On "Quantum interference with slits" and its "revisited"

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Marcella in 2002 published a "quantum-mechanical" treatment of the famous single- and double-slit interference experiment in classical wave optics by a simple assumption that quantum mechanical wave function is a constant anywhere within a slit. Rothman and Boughn in 2011 commented that Marcella introduced no quantum physics into the problem other than a symbol substitution \( p = \hbar k \), and the used essentially the classical wave optics. In present comment, we point out that though Marcella made a fundamental mistake, the problem is nevertheless remediable to give the satisfactory quantum mechanical results which are in qualitatively agreement with experimental ones with material particles.

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In order to get a suggestive understanding of the famous uncertainty relation \( \Delta x_i \Delta p_i \simeq \hbar \) (\( i = x, y, z \)), it is a typical treatment in the quantum mechanics textbooks that use the well-known single- and double-slit interference experiment in classical wave optics to illustrate the relation. This is advantageous because there has not yet been a consistent, full quantum–mechanical treatment of the slit experiment. In a 2002 issue of present journal, Marcella \[1\] reported a "quantum-mechanical" treatment of slit experiment, which is now widely cited as a convincing quantum mechanical method reproducing exactly the same expressions originally given by the classical wave optics, and the number recorded by the Google scholar citation up to today (Jan. 24, 2019) is 24 and 7 of them are done after 2011. Unfortunately, Marcella by and large misunderstood both his derivations and results, as correctly pointed out by Rothman and Boughn in a 2011 paper published also in present journal, which attracts less attention and the number recorded by the Google scholar citation is only two. We agree with Rothman and Boughn on their comments on the Marcella’s paper: "while Marcella’s procedure is useful in giving students practice with the Dirac formalism, it has introduced no quantum physics into the problem other than setting \( p = \hbar k \), and has implicitly made all the assumptions that show this is indeed a problem of classical optics." \[2\] The situation is even worse for both the original paper \[1\] and its comment \[2\] fail to point out that the wave function Marcella introduced at very beginning of his treatment is incompatible with quantum mechanics. Based on this observation, we are going to show that Marcella’s approach contains a truly quantum mechanical treatment with slits, though qualitative but sufficient. We focus on the single-slit experiment in present comment for sake of the simplicity, and the same manner can be directly used to deal with the multi-slit one.

It is accepted that from the well-known single- and double-slit interference experiment in classical wave optics, we can really reach the result \( \Delta x_i \Delta p_i \simeq \hbar \) with an Einstein relation \( p \simeq \hbar /\lambda \). To see it, let us consider a setup in Fig.1.

The diffraction angle \( \theta \) is drawn from the centre of the single slit of openness \( a \). In classical wave optics, a plane wave of wave length \( \lambda \) is initially incident to the plane \( z = 0 \), and can be slightly diffracted to a small angle \( \theta \) after through the slit, the diffraction amplitude on the distant screen is proportional to 

\[
\sin \left( \frac{a \pi \sin \theta}{\lambda} \right) \sim \sin \left( \frac{a \pi \theta}{2\hbar} \right). \tag{1}
\]

The first minimum of \( \sin \left( \frac{a \pi \sin \theta}{\lambda} \right) \) occurs at

\[
\frac{a \pi \sin \theta_{\text{min}}}{\lambda} = \pm \pi, \text{ i.e., } \sin \theta_{\text{min}} = \pm \frac{\lambda}{a}. \tag{2}
\]

Now, let the incident plane wave represent a beam of matter particles of momentum \( p \), the momentum uncertainty along the direction of measuring the width \( a \) of slit is

\[
\delta p_y \simeq \left| p \sin \theta_{\text{min}} \right| = \frac{p \lambda}{a}. \tag{3}
\]

Since the position uncertainty through the slit can be \( \delta y \simeq a \), we have then with \( p = \hbar /\lambda \),

\[
\delta y \delta p_y \simeq \lambda p = \hbar. \tag{4}
\]

Thus, in symbols of quantum mechanics, Eq. (1) is

\[
\frac{\sin \left( \frac{a p_y}{2\hbar} \right)}{a p_y /2\hbar}. \tag{5}
\]
where \( p_y \approx p \sin \theta \), and \( \theta \approx 0 \) for guaranteeing the validness of the paraxial approximation.

The key step of Marcella’s work was a seemingly reasonable assumption that the wave function of the particle is along \( y \)-axes \([1]\)

\[
\psi(y) = 1/\sqrt{a}, \text{ for } y \in \text{slit, and } \psi(y) = 0, \text{ for } y \notin \text{slit.} \tag{6}
\]

The corresponding wave function in momentum space is

\[
\phi(p_y) = \sqrt{\frac{a}{2\pi \hbar}} \frac{\sin \left( \frac{ap_y}{2\hbar} \right)}{ap_y/2\hbar}, \tag{7}
\]

which differs from result (5) only by a proportional factor. The diffraction pattern on the screen, given by \(|\phi(p_y)|^2\), is the momentum space image of the slit represented by (6) in position space. However, according to quantum mechanics, the wave function (6) is fundamentally flawed for the kinetic energy is divergent for we have

\[
\int_{-\infty}^{\infty} p_y^2 |\phi(p_y)|^2 \, dp_y = \frac{2\hbar}{\pi a} \int_{-\infty}^{\infty} \sin^2 \left( \frac{ap_y}{2\hbar} \right) \, dp_y \to \infty. \tag{8}
\]

If it is true, the particle would get an infinitely large energy via the slit after through it, which is ridiculous. The classical wave optics suggests a relation \( \delta x \delta p_x \to \infty \) instead of \( \delta x \delta p_x \approx \hbar \) that is what Marcella implicitly assumed to be true for sure but he actually failed to achieve. \([1]\) So, the wave function (6) Marcella \([1]\) introduced is not a quantum mechanical object but a classical wave, as Rothman and Boughn commented. \([2]\)

In the following, we try to fix the problem associated with the wave function (6) and resort to a simple quantum mechanical model of the slit. Note that the slit serves as nothing but a configurational confinement on the motion along \( y \)-direction. According to quantum mechanics, there must be a minimum energy coming from the motion in the \( y \)-direction along the slit. The particle before the slit is \( E_0 = p^2/2\mu \), and after the slit, the energy conservation requires

\[
E_0 = E_y + \frac{p_y^2}{2\mu}, \tag{9}
\]

where \( E_y \) stands for permissible energies due to the slit confinement placed along \( y \)-direction, which must have a minimum value, and the motion along \( z \)-direction is free between the slit and screen and its energy spectrum is continuous. A rough approximation, without loss of the physics content, is to treat motion along the slit as an infinitely deep well, and the ground state in it is simply

\[
\psi(y) = \sqrt{\frac{2}{a}} \sin \left( \frac{\pi (y + a/2)}{a} \right). \tag{10}
\]

If the particle has mass \( \mu \), the minimum energy is therefore

\[
E_y = \frac{1}{2\mu} \left( \frac{\pi \hbar}{a} \right)^2. \tag{11}
\]

The wave function of (10) in momentum space is \([3,4]\)

\[
\varphi(p_y) = 2 \sqrt{\frac{a\pi}{\hbar}} \frac{\cos \left( \frac{ap_y}{2\hbar} \right)}{\left( \pi^2 - a^2p_y^2/\hbar^2 \right)^{1/2}}. \tag{12}
\]

The first minimum of \( \cos \left( ap_y/2\hbar \right) \) occurs at \( p_y = \frac{a}{2\hbar} = \pm \pi \), i.e., \( p_y = \pm \pi \hbar/2\), which differs from that given (7) by a factor 3/2. The maxima beyond the zeroth one, given by (12), are much less appreciable than those given by (7). It is worth stressing that in the real single slit experiment with a beam of material particles such as \( C_70 \) or \( C_{60} \) molecules, \([5,6]\) the diffraction patterns do not exhibit the presence of the maxima beyond the zeroth one. The diffraction patterns on the screen are plotted in Fig.2.

Three results above (10)-(12) are from purely quantum mechanical considerations. They are very important in the following three aspects. 1, No particle passes through the slit once \( E_y \ll (\pi \hbar/2\hbar)^2/2\mu \) for there is no state of energy less than this one. Note that the state of energy \( E_y = 0 \) is prohibited by quantum mechanics. 2, The quantum mechanics gives the exact result for the \( y \)-momentum uncertainty, \( \Delta p_y = \pi \hbar/2\), which can also be estimated from result (11). 3, The quantum mechanics gives the \( y \)-position uncertainty \( \Delta y \approx 0.18a \), so we have the quantum mechanical
uncertainty relation $\Delta y \Delta p_y \simeq 0.56h > h/2$, completely compatible with $\delta y \delta p_y \simeq h$ suggested by the classical wave optics. Therefore, although we do not know dynamical details how the slit makes the propagation direction of the particle slightly deviated from the original direction, the single slit diffraction pattern can be understood within quantum mechanics.

In conclusion, Marcella’s approach contains a truly quantum mechanical treatment with slits, with the erroneous wave function be replaced by the ground state of the infinitely deep well modelling the slit.

Acknowledgments

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[1] Marcella Thomas V 2002 *Eur. J. Phys.* 23 615-621
[2] Rothman Tony and Boughn Stephen 2011 *Eur. J. Phys.* 32 107-113
[3] Zhang Yongde 2006 *Quantum Mechanics* (Beijing, Science Press) p.57 (in Chinese)
[4] Cohen-Tannoudji C, Diu B and Laloe F 1977 *Quantum Mechanics* (vol. one), (Paris: Hermann) p.269-274
[5] Nairz O, Arndt M, Zeilinger A 2002 *Phys. Rev. A* 65, 032109
[6] Hackermüller L, Uittenhaler S, Hornberger K, Reiger E, Brezger B, Zeilinger A, Arndt M 2003 *Phys. Rev. Lett.* 91, 090408

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FIG. 1: Marcella’s setup for the one- and two-slit experiment. After passing through the slit(s), the particle is assumed to be scattered at an angle $\theta$ with a momentum $p$ and a $y$-momentum of $p_y = p \sin \theta$. Note that i) the source is placed sufficiently away from the slit such that the wave front near the slit is a plane; and ii) the plane wave propagates along the $z$-axis and the slit is situated on the plane $z = 0$; and the diffracted angle $\theta$ is measured from the centreline.
FIG. 2: Diffraction distributions given by (7) (dashed line) and (12) (solid line) respectively. In quantum mechanics, the momentum uncertainty for dashed line is divergent but in classical wave optics $\Delta p_y \simeq h/a$. 