TWISTED GENERALIZED WEYL ALGEBRAS, POLYNOMIAL CARTAN MATRICES AND SERRE-TYPE RELATIONS

Jonas T. Hartwig
Institute of Mathematics, University of São Paulo, São Paulo, Brazil

A twisted generalized Weyl algebra (TGWA) is defined as the quotient of a certain graded algebra by the maximal graded ideal $I$ with trivial zero component, analogous to how Kac–Moody algebras can be defined. In this article we introduce the class of locally finite TGWAs and show that one can associate to such an algebra a polynomial Cartan matrix (a notion extending the usual generalized Cartan matrices appearing in Kac–Moody algebra theory) and that the corresponding generalized Serre relations hold in the TGWA. We also give an explicit construction of a family of locally finite TGWAs depending on a symmetric generalized Cartan matrix $C$ and some scalars. The polynomial Cartan matrix of an algebra in this family may be regarded as a deformation of the original matrix $C$ and gives rise to quantum Serre relations in the TGWA. We conjecture that these relations generate the graded ideal $I$ for these algebras, and prove it in type $A_2$.

Key Words: Generalized Weyl algebra; Quantum group; Serre relation.

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1. INTRODUCTION

Generalized Weyl algebras (GWAs) were introduced by Bavula [1, 2] and Rosenberg [18] under the name hyperbolic rings. Their structure and representation theory have been extensively studied in varying degrees of generality, see [3–6, 8, 11] and references therein. Examples of generalized Weyl algebras include the $n$th Weyl algebra $A_n$, the enveloping algebra $U(\mathfrak{sl}_2)$, and the quantum group $U_q(\mathfrak{sl}_2)$, as well as many other interesting families of algebras (see for example [7] and references therein).

In an attempt to enlarge the class of GWAs to include also the enveloping algebras of semisimple Lie algebras of higher rank, Mazorchuk and Turowska [15] defined the notion of a twisted generalized Weyl algebra (TGWA). They are natural generalizations of the GWAs but their structure is more complicated. Representations of TGWAs were investigated in [10, 14–16].

In [15] a first indication of a relation to higher rank Lie algebras was found (in the support of weight modules). Later, in [14], it was shown that the Mickelsson step...
algebras $Z(\mathfrak{gl}_{n+1}, \mathfrak{gl}_n \oplus \mathfrak{gl}_1)$ as well as some extended orthogonal Gelfand–Zetlin (OGZ) algebras associated to $\mathfrak{gl}_n$ are examples of TGWAs. It is still unknown whether enveloping algebras of higher rank Lie algebras are examples of TGWAs, but it is known that a certain localization of $U(\mathfrak{gl}_n)$ is isomorphic to an extended OGZ algebra, and hence is a TGWA.

In [9] another natural class of algebras was defined, which contains both generalized Weyl algebras and the enveloping algebra of $\mathfrak{gl}_n$.

In this article we further strengthen the link between TGWAs and enveloping algebras of semisimple Lie algebras. Namely, we define a natural subclass of TGWAs, which we call locally finite TGWAs (Definition 4.1). We show (Theorem 4.4) that in those algebras, relations of the following type hold:

$$X_i^{m_i}X_j + \eta_{ij}^{(1)} X_i^{m_i-1}X_jX_i + \cdots + \eta_{ij}^{(m_{ij})}X_jX_i^{m_{ij}} = 0,$$

where $\eta_{ij}^{(k)}$ are scalars. Such relations may be regarded as generalized Serre relations. Furthermore, we construct examples of locally finite TGWAs, denoted $\mathcal{T}_{q,\mu}(C)$, were these relations are the quantum Serre relations associated to a symmetric generalized Cartan matrix $C$. More precisely, Theorem 5.2(c) says that there exist nonzero algebra homomorphisms

$$\varphi_{\pm} : U_q(n_{\pm}) \to \mathcal{T}_{q,\mu}(C),$$

where $n_{\pm}$ are the positive and negative part of the Kac–Moody algebra $\mathfrak{g}(C)$ associated to $C$. We conjecture that the maps $\varphi_{\pm}$ are injective and prove it in the case when $C$ is of type $A_2$ (Corollary 6.4). We believe that the algebras in this family deserve further study, and we plan to investigate their structure and representations in forthcoming articles.

The plan of this article is as follows. In Section 2, we recall the definition of TGWAs and give some examples. After establishing some preliminary results in Section 3, we define the class of locally finite TGWAs in Section 4 and show how to associate polynomial Cartan matrices to them such that the corresponding Serre relations hold. Then, in Section 5, we give the construction of the locally finite TGWAs, $\mathcal{T}_{q,\mu}(C)$, associated to a symmetric generalized Cartan matrix and some parameters $q$, $\mu = (\mu_{ij})$. Finally, in Section 6, we give sufficient conditions for the Serre-type relations to generate the maximal graded ideal appearing in the definition of TGWAs, which in particular can be used to obtain an explicit presentation of the algebra $\mathcal{T}_{q,\mu}(C)$ when $C$ is of type $A_2$.

2. NOTATION AND DEFINITIONS

We fix an arbitrary ground field $\mathbb{K}$. All algebras are associative unital $\mathbb{K}$-algebras. Let us recall the definition of a twisted generalized Weyl algebra [14, 15]. The input for this construction is a positive integer $n$, a commutative $\mathbb{K}$-algebra $R$, an $n$-tuple $\sigma = (\sigma_1, \ldots, \sigma_n)$ of commuting $\mathbb{K}$-algebra automorphisms of $R$, an $n$-tuple $t = (t_1, \ldots, t_n) \in (R \setminus \{0\})^n$, and a symmetric matrix $\mu = (\mu_{ij})_{i,j=1}^n$ with entries
from $\mathbb{K}\setminus\{0\}$ but diagonal elements $\mu_i$ left undefined (they are irrelevant). The following consistency relations are required:

$$t_it_j = \mu_{ij}\mu_{ji}\sigma^{-1}_j(t_i)\sigma^{-1}_j(t_j) \quad \forall i \neq j. \quad (2.1a)$$

$$\sigma_i\sigma_j(t_k) = \sigma_j(t_k)\sigma_i(t_j) \quad \forall i \neq j \neq k \neq i. \quad (2.1b)$$

The associated twisted generalized Weyl construction (TGWC), $A' = A'(\sigma, t, \mu)$, is the algebra obtained from $R$ by adjoining new noncommuting generators $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ with the following defining relations for $i, j = 1, \ldots, n, i \neq j$:

$$X_iR = \sigma_i(r)X_i, \quad Y_iR = \sigma_i^{-1}(r)Y_i, \forall r \in R, \quad (2.2a)$$

$$Y_iX_i = t_i, \quad X_iY_i = \sigma_i(t_i), \quad (2.2b)$$

$$X_iY_j = \mu_{ij}Y_jX_i. \quad (2.2c)$$

One can show [19, Corollary 2.17] that relations (2.1) are sufficient and necessary for the natural map $R \to A'$ to be injective, if the $t_i$ are not zero-divisors in $R$. $A'$ has a $\mathbb{Z}^n$-gradation $\{A'_i\}_{g \in \mathbb{Z}^n}$ given by $\deg X_i = e_i$, $\deg Y_i = -e_i$, $\deg r = 0, \forall r \in R$, where $\{e_i\}_{i=1}^n$ is the standard $\mathbb{Z}$-basis in $\mathbb{Z}^n$. One can check that $A'_0 = R$. Let $I \subseteq A'$ be the unique maximal graded ideal intersecting $R$ trivially. Equivalently, $I$ is the sum of all graded ideals intersecting $R$ trivially. The twisted generalized Weyl algebra (TGWA), $A = A(R, \sigma, t, \mu)$, is defined as the quotient $A = A'/I$. Since $I$ is graded, $A$ inherits a $\mathbb{Z}^n$-gradation from $A'$. The images of the elements $X_i, Y_i$ in $A$ will be denoted by the same letters.

In this article, the assumption that $\mu$ is symmetric is a matter of convenience, because then relations (2) imply that $A'$ carries an anti-involution $*$, (that is, a $\mathbb{K}$-linear map $A' \to A'$ with $a^{**} = a$, $(ab)^* = b^*a^*$) determined by

$$X_i^* = Y_i, \quad Y_i^* = X_i, \quad i = 1, \ldots, n, \quad r^* = r \quad \forall r \in R. \quad (2.3)$$

Clearly, $I^* = I$ so $*$ descends to an anti-involution on $A$.

By a homogenous element in $A'$ (resp., $A$) we mean an element of $\bigcup_{g \in \mathbb{Z}^n} A'_g$ (resp., $\bigcup_{g \in \mathbb{Z}^n} A_g$). By a monic monomial in $A'$ (or $A$) we mean a product $Z_1 \cdots Z_i$, where $Z_j \in \{X_i\}_{i=1}^n \cup \{Y_i\}_{i=1}^n$ for each $j$. The group $\mathbb{Z}^n$ acts on $R$ via the automorphisms $\sigma_i : g(r) = (\sigma_i^g_1 \cdots \sigma_i^g_n)(r)$ for $g = (g_1, \ldots, g_n) \in \mathbb{Z}^n$ and $r \in R$. Using this action and (2.2a), we have

$$a \cdot r = (\deg a)(r) \cdot a \quad \text{for any homogenous } a \in A \text{ and any } r \in R. \quad (2.4)$$

**Example 2.1** (The TGWA of “Type $A_2$” from Example 3 in [15]). Let $n = 2, R = \mathbb{K}[H], \sigma_1(H) = H + 1, \sigma_2(H) = H - 1, t_1 = H, t_2 = H + 1$, and $\mu_{12} = \mu_{21} = 1$. Then the consistency relation $t_1t_2 = \sigma_1^{-1}(t_1)\sigma_2^{-1}(t_2), i \neq j$ holds. Let $A = A(R, \sigma, t, \mu)$ be the corresponding TGWA. In [15] this example was associated to the coxeter graph of type $A_2$, but the ideal $I$ was not described. We will come back to this example and eventually exhibit a set of generators for the ideal $I$ (Examples 4.5 and 6.3).
Example 2.2 (Quantized Weyl Algebras). Let $\tilde{q} = (q_1, \ldots, q_n)$ be an $n$-tuple of elements of $K \setminus \{0, 1\}$. Let $\Lambda = (\lambda_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with nonzero entries from $K$ such that $\lambda_{ij} = \lambda_{ji}^{-1}$ for $i < j$. The $n$th quantized Weyl algebra $A_{n}^{\tilde{q},\Lambda}$ is the $K$-algebra with generators $x_i, y_i$, $1 \leq i \leq n$, and relations

\begin{align}
x_i x_j &= q_{ij} \lambda_{ij} x_j x_i, \\
y_i y_j &= \lambda_{ij} y_j y_i, \\
x_i y_j &= \tilde{\lambda}_{ij} y_j x_i, \\
y_i x_j &= q_{ij} \tilde{\lambda}_{ij} x_j y_i,
\end{align}

for $1 \leq i < j \leq n$, and

\begin{align}
x_i y_i - q_i y_i x_i &= 1 + \sum_{k=1}^{i-1} (q_k - 1) y_k x_k,
\end{align}

for $i = 1, \ldots, n$. This algebra was introduced in [17] (for special parameters) and investigated since then by many authors. Its representation theory has been studied from the point of view of TGWAs in [10, 16]. To realize it as a TGWA, let $R = K[t_1, \ldots, t_n]$ be the polynomial algebra in $n$ variables and $\sigma_i$ the $K$-algebra automorphisms of $R$ defined by

\begin{align}
\sigma_i(t_j) &= \begin{cases} t_j, & j < i, \\
1 + q_i t_i + \sum_{k=1}^{i-1} (q_k - 1) t_k, & j = i, \\
q_i t_j, & j > i.
\end{cases}
\end{align}

One can check that the $\sigma_i$ commute. Let $\mu = (\mu_{ij})_{i,j=1}^n$ be defined by $\mu_{ij} = \lambda_{ji}$ and $\mu_{ji} = q_i \lambda_{ij}$ for $i < j$ (so if the involution (2.3) is required, one needs to impose also that $\lambda_{ji} = q_i \lambda_{ij}$, $i < j$). Put $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $t = (t_1, \ldots, t_n)$. One can show that the maximal graded ideal of the TGWC $A'(R, \sigma, t, \mu)$ intersecting $R$ trivially is generated by the elements

\begin{align*}
X_i X_j - q_i \lambda_{ij} X_j X_i, \\
Y_i Y_j - \lambda_{ij} Y_j Y_i, \\
1 \leq i < j \leq n
\end{align*}

so that $A_{n}^{\tilde{q},\Lambda}$ is isomorphic to the TGWA $A(R, \sigma, t, \mu)$ via $x_i \mapsto X_i$, $y_i \mapsto Y_i$.

3. PRELIMINARY RESULTS

The following observation about the ideal $I$ in a TGWC is very useful. Recall that an ideal $J$ in a graded algebra $B$ is called completely gr-prime if $ab \in J$ implies $a \in J$ or $b \in J$ for any homogenous elements $a, b \in B$.

Proposition 3.1. Let $A' = A'(R, \sigma, t, \mu)$ be a TGWC, where $R$ is a domain. Then $I$ is completely gr-prime.

Proof. Let $a, b \in A'$ be homogenous and suppose $ab \in I$. First we assume that $a = b^*$. Then, since $\deg(b^*) = -\deg(b)$, we have $b^* b \in I \cap R = 0$. Let $c_1, c_2 \in A'$ be any homogenous elements such that $\deg(c_1 c_2) = 0$. Then $(c_1 c_2)^* \cdot (c_1 c_2)$ =
Proposition 3.4. Let \( R \) be a TGWC and certain relations in \( R \) and certain elements in the ideal \( I \) of a TGWC.

Corollary 3.3. Let \( A' = A'(R, \sigma, t, \mu) \) be a TGWC, where \( R \) is a domain. Then no monic monomial belongs to \( I \).

Using Proposition 3.1, we can prove the following explicit correspondence between certain relations in \( R \) and certain elements in the ideal \( I \) of a TGWC.

Proposition 3.4. Let \( A' = A'(R, \sigma, t, \mu) \) be a TGWC and suppose \( R \) is a domain. Fix \( i, j \in \{1, \ldots, n\}, i \neq j \). Let \( m \in \mathbb{Z}_{\geq 0} \) and \( r_0, \ldots, r_m \in R \). Then the following are equivalent:

(i) \( r_0 X_i^m X_j + r_1 X_j X_i + \cdots + r_m X_i X_j \in I \);

(ii) \( Y_j Y_i r_0 + Y_j Y_i r_1 + \cdots + Y_j Y_i r_m \in I \);

(iii) \( s_0 \sigma_i^m(t_j) + s_i \sigma_j^{m-1}(t_j) + \cdots + s_m t_j = 0 \), where \( s_k = \mu_i^k \sigma_j^{-1}(r_k) \) for \( k = 0, \ldots, m \).

Proof. By applying the involution \(*\), (i) is clearly equivalent to (ii). Put \( a = \sum_{k=0}^m r_k X_i^{m-k} X_j^k \). Then \( a \) is homogenous of degree \( me_i + e_j \), where \( e_i = (0, \ldots, 1, \ldots, 0) \) (1 on \( i \) th position). So if \( a \in I \) then \( aY_j Y_i = 0 \) by definition of \( I \). Conversely, if \( aY_j Y_i = 0 \), then \( a \in I \) since \( I \) is completely gr-prime and does not contain the monic monomial \( Y_j Y_i \). On the other hand,

\[
a Y_j Y_i = \left( \sum_{k=0}^m r_k X_i^{m-k} X_j^k \right) Y_j Y_i = \left( \sum_{k=0}^m \mu_i^k r_k \sigma_j^{-1}(\sigma_j(t_j)) \right) X_i^m Y_i^m
\]

\[= \sigma_j \left( \sum_{k=0}^m \mu_i^k \sigma_j^{-1}(r_k) \cdot \sigma_i^{m-k}(t_j) \right) X_i^m Y_i^m,
\]

which is zero iff (iii) holds, since \( X_i^m Y_i^m \in R \) and is nonzero, and \( R \) has no zero-divisors.

3.1. A Note on the Case of Noetherian Base Ring \( R \)

This subsection is independent of the rest of the article. We include it to show that one can say something about the ideal \( I \) even under very mild conditions on the data \( R, \sigma, t, \mu \).
Proposition 3.5. Let $A' = A'(R, \sigma, t, \mu)$ be a TGWC, and assume that $R$ is a Noetherian domain. Then for any $i, j \in \{1, \ldots, n\}$, $i \neq j$, there exist $m_{ij}, n_{ij} \in \mathbb{Z}_{>0}$ and $r_{ij}^{(1)}, \ldots, r_{ij}^{(m_{ij})} \in R$ and $s_{ij}^{(1)}, \ldots, s_{ij}^{(n_{ij})} \in R$, where $r_{ij}^{(m_{ij})}, s_{ij}^{(n_{ij})} \neq 0$, such that

$$X_i^{m_{ij}} X_j + r_{ij}^{(1)} X_i^{m_{ij}-1} X_j X_i + \cdots + r_{ij}^{(m_{ij})} X_j X_i^{m_{ij}} \in I$$

(3.1)

and

$$X_j X_i^{n_{ij}} + s_{ij}^{(1)} X_i X_j X_i^{n_{ij}-1} + \cdots + s_{ij}^{(n_{ij})} X_j X_i^{n_{ij}} \in I.$$  (3.2)

By applying the anti-involution $^*$ we have analogous identities for the $Y_i$'s.

Remark 3.6. Note that the leftmost coefficients in (3.1),(3.2) are 1. Otherwise, the statement would be trivial due to the identity $\sigma_j(t_j)X_jX_i = \mu_{ij}^{-1}X_jX_i$, which follows from (2).

Proof. Since $R$ is Noetherian, the ascending chain of ideals

$$(t_j) \subseteq (t_j, \sigma_j(t_j)) \subseteq (t_j, \sigma_j(t_j), \sigma_j^2(t_j)) \subseteq \cdots$$

in $R$ must stabilize. Thus there is a positive integer $k$ and $r_1', \ldots, r_k' \in R$ such that

$$\sigma_k^j(t_j) = r_1' \sigma_k^j(t_j) + r_2' \sigma_k^{k-1}(t_j) + \cdots + r_k' t_j.$$  (3.3)

If $k$ was chosen minimal, then $r_k' \neq 0$. Indeed, otherwise, apply $\sigma_k^{-1}$ to (3.3) to contradict minimality. By Proposition 3.4, we get a relation of the form (3.1) with minimal $m_{ij}$. For (3.2), look instead on the chain $(t_j) \subseteq (t_j, \sigma_j^{-1}(t_j)) \subseteq \cdots$ and apply a positive power of $\sigma$. $\square$

4. LOCALLY FINITE TGWAs AND THEIR POLYNOMIAL CARTAN MATRICES

In view of Proposition 3.4, the following class of TGWAs seems natural to consider.

Definition 4.1. (a) Let $A = A(R, \sigma, t, \mu)$ be a TGWA. Define the following $K$-linear subspaces of $R$ for $i, j = 1, \ldots, n$:

$$V_{ij} = \text{Span}_K(\sigma^k_i(t_j) \mid k \in \mathbb{Z}).$$

(4.1)

We say that $A$ is locally finite if $\dim_K V_{ij} < \infty$ for all $i, j$.

(b) To a locally finite TGWA, $A$, we associate the matrix of minimal polynomials $P_A = (p_{ij})_{i,j=1}^n$ by letting $p_{ij} \in K[x]$ be the minimal polynomial for $\sigma_i$ acting on the finite-dimensional space $V_{ij}$. Equivalently, $p_{ij}$ is the monic polynomial of minimal degree such that $(p_{ij}(\sigma_i))(t_j) = 0$.

Remark 4.2. If the algebra $R$ is generated, as a $K$-algebra, by the set $\{\sigma_i^k(t_j) \mid i, j = 1, \ldots, n, k \in \mathbb{Z}\}$, then locally finiteness of the TGWA is equivalent to that the automorphisms $\sigma_i$ act locally finitely on $R$, hence the choice of terminology.
Recall that a generalized Cartan matrix is a matrix \( C = (c_{ij})_{i,j=1}^n \) with \( c_{ij} \in \mathbb{Z} \) \forall i, j which satisfies i) \( c_{ij} = 2 \forall i \), ii) \( c_{ij} \leq 0 \forall i \neq j \), iii) \( c_{ij} = 0 \iff c_{ji} = 0 \forall i \neq j \). We make the following definition.

**Definition 4.3.** A polynomial Cartan matrix, \( P = (p_{ij})_{i,j=1}^n \), is a matrix of polynomials \( p_{ij} \in \mathbb{K}[x] \) such that its shifted degree matrix \( (1 - \deg p_{ij})_{i,j=1}^n \) coincides with a generalized Cartan matrix away from the diagonal. Equivalently, \( P \) is a polynomial Cartan matrix iff for any \( i \neq j \) we have \( \deg p_{ij} \geq 1 \) and \( \deg p_{ij} = 1 \) iff \( \deg p_{ji} = 1 \). We will denote the Cartan matrix obtained from \( P \) by \( C(P) \).

The point in making these definitions is the following theorem.

**Theorem 4.4.** Let \( A = A(R, \sigma, t, \mu) \) be a locally finite TGWA, where \( R \) is a domain. Then:

(a) The matrix of minimal polynomials \( P_A = (p_{ij})_{i,j=1}^n \) of \( A \) is a polynomial Cartan matrix;

(b) Writing

\[
p_{ij}(x) = x^{m_{ij}} + \lambda_{ij}^{(1)} x^{m_{ij}-1} + \cdots + \lambda_{ij}^{(m_{ij})},
\]

where all \( \lambda_{ij}^{(k)} \in \mathbb{K} \), the following identities hold in \( A \), for any \( i \neq j \):

\[
X_i^{m_{ij}} X_j + \lambda_{ij}^{(1)} \mu_{ij} X_i^{m_{ij}-1} X_j + \cdots + \lambda_{ij}^{(m_{ij})} \mu_{ij} X_i X_j^{m_{ij}} = 0 \tag{4.2}
\]

and

\[
Y_j Y_i^{m_{ij}} + \lambda_{ij}^{(1)} \mu_{ij} Y_j Y_i^{m_{ij}-1} + \cdots + \lambda_{ij}^{(m_{ij})} \mu_{ij} Y_j Y_i^{m_{ij}} = 0. \tag{4.3}
\]

Moreover, for any \( i \neq j \) and \( m < m_{ij} \), the sets \( \{X_i^{m_{ij}} X_j X_k\}_{k=0}^m \) and \( \{Y_j^{m_{ij}} Y_i Y_k\}_{k=0}^m \) are linearly independent over \( \mathbb{K} \).

**Proof.** (a) Since \( t_i \neq 0 \) for each \( i \) it is clear that \( \deg p_{ij} \geq 1 \) for all \( i, j \). And if \( p_{ij} \) has degree one for some \( i \neq j \), then \( \sigma_i(t_j) = \lambda t_j \) for some \( \lambda \in \mathbb{K} \backslash \{0\} \). Then the consistency relations (2.1) and that \( t_i \) is not a zero-divisor imply that \( \sigma_i(t_j) = \lambda t_i \) for some \( \lambda \in \mathbb{K} \backslash \{0\} \). Thus \( \deg(p_{ji}) = 1 \).

(b) This is immediate from Proposition 3.4. \( \square \)

Note that the constant terms \( \lambda_{ij}^{(m_{ij})} \) are all nonzero, since the \( p_{ij} \) are minimal polynomials for invertible linear transformations.

**Example 4.5** (The TGWA of “Type \( A_n \)”, Contd.). Recall that \( n = 2 \), \( R = \mathbb{K}[H] \), \( \sigma_j(H) = H + 1 \), \( \sigma_i(H) = H - 1 \), and \( t_1 = H, t_2 = H + 1 \), and \( \mu_{12} = \mu_{21} = 1 \) and \( A = A(R, \sigma, t, \mu) \) is the corresponding TGWA. Clearly, \( A \) is locally finite with \( V_{ij} = \mathbb{C} H \oplus \mathbb{C} I \) for \( i, j = 1, 2 \). Observing that \( \sigma_2(t_1) \) and \( t_1 \) are linearly independent and that

\[
\sigma_2^2(t_1) - 2\sigma_2(t_1) + t_1 = H - 2 - 2(H - 1) + H = 0
\]
we see that the minimal polynomial $p_{21}$ for $\sigma_1$ acting on $V_{21}$ is given by $p_{21}(x) = x^2 - 2x + 1 = (x - 1)^2$. Similarly, one checks that in fact $p_j(x) = (x - 1)^2$ for all $i, j = 1, 2$. Thus, by Theorem 4.4(b), we have the relations

$$X_1^2 X_2 - 2X_1 X_2 X_2 + X_2 X_1^2 = 0, \quad Y_1^2 Y_2 - 2Y_1 Y_2 Y_2 + Y_2 Y_1^2 = 0,$$

in $A$, which are precisely the Serre relations in the enveloping algebra of $\mathfrak{sl}_3(\mathbb{K})$. Also, $X_1 X_2$, $X_2 X_1$, and $Y_1 Y_2$, $Y_2 Y_1$ are linearly independent. Note that $C(P_A) = \left[ \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right]$ is the Cartan matrix of type $A_2$, thus really motivating us to call this algebra a TGWA of type $A_2$.

**Example 4.6.** From (2.8) it is easy to see that the quantized Weyl algebra $A^{\mathbb{K},A}_3$ is also locally finite, and that the Cartan matrix $C(P)$ associated to its matrix of minimal polynomials $P$ is of the type $(A_1)^n = A_1 \times \cdots \times A_1$ (i.e., all zeros outside the diagonal).

The polynomial Cartan matrices should be regarded as a refinement of the notion of generalized Cartan matrices in the following sense.

**Example 4.7.** Let $C$ be any generalized Cartan matrix. To $C$ we can associate the matrix $P = (p_{ij})$ given by $p_{ij}(x) = (x - 1)^{1-a_{ij}}$ for $i \neq j$ and anything on the diagonal. Then $P$ is a polynomial Cartan matrix and $C(P) = C$. More generally, we could take $p_{ij}(x) = (x - q^{a_{ij}})(x - q^{a_{ij}+2}) \cdots (x - q^{a_{ij}+n})$, where $q \in \mathbb{K}\setminus\{0\}$. Assuming that we had a locally finite TGWA, with $R$ a domain, whose polynomial Cartan matrix was equal to such a $P$, and if all $\mu_{ij} = 1$, relations (4.2), (4.3) would become the usual (quantum) Serre relations occurring in the (quantum) enveloping algebra of the Kac–Moody algebra associated to $C$. In the next section, we construct such an algebra in the case when $C$ is symmetric.

It would be interesting to find conditions under which a locally finite TGWA $A$ is determined up to isomorphism by its polynomial Cartan matrix $P_A$. Part of this question would be to determine under what conditions the generalized Serre relations (4.2), (4.3) generate the ideal $I$. In Section 6 we give some sufficient conditions for this when $n = 2$.

5. A CONSTRUCTION OF LOCALLY FINITE TGWAS WITH SPECIFIED POLYNOMIAL CARTAN MATRIX

Let $C = (a_{ij})_{i,j=1}^n$ be a symmetric generalized Cartan matrix, let $q \in \mathbb{K}\setminus\{0\}$, and let $\mu = (\mu_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix without diagonal with $\mu_{ij} \in \mathbb{K}\setminus\{0\}$.

Take $R$ to be the following polynomial algebra over $\mathbb{K}$:

$$R = \mathbb{K}[H_{ij}^{(k)} | 1 \leq i < j \leq n, \text{ and } k = a_{ij}, a_{ij} + 2, \ldots, -a_{ij}].$$

Define $\sigma_1, \ldots, \sigma_n \in \text{Aut}_{\mathbb{K}}(R)$ by setting, for all $i < j$ and $k = a_{ij}, a_{ij} + 2, \ldots, -a_{ij}$:

$$\sigma_j(H_{ij}^{(k)}) = \mu_{ij} q^k H_{ij}^{(k)} + H_{ij}^{(k-2)}, \quad \text{where } H_{ij}^{(a_{ij}+2)} := 0,$$

\hspace{1cm} (5.1)
For notational purposes, put
\[ H_{ij} = H_{i,j}^{(-a_{ij})}, \quad H_{ji} = \sigma_j^{-1}(H_{ij}) \quad \text{for all } i < j, \quad \text{and } H_{ii} = 1 \quad \text{for all } i \]
and define
\[ t_i = H_{1i} H_{2i} \cdots H_{ni}, \quad \text{for } i = 1, \ldots, n. \]

**Proposition–Definition 5.1.** The data \((R, \sigma, t, \mu)\) satisfies the conditions required in the definition of a TGWA, namely, the \(\sigma_i\) commute with each other, and the consistency relations (2.1) hold. We let \(\mathcal{T}'_{q,R}(C) = A'(R, \sigma, t, \mu)\) denote the associated twisted generalized Weyl construction and \(\mathcal{T}_{q,R}(C) = A(R, \sigma, t, \mu)\) the corresponding twisted generalized Weyl algebra.

**Proof.** It is easy to check that the automorphisms \(\sigma_i\) commute, since on each \(H_{ij}\) either one of the automorphisms is the identity, or their composition is a multiple of the identity. To prove the consistency relations
\[ t_i t_j = \mu_{ij}^2 \sigma_j^{-1}(t_i) \sigma_i^{-1}(t_j) \quad \forall i \neq j, \]
we can assume by symmetry that \(i < j\). Then the right-hand side equals
\[ \mu_{ij}^2 H_{i1} \cdots \sigma_j^{-1}(H_{ij}) \cdots H_{im} \cdot H_{j1} \cdots \sigma_i^{-1}(H_{ji}) \cdots H_{jm}, \]
which equals \(t_i t_j\) since \(\sigma_i^{-1}(H_{ji}) = H_{ij}\) by choice of notation and
\[ \mu_{ij}^2 \sigma_j^{-1}(H_{ij}) = \mu_{ij}^2 \sigma_i^{-1} \sigma_j^{-1}(H_{ij}) = H_{ij} \]
by (5.2). Similarly one checks the other relations in (2.1).

To formulate the next theorem, recall that for any \(q \in \mathbb{K} \setminus \{0\}\), the \(q\)-binomial coefficients \(\begin{bmatrix} n \end{bmatrix}_q\) may be defined as the elements in the subring \(\mathbb{Z}[q, q^{-1}]\) of \(\mathbb{K}\) given by requiring the identity
\[ (x + q^{-m+1})(x + q^{-m+2}) \cdots (x + q^{-1}) = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q x^k \]  
(5.4)
in \(\mathbb{K}[x]\) for any \(m \in \mathbb{Z}_{\geq 0}\), and \(\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1\) by convention. If \(q\) is not a root of unity and \(\mathbb{K}\) has characteristic zero, we have
\[ \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q ! [k]_q !}, \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q, \quad [n]_q = q^{-m+1} + q^{-m+2} + \cdots + q^{-1}, \]
(5.5)
which can be proved by verifying that both definitions solve the same recursion relation

\[
\begin{bmatrix} m \\ k \end{bmatrix}_q = q^{-k} \begin{bmatrix} m - 1 \\ k \end{bmatrix}_q + q^{m-k} \begin{bmatrix} m - 1 \\ k - 1 \end{bmatrix}_q.
\]

We now come to the main theorem in this section, which relates the algebra \( T_{q,u} (C) \) to the quantum group \( U_q (\mathfrak{g}(C)) \) associated to the Kac–Moody algebra \( \mathfrak{g}(C) \) (see for example [13] for an introduction to Drinfel’d–Jimbo quantum groups).

**Theorem 5.2.**

(a) \( T_{q,u} (C) \) is locally finite.

(b) The off-diagonal entries of the polynomial Cartan matrix \( P = (p_{ij}) \) of \( T_{q,u} (C) \) are given by

\[
p_{ij}(x) = (x - \mu_{ij} q^{a_{ij}})(x - \mu_{ij} q^{a_{ij} + 2}) \cdots (x - \mu_{ij} q^{-a_{ij}})
\]

for any \( i \neq j \).

(c) In \( T_{q,u} (C) \), the quantum Serre relations of the quantum group \( U_q (\mathfrak{g}(C)) \) hold. That is, for any \( i \neq j \),

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_q x_i^{1-a_{ij}-k} x_j x_i^k = 0,
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_q y_i^k y_j y_i^{1-a_{ij}-k} = 0.
\]

Moreover, for any \( i \neq j \) and any \( m < 1 - a_{ij} \), the elements \( x_i^{m-k} x_j x_i^k \) and \( y_i^{m-k} y_j y_i^k \), \( k = 0, 1, \ldots, m \), are linearly independent over \( \mathbb{K} \).

**Proof.** (a) Giving \( R \) the natural \( Z_{\geq 0} \)-gradation by letting all \( H_{ij}^{(k)} \) be of degree one, we observe that all automorphisms \( \sigma_i \) preserve the degree one subspace of \( R \). Since the \( t_i \) have degree \( n \), each \( V_{ij} \) is, therefore, contained in the finite-dimensional subspace of \( R \) of elements of degree \( n \), so \( T_{q,u} (C) \) is locally finite.

(b) Let \( 1 \leq i < j \leq n \), and consider the following linear subspace of \( R \):

\[
W_{ij} = \mathbb{K} H_{ij}^{(a_{ij})} \oplus \mathbb{K} H_{ij}^{(a_{ij} + 2)} \oplus \cdots \oplus \mathbb{K} H_{ij}^{(-a_{ij})}.
\]

It has dimension \( 1 - a_{ij} \). By (5.1) the \( \mathbb{K} \)-algebra automorphism \( \sigma_j \) preserves the subspace \( W_{ij} \) and the matrix of \( \sigma_j |_{W_{ij}} \) in the ordered basis \( (H_{ij}^{(a_{ij})}, H_{ij}^{(a_{ij} + 2)}, \ldots, H_{ij}^{(-a_{ij})}) \)
is given by
\[
\begin{bmatrix}
\mu_j q^{a_{ij}} & 1 \\
\mu_j q^{a_{ij}+2} & 1 \\
& \ddots \\
\mu_j q^{-a_{ij}} & 1 \\
\end{bmatrix}
\]
(zeros omitted). Thus the minimal polynomial \(p_{ij}(x)\) of \(\sigma_j|_{W_{ij}}\) equals
\[
p_{ij}(x) = (x - \mu_j q^{a_{ij}})(x - \mu_j q^{a_{ij}+2}) \cdots (x - \mu_j q^{-a_{ij}}).
\]
Since \(\sigma_j(H^{(k)}_{ij}) = H^{(k)}_{ij}\) if \(r, s \neq j\) it follows that \(p_{ij}\) is also the minimum polynomial for \(\sigma_j\) acting on the space \(V_j\) spanned by all \(\sigma_j^k(t_i)\) \((k \in \mathbb{Z})\). On \(W_{ij}\) we also have, using (5.2),
\[
p_{ij}(\sigma_i)|_{W_{ij}} = (\mu_j^2 \sigma_j^{-1} - \mu_j q^{a_{ij}})(\mu_j^2 \sigma_j^{-1} - \mu_j q^{a_{ij}+2}) \cdots (\mu_j^2 \sigma_j^{-1} - \mu_j q^{-a_{ij}})
\]
\[
= \mu_j^{1-a_{ij}} \sigma_j^{-a_{ij}-1}(\mu_j q^{-a_{ij}} - \sigma_j)(\mu_j q^{-a_{ij}}-2 - \sigma_j) \cdots (\mu_j q^{a_{ij}} - \sigma_j)
\]
\[
= (-\mu_j)^{1-a_{ij}} \sigma_j^{-a_{ij}-1}(p_{ij}(\sigma_j)) = 0.
\]
On the other hand, \(\sigma_i|_{W_i}\) cannot satisfy any polynomial of lower degree since it is a multiple of the inverse of \(\sigma_j|_{W_j}\). This shows that \(p_{ij}(x)\) is also the minimal polynomial of \(\sigma_i|_{W_i}\) and thus the minimal polynomial for \(\sigma_i|_{W_i}\). This proves that the polynomial Cartan matrix \(P\) of \(\mathcal{T}_{q,R}(C)\) is given by (5.6). The equality (5.7) follows from the definition (5.4) of the \(q\)-binomial coefficients.

(c) This follows from part b) and Theorem 4.4. \(\square\)

We believe that the quantum Serre relations (5.8), (5.9) in fact generate the ideal \(I\) of the TGWC \(\mathcal{T}_{q,R}(C)\). This would yield a complete presentation of \(\mathcal{T}_{q,R}(C)\) by generators and relations. The result in the next section implies that this is true in the case when \(C\) is the Cartan matrix of type \(A_2\).

6. SUFFICIENT CONDITIONS FOR SERRE-TYPE RELATIONS TO GENERATE THE IDEAL \(I\) OF A TGWC

**Theorem 6.1.** Assume that \(A' = A'(R, \sigma, t, \mu)\) is a twisted generalized Weyl construction of rank \(n = 2\) where \(R\) is a domain, with the following properties:

(P1) For any positive integer \(k\) and any \(r \in R\) the following implication holds:
\[
r \cdot (\sigma_1 \sigma_2)^k(t_1) \cdot (\sigma_1 \sigma_2)^{k-1}(t_1) \cdots (\sigma_1 \sigma_2)(t_1) \in R \cdot \sigma_1(t_1) \implies r \in R \cdot \sigma_1(t_1);
\]

(P2) The ideal \(I\) of \(A'\) contains elements of the form
\[
s_1 := X_2X_1^2 - \xi_1X_1X_2X_1 - \xi_2X_1^2X_2,
\]
\[ s_2 := X_2^2X_1 - \eta_1X_2X_1X_2 - \eta_2X_2X_2^2, \]

where \( \xi_i, \eta_i \) are nonzero elements of \( \mathbb{K} \).

Then \( I \) is generated by \( s_1, s_2, s_1', s_2' \).

**Remark 6.2.** Condition (P1) is satisfied if the following two properties hold:

(P1a) \( \sigma_1\sigma_2(t_1) = \lambda t_1 \) for some \( \lambda \in \mathbb{K} \setminus \{0\} \); and

(P1b) \( R \cdot t_i \) is a prime ideal in \( R \) and the ideals \( R \cdot t_1 \) and \( R \cdot \sigma_i(t_i) \) are coprime, i.e., their sum is all of \( R \).

**Proof.** We begin with a simple observation. Let \( g = (g_1, g_2) \in \mathbb{Z}^2 \) and suppose \( a \in I \cap A \), \( a \neq 0 \). If \( g_1 \geq 0 \) and \( g_2 \leq 0 \), then \( a = rX_{1i}^rY_{2}^{\sigma} \) for some \( r \in R \). Since \( I \) is completely gr-prime and contains no monic monomials (Proposition 3.1 and Corollary 3.3), we get \( r \in I \). But \( I \cap R = 0 \) so \( r = 0 \), contradicting \( a \neq 0 \). Similarly, we cannot have \( g_1 \leq 0 \) and \( g_2 \geq 0 \). Thus, if \( I \cap A \neq 0 \), then either \( g_1 > 0 \) and \( g_2 > 0 \), or \( g_1 < 0 \) and \( g_2 < 0 \).

Now let \( I_\delta \) denote the ideal in \( A' \) generated by \( s_1, s_2, s_1', s_2' \). By Property (P2) we know \( I_\delta \subseteq I \). Assume that \( I_\delta \subseteq I \). Then, among all homogenous elements in \( I \setminus I_\delta \), choose one, \( a \), with minimal total degree \( |g_1 + g_2| \), where \( (g_1, g_2) = \deg a \in \mathbb{Z} \). After applying the involution \( * \) if necessary, we can, by the previous paragraph, assume that \( g_1, g_2 \geq 0 \).

For a monic monomial \( w = Z_1 \cdots Z_k \in A' \) where \( Z_i \in \{X_1, X_2\} \ \forall i \), define the length \( \ell(w) \) as the number of pairs \((i, j)\), \( 1 \leq i < j \leq k \) such that \( Z_i = X_2 \) and \( Z_j = X_1 \). The element \( a \) can be written as a sum of monomials in the noncommuting elements \( X_1, X_2 \) with coefficients from \( R \) written on the left. Some of these monomials can be reduced mod \( I_\delta \) with the reductions

\[ X_2X_1^2 \rightarrow \xi_1X_1X_2X_1 + \xi_2X_1^2X_2, \quad (6.1) \]

\[ X_2^2X_1 \rightarrow \eta_1X_2X_1X_2 + \eta_2X_1X_2^2. \quad (6.2) \]

At each reduction a monomial \( w \) is replaced by a sum of two monomials each having strictly lower length than \( w \). Thus after finite number of steps we obtain an element of the form

\[ a' = \sum_{i=0}^{\min(g_1, g_2)} \beta_iX_{1i}^{e_1-i}(X_2X_1)^{e_2-i}, \quad (6.3) \]

where \( \beta_i \in R \). We can without loss of generality assume \( a = a' \) since we only added elements from \( I_\delta \) of degree \((g_1, g_2)\). Then we have

\[ aY_2^{\sigma} = \sum_{i=0}^{\min(g_1, g_2)} \alpha_iX_{1i}^{e_1-i} \cdot (\sigma_1\sigma_2)(t_1) \cdot (\sigma_1\sigma_2)^{-1} \cdots \sigma_i\sigma_2(t_i) \cdot Y_2^{\sigma}X_2^2, \]

where \( \alpha_i \) equals \( \beta_i \) times some power of \( \mu_{21} \). Multiplying further by \( Y_2^{\sigma} \) from the right we get zero since \( a \in I \). After that we can cancel \( X_2^{e_2}Y_2^{\sigma} \) since it is nonzero and
Corollary 6.4. Let $C$ be the Cartan matrix of type $A_n$. Then the maximal graded ideal with trivial zero component, $I$, of $\mathcal{F}_{q,n}(C)$ is generated by the elements

\begin{align*}
X_1^2X_2 - (q + q^{-1})X_1X_2X_1 + X_2X_1^2, \\
X_2^2X_1 - (q + q^{-1})X_2X_1X_2 + X_1X_2^2, \\
Y_1^2Y_2 - (q + q^{-1})Y_1Y_2Y_1 + Y_2Y_1^2, \\
Y_2^2Y_1 - (q + q^{-1})Y_2Y_1Y_2 + Y_1Y_2^2.
\end{align*}
Proof. Properties (P1a) and (P1b) are easily verified for the TGWC $\mathcal{T}_{q,\mu}^\prime(C)$, and Property (P2) holds by Theorem 5.2(c), so Theorem 6.1 can be applied and yields the result. □

It is now straightforward to write down a set of generators and relations for $\mathcal{T}_{q,\mu}^\prime(C)$ when $C$ is of type $A_2$.

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