SPECTRAL PROPERTIES OF $k$-QUASI-2-ISOMETRIC OPERATORS

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Abstract. Let $T$ be a bounded linear operator on a complex Hilbert space $H$. For a positive integer $k$, an operator $T$ is said to be a $k$-quasi-2-isometric operator if $T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k = 0$, which is a generalization of 2-isometric operator. In this paper, we consider basic structural properties of $k$-quasi-2-isometric operators. Moreover, we give some examples of $k$-quasi-2-isometric operators. Finally, we prove that generalized Weyl’s theorem holds for polynomially $k$-quasi-2-isometric operators.

1. Introduction

Let $H$ be an infinite dimensional separable Hilbert space. We denote by $B(H)$ the algebra of all bounded linear operators on $H$, write $N(T)$ and $R(T)$ for the null space and range space of $T$, and also, write $\sigma(T)$, $\sigma_a(T)$ and $\text{iso}\sigma(T)$ for the spectrum, the approximate point spectrum and the isolated point spectrum of $T$, respectively.

An operator $T$ is called Fredholm if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim H/R(T) < \infty$. Moreover if $i(T) = \alpha(T) - \beta(T) = 0$, then $T$ is called Weyl. The Weyl spectrum $w(T)$ of $T$ is defined by $w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$. Following [12], we say that Weyl’s theorem holds for $T$ if $\sigma(T) \setminus w(T) = \pi_{00}(T)$, where $\pi_{00}(T) := \{\lambda \in \text{iso}\sigma(T) : 0 < \dim N(T - \lambda) < \infty\}$.

More generally, Berkani investigated $B$-Fredholm theory (see [4, 5, 6]). An operator $T$ is called $B$-Fredholm if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator $T_{[n]} : R(T^n) \ni x \mapsto Tx \in R(T^n)$ is Fredholm, i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\alpha(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = \dim R(T^n)/R(T_{[n]}) < \infty$.
Similarly, a $B$-Fredholm operator $T$ is called $B$-Weyl if $i(T_{[n]}) = 0$. The $B$-Weyl spectrum $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda$ is not $B$-Weyl$\}$. We say that generalized Weyl’s theorem holds for $T$ if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues (no restriction on multiplicity). Note that, if generalized Weyl’s theorem holds for $T$, then so does Weyl’s theorem [5].

In [1] Agler obtained certain disconjugacy and Sturm-Liouville results for a subclass of the Toeplitz operators. These results were suggested by the study of operators $T$ which satisfies the equation,

$$T^* T^2 - 2T^* T + I = 0.$$ 

Such $T$ are natural generalizations of isometric operators ($T^* T = I$) and are called 2-isometric operators. It is known that an isometric operator is a 2-isometric operator. 2-isometric operators have been studied by many authors and they have many interesting properties (see [2, 3, 7, 8, 9, 14]).

In order to extend 2-isometric operators we introduce $k$-quasi-2-isometric operators defined as follows:

**Definition 1.1.** For a positive integer $k$, an operator $T$ is said to be a $k$-quasi-2-isometric operator if

$$T^{*k}(T^* T^2 - 2T^* T + I)T^k = 0.$$ 

It is clear that each 2-isometric operator is a $k$-quasi-2-isometric operator and each $k$-quasi-2-isometric operator is a $(k + 1)$-quasi-2-isometric operator.

In this paper we give a necessary and sufficient condition for $T$ to be a $k$-quasi-2-isometric operator. Moreover, we study characterizations of weighted shift operators which are $k$-quasi-2-isometric operators. Finally, we prove polynomially $k$-quasi-2-isometric operators satisfy generalized Weyl’s theorem.

2. Main Results

We begin with the following theorem which is the essence of this paper; it is a structure theorem for $k$-quasi-2-isometric operators.

**Theorem 2.1.** If $T^k$ does not have a dense range, then the following statements are equivalent:

1. $T$ is a $k$-quasi-2-isometric operator;
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(2) $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$, where $T_1$ is a 2-isometric operator and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. (1) $\Rightarrow$ (2) Consider the matrix representation of $T$ with respect to the decomposition $H = \overline{R(T^k)} \oplus N(T^{*k})$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$ 

Let $P$ be the projection onto $\overline{R(T^k)}$. Since $T$ is a $k$-quasi-2-isometric operator, we have

$$P(T^*2T^2 - 2T^*T + I)P = 0.$$

Therefore

$$T^*_1T_1^2 - 2T^*_1T_1 + I = 0.$$

On the other hand, for any $x = (x_1, x_2) \in H$, we have

$$(T^*_3x_2, x_2) = (T^k(I - P)x, (I - P)x) = ((I - P)x, T^{*k}(I - P)x) = 0,$$

which implies $T_3^k = 0$.

Since $\sigma(T) \cup M = \sigma(T_1) \cup \sigma(T_3)$, where $M$ is the union of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_3)$ by Corollary 7 of [11], and $\sigma(T_1) \cap \sigma(T_3)$ has no interior point and $T_3$ is nilpotent, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) $\Rightarrow$ (1) Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$, where $T^*_1T_1^2 - 2T^*_1T_1 + I = 0$ and $T_3^k = 0$. Since

$$T^k = \begin{pmatrix} T_1^k \\ 0 \\ \sum_{j=0}^{k-1} T^j_1T_2T_3^{k-1-j} \\ 0 \end{pmatrix},$$

we have

$$T^{*k}(T^*2T^2 - 2T^*T + I)T^k = \begin{pmatrix} T_1^* & T_2^* \\ 0 & T_3^* \end{pmatrix}.$$
\[
\times \left( \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \right)^{2} \left( \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \right)^{2} - 2 \left( \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \right)^{\ast} \left( \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \right) + I
\]
\[
\times \left( \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \right)^{k}
\]
\[
= \begin{pmatrix} T_{1}^{\ast k} DT_{1}^{k} & T_{1}^{\ast k} D \sum_{j=0}^{k-1} T_{1}^{j} T_{2} T_{3}^{k-1-j} \\ (\sum_{j=0}^{k-1} T_{1}^{j} T_{2} T_{3}^{k-1-j})^{\ast} DT_{1}^{k} & (\sum_{j=0}^{k-1} T_{1}^{j} T_{2} T_{3}^{k-1-j})^{\ast} D \sum_{j=0}^{k-1} T_{1}^{j} T_{2} T_{3}^{k-1-j} \end{pmatrix}
\]
where \( D = T_{1}^{\ast 2} T_{1}^{2} - 2 T_{1}^{\ast} T_{1} + I \). It follows that \( T^{\ast k} (T_{1}^{\ast 2} T_{1}^{2} - 2 T_{1}^{\ast} T_{1} + I) T^{k} = 0 \) on \( H = \mathbb{R}(T^{k}) \oplus N(T^{k}) \). Thus \( T \) is a \( k \)-quasi-2-isometric operator. \( \square \)

**Corollary 2.2.** If \( T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \) is a \( k \)-quasi-2-isometric operator and \( T_1 \) is invertible, then \( T \) is similar to a direct sum of a 2-isometric operator and a nilpotent operator.

**Proof.** Since \( T_1 \) is invertible, we have \( \sigma(T_1) \cap \sigma(T_3) = \emptyset \). Then there exists an operator \( S \) such that \( T_1 S - S T_3 = T_2 [15] \). Since \( \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -S \\ 0 & I \end{pmatrix} \), it follows that
\[
T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}.
\]

\( \square \)

**Corollary 2.3.** If \( T \) is a \( k \)-quasi-2-isometric operator and \( R(T^{k}) \) is dense, then \( T \) is a 2-isometric operator.

**Proof.** This is a result of Theorem 2.1. \( \square \)

**Corollary 2.4.** If \( T \) is a \( k \)-quasi-2-isometric operator, then so is \( T^n \) for every natural number \( n \).

**Proof.** We decompose \( T \) as
\[
T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \mathbb{R}(T^{k}) \oplus N(T^{k}).
\]
Then by Theorem 2.1, \( T_{1}^{\ast 2} T_{1}^{2} - 2 T_{1}^{\ast} T_{1} + I = 0 \). Hence \( T_1 \) is a 2-isometric operator, by [14, Theorem 2.1], \( T_{1}^{n} \) is a 2-isometric operator. Since
\[
T^n = \begin{pmatrix} T_{1}^{n} & \sum_{j=0}^{n-1} T_{1}^{j} T_{2} T_{3}^{n-1-j} \\ 0 & T_{3}^{n} \end{pmatrix} \text{ on } H = \mathbb{R}(T^{k}) \oplus N(T^{k}),
\]
$T^n$ is a $k$-quasi-2-isometric operator for every natural number $n$ by Theorem 2.1. □

**Lemma 2.5.** $T$ is a $k$-quasi-2-isometric operator if and only if
\[ ||T^{k+2}x||^2 + ||T^kx||^2 = 2||T^{k+1}x||^2 \]
for every $x \in H$.

**Theorem 2.6.** Let $T$ be a $k$-quasi-2-isometric operator and $M$ be an invariant subspace for $T$. Then the restriction $T|_M$ is also a $k$-quasi-2-isometric operator.

**Proof.** For $x \in M$, we have
\[
2||(T|_M)^{k+1}x||^2 = 2||T^{k+1}x||^2 \\
= ||T^{k+2}x||^2 + ||T^kx||^2 = ||(T|_M)^{k+2}x||^2 + ||(T|_M)^kx||^2.
\]
Thus $T|_M$ is a $k$-quasi-2-isometric operator. □

**Example 2.7.** Given a bounded sequence $\alpha : \alpha_0, \alpha_1, \alpha_2, \ldots$ (called weights), the unilateral weighted shift $W_\alpha$ associated with $\alpha$ is the operator on $l^2$ defined by
\[
W_\alpha e_n = \alpha_n e_{n+1}
\]
for all $n \geq 0$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthogonal basis for $l^2$ and $|\alpha_n| \neq 0$ for each $n \geq 0$. Then the following statement holds: $W_\alpha$ is a $k$-quasi-2-isometric operator if and only if
\[
|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0 \quad (n = k, k + 1, k + 2, \ldots).
\]

**Proof.** By calculation, $W_\alpha^2 W_\alpha = |\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 + \cdots$ and $W_\alpha^2 W_\alpha^2 = |\alpha_0|^2|\alpha_1|^2 + |\alpha_1|^2|\alpha_2|^2 + |\alpha_2|^2|\alpha_3|^2 + \cdots$, by definition, $W_\alpha$ is a $k$-quasi-2-isometric operator if and only if $|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0 \quad (n = k, k + 1, k + 2, \ldots)$. □

**Remark 2.8.** Let $W_\alpha$ be the unilateral weighted shift with weight sequence $(\alpha_n)_{n \geq 0}$ and $|\alpha_n| \neq 0$ for each $n \geq 0$. From Example 2.7 we obtain the following characterizations:

1. $W_\alpha$ is a $k$-quasi-2-isometric operator if and only if
\[
|\alpha_n|^2 = \frac{(n - k + 1)|\alpha_k|^2 - (n - k)}{(n - k)|\alpha_k|^2 - (n - k - 1)}
\]
for \( n \geq k \).

2. \( \{ |\alpha_n| \} \) is a decreasing sequence of real numbers converging to 1 for \( n \geq k \).

3. \( \sqrt{2} \geq |\alpha_n| \geq 1 \) for \( n \geq k + 1 \).

4. Let \( 2 = |\alpha_k|, 1 = |\alpha_{k+1}| = |\alpha_{k+2}| = |\alpha_{k+3}| = \cdots \). Then \( W_\alpha \) is a \((k+1)\)-quasi-2-isometric operator but not a \(k\)-quasi-2-isometric operator.

In the sequel, we focus on polynomially \( k \)-quasi-2-isometric operators.

We say that \( T \) is a polynomially \( k \)-quasi-2-isometric operator if there exists a nonconstant complex polynomial \( p \) such that \( p(T) \) is a \( k \)-quasi-2-isometric operator. It is clear that a \( k \)-quasi-2-isometric operator is a polynomially \( k \)-quasi-2-isometric operator. The following example provides an operator which is a polynomially \( k \)-quasi-2-isometric operator but not a \( k \)-quasi-2-isometric operator.

**Example 2.9.** Let \( T = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \in B(l_2 \oplus l_2) \). Then \( T \) is a polynomially \( k \)-quasi-2-isometric operator but not a \( k \)-quasi-2-isometric operator.

**Proof.** Since 
\[
T^* = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix},
\]
we have
\[
T^*T^2 - 2T^*T + I = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then 
\[
T^k(T^*T^2 - 2T^*T + I)T^k = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix} \neq 0.
\]

Therefore \( T \) is not a \( k \)-quasi-2-isometric operator.

On the other hand, consider the complex polynomial \( h(z) = (z - 1)^2 + 1 \). Then \( h(T) = I \), and hence \( T \) is a polynomially \( k \)-quasi-2-isometric operator. \( \square \)

Recall that an operator \( T \) is said to be isoloid if every isolated point of \( \sigma(T) \) is an eigenvalue of \( T \) and polaroid if every isolated point of \( \sigma(T) \) is a pole of the resolvent of \( T \). In general, if \( T \) is polaroid, then it is isoloid. However, the converse is not true.

**Theorem 2.10.** Let \( T \) be a polynomially \( k \)-quasi-2-isometric operator. Then \( T \) is polaroid.

**Proof.** We first show that a \( k \)-quasi-2-isometric operator is polaroid. We consider
the following two cases: Case I: If the range of $T^k$ is dense, then $T$ is a 2-isometric operator, $T$ is polaroid. Since an invertible 2-isometric operator is a unitary operator by [2, Proposition 1.23], and if $T$ is a non-invertible 2-isometric operator, then $\text{iso}(T)$ is empty.

Case II: If the range of $T^k$ is not dense, by Theorem 2.1, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = \overline{R(T^k)} \oplus N(T^{*k}).$$

Let $\lambda \in \text{iso}(T)$. Suppose that $T_1$ is a non-invertible 2-isometric operator. Then $\sigma(T) = D$, where $D$ is the closed unit disk. Since $\sigma(T) = \sigma(T_1) \cup \{0\}$, we have $\text{iso}(T)$ is empty; thus $T_1$ is a invertible 2-isometric operator and $\lambda \in \text{iso}(T_1)$ or $\lambda = 0$, $T_1$ is a unitary operator, $T_3$ is nilpotent. It is easy to prove that $T - \lambda$ has finite ascent and descent, i.e., $\lambda$ is a pole of the resolvent of $T$, therefore $T$ is polaroid.

Next we show that a polynomially $k$-quasi-2-isometric operator is polaroid. If $T$ is a polynomially $k$-quasi-2-isometric operator, then $p(T)$ is a $k$-quasi-2-isometric operator for some nonconstant polynomial $p$. Hence it follows from the first part of the proof that $p(T)$ is polaroid. Now apply [10, Lemma 3.3] to conclude that $p(T)$ polaroid implies $T$ polaroid.

**Corollary 2.11.** Let $T$ be a polynomially $k$-quasi-2-isometric operator. Then $T$ is isoloid.

An operator $T$ is said to has the single valued extension property (abbreviated SVEP) if, for every open subset $G$ of $\mathbb{C}$, any analytic function $f : G \to H$ such that $(T - z)f(z) \equiv 0$ on $G$, we have $f(z) \equiv 0$ on $G$.

**Theorem 2.12.** Let $T$ be a polynomially $k$-quasi-2-isometric operator. Then $T$ has SVEP.

**Proof.** We first suppose that $T$ is a $k$-quasi-2-isometric operator. We consider the following two cases:

Case I: If the range of $T^k$ is dense, then $T$ is a 2-isometric operator, $T$ has SVEP by [8, Theorem 2].

Case II: If the range of $T^k$ is not dense, by Theorem 2.1, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = \overline{R(T^k)} \oplus N(T^{*k}).$$

Suppose $(T - z)f(z) = 0$, $f(z) = f_1(z) \oplus f_2(z)$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$. Then we
can write
\[
\begin{pmatrix}
T_1 - z & T_2 \\
0 & T_3 - z
\end{pmatrix}
\begin{pmatrix}
f_1(z) \\
f_2(z)
\end{pmatrix}
= \begin{pmatrix}
(T_1 - z)f_1(z) + T_2f_2(z) \\
(T_3 - z)f_2(z)
\end{pmatrix}
= 0.
\]
And \(T_3\) is nilpotent, \(T_3\) has SVEP, hence \(f_2(z) = 0, (T_1 - z)f_1(z) = 0\). Since \(T_1\) is a 2-isometric operator, \(T_1\) has SVEP by [8, Theorem 2], then \(f_1(z) = 0\). Consequently, \(T\) has SVEP.

Now suppose that \(T\) is a polynomially \(k\)-quasi-2-isometric operator. Then \(p(T)\) is a \(k\)-quasi-2-isometric operator for some nonconstant complex polynomial \(p\), and hence \(p(T)\) has SVEP. Therefore, \(T\) has SVEP by [13, Theorem 3.3.9].

Since the SVEP for \(T\) entails that generalized Browder’s theorem holds for \(T\), i.e. \(\sigma_{BW}(T) = \sigma_D(T)\), where \(\sigma_D(T)\) denotes the Drazin spectrum, a sufficient condition for an operator \(T\) satisfying generalized Browder’s theorem to satisfy generalized Weyl’s theorem is that \(T\) is polaroid. In [14], Patel showed that Weyl’s theorem holds for 2-isometric operator. Then we have the following result:

**Theorem 2.13.** If \(T\) is a polynomially \(k\)-quasi-2-isometric operator, then generalized Weyl’s theorem holds for \(T\), so does Weyl’s theorem.

**Proof.** It is obvious from Theorem 2.10, Theorem 2.12 and the statements of the above.

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