Graviton Mode Function in Inflationary Cosmology

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We consider the production of gravitons in an inflationary cosmology by approximating each epoch of change in the equation of state as sudden, from which a simple analytic graviton mode function has been derived. We use this mode function to compute the graviton spectral energy density and the tensor-induced cosmic microwave background anisotropy. The results are then compared to the numerical calculations which incorporate a smooth radiation-matter phase transition. We find that the sudden approximation is a fairly good method. Besides, in determining the frequency range and amplitude of the mode function, we introduce a pre-inflationary radiation-dominated epoch and use a physically sensible regularization method.

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I. Introduction

The spatial flatness and homogeneity of the present Universe strongly suggest that a period of de Sitter expansion (inflation) had occurred in the early Universe \[1\]. During inflation quantum fluctuations of the inflaton field may give rise to energy density (scalar) perturbations \[2\], which can serve as the seeds for structure formation of the Universe. Also, a well-defined spectrum of gravitational waves could be produced from the de Sitter vacuum \[3\]. Since the waves decouple from matter very early, they remain as a stochastic background of gravitational waves at present and provide a potentially important probe of the early time. Unfortunately, the detection of these primordial waves by using terrestrial wave detectors or the timing of millisecond pulsars \[4\] (this is only sensitive to short-wavelength waves of period less than order years) is still several orders of magnitude below the present experimental sensitivity. However, being similar to energy density perturbations, horizon-sized gravitational waves can induce distortions of the cosmic microwave background (CMB) \[5–11\] via the Sachs-Wolfe effect \[12\]. In fact, large-angular-scale temperature anisotropy of the CMB has been recently detected \[13\]. So, it is worthwhile to reexamine in detail the graviton production from inflation.

Production of gravitational waves from quantum fluctuations during inflationary period have been considered \[8,14–17\]. In transverse traceless (TT) gauge, a classical gravitational wave has two independent polarization states. As noticed by Grishchuk \[18\], each polarization state of the wave behaves as a minimally coupled, massless, real scalar field, with a normalization factor of \(\sqrt{16\pi G}\) linking the two cases (we will show this in Section III). Thus, the study of graviton production during inflation reduces to considering the quantum fluctuations of a scalar field in the de Sitter space-time. The result is the well-known scale-invariant spectrum. The spectrum can be derived by relating the scalar quantum-mechanical two-point function defined in the de Sitter background to a two-point statistical average of an ensemble of classical gravitational fields \[8\]. In evaluating the two-point function, the de Sitter invariant vacuum has been chosen. The subsequent evolution of the fields is then governed by the classical equation of motion. The spectrum can also be derived by a sequence of Bogoliubov transformations between creation and annihilation operators defined in various stages of the Universe: inflationary, radiation-, and matter-dominated. This consideration
reveals that the results are independent of state over a wide range of initial states in the inflationary period, thus allowing one to select the de Sitter invariant vacuum \[14\].

In this paper, we will reexamine in detail the graviton production during inflation. We assume four sequential phases of the Universe: radiation-, inflationary, radiation-, and matter-dominated. Each phase change is approximated as a sudden process (i.e., assuming the transition from one phase to another phase to be instantaneous). We begin with a scalar field in a radiation-dominated period prior to inflation, and work out the dynamics of the growing quantum fluctuations of the field during inflation. In evaluating the scalar two-point function, we avoid using any formal renormalization schemes to remove divergences, instead, we will adopt a physically sensible method introduced by Vilenkin to regularize the infinities \[19\]. This allows us to determine the frequency range and amplitude of the gravitons, and also show explicitly the validity of selecting the de Sitter invariant vacuum during the inflationary period. Based on the sudden approximation of the phase changes, we will work out the graviton mode function for a wide range of frequency which is relevant to astrophysical measurements. This mode function is then applied to compute the spectral energy density and the tensor-induced CMB anisotropy. Since in a realistic phase transition the phase change is rather smooth than abrupt, we have to examine how good the sudden approximation can reproduce the actual situation. In order to do so we will compare the results obtained by taking the sudden approximation to the numerical calculations which incorporate a smooth phase transition.

II. Background Metric of the Early Universe

Let us suppose an initially radiation-dominated universe. The Universe expanded and became dominated by the energy density resided in the vacuum which drove the inflation. After inflation, the Universe was reheated by the latent heat released from the vacuum and resumed radiation-dominated. Later, matter dominated the energy density and the Universe is currently in the matter-dominated phase. This supposition makes sense only if inflation did occur below the Planck scale (for instance, grand unified theories (GUT) inspired inflationary models), since a temperature above the Planck scale may not be defined. At temperatures well below the Planck scale but above the GUT scale, radiation most likely dominates the energy density of the Universe. For Planck-scale inflation (e.g., chaotic inflation \[20\]) or
eternal inflation (inflationary period extends to negative infinity), it reduces to a three-step calculation in which the de Sitter invariant vacuum in the inflationary epoch is reasonably singled out. In the following, we will assume an inflation which is an exponential spatial expansion. In fact, our results can be easily extended to other possibility like power-law inflation. Since the curvature term is sub-dominated at very early times and the Universe is essentially spatially flat after inflation, the space-time is well depicted by the $k = 0$ Robertson-Walker metric,

$$\begin{align*}
    ds^2 &= dt^2 - a^2(t)dx^2 \\
        &= a^2(\eta) \left( d\eta^2 - dx^2 \right),
\end{align*}$$

where $a(\eta)$ and $d\eta = dt/a(t)$ are the scale factor and conformal time respectively. Here we are using the signature $(-1,-1,-1,1)$.

Let us consider four phases: pre-inflation radiation-dominated, inflationary, radiation-, and matter-dominated. The scale factor for each phase is given by $a(\eta) = A(\eta + B)^{2/(1+3q)}$ according to the equation of state $p = q\rho$ of that phase, where $A$ and $B$ are constants. For our purposes, we normalize $a(\eta) = 1$ and choose $\eta = -1/H$ at $t = 0$ when inflation begins, where $H$ is the Hubble constant during inflation. Here we will assume sudden phase transitions. By requiring that $a(\eta)$ and $\dot{a}(\eta)$ (where dot means taking derivative with respect to $\eta$) are continuous functions in the cosmological history, we obtain

$$a(\eta) = \begin{cases} 
2 - \eta/\eta_3 , & \eta < \eta_3 \\
\eta_3/\eta , & \eta_3 < \eta < \eta_2 \\
-\eta_3(\eta - 2\eta_2)/\eta_2^2 , & \eta_2 < \eta < \eta_1 \\
-\eta_3(\eta + \eta_1 - 4\eta_2)^2/4\eta_2^2(\eta_1 - 2\eta_2) , & \eta_1 < \eta
\end{cases},$$

where $\eta_3$, $\eta_2$, and $\eta_1$ are respectively the conformal times at which the inflationary, radiation-, and matter-dominated eras begin. We have $\eta_3 = -1/H$ and $\eta_2 = -1/(He^{H\tau})$ (where $\tau$ is the duration of inflation). Note that $\eta \in (2\eta_3, +\infty)$ and today will be denoted by $\eta_0$. It can be easily shown that

$$\eta_1 \simeq -\frac{a(\eta_1)}{a(\eta_2)}\eta_2 , \quad \eta_0 \simeq 2 \left[ \frac{a(\eta_0)}{a(\eta_1)} \right]^{\frac{1}{2}} \eta_1 ,$$

provided that $a(\eta_3)/a(\eta_2), a(\eta_0)/a(\eta_1) >> 1$. 

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III. Gravitational Wave Equation

Weak gravitational waves on a Robertson-Walker universe has been investigated by Lifshitz [21]. As we restrict ourselves to a conformally flat universe, this conformal property allows us to rederive the wave equation in a much faster way as follows.

The theory of pure gravity is given by the Einstein-Hilbert action

\[ I_G = \frac{1}{16\pi G} \int d^4x \sqrt{g} R, \]

where \( G \) is the gravitational constant. In the weak field approximation, small metric perturbations are ripples on the background metric:

\[ g_{\mu\nu} = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu}), \quad h_{\mu\nu} << 1, \]

where \( \eta_{\mu\nu} \) is the Minkowski metric, and Greek indices run from 0 to 3. In this section, we will leave \( a(\eta) \) as a dimensionless and arbitrary scale function. Therefore, without any loss of generality, we choose \( a(\eta) \) very near to unity and write it as

\[ a(\eta) = e^{\sigma(\eta)}; \quad \sigma(\eta) << 1. \]

By expanding Eq. (4) and keeping terms up to quadratic in \( h_{\mu\nu} \) and \( \sigma \), we obtain

\[
I_G = \frac{1}{16\pi G} \int d^4x \ a^2 \left( \frac{1}{4} \partial_\mu h_{\alpha\beta} \partial^\mu h^{\alpha\beta} - \frac{1}{4} \partial_\mu h \partial^\mu h + \frac{1}{2} \partial_\alpha h^{\alpha\beta} \partial_\beta h - \frac{1}{2} \partial_\alpha h^\alpha h^\beta_{\beta\mu} - 2\partial_\alpha \sigma \partial^\mu h + 2\partial_\alpha \sigma \partial_\beta h^{\alpha\beta} - 6\partial_\mu \sigma \partial^\mu \sigma \right),
\]

where all indices are lowered and raised with \( \eta_{\mu\nu} \) and \( h = \eta^{\mu\nu} h_{\mu\nu} \). In synchronous gauge, \( h_{00} = h_{0i} = 0 \), where \( i \) runs from 1 to 3. The remaining \( h_{ij} \) contain a transverse, traceless tensor which corresponds to a gravitational wave. Here we are only interested in this tensor mode. Henceforth we will work in the TT gauge, i.e., \( h^k_k = \partial_i h^{ij} = 0 \) and denote the two independent polarization states of the wave as +, \( \times \). Then, we obtain from Eq. (7) the action of graviton as

\[ I_{\text{graviton}} = \frac{1}{16\pi G} \int d^4x \ a^2 \ \frac{1}{4} \partial_\mu h_{ij} \partial^\mu h^{ij}. \]

We can write a monochromatic wave with a wave vector \( k \) as
\[ h_{ij}(x) = h(x; k, \lambda) \epsilon_{ij}(k; \lambda), \]  

(9)

where \( \epsilon_{ij}(k; \lambda) \) is the polarization tensor and \( \lambda = +, \times \). The polarization tensor satisfies

\[ \epsilon_{ij}(k; \lambda)\epsilon^{ij}(k; \lambda') = 2\delta_{\lambda\lambda'} . \]  

(10)

Hence, for this wave Eq. (8) becomes

\[ I_{\text{graviton}} = \frac{1}{16\pi G} \int d^4 x a^2(\eta) \frac{1}{2} \left[ (\partial_\mu h(x; k, +))^2 + (\partial_\mu h(x; k, \times))^2 \right] , \]  

(11)

which is in fact the action for two real, massless free scalar fields \( \phi(x; k, \lambda) = (16\pi G)^{-1/2} h(x; k, \lambda) \) in the background space-time. As a result, each polarization state of the wave behaves as a real, massless, minimally coupled scalar field, with a normalization factor \( \sqrt{16\pi G} \) relating the two cases. Writing

\[ h(x; k, \lambda) = (2\pi)^{-\frac{3}{2}} h_\lambda(\eta; k) e^{ikx} + \text{h.c.} \]  

(12)

one finds from Eq. (11) that the equation of motion for the wave amplitude \( h_\lambda(\eta; k) \) is

\[ \ddot{h}_\lambda + 2 \frac{\dot{a}}{a} \dot{h}_\lambda + k^2 h_\lambda = 0 , \]  

(13)

where the dot denotes taking differentiation with respect to \( \eta \). This is just the Klein-Gordon equation for a massless plane wave in the background space-time [18]. In what follows we will simply consider a massless real scalar field \( \phi(x) \).

IV. Solutions of Wave Equation

We consider the plane-wave mode expansion of a massless real scalar field in the background metric (1),

\[ \phi(x) = (2\pi)^{-\frac{3}{2}} \int d^3 k \left[ a_k \psi_k(\eta) e^{ikx} + \text{h.c.} \right] , \]  

\[ [a_k, a_{k'}^\dagger] = \delta(k - k') , \]  

(14)

where the mode function \( \psi_k(\eta) \), which defines the vacuum state, satisfies Eq. (13). When the scale factor \( a(\eta) \), which defines the vacuum state, satisfies Eq. (13), we find:

(i) When \( \eta < \eta_3 \),
\[ \psi_k(\eta) = a^{-1}(\eta) \frac{1}{2} (\pi |\xi|)^{\frac{1}{2}} \left[ \alpha_1 H^{(1)}_{\frac{1}{2}}(k\xi) + \alpha_2 H^{(2)}_{\frac{1}{2}}(k\xi) \right], \]  

where \( \xi = 2/H + \eta \), \( H^{(1)}_{\frac{1}{2}} \) and \( H^{(2)}_{\frac{1}{2}} \) are the Hankel functions. The coefficients \( \alpha_1 \) and \( \alpha_2 \) are functions of \( k \), subject to the normalization condition:

\[ |\alpha_2|^2 - |\alpha_1|^2 = 1. \]  

(ii) When \( \eta_3 < \eta < \eta_2 \),

\[ \psi_k(\eta) = a^{-1}(\eta) \frac{1}{2} (\pi |\eta|)^{\frac{1}{2}} \left[ \beta_1 H^{(1)}_{\frac{1}{2}}(k\eta) + \beta_2 H^{(2)}_{\frac{1}{2}}(k\eta) \right]. \]  

(iii) When \( \eta_2 < \eta < \eta_1 \),

\[ \psi_k(\eta) = a^{-1}(\eta) \frac{1}{2} (\pi |\xi|)^{\frac{1}{2}} \left[ \gamma_1 H^{(1)}_{\frac{1}{2}}(k\xi) + \gamma_2 H^{(2)}_{\frac{1}{2}}(k\xi) \right], \]  

where \( \xi = \eta - 2\eta_2 \).

(iv) When \( \eta_1 < \eta \),

\[ \psi_k(\eta) = a^{-1}(\eta) \frac{1}{2} (\pi |\xi|)^{\frac{1}{2}} \left[ \delta_1 H^{(1)}_{\frac{1}{2}}(k\xi) + \delta_2 H^{(2)}_{\frac{1}{2}}(k\xi) \right], \]  

where \( \xi = \eta + \eta_1 - 4\eta_2 \). Similar to the case (i), the Greek coefficients in cases (ii)-(iv) are \( k \) dependent and each pair is subject to the same normalization condition (16). We list below the Hankel functions and their properties which will be useful for determining the coefficients:

\[ H^{(1,2)}_{\nu}(z) = J_{\nu}(z) \pm i N_{\nu}(z) = \sqrt{\frac{2z}{\pi}} h^{(1,2)}_{\nu}(z) = \sqrt{\frac{2z}{\pi}} \left[ j_{\nu-\frac{1}{2}}(z) \pm in_{\nu-\frac{1}{2}}(z) \right], \]

\[ H^{(1)}_{\nu}(-z) = e^{-\pi i} H^{(1)}_{\nu}(z) - 2e^{-\pi i} J_{\nu}(z), \]

\[ H^{(2)}_{\nu}(-z) = e^{-\pi i} H^{(2)}_{\nu}(z) + 2e^{\pi i} J_{\nu}(z), \]

\[ H^{(1)}_{\frac{1}{2}}(z) = -i \sqrt{\frac{2}{\pi z}} e^{iz}, \quad H^{(2)}_{\frac{1}{2}}(z) = i \sqrt{\frac{2}{\pi z}} e^{-iz}, \]

\[ H^{(1)}_{\frac{3}{2}}(z) = -\sqrt{\frac{2}{\pi z}} (1 + i \frac{z}{z}) e^{iz}, \quad H^{(2)}_{\frac{3}{2}}(z) = -\sqrt{\frac{2}{\pi z}} (1 - i \frac{z}{z}) e^{-iz}, \]  

where \( J_{\nu} \) and \( N_{\nu} \) are Bessel functions, \( j_{\nu-\frac{1}{2}} \) and \( n_{\nu-\frac{1}{2}} \) are spherical Bessel functions, \( z > 0 \) is real, and \( \nu \) is complex.

To determine the coefficients, we require that \( \psi_k \) and its time derivative \( \dot{\psi}_k \) are continuous at \( \eta \) equal to \( \eta_3, \eta_2, \) and \( \eta_1 \). Thus, referring to Eqs. (15-19), we have six equations with eight
unknowns. However, for $\eta < \eta_3$, we choose the conformal vacuum state, i.e., $\alpha_1 = 0$ and $\alpha_2 = 1$ in Eq. (15), which corresponds to the positive-frequency mode functions, $\psi_k(\eta) = i a^{-1}(\eta) e^{-i k \xi / \sqrt{2k}}$, where $\xi = 2/H + \eta$. Note that these mode functions are conformal transforms of the positive-frequency mode functions defined in Minkowski space-time. As a result, the remaining six coefficients can be uniquely determined. In terms of

$$z_3 \equiv -k \eta_3 = \frac{k}{H}, \quad z_2 \equiv -k \eta_2 = \frac{k}{H} \frac{1}{a(\eta_2)}, \quad z_1 \equiv k(\eta_1 - 2\eta_2) = \frac{k}{H} \frac{a(\eta_1)}{a^2(\eta_2)},$$

we find that

$$\beta_1 = -\frac{1}{2 z_3^2},$$
$$\beta_2 = \left(1 - i \frac{z_2}{\eta_2} - \frac{1}{2 z_3^2}\right) e^{-2 i z_3},$$
$$\gamma_1 = -\beta_1 \left(1 - i \frac{z_2}{\eta_2} - \frac{1}{2 z_2^2}\right) e^{-2 i z_2} - \beta_2 \frac{1}{2 z_2^2},$$
$$\gamma_2 = \beta_1 + \beta_2 \left(1 + i \frac{z_2}{\eta_2} - \frac{1}{2 z_2^2}\right) e^{2 i z_2},$$
$$\delta_1 = i e^{-i z_1} \gamma_1 \left(1 - i \frac{z_1}{8 z_1^2} - \frac{1}{8 z_1^2}\right) + i e^{-3 i z_1} \gamma_2 \frac{1}{8 z_1^2},$$
$$\delta_2 = -i e^{3 i z_1} \gamma_1 \frac{1}{8 z_1^2} - i e^{i z_1} \gamma_2 \left(1 + i \frac{z_1}{8 z_1^2} - \frac{1}{8 z_1^2}\right).$$

These results are obtained for any arbitrary $k$. In the next section, by examining the dynamics of the growing quantum fluctuations of the scalar field during inflationary period, we will show that only plane-wave modes of $k$ within certain range will be excited and generated from the de Sitter vacuum.

V. Scalar Field Fluctuations

The general formalism for calculating the quantum fluctuation $\langle \phi^2 \rangle$ in the de Sitter space-time has been developed by Bunch and Davies [23]. Using a point-splitting regularization scheme they calculate a two-point function for $\phi(x)$ in Eq. (14),

$$\langle \phi(x'') \phi(x') \rangle = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i k \cdot (x'' - x')} \psi_k(\eta'') \psi_k^*(\eta') d^3k,$$

where $\psi_k(\eta)$, which defines the vacuum state, is given by Eq. (17). Furthermore, they specialize to the de Sitter invariant vacuum such that $\beta_1 = 0$ and $\beta_2 = 1$. With this
choice, Eq. (23) is singular for $x'' \to x'$. This singularity can be removed by an appropriate renormalization scheme. In the present paper, we will treat $\langle \phi^2 \rangle$ based on an alternative procedure introduced by Vilenkin [19].

In an inflationary universe one has to carefully consider the physically relevant quantity to be calculated. As has been discussed by Vilenkin, in the inflationary universe the modes of interest are only those with wavelength greater than the horizon, as these are those responsible for the growth of $\langle \phi^2 \rangle$. These are the modes with $k|\eta| \ll 1$, so that the integration over $k$ should be cut off at $k \simeq H e^{H \tau}$. When $k \leq H$, the magnitude of the fluctuation depends on the initial conditions of the universe, but with a power-law expansion rate of the universe prior to inflation there is no reason to believe that $\langle \phi^2 \rangle$ exceeds the anomalous de Sitter fluctuations [19]. Hence we can write $\langle \phi^2 \rangle$ at time $t$ as

$$\langle \phi^2 \rangle = \langle \phi^2 \rangle_0 + \frac{1}{2\pi^2} \int_H^{H e^{Ht}} dk \ k^2 |\psi_k(\eta)|^2,$$  \hspace{1cm} (24)

where $\langle \phi^2 \rangle_0$ is the initial value and $t \leq \tau$. Although Eq. (24) is not without arbitrariness, Vilenkin’s regularization technique makes physical sense as well as being useful from a calculational point of view. In fact, this method can easily reproduce previous results. For example, for a massless scalar field, the growth of quantum fluctuations during inflation in this formalism, by using Eqs. (17), (20)-(22), and (24), is calculated as

$$\langle \phi^2 \rangle = \frac{H^2}{8\pi} |\eta|^3 \int_H^{H e^{Ht}} dk \ k^2 |\beta_1 H^{(1)}_2(k \eta) + \beta_2 H^{(2)}_2(k \eta)|^2 \
\to \frac{H^2}{4\pi^2} H t \text{ as } t >> H^{-1}.$$ \hspace{1cm} (25)

This is the well-known result of the linear growth of fluctuations [2|19,24]. In essence, short-wavelength quantum fluctuations in the scalar field $\psi_k$ of wavenumbers, $H < k < H e^{H \tau}$, will get red-shifted out of the horizon during the inflationary period, after which they freeze in, remaining with constant amplitude (see below). The superposition of these frozen modes therefore constitutes the coherent scalar field. When the mode re-enters the horizon much later during the radiation- or matter-dominated era it appears as a long-wavelength, classical wave.

VI. Graviton Mode Function
We write the graviton field in terms of the plane-wave modes given in Eqs. (11) and (12) as

\[ h_{ij}(x) = (2\pi)^{-3/2} \sum_{\lambda} \int d^3k \left[ a_{\lambda}(k) h_{\lambda}(\eta; k) e^{ik \cdot x} \epsilon_{ij}(k; \lambda) + \text{h.c.} \right], \]

\[ [a_{\lambda}(k), a_{\lambda'}^+(k')] = \delta(k - k') \delta_{\lambda\lambda'}. \]  

(26)

As explained above, the gravitational wave is related to the scalar field by \( h_{\lambda}(\eta; k) = (16\pi G)^{1/2} \psi_k(\eta). \) In the background metric (1), \( \psi_k(\eta) \) are given by Eqs. (13)-(19) with the coefficients given by Eqs. (21)-(22). Similar to the scalar case, only quantum fluctuations in the gravitational field \( h_{\lambda}(\eta; k) \) of short wavelengths, \( H < k < H e^{H_\tau}, \) will get pushed outside the horizon during the inflationary period. Therefore, from Eq. (21), \( z_3, z_2, \) and \( z_1 \) are such that

\[ 1 < z_3 < a(\eta_2), \quad \frac{1}{a(\eta_2)} < z_2 < 1, \quad \frac{a(\eta_1)}{a(\eta_2)} < z_1 < \frac{a(\eta_1)}{a(\eta_2)}. \]  

(27)

In an inflationary cosmology, the exponential expansion factor \( a(\eta_2) \) is at least \( 10^{28} \) in order to circumvent the cosmological problems [1]. Let us assume \( a(\eta_2) = 10^{28}. \) If the reheating temperature after inflation is about the GUT scale of order \( 10^{15}\text{GeV}, \) then the ratio of the scale factors at the beginning of matter-dominated era and at the end of inflation, \( a(\eta_1)/a(\eta_2), \) would be about \( 10^{23}. \) In addition, the present scale factor \( a(\eta_0) \simeq 10^4 a(\eta_1) [25]. \) For definiteness, we fix

\[ a(\eta_2) = 10^{28}, \quad \frac{a(\eta_1)}{a(\eta_2)} = 10^{23}, \quad \frac{a(\eta_0)}{a(\eta_1)} = 10^4. \]  

(28)

Hence, we find from Eq. (27) that

\[ 1 < z_3 < 10^{28}, \quad 10^{-28} < z_2 < 1, \quad 10^{-5} < z_1 < 10^{23}. \]  

(29)

It is useful to compare the size of a wave to the horizon size at time \( \eta. \) For the background metric (1), after inflation, the horizon size \( l \) grows as

\[ l = a(\eta) \int_{\eta_2}^{\eta} d\eta' \simeq a(\eta)\eta, \quad \text{for} \eta \gg \eta_2. \]  

(30)

If we consider a wave with physical wavelength \( \lambda_{\text{phys}} \) which is just entering the horizon at time \( \eta, \) then its wavenumber \( k \) must satisfy
\[ k = \frac{2\pi a(\eta)}{\lambda_{\text{phys}}} = \frac{2\pi a(\eta)}{l} = \frac{2\pi}{\eta}. \]  

(31)

This condition separates two limiting cases: a wave with \( k\eta << 2\pi \) is well outside the horizon whereas a wave with \( k\eta >> 2\pi \) is well within the horizon.

Of particular interest are short-wavelength gravitational waves of period which is within the measurable range of terrestrial or astrophysical wave detector, as well as waves of long wavelengths comparable to the present horizon size. We can express the wavenumbers \( k \) of these waves in terms of the present time \( \eta_0 \) by using Eq. (31) as

\[ k = \frac{2n\pi}{\eta_0}, \]  

(32)

where \( n = 1 \) corresponds to a wave just entering the present horizon, and the wave with \( n \geq 10^{10} \) (this wave re-enters the horizon during the radiation-dominated epoch at time \( \eta \leq 10^{-10}\eta_0 \)) has present period less than about a year. Substituting Eq. (32) in Eq. (21) and then using Eqs. (28) and (3), we find for such waves that

\[ z_3 \simeq 10^3 n\pi, \quad z_2 \simeq 10^{-25} n\pi, \quad z_1 \simeq 10^{-2} n\pi. \]  

(33)

Also, Eq. (29) implies \( 10^{-3} < n < 10^{25} \). Therefore, in either case, \( z_3 >> 1 \) and \( z_2 << 1 \). This leads to approximation of expressions (22) as

\[
\begin{align*}
\beta_1 & \simeq 0, \quad \beta_2 \simeq e^{-2i z_3}, \\
\gamma_1 & \simeq \gamma_2 \simeq -\frac{\beta_2}{2 z_2^2}, \\
\delta_1 & \simeq \gamma_1 \left[ i e^{-i z_1} \left( 1 - i \frac{1}{2 z_1} - \frac{1}{8 z_1^2} \right) + i e^{-3i z_1} \frac{1}{8 z_1^2} \right], \\
\delta_2 & \simeq \gamma_1 \left[ -i e^{i z_1} \left( 1 + i \frac{1}{2 z_1} - \frac{1}{8 z_1^2} \right) - i e^{3i z_1} \frac{1}{8 z_1^2} \right],
\end{align*}
\]  

(34)

where \( z_3, z_2 \) and \( z_1 \) are given in Eq. (33). By using this and Eqs. (17)-(19), we can construct the wave form of the gravitational wave for any particular value of \( n \). For example, at time \( \eta >> \eta_1 \) in the matter-dominated epoch, the wave amplitude is found to be given by

\[
\begin{align*}
h_{\lambda}(\eta; k) \simeq & \ - (32\pi G)^{\frac{3}{2}} H \frac{k \eta}{k_\eta} e^{-2i z_3} \left\{ \left[ i e^{-i z_1} \left( 1 - i \frac{1}{2 z_1} - \frac{1}{8 z_1^2} \right) + i e^{-3i z_1} \frac{1}{8 z_1^2} \right] h_1^{(1)}(k\eta) \\
& + \left[ -i e^{i z_1} \left( 1 + i \frac{1}{2 z_1} - \frac{1}{8 z_1^2} \right) - i e^{3i z_1} \frac{1}{8 z_1^2} \right] h_1^{(2)}(k\eta) \right\}.
\end{align*}
\]  

(35)
where we have used $\xi = \eta + \eta_1 - 4\eta_2 \simeq \eta$ in Eq. (19) since $\eta >> \eta_1 >> \eta_2$. In next section we will use this expression to calculate the effects of primordial gravitational waves on the CMB. We note that Eq. (53) is at variance with previous results as found in Refs. [5–9]. However, we will show shortly that those results can be in fact reproduced by taking different limits of Eq. (53). Also, similar expressions can be found in Refs. [14,15,26,27]. In the radiation-dominated epoch, for $\eta >> \eta_2$, we find that

$$
|h_\lambda(\eta; k)|^2 \simeq - \left(8\pi G\right)^{1/2} H e^{-2i\eta} j_0(k\eta),
$$

which agrees with the result given in Refs. [14,9].

As mentioned above, a gravitational wave well outside the horizon will remain with constant amplitude until much later it re-enters the horizon as a propagating classical wave. To illustrate this, we consider a gravitational wave with wavelength larger than the present horizon size ($n < 1$). For this wave, $z_1 < 10^{-2} \pi << 1$. Thus, further approximation of Eq. (34) can be made:

$$
\delta_1 \simeq \delta_2 \simeq \gamma_1 \frac{3}{4\eta_1},
$$

where $\gamma_1$ is given in Eq. (34). By using Eqs. (19) and (37), we obtain the amplitude squared of the wave in the matter-dominated epoch as

$$
|h_\lambda(\eta; k)|^2 \simeq 16\pi G a^{-2}(\eta) \frac{\pi}{4} |\delta_1|^2 \frac{2k\xi}{\pi} j_1^2(k\xi),
$$

where $\xi = \eta + \eta_1 - 4\eta_2$. For $\eta >> \eta_1$, this can be simplified as

$$
|h_\lambda(\eta; k)|^2 k^3 \simeq 8\pi G H^2 \left[ \frac{3j_1(k\eta)}{k\eta} \right]^2,
$$

where

$$
\frac{3j_1(k\eta)}{k\eta} \simeq 1 \quad \text{as} \quad n << 1.
$$

The expression (39) which is the long-wavelength limit of Eqs. (19) or (55) coincides with the well-known result of the scale-invariant spectrum of gravitational waves generated by inflation [5–9,14]. After a wave has entered into the horizon, its wave amplitude decreases with time. To show this, we consider a wave with $n = 10^{10}$, which corresponds to $z_1 = 10^8 \pi >> 1$. Then, for $\eta >> \eta_1$, we find from Eq. (55) that
\[ |h_\lambda(\eta; k)|^2 k^3 \simeq 128\pi G H^2 (k\eta_1)^2 \left[ \frac{n_1(k\eta)}{k\eta} \right]^2, \tag{41} \]

which agrees with the result given in Ref. [7] except differing by a factor of 16. Note also that the \( k \)-dependence in Eq. (41) differs from that in Eq. (39), namely, the former has less power suppression in \( k \). This behavior has also been found by using transfer function methods [28].

VII. Sudden Transition Approximation

The derivation of the graviton mode functions in Sec. VI is based on the sudden approximation of the phase changes. One should justify this approximation before he can trust the result in Eq. (33). As seen in Eq. (2), we have approximated each phase transition as being instantaneous. In reality, changing from one phase to another phase must take some finite time. The phase transition from the pre-inflationary radiation-dominated era to the inflationary era and the reheating at the end of inflation should involve very short time scales, hence only affecting the extremely high frequency part of the graviton spectrum. Discussion of the influence of the reheating on the high frequency cutoff of the spectrum can be found in Ref. [29]. Here we will mainly concern the affect of the radiation-matter phase transition on the comparatively much lower frequency region of the spectrum. We will see that this phase transition would affect the spectrum in the frequency range that is relevant for CMB anisotropy calculation.

Many attempts have been made in the past in finding an accurate solution for the graviton mode function across the radiation-matter transition. In a two-component (radiation plus dust) cosmology which has a smooth radiation-matter phase transition, the equation of motion (13) can be solved exactly in terms of complicated spheroidal wave functions [16,30]. In addition, the equation can be solved approximately by the time-independent [28] or time-dependent [31] transfer function method, or by the WKB approximation [32]. Of course, the equation can be easily solved via numerical method. However, sometimes it is much more convenient to have a simple yet reasonably accurate graviton mode function. Undoubtedly, the sudden approximation is the most economical way to obtain such a simple analytic solution.
To justify the accuracy of the sudden approximation method, we compare the period of a wave, $2\pi/k$, with the characteristic duration of the radiation-matter phase transition, $\Delta \eta$, at time $\eta_1$. Unfortunately, the phase transition, being rather gradual than sudden, does not bear a characteristic timing. However, $\Delta \eta$ should be comparable to $\eta_1$. As such, the condition that the phase transition is sudden as seen by the wave is

$$\frac{2\pi}{k} >> \Delta \eta \simeq \eta_1 \quad \text{i.e.,} \quad z_1 << 2\pi .$$

Therefore, Eq. (35) is valid for $z_1 << 2\pi$, i.e., for waves entering the horizon during the matter-dominated era. However, we should look into this matter in more detail. Let us recast Eq. (13) in terms of the function $g_\lambda \equiv a(\eta)h_\lambda$ in the form

$$\ddot{g}_\lambda + \left( k^2 - \frac{\ddot{a}}{a} \right) g_\lambda = 0 ,$$

where

$$\frac{\ddot{a}}{a} = \begin{cases} 0 , & \eta < \eta_1 ; \\ 2/\eta^2 , & \eta > \eta_1 , \end{cases}$$

for a sudden radiation-matter phase transition. In the radiation-dominated era, the solution to Eq. (13) which matches the wave function in the inflationary phase is given by $\sin(k\eta)$. Hence, $h_\lambda k^\frac{3}{2} \propto \sin(k\eta)/(ka)$ (where $a \propto \eta$), which is in fact the result in Eq. (31). In the matter-dominated era, the modes whose $k$’s satisfy $k\eta \geq z_1 >> \sqrt{2}$ will not be affected by the term $\ddot{a}/a$. Then, the solution to Eq. (13) for these modes is again given by $\sin(k\eta)$. Hence, the matter-dominated mode function $h_\lambda$ is given by the same solution in the radiation-dominated era with $a \propto \eta^2$ instead. This actually reproduces the result in Eq. (14). Although the phase transition as seen by the short-wavelength modes is gradual, they are not affected by the sudden phase change at all and only scale as $a(\eta)$. This is actually an illustration of the adiabatic theorem. Therefore, the sudden approximation can give good approximate graviton mode function across the radiation-matter phase transition for modes with $z_1 << 2\pi$ or $z_1 >> \sqrt{2}$. In the next section, we will use Eq. (33) to calculate the present spectral energy density of primordial gravitational waves and the induced CMB anisotropy. These will be compared to the results obtained by considering an equation of state which smoothly interpolates between the radiation- and matter-dominated eras. We will find that the sudden approximation can actually give fairly good approximation of these quantities even in the regime $z_1 \sim 1$. 

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VIII. Spectral Energy Density and CMB Anisotropy

To incorporate a smooth radiation-matter phase transition, we consider a two-component universe containing radiation and dust. For convenience, we define the new variables

$$\eta' \equiv (\sqrt{2} - 1) \frac{\eta}{\eta_1}, \quad k' \equiv \frac{k\eta_1}{\sqrt{2} - 1}. \quad (45)$$

It can be easily shown that

$$a(\eta') = a(\eta'_1)\eta'(\eta' + 2). \quad (46)$$

Note that the radiation-matter equality time $\eta'_1 \simeq 0.41$. For $a(\eta'_0)/a(\eta'_1) = 6000$, the present time $\eta'_0 \simeq 76.5$. (Note that $\eta'_0 \simeq 63.5$ if we have used Eq. (3).) Then, the gravitational wave equation (13) can be exactly solved in terms of complicated spheroidal wave functions [16,30]. Here we simply solve it numerically, using the following initial conditions for $h_\lambda$:

$$h_\lambda k' \eta' \equiv -e^{-2iz_3}(8\pi G)^{1/2}H \quad \text{and} \quad \frac{dh_\lambda}{d\eta'} = 0 \quad \text{as} \quad \eta' \to 0, \quad (47)$$

which is consistent with Eq. (36).

From Eqs. (11) and (26), the spectral energy density of gravitational waves can be written as

$$\rho_g \equiv \sum_{\lambda=\pm} \frac{k^2|d\rho_\lambda|}{dk'} = \sum_{\lambda=\pm} \frac{1}{32\pi G\langle a^2(\eta') \rangle} \left( \frac{k'}{2\pi} \right)^3 \left[ k'^2|h_\lambda|^2 + \left| \frac{dh_\lambda}{d\eta'} \right|^2 \right]. \quad (48)$$

It is useful to define a dimensionless parameter which is the spectral energy density divided by the closure density of the Universe,

$$\Omega_g \equiv \frac{\rho_g}{\rho_c}, \quad \rho_c = \frac{3H^2(\eta')}{8\pi G}, \quad (49)$$

where the Hubble parameter $H(\eta') = (da/d\eta')/a^2$. As the graviton production rate during inflation for each polarization state is statistically equal, it is expected that $h_+ = h_\times \equiv -e^{-2iz_3}(8\pi G)^{1/2}Hk'^{-3/2}h$, where $h$ is real and normalized to unity as $\eta' \to 0$. Hence,

$$\Omega_g = \frac{v}{9\pi} \eta'^2 \left( \frac{\eta' + 2}{\eta' + 1} \right)^2 \left[ k'^2h^2 + \left( \frac{dh}{d\eta'} \right)^2 \right], \quad (50)$$
where \( v = 3GH^2/8\pi = V_0/m_{Pl}^4 \) is the inflation parameter (\( V_0 \) and \( m_{Pl} \) are the de Sitter vacuum energy and Planck mass respectively). To estimate the amplitude of this quantity, one typically makes use of the following physical argument. Before a gravitational wave enters the horizon, its amplitude is constant. After it enters the horizon, on the other hand, it behaves effectively as radiation and will scale with \( a \) as such. This gives a \( \Omega_g \sim 10^{-13} \) for \( v \sim 10^{-9} \) \[13,16\]. Since we have already derived the graviton mode function, we can directly calculate \( \Omega_g \) at any time. Fig. 1 shows the present time \( \Omega_g \) in units of \( v \) verus the wavenumber in terms of \( n \). Note again that \( n = 1 \) corresponds to present horizon-sized waves. The curves denoted respectively by the symbols \( h_n \), \( h_s \), and \( h_m \) are obtained by using the numerical solution of Eq. (13) with the initial conditions (17), the sudden approximation mode function (15), and the matter-dominated mode function (39). We see that in the horizon- and super horizon-sized wavelength region, the three curves are equal. While \( h_m \) severely underestimates the spectral energy density at large \( n \)'s, \( h_s \) is a fairly good approximation for all \( n \)'s. The worst underestimation by \( h_s \) is by a factor of 2.5 at \( n \approx 35 \). At large \( n \)'s, it approaches to a constant plateau and then falls off at the high frequency cutoff [25,29], being consistent with the numerical result.

In the last section, we have argued that the sudden approximation should be valid for any \( n \) which is much smaller or larger than about 100 (corresponding to \( z_1 \approx \pi \)). This is verified in Fig. 1. It has been suggested that \( \delta_1 \) and \( \delta_2 \) in Eq. (34) for \( z_1 > 2\pi \) (or \( n > 200 \)) should be modified to exponential forms (see Eqs. (4.22) and (4.23) of Ref. [27]) due to the adiabatic theorem. However, we find that the spectral energy density obtained by using the sudden approximation (35) differs by less than 34% for \( n \geq 200 \) from the numerical result.

We now turn to calculate the CMB anisotropy induced by the primordial gravitational wave background. The temperature anisotropy is induced via the Sachs-Wolfe effect,

\[
\frac{\delta T}{T}(e) = -\frac{1}{2} \int_e^{\infty} d\eta' e^\eta e^{\eta' \partial h_{ij}(\eta', \vec{x})},
\]

where \( e \) is the propagation direction of the photon, \( h_{ij} \) is a field operator given by Eq. (26), and the lower (upper) limit of integration represents the point of emission (reception) of the photon. In CMB measurements, the measured temperature anisotropy is usually expanded in terms of spherical harmonics,
$$\frac{\delta T}{T}(e) = \sum_{l,m} a_{lm} Y_{lm}(e).$$  \hspace{1cm} (52)

Since $\delta T/T$ arises from quantum fluctuations generated during inflation, it is a Gaussian random field. This implies that $a_{lm}$'s are independent Gaussian random variables satisfying

$$\langle a^\dagger_{lm} a_{l'm'} \rangle = \frac{C_l}{2l+1} \delta_{ll'} \delta_{mm'},$$ \hspace{1cm} (53)

where $C_l$ is the anisotropy power spectrum, from which we can construct the two-point temperature correlation function,

$$\langle \frac{\delta T}{T}(e_1) \frac{\delta T}{T}(e_2) \rangle = \frac{1}{4\pi} \sum_l C_l P_l(e_1 \cdot e_2),$$ \hspace{1cm} (54)

where $P_l$ is a Legendre polynomial. The formula for the power spectrum is given by

$$C_l = \frac{32\pi v^3}{3} (2l+1)(l+2)(l+1)l(l-1) \int_0^{\infty} \frac{dk'}{k'} \left[ \int_{\eta_{dec}}^{\eta'_0} d\eta' \frac{d\tilde{h}(\eta')}{d\eta'} \frac{j_l(k'\eta'_0 - \eta')}{k'^2(\eta'_0 - \eta')^2} \right]^2,$$ \hspace{1cm} (55)

where $j_l$ is a spherical Bessel function. We will use the decoupling time $\eta_{dec} \approx 1.54$ corresponding to $a(\eta'_0)/a(\eta_{dec}) = 1100$. In Fig. 2, we plot $C_l$ for $l = 2 - 250$ by using the three different mode functions for $h$ as in the calculation of the spectral energy density. Since the time integration in Eq. (55) is getting close to the radiation-matter equality time, we should keep $\xi = \eta + \eta_1$ in Eqs. (35) and (39) (i.e., replacing $\eta$ in these equations by $\eta + \eta_1$). We see from Fig. 2 that the three curves almost match for $l < 20$. The matter-dominated mode function $h_m$ severely underestimates the $C_l$ for large $l$. Using the sudden approximation mode function $h_s$, on the other hand, we obtain much better accuracy; the $C_l$ we find differs by less than 2% for $l \leq 30$, less than 4% for $l \leq 130$, less than 10% for $l \leq 150$, and less than 25% for $l \leq 250$ from the numerical result.

For a fixed $l$, the main contribution to the integral (55) for $C_l$ comes from the mode of wavenumber $k' \approx l/\eta'_0$ (or $l \approx 2n\pi$) at the horizon crossing time $\eta'_c \approx \pi/k'$ \hspace{1cm} (33). In virtue of this, the mode functions $h_s$ and $h_m$, while underestimating the spectral energy density at large $n$, underestimate the anisotropy power spectrum at large $l$. It is interesting to note that in Fig. 2 all three curves have minima at $l \approx 170$. While both $h_n$ and $h_s$ have maxima at $l \approx 220$, $h_m$ has a maximum at a lower $l \approx 210$. It has been pointed out that it is the temporal phase of the gravitational wave (the phase of $h$) which determines the overall shape
of the power spectrum at large $l$ [32]. Specifically, the location of a maximum or a minimum is controlled by the phase of the wave. We thus plot $h_n$, $h_s$ and $h_m$ each of wavenumber $k' \simeq 2.2$ corresponding to $l \simeq 170$ at near the horizon crossing time $\eta_\text{c} \simeq 1.4$ in Fig. 3. We find that both $h_s$ and $h_m$, while underestimating the amplitude, reproduce accurately the phase of the wave, in accordance with the behavior of the curves at $l \simeq 170$ in Fig. 2. In Fig. 4, we plot again the three mode functions with wavenumber $k' \simeq 2.9$ corresponding to $l \simeq 220$ at near $\eta_\text{c} \simeq 1.1$. We find that $h_s$ reproduces rather good the phase of the wave and the location of the maximum at $l \simeq 220$. But, at this $l$, not only $h_m$ badly underestimates both the amplitude of the wave and the power spectrum, but also its $C_l$ curve rises up too quickly from the minimum to the maximum.

The tensor-induced CMB anisotropy power spectrum has been calculated by using numerical codes to evolve the photon distribution function using the general relativistic Boltzmann equation for radiative transfer [11,33], and by semi-analytic approaches using the Sachs-Wolfe integral (55). To evaluate the integral, different graviton mode functions of varying accuracy have been used [27,28,30–32]. We find that the $h_n$ curve in Fig. 2 is very close to the result from Boltzmann codes [33] as well as the result from spheroidal wave functions [30]. Note that we have used $\eta'_\text{dec} \simeq 1.54$ in Eq. (55) for all three cases. But, in the sudden approximation, Eq. (3) implies that $\eta'_\text{dec}$ should have been equal to 1.15. However, in order to make comparison between the three curves, we insist in using a common $\eta'_\text{dec} \simeq 1.54$. If we have used $\eta'_\text{dec} \simeq 1.15$ to evaluate the integral (55), the $h_s$ curve should have been similar to the results in [27,30]. There they have also used a similar sudden approximation mode function to calculate the power spectrum. Ref. [31] has proposed an approximate graviton mode function which in fact has an accuracy comparable to this work, but has to involve more complicated time-dependent transfer functions.

VIII. Discussions and Conclusions

In this paper, we have attempted to derive a simple yet reasonably accurate analytic mode function for the gravitational waves generated in inflationary cosmology. The wave amplitude in Eq. (35) is our main result. To derive this result, firstly, we have assumed that the Universe underwent four phases in sequence: pre-inflation radiation-dominated,
inflationary, radiation-, and matter-dominated. This is quite natural as long as one considers GUT-scale inflation. To the first order approximation, we have treated each phase transition as being instantaneous. Secondly, we follow Vilenkin’s physically sensible method to regularize the infinities in the evaluation of the scalar two-point function. In essence, the regularization scheme simply introduces an infrared cutoff and an ultraviolet cutoff. As a result, only those quantum fluctuations in the gravitational field with wavenumbers lying between the two cutoffs get pushed outside the horizon during the inflationary period, after which they freeze out and become collective modes. Our four-phase calculation explicitly shows that (see Eq. (22)) it is too naive to select the de Sitter vacuum ($\beta_1 = 0, \beta_2 = 1$) during the inflationary phase. Nevertheless, it could be a fairly good approximation since $z_3 > 1$ (see Eq. (29)). Moreover, for gravitational waves which are relevant to CMB observations, pulsar timing measurements or terrestrial wave detectors, it is perfectly fine to choose the de Sitter vacuum (see Eq. (34)). Besides, we can reproduce previous results by taking certain limits of Eq. (35). For gravitational waves with large wavelength comparable to the present horizon size, we have obtained Eq. (39) which is the well-known scale-invariant spectrum. For shorter waves, the wave form is different from Eq. (39), and only by choosing some special value of the wavenumber can one obtain the result given in Eq. (41).

We have used the sudden approximation mode function (35) to calculate the present graviton spectral energy density and the induced CMB anisotropy. The results are compared to the results obtained by using the numerical solution for the graviton mode function in a two-component universe which incorporates a smooth radiation-matter phase transition. We have found that the approximation of the phase change as a sudden process is a fairly good method. Although the mode function (35) mildly underestimates the spectral energy density and the anisotropy power spectrum, it reproduces accurately the overall shape of the power spectrum. This simple analytic mode function should be quite useful in studying tensor perturbations from inflationary models.

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Captions

Figure 1. Present graviton spectral energy density in units of $v$. The symbols $h_n$, $h_s$, and $h_m$ denote respectively the curves obtained by using the numerical mode function, the sudden approximation mode function, and the matter-dominated mode function. Note that $n = 1$ corresponds to a wavelength of the present horizon size.

Figure 2. Anisotropy power spectra by using $h_n$, $h_s$, and $h_m$ respectively.

Figure 3. Graviton mode functions each of wavenumber $k' \simeq 2.2$ corresponding to $l \simeq 170$ at near the horizon crossing time $\eta'_c \simeq 1.4$. The curves in descending amplitude are $h_n$, $h_s$, and $h_m$ respectively.

Figure 4. Graviton mode functions each of wavenumber $k' \simeq 2.9$ corresponding to $l \simeq 220$ at near the horizon crossing time $\eta'_c \simeq 1.1$. The curves in descending amplitude are $h_n$, $h_s$, and $h_m$ respectively.