On mixed problems of control of hypersonic boundary layer on the fragments of segment of control

G G Bilchenko¹,² and N G Bilchenko¹,³
¹ MPTP Laboratory, Department of Heat Engineering and Power Engineering Machinery, Kazan National Research Technical University (KNRTU-KAI) named after A N Tupolev, 10, K Marx St., Kazan, Tatarstan 420111, Russia
E-mail: ² ggbil2@gmail.com and ³ bilchnat@gmail.com

Abstract. The problems of mathematical modeling of effective control ensuring the necessary combination of heat protection and aerodynamic properties of permeable cylindrical and spherical surfaces of hypersonic aircraft are considered, taking into account constructive and gas dynamics restrictions. The blowing into boundary layer, temperature factor and magnetic field are used as controls. Integration, observation and control meshes and fragmentation of the control segment are introduced. The statements of mixed extreme and non-extreme problems are given. A hybrid objective function is clarified. The conditions of choice of the parameters at the junction points of the fragments are proposed. Questions of proximity in mixed problems are discussed. The statements of simple and complex mixed problems are given. A classification of mixed problems of control of heat and mass transfer and friction in a laminar boundary layer on the permeable cylindrical and spherical surfaces of hypersonic aircraft on fragments of the control segment is carried out. The definition of partial restrictions on fragments of control segment is given similar to total restrictions on the entire control segment. Example illustrating the introduced restrictions is given. The scheme of the algorithm for choice the variant of the problem being solved is proposed.

1. Introduction
As a continuation of article [1] this paper preserves the notation introduced in it. Previously, direct (both non-extreme and extreme) and inverse problems of control LBL on permeable surfaces of HA were considered separately and along the entire segment of control. However, to ensure the necessary combination of heat protection and aerodynamic properties of the surfaces of HA intended for flight in dense layers of the atmosphere it is necessary to take into account the constructive and gas dynamics restrictions [2] that imply both the fragmentation of the segment of control and the application of various controls on its fragments.

2. Fragmentation
2.1. Fragmentation of meshes, partial values and problems on fragment
Let \( r \geq 1 \) fragments \( X_\ell = [x_{\ell,1}^\wedge; x_{\ell}^\wedge] \) (\( \ell = 1, \ldots, r \)) be allocated in the segment \( X_{tot} = [0; 1] \) by selection of indices \( i_0^\wedge = 0 < i_1^\wedge < \ldots < i_r^\wedge = n^\wedge \) of the mesh of control \( X^\wedge = (x_j^\wedge)_{j=0, \ldots, n^\wedge} \) nodes. In conditions (9) of [1] such indices \( i_0^\vee < i_1^\vee < \ldots < i_r^\vee = n^\vee \) of the mesh of observation
\[ X^\nu, (x_j^\nu)_{j=0, \ldots, n^\nu} \] nodes exist that \( x_j^\nu = x_{j+1}^\nu \), i.e. \( X_\ell = [x_\ell^{\nu, i}; x_\ell^{\nu, f}] \) for \( \ell = 1, \ldots, r \). Denoting \( X^\nu_\ell = X^\nu \cap X_\ell = (x_j^\nu)_{j=i_\ell^{\nu, i}, \ldots, i_\ell^{\nu, f}} \) for \( \ell = 1, \ldots, r \) we can introduce the fragmentations \( X^\nu_\ell = (X^\nu_\ell)_{r=1, \ldots, r} \), \( X^{\nu, r} = (X^{\nu, r})_{r=1, \ldots, r} \) for \( \ell = 1, \ldots, r \) of \( X^\nu, X^\nu, X_{\text{tot}} \).

The integral functionals \( Q, F, N \) computed on the segments \( X_{\text{tot}} \) and \( X_\ell \) are denoted by \( Q_{\text{tot}}, F_{\text{tot}} \), \( N_{\text{tot}} \) and \( Q_\ell, F_\ell, N_\ell \), respectively. \( Q_{\text{tot}}, F_{\text{tot}}, N_{\text{tot}} \) will be called total, and \( Q_\ell, F_\ell, N_\ell \) will be called partial. The following relations hold \( Q_{\text{tot}} = \sum_{\ell=1}^r Q_\ell, F_{\text{tot}} = \sum_{\ell=1}^r F_\ell, N_{\text{tot}} = \sum_{\ell=1}^r N_\ell \).

A set of elements [2] of a function on \( X_\ell \) will be called a fragment of a function. Problems with controls on \( X^\nu_\ell \) and with computed and observed parameters on \( X^{\nu, r} \) will be called problems on fragmentations. The problem (DP, IPIS, IPAS, IPMS, HDEP, IEP) determined by the introduction of indices \( (i_\ell^A) \) and \( (i_\ell^B) \) is transformed into a set of problems on fragmentations. The set of problems on fragments considered on \( X_{\text{tot}} \) will be called a mixed problem (MP). In this case, a set without HDEP and IEP will be called a mixed non-extreme problem (MNEP), and a set with HDEP and/or IEP will be called a mixed extreme problem (MEP). MP allows two statements that are considered in sections 3.3 and 3.4, and for MEP in paragraphs 4.3 and 4.4.

Example 1. Let for \( n^\nu \geq 2 \) on \( X^\nu \) and \( X^\nu \) two \( \text{IPIS}_m^\nu \) (with \( \nu \geq 0, p = +\infty, r^\nu \geq 1 \)) be given: 
\[
(q_j^A)_{j=0, \ldots, n^\nu} \rightarrow (q_j^A)_{j=0, \ldots, i_1^A}, (q_j^B)_{j=i_1^A, \ldots, n^\nu} \rightarrow (q_j^B)_{j=i_1^A, \ldots, n^\nu} ; \\
(q_j^C)_{j=0, \ldots, n^\nu} \rightarrow (q_j^C)_{j=0, \ldots, i_1^C}, (q_j^D)_{j=i_1^C, \ldots, n^\nu} \rightarrow (q_j^D)_{j=i_1^C, \ldots, n^\nu} ; \\
(q_j^{IM})_{j=0, \ldots, n^\nu} \rightarrow (q_j^{IM})_{j=0, \ldots, i_1^{IM}}, (q_j^{IM})_{j=i_1^{IM}, \ldots, n^\nu} \rightarrow (q_j^{IM})_{j=i_1^{IM}, \ldots, n^\nu} ;
\]
Obviously \( q_i^A = q_i^B, q_i^C = q_i^D, (\tau(k))_i^A = (\tau(k))_i^B, (\tau(k))_i^C = (\tau(k))_i^D \) for \( k = 0, \ldots, r^\nu \).

2.2. Condition at the junction point

For each \( \ell = 1, \ldots, r - 1 \) the value of the observed on \( X_{\ell+1} \) parameter \( q^\nu \) or \( f^\nu \) at the junction point \( x_{\ell}^{\nu, i} \) of fragments \( X_\ell \) and \( X_{\ell+1} \) must be agreed upon with value of the same parameter, computed or observed on \( X_\ell \) in this point. In condition of Example 1 let’s form an \( \text{IPIS}_m^\nu \) on \( X_\ell \): \( (q_j^A)_{j=0, \ldots, n^\nu} \rightarrow (q_j^A)_{j=0, \ldots, i_1^A}, (q_j^B)_{j=i_1^A, \ldots, n^\nu} \rightarrow (q_j^B)_{j=i_1^A, \ldots, n^\nu} ; \\
(q_j^C)_{j=0, \ldots, n^\nu} \rightarrow (q_j^C)_{j=0, \ldots, i_1^C}, (q_j^D)_{j=i_1^C, \ldots, n^\nu} \rightarrow (q_j^D)_{j=i_1^C, \ldots, n^\nu} ; \\
(q_j^{IM})_{j=0, \ldots, n^\nu} \rightarrow (q_j^{IM})_{j=0, \ldots, i_1^{IM}}, (q_j^{IM})_{j=i_1^{IM}, \ldots, n^\nu} \rightarrow (q_j^{IM})_{j=i_1^{IM}, \ldots, n^\nu} ;
\]
For the control parameters \( m, \tau \) the matching conditions are given in \( I^m, I^\tau, \Delta m, \Delta \tau \). If the parameters \( m^\nu \) or/and \( \tau^\nu \) are sought on \( X_{\ell+1} \), and \( \Delta m_{j,k} \neq \Delta m_{j,k} \) or/and \( \Delta \tau_{j,k} \neq \Delta \tau_{j,k} \), then the MP is underdetermined. If the parameters \( m \) or/and \( \tau \) are given on \( X_{\ell+1} \), and \( \Delta m_{j,k} = \Delta m_{j,k} \) or/and \( \Delta \tau_{j,k} = \Delta \tau_{j,k} \), then the MP may be overdetermined. In Example 1 if \( (\tau(k))_i^A = (\tau(k))_i^D \) and such \( k \) exist that \( \tau(k)_i^A \neq \tau(k)_i^D \), but \( \Delta \tau_{i_1^A,k} = \Delta \tau_{i_1^D,k} \), then MP is unsolvable.

Conclusions. For \( X_{\ell+1} \) the observed parameters should be specified at the points \( (x_j^\nu)_{j=i_\ell^{\nu, i} + 1, \ldots, i_{\ell+1}^{\nu, i}} \). This approach avoids overdetermination just at the junction point: the problem on the fragment \( X_{\ell+1} \) in case of a large value \( |q_j^\nu - q_j^{\nu, +1}| \) or/and \( |f_j^\nu - f_j^{\nu, +1}| \) may be unsolvable. If the parameters \( m^\nu \) or/and \( \tau^\nu \) are sought on \( X_{\ell+1} \), then we consider the case of a fixed size of jumps only (see paragraph 3.2 [2]): in this case the values on the left boundary \( X_{\ell+1} \) are uniquely determined by the values on the right boundary \( X_\ell \).
3. Mixed non-extreme problems

3.1. Mixing conditions

Suppose that the fragmentation $X^\wedge_r$, the control $s$, the numbers $p \in [1; +\infty]$ and $\varepsilon > 0$ are given. Let’s consider the case of proximity indicator (for IP realized on different fragments) equal to a given value $p$, and the case of accuracy (for IPIS and MIPS) equal to a given value $\varepsilon$.

Suppose that for each $\ell = 1, \ldots, r$ two of the four parameters $m, \tau, q, f$ are given ("1") on the fragment $X_\ell$ ($m, \tau$ on $X^\wedge_\ell$, $q, f$ on $X^\vee_\ell$), and two others on it are free ("0"). Suplementing $\delta$ from table 1 [1] by the index $\ell$, we obtain

$$\delta^m_\ell + \delta^q_\ell + \delta^f_\ell = 2.$$  \hspace{1cm} (1)

Let for each of the given observed parameters either an interpolation or an approximation statement be indicated, i.e. $\delta^q_\ell, \delta^a_q, \delta^f_\ell, \delta^a_f$ are given, satisfying (17) [1]. Let’s find

$$\delta^\text{IPIS}_\ell = \delta^m_\ell \cdot \delta^q_\ell, \quad \delta^\text{IPIS}_\ell = \delta^m_\ell \cdot \delta^q_\ell, \quad \delta^\text{IPMS}_\ell = \delta^m_\ell \cdot \delta^q_\ell, \quad \delta^\text{ODIP}_\ell = \delta^m_\ell \cdot \delta^q_\ell,$$

where $\delta^q_\ell, \delta^a_q$ are defined from (16) [1]. In conditions (1) on each fragment one of the 12 IPs listed in tables 2 and 5 [1] or DP is realized, i.e. in (2) exactly one $\delta^q_\ell$ equals to 1.

3.2. Distances on $X_\ell$

The statements of IP on $X_\ell$ assume the introduction of distances (where $y = q$ or $y = f$)

$$R_{\infty, \ell}(\tilde{y}; y^\gamma) = \max_{j=i_{\ell-1}^\gamma, \ldots, i^\gamma_\ell} |y_j^\gamma - y^\gamma_j| \quad \text{for } \ p = +\infty,$$

$$R_{p, \ell}(\tilde{y}; y^\gamma) = \left( \sum_{j=i^\gamma_{\ell-1}}^{i^\gamma_\ell} |y^\gamma_j - y^\gamma_j|^p \right)^{1/p} \quad \text{for } \ p \in [1; +\infty).$$

For $\sigma \in \{a, q\}$ and $\delta^q_\ell, \delta^a_q, \delta^f_\ell, \delta^a_f$ from tables 2 and 5 of [1] let’s introduce

$$D^\sigma_{\infty, \ell} = \max \{ \delta^\sigma q \cdot R_{\infty, \ell}(q^\gamma; q^\gamma), \delta^\sigma f \cdot R_{\infty, \ell}(f^\gamma; f^\gamma) \} \quad \text{for } \ p = +\infty,$$

$$R^\sigma_{\infty, \ell}((q^\gamma, f^\gamma); (q^\gamma, f^\gamma)) = \max_{\ell=1, \ldots, r} D^\sigma_{\infty, \ell} \quad \text{for } \ p = +\infty,$$

$$D^\sigma_{p, \ell}((q^\gamma, f^\gamma); (q^\gamma, f^\gamma)) = \left( \delta^\sigma q \cdot R^p_{p, \ell}(q^\gamma; q^\gamma) + \delta^\sigma f \cdot R^p_{p, \ell}(f^\gamma; f^\gamma) \right)^{1/p} \quad \text{for } \ p \in [1; +\infty),$$

$$E^\sigma_{p, \ell} = \left( \delta^\sigma q \cdot \delta^\sigma q_{\ell+1} \cdot |q^\gamma_{\ell+1} - q^\gamma_{\ell+1}|^p + \delta^\sigma f \cdot \delta^\sigma f_{\ell+1} \cdot |f^\gamma_{\ell+1} - f^\gamma_{\ell+1}|^p \right)^{1/p} \quad \text{for } \ p \in [1; +\infty),$$

$$R^p_{p, \ell}((q^\gamma, f^\gamma); (q^\gamma, f^\gamma)) = \left( \sum_{\ell=1}^{r} (D^\sigma_{p, \ell})^p - \sum_{\ell=1}^{r-1} (E^\sigma_{p, \ell})^p \right)^{1/p} \quad \text{for } \ p \in [1; +\infty).$$

Formulas (6) and (8) allow us to reduce formally the statements of various IP (including the DP for which $\delta^q_\ell = \delta^a_q = \delta^f_\ell = \delta^a_f = 0$) on $X_\ell$ to IP$^{(q,f)}_{(m,\tau)}$. Formulas (7) and (10) allow us to reduce formally the statements of various IP (including DP) on $X_\ell$ to IP$^{(q,f)}_{(m,\tau)}$. 

3
3. Simple MNEP

A simple MNEP assumes a sequential solution of problems on fragments from \( \ell = 1 \) to \( \ell = r \):

- if an DP (\( \delta^m_\ell = \delta^s_\ell = 1 \)) is given on \( X_\ell \), then \( q \) and \( f \) are computed on it;
- if an IPIS (\( \delta^s_\ell = 1, \delta^m_\ell = 0 \)) is given on \( X_\ell \), then the parameters \( m^\sim \) or/and \( \tau^\sim \) being free on it are sought in conditions (10) or/and (11) from [1], and also

\[
D^i_{p,\ell} \leq \varepsilon; \tag{11}
\]

- if an IPAS (\( \delta^m_\ell = 0, \delta^s_\ell = 1 \)) is given on \( X_\ell \), then the parameters \( m^\sim \) or/and \( \tau^\sim \) being free on it are sought in conditions (10) or/and (11) from [1] as an approximate solution of the extreme problem

\[
\inf_{m^\sim} D^a_{p,\ell} \text{ for } \delta^{m^\sim}_\ell = 0, \quad \delta^s_\ell = 1, \quad \tag{12}
\]

\[
\inf_{\tau^\sim} D^a_{p,\ell} \text{ for } \delta^{s}_\ell = 1, \quad \delta^{m^\sim}_\ell = 0, \quad \tag{13}
\]

\[
\inf_{m^\sim, \tau^\sim} D^a_{p,\ell} \text{ for } \delta^{m^\sim}_\ell = 0, \quad \delta^{s}_\ell = 0; \quad \tag{14}
\]

- if an IPMS (\( \delta^m_\ell = \delta^s_\ell = 1 \)) is given on \( X_\ell \), then the parameters \( m^\sim, \tau^\sim \) being free on it are sought as an approximate solution of the extreme problem (14) in conditions (10), (11) of [1] and (11).

3.4. Complex MNEP

A complex MNEP assumes simultaneous search for controls \( m^\sim \) and \( \tau^\sim \) satisfying conditions (10) and (11) from [1] on all those fragments \( X^\wedge_\ell \), where they are not given (\( \delta^m_\ell = 0 \) and/or \( \delta^s_\ell = 0 \)), as approximate solutions of the extreme problem

\[
\inf_{m^\sim, \tau^\sim} R^a_p((q^\sim, f^\sim); (q^\vee, f^\vee)) \quad \tag{15}
\]

in condition

\[
R^i_p((q^\sim, f^\sim); (q^\vee, f^\vee)) \leq \varepsilon. \tag{16}
\]

4. Mixed extreme problems

4.1. Mixing condition

Suppose that the fragmentation \( X^\wedge_\ell \), the control \( s \), the numbers \( p \in [1; +\infty] \) and \( \varepsilon > 0 \) are given. Suppose that for each \( \ell = 1, \ldots, r \) one or two of the four parameters \( m, \tau, q, f \) are given on fragment \( X_\ell \), and the others are free on it, i.e. unlike paragraph 3.1

\[
\delta^{m^\sim}_\ell + \delta^s_\ell + \delta^q_\ell + \delta^f_\ell \in \{1; 2\}. \tag{17}
\]

Let’s assume that an extreme problem (EP) is presented at least on one fragment:

\[
\delta^{m^\sim}_\ell + \delta^s_\ell + \delta^q_\ell + \delta^f_\ell = 1. \tag{18}
\]

Let for each of the given observed parameters either an interpolation (in particular, for the IEP) or an approximation statement be indicated, i.e. \( \delta^{c^\sim}_{\ell q}, \delta^{c^\sim}_s, \delta^{c^\sim}_s, \delta^{c^\sim}_f \) satisfying (16), (17) from [1], are given. We find \( \delta^m_\ell \) from (2) and

\[
\delta^{HDEP^{QF}}_\ell = (1 - \delta^{m^\sim}_\ell) \cdot \delta^{s^\sim}_\ell \cdot (1 - \delta^{s^\sim}_\ell) \cdot (1 - \delta^{q^\sim}_\ell) \cdot (1 - \delta^{q^\sim}_\ell) \cdot (1 - \delta^{f^\sim}_\ell); \tag{19}
\]

\[
\delta^{HDEP^{QF}}_\ell = (1 - \delta^{m^\sim}_\ell) \cdot (1 - \delta^s_\ell) \cdot (1 - \delta^{q^\sim}_\ell) \cdot (1 - \delta^{q^\sim}_\ell) \cdot (1 - \delta^{f^\sim}_\ell). \tag{19}
\]

In conditions (17) \( \delta^{HDEP} + \delta^{IEP^{QF}} = 1 \), and on each fragment either one of the 13 NEP listed in 3.1, or one of the two HDEP generalizing the DEP from table 3 [1], or one of the two IEP from table 6 [1] can be given. Thus, in (2) and (19) exactly one \( \delta^i_\ell \) equals to 1.
4.2. Hybrid objective function for introduced fragmentation

If fragmentation $X^\land_{(r)}$ is given, then for hybrid objective function $\Psi$ defined by (38) [1] holds

$$\Psi_{tot} = \sum_{\ell=1}^{r} \Psi_{\ell}, \quad \text{where} \quad \Psi_{\ell} = \sum_{j=i_{\ell-1}+1}^{i_{\ell}} \Psi([x_{j-1}^\land; x_j^\land]; m, \tau, s; \varphi).$$  

(21)

To take into account only fragments with EP, we introduce a clarified objective function

$$\Psi_{tot}^* = \sum_{\ell=1}^{r} \delta_{\ell}^{EPQF} \cdot \Psi_{\ell}.$$  

(22)

The formulas (38) [1], (21) and (22) make it possible to reduce formally the HDEP to the IEP (for $\delta_{\ell}^{HDEP} = 0$ there are no restrictions on the number of solutions (41), (42) [1], and the formally free second parameter $m$ or $\tau$ should be fixed with the use of (22)-(24) [1]). Suppose that in addition to the parameters listed in paragraph 4.1, by means of elements, there is given a continuous on $X_{tot}$ and bounded on $X_{tot}$ function $\varphi(x)$, and for all $\ell = 1, \ldots, r$ the numbers of solutions (41) or (42) [1] on $X_{\ell}$ for $\delta_{\ell}^{IEPQF} = 1$ or for $\delta_{\ell}^{IEPQF} = 1$, respectively, are finite.

4.3. Simple MEP

The statement of basic simple MEP $(Q,F)$ with piecewise-continuous $\varphi$ implies a sequential solution of problems on fragments from $\ell = 1$ to $\ell = r$. In other words, if DP or IP is given on $X_{\ell}$ ($\delta_{\ell}^{NEP} = 1$), then it is necessary to act similarly to paragraph 3.3; if HDEP$_m^{(Q,F)}$ or HDEP$_\tau^{(Q,F)}$ is given on $X_{\ell}$, then it is required in conditions (10) or (11) from [1] to find the controls $m^\sim$, $\tau^\sim$ and the value

$$\Psi_{\ell}^\sim = \Psi(X_{\ell}; m^\sim, \tau^\sim, s; \varphi) \quad \text{for} \quad \delta_{\ell}^{HDEP_m^{QF}} = 1 \quad \text{or} \quad \Psi_{\ell}^\sim = \Psi(X_{\ell}; m, \tau^\sim, s; \varphi) \quad \text{for} \quad \delta_{\ell}^{HDEP_\tau^{QF}} = 1$$  

(23)

as an approximate solution of the problem minimizing of functional (38) [1] on $X_{\ell}$

$$\inf_{m^\sim} \Psi(X_{\ell}; m^\sim, \tau^\sim, s; \varphi) \quad \text{or} \quad \inf_{\tau^\sim} \Psi(X_{\ell}; m, \tau^\sim, s; \varphi);$$  

(24)

if IEP$_{m,\tau}^{(q,F)}$ or IEP$_{m,\tau}^{(Q,F)}$ ($\delta_{\ell}^{IEP_m^{QF}} = 1$ or $\delta_{\ell}^{IEP_{\tau}^{QF}} = 1$) is specified for $X_{\ell}$, then it is required in conditions (10), (11) of [1] and (11) to find the controls $m^\sim$, $\tau^\sim$ and the value

$$\Psi_{\ell}^\sim = \Psi(X_{\ell}; m^\sim, \tau^\sim, s; \varphi),$$  

(25)

as an approximate solution of the problem minimizing of functional (38) [1] on $X_{\ell}$

$$\inf_{m^\sim, \tau^\sim} \Psi(X_{\ell}; m^\sim, \tau^\sim, s; \varphi).$$  

(26)

4.4. Complex MEP

For MEP$_{(Q,F)}$ in the set of problems on fragments there must be HDEP or IEP, hence, statement of the complex MEP should include optimization (22) on $X_{tot}$. According to paragraph 3.4, if IPAS or IPMS is present in the set of problems on the fragments, then the statement of the complex MEP must include optimization (15) on $X_{tot}$. But function (15) is nonnegative in contrast to function (22). So, the result of mixing problems with (15) and (22) is the problem of two-criteria optimization. Let’s consider single-criterion problems only (without IPAS or IPMS).
Statement of basic complex (one-criterion) MEP\((Q,F)\) with piecewise-continuous \(\varphi\) implies simultaneous search for controls \(m^\sim\) and \(\tau^\sim\) satisfying conditions (10), (11) from [1] and (16) on all those fragments \(X^r_\ell\), where they are not given (\(\delta^m_\ell = 0\) and/or \(\delta^r_\ell = 0\)), as approximate solutions of the extreme problem

\[
\inf_{m^\sim,\tau^\sim} \Psi^*_\text{tot}.
\]  

Note that
1) three MEP combining IP\(_m^Q\) and DEP\(_q^Q\) for the case \(X^V=X^V\) are constructed in [3];
2) a simple MEP solution can be used as an initial approximation to solve a complex MEP.

5. Partial restrictions

Analogously to the total restrictions (29)-(31) of [1] defined on \(X_\text{tot}\), the fragmentation allows us to introduce partial restrictions on \(X_\ell\) for \(Q_\ell\), \(F_\ell\), \(N_\ell\).

5.1. Multiple case

The basic statements of HDEP, IEP, IPAS, IPMS considered in [1] on \(X_\text{tot}\) can be supplemented by restrictions on fragments. If the fragmentation \(X^r_\ell\) and segments \(I^{N_\text{tot}} = [\overline{N}_\ell; \overline{N}_\text{tot}]\), \(I^N_1 = [\overline{N}_1; \overline{N}_1]\), \ldots, \(I^N_r = [\overline{N}_r; \overline{N}_r]\) are given, then for example in DEP\(_Q^Q\) it is required in conditions (10), (29) [1] and

\[
N_1 \in I^{N_1}, \ldots, N_r \in I^{N_r},
\]
where \(N_\ell = N(X_\ell; m, \tau, s)\), to find \(m^\sim\) as an approximate solution of problem (27) [1] on \(X_\text{tot}\). For DEP\(_Q^F\) the same. For DEP\(_Q^Q\) and DEP\(_Q^F\) \(N_\ell = N(X_\ell; m, \tau, s)\) is used. Incorrect choice of \(I^{N_\text{tot}}\), \(I^{N_1}, \ldots, I^{N_r}\) leads to an empty domain of admissible solutions. In particular the following conditions must hold

\[
\overline{N}_\text{tot} \leq S^+_0, \quad S^+_0 \leq \overline{N}_\text{tot},
\]
where denoted

\[
S^+_k = \sum_{j=k+1}^{r} \overline{N}_j, \quad S^+_k = \sum_{j=k+1}^{r} \overline{N}_j
\]
for \(k = 0, \ldots, r - 1\). For \(Q\) and \(F\) similarly. Note that conditions (28\(\_1\)), \ldots, (28\(\_r\)) are simpler than (34) [1].

5.2. Polyhedron

Nondegenerate (i.e. \(N_\ell < \overline{N}_\ell\) for \(\ell = 1, \ldots, r\)) conditions (28\(\_1\)), \ldots, (28\(\_r\)) define the interior of the 2r-hedron \(I^{N_1} \times \cdots \times I^{N_r}\), including its boundary, in the space of \(r\) variables \(N_1, \ldots, N_r\).

Example 2. On figure 1 for \(r = 2\) total restrictions \(N_\text{tot} \in I^{N_\text{tot}}\) for \(I^{N_\text{tot}} = [0.6; 1.6]\) and partial restrictions \(N_1 \in I^{N_1}\) and \(N_2 \in I^{N_2}\) for \(I^{N_1} = [0.225; 1.225]\) and \(I^{N_2} = [0.175; 0.575]\) are presented. Lines 1-10 correspond to

1: \(N_1 + N_2 = \overline{N}_\text{tot}\); 2: \(N_1 + N_2 = \overline{N}_\text{tot}\);
3: \(N_1 = \overline{N}_1\); 4: \(N_1 = \overline{N}_1\); 5: \(N_2 = \overline{N}_2\); 6: \(N_2 = \overline{N}_2\);
7: \(N_1 + N_2 = \overline{N}_1 + \overline{N}_2\); 8: \(N_1 + N_2 = \overline{N}_1 + \overline{N}_2\);
9: \(N_1 + N_2 = \overline{N}_1 + \overline{N}_2\); 10: \(N_1 + N_2 = \overline{N}_1 + \overline{N}_2\).

Arrows at 1-8 indicate the position of the half-planes being solutions corresponding to equality in inequalities.

Figure 1. The conditions on \((N_1, N_2)\).
The mutual position of lines 9 and 10 depends on the sign \((\overline{N}_1 + \overline{N}_2) - (N_1 + \overline{N}_2)\). The mutual positions of lines 1 and 8, and lines 7 and 2 correspond to inequalities (29). The positions of line 1 between lines 7 and 9 and line 2 between 8 and 10 correspond to the case of all active restrictions: (29) [1] and both (28_1),(28_2). The position of points \((N_1; N_2)\), satisfying all restrictions, is a convex closed domain being the interior of the polygon \(BDEGKL\), including its boundary.

Nondegenerate (i.e. \(\overline{N}_{\text{tot}} < \overline{N}_{\text{tot}}\)) conditions (29) [1] for

\[
S^+_{0,j} < \overline{N}_{\text{tot}} \quad \text{and} \quad \overline{N}_{\text{tot}} < S^+_{0,j},
\]

(30)
define two parallel hyperplanes that cut off vertices with coordinates \((\overline{N}_1, \ldots, \overline{N}_r)\) and \((\overline{N}_1, \ldots, \overline{N}_r)\), i.e. two \((r-1)\)-dimensional faces are added. If for some \(j \in \{1, \ldots, r\}\) hold

\[
\overline{N}_j < \overline{N}_{\text{tot}} - S^+_{0,j} + \overline{N}_j \quad \text{or/and} \quad \overline{N}_{\text{tot}} - S^+_{0,j} + \overline{N}_j < \overline{N}_j,
\]

(31)

then the part \(\overline{N}_j \leq N_j\) or/and \(N_j \leq \overline{N}_j\) of the condition (28_\(j\)) is/are satisfied, and the corresponding to the equation \(N_j = \overline{N}_j\) or/and \(N_j = \overline{N}_j\) hyperface is/are cut off, respectively.

Lines 11, 12 correspond to equations 11: \(N_1+\overline{N}_2=\overline{N}'_{\text{tot}}\); 12: \(N_1+\overline{N}_2=\overline{N}'_{\text{tot}}\), where \(\overline{N}'_{\text{tot}} = 0.9\) and \(\overline{N}'_{\text{tot}} = 1.3\): the arrows at 11 and 12 determine the position of the strip \(N_{\text{tot}} \in I^N_{\text{tot}}\). For \(j = 1\) both conditions (31) hold for \(I^N_{\text{tot}}\): lines 11 and 12 cut off the faces \(AK\) and \(DF\).

5.3. Repeated case

When conditions (29) [1] are used in the MP in a simple statement, if

\[
\overline{N}_{\text{tot}} \leq S^+_{0,N} \quad \text{and} \quad S^+_{0,N} \leq \overline{N}_{\text{tot}},
\]

(32)

then total conditions are fulfilled automatically, and on each fragment \(X_{\ell}\) for \(\ell\) from 1 to \(r\) only one partial condition (28_\(\ell\)) is used. If (30) is satisfied instead of (32), then on each fragment \(X_{\ell}\) for \(\ell\) from 1 to \(r\) it is required to determine the boundaries of admissible \(N_{\ell}\), i.e. to solve problems \(\overline{N}^*_{\ell} = \min_{N_{\ell}, \ldots, N_r} N_{\ell}\) or/and \(\overline{N}^*_{\ell} = \max_{N_{\ell}, \ldots, N_r} N_{\ell}\) with restrictions (28_\(\ell\)), \ldots, (28_\(r\)) and

\[
\overline{N}_{\text{tot}} \leq S^+_{\ell,N_{\text{tot}}} + \sum_{j=\ell}^{r} N_j \quad \text{for} \quad S^+_{\ell,N_{\text{tot}}} = \sum_{j=\ell}^{r} N_j \quad \text{for} \quad S^+_{\ell,N_{\text{tot}}} = \sum_{j=\ell}^{r} N_{\ell}^\sim,
\]

(33)

where \(N_{\ell}^\sim\) for \(j\) from 1 to \(\ell - 1\) are found solving the DP, IP, HDEP or IEP on \(X_{\ell}\). Then

\[
\overline{N}^*_{\ell} = \max\{\overline{N}_{\ell}; \overline{N}_{\text{tot}} - S^+_{\ell,N_{\text{tot}}}; \overline{N}^*_{\ell}\}, \quad \overline{N}^*_{\ell} = \min\{\overline{N}_{\ell}; \overline{N}_{\text{tot}} - S^+_{\ell,N_{\text{tot}}}; \overline{N}^*_{\ell}\}.
\]

(34)

In conditions of Example 2 if the value \(N_{\ell}^\sim \in [0.425; 1.025]\), then the point \((N_{\ell}^\sim; N_2)\) is in the domain \(BCGH\) on figure 1, and \(N_2 \in I^N_2\); if \(N_{\ell}^\sim \in [0.225; 0.425]\), then the point is in the domain \(BHKL\), and \(N_2 \in [\overline{N}_{\text{tot}} - N_{\ell}^\sim; \overline{N}_2]\); if \(N_{\ell}^\sim \in [1.025; 1.225]\), then the point is in the domain \(CDEG\), and \(N_2 \in [\overline{N}_2; \overline{N}_{\text{tot}} - N_{\ell}^\sim]\).

6. Algorithm of determination the variant of solved problem

We give the scheme of the algorithm.

1. Read unchanged parameters: body type (cylinder, sphere), body radius \(R\) [m], flight altitude \(H\) [km], flight speed \(M_\infty\). Compute the standard atmosphere parameters according to \(H\). Select the gas model (perfect, dissociating, ionized) according to \(M_\infty\) [4].

2. Read the integration mesh \(X^\times\) and meshes \(X^\wedge, X^\vee, X^\wedge\). Check \(X^\wedge\) and \(X^\times \subseteq X^\vee \subseteq X^\times\).
3. Read the restrictions $I^Q$, $I^F$, $I^N$, $I^q$, $I^f$, $I^q_m$, $I^f_m$, $I^q_r$, $I^f_r$ for $X_{int}$ and for each fragment. Read conditions $\Delta m$, $\Delta m$, $\Delta r$, $\Delta r$ for jumps [2].

4. Read $p \in [1; +\infty]$, $\varepsilon > 0$ and the set $(\delta^m, \delta^r, \delta^q, \delta^a_q, \delta^a_f, \delta^a_r)$. Compute (2), (19) for determination of problem type on each fragment.

5. Check of admissible controls set nonemptiness (only conditions of order $k = 0$ and 1 are taken into account); construct test $m$, $m$, $r$, $r$.

6. Read fragments of functions $m$, $\tau$, $q^\varphi$, $f^\varphi$, $\varphi$. Check the elements for correspondence to $I^m$, $I^\tau$, $I^q$, $I^f$. Check $\varphi$.

7. If $M_{0\infty} \geq 10$, then read $s$, else assign $s = 0$.

8. Compute [4, 5] for given fragments of functions $m$, $\tau$, supplemented by the fragments $m$, $m$, $r$, $r$, and $s$ of the estimates $lQ, \ell Q, \ell F, \ell F, lN, \ell N$ on $X_{int}$ and $X_{fr}$. Check values $Q$, $Q$, $F$, $F$, $N$, $N$ for correspondence to $[lQ, \ell Q], [lF, \ell F], [lN, \ell N]$.

9. If $r > 1$, then read the MP statement: simple or complex. Check $q^\delta_r$ for complex IEP.

10. If $r = 1$, then solve problem as in [1], else solve MIP as in paragraphs 3.3, 3.4, 4.3, 4.4.

Remarks.

1) The saved solution $m^\sim$ and/or $\tau^\sim$ can be read in Step 6 as the initial approximation for the problem with the modified $R$, $H$, $M_{0\infty}$, $X^X$, $X^{(r)}$, $I^Q$, $I^q$, $I^m$, $I^\tau$, $\Delta^m$, $\Delta^r$, $\varepsilon$, $p$, $m$, $m$, $r$, $r$, $s$, $q^\varphi$, $f^\varphi$, $\varphi$ and/or type of statement (simple/complex, interpolation/approximation).

2) The saved results $m^\sim$, $\tau^\sim$, $q^\sim$, $f^\sim$, $\eta^\sim$, $Q^\sim$, $F^\sim$, $N^\sim$ are used to elaborate models of control parameters $m$, $\tau$, models of observed parameters $q^\varphi$, $f^\varphi$, $\eta^\varphi$ and models of restrictions $I^m$, $I^\tau$, $I^q$, $I^f$, $I^q_m$, $I^f_m$, $I^q_r$, $I^f_r$.

3) The saved $m^\sim$, $\tau^\sim$, $q^\sim$, $f^\sim$, $\eta^\sim$ are defined for the meshes $X^\sim$, $X^\sim$ on which were obtained. To prepare them for use on other meshes a special program of element conversion is applied [2].

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