Asymptotic Solutions and Estimations of Some Three-dimensional Problems for Bodies Weakened by Plane Crack Systems

B. V. Sobol*

Don State Technical University, Gagarin Square, 1, Rostov-on-Don, 344000, Russia

Abstract

A selection of the author’s [1, 3, 4, 8] and some other researchers’ works deal with the formulation and investigation of a series of problems relating to elastic space, half-space and layer equilibrium, containing systems of plane cracks. A few cases where such cracks are perpendicular or parallel to the boundary have been considered. There have also been studied some issues of interaction between two cracks aligned symmetrically, an infinite chain and a twice-periodical system of cracks lying in the same plane. A case where a three-periodical system of cracks fills the elastic space has been discussed as well.

It is important to note that over the last decades there have been developed a few highly effective numerical methods for the solution of mechanical problems [2]. However, analytical methods are still of equal importance as they support not only quantitative, but also qualitative investigations of these and other effects.

Here we are focusing on the numerical and comparative qualitative analysis of the above problems. We have established some features of mutual interaction between the cracks and the borders on the stress-intensity factor in the contour vicinity for each particular case.

Keywords: Crack, contour, generalized Fourier-transform, singular integro-differential equation, stress-intensity factor.

1. Introduction

In general, our studies of the problems of equilibrium cracks in elastic bodies are based on the following approach. We are to consider the superposition of two problems: 1. Problem of equilibrium of an elastic body without cracks, a definition of the stress-strain state in the region of the cracks. 2. Problem of balance of an elastic environment weakened by cracks in a view of corresponding boundary conditions on their coasts.

Let us assume that a crack occupies an elastic space area \( \Omega \) in a plane \( z=0 \). The crack is in opened state influenced by a loading \( \sigma_z = -p(x, y), z = \pm 0, (x, y) \in \Omega \). It is assumed that stresses and displacements decrease at infinity. The symmetry of the problem against the location of the crack plane and a lack (or equality) of tangential stresses on its coasts may reduce it to the solution of a singular integro-differential equation of first kind for the function of crack opening. To achieve this, we use a two-dimensional Fourier integral transform in variables \( x \) and \( y \), and then a
general solution of the resulting system of ordinary differential equations satisfying the boundary conditions on the crack coasts.

The same results may be achieved otherwise. The solution of elasticity problem for a half space with mixed boundary conditions (due to symmetry) may be constructed in terms of the Papkovich-Neyber functions. Then we obtain the Dirichlet problem for a half space with the solution shaped as a double-layer potential.

As a result, we obtain the following equation for definition of the crack opening function $\gamma(\xi, \eta)$:

\[
\Delta \int_{\Omega} \gamma(\xi, \eta) \frac{d\xi d\eta}{R} = -\frac{2\pi}{\theta} p(x, y), \quad (x, y) \in \Omega, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},
\]

\[R = \sqrt{(x - \xi)^2 + (y - \eta)^2}, \quad \theta = E /[2(1 - \nu)], \quad E - Young \ modulus, \quad \nu - Poisson \ ratio.
\]

Another approach is also available, i.e. reduction of the problem to dual integral equations [4]. The work [3] deals with studying the concentration of pressure in the vicinity of the cracks in semi-limited and limited elastic bodies. In such cases, the issues of mutual influence of the systems of cracks and the borders of the bodies are reduced to the analysis of some corresponding integro-differential equations having a similar, but more complicated structure.

2. Statement of problems

We have received some integral equations of the problems for cracks with arbitrarily symmetrical form in the plane. All of them have the following structure:

\[
\Delta \int_{\Omega} \gamma(\xi, \eta) \frac{d\xi d\eta}{R} + \int_{\Omega} \gamma(\xi, \eta) S(x, y, \xi, \eta) d\xi d\eta = -\frac{2\pi}{\theta} p(x, y)
\]

These equations result from applying the generalized Fourier integral transform [1] or the generalized Fourier decompositions [3]. The singular part of the kernel corresponds to the problem of isolated crack in infinite space. The regular part corresponds to any geometrical or physical factors.

Thus, with two symmetrical cracks in the infinite body (Fig.1a) it has the following structure (problem 1):

\[S(x, \xi, y, \eta) = q(x + \xi + 2h, \eta - y), \quad q(\alpha, \beta) = (\alpha^2 + \beta^2)^{-3/2}
\]

For the problem of a periodical (chain), or a two-periodical system of cracks (Fig.2, 3a) (problems 2, 3). The following periodicity of the function $p(x, y)$: $p(x, y) = p(x + 2mh_1, y + 2nh_2)$, $(m, n = \pm 1, \pm 2, \ldots)$ is assumed. As all the cracks are in identical conditions, the function $\gamma(x, y)$ is also two-periodical with the according periods, hence the system of integro-differential equations may be reduced to a single equation (2) in the area occupied by one of cracks.

\[S(x, \xi, y, \eta) = s(x - \xi, y - \eta); \quad s(\alpha, \beta) = \sum_{m, n = -\infty}^{\infty} q(\alpha + 2mh_1, \beta - 2nh_2) + q(\alpha + 2mh_1, \beta + 2nh_2)
\]

For the problem of the crack (or a system of cracks) in the elastic half-space or the layer (Fig.4, 5).
\[ S(x, \xi, y, \eta) = s(\xi - x, y, \eta); \quad s(\alpha, y, \eta) = 18y\eta J - \frac{\varepsilon_\alpha^2 - 7\varepsilon_\alpha^2}{2\varepsilon_2} J - \]
\[ - \frac{3(y + \eta)}{\varepsilon_1^2\varepsilon_2^2\alpha^2} \left[ \frac{1}{J} - \frac{(y + \eta)^5}{2} - \frac{\eta^2(y + \eta)^3}{2} \right] - C_1(y + \eta)^2, \quad J = \left[ (y + \eta)^2 + \alpha^2 \right]^{5/2}, \]
\[ C_1 = (4\varepsilon_1^2\varepsilon_2 - 18\varepsilon_1 - 43\varepsilon_2)(4\varepsilon_1^2\varepsilon_2)^{-1}; \quad \varepsilon = \frac{1 - 2\nu}{2(1 - \nu)}; \quad \varepsilon_1 = \frac{1 - 2\nu}{2(1 - \nu)}; \quad \varepsilon_2 = \frac{1 - 2\nu}{2(1 - \nu)}; \quad \varepsilon_3 = \frac{\nu}{1 - \nu}. \]

It noteworthy that the crack is located in the plane lying perpendicular to the border. This equation was obtained by using a generalized two-dimensional Fourier-transform [1].

For the case where the system of two symmetrical cracks lies in the same plane (problem 6, scheme in Fig.8):

\[ S(x, \xi, y, \eta) = s(x + \xi + 2h_2, \eta - y) + s(\xi - x, y + h_1, \eta + h_1) + s(\xi + x + 2h_2, y + h_1, \eta + h_1) \]

(6)

For the problem of crack lying in the middle plane of the layer (Fig. 6a).

\[ S(x, \xi, y, \eta) = \frac{F(R/h)}{h^3} F(t) = \int_0^\infty [\mathcal{H}(u) - u]J_0(ut)du, \quad H(u) = u(\cosh 2u + 2u)(\cosh 2u - 1)^{-1} \]

(7)
In the cases with two symmetrical or periodical systems of cracks in the layer, corresponding regular kernels are formed from (4), (5), like in (2), (3).

If we carry out some limiting transitions in the equations (1) – (7) by directing the values of one semiaxe of cracks to infinity and fixing the other, then integro-differential equations have to be transformed to some corresponding equations for the flat analogues of the problems under consideration [9], (Fig. 1 – 6b).

3. Treatment

In solving the integro-differential equations of kind (2), a well-known method of “big $\lambda$” may be applied [1]. For this purpose, we imbed the dimensionless parameter $\lambda = h / a$ ($\lambda > 1$) and also present the regular part of the kernel as the following decomposition: $S(x, y, \xi, \eta) = \sum_{n=3}^{\infty} s_n (x, \xi, y, \eta) \lambda^{-n}$.

The solution of this integro-differential equation can be found in the form of a similar asymptotical decomposition $\gamma(x, y) = \sum_{m=0}^{\infty} \gamma_m (x, y) \lambda^{-m}$.

Substitution the mentioned decompositions in the equation (2) and equating of the members at identical degrees of $\lambda$ reduces the problem to an infinite system of consistently resolved integro-differential equations:

$$\begin{align*}
\Delta \iint_{\Omega_i} \gamma_0(\xi, \eta) \frac{d\xi d\eta}{R} = -\frac{2\pi}{\theta} p(x, y), & \quad \gamma_0(x, y) = 0 \text{ at } (x, y) \in L_1 \text{ (contour of crack)}, \quad \gamma_1 = \gamma_2 = 0 \\
\Delta \iint_{\Omega_i} \gamma_3(\xi, \eta) \frac{d\xi d\eta}{R} = -\iint_{\Omega_i} \gamma_0(\xi, \eta) s_3(x, \xi, y, \eta) d\xi d\eta, & \quad \gamma_3(x, y) = 0 \text{ at } (x, y) \in L_1, \quad \text{etc.} \quad (8)
\end{align*}$$

In the cases of more complicated problems the solution is constructed in the form of decomposition on system of parameters.

4. Numerical results and discussions

If the relative distances between cracks or between the crack and the boundary of the body, or any other physical or geometrical factors are sufficiently large, it is possible to obtain asymptotic solutions in the form of decomposition of a small parameter. We have received the values of the stress – intensity factor in all the cases for
elliptical-shaped cracks. The solutions obtained have a sufficiently simple structure. However, direct calculations convince us that within this range of geometric parameters, the interaction between cracks (or influence of boundary) is sufficiently small. However, they are usually not effective where the distances between cracks (or between a crack and the boundary of the body) are approximately equal to their sizes or bigger.

Let's define the stress-intensity factor $K_{Ia}$ at point A of the contour of a crack by applying the following formula:

$$K_{Ia} = K_{Ic}N,$$

$$K_{IIa} = \frac{pA\sqrt{\pi}}{E(k)\sqrt{ab}}(a^2\sin^2 \varphi + b^2\cos^2 \varphi)^{1/4},$$

$$N = 1 + \sum_{k=3}^{\infty} A_k A^{-k}, \quad (9)$$

where $K_{Ic}$ – the corresponding value for an isolated elliptical crack in an infinite elastic body [7]. $N$ – stress-intensity magnification factor, that shows the effect of other cracks or the boundaries of the body. The coefficients $A_k$ for each problem are derived from corresponding expressions.

Thus, the construction of the solutions of the problems listed above for the cases of small relative distances remain very essential.

From the viewpoint of the fracture strength criterion [6], the points $A$, where the stress intensity factor becomes maximal, are of special interest. Another non-dimensional parameter $\mu = \delta / \rho$, was introduced to reflect the relative distance in this case ($\delta$ – minimum distance between crack-borders in these points and their neighborhoods, $\rho$ - radius of curvature of a contour). The described problems (a) and the correspondence plane analogs (b) are schematically presented in Fig. 1–6.

It should be noted that in the plane analogues of the problems listed [9], the corresponding decompositions (8) begins with the members $k=2$ ($k=3$ in a spatial case). On the other hand, the contours of stripe cracks contain the inside elliptical cracks, therefore these asymptotic estimations for small relative distances, like in the case of big ones, will be estimations from the above.

The asymptotic solutions received for big relative distances together with the aforementioned asymptotic estimations for small relative distances, make it possible to completely describe the picture of interaction between the systems of cracks and boundaries. Direct calculations, made by the author, and comparing theorems [5] illustrate the above reasoning.

To illustrate the results of the investigation of the problems discussed in Fig.7, we present the values of parameter $N$, in the problems of two symmetrical cracks (1), about the chain (2) and twice-periodical system of cracks (3). Calculations have been made for the point of contour A with minimal curvature. $N_1, N_3$ are the corresponding values of the flat analogues of the problems.

It is easy to notice that the solution of flat problem $N_3$ can be made by extrapolating the curve 3, and $N_1$ – curves 1 and 2. The same conclusion can be reached, if we make a corresponding limiting transition directly in the integro-differential equations (2, 3 and 4).

Let's notice that at the equality of all the corresponding values of the geometrical parameters, from the viewpoint of the power criterion of destruction [6], a greatest danger comes from the twice-periodic system of cracks. For comparison, the hatched line in the drawing presents the numerical solution of the problem 1 by the “mass forces” method [10]. The divergence of results does not exceed 2 %.

The results of the calculations of parameter $N$ at points $A$ and $B$ of the contour of a crack co-operating with another similar crack and the border of semi-space (problem 6) are presented on Fig. 8. In the calculations it is accepted that: $b/a=0.5; \lambda_1 = \lambda_2 = 2; \nu = 0.5$.

Analyzing the values of the parameters $N$ and $K$ at points $A$ and $B$, we may conclude that in this case, at achieving the critical value by loading, it is necessary to expect an initial association of cracks into the one that comes out on the surface. The leading role here is obviously played by the values of the stress-intensity factor at the points under consideration: $K_{L4} = \sqrt{2}K_{L5}; \quad N_A \approx N_B$. 
5. Conclusions

- Along with direct numerical methods, asymptotic methods of solving the problems of fracture mechanics are rather actual, as they allow applying both quantitative and qualitative analysis of the solutions of similar problems.
- A uniform approach to the solution of a wide range of problems of destruction mechanics has been shown. The kernel of the integrated equation in all the cases being considered has a uniform structure. The nature of the mutual influence of the systems of cracks and borders on the concentration of pressure in the contour vicinity has been establishment.
- Application of the solutions of flat analogues of the problems being considered as asymptotic solutions in the cases of small relative distances, allows obtaining estimations of the aforementioned solutions from the above.

Acknowledgements

This work was supported by RFBR (project № 10-08-00839-a).

References

[1] Alexandrov VM, Smetanin BI, Sobol BV. Thin stress concentrators in elastic bodies. Moscow: Nauka Publishers, General Editorial board for literature on Physics and Mathematics, 1993, 224 p.
[2] Stress intensity factors handbook (in 2 volumes). Editor-in-Chief Murakami Y. Pergamon press, 1987.
[3] Vatuljan AO, Sobol BV. About one effective way of construction of explosive solutions of mechanical problems for bodies of the finite sizes. Izv. Ros. Akad. Nauk, 1995, No. 6, p. 62-69.
[4] Aizikovich SM, Krenev LI, Sobol BV, Trubchik IS. An equilibrium penny-shaped crack in an inhomogeneous elastic medium. Prikl. Mat. Mekh., 2010, Vol. 74, No. 2, p. 324-335.
[5] Shifrin EI. Spatial problems of linear fracture mechanics. Moscow: Nauka Publishers, General Editorial board for literature on Physics and Mathematics, 2002, 368 p.
[6] Irwin GR. Crack-extension force for a parthrough crack I a plate. Trans. ASME, E, 1962, Vol. 29, No. 4, p. 651-654.
[7] Sneddon IN. The distribution of stress neighborhood of cracks in elastic solid. Proc. Roy. Soc. ser. A. 1946. V. 187.
[8] Sobol BV. The equilibrium of elastic space weakens by the plane system of cracks. Izv. VUZ. Sev-Kavkaz, Region. Estestv. Nauki, 1984, No. 1, p. 47-51.
[9] Savruk M. Two-dimensional problems of elasticity for bodies with cracks. Kiev: Nauk. Dumka, 1986, 324 p.
[10] Nisitani A, Murakami Y. Stress intensity factors of an elliptical crack or a semi-elliptical crack subjected to tension. Int. J. Fract., 1974, 10, No. 3, p. 353-368.