Reducing the Spectral Radius of a Torus Network by Link Removal

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Abstract

The optimal link removal (OLR) problem aims at removing a given number of links of a network so that the spectral radius of the residue network obtained by removing the links from the network attains the minimum. Torus networks are a class of regular networks that have witnessed widespread applications. This paper addresses three subproblems of the OLR problem for torus networks, where two or three or four edges are removed. For either of the three subproblems, a link-removing scheme is described. Exhaustive searches show that, for small-sized tori, each of the proposed schemes produces an optimal solution to the corresponding subproblem. Monte-Carlo simulations demonstrate that, for medium-sized tori, each of the three schemes produces a solution to the corresponding subproblem, which is optimal when compared to a large set of randomly produced link-removing schemes. Consequently, it is speculated that each of the three schemes produces an optimal solution to the corresponding subproblem for all torus networks. The set of links produced by each of our schemes is evenly distributed over a network, which may be a common feature of an optimal solution to the OLR problem for regular networks.

1 Introduction

The epidemic modeling is recognized as an effective approach to the understanding of propagation process of objects over a network [1, 2]. For instance, epidemic models help understand the prevalence of malware [3–9]. The speed and extent of spread of an epidemic over a network depend largely on the structure of the network; whether the epidemic tends to extinction is determined by the spectral radius of the network [8, 10, 11]. As a smaller spectral radius contributes to the containment of an undesirable epidemic, a natural option to mitigate the negative effect of the epidemic is to reduce the spectral radius of the underlying network.

Preciado and Jadabaie [12] suggested to remove links from a network so as to reduce its spectral radius. The optimal link removal (OLR) problem aims at removing a given number of links of a network so that the spectral radius of the residue network obtained by removing the links from the network attains the minimum. As the OLR problem is NP-hard [13], it is much unlikely that there exist an efficient algorithm for solving the problem. As a result, multifarious
heuristics for solving the problem have been proposed [13–19]. In most situations, however, these heuristics produce non-optimal solutions.

Torus networks are a class of regular networks [20]. Due to the superiority in unicast routing [21–26], multicast routing [27] and fault tolerance [28], torus networks frequently serve as the underlying interconnection network of a multicomputer system. To our knowledge, there is no report in literature on the OLR problem for torus networks.

This paper addresses three subproblems of the OLR for two-dimensional torus networks, where two or three or four edges are removed, respectively. For either of the three subproblems, a link-removing scheme is described. Exhaustive searches show that, for small-sized tori, each of the proposed schemes produces an optimal solution to the corresponding subproblem. Monte-Carlo simulations demonstrate that, for medium-sized tori, each of the three schemes produces a solution to the corresponding subproblem, which is optimal when compared to 10,000 randomly produced sets of links. Consequently, it is speculated that each of the three schemes produces an optimal solution to the corresponding subproblem for all torus networks.

The remaining materials are organized in this fashion: The preliminary knowledge is provided in Section 2. Section 3 presents the main results of this work. Finally, Section 4 outlines this work.

2 Preliminaries

For fundamental knowledge on the spectral radius of a network, see Ref. [29]. The optimal link removal (OLR) problem is formulated as follows: Given a network $G = (V, E)$ and a positive integer $k$, find a set of $k$ links of $G$ so that the spectral radius of the residue network obtained by removing the links from the network attains the minimum.

An $N \times N$ torus network, denoted $T_N$, is a network with node set $V = \{(i, j): i, j = 0, 1, \ldots, N - 1\}$, which has the following two kinds of links:

- $\{(i, j), ((i + 1) \mod N, j)\}$, which is abbreviated as $(i, j) \rightarrow$.
- $\{(i, j), (i, (j + 1) \mod N)\}$, which is abbreviated as $(i, j) \uparrow$.

Fig 1 gives a schematic diagram of $T_N$.

3 Main results

This section considers the optimal scheme of removing two or three or four links from $T_N$.

3.1 Removing two links

Let us consider a subproblem of OLR problem, where the network is torus, and two edges are removed. Denote the subproblem by OLR-T2.

In view of the symmetry of $T_N$, we may assume that one of the two links to be removed from $T_N$ is $e_1 = (0, 0) \rightarrow$. Now, let us choose the second link $e_2$ to be removed from $T_N$ in the following way.

If $N$ is even, then

$$e_2 = \left(\frac{N}{2}, \frac{N}{2}\right) \rightarrow.$$
If $N$ is odd, then
\[ e_2 = \left( \frac{N + 1}{2}, \frac{N - 1}{2} \right) \uparrow. \]

Figs 2 and 3 show the two edges determined by this scheme for $T_9$ and $T_{10}$, respectively. Exhaustive search gives the spectral radii of all residue networks obtained by removing two links from $T_N$, $3 \leq N \leq 20$, and Figs 4 and 5 exhibit the spectral radii of all residue networks of $T_9$ and $T_{10}$, respectively. It can be seen that the larger the distance between the two links, the smaller the spectral radius of the residue network. At the extreme, the spectral radius of the residue network $T_N - \{e_1, e_2\}$ attains the minimum among all residue networks obtained by removing two links from $T_N$. That is, the proposed scheme produces an optimal solution to the OLR-T2 problem.

For $21 \leq N \leq 30$, the above scheme is compared with 10,000 randomly produced schemes of removing two links in terms of the spectral radius of the residue network, see Fig 6. Clearly,
the proposed scheme produces an optimal solution as compared to the 10,000 schemes. Therefore, we propose the following conjecture.

**Conjecture 1.** For \( N \geq 3\), the proposed scheme produces an optimal solution to the OLR-T2 problem.

### 3.2 Removing three links

Let us consider a subproblem of OLR problem, where the network is torus, and three links are removed. Denote the subproblem by OLR-T3.

In view of the symmetry of \( T_N \), we may assume that one of the three links to be removed from \( T_N \) is \( e_1 = (0, 0) \rightarrow \). Now, let us choose the other two links, \( e_2 \) and \( e_3 \), to be removed from \( T_N \) in the following way.

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**Fig 2.** The two edges in \( T_{9} \).  
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If \( N \equiv 0 \mod 3 \), then

\[
e_2 = \left( \frac{N}{3}, \frac{N}{3} \right) \to, \quad e_3 = \left( \frac{2N}{3}, \frac{2N}{3} \right) \to.
\]

If \( N \equiv 1 \mod 3 \), then

\[
e_2 = \left( \left\lfloor \frac{N + 1}{3} \right\rfloor, \frac{2N + 1}{3} - \left\lfloor \frac{N + 1}{3} \right\rfloor \right) \to,
\]

\[
e_3 = \left( \left\lfloor \frac{N + 1}{3} \right\rfloor + \frac{N + 2}{3}, N - 1 - \left\lfloor \frac{N + 1}{3} \right\rfloor \right) \uparrow.
\]
If \( N \equiv 2 \) mod 3 and \( N > 5 \), then
\[
\begin{align*}
  e_2 &= \left( \left\lfloor \frac{N+2}{3} \right\rfloor, \frac{2N+2}{3} - \left\lfloor \frac{N+2}{3} \right\rfloor \right) \\
  e_3 &= \left( \left\lfloor \frac{N+2}{3} \right\rfloor + \frac{N+1}{3}, N - \left\lfloor \frac{N+2}{3} \right\rfloor \right)
\end{align*}
\]

If \( N = 5 \), then
\[
  e_2 = (0, 2) \quad \text{and} \quad e_3 = (3, 3)
\]

Figs 7–9 show the distribution of the three edges in \( T_{12}, T_{13}, \) and \( T_{11} \), respectively. For \( 3 \leq N \leq 20 \), exhaustive search shows that the spectral radius of the residue network \( T_N \) \( \setminus \{ e_1, e_2, e_3 \} \) attains the minimum among all residue networks obtained by removing three links from \( T_N \). That is, the proposed scheme produces an optimal solution to the OLR-T3 problem.

Fig 4. The spectral radii of all residue networks of \( T_9 \) obtained by removing (a) \( e_2 \), or (b) \( e_2 \).

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Fig 5. The spectral radii of all residue networks of \( T_{10} \) obtained by removing (a) \( e_2 \), or (b) \( e_2 \).

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Fig 6. A comparison between the proposed scheme and 10,000 randomly produced schemes.

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For $21 \leq N \leq 30$, the above scheme of choosing three links is compared with 10000 randomly produced schemes of choosing three links with respect to the spectral radius of the residue network obtained by removing the three chosen links from $T_N$, see Fig 10. It can be seen from this figure that the proposed scheme produces a residue network with minimum spectral radius among the 10001 schemes (the minimum spectral radius as compared with the 10,000 randomly produced schemes). Therefore, we propose the following conjecture.

**Conjecture 2.** For $N \geq 3$, the proposed scheme produces an optimal solution to the OLR-T3 problem.

### 3.3 Removing four links

Let us consider a subproblem of OLR problem, where the network is torus, and four edges are removed. Denote the subproblem by OLR-T4.
In view of the symmetry of $T_N$, we may assume that one of the four links to be removed from $T_N$ is $e_1 = (0, 0) \rightarrow$. Now, let us choose the other three links, $e_2$, $e_3$ and $e_4$, to be removed from $T_N$ in the following way.

If $N$ is even, then

$$e_2 = \left( 0, \frac{N}{2} \right) \rightarrow,$$

$$e_3 = \left( \frac{N}{2}, \left\lfloor \frac{2N - 4 \left\lfloor \frac{N}{4} \right\rfloor}{4} \right\rfloor \right) \rightarrow,$$
If \( N \) is odd and \( N > 3 \), then

\[
\epsilon_1 = \left( \frac{N}{2}, \left\lfloor \frac{4N - 4}{4} \frac{N}{2} \right\rfloor \right) \uparrow.
\]

If \( N \) is odd and \( N > 3 \), then

\[
\epsilon_2 = \left( \left\lfloor \frac{2N - 4}{4} \frac{N}{2} \right\rfloor, \frac{N}{2} \right) \uparrow,
\]

\[
\epsilon_3 = \left( \frac{N + 1}{2}, 0 \right) \uparrow,
\]
Fig 10. A comparison between our proposed scheme and 10,000 randomly produced schemes.

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If $N = 3$, then

$$e_1 = \left( \left\lfloor \frac{2N - 4}{4} \right\rfloor + \left\lfloor \frac{N}{2} \right\rfloor, N - \left\lfloor \frac{N}{2} \right\rfloor \right) 
\rightarrow .$$

If $N = 3$, then

$$e_2 = (2, 0) \uparrow, \quad e_3 = (0, 1) \rightarrow, \quad e_4 = (0, 2) \rightarrow .$$

Figs 11 and 12 show the distribution of the four edges in $T_{10}$ and $T_{11}$, respectively.

For $3 \leq N \leq 11$, exhaustive search shows that the spectral radius of the residue network $T_N - \{e_1, e_2, e_3, e_4\}$ attains the minimum among all residue networks obtained by removing four links from $T_N$. That is, the proposed scheme produces an optimal solution to the OLR-T4 problem.

For $12 \leq N \leq 20$, the above scheme of choosing four links is compared with 10,000 randomly produced schemes of choosing three links with respect to the spectral radius of the residue network obtained by removing the four chosen links from $T_N$, see Fig 13. It can be seen from this figure that the proposed scheme produces a residue network with minimum spectral
radius among the 10,001 schemes (the minimum spectral radius as compared with the 10,000 randomly produced schemes). Therefore, we propose the following conjecture.

**Conjecture 3.** For $N \geq 3$, the proposed scheme produces an optimal solution to the OLR-T4 problem.

### 4 Conclusions

In this paper, we have studied the problem of how to delete two, three or four links from an $N \times N$ torus so that the spectral radius of the residue network is minimized. Given the number of links to be removed, a scheme of removing links has been presented. For smaller $N$, exhaustive search shows that the proposed scheme is optimal among all possible schemes, because it produces a residue network with minimum spectral radius. For medium-sized $N$, Monte-Carlo experiment shows that the proposed scheme is optimal among 10,000 randomly produced schemes. We guess that the proposed scheme is optimal for all $N$ and among all schemes. From the distribution of the links determined by our schemes, we guess that an optimal scheme tends to remove a set evenly distributed set of edges.

In our opinion, similar work should be done for other kinds of regular networks such as the bijective connection networks [30–32].
Author Contributions
Conceived and designed the experiments: PL XY LXY. Performed the experiments: PL LXY. Analyzed the data: XY PL. Contributed reagents/materials/analysis tools: XY YW. Wrote the paper: XY LXY PL YW.

References
1. Ma ZE, Zhou YC, Wu JH. Modeling and Dynamics of Infectious Diseases. Higher Education Press; 2009.
2. Draief M, Massoulié L. Epidemics and Rumours in Complex Networks. Cambridge University Press; 2010.
3. Kephart JO, White SR. Directed-graph epidemiological models of computer viruses. IEEE Comput. Soc. Symp. Res. Secur. Privacy 1991; 343–359.
4. Kephart JO, White SR, Chess DM. Computers and epidemiology. IEEE Spectrum 1993; 30: 20–26. doi: 10.1109/6.275061
5. Gao C, Liu J, Zhong N. Network immunization with distributed autonomy-oriented entities. IEEE Trans. Parallel Distrib. System. 2011; 22: 1222–1229. doi: 10.1109/TPDS.2010.197
6. Gao C, Liu J. Modeling and restraining mobile virus propagation. IEEE Trans. Mobile Comput. 2011; 22: 1222–1229.

Fig 13. A comparison between our proposed scheme and 10,000 randomly produced schemes.
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7. Yang LX, Yang X, Liu J, Zhu Q, Gan C. Epidemics of computer viruses: A complex-network approach. Appl. Math. Comput. 2013; 219: 8705–8717.
8. Yang LX, Draief M, Yang X. The impact of network topology on the viral prevalence: a node-based approach. PLoS ONE 2015; 10: e0134507. doi: 10.1371/journal.pone.0134507
9. Yang LX, Yang X. A novel virus-patch mixed spreading model. PLoS ONE 2015; 10: e0137858. doi: 10.1371/journal.pone.0137858
10. Draief M, Ganesh A, Massoulie L. Thresholds for virus spread on networks. Ann. Appl. Probab. 2008; 18: 359–378.
11. Mieghem PV, Omic J, Kooij R. Virus spread in networks. IEEE/ACM Trans. Networking 2009; 17: 1–14. doi: 10.1109/TNET.2008.925623
12. Preciado VM, Jadabaie A. Spectral analysis of virus spreading in random geometric networks. IEEE Conf. Decision Control 2009; 4802–4807.
13. Mieghem PV, Stevanovic D, Kuiper F, Li C, Van de Bovenkamp R. Decreasing the spectral radius of a graph by link removals. Phys. Rev. E 2011; 84: 016101. doi: 10.1103/PhysRevE.84.016101
14. Restrepo JG, Ott E, Hunt BR. Approximating the largest eigenvalue of network adjacency matrices. Phys. Rev. E 2007; 76: 056119. doi: 10.1103/PhysRevE.76.056119
15. Milanese A, Sun J, Nishikawa T. Approximating spectral impact of structural perturbations in large networks. Phys. Rev. E 2010; 81: 046112. doi: 10.1103/PhysRevE.81.046112
16. Li C, Wang H, Mieghem PV. Degree and Principal Eigenvectors in Complex Networks. Networking 2012; 7289: 149–160.
17. Pastor-Satorras R, Vespignani A. Immunization of complex networks. Phys. Rev. E 2002; 65: 036104. doi: 10.1103/PhysRevE.65.036104
18. Mieghem PV, Wang H, Ge X, Tang S, Kuipers FA. Influence of assortativity and degree-preserving rewiring on the spectra of networks. The European Phys. J. B 2010; 76: 643–652. doi: 10.1140/epjb/e2010-00219-x
19. Chung F, Horn P, Tsiatas A. Distributing antidote using PageRank vectors. Internet Math. 2009; 6: 237–254. doi: 10.1080/15427951.2009.10129184
20. Grama A, Gupta A, Karypis G, Kumar V. Introduction to Parallel Computing. The Benjamin/Cummings Publishing Company, Inc; 1994.
21. Xiang D, Luo W. An efficient adaptive deadlock-free routing algorithm for torus networks. IEEE Trans. Parallel Distrib. System. 2012; 23: 800–808. doi: 10.1109/TPDS.2011.145
22. Yu Z, Xiang D, Wang X. Fully adaptive routing in torus networks based on center distance. Acta Elect. Sinica 2013; 41: 2113–2119.
23. Yu Z, Xiang D, Wang X. VCBR: virtual channel balanced routing in torus networks. in Proceedings of IEEE 10th International Conference on High Performance Computing and Communications & IEEE 10th International Conference on Embedded and Ubiquitous Computing, China, 2013; 1359-1365.
24. Singh A, Dally WJ, Towles B, Gupta AK. Locality-preserving randomized oblivious routing on torus networks. in Proceedings of 4th annual ACM symposium on Parallel Algorithms and Architectures, 2002.
25. Ramanujam RS. Weighted random routing on torus networks. Comput. Arch. Lett. 2009; 8: 1–4. doi: 10.1109/L-CA.2008.14
26. Chen XB. Panconnectivity and edge-pancyclicity of multidimensional torus networks. Discrete Appl. Math. 2014; 178: 33–45.
27. Hasunuma T, Morisaka C. Completely independent spanning trees in torus networks. Networks 2012; 60: 59–69. doi: 10.1002/net.20460
28. Kim HC, Lim HS, Park JH. An approach to conditional diagnosability analysis under the PMC model and its application to torus networks. Theo. Comput. Sci. 2012; 548: 98–116. doi: 10.1016/j.tcs.2014.07.006
29. Mieghem PV. Graph Spectra for Complex Networks. Cambridge University Press; 2012.
30. Fan J, Lin X. The Vk-Diagnosability of the BC Graphs. IEEE Trans. Comput. 2005; 54: 176–184. doi: 10.1109/TC.2005.33
31. Fan J, Jia X, Liu X, Zhang S, Yu J. Efficient unicast in bijective connection networks with the restricted faulty node set. Infor. Sci. 2011; 181: 2303–2315. doi: 10.1016/j.ins.2010.12.011
32. Fan J, Jia X, Cheng B, Yu J. An efficient fault-tolerant routing algorithm in bijective connection networks with restricted faulty edges. Theo. Comput. Sci. 2011; 412: 3440–3450. doi: 10.1016/j.tcs.2011.02.014