Nonclassical correlations in damped $N$-solitons

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The quantum statistics of damped higher-order optical solitons are analyzed numerically, using cumulant-expansion techniques in Gaussian approximation. A detailed analysis of nonclassical properties in both the time and the frequency domain is given, with special emphasis on the role of absorption. Highly nonclassical broadband spectral correlation is predicted.

From classical optics it is well known that nonlinearities can compensate for the dispersion-assisted pulse spreading [1,2] or for diffraction-assisted beam broadening (see, e.g., [3]). In the two cases, the undamped motion of the (slowly varying) bosonic field variables $\hat{a}(x, t)$ is governed by the Hamiltonian

$$\hat{H} = \hbar \int dx \left[ \frac{1}{2} \omega^{(2)} (\partial_x \hat{a}^\dagger)(\partial_x \hat{a}) + \frac{1}{2} \chi \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \right],$$

$$[\hat{a}(x, t), \hat{a}^\dagger(x', t)] = \delta(x - x')$$

with $t$, propagation variable; $x$, “transverse” coordinate; $\omega^{(2)}$, second order dispersion or diffraction constant; $\chi$ nonlinearity constant; see, e.g., [4,5]. Note that bright temporal solitons can be formed either in focusing media with anomalous dispersion ($\chi < 0$, $\omega^{(2)} > 0$) or in defocusing media with normal dispersion ($\chi > 0$, $\omega^{(2)} < 0$), whereas spatial solitons require always focusing nonlinearity. The effect of absorption is described in terms of ordinary Markovian relaxation theory resulting, in the low temperature limit, in the master equation

$$i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}] + i\gamma \hbar \int dx \left( 2\hat{a}\hat{a}^\dagger \hat{\rho} - \hat{\rho} \hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} \hat{\rho} \right)$$

with $\gamma$, damping constant).

The master equation (3) is converted, after spatial discretization, into a pseudo-Fokker-Planck equation for an $s$-parametrized multi-dimensional phase-space function, which is solved numerically using cumulant expansion in Gaussian approximation [5]. The initial condition is realized by a multimode coherent state without internal entanglement, and it is assumed that the field expectation value corresponds to the classical $N$-soliton solution, $\langle \hat{a}(x, t_0) \rangle = Na_0 \text{sech}(x/x_0)$, $N = 1, 2, \ldots$ ($a_0$ and $x_0$, mean amplitude and width of the fundamental soliton, respectively).

Spectral properties can be studied introducing the Fourier-component operators

$$\hat{a}(\omega, t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dx e^{i\omega x} \hat{a}(x, t).$$

Here we restrict our attention to correlations of photon number fluctuations. In the case of fiber soliton pulses the correlations in the $\omega$-domain can be measured using appropriate spectral filtering (see, e.g., in [6]). In the case of spatial solitonic beams the correlations in both the $x$ and $\omega$-domains, respectively, can be measured by filtering the field in the near- and far-field zones of the output beam (see, e.g., [4]).
FIG. 1. The evolution of the mean photon number $\langle \hat{n}_i \rangle$ and the correlation coefficient $\eta_{ii}$ of an undamped soliton, $N = 2$, is plotted in the $x$-domain, $\Delta x = 0.05 x_0$ [(a),(b)], and the $\omega$-domain, $\Delta \omega = 0.25 \omega_0$ [(c),(d)]. The plots $(e_1) - (e_5)$ ($x$-domain) and $(f_1) - (f_5)$ ($\omega$-domain) show the correlation coefficient $\eta_{ij}$ for typical propagation lengths ($\omega_0 = 1/x_0$, $t_d = |x_0^2/\omega|^2$, $\int dx \hat{a}^\dagger(x,0)\hat{a}(x,0) = 8 \times 10^9$).

The output can be given by (see, e.g., [7])

$$\hat{b}(\nu, t) = G(\nu, t)\hat{a}(\nu, t) + \sqrt{1 - |G(\nu, t)|^2} \hat{f}(\nu, t),$$

(5)

where, according to the domain considered, $\nu$ stands for $x$ or $\omega$, and $G(\nu, t)$, $|G(\nu, t)| \leq 1$, is the (complex) transmittance of the filter and $\hat{f}(\nu, t)$ is a bosonic noise operator. The photon number operator of the detected light is $\hat{n} = \int d\nu \hat{b}^\dagger(\nu, t)\hat{b}(\nu, t)$. Assuming square bandpass filters with $G_i(\nu, t) = 1$ if $|\nu - \Omega_i| \leq \Delta \Omega$ and $G_i(\nu, t) = 0$ otherwise, we consider the correlation coefficient
Violation of the Cauchy–Schwarz inequality

Photon number squeezing with optimized filter, -10log_{10} F

FIG. 2. The maximal violation of the Cauchy–Schwarz inequality for the photon number fluctuation [plots (a)–(d)] and the smallest Fano factor $F = \langle \Delta \hat{n}^2 \rangle / \langle \hat{n} \rangle$ (strongest photon number squeezing) achievable with optimized filters [plots (e)–(h)] are shown for the fundamental soliton, $N = 1$, (dotted line) and the soliton with $N = 2$ (full line) [x-domain: plots (a), (b), (e), (f), ω-domain: plots (c), (d), (g), (h); $\gamma = 0$: plots (a), (c), (e), (g), $\gamma t_d = 0.03$: plots (b), (d), (f), (h); other parameters as in Fig. 1].

$$\eta_{ij} = \frac{\langle \Delta \hat{n}_i \Delta \hat{n}_j \rangle}{\sqrt{\langle \Delta \hat{n}_i^2 \rangle \langle \Delta \hat{n}_j^2 \rangle}} = \frac{c_{ij}}{(c_{ii} + m_i)(c_{jj} + m_j)}$$

$m_i = \langle \hat{n}_i \rangle$, $c_{ij} = \langle \Delta \hat{n}_i \Delta \hat{n}_j \rangle$, $\Delta \hat{n}_i = \hat{n}_i - m_i$, where $\Delta n$ introduces normal ordering. It can be shown that $\eta_{ii} \leq 1$, and $|\eta_{ij}| \leq 1$ for nonoverlapping intervals. A negative sign of the coefficient $\eta_{ii}$ or a value smaller than unity of the Fano factor $F_i = \langle \Delta \hat{n}_i^2 \rangle / \langle \hat{n}_i \rangle = (1 - \eta_{ii})^{-1}$ indicates photon number squeezing of the filtered light.

From Fig. 1 it is seen that typical changes in the evolution of $\langle \hat{n}_i \rangle$ [Figs. 1(a), (c)] and those of $\eta_{ii}$ [Figs. 1(b), (d)] and $\eta_{ij}$ [Figs. 1(e1)–(e5), (f1)–(f5)] are closely related to each other. Near the points of soliton compression [maxima of $\langle \hat{n}_i \rangle$ in Fig. 1(a)] the formation of strong-correlation patterns is observed [Figs. 1(e2)–(e4), (f2), (f4)]. In contrast to the $x$-domain [Fig. 1(b)], sub-Poissonian statistics is observed in the $\omega$-domain [Fig. 1(d)]. Moreover, the correlation in the $\omega$-domain extends over a larger interval (relative to the corresponding initial pulse width) than the correlation in the $x$-domain. One possible explanation of such strong, almost perfect correlation ($\langle |\eta_{ij}| \rangle \rightarrow 1$) can be seen in the instability of the classical $N$-soliton solution. From a linearization approach [8], the internal noise of a quantum soliton should be associated with interferences between the soliton components and the continuum part of the solution to the classical nonlinear Schrödinger equation, as obtained by means of inverse scattering method (see, e.g., [10]). The qualitative changes observed for turning from the fundamental soliton to higher-order solitons ($N = 1 \rightarrow N = 2, 3, \ldots$) are due to the presence of more than one soliton component. Discrepancies between the parameters
(amplitude, group velocity, etc.) of the soliton components of the $N$-soliton solution play the central role in establishing very strong internal correlations.

Nonclassical correlation can be detected, e.g., by testing the Cauchy-Schwarz inequality for the normally ordered photon number variances. When it is violated, i.e.,

$$c_{ij}c_{jj} - c_{ij}^2 < 0,$$

then the photon number noise in the intervals $i$ and $j$ is nonclassically correlated. Figures 2(a)−(d) reveal that the nonclassical correlation of the 2-soliton is substantially stronger than that of the fundamental soliton even for an absorbing fiber. Such an increase cannot be explained by a simple intensity scaling. The effect is obviously related to the mentioned instability of higher-order solitons. It is remarkable that there exist propagation distances for which the nonclassical correlation is stronger for an absorbing fiber than a nonabsorbing one.

The strongest photon number squeezing (smallest Fano factor) achievable with an optimized broadband filter is illustrated in Figs. 2(e)−(h). Compared with the fundamental soliton, only a small increase of the effect is observed for the 2-soliton in the $\omega$-domain [Fig. 2(e), 6.6 → 8.4dB]. On the contrary, a rather strong increase of the effect can be observed in the $x$-domain [Fig. 2(g), 3.3 → 9.6dB], provided that losses can be disregarded. It is worth noting that the best photon number squeezing is achieved in the $\omega$-domain for the fundamental soliton and in $x$-domain for the 2-soliton. The results show that the degree of squeezing sensitively depends on the domain considered. Hence, replacing the Fourier transformation in Eq. (4) [including Eq. (5)] with more general transformation that relates the fields in the two domains, may offer possibilities of further optimization. In particular, when we restrict our attention to linear transformations which can be realized experimentally by passive linear optical elements, then we are left with a two-dimensional integral kernel function to be optimized. In this way we may hope that also for other nonlinear quantum objects a considerable improvement of nonclassical features can be achieved.

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