From Sampling to Optimization on Discrete Domains with Applications to Determinant Maximization

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Abstract

We show a connection between sampling and optimization on discrete domains. For a family of distributions $\mu$ defined on size $k$ subsets of a ground set of elements that is closed under external fields, we show that rapid mixing of natural local random walks implies the existence of simple approximation algorithms to find $\max \mu(\cdot)$. More precisely we show that if (multi-step) down-up random walks have spectral gap at least inverse polynomially large in $k$, then (multi-step) local search can find $\max \mu(\cdot)$ within a factor of $k^{O(k)}$. As the main application of our result, we show a simple nearly-optimal $k^{O(k)}$-factor approximation algorithm for MAP inference on nonsymmetric DPPs. This is the first nontrivial multiplicative approximation for finding the largest size $k$ principal minor of a square (not-necessarily-symmetric) matrix $L$ with $L + L^T \succeq 0$.

We establish the connection between sampling and optimization by showing that an exchange inequality, a concept rooted in discrete convex analysis, can be derived from fast mixing of local random walks. We further connect exchange inequalities with composable core-sets for optimization, generalizing recent results on composable core-sets for DPP maximization to arbitrary distributions that satisfy either the strongly Rayleigh property or that have a log-concave generating polynomial.

1 Introduction

Sampling and optimization are fundamental tasks in mathematics, statistical physics, and various subfields of computer science such as cryptography, differential privacy, machine learning, and artificial intelligence. In continuous settings, sampling and optimization are known to be intimately connected; convex sets, and more generally log-concave distributions, are the natural domains where either task is algorithmically tractable. For a more formal treatment of this connection in continuous settings see [LV06; LSV18].

On discrete/combinatorial domains, the relationship between sampling and optimization is less clear. For example, the intersection of two matroids is easy to optimize over, but not known to be easy to sample from, and the opposite holds for determinantal point processes, which are easy to sample from [see, e.g., AOR16] and hard to optimize [CM10].

The goal of this work is to establish a new connection between sampling and optimization in discrete settings. For a family of distributions $\mu$ defined on size $k$ subsets of a ground set of elements¹ that is closed under external fields, we show that rapid mixing of natural local ran-

¹The restriction of the domain to size $k$ subsets of a ground set should be thought of as a “canonical form”; many
dom walks implies the existence of simple approximation algorithms to find $\max \mu(\cdot)$. More specifically, we show that local search can approximately find $\max \mu(\cdot)$ within a nearly-optimal approximation factor.

We study a family of natural local search algorithms (Algorithm 2) to find $\max \mu(\cdot)$. These algorithms start with a set $S$, and repeatedly try to increase $\mu(S)$ by swapping a constant number of elements in $S$ with elements outside of $\hat{S}$ until no more improvements can be made.

More formally, suppose that the domain of the objective $\mu$ is the collection of size $k$ subsets of the ground set $[n] = \{1, \ldots, n\}$, which we denote by $\binom{[n]}{k}$. Then, local search is defined with a parameter $r \geq 0$ which specifies the “local neighborhood” the algorithm searches over in each iteration. The $r$-neighborhood of $S \in \binom{[n]}{k}$ are all the sets that can be reached by swapping at most $r$ elements:

$$\mathcal{N}_r(S) := \left\{ T \in \binom{[n]}{k} \bigg| |S - T| \leq r \right\}.$$

Each iteration of local search goes from a set $S$ to $\hat{S} \in \mathcal{N}_r(S)$ which maximizes $\mu(\hat{S})$. If we reach a local optimum, i.e., $S = \hat{S}$, then $\mu(S) = \max \{ \mu(T) \mid T \in \mathcal{N}_r(S) \}$.

We show in this work that rapid mixing of natural local random walks, the (multi-step) down-up random walks, designed to sample from $\mu$ and related distributions, implies that local maxima of $\mu$ are approximate global maxima.

**Definition 1** (Down-Up Random Walks). For a density $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$, and an integer $\ell \leq k$, we define the $k \leftrightarrow \ell$ down-up random walk as the sequence of random sets $S_0, S_1, \ldots$ generated by the following algorithm:

```plaintext
for $t = 0, 1, \ldots$ do
    Select $T_t$ uniformly at random from subsets of size $\ell$ of $S_t$.
    Select $S_{t+1}$ with probability $\propto \mu(S_{t+1})$ from supersets of size $k$ of $T_t$.
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This random walk is time-reversible, always has $\mu$ as its stationary distribution, and moreover has positive real eigenvalues [see, e.g., ALO20]. This random walk, specially for the case of $\ell = k - 1$, has received a lot of attention in the literature on high-dimensional expanders [see, e.g., LLP17; KO18; DK17; KM16; AL20; ALO20]. Each step of this random walk can be efficiently implemented as long as $k - \ell = O(1)$ and we have oracle access to $\mu$. We remark that down-up walks generalize other well-known local random walks like the Glauber dynamics [see, e.g., ALO20]. Note that the down-up random walk is **local** in the sense that $S_{t+1} \in \mathcal{N}_{k-\ell}(S_t)$. Naturally, we tie mixing of these random walks to local search with $r = k - \ell$ neighborhoods.

There has been a recent surge of interest in analyzing the mixing properties of down-up random walks due to a number of breakthrough applications to open problems in sampling and counting [Ana+19; AL20; ALO20; Ali+21; Che+21b; CLV21; Fen+21; Liu+21; JPV21; Bla+21; Che+21a; Ana+21b; Ana+21a]. Key to many of these works was the notion of spectral independence. Alimohammadi, Anari, Shiragur, and Vuong [Ali+21] introduced a stronger notion called fractional log-concavity, and showed that it implies a $k^{-O(1)}$ lower bound on the spectral gap of $k \leftrightarrow (k - O(1))$-down-up random walks on $\mu$. We remark that fractional log-concavity, unlike spectral independence, is preserved under external fields, formally defined as follows.

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*other discrete domains can be naturally transformed into this form.*
For a distribution $\mu$ on $\binom{[n]}{k}$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_{\geq 0}^n$, the $\lambda$-external field applied to $\mu$ is another distribution on $\binom{[n]}{k}$, denoted by $\lambda \ast \mu$, defined up to normalization as follows:

$$\mathbb{P}_{\lambda \ast \mu}[S] \propto \mu(S) \cdot \prod_{i \in S} \lambda_i.$$  

As established in [Ali+21], various distributions of interest involving determinants are fractionally log-concave. For a fractionally log-concave distribution $\mu$, the $k \leftrightarrow (k - O(1))$-down-up walk on $\lambda \ast \mu$ has inverse-polynomially large spectral gap, even when an arbitrary external field $\lambda \in \mathbb{R}_{\geq 0}^n$ is applied to $\mu$. We show that this property\(^2\) implies nearly optimal approximation for (multi-step) local search on $\mu$.

**Theorem 2.** Consider a distribution $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$. Suppose that for some $r = O(1)$, the $k \leftrightarrow (k - r)$ down-up random walk on $\lambda \ast \mu$ has spectral gap at least $k^{-O(1)}$ for all external fields $\lambda \in \mathbb{R}_{\geq 0}^n$. Then any approximate local maximum, that is a set $S \in \binom{[n]}{k}$ such that

$$\mu(S) \geq \Omega(1) \cdot \max \{\mu(T) \mid T \in \mathcal{N}_r(S)\}$$

is a $k^{O(k)}$-approximate global maximum, that is

$$\mu(S) \geq k^{-O(k)} \cdot \max \left\{ \mu(T) \mid T \in \binom{[n]}{k} \right\}.$$  

Moreover, such an approximate local maximum can be found efficiently given oracle access to $\mu$ and a starting point in the support of $\mu$.

In particular, combined with rapid mixing results of [Ali+21], Theorem 2 implies that local search is an efficient $k^{O(k)}$-approximation algorithm for the optimization problem on nonsymmetric determinantal point processes (see Section 1.1), and on the intersection of a strongly Rayleigh distributions over $\binom{[n]}{k}$ and constantly many partition constraints (Corollary 32). Our approximation algorithm for nonsymmetric determinantal point processes is the first unconditional multiplicative approximation algorithm for this problem.

**Remark 3.** We remark that the approximation factor of $k^{O(k)}$ is nearly optimal amongst efficient algorithms. The special case of symmetric determinantal point processes was shown to be hard to approximate within a factor of $c^k$ for some constant $c > 1$ [CM10]. Further, the factor of $k^{O(k)}$ is tight for local search, even in the special case of symmetric determinantal point processes [see, e.g., AV20].

### 1.1 MAP Inference on Nonsymmetric DPPs

Determinantal point processes (DPPs) have found many applications in machine learning, such as data summarization [Gon+14; LB12], recommender systems [GPK16; Wil+18], neural network compression [MS15], kernel approximation [LJS16], multi-modal output generation [Elf+19], etc.

Formally, a DPP on a set of $[n]$ items is a probability distribution over subsets $Y \subseteq [n]$ parameterized by a matrix $L \in \mathbb{R}^{n \times n}$ where $Y$ is chosen with probability proportional to the determinant of the principal submatrix $L_Y$ whose columns are rows are indexed by $Y$:

$$\mathbb{P}[Y] \propto \det(L_Y).$$

\(^2\)Curiously, in continuous settings applying an external field also preserves log-concavity, the standard of algorithmic tractability for sampling; applying an external field is the same as multiplication by a log-linear function.
A related and perhaps more widely used model, is a $k$-DPP, where the size of $Y$ is restricted to be exactly $k$. In applications, usually $k$ is set to be much smaller than $n$. We study this model in this paper.

A fundamental optimization problem associated to probabilistic models, including DPPs, is to find the most likely, or the maximum a posteriori (MAP) configuration [GKT12]:

$$\max \left\{ P[S] \mid S \in \binom{[n]}{k} \right\}.$$  \hfill (1)

MAP inference is particularly useful when the end application requires outputting a single set; e.g., in recommender systems, the task is to produce a fixed-size subset of items to recommend to the user.

Most prior work on DPPs requires the kernel matrix $L$ to be symmetric, but such symmetric kernels are known to be able to only encode repulsive (negatively correlated) interactions between items [BBL09]. This severely limits their modeling power in practical settings. For example, a good recommender system for online shopping should model iPads and Apple Pencils as having positive interactions, since these are complementary items and tend to be bought together. To remedy this, recent work has considered the more general class of nonsymmetric DPPs (NDPPs) and shown that these have additional useful modeling power [Bru18; Gar+19]. Gartrell, Brunel, Dohmatob, and Krichene [Gar+19] consider NDPPs parameterized by nonsymmetric positive semi-definite (nPSD) kernel matrices $L$, i.e., those matrices where $L + L^\top \succeq 0$, and show efficient algorithms for learning such NDPPs.

**Definition 4.** A (not-necessarily-symmetric) matrix $L \in \mathbb{R}^{n \times n}$ is nonsymmetric positive semidefinite (nPSD) if $L + L^\top \succeq 0$.

Throughout, we will consider only NDPPs with nPSD kernels (nPSD-NDPPs) [see KT12, for a survey on fixed-size DPPs and their applications]. Alimohammadi, Anari, Shiragur, and Vuong [Ali+21] showed how to efficiently sample from fixed-size nPSD-NDPPs using natural Markov chains. Gartrell, Han, Dohmatob, Gillenwater, and Brunel [Gar+20] proposed a new learning algorithm, and showed how to efficiently implement and analyze the natural greedy MAP inference heuristic for symmetric DPPs on nPSD-NDPPs. This greedy heuristic (Algorithm 1) starts from an empty set and runs for $k$ iterations, in each iteration adding the item that most increases the DPP score.

Though this greedy algorithm is guaranteed to obtain a $k^{O(k)}$-approximation for symmetric DPPs [ÇM10], it could not achieve even a finite approximation factor for nPSD-NDPPs. For example, on a skew-symmetric matrix $X$, i.e., $X = -X^\top$, since all odd-sized principle minors of $X$ are zero, Algorithm 1 would necessarily resort to picking an arbitrary/random item at every other iteration, which can result in an arbitrarily bad final answer. Consider a concrete example, which helps build intuition on why greedy fails to achieve a meaningful approximation factor. This example also shows that local search greedy [KD16]3, another candidate MAP inference algorithm with theoretical performance guarantees for symmetric DPPs, also used by Gartrell, Han, Dohmatob, Gillenwater, and Brunel [Gar+20] as a baseline to compare their greedy method, also fails to achieve a meaningful approximation factor.

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3This algorithm starts with the output $S$ of Algorithm 1, then continuously swaps out an element in $S$ with one outside $S$ to increase the DPP score, until either a local maximum is reached or $k^2 \log k$ swaps have been performed.
Algorithm 1 first picks item 1 since \( \det(L_{\{1\}}) = c_1 \) is maximum among all \( L_{\{i\}} \), then picks item 2 since \( \det(L_{\{1,2\}}) = c_2^2 + x_1^2 \gg \det(L_{\{1,i\}}) = c_1c_i \forall i \neq 1 \), and so on. On the other hand, the optimal subset is \( \{n-2t+1, \ldots, n\} \) by our choice of \( x_i \), and this could be arbitrarily better than Algorithm 1’s solution. We may think of items \( 2i-1 \) and \( 2i \) as complementary items, say, e.g., toothpaste and toothbrush proposed in a recommender system. The conditions on \( c_i \)'s and \( x_i \)'s mean that the degree of complementarity between these pairs increases with \( i \). So \( 2n-1 \) and \( 2n \) are the most likely pair to appear together, but each one of \( 2n-1 \) and \( 2n \) is most unlikely to appear as a singleton, and the opposite holds for item 1 and 2; for example, think of \( 2n-1 \) and \( 2n \) as a tea cup and tea cup lid, which are almost always bought together, but 1 and 2 as toothpaste and toothbrush, which are sometimes purchased separately.

Furthermore, switching out any item \( 2i-1 \) or \( 2i \) in \( S \) for an item \( 2j-1 \) or \( 2j \) outside of \( S \) reduces the determinant by \( (c_i^2 + x_i^2)/c_i c_j > c_i > 1 \), so \( S \) is also maximum among its 1-neighborhood. Thus local search greedy, or equivalently, local search initialized at \( S \), will simply output \( S \) itself.

We remark that it is easy to construct an example where Algorithm 1 produces a subset with zero determinant, whereas the optimal subset can have arbitrarily large determinant. E.g., in Example 5, we can make all diagonal entries except for \( L_{1,1} \) zero; then, Algorithm 1 with even \( k \) will necessarily produces a zero determinant.

As our main application, we show the first efficient algorithm for MAP inference on nPSD-NDPPs that gives a multiplicative factor approximation for \( \max \{ \det(L_S) \mid S \in \binom{[n]}{k} \} \), without requiring any additional assumption on the kernel matrix \( L \). Further, we obtain multiplicative approximation guarantees for \( \det(L_S) \), unlike prior related work [Gar+20] which obtained multiplicative approximations for \( \log \det(L_S) \); this is often a stronger guarantee when \( \text{OPT} \) is sufficiently large – roughly super-exponentially large in \( k \). The assumptions behind prior work often implicitly imply that \( \text{OPT} \) is at least exponentially large in \( k \), making our approximation guarantees attractive.

**Theorem 6.** There is a polynomial time algorithm that on input \( L \in \mathbb{R}^{n \times n} \) that is nPSD, outputs a set of indices \( S \in \binom{[n]}{k} \) guaranteeing

\[
\det(L_S) \geq k^{-O(k)} \cdot \max \left\{ \det(L_S) \mid S \in \binom{[n]}{k} \right\}.
\]

Moreover, the algorithm runs in \( O(n^4k + n^2k^5 \log n) \) time given the entries of \( L \), and \( O(n^2d^2k + n^2d^2k^3 \log n) \) time given a rank-\( d \) decomposition of \( L \), i.e., \( L = BCB^\top \) with \( B \in \mathbb{R}^{n \times d} \), \( C \in \mathbb{R}^{d \times d} \).

Our approximation factor matches that of the standard greedy heuristic on symmetric DPPs, as well as the guarantee of other simple heuristics proposed for symmetric DPPs [ČM10; KD16]. As mentioned earlier, Çivril and Magdon-Ismail [ČM10]’s greedy and Kathuria and Deshpande
KD16’s local search algorithm do not achieve any finite approximation factor for nPSD-NDPPs. Our result is incomparable to Gartrell, Han, Dohmatob, Gillenwater, and Brunel [Gar+20] as

i. multiplicative approximations for maximizing \( \log \det(L_S) \) do not imply similar results for \( \det(L_S) \),

ii. we place no additional assumption on \( L \). As demonstrated earlier, our approximation guarantees hold for matrices \( L \) where Algorithm 1 fails to achieve even a finite approximation factor.

Our local search algorithm for nPSD-NDPPs searches over 2 neighborhoods, unlike most prior related works which typically use 1 neighborhoods; using 2 neighborhoods is necessary, and is compatible with intuition from prior work of Anari and Vuong [AV20] who first studied 2 neighborhood local search for the related problem of finding the maximum \( k \times k \) subdeterminant of a rectangular matrix. Unlike [AV20], our analysis of local search is not based on algebraic identities, which we believe do not have a counterpart in the world of nPSD-DPPs, but rather mixing properties of random walks.

Algorithm 1: Standard GREEDY for DPPs

Initialize \( S \leftarrow \emptyset \).

while \(|S| < k\) do

Pick \( i \notin S \) that maximizes \( \det(L_{S \cup \{i\}}) \), and update \( S \leftarrow S \cup \{i\} \).

1.2 Composable Core-Sets for Strongly Rayleigh Distributions and Log-Concave Polynomials

As further application of our methods, we extend prior work of Mahabadi, Indyk, Gharan, and Rezaei [Mah+19] on the construction of composable core-sets for maximizing symmetric DPPs to the more general class of distributions that satisfy the strongly Rayleigh property [BBL09] or have a log-concave generating polynomial [AOV18].

Composable core-sets are a tool [Ind+14] to handle computational problems involving large amounts of data. Roughly speaking, a core-set is a summary of a dataset that is enough to solve the computational problem at hand; a composable core-set has the additional property that the union of summaries for multiple datasets is itself a good summary for union of all datasets. More precisely, in the context of the optimization problem on \( \mu : \binom{[n]}{k} \rightarrow \mathbb{R}_{\geq 0} \), a function \( c \) that maps any set \( P \subseteq [n] \) to one of its subsets is called an \( \alpha \)-composable core-set ([Mah+19]) if it satisfies the following condition: given any integer \( m \) and any collection of sets \( P_1, \ldots, P_m \subseteq [n] \)

\[
\alpha \cdot \max \left\{ \mu(S) \left| S \subseteq \bigcup_{i=1}^{m} c(P_i) \right. \right\} \geq \max \left\{ \mu(S) \left| S \subseteq \bigcup_{i=1}^{m} P_i \right. \right\}.
\]

We also say \( c \) is a core-set of size \( t \) if \( |c(P)| \leq t \) for all sets \( P \). Composable core-sets are very versatile; when a composable core-set is designed for a task, they automatically imply efficient streaming and distributed algorithms for the same task.

One strategy for constructing composable core-sets is local search. Mahabadi, Indyk, Gharan, and Rezaei [Mah+19] showed that for \( k \)-DPP parameterized by symmetric PSD matrix \( L \), (1-step)-local
search (Algorithm 2 with \( r = 1 \)) gives a \( k^{O(k)} \)-composable core-sets of size \( k \). The approximation factor of \( k^{O(k)} \) is nearly optimal.

Recall that \( k \)-DPP parameterized by symmetric PSD matrix \( L \) belongs to the family of homogeneous strongly Rayleigh distributions, i.e., distributions \( \mu \) whose generating polynomial \( g_{\mu} \) is nonvanishing on the upper half plane [BBL09]. An even more general family of distributions is the family of log-concave distributions [AOV18]. We extend [Mah+19]’s result to any distribution \( \mu : [n] \to \mathbb{R}_{\geq 0} \) that is strongly Rayleigh or has a log-concave generating polynomial.

**Theorem 7.** Given a distribution \( \mu : [n] \to \mathbb{R}_{\geq 0} \), let \( c \) be a map that takes \( P \subseteq [n] \) to some \( c(P) \in \binom{[n]}{k} \) that is an \( \zeta \)-approximate local maximum in the \( 1 \)-neighborhood with respect to \( \mu \), for some fixed constant \( \zeta \in (0, 1) \):
\[
\mu(c(P)) \geq \zeta \cdot \max \{ \mu(S) \mid S \in \mathcal{N}_1(c(P)) \}.
\]
Then \( c \) is an \( \alpha \)-composable core-set of size \( k \) for the MAP-inference problem on \( \mu \) with \( \alpha = k^{O(k)} \) for strongly Rayleigh \( \mu \), and \( \alpha = 2^{O(k)} \) when \( \mu \) has a log-concave generating polynomial.

### 1.3 Techniques

Our main tool for proving Theorem 2 is a form of (approximate) exchange inequality. Exchange inequalities have been traditionally been studied in discrete convex analysis [Mur+16], but have recently been extended and used in sampling [Ana+21b] and optimization [AV20] problems beyond the reach of traditional discrete convex analysis. Unlike prior works, here we go in the opposite direction and show that efficient sampling implies a form of exchange inequality. To prove Theorem 2, we set the external field \( \lambda \) appropriately, and use the lower bound on the spectral gap of the down-up walk on \( \lambda * \mu \) to derive our approximate exchange property Lemma 30.

We then show that this approximate exchange property implies the desired approximation factor for local search (Proposition 31). Since nonsymmetric DPPs are 1/4-fractionally log-concave [Ali+21], Theorem 2 already implies an efficient algorithm (Algorithm 2 with \( r = 4 \)) to get \( k^{O(k)} \)-approximation factor for the MAP inference problem on nonsymmetric DPPs. We can further improve the the local search radius \( r \) to 2, and get a faster algorithm that matches the runtime stated in Theorem 6 by showing a stronger approximate exchange property (Theorem 34).

To prove Theorem 7, we use the approximate exchange property introduced by [Ana+21b] that is satisfied by strongly Rayleigh and log-concave distributions. This exchange property is a quantitative version of the strong basis exchange axiom for matroids. We rename it the strong approximate basis exchange property (Definition 40), to distinguish it from weaker exchange properties that we show in this paper. We show that the strong approximate basis exchange implies that approximate local maxima in the \( 1 \)-neighborhood is a size-\( k \) core-set with the desired approximation factor (Lemma 41).

### 2 Preliminaries

We use \([n]\) to denote the set \( \{1, \ldots, n\} \) and \( \binom{[n]}{k} \) to denote the family of size \( k \) subsets of \([n]\). We use \( \mathbb{1} \) to denote the all 1 vector. When \( n \) is clear from context, we use \( \mathbb{1}_S \in \mathbb{R}^n \) to denote the indicator vector of the set \( S \subseteq [n] \), having a coordinate of 0 everywhere except for elements of \( S \), where the coordinate is 1. For sets \( S, T \) of the same size we define their distance to be 
\[
d(S, T) := |S \Delta T|/2 = |S \setminus T| = |T \setminus S|.
\]
With this notion of distance, we can define neighborhoods:
We say a NDPP kernel \( L \) is automatic ally written as \( A \) DPP on a set of \( n \) items defines a probability distribution over subsets \( Y \subseteq [n] \). It is parameterized by a matrix \( L \in \mathbb{R}^{n \times n} \): \( \mathbb{P}_L[Y] \propto \det(L_Y) \), where \( L_Y \) denote the principle submatrix whose columns and rows are indexed by \( Y \). We call \( L \) the kernel matrix.

For \( Y \subseteq [n] \), if we condition the distribution \( \mathbb{P}_L \) on the event that items in \( Y \) are included in the sample, we still get a DPP; the new kernel is given by the Schur complement \( L^Y = L_\hat{Y} - L_{\hat{Y},Y}L_\hat{Y}^{-1}L_{Y,\hat{Y}} \) where \( \hat{Y} = [n] \setminus Y \).

Given a cardinality constraint \( k \), the \( k \)-DPP parametrized by \( L \) is a distribution over subsets of size \( k \) of \( Y \) defined by \( \mathbb{P}_L^k[Y] = \frac{\det(L_Y)}{\sum_{|Y|=k} \det(L_Y)} \).

To ensure that \( \mathbb{P}_L \) defines a probability distribution, all principal minors of \( L \) must be non-negative: \( \det(L_S) \geq 0 \). Matrices that satisfy this property are called \( P_0 \)-matrices [Fan89, Definition 1]. Any nonsymmetric (or symmetric) PSD matrix is automatically \( P_0 \)-matrix [Gar+19, Lemma 1].

We say a NDPP kernel \( L \in \mathbb{R}^{n \times n} \) has a low-rank decomposition [Gar+19; Gar+20] if \( L \) can be written as \( L = BCB^\top \) for some \( d \leq n \), \( B \in \mathbb{R}^{n \times d} \), \( C \in \mathbb{R}^{d \times d} \). Clearly, \( \text{rank}(L) = d \), and we say \( L = BCB^\top \) is a rank-\( d \) decomposition of \( L \). We will need the following identity, which is derived from Schur complements; it has previously appeared in [Gar+20]. For \( S \subseteq [n] \), let \( B_S \) denote the sub-matrix of \( B \) consisting of rows in \( S \); then \( L_S = B_SB_S^\top \) and

\[
\det(L_{Y \cup D}) = \det(L_Y) \det(L_D - L_{D,Y}L_Y^{-1}L_{Y,D}) \\
= \det(L_Y) \det(L_D - B_D C (B_Y^\top L_Y^{-1}B_Y) C B_D^\top ) .
\]

Given \( \det(L_Y) \) and \( L_Y^{-1} \), we can compute \( \det(L_{Y \cup D}) \) in \( O(|D|d^2 + |D|^2d + |D|^3) \) time.

### 2.2 MAP Inference

Given a density \( \mu : \binom{[n]}{k} \rightarrow \mathbb{R}_{\geq 0} \), the optimization with respect to \( \mu \) or MAP inference on \( \mu \) is to find

\[
S^* := \arg \max_{S \in \binom{[n]}{k}} \mu(S).
\]

Throughout the paper, we let \( \text{OPT} := \max_{S \in \Omega} \mu(S) \).

We say an algorithm gives a factor \( c \)-approximation for MAP inference on \( \mu \) if its output \( \hat{S} \in \binom{[n]}{k} \) such that \( c\mu(\hat{S}) \geq \text{OPT} \).

When \( \mu \) is defined by a DPP, i.e. \( \mu(S) = \det(L_{S,S}) \) for a \( n \times n \) matrix \( L \), MAP inference on \( \mu \) is also called the determinant maximization problem [e.g., see Mah+19].
2.3 Markov Chains

For two measures $\mu, \nu$ defined on the same state space $\Omega$, we define their total variation distance as

$$d_{TV}(\mu, \nu) = \frac{1}{2} \sum_{\omega \in \Omega} |\mu(\omega) - \nu(\omega)| = \max \{\mathbb{P}_\mu[S] - \mathbb{P}_\nu[S] \mid S \subseteq \Omega\}.$$

A Markov chain on a state space $\Omega$ is defined by a row-stochastic matrix $P \in \mathbb{R}^{\Omega \times \Omega}$. We view distributions $\mu$ on $\Omega$ as row vectors, and as such $\mu P$ would be the distribution after one transition according to $P$, if we started from a sample of $\mu$. A stationary distribution $\mu$ for the Markov chain $P$ is one that satisfies $\mu P = \mu$. Under mild assumptions on $P$ (ergodicity), stationary distributions are unique and the distribution $\nu P^t$ converges to this stationary distribution as $t \to \infty$ [LP17]. We refer the reader to [LP17] for a detailed treatment of Markov chain analysis.

In this paper, we will only consider reversible Markov chain. We say a Markov chain with transition matrix $P$ is reversible if

$$\mu(x) P(x, y) = \mu(y) P(y, x) \forall x, y \in \Omega.$$

The conductance$^4$ of a subset $S$ of states in a Markov chain is

$$\Phi(S) = \frac{Q(S, \Omega \setminus S)}{\mu(S)}$$

where $Q(S, \Omega \setminus S) = \sum_{x \in S, y \in \Omega \setminus S} \mu(x) P(x, y)$ is the ergodic flow between $S$ and $\Omega \setminus S$, and $\mu(S) = \sum_{x \in S} \mu(x)$.

The conductance of a Markov chain is defined as the minimum conductance over all subsets $S$ with $\mu(S) \leq 1/2$, i.e.

$$\Phi = \min_{S: \mu(S) \leq 1/2} \Phi(S)$$

Theorem 9 ([see, e.g., LP17, Thm. 13.10]). Let $\lambda_2$ be the second largest eigenvalue of the transition matrix $P$, then

$$\frac{\Phi^2}{2} \leq 1 - \lambda_2 \leq 2\Phi.$$

For a Markov chain $P$, we define the mixing time from a starting distribution $\nu$ as the first time $t$ such that $\nu P^t$ gets close to the stationary distribution $\mu$.

$$t_{mix}(P, \nu, \epsilon) = \min \{t \mid d_{TV}(\nu P^t, \mu) \leq \epsilon\}.$$ 

We drop $P$ and $\nu$ if they are clear from context. If $\nu$ is the Dirac measure on a single point $\omega$, we write $t_{mix}(P, \omega, \epsilon)$ for the mixing time. When mixing time is referenced without mentioning $\epsilon$, we imagine that $\epsilon$ is set to a reasonable small constant (such as $1/4$). This is justified by the fact that the growth of the mixing time in terms of $\epsilon$ can be at most logarithmic [LP17].

We can relate the mixing time and conductance as follow.

Theorem 10 ([see, e.g., LP17, Thm. 7.4]). For a reversible Markov chain $P$ with conductance $\Phi$, we have

$$t_{mix}(P, 1/4) \leq 4\Phi.$$

$^4$also known as bottleneck ratio in [LP17]
2.4 The Down-Up Random Walk

Consider a distribution \( \mu : \binom{[n]}{k} \rightarrow \mathbb{R}_{\geq 0} \). The down-up walk is given by the composition of two row-stochastic operators, known as the down and up operators.

**Definition 11 (Down Operator).** For a ground set \( \Omega \), and \(|\Omega| \geq k \geq \ell\), define the down operator \( D_{k \to \ell} \in \mathbb{R}^{(\ell \setminus k)} \times (\ell) \) as

\[
D_{k \to \ell}(S, T) = \begin{cases} 
\binom{\ell}{k} & \text{if } T \subseteq S, \\
0 & \text{otherwise}.
\end{cases}
\]

Note that \( D_{k \to \ell} D_{\ell \to m} = D_{k \to m} \).

**Definition 12 (Up Operator).** For a ground set \( \Omega \), \(|\Omega| \geq k \geq \ell\), and density \( \mu : \binom{[\ell]}{\ell} \rightarrow \mathbb{R}_{\geq 0} \), define the up operator \( U_{\ell \to k} \in \mathbb{R}^{(\ell \setminus k)} \times (\ell) \) as

\[
U_{\ell \to k}(T, S) = \begin{cases} 
\frac{\mu(S)}{\sum_{S' \supseteq T} \mu(S')} & \text{if } T \subseteq S, \\
0 & \text{otherwise}.
\end{cases}
\]

If we define \( \mu_k = \mu \) and more generally let \( \mu_\ell \) be \( \mu_k D_{k \to \ell} \), then the down and up operators satisfy

\[
\mu_k(S) D_{k \to \ell}(S, T) = \mu_\ell(T) U_{\ell \to k}(T, S).
\]

This property ensures that the composition of the down and up operators have the appropriate \( \mu \) as a stationary distribution, are reversible, and have nonnegative real eigenvalues.

**Proposition 13** ([see, e.g., KO18; AL20; ALO20]). The operators \( D_{k \to \ell} U_{\ell \to k} \) and \( U_{\ell \to k} D_{k \to \ell} \) both define Markov chains that are time-reversible and have nonnegative eigenvalues. Moreover \( \mu_k \) and \( \mu_\ell \) are respectively their stationary distributions.

**Definition 14 (Down-Up Walk).** For a ground set \( \Omega \), \(|\Omega| \geq k \geq \ell\), and density \( \mu : \binom{[\ell]}{\ell} \rightarrow \mathbb{R}_{\geq 0} \), the \( k \leftrightarrow \ell \) down-up walk is defined by the row-stochastic matrix \( U_{\ell \to k} D_{k \to \ell} \).

2.5 Real-Stable and Sector-Stable Polynomials

We use \( \mathbb{F}[z_1, \ldots, z_n] \) to denote \( n \)-variate polynomials with coefficients from \( \mathbb{F} \), where we usually take \( \mathbb{F} \) to be \( \mathbb{R} \) or \( \mathbb{C} \). We denote the degree of a polynomial \( g \) by \( \text{deg}(g) \). We call a polynomial homogeneous of degree \( k \) if all nonzero terms in it are of degree \( k \).

**Definition 15 (Stability).** For an open subset \( U \subseteq \mathbb{C}^n \), we call a polynomial \( g \in \mathbb{C}[z_1, \ldots, z_n] \) \( U \)-stable iff

\[
(z_1, \ldots, z_n) \in U \implies g(z_1, \ldots, z_n) \neq 0.
\]

We also call the identically 0 polynomial \( U \)-stable. This ensures that limits of \( U \)-stable polynomials are \( U \)-stable. For convenience, when \( n \) is clear from context, we abbreviate stability w.r.t. regions of the form \( U \times U \times \cdots \times U \) where \( U \subseteq \mathbb{C} \) simply as \( U \)-stability.

Our choice of the region \( U \) in this work is the product of open sectors in the complex plane.

**Definition 16 (Sectors).** We name the open sector of aperture \( \alpha \pi \) centered around the positive real axis \( \Gamma_a \):

\[
\Gamma_a := \{ \exp(x + iy) \mid x \in \mathbb{R}, y \in (-\alpha \pi/2, \alpha \pi/2) \}.
\]
Note that $\Gamma_1$ is the right-half-plane, and $\Gamma_1$-stability is the same as the classically studied Hurwitz-stability [see, e.g., Brä07]. Another closely related notion is that of real-stability where the region $U$ is the upper-half-plane $\{z \mid \Im(z) > 0\}$ [see, e.g., BBL09]. Note that for homogeneous polynomials, stability w.r.t. $U$ is the same as stability w.r.t. any rotation/scaling of $U$; so Hurwitz-stability and real-stability are the same for homogeneous polynomials.

We use $\alpha$-sector-stable as a shorthand for $\Gamma_\alpha$-stable. Naturally, we call a distribution $\alpha$-sector-stable if its generating polynomial is $\alpha$-sector-stable.

**Proposition 17 ([Ali+21]).** The following operations preserve $S_\alpha$-sector-stability on homogeneous multi-affine polynomials:

1. Specialization: $g \mapsto g(a, z_2, \ldots, z_n)$, for $a \in \bar{S}_\alpha$.
2. Derivative: $g \mapsto \frac{\partial}{\partial z_1} g(z_1, \ldots, z_n)$.
3. Scaling: $g \mapsto g(\lambda_1 z_1, \ldots, \lambda_n z_n)$, for $\lambda \in \mathbb{R}^n_{\geq 0}$.

We state some examples of sector stable distributions.

**Lemma 18 ([Ali+21]).** Consider $L \in \mathbb{R}^{n \times n}$ that is nPSD, i.e., $L + L^T \succeq 0$, then $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ defined by $\mu(S) = \det(L_{S,S})$ is $1/2$-sector-stable.

**Lemma 19 ([Ali+21]).** Given a density $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ and a partition $T_1 \cup T_2 \cup \cdots \cup T_s = [n]$, and numbers $c_1, \ldots, c_s \in \mathbb{Z}_{\geq 0}$, let the partition constraint density $\mu_{T,c}$ be $\mu$ restricted to sets $S \in \binom{[n]}{k}$ where $|S \cap T_i| = c_i$. When $\mu$ is strongly Rayleigh, $\mu_{T,c}$ is $1/2^c$-sector-stable.

### 2.6 Log-Concavity and Fractional Log-Concavity

We now formally introduce log-concavity for distributions over size-$k$ subsets of $n$ elements, and its direct generalization, $\alpha$-fractional-log-concavity.

**Definition 20.** A function $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ is log-concave if $\log f(z_1, \ldots, z_n)$ is concave over $\mathbb{R}^n_{\geq 0}$, i.e. for all $x, y \in \mathbb{R}^n_{\geq 0}$ and $\lambda \in [0, 1]$, we have:

$$g(\lambda x - (1 - \lambda)y) \geq g(x)^{\lambda} \cdot g(y)^{1-\lambda} \iff \log g(\lambda x - (1 - \lambda)y) \geq \lambda \log g(x) + (1 - \lambda) \log g(y)$$

We say a probability distribution $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ is log-concave if $\log g(z_1, \ldots, z_n)$ is concave over $\mathbb{R}_{\geq 0}$, or in other words, that its generating polynomial is a log-concave function over $\mathbb{R}_{\geq 0}$.

[Ana+19; BH19] shows that for homogeneous multiaffine polynomials, real-stability implies log-concavity. A similar relationship holds for sector stability and fractional log-concavity.

**Lemma 21.** (Lemma 67 from [Ali+21]) If a polynomial $g$ is $\alpha$-sector-stable, then it is $\frac{1}{2}$-fractionally-log-concave.

We note that scaling preserves $\alpha$-log-concavity of homogeneous distributions [Ali+21] i.e. if $\mu$ is $\alpha$-log-concave, then so is $\lambda * \mu$ for all $\lambda \in \mathbb{R}^n_{\geq 0}$.

**Theorem 22 ([Ali+21; Ana+21a]).** Suppose $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ is $\alpha$-fractional-log-concave. The $k \leftrightarrow (k - \lceil 1/\alpha \rceil)$-down-up-walk on $\mu$ has spectral gap at least $\Omega(k^{-1/\alpha})$. 

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2.7 Composable Core-Set

**Definition 23** ([Mah+19, Definition 2.2]). A function \( c(P) \) that maps the input set \( P \subseteq \mathbb{R}^d \) to one of its subsets is called an \( \alpha \)-composable core-set for a function \( f : 2^{\mathbb{R}^d} \to \mathbb{R} \) if, for any collection of sets \( P_1, \ldots, P_n \subseteq \mathbb{R}^d \) we have \( f(C) \geq f(P)/\alpha \) where \( P = \bigcup_{i \leq n} P_i \) and \( C = \bigcup_{i \leq n} c(P_i) \).

3 MAP Inference via Local Search

In this section, we show how to efficiently find a local optima\(^5\) of a given distribution \( \mu \). We run a two stage algorithm:

(i) first, we find some “good” initial subset \( S_0 \in \binom{[n]}{k} \), i.e., one such that the ratio \( \text{OPT}/\mu(S_0) \) is bounded by \( 2^{\text{poly}(n,k)} \) (see Lemma 26),

(ii) then, for a suitably chosen radius \( r \in \mathbb{N}_{>1} \), we run a simple local search (Algorithm 2) that starts with \( S \leftarrow S_0 \), and find better and better solutions by swapping at most \( r \) elements in \( S \) for elements outside of \( S \) until no more improvement in term of \( \mu(S) \) can be found.

To ensure that our algorithm terminates within polynomial time, we will only take improvements that increase the determinant by at least a lower multiplicative threshold, say, by a factor of 2.

**Algorithm 2**: LOCAL-SEARCH-\( r \) (LS\(_r\))

**Input**: \( \alpha \leq 1, S_0 \in \binom{[n]}{k} \) with \( \mu(S_0) > 0 \).

Initialize \( S \leftarrow S_0 \).

while \( \mu(S) < \zeta \cdot \mu(T) \) for some \( T \in \mathcal{N}_r(S) \) do

\[ \text{Update } S \leftarrow \arg \max_{T \in \mathcal{N}_r(S)} \mu(T). \]

We prove the algorithmic part of Theorem 2, that with a suitable choice for \( S_0 \), Algorithm 2 runs in polynomial time.

**Proposition 24.** The number of steps taken by Algorithm 2 with \( r = O(1) \) starting from \( S_0 \) is at most

\[ \log_{1/\alpha} \left( \text{OPT}/\mu(S_0) \right). \]

Each step can be implemented using \( O((nk)^r) \) oracle access to \( \mu \).

**Proof.** Each iteration improves \( \mu(S) \) by a factor of at least \( 1/\alpha \). On the other hand, this value can never exceed \( \text{OPT} \), and it starts as \( \mu(S_0) > 0 \).

Clearly, to perform local search in the \( r \)-neighborhood of a set \( S \), we only need to query \( \mu((S \setminus U_1) \cup U_2) \) for \( U_1 \in \binom{[n]}{\leq r} \) and \( U_2 \in \binom{[n]}{\leq r} \). The total number of such queries is \( O((nk)^r) \).

\[ \square \]

**Definition 25.** For \( \mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0} \) and \( \alpha > 0 \), we say \( S \in \binom{[n]}{k} \) is a \((r, \zeta)\)-local maximum w.r.t. \( \mu \) if \( \mu(S) \geq \zeta \mu(T) \) for all \( T \in \mathcal{N}_r(S) \).

\(^5\)More precisely, we show how to find an approximate local optima, which is sufficient for our purpose.
Clearly, when Algorithm 2 terminates, the output is a \((r, \zeta)\)-local maximum.

Next, we show how to obtain a “good” initialization \(S_0\) by a simple greedy algorithm, which we call INDUCED-GREEDY, that is based on maximizing the marginal gain defined by the distribution on size \(\leq k\) subsets. This gain is induced by the distribution \(\mu\), as defined below.

For subset \(T\) of \([n]\) of size \(\leq k\), let \(\mu(T) = \sum_{S \in [n]: S \supseteq T} \mu(S)\).

**Algorithm 3:** INDUCED-GREEDY

1. Initialize \(S \leftarrow \emptyset\).
2. While \(|S| < k\) do
   - Pick \(i \notin S\) that maximizes \(\mu(S \cup \{i\})\) and update \(S \leftarrow S \cup \{i\}\).

**Lemma 26.** Algorithm 3 returns \(S_0\) with

\[
O(n^k) \cdot \det(L_{S_0}) \geq \text{OPT}.
\]

**Proof.** For \(j \in [k]\), let \(i_j\) be the element added to \(S\) at the \(j\)-th iteration of the while loop. Let \(S_0 = \emptyset, S_j = S_{j-1} \cup \{i_j\}\). Observe that \(|S_j| = j\) and for each \(j \geq 0\)

\[
\mu(S_j) = \frac{1}{k - |S_j|} \sum_{i \in S_j} \mu(S_j \cup \{i\}) \leq \frac{n - j}{k - j} \mu(S_{j+1})
\]

thus 

\[
\binom{n}{k} \mu(S_k) \geq \mu(S_0) = \mu(\emptyset) = \sum_{S' \subseteq [n]} \mu(S') \geq \text{OPT}.
\]

\[\square\]

**Remark 27.** In Algorithm 3, it is enough to find \(i\) that approximately maximizes \(\mu(S \cup i)\) i.e. for some constant \(\zeta \in (0, 1)\), \(\mu(S \cup i) \geq \zeta \mu(S \cup j)\) for all \(j \notin S\). In that case, Lemma 26 still holds, and Algorithm 3 can be efficiently implemented given access to efficient algorithms that approximately sample from \(\lambda \ast \mu\) for \(\lambda \in \mathbb{R}_{\geq 0}^n\). Indeed, note that \(\mu(S \cup i)/\mu(S)\) is the marginal of \(\lambda \ast \mu\) where \(\lambda_i = \begin{cases} \infty & \text{for } i \in S \\ 1 & \text{else} \end{cases}\). Thus, \(\mu(S \cup i)\) can be approximate within some small constant factor.

### 4 From Sampling to Optimization via Local Search

In this section, we prove Theorem 2.

**Definition 28** \((r, \text{-exchange})\). For \(\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}, r \in \mathbb{N}\) and \(S, T \subseteq \binom{[n]}{k}\), we let

\[
\mathcal{E}^r(S, T) := \{U \subseteq S \Delta T : |U \cap S| = |U \cap T| = r\}
\]

be the set of all \(r\)-exchanges between \(S\) and \(T\).

**Definition 29** (Weak \((r, \beta)\)-approximate exchange). We say a distribution \(\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}\) satisfies weak \((r, \beta)\)-approximate exchange if for any \(S, T \subseteq \binom{[n]}{k}\), there exists \(s \in \{1, \ldots, r\}\) and \(U \in \mathcal{E}^{s}(S, T)\) such that

\[
\mu(S) \leq \beta \cdot \mu(S \Delta U) \left(\frac{\mu(S)}{\mu(T)}\right)^{s/d(S, T)}
\]
Lemma 30. Consider $\mu : \{[n]\} \to \mathbb{R}_{\geq 0}$ such that for all external field $\lambda \in \mathbb{R}^{\binom{n}{2}}_{\geq 0}$, the conductance of the $k \leftrightarrow (k-r)$-down-up walk on $\lambda \ast \mu$ is at least $\Omega(k^{-c})$. Then $\mu$ satisfies weak $(r, O(k^{c}+c))$-approximate exchange.

Proposition 31. If $\mu : \{[n]\} \to \mathbb{R}_{\geq 0}$ satisfies weak $(r, \beta)$-approximate exchange then any $(r, \zeta)$-local max with respect to $\mu$ is also an $O((\beta/\zeta)^{k})$-approximate global max.

In particular, when $\mu$ is $\alpha$-fractionally log-concave, Lemma 30 and Proposition 31 hold with $r = \lceil 1/\alpha \rceil$ and $c = 1/\alpha$ and $\beta = O(k^{c})$.

Corollary 32. Let $\mu : \{[n]\} \to \mathbb{R}_{\geq 0}$ be strongly Rayleigh. Given a density $\mu : \{[n]\} \to \mathbb{R}_{\geq 0}$ and a partition $T_{1} \cup T_{2} \cup \cdots \cup T_{s} = \{[n]\}$ and numbers $c_{1}, \ldots, c_{s} \in \mathbb{Z}_{\geq 0}$, let the partition constraint density $\mu_{T, c}$ be $\mu$ restricted to sets $S \in \binom{[n]}{k}$ where $|S \cap T_{i}| = c_{i}$. When $\mu$ is strongly Rayleigh and $c = O(1)$, one can efficiently finds a $k^{O(k)}$-approximation for $\max \mu_{T, c} \cdot \cdot \cdot$ using Algorithm 2 with $r = 2^{c}$.

The local search guarantee in Theorem 2 follows from Theorems 9 and 22, Lemma 30, and Proposition 31, and the runtime bound follows from Remark 27 and Proposition 24.

Proof of Lemma 30. If $d(S, T) \leq r$ then the lemma holds trivially by setting $U = S \Delta T$. In what follows, we assume $d(S, T) \geq r$. Wlog we can assume that $S = \{1, \ldots, t\} \cup C$ and $T = \{t + 1, \ldots, 2t\} \cup C$ with $C = \{2t + 1, \ldots, t + k\}$ and $t = d(S, T)$.

Consider distribution $\mu' = \lambda \ast \mu$ with $\lambda_{i} = \begin{cases} 1 & \text{if } 1 \leq i \leq t \\ (\mu(S)/\mu(T))^{t} & \text{if } t + 1 \leq i \leq 2t \\ \infty & \text{if } 2t + 1 \leq i \leq t + k \\ 0 & \text{else} \end{cases}$.

Note that $\mu'$ is supported on $W \in \binom{[n]}{k}$ where $(S \cap T) = C \subseteq W \subseteq (S \cup T)$. Let $\Phi$ be the conductance of the $k \leftrightarrow (k-r)$-down-up walk on $\mu'$, then $\Phi \geq \Omega(k^{-c})$. On the other hand, since $\mu'(S) = \mu'(T) = \mu(S) \leq \frac{\sum_{W} \mu'(W)}{2}$, we have that by definition of $\Phi$

$$\Phi = \min_{\mu'(S) \leq \mu'(S)/2} \frac{Q(S, \Omega \setminus S)}{\mu'(S)} \leq \frac{Q(\{S\} \cup \{S\})}{\mu'(S)}$$

where we can rewrite $Q(\{S\} \cup \{S\})$ as

$$Q(\{S\} \cup \{S\}) = \mu'(S) \frac{1}{\binom{r}{T}} \sum_{U_{1} \in \binom{[r]}{t}} \sum_{W \supseteq S \setminus U_{1}} \frac{\mu'(W)}{\mu'(S \setminus U_{1})}$$

where $\mu'(S \setminus U_{1}) = \sum_{W \in \binom{[n]}{k}, W \supseteq S \setminus U_{1}} \mu'(W)$.

Note that

$$\{W \in \supp(\mu') \setminus \{S\} \mid W \supseteq S \setminus U_{1}\} \subseteq \left\{(S \setminus U_{1}) \cup U_{2} \mid U_{2} \in \binom{T \cup U_{1}}{r} \setminus \{U_{1}\}\right\}$$

thus

$$\bigcup_{U_{1} \in \binom{[r]}{t}} \left\{W \in \supp(\mu') \setminus \{S\} \mid W \supseteq S \setminus U_{1}\right\} \subseteq \left\{S \Delta U \mid U \in \bigcup_{s=1}^{r} \mathcal{E}(S, T)\right\}.$$
Moreover, \(|\binom{k}{r}| = \binom{k}{r}^\star\) and for each \(U_1 \in \binom{k}{r}\), the cardinality of \(\{W \in \text{supp}(\mu') \setminus \{S\} \mid W \supseteq S \setminus U_1\}\) is at most \(\binom{k+r}{r} - 1 \leq k'\). Hence, there must exist \(r \in [s]\) and \(U \in \mathcal{E}^s(S, T)\) such that

\[
\frac{\mu'(S \Delta U)}{\mu'(S \setminus U)} \geq \frac{1}{k'} \frac{Q(\{S\}, \Omega \setminus \{S\})}{\mu'(S)} \geq \Omega(k^{-(r+c)}).
\]

Thus

\[
\mu(S) = \mu'(S) \leq \mu'(S \setminus U) \leq O(k^{r+c})\mu'(S \Delta U) = O(k^{r+c})\mu(S \Delta U)(\frac{\mu(S)}{\mu(T)})^{s/t}.
\]

\[\square\]

**Proof of Proposition 31.** Apply Lemma 30 for \(S\) being a \((r, \zeta)\)-local max and \(T := \arg \max \mu(W)\). Let \(t = d(S, T)\). For some \(s \in [r]\) and \(U \in \mathcal{E}^s(S, T)\)

\[
\mu(S) \leq O(k^{r+c})\mu(S \Delta U)(\frac{\mu(S)}{\mu(T)})^{s/t} \leq O(k^{r+c}/\zeta)\mu(S)(\frac{\mu(S)}{\mu(T)})^{s/t}
\]

where the inequality follows from definition of \((r, \zeta)\)-local max. Divide both sides by \(\mu(S) > 0\), we get

\[
\mu(T) \leq O(k^{r+c}/\zeta)^{t/s}\mu(S) \leq O(k^{r+c}/\zeta)^k \mu(S).
\]

where we use the fact that \(t/s \leq k\).

\[\square\]

### 5 Improved Local Search for Sector-Stable Distributions

By Lemmas 21 and 30, for any \(\alpha\)-sector-stable distribution \(\mu : \binom{n}{k} \rightarrow \mathbb{R}\), Algorithm 2 with \(r = \lceil \frac{\alpha}{\alpha} \rceil\) finds a \(k^{O(k/\alpha)}\)-approximation of OPT. In this section, we show how to improve the local search radius \(r\) to \(\lceil \frac{1}{\alpha} \rceil\) for \(\alpha \in [1/2, 1]\). As an application, we prove Theorem 6.

When \(\alpha = 1\), the distribution \(\mu\) is real stable, thus log-concave, and Proposition 31 already shows \(LS_1\) gives a \(k^{O(k)}\)-approximation for MAP inference. Clearly, for \(\alpha \in [1/2, 1]\), any \(\alpha\)-sector-stable \(\mu\) is also 1/2-sector-stable, and \(\lceil 1/\alpha \rceil = 2\), so we only need to consider the case \(\alpha = 1/2\).

**Definition 33** \((r, \beta)\)-approximate exchange. For \(r \in \mathbb{N}_{\geq 1}\) and \(\beta > 0\), we say \(\mu : \binom{n}{k} \rightarrow \mathbb{R}_{\geq 0}\) satisfies \((r, \beta)\)-approximate exchange if for any \(S, T \in \binom{n}{k}\)

\[
\mu(S)\mu(T) \leq \max_{i=1}^r \left\{ \beta^i M^i(S \rightarrow T)M^i(T \rightarrow S) \right\}
\]

where \(M^i(S \rightarrow T) := \max_{U \in \mathcal{E}^i(S, T)} \mu(S \Delta U)\).

**Theorem 34.** Suppose \(\mu : \binom{n}{k} \rightarrow \mathbb{R}_{\geq 0}\) is 1/2-sector stable. For any \(S, T \in \binom{n}{k}\)

\[
\mu(S)\mu(T) \leq \max_{i=1}^2 \left\{ \left( \sum_{U \in \mathcal{E}^i(S, T)} \mu(S \Delta U) \right) \left( \sum_{U \in \mathcal{E}^i(S, T)} \mu(T \Delta U) \right) \right\}
\]

\[
\leq \sum_{i=1}^2 \left\{ k^i M^i(S \rightarrow T)M^i(T \rightarrow S) \right\}. \tag{3}
\]

Consequently, \(\mu\) satisfies \((2, k^4)\)-approximate exchange.
We prove the approximate exchange property by relying on the following theorem.

**Theorem 35** ([Asn70]). Consider a univariate 1-sector-stable (Hurwitz-stable) polynomial \( f(z) = \sum_{i=0}^{n} a_i z^i \) with \( a_i \geq 0 \forall i \). Its Hurwitz matrix \( H = (h_{ij}) \in \mathbb{R}^{n \times n} \) is defined by \( h_{ij} = a_{2j-i} \) when \( 0 \leq 2j-i \leq n \), otherwise \( h_{ij} = 0 \). \( H \) is totally nonnegative, in the sense that all its minors are nonnegative.

As an immediate consequence, we obtain the following lemma about coefficients of univariate Hurwitz stable polynomial.

**Lemma 36.** If \( f(z) = a_n z^n + \cdots + a_1 z + a_0 \) with \( a_i \geq 0 \forall i \) is 1-sector stable, then \( a_n a_0 \leq \max \{ a_1 a_{n-1}, a_2 a_{n-2} \} \)

**Proof.** If \( n \leq 2 \) then the claim is trivially true. Below, we assume \( n \geq 3 \). We consider two cases, when \( n \) is odd and when \( n \) is even. Suppose \( n = 2t - 1 \) for \( t \in \mathbb{N} \). By Theorem 35, all minors of \( H \) are non-negative, hence

\[
\det \begin{bmatrix} h_{1,1} & h_{1,t} \\ h_{2,1} & h_{2,t} \end{bmatrix} = \det \begin{bmatrix} a_1 & a_{2t-1} \\ a_0 & a_{2t-2} \end{bmatrix} = a_1 a_{2t-2} - a_0 a_{2t-1} = a_1 a_{n-1} - a_0 a_n \geq 0.
\]

Suppose \( n = 2t \) for \( t \in \mathbb{N} \). Again, Theorem 35 implies

\[
\det \begin{bmatrix} h_{2,2} & h_{2,t+1} \\ h_{4,2} & h_{4,t+1} \end{bmatrix} = \det \begin{bmatrix} a_2 & a_{2t} \\ a_0 & a_{2t-2} \end{bmatrix} = a_2 a_{2t-2} - a_0 a_{2t} = a_2 a_{n-2} - a_0 a_n \geq 0.
\]

**Lemma 36,** in turn implies the following fact about coefficients of 1/2-sector-stable univariate polynomials that only have even-degree terms.

**Corollary 37.** If \( f(z) = \sum_{i=0}^{t} a_{2i} z^{2i} \) with \( a_{2i} \geq 0 \forall i \) is 1/2-sector stable, then \( a_0 a_{2t} \leq \max \{ a_2 a_{2t-2}, a_4 a_{2t-4} \} \)

**Proof.** Let \( g(z) = f(z^{1/2}) = \sum_{i=0}^{t} a_{2i} z^i \) then \( g(z) \) is 1-sector stable, and the claim follows from Lemma 36.

**Proof of Theorem 34.** Let \( f(z_1, \cdots, z_n) = \sum_{W \in \mathcal{P}(T)} \mu(W) z_W \) be the generating polynomial of \( \mu \). We deal with the case where \( S \cap T = \emptyset \) and \( |T| = S \cup T \). Other cases can be reduced to this scenario by setting \( z_i \) to 0 for \( i \notin S \cup T \), and taking derivative(s) of \( f \) with respect to \( i \in S \cap T \). Recall that setting variables to 0 and taking derivative(s) preserve 1/2-sector-stability and homogeneity of polynomials (see Proposition 17). W.l.o.g., assume \( S = [t] \) and \( T = \{ t+1, \cdots, 2t \} \). We can rewrite \( f \) as \( f(z_1, \cdots, z_{2t}) = \sum_{W \in \mathcal{P}(T)} \mu(W) z_W \).

In \( f \), set \( z_i = z \) if \( i \in S \) and \( z_i = z^{-1} \) if \( i \in T \). We obtain a single variate 1/2-stable polynomial

\[
\tilde{f}(z) = z^t f(z, \cdots, z, z^{-1}, \cdots, z^{-1}) = \sum_{i=0}^{2t} b_i z^i = \sum_{i=0}^{t} b_{2i} z^{2i}
\]

Note that a term \( \mu(W) z_W \) contribute to \( b_{2i} \) if and only if \( i = |W \cap S| + (t - |W \cap T|) = 2 |W \cap S| \).

In particular, \( b_{2i+1} = 0 \) for all \( i \in \mathbb{N} \) and

\[
b_{2i} = \sum_{W : |W \cap S| = i} \mu(W) = \sum_{U \in \mathcal{E}^t(S,T)} \mu(T \Delta U) = \sum_{U \in \mathcal{E}^t+1(S,T)} \mu(S \Delta U).
\]
In particular, $b_2t = \mu(S)$ and $b_0 = \mu(T)$. The first line of Eq. (3) follows by applying Corollary 37 to $\tilde{f}$, and the second line follows by observing that $\sum_{U \in \mathcal{E}(T)} \mu(T \Delta U) \leq \left(\frac{|S\Delta T|}{i}\right)^2 \max_{U \in \mathcal{E}(S,T)} \mu(T \Delta U)$.

**Lemma 38.** Suppose $\mu : \binom{[n]}{k} \rightarrow \mathbb{R}_{\geq 0}$ satisfies $(r, \beta)$-exchange and $S$ is a $(r, \zeta)$ local maximum with $\alpha \leq 1$ and $\mu(S) > 0$. Then $S$ is a $(\beta / \zeta)^k$-approximate global optimum:

$$
(\beta / \zeta)^k \mu(S) \geq \max_{T \in \binom{[n]}{k}} \mu(T).
$$

**Proof.** Let $S : \binom{[n]}{k}$ be a $(r, \alpha)$-local maximum with $\mu(S) > 0$ and let $S^* := \arg \max_{T \in \binom{[n]}{k}} \mu(T)$. We first prove the following claim.

**Claim 39.** For any $T : \binom{[n]}{k}$ where $T \neq S$, there exists $i \in [r]$ and $W \in \binom{[n]}{k}$ such that $d(S,W) = d(S,T) - i$ and $\mu(T) \leq \frac{\beta^i}{\zeta} \mu(W)$.

**Proof of Claim 39.** By Definition 33, for some $i \in [r]$, there exists $U_1, U_2 \in \mathcal{E}(S, T)$ such that

$$
\mu(S) \mu(T) \leq \beta^i \mu(S \Delta U_1) \mu(T \Delta U_2) \leq \beta^i \frac{\mu(S)}{\zeta} \mu(T \Delta U_2)
$$

where the last inequality follows from the definition of $(r, \zeta)$-local maximum.

Note that $d(S, T \Delta U_2) = d(S, T) - i$. Setting $W = T \Delta U_2$ and dividing both sides by $\mu(S) > 0$ gives the desired inequality. \qed

Note that initially $d(S, S^*) \leq k$. We can iteratively apply Claim 39 for up to $k$ times to obtain the desired inequality. Indeed, let $T_0 = S^*$, and for $j \geq 1$ let $i_j \in [r]$ and $T_j \in \binom{[n]}{k}$ be such that $\mu(T_{j-1}) \leq \frac{\beta^{i_j}}{\zeta} \mu(T_j)$ and $d(S, T_j) = d(S, T_{j-1}) - i_j$. Claim 39 guarantees the existence of such $i_j$ and $T_j$, as long as $T_{j-1} \neq S$. Let $s$ be the minimum index such that $d(S, T_s) = 0$. Note that $s \leq k$ and $T_s = S$. We have

$$
\mu(S^*) = \mu(T_0) \leq \frac{\beta^{i_1}}{\zeta} \cdot T_1 \leq \frac{\beta^{i_2}}{\zeta} \cdot T_2 \leq \cdots \leq \prod_{j=1}^{s} \frac{\beta^{i_j}}{\zeta} \mu(T_s) \leq \frac{\beta^k}{\zeta^k} \mu(S)
$$

where the last inequality follows from the facts that $\sum_{j=1}^{s} i_j = d(S, T_0) - d(S, T_s) \leq k$ and $(\frac{\beta}{\zeta})^s \leq \left(\frac{1}{\zeta}\right)^k$. \qed

Now, we are ready to prove Theorem 6.

**Proof of Theorem 6.** We let $\mu(S) = \det(L_S)$ and run the two stage algorithm in Section 3 with $r = 2$. The approximation guarantee is a direct consequence of Lemmas 18 and 38 and Theorem 34.

Suppose we are given access to the entries of $L$. Each iteration of Algorithm 2 clearly runs in $O(n^2k^5)$ time, since $N_2(S)$ has at most $O(k^2n^2)$ elements and computing the determinant of $k \times k$ matrices costs $O(k^3)$ time. The cost of $LS_2$ can be reduced to $O(n^2k^4)$ time using Schur
complements to compute all \(\det(L_{Y \cup D})\) for each fixed \(Y\) and all \(D\) of size \(\leq r\) in \(O(k^3 + n^2k^2)\) time. [see Eq. (2) or Gar+20, for example]. If we are only given \(B, C\), then each of these submatrices and their determinant can be computed in \(O(d^2)\) time, so that each iteration takes \(O(n^2d^2k^2)\) time.

Now, we bound the runtime of Algorithm 3. To implement each iteration of Algorithm 3, we need to compute \(\mu(Y) = \sum_{S \in \binom{[n]}{\leq Y}} \det(L_Y)\), which is the coefficient of \(\lambda^{n-k}\) in \(g(\lambda) = \det(L + \lambda \cdot \text{diag}I_Y)\) where \(\bar{Y} = [n] \setminus Y\).

There are several ways to compute \(\mu(Y)\). To compute the coefficients of polynomial \(g(\lambda)\) of degree \(\leq n\), we can evaluate \(g\) at \(n + 1\) distinct points \(\lambda\) and use polynomial interpolation, i.e., solve a linear system of equations involving the the Vandermonde matrix. A more efficient way, which costs \(O(n^3)\) per computation of \(\mu(T)\), for a total runtime of \(O(n^4k)\), is as follow:

(i) Let \(D = \text{diag}I_Y\). We use the QZ decomposition algorithm [GV96, Section 7.7, p. 313] to compute unitary matrices \(Q, Z\) such that

\[
L = Q\bar{A}Z^*, D = Q\bar{D}Z^*
\]

where \(\bar{A}, \bar{D}\) are both upper triangular. Note that \(\deg(g) \leq n - |T|\).

Compute the roots of \(g(\lambda) = \det(L + \lambda D)\), which are exactly the generalized eigenvalues \(\lambda_1, \ldots, \lambda_{\deg(g)}\) defined by \(\lambda_i = \frac{A_{ij}}{B_{ij}}\) where we may assume w.l.o.g. that \(\bar{D}_{ij} \neq 0\) for \(i = 1, \ldots, \deg(g)\), and is zero otherwise. Let \(c := \prod_{i \in [n]: B_{ij} \neq 0} \bar{D}_{ij} \prod_{i \in [n]: D_{ij} = 0} \bar{A}_{ij}\). Then

\[
g(\lambda) = c \prod_{i \in [\deg(g)]} (\lambda - \lambda_i)
\]

(ii) We then compute the \((k - n + n')\text{-th}\) symmetric polynomial of \(\lambda_1, \ldots, \lambda_{n - |T|}\) where \(e_t = \sum_{W \in \{k - |T|\}} \prod_{j \in W} \lambda_j\) using the recursion [KT12]

\[
et_t = e_{t-1}p_1 - e_{t-2}p_2 + e_{t-3}p_3 - \cdots \pm p_k
\]

with \(p_t = \sum \lambda_i^t\), and output \(\mu(Y) = |ce_{k-\deg(g)}(g)|\).

Given the low-rank decomposition \(L = BCB^T\), we can further optimize by reducing the cost of step (i) to \(O(nd^2)\). Then the total runtime will be \(O(n^2kd^2)\).

Let \(L_Y\) be the kernel of \(P_L\) conditioned on the inclusion of items in \(Y\). The eigenvalues of \(L_Y\) are exactly the roots of \(g(\lambda)\). By Eq. (2), \(L_Y\) can be rewritten as product of two matrices of rank \(\leq d\), thus the nonzero eigenvalues of \(L_Y\) can be computed in \(O(d^3)\) time. Indeed, let \(D_Y := B_Y^T(B_YCB_Y^T)^{-1}B_Y\) then \(L_Y = B_Y(C - CD_YC)B_Y^T\) and \(\text{rank}(D_Y) \leq k\) and \(D_Y\) can be computed in \(O(kd^2)\) time (see Eq. (2)). The matrix \(F_Y := \left((C - CD_YC)B_Y^T\right)B_Y\) has the same characteristic polynomial and nonzero eigenvalues as \(L_Y\). Clearly, \(\text{rank}(F_Y) \leq \text{rank}(B) \leq d\), so \(F_Y\) and its eigenvalues can be computed in \(O(nd^2)\) time.

6 Composable Core-Sets via Local Search

Here we prove that local search yields composable core-sets for distributions that satisfy a strong form of exchange.
Claim 42 guarantees the existence of such $W$, and the fact that strongly Rayleigh (log concave resp.) distributions satisfy $k^{O(k)}$-strong approximate basis exchange (2$^{O(k^2)}$-strong approximate basis exchange resp.) [Ana+21b].
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