CHEMICAL TRANSPORT AND SPONTANEOUS LAYER FORMATION IN FINGERING CONVECTION IN ASTROPHYSICS

JUSTIN M. BROWN\textsuperscript{1}, PASCAL GARAUD\textsuperscript{2}, AND STEPHAN STELLMACHT\textsuperscript{3}

\textsuperscript{1} Department of Astronomy and Astrophysics, University of California Santa Cruz, 201 Interdisciplinary Sciences Building, Santa Cruz, CA 95064, USA; jumbrown@ucsc.edu
\textsuperscript{2} Department of Applied Mathematics and Statistics, Baskin School of Engineering, 1156 High Street, Mail Stop SOE2, Santa Cruz, CA 95064, USA
\textsuperscript{3} Institut für Geophysik, Westfälische Wilhelms-Universität Münster, D-48149 Münster, Germany

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ABSTRACT

A region of a star that is stable to convection according to the Ledoux criterion may nevertheless undergo additional mixing if the mean molecular weight increases with radius. This process is called fingering (thermohaline) convection and may account for some of the unexplained mixing in stars such as those that have been polluted by planetary infall and those burning $^3$He. We propose a new model for mixing by fingering convection in the parameter regime relevant for stellar (and planetary) interiors. Our theory is based on physical principles and supported by three-dimensional direct numerical simulations. We also discuss the possibility of formation of thermocompositional staircases in fingering regions, and their role in enhancing mixing. Finally, we provide a simple algorithm to implement this theory in one-dimensional stellar codes, such as KEPLER and MESA.

Key words: convection – diffusion – hydrodynamics – instabilities – planet–star interactions – stars: evolution

Online-only material: color figures

1. INTRODUCTION

The phenomenon of fingering convection (otherwise known as thermohaline convection) occurs in regions of stellar and planetary interiors where the mean molecular weight, $\mu$, increases upward but which are nevertheless stable to the Ledoux criterion. If not for the effects of diffusion, such a system would be stable to small perturbations. In the presence of diffusion, however, instability can occur. Indeed, when a high $\mu$, high entropy parcel is displaced downward, heat rapidly leaks via thermal diffusion. Since compositional diffusion is much weaker, the parcel remains denser than its surroundings and proceeds to sink. The instability takes the form of finger-like structures and can significantly increase the flux of heat and composition above that of molecular or radiative diffusion. Fingering convection appears in several key problems in astrophysics in which an inverse $\mu$-gradient is observed, the most notable of which are planet infall and $^3$He fusion.

Take for instance the problem of planet infall. Stars with detected planets have higher metallicities than those without (e.g., Santos et al. 2001; Fischer & Valenti 2005). The relevant question is then whether this observation results from a superior ability of high-metallicity gas to form planets or from pollution by planet infall. In the latter scenario, it is thought that stars accrete planets onto their outer convective zones, raising the metallicity at the surface. Vauclair (2004) was the first to argue that fingering convection may play an important role in this problem. Because the mean molecular weight in the convective zone rises upon absorbing a planet, the radiative region just beneath experiences an inverse $\mu$-gradient and becomes unstable to the fingering instability. This drains the excess metallicity from the convective zone into the radiative interior. However, the timescale for mixing by fingering convection was until recently unknown and is needed to determine how long the post-infall excess metallicity of the convective zone remains observable. Thus, in order to determine whether the planet–metallicity connection is an effect of planet formation or infall, we must quantify mixing by fingering convection.

During the process of $^3$He fusion, in which $^3$He + $^3$He $\rightarrow$ $^4$He + 2$^1$H, the total number of particles increases, so the mean molecular weight decreases (Ulrich 1972). Because fusion occurs more frequently at higher temperatures and densities, this reaction preferentially decreases the molecular weight deeper in the star, and builds up an inverse $\mu$-gradient. This is thought to occur in red giant branch stars. Indeed, these stars are observed to undergo a process known as dredge-up, in which particular isotopes (such as $^3$He) produced in the deep stellar interior are advected outward (e.g., Gilroy 1989; Charbonnel 1994). Simulations by Charbonnel (1994) have produced surface abundances that agree well with observations for the so-called first dredge-up. However, she noted that some additional mixing in low-mass red giants is needed to account for the observed carbon isotope ratios. Eggleton et al. (2006) initially suggested that a Rayleigh–Taylor instability caused by the inverse $\mu$-gradient could provide this additional mixing. However, Charbonnel & Zahn (2007) later proposed that a fingering instability caused by $^3$He burning was more likely but were unable to quantify the mixing due to the lack of constraints on the efficiency of fingering convection. Further progress on both subjects—planetary infall and $^3$He fusion—requires the development of improved mixing theories.

Although the applications in which we are interested are astrophysical, much of the formalism of fingering convection was first developed to address the phenomenon of salt fingers in the ocean. There, temperature and salt play the roles of the entropy and metallicity, and salt diffuses about 100 times more slowly than temperature. Stern (1960) first proposed the theory of a “salt-fountain” to explain the persistent, small-scale motions observed in warm, salty water lying above cool, fresh water even though density decreases upward. For a recent review of the field, see Kunze (2003). By contrast with astrophysical fingering, it is possible to run laboratory experiments of fingering convection in saltwater and measure its turbulent transport efficiency (e.g., Turner 1967; Stern & Turner 1969; Schmitt 1979). Unfortunately, none of these laboratory experiments are applicable to the parameter regime relevant for stars.
to astrophysics. Because of this, theoretical models of fingering convection in astrophysics have remained, until recently, mostly phenomenological.

Several prescriptions have been suggested over the past four decades. Ulrich (1972) and Kippenhahn et al. (1980) attempted to constrain the dimensions of fingers using stability arguments and determined the resulting thermal and compositional mixing timescales using dimensional analysis. Both these prescriptions have a free parameter: in the former, the free parameter is the "effective inertia of the flow"; whereas in the latter, it is the aspect ratio of the fingers. Later, Schmitt (1983) used linear theory to predict the ratio of the heat to the compositional turbulent fluxes in astrophysical fingering but could not determine the absolute fluxes directly from this method.

Only in the last few years have numerical simulations of fingering convection approaching the astrophysical parameter regime become more readily available (e.g., Denissenkov 2010; Traxler et al. 2011a). Denissenkov (2010) ran two-dimensional (2D) simulations to measure the aspect ratio of fingers and used the previous literature and a dimensional argument to find a prescription for mixing by fingering convection in red giant branch stars. He concluded that while this process could provide some of the required mixing discussed by Charbonnel & Zahn (2007), it alone was insufficient to account for the observed abundances of low-mass red giant branch stars. Traxler et al. (2011a) presented three-dimensional (3D) numerical simulations of fingering convection and generated an empirical prescription for the turbulent fluxes of heat and composition in the astrophysical regime that can resolve both the fingers and these large-scale instabilities. From a theoretical point of view, Traxler et al. (2011a) predicted that layers could not form by the \( \gamma \)-instability in astrophysics, but we now investigate this more thoroughly and also pose the question of whether layers may form by the collective instability instead.

In Section 2, we describe our model. In Section 3, we present the results of existing and new numerical simulations and compare them to the model described in Traxler et al. (2011a); we find that the latter does not fit our data as the system approaches overturning convection. In Section 4, we then provide a new model that more precisely fits the simulations and that is constructed from physical principles rather than empirical fits. We study the conditions for layer formation in Section 5 and conclude in Section 6 by discussing this new model and its implications in stellar astrophysics.

2. MODEL

As discussed in Traxler et al. (2011a), the characteristic length scale of the fingering instability in astrophysical objects is much smaller than a pressure scale height, which in turn is generally small enough to ensure that the curvature of the star plays little role in the system dynamics. We therefore use a Cartesian grid \((x, y, z)\), with gravity given by \( g = -g e_z \) to model a small region of the star. In addition, the velocities within these fingers are much slower than the sound speed of the plasma, so we can treat the fluid with the Boussinesq approximation (Spiegel & Veronis 1960). We assume that there is a background temperature profile, \( T_0(z) \), and a mean molecular weight profile, \( \mu_0(z) \), which depend only on \( z \). If the region considered is small enough, these background profiles can be approximated by linear functions with constant slopes, \( T_{0c} \) and \( \mu_{0c} \), respectively. We therefore assume that each profile is comprised of a linear background state and a perturbation, e.g., \( \rho_{0c} = \zeta T_{0c} + T \).

The equation of state for the density perturbation, \( \rho \), within the Boussinesq approximation, is

\[
\frac{\rho}{\rho_0} = -\alpha T + \beta \mu, \tag{1}
\]

where \( \rho_0 \) is the mean density of the region. The coefficients \( \alpha \) and \( \beta \) are those of thermal expansion and compositional contraction and can be obtained by linearizing the equation of state around \( \rho_0 \). The remaining governing equations for a Boussinesq system come from Spiegel & Veronis (1960):

\[
\nabla \cdot \mathbf{u} = 0, \tag{2}
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla \rho}{\rho_0} + g(\alpha T - T_{0c})e_z + \nu \nabla^2 \mathbf{u}, \tag{3}
\]

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T + w(T_{0c} - T_{0c}) = \kappa \nabla^2 T, \quad \text{and} \tag{4}
\]
where \( \mathbf{u} = (u, v, w) \) is the fluid velocity, \( p \) is the pressure, \( v \) is the kinematic viscosity, \( \kappa_T \) is the thermal diffusivity, \( \kappa_\mu \) is the compositional diffusivity, and \( T_0^{\text{ad}} \) is the background adiabatic temperature gradient, assumed constant within the considered domain. It should be noted here that \( v, \kappa_T, \) and \( \kappa_\mu \) are assumed to be constant as additional requirements of the Boussinesq approximation. Other interesting dynamics may arise for variable \( v, \kappa_T, \) and \( \kappa_\mu \), but these are not addressed here.

In all that follows, we consider stably stratified regions so that \( T_0 - T_0^{\text{ad}} > 0 \). We then non-dimensionalize the governing equations, taking our length scale to be the anticipated finger width, \( d = (\kappa_T v/\partial T_0 - T_0^{\text{ad}})^{1/4} \) (Stern 1960; Kato 1966).

We scale time, temperature, and composition as:

\[
\left[ \begin{array}{c} t \\ T \\ \mu \\ \rho \\ \gamma \\ T_{0}^{\text{ad}} \end{array} \right] = \left[ \begin{array}{c} \tau \\ \frac{T}{T_{0}} \\ \frac{\mu}{\mu_{0}} \\ \frac{\rho}{\rho_{0}} \\ \frac{\gamma}{\gamma_{0}} \\ \frac{T_{0} - T_{0}^{\text{ad}}}{T_{0}^{\text{ad}}} \end{array} \right]
\]

In all that follows, we consider stably stratified regions so that \( T_0 - T_0^{\text{ad}} > 0 \). We then non-dimensionalize the governing equations as follows:

\[
\frac{\partial \mu}{\partial \tau} + \mathbf{u} \cdot \nabla \mu + w \mu_{0c} = \kappa_\mu \nabla^2 \mu, \tag{5}
\]

If \( R_0 < 1 \), the destabilizing compositional gradient dominates, and the system is unstable to the Ledoux criterion. One expects the region to be fully mixed by overturning convection. For \( R_0 \gg 1 \), the stabilizing entropy gradient dominates, and the system is stable. The only possible transport is via diffusion. For intermediate values, \( 1 < R_0 < 1/\tau \), Baines & Gill (1969) showed that the system is unstable to finger convection. It is this region of parameter space to which we now focus our attention. As in Traxler et al. (2011a), we define a reduced density ratio,\[ r = \frac{R_0 - 1}{\tau - 1 - 1}, \tag{11} \]

which remaps the “fingering regime” to the interval of \( r \in [0, 1] \) regardless of the value of \( \tau \).

We model fingering convection numerically, and solve Equations (6)-(9) using a pseudo-spectral double-diffusive convection code as in Traxler et al. (2011a, 2011b) and Stellmach et al. (2011). To avoid spurious boundary effects, we apply triply periodic boundary conditions on all perturbations \( T, \mu, \) and \( \mathbf{u} \). The code uses no sub-grid scale model, so the scale on which energy is dissipated must always be fully resolved. In all simulations described below, unless specifically noted, the computational domain is taken to be a cube with side length 100d. Since the wavelength of the fastest growing mode ranges from 6d to 15d in the parameter regime considered in this paper, this implies that there are at least 5–10 wavelengths of the fastest growing mode per 100d. We test the required resolution by inspecting the compositional and vorticity fields, which are in general the least resolved ones given the small values of \( Pr \) and \( \tau \) selected.

### 3. RESULTS

Traxler et al. (2011a) explored a significant sample of parameter space, with \( Pr \) and \( \tau \) ranging from 1/3 down to 1/30, as seen in Table 1. However, they only tested a few different density ratios in the cases with \( Pr \), \( \tau \sim 1/30 \). Furthermore, owing to computational limitations, they did not explore the low \( R_0 \) limit, the regime where we believe their prescription is most questionable (see Section 1). Here, we explore more comprehensively in \( R_0 \) the parameters chosen by Traxler et al. (2011a) and also run simulations down to \( Pr \), \( \tau \sim 1/100 \). This enables us to test the validity of their prescription in more detail. It is currently computationally prohibitive to reduce \( Pr \), \( \tau \) much lower than this, particularly at low \( R_0 \), where the simulations become increasingly turbulent.4

As in Traxler et al. (2011a), we report our results in the form of the Nusselt numbers, which are defined as the ratio of the total vertical flux to the diffused flux and expressed here in terms of the non-dimensional fields as:

\[
\text{Nu}_T = 1 - \langle wT \rangle, \tag{12}
\]

\[
\text{Nu}_\mu = 1 - \frac{R_0}{\tau} \langle w\mu \rangle. \tag{13}
\]

The vertical turbulent fluxes, \( \langle wT \rangle \) and \( \langle w\mu \rangle \), are measured by taking the average of the products \( wT \) and \( w\mu \) over the entire domain.

When we discuss layer formation, we will also use the ratio of the total non-dimensional fluxes:

\[
\gamma = \frac{-1 + \langle wT \rangle}{-\frac{\phi}{\delta} + \langle w\mu \rangle} = \frac{R_0 \text{Nu}_T}{\text{Nu}_\mu}, \tag{14}
\]

and the ratio of the turbulent non-dimensional fluxes:

\[
\gamma_{\text{urb}} = \frac{\langle wT \rangle}{\langle w\mu \rangle} = \frac{R_0 (\text{Nu}_T - 1)}{\tau (\text{Nu}_\mu - 1)}. \tag{15}
\]

### 3.1 Typical and Atypical Simulations

In Figure 1, we present the temporal evolution of the thermal and compositional Nusselt numbers in a typical numerical simulation of fingering convection, taking \( Pr = 1/10, \tau = 1/30 \), and \( R_0 = 3 \). This evolution begins with a period of

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4 We note that simulations at much lower values of \( Pr \) and \( \tau \) have been reported by Denissenkov (2010). Even though the latter are two-dimensional, it is unlikely that they are fully resolved. Whether underresolved simulations yield satisfactory estimates for the fluxes is a different question that we shall address in a subsequent publication. The results from the simulations of Denissenkov (2010) (\( \text{Nu}_\mu = 997.6 \) for RGB interiors) are consistent with our theoretical model (\( \text{Nu}_\mu = 1294.7 \) for the same parameters, see Section 4) within 30%.

5 The form in terms of dimensional variables \( w \) and \( T \) is \( \text{Nu}_T = 1 - \langle wT \rangle/\tau (T_0 - T_0^{\text{ad}}) \). Note that this definition of the Nusselt number describes the flux of the potential temperature and not that of temperature. The two are only equal when \( T_0^{\text{ad}} = 0 \). For a more complete discussion, see Mirouh et al. (2012).
Table 1
Turbulent Flux Measurements and Respective Errors for Low Pr, Low τ Simulations of Thermohaline Convection

| Pr  | τ   | R₀   | Nuₜ   | σₜ   | Nuᵤ   | σᵤ   | Layers |
|-----|-----|------|-------|------|-------|------|--------|
| 1/3 | 1/3 | 1.02003ᵇ | 31.0  | 4.0  | 120.0 | 10.0 | N      |
|     |     | 1.08012ᵃ | 23.0  | 1.0  | 92.0  | 5.0  | N      |
|     |     | 1.10015ᵇ | 20.0  | 1.0  | 83.0  | 5.0  | N      |
|     |     | 1.2003ᵃ | 13.5  | 0.6  | 60.0  | 3.0  | N      |
|     |     | 1.3004ᵇ | 9.9   | 0.5  | 46.0  | 2.0  | N      |
|     |     | 1.5007ᵇ | 5.7   | 0.2  | 28.3  | 0.9  | N      |
|     |     | 2.0015ᵇ | 2.16  | 0.05 | 9.5   | 0.3  | N      |
|     |     | 2.4021ᵃ | 1.3   | 0.01 | 3.5   | 0.09 | N      |
|     |     | 2.8027ᵇ | 1.037 | 0.003| 1.33  | 0.02 | N      |
|     |     | 2.9028ᵇ | 1.0159| 0.0009| 1.143 | 0.008| N      |

1/3 1/10 1.01 28.6 0.4 428.0 6.0 Y
1.54ᵇ 9.0 0.2 197.0 4.0 N
3.97ᵇ 1.62 0.02 38.7 1.0 N
7.03ᵇ 1.075 0.003 7.6 0.2 N
9.46ᵇ 1.0041 0.0004 1.41 0.04 N
1/10 1/3 1.10015ᵇ | 8.0 | 0.5 | 33.0 | 2.0 | N |
1.71015ᵇ | 2.16 | 0.05 | 8.7 | 0.3 | N |
2.30195ᵇ | 1.24 | 0.01 | 2.9 | 0.1 | N |
2.8027ᵇ | 1.019 | 0.0007 | 1.17 | 0.006 | N |
2.9028ᵇ | 1.0065 | 0.0003 | 1.059 | 0.003 | N |
1/10 1/10 1.003 | 11.36 | 0.09 | 169.0 2.0 | N |
1.09ᵇ | 8.3 | 0.7 | 143.0 | 9.0 | N |
1.45ᵇ | 4.3 | 0.1 | 86.0 | 3.0 | N |
1.99ᵇ | 2.5 | 0.05 | 54.0 | 2.0 | N |
2.8ᵇ | 1.73 | 0.02 | 34.8 | 1.0 | N |
2.98ᵇ | 1.62 | 0.01 | 31.3 | 0.6 | N |
3.25ᵇ | 1.51 | 0.02 | 27.4 | 0.8 | N |
4.96ᵇ | 1.148 | 0.003 | 11.6 | 0.2 | N |
7.03ᵇ | 1.032 | 0.001 | 3.9 | 0.1 | N |
9.1ᵇ | 1.0036 | 0.0001 | 1.36 | 0.01 | N |
1/10 1/30 1.1 | 7.6 | 0.1 | 500.0 | 7.0 | Y |
1.5 | 4.24 | 0.06 | 347.0 | 5.0 | N |
2.0 | 2.77 | 0.02 | 252.0 | 2.0 | N |
3.0 | 1.85 | 0.02 | 174.0 | 3.0 | N |
6.0 | 1.233 | 0.004 | 95.0 | 1.0 | N |
8.0 | 1.1382 | 0.0006 | 66.0 | 0.3 | N |
10.96³ᵇ | 1.059 | 0.002 | 32.4 | 0.7 | N |
20.34ᵇ | 1.0075 | 0.0005 | 7.1 | 0.4 | N |
1/10 1/100 1.1ᵇ | 7.4812 | … | 1584.517 | … | N |
1.3ᵇ | 5.1979 | … | 1285.765 | … | N |
1.5ᵇ | 4.1962 | … | 1124.055 | … | N |
1.7ᵇ | 3.581 | … | 1031.832 | … | N |
1.9ᵇ | 2.914 | … | 884.519 | … | N |
2.1ᵇ | 2.6215 | … | 816.522 | … | N |
2.3ᵇ | 2.4437 | … | 789.866 | … | N |
2.5ᵇ | 2.2994 | … | 761.9 | … | N |
2.7ᵇ | 2.0685 | … | 684.936 | … | N |
2.9ᵇ | 1.9298 | … | 636.724 | … | N |
6.0 | 1.316 | 0.002 | 416.0 | 2.0 | N |
12.0 | 1.088 | 0.002 | 214.0 | 4.0 | N |
1/30 1/10 1.1 | 4.92 | 0.08 | 77.0 | 1.0 | N |
1.5 | 2.26 | 0.04 | 37.8 | 0.9 | N |
2.0 | 1.63 | 0.01 | 24.8 | 0.3 | N |
3.97ᵇ | 1.141 | 0.005 | 9.9 | 0.3 | N |
7.03ᵇ | 1.021 | 0.001 | 2.83 | 0.09 | N |
1/100 1/100 5.0 | 1.072 | 0.001 | 102.0 | 1.0 | N |
10.0 | 1.02569 | 3 \times 10^{-5} | 62.9 | 0.2 | N |

Notes.
-ᵇ Data from Traxler et al. (2011a).
-ᵇ Data from Radko & Smith (2012) with a domain size of 38.2 × 38.2 × 76.3 and a resolution of 256 × 256 × 512. The uncertainties of these fluxes are not known to us.

Figure 1. A typical example of the evolution of the thermal (top) and compositional (bottom) Nusselt numbers, Nuₜ and Nuᵤ (see Equation (12)), in a simulation with Pr = 1/10, τ = 1/30, and R₀ = 3. Time is measured in units of the thermal diffusion time across a length d. We note that the Nusselt curves grow exponentially, peak, and fall to a quasi-steady equilibrium state, which we call the saturated regime. The vertical lines indicate the times of the snapshots in Figure 2.

(A color version of this figure is available in the online journal.)

explosive growth from \( t = 0 \) to \( t = 160 \). The observed growth rate is found to be twice the growth rate of the fastest growing mode according to a linear stability analysis of the governing equations. This is as expected, as shown in Section 4.1. The transport peaks around \( t = 165 \), after which the fluxes of temperature and composition saturate (at a level slightly below the peak) and remain roughly constant for the remainder of the simulation.

In order to visualize the saturation process, we present snapshots of the compositional field of the simulation at three stages before and after saturation in Figures 2(a)–(c), marked with vertical lines in Figure 1. In Figure 2(a), which shows the state just as the primary fingering instability begins to develop, we see tall and thin vertical structures, called “elevator modes,” with high \( μ \) plumes sinking and low \( μ \) plumes rising. In Figure 2(b), which is taken just before the peak in Figure 1, we recognize the same vertical structures but note that smaller-scale instabilities have distorted them. The latter eventually destroy the elevator modes and the system achieves saturation in Figure 2(c). At this point, very little of the original elevator modes remain discernible, and quasi-steady, homogeneous turbulence dominates the dynamics.

While most simulations behave exactly as Figure 1, in some cases where \( R₀ \) is chosen to be very close to the onset of overturning convection (i.e., for small \( r \), see Equation (11)), the Nusselt numbers begin to increase gradually again after saturation. This behavior appears clearly in Figure 3, for which \( Pr = 1/10, \tau = 1/30, \) and \( R₀ = 1.1 \). We distinguish two phases post-saturation: from \( t = 100 \) to \( t = 600 \), the mean Nusselt numbers increase slowly and undergo quasi-periodic oscillations. We find that these oscillations have twice the buoyancy frequency, which is what we expect for gravity waves. Since linear gravity waves do not lead to an increase in the mean flux, the gradual rise of the mean Nusselt numbers indicates that the waves are highly nonlinear. At \( t = 600 \), the transport suddenly increases dramatically. Meanwhile, the mean
Figure 2. Snapshots of the compositional perturbation in the simulation shown in Figure 1 at the characteristic times marked in Figure 1. In (a) at \( t \approx 100 \), prior to saturation, the composition field is dominated by tall “elevator mode” structures characteristic of linear fingering convection. In (b) at \( t \approx 155 \), though the original elevator modes are still recognizable, they have been disrupted by secondary instabilities. At \( t \approx 180 \), (c) illustrates the “saturated regime” discussed in Figure 1. Note that the elevator modes have been completely destroyed by the secondary instabilities, leaving a domain filled completely with homogeneous turbulence.

(A color version of this figure is available in the online journal.)

density profile in part of the domain inverts, suggesting the formation of a fully convective layer (see Section 5.2). This is reminiscent of the formation of gravity waves and layer development in oceanic fingering convection (Stellmach et al. 2011). We will discuss these large-scale instabilities and related transport properties in Section 5.

3.2. Extraction of the Fluxes

To calculate the saturated fluxes in the homogeneous phase of all simulations, i.e., prior to the onset of any large-scale dynamics discussed above, we use the following method. We identify the first local minimum in \( \text{Nu}_T(t) \) after saturation and set this to be the beginning of the saturated regime. We then fit the remainder of the time series to a linear function of time with a least-squares fit, which yields a slope and a \( y \)-intercept. We use regression analysis to estimate an uncertainty for each parameter. If the uncertainty in the slope is greater than its magnitude, we then say that the Nusselt curve after saturation is flat and set the end of the saturated regime at the end of the simulation. On the other hand, for a simulation like the one shown in Figure 3, where the flux saturates initially but gravity waves begin enhancing the transport again thereafter, we are only interested in the short interval of time just after saturation. If the linear fit is not sufficiently flat, we reduce the time interval. The error is taken to be the rms of the fluxes.

3.3. Numerical Simulations

A compilation of our new results with published data from Traxler et al. (2011a) and Radko & Smith (2012) is given in Table 1. Figure 4 shows the corresponding values of the
mean Nusselt numbers of both composition and temperature, in the homogenous saturated phase, as a function of the reduced density ratio \( r \) (see Equation (11)). As expected, we see that both \( \text{Nu}_T - 1 \) and \( \text{Nu}_u - 1 \) tend to 0 as \( r \rightarrow 1 \). In contrast, as \( r \rightarrow 0 \), both \( \text{Nu}_T - 1 \) and \( \text{Nu}_u - 1 \) increase rapidly, presumably approaching values of Rayleigh–Benard convection in a triply periodic domain (Calzavarini et al. 2005; Garaud et al. 2010).

As an additional note, it is clear from Figure 4 that \( \text{Nu}_u \) does not depend on \( \text{Pr} \) and \( \tau \) individually, but rather on their ratio. This effect had already been observed in Traxler et al. (2011a), and led them to propose their empirical scaling laws:

\[
\text{Nu}_T - 1 = \text{Pr}^{1/2} \tau^{3/2} f(r),
\]

\[
\text{Nu}_u - 1 = \sqrt{\frac{\text{Pr}}{\tau}} g(r),
\]

where \( g(r) \) and \( f(r) \) have the form \( a e^{-b(1 - r)^c} \), where for \( f(r) \), \( a = 264 \pm 1, b = 4.7 \pm 0.2, c = 1.1 \pm 0.1 \), and for \( g(r) \), \( a = 101 \pm 1, b = 3.6 \pm 0.3, c = 1.1 \pm 0.1 \).

As discussed in Section 1, the prescription given by Traxler et al. (2011a) agrees well with the data for high \( R_0 \); however, these scalings underestimate the transport at low \( R_0 \), particularly as \( \text{Pr} \) and \( \tau \) decrease. It is clear from Figure 4 that their prescription breaks down at very low \( r \), particularly as \( \text{Pr} \) decreases. For this reason we now propose a theory that better models the data at both large and small \( r \). We derive this new theory with physical justification.

4. THEORETICAL MODEL FOR FLUXES AT SATURATION

Recently, Radko & Smith (2012) proposed a theory that characterizes the saturation of the fingering instability and predicts the thermal and compositional fluxes at saturation. The fundamental idea of this theory is straightforward: the saturation is thought to be caused by a secondary instability (or “parasitic” instability) of the elevator mode and occurs when the growth rate of the secondary instability, \( \sigma \), is of the order of the growth rate of the elevator mode, \( \lambda \). The latter can be obtained by linearizing the governing equations and deducing the properties of the fastest growing mode. Radko & Smith (2012) then calculate the growth rate of the secondary instability using Floquet theory. Unfortunately, their procedure turns out to be computationally quite expensive. In stellar evolution codes, fluxes need to be calculated for different values of \( \text{Pr}, \tau, \) and \( R_0 \) corresponding to each radial mesh point considered at each time step. Using the Radko & Smith (2012) method would be prohibitive for this application.

We pursue a theoretical model that is effectively a simplified form of the Radko & Smith (2012) theory but has the advantage of being much more straightforward and can thus be used in real time in stellar evolution codes. In what follows, we first review the derivation of the growth rate of the fastest growing mode, \( \lambda \), in Section 4.1 (Baines & Gill 1969). In Section 4.2, we then propose a very simple analytical formula for the growth rate of the secondary instability, \( \sigma \), and finally determine the Nusselt numbers expected when \( \sigma \sim \lambda \).

4.1. Fastest Growing Modes

The properties of the fastest growing fingering mode can be determined by linearizing the governing equations and assuming exponential solutions of the form

\[ q = \hat{q}e^{\lambda \tau + i(kx)}, \]

for each of the velocity, temperature, composition, and pressure perturbations (Baines & Gill 1969). This yields a cubic equation for the growth rate \( \lambda \), with coefficients that depend on the wave vector, \( k \), and the governing non-dimensional parameters, \( \text{Pr}, \tau, \) and \( R_0 \). It can be shown that all properties of the fastest growing mode (e.g., the velocity, temperature, and salinity fields) are independent of \( z \), hence the term “elevator” mode used to describe them. Since our equations are symmetric in \( x \) and \( y \), \( \lambda \) only depends on the magnitude \( l \) of the horizontal wavenumber and not on its direction.

The resulting cubic then reads (Baines & Gill 1969):

\[
\lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0,
\]

where

\[
a_2 = l^2(1 + \text{Pr} + \tau),
\]

\[
a_1 = l^4(\tau \text{Pr} + \text{Pr} + \tau) + Pr \left( 1 - \frac{1}{R_0} \right),
\]

\[
a_0 = l^6 \tau \text{Pr} + l^2 \text{Pr} \left( \frac{\tau}{R_0} - 1 \right).
\]

To identify the fastest growing mode, we maximize \( \lambda \) with respect to the wavenumber \( l \). This yields a quadratic for \( \lambda \):

\[
a_2 \lambda^2 + a_1 \lambda + a_0 = 0,
\]

where

\[
a_2 = 1 + \tau \text{Pr} + \tau,
\]

\[
a_1 = 2l^2(\tau \text{Pr} + \text{Pr} + \tau),
\]

\[
a_0 = 3l^4 \text{Pr} + Pr \left( \frac{\tau}{R_0} - 1 \right).
\]

Equations (19) and (20) can be solved numerically simultaneously to determine the wavelength and growth rate of the fastest growing mode. More interestingly for anyone interested...
in quick analytical estimates, it can be shown that in the limit of $Pr, \tau \ll 1$,

$$
\lambda \approx \begin{cases} 
\sqrt{Pr} & \text{if } r \ll Pr \ll 1 \\
\sqrt{Pr} / \tau & \text{if } Pr \ll r \ll 1
\end{cases},
$$

and

$$
l^2 \approx \frac{1}{\sqrt{1 + \tau / Pr}}.
$$

A detailed derivation of these asymptotic limits (and higher-order terms) is presented in Appendix B.

### 4.2. Estimating the Nusselt Number

Radko & Smith (2012) consider all possible sources of secondary instabilities for the elevator modes. Here, for simplicity, we restrict our analysis to instabilities arising from the shear between adjacent elevators and neglect viscosity. As we demonstrate below, this yields very satisfactory results, and allows for a much simpler solution to the problem.

The elevator modes are characterized by a vertically invariant flow field of the kind

$$
U(x, t) = w_0 e^{\lambda t} \sin(lx) e_z \equiv w_E(t) \sin(lx) e_z,
$$

where $\lambda$ and $l$ are now strictly defined as the growth rate and horizontal wavenumber of the fastest growing mode, introduced in the previous section. Without loss of generality, we have selected the horizontal phase of the mode to be $\sin(lx)$. This flow field is associated with a temperature and composition field

$$
T_E(x, t) = T_0 e^{\lambda t} \sin(lx) \quad \text{and} \quad \mu_E(x, t) = \mu_0 e^{\lambda t} \sin(lx),
$$

where $w_0$, $T_0$, and $\mu_0$ are related via Equations (8) and (9):

$$
\lambda T_0 + w_0 = -l^2 T_0 \Rightarrow T_0 = -\frac{w_0}{\lambda + l^2},
$$

$$
\lambda \mu_0 + R_0^{-1} w_0 = -\tau l^2 \mu_0 \Rightarrow \mu_0 = -\frac{R_0^{-1} w_0}{\lambda + \tau l^2},
$$

since elevator modes are exact solutions of the full set of nonlinear equations. At early times, the velocity shear, $w_E(t)$, is weak and the elevator modes grow unimpeded. Using the definitions of the Nusselt numbers given in Equation (12) with Equations (23)–(25), we find that

$$
\text{Nu}_\tau(t) = 1 - \frac{1}{L_x L_y L_z} \int_x \int_y \int_z w_E(t) T_E(t) \sin(lx) dx dy dz
= 1 - \frac{1}{2} \frac{w_E(t) T_E(t)}{1 + \frac{w_0^2}{2(\lambda + l^2)} e^{2\lambda t}}.
$$

A similar expression can be obtained for $\text{Nu}_\mu$. This shows that the Nusselt numbers initially grow exponentially with a growth rate that is twice that of the fastest growing mode, as mentioned in Section 3.1, prior to saturation.

We now consider secondary shearing instabilities on the elevator mode flow. Since $Pr \ll 1$, we assume that viscosity is negligible. In that case, the growth rate of the shearing modes, $\sigma$, is a simple increasing function of the velocity within the fingers. Since the latter increases exponentially, $\sigma$ also increases, while the growth rate of the fingering instability remains constant. We therefore expect that there will be a time where the two are of the same order of magnitude. At this point, the elevator modes are destroyed and saturation occurs.

As in Radko & Smith (2012), we neglect the temporal variation of the primary elevator modes when evaluating the growth rate of the secondary shearing modes, and simply write

$$
U(x, z) = \hat{w}_E \sin(lx) e_z.
$$

The growth rate of the shearing instability can be found through dimensional analysis or through Floquet theory (see Appendix A). The dimensional argument goes as follows: the only relevant velocity in the system is that of fluid within the fingers, $\hat{w}_E$, and the only relevant length scale is $1/l$, so the only relevant growth rate is $\hat{w}_E l$. Hence,

$$
\sigma = K \hat{w}_E l,
$$

where $K$ is a universal constant of proportionality. By “universal,” we imply that this constant is independent of any system parameter ($\tau$, Pr, $R_0$) or any property of the elevator mode. All information about the latter is contained in $\lambda$ and $\hat{w}_E$.

Assuming that saturation occurs when the shearing instability growth rate is of the order of the fingering growth rate can be written mathematically as

$$
\sigma = K \hat{w}_E l = c \lambda,
$$

where $c$ is independent of the fundamental parameters of the system. This equation uniquely determines the velocity within the fingers at saturation to be

$$
\hat{w}_E = \frac{c \lambda \sqrt{\frac{\tau}{l}}}{l},
$$

where $C = c/K \sqrt{\frac{\tau}{l}}$ is another universal constant (to be determined). Equation (31) is similar to the one used by Denissenkov (2010) to estimate the effective diffusivity by dimensional analysis. Our own model, however, departs from his analysis at this point and produces a result that more accurately fits the results of simulations.

A given elevator mode with velocity $w_E(x, z) = \hat{w}_E \sin(lx)$ has a temperature profile $T_E(x, z) = T_E(x) \sin(lx)$ with $T_E = -\hat{w}_E / (\lambda + l^2)$, for the same reasons that led us to Equation (25). Hence, at saturation,

$$
\text{Nu}_T = 1 + \frac{\hat{w}_E^2}{2(\lambda + l^2)} = 1 + C^2 \frac{\lambda^2}{l^2(\lambda + l^2)}
$$

using Equation (31). And by similar arguments, it can be shown that

$$
\text{Nu}_\mu = 1 + C^2 \frac{\lambda^2}{\tau l^2(\lambda + l^2)},
$$

where $C$ is the same constant as in Equation (32). The physical interpretation of $C$ is now clearer. Since $C$ is related to the relative importance of the shearing and fingering instabilities at saturation, a large value of $C$ indicates that the shearing instability cannot easily disturb the initial fingers, resulting in a large saturated flux. Conversely, a small value suggests that shear easily destroys fingers, resulting in a small saturated flux.

We compare this prescription with data from this study and those simulations from Traxler et al. (2011a) and Radko & Smith (2012) that approach the astrophysical regime. We fit $C$ by using
a chi-squared statistical test and find a best fit at \( C = 7.0096 \). As illustrated in Figure 5, for a value of \( C = 7 \) and \( \tau \ll Pr \), we find that our theoretical results match all simulations remarkably well. It is important to note that, while \( C \) needs to be fitted, it is a universal constant and cannot depend on \( Pr, \tau \), or \( R_0 \). The fact that we are able to use only one value of \( C \) to correctly fit the data suggests that the premises of this theory are correct.

In Figure 6, we show that in the opposite case (\( \tau > Pr \)) we do not observe the same quality of fit. The full analysis by Radko & Smith (2012) is also unable to explain these results, so it is likely that saturation in this case operates somewhat differently. However, since stars are always in the former situation, the poorness of fit of the latter case is interesting but not relevant to our ultimate purpose.

### 4.3. Asymptotic Expansions

As such, Equations (32) and (33) yield the thermal and compositional Nusselt numbers provided \( \lambda \) and \( l \) are known. Calculating these quantities requires the simultaneous solution of a cubic and a quadratic (see Section 4.1), which can be done numerically quite easily. However, analytical approximations to \( Nu_T \) and \( Nu_\mu \) can also be obtained using asymptotic analysis (see Appendix B), to find that for \( r \ll (\tau, Pr) \ll 1 \),

\[
Nu_T - 1 \approx C^2 Pr \left( \frac{1}{Pr} + \frac{\tau}{Pr} \right), \quad (34)
\]

\[
Nu_\mu - 1 \approx C^2 \sqrt{\frac{Pr}{\tau}} \sqrt{1 + \frac{\tau}{Pr}}. \quad (35)
\]

For the case with \( (\tau, Pr) \ll r \ll 1 \) and \( \tau / Pr \sim 1 \),

\[
Nu_T - 1 \approx C^2 Pr \frac{1}{r}, \quad (36)
\]

\[
Nu_\mu - 1 \approx C^2 \sqrt{\frac{Pr}{\tau}} \sqrt{1 + \frac{\tau}{Pr}}. \quad (37)
\]

Let us discuss these results in more detail. First note that in the case \( (\tau, Pr) \ll r \ll 1 \), we find \( Nu_T - 1 \propto Pr^2 r^{-1/2} \) and \( Nu_\mu - 1 \propto Pr^1 r^{-1/2} \). The discrepancy is not so surprising given that they did not differentiate between different regimes. Second, note that in astrophysical cases, \( Pr \ll 1 \), which implies that turbulent heat transport by homogeneous fingering convection is negligible. Such is not the case for \( Nu_\mu - 1 \), which can become quite large, especially in the limit of \( r \to 0 \). We will discuss the implications of these results more in Section 6. Finally, note that these asymptotic formulae should work well for the true astrophysical regime where \( Pr, \tau \ll 1 \). However, since none of our simulations are particularly close to this regime, the asymptotic approximation does not compare well with the numerical simulations we have run so far.

### 5. THE DEVELOPMENT OF LAYERS

As discussed in Section 1, substantial evidence from oceanography exists to suggest that mixing rates beyond the levels discussed in the previous section are possible. In fingering regions in the ocean, thermohaline staircases—layers of overturning convection separated by thin fingering interfaces—have been observed and have been linked with dramatic increases in the transport of salt and heat (e.g., Schmitt et al. 2005). Thus, it is important to consider whether layers can form in the astrophysical case, and whether they also lead to increased transport.

#### 5.1. The \( \gamma \)-instability

Radko (2003) developed the so-called \( \gamma \)-instability theory to explain the formation of thermohaline staircases in fingering
regions of the ocean. The theory was found to agree with direct numerical simulations of staircase formation in two and three dimensions (Radko 2003; Stellmach et al. 2011) and has since become well accepted in the physical oceanography community.

The original $\gamma$-instability theory is a linear mean-field instability theory. Radko (2003) first averaged the governing equations of fingering convection (see Equations (6)–(9)) spatially, over length scales that span several fingers, and temporally, over multiple finger lifetimes. He then performed a linear stability analysis of the averaged, or “mean-field,” equations. He argued that the stability of the system depends on the quantity $\gamma_{\text{turb}}$, defined as the ratio of the turbulent thermal flux, $\text{Nu}_T$, $-1$, to the turbulent compositional flux, $\nu (N_u - 1)/R_0$. More specifically, he showed that homogeneous fingering convection is unstable to layering if and only if $\partial \gamma_{\text{turb}}/\partial R_0 < 0$. Traxler et al. (2011a) later improved this theory and showed that in the astrophysical case, this result still holds as long as $\gamma$ is redefined as the ratio of the total fluxes, diffusive and turbulent; in other words, a system is unstable to layering if and only if

$$\frac{\partial \gamma}{\partial R_0} < 0 \quad \text{with} \quad \gamma = \frac{R_0 \text{Nu}_T}{\nu N_u}. \tag{40}$$

As mentioned in Section 3.2, some of our simulations at very low $r$ exhibit properties consistent with layering. In order to determine whether these layers form via the $\gamma$-instability, we plot in Figure 7 the variation of $\gamma$ with $R_0$ for various values of $\text{Pr}$ and $\tau$. Unlike $\gamma_{\text{turb}}$, which does decrease for some $r$ at low $\text{Pr}$, $\gamma$ always seems to increase. As discussed by Traxler et al. (2011a), this is because $\text{Nu}_T$ and $N_u$ are both dominated by diffusive fluxes at low $\text{Pr}$ and $\tau$. The diffusive flux ratio, $R_0/\tau$, increases strongly with $R_0$ because $\tau$ is asymptotically small and $R_0$ approaches $1/\tau$ as the system becomes increasingly stable. This effect dominates the total flux ratio. Since $\gamma$ is always an increasing function of $R_0$, we conclude that the observed layers cannot be forming by the $\gamma$-instability. Thus, while the $\gamma$-instability can lead to the development of staircases in high $\text{Pr}$

Figure 7. The total flux ratio, $\gamma$, using the same data as in Figure 4. Note how $\gamma$ increases monotonically with $R_0$, which implies that thermocompositional staircases cannot form by the $\gamma$-instability.

(A color version of this figure is available in the online journal.)

The discussion of Denissenkov (2010) on layer formation uses $\gamma_{\text{turb}}$ rather than $\gamma$, which Traxler et al. (2011a) showed to be incorrect for the astrophysical case.

5 Despite the fact that $\gamma$ is a strictly increasing function of $r$, we do observe layer formation in some simulations. We now study these results in more detail, focusing on the case presented in Figure 3, which has $\text{Pr} = 1/10$, $\tau = 1/30$, and $R_0 = 1.1$ and shows strong evidence for the formation of a convective layer. Figure 8 shows a snapshot of the compositional perturbation at $t = 825$; large-scale plumes characteristic of overturning convection are clearly visible. We can check to see whether these plumes are associated with an inversion of the horizontally averaged total density profile, which is given by

$$\langle \rho_{\text{tot}} \rangle = -(1 - R_0^{-1}) z - (T) + \langle \mu \rangle. \tag{41}$$

Figure 9 shows $\langle \rho_{\text{tot}} \rangle$ as a function of height for the simulation shown in Figure 3 at four different times selected during the later stages, where the Nusselt numbers increase significantly. The solid line indicates the linear background gradient for reference. Note how each of the four density profiles shows a sharp, stably stratified transition region between $z = 25$ and $z = 45$, the interface. Three of the four profiles also have a region where the total density increases with $z$, the convective layer. This demonstrates that the homogeneous fingering has indeed transitioned into a layered system with convective layers separated by stable interfaces. At $t = 725$, shown in Figure 9, the convective layer is just beginning to develop, and a slight inversion of the density profile appears in Figure 9. As the simulation approaches the stage of maximum transport at $t = 750$ (see Figure 3), the density inverts more strongly. The ensuing period of active mixing thickens the interface again and flattens the density profile. At $t = 775$, the convective mixing briefly shuts down. This process repeats, as can be seen at $t = 800$, when the density profile once more is beginning to invert. Note that the interface appears to move up and down during this period, which we attribute to the advection of a large quantity of compositionally dense material by convective plumes.

Figure 8. A snapshot of the compositional perturbation from the same simulation as in Figure 3 ($\text{Pr} = 1/10$, $\tau = 1/30$, and $R_0 = 1.1$) at $t = 825$. Large convective plumes are clearly visible and span a significant fraction of the compositional domain. The interface is harder to identify but can be seen near the bottom of the plot, where finger structures are still apparent. Note that it is far from “flat,” and instead is highly distorted by the large-scale plumes in the convective layer.

(A color version of this figure is available in the online journal.)
This is not the only such simulation in which we observe the formation of staircases: we see this behavior at low $r$ ($\leq 0.003$) for several values of $Pr$ and $\tau$, including $Pr = 1/10$, $\tau = 1/10$ and $Pr = 1/3$, $\tau = 1/10$. In all these cases, the $Nu_T$ and $Nu_\mu$ time series as well as the properties of the layered state are qualitatively similar to the ones shown in Figures 3, 8, and 9.

Having established that this layering transition cannot be due to the $\gamma$-instability, we look instead at the original mechanism proposed by Stern (1969) (see also Stern et al. 2001) for the formation of thermohaline staircases in the ocean. In addition to the $\gamma$-instability, fingering convection is also known to excite large-scale gravity waves through another mean-field instability, the “collective instability.” Stern (1969) studied this effect, and showed that large-scale gravity waves can be excited under some circumstances and grow in amplitude exponentially. These waves transport momentum, heat, and heavy elements until they begin to “break,” at which point the advected material mixes with the background. He went on to suggest that if the waves break on large enough scales, this could cause enough local mixing to trigger the formation of staircases.

In related numerical simulations at oceanographically relevant parameters ($Pr = 7$, $\gamma = 0.3$), Stellmach et al. (2011) found that gravity waves are indeed excited by the collective instability but do not trigger layers. Layer formation in their simulations—and thus presumably in the ocean—is caused by the $\gamma$-instability rather than the collective instability. Our simulations suggest, however, that the converse may be true in the astrophysical parameter regime. As discussed in Figure 3, gravity waves are observed to dominate the dynamics of the system between $t = 100$ and $t = 600$. Furthermore, the gradual increase in $Nu_T$ and $Nu_\mu$ during that phase suggests that the waves have high enough amplitudes to interact nonlinearly with each other, and with the background, by contrast with the oceanographic case. Indeed, linear waves cause oscillations in $Nu_T$ and $Nu_\mu$, without changing their mean values—only nonlinear waves can affect the mean. Since the waves in Figure 3 are clearly strong enough to interact nonlinearly, we can hope that they may also break and cause mixing on larger scales, following the original hypothesis proposed by Stern (1969).

Checking this hypothesis in detail is difficult with the available simulations. However, we can at least determine whether our observed waves do indeed appear in accordance with the collective-instability theory. A system is unstable to the collective instability if the Stern number, in our notation defined as

$$A = \frac{(Nu_T - 1)(\gamma^{-1}_{\text{turb}} - 1)}{Pr(1 - R_0^{-1})},$$

exceeds order unity (Stern et al. 2001). Note that this expression uses $\gamma_{\text{turb}} = R_0(Nu_T - 1)/\tau(Nu_\mu - 1)$ instead of $\gamma$. For the simulation shown in Figures 3 and 9, $Nu_T = 6.55$ and $\gamma_{\text{turb}} = 0.438$, so that the Stern number is 924. This implies that the system is indeed unstable to the collective instability.

More generally, we find that all the simulations in which we observe layer formation have Stern numbers of the order of $10^5$ or more. However, not all such simulations go into layers, as can be seen in Figure 10, so it is likely that the condition for layer formation through the collective instability also depends on the value of $r$. It is also possible that these gravity waves grow in all simulations with $A > 1$, but since the growth rate depends on the Stern number, gravity waves may be too weak to detect if $A$ is too low. If this is the case, it will be difficult to see layer formation in models with smaller $A$ because these simulations are expensive and difficult to integrate for long. We discuss this more in Section 6.3.

Much work still needs to be done to identify the parameters for which we expect layer formation in the astrophysical regime and to quantify transport in the layered case (see Wood et al. 2012 for related efforts in the case of semi-convection). We have only three simulations that develop into layers and thus require a more extensive sample of simulations at low $r$ in order to better understand the conditions of layer formation and their properties as a function of $Pr$, $\tau$, and $r$. In every simulation that exhibits layers, we find that a single convective plume spans the majority
of the domain and do not see the development of multiple layers, as normally seen in simulations of the fingering regime in the oceanographic case (e.g., Stellmach et al. 2011) and in the case of semi-convection (Rosenblum et al. 2011; Mirouh et al. 2012). To characterize this process, we must increase the size of the domain and extend the integration time to let the layers develop more naturally. This will allow us to describe more completely the size scales and transport of thermocompositional layers in astrophysical fingering convection. This problem, however, is beyond the scope of this paper and is deferred to a later publication.

6. DISCUSSION

In the preceding sections, we have used the results from Section 3 to formulate a theory that models the Nusselt numbers of homogeneous fingering convection in the astrophysical regime. We now provide guidance on how to apply our theory to calculate the effective diffusivities in a stellar code. We quantify the range of parameters where layer formation via the collective instability may be possible. To provide perspective on the impact of these results, we conclude by discussing the predictions from this theory in several astrophysical systems.

6.1. A Method for Applying the Model

The first step toward using our model in stellar evolution calculations is to identify the non-dimensional parameters of the system in question at each grid point and/or time step. Recall that Pr = ν/κT and τ = κμ/κT. The expressions for the microscopic viscosity and diffusivities can be found in many textbooks of plasma physics (e.g., Chapman & Cowling 1970). In stars, the value of Pr typically ranges from 10^{-7} to 10^{-4} and that of τ from 10^{-8} down to 10^{-6}, with Pr > τ.

The density ratio, R0, is given by

\[ R_0 = \frac{\alpha (T_{oc} - T_{ad})}{\beta \mu_{oc}} = \frac{V - V_{ad}}{S \frac{d}{d}} \frac{\mu}{\mu}, \]

where V = (d ln T/d ln P) is the actual temperature gradient, V_{ad} = (d ln T/d ln P)_S is the adiabatic temperature gradient, \( \nu \) = (d ln \mu/d ln P) is the mean molecular weight gradient, and \( \omega \) = (d ln \rho/\delta ln T)_\mu, and \( \phi \) = (d ln \rho/\delta ln \mu)_T (see Kippenhahn & Weigert 1990 for a detailed description of these gradients). The quantities \( \nabla \) and \( \nabla_{\mu} \) can be calculated from the stellar model grid. The other values, \( \nabla_{ad} \), \( \phi \), and \( \delta \), can be calculated from the equation of state. Note that R0 can take a broad range of values in stellar interiors, but the fluid is fingering-unsatable only when 1 < R0 < τ^{-1}.

Once Pr, \( \tau \), and R0 are known, one can then calculate NuT and Nu_w according to Equations (32) and (33) for a numerically precise answer or by the asymptotic expressions in Section 4.3 if a rough estimate is all that is needed. Since Nu_w is the total flux of composition divided by the background diffusive flux, the effective compositional diffusivity for modeling transport by fingering convection is given by

\[ D_{\mu} = Nu_w \kappa_{mu}, \]

where

\[ F_{\mu} = -D_{\mu} \mu_{oc} \]

is the total dimensional flux of the mean molecular weight through the fingering region. An analogous expression can be derived for the heat transport, but this term is nearly always negligible (see Section 4.2) when Pr \ll 1 unless layers form. We ignore it from here onward. By contrast, the turbulent compositional transport can be significant, as can be seen in Figure 11, where we plot the logarithm of Nu_w against r/τ, assuming Pr \approx 10^{-6}. Near the marginal stability limit, r = 1, the compositional turbulent transport is negligible as well, and Nu_w drops to zero. However, for low r, particularly for low τ/Pr, the turbulent compositional transport increases by many orders of magnitude. This plot is intended as a quick estimate for a large range of parameters; more accurate estimates can be obtained from the expressions given in Sections 4.2 and 4.3.

The results presented so far concern only the case of homogeneous fingering convection. As discussed in Section 5.2, however, layer formation seems to be possible in some cases and leads to a significant increase in transport compared with the homogeneous levels. To determine when this may occur, we use the theory from Section 4 to estimate the Stern number as a function of τ/Pr = κμ/ν and r, and show the results in Figure 12. We find that the Stern number is largest for small r and large τ/Pr. Since we observe layer formation for a Stern number near or above 10^3 (see Section 5.2), the same criterion suggests naively that layer formation will only be relevant in stellar interiors (where τ/Pr < 0.1) for r \lesssim 10^{-5}. This is so close to actual overturning convection that it may in fact be irrelevant for practical purposes. Of course, as mentioned in Section 5.2, much work still needs to be done to confirm these results.

6.2. Impact of the New Model on Astrophysical Applications

We now reexamine several problems that have been recently discussed in the context of fingering convection, namely, planet pollution (Vauclair 2004; Garaud 2011) and the impact of 3^He fusion in red giant branch stars (Charbonnel & Zahn 2007; Denissenkov 2010).

As first discussed by Vauclair (2004), fingering convection determines the timescale during which planetary material
remains detectable on the surface of a star after infall. Garaud (2011) used the fingering transport prescription from Traxler et al. (2011a) to study the problem and determined that this timescale is too short for the excess metallicity of planet-host stars to be related to planet infall. Since we have found that the prescription by Traxler et al. (2011a) only underestimates the transport at low $r$ while still recovering the scalings at high $r$, the use of our new, improved model can only increase the turbulent mixing rates and decrease the mixing time, so the qualitative conclusion of Garaud (2011) still holds: the fact that planets are more readily found around high metallicity host stars must be a primordial effect.

To address the case of red giant branch stars, we compare our results to the numerical simulations of Denissenkov (2010). For his choice of governing parameters, $Pr = 4 \times 10^{-6}$, $\tau = 2 \times 10^{-6}$, and $R_0 = 1700$, he finds $Nu_r = 998$. He concludes from this result that fingering convection cannot provide all the additional mixing needed in red giant stars. Our model finds a remarkably similar value of $Nu_r = 1294$ for the same parameters, and thus we confirm his conclusion that additional mixing is necessary to explain observations.

6.3. Summary and Future Work

We have developed a simple semi-analytical model for the vertical transport of heat and composition by homogeneous fingering convection that reproduces the results of numerical simulations from this study and from previous studies and can easily be implemented in a stellar evolution code. We have found that, under conditions relevant for stellar interiors, the turbulent heat transport is negligible, but compositional transport can be important. In particular, this model predicts more efficient transport for weakly stratified systems than was initially suggested in previous work by Traxler et al. (2011a).

We have also found the first evidence for layer formation by fingering convection in the astrophysical parameter regime. We have identified the collective instability as the origin of layer formation. By contrast, in the oceanographic and semi-convective cases, the $\gamma$-instability causes layer formation (e.g., Stellmach et al. 2011; Mirouh et al. 2012). Much work still needs to be done to characterize the conditions for layer formation and transport by layered convection in this case. With our limited computational resources, we have found that layer formation appears to require high Stern numbers (see Equation (42)) to occur. If true, this would imply that layers are only likely to form in regions that are very close to being fully convective anyway. However, it remains to be seen whether lower Stern numbers can also result in layer formation. To study this will require additional simulations and longer integration times than have been covered in this paper. In addition to the conditions of layer formation, it is also important to develop a theoretical or empirical model for the transport of this case. Our simulations show that transport increases significantly when layers form. Concurrent work by Wood et al. (2012) reveals that thermal and compositional transport in the layered semi-convective case scales with the layer height to the power of 4/3. If this scaling also applies here, we may expect that very significant transport rates could occur for large enough layer heights. What determines the latter, however, will also require new theories.

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time-dependence of the elevator mode velocity field $U$, setting $w_E(t) = \dot{w}_E$ where $\dot{w}_E$ is constant. Finally, we neglect the effect of viscosity entirely (which is justified in the low Prandtl number astrophysical limit).

We then consider perturbations $u = (u', 0, w')$ to this basic flow, driven by secondary shearing instabilities. As in Radko & Smith (2012), we work with a stream function $\psi$, defined so that $u' = -\partial \psi / \partial z$ and $w' = \partial \psi / \partial x$. The linearized momentum equation reads:

$$\frac{\partial u'}{\partial t} + u' \cdot \nabla U + U \cdot \nabla u' = -\nabla \cdot \bar{\Pi} \rho,$$

(A2)

where the nonlinear term $U \cdot \nabla U$ naturally vanishes, since the elevator modes are fully nonlinear solutions of the governing equations. Written in terms of the stream function, this becomes

$$\frac{\partial}{\partial t} (\nabla^2 \psi) + \dot{w}_E \sin(lx) \frac{\partial}{\partial z} (\nabla^2 \psi) + \dot{w}_E l^2 \sin(lx) \frac{\partial \psi}{\partial z} = 0.$$  

(A3)

This is a linear partial differential equation for $\psi$, with coefficients that are constant in time and in the coordinate $z$. We may write the solutions as

$$\psi(x, z, t) = \hat{\psi}(x) e^{i\sigma t + \sigma t}$$

(A4)

to satisfy periodic boundary conditions in $z$, as in the original problem. Floquet theory further shows that the $x$-dependence of the solution must be in the form

$$\hat{\psi}(x) = e^{i\beta x} \sum_{n=-N}^{N} \psi_n e^{i\alpha x},$$

(A5)

where we have used for the sake of clarity the same notation as Radko & Smith (2012) for the Floquet term $e^{i\beta x}$, where $\beta$ is the Floquet coefficient. Since $\beta$ need not be an integer, the shearing modes are usually quasi-periodic rather than periodic. Note also that the summation term should in reality be taken from $n = -\infty$ to $+\infty$, but is written here already in its approximate truncated form, in view of a numerical implementation of the problem.

Plugging this expression into Equation (A3), and projecting the result onto individual Fourier modes, we obtain the following tri-diagonal linear system: for each value of $n$ ranging from $-N$ to $N$, we have

$$\begin{align*}
-\frac{\dot{w}_E}{2} \psi_{n-1} [m^2 - l^2 + l^2 (f + n - 1)^2] \\
- \sigma \psi_n [m^2 + l^2 (f + n)^2] \\
+ \frac{\dot{w}_E}{2} \psi_{n+1} [m^2 - l^2 + l^2 (f + n + 1)^2] = 0,
\end{align*}$$

(A6)

with the implicit assumption that $\psi_{-N-1} = \psi_{N+1} = 0$. Each secondary mode of instability is identiﬁed by the pair $(f, m)$, and grows with rate $\sigma(f, m)$ obtained by setting the determinant of the linear system (Equation (A6)) to zero. As in Radko & Smith (2012), we can then scan over all possible values of $m$ and $f$ to find the growth rate of the fastest growing secondary instability.

The beauty of this method is that, by contrast with the more general system studied by Radko & Smith (2012), the scalings for $\sigma$ can be obtained without solving for it at all! Let us deﬁne the rescaled vertical wavenumber $\hat{m} = m/l$ and the rescaled growth rate $\hat{\sigma} = \sigma/\dot{w}_E l$. The system of equations described by Equation (A6) for $n = -N..N$ can then be re-cast in the form

$$\begin{align*}
-\frac{1}{2} \psi_{n-1} [\hat{m}^2 - 1 + (f + n - 1)^2] - \hat{\sigma} \psi_n [\hat{m}^2 + (f + n)^2] \\
+ \frac{1}{2} \psi_{n+1} [\hat{m}^2 - 1 + (f + n + 1)^2] = 0.
\end{align*}$$

(A7)

The rescaled growth rate $\hat{\sigma}$ is now a function of $\hat{m}$ and $f$ only, so that maximizing $\hat{\sigma}$ over all possible values of $\hat{m}$ and $f$ yields one universal constant. In other words, the fastest growing secondary shearing mode satisfies $\hat{\sigma} = K$, where $K$ is a universal constant, independent of $l$ or $\dot{w}_E$. This ﬁnally implies, as discussed in the main text, that

$$\sigma = K \dot{w}_E l,$$

(A8)

where the value of $K$ is somewhat irrelevant since it can be folded into the constant $C$ defined in Section 4.

APPENDIX B

ASYMPTOTIC ANALYSIS

In Section 4, we derived an analytical expression for the thermal and compositional Nusselt numbers (see Equations (32) and (33)), in terms of the growth rate $\lambda$ and wavenumber $l$ of the fastest growing elevator mode. The latter can be found semi-analytically by solving a cubic and quadratic simultaneously, given by Equations (19) and (20).

While this can be done exactly with a Newton–Raphson relaxation method (or any other numerical method), one may be interested in a quick fully analytical estimate of these ﬂuxes instead. In this Appendix, we derive an approximate formula for the turbulent ﬂuxes, which is increasingly accurate in the asymptotic limit of $Pr, \tau \to 0$. To do so, we perform an asymptotic analysis of Equations (19) and (20) in these limits. Since $\tau$ is always of the order the Prandtl number in most astrophysical applications, we simplify our analysis by considering the two limits at the same time, deﬁning the quantity $\phi \equiv \tau/Pr$, and requiring it to be of order unity.

Asymptotic analyses are always much easier to do if one has a good idea of the behavior of the exact solution. The numerical solutions to Equations (19) and (20) are shown in Figure 13 (see data points). We see that the growth rate of the fastest growing mode $\lambda$ as a function of the reduced density ratio $r$ follows different asymptotic laws depending on the range of $r$ considered. For $r \ll (Pr, \tau)$, $\lambda$ appears to be constant and proportional to $\sqrt{Pr}$. For $r \ll 1$ but not in the previous limit, $\lambda$ appears to be proportional to $Pr$, and decreases as $r^{-1/2}$. Finally, in the limit of $r \to 1$, $\lambda$ drops to 0. We now investigate these three limits in more detail.

B.1. First Regime: $r \ll (Pr, \tau) \ll 1$

In the limit of $r \ll (Pr, \tau)$, we find numerically that $\lambda$ is independent of $r$. The same is true for the wavenumber $l$ of the fastest growing mode. This suggests that both $\lambda$ and $l$ are continuous in the limit $r \to 0$, and that one may study this ﬁrst regime simply by solving Equations (19) and (20) with $r \equiv 0$ (alternatively, $R_0 = 1$). The resulting equations are (using the deﬁnition of $\phi$ given above)

$$\begin{align*}
\lambda^3 + \lambda^2 l^2 (1 + Pr(1 + \phi)) + \lambda l^2 Pr(1 + \phi(Pr + 1)) \\
+ l^2 Pr^2 \phi + l^2 Pr(Pr\phi - 1) = 0,
\end{align*}$$

(B1)
The asymptotic expansion switches between two regimes of low $r$, $r \gg (r, \tau)$ and $r \ll (r, \tau)$, at $r = (r, \tau)$. The data points represent the numerical solution and the curves represent the asymptotic expansion. Note that the asymptotic expansion fits the data remarkably well in the respective limits considered.

(A color version of this figure is available in the online journal.)

$$\lambda = \sqrt{\Pr - \Pr \sqrt{1 + \phi} + \cdots}, \quad \text{and} \quad l^2 = \frac{1}{\sqrt{1 + \phi}} \sqrt{\Pr \left(1 + \frac{\phi}{(1 + \phi)^2}\right) + \cdots}. \quad (B5)$$

As can be seen in Figure 13, this approximation does well for small $\Pr$ and $\tau$ for $r \gg (\Pr, \tau)$. Note that although the only illustrated cases are those with $\Pr = \tau$, we have tested cases down to $\phi \sim 10^{-2}$.

We may then use Equations (32) and (33) to estimate the thermal and compositional Nusselt numbers. Keeping only the leading order terms yields

$$\text{Nu}_\tau \approx C^2 \left(\Pr (1 + \phi) - \Pr^{3/2} \frac{1 + \phi^2}{\sqrt{1 + \phi}} + \cdots\right), \quad (B6)$$

This too is compared to the numerical solution in Figure 14.

**B.2. Second Regime: $(Pr, \tau) \ll r \ll l$**

In this regime, since $(Pr, \tau) \ll r$, we expand Equations (19) and (20) first in $Pr$ and then in $r$. Inspection of the numerical solutions suggests that $\lambda$ is proportional to $Pr$, while $l^2$ is more-or-less independent of it. Seeking solutions of Equations (19) and (20) of the form $\lambda = Pr \hat{\lambda}$, to the lowest order in $Pr$, yields

$$l^2 \hat{\lambda}^2 + [l^4 (1 + \phi) + 1] \hat{\lambda} + l^6 \phi + l^2 \phi \frac{r - 1}{r} = 0, \quad \text{and} \quad (B8)$$

Eliminating the $\hat{\lambda}^2$ term yields

$$\hat{\lambda} [l^4 (1 + \phi) - 1] + 2l^6 \phi = 0, \quad \text{B10}$$

which can be easily solved to find

$$\hat{\lambda} = -\frac{2l^6 \phi}{l^4 (1 + \phi) - 1}. \quad \text{B11}$$

Plugging this into Equation (B9) and collecting the results in terms of $l$ yields

$$-l^2 l^4 (1 - \phi^2 + l^8 (1 + \phi) \left[\frac{\phi - 1}{r} - \frac{1 + \phi}{r}\right]$$

$$+ l^4 \left[1 - 2 \phi + \frac{2}{r} (1 + \phi)\right] + \frac{r - 1}{r} = 0, \quad \text{B12}$$

which is a cubic equation for $l^4$. Unfortunately, this cubic has no simple solution, so we look for an expansion in the other asymptotic variable, $r$. Inspection of the numerical solutions suggests that the expansion needs to be in terms of $r^{1/2}$, so we
set \( l^4 = l_0^4 + l_1^4 \sqrt{r} + l_2^4 r + \cdots \). Using this ansatz in Equation (B12) yields, to lowest order,
\[
l_0^4 (1 + \phi)^2 - 2l_0^3 (1 + \phi) + 1 = 0,
\]
which can be solved to find \( l_0^4 = 1/(1 + \phi) \). This, on its own, causes \( \lambda \) to diverge (see Equation (B11)), so we need to go to the next order in \( \sqrt{r} \). By choosing the growing solution, we then find that
\[
l^4 = \frac{1}{1 + \phi} - 2 \frac{\sqrt{r \phi}}{(1 + \phi)^{3/2}} + \cdots,
\]
which eventually yields
\[
\lambda = \Pr \sqrt{\frac{\phi}{r}} - \Pr \sqrt{1 + \phi} + \cdots,
\]
keeping the first two terms in the expansion only. As expected from the numerical solutions discussed above, \( \lambda \) scales like \( r^{-1/2} \Pr \) to lowest order.

As in the previous section, we use Equations (32) and (33) along with these expansions to find approximate expressions for \( \text{Nu}_T \) and \( \text{Nu}_\mu \):
\[
\begin{align*}
\text{Nu}_T - 1 & \approx C^2 \Pr^2 \left( \frac{1 + \phi}{r} - 2 \frac{\phi + \phi^2 + \phi^3}{r (1 + \phi)} \right), \\
\text{Nu}_\mu - 1 & \approx C^2 \left( \frac{1 + \phi}{r \phi} - \frac{1 + 2 \phi + 2 \phi^2}{\phi (1 + \phi)} \right).
\end{align*}
\]

We compare the results of the asymptotic expansion to the numerical solution in Figures 13 and 14, where we look at \( \lambda \) and \( \text{Nu}_\mu - 1 \), respectively. We choose to set the transition between the two regimes at \( r = (\tau, \Pr) \). For those simulations where \( \tau \) is several orders of magnitude smaller than \( \Pr \), we recommend choosing this point at \( r = \tau \). It is clear that the asymptotic expansion does an excellent job of reproducing the true numerical solution in the limit considered. We also see that, as suggested by the numerical data, \( \text{Nu}_\mu \) only depends on \( \phi \) rather than on \( \Pr \) or \( \tau \) individually as was noticed by Traxler et al. (2011a).

### B.3. Third Regime: \( r \to 1 \)

Equations (B11) and (B12) were derived without any assumptions on the value of \( r \), aside from the fact that it needs to be much larger than \( \Pr \) and \( \tau \). We can therefore use them directly to study the limit of \( r \to 1 \). To do so, we set \( \epsilon = 1 - r \), and rewrite Equation (B12) in terms of the small parameter \( \epsilon \):
\[
l^{12} (1 - \phi)^2 (\epsilon - 1) - l^8 (1 + \phi) [2 + 2 \epsilon (\phi - 1)]
+ l^4 [3 + \epsilon (2 \phi - 1)] - \epsilon = 0.
\]

Inspection of the solutions for \( l^4 \) in the limit of small \( \epsilon \) suggests the expansion \( l^4 = \epsilon l^4_0 + \epsilon^2 l^4 + \cdots \). Plugging this into Equation (B18) then yields, to lowest order,
\[
l^4 = \frac{1}{3} (1 - r) + \cdots \tag{B19}
\]
Plugging this expression into Equation (B11), we get
\[
\lambda = 2 \Pr \phi \left[ \frac{1}{3} (1 - r) + \cdots \right]^{3/2} \left[ 1 - \frac{1 + \phi}{3} (1 - r) + \cdots \right]^{-1}.
\]

As before, we use Equations (B19) and (B20) with Equations (32) and (33) to find expansions in \( \Pr \) and \( r \) for \( \text{Nu}_T \) and \( \text{Nu}_\mu \) in the limits \( \Pr \to 0 \) and \( r \to 1 \). We find
\[
\begin{align*}
\text{Nu}_T - 1 & \approx C^2 \Pr^2 \frac{4}{9} \phi^2 (1 - r)^2 - \frac{8}{81} \phi^2 (\phi - 8) (1 - r)^3 + \cdots, \\
\text{Nu}_\mu - 1 & \approx C^2 \frac{4}{9} (1 - r)^2 - \frac{8}{81} (\phi - 5) (1 - r)^3 + \cdots.
\end{align*}
\]

We compare the asymptotic expansions to the numerical solutions for \( \lambda \) and \( \text{Nu}_\mu \) in Figures 15 and 16. The expansion clearly works best for \( r \gtrsim 0.5 \). Note that these asymptotic expansions are designed to work only when \( \phi \sim 1 \) and should be used with caution if \( \phi \ll r \).
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