A LITTLEWOOD-RICHARDSON RULE FOR THE MACDONALD INNER PRODUCT AND BIMODULES OVER WREATH PRODUCTS.

ERIK CARLSSON AND ANTHONY M. LICATA

Abstract. We prove a Littlewood-Richardson type formula for \((s_{\lambda/\mu}, s_{\nu/\kappa})_{t^k,t}\), the pairing of two skew Schur functions in the MacDonald inner product at \(q = t^k\) for positive integers \(k\). This pairing counts graded decomposition numbers in the representation theory of wreath products of the algebra \(\mathbb{C}[x]/x^k\) and symmetric groups.

1. Introduction

Let \(\Lambda\) denote the algebra of symmetric functions, endowed with the standard bilinear form with respect to which the Schur basis \(\{s_\lambda\}\) is orthonormal. The Littlewood-Richardson rule gives an enumerative formula for the inner products

\[c_{\mu\nu}^\lambda = (s_{\nu}^* s_{\lambda}, s_{\mu}),\]

which are known as Littlewood-Richardson coefficients. Here \(s_{\nu}^*\) is the linear operator on symmetric functions adjoint to multiplication by the Schur function \(s_{\nu}\). (We refer to [3, 5] for detailed treatments of the Littlewood-Richardson rule). The \(c_{\mu\nu}^\lambda\) are non-negative integers, and they enumerate tableaux satisfying certain conditions. The integrality of the Littlewood-Richardson coefficients is also manifest in their appearance as tensor product multiplicities in the representation theory of symmetric groups and as intersection numbers in the Schubert calculus of Grassmannians. A mild generalization of the Littlewood-Richardson rule adds a fourth partition to the picture: consider the algebra \(H_\Lambda\) of operators on symmetric functions spanned by the operators \(\{s_{\mu} s_{\nu}^*\}_{\mu,\nu}\). Define \(c_{\mu\nu}^{\lambda\kappa}\) to be structure constants in the expansion

\[s_{\nu}^* s_{\lambda} = \sum_{\mu,\kappa} c_{\mu\nu}^{\lambda\kappa} s_{\mu} s_{\nu}^*,\]

For \(\kappa = \emptyset\) these are the ordinary Littlewood-Richardson coefficients; an enumerative formula for the general case was found by Zelevinsky [7] in the language of pictures. The algebra \(H_\Lambda\) is a Hopf algebra; in fact it is the Heisenberg double of the Hopf algebra \(\Lambda\). Thus the Littlewood-Richardson coefficients may also be thought of as structure constants in the canonical basis of the Hopf algebra \(H_\Lambda\).

Let \(\Lambda_{q,t}\) denote the algebra of symmetric polynomials over the two-variable coefficient ring \(\mathbb{C}(q,t)\), together with the MacDonald inner product \((\cdot,\cdot)_{q,t}\), which specializes to the standard inner product at \(q = t\). We will be interested in the specialization \(\Lambda_{t^k,t}\) for some integer \(k \geq 1\). The ring \(\Lambda_{t^k,t}\) appears in several other mathematical contexts, including the representation theory of quantum affine algebras. In particular, several important bases of \(\Lambda_{t^k,t}\), such as the Schur basis, should be related to important bases in the representation theory of quantum affine algebras and in the geometry of quiver varieties of affine type. As a result, it is natural to suspect that much of the positive integral structure appearing the ordinary theory of symmetric functions will admit an interesting generalization from \(\Lambda\) to \(\Lambda_{t^k,t}\). The first goal of the present paper is to extend [1] to the ring \(\Lambda_{t^k,t}\), in which the dual
We consider the Heisenberg algebra, as presented in [1]. The structure constants of the algebra are given by computing the structure constants for multiplication in the canonical basis of the quantum group. Similarly, the generalized Littlewood-Richardson coefficients of Theorem A. These coefficients play a crucial role in our original motivation for considering the polynomials of symmetric groups [4].

A well-known isomorphism between $\Lambda$ and the Grothendieck group of representations of $S_n$ is established in [4]. This isomorphism does not preserve an interesting grading on the module category, and the identification above reduces to the usual Littlewood-Richardson rule. The proof of this theorem is given in Section 2, which deals only with combinatorics. We also show in proposition 4 that these coefficients are symmetric and unimodal, something that is not obvious from enumerative description of Theorem A.

In Section 3, we identify an integral form $\Lambda_{t,k}$ of $\Lambda_{t,k}$ with the Grothendieck group of graded projective modules over an $S_n$-equivariant graded ring. Under this identification, the Schur polynomial $s_\lambda$ corresponds to a certain indecomposable projective module $S_\lambda$, and $(s_\mu, s_\nu)_t$ measures the graded dimensions of the Hom$(S_\mu, S_\nu)$ up to a grading shift. In particular, this implies that this graded dimension is a Laurent polynomial in $t$ with nonnegative integer coefficients, explaining the positive-integral structure of the generalised Littlewood-Richardson coefficients $c_{\mu\nu}^\kappa(t)$. Specifically, let $A_k = C[x]/x^k$, and let $A_k^{[n]}$ denote the smash product of $A_k$ with $C[S_n]$. The algebras $A_k^{[n]}$ are graded, and we consider the Grothendieck group $K(A_k^{[n]} - \text{gmod})$ of finitely generated projective $A_k^{[n]}$ modules. This space is a free $\mathbb{Z}[t, t^{-1}]$ module, where multiplication by $t$ corresponds to a shift in the grading. Since the Hom pairing in the category of graded $A_k^{[n]}$ modules induces a semi-linear pairing on the Grothendieck group, we slightly modify the bilinear form on $\Lambda_{t,k}$ to be semi-linear in $t$. Our second main theorem is then the following.

**Theorem B.** There is an isometric isomorphism

$$\Phi : \bigoplus_{n=0}^{\infty} K(A_k^{[n]} - \text{gmod}) \rightarrow \Lambda_{t,k},$$

where the bilinear form on the left hand side is induced from the Hom bifunctor.

As a result, we obtain an interpretation of the polynomials $c_{\mu\nu}^\kappa(t)$ as decomposition numbers in the convolution product of explicit bimodules over the rings $A_k^{[n]}$. When $k = 1$ there is no interesting grading on the module category, and the identification above reduces to the well-known isomorphism between $\Lambda$ and the Grothendieck group of representations of all symmetric groups $\mathfrak{S}_n$.

The graded rings $A_k^{[n]}$ appear in several other representation theoretic contexts. For example, these algebras for $k = 2$ play a central role in the categorification of the Heisenberg double of $\mathfrak{S}_2$ and in the level one quantum affine categorifications of $\mathfrak{sl}_2$; the algebras $A_k^{[n]}$ for $k > 2$ should appear in the higher level analogs of those constructions. In fact, part of our original motivation for considering the polynomials $c_{\mu\nu}^\kappa(t)$ was to combinatorially compute the structure constants for multiplication in the canonical basis of the quantum Heisenberg algebra considered in [4]; these structure constants are given by the $k = 2$ case of Theorem A. Similarly, the generalized Littlewood-Richardson coefficients $c_{\mu\nu}^\kappa(t)$ should...
all appear as structure constants for multiplication in the canonical basis of certain infinite dimensional Hopf algebras. Another closely related appearance of $A^n_k$ involves category $O$ for the rational Cherednik algebra of the complex reflection group $(\mathbb{Z}/k\mathbb{Z}) \wr S_n$ at integral parameters. The algebra $A^n_k$ is an example of an Ariki-Koike algebra, and the study of $O$ via the KZ functor involves mapping $O$ to a category of $A^n_k$ modules. From this point of view, Theorems A and B together have applications to combinatorial description of hom spaces between projective modules in $O$, though we have not fully explored this here. Relationships between higher-level Heisenberg categorification and category $O$ for rational Cherednik algebras appear in [6].

The bilinear form on $\Lambda_{k^d, t}$, the ring $A^n_k$, and the generalized Littlewood-Richardson coefficients $c_{\mu \lambda}^\nu(t)$ all depend on the choice of positive integer $k$. It might be interesting to formulate versions of Theorems A and B in a way that does not require $k$ to be a positive integer and thus recover the variable $q$ in the Macdonald theory. Notice that a rational function in $\mathbb{C}(q, t)$ is determined by its values at $q = t^k$ at all positive integers $k$, so in a sense we lose no information by specializing.

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2. The Littlewood-Richardson Rule

2.1. Notations. For all notations in this section, we have followed MacDonald’s book [5].

Given partitions $\kappa, \lambda, \mu, \nu$, let $\text{Tab}(\lambda - \mu, \nu)$ denote the set of semi-standard Young tableaux of shape $\lambda - \mu$, and content $\nu$. The word $w(T)$ of a tableau $T$ is the set of numbers in the diagram read from right to left, top to bottom. For instance, the word of the tableau

```
1 1 2 4 4 4
3 4
1 2 2
```

is 4442143221. The word of a partition $w(\lambda)$ is defined as the word of the tableau of shape $\lambda$ in which row $i$ is filled with the number $i$. Let $\text{Tab}^0(\lambda - \mu, \nu)$ denote the subset of tableau whose word $a_1 \cdots a_n$ is a lattice permutation, meaning that for each $i, k$, the number of occurrences of $i$ in $a_1 \cdots a_k$ is greater than or equal to the number of occurrences of $i + 1$. More generally, we define $\text{Tab}^0(\lambda - \mu, \nu - \kappa)$ to be the set of tableaux $T$ with content $\nu - \kappa$ such that the concatenated word $w(\kappa)w(T)$ is a lattice permutation.

Fix an integer $k \geq 1$, and define a $k$-tableau of shape $\lambda - \mu$ to be a labeling of the boxes of $\lambda - \mu$ with monomials of of the form $at^b$ for $a \geq 1, 0 \leq b \leq k - 1$. We define a total order on these monomials by

$$at^b \leq ct^d \iff b < d \text{ or } b = d \text{ and } a \leq c,$$

which is the same as the ordering obtained by replacing $t$ by a large positive number. Call $T'$ the tableau obtained from $T$ by setting $t = 1$, and let $\text{Tab}_k(\lambda - \mu, \nu - \kappa)$ be the set of $k$-tableaux which are semistandard with respect to (2), such that content of $T'$ is $\nu - \kappa$. We also define a statistic on $k$-Tableau by

$$c(T) = \prod_{at^b \in T} t^b.$$
Any $k$-tableau $T$ corresponds to a sequence of regular tableaux $T^i$ for $i \geq 0$, defined as the subdiagram of coefficients in all boxes containing $at^i$ for some $a$. For instance, if

$$T = \begin{array}{ccc} 3 & 2t & 2t \\
1 & t & t^2 \\
1^2 & 1^2 \\
\end{array}$$

then

$$T^0 = \begin{array}{ccc} 3 \\
1 \\
1^2 \\
\end{array}, \quad T^1 = \begin{array}{ccc} 2 & 2 \\
1 \\
1 \\
\end{array}, \quad T^2 = \begin{array}{ccc} 1 & 1 \\
1 \\
1 \\
\end{array}.$$  

Let $\text{Tab}^0_k(\lambda - \mu, \nu - \kappa)$ denote the subset of $k$-tableaux such that $w(\kappa)w(T^0)w(T^1) \cdots$ is a lattice permutation. For instance,

$$\begin{array}{ccc} 1 & 2 & 3 & t^2 \\
1 & t & t \\
1^2 & 1^2 \\
\end{array} \in \text{Tab}^0_3([532] - [21], [442] - [21])$$

because 1123211122 is a lattice permutation, and the semistandardness condition is satisfied. The statistic is $c(T) = t^6$.

2.2. **The main theorem.** Consider the space of symmetric polynomials $\Lambda$ in infinitely many variables, and let $p_\mu$, $e_\mu$, $h_\mu$, and $s_\mu$ denote the power sum, elementary, complete, and Schur bases respectively. We denote by $\Lambda^{\square} = \text{span}_{\mathbb{Q}[t,t^{-1},q,q^{-1}]} \{s_\mu\}$ the integral form of $\Lambda$ spanned by the Schur functions. The MacDonald inner product on $\Lambda$ is defined in the power sum basis by

$$(p_\mu, p_\nu)_{q,t} = \delta_{\mu\nu} \delta(\mu) \prod_j \frac{1 - q^{\mu_j}}{1 - p_j},$$

where

$$p_\mu = \prod_j p_j, \quad p_j = \sum_i x_i^j, \quad \delta(\mu) = \text{aut}(\mu) \prod_j \mu_j.$$

For a fixed integer $k$, and any symmetric polynomial $f \in \Lambda$, define a dual multiplication operator by

$$(f^* g, h)_{t^k,s} = (g, fh)_{t^k,s}.$$  

We may now state the main theorem:

**Theorem 1.** Fix a positive integer $k$. There exist unique coefficients $c^\kappa_\mu(\lambda)$ satisfying

$$s^*_\nu s_\lambda = \sum_{\mu, \kappa} c^\kappa_\mu(\lambda) s_\mu s^*_\kappa.$$  

Furthermore,

$$(4) \quad c^\kappa_\mu(\lambda) = \sum_{T \in \text{Tab}^0_k(\lambda - \mu, \nu - \kappa)} c(T),$$

if $\mu \subset \lambda$ and $\kappa \subset \nu$, or zero otherwise.
Example 1. Take $k = 2, \kappa = \{1\}, \lambda = \{32\}, \mu = \{1\}, \nu = \{32\}$. By lemma below we have
\[
c_{\kappa \lambda \mu \nu}(t) = (s_{\lambda / \mu}, s_{\nu / \kappa})_{t^2, t} = 2 + 5t^2 + 5t^3 + 2t^4.
\]
Which equals 21 at $t = 1$. On the other hand, there are 25 elements of $\text{Tab}_2([32] - [1], [32] - [1])$. The remaining four that do not satisfy the lattice word condition are
\[
\begin{array}{cccc}
1 & 2 & 2 & 1 \\
1 & 2 & 2 & t \\
2 & t & 2 & t \\
2 & t & 2 & 2t
\end{array}
\]
Example 2. If $\nu = (n), \lambda = (m)$ consist of a horizontal strip, then $c_{\kappa \lambda \mu \nu}(t)$ is zero unless $\kappa = (n - l), \mu = (m - l)$ for some $l \geq 0$. In this case
\[
\text{Tab}_k^0(\lambda - \mu, \nu - \kappa) = \text{Tab}_k(\lambda - \mu, \nu - \kappa)
\]
because the coefficients of every box are one, so that the lattice word condition is always satisfied. By stars and bars, we have
\[
c_{(n - l)(m)}^{(n - l), (m)}(t) = \frac{(1 - t^{k+l-1}) \cdots (1 - t^k)}{(1 - t) \cdots (1 - t^l)}.
\]
Before proving the theorem, we need a couple of lemmas. Let $s_{\lambda / \mu}$ denote the skew Schur polynomial. More generally, if $u_\mu$ is any basis of $\Lambda$, we define
\[
u_{\lambda / \mu} = u_\mu^* u_\lambda
\]where the dual is taken with respect to the standard inner product, $k = 1$.

Lemma 1. We have
\[
c_{\kappa \lambda \mu \nu}^{\kappa \lambda}(t) = (s_{\lambda / \mu}, s_{\nu / \kappa})_{t^2, t}.
\]
Proof. It is straightforward to check this relation when the Schur basis is replaced by the power sums on the right hand side, and in the definition of $c_{\kappa \lambda \mu \nu}^{\kappa \lambda}(t)$. Then simply apply the change of basis matrix from $p$ to $s$ to each of the coordinates in the tensor $c_{\kappa \lambda \mu \nu}^{\kappa \lambda}(t)$.

Lemma 2. We have
\[
(s_{\lambda / \mu}, h_{\nu})_{t^2, t} = \sum_{T \in \text{Tab}_k(\lambda - \mu, \nu)} c(T).
\]
Proof. We begin by rewriting
\[
(s_{\lambda / \mu}, h_{\nu})_{t^2, t} = \sum_{T \in \text{Tab}_k(\lambda - \mu, \nu)} c(T).
\]
where $g_\mu$ is the image of $h_\mu$ under the homomorphism defined on generators by
\[
\rho_{t^2, t} : p_j \mapsto \frac{1 - t^{k_j}}{1 - t^j} p_j.
\]
Now let us proceed by induction on the length of $\nu$. We start with the base case in which $\nu$ only has one component, say $\nu_1 = m$. We first find the coefficients in the expansion
\[
g_m = \sum_{T \in \text{Tab}_k(\lambda - \mu, \nu)} c(T).
\]
We claim that
\[
a_\pi = \sum_{\beta \to \pi} \sum_{0 \leq a_1 < \cdots < a_k \leq k-1} t^{a_1 \beta_1 + \cdots + a_k \beta_k}.
\]
where $\ell = \ell(\pi)$ is the length of $\pi$, and $\beta$ ranges over all rearrangements of $\pi$, i.e. over the orbit of $\pi$ under $S_\ell$. The easiest way to see this is to notice that

$$g_\mu(x_1, x_2, \ldots) = h_\mu(x_1, x_2, \ldots; t x_1, t x_2, \ldots; t^2 x_1, t^2 x_2, \ldots),$$

and compare the coefficient of $x^\nu$ with the right hand side of (6).

Inserting (6) and (7) into (5), and using the Pieri rule, we have

$$\sum_{\pi} (s_\lambda/\mu, s_\pi) s_{\mu\nu} = \sum_{\beta} \sum_{a_1 < \cdots < a_\ell} t^{a_1 \beta_1 + \cdots + a_\ell \beta_\ell} |\operatorname{Tab}(\lambda - \mu, \beta)|.$$

In particular, the tableau count is independent of the ordering of $\beta$. Finally, there is a bijection

$$\bigcup_{\beta} \{ a_1 < \cdots < a_\ell \} \times \operatorname{Tab}(\lambda - \mu, \beta) \longleftrightarrow \operatorname{Tab}_k(\lambda - \mu, m),$$

in which a box label $j$ maps to $a_j$. The statistic $c(T)$ corresponds to the power of $t$ in $\mathbb{F}$, proving the base case.

Now for the induction step, let $\nu = \nu' \cup \nu''$ be any decomposition of $\nu$, so that $g_\nu = g_{\nu'} g_{\nu''}$. Let us also fix an identification of the rows of $\nu', \nu''$ with the corresponding row of $\nu$. Since

$$(s_{\lambda, \nu} s_{\mu}) = \sum_{\pi} (s_\lambda, g_{\nu'} s_{\pi})(s_{\pi, \nu''} s_{\mu}),$$

it suffices to prove that there is a bijection

$$(\operatorname{Tab}_k(\lambda - \mu, \nu) \leftrightarrow \bigcup_{\pi} \operatorname{Tab}_k(\lambda - \pi, \nu') \times \operatorname{Tab}_k(\pi - \mu, \nu''),)$$

such that $c(T) = c(U)c(V)$ whenever $T$ maps to $(U, V)$. To do this, partition the left side into groups $\operatorname{Tab}_k(\lambda - \mu, \nu)_{\gamma}$ indexed by the multiset $\gamma$ of all monomials $a t^b$ in $T$. Since $\gamma$ is a total ordering, any choice of content $\gamma$ induces a multiset $c \in \mathbb{Z}_{\geq 0}$ such that

$$|\operatorname{Tab}_k(\lambda - \mu, \nu)_{\gamma}| = |\operatorname{Tab}(\lambda - \mu, \nu)|.$$

The choice of $\gamma$ induces a corresponding decomposition $(\gamma', \gamma'')$ on the right side of (9), by taking the elements of $\gamma', \gamma''$ to be the monomials $a t^b \in \gamma$ such that the row $\nu_a$ is in $\nu', \nu''$ respectively. The argument of the preceding paragraph reduces the statement to the case $k = 1$, which is true by the usual Littlewood-Richardson rule.

We may now prove the theorem.

**Proof.** The existence and uniqueness statement follow from lemma 1.

We first prove the case $\kappa = \emptyset$, by extending the proof of the usual Littlewood-Richardson rule, as it is explained in [2]. Using the expansion of $s_\nu$ in the monomial basis, which is well known to be the dual basis to $h_\nu$ under the standard inner product, we have

$$(s_\lambda/\mu, h_\nu)_{k, \ell} = \sum_{\pi} (s_\lambda/\mu, s_{\pi})_{k, \ell} |\operatorname{Tab}(\pi, \nu)|.$$

Then by lemma 2 and the invertibility of the triangular matrix $|\operatorname{Tab}(\pi, \nu)|$, it suffices to prove that there is a bijection

$$(\operatorname{Tab}_k(\lambda - \mu, \nu) \leftrightarrow \bigcup_{\pi} \operatorname{Tab}_k(\lambda - \mu, \pi) \times \operatorname{Tab}(\pi, \nu),$$

producing the correct statistic $c(T)$.

Each tableau $T \in \operatorname{Tab}_k(\lambda - \mu, \nu)$ has an associated filtration

$$F : \mu = \mu^1 < \cdots < \mu^{k+1} = \lambda,$$
so that $\mu_{i+1} - \mu_i$ is the shape of $T_i$. We obtain a decomposition

$$
\text{Tab}_k(\lambda - \mu, \nu) \leftrightarrow \bigcup_F \text{Tab}_k(\lambda - \mu, \nu)_F,
$$

and similarly for $\text{Tab}_0^0(\lambda - \mu, \nu)$, by restriction. Given such a filtration $F$, let $\lambda_F$ denote any skew diagram which is a disconnected union of the shapes $\mu_{i+1} - \mu_i$, positioned in the plane in some way in order from upper right to lower left. For instance, one choice of $\lambda_F$ might be

$$
F = [1] \subset [3, 2] \subset [5, 2] \subset [5, 3, 1], \quad \lambda_F = \square.
$$

For any choice of $\lambda_F$, we obtain a bijection

$$
\text{Tab}_k(\lambda - \mu, \nu)_F \leftrightarrow \text{Tab}(\lambda_F, \nu),
$$

that respects the corresponding lattice word conditions. This reduces (10) to the case $k = 1$, which follows from the usual algorithm of Littlewood-Robinson-Schensted.

We now prove the general case. Using the usual Littlewood-Richardson rule to expand $s_{\nu/\kappa}$, we have

$$
(s_{\lambda/\mu}, s_{\nu/\kappa})_t^k = \sum_{\pi} (s_{\lambda/\mu}, s_{\pi})_t^k \text{Tab}_0^0(\nu - \kappa, \pi).
$$

Then applying the special case we just proved, it suffices to find a suitable bijection

$$
\text{Tab}_k^0(\lambda - \mu, \nu - \kappa) \leftrightarrow \bigcup_{\pi} \text{Tab}_k^0(\lambda - \mu, \pi) \times \text{Tab}_k^0(\nu - \kappa, \pi).
$$

Once again, the decomposition (11) reduces this statement to the case $k = 1$, which was proved by Zelevinsky [7].

Now consider the semi-linear form on $\Lambda_{t, k, t}$ defined by

$$
\langle f, g \rangle_k = (f, t^{(1-k)}g)_{2k, t^2},
$$

Where the conjugation takes $t \mapsto t^{-1}$. Its extension to the rationals takes the form

$$
\langle p_{\mu}, p_{\nu} \rangle_k = \delta_{\mu, \nu} \delta(\mu) \prod_i \left( t^{(1-k)\mu_i} + t^{(3-k)\mu_i} + \cdots + t^{(k-1)\mu_i} \right),
$$

establishing that it is Hermitian. With respect to this inner product, we then have

$$
\sum_{\mu, \kappa} C_{\mu \nu}^{\kappa \lambda}(t) s_{\mu} s_{\kappa}^*,
$$

where

$$
C_{\mu \nu}^{\kappa \lambda}(t) = t^{(1-k)(|\lambda| - |\mu|)} E_{\mu \nu}^{\kappa \lambda}(t^2).
$$

Call a Laurent polynomial

$$
f(t) = a_{-1}t^{-1} + a_2 t^2 + \cdots + a_{t-2} t^{t-2} + a_{t} t^t
$$

symmetric unimodal if for all $f(t) = f(t^{-1})$, and $a_i \leq a_{i+2}$ for $i < 0$.

**Proposition 1.** The polynomials $C_{\mu \nu}^{\kappa \lambda}(t)$ are symmetric unimodal.
3.1. Wreath products of $C$ of partitions of this way, or a representation-theoretic proof, along the lines of the next section.

Since the sum of unimodal expressions is unimodal, it suffices to check the unimodality of $\langle s_\mu, s_\nu \rangle_k$, by lemma [1] and the Schur-positivity of the skew Schur functions. We have

$$\langle s_\mu, s_\nu \rangle_k = (s_\mu, t^{(1-k)d_{12k}}) \sum_{\kappa, \lambda} a_{\kappa, \lambda} s_\kappa(t^{1-k}, t^{3-k}, ..., t^{k-1})(s_\mu, s_\lambda),$$

where $a_{\kappa, \lambda}$ are the multiplicities of the decomposition into irreducibles

$$S_\nu(U \boxtimes V) = \bigoplus_{\kappa, \lambda} a_{\kappa, \lambda} S_\kappa(U) \boxtimes S_\lambda(V)$$

over $GL(U) \times GL(V)$. In particular, they are nonnegative integers. The answer now follows from the unimodality of $s_\kappa(t^{1-k}, t^{3-k}, ..., t^{k-1})$, see [3] chapter I, section 8, example 4.

A notable feature of Proposition [1] is that neither symmetry nor unimodality is immediately clear from theorem [1]. It would be interesting to give a purely enumerative proof in this way, or a representation-theoretic proof, along the lines of the next section.

3. Categorification

3.1. Wreath products of $H^*(\mathbb{P}^{k-1})$. Fix the integer $k \geq 1$, and let $A_k = H^*(\mathbb{P}^{k-1}, \mathbb{C}) \cong \mathbb{C}[x]/x^k$. Denote by $A_k^{[n]}$ the smash product of $A_k$ with the group algebra of the symmetric group $S_n$.

$$A_k^{[n]} = A_k \# \mathbb{C}[S_n].$$

As a vector space, $A_k^{[n]} = A_k^{\otimes n} \otimes \mathbb{C}[S_n]$, and by convention we take $A_k^{[0]} = \mathbb{C}$. We give $A_k^{[n]}$ a grading by declaring the degree of $x$ to be 1, and putting $\mathbb{C}[S_n]$ in degree 0.

Let $K(A_k^{[n]})$ denote the Grothendieck group of the category of $\mathbb{Z}$-graded finitely-generated projective left $A_k^{[n]}$ modules. The $\mathbb{Z}$-grading on $A_k^{[n]}$ endows $K(A_k^{[n]})$ with the structure of a free $\mathbb{Z}[t, t^{-1}]$ module, where shifting by 1 in the internal grading of a module corresponds to multiplication by $t$ on the class in the Grothendieck group,

$$[M \{\pm 1\}] = t^\pm 1[M] \in K(A_k^{[n]}).$$

If we choose a complete set of minimal idempotents $e_\lambda \in \mathbb{C}[S_n]$, so that $\{\mathbb{C}[S_n]e_\lambda\}_{\lambda \vdash n}$ are representatives of the isomorphism classes of irreducible $\mathbb{C}[S_n]$ modules, then $\{A_k^{[n]}e_\lambda\}$ are representatives of the isomorphism classes of indecomposable projective $A_k^{[n]}$ modules, up to grading shift. It follows that the rank of $K(A_k^{[n]})$ as a free $\mathbb{Z}[t, t^{-1}]$ module is the number of partitions of $n$. Thus the free $\mathbb{Z}[t, t^{-1}]$ module $K(A_k^{[n]})$ comes equipped with a canonical basis, namely, the classes $\{[A_k^{[n]}e_\lambda]\}_{\lambda \vdash n}$ of the indecomposable projective modules.

Let $\mathcal{F}_k = \oplus_{n=0}^\infty K(A_k^{[n]})$. Homomorphisms in the module category endow $\mathcal{F}$ with a bilinear form,

$$\langle [X], [Y] \rangle_{\mathcal{F}_k} = gdim \text{ Hom}(X, Y) \in \mathbb{Z}[t, t^{-1}],$$

where $gdim V$ denotes the graded dimension of a $\mathbb{Z}$-graded vector space $V$. By convention, the individual summands $K(A_k^{[n]})$ for different $n$ are orthogonal with respect to this bilinear form. This form is semi-linear with respect to $t$,

$$\langle t^\pm 1[M], [N] \rangle_{\mathcal{F}_k} = t^{\pm 1}\langle [M], [N] \rangle_{\mathcal{F}_k} = \langle [M], t^{\mp 1}[N] \rangle_{\mathcal{F}_k}.$$
The following Lemma can be checked easily using the fact that $A_k^{[n]}$ has a nondegenerate trace $tr : A_k^{[n]} \to \mathbb{C}$.

**Lemma 3.** With respect to the basis $\{[A_k^{[n]}e_\lambda]\}$, the matrix of the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{F}_k}$ is symmetric.

(The above symmetry together with the semi-linearity together can be thought of as defining a Hermitian inner product on $\mathcal{F}_k$ after specialising $t$ to a point on the unit circle.)

3.2. Bimodules. The standard embeddings $S_m \subset S_n, m \leq n$ of small symmetric groups into larger ones give rise to embeddings of algebras $A_k^{[m]} \subset A_k^{[n]}$. Let $\lambda \vdash m$ a partition, and let $e_\lambda \in \mathbb{C}[S_m]$ be an associated minimal idempotent in the group algebra. We view $e_\lambda$ as an element of $A_k^{[n]}$ for any $n > m$ via the embeddings $\mathbb{C}[S_m] \subset A_k^{[m]} \subset A_k^{[n]}$, and set

$$P^\lambda = A_k^{[n]}e_\lambda$$

and $Q^\lambda = e_\lambda A_k^{[n]}\{m\}$. The internal grading shift $\{m\}$ in the definition of $Q^\lambda$ is for convenience; it ensures that various hom spaces occurring later have a grading which is symmetric about the origin.

The space $P^\lambda$ is naturally an $(A_k^{[n]}, A_k^{[n-m]})$ bimodule, while the space $Q^\lambda$ is naturally a $(A_k^{[n-m]}, A_k^{[n]})$ bimodule. By convention, we set $P^\lambda = Q^\lambda = 0$ when $m > n$, and $P^\emptyset = Q^\emptyset = A_k^{[n]}$ as an $(A_k^{[n]}, A_k^{[n]})$ bimodule.

We denote tensor products of these bimodules by concatenation, where the tensor product is understood to be over the algebra $A_k^{[l]}$ acting on both sides of the tensor product. So, for example, if $\lambda \vdash m$ and $\mu \vdash l$, then for each $n \geq \max\{m, l\}$,

$$P^\lambda Q^\mu = P^\lambda \otimes A_k^{[n-m]} Q^\mu$$

is an $(A_k^{[n]}, A_k^{[n-m+l]})$ bimodule. On the other hand,

$$Q^\mu P^\lambda = Q^\mu \otimes A_k^{[n]} P^\lambda$$

is an $(A_k^{[n-l]}, A_k^{[n-m]})$ bimodule.

Denote by $1_0$ the trivial module over $A_k^{[0]} = \mathbb{C}$. The $\mathbb{Z}[t, t^{-1}]$-module $\mathcal{F}_k$ comes equipped with various natural bases indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$. Examples include

1. $S^\lambda := P^\lambda 1_0$ (This module was denoted $A_k^{[n]}e_\lambda$ in the previous subsection),
2. $E^\lambda := P^{(1^{\lambda_1})} P^{(1^{\lambda_2})} \ldots P^{(1^{\lambda_r})} 1_0$, and
3. $H^\lambda := P^{(\lambda_1)} P^{(\lambda_2)} \ldots P^{(\lambda_r)} 1_0$.

3.3. The character map. For the remainder of this paper, we consider the integral form $\Lambda_{\ell, t}$ of $\Lambda_{\ell, t}$, by definition $\Lambda_{\ell, t}$ is the free $\mathbb{Z}[t, t^{-1}]$-module spanned by the Schur functions. We define a map of $\mathbb{Z}[t, t^{-1}]$ modules

$$\Phi : \mathcal{F}_k \to \Lambda_{\ell, t}, \quad \Phi([S_\lambda]) = s_\lambda$$

by sending each canonical basis vector to the corresponding Schur function. It is straightforward to check that $\Phi([E^\lambda])$ is the elementary symmetric function $e_\lambda$, while $\Phi([H^\lambda])$ is the complete symmetric function $h_\lambda$.

The proof of the following theorem will be given at the end of this section.

**Theorem 2.** The map $\Phi$ is an isometry with respect to $\langle \cdot, \cdot \rangle_{\mathcal{F}_k}$, $\langle \cdot, \cdot \rangle_k$. 
The above theorem translates to the statement that the module categories for the algebras $A_k^{[n]}$ for all $n$ together categorify Macdonald’s ring of symmetric functions at $q = t^k$. When $k = 1$, the map $\Phi$ is just the Frobenius character map, and the above theorem is of course well known. Note that $A_{1,t}$ is the ring of symmetric functions (over $\mathbb{Z}[t, t^{-1}]$) endowed with a bilinear form with respect to which the Schur functions $\{s_\lambda\}$ are orthonormal basis. On the other hand, $A_k^{[n]} = \mathbb{C}[S_n]$ is a semi-simple algebra, whence the classes of indecomposable projective (irreducible) modules give an orthonormal basis in the Grothendieck group.

### 3.4. Representation-theoretic interpretation of $c^\kappa_\mu_\lambda(t)$.

As a consequence of Theorem 2 the generalized Littlewood-Richardson coefficients $c^\kappa_{\mu\nu}(t)$ inherit an interpretation as the graded dimension of vector spaces arising in the representation theory of the algebras $A_k^{[n]}$. To explain this interpretation, we recall the following.

**Proposition 2.** The bimodules $P^\lambda Q^n$ are indecomposable. Any bimodule of the form $Q^\alpha P^\beta$ decomposes as a direct sum of (graded shifts of) the indecomposable bimodules $(P^\lambda Q^n)_{\lambda,\mu}$.

**Proof.** As explained in, for example, [1, Proposition 6], it is straightforward to check that $\text{End}(P^\lambda Q^n)$ is a non-negatively graded algebra whose degree 0 piece is one-dimensional. Thus $P^\lambda Q^n$ is indecomposable. The fact that all indecomposable bimodules are of the form $P^\lambda Q^n$ also follows just as in [1, Proposition 6].

For partitions $\kappa, \lambda, \mu, \nu$, we may therefore define a $\mathbb{Z}$ graded vector $C^\kappa_{\mu\nu}$ as the multiplicity space of $P^\mu Q^\kappa$ in the decomposition of $Q^\nu P^\lambda$ into indecomposable bimodules:

$$Q^\nu P^\lambda = \bigoplus_{\mu, \kappa} P^\mu Q^\kappa \otimes_{\mathbb{C}} C^\kappa_{\mu\nu}.$$  

**Theorem 3.** The graded dimension of $C^\kappa_{\mu\nu}$ is equal to the generalized Littlewood-Richardson coefficient $c^\kappa_{\mu\nu}(t)$.

**Proof.** This follows immediately from Theorems 1 and 2 and the normalization [13].

In light of Proposition 1 and Theorem 3 above, it is tempting to speculate that the graded vector space $C^\kappa_{\mu\nu}$ can be endowed with a linear action of the Lie algebra $\mathfrak{gl}_2$ in a way that aligns the weight space decomposition with the grading; this would give a more conceptual explanation of the symmetry and unimodality of these coefficients.

### 3.5. Proof of Theorem 2.

We now give the proof of Theorem 2. For each $n \geq m$ the $(A_k^{[n]}, A_k^{[n-m]})$ bimodule $P^\lambda$ is flat, as is the bimodule $Q^\lambda$. Summing over $n$, we have induced endomorphisms of the Grothendieck group

$$[P^\lambda], [Q^\lambda] : \mathcal{F}_k \to \mathcal{F}_k.$$  

The proof of the following proposition is given in [1]. That reference was concerned with the particular case $k = 2$, although the proof carries over with easy modification to general $k \geq 1$.

**Proposition 3.** [1, Proposition 2] The operators $[P^\lambda], [Q^\lambda]$ satisfy the following properties.

1. $[P^\lambda] \circ [P^\nu] = \sum_{\nu} d^\nu_{\lambda\mu} [P^\nu], \text{ where } d^\nu_{\lambda\mu} \geq 0 \text{ is the (ordinary) Littlewood-Richardson coefficient.}$
2. $[Q^m] \circ [P^n] = \sum_{l \geq 0} \binom{k+t-1}{t} [P^{m-l}] \circ [Q^{n-l}], \text{ where } \binom{t}{s} \text{ is the quantum binomial coefficient.}$
In the above proposition, the quantum binomial coefficients are normalized to be symmetric about the origin, e.g. \( \binom{t}{1} = t^{-4} + t^{-2} + 2 + t^2 + t^4 \).

Now, considering symmetric functions instead of Grothendieck groups, we define endomorphisms

\[ p^\lambda, q^\lambda : \Lambda_{t^k, t} \to \Lambda_{t^k, t} \]

by letting \( p^\lambda \) be multiplication by the Schur function \( s_\lambda \) and letting \( q^\lambda \) be its adjoint with respect to \( \langle \cdot, \cdot \rangle_{t^k, t} \).

**Proposition 4.** The operators \( p^\lambda, q^\lambda \) satisfy the following properties.

1. \( p^\lambda \circ p^\mu = \sum_\nu d^\nu_{\lambda \mu} p^\nu \), where \( d^\nu_{\lambda \mu} \geq 0 \) is the (ordinary) Littlewood-Richardson coefficient.
2. \( q^{(n)} \circ p^{(m)} = \sum_{l \geq 0} (k+l-1) p^{(m-l)} q^{(n-l)} \).

**Proof.** The first statement is clear. The second follows from example 2 and (13). \( \square \)

Now Propositions 3 and 4 imply the theorem. For, the inner product \( \langle [H^\lambda], [H^\nu] \rangle_{F_k} \) can be computed as

\[
\langle [P^{(\lambda_1)}][P^{(\lambda_2)}] \ldots [P^{(\lambda_r)}] 1_0, [P^{(\mu_1)}][P^{(\mu_2)}] \ldots [P^{(\mu_s)}] 1_0 \rangle_{F_k} = \\
\langle [Q^{(\mu_1)}][P^{(\lambda_1)}][P^{(\lambda_2)}] \ldots [P^{(\lambda_r)}] 1_0, [P^{(\mu_2)}] \ldots [P^{(\mu_s)}] 1_0 \rangle_{F_k}
\]

where the last equality used the adjointness of \( P^{(\mu_1)} \) and \( Q^{(\mu_1)} \). Now we use the second part of Proposition 3 to write \( Q^{(\mu_1)}[P^{(\lambda_1)}][P^{(\mu_2)}] \ldots [P^{(\mu_s)}] 1_0 \) as a sum of \( [H^\kappa] \)'s for smaller \( \kappa \), inductively determining the \( \langle [H^\lambda], [H^\nu] \rangle_{F_k} \) in terms of inner products involving smaller partitions.

Similarly, the inner product

\[
\langle h_\lambda, h_\mu \rangle_k = \langle p^{\lambda_1} p^{\lambda_2} \ldots p^{\lambda_r} 1_0, p^{\mu_1} p^{\mu_2} \ldots p^{\mu_s} 1_0 \rangle_k
\]

can be computed using the adjointness of \( p^{(\mu_1)} \) and \( q^{(\mu_1)} \), together with the second part of Proposition 4. Since the structure constants in Propositions 3 and 4 agree, we conclude by induction that

\[
\langle [H^\lambda], [H^\mu] \rangle_{F_k} = \langle h_\lambda, h_\mu \rangle_k = \langle \Phi([H^\lambda]), \Phi([H^\mu]) \rangle_k,
\]

as desired.

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