A NEW CLASS OF MINIMAL ASYMPTOTIC BASES

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Abstract. A set $A$ of nonnegative integers is an asymptotic basis of order $h$ if every sufficiently large integer can be represented as the sum of $h$ not necessarily distinct elements of $A$. An asymptotic basis $A$ is minimal if removing any element of $A$ destroys every representation of infinitely many integers. In this paper, a new class of minimal asymptotic bases is constructed.

1. $G$-adic asymptotic bases

Let $N_0 = \{0, 1, 2, 3, \ldots \}$ be the set of nonnegative integers and let $h$ be a positive integer. Let $A_0, A_1, \ldots, A_h$ be subsets of $N_0$. We define the sumset

$$A_1 + \cdots + A_h = \{ a_1 + \cdots + a_h : a_i \in A_i \text{ for all } i = 1, 2, \ldots, h \}$$

and the $h$-fold sumset

$$hA = A + \cdots + A = \{ a_1 + \cdots + a_h : a_i \in A \text{ for all } i = 1, 2, \ldots, h \}.$$

The set $A$ is a basis of order $h$ if every nonnegative integer can be represented as the sum of $h$ not necessarily distinct elements of $A$, that is, if $hA = N_0$. The set $A$ is an asymptotic basis of order $h$ if $hA$ contains all sufficiently large integers. An asymptotic basis of order $h$ is minimal if no proper subset of $A$ is an asymptotic basis of order $h$. Thus, if $A$ is a minimal asymptotic basis of order $h$, then, for all $a \in A$, there are infinitely many integers $n$ such that removing $a$ from $A$ destroys every representation of $n$ as a sum of $h$ elements of $A$.

Minimal asymptotic bases are extremal objects in additive number theory, and are related to the conjecture of Erdős and Turán [4] that the representation function of an asymptotic basis of order $h$ must be unbounded.

At this time there are few explicit constructions of minimal asymptotic bases. Nathanson [11, 12] used a 2-adic construction to produce the first examples of minimal asymptotic bases. This method was extended to $g$-adically defined sets by Chen [2], Chen and Chen [1], Chen and Tang [3], Jia [5], Jia and Nathanson [6], Lee [7], Li and Li [8], Ling and Tang [9, 10], Sun [15, 16], and Sun and Tao [17]. This paper constructs a new class of minimal asymptotic bases.

An interval of integers of length $t$ is a set of $t$ consecutive integers. For $u, v \in N_0$ with $u \leq v$, the set

$$[u, v] = \{ x \in N_0 : u \leq x \leq v \}$$

is an interval of integers of length $v - u + 1$. Thus, $[1, h] = \{1, 2, \ldots, h\}$.

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A $G$-adic sequence is a strictly increasing sequence of positive integers $G = (g_i)_{i=0}^\infty$ such that $g_0 = 1$ and $g_{i-1}$ divides $g_i$ for all $i \geq 1$. Let $(d_i)_{i=1}^\infty$ be the sequence of positive integers defined by

$$d_i = \frac{g_i}{g_{i-1}}.$$ 

For all $i \geq 1$, we have

(1) \hspace{1cm} d_i \geq 2 \\

and

(2) \hspace{1cm} g_i = d_1 d_2 \cdots d_i.

For all $i$ and $j$ with $0 \leq i < j$, we have

(3) \hspace{1cm} \frac{g_{i+j}}{g_i} = d_{i+1} d_{i+2} \cdots d_{i+j}.

Every positive integer $n$ has a unique $G$-adic representation

$$n = \sum_{j=0}^\infty x_j g_j$$

where

$$x_j \in [0, d_{j+1} - 1]$$

for all $j \in \mathbb{N}_0$ and $x_j = 0$ for all sufficiently large $j$ (Nathanson [13, 14]). Equivalently, for every positive integer $n$, there is a unique nonempty finite set $F \subseteq \mathbb{N}_0$ and a unique set $\{x_j : j \in F\}$ such that

(4) \hspace{1cm} n = \sum_{j \in F} x_j g_j

where $x_j \in [1, d_{j+1} - 1]$ for all $j \in F$.

For every integer $g \geq 2$, the usual $g$-adic representation uses the $G$-adic sequence $G = (g^i)_{i=0}^\infty$ with quotients $d_i = g$ for all $i \geq 1$.

**Lemma 1.** Let $G = (g_i)_{i=0}^\infty$ be a $G$-adic system. If $n = \sum_{j \in F} x_j g_j$ is a positive integer with $x_j \in [1, d_{j+1} - 1]$ for all $j \in F$, then

$$g_M \leq n < g_{M+1}$$

if and only if

$$\text{max}(F) = M.$$ 

**Proof.** If $\text{max}(F) = M$, then $F \subseteq [0, M]$ and

$$g_M \leq x_M g_M \leq n = \sum_{j \in F} (d_{j+1} - 1) g_j$$

$$\leq \sum_{j=0}^M (d_{j+1} - 1) g_j = \sum_{j=0}^M (g_{j+1} - g_j)$$

$$= g_{M+1} - 1 < g_{M+1}.$$ 

Conversely, if $g_M \leq n < g_{M+1}$ and $\text{max}(F) = M'$, then the inequality

$$g_{M'} \leq x_{M'} g_{M'} \leq n < g_{M+1}$$

is satisfied.
implies $M' \leq M$. If $M' \leq M - 1$, then

$$n \leq \sum_{j=0}^{M'} (d_{j+1} - 1) g_j < g_{M'+1} \leq g_M$$

which is absurd. Therefore, $g_M \leq n < g_{M+1}$ implies $\max(F) = M$. This completes the proof. \hfill \Box

Let $W$ be a nonempty set of nonnegative integers, and let $F^*(W)$ be the set of all nonempty finite subsets of $W$. Let $G = (g_i)_{i=0}^\infty$ be a $G$-adic sequence. We define the set of positive integers

$$A_G(W) = \left\{ \sum_{j \in F} x_j g_j : F \in F^*(W) \text{ and } x_j \in [d_{j+1} - 1] \right\}.$$ 

Note that $0 \notin A_G(W)$ because $\emptyset \notin F^*(W)$.

Let $h \geq 2$. A partition of $\mathbb{N}_0$ is a sequence $W = (W_i)_{i=0}^{h-1}$ of nonempty pairwise disjoint sets such that $\mathbb{N}_0 = W_0 \cup W_1 \cup \ldots \cup W_{h-1}$.

**Theorem 1.** Let $h \geq 2$ and let $W = (W_i)_{i=0}^{h-1}$ be a partition of $\mathbb{N}_0$. Let $G = (g_i)_{i=0}^\infty$ be a $G$-adic sequence. The set

$$hA_G(W) = \bigcup_{i=0}^{h-1} A_G(W_i)$$

is an asymptotic basis of order $h$ with $h$-fold sumset $hA_G(W) = \{ n \in \mathbb{N}_0 : n \geq h \}$.

**Proof.** The smallest integer in the set $A_G(W)$ is $1 = 1 \cdot g_0$. It follows that $h \in hA_G(W)$ but $[0, h-1] \cap hA_G(W) = \emptyset$.

Every positive integer $n$ has a unique $G$-adic representation $n = \sum_{j \in F} x_j g_j$, where $F$ is a nonempty finite set of nonnegative integers and $x_j \in [0, d_{j+1} - 1]$. For $i \in [0, h-1]$, let

$$F_i = F \cap W_i$$

and

$$n_i = \sum_{j \in F_i} x_j g_j.$$ 

If $F_i = \emptyset$, then $n_i = 0 \notin A_G(W)$. If $F_i \neq \emptyset$, then

$$n_i = \sum_{j \in F_i} x_j g_j \in A_G(W_i) \subseteq A_G(W).$$

Let $L = \{ i \in [0, h-1] : F_i \neq \emptyset \} = \{ i \in [0, h-1] : n_i \geq 1 \}$ and $|L| = \ell_0$.

We have $\ell_0 \in [1, h]$ and

$$n = \sum_{i \in L} n_i \in \ell_0 A_G(W).$$

Let $\ell$ be the largest integer such that $\ell \leq h$ and $n \in \ell A_G(W)$. We must prove that $\ell = h$. 

It follows that $n = n_1 + \cdots + n_{k-1} + n_k + n_{k+1} + \cdots + n_\ell$.

For each $i \in [1, \ell]$ there is an integer $s_i \in [0, h-1]$ and a set $F_{s_i} \in \mathcal{F}^*(W_{s_i})$ such that $n_i$ has the $G$-adic representation

$$n_i = \sum_{j \in F_{s_i}} x_j g_j \in A_G(W_{s_i}).$$

Suppose that $\ell < h$. If $|F_{s_i}| \geq 2$ for some $k \in [1, \ell]$, then there are nonempty sets $F'_{s_i}$ and $F''_{s_i}$ such that

$$F_{s_k} = F'_{s_k} \cup F''_{s_k} \quad \text{and} \quad F'_{s_k} \cap F''_{s_k} = \emptyset.$$ 

The integers

$$n'_k = \sum_{j \in F'_{s_k}} x_j g_j \in A_G(W_{s_k}) \quad \text{and} \quad n''_k = \sum_{j \in F''_{s_k}} x_j g_j \in A_G(W_{s_k})$$

satisfy

$$n_k = n'_k + n''_k$$

and so

$$n = n_1 + \cdots + n_{k-1} + n'_k + n''_k + n_{k+1} + \cdots + n_\ell \in (\ell+1)A_G(W).$$

This contradicts the maximality of $\ell$, and so $|F_{s_i}| = 1$ for all $i \in [1, \ell]$ and

$$n_i = x_j g_j.$$

for some $g_j \in W_{s_i}$ and $x_j \in [0, d_j+1 - 1]$.

If $x_j \geq 2$ for some $k \in [1, \ell]$, then

$$g_{jk} \in A_G(W_{s_k}) \quad \text{and} \quad (x_j - 1)g_{jk} \in A_G(W_{s_k})$$

and

$$n_k = g_{jk} + (x_j - 1)g_{jk}.$$ 

It follows that

$$n = n_1 + \cdots + n_{k-1} + g_{jk} + (x_j - 1)g_{jk} + n_{k+1} + \cdots + n_\ell \in (\ell+1)A_G(W),$$

which again contradicts the maximality of $\ell$. Therefore, $x_j = 1$ for all $i \in [1, \ell]$, and

$$n = g_1 + \cdots + g_{jk} + \cdots + g_{j_\ell}.$$

If $j_k \geq 1$ for some $k \in [1, \ell]$, then

$$g_{jk} = g_{jk-1} + (d_{jk} - 1)g_{jk-1}$$

and

$$n = g_1 + \cdots + g_{jk-1} + (d_{jk} - 1)g_{jk-1} + \cdots + g_{j_\ell} \in (\ell+1)A_G(W),$$

which also contradicts the maximality of $\ell$. Therefore, $j_k = 0$ and $g_{jk} = g_0 = 1$ for all $k \in [1, \ell]$, and so

$$n = \underbrace{g_0 + \cdots + g_0}_{\ell \text{\ summands}} = \underbrace{1 + \cdots + 1 + \cdots + 1}_{\ell \text{\ summands}} = 1 \leq h - 1,$$

which is absurd. This completes the proof. $\Box$
Theorem 2. Let \( h \geq 2 \) and let \( W = (W_i)_{i=0}^{h-1} \) be a partition of \( \mathbb{N}_0 \). The set
\[
A = \{0\} \cup A_G(W)
\]
is a basis of order \( h \) but not a minimal asymptotic basis of order \( h \).

Proof. Because \( 0 \in A \) and \( 1 \in A_G(W) \subseteq A \), we have
\[
\ell = (h - \ell) \cdot 0 + \ell \cdot 1 \in hA
\]
for all \( \ell \in [0, h - 1] \). By Theorem \([4]\)
\[
\{n \in \mathbb{N}_0 : n \geq h\} = hA_G(W) \subseteq hA
\]
and so \( A \) is a basis of order \( h \).

The set \( A \) is not a minimal asymptotic basis of order \( h \) because \( 0 \in A \) and the removal of \( 0 \) from \( A \) gives the set \( A \setminus \{0\} = A_G(W) \), which is still an asymptotic basis of order \( h \). This completes the proof. \( \square \)

2. Minimal asymptotic bases

The following lemma generalizes a result of Jia \([5]\).

Lemma 2. Let \( G = (g_i)_{i=0}^{\infty} \) be a \( G \)-adic sequence. Let \( (u_i)_{i=1}^{p} \) be a strictly increasing finite sequence of nonnegative integers, and let \( (v_j)_{j=1}^{q} \) be a finite sequence of not necessarily distinct nonnegative integers. Let
\[
x_i \in [1, d_{u_i+1} - 1]
\]
for all \( i \in [1, p] \) and
\[
y_j \in [1, d_{v_j+1} - 1]
\]
for all \( j \in [1, q] \). If
\[
n = \sum_{i=1}^{p} x_i g_{u_i} = \sum_{j=1}^{q} y_j g_{v_j}
\]
then
\[
\sum_{u_i \leq u_k} x_i g_{u_i} \leq \sum_{v_j \leq u_k} y_j g_{v_j}
\]
for all \( k \in [1, p] \).

Proof. Because the sequence \( (u_i)_{i=1}^{p} \) is strictly increasing, Lemma \([4]\) implies
\[
\sum_{u_i \leq u_k} x_i g_{u_i} = \sum_{i=1}^{k} x_i g_{u_i} < g_{u_{k+1}}
\]
for all \( k \in [1, p] \). Choosing \( k = p \) gives
\[
n = \sum_{i=1}^{p} x_i g_{u_i} < g_{u_{p+1}}.
\]
Relation \([6]\) implies
\[
g_{v_j} \leq n < g_{u_{p+1}}
\]
an so \( v_j \leq u_p \) for all \( j \in [1, q] \). This implies \([6]\) for \( k = p \).
In the sequence $G = (g_i)_{i=0}^{\infty}$, the integer $g_i$ divides $g_j$ for all $i \leq j$. Let $k \in [1, p-1]$. Because the sequence $(u_i)_{i=1}^p$ is strictly increasing, for all $i \in [k+1, p]$ we have

$$u_k < u_k + 1 \leq u_{k+1} \leq u_i$$

and $g_{u_{k+1}}$ divides $g_{u_i}$, that is,

$$g_{u_i} \equiv 0 \pmod{g_{u_{k+1}}}. $$

If $v_j \geq u_k + 1$, then

$$g_{v_j} \equiv 0 \pmod{g_{u_{k+1}}}. $$

Rearranging (6), we obtain

$$\sum_{u_i \leq u_k} x_i g_{u_i} - \sum_{v_j \leq u_k} y_j g_{v_j} = \sum_{v_j \geq u_k + 1} y_j g_{v_j} - \sum_{u_i > u_k} x_i g_{u_i} \equiv 0 \pmod{g_{u_{k+1}}}. $$

If

$$\sum_{u_i \leq u_k} x_i g_{u_i} > \sum_{v_j \leq u_k} y_j g_{v_j}$$

then Lemma 3 gives

$$0 < \sum_{i=1}^k x_i g_{u_i} - \sum_{v_j \leq u_k} y_j g_{v_j} \leq \sum_{i=1}^k x_i g_{u_i} < g_{u_{k+1}}. $$

This inequality contradicts congruence (7). This completes the proof.

**Theorem 3.** Let $h \geq 2$ and let $t$ be an integer such that

$$t \geq 1 + \frac{\log h}{\log 2}. $$

Let $W = (W_i)_{i=0}^{h-1}$ be a partition of $\mathbb{N}_0$ such that, for all $i \in [0, h-1]$, there is an infinite set $M_i$ of positive integers such that

$$[M_i - t + 1, M_i] \subseteq W_i$$

for all $M_i \in M_i$. Let $G = (g_i)_{i=0}^{\infty}$ be a $G$-adic sequence. The set

$$A_G(W) = \bigcup_{i=0}^{h-1} A_G(W_i)$$

is a minimal asymptotic basis of order $h$.

**Proof.** By Theorem 1 the set $A_G(W)$ is an asymptotic basis of order $h$.

Let $a \in A_G(W)$. Without loss of generality, we can assume that $a = a_0 \in A_G(W_0)$ and

$$a_0 = \sum_{j \in F_0} x_{0,j} g_j$$

where $F_0 = F^*(W_0)$ and $x_{0,j} \in [1, d_{j+1} - 1]$ for all $j \in F_0$. Let $M_0 = \max(F_0)$. By Lemma 3

$$g_{M_0} \leq a_0 < g_{M_0+1}. $$

For all $i \in [1, h-1]$, choose an integer $M_i$ in the infinite set $M_i$ such that

$$M_i \geq M_0 + t.$$
and let
\[(12) \quad a_i = \sum_{j \in W_i, j < M_0} (d_{j+1} - 1) g_j + g_{M_i} \in A_G(W_i).\]
This is the $G$-adic representation of $a_i$. Let
\[(13) \quad n = \sum_{i=0}^{h-1} a_i = a_0 + \sum_{i=1}^{h-1} \sum_{j \in W_i, j < M_0} (d_{j+1} - 1) g_j + \sum_{i=1}^{h-1} g_{M_i}.\]
This is the $G$-adic representation of $n$.

Let
\[(14) \quad b_{k_i} = \sum_{j \in E_{k_i}} y_{i,j} g_j \in A_G(W_{k_i})\]
be any representation of $n$ as the sum of $h$ elements of $A_G(W)$, where $k_i \in [0, h-1]$ for all $i \in [0, h-1]$ and $b_{k_i} \in A_G(W_{k_i})$. We must prove that $b_{k_i} = a_0$ for some $i \in [0, h-1]$.

Each integer $b_{k_i}$ is of the form
\[b_{k_i} = \sum_{j \in E_{k_i}} y_{i,j} g_j \in A_G(W_{k_i})\]
where $E_{k_i} \in F^*(W_{k_i})$ and $y_{i,j} \in [1, g_{j+1} - 1]$ for all $j \in E_{k_i}$. The uniqueness of the $G$-adic representation implies that if \{\(k_0, k_1, \ldots, k_{h-1}\)\} $\neq [0, h-1]$, then, after rearrangement, $k_i = i$ and $a_i = b_i$ for all $i \in [0, h-1]$.

If \{\(k_0, k_1, \ldots, k_{h-1}\)\} $\neq [0, h-1]$, then there exists $s \in [0, h-1]$ such that $s \notin \{k_0, k_1, \ldots, k_{h-1}\}$. Suppose that $s \neq 0$. Recall that
\[M_s \geq M_0 + t\]
and
\[\left[M_s - t + 1, M_s\right] \subseteq W_s.\]
Because $k_i \neq s$ for all $i \in [0, h-1]$, we have
\[\left[M_s - t + 1, M_s\right] \cap E_{k_i} \subseteq W_s \cap W_{k_i} = \emptyset.\]
We construct the partition
\[E_{k_i} = E'_{k_i} \cup E''_{k_i}\]
with
\[E'_{k_i} = \{j \in E_{k_i} : j \leq M_s - t\}\]
and
\[E''_{k_i} = \{j \in E_{k_i} : j \geq M_s + 1\}.\]
The sets $E'_{k_i}$ and $E''_{k_i}$ are not necessarily nonempty. Let
\[b_{k_i} = b'_{k_i} + b''_{k_i}\]
where
\[b'_{k_i} = \sum_{j \in E'_{k_i}} y_{i,j} g_j \quad \text{and} \quad b''_{k_i} = \sum_{j \in E''_{k_i}} y_{i,j} g_j.\]
Note that $b'_{k_i} = 0$ if $E'_{k_i} = \emptyset$ and $b''_{k_i} = 0$ if $E''_{k_i} = \emptyset$.

By Lemma 1
\[(14) \quad b'_{k_i} < g_{M_s - t + 1} \]
and
\[ b''_{k_i} = 0 \quad \text{or} \quad b''_{k_i} \geq g_{M_s+1}. \]

Let
\[ n' = \sum_{i=0}^{h-1} b'_{k_i} \quad \text{and} \quad n'' = \sum_{i=0}^{h-1} b''_{k_i}. \]

Recall inequalities (1) and (8):
\[ d_i \geq 2 \quad \text{and} \quad h \leq 2^{t-1}. \]

From (15), (14), and (3), we obtain
\[ n' < h g_{M_s-t+1} \leq 2^{t-1} g_{M_s-t+1} \leq g_{M_s-t+1} \prod_{i=1}^{t-1} d_{M_s-t+i} = g_{M_s}. \]

Therefore, the \( G \)-adic representation of \( n' \) is of the form
\[ n' = \sum_{j=0}^{M_s-1} z_j g_j \]
with \( z_j \in [0, d_{j+1} - 1] \). Because
\[ n'' = 0 \quad \text{or} \quad n'' \geq g_{M_s+1}, \]
the \( G \)-adic representation of \( n'' \) is of the form
\[ n'' = \sum_{j=M_s+1}^{\infty} z_j g_j \]
with \( z_j \in [0, d_{j+1} - 1] \) and \( z_j \geq 1 \) for only finitely many \( j \). Therefore,
\[ n = n' + n'' = \sum_{i=0}^{M_s-1} z_i g_i + \sum_{i=M_s+1}^{\infty} z_i g_i \]
is the \( G \)-adic representation of \( n \). In this representation, the coefficient of \( g_{M_s} \) is 0, which contradicts the construction of \( n \). It follows that
\[ [1, h - 1] \subseteq \{k_0, k_1, \ldots, k_{h-1}\}. \]

Renumbering the integers \( b_i \), we can assume that \( k_i = i \) and \( b_i \in A_G(W_i) \) for all \( i \in [1, h - 1] \).

We must prove that \( k_0 = 0 \), or, equivalently, that \( b_0 \in A_G(W_0) \). If not, then \( b_0 \in A_G(W_r) \) for some \( r \in [1, h - 1] \). Because
\[ M_0 = \max(F_0) \in F_0 \subseteq W_0 \]
we have
\[ M_0 \notin \bigcup_{i=0}^{h-1} E_{k_i} \subseteq \bigcup_{i=1}^{h-1} W_i. \]

Identity (10) and Lemma 1 give
\[ g_{M_0} \leq a_0 = \sum_{j \in F_0} x_{0,j} g_j < g_{M_0+1}. \]
From (12) we have
\[ n = a_0 + \sum_{i=1}^{h-1} \sum_{j \in W_i, j < M_0} (d_{j+1} - 1)g_j + \sum_{i=1}^{h-1} g_{M_i} \]
\[ = \sum_{i=0}^{h-1} \sum_{j \in E_i} y_{i,j} g_j. \]

Summing only over terms \( g_j \) with \( j \leq M_0 \) and applying Lemma 2, we obtain
\[ a_0 + \sum_{i=1}^{h-1} \sum_{j \in W_i, j < M_0} (d_{j+1} - 1)g_j \leq \sum_{i=0}^{h-1} \sum_{j \in E_i} y_{i,j} g_j + \sum_{j \in E_0} y_{0,j} g_j + \sum_{i=1}^{h-1} \sum_{j \in E_i, j < M_0} y_{i,j} g_j \]
\[ < g_{M_0} + \sum_{i=1}^{h-1} \sum_{j \in W_i, j < M_0} (d_{j+1} - 1)g_j \]
\[ \leq a_0 + \sum_{i=1}^{h-1} \sum_{j \in W_i, j < M_0} (d_{j+1} - 1)g_j. \]

This is absurd, and so \( b_0 \in A_G(W_0) \). It follows that the integer \( n \) defined by (13) has a unique representation as the sum of \( h \) elements of \( A_G(W) \). Therefore,
\[ n \notin h (A_G(W) \setminus \{a\}). \]

For all \( i \in [1, h-1] \), there are infinitely many integers \( M_i \in \mathcal{M}_i \) with \( M_i \geq M_0 + t \), and so infinitely many positive integers \( n \) satisfying (16). It follows that \( A_G(W) \setminus \{a\} \) is not an asymptotic basis of order \( h \) for all \( a \in A_G(W) \). Equivalently, \( A_G(W) \) is a minimal asymptotic basis of order \( h \). This completes the proof. \( \square \)

**Corollary 1.** Let \( W = W_0 \cup W_1 \) be a partition of \( \mathbb{N}_0 \) such that both \( W_0 \) and \( W_1 \) contain infinitely many pairs of consecutive integers. Let \( G = (g_i)_{i=0}^{\infty} \) be a \( G \)-adic sequence. The set
\[ A_G(W) = A_G(W_1) \cup A_G(W_2) \]
is a minimal asymptotic basis of order 2.

**Proof.** This is the case \( h = 2 \) of Theorem 3. \( \square \)

### 3. Open problems

1. Let \( h \geq 2 \). By Theorem 1 for every partition \( W = (W_i)_{i=0}^{h-1} \) of \( \mathbb{N}_0 \) and every \( G \)-adic sequence, the set
\[ A = \{0\} \cup A_G(W) \]
is a basis of order \( h \) that is not a minimal asymptotic basis of order \( h \).
   (a) Determine the set of all integers \( a \in A \) such that \( A \setminus \{a\} \) is an asymptotic basis of order \( h \).
(b) Determine the set of all integers $a \in A$ such that $A \setminus \{a\}$ is a minimal asymptotic basis.

(2) Let $G = (g_i)_{i=0}^\infty$ be a $G$-adic sequence. Let $h \geq 2$.

(a) Construct partitions $W = (W_i)_{i=0}^{h-1}$ of $\mathbb{N}_0$ such that the set $A_G(W)$ is a minimal asymptotic basis of order $h$.

(b) Construct partitions $W = (W_i)_{i=0}^{h-1}$ of $\mathbb{N}_0$ such that the set $A_G(W)$ is not a minimal asymptotic basis of order $h$.

(3) Let $(d_i)_{i=1}^\infty$ be a sequence of 2s and 3s. Let $G = (g_i)_{i=0}^\infty$ be the $G$-adic sequence defined by $g_0 = 1$ and $g_i = \prod_{j=1}^{i} d_i$ for $i \geq 1$. Consider problem (2) with respect to this $G$-adic sequence. Of particular interest are the infinitely many $G$-adic sequences $G = (g_i)_{i=0}^\infty$ with quotients $\{d_{2i-1}, d_{2i}\} = \{2, 3\}$ for all $i = 1, 2, 3, \ldots$ In this case, $g_{2i} = 6^i$ for all $i$.

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