Noncommutative ampleness from finite endomorphisms

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Abstract

Let $X$ be a projective integral scheme with endomorphism $\sigma$, where $\sigma$ is finite, but not an automorphism. We examine noncommutative ampleness of bimodules defined by $\sigma$. In contrast to the automorphism case, one-sided ampleness is possible. We also find that rings and bimodule algebras associated with $\sigma$ are not noetherian.

Keywords: Ampleness, vanishing theorem, finite endomorphism, twisted homogeneous coordinate ring

1. Introduction

Let $X$ be a projective integral scheme over a field $k$. A homogeneous coordinate ring $R$ can be built from global sections of $L^{\otimes n}$, where $L$ is an ample invertible sheaf. The sheaf $L$ can be used to prove the Serre Correspondence Theorem, showing a category equivalence between the tails of finitely generated graded $R$-modules and coherent sheaves on $X$.

In [AV], Artin and Van den Bergh defined a twisted homogeneous coordinate ring $B$ by using an automorphism $\sigma$ of $X$ to twist the usual multiplication. Via a noncommutative definition of ampleness (see Definition 4.2), they again showed a category equivalence between the tails of graded (right) $B$-modules and coherent sheaves on $X$.

However, this definition was not clearly equivalent on the left and right. In [K1], the first author showed that the left and right definitions are equivalent. A key technique was to study the behavior of $\sigma$ on the numerical equivalence classes of divisors.

In [VdB], Van den Bergh generalized these definitions to include the possibility that $\sigma$ was not an automorphism, but only a finite endomorphism. In this paper, we examine this noncommutative ampleness in the general finite...
endomorphism case. Unlike the automorphism case, one never has ampleness on the left, but it is possible to have ampleness on the right. In Corollary 4.13 and Proposition 4.14 we have (in simplified form)

**Theorem 1.1.** Let $X$ be a regular projective integral scheme with finite endomorphism $\sigma$ and invertible sheaf $L$. Suppose $\sigma$ is not an automorphism. Then the sequence of $\mathcal{O}_X$-bimodules defined by $\sigma$ and $L$ is not left ample.

However, if $L$ is ample (in the commutative sense) and $\sigma^*L \cong L^\otimes r$ for some $r \in \mathbb{Z}$, then the sequence is right ample.

We also show that in case the sequence is right ample, the resulting twisted homogeneous coordinate ring is not noetherian, as is the bimodule algebra which defines the ring. See Theorem 4.15 and Corollary 4.16. As a specific example, in Section 5 we examine the case of $X = \mathbb{P}^m$ and $\sigma$ the relative Frobenius endomorphism. We find

**Proposition 1.2.** Let $X = \mathbb{P}^m$ over a perfect field of characteristic $p > 0$. Let $f$ be the relative Frobenius endomorphism. Then the twisted homogeneous coordinate ring $F = \oplus \Gamma \left( \mathcal{O} \left( \frac{p^n - 1}{p - 1} \right) \right)$ is not finitely generated, unless $p = 2$ and $m = 1$. In that case, $F$ is generated by $F_1 = \Gamma(\mathcal{O}(1))$, but $F$ is not noetherian.

The proofs of category equivalences in \[AV, VdB\] rely on noetherian conditions. Thus we have not generalized those results in this paper.

2. Finite Endomorphism Properties

Throughout this paper, $X$ will be an integral scheme of finite-type over a field $k$. Often we will also assume $X$ is projective, normal, or regular, in which case we will make these assumptions clear.

We begin by verifying some basic properties of a finite endomorphism $f$. Presumably these are all well-known.

**Lemma 2.1.** Let $X$ be an integral scheme. Let $f : X \to X$ be a finite morphism. Then for any open affine set $U$, the corresponding map of rings $f_{|U}^# : \mathcal{O}(f^{-1}(U)) \to \mathcal{O}(U)$ is injective. In other words, when $\text{Spec} \ A = f^{-1}(\text{Spec} \ B)$ we may consider $B$ as a subring of $A$ (up to isomorphism).

**Proof.** Since $f$ is finite, for any open affine set $U = \text{Spec} \ B$, the pre-image $f^{-1}(U) = \text{Spec} \ A$ where $A$ is a $B$-algebra which is finitely generated as a $B$-module. Now $A$ is a $B$-module via the ring homomorphism $\phi = f_{|U}^#$ from $B$ to $A$ corresponding to $f$.

Thus, the map $\phi : B \to A$ makes $A$ integral over $B$ (technically over $\phi(B)$). Then by \[E, Proposition 9.2\], $\dim A = \dim(B/\ker \phi)$. As $X$ is irreducible (being integral), every nonempty open subset of $X$ is dense in $X$. Hence the dimension of every nonempty open subset of $X$ equals the dimension of $X$. In particular, $\dim U = \dim f^{-1}(U)$ and so $\dim B = \dim(B/\ker \phi)$. Since $X$ is integral, $B$ is a domain (and $\{0\}$ is prime) and so the only way to have $\dim B = \dim(B/\ker \phi)$ is for $\ker \phi = \{0\}$. Thus $\phi$ must be injective and we may consider $B$ a subset of $A$. \[\square\]
Lemma 2.2. [Sh, Theorem 4, p. 61] Let $X$ be an integral scheme with $f$ a finite endomorphism. Then $f$ is surjective.

Corollary 2.3. Let $X$ be a normal, integral scheme. Let $f : X \rightarrow X$ be a finite morphism. Then $f$ is an open morphism.

Proof. By [D, 2.8 Theorem, p. 220], a finite dominant morphism $f : X \rightarrow Y$, where $Y$ is normal, is open.

The following connection between the degree of $f$ and the possible invertibility of $\sigma$ will be important for our study of ampleness. See [Kl, p. 299] for the definition of degree and its basic properties.

Lemma 2.4. Let $X$ be a normal, integral scheme and $f : X \rightarrow X$ a finite morphism of degree 1. Then $f$ is an automorphism.

This fact reflects the description in [D, p. 219] that “an algebraic variety $X$ is said to be normal if every finite birational morphism $X' \rightarrow X$ is an isomorphism.”

The following argument is based on the proof of [Sc, Proposition 1.2.1.12].

Proof. To show that $f$ is an isomorphism, it suffices to prove that $f$ is a homeomorphism and that for any open affine $U \subset X$, we have $f^\#_{|U}$ is an isomorphism of sheaves. By [H1, Exercise II.2.18], the homeomorphism property follows from the isomorphism of sheaves.

First, since $X$ is reduced and $f$ has degree 1, the map $f$ must be birational by [Kl, p. 299, Example 2].

Next, we want that for any open affine $U \subset X$, the map $f^\#_{|U}$ is an isomorphism of sheaves. Since $U$ is affine, we have $U = \text{Spec } B$ for some $k$-algebra $B$. Since $f$ is finite, $f^{-1}(U) = \text{Spec } A$ where $A$ is a $B$ algebra which is a finitely generated $B$-module. As in Lemma 2.1, the map $f^\#_{|f^{-1}(U)}$ corresponds to some map $\phi : B \rightarrow A$ which induces an inclusion $B \subset A$. Since $f$ is birational, we have that $A$ and $B$ have the same field of fractions.

As $X$ is normal, we know that $\bar{B}$ is integrally closed; however, since $A$ is a finite $\bar{B}$-algebra, $A$ is integral over $B$. Thus $A = B$ because $A$ and $B$ have the same field of fractions. So the sheaf map induced from $\phi$ is an isomorphism of sheaves.

We must now strengthen the hypothesis on $X$ from normal to regular. By regular we mean that the stalks of $O_X$ are regular local rings. Recall that regular implies normal [H1, Exercise I.5.13].

Lemma 2.5. Let $X$ be a regular, integral scheme and $f : X \rightarrow X$ a finite morphism. Then $f$ is flat.

Proof. Since $X$ is regular, $X$ is Cohen-Macaulay [H1 Theorem II.8.21A]. Thus $f$ is flat by [H1, Exercise III.10.9].
Note that the regular hypothesis is necessary in general. In [Ku], it was shown that in characteristic $p$, a local ring $R$ is regular if and only if $R$ is reduced and flat over $R^p$, the image of $R$ under the Frobenius morphism. Thus the Frobenius endomorphism of an integral scheme $X$ is flat if and only if $X$ is regular.

By definition, a morphism $f$ is \textit{faithfully flat} if $f$ is surjective and flat. We then have the following by Lemmas 2.2 and 2.5.

\textbf{Corollary 2.6.} Let $X$ be a regular, integral scheme. Then any finite morphism $f : X \to X$ is faithfully flat.

3. Bimodule algebras

In this section we examine bimodule algebras in the sense of [VdB], while verifying some basic properties that were omitted.

Recall the following definitions from [VdB, Definitions 2.1 and 2.3].

\textbf{Definition 3.1.} Given $f : Y \to X$ a morphism of finite-type between noetherian schemes and $M$ a quasi-coherent $\mathcal{O}_Y$-module, we say that $M$ is \textit{relatively locally finite (rlf)} for $f$ if for all coherent $M' \subset M$ the restriction $f|_{\text{Supp} M'}$ is finite.

Given $X$ and $Y$ noetherian $S$-schemes of finite-type, an $\mathcal{O}_S$-central $\mathcal{O}_X - \mathcal{O}_Y$-bimodule is a quasi-coherent $\mathcal{O}_X \times S Y$-module, relatively locally finite for the projections $\text{pr}_{1,2} : X \times S Y \to X,Y$.

\textbf{Definition 3.2.} Let $k$ be field. Let $L$ be a quasi-coherent sheaf on a noetherian scheme $X$ over $S = \text{Spec } k$ and $\sigma$ a finite $S$-homomorphism $X \to X$. Let $L_0 = \mathcal{O}_X$. Then for all integers $j \geq 1$, define

$$L_j := L \otimes_{\mathcal{O}_X} \sigma^* L \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} (\sigma^{j-1})^* L.$$

Define $B_0 = \mathcal{O}_X$ and, for all $j \geq 1$, define $B_j$ to be the bimodule $(L_j)_\sigma$. Then let $B = \bigoplus_{j=0}^\infty B_j$ (alternatively, let $B_j = 0$ for all $j < 0$ and let $B = \bigoplus_{j \in \mathbb{Z}} B_j$).

Then let $B_j = \Gamma(B_j) = H^0(X,B_j)$ and define $B = \bigoplus_{n=0}^\infty B_j = \Gamma(B)$.

As in [AV] and [VdB], this construction produces an $\mathbb{N}$-graded $k$-algebra $B$ and a graded bimodule algebra $B$, as Theorem 3.5 will show shortly.

\textbf{Lemma 3.3.} Let $S = \text{Spec } k$ and $V, X, Y$ be proper $S$-schemes with finite $S$-maps $\alpha : V \to X, \beta : V \to Y$. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_V$-module. Then $\alpha \mathcal{M}_\beta$ is an $\mathcal{O}_S$-central $\mathcal{O}_X - \mathcal{O}_Y$-bimodule.

In particular, for all $j \in \mathbb{Z}$, the $\mathcal{O}_X$-bimodule $B_j$ is an $\mathcal{O}_S$-central $\mathcal{O}_X - \mathcal{O}_X$ bimodule.

\textbf{Proof.} Since $X \to S, Y \to S$ are proper, so is $X \times_S Y \to S \times_S S \cong S$ [HI Corollary II.4.8(d)]. Since $\alpha, \beta$ are finite, so is $(\alpha, \beta)$, as finite maps satisfy the properties of [HI Exercise II.4.8]. Since every finite map is proper [HI]
Exercise II.4.1], we have that $(\alpha, \beta)$ is closed. Hence $W := (\alpha, \beta)(V)$ is a closed subscheme of $X \times_S Y$ and hence proper [H1, Corollary II.4.8(a)].

So consider the commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\alpha \times \beta} & W \\
\downarrow{\alpha} & & \downarrow{\text{pr}_1|_W} \\
X.
\end{array}
$$

Since $W$ and $X$ are proper over Spec $k$, the map $\text{pr}_1|_W$ is proper by [H1, Corollary II.4.8(e)]. Since $(\alpha, \beta)$ surjects onto $W$ and $\alpha$ is finite, we have that $\text{pr}_1|_W$ is quasi-finite (that is, $\text{pr}_1|_W$ has finite fibers). Since $\alpha$ is finite, we have that $\text{pr}_1|_W$ is quasi-finite (that is, $\text{pr}_1|_W$ has finite fibers). Since $\text{pr}_1|_W$ is proper and quasi-finite, the map is finite [EGA, IV.3, Théorème 8.11.1].

Now let $M$ be a quasi-coherent $O_X$-module and $F$ be a coherent subsheaf of $(\alpha, \beta)^* M$. Then $\text{Supp} F$ is a closed subscheme of $W$ [H1, Caution II.5.8.1].

So we have $\text{pr}_1|_{\text{Supp} F}$ is finite. Thus $(\alpha, \beta)^* M$ is rlf for $\text{pr}_1$ and similarly for $\text{pr}_2$. And thus $(\alpha, \beta)^* M$ is an $O_S$-central $O_X - O_Y$ bimodule by definition.

The claim regarding $B_j$ then follows as a special case.

\begin{corollary}
Let $X$ be a $k$-scheme of finite-type; recall Definition 3.2. For any $j \in \mathbb{Z}$ and any $O_X$-module $M$, the bimodule tensor product $1_M \otimes_{O_X} B_j$ is an $O_S$-central $O_X - O_Y$ bimodule where $S = \text{Spec} k$.

\end{corollary}

\begin{proof}
By Lemma 3.3 we have that $1_M$ and $B_j$ are both $O_S$-central $O_X - O_X$-bimodules. By [VdB, p. 442], so is $1_M \otimes_{O_X} B_j$.

Technically each $B_j$ is defined on $X \times \text{Spec} X$; since $\text{Spec} k$ is a single point, one may identify $X \times \text{Spec} k$ with the Cartesian product $X \times X$.

Looking specifically at the right-hand and left-hand module structure, we note that on the right, we have

$$
B_n(X \times U) = (1, \sigma^n)_* \mathcal{L}_n(X \times U) = \mathcal{L}_n((1, \sigma^n)^{-1}(X \times U)) = \mathcal{L}_n((\sigma^n)^{-1}U) = (\sigma^n)_* \mathcal{L}_n(U).
$$

Similarly, on the left, we have

$$
B_n(U \times X) = (1, \sigma^n)_* \mathcal{L}_n(U \times X) = \mathcal{L}_n((1, \sigma^n)^{-1}(U \times X)) = \mathcal{L}_n(U).
$$

So the left module structure in this case is always one from an invertible sheaf. However, it is possible in general for $B_j$ to have an invertible left module structure, but not an invertible right module structure since in general the pushforward of an invertible sheaf is not invertible. In fact, $B_j$ is an invertible bimodule if and only if $\sigma$ is an automorphism [AV, Proposition 2.15].

On the other hand, by [H2] Proposition 4.5 if $\sigma$ is a faithfully flat morphism and $F$ a locally free sheaf of finite rank, then $\sigma_* F$ is also locally free of finite rank. Thus these bimodules still have nice tensor properties, as in Lemma 4.3.

\end{proof}
Theorem 3.5. Let $L$ be a quasi-coherent sheaf on a noetherian scheme $X$ of finite-type and $\sigma$ a finite homomorphism $X \to X$. Then

- the $B$ from Definition 3.2 is a graded bimodule algebra;
- the $B$ from Definition 3.2 is a $\mathbb{N}$-graded $k$-algebra.

Proof. The set $B$ automatically has a well-defined addition. Having a (compatible) well-defined multiplication will follow immediately from the product map $\mu : B \otimes_{O_X} B$ that is part of $B$’s bimodule-algebra structure.

The composition of inclusions $O_X \hookrightarrow B_0 \hookrightarrow B$ gives the unit map. So all that is left to verify from [VdB, Definition 3.1(1)] is the product map.

The key step in proving this is Lemma 3.6, which shows that

$$1_L \otimes 1_M \cong 1_{(L \otimes f^*M)}_{gf}$$

for any finite homomorphisms $f$ and $g$ from $X$ to itself and any quasi-coherent sheaves $L$ and $M$. Then

$$1(L_j)_{\sigma^i} \otimes_{O_X} 1(L_n)_{\sigma^i} \cong 1(L_j \otimes_{O_X} (\sigma^i)^*L_n)_{\sigma^i+n} \cong 1(L_j+n)_{\sigma^i+n}$$

which is $B_j \otimes B_n \cong B_{j+n}$. For the ring $B$, that its multiplication is defined and has the usual compatibilities follows from the induced map on global sections $B_j(X) \otimes_{O_X(X)} B_n(X) \to B_{j+n}(X)$. $\square$

We now examine a special case of [VdB Lemma 2.8(2)].

Lemma 3.6. Let $X, Y, Z$ be noetherian schemes and $\beta : X \to Y, \delta : Y \to Z$ finite maps. Let $L$ (respectively $M$) be quasicoherent $O_X$-module (respectively $O_Y$-module). Then

$$\text{id}_X L_\beta \otimes_{O_Y} \text{id}_Y M_\delta \cong \text{id}_X (L \otimes_{O_X} \beta^*M)_{\delta \beta}$$

as $O_S$-central $O_X - O_Z$-bimodules. Therefore, using the notation from Definition 3.2, we have that $B_j \otimes_{O_X} B_\ell \cong B_{j+\ell}$ for all nonnegative integers $j$ and $\ell$.

Proof. The first claim is [VdB Lemma 2.8(2)] in the special case of $X = V, Y = W, \alpha = \text{id}_X, \gamma = \text{id}_Y$. To follow that proof, we let $p, q : X \times_Y Y \to X, Y$ be the two projections. Because $\gamma = \text{id}_Y$, we have that $p : X \times_Y Y \to X$ is an isomorphism, induced by the ring isomorphism $B \to B \otimes_A A$. We then have the following commutative diagram, as in the proof of [VdB Lemma 2.8(2)].
Then [VdB, Lemma 2.8(2)] gives

\[
\text{Id}_X \otimes_{\mathcal{O}_Y} \text{Id}_Y \cong p(p^* \mathcal{L} \otimes_{\mathcal{O}_{X \times Y}} q^* \mathcal{M})_{\delta \circ q}.
\]

Note that \(\alpha'' = p \circ p^{-1} = \text{Id}_X\) and \(\beta'' = \delta \circ q \circ p^{-1}\). But the diagram above also shows \(q = \beta \circ p\). Thus \(\beta'' = \delta \circ \beta\). Taking \(\epsilon = p^{-1}\), we are in the special case discussed below [VdB, Equation 2.2]. We thus have

\[
p(p^* \mathcal{L} \otimes_{\mathcal{O}_{X \times Y}} q^* \mathcal{M})_{\delta \circ q} \cong \text{Id}_X ((p^{-1})^*p^* \mathcal{L} \otimes_{\mathcal{O}_{X \times Y}} (p^{-1})^*q^* \mathcal{M})_{\delta \circ \beta}
\]

as desired.

The final claim follows when \(X = Y = Z\).

\[\square\]

4. Ampleness and noetherianity

In this section we focus on cohomological vanishing related to the bimodules \(B_n = \mathcal{O}(\mathcal{L}_n)_{\sigma^n}\) where \(\mathcal{L}\) is an invertible \(\mathcal{O}_X\)-module. We begin with defining ampleness. Following [VdB, Definition 5.1], we have the following definition, with the introduction of the modifiers \textit{left} and \textit{right}. We also omit the requirement that the sheaves are eventually generated by global sections since that is implied when \(\sigma\) is faithfully flat; see Proposition [4.6]

\textbf{Definition 4.1.} A sequence \((B_n)_{n}\) of coherent \(\mathcal{O}_X\) bimodules is \textit{left ample} if for any coherent \(\mathcal{O}_X\)-module, one has that for \(n \gg 0\), \(H^q(X, B_n \otimes_{\mathcal{O}_X} \mathcal{M}) = 0\) for \(q > 0\).

A sequence \((B_n)_{n}\) of coherent \(\mathcal{O}_X\) bimodules is \textit{right ample} if for any coherent \(\mathcal{O}_X\) module, one has that for \(n \gg 0\), \(H^q(X, \mathcal{M} \otimes_{\mathcal{O}_X} B_n) = 0\) for \(q > 0\).

\textbf{Definition 4.2.} Let \(\sigma\) be a map from \(X\) to \(X\); let \(\mathcal{L}\) be a quasi-coherent sheaf. Say that \(\mathcal{L}\) is \textit{left} \(\sigma\)-\textit{ample} when the sequence \(B_n\), as defined in Definition [4.1], is left ample; say that \(\mathcal{L}\) is \textit{right} \(\sigma\)-\textit{ample} when that sequence \(B_n\) is right ample.

Since ampleness depends on tensor products, it is useful to know that tensoring is exact.
Lemma 4.3. Let \( X \) be an integral scheme with locally free sheaf \( L \). Let \( \alpha, \beta \) be finite faithfully flat endomorphisms of \( X \). Then tensoring the \( O_X \)-biomodule \( \alpha(L)_\beta \) with quasicoherent \( O_X \)-modules is exact.

Thus with \( B_j \) as in Definition 3.2, we have that tensoring \( O_X \)-modules with \( B \) is exact.

Proof. By \[VdB\] Proposition 2.6, tensor products of bimodules is right exact, but it will be straightforward to prove exactness directly. Consider the exact sequence of quasicoherent \( O_X \)-modules

\[ M_1 \rightarrow M_2 \rightarrow M_3. \]

By \[VdB\] Lemma 2.8(1), we have

\[ M_i \otimes_{O_X} \alpha(L)_\beta \cong \alpha(\alpha^*M_i \otimes_{O_X} L)_\beta. \]

Since \( \alpha \) is flat, pullback by \( \alpha \) is exact \[EGA\] IV, Théorème 2.1.3. Since \( L \) is locally free, tensoring with \( L \) is exact. Finally, since \( (\alpha, \beta) \) is a finite map, pushforward by \( (\alpha, \beta) \) is exact.

Since \( B_j \) is of the form \( \alpha(L)_\beta \) and \( B \) is a direct sum of the \( B_j \), we have that tensoring with \( B \) is exact \[Stn\] Proposition I.10.1.

When considering the ampleness of the sequence \( B_n \), it will be convenient to examine the related \( O_X \)-modules \( F \otimes L_n \) and \( L_n \otimes (\sigma^*)^n F \), as these have the same cohomology as \( F \otimes B_n \) and \( B_n \otimes F \) since \( (1, \sigma^n) \) is finite.

Also, when \( f : X \rightarrow X \) is finite and faithfully flat, and \( E \) is a locally free coherent sheaf, we have that \( f_*E \) is a locally free coherent sheaf \[H2\] Proposition 4.5]. Hence tensoring with \( f_*E \) is exact. Further, if \( F \) is any coherent sheaf, then we have the projection formula

\[ f_*(E) \otimes F \cong f_*(E \otimes f^*F). \] (4.4)

This holds for any coherent \( F \) because \( f \) is finite \[A\] Lemma 5.7]. Thus we may study the left ampleness of \( B_n \) by tensoring with the \( O_X \)-modules \( \sigma^n(L_n) \), allowing us to ignore the bimodule structure. Therefore we may apply the following proposition.

Proposition 4.5. \[K2\] Theorem 7.2, Proposition 7.3] Let \( X \) be a projective scheme with \( E_n \) a sequence of locally free coherent sheaves. Consider the following properties.

1. For any coherent sheaf \( F \), there exists \( n_0 \) such that \( H^q(F \otimes E_n) = 0 \) for \( q > 0, n \geq n_0 \).
2. For any coherent sheaf \( F \), there exists \( n_0 \) such that \( F \otimes E_n \) is generated by global sections for \( n \geq n_0 \).
3. For any invertible sheaf \( H \), there exists \( n_0 \) such that \( H \otimes E_n \) is an ample locally free sheaf for \( n \geq n_0 \).

Then \[1] \implies [2] \implies [3].

If the \( E_n \) are invertible, then the conditions are equivalent. \(\square\)
Note that the conditions above are not equivalent for general locally free sheaves [K2 Remark 7.4].

In our specific situation, we obtain the following.

Proposition 4.6. Let $X$ be a projective integral scheme with finite faithfully flat endomorphism $\sigma$. Define $B_n$ and $L_n$ as in Definition 3.2 with $L$ invertible. Let $F$ be a coherent sheaf and let $H$ be an invertible sheaf. Then

1. If $B_n$ is right ample, then for all $n \gg 0$, $F \otimes L_n$ is generated by global sections and $H \otimes L_n$ is an ample invertible sheaf.

2. If $B_n$ is left ample, then for all $n \gg 0$, $L_n \otimes (\sigma^n)^* F$ is generated by global sections and $L_n \otimes (\sigma^n)^* H$ is an ample invertible sheaf.

Proof. The first claim follows immediately from Proposition 4.5.

For the second claim, consider the sequence of coherent sheaves $E_n = \sigma_n^* (L_n)$, which are locally free [H2 Proposition 4.5]. Then $E_n$ satisfies (1) of Proposition 4.5 because of the isomorphism (4.4) and the fact that push forward by a finite morphism preserves cohomology [H1 Exercise III.4.1].

Therefore $\sigma_n^* (L_n \otimes (\sigma^n)^* F)$ is generated by global sections and $\sigma_n^* (L_n \otimes (\sigma^n)^* H)$ is an ample locally free sheaf for $n \gg 0$ by Proposition 4.5. As in the proof of [H2 Proposition 4.5], we can pull back

$$\oplus O_x \to \sigma_n^* (L_n \otimes (\sigma^n)^* F) \to 0$$

by $\sigma^n$ to get that

$$(\sigma^n)^* \sigma_n^* (L_n \otimes (\sigma^n)^* F)$$

is generated by global sections. But since $\sigma^n$ is affine, the natural map

$$(\sigma^n)^* \sigma_n^* (L_n \otimes (\sigma^n)^* F) \to L_n \otimes (\sigma^n)^* F$$

is surjective. Hence $L_n \otimes (\sigma^n)^* F$ is generated by global sections.

Since $\sigma_n^* (L_n \otimes (\sigma^n)^* H)$ is an ample locally free sheaf for $n \gg 0$, we have that $L_n \otimes (\sigma^n)^* H$ is an ample invertible sheaf by [H2 Proposition 4.5].

Due to Proposition 4.6, we will examine the ampleness of the invertible sheaves $H \otimes L_n$ and $L_n \otimes (\sigma^n)^* H$ for arbitrary invertible sheaves $H$. As in [K1], the key idea is to examine the action of $\sigma^*$ on the numerical equivalence classes of invertible sheaves, or equivalently Cartier divisors.

Let $\text{Num}(X) = A_1^\text{num}(X)$ be the set of Cartier divisors of $X$, modulo numerical equivalence. We follow the work of [K1] Sections 3-4. Recall that $\text{Num}(X) \cong \mathbb{Z}^\ell$, for some $\ell$, as a group and so we can take $P \in M_\ell(\mathbb{Z})$ where $P$ is the action of $\sigma^*$ on $\text{Num}(X)$ [K1 p. 305, Remark 3].

Lemma 4.7. Let $X$ be a projective integral scheme with $\sigma : X \to X$ finite. Let $P$ be the action of $\sigma^*$ on $\text{Num}(X)$. Then the spectral radius $r$ of $P$ is an eigenvalue of $P$ and $r \geq 1$. If the largest Jordan block containing $r$ is $m \times m$ and $\lambda$ is an eigenvalue with $|\lambda| = r$, then any Jordan block containing $\lambda$ has size less than or equal to $m \times m$.

If $X$ is normal and $\sigma$ is not an automorphism, then $r > 1$. 9
Proof. We have that $\sigma$ is surjective by Lemma 2.2. So $\sigma^*$ is an automorphism of $\text{Num}(X)$ [K1, p. 331, Remark 1]. Thus $P$ is invertible as a matrix (over $\mathbb{Q}$).

Let $\ell$ be the rank of $\text{Num}(X)$. Since $P \in M_{\ell}(\mathbb{Z})$, we have that $\det(P) \in \mathbb{Z}$ and $|\det(P)| \geq 1$ since $P$ is invertible over $\mathbb{Q}$. Thus we have that the spectral radius $r \geq 1$.

Since $\sigma$ is surjective, $\sigma^*$ preserves the cone of nef divisors in $\mathbb{R}^\ell$ [K1, p. 303, Proposition 1] and that cone has non-empty interior (the ample cone) [K1, p. 325, Theorem 1]. Thus $r$ is an eigenvalue of $P$ and we also have the claim regarding Jordan blocks [V, Theorem 3.1].

Now suppose $X$ is normal and $\sigma$ is not an automorphism. Then $\text{deg } \sigma > 1$ by Lemma 2.4.

Let $D$ be an ample divisor, or more generally a divisor with self-intersection number $(D \dim X) > 0$. Consider the integer valued function $g(m) = (\sigma^m D)^\dim X = (P^m D)^\dim X$ with $m \in \mathbb{N}$. By [K1, p. 299, Lemma 2, Proposition 6],

$$(P^m D)^\dim X = (\deg \sigma^m) (D^\dim X) = (\deg \sigma)^m (D^\dim X).$$

Since $\deg \sigma > 1$, we have that $g$ grows exponentially.

Suppose for contradiction that $r = 1$. Then $\det(P) = \pm 1$ and every eigenvalue has absolute value 1. This implies that every eigenvalue is a root of unity since $P \in \text{GL}_{\ell}(\mathbb{Z})$ [AV, Lemma 5.3]. That is, we say that $P$ is quasi-unipotent. We may replace $P$ with a power of $P$ and assume $P = I + N$ where $I$ is the identity matrix and $N$ is a nilpotent matrix. And so $g$ has polynomial growth since $P^m D = (I + N)^m D$ can be expressed as a “polynomial” in $m$ with divisors as coefficients, as in [K1, Equation 4.2]. Thus the self-intersection number is a polynomial with integer coefficients.

But $g$ grows exponentially, so it must be that $r \neq 1$. Thus $r > 1$.

Note that it is possible for an automorphism to yield a spectral radius $r > 1$ [K1, Example 3.9].

The spectral radius $r$ is key to determining the ampleness of the appropriate sequences. When $\sigma$ is an automorphism, we have an exact characterization.

**Proposition 4.8.** [K1, Theorem 4.7, Corollary 5.1] Let $X$ be a projective integral scheme with automorphism $\sigma$ and invertible sheaf $L$. Define $B_n, L_n$ as in Definition 3.2. Then $B_n$ is right ample if and only if $B_n$ is left ample. This occurs if and only if $L_n$ is an ample invertible sheaf for $n \gg 0$ and $r = 1$ with $r$ the spectral radius of the numerical action of $\sigma^*$.

This leaves us to consider the case of $r > 1$. To do so, we determine the growth of intersection numbers.

**Lemma 4.9.** Let $X$ be a projective integral scheme with finite endomorphism $\sigma$. Let $D$ be a divisor. Let $P$ be the action of $\sigma^*$ on $\text{Num}(X)$ with spectral radius $r$ and largest Jordan block associated to $r$ of size $(j + 1) \times (j + 1)$.

Then
1. For all curves $C$, there exists $c_1 > 0$ such that 
\[(P^m D.C) \leq c_1 m^j r^m \text{ for all } m > 0.\]

2. If $D$ is ample, then there exists a curve $C$ and $c_2 > 0$ such that 
\[(P^m D.C) \geq c_2 m^j r^m \text{ for all } m > 0.\]

Proof. Let $v_1, v_2, \ldots, v_\rho$ be a basis for $\text{Num}(X) \otimes \mathbb{C}$ associated to a Jordan canonical form of $P$, with an $(j+1) \times (j+1)$ block for $r$ in the upper left. Since $r$ is real, the vectors $v_1, v_2, \ldots, v_{j+1}$ have real components.

Applying $P^m$ to $v_{i+1}$ for $i = 0, \ldots, j$ gives 
\[P^m v_{i+1} = \left(\begin{array}{c} m \\ i \end{array}\right) r^{m-i} v_1 + \left(\begin{array}{c} m \\ i-1 \end{array}\right) r^{m-i+1} v_2 + \cdots + r^m v_{i+1}. \tag{4.10}\]

Recall that every eigenvalue $\lambda$ of $P$ has $|\lambda| < r$ or the Jordan blocks of $\lambda$ are no larger than $(j+1) \times (j+1)$ by Lemma 4.7. Thus for all $i = 1, \ldots, \rho$, the absolute values of the scalar coefficients of $P^m v_1$ cannot grow faster than $r^{m-j}(m)$. 

Let $D = \sum a_i v_i$. Since $D \in \text{Num}(X)$, we have $a_i \in \mathbb{R}$ whenever $v_i$ corresponds to a real eigenvalue. If $v_i$ corresponds to a complex eigenvalue $\lambda$, then the complex conjugate $\overline{\lambda}$ is also an eigenvalue with corresponding vector $\overline{v_i}$ and coefficient $\overline{a_i}$. Letting $v'_i = a_i v_i + \overline{a_i} \overline{v_i}$ we write $D$ as the sum of real vectors. Considering $P^m D$, we see that no scalar coefficient of a $v_i$ or $v'_i$ can grow faster than $r^{m-j}(m)$ by Equation (4.10). Thus intersecting with any curve $C$ we have 
\[(P^m D.C) \leq c_1 m^j r^m \text{ for some } c_1 > 0 \text{ and all } m > 0. \] (We can choose $c_1$ large enough to handle the case of small $m$.)

Now suppose $D$ is ample. Then $D$ is an element of the interior of the nef cone \cite[p. 1209]{K} and so there exists $\alpha > 0$ such that $\alpha D - v_{j+1}$ is an element of the ample cone \cite[p. 1209]{V}. Since $\sigma$ is finite, $P^m(\alpha D - v_{j+1})$ is ample for all $m \geq 0$ \cite[Proposition 4.3]{H}. Thus for all curves $C$ and all $m > 0$, $\alpha(P^m D.C) > (P^m v_{j+1}.C)$ since the intersection number of an ample divisor and any curve is positive.

Since $v_1 \neq 0 \in \text{Num}(X)$, there exists a curve $C$ such that $(v_1.C) \neq 0$. Replacing $v_1$ with $-v_1$ if necessary, we can assume $(v_1.C) > 0$. Then by Equation (4.10), we have $\alpha(P^m D.C) > c_3 m^{j+1} r^m$ for some $c_3 > 0$ and all $m \gg 0$. We can choose $c_3$ small enough to make the inequality true for all $m > 0$. Then take $c_2 = c_3/\alpha$ to complete the proof. \hfill \Box

While the lemma above holds for $r = 1$, we have different behavior for the intersection numbers of sums of $P^k D$, depending on $r$. For $r = 1$, the intersection numbers $(D + PD + \cdots + P^{m-1} D.C)$ grow at most like the polynomial $m^{j+1}$ \cite[Equation (4.3)]{K}. However, when $r > 1$, these numbers grow at most like $m^j r^m$.

Lemma 4.11. Use the hypotheses of Lemma 4.9 with $D$ any divisor. Suppose $r > 1$. Then for all curves $C$, there exists $c_3 > 0$ such that
\[(D + PD + P^2 D + \cdots + P^{m-1} D.C) \leq c_3 m^j r^m \text{ for all } m > 0.\]
Proof. Consider \( \sum_{i=0}^{m-1} i^r r^i = \frac{r^{m+1} - r^i}{r - 1} \). By differentiating \( j \) times with respect to \( r \), we see that \( \sum_{i=0}^{m-1} i^r r^i \) is bounded above by \( c_4 m^j r^m \) for some \( c_4 > 0 \). The claim then follows from Lemma 4.9(1). \( \square \)

We can now prove that left ampleness never occurs when \( \sigma \) is not an automorphism and \( X \) is normal. For simplicity, define

\[ \Delta_m = D + \sigma^* D + (\sigma^*)^2 D + \cdots + (\sigma^*)^{m-1} D. \]

**Proposition 4.12.** Let \( X \) be a projective integral scheme with \( \sigma : X \to X \) finite. Let \( P \) be the action of \( \sigma^* \) on \( \text{Num}(X) \) and let \( r \) be the spectral radius of \( P \). Then there exists an ample divisor \( H \) such that \( \Delta_m - (\sigma^*)^m H \) is not ample for all \( m > 0 \).

Proof. Let \( H \) be an ample divisor. Choose a curve \( C \) and \( c_2 > 0 \) such that \( (P^m H.C) \geq c_2 m^j r^m \) for all \( m > 0 \), with \( j \) as in Lemma 4.9. Then there exists \( c_3 > 0 \) such that \( (\Delta_m.C) \leq c_3 m^j r^m \) for all \( m > 0 \) by Lemma 4.11. Replacing \( H \) with an integer multiple, we can assume \( c_2 > c_3 \). Thus \( (\Delta_m - (\sigma^*)^m H.C) < 0 \) for all \( m > 0 \). Since ample divisors have positive intersection with every curve, we are done. \( \square \)

Combining Propositions 4.6 and 4.12 we immediately have the following. Recall that if \( X \) is normal and \( \sigma \) is not an automorphism, then \( r > 1 \) by Lemma 4.7. Note that by Corollary 2.6, the following holds for any finite endomorphism when \( X \) is regular.

**Corollary 4.13.** Let \( X \) be a projective scheme with finite faithfully flat endomorphism \( \sigma \). Let \( P \) be the action of \( \sigma^* \) on \( \text{Num}(X) \) and let \( r \) be the spectral radius of \( P \), with \( r > 1 \). Then \( B_n \) is not left ample, where \( B_n \) is as in Definition 3.2 with \( L \) an invertible \( \mathcal{O}_X \)-module.

It is however possible for the sequence \( B_n \) to be right ample when \( \sigma \) is not an automorphism. This is in contrast to the automorphism case, where left and right ampleness are equivalent \([K1\text{ Corollary 5.1]}\]. Note that we do not need \( \sigma \) to be faithfully flat due to the equivalences of Proposition 4.5.

**Proposition 4.14.** Let \( X \) be a projective integral scheme with finite endomorphism \( \sigma \) and \( P \) the action of \( \sigma^* \) on \( \text{Num}(X) \). Suppose there exists an ample invertible sheaf \( L = \mathcal{O}_X(D) \) such that \( D \) is an eigenvector of \( P \). Then the sequence \( B_n = 1(L_n)_\sigma^{*} \) is right ample.

Proof. Note that due to Lemma 4.9, the eigenvalue related to \( D \) must be the spectral radius \( r \). For any divisor \( H \), there exists \( \alpha_0 \in \mathbb{R} \) such that \( \alpha D + H \) is in the ample cone for all \( \alpha \geq \alpha_0 \) \([V\text{ p. 1209]}\]. Thus \( \Delta_m + H \) is ample for all \( m \gg 0 \). So by Proposition 4.5, the sequence \( B_n \) is right ample. \( \square \)

Note that since \( D \in \mathbb{Z}^I \), we must have \( r \in \mathbb{Q} \). Also, we have self-intersection number

\[ (\sigma^* D)^{\dim X} = r^{\dim X} (D)^{\dim X} = (\deg \sigma)(D)^{\dim X}. \]
Since $X$ is integral, $\deg \sigma \in \mathbb{N}$. Thus $r \in \mathbb{N}$. In fact, it is known that when an ample divisor is an eigenvector, we can write $P = r(I \oplus O)$ where $I$ is the $1 \times 1$ identity matrix and $O$ is an orthogonal matrix with $\mathbb{Q}$ coefficients \cite[p. 330, Proposition 3]{Kl}. But we will not need this.

One important example of a finite endomorphism satisfying Proposition 4.14 is the relative Frobenius endomorphism $f$ for characteristic $p > 0$. In this case, $P$ is just the scalar matrix given by multiplication by $p$ \cite[Lemma 2.4]{A}. In the next section we examine the twisted ring obtained from the Frobenius endomorphism on $\mathbb{P}^n$.

It is also trivial that Proposition 4.14 holds when $\text{Pic}(X) \cong \mathbb{Z}$.

We end this section by showing that the bimodule algebras obtained from non-automorphisms are not noetherian.

**Theorem 4.15.** Let $X$ be a projective normal integral scheme with finite endomorphism $\sigma$, with $\sigma$ not an automorphism. Let $L$ be an invertible $\mathcal{O}_X$-module and define $B_n$ and $B = \oplus \Gamma(B_n)$ as in Definition 3.2. Suppose $B_n$ is a right ample sequence. Then $B$ is neither left nor right noetherian.

**Proof.** Let $L = \mathcal{O}_X(D)$ and let $P$ be the action of $\sigma^*$ on $\text{Num}(X)$, with spectral radius $r$. We have $r > 1$ by Lemma 4.7.

By Lemma 4.9, choose a curve $C$ such that $(P^n D.C) \geq cr^n$ for some $c > 0$. We have an exact sequence of coherent sheaves

$$0 \to I \to \mathcal{O}_X \to \mathcal{O}_C \to 0.$$  

Tensoring with $B_n$ on the right, we have $H^1(X, I \otimes B_n) = 0$ for $n \gg 0$. Thus $\Gamma(B_n) \to \Gamma(\mathcal{O}_C \otimes B_n)$ is surjective for $n \gg 0$. By the Riemann-Roch formula for curves, $\Gamma(\mathcal{O}_C \otimes B_n)$ grows exponentially and hence so does $\Gamma(B_n)$.

Since $B$ has exponential growth, $B$ is neither left nor right noetherian by \cite[Theorem 0.1]{SZ}.

We also find that the bimodule algebra $B$ is not right noetherian. See \cite{VdB} for details on the category of right $B$-modules.

**Corollary 4.16.** Let $X$ be a regular projective integral scheme over an algebraically closed field $k$. Let $\sigma$ be a finite endomorphism which is not an automorphism. Define $B_n$ and $B$ as in Definition 3.2. Suppose $B_n$ is a right ample sequence. Then $B$ is not right noetherian in the category of right $B$-modules.

**Proof.** Suppose $B$ is right noetherian. First, note that the $B_n$ are left flat by Lemma 4.3. Thus the category of coherent right $B$-modules is abelian \cite[Corollary 3.8]{VdB}.

Therefore, all hypotheses of \cite[Theorem 5.2]{VdB} hold. Thus $B = \Gamma(B)$ is right noetherian. But this contradicts Theorem 4.15. So $B$ cannot be right noetherian.
5. Frobenius endomorphism on projective space

In this section, we examine more closely a specific case of a twisted homogeneous coordinate ring, specifically the one created from the relative Frobenius morphism \( f \) (the \( k \)-algebra homomorphism induced by \( x_i \mapsto x_i^p \)) on \( \mathbb{P}_k^m \) where \( \text{char } k = p > 0 \) and the standard coordinate ring of \( \mathbb{P}^m \) is written \( k[x_0, \ldots, x_m] \). Note that “relative” means that \( f \) is the identity on \( k \), so that \( f \) is a \( k \)-morphism. This is a modification of the absolute Frobenius morphism, in which \( a \mapsto a^p \) for \( a \in k \), exploiting the fact that the Frobenius morphism is an isomorphism of a perfect field \( k \).

**Definition 5.1.** Let \( k \) be a perfect field of characteristic \( p > 0 \). Let \( f \) be the relative Frobenius map of \( \mathbb{P}_k^m \), i.e., the map generated by the Frobenius homomorphism described by \( x_i \mapsto x_i^p \) on \( k[x_0, x_1, \ldots, x_m] \). Let \( F = \bigoplus F_n \) where

\[
F_n = \Gamma(\mathcal{O}(1) \otimes f^* \mathcal{O}(1) \otimes (f^2)^* \mathcal{O}(1) \otimes \cdots \otimes (f^{n-1})^* \mathcal{O}(1)).
\]

As with the standard construction, for \( a \in F_i \) and \( c \in F_j \), the multiplication of \( a \) and \( c \) is given by \( a f^j(c) \).

As \( (f^j)^* \mathcal{O}(1) \cong \mathcal{O}(p^j) \), we have that \( F_n = \Gamma(\mathcal{O}(1 + p + p^2 + \cdots + p^{n-1})) \).

For convenience, let

\[
e_n = 1 + p + p^2 + \cdots + p^{n-1} = \frac{p^n - 1}{p - 1}.
\]

In this notation, we have that \( \dim_k F_n = \binom{e_n + m}{m} \), which grows like \( (p^m)^n \). Hence, \( \dim_k F_n \) grows exponentially with \( n \) and \( F \) cannot be either left or right noetherian by [SZ, Theorem 0.1], just as in Theorem 4.15.

Moreover, \( F \) is not finitely generated as a \( k \)-algebra, except in the case when both \( m = 1 \) and \( p = 2 \), as we will now show. The proof breaks into two cases, one where \( p > 2 \) and one where \( p = 2 \).

Before getting to the proofs, a few words about notation. Given any homogeneous element \( a \) of \( F \), it has the usual total degree as a polynomial and the degree referring to which \( F_j \) the element \( a \) belongs. We adopt the following convention for this section, \( \deg a \) is the total degree of \( a \) considered as a polynomial in \( k[x_0, \ldots, x_m] \) and \( \deg_x a \) is the exponent of \( x_i \) when \( a \) is written as a polynomial in \( k[x_0, \ldots, x_m] \). So \( \deg x_0^{p+1} = p + 1 \) even though \( x_0^{p+1} \in F_2 \).

Also, given two elements \( a_1 \) and \( a_2 \) in \( F \), their product as elements of \( F \) will be denoted \( a_1 \cdot a_2 \) while if \( a_1 \) and \( a_2 \) are considered solely as polynomials, the notation \( a_1 a_2 \) will denote the standard multiplication of polynomials. Similarly, given sets of elements \( A_1 \) and \( A_2 \), we define \( A_1 \cdot A_2 = \{ \sum_{\text{finite}} a_1 \cdot a_2 | a_i \in A_i \} \) and \( A_1 A_2 = \{ \sum_{\text{finite}} a_1 a_2 | a_i \in A_i \} \).

### 5.1. The case \( p > 2 \)

First, the case when \( p > 2 \). To show that \( F \) is not finitely generated, we will show that any element \( z \) of \( F_n \) such that \( \deg_{x_0} z = p^{n-1} - 1 \) will not be
the product of elements of strictly smaller degree for any \( n > 1 \). The proof is by contradiction. So assume that there is some \( z \in F_n \) with \( \deg_{x_0} z = p^n - 1 - 1 \) such that \( z = uv \) where \( u \in F_a \) and \( v \in F_b \) for some integers \( a \) and \( b \) such that \( a + b = n \) and \( 1 < a, b < n \). (As \( z \) is a monomial, \( u \) and \( v \) are also monomials.) Then \( \deg_{x_0} u = c \) for some integer \( c \leq \frac{p^n - 1}{p^p - 1} \) and \( \deg_{x_0} v = d \) for some integer \( d \leq \frac{p^b - 1}{p^b - 1} \). By definition of the multiplication, we now have \( \deg_{x_0}(uv) = c + p^d d \).

Since \( z = uv \), this means \( c + p^d d = p^n - 1 - 1 \). As \( a \leq n - 1 \), we have \( c \equiv -1 \) (mod \( p^a \)). Since \( 0 \leq c \), this means \( c \geq p^a - 1 \), but as \( c \leq \frac{p^n - 1}{p^p - 1} \) this is impossible for \( p > p \). Hence \( z \) cannot be generated by elements of smaller degree.

5.2. The case \( p = 2 \)

When \( m = 1 \), the algebra \( F \) is in fact generated by \( F_1 \) as follows. In this case

\[
F_n = \mathcal{O}(\frac{2^n - 1}{2 - 1})(X) = \mathcal{O}(2^n - 1)(X) = \bigoplus_{j=0}^{2^n - 1} kx^{2^n - 1 - j}y^j.
\]

For any \( z \in F_n \), we have \( x \cdot z = xz^2 \) and \( y \cdot z = yz^2 \) (where the right-hand side is written as an element of \( F_{n+1} \)). Therefore \( x \cdot F_n = \bigoplus_{j=0}^{2^n - 1} kx^{2^n + 1 - 2j}y^{2j} \) and \( y \cdot F_n = \bigoplus_{j=0}^{2^n - 1} kx^{2^n + 1 - 2j}y^{2j+1} \). Hence, \( F_1 \cdot F_n = x \cdot F_n + y \cdot F_n = F_{n+1} \) and, by induction, \( F_1 \) must be generated by \( F_1 \) for all \( \ell \geq 1 \).

However, when \( m \geq 2 \), the algebra \( F \) is not finitely generated. In this case, we will show that the element

\[
z = x_0^{2^n - 1}x_1^{2i+1}x_2^{2j+1} \in F_n
\]

is not generated by elements of smaller degree for any nonnegative integers \( i \) and \( j \) (here \( 2i + 1 + 2j + 1 = 2^{n-1} \)). As before, the proof proceeds by contradiction. Suppose that \( z = uv \) for some \( u \in F_r \) and \( v \in F_s \) where \( r + s = n \) and \( 1 < r, s < n \). Since \( z \) is a monomial only in \( x_0, x_1 \) and \( x_2 \), so are \( u \) and \( v \). Write \( u = x_0^a x_1^b x_2^c \) and \( v = x_0^{d} x_1^{e} x_2^{f} \); here \( a + b + c = 2^r - 1 \) and \( d + e + f = 2^s - 1 \).

By definition of the multiplication \( uv = x_0^d x_1^e x_2^f \). Since \( z = uv \), this means \( 2^r d + a = 2^n - 1 \), \( b + 2^s e = 2i + 1 \), and \( c + 2^s f = 2j + 1 \). Then \( a \) and \( c \) are even (mod \( 2^r \)). Thus \( a = 2^r g - 1 \) for some positive integer \( g \). Then \( 2^r - 1 = a + b + c = 2^r g - 1 + b + c \); the only way this is possible is for \( b = c = 0 \) and \( a = 2^r - 1 \). Now, the other two equations become \( 2^r e = 2i + 1 \) and \( 2^r f = 2j + 1 \), neither of which is possible. Hence \( z \) cannot be generated by elements of smaller degree.

Summing up our work, we have the following.

**Proposition 5.2.** Let \( X = \mathbb{P}^m \) over a perfect field of characteristic \( p > 0 \). Let \( f \) be the relative Frobenius endomorphism. Then the twisted homogeneous coordinate ring \( F = \oplus \Gamma \left( \mathcal{O} \left( \frac{z^n - 1}{p^n - 1} \right) \right) \) is not finitely generated, unless \( p = 2 \) and \( m = 1 \). In that case, \( F \) is generated by \( F_1 = \Gamma(\mathcal{O}(1)) \), but \( F \) is not noetherian. \( \square \)
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