Dissipation-relaxation dynamics of a spin-1/2 particle with a Rashba-type spin-orbit coupling in an ohmic heat bath

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Spin-orbit coupling (SOC), which is inherent to a Dirac particle that moves under the influence of electromagnetic fields, manifests itself in a variety of physical systems including non-relativistic ones. For instance, it plays an essential role in spintronics developed in the past few decades, particularly by controlling spin current generation and relaxation. In the present work, by using an extended Caldeira-Leggett model, we elucidate how the interplay between spin relaxation and momentum dissipation of an open system of a single spin-1/2 particle with a Rashba type SOC is induced by the interactions with a spinless, three-dimensional environment. Staring from the path integral formulation for the reduced density matrix of the system, we have derived a set of coupled nonlinear equations that consists of a quasi-classical Langevin equation for the momentum with a frictional term and a spin precession equation. The spin precesses around the effective magnetic field generated by both the SOC and the frictional term. It is found from analytical and numerical solutions to these equations that a spin torque effect included in the effective magnetic field causes a spin relaxation and that the spin and momentum orientations after a long time evolution are largely controlled by the Rashba coupling strength. Such a spin relaxation mechanism is qualitatively different from, e.g., the one encountered in semiconductors where essentially no momentum dissipation occurs due to the Pauli blocking.

I. INTRODUCTION

Spin-orbit coupling (SOC) is ubiquitous in physics, ranging from atomic fine structure [1] to nuclear shell structure [2, 3], spin dynamics in semiconductors [4], etc. The SOC originates from the $O(m^{-2})$ correction in non-relativistic reduction of a charged Dirac particle of mass $m$ under electromagnetic fields [5]; e.g., for an electron of mass $m_e$, it reads

$$H_{\text{so}} = \frac{1}{2} \sigma \cdot (\alpha \times \hat{p}),$$

where $\sigma$ are the Pauli matrices, $\alpha = -\frac{e}{2m_e c^2} E$ with $E = -\nabla \phi$ is an external electric field from a static potential $\phi$, and no magnetic field is applied. For simplicity, we will hereafter use natural units where $\hbar = 1$. For the past few decades, study of electrons pertaining to electron spin currents, i.e., spintronics, has developed significantly for possible application to novel devices of information technology [6], where the spin Hall effect due to the SOC plays a crucial role in controlling the spin currents [7]. The platform for such devices is provided by semiconductors, for instance, GaAs in which the transition between electrons in conducting and valence bands leads to an effective mass $m^*$ that is much smaller than $m_e$ and hence enhances the SOC at a level that cannot be ignored in comparison to that in vacuum [8]. In such semiconductors, the conducting electrons are moving effectively in a quasi-2 dimensional well confined in $z$ direction, and two types of the SOC can be realized: one is the Rashba type $H_{\text{so}} \approx \sigma_x p_y - \sigma_y p_x$ [9], and the other the Dresselhaus type $H_{\text{so}} \approx \sigma_x p_z - \sigma_y p_y$ [10]. Obviously, the SOC constitutes a part of the single-body Hamiltonian and leads to an energy splitting in spin states given a finite momentum, in addition to the Zeeman splitting by an external magnetic field [11], and/or the Landau splitting by a spatial rotation [12]. Such a spin state, however, does not last long in a coherent manner, but relaxes to a lower energy state in the presence of various interactions with environmental degrees of freedom. Thus, to know the spin relaxation process is important particularly for spintronics.

Recently, the SOC in cold atomic many-body systems has also attracted much attention. A well-designed laser geometry provides these systems with artificial electromagnetic fields, which act on the hyperfine states of trapped atoms and help a pair of such pseudo-spin states with different momenta to couple with each other so as to mimic the SOC of spin-1/2 electrons [13]. This technique was applied to experimental realization of the Bose-Einstein condensation of spin-orbit coupled bosonic atoms [14]. More interestingly, such atoms with or without the SOC can be confined in anisotropic traps as mi-
minority (impurity) atoms, together with majority atoms as an environment, which simulates a polaron problem for atomic impurities. This kind of atomic impurities, referred to as Fermi or Bose polarons according to whether the majority atoms are fermions or bosons, has been observed in cold atomic experiments. Indeed, the Pauli blocking only allows the electron that the latter involves essentially no momentum dissipation and spin relaxation dynamics in semiconductors in the sense that the majority atoms are fermions or bosons, has been observed in cold atomic experiments.

Theoretically, such an atomic impurity without the SOC has been treated as a quantum open system, where the dissipation of energy and momentum of the impurity occurs due to the interaction with a cold or hot bosonic environment. In the presence of the SOC, however, the interplay between momentum dissipation and spin relaxation of the impurity has yet to be investigated. Here it should be noted that the dissipative dynamics of a mobile atomic impurity is different from the electron-spin relaxation dynamics in semiconductors in the sense that the latter involves essentially no momentum dissipation. Indeed, the Pauli blocking only allows the electron momentum to change, during scattering processes with environmental degrees of freedom like phonons, from \( k \) to \( k' \) just on the Fermi surface, i.e., \( |k| = |k'| = k_F \) with the Fermi momentum \( k_F \), which leads to no dissipation. More intriguingly, such momentum changing processes cause effective magnetic-field fluctuations to act on the spin degrees of freedom through the SOC in such a way as to relax the spin orientation towards a possible lower-lying energy state, as in nuclear magnetic resonances where effective magnetic-field fluctuations are provided by environmental electron spins.

In the present study we demonstrate the spin relaxation of a spin-1/2 particle, which occurs together with the momentum dissipation, by employing an extended version of the one-dimensional Caldeira-Leggett (CL) model in such a way as to be applicable to the particle that moves with a Rashba type SOC in a three-dimensional environment. In particular, we figure out a possible mechanism for the interplay between the spin relaxation and momentum dissipation by simultaneously analyzing quasi-classical equations for the spin orientation and for the momentum. The CL model is known to derive the Langevin equation for a quantum Brownian particle: Starting from the von Neumann equation for the full density matrix and integrating out the environmental degrees of freedom, one can read off the Langevin equation from the resulting effective action in the path integral formulation. In Sec. III we present numerical simulations for the spin relaxation and momentum dissipation, which in turn are classified into typical patterns of the dynamics according to the model parameters that govern the relaxation and dissipation (damping) time scales. Section IV is devoted to summary and outlooks.

II. FORMULATION

In this section, we present the system-plus-environment-plus-interaction Hamiltonian, the eigen energy of the system, the Feynman-Vernon influence functional, the quasi-classical dynamical equations, and the asymptotic state of the system.

A. A model Hamiltonian

We consider a system-plus-environment-plus-interaction model described by the Hamiltonian

\[ H = H_S + H_B + H_I, \]

where

\[ \hat{H}_S = \frac{\hat{p}^2}{2m} + \hat{s} \cdot (\alpha \times \hat{p} + \hat{B}), \]

\[ \hat{H}_B = \frac{1}{2} \sum_{k=0}^{\infty} \left[ \hat{p}_k^2 + \omega_k^2 \hat{X}_k^2 \right], \]

\[ \hat{H}_I = -\hat{\mathbf{x}} \cdot \left( \sum_{k=0}^{\infty} c_k \hat{X}_k \right). \]

\( \hat{H}_S \) denotes the Hamiltonian of the system of a spin-1/2 particle moving with a Rashba type SOC, which is characterized by the spin operator \( \hat{s} = \frac{1}{2} \hat{s} \), the momentum operator \( \hat{p} \), and a constant vector \( \alpha = (0, 0, \alpha) \) whose size determines the Rashba coupling strength. In addition, we assume that an external field \( \hat{B} \) brings about the Zeeman term \( \hat{s} \cdot \hat{B} \), which appears in general for particles having a nonzero spin and an intrinsic dipole magnetic moment parallel to the spin. Note that in the present study we employ units in which the size of the dipole magnetic moment is unity. We also assume throughout the present study that \( \hat{B} \) is parallel to the z axis, i.e., \( \alpha \).
as in semiconductor experiments, and that the \( z \) component of the particle’s momentum is always zero. \( \hat{H}_B \) is the Hamiltonian of the environment composed of an infinite number of harmonic oscillators, which is characterized by the angular frequency \( \omega_k \), momentum operator \( \hat{P}_k \), and coordinate operator \( \hat{X}_k \) of each mode \( k \). \( \hat{H}_I \) describes the interaction between the system and the environment, which is characterized by the linear coupling between their coordinate operators, i.e., \( \hat{X}_k \cdot \hat{X}_k \), with the strength \( c_k \) for each mode \( k \). The operators satisfy the canonical relations \( [\hat{x}_i, \hat{p}_j] = i\delta_{ij} \) and \( [\hat{X}_k, \hat{P}_{k'}] = i\delta_{kk'} \). It should be noted that \( \hat{H}_B \) together with \( \hat{H}_I \) constitutes the Caldeira-Leggett type heat bath model.

**B. Single particle energies**

In order to clarify conserved quantities of the system, which eventually undergo dissipation and relaxation under the influence of the environment, we first obtain the solution of the eigen value problem solely for the system’s Hamiltonian as

\[
\hat{H}_S|p, s\rangle = E_s(p)|p, s\rangle,
\]

with

\[
E_s(p) = \frac{p^2}{2m} + \frac{1}{2}|\alpha \times p + B|,
\]

where \( |p, s\rangle \) represents the eigen state with the eigen values of the momentum \( p \) and the spin doublet \( s = \pm 1 \) with respect to the quantization axis parallel to \( \alpha \times p + B \). The system particle thus keeps having a constant momentum \( p \) once given, and its spin expectation value precesses about the constant vector \( \alpha \times p + B \), if there is no influence from the environment. Note that since we have taken \( B = (0, 0, B) \), \( \hat{H}_S \) possesses the rotational symmetry about the \( z \) axis. The single particle energies are depicted in Fig. 1 for some characteristic values of \( B \); there exists a critical value of the magnetic field,

\[
B_c = ma^2/2,
\]

above which the degenerate minima of \( E_{-1} \) merge into one.

**C. Feynman-Vernon influence functional**

We now proceed to consider the time evolution of the system’s density matrix \( \hat{\rho}_0^S \) (the reduced density matrix) under the influence of the environment. To this end, we employ the path integral formalism, from which the effective action of the system and the corresponding quasi-classical dynamical equations can be exploited.

We start with the time evolution of the full density matrix \( \hat{\rho}_t \), which is governed by the von Neumann equation,

\[
\frac{d\hat{\rho}_t}{dt} = -i[\hat{H}, \hat{\rho}_t],
\]

with

\[
\hat{\rho}_t = \exp(-i\hat{H}t)\hat{\rho}_0 \exp(i\hat{H}t),
\]

where \( \hat{\rho}_0 \) represents the initial density matrix at \( t = 0 \). We assume that \( \hat{\rho}_0 \) is given by the direct product of the initial density matrices of the system and environment as

\[
\hat{\rho}_0 = \hat{\rho}_0^S \otimes \hat{\rho}_0^B,
\]

where the environmental part is in thermal equilibrium of temperature \( T \), i.e., \( \hat{\rho}_0^B = e^{-\beta H_B}/Z \) with \( \beta = 1/k_B T \).

The element of the reduced density matrix with respect to the particle’s coordinate \( x \) and spin state \( g \) is given by taking a trace over the environmental coordinates as

\[
\rho_t^S(xg, x'g') = \int_X \langle x, g; X|\hat{\rho}_t|x', g'; X\rangle,
\]

where \( \int_X \equiv \Pi_k \int dX_k \), and we represent the spin state by a general \( SU(2) \) rotation of the highest weight spin eigen state [38] as

\[
|g\rangle = e^{-i\phi\hat{z}} e^{-i\theta\hat{y}} e^{-i\psi\hat{x}} |+1\rangle
\]

with the Euler angles \( \phi, \theta, \) and \( \psi \). This representation is useful to express the reduced density matrix [41] in the path integral formalism, where the path integral coordinates of the spin state can be represented by continuous compact parameters, i.e., the Euler angles, on the \( S^3 \sim SU(2) \) manifold, and the decomposition of the identity used in the path integral is given in terms of the Haar measure in \( SU(2) \) as

\[
\int_{S^3} dg |g\rangle\langle g| = 1,
\]

**FIG. 1.** Single particle energies \( E_{\pm 1}(p) \) as functions of \( p_x \) for \( B = 0 \) (solid lines), \( B < B_c \) (dashed), and \( B > B_c \) (dotted). The upper (lower) lines correspond to \( E_{-1} \) (\( E_{+1} \)). Note that these energies have the rotation symmetry in \( p_x - p_y \) plane, while the plotted values of the momenta and energies are normalized, respectively, by the asymptotic value (44) at \( B = 0 \) and the absolute value of \( E_{\min} = -m\alpha^2/8 \) corresponding to \( E_{-1}(p(\infty)) \) at \( B = 0 \).
where \( I_s \) dq = \( \frac{1}{\sin} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \int_0^4 \psi d\psi \), and I is the identity operator. For the spin coherent state (12), the spin expectation value can be expressed in terms of the Bloch sphere coordinates as

\[
\mathbf{\rho} \equiv \mathbf{\langle g| s| g \rangle} = \frac{1}{2} (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\]

The remaining angle \( \psi \) is hidden in \( |g\rangle \) as an overall \( U(1) \) gauge factor. Thus we will denote the Bloch sphere co-
ordinates with respect to the forward and backward path integrals, i.e.,

\[
\mathbf{\rho} \equiv \mathbf{\langle g| s| g \rangle} = \langle \mathbf{x}, g| g \rangle \mathbf{\rho}_0 \langle g| g \rangle.
\]

\[ \mathbf{\rho}_0 = \mathbf{\rho}^B(\mathbf{\bar{x}}, \mathbf{\bar{X}}), \]

where \( \mathbf{\rho}_0 = \langle \mathbf{x}, g| g \rangle \mathbf{\rho}_0 \langle g| g \rangle \), and the propagator has the forward-backward path integral representation

\[
G_t(\mathbf{x}, \mathbf{g}; \mathbf{x}', \mathbf{g}') = \int_{x, \bar{g}, x', \bar{g}'} G_t(\mathbf{x}, \mathbf{g}; \mathbf{x'}, \mathbf{g}') \mathbf{\rho}_0 \langle g| g \rangle,
\]

where \( \mathbf{\rho}_0 = \langle \mathbf{x}, \mathbf{g}| \mathbf{\rho}_0 \langle g| g \rangle \rangle \), and the propagator has the forward-backward path integral representation

\[
\mathbf{\rho}_0 = \mathbf{\rho}^B(\mathbf{\bar{x}}, \mathbf{\bar{X}}),
\]

where \( \mathbf{\rho}_0 = \langle \mathbf{x}, g| g \rangle \mathbf{\rho}_0 \langle g| g \rangle \rangle \), and the effective action of the system is given by

\[
W(\mathbf{x}, \mathbf{g}; \mathbf{x}', \mathbf{g}') = \mathcal{A}^S(\mathbf{x}, \mathbf{g}) - \mathcal{A}^S(\mathbf{x}', \mathbf{g}')
\]

\[
+ W_1(\mathbf{x}, \mathbf{g}) + W_2(\mathbf{x}, \mathbf{g}).
\]

Here, \( \mathcal{A}^S \) is the action of the particle alone,

\[
\mathcal{A}^S(\mathbf{x}, \mathbf{g}) = \int_0^t du \mathcal{L}^S(\mathbf{x}(u), \mathbf{g}(u))
\]

with

\[
\mathcal{L}^S = \frac{m}{2} (\dot{\mathbf{x}} - \mathbf{n} \times \mathbf{\alpha})^2 + \frac{1}{2} \mathbf{\omega} \mathbf{\phi} \cos \theta - \mathbf{n} \cdot \mathbf{B},
\]

which satisfies the boundary conditions at the initial time, i.e., \( u = 0 \),

\[
\{ \mathbf{x}(0), \mathbf{g}(0) \} = \{ \mathbf{x}, \mathbf{g} \}, \quad \{ \mathbf{x}'(0), \mathbf{g}'(0) \} = \{ \mathbf{x}', \mathbf{g}' \},
\]

and at the final time, i.e., \( u = t \),

\[
\{ \mathbf{x}(t), \mathbf{g}(t) \} = \{ \mathbf{x}, \mathbf{g} \}, \quad \{ \mathbf{x}'(t), \mathbf{g}'(t) \} = \{ \mathbf{x}', \mathbf{g}' \}.
\]

Note that the dot in Eq. (21) denotes the time derivative. Also, \( W_1 \) and \( W_2 \) are respectively the imaginary and real parts of the influence functional, i.e., the contribution to the effective action from \( H_B \) and \( H_1 \), as given by

\[
W_1(\mathbf{x}, \mathbf{g}) = i \int_0^t \int_0^t du \int_0^\infty \mathbf{J}(\omega) \coth \frac{\text{coth}(\frac{\omega}{2})}{2} \times \mathbf{\cos}(u - u') \mathbf{x}(-u) \cdot \mathbf{x}(-u'),
\]

\[
W_2(\mathbf{x}, \mathbf{g}) = 2 \int_0^t \int_0^t du \int_0^\infty \mathbf{J}(\omega) \times \mathbf{\sin}(u - u') \mathbf{x}(-u) \cdot \mathbf{x}(u'),
\]

where

\[
\mathbf{J}(\omega) = \sum_k \frac{c_k^2}{\omega_k^2} \delta(\omega - \omega_k),
\]

is the spectral density function, and

\[
\mathbf{x}_+(u) = \frac{\mathbf{x}(u) + \mathbf{x}'(u)}{2},
\]

\[
\mathbf{x}_-(u) = \mathbf{x}(u) - \mathbf{x}'(u)
\]

represent respectively the center of mass and relative co-
ordinates with respect to the forward and backward path integrals, i.e., the diagonal and off-diagonal elements of the reduced density matrix \( \rho_0(\mathbf{x}, \mathbf{g}; \mathbf{x}', \mathbf{g}') \) at each \( u \) with \( 0 \leq u \leq t \).

**D. Quasi-classical dynamical equation**

From here on, as in the Caldeira-Leggett model, we assume that spectral density is ohmic, i.e.,

\[
\mathbf{J}(\omega) = \frac{c \omega}{\pi},
\]

where \( c \) is a constant. 2

and then we obtain

\[
W_2(\mathbf{x}, \mathbf{g}) = -c \int_0^t \mathbf{du} \mathbf{x}_+(u) \cdot \mathbf{x}_-(u)
\]

\[
+ c' \int_0^t \mathbf{du} \mathbf{x}_+(u) \cdot \mathbf{x}_-(u),
\]

where the coefficient in the second term is interpreted as \( c' = \sum_k \frac{c_k^2}{\omega_k^2} \). The last two terms will be ignored hereafter because the second term can be renormalized into an external potential of the particle, which we suppose to be absent in the present study 2, and the last one is a surface term irrelevant for the dynamics.

1 It should be noted that in the non-ohmic case, i.e., \( \mathbf{J}(\omega) \sim \omega^\alpha \) with \( \alpha \neq 1 \), the non-locality in \( W_2 \) is maintained so that the memory effect comes out in the resultant Langevin equation.

2 The second term, which comes from the original form (20) of the spectral function, would add an elastic force with negative spring constant and hence acts to destabilize the system, but one could counteract such a possible instability by adding a harmonic trap potential.
Here we also assume that the environment temperature is sufficiently higher than the excitation energies, i.e., $k_B T \gg \omega_k$, so that $W_1$ can be approximated by

$$W_1 [x, x'] \simeq i c k_B T \int_0^t du \, x^2 (u). \quad (31)$$

This approximation is consistent with the quasi-classical description that we will develop below.

Under these assumptions just made above, let us now take the optimal condition for the effective action, i.e., \(\delta W = 0\), with respect to the off-diagonal variables \(x_+\), \(\theta_+ = \theta - \theta'\), and \(\phi_+ = \phi - \phi'\) at \(x_- = \theta_- = \phi_- = 0\), to obtain the equations for the quasi-classical (diagonal) variables:

\[
\begin{align*}
\dot{x}_+ + \gamma \dot{x}_+ - (n_+ \times \alpha) &= 0, \quad (32) \\
\dot{n}_+ &= [\alpha \times m (x_+ - n_+ \times \alpha) + B] \times n_+, \quad (33)
\end{align*}
\]

where \(\gamma = c/m\) is the friction coefficient, and \(n_+ = n_{\theta=\theta', \phi=\phi'}\) with the diagonal Euler angles \(\theta_+ = (\theta + \theta')/2\) and \(\phi_+ = (\phi + \phi')/2\). The first equation \((32)\) corresponds to the quasi-classical dynamical equation with the friction term \(\gamma \dot{x}_+\) and with the additional term involving \(\alpha\) that comes from the SOC, while the second equation \((33)\) mainly governs the spin dynamics that can be seen as an instantaneous spin precession about a velocity-dependent effective magnetic field. The precession equation is equivalent to the one derived from the Heisenberg equation of motion in terms of the canonical momentum as will be shown in Eq. \((38)\) below.

Here it should be noted that the optimal condition to derive Eqs. \((32)\) and \((33)\) corresponds to tracing the diagonal path \(x_+\), that is, not affected by off-diagonal fluctuations \(x_-\) and \(g_- \equiv g - g'\) at any \(u\) that satisfies \(0 < u < t\).

In this sense these equations determine the quasi-classical trajectory of the particle, i.e., the wave packet of width \(|x_-|\), which is in turn influenced systematically by its own quasi-classical spin dynamics through the SOC, and vice versa. This sort of classical description is justified at high temperatures, because the time evolution of the reduced density matrix is dominated by the imaginary part \((31)\) as

\[
\partial_u \rho^S (x, x') \simeq -2\gamma \left( \frac{x}{\lambda_T} \right)^2 \rho^S (x, x'), \quad (34)
\]

which implies that the probability of having a finite off-diagonal fluctuation of size \(|x_-| = |x - x'|\) diminishes exponentially fast and the corresponding decoherence time scale

\[
\tau_{\text{dec}} = (2\pi \gamma)^{-1} \left( \frac{x}{\lambda_T} \right)^{-2} \quad (35)
\]

becomes smaller than the damping time scale \(\gamma^{-1} = m/c\) for sufficiently large width \(|x_-|\) of the particle wave packet compared with the thermal de Broglie wavelength \(\lambda_T = h \sqrt{2\pi/mk_B T} \). Thus the decoherence of the superposition between different coordinates is quickly achieved. We can also expect a fast decoherence with respect to the spin variable \(g_-\) as in the case of \(x_-\). Discussion of this point is given in appendix A.

In order to make Eqs. \((32)\) and \((33)\) more transparent, it is instructive to introduce the canonical momentum of the particle, which can be obtained from the Lagrangian \((31)\) as

\[
p = \frac{\partial L_S}{\partial \dot{x}} = m (\dot{x} - n \times \alpha), \quad (36)
\]

and then to represent the equations as

\[
\dot{p} + \gamma \dot{p} = m \gamma (\alpha \times n), \quad (37)
\]

\[
\dot{n} = (\alpha \times p + B) \times n, \quad (38)
\]

where we have simplified the notation as \(x_+ \rightarrow x\) and \(n_+ \rightarrow n\). The dynamics of \(p\) is mainly controlled by Eq. \((37)\), which includes the inhomogeneous (source) term attributable to both of the dissipation (\(\gamma\)) and SOC (\(\alpha\)) effects. The source term alters the orientation of \(p\) during the momentum dissipation as will be directly observed from numerical results in the next section. On the other hand, Eq. \((38)\) leads to the spin precession about the axis of \(B\) plus the effective magnetic field defined by \(b_{\text{eff}} = \alpha \times p\). Since \(b_{\text{eff}}\) changes its direction and magnitude with time in accordance with the dynamics of \(p\), the spin precession around \(b_{\text{eff}} + B\) is only instantaneous. We can see this situation more explicitly by rewriting the equations in terms of the effective magnetic field as

\[
\dot{b}_{\text{eff}} + \gamma b_{\text{eff}} = m \gamma \alpha \times (\alpha \times n), \quad (39)
\]

\[
\dot{n} = (b_{\text{eff}} + B) \times n. \quad (40)
\]

In what follows, we will simultaneously solve the set of equations, i.e., Eqs. \((39)\) and \((40)\), and discuss the dissipation-relaxation dynamics of the spin and momentum up to the possible final state in a manner that depends on the parameters \(\alpha\) and \(\gamma\) as well as on the initial condition.

As a first step, for given \(n(u)\), we solve Eq. \((39)\) with respect to \(b_{\text{eff}}\) analytically using the retarded Green function with the boundary condition at \(u = 0\) as

\[
b_{\text{eff}} (u) = b_{\text{eff}} (0) e^{-\gamma u} + m \gamma \int_0^u du' e^{-\gamma (u-u')} \times [\alpha \cdot n(u')] \alpha - \alpha^2 n(u')] \quad (41)
\]

Plugging the above result back to Eq. \((40)\), we can observe that the first term in the right side of Eq. \((41)\) is responsible for the spin precession around the direction of \(b_{\text{eff}}(0)\) that lies on \(x-y\) plane, although it will damp exponentially. The term proportional to \(\alpha\) in the square bracket in Eq. \((41)\), together with \(B\), also leads to the spin precession around \(z\) axis. In total, therefore, the axis of the spin precession at a given snapshot deviates from \(b_{\text{eff}}(0)\) axis, as illustrated in Fig. 4. More importantly, the term proportional to \(\alpha^2 n(u)\) in Eq. \((41)\) plays the role of the spin torque that will bend the spin towards the vertical direction to both the past and present spin
The fate of the dynamical system is characterized by the asymptotic behavior of the variables, i.e., \( n(\infty) \) and \( p(\infty) \) (or, equivalently, \( b_{\text{eff}}(\infty) \)). Such behavior can be deduced from the static limit of Eqs. (42) and (43), leading to

\[
\begin{align*}
p &= m \alpha \times n, \quad [\alpha \times p + B] \times n = 0 \\
&\rightarrow [m (\alpha \cdot n \alpha - \alpha^2 n) + B] \parallel n. \quad (42)
\end{align*}
\]

In the case of our interest, where \( B = (0, 0, B) \) and \( 0 \leq B < B_c \), the momentum becomes asymptotically vertical to the spin as

\[
p(\infty) = m \alpha \times n(\infty), \quad (43)
\]

and its magnitude can be obtained from the extreme condition \( \partial_p E_{-1} = 0 \) as

\[
|p(\infty)| = \frac{1}{\alpha} \sqrt{B_c^2 - B^2}. \quad (44)
\]

The extreme condition above implies that the group velocity vanishes in the end, i.e., \( \dot{z}(\infty) = 0 \); see Fig. 2 in appendix B. Here it should be reminded again that \( p \) is always confined in \( x-y \) plane during the evolution. The asymptotic spin, on the other hand, has nonzero parallel and vertical components to \( z \) axis when \( B \neq 0 \). The \( z \) component of the spin can be determined from the condition (42) as

\[
m \alpha \cdot n(\infty) \alpha + B = 0 \]

\[
\rightarrow n_z(\infty) = -\frac{B}{2B_c}, \quad (45)
\]

while the magnitude of the spin projection on \( x-y \) plane can be determined from \( n^2 = 1/4 \). Note that the asymptotic behavior at zero magnetic field can be obtained simply by taking the limit of \( B \to 0 \) in the above results. In this case, the asymptotic spin has no \( z \) component. Incidentally, in the case of \( B > B_c \), we obtain \( p(\infty) = 0 \) as the minimum of the single particle energy \( E_{-1} \) (see Fig. 1), and the spin eventually gets antiparallel to \( B \), i.e., \( n_z(\infty) = -1/2 \).

The above analysis shows that only the relative angles among \( \alpha, p \), and \( n \) are fixed asymptotically irrespective of the initial condition. Since the Hamiltonian has the rotational symmetry around \( z \) axis, the direction of \( p(\infty) \) on \( x-y \) plane is determined by \( p(0) \) and \( n(0) \). We will demonstrate this situation by numerical simulations below and then classify the resultant asymptotic behavior into two characteristic cases.

### III. Numerical Simulations and Discussion

Before going into detailed calculations, we define the initial precession period and the damping time as

\[
\tau_{\text{prec}} = \frac{2\pi}{\sqrt{\alpha^2 p^2(0) + B^2}}, \quad (46)
\]

\[
\tau_{\text{damp}} = \gamma^{-1} = m/c, \quad (47)
\]

both of which are assumed much longer than the decoherence time scale (65) for a typical size of the particle wave packet. In numerical calculations, we will use them as the reference time scales for classification of dynamical domains.

We always set the initial momentum parallel to \( x \) axis as \( p(0) = (p_0, 0, 0) \) without loss of generality and then observe the time evolution of the variables to see how their final state depends on the initial condition of \( n(0) \) for some typical values of \( \alpha \) and \( \gamma \) as well as a fixed value of \( p_0 \). The spin direction is specified by the angles \( \theta_n \) and \( \phi_n \), as depicted in Fig. 2. The zenith axis is given along the instantaneous precession axis \( b_{\text{eff}} + B \), the zenith angle \( \theta_n \) is taken between \( b_{\text{eff}} + B \) and \( n \), and the azimuthal angle \( \phi_n \) between \( n \) and \( \alpha \) projected on the plane normal to \( b_{\text{eff}} + B \). The angle between \( p(0) \) and \( p(t) \) is denoted by \( \phi_p \).

#### A. Zero Magnetic Field: \( B = 0 \)

We first examine the case of \( B = 0 \) numerically. In Fig. 3, we show the result for the asymptotic value of the
momentum angle $\phi_p(\infty)$ as functions of the initial values of the spin angles $\theta_n(0)$ and $\phi_n(0)$ both for $\tau_{\text{prec}} \gg \tau_{\text{damp}}$ and $\tau_{\text{prec}} \ll \tau_{\text{damp}}$. Since we have fixed the initial values of $p_0$ and $\gamma$ to some specific values, the precession period $\tau_{\text{prec}}$ is determined solely by the strength of the Rashba coupling $\alpha$. Figure 3(a) shows that for strong Rashba coupling, that is, for $\tau_{\text{prec}} \ll \tau_{\text{damp}}$, the asymptotic momentum direction does not deviate so much from its initial one for a wide range of the initial spin angles satisfying $\theta_n(0) \gtrsim \pi/4$. For weak Rashba coupling, that is, for $\tau_{\text{prec}} \gg \tau_{\text{damp}}$, on the other hand, the asymptotic direction of the momentum fluctuates significantly as the initial spin angles change only slightly. This feature can be seen clearly from Fig. 3(b).

To observe explicitly what happens in between, we show in Fig. 4 the whole time evolution of the spin and momentum by taking two sets of the initial spin angles, $\{\phi_n(0), \theta_n(0)\} = \{\pi/4, \pi/12\}$ and $\{\pi/4, 2\pi/3\}$, which lead to different intermediate behaviors. For $\tau_{\text{prec}} \ll \tau_{\text{damp}}$, as can be seen from Figs. 4(a) and (b), the spin precesses repeatedly as it should. Then, the spin torque is effective at bending the spin direction almost completely to $\theta_n = \pi$ within the damping time scale $\gamma = O(1)$. Simultaneously, as shown in Fig. 4(c), the magnitude of $p$ reaches $|p(\infty)|$ almost completely, which gives the energy minimum of $E_{\text{min}}$. We observe from Figs. 4(a) and (c) that a weird behavior happens when the spin angle $\theta_n$ passes through $\pi/2$: The precession slows temporarily, while the momentum orientation changes abruptly. The eventual momentum orientation corresponds to a point just on the ridge that appears in Fig. 3(a). This kind of behavior arises presumably from the nonlinearity of the quasi-classical dynamical equations.

For $\tau_{\text{prec}} \gg \tau_{\text{damp}}$, on the other hand, we can observe from Figs. 4(d) and (e) that the spin precesses only once or less within the damping time scale and that it takes a long time for the spin orientation to reach $\theta_n = \pi$. In this case, as shown in Fig. 4(f), the magnitude of the momentum decays and almost reaches $|p(\infty)|$ within the damping time scale, whereas its orientation keeps changing continuously even after that in conjunction with the nonlinear spin dynamics. This is the reason why the asymptotic orientation of the momentum fluctuates significantly as a function of the initial spin orientations as illustrated in Fig. 3(b). Note also that in the case in which the initial spin is provided along the precession axis, no spin torque is activated, so that only the momentum relaxes to the point of the energy minimum. Numerical results for other initial spin angles and also for the time evolution of the velocity are given in appendix B.

**B. Non zero magnetic field: $B \neq 0$**

In the presence of a nonzero magnetic field, numerical results are qualitatively similar to the $B = 0$ case. There are still minor differences. For $B \neq 0$, the precession axis deviates from the $x$-$y$ plane, and the precession period $\tau_{\text{prec}}$ given by Eq. (46) gets shorter than the $B = 0$ case at a given Rashba coupling strength. Accordingly, the dynamics of the spin and momentum is modified. We relegate the numerical results for $B \neq 0$ to appendix B.

**IV. SUMMARY AND OUTLOOKS**

In the present study we have studied the open-system dynamics of a single spin-1/2 particle with a Rashba-type SOC in a three-dimensional ohmic heat bath by employing the extended version of the Calderia-Leggett model. At sufficiently high temperature, we have succeeded in deriving the quasi-classical Langevin equation for the momentum with a friction term and the dynamical equation for the instantaneous spin precession; these equations are nonlinearly coupled with each other, leading to a complex relaxation-dissipation dynamics until the spin and momentum settle down in one of the minima of the spin-down eigen energy $E_{-1}$. 
By obtaining the analytical and numerical solutions to these equations, we have found that when the precession period $\tau_{\text{prec}}$ is much shorter than the damping time $\tau_{\text{damp}}$, i.e., $\tau_{\text{prec}} \ll \tau_{\text{damp}}$, the initial and final momenta point to almost the same direction for a wide range of the initial spin direction. An interesting implication of this finding is that one can control the final state of the spin direction by increasing the Rashba coupling strength $\alpha$. In the opposite case of $\tau_{\text{prec}} \gg \tau_{\text{damp}}$, however, the final momentum significantly fluctuates around the initial one, and so does the final spin direction. Since the spin dynamics in our model is always accompanied by the momentum dissipation, the spin relaxation mechanism elucidated in this study is qualitatively different from that encountered in semiconductors where the Pauli blocking prevents momentum dissipation and also in nuclear magnetic resonances where the spin is localized.

Throughout the present study we have restricted ourselves to the quasi-classical dynamics that can be described by deterministic local differential equations by virtue of the ohmic heat bath. This is a rather ideal case but in general we have to consider memory effects and quantum fluctuations. For possible application to more realistic cases, e.g., a mobile atomic impurity in trapped cold atoms, it is not always good to assume high temperatures and ohmic environments, but dynamical description of the spin and momentum of the impurity may require the fully quantal time evolution of the reduced density matrix. For such purpose, the path integral formulation with spin degrees of freedom that we have developed here may be utilized. It would be also interesting to use it to explain a global spin polarization of heavy hadrons observed in relativistic heavy-ion collision experiments [30].

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Appendix A: Off-diagonal fluctuation effects

We rewrite the effective action \( iW \) for the ohmic case in terms of \( x_{\pm} \) and \( g_{\pm} \) as

\[
iW[x_+, g_+, g_-] = \int_0^t \text{d}u \left\{ m (\dot{x}_+ - d\mathbf{n}_+ \times \alpha) \cdot (\dot{x}_- - d\mathbf{n}_- \times \alpha) + \frac{1}{2} \dot{x}_+ \sin \theta_+ \sin \frac{\theta_+}{2} + \frac{1}{2} \dot{x}_- \cos \theta_+ \cos \frac{\theta_+}{2} - d\mathbf{n}_- \cdot B + c \dot{x}_+ \cdot x_- \right\}
- \frac{1}{2} \int_0^t \text{d}u \int_0^t \text{d}u' L(u-u') x_-(u) \cdot x_-(u'), \tag{A1}
\]

where \( d\mathbf{n}_+ = \{n(g) + n(g')\}/2, \ d\mathbf{n}_- = n(g) - n(g') \), and

\[
L(u-u') = \int_0^\infty \text{d}\omega J(\omega) \coth \frac{\omega}{2} \cos \omega(u-u') \tag{A2}
\]

Since we are interested in the probability functional of the quasi-classical path \( x_- \), we integrate out the off-diagonal fluctuation path \( x_+ \), which is suppressed by the Gaussian term including \( L(u-u') \), to obtain the probability functional up to an irrelevant constant as

\[
iW[x_+, g_+, g_-] = \int_0^t \text{d}u \left\{ m (\dot{x}_+ - d\mathbf{n}_+ \times \alpha) \cdot (\dot{x}_- - d\mathbf{n}_- \times \alpha) + \frac{1}{2} \dot{x}_+ \sin \theta_+ \sin \frac{\theta_+}{2} + \frac{1}{2} \dot{x}_- \cos \theta_+ \cos \frac{\theta_+}{2} \right.
- d\mathbf{n}_- \cdot B
\]
\[
+ \frac{1}{2} \int_0^t \text{d}u \int_0^t \text{d}u' L^{-1}(u-u') \times \xi \left[ x_+(u), d\mathbf{n}_+(u) \right] \cdot \xi \left[ x_+(u'), d\mathbf{n}_+(u') \right], \tag{A3}
\]

where we have introduced a noise functional

\[
\xi \left[ x_+, d\mathbf{n}_+ \right] \equiv m [\dot{x}_+ + \gamma x_+ - d\mathbf{n}_+ \times \alpha]. \tag{A4}
\]

The above result can be interpreted as follows: If the functional \( \xi \) is used instead of \( x_+ \) in the path integral of \( e^{iW} \), i.e., \( \mathcal{D}x_+ = \mathcal{J} \mathcal{D}\xi \) with \( \mathcal{J} = \{\mathcal{D}x_+ / \mathcal{D}\xi\} \) the functional Jacobian, \( x_+ \) becomes inversely the functional of \( \xi \) whose path probability is given by \( \mathcal{J} e^{iW} \). The Jacobian in our case becomes a constant. Once a noise fluctuation path \( \xi \) is given in accordance with this probability, therefore, Eq. \( \text{(A3)} \) can be regarded as a Langevin equation, whose solution for \( \dot{x}_+ \) is given by

\[
\dot{x}_+(u) = \dot{x}_+(0) e^{-\gamma u} + \int_0^u \text{d}u' e^{-\gamma(u-u')} \left[ \frac{\xi(u')}{m} + d\mathbf{n}_+(u') \times \alpha \right] = d\mathbf{n}_+(u) \times \alpha + e^{-\gamma u} [\dot{x}_+(0) - d\mathbf{n}_+(0) \times \alpha]
\]

\[
- \gamma \int_0^u \text{d}u' e^{-\gamma(u-u')} d\mathbf{n}_+(u') \times \alpha
+ \frac{1}{m} \int_0^u \text{d}u' e^{-\gamma(u-u')} \xi(u'), \tag{A5}
\]

Plugging the above expression into Eq. \( \text{(A3)} \), we obtain the functional in terms of \( \xi \) and \( g_{\pm} \) as

\[
iW[\xi; g_\pm] = -i \int_0^t \text{d}u \left\{ e^{-\gamma u} p_+(0) \cdot (d\mathbf{n}_- \times \alpha) + \frac{1}{2} \dot{x}_+ \sin \theta_+ \sin \frac{\theta_+}{2} + \frac{1}{2} \dot{x}_- \cos \theta_+ \cos \frac{\theta_+}{2} + d\mathbf{n}_- \cdot B \right\}
+ \frac{1}{2} \int_0^t \text{d}u \int_0^t \text{d}u' G(u-u') \times \{d\mathbf{n}_+(u') \times \alpha\} \cdot \{d\mathbf{n}_-(u) \times \alpha\}
- \frac{1}{2} \int_0^t \text{d}u \int_0^t \text{d}u' L^{-1}(u-u') \xi(u') \cdot \xi(u'), \tag{A6}
\]

where \( G(t) = e^{-\gamma t} \theta(t) \) is the retarded Green function. Note that the path \( \xi(0) = 0 \) corresponds to the quasi-classical path of \( x_+ \) that satisfies the homogeneous equation of Eq. \( \text{(A4)} \). From the Gaussian form for \( \xi \) in Eq. \( \text{(A3)} \) we expect this quasi-classical path to dominate the path integral when the off-diagonal Euler-angle fluctuations are consistently small, i.e., \( g_- \sim 0 \). In this case the noise fluctuation satisfies \( \langle \xi \rangle = 0 \) and a Brownian (Kubo’s second) fluctuation-dissipation relation \( \langle \xi(t) \xi(t') \rangle = \delta_{ij} L(t-t') \).

Finally, performing the Gaussian path integral with respect to \( \xi \), we obtain the effective action for the spin variables \( g_{\pm} \) as

\[
iW[g_\pm] = -i \int_0^t \text{d}u \left\{ \dot{x}_+ \sin \theta_+ \sin \frac{\theta_+}{2} - \frac{1}{2} \dot{x}_+ \cos \theta_+ \cos \frac{\theta_+}{2} + d\mathbf{n}_- \cdot B \right\}
+ \frac{1}{m} \int_0^t \text{d}u' G(u-u') \times \{d\mathbf{n}_+(u') \times \alpha\} \cdot \{d\mathbf{n}_-(u-0) \times \alpha\} + \frac{1}{2} \int_0^t \text{d}u \int_0^t \text{d}u' L^{-1}(u-u') \xi(u') \cdot \xi(u'), \tag{A7}
\]

where we have introduced the following functional

\[
\beta_-(u) = \int_0^t \text{d}u' G(u'-u) \{d\mathbf{n}_-(u') \times \alpha\}. \tag{A8}
\]
In Eq. (31) we again encounter the Gaussian suppression characterized by \( L(u-u') \) also for the off-diagonal spin variable \( g_- \). At high temperatures where \( L(u-u') \approx 2\gamma mk_B T \delta(u-u') \), this Gaussian term can be approximated by

\[
\frac{1}{2} \int_0^t \int_0^t du' L(u-u') \\
\times \{ \beta_-(u) - G(t-u)x_-(t) \} \cdot \{ \beta_-(u') - G(t-u')x_-(t) \}
\approx -m\gamma k_B T \int_0^t \int_0^t ds \int_0^t ds' \int_0^t du G(s-u)G(s'-u) \\
\times \{dn_-(s)\alpha - 2\delta(t-s)x_-(s)\} \\
\times \{dn_-(s')\alpha - 2\delta(t-s')x_-(s')\}.
\]

Indeed the above expression is of Gaussian form, but it is nonlocal in time unlike the case of \( x_- \) as depicted in Eq. (31). This nonlocality prevents us from explicitly proving the decoherence of the off-diagonal spin variable \( g_- \). Nevertheless, one can show that after integrating out the orbital variables \( x_- \), the path of \( g_- = 0 \), i.e., \( \beta_- = 0 \), is not suppressed by any Gaussian factor, implying that it gives the most probable one, i.e., the quasi-classical path. In fact, the variation of \( \mathcal{W} \) with respect to \( g_- \) at \( g_- = 0 \) restores Eq. (10) that has Eq. (11) incorporated. We postpone further detailed analysis about nonlinear fluctuation effects of \( g_- \) on the quasi-classical dynamics elsewhere.

### Appendix B: Time evolution of spin and momentum

In Sec. III, we have presented numerical results only in the case of \( \phi_0(0) = \pi/4 \) and \( B = 0 \), which are in this appendix supplemented as follows: Figs. 3 and 4 show results for \( \phi_0(0) = 0, \pi/2 \) at \( B = 0 \), and Figs. 8 and 11 for \( \phi_0(0) = 0, \pi/4, \pi/2 \) at \( B \neq 0 \). The values of \( \alpha \) and \( p_0 \) are the same as in Fig. 3. We also present the time evolution of the velocity in Fig. 7, where the initial condition and parameter values are the same as in Fig. 4.
FIG. 6. Time evolution of the spin and momentum at \( B = 0 \), as plotted for \( \tau_{\text{prec}} \gg \tau_{\text{damp}} \) under the initial conditions \( \{ \phi_n(0), \theta_n(0) \} = \{ 0, \{ \pi/12, \pi/4, 2\pi/3 \} \} \) ((a), (b), (c), (d)) and \( \{ \phi_n(0), \theta_n(0) \} = \{ \pi/2, \{ \pi/12, \pi/4, 2\pi/3 \} \} \) ((e), (f), (g), (h)).

FIG. 7. Time evolution of the velocity, \( v = \dot{\mathbf{x}} \), at \( B = 0 \), as plotted for \( \tau_{\text{prec}} \ll \tau_{\text{damp}} \) ((a), (b)) and \( \tau_{\text{prec}} \gg \tau_{\text{damp}} \) ((c), (d)) under the initial conditions \( \{ \phi_n(0), \theta_n(0) \} = \{ \pi/4, \pi/12 \} \). \( \phi_v \) denotes the angle between \( v(0) \) and \( v(t) \) in the \( x-y \) plane. The values \( p_0 \) and \( \alpha \) are the same as in Fig. 3.
FIG. 8. Asymptotic values of the momentum angle $\phi_p(\infty)$ as a function of the initial values of the spin angles $\theta_n(0)$ and $\phi_n(0)$, as plotted for $\tau_{\text{prec}} \ll \tau_{\text{damp}}$ (a) and $\tau_{\text{prec}} \gg \tau_{\text{damp}}$ (b) at $B = 0.027563 \gamma < B_c$. 

FIG. 9. Time evolution of the spin and momentum at $B = 0.027563 \gamma < B_c$, calculated under the initial conditions $(\phi_n(0), \theta_n(0)) = \{\pi/4, \{\pi/12, 2\pi/3\}\}$. Panels (a), (b), (c) depict the results for $\tau_{\text{prec}} \ll \tau_{\text{damp}}$, and (d), (e), (f) for $\tau_{\text{prec}} \gg \tau_{\text{damp}}$. 
FIG. 10. Time evolution of the spin and momentum at $B = 137.42 \gamma > B_c$, calculated under the initial conditions $\{\phi_n(0), \theta_n(0)\} = \{\pi/4, \{\pi/12, 2\pi/3\}\}$. Panels (a), (b), (c), (d) depict the results for $\tau_{\text{prec}} \ll \tau_{\text{damp}}$, and (e), (f), (g), (h) for $\tau_{\text{prec}} \gg \tau_{\text{damp}}$. 

$(a) \phi_n(0)=\pi/4, \theta_n(0)=\pi/12$

$(b) \phi_n(0)=\pi/4, \theta_n(0)=2\pi/3$

$(c) \phi_n(0)=\pi/4, \theta_n(0)=\pi/12$

$(d) \phi_n(0)=\pi/4, \theta_n(0)=2\pi/3$

$(e) \phi_n(0)=\pi/4, \theta_n(0)=\pi/12$

$(f) \phi_n(0)=\pi/4, \theta_n(0)=2\pi/3$

$(g) \phi_n(0)=\pi/4, \theta_n(0)=\pi/12$

$(h) \phi_n(0)=\pi/4, \theta_n(0)=2\pi/3$