Feasibility study of model-independent approach to $\phi_3$ measurement using Dalitz plot analysis.

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In this paper, we present results of a feasibility study of a model-independent way to measure the angle $\phi_3$ of the unitarity triangle. The method involves $B^{\pm} \to DK^{\pm}$ decays where the neutral $D$ decays to the $K^0_S \pi^+ \pi^-$ final state, together with the sample of decays of $CP$-tagged $D$ mesons (produced, e.g. in $\psi(3770) \to DD$ process) to the same final state. We consider different approaches to the extraction of $\phi_3$ and obtain the expected statistical accuracy of the $\phi_3$ measurement as a function of $B$ and $D_{CP}$ statistics.

1. INTRODUCTION

Determinations of the Cabibbo-Kobayashi-Maskawa (CKM) [1] matrix elements provide important checks on the consistency of the Standard Model and ways to search for new physics. One of the ways to check the consistency of the CKM model is measurement of all angles of the unitarity triangle in decays of $B$ mesons. One angle, $\phi_1$, has been measured with high precision at modern $B$ factories by the experiments BaBar [2] and Belle [3]. The measurement of the angle $\phi_2$ is more difficult due to theoretical uncertainties in calculation of the penguin diagram contribution. Precise determination of the third angle, $\phi_3$, is possible, e.g. in the decays $B^{\pm} \to DK^{\pm}$. Although it requires a lot more data than for the other angles, it is theoretically clean due to the absence of loop contributions. Finding the way to use the available experimental data most efficiently is therefore essential for the $\phi_3$ measurement.

Various approaches have been proposed to measure $\phi_3$ in $B^{\pm} \to DK^{\pm}$ decays [4-7] by utilizing the interference between $\bar{b} \to \bar{c}u\bar{s}$ and $\bar{b} \to \bar{u}c\bar{s}$ transitions. The $\phi_3$ can be measured by studying the interference of $B^{\pm} \to D^0 K^{\pm}$ and $B^{\pm} \to \bar{D}^0 K^\pm$, which implies the use of neutral $D$ final states common to $D^0$ and $\bar{D}^0$. This is provided by using $CP$ eigenstates of $D$ meson (GLW method [4]) or Cabibbo-favored and doubly Cabibbo-suppressed modes (ADS method [7]).

The Dalitz plot analysis of the three-body $D^0$ decay from $B^{\pm} \to DK^{\pm}$ processes [8-15] provides today the best measurement of the angle $\phi_3$. It was proposed by Giri et al. [8] and independently by the Belle collaboration [9]. While Giri et al. mainly discussed a model-independent way of determining $\phi_3$ using a binned Dalitz analysis of $D^0 \to K^0_S \pi^+ \pi^-$ decay from $B^{\pm} \to DK^{\pm}$, the Belle collaboration proposed an unbinned Dalitz plot fit of this decay using the $D^0 \to K^0_S \pi^+ \pi^-$ model based on a coherent sum of quasi-two-body amplitudes. The latter technique has been used so far by BaBar and Belle in their $\phi_3$ measurement. The most recent BaBar result is $\phi_3 = 67^\circ \pm 29^\circ$ (stat) $\pm 13^\circ$ (syst) $\pm 11^\circ$ (model) using 211 $fb^{-1}$ data sample [15], the result of the Belle measurement is $\phi_3 = 53^\circ \pm 15^\circ$ (stat) $\pm 3^\circ$ (syst) $\pm 9^\circ$ (model) with 357 $fb^{-1}$ data sample [13]. The use of Cabibbo-favored mode $D^0 \to K^0_S \pi^+ \pi^-$ allows to increase the statistics compared to GLW or ADS methods. However, this approach suffers from the model uncertainty introduced by the unknown complex phase of $D^0$ decay. Currently, this uncertainty is estimated to be of order of 10$^\circ$, which is lower than the statistical error. In the future, as the statistics of $B$ decays collected at $B$-factories increases, this uncertainty will start to dominate the $\phi_3$ measurement. While the estimations of the model uncertainty can also improve, they can only be based on arbitrary variations of the decay amplitude description within phenomenological models (e.g. changing the list of resonances or using K-matrix formalism). However these models are only approximations of the 3-body dynamics and thus any model uncertainty estimates cannot be treated as absolutely reliable.

It has been shown [8] that the $D^0$ model uncertainty can be eliminated using the sample of neutral $D$ mesons in $CP$ eigenstates. This sample can be obtained at charm facilities, like currently running CLEO-c [16], in the decays of $\psi(3770)$ into two neutral $D$ mesons. In this paper, we report the results of a feasibility study of such an approach using toy Monte-Carlo (MC) simulation based on the realistic description of $D$ decay amplitude. Our main goal is the determination of the amount of $D_{CP}$ decays needed for the model-independent measurement.

The sensitivity to the angle $\phi_3$ comes from the interference of two amplitudes producing opposite flavors of neutral $D$ meson. Such a mixed state of neutral $D$ will be called $\hat{D}$. For example, in the case of $B^+$ decay, the mixed state is $\hat{D}_+ = D^0 + r_B e^{i\theta_B} D^0$. Here $r_B$ is the ratio of the suppressed and favored amplitudes, the total phase between the $D^0$ and $\bar{D}^0$ is $\theta_\perp = \phi_3 + \delta_B$, where $\delta_B$ is the strong phase difference between suppressed and favored $B$ decays. Analogously, for the decay of $B^-$, one can write $\hat{D}_- = D^0 + r_B e^{i\theta_B} \bar{D}^0$ with $\theta_- = -\phi_3 + \delta_B$. The ratio $r_B$ is expected to be of order of $0.1-0.2$ [17]. The BaBar ($r_B = 0.12 \pm 0.08 \pm 0.03$(syst) $\pm 0.04$(model), [14]) and Belle ($r_B = 0.16 \pm 0.05 \pm 0.01$(syst) $\pm 0.05$(model), [13]) measurements of the $r_B$ in Dalitz analysis of $D^0$ from $B^{\pm} \to DK^{\pm}$ decay confirm these estimates.
The Dalitz plot density of $\bar{D}$ gives immediate information about $r_B$ and $\theta_\pm$, once the amplitude of the $\bar{D}^0$ decay is known. The amplitude of the $\bar{D}^0_\pm$ decay as a function of Dalitz plot variables $m^2_+ = m^2_{K_S^0\pi^+}$ and $m^2_- = m^2_{K_S^0\pi^-}$ is

$$f_{B\pm} = f_D(m^2_+, m^2_+) + r_B e^{i\phi_3} f_D(m^2_-, m^2_-),$$

where $f_D(m^2_+, m^2_-)$ is an amplitude of the $\bar{D}^0 \to K_{S}^{0}\pi^{+}\pi^{-}$ decay. Here, we neglect possible effects of charm mixing and $CP$ violation in $D$ decay. It has been shown [18], that the mixing effects, which are suppressed in the Standard Model, lead to a bias in $\phi_3$ only at a second order in the small parameters $\Delta m_D/\Gamma_D$ and $\Delta\Gamma_D/\Gamma_D$ (where $\Delta m_D$ and $\Delta\Gamma_D$ are the mass and decay width differences between the two $D$ mass eigenstates, $\Gamma_D$ is the average decay width of mass eigenstates), and therefore can be safely neglected.

The $\bar{D}^0 \to K_{S}^{0}\pi^{+}\pi^{-}$ decay model can be determined from a large sample of flavor-tagged $\bar{D}^0 \to K_{S}^{0}\pi^{+}\pi^{-}$ decays produced in continuum $e^+e^-$ annihilation. Once that is known, a simultaneous fit of $\phi_3$ and $\delta_B$. However, flavor-tagged $\bar{D}^0 \to K_{S}^{0}\pi^{+}\pi^{-}$ decays only provide information about the absolute value of the $f_D$ amplitude. The phase term can only be obtained based on some model assumptions (such as Breit-Wigner amplitude dependence of the underlying resonances) which results in the model uncertainty of the $\phi_3$ measurement.

In the original publication by Giri et al. [8], an approach based on the binned Dalitz plot analysis was proposed which allows to extract $\phi_3$ without model uncertainties. The idea of the approach is based on the fact that, to measure $\phi_3$, complete knowledge of the phase term of $f_D$ amplitude is not necessary. In the $\bar{D}^0 \to K_{S}^{0}\pi^{+}\pi^{-}$ decay amplitude, we emphasize the absolute value $|f|$ and phase $\delta_D$:

$$f_D(m^2_+, m^2_-) = |f_D(m^2_+, m^2_-)| e^{i\delta_D(m^2_+, m^2_-)}$$

Since the phase of $f_B$ is arbitrary, it can be expressed as

$$f_{B\pm} = |f_D(m^2_+, m^2_-)| + r_B e^{\pm i\phi_3 + i\delta_B} |f_D(m^2_+, m^2_-)| \exp(i\delta_D(m^2_+, m^2_-) - i\delta_D(m^2_-, m^2_-)),$$

i.e. only the difference of phase terms $\delta_D(m^2_+, m^2_-) - \delta_D(m^2_-, m^2_-)$ between symmetric points of the phase space is relevant.

This difference can be obtained either by studying the decays of neutral $D$ in $CP$ eigenstate or treating them as free parameters in the extraction of the $\phi_3$ from the $B$ decays (see Section 5). Decays of $D$ mesons in $CP$ eigenstate to $K_{S}^{0}\pi^{+}\pi^{-}$ can be obtained at $e^+e^-$ machine in the process, e.g. $e^+e^- \to \psi(3770) \to DD$, where the other (tag-side) $D$ meson is reconstructed in $CP$ eigenstate, such as $K^+K^-$ or $K_S^0\omega$. The binned Dalitz plot analysis allows to determine $\phi_3$ without the need to parametrize the dynamics of the $\bar{D}^0 \to K_{S}^{0}\pi^{+}\pi^{-}$ decay (e.g. by introducing resonances) and thus allows for a truly model independent measurement.

Different approaches to the extraction of $\delta_D$ phase and different Dalitz plot binnings are tested in our feasibility study and compared to the model-dependend approach performed in an unbinned fashion. We consider only the approaches described in [8]. However, another possibility exists to access the phase information, which involves the process $e^+e^- \to \psi(3770) \to DD$ with both neutral $D$ mesons decaying to $K_{S}^{0}\pi^{+}\pi^{-}$ final state [19]. This approach requires to study the correlations of the two Dalitz plots.

In the next section, we follow the technique proposed in [8], to introduce the notation which will be used in the rest of the paper.

### 2. BASIC IDEA OF THE MODEL-INDEPENDENT TECHNIQUE

The amplitude of $\bar{D}^0 \to K_{S}^{0}\pi^{+}\pi^{-}$ decay from $CP$ eigenstate of neutral $D$ is

$$f_{CP\pm}(m^2_+, m^2_-) = \frac{1}{\sqrt{2}} [f_D(m^2_+, m^2_-) \pm f_D(m^2_-, m^2_+)],$$

for $CP$-even and $CP$-odd states of neutral $D$, respectively.

Consider the $\bar{D}^0 \to K_{S}^{0}\pi^{+}\pi^{-}$ Dalitz plot is divided into $2N$ bins symmetrically over exchange $m^2_+ \leftrightarrow m^2_-$. The bins will be denoted by the index $i$ ranging from $-N$ to $N$ (excluding 0); the exchange $m^2_+ \leftrightarrow m^2_-$ corresponds to the exchange $i \leftrightarrow -i$. The number of events in the $i$-th bin of the Dalitz plot, given by the region $D_i$, is

$$K_i = A_D \int_{D_i} |f_D(m^2_+, m^2_-)|^2 dm^2_- dm^2_+ \equiv A_D F_i,$$

where $A_D$ is the normalization which depends on the luminosity integral and definition of amplitude $f_D$. 

Similarly, as follows from Eq. 1, for $D^0 \rightarrow K^0_\pi^+\pi^-$ decays from $B^\pm \rightarrow D K^\pm$ mode the number of events in $i$-th bin is
\[
N_i^\pm = A_B \int_{D_i} |f_{B^\pm}(m_+^2, m_-^2)|^2 dm_+^2 dm_-^2
\]
\[
= A_B \left[ \int_{D_i} |f_D(m_+^2, m_-^2)|^2 dm_+^2 dm_-^2 + r_B^2 \int_{D_i} |f_D(m_+^2, m_-^2)|^2 dm_+^2 dm_-^2 \right]
+ 2r_B Re \left( e^{i\theta^2} \int_{D_i} f_D(m_+^2, m_-^2) f_D^*(m_+^2, m_-^2) dm_+^2 dm_-^2 \right)
\equiv A_B \left[ F_{\pm i} + 2r_B (e^{i\theta} G_i + r_B^2 F_{\mp i}) \right]
= A_B \left[ F_{\pm i} + 2r_B (\cos \theta \pm Re G_i - \sin \theta \pm Im G_i) + r_B^2 F_{\mp i} \right].
\]

Thus in order to extract information about $\theta^\pm$, one needs the values of $F_{\pm i}$, which are obtained from the decays of flavor tagged $D$ mesons, and also the values of $Re G_i$ and $Im G_i$.

The number of events in $i$-th bin for $CP$ eigenstate of neutral $D$ is
\[
M_i^\pm = A_{CP^\pm} \int_{D_i} |f_{CP^\pm}(m_+^2, m_-^2)|^2 dm_+^2 dm_-^2
\]
\[
= 2A_{CP^\pm}(F_i \pm 2 Re G_i + F_{\mp i}).
\]

Substituting the integrals $F_i$ from Eq. 5 and introducing the parameters $c_i$ and $s_i$ which depend on the phase term $\phi(m_+^2, m_-^2)$, we finally obtain the following system of equations which relates the number of events in bins of flavor $D$, $D_{CP}$ and $D$ from $B$ decays:
\[
N_i^\pm = h_B[K_{\pm i} + r_B^2 K_{\mp i} + 2r_B \sqrt{K_i K_{\mp i}}(c_i \cos(\pm \phi_3 + \delta_B) - s_i \sin(\pm \phi_3 + \delta_B))],
\]
\[
M_i^\pm = h_{CP^\pm}[K_i + K_{-i} \pm 2 \sqrt{K_i K_{-i}} c_i],
\]
where $h_B$ and $h_{CP^\pm}$ are the normalization factors which take into account different branching ratios, luminosity integrals and detection efficiencies for $B$, $D^0$ and $D_{CP^\pm}$ decays,
\[
c_i = \frac{1}{\sqrt{F_i F_{-i}}} \int_{D_i} |f_D(m_+^2, m_-^2)| |f_D(m_+^2, m_-^2)| \cos(\delta_D(m_+^2, m_-^2) - \delta_D(m_+^2, m_-^2)) dm_+^2 dm_-^2,
\]
\[
s_i = \frac{1}{\sqrt{F_i F_{-i}}} \int_{D_i} |f_D(m_+^2, m_-^2)| |f_D(m_+^2, m_-^2)| \sin(\delta_D(m_+^2, m_-^2) - \delta_D(m_+^2, m_-^2)) dm_+^2 dm_-^2.
\]

Note that $c_i = c_{-i}$ and $s_i = -s_{-i}$. The meaning of these parameters is sine and cosine of the phase difference, averaged over the bin area with the weight proportional to $|f_D(m_+^2, m_-^2)|^2 |f_D(m_+^2, m_-^2)|$. It is clear that $s_i^2 + c_i^2 \leq 1$.

The parameters $c_i$ and $s_i$ are in general unknown, they can be determined either from $D_{CP}$ decays, or by leaving them as free parameters in the fit to $D$ Dalitz plot from $B^\pm \rightarrow D K^\pm$ decay. Depending on the way these parameters are obtained, several approaches of $\phi_i$ extraction are possible. Below we discuss these approaches, together with the toy MC simulation results for each of them.

In our toy MC studies, we assume that the statistics of flavor-tagged $D$ mesons is practically infinite, therefore, the absolute value of the $f_D$ amplitude is known to high precision. This is provided by the large cross-section of $D^{\pm}$ meson production in $e^+e^- \rightarrow c\bar{c}$ process. We also assume that the number of $D_{CP}$ decays is much larger than the number of $B$ decays, due to the fact that the cross-section of $e^+e^- \rightarrow \psi(3770) \rightarrow D^0 D^0_{CP}$ production is approximately two orders of magnitude larger than that of $e^+e^- \rightarrow \Upsilon(4S) \rightarrow B\bar{B}_{DK}$ process.

3. APPROACH WITH TAGGED $D_{CP}$ AND CONSTRAINT ON $s_i$

It is clear that for the most precise measurement of $\phi_3$ given the limited statistics of $B$ decays, all the available information has to be used. In particular, the unknown quantities $c_i$ and $s_i$ have to be extracted from the higher-statistics $D_{CP}$ data.
The coefficient $c_i$ can be obtained directly from the $D^0$ and $D_{CP}$ decays. For example, if the numbers of $D_{CP+}$ and $D_{CP-}$ decays are proportional to the branching fractions of $\psi(3770) \rightarrow D D_{CP+}$, and $\psi(3770) \rightarrow D D_{CP-}$, respectively, i.e. if $h_{CP+} = h_{CP-}$,

$$c_i = \frac{1}{2} \frac{M_i^+ - M_i^-}{M_i^+ + M_i^-} \frac{K_i + K_{-i}}{\sqrt{K_i K_{-i}}}$$

(12)

(see Eq. 9). However, $c_i$ can be extracted even in $D$ decays of only one $CP$ parity.

If the binning is fine enough so that both $f(m_{i+}^2, m_{i+}^2)$ and $f(m_{i-}^2, m_{i-}^2)$ are approximately constant over $(m_{i+}^2, m_{i+}^2) \in D_i$, the expressions for $c_i$ (10) and $s_i$ (11) reduce to $c_i = \cos(\delta_D(m_{i+}^2, m_{i+}^2) - \delta_D(m_{i-}^2, m_{i-}^2))$, $s_i = \sin(\delta_D(m_{i+}^2, m_{i+}^2) - \delta_D(m_{i-}^2, m_{i-}^2))$. In such a case, $s_i$ can be directly obtained as $s_i = \pm \sqrt{1 - c_i^2}$. The sign of $s_i$ can be chosen either based on the continuity requirement (it should vary smoothly for adjacent bins) or under the assumption that the $D^0$ decay proceeds via two-body amplitudes. For simplicity, in our study we take the “correct” $s_i$ sign (calculated from the input $D^0$ model).

Once the coefficients $c_i$ and $s_i$ are determined, the parameters $r_B$, $\phi_3$ and $\delta_B$ can be obtained from $B$ decays by minimizing the negative logarithmic likelihood

$$-2 \log L = -2 \log L_+ - 2 \log L_-,$$

(13)

where

$$-2 \log L_\pm = \sum_{i=1}^{N} (N_i^\pm - h_B[K_{\pm i} + r_B^2 K_{\mp i} + 2r_B \sqrt{K_i K_{-i}}(c_i \cos(\pm \phi_3 + \delta_B) - s_i \sin(\pm \phi_3 + \delta_B)])^2$$

(14)

is the negative logarithmic likelihood for one $B$ flavor ($B^+$ or $B^-$), $N_i^\pm$ is the number of events in $i$-th bin for corresponding $B$ flavor. Here, Gaussian behavior of the number of events in each bin is assumed.

To avoid possible biases due to non-constant $f(m_{i+}^2, m_{i+}^2)$, the bin size should be relatively small. The size of the bin is determined by the width of the most narrow structure in the Dalitz plot; in our case, it is the $\omega$ meson with $\Gamma = 5.4$ MeV/$c^2$; therefore, typical scale of phase variation is $\Delta m^2 = 0.013$ GeV$^2$/c$^4$. However, since we assume that the number of $B$ decays is much smaller than the number of $D_{CP}$ decays, the number of $B$ events per bin for such a fine binning can be small. This can affect the fit procedure since we expect Gaussian behavior of the number of events. Therefore, for our study we take relatively rough binning for the fit of $B$ decays, and introduce a sub-binning of each of “$B$-bins” into several “$D$-bins”. As follows from definitions (10), (11) and (5), the parameters $c_i$ and $s_i$ for “$B$-bins” can be calculated from finer “$D$-bins” as

$$c_i = \sum_\alpha c_{i,\alpha} \sqrt{K_{i,\alpha} K_{-i,-\alpha} / K_i K_{-i}},$$

(15)

$$s_i = \sum_\alpha s_{i,\alpha} \sqrt{K_{i,\alpha} K_{-i,-\alpha} / K_i K_{-i}},$$

(16)

where the index $\alpha$ corresponds to the sub-binning of the bin $D_i$.

A toy MC study has been performed to obtain the accuracy of $\phi_3$ measurement as a function of $B$ and $D_{CP}$ decay statistics, and to investigate the influence of the finite bin size on the measurement result. The Dalitz plot binning used for our MC study is shown in Fig. 1. The phase space $\Delta m^2 = m_{i+}^2 - m_{i-}^2 \in (-3$ GeV$^2$/c$^4, 3$ GeV$^2$/c$^4)$ and $m_{i+}^2 = 0.02$ GeV$^2$/c$^4$ was uniformly divided into rectangular bins, symmetrically over the exchange of the sign of the final state pions. The bin size should certainly affect the statistical precision of the measurement. Since the structure of the $D^0 \rightarrow K^0_S \pi^+ \pi^-$ decay matrix element is quite complex, the variations of the event density due to $CP$ violation change rapidly across the phase space, and, as the bin size becomes larger, the average effect of $CP$ violation on each bin is reduced. Three different binnings were studied: depending on the number of bins in half of the phase space they are referred to as 3x3 (the corresponding number of bins which enter the half of the kinematically allowed phase space is $N = 8$), 5x5 ($N = 19$) and 7x7 ($N = 32$) binnings.

The parameters $r_B$ and $\delta_B$ were taken to be 0.2 and 180°, respectively: $\phi_3$ was scanned in a range from 0° to 180°. For the $D^0$ decay model, the recent Belle measurement was used [12]. The efficiency variations over the phase space and background contribution were not taken into account in the simulation. The statistical precision of the method certainly depends on the actual $D^0$ model, therefore, the real experimental precision can somewhat differ from our estimate. The error is also inversely proportional to the magnitude $r_B$ of the opposite $D$ flavor contribution. The estimates of this value vary in the range 0.1–0.2, the current measurements of $r_B$ [13, 15] agree with these numbers,
allow a measurement of $\phi - D$ sample of line. $\phi$ even without this constraint. In this case, there are 8 dependent on the width of the Dalitz plot structures of $B$. The dependences on both should also be noted, therefore, the actual most optimal.

There is no significant difference in the statistical accuracy for 5x5 and 7x7 binnings, while in the case of 3x3 binning the error scales inversely proportional to $r$. The correction should be negligible. The weak dependence of the statistical error on the phases $B$ differs from our expectation, the results of our study should be scaled accordingly.

The statistical error of the $\phi_3$ measurement as a function of $\phi_3$ for the different binnings is shown in Fig. 2. For each $\phi_3$ value, 200 pseudo-experiments with the statistics of $10^4 B^\pm \rightarrow DK^\pm$ decays were generated. The $\phi_3$ error was obtained from the spread of the reconstructed values. To keep the contribution of finite $D_{CP}$ statistics small, we used the large number of $D_{CP}$ decays ($2 \times 10^5$ decays of each CP-parity). The $\phi_3$ accuracy depends only weakly on the values of $\phi_3$ or $\delta_B$ since both $c_i$ and $s_i$ are obtained from the same $D_{CP}$ statistics and have comparable accuracies. There is no significant difference in the statistical accuracy for 5x5 and 7x7 binnings, while in the case of 3x3 binning the accuracy is worse by 30-40%. We have used the 5x5 binning for all further studies (excluding the study of the approach without $D_{CP}$ decays, see Section 5). The precision of $\phi_3$ extracted from a model-dependent unbinned Dalitz plot fit with the same input parameters is also shown for comparison. The uncombined fit is the optimal technique from the statistical point of view and represents the best limit of the statistical precision. The resolution in that case is approximately 30% better than for the case of binned approach. The optimal choice of binning can possibly improve the accuracy of the binned technique. We leave that question for the future studies.

The results of the toy MC study of $\phi_3$ bias due to finite $D$-bin size are shown in Fig. 3. The angle $\phi_3$ was scanned from 0 to 180°, other parameters were taken to be $r_B = 0.2$, $\delta_B = 180^\circ$. Points show the $\phi_3$ bias for no sub-binning, 3x3 sub-binning and 5x5 sub-binning (shown in Fig. 1 for one of the bins). For our further study, we have used 5x5 sub-binning, for which the bias does not exceed 2°. In a real experiment the bias can be corrected, e.g. based on the quasi two-body model. Since the correction is comparatively small, the uncertainty introduced by a model-based correction should be negligible.

The results of the study of $\phi_3$ sensitivity dependence on the number of reconstructed $B^\pm \rightarrow DK^\pm$ and $D_{CP}$ decays are shown in Fig. 4. The graph shows the $\phi_3$ error for $r_B = 0.2$, $\phi_3 = 70^\circ$, $\delta_B = 180^\circ$. For different values of $r_B$, the error scales inversely proportional to $r_B$. The weak dependence of the statistical error on the phases $\phi_3$ and $\delta_B$ should also be noted, therefore, the actual $\phi_3$ error can differ by 30-50% depending on the true values of these phases. The dependences on both $B$ and $D_{CP}$ statistics show a square root scaling. For comparison, the statistical error of $\phi_3$ extracted by model-dependent unbinned Dalitz plot fit with the same input parameters is shown with the dotted line.

According to our estimations, the proposed super-$B$ factory with its design integrated luminosity of 50 ab$^{-1}$, would allow a measurement of $\phi_3$ with accuracy below 2°. To keep the contribution to the statistical error from the finite sample of $D_{CP}$ decays below that level, one needs about $10^4$ tagged $D_{CP}$ decays, corresponding to approximately 10 fb$^{-1}$ of integrated luminosity at $\psi(3770)$ peak. In the following sections we consider possible alternative approaches of the binned analysis and compare their precision with the approach described above, which we have found to be the most optimal.

4. APPROACH WITHOUT $s_i$ CONSTRAINT

Constraining $s_i$ from $c_i$ introduces a slight model dependence to the measurement result, since the bin size becomes dependent on the width of the Dalitz plot structures of $D^0$ decay. In principle, the system of equations is solvable even without this constraint. In this case, there are $8N$ equations (8, 9) for $2N + 6$ unknowns ($c_i$, $s_i$, $r_B$, $\phi_3$, $\delta_B$, $\epsilon_i$).
FIG. 2: $\phi_3$ error as a function of $\phi_3$, for different Dalitz plot binnings. Results of the toy MC study with $2 \times 10^8 B^\pm \to DK^\pm$ decays, $4 \times 10^5 D_{CP}$ decays, $r_B = 0.2$, $\delta_B = 180^\circ$.

FIG. 3: $\phi_3$ bias due to finite size of the bin as a function of $\phi_3$, for the cases of no sub-binning (circles), 3x3 sub-binning (triangles), 5x5 sub-binning (squares).

FIG. 4: Statistical error of $\phi_3$ as a function of number of reconstructed $B^\pm \to DK^\pm$ decays (open circles) and $D_{CP}$ decays (dots). Results of toy MC study with $r_B = 0.2$, $\phi_3 = 70^\circ$, $\delta_B = 180^\circ$. $\phi_3$ error from model-dependent unbinned Dalitz plot fit with the same input parameters is also shown (dotted line).
limitation on the maximum number of free parameters in the software used for minimization. This can lead to significant changes of the fit result. Binning finer than 5x5 has not been investigated due to an internal incompatibility.

Practically, however, this approach offers too low sensitivity for any reasonable B decay statistics. A toy MC study has been performed with $10^5 B^\pm \to DK^\pm$ decays of each flavor. The parameters $r_B$ and $\delta_B$ were taken to be 0.2 and 180°, respectively; $\phi_3$ was scanned in a range from 30° to 140°. The results of the toy MC study are presented in Fig. 6 for 3x3 binnings. Statistical precision with $D_{CP}$ data and $s_i$ constraint is also shown for comparison. It can be seen from the plots, that the accuracy of the $\phi_3$ determination is quite poor compared to the method which makes use of $D_{CP}$ decays. Reaching the accuracy of $2^\circ$ would require about $5 \times 10^6$ reconstructed $B^\pm \to DK^\pm$ decays, which corresponds to the currently unrealistic luminosity integral of 5000 ab$^{-1}$ at B-factory.

Taking finer 5x5 binning could slightly improve the sensitivity, however, due to large number of free parameters, the fit converges poorly even with a number of B decays as large as $10^5$. Variations of the initial parameters in the fit can lead to significant changes of the fit result. Binning finer than 5x5 has not been investigated due to an internal limitation on the maximum number of free parameters in the software used for minimization.

5. APPROACH WITHOUT TAGGED $D_{CP}$

Even if the $D_{CP}$ decays are not available, it is in principle possible to extract $\phi_3$ in a model-independent way using only $B^\pm \to DK^\pm$ and flavor-tagged D decays. For $N$ pairs of bins, there are $4N$ equations $(8)$ $(2N$ for each flavor of $B)$, and $2N + 4$ unknowns $(c_i, s_i, r_B, \phi_3, \delta_B, h_B)$. Therefore, the system is solvable for $N \geq 2$. Technically (if the number of equations is greater than the number of unknowns), it can be done by minimizing the negative logarithmic likelihood (Eq. 13, 14) with $c_i$ and $s_i$ coefficients taken as free parameters.

Practically, however, this approach offers too low sensitivity for any reasonable B decay statistics. A toy MC study has been performed with $10^5 B^\pm \to DK^\pm$ decays of each flavor. The parameters $r_B$ and $\delta_B$ were taken to be 0.2 and 180°, respectively; $\phi_3$ was scanned in a range from 30° to 140°. The results of the toy MC study are presented in Fig. 6 for 3x3 binnings. Statistical precision with $D_{CP}$ data and $s_i$ constraint is also shown for comparison. It can be seen from the plots, that the accuracy of the $\phi_3$ determination is quite poor compared to the method which makes use of $D_{CP}$ decays. Reaching the accuracy of $2^\circ$ would require about $5 \times 10^6$ reconstructed $B^\pm \to DK^\pm$ decays, which corresponds to the currently unrealistic luminosity integral of 5000 ab$^{-1}$ at B-factory.

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6. CONCLUSION

We have performed a toy Monte-Carlo study of a model-independent way to measure the angle $\phi_3$ of the unitarity triangle, suggested by Giri et al. [8]. The method involves $B^{\pm} \rightarrow DK^{\pm}$ decays with neutral $D$ decaying to $K_S^0\pi^+\pi^-$ final state, together with the decays of $D$ mesons in CP eigenstate to the same final state. Different approaches to the extraction of $\phi_3$ are considered. Although technically the model-independent measurement of $\phi_3$ can be performed without decays of $D$ in CP eigenstate, the statistics of $B^{\pm} \rightarrow DK^{\pm}$ decays required to obtain a reasonable $\phi_3$ accuracy in this approach is unrealistically high.

We have shown that for the optimal strategy (which involves constraining $s_i$ coefficients from $c_i$), the accuracy of the model-independent binned approach is only 30% worse compared to the unbinned technique. Possibly the choice of the optimal binning (e.g. using non-rectangular bins) can improve the accuracy of the model-independent measurement. The statistical error of $\phi_3$ does not depend significantly on the $\phi_3$ and strong phase values.

According to our estimations, the proposed super-$B$ factory [20] with its design integrated luminosity of 50 ab$^{-1}$, would allow a measurement of $\phi_3$ with statistical accuracy below 2$^\circ$. To keep the contribution to the statistical error from the finite sample of $D_{CP}$ decays below that level, around $10^4$ tagged $D_{CP}$ decays are needed, corresponding to approximately 10 fb$^{-1}$ of integrated luminosity collected at $\psi(3770)$ peak. This task can be accomplished by CLEO-c experiment at CESR collider [16] and by upgraded BESIII/BEPCII complex at Beijing with design luminosity of $10^{33}$ cm$^{-2}$s$^{-1}$ [21].

The systematic errors due to variations of the detection efficiency or background density over the Dalitz plot and other related factors were not considered in this study. They will certainly depend on the specific detector parameters, but the current experience with model-dependent Dalitz analyses at $B$ factories allows us to believe that these contributions will also decrease with increasing statistics.

7. ACKNOWLEDGMENTS

We would like to thank Tim Gershon and David Asner for helpful discussions and suggestions on improving this paper. We are grateful to the Belle collaboration for the interest expressed to our study.

This work is supported by the Federal Agency for Science and Innovations of the Russian Federation under contract 02.434.11.7075.

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