SOME TWISTED SECTORS FOR THE MOONSHINE MODULE

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ABSTRACT. The construction of twisted sectors, or \( g \)-twisted modules, for a vertex operator algebra \( V \) and automorphism \( g \), is a fundamental problem in algebraic conformal field theory and the theory of orbifold models. For the moonshine module \( V^\natural \), whose automorphism is the Monster \( M \), Tuite has shown that this problem is intimately related to the generalized moonshine conjecture which relates Hauptmoduln to twisted sectors.

In this paper we show how to give a uniform existence proof for irreducible \( g \)-twisted \( V^\natural \)-modules for elements of type 2A, 2B and 4A in \( M \). The most interesting of these is the twisted sector \( V^\natural(2A) \), whose automorphism group is essentially the centralizer of 2A in \( M \). This is a 2-fold central extension of the Baby Monster, the second largest sporadic simple group.

We also establish uniqueness of the twisted sectors and the Hauptmodul property for the graded traces of automorphisms of odd order, as predicted by conformal field theory.

1. Introduction

The purpose of this paper is to give a uniform existence proof for irreducible \( g \)-twisted \( V^\natural \)-modules for elements \( g \) in \( M \) of type 2A, 2B and 4A. Here \( V^\natural \) is the moonshine module [FLM] with automorphism group the Monster \( M \), and the notation for elements in \( M \) follows the conventions of [Cal]. We assume the reader to be familiar with the definition of vertex operator algebra (VOA) (see [B1] and [FLM]) which we do not repeat here, noting only that \( V^\natural \) is a particularly famous example of a VOA.

If \( V \) is a VOA and \( g \) an automorphism of finite order, there is the notion of a (weak) \( g \)-twisted \( V \)-module. Briefly, a weak \( g \)-twisted \( V \)-module is a pair \((M, Y_M)\) consisting of a \( \mathbb{C} \)-graded linear space \( M = \oplus_{n \in \mathbb{C}} M_n \) locally truncated below in the sense that

1991 Mathematics Subject Classification. Primary 17B69; Secondary 17B68, 81T40.
This paper is in final form and no version of it will be submitted for publication elsewhere.
C.D. and G.M. are partially supported by NSF grants and a research grant from the Committee on Research, US Santa Cruz.
$M_{n+q} = 0$ for fixed $n \in \mathbb{C}$ and sufficiently small $q \in \mathbb{Q}$, together with a linear map

$$V \to (\text{End } M)[[z^{1/T}, z^{-1/T}]],$$

$$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} (v_n \in \text{End } M),$$

where $T$ is the order of $g$. One requires a certain Jacobi identity to be satisfied by these maps, as well as an appropriate action of the Virasoro algebra. See [DM1] for the complete definition. A $g$-twisted $V$-module is a weak $g$-twisted $V$-module such that all of the homogeneous spaces $M_n$ are of finite dimension. We should point out that the definition in [DM1] differs slightly from that used here. Namely, we allow $M$ to be $\mathbb{C}$-graded in the present paper (in order to be able to apply results in [DM2]) and we admit the possibility of infinite-dimensional homogeneous spaces.

It can be easily shown that any weak $g$-twisted module is a direct sum of submodules of type

$$N = \bigoplus_{n=0}^{\infty} N_{c+n/T}$$

where $c \in \mathbb{C}$ such that $N_c \neq 0$. The subspace $N_c$ is called the top level of $N$. Note that the grading of a simple weak $g$-twisted module always has the form (1.1).

**Theorem 1.** Let $g \in \mathfrak{M}$ be of type $2A$, $2B$ or $4A$. The following hold:

(i) Every simple weak $g$-twisted $V^2$-module is a $g$-twisted $V^2$-module.

(ii) Up to isomorphism there is exactly one simple $g$-twisted $V^2$-module.

(iii) If $g$ has type $2A$ then every $g$-twisted $V^2$-module is completely reducible.

Twisted sectors for $V^2$ of type $2B$ have been constructed by [Hu] and also by two of us [DM2], but until now that has been the extent of our knowledge of the existence of twisted sectors for $V^2$.

Continuing with earlier notation, we define an extended automorphism of $(M, Y_M)$ to be a pair $(x, \alpha(x))$ where $x : M \to M$ and $\alpha(x) : V \to V$ are invertible linear maps satisfying

$$xY_M(v, z)x^{-1} = Y_M(\alpha(x)v, z)$$

$$\alpha(x)g = g\alpha(x), \alpha(x)1 = 1, \alpha(x)\omega = \omega$$

for $v \in V$ where $1, \omega$ are the vacuum and conformal element respectively. This definition is a slight modification of [DM4], where it is explained that if $V$ and $M$ are both simple then $x \mapsto \alpha(x)$ is a group homomorphism from the group $\text{Aut}^e(M)$ of extended automorphisms of $M$ (which we identify with the group of linear maps $x$) into $\text{Aut}(V)$. Moreover, the kernel is a central subgroup of $\text{Aut}^e(M)$ consisting of the scalar operators.

We will show that if $V^2(2A)$ is the simple $2A$-twisted $V^2$-module whose existence is guaranteed by Theorem 1, then $\text{Aut}^e(V^2(2A))$ is isomorphic to $\mathbb{C}^* \times 2\text{Baby}$ where $2\text{Baby}$ denotes the 2-fold central extension of the Baby Monster which is known to
be isomorphic to the centralizer of $2A$ in $\mathbb{M}$ (cf. [Cal] and references therein). So in this case, the map $x \mapsto \alpha(x)$ splits, though this is not always the case. (E.g., for $g = 2B$ it does not. See Section 6 for details.) We identify $2\text{Baby} \subset \text{Aut}^e(V^\natural(2A))$ with the corresponding centralizer in $\mathbb{M}$. If $V^\natural$ has grading
\[ V^\natural = \bigoplus_{n=0}^{\infty} V^\natural_n \] (1.3) then for $h \in \mathbb{M}$ we define
\[ Z(1, h, \tau) = q^{-1} \sum_{n=0}^{\infty} \text{tr}(h|V^\natural_n)q^n, \] (1.4)
sometimes called the McKay-Thompson series of $h$. Here $q = e^{2\pi i \tau}$ as usual. It is known (the Conway-Norton conjecture = Borcherds’ theorem [B2]) that each $Z(1, h, \tau)$ is a so-called hauptmodul. For example, $Z(1, 1, \tau) = J(q) = q^{-1} + 196884q + \cdots$ is the absolute modular invariant with constant term 0.

It transpires that if $M = V^\natural(2A)$ then the constant $c$ in (1.1) is equal to $1/2$. Then for $g \in \mathbb{M}$ of type of $2A$ and $h \in C_\mathbb{M}(g) \simeq 2\text{Baby}$ we define
\[ Z(g, h, \tau) = q^{-1} \sum_{n=1}^{\infty} \text{tr}(h|V^\natural(2A)_{n/2})q^{n/2}. \] (1.5)

Remark that from (1.2) with $v$ equal to the conformal element $\omega$, it follows that $\text{Aut}^e(V^\natural(2A))$ preserves the $\mathbb{Q}$-grading on $V^\natural(2A)$, so that (1.5) makes sense. We prove

**Theorem 2.** The following hold:

(i) If $h$ has odd order then
\[ Z(g, h, 2\tau) = Z(1, gh, \tau). \] (1.6)

(ii) If $h$ either has odd order or satisfies $g \in \langle h \rangle$, then $Z(g, h, \tau)$ is a hauptmodul.

**Remark 3.** 1. These results amount to establishing the generalized moonshine conjecture due to Norton (cf. the appendix to [M]) for the commuting pairs $(g, h)$ such that $\langle g, h \rangle$ is cyclic and $g = 2A$.

2. If $g \in \langle h \rangle$ there are formulas which are analogous to (1.6), though more complicated.

3. Similar results also hold for the twisted sectors of type $2B$ and $4A$.

4. We can compute $Z(g, h, \tau)$ in other cases too. For example if $\langle g, h \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ has all 3 involutions of type of $2A$ then we will show in Section 5 that $Z(g, h, \tau)$ is precisely the hauptmodul $t_{2/2} = \sqrt{J(q) - 984}$ as predicted by Conway-Norton [CN].
The proofs of the two theorems depend heavily on the papers [DM2] and [DMZ]. In [DMZ] it is explained that $V^\natural$ contains a sub VOA which is isomorphic to $L = L(\frac{1}{2}, 0)^{\otimes 48}$, the tensor product of 48 vertex operator algebras $L(\frac{1}{2}, 0)$ associated with the highest weight unitary representation of the Virasoro algebra with central charge 1/2. Moreover, $L$ is a rational VOA.

There are many sub VOAs of $V^\natural$ isomorphic to $L$; they may be constructed from 48-dimensional associative subalgebras $A$ of the Griess algebra (cf. [MN] and [Mi] for example) which themselves are related to what we have called marked Golay codes [CM]. One can show easily that if $g \in M$ has type $2A$, $2B$ or $4A$ then $g$ fixes a suitable $A$ (element-wise), so that we may take $L$ to lie in the sub VOA $(V^\natural)_g$ of $g$-invariants. Then we are in a position to invoke several results from [DM2] to conclude Theorem 1.

Theorem 1 is purely an existence theorem. To get further information one needs to know the representation of the extended automorphism group on the top level $M_c$ of the corresponding twisted sector. This can vary greatly: for $M = V^\natural(2B)$ the top level is of dimension 24, whereas for $M = V^\natural(2A)$ it is 1. In each case the extended automorphism group acts irreducibly. To establish Theorem 2 we argue first that the top level of $V^\natural(2A)$ is a trivial module for 2Baby, using the structure of the Leech lattice as well as the fusion rules for $L(\frac{1}{2}, 0)$ and its modules established in [DMZ]. Then we can prove that the dimension is 1 by combining some monstrous calculation of Norton [N] together with further properties of the Virasoro algebra.

At this point we are in a position to combine the modular-invariance properties of [DM2], which relate graded traces of elements on $V^\natural(2A)$ to those on $V^\natural$, with Borcherds’ proof of the original moonshine conjecture. This leads to the proof of Theorem 2.

We thank the referee for useful comments.

2. TRANSPOSITIONS AND VIRASORO ALGEBRAS

With the conformal grading (1.3) of the moonshine module one knows that $V^\natural_0 = \mathbb{C}1$ is spanned by the vacuum and $V^\natural_1 = 0$. $V^\natural_2$ carries the structure of a commutative non-associative algebra which we denote by $B$. As Monster module, $B$ is the direct sum of two irreducible modules $1_2 \omega \oplus B_0$. We refer to both $B$ and $B_0$ as ‘the’ Griess algebra. Note that $\frac{1}{2} \omega$ is the identity of $B$.

A transposition of $M$ is an involution of type $2A$. If $x$ is a transposition then $C_M(x) \simeq 2Baby$. We fix such an $x$ and let $H = C_M(x)$. The space $B^H$ of $H$-invariants on $B$ is 2-dimensional and spanned by $\omega$ together with the so-called transposition axis $t_x$ (cf. [C] or [MN]). Moreover $H$ is the subgroup of $M$ leaving $t_x$ invariant. From this we easily conclude

**Lemma 2.1.** Let $x_1, ..., x_k$ be transpositions, let $A = \langle t_{x_1}, ..., t_{x_k}, \omega \rangle$ be the subalgebra of $B$ spanned by $w$ and the corresponding transpositions axes, and let $E = \langle x_1, ..., x_k \rangle$
be the subgroup of \( \mathbb{M} \) generated by the \( x_i \). Then the subgroup \( C_\mathbb{M}(E) \) of elements of \( \mathbb{M} \) commuting with \( E \) coincides with the subgroup \( C_\mathbb{M}(A) \) of \( \mathbb{M} \) fixing \( A \) (elementwise).
\[ \square \]

The subalgebra \( A \) of Lemma 2.1 is associative if, and only if, each product \( x_ix_j \) (\( i \neq j \)) is an involution of type 2B (Theorem 5, Corollary 1 of [MN]). In particular, \( E \) is elementary abelian in this case.

It was shown in [MN] that we may choose an associative algebra \( A \) of dimension 48 satisfying the conclusions of Lemma 2.1. It is sufficient to give 24 mutually orthogonal elements \( \pm \) commuting with \( E \). Let \( x_{i1}, \ldots, x_{i24} \) be the subgroup of \( \mathbb{M} \) generated by the \( x_i \). Then

\[
\text{code} \ [CM].
\]

These are the three types of Monster elements (apart from 1) contained in \( \mathbb{M} \).

**Lemma 2.2.** We may choose the \( v_i \) so that the corresponding elementary abelian group \( E \) satisfies \(|E| \leq 2^9\).

**Proof.** We may take \( E \leq Q = O_2(C) \simeq 2^{1+24}_2 \) where \( C \) is the centralizer of an involution of type 2B in \( M \). We identify \( Q/Z(Q) \) with \( \Lambda/2\Lambda \), so that each \( \pm v_i \) becomes an involution (of type 2A) in \( Q \). Clearly \( E = \langle \pm v_i \rangle = \langle -1, v_i \rangle \). We have \( v_1 + v_2 = (8, 0^{23}) \) and for \( i \geq 1, v_{2i+1} + v_{2i+2} = (0^{2i}, 8, 0^{23-2i}) = v_1 + v_2 + 2(-4, 0^{2i-1}, 4, 0^{23-2i}) \equiv v_1 + v_2 \pmod{2\Lambda} \). So \( E = \langle -1, v_2, v_{2i-1}, 1 \leq i \leq 12 \rangle \).

Suppose now that we mark our Golay Code so that the six 4-element sets \( \{1, 2, 3, 4\}, \ldots, \{21, 22, 23, 24\} \) constitute a sextet i.e., the union of any two of them is an octad (=block in the Witt design). Then \( v_1 + v_3 + v_5 + v_7 = (4^8, 0^{16}) = 2(2^8, 0^{16}) \in 2\Lambda \) and similarly \( v_1 + v_3 + v_{4i+1} + v_{4i+3} \in 2\Lambda \) for \( 1 \leq i \leq 5 \). Hence \( E = \langle -1, v_2, v_3, v_5, v_9, v_13, v_{17}, v_{21} \rangle \).
\[ \square \]

**Corollary 2.3.** With the choice made in Lemma 2.2, the corresponding associative algebra \( A \) of dimension 48 is fixed (pointwise) by elements in \( \mathbb{M} \) of types 2A, 2B and 4A.

**Proof.** These are the three types of Monster elements (apart from 1) contained in \( Q \). Since \( Q \) is extra-special of order \( 2^{25} \) and \( E \leq Q \) has order less than or equal to \( 2^9 \), then certainly \( C_Q(E) \) contains elements of these three types. Now apply Lemma 2.1.
\[ \square \]

It was shown in [DMZ] that corresponding to the 48 transpositions \( x_1, \ldots, x_{48} \) with transposition axes spanning a 48-dimensional associative algebra we obtain a certain
Lemma 2.4. Let $\omega_1$ and $\omega_2$ be two orthogonal idempotents such that the components of the vertex operator $Y(\omega_i, z)$ generate a copy Virasoro algebra with central charge $1/2$. Then the actions of these two Virasoro algebras are commutative on $V^2$.

Proof. Set $Y(\omega_i, z) = \sum_{n \in \mathbb{Z}} L_i(n) z^{-n-2}$ for $i = 1, 2$. Then the inner product of $\omega_1$ with $\omega_2$ is given by $L^1(0)\omega_2$, which is 0 by assumption. By Norton’s inequality (cf [C] or [MN]) $0 = (\omega_1, \omega_2) = (\omega_1^2, \omega_2^2) \geq (\omega_1\omega_2, \omega_1\omega_2) \geq 0$, the product $\omega_1\omega_2$ (which is $L^1(0)\omega_2$) in the Griess algebra is also 0. Thus $L^1(n)\omega_2 = 0$ for all nonnegative integrals $n$ and $\omega_2$ is a highest weight vector with highest weight 0 for the Virasoro algebra $\text{Vir}_1$ generated by $L^1(m)$. The submodule of $V^2$ for $\text{Vir}_1$ generated by $\omega_2$ is necessarily isomorphic to $L(1/2, 0)$. Here $L(k, h)$ is the simple highest weight representation of $\text{Vir}$ with central change $k$ and highest weight $h$. From the module structure of $L(1/2, 0)$ we see immediately that $L^1(1/2)\omega_2 = 0$. Now use the commutator formula

$$\left[ L^1(m), L^2(n) \right] = \sum_{i=-1}^{\infty} \binom{m+1}{i+1} L^1(i)\omega_2 |_{m+n+1-i} = 0$$

to complete the proof.

Some multiple $\omega_i$ of the transposition axis $t_{x_i}$ is an idempotent such that the components of the vertex operator $Y(\omega_i, z)$ generate a copy Virasoro algebra with central charge $1/2$. As the $\omega_i$ are orthogonal the algebras mutually commute as operators on $V^2$ by Lemma 2.4, yielding an action of the sub VOA

$$L = L\left(\frac{1}{2}, 0\right)^{\otimes 48}$$

on $V^2$.

Note that if $g \in \mathbb{M}$ fixes the associative algebra $A$ spanned by the $t_{x_i}$, then $g$ fixes each $\omega_i$. Hence we get

Lemma 2.5. If $g$ is of type $2A$, $2B$ or $4A$ then the fixed subalgebra $(V^2)^g$ contains $L$.

The sum of all the $\omega_i$ is the conformal vector $\omega$. If $H = 2\text{Baby}$ fixes $\omega_1$, say, then it also fixes $\omega_0$ where $\omega = \omega_1 + \omega_0$. Now the component operators of $Y(\omega_0, z)$ generate a copy of the Virasoro algebra of central charge $47/2$, and in any case we get a subspace of $(V^2)^H$ corresponding to $Y(\omega_1, z)$ and $Y(\omega_0, z)$, that is to say the subspace $L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{47}{2}, 0\right)$ which is again a vertex operator algebra [FHL].
Now one knows (cf. [KR] for example) that the $q$-characters of these two algebras are as follows:

\[
ch_q L\left(\frac{1}{2}, 0\right) = \frac{1}{2} \left( \prod_{n=1}^{\infty} (1 + q^{n-1/2}) + \prod_{n=1}^{\infty} (1 - q^{n-1/2}) \right)\\
ch_q L\left(\frac{47}{2}, 0\right) = \prod_{n=2}^{\infty} (1 - q^n)^{-1}.
\]

We thus calculate that the subspace $L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{47}{2}, 0\right) \subset (V^\natural)^H$ has $q$-character

\[
1 + 2q^2 + 2q^3 + 5q^4 + 6q^5 + 12q^6 + \cdots.
\]

**Lemma 2.6.** Up to weight six, all elements of $(V^\natural)^H$ lie in $L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{47}{2}, 0\right)$.

**Proof.** Norton has calculated (Table 5 of [N]) the decomposition of the permutation representation of $M$ on the conjugacy class of transpositions into simple characters for $M$. The representation is multiplicity-free and the characters occurring are precisely $\chi_1, \chi_2, \chi_4, \chi_5, \chi_9, \cdots$ where we order the simple characters according to their degrees as in [Cal].

By elementary character theory, if $V_i$ is the $M$-module affording $\chi_i$, then the dimension of the space of $H$ (=2Baby) invariants $V_i^H$ is precisely the multiplicity of $\chi_i$ above. Hence it is either 0 or 1 and is 1 precisely if $i = 1, 2, 4, 5, 9, \cdots$.

Now the decomposition of the first few homogeneous spaces $V_n^\natural$ into simple Monster module is known (e.g. [MS]). We have

\[
V_0^\natural = V_1, \quad V_1^\natural = 0\\
V_2^\natural = V_1 \oplus V_2, \quad V_3^\natural = V_1 \oplus V_2 \oplus V_3\\
V_4^\natural = 2V_1 \oplus 2V_2 \oplus V_3 \oplus V_4\\
V_5^\natural = 2V_1 \oplus 3V_2 \oplus 2V_3 \oplus V_4 \oplus V_6\\
V_6^\natural = 4V_1 \oplus 5V_2 \oplus 3V_3 \oplus 2V_4 \oplus V_5 \oplus V_6 \oplus V_7.
\]

So to compute $\sum_{n=0}^{6} \dim(V_n^\natural)^H q^n$, from the foregoing it is equivalent to counting the number of occurrences of $\chi_1, \chi_2, \chi_4, \chi_5$ in (2.5). We find that the multiplicities are precisely those of (2.4). \(\square\)
3. Proof of Theorem 1

We need to quote some results from [DM2]. First, following [Z] and [DM2], we say that the VOA $V$ satisfies the Virasoro condition if $V$ is a sum of highest weight modules for the Virasoro algebra generated by the components of $Y(\omega, z)$; we say that $V$ satisfies the $C_2$ condition if $V/C_2(V)$ is finite-dimensional, where $C_2(V)$ is spanned by $u_{-2}v$ for $u, v \in V$. What is important for us is that because $V^\natural$ contains $L$ as a rational subalgebra, $V^\natural$ satisfies the $C_2$ condition as well as the Virasoro condition. See [Z] for more information on this point.

**Theorem 3.1.** [DM2] Suppose that $V$ is a holomorphic VOA satisfying both Virasoro and $C_2$ conditions. Let $g \in \text{Aut}(V)$ have finite order. Then the following hold:

(i) $V$ has at most one simple $g$-twisted module.

(ii) $V$ has at least one weak simple $g$-twisted module.

So to prove parts (i) and (ii) of Theorem 1, it is enough to prove that a weak simple $g$-twisted $V^\natural$-module for $g$ of type $2A$, $2B$ or $4A$ has finite-dimensional homogeneous spaces. Let $M$ be such a module.

Let $W = (V^\natural)^g$. Then $M$ is an ordinary weak $W$-module, in fact it is the sum of a finite number of simple weak $W$-modules (see [DM3], for example). So it suffices to prove that a weak simple $W$-module $N$, say, has finite-dimensional homogeneous spaces.

It is shown in [DMZ] that $L(\frac{1}{2}, 0)$ is a rational VOA with three irreducible modules $L(\frac{1}{2}, h_i), h_i = 0, 1/2, 1/16$ which are exactly the highest weight unitary representations of $Vir$ with central charge $1/2$. Moreover all weak modules are ordinary modules. The characters of these modules are as follows: as well as $L(\frac{1}{2}, 0)$ given in (2.3) we have

$$ch_q L\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} \left( \prod_{n=1}^{\infty} (1 + q^{n-1/2}) - \prod_{n=1}^{\infty} (1 - q^{n-1/2}) \right)$$

$$ch_q L\left(\frac{1}{2}, \frac{1}{16}\right) = q^{1/16} \prod_{n=1}^{\infty} (1 + q^n).$$

The fusion rules are as follows:

$$L\left(\frac{1}{2}, \frac{1}{2}\right) \times L\left(\frac{1}{2}, \frac{1}{2}\right) = L\left(\frac{1}{2}, 0\right)$$

$$L\left(\frac{1}{2}, \frac{1}{2}\right) \times L\left(\frac{1}{2}, \frac{1}{16}\right) = L\left(\frac{1}{2}, \frac{1}{16}\right)$$

$$L\left(\frac{1}{2}, \frac{1}{16}\right) \times L\left(\frac{1}{2}, \frac{1}{16}\right) = L\left(\frac{1}{2}, 0\right) + L\left(\frac{1}{2}, \frac{1}{2}\right)$$

where the “product” $\times$ is commutative and $L\left(\frac{1}{2}, 0\right)$ is the identity.
From these facts we conclude (cf. [FHL]) that $L$ has just $3^{48}$ simple modules, namely

$$L(h_1, ..., h_{48}) = L\left(\frac{1}{2}, h_1\right) \otimes \cdots \otimes L\left(\frac{1}{2}, h_{48}\right)$$

(3.3)

where $h_i \in \{0, 1/2, 1/16\}$. The corresponding fusion rules (see Proposition 2.10 of [DMZ]) are

$$L(h_1, ..., h_{48}) \times L(h'_1, ..., h'_{48}) = (L\left(\frac{1}{2}, h_1\right) \times L\left(\frac{1}{2}, h'_1\right)) \otimes \cdots \otimes (L\left(\frac{1}{2}, h_{48}\right) \times L\left(\frac{1}{2}, h'_{48}\right))$$

(3.4)

The fusion rules can be interpreted in terms of a tensor product $\boxtimes$ of modules for vertex operator algebras (see [HL] and [L]). Then the product $\times$ in (3.2) and (3.4) can be replaced by the tensor product $\boxtimes$ to get the tensor product decomposition. We refer the reader to [L] for more details.

Now if $0 \neq n \in \mathbb{N}$ is a highest weight vector with highest weight $(h_1, ..., h_{48})$ for the Virasoro algebra generated by the components of $Y(\omega_i, z)$, $i = 1, ..., 48$, then the $L$-module generated by $n$ is isomorphic to $L(h_1, ..., h_{48})$. Let $N_{h'_1, ..., h'_{48}}$ be the multiplicity of $L(h'_1, ..., h'_{48})$ in $W$. Then as an $L$-module $W$ has decomposition

$$W = \bigoplus_{h'_i \in \{0, 1/2, 1/16\}} N_{h'_1, ..., h'_{48}} L(h'_1, ..., h'_{48}).$$

Then we have the tensor product of $L$-modules

$$W \boxtimes L(h_1, ..., h_{48}) = \bigoplus_{h'_i, h'_j \in \{0, 1/2, 1/16\}} N_{h'_1, ..., h'_{48}} L(h'_1, ..., h'_{48}) \boxtimes L(h_1, ..., h_{48})$$

which is an ordinary $L$-module with finite-dimensional homogeneous subspaces. From Proposition 4.1 of [DM3] we see that $N$ is spanned by $w_mn$ for $w \in W$ and $m \in \mathbb{Z}$. Now using the universal property of tensor product, we conclude that $N$ is a submodule of $W \boxtimes L(h_1, ..., h_{48})$ as $L$-modules. Thus each homogeneous subspace of $N$ is finite-dimensional. This completes the proof of parts (i) and (ii) of Theorem 1.

If $H = 2\text{Baby}$ is the centralizer of $g = 2A$ in $\mathbb{M}$ then, as explained in [DM1] and [DM4], the uniqueness of $V^2(2A)$ yields a projective representation of $H$ on $V^2(2A)$. This must be an ordinary and faithful representation since $H^2(\text{Baby}, \mathbb{C}^*) \simeq \mathbb{Z}_2$. Thus the group of extended automorphisms of $V^2(2A)$ is precisely $\mathbb{C}^* \times H$, as claimed in the introduction.

4. Top level of $V^2(2A)$

First we review some further results from [DM2]. Under the assumptions of the Virasoro condition and the $C_2$ condition, which we know hold for $V^2$, together with the complete reducibility of $V^2$-modules [D], the modular-invariance properties established in [DM2] may be stated as follows:

$$Z(g, h, \gamma \tau) = \sigma(\gamma^{-1}, g, h)Z((g, h)\gamma, \tau).$$

(4.1)
Here \((g, h)\) is a pair of elements which generate a cyclic group, \(Z(g, h, \tau)\) is a function in the so-called \((g, h)\)-conformal block, \(\gamma \in SL(2, \mathbb{Z})\) and \(\sigma(\gamma^{-1}, g, h)\) is a nonzero constant. The notation \((g, h)\gamma\) is the action of \(SL(2, \mathbb{Z})\) on pairs of commuting elements:

\[
(g, h) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (g^a h^c, g^b h^d).
\]

(4.2)

What is important is that if \(g\) is either \(1A\) or one of \(2A, 2B\) or \(4A\), so that Theorem 1 applies, then \(qZ(g, h, \tau)\) is precisely the graded trace of \(h\) on the \(g\)-twisted sector \(V^g(g)\). In particular, taking \(g = 1, h = 2A\) and \(\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) = \(S\) in (4.1) yields

\[
Z(1, 2A, S\tau) = \sigma(\gamma^{-1}, 1, 2A)Z(2A, 1, \tau).
\]

(4.3)

By Borcherds’ theorem [B2], \(Z(1, 2A, \tau)\) is a hauptmodul, specifically the one denoted by 2+ in [CN]. This means that \(Z(1, 2A, \tau)\) is invariant under the Fricke involution \(W_2 = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} = S \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\), so that (4.3) yields

\[
Z(1, 2A, \tau/2) = Z(1, 2A, W_2(\tau/2)) = Z(1, 2A, S\tau) = \sigma(\gamma^{-1}, 1, 2A)Z(2A, 1, \tau).
\]

Now \(Z(1, 2A, \tau/2) = q^{-1/2} + O(q^{1/2})\), and \(qZ(2A, 1, \tau)\) is the \(q\)-character of \(V^g(2A)\). We conclude that

**Lemma 4.1.** The \(q\)-character of \(V^g(2A)\) has the form

\[
k(q^{1/2} + O(q^{3/2}))
\]

for some constant \(k \neq 0\).

We will establish

**Theorem 4.2.** The constant \(k\) is equal to 1.

In the following we let \(M = V^g(2A)\) with \(M_{1/2}\) the top level of \(M\). Now \(M_{1/2}\) is spanned by highest weight vectors corresponding to simple \(L\)-modules \(L(h'_1, \ldots, h'_{48}) \subset M\) satisfying \(\sum_i h'_i = 1/2\). Then \(M^1 = \sum_{n \in \mathbb{Z}} M_{n+\frac{1}{2}}\) and \(M^0 = \sum_{n \in \mathbb{Z}} M_n\) are irreducible \(W\)-modules where \(W = (V^g)^{(2A)}\) (see Theorem 6.1 of [DM4]). Since we are mainly concerned with the top level \(M_{1/2}\) we will only pay attention to the \(W\)-module \(M^1\). We have already explained that \(W\) is the sum of simple \(L\)-modules \(W = W_1 + \cdots + W_i\) and \(M^1\) is spanned by \(w_l m\) for \(w \in W, l \in \mathbb{Z}\), and fixed \(0 \neq m \in M_{1/2}\). So if we choose \(m\) to be a highest weight vector in the simple \(L\)-module \(N = L(h'_1, \ldots, h'_{48}) \subset M^1\) we get information about \(M^1\) by considering the fusion rules \(W_i \times N\) or the tensor product \(W_i \otimes N\).
In particular, we can bound \( \dim M_{1/2} \) by estimating how many indices \( i \) are such that \( W_i \otimes N \) contains a simple \( L \)-module of highest weight \( 1/2 \). Let \( W_i = L(h_1, \ldots, h_{48}) \) and assume that \( W_i \otimes N \supset L(k_1, \ldots, k_{16}) \) with \( \sum k_i = 1/2 \).

**Lemma 4.3.** If \( N = L(\frac{1}{2}, 0^{47}) \) with ‘1/2’ in any position then Theorem 4.2 holds.

**Proof.** Without loss take ‘1/2’ in the first position. Then we have \( k_j = h_j, 2 \leq j \leq 48 \), and \( k_1 = 1/2 \) if \( h_1 = 0; k_1 = 0 \) if \( h_1 = 1/2; k_1 = 1/16 \) if \( h_1 = 1/16 \). This follows from the fusion rules (3.2).

Since \( \sum h_j \) is a nonnegative integer not equal to 1 (since \( V_n^2 = 0 \) for \( n < 0 \) or \( n = 1 \)), the condition \( \sum k_i = 1/2 \) forces \( h_1 = \cdots h_{48} = 0 \), that is \( W_i = L \). Since \( L \) has multiplicity 1 in \( V^2 \) and \( L \otimes N = N \), the lemma follows. \( \square \)

From now on we assume that all highest weight vectors in \( M_{1/2} \) generate \( L \)-modules of type \( L((\frac{1}{16})^8, 0^{40}) \) for some distribution of the 1/16’s. For now we take \( N = L((\frac{1}{16})^8|0^{40}) \), meaning that the 1/16’s are in the first 8 coordinate positions (this is purely for notational convenience).

**Lemma 4.4.** If \( W_i \) is such that \( W_i \times N \supset N' = L(k_1, \ldots, k_{48}) \) with \( \sum k_i = 1/2 \) then one of the following holds:

- (a) \( W_i = L(0, (\frac{1}{2})^3, (\frac{1}{16})^4|0^{36}) \) and \( N' = L((\frac{1}{16})^4, 0^4|0^{36}) \);
- (b) \( W_i = L((\frac{1}{2})^8|0^{40}) \) and \( N' = N \);
- (c) \( W_i = L((\frac{1}{2})^6, 0^2|0^{40}) \) and \( N' = N \);
- (d) \( W_i = L((\frac{1}{2})^4, 0^4|0^{40}) \) and \( N' = N \);
- (e) \( W_i = L \) and \( N' = N \).

(No specific ordering of the first 8 coordinates or the last 40 entries is implied.)

**Proof.** Let \( W_i = (0^a, (\frac{1}{2})^b, (\frac{1}{16})^c|h_9, \ldots, h_{48}) \). Using the fusion rules (3.2), we see that \( N' = ((\frac{1}{16})^a+b, 0^c-d, (\frac{1}{2})^d|h_9, \ldots, h_{48}) \) for some \( 0 \leq d \leq c \).

We have, setting \( s = \sum_{i=9}^{48} h_i \), that

\[
\frac{a+b}{16} + \frac{d}{2} + s = \frac{1}{2},
\]

\[
\frac{b}{2} + \frac{c}{16} + s = 0 \text{ or } \geq 2, \text{ lies in } \mathbb{Z}
\]

\[
a + b + c = 8.
\]

It follows that \( c \equiv 0 \pmod{4} \) and \( d = 0 \). If \( c = 8 \) then \( a + b = 0 \), so \( s = 1/2 \) and \( c/16 + s = 1 \), contradiction. So \( c = 0 \) or \( 4 \). If \( c = 4 \) then \( a + b = 4 \), \( s = 1/4 \), and then \( b = 3 \). If \( c = 0 \) then \( a + b = 8 \), \( s = 0 \), \( b = 4, 6 \) or 8. The lemma follows. \( \square \)

**Lemma 4.5.** In the notation of Lemma 4.4, if \( W_i \times N = N' \subset M^1 \) then \( w_{wtw-1}m \neq 0 \) where \( w \) is a nonzero highest weight vector of \( W_i \). Moreover \( v_{wtw-1}m \) is a scalar multiple of \( w_{wtw-1}m \) for any homogeneous \( v \in W_i \).
Proof. We need to use results on the Zhu algebra $A(V)$ and its bimodule $A(W_i)$ (which is a quotient of $W_i$ by a subspace) to prove this result. We refer the reader to [Z] and [FZ] for the definitions. Let $m'$ be a nonzero highest weight vector of $N'$. By Theorem 1.5.3 of [FZ] and Theorem 4.2.4 of [L], $A(W') \otimes_{A(L)} \mathbb{C}m$ is isomorphic to $\mathbb{C}m'$ as $A(L)$-modules under the map $\bar{v} \otimes m \rightarrow v_{wt_v - 1}m$ where $\bar{v}$ is the image of $v \in W_i$ in $A(W_i)$. By Lemma 2.8 and Proposition 3.3 of [DMZ], $A(L)$ is isomorphic to the associative commutative algebra $\mathbb{C}[t_j | j = 1, ..., 48]/I$ where $I$ is the ideal generated by $t_j(t_j - \frac{1}{2})(t_j - \frac{1}{16})$. By Lemma 2.9, Propositions 3.1 and 3.4 of [DMZ], $A(W_i)$ is isomorphic to $\mathbb{C}[x_j, y_j | j = 1, ..., 48]/I_i$ where $I_i$ is a certain ideal of $\mathbb{C}[x_j, y_j | j = 1, ..., 48]$ and the left and right actions of $t_j + I$ on $A(W_i)$ are multiplications by $x_j$ and $y_j$ respectively. Moreover, under this identification, $\bar{w}$ is mapped to $1 + I_i$. Since $A(W_i)$ is generated by $\bar{w}$ as $A(L)$-bimodule, the lemma follows immediately. □

Lemma 4.6. $M_{1/2}$ is a trivial module for $H = 2$Baby.

Proof. The minimal nontrivial representation of $H$ has degree 4371 [Cal] so it suffices to show that $\dim M_{1/2}$ is less than this. Now the multiplicity of $L(h_{1}, ..., h_{48})$ in $V^2$ is less than or equal to 1 if all $h_{i} \in \{0, 1/2\}$ by Proposition 5.1 of [DMZ]. So by Lemma 4.4 we conclude that the multiplicity of $N = L((\frac{1}{16})^{8}(0^{40}))$ in $M$ is at most $\binom{8}{0} + \binom{8}{2} + \binom{8}{4} + \binom{8}{6} = 100$. All other simple $L$-submodules of $M$ of weight 1/2 arise from the action of $L(0, (\frac{1}{2})^{3}, (\frac{1}{16})^{4}, (\frac{1}{16})^{4}, 0^{36}) \subset V^2$, and the multiplicity$^1$ $\mu$ of all modules of this type in $V^2$ is given in Theorem 6.5 [DMZ]. Since $\mu = 24 \cdot 2^6$, the lemma follows. □

Lemma 4.7. $M_{1/2}$ is contained in the span of $v_{wt_v - 1}m$ for $v \in V^2_n$ with $n \leq 4$.

Proof. First we observe from Lemma 4.4 that the highest weight vectors in $W_i$ have weights 0, 2, 3 or 4. By Corollary 4.2 of [DM3], $M_{1/2}$ is spanned by $v_{wt_v - 1}m$ for $v \in W_i$ with $W_i$ occurring in Lemma 4.4. So it is enough to show that the span of $v_{wt_v - 1}m$ for $v \in W_i$ is contained in $w_{wt_w - 1}m$ where $w$ is a nonzero highest weight vector of $W_i$. This follows from Lemma 4.5. □

Now we can complete the proof of Theorem 4.2. We may represent the conclusion of Lemma 4.7 by the containment $(\bigoplus_{n=0}^{4} V^2_n)m \supset M_{1/2}$. Since $M_{1/2}$ is a trivial $H$-module by Lemma 4.6, it follows that $M_{1/2} \subset (\bigoplus_{n=0}^{4} (V^2_n)^H)m$.

Now use Lemma 2.6 to see that $M_{1/2}$ lies in the space spanned by $u_{wt_u - 1}m$ for $u \in L(\frac{1}{2}, 0) \otimes L(\frac{47}{2}, 0)$ homogeneous. But $m$ is an eigenvector for such operators. So we get $M_{1/2} = \mathbb{C}m$, as required. □

We are now ready for the proof of Theorem 1 (iii). Let $M$ be an irreducible $g$-twisted $V^2$-module so that the top level of $M$ is $\mathbb{C}m$. Let $N$ be the $L$-submodule of $M$

$^1$Some of the multiplicities stated in Theorem 6.5 of [DMZ] are inaccurate. See [Ho] for corrections.
generated by $m$. Then by Lemmas 4.3 and 4.4, $N$ is either $L(\frac{1}{2}, 0^{47})$ or $L((\frac{1}{16})^8|0^{40})$. Then $W_i \times N = N$ if, and only, if in the former case $W_i = L$, and in the latter case $W_i$ appears in (b)-(e) of Lemma 4.4. In either case, since $M$ is simple we see that the subspace of $N$ spanned by $v_l m$ for $v \in W_i$ and $l \in \mathbb{Z}$ is $N$ (by Proposition 11.9 of [DL]). Again by Lemma 4.5, $w_{w+1} w \neq 0$.

Let $X$ be a $g$-twisted $V^\natural$-module. Then $X$ has a finite composition series (see [DLM]). Using induction on the number of composition-factors of $X$, we only need to prove that $X$ is completely reducible if $X$ has two factors. It is shown in [DLM] that $X$ is a completely reducible $g$-twisted $V^\natural$-module if, and only, if $X_{1/4}$ is a semisimple $A_g(V^\natural)$-module via the action $v_{w+1} w$ for homogeneous $v \in V^\natural$. If $N = L(\frac{1}{2}, 0^{47})$ it is clear that $X_{1/4}$ is a semisimple $A_g(V^\natural)$-module from the discussion above. Here $A_g(V^\natural)$ is the twisted Zhu algebra as defined and used in [DLM].

Now we assume that $N = L((\frac{1}{16})^8|0^{40})$. Let $X_{1/4} = \mathbb{C} x_1 + \mathbb{C} x_2$ such that $x_1$ generates an irreducible $g$-twisted $V^\natural$-module $X_1$ (which is necessarily isomorphic to $M$). Using the associativity of vertex operators (see the proof of Proposition 4.1 of [DM3], for example) we see that $w_{w+1} w w_{w+1} x_2$ is a nonzero multiple of $x_2$. Thus $w_{w+1} w w_{w+1} x_2$ acts semisimply on $X_{1/4}$, hence acts as a scalar. Thus for all homogeneous $v \in W_i$ the action of $w_{w+1} w w_{w+1} x_2$ on $X_{1/4}$ is semisimple. Note that the image of $A_g(V^\natural)$ in $End(X_{1/4})$ is a subalgebra of dimension less than or equal to 2. So we conclude that $X_{1/4}$ is indeed a semisimple $A_g(V^\natural)$-module.

5. Proof of Theorem 2

Let $g \in M$ be of type 2A and let $h$ be an element of $C_{2A}(g)$ of odd order $N$. We use the notation of (4.1)-(4.2).

**Lemma 5.1.** Let $F \leq SL(2, \mathbb{R})$ be the fixing group of $Z(1, gh, \tau)$. Then $F$ contains the Atkin-Lehner involution $W_2$.

**Proof.** By Borcherds’ theorem [B2], each $Z(1, x, \tau)$ for $x \in M$ is a hauptmodul on a discrete group $F = F_x \leq SL(2, \mathbb{R})$. Then $F$ is precisely the group conjectured in [CN], Table 2. This has been established by the work of several authors; see [CN] and [F] for further references.

It is a fact that if $x = gh$ then $W_2$ always lies in $F$. In fact, the group $F$ is of the form $2N+$, or $2N + 2$ in all but one case. The exception is the element $30F$, where the group is $30 + 2$, $15, 30$. In any case, the lemma follows. □
Now we may take $W_2 = \begin{pmatrix} a & b \\ cN & 2d \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ where $\gamma = \begin{pmatrix} a & b \\ cN & 2d \end{pmatrix} \in SL(2, \mathbb{Z})$. Then we have from (4.1) that

$$Z(1, gh, \tau/2) = Z(1, gh, W_2(\tau/2)) = \sigma Z((1, gh)\gamma, \tau) = \sigma Z((gh)^cN, (gh)^{2d}, \tau) = \sigma Z(g, h', \tau).$$

Here, $\sigma$ is a constant and $h' = h^{a'}$ where $aa' \equiv 1 \pmod{N}$. But $Z(1, gh, \tau/2) = q^{-1/2} + \cdots$ and $Z(g, h', \tau) = q^{-1/2} + \cdots$ by Theorem 4.2. So $\sigma = 1$. Now Theorem 2, part (i) follows immediately.

As for part (ii) of Theorem 2, if we now take $h \in M$ such that $g \in \langle h \rangle$ then we can find $\gamma \in SL(2, \mathbb{Z})$ such that $(g, h) = (1, h)\gamma$. Then (4.1) yields

$$Z(g, h, \tau) = \sigma Z(1, h, \gamma\tau)$$

for some constant $\sigma$. As $Z(1, h, \tau)$ is a hauptmodul by Borcherds’ theorem then so is $Z(g, h, \tau)$ by (5.1).

Now let us consider $Z(g, h, \tau)$ where $g, h$ and $gh$ are all of type $2A$. From [DM2] we know that $Z(g, h, \tau)$ is holomorphic on the upper half-plane and meromorphic at the cusps. Furthermore (4.1) still holds because of Theorem 1 (iii). We see that if $\gamma \in SL(2, \mathbb{Z})$ then

$$Z(g, h, \gamma\tau) = \sigma(\gamma)Z(g, h, \tau)$$

for some constant $\sigma(\gamma)$. So $\sigma$ is a character of $SL(2, \mathbb{Z})$. As $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ covers the abelianization of $SL(2, \mathbb{Z})$ then the kernel $K$ of $\sigma$ is the subgroup of $SL(2, \mathbb{Z})$ of index 2. As $i\infty$ is the unique cusp for $K$ we see easily that $K$ is indeed of genus zero with hauptmodul $Z(g, h, \tau)$. It is in fact the function denoted $t_{2/2} = \sqrt{J(q)} - 984 = q^{-1/2} - 492q^{1/2} - 22590q^{3/2} + \cdots$ in [CN]. To see this, note that from Theorem 2 (i) with $h = 1$, combined with tables in [CN] and the results of [B2], we see that $Z(g, 1, \tau) = q^{-1/2} + 4372q^{1/2} + \cdots$. This tells us that the weight $3/2$ subspace of $V^2(2A)$ is the module $1 \oplus 4371$ for 2Baby (see [Cal]), on which $h$ has trace $-492$ (ibid). Thus $Z(g, h, \tau) = q^{-1/2} - 492q^{1/2} + \cdots$ as claimed.

6. Final comments

We have rather ignored the twisted sectors $V^2(2B)$ and $V^2(4A)$. As we have said, there is a construction of $V^2(2B)$ in [Hu], and its existence also follows from [DM2] without the necessity of the effort we needed to understand $V^2(2A)$. Huang also constructs the $2B$-orbifold, i.e., puts an abelian intertwining algebra structure [DL] on $V^2 \oplus V^2(2B)$. This is closely related to the construction of $V^2$ in [FLM].
Concerning the extended automorphism group it is known \([G]\) that the centralizer \(C\) of \(2B\) in \(M\) is a non-split extension of \(\cdot 1\) (largest simple Conway group) by the extra-special group \(2^{1+24}_+\). Furthermore (loc.cit.) \(H^2(C, C^*) \simeq \mathbb{Z}_2\). In fact if \(\cdot 0\) is the 2-fold cover of \(\cdot 1\) (=automorphism group of the Leech lattice) there is a diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & 2^{1+24}_+ & \rightarrow & C & \rightarrow & \cdot 1 & \rightarrow & 1 \\
\| & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & \rightarrow & 2^{1+24}_+ & \rightarrow & \hat{C} & \rightarrow & \cdot 0 & \rightarrow & 1
\end{array}
\]

and \(\hat{C}\) is the universal central extension of \(C\).

Griess also shows that \(\hat{C}\) has a simple module of degree \(2^{12}\), whereas \(C\) has no such representation. The smallest faithful irreducible representation for \(C\) has dimension \(24 \cdot 2^{12}\).

Now the weight \(3/2\) subspace of \(V^2(2B)\) has dimension \(2^{12}\) (see [Hu] and [DM2]) and there is a projective representation of \(C\) on \(V^2(2B)\) by [DM1] and [DM4]. The conclusion is thus

**Lemma 6.1.** The extended automorphism group of \(V^2(2B)\) is a non-split extension

\[
1 \rightarrow C^* \rightarrow Aut^e(V^2(2B)) \rightarrow C \rightarrow 1.
\]

It has commutator subgroup isomorphic to \(\hat{C}\).

**Theorem 6.2.** Let \(h \in M\) be such that \(\langle h \rangle\) contains \(g = 2B\) or \(4A\). Then \(Z(g, h, \tau)\) is a hauptmodul.

**Proof.** Same as the proof of Theorem 2 (ii).

Finally, we consider the \(q\)-characters of \(V^2(2B)\) and \(V^2(4A)\). Note by [CN] or [FLM] that we have for \(g \in M\) of type \(2B\) that

\[
Z(1, g, \tau) = 24 + q^{-1} \prod_{n \text{ odd}} (1 - q^n)^{24} = 24 + \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}}
\]

where \(\eta(\tau)\) is the Dedekind eta-function. By (4.1) and the transformation law for \(\eta(\tau)\) [K] we get the \(q\)-character of \(V^2(2B)\) equal to

\[
Z(g, 1, \tau) = \sigma Z(1, g, S\tau) = \sigma \left\{ 24 + \frac{212\eta(\tau)^{24}}{\eta(\tau/2)^{24}} \right\}.
\]

In fact \(\sigma = 1\), as we know from [Hu] and [DM2].

Similarly for \(t\) of type \(4A\) we have

\[
Z(1, t, \tau) = -24 + \frac{\eta(2\tau)^{48}}{\eta(\tau)^{24}\eta(4\tau)^{24}}
\]
which leads to
\[ Z(t, 1, \tau) = \sigma \left\{ -24 + \frac{\eta(\tau/2)^{48}}{\eta(\tau)^{24}\eta(\tau/4)^{24}} \right\}. \]

Presumably \( \sigma = 1 \) in this case too, but a proof would be more difficult than that for 2A given above.

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