Possibility of Reconstructing Nonlinear Dynamical System from Marked Point Process

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Abstract

We can observe marked point processes in a wide variety of natural phenomena. Thus, it is important to establish theories and methods for analyzing such observed marked point processes. In a nonlinear time series analysis, the theories are the embedding theorems and the methods are the state space reconstruction from the observed time series. Although it has been revealed that we can reconstruct a state space from a marked point process. Therefore, in this study, we numerically investigated whether a state space could be reconstructed from a marked point process without marked values, it has not been clarified yet whether we can reconstruct the state space from a marked point process. Therefore, in this study, we numerically investigated whether a state space could be reconstructed from a marked point process using a delayed coordinate system. To assess the issue, we evaluated the similarity between the interpoint distance distributions on an attractor of the original dynamical system that generated the marked point process and an attractor reconstructed from the marked point process. Results of numerical experiments show that the interpoint distance distributions in the original and reconstructed state spaces are similar, implying that it is possible to reconstruct nonlinear dynamical systems from the observed marked point processes.

1. Introduction

In recent years, it has become possible to observe various time series data with high precision because observational technologies have been much improved. Then, we can observe marked point processes that are observed at irregular time intervals with a marked value. Typical examples of marked point process data include spike trains derived from neural activities, occurrence timings of seismic activities, and terrestrial fluctuations of economic indices.

Identifying causal relations only from the observed data would be useful to understand various phenomena. For example, using a method for detecting causality, we can estimate the network structure only from a multi-dimensional observed time series [1]. Thus, various methods for estimating causality from the observed time series, such as the Granger causality [2] and the transfer entropy [3], have been proposed. However, it has been pointed out that these methods cannot distinguish actual causality from spurious causality and that they require a large number of data points [4]. On the other hand, a method called convergent cross mapping [4], based on the nonlinear dynamical system theory, has been proposed to resolve these issues.

We have already proposed a causality detection method for point processes [5–7] by modifying the convergent cross mapping. The proposed method detects causality based on information in the reconstructed state space. Then, we need to reconstruct an attractor from the marked point process. Although it is already known that we can reconstruct the attractor from event intervals of the point process [8–10], it has not been clarified yet whether we can reconstruct the attractor from a marked point process.

Therefore, in this study, we investigate the possibility of reconstructing the attractor of a nonlinear dynamical system from a marked point process. Considering the marked point process as a two-dimensional time series of event intervals and marked values, we evaluate the validity of the method of reconstructing the attractor from this two-dimensional time series.

2. Method for Generating Marked Point Processes

In this section, we describe how to generate a marked point process. We generated a marked point process by extracting local maxima of a continuous time series (Fig. 1) [11]. In Fig. 1, the nth event timing \( S(n) \) is defined as that when the continuous time series takes the nth local maximum and the marked value \( M(n) \) is its state value. Then, we define the nth interevent interval (IEI) as \( T(n) = S(n+1) - S(n) \). To generate the continuous time series and marked point processes, we use the Lorenz system [12] as shown in Eq. (1).

\[
\begin{align*}
\dot{x} &= \sigma(-x + y) \\
\dot{y} &= -xz + rx - y \\
\dot{z} &= xy - bz
\end{align*}
\]

(1)

where we use the parameter values \( r = 28, \sigma = 10, \) and \( b = 8/3 \).
b = 8/3 in this paper. These parameters are the most well-known parameters for generating a typical chaotic behavior, as first studied by Lorenz [12]. The reason why these parameters were used in Ref. [12] is that for r > 24.74, a chaotic response was obtained.

3. Evaluation of Reconstructed State Space from Marked Point Process

3.1 State space reconstruction from marked point process

We describe how to reconstruct a state space from the marked point process. First, we normalize the IEI time series $T(n)$ and marked value time series $M(n)$ to the range $[0, 1]$ by Eqs. (2) and (3).

$$\hat{T}(n) = \frac{T(n) - T_{\min}}{T_{\max} - T_{\min}}$$

(2)

$$\hat{M}(n) = \frac{M(n) - M_{\min}}{M_{\max} - M_{\min}}$$

(3)

where $T_{\max}$ and $T_{\min}$ are the maximum and minimum values of $T(n)$, and $M_{\max}$ and $M_{\min}$ are the maximum and minimum values of $M(n)$. $T(n)$ and $M(n)$ are the normalized IEIs and marked values. Then, the state space can be reconstructed using the delay coordinate system [13] as follows:

$$V(n) = (\hat{M}(n), \hat{T}(n), \cdots, \hat{M}(n + (m - 1)\tau), \hat{T}(n + (m - 1)\tau))$$

(4)

where $2m$ is the reconstruction dimension and $\tau$ is the time delay. It seems reasonable to reconstruct the state space from the time series of IEIs and marked values because it has already been shown that the state space can be reconstructed from a multi-dimensional time series [14, 15].

If a nonlinear dynamical system can be reconstructed from the marked point process, several properties of the nonlinear dynamical system should be preserved in the reconstructed state space: the orbital instability characterized by the Lyapunov spectrum and the self-similarity of attractors characterized by the fractal dimension. If the topological characteristics of the dynamical system are preserved, the interpoint distance distributions of the original and reconstructed attractors will be similar. Therefore, to evaluate whether the original dynamical system can be reconstructed, we calculate the Jensen–Shannon (JS) divergence [16] between the interpoint distance distributions in the original state space and the state space reconstructed from the marked point process obtained from the Lorenz system.

Then, we investigate whether the JS divergence between the interpoint distance distributions of the original state space and the state space reconstructed from the marked point process is sufficiently small. We also generate the random shuffle surrogate data (RSSD) of the marked point process and evaluate the JS divergence between the interpoint distance distributions of the original state space and the state space reconstructed from RSSD as a baseline in our analysis. We generate RSSD by randomly shuffling the temporal order of the normalized IEIs and marked values independently. We respectively define $T'(n)$ and $M'(n)$ as the randomly shuffled IEIs and marked values.

3.2 Jensen–Shannon (JS) divergence

The JS divergence [16] is defined as the sum of the Kullback–Leibler (KL) divergences. The KL divergence is a measure used to quantify the dissimilarity of two probability distributions. The KL divergence of a probability distribution $P$ from a probability distribution $Q$ is defined by

$$D_{KL}(P||Q) = \sum_{i=1}^{N} P(k_i) \log \left( \frac{P(k_i)}{Q(k_i)} \right)$$

(5)

where $P(k_i)$ and $Q(k_i)$ are the probabilities of the ith class of the probability distributions $P$ and $Q$ in ascending order and $N$ is the total number of classes. In this study, we set $N = 20$ in all the numerical experiments. From Eq. (5), the KL divergence is an asymmetric measure, namely $D_{KL}(P||Q) \neq D_{KL}(Q||P)$ except in the case that $P(k_i) = Q(k_i)$ for all $i$.

The JS divergence for the probability distributions $P$ and $Q$ is defined by

$$D_{JS}(P, Q) = \frac{1}{2}(D_{KL}(P||R) + D_{KL}(Q||R))$$

(6)

where the probability distribution $R$ is the average of the probability distributions $P$ and $Q$, and is defined by $R(k_i) = \frac{1}{2}(P(k_i) + Q(k_i))$.

3.3 Results

First, we numerically investigate whether we can evaluate the similarity between the interpoint distance distributions of the original attractor and an attractor reconstructed from the time series by calculating the JS divergence. Figure 2 shows the JS divergence between the interpoint distance distributions of the original Lorenz attractor and the attractors reconstructed from the time series of variables $x$, $y$, and $z$ of the Lorenz system, where the reconstruction dimension varies in the range of [2, 20] at intervals of 1. Here, we set the
time delay \( \tau \) as four when we reconstruct the attractor. In Fig. 2, \( D_{JS}(P_o, P_i) \) \((i \in \{x, y, z\})\) represents the JS divergence between the original state space of the Lorenz system and the state space reconstructed from the time series of the variable \( i \) \((i \in \{x, y, z\})\). In this study, we numerically solved the Lorenz equations using the fourth-order Runge–Kutta method with step size \( h = 0.01 \), where we excluded the first 100,000 data points as a transient state. Then, we used 10,000 data points to calculate the interpoint distance distribution. Namely, the number of interpoint distances is 49,995,000.

From Fig. 2, we found that the JS divergence takes minimum values at the reconstruction dimensions of five for variable \( x \) and seven for variables \( y \) and \( z \). The box-counting dimension of the Lorenz attractor is approximately 2.06 [17]; thus the reconstruction dimension that satisfies the condition for an embedding is five or more. Therefore, from Fig. 2, it is reasonable that the JS divergence is minimized at reconstruction dimensions of five for variable \( x \) and seven for variables \( y \) and \( z \). These results suggest that the interpoint distance distributions of the original and reconstructed attractors are similar when the transformation by the delay coordinate can be an embedding as the reconstruction dimension is increased.

To discuss the reason why the JS divergence increases with the reconstruction dimension in Fig. 2, we show examples of the interpoint distance distributions of the original Lorenz attractor and the attractor reconstructed from the time series of variable \( z \) at reconstruction dimensions of 7 and 20 in Fig. 3. From Fig. 3, we found that the interpoint distance distributions of the original and reconstructed attractors are similar when the reconstruction dimension is seven. However, the peak of the frequency distribution of the interpoint distances of the reconstructed attractor (green) is larger when the reconstruction dimension is 20 (Fig. 3(b)). This is because the reconstructed state space becomes sparser as the dimension increases, and the interpoint distance becomes larger.

We next investigate whether the state space can be reconstructed from a marked point process by calculating the JS divergence of the interpoint distance distributions between the original and reconstructed attractors. First, we show the state spaces with and without trajectories reconstructed from the pair \((\hat{T}_x(n), \hat{M}_x(n))\) and the pair \((T'_x(n), M'_x(n))\) in Fig. 4. Here, we defined the normalized marked point process obtained by variable \( x \) of the Lorenz system as \( M_x \), and the normalized IEIs and marked values of \( M_x \) as \( \bar{T}_x(n) \) and \( \hat{M}_x(n) \), respectively. In addition, we defined the randomly shuffled normalized IEIs and marked values of \( M_x \) as \( T'_x(n) \) and \( M'_x(n) \), respectively. From Figs. 4(a) and (b), we found that the trajectory of \( M_x \) has a recurrence structure and that the points on the reconstructed state space follow deterministic dynamics. In contrast, from Figs. 4(c) and (d), the trajectory of RSSD has no structure and the points on the reconstructed state space do not follow any deterministic dynamics.

Then, we investigated the relationship between the JS divergence and the reconstruction dimension. Figure 5 shows the results of the JS divergence between the interpoint distance distribution of the original dynamical system and the interpoint distance distribution reconstructed from the marked point process in the case of changing the reconstruction dimension. In Fig. 5, we also show the range between the minimum and maximum values of the JS divergence of RSSD.
We generated ten RSSD for each value of \( m \). We varied the reconstruction dimension in the range of \([2, 20]\) at intervals of 2. We set the time delay \( \tau \) to unity.

In Fig. 5, the horizontal and vertical axes represent the reconstruction dimension and the JS divergence, respectively. In Fig. 5, \( D_{JS}(P_0, P_{M_0}) \) represents the JS divergence between the original state space of the Lorenz system and the state space reconstructed from the marked point process of variable \( i \ (i \in \{x, y, z\}) \), and \( D_{JS}(P_0, P'_{M_i}) \) (\( i \in \{x, y, z\} \)) represents the JS divergence between the original state space of the Lorenz system and the state space reconstructed from RSSD for each variable.

![Graph](image)

Figure 5: The JS divergence between the original Lorenz system and the system reconstructed from the marked point process \( M_i \ (i \in \{x, y, z\}) \) when \( m \) changes: The grey, red, and blue areas show the range from the minimum to maximum values when calculating the JS divergence of RSSD.

From Fig. 5, we found that \( D_{JS}(P_0, P_{M_i}) \) has a small value of less than 0.02 when \( 2m = 8 \). Moreover, the values of \( D_{JS}(P_0, P_{M_0}) \) are smaller than that of \( D_{JS}(P_0, P'_{M_i}) \). In other words, the interpoint distance distribution reconstructed from the marked point process is similar to the interpoint distance distribution of the original state space. In addition, although \( D_{JS}(P_0, P_{M_0}) \) and \( D_{JS}(P_0, P_{M_y}) \) are larger than \( D_{JS}(P_0, P_{M_x}) \), they are smaller than the JS divergences of RSSD, namely \( D_{JS}(P_0, P'_{M_x}) \) and \( D_{JS}(P_0, P'_{M_y}) \), when \( 2m \geq 6 \). This condition \( 2m > 6 \) might be closely related to the theorem [14] that a reconstruction dimension \( > 2D_0 \) is sufficient, where \( D_0 \) is the box-counting dimension. Because the box-counting dimension of the Lorenz attractor is approximately 2.06 [17], the minimum value of \( 2m \) that satisfies \( D_{JS}(P_0, P_{M_x}) < D_{JS}(P_0, P_{M_y}) \) (\( i \in \{x, y, z\} \)) seems to satisfy the condition that \( 2m > 2D_0 \approx 4.12 \).

On the other hand, in the cases of \( D_{JS}(P_0, P_{M_x}) \) and \( D_{JS}(P_0, P_{M_y}) \), the values of the JS divergence calculated from the original \( M_x \) and \( M_y \) are not significantly smaller than that of RSSD when the reconstruction dimension is small. The reason for this is presumably that for lower dimensions, especially for \( 2m \leq 4 \), a one-to-one correspondence between the points on the reconstructed state space and the points on \( M_z \) does not exist; as shown in Fig. 4 (b), several points are not unfolded at lower dimensions. These points lead to a higher frequency of small distances, allowing the value of the JS divergence to be larger than that of RSSD, even though there are no folded points in the original Lorenz attractor. Different from Fig. 2, the values for which the JS divergence is minimized do not appear clearly for all variables in Fig. 5. However, even though the peak value of the JS divergence does not appear, using the JS divergence of RSSD as a baseline, we might be able to infer an appropriate reconstruction dimension for an embedding when we only obtain a finite number of points of marked point processes.

4. Conclusion

We investigated whether a nonlinear dynamical system can be reconstructed from an observed marked point process using a delay-coordinate system by calculating the JS divergence between the interpoint distance distributions in the original state space and in the state space reconstructed from a marked point process and its randomly shuffled marked point process. As a result, it is shown that the trajectories of the attractors reconstructed from the marked point process had a more recurrence structure than those of the attractors reconstructed from RSSD. It is also shown that the interpoint distance distribution in the state space reconstructed from the marked point process is similar to that in the state space of the original Lorenz system.

In this paper, we evaluated the similarity between the reconstructed and original attractors in terms of the similarity of the interpoint distance distribution. However, it should also be evaluated in terms of dynamics. To evaluate the possibility of reconstructing the state space from a marked point process in terms of dynamics, it is an important issue to investigate the reconstructed attractor by evaluating the prediction accuracy or estimating the Lyapunov exponents. In addition, we investigated the similarity between the original and reconstructed Lorenz attractors for the parameter values of \( r = 28, \sigma = 10, \) and \( b = 8/3 \); however, it is also important in the future to investigate the similarity when we change the parameter values.

In this study, we did not theoretically clarify whether an embedding was satisfied by observing the marked point process from nonlinear dynamical systems. Then, another important future issue is to prove that the delay coordinates by marked point processes observed from nonlinear dynamical systems can be an embedding.

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References

[1] K. Sawada, Y. Shimada and T. Ikeguchi: Network structure detection using convergent cross mapping on multivariate time series, NOLTA, IEICE, Vol. 11, No. 4, pp. 422-432, 2020.

[2] C. W. J. Granger: Investigating causal relations by econometric models and cross-spectral methods, Econometrica, Vol. 37, No. 3, pp. 424-438, 1969.

[3] T. Schreiber: Measuring information transfer, Phys. Rev. Lett., Vol. 85, No. 2, pp. 461-464, 2000.

[4] G. Sugihara, R. May, H. Ye, C. Hsieh, E. R. Deyle, M. Fogarty and S. Munch: Detecting causality in complex ecosystems, Science, Vol. 338, No. 6106, pp. 496-500, 2012.

[5] K. Sawada, Y. Shimada and T. Ikeguchi: Estimation of connections between neurons only from inter-spike-interval, Proc. 5th Int. Conf. on Mathematical Neuroscience, p. 80, 2019.

[6] K. Sawada, Y. Shimada and T. Ikeguchi: Estimation of neural network structure using inter-spike intervals, Proc. 2019 Int. Symp. Nonlinear Theory and its Applications, pp. 591-594, 2019.

[7] K. Sawada, Y. Shimada and T. Ikeguchi: Reconstruction of dynamical system from marked point process and its application to causality detection, Proc. RISP Int. Workshop Nonlinear Circuits, Communications and Signal Processing 2021, No. IAM1–1–4, pp. 13-16, 2021.

[8] T. Sauer: Reconstruction of dynamical systems from interspike intervals, Phys. Rev. Lett., Vol. 72, No. 24, pp. 3811-3814, 1994.

[9] T. Sauer: Interspike interval embedding of chaotic signals, Chaos, Vol. 5, No. 1, pp. 127-132, 1995.

[10] T. Sauer: Reconstruction of integrate-and-fire dynamics, Nonlinear Dynamics and Time Series, Vol. 11, pp. 63-74, 1997.

[11] N. Yabuta and T. Ikeguchi: Prediction of high-dimensional multivariate information as an amplitude-event dynamical system, Proc. 2007 Int. Symp. Nonlinear Theory and its Applications, pp. 16-19, 2007.

[12] E. N. Lorenz: Deterministic nonperiodic flow, J. Atmos. Sci., Vol. 20, No. 2, pp. 130-141, 1963.

[13] F. Takens: Detecting strange attractors in turbulence, D. Rand, L.-S. Young (Eds.), Dynamical Systems and Turbulence, Lecture Notes in Mathematics, Vol. 898, Springer, Berlin, pp. 366-381, 1981.

[14] T. Sauer, J. A. Yorke and M. Casdagli: Embedology, J. Stat. Phys., Vol. 65, No. 3, pp. 579-616, 1991.

[15] E. R. Deyle and G. Sugihara: Generalized theorems for nonlinear state space reconstruction, PLOS ONE, Vol. 6, No. 3, p. e18295, 2011.

[16] J. Lin: Divergence measures based on the Shannon entropy, IEEE Trans. Inf. Theory, Vol. 37, No. 1, pp. 145-151, 1991.

[17] P. Grassberger and I. Procaccia: Characterization of strange attractors, Phys. Rev. Lett., Vol. 50, No. 5, pp. 346-349, 1983.