A QUATERNIONIC PROOF OF THE REPRESENTATION FORMULA OF A QUATERNARY QUADRATIC FORM

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Abstract. The celebrated Four Squares Theorem of Lagrange states that every positive integer is the sum of four squares of integers. Interest in this Theorem has motivated a number of different demonstrations. While some of these demonstrations prove the existence of representations of an integer as a sum of four squares, others also produce the number of such representations. In one of these demonstrations, Hurwitz was able to use a quaternion order to obtain the formula for the number of representations. Recently the author has been able to use certain quaternion orders to demonstrate the universality of other quaternary quadratic forms besides the sum of four squares. In this paper we develop results analogous to Hurwitz’s above mentioned work by delving into the number theory of one of these quaternion orders, and discover an alternate proof of the representation formula for the corresponding quadratic form.

1. Introduction

Four variable quadratic forms have been of interest since Lagrange proved the theorem that every positive integer is the sum of four integer squares. A quadratic form over \( \mathbb{Z} \) is called universal if it represents each positive integer at least once. It has been shown that there are a total of 54 quaternary quadratic forms without cross product terms that are universal (see Duke [2] and Ramanujan [8]). Hurwitz was able to demonstrate the universal property for the quadratic form that is the sum of four squares by use of a special quaternion order. Hurwitz’s study of the number theory of this order additionally yielded the formula for the number of representations of any positive integer as a sum of four squares. In previous work, the author has been able to show the universal property of seven other quadratic forms by consideration of appropriate quaternion orders. Here we develop the number theory of the quaternion order related to \( x^2 + y^2 + 2z^2 + 2w^2 \), thus creating an alternate proof of the representation formula and the universality for this quadratic form. In addition we find formulas for the number of representations of four and eight times an odd number by this quadratic form with certain restrictions on the parity of the variables \( x, y, z, \) and \( w \). See Hurwitz [6] and Deutsch [1] for further background.

Using the notation of Deutsch [1], we let \( i, j, k \) be the standard noncommutative basis elements of the quaternions. Bold type is used to represent quaternions, so a typical one is of the form

\[
q = q_1 + q_2i + q_3j + q_4k, \quad q_1, q_2, q_3, q_4 \in \mathbb{R}
\]  

(1.1)
To each such \( q \) there corresponds a conjugate quaternion denoted \( \overline{q} \) which has the value \( q_1 - q_2i - q_3j - q_4k \). The quaternionic norm, \( N(q) \) is simply the product \( q \cdot \overline{q} \).

Recall that a ring of quaternions is norm Euclidean if there exists a nonnegative \( \delta < 1 \) such that for any quaternion \( q \) there exists a quaternion \( a \) in the ring for which \( N(q - a) \leq \delta \).

In general, a unit of a ring with unity is an invertible element of the ring. An associate of an element of the ring is that element multiplied by a unit on the left or right.

For norm Euclidean rings of quaternions, there exist left and right greatest common divisors of any two nonzero elements. These are unique up to associates. The right greatest common divisor can be written as a right linear combination of the two nonzero elements under consideration. An analogous result holds for left greatest common divisors. For details and demonstrations, refer to Hardy and Wright [5, Ch. XX].

The relevant quaternion order for the quadratic form \( x^2 + y^2 + 2z^2 + 2w^2 \) is denoted \( H_{1,2,2} \). It is the \( \mathbb{Z} \) module with generators

\[
\begin{align*}
v_1 &= 1, \\
v_2 &= i, \\
v_3 &= \frac{1}{2} \left( 1 + i + \sqrt{2}j \right), \\
v_4 &= \frac{1}{2} \left( 1 + i + \sqrt{2}k \right).
\end{align*}
\]

This module is a norm Euclidean ring with a total of 24 units, namely

\[
\pm \begin{cases}
v_1, v_2, v_3, v_4, \\
v_3 - v_1, v_3 - v_2, v_4 - v_3, v_4 - v_1, v_4 - v_2, \\
v_3 - v_2 - v_1, v_4 - v_2 - v_1, v_4 + v_3 - v_2 - v_1
\end{cases}
\]

(see Deutsch [1]). \( H_{1,2,2} \) is closed under conjugation, and the norm maps \( H_{1,2,2} \) into the nonnegative rational integers. In addition to \( H_{1,2,2} \), we shall also make use of the \( \mathbb{Z} \) module generated by \( \{1, i, \sqrt{2}j, \sqrt{2}k\} \). This module will be denoted \( H_{1,2,2}^0 \).

For related notation and further properties of the above quaternion order see Deutsch [1]. More background material on quaternion orders and algebras is available in Pierce [7], Scharlau [9] and Vignéras [10]. An alternate proof of the representation formula using analytic techniques is given in Fine [3].

2. Residues modulo powers of \( 1 + i \)

In Hurwitz’s study of the quaternion order for the sum of four squares, elements of norm 2 were very significant. In particular, \( 1 + i \) had the special property that every element of the order with norm divisible by 2 was also divisible by \( 1 + i \) on both the left and the right. This property carries over to the case of \( H_{1,2,2} \) with some work. Of course \( 1 + i = v_1 + v_2 \in H_{1,2,2} \). We start by noting that \( N(v_1 + v_3 - v_4) = 2 \). Also, in \( H_{1,2,2} \) we have

\[
\begin{align*}
v_1 + v_3 - v_4 &= \left( \frac{1}{2} - \frac{1}{2}i + \frac{\sqrt{2}}{2}j \right) \cdot (1 + i) \\
&= (1 + i) \cdot \left( \frac{1}{2} - \frac{1}{2}i - \frac{\sqrt{2}}{2}k \right).
\end{align*}
\]
Lemma 1. The coset representatives of $H_{1,2,2}$ modulo $(1+i)$ are \{0, 1, $v_3$, 1+$v_3$\}. This holds whether $(1+i)$ is considered a left or a right ideal.

Proof. Consider any $g = g_1v_1 + g_2v_2 + g_3v_3 + g_4v_4$ in $H_{1,2,2}$. Using equation (2.1), and letting $h$ be an appropriate element in this order that may be different on each line, we have

$$g = g_1 + g_2i + g_3v_3 + g_4(v_3+1) + (1+i)h = (g_1 + g_4) + g_2i + v_3(g_3 + g_4) + (1+i)h \quad (2.2)$$

The last line follows from the elementary identity $i-1 = i(1+i) = (1+i)i$. Since $1+i$ divides 2, the only possibilities for the integer quantities $g_1 + g_2 + g_4$ and $g_3 + g_4$ are 0 and 1. These possibilities generate the coset representatives of Lemma 1. These coset representatives are not divisible by $1+i$ since the norm of the differences of any two of them is odd, while the norm of $1+i$ is even.

The same argument hold in the case where $h$ is on the left in (2.2). \[\square\]

Lemma 2. The coset representatives of $H_{1,2,2}$ modulo $(1+i)$ can be chosen as \{0, 1, $v_3$, $v_3^2$\}.

Proof. This follows from

$$v_3^2 = v_3 - 1 = v_3 + 1 - 2 = v_3 + 1 - (1+i)(1-i) = v_3 + 1 - (1-i)(1+i). \quad (2.3)$$

We have already used the fact that if $1+i$ divides an element of $H_{1,2,2}$ then 2 must divide the norm of that element. It is nontrivial that the converse is also true.

Theorem 3. Suppose $g \in H_{1,2,2}$ with $N(g)$ even. Then $g$ has $1+i$ as a left factor in $H_{1,2,2}$. Also $g$ has $1+i$ as a right factor in $H_{1,2,2}$.

Proof. Let us show that $g$ has $1+i$ as a right factor. Considering the coset representatives of $H_{1,2,2}$ modulo $1+i$, $g$ must have one of the forms

$$0 + h(1+i) \; \; \; \; \; 1 + h(1+i) \; \; \; \; \; v_3 + h(1+i) \; \; \; \; \; 1 + v_3 + h(1+i). \quad (2.4)$$

for some $h \in H_{1,2,2}$. Write $g = m + h(1+i)$. Then

$$g \cdot \overline{g} = (m + h(1+i)) \cdot (\overline{m} + (1-i)\overline{h}) = m \cdot \overline{m} + h(1+i)\overline{m} + m(1-i)\overline{h} + 2h \cdot \overline{h}. \quad (2.5)$$

For the center two terms, note that

$$\overline{(h(1+i)\overline{m})} = m(1-i)\overline{h}, \quad (2.6)$$

and that the sum of a quaternion and its conjugate is twice its real coefficient when written in the standard basis, i.e. for $q$ as in (1.1) we have $\Re(q) = q + \overline{q} = 2q_1$. We must therefore find the real coefficient in the standard basis of $h(1+i)\overline{m}$. Write $h$ and $m$ in the standard quaternion basis

$$h = h_1 + h_2i + h_3j + h_4k, \; \; \; \; m = m_1 + m_2i + m_3j + m_4k. \quad (2.7)$$
Computation shows that
\[ \Re(\mathbf{h} (1 + i) \mathbf{m}) = (h_1 - h_2)m_1 + (h_1 + h_2)m_2 + (h_3 + h_4)m_3 + (h_4 - h_3)m_4. \]  
(2.8)
However, \( \mathbf{m} \) is restricted to the four values of the coset representatives listed in Lemma 1. We wish to show this real part must always be a rational integer.

If \( \mathbf{m} = 0 \) then \( \Re(\mathbf{h} (1 + i) \mathbf{m}) = 0 \). If \( \mathbf{m} = 1 \) then \( \Re(\mathbf{h} (1 + i) \mathbf{m}) = h_1 - h_2 \).

Write \( \mathbf{h} \) in terms of the basis for \( H_{1,2,2} \).
\[
\mathbf{h} = t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + t_3 \mathbf{v}_3 + t_4 \mathbf{v}_4
\]
\[
= \left( t_1 + \frac{1}{2} t_3 + \frac{1}{2} t_4 \right) + \left( t_2 + \frac{1}{2} t_3 + \frac{1}{2} t_4 \right) i + \frac{t_3}{2} \sqrt{2} j + \frac{t_4}{2} \sqrt{2} k.
\]  
(2.9)
A simple calculation shows that \( h_1 - h_2 = t_1 - t_2 \) which is an element of \( \mathbb{Z} \).

For \( \mathbf{m} = \mathbf{v}_3 \) we similarly find \( \Re(\mathbf{h} (1 + i) \mathbf{m}) = t_1 + t_3 + t_4 \in \mathbb{Z} \). In the case of \( \mathbf{m} = 1 + \mathbf{v}_3 \), \( \Re(\mathbf{h} (1 + i) \mathbf{m}) = 2t_1 - t_2 + t_3 + t_4 \in \mathbb{Z} \).

In all cases we find \( \Re(\mathbf{h} (1 + i) \mathbf{m}) \) is a rational integer, so twice this quantity is an even integer. Thus
\[
N(\mathbf{g}) = \mathbf{g} \cdot \mathbf{\bar{g}} \equiv \mathbf{m} \cdot \mathbf{\bar{m}} \pmod{2}
\]  
(2.10)
in the rational integers. As \( \mathbf{m} \) varies through the cases listed above, the norm of \( \mathbf{m} \) takes on the values 0, 1, 1, 3 in order. Thus if \( N(\mathbf{g}) \) is even, \( \mathbf{m} \) must equal 0. Hence \( \mathbf{g} = \mathbf{h} (1 + i) \) so we find \( 1 + i \) is a right factor of \( \mathbf{g} \).

The demonstration that \( 1 + i \) is also a left factor of \( \mathbf{g} \) is entirely analogous to the above proof. \( \square \)

A simple induction argument yields the following.

**Corollary 4.** Suppose \( \mathbf{g} \in H_{1,2,2} \) and \( 2^s \) divides \( N(\mathbf{g}) \) for some positive integer \( s \). Then \( (1 + i)^s \) divides \( \mathbf{g} \) on each of the left side and the right side.

Since every real number is in the centralizer of the quaternions, we do not have to distinguish left and right cosets when considering \( H_{1,2,2} \) modulo a rational integer.

**Lemma 5.** The set of coset representatives for \( H_{1,2,2} / (2) \) can be chosen as the twelve units listed in (1.3) taken with the positive sign, and the four nonunits 0, 1 + \( i \), 1 + \( \mathbf{v}_3 + \mathbf{v}_4 \), \( i + \mathbf{v}_3 + \mathbf{v}_4 \).

**Proof.** A typical \( \mathbf{g} \in H_{1,2,2} \) can be written \( \mathbf{g} = g_1 + g_2 i + g_3 \mathbf{v}_3 + g_4 \mathbf{v}_4 \) with \( g_1, \ldots, g_4 \in \mathbb{Z} \). Taken modulo 2 we need only consider the values \( g_1, \ldots, g_4 \in \{0, 1\} \). Shifting appropriate \( g's \) by 2 produces the listing of the Lemma. Note that \( N(0) = 0, N(1 + i) = 2, N(1 + \mathbf{v}_3 + \mathbf{v}_4) = 6, N(i + \mathbf{v}_3 + \mathbf{v}_4) = 6 \). Hence these elements cannot be units.

Since as free abelian groups \( H_{1,2,2} / (2) \simeq \mathbb{Z}^4 / 2\mathbb{Z}^4 \simeq (\mathbb{Z} / 2\mathbb{Z})^4 \), a group of order 16, the Lemma lists the correct number of elements. \( \square \)

**Definition 6.** An element of \( H_{1,2,2} \) is an odd quaternion if its norm is odd.

**Lemma 7.** Any odd quaternion in \( H_{1,2,2} \) is congruent to a unit modulo 2.

**Proof.** Let \( \mathbf{b} \) be odd. If \( \mathbf{b} \) were congruent to one of the non-units then we could write \( \mathbf{b} = \mathbf{r} + 2 \mathbf{h} \) with \( N(\mathbf{r}) \) even. Thus \( 1 + i \) divides \( \mathbf{r} \) on the left. But \( 1 + i \) divides \( 2 \) also, so \( 1 + i \) divides \( \mathbf{b} \). Hence the norm of \( \mathbf{b} \) would be even, a contradiction. \( \square \)
Lemma 8. Multiplication by an odd quaternion on the left permutes the twelve units of Lemma 5 modulo 2. The same holds for multiplication on the right.

Proof. Let $b$ be odd, and let $u$ be one of the above mentioned twelve units modulo 2. Then $N(bu) = N(b)N(u)$ is odd so $bu$ is congruent to one of these twelve units modulo 2.

Let $u, u'$ be units and suppose $bu ≡ bu'(\text{mod } 2)$ in $H_{1,2}$. Then $2 | N(b - u')$. Hence $4 | N(u - u')$ as $b$ is odd. But this implies that $(1 + i)^2 | u - u'$. Thus it follows that $2 | u - u'$, so $u$ and $u'$ are equivalent modulo 2.

Hence multiplication by odd $b$ on the left induces a one-to-one onto mapping of the twelve units modulo 2 to themselves. The same argument holds for multiplication on the right. □

Corollary 9. Let $b$ be an odd quaternion in $H_{1,2}$. There exist two units $u$ and $u_1$ which satisfy the congruences

$$bu ≡ u_1b ≡ 1 \pmod{2}.$$ (2.11)

Proof. Since multiplication by $b$ induces a permutation among the units modulo 2, and 1 is a unit, the Corollary follows. □

3. Primary quaternions

Hurwitz used the concept of primary quaternions to enable the counting of quaternions with requisite properties for the order related to the quadratic form $x^2 + y^2 + z^2 + w^2$. It turns out that the exact analogue of Hurwitz’s definition accomplishes the same goal for the order $H_{1,2}$.

Lemma 10. Let $a ∈ \mathbb{Z}, a ≥ 0$. Then the left ideal generated by $2^a(1+i)$ in $H_{1,2}$ is the same as the right ideal generated by this element. This set is the same as the two sided ideal generated by $2^a(1+i)$.

Proof. Let $q = h · 2^a(1+i)$. Then $2^{2a+1} | N(q)$. By a Lemma above, $(1+i)^{2a+1}$ divides $q$ on the left. Since $(1+i)^2$ is $2i$, we find there exists $h'$ such that $q = 2^a(1+i) · h'$. A similar argument holds in the other direction.

Any element in a one sided ideal is contained in the two sided ideal. In the situation of this Lemma, we again have that $2^{2a+1}$ divides the norm of any element of the two sided ideal. Thus $2^a(1+i)$ divides any element of the two sided ideal from the left or right. □

In particular, it makes sense to use the notation $(2(1+i))$ for any of the one sided or two sided ideals since they are all the same.

Lemma 11. The set $\{1, 1+2v_3\}$ forms a multiplicative subgroup of $H_{1,2,2}$ modulo the ideal $(2(1+i))$.

Proof. The only nontrivial product to check is $(1 + 2v_3)^2$. Consider the difference $(1 + 2v_3)^2 - 1$. This equals $4 · (v_3 + v_3^2)$. Since $2(1+i)$ divides 4, $(1 + 2v_3)^2$ is congruent to 1 modulo $2(1+i)$. □
Definition 12. A primary quaternion is an element of $H_{1,2,2}$ which is congruent to 1 or $1 + 2v_3$ modulo the ideal $(2(1 + i))$.

By Lemma 11 we immediately see that the product of two primary quaternions is primary.

Lemma 13. Primary quaternions are elements of $H^0_{1,2,2}$.

Proof. Let $q$ be a primary quaternion. Then $q$ is congruent to 1 or $1 + 2v_3$ modulo $2(1 + i)$. In particular $q$ can be written in the form $1 + 2g$ for some $g \in H_{1,2,2}$. Thus

$$q = 1 + 2(g_1v_1 + g_2v_2 + g_3v_3 + g_4v_4)$$
$$= 1 + 2g_1 + 2g_2i + g_3(1 + i + \sqrt{2}j) + g_4(1 + i + \sqrt{2}k)$$

an element of $H^0_{1,2,2}$. □

Lemma 14. The set of 24 right sided associates of an odd quaternion always contains one which is primary. The same holds for the left sided associates.

Proof. Start with an arbitrary quaternion congruent to 1 modulo 2. It must be of the form $1 + 2g$, $g \in H_{1,2,2}$. To consider this quantity modulo $2(1 + i)$ we need only look at the possible values of $g$ modulo $1 + i$, namely $\{0, 1, v_3, v_3^2\}$. Thus the possible values of $1 + 2g$ are

$$1 + 2(1 + i)g', \quad 1 + 2((1 + i)g' + 1), \quad 1 + 2((1 + i)g' + v_3),$$
$$1 + 2((1 + i)g' + v_3^2)$$

for some $g' \in H_{1,2,2}$. Taking this modulo $2(1 + i)$ the possible values become $1, 1 + 2, 1 + 2v_3, 1 + 2v_3^2$. Since $2(1 + i)$ divides 4, we observe $1 + 2 = 3 \equiv -1 (\text{mod } 4)$ so $3 \equiv -1 (\text{mod } 2(1 + i))$. Also

$$1 + 2v_3^2 \equiv 1 + 2(v_3 - 1) \equiv -1 + 2v_3 \equiv -1 - 2v_3 \quad (\text{mod } 4).$$

Clearly these equivalences also hold modulo $2(1 + i)$. The possible values for $1 + 2g$ modulo $2(1 + i)$ reduce to $1, -1, 1 + 2v_3, -1 - 2v_3$.

Fix an odd quaternion $b$. There exists a unit $u$ such that $bu \equiv 1 (\text{mod } 2)$. Also $b(-u) \equiv -1 \equiv 1 (\text{mod } 2)$. We conclude that one of $bu$ and $b(-u)$ is congruent to a member of the set $\{1, 1 + 2v_3\}$ modulo $2(1 + i)$.

For left sided associates, we need only consider $u_1b$ and $-u_1b$. □

Lemma 15. Exactly one of the right sided associates of an odd quaternion is primary. The same holds for the left sided associates.

Proof. Let $b$ be as above. Suppose $bu$ and $bu_1$ are both primary for $u$ and $u_1$ units. Then $bu$ and $bu_1$ are both congruent to 1 modulo 2 in $H_{1,2,2}$. Let $u_2 = u^{-1}u_1$. Then 2 divides $bu(1 - u_2)$. Taking norms, we see that in the rational integers 4 must divide $N(b)N(u)N(1 - u_2)$. Thus 4 evenly divides $N(1 - u_2)$. But $u_2$ is a unit, and computation shows that of the 24 possible units, 4 divides $N(1 - u_2)$ only in the case $u_2 = \pm 1$.

If $u_2 = 1$ then $u = u_1$ and the Lemma is proven. If $u_2 = -1$ then $-u = u_1$ thus

$$bu_1 \equiv -1 \text{ or } -1 - 2v_3 \quad (\text{mod } 2(1 + i)).$$

(3.4)
Computation shows that \( \{1, 1+2v_3, -1, -1-2v_3\} \) are all distinct modulo \( 2(1+i) \) as the norms of all pairs of differences are divisible by 4 but never by 8. Thus \( bu_1 \) in the above equation can never be primary.

A very similar proof holds for the products \( ub \) where \( u \) is a unit. □

**Lemma 16.** Let \( b \) be a primary quaternion. If \( b \equiv 1 \pmod{2(1+i)} \) then the conjugate of \( b \), \( \overline{b} \) is also primary. If \( b \equiv 1 + 2v_3 \pmod{2(1+i)} \) then \( -\overline{b} \) is also primary.

**Proof.** We start from the congruence of the left ideal

\[
b \equiv 1 \text{ or } 1 + 2v_3 \pmod{2(1+i)} \tag{3.5}
\]

and by taking conjugates we have the congruence of a right ideal

\[
\overline{b} \equiv 1 \text{ or } 1 + 2\overline{v}_3 \pmod{2(1-i)}. \tag{3.6}
\]

However \( 1-i = -i(1+i) = (1+i)(-i) \) so we have

\[
\overline{b} \equiv 1 \text{ or } 1 + 2\overline{v}_3 \pmod{2(1+i)}. \tag{3.7}
\]

Thus we are back again in the case where the left, right and two sided ideals are the same. At this point we need only show

\[
1 + 2v_3 \equiv -(1 + 2\overline{v}_3) \pmod{2(1+i)}. \tag{3.8}
\]

This is equivalent to

\[
2 + 2(v_3 + \overline{v}_3) \equiv 0 \pmod{2(1+i)}. \tag{3.9}
\]

As the left side reduces to 4, the equation is obviously valid. □

### 4. The Correspondence Theorem

A quaternion algebra over a field is known to be isomorphic to either a division algebra or the ring of \( 2 \times 2 \) matrices over the field (see Pierce [7]). Hurwitz’s Correspondence Theorem is an analogue of this fact for a certain quaternion order over the ring of rational integers modulo an odd integer \( m \). In particular, the Correspondence theorem states there is an isomorphism of the Hurwitz quaternions taken modulo odd \( m \) with the \( 2 \times 2 \) matrices over \( \mathbb{Z}/m\mathbb{Z} \). The Correspondence Theorem comes over in its entirety in the case of the order \( H_{1,2,2} \), as will be seen in this section.

Let \( m \) be an odd positive integer. Since \( m \) is in the centralizer, the left ideal, right ideal and two sided ideal generated by \( m \) are all the same.

**Lemma 17.** Each quaternion \( q \in H_{1,2,2} \) is congruent modulo \( m \) to an element of \( H^0_{1,2,2} \).

**Proof.** Set \( q = q_1v_1 + q_2v_2 + q_3v_3 + q_4v_4 \). Then

\[
q \equiv q_1v_1 + q_2v_2 + (1+m)q_3v_3 + (1+m)q_4v_4 \pmod{m}. \tag{4.1}
\]

Since \( 1+m \) is even \( (1+m)v_3 \) and \( (1+m)v_4 \) are in \( H^0_{1,2,2} \). □
Lemma 18. The \( m^4 \) quaternions

\[
q = q_1 v_1 + q_2 v_2 + q_3 \sqrt{2} j + q_4 \sqrt{2} k, \quad q_1, \ldots, q_4 \in \{0, 1, \ldots, m-1\} \quad (4.2)
\]
form a complete residue system for \( H_{1,2,2}/(m) \).

Proof. By the previous Lemma, each element of \( H_{1,2,2} \) is congruent modulo \( m \) to a quaternion listed in (4.2). Considered as free abelian groups, the quotient group, \( H_{1,2,2}/(m) \) has order \( m^4 \) so we have the proper number of elements. □

Lemma 19. For an odd rational integer \( m \), there exists \( r, s \in \mathbb{Z} \) such that

\[
2^{-1} + r^2 + s^2 \equiv 0 \pmod{m} \quad (4.3)
\]
where \( 2^{-1} \) is the multiplicative inverse of 2 modulo \( m \).

Proof. As in Hurwitz [6], it can be shown by elementary number theory that for odd \( m \) there exist \( r, s \in \mathbb{Z} \) such that \( 2 + r^2 + s^2 \equiv 0 \pmod{m} \). Multiply this equation through by the square of \( 2^{-1} \) to prove the Lemma. □

At this point we wish to prove the analogue of Hurwitz’s Correspondence Theorem for the case of the quadratic form \( x^2 + y^2 + 2z^2 + 2w^2 \). It turns out to be advantageous to start with appropriate versions of the Hurwitz variables \( \xi_1, \xi_2, \xi_3, \xi_4 \). Using these we then produce the \( \alpha, \beta, \gamma, \delta \) and the coefficients \( q_1, q_2, q_3, q_4 \) of the quaternion \( q \). See Hurwitz [6, Vorlesung 8] for details in the case of the form \( x^2 + y^2 + z^2 + w^2 \).

For the remainder of this section, \( m \) will be an odd rational integer. We fix such an \( m \), and choose \( r, s \) as in Lemma 19.

Definition 20.

\[
\begin{align*}
\xi_1 &= 1 + r\sqrt{2} j + s\sqrt{2} k, & \xi_2 &= i + s\sqrt{2} j - r\sqrt{2} k \\
\xi_3 &= -i + s\sqrt{2} j - r\sqrt{2} k, & \xi_4 &= 1 - r\sqrt{2} j - s\sqrt{2} k. \quad (4.4)
\end{align*}
\]

Lemma 21. The following orthogonal relations hold modulo \( m \).

\[
\begin{align*}
\xi_1 \xi_3 &\equiv \xi_1 \xi_4 \equiv 0, & \xi_2 \xi_1 &\equiv \xi_3 \xi_4 \equiv 0, \\
\xi_4 \xi_1 &\equiv \xi_4 \xi_2 \equiv 0, & \xi_2^2 &\equiv \xi_3^2 \equiv 0. \quad (4.5)
\end{align*}
\]

In addition, the non-orthogonal relations below also are valid modulo \( m \).

\[
\begin{align*}
\xi_1^2 &\equiv \xi_2 \xi_3 \equiv 2\xi_1, & \xi_1 \xi_2 &\equiv \xi_2 \xi_4 \equiv 2\xi_2, \\
\xi_3 \xi_1 &\equiv \xi_4 \xi_3 \equiv 2\xi_3, & \xi_3 \xi_2 &\equiv \xi_1^2 \equiv 2\xi_4. \quad (4.6)
\end{align*}
\]

Proof. Verified by computer algebra. □
**Definition 22.** Given a four-tuple of integers \( \alpha, \beta, \gamma, \delta \), modulo odd \( m \), the corresponding quaternion \( q \in H_{1,2,2}/(m) \) is defined by

\[
2q = 2q_1 + 2q_2 i + 2q_3 \sqrt{2}j + 2q_4 \sqrt{2}k \\
\equiv \alpha \xi_1 + \beta \xi_2 + \gamma \xi_3 + \delta \xi_4 \pmod{m}.
\]

(4.7)

It is clear from Definition 20 that the quantities \( 2q_1, \ldots, 2q_4 \) are well defined modulo \( m \). Since \( m \) is odd that means \( q \) is well defined modulo \( m \). Additionally we have the following.

**Lemma 23.** With \( q \), and \( \alpha, \beta, \gamma, \delta \) as above, we have modulo \( m \)

\[
2q_1 \equiv \alpha + \delta, \quad 2q_3 \equiv r(\alpha - \delta) + s(\beta + \gamma) \\
2q_2 \equiv \beta - \gamma, \quad 2q_4 \equiv s(\alpha - \delta) - r(\beta + \gamma)
\]

(4.8)

**Proof.** This is easily verified by hand computation or computer algebra.

We now produce the inverse mapping from quaternions \( q \in H_{1,2,2}/(m) \) to the four-tuple that are coefficients of \( \xi_1, \xi_2, \xi_3, \xi_4 \). This will later help demonstrate the one to one correspondence between modulo \( m \) quaternions and certain \( 2 \times 2 \) integer matrices.

**Lemma 24.** The inverse mapping from the quaternion \( q \) of (4.7) to the modulo \( m \) four-tuple \( \alpha, \beta, \gamma, \delta \) is given by

\[
\alpha \equiv q_1 - 2r q_3 - 2s q_4, \quad \beta \equiv q_2 - 2s q_3 + 2r q_4, \\
\delta \equiv q_1 + 2r q_3 + 2s q_4, \quad \gamma \equiv -q_2 - 2s q_3 + 2r q_4,
\]

(4.9)

where all congruences are taken modulo \( m \).

**Proof.** Given the relations of the previous Lemma we compute

\[
2r q_3 + 2s q_4 \equiv r^2(\alpha - \delta) + rs (\beta + \gamma) + s^2(\alpha - \delta) - sr (\beta + \gamma) \\
\equiv (r^2 + s^2)(\alpha - \delta) \equiv -2^{-1}(\alpha - \delta) \pmod{m}.
\]

(4.10)

Thus \(-4r q_3 - 4s q_4 \equiv \alpha - \delta \pmod{m} \). Using the relationship for \( q_1 \) we find

\[
2q_1 - 4r q_3 - 4s q_4 \equiv 2\alpha \pmod{m} \\
\implies \alpha \equiv q_1 - 2r q_3 - 2s q_4 \pmod{m}
\]

(4.11)

Similarly we find \( \delta \equiv q_1 + 2r q_3 + 2s q_4 \pmod{m} \).

The formulas for \( \beta \) and \( \gamma \) are deduced in a similar fashion. \( \square \)

**Lemma 25.** Let the quaternion \( q \) and \( \alpha, \beta, \gamma, \delta \) be related as in the previous Lemma. Then

\[
N(q) \equiv \alpha \delta - \beta \gamma \pmod{m}.
\]

(4.12)

**Proof.** For the left side we have, modulo \( m \), \( q \equiv q_1 + q_2 i + q_3 \sqrt{2}j + q_4 \sqrt{2}k \) which gives \( N(q) \equiv q_1^2 + q_2^2 + 2q_3^2 + 2q_4^2 \). On the other side of the equation

\[
\alpha \delta - \beta \gamma \equiv [q_1^2 - (2r q_3 + 2s q_4)^2] - [(2s q_3 + 2r q_4)^2 - q_2^2] \\
\equiv q_1^2 + q_2^2 - (4r^2 + 4s^2)q_3^2 - (4r^2 + 4s^2)q_4^2
\]

(4.13)

but \( r^2 + s^2 \equiv -2^{-1} \) so \( \alpha \delta - \beta \gamma \) reduces to \( q_1^2 + q_2^2 + 2q_3^2 + 2q_4^2 \) modulo \( m \). \( \square \)
Theorem 26 (Hurwitz Correspondence Theorem). There is a ring isomorphism from the quaternions of $H_{1,2,2}$ modulo $m$ to the ring of $2 \times 2$ matrices of rational integers modulo $m$, $M_2(\mathbb{Z}/m\mathbb{Z})$. Using the notation of the previous few Lemmas, the isomorphism is given using equation (4.9):

$$\tau : q \rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \quad (4.14)$$

This isomorphism maps the quaternion norm into the determinant of the corresponding matrix.

Proof. Since a complete residue system modulo $m$ for $H_{1,2,2}$ is given by (4.2), both $H_{1,2,2}/(m)$ and the ring of $2 \times 2$ matrices over $\mathbb{Z}/m\mathbb{Z}$ contain $m^4$ elements. Thus the equations (4.8) and (4.9) produce mappings between $H_{1,2,2}/(m)$ and $M_2(\mathbb{Z}/m\mathbb{Z})$. Since the mapping corresponding to equation (4.9) returns to the preimage of the map of (4.8), the former mapping is surjective. As the sets in question have the same number of elements, the mapping of (4.9) is a bijection. The same holds for the mapping of (4.8). Note $\tau(1 + 0i + 0\sqrt{2}j + 0\sqrt{2}k)$ is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The correspondence between the norm and determinant was demonstrated in Lemma 25. Suppose

$$q \equiv q_1 + q_2 i + q_3 \sqrt{2} j + q_4 \sqrt{2} k, \quad q' \equiv q'_1 + q'_2 i + q'_3 \sqrt{2} j + q'_4 \sqrt{2} k \quad (4.15)$$

and

$$\tau(q) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \tau(q') = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}. \quad (4.16)$$

Then

$$\tau(q + q') = \tau\left((q_1 + q'_1) + (q_2 + q'_2)i + (q_3 + q'_3)\sqrt{2}j + (q_4 + q'_4)\sqrt{2}k\right) \quad (4.17)$$

The linearity of the equations (4.9) implies $\tau(q + q') = \tau(q) + \tau(q')$. To find $\tau(qq')$ consider

$$2q \equiv \alpha \xi_1 + \beta \xi_2 + \gamma \xi_3 + \delta \xi_4, \quad 2q' \equiv \alpha' \xi_1 + \beta' \xi_2 + \gamma' \xi_3 + \delta' \xi_4 \pmod{m} \quad (4.18)$$

Multiplying and simplifying using the relations (4.5) and (4.6) we find

$$4qq' \equiv 2 \left[ (\alpha \alpha' + \beta \gamma') \xi_1 + (\alpha \beta' + \beta \delta') \xi_2 \\ + (\gamma \alpha' + \delta \gamma') \xi_3 + (\gamma \beta' + \delta \delta') \xi_4 \right] \pmod{m}. \quad (4.19)$$

Thus

$$\tau(qq') = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \tau(q) \tau(q'), \quad (4.20)$$

so $\tau$ is indeed a ring isomorphism. \qed

We slightly modify Hurwitz’s definition of a quaternion primitive to $m$ for the sake of clarity.
Definition 27. A quaternion $g = g_1 v_1 + g_2 v_2 + g_3 v_3 + g_4 v_4 \in H_{1,2,2}$ is primitive to $m$ if $\gcd(g_1, g_2, g_3, g_4, m) = 1$.

Clearly any quaternion congruent modulo $m$ to $g$ in $H_{1,2,2}$ is primitive to $m$ if and only if $g$ is primitive to $m$. Therefore primitivity to $m$ is well defined on $H_{1,2,2}/(m)$.

Lemma 28. A quaternion $q = q_1 + q_2 i + q_3 \sqrt{2} j + q_4 \sqrt{2} k \in H_{0,1,2,2}$ is primitive to $m$ if and only if $\gcd(q_1, q_2, q_3, q_4, m) = 1$.

Proof. In terms of the basis for $H_{1,2,2}$
\[
q = q_1 + q_2 i + q_3 \sqrt{2} j + q_4 \sqrt{2} k = g_1 v_1 + g_2 v_2 + g_3 v_3 + g_4 v_4,
\]
thus it follows that
\[
2q_1 = 2g_1 + g_3 + g_4, \quad 2q_3 = g_3, \\
2q_2 = 2g_2 + g_3 + g_4, \quad 2q_4 = g_4.
\]
\[
(4.21)
\]
For $m$ odd
\[
\gcd(g_1, g_2, g_3, g_4, m) = \gcd(2g_1, 2g_2, g_3, g_4, m) \\
= \gcd(2g_1 + g_3 + g_4, 2g_2 + g_3 + g_4, g_3, g_4, m) \\
= \gcd(2q_1, 2q_2, g_3, g_4, m) \\
= \gcd(2q_1, 2q_2, 2q_3, 2q_4, m) \\
= \gcd(q_1, q_2, q_3, q_4, m).
\]
\[
(4.23)
\]
The Lemma follows immediately. The definition of primitivity to $m$ used here is thus equivalent to Hurwitz’s definition. □

Definition 29. An element $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z}/m\mathbb{Z})$ is primitive to $m$ if the greatest common divisor $\gcd(\alpha, \beta, \gamma, \delta, m) = 1$.

Theorem 30. For $m$ odd, a quaternion primitive to $m$ corresponds to a matrix primitive to $m$, and conversely.

Proof. From the transformation formulas (4.8) and (4.9), this is clear. □

Theorem 31. The number of incongruent quaternions in $H_{1,2,2}/(m)$ that are primitive to $m$ and satisfy $N(q) \equiv 0 \pmod{m}$ is
\[
\psi(m) = m^3 \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \left(1 + \frac{1}{p}\right)
\]
\[
(4.24)
\]
where $p$ runs through all primes dividing $m$.

Proof. By the Hurwitz Correspondence Theorem, we need to count the same matrices as in Hurwitz [6, Vorlesung 8]. Thus we get the same result. □
Theorem 32. The number of incongruent quaternions in $H_{1,2,2}/(m)$ that satisfy $N(q) \equiv 1 \pmod{m}$ is

$$m^3 \prod_{p|m} \left(1 - \frac{1}{p^2}\right)$$

(4.25)

where $p$ runs through all primes dividing $m$.

**Proof.** Same proof as in the previous Theorem. □

Lemma 33. Two distinct units of $H_{1,2,2}$ cannot be congruent to each other modulo $m$ for odd $m > 1$.

**Proof.** Suppose not. Then $u_1 \equiv u_2 \pmod{m}$ for units $u_1$ and $u_2$. Let $n = u_1 \cdot u_2^{-1}$. Then $n$ is also a unit of $H_{1,2,2}$ and $n \equiv 1 \pmod{m}$. Since $m \mid (n - 1)$ it follows that $m^2 \mid N(n - 1)$. Computation shows that the values of $N(n - 1)$ are in the set $\{0, 1, 2, 3, 4\}$ as $n$ ranges through the units. As none of these values are divisible by the square of an odd number, the Lemma follows. □

Theorem 34. Let $p$ be an odd prime and $f$ any of the $\psi(p) = (p^2 - 1)(p + 1)$ quaternions of $H_{1,2,2}$ modulo $p$ which satisfy $N(f) \equiv 0 \pmod{p}$ but have at least one component not divisible by $p$. The number of distinct quaternion solutions $x$ modulo $p$ which satisfy $xf \equiv 0 \pmod{p}$ is $p^2$.

**Proof.** The proof is the same as in Theorem 31. □

5. Prime Quaternions

We note that $H_{1,2,2}$ is a division algebra as it is a noncommutative subring of the set of all quaternions. It is easy to see that for any nonzero quaternion, the left multiplicative inverse and the right multiplicative inverse are equal. We recall that an element of a ring is a unit if it has an inverse in the ring. The units of $H_{1,2,2}$ are the elements of norm 1 and are listed in (1.3).

Definition 35. A prime of $H_{1,2,2}$ is a nonzero nonunit $\pi$ such that if $\pi = a \cdot b$ in $H_{1,2,2}$ then at least one of $a$ or $b$ is a unit.

Theorem 36. A rational prime $p \in \mathbb{Z}$ is not a prime of $H_{1,2,2}$.

**Proof.** Clearly 2 is not prime as $2 = (1 + i)(1 - i)$. Consider the case of $p$ an odd rational prime. By previous results we may choose $q \in H_{1,2,2}$ with $q$ primitive to $p$ and $N(q) \equiv 0 \pmod{p}$. Thus $q \not\equiv 0 \pmod{p}$.

Set $d$ equal to the right greatest common divisor of $p$ and $q$. Thus there exists $b, c, d_1, d_2$ in $H_{1,2,2}$ such that

$$d = cp + bq, \quad p = d_1d, \quad q = d_2d.$$  

(5.1)

Taking conjugates we have $\overline{d} = p\overline{c} + \overline{q}\overline{b}$ so

$$dd = p^2c\overline{c} + pcq\overline{b} + pbq\overline{c} + bq\overline{q}\overline{b}$$

(5.2)
The proof is similar to that in Hurwitz [6, Vorlesung 9] though the norm

\[ N(d) = d \overline{d} \equiv 0 \pmod{p}. \]

If \( d \) were a unit, then \( N(d) = 1 \neq 0 \pmod{p} \). Hence \( d \) is not a unit.

If \( d_1 \) were a unit, then from \( p = d_1 d \) it follows that \( d = d_1^{-1} p \). Hence \( q = d_2 d = d_2 d_1^{-1} p \). Thus \( q \equiv 0 \pmod{p} \) in \( H_{1,2,2} \) so \( q \) is not primitive to \( p \), a contradiction. Thus \( p = d_1 d \) is a product of two elements neither of which is a unit, so \( p \) cannot be prime in \( H_{1,2,2} \). \( \square \)

**Theorem 37.** An element \( \pi \in H_{1,2,2} \) is prime iff \( N(\pi) \) is a rational prime.

**Proof.** Suppose \( N(\pi) = p \), a prime of \( \mathbb{Z} \), and that \( \pi \) factors as \( ab \) in \( H_{1,2,2} \). Then \( p = N(a)N(b) \) in \( \mathbb{Z} \) so one of \( N(a) \) or \( N(b) \) is 1. Hence one of \( a \) or \( b \) is a unit, so \( \pi \) is a prime.

In the other direction, suppose \( \pi \in H_{1,2,2} \) is prime. Since \( \pi \) is not a unit, \( N(\pi) \) is not 1 so there exists some rational prime \( p \) dividing \( N(\pi) \). Set \( d \) to the one sided greatest common divisor of \( p \) and \( \pi \) as in the previous theorem. As above, we find that \( N(d) \equiv 0 \pmod{p} \), hence \( d \) cannot be a unit. Also we have

\[ p = d_1 d, \quad \pi = d_2 d \]

(5.3)

for \( d_1, d_2 \in H_{1,2,2} \). Since \( \pi \) is prime, \( d_2 \) must be a unit. Thus \( p = d_1 d_2^{-1} \pi \).

Taking the norms, \( p^2 = N(d_1) \cdot 1 \cdot N(\pi) \).

Since \( p \) divides \( N(\pi) \), that means \( N(d_1) \) can be either 1 or \( p \). If it were 1 then \( d_1 \) would be a unit implying that \( p \) is a unit times the prime \( \pi \). Thus \( p \) would be a prime of \( H_{1,2,2} \), a contradiction. We conclude that \( N(d_1) = p \) and consequently that \( N(\pi) = p \). \( \square \)

We now wish to find all prime quaternions dividing a prime rational number.

**Lemma 38.** Let \( \pi \) be a prime rational quaternion dividing 2 in \( H_{1,2,2} \). Then \( \pi \) equals \( 1 + i \) times a unit.

**Proof.** The demonstration is the same as in Hurwitz [6, Vorlesung 9]. \( \square \)

**Lemma 39.** Let \( f \in H_{1,2,2} \) be a quaternion primitive to a rational prime \( p \neq 2 \). Suppose \( N(f) \equiv 0 \pmod{p} \). Then there exists \( \tilde{f} \in H_{1,2,2} \) such that \( \tilde{f} \equiv f \pmod{p} \) and \( N(\tilde{f}) \equiv 0 \pmod{p^2} \) while \( N(\tilde{f}) \neq 0 \pmod{p} \).

**Proof.** The proof is similar to that in Hurwitz [6, Vorlesung 9] though the norm form is more complicated. In this case we have

\[
N(a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4) = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_1a_3 + a_2a_3 \\
+ a_1a_4 + a_2a_4 + a_3a_4.
\]

(5.4)

Set \( f = f_1v_1 + f_2v_2 + f_3v_3 + f_4v_4 \) with \( f_1, f_2, f_3, f_4 \in \mathbb{Z} \) and \( \tilde{f} = f + p(t_1v_1 + t_2v_2 + t_3v_3 + t_4v_4) \) with \( t_1, t_2, t_3, t_4 \in \mathbb{Z} \). Computer algebra shows

\[
N(\tilde{f}) = N(f) + p(2f_3t_4 + f_3t_4 + f_2t_4 + f_1t_4 + f_4t_3 \\
+ 2f_3t_3 + f_2t_3 + f_1t_3 + f_4t_2 + f_3t_2 \\
+ 2f_2t_2 + f_4t_1 + f_3t_1 + 2f_1t_1) \pmod{p^2}.
\]

(5.5)
We need only solve for rational integer \( t_1, t_2, t_3, t_4 \) such that
\[
(f_4 + f_3 + 2 f_1) t_1 + (f_4 + f_3 + 2 f_2) t_2 \\
+ (f_4 + 2 f_3 + f_2 + f_1) t_3 + (2 f_4 + f_3 + f_2 + f_1) t_4 \equiv 1 \pmod{p}.
\] (5.6)
If any of the coefficients of \( t_1, t_2, t_3, t_4 \) are not congruent to zero modulo \( p \) then set the other \( t \)'s to zero and solve for the value of the remaining \( t \) modulo \( p \). This produces an \( \hat{f} \) satisfying the condition of the Lemma.

The remaining possibility is if all four of the coefficients of the \( t \)'s are congruent to zero modulo \( p \). From the coefficients of \( t_1 \) and \( t_2 \) we find
\[
2 f_1 \equiv 2 f_2 \equiv -(f_3 + f_4) \pmod{p}.
\] (5.7)
Adding the coefficients of \( t_3 \) and \( t_4 \)
\[
3(f_3 + f_4) + 2f_2 + 2f_1 \equiv 0 \pmod{p}.
\] (5.8)
Combining (5.7) with (5.8) results in \( f_3 + f_4 \equiv 0 \pmod{p} \). This implies \( f_1 \equiv f_2 \equiv 0 \pmod{p} \). The coefficient of \( t_3 \) now reduces to \( f_4 + 2f_3 \). This being congruent to 0 by hypothesis, implies that \( f_3 \), and then \( f_4 \) are congruent to 0 modulo \( p \). Hence \( f \) is not primitive to \( p \), a contradiction. □

**Lemma 40.** Let \( f \in H_{1,2,2} \) be a quaternion primitive to a rational prime \( p \neq 2 \). Suppose \( N(f) \equiv 0 \pmod{p} \). Then we can uniquely associate \( f \) to a primary prime quaternion \( \pi \) of norm \( p \) by the following algorithm.

Find \( \hat{f} \in H_{1,2,2} \) such that \( \hat{f} \equiv f \pmod{p} \) and \( N(\hat{f}) \equiv 0 \pmod{p} \) while \( N(\hat{f}) \neq 0 \pmod{p^2} \). Compute the right greatest common divisor of \( \hat{f} \) and \( p \). This is a prime quaternion of norm \( p \). Let \( \pi \) be the primary associate of this prime quaternion.

**Proof.** We must show \( \pi \) exists with the requisite properties and is unique. \( \hat{f} \) exists by the Lemma above. Let \( \tau \) be the right greatest common divisor of \( \hat{f} \) and \( p \). By an argument very similar to (5.1) and (5.2) we find \( N(\tau) \equiv 0 \pmod{p} \). Hence \( \tau \) cannot be a unit. Additionally \( N(\tau) \) divides \( N(p) = p^2 \), so \( \tau \) is odd. We may take \( \pi \) to be the unique primary associate of \( \tau \).

We have \( \hat{f} = a \pi \) for some \( a \in H_{1,2,2} \). However \( N(a) N(\tau) \equiv N(\hat{f}) \equiv 0 \pmod{p^2} \). Hence \( N(\tau) \) cannot equal 0 modulo \( p^2 \). Thus \( \tau \) is of norm \( p \), and must be a prime. Since \( N(\pi) = N(\tau) \) it follows that \( \pi \) is also prime, and indeed a primary prime of norm \( p \).

We must show that the \( \pi \) constructed above is unique. Let \( g \equiv \hat{g} \pmod{p} \) with \( N(g) \equiv N(\hat{g}) \equiv 0 \pmod{p} \) while both of \( N(g) \) and \( N(\hat{g}) \) are not congruent to 0 modulo \( p^2 \). Let the right greatest common divisor mapping described above produce \( \pi \) for \( g \) and \( \tau \) for \( \hat{g} \) where \( \pi \) and \( \tau \) are both primary primes of norm \( p \). Then \( g = a \pi, \hat{g} = b \tau \) for some \( a, b \in H_{1,2,2} \).

From \( N(g) = N(a)N(\pi) \) it follows that \( N(a) \neq 0 \pmod{p} \). If either one sided greatest common divisor of \( a \) and \( p \) had norm greater than 1, then \( a \) would have a divisor of norm \( p \) or \( p^2 \). This would imply that \( N(a) \equiv 0 \pmod{p} \), a contradiction. Hence each one sided greatest common divisor of \( a \) and \( p \) is a unit. In particular there exists \( w, x \in H_{1,2,2} \) such that \( wa + xp = 1 \). Thus \( wa\pi + xp = \pi \). This implies \( wg + x\pi p = \pi \). By hypothesis, \( g = \hat{g} + cp \pmod{p} \) for some \( c \in H_{1,2,2} \). Thus
\[
\pi = w\hat{g} + wc p + x\pi p = w\pi^2 + (wc \pi + x\pi \pi)\tau. \tag{5.9}
\]
Thus \( \tau \) divides \( \pi \), but as both are primary primes, they are equal. □
Lemma 41. The mapping of the previous Lemma is surjective.
Proof. Let \( \pi \) be a primary prime quaternion of norm \( p \). Then we may take \( f = \pi \) as \( \pi \) is primitive to \( p \) (else the norm of \( \pi \) would be divisible by \( p^2 \)). Since \( N(\pi) = p \neq 0 \) (mod \( p^2 \)) we may take \( \hat{f} = \pi \). As \( p = N(\pi) = \overline{\pi} \pi \), when we take the primary form of the right greatest common divisor of \( p \) and \( \pi \) we merely reproduce \( \pi \). \( \square \)

Lemma 42. A prime \( \pi \) divides a rational prime \( p \) iff \( N(\pi) = p \).
Proof. \( N(\pi) = p \) implies \( p = \overline{\pi} \pi = \pi \overline{\pi} \). Thus \( \pi \) divides \( p \) on the left and the right.
In the other direction, if \( \pi \) divides a rational prime \( p \) then \( N(\pi) \mid N(p) \) so \( N(\pi) \) divides \( p^2 \). Since \( \pi \) is prime, its norm must be a rational prime, and hence equals \( p \). \( \square \)

Lemma 43. Under the conditions of Lemma 40 two quaternions primitive to \( p \), \( f \) and \( f^{(1)} \), produce the same primary prime quaternion under the right greatest common divisor algorithm iff there exists \( q \in H_{1,2,2} \) such that \( f^{(1)} \equiv q f \) (mod \( p \)).
Proof. By previous results we may construct quaternions in \( H_{1,2,2} \) congruent to each of \( f \) and \( f^{(1)} \) modulo \( p \) such that the norms are divisible by \( p \) but not by \( p^2 \). We may replace \( f \) and \( f^{(1)} \) by the constructed quaternions. and it is clear that the Lemma will be valid for the original quaternions iff it is valid for the constructed quaternions.

Suppose \( f \) and \( f^{(1)} \) produce the same quaternion \( \pi \) under the right gcd algorithm. Then there exists \( \alpha, \alpha^{(1)} \in H_{1,2,2} \) such that
\[
f = \alpha \pi, \quad f^{(1)} = \alpha^{(1)} \pi
\]
(5.10)
with \( N(\pi) = p \). Thus \( N(\alpha) \) and \( N(\alpha^{(1)}) \) are relatively prime to \( p \) since otherwise \( N(f) \) or \( N(f^{(1)}) \) would be divisible by \( p^2 \). Thus we may solve the following congruence for the unknown quaternion \( q \)
\[
q N(\alpha) \equiv \alpha^{(1)} \overline{\alpha} \pmod{p}
\]
(5.11)
by merely considering coefficients of basis elements of \( H_{1,2,2} \). Since \( N(\alpha) = \alpha \overline{\alpha} \) we find
\[
\left( q \alpha - \alpha^{(1)} \right) \overline{\alpha} \equiv 0 \pmod{p}.
\]
(5.12)
Consider either one sided gcd of \( \overline{\alpha} \) and \( p \). If this gcd were not a unit then \( p \) would divide \( N(\overline{\alpha}) \) and thus \( N(\alpha) \), a contradiction. Thus each one sided gcd is a unit and thus there exists one sided linear combinations of \( \overline{\alpha} \) and \( p \) that sum up to 1. In particular there exists \( b \in H_{1,2,2} \) such that \( \overline{\alpha} b \equiv 1 \pmod{p} \). Multiplying (5.12) by \( b \) on the right results in \( q \alpha - \alpha^{(1)} \equiv 0 \pmod{p} \). Multiplying by \( \pi \) on the right results in \( q \alpha \pi - \alpha^{(1)} \pi \equiv 0 \pmod{p} \). Thus \( q f \equiv f^{(1)} \pmod{p} \).

In the other direction suppose that there exists \( q \in H_{1,2,2} \) such that \( f^{(1)} \equiv q f \pmod{p} \) and that \( f \) corresponds to \( \pi \) under the right gcd algorithm. Then there exists \( \alpha \in H_{1,2,2} \) such that \( f = \alpha \pi \) and \( N(\pi) = p \). For some \( c \in H_{1,2,2} \)
\[
f^{(1)} = q f + c p = q \alpha \pi + c \overline{\alpha} \pi.
\]
(5.13)
Thus \( \pi \) divides the right greatest common divisor of \( f^{(1)} \) and \( p \). Yet this right gcd has norm \( p \) and the algorithm chooses the primary associate. The primary associate must therefore be \( \pi \) and hence \( f^{(1)} \) also corresponds to \( \pi \). \( \square \)
Lemma 44. Let \( \pi \) be a primary prime quaternion of norm \( p \neq 2 \) in \( H_{1,2,2} \). The number of incongruent quaternions in \( H_{1,2,2}/(p) \) primitive to \( p \) that produce \( \pi \) under the right gcd algorithm is \( p^2 - 1 \).

Proof. The demonstration is the same as in Hurwitz [6, Vorlesung 9]. \( \square \)

Corollary 45. The number of primary prime quaternions in \( H_{1,2,2} \) of norm \( p \neq 2 \) is \( p + 1 \).

Proof. The demonstration is the same as in Hurwitz [6, Vorlesung 9]. \( \square \)

Corollary 46. There are exactly \( 24 (p + 1) \) prime quaternions dividing an odd rational prime \( p \neq 2 \) in \( H_{1,2,2} \).

Proof. The demonstration is the same as in Hurwitz [6, Vorlesung 9]. \( \square \)

6. DECOMPOSITION THEOREM IN \( H_{1,2,2} \)

We wish to study the factorization into prime quaternions in \( H_{1,2,2} \). We also would like to discover the number of primary quaternions of given norm \( m \) in this quaternion order. As in Hurwitz [6, Vorlesung 10] any \( a \in H_{1,2,2} \) may be written as \( a = (1 + i) r b \) with nonnegative integer \( r \) and \( b \) an odd quaternion. We note that \( b \) is only a unit factor from an odd primary quaternion.

Definition 47. A quaternion \( c = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 \) in \( H_{1,2,2} \) is primitive if it is primary and \( \gcd(c_1, c_2, c_3, c_4) = 1 \) in \( \mathbb{Z} \).

Lemma 48. Let \( b \) be a primary quaternion and let \( m \) equal the greatest common divisor of the components of \( b \) with respect to the basis \( \{v_1, v_2, v_3, v_4\} \). There exists a primitive quaternion \( c \) such that \( b = m c \) or \( b = -m c \).

Proof. Since \( b \) is primary, \( b \equiv 1 \) or \( 1 + 2v_3 \) modulo \( 2(1 + i) \). Thus \( b \equiv 1 \) (mod 2) which implies \( b \equiv 1 \) (mod \( 1 + i \)). Thus \( b \) is odd and hence \( m \) is odd. The proof now proceeds as in Hurwitz [6, Vorlesung 10]. \( \square \)

It is simple to observe that a primary prime quaternion is also primitive. While Hurwitz redefines conjugate at this point, we introduce the definition of a variation of conjugate.

Definition 49. Let \( \pi \) be a primary prime quaternion. The \( p \)-conjugate of \( \pi \), denoted \( \overline{\pi}_p \), is

\[
\overline{\pi}_p = \begin{cases} 
\overline{\pi} & \text{if } \pi \equiv 1 \pmod{2(1+i)} \\
-\overline{\pi} & \text{if } \pi \equiv 1 + 2v_3 \pmod{2(1+i)}
\end{cases}
\] (6.1)

Lemma 50. Let \( \pi \) be a primary prime quaternion. The \( p \)-conjugate of \( \pi \) is \( \overline{\pi} \) or \( -\overline{\pi} \) depending on whether \( p = N(\pi) \equiv 1 \) or \( -1 \) (mod 4) respectively.

Proof. We first note that if \( k \in \mathbb{Z} \) is divisible by \( 1 + i \) then \( k \) is even, for if \( k \) were odd then \( N(k) \) would be odd but divisible by \( N(1+i) = 2 \), a contradiction.

If \( \pi \equiv 1 \pmod{2(1+i)} \), then \( \overline{\pi} \equiv 1 \pmod{2(1+i)} \) so \( p = \pi \overline{\pi} \equiv 1 \pmod{2(1+i)} \). Thus \( (p-1)/2 \) is a rational integer divisible by \( 1 + i \) and hence even, so \( p \equiv 1 \pmod{4} \).
If \( \pi \equiv 1 + 2v_3 \pmod{2(1+i)} \) then using equation (3.8)
\[
-\pi \equiv -(1 + 2v_3) \equiv 1 + 2v_3 \quad (\text{mod } 2(1+i)).
\] (6.2)
Thus \( -p = -\pi \pi \equiv (1 + 2v_3)^2 \equiv 1 \pmod{2(1+i)} \) so \((-p - 1)/2\) is a rational integer divisible by \(1 + i\). We conclude it is even, so \( p \equiv -1 \pmod{4} \). \( \square \)

Note that the \( p \)-conjugate of a primary prime quaternion is also primary by equation (3.8).

**Lemma 51.** Let \( c \) be a primitive quaternion and suppose the rational prime \( p \) divides \( N(c) \). Then the left greatest common divisor of \( c \) and \( p \) is a prime quaternion \( \pi \in H_{1,2,2} \). If chosen as primary, \( \pi \) is uniquely determined.

**Proof.** Let \( \lambda \) be the left greatest common divisor of \( c \) and \( p \). Then \( N(\lambda) \) divides \( N(p) \) in \( \mathbb{Z} \) so \( N(\lambda) \) can take only the values \( 1, p \) or \( p^2 \).

Suppose \( N(\lambda) = p^2 \). Let \( p = \lambda \tau \) for some \( \tau \in H_{1,2,2} \). Then \( N(\tau) = 1 \) so \( \tau \) is a unit. Thus \( \lambda \) is a unit, which implies \( p \) divides \( c \) in \( H_{1,2,2} \). So \( c = p(g_1v_1 + g_2v_2 + g_3v_3 + g_4v_4) \) in \( H_{1,2,2} \). We observe that \( p \) divides the greatest common divisor of the components of \( c \) which implies \( c \) is not primitive, a contradiction.

Suppose \( N(\lambda) = 1 \). Then \( \lambda \) is a unit. There exists \( a, b \in H_{1,2,2} \) such that \( 1 = c_0a + p_0b \) from the left greatest common divisor hypothesis. Thus \( 1 \equiv c_0a \pmod{p} \). Taking conjugates we find \( 1 \equiv \bar{a}\bar{c} \pmod{p} \). Multiplying together \( 1 \equiv N(c)N(a) \pmod{p} \). Thus \( p \) does not divide \( N(c) \), a contradiction.

The only possibility remaining is \( N(\lambda) = p \) so \( \lambda \) is a prime quaternion. \( \square \)

**Theorem 52 (Decomposition Theorem).** Let \( c \) be a primitive quaternion of \( H_{1,2,2} \) whose norm has the rational prime decomposition \( N(c) = p_1 p_2 p_3 \cdots \). Then \( c \) can be uniquely represented in the form \( c = \pi_1 \pi_2 \pi_3 \cdots \) where \( \pi_1, \pi_2, \ldots \) are primary prime quaternions of norm \( p_1, p_2, \ldots \) respectively. Furthermore, this can be done for any rearrangement of the order of the primes \( p_1, p_2, \ldots \).

**Proof.** The proof follows as in Hurwitz [6, Vorlesung 10]. \( \square \)

**Corollary 53.** Let \( c \) be a primitive quaternion of \( H_{1,2,2} \) whose norm has the rational prime decomposition \( N(c) = p^h q^k \cdots \), where \( p, q \cdots \) are distinct primes. Then \( c \) can be uniquely factored into the form
\[
\pi_1 \pi_2 \cdots \pi_h \chi_1 \chi_1 \cdots \chi_k \cdots
\] (6.3)
where the \( \pi \)'s are primary prime quaternions of norm \( p \), the \( \chi \)'s are primary prime quaternions of norm \( q \), etc.

**Proof.** The proof follows, as in Hurwitz [6, Vorlesung 10], from the Decomposition Theorem. \( \square \)

We note that in the primary prime decomposition of \( c \) as above, a prime \( \pi \) and its \( p \)-conjugate cannot appear adjacent to each other. If they were to do so, then \( p = N(\pi) \) would divide \( c \), and thus all the components of \( c \) written in the basis \( \{v_1, v_2, v_3, v_4\} \). Thus \( c \) would not be primitive.
Theorem 54. Let \( p, q \ldots \) be rational primes and \( \pi_s, \chi_s, \ldots \) be primary prime quaternions in \( H_{1,2,2} \) of norm \( p, q, \ldots \) respectively for all positive integers \( s \). Then the product \( \pi_1 \pi_2 \cdots \pi_h \chi_1 \chi_2 \cdots \chi_k \cdots \) is a primitive quaternion as long as two \( p \)-conjugate prime quaternions never appear adjacent to each other.

Proof. The proof is the same as in Hurwitz [6, Vorlesung 10]. □

Theorem 55. Let \( m \) be an odd rational integer with the decomposition into distinct primes \( m = p_1^{b_1} p_2^{b_2} \cdots \). Then the number of primitive quaternions of norm \( m \), denoted \( Q(m) \), is

\[
Q(m) = m \prod_s \left( 1 + \frac{1}{p_s} \right)
\]

(6.4)

Proof. The proof is the same as in Hurwitz [6, Vorlesung 10]. □

For computational reasons we define \( Q(1) = 1 \). It is easy to see that for relatively prime odd rational integers \( m_1, m_2 \) we have \( Q(m_1 m_2) = Q(m_1) Q(m_2) \).

Theorem 56. Let \( m \) be an odd rational integer. The number of primary quaternions in \( H_{1,2,2} \) of norm \( m \) is \( \sigma(m) \), the sum of the divisors of \( m \).

Proof. The proof is the same as in Hurwitz [6, Vorlesung 10]. □

7. Counting solutions

We have now developed enough number theory in the order \( H_{1,2,2} \) to determine the number of representations in \( \mathbb{Z} \) by the norm form. As the expansions of the units of this order in the standard quaternion basis are relevant to counting representations, it is convenient to list them, namely (see Deutsch [1])

\[
\pm \left\{ 1, \ i, \ \frac{1}{2} \pm \frac{1}{2} i \pm \frac{\sqrt{2}}{2} j, \ \frac{1}{2} \pm \frac{1}{2} i \pm \frac{\sqrt{2}}{2} k, \ \frac{\sqrt{2}}{2} j \pm \frac{\sqrt{2}}{2} k \right\}.
\]

(7.1)

Theorem 57. Let \( n \) be a positive integer, \( n = 2^r m \) with \( m \) an odd rational integer. The number of representations of \( n \) by the quadratic form \( x^2 + y^2 + 2 z^2 + 2 w^2 \) is

\[
\begin{cases}
4 \sigma(m) & \text{if } n \text{ is odd} \\
8 \sigma(m) & \text{if } r = 1 \\
24 \sigma(m) & \text{if } r \geq 2 
\end{cases}
\]

(7.2)

Proof. We wish to find the number of quaternions \( x \in H_{1,2,2}^0 \) such that \( N(x) = n \). Since \( n = 2^r m \) we may write \( x = (1 + i)^r y \) where \( y \) is an odd quaternion in \( H_{1,2,2} \). Being odd, \( y \) can be uniquely be factored as a unit times a primary quaternion. Let \( y = u c \) where \( u \) is a unit and \( c \) is primary. Then \( c \equiv 1 \pmod{2} \) so there exists \( g \in H_{1,2,2} \) such that \( c = 1 + 2g \) Thus

\[
x = (1 + i)^r u c = (1 + i)^r u (1 + 2 g) = (1 + i)^r u + 2 (1 + i)^r u g.
\]

(7.3)
Since \((1 + i)^r u g \in H_{1,2,2}\), twice this quantity is in \(H_{1,2,2}^0\). Thus \(x \in H_{1,2,2}^0\) if and only if \((1 + i)^r u \in H_{1,2,2}^0\). This is amenable to computation as there are only 24 units.

For \(r = 0\) the only units in \(H_{1,2,2}^0\) are \(\pm\{1, i\}\). Hence the number of solutions \(x\) is four times the number of primary quaternions of norm \(m\), i.e. \(4\sigma(m)\).

For \(r = 1\) computation shows that the only units \(u\) such that \((1 + i)u\) are in \(H_{1,2,2}^0\) are
\[
\pm \left\{1, i, \frac{\sqrt{2}}{2}j \pm \frac{\sqrt{2}}{2}k\right\}.
\]
Thus the number of solutions \(x\) is \(8\sigma(m)\).

For \(r \geq 2\) note that \((1 + i)^r\) contains a factor of \(2\). Hence \((1 + i)^r u\) is in \(H_{1,2,2}^0\) for all units \(u\). The number of solutions \(x\) is therefore \(24\sigma(m)\). \(\Box\)

In the above Theorem we have cases where only 4 or 8 of the 24 units of \(H_{1,2,2}\) are accounted for. The complementary set of units appear when counting the number of representations of certain multiples of an odd integer \(m\) under parity restrictions.

**Theorem 58 (Theorem on Complementary Representations).** Let \(m\) be an odd positive rational integer. We consider representations by the quadratic form \(x^2 + y^2 + 2z^2 + 2w^2\).

(i) The number of representations of \(4m\) by this quadratic form with \(x\) and \(y\) even while \(z\) and \(w\) are odd is \(4\sigma(m)\).

(ii) The number of representations of \(8m\) by this quadratic form where \(x\) and \(y\) are even while \(z\) and \(w\) are odd is \(16\sigma(m)\).

(iii) The number of representations of \(4m\) by this quadratic form where \(x\) and \(y\) are odd while \(z\) and \(w\) have different parity is \(16\sigma(m)\).

**Proof.** Consider the equation in rational integers
\[
4m = x^2 + y^2 + 2z^2 + 2w^2, \quad x, y \text{ even, } z, w \text{ odd.} \tag{7.5}
\]
Dividing by 4 we find \(w = (x/2)^2 + (y/2)^2 + 2(z/2)^2 + 2(w/2)^2\). Set
\[
\frac{x}{2} = x', \quad \frac{y}{2} = y', \quad \frac{z}{2} = \frac{1}{2} + z', \quad \frac{w}{2} = \frac{1}{2} + w'. \tag{7.6}
\]
Then \(x', y', z', w' \in \mathbb{Z}\) and
\[
m = N\left(x' + y'i + \left(\frac{1}{2} + z'\right)\sqrt{2}j + \left(\frac{1}{2} + w'\right)\sqrt{2}k\right). \tag{7.7}
\]
Note \(\sqrt{2}k = 2v_4 - v_1 - v_2\), \(\sqrt{2}j = 2v_3 - v_1 - v_2\) and \(v_3 + v_4 = v_1 + v_2 + \sqrt{2}(j + k)\). The quaternion inside the norm symbol of (7.7) can be written
\[
v_3 + v_4 - v_1 - v_2 + x'v_1 + y'v_2 + z'(2v_3 - v_1 - v_2) + w'(2v_4 - v_1 - v_2). \tag{7.8}
\]
This is clearly an element of \(H_{1,2,2}\) of norm \(m\). It has integer coefficients for 1 and \(i\) and half integer coefficients for \(\sqrt{2}j\) and \(\sqrt{2}k\).

Any such element of \(H_{1,2,2}\) of norm \(m\) corresponds to a solution of (7.5). Let \(x\) be such an element, then \(x\) can be uniquely factored as \(uc\) for \(u\) a unit and \(c\)
primary of norm \(m\). Arguing as in Theorem 57, \(x\) has the requisite coefficients if and only if \(u\) has. The number of units in \(H_{1,2,2}\) with half integer coefficients only for \(\sqrt{2}j\) and \(\sqrt{2}k\) is 4, namely

\[
\pm \left\{ \frac{\sqrt{2}}{2} j \pm \frac{\sqrt{2}}{2} k \right\}.
\] (7.9)

Thus the total number of solutions is \(4\sigma(m)\).

Now consider the equation in rational integers

\[
8m = x^2 + y^2 + 2z^2 + 2w^2, \quad x, y \text{ even, } z, w \text{ odd.} \quad (7.10)
\]

The same substitution as in (7.6) yields the following variant of (7.7):

\[
2m = N \left( x' + y'i + \left( \frac{1}{2} + z' \right) \sqrt{2}j + \left( \frac{1}{2} + w' \right) \sqrt{2}k \right).
\] (7.11)

As in the previous case, the term inside the norm symbol is an element of \(H_{1,2,2}\) with integer coefficients for 1 and \(i\) and half integer coefficients for \(\sqrt{2}j\) and \(\sqrt{2}k\). Again, any such quaternion \(x\) is a solution to (7.10). Such an \(x\) has a unique factorization as \((1 + i)\text{ times a unit } u\text{ times a primary quaternion } c\) of norm \(m\). As in Theorem 57, \(x\) has the requisite type of coefficients if and only if \((1 + i)u\) has. Computation shows that the number of units where \((1 + i)u\) have half integer coefficients only for \(\sqrt{2}j\) and \(\sqrt{2}k\) is 16. Thus the total number of solutions to (7.10) is \(16\sigma(m)\).

The remaining result breaks into two cases. Consider the equation

\[
4m = x^2 + y^2 + 2z^2 + 2w^2, \quad x, y, z \text{ odd, } w \text{ even.} \quad (7.12)
\]

Dividing by 4 and setting

\[
\frac{x}{2} = \frac{1}{2} + x', \quad \frac{y}{2} = \frac{1}{2} + y', \quad \frac{z}{2} = \frac{1}{2} + z', \quad \frac{w}{2} = w'.
\] (7.13)

we have \(x', y', z', w' \in \mathbb{Z}\) and

\[
m = N \left( \frac{1}{2} + x' + \left( \frac{1}{2} + y' \right) i + \left( \frac{1}{2} + z' \right) \sqrt{2}j + w' \sqrt{2}k \right).
\] (7.14)

The quaternion inside the norm symbol equals

\[
v_3 + x'v_1 + y'v_2 + z'(2v_3 - v_1 - v_2) + w'(2v_4 - v_1 - v_2).
\] (7.15)

This is clearly an element of \(H_{1,2,2}\) of norm \(m\). It has half integer coefficients for 1, \(i\) and \(\sqrt{2}j\) but an integer coefficient for \(\sqrt{2}k\).

The case of the equation

\[
4m = x^2 + y^2 + 2z^2 + 2w^2, \quad x, y, w \text{ odd, } z \text{ even,} \quad (7.16)
\]

is completely analogous, with solutions corresponding to those elements of norm \(m\) with half integer coefficients for 1, \(i\) and \(\sqrt{2}k\) but an integer coefficient for \(\sqrt{2}j\).

Any \(x \in H_{1,2,2}\) can be uniquely written in the form of a unit \(u\) times a primary quaternion \(c\) of norm \(m\). Then as above, \(x\) has the coefficients of the proper type if and only if \(u\) has. Since the number of units with half integer coefficients for 1, \(i\) and only one of \(\sqrt{2}j\) or \(\sqrt{2}k\) is 16, the total number of solutions is \(16\sigma(m)\). □
8. Further Directions

It is very plausible that a similar study of the appropriate norm Euclidean quaternionic rings could lead to results for the quadratic forms \(x^2 + 2y^2 + 3z^2 + 6w^2\) and \(x^2 + y^2 + 3z^2 + 3w^2\). However, due to the significance of the prime 3, the definition of primary would have to be modified in an as yet undetermined fashion. Also the formula for the number of representations for the latter quadratic form tends to imply that additional complications will have to be surmounted. This formula for the number of representations of \(N\) is

\[
(-1)^{N-1} 4 \sum_{d \mid N} d \cdot \chi(d), \quad \chi(n) = \begin{cases} 
1 & \text{if } n \equiv 1, 5 \pmod{6} \\
-1 & \text{if } n \equiv 2, 4 \pmod{6} \\
0 & \text{if } n \equiv 0, 3 \pmod{6}.
\end{cases} \quad (8.1)
\]

In particular the sum of divisors function does not appear in (8.1). For a proof of the formula, see Fine [3].

9. The Computation

The PUNIMAX version of MAXIMA was used on a LINUX partition of a Pentium 133 chip personal computer with 32 megabytes of RAM. The operating system was Linux 2.0.35. PUNIMAX was built using CLISP 1997-05-03 and GCC 2.7.2.3. A short C program was written to numerically verify the Theorem on Complementary Representations.

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