THE PBW FILTRATION

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ABSTRACT. In this paper we study the PBW filtration on irreducible integrable highest weight representations of affine Kac-Moody algebra \( \hat{g} \). The \( n \)-th space of this filtration is spanned with the vectors \( x_1 \ldots x_nv \), where \( x_i \in \hat{g}, \ s \leq n \) and \( v \) is a highest weight vector. For the vacuum module we give a conjectural description of the corresponding adjoint graded space in terms of generators and relations. For \( g \) of the type \( A_1 \) we prove our conjecture and derive the fermionic formula for the graded character.

INTRODUCTION

Let \( g \) be a Kac-Moody Lie algebra of finite or affine type and \( U(g) \) be the universal enveloping algebra of \( g \). For the dominant integral weight \( \lambda \) let \( L_\lambda \) be the corresponding irreducible highest weight representation with highest weight \( \lambda \) and highest weight vector \( v_\lambda \). We consider a filtration \( U(g)_s \) on \( U(g) \) defined by

\[
U(g)_0 = \mathbb{C}1, \ U(g)_{s+1} = U(g)_s + \text{span}\{gu : g \in g, u \in U(g)_s\}.
\]

This filtration induces a filtration \( F_s = U(g)_s \cdot v_\lambda \) on \( L_\lambda \). Define an associated graded space

\[
L^{gr}_\lambda = F_0 \oplus F_1/F_0 \oplus F_2/F_1 \oplus \ldots
\]

We define the graded \( u \)-character by

\[
\text{ch}_u L^{gr}_\lambda = \sum_{s \geq 0} u^s \text{ch}(F_s/F_{s-1}),
\]

where we set \( F_{-1} = 0 \) and \( \text{ch} \) denotes the usual character with respect to the Cartan subalgebra of \( g \). For example for \( g \) of the type \( A_1 \) and its arbitrary finite-dimensional representation all the spaces \( F_s/F_{s-1} \) are one-dimensional. The PBW-filtration for \( g \) of the type \( A_1^{(1)} \) and its level 1 representations was used in [FFJMT] for the study of \( \phi_{1,3} \)-field in Virasoro minimal theories.

Let us briefly describe our approach to the study of the spaces \( L^{gr}_\lambda \). Let \( n_- \hookrightarrow g \) be the nilpotent subalgebra of creating operators, i.e.

\[
L_\lambda = U(n_-) \cdot v_\lambda.
\]

The action of \( n_- \) on \( L_\lambda \) induces the action of the abelian algebra \( n_{ab}^\top \) on \( L^{gr}_\lambda \) (\( n_{ab}^\top \) is isomorphic to \( n_- \) as a vector space). Therefore the space \( L^{gr}_\lambda \) is generated from the vector \( v_\lambda \) with the action of \( U(n_{ab}^\top) \), which is isomorphic
to the polynomial algebra. This allows to describe $L^g_k$ as a quotient of the polynomial algebra on $n_-$ by some ideal.

In this paper we study the case of an affine Kac-Moody algebra $\hat{g}$ and its vacuum representations. Let $g$ be a finite-dimensional simple Lie algebra and $\hat{g}$ be the corresponding affine algebra, i.e. the central extension of $g \otimes \mathbb{C}[t, t^{-1}]$. We fix its vacuum level $k$ representation $L_k$ with a highest weight vector $v_k$. In this case one has $(g \otimes \mathbb{C}[t]) \cdot v_k = 0$ and therefore the space $L^g_k$ is generated from the highest weight vector with an action of the universal enveloping algebra of the abelian algebra $g_{ab} \otimes t^{-1} \mathbb{C}[t]$. This gives

$$L^g_k \simeq U(g_{ab} \otimes t^{-1} \mathbb{C}[t]) / I_k,$$

where $I_k$ is some ideal. We describe the ideal $I_k$ in the following way. We note that $g = g \otimes 1$ annihilates the highest weight vector $v_k$. Therefore the action of $g$ on $L_k$ induces a structure of $g$-module on $L^g_k$ and $I_k$. (This structure plays an important role in our approach, see the conjecture below. For non vacuum modules $L^g_{\lambda}$ is not $g$-module anymore. This means that the ideal of relations for general highest weight is to be described in other terms then in the vacuum case.) We give a conjectural description of $I_k$ below. The proof is given in the paper for the case $g = \mathfrak{sl}_2$.

The adjoint action of $g$ on itself endows the space $U(g_{ab} \otimes t^{-1} \mathbb{C}[t])$ with a structure of $g$-module. Let $\theta$ be the longest root of $g$, $f_\theta \in g$ be an element of the weight $\theta$ and

$$f_\theta(z) = \sum_{n \geq 0} z^n (f_\theta \otimes t^{-n-1})$$

be the corresponding current. We recall (see for example [BF]) that the series $f_\theta(z)^{k+1}$ acts by zero on $L_k$ (the coefficients of the $z$-expansion of $f_\theta(z)^{k+1}$ are acting by zero).

**Conjecture 0.1.** We have an equality

$$I_k = (U(g) \oplus U(g_{ab} \otimes t^{-1} \mathbb{C}[t^{-1}])) \cdot \text{span}\{ \text{coefficients of } f_\theta(z)^{k+1} \},$$

i.e. $I_k$ is the minimal $U(g)$-stable ideal which contains the coefficients of $f_\theta(z)^{k+1}$.

We verify this conjecture for $g = \mathfrak{sl}_2$ (the level 1 case was studied in [FEJMT] by different method). Combining the description of $I_k$ with the vertex operator realization technique we obtain the fermionic formula for the $u$-character of $L^g_k$ for $g = \mathfrak{sl}_2$.

The paper is organized in the following way:

In Section 1 we fix our notations and recall some constructions from the representation theory of affine Kac-Moody algebras and bosonic vertex operator algebras.

In Section 2 we study the adjoint graded space $L^g_1$.

In Section 3 we describe the ideal of relations in $L^g_k$ for the general level and obtain the fermionic formula for the graded character.
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1. Preliminaries and definitions

1.1. Kac-Moody Lie algebras, representations and filtrations. Let \( \mathfrak{g} \) be a finite-dimensional simple Lie algebra with a Cartan decomposition \( \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_- \). We consider the corresponding affine Kac-Moody algebra

\[
\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} K \oplus \mathbb{C} d,
\]

where \( K \) is a central element and \( d \) is a degree operator, \( [d, x \otimes t^i] = -ix \otimes t^i \) for all \( x \in \mathfrak{g} \). The affine algebra \( \hat{\mathfrak{g}} \) admits the Cartan decomposition \( \hat{\mathfrak{g}} = \hat{\mathfrak{n}} \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_- \), where

\[
\hat{\mathfrak{n}} = \mathfrak{n} \otimes 1 \oplus \mathfrak{g} \otimes t\mathbb{C}[t],
\]

\[
\hat{\mathfrak{h}} = \mathfrak{h} \otimes 1 \oplus \mathbb{C} K \oplus \mathbb{C} d,
\]

\[
\hat{\mathfrak{n}}_- = \mathfrak{n}_- \otimes 1 \oplus \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}].
\]

Let \( \lambda \in \hat{\mathfrak{h}}^* \) be an integral dominant weight and \( L_\lambda \) be the corresponding irreducible highest weight representation of \( \hat{\mathfrak{g}} \) with highest weight \( \lambda \) and highest weight vector \( v_\lambda \). Then

\[
\hat{\mathfrak{n}} \cdot v_\lambda = 0, \quad hv_\lambda = \lambda(h)v_\lambda \quad \forall h \in \hat{\mathfrak{h}}, \quad L_\lambda = U(\hat{\mathfrak{n}}_-) \cdot v_\lambda,
\]

where \( U(\hat{\mathfrak{n}}_-) \) is a universal enveloping algebra. The number \( \lambda(K) \) is called the level of \( L_\lambda \). We let \( L_k \) to denote the level \( k \) vacuum representation (the restriction of the highest weight of \( L_k \) to \( \mathfrak{h} \otimes 1 \) vanishes).

We define an increasing filtration \( U(\hat{\mathfrak{n}}_-)_s \) on the universal enveloping algebra by the following rule:

\[
U(\hat{\mathfrak{n}}_-)_0 = \mathbb{C} \cdot 1, \quad U(\hat{\mathfrak{n}}_-)_{s+1} = U(\hat{\mathfrak{n}}_-)_s + \hat{\mathfrak{n}}_- U(\hat{\mathfrak{n}}_-)_s.
\]

This filtration induces an increasing filtration \( F_s \) on \( L_\lambda \):

\[
F_s = U(\hat{\mathfrak{n}}_-)_s \cdot v_\lambda.
\]

The filtration \( F_\bullet \) is called the Poincare-Birkhoff-Witt filtration (the PBW filtration for short).

**Definition 1.1.** Define \( L_\lambda^{gr} \) as an adjoint graded space with respect to the filtration \( F_s \), i.e.

\[
L_\lambda^{gr} = F_0 \oplus \bigoplus_{s>0} F_s/F_{s-1}.
\]

Define the \( u \)-character of \( L_\lambda^{gr} \) as

\[
\text{ch}_u L_\lambda^{gr} = \sum_{s \geq 0} u^s \text{ch}(F_s/F_{s-1}),
\]

where we set \( F_{-1} = 0 \) and \( \text{ch} \) denotes the usual character with respect to \( \hat{\mathfrak{h}}^* \).
Remark 1.1. Let $\hat{n}^{ab}_-$ be an abelian Lie algebra which coincides with $\hat{n}_-$ as a vector space. Then $L^{gr}_\lambda$ carries a natural structure of representation of $\hat{n}^{ab}_-$.

Remark 1.2. The definition above is valid in the case of finite-dimensional algebras as well, providing a filtration on finite-dimensional modules.

Remark 1.3. For the general weight $\lambda$ the spaces $F_s$ are not stable under the action of an algebra $g = g \otimes 1 \hookrightarrow \hat{g}$. This motivates the following modification of the filtration above. Let

$$\tilde{F}_0 = U(n_- \otimes 1) \cdot v_\lambda, \quad \tilde{F}_{s+1} = \tilde{F}_s + \hat{n}_- F_s.$$ 

Here $F_0 = C v_\lambda$ is replaced by $\tilde{F}_0$, which is a finite-dimensional irreducible representation of $g$. We denote the corresponding adjoint graded object by $\tilde{L}^{gr}_\lambda$. It is obvious that for any $s$ the space $\tilde{F}_s$ is $g$-invariant. We hope to return to the study of this filtration elsewhere.

1.2. The $\mathfrak{sl}_2$ case. Let $g = \mathfrak{sl}_2$ and let $e, h, f$ be its standard basis. Then $f$ spans $n_-$, $h$ spans $h$ and $e$ spans $n$. For $x \in \mathfrak{sl}_2$ we set $x_i = x \otimes t^i$. Let $v_k$ be a highest weight vector of the vacuum level $k$ module $L_k$.

We now recall a construction of the $ehf$-basis of $L_k$ from [FKLMM].

Definition 1.2. A monomial of the form

$$\ldots f^{a_n} h^{b_n} e^{c_n} \ldots f^{a_1} h^{b_1} e^{c_1}$$

is called an ordered monomial. In addition it is called a $ehf$-monomial if it satisfies the following conditions:

(a) $a_i + a_{i+1} + b_{i+1} \leq k$ for $i > 0$,
(b) $a_i + b_{i+1} + c_{i+1} \leq k$ for $i > 0$,
(c) $a_i + b_i + c_{i+1} \leq k$ for $i > 0$,
(d) $b_i + c_i + c_{i+1} \leq k$ for $i > 0$.

Theorem 1.1. The set $\{m \cdot v_k\}$, where $m$ runs over the set of $ehf$-monomials provides a basis of $L_k$.

Remark 1.4. The following picture from [FKLMM] illustrates the set of $ehf$-monomials:

Namely one considers a set of monomials \ref{eq:7} such that the sum of exponents over any triangle (corresponding to the conditions (a)–(d)) is less than or equal to $k$. 

\begin{equation}
\begin{array}{cccc}
& c_1 & c_2 & \ldots \\
& a_2 & a_1 & \ldots \\
& & & \ldots
\end{array}
\end{equation}
1.3. Lattice vertex operator algebras and affine Kac-Moody algebras.

In this section we recall main properties of lattice vertex operator algebras (VOA for short) and their principal subspaces. The main references are [K2], [BF], [D], [FK].

Let $Q$ be a lattice of finite rank equipped with a symmetric bilinear form $(\cdot, \cdot) : Q \times Q \to \mathbb{Z}$ such that $(\alpha, \alpha) > 0$ for all $\alpha \in Q \setminus \{0\}$. Let $\mathfrak{h} = Q \otimes \mathbb{Z} \mathbb{C}$. The form $(\cdot, \cdot)$ induces a bilinear form on $\mathfrak{h}$, for which we use the same notation. Let $\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} K$ be the corresponding multi-dimensional Heisenberg algebra with the bracket

$$[\alpha \otimes t^i, \beta \otimes t^j] = i \delta_{i, -j} (\alpha, \beta) K, \quad [K, \alpha \otimes t^i] = 0, \quad \alpha, \beta \in \mathfrak{h}.$$ 

For $\alpha \in \mathfrak{h}$ define the Fock representation $\pi_\alpha$ of the Heisenberg algebra generated by a vector $|\alpha\rangle$ such that $$(\beta \otimes t^n)|\alpha\rangle = 0, \quad n > 0; \quad (\beta \otimes 1)|\alpha\rangle = (\beta, \alpha)|\alpha\rangle; \quad K|\alpha\rangle = |\alpha\rangle.$$ 

We now define a VOA $V_Q$ associated with $Q$. We deal only with an even case, i.e. $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$ (in the general case the construction leads to the so called super VOA). As a vector space

$$V_Q \simeq \bigoplus_{\alpha \in Q} \pi_\alpha.$$ 

The $q$-degree on $V_Q$ is defined by

$$\deg_q |\alpha\rangle = \frac{(\alpha, \alpha)}{2}, \quad \deg_q (\beta \otimes t^n) = -n.$$ 

The main ingredients of the VOA structure on $V_Q$ are bosonic vertex operators $\Gamma_\alpha(z)$ which correspond to highest weight vectors $|\alpha\rangle$. One sets

$$\Gamma_\alpha(z) = S_\alpha \exp(-\sum_{n<0} \frac{\alpha \otimes t^n}{n} z^{-n}) \exp(-\sum_{n>0} \frac{\alpha \otimes t^n}{n} z^{-n}),$$ 

where $z^\otimes 1$ acts on $\pi_\beta$ by $z^{(\alpha, \beta)}$ and the operator $S_\alpha$ is defined by

$$S_\alpha|\beta\rangle = c_{\alpha, \beta}|\alpha + \beta\rangle; \quad [S_\alpha, \beta \otimes t^n] = 0, \quad \alpha, \beta \in \mathfrak{h},$$

where $c_{\alpha, \beta}$ are some non vanishing constants. The Fourier decomposition is given by

$$\Gamma_\alpha(z) = \sum_{n \in \mathbb{Z}} \Gamma_\alpha(n) z^{-n - (\alpha, \alpha)/2}.$$ 

In particular,

$$\Gamma_\alpha(-(\alpha, \alpha)/2 - (\alpha, \beta))|\beta\rangle = c_{\alpha, \beta}|\alpha + \beta\rangle.$$ 

One of the main properties of vertex operators is the following commutation relation:

$$[\alpha \otimes t^n, \Gamma_\beta(z)] = (\alpha, \beta) z^n \Gamma_\beta(z).$$
Another important formula describes the product of two vertex operators

\[(12) \quad \Gamma_\alpha(z)\Gamma_\beta(w) = (z - w)^{(\alpha, \beta)}S_\alpha S_\beta z^{(\alpha + \beta) \otimes 1} \times \\
\exp\left(-\sum_{n<0} \frac{\alpha \otimes t^n}{n} z^{-n} + \frac{\beta \otimes t^n}{n} w^{-n}\right) \exp\left(-\sum_{n>0} \frac{\alpha \otimes t^n}{n} z^{-n} + \frac{\beta \otimes t^n}{n} w^{-n}\right)\].

This leads to the proposition:

**Proposition 1.1.**

\[(13) \quad \left(\Gamma_\alpha(z)\right)^{(k)}\left(\Gamma_\beta(z)\right)^{(l)} = 0 \quad \text{if} \quad k + l < (\alpha, \beta),\]

where the superscript \((k)\) denotes the \(k\)-th derivative of the corresponding series. In addition if \((\alpha, \beta) = 0\) then

\[\Gamma_\alpha(z)\Gamma_\beta(z) \text{ is proportional to } \Gamma_{\alpha + \beta}(z)\].

We now recall the Frenkel-Kac construction which provides a vertex operator realization of the basic representation of the affine algebra \(\hat{g}\) for \(g\) of the types \(A\), \(D\) and \(E\). Let \(\vartriangle\) be the root system of \(g\), \(Q\) be the root lattice and \(g = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \vartriangle} C e_\alpha)\) be the weight decomposition. For any \(\alpha \in Q\) we have a vertex operator

\[\Gamma_\alpha(z) = \sum_{n \in \mathbb{Z}} \Gamma_\alpha(n) z^{-n-1}\]

acting on the space \(V_Q\). We let \(V(\mathfrak{g})\) to denote the vertex operator algebra associated with \(\hat{g}\).

**Theorem 1.2.** The identification \(\Gamma_\alpha(n) \mapsto e_\alpha \otimes t^n\) defines an isomorphism \(V(\mathfrak{g}) \simeq V_Q\), which sends a highest weight vector of \(L_k\) to \(|0\rangle\).

We finish this section with the description of principal subspaces of vertex operator algebras (see [FS]). Let \(\alpha_1, \ldots, \alpha_N\) be a set of of linearly independent vectors generating the lattice \(Q\). Let \(M = (m_{i,j})_{1 \leq i,j \leq N}\) be the non degenerate matrix of the scalar products of \(\alpha_i\) \((m_{i,j} = (\alpha_i, \alpha_j))\) such that \(m_{i,i} = 2\) for all \(i\). Consider the principal subspace \(W_Q \hookrightarrow V_Q\) generated from the vector \(|0\rangle\) with an action of operators \(\Gamma_{\alpha_i}(-n_i)\) with \(n_i \geq 1\) \((1 \leq i \leq N)\). Note that because of \((10)\) one has \(\Gamma_{\alpha_i}(-n)|0\rangle = 0\) for \(n < 1\).

Our goal is to describe \(W_Q\) (in particular to find its character). We first realize this subspace as a quotient of a polynomial algebra. Namely define \(W'_Q\) as a quotient of \(\mathbb{C}[a_i(-n)]\) with \(1 \leq i \leq N, \ n \geq 1\), by the ideal of relations generated with

\[a_i(z)^{(k)} a_j(z)^{(l)}, \ k + l < m_{ij},\]

where \(a_i(z) = \sum_{n \geq 1} z^n a_i(-n)\). We note that \(W'_Q = \bigoplus_{n \in \mathbb{Z}^N_{>0}} W'_{Q,n}\), where \(W'_{Q,n}\) is a subspace spanned by monomials in \(a_i(k)\) such that the number
of factors of the type $a_{i_0}(k)$ with fixed $i_0$ is exactly $n_{i_0}$. The $q$-character of $W_{Q,n}$ is naturally defined by $\deg_q a_i(k) = -k$.

The following lemma is standard (see for example [FF]).

**Lemma 1.1.**

\[(14) \text{ch}_q W_{Q,n} = \frac{q^{nMn/2}}{(q)_n},\]

where $(q)_n = \prod_{j=1}^N(q)_{n_j}$, $(q)_n = \prod_{j=1}^n(1-q^j)$.

The next proposition states that the spaces $W_Q$ and $W'_Q$ are isomorphic (see for example [FF]).

**Proposition 1.2.** The map $|0\rangle \mapsto 1$, $\Gamma a_i(n) \mapsto a_i(n)$ induces the isomorphism $W_Q \cong W'_Q$.

In particular for any $n = (n_1, \ldots, n_N) \in \mathbb{Z}_{\geq 0}^N$

\[(15) \text{ch}_q (W_Q \cap \pi_{n_1\alpha_1+\ldots+n_N\alpha_N}) = \frac{q^{nMn/2}}{(q)_n}.\]

**2. Level 1 case**

### 2.1. Algebras $A_1$ and $B_1$.

We start with the description of the adjoint graded space $L^{gr}_{1}$ for vacuum level 1 $\hat{\mathfrak{sl}}_2$-module. Note that this case was considered in [FFJMT] by the different method. We make a connection to their approach in the end of the section.

The space $L^{gr}_{1}$ carries the natural structure of the representation of abelian Lie algebra $\mathfrak{sl}_{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ (see Remark [TT], where $\mathfrak{sl}_{ab}$ is an abelinization of $\mathfrak{sl}_2$, i.e. the 3-dimensional abelian Lie algebra. For $x \in \mathfrak{sl}_2$ we denote the corresponding element of $\mathfrak{sl}_{ab}$ by $\tilde{x}$. We also set

\[\tilde{x}(z) = \sum_{i<0} \tilde{x}_i z^{-i-1}.\]

The space $L^{gr}_{1}$ is isomorphic to a quotient of the polynomial algebra

\[\mathbb{C}[\tilde{e}, \tilde{h}, \tilde{f}]_{i<0}\]

by some ideal. Our goal is to show that the ideal of relations is generated with coefficients of the following series

\[(16) \tilde{e}(z)^2, \tilde{e}(z)\tilde{h}(z), 2\tilde{e}(z)\tilde{f}(z) - \tilde{h}(z)^2, \tilde{h}(z)\tilde{f}(z), \tilde{f}(z)^2,\]

i.e. by the $\mathfrak{sl}_2$ consequences of the relation $\tilde{e}(z)^2$ ($\mathfrak{sl}_2$ acts on $\mathfrak{sl}_{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ via the adjoint action on the space $\mathfrak{sl}_{ab}$).

**Definition 2.1.** Let $A_1$ be an algebra generated with the commuting variables $\tilde{x}_i, x = e, h, f, i < 0$ modulo the relations [TT].

In order to make a connection between $A_1$ and $L^{gr}_{1}$ we need a modification of $A_1$. 
**Definition 2.2.** Let $B_1$ be an algebra generated with the abelian variables $\tilde{x}_i$, $x = e, h, f$, $i < 0$ modulo the relations

\[ e(z)^2, \tilde{e}(z)\tilde{h}(z), \tilde{h}(z)^2, \tilde{h}(z)f(z), \tilde{f}(z)^2. \]

We define the $(q, z, u)$-characters of $A_1$ and $B_1$ assigning the $(q, z, u)$-degree to each generator $\tilde{x}_i$:

\[ \deg_q \tilde{x}_i = -i, \ \deg_z \tilde{e}_i = 2, \ \deg_z \tilde{f}_i = -2, \ \deg_u \tilde{h}_i = 0, \ \deg_u \tilde{x}_i = 1. \]

**Lemma 2.1.** $\text{ch}_{q, z, u} L_1^{gr} \leq \text{ch}_{q, z, u} A_1 \leq \text{ch}_{q, z, u} B_1$ (the inequalities are true in each weight component).

**Proof.** To show that $\text{ch}_{q, z, u} L_1^{gr} \leq \text{ch}_{q, z, u} A_1$ it suffices to verify that relations (16) hold in $L_1^{gr}$. In fact, $e(z)^2 = 0$ in $L_1$ and $sl_2 = sl_2 \otimes 1 \hookrightarrow \hat{sl}_2$ is acting on $L_1^{gr}$ as well as on $L_1$. In addition all of the relations (16) can be produced from $e(z)^2$ by applying the operator $f$.

To prove that $\text{ch}_{q, z, u} A_1 \leq \text{ch}_{q, z, u} B_1$ we introduce a filtration $G_s$ on $A_1$ by setting $G_0$ to be the subspace generated with variables $\tilde{e}_i, \tilde{f}_i$ and

\[ G_{s+1} = \text{span}\{\tilde{h}_i w : i < 0, w \in G_s\}. \]

Then the relation $2\tilde{e}(z)\tilde{f}(z) - \tilde{h}(z)^2 = 0$ (which holds in $A_1$) gives $\tilde{h}(z)^2 = 0$ in the adjoint graded space. Lemma is proved. \hfill \Box

In the following subsections we give two proofs of the equality

\[ \text{ch}_{q, z, u} L_1 = \text{ch}_{q, z, u} B_1. \]

2.2. **A basis of $B_1$.** Recall the $ehf$-basis of $L_1$ (see Theorem 1.1):

\[ \ldots e_2^{c_2} f_1^{a_1} h_1^{b_1} e_1 \]

with

\begin{align*}
(a) & \quad a_i + a_{i+1} + b_{i+1} \leq 1, \\
(b) & \quad a_i + b_{i+1} + c_{i+1} \leq 1, \\
(c) & \quad a_i + b_i + c_{i+1} \leq 1, \\
(d) & \quad b_i + c_i + c_{i+1} \leq 1.
\end{align*}

We now consider a modified set of restrictions

\begin{align*}
(a') & \quad a_i + a_{i+1} + b_{i+1} \leq 1, \\
(b') & \quad a_i + b_i + b_{i+1} \leq 1, \\
(c') & \quad b_i + b_{i+1} + c_{i+1} \leq 1, \\
(d') & \quad b_i + c_i + c_{i+1} \leq 1,
\end{align*}

and add one additional restriction

\[ (N) \quad b_i + a_{i+1} + c_{i+2} \leq 2. \]

We refer to the monomials (18) with restrictions $(a')$, $(b')$, $(c')$, $(d')$ and $(N)$ as $ehf'$-monomials.

Conditions $(a') - (d')$ and $(N)$ can be expressed as follows (the first picture explains $(a') - (d')$ and the second explains $(N)$):
The *ehf'-monomials are monomials \[m\] such that the sum of exponents over any triangle in the first picture is less than or equal to 1 and the sum of exponents over any triangle in the second picture is less than or equal to 2.

Lemma 2.2. The characters of *ehf- and *ehf'-monomials coincide,

Proof. In order to prove our Lemma we construct a \((q, z, u)\)-degree preserving bijection \(\phi\) from the set of *ehf-monomials to the set of *ehf'-monomials. Fix some *ehf-monomial \(m\) of the form \[m\]. If \(m\) satisfies conditions \((a'), \quad (b'), \quad (c')\) and \((d')\) then we set \(\phi(m) = m\). Now suppose that \(j\) is a smallest number such that \((b')\) or \((c')\) is violated, i.e.

\[a_j + b_j + b_{j+1} = 2 \quad \text{or} \quad b_j + b_{j+1} + c_{j+1} = 2.\]

This means

\[b_j = b_{j+1} = 1, \quad a_j = c_{j+1} = 0.\]

We construct a new monomial

\[m' = \ldots f_{-n} a_n h_n e_{-n} \ldots f_{-1} h_{-1} e_{-1},\]

which differs from \(m\) only in terms \(a_j, b_j, b_{j+1}, c_{j+1}\). The new values are given by

\[a'_j = c'_{j+1} = 1, \quad b'_j = b'_{j+1} = 0.\]

The new monomial \(m'\) satisfies conditions \((a')\)-(\(d'\)) for all \(i \leq j\). In addition it satisfies the condition \((N)\) for all \(i < j\). In fact the violation of \((N)\) means that for some \(i < j\) \(b'_i = a'_{i+1} = c'_{i+2} = 1\). Therefore \(i = j - 1\) and \(b_{j-1} = b_j = 1\). This gives the violation of the conditions \((b')\) and \((c')\) for \(i = j - 1\), which contradicts with the choice of \(j\).

We repeat the procedure until all the conditions \((a')-(d')\) are satisfied for all \(i\). Denote the result by \(\phi(m)\). We note that by the same reason as above the condition \((N)\) is satisfied by \(\phi(m)\). Therefore \(\phi(m)\) is an *ehf'-monomial.

We now show that \(\phi\) is a bijection. The inverse map \(\phi^{-1}\) can be constructed in the following way. Fix some \(m'\), which is *ehf'- but not *ehf-monomial. Then for some \(j\)

\[a'_j + b'_j + c'_{j+1} = 2 \quad \text{or} \quad a'_j + b'_{j+1} + c'_{j+1} = 2,\]
which means \( a_j' = c_j' = 1, b_j' = b_{j+1}' = 0 \). Then the inverse transformation to (19) is given by

\[
a_j = c_{j+1} = 0, \quad b_j = b_{j+1} = 1.
\]

Lemma is proved. \( \square \)

**Lemma 2.3.** The set of \( ehf' \)-monomials spans \( B_1 \).

**Proof.** For any ordered monomial \( m \) of the form

\[
(20) \quad \ldots f^n_{-n} h^b_{-n} e^c_{-n} \ldots f^1_{-1} h^1_{-1} e^1_{-1}
\]

we denote by \( \deg m \) the sum of all the exponents \( a_i, b_i \) and \( c_i \). Following [PKLMM] we define a complete lexicographic ordering on the set of ordered monomials by the following rule. If \( \deg m > \deg m' \) then \( m > m' \). Suppose \( \deg m = \deg m' \). Then if \( c_1 < c_1' \) then \( m > m' \). If \( \deg m = \deg m', c_1 = c_1' \) and \( b_1 < b_1' \) then \( m > m' \). Next we compare \( a_1 \) and \( a_1' \) and so on.

We now show that any monomial \( m \) which violates one of the conditions (a’)-(d’), (N) can be rewritten as a sum of smaller monomials. Suppose that for some \( i \) the condition (a’) is violated, i.e. \( a_i + a_{i+1} + b_{i+1} > 1 \). This means that at least two of numbers among \( a_i, a_{i+1}, b_{i+1} \) are greater than zero. But each of the products \( a_i a_{i+1}, a_i b_{i+1}, a_{i+1} b_{i+1} \) can be written as a linear combination of smaller monomials using the relations

\[
\sum_{\alpha + \beta = 2i+1} \tilde{f}_\alpha \tilde{f}_\beta = 0, \quad \sum_{\alpha + \beta = 2i+1} \tilde{f}_\alpha \tilde{h}_\beta = 0, \quad \sum_{\alpha + \beta = 2i+2} \tilde{f}_\alpha \tilde{h}_\beta = 0
\]

respectively. By the similar reason the violation of (b’), (c’) and (d’) allows to rewrite the corresponding monomial in terms of the smaller ones. So we finish the proof with rewriting a monomial which satisfies (a’) – (d’) but not (N). This reduces to rewriting a monomial \( \tilde{h}_i \tilde{e}_{i+1} \tilde{f}_{i+2} \) as a linear combination of smaller ones. We have

\[
(21) \quad \tilde{h}_i \tilde{e}_{i+1} \tilde{f}_{i+2} = -\tilde{e}_i \tilde{h}_{i+1} \tilde{f}_{i+2} + \cdots = \tilde{e}_i \tilde{h}_{i+2} \tilde{f}_{i+1} + \ldots,
\]

where we use \( \ldots \) to denote a linear combination of the monomials, which are smaller than \( \tilde{h}_i \tilde{e}_{i+1} \tilde{f}_{i+2} \) (the relations \( \tilde{e}(z) \tilde{h}(z) = 0 \) and \( \tilde{h}(z) \tilde{f}(z) = 0 \) are used in (21)). Now it is enough to note that the last expression is smaller than \( \tilde{h}_i \tilde{e}_{i+1} \tilde{f}_{i+2} \). \( \square \)

**Corollary 2.1.** \( ch_{q,z,u} B_1 = ch_{q,z,u} L_1^{0r} \).

**Proof.** From Lemmas 2.2 and 2.3 we know that the character of the set of \( ehf' \)-monomials is greater than or equal to the character of \( B_1 \) and coincides with the character of \( L_1 \). Now our Corollary follows from Lemma 2.1 \( \square \)

2.3. **The degeneration procedure.** In this section we give an alternative proof of Corollary 2.1. We involve the degeneration procedure which will be used later in the case of general level \( k \).

Let \( p, q, r \in \mathbb{R}^3 \) be linearly independent vectors with the scalar products

\[
(p, p) = (q, q) = (r, r) = 2, (p, q) = (q, r) = (p, r) = 0.
\]
Let $Q_1$ be a lattice generated by $p, q, r$ and $\Gamma_p(z), \Gamma_q(z), \Gamma_r(z)$ be the corresponding vertex operators.

**Lemma 2.4.** The identification

\[
\Gamma_p(n) \rightarrow \tilde{e}_n, \quad \Gamma_q(n) \rightarrow \tilde{h}_n, \quad \Gamma_r(n) \rightarrow \tilde{f}_n
\]

provides an isomorphism $W_{Q_1} \simeq B_1, |0\rangle \rightarrow 1$.

**Proof.** Follows from the Proposition 1.2. \qed

We now consider the Lie algebra $\hat{\mathfrak{sl}}_4$ and its vertex operator realization provided by the Frenkel-Kac construction. For $1 \leq i, j \leq 4$ we set

\[
E_{i,j}(z) = \sum_{n \in \mathbb{Z}} (E_{i,j} \otimes t^n) z^{-n-1}.
\]

Using Theorem 1.2 and Lemma 2.4 we obtain the following Lemma.

**Lemma 2.5.** The identification

\[
\tilde{e}_n \rightarrow E_{2,3} \otimes t^n, \quad \tilde{h}_n \rightarrow E_{2,4} \otimes t^n, \quad \tilde{f}_n \rightarrow E_{1,4} \otimes t^n
\]

provides an isomorphism between $B_1$ and the subspace of $V(\mathfrak{sl}_4)$ generated from the highest weight vector with the fields $E_{2,3}(z), E_{2,4}(z)$ and $E_{1,4}(z)$.

We want to show that the character of $B_1$ is smaller than or equal to the character of $L_1$. In order to do this we construct a deformation of the Lie algebra $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_4$ to the subspace spanned by $E_{2,3}, E_{2,4}$ and $E_{1,4}$.

**Proposition 2.1.** There exists a continuous family of Lie subalgebras $S(\varepsilon)$ of $\mathfrak{sl}_4$, $0 \leq \varepsilon \leq 1$ such that

a). $S(\varepsilon) \simeq \mathfrak{sl}_2$ for $0 \leq \varepsilon < 1$,

b). $S(1)$ is spanned with $E_{2,3}, E_{2,4}$ and $E_{1,4}$,

c). There exist standard bases $e(\varepsilon), h(\varepsilon), f(\varepsilon)$ of $S(\varepsilon)$ ($0 \leq \varepsilon < 1$) such that

\[
\lim_{\varepsilon \to 1} C e(\varepsilon) = CE_{1,4}, \quad \lim_{\varepsilon \to 1} C h(\varepsilon) = CE_{2,4}, \quad \lim_{\varepsilon \to 1} C f(\varepsilon) = CE_{2,3}.
\]

**Proof.** Let $v_1, v_2, v_3, v_4$ be a basis of $\mathbb{C}^4$. For $0 \leq \varepsilon < 1$ we let $S(\varepsilon)$ to denote the subalgebra determined by the following conditions:

- $S(\varepsilon)$ preserves span$\{v_1, v_2\}$,
- $S(\varepsilon)$ annihilates span$\{(1-\varepsilon)v_3 + \varepsilon v_1, (1-\varepsilon)v_4 + \varepsilon v_2\}$,
- $S(\varepsilon)$ contains only traceless matrices.

Then $S(\varepsilon)$ consists of the matrixes of the form

\[
\begin{pmatrix}
x & y & -xe & -ye \\
z & -x & \sqrt{1-\varepsilon} & \frac{y}{1-\varepsilon} \\
0 & 0 & \sqrt{1-\varepsilon} & \frac{xe}{1-\varepsilon} \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
Let $e(\varepsilon)$, $h(\varepsilon)$, $f(\varepsilon)$ be the standard basis of $S(\varepsilon) \cong \mathfrak{sl}_2$ (we fix the identification via the upper left $2 \times 2$ corner of $S(\varepsilon)$). Then

$$\lim_{\varepsilon \to 1}(\varepsilon - 1)e(\varepsilon) = E_{1,4}, \quad \lim_{\varepsilon \to 1}(1 - \varepsilon)h(\varepsilon) = E_{2,4}, \quad \lim_{\varepsilon \to 1}(1 - \varepsilon)^{1/2}f(\varepsilon) = -E_{2,3}.$$ 

This finishes the proof of the Proposition.

**Corollary 2.2.** There exists a continuous family of subspaces $M(\varepsilon) \hookrightarrow V(\mathfrak{sl}_4)$, $0 \leq \varepsilon \leq 1$ such that $M(\varepsilon) \cong L_1$ for $0 \leq \varepsilon < 1$ and $M(1) \cong B_1$.

**Corollary 2.3.** $L^q_1$ is isomorphic to $A_1$ as a representation of the algebra $\mathfrak{sl}_2^a \otimes t^{-1}\mathbb{C}[t^{-1}]$.

**Proof.** Follows from Lemma 2.1 and Lemma 2.1.

Recall the $(q, z, u)$-character of $L^q_{0,1}$ by

$$\text{ch}_{q,z,u}L^q_{1} = \sum_{s=0}^{\infty} u^s \text{ch}_{q,z}(F_s/F_{s-1}).$$

**Proposition 2.2.**

$$\text{ch}_{q,z,u}L^q_{1} = \sum_{n^+,n^0,n^- \geq 0} u^{n^+ + n^0 + n^-} z^{2(n^+ - n^-)} q^{(n^+)^2 + (n^0)^2 + (n^-)^2 + n^0 n^-} \frac{(q)_{n^+}(q)_{n^0}(q)_{n^-}}{(q)_{n^+}(q)_{n^0}(q)_{n^-}},$$

where $(q)_n = \prod_{i=1}^{n} (1 - q^i)$.

**Proof.** We first calculate the character of $B_1$. Consider the map $\phi$ from $B_1^*$ to the space of polynomials in 3 groups of variables:

$$(\phi(\theta))(x_1^-, \ldots, x_n^-, x_1^0, \ldots, x_n^0, x_1^+, \ldots, x_n^+) = \theta(\hat{f}(x_1^-) \ldots \hat{f}(x_n^-) \hat{h}(x_1^0) \ldots \hat{h}(x_n^0) \hat{e}(x_1^+) \ldots \hat{e}(x_n^+)).$$

Because of the relations (17) the image of $\phi$ coincides with the the space of polynomials of the form

$$\prod_{\alpha = +, 0, -} x_1^\alpha \prod_{1 \leq i < j \leq n^0} (x_i^\alpha - x_j^\alpha)^2 \prod_{1 \leq i \leq n^0} (x_i^+ - x_i^0) \prod_{1 \leq i \leq n^-} (x_i^0 - x_i^-) \times F(x_1^-, \ldots, x_n^-, x_1^0, \ldots, x_n^0, x_1^+, \ldots, x_n^+),$$

where $F$ is an arbitrary polynomial in $x_i^\alpha$, symmetric in each group $(+, 0, -)$ of variables. The natural $(q, z, u)$-grading on the space $B_1^*$ comes from the grading on $B_1$. It is easy to see that the corresponding character of the space of polynomials (24) is given by

$$u^{n^+ + n^0 + n^-} z^{2(n^+ - n^-)} q^{(n^+)^2 + (n^0)^2 + (n^-)^2 + n^0 n^-} \frac{(q)_{n^+}(q)_{n^0}(q)_{n^-}}{(q)_{n^+}(q)_{n^0}(q)_{n^-}}.$$ 

Now our Lemma follows from Corollary 2.1. □
Corollary 2.4.

\( \text{ch}_{q,z}L_1 = \sum_{n^+,n^0,n^- \geq 0} z^{2(n^+ - n^-)} \frac{q^{(n^+)^2 + (n^0)^2 + (n^-)^2} + n^0n^-}{(q)^{n^+}(q)^{n^0}(q)^{n^-}}. \)

In particular the right hand side equals to the well known expression for the character of \( L_1 \):

\[ \text{ch}_{q,z}L_1 = \frac{1}{(q)_{\infty}} \sum_{n \in \mathbb{Z}} z^{2n} q^{n^2}, \]

where \( (q)_{\infty} = \prod_{i \geq 1} (1 - q^i) \).

We now compare our formula with one from [FFJMT], which is given in terms of supernomial coefficients (see [SW]).

Corollary 2.5.

\( \text{ch}_{q,z,u}L_{1gr} = \sum_{m \geq 0} u^m q^m \sum_{-m \leq l \leq m} z^{2l} S_{m,l}(q), \)

where

\[ S_{m,l}(q) = \sum_{m \geq \nu \geq m-l-\nu} q^{(\nu+l-m)(\nu+l)+\nu(\nu-m)} \binom{m}{\nu} \binom{\nu}{m-l-\nu} q \]

and \( \binom{n}{m}_q = \frac{(q)_n}{(q)_m(q)_{n-m}}. \)

Proof. We note that

\[ \frac{1}{(q)_m} \binom{m}{\nu}_q \binom{\nu}{m-l-\nu} = \frac{1}{(q)_{m-\nu}(q)_{2\nu+m+l}(q)_{m-l-\nu}}. \]

The change of variables

\[ m - \nu = n_+, \hspace{1em} 2\nu - m + l = n_0, \hspace{1em} m - l - \nu = n_- \]

identifies formulas (26) and (23). \( \square \)

3. The general case

In this section we consider the PBW-filtration on the vacuum level \( k \) representation \( L_k \) and the corresponding adjoint graded space \( L_k^{gr} \).

3.1. Algebras \( A_k \) and \( B_k \). We introduce 2 series of algebras \( A_k \) and \( B_k \) generated with Fourier coefficients of the abelian currents \( \tilde{e}(z), \tilde{f}(z) \) and \( \tilde{h}(z) \), where

\[ \tilde{x}(z) = \sum_{i \geq 1} z^{-i-1} \tilde{x}_i. \]

Definition 3.1. An algebra \( A_k \) is a quotient of the polynomial algebra in variables \( \tilde{e}_i, \tilde{h}_i, \tilde{f}_i, i < 0 \) by the ideal generated by Fourier coefficients of \( \tilde{e}(z)^{k+1} \) and all its \( \mathfrak{sl}_2 \) consequences, i.e. by coefficients of \( 2k + 3 \) series:

\( \tilde{e}(z)^{k+1}, \tilde{e}(z)^k \tilde{h}(z), \tilde{e}(z)^{k-1} \tilde{h}^2(z) - 2\tilde{e}(z)^k \tilde{f}(z), \ldots, \tilde{f}(z)^{k+1}. \)
In other words these relations can be described as follows. Identify \( \mathfrak{sl}_2 \) with its 3-dimensional irreducible representation \( \pi_2 \) with \( e \) being the highest weight vector. Then we have an embedding

\[
i : \pi_{2k+2} \rightarrow \pi_{2}^{\otimes (k+1)}
\]

of \((2k + 3)\)-dimensional irreducible \( \mathfrak{sl}_2 \) module \( \pi_{2k+2} \) into the tensor power \( \pi_{2}^{\otimes (k+1)} \) (the image of \( \pi_{2k+2} \) is generated from \( e^{\otimes (k+1)} \in \pi_{2}^{\otimes (k+1)} \) by the action of universal enveloping algebra of \( \mathfrak{sl}_2 \)). Define an affinization map \( \alpha \), which sends an element of the tensor power \( \pi_{2}^{\otimes (k+1)} \) to the product of the corresponding series:

\[
\alpha(x^1 \otimes \ldots \otimes x^{k+1}) = \tilde{x}^1(z) \ldots \tilde{x}^{k+1}(z), \ x^i = e, h, f.
\]

Then the defining relations of \( A_k \) are coefficients of \( \alpha(i(\pi_{2k+2})) \).

We note that \((q, z, u)\)-character of \( A_k \) is naturally defined by (28).

**Lemma 3.1.** \( \text{ch}_{q,z,u} L_{gr}^g \leq \text{ch}_{q,z,u} A_k \).

**Proof.** Follows from the equality \( e(z)^{k+1} = 0 \) in \( L_k \). \( \square \)

**Definition 3.2.** An algebra \( B_k \) is a quotient of the polynomial algebra in variables \( \tilde{e}_i, \tilde{h}_i, \tilde{f}_i, i \leq -1 \) by the ideal generated by Fourier coefficients of \((2k + 3)\) series

\[
\tilde{e}(z)^i \tilde{h}(z)^{k+1-i}, 1 \leq i \leq k + 1; \tilde{h}(z)^i \tilde{f}(z)^{k+1-i}, 0 \leq i \leq k + 1.
\]

We note that \((q, z, u)\)-character of \( B_k \) is naturally defined by (28).

**Lemma 3.2.** \( \text{ch}_{q,z,u} A_k \leq \text{ch}_{q,z,u} B_k \).

**Proof.** We introduce a filtration \( G_s \) on \( A_k \) by setting \( G_0 \) to be the subspace generated by variables \( \tilde{e}_i, \tilde{f}_i \) (but not \( \tilde{h}_i \)) and

\[
G_{s+1} = \text{span}\{\tilde{h}_iw : i < 0, w \in G_s\}.
\]

Then for \( 0 \leq i \leq k + 1 \) the relation

\[
\alpha(i(f^i \cdot e^{\otimes (k+1)})) = 0
\]

which holds in \( A_k \) contains a term \( \tilde{e}(z)^{k+1-i}\tilde{h}(z)^i \) and the coefficients of the difference

\[
\alpha(i(f^i \cdot e^{\otimes (k+1)})) - \tilde{e}(z)^{k+1-i}\tilde{h}(z)^i
\]

belongs to \( G_{i-1} \). This means that the relation \( \tilde{e}(z)^{k+1-i}\tilde{h}(z)^i = 0 \) holds in the adjoint graded space \( G_0 \oplus \bigoplus_{s>0}(G_s/G_{s-1}) \). Similarly the rest of the relations (29) are true in the adjoint graded space. Lemma is proved. \( \square \)

**Corollary 3.1.** \( \text{ch}_{q,z,u} L_{gr}^g \leq \text{ch}_{q,z,u} B_k \).
3.2. A quadratic algebra $C_k$ and principal subspace $D_k$. We consider a set of commuting variables
\[ \tilde{x}_i^{[l]}, \ x = e, h, f, \ 1 \leq l \leq k, \ i \leq -l \]
and the corresponding currents $\tilde{x}_i^{[l]}(z) = \sum_{i<0} z^{-i-l} \tilde{x}_i^{[l]}$ (we set $\tilde{x}_i^{[l]} = 0$ for $i > -l$). For the series $p(z)$ let $p(z)^{(r)}$ be the $r$-th derivative.

**Definition 3.3.** Let $C_k$ be the quotient of the polynomial algebra in commuting variables $\tilde{x}_i^{[l]}, \ 1 \leq l \leq k, \ i \leq -l$ by the ideal of relations generated by coefficients of currents
\begin{align*}
(30) & \quad \tilde{x}_i^{[l]}(z)^{(\alpha)} \tilde{x}_j^{[m]}(z)^{(\beta)} \text{ for } x = e, h, f, \ \alpha + \beta < 2 \min(l, m), \\
(31) & \quad \tilde{e}_i^{[l]}(z)^{(\alpha)} \tilde{h}_j^{[m]}(z)^{(\beta)} \text{ for } \alpha + \beta < \max(0, l + m - k), \\
(32) & \quad \tilde{h}_i^{[l]}(z)^{(\alpha)} \tilde{f}_j^{[m]}(z)^{(\beta)} \text{ for } \alpha + \beta < \max(0, l + m - k).
\end{align*}

We define the $(q, z, u)$-degree on $C_k$ by the formulas
\[ \deg_q \tilde{x}_i^{[l]} = -i, \ \deg_z \tilde{x}_i^{[l]} = 2l, \ \deg_z \tilde{h}_i^{[l]} = 0, \ \deg_u \tilde{x}_i^{[l]} = l. \]

In what follows we show that the $(q, z, u)$-characters of $B_k$ and $C_k$ coincide.

We will need the following Lemma from [FS] [FJKLM].

**Lemma 3.3.** Consider the quotient of the polynomial algebra $\mathbb{C}[a_0, a_{-1}, \ldots]$ by the ideal generated with the coefficients of the series $a(z)^{k+1}$. Then there exists a filtration $F_k$ of this quotient (labeled by Young diagrams $\mu$) such that the adjoint graded algebra is generated with the coefficients of series $a^{[i]}(z)$, which are the images of powers $a(z)^i$, $1 \leq i \leq k$. In addition defining relations in the adjoint graded algebra are of the form
\[ a^{[i]}(z)^{(l)} a^{[j]}(z)^{(r)} = 0 \text{ if } l + r < 2 \min(i, j). \]

**Lemma 3.4.** $\text{ch}_{q,z,u}B_k \leq \text{ch}_{q,z,u}C_k$.

**Proof.** Using Lemma 3.3 we define a filtration on $B_k$ such that the adjoint graded space is generated by the images of coefficients of the series $\tilde{x}_i^{[l]}$, $x = e, h, f, \ 1 \leq l \leq k$. Denote the corresponding series by $\tilde{x}_i^{[l]}(z)$ and the corresponding Fourier coefficients by $\tilde{x}_i^{[l]}$. Then from Lemma 3.3 we obtain the relations (30). The relations (31), (32) in the adjoint graded space follow from the relations
\begin{align*}
(34) & \quad \tilde{e}(z)^{(\alpha)} \tilde{h}(z)^{(\beta)} = 0 \text{ for } \alpha + \beta < \max(0, l + m - k), \\
(35) & \quad \tilde{h}(z)^{(\alpha)} \tilde{f}(z)^{(\beta)} = 0 \text{ for } \alpha + \beta < \max(0, l + m - k),
\end{align*}
which hold in $B_k$. We thus obtain that all of the relations of $C_k$ are true in the adjoint graded space of $B_k$ with respect to the certain filtration. Lemma is proved. \[\square\]
Lemma 3.5. The equation (38) proves the first part of our Lemma. To show the second enequlity we use the degeneration procedure. Namely we construct a deformation $D_k(\varepsilon)$ of $D_k$ such that $D_k(\varepsilon) \simeq D_k$ for $1 \geq \varepsilon > 0$ and $D_k(0)$ contains $C_k$. Namely let $D_k(\varepsilon)$ be the subspace generated from the highest weight vector by the Fourier coefficients of $V_e(\varepsilon)$, $V_h(\varepsilon)$, $V_f(\varepsilon)$. Then the limit $\lim_{\varepsilon \to 0} D_k(\varepsilon)$ contains the Fourier coefficients of the series

$$
\Gamma_{p_1, \ldots, p_i}(z) = \lim_{\varepsilon \to 0} \varepsilon^{i(i-1)/2} (V_e(\varepsilon))^i, \\
\Gamma_{q_1, \ldots, q_i}(z) = \lim_{\varepsilon \to 0} \varepsilon^{i(i-1)/2} (V_h(\varepsilon))^i, \\
\Gamma_{r_1, \ldots, r_i}(z) = \lim_{\varepsilon \to 0} \varepsilon^{i(i-1)/2} (V_f(\varepsilon))^i.
$$

We note that

$$
(p_1 + \cdots + p_i, p_1 + \cdots + p_j) = (q_1 + \cdots + q_i, q_1 + \cdots + q_j) = 2 \min(i, j), \\
(r_1 + \cdots + r_i, r_1 + \cdots + r_j) = 2 \min(i, j), \\
(p_1 + \cdots + p_i, q_1 + \cdots + q_j) = \max(0, i + j - k), \\
(r_1 + \cdots + r_i, q_1 + \cdots + q_j) = \max(0, i + j - k).
$$

We want to show that $\text{ch}_{q,z,u} B_k \geq \text{ch}_{q,z,u} C_k$. We use the vertex operator technique. Fix an integer $N$ such that there exists a set of linearly independent vectors $p_i, q_i, r_i \in \mathfrak{h} = \mathbb{R}^N$, $1 \leq i \leq k$ with the scalar products

$$
(p_i, p_j) = (q_i, q_j) = (r_i, r_j) = 2 \delta_{i,j}, \\
(p_i, q_j) = (q_i, r_j) = \delta_{i,k+1-j}, \\
(p_i, r_j) = 0.
$$

For example, setting $N = 3k$ and fixing some orthonormal basis $e_i$ with respect to $(\cdot, \cdot)$ one can define

$$
p_i = \sqrt{2} e_i, \\
q_i = \frac{1}{\sqrt{2}} e_{k+1-i} + \sqrt{\frac{3}{2}} e_{k+i}, \\
r_i = \sqrt{2} e_{2k+1-i} + \sqrt{\frac{4}{3}} e_{2k+i}.
$$

We consider a lattice $Q$ generated by the vectors $p_i, q_i, r_i$ and the corresponding VOA $V_Q$. Set

$$
V_e(z) = \sum_{i=1}^{k} \Gamma_{p_i}(z), \\
V_h(z) = \sum_{i=1}^{k} \Gamma_{q_i}(z), \\
V_f(z) = \sum_{i=1}^{k} \Gamma_{r_i}(z).
$$

Note that due to the Proposition [11] one has

$$
\Gamma_{p_i}(z) \Gamma_{q_{k+1-i}}(z) = 0 = \Gamma_{q_i}(z) \Gamma_{r_{k+1-i}}(z).
$$

Therefore for any $0 \leq i \leq k + 1$

$$
V_e(z)^i V_h(z)^{k+1-i} = V_h(z)^i V_f(z)^{k+1-i} = 0.
$$

We let $D_k$ to denote the space generated with the Fourier coefficients of $V_e(z)$, $V_h(z)$, $V_f(z)$ from the highest weight vector.

Lemma 3.5. $\text{ch}_{q,z,u} B_k \geq \text{ch}_{q,z,u} D_k \geq \text{ch}_{q,z,u} C_k$. 

Proof. The equation (38) proves the first part of our Lemma. To show the second enequlity we use the degeneration procedure. Namely we construct a deformation $D_k(\varepsilon)$ of $D_k$ such that $D_k(\varepsilon) \simeq D_k$ for $1 \geq \varepsilon > 0$ and $D_k(0)$ contains $C_k$. Namely let $D_k(\varepsilon)$ be the subspace generated from the highest weight vector by the Fourier coefficients of $V_e(\varepsilon)$, $V_h(\varepsilon)$, $V_f(\varepsilon)$. Then the limit $\lim_{\varepsilon \to 0} D_k(\varepsilon)$ contains the Fourier coefficients of the series

$$
\Gamma_{p_1, \ldots, p_i}(z) = \lim_{\varepsilon \to 0} \varepsilon^{i(i-1)/2} (V_e(\varepsilon))^i, \\
\Gamma_{q_1, \ldots, q_i}(z) = \lim_{\varepsilon \to 0} \varepsilon^{i(i-1)/2} (V_h(\varepsilon))^i, \\
\Gamma_{r_1, \ldots, r_i}(z) = \lim_{\varepsilon \to 0} \varepsilon^{i(i-1)/2} (V_f(\varepsilon))^i.
$$

We note that

$$
(p_1 + \cdots + p_i, p_1 + \cdots + p_j) = (q_1 + \cdots + q_i, q_1 + \cdots + q_j) = 2 \min(i, j), \\
(r_1 + \cdots + r_i, r_1 + \cdots + r_j) = 2 \min(i, j), \\
(p_1 + \cdots + p_i, q_1 + \cdots + q_j) = \max(0, i + j - k), \\
(r_1 + \cdots + r_i, q_1 + \cdots + q_j) = \max(0, i + j - k).
$$
Let \( Q_k \) be the lattice generated by the vectors
\[
p_1 + \cdots + p_l, \ q_1 + \cdots + q_l, \ r_1 + \cdots + r_l, \ 1 \leq l \leq k.
\]
Then using Proposition 1.2 we obtain that the principal subspace \( W_{Q_k} \) is isomorphic to \( C_k \). The isomorphism is given by the identification
\[
\Gamma_{p_1 + \cdots + p_l(i)} \mapsto \tilde{e}_l^{[l]}, \ \Gamma_{q_1 + \cdots + q_l(i)} \mapsto \tilde{h}_l^{[l]}, \ \Gamma_{r_1 + \cdots + r_l(i)} \mapsto \tilde{f}_l^{[l]}.
\]
This gives
\[
\text{ch}_{q,z,u} D_k \geq \text{ch}_{q,z,u} C_k.
\]
Lemma is proved. \( \square \)

Lemmas 3.4 and 3.5 give the following Corollary:

**Corollary 3.2.** \( \text{ch}_{q,z,u} B_k = \text{ch}_{q,z,u} C_k = \text{ch}_{q,z,u} D_k \).

**Proposition 3.1.** \( \text{ch}_{q,z,u} L_k \geq \text{ch}_{q,z,u} D_k \).

**Proof.** We recall that there exists an embedding \( L_k \hookrightarrow L_k^\otimes k \) such that the highest weight vector \( v_k \) of \( L_k \) maps to the tensor power \( v_k^\otimes k \) and for any \( x \in \mathfrak{sl}_2 \) the current \( x(z) \) on \( L_k \) corresponds to the sum \( \sum_{i=1}^k x^{(i)}(z) \), where
\[
x^{(i)}(z) = \text{Id} \otimes \ldots \otimes x(z) \otimes \ldots \otimes \text{Id}
\]
(\( x(z) \) on the \( i \)-th place). From the definition of \( D_k \) we also have an embedding \( D_k \hookrightarrow D_k^\otimes k \) (see the definition (37)). Now the degeneration from the Corollary 2.2 gives the degeneration of \( L_k \). From the part c) of the Proposition 2.1 we conclude that the limit of this degeneration contains \( D_k \). Proposition is proved. \( \square \)

**Theorem 3.1.**

- The \((q, z, u)\) characters of \( L_k^{gr}, A_k, B_k, C_k \) and \( D_k \) coincide.
- \( L_k^{gr} \cong A_k \) as the modules over the abelian algebra with a basis \( \tilde{e}_i, \tilde{h}_i, \tilde{f}_i, i \leq -1 \).

**Proof.** Follows from Lemmas 3.1, 3.2, Corollary 3.2 and Proposition 3.1. \( \square \)

### 3.3. The character formula

In this section we compute the \((q, z, u)\)-character of \( L_k^{gr} \).

**Proposition 3.2.**

\[
(39) \quad \text{ch}_{q,z,u} C_k = \sum_{n^{-}, n^{0}, n^{+} \in \mathbb{Z}_{\geq 0}^k} q^{n^{-} \cdot \left(n^{+} + n^{0} - \frac{1}{2}(n^{+} - n^{0}) \right) + n^{-} \cdot \left(n^{+} + n^{0} - \frac{1}{2}(n^{+} - n^{0}) \right)} x_{n^{-}, n^{0}, n^{+}} \times \\
q^{4(n^{+} An^{+} + n^{0} An^{0} + n^{-} An^{-}) + n^{+} Bn^{0} + n^{0} Bn^{-}} \frac{(q)_{n^{+}} (q)^{n^{0}}} {(q)^{n^{-}}},
\]

where for \( n \in \mathbb{Z}_{\geq 0}^k \) we set \( |n| = \sum_{i=1}^k i n_i \) and matrices \( A \) and \( B \) are defined by
\[
A_{i,j} = 2 \min(i, j), \quad B_{i,j} = \max(0, i + j - k).
\]
Proof. Follows from the vertex operator realization of $C_k$ constructed in Corollary 3.2 and Proposition 1.2. □

Theorem 3.2. The $(q, z, u)$-character of $L^p_k$ is given by the right hand side of (32).

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