Geometrical theory of whispering gallery modes
Michael L. Gorodetsky, Aleksey E. Fomin

Abstract—Using quasiclassical approach rather precise analytical approximations for the eigenfrequencies of whispering gallery modes in convex axisymmetric bodies may be found. We use the eikonal method to analyze the limits of precision of quasiclassical approximation using as a practical example spheroidal dielectric cavity. The series obtained for the calculation of eigenfrequencies is compared with the known series for dielectric sphere and with numerical calculations. We show how geometrical interpretation allows expansion of the method on arbitrary shaped axisymmetric bodies.

I. INTRODUCTION
Submillimeter size optical microspheres made of fused silica with whispering gallery modes (WGM) [1] can have extremely high quality-factor, up to $10^{10}$ that makes them promising devices for applications in optoelectronics and experimental physics. Historically Richtmyer [2] was the first to suggest that whispering gallery modes in axisymmetric dielectric body should have very high quality-factor. He examined the cases of sphere and torus. However only recent breakthroughs in technology in several labs allowed producing not only spherical and not only fused silica but spheroidal, toroidal [3], [4] or even arbitrary form axisymmetrical optical microcavities from crystalline materials preserving or even increasing high quality factor [5]. Especially interesting are devices manufactured of nonlinear optical crystals. Microresonators of this type can be used as high-finesse cavities for laser stabilization, as frequency discriminators and high-sensitive displacement sensors, as sensors of ambient medium and in optoelectronical RF high-stable oscillators. (See for example materials of the special LEOS workshop on WGM microresonators [6]).

The theory of WGMs in microspheres is well established and allows precise calculation of eigenmodes, radiative losses and field distribution both analytically and numerically. Unfortunately, the situation changes drastically even in the case of simplest axisymmetric geometry, different from ideal sphere or cylinder. No closed analytical solution can be found in this case. Direct numerical methods like finite elements method are also inefficient when the size of a cavity is several orders larger than the wavelength. The theory of quasiclassical methods of eigenfrequencies approximation starting from pioneering paper [7] have made a great progress lately [8]. For the practical evaluation of precision that these methods can in principal provide, we chose a practical problem of calculation of eigenfrequencies in dielectric spheroid and found a series over angular mode number $l$. This choice of geometry is convenient due to several reasons: 1) other shapes, for example toroids [4] may be approximated by equivalent spheroids; 2) the eikonal equation as well as scalar Helmholtz equation (but not the vector one!) is separable in spheroidal coordinates that gives additional flexibility in understanding quasiclassical methods and comparing them with other approximations; 3) in the limit of zero eccentricity spheroid turns to sphere for which exact solution and series over $l$ up to $l^{8+3}$ is known [9].

The Helmholtz vector equation is unseparable [10] in spheroidal coordinates and no vector harmonics tangential to the surface of spheroid can be build. That is why there are no pure TE or TM modes in spheroids but only hybrid ones. Different methods of separation of variables (SVM) using series expansions with either spheroidal or spherical functions have been proposed [11], [12], [13]. Unfortunately they lead to extremely bulky infinite sets of equations which can be solved numerically only in simplest cases and the convergence is not proved. Exact characteristic equation for the eigenfrequencies in dielectric spheroid was suggested [14] without provement that if real could significantly ease the task of finding eigenfrequencies. However, we can not confirm this claim as this characteristic equation contradicts limiting cases with the known solutions i.e. ideal sphere and axisymmetrical oscillations in a spheroid with perfectly conducting walls [15].

Nevertheless, in case of whispering gallery modes adjacent to equatorial plane the energy is mostly concentrated in tangential or normal to the surface electric components that can be treated as quasi-TE or quasi-TM modes and analyzed with good approximation using scalar wave equations.

II. SPHEROIDAL COORDINATE SYSTEM
There are several equivalent ways to introduce prolate and oblate spheroidal coordinates and corresponding eigenfunctions [16], [17], [18]. The following widely used system of coordinates allows to analyze prolate and oblate geometries simultaneously:

$$
x = \frac{d}{2} [ (s^2 - 1 + 2l) \frac{1}{2} j^{l=2} \cos ( ) ]
$$

$$
y = \frac{d}{2} [ (s^2 - 1 + 2l) \frac{1}{2} j^{l=2} \sin ( ) ]
$$

$$
z = \frac{d}{2} ;
$$

where we have introduced a sign variable $s$ which is equal to 1 for the prolate geometry with $2 \ [1;1]$ determining spheroids and $2 \ [1;1]$ describing two-sheeted hyperboloids of revolution (Fig.1, right). Consequently, $s = 1$ gives oblate spheroids for $2 \ [1;1]$ and one-sheeted hyperboloids of revolution (Fig.1, right). $d=2$ is the semidistance between focal points. We are interested in the modes inside a spheroid adjacent to its surface in the equatorial plane. It is convenient to designate a semiaxis in this plane as $a$ and in the $z$-axis of rotational symmetry of the body as $b$. In this case...
\[ d^2 = \sum\frac{a^2}{x^2} \] and eccentricity \( e = \frac{b}{a} \). The scale factors for this system are the following:

\[
\begin{align*}
\frac{\partial}{\partial r} & \left( \frac{1}{r^2} \right) + \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta \cos \phi} \right) = 0; \\
\end{align*}
\]

The scalar Helmholtz differential equation is separable

\[
+ \kappa^2 = 0; \quad \text{(3)}
\]

where \( \kappa = k d = 2 \). The solution is \( R_n(x) \), \( S_n(x) \), \( e^{i n \theta} \) where radial and angular functions are determined by the following equations:

\[
\begin{align*}
\frac{\partial}{\partial r} & \left( \frac{1}{r^2} \right) R_n + n^2 R_n = 0; \\
\end{align*}
\]

Here \( n_1 \) is the separation constant of the equations which should be independently determined and it is a function on \( m_1 \) and \( c \). With substitution \( m = 2 \pi \), \( n = 2 \pi \) the first equation transforms to the equation for the spherical Bessel function \( n_1 \) of \( \kappa r \) if \( m = 2 \). In which case the second equation immediately turns to the equation for the associated Legendre polynomials \( P_m^l(\cos \theta) \) with \( l = n_1 + 1 \). That is why spheroidal functions are frequently analyzed as decomposition over these spherical functions.

The calculation of spheroidal functions and of \( n_1 \) is not a trivial task [19], [20]. The approximation of spheroidal functions and their zeroes may seem more straightforward for the calculation of eigenfrequencies of spheroids, however we found that another approach that we develop below gives better results and may be easily generalized to other geometries.

### III. EIKONAL APPROXIMATION IN SPHEROID

The eikonal approximation is a powerful method for solving optical problems in inhomogeneous media where the scale of the variations is much larger than the wavelength. It was shown by Keller and Rubinow [7] that it can also be applied to eigenfrequency problems and that it has very clear quasiclassical ray interpretation. It is important that this quasiclassical ray interpretation requiring simple calculation of the ray paths along the geodesic surfaces and application of phase equality (quantum) conditions gives precisely the same equations as the eikonal equations. Eikonal equations allow, however, to obtain more easily not only eigenfrequencies but field distribution also.

In the eikonal approximation the solution of the Helmholtz scalar equation is found as superposition of straight rays:

\[
\psi(r) = A_0 (r) e^{ik \psi(r)}; \quad \text{(7)}
\]

The first order approximation for the phase function \( \psi \) called eikonal is determined by the following equation.

\[
(\kappa \psi)^2 = \psi; \quad \text{(8)}
\]

where \( \kappa \) is optical susceptibility. For our problem of searching for eigenfrequencies does not depend on coordinates, \( \psi \) inside the cavity and \( \psi = 1 - \text{outside} \). Though the eikonal can be found as complex rays in the external area and stitched on the boundary as well as ray method of Keller and Rubinow [7], [8], [21] can be extended for whispering gallery modes in dielectric bodies in a more rigorous way [22]. To do so we must account for an additional phase shift on the dielectric boundary. Fresnel amplitude coefficient of reflection [23]:

\[
R = \frac{\cos \frac{\pi}{2} \cos^2 \gamma}{\cos \frac{\pi}{2} \sin^2 \gamma} \quad \text{for quasi-TE modes and} \quad \gamma = n_2 \sin \gamma \text{for quasi-TM modes.}
\]

However direct use of this phase shift in the equations for internal rays as suggests in [22] leads to incorrect results. The reason is a well known Goos-Hänchen effect — the shift of the reflected beam along the surface. The beams behave as if they are reflected from the surface hold away from the real boundary at \( r = \frac{\pi}{2 \kappa \cos \gamma} \). That is why we may substitute the problem for a dielectric body with the problem for an equivalent body enlarged on \( r \) with the totally reflecting boundaries. The parameters of equivalent spheroid are marked below with overbars.

The eikonal equation separates in spheroidal coordinates if we choose \( S = S_1(\theta) + S_2(\theta) + S_3(\theta) \):

\[
\begin{align*}
\frac{2}{s} \frac{\partial S_1(\theta)}{\partial \theta} & + \frac{1}{s} \frac{\partial S_2(\theta)}{\partial \theta} + \frac{\partial^2 S_3(\theta)}{\partial \theta^2} = \frac{n^2 \kappa^2}{4}; \quad \text{(11)}
\end{align*}
\]

After immediate separation of \( \frac{4s}{\kappa} = \) we have:

\[
\begin{align*}
\frac{2}{s} \frac{\partial S_1(\theta)}{\partial \theta} & + \frac{1}{s} \frac{\partial S_2(\theta)}{\partial \theta} + \frac{\partial^2 S_3(\theta)}{\partial \theta^2} = \frac{n^2 \kappa^2}{4}; \quad \text{(12)}
\end{align*}
\]
Introducing another separation constant solutions:

\[
\begin{align*}
\frac{\partial S_1}{\partial \vartheta} &= \frac{n^2 \varpi^2}{4} \left( \frac{2}{s} \right) \frac{2}{\vartheta} \frac{s^2}{s} \left( \frac{s^2}{s} \right)^{1/2} \frac{(1 - s^2)}{s} \left( \varphi \right) \\
\frac{\partial S_2}{\partial \vartheta} &= \frac{n^2 \varpi^2}{4} \left( \frac{2}{s} \right) \frac{2}{\vartheta} \frac{s^2}{s} \left( \frac{s^2}{s} \right)^{1/2} \frac{(1 - s^2)}{s} \left( \varphi \right)
\end{align*}
\]

which after some manipulations transform to:

\[
\begin{align*}
\frac{\partial S_1}{\partial \vartheta} &= \frac{n d}{2} \left( \frac{2}{r} \right) \left( \frac{2}{s} \right) \left( \frac{s^2}{s} \right) \\
\frac{\partial S_2}{\partial \vartheta} &= \frac{n d}{2} \left( \frac{2}{r} \right) \left( \frac{2}{s} \right) \left( \frac{s^2}{s} \right)
\end{align*}
\]

where

\[
\begin{align*}
\left( \frac{2}{r} \right) &= \frac{(1 + s)}{4s} \\
\left( \frac{2}{r} \right) &= \frac{(1 + s)}{4s} + \frac{2s}{4s} \frac{2s}{s}
\end{align*}
\]

\[
\begin{align*}
\left( \frac{2}{r} \right) &= \frac{1 + s}{s} \left( \frac{s^2}{s} \right)
\end{align*}
\]

where \( s^2 = \frac{n^2 \varpi^2}{4}, \) \( s^2 = 1 \) and \( s^2 = 2. \) It is now the time to turn to the quasiclassical ray interpretation [7], [8]. The equation for the eikonal describes the rays that can spread inside spheroid along the straight line. These are the rays that freely go inside spheroid than touch the surface and reflect. For the whispering gallery modes the angle of reflection is close to \( = 2. \) The closest to the center points of these rays form the caustic surface which is the ellipsoid determined by a parameter \( = 2. \) The rays are the tangents to this internal ellipsoid and follow along the geodesic lines on it. In case of ideal sphere all the rays of the same family lie in the same plane. However, even a slightest eccentricity removes this degeneracy and inclined closed circular modes which should be more accurately called quasimodes [24] are turned into open-ended helices winding up on caustic spheroid precessing [25], and filling up the whole region as in a clew. The upper and lower points of these trajectories determine other caustic surface with a parameter \( = 2, \) which should be conserved as well as the kinetic energy (velocity). That is why \( = 2, \) is simply equal to the sine of the angle between the equatorial plane and the trajectory crossing the equator and at the same time it determines the maximum elongation of the trajectory from the equator plane. If all the rays touch the caustic or boundary surface with phases that form stationary distribution (that means that the phase difference along any closed curve on them is equal to integer times \( 2 \)), then the eigenfunction and hence eigenfrequency is found.

To find the circular integrals of phases \( k S \) we should take into account the properties of phase evolutions on caustic and reflective boundary. Every touching of caustic adds \( = 2 \) (see for example [8]) and reflection adds \( = 2. \) Thus for \( S_1 \) we have one caustic shift of \( = 2 \) at \( = 2 \) and one reflection from the equivalent boundary surface \( s \) (at the distance from the real surface), for \( S_2 \) – two times \( = 2 \) due to caustic shifts at \( = 2, \) and we should add nothing for \( S_3. \)

\[
\begin{align*}
k S_1 &= 2k \left( \frac{Z}{S_1} \right) - 2 \left( q + 1 = 2 \right) \\
k S_2 &= 2k \left( \frac{Z}{S_2} \right) - 2 \left( q + 1 = 2 \right) \\
k S_3 &= k \left( \frac{Z}{S_3} \right) - 2 \left( q + 1 = 2 \right)
\end{align*}
\]

where \( q = 1, 2, 3, \ldots \) is the order of the mode, showing
the number of the zero of the radial function on the surface, and 
\( p = 1 \) \( \text{for} \) 0; 1; 2; \ldots. These conditions plus integrals (14) completely coincide with those obtained by Bykov [26], [27], [28] if we transform ellipsoidal to spheroidal coordinates, and have clear geometrical interpretation. The integral for \( S_1 \) corresponds to the difference in lengths of the two geodesic curves on \( r \) between two points \( P_1 = (r_1; \theta_1; \varphi_1) \) and \( P_2 = (r_2; \theta_2; \varphi_2) \). The first one goes from the caustic circle of intersection between \( r \) and \( r \) along \( r \) to the boundary surface \( s \), reflects from it, and returns back to the same circle. The second is simply the arc of the circle between \( P_1 \) and \( P_2 \). The integral for \( S_2 \) corresponds to the length of a geodesic line going from \( P_1 \) along \( r \), lowering to \( r \) and returning to \( r \) at \( P_2 \) minus the length of the arc of the circle between \( P_1 \) and \( P_2 \). The third integral is simply the length of the circle of intersection of \( r \) and \( r \).

These are elliptic integrals. For the whispering gallery modes when \( r = 1 \) and \( r \), \( S_2 \) may be expanded into series over \( r \) and \( r \) and integrated with the substitutions \( \frac{r}{r} = \sin \), \( \frac{r}{r} = \cos \) and Finally, expressing spheroidal coordinates \( r \) and expressing \( 0 \) through parameters of spheroid, we have:

\[
S_1 = \frac{\pi b}{2a^2} \int_{\pi}^{0} 1 + \left( \frac{2}{3} \right)^2 \frac{1}{\sin^2 \theta \sin^2 \varphi} \, dp \quad \text{and} \quad S_2 = \frac{\pi b}{2a^2} \int_{\pi}^{0} 1 + \left( \frac{2}{3} \right)^2 \frac{1}{\sin^2 \theta \sin^2 \varphi} \, dp
\]

Finally, expressing \( \frac{r}{r} \) and \( \frac{r}{r} \) with totally reflecting boundaries for eigenfrequencies of TE modes in spheroids with different eccentricities with totally reflecting boundaries for \( l = m = 100 \) (Fig.2). Significant improvement of our series is evident. The divergence of the series for large eccentricities is explained by the fact that the approximation that we used to calculate the integrals [16], but not the method itself breaks down in this case. Namely \( r \) becomes comparable to \( \text{should not be treated as a small parameter.}

Using the method of sequential iterations, starting for example from \( n(a) = 1 \) \( \text{for} \) 0; 1; 2; \ldots. This system may be resolved:

\[
\begin{align*}
2 = \frac{(2p+1)a}{b} & \quad \text{and} \quad \frac{q \sqrt{2}}{b} = \frac{2}{b^2} \quad \frac{1}{2} + O \left( l \right)^3 \\
0 = \frac{q \sqrt{2}}{b} & \quad \text{and} \quad \frac{q \sqrt{2}}{b} = \frac{2}{b^2} \quad \frac{1}{2} + O \left( l \right)^3 \\
2 = \frac{q \sqrt{2}}{b} & \quad \text{and} \quad \frac{q \sqrt{2}}{b} = \frac{2}{b^2} \quad \frac{1}{2} + O \left( l \right)^3
\end{align*}
\]

Now we should solve the following system of equations:

\[
\begin{align*}
k S_1 &= 2k S_1 (0) \\
&= \frac{2p \pi b}{3a^2} \int_{\pi}^{0} 1 + \left( \frac{2}{3} \right)^2 \frac{1}{\sin^2 \theta \sin^2 \varphi} \, dp \\
&= 2 (q = 1) = 4 \\
k S_2 &= k S_2 (0) \\
&= \frac{2p \pi b}{3a^2} \int_{\pi}^{0} 1 + \left( \frac{2}{3} \right)^2 \frac{1}{\sin^2 \theta \sin^2 \varphi} \, dp \\
&= 2 (q = 1) = 4 \\
k S_3 &= k = 2 \frac{\pi b}{3a^2} \int_{\pi}^{0} 1 + \left( \frac{2}{3} \right)^2 \frac{1}{\sin^2 \theta \sin^2 \varphi} \, dp \\
&= 2 (q = 1) = 4
\end{align*}
\]

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k S_2 &= k S_2 (0) \\
&= \frac{2p \pi b}{3a^2} \int_{\pi}^{0} 1 + \left( \frac{2}{3} \right)^2 \frac{1}{\sin^2 \theta \sin^2 \varphi} \, dp \\
&= 2 (q = 1) = 4 \\
k S_3 &= k = 2 \frac{\pi b}{3a^2} \int_{\pi}^{0} 1 + \left( \frac{2}{3} \right)^2 \frac{1}{\sin^2 \theta \sin^2 \varphi} \, dp \\
&= 2 (q = 1) = 4
\end{align*}
\]
solution even better we may just formally use \( q \) instead of \( q^* \). 2) Minor difference in the last term is caused, we think, by misprint in [9], where in our designations instead of \( c^2 = 2^{3/2}h^2 (3 + 2^2) \) \( q=6 \) should be \( c^2 = 2^{3/2} (3r^2 + 2^2) \). The eikonal equation for the sphere may be solved explicitly and the expansion of the solution shows that quasiclassical approximation breaks down on a term \( O (L^{-1}) \), and of the same order should be the error introduced by substitution of vector approximations.

It is interesting to note that when \( a = 2b \) (oblate spheroid with the eccentricity \( e = \frac{c}{a} \)), the eigenfrequency separation in the first order of approximation between modes with the same \( l \) determined by the third term becomes equal to the separation between modes with different \( l \) and the same \( m \) (free spectral range). This difference appears only in the term proportional to \( O (L^{-2}) \). This situation is close to the case that was experimentally observed in [3]. This new degeneracy has simple quasigeometrical interpretation – like in case of a sphere geodesic lines inclined to the equator plane on such spheroid are closed curves returning at the same point of the equator after the whole revolution, crossing, however, the equator not twice as big circles on a sphere but four times.

IV. ARBITRARY CONVEX BODY OF REVOLUTION

To find eigenfrequencies of whispering gallery modes in arbitrary body of revolution one may use directly the results of the previous section by fitting the shape of the body in convex equatorial area by equivalent spheroid. In fact the body should be convex only in the vicinity of WG mode itself. For example a torus with a circle of radius \( R_0 \) with its center at a distance \( R_0 \) from the \( z \) axis as a generatrix may be approximated by a spheroid with \( a = R_0 + R_0 \) and \( b = \frac{R_0 + R_0}{2} \). Nevertheless, more rigorous approach may be developed.

The first step is to find families of caustic surfaces. This is not a trivial task in general but it is equivalent to finding caustic curves for the plane curve forming the body of revolution which is in fact the so-called biinvolute curve (the difference in length between a sum of two tangent lines from a point on a curve to a biinvolute curve and an arc between these lines is constant). Unfortunately we can not give now the formal proof of this statement but it looks true. Another family form curves orthogonal to the first family. For example in case of torus these families are concentric circles and radii, that is why caustic surfaces for a torus are also concentric toruses.

In general case the following approximation may be used to find the first family of biinvolute curves [8]:

\[
n(s) = 1 + s^3 \quad (s) + \varnothing (s^2) \quad (20)
\]

where \( n(s) \) is the normal distance from a point \( s \) on the curve to a biinvolute curve, \( s \) is a parameter of a family, and \( \varnothing \) is the radius of curvature of the initial curve at \( s \).

Let we have found a caustic surface from the first family, parametrized as

\[
\begin{align*}
z &= u \\
x &= g(u) \cos \\
y &= g(u) \sin
\end{align*}
\]

A geodesic line for this surface is given by the following integral:

\[
d s = \sqrt{1 + g''(u)^2} \quad \frac{d u}{q(u)} = \frac{p}{q(u)} \quad (22)
\]

where \( q(u) \) is some constant, which is equal in our case \( \tau = g(\omega, x) \) - the radius of caustic circle at maximum distance from equatorial plane. The length of geodesic line:

\[
L_1 = 2 \int_{u_1}^{u_2} \sqrt{1 + g''(u)^2} \quad \frac{d u}{q(u)} = \frac{p}{q(u)} \quad (23)
\]

The length of geodesic line, connecting points \( 1 \) and \( 2 \):

\[
L_2 = 2 \int_{u_1}^{u_2} \sqrt{1 + g''(u)^2} \quad \frac{d u}{q(u)} = \frac{p}{q(u)} \quad (24)
\]

The length of arc from \( 0 = 0 \) to \( c = 2R_0 \quad r \quad d u \quad d u \) is equal to \( L_2 = \tau c \).

\[
L_2 = 2 \int_{u_1}^{u_2} \sqrt{1 + g''(u)^2} \quad \frac{d u}{q(u)} = \frac{p}{q(u)} \quad (25)
\]

Finally:

\[
nk(L_1^2, L_2^2) = 2nk \quad \frac{\tau}{u_1} \quad \frac{\tau}{u_2} \quad \frac{\tau}{(p + 1=2)} \quad (26)
\]

In analogous way for another geodesic line on a caustic surface from the other family:

\[
\begin{align*}
z &= v \\
x &= h(v) \cos \\
y &= h(v) \sin
\end{align*}
\]

we have

\[
nk(L_1^2, L_2^2) = 2nk \quad \frac{h''}{h} \quad \frac{h''}{h} \quad (27)
\]

The third condition is:

\[
2nk \quad \tau = 2 \quad m \quad (29)
\]

With the substitution \( u = \frac{\tau}{p} \quad v = \frac{\tau}{p} \quad g(u) = \frac{\tau}{p} \quad s \quad \frac{\tau}{p} \quad (20,22,23) \) we again obtain expressions for spheroid obtained before.

For a torus caustic surfaces are toruses and cones and are determined by the equations:

\[
\begin{align*}
g(u) &= R_0 + \frac{p}{r} \quad \frac{r}{2} \\
h(v) &= R_0 + \frac{v}{r} \quad (r, R_0)
\end{align*}
\]

In conclusion. We have analyzed quasiclassical method of calculation of eigenfrequencies in spheroidal cavities and found that it gives approximations correct up to the term proportional to \( L^{-2} \). This method may be easily expanded on arbitrary convex bodies of revolution.
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