FUNCTIONAL EQUATION, UPPER BOUNDS AND ANALOGUE OF LINDELÖF HYPOTHESIS FOR THE BARNES DOUBLE ZETA-FUNCTION

TAKASHI MIYAGAWA

Abstract. The functional equations of the Riemann zeta function, the Hurwitz zeta function, and the Lerch zeta function have been well known for a long time and there are great importance when studying these zeta-functions. For example, fundamental properties of the upper bounds, the distribution of zeros, the zero-free regions in the Riemann zeta function start from functional equations.

In this paper, we prove a functional equations of the Barnes double zeta-function

\[ \zeta(2, \alpha; v, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(\alpha + vm + wn)^s}. \]

Also, applying this functional equation and the Phragmén-Lindelöf convexity principle, we obtain some upper bounds for \( \zeta_{2}(\sigma + it, \alpha; v, w) \) (0 \( \leq \) \( \sigma \) \( \leq \) 2) with respect to \( t \) as \( t \to \infty \).

1. Introduction

In this section, we introduce the Barnes double zeta-function, the functional equations of the Hurwitz zeta-function and the Lerch zeta-function, and some upper bounds for the Riemann zeta-function.

Let \( s = \sigma + it \) be a complex variable. For \( \theta \in \mathbb{R} \) let \( H(\theta) = \{ z = re^{i(\theta + \phi)} \in \mathbb{C} | r > 0, -\pi/2 < \phi < \pi/2 \} \) be the open half plane whose normal vector is \( e^{i\theta} \). The Barnes double zeta-function is defined by

\[ \zeta_r(s, \alpha; v, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(\alpha + vm + wn)^s} \] (1.1)

where \( \alpha, v, w \in H(\theta) \) for some \( \theta \). Then it is known that this series converges absolutely uniformly on any compact subset in \( \{ s \in \mathbb{C} | \text{Re}(s) > 2 \} \). This function was introduced by E. W. Barnes \( [2] \) in the theory of double gamma function, and double series of the form (1.1) is introduced in \( [3] \). Furthermore in \( [4] \), in connection with the theory of the multiple gamma function, and \( r \)-multiple series of the form (2.3) was introduced, which we will mention in a Remark \( [2] \) in the next section.

Above series (1.1) is a double series version of the Hurwitz zeta-function

\[ \zeta_H(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s} \] (0 < \( \alpha \) \( \leq \) 1). (1.2)

Also, as a generalization of this series in another direction, the Lerch zeta-function

\[ \zeta_L(s, \alpha, \lambda) = \sum_{n=0}^{\infty} \frac{e^{2\pi in\lambda}}{(n + \alpha)^s} \] (0 < \( \alpha \) \( \leq \) 1, 0 < \( \lambda \) \( \leq \) 1) (1.3)

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is also an important research subject. These series are absolutely convergent for $\sigma > 1$. Also, if $0 < \lambda < 1$, then the series (1.3) is convergent even for $\sigma > 0$. Further, series (1.2), (1.3) satisfy the following functional equations

$$\zeta_H(s, \alpha) = \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \left\{ e^{\pi i (1-s)/2} \sum_{n=1}^{\infty} \frac{e^{2\pi in(1-\alpha)}}{n^{1-s}} + e^{-\pi i (1-s)/2} \sum_{n=1}^{\infty} \frac{e^{2\pi in\alpha}}{n^{1-s}} \right\} (1.4)$$

as $\sigma < 0$, and

$$\zeta_L(s, \alpha, \lambda) = \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \left\{ e^{(1-s)/2-2\alpha\lambda}\pi i \sum_{n=0}^{\infty} \frac{e^{2\pi in(1-\alpha)}}{(n + \lambda)^{1-s}} + e^{(-1-s)/2+2\alpha(1-\lambda)}\pi i \sum_{n=0}^{\infty} \frac{e^{2\pi in\alpha}}{(n + 1 - \lambda)^{1-s}} \right\} (1.5)$$

with $\sigma < 0, 0 < \lambda < 1$, respectively, which are well known.

**Remark 1.** Furthermore, using the Lerch zeta-function $\zeta_L(s, \alpha, \lambda)$ two functional equations (1.4) and (1.5) can be expressed as follows

$$\zeta_H(s, \alpha) = \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \left\{ e^{\pi i (1-s)/2} \zeta_L(1 - s, 1, 1 - \alpha) + e^{-\pi i (1-s)/2} \zeta_L(1 - s, 1, \alpha) \right\},$$

$$\zeta_L(s, \alpha, \lambda) = \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \left\{ e^{(1-s)/2-2\alpha\lambda}\pi i \zeta_L(1 - s, \lambda, 1 - \alpha) + e^{(-1-s)/2+2\alpha(1-\lambda)}\pi i \zeta_L(1 - s, 1, \lambda, \alpha) \right\}.$$

Also, since it is well known that $\zeta_L(s, \alpha, \lambda)$ is analytically continued to meromorphic function for the whole complex plane, the above two equations holds for the whole complex $s$-plane.

Next, we will introduce the some results of the most fundamental order evaluations for the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$ 

These evaluations are order of $\zeta(\sigma + it)$ when the imaginary part $t$ as $t \to \infty$. In particular, research on order evaluation for $\zeta(1/2 + it)$ as $t \to \infty$ is most prominent.

As a classical asymptotic formula,

$$\zeta(\sigma + it) \ll |t|^{(1-\sigma)/2+\epsilon} \quad (0 \leq \sigma \leq 1, |t| \geq 2)$$

is known, especially when $\sigma = 1/2$ is

$$\zeta(1/2 + it) \ll t^{1/4+\epsilon}.$$ 

This result is obtained using the $\zeta(s)$ functional equation and the Phragmén-Lindelöf convexity principle. Another classical asymptotic formula of $\zeta(s)$ was proved by Hardy and Littlewood using the following formula;

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \quad (x \to \infty), \quad (1.6)$$

uniformly for $\sigma \geq \sigma_0 > 0, |t| < 2\pi x/C$, when $C > 1$ is a constant. This formula gives an indication in the discussion in the critical strip of $\zeta(s)$. Also, Hardy and Littlewood proved
the following asymptotic formula (§4 in [15]); suppose that $0 \leq \sigma \leq 1$, $x \geq 1$, $y \geq 1$ and $2\pi i xy = |t|$ then

$$
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{1/2-\sigma}y^{\sigma-1}),
$$

(1.7)

where $\chi(s) = 2\Gamma(1-s)\sin(\pi s/2)/(2\pi)^{s-1}$ and note that the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ holds. This formula (1.7) is called the approximate functional equation. Hardy-Littlewood improve to

$$
\zeta(1/2 + it) \ll t^{1/6+\varepsilon}
$$

(1.8)

by the van der Corput method, applying to (1.7). In 1988, $\zeta(1/2 + it) \ll t^{9/56+\varepsilon}$ was proved by Bombieri and Iwaniec, after which many mathematicians gradually improved, and now $\zeta(1/2 + it) \ll t^{92/205+\varepsilon}$ has been proved by Huxley in 2005. Furthermore in 2017, $\zeta(1/2 + it) \ll t^{13/84+\varepsilon}$ was proved by Bourgain (in see [5]).

Improvements in order estimation of $\zeta(1/2 + it)$ are still ongoing, and the true order is expected to be

$$
\zeta(\frac{1}{2} + it) \ll t^{\varepsilon}
$$

(1.9)

for any $\varepsilon > 0$. This conjecture is called the Lindelöf hypothesis. Furthermore, it is well known that (1.9) holds is equivalent to

$$
\int_2^T |\zeta(\sigma + it)|^{2k} dt = O(T^{1+\varepsilon})
$$

for any $k \in \mathbb{N}$ and any $\varepsilon > 0$.

Author prove the analogue of (1.6) for the case of the Barnes double zeta-functions as follows.

**Proposition 1 (Theorem 3 in [13]).** Let $0 < \sigma_1 < \sigma_2 < 2$, $x \geq 1$ and $C > 1$. Suppose $s = \sigma + it \in \mathbb{C}$ with $\sigma_1 < \sigma < \sigma_2$ and $|t| \leq 2\pi x/C$. Then

$$
\zeta_2(s, \alpha; v, w) = \sum_{0 \leq m, n \leq x} \frac{1}{(\alpha + vm + wn)^s}
$$

$$
+ \frac{(\alpha + vx)^2 - s + (\alpha + wx)^2 - s - (\alpha + vx + wx)^2 - s}{vw(s-1)(s-2)} + O(x^{1-\sigma})
$$

(1.10)

as $x \to \infty$.

2. Statements of Main results

In this section, we give statements of the functional equations, the upper bounds, and other results for the Barnes double zeta function (1.11). In particular, in Theorem 2 the results when the complex parameter $v, w$ linearly independent are different from the results when $v, w$ are linearly dependent over $\mathbb{Q}$. Also, we obtain some upper bounds for $\zeta_2(\sigma + it, \alpha; v, w)$ ($0 \leq \sigma \leq 2$) with respect to $t \to \infty$, and analogue of the Lindelöf hypothesis for the Barnes double zeta-function.

If $\alpha, v, w \in H(\theta)$, then $\alpha + vm + wn \in H(\theta)$. Here, let $\alpha = v(1 - y_1) + w(1 - y_2)$ and consider only the case where $0 \leq y_1, y_2 < 1$. 


Theorem 2. Suppose that $\alpha, v, w \in H(\theta)$. We have follows;

(i) If $v/w$ is imaginary or irrational real algebraic number;

\[
\zeta_2(s, \alpha; v, w) = -\frac{\Gamma(1-s)}{(2\pi i)^{1-s}e^{\pi is}} \left\{ \frac{1}{v^s} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n(v_1+w_2)/v}}{(e^{2\pi i n/v} - 1)n^{1-s}} + \frac{1}{w^s} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n(v_1+w_2)/w}}{(e^{2\pi i n/w} - 1)n^{1-s}} \right\}, \tag{2.1}
\]

where $-\theta < \arg(2\pi i/n/v), \arg(2\pi i/n/w) < -\theta + \pi$ and two finite series of the right-hand side converges absolutely uniformly on the whole space $\mathbb{C}$ if $(y_1, y_2) \neq (0, 0)$, and on the region $\Re(s) < 0$ if $y_1 = 0$ or $y_2 = 0$.

(ii) If $v/w$ is rational number, i.e. $v, w$ are linearly dependent over $\mathbb{Q}$, exist $p, q \in \mathbb{N}$ such as $pv = qw$ and $(p, q) = 1$. Then we have

\[
\zeta_2(s, \alpha; v, w) = -\frac{\Gamma(1-s)}{(2\pi i)^{1-s}e^{\pi is}} \left\{ \frac{1}{v^s} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n(v_1+w_2)/v}}{(e^{2\pi i n/v} - 1)n^{1-s}} + \frac{1}{w^s} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n(v_1+w_2)/w}}{(e^{2\pi i n/w} - 1)n^{1-s}} + \frac{q^{s-1}}{2\pi i pu^s} (s - 1) \left( \zeta_L \left( 2 - s, 1, -\frac{q\alpha}{v} \right) + e^{\pi is} \zeta_L \left( 2 - s, 1, \frac{q\alpha}{v} \right) \right) - \left( \frac{\alpha q}{pv^2} - \frac{p + q}{pv} + \frac{wp}{2q} + \frac{v}{2} \right) \left( \frac{q\alpha}{v} \right)^{s-1} \left( \zeta_L \left( 1 - s, 1, -\frac{q\alpha}{v} \right) - e^{\pi is} \zeta_L \left( 1 - s, 1, \frac{q\alpha}{v} \right) \right) \right\}, \tag{2.2}
\]

where the branch of $\arg(2\pi i/n/v), \arg(2\pi i/n/w)$ are the same as in (i), and convergence of the finite series on the first and second terms on the right-hand side are the samely as in (i).

Remark 2. The functional equations for Theorem 2(i) was given by Y. Komori, K. Matsumoto, H. Tsumura \cite{7} for the general $r$-multiple zeta function $\zeta_r(s, \alpha; w_1, \ldots, w_r)$. The definition of that function is given as follows. Let $r$ be a positive integer and $w_j \in H(\theta) (j = 1, \ldots, r)$ are complex parameters. The Barnes multiple zeta-function is defined by

\[
\zeta_r(s, \alpha; w_1, \ldots, w_r) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{1}{(\alpha + w_1 m_1 + \cdots + w_r m_r)^s}, \tag{2.3}
\]

where the series on the right hand-side is absolutely convergence for $\Re(s) > r$, and is continued meromorphically to $\mathbb{C}$ and its only singularities are the simple poles located at $s = j (j = 1, \ldots, r)$. In their result in \cite{7}, the complex parameters $w_1, w_2, \ldots, w_r$ are conditionally given to satisfies $\text{Im}(w_j/w_k) \neq 0 (j \neq k)$. But in the results of Theorem 2 $v$ and $w$ are divided linearly independent and linearly dependent over $\mathbb{Q}$, giving more general results.
We can rewrite the formula (2.2) in Theorem 2 as follows.

**Corollary 3.** If \(v, w\) are linearly dependent over \(\mathbb{Q}\), exist \(p, q \in \mathbb{N}\) such as \(pv = qw\) and \((p, q) = 1\). Then we have

\[
\zeta_2(s, \alpha; v, w) = \frac{\Gamma(1-s)e^{\pi i(1-s)/2}}{(2\pi)^{1-s}} \left\{ \sum_{n=\infty}^{\infty} \frac{e^{2\pi i ny_1 + wy_2}/v}{(e^{2\pi i n}/v - 1)n^{1-s}} + \sum_{n=\infty}^{\infty} \frac{e^{2\pi i ny_1 + wy_2}/w}{(e^{2\pi i n}/w - 1)n^{1-s}} \right\}
\]

\[
+ \left( \frac{p+q}{pv} - \frac{vp}{2q} - \frac{v}{2} - \frac{aq}{pv^2} \right) \left( \frac{q}{v} \right)^{s-1} \zeta_H \left( s, \frac{q\alpha}{v} \right) + \frac{q^{s-1}}{pv^s} \zeta_H \left( s - 1, \frac{q\alpha}{v} \right),
\]

(2.4)

where \(-\theta < \arg (2\pi in/v), \arg (2\pi in/w) < -\theta + \pi\) and two finite series of the right-hand side converges absolutely uniformly on the whole space \(\mathbb{C}\) if \((y_1, y_2) \neq (0, 0)\), and on the region \(\text{Re}(s) < 0\) if \(y_1 = 0\) or \(y_2 = 0\).

**Remark 3.** The above functional equations (2.4) is a generalization that includes the well-known equation

\[
\zeta_2(s, \alpha; 1, 1) = (1 - \alpha)\zeta_H(s, \alpha) + \zeta_H(s - 1, \alpha)
\]

for the double Hurwitz zeta-function as a special case (See in [11], p. 86). Actual, considering the special case of \(v = w = 1\) in Corollary 3.

**Theorem 4 (Convexity bound).** Assume \(v > 0, w > 0\) and \(v/w\) is algebraic number. For \(0 \leq \sigma \leq 2\) and any \(\varepsilon > 0\), we have

\[
\zeta_2(\sigma + it, \alpha; v, w) \ll \begin{cases} |t|^{(2-\sigma)/4+\varepsilon} & (v/w \in \overline{\mathbb{Q}} \setminus \mathbb{Q}), \\ |t|^{3(2-\sigma)/4+\varepsilon} & (v/w \in \mathbb{Q}). \end{cases}
\]

Taking \(\sigma = 1, 3/2\) in this theorem, we obtain the following corollary.

**Corollary 5.** If \(v > 0, w > 0\) and \(v/w \in \overline{\mathbb{Q}} \setminus \mathbb{Q}\), we have

\[
\zeta_2(1 + it, \alpha; v, w) \ll |t|^{1/4+\varepsilon}
\]

and

\[
\zeta_2\left(\frac{3}{2} + it, \alpha; v, w\right) \ll |t|^{1/8+\varepsilon}.
\]

(2.5)

**Theorem 6.** Suppose that \(v > 0, w > 0\) and \(v/w \in \overline{\mathbb{Q}} \setminus \mathbb{Q}\). For any \(\varepsilon > 0\),

\[
\zeta_2(\sigma + it, \alpha; v, w) = O(t^\varepsilon) \quad \left( \frac{1}{2} \leq \sigma \leq 2 \right)
\]

holds is equivalent to

\[
\int_{\frac{1}{2}}^{T} |\zeta_2(\sigma + it, \alpha; v, w)|^{2k} dt = O(T^{1+\varepsilon}) \quad \left( \frac{1}{2} \leq \sigma \leq 2 \right)
\]

holds for any \(k \in \mathbb{N}\) and any \(\varepsilon > 0\). However, \(O\)-constant depends only on \(\varepsilon\) and \(k\).
3. SOME LEMMATA

In this section, we introduce five lemmas (Lemma 7 to Lemma 11). Lemma 7, Lemma 8 and Lemma 9 are supplementary on the absolutely convergence of infinite series involving exponential forms. Lemma 10 is called the Phragmén-Lindelöf convexity principle. Using this principle, we can obtain a simple upper bound. By Lemma 11, we can give the equivalence of the Lindelöf conjecture for the Barnes double zêta-function.

Lemma 7 (Lemma 1 in [II]). For any irrational real algebraic number $\beta_0$ and any pair $(p_1, p_2)$ of real numbers, the infinite series

$$
\eta(\beta_0, s, p_1, p_2) = \sum_{n=1}^{\infty} \frac{e^{2\pi in(p_1\beta_0+p_2)}}{(1 - e^{2\pi in\beta_0})n^{1-s}}
$$

(3.1)
is absolutely convergent if $\text{Re}(s) < 0$.

Proof. For any $x \in \mathbb{R}$ denote by $\langle x \rangle$ the unique real number with the conditions $-1/2 < \langle x \rangle \leq 1/2$ and $x - \langle x \rangle \in \mathbb{Z}$. Take any positive number $\varepsilon$ with $0 < 2\varepsilon < -\sigma$, and let $\varepsilon$ be fixed. By using the Thue-Siegel-Roth theorem in the diophantine approximation theory, there exists a constant $C_1 = C_1(\beta_0, \varepsilon) > 0$ such that

$$
|\beta_0 - \frac{m}{n}| > \frac{C_1(\beta_0, \varepsilon)}{n^{2+\varepsilon}}
$$

holds for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Therefore,

$$
n|\langle n/\beta_0 \rangle| > \frac{C(\beta_0, \varepsilon)}{n^\varepsilon}
$$

for all $n \in \mathbb{N}$. By using

$$
\sum_{n=1}^{m} \frac{1}{n|\langle n/\beta_0 \rangle|} = O\left(\Psi(2m) \log m + \sum_{n=1}^{m} \frac{\Psi(2n) \log n}{n}\right)
$$

(Lemma 3.3 in p. 123 of [8]) and taking $\Psi(n) = C_1n^\varepsilon$, then

$$
\sum_{n=1}^{m} \frac{1}{n|\langle n/\beta_0 \rangle|} = O(m^{2\varepsilon}).
$$

Immediately, we have

$$
\sum_{n=1}^{\infty} \frac{1}{n^{1+2\varepsilon}|\langle n/\beta_0 \rangle|} = O(1).
$$

Further there exist a positive constant $C_2$ such that

$$
\frac{|x|}{|1 - e^{2\pi ix}|} \leq C_2
$$

for any $x \in \mathbb{R}$ with $|x| \leq 1/2$. Then it easily follows that

$$
\sum_{n=1}^{\infty} \left|\frac{e^{2\pi in(p_1\beta_0+p_2)}}{(1 - e^{2\pi in\beta_0})n^{1-s}}\right| \leq C_2 \sum_{n=1}^{\infty} \frac{1}{n^{1-\sigma}|\langle n/\beta_0 \rangle|},
$$

and above series on the right-hand side is converges when $\sigma < 0$. Therefore, $\eta(\beta_0, s, p_1, p_2)$ is absolutely convergent for $\sigma < 0$. \qed
Lemma 8. Let $\alpha, v, w \in H(\theta)$, and expressed as $\alpha = v(1-y_1) + w(1-y_2)$ ($0 \leq y_1, y_2 < 1$). Also, $v/w$ is either an imaginary or non-transcendental real number, i.e. $\beta \in (\mathbb{C}\setminus\mathbb{R}) \cup (\mathbb{R} \cap \mathbb{Q})$. The convergence of the infinite series

$$
\sum_{n=1}^{\infty} \frac{e^{2\pi i n(vy_1 + wy_2)/v}}{(e^{2\pi i n/v} - 1)n^{1-s}} \quad (3.3)
$$

is as follows;

(i) If $w/v$ is an algebraic real irrational number, (3.3) is absolutely convergence for $\sigma < 0$.

(ii) If $w/v$ is imaginary, (3.3) is absolutely convergence on the whole space $\mathbb{C}$ if $0 < y_2 < 1$, and for $\sigma < 0$ if $y_2 = 0$.

Proof. For $\sigma > 0$,

$$
\sum_{n=1}^{\infty} \frac{e^{2\pi i n(vy_1 + wy_2)/v}}{(e^{2\pi i n/v} - 1)n^{1-s}} = \sum_{n=1}^{\infty} \left\{ \frac{e^{2\pi i n(vy_1 + wy_2)/v}}{(e^{2\pi i n/v} - 1)n^{1-s}} + (-1)^{s-1} \frac{e^{-2\pi i n(vy_1 + wy_2)/v}}{(e^{-2\pi i n/v} - 1)n^{1-s}} \right\}
$$

$$= -\eta(w/v, s, y_2, y_1) + (-1)^s \eta(-w/v, s, y_2, -y_1) \quad (3.4)
$$

holds.

(i) If $w/v$ is an algebraic real irrational number, By the Lemma 7 both the first and second terms of (3.4) are absolutely convergence when $\sigma > 0$, so the series (3.3) is absolutely convergence for $\sigma > 0$.

(ii) If $w/v$ is an imaginary, we consider separately when Im$(w/v) > 0$ and Im$(w/v) < 0$. When Im$(w/v) > 0$, we put Im$(w/v) = I$. From here, for $z \in \mathbb{C}$ and $\varepsilon > 0$, let $D(z, \varepsilon)$ be the closed disk whose center in $z$ with fadious $\varepsilon$. We use the notation $x_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$. We first note that for any $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ such that for $z \in \mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} D(2\pi im, \varepsilon)$ the inequality

$$
\left| \frac{1}{e^{z^2} - 1} \leq Me^{-(\Re(z))^+}
$$

holds. Then there exist $M_0 > 0$ such that

$$
\left| \frac{e^{2\pi i n(vy_1 + wy_2)/v}}{e^{2\pi i n/v} - 1} \right| \leq M_0 e^{-2\pi nI} \left( \begin{array}{l} M_0 e^{2\pi nIy_2} & (n > 0), \\ M_0 e^{2\pi nI(1-y_2)} & (n < 0). \end{array} \right)
$$

When Im$(w/v) < 0$, we put Im$(w/v) = -I$ in the same way as above, then there exist $M_0 > 0$ such that

$$
\left| \frac{e^{2\pi i n(vy_1 + wy_2)/v}}{e^{2\pi i n/v} - 1} \right| \leq \left( \begin{array}{l} M_0 e^{-2\pi nIy_2} & (n > 0), \\ M_0 e^{-2\pi nI(1-y_2)} & (n < 0). \end{array} \right)
$$
If $0 < y_2 < 1$ then,
\[
\sum_{n=-\infty}^{\infty} \left| \frac{e^{2\pi in(v_1 + w_2)}/v}{(e^{2\pi inw/v} - 1)n^{1-s}} \right| \leq M_0 \sum_{n=1}^{\infty} \left( \frac{e^{-2\pi nIy_2}}{n^{1-\sigma}} + \frac{e^{-2\pi nI(1-y_2)}}{n^{1-\sigma}} \right)
\]
\[
\ll \begin{cases}
\sum_{n=1}^{\infty} \frac{1}{n^{1-\sigma}} & (\sigma < 0) \\
\sum_{n=1}^{\infty} e^{-2\pi nIy_2} & (\sigma = 0) \\
(2\pi Iy_2)^{-\sigma} \Gamma(\sigma) & (\sigma > 0)
\end{cases}
\ll 1 \quad (\sigma \in \mathbb{R}).
\]

On the other hand, if $y_2 = 0$ then
\[
\sum_{n=-\infty}^{\infty} \left| \frac{e^{2\pi in(v_1 + w_2)}/v}{(e^{2\pi inw/v} - 1)n^{1-s}} \right| \leq M_0 \sum_{n=1}^{\infty} \left( \frac{1}{n^{1-\sigma}} + \frac{e^{-2\pi nI}}{n^{1-\sigma}} \right) \ll 1 \quad (\sigma < 0).
\]

\[\square\]

**Lemma 9.** Let $\alpha, v, w \in H(\theta)$, and expressed as $\alpha = v(1-y_1) + w(1-y_2)$ ($0 \leq y_1, y_2 < 1$). Also, if $v, w$ can be expressed as $pv = qw$ with $p, q \in \mathbb{N}$ and $(p, q) = 1$. The convergence of the infinite series
\[
\sum_{n=-\infty}^{\infty} \frac{e^{2\pi in(v_1 + w_2)}/v}{(e^{2\pi inp/q} - 1)n^{1-s}}
\]  
(3.5)
is absolutely convergence on the whole space $\mathbb{C}$ if $0 < y_2 < 1$, and for $\sigma < 0$ if $y_2 = 0$.

This lemma can be proved in much the same way as Lemma 8.

**Lemma 10** (Phragmén-Lindelöf convexity principle). Let $\sigma_1, \sigma_2 > 0$ and $B = \{s = \sigma + it \in \mathbb{C} | \sigma_1 \leq \sigma \leq \sigma_2, t \geq t_0\}$. Assume that $f$ is continuous over $B$, holomorphic inner points of $B$, and satisfies to
\[
|f(s)| \leq c_1 \exp(\exp(c_2t))
\]
where $c_1, c_2 > 0$ are constant and $c_2 < \pi/(\sigma_2 - \sigma_1)$. If $|f(s)| \leq A$ in $\partial B$, then
\[
|f(s)| \leq A \quad (s \in B)
\]
holds.

**Lemma 11.** Suppose that $\alpha, v, w \in H(\theta)$, and $v, w$ are linearly independent over $\mathbb{Q}$. Let $1/2 \leq \sigma \leq 2, t \leq 2$ and $\ell \in \mathbb{N}$. Then
\[
|\zeta_2(\sigma + it, \alpha; v, w)|^\ell \ll (\log t) \int_{-\delta}^{\delta} |\zeta_2(\sigma + ix, \alpha; v, w)|^\ell dx + t^{-A} \quad (3.6)
\]
for any fixed $0 < \delta < 1/2$ and $A > 0$. 
Proof. Let $B, C > 0$, and $r = [C \log t]$ with $s = \sigma + it$. By residue theorem,

$$
\int_0^B \cdots \int_0^B \int_{|z| = \delta} \zeta_2(s + z, \alpha; v, w)^{\ell} e^{(u_1 + \cdots + u_r)z} z^{-1} dz du_1 \cdots du_r
$$

$$
= 2\pi i \zeta_2(s, \alpha; v, w)^{\ell} \int_0^B \cdots \int_0^B du_1 \cdots du_r
$$

$$
= 2\pi i B^r \zeta_2(s, \alpha; v, w)^{\ell}
$$

holds. On the other hand, the circular path $|z| = \delta$ is divided into two parts $\text{Re}(z) < 0$ and $\text{Re}(z) \geq 0$, which are denoted by $R_1$ and $R_2$, respectively. The integral for $R_1$ is

$$
\int_{R_1} \zeta_2(s + z, \alpha; v, w)^{\ell} \int_0^B e^{zu_1} du_1 \int_0^B e^{zu_2} du_2 \cdots \int_0^B e^{zu_r} du_r \frac{dz}{z}
$$

$$
= \int_{R_1} \zeta_2(s + z, \alpha; v, w)^{\ell} \left( \frac{e^{Bz} - 1}{z} \right)^r \frac{dz}{z},
$$

where $|e^{Bz}| \leq 1$ on $R_1$, satisfies $|e^{Bz} - 1| \leq 2$. We denote the right-hand side by $I_0$. Also, since $1/2 \leq \sigma \leq 2$ and $-1/2 \leq \text{Re}(z) < 0$ on $R_1$, so $0 \leq \text{Re}(s + z) < 2$. Then we have

$$
\zeta_2(s + z, \alpha; v, w) \ll t^{(2-\text{Re}(s+z))/4+\epsilon} \ll t^{1/2}
$$

Then, the above equation can be evaluated as

$$
\ll \int_{R_1} t^{\ell/2} \left( \frac{2}{\delta} \right)^r |dz| \ll t^{\ell/2} \left( \frac{2}{\delta} \right)^r.
$$

Also, let $U = \exp(u_1 + u_2 + \cdots + u_r)$. We consider separately as

$$
\frac{U^z}{z} = \frac{U^z - U^{-z}}{z} + \frac{U^{-z}}{z}.
$$

The integral for $R_2$ is

$$
\int_0^B \cdots \int_0^B \int_{R_2} \zeta_2(s + z, \alpha; v, w)^{\ell} e^{(u_1 + \cdots + u_r)z} z^{-1} dz du_1 \cdots du_r
$$

$$
= \int_{R_2} \int_0^B \cdots \int_0^B \int_0^B \zeta_2(s + z, \alpha; v, w)^{\ell} \left( \frac{U^z - U^{-z}}{z} + \frac{U^{-z}}{z} \right) du_1 \cdots du_r dz
$$

$$
= \int_{R_2} \int_0^B \cdots \int_0^B \int_0^B \int_0^B \zeta_2(s + z, \alpha; v, w)^{\ell} \left( \frac{U^z - U^{-z}}{z} \right) du_1 \cdots du_r dz
$$

$$
+ \int_{R_2} \int_0^B \cdots \int_0^B \int_0^B \int_0^B \int_0^B \zeta_2(s + z, \alpha; v, w)^{\ell} \left( \frac{U^{-z}}{z} \right) du_1 \cdots du_r dz.
$$

We denote the right-hand side by $I_1 + I_2$. Since $|e^{-Bz} - 1| \leq 2$ on $z \in R_2$. Also, since $1/2 \leq \sigma \leq 2$ and $0 \leq \text{Re}(z) \leq 1/2$ on $R_2$, so $1/2 \leq \text{Re}(s + z) \leq 3/2$. Then we have

$$
\zeta_2(s + z, \alpha; v, w) \ll t^{(2-\text{Re}(s+z))/4+\epsilon} \ll t^{3/8}.
$$

Then, $I_1$ can be evaluated as

$$
I_1 = \int_{R_2} \zeta_2(s + z, \alpha; v, w)^{\ell} \left( \frac{1 - e^{-Bz}}{z} \right) \frac{1}{z} dz \ll \int_{R_1} t^{3\ell/8} \left( \frac{2}{\delta} \right)^r |dz|
$$

$$
\ll t^{3\ell/8} \left( \frac{2}{\delta} \right)^r.
$$
Next, we consider $I_2$. Since $(U^z - U^{-z})/z$ has a removable singularity at $z = 0$, so it is holomorphic at $z = 0$. By using Cauchy's integral theorem, a integral path $R_2$ can be change to integral path form $-i\delta$ to $i\delta$ on the imaginary axis. Then we have

$$
\int_0^B \cdots \int_0^B \int_{-i\delta}^{i\delta} \zeta_2(s + z, \alpha; v, w) \ell \left( \frac{U^z - U^{-z}}{z} \right) dz \, du_1 \cdots du_r
$$

$$
\ll \int_0^B \cdots \int_0^B \int_{-i\delta}^{i\delta} \zeta_2(s + z, \alpha; v, w) \ell \left( \frac{U^z - U^{-z}}{z} \right) dz \, du_1 \cdots du_r,
$$

Here, if $z = ix (-\delta \leq x \leq \delta)$, then

$$
\left| \frac{U^z - U^{-z}}{z} \right| = \left| \frac{U^{ix} - U^{-ix}}{x} \right| = \frac{2}{x} \left| \frac{e^{ix\log U} - e^{-ix\log U}}{2i} \right| = 2 \left| \frac{\sin (x \log U)}{x} \right|
$$

$$
\ll \log U = u_1 + u_2 + \cdots + u_r \leq Br
$$

$$
\ll BC \log t.
$$

Then we have

$$
I_2 \ll Br+1C \int_{-\delta}^{\delta} |\zeta_2(\sigma + it + ix, \alpha; v, w)|^\ell (\log t) \, dx.
$$

Using the evaluation results of $I_0$, $I_1$ and $I_2$, we obtain

$$
|\zeta_2(\sigma + it, \alpha; v, w)|^\ell \leq \frac{1}{2\pi i} \cdot \frac{1}{B^r} (I_2 + I_1 + I_0)
$$

$$
\ll BC \int_{-\delta}^{\delta} |\zeta_2(\sigma + it + ix, \alpha; v, w)|^\ell (\log t) \, dx + t^{\ell/2} \left( \frac{2}{B\delta} \right)^r.
$$

By setting $B = 4/\delta$, the second term on the rightmost side of the above equation can be evaluated as

$$
t^{\ell/2} \left( \frac{2}{B\delta} \right)^r = t^\ell \cdot 2^{-r} \ll t^{\ell/2-C\log 2}.
$$

Furthermore, if we choose $C$ such that $C > \ell/\log 4$, we obtain

$$
|\zeta_2(\sigma + it, \alpha; v, w)|^\ell \ll (\log t) \int_{-\delta}^{\delta} |\zeta_2(\sigma + it + ix, \alpha; v, w)|^\ell (\log t) \, dx + t^{-A}.
$$

as $A = C \log 2 - \ell/2$. $\square$

4. Proof of Main theorems

In this section, we give the proof of Theorem 2, Theorem 4 and Theorem 6. First, we give a proof of the theorem 2. In the course of this proof, we use Lemmas 8 and Lemma 9 for the discussion of absolutely convergence of some infinite series. Finally, we give proofs of Theorem 4 and Theorem 6 which use Lemma 10 and Lemma 11, respectively.

Proof of Theorem 2

Assume that $\text{Re}(s) > 2$. For $x \in H(\theta)$, we have the formula for the gamma function

$$
x^{-s} = \frac{1}{\Gamma(s)} \int_0^{e^{-i\theta} \infty} e^{-xt} t^{s-1} dt.
$$
Then we have

$$\zeta_2(s, \alpha; v, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(\alpha + vm + wn)^s} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\Gamma(s)} \int_0^{e^{-i\theta}} z^{s-1} e^{-(\alpha + vm + wn)z} dz = \frac{1}{\Gamma(s)} \int_0^{e^{-i\theta}} \frac{z^{s-1} e^{(v + w - \alpha)z}}{(e^{vz} - 1)(e^{wz} - 1)} dz,$$

(4.1)

where the argument of $t$ is taken in $-\theta \leq \arg t \leq -\theta + 2\pi$ and $C(\theta)$ is contour which starts at $e^{-i\theta} \infty$, goes counterclockwise around the origin with sufficiently small radius, and ends at $e^{-i\theta} \infty$.

Next, we consider changing the small circle of integration path $C(\theta)$ to integration path $C_R(\theta)$ with a large circle of radius $R$, and residues of integrand of (4.1)

$$F(z) = \frac{z^{s-1} e^{(v + w - \alpha)z}}{(e^{vz} - 1)(e^{wz} - 1)};$$

where, $R$ is a sufficiently large positive real number so that $C_R(\theta)$ does not pass through the singular point of $F(z)$. Note that these residues of $F(z)$ depends on whether the parameters $v, w$ are linearly independent or linearly dependent over $\mathbb{Q}$.

(i) $v, w$: linear independent over $\mathbb{Q}$
(ii) $v, w$: linear dependent over $\mathbb{Q}$
(i) In the case when \( v, w \) are linear independent over \( \mathbb{Q} \), and \( v/w \) is imaginary or irrational real algebraic number. \( F(z) \) has simple poles at
\[
z = \frac{2\pi in}{v}, \frac{2\pi in}{w} \quad (n = \pm 1, \pm 2, \cdots).
\]
Since
\[
\lim_{z \to 2\pi in/v} \left( z - \frac{2\pi in}{v} \right) F(z) = \lim_{z \to 2\pi in/v} \left( z - \frac{2\pi in}{v} \right) \frac{z^{s-1} e^{(v+w-\alpha)z}}{(e^{vz} - 1)(e^{wz} - 1)}
\]
\[
= \lim_{z \to 2\pi in/v} \left( \frac{e^{vz} - e^{2\pi in/v}}{z - 2\pi in/v} \right) \frac{z^{s-1} e^{(v+w-\alpha)z}}{e^{wz} - 1}
\]
\[
= \frac{1}{v} \left( \frac{2\pi in}{v} \right)^{s-1} \frac{e^{(w-\alpha)2\pi in/v}}{e^{2\pi in/v} - 1},
\]
then we have
\[
\text{Res}_{z=2\pi in/v} F(z) = \frac{1}{v} \left( \frac{2\pi in}{v} \right)^{s-1} \frac{e^{(w-\alpha)2\pi in/v}}{e^{2\pi in/v} - 1},
\]
\[
\text{Res}_{z=2\pi in/w} F(z) = \frac{1}{w} \left( \frac{2\pi in}{w} \right)^{s-1} \frac{e^{(v-\alpha)2\pi in/w}}{e^{2\pi in/w} - 1}
\]
and we obtain
\[
\zeta_2(s, \alpha; v, w) \quad \Gamma(1-s) \quad (2\pi i)^{1-s} e^{\pi i s} \left\{ \frac{1}{v^s} \sum_{0 < |n| \leq K} \frac{e^{2\pi in(w-\alpha)/v}}{(e^{2\pi in/v} - 1)n^{1-s}} + \frac{1}{w^s} \sum_{0 < |n| \leq L} \frac{e^{2\pi in(v-\alpha)/w}}{(e^{2\pi in/w} - 1)n^{1-s}} \right\}
\]
\[
+ \frac{1}{\Gamma(s)(e^{2\pi i s} - 1)} \int_{C_K(\theta)} \frac{z^{s-1} e^{(v+w-\alpha)z}}{(e^{vz} - 1)(e^{wz} - 1)} dz,
\]
where \( K = ||v|R/(2\pi)|, L = ||w|R/(2\pi)\). Hence for
\[
z \in \mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} D(2\pi im/v, \varepsilon) \cup D(2\pi im/w, \varepsilon)
\]
we have
\[
\left| \frac{z^{s-1} e^{(v+w-\alpha)z}}{(e^{vz} - 1)(e^{wz} - 1)} \right| \leq M_1 e^{-\text{Re}(vz)+} \cdot M_2 e^{-\text{Re}(wz)+} e^{\text{Re}((v+w-\alpha)z)|z|^{\sigma-1}}
\]
\[
= M_1 M_2 \exp \{ \text{Re}((v + w - \alpha)z) - \text{Re}(vz)+ - \text{Re}(wz)+ \}|z|^{\sigma-1}
\]
with a certain \( M_1 = M_1(\varepsilon, v) > 0 \) and \( M_2 = M_2(\varepsilon, w) > 0 \). Here, let \( 0 \leq y_1, y_2 < 1 \) and, put
\[
\alpha = \alpha(y_1, y_2) = v(1 - y_1) + w(1 - y_2) \in H(\theta)
\]
and \( z' = z/|z| \). Then we see that there exists \( T = T(\varepsilon) \geq 0 \) such that for
\[
\text{Re}((v + w - \alpha)z') - \text{Re}(vz')_+ - \text{Re}(wz')_+
\]
\[
= \text{Re}((vy_1 + wy_2)z') - \text{Re}(vz')_+ - \text{Re}(wz')_+
\]
\[
\leq \text{Re}(vz')y_1 + \text{Re}(wz')y_2 - \text{Re}(vz')_+ - \text{Re}(wz')_+
\]
\[
\leq -T,
\]
and so
\[ \Re((v + w - \alpha)z) - \Re(vz)_+ - \Re(wz)_+ \leq -T|z|. \]

Hence we see that for all \( z \in \mathbb{C} - \bigcup_{m \in \mathbb{Z}} D(2\pi im/v, \varepsilon) \cup D(2\pi im/w, \varepsilon), \)
\[
\left| \frac{z^{s-1}e^{(v+w-\alpha)z}}{(e^{vz} - 1)(e^{wz} - 1)} \right| \leq M_1M_2 \cdot |z^{s-1}|e^{-T|z|} \tag{4.3}
\]

If \( 0 < y_1, y_2 < 1 \) or \( y_1 = y_2 = 0 \), that we can choose \( T > 0 \). In fact, since \( v, w \) are linear independent over \( \mathbb{Q} \), for any \( z' \) with \( |z'| = 1 \) at least one of \( \Re(vz') \neq 0 \) and \( \Re(wz') \neq 0 \) holds. If \( 0 < y_1 < 1 \) and \( y_2 = 0 \), that we can choose \( T > 0 \). In fact,
\[
\Re((vy_1 + wy_2)z') - \Re(vz')_+ - \Re(wz')_+ \leq \Re(vz'y_1 - \Re(vz')_+ - \Re(wz')_+ < 0
\]

Similarly in the case of \( y_1 = 0 \) and \( 0 < y_2 < 1 \), we can choose \( T > 0 \).

From [12], we see that integral term on the rightmost side of (4.2) converges to 0 when the radius of the contour goes to infinity if \( 0 < y_1, y_2 < 1 \) and \( 0 < y_1, y_2 = 0 \) and \( y_1 = 0, 0 < y_2 < 1 \) or \( y_1 = y_2 = 0 \) with \( \Re(s) < 0 \). Namely,
\[
\left| \int_{C_R(\theta)} \frac{z^{s-1}e^{(v+w-\alpha)z}}{(e^{vz} - 1)(e^{wz} - 1)} \, dz \right| \leq \int_{|z|=R} \left| \frac{z^{s-1}e^{(v+w-\alpha)z}}{(e^{vz} - 1)(e^{wz} - 1)} \right| \, |dz|
\leq \int_{|z|=R} M_1M_2 |z^{s-1}|e^{-T|z|} |dz| \to 0 \quad (R \to \infty)
\]

Hence we can calculate the integral by counting all the residuees on the whole space. Since by the assumption the poles of the integrand are all simple except the origin, we obtain
\[
\zeta_2(s, \alpha(y_1, y_2); v, w) = -\frac{\Gamma(1 - s)}{(2\pi i)^{1-s}e^{\pi is}} \left\{ \frac{1}{v^s} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi in(y_1 + wy_2)/v}}{(e^{2\pi inv/v} - 1)n^{1-s}} + \frac{1}{w^s} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi in(y_1 + wy_2)/w}}{(e^{2\pi inw/w} - 1)n^{1-s}} \right\}.
\]

Also, two finite series of the right-hand side convergers absolutely uniformly on the whole space \( \mathbb{C} \) if \( (y_1, y_2) = (0, 0) \), and on the region \( \Re(s) < 0 \) if \( y_1 = 0 \) or \( y_2 = 0 \), by Lemma [8].

(ii) In the case when \( v, w \) are linear dependent over \( \mathbb{Q} \), that is, there exist \( p, q \in \mathbb{N} \) such that \( pv = qw \) and \( (p, q) = 1 \). \( F(z) \) has simple poles at
\[
z = \frac{2\pi in}{v} \quad (n \in \mathbb{Z} \setminus \{0\}, \, q \nmid n), \quad \frac{2\pi in}{w} \quad (n \in \mathbb{Z} \setminus \{0\}, \, p \nmid n).
\]

Also, residue of \( F(z) \) at simple poles are
\[
\text{Res}_{z=2\pi in/v} F(z) = \frac{1}{v} \left( \frac{2\pi in}{v} \right)^{s-1} \frac{e^{(v-\alpha)2\pi in/v}}{e^{2\pi inv/v} - 1},
\]
\[
\text{Res}_{z=2\pi in/w} F(z) = \frac{1}{w} \left( \frac{2\pi in}{w} \right)^{s-1} \frac{e^{(v-\alpha)2\pi in/w}}{e^{2\pi inw/w} - 1}.
\]
On the other hand here for $w = pv/q$,

\[
\lim_{z \to 2q\pi in/v} \frac{d}{dz} \left\{ \left( z - \frac{2q\pi in}{v} \right)^2 \frac{z^{s-1}e^{(v+pv/q-\alpha)z}}{(e^{vz} - 1)(e^{pvz/q} - 1)} \right\}
\]

\[
= \lim_{z_0 \to 0} \frac{d}{dz_0} \left\{ z_0^2 \left( z_0 + \frac{2q\pi in}{v} \right)^{s-1} \frac{e^{(v+pv/q-\alpha)z_0} e^{-2q\pi in/v}}{(e^{vz_0} - 1)(e^{pvz_0/q} - 1)} \right\}
\]

\[
= \lim_{z_0 \to 0} \frac{d}{dz_0} \left\{ \frac{z_0^2}{z_0} \left( z_0 + \frac{2q\pi in}{v} \right)^{s-1} \frac{e^{(v+pv/q-\alpha)z_0} e^{-2q\pi in/v}}{\left( v z_0 + \frac{v^2}{2!} z_0^2 + O(z_0^3) \right) \left( \frac{pv}{q} z_0 + \frac{1}{2!} \left( \frac{pv}{q} \right)^2 z_0^2 + O(z_0^3) \right)} \right\}
\]

\[
= \frac{q}{pv^2} (s-1) \left( \frac{2q\pi in}{v} \right)^{s-2} e^{-2q\pi in/v}
\]

\[
- \left( \frac{\alpha q}{pv^2} - \frac{p + q}{pv} + \frac{vp}{2q} + \frac{v}{2} \right) \left( \frac{2q\pi in}{v} \right)^{s-1} e^{-2q\pi in/v}
\]

Therefore $F(z)$ has double poles at

\[
z = \frac{2\pi i q n}{v} = \frac{2\pi i p n}{w} \quad (n \in \mathbb{Z} \setminus \{0\}).
\]

Then, we calculate the following residue sum;

\[
\sum_{0 < |n| \leq M} \text{Res} F(z) = \sum_{0 < |n| \leq L} \text{Res}_{z=2q\pi in/v} F(z) + \sum_{0 < |n| \leq K} \text{Res}_{z=2q\pi in/w} F(z)
\]

\[
+ \sum_{0 < |k| \leq K} \text{Res}_{z=2\pi i q k/v} F(z)
\]

therefore, we obtain

\[
\zeta_2(s, \alpha; v, w)
\]

\[
= -\frac{\Gamma(1-s)}{(2\pi i)^{1-s} e^{\pi i s}} \left\{ \frac{1}{v^s} \sum_{q \nmid n}^{\infty} \frac{e^{2\pi in(v-\alpha)/w}}{(e^{2\pi inw/v} - 1)n^{1-s}} + \frac{1}{w^s} \sum_{p \nmid n}^{\infty} \frac{e^{2\pi ivn(v-\alpha)/w}}{(e^{2\pi ivn/w} - 1)n^{1-s}}
\]

\[
+ \frac{q^{s-1}}{2\pi iv^s} \left( 1 - s \right) \sum_{n=\infty \atop n \neq 0}^{\infty} \frac{e^{-2\pi inq/v}}{n^{2-s}}
\]

\[
- \left( \frac{\alpha q}{pv^2} - \frac{p + q}{pv} + \frac{vp}{2q} + \frac{v}{2} \right) \left( \frac{q}{v} \right)^{s-1} \sum_{n=\infty \atop n \neq 0}^{\infty} \frac{e^{-2\pi inq/v}}{n^{1-s}} \right\}
\]
Also, first and second finite series of the right-hand side converges absolutely uniformly on the whole space $\mathbb{C}$ if $(y_1, y_2) = (0, 0)$, and on the region $\text{Re}(s) < 0$ if $y_1 = 0$ or $y_2 = 0$, by Lemma 9. Furthermore, sum in the third and fourth term of the above are equation to
\[
\sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{e^{-2\pi inq/v}}{n^{2-s}} = \zeta_L \left( 2 - s, 1, -\frac{q\alpha}{v} \right) + e^{\pi i s} \zeta_L \left( 2 - s, 1, \frac{q\alpha}{v} \right),
\]
\[
\sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{e^{-2\pi inq/v}}{n^{1-s}} = \zeta_L \left( 1 - s, 1, -\frac{q\alpha}{v} \right) + e^{\pi i s} \zeta_L \left( 1 - s, 1, \frac{q\alpha}{v} \right)
\]
respectively.

Hence proof of Theorem 2 is complete. \qed

**Proof of Theorem 4** Let $B = \{ s \in \mathbb{C} \mid \sigma_1 \leq \sigma \leq \sigma_2, |t| \geq t_0 (> 2) \}$. And let $g(s)$ be continuous in $B$, regular in $B$, and satisfies $g(s) \leq c_1 \exp (\exp (c_2 t))$ where $c_1, c_2 > 0$ are constant and $c_2 < \pi / (\sigma_2 - \sigma_1)$. Also if $v, w$ are linearly independent over $\mathbb{Q}$, applying the functional equation of $\zeta_2(s, \alpha; v, w)$ (Theorem 2(i)), and by using the Stirling’s formula
\[
\Gamma(s) = (2\pi)^{1/2}|t|^{-s/2} e^{-\pi|t|/2} (1 + O(|t|^{-1}))
\]
uniformly holds in $B$. By Lemma 8 two series
\[
\sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{e^{2\pi in(w-\alpha)/v}}{(e^{2\pi inw/v}-1)n^{1-s}}, \sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{e^{2\pi in(w-\alpha)/w}}{(e^{2\pi inw/w}-1)n^{1-s}}
\]
in (4.4) are absolutely converge to $\sigma < 0$ if $v/w \in \mathbb{Q} \setminus \mathbb{Q}$, so we obtain
\[
\zeta_2(s, \alpha; v, w) \ll \frac{\Gamma(1-s)}{(2\pi i)^{-s} e^{\pi is}} \ll |t|^{1-s} \quad (\sigma < 0).
\]
From above results, $a = \varepsilon + 1/2, b = 0$ and
\[
k(\sigma) = \frac{2 - \sigma}{4} + \frac{\varepsilon}{2}.
\]
Therefore, if \( v/w \in \mathbb{Q}\backslash\mathbb{Q} \) we have
\[
\zeta_2(\sigma + it, \alpha; v, w) \ll |t|^{(2-\sigma)/4+\varepsilon}
\]

On the other hand, \( v/w \in \mathbb{Q} \), applying the functional equation of \( \zeta_2(s, \alpha; v, w) \) (Theorem 2(ii)), Stirling’s formula and absolutely convergence for \( \zeta_L(2-s, 1, -q\alpha/v) \) \((\sigma < 0)\), we obtain
\[
\zeta_2(s, \alpha; v, w) \ll -\frac{\Gamma(1-s)}{(2\pi i)^{1-s} e^{\pi i s}} \cdot \frac{q^{s-1}}{2\pi iv^s} (1-s) \zeta_L(2-s, 1, -\frac{q\alpha}{v})
\]
\[
\ll \frac{\Gamma(2-s)}{(2\pi i)^{2-s} e^{\pi i (s-1)}} \cdot \frac{q^{(s-1)}}{pv^s} \zeta_L(2-s, 1, -\frac{q\alpha}{v})
\]
\[
\ll |t|^{-\sigma+3/2} \quad (\sigma < 0).
\]
From above results, \( a = 3\varepsilon + 3/2 \), \( b = 0 \) and
\[
k(\sigma) = \frac{2-2\varepsilon-\sigma}{2+4\varepsilon} \left( 3\varepsilon + \frac{3}{2} \right) = \frac{3(2-\sigma)}{4} + \frac{\varepsilon}{2}.
\]
Therefore, if \( v, w \) are linearly dependent over \( \mathbb{Q} \), we have
\[
\zeta_2(\sigma + it, \alpha; v, w) \ll |t|^{3(2-\sigma)/4+\varepsilon}.
\]
Hence proof of Theorem 4 is complete. \( \square \)

**Proof of Theorem 6** Assume that
\[
\zeta_2(\sigma + it, \alpha; v, w) = O(|t|^{\varepsilon/2k}) \quad \left( \frac{1}{2} \leq \sigma \leq 2 \right)
\]
holds for any \( \varepsilon > 0 \) and \( k \in \mathbb{N} \). Then,
\[
\int_1^T |\zeta_2(\sigma + it, \alpha; v, w)|^{2k} dt \ll \int_1^T t^{\varepsilon} dt = \frac{1}{1+\varepsilon} (T^{1+\varepsilon} - 1) \ll T^{1+\varepsilon}.
\]
On the other hand, taking \( l = 2k \) for (3.6) in Lemma III then
\[
|\zeta_2(\sigma + it, \alpha; v, w)|^{2k} \ll (\log t) \int_{-\delta}^\delta |\zeta_2(\sigma + it + ix, \alpha; v, w)|^{2k} dx + t^{-A}
\]
\[
\ll (\log t) \cdot t^{1+\varepsilon/2} + t^{-A}
\]
\[
\ll t^{1+\varepsilon}.
\]
Hence,
\[
\zeta_2(\sigma + it, \alpha; v, w) \ll t^{1/2k+\varepsilon}
\]
holds for any \( k \in \mathbb{N} \) and \( \varepsilon > 0 \). Therefore, we have
\[
\zeta_2(\sigma + it, \alpha; v, w) = O(|t|^{\varepsilon})
\]
for \( 1/2 \leq \sigma \leq 2 \). \( \square \)

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Takashi Miyagawa:
Faculty of Economics, Management and Information Science,
Onomichi City University,
1600-2 Hisayamada-cho Onomichi, Hiroshima 722-8506, Japan
E-mail: miyagawa@onomichi-u.ac.jp