TORSION-FREE $G_2$-STRUCTURES WITH IDENTICAL RIEMANNIAN METRIC

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Abstract

Based on a general formula due to R. Bryant, we work out the topological structure of the space of torsion-free $G_2$-structures generating the same associated Riemannian metric on a compact 7-manifold. We also identify a corresponding Lie group-theoretic structure of the space. These observations are then used to describe the moduli space of torsion-free $G_2$-structures in certain cases - by way of covering spaces.

1 Introduction

The group $G_2$ can be defined as the group of automorphisms of the Octonions $\mathbb{O}_8$. This definition can be shown to be equivalent to the subgroup of $SO(7)$ that preserves the 3-form

$$\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}$$

in $\mathbb{R}^7$, where $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$. On a 7-dimensional manifold $M$, a $G_2$-structure is a principal sub-bundle of the $GL(7, \mathbb{R})$-frame bundle over $M$, with the reduced structure group $G_2$. Analytically, a $G_2$-structure is described by a smooth 3-form $\varphi$ on $M$ resembling $\varphi_0$ in appropriate local frames. The switching of thinking between the two equivalent ways of describing $G_2$-structures is a theme throughout this paper.

Because $G_2$ is a subgroup of $SO(7)$, a $G_2$-structure is contained in a unique $SO(7)$-structure, i.e. it is associated to a unique Riemannian metric together with a unique orientation. In the analytic language, we say that a $G_2$-structure $\varphi$ generates a unique Riemannian metric $g_\varphi$. However, different $G_2$-structures may generate the same metric. It is therefore natural to ask: for a fixed Riemannian metric generated by some $G_2$-structure, what are the distinct $G_2$-structures that generate the same metric? Equivalently, we are asking for a description of the space of $G_2$-structures contained in a single $SO(7)$-structure over a 7-manifold. R. Bryant gave an answer to this general question in [1]: given a $G_2$-structure $\varphi$ on a 7-manifold $M$, for any $f \in C^\infty(M)$ and $\omega \in \Omega^1(M)$ such that $f^2 + |\omega|^2 = 1$ the 3-form $\tilde{\varphi}$ given by

$$\tilde{\varphi} = (f^2 - |\omega|^2)\varphi + 2f * (\omega \wedge \varphi) + 2\omega \wedge *(\omega \wedge *\varphi),$$
is a $G_2$-structure that generates the same Riemannian metric and orientation as $\varphi$, where the Hodge star $\ast$ and norm $| \cdot |$ are induced by the metric $g_\varphi$. Moreover, any $G_2$-structure $\tilde{\varphi}$ that induces the same metric and orientation as $\varphi$ can be written as (2) for some $f \in C^\infty(M)$ and $\omega \in \Omega^1(M)$ such that $f^2 + |\omega|^2 = 1$, unique up to replacement by $(-f, -\omega)$. Therefore, the space of $G_2$-structures that generate the same Riemannian metric is equivalent to the space of sections of a $\mathbb{RP}^7$-bundle over $M$.

On the other hand, a torsion-free $G_2$-structure is one which is preserved by parallel translation with respect to the metric it generates. Therefore there should be a more elegant description of the finer space of all torsion-free $G_2$-structures that generate the same metric and orientation. To this end, let $\varphi$ be a torsion-free $G_2$-structure on $M$ and we define the space

$$\Gamma = \{ \text{torsion-free } G_2\text{-structures on } M \text{ that generate } g_\varphi, \text{ with the same orientation as } \varphi \}.$$  

Canonically we think of $\Gamma$ as a set of 3-forms on $M$. It turns out that the topological structure of $\Gamma$ is independent of $\varphi$. Since this paper mainly concerns the topology of $\Gamma$, we omit $\varphi$ in the notation for $\Gamma$.

The main goal of this paper is to derive an analytic description of $\Gamma$ from (2) that seems to be missing in the literature. For this part of the paper we shall assume that $M$ is compact. In particular, we will see that this description implies $\Gamma$ is topologically $\mathbb{RP}^{b_1}$, where $b_1$ is the first Betti number of $M$. In fact we will show that $\Gamma$ is $C^1$-diffeomorphic to $\mathbb{RP}^{b_1}$ (see Proposition 3.2). It turns out that the topology of $\Gamma$, along with a particular group of isometries $S_0$ to be defined later, determines the topology in a particular direction on the moduli space $\mathcal{M}$ of torsion-free $G_2$-structures on a compact 7-manifold. Denote by $X$ the projected image of $\Gamma$ into $\mathcal{M}$, we will prove the following result.

**Theorem 1.1.** The projection $\Gamma \rightarrow X$ is a covering map.

Various consequences of Theorem 1.1 on the moduli space $\mathcal{M}$ will be discussed for special cases. As $\Gamma$ is homeomorphic to real projective space, understanding the topology of $X$ is inevitably related to the classic problem of classifying free group actions on spheres. If $b_1 > 1$, we will see that covering space theory also implies that, provided $\Gamma \rightarrow X$ is a normal covering, the fundamental group of $X$ must be a central extension of the group of deck transformations of $\Gamma \rightarrow X$ by $\mathbb{Z}_2$ - the fundamental group of real projective space of dimension bigger than 1. A complete understanding of the topology of $X$ hinges on a full understanding of these problems in algebra/algebraic-topology.

We would like to point out that the global topology of $\mathcal{M}$ is still unknown, and is a very important problem. In particular, geometers and physicists are very interested in the case when the underlying $M$ has full $G_2$-holonomy. In this case, $b_1 = 0$ and $X$ is just a point. Thus our results are only interesting when the underlying manifold has holonomy group properly contained in $G_2$. However, because of the availability of concrete examples when the holonomy group is properly contained in $G_2$, we believe our results still have merit in understanding the subject as a whole.

On the other hand, there is a general Lie group-theoretic description of $\Gamma$ which does not require $M$ to be compact. To be more precise, we choose a point $p \in M$ over which a frame $f$ is adapted to a torsion-free $G_2$-structure in $\Gamma$. Then we define the set

$$N_f = \{ g \in SO(7) \mid g^{-1}\text{Hol}_f(D)g \subset G_2 \},$$

(3)
where Hol\(_f(D)\) denotes the holonomy group (based at \(f\)) of the Levi-Civita connection \(D\) associated to the unique Riemannian metric of \(\Gamma\). For any chosen point-wise frame \(f\), denote by \([f]\) the orbit of the right-action by elements of \(G_2\). Then \([f]\) represents a torsion-free \(G_2\)-structure (hence an element of \(\Gamma\)) by parallel translation with respect to the induced Riemannian metric. Then every element in \(\Gamma\) can be denoted by \([f,h]\) for some \(h \in SO(7)\), where \(f,h\) denotes the canonical right-action of \(h\) on the frame \(f\). We shall demonstrate explicitly in Section 4 that

Proposition 1.2. The map \(N_f/G_2 \rightarrow \Gamma\) given by \(h G_2 \mapsto [f,h]\) is a bijection.

In other words, \(\Gamma\) is parameterized by the coset space \(N_f/G_2\). It should be pointed out that Proposition 1.2 is actually a special case of Proposition 3.1.8 in [4], although it is often ignored. Via our first result, we will show that \(N_f/G_2\) is in fact also homeomorphic to \(\mathbb{RP}^{14}\) when \(M\) is compact. An important message here is that the respective topologies of \(\Gamma\) and \(N_f/G_2\) only depend on the topology of the underlying manifold \(M\).

The organization of this paper is as follows. In the next section, we will discuss the necessary background concerning \(G_2\)-structures on 7-manifolds and the associated moduli space. In Section 3 we will derive the analytic description of \(\Gamma\) from Bryant’s formula (2) and identify the topological structure of \(\Gamma\). In Section 4, we will clarify the characterization of \(\Gamma\) that uses frames, prove the homeomorphism of \(\Gamma\) and \(N_f/G_2\) and discuss its relation to the analytic description. The background material in Section 2 is presented from the analytic point of view, while the equivalent principal bundle version will appear in Section 4 since it is more relevant there. Finally, in the last section we will prove Theorem 1.1 and discuss the insights it provides into the topology of the moduli space \(M\) in special cases. The associated free action on spheres and group extension problems will highlight the end of the paper.

2 Preliminaries on \(G_2\)-structures

A \(G_2\)-structure on a 7-manifold \(M\) is a 3-form \(\varphi\) such that there are local tangent frames \(f\) for which \(\varphi = f^* \varphi_0\), with \(\varphi_0\) as in (1). Throughout the paper \(M\) is always connected. For each such local frame \(f\) and a point \(p \in M\) in the relevant trivializing neighborhood, denote by \(f_p\) as the value of the frame at \(p\). Here we are treating \(f_p = \{e_i\}_{i=1}^{7}\) as the point-wise linear map \(f : T_p M \rightarrow \mathbb{R}^7\) given by \(f(e_i) = \bar{e}_i\), where \(\{e_i\}_{i=1}^{7}\) is a basis \(T_p M\) and \(\bar{e}_i\) is the natural unit vector. The transition functions of these local frames lie exactly in the group \(G_2\) as defined via (1). \(G_2\) acts naturally on \(\mathbb{R}^7\) and its exterior algebra, giving rise to irreducible representations. Given a \(G_2\)-structure \(\varphi\) on \(M\), these irreducible representations pass onto the spaces of differential forms. In particular, we have

\[
\Omega^2(M) = \Omega^2_1 \oplus \Omega^2_{14}
\]
\[
\Omega^3(M) = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}
\]
∇ is said to be torsion-free if
\[ \Omega_2^2 = \{ X \cdot \varphi \mid X \in \Gamma(TM) \} = \{ \beta \in \Omega^2(M) \mid * (\varphi \wedge \beta) = -2\beta \} \]
\[ \Omega^2_{14} = \{ \beta \in \Omega^2(M) \mid \beta \wedge * \varphi = 0 \} \]
\[ \Omega_3^2 = \{ f \cdot \varphi \mid f \in C^\infty(M) \} \]
\[ \Omega_3^3 = \{ X \cdot \varphi \mid X \in \Gamma(TM) \} \]
\[ \Omega_2^3 = \{ \eta \circ \varphi \mid \eta \text{ trace-less symmetric 2-tensor} \}, \quad (4) \]
where \( \circ \) is defined by the canonical linear map \( T^* M \otimes T^* M \xrightarrow{} \Omega^3(M) \) given by
\[ (b \circ \varphi)(u, v, w) = \varphi(bu, v, w) + \varphi(u, bw) + \varphi(u, v, bw) \quad (5) \]
with \( b \in \text{End}(TM) \) the \( g_\varphi \)-dual of \( b \in T^* M \otimes T^* M \). The corresponding decompositions for \( \Omega^3(M) \) and \( \Omega^2(M) \) are defined by taking the Hodge star of the irreducible components of \( \Omega^3(M) \) and \( \Omega^2(M) \), respectively. More generally, the map (5) encodes a fundamental relation between \( \Omega^2(M) \) and \( \Omega^3(M) \), as seen by the decomposition
\[ T^* M \otimes T^* M = \text{Sym}^2(TM) \oplus \Omega^2(M) \]
\[ = \left( C^\infty(M) \otimes g_\varphi \right) \oplus \text{Sym}^2_0(TM) \oplus \Omega^2_2 \oplus \Omega^2_{14} \xrightarrow{} \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}, \quad (6) \]
where \( C^\infty(M) \otimes g_\varphi \) is mapped isomorphically onto \( \Omega^3_1 \), the space of trace-less symmetric 2-tensors \( \text{Sym}^2_0(TM) \) is mapped isomorphically onto \( \Omega^3_{27} \), \( \Omega^2_2 \) is mapped isomorphically onto \( \Omega^3_7 \), and \( \Omega^2_{14} \) is the kernel of the map. In fact, \( \Omega^2_{14} \) is isomorphic to the Lie Algebra of \( G_2 \). We would also like to point out that with respect to any (point-wise) frame \( \{ e_i \}_{i=1} \) adapted to the \( G_2 \)-structure, it can be readily verified from (5) that
\[ b \circ \varphi = b_{ij} \omega_i \wedge (e_j \cdot \varphi) \]
\[ = b_{ij} \omega_i \wedge *(\omega_j \wedge \varphi), \quad (7) \]
where \( \omega_i \) is the metric dual 1-form of \( e_i \) and \( b_{ij} = b(e_i, e_j) \). This local expression is useful in many situations. In fact, we will make use of it in Section 3.

The necessary and sufficient conditions for the existence of a \( G_2 \)-structure on a 7-manifold is that it be orientable and spin, which are relatively mild topological conditions. What we are interested in are torsion-free \( G_2 \)-structures, which are a lot harder to come by. A \( G_2 \)-structure \( \varphi \) is said to be torsion-free if \( \nabla \varphi = 0 \) on \( M \), where \( \nabla \) is the Levi-Civita connection corresponding to \( g_\varphi \). It follows immediately that \( d\varphi = 0 = \delta \varphi \) if \( \varphi \) is torsion-free. It was the work of Fernandez and Gray [2] that \( d\varphi = 0 = \delta \varphi \) also implies \( \nabla \varphi = 0 \). If \( M \) is compact \( d\varphi = 0 = \delta \varphi \) is also equivalent to \( \Delta \varphi = 0 \), where \( \Delta = d\delta + \delta d \) is the Hodge Laplacian induced by \( g_\varphi \). The torsion-free condition on \( \varphi \) is also equivalent to the Riemannian holonomy group of \( g_\varphi \) being contained in \( G_2 \). We say \( M \) is a \( G_2 \)-manifold if it admits a torsion-free \( G_2 \)-structure. It can be shown that a \( G_2 \)-manifold is always Ricci-flat.

The most important consequence of having a torsion-free \( G_2 \)-structure is that the decomposition (1) descends to cohomology when \( M \) is compact. Throughout this paper, all cohomology groups have coefficients in \( \mathbb{R} \). If \( \varphi \) is torsion-free, then the corresponding Hodge Laplacian \( \Delta \) maps \( \Omega^k \) to itself. Therefore via the Hodge Theorem we can define the cohomology group
We then have

\[ H^2(M) = H^2_1(M) \oplus H^2_{14}(M) \]
\[ H^3(M) = H^3_1(M) \oplus H^3_2(\mathcal{M}) \oplus H^3_{27}(M) \]  

with the other cohomology groups defined via the Hodge star operator. It can be shown that \( H^1(M) \cong H^k_2(M) \) for \( k = 2, 3, 4, 5 \). Defining the reduced Betti numbers \( b^k_l = \dim H^k_l(M) \), we thus have \( b^k_1 = b^1 \) and that the only unknown ones are essentially \( b^2_{14} \) and \( b^3_{27} \). Note that \( b^1_1 = 1 \) always. It follows that \( b^2_{14} \) and \( b^3_{27} \) are topological as well. We also know that for a compact \( G_2 \)-manifold, \( \text{Hol}_f(D) = G_2 \) if and only if the fundamental group of \( M \) is finite (see [4]), and therefore \( \text{Hol}_f(D) = G_2 \) implies \( b^1 \) = 0.

One of the most seminal results in \( G_2 \)-geometry was D. Joyce’s construction of compact \( G_2 \)-manifolds of full \( G_2 \)-holonomy (see his book [4]). The examples he painstakingly constructed were non-explicit, due to the fact that such manifolds do not have any “\( G_2 \)-symmetry” (see [7] for an explanation). On the other hand, there are well-known concrete examples when the holonomy group is properly contained in \( G_2 \):

1. \( S^1 \times Y \) with \( \varphi = d\theta \wedge \omega + \text{Re} \Omega \), where \( Y \) is a Calabi-Yau 3-fold with Kähler form \( \omega \) and \( \Omega \) its holomorphic volume form. The holonomy group is \( SU(3) \).
2. \( T^3 \times Y \) with \( \varphi = dx_{123} + dx_1 \wedge \omega + dx_2 \wedge \text{Re} \Omega - dx_3 \wedge \text{Im} \Omega \), where \( Y \) is a Calabi-Yau 2-fold with Kähler form \( \omega \) and \( \Omega \) its holomorphic volume form. The holonomy group is \( SU(2) \).
3. \( T^7 \) with the inherited \( G_2 \)-structure \( \Box \) from \( \mathbb{R}^7 \). The holonomy group is \( \{1\} \).

As shown in [7], the dimension of \( G_2 \)-symmetries of a torsion-free \( G_2 \)-structure on a compact manifold is equal to the first Betti number. For the three examples above they are 1, 3, 7 in order, and such symmetries are exhibited explicitly by the respective torus parts of the manifold.

Let us end this section by recalling the moduli space of torsion-free \( G_2 \)-structures. In [4], D. Joyce defined the Moduli space \( \mathcal{M} = \{\varphi \text{ torsion-free of same orientation}\}/\text{Diff}_0(M) \) on a compact 7-manifold \( M \), where \( \text{Diff}_0(M) \) denotes the group of diffeomorphisms of \( M \) to itself that are isotopic to the identity. Let \( \{\varphi\} \in \mathcal{M} \) denote the orbit of the action of \( \text{Diff}_0(M) \) on \( \varphi \) by pull-back. Note that the action by diffeomorphisms isotopic to the identity ensures that all elements in \( \{\varphi\} \) belong to the same cohomology class \( [\varphi] \) in \( H^3(M) \). It is proved in [4] that \( \mathcal{M} \) is locally diffeomorphic to \( H^3(M) \) through the map \( \{\varphi\} \mapsto [\varphi] \). However, the global topology of \( \mathcal{M} \) is unknown. On the other hand, \( c \varphi \) is still a torsion-free \( G_2 \)-structure with the same orientation as \( \varphi \), for any \( c > 0 \). Therefore any local chart given by Joyce’s map above can be extended towards \( 0 \in H^3(M) \) and away to infinity along the \( H^3_1(M) \)-direction tangential to \( \mathcal{M} \). Hence \( \mathcal{M} \) actually has a cone structure - with the unknown topology being in directions tangent to the subspace \( H^3_2(M) \oplus H^3_{27}(M) \) (the “collars” of the cone) at each point on \( \mathcal{M} \). What is also unknown is the number of connected components of \( \mathcal{M} \). We will eventually see in Section 5 that the topology in the \( H^3_2(M) \)-direction at every point on \( \mathcal{M} \) is manifested as the orbit space of a group consisting of isometries acting on \( \Gamma \).
3 Analytic description based on Bryant’s formula

In this section, we will derive the following analytic description of \( \Gamma \) from formula (2).

**Proposition 3.1.** On a compact 7-manifold endowed with a torsion-free \( G_2 \)-structure \( \varphi \), there is the bijection

\[
\Gamma \cong \left\{ (c, \omega) \in [-1, 1] \times H^1(M) \mid c^2 + |\omega|^2 = 1 \right\} / \left\{ (c, \omega) \sim (-c, -\omega) \right\},
\]

where \( \omega \) denotes the harmonic representative of an element of \( H^1(M) \) with respect to \( g_\varphi \).

Note that since \( \omega \) is a harmonic 1-form on a compact, Ricci-flat manifold, it is also parallel. Therefore the point-wise norm \( |\omega| \) is constant on \( M \), and hence the condition \( c^2 + |\omega|^2 = 1 \) makes sense on \( M \). In particular, up to a constant multiple \( |\omega| \) serves as a norm on \( H^1(M) \).

Before giving the proof of Proposition 3.1, let us provide some extra motivation. Let \( \varphi(t) \) be a smooth family of \( G_2 \)-structures. Then according to (4), (5), and (6) we can write

\[
\frac{\partial \varphi}{\partial t} = b(t) \odot \varphi(t) + X(t) \ast \varphi(t)
\]

for a unique family of symmetric 2-tensors \( b(t) \) and a unique family of vector fields \( X(t) \), and where \( \odot \) and \( \ast \) are induced by \( \varphi(t) \). It can be shown (see [5]) that

\[
\frac{\partial g}{\partial t} = 2b(t),
\]

where \( g(t) = g_{\varphi(t)} \) is the corresponding family of metrics induced. Now suppose \( \varphi(t) \) generate the same metric for all \( t \), then it follows that \( \frac{\partial \varphi}{\partial t} \in \Omega^3_M \) for all \( t \). If in addition \( \varphi(t) \) is torsion-free for all \( t \), we see that

\[
d \frac{\partial \varphi(t)}{\partial t} = \frac{\partial}{\partial t} d\varphi(t) = 0
\]

and

\[
d \ast \frac{\partial \varphi(t)}{\partial t} = \frac{\partial}{\partial t} d\ast \varphi(t) = 0,
\]

where the fixed metric also fixes the Hodge star \( \ast \). It follows that \( \frac{\partial \varphi}{\partial t} \) is harmonic for all \( t \) and thus represents an element in \( H^3_M(M) \). On the moduli space \( \mathcal{M} \) this means that variations which preserve the metric are always in the \( H^3_M(M) \)-directions. It follows that \( \Gamma \) is a discrete set in \( \mathcal{M} \) if \( H^1(M) = 0 \) (which happens when \( M \) has full \( G_2 \)-holonomy). This local result on \( \mathcal{M} \) is refined by Proposition 3.1, which says that in fact \( \Gamma \) consists of a single point. In the next section, we will see yet another way to arrive at this fact.

On the other hand, suppose \( H^3_M(M) \neq 0 \). At any point on \( \mathcal{M} \) and given a nonzero element \([\eta]\) of \( H^3_M(M) \), is there always a variation of torsion-free \( G_2 \)-structures preserving the metric in the direction of \([\eta]\)? After the proof of Proposition 3.1 below, we will prove that the bijection there is actually a \( C^1 \)-diffeomorphism, which consequently means that \( \Gamma \) is homeomorphic to \( \mathbb{RP}^{b_1} \). In the last section we will show that \( \Gamma \) projects into \( \mathcal{M} \) via a covering map. Thus locally, the image of \( \Gamma \) in \( \mathcal{M} \) still has dimension \( b_1 \). Then since \( b_2^2 = b_1 \), this answers the question above in the affirmative.
Proof of Proposition 3.1. First note that the first two terms on the right-hand side of (2) are respectively in $\Omega^3_1$ and $\Omega^3_2$. For the third term however, we note that
\[
\begin{align*}
\omega \wedge *(\omega \wedge \varphi) \wedge (X_{\perp} \wedge \varphi) &= -\omega \wedge *(\omega \wedge \varphi) \wedge X_{\perp} \wedge \varphi \\
&= -\omega \wedge X_{\perp} \wedge \varphi \wedge *(\omega \wedge \varphi) \\
&= 2\omega \wedge X_{\perp} \wedge \omega \wedge \varphi \\
&= 0, 
\end{align*}
\]
where in arriving at the third equality we observed that $*(\omega \wedge \varphi) \in \Omega^3_2$ (see (3)). It follows that the third term has no $\Omega^3_1$-component. On the other hand, let $\pi_1(\omega \wedge *(\omega \wedge \varphi)) = \alpha \varphi$ for some function $\alpha$ on $M$, and we see that
\[
7\alpha dv_\varphi = \alpha \varphi \wedge \varphi \\
= \omega \wedge *(\omega \wedge \varphi) \wedge \varphi \\
= \omega \wedge 3\omega \\
= 3|\omega|^2 dv_\varphi,
\]
where in arriving at the third equality we used the standard identity $*\varphi \wedge *(\varphi \wedge \alpha) = 3*\alpha$ for any $\alpha \in \Omega^1(M)$ (see Proposition A.3 of [5]). Thus $\alpha = \frac{3}{7}|\omega|^2$, which is in general nonzero. Hence $\omega \wedge *(\omega \wedge \varphi) \in \Omega^3_1 \oplus \Omega^3_{27}$. We can then rewrite (2) explicitly in its $\Omega^3_k$-components as
\[
\begin{align*}
\tilde{\varphi} &= (f^2 - |\omega|^2)\varphi + 2f*(\omega \wedge \varphi) + 2\left(\frac{3}{7}|\omega|^2\varphi + \pi_{27}(\omega \wedge *(\omega \wedge \varphi))\right) \\
&= \frac{1}{7}(8f^2 - 1)\varphi + 2f*(\omega \wedge \varphi) + 2\pi_{27}(\omega \wedge *(\omega \wedge \varphi)), 
\end{align*}
\]
where we have also used $f^2 + |\omega|^2 = 1$.

Now, $\tilde{\varphi}$ is torsion-free, and therefore harmonic with respect to $g_{\tilde{\varphi}} = g_\varphi$. Taking the Laplacian with respect to $g_\varphi$ on (10) and using the fact that the Laplacian commutes with projections to irreducible subspaces, it follows that $8f^2 - 1$ must be harmonic and hence $f$ must be constant on the compact manifold $M$.

On the other hand, because $f = c$ is constant and $\varphi$ is torsion-free, we also see that
\[
0 = \Delta*(f\omega \wedge \varphi) \\
= *c \Delta(\omega \wedge \varphi) \\
= c*(\Delta\omega \wedge \varphi).
\]
Thus $\Delta\omega = 0$ necessarily.

Next, let $(c, \omega) \in [-1, 1] \times H^1(M)$. Then by taking the Laplacian again, it is clear that the first two terms of the right-hand side of (2) go to zero. For the last term, since $\Delta\omega = 0$ on $M$ and $M$ is compact, it follows again by the Ricci-flatness of $M$ that $\nabla \omega = 0$ on $M$. Since $\nabla \varphi = 0$ on $M$ as well, by the product rule and the fact that $\nabla$ commutes with $*$, it follows that $\omega \wedge *(\omega \wedge \varphi)$ is parallel as well. This then implies that $\omega \wedge *(\omega \wedge \varphi)$ must be both closed and co-closed, hence harmonic. Therefore $(c, \omega)$ indeed represents an element in $\Gamma$.

Finally, suppose $(c_1, \omega_1), (c_2, \omega_2) \in [-1, 1] \times H^1(M)$ both determine the same $\tilde{\varphi} \in \Gamma$. Note from (10) that $\pi_1(\tilde{\varphi}) = \frac{1}{7}(8c^2 - 1)\varphi$ and $\pi_7(\tilde{\varphi}) = 2c*(\omega \wedge \varphi)$ for some $c \in [-1, 1]$ and $\omega \in H^1(M)$. 7
By comparing the corresponding irreducible components we see that \( c_1 = \pm c_2 \) and \( \omega_1 = \pm \omega_2 \), therefore we indeed have a bijection between \( \Gamma \) and the set in question.

Next we investigate the regularity of the map above.

**Proposition 3.2.** The map in Proposition 3.1 is a \( C^1 \)-diffeomorphism, hence a homeomorphism.

**Proof.** The map \((c, \omega) \mapsto \varphi(c, \omega)\) in Proposition 3.1 given by

\[
\varphi(c, \omega) = (c^2 - |\omega|^2) \varphi + 2c * (\omega \wedge \varphi) + 2\omega \wedge * (\omega \wedge * \varphi),
\]

was shown to be bijective onto \( \Gamma \) up to the equivalence \((c, \omega) \sim (-c, -\omega)\), and is clearly \( C^1 \) as a map into the finite-dimensional space of harmonic 3-forms with respect to \( g_\varphi \). We will show that the induced derivative map from the tangent space of any point on the \( b^3 \)-dimensional sphere

\[
\left\{ (c, \omega) \in [-1, 1] \times H^1(M) \mid c^2 + |\omega|^2 = 1 \right\}
\]

is injective. Then by the inverse function theorem, this shows that the map (11) is a \( C^1 \)-diffeomorphism onto \( \Gamma \) up to the equivalence \((c, \omega) \sim (-c, -\omega)\) with a \( C^1 \) inverse. Ultimately the map is a topological embedding and hence a homeomorphism onto \( \Gamma \).

Let \((c_0, \omega_0)\) be a point on the sphere above, and let \((c(t), \omega(t))\) be a \( C^1 \)-curve on the sphere such that \((c(0), \omega(0)) = (c_0, \omega_0)\). In view of formula (11), suppose

\[
\frac{\partial}{\partial t} \varphi(c(t), \omega(t)) \bigg|_{t=0} = 4c_0 \dot{c}(0) \varphi + 2\dot{c}(0) * (\omega(0) \wedge \varphi) + 2c_0 * (\dot{\omega}(0) \wedge \varphi) + 2\dot{\omega}(0) \wedge * (\omega(0) \wedge * \varphi) + 2\omega_0 \wedge * (\dot{\omega}(0) \wedge * \varphi)
\]

\[
= 0.
\]

(12)

Note that we have used the condition \( c^2 + |\omega|^2 = 1 \) in deriving the term \( 4c_0 \dot{c}(0) \) above. From the proof of Proposition 3.1 we see that \((c(t))^2 - |\omega(t)|^2) \varphi \in \Omega^3_1, 2c(t) * (\omega(t) \wedge \varphi) \in \Omega^3_2, \) and \( 2\omega(t) \wedge * (\omega(t) \wedge * \varphi) \in \Omega^3_3 \otimes \Omega^2_{27} \) with respect to \( \varphi \), for all \( t \). Therefore derivatives will preserve the same irreducible subspaces induced by \( \varphi \). Applying this fact, the \( \Omega^3_2 \)-term in (12) must be zero, which means that \( \dot{c}(0) \omega_0 + c_0 \dot{\omega}(0) = 0 \) by the definition of \( \Omega^2_2 \).

Consider points \((c_0, \omega_0)\) where \( c_0 \neq 0 \). It follows that

\[
\dot{\omega}(0) = -\frac{\dot{c}(0)}{c_0} \omega_0.
\]

(13)

Note that this means if \( \dot{c}(0) = 0 \) then \( \dot{\omega}(0) = 0 \) as well, and we would be done with this case. Thus we assume \( \dot{c}(0) \neq 0 \). Differentiating \( c(t)^2 + |\omega(t)|^2 = 1 \) and using (13) gives us

\[
0 = c_0 \dot{c}(0) + \langle \dot{\omega}(0), \omega_0 \rangle
\]

\[
= c_0 \dot{c}(0) - \frac{\dot{c}(0)}{c_0} |\omega_0|^2,
\]

or \( c_0^2 = |\omega_0|^2 \). On the other hand, from (13) and the proof of Proposition 3.1 we also have

\[
\dot{\omega}(0) \wedge * (\omega_0 \wedge * \varphi) + \omega_0 \wedge * (\dot{\omega}(0) \wedge * \varphi) = -2 \frac{\dot{c}(0)}{c_0} \omega_0 \wedge * (\omega_0 \wedge * \varphi)
\]

\[
= -2 \frac{\dot{c}(0)}{c_0} \left( \frac{3}{2} |\omega_0|^2 \varphi + \pi_{27}(\omega_0 \wedge (\omega_0 \wedge * \varphi)) \right).
\]

(14)
Since the $\Omega^3_1$-term in (12) must also be zero, it follows that
\[ c_0 \dot{c}(0) - \frac{3\dot{c}(0)}{7c_0} |\omega_0|^2 = 0, \]
or $c_0^2 = 3|\omega_0|^2/7$. We now get a contradiction to the previous conclusion of $c_0^2 = |\omega_0|^2$ unless $\omega_0 = 0$. However, $\omega_0 = 0$ would imply $c_0 = 0$. Therefore at $(c_0, \omega_0)$ where $c_0 \neq 0$, (12) implies $(\dot{c}(0), \dot{\omega}(0)) = (0, 0)$ necessarily.

Next we consider the case $c_0 = 0$. Then it follows by differentiating $c(t)^2 + |\omega(t)|^2 = 1$ that $\langle \dot{\omega}(0), \omega_0 \rangle = 0$. Then by the standard identity in Proposition A.3 of [5] again, we see that
\[ \dot{\omega}(0) \wedge *(\omega_0 \wedge *\varphi) \wedge *\varphi = \dot{\omega}(0) \wedge *3\omega_0 \]
\[ = 3(\dot{\omega}(0), \omega_0) dv_\varphi \]
\[ = 0. \]
Therefore $\pi_1(\dot{\omega}(0) \wedge *(\omega_0 \wedge *\varphi)) = 0$, and hence $\dot{\omega}(0) \wedge *(\omega_0 \wedge *\varphi) \in \Omega^3_{27}$. The identical argument shows that $\omega_0 \wedge *(\dot{\omega}(0) \wedge *\varphi) \in \Omega^3_{27}$ as well. Now assuming (12), it follows that
\[ \dot{\omega}(0) \wedge *(\omega_0 \wedge *\varphi) + \omega_0 \wedge *(\dot{\omega}(0) \wedge *\varphi) = 0. \quad (15) \]
Recall that (12) also implies $\dot{c}(0) \omega_0 + c_0 \dot{\omega}(0) = 0$, and so our assumption of $c_0 = 0$ immediately implies $\dot{c}(0) = 0$. It remains to show that $\dot{\omega}(0) = 0$ follows from (15). To this end, we first observe that
\[ \dot{\omega}(0) \wedge *(\omega_0 \wedge *\varphi) + \omega_0 \wedge *(\dot{\omega}(0) \wedge *\varphi) = h \circ \varphi, \quad (16) \]
where $h = \dot{\omega}(0) \otimes \omega_0 + \omega_0 \otimes \dot{\omega}(0)$. To verify (16), let $\{e_i\}_{i=1}^7$ be a point-wise frame adapted to the $G_2$-structure and $\{\omega_i\}_{i=1}^7$ its metric dual frame. Then write $\dot{\omega}(0) = \alpha_i \omega_i$ and $\omega_0 = \beta_i \omega_i$, from which it follows that $h_{ij} = \alpha_i \beta_j + \beta_i \alpha_i$. Then (16) follows directly from (7) in Section 2.

Now, by the isomorphism of $\text{Sym}^2(TM)$ onto $\Omega^3_{27}$ via the map $h \mapsto h \circ \varphi$, (15) implies $h = 0$. Then we compute
\[ h(\dot{\omega}(0)^\#, e_j) = h_{ij} \alpha_i \]
\[ = (\alpha_i \beta_j + \beta_i \alpha_i) \alpha_i \]
\[ = |\dot{\omega}(0)|^2 \beta_j + \langle \omega_0, \dot{\omega}(0) \rangle \alpha_j = |\dot{\omega}(0)|^2 \beta_j. \]
Since $\omega_0 \neq 0$, there must be some $\beta_j \neq 0$, which implies that $\dot{\omega}(0) = 0$ necessarily.

Lastly, we want to reiterate that the diffeomorphism $\Gamma \cong \mathbb{RP}^{b_1}$ is independent of the choice of $\{\varphi\}$ on $M$, since it only depends on the first Betti number $b_1$ of $M$. Thus only the topology of the underlying manifold $M$ is relevant to the topology of $\Gamma$.

4 The parameter space $N_f/G_2$.

For those familiar with the theory and language of principal bundles, it will be self-evident that most of the discussion in this section apply to all other holonomy groups. Let $f$ be a frame over $p \in M$ that is adapted to some $G_2$-structure $\varphi$ - here thought of as a principal sub-bundle of the $GL(7, \mathbb{R})$-frame bundle over $M$ with structure group $G_2$. It is well-known that $\varphi$ is
torsion-free if and only if $\text{Hol}_f(D) \subset G_2$, where $D$ is the Levi-Civita connection associated to $g_{\varphi}$. An equivalent characterization of $\varphi$ being torsion-free is that the $G_2$-structure is preserved by parallel-translation using $D$. Therefore assuming $\text{Hol}_f(D) \subset G_2$, we can parallel translate $f$ to reconstruct the corresponding $G_2$-structure due to $M$ being connected. In particular, a torsion-free $G_2$-structure (over $M$ connected) is determined by the fiber of the associated principal sub-bundle over a single point $p \in M$. We denote this fiber by $\{f, h \mid h \in G_2\}$ with respect to the free and transitive right-action of the general linear group on the set of all (tangent) frames over a point. Then note that two distinct $G_2$-structures over $M$ that generate the same Riemannian metric share the same Levi-Civita connection, which must preserve different frames due to the uniqueness property of parallel-translation. Therefore the two $G_2$-structures must have distinct fibers over every point on $M$.

Any frame over $p \in M$ adapted to the corresponding unique $SO(7)$-structure (the metric $g_{\varphi}$) can be written as $f.h$ for some unique $h \in SO(7)$. Then for the frame $f$ above, by the well-known formula $\text{Hol}_f, h(D) = h^{-1}\text{Hol}_f(D)h$ it follows that $[f, h]$ represents a torsion-free $G_2$-structure also generating $g_{\varphi}$ if and only if $h^{-1}\text{Hol}_f(D)h \subset G_2$. Therefore $[f, h] \in \Gamma$ if and only if $h^{-1}\text{Hol}_f(D)h \subset G_2$, and hence definition (13) makes sense.

$N_f$ is a parameterization, based at $f$, of the set of all frames $\tilde{f}$ adapted to $g_{\varphi}$ (treated as an $SO(7)$-structure) such that $\text{Hol}_f(D) \subset G_2$. Suppose we had chosen a different frame $\tilde{f}$ adapted to another torsion-free $G_2$-structure that also generates $g_{\varphi}$, then by definition $\tilde{f} = f.g$ for some $g \in N_f$ and we see that

$$h \in N_{\tilde{f}} \iff (gh)^{-1}\text{Hol}_f(D)gh = h^{-1}\text{Hol}_f(D)h \subset G_2.$$

$$\iff gh \in N_f.$$  \hspace{1cm} (17)

The map $N_f \rightarrow N_{\tilde{f}}$ given by $h \mapsto g^{-1}h$ is clearly bijective, and it furnishes a change of parameterization of the same set of frames via “shifting” by the element $g$.

We define the map $N_f \rightarrow \Gamma$ by $h \mapsto [f, h]$, and it is clearly surjective by definition. Next we note that

$$[f, h] = [f, g] \text{ for } h, g \in N_f$$

$$\iff f.h = (f.g).\tilde{h} = f.g\tilde{h} \text{ for some } \tilde{h} \in G_2$$

$$\iff g^{-1}h = \tilde{h} \in G_2$$

$$\iff hG_2 = gG_2.$$  \hspace{1cm} (18)

The second equivalence above follows from the fact that the action of the general linear group on ordered basis is free. It then follows that the induced map $N_{\tilde{f}}/G_2 \rightarrow \Gamma$ given by $hG_2 \mapsto [f, h]$ is well-defined and bijective. Proposition (12) is now verified. We want to reiterate that in general $N_{\tilde{f}}/G_2$ is only a coset space, since $N_f$ may not be a group. There is also the induced map $N_{\tilde{f}}/G_2 \rightarrow N_{f, g}/G_2$, where the parametrization of $\Gamma$ by $N_{f, g}/G_2$ is equivalent to that by $N_{\tilde{f}}/G_2$ up to a shifting of cosets via left-multiplication by $g^{-1}$, for any $g \in G_2$.

Note that according to (13), when $\text{Hol}_f(D) = G_2$, $N_f$ is exactly the normalizer $N(G_2)$ of $G_2$ in $SO(7)$. Since normalizers are Lie groups, $N(G_2)/G_2$ is also a Lie group. In (10), it was shown by purely algebraic means that $N(G_2) = G_2$. Therefore by Proposition (12) we have the following result.
Corollary 4.1. Let $M$ be a connected 7-dimensional manifold admitting a $G_2$-structure $\varphi$ with full $G_2$-holonomy. Then $\varphi$ is the only torsion-free $G_2$-structure on $M$ that generates the metric $g_\varphi$.

This is the same result that was obtained in the previous section from Bryant’s formula [2]. It is also interesting here that a result in pure algebra implies a result in moduli spaces by way of the geometric structures involved.

On the other hand, we can also turn the argument around and give a new moduli space argument of why $N(G_2) = G_2$. Set-theoretically, we have $N_f/G_2 \cong \Gamma \cong \mathbb{R}^b_1$ by incorporating the result from Section 3. By taking any one of the compact $G_2$-manifolds with full $G_2$-holonomy constructed by Joyce (see [4]), we have $b_1 = 0$ and $N_f = N(G_2)$. Then $N(G_2)/G_2 \cong \mathbb{R}^0$ = \{single point\}, which implies that $N(G_2) = G_2$ necessarily.

It can be shown that $N_f$ is a submanifold of $SO(7)$. Therefore $N_f/G_2$, which is the orbit space of $G_2$ acting on $N_f$ by right-multiplication, is also a manifold. We now argue that $N_f/G_2$ is diffeomorphic, and hence homeomorphic to $\Gamma$.

**Proposition 4.2.** The map $N_f/G_2 \longrightarrow \Gamma$ given by $hG_2 \mapsto [f,h]$ is a diffeomorphism.

**Proof.** We already saw that $hG_2 \mapsto [f,h]$ is a bijection. We will show that this map is in fact locally a diffeomorphism, via the inverse function theorem.

Consider $\bar{f} = f.g$ for any $g \in N_f$. Let $h(t)$ be a smooth family of elements in $N_f$ such that $h(0) = I$. Up to parallel translation on $M$, we can write $\bar{f}(h(t)) = (\bar{f}(t))^{*}\varphi_0 = \varphi(t)$ as elements in $\Gamma$. We also denote $\varphi = \bar{f}^{*}\varphi_0$. By the natural action of the general linear group on frames and the fact that $h(t) \in SO(7)$, it follows that $\bar{f}(h(t)) = h(t)^{-1}\bar{f}$, where the right-hand side denotes a composition of linear maps. Then for tangent vectors $u, v, w$ we see that

$$
\varphi(t)(u,v,w) = (\bar{f}(h(t)))^{*}\varphi_0(u,v,w)
= \varphi_0(h(t)^{-1}\bar{f}(u),h(t)^{-1}\bar{f}(v),h(t)^{-1}\bar{f}(w))
= \bar{f}^{*}\varphi_0(\bar{f}^{-1}h(t)^{-1}\bar{f}(u),\bar{f}^{-1}h(t)^{-1}\bar{f}(v),\bar{f}^{-1}h(t)^{-1}\bar{f}(w))
= \varphi(\bar{f}^{-1}h(t)^{-1}\bar{f}(u),\bar{f}^{-1}h(t)^{-1}\bar{f}(v),\bar{f}^{-1}h(t)^{-1}\bar{f}(w)).
$$

(18)

Then

$$
\frac{\partial \varphi(t)}{\partial t}(u,v,w) \bigg|_{t=0} = \varphi(-\bar{f}^{-1}h'(0)\bar{f}(u),v,w) + \varphi(u,-\bar{f}^{-1}h'(0)\bar{f}(v),w)
+ \varphi(u,v,-\bar{f}^{-1}h'(0)\bar{f}(w))
= (-\bar{f}^{-1}h'(0)\bar{f}) \odot \varphi(u,v,w),
$$

(19)

where $(h^{-1})'(0) = -h'(0)$ follows from $h(0) = I$.

Now, suppose $\frac{\partial \varphi(t)}{\partial t} \bigg|_{t=0} = 0$. By (19) and the discussion following (6) in Section 2 we conclude that $h'(0)$ must be an element of the Lie algebra of $G_2$. This shows that for any transverse slice through $I \in G_2$ in $N_f$, the derivative of the map $h \mapsto [f,h] = (\bar{f}(h))^{*}\varphi_0$ is injective. By the inverse function theorem this shows that the map is locally invertible onto the transverse slice, with the
inverse also differentiable. Passing to cosets this shows that the induced map $N_f/G_2 \longrightarrow \Gamma$ is locally invertible about the identity element $G_2$ with a differentiable inverse.

Next we refocus on the map $N_f/G_2 \longrightarrow \Gamma$. For any $g \in N_f$, we have $gG_2 \mapsto [f,g] = [\tilde{f}] = \tilde{f}^*\varphi_0 = \tilde{\varphi}$, using the notations above. Then note that the assignment above can be factored via the composition

$$N_f/G_2 \longrightarrow N_f/G_2 \longrightarrow \Gamma$$

of maps, where we recall that $N_f/G_2 \longrightarrow N_f/G_2$ is given by left-multiplication by $g^{-1}$ and is clearly a diffeomorphism. Then since we saw that the map $N_f/G_2 \longrightarrow \Gamma$ is a local diffeomorphism about $G_2$, this shows that $N_f/G_2 \longrightarrow \Gamma$ is a diffeomorphism. □

Note that a-priori, $N_f$ (and hence $N_f/G_2$) could have multiple connected components. The work above however, shows that the manifold $N_f/G_2$ actually has exactly one component. Also, note that we identified the topology of $N_f/G_2$ by analyzing relevant geometric structures, and not through direct Lie group-theoretic means.

## 5 Implications on the Moduli Space

Consider the continuous projection map $\Gamma \longrightarrow \mathcal{M}$ given by $\tilde{\varphi} \mapsto \{\tilde{\varphi}\}$. If this map is injective, then the topology in the $H^2_f(M)$-directions on $\mathcal{M}$ is given by a space homeomorphic to $\mathbb{R}^{\dim}$. However, it is conceivable that for some $\tilde{\varphi} \in \Gamma$, $f^*\tilde{\varphi}$ is also in $\Gamma$ for some $f \in \text{Diff}_0(M)$ such that $f^*\tilde{\varphi} \neq \tilde{\varphi}$. Note that in this case, $f$ is necessarily an isometry of $g_{\tilde{\varphi}}$ since it can be shown that $g_f^{-1} \tilde{\varphi} = f^*g_{\tilde{\varphi}}$ for any diffeomorphism $f$ of $M$ and any $G_2$-structure $\tilde{\varphi}$. In other words, $\Gamma$ could intersect a given orbit of $\mathcal{M}$ more than once. Throughout this section $f$ denotes a diffeomorphism of $M$, and not a point-wise frame as in Section 4.

The observation above points to the structure of covering spaces. Let $X$ denote the image of the map $\Gamma \longrightarrow \mathcal{M}$ above. The topology of $X$ is exactly what is meant by the topology of $\mathcal{M}$ in directions tangent to $H^2_f(M)$. Since $\tilde{\varphi}$ and $f^*\tilde{\varphi}$ projects onto the same point in $X$ for any isometry of $g_{\tilde{\varphi}}$ in $\text{Diff}_0(M)$, it is natural to expect that $X$ is the orbit space of the action by the pull-back of such isometries on $\Gamma$, and that $\Gamma \longrightarrow X$ is a covering map.

We begin with the following definition and the ensuing basic result. Given a differential form $\eta$, we say that a smooth family $f_t$ of diffeomorphisms of $M$ is regular (relative to $\eta$) if $\frac{\partial}{\partial t} f^*_t \eta \neq 0$ for all $t$.

**Lemma 5.1.** On a compact 7-manifold with torsion-free $G_2$-structure $\varphi$, there does not exist any smooth, regular (relative to $\varphi$) family of isometries of $g_{\varphi}$.

**Proof.** Let $f_t$ be a smooth family of isometries of $g_{\varphi}$. Then $f_t^* \varphi$ is a family of torsion-free $G_2$-structures on $M$. From the discussion in Section 3, we know that the velocity vectors of the curve $f_t^* \varphi$ in $\Omega^3(M)$ are harmonic $\Omega^3_f$-forms. On the other hand, we see that

$$\frac{\partial f_t^* \varphi}{\partial t} = f_t^* L_V \varphi = L(f_t)_* V \varphi = d((f_t)_* V \varphi),$$

which is exact and hence equals 0 if harmonic. This simple calculation implies that $f_t^* \varphi = f_0^* \varphi$ for all $t$. Thus $f_t^*$ cannot be a regular family relative to $\varphi$. □
We denote by \( S_0 \) the set of all isometries of \( g \) in \( \text{Diff}_0(M) \). Note that because \( g_{f^* \tilde{\varphi}} = f^* g_{\tilde{\varphi}} \), we have \( f^* : \Gamma \rightarrow \Gamma \) for every \( f \in \Sigma \). Define the binary operation \( f \# \tilde{f} = \tilde{f} \circ f \) for all \( f, \tilde{f} \in S_0 \). The reversing of order of functions is tailored to giving a well-defined left group action of \( S_0 \) on \( \Gamma \), as we shall see below.

**Lemma 5.2.** \( S_0 \) is a compact Lie group with respect to the operation \( \# \). Moreover, pull-back of elements in \( \Gamma \) by elements in \( S_0 \) is a left group action, so that \( X \) is exactly the orbit space of this action.

**Proof.** First we verify the group structure of \( S_0 \). Clearly, the identity \( I \in S_0 \). Next, \( f \# \tilde{f} = \tilde{f} \circ f \) is still an isometry for any \( f, \tilde{f} \in S_0 \). Furthermore, \( f \# \tilde{f} \in \text{Diff}_0(M) \) because \( \text{Diff}_0(M) \) is a group. Note that \( f \# \tilde{f} \) is not necessarily isotopic to the identity through isometries - as only diffeomorphisms are required by the definition of \( S_0 \). Thus \( f \# \tilde{f} \in S_0 \) as well. Clearly the identity \( I \) is in \( S_0 \), and that \( I \# f = f \# I = f \) for all \( f \in S_0 \). For any \( f \in S_0 \), its inverse \( f^{-1} \) is also an isometry and simultaneously a member of \( \text{Diff}_0(M) \), and so \( S_0 \) is also closed under taking inverses. Lastly, the associativity of \( \# \) is inherited from \( \text{Diff}_0(M) \) as well. Therefore \( S_0 \) is indeed a group.

Next we establish the smooth structure on \( S_0 \). Recall that the 7-manifold \( M \) is compact. It is well-known that the isometry group \( \text{Isom}(g) \) of any metric \( g \) on a compact manifold \( M \) is a compact Lie group. Therefore \( \text{Isom}(g_{\varphi}) \) is a compact Lie group. Also note that since \( \text{Diff}_0(M) \) is the connected component of \( \text{Diff}(M) \) containing the identity, it follows from general topology that \( \text{Diff}_0(M) \) must be a closed subset of \( \text{Diff}(M) \). On the other hand, \( \text{Diff}(M) \) is also Hausdorff, therefore the compact subset \( \text{Isom}(g_{\varphi}) \) must also be closed in \( \text{Diff}(M) \) as well. This means \( S_0 = \text{Isom}(g_{\varphi}) \cap \text{Diff}_0(M) \) is also closed in the compact set \( \text{Isom}(g_{\varphi}) \), hence \( S_0 \) is also compact. Finally, since \( S_0 \) is a (topologically) closed subgroup of \( \text{Isom}(g_{\varphi}) \), it inherits the smooth structure of \( \text{Isom}(g_{\varphi}) \). This completes the proof that \( S_0 \) is a compact Lie group.

We define a left group action of \( S_0 \) on \( \Gamma \) by \( f.\tilde{\varphi} = f^* \tilde{\varphi} \). Let us check that this is indeed a left group action. Consider any \( \tilde{\varphi} \in \Gamma \). Clearly \( I.\tilde{\varphi} = \tilde{\varphi} \). Then we see that

\[
\tilde{f}.(f.\tilde{\varphi}) = \tilde{f}.(f^* \tilde{\varphi}) = \tilde{f}^* f^* \tilde{\varphi} = (f \circ \tilde{f})^* \tilde{\varphi} = (\tilde{f} \# f).\tilde{\varphi}
\]

for any \( f, \tilde{f} \in S_0 \). This verifies the group action. The space \( X \) is then the orbit space of this action by the definition of the projection \( \Gamma \rightarrow X \).

**Corollary 5.3.** Let \( M \) be a compact 7-manifold with torsion-free \( G_2 \)-structure \( \varphi \). If the only isometry of \( g_{\varphi} \) in \( \text{Diff}_0(M) \) is the identity, then \( X \) is homeomorphic to \( \mathbb{RP}^{b_1} \).

To show that \( \Gamma \rightarrow X \) is a covering map, it would be sufficient to show that \( S_0 \) acts properly discontinuously on \( \Gamma \), which would then imply that \( S_0 \) is the group of deck transformations. However, when \( b_1 \geq 1 \) there exist symmetries of the underlying \( G_2 \)-structure (see [7]), which means the action by \( S_0 \) is not free and hence cannot be properly discontinuous. We will elaborate on this in more detail later, but in spite of it we can still show that \( \Gamma \rightarrow X \) is a covering map. To do so, we will examine the nature of each orbit of the action by \( S_0 \). In what follows, we will denote by “\( \text{orb}_{S_0}(\tilde{\varphi}) \)” as the orbit containing \( \tilde{\varphi} \) of \( S_0 \) acting on \( \Gamma \), and denote by “\( \text{stab}_{S_0}(\tilde{\varphi}) \)” as the stabilizer subgroup of \( S_0 \) fixing \( \tilde{\varphi} \).

**Lemma 5.4.** For every \( \tilde{\varphi} \in \Gamma \), \( \text{orb}_{S_0}(\tilde{\varphi}) \) is a finite subset of \( \Gamma \) and \( \text{stab}_{S_0}(\tilde{\varphi}) \) is a compact Lie subgroup of \( S_0 \) with dimension \( b_1 \). In particular, \( S_0 \) is also of dimension \( b_1 \).
Proof. Since $S_0$ is compact by Lemma 5.2, it is also a proper action. Then by the standard fact that orbits for a proper action on any manifold are closed submanifolds, it follows that $\text{orb}_{S_0}(\tilde{\varphi})$ is a closed submanifold of $\Gamma$ for any $\tilde{\varphi} \in \Gamma$. Then by Lemma 5.1, $\text{orb}_{S_0}(\tilde{\varphi})$ must be a 0-dimensional submanifold of $\Gamma$. In particular, $\text{orb}_{S_0}(\tilde{\varphi})$ is just a finite collection of points in $\Gamma$, otherwise (via the compactness of $\Gamma$) a limit point would violate the definition of $\text{orb}_{S_0}(\tilde{\varphi})$ being a submanifold of $\Gamma$.

The stabilizers of proper actions are compact, so it follows that $\text{stab}_{S_0}(\tilde{\varphi})$ is compact. In particular, since $\text{stab}_{S_0}(\tilde{\varphi})$ is also a closed subset of of the Lie group $S_0$, it must also be a Lie subgroup of $S_0$. Recall that $\text{orb}_{S_0}(\tilde{\varphi}) \cong S_0/\text{stab}_{S_0}(\tilde{\varphi})$ as sets. Thus the cosets of $\text{stab}_{S_0}(\tilde{\varphi})$ must be a finite collection of mutually-disjoint compact submanifolds of $\text{Isom}(g_\varphi)$, and each coset is a translation of (hence homeomorphic to) $\text{stab}_{S_0}(\tilde{\varphi})$. In [7] it was shown that $b^1$ is the dimension of the vector space of (continuous) symmetries of $\varphi$. Therefore by the standard translation argument we see that as a manifold, $\text{stab}_{S_0}(\tilde{\varphi})$ (hence each of its cosets) must also have dimension equal to $b^1$. Finally, since $S_0$ is also the disjoint union of the cosets of $\text{stab}_{S_0}(\tilde{\varphi})$, it follows that $S_0$ must also have the same dimension $b^1$.

Proof of Theorem 1.1. Let $\{\tilde{\varphi}\} \subset X$. Then by definition $\text{orb}_{S_0}(\tilde{\varphi})$ is the pre-image of the projection $\Gamma \twoheadrightarrow X$. The finiteness of $\text{orb}_{S_0}(\tilde{\varphi})$ from Lemma 5.4 allows us to choose an open neighborhood $U$ in $X$ containing $\{\tilde{\varphi}\}$, so that there are mutually-disjoint open neighborhoods in $\Gamma$ containing each point in $\text{orb}_{S_0}(\tilde{\varphi})$ which project onto $U$. This concludes the proof.

Note that by the smoothness of $\mathcal{M}$, the number of sheets in the covering map $\Gamma \twoheadrightarrow X$ is constant over each connected component of $\mathcal{M}$, but may vary across different components.

When $b^1 = 0$, both $\Gamma$ and $X$ are respectively just the single point $\varphi$. Then $\Gamma \twoheadrightarrow X$ is the trivial covering map with the identity as its only deck transformation. Since $\text{orb}_{S_0}(\tilde{\varphi}) \cong S_0/\text{stab}_{S_0}(\tilde{\varphi})$ as sets, this implies that every element of $S_0$ fixes a $G_2$-structure $\varphi$ with full $G_2$-holonomy. Moreover, $S_0$ is a 0-dimensional submanifold of the compact Lie group $\text{Isom}(\mathcal{M})$ by Lemma 5.4, so it must be finite. Next, we investigate the situation where $b^1 = 1$. From this point on, we also denote by $\Sigma$ the group of deck transformations of $\Gamma \twoheadrightarrow X$.

Proposition 5.5. If $b^1 = 1$, then $X$ is homeomorphic to $S^1$ for every $\varphi$ that represents a point on $\mathcal{M}$. Moreover, $\Sigma \cong \mathbb{Z}_q$ for some positive integer $q$. If in addition $b^2_{27} = 0$, then each connected component of $\mathcal{M}$ is homeomorphic to the punctured-plane.

Proof. Since $\mathbb{R}P^1$ is homeomorphic to $S^1$, the corresponding covering map can be written as $S^1 \twoheadrightarrow X$. The covering map is continuous, so $X$ must be compact, connected, and 1-dimensional (locally as a differentiable manifold). Because the local deformation of $\Gamma$ is tangent to the 1-dimensional subspaces $H^2_3(M)$, there cannot be any self-intersections of $X$ as a submanifold in $\mathcal{M}$. Thus $X$ must be homeomorphic to $S^1$ for every $\varphi$ that represents a point on $\mathcal{M}$. It follows that if in addition $b^2_{27} = 0$, then $\mathcal{M}$ is a 2-dimensional manifold necessarily homeomorphic to $(0, \infty) \times S^1$, or the punctured-plane.

Let $H$ denote the (injective) push-forward of $\pi_1(\Gamma)$ under the covering map $\Gamma \twoheadrightarrow X$. Recall from the theory of covering spaces that the group of deck transformations is isomorphic to the normalizer of $H$ in $\pi_1(X)$ modulo $H$. Since in our case $\pi_1(X) \cong \pi_1(S^1) \cong \mathbb{Z}$, hence abelian, the group of deck transformations is isomorphic to $\pi_1(X)/H$. In other words, the covering map $\Gamma \twoheadrightarrow X$ is normal (see [3]). On the other hand, $\pi_1(\Gamma) \cong \pi_1(S^1) \cong \mathbb{Z}$ as well, so $H$ must also
be isomorphic to a nontrivial subgroup of \( \pi_1(X) \cong \mathbb{Z} \). Elementary group theory tells us that such a subgroup must be \( q\mathbb{Z} \) for some positive integer \( q \). Taking the quotient, it follows that the group of deck transformations is isomorphic to \( \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}_q \).

Note that a-priori \( \text{stab}_{S_0}(\varphi) \), and hence \( S_0 \), could be disconnected. Based on Lemma 5.4 and Lemma 5.22 we expect \( S_0 \) to be disjoint copies of circles. Therefore we make the following conjecture:

**Conjecture.** If \( b^1 = 1 \), then \( S_0 \) is isomorphic to \( S^1 \times \mathbb{Z}_q \) for some positive integer \( q \).

For \( b^1 > 1 \), it is natural to consider the composition \( S^{b^1} \longrightarrow \mathbb{R}P^{b^1} \longrightarrow X \), with the standard covering map \( S^{b^1} \longrightarrow \mathbb{R}P^{b^1} \cong \Gamma \) that has \( \mathbb{Z}_2 \) as its group of deck transformations. Since \( S^{b^1} \) is simply-connected, the resulting universal covering \( S^{b^1} \longrightarrow X \) is normal, and its group \( G \) of deck transformations is isomorphic to \( \pi_1(X) \). Moreover, \( X \) is homeomorphic to the orbit space \( S^{b^1}/G \), where \( G \) acts freely on \( S^{b^1} \).

Note that if \( \Gamma \longrightarrow X \) is also a normal covering, then \( \Sigma \) is isomorphic to \( \pi_1(x) \) modulo the injective push-forward of \( \pi_1(\Gamma) \cong \pi_1(\mathbb{R}P^{b^1}) \cong \mathbb{Z}_2 \). It then follows that \( G \) is a group extension of \( \Sigma \) by \( \mathbb{Z}_2 \), i.e. there exists a short exact sequence

\[
0 \longrightarrow \mathbb{Z}_2 \longrightarrow G \longrightarrow \Sigma \longrightarrow 0.
\]

If \( \Gamma \) is a normal covering space over \( X \), the image of the 2-element group \( \mathbb{Z}_2 \) must be contained in the center of \( G \). Therefore \( G \) is a central extension - an algebraic object classified by the group cohomology \( H^2(\Sigma, \mathbb{Z}_2) \). Note that if \( G \) splits, it can only do so as the trivial extension \( \mathbb{Z}_2 \times \Sigma \) since the automorphism group of \( \mathbb{Z}_2 \) is trivial, and this corresponds to the zero element in \( H^2(\Sigma, \mathbb{Z}_2) \).

Next we observe that in the proof of Lemma 5.4 orb_{S_0}(\tilde{\varphi}) is actually diffeomorphic to the quotient manifold \( S_0/\text{stab}_{S_0}(\tilde{\varphi}) \). Therefore if we assume \( \text{stab}_{S_0}(\tilde{\varphi}) \) is a normal subgroup of \( S_0 \), then \( S_0/\text{stab}_{S_0}(\tilde{\varphi}) \) is a Lie group that acts transitively on orb_{S_0}(\tilde{\varphi}). It would then follow that \( X \) is diffeomorphic to the orbit space of \( S_0/\text{stab}_{S_0}(\tilde{\varphi}) \) acting on \( \Gamma \), which also means that \( \Gamma \longrightarrow X \) would be a normal covering. In particular, it follows that \( S_0/\text{stab}_{S_0}(\tilde{\varphi}) \) is exactly the group \( \Sigma \) of deck transformations of \( \Gamma \longrightarrow X \). Although it certainly seems artificial at the present, we shall conveniently assume that \( \text{stab}_{S_0}(\tilde{\varphi}) \) is a normal subgroup of \( S_0 \) for the cases \( b^1 > 1 \).

There are some partial results that can be obtained regarding the group extension problem above. For example, by the Schur-Zassenhaus Lemma (see [8] for reference), if \( \Sigma \) is finite and whose order \( |\Sigma| \) is relatively prime with \( |\mathbb{Z}_2| = 2 \), then \( H^2(\Sigma, \mathbb{Z}_2) = 0 \). Then since \( \Sigma \) has to be a finite group due to Lemma 5.4 we can draw the following conclusion.

**Proposition 5.6.** Suppose \( b^1 > 1 \), \( \text{stab}_{S_0}(\tilde{\varphi}) \) is normal in \( S_0 \), and \( \Sigma \) has odd order. Then \( X \) is homeomorphic to the orbit space of an action of \( \mathbb{Z}_2 \times \Sigma \) on \( S^{b^1} \).

One can also approach the problem of describing \( X \) by asking which groups can act freely on spheres. For finite groups it is known that such a group must have periodic cohomology - a condition equivalent to all abelian subgroups of the group being cyclic. Via the Künneth formula it can be shown that \( \mathbb{Z}_p \times \mathbb{Z}_p \) does not have periodic cohomology for any prime \( p \). For a survey of these results we refer to [9].
Proposition 5.7. Suppose $b^1 > 1$, $\text{stab}_{S_0}(\tilde{\varphi})$ is normal in $S_0$, and $\Sigma \cong \mathbb{Z}_2$. Then $X$ is homeomorphic to the orbit of an action of $\mathbb{Z}_4$ on $S^{b^1}$.

Proof. $\Sigma \cong \mathbb{Z}_2$ implies that $G$ must be of order 4, since $G/\mathbb{Z}_2 \cong \Sigma$. The only possibilities are then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$. The former can be ruled out because not all its subgroups are cyclic. Therefore the proposition follows.

On the other hand, the author does not know whether or not $\mathbb{Z}_4$ can act freely on spheres of odd dimension at least 3. The assumptions in Propositions 5.6 and 5.7 also seem artificial at the present. In particular, we would like there to be a way to determine whether or not $\text{stab}_{S_0}(\tilde{\varphi})$ is normal in $S_0$. It would also be very helpful if there is some way of determining $\Sigma$ independently. Ultimately, we should be able to understand the topology of $X$ more if our knowledge about free actions on spheres or the structure of $H^2(\Sigma, \mathbb{Z}_2)$ advances.

The remaining mystery concerning the topology of $\mathcal{M}$ lies in directions tangent to $H^2_{27}(M)$. If $b^2_{27} \neq 0$, there may be some associated global topology like that of $\Gamma$ described in this paper. On the other hand, according to (9) we have $b^2_{27} = b^3 - 1 - b^1$. In view of Proposition 5.5, the moduli space $\mathcal{M}$ of the example of $S^1 \times Y$ in Section 2 contains families of circles. Note that by the K"unneth formula, we have $H^3(S^1 \times Y) = H^2(Y) \oplus H^3(Y)$. We may be able to find a Calabi-Yau 3-fold such that $b^2(Y) + b^3(Y) = 2$, so that the moduli space $\mathcal{M}$ of $S^1 \times Y$ is homeomorphic to the punctured plane. This, and the many questions raised above, serve as a good source for future research.

References

[1] Bryant, Robert Some remarks on $G_2$-structures. Proceedings of Gokova Geometry-Topology Conference 2005, edited by S. Akbulut, T. Onder, and R. J. Stern (2006), International Press, 75-109.

[2] M. Fernández, A. Gray Riemannian manifolds with structure group $G_2$, Ann. Mat. Pura Appl (IV) 32 (1982), 19-45.

[3] Hatcher, Allen. Algebraic Topology, www.math.cornell.edu/ hatcher/AT/ATchapters.html

[4] Joyce, Dominic D. Compact Manifolds with Special Holonomy, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[5] Karigiannis, Spiro. Flows of $G_2$-structures, I., Quarterly Journal of Mathematics 60 (2009), 487 - 522.

[6] Katz, Nicholas M. Notes on $G_2$, determinants, and equidistribution., Finite Fields and thier Applications 10 (2004), 221-269.

[7] Lin, Christopher. Laplacian Solitons and Symmetry in $G_2$-geometry., J. Geom. Phys., vol. 64 (2013), 111-119.

[8] Rotman, Joseph J. An Introduction to the Theory of Groups 3rd ed, Wm. C. Brown Publishers, 1988.
[9] Wall, C.T.C. *Free Actions of Finite Groups on Spheres* Proceedings of Symposia in Pure Mathematics, Vol. 32 (1978), Amer. Math. Soc., Providence, RI 1978, 145-154.

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