Semiclassical and quantum Liouville theory

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Abstract. We develop a functional integral approach to quantum Liouville field theory completely independent of the hamiltonian approach. To this end on the sphere topology we solve the Riemann-Hilbert problem for three singularities of finite strength and a fourth one infinitesimal, by determining perturbatively the Poincaré accessory parameters. This provides the semiclassical four point vertex function with three finite charges and a fourth infinitesimal. Some of the results are extended to the case of $n$ finite charges and $m$ infinitesimal. With the same technique we compute the exact Green function on the sphere on the background of three finite singularities. Turning to the full quantum problem we address the calculation of the quantum determinant on the background of three finite charges and of the further perturbative corrections. The zeta function regularization provides a theory which is not invariant under local conformal transformations. Instead by employing a regularization suggested in the case of the pseudosphere by Zamolodchikov and Zamolodchikov we obtain the correct quantum conformal dimensions from the one loop calculation and we show explicitly that the two loop corrections do not change such dimensions. We then apply the method to the case of the pseudosphere with one finite singularity and compute the exact value for the quantum determinant. Such results are compared to those of the conformal bootstrap approach finding complete agreement.

1. Introduction
Liouville theory $^{11}$ $^{12}$ $^{13}$ $^{14}$ plays an important role in several branches of physics and mathematics. It is deeply related to the problem of uniformization of Riemann surfaces; it plays a major role in 2+1 dimensional gravity also in presence of matter; it appears in two dimensional gravity as an outcome of the conformal anomaly, in non critical string theory, in special models of 2D critical string theory and in the AdS-CFT correspondence $^{5}$ and recently in some D-brane models $^{6}$.

There are practically three approaches to the quantum problem: the hamiltonian, the functional and the conformal bootstrap approach.

The hamiltonian approach (Minkowski space) was first pursued by Curtright and Thorn $^{11}$ and by D’Hoker, Friedman and Jackiw $^{21}$ $^{13}$. In papers $^{21}$ $^{13}$ it was shown that in presence of a ground state the theory develops a spontaneous symmetry breaking of Poincaré invariance and in particular of translational invariance. As a result the theory is defined on the half-line and of the whole conformal group, present in the classical action, only the subgroup $SO(2,1)$ survives. In $^{11}$ (see also $^{7}$ $^{8}$) the theory is compactified on a circle, there is no ground state, but by properly modifying the energy momentum tensor it is possible to show that the emerging quantum theory is invariant under the whole conformal group. The central charge of the theory turns out to be $c = 1 + 6Q^2$, where $Q = 1/b + b$ being $b$ the coupling constant; the dimension of the vertex function $V_\alpha = \exp(2\alpha\phi)$ is $\Delta_\alpha = \alpha(Q - \alpha)$. 


The functional approach \textsuperscript{[9] [10] [11] [12]} has been used mostly at the formal level in combination with the conformal bootstrap approach. One accepts a priori the invariance of the theory under the full conformal group. Some results are borrowed from the hamiltonian approach; the integral computed at some special point in the space of the charges and interpolation formulae devised to extend the treatment to the general case. Four point functions on the sphere (or two point function on the pseudosphere) are computed when one of the vertex function is a degenerate field (Teschner trick). Consistency with the formulation in the crossed channel originates some difference equation. After imposing a further symmetry on the result one gets the final answer.

The most important results are the exact expressions for the three point function on the sphere \textsuperscript{[13] [11] [10]}, the one point function on the pseudosphere (ZZ-brane) \textsuperscript{[12]}, and several similar results in the case of boundary conformal Liouville theory (FZZT-brane) \textsuperscript{[14] [15]}. The subject of this talk is to widen the range of applications of the standard functional approach. By this we mean the formulation in which one first computes a stable background and then integrates over the fluctuations around it. We shall see that it is possible to develop techniques which allow the resummation of infinite classes of Feynman graphs and compare such results with formulas derived in the conformal bootstrap approach. There is a more general aspect in this kind of research. It is well known that a quantum field theory is characterized non only by an action but also by a regularization and renormalization procedure. Thus there is a non trivial question to answer i.e.: Which is the correct action to start with and which regularization procedure has one to adopt in order the produce perturbatively result which are consistent with invariance under the full conformal group? As we shall see not all regularization procedures give rise to a field theory satisfying such requirements.

2. Classical Liouville theory

We start with the sphere topology. The regularized classical action for the Liouville theory on the sphere in presence of $N$ sources is given by \textsuperscript{[11]}

$$S_L[\phi] = \lim_{R \to \infty} \left\{ \int_{\Gamma_{e,R}} \left[ \frac{1}{2\pi i} \partial_z \phi \partial_{\bar{z}} \phi + \mu e^{2\phi} \right] \right\} dz \wedge d\bar{z}$$

\[= \frac{Q}{2\pi i} \oint_{\partial \Gamma_R} \phi \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + Q^2 \log R^2 - \frac{1}{2\pi} \sum_{n=1}^{N} \alpha_n \oint_{\partial \Gamma_n} \phi \left( \frac{dz}{z-n} - \frac{d\bar{z}}{\bar{z}-n} \right) - \sum_{n=1}^{N} \alpha_n^2 \log \varepsilon_n^2 \]

where $z_n$ and $\alpha_n$ are position and charge of the $n$-th source. The domain of integration is the region $\Gamma_{e,R} = \{ |z| < R \} \setminus \bigcup_n \{ |z-z_n| < \varepsilon_n \}$, $\partial \Gamma_R$ is the border around infinity while $\partial \Gamma_n$ is the border around the $n$-th source. Here $Q$ is a parameter linked to the transformation law of the Liouville field. Classically its value is $Q = 1/b$.

In order to examine the semiclassical limit of the $N$-point function it is useful \textsuperscript{[11]} to go over to the field $\varphi = 2b\phi$. The corresponding charges are $\eta_n = \alpha_n b$ and the action takes the form

$$S[\varphi] = b^2 S_L[\phi] = \lim_{R \to \infty} \int_{\Gamma_{e,R}} \left[ \frac{1}{4\pi i} \partial_z \varphi \partial_{\bar{z}} \varphi + b^2 \mu e^{b\varphi} \right] dz \wedge d\bar{z}$$

\[= \frac{bQ}{4\pi i} \oint_{\partial \Gamma_R} \varphi \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + (bQ)^2 \log R^2 - \frac{1}{4\pi} \sum_{n=1}^{N} \eta_n \oint_{\partial \Gamma_n} \varphi \left( \frac{dz}{z-n} - \frac{d\bar{z}}{\bar{z}-n} \right) - \sum_{n=1}^{N} \eta_n^2 \log \varepsilon_n^2. \]

The field $\varphi$ behaves like

$$\begin{align*}
\varphi(z) &= -2\eta_n \log |z-z_n|^2 + O(1) \quad \text{for } z \to z_n \\
\varphi(z) &= -2bQ \log |z|^2 + O(1) \quad \text{for } z \to \infty.
\end{align*}$$
We decompose the field $\phi$ into the sum of a classical background $\varphi_B$ and a quantum fluctuation $\varphi = \varphi_B + 2b\chi$. The action becomes $S_L[\varphi_B, \chi] = S_{cl}[\varphi_B] + S_q[\varphi_B, \chi]$ where

$$S_{cl}[\varphi_B] = \lim_{\epsilon_n \to 0} \frac{1}{b^2} \left[ \frac{1}{8\pi} \int_\Gamma \left( \frac{1}{2} (\partial_\lambda \varphi_B)^2 + 8\pi \mu b^2 e^{\varphi_B} \right) \right] d^2z$$

$$- \sum_{n=1}^N \left( \eta_n \frac{1}{4\pi i} \oint_{\partial \Gamma_n} \varphi_B \frac{dz}{z - z_n} - \frac{dz}{\bar{z} - \bar{z}_n} \right) + \eta_n^2 \log \varepsilon_n^2 + \frac{1}{4\pi i} \oint_{\partial \Gamma_R} \varphi_B \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + \log R^2$$

and

$$S_q[\varphi_B, \chi] = \lim_{R \to \infty} \frac{1}{4\pi} \int_\Gamma \left( (\partial_\lambda \chi)^2 + 4\pi \mu e^{\varphi_B} (e^{2b\chi} - 1 - 2b\chi) \right) d^2z$$

$$+ (2 + b^2) \ln R^2 + \frac{1}{4\pi i} \oint_{\partial \Gamma_R} \varphi_B \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + \frac{b}{2\pi i} \oint_{\partial \Gamma_R} \chi \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right).$$

The terms in the second row arise from having chosen $Q = 1/b + b$. The main difficulty with the sphere topology is that there is no stable solution to the Liouville equation derived from the classical action in absence of sources. This is due to some inequalities which go back to Picard [17], i.e. we must have $\eta_n \leq 1/2$ and $\sum \eta_n > 1$ which implies that at least three singularities have to be present and sufficiently strong. Thus there cannot be a usual perturbative expansion on the sphere with weak sources, unless one uses the fixed area approach; but functionally integrating with constraints is more difficult.

From the classical action [14] one derives the Liouville equation

$$- \Delta \phi + 8\pi \mu b^2 e^\phi = 8\pi \sum_{n=1}^N \eta_n \delta^2(z - z_n)$$

whose solutions can be reduced to the solution of the Fuchsian equation

$$y''(z) + Q(z)y(z) = 0$$

where

$$Q(z) = \sum_{n=1}^N \left( \frac{1 - \lambda_n^2}{4(z - z_n)^2} + \frac{\beta_n}{2(z - z_n)} \right).$$

Here in addition to the parameters $\lambda_n = 1 - 2\eta_n$ related to the charges, also the Poincaré accessory parameters $\beta_n$ appear. These accessory parameters must satisfy three constraints known as Fuchs relations which in the case of only three singularities are sufficient to determine the three accessory parameters. In the case of more than three singularities the accessory parameters have to be fixed by imposing the monodromy of the conformal factor $\phi$.

The solution of equation (6) is given by

$$e^{\phi_c} = \frac{1}{\pi \mu b^2} \frac{|w_{12}|^2}{(y_2\bar{y}_2 - y_1\bar{y}_1)^2}$$

where $w_{12} = y_1y_2' - y_1'y_2$ is the constant wronskian and the two solutions $y_1$ and $y_2$ of (7) must be chosen in such a way that their monodromy group is $SU(1, 1)$ in order to ensure that the Liouville field $\phi(z)$ is one-valued on the whole complex plane. In the case of only three singularities the conformal factor is given in terms of hypergeometric functions.
The classical action \( \mathcal{S}_{cl} \) has very simple transformation properties under \( SL(2,C) \)\[16\] [15]. Moreover on the solution of Liouville equation it satisfies two important relations. The first is easily derived from the form of the action and reads

\[
\frac{\partial \mathcal{S}_{cl}}{\partial \eta_i} = -X_i \tag{10}
\]

where \( X_i \) is the finite part of the field \( \varphi_c \) at \( z_i \)

\[
\varphi_c(z) = -2\eta_i \log |z - z_i|^2 + X_i + o(|z - z_i|). \tag{11}
\]

The second relation is the so called Polyakov relation \[19\] [20] [21]

\[
\frac{\partial \mathcal{S}_{cl}}{\partial z_i} = -\frac{\beta_i}{2} \tag{12}
\]

which directly relates the accessory parameters to the classical Liouville action. These two relations properly rewritten and interpreted, contain all the hamiltonian structure of 2+1 dimensional gravity \[22\] [23]. Using relations \[10\] [12] it is possible to compute the semiclassical limit of the three-point function, which is related to the value of the classical action. Integrating the differential system \[10\] [14] one obtains \[11\]

\[
\mathcal{S}_{cl}[z_1, z_2, z_3; \eta_1, \eta_2, \eta_3] = (\delta_1 + \delta_2 - \delta_3) \log |z_1 - z_2|^2 + (\delta_2 + \delta_3 - \delta_1) \log |z_2 - z_3|^2 + (\delta_3 + \delta_1 - \delta_2) \log |z_3 - z_1|^2 + \mathcal{S}_{cl}[0, 1, \infty; \eta_1, \eta_2, \eta_3] \tag{13}
\]

where \( \delta_i = \eta_i(1 - \eta_i) \) are the semiclassical dimensions and

\[
\mathcal{S}_{cl}[0, 1, \infty; \eta_1, \eta_2, \eta_3] = S_0 + \left( \eta_1 + \eta_2 + \eta_3 - \frac{3}{2} \right) \log(\pi\mu b^2) + 3F(1) - F(2\eta_1) - F(2\eta_2) - F(2\eta_3) + F(\eta_1 + \eta_2 + \eta_3 - 1) + F(\eta_3 + \eta_2 - \eta_1) + F(\eta_2 + \eta_1 - \eta_3) + F(\eta_3 + \eta_1 - \eta_2). \tag{14}
\]

The function \( F \) is given by

\[
F(x) = \int_{1/2}^{x} \log \gamma(s) \, ds. \tag{15}
\]

where as usual \( \gamma(x) = \Gamma(x)/\Gamma(1-x) \). Notice that a regulation procedure of the action is necessary also at the classical level and it gives rise to the semiclassical dimension \( \delta_i = \eta_i(1 - \eta_i) \). These are not yet the quantum dimensions obtained in the hamiltonian approach. To proceed one needs the Green function on the background of three (not small) sources.

**3. The semiclassical four point function**

In this section we shall determine the classical action in presence of three finite singularities and a fourth infinitesimal; such a calculation gives the semiclassical four point function for vertices with three finite charges and the fourth small \[24\]. The procedure we shall use in presence of a fourth weak singularity is to solve perturbatively the fuchsian equation associated to the Liouville equation leaving the fourth small accessory parameter \( \beta_4 \) free, and then determine it by imposing the monodromy condition on the conformal factor. Given four singularities, by means of an \( SL(2,C) \) transformation we can take three of them in 0, 1, \( \infty \). The position of the fourth will be called \( t \) and the coefficient \( Q \) in the fuchsian equation becomes

\[
Q(z) = \frac{1 - \lambda_1^2}{4z^2} + \frac{1 - \lambda_2^2}{4(z - 1)^2} + \frac{1 - \lambda_3^2}{4(z - t)^2} + \frac{\beta_1}{2z} + \frac{\beta_2}{2(z - 1)} + \frac{\beta_4}{2(z - t)}. \tag{16}
\]
For the source in $t$ of infinitesimal strength we shall write $\lambda_4 = 1 - 2\varepsilon$ and $\beta_4 = \varepsilon \beta$ and our aim will be to determine $\beta$. Using Fuchs relations we have

$$
\beta_1 = \frac{1 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2}{2} + \varepsilon \left[(t-1)\beta + 2\right] + O(\varepsilon^2)
$$
$$
\beta_2 = \frac{-1 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2}{2} - \varepsilon \left[2 + t\beta\right] + O(\varepsilon^2)
$$

(17)

and we write

$$
Q(z) = Q_0(z) + \varepsilon q(z)
$$

(18)

where $Q_0(z)$ stays for the coefficient of the three singularity problem, while $q(z)$ is the perturbation

$$
q(z) = \frac{1}{2} \left[\frac{(t-1)\beta + 2}{z} - \frac{2 + t\beta}{z-1} + \frac{\beta}{z-t} + \frac{2}{(z-t)^2}\right].
$$

(19)

After writing $y = y_0 + \varepsilon \delta y$, being $y_0$ a solution of the unperturbed equation, we have to first order in $\varepsilon$ the inhomogeneous equation

$$
(\delta y)'' + Q \delta y = -q y_0.
$$

(20)

Such an equation can be solved by a well known method and in our case we have

$$
\delta y_i = -\frac{1}{w_{12}} \int_{z_0}^z dx \left[y_1(x)y_2(z) - y_1(z)y_2(x)\right] q(x)y_i(x)
$$

(21)

being $w_{12} = y_1 y_2' - y_1' y_2$ the constant wronskian and $z_0$ an arbitrary base point in the complex plane. It will be useful to define the following integrals

$$
I_{ij}(z) \equiv \int_{z_0}^z dx \ y_i(x)y_j(x) q(x).
$$

(22)

We must now compute the monodromy matrices around 0, 1, $t$ and impose on them the $SU(1,1)$ nature. This will determine uniquely the parameter $\beta$. The calculation gives [24]

$$
\beta = -4 \kappa \frac{\bar{y}_1 y'_1 - \bar{y}_2 y'_2}{\kappa} (t) = 2 \partial_x \varphi_c^0(z)|_{z=t}
$$

(23)

being $\kappa = |k_0|^4$ with $k_0$ the parameter which enters the three-singularity conformal factor

$$
e^{2b\phi_c} = \frac{1}{\pi \mu b^2} \frac{w_{12}^2}{(|k_0|^2 y_1 \bar{y}_1 - |k_0|^{-2} y_2 \bar{y}_2)^2}
$$

(24)

and

$$
e^{2b\phi_c} = e^{\varphi_c} = \frac{1}{\pi \mu b^2} \frac{w_{12}^2}{(Z_1 \bar{Z}_1 - Z_2 \bar{Z}_2)^2}
$$

(25)

with

$$
Z_1(z) = k_0 \left[1 + \varepsilon \frac{I_{12}(z)}{w_{12}} + \frac{h}{w_{12}}\right] y_1(z) - \varepsilon \frac{I_{11}(z)}{w_{12}} y_2(z)
$$
$$
Z_2(z) = \frac{1}{k_0} \left[\varepsilon \frac{I_{22}(z)}{w_{12}} y_1(z) + \left(1 - \varepsilon \frac{I_{12}(z)}{w_{12}} + \frac{h}{w_{12}}\right) y_2(z)\right]
$$

(26)
where the $h$ can also be computed \cite{24}. The functions $Z_1, Z_2$ have $SU(1, 1)$ monodromies around all singularities and as such determine a globally monodromic conformal factor satisfying the Liouville equation. We can now compute the conformal factor in presence of our four sources to first order in $\varepsilon$

$$e^{\phi_c} = e^{\phi_c^0} \left\{ 1 - \varepsilon \frac{2}{w_{12} (\kappa y_1 y_1 - y_2 y_2)} \right.$$  
$$\left[ (\kappa y_1 y_1 + y_2 y_2) \left( I_{12} + \bar{I}_{12} + h + \bar{h} \right) - y_1 y_2 \left( I_{22} + \kappa \bar{I}_{11} - \tilde{y}_1 y_2 \left( I_{22} + \kappa I_{11} \right) \right) + O(\varepsilon^2) \right\} \equiv e^{\phi_c^0} (1 + \varepsilon \chi + O(\varepsilon^2)).$$

Eq. \ref{23} gives the value of $\beta_4$ to first order $\beta_4 = \varepsilon \beta$. Recalling the expression of the unperturbed conformal factor $e^{\phi_c^0}$ with only three sources we have

$$\beta_4 = -4 \varepsilon e^{\phi_c^0/2} \frac{\partial z}{\partial z} e^{-\phi_c^0/2} \bigg|_{z=t} = 2 \varepsilon \left. \frac{\partial \varphi_c^0 (z)}{\partial t} \right|_{z=t}.$$  

The above obtained relation can be understood by expanding Liouville equation around the unperturbed solution. We can exploit such a result and Polyakov relation to compute to order $\varepsilon$ the classical action for the new solution

$$\frac{\partial S_{cl}[\eta_1, \eta_2, \eta_3, \varepsilon]}{\partial t} = -\frac{\beta_4}{2} = -\varepsilon \frac{\partial \varphi_c^0}{\partial t}$$

and using again eq. \ref{10} we reach for the semiclassical four point function with small $\alpha_4$

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) V_{\alpha_4}(t) \rangle_{sc} = \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{sc} e^{2\alpha_4 \phi_c^0(t)}.$$  

It is easily checked that the four point function \ref{30} has the correct transformation properties with dimensions $\alpha_4/b$ for the vertex field $V_{\alpha_4}(z_4)$ in agreement with the semiclassical dimensions $\alpha_4 (1/b - \alpha_4)$ keeping in mind that we have been working to first order in $\alpha_4$, and thus we can write to first order in $\alpha_4$

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(z_4) \rangle_{sc} = \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle_{sc} e^{2\alpha_4 \phi_c^0(z_4)}.$$  

4. Generalization to $n$-point functions

We can generalize some of the results obtained above to $n$ arbitrary sources and $m$ infinitesimal sources. The discussion we have been performing in the case of three sources which leads to the inhomogeneous equation \ref{20} remains valid also in this case; the only difference is that now we do not know the explicit form of the unperturbed solutions $y_1, y_2$. The accessory parameter $\beta_4 = \varepsilon \beta$ is again given by eq. \ref{28}

$$\beta = 2 \partial_z \varphi_c^0 (z) \bigg|_{z=t}$$

where now $\varphi_c^0$ is the conformal field which solves the problem in presence of the $n$ finite sources. Thus we have a general relation between the value of the accessory parameter relative to the infinitesimal source in $t$ and the conformal factor for the unperturbed background and thus we can extend the result \ref{31} to $n$ finite sources plus an infinitesimal one. Finally due to the additive nature of the perturbation with $m$ infinitesimal sources we have for the $n + m$ semiclassical correlation function

$$\langle V_{\alpha_1}(z_1) \ldots V_{\alpha_n}(z_n) V_{\gamma_1}(t_1) \ldots V_{\gamma_m}(t_m) \rangle_{sc} = \langle V_{\alpha_1}(z_1) \ldots V_{\alpha_n}(z_n) \rangle_{sc} \prod_{j=0}^{m} e^{2\gamma_j \phi_c^0(t_j)}.$$  

Such relations were already argued in \ref{11}.
5. The Green function on the sphere with three singularities
From the above derived results we can extract the exact Green function on the sphere in presence of three finite singularities. The equation for the Green function is

\[- \Delta g(z, t) + 8\pi \mu b^2 e^{\varphi_B(z)} g(z, t) = 2\pi \delta^2(z - t)\]  

(34)

where \(\varphi_B\) is the classical solution in presence of three finite singularities. Such a Green function can be computed from the result obtained in Sect.3. In fact we have found a solution to

\[- \Delta \varphi + 8\pi \mu b^2 e^{\varphi_B} = 8\pi \sum_{i=1}^{3} \eta_i \delta^2(z - z_i) + 8\pi \varepsilon \delta^2(z - t)\]  

(35)

for infinitesimal \(\varepsilon\) i.e. \(\varphi = \varphi_B + \varepsilon \chi\). Substituting we obtain

\[- \Delta \chi + 8\pi \mu b^2 e^{\varphi_B} \chi = 8\pi \delta^2(z - t)\]  

(36)

i.e. we have \(g(z, t) = \frac{\chi}{4}\). From eq.(37) we have

\[g(z, t) = - \frac{1}{2w_{12}} \left[ \kappa y_1(z) \tilde{y}_1(z) - y_2(z) \tilde{y}_2(z) \right] \left\{ \left[ \kappa y_1(z) \tilde{y}_1(z) + y_2(z) \tilde{y}_2(z) \right] \cdot \left[ I_{12}(z, t) + \tilde{I}_{12}(\tilde{z}, \tilde{t}) + h(t) + \tilde{h}(\tilde{t}) \right] 

- y_1(z) \tilde{y}_2(z) \left[ I_{22}(z, t) + \kappa \tilde{I}_{11}(\tilde{z}, \tilde{t}) \right] 

- \tilde{y}_1(z) y_2(z) \left[ \tilde{I}_{22}(\tilde{z}, \tilde{t}) + \kappa I_{11}(z, t) \right] \right\}.\]  

(37)

It is possible to verify directly that (37) satisfies eq. (34) and is regular on the three finite sources. Actually expression (37) is completely general, i.e. it applies also for the case of a background given by \(n\) finite sources with \(g_i\) solutions of the related fuchsian equation. In the case of \(n = 3\) we know the explicit form of \(y_i\). One would expect the Green function \(g(z, t)\) to be symmetric in the arguments. This is far from evident from the expression (37). The differential operator \(D = -\Delta_{LB} + 1\) is hermitean in the background metric \(e^{\varphi_B} d^2 z\). As a result also its inverse \(G = D^{-1}\) is hermitean \(G = G^+\). \(G\) is represented by \(g(x, t)\) which is also real and thus we have \(g(z, t) = g(t, z)\).

6. The quantum determinant
The complete action is given by eqs.44 and 51 and the quantum \(n\)-point function by

\[\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2) \ldots V_{\alpha_n}(z_n) \rangle = e^{-S_{cl}[\phi_B]} \int D[\chi] \ e^{-S_q}.\]  

(38)

We recall that \(S_{cl}\) is \(O(1/b^2)\) while the first integral appearing in the quantum action (5) can be expanded as

\[\frac{1}{4\pi} \int_G \left( (\partial_\tau \chi)^2 + 8\pi \mu b^2 e^{\varphi_B} \chi^2 + 8\pi \mu b^2 e^{\varphi_B} \left( \frac{4b^2 \chi^3}{3!} + \frac{8b^2 \chi^4}{4!} + \ldots \right) \right) d^2 z.\]  

(39)

From now on we shall denote by \(\varphi_B\) the classical solution with three singularities at \(z_1, z_2, z_3\) and with charges \(\eta_1, \eta_2, \eta_3\). In performing the perturbative expansion in \(b\) we have to keep the \(\eta_1, \eta_2, \eta_3\) constant [11]. The \(O(b^0)\) contribution to the three point function is given by

\[\langle \text{ Det} D \rangle^{-\frac{1}{2}} = \int D[\chi] e^{-\int \chi(z) D \chi(z) f(z) d^2 z}\]  

(40)
where \( f(z) = 8\pi\mu b^2 e^{2\phi_B(z)} \) and \( D = (-\Delta_{LB} + 1)/4\pi \) being \( \Delta_{LB} = f^{-1} \Delta \) the Laplace-Beltrami operator on the background \( f(z) \) of the three charges. It provides the one loop quantum correction to the semiclassical results we have been discussing above.

When confronted to the computation of a functional determinant the first idea is to use the \( Z \)-function regularization; but that does not work because such a regularization is invariant under conformal transformations and as such leaves the dimensions of the vertex functions at their semiclassical values; in particular the weights of the cosmological term remain \((1-b^2, 1-b^2)\) instead of becoming \((1,1)\) and further quantum corrections which are higher order in \( b^2 \) cannot mend such a discrepancy. The situation is similar to the one discussed by D’Hoker, Freedman and Jackiw [2,3] and the one considered by Takhtajan [16] with the use of an invariant regularization of the Green function (Hadamard regularization) and \( Q = 1/b \). Thus a different way to define the regularized determinant has to be devised. One can take the derivative of the logarithm of the determinant thus exposing the role of the regularized Green function at coincident points. We have

\[
\frac{\partial}{\partial \eta_j} \left( \log(\text{Det} D)^{-\frac{1}{2}} \right) = -2\mu b^2 \int \frac{\partial \varphi_B}{\partial \eta_j}(z) g(z, z) e^{\varphi_B(z)} d^2 z. \tag{41}
\]

In the above equation the Green function at coincident points appears and such a quantity has to be regularized. We have already seen that the invariant regularization gives rise to a theory in which the cosmological term \( e^{2\phi_B(z)} \) does not have weights \((1,1)\) and as such does not give rise to a theory invariant under the whole (infinite dimensional) conformal group.

We shall adopt here the regularization proposed by Zamolodchikov and Zamolodchikov [12] (ZZ regulator) for perturbative calculations on the pseudosphere i.e.

\[
g(z, z) = \lim_{z' \to z} \left( g(z, z') + \log |z - z'| \right). \tag{42}
\]

As under an \( SL(2, C) \) transformation \( w = (az + b)/(cz + d) \) the Green function is invariant in value \( g^w(w, w') = g(z, z') \) and we have

\[
g^w(w, w) = g(z, z) + \log \left| \frac{\partial w}{\partial z} \right| = g(z, z) + \log \left( \frac{1}{|cz + d|^2} \right). \tag{43}
\]

Using the above relation one can compute the change of \( (41) \) under \( SL(2, C) \) transformations. Taking into account the contribution of the boundary term we have for \( j = 1, 2, 3, k = 2, 3, 1, l = 3, 1, 2 \)

\[
\frac{\partial}{\partial \eta_j} \log((\text{Det} D)^{-\frac{1}{2}}) - \frac{\partial X_{\infty\infty}}{\partial \eta_j} = f_j(\eta_1, \eta_2, \eta_3) - 2 \log \left| \frac{(z_j - z_k)(z_j - z_l)}{z_k - z_l} \right|. \tag{44}
\]

Integration of the above equation gives

\[
c(\eta_1, \eta_2, \eta_3) - 2(\eta_1 + \eta_2 - \eta_3) \log |z_1 - z_2| - 2(\eta_2 + \eta_3 - \eta_1) \log |z_2 - z_3| - 2(\eta_3 + \eta_1 - \eta_2) \log |z_3 - z_1| \tag{45}
\]

as \( O(b^0) \) correction i.e. the one loop correction, and we have obtained the three-point function with the correct quantum dimensions \( \Delta_j = \eta_j/(1 - \eta_j)/b^2 + \eta_j \). Thus the situation is very similar to what happens on the pseudosphere, where the one loop corrections with the ZZ regulator provide the exact quantum dimensions \([12, 13]\).

7. Two loop contributions
The graphs contributing to two loop are shown in Figure.1. Of them graph (c) is convergent and invariant under under \( SL(2, C) \); (a) and (b) are not due to the appearance of the regularized Green function \( g(z, z) \), while (d) and (e) arise from the boundary term. Integrating by parts a
number of times and using the equation for the Green function one can prove that the sum is invariant under translations, dilatations and inversion and thus under the whole global conformal group \[24\]. This shows that the dimension are not altered at two loop. One also envisages a general procedure to higher loop but that has not yet been done explicitly; on the pseudosphere one can give an explicit procedure to all orders \[29\].

\[\text{Figure 1. Two loop contributions}\]

8. The pseudosphere
In this section we shall apply the developed technique to the pseudosphere \[26\]. For the one point function, this analysis goes well beyond the previous perturbative expansions performed in \[12, 25, 18\] where \(\alpha\) has been taken small; in fact our result corresponds to the summation of an infinite class of perturbative graphs. Thus, we obtain a strong check of the ZZ bootstrap formula for the one point function \[12\], which includes all the previous perturbative checks. We start from the Liouville action on the pseudosphere in presence of \(N\) sources characterized by heavy charges \(\eta_1, \ldots, \eta_N\), given in \[18\]. The standard representations of the pseudosphere are the unit disk \(\Delta\) and the upper half plane \(H\). Here we shall mostly use the \(\Delta\) representation. Decomposing the Liouville field as before the Liouville action separates into a classical part, depending only on the background field \(\varphi_B\), and a quantum action for the quantum field \(\chi\), 
\[
S_{\Delta,N}[\phi] = S_{cl}[\varphi_B] + S_q[\varphi_B, \chi]
\]
which have expressions similar to the ones appearing in the case of the sphere \[18, 26\]. The coupling constant \(b\) is still related to the parameter \(Q\) occurring in the central charge \(c = 1 + 6Q^2\) by \(Q = 1/b + b/1\). Again at semiclassical level, we have
\[
\langle V_{\alpha_1}(z_1) \cdots V_{\alpha_N}(z_N) \rangle_{sc} = \frac{e^{-S_{cl}(\eta_1, z_1; \ldots; \eta_N, z_N)}}{e^{-S_{cl}(0)}}
\]
where \(S_{cl}(\eta_1, z_1; \ldots; \eta_N, z_N)\) is the classical action \(S_{cl}[\varphi_B]\) computed on the solution \(\varphi_B\) of the Liouville equation with sources. One can see that the transformation law of \(S_{cl}[\varphi_B]\) assigns to the vertex operator \(V_\alpha(z)\) the semiclassical dimensions \(\alpha (1/b - \alpha) = \eta (1 - \eta) / b^2\) \[18\] as already found in Sect. [2] on the sphere.

For the one point function, we have a single heavy charge \(\eta_1 = \eta\), which can be placed in \(z_1 = 0\), and the explicit solution of the Liouville equation is \(\varphi_B = \varphi_{cl}\), given by \[27\]
\[
e^{\varphi_{cl}} = \frac{1}{\pi \mu b^2} \frac{(1 - 2\eta)^2}{((z \bar{z})^\eta - (z \bar{z})^{1-\eta})^2}.
\]
The classical action computed on this background gives the semiclassical one point function
\[
\langle V_{\eta/b}(0) \rangle_{sc} = \exp \left\{ -\frac{1}{b^2} \left( \eta \log \left( \pi b^2 \mu \right) + 2\eta + (1 - 2\eta) \log(1 - 2\eta) \right) \right\}.
\]
It is fixed by requiring the vanishing of monodromy arguments because it is the coefficient of a solution of the homogeneous equation.

What done for the sphere and $B_{ij}$ are defined similarly to what done for the sphere and $h_0$ is a free real parameter which cannot be determined through monodromy arguments because it is the coefficient of a solution of the homogeneous equation. It is fixed by requiring the vanishing of $\chi(z,t)$ at infinity, i.e. when $|z| \to 1$, in order to respect the boundary condition on the pseudosphere. The Green function on the background $\varphi_{cl}(z)$ is given by $g(z,t) = \chi(z,t)/4$. By exploiting the invariance under rotation, we can write our result for a generic complex $t \in \Delta$. The final expression of the exact Green function in the explicit symmetric form is

$$g(z,t) = -1 + \frac{1}{2} \left( \frac{1 + (z \bar{z})^{1-2\eta}}{1 - (z \bar{z})^{1-2\eta}} \right) \log \omega(z,t) - \frac{1}{1 - 2\eta}$$

where $\omega(z,t)$ is the $SU(1,1)$ invariant ratio

$$\omega(z,t) = \frac{(z - t) (\bar{z} - \bar{t})}{(1 - z t) (1 - \bar{z} \bar{t})}$$

and $B_x(a,0)$ is a particular case of the incomplete Beta function $B_x(a,b)$

$$B_x(a,0) = \frac{x^a}{a} F(a,1;a+1; x) = \int_0^x \frac{y^{a-1}}{1 - y} dy = \sum_{n=0}^{+\infty} \frac{x^{a+n}}{a + n}.$$ 

Moreover, in the limit $\eta \to 0$, we recover the propagator on the pseudosphere without sources given in [3, 12]. From [31], we find

$$g(z,z) \equiv \lim_{t \to z} \left\{ g(z,t) + \frac{1}{2} \log |z - t|^2 \right\}$$
\[
\left( \frac{1 + (zz^{-1})^{1-2\eta}}{1 - (zz^{-1})^{1-2\eta}} \right)^2 \log (1 - zz^{-1}) - \frac{1}{1 - 2\eta} \left( \frac{1 + (zz^{-1})^{1-2\eta}}{1 - (zz^{-1})^{1-2\eta}} \right)
\]
\[\frac{2(zz^{-1})^{1-2\eta}}{(1 - (zz^{-1})^{1-2\eta})^2} \left( B_{zz}(2\eta, 0) + B_{zz}(2 - 2\eta, 0) \right)
\]
\[+ 2\gamma_E + \psi(2\eta) + \psi(2 - 2\eta) - \log zz^{-1}\]

(55)

where \(\gamma_E\) is the Euler constant and \(\psi(x) = \Gamma'(x)/\Gamma(x)\).

In the case of the one point function, expression (41) can be explicitly computed and the result is
\[\frac{\partial}{\partial \eta} \log (\text{Det} D(\eta, 0))^{-1/2} = 2\gamma_E + 2\psi(1 - 2\eta) + \frac{3}{1 - 2\eta}.\]

(56)

Integrating back in \(\eta\) with the initial condition given in [12], i.e. \(\left(\text{Det} D(\eta, 0)\right)^{-1/2} |_{\eta = 0} = 1\), we find
\[\log (\text{Det} D(\eta, 0))^{-1/2} = 2\gamma_E \eta - \log \Gamma(1 - 2\eta) - \frac{3}{2} \log(1 - 2\eta).\]

(57)

Putting this result together with the classical contribution (58), we have
\[\log \langle V_{\eta/b}(0) \rangle = -\frac{1}{b^2} \left( \eta \log \left[ \pi b^2 \mu \right] + 2\eta + (1 - 2\eta) \log(1 - 2\eta) \right)
\]
\[+ \left( 2\gamma_E \eta - \log \Gamma(1 - 2\eta) - \frac{3}{2} \log(1 - 2\eta) \right) + O(b^2)\]

(58)

to all orders in \(\eta\). We can compare (58) with the result obtained by ZZ within the bootstrap approach [12]
\[\langle V_{\alpha}(z_1) \rangle = \frac{U(\alpha)}{(1 - z_1 \bar{z}_1)^{2\alpha(Q - \alpha)}}\]

(59)

where \(U(\alpha)\) has been determined through the bootstrap method [12] with the result for the basic vacuum
\[U(\alpha) = U_{1,1}(\alpha) = (\pi\mu \gamma(b^2))^{-\alpha/b} \frac{\Gamma(Q) \Gamma(Q/b) Q}{\Gamma((Q - 2\alpha) b) \Gamma((Q - 2\alpha)/b) (Q - 2\alpha)}\]

(60)

where \(\gamma(x) = \Gamma(x)/\Gamma(1 - x)\). Our result (58) agrees with the expansion in \(b^2\) of \(U(\eta/b)\) and it corresponds to the summation of an infinite class of graphs of the usual perturbative expansion [12] [25] [18]. With some more work one could compute the two loop correction to eq.(58). The technique developed above can be applied to compute the two point function with one arbitrary charge \(\eta\) and another charge \(\varepsilon\) to first order in \(\varepsilon\) [26] and compared successfully with the results of the conformal bootstrap approach.

9. Conclusions

We have pursued the functional approach to Liouville quantum field theory, in the usual meaning of computing a classical stable background and integrating over the quantum fluctuations around it. We found that the standard technique of regulating the quantum determinants (Z-function) violates conformal invariance. A non conventional technique i.e. the ZZ regularization is necessary both in computing the functional determinant and higher order graphs. The boundary terms term play an essential role in the proof of the invariance at one and two loop order. The
exact Green function on the background of the sphere with three finite singularities has been obtained in terms of quadratures. The explicit form of the Green function on the background of the pseudosphere in presence of a singularity has been given and the exact quantum determinant computed. In corresponds to the resummation of an infinite family of graphs and the results agree with the conformal bootstrap results of A.B. Zamolodchikov and Al.B. Zamolodchikov. Applications to boundary Liouville theory on the disk and to higher loop are underway [29].

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