Hawking Radiation of Black Shells

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Abstract

A black shell consists of a massive thin spherical shell contracting toward its gravitational radius, coinciding with 't Hooft’s brick wall, from the point of view of an external observer far from the shell. This object was conceived by J.R. Arenas and W. Israel in order to effectively model the gravitational collapse dynamics and the thermodynamics of a black hole [5, 6, 21, 1]. We show in this article that a black shell presents the same emission rate of a black hole when we consider Klein-Gordon equation in the near horizon limit. We will use Parikh-Wilczek tunneling approach to obtain the black shell emission rate [2].

1 Introduction

Bekenstein-Hawking entropy $S_{BH}$ has been derived from different points of view [12, 13, 14], and all agree that it is proportional to surface area $A$ (in natural units):

$$S_{BH} = \frac{A}{4}.$$ (1)

This suggests that the black hole thermal energy is strongly concentrated near the horizon. With this idea, was conceived the black shell model described above and the entanglement entropy of black shells was obtained [5]. Afterwards, physical entropy of a black shell was derived from Gibbons-Hawking Euclidean approach [6].

In 1975, Hawking showed that black holes emit thermal radiation with temperature (in natural units):

$$T_H = \frac{1}{4\pi R_{Sch}}.$$ (2)

Where $R_{Sch}$ is Schwarzschild radius [10]. In the same paper, Hawking gave an heuristic picture of the black hole radiation as tunnelling of virtual particles across the horizon.

In this context Parikh and Wilczek presented a derivation of black hole evaporation as a tunneling process [2]. We show in this article that a black shell presents the same emission rate of a black hole considering Parikh-Wilczek tunneling approach [2].
2 Hawking Radiation of Black Shells

In 1975 Stephen Hawking presented an heuristic image for the process of radiation in terms of quantum tunnel effect of virtual particles crossing the event horizon [10]. Subsequently, Parihk and Wilczek presented a semiclassic derivation where the tunneling occur near the horizon under energy conservation condition[2].

We consider in this section, a static thin shell of radius $R_0$ outside and near the horizon according to Israel-Arenas black shell model described above [5, 6]. In this context, we employ the Darmois-Israel formalism [3, 1, 15] for a spherical thin shell $\Sigma$ which divides the space-time in two regions: the interior region $M^-$, described by flat Minkowskian geometry and $M^+$, the exterior geometry described by Schwarzschild spacetime. In what follows we will use geometric units where: $C = G = \hbar = 1$, and signature: $(-, +, +, +)$.

The exterior region $M^+$ is described by coordinates: $X^\alpha = (t, r, \theta, \phi)$, and the line element:

$$dS^2 = -f(r) \, dt^2 + f^{-1}(r) \, dr^2 + r^2 \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2. \quad (3)$$

Where: $f(r) = 1 - \frac{R_{sch}}{r}$ and $R_{sch} = 2M$ is the Schwarzschild radius for a shell of mass $M$.

The interior region $M^-$ is described by coordinates: $Y^\alpha = (T, r, \theta, \phi)$, and the line element:

$$dS^2 = -dT^2 + dr^2 + r^2 \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2. \quad (4)$$

The interior and exterior coordinates do not join smoothly on the hypersurface $\Sigma$, but this doesn’t matter because the junction conditions are coordinate independent tensor equations [1].

The shell hypersurface $\Sigma$ is described using coordinates: $\xi^i = (\tau, \theta, \phi)$, and the 3-metric:

$$dS^2 = -d\tau^2 + R_0^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right). \quad (5)$$

where $\tau$ is the proper time of the shell particles.

Lets consider the massless Klein-Gordon equation for a scalar field $\varphi_g$, in the background metric $g_{\mu\nu}$:

$$\frac{1}{\sqrt{|g|}} \partial_{\mu} \left( \sqrt{|g|} g^{\nu\rho} \partial_{\nu} \right) \varphi_g = 0. \quad (6)$$
This equation can be written:

\[
\frac{1}{\sqrt{|g|}} \partial_{\mu} \sqrt{|g|} g^{\mu\nu} \partial_{\nu} \varphi_g + \partial_{\mu} \left( g^{\mu\nu} \partial_{\nu} \right) \varphi_g = 0. \tag{7}
\]

Near the horizon and because tunneling occurs in radial direction, the left term on the previous equation vanishes when \( \omega \ll M \), which gives:

\[
\partial_{\mu} \left( g^{\mu\nu} \partial_{\nu} \right) \varphi_g = 0. \tag{8}
\]

Expressing the field in terms of action \( S \):

\[
\varphi_g = e^{-\frac{i}{\hbar} S}, \tag{9}
\]

and replacing in (8), can be obtained:

\[
\frac{i}{\hbar} \partial_{\mu} \left( g^{\mu\nu} \partial_{\nu} S_g \right) - \frac{1}{\hbar^2} \left( g^{\mu\nu} \partial_{\mu} S_g \partial_{\nu} S_g \right) = 0. \tag{10}
\]

Solving for \( \partial_r S_g \), in the real part of equation (10), and because tunneling occurs in radial direction, we obtain:

\[
\partial_r S_g = \sqrt{-g_{rt} g_{rr}} \omega. \tag{11}
\]

Being \( \partial_t S = \omega \), the energy of the emitted particle.

This solution also solves the complex part of the equation (10), which can be verified by a direct calculation replacing the solution (11):

\[
\partial_r \left( \sqrt{-g^{rt} g^{rr}} \omega \right) = 0 \tag{12}
\]

For a black shell, as previously stated, space-time is divided into 2 regions:

The outer region \( M^+ \) is described by Schwarzschild space-time with metric \( g \).

The equation to be solved for \( M^+ \) and according to previous considerations, is reduced to:

\[
\Box_g \varphi_g = \partial_{\mu} \left( g^{\mu\nu} \partial_{\nu} \right) \varphi_g = 0. \tag{13}
\]

Whose solution is obtained by integrating the equation (11).

The inner region \( M^- \) is described with Minkowskian metric \( \eta \).

And the equation to be solved is:

\[
\frac{1}{\sqrt{|\eta|}} \partial_{\mu} \left( \sqrt{|\eta|} \eta^{\mu\nu} \partial_{\nu} \right) \varphi_\eta = 0. \tag{14}
\]

Or equivalently:
\[ \Box_{\eta} \varphi_{\eta} = \partial_{\mu} \left( \eta^{\mu\nu} \partial_{\nu} \right) \varphi_{\eta} = 0. \]  \hspace{1cm} (15)

The previous equation is linear, which allows to write it as follows:

\[ \partial_{\mu} \left( \left( \eta^{\mu\nu} - g^{\mu\nu} \right) \partial_{\nu} \right) \varphi_{\eta} + \partial_{\mu} \left( g^{\mu\nu} \partial_{\nu} \right) \varphi_{\eta} = 0. \]  \hspace{1cm} (16)

Or equivalently:

\[ \left( \Box_{\eta-g} + \Box_g \right) \varphi_{\eta} = 0. \]  \hspace{1cm} (17)

Is important to observe that the following equation is valid regardless if \( \eta^{\mu\nu} - g^{\mu\nu} \) are well defined metric or solutions of Einstein field equations:

\[ \Box_{\eta} = \Box_{\eta-g} + \Box_g \]  \hspace{1cm} (18)

Suppose that solutions of the previous equation, can be written as: \( \varphi_{\eta} = \varphi_g + \varphi_{\eta-g} + \varphi_x \), then replacing it in (17) shows that solutions of \( \varphi_{\eta} \) are linear combination of solutions of the following equations:

\[ \Box_g \varphi_g = 0 \]  \hspace{1cm} (19)

\[ \Box_{\eta-g} \varphi_{\eta-g} = 0 \]  \hspace{1cm} (20)

\[ \Box_{\eta} \varphi_x = -\Box_{\eta-g} \varphi_g - \Box_g \varphi_{\eta-g} \]  \hspace{1cm} (21)

The contribution to semiclassical emission rate of \( \varphi_{\eta-g} \), is obtained by solving the real part of:

\[ S = -Im \left( \int_{r_1}^{r_f} \sqrt{\frac{(\eta^{tt} - g^{tt})}{(\eta^{rr} - g^{rr})}} \omega dr \right). \]  \hspace{1cm} (22)

Where: \( \omega \) is the energy of the emitted particle.

Replacing the inverse metric components:

\[ g^{tt} = -\frac{1}{f(r)}, \]
\[ g^{rr} = f(r), \]
\[ \eta^{tt} = -1, \]
\[ \eta^{rr} = 1. \]

Leads to:
\[ S_{\eta-g} = \int_{r_i}^{r_f} \sqrt{\frac{(\eta_{tt} - g_{tt})}{(\eta_{rr} - g_{rr})}} \omega dr = \int_{r_i}^{r_f} \sqrt{\frac{-r}{r - R_{Sch}}} \omega dr \]  

(23)

With the change of variable: \( r = R_{Sch} + \epsilon e^{i\theta}; \) \( dr = i\epsilon e^{i\theta}, \) the solution is:

\[ S_{\eta-g} = \lim_{\epsilon \to 0} \int_0^\pi -\sqrt{\epsilon} e^{i\theta} \sqrt{r_p + \epsilon e^{i\theta}} \omega d\theta = 0 \]  

(24)

This result shows that in the integration region \( R_{Sch} - \epsilon \leq r \leq R_{Sch} + \epsilon, \) the solution \( \varphi_{\eta-g} \) vanishes in the limit \( \epsilon \to 0: \)

\[ \varphi_{\eta-g} = 0. \]  

(25)

Replacing (25) and (11) in (21) shows that \( \varphi_x = \varphi_g, \) and by this reason \( \varphi_g \) is the only solution for the inner region \( M^- \). We must note that the interior and exterior coordinates do not join smoothly on \( \Sigma, \) but they are both solutions of coordinate independent tensor equations. Therefore, the junction conditions allow us to join the solutions of both regions and say that \( \varphi_g \) is solution for the entire space: \( M^- \cup M^+. \)
3 Conclusions

Using the Parikh-Wilczek tunneling method \[2\] and solving the Klein-Gordon equation in the near horizon limit, we show that semiclassical emission rate of a black shell is the same as that of a black hole. Israel junction conditions \[1] guarantees that this solution is valid for the entire space formed by \(M^- \cup M^+\), and we can conclude that Israel-Arenas black shell emits thermal radiation with Hawking temperature. The previous arguments could be extended to Reissner-Nordström space-time.

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