Beyond Sperner’s lemma

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Preface

In 1967 H. Scarf [Sc2] suggested a new proof of Brouwer’s fixed point theorem. In outline, his proof is similar to the well known proof based on Sperner’s lemma and Knaster–Kuratowski–Mazurkiewicz [KKM] argument. But instead of triangulations Scarf uses (sufficiently dense) finite subsets of a simplex. This forces a modification of the KKM argument: the inequalities in the coloring rule must be reversed. Also, instead of using the fundamental fact that an odd number cannot be equal zero, Scarf use a path-following algorithm motivated by the game theory and the linear programming. From Scarf’s point of view, the algorithmic nature of his proof was especially important. But, as explained in [I1], the proof based on Sperner’s lemma can be easily turned to a path-following algorithm. Moreover, path-following arguments we used already in 1929 by Hurewicz [H].

The most striking feature of Scarf’s proof is a combinatorial theorem, namely, an analogue of Sperner’s lemma dealing with finite subsets of a simplex instead of triangulations. See [Sc2], Theorem 1. The origins of this theorem and its proof also belong to the linear programming and the game theory. Actually, Scarf’s proof of Brouwer’s theorem is a

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byproduct of Scarf’s fundamental paper [Sc_1] in game theory. A topological interpretation of Scarf’s combinatorial theorem was suggested by H. Kuhn. See [Ku], SCARF’S THEOREM.

Scarf’s combinatorial theorem is a truly combinatorial result, but the style of his papers is very geometric, and he always puts forward the geometric underpinnings of combinatorial arguments. For example, Scarf orders subsets of a simplex by the values of one of barycentric coordinates, but instead of taking the minimal element of a set he moves a face of a simplex parallely to itself (see [Sc_2], the proof of Lemma 1, for example).

Recently H. Petri and M. Voorneveld [PV] published an almost geometry-free version of Scarf’s proof of Brouwer’s theorem. Eliminating geometry is an explicitly stated goal of [PV]. See [PV], Concluding remarks. This is achieved not without a cost, Petri–Voorneveld version of Scarf’s proof is fairly tricky. A minimal amount of geometry is inevitable since Brouwer’s theorem is about a simplex. But the geometry of simplices is eliminated by dealing with a family of arbitrary linear orders on a finite set instead of the linear orders induced by barycentric coordinates on a subset of a simplex. Scarf himself had applied his methods to other orders and was aware that they work for arbitrary orders. See [Sc_3], Chapter 6 and, especially, the discussion at the bottom of p. 146. Rewriting Scarf’s arguments in such an abstract setting is only natural and makes their combinatorial nature transparent.

But Petri and Voorneveld went further and eliminated also only implicitly geometric aspects of the proof. Like Sperner [S], Petri and Voorneveld [PV] start with a coloring of a finite set. In Sperner’s context this finite set is the set of vertices of a triangulation, and this triangulation is present before any coloring is introduced and even before the proof begins. A similar structure on an arbitrary finite set of points in a simplex (or a finite set with a family of linear orders) was discovered by Scarf [Sc_1, Sc_2]. It is the collection of so-called primitive sets, and it was interpreted by Kuhn [Ku] as the structure of a simplicial complex. See Section 3 below. Petri and Voorneveld use only (a version of) primitive sets closely related to the coloring in question, and this renders the Scarf–Kuhn structure invisible.

The present paper originated in a text written simply to verify Petri–Voorneveld proof. After the verification, the proof was rearranged in order to stress similarities with Sperner’s proof of his lemma. The rearranged proof turned out to be equally well suited for comparing with Scarf’s one. After such a comparison it became clear that the Scarf–Kuhn structure should be separated from colorings.

The main goal of this paper is to present the resulting proof of Scarf’s combinatorial theorem in the setting of a family of orders on a finite set. This is done in Section 1. Section 2 is devoted to a deduction of Brouwer’s fixed point theorem from Scarf’s theorem. This deduction follows Petri and Voorneveld [PV] version. Since the geometry is avoided, their version is a little more complicated than Scarf’s one. Finally, Section 3 explains how the proofs in [PV] and in the present paper are related to Scarf’s proofs. In a related paper [I_2] Scarf’s results are discussed from the point of view of the combinatorial topology, and, in particular, some ideas of Kuhn [Ku] are refined. See [I_2], Sections 1 – 3.
1. Scarf’s combinatorial theorem in an abstract setting

**Notations.** Let $A$ be a set. For $a \in A$ we denote by $A - a$ the set $A \setminus \{a\}$, and for $b \not\in A$ we denote by $A + b$ the set $A \cup \{b\}$. The set $A - a$ is defined only if $a \in A$, and $A + b$ is defined only if $b \not\in A$. By $|A|$ we denote the number of elements of $A$.

**Linear orders and dominant sets.** Let $T$ be a finite set. Suppose that a family of linear orders $<_i$ on $T$, labeled by elements $i$ of a finite set $I$, is given. For a non-empty subset $\sigma \subset T$ let $\min_i \sigma$ be the minimal element of $\sigma$ with respect to the order $<_i$. A subset $\sigma \subset T$ is said to be **dominant with respect to** a non-empty subset $C$ of $I$ if there is no element $y \in T$ such that $\min_i \sigma <_i y$ for all $i \in C$.

It is convenient to agree that $\emptyset \subset T$ is dominant with respect to every non-empty $C \subset I$. The basic properties of dominant sets are summarized in the following lemma.

**1. Lemma.** If $\sigma \subset T$ is dominant with respect to $C \subset I$, then $\sigma = \{\min_i \sigma \mid i \in C\}$ and hence $|\sigma| \leq |C|$. If $\tau \subset \sigma$, then $\tau$ is dominant with respect to $C$, and if $C \subset D$, then $\sigma$ is dominant with respect to $D$.

**Proof.** Clearly, $\sigma$ contains all minima $\min_i \sigma$. Suppose that $x \in \sigma$ is different from all $\min_i \sigma$ with $i \in C$. Then $\min_i \sigma <_i x$ for all $i \in C$, contrary to the assumption. This proves the first statement of the lemma. The second statement follows from the fact that $\min_i \sigma \leq_i \min_i \tau$ for every $i \in I$, and the third statement is obvious. ■

**Cells, rooms, and doors.** A cell is defined as a pair $(\sigma, C)$ of subsets $\sigma \subset T$ and $C \subset I$ such that $C$ is non-empty and $\sigma$ is dominant with respect to $C$. If $(\sigma, C)$ is a cell, then $|\sigma| \leq |C|$ by Lemma 1. A cell $(\sigma, C)$ is called a **room** if $|C| = |\sigma|$, and a **door** if $|C| = |\sigma| + 1$.

A pair $(\tau, D)$ of subsets $\tau \subset T$ and $D \subset I$ is said to be a door of a cell $(\sigma, C)$ if either $(\tau, D) = (\sigma - x, C)$ for some $x \in \sigma$, or $(\tau, D) = (\sigma, C + i)$ for some $i \in I \sim C$. By Lemma 1 every such pair $(\tau, D)$ is a a cell. It follows that every room has $|I|$ doors.

Equivalently, a pair $(\tau, D)$ is a door of a cell $(\sigma, C)$ if either $(\sigma, C) = (\tau + y, D)$ for some $y \in T \sim \sigma$, or $(\sigma, C) = (\tau, D - j)$ for some $j \in D$.

Clearly, if $(\tau, D)$ is a door of $(\sigma, C)$, then $|D| - |\tau| = |C| - |\sigma| + 1$. It follows that if a cell $(\sigma, C)$ is a room, then $(\sigma, C)$ is not a door of any cell. Similarly, if $(\tau, D)$ is a door of a cell $(\sigma, C)$, then $(\sigma, C)$ is a room. A door $(\tau, D)$ is called an **outside door** if $\tau = \emptyset$ and an **internal door** otherwise.
One can imagine that $T$ together with the rooms and doors is some sort of a building. As we will see, an outside door is a door of exactly one room, and an internal door is a common door of exactly two rooms. These properties are the main properties of this building.

2. **Lemma.** Suppose that $|D| = 1$. Then $(\emptyset, D)$ is an outside door and is a door of exactly one room. Every outside door has this form.

**Proof.** Since $\emptyset$ is dominant with respect to every non-empty subset of $I$, the pair $(\emptyset, D)$ is a cell. Since $|D| = 1$, this cell is a door and hence an outside door. Also, $|D| = 1$ implies that $D = \{i\}$ for some $i \in I$. If $(\emptyset, \{i\})$ is a door of a cell $(\sigma, C)$, then either $(\sigma, C) = (\emptyset, \{i\} + j)$ for some $j \notin \{i\}$, i.e. for some $j \neq i$, or $(\sigma, C) = (\{x\}, \{i\})$ for some $x \in T$. In the first case $|\sigma| = 0$ and $|C| = 2$ and hence $(\sigma, C)$ is not a room. In the second case $(\sigma, C)$ is a cell if and only if $x$ is dominant with respect to $\{i\}$, i.e. if and only if $x$ is the maximal element of $T$ with respect to $<_{i}$. It follows that $(\emptyset, \{i\})$ is a door of exactly one room. Clearly, $(\emptyset, D)$ is a door only if $|D| = 1$. □

3. **Lemma.** An internal door $(\tau, D)$ is a door of exactly two rooms.

**Proof.** Since $(\tau, D)$ is an internal door, $|\tau| \geq 1$ and $|D| = |\tau| + 1 \geq 2$. Suppose that $(\tau, D)$ is a door of $(\sigma, C)$. Then either $(\sigma, C) = (\tau + x, D)$ for some $x \notin \tau$, or $(\sigma, C) = (\tau, D - i)$ for some $i \in D$. Since $|D| \geq 2$, the set $C$ is non-empty. Therefore, if $\sigma$ is dominant with respect to $C$, then $(\sigma, C)$ is a cell and hence is a room. Let us find out when $\tau + x$ is dominant with respect to $D$ and $\tau$ is dominant with respect to $D - i$. Since $|\tau| = |D| + 1$, Lemma 1 implies that there is a unique pair $\{a, b\} \subset D$ such that

$$\min_{a} \tau = \min_{b} \tau$$

and $a \neq b$. For $i = a$ or $b$ let

$$\mathbb{M}_{i} = \{ y \in T \mid \min_{k} \tau <_{k} y \text{ for all } k \in D - i \}.$$  

When $\mathbb{M}_{i} \neq \emptyset$, we will denote by $m_{i}$ the maximal element of $\mathbb{M}_{i}$ with respect to $<_{i}$. Since $\tau$ is dominant with respect to $D$, the intersection $\mathbb{M}_{a} \cap \mathbb{M}_{b}$ is empty. Therefore, if both $\mathbb{M}_{a}$ and $\mathbb{M}_{b}$ are non-empty, then $m_{a} \neq m_{b}$ and hence the pairs $(\tau + m_{a}, D)$ and $(\tau + m_{b}, D)$ are different. This reduces the lemma to the following two sublemmas.

3.1. **Sublemma.** The set $\tau$ is dominant with respect to $D - i$ if and only if $i \in \{a, b\}$ and $\mathbb{M}_{i} = \emptyset$.

**Proof.** If $i \neq a, b$, then the set $\{ \min_{k} \tau \mid k \in D - i \}$ has $|D| - 2 = |\tau| - 1$ elements and hence $\tau$ is not dominant with respect to $D - i$ by Lemma 1. If $i = a$ or $b$, then $\tau$ is dominant with respect to $D - i$ if and only if $\mathbb{M}_{i} = \emptyset$. □
3.2. Sublemma. \( \tau + x \) is dominant with respect to \( D \) if and only if \( x = m_i \) for some \( i \in \{a, b\} \) such that \( \mathbb{M}_i \neq \emptyset \).

Proof. To begin with, let us observe that

1. \( \min_i (\tau + x) = x \) if \( x <_i \min_i \tau \), and
2. \( \min_i (\tau + x) = \min_i \tau \) if \( \min_i \tau <_i x \).

In particular, \( \min_i (\tau + x) = \min_i \tau \) or \( x \) for every \( i \in D \). Lemma 1 implies that

\[
\{ \min_i \tau \mid i \in D \} = \tau \quad \text{and that} \quad \{ \min_i (\tau + x) \mid i \in D \} = \tau + x
\]

if \( \tau + x \) is dominant with respect to \( D \). This may happen only if \( \min_i (\tau + x) = \min_i \tau \) for all \( i \in D \sim \{a, b\} \) and for \( i \) equal to one of the elements of the pair \( \{a, b\} \), and if \( \min_i (\tau + x) = x \) for \( i \) equal to the other element of \( \{a, b\} \).

Therefore, if \( \tau + x \) is dominant with respect to \( D \), we may assume that

1. \( \min_i (\tau + x) = \min_i \tau \) for all \( i \in D \sim a \) and
2. \( \min_a (\tau + x) = x \).

By (1) and (2) in this case \( \min_i \tau <_i x \) for all \( i \in D \sim a \) and \( x <_a \min_a \tau \). It follows that \( x \in \mathbb{M}_a \) and that \( \tau + x \) can be dominant with respect to \( D \) only if \( x \) is the maximal element of \( \mathbb{M}_a \) with respect to \( <_a \), i.e. only if \( x = m_a \).

Conversely, if, say, \( \mathbb{M}_a \neq \emptyset \) and \( x \in \mathbb{M}_a \), then

\[ \min_i \tau <_i x \quad \text{for all} \quad i \in D \sim a. \]

If also \( \min_a \tau <_a x \), then \( \tau \) is not dominant with respect to \( D \), contrary to the assumption. Therefore \( x <_a \min_a \tau \). By applying (1) and (2) again, we see that (3) and (4) hold. It follows that if \( x = m_a \), then \( \tau + x \) is dominant with respect to \( D \). \( \square \)

Colorings. A coloring is arbitrary map \( T \longrightarrow I \). Let us fix a coloring \( c \). A cell \((\sigma, C)\) is called colorful if \( C = c(\sigma) \). A colorful cell is automatically a room. Scarf’s combinatorial theorem asserts that for every coloring there exists a colorful room. See Theorem 8.

A cell \((\sigma, C)\) is called nearly colorful if \( |C \sim c(\sigma)| = 1 \). If \((\sigma, C)\) is nearly colorful, then \( C \sim c(\sigma) \) consists of one element, called the type of \((\sigma, C)\).
4. Lemma. For every \( i \in I \) there is exactly one outside door of the type \( i \). Every door of a colorful room is nearly colorful and there is exactly one door of each type among them.

Proof. The outside doors are the cells of the form \( (\emptyset, \{i\}) \), where \( i \in I \). Clearly, the door \( (\emptyset, \{i\}) \) is nearly colorful and its type is \( i \). This proves the first statement.

Let \( (\sigma, C) \) be a colorful room. If \( x \in \sigma \), then \( |C \setminus c(\sigma - x)| = 1 \) and hence the door \((\sigma - x, C)\) is nearly colorful. If \( i \in I - C \), then \(|(C + i) \setminus c(\sigma)| = 1\) and hence the door \((\sigma, C + i)\) is nearly colorful. Obviously, the type of \((\sigma - x, C)\) is \(c(x) \in c(\sigma)\), and the type of \((\sigma, C + i)\) is \(i \in I - C\). Since \((\sigma, C)\) is colorful, the map \(c\) induces a bijection \(\sigma \rightarrow C\). The second statement follows.

5. Lemma. Suppose that \((\tau, D)\) is a nearly colorful door of a room \((\sigma, C)\). Then \((\sigma, C)\) is either colorful, or nearly colorful of the same type as \((\tau, D)\).

Proof. Clearly, \(C \setminus c(\sigma) \subset D \setminus c(\tau)\). Together with \(|D \setminus c(\tau)| = 1\) this implies that \(|C \setminus c(\sigma)| \leq 1\). Since \(|c(\sigma)| \leq |\sigma| \leq |C|\), if \(|C \setminus c(\sigma)| = 0\), then \(C = c(\sigma)\) and hence \((\sigma, C)\) is colorful. If \(|C \setminus c(\sigma)| = 1\), then \(C \setminus c(\sigma) = D \setminus c(\tau)\) and hence \((\sigma, C)\) is nearly colorful and has the same type as \((\tau, D)\).

6. Lemma. If \((\sigma, C)\) is a nearly colorful room, then \(|c(\sigma)|\) is equal either to \(|\sigma|\) or to \(|\sigma| - 1\). If \((\sigma, C)\) is a nearly colorful door, then \(c(\sigma) \subset C\).

Proof. If \((\sigma, C)\) is a nearly colorful room, then \(|C| \leq |c(\sigma)| + 1\) and hence

\[|\sigma| \leq |C| \leq |c(\sigma)| + 1 \leq |\sigma| + 1.\]

It follows that \(|c(\sigma)| = |\sigma|\) or \(|\sigma| - 1\). If \((\sigma, C)\) is a nearly colorful door, then

\[|\sigma| \geq |c(\sigma)| \geq |c(\sigma) \cap C| = |C| - 1 = |\sigma|\]

and hence \(|c(\sigma)| = |c(\sigma) \cap C|\) and \(c(\sigma) = c(\sigma) \cap C\). It follows that \(c(\sigma) \subset C\).

7. Lemma. If \((\sigma, C)\) is a nearly colorful room, then \((\sigma, C)\) has two nearly colorful doors.

Proof. Lemma 6 implies that either \(|c(\sigma)| = |\sigma|\) or \(|c(\sigma)| = |\sigma| - 1 = |C| - 1\).

Suppose first that \(|c(\sigma)| = |\sigma| - 1 = |C| - 1\). Then \(c(\sigma) \subset C\). Therefore, if also \(i \not\in C\), then \(|(C + i) \setminus c(\sigma)| = 2\) and hence \((\sigma, C + i)\) is not nearly colorful. Since \(|c(\sigma)| = |\sigma| - 1\), there are different elements \(x, y \in \sigma\) such that \(c(x) = c(y)\). Moreover, the coloring \(c\) is injective on both \(\sigma - x\) and \(\sigma - y\). If \(z \in C\) and \(z \neq x, y\), then \(c(\sigma - z)\) is properly contained in \(c(\sigma)\). Since \(c(\sigma)\) is properly contained in \(C\), it follows
that \(|C \sim c(\sigma - z)| = 2\) and hence \((\sigma - z, C)\) is not nearly colorful. On the other hand, if \(z = x\) or \(y\), then \(c(\sigma - z) = c(\sigma)\) and hence \(|C \sim c(\sigma - z)| = 1\). In this case \((\sigma - z, C)\) is nearly colorful. Therefore \((\sigma, C)\) has two nearly colorful doors.

Suppose that \(|c(\sigma)| = |\sigma|\). Then \(c\) is injective on \(\sigma\). Since \((\sigma, C)\) is nearly colorful, this implies that \(|c(\sigma) \sim C| = 1\) and hence there is a unique \(y \in \sigma\) such that \(c(y) \not\in C\). If \(x \in \sigma\), then \(c(x) \not\in c(\sigma - x)\) because \(c\) is injective on \(\sigma\). If also \(c(x) \in C\), then

\[
C \sim c(\sigma - x) = (C \sim c(\sigma)) + c(x)
\]

and hence \(|C \sim c(\sigma - x)| = 2\). Therefore in this case \((\sigma - x, C)\) is not nearly colorful.

On the other hand, if \(c(x) \not\in C\), i.e. if \(x = y\), then

\[
C \sim c(\sigma - x) = C \sim c(\sigma)
\]

and hence \(|C \sim c(\sigma - x)| = |C \sim c(\sigma)| = 1\). Therefore \((\sigma - x, C)\) is nearly colorful if and only if \(x = y\). If \(i \in I \sim C\) and \(i \not\in c(\sigma)\), then

\[
(C + i) \sim c(\sigma) = (C \sim c(\sigma)) + i
\]

and hence \(|(C + i) \sim c(\sigma)| = |C \sim c(\sigma)| + 1 = 2\). Therefore in this case \((\sigma, C + i)\) is not nearly colorful. On the other hand, if \(i \in c(\sigma)\), i.e. if \(i = c(y)\), then

\[
(C + i) \sim c(\sigma) = C \sim c(\sigma)
\]

and hence \(|(C + i) \sim c(\sigma)| = |C \sim c(\sigma)| = 1\). Therefore \((\sigma, C + i)\) is nearly colorful if and only if \(i = c(y)\). Hence in this case \((\sigma, C)\) also has two nearly colorful doors.

8. Scarf’s combinatorial theorem. There is at least one colorful room. Moreover, the number of such rooms is odd.

An informal proof. The following informal explanation of the proof is based on Scarf’s outline (see [Sc3], p. 48). Actually, the terms rooms and doors were suggested by this outline, although Scarf used them only informally. Let us think about \(T\) and its rooms and doors as if it is a building. The above lemmas tell us all what we need to know about this building. We are interested only in colorful and nearly colorful rooms and in nearly colorful doors. One can imagine that all other doors and rooms are permanently locked.

Our plan is to enter this building through an outside door and then follow rooms and doors, never turning back. An outside door leads to a particular room, which is either colorful or has only one other unlocked door. This unlocked door leads us to another room, which is
also either colorful or has only one other unlocked door. We will continue to explore the
building in this way until we enter a colorful room. Since we do not turn back, we cannot
pass through the same door or enter the same room twice (a room has only two unlocked
doors). Potentially, our journey may end at another outside door. But Lemma 5 implies
that the types of doors and rooms stay the same during our journey, and by Lemma 4 there
is only one outside door of any given type. Since we cannot return to this door and there
are only finitely many rooms, we will eventually reach a colorful room.

We found a particular colorful room. If there are other, then the same argument shows that
they are pairwise related by routes consisting of nearly colorful rooms and doors of the type
$i$, where $i$ is the type of the outside door used to enter the building. Therefore the number
of other colorful rooms is even and the total number of colorful rooms is odd.

**The graphs of rooms and doors.** In order to turn the above informal proof in a formal one,
it is convenient to represent journeys through the building by paths in appropriate graphs.
Let us fix some element $i \in I$ and restrict our attention by the colorful rooms and by the
nearly colorful rooms and doors of the type $i$. The relation

$$(\tau, D) \text{ is a door of } (\sigma, C)$$

between such rooms and doors can be encoded in terms of a graph $G_i$, having as the ver-
tices the colorful rooms and the nearly colorful rooms and doors of the type $i$. Each such
room is connected by an edge to each of its nearly colorful doors. There are no other edges.
Clearly, Scarf’s combinatorial theorem immediately follows from the following fact.

**9. Theorem.** *The graph $G_i$ consists of several disjoint paths and cycles. With one exception,
the endpoints of these paths are properly colored cells. The only exception is $(\emptyset, \{i\})$.*

**Proof.** By Lemma 4 there is only one outside door with the type $i$. By Lemma 2 it is a
door of exactly one room, which is nearly colorful and has the type $i$ by Lemma 5. There-
fore this outside door is an endpoint of exactly one edge. Also by Lemma 4 every colorful
room has exactly one nearly colorful door with the type $i$ and hence is also an endpoint of
exactly one edge. Lemmas 3 and 5 imply that every internal door is an endpoint of two
edges. Finally, Lemmas 7 and 5 imply that every nearly colorful room is an endpoint of
two edges. To sum up, every vertex of $G_i$ is an endpoint of one or two edges. A vertex is
an endpoint of only one edge if and only if it is either a colorful room or the unique outside
door of the type $i$. The theorem follows. ■

**Remarks.** The above proof can be easily turned into an algorithm for finding a properly col-
ored room. In this circle of questions such *path-following algorithms* originate in the work
of Scarf [Sc$_2$] and are very popular. Such algorithms were actually used, already by Scarf,
to compute approximations to fixed points of self-maps of a simplex. See [Sc$_2$], [Sc$_3$].
2. Brouwer’s fixed point theorem

The standard simplex. Let us turn to the non-combinatorial part of the proof of Brouwer’s theorem. Let us fix a non-negative integer $n$, and let $I = \{0, 1, \ldots, n\}$. Let us number the coordinates in $\mathbb{R}^{n+1}$ by elements of $I$. For a point $x \in \mathbb{R}^{n+1}$ let $x_i$ be the $i$-th coordinate of $x$, so that $x = (x_0, x_1, \ldots, x_n)$. Let $\Delta^n \subset \mathbb{R}^{n+1}$ be the standard $n$-simplex defined by the equation $x_0 + x_1 + \ldots + x_n = 1$ and the inequalities $x_i \geq 0$ with $i \in I$. Let $l \geq 1$ be another integer and let $T = T_l$ be the set of all $x \in \Delta^n$ such that every $x_i$ is an integer multiple of $1/l$. The set $T$ will serve as a discrete approximation to $\Delta^n$.

The linear orders on $T$. For each $i \in I$ let us choose a linear order $<_i$ on $T$ such that

(5) $x_i < y_i$ implies $x <_i y$

for every $x, y \in T$ (obviously, such orders exist).

10. Theorem. Let $\sigma \subset T$ and $C \subset I$. If $\sigma$ is dominant with respect to $C$, then

$$|x_i - y_i| < 2(n + 1)/l$$

for every $x, y \in \sigma$ and $i \in I$ and $x_i < (n + 1)/l$ for every $x \in \sigma$ and $i \in I \setminus C$.

Proof. For each $i \in I$ let $m(i) = \min_i \sigma$. Let $m_i = m(i)_i$ be the $i$-th coordinate of $m(i)$ for $i \in C$ and let $m_i = 0$ for $i \in I \setminus C$. By using the triangle inequality and the fact that $|C| \leq n + 1$, we see that it is sufficient to prove that

$$0 \leq x_i - m_i < |C|/l$$

for every $x \in \sigma$ and $i \in I$. The inequalities $0 \leq x_i - m_i$ hold by the definition of $m_i$. As the first step toward the inequalities $x_i - m_i < |C|/l$, let us prove that

(6) $1 - \sum_{k \in C} m_k < |C|/l$.

If this is not the case, then

$$\sum_{k \in C} \left( m_k + (1/l) \right) = \left( \sum_{k \in C} m_k \right) + |C|/l \leq 1.$$

Since every $m_i$ is a multiple of $1/l$, This implies that there exists a point $M \in T$ such that $M_k \geq m_k + (1/l)$ and hence $\min_k \sigma <_k M$ for every $k \in C$. The contradiction with $\sigma$ being dominant with respect to $C$ proves (6).
Let $x \in \sigma$ and $i \in I$. If $i \in C$, then

$$x_i - m_i \leq \sum_{k \in C} (x_k - m_k) \leq \sum_{k \in I} m_k = 1 - \sum_{k \in C} m_k.$$ 

If $i \notin C$, then $m_i = 0$ and hence

$$x_i - m_i = x_i \leq \sum_{k \notin C} x_k = 1 - \sum_{k \in C} m_k.$$ 

Together with (6) these inequalities imply that $x_i - m_i < |C|/l$ for every $i \in I$. □

**Continuous self-maps of** $\Delta^n$. Now we turn to the final part of the proof of Brouwer's theorem. Let $f : \Delta^n \to \Delta^n$ be a continuous map. Recall that $T = T_l$ depends on $l$. If

$$x = (x_0, x_1, \ldots, x_n) \in T_l \quad \text{and} \quad y = (y_0, y_1, \ldots, y_n) = f(x),$$

then $x_0 + x_1 + \ldots + x_n = y_0 + y_1 + \ldots + y_n = 1$ and hence $y_i \geq x_i$ for some $i \in I$. Let $c(x)$ be equal to any such $i$. This rule defines a coloring of $T_l$, i.e. a map $c : T_l \to I$.

By Theorem 8 for every $l$ there exists a properly colored cell, i.e. a pair $(\sigma_l, C_l)$ such that $C_l$ is non-empty, $\sigma_l \subset T_l$ is dominant with respect to $C_l$, and $C_l = c(\sigma_l)$. By Theorem 10 the diameter of the sets $\sigma_l$ tends to 0 when $l \to \infty$. Therefore, after passing to a subsequence, still denoted by $\sigma_l$, we can assume that all elements of $\sigma_l$ converge to the same point $z = (z_0, z_1, \ldots, z_n) \in \Delta^n$ when $l \to \infty$. Let

$$w = (w_0, w_1, \ldots, w_n) = f(z).$$

Since there are only finitely many subsets of $I$, after passing to a further subsequence we can assume that $C_l = C$ for some non-empty subset $C \subset I$ independent of $l$. Then $C = C_l = c(\sigma_l)$ for every $l$ and hence for every $l$ and every $i \in C$ there is a point $z(i, l) \in \sigma_l$ such that $c(z(i, l)) = i$. By the choice of the colorings $c$ passing to the limit $l \to \infty$ shows that $w_i \geq z_i$ for every $i \in C$.

At the same time Theorem 5 implies that $x_i < (n + 1)/l$ for every $x \in \sigma$ and $i \in I \sim C$. By passing to the limit $n \to \infty$ we conclude that $z_i = 0$ for every $i \in I \sim C$. Therefore

$$\sum_{i \in C} z_i = 1.$$ 

Since $w_i \geq z_i$ for every $i \in C$, this equality implies that $\sum_{i \in C} w_i \geq 1$. But

$$\sum_{i \in I} w_i = 1.$$ 

It follows that $w_i = 0 = z_i$ for every $i \in I \sim C$ and $\sum_{i \in C} w_i = 1 = \sum_{i \in C} z_i$. 

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Since \( w_i \geq z_i \) for every \( i \in C \), this equality implies that \( w_i = z_i \) for every \( i \in \mathcal{C} \) also. Therefore \( w_i = z_i \) for every \( i \in I \) and hence \( f(z) = w = z \), i.e. \( z \) is a fixed point of the map \( f \). This completes the proof of Brouwer’s fixed point theorem. ■

3. Scarf’s proof and its versions

Preferences and utility functions. In the context of the mathematical economics and the game theory the elements of the set \( T \) can be interpreted as goods, and the elements of the set \( I \) as traders on a market. From this point of view, the order \( <_i \) on the set of goods reflects the preferences of the trader \( i \). It is quite natural to think that the preferences of each trader \( i \) arise from a utility function \( u_i : T \rightarrow R^+ = (0, \infty) \) such that the inequalities \( x < y \) and \( u_i(x) < u_i(y) \) are equivalent. In any case, such utility functions always exist. By numbering the traders one can arrange utility functions into a map \( u : T \rightarrow R_+^{n+1} \) taking \( x \in T \) to the point \( u(x) \in R_+^{n+1} \) having the utilities \( u_i(x) \) as its coordinates. By perturbing a little the utility functions, if necessary, one can assume that \( u \) is injective, and then identify the set \( T \) with its image \( u(T) \). Therefore, without any loss of generality one can assume that \( T \) is a finite subset of \( R_+^{n+1} \) and the orders \( <_i \) are determined by the values of coordinates. For applications to Brouwer’s fixed point theorem the case of \( T \subset \Delta^n \) is sufficient. This is the framework of Scarf’s papers [Sc1] – [Sc3].

Slack vectors. Scarf enlarges \( T \) by adding to it vectors representing the \((n-1)\)-faces of \( \Delta^n \). The face defined by the equation \( x_i = 0 \) is represented by

\[ s(i) = (M_i, \ldots, M_i, 0, M_i, \ldots, M_i) \in R^{n+1}, \]

the vector with the \( i \)th coordinate equal to 0 and other coordinates equal to some real number \( M_i \) bigger than the \( i \)th coordinate of every vector \( x \in T \). If \( T \subset \Delta^n \), it is sufficient to assume that \( M_i > 1 \). To stress an analogy with linear programming, Scarf calls these vectors slack vectors. Let \( \mathbb{I} \) be the set of the slack vectors.

It is convenient to identify the \( i \)th trader with the \( i \)th slack vector \( s(i) \) for every \( i \in I \) and therefore identify \( I \) with \( \mathbb{I} \). The enlarged set \( T \) is the union \( T \cup \mathbb{I} \). If \( M_i \) are pairwise different, then the values of coordinates define linear orders on \( T \cup \mathbb{I} \). A subset \( X \subset T \cup \mathbb{I} \) is called primitive if \( |X| = n + 1 \) and \( X \) is a dominant subset of \( T \cup \mathbb{I} \) with respect to the set \( I = \mathbb{I} \) of all orders.

By the choice of slack vectors a subset \( \sigma \subset T \) is dominant with respect to \( \mathcal{C} \subset \mathbb{I} \) if and only if \( \sigma \cup (\mathbb{I} \setminus \mathcal{C}) \) is a primitive set. Conversely, \( X \subset T \cup \mathbb{I} \) is a primitive subset if and only if the intersection \( X \cap T \) is dominant with respect to \( \mathbb{I} \setminus X \). Therefore the notions of primitive and dominant sets are equivalent and one can use either of them.
**Scarf’s main lemma.** Suppose that \( X \subset T \cup \mathbb{I} \) is a primitive set, and \( x \in X \). Then either \( X - x \subset \mathbb{I} \), or there is a unique \( y \in (T \cup \mathbb{I}) \sim X \) such that \( X - x + y \) is a primitive set. For every \( i \in \mathbb{I} \) there is exactly one primitive set containing \( \mathbb{I} - i \).

See \([Sc_2]\), Lemma 1. The analogy between primitive sets and \( n \)-simplices of a triangulation of \( \Delta^n \) suggests to state this lemma differently. Let us call a subset \( Y \subset T \cup \mathbb{I} \) almost primitive if \(|Y| = n\) and there is a primitive set containing \( Y \). The set \( Y = X - x \) from Scarf’s main lemma is almost primitive, and hence we can restate this lemma as follows.

**Another form of Scarf’s main lemma.** Suppose that \( Y \subset T \cup \mathbb{I} \) is an almost primitive set. Then either \( Y \subset \mathbb{I} \), or \( Y \) is a subset of exactly two primitive sets. For every \( i \in \mathbb{I} \) the set \( \mathbb{I} - i \) is almost primitive and is a subset of only one primitive set.

By the definitions, if \( X \subset T \cup \mathbb{I} \) is a primitive set, then \((X \cap T, \mathbb{I} \sim X)\) is a room, and if \( X \) is an almost primitive set, then this pair is a door. Moreover, if \( Y \) is an almost primitive set, \( X \) is a primitive set, and \( Y \subset X \), then \((Y \cap T, \mathbb{I} \sim Y)\) is a door of the room \((X \cap T, \mathbb{I} \sim X)\). Conversely, if \((\sigma, C)\) is a room, then \( \sigma \cup (\mathbb{I} \sim C) \) is a primitive set. The corresponding statement for the doors is only a little less obvious. By Lemmas 2 and 3 every door is a door of a room (one can also see this directly). It follows that if \((\tau, D)\) is a door, then \( \tau \cup (\mathbb{I} \sim D) \) is contained in a primitive set and hence is an almost primitive set. So, the notions of primitive and almost primitive sets are equivalent to the notions of rooms and doors, and Scarf’s main lemma is equivalent to Lemmas 2 and 3 together.

**Primitive sets and rooms.** An obvious advantage of working with slack vectors and primitive sets is the fact that an almost primitive set is always obtained by removing a point from a primitive set. In contrast, a door can be obtained from a room \((\sigma, C)\) either by removing a point from \(\sigma\) or by adding a point to \(C\). Still, traders and goods are inherently different, and it is only natural to keep track of this distinction, as it is done in the notion of a room.

Scarf’s proof of his combinatorial theorem uncovered a structure hidden in a family of orders on a given set, namely, the collection of primitive sets. H. Kuhn \([Ku]\) interpreted this structure as the structure of a simplicial complex. The \( n \)-simplices of this complex are primitive sets and Scarf’s main lemma means that this complex is a pseudo-manifold. Sperner’s lemma and its proofs naturally extend to pseudomanifolds and Scarf’s combinatorial theorem can be proved by applying such an extension to this pseudo-manifold.

By keeping the distinction between goods and traders one can see an even more rich combinatorial structure behind the family of orders. Namely, a room \((\sigma, C)\) can be thought as “belonging” to the face \(C\) of the abstract simplex having \(\mathbb{I}\) as the set of vertices. This leads to a refined version of Kuhn’s interpretation and to a proof of Scarf’s combinatorial theorem using Scarf’s main lemma as the starting point and then proceeding in the same way as in combinatorial or homological proofs of Sperner’s lemma. See \([I_2]\), Section 1.
Petri–Voorneveld version. In contrast with Scarf [Sc2] and with Section 1, H. Petri and M. Voorneveld [PV] start with fixing a coloring \( c : T \rightarrow I \). In the language of the present paper, they work only with pairs \((\sigma, C)\) such that \( |C \sim c(\sigma)| \leq 1 \). Petri and Voorneveld call them candidates. One can easily check that every such pair is either a room or a door, but other rooms and doors are missing. Of course, such pairs are sufficient for proving Scarf combinatorial theorem (called in [PV] No-bullying lemma), but the structure uncovered by Scarf remains hidden. Perhaps, this reflects the intention to eliminate geometry.

One has to admit that the previous paragraph is not quite faithful to [PV]. After fixing a coloring, Petri and Voorneveld replace each color (i.e. each of the orders) by as many of its copies as there are elements of \( T \) with this color. This trick allows to turn the coloring \( c \) into a bijection. Petri and Voorneveld go one step further and use this bijective coloring to identify \( T \) and \( I \). This erases the difference between goods and traders completely (actually, Petri and Voorneveld speak about toys and kids). For the present author this identification was the main stumbling block in reading [PV].

The paths to primitive sets and colorful rooms. Let \( c : T \rightarrow I \) be a coloring, and let us extend \( c \) to a coloring \( T \cup I \rightarrow I \) equal to the identity on \( I \). There is no danger in denoting this extension also by \( c \). In this language Scarf’s combinatorial theorem asserts that there is a primitive set \( X \) such that \( c(X) = I \). In order to prove this, Scarf’s uses as the main tool the operation of replacing \( X \) by \( X - x + y \) as in the main lemma. This replaces in \( X \) the element \( x \) by a new element \( y \) and is called a replacement step. The proof starts with fixing a color \( i \in I \) and considering the set \( I - i \). Clearly, the set \( I - i \) is almost primitive and by the main lemma is contained in a unique primitive set \( X \). If \( c(X) = I \), then we are done. Otherwise \( c(X) = I - i \) and one can apply to \( X \) a sequence of replacement steps. The elements to replace are chosen in such a way that after each step either \( c(X) = I \) or \( c(X) = I - i \). Scarf proves that such sequence of replacement steps eventually arrives at a primitive set \( X \) such that \( c(X) = I \), when it stops.

One can split each replacement step into two simpler steps. Namely, instead of passing from a primitive set \( X \) to \( X - x + y \) as in the main lemma, one can first pass from \( X \) to an almost primitive set \( X - x \) and then pass from \( X - x \) to \( X - x + y \). Let us split in this way every replacement step in Scarf’s sequence and add \( I - i \) at the beginning. The resulting sequence alternates between almost primitive and primitive sets and \( c(X) = I - i \) for every its term \( X \) except of the last one. The transformation \( Z \mapsto (Z \cap T, I \sim Z) \) turns this sequence into a sequence of rooms and doors starting with \((\emptyset, \{i\})\) and ending with a colorful room. The condition \( c(X) = I - i \) implies that doors and rooms in this sequence are nearly colorful of type \( i \), except the last one. Moreover, this sequence is a path in the graph \( G_i \) connecting \((\emptyset, \{i\})\) with a colorful room. But Theorem 9 implies that there is only one such path. Hence the transformation \( Z \mapsto (Z \cap T, I \sim Z) \) turns Scarf’s sequence into the path to a colorful room from Section 1. In other words, the arguments of Section 1 lead to essentially the same the path to a colorful room as Scarf’s ones. Of course, the same is true with respect to the arguments of Petri and Voorneveld [PV].
References

[H] W. Hurewicz, Über ein topologisches Theorem, Mathematische Annalen, V. 101 (1929), 210–218.

[I1] N.V. Ivanov, The lemmas of Alexander and Sperner, 2019, 55 pp. arXiv:1909.00940.

[I2] N.V. Ivanov, Scarf theorems, simplicial complexes, and oriented matroids, 2019, 63 pp. arXiv:2207.10832.

[KKM] B. Knaster, C. Kuratowski, S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe, Fundamenta Mathematicae, V. 14 (1929), pp. 132–137.

[Ku] H.W. Kuhn, Simplicial approximation of fixed points, Proceedings of the National Academy of Sciences USA, V. 61 (1968), 1238–1242.

[PV] H. Petri, M. Voorneveld, No bullying! A playful proof of Brouwer’s fixed point theorem, Journal of Mathematical Economics, V. 78, No. 1 (2018), pp. 1–5.

[Sc1] H. Scarf, The core of an N person game, Econometrica, V. 35, No. 1 (1967), 50–69.

[Sc2] H. Scarf, The approximation of fixed points of a continuous mapping, SIAM Journal of Applied Mathematics, V. 15, No. 5 (1967), 1328–1343.

[Sc3] H. Scarf, with the collaboration of T. Hansen, The computation of economic equilibria, Yale University Press, 1973, x, 249 pp.

[S] E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, Abh. Math. Semin. Hamburg. Univ., Bd. 6 (1928), pp. 265–272.

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