Gravitational Collapse in General Relativity and in $R^2$-gravity: A Comparative Study

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We compare the gravitational collapse of homogeneous perfect fluid with various equations of state in the framework of General Relativity and in $R^2$-gravity. We make our calculations using dimensionless time with characteristic timescale $t_g \sim (G\rho)^{-1/2}$ where $\rho$ is a density of collapsing matter. The cases of matter, radiation and stiff matter are considered. We also account the possible existence of vacuum energy and its influence on gravitational collapse. In a case of $R^2$ gravity we have additional degree of freedom for initial conditions of collapse. For barotropic equation of state $p = w\rho$ the result depends from the value of parameter $w$: for $w > 1/3$ the collapse occurs slowly in comparison with General Relativity while for $w < 1/3$ we have opposite situation. Vacuum energy as expected slows down the rate of collapse and for some critical density gravitational contraction may change to expansion. It is interesting to note that for General Relativity such expansion is impossible. We also consider the collapse in the presence of so-called phantom energy. For description of phantom energy we use Lagrangian in the form $-X - V$ (where $X$ and $V$ are the kinetic and potential energy of the field respectively) and consider the corresponding Klein-Gordon equation for phantom scalar field.

Keywords: gravitational collapse; modified gravity.

1. Introduction

Gravitational collapse has become a major topic of research recently. It is believed to be the basic mechanism for the structure formation of our universe. The problem of a star contracting under the influence of its own gravitational field was firstly studied by J.R. Oppenheimer and his student H. Snyder in 1939 [1], some time after Einstein developed the idea of General Relativity. They solved the Einstein’s field equation for the case of a star consisting of pressureless fluid having uniform density and collapsing to a black hole. By using the two matching conditions, they tried to answer whether it is possible to maintain smoothness of the metric at the common boundary. Using the condition of smoothness allows to describe the behavior of the components of the metric. Then gravitational collapse in frames of General
Relativity has been studied in many papers (see [2] and references therein). For simple case of dust matter with zero pressure it is possible to obtain that time of collapse is \( \sim \rho_0^{-1/2} \) where \( \rho_0 \) is initial density of collapsing object. For more realistic case of collapsing star one needs to know the equation of state for dense matter.

It is necessary to note that General Relativity is very successful theory [2,3,4]. The solar system tests like the precession of the perihelion of Mercury, bending of light in gravitational field of sun and the gravitational red-shift are all in agreement with the prediction of General Relativity. Gravitational waves, which were first proposed by Henri Poincare and predicted by General Relativity were recently observed by LIGO collaboration [5].

In spite of this achievements there have been speculations among the scientific community regarding of the validity of the General Relativity. Motivations to doubt this theory come from its inability to explain certain phenomena, like the inflation proposed by Alan Guth in 1979 [6,7], which happened soon after the Big Bang, and the late cosmic acceleration observed by Perlmutter et al. [8] and Riess et al. [9] in 1998. This acceleration cannot be explained in General Relativity with usual matter sources such as dust matter and radiation. One needs to postulate existence of so called dark energy i.e. a matter field having the equation of state parameter \( w < -\frac{1}{3} \). Such fluid can lead to repulsing effect and corresponding acceleration. But physical nature of dark energy is unclear. The first tentative candidate of dark energy having the potential to explain the accelerated expansion of the universe is the cosmological constant \( \Lambda \) (nonzero vacuum energy) [10,11,12]. However, such explanation contains the problem related with the fine tuning problem which is not yet resolved [13,14]. Other candidates of dark energy like k-essence, holographic energy, phantom energy, chameleon scalar fields have also been explored in literature [15,16,17,18,19,20].

An alternative approach to explain the cosmic acceleration is to reformulate the theory of gravity such that it could provide an explanation of observational data without any exotic dark energy components. Of course this modification is restricted from observational and local gravity constraints. The most simple modification of gravity is to include the functions of the scalar curvature in the gravitational action \( f(R) \) theories of modified gravity (for review see [21,22,23,24,25]). These theories in principle have the capability to explain the acceleration era without the existence of exotic matter fields. Other models of modified gravitational theory include \( f(R,T) \) gravity [26], scalar-tensor theories [27,28], Gauss-Bonnet gravity [29,30] etc, each having its own merits and demerits being widely discussed in literature.

Gravitational collapse in frames of General Relativity has been recently investigated in many papers [31,32,33,34,35,36,37,38,39]. In a case of modified gravity interesting results are obtained in [40,41,42,43,44,45,46,47]. For example the authors of [40] obtained that constant scalar curvature term in the action led to slowdown of collapse. In [41] the general \( f(R) \) model is analyzed for uniformly collapsing
cloud of self-gravitating dust particles. According to calculations for viable \( f(R) \) models we have initial epoch with higher contraction than in General Relativity. E. Santos \[42\] paid the attention on theoretical arguments for the collapse of massive stars, which as expected is unavoidable in General Relativity. As shown this is not necessary for the theory with Einstein-Hilbert action involving a function of \( R^2 - R_{\mu \nu}R^{\mu \nu}/2 \) added to the Ricci scalar. Authors of \[43\] found exact nonstatic dust solutions in metric \( f(R) \) gravity, imposed by the constant scalar curvature and Yang-Mills gauge theory, which describes the gravitational collapse of pressureless dust in (anti-)de Sitter higher-dimensional background. The interesting question about curvature singularities in \( f(R) \) gravity is considered in \[44\]. Authors investigated a curvature singularity appearing in the star collapse process and concluded that addition of term \( \sim R^\alpha \) \((1 < \alpha < 2)\) could cure the curvature singularity.

Our primary purpose is to compare process of the collapse in General Relativity and in \( R^2 \) gravity. Paper is organized as follows. In section 2, we recall the description of collapse in frames of General Relativity. Three different types of fluid (dust matter, radiation and stiff matter) are considered. We study also the gravitational collapse in the presence of vacuum energy. In our calculations the dimensionless time is used for simplicity. The characteristic gravitational time-scale is \( \sim (G\rho)^{-1/2} \) where \( \rho \) is initial energy density of collapsing fluid. In the next section the influence of phantom dark energy on collapse is studied. We considered dynamical equation for the scale factor and Klein-Gordon equation for phantom scalar field. Then we discuss the collapse of the perfect fluid in the presence of a scalar field dark energy model called quintessence. Section 4 is devoted to study the collapse in frames of \( f(R) = R + \beta R^2 \) gravity. Lastly, we end the paper with conclusions derived from the calculations in the previous sections.

2. Gravitational collapse in General Relativity

We intend to study the gravitational collapse of homogeneous and isotropic fluid having a spherically symmetric Lemaitre-Tolman-Bondi metric (whose components are separable) given by

\[
ds^2 = -dt^2 + A^2(t)h(r)dr^2 + A^2(t)r^2d\Omega^2.
\]

The collapsing fluid is taken as a perfect fluid with stress-energy tensor \( T^\nu_{\mu} = \text{diag}(-\rho, p, p, p) \). Hereafter we use system of units in which \( G = c = 1 \). The components \( tt, rr, \theta \theta \) of the Einstein field equations are written in terms of \( A(t) \) and \( h(r) \) as follows:

\[
3\frac{\ddot{A}}{A} = -8\pi\rho + \frac{R}{2},
\]

\[
\frac{\dot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{h'}{A^2 h^2 r} = 8\pi p + \frac{R}{2},
\]

\[
\frac{\ddot{A}}{A} + 2\frac{\dot{A}^2}{A^2} + \frac{h'}{A^2 h^2 r} - \frac{1}{A^2 h^2 r} + \frac{1}{A^2 r^2} = 8\pi p + \frac{R}{2}.
\]
Here dot means time derivative whereas comma is derivative on radial coordinate $r$. From last two equations one can derive the following:

$$\frac{h'}{2h^2r} + \frac{1}{hr} - \frac{1}{r^2} = 0 \quad (5)$$

which is satisfied by

$$h(r) = \frac{1}{1 + C_1 r^2}. \quad (6)$$

This gives us the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + A^2(t) \left( \frac{dr^2}{1 + C_1 r^2} + r^2 d\Omega^2 \right). \quad (7)$$

In order to determine the collapse dynamics of the star, we will have to find how the scale factor $A(t)$ behaves. Substituting for $h(r)$ in Eq. (3) or (4), using Eq. (2) to eliminate the second derivative of scale factor and relation for scalar curvature $R = -8\pi T = 8\pi (\rho - 3p)$ we obtain the following first-order differential equation for $A(t)$:

$$\dot{A}^2 = \frac{8\pi}{3} A^2 \rho + C_1. \quad (8)$$

Without loss of generality one can assume that $A(0) = 1$. For first derivative one put $\dot{A}(0) = 0$. For barotropic equation of state in the form

$$p = w\rho$$

we have simple dependence of energy density from the scale factor

$$\rho = \rho_0 A^{-3(1+w)}. \quad (9)$$

Here $\rho_0$ means the value of energy density at $t = 0$. The non-trivial solution for above mentioned conditions realizes for $C_1 = 8\pi \rho_0$. Therefore for solution describing gravitational collapse we have equation:

$$\dot{A} = -\sqrt{k(-1 + A^{-1 - 3w})}, \quad k = \frac{8\pi}{3} \rho_0. \quad (10)$$

Introducing dimensionless time according to relation

$$\tau = \sqrt{k} t$$

allows to rewrite the Eq. (10) in the following form:

$$\frac{dA}{d\tau} = -\sqrt{-1 + A^{-1 - 3w}}. \quad (11)$$

For example for $w = 0$ (dust) we have following solution for $\tau$ as function of scale factor:

$$\tau = \frac{\pi}{2} - \arcsin \sqrt{A + \sqrt{A - A^2}}.$$
The moment of singularity corresponds to $A = 0$ and therefore we have $\tau_s = \frac{\pi}{2}$.

The radiation collapse corresponds to $w = 1/3$. In this case one can derive scale factor in the explicit form:

$$A(\tau) = \sqrt{1 - \tau^2}.$$  \hspace{1cm} (12)

The moment of singularity corresponds to $\tau_s = 1$. For stiff fluid with $w = 1$ we found that moment of singularity is

$$\tau_s = \frac{1}{4} B\left(\frac{1}{2}, \frac{3}{4}\right) \approx 0.6$$

where $B$ means beta-function. The dynamics of collapse for considered cases is presented on Fig. 1.

Eq. (10) corresponding to cosmological constant $\Lambda$ ($w = -1$) is

$$\frac{dA}{d\tau} = -\sqrt{1 + A^2}. \hspace{1cm} (11)$$

solving which we get

$$A(\tau) = \cosh(\tau + C_1). \hspace{1cm} (12)$$
Proceeding the same way as we did above, i.e. using \( A(0) = 1 \), we get \( C_1 = 0 \) giving us:

\[
A(\tau) = \cosh(\tau).
\]

(13)

We can conclude from the above equation that \( A(\tau) \) can never be 0 at any finite time, which means that the cosmological constant never collapses to form a singularity, as expected.

Finally we consider the case of fluid with barotropic EoS and cosmological constant. The corresponding equation in dimensionless units in this case is

\[
\frac{dA}{d\tau} = -\sqrt{-\Omega_m^{-1} + \frac{\Omega_A}{\Omega_m} A^2 + A^{-1-3w}}.
\]

(14)

Here \( \Omega_m \) and \( \Omega_A \) are fractions of energy density of matter and cosmological constant correspondingly in the moment \( \tau = 0 \), i.e.

\[
\Omega_m = \frac{\rho_0}{\rho_0 + \Lambda}, \quad \Omega_A = \frac{\Lambda}{\rho_0 + \Lambda}.
\]

The gravitational collapse due to repulsion of vacuum energy occurs more slowly in comparison with the case of one fluid (see Fig. 1).

As suggested in the introduction, there have been attempts to explain the current observations and the inflationary era by either modifying the general theory of relativity or by adding dark energy in general relativity. One such possible form of dark energy is called phantom energy. We focus on investigating the collapse of a gravitationally bound fluid in the framework of General Relativity in presence of phantom energy.

3. Phantom fluid

The action for scalar field with non-canonical kinetic term is given by

\[
S = \int \left( \frac{R}{16\pi} + P(\phi, X) \right) \sqrt{-g} d^4x + S_m.
\]

(15)

Here \( P(\phi, X) \) is an arbitrary function of scalar field \( \phi \) and \( X \), i.e. the kinetic energy of the field given by \(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\). From the above action, the energy-momentum tensor of the scalar field can be evaluated as

\[
T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} P)}{\delta g^{\mu\nu}} = P,_{\mu} \phi,_{\nu} \phi + g_{\mu\nu} P.
\]

(16)

We consider the scalar field with energy-momentum tensor \( T_{\mu\nu}^{(\phi)} = (\rho_\phi + p_\phi) u_\mu u_\nu + g_{\mu\nu} P \) giving us \( p_\phi = P \) and \( \rho_\phi = 2XP_{,X} - P \). Hence we obtain the equation of state parameter \( w_\phi \) as

\[
w_\phi = \frac{P}{2XP_{,X} - P}.
\]

(17)

Phantom energy is a particular case of k-essence in which \( w_\phi < -1 \) which is obtained by imposing the condition \( P_{,X} < 0 \). The simplest model being a scalar field
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$P(X, \phi) = -X - V(\phi)$ where $V(\phi)$ is the potential of scalar field. We obtain in this case $p_\phi = -\dot{\phi}^2 - V(\phi)$ and $\rho_\phi = -\frac{\dot{\phi}^2}{2} + V(\phi)$ from which we obtain the equation of state parameter

$$w_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}$$

When the potential energy dominates the kinetic energy, we get the situation $w_\phi < -1$. Behavior of various potential has already been studied and are available in literature [48,49,50,51,52], one of which is a linear model with power law potential $V(\phi) = V_0 \phi^n$. In the mathematical tool which we are going to apply, the collapse dynamics becomes independent of the coefficient of $\phi^n$ in the above mentioned linear model with power law potential. In order to make the coefficient of $\phi^n$ to play a significant role in determining the astrophysical scenario, we propose a model of phantom energy of the form $P(X, V) = -X - V$ having the slope of the associated potential energy

$$\frac{dV}{d\phi} = V_0(t)\phi^n, \quad V_0(t) = V_0 + \alpha t^m.$$  

We consider the collapse scenario of an ordinary matter perfect fluid in the presence of phantom scalar field. The Einstein’s field equations are:

$$\frac{\dot{A}^2}{A^2} = \frac{8\pi}{3} \left( \rho_m - \frac{\dot{\phi}^2}{2} + V(\phi) \right),$$

(19)

$$\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} = -4\pi \left( -\dot{\phi}^2 + \rho_m + p_m \right).$$

(20)

The equation of continuity for matter field is given by

$$\dot{\rho} + 3\frac{\dot{A}}{A}(\rho_m + p_m).$$

(21)

For phantom field we obtain the Klein-Gordon equation

$$\ddot{\phi} + 3\frac{\dot{A}}{A}\dot{\phi} - \frac{dV}{d\phi} = 0.$$  

(22)

We use the integrability condition on anharmonic oscillator [53] which states that for the differential equation of the form:

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^n = f_4(t)$$

(23)

which governs the time evolution of the space variable $x(t)$ of an anharmonic oscillator (where $x$ and $f_k(t)$ are continuously differentiable real functions defined on some interval), we have the following statement for $n \neq -3, -1, 0, 1$:

The coefficients of Eq. (23) satisfy the differential equation:

$$\frac{\ddot{f}_3}{(n+3)f_3} - \frac{n+4}{(n+3)^2 f_3^2} \frac{\dot{f}_3^2}{f_3} + \frac{n-1}{(n+3)^2 f_3} f_3 f_1 + \frac{2\dot{f}_1}{n+3} + \frac{2(n+1)f_1^2}{(n+3)^2} = f_2$$  

(24)
if and only if Eq. (23) can be transformed into an integrable form

\[ \ddot{X}(T) + X^n(T) = 0. \]  

(25)

where

\[ X(T) = Cx(t)f_{\frac{n+3}{3}}(t)e^{ \frac{2}{n+3} \int f_1(\eta)d\eta}, \]  

(26)

\[ T(t) = C^{\frac{1}{n-2}} \int f_{\frac{n+3}{3}}(\zeta)e^{ \frac{2}{n+3} \int f_3(\eta)d\eta}d\zeta. \]  

(27)

Comparing Eq. (23) with the Klein-Gordon Eq. (22) corresponding to the phantom field having slope of the potential energy \( \frac{dV}{d\phi} = (V_0 + \alpha t^m) \phi^n \), we have \( f_1 = \frac{3A}{A} \), \( f_2 = 0 \) and \( f_3 = -(V_0 + \alpha t^m) \), for which the Eq. (24) becomes

\[
\frac{1}{n+3} \frac{\alpha m(m-1)t^{m-2}}{V_0 + \alpha t^m} - \frac{n + 4}{(n+3)^2} \left( \frac{m \alpha t^{m-1}}{V_0 + \alpha t^m} \right)^2 + 
+ \frac{n - 1}{(n+3)^2} \left( \frac{m \alpha t^{m-1}}{V_0 + \alpha t^m} \right) \frac{3A}{A} + \frac{6}{n+3} \left( \frac{A}{A} - \frac{\dot{A}^2}{A^2} \right) + \frac{18(n+1)}{(n+3)^2} \frac{\dot{A}^2}{A^2} = 0.
\]

(28)

For simplicity, we consider a particular case corresponding to \( m = 1 \). The Eq. (24) becomes

\[
- \frac{n + 4}{n+3} \left( \frac{\alpha}{V_0 + \alpha t} \right)^2 + \frac{n - 1}{n+3} \left( \frac{\alpha}{V_0 + \alpha t} \right) \frac{3A}{A} + \frac{6}{n+3} \frac{\dot{A}^2}{A^2} + \frac{12n}{n+3} \frac{\dot{A}^2}{A^2} = 0.
\]

(29)

On solving the above differential equation, we get the equation of scale factor governing the collapse of the fluid

\[ A(t) = C_2 (V_0 + \alpha t)^{-1/6} \left[ 1 + C_1 (V_0 + \alpha t)^{3/2} \right]^{\frac{n+3}{3(n+1)}}. \]  

(30)

In the above equation, we have unknown constants \( C_1, C_2 \) for given parameters \( V_0, \alpha \) and \( n \). We use the initial conditions \( A(0) = 1 \) and \( \dot{A}(0) = 0 \) and get the following values for \( C_1 \) and \( C_2 \):

\[ C_1 = \frac{V_0^{-\frac{1}{2}}(n+1)}{2(n+4)}; \quad C_2 = V_0^{1/6} \left[ 1 + C_1 V_0^{3/2} \right]^{\frac{n+3}{3(n+1)}}. \]

(31)

For \(-3 < n < -1\) value of \( C_1 \) lies between -1 and 0. Therefore \( A \) diverges at some moment of time (backward collapse). From Eq. (31) one can see that for \( n > -1 \) and \( n < -3 \) on late times

\[ A \sim 1^{\frac{n+3}{3(n+1)}}, \quad t \to \infty \]

(32)

From this asymptotics one can see that in the presence of phantom energy field having slope of the potential of the form \( \frac{dV}{d\phi} = (V_0 + \alpha t) \phi^n \), the fluid collapses for \(-4 < n < -3\). The collapse becomes faster with increasing of value of \( n \). For a given \( n \) in the above mentioned domain, increasing \( V_0 \) slows down the collapse, while increasing \( \alpha \) fastens the process.
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Figure 2. Dynamics of scale factor $A(\tau)$ in presence of phantom energy, having Lagrangian $P(X, \phi) = -X - V(\phi)$ where the potential has the slope $\frac{dV(\phi)}{d\phi} = (V_0 + \alpha t)\phi^n$, for various values of $n$ and $V_0$.

For $n < -4$ and $n > -1$ there is no collapse. This could be interpreted as if the dark energy has become so dominant that it doesn’t allow any inward gravitational pull at all. For these cases increasing of $n$ reduces the expansion rate. Increasing $V_0$ ($\alpha$) slows down (speeds up) the expansion of the fluid. For $n$ close to $-1$ from above, the phantom field becomes too strong. For example for $V_0 = \alpha = 1$ and for $n = -0.99$, the scale factor reaches up to $3.5 \times 10^{107}$ for time $t = 10^3$. Comparing it with the outcome for $n = 5$, the scale factor increases 18 times for the same time period. Dependence of the expansion of the fluid from the values $V_0$ and $\alpha$ is similar to the previous case.

If we consider a more general dark energy model of similar kind having slope of the form $\frac{dV}{d\phi} = (V_0 + \alpha t^m)\phi^n$, the equation of state parameter $w$ is given by Eq.(17). Analysis shows that for positive $m$ the gravitational collapse is expected to slow down and if there is expansion, then it speeds up.
3.1. Comment on quintessence

As is discussed in the introduction, an alternative to the cosmological constant is necessary in order to avoid the coincidence problem. The real candidate for dark energy is scalar field called quintessence. Unlike the cosmological constant case, there is no restriction as to how small the energy density need to be with respect to matter or radiation density for the very early phase of the universe.

The action for a canonical scalar field, called the quintessence, having potential $V(\phi)$ is given by

$$S = \int \left( \frac{R}{16\pi} + X - V(\phi) \right) \sqrt{-g} d^4x + S_m \quad (32)$$

where $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, i.e. the kinetic energy of the scalar field. From the above action, the energy-momentum tensor of the scalar field can be evaluated as

$$T^{(\phi)}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu} \left( \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi) \right), \quad (33)$$

We consider the scalar field having the behavior of a perfect fluid following the equation $T^{(\phi)}_{\mu\nu} = (\rho_\phi + P_\phi)u_\mu u_\nu + g_{\mu\nu}P$, giving us $P_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and $\rho_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. Hence we obtain the equation of state parameter $w$ as

$$w_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}, \quad (34)$$

We consider the slope in the potential as $\frac{dV}{d\phi} = V_0(t)\phi^n$ where $V_0(t) = V_0 + \alpha t^n$. Using the field equations corresponding to action Eq.(32) and the equation of continuity for matter field Eq.(21), we obtain the Klein-Gordon equation for quintessence given by

$$\ddot{\phi} + \frac{3A}{\hat{A}} \dot{\phi} + \frac{dV}{d\phi} = 0. \quad (35)$$

Using the integrability condition, we obtain a differential equation of $A(t)$ given by Eq.(28). We can conclude that the equations which govern the dynamics of the fluid in the presence of quintessence is same as that in presence of phantom energy for the case where the slope of the potential is given by $V_0 + \alpha t^n$.

4. Gravitational collapse in $f(R) = R + \beta R^2$ gravity

Let’s consider the gravitational theory with action for gravitational field

$$S_g = \int \frac{f(R)}{16\pi} \sqrt{-g} d^4x, \quad (36)$$

where $f(R)$ is arbitrary differentiable function of Ricci scalar $R$. For our purposes we write function $f(R)$ as

$$f(R) = R + h(R),$$
extracting contributions with respect to GR, $h(R)$. The field equations obtained from Eq. (36) are

$$(1 + h_R)G_{\mu\nu} - \frac{1}{2}(h - h_R R)g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box)h_R = 8\pi T_{\mu\nu}. \quad (37)$$

Here $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor and $h_R = \frac{dh}{dR}$, $\Box = g^{\mu\nu}\nabla_\mu \nabla_\nu$ is the covariant D’Alamber operator. For the Ricci curvature scalar one can get the following equation:

$$3h_{RR} \Box R = 8\pi T - 3h_{RRR}(\nabla R)^2 + R + 2h - Rh_R \quad (38)$$

For FLRW metric with positive curvature we have following equations for $A(t)$:

$$H^2 - \frac{C_1}{A^2} = \frac{8\pi \rho_0}{3(1 + h_R)} + \frac{1}{3(1 + h_R)} \left( \frac{1}{2} h_R R - 3h_{RR} H \dot{R} \right). \quad (39)$$

Finally for Ricci scalar equation can be rewritten as:

$$3h_{RR} (\ddot{R} + 3H \dot{R}) = -3h_{RRR} \dot{R}^2 + 8\pi T + 2h - h_R R. \quad (40)$$

Here $H = \frac{\dot{A}}{A}$. From metric one can obtain the following relation for Ricci scalar

$$R = 6 \left( \dot{H} + 2H^2 - \frac{C_1}{A^2} \right). \quad (41)$$

The equation of continuity remains the same:

$$\dot{\rho} + 3H (\rho + p) = 0. \quad (42)$$

Scalar curvature depends from second derivatives of metric $g_{\mu\nu}$ and therefore from fourth derivative of $A(t)$. However one can solve equation for scalar curvature as independent variable with equations for $A$ and $\rho$. We use equations (39), (40), (41) with definition for $H$ and integrate it for some initial conditions $\rho_0, R_0, \dot{R}_0, A_0, H_0$. Note that equation (39) is consistent with another equations and therefore give no additional information. But this equation can be used to fix initial data. For example for given $H(0), \rho(0), A(0)$ and $\dot{R}(0)$ one can derive the condition on scalar curvature at $t = 0$. In dimensionless units we put $C_1 = -8\pi \rho_0/3$ and rewrite (39), (40), (41) in following form:

$$H^2 = -\frac{1}{A^2} + \frac{\rho}{(1 + h_R)} + \frac{1}{(1 + h_R)} \left( \frac{1}{2} h_R R - h_{RR} H \dot{R} \right), \quad (43)$$

$$3h_{RR} (\ddot{R} + 3H \dot{R}) = -3h_{RRR} \dot{R}^2 + 3T + 3R + 3(2h - h_R R), \quad (44)$$

$$R = 2 \left( \dot{H} + 2H^2 + \frac{1}{A^2} \right). \quad (45)$$

Here we adopted the following dimensionless units for density, pressure, time and curvature:

$$\rho \rightarrow \rho \rho_0, \quad p \rightarrow p \rho_0.$$
Let’s consider simple model of $f(R)$ with $h = \beta R^2$ where $\beta$ is constant parameter ($R^2$ gravity) \cite{21}. One can consider the various cases of initial conditions. Let’s choose that $H(0) = 0, \rho(0) = 1$ and $\dot{R}(0) = 0$. For given $A(0)$ from (43) one can define initial condition for scalar curvature $R_0$. Then integration of (44), (45) with equation of continuity gives the solution for unknown functions. We consider especially case $A(0) = 1$. For scalar curvature we have that $R(0) = 0$. Results for $A$ are given on Fig. 3 for matter ($w = 0$) and stiff matter ($w = 1$). For radiation ($w = 1/3$) at given initial conditions we have no significant difference from General Relativity (scalar curvature is zero as in General Relativity for radiation). The time of collapse increases with $\beta$ for stiff matter and decreases for dust matter. For $\beta > \sim 0.1$ in dimensionless units (this value corresponds to $(8\pi \rho_0)^{-1}$) $t \to 1$ for both cases. Varying equation of state parameter $w$ allows to conclude that for $0 \leq w < 1/3$ collapsing time decreases with $\beta$.

The next step is to add into consideration the lambda-term and study its influence on process of collapse. Evolution in this case depends from the fraction of vacuum energy (see Fig. 4). For some $\Omega_{\Lambda}^{\text{crit}}$ initial contraction turns into expansion. The value of $\Omega_{\Lambda}^{\text{crit}}$ depends from $\beta$ and equation of state. One notes that in General Relativity the vacuum energy only slows down gravitational collapse and contraction never turns into expansion.

We also considered another class of initial conditions. Assuming $H(0) = 0, \dot{R} = 0, \rho(0) = 1$ one can vary $A(0)$. Additional degree of freedom (curvature $R(0)$) allows to satisfy condition $H(0) = 0$. The results are presented on Fig. 4. One can see the following features. For small values of $\beta$ $A(t)$ increases and then decreases. The time of collapse grows with $\beta$ in narrow interval. But for any value of $A(0)$...
there is a minimal value of parameter $\beta$ at which the process of collapse become similar to previous case for $\beta > \sim 0.1$. Time of collapse asymptotically tends to $\sim A(0)$ (for previous case we have $A(0) = 1$ and therefore $\tau_f = 1$).
5. Conclusion

We compared the collapse dynamics of homogeneous perfect fluid in General Relativity and in $R^2$ gravity and formulated the equations governing various types of fluids in this category, namely stiff fluid, radiation, dust and the cosmological constant. The general features are the same for General Relativity and its above simple modification. As expected stiff fluid collapses to a singularity faster than the radiation fluid, which in turn, is faster than the dust fluid collapse. Vacuum energy reduces the rate of collapse.

We also investigated the collapse of a perfect fluid in the presence of phantom fluid, one of the candidate of the dark energy. We choose case $dV(\phi) = (V_0 + \alpha t^m)\phi^n$ for potential of scalar field in order to make the coefficient of $\phi^n$ to play a significant role in determining of the astrophysical scenario. We obtained the Klein-Gordon equation which we compared with a general differential equation of the form Eq.(23) and used a theorem of integrability condition for the anharmonic oscillator to get an equation which governs the collapse dynamics of the perfect fluid in phantom field.
Comparing the gravitational collapse of ordinary perfect fluid with that in the presence of phantom fluid we concluded that for values of \( n \) outside the interval \((-4, -3)\), there is no collapse. This observation could be interpreted as follows. The fluid cannot to collapse if the dark energy has completely dominated the process. Except for \( n \in (-3, -1) \) we have no singularities for finite time (for \( n \in (-3, -1) \) there is a Big Rip singularity).

When \( n \in (-4, -3) \), the collapse is very different from the usual collapse in the sense that the scale factor \( A(t) \) asymptotically tends to zero and singularity takes place at infinite time.

The role of initial value \( V_0 \) is such that for greater values of \( V_0 \) for \( n < -4 \) and \( n \in (-3, -1) \) the scale factor increases more slowly. Similarly, for \(-4 < n < -3\) it decreases more slowly with growing of \( V_0 \).

In frames of \( f(R) = R + \beta R^2 \) theory of gravity gravitational collapse has some features in comparison with General Relativity. Firstly due to additional degree of freedom (value of scalar curvature at the initial moment) one can consider various initial conditions for collapse. For time derivatives of scalar curvature and scale factor we take zero values. General feature is that the collapsing time decreases with increasing of parameter \( \beta \) in narrow interval for equation of state parameter \( w < 1/3 \) and increases for \( w > 1/3 \). But for \( \beta > \beta_0 \) (\( \beta_0 \) depends from density of collapsing volume) the dynamics of collapse doesn’t depend significantly from \( w \). Addition of vacuum energy leads to increasing of collapsing time as expected. Depending on value of vacuum energy fraction we have various dynamics: simple contraction, collapse including several consecutive stages of expansion and final contraction or expansion (backward collapse). Non-zero initial curvature leads to dynamics similar in a case of \( R(0) = 0 \) but for small values of parameter \( \beta \) initial small expansion takes place and then contraction.

There is number of candidates of dark energy which tries to fix the gap between the observation and theory. However, not all are expected to be correct and most of them will be discarded sooner or later. Even in the case of phantom fluid, there are multiple candidates (different functions \( P(X, V) \) in addition to different potential functions \( V(\phi) \)) each of them can change the collapse scenario. The same is true for different candidates of modified gravity, like the \( f(R) \) gravity. Of course for detailed description of realistic collapse for example in a case of massive stars or central areas of galaxies one needs to know the equation of state for collapsing matter. Investigating the dynamics of gravitational collapse could possibly help us by shedding some light on the viability of the dark energy as well as modified gravity models so that some of them could be discarded, thereby reducing the set of viable models.

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