NRQED APPROACH TO THE HYPERFINE STRUCTURE OF THE MUONIUM GROUND STATE

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ABSTRACT

The method of NRQED is an adaptation of QED to bound systems. While it is fully equivalent to QED, it enables us to explore the QED bound states systematically as an expansion in the fine structure constant $\alpha$ and velocity ($\sim Z\alpha$) of the bound electron. I describe how to construct the NRQED Hamiltonian choosing recent works on the $\alpha(Z\alpha)$, $\alpha^2(Z\alpha)$, $\alpha(Z\alpha)^2$, $\alpha(Z\alpha)^3$, and $\alpha^2(Z\alpha)^2$ radiative corrections to the hyperfine structure of the muonium ground state as examples.

1. Introduction

The most widely used approach to the relativistic bound state problem is that based on the Bethe-Salpeter equation. However, this tends to become very cumbersome when applied to higher-order radiative corrections, while the result reflects mostly the non-relativistic features of the system, which are well described by the Schrödinger equation. The problem is how to relate the latter to the relativistic quantum field theory. In the case of the electromagnetic interaction, a solution was given by the theory called non-relativistic quantum electrodynamics, or NRQED.

In this talk, I should like to describe the main features of NRQED choosing as an example our recent work on the radiative corrections to the hyperfine structure of the muonium ground state. This quantity, which is one of the most precisely measured quantities, is particularly suitable for a detailed examination of NRQED, being relatively free from the effects of hadronic interaction. The latest measurement gives

$$\Delta\nu(\text{exp}) = 4463302.776 \text{ (51 kHz) (11 ppb)},$$

which includes the earlier result and reduces its uncertainty by a factor of three.

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2. The NRQED method

The NRQED is a rigorous and systematic adaptation of QED to bound states. It is described by a Hamiltonian derived from QED by expanding it in the velocity of the bound electron. The equivalence of NRQED and QED is guaranteed by requiring that the NRQED scattering amplitude coincides with the QED scattering amplitude at some chosen momentum scale, e.g., at the threshold of the external on-shell particles. The coupling constant $\alpha$ and mass, renormalized on-shell, are identified with the observed values and used as the input coupling constant and mass of NRQED.

The NRQED Hamiltonian consists of all possible local interactions satisfying the required symmetries, such as Galileian invariance, gauge invariance, parity invariance, time reversal invariance, hermiticity, and locality. In the absence of dynamical photons it is nothing but the Foldy-Wouthuysen-Tani (FWT) transform of the Dirac Hamiltonian. Since the photon is always relativistic, its Lagrangian is the same as that of QED. When the interaction with photons is taken into account, the NRQED Hamiltonian breaks up into two parts: $H^\Lambda_{\text{main}}$ and $H^\Lambda_{\text{contact}}$. $H^\Lambda_{\text{main}}$ is bilinear in fermion operators as well as photon operators. In addition, new photon interaction terms, such as vacuum polarization and light-by-light scattering, are introduced to represent insertion of the fermion loop. $H^\Lambda_{\text{contact}}$ represents the remainder.

The crucial feature of NRQED is that all operators (terms of the Hamiltonian) are restricted to small momentum transfer, defined by some cut-off $\Lambda$, while effects of large momentum transfer are represented by coefficients of these operators and terms of $H^\Lambda_{\text{contact}}$. Since this theory is meant to apply to non-relativistic systems, the cut-off $\Lambda$ may be chosen as typical mass scale of the system, e.g., the rest mass of the electron.

For the calculation of non-recoil terms in which we can take the limit $m/M \to 0$, $H^\Lambda_{\text{main}}$, relevant to the muonium calculation can be written, to order $\alpha$, as

$$
H^\Lambda_{\text{main}} = \psi^\dagger(p') \left[ \frac{\bar{p}^2}{2m} + eA^0 - \frac{(\bar{p}^2)^2}{8m^3} - \frac{e}{2m}(\bar{p}^2 + \bar{p}) \cdot \bar{A} \right] 
+ \frac{e^2}{2m} \bar{A} \cdot \bar{A} - \frac{ie}{2m} c_F \bar{\sigma} \cdot (\bar{q} \times \bar{A}) - \frac{e}{8m^2} c_D \bar{q}^2 A^0 + \frac{ie}{4m^2} c_S \bar{\sigma} \cdot (\bar{p}' \times \bar{p}) A^0 
- \frac{ie^2}{4m^2} c_S \bar{\sigma} \cdot (\bar{q}_1 \times \bar{A}(q_1)) A^0(q_2) + \frac{ie}{8m^2} c_S q^0 \bar{\sigma} \cdot (\bar{p}' + \bar{p}) \bar{A} 
+ \frac{ie}{8m^3} c_W(\bar{p}'^2 + \bar{p}^2) \bar{\sigma} \cdot (\bar{q} \times \bar{A}) + \frac{ie}{8m^3} c_F \bar{q}^2 \bar{\sigma} \cdot (\bar{q} \times \bar{A}) 
+ \frac{ie}{8m^3} c_F \{ \bar{p} \cdot (\bar{q} \times \bar{A})(\bar{\sigma} \cdot \bar{p}') + \bar{p}' \cdot (\bar{q} \times \bar{A})(\bar{\sigma} \cdot \bar{p}) \} \ldots \right] \psi(p) 
+ c_v A^j(q) \frac{\bar{q}^4}{m^2} A^j(q)(\bar{q}^j - \frac{q^j q^j}{q^2}) + c_v A^0(q) \frac{-\bar{q}^4}{m^2} A^0(q), \tag{2}
$$

where $\psi$ is the Pauli two-component spinor, $\bar{p}'$ and $\bar{p}$ are the outgoing and incoming fermion momenta, respectively, and $q = (q^0, \bar{q})$ is the incoming photon momentum. In the seagull vertex (on the third line), $\bar{q}_1$ is the incoming momentum of the vector potential $\bar{A}$. The large momentum-transfer effect is represented by the NRQED “renormalization” coefficients which, to order $\alpha$, are found to be

$$
c_F = 1 + a_e ,
$$
\[ c_D = 1 + \frac{\alpha}{\pi} \frac{8}{3} \left[ \ln \left( \frac{m}{2\Lambda} \right) - \frac{3}{8} + \frac{5}{6} \right] + 2a_e, \]
\[ c_S = 1 + 2a_e, \]
\[ c_W = 1, \]
\[ c_{q^2} = \frac{\alpha}{\pi} \frac{4}{3} \left[ \ln \left( \frac{m}{2\Lambda} \right) - \frac{3}{8} + \frac{5}{6} + \frac{1}{4} \right] + \frac{a_e}{2}, \]
\[ c_{p_p} = a_e, \]
\[ c_{c_p} = \frac{\alpha}{15\pi}. \quad (3) \]

Note that \( c_i \)'s do not have coefficients involving \( Z\alpha \) caused by the binding effect because they are determined solely by comparison of the NRQED and QED scattering amplitudes without referring to bound states.

\( H^\Lambda_{\text{main}} \) alone cannot reproduce the amplitudes of QED. To make NRQED fully equivalent to QED, we must add terms of contact interaction type:

\[ H^\Lambda_{\text{contact}} = -d_1 \frac{1}{mM}(\psi^\dagger \vec{\sigma}\psi) \cdot (\chi^\dagger \vec{\sigma}\chi) - d_2 \frac{1}{mM}(\psi^\dagger \psi)(\chi^\dagger \chi) \]
\[ -d_3 \frac{1}{mM}(\psi^\dagger \chi)(\chi^\dagger \psi) - d_4 \frac{1}{mM}(\psi^\dagger \sigma^2 \psi) \cdot (\chi^\dagger \sigma \chi) - \cdots, \quad (4) \]

where \( \chi \) represents the muon (of mass \( M \)). The coefficients \( d_i \) are determined completely by the requirement that the contact terms make up the difference between the QED electron-muon scattering amplitude and the corresponding NRQED scattering amplitude generated by \( H^\Lambda_{\text{main}} \) and lower order \( H^\Lambda_{\text{contact}} \). Several terms of \( H^\Lambda_{\text{contact}} \) have been determined:

\[ d_1 = \alpha(Z\alpha)^2 \frac{2}{3} \left( \ln 2 - \frac{13}{4} + \frac{3}{4} \right) + \alpha^2(Z\alpha)^2 \frac{2}{3}(0.7679(79)) + \cdots, \]
\[ d_2 = \alpha(Z\alpha)^2 \frac{4}{3} \left( \frac{139}{128} + \frac{5}{192} - \frac{1}{2} \ln 2 \right) + \cdots, \]
\[ d_3 = d_4 = 0. \quad (5) \]

The \( Z\alpha \) factor in (5) comes from exchange of photons between the electron and the muon (of charge \( -Ze \)) and has nothing to do with the bound-state effect.

Once the NRQED Hamiltonian is constructed the corrections to the energy and wave function can be determined by the usual bound state perturbation theory.

### 3. Outline of non-recoil term evaluation

Thus far, non-recoil terms of order \( \alpha(Z\alpha), \alpha(Z\alpha)^2, \) and \( \alpha^2(Z\alpha) \) have been evaluated explicitly in NRQED. They agree with evaluation by other methods. Evaluation of \( \alpha(Z\alpha)^3 \) and \( \alpha^2(Z\alpha)^2 \) terms within the framework of NRQED is in progress. Instead of discussing these calculations in detail, I shall focus on their qualitative features to illustrate how the NRQED method works. For simplicity let us discuss here only diagrams with radiative photons, leaving out the vacuum-polarization contribution.

In general, radiative corrections of order \( \alpha^n(Z\alpha)^n \) to the hyperfine splitting are of two types: (A) First-order perturbation of contact terms of order \( \alpha^n(Z\alpha)^{n+1} \). (B) Higher-order perturbation of lower-order terms of \( H^\Lambda_{\text{main}} \) and \( H^\Lambda_{\text{contact}} \). The type (A) originates
from a set of QED scattering diagrams in which \( n + 1 \) photons are exchanged between the electron and the muon and the electron line is dressed by \( m \) virtual photon lines.

Since operators of the NRQED Hamiltonian describe soft (low momentum-transfer) effects only, the type \( (B) \) scattering amplitude will reproduce the low energy behavior (including renormalization effects) but not the complete QED scattering amplitude. The difference between the QED scattering amplitude and the NRQED scattering amplitude is due to large momentum-transfer effect and leads to the type \( (A) \) contact term.

**The \( \alpha(Z\alpha) \) term.** The Hamiltonian \( H_{\text{main}}^\Lambda \) consists only of terms with even parity. Their expectation values with respect to the Coulomb wave function are even in \( Z\alpha \), too. Thus type \( (B) \) terms cannot generate terms proportional to \( Z\alpha \). The only contribution is from the type \( (A) \) term, which arises from the difference of QED and NRQED scattering amplitudes:

\[
-d_1 \frac{1}{mM} (\psi^\dagger \vec{\sigma}_e \psi) (\chi^\dagger \vec{\sigma}_\mu \chi) \equiv i \mathcal{T}_{\text{QED}} - i \mathcal{T}_{\text{NRQED}}.
\]

The QED amplitude \( \mathcal{T}_{\text{QED}} \) is derived from the electron-muon scattering diagrams in which two photons are exchanged between the electron and the muon and the electron line is dressed by a virtual photon line. Both \( \mathcal{T}_{\text{QED}} \) and \( \mathcal{T}_{\text{NRQED}} \) have threshold singularities in the limit of vanishing Coulomb photon momentum, which cancel in the difference (6).

The origin of the threshold singularity may be understood as follows: In order to contribute to the hyperfine splitting, one of the two exchanged photons in the bound state perturbation theory must be Coulomb-like while the other is transverse. The Coulomb photon can be absorbed in the wave function or the Green function. Such a diagram is reduced to lower order in \( Z\alpha \), or, equivalently, multiplied by \( 1/(Z\alpha) \). In the corresponding scattering state, this factor manifests itself as an IR-divergent factor. Since the bound state has no threshold singularity all these singularities must cancel out in the end.

The expectation value of the contact term (6) can be evaluated in the first order bound state perturbation theory. Only the value of the Coulomb wave function at the origin contributes to the integral, leading to

\[
\Delta \nu = |\phi(0)|^2 - d_1 \frac{1}{mM} (\vec{\sigma}_e \vec{\sigma}_\mu)|_J=1^J=0^J=1^J=0.
\]

where \( |\phi(0)|^2 = \gamma^3/\pi \) for the ground state. The difference between \( J = 1 \) and \( J = 0 \) spin states is taken. Together with the vacuum-polarization contribution this gives the first term of \( d_1 \) in (5).

**The \( \alpha^2(Z\alpha) \) term.** For the same reason as in the \( \alpha(Z\alpha) \) case this correction comes only from the contact term of type \( (A) \). It is defined as the difference between the gauge invariant set of QED scattering diagrams in which the electron and the muon exchange two photons and the electron line is dressed by two radiative photons and the corresponding NRQED scattering diagrams.

**The \( \alpha^2(Z\alpha)^2 \) term.** One contribution comes from a type \( (A) \) contact term of order \( \alpha(Z\alpha)^3 \), which arises from QED scattering diagrams that exchange three photons between the electron and the muon, dressed by one radiative photon on the electron line. These amplitudes have threshold singularity of up to order \( (m/\lambda)^2 \). The type \( (B) \) NRQED scattering diagrams which have the same leading threshold singularity are of two kinds: One is the first term of \( d_1 \) in (5) combined with one Coulomb potential and the other is the Fermi potential times the anomalous magnetic moment combined with two Coulomb potentials.
The $\alpha^2(Z\alpha)^2$ term. The relevant QED diagrams are those that exchange three photons between the electron and the muon, dressed by two radiative photons on the electron line. The corresponding NRQED diagrams have two radiative photons, too. However, these diagrams contribute only to order $\alpha^2(Z\alpha)^4$, which is too small by a factor $(Z\alpha)^2$. Thus the radiative correction of order $\alpha^2(Z\alpha)^2$ comes from the QED diagrams only. The contact term of NRQED derived from these QED diagrams does not have the $\ln(m/\gamma)$ dependence. Thus logarithmic terms of order $\alpha^2(Z\alpha)^2$ can come only from the NRQED operators of $H_\text{main}^K$, contributing to the $\alpha(Z\alpha)^2$ term combined with an additional factor of $\alpha$ found in the “renormalization” effect. This leads to the result reported in (14).

The $\alpha(Z\alpha)^3$ term. The status of the theory of this term will be discussed in detail by Dr. Nio in her talk presented at this Workshop.

4. Current status of theory of muonium hfs

As is well known, the bulk of the hyperfine splitting is given by the Fermi formula

$$E_F = \frac{16}{3} \alpha^2 c R_\infty \frac{m}{M} \left[1 + \frac{m}{M}\right]^{-3} = 4453839.405 (518) \text{ kHz},$$

where $R_\infty$ is the Rydberg constant for infinite nuclear mass, and $m$ and $M$ are the electron and muon masses, respectively. The values used above for $\alpha$, $R_\infty$, and $M/m$ are $\alpha^{-1} = 137.035\ 999\ 58\ (52)\ (3.8\ \text{ppb})$, $R_\infty = 10973\ 731.568\ 639\ (91)\ \text{m}^{-1}$, $\frac{M}{m} = 206.768\ 273\ (24)\ (117\ \text{ppb})$. (9)

Various theoretical corrections to $E_F$ have been calculated over the last 50 years. They may be classified into non-recoil and recoil terms, both including radiative and binding effects. In addition there are contributions of hadronic vacuum-polarization and weak interaction effect. Thus one may write

$$\Delta \nu(\text{theory}) = \Delta \nu(\text{non-recoil}) + \Delta \nu(\text{recoil}) + \Delta \nu(\text{hadron}) + \Delta \nu(\text{weak}).$$

A purely radiative non-recoil term of order $\alpha(Z\alpha)$ and an approximate value of the $\alpha(Z\alpha)^2$ term have been known for some time. Recently radiative corrections of orders $\alpha(Z\alpha)^2$ and $\alpha^2(Z\alpha)$ have been evaluated with high precision. Including these results and the Fermi term $E_F$ itself, the non-recoil term of the muonium hyperfine structure can be written as

$$\Delta \nu(\text{non-recoil}) = E_F(1 + a_{\nu})(1 + b + a_e + \alpha(Z\alpha) \left(\ln 2 - \frac{5}{2}\right))$$

$$= 169042(11)\frac{\alpha(Z\alpha)^2}{\pi} + 0.7717(4)\frac{\alpha^2(Z\alpha)}{\pi}$$

$$= 4464098.677\ (518)\ \text{kHz}.$$ (11)
Here $a_e$ and $a_\mu$ are the anomalous magnetic moments of the electron and muon, respectively, and $b$ is the Breit term

$$b = \frac{1}{\beta(2\beta - 1)} - 1, \quad \beta = (1 - (Z\alpha)^2)^{1/2}. \quad (12)$$

The appearance of the factor $(1 + a_\mu)$ in (11) is in accord with the definition (8) of $E_F$. The coefficient 16.9042 of the $\alpha(Z\alpha)^2$ term is different from 15.39 given in (3). This is because the latter contains, besides the $\alpha(Z\alpha)^2$ term, terms of order $\alpha(Z\alpha)^3$ and higher. More recent evaluation of this term without expansion in $Z\alpha$ is given in (13). For a detailed discussion see (14).

In order to match the precision of the new measurement (2), however, it is necessary to improve the theoretical prediction of terms of order $\alpha^4$. This is because, when enhanced by the factor $\ln^2(m/\gamma)$ or $\ln(m/\gamma)$ where $m/\gamma = 1/Z\alpha$, corrections of order $\alpha(Z\alpha)^3$ and $\alpha^2(Z\alpha^2)$ contribute to the muonium hyperfine structure as much as the $\alpha^2(Z\alpha)$ term. (There is no $\ln(Z\alpha)$ enhancement for $\alpha^4$, $\alpha^3(Z\alpha)$ and $(Z\alpha)^4$ terms.)

The leading log contribution of the $\alpha(Z\alpha)^3$ term is known exactly (3, 12, 13, 14). The non-log term has also been estimated (4). Altogether we have

$$\Delta \nu(\alpha(Z\alpha)^3) = \alpha(Z\alpha)^3E_F \ln\left(\frac{m}{\gamma}\right) \left[5 \ln 2 - \frac{191}{16} + \frac{13}{24}\right]$$

$$= (12 \pm 2)\alpha(Z\alpha)^3E_F/\pi$$

$$= -0.542 (8) \text{ kHz}, \quad (13)$$

where the first term in the brackets is the contribution from the radiative photon and the second is due to the vacuum polarization. The uncertainty is an estimate based on the non-log part of the $\alpha(Z\alpha)^3$ term given in (13). We are trying to improve this uncertainty.

For the $\alpha^2(Z\alpha)^2$ term the leading log part together with an estimate of non-log part gives the value (3, 12, 16)

$$\Delta \nu(\alpha^2(Z\alpha)^2) = \frac{\alpha^2(Z\alpha)^2}{\pi^2}E_F \ln^2\left(\frac{m}{\gamma}\right) \left(-\frac{4}{3}\right) \left(1 - \frac{4\pi m}{\alpha M}\right)$$

$$+ \frac{\alpha^2(Z\alpha)^2}{\pi^2}E_F \ln\left(\frac{m}{\gamma}\right) \left[-2\left(-\frac{4358}{1296} - \frac{10}{27}\pi^2 + \frac{3}{2}\pi^2 \ln 2 - \frac{9}{4}\zeta(3)\right)\right]$$

$$- \frac{3}{2}\left(\zeta(2) + 1 + 8\ln 2\right) \zeta(2) + \frac{3}{4}\zeta(3)$$

$$+ \left(1 - \frac{4\pi m}{\alpha M}\right) \frac{8}{3} \left(-\ln 2 + \frac{3}{4}\right) + \frac{1}{2}\left(-\frac{11}{9} - 1 + \frac{8}{3}\right)$$

$$+ \frac{\alpha^2(Z\alpha)^2}{\pi^2}E_F \left(1 - \frac{4\pi m}{\alpha M}\right)(10 \pm 2.5)$$

$$= 0.193 (24) \text{ kHz}, \quad (14)$$

where the factor $m/M$ originates from the reduced mass effect. An error in (3) due to an incorrect treatment of low energy contribution is rectified in (14).

The pure recoil and order $\alpha$ radiative recoil corrections, except for the last term of order $(m/M)^2$ obtained recently (17), have been known for some time (3, 18, 19)

$$\Delta \nu(\text{recoil}) = E_F \left(-\frac{3Z\alpha}{\pi} \frac{mM}{M^2 - m^2} \ln \frac{M}{m} + \frac{\gamma^2}{mM} \left[2\ln \left(\frac{m_e}{2\gamma}\right) - 6\ln 2 + \frac{65}{18}\right]\right)$$

6
\[ E_F \alpha(Z\alpha) \frac{m}{M} \left( -2 \ln^2 \frac{M}{m} + \frac{13}{12} \ln \frac{M}{m} \right) \]
\[ + 6\zeta(3) + \zeta(2) - \frac{71}{72} + 3\pi^2 \ln 2 + Z^2 \left[ \frac{9}{2} \zeta(3) + \frac{39}{8} - 3\pi^2 \ln 2 \right] \]
\[ + \frac{\alpha}{\pi} \left[ -\frac{4}{3} \ln^3 \frac{M}{m} + \frac{4}{3} \ln^2 \frac{M}{m} \right] \]
\[ + 6\zeta(3) + \zeta(2) - \frac{71}{72} + 3\pi^2 \ln 2 + Z^2 \left[ \frac{9}{2} \zeta(3) + \frac{39}{8} - 3\pi^2 \ln 2 \right] \]
\[ = -795.228 \text{ kHz}, \quad (15) \]

where \( \gamma \equiv Z\alpha m_r, \ m_r = \frac{mM}{m + M} \). The \( \ln^3 \) and \( \ln^2 \) parts of the \( \alpha^2(Z\alpha) \) term were evaluated by Eides et al. \cite{19}. Further refinement of \( \Delta(\text{recoil}) \) comes from evaluation of \( (Z\alpha)^3(m/M) \) terms. One of the leading log term of this contribution arises from the recoil hfs potential (the first line of (15)) and the 1-photon exchange part of the Coulomb Green function:

\[ \Delta \nu^{(1)}((Z\alpha)^3(m/M)) = 2 < V_{\text{hfs}} G_{1p} V_K > + 2 < V_{\text{hfs}} G_{1p} V_D > \]
\[ \approx -\frac{3(Z\alpha)^2}{M} E_F \ln \left( \frac{M}{m} \right) \ln \left( \frac{m}{\gamma} \right) \]
\[ = -0.210 \ (43) \text{ kHz}, \quad (16) \]

where \( V_K \) and \( V_D \) are the \( k^4 \) kinetic energy and Darwin terms, respectively. The uncertainty is estimated assuming that it is a factor \( \ln(Z\alpha) \) smaller than the leading term.

Another log term arises from the Salpeter recoil correction to the Lamb shift \cite{20}. This contribution involves both relativistic and non-relativistic regions. Unfortunately, the latter was handled incorrectly in \cite{8}. Correcting this error we find \cite{12, 16}

\[ \Delta \nu^{(2)}((Z\alpha)^3(m/M)) = \frac{(Z\alpha)^3}{\pi} E_F \frac{m}{M} \ln^2 \left( \frac{m}{\gamma} \right) \left[ -\frac{2}{3} \right] \]
\[ + \frac{(Z\alpha)^3}{\pi} E_F \frac{m}{M} \ln \left( \frac{m}{\gamma} \right) \left[ -2C_S + \frac{32}{3} \left( -\ln 2 + \frac{3}{4} \right) \right] \]
\[ + \frac{(Z\alpha)^3}{\pi} E_F \frac{m}{M} (40 \pm 22) \]
\[ = -0.194 \ (59) \text{ kHz}, \quad (17) \]

where one factor of \( Z\alpha \) is radiative correction due to exchange of transverse photon between the electron and the muon and \( C_S \) is given by

\[ C_S = -\frac{1}{9} \left( \frac{7}{3}(2 \ln 2 + 3) \right) - \frac{2}{1 - (m/M)^2} \left( \ln(1 + m/M) - (m/M)^2 \ln(1 + M/m) \right). \quad (18) \]

The hadronic vacuum polarization contributes \cite{21}

\[ \Delta \nu(\text{hadron}) = \frac{\alpha(Z\alpha) m M}{\pi^2} \frac{m}{m^2_{\pi}} (3.75 \pm 0.24) E_F \]
\[ = 0.250 \ (16) \text{ kHz}, \quad (19) \]

where \( m_{\pi} \) is the charged pion mass.
Finally, there is a contribution due to the $Z^0$ exchange\cite{22,23,24}

$$\Delta \nu(\text{weak}) = -G_F \frac{3\sqrt{2}mM}{8\alpha\pi} - E_F \approx -0.065 \text{ kHz},$$

(20)

where $G_F$ is the Fermi weak coupling constant.

Collecting all contributions we obtain the best current estimate

$$\Delta \nu(\text{theory}) = 4463\ 302.649\ (517)\ (34)\ \leq (100) \text{ kHz},$$

(21)

where the first and second errors come from those of $M/m$ and $\alpha$ in (4) and the last is an estimate of the theoretical uncertainty from the yet-to-be calculated terms. This is in excellent agreement with the measurement (1).

5. Concluding remarks

Current uncertainties in (13), (14), (16), and (17) are 0.008 kHz, 0.024 kHz, 0.043 kHz, and 0.059 kHz, respectively. For further improvement it is particularly important to evaluate the $O(\alpha^3(m/M))$ recoil terms.

Study of positronium in the framework of NRQED requires additional terms to the Hamiltonian. This is discussed by P. Labelle in this Workshop. Adaptation of NRQED method to the chiral perturbation theory is of great interest at present in view of the fine experimental works on the $\pi^+\pi^-$ bound system.

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