^z(X(T(\Lambda_{s_{k+1}})) \in dy)$$

for $x \in \Lambda_{s_k}$. The boundary Harnack principle implies that

$$\frac{P_{x_1}^z(X(T(\Lambda_{s_{k+1}})) \in dy)}{P_{x_2}^z(X(T(\Lambda_{s_{k+1}})) \in dy)} \cdot \frac{P_{x_2}^z(T(\Lambda_{s_{k+1}}) < \infty)}{P_{x_1}^z(T(\Lambda_{s_{k+1}}) < \infty)} < c_3 < \infty$$

for $x_1, x_2 \in \Lambda_{s_k}$. Let $z_k = (0, \ldots, 0, s_k)$. It is easy to see that there is $c_4 > 0$ such that for all $x \in \Lambda_{s_k}$ with $|\tilde{x}| > (1 - c_4)f(s_k)$, we have

$$P_x^z(T(\Lambda_{s_{k+1}}) < \infty) < (c_4^{-1}/2)P_{z_k}^z(T(\Lambda_{s_{k+1}}) < \infty).$$

This, (2.17) and (2.18) imply that $h(x) \leq h(z_k)/2$ for $x \in \Lambda_{s_k}$ with $|\tilde{x}| > (1 - c_4)f(s_k)$. It follows that the maximum of $h$ on $\Lambda_{s_k}$ is attained at a point in the set

$$A_k \overset{\text{df}}{=} \{x \in \Lambda_{s_k} : |\tilde{x}| \leq (1 - c_4)f(s_k)\}.$$ 

Let $a_k$ be the maximum of $h$ over $\Lambda_{s_k}$. Since

$$P_x^z(T(\Lambda_{s_{k+1}}) \leq T(\partial D_n)) < c_5 < 1$$

for $x \in \Lambda_{s_k}$, we have $a_k < c_5a_{k+1}$ assuming $a < s_k < s_{k+1} < b$. It follows that $a_k < c_5^j a_{k+j}$. By the Harnack principle, $h(x) > c_6a_k$ for some $c_6 > 0$ and all $x \in A_k$. Let $m$ be so large that $c_5c_7^m > 2$. Then $a_k < h(x)/2$ for all $x \in A_{k+m}$ provided $a < s_k < s_{k+m} < b$. We obtain

$$P_h^x(T(\Lambda_{s_j}) < \infty) = \int_{\Lambda_{s_j}} \frac{h(y)}{h(x)}P_x^z(X(T(\Lambda_{s_j})) \in dy) \leq 1/2$$
for \( x \in A_{j+m} \). Here and later in the proof we assume that \( a < s_j < s_{j+m} < b \). This assumption could be easily disposed of. We have
\[
P^{x_k}(T(A_{k+1}) < T(\partial D_n \cup \Lambda_{s_{k-1}})) > c_7 > 0
\]
and an application of the Harnack principle shows that
\[
P^{x_k}(T(A_{k+1}) < T(\Lambda_{s_{k-1}})) > c_8 > 0.
\]
By Lemma 2.3,
\[
P^x_h(T(A_{k+1}) < T(\Lambda_{s_{k-1}})) > c_9 > 0
\]
for all \( x \in \Lambda_{s_k} \). By the strong Markov property applied at the hitting times of \( A_i \),
\[
P^{x_k}(T(A_{j+m}) < T(\Lambda_{s_{j}})) > c_9^{m-1}
\]
for all \( x \in \Lambda_{s_{j+1}} \). Let
\[
U_1 = \inf\{t > T(\Lambda_{s_{j+1}}) : X_t \in A_{j+m}\},
U_2 = \inf\{t > T(\Lambda_{s_{j+1}}) : X_t \in \Lambda_{s_{j}}\},
U_3 = \inf\{t > U_1 : X_t \in \Lambda_{s_{j}}\}.
\]
Then (2.19)-(2.21) imply that for \( x \in \Lambda_{s_j} \)
\[
P^x_h(T^2 = \infty) \geq P^x_h(T(\Lambda_{s_{j+1}}) < T(\Lambda_{s_{j-1}}), U_1 < U_2, U_3 = \infty) > c_9^m/2 > 0
\]
for \( x \in \Lambda_{s_j} \). Both conclusions of the lemma now follow by the repeated application of the strong Markov property at the stopping times \( T^k \).

\((2.22)\) Lemma. \( \) For all \( x_1 \in D_f \) such that \( s_{k+1} \leq x_1^d \leq s_{k+2} \) and \( x_2 \in \Lambda_{s_k} \) we have \( E^{x_1}_{x_2} \zeta < c_1 f^2(s_k) \) where \( E^{x_1}_{x_2} \) refers to the conditioned Brownian motion in \( D_f \).

\( \) Proof. \( \) We will suppose that \( x_1 \in \Lambda_{s_{k+1}} \). The modifications needed for the general case are obvious.

By Brownian scaling (2.1), we may assume that \( f(s_k) = 1 \) and prove that \( E^{x_1}_{x_2} \zeta < c_1 \). Note that then \( |x_1^d - x_2^d| = 1/2 \).

We have
\[
E^{x_1}_{x_2} \zeta = c_2 \int_{D_f} \frac{G_{D_f}(x_1, z) G_{D_f}(z, x_2)}{G_{D_f}(x_1, x_2)} dz.
\]

In view of Lemma 2.9 it will suffice to prove the lemma for \( x_1 \in \Lambda_{s_{k+1}}, |\bar{x}_1| < c_3 f(s_{k+1}) \), and \( x_2 \in \Lambda_{s_k}, |\bar{x}_2| < c_3 \) for some \( c_3 < 1 \). Under this additional assumption, \( x_1 \) and \( x_2 \) may be connected in \( D_f \) by a Harnack chain of balls of bounded length and this implies that \( G_{D_f}(x_1, x_2) > c_4 > 0 \). Hence,
\[
E^{x_1}_{x_2} \zeta < c_5 \int_{D_f} G_{D_f}(x_1, z) G_{D_f}(z, x_2) dz.
\]
Let

\[ A_j = \{ z \in D_f : |z - x_j| < 5, |z - x_{3-j}| > |x_1 - x_2|/2 \}, \quad j = 1, 2, \]

\[ A_3 = \{ z \in D_f : |z - x_1| \geq 5, z^d < s_k \}, \]

\[ A_4 = \{ z \in D_f : |z - x_1| \geq 5, z^d > s_{k+1} \}. \]

Assume for now that \( d \geq 3 \), and recall that \( G(x, y) \) \( \overset{df}{=} G_{\mathbb{R}^d}(x, y) = c_6|x - y|^{2-d} \).

For \( j = 1, 2 \) we obtain

\[
\int_{A_j} G_{D_f}(x_1, z)G_{D_f}(z, x_2)dz \leq \int_{A_j} G(x_1, z)G(z, x_2)dz \\
\leq c_7 \int_{A_j} (|x_1 - x_2|/2)^{2-d}|z - x_j|^{2-d}dz \\
\leq c_7(|x_1 - x_2|/2)^{2-d} \int_0^5 r^{2-d}r^{d-1}dr < c_8 < \infty.
\]

Let \( x_0 = (0, \ldots, 0, s_k) \),

\[ \tilde{D} = \{ x \in \mathbb{R}^d : x^d < s_k \}, \]

\[ D_* = D_f \cup \{ x \in \mathbb{R}^d : x^d \in (-\infty, s_k) \cup (s_{k+1}, \infty) \}, \]

\[ M = \{ x \in \tilde{D} : |x - x_1| = 4 \}. \]

The Poisson kernel \( K(x) \) in \( \tilde{D} \) with the pole at \( x_0 \) has the form \( c_9|x^d - s_k|/|x - x_0|^d \) (Doob (1984) 1.VIII.9). By the boundary Harnack principle,

\[ G_{D_*}(x_1, x) \leq c_{10}K(x) \]

for \( x \in M \) and, therefore, for all \( x \in \tilde{D} \) such that \( |x - x_1| \geq 4 \), in particular, for \( x \in A_3 \). Hence, for \( x \in A_3 \),

\[ G_{D_*}(x_1, x) \leq c_{11}|x^d - s_k|/|x - x_0|^d \leq c_{11}|x - x_0|^{1-d} \]

and the same estimate holds for \( G_{D_*}(x_2, x) \). It follows that

\[
\int_{A_3} G_{D_f}(x_1, z)G_{D_f}(z, x_2)dz \leq \int_{A_3} G_{D_*}(x_1, z)G_{D_*}(z, x_2)dz \\
\leq \int_{A_3} (c_{11}|z - x_0|^{1-d})^2dz \\
\leq c_{12} \int_2^\infty r^{2(1-d)}r^{d-1}dr < c_{13} < \infty
\]

and a similar estimate holds for \( A_4 \). Since \( D_f \subset A_1 \cup A_2 \cup A_3 \cup A_4 \), the lemma follows from (2.23)-(2.25).

If \( d = 2 \), an argument similar to the above could be given. In this case, \( \tilde{D} \) should be replaced by a suitable wedge with angle \( \alpha < \pi \). The Green function in such a wedge decays like \( r^{-\pi/\alpha} \), and this is sufficient to make the bounding integrals finite. \( \square \)
(2.26) Lemma. For $x \in D_f$ and $y \in \Lambda_{s_k}$, let

$$g^k_x(y)dy \overset{df}{=} P^x_h(X(T(\Lambda_{s_k})) \in dy).$$

Then there exist $c_1 < \infty$ and $c_2 < 1$ such that

$$\frac{g^n_{x_1}(y_1)}{g^n_{x_1}(y_2)} \geq a_i \frac{g^n_{x_2}(y_1)}{g^n_{x_2}(y_2)}$$

and

$$a_i \geq 1 - c_1 c_2^i$$

for all $i > 0$, all $n$, where $x_1, x_2 \in D_{n-i}$ and $y_1, y_2 \in \Lambda_{s_n}$.

Proof. A standard application of the boundary Harnack principle in the spirit of Lemma 2.3 shows that (2.27) holds for $i = 1$ with some $a_1 > 0$.

Assume that (2.27) holds for all $n$ and for some $i$; we will show that it holds for $i + 1$ as well. Let $j = n - i$. By the strong Markov property applied at $T(\Lambda_{s_{n-1}})$,

$$g^n_x(y) = \int_{\Lambda_{s_{n-1}}} g^{n-1}_{x_1}(v)g^n_v(y)dv$$

for $y \in D_{j-1}$. Now apply Lemma 6.1 of Burdzy, Toby and Williams (1989). Set in that lemma $V = W = \Lambda_{s_{n-1}}$ and $U = \emptyset$, set $f_1$ and $f_2$ equal to our $g^{n-1}_{x_1}$ and $g^{n-1}_{x_2}$, set $g_z(v)$ equal to our $g^n_v(z)$, and take $c = a_i$, $d = a_1$, and $b = 1$. The aforementioned lemma implies that

$$\frac{g^n_{x_1}(y_1)}{g^n_{x_1}(y_2)} \geq a_{i+1} \frac{g^n_{x_2}(y_1)}{g^n_{x_2}(y_2)}$$

for all $y_1, y_2 \in D_{j-1}$, where

$$a_{i+1} = a_i + a_1^2(1 - a_i).$$

Hence

$$1 - a_{i+1} = 1 - a_i - a_1^2(1 - a_i) = (1 - a_i)(1 - a_1^2)$$

and, by induction,

$$1 - a_{i+1} \leq c_1 c_2^i,$$

with $c_2 \overset{df}{=} 1 - a_1^2 < 1$. □
(2.28) **Corollary.** With the notation of Lemma 2.26,

\[ a_{n-j}^{-1} \geq \frac{g^n_{x_1}(y)}{g^n_{x_2}(y)} \geq a_{n-j} \]

for every \( j < n \), \( x_1, x_2 \in D_j \) and \( y \in \Lambda_{s_n} \).

**Proof.** Let \( M \) and \( m \) be the supremum and infimum of \( g^n_{x_1}(y)/g^n_{x_2}(y) \) over \( y \in \Lambda_{s_n} \). By Lemma 2.26, \( m \geq a_{n-j} M \), and

\[ M g^n_{x_2}(y) \geq g^n_{x_1}(y) \geq m g^n_{x_2}(y). \]

Integrating with respect to \( y \) shows that \( M \geq 1 \geq m \), from which the desired conclusion follows. □

**Proof of Theorem 1.6.** (i) We will first prove the lower bound in (1.7).

Suppose that \( s_{j_0} \leq x^d < s_{j_0+1} < s_{j_0+2} < b \). The other cases are left to the reader. Let \( T_j = T(\Lambda_{s_j}) \). For each \( j > j_0 + 2 \) the process \( \{X_t, t \in [T_{j-1}, T_j]\} \) under \( P^x_h \) is a conditioned Brownian motion in \( D_j \) starting from a (random) point in \( \Lambda_{s_{j-1}} \) and converging to \( \Lambda_{s_j} \) at its lifetime. By Lemma 2.13, for \( j \in [j_0 + 2, k_f - 1] \),

\[ E^x_h(T_j - T_{j-1}) \geq c_1 f^2(s_{j-1}) \]

and, therefore,

\[ E^x_h \zeta \geq \sum_{j=j_0+2}^{k_f-1} E^x_h(T_j - T_{j-1}) \geq \sum_{j=j_0+2}^{k_f-1} c_1 f^2(s_{j-1}). \]

Since

\[ c_2 f^2(s_{j-1}) < \int_{s_{j-1}}^{s_j} f(v) dv < c_3 f^2(s_{j-1}), \]

the sum on the right hand side of (2.29) is bounded below by \( c_4 \int_{s_{j_0+1}}^{s_{j_0+2}} f(v) dv \). Note that

\[ \int_{x^d}^{s_{j_0+1}} f(v) dv < c_5 \int_{s_{j_0+1}}^{s_{j_0+2}} f(v) dv \]

and

\[ \int_{k_f-2}^{b} f(v) dv < c_5 \int_{k_f-3}^{k_f-2} f(v) dv. \]

Hence

\[ \int_{x^d}^{b} f(v) dv < c_6 \int_{s_{j_0+1}}^{s_{j_0+2}} f(v) dv \]
and, therefore,

\[ E_h^x \zeta \geq c_7 \int_{x_d}^b f(v)dv. \]

(ii) Next we will prove (ii) of Theorem 1.6.

First note that \( k_f = \infty \). Recall the definitions of \( j_0 \) and the \( T_j \)'s from part (i) of the proof. By Lemma 2.13 and the strong Markov property applied at \( T_j \)'s, there exist non-negative (not necessarily independent) random variables \( Z_j \) and i.i.d. non-negative random variables \( Y_j \) such that

\[ \sum_{j=j_0+2}^{\infty} (T_j - T_{j-1}) \]

has the same distribution as

\[ \sum_{j=j_0+2}^{\infty} (Z_j + f^2(s_j-1)Y_j). \]

For later use, note that, as in the proof of Lemma 2.13, we can write \( Y_j = W_j Y'_j \), where the \( Y'_j \) are independent of the \( Z \)'s and \( W \)'s, with some common mean \( \mu \) and variance \( \sigma^2 \). Each \( W_j \) takes values 0 or 1, and \( W_j = 1 \) with some common probability \( p \), even if conditioned on the preceding \( W \)'s and on \( \{X_t, t \in [0, T_{j-1} - 1]\} \). Thus the \( W_j \) are i.i.d., though they may not be independent of the \( Z_j \).

It is elementary to check that \( \sum_{j=j_0+2}^{\infty} f^2(s_j-1) = \infty \) because \( \int_{x_d}^b f(v)dv = \infty \). Hence,

\[ \sum_{j=j_0+2}^{\infty} E(f^2(s_j-1)Y_j) = \infty. \]

Recalling that each \( Y_j \) is non-negative, non-constant and bounded, the three-series theorem now easily implies that a.s.

\[ \sum_{j=j_0+2}^{\infty} f^2(s_j-1)Y_j = \infty. \]

It follows that the sums in (2.31), and therefore in (2.30), must be infinite a.s.

(iii) We are going to prove the lower bound in (1.8).

Let \( j_0 \), the \( Y'_j \)'s, etc. be as in part (ii) of the proof. By adjusting the first and last \( Z \), if necessary, we can guarantee that

\[ \zeta = \sum_{j=j_0+2}^{k_f-1} (Z_j + f^2(s_j-1)Y_j) \]

\[ = \sum_{j=j_0+2}^{k_f-1} (Z_j + f^2(s_j-1)\mu W_j) + \sum_{j=j_0+2}^{k_f-1} (f^2(s_j-1)W_j(Y'_j - \mu)). \]
Therefore by independence,
\[
\text{Var}_h^x \zeta = \text{Var}_h^x \left( \sum_{j=j_0+2}^{k_f-1} (Z_j + f^2(s_{j-1}) \mu W_j) \right) + \sum_{j=j_0+2}^{k_f-1} E_h^x \left( (f^2(s_{j-1}) W_j (Y'_j - \mu))^2 \right)
\]
\[
\geq \sum_{j=j_0+2}^{k_f-1} E_h^x \left( (f^2(s_{j-1}) W_j (Y'_j - \mu))^2 \right)
\]
\[
\geq \sum_{j=j_0+2}^{k_f-1} f^4(s_{j-1}) \sigma^4 \geq c_3 \int_{x^d}^b f^3(v) dv.
\]

(iv) We will now prove part (v) of Theorem 1.6.
We will again invoke the \(Y_j\)'s and \(Z_j\)'s of part (ii) of the proof. Suppose that 
\[
\int_{x^d}^b f^3(v) dv = \infty.
\]
Then necessarily \(b = \infty\). Let us assume that
\[
\limsup_{v \to \infty} f(v) < \infty.
\]
In order to simplify the notation, suppose that \(x^d = s_{j_0}\).
First, let \(w_1, w_2, \ldots\) be any sequence of 0's and 1's, such that
\[
\sum_{j > j_0} f^4(s_{j-1}) w_j = \infty.
\]
Consider
\[
\tilde{Y}_k = \sum_{j=j_0+1}^k f^2(s_{j-1}) w_j (Y'_j - \mu) \quad \text{and} \quad \hat{Y}_k = \tilde{Y}_k / (\text{Var} \tilde{Y}_k)^{1/2}.
\]
Since the \(Y'_j\)'s are uniformly bounded, the Lindeberg-Feller condition can be easily verified using (2.33) and it follows that the distributions of \(\hat{Y}_k\) converge to the standard normal distribution as \(k \to \infty\). In fact it is simple to show, using (2.33) and the Berry-Eseen theorem, that for every \(c_1 < \infty\) and \(c_2 > 0\) there exists a \(c_3 < \infty\) such that
\[
P(\hat{Y}_k \in (u, u + c_1)) < c_2/2 \quad \text{for every } u \in \mathbb{R}, \text{ if } \text{Var}\hat{Y}_k > c_3.
\]
Since \(\sum_{j > j_0} f^4(s_{j-1}) W_j = \infty\) almost surely, we can choose a \(k_0 < \infty\) such that
\[
P_h^x \left( \sum_{j=j_0+1}^k f^4(s_{j-1}) W_j > c_3 \right) > 1 - c_2/2
\]
for every $k \geq k_0$. Also, as in (2.32) we have that

$$T(\Lambda_{s_k}) = \sum_{j=j_0+2}^{k} (Z_j + f^2(s_{j-1})\mu W_j) + \sum_{j=j_0+2}^{k} (f^2(s_{j-1})W_j(Y'_j - \mu)).$$

Therefore, conditioning on the values of $W_j, j > j_0$ yields that

$$P_h^x(T(\Lambda_{s_k}) \in (u, u + c_1)) < c_2$$

for every $u \in \mathbb{R}$.

The case when (2.33) fails is not hard and is left to the reader.

(v) Next we prove the upper bound in (1.7).

Suppose that $s_{n+1} \leq x^d \leq s_{n+2}$. Let $L$ be the last exit time from $\Lambda_{s_n}$. Under $P_h^x$, the process $\{X_t, t \in [0, L]\}$ is a conditioned Brownian motion in $D_f$ starting from $x$ and converging to a (random) point of $\Lambda_{s_n}$. Lemma 2.22 implies that $E_h^x L < c_1 f^2(s_n)$ and this in turn implies that

(2.35) \[ E_h^x L < c_2 \int_{x^d}^{s_{n+3}} f(v) dv. \]

For every $\varepsilon > 0$, the process $\{X_{t+L+\varepsilon}, t \geq 0\}$ under $P_h^x$ is an $h$-process in the domain $D_g$ where $g(s) = f(s)1_{(s_n, \infty)}(s)$. This and (2.35) show that (1.7) will follow once we prove that

$$E_h^x \zeta < c_3 \int_a^b f(v) dv.$$ 

Let $M_k = \{y \in D_f : s_{k-1} < y^d < s_{k+1}\}$ and consider an $h_0$-process in $M_k$ for some positive harmonic function $h_0$ in $M_k$. A variation of Lemma 2.6 shows that

(2.36) \[ E_{h_0}^y \zeta < c_4 \]

for all $y \in M_k$, provided $f(s_k) = 1$. By scaling,

(2.37) \[ E_{h_0}^y \zeta < c_4 f^2(s_k) \]

for any value of $f(s_k)$.

Recall the stopping times $S^k_j$ and $T^k_j$ from Lemma 2.16 and let $F^k_j \overset{df}{=} \{T^k_j < \infty\}$. Let $T_0$ be the hitting time of $\bigcup_k \Lambda_{s_k}$. We have

(2.38) \[ \zeta = T_0 + \sum_{j,k} (S^k_j - T^k_j)1_{F^k_j}. \]

Given $T^k_j < \infty$, the process $\{X_t, t \in [T^k_j, S^k_j]\}$ is a conditioned Brownian motion in $M_k$ and, therefore,

$$E_h^x [(S^k_j - T^k_j) \mid F^k_j] < c_4 f^2(s_j).$$
By Lemma 2.16,

\begin{equation}
\sum_k E^x_h(S^k_j - T^k_j)1_{F^k_j} < c_5 f^2(s_j).
\end{equation}

Recall that \( s_{n+1} \leq x^d \leq s_{n+2} \). Hence \( E^x_h T_0 < c_4 f^2(s_n) \). This and (2.38)-(2.39) yield

\[ E^x_h \zeta \leq c_6 \sum_j f^2(s_j). \]

It is easy to check that the last quantity is bounded by \( c_7 \int_a^b f(v)dv \).

(vi) We will now prove the upper bound for the variance in (1.8). Recall \( M_k \) and the use of an \( h_0 \)-process in \( M_k \) from part (v) of the proof. The Chebyshev inequality and (2.36) show that \( P^x_{h_0}(\zeta > c_1) < c_2 \) for some \( c_1 < c_2 \) and all \( x \in M_k \) provided \( f(s_k) = 1 \). By the Markov property applied repeatedly at the multiples of \( c_1 \), \( P^x_{h_0}(\zeta > jc_1) < c_2^j \). Hence \( E^x_{h_0} \zeta^2 < c_3 \) in the case \( f(s_k) = 1 \) and, by scaling,

\begin{equation}
E^x_{h_0} \zeta^2 < c_3 f^4(s_k)
\end{equation}

for any value of \( f(s_k) \), all \( x \in M_k \) and all harmonic functions \( h_0 \) in \( M_k \).

Let \( S^k_j \) and \( T^k_j \) be as in Lemma 2.16. Let \( F^k_j \) be \( \{ T^k_j < \infty \} \). Given \( F^k_j \), the process \( \{ X_t, t \in [T^k_j, S^k_j] \} \) is a conditioned Brownian motion in \( M_k \) and this implies in view of (2.37) and (2.40), that

\begin{equation}
E^x_h[(S^k_j - T^k_j) \mid F^k_j] < c_4 f^2(s_j) \quad \text{and}
\end{equation}

\begin{equation}
E^x_h[(S^k_j - T^k_j)^2 \mid F^k_j] < c_3 f^4(s_j).
\end{equation}

Let \( \Theta^k_j \equiv (S^k_j - T^k_j)1_{F^k_j} \). Define \( q \) by the condition that \( s_{q-1} < x^d \leq s_q \), and recall from Lemma 2.16 that

\begin{equation}
P^x_h(F^k_j) \leq \begin{cases} 
    c_5 c_6^{k+q-j}, & j < q \\
    c_5 c_6^k, & j \geq q,
\end{cases}
\end{equation}

where \( c_6 < 1 \). This and (2.41) imply that

\begin{equation}
E^x_h[\Theta^k_j] \leq \begin{cases} 
    c_4 c_5 c_6^{k+q-j} f^2(s_j), & j < q \\
    c_4 c_5 c_6^k f^2(s_j), & j \geq q,
\end{cases}
\end{equation}

and

\begin{equation}
E^x_h[(\Theta^k_j)^2] \leq \begin{cases} 
    c_3 c_5 c_6^{k+q-j} f^4(s_j), & j < q \\
    c_3 c_5 c_6^k f^4(s_j), & j \geq q,
\end{cases}
\end{equation}
Now assume that \( j < n \), and let

\[
A = \{ T_j^k < T_n^1 \}, \quad B = \{ T_n^1 < T_j^k \}, \quad B_i = \{ T_j^{i-1} < T_n^1 < T_j^i \}
\]

where \( T_j^0 \) is taken to be 0. Then

\[
\text{Cov}^\tau_h(\Theta_j^k, \Theta_n^m) = E^\tau_h((\Theta_j^k - E^\tau_h \Theta_j^k)(\Theta_n^m - E^\tau_h \Theta_n^m))
\]

\[
= E^\tau_h((\Theta_j^k - E^\tau_h \Theta_j^k)(\Theta_n^m - E^\tau_h \Theta_n^m)1_A) +
\]

\[
+ E^\tau_h((\Theta_j^k - E^\tau_h \Theta_j^k)(\Theta_n^m - E^\tau_h \Theta_n^m)1_B)
\]

\[*\]

Consider term \( I \) of (2.45). If \( q > n \) then \( I = 0 \) automatically. So suppose that \( q \leq j \). By Corollary 2.28 and the strong Markov property at \( T_n^1 \),

\[
| E_h^\tau \Theta_n^m - E_h^\tau \Theta_n^m | \leq c_\tau c_8^{n-j} E_h^\tau \Theta_n^m
\]

for any \( y \in D_j \), where \( c_8 < 1 \). In particular,

\[
| E_h^\tau(\Theta_n^m | F_{S_j^k}) - E_h^\tau \Theta_n^m | \leq c_\tau c_8^{n-j} E_h^\tau(\Theta_n^m)
\]

on \( A \). Thus, by (2.43),

\[
I = E_h^\tau \left[ (\Theta_j^k - E_h^\tau \Theta_j^k)1_A E_h^\tau(\Theta_n^m - E_h^\tau \Theta_n^m) | F_{S_j^k} \right]
\]

\[
\leq E_h^\tau \left[ |\Theta_j^k - E_h^\tau \Theta_j^k| \cdot 1_A \cdot | E_h^\tau(\Theta_n^m | F_{S_j^k}) - E_h^\tau \Theta_n^m | \right]
\]

\[
\leq 2c_\tau c_8^{n-j} E_h^\tau(\Theta_n^m) E_h^\tau(\Theta_j^k) \leq c_9 c_8^{n-j} c_6^{k+m} f^2(s_j) f^2(s_n).
\]

If, on the other hand, we have \( j < q \leq n \), then by a similar argument,

\[
| E_h^\tau(\Theta_n^m | F_{S_j^k}) - E_h^\tau \Theta_n^m | \leq c_\tau c_8^{n-q} E_h^\tau(\Theta_n^m)
\]

on \( A \), and

\[
I \leq 2c_\tau c_8^{n-q} E_h^\tau(\Theta_n^m) E_h^\tau(\Theta_j^k) \leq c_9 c_8^{n-q} c_6^{k+m+q-j} f^2(s_j) f^2(s_n).
\]

Taking \( c_{11} = \max(c_8, c_6) \), it follows that

\[
(2.46) \quad I \leq c_9 c_{11}^{n-j} c_6^{k+m} f^2(s_j) f^2(s_n),
\]

regardless of the value of \( q \).
Consider now the term $II$ of (2.45). By (2.44), and by Lemma 2.16 again,
\[
E_h^x((\Theta_j^k)^2 1_B) = \sum_{i=1}^{k} E_h^x((\Theta_j^k)^2 1_{B_i}) \\
= \sum_{i=1}^{k} E_h^x(E_h^x((\Theta_j^k)^2 1_{B_i} | F_{S_i}^1)) \\
\leq \sum_{i=1}^{k} f^4(s_j)c_0c_3c_6^{n-j+k-i+1}P_h^x(B_i) \\
\leq \sum_{i=1}^{k} f^4(s_j)c_0c_3c_6^{n-j+k-i+1}P_h^x(F_j^{i-1}) \\
\leq k c_0c_3c_6^{n-j+k} f^4(s_j) \leq c_{12}c_{13}^{n-j+k} f^4(s_j),
\]
where $c_{13} < 1$. As a result,
\[
II \leq (E_h^x((\Theta_j^k)^2 1_B))^{1/2}(E_h^x((\Theta_n^m)^2))^{1/2} \\
\leq f^2(s_j)f^2(s_n)(c_{12}c_{13}^{n-j+k} c_0c_3c_6^m)^{1/2} \\
\leq c_{14}c_{15}^{k+m+n-j} f^2(s_j)f^2(s_n),
\]
where $c_{14} < 1$. Combining this with (2.45) and (2.46), it follows that
\[
\text{Cov}_h^x(\Theta_j^k, \Theta_n^m) \leq c_{16}c_{17}^{k+m+n-j} f^2(s_j)f^2(s_n), 
\]
for $j < n$, where $c_{17} < 1$. By symmetry, the same is true for $j > n$, and the inequality is even simpler to prove if $j = n$ ((2.46) is no longer needed). Thus, (2.47) holds for every $j, k, m, n$.

If $\int_a^b f^3(v)dv = \infty$ then the upper bound in (1.8) is trivial. Assume therefore that $\int_a^b f^3(v)dv < \infty$. Then for each $\varepsilon > 0$ there are only finitely many $j$ such that $f(s_j) > \varepsilon$. Hence we may choose an ordering $\{j_i\}_{i \geq 1}$ of the set $\{k : a < s_k < b\}$ which satisfies $f(s_{j_i+1}) \leq f(s_{j_i})$ for all $i$. By (2.47)
\[
\text{Var}_h^x \zeta = \text{Var}_h^x \left( \sum_{j,k} \Theta_j^k \right) = \sum_{j,k,n,m} \text{Cov}_h^x(\Theta_j^k, \Theta_n^m) \\
\leq 2 \sum_{i \geq i} \sum_{k \geq k} \sum_{m \geq m} \text{Cov}_h^x(\Theta_j^k, \Theta_j^m) \\
\leq 2 \sum_{i \geq i} \sum_{k \geq k} \sum_{m \geq m} c_{13}^{k+m+n-j} f^2(s_{j_i})f^2(s_{j_n}) \\
\leq \sum_{i \geq i} \sum_{n \geq n} c_{14}^{j_n-j_i} f^4(s_{j_i}) \\
\leq \sum_j c_{15}^{j} f^4(s_j) \leq c_{16} \int_a^b f^3(v)dv.
\]
(vii) Next we will prove part (iv) of Theorem 1.6.

Fix some \( x \in D_f \) and suppose for convenience that \( x^d = s_q \) for some \( q \). Recall \( S_j^k, T_j^k, F_j^k \) and \( \Theta_j^k \) from part (v) of the proof. With slightly more work, the argument for (2.47) can be seen to yield the following improved estimate:

\[
\text{Cov}_h(x) \leq \begin{cases} 
  c_1 c_2^{k+m+|n-j|} f^2(s_j) f^2(s_n), & j, n \geq q \\
  c_1 c_2^{k+m+|n-j|} c_3^{q-j} f^2(s_j) f^2(s_n), & j < q \leq n \\
  c_1 c_2^{k+m+|n-j|} c_3^{q-j} c_3^{q-n} f^2(s_j) f^2(s_n), & j, n < q,
\end{cases}
\]

where \( c_2, c_3 < 1 \).

Now we assume that the Lipschitz constant of \( f \) is so small that for each \( j \),

\[
\frac{f^2(s_{j-1})}{f^2(s_j)} < \frac{c_4^{-1} + 1}{2}.
\]

Therefore

\[
(2.49) \quad \text{Cov}_h^x(\Theta_j^k, \Theta_n^m) \leq \begin{cases} 
  c_1 c_2^{k+m+|n-j|} f^2(s_j) f^2(s_n), & j, n \geq q \\
  c_1 c_2^{k+m+|n-j|} c_4^{q-j} f^2(s_q) f^2(s_n), & j < q \leq n \\
  c_1 c_2^{k+m+|n-j|} c_4^{q-j} c_4^{q-n} f^4(s_q), & j, n < q,
\end{cases}
\]

for some \( c_4 < 1 \).

If \( \int_{x^d}^b f^3(v) dv = \infty \) then (1.9) obviously holds. Assume that \( \int_{x^d}^b f^3(v) dv < \infty \). Then we may choose an ordering \( \{j_i\}_{i \geq 1} \) of the set \( \{k : x^d \leq s_k < b\} \) which satisfies
\[ f(s_{j_{i+1}}) \leq f(s_{j_i}) \text{ for all } i. \] Let \( j_{i_0} = q \). Then in view of (2.49),

\[
\begin{align*}
\text{Var}_h^x \zeta &= \text{Var}_h^x \left( \sum_{j,k} \Theta^k_j \right) = \sum_{j,k,n,m} \text{Cov}_h^x(\Theta^k_j, \Theta^m_n) \\
&\leq 2 \sum_{i} \sum_{n \geq i} \sum_{k} \sum_{m} \text{Cov}_h^x(\Theta^k_j, \Theta^m_j) \\
&\quad + 2 \sum_{j \leq n < q} \sum_{k} \sum_{m} \text{Cov}_h^x(\Theta^k_j, \Theta^m_n) \\
&\quad + 2 \sum_{j < q} \sum_{i \geq i_0} \sum_{k} \sum_{m} \text{Cov}_h^x(\Theta^k_j, \Theta^m_j) \\
&\quad + 2 \sum_{j < q < i_0} \sum_{k} \sum_{m} \text{Cov}_h^x(\Theta^k_j, \Theta^m_j) \\
&\leq 2 \sum_{i} \sum_{n \geq i} \sum_{k} \sum_{m} c_1 c_2^{k+m+|j_n-j_i|} f^2(s_{j_i}) f^2(s_{j_n}) \\
&\quad + 2 \sum_{j \leq n < q} \sum_{k} \sum_{m} c_1 c_2^{k+m-|n-j_i|} c_4^{q-j} c_4^{-n} f^4(s_q) \\
&\quad + 2 \sum_{j < q} \sum_{i \geq i_0} \sum_{k} \sum_{m} c_1 c_2^{k+m+|j_i-j_q|} c_4^{q-j} f^2(s_q) f^2(s_{j_i}) \\
&\quad + 2 \sum_{j < q < i_0} \sum_{k} \sum_{m} c_1 c_2^{k+m+|j_i-j_q|} c_4^{q-j} f^2(s_q) f^2(s_{j_i}) \\
&\leq \sum_{i} c_5 f^4(s_{j_i}) + c_6 f^4(s_q) + c_7 f^4(s_q) + \sum_{i} c_8 f^4(s_{j_i}) \\
&\leq c_9 \sum_{j \geq q} f^4(s_j) \leq c_{10} \int_{x^d}^{b} f^3(v) dv. \quad \Box
\end{align*}
\]

Because they use similar arguments to those just given, we include the following two subsidiary results in this section.

(2.51) **Corollary.** Suppose that \( D_f \) and \( h \) are as in Theorem 1.6. Assume that \( \int_a^b f^3(v) dv < \infty \). Then

\[
\lim_{x^d \to \infty} \sup\{ \text{Var}_h^x T(\Lambda_u) : u > x^d \} = 0.
\]

**Proof.** Recall the notation from the proof of Theorem 1.6. As in the proof of (2.48), for every \( x \) and for every \( u = s_i \),

\[
\text{Var}_h^x T(\Lambda_u) = \sum_{j,k,n,m} \text{Cov}_h^x(\Theta^k_j 1_{\{T^k_j < T(\Lambda_u)\}}, \Theta^m_n 1_{\{T^m_n < T(\Lambda_u)\}}).
\]
An examination of the proof of (2.48) shows that the terms of this sum are bounded by the terms of an absolutely convergent series, uniformly in \(x\) and in \(u = s_i\). With a little more work, it is easy to see that this domination holds for \(u \in (a, b)\) as well. For fixed \(j, k, m, n,\)

\[
\text{Cov}_h^k(\Theta_j^k 1_{\{T_j^k < T(\Lambda_u)\}}, \Theta_n^m 1_{\{T_n^m < T(\Lambda_u)\}}) \to 0
\]
as \(x^d \to \infty\), uniformly in \(u\), because of (2.42). This easily implies (2.52). \(\square\)

\textbf{(2.53) Lemma.} Assume that \(D_f\) and \(h\) are as in Theorem 1.6. Set

\[
f_*(v) \overset{\text{def}}{=} \sup_{u \geq v} f(u).
\]

There exists a \(c_1 < \infty\) such that for all \(u\) and all \(x_1, x_2 \in D_f\) with \(x_1^d = x_2^d < u\) we have

\[
|E_h^{x_1} T(\Lambda_u) - E_h^{x_2} T(\Lambda_u)| \leq c_1 f^2(x_1^d).
\]

\textbf{Proof.} We will use an argument from part (v) of the proof of Theorem 1.6. Suppose that \(s_{n+1} \leq x_1^d \leq s_{n+2}\) and let \(L\) be the last exit from \(\Lambda_{s_n}\). It has been proved that

\[
E_h^{x_k} L < c_2 f^2(s_n)
\]

for \(k = 1, 2\) (see the paragraph preceding (2.35)). Recall the definitions of \(T_0, S_j^k, T_j^k\) and \(F_j^k\) from the same proof, and set

\[
G_j^k \overset{\text{def}}{=} F_j^k \cap \{T_j^k < T(\Lambda_u)\}.
\]

We have

\[
E_h^{x_k} T_0 < c_3 f^2(s_{n+1})
\]

by an argument analogous to that proving (2.37). By Lemma 2.26, the Radon-Nikodym derivative of the initial distributions of \(\{X_t, t \in [T_j^k, S_j^k]\}\) under \(P_h^{x_k}(\cdot \mid G_j^k)\) and \(P_h^{x_k}(\cdot \mid G_j^k)\) differs from 1 by no more than \(c_4 c_5^{[n-j]}\) where \(c_5 < 1\). It follows that

\[
|E_h^{x_1} [(S_j^k - T_j^k)1_{G_j^k}] - E_h^{x_2} [(S_j^k - T_j^k)1_{G_j^k}]| \leq c_4 c_5^{[n-j]} E_h^{x_1} [(S_j^k - T_j^k)1_{G_j^k}].
\]

Now (2.39) implies that

\[
\left| \sum_k E_h^{x_1} (S_j^k - T_j^k)1_{G_j^k} - \sum_k E_h^{x_2} (S_j^k - T_j^k)1_{G_j^k} \right|
\]

\[
\leq c_4 c_5^{[n-j]} \sum_k E_h^{x_1} (S_j^k - T_j^k)1_{G_j^k}
\]

\[
\leq c_4 c_5^{[n-j]} c_6 f^2(s_j).
\]
Since
\[ \sum_{k \geq 1} (S_j^k - T_j^k) 1_{G_j^k} \leq T(\Lambda_u) \leq T_0 + L + \sum_{j \geq n} \sum_{k \geq 1} (S_j^k - T_j^k) 1_{G_j^k}, \]
we obtain from (2.54)-(2.56) that
\[ |E_{x_1}^T(\Lambda_u) - E_{x_2}^T(\Lambda_u)| \]
\[ \leq 2c_2 f^2(s_n) + 2c_3 f^2(s_{n+1}) + \sum_{j \geq n} c_4 c_5^{n-j} c_6 f^2(s_j) \]
\[ \leq 2c_2 f^2(s_n) + 2c_3 f^2(s_n) + \sum_{j \geq n} c_4 c_5^{n-j} c_6 f^2(s_n) \leq c_7 f^2(s_n). \]
\[ \square \]

3. Disintegration of harmonic functions.

The purpose of this section is to prove Theorem (1.3). Unless otherwise indicated, the notation and general hypotheses of Theorem (1.3) will be assumed throughout this section.

Fix some \( x_0 \in D_f \) and let \( g(u) \) \( \triangleq -E_{x_0}^T(\Lambda_u) \). Recall that \( f^*(v) = \sup_{u \geq v} f(u) \). Note that in either case (a) or (b) of Theorem (1.3) (i), we have that \( f(v) \to 0 \) as \( v \to \infty \).

**3.1 Lemma.** Suppose that one of the assumptions (a) or (b) of Theorem (1.3) (i) is satisfied. Then
\[ \lim_{u \to \infty} (T(\Lambda_u) + g(u)) \] exists \( P_{x_0}^T \)-a.s.

**Proof.** Lemma 2.53 and Corollary (2.51) show that for \( k \geq 1 \), we can choose \( u_k \) such that
\[ |E_{x_1}^T(\Lambda_u) - E_{x_2}^T(\Lambda_u)| \leq c_1 f^2(u_k) \leq 1/k^2 \]
for all \( x_1, x_2 \in D_f \) and \( u \) with \( u_k \leq x_1^d = x_2^d < u \). We may also assume that
\[ \text{Var}_{x}^T(\Lambda_u) \leq 1/k^6 \]
for \( x \in D_f \) and \( u \) with \( u_k \leq x^d < u \).

Suppose \( u \in [u_k, u_{k+1}) \). Since
\[ T(\Lambda_{u_{k+1}}) = (T(\Lambda_{u_{k+1}}) - T(\Lambda_u)) + T(\Lambda_u), \]
we have
\[ g(u_{k+1}) = -E_{x_0}^T(\Lambda_{u_{k+1}}) - T(\Lambda_u)) + g(u). \]
This, (3.2), and the strong Markov property applied at \( T(\Lambda_u) \) imply that
\[
(3.4) \quad |E_h^x T(\Lambda_{u_{k+1}}) + (g(u_{k+1}) - g(u))| \leq 1/k^2
\]
for all \( x \in D_f \) such that \( x^d = u \). The Chebyshev inequality and (3.3) yield that
\[
P_h^x(|T(\Lambda_{u_{k+1}}) - E_h^x T(\Lambda_{u_{k+1}})| \geq 1/k^2) \leq k^4 \text{Var}_h^x T(\Lambda_{u_{k+1}}) \leq 1/k^2,
\]
if \( x^d = u \). This and (3.4) give
\[
P_h^x(|T(\Lambda_{u_{k+1}}) - E_h^x T(\Lambda_{u_{k+1}})| \geq 2/k^2) \leq 1/k^2,
\]
for \( x \in D_f \) such that \( x^d = u \). By the strong Markov property applied at \( T(\Lambda_u) \),
\[
(3.5) \quad P_h^x(|T(\Lambda_{u_{k+1}}) - T(\Lambda_u) + (g(u_{k+1}) - g(u))| \geq 2/k^2) \leq 1/k^2,
\]
for any \( x \in D_f \) with \( x^d \leq u \). In particular,
\[
(3.6) \quad P_h^x(|T(\Lambda_{u_{k+1}}) - T(\Lambda_u) + (g(u_{k+1}) - g(u_k))| \geq 2/k^2) \leq 1/k^2,
\]
if \( x^d \leq u \).

Fix some \( c_2 > 0 \) and find \( j_0 \) so large that \( \sum_{j > j_0} 2/j^2 < c_2 \). Suppose that
\( k > j_0, x^d \leq u \), and recall that \( u \in [u_k, u_{k+1}) \). Then (3.5)-(3.6) imply that with
\( P_h^x \)-probability larger than \( 1 - c_2 \), the event
\[
(3.7) \quad \{ |T(\Lambda_{u_{k+1}}) - T(\Lambda_u) + (g(u_{k+1}) - g(u))| \leq 2/k^2 \}
\]
\[
\cap \bigcap_{j \geq j_0} \{ |T(\Lambda_{u_{j+1}}) - T(\Lambda_{u_j}) + (g(u_{j+1}) - g(u_j))| \leq 2/j^2 \}
\]
occurs. Let
\[
A_v \overset{df}{=} \{ |(T(\Lambda_{u_m}) + g(u_m)) - (T(\Lambda_v) + g(v))| < c_2 \ \forall u_m \geq v \}.
\]
If the event in (3.7) holds then \( A_v \) holds, because in such a case we have
\[
|(T(\Lambda_{u_m}) + g(u_m)) - (T(\Lambda_u) + g(u))| \\
\leq |(T(\Lambda_{u_{k+1}}) + g(u_{k+1})) - (T(\Lambda_u) + g(u))| \\
+ \sum_{j = k+1}^{m-1} |(T(\Lambda_{u_{j+1}}) - T(\Lambda_{u_j})) + (g(u_{j+1}) - g(u_j))| \\
\leq 2/k^2 + \sum_{j = k+1}^{m-1} 2/j^2 < c_2.
\]
Let

\[ W = W(u) \overset{\text{df}}{=} \inf\{v > u : |(T(A_v) + g(v)) - (T(A_u) + g(u))| \geq 2c_2\} . \]

By the strong Markov property applied at \( T(A_W) \) we have \( P^x_{\Phi_v}(A_W \mid W < \infty) > 1 - c_2 \). Since \( A_u \cap \{ W < \infty \} \cap A_W = \emptyset \), it follows that \( P^x_{\Phi_v}(A_W \cap \{ W < \infty \}) < c_2 \), and hence \( P^x_{\Phi_v}(W < \infty) < c_2/(1 - c_2) \). This proves the Lemma, since we may assume that \( c_2 > 0 \) is arbitrarily small by choosing \( u \) sufficiently large. \( \square \)

We now make some general observations about parabolic Martin boundaries. Let \( D \) be a domain. For \( \phi \) a parabolic function on \( \hat{D} \), and \( v < 0 \), define

\[ \phi_v(x, t) \overset{\text{df}}{=} \phi(x, t + v) . \]

Then \( \phi_v \) is also parabolic. Moreover, if \( \phi \) is minimal then \( \phi_v \) is either minimal or \( \phi_v \equiv 0 \) (see Doob (1984) 1.XV.17).

**Lemma.** Let \( D \) be a domain. Let \( \phi \) be parabolic on \( \hat{D} \), and let \( v < 0 \). Then the laws of \( X \) under \( P^x_{\phi_v} \) and \( P^x_{\phi} \) are the same.

**Proof.** It suffices to show that \( P^x_{\phi_v}(A) = P^x_{\phi}(A) \), for \( A \) an event of the form \( \{ X(t_1) \in A_1, \ldots, X(t_n) \in A_n \} \), where \( t_1 < t_2 < \cdots < t_n \). But

\[
P^x_{\phi_v}(A) = \frac{1}{\phi_v(x, t)} E^{x, t}[1_A \phi_v(X_{t_n}, \tau_{t_n})] = \frac{1}{\phi(x, t + v)} E^{x, t}[1_A \phi(X_{t_n}, \tau_{t_n} + v)] = \frac{1}{\phi(x, t + v)} E^{x, t+v}[1_A \phi(X_{t_n}, \tau_{t_n})] = P^{x,v}_{\phi}(A). \]

Now, if \( (y_k, t_k) \in \hat{D} \), \( (y_k, t_k) \to z \in \partial^M \hat{D} \), and each \( t_k < v \), then

\[
\dot{K}((x, t), (y_k, t_k - v)) = \frac{p_{t-t_k+v}^D(x, y_k)}{p_{t}^D(x_0, y_k)} = \frac{p_{t-t_k+v}^D(x, y_k)}{p_{t-t_k}^D(x_0, y_k)} \cdot \frac{p_{-t}^D(x_0, y_k)}{p_{-t_k+v}^D(x_0, y_k)} \\
\to \dot{K}((x, t + v), z) = \frac{\dot{K}((x, t + v), z)}{\dot{K}((x, t), z)}. 
\]

Thus, provided \( \dot{K}((x_0, v), z) > 0 \), it follows that \( (y_k, t_k - v) \) converges in \( \hat{D}^M \) to a point \( \Phi_{v,z} \in \partial^M \hat{D} \) with

\[
\dot{K}(\cdot, \Phi_{v,z}) = \frac{\dot{K}_v(\cdot, z)}{\dot{K}_v((x_0, 0), z)}. \]

**Proof.** It suffices to show that \( \phi_v(x, t) \) is minimal then \( \phi_v \) is either minimal or \( \phi_v \equiv 0 \) (see Doob (1984) 1.XV.17).
Of course, it may happen that $\Phi_v z = z$. Note also that

$$
\dot{K}(x_0, 0), \Phi_v z) = 1,
$$

so that $\Phi_v z$ is a minimal point, if and only if $\dot{K}_v(\cdot, z)$ is a minimal function.

It would simplify several future arguments, if the map $\Phi_v$ could be defined for $v > 0$ as well. A natural way of doing this would be to set

$$
\phi_v(x, t) \overset{df}{=} \begin{cases} 
\phi(x, t + v), & t + v \leq 0 \\
\int p^D_{t+v}(x, y) \phi(y, 0) dy, & t + v > 0.
\end{cases}
$$

The obstacle to this approach is that in general, this integral need not converge.

The following result is well known. See, for example, Theorems C and E of Aronson (1968).

(3.11) Lemma. Let $D$ be a domain, and let $A \subset \dot{D}$ be compact.

(i) Let $\varepsilon > 0$ and $M < \infty$. There exists a $\delta > 0$ such that if $u$ is parabolic on $\dot{D}$ and $u \leq M$, then $|u(z) - u(z')| < \varepsilon$ whenever $z, z' \in A$ and $|z - z'| < \delta$.

(ii) Let $x \in D$. There exists an $M < \infty$ such that if $u$ is parabolic on $\dot{D}$, and $u(x, 0) \leq 1$, then $u \leq M$ on $A$.

(3.12) Lemma. Let $D$ be a domain. Suppose that $(y_k, t_k) \in \dot{D}$ converge to some $z \in \partial M \dot{D}$, and that $a_k \to 0$. Let $(x, t) \in \dot{D}$ (so that, in particular, $t < 0$) and suppose that $\dot{K}((x, t), z) > 0$. Then

$$
\frac{p^D_{t-t_k}(x, y_k)}{p^D_{t-t_k-a_k}(x, y_k)} \to 1
$$
as $k \to \infty$. Moreover,

$$
(y_k, t_k + a_k - t) \to \Phi_t z.
$$

Proof. If $\dot{K}((x, t), z) > 0$, then the $\dot{K}((x, t), (y_k, t_k))$ are bounded away from 0. Since $\dot{K}((x_0, 0), (y_k, t_k)) = 1$, (ii) of Lemma 3.11 shows that the $\dot{K}((\cdot, (y_k, t_k))$ are uniformly bounded, on a suitable neighbourhood of $(x, t)$. Applying (i) of Lemma 3.11 on this neighbourhood shows that

$$
\frac{p^D_{t-t_k}(x, y_k)}{p^D_{t-t_k-a_k}(x, y_k)} = \frac{\dot{K}((x, t), (y_k, t_k))}{\dot{K}((x, t-a_k), (y_k, t_k))} \to 1,
$$
as $k \to \infty$, showing (3.13).

To prove (3.14), we must show that

$$
\lim_{k \to \infty} \dot{K}((x, s), (y_k, t_k + a_k - t)) = \lim_{k \to \infty} \dot{K}((x, s), (y_k, t_k - t))
$$
for every \((x, s) \in \hat{D}\). But as before,
\[
\hat{K}((x, s), (y_k, t_k + a_k - t)) = \frac{p_{s + t - t_k - a_k}(x, y_k)}{p_{t - t_k - a_k}(x_0, y_k)}
\]
\[
= \frac{\hat{K}((x, s + t - a_k), (y_k, t_k))}{\hat{K}((x_0, t - a_k), (y_k, t_k))}
\]
\[
\rightarrow \frac{\hat{K}((x, s + t), z)}{\hat{K}((x_0, t), z)} = \hat{K}((x, s), \Phi_t z). \quad \Box
\]

**(3.15) Lemma.** Assume that \(f(u) \to 0\) as \(u \to \infty\). Let \((z_k, t_k) \in \hat{D}_f\) converge to \(z \in \partial^M \hat{D}_f\), and suppose that \(\hat{K}((x, t), z) > 0\) for every \((x, t) \in \hat{D}_f\). If \(y_k \in D_f\) and \(z_k^d = y_k^d\) for each \(k\), then for some \(c_1 < \infty\) and \(c_2 > 0\), and for every \(q < 0\) and \((x, t) \in \hat{D}_f\),
\[
\limsup_{k \to \infty} \hat{K}((x, t), (y_k, t_k - q)) \leq c_1 \hat{K}((x, t), \Phi_q z),
\]
\[
\liminf_{k \to \infty} \hat{K}((x, t), (y_k, t_k - q)) \geq c_2 \hat{K}((x, t), \Phi_q z).
\]

**Proof.** Let \(r_0 > 0\) be so small that for each \(w \in \partial D_f\), the set \(\partial D_f \cap B(w, r_0 f(w^d))\) is the graph of a Lipschitz function \(F\), with Lipschitz constant \(\lambda_0\) in some orthonormal coordinate system \(CS_w\). Let the coordinates of \(x\) in \(CS_w\) be \((\hat{x}, x')\), so that
\[
D_f \cap B(w, r_0 f(w^d)) = \{ (\hat{x}, x') : x' > F(\hat{x}) \} \cap B(w, r_0 f(w^d)).
\]

Let
\[
\Psi_r(w, s) = \{(x, t) \in \hat{D}_f : |x - w| < r, |s - t| < r^2\},
\]
\[
A_r(w) = (\hat{w}, w' + r) \quad \text{in} \ CS_w.
\]

We fix a suitable \(\bar{s} < 0\) and apply Theorem 1.6 of Fabes et al. (1986) to some \(\Psi_{r/8}(w, \bar{s})\), to see that if \(x_1, x_2 \in D_f\), \(w \in \partial D_f\), \(r < r_0 f(w^d)/2\), \(s, s' < \bar{s}\) and \(y \in B(w, r/8)\), then
\[
\frac{p_{D_f}^{D_f}(x_2, y)}{p_{D_f}^{D_f}(x_1, y)} = \frac{\hat{G}_{D_f}((y, \bar{s}), (x_2, \bar{s} + s))}{\hat{G}_{D_f}((y, \bar{s}), (x_1, \bar{s} + s'))}
\]
\[
\leq c_1 \frac{\hat{G}_{D_f}((A_r(w), \bar{s} + 2r^2), (x_2, \bar{s} + s))}{\hat{G}_{D_f}((A_r(w), \bar{s} - 2r^2), (x_1, \bar{s} + s'))}
\]
\[
= c_1 \frac{p_{D_f}^{D_f}(x_2, A_r(w))}{p_{D_f}^{D_f}(x_1, A_r(w))}.
\]
Note that, although Theorem 1.6 of Fabes et al. (1986) would in principle allow the above constant \( c_1 \) to depend on \( f(w^d) \), in fact a scaling argument shows that it does not.

Fix \((x,t) \in \hat{D}_f\) and \( q < 0 \). Let \( M = \bigcup_{w \in \partial D_f} B(w, r_0 f(w^d)/32) \). If \( y_k \in M \), choose \( w \) so that \( y_k \in B(w, r/8) \), where \( r = r_0 f(w^d)/4 \). With this choice of \( r \), set

\[
\bar{y}_k = A_r(w), \quad a_k = 2r^2.
\]

If \( y_k \notin M \), set

\[
\bar{y}_k = y_k, \quad a_k = 0.
\]

The assumption that \( \hat{K}((x,t), z) > 0 \) for every \((x,t) \in \hat{D}_f\) easily implies that \( t_k \to -\infty \). By (3.18),

\[
\frac{p_{t+q-t_k}^{D_f}(x, y_k)}{p_{q-t_k}^{D_f}(x_0, y_k)} \leq c_1 \frac{p_{t+q-t_k+a_k}^{D_f}(x, \bar{y}_k)}{p_{q-t_k-a_k}^{D_f}(x_0, \bar{y}_k)},
\]

for \( k \) so large that \( t_k - t - q < \bar{s} \).

Let \( b_k = f^2(z_k^d) \). A precise version of the parabolic Harnack principle (see Theorem 0.2 of Fabes et al. (1986)) implies that for \( k \) large and for every \( v \in D_f \) with \(|v^d - z_k^d| < f(z_k^d)\) and \( v \notin M \), we have

\[
\frac{p_{t+q-t_k+a_k+b_k}^{D_f}(x, \bar{y}_k)}{p_{q-t_k-a_k-b_k}^{D_f}(x_0, \bar{y}_k)} \leq c_2 \frac{p_{t+q-t_k+a_k+b_k}^{D_f}(x, v)}{p_{q-t_k-a_k-b_k}^{D_f}(x_0, v)}.
\]

As above, take \( \bar{z}_k \) equal to either \( z_k \) (if \( z_k \notin M \)), or an \( A_r(w) \) (if \( z_k \in B(w, r/8) \), where \( r = r_0 f(w^d)/4 \)). Take \( d_k \) equal to 0 or \( 2r^2 \) respectively. Therefore

\[
\frac{p_{t+q-t_k+a_k+b_k}^{D_f}(x, \bar{y}_k)}{p_{q-t_k-a_k-b_k}^{D_f}(x_0, \bar{y}_k)} \leq c_1 \frac{p_{t+q-t_k+a_k+b_k+d_k}^{D_f}(x, \bar{z}_k)}{p_{q-t_k-a_k-b_k-d_k}^{D_f}(x_0, \bar{z}_k)},
\]

for \( k \) large, as before. Since \( q < 0, a_k \to 0, b_k \to 0, \) and \( d_k \to 0 \), it follows from (3.13) that

\[
\lim_{k \to \infty} \frac{p_{t+q-t_k+a_k+b_k+d_k}^{D_f}(x, \bar{z}_k)}{p_{q-t_k-a_k-b_k-d_k}^{D_f}(x_0, \bar{z}_k)} = \lim_{k \to \infty} \frac{p_{t+q-t_k}^{D_f}(x, \bar{z}_k)}{p_{q-t_k}^{D_f}(x_0, \bar{z}_k)} = \lim_{k \to \infty} \hat{K}((x,t),(\bar{z}_k,t_k-q)) = \hat{K}((x,t), \Phi q z).
\]

Thus, taking \( v = \bar{z}_k \), it follows from this and (3.19)-(3.21) that

\[
\lim_{k \to \infty} \hat{K}((x,t),(y_k,t_k-q)) = \lim_{k \to \infty} \frac{p_{t+q-t_k}^{D_f}(x, y_k)}{p_{q-t_k}^{D_f}(x_0, y_k)} \leq c_3 \hat{K}((x,t), \Phi q z)
\]

as well, proving (3.16). The argument for (3.17) is similar. □

We may improve upon the conclusion of Lemma 3.15, by assuming that \( z \) is minimal:
Lemma. Assume that $f(u) \to 0$ as $u \to \infty$. Let $(z_k, t_k) \in \dot{D}_f$ converge to minimal point $z \in \partial^M_0 \dot{D}_f$, and suppose that $\dot{K}((x,t),z) > 0$ for every $(x,t) \in \dot{D}_f$. If $y_k \in \dot{D}_f$ satisfy $z^d_k = y^d_k$ for each $k$, and $q_k \to q < 0$, then $(y_k, t_k - q_k) \to \Phi_q z$.

That is,

$$\lim_{k \to \infty} \dot{K}((x,t),(y_k,t_k - q_k)) = \dot{K}((x,t),\Phi_q z)$$

for every $(x,t) \in \dot{D}_f$.

Proof. We first consider the limit of $(y_k, t_k - q)$. If $w$ is any limit point of this sequence, then by (3.16) we have that

$$\dot{K}(\cdot, w) \leq c_1 \dot{K}(\cdot, \Phi_q z).$$

By minimality of $z$ (and hence $\Phi_q z$), in fact

$$\dot{K}(\cdot, w) = c \dot{K}(\cdot, \Phi_q z)$$

for some $c < \infty$. By (3.17) we must have $c > 0$, so $w \neq 0$.

Let $k_i$ be a subsequence along which $(y_{k_i}, t_{k_i} - q) \to w$. By passing to a further subsequence, if necessary, we may also ensure that $(y_{k_i}, t_{k_i} - q/2)$ converges to some $w' \neq 0$. Then $w = \Phi_{q/2} w'$, so by (3.10),

$$\dot{K}((x_0,0), w) = 1 = \dot{K}((x_0,0), \Phi_q z).$$

Thus $c = 1$, and so $w = z$. Since $\Phi_q z$ is the only limit point of $(y_k, t_k - q)$, it follows that the sequence itself converges to $\Phi_q z$.

Similarly, $(y_k, t_k - q/2) \to \Phi_{q/2} z$. Since $\Phi_q z = \Phi_{q/2} (\Phi_{q/2} z)$, we may set $a_k = q - q_k$, and apply (3.14) (with $t = q/2$), to obtain in addition that $(y_k, t_k - q_k) \to \Phi_q z$, as required. \qed

Proof of Theorem 1.3.

(i) Assume either (a) or (b) of (i) of the Theorem, and recall that this implies that $f(u) \to 0$ as $u \to \infty$.

Let $\mathcal{H}_s$ denote the set of points $z$ of the minimal Martin boundary $\partial^M_0 \dot{D}_f$, such that $g(u) - \tau(\Lambda_u) \rightarrow s$, $P_{x_0}^{\tau_0,0}$-a.s. Set $\mathcal{H} = \bigcup_{s \in \mathbb{R}} \mathcal{H}_s$. Recall that if $\phi$ is a minimal parabolic function, then the tail $\sigma$-field of every $\phi$-transform of space-time Brownian motion is trivial. By Lemma (3.1), the random variable $\lim_{u \to \infty} g(u) - \tau(\Lambda_u)$ is well defined $P_{h}^{\tau_0,0}$-a.s. It is clearly measurable with respect to the tail $\sigma$-field of $\dot{X}_t$, and so

$$h(x) = \int_{\mathcal{H}} \dot{K}(\cdot, z) \mu(dz),$$

for some measure $\mu$ concentrated on $\mathcal{H}$. In particular, it follows that $\mathcal{H}_s$ is non-empty, for some $s \in \mathbb{R}$. We will work towards proving that, in fact,

\begin{equation}
(3.23)\quad \text{every } \mathcal{H}_s \text{ consists of a single point,}
\end{equation}
namely the $z_s$ of (A).

In fact, the conclusion of (B) will follow immediately from (3.23), since $\mathcal{H}_{s_1}$ and $\mathcal{H}_{s_2}$ are disjoint if $s_1 \neq s_2$.

For $z \in \mathcal{H}_s$, we have that $g(u) - \tau(\Lambda_u) \to s$, $P_{\tau}^{x,0}$-a.s. A standard argument now shows that the same is true $P_{\tau}^{x,t}$-a.s., for every $(x, t) \in \hat{D}_f$. Thus (D) will also follow immediately, once (3.23) is proven.

(ii) It is a routine matter to prove that if

$$P_{\tau}^{x,0} \left( \lim_{u \to \infty} (g(u) - \tau(\Lambda_u)) \in (s_1, s_2) \right) > 0$$

then for every $s_3 \in \mathbb{R}$,

$$P_{\tau}^{x,0} \left( \lim_{u \to \infty} (g(u) - \tau(\Lambda_u)) \in (s_1 + s_3, s_2 + s_3) \right) > 0.$$

Hence, (3.24) holds for all $-\infty < s_1 < s_2 < \infty$. Therefore

$$\mu \left( \bigcup_{s \in (s_1, s_2)} \mathcal{H}_s \right) > 0,$$

for every such $s_1, s_2$. This will establish (C). Moreover, it shows that

$$\exists \{s_k\}_{k \geq 1} \text{ such that } \lim_{k \to \infty} s_k = \infty \text{ and for every } k, \mathcal{H}_{s_k} \neq \emptyset.$$

If $\phi = \hat{K}(\cdot, z)$, where $z \in \mathcal{H}_s$, and $v < 0$, then by Lemma 3.8,

$$1 = \frac{P_{\tau}^{x,t+v}(g(u) - \tau(\Lambda_u) \to s)}{P_{\tau}^{x,t}(g(u) + T(\Lambda_u) - t - v \to s)} = \frac{P_{\tau}^{x,t}(g(u) + T(\Lambda_u) - t - v \to s)}{P_{\tau}^{x,t}(g(u) - \tau(\Lambda_u) \to s + v)}.$$

That is, the pole of $\phi_v$ belongs to $\mathcal{H}_{s+v}$. Thus, $\Phi_v$ maps $\mathcal{H}_s$ into $\mathcal{H}_{s+v}$. Appealing to (3.25), we conclude that $\mathcal{H}_s$ is nonempty, for every $s \in \mathbb{R}$.

(iii) Let $s \in \mathbb{R}$, and pick $z \in \mathcal{H}_s$. For any sequence $u_k \to \infty$, we may set $y_k = X(T(\Lambda_{u_k}))$, and $t_k = \tau(\Lambda_{u_k})$. Because $X(T(\Lambda_{u_k})) \to z$ in the Martin topology, $P_{\tau}^{x,0}$-a.s., it follows that we have constructed a sequence $(y_k, t_k) \to z$ as in part (A), with $s_k \overset{df}{=} g(y_k^d) - t_k \to s$.

Next we will show that $\mathcal{H}_s$ consists of a single point, for each $s$. Suppose to the contrary that $z, \tilde{z} \in \mathcal{H}_s$ for some $s$. It is easy to see that we must have $\Phi_{s'-s}z \neq \Phi_{s'-s}\tilde{z}$ for some $s' < s$. Fix any sequence $u_k \to \infty$. Consider any sequence $(y_k, t_k) \to z$, with $g(y_k^d) - t_k \to s$ and $y_k^d = u_k$, constructed as in the previous paragraph. Let $(\tilde{y}_k, \tilde{t}_k)$ be the analogous sequence with $(\tilde{y}_k, \tilde{t}_k) \to \tilde{z}$, $\tilde{y}_k^d = u_k$ and $g(\tilde{y}_k^d) - \tilde{t}_k \to s$. Note that $t_k - \tilde{t}_k \to 0$ because $y_k^d = \tilde{y}_k^d = u_k$, $g(y_k^d) - t_k \to s$, and $g(\tilde{y}_k^d) - \tilde{t}_k \to s$. Lemma 3.22 implies that $(y_k, t_k + s - s') \to \Phi_{s'-s}z$. But it
also implies that \((y_k, t_k + s - s') \to \Phi_{s'-s} \tilde{z}\), because \(y_k^d = \tilde{y}_k^d\) and \(t_k - \tilde{t}_k \to 0\). This contradicts the fact that \(\Phi_{s'-s} \tilde{z} \neq \Phi_{s'-s} \tilde{z}\) and so it proves our claim, and establishes (3.23).

Now let \(r_k \to s\), and consider any sequence \(x_k\) such that \(x_k^d \to \infty\). Our goal is to show that \((x_k, g(x_k^d) - r_k) \to z\), where \(z\) is the only element of \(\mathcal{H}_s\). Set \(u_k = x_k^d\), and this time choose \(s' > s\). Let \(z'\) be the element of \(\mathcal{H}_{s'}\). Note that \(\Phi_{s'-s} z' = z\).

By the argument of the first paragraph of (iii), we may choose \((y_k, t_k') \to z'\) with \(y_k^d = u_k = x_k^d\) and \(g(y_k^d) - t_k' \to s'\). Since \(t_k' - g(x_k^d) - r_k \to -s' + s\), we may apply Lemma 3.22 and obtain that

\[(x_k, g(x_k^d) - r_k) \to \Phi_{s'-s} z' = z.\]

This finishes the proof of (A). Thus, part (i) of Theorem 1.3 is proven.

(iv) Turning to part (ii) of Theorem 1.3, suppose that \(\int u^\infty f^3(v)dv = \infty\) for all \(u < \infty\). We also assume, as it simplifies the proof, that \(f(u) \to 0\) as \(u \to \infty\). At the end we will sketch out how to extend the argument to the general case, that \(\limsup_{u \to \infty} f(u) < \infty\).

We use a coupling argument. Fix \(x_1, x_2 \in D_f\), and \(s \leq 0\). Let \(X_1\) and \(X_2\) be independent processes, under a probability measure \(P\), with the same distributions as \(X\) under \(P_{h_1}^{x_1,s}\) and \(P_{h_2}^{x_2,s}\) respectively. Thus, \(\hat{X}_1(t) = (X_1(t), \tau_t)\) and \(\hat{X}_2(t) = (X_2(t), \tau_t)\) are versions of \(\hat{X}\), where \(\tau(t) = s - t\). Define

\[W = \inf\{t > 0 : X_1^d(t) = X_2^d(t)\}.\]

We will show that

\[(3.26) \quad P(W < \infty) = 1.\]

Write \(T_j(\Lambda_u)\) for the hitting time of \(\Lambda_u\) by \(X_j\). We may assume, without loss of generality, that \(x_1^d \leq x_2^d\). Set \(u_0 = x_2^d + f(x_2^d), Y_j = T_j(\Lambda_{u_0})\) and \(Z_j = T_j(\Lambda_u) - T_j(\Lambda_{u_0})\), where the value of \(u\) will be chosen later. A standard application of the boundary Harnack principle 2.2 shows that the Radon-Nikodym derivative of the hitting distributions of \(\Lambda_{u_0}\) under \(P_{h_1}^{y_1}\) and \(P_{h_2}^{y_2}\) is bounded below by \(c_1 > 0\) for all \(y_1, y_2 \in \Lambda_{x_2^d}\).

Let \(c_2\) be so large that

\[(3.27) \quad P(Y_1 - Y_2 \geq c_2) < c_1/16.\]

Use Theorem (1.6) (v) to find \(u\) so large that for every \(v \in \mathbb{R}\) we have

\[(3.28) \quad P(Z_2 \in (v, v + c_2)) < c_1/8.\]

Let \(v_1\) be the median of \(Z_1\), in other words,

\[(3.29) \quad P(Z_1 \leq v_1) \geq 1/2, \quad P(Z_1 \geq v_1) \geq 1/2.\]
By applying the strong Markov property at \( T(\Lambda_{u_0}) \), and by our choice of \( c_1 \), we have 
\[ P(Z_2 \geq v_1) \geq c_1/2. \]
Now we use (3.28) to obtain that 
\[ P(Z_2 \geq v_1 + c_2) \geq 3c_1/8. \]
This, (3.29) and the independence of \( Z_1 \) and \( Z_2 \) show that 
\[ P(Z_2 - Z_1 \geq c_2) \geq P(Z_1 \leq v_1, Z_2 \geq v_1 + c_2) \geq 3c_1/16. \]

Inequality (3.27) now implies that 
\[ P(T_1(\Lambda_u) < T_2(\Lambda_u)) = P(Y_1 + Z_1 < Y_2 + Z_2) \]
\[ \geq P(Y_1 - Y_2 < c_2 \leq Z_2 - Z_1) \]
\[ \geq P(Z_2 - Z_1 \geq c_2) - P(Y_1 - Y_2 \geq c_2) \geq c_1/8. \]

Let \( V_j^0 = x_j, \tau^0 = s \), \( T^1 = \max(T_1(\Lambda_u), T_2(\Lambda_u)) \), \( \tau^1 = \tau(T^1), V_j^1 = X(T_j(\Lambda_u)) \), \( U^1 = u \). Repeat the above argument, starting from \( (V_j^1, \tau^1) \) in place of \( (V_j^0, \tau^0) \), and ensuring that \( U^2 \) is chosen so large that each \( T_j(\Lambda_{U^2}) > T^1 \). Then continue this procedure inductively, to obtain sequences of random variables \( V_j^k, T^k, \tau^k \), and \( U^k \). By the strong Markov property, (3.30) becomes that 
\[ P(T_1(\Lambda_{U^{k+1}}) < T_2(\Lambda_{U^{k+1}}) \mid \mathcal{F}_{T^k}) \geq c_1/8, \]
where \( \mathcal{F}_t \) is the filtration of \( (X_1(t), X_2(t)) \). It follows that an infinite number of these events will occur, \( P \)-a.s. The same is true when the roles of \( X_1 \) and \( X_2 \) are reversed. Thus (3.26) holds.

(v) According to (3.26), used repeatedly, there are points \((x_{j,k}, t_k)\) on the paths of \( \hat{X}_j \) such that \( x_{1,k}^d = x_{2,k}^d \rightarrow \infty \). Using Lemma 3.22, as in the argument of section (iii) above, we get that \((x_{1,k}, t_k)\) and \((x_{2,k}, t_k)\) have the same limit in \( \partial_0^M \hat{D}_f \). Thus, the limits of \( \hat{X}_1(t) \) and \( \hat{X}_2(t) \) in \( \partial_0^M \hat{D}_f \), as \( t \rightarrow \infty \), are the same. Since \( \hat{X}_1 \) and \( \hat{X}_2 \) are independent, the measure \( \mu \) such that \( h(x) = \int_{\partial_0^M \hat{D}_f} \hat{K}((x, 0), z) \mu(dz) \) must actually be supported on a singleton. That is, \( h \) must be minimal as a parabolic function.

It is the use of Lemma 3.22 that requires the assumption that \( f(u) \rightarrow 0 \). If only \( \limsup_{u \rightarrow \infty} f(u) < \infty \), we modify the argument as follows. For any \( \varepsilon > 0 \),
\[ P_{h,f}^{x,t}(X((f(x^d)^2)) \in B((0, \ldots, 0, x + f(x^d)), \varepsilon f(x^d))) \geq c(\varepsilon) > 0, \]
for every \((x, t) \in \hat{D}_f \). Let
\[ W_{\varepsilon} \overset{df}{=} \inf\{t > 0 : X_1^d(t), X_2^d(t) \in B((0, \ldots, 0, u + f(u)), \varepsilon f(u)) \text{ for some } u\}. \]
Applying (3.31) to \( x = X_j(W) \) and using another iterative argument, one can show that \( P(W_{\varepsilon} < \infty) = 1 \) for every \( \varepsilon > 0 \). Taking a sequence \( \varepsilon_k \rightarrow 0 \), this now gives sequences \((x_{j,k}, t_k)\) on the paths of \( \hat{X}_j \), such that
\[ x_{j,k} \in B((0, \ldots, 0, u_k + f(u_k), \varepsilon_k f(u_k))), \]
where \( u_k \rightarrow \infty \). An argument as in the proof of Lemmas 3.15 and 3.22 now shows that the \((x_{1,k}, t_k)\) and \((x_{2,k}, t_k)\) have the same limit in \( \partial_0^M \hat{D}_f \). As before, this shows that \( h \) is parabolically minimal. \( \square \)
References

1. A. Ancona, *Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien*, Ann. Inst. Fourier 28 (1978), 169–213.

2. D.G. Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa 22 (1968), 607–694.

3. R. Bañuelos, R. Bass and K. Burdzy, *Hölder domains and the boundary Harnack principle*, Duke Math. J. 64 (1991), 195–200.

4. R. Bañuelos and B. Davis, *A geometrical characterization of intrinsic ultracontractivity for planar domains with boundaries given by the graphs of functions*, Indiana U. Math. Jour. 41 (1992), 885–912.

5. R. Bass and K. Burdzy, *A boundary Harnack principle in twisted Hölder domains*, Ann. Math. 134 (1991), 253–276.

6. R. Bass and K. Burdzy, *Lifetimes of conditioned diffusions*, Probab. Th. Rel. Fields 91 (1992), 405–443.

7. K. Burdzy, E. Toby and R.J. Williams, *On Brownian excursions in Lipschitz domains. Part II. Local asymptotic distributions*, Seminar on Stochastic Processes 1988 (E. Cinlar, K.L. Chung, R. Getoor, J. Glover, eds.), Birkhäuser, Boston, 1989, pp. 55–85.

8. M. Cranston, *Lifetime of conditioned Brownian motion in Lipschitz domains*, Z. Wahrschein. Verw. Gebiete 70 (1985), 335–340.

9. M. Cranston and T.R. McConnell, *The lifetime of conditioned Brownian motion*, Z. Wahrschein. Verw. Gebiete 65 (1983), 1-11.

10. B. Dahlberg, *Estimates of harmonic measure*, Arch. Rat. Mech. Anal. 65 (1977), 275–288.

11. B. Davis, *Conditioned Brownian motion in planar domains*, Duke Math. J. 57 (1988), 397–421.

12. B. Davis and B. Zhang, *Moments of the lifetime of conditioned Brownian motion in cones*, Proc. AMS 121 (1994), 925–929.

13. J.L. Doob, *Classical Potential Theory and Its Probabilistic Counterpart*, Springer, New York, 1984.

14. E.B. Fabes, N.Garofalo and S. Salsa, *A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations*, Illinois J. Math. 30 (1986), 536–565.

15. B. Fristedt and S. Orey, *The tail σ-field of one-dimensional diffusions*, Stochastic Analysis (A. Friedman and M. Pinsky, eds.), Academic Press, New York, 1978, pp. 127–138.

16. U. Küchler and U. Lunze, *On the tail σ-field and minimal parabolic functions for one-dimensional quasi-diffusions*, Z. Wahrschein. Verw. Gebiete 51 (1980), 303–322.

17. P.A. Meyer, R.T. Smythe and J.B. Walsh, *Birth and death of Markov processes*, Proc. 6-th Berkeley Symp. Math. Stat. Prob., vol. III, Univ. of California Press, Berkeley, CA, 1972, pp. 295–305.

18. R. Pinsky, *Positive Harmonic Functions and Diffusion*, Cambridge Univ. Press, Cambridge, 1995.

19. L.C.G. Rogers, *Coupling and the tail σ-field of a one-dimensional diffusion*, Stochastic calculus in application (J.R. Norris, ed.), Pitman Res. Notes Math., vol. 197, Longman Sci. Tech., Harlow, England, 1988, pp. 78–88.

20. U. Rösler, *The tail σ-field of a time-homogeneous one-dimensional diffusion process*, Ann. Prob. 7 (1979), 847–857.

21. J.-M. G. Wu, *Comparison of kernel functions, boundary Harnack principle, and relative Fatou theorem on Lipschitz domains*, Ann. Inst. Fourier Grenoble 28 (1978), 147–167.

22. J. Xu, *The lifetime of conditioned Brownian motion in domains of infinite area*, Prob. Th. Rel. Fields 87 (1991), 469–487.

23. B. Zhang, *On the variances of occupation times of conditioned Brownian motion*, Trans. Amer. Math. Soc. 348 (1996), 173–185.
98195-4350

E-mail address: burdzy@math.washington.edu

Department of Mathematics and Statistics, York University, North York, Ontario, Canada M3J 1P3
E-mail address: salt@nexus.yorku.ca