ON TRANSFORMATIONS OF POTAPOV’s
FUNDAMENTAL MATRIX INEQUALITY

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According to V.P. Potapov, a classical interpolation problem can be reformulated in terms of a so-called Fundamental Matrix Inequality (FMI). To show that every solution of the FMI satisfies the interpolation problem, we usually have to transform the FMI in some special way. In this paper the number of transformations of the FMI which come into play are motivated and demonstrated by simple, but typical examples.

0. PREFACE

V.P. Potapov’s approach to classical interpolation problems research consists in the following. Instead of original interpolation problem (or problem on integral representation), an inequality for analytic functions is considered in an appropriate domain. This inequality is said to be the Fundamental Matrix Inequality (FMI) for the considered interpolation problem. Here two problems appear. The first problem is how to “solve” this inequality. The second problem is to prove that this inequality is equivalent to the original interpolation problem.

The study of the second problem consist of two parts. First, we have to prove that any function which is a solution of the original problem is also a solution of the FMI. Usually this part is not difficult. Secondly, we have to extract the full interpolation information from the FMI. This means that we have to prove that any analytic function which satisfies the FMI is also a solution of of the original interpolation problem. In simple situations it is not difficult to obtain the interpolation information from the FMI. However, in the general case this is not easy, and we have to apply a special transformation to the FMI. Such a transformation can be applied to every FMI. However, in the simplest situations it is possible to do without such a transformation. The development of Potapov’s method began with consideration of the simplest interpolation problem, i.e. the Nevanlinna-Pick (NP) problem. The equivalence of the NP problem to its FMI is clear. Because of this, this transform was camouflaged in the beginning of the theory. However, by the study of of the power moment problem we already can not do without it. In the paper [KKY] such a transform was used in the very general setting of the so called Abstract Interpolation Problem. Namely, such a transformation was used in considerations related to Theorem 1 of this paper. Of course, the authors of [KKY] took into account the experience which was accumulated by previous work with concrete problems. However, this transformation...
was introduced in [KKY] in a formal way, without any motivation. As result, the proof of Theorem 1 of [KKY] looks like a trick. This is not satisfactory, because the transformation of FMI lies at the heart of the FMI business. The main goal of the present paper is to motivate and to demonstrate the transformation of the FMI by the simplest but typical example of the power moment problem. For contrast, the \( \mathcal{NP} \) problem and the FMI for it are considered as well. We would like to demonstrate the algebraic side of the matter. Therefore, we will avoid the entourage of general vector spaces and Hilbert spaces in the generality of the paper [KKY]. All our spaces are finite-dimensional. Instead of abstract kernels and operators, we will consider matrices.

1. THE FMI AND ITS STRUCTURE

Classical interpolation problems can be considered for various function classes in various domains. Here we consider two function classes related to the unit disc \( \mathbb{D} \) and to the upper half plane \( \mathbb{H} \).

DEFINITION 1.1. I. The class \( \mathcal{C}(\mathbb{D}) \) is the class of functions \( w \) which are holomorphic outside the unit circle \( \mathbb{T} \), satisfy the symmetry condition

\[ w(z) = -w^*(1/z) \quad (z \in \mathbb{C} \setminus \mathbb{T}) \]  
and the positivity condition

\[ \frac{w(z) + w^*(z)}{1 - |z|^2} \geq 0 \quad (z \in \mathbb{C} \setminus \mathbb{T}). \]  

II. The class \( \mathcal{R}(\mathbb{H}) \) is the class of functions \( w \) which are holomorphic outside the real axes \( \mathbb{R} \) and satisfies the symmetry condition

\[ w(z) = w^*(\overline{z}) \quad (z \in \mathbb{C} \setminus \mathbb{R}) \]  
and the positivity condition

\[ \frac{w(z) - w^*(z)}{z - \overline{z}} \geq 0 \quad (z \in \mathbb{C} \setminus \mathbb{R}). \]  

III. The class \( \mathcal{R}_0(\mathbb{H}) \) is the subclass of the class \( \mathcal{R}(\mathbb{H}) \) which is singled out by the condition

\[ \lim_{y \to \infty} y |w(iy)| < \infty. \]  

The FMI of a classical interpolation problem has the form

\[
\begin{bmatrix}
A & B_w(z) \\
B_w^*(z) & C_w(z)
\end{bmatrix} \geq 0,
\]

where \( A \) is some hermitian matrix, constructed from the interpolation data (interpolation points and interpolating values) only. It is nonnegative if and only if the considered interpolation problem is solvable. The entry \( C_w(z) \) contains the function \( w \) only, but not the
interpolation data. Its form depend on the function class to which the function $w$ belongs. For an interpolation problem in the class $C(D)$ the entry $C_w(z)$ has the form

$$C_w(z) = \frac{w(z) + w^*(z)}{1 - |z|^2}. \quad (1.7)$$

For an interpolation problem in the class $R(H)$ the entry $C_w(z)$ has the form

$$C_w(z) = \frac{w(z) - w^*(z)}{z - \overline{z}}. \quad (1.8)$$

In the entry $B_w(z)$ both the interpolation data and the function $w$ are combined. This entry looks like

$$B_w(z) = (zI - T)^{-1}(u \cdot w(z) - v), \quad (1.9)$$

or like

$$B_w(z) = T(I - zT)^{-1}(u \cdot w(z) - v) \quad (1.10)$$

To each classical interpolation problem the following objects are related:

1. The hermitian matrix $A$, which is nonnegative iff the problem is solvable.
2. The matrix $T$ which “determines” the interpolation nodes.
3. The vectors $u$ and $v$ which determine the interpolation values.

The terms $A, T, u, v$ satisfy the so called Fundamental Identity (FI). The form of the FI depends on the function class in which the interpolation problem is considered. For the function class $C(D)$, FI has the form

$$A - TAT^* = uv^* + vu^*. \quad (1.11)$$

For the class $R(H)$, FI has the form

$$TA - AT^* = uv^* - vu^*. \quad (1.12)$$

If the FMI (1.6) is satisfied (for some $z$), and if $M$ is a matrix of an appropriate size, then the inequality

$$M \begin{bmatrix} A & B_w(z) \\ B_w^*(z) & C_w(z) \end{bmatrix} M^* \geq 0 \quad (1.13)$$

holds as well. If the matrix $M$ is invertible, then both the inequalities (1.6) and (1.13) are equivalent.

2. FMI FOR THE NEVANLINNA – PICK PROBLEM.

Now we obtain the FMI for the $NP$ problem in the function class $C(D)$.

DEFINITION 2.1. Given $n$ points $z_1, z_2, \ldots, z_n$ in the unit disc $D$ (interpolation nodes) and $n$ complex numbers $w_1, w_2, \ldots, w_n$ (interpolation values). A holomorphic function $w(z)$ from the class $C(D)$ is said to be a solution of the Nevanlinna – Pick problem with interpolation data \{z_1, w_1\}, \{z_2, w_2\}, \ldots, \{z_n, w_n\}, if the interpolation conditions

$$w(z_k) = w_k \quad (k = 1, 2, \ldots, n) \quad (2.1)$$
are satisfied.

Let us associate with the \(NP\) problem two \(n \times 1\) vectors, which characterize the interpolation values:

\[
  u = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}.
\]

(2.2)

The matrix \(T\), which characterize the interpolation nodes, has the form

\[
  T = \text{diag } [z_1, z_2, \cdots, z_n].
\]

(2.3)

The matrix \(A\), the so called Pick matrix for the problem, has the form

\[
  A = \|a_{kl}\|_{1 \leq k,l \leq n}, \quad a_{kl} = \frac{w_k + \bar{w}_l}{1 - z_k \bar{z}_l}.
\]

(2.4)

The Fundamental Identity (1.11) for this choice of \(u, v, T,\) and \(A\) can be checked directly.

The Fundamental Matrix Inequality for the Nevanlinna-Pick problem (FMI(\(NP\))) has the form (1.6) with \(A\) from (2.4), \(C_w(z)\) from (1.7) and \(B_w(z)\) from (1.9), (2.2) (2.3).

THEOREM 2.1. (From FMI(\(NP\)) to interpolation conditions.) Let \(w(z)\) be a function which is holomorphic in the unit disc \(D\) and which satisfies the FMI(\(NP\)) for every \(z \in D\). Then the function \(w\) satisfies the condition \(w(z) + w^*(z) \geq 0 (z \in D)\) and the interpolation conditions (2.1).

PROOF. Since the entry \(C_w(z)\) must be nonnegative for \(z \in D\), the real part of the function \(w\) is nonnegative in \(D\). Now we take into account the concrete form of the entry \(B_w(z)\):

\[
  B_w(z) = \begin{bmatrix} b_{1,w}(z) \\ b_{2,w}(z) \\ \vdots \\ b_{n,w}(z) \end{bmatrix},
\]

(2.5)

where

\[
  b_{k,w}(z) = \frac{w(z) - w_k}{z - z_k} \quad (k = 1, 2, \cdots, n).
\]

(2.6)

Because the “full” matrix (1.6) is nonnegative, its appropriate submatrices are nonnegative all the more:

\[
  \begin{bmatrix} a_{kk} & b_{w,k}(z) \\ b_{w,k}^*(z) & C_w(z) \end{bmatrix} \geq 0,
\]

(2.7)

Since the function \(w\) is holomorphic in \(D\), the entry \(C_w(z)\), (1.7), is locally bounded in \(D\). Thus, from (2.7) it follows, that the entry \(b_{w,k}(z)\) is locally bounded in \(D\) as well. However, if function \(b_k\) is bounded even near the point \(z_k\), then the interpolation conditions

\[\text{If we continue the function } w, \text{ which is defined originally in } \mathbb{D} \text{ only, into the exterior of the unit circle according to the symmetry (1.1), then the function which is continued in this way will satisfy the condition (1.2).}\]
(2.1) are satisfied.

Thus, for the \( \mathcal{NP} \) interpolation problem it is not difficult to extract the interpolation information from its FMI.

It is worth mentioning, that the inequality (2.7) can be consider as an inequality of the form (1.13), with

\[
M = \begin{bmatrix}
0 & 0 & \cdots & 1 & \cdots & 0 & \vdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & \vdots & 1 \\
\end{bmatrix}.
\]  

(2.8)

3. DERIVATION OF THE FMI (\( \mathcal{NP} \))

A crucial role in deriving of the FMI for the \( \mathcal{NP} \) problem is played by the Riesz-Herglotz theorem. Given a nonnegative measure \( \sigma \) and a real number \( c \), we associate with them the function \( w_{\sigma,c} \):

\[
w_{\sigma,c}(z) = ic + \frac{1}{2} \int_{\mathcal{T}} \frac{t + z}{t - \bar{z}} d\sigma(t), \quad (z \in \mathbb{C} \setminus \mathbb{T}).
\]  

(3.1)

The function \( w_{\sigma,c} \) belongs to the class \( \mathcal{C}(\mathbb{D}) \).

THEOREM (RIESZ-HERGLOTZ). Let \( w \) be a function which belongs to the class \( \mathcal{C}(\mathbb{D}) \). Then this function \( w \) is of the form (3.1) for some \( \sigma \) and \( c \). Such \( \sigma \) and \( c \) are determined from the given \( w \) uniquely.

Let us start to derive the FMI(\( \mathcal{NP} \)). Given a measure \( \sigma \geq 0 \) on \( \mathbb{T} \), a real number \( c \) and points \( z_1, z_2, \ldots, z_n; z \in \mathbb{D} \). Let \( u \) be defined by (2.2), \( T \) be defined by (2.3). Then the following inequality \( (z_1, z_2, \ldots, z_n \text{ appear in } T) \) holds:

\[
\int_{\mathcal{T}} \left[ \begin{array}{c}
(tI - T)^{-1}u \\
\vdots \\
\frac{t}{\bar{t} - \bar{z}}^{-1}
\end{array} \right] \cdot d\sigma(t) \cdot \left[ \begin{array}{c}
u^*(\bar{t}I - T^*)^{-1} \\
\vdots \\
\frac{t}{t - z}
\end{array} \right] \geq 0.
\]  

(3.2)

This is a block-matrix inequality of the form

\[
\begin{bmatrix}
A_{\sigma} & B_{\sigma}(z) \\
B_{\sigma}(z) & C_{\sigma}(z)
\end{bmatrix} \geq 0.
\]  

(3.3)

We consider also the function \( w_{\sigma,c} \), (3.1), associated with \( \sigma \) and \( c \).

Now we will discuss the entries of the block-matrix on the right-hand side of the inequality (3.3). Originally these entries were defined by means of an integral representation. However, they can be expressed in terms of the function \( w_{\sigma,c} \). Let us consider the block \( A_{\sigma} \):

\[
A_{\sigma} = \int_{\mathcal{T}} (tI - T)^{-1}u \cdot d\sigma(t) \cdot u^*(\bar{t}I - T^*)^{-1},
\]  

(3.4)
or, for the entries $A_\sigma = \|a_{\sigma,kl}\|_{1 \leq k,l \leq n}$:

$$a_{\sigma,kl} = \int_{\mathbb{T}} (t - z_k)^{-1} \cdot d\sigma(t) \cdot (\bar{t} - \bar{z}_l)^{-1}, \quad (1 \leq k, l \leq n).$$

According to the well known identity for the Schwarz kernel $\frac{1}{2} (t + z)(t - z)^{-1}$,

$$A_\sigma = \left\| \frac{w_{\sigma,c}(z_k) + w_{\sigma,c}(z)}{1 - z_k \bar{z}_l} \right\|_{1 \leq k,l \leq n}. \quad (3.5)$$

(The constant $c$ does not appear in (3.5).) The block $B_\sigma$ has the following form:

$$B_\sigma = \int_{\mathbb{T}} (tI - T)^{-1} u \cdot \frac{t}{t - z} \cdot d\sigma(t). \quad (3.6)$$

The block $B_\sigma$ (which does not depend on $c$) can be transformed in the following way. Integrating the identity

$$(tI - T)^{-1} \frac{t}{t - z} = (zI - T)^{-1} \cdot \frac{1}{2} \frac{t + z}{t - z} - (zI - T)^{-1} \cdot \frac{1}{2} \frac{tI + T}{tI - T}$$

with respect the measure $d\sigma$, we obtain:

$$B_\sigma = (zI - T)^{-1} \left( uw_{\sigma,c}(z) - v_{\sigma,c} \right), \quad (3.7)$$

where

$$v_{\sigma,c} = icu + \frac{1}{2} \int_{\mathbb{T}} \frac{tI + T}{tI - T} d\sigma(t). \quad (3.8)$$

It can be checked that

$$A_\sigma - TA_\sigma T^* = u \cdot v_{\sigma,c}^* - v \cdot u_{\sigma,c}^*. \quad (3.9)$$

According to (3.1) and to (2.3),

$$v_{\sigma,c} = \begin{bmatrix} w_{\sigma,c}(z_1) \\ w_{\sigma,c}(z_2) \\ \vdots \\ w_{\sigma,c}(z_n) \end{bmatrix}. \quad (3.10)$$

Of course,

$$C_\sigma(z) = \int_{\mathbb{T}} \frac{d\sigma(t)}{|t - z|^2} = \frac{w_{\sigma,c}(z) + \overline{w_{\sigma,c}(z)}}{1 - |z|^2}. \quad (3.11)$$

Now let the function $w_{\sigma,c}$ satisfy the interpolation conditions (2.1), i.e. let

$$w_{\sigma,c}(z_k) = w_k \quad (k = 1, 2, \ldots, n). \quad (3.12)$$

Comparing (3.5) and (2.4), we obtain that

$$A_\sigma = A. \quad (3.13)$$
From (3.10) and (2.2),
\[ v_{\sigma,c} = v. \]  
(3.14)

Comparing now (3.7) with (1.9), we obtain that
\[ B_\sigma(z) = B_{w_{\sigma,c}}(z). \]  
(3.15)

Of course, (3.11), \( C_\sigma(z) = C_{w_{\sigma,c}}(z) \). Thus, we obtain the following statement:

**LEMMA 3.1.** If the function \( w_{\sigma,c} \), defined by (3.1), satisfies the interpolation conditions (3.12), then the FMI (1.6) (with \( w \) replaced by \( w_{\sigma,c} \)) is satisfied for every \( z \in \mathbb{C} \setminus \mathbb{T} \), where \( A \) is defined by (2.4), \( B_w \) is defined by (1.9), (2.2), (2.3) and \( C_w \) is defined by (1.7).

According to the Riesz-Herglotz theorem, each function \( w \) from the considered class has the representation \( w = w_{\sigma,c} \). Thus, the following result holds:

**THEOREM 3.1.** (From interpolation conditions to FMI(NP)). Let interpolation data for NP problem be given. Let \( w \) be a function, which belongs to the class \( \mathcal{C}(\mathbb{D}) \). If the function \( w \) satisfies the interpolation conditions (2.1), then the FMI(NP) for this function (with \( A \) and \( v \) constructed from the given interpolation data) is satisfied for every \( z \in \mathbb{C} \setminus \mathbb{T} \).

We have stated this (well known) derivation of the FMI (NP) because the formulas (3.4) and (3.6) are a very convenient starting point to guess formulas for transformations of FMI.

4. THE HAMBURGER MOMENT PROBLEM AS A CLASSICAL INTERPOLATION PROBLEM

This problem can be considered as a classical interpolation problem in the class \( \mathfrak{R}(\mathbb{H}) \).

**FORMULATION OF THE HAMBURGER MOMENT PROBLEM.** The data of the Hamburger problem is a finite sequence \( s_0, s_1, \ldots, s_{2n-1}, s_{2n} \) of real numbers. A nonnegative measure \( \sigma \) on the real numbers is said to be a solution of the Hamburger moment problem (with these data), if its power moments
\[ s_k(\sigma) = \int_{\mathbb{R}} \lambda^k d\sigma(\lambda) \quad (k = 0, 1, \ldots, 2n - 1, 2n) \]  
(4.1)
exist and satisfy the moment conditions
\begin{align*}
\text{i). } s_k(\sigma) = s_k & \quad (k = 0, 1, \ldots, 2n - 1); \\
\text{ii). } s_{2n}(\sigma) & \leq s_{2n}.
\end{align*}  
(4.2)

Measures \( \sigma \) satisfying these moment conditions are sought.

At first glance the formulated moment problem does not look like an interpolation problem. However, this problem can be reformulated as a classical interpolation problem.

Namely, let \( \sigma \) be a nonnegative measure on \( \mathbb{R} \) which is finite: \( s_0(\sigma) < \infty \). We associate with this measure \( \sigma \) the function \( w_\sigma \):
\[ w_\sigma(z) = \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{\lambda - z} \quad (z \in \mathbb{C} \setminus \mathbb{R}) \]  
(4.3)
This function \( w_\sigma \) belongs to the class \( \mathfrak{R}_0(\mathbb{H}) \).
The following result is a version of the Riesz - Herglotz theorem for the upper half-plane.

**THEOREM (Nevanlinna).** Let \( w \) be a function from the class \( \mathcal{R}_0 (\mathbb{H}) \). Then this function \( w \) is representable in the form (4.3), with some finite nonnegative measure \( \sigma : \sigma \geq 0, s_0(\sigma) < \infty \). This measure \( \sigma \) is determined from the function \( w \) uniquely.

It turns out that if a measure \( \sigma \) solves the Hamburger moment problem (4.2), then the function \( w_\sigma \), associated with this measure \( \sigma \), satisfies some asymptotic relation. To obtain such a relation, we consider the functions \( w_{\sigma,k} \):

\[
w_{\sigma,k}(z) = \int_{\mathbb{R}} \frac{\lambda^k \, d\sigma(\lambda)}{\lambda - z} \quad (k = 0, 1, 2, \ldots, 2n).
\]

(In this notation, \( w_\sigma = w_{\sigma,0} \)). Assume that a measure \( \sigma \geq 0 \) on \( \mathbb{R} \) has the moment \( s_{2n}(\sigma) \) (and hence, also the moments \( s_0(\sigma), \ldots, s_{2n-1}(\sigma) \)). Integrating the identity

\[
\frac{\lambda^k}{\lambda - z} = \frac{z^k}{\lambda - z} + \sum_{0 \leq j \leq k-1} z^{k-1-j} \lambda^j
\]

with respect to the measure \( \sigma \), we come to the equality

\[
w_{\sigma,k}(z) = z^k \left( w_\sigma(z) + \sum_{0 \leq j \leq k-1} \frac{s_j(\sigma)}{z^{j+1}} \right) \quad (k = 0, 1, 2, \ldots, 2n).
\]

Since

\[
w_{\sigma,2n}(z) = -\frac{s_{2n}(\sigma)}{z}(1 + o(1)) \quad (|z| \to \infty, \ z = iy),
\]

it follows from (4.6) (with \( k = 2n \)) that

\[
z^{2n} \left( w_\sigma(z) + \sum_{0 \leq j \leq 2n-1} \frac{s_j(\sigma)}{z^{j+1}} \right) = -\frac{s_{2n}(\sigma)}{z}(1 + o(1)) \quad (|z| \to \infty, \ z = iy).
\]

The asymptotic relation (4.8), together with (4.2),(4.6) suggests the following:

Given the function \( w \) of the class \( \mathcal{R}(H) \) and a set of real numbers \( s_0, s_1, \ldots, s_{2n-1} \), it has to be profitable to consider the functions \( b_{w,k}(z) = b_{w,k}(z; s_0, s_1, \ldots, s_{k-1}) \):

\[
b_{w,k}(z) = z^k w(z) + \sum_{0 \leq j \leq k-1} z^{k-1-j} s_j \quad (k = 0, 1, 2, \ldots, 2n)
\]

and the asymptotic relation of the form

\[
|b_{w,k}(z)| = O(|z|^{-1}) \quad (|z| \to \infty, \ z = iy).
\]

In this notation the equality (4.6) means that

\[
w_{\sigma,k}(z) = b_{w_{\sigma,k}}(z; s_0(\sigma), \ldots, s_{k-1}(\sigma))
\]

From (4.8) and (4.11) it follows that:
If a measure $\sigma \geq 0$ on $\mathbb{R}$ satisfies the moment conditions (4.2), then the asymptotic relation
\[
|b_{w,2n}(z; s_0, \ldots, s_{2n-1})| \leq \frac{s_{2n}}{|z|} (1 + o(1)) \quad (|z| \to \infty, \ z = iy)
\] (4.12)
holds.

It is remarkable that the last statement can be inverted.

THEOREM (Hamburger). Let $w$ be a function which belongs to the class $\mathcal{R}(\mathbb{H})$ and let $s_0, s_1, \ldots, s_{2n-1}$ be real numbers. Assume that the function $w$ satisfies the asymptotic condition
\[
|b_{w,2n}(z; s_0, \ldots, s_{2n-1})| = O(|z|^{-1}) \quad (|z| \to \infty, \ z = iy)
\] (4.13)
(where $b_{w,2n}$ is defined in (4.9)). Then the function $w$ has the representation of the form (4.3), with a nonnegative measure $\sigma$, which has $2n$-th moment: $s_{2n}(\sigma) < \infty$. Moreover,
\[
s_0(\sigma) = s_0, s_1(\sigma) = s_1, \ldots, s_{2n-1}(\sigma) = s_{2n-1}, \quad (4.14)
\]
\[
s_{2n}(\sigma) = \lim_{|z| \to \infty} (-z)b_{w,2n}(z; s_0, s_1, \ldots, s_{2n-1}) \quad (4.15)
\]

This theorem was proved by Hamburger ([H], Theorem IX). It is reproduced in the monograph by N. Akhiezer ([A], Theorem 2.3.1). The proof which was presented by Hamburger is based on a “step by step” algorithm. Another proof of this theorem, and its far reaching generalizations, is presented in [K1].

Thus the Hamburger moment problem can be reformulated as the following interpolation problem:

**Function class:** the class $\mathcal{R}(\mathbb{H})$.

**Interpolation data:** a finite sequence $s_0, s_1, \ldots, s_{2n}$ of real numbers.

The asymptotic relation
\[
|z^{2n} \left( w(z) + \sum_{0 \leq j \leq 2n-1} \frac{s_j}{z^{j+1}} \right) | \leq \frac{s_{2n}}{|z|} (1 + o(1)) \quad (|z| \to \infty, \ z = iy)
\] (4.16)

is considered as an interpolation condition. (The point $z = \infty$ is a multiple interpolation node which lies on the boundary of the upper half-plane $\mathbb{H}$. Its multiplicity\(^2\) equals $2n$). We seek functions $w$ from this class which satisfy the condition (4.16).

**REMARK 4.1. i).** Assume that a function $w$ from the class $\mathcal{R}(\mathbb{H})$ satisfies the condition (4.13). Suppose that we also know (for example, from the Hamburger theorem), that $w = w_\sigma$, where $s_{2n}(\sigma) < \infty$. Then we can construct the function $w_{\sigma,2n}$ by (4.6). Comparing the asymptotics (4.13) and (4.7), we conclude, that $b_{w,2n} = w_{\sigma,2n}$. Hence, the moment condition (4.2.i) is satisfied, as well as the condition
\[
|z^{2n} \left( w(z) + \sum_{0 \leq j \leq 2n-1} \frac{s_j}{z^{j+1}} \right) | \leq \frac{s_{2n}(\sigma)}{|z|} (1 + o(1)) \quad (|z| \to \infty, \ z = iy).
\] (4.17)

\(^2\) Strictly speaking, the considered problem has two interpolation nodes which are symmetric with respect to the real axis and are located at the points $+i \cdot \infty$ and $-i \cdot \infty$. The multiplicity of each of them equals $n$. 
Moreover, the function \( b_{w,2n}(z; s_0, \ldots, s_{2n-1}) \) belongs to the class \( \mathcal{R}_0(\mathbb{H}) \). (If \( d\sigma(\lambda) \) is a measure which represents \( w \), then the measure \( \lambda^{2n}d\sigma(\lambda) \) represents the function \( b_{w,2n} \).

\[ ii) \text{ Assume now that the function } b_{w,2n}(z; s_0, \ldots, s_{2n-1}) \text{ belongs to the class } \mathcal{R}_0(\mathbb{H}) \text{. Then, by the Nevanlinna'sn theorem, the function } b_{w,2n} \text{ has the form } w, \text{ for some } d\tau \geq 0, s_0(\tau) < \infty. \text{ Thus,} \]

\[
\int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - z} = z^{2n} \int_{\mathbb{R}} \frac{d\sigma(\lambda)}{\lambda - z} + \sum_{0 \leq j \leq 2n-1} s_j z^{2n-1-j}
\]

Applying the generalized Stieltjes inversion formula ([KaKr], §2), we conclude that \( d\tau(\lambda) = \lambda^{2n}d\sigma(\lambda) \). Hence, \( \int_{\mathbb{R}} \lambda^{2n}d\sigma(\lambda) = \int_{\mathbb{R}} d\tau(\lambda) < \infty \). Thus, \( b_{w,2n} = w_{\sigma,2n} \); and (4.17) is satisfied.

### 5. Derivation of the FMI (\( \mathcal{H} \))

Given the Hamburger moment problem with data \( s_0, s_1, \ldots, s_{2n} \), we associate with this problem the Pick matrix

\[
A = \begin{bmatrix}
    s_0 & s_1 & \cdots & s_n \\
    s_1 & s_2 & \cdots & s_{n+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    s_{n-1} & s_n & \cdots & s_{2n-1} \\
    s_n & s_{n+1} & \cdots & s_{2n}
\end{bmatrix}, \quad (5.1)
\]

and the vectors of the interpolation data

\[
u = \begin{bmatrix}
    1 \\
    0 \\
    \vdots \\
    0
\end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix}
    0 \\
    -s_0 \\
    \vdots \\
    -s_{n-1}
\end{bmatrix}. \quad (5.2)
\]

The matrix, which is responsible for interpolation knots (with multiplicity) is:

\[
T = \begin{bmatrix}
    0 & 0 & \cdots & 0 & 0 & 0 \\
    1 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 1 & \cdots & 0 & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & \cdots & 1 & 0 & 0 \\
    0 & 0 & \cdots & 0 & 1 & 0 \\
\end{bmatrix}^{(n+1)}. \quad (5.3)
\]

The Fundamental Identity (1.12) for this choice of \( u, v, T \) and \( A \) can be checked straightforwardly.

Now we derive th Fundamental Matrix Inequality for the Hamburger Moment Problem (FMI (\( \mathcal{H} \))). Let \( d\sigma(\lambda) \) be a nonnegative measure on \( \mathbb{R} \) for which the \( 2n \) th moment is finite: \( s_{2n}(\sigma) < \infty \). The following inequality is clear:

\[
\int_{\mathbb{R}} \begin{bmatrix}
    (I - \lambda T)^{-1}u \\
    \vdots \\
    (\bar{\lambda} - \bar{z})^{-1}
\end{bmatrix} \cdot d\sigma(\lambda) \cdot \begin{bmatrix}
    u^*(I - \bar{\lambda} T^*)^{-1} \\
    \vdots \\
    (\bar{\lambda} - z)^{-1}
\end{bmatrix} \geq 0. \quad (5.4)
\]
This inequality has the form

\[
\begin{bmatrix}
A_\sigma & B_\sigma(z) \\
B_\sigma^*(z) & w_\sigma(z) - w_\sigma^*(z)
\end{bmatrix} \geq 0,
\]

where the function \(w_\sigma\) is defined by (4.3). It is clear that

\[
A_\sigma = \int_\mathbb{R} \frac{(I - \lambda T)^{-1} u \cdot d\sigma(\lambda)}{\lambda - z} u^* (I - \lambda T^*)^{-1},
\]

(5.6)

where \(A_\sigma = \|a_{\sigma,kl}\|_{0 \leq k, l \leq n}, \quad a_{\sigma,kl} = s_{k+1}(\sigma) \quad (0 \leq k, l \leq n).\)

(5.7)

It is also clear, that

\[
B_\sigma(z) = \int_\mathbb{R} \frac{(I - \lambda T)^{-1} u}{\lambda - z} d\sigma(\lambda).
\]

(5.8)

Since

\[
\frac{(I - \lambda T)^{-1}}{\lambda - z} = (I - zT)^{-1} \left( \frac{1}{\lambda - z} + T(I - \lambda T)^{-1} \right),
\]

(5.9)

it follows that

\[
B_\sigma(z) = (I - zT)^{-1} (u \cdot w_\sigma(z) - v_\sigma),
\]

(5.10)

where

\[
v_\sigma = - \int_\mathbb{R} T(I - \lambda T)^{-1} u d\sigma(\lambda).
\]

(5.11)

From the concrete expressions (5.2) and (5.3) for \(u\) and \(T\) it is not difficult to see that

\[
v_\sigma = \begin{bmatrix}
0 \\
-s_0(\sigma) \\
\vdots \\
-s_{n-2}(\sigma) \\
-s_{n-1}(\sigma)
\end{bmatrix}.
\]

(5.12)

Assume now, that the measure \(\sigma\) satisfies the moment conditions (4.2). Then, according to (5.2) and (5.12), \(v_\sigma = v\), and according to (5.1) and (5.7), \(a_{\sigma,kl} = a_{kl} \quad (0 \leq k + l < 2n, \quad a_{\sigma,nn} \leq a_{nn}, \quad \text{hence}, \quad A_\sigma \leq A). \) Thus, we obtain

**THEOREM 5.1.** (From the moment conditions to the FMI \((\mathcal{H})\)). Let interpolation data for the Hamburger moment problem be given. Let \(w\) be a function of the form (4.3), where the measure \(\sigma\) satisfies the moment conditions (4.2) (or, what is the same according to Hamburger, the interpolation condition (4.16) is satisfied). Then the FMI\((\mathcal{H})\) (1.6) holds for this function \(w\) at every point \(z \in \mathbb{C} \setminus \mathbb{R}\), where \(A\) is defined by (5.1), \(C_w\) is defined by (1.8) and \(B_w\) is defined by (1.10), (5.2), (5.3).
6. TRANSFORMATION OF THE FMI ($\mathcal{H}$)

Let $s_0, \ldots, s_{2n}$ be interpolation data for the Hamburger moment problem. Then the Pick matrix $A$ is defined by (5.1), the interpolation nodes matrix $T$ be defined by (5.3) and interpolation values vectors $u$ and $v$ are defined by (5.2). Given a function $w$, which is holomorphic in $\mathbb{C} \setminus \mathbb{R}$ and satisfies the symmetry conditions (1.3), assume that the \text{FMI (H)}

\[
\begin{bmatrix}
A & B_w(z) \\
- & - \\
B^*_w(z) & w(z) - w^*(z) / z - z
\end{bmatrix} \geq 0
\]

(6.1)
is satisfied for every $z \in \mathbb{C} \setminus \mathbb{R}$. Here $B_w$ is defined by (1.10), (5.2), (5.3), or in detail,

\[
B_w(z) = \begin{bmatrix}
0 \\
\vdots \\
b_{w, n-1}(z)
\end{bmatrix}
\]

(6.2)

Our goal is to extract interpolation information from this FMI. Of course, from (6.1) it follows, that the function $w$ satisfies the positivity condition (1.4). Proceeding in the same way, as in the Proof of Theorem 2.1, we have to consider the “subinequalities” (2.7) of the inequality (6.1). The most information which we can obtain in this way from (6.1) is contained in the subinequality

\[
\begin{bmatrix}
s_{2n} \\
b^*_{w, n-1} \\
b_{w, n-1}
\end{bmatrix} \frac{w(z) - w^*(z)}{z - z} \geq 0.
\]

(6.3)

First and foremost, from (6.3) we obtain the estimate (1.5) for $w$. By the Nevanlinna Theorem, the function $w$ has the form $w_\sigma$ for some nonnegative measure $\sigma$ with $s_0(\sigma) < \infty$. Moreover, the estimate $|b_{w, n-1}(iy)| = O(|y|^{-1})$ as $y \uparrow \infty$ follows from (6.3). This is not enough since the function $b_{w, n-1}$ contains the interpolation data $s_0, s_1, \ldots, s_{n-1}$ only, and does not contain the data $s_n, s_{n+1}, \ldots, s_{2n-1}$ at all. We need to obtain the condition (4.16) from (6.1). Clearly, it is impossible to extract the condition (4.16) by considering “subinequalities” of the inequality (6.1). More generally, it is impossible to obtain (4.16) from any inequality of the form (1.13) when the framing matrix $M$ does not depend on $z$ because the data $s_n, s_{n+1}, \ldots, s_{2n-1}$ appear in the block $A$ only, which does not depend on $z$.

Therefore, in order to extract (4.16) from (6.1) (if it is at all possible), we have to choose a matrix $M$ in (1.13), which depends on $z$. To understand how to do this we return to the derivation of the FMI (H). Let us consider the inequality (5.5). It contains the functions $w_{\sigma,k} = b_{\sigma,k}$ with $k = 0, 1, \ldots, n - 1$ only. However, we need the function $w_{\sigma,2n-1}$. The only information which is available for us is the block $A_\sigma$, which is defined by (5.6) and (5.7). The Hankel matrix $A_\sigma$ is related to the Hankel matrix

\[
W_\sigma(z) = \|w_{\sigma,kl}(z)\|_{0 \leq k, l \leq n}.
\]

(6.4)
with entries

$$w_{\sigma,kl}(z) = \int_{\mathbb{R}} \lambda^k \cdot \frac{d\sigma(\lambda)}{\lambda - z} \cdot \lambda^l \quad (0 \leq k, l \leq n).$$

(6.5)

$k, l$-entries of the matrix $W_{\sigma}$ with $k + l < n$ are the same functions which appear in the column $B_\sigma$. The entries with $n \leq k + l \leq 2n$ are exactly those which we need. Thus, the problem is to obtain the matrix $W_{\sigma}$ from the matrix $A_\sigma$. According to (6.5), (5.2) and (5.3),

$$W_{\sigma}(z) = \int_{\mathbb{R}} (I - \lambda T)^{-1} u \cdot \frac{d\sigma(\lambda)}{\lambda - z} \cdot u^*(I - \lambda T^*).$$

(6.6)

Comparing (6.6) with (5.6) we see that we have to replace $(I - \lambda T)^{-1}$ with $(I - \lambda T)^{-1}$ in (5.6). Let us turn to the identity (5.9):

$$T(I - zT)^{-1} (I - \lambda T)^{-1} u = \frac{(I - \lambda T)^{-1}}{\lambda - z} u - \frac{(I - zT)^{-1}}{\lambda - z} u .$$

(6.7)

From (6.6) and (6.7) it follows that

$$T(I - zT)^{-1} A_\sigma = W_\sigma(z) - (I - zT)^{-1} u \cdot \int_{\mathbb{R}} d\sigma(\lambda) \frac{u^*(I - \lambda T^*)^{-1}}{\lambda - z} .$$

(6.8)

Taking into account (5.8), we obtain the equality

$$W_\sigma(z) = T((I - zT)^{-1}) A_\sigma + (I - zT)^{-1} u \cdot B_\sigma^*(\bar{z}) .$$

(6.9)

The equality (6.9) provide us a heuristic reason for the following

DEFINITION 6.1. Given a Hermitian matrix $A$, a matrix $T$ and vectors $u$ and $v$, which satisfy the Fundamental Identity (1.12), we associate with each function $w$, which is holomorphic in $C \setminus \mathbb{R}$ and satisfies the symmetry condition (1.3), the function $W_w$:

$$W_w(z) = T((I - zT)^{-1}) A + (I - zT)^{-1} u \cdot B_w^*(\bar{z}).$$

(6.10)

or, in detail,

$$W_w(z) = T(I - zT)^{-1} A - (I - zT)^{-1} u \cdot v^*(I - zT^*)^{-1}$$

$$+ (I - zT)^{-1} u \cdot w(z) \cdot u^*(I - zT^*)^{-1} .$$

(6.11)

LEMMA 6.1. The matrix function $W_w$ satisfies the same symmetry condition as that the function $w$:

$$W_w(z) = W^*_w(\bar{z}) \quad (z \in C \setminus \mathbb{R}).$$

(6.12)

Straightforward calculation gives us the explicit expression for $W_w(z)$:

$$W_w(z) = \|b_{w,k+l}(z)\|_{0 \leq k, l \leq n}$$

(6.13)
Thus, the matrix-function $W_w$ is exactly what we need: it contains the function $b_{w,2n}$. In particular, from the formula it follows that the matrix $W_w(z)$ is a Hankel matrix. However, the Hankel structure of the matrix $W_w(z)$ can be obtained in a less special way, i.e. by using the FI (1.12) only:

**Lemma 6.2.** The matrix $W_w(z)$ satisfies the following identity\(^3\):\n
$$TW_w(z) - W_w(z)T^* = u \cdot \varphi_w^*(z) - \varphi_w(z) \cdot u^*, \quad \text{where} \quad \varphi_w(z) = -T(I - zT)^{-1}(u \cdot w(z) - v).$$

(6.14)

**Lemma 6.3.** For the Hamburger moment problem, the function $w(z)$ and the column $B_w(z)$ can be recovered from the matrix-function $W_w(z)$ in the following way:

$$w(z) = e_0 \cdot W_w(z) \cdot e^*_0, \quad B_w(z) = W_w(z) \cdot e^*_0.$$ \hspace{1cm} (6.15)

where $e_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ is a $(n + 1) \times 1$ vector.

**Proof.** The formulas in (6.15) follow from the equalities

$$e_0 \cdot T = 0, \quad e_0 \cdot u = 1 \quad \text{and} \quad e_0 \cdot v = 0.$$ \hspace{1cm} (6.16)

**Remark 6.1.** The proof of the lemma depends on the equalities (6.16), not on the FI (1.12). It is specific for the problem in question.

Let us turn to the FMI (6.1). It is clear that the matrix $W_w(\bar{z})$ appears in the product

$$\left[ T(I - \bar{z}T)^{-1} : (I - \bar{z}T)^{-1}u \right] \cdot \begin{bmatrix} A & B_w(z) \\ \vdots & \vdots \\ B_w^*(z) & w(z) - w^*(z) \end{bmatrix}.$$ \hspace{1cm} (6.17)

In order to transform the FMI (6.1), we have to “frame” it according to (1.13), where now the matrix $M$ depends on $z$. It is clear that the row $\left[ T(I - \bar{z}T^{-1}) : (I - \bar{z}T)^{-1}u \right]$ ought to be one of the rows of the matrix $M(z)$. There are two main possibilities. Either the mentioned row is the first row of the matrix $M$:

$$M_1(z) = \begin{bmatrix} T(I - \bar{z}T)^{-1} & (I - \bar{z}T)^{-1}u \\ 0 & 1 \end{bmatrix},$$ \hspace{1cm} (6.18)

or the mentioned row is the second row of the matrix $M$:

$$M_2(z) = \begin{bmatrix} I & 0 \\ T(I - \bar{z}T)^{-1} & (I - \bar{z}T)^{-1}u \end{bmatrix},$$ \hspace{1cm} (6.19)

---

\(^3\) The equality (6.14), considered as an equation with respect to the matrix $W_w(z)$, can be used to calculate this matrix.
Upon performing the matrix multiplications, we obtain (after some calculations with the matrix entries):

\[
M_1(z) \cdot \begin{bmatrix}
A & B_w(z) \\
B_w^*(z) & w(z) - w^*(z)
\end{bmatrix} \cdot M_1^*(z) = \begin{bmatrix}
\frac{W_w(z) - W_w^*(z)}{z - \bar{z}} & \frac{B_w(z) - B_w(\bar{z})}{z - \bar{z}} \\
\frac{B_w^*(\bar{z}) - B_w^*(z)}{z - \bar{z}} & \frac{w(z) - w^*(z)}{z - \bar{z}}
\end{bmatrix}
\]  

(6.20)

and

\[
M_2(z) \cdot \begin{bmatrix}
A & B_w(z) \\
B_w^*(z) & w(z) - w^*(z)
\end{bmatrix} \cdot M_2^*(z) = \begin{bmatrix}
A & W_w(z) \\
W_w^*(z) & \frac{W_w(z) - W_w^*(z)}{z - \bar{z}}
\end{bmatrix}
\]  

(6.21)

The calculations with the matrix entries are based essentially on the following consequence of the FI (1.12):

**Lemma 6.4.** The identity

\[
T(I - zT)^{-1} \cdot A \cdot (I - \bar{z}T^*)^{-1}T^* =
\]

\[
= \frac{T(I - zT)^{-1} A - A(I - \bar{z}T^*)^{-1}T^*}{z - \bar{z}} - (I - zT)^{-1} \cdot \frac{w^* - v^*}{z - \bar{z}} \cdot (I - \bar{z}T^*)^{-1}
\]

(6.22)

holds.

### 7. USING OF THE TFMI (H) – FROM THE FMI (H) TO INTERPOLATION INFORMATION

We consider two kinds of Transformed Fundamental Matrix Inequalities (for the Hamburger problem): TFMI_I(H) and TFMI_II(H).

The TFMI_I(H) is of the form

\[
\begin{bmatrix}
\frac{W_w(z) - W_w^*(z)}{z - \bar{z}} & \frac{B_w(z) - B_w(\bar{z})}{z - \bar{z}} \\
\frac{B_w^*(\bar{z}) - B_w^*(z)}{z - \bar{z}} & \frac{w(z) - w^*(z)}{z - \bar{z}}
\end{bmatrix} \geq 0.
\]  

(7.1)

The TFMI_II(H) is of the form

\[
\begin{bmatrix}
A & W_w(z) \\
W_w^*(z) & \frac{W_w(z) - W_w^*(z)}{z - \bar{z}}
\end{bmatrix} \geq 0.
\]  

(7.2)

We see that both of the TFMI’s contain the function \(W_w(z)\). Now the problem of extracting interpolation information from the TFMI arises.
Now we will discuss the extent to which the FMI ($\mathcal{H}$) and the TFMI ($\mathcal{H}$) are equivalent. In view of (6.20) and (6.21), it is clear that

$$\text{FMI (}$\mathcal{H}$\text{)} \Rightarrow \text{TFMI}_I (\mathcal{H}) \quad (7.3)$$

and

$$\text{FMI (}$\mathcal{H}$\text{)} \Rightarrow \text{TFMI}_II (\mathcal{H}). \quad (7.4)$$

More formally:

**Lemma 7.1.** If the FMI ($\mathcal{H}$) is satisfied for some $z \in \mathbb{C} \setminus \mathbb{R}$, then both TFMI$_I (\mathcal{H})$ and TFMI$_II (\mathcal{H})$ are satisfied for the same $z$ as well.

The opposite implications (with respect to (7.3), (7.4)) may be false, because the matrices $M_1(z)$ and $M_2(z)$ are not invertible: $e_0^T T = 0$, and the matrix $M_2(z)$ is not even square. Actually,

$$\text{FMI (}$\mathcal{H}$\text{)} \not\Rightarrow \text{TFMI}_I (\mathcal{H}) \quad (7.5)$$

Indeed, the product in the left hand side does not contain the $nn$-th entry $s_{2n}$ of the matrix $A$ at all, and the positivity of the matrix $A$ (and hence, the positivity of the matrix of the FMI ($\mathcal{H}$)) depends essentially on this entry. However, the FMI ($\mathcal{H}$) and the TFMI$_I (\mathcal{H})$ are “almost equivalent”: the matrix $M_1(z)$ (6.9) is “almost invertible”. Since $T^* T = P$, where $P$ is a projector matrix: $P = \text{diag} [1, \ldots, 1, 0]$ ($p_{kk} = 1$, $k = 0, 1, \ldots, n - 1; p_{nn} = 0$), then

$$\begin{bmatrix} T^*(I - zT)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \cdot M_1(z) = \begin{bmatrix} P_{n-1} & 0 \\ 0 & 1 \end{bmatrix}. \quad (7.6)$$

Hence, the inequality, which is obtained from the inequality (6.1) by replacing$^4$ the matrix $A$ by the matrix $PA P$ and the column $B_w(z)$ by the column $PB_w(z)$, holds.

The inequalities FMI ($\mathcal{H}$) and TFMI$_II (\mathcal{H})$ are equivalent, because there exists a left inverse matrix to the matrix $M(z)$:

$$N(z) = \begin{bmatrix} I & 0 \\ 0 & e_0 (I - zT) \end{bmatrix}, \quad N(z) \cdot M_2(z) = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}. \quad (7.7)$$

Thus, we have proved that

$$\text{FMI (}$\mathcal{H}$\text{)} \iff \text{TFMI}_II (\mathcal{H}). \quad (7.8)$$

More formally:

**Lemma 7.2.** The inequality FMI ($\mathcal{H}$) is satisfied at some point $z \in \mathbb{C} \setminus \mathbb{R}$ if and only if the inequality TFMI$_II (\mathcal{H})$ is satisfied for the same $z$.

The matrix of the TFMI$_I (\mathcal{H})$ is invariant with respect to the change $z \rightarrow \bar{z}$. Thus:

If the inequality TFMI$_I (\mathcal{H})$ is satisfied at some point $z \in \mathbb{C} \setminus \mathbb{R}$, then it is satisfied also at the conjugate point $\bar{z}$.\n
$^4$The last inequality is nothing more than the FMI of the form (6.1), which is constructed from the “truncated” date $s_0$, $s_1$, ..., $s_{n-2}$. (The FMI (6.1) is constructed from the data $s_0$, $s_1$, ..., $s_{2n}$.)
The following statement is not so evident:

**Lemma 7.3.** If the FMI \((\mathcal{H})\) is satisfied at some point \(z \in \mathbb{C} \setminus \mathbb{R}\), then it is satisfied also at the conjugate point \(\bar{z}\).

**Proof.** The FMI \((\mathcal{H})\) can be written in the form

\[
\begin{pmatrix}
(I - zT)A(I - \bar{z}T^*) & u \cdot w(z) - v \\
w^* \cdot u^* - v^* & \frac{w(z) - w^*(z)}{z - \bar{z}}
\end{pmatrix} \geq 0.
\]

The claim of the lemma follows from the matrix identity

\[
\begin{pmatrix}
I & (\bar{z} - z)w \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
(I - zT)A(I - \bar{z}T^*) & u \cdot w - v \\
w^* \cdot u^* - v^* & \frac{w(z) - w^*(z)}{z - \bar{z}}
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
(z - \bar{z})w & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
(I - \bar{z}T)A(I - zT^*) & u \cdot w^* - v \\
w \cdot u^* - v^* & \frac{w(z) - w^*(z)}{z - \bar{z}}
\end{pmatrix}
\]

(7.9)

(where \(w\) is an arbitrary complex number; we have to put \(w = w(z)\), then \(w^* = w(\bar{z})\)).

To obtain the identity (7.9), we perform the matrix multiplication and use the identity

\[
(I - \bar{z}T)A(I - zT^*) - (z - \bar{z})(u \cdot v^* - v \cdot u^*) = (I - zT)A(I - \bar{z}T^*),
\]

(7.10)

which is equivalent to the Fundamental Identity (1.12).

Now we turn to the extraction of interpolation information from the FMI \((\mathcal{H})\).

**Theorem 7.1.** (From the FMI \((\mathcal{H})\) to the moment conditions). Let the interpolation data \(s_0, s_1, \ldots, s_{2n-1}, s_{2n}\) for the Hamburger moment problem be given. Let \(w\) be a function of the class \(R(\mathbb{H})\) and let the FMI \((\mathcal{H})\) (6.1) for this \(w\) be satisfied at every point \(z\) in the upper half plane. Then the function \(w\) is representable in the form \(w = w_\sigma\) for some (uniquely determined) measure \(\sigma\). This measure satisfies the moment conditions (4.2); the interpolation conditions (4.16) are satisfied as well.

**Proof.** According to Lemma 7.3, the FMI \((\mathcal{H})\) is satisfied for every \(z \in \mathbb{C} \setminus \mathbb{R}\). By Lemma 7.2, the TFMI \((\mathcal{H})\) is satisfied for every \(z \in \mathbb{C} \setminus \mathbb{R}\). First, from the TFMI \((\mathcal{H})\) we obtain the positivity condition

\[
\frac{W_w(z) - W_w^*(z)}{z - \bar{z}} \geq 0 \quad (\forall z \in \mathbb{C} \setminus \mathbb{R}).
\]

(7.11)

Secondly, we derive the estimate

\[
y W_w(iy) = O(1) \quad (\text{as } y \uparrow \infty).
\]

(7.12)

According to the matrix version of Nevanlinna’s theorem, the matrix function \(W_w(z)\) is representable in the form

\[
W_w(z) = \int_\mathbb{R} \frac{d\Sigma(\lambda)}{\lambda - z} \quad (\forall z \in \mathbb{C} \setminus \mathbb{R}),
\]

(7.13)
where \( d\Sigma(\lambda) \) is a nonnegative matrix-valued measure and the integral

\[
s_0(\Sigma) = \int d\Sigma(\lambda) \tag{7.14}
\]

exists in the proper sense. Moreover,

\[
\lim_{y \uparrow \infty} -iy W_w(iy) = s_0(\Sigma). \tag{7.15}
\]

From the TFMI\( \Pi(\mathcal{H}) \) (7.2) (for \( z = iy, y \to \infty \)) and from (7.15) it now follows, that

\[
A - s_0(\Sigma) \geq 0. \tag{7.16}
\]

Of course, the condition (1.5) for \( w \) (see (6.15)) follows from the inequality (6.15). Thus, \( w = w_\sigma \) for some \( \sigma : s_0(\sigma) < \infty \). Let us clarify the structure of the measure \( d\Sigma \). We can expect that

\[
d\Sigma(\lambda) = (I - \lambda T)^{-1} u \cdot d\sigma(\lambda) \cdot u^*(I - \lambda T^*)^{-1}. \tag{7.17}
\]

This is the case indeed. To prove (7.17), we turn to the formula (6.11). The functions \( (I - zT)^{-1} \) and \( (I - zT^*)^{-1} \) are holomorphic near the real axis (actually, these function are entire). Applying the generalized Stieltjes inversion formula ([KaKr], §2) to (6.11), we obtain (7.17). In particular (see (5.6) and (7.17)), the equality

\[
s_0(\Sigma) = A_\sigma \tag{7.18}
\]

holds. Now (7.16) takes the form

\[
A - A_\sigma \geq 0. \tag{7.19}
\]

The inequality (7.19) itself ensures the condition (4.2.ii), but it does not ensure the condition (4.2.i). However, we can also exploit the asymptotics (7.15). Taking into account the concrete structure (6.13) of the matrix-function \( W_w \), we see that the asymptotic (7.15) together with (4.2.ii) leads to the condition (4.16). From (4.16) of course follow the moment condition (4.2.i).

Another way to obtain these results is to multiply the equality (6.11) by \((I - zT)\) from the left and by \((I - zT^*)\) from the right and then upon comparing the asymptotics of both sides, we see that

\[
T(A - A_\sigma)T^* = 0. \tag{7.20}
\]

Thus, the nonnegative matrix \( A - A_\sigma \) vanishes at all vectors from the image of the matrix \( T \). The orthogonal complement to this image is generated by the \((n + 1) \times 1\) vector

\[
e_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.
\]

Hence,

\[
A = A_\sigma + \rho \cdot e_n^* e_n, \quad \text{where } \rho \text{ is a nonnegative number}. \tag{7.22}
\]

In view of (5.1) and (5.7), the representation (7.22) is equivalent to the moment conditions (4.2).
REMARK 7.1. To obtain the estimate for the function \( b_{w,2n} \), we could restrict ourselves to the subinequality of the inequality (7.2):

\[
\begin{bmatrix}
s_{2n} & b_{w,2n}(z) \\
b^*_{w,2n}(z) & b_{w,2n}(z) - b^*_{w,2n}(z)
\end{bmatrix}
\begin{bmatrix}
z - \bar{z}
\end{bmatrix} \geq 0.
\] (7.23)

We can obtain this inequality from the inequality (7.2), by “framing” it with the matrix

\[
\begin{bmatrix}
0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}.
\]

Combining this with (6.21), we obtain the following “truncated” transformation:

\[
m(z) \cdot \begin{bmatrix}
A & B_w(z) \\
B^*_w(z) & w(z) - w^*(z)/z - \bar{z}
\end{bmatrix} \cdot m^*(z) = \begin{bmatrix}
s_{2n} & b_{w,2n}(z) \\
b^*_{w,2n}(z) & b_{w,2n}(z) - b^*_{w,2n}(z)/z - \bar{z}
\end{bmatrix},
\] (7.24)

where

\[
m(z) = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
z^{n-1} & z^{n-2} & \cdots & 1 & 0 & \cdots & z^n
\end{bmatrix}.
\] (7.25)

A transformation of the FMI of approximately the form (7.25) appeared in the paper [Kov] by I.Kovalishina (see pages 460-461 of the Russian original or pages 424-425 of the English translation). (I.Kovalishina used a step by step algorithm, and did not introduce the matrix (7.25) explicitly, but it is possible to extract this matrix from her considerations.) Starting from\(^5\) [Kov], the author considered transformations of the FMI for various problems on integral representations, both discrete and continuous in [K2]. The nontruncated transformation FMI (\( \mathcal{H} \)) \( \rightarrow \) TFMI\( _{II} \) (\( \mathcal{H} \)) was considered by author in [K3]. Such a transformation was considered also by T.Ivanchenko and L.Sakhnovich [IS1], [IS2]. The nontruncated transformation FMI (\( \mathcal{H} \)) \( \rightarrow \) TFMI\( _{I} \) (\( \mathcal{H} \)) was considered (for other classes of functions and in different notation) in [KKY]. Systematic development of transformations of the FMI was also presented in the preprint [K4], but [K4] is not easily available.

**8. TRANSFORMATION OF FMI (\( \mathcal{NP} \)).**

It is very easy to extract interpolation information from the FMI (\( \mathcal{NP} \)). For this goal we need not transform the FMI. However, we have already learned that such transformations and related structures are objects which are interesting in themselves. Therefore, we will discuss transformations of the FMI (\( \mathcal{NP} \)). (We know, that to a large extent such transformations depend only on Fundamental Identity for the considered problem and not on the concret expression for the entries in this identity.) Thus, we consider a FMI of the form (1.6) with \( B_w \) and \( C_w \) of the forms (1.9), (2.2), (2.3) and (1.7), respectively, and we assume, that the Fundamental Identity (1.11) is satisfied.

\(^5\)The paper [Kov] was published in 1983 only, but author was aware of its content much earlier.
Let the function $w$ which appears in FMI ($\mathcal{NP}$) be of the form $w = w_{\sigma,c}$ as in (3.1). To guess formulas for transformations of the FMI, we first consider the matrix function

$$W_\sigma(z) = \int_\mathbb{T} (tI - T)^{-1} \cdot \frac{1}{2} \frac{t + z}{t - z} d\sigma(t) \cdot (iI - T^*)^{-1},$$

(8.1)

which is obtained by inserting the Schwarz kernel into the formula (3.4) for $A_\sigma$. We would like to obtain $W_\sigma$ from $A_\sigma$. For this goal we use the identity

$$\frac{1}{2} \frac{T + zI}{T - zI} (tI - T)^{-1} = \frac{1}{2} \frac{t + z}{t - z} (tI - T)^{-1} + \frac{z}{z - t} (zI - T)^{-1},$$

(8.2)

which was constructed with formulas (3.4) and (3.6) for $A_\sigma$ and $B_\sigma$ in mind. Now we multiply the identity (8.2) by $u \cdot d\sigma(t) \cdot u^*(iI - T^*)^{-1}$ and integrate over $\mathbb{T}$. Taking into account (3.4) and (3.6), we obtain

$$W_\sigma(z) = \frac{1}{2} \frac{T + zI}{T - zI} A_\sigma - (zI - T)^{-1} u \cdot B_{\sigma,c}^*(1/\bar{z}).$$

(8.3)

The last formula is a heuristic reason for the following

**DEFINITION 8.1.** Given a Hermitian matrix $A$, a matrix $T$ and vectors $u$ and $v$ which satisfy the FI (1.11), we associate with each function $w$, which is holomorphic in $\mathbb{C} \setminus \mathbb{T}$ and satisfies the symmetry condition (1.1), the function $W_w$:

$$W_w(z) = \frac{1}{2} \frac{T + zI}{T - zI} A - (zI - T)^{-1} u \cdot B_{w,c}^*(1/\bar{z}).$$

(8.4)

or, in detail,

$$W_w(z) = \frac{1}{2} \frac{T + zI}{T - zI} A + (zI - T)^{-1} u \cdot v^* (z^{-1} I - T^*)^{-1}$$

$$+ (zI - T)^{-1} u \cdot w(z) \cdot u^* (z^{-1} I - T^*)^{-1}.$$  (8.5)

Using the FI (1.11), we obtain also another representation for $W_w(z)$:

$$W_w(z) = \frac{1}{2} A \frac{I + zT}{I - zT} + B_w(z) \cdot u \frac{z}{I - zT^*},$$

(8.6)

or, in detail,

$$W_w(z) = \frac{1}{2} A \frac{I + zT}{I - zT} - (zI - T)^{-1} u \cdot v^* (z^{-1} I - T^*)^{-1}$$

$$+ (zI - T)^{-1} u \cdot w(z) \cdot u^* (z^{-1} I - T^*)^{-1}.$$  (8.7)

In other words:

**LEMMA 8.1.** The matrix-function $W_w$ satisfies the symmetry condition

$$W_w(z) = -W_w^*(1/\bar{z}) \quad (\forall z \in \mathbb{C} \setminus \mathbb{T}).$$

(8.8)
Using the FI (1.11), we obtain also the following result:

**LEMMA 8.2.** The matrix-function $W_w$ satisfies the identity

$$W_w(z) - T W_w(z) T^* = u \cdot \varphi_w^* (1/\bar{z}) - \varphi_w(z) \cdot u^*, \quad (8.9)$$

where

$$\varphi_w(z) = \frac{1}{2} \frac{T + zI}{T - zI} (u \cdot w(z) - v). \quad (8.10)$$

**REMARK 8.1.** For $z = 0$, the expression on the left hand side of (8.9) is equal to

$$\frac{1}{2} (A - TAT^*),$$

and the expression on the right hand side is equal to

$$\frac{1}{2} (u \cdot v^* + v \cdot u^*).$$

Thus, the formula (8.9) is in some sense an analytic continuation of the FI (1.11)

**REMARK 8.2.** The equality (8.9), considered as an equation with respect to the matrix $W_w(z)$, can be used to calculate this matrix.

Let us calculate the matrix $W_w(z)$ for the $NP$ problem with data given by (2.2) and (2.3). From the equation (8.9), we obtain the following formula:

$$W_w(z) = \frac{1}{2} \left[ \begin{array}{c} \frac{z_k + z}{z_k - z} (w_k - w(z)) + \frac{1 + z\bar{z}_l}{1 - z\bar{z}_l} (w(z) + w_1) \\ 1 - z_k \bar{z}_l \end{array} \right] \cdot \left[ \begin{array}{c} 1 \leq k, l \leq n \end{array} \right]. \quad (8.11)$$

Let us introduce the matrices

$$M_1(z) = \left[ \begin{array}{cc} (I - zT)^{-1} & \bar{z} (I - zT)^{-1} u \\ 0 & 1 \end{array} \right] \quad (8.12)$$

and

$$M_2(z) = \left[ \begin{array}{cc} I & 0 \\ (I - zT)^{-1} & \bar{z} (I - zT)^{-1} u \end{array} \right]. \quad (8.13)$$

Performing the matrix multiplication, we obtain (after some calculations with the entries):

$$M_1(z) \cdot \left[ \begin{array}{cc} A & B_w(z) \\ B_w^*(z) & w(z) - w^*(z) / z - \bar{z} \end{array} \right] \cdot M_1^*(z) = \left[ \begin{array}{cc} W_w(z) + W_w^*(z) / 1 - z\bar{z} & B_w(z) - B_w(1/\bar{z}) / 1 - z\bar{z} \\ B_w^*(z) - B_w^*(1/\bar{z}) / 1 - z\bar{z} & w(z) + w^*(z) / 1 - z\bar{z} \end{array} \right] \quad (8.14)$$

and

$$M_2(z) \cdot \left[ \begin{array}{cc} A & B_w(z) \\ B_w^*(z) & w(z) - w^*(z) / z - \bar{z} \end{array} \right] \cdot M_2^*(z) = \left[ \begin{array}{cc} A & W_w(z) + \frac{A}{2} \\ W_w^*(z) + \frac{A}{2} & W_w(z) + W_w^*(z) / 1 - z\bar{z} \end{array} \right]. \quad (8.15)$$
The calculations mentioned above are based essentially on the following consequence of the FI (1.11):

\[
(z - T)^{-1} A (\bar{z} - T^*)^{-1} = \frac{1}{1 - z\bar{z}} \begin{pmatrix} 1 & T + zI \\ \frac{1}{2} T^* + \bar{z}I \end{pmatrix} \begin{pmatrix} \frac{1}{2} T + zI & A + \frac{1}{2} A \frac{1}{2} A \end{pmatrix} + (zI - T)^{-1} \cdot \frac{u}{1 - z\bar{z}} \cdot (\bar{z}I - T^*)^{-1}.
\]

(8.16)

We consider two variants of the Transformed Fundamental Matrix Inequality (for the Nevanlinna-Pick problem): the TFMI\(_I(\mathcal{NP})\) and the TFMI\(_{II}(\mathcal{NP})\).

TFMI\(_I(\mathcal{NP})\) has the form

\[
\begin{bmatrix}
\frac{W_w(z) + W_w^*(z)}{1 - z\bar{z}} & \frac{B_w(z) - B_w(1/\bar{z})}{1 - z\bar{z}} \\
\frac{B_w^*(z) - B_w^*(1/\bar{z})}{1 - z\bar{z}} & \frac{w(z) + w^*(z)}{1 - z\bar{z}}
\end{bmatrix} \geq 0.
\]

(8.17)

TFMI\(_{II}(\mathcal{NP})\) has the form

\[
\begin{bmatrix}
\frac{A}{2} & \frac{W_w(z) + A}{2} \\
\frac{W_w^*(z) + A}{2} & \frac{W_w(z) + W_w^*(z)}{1 - z\bar{z}}
\end{bmatrix} \geq 0.
\]

(8.18)

We see that both of these TFMI's contain the function \(W_w(z)\).

**DEFINITION 8.2.** Given a \(\mathcal{NP}\) problem with interpolation nodes \(z_1, z_2, \ldots, z_n\) in the unit disc \(\mathbb{D}\), the point \(z \in \mathbb{C} \setminus \mathbb{T}\) is said to be nonsingular, if \(z \neq 0, \infty; z_1, z_2, \ldots, z_n; z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1}\).

If \(z\) is a nonsingular point, then the matrices \((zI - T)^{-1}, (I - \bar{z}T)^{-1}\) are defined (and, of course, invertible). (Strictly speaking, we can define the matrices \(W_w(z), M_1(z)\) and \(M_2(z)\) for nonsingular \(z\) only). For nonsingular \(z\), the matrix \(M_1(z)\) is invertible and the matrix \(M_2(z)\) has a left inverse.

**LEMMA 8.3.** Let \(z \in \mathbb{C} \setminus \mathbb{T}\) be a nonsingular point. Then the FMI(\(\mathcal{NP}\)) is satisfied at this point \(z\) if and only if each of two inequalities TFMI\(_I(\mathcal{H})\) and TFMI\(_{II}(\mathcal{H})\) is satisfied at this point.

**LEMMA 8.4.** Let \(z \in \mathbb{C} \setminus \mathbb{T}\) be a nonsingular point. Then the FMI(\(\mathcal{NP}\)) is satisfied at this point \(z\) if and only if it is satisfied at the “symmetric” point \(z^{-1}\) as well.

**PROOF.** The FMI(\(\mathcal{NP}\)) is equivalent to the inequality

\[
\begin{bmatrix}
(zI - T) A (\bar{z}I - T^*) & u \cdot w(z) - v \\
w^*(z) \cdot u^* - v^* & \frac{w_w(z) + w_w^*(z)}{1 - z\bar{z}}
\end{bmatrix} \geq 0.
\]

(8.19)
The claim of the lemma follows from the matrix identity

\[
\begin{bmatrix}
I & -(1 - \bar{z}z) u \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
(I - z^T) A (I - \bar{z}^T) & u \cdot w - v \\
\bar{w} \cdot u - v^* & \frac{w(z) + \bar{w}(z)}{1 - z\bar{z}}
\end{bmatrix}
\begin{bmatrix}
I \\
0 & -(1 - \bar{z}z) u^* \\
w \cdot u - v^* & \frac{w(z) + \bar{w}(z)}{1 - z\bar{z}}
\end{bmatrix}
= \\
\begin{bmatrix}
(I - \bar{z}^T) A (I - z^T) & u \cdot w^* - v \\
w \cdot u^* - v^* & \frac{w(z) + \bar{w}(z)}{1 - z\bar{z}}
\end{bmatrix},
\tag{8.20}
\]

where \( w = w(z) \) and \( w^* = -w(1/\bar{z}) \). To obtain the identity (8.20), we perform the matrix multiplication and use the identity

\[
(zI - T) A (\bar{z}I - T^*) - (1 - z\bar{z}) (u \cdot v^* + v \cdot u^*) = (I - \bar{z}T) A (I - zT^*),
\tag{8.21}
\]

which is equivalent to the FI (1.11).

**Lemma 8.5.** The TFMI\textsubscript{II}(\mathcal{NP}) (8.18) holds for every point \( z \in \mathbb{D} \) if and only if the function \( W_w(z) \) satisfies the positivity condition:

\[
W_w(z) + W_w^*(z) \geq 0 \quad (\forall z \in \mathbb{D}).
\tag{8.22}
\]

**Proof.** The implication TFMI\textsubscript{II} \( \Rightarrow \) (8.22) is evident. The opposite implication is nothing more that the Schwarz-Pick inequality for the function \( W_w(z) \) for the points: 0 and \( z \) (because \( W_w(0) = \frac{A}{z} \)).

From Lemmas 8.3 and 8.5 we obtain the following conclusion:

**Theorem 8.1.** A function \( w \), holomorphic in \( \mathbb{C} \setminus \mathbb{T} \) and satisfying the symmetry condition (1.1), satisfies the FMI (\mathcal{NP}) for all \( z \in \mathbb{D} \) (or, what the same for all \( z \in \mathbb{C} \setminus \mathbb{T} \)) if and only if the function \( W_w(z) \) which is defined by (8.4) satisfies the positivity condition (8.22).

Taking into account the concrete form (8.11) of the matrix \( W \) for the \( \mathcal{NP} \) problem, we obtain:

**Theorem 8.2.** Let the interpolation data for the \( \mathcal{NP} \) problem (2.1) in the function class \( \mathcal{C}(\mathbb{D}) \) be given by (2.2) and (2.3). A function \( w \), which is holomorphic in \( \mathbb{D} \), is a solution of the \( \mathcal{NP} \) problem (with these data) if and only if the real part of the matrix on the right hand side of (8.11) is nonnegative for every \( z \in \mathbb{D} \).

**Remark 8.3.** The matrix in (8.11) is an orthogonal projection of the operator \( \frac{1}{2} (I + zU) (I - zU)^{-1} \), where \( U \) is a generalised unitary extension of some isometric operator, related to the considered problem.

This is a consequence of the TFMI\textsubscript{II}(\mathcal{NP}). A consequence of the TFMI\textsubscript{I}(\mathcal{NP}) also may be interesting. The inequality (8.17) is equivalent to the inequality

\[
\begin{bmatrix}
W_w(z) + W_w^*(z) & B_w(z) - B_w(1/\bar{z}) \\
B_w^*(z) - B_w^*(1/\bar{z}) & w(z) + w^*(z)
\end{bmatrix} \geq 0 \quad (\forall z \in \mathbb{D}).
\tag{8.23}
\]
The matrix function on the left hand side of (8.23) is harmonic and nonnegative in $\mathbb{D}$ and hence it admits a Riesz-Herglotz representation. Let
\[
\begin{bmatrix}
d \Sigma(t) & d \mu(t) \\
d \mu^*(t) & d \sigma(t)
\end{bmatrix}
\]
be the block decomposition of the representing measure. Now we can apply Šmul’yan’s results from\(^6\) [S], to obtain the inequality
\[
\int_T d \mu(t) (d \sigma(t))^{-1} d \mu^*(t) \leq \int_T d \Sigma(t),
\]
where the integral on the left hand side is the so called Operator Hellinger Integral. Because
\[
W_w(0) + W_w^*(0)) = A,
\]
it follows that $\int_T d \Sigma = A$.

Thus
\[
\int_T d \mu(t) (d \sigma(t))^{-1} d \mu^*(t) \leq A.
\]

It is not difficult to show that in the considered case (the $\mathcal{NP}$ problem with finitely many interpolation nodes located inside $\mathbb{D}$) the equality holds in (8.26). In the general situation, $A$ is a nonnegative Hermitian form in some vector space. Then, the TFMI$_T(\mathcal{NP})$ leads to the representation of a nonnegative Hermitian form by the Hellinger Integral. It is worthy to mention that it was the Hellinger integral, which was used for the integral representation of Hermitian kernels early in the development of the theory. In more recent time, the Stieltjes integral ousted the Hellinger integral from this circle of problem. However, the use of the Stieltjes integral leads to difficulties. It may not exist, and we have to use rigged Hilbert spaces and all that. And the Hellinger integral exists always (and under some conditions it may be reduced to the Stieltjes integral). By our opinion, the use of the Hellinger integral lies in the essence of matter. The moral is clear:

GO BACK TO THE CLASSICS.

\(^6\)The paper [S] by Yu.L. Šmul’yan looks as it was written especially to be used in this paper.
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