Generalising Wigner’s theorem

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Abstract
We analyse linear maps of operator algebras $B_\mathcal{H}(\mathcal{H})$ mapping the set of rank-$k$ projectors onto the set of rank-$l$ projectors surjectively. A complete characterisation of such maps for prime $n = \dim \mathcal{H}$ is provided. A particular case corresponding to $k = l = 1$ is well known as Wigner’s theorem. Hence our result may be considered as a generalisation of this celebrated Wigner’s result.

Keywords: Wigner’s theorem, linear preservers, projectors, maps between operator algebras

1. Introduction

The celebrated Wigner’s theorem [1] in its original formulation says that any map $\Phi$ between rank-1 projectors in a Hilbert space preserving the Hilbert–Schmidt product, i.e. $(\Phi(P_1), \Phi(P_2))_{\text{HS}} = (P_1, P_2)_{\text{HS}}$, is of the form:

$$\Phi(X) = UXU^\dagger \quad \text{or} \quad \Phi(X) = UX^*U^\dagger,$$

where $X^\dagger$ denotes a transposition with respect to a fixed orthonormal basis in $\mathcal{H}$ and $U$ is a unitary operator (see also [2] and recent analysis in [3]). It is clear that any such map induces a unitary or antunitary operation in the original Hilbert space. Wigner’s theorem is sometimes reformulated as follows [6]: one restricts to linear maps $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ which are not necessarily Hilbert–Schmidt isometries. Now, any such map that maps bijectively rank-1 projectors to rank-1 projectors is of the form (1). Clearly, a map mapping rank-1 projectors to rank-1 projectors is by construction a positive map [4, 5] and hence Wigner’s theorem states that a positive trace-preserving map $\Phi$ has a positive inverse if and only if it has the form (1).

In this paper we consider linear maps mapping surjectively rank-$k$ projectors to rank-$l$ projectors. We show for which $k$, $l$ such maps exist and that for prime dimensions of the Hilbert space they have exactly the form (1). We also provide an example of a map which is
not of the form (1). Interestingly, linear maps acting surjectively between sets of projectors of fixed rank maps have recently attracted attention in the problem of entanglement detection [7].

Let us consider a real space of self-adjoint operators $B_{d\ell}({\mathcal{H}})$. We denote the set of rank-$k$ projectors supported on a subspace $V \subset {\mathcal{H}}$ by $\mathcal{P}_k(V) \subset B_{d\ell}$. It is a smooth manifold of dimension $k(n-k)$. Let $\Phi$ be a linear endomorphism of $B_{d\ell}({\mathcal{H}})$ and let $\Phi (\mathcal{P}_k(V)) = \mathcal{P}_k(V)$. From the fact that $\mathcal{P}_k(V)$ spans the whole $B_{d\ell}({\mathcal{H}})$, one gets that $\Phi$ is a surjective endomorphism —hence a bijection, hence also a bijection between sets $\mathcal{P}_k(V)$ and $\mathcal{P}_k(V)$. This implies that $l = k$ or $l = n - k$. There is a natural linear isomorphism between $\mathcal{P}_k(V)$ and $\mathcal{P}_{n-k}(V)$:

$$R_k(X) = \frac{1}{k} \operatorname{tr}X - X.$$  

Hence any bijection $\Phi : \mathcal{P}_k(V) \to \mathcal{P}_{n-k}(V)$ can be represented as a composition of $R_k$ and a bijective endomorphism of $\mathcal{P}_k(V)$. Therefore, it suffices to characterise all bijective endomorphisms of $\mathcal{P}_k(V)$.

Note that this problem belongs to the so-called linear preserver problem which deals with the characterisation of linear operators that leave certain properties or certain subsets in their domain invariant. This programme has already been started by Frobenius [9]. Well known examples of linear preservers are rank preservers, nilpotency preservers and spectrum preservers: $\Phi$ is a rank preserver iff $\Phi(X) = MXN$ or $\Phi(X) = MX'N$, where $M, N$ are invertible elements from $M_n({\mathbb{C}})$. $\Phi$ provides a preserver of nilpotency iff $\Phi(X) = MXN$ or $\Phi(X) = MX'N$, where $M, N \in M_n({\mathbb{C}})$ such that $MN = cI_n$ and $c \in {\mathbb{C}}$. Finally, $\Phi$ is a spectrum preserver iff $\Phi(X) = MXM^{-1}$ or $\Phi(X) = MX'M^{-1}$ (see for example [8, 10]).

Clearly, the $\Phi$ defined in (1) is an example of a rank preserver and a nilpotency preserver, but in our problem the restriction is weaker—we demand preserving the nilpotency only for one given value of the rank (the set of nilpotent Hermitian operators splits into connectivity components grouping projectors of the same rank). With a weaker assumption one can expect that the resulting set of linear operations can be in general greater. Indeed, for a special choice of $n$ and $k$ we provide an example of an invertible map preserving a rank-$k$ projector not of the form (1).

Recently Marciniak [11] considered a related problem and showed that every positive map $\Phi$ such that rank $\Phi (P) \leq 1$ for any rank-$1$ projector $P$ is the rank-$1$ preserver and has the form (1).

2. Main results

This section provides the main results of the paper. Let us start with the following:

**Proposition 1.** Any $\Phi : B_{d\ell}({\mathcal{H}}) \to B_{d\ell}({\mathcal{H}})$ mapping bijectively $\mathcal{P}_k(V)$ onto itself preserves the orthogonality.

**Proof.** If $2k > n$, then there are no non-zero mutually orthogonal elements in $\mathcal{P}_k(V)$ and the proposition is true in a trivial way. We will consider the case when $2k \leq n$. Let $P_V$ be the projector onto a $2k$-dimensional subspace $V \subset {\mathcal{H}}$. $P_V$ can be decomposed in various ways into the sum of two rank-$k$ orthogonal projectors $P_1$ and $P_2$. They are mapped via $\Phi$ onto two rank-$k$ projectors $Q_1 = \Phi (P_1)$ and $Q_2 = \Phi (P_2)$. From the positivity of $Q_1$ and $Q_2$ one has $\operatorname{Im} Q_1 = \operatorname{Im} \Phi (P_1) \subset \operatorname{Im} \Phi (P_V)$. One can repeat it for any choice of $P_1$ and $P_2$; hence
∀P ∈ P₂(V) Im Φ(P) ⊂ Im Φ(P₂) and thus Φ(P₂(V)) ⊂ B_H(Im Φ(P₂)). Because P₂(V) spans the whole B_H(V), we have Φ(B_H(V)) ⊂ B_H(Im Φ(P₂)).

While Φ is bijective on P₂(H), it is also bijective on B_H(H). Hence dim B_H(V) ≤ dim B_H(Im Φ(P₂)) and hence dim V ≤ dim Im Φ(P₂). While Φ(P₂) is a sum of two rank-k projectors, dim Im Φ(P₂) ≤ 2k = dim V and hence dim Im Φ(P₂) = dim V. Φ establishes a bijection between B_H(V) and B_H(Im Φ(P₂)) and hence between P₂(V) and P₂(Im Φ(P₂)). Any rank-k projector Q ∈ P₂(Im Φ(P₂)) can be realised as Φ(P) for some P ∈ P₂(V).

Now we choose the basis of Im Φ(P₂) to make Φ(P₂) diagonal. Take a rank-k dimensional projector Q ∈ P₂(Im Φ(P₂)), diagonal in this base (commuting with Φ(P₂)). The operator Φ(P₂) − Q = Φ(P₂) − Φ(P) = Φ(P₂ − P) is a rank-k projector, diagonal in the chosen basis. One can easily find that this implies that the only possible values on the diagonal of Φ(P₂) are 1s and 2s. But the rank of Φ(P₂) and its trace are equal 2k, so Φ(P₂) is the projector onto Im Φ(P₂).

For any two orthogonal projectors P₁ and P₂ the operator Φ(P₁ + P₂) = Φ(P₁) + Φ(P₂) is a rank-2k dimensional projector; hence the projectors Φ(P₁) and Φ(P₂) are orthogonal. □

One immediately obtains the following:

**Corollary 1.** Any Φ : B₇(H) → B₂(H) mapping bijectively P₂(H) onto itself maps bijectively P₇(H) onto itself for q ∈ N.

**Remark 1.** Note that Wigner’s theorem immediately follows from preserving the orthogonality relation and the properties of a spectrum preserver. Indeed, preserving the orthogonality relation of rank-1 projectors implies that a Schatten decomposition ∑ₜ λₜPᵣ of a Hermitian operator is mapped to another Schatten decomposition with the same spectrum. Such a map is therefore a spectrum preserver and hence has the form Φ(X) = MXM⁻¹ or Φ(X) = MXᵣM⁻¹. Finally, preservation of orthogonality implies that M is unitary.

**Proposition 2.** Assume that k = n mod l. If any linear map Ψ : B₇(H) → B₂(H) transforming P₂(H) onto itself bijectively is of the form (1), then also each map Φ : B₇(H) → B₂(H) transforming P₇(H) onto itself bijectively is of the form (1).

**Proof.** Let Φ : B₇(H) → B₂(H) map bijectively P₇(H) onto itself. Let n = k + q · l. Then given corollary 1, for any P ∈ P₇(H) one has Φ(P) ∈ Pₗ(H). Now, any projector from Pₗ(H) may be written as I − Pₖ with Pₖ ∈ Pₖ(H) and hence

\[
\Phi(I − Pₖ) = \Phi(I) − \Phi(Pₖ) = I − Qₖ,
\]

for some Qₖ ∈ Pₖ(H). One therefore has

\[
Qₖ = \Phi(Pₖ) − \Phi(I) + I.
\]

Let us consider a linear map Φ(X) = Φ(X) − \frac{1}{k}Tr(X)(Φ(I) − I). The above relation implies that Φ transforms P₇(H) onto itself bijectively, so given our assumption it can be written as

\[
\Phi(X) = UXₖU^†,
\]
where $\widetilde{X} = X$ or $\widetilde{X} = X'$. Thus,

$$\Phi(X) = \frac{1}{k}[\Phi(I) - I]\Tr X + U\widetilde{X}U^\dagger. \quad (6)$$

In particular if $X = P_l \in \mathcal{P}_l(\mathcal{H})$ one has

$$\Phi(P_l) = \frac{1}{k}[\Phi(I) - I] + Q_l = D + Q_l,$$  \quad (7)

where $Q_l = U\overline{P}_l U^\dagger \in \mathcal{P}_l(\mathcal{H})$. To complete the proof we need to show that $D = 0$. Since $D + Q_l$ is a projector one has $(D + Q_l)^2 = D + Q_l$ and hence

$$DQ_l + Q_lD = D - D^2. \quad (8)$$

Taking into account that $\Phi$ is trace-preserving one has $\Tr D = 0$ which implies $2\Tr(DQ_l) = -\Tr D^2$ for all $Q_l \in \mathcal{P}_l(\mathcal{H})$. Now, if $Q_l$ is a projector on the $l$-dim. subspace spanned by eigenvectors of $D$ corresponding to the $l$ largest eigenvalues of $D$, i.e. $d_1 \geq d_2 \geq \ldots \geq d_l$, then $2\Tr(QD) = 2\sum_{i=1}^{l} d_i = -\Tr D^2 \leq 0$ but since $D$ is traceless one has $\sum_{i=1}^{l} d_i \geq 0$ which proves that $D = 0$. \hfill $\square$

The main result of the paper is provided by the following:

**Theorem 1.** If $n$ is prime, then any $\Phi$ mapping surjectively rank-$k$ projectors onto rank-$k$ projectors is of the form $(1)$.

**Proof.** Let us define a sequence via formulæ $k_{i+1} = n \mod k_i$ and $k_0 = k$. This is a strictly decreasing, finite sequence, and it terminates at $0$ and let $0 = n \mod k_0$, that is, the sequence reads $\{k_0 = k > k_1 > k_2 > \ldots > k_s > 0\}$. Given proposition 2, if any $\Psi : B_l(\mathcal{H}) \rightarrow B_l(\mathcal{H})$ mapping bijectively $\mathcal{P}_{k_s}(\mathcal{H})$ onto itself is of the form $(1)$, then any $\Psi : B_l(\mathcal{H}) \rightarrow B_l(\mathcal{H})$ mapping bijectively $\mathcal{P}_{k_s}(\mathcal{H})$ onto itself is of the form $(1)$ as well. Now, given Wigner’s theorem, if $k_s = 1$ then for any $k_i < k_s$ in the sequence, any $\Psi : B_l(\mathcal{H}) \rightarrow B_l(\mathcal{H})$ mapping bijectively $\mathcal{P}_{k_i}(\mathcal{H})$ onto itself is of the form $(1)$, particularly $k_0 = k$. Note that $k_s$ is by construction a divisor of $n$ and hence if $n$ is prime, then $k_s = 1$ which completes the proof. \hfill $\square$

Observe that $n = g_k k_i + k_{i+1}$ and if for some number $d$ one has $d|n$ and $d|k_i$ then $d|k_{i+1}$, so the common divisors of $n$ and $k_0 = k$ are also the common divisors of all elements of the sequence. Thus this method does not give a conclusive answer if the starting point $k_0 = k$ is not relatively prime to $n$. If $n$ and $k$ are relatively prime then a conclusive answer is not guaranteed. Indeed, if $k = 3$ and $n = 10$ one has a conclusive answer, but for $k = 3$ and $n = 8$ the method does not give a conclusive answer. Considering that $k_s$ is by construction a divisor of $n$, one has the following:

**Remark 2.** To complete the characterisation of surjective maps from $\mathcal{P}_k(\mathcal{H})$ into $\mathcal{P}_k(\mathcal{H})$ it is enough to characterise these maps for $k|n$.

**Remark 3.** Let $n = 2k$. Consider

$$\Phi(X) = \frac{1}{k}[2k-I]\Tr X - X. \quad (9)$$
It is clear that $\Phi$ maps rank-$k$ projectors onto rank-$k$ projectors but evidently they do not have the form (1). Indeed, if $P$ is a rank-$1$ projector then $\Phi(P)$ has rank $n - 1$ and hence it is not a rank-$1$ projector. Note that $\Phi^{-1} = \Phi$.

Such maps have already been defined (2) and because $\mathcal{P}_k(\mathcal{H})$ and $\mathcal{P}_{n-k}(\mathcal{H})$ are the same set, these maps are endomorphisms. This encourages us to claim that these are the only additional endomorphisms. Therefore we propose the following:

**Conjecture 1.** Let $n = 2k$ and $\Phi : \mathcal{P}_k(\mathcal{H}) \to \mathcal{P}_k(\mathcal{H})$ surjectively. Then the map is of the form $\Phi$ or $\Phi \circ R_k$, where $\Phi$ is of the form (1) and $R_k$ is defined by (2).

This conjecture allows us to perform perfect characterisation of surjective maps $\mathcal{P}_k(\mathcal{H}) \to \mathcal{P}_k(\mathcal{H})$.

**Proposition 3.** Let us assume that Conjecture 1 is true. If $k|n$, $n/k > 2$ and $\Phi : \mathcal{P}_k(\mathcal{H}) \to \mathcal{P}_k(\mathcal{H})$ is surjective, then $\Phi$ has the form (1).

**Proof.** Let $\{e_i\}_{i=0}^{n-1}$ be the standard basis of $\mathcal{H}$ and let $P_i$ be a rank-$k$ projector defined by

$$P_i = \sum_{j=0}^{k-1} |e_{ik+j}\rangle \langle e_{ik+j}|, \quad i = 0, 1, \ldots, n/k - 1.$$  

It is clear that $P_i$ and $P_j$ are mutually orthogonal for $i \neq j$. Now, given proposition 1, a set of projectors $\{P_i\}$ is mapped onto the set of pairwise orthogonal rank-$k$ projectors. Now, since

$$P_0 + \ldots + P_n/k = \mathbb{I} = \Phi(P_0) + \ldots + \Phi(P_{n/k})$$

projectors $\{\Phi(P_0), \ldots, \Phi(P_{n/k})\}$ are unitarily equivalent to $\{P_0, \ldots, P_{n/k}\}$; that is, $\Phi(P_i) = U P_i U^\dagger$ for some unitary $U$. Without losing generality we may assume that $U = \mathbb{I}$; that is, $\Phi$ maps $P_0$ to $P_i$. Let $V_i = \text{span}\{e_{ik}, \ldots, e_{ik+(k-1)}\}$ be the range of $P_i$ and let $I_{ij}$ be the projector onto the subspace $V_i \oplus V_j$. Let

$$\Phi_{ij} = I_{ij} \circ \Phi \circ I_{ij},$$

be a restriction of $\Phi$ to the subspace $V_i \oplus V_j$. By conjecture 1 it is of the form

$$\Phi_{ij}(X) = U_{ij} \Phi_{ij}(X) U_{ij}^\dagger,$$

where $\Phi_{ij}$ maps $X$ onto $X, X^\dagger, \frac{1}{2}I_{ij} \text{Tr}X - X$ or $\frac{1}{2}I_{ij} \text{Tr}X - X'$. One has that for all $i, j$ $\Phi_{ij}(P_i) = P_i$ and $\Phi_{ij}(P_j) = P_j$, and hence for all $\alpha, \beta$

$$\begin{bmatrix} \alpha P_i \\ \beta P_j \end{bmatrix} = U_{ij} \begin{bmatrix} \alpha P_i \\ \beta P_j \end{bmatrix} U_{ij}^\dagger = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \alpha P_i \\ \beta P_j \end{bmatrix}$$

which implies that $U_{ij}$ is block-diagonal if $\Phi_{ij}(X) = X$ or $X'$ and $U_{ij}$ is block-antidiagonal if $\Phi_{ij}(X) = \frac{1}{2}I_{ij} \text{Tr}X - X$ or $\frac{1}{2}I_{ij} \text{Tr}X - X'$.

Moreover, if $\psi \in V_i$ then

$$\Phi_{ij}(|\psi\rangle\langle\psi|) = I_{ij} \Phi(|\psi\rangle\langle\psi|)$$

does not depend on $j$, and similarly if $\phi \in V_j$ then

$$\Phi_{ij}(|\phi\rangle\langle\phi|) = I_{ij} \Phi(|\phi\rangle\langle\phi|)$$

does not depend on $i$. It follows that all $\Phi_{ij}$ do not depend on $i, j$ and hence $\Phi_{ij}(X) = X$ or $X'$ (otherwise it could not give the same result for different $j$ if $n/k > 2$, as the reduction map has the information about the trace of the second block). Now we know that all $U_{ij}$ are block-
diagonal and again, because $\Phi_j(\langle \Psi \rangle \langle \Psi \rangle)$ has to give the same result for all $j$ one gets that $U_j = U_i \oplus U_j$.

Finally we get that

$$\Phi(X) = U X U^\dagger \quad \text{or} \quad \Phi(X) = U X' U^\dagger$$

with $U = \bigoplus_{i=1}^n U_i$, which ends the proof. \qed

3. Conclusions

Let us summarise the paper by the following remarks:

**Remark 4.** Let us observe that if we relax the condition that the map $\Phi$ is invertible then one may have maps from $\mathcal{P}_k(\mathbb{C}^n)$ to $\mathcal{P}_l(\mathbb{C}^p)$ with $l = n - k$. A well known example is provided by the Breuer–Hall map $\Phi_{BH} : M_{2n}(\mathbb{C}) \to M_{2n}(\mathbb{C})$ defined as follows [12–14]

$$\Phi_{BH}(X) = \frac{1}{2(n - 1)}(I_{2n} \text{Tr} X - UXU^\dagger), \quad (10)$$

where $U$ is an arbitrary anti-symmetric $2n \times 2n$ matrix. $\Phi_{BH}$ maps rank-1 projectors onto projectors of rank $2(n - 1)$. It is evident that $\Phi_{BH}$ is not invertible.

**Remark 5.** Let us observe that maps (1) are characterised by the following property: $\Phi$ is positive and trace-preserving and $\Phi^{-1} = \Phi^*$, where the dual map $\Phi^*$ is defined by $(X, \Phi(Y))_{HS} = (\Phi^*(X), Y)_{HS}$. Interestingly, map (9) is characterised by the following property: $\Phi$ is trace-preserving and $\Phi^{-1} = \Phi^* = \Phi$. It means that both (1) and (9) are isometries with respect to the Hilbert–Schmidt product and hence the corresponding eigenvalues satisfy $|\lambda_i| = 1$.

On the other hand, any rank-1 projector $P$ can be decomposed as a combination of rank-$k$ projectors commuting with $P$ and hence any Hermitian operator can be decomposed as a combination of $n$ commuting rank-$k$ projectors. This combination is mapped onto another combination of rank-$k$ projectors (in general not commuting) with the same coefficients and hence there exists a common upper bound for the maximal eigenvalue of all invertible maps mapping $\mathcal{P}_k(\mathcal{H})$ onto themselves. While such maps form a group, their eigenvalues have to lie therefore on the unit circle and at least one of them is equal 1. We stress that this property is not equivalent to a Hilbert–Schmidt isometry. If one could prove that for any invertible map $\Phi$ mapping $\mathcal{P}_k(\mathcal{H})$ onto itself its adjoint map $\Phi^*$ has the same property, then it would imply that this class is a subclass of Hilbert–Schmidt isometries.

**Remark 6.** The crucial difference between (1) and (9) is positivity. Let us recall [5] that a trace-preserving map is positive iff

$$\|\Phi(X)\|_1 \leq \|X\|_1, \quad (11)$$

for all self-adjoint elements $X$. It is clear that (9) is not positive and hence violates (11). Interestingly, for $X$ that is traceless one has

$$\|\Phi(X)\|_1 = \|-X\|_1 = \|X\|_1. \quad (12)$$

In particular taking two density operators $\rho_1$ and $\rho_2$ one has

$$\|\Phi(\rho_1 - \rho_2)\|_1 = \|\rho_1 - \rho_2\|_1. \quad (13)$$
which means that (1) preserves the distinguishability between arbitrary quantum states. Clearly, (1) enjoys the same property.

Finally, it is hoped that the presented result finds applications in quantum information theory (e.g. entanglement detection) and the analysis of symmetries of quantum systems (e.g. the evolution of quantum systems).

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