Radiation intensity and polarization in atmosphere with chaotic magnetic field

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Abstract
We consider the influence of Faraday rotation in chaotic magnetic field on the intensity and linear polarization of multiple scattered radiation. All fluctuations are assumed as Gaussian type and isotropic. The values of magnetic field are less $10^5$ Gauss. In this case the parameter $\omega_B/\omega = (eB/m_ee)/\omega \simeq 0.93 \cdot 10^{-8}(\mu G)$ is small and the radiation scatterings are the usual scatterings on free electrons. Only the Faraday rotation influences on the intensity and polarization of radiation. We consider the Milne problem in the electron magnetized atmosphere without the mean magnetic field. We found that chaotic magnetic field does not increase the number of $H$-functions, describing the Milne problem. There are four $H$-functions as in the Milne problem for non-magnetized case. The calculations demonstrate that the polarization of outgoing radiation diminishes strongly with the increasing of the level of magnetic fluctuations. The calculations can be used for estimations of inclination angle $i$ of accretion discs and the level of magnetic fluctuations.

Key words: radiative transfer, scattering, polarization, Faraday rotation

1 Introduction
The turbulent motions in stellar magnetized atmospheres, accretion discs and other objects are widespread phenomena (see, for example, Rothshtein & Lovelace 2008 and Penna et al. 2010). It is known that in electron plasma the frozen magnetic field occurs (see Landau & Lifshitz 1984). There are many MHD waves in magnetized plasma, which also acquire stochastic character (see Priest 1982). The Milne problem corresponds to multiple scattering of non-polarized thermal radiation going from the optically thick level of atmosphere. The polarization occurs by the last scatterings before escape the atmosphere.

The Milne problem for polarized radiation in non-magnetized electron atmosphere firstly was solved by Chandrasekhar 1947 and Sobolev 1949 (see also Chandrasekhar 1960 and Sobolev 1969). For regular magnetic field perpendicular to the surface the Milne problem was derived by Silant’ev 1994; Agol & Blues 1996; Agol et al. 1998; Shternin et al. 2003. The considered problems are axially symmetric.

The general theory of radiative transfer in stochastic magnetized atmosphere was given by Silant’ev 2005. More detailed the problem of the derivation of radiative transfer in stochastic media is presented in Silant’ev et al. 2017b. The derivation of the system of nonlinear equations for $H$-functions in stochastic magnetic fields on the base of the invariance principle was given in Silant’ev 2007.

Here we derived the system of 4 $H$-functions following to generalized Sobolev’s method (see Sobolev 1969; Silant’ev et al. 2017a). Note that the Sobolev technique (the resolvent method) also works for the general case of given sources, which does not be calculated by the principle of invariance method. We present detailed data for angular distribution and degree of polarization for many values of magnetic field fluctuations. They can be used to estimate the inclination angle of accretion discs and the level of magnetic fluctuations.

We observe the intensity $I(\mu, \tau)$ and polarization $Q(\mu, \tau)$ in stochastic atmosphere as mean values over the time of observation and over the area of observation. Here $\mu = n \cdot N$ is the cosine of the angle between line of sight $n$ and the normal $N$ to the surface, $\tau$ is the optical depth in an atmosphere directed along the normal $N$. These observed values correspond to the average over statistical ensemble of realizations. The stochastic values are assumed to be of Gaussian type. We denote the averaged values by brackets $\langle \cdot \rangle$. For example: $\langle I(\mu, \tau) \rangle \equiv I_0(\mu, \tau)$, $\langle Q(\mu, \tau) \rangle \equiv Q_0(\mu, \tau)$ and $\langle U(\mu, \tau) \rangle \equiv U_0(\mu, \tau)$. The fluctuations we denote by primes: $I'(\mu, \tau)$, $Q'(\mu, \tau)$ and $U'(\mu, \tau)$. The averaged fluctuations are equal to zero, $\langle I' \rangle = 0$, $\langle Q' \rangle = 0$ and $\langle U' \rangle = 0$. 

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Our aim is to derive the radiative transfer equations for $I_0(\mu, \tau)$, $Q_0(\mu, \tau)$ and $U_0(\mu, \tau)$ from exact transfer equations, where all values are assumed to be stochastic - the absorption factor $\alpha = \alpha_0 + \alpha'$ and magnetic field $B = B_0 + B'$, etc. The procedure of derivation is presented in Silant'ev et al. 2017b in detail. According to this method, we average the exponential solutions of the transfer equations without the integral terms. This gives rise to effective absorption factors $\alpha_{eff}(l)$ for the mean intensity $I_0(\mu, \tau)$ and the effective absorption factor $\alpha_{pol}(l)$ for the Stokes parameters $Q_0(\mu, \tau)$ and $U_0(\mu, \tau)$. Using the Gaussian type of the ensemble of fluctuations, we obtain $\langle \exp[-(\alpha_0 + \alpha' z)] \rangle = \exp(-\alpha_{eff}z)$. Below we give the results of such procedure.

Note that the every turbulence is characterized by the mean length of correlations $R_l$ (the mean radius of curls in the fluid turbulence) and by the dependence of correlation on the distance $R$ between two considered points, characterized by the function $A(R)$. Our theory depends on the integrals from the function $A(R)$. These integrals we denote as $f_{0s}$, $f_{0t}$ etc. They depend weakly on the particular forms $\alpha I$, $\alpha Q$, $\alpha U$. The value $\alpha_{eff}(l)$ is taken in microns and the magnetic field $\mathbf{B}$ is the Thomson scattering cross-section, $e$ is the electron charge. Note that the number density $N_e = N_e^0 + N_e'$. Parameter $\delta$ describes the Faraday rotation of the wave electric field. The angle of rotation $\chi$ is equal to:

$$\chi = \frac{1}{2} \delta B_l || \tau_T,$$

$$\delta B_l = \frac{3 \lambda^4}{4 \pi e c} \cdot \frac{B_l}{m_e c c \omega} \approx 0.8 \lambda^2 (\mu m) B_l || (G),$$

where wavelength $\lambda$ is taken in microns and the magnetic field in gauss. $B_l || = B \cdot n$ is part of magnetic field along the direction of light propagation $n$, $\tau_T = N_e \sigma_T z$ is the Thomson optical length of $z$. We assume $\langle B^2_l \rangle = \langle B^2 \rangle / 3$.

Eq.(1) shows that absorption factor $\alpha_{eff}(l)$ is less than the mean absorption factor $\alpha_0$, i.e. the atmosphere with chaotic motions is more transparent than non-turbulent one. In opposite, the absorption factor of polarization parameters $Q$ and $U$ due to the chaotic Faraday rotations becomes greater than those in non-magnetized medium. In the Milne problem, where the semi-infinite atmosphere is considered, the first effect is not displayed, but the second is showed strongly.

2 RADIATIVE TRANSFER EQUATION

According to Dolginov et al. 1995 we have the transfer equations:

\[
\begin{align*}
(n \nabla) I(\mathbf{n}, \mathbf{r}) &= -\alpha I(\mathbf{n}, \mathbf{r}) + N_e \sigma_T B_l(\mathbf{n}, \mathbf{r}), \\
(n \nabla) Q(\mathbf{n}, \mathbf{r}) &= -\alpha Q(\mathbf{n}, \mathbf{r}) - N_e \sigma_T \mathbf{B} \cdot \mathbf{n} U(\mathbf{n}, \mathbf{r}) + N_e \sigma_T B_Q(\mathbf{n}, \mathbf{r}), \\
(n \nabla) U(\mathbf{n}, \mathbf{r}) &= -\alpha U(\mathbf{n}, \mathbf{r}) + N_e \sigma_T \mathbf{B} \cdot \mathbf{n} Q(\mathbf{n}, \mathbf{r}) + N_e \sigma_T B_U(\mathbf{n}, \mathbf{r}).
\end{align*}
\]

Here terms $B_l, B_Q$ and $B_U$ describe the scattering on free electrons. The particular form of these terms is given in Chandrasekhar 1960. Considering the Milne problem, we do not write the source terms in this system of transfer equations.

To obtain the system of equations for averaged values $I(\mathbf{n}, \mathbf{r}), Q(\mathbf{n}, \mathbf{r})$ and $U(\mathbf{n}, \mathbf{r})$, we take the value $R(\mathbf{n}, \mathbf{r}) = -Q(\mathbf{n}, \mathbf{r}) + iU(\mathbf{n}, \mathbf{r})$:

\[
\begin{align*}
(n \nabla) R(\mathbf{n}, \mathbf{r}) &= -[\alpha + i N_e \sigma_T \mathbf{B} \cdot \mathbf{n}] R(\mathbf{n}, \mathbf{r}) + N_e \sigma_T B_R(\mathbf{n}, \mathbf{r}).
\end{align*}
\]

Note that the term $(n \nabla) R(\mathbf{n}, \mathbf{r}) \equiv \mu dR/ds$ characterize the variations along the line of sight $n$. It is clear, that the characteristic length along the line of sight is the free scattering length $s \approx 1/N_e \sigma_T$. We consider the statistical ensemble of fluctuations on this length. The solution of Eq.(4) without term $B_R(\mathbf{n}, \mathbf{r})$ has the form:

\[
R(\mathbf{n}, s) = R(0) \exp \left[ - \int_0^s ds' \left( \alpha(s') + i N_e \sigma_T \mathbf{B}(s') \cdot \mathbf{n} \right) \right].
\]

This exponential function can be simply averaged according to the technique described in Silant'ev et al. 2017b. To obtain the total averaged equations for $I_0(\mathbf{n}, \mathbf{r}), Q_0(\mathbf{n}, \mathbf{r})$ and $U_0(\mathbf{n}, \mathbf{r})$ we derived the averages of $\langle N_e \sigma_T B_l(\mathbf{n}, \mathbf{r}) \rangle = \alpha_{eff} B_l^{(0)}$ and $\langle N_e \sigma_T R(\mathbf{n}, \mathbf{r}) \rangle = \alpha_{eff} (1 + h) R_0$. As a result, the system (3) transforms to the system:

\[
\begin{align*}
(n \nabla) I_0(\mathbf{n}, \mathbf{r}) &= -\alpha_{eff} I_0(\mathbf{n}, \mathbf{r}) + \alpha_{eff} B_l(\mathbf{n}, \mathbf{r}), \\
(n \nabla) Q_0(\mathbf{n}, \mathbf{r}) &= -\alpha_{eff}(1 + h) Q_0(\mathbf{n}, \mathbf{r}) + \alpha_{eff} B_Q(\mathbf{n}, \mathbf{r}), \\
(n \nabla) U_0(\mathbf{n}, \mathbf{r}) &= -\alpha_{eff}(1 + h) U_0(\mathbf{n}, \mathbf{r}) + \alpha_{eff} B_U(\mathbf{n}, \mathbf{r}).
\end{align*}
\]
\[ \frac{1}{2} \hat{A}(\mu) \int_{-1}^{1} d\mu' \, \hat{A}^T(\mu') I_0(\mu', \tau). \]  

(7)

Here we use the vector (column) notation for \( I_0(\mu, \tau) \) and \( Q_0(\mu, \tau) \). The optical depth is equal to \( \tau = \frac{\alpha_0}{\mu} \int d\zeta \). The superscript \( T \) stands for the matrix transpose. We use the factorization with the matrix (see Silant’ev et al.2017a):

\[ \hat{A}(\mu) = \hat{A}_1(\mu) + \hat{A}_2(\mu) = 
\begin{pmatrix} 1, \sqrt{C(1 - 3\mu^2)} \\ 0, 3\sqrt{C(1 - \mu^2)} \end{pmatrix} \equiv \begin{pmatrix} 1, a(\mu) \\ 0, b(\mu) \end{pmatrix}. \]  

(8)

Here \( C = 1/\sqrt{8} = 0.353553 \), \( a(\mu) = \sqrt{C(1 - 3\mu^2)} \), \( b(\mu) = 3\sqrt{C(1 - \mu^2)}. \)

The matrices \( \hat{A}_1(\mu) \) and \( \hat{A}_2(\mu) \) have the forms:

\[ \hat{A}_1(\mu) = \begin{pmatrix} 1, a(\mu) \\ 0, 0 \end{pmatrix}, \quad \hat{A}_2(\mu) = \begin{pmatrix} 0, 0 \\ 0, b(\mu) \end{pmatrix}. \]  

(9)

It is useful to introduce the vector \( K(\tau) \):

\[ K(\tau) \equiv \begin{pmatrix} K_1(\tau) \\ K_0(\tau) \end{pmatrix} = \frac{1}{2} \int_{-1}^{1} d\mu \, \hat{A}^T(\mu) I_0(\mu, \tau). \]  

(10)

The matrix product of matrices \( \hat{A}_1 \) and \( \hat{A}_2 \) is denoted as \( \hat{A}_1 \hat{A}_2 \). It is easy to verify that Eq.(7) gives rise to the conservation law of radiative flux.

3 THE INTEGRAL EQUATION FOR \( K(\tau) \)

Formal solution of Eq.(7) gives the integral dependence of \( I_0(\mu, \tau) \) on \( K(\tau) \). Substitution of this solution in Eq.(10) gives homogeneous integral equation for \( K(\tau) \):

\[ K(\tau)|_{hom} = \int_{0}^{\infty} d\tau' \hat{L}(|\tau - \tau'|) K(\tau')|_{hom}. \]

(11)

We are interested in the non-zero solution of this equation. The kernel \( \hat{L}(|\tau - \tau'|) \) has the form:

\[ \hat{L}(|\tau - \tau'|) = \int_{0}^{1} d\mu' \left[ \exp \left( -\frac{|\tau - \tau'|}{\mu} \right) \hat{\Psi}_1(\mu') + \exp \left( -\frac{|\tau - \tau'|}{\mu + h} \right) \hat{\Psi}_2(\mu') \right]. \]  

(12)

The matrices \( \hat{\Psi}_1(\mu) \) and \( \hat{\Psi}_2(\mu) \) have the forms:

\[ \hat{\Psi}_1(\mu) = \frac{1}{\mu} \hat{A}^T_1(\mu) \hat{A}_1(\mu), \quad \hat{\Psi}_2(\mu) = \frac{1}{\mu} \hat{A}^T_2(\mu) \hat{A}_2(\mu). \]  

(13)

Note that \( \hat{L}^T = \hat{L}, \hat{\Psi}_1^T = \hat{\Psi}_1 \) and \( \hat{\Psi}_2^T = \hat{\Psi}_2 \).

The general theory to calculate the vector \( K(\tau) \) was presented in Silant’ev et al. 2015. Recall, that according to this theory the vector \( K(\tau) \) has solution through the resolvent matrix \( \hat{R}(\tau, \tau') \):

\[ K(\tau) = \int_{0}^{\infty} d\tau' \hat{R}(\tau, \tau') \]  

(14)

Here and in what follows we omit the subscription "hom". The resolvent matrix obeys the equation:

\[ \hat{R}(\tau, \tau') = \hat{L}(|\tau - \tau'|) + \int_{0}^{\infty} d\tau'' \hat{L}(|\tau - \tau''|) \hat{R}(\tau'', \tau'). \]  

(15)

The kernel \( \hat{L}(|\tau - \tau'|) \) equation for \( \hat{R}(\tau, \tau') \) is symmetric: \( \hat{L} = \hat{L}^T \). This gives rise to the relation \( \hat{R}(\tau, \tau') = \hat{R}^T(\tau', \tau) \). The double Laplace transform of \( \hat{R}(\tau, \tau') \) over parameters \( a \) and \( b \) is equal to:

\[ \tilde{R}(a, b) = \frac{1}{a + b} [ \tilde{R}(a, 0) + \tilde{R}^T(0, b) + \tilde{R}(0, 0) \tilde{R}^T(b, 0)]. \]  

(16)

This property demonstrates that the matrix \( \hat{R}(\tau, \tau') \) can be calculated, if we know the matrices \( \hat{R}(\tau, 0) \) and \( \hat{R}(0, \tau) \) (see Silant’ev et al. 2015). For this reason it is useful to study the equation for \( \hat{R}(\tau, 0) \), which follows from Eq.(15):

\[ \hat{R}(\tau, 0) = \hat{L}(\tau) + \int_{0}^{\infty} d\tau' \hat{L}(|\tau - \tau'|) \hat{R}(\tau', 0) \equiv \hat{L}(\tau) + \int_{0}^{\infty} d\tau' \hat{R}(\tau', \tau) \hat{L}(\tau'). \]  

(17)

Using the particular form of \( \hat{L}(|\tau - \tau'|) \) and Eq.(16), we obtain the relation:

\[ \hat{R} \left( \frac{1}{a}, 0 \right) = \left( \hat{E} + \hat{R} \left( \frac{1}{a}, 0 \right) \right) \int_{0}^{1} d\mu' \mu' \times \left[ \left( \hat{E} + \hat{R}^T \left( \frac{1}{a}, 0 \right) \right) \hat{\Psi}_1(\mu') + \left( \hat{E} + \hat{R}^T \left( \frac{1}{a} + \frac{1}{h} \right) \right) \hat{\Psi}_2(\mu') \right]. \]  

(18)

Here \( \hat{E} \) is the unit matrix. Further in our theory we use \( a = \mu \) and \( \alpha = \mu/(1 + h) \). Introducing the new values:

\[ \hat{R} \left( \frac{1}{\mu}, 0 \right) + \hat{E} \equiv \hat{H}(\mu) = \left( \begin{array}{cc} A(\mu), & C(\mu) \\ D(\mu), & G(\mu) \end{array} \right), \]

\[ \hat{R} \left( \frac{1 + h}{\mu}, 0 \right) + \hat{E} \equiv \hat{\Phi}(\mu) = \left( \begin{array}{cc} E(\mu), & K(\mu) \\ M(\mu), & N(\mu) \end{array} \right), \]

(19)

we obtain the following equations:

\[ \hat{H}(\mu) = \hat{E} + \mu \hat{H}(\mu) \int_{0}^{1} d\mu' \frac{\hat{H}^T(\mu')(\hat{\Psi}_1(\mu') + \hat{\Psi}_2(\mu'))}{\mu' + \mu + (1 + h)\mu}, \]  

(20)

\[ \hat{\Phi}(\mu) = \hat{E} + \mu \hat{\Phi}(\mu) \int_{0}^{1} d\mu' \frac{\hat{H}^T(\mu')(\hat{\Psi}_1(\mu') + \hat{\Psi}_2(\mu'))}{(1 + h)\mu' + (1 + h)(\mu' + \mu)}. \]  

(21)

Note, that the \( H \)-functions \( E(\mu) \) and \( K(\mu) \) can be obtained from algebraic system of equations, if we know the other functions - \( A(\mu), C(\mu), D(\mu), G(\mu), M(\mu) \) and \( N(\mu) \). These functions do not occur in the Milne problem.

4 FORMULAS FOR THE MILNE PROBLEM

The specific feature of the Milne problem is that we are to solve integral equation (see Eq.(11)) for \( K(\tau) \), which has not the free term. Using the vector \( K(\tau) \), one can obtain the solution of transfer equation (7). The vector \( I_0(\mu, \tau) \) describes the emerging radiation. This vector has the form:

\[ I_0(0, \mu) = \int_{0}^{\infty} d\tau \left[ \hat{A}_1(\mu) \exp \left( -\frac{\tau}{\mu} \right) + \hat{A}_2(\mu) \exp \left( -\frac{\tau}{\mu + h} \right) \right]. \]
\[
\hat{A}_2(\mu) \exp \left( -\frac{1 + h}{\mu} \right) K(\tau),
\]
\[ (22) \]
i.e. this expression depends on the Laplace transforms of \( K(\tau) \) over variable \( \tau \).

From Eq. (22) we obtain the particular formulas:
\[
I_0(0, \mu) = \frac{1}{\mu} \left[ \tilde{K}_1 \left( \frac{1}{\mu} \right) + a(\mu)\tilde{K}_0 \left( \frac{1}{\mu} \right) \right],
\]
(23)
\[
Q_0(0, \mu) = b(\mu)\frac{1}{\mu} \tilde{K}_0 \left( \frac{1 + h}{\mu} \right).
\]
(24)

Further we follow to simple approach of Sobolev 1969, generalizing his method for the vector case and magnetized atmosphere. The final formulas depend on \( K(0) \). Taking \( \tau = 0 \) in Eq. (11), we obtain:
\[
K(0) = \int_0^\infty d\tau \hat{L}(\tau) K(\tau) \equiv \frac{1}{\mu} \int_0^1 \frac{d\mu}{\mu} \left[ \hat{\Psi}_1(\mu) K \left( \frac{1}{\mu} \right) + \hat{\Psi}_2(\mu) K \left( \frac{1 + h}{\mu} \right) \right].
\]
(25)

Now we find the relations \( K \left( \frac{1}{\mu} \right) \) and \( K \left( \frac{1 + h}{\mu} \right) \) with the matrices \( \hat{H}(\mu) \) and \( \hat{F}(\mu) \). Let us get the equation for derivative \( dK(\tau)/d\tau \), taking into account that the kernel \( \hat{L}(|\tau - \tau'|) \) depends on the difference \( (\tau - \tau') \):
\[
\frac{dK(\tau)}{d\tau} = \hat{L}(\tau) K(0) + \int_0^\infty d\tau' \hat{L}(\tau - \tau') \frac{K(\tau')}{d\tau'}. \tag{26}
\]

Eq. (26) is non-homogeneous integral equation. According to the general theory of integral equations (see Tricomi 1957, Smirnov 1964) the general solution of Eq. (26) consists of the sum of two terms - the nonzero solution of homogeneous Eq. (11) with some constant \( k \), i.e. \( k\hat{K}(\tau) \), and the solution of non-homogeneous equation \( \hat{K}(\tau) \) with free term \( \hat{L}(\tau) K(0) \). The latter is proportional to \( K(0) \) with some factor. This factor obeys Eq. (17) for \( \hat{R}(\tau, 0) \), i.e. the general solution of Eq. (26) has the form:
\[
\frac{dK(\tau)}{d\tau} = kK(\tau) + \hat{R}(\tau, 0) K(0). \tag{27}
\]

Now let us derive the Laplace transforms of this equation. First transform corresponds to \( -\tau/\mu \), and the second one corresponds to \( -\left(1 + h\right)/\tau/\mu \). Let us consider the first transform in detail. The Laplace transform of the left part of Eq. (27) is equal to:
\[
\int_0^\infty d\tau \exp \left( -\frac{\tau}{\mu} \right) \frac{dK(\tau)}{d\tau} = -k K(0) + \frac{1}{\mu} \tilde{K} \left( \frac{1}{\mu} \right). \tag{28}
\]

The Laplace transform of the right part of Eq. (27) has the form:
\[
k \tilde{K} \left( \frac{1}{\mu} \right) + \tilde{R} \left( \frac{1}{\mu} 0 \right) K(0). \tag{29}
\]

The equality of Eq. (28) with Eq. (29) gives rise to the relation:
\[
\frac{1}{\mu} \tilde{K} \left( \frac{1}{\mu} \right) = \frac{\hat{H}(\mu) K(0)}{1 - k\mu}.
\]
(30)

Here we used the relation \( \tilde{R}(1/\mu, 0) = (\hat{H}(\mu) - \hat{E}) \) (see Eq. (19)).

Analogously we obtain the another relation:
\[
\frac{1}{\mu} \tilde{K} \left( \frac{1 + h}{\mu} \right) = \hat{F}(\mu) K(0) \frac{1}{1 + h - k\mu}. \tag{31}
\]

Substituting these formulas into Eqs. (23) and (24), we obtain the final expression for \( I_0(0, \mu) \):
\[
I_0(0, \mu) = \left[ A_1(\mu)\hat{H}(\mu) + A_2(\mu)\hat{F}(\mu) \right] K(0). \tag{32}
\]

The detailed expressions from Eq. (32) are:
\[
I_0(0, \mu) = \beta(\mu) K_1(0) + \gamma(\mu) K_0(0), \tag{33}
\]

\[ Q_0(0, \mu) = b(\mu) [M(\mu) K_1(0) + N(\mu) K_0(0)], \tag{34} \]

where
\[
\beta(\mu) = A(\mu) + a(\mu) D(\mu), \tag{35}
\]
\[ \gamma(\mu) = C(\mu) + a(\mu) G(\mu). \tag{36} \]

Thus, the solution of the Milne problem depends on 4 H-functions \( \beta(\mu), \gamma(\mu), M(\mu), N(\mu) \) and the components \( K_1(0) \) and \( K_0(0) \).

From Eqs. (25), (30) and (31) we obtain the homogeneous algebraic system:
\[
K(0) = \int_0^1 \frac{d\mu}{\mu} \left[ \hat{\Psi}_1(\mu) \hat{H}(\mu) + \hat{\Psi}_2(\mu) \hat{F}(\mu) \right] K(0). \tag{37}
\]

This homogeneous equation allows us to obtain only the ratio \( K_1(0)/K_2(0) \). So, the expression \( I_0(0, \mu) \) contains an arbitrary Const. This Const can be expressed through the observed flux of outgoing radiation. Note that the angular distribution \( J(\mu) = I(0, \mu)/I(0, 0) \) and the degree of polarization \( p(\mu) = Q(0, \mu)/I(0, 0) \) are independent of Const. Note that negative \( Q(0, \mu) \) denotes that the wave electric field oscillations are perpendicular to the plane \( u_{\mathbf{n}} \).

The necessary condition to obtain \( K(0) \) is zero of the determinant of expression (35). We consider the Milne problem in conservative atmosphere. In such case \( k = 0 \).

The system of equations for 4 H-functions \( \beta(\mu), \gamma(\mu), M(\mu), N(\mu) \) is:
\[
\beta(\mu) = 1 + \frac{\mu}{2} \int_0^1 d\mu' \frac{[\beta(\mu')\beta(\mu' + \gamma(\mu')\gamma(\mu')]}{\mu + \mu'} \tag{38}
\]
\[
\gamma(\mu) = a(\mu) + \frac{\mu}{2} \int_0^1 d\mu' \left( a(\mu')[\beta(\mu')\beta(\mu') + \gamma(\mu')\gamma(\mu')] + \frac{b(\mu)[\beta(\mu)\beta(\mu') + \gamma(\mu)\gamma(\mu')]}{1 + h\mu + \mu'} \right), \tag{39}
\]
\[
M(\mu) = \frac{\mu}{2} \int_0^1 d\mu' \frac{M(\mu')\beta(\mu') + N(\mu')\gamma(\mu')}{1 + h\mu + \mu'}, \tag{40}
\]
\[
N(\mu) = 1 + \frac{\mu}{2} \int_0^1 d\mu' \frac{[(a(\mu')][M(\mu')\beta(\mu') + N(\mu')\gamma(\mu')] + \frac{b(\mu)[M(\mu')M(\mu') + N(\mu')N(\mu')]}{(1 + h)(\mu + \mu')}}{1 + h\mu + \mu'} \right). \tag{41}
\]

These equations can be obtained from Eqs. (20) and (21) by transforms to matrices \( A_1(\mu)\hat{H}(\mu) \) and \( A_2(\mu)\hat{F}(\mu) \). The Eq. (36) gives rise to the following equation for zeros moments \( \tau_0 \) and \( \tau_0' \):
\[
\tau_0 = 1 + \frac{1}{4}(\tau_0^2 + \tau_0'). \tag{42}
\]
The result of calculations of Eqs.(35)-(38) demonstrates that the linear polarization, emerging from the optically thick magnetized atmosphere, depends very strongly on the parameter \( h \) (see Eq.(1)). We calculated the angular distribution \( J(\mu) = I(\mu)/I(\mu = 0) \) and polarization degree \( p(\mu) = -Q(\mu)/I(\mu) \). The peak values of the polarization reach at \( \mu = 0 \) and the angular distribution at \( \mu = 1 \). The position angle of emerging radiation corresponds to the wave electric field oscillations parallel to the accretion disc plane, i.e. as in the case of the standard Milne problem.

Our equations at \( h = 0 \) give rise to the standard angular distribution and polarization degree (see Chandrasekhar 1960) with \( p_{\text{max}} = 11,713\% \) and \( J_{\text{max}} = 3.063 \). At \( h = 0.5 \) these values are 7.169% and 3.046, correspondingly (see Table 1). With the increasing of \( h \) the polarization tends to zero and the angular distribution tends to standard value in the Milne problem for the intensity without taking into account of polarization (\( J_{\text{max}}(1) = 3.021 \)). From Tables 2 and 3 we see, that calculations confirm the exact values \( \overline{\beta}_n = 2 \) and \( \overline{\gamma}_n = 0 \) for all values of parameter \( h \).

Table 4 gives the \( J(\mu) \) and \( p(\mu) \) - values for \( h = 0,0.5 \) and 1 . In Fig.1 we give the detailed dependence of polarization degree \( p(\mu) \) on parameter \( \mu \) for many values of \( h \). The angular distribution \( J(\mu) \) is given only for \( h = 0 \) and 20. The value \( J(1) \) at \( h = 20 \) is equal to 3.021. Thus, it is seen that \( J(\mu) \) practically is independent of parameter \( h \).

According to Eqs.(1) and (2), we have:

\[
h(\lambda_2) = \left( \frac{\lambda_2}{\lambda_1} \right)^4 h(\lambda_1),
\]

i.e. the dependence of the depolarizing factor \( h \) on the wavelength \( \lambda \) is very strong. This gives strong decreasing of polarization degree with increasing of parameter \( h \) (see Fig.1). It means also that the polarization decreases strongly with the increase of the wavelength \( \lambda \).

There are many methods to estimate the inclination angle \( i \) (cos \( i = \mu \)) of an accretion disc plane (see Axon et al. 2008; Marin 2014). All methods assume the different suggestions about physical picture of considering objects.

Thus, Afanasiev et al. (2018) assume that the optically thick accretion discs radiate according to the Milne problem in non-magnetized electron atmosphere. In this case the polarization of continuum radiation does not depend on the wavelength. They considered the objects with such polarization and used the standard Chandrasekhar’s dependence of polarization degree on the parameter \( \mu \) (see Chandrasekhar 1960).

### Table 1.

| \( h \) | \( K_0(0)/K_1(0) \) | \( J(\mu = 1) \) | \( p(\mu = 0) \) |
|---|---|---|---|
| 0  | -0.10628 | 3.063 | 11.713 |
| 1  | -0.10937 | 3.058 | 10.383 |
| 2  | -0.10879 | 3.054 | 9.332 |
| 3  | -0.10023 | 3.051 | 8.478 |
| 4  | -0.09895 | 3.048 | 7.769 |
| 5  | -0.09788 | 3.046 | 7.169 |
| 6  | -0.09697 | 3.044 | 6.656 |
| 7  | -0.09617 | 3.042 | 6.211 |
| 8  | -0.09547 | 3.041 | 5.822 |
| 9  | -0.09486 | 3.040 | 5.479 |
| 10 | -0.09431 | 3.039 | 5.174 |
| 11 | -0.09388 | 3.033 | 4.920 |
| 12 | -0.09267 | 3.029 | 4.704 |
| 13 | -0.09176 | 3.026 | 4.503 |
| 14 | -0.09176 | 3.026 | 4.503 |
| 15 | -0.09176 | 3.026 | 4.503 |
| 16 | -0.09176 | 3.026 | 4.503 |
| 17 | -0.09176 | 3.026 | 4.503 |
| 18 | -0.09176 | 3.026 | 4.503 |
| 19 | -0.09176 | 3.026 | 4.503 |
| 20 | -0.09176 | 3.026 | 4.503 |

### Table 2.

| \( n \) | \( \overline{n}_n \) | \( \overline{\tau}_n \) | \( \overline{M}_n \) | \( \overline{N}_n \) |
|---|---|---|---|---|
| 0  | 2.00000 | - 0.00000 | 0.08345 | 1.36024 |
| 1  | 1.13822 | - 0.12202 | 0.04963 | 0.70915 |
| 2  | 0.80123 | - 0.13230 | 0.03505 | 0.47951 |
| 3  | 0.61807 | - 0.14297 | 0.02699 | 0.36137 |
| 4  | 0.50248 | - 0.11462 | 0.02189 | 0.28935 |
| 5  | 0.42276 | - 0.10454 | 0.01837 | 0.24084 |

### Table 3.

| \( n \) | \( \overline{\beta}_n \) | \( \overline{\gamma}_n \) | \( \overline{M}_n \) | \( \overline{N}_n \) |
|---|---|---|---|---|
| 0  | 2.00000 | - 0.00000 | 0.00576 | 1.00606 |
| 1  | 1.14062 | - 0.09676 | 0.00362 | 0.49862 |
| 2  | 0.80414 | - 0.10262 | 0.00263 | 0.33991 |
| 3  | 0.62098 | - 0.09571 | 0.00207 | 0.24700 |
| 4  | 0.50526 | - 0.08706 | 0.00169 | 0.19663 |
| 5  | 0.42536 | - 0.07884 | 0.00143 | 0.16305 |
Following their method (with the same assumptions) we can consider the accretion discs, where continuum radiation has polarization degree rapidly decreasing with the increase of wavelength $\lambda$. Now the problem is to seek such two parameters $\mu$ and $h$ in order to obtain the observed polarization degree for many values of wavelength $\lambda$. Below we will show that sometimes this problem can be solved.

Table 5 shows that in the atmosphere with chaotic magnetic field the observed degrees of polarization $p(i)\%$ occur inside various intervals of parameter $h$. Thus, the polarization $p(i) = 2\%$ occurs according the Chandrasekhar table only at the inclination $i = 57^\circ$. But in atmosphere with chaotic magnetic field this polarization can exist in all values of $h$ from zero up to $h = 3$. The existence of these intervals gives possibility to estimate the inclination angle $i$ from two or more observed degrees of polarization.

As an example, we obtain the inclination angle $i$ for Seyfert 1 nucleus Mrk 231 (see Smith et al. 2004, Fig. 4). While Smith et al. interpreted the polarization as arising in a jet (polar scattering region), we explore the possibility that the polarization is due to the Milne problem in optically thick accretion disc with chaotic magnetic field. From Fig.4 in this paper we have: $p(\lambda = 0.51\mu m) = 5.5\%$, $p(\lambda = 0.55\mu m) = 4.8\%$, $p(\lambda = 0.58\mu m) = 4\%$. Using our Fig.1, we obtain $i \simeq 85^\circ$. The corresponding values of parameter $h$ are: 0.36, 0.5, 0.62.

Note that our aim is to demonstrate that the Milne problem in an atmosphere with chaotic magnetic fields can be used in consideration of observed polarization data.

6 CONCLUSION

The Sobolev’s technique is generalized to describe the Milne problem in the atmosphere with chaotic magnetic fields without the mean value. The radiation equation for this case has different absorption factors for intensity $I$ and the Stokes parameter $Q$. This case is described by the system of four nonlinear equations for $H$-functions $\beta(\mu), \gamma(\mu), M(\mu)$ and $N(\mu)$, as it occurs for the Milne problem without magnetic field. The solution of these equations by the method of successive approximations is very effective, if we use the exact moments of $H$-functions $\beta_0 = 2$ and $\gamma_0 = 0$. The calculations demonstrate that the chaotic Faraday rotations of the wave electric field diminishes very effectively the polarization. The results can be used by consideration of linear polarization of radiation in the optically thick chaotically magnetized atmospheres of stars, accretion discs etc. Note that in spherical stars with homogeneous atmosphere the total polarization from the star is equal to zero. The total polarization from an star can be observed if the distribution of turbulent magnetic fields is non-homogeneous. The strong dependence of polarization on the wave length $\lambda$ may
The angular distribution $J(\mu)$ and degree of polarization $p(\mu) = -Q(\mu)/I(\mu)$ in % for $h = 0, 0.5$ and 1.

| $\mu$ | $J_0$ | $p_0$ | $J_0,5$ | $p_0,5$ | $J_1$ | $p_1$ |
|-------|-------|-------|---------|---------|-------|-------|
| 0.01  | 1.036 | 10.878| 1.037   | 6.692   | 1.037 | 4.851 |
| 0.02  | 1.066 | 10.295| 1.067   | 6.351   | 1.067 | 4.618 |
| 0.03  | 1.094 | 9.805 | 1.094   | 6.061   | 1.094 | 4.418 |
| 0.04  | 1.120 | 9.374 | 1.121   | 5.805   | 1.121 | 4.240 |
| 0.05  | 1.146 | 8.986 | 1.146   | 5.573   | 1.146 | 4.079 |
| 0.06  | 1.170 | 8.631 | 1.171   | 5.360   | 1.171 | 3.930 |
| 0.07  | 1.194 | 8.304 | 1.195   | 5.164   | 1.195 | 3.793 |
| 0.08  | 1.218 | 8.000 | 1.218   | 4.980   | 1.218 | 3.664 |
| 0.09  | 1.241 | 7.716 | 1.242   | 4.809   | 1.242 | 3.543 |
| 0.10  | 1.264 | 7.449 | 1.264   | 4.647   | 1.264 | 3.429 |
| 0.15  | 1.375 | 6.312 | 1.375   | 3.956   | 1.375 | 2.939 |
| 0.20  | 1.483 | 5.410 | 1.482   | 3.404   | 1.482 | 2.544 |
| 0.25  | 1.587 | 4.667 | 1.586   | 2.947   | 1.585 | 2.214 |
| 0.30  | 1.690 | 4.041 | 1.688   | 2.560   | 1.687 | 1.932 |
| 0.35  | 1.791 | 3.502 | 1.789   | 2.225   | 1.787 | 1.688 |
| 0.40  | 1.892 | 3.033 | 1.888   | 1.933   | 1.887 | 1.472 |
| 0.45  | 1.991 | 2.619 | 1.987   | 1.674   | 1.985 | 1.280 |
| 0.50  | 2.091 | 2.252 | 2.085   | 1.443   | 2.083 | 1.107 |
| 0.55  | 2.189 | 1.923 | 2.183   | 1.235   | 2.180 | 0.951 |
| 0.60  | 2.287 | 1.627 | 2.280   | 1.047   | 2.276 | 0.809 |
| 0.65  | 2.385 | 1.358 | 2.376   | 0.876   | 2.373 | 0.679 |
| 0.70  | 2.483 | 1.113 | 2.473   | 0.720   | 2.469 | 0.559 |
| 0.75  | 2.580 | 0.888 | 2.570   | 0.575   | 2.564 | 0.448 |
| 0.80  | 2.677 | 0.682 | 2.665   | 0.443   | 2.660 | 0.346 |
| 0.85  | 2.774 | 0.492 | 2.760   | 0.320   | 2.755 | 0.251 |
| 0.90  | 2.870 | 0.316 | 2.856   | 0.206   | 2.850 | 0.162 |
| 0.95  | 2.960 | 0.232 | 2.875   | 0.184   | 2.869 | 0.144 |
| 1.00  | 3.090 | 0.249 | 2.894   | 0.162   | 2.888 | 0.128 |
| 1.05  | 3.228 | 0.216 | 2.913   | 0.141   | 2.907 | 0.111 |
| 1.10  | 3.364 | 0.184 | 2.932   | 0.120   | 2.925 | 0.094 |
| 1.15  | 3.497 | 0.152 | 2.951   | 0.099   | 2.944 | 0.078 |
| 1.20  | 3.626 | 0.121 | 2.970   | 0.079   | 2.963 | 0.062 |
| 1.25  | 3.750 | 0.090 | 2.989   | 0.059   | 2.982 | 0.046 |
| 1.30  | 3.872 | 0.060 | 3.008   | 0.039   | 3.001 | 0.031 |
| 1.35  | 4.000 | 0.030 | 3.027   | 0.019   | 3.020 | 0.015 |
| 1.40  | 4.000 | 0.030 | 3.046   | 0.019   | 3.039 | 0.015 |

The angular distribution $P(\mu)$ is given by $J(\mu) = \frac{Q(\mu)}{I(\mu)} \times 100$.

Table 5. The dependence of observed degree of polarization $p/\%$ from values $h$ and the angle of accretion disc inclination $i^\circ$.

| $h$ | 0 | 0.5 | 1.0 | 2.0 | 3.0 | 4.0 | 5.0 | 6.0 |
|-----|---|-----|-----|-----|-----|-----|-----|-----|
| $p = 5\%$ | 77$^\circ$ | 86$^\circ$ | 90$^\circ$ | – | – | – | – | – |
| 4\% | 72$^\circ$ | 81$^\circ$ | 87$^\circ$ | – | – | – | – | – |
| 3\% | 66$^\circ$ | 76$^\circ$ | 82$^\circ$ | 89$^\circ$ | – | – | – | – |
| 2\% | 57$^\circ$ | 67$^\circ$ | 73$^\circ$ | 82$^\circ$ | 88$^\circ$ | – | – | – |
| 1\% | 45$^\circ$ | 53$^\circ$ | 59$^\circ$ | 66$^\circ$ | 72$^\circ$ | 78$^\circ$ | 81$^\circ$ | 85$^\circ$ |
| 0.5\% | 32$^\circ$ | 40$^\circ$ | 44$^\circ$ | 49$^\circ$ | 57$^\circ$ | 60$^\circ$ | 63$^\circ$ | 67$^\circ$ |

demonstrate that chaotic magnetic field really exists in an atmosphere.

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