POISSON–LIE T–DUALITY AND SUPERSYMMETRY

Konstadinos Sfetsos

Institute for Theoretical Physics, Utrecht University
Princetonplein 5, TA 3508, The Netherlands

ABSTRACT

We review aspects of Poisson–Lie T–duality which we explicitly formulate as a canonical transformation on the world–sheet. Extensions of previous work on T–duality in relation to supersymmetry are also discussed.

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†e–mail address: sfetsos@fys.ruu.nl
‡Presently serving in the Greek Armed Forces.
1 Introduction

Duality symmetries in string theory reveal phenomena with no field theoretical analogs. As such they should be used in order to resolve long standing problems in fundamental Physics. Unlike the inherently non-perturbative S–duality and the dualities between apparently different string theories, T–duality \[1\] is rather well understood. Its interplay with supersymmetry served as a useful tool in discovering such phenomena, since in certain cases Abelian as well as non–Abelian T–duality are in conflict with supersymmetry from a field theoretical point of view \[2, 3\] but in complete harmony in a string theoretical setting \[4, 3\] (see also \[3, 3\]). T–duality as a supersymmetry breaking–restoration mechanism has been discussed in \[7\] in connection also with revised versions of some key theorems in 2-dim $\sigma$–models with extended supersymmetry \[8\]. All these should have implications, yet unexplored, for supersymmetry breaking scenarios in string phenomenology and they might hint the mechanism that resolves, in a string theoretical framework, the information loss paradox in black hole Physics.

In order to be able to compare with more general cases to be discussed in the rest of the paper we will review some results concerning world–sheet supersymmetry and T–duality leaving aside the case of target space supersymmetry where phenomena of similar origin also occur \[2, 3, 2, 3, 7\]. The case of $N = 1$ supersymmetry presents no problem from the point of view of its behaviour under T–duality since both dual backgrounds can be made $N = 1$ supersymmetric \[10, 5\]. Interesting situations arise only when we consider $N = 2$ and especially $N = 4$ extended supersymmetry. The conventional \[8\] definition of $N$–extended world–sheet supersymmetry implies that there exist $N – 1$ complex structures $F_{I}^{\pm}$, $I = 1, 2, \ldots, N – 1$ in the two chiral sectors separately which are covariantly constant. It is useful to assign the complex structures into representations of the isometry group $G$ with respect to which the duality is performed. Let’s denote by $\{v^{a}\}$ the set of vector fields generating the algebra for $G$ and by $\mathcal{L}_{v}$ the corresponding Lie–derivative. We concentrate on the case of $N = 4$ world–sheet supersymmetry and distinguish the following interesting cases \[4, 3\]:

**case (i):** The duality group is $G \simeq U(1)$ and all three complex structures are singlets. Then the corresponding Abelian T–duality preserves $N = 4$. Symbolically we write

\[
\left\{ \mathcal{L}_{v} F_{I}^{\pm} = 0 , \quad I = 1, 2, 3 \right\} \xRightarrow{\text{duality}} N = 4.
\]

**case (ii):** The duality group is $G \simeq U(1)$ but only one complex structure is a singlet, whereas the other two form a doublet. Then the Abelian T–duality preserves the part of the original $N = 4$ generated by the complex structure which is a singlet. Hence we have

\[
\left\{ \begin{array}{c}
\mathcal{L}_{v} F_{3}^{\pm} = 0 \\
\mathcal{L}_{v} F_{1}^{\pm} = F_{2}^{\pm} \\
\mathcal{L}_{v} F_{2}^{\pm} = - F_{1}^{\pm}
\end{array} \right\} \xRightarrow{\text{duality}} N = 2.
\]

The rest of supersymmetry is realized non–locally with corresponding complex structures that are non–local functionals of the dual target space variables. Hence, although super-
symmetry seems to be broken after T–duality from the low energy field theory point of view, it is nevertheless restored in a string theoretical framework.

Case (iii): The duality group is \( G \simeq SO(3) \) and all three complex structures are singlets. Then the corresponding non–Abelian T–duality preserves \( N = 4 \), namely
\[
\left\{ \mathcal{L}_{\psi^a} F^\pm_b = 0 , \quad a, b = 1, 2, 3 \right\} \underset{\text{duality}}{\Rightarrow} N = 4 .
\] (3)

Case (iv): The duality group is \( G \simeq SO(3) \) but the complex structures belong to the triplet representation of it. Then the non–Abelian T–duality breaks completely the extended supersymmetry from the low energy effective field theory point of view. Symbolically
\[
\left\{ \mathcal{L}_{\psi^a} F^\pm_b = \epsilon_{abc} F^\pm_c , \quad a, b, c = 1, 2, 3 \right\} \underset{\text{duality}}{\Rightarrow} N = 1 .
\] (4)

Needless to say that supersymmetry is restored in full at the string level and is realized with non–local complex structures. Prototype examples realizing all of the above cases have been explicitly worked out \([4, 3]\) and are hyper–kahler metrics with translational or rotational groups of isometries, which include the Eguchi–Hanson, the Taub–NUT and the Atiyah–Hitchin metrics.

In addition it has been shown \([1, 4, 3]\) that in some cases where the exact conformal field theory corresponding to the dual backgrounds is known, the non–local realizations of supersymmetry after duality are naturally represented using classical parafermions \([11]\). Hence, direct contact with corresponding realizations of the \( N = 4 \) superconformal algebra \([12]\) has been made. The most elementary example for both Abelian and non–Abelian duality is the background corresponding to the WZW model for \( SU(2) \otimes U(1) \).

Finally, let us mention that all theorems which have been proved in the past concerning extended world–sheet supersymmetry, implicitly assumed local realizations of it \([8]\) and they are not valid when it is non–locally realized. For instance, it is not always true that when the string torsion vanishes \( N = 4 \) extended supersymmetry implies that the manifold is Ricci flat. The interesting reader will find revised versions of some of these theorems, including the one we just mentioned, in \([4]\). Here we only mention that in general one can show that when the complex structures are non–local functionals of the target space variables they do not have to be covariantly constant in order to define an extended supersymmetry but they should obey instead \([7]\)
\[
D^\pm_\mu F^\pm_{\nu\rho} \partial_\mp X^\mu + \tilde{\partial}_\mp F^\pm_{\nu\rho} = 0 ,
\] (5)
where the tilded world–sheet derivative acts only on the non–local part of the complex structure.

In order to further improve our picture it remains to examine in this context the so called Poisson–Lie T–duality \([13]\), which generalizes the concept of T–duality as it can be performed even in the absence of isometries. Hence, it is an advancement both conceptually as well as in practice since isometries are not always preserved under T–duality anyway. In this paper we address issues involved in the interplay between Poisson–Lie T–duality and supersymmetry. In particular, we find the modified version of \([1]–[3]\) and remark on the potential physical applications.
The organization of this paper is as follows: In section 2 we review aspects of Poisson–Lie T–duality [13, 14] which are directly relevant in our context and in addition we explicitly formulate it as a canonical transformation on the world–sheet. In this respect we put Poisson–Lie T–duality in equal footing with T–duality in the presence of isometries as the latter has already been given an analogous formulation [15, 16, 3]. In section 3 we start examining supersymmetric Poisson–Lie T–duality by presenting its on shell formulation in $\sigma$–models with $N = 1$ as well as extended world–sheet supersymmetry. We end the paper in section 4 with concluding remarks and discussion on the future directions of this work as well as on its relevance in resolving some fundamental problems in Physics.

## 2 Poisson–Lie T–duality

We consider classical closed string propagation in $d$-dimensional backgrounds. Some of the target space variables are chosen to parametrize an element $g$ of a group $G$ (with algebra $\mathcal{G}$) and will be denoted by $X^\mu$, $\mu = 1, 2, \ldots, \text{dim}(G)$, whereas the rest are the so called spectators and will be denoted by $Y^i$, $i = 1, 2, \ldots, d - \text{dim}(G)$. We also introduce representation matrices $\{T_a\}$, with $a = 1, 2, \ldots, \text{dim}(G)$ and the components of the left–invariant Maurer–Cartan forms $L_a^\pm = L_a^\mu \partial_{\pm} X^\mu$. The inverse of $L_a^\mu$ will be denoted by $L_\mu a$ and for notational convenience we will also use $L_i^\pm \equiv \partial_{\pm} Y^i$. Then the most general $\sigma$–model action is given by (the light–cone coordinates on the world–sheet are $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$)

$$S = \frac{1}{2} \int L_A^A E_{AB} L_B^B$$

$$= \frac{1}{2} \int L^A_{\pm} E^{+}_{AB} L^B_{\pm} + \partial_+ Y^i \Phi^+_{ia} L^a_{\pm} + L^a_+ \Phi^+_{ai} \partial_- Y^i + \partial_- Y^i \Phi_{ij} \partial_- Y^j ,$$ \hspace{1cm} (6)

where the index $A = \{a, i\}$. The couplings $E^{+}_{ab}$, $\Phi^+_{ia}$, $\Phi^+_{ai}$ (for later use we also define $E^{-}_{ab} = E^+_{ba}$, $\Phi^-_{ia} = \Phi^+_{ia}$ and $\Phi^-_{ai} = \Phi^+_{ia}$) and $\Phi_{ij}$ may depend on all variables $X^\mu$ and $Y^i$. Hence, we do not require any isometry associated with the group $G$.

Another $\sigma$–model (denoted as usual with tilded symbols) is said to be dual to (6 ) in the sense of Poisson–Lie T–duality [13] if the algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ form a pair of maximally isotropic subalgebras the Lie algebra $\mathcal{D}$ of a Lie group $D$ known as the Drinfeld double can be decomposed to. Leaving aside the mathematical details (see for instance [17]), this implies the non–trivial commutator

$$[T_a, \tilde{T}^b] = if^{bc}_a T_c - i f_{ac}^b \tilde{T}^c .$$ \hspace{1cm} (7)

The Jacobi identity $[[T_a, T_b], \tilde{T}^c] + \text{cyclic} = 0$ relates the structure constants of the two algebras as [17, 13]

$$f_{ab}^d \tilde{f}^{dc}_a + f_{d[a}^e \tilde{f}^{de} b] - f_{d[a}^e \tilde{f}^{dc} b] = 0 .$$ \hspace{1cm} (8)

This restricts severely the candidate algebras to form Drinfeld doubles. Besides the trivial solution to (8), when the group $G$ or $\tilde{G}$ is Abelian, other non–trivial solutions exist and
typically involve the decomposition of certain semi–simple Lie groups into factors that contain Borel subgroups \([17]\). In addition to (7), there is a bilinear invariant \(\langle \cdot|\cdot \rangle\)

\[
\langle T_a|T_b \rangle = \langle \tilde{T}^a|\tilde{T}^b \rangle = 0, \quad \langle T_a|\tilde{T}^b \rangle = \delta_a^b.
\]

We also define matrices \(a(g), b(g)\) and \(\Pi(g)\) as

\[
g^{-1}T_ag = a^b T_b, \quad g^{-1}\tilde{T}^a g = b_{ab} T_b + (a^{-1})^a_b \tilde{T}^b, \quad \Pi^{ab} = b^a a^b_c.
\]

Consistency then requires that

\[
a(g^{-1}) = a^{-1}(g), \quad b^T(g) = b(g^{-1}), \quad \Pi^T(g) = -\Pi(g).
\]

Then the various couplings in the \(\sigma\)–model action (6) are restricted to be of the form \([13, 18]\) (although occasionally suppressed in the rest of the paper, the index structure should be clear)

\[
\Phi^\pm = E^\pm (E_0^\pm)^{-1} F^\pm, \quad E^\pm = \left((E_0^\pm)^{-1} \pm \Pi\right)^{-1},
\]

\[
\Phi = F - F^+ \Pi E^+ (E_0^+)^{-1} F^+ ,
\]

where the new couplings \(F^+_ia = F^i_a\), \(F^+_ai = F^i_a\) and \((E^+_i)_{ab} = (E^-_0)_{ba}\) maybe functions of the spectator variables \(Y^i\) only. The couplings of the dual action are also determined in a similar fashion \([13, 18]\)

\[
\tilde{\Phi}^\pm = \pm \tilde{E}^\pm F^\pm, \quad \tilde{E}^\pm = \left(\tilde{E}_0^\pm \pm \tilde{\Pi}\right)^{-1},
\]

\[
\tilde{\Phi} = F - F^+ \tilde{E} F^+ .
\]

The traditional non–Abelian duality is recoved if \(\tilde{G}\) (equivalently \(G\)) is Abelian. Then it is easy to see using (10) that

\[
b_{ab} = \Pi^{ab} = 0, \quad \tilde{a}^a_b = \delta_b^a, \quad \tilde{b}_{ab} = f_{ab}^c \chi_c = -\tilde{\Pi}_{ab}, \quad (\tilde{L}_\pm)_a = \partial_\pm \chi_a ,
\]

where the \(\chi_a\)'s parametrize the group element \(\tilde{g} = e^{i\chi_a \tilde{T}^a} \in \tilde{G}\). Then action (8) becomes invariant under left transformations of \(g \in G\), i.e. there is an isometry and the dual action reduces to the one computed in \([19]\).

**Poisson–Lie T–duality as a canonical transformation**

Poisson–Lie T–duality is by definition a canonical transformation and an expression for the generating functional has been given \([14]\). Here we formulate Poisson–Lie T–duality as an explicit transformation between the canonical variables of the two dual \(\sigma\)–models. This was done for the case of Abelian duality in \([15]\), for non–Abelian duality on Principal Chiral models in \([16]\) and for the general \(\sigma\)–model \([4]\) when it is invariant under the left (or right) action of the group \(G\), i.e. when (14) holds, in \([3]\). It is instructive to see how this
can be used to make an educated guess for the corresponding canonical transformation for Poisson–Lie T–duality. We recall that when (6) is invariant under the left action of the group $G$ the transformation between the canonical pairs of variables $(L^a_\sigma, P_a)$ and $(\partial_\sigma \chi_a, \tilde{P}^b)$ is given by

\begin{align}
L^a_\sigma &= \tilde{P}^a, \\
P_a &= - f_{ab}^{\quad c} \chi_c \tilde{P}^b + \partial_\sigma \chi_a.
\end{align}

(15) (16)

where $L^a_\sigma \equiv L^a_\mu \partial_\sigma X^\mu$, $P_a \equiv L^a_\mu P_\mu$ and similarly for $(\tilde{L}_\sigma)_a$ and $\tilde{P}^a$. Rewriting (16) with the help of (14) suggests the following ansatz for a canonical transformation corresponding to Poisson–Lie T–duality

\begin{align}
L^a_\sigma &= (\delta^a_b - \Pi^{ac} \tilde{\Pi}^{cb}) \tilde{P}^b - \Pi^{ab} (\tilde{L}_\sigma)_b, \\
P_a &= \tilde{\Pi}^{ab} \tilde{P}^b + (\tilde{L}_\sigma)_a.
\end{align}

(17) (18)

This ansatz is duality invariant since it implies a transformation similar to (18) with tilded and untilded symbols exchanged\footnote{It is also manifest when we cast (17), (18) into the form

\begin{align}
P &= (I - \tilde{\Pi})^{-1}(\tilde{L}_\sigma + \tilde{\Pi}L_\sigma), \\
\tilde{P} &= (I - \Pi \tilde{\Pi})^{-1}(L_\sigma + \Pi \tilde{L}_\sigma).
\end{align}

(19)} It is important to check whether or not (17), (18) indeed constitute a canonical transformation since there are obviously infinitely many transformations that reduce to (15), (16) when (14) holds. Notice that the right hand side of (17) depends on tilded as well as untilded variables. Hence, it is not immediately obvious that the $\sigma$–model arising after such a transformation will be well defined, i.e. local in the dual target space variables. For simplicity we will not consider spectator fields, since in any case they enter the canonical transformation implicitly via the definitions of $P_a$ and $\tilde{P}^a$. Then, the Hamiltonian associated with the action (6) is (we omit writing the $\sigma$–integration)

\[H = \frac{1}{2}(P_a - B_{ac} L^c_\sigma) G^{ab} (P_b - B_{bd} L^d_\sigma) + \frac{1}{2} L^a_\sigma G_{ab} L^b_\sigma,\]

(20)

where $G_{ab}$ and $B_{ab}$ are the symmetric and antisymmetric parts of $E^+_a$

\begin{align}
G &= (I + E^+_0 \Pi)^{-1} G_0 (I - \Pi E^-_0)^{-1}, \\
B &= (I + E^+_0 \Pi)^{-1} (B_0 - E^+_0 \Pi E^-_0) (I - \Pi E^-_0)^{-1}.
\end{align}

(21)

It is a straightforward, although a technically laborious, computation to verify that the transformation (17), (18) gives the Hamiltonian corresponding to the dual to (6) action. Hence, it is defined entirely in the dual model and potential non–localities associated with the particular form of (17), have not arisen. It also turns out that (17), (18) are valid even when we turn on the spectator fields which are then left invariant under
the canonical transformation. In principle in order for (17), (18) to be a canonical transformation they have, in addition to generating the correct Hamiltonian for the dual model, to preserve the canonical Poisson brackets for the conjugate pair of variables \{L^a_\mu, P_a\} (see for instance [3]). This is equivalent to finding a generating functional for the transformation. As already mentioned an implicit expression for such a generating functional has been given in [14]. It is important to check whether or not it indeed reproduces our explicit transformation (17), (18).

Poisson–Lie T–duality at the Drinfeld double

Next we present a manifest formulation of Poisson–Lie T–duality directly at the level of the Drinfeld double. We will basically follow [14] but we will also include the spectator fields in our discussion explicitly. At the end of this subsection we will comment on an alternative formulation [18].

We introduce a basis of vectors \( R^\pm_a \) satisfying
\[
\langle R^\pm_a | R^\pm_b \rangle = \pm \eta_{ab} , \quad \langle R^+_a | R^-_b \rangle = 0 ,
\] (22)
and the completeness relation
\[
| R^+_a \rangle \eta^{ab} \langle R^+_b | - | R^-_a \rangle \eta^{ab} \langle R^-_b | = I ,
\] (23)
where \( \eta_{ab} \) and \( \eta^{ab} \) are some metric and its inverse. We also introduce an operator \( R \) as
\[
R = | R^+_a \rangle \eta^{ab} \langle R^+_b | + | R^-_a \rangle \eta^{ab} \langle R^-_b | .
\] (24)
A useful representation in terms of \( T_a \) and \( \tilde{T}^a \) is
\[
R^\pm_a = T_a \pm (E^\pm_0)_{ab} \tilde{T}^b , \quad \eta_{ab} = (E^+_0)_{ab} + (E^-_0)_{ab} .
\] (25)

Then consider an action defined in the Drinfeld double as
\[
S = I_0(l) + \frac{1}{2\pi} \int \langle l^{-1} \partial_\sigma l | R | l^{-1} \partial_\sigma l \rangle \\
-2i \partial_+ Y F^+ \eta^{-1} \langle l^{-1} \partial_\sigma l | R^- \rangle - 2i \langle l^{-1} \partial_\sigma l | R^+ \rangle \eta^{-1} F^+ \partial_- Y \\
+ \partial_+ Y F \partial_- Y - \frac{1}{2}(\partial_+ Y F^+ + \partial_- Y F^-) \eta^{-1}(F^- \partial_+ Y + F^+ \partial_- Y) .
\] (26)
The first line is the action introduced in [14], where \( I_0(l) \) is the WZW action for a group element \( l \in D \) with the world–sheet variables \( \sigma \) and \( \tau \) playing the role of light–cone variables. The second line contains natural coupling terms between the spectator fields and the 1–form in the double \( l^{-1} \partial_\sigma l \). The third line depends purely on spectator fields. With the help of the Polyakov–Wiegman formula is easily seen that (26) is invariant under \( l(\tau, \sigma) \mapsto l_0(\tau)l(\tau, \sigma) \) for some \( \tau \)–dependent element \( l_0 \in D \). This local invariance can be used to cast the equations of motion for (26) with respect to variations of \( l \) into the form
\[
\langle l^{-1} \partial_\pm l | R^\pm_a \rangle = \pm i \partial_\pm Y F^\pm_\alpha .
\] (27)
In the vicinity of the unit element of \( D \) we may decompose \([7]\) the group element \( l \in D \) as \( l = \tilde{h}g \) or as \( l = h\tilde{g} \), where \( h, g \in G \) and \( \tilde{h}, \tilde{g} \in \hat{G} \). Using the first decomposition we may solve for the “gauge field” \( A_\pm = \tilde{h}^{-1}\partial_\pm \tilde{h} \in \hat{G} \) with result

\[
A_\pm = \left( \pm i\partial Y F^+ - \langle g^{-1}\partial g|R^+ \rangle \right) M_\pm^{-1}, \quad M_\pm = \langle g^{-1}\tilde{T}g|R^\pm \rangle
\]

\[
= \pm iaE^\mp \left( L_\pm + (E_0^\mp)^{-1}F^\mp \partial_\pm Y \right),
\]

(28)

where in order to write the second line we have used (25). Using the second decomposition it is clear that we may solve for \( \tilde{A}_\pm = h^{-1}\partial_\pm h \in \hat{G} \) with result

\[
\tilde{A}_\pm = \pm ia\tilde{E}^\mp \left( \tilde{L}_\pm \mp F^\mp \partial_\pm Y \right).
\]

(29)

We also note that the vanishing of the curvature associated with \( A_\pm \) or \( \tilde{A}_\pm \) boils down to condition (8) on the structure constants of the bialgebra \( D \). In order to recover (3) with couplings (12) we insert the decomposition \( l = \tilde{h}g \) into (26) and we use the Polyakov–Wiegman formula and (3). It turns out that the action depends quadratically on \( A_\sigma = \frac{1}{2}(A_+ - A_-) \) but not on \( A_\tau = \frac{1}{2}(A_+ + A_-) \). Hence, we may replace \( A_\sigma \) by its on shell value using (28). The result is action (3) with couplings (12). Needless to say that if we use the second decomposition \( l = h\tilde{g} \) then we obtain the dual \( \sigma \)-model action with couplings (13).

It is instructive to see the meaning of the canonical transformation (18) in this formalism. We evaluate \( \langle l^{-1}\partial_\pm l|\tilde{T}^a \rangle \) in the alternative ways suggested by the two possible parametrizations of \( l \in D \)

\[
\langle l^{-1}\partial_\pm l|\tilde{T}^a \rangle|_{A^\pm}^{l=\tilde{h}g} = i(E_0^-)^{-1}E^\mp(L_\pm \pm \Pi F^\mp \partial_\pm Y),
\]

(30)

\[
\langle l^{-1}\partial_\pm l|\tilde{T}^a \rangle|_{\tilde{A}^\pm}^{l=h\tilde{g}} = \pm i\tilde{E}^\pm(\tilde{L}_\pm \mp F^\mp \partial_\pm Y),
\]

(31)

where as indicated, we have accordingly substituted for \( A_\pm \) or \( \tilde{A}_\pm \). Then let us consider the transformation between variables of the two dual models defined as

\[
\langle l^{-1}\partial_\pm l|\tilde{T}^a \rangle|_{A^\pm}^{l=\tilde{h}g} = \langle l^{-1}\partial_\pm l|\tilde{T}^a \rangle|_{\tilde{A}^\pm}^{l=h\tilde{g}}.
\]

(32)

It is a straightforward computation to show that this implies the canonical transformation (18), (17). Vice versa, it can be shown that similarly to the cases of Abelian and non–Abelian duality \([20, 3]\), the transformations (17), (18) and the requirement for 2–dim Lorentz invariance on the world–sheet implies (32) as well as (12) and (13).

At this point we shall very briefly mention an alternative formulation of Poisson–Lie T–duality in the Drinfeld double \([18]\). In this construction the WZW action \( I_0(l) \) is

\[2\] Using (25) and the equations of motion (27) we may cast (32) into the equivalent form

\[
\langle l^{-1}\partial_\pm l|R^a_\pm \rangle|_{A^\pm}^{l=\tilde{h}g} = \langle l^{-1}\partial_\pm l|R^a_\pm \rangle|_{\tilde{A}^\pm}^{l=h\tilde{g}}.
\]

(33)
also involved but in contrast to (26) it is defined with the usual light–cone variables $\sigma^\pm$.
Also in contrast to (26) the entire action of [18] is manifestly 2–dim Lorentz invariant. However, in order to recover the dual $\sigma$–model actions one of the equations in (27) is imposed as a constraint. A clarification of the relation between the two constructions is an open problem and it might help to find the off shell formulation of supersymmetric Poisson–Lie T–duality as we shall soon argue.

3 Supersymmetric Poisson–Lie T–duality

Extending our previous discussion to $N = 1$ supersymmetric models is not entirely trivial. One approach is to start with the supersymmetric versions of the bosonic equations of motion (27). They are given by

$$\langle L^{-1} D_\pm L | R^\pm_a \rangle = \pm i D_\pm Z^i F^\pm_{ia} \quad. \quad (34)$$

where $L$ and $Z^i$ are superfields corresponding to the bosonic group element $l \in D$ and the spectator field $Y^i$ and of course $F^\pm_{ia}$ depends on the $Z^i$’s. The superfield $L$ and its inverse have expansions in terms of anticommuting Grassman variables $\theta_{\pm}$ given by (see for instance [21])

$$L = l - i \theta_+ \chi_- l + i \theta_- l \chi_+ - i \theta_+ \theta_- f \quad, \quad L^{-1} = l^{-1} + i \theta_+ l^{-1} \chi_- - i \theta_- l^{-1} \chi_+ - i \theta_+ \theta_- f^+ \quad, \quad$$

$$f^+ = -l^{-1} ft^{-1} + i(\chi_+ l^{-1} \chi_- - l^{-1} \chi_- \chi_+ l^{-1}) \quad. \quad (35)$$

where $\chi_{\pm} \in D$ are world–sheet fermions and $f$ is the highest component of the superfield. Also the world–sheet superderivatives are defined as $D_\pm = \mp i \partial_{\theta_{\pm}} \mp \theta_{\mp} \partial_{\pm}$. For the superfield $Z^i$ an analogous expansion holds

$$Z^i = Y^i - i \theta_+ \Psi^i_- + i \theta_- \Psi^i_+ - i \theta_+ \theta_- F^i \quad. \quad (36)$$

Similarly to the bosonic case, we may use two alternative decompositions $L = \tilde{H}G$ or $L = H\tilde{G}$, where the superfields $H, G \in G$ and $\tilde{H}, \tilde{G} \in \tilde{G}$ have a similar to (35) expansion in terms of Grassman variables. Then after solving for the corresponding “gauge fields” it is obvious that the $N = 1$ dual supersymmetric $\sigma$–models for the superfields $(G, Z^i)$ and $(\tilde{G}, \tilde{Z}^i)$ are given by the straightforward $N = 1$ supersymmetrized bosonic actions, namely by replacing the various fields and derivatives by their supersymmetric counterparts.

We have just presented an on shell formulation of supersymmetric Poisson–Lie T–duality based on equations (34). However, an off shell formulation is not as straightforward as one might expect since it requires an $N = 1$ supersymmetric action with a local invariance similar to the one for the bosonic action (26) which then can be used to cast the corresponding equations of motion into the form (34). Although the supersymmetrization of the WZW action $I_0(l)$ in (26) is known [21], there are certain technical difficulties
associated with the other terms. A possible way around this problem could be to use the alternative formulation \[18\] of Poisson–Lie T–duality in the Drinfeld double we have mentioned. Since it is manifestly 2–dim Lorentz invariant the supersymmetrization of the action and the constraint causes no problem. Hence, finding the precise relationship between the two formulations of Poisson–Lie T–duality in \[18\] and \[14\] seems important also in a supersymmetric context. We hope to report progress along these lines in the future \[22\].

Next we turn into the behaviour of extended \(N \geq 2\) supersymmetry under Poisson–Lie T–duality. A complete discussion at the level of (super)Drinfeld double requires knowledge of the \(N = 1\) supersymmetric version of the bosonic action (26) which as mentioned we lack. Nevertheless, we may still discuss the situation at the level of the \(\sigma\–models whose bosonic parts are given by (3) with couplings (12) and by its dual action with couplings (13). Hence, we are looking for (generally non–local) complex structures satisfying (5). Quite generally they have the form

\[
F^\pm = C^\pm_{AB}(Y) f^A_\pm \wedge f^B_\pm = C^\pm_{ab} f^a_\pm \wedge f^b_\pm + 2C^\pm_{ia} \partial_\pm Y^i \wedge f^a_\pm + C^\pm_{ij} \partial_\pm Y^i \wedge \partial_\pm Y^j,
\]

(37)

where \(f^a_\pm\) and \(f^a_-\) are some \((1,0)\) and \((0,1)\) forms on the world–sheet. A similar to (37) expression holds for \(\tilde{F}^\pm\) as well. Various interesting cases are:

**case (i):** The \(f^a_\pm\)’s have the form

\[
f^a_\pm = \langle l^{-1} \partial_\pm | \tilde{T}^a \rangle |_{A_\pm},
\]

(38)

where the right hand side is given explicitly by (30). In the dual model \(\tilde{f}^a_\pm\) is given instead by (31). Hence, in this case extended supersymmetry is local in both dual models even though the dual complex structures are not invariant under left(and right) transformations of \(g \in G\).

**case (ii):** The \(f^a_\pm\)’s are simply given by

\[
f^a_\pm = L^a_\pm.
\]

(39)

In the dual model \(L^a_\pm\) is expressed using (30)–(32) as

\[
L^A_\pm = Q^A_{\pm B} \tilde{L}^B_\pm,
\]

(40)

where

\[
Q_\pm = \begin{pmatrix}
\pm(E^\mp)^{-1} E^\mp \tilde{E}^\mp - (I - \Pi \tilde{\Pi}) E^\mp F^\mp \\
0 & I
\end{pmatrix}.
\]

(41)

In order to find the precise mapping under Poisson–Lie T–duality one has to solve (40) for \(g \in G\). We haven’t been able to do so but it is certain that it will be a non–local functional of the dual model target space variables. Hence, the corresponding extended
supersymmetry will be non–locally realized in the dual model, even though the original complex structure \((37), (39)\) was invariant under the left action of the group \(G\). The non–local complex structure will be of the form \((37)\) but with \(\tilde{C}^\pm_{AB}\) given by

\[
\tilde{C}^\pm_{AB} = Q^C_{\pm A} Q^D_{\pm B} C^\pm_{CD} .
\] (42)

case (iii): In this case we consider simultaneously more than one complex structures of the particular form

\[
(F^\pm) = \langle l^{-1} \tilde{T} a l | T b \rangle |_{A^\pm}^{l^{-1} h} (F^\pm_0)^b ,
\] (43)

where \((F^\pm_0)\)'s are 2–forms similar to \((37)\) with \(f^a_\pm\)'s given as in case (i) by \((38)\). We compute

\[
\langle l^{-1} \tilde{T} l | T \rangle |_{A^\pm}^{l^{-1} h} = \tilde{a}(h)(a^T(g))^{-1} , \quad \tilde{h} = P e^{\int^\sigma A_x} ,
\] (44)

where \(P\) stands for path ordering and \(A_x \equiv \frac{1}{2}(A_+ - A_-)\) is found using \((28)\). In the dual decomposition \(l = h\tilde{g}\), the 2–form \((F^\pm_0)^a\) remains local in the dual space target space variables as well, but analogously to \((44)\) we have

\[
\langle l^{-1} \tilde{T} l | T \rangle |_{A^\pm}^{l^{-1} h} = b(h)\tilde{b}(\tilde{g}) + (a^T(h))^{-1}\tilde{a}(\tilde{g}) , \quad h = P e^{\int^\sigma \tilde{A}_x} .
\] (45)

Hence, in both models extended supersymmetry is realized non–locally and this is a novel characteristic of Poisson–Lie T–duality. A similar conclusion is reached in a variation of case (iii), namely when \(f^a_\pm\) is given by \((39)\) instead of \((38)\).

4 Discussion and concluding remarks

We have reviewed aspects of Poisson–Lie T–duality and explicitly formulated it as a canonical transformation on the string world–sheet. We gave an on–shell formulation of supersymmetric \(\mathcal{N} = 1\) Poisson–Lie T–duality at the level of the two dual \(\sigma\)–models and examined the behaviour of extended world–sheet supersymmetry under Poisson–Lie T–duality. An open interesting problem is to find the corresponding off–shell formulation fully at the level of the Drinfeld double.

This work is part of a program whose aim is to use duality symmetries in order to attack problems in fundamental Physics from a string theoretical point of view. An outstanding such problem is the information loss paradox in black hole Physics via Hawking radiation. A preliminary exploratory step in using Poisson–Lie T–duality in this direction would be to search for non–trivial backgrounds, in the sense of \((8)\), that are dual to black holes. In addition, discovering unconventional non–locally realized supersymmetry in black holes or equivalently their duals might stabilize them against Hawking radiation similarly to the case of extremally charged black holes whose stability may be attributed to their supersymmetric properties. We hope to be able to report work in this direction in the future.
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