GLOBAL WELL-POSEDNESS FOR THE 2D BOUSSINESQ EQUATIONS WITH A VELOCITY DAMPING TERM

RENHUI WAN
Institute of Mathematics, School of Mathematical Sciences
Nanjing Normal University, Nanjing 210023, China
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Abstract. In this paper, we prove global well-posedness of smooth solutions to the two-dimensional incompressible Boussinesq equations with only a velocity damping term when the initial data is close to a nontrivial equilibrium state $(0, x_2)$. As a by-product, under this equilibrium state, our result gives a positive answer to the question proposed by [1] (see P.3597).

1. Introduction. In this paper, we investigate the global existence and uniqueness of solutions to the two-dimensional (2D) incompressible Boussinesq equations with a velocity damping term, namely,

$$
\begin{aligned}
\begin{cases}
\partial_t u + u \cdot \nabla u + \nu u + \nabla p = \Psi e_2, \\
\partial_t \Psi + u \cdot \nabla \Psi = 0, \\
div u = 0,
\end{cases}
\end{aligned}
$$

where $u = (u_1, u_2)$ stand for the 2D velocity field, $p$ the pressure and $\Psi$ the temperature in thermal convection or the density in geophysical flows, $\nu > 0$ is a parameter and $e_2 = (0, 1)$ is the unit vector in the vertical direction. $u_0$ satisfies $div u_0 = 0$.

Many geophysical flows such as atmospheric fronts and ocean circulations (see, e.g., [3, 5, 8, 9]) can be modeled by the Boussinesq equations. Mathematically we regard the 2D Boussinesq equations as a lower-dimensional model of the 3D hydrodynamics equations. It is an open question whether the solutions to the inviscid case ($\nu = 0$) exist for all time or blow-up in a finite time. Indeed, adding only a velocity damping term seems not helpful to get the global solution even under the assumption that the initial data is small enough.

When adding the temperature damping term $\eta \Psi$ ($\eta > 0$) to (1), that is,

$$
\begin{aligned}
\begin{cases}
\partial_t u + u \cdot \nabla u + \nu u + \nabla p = \Psi e_2, \\
\partial_t \Psi + u \cdot \nabla \Psi + \eta \Psi = 0, \\
div u = 0,
\end{cases}
\end{aligned}
$$

Adhikar et al. [1] proved global well-posedness to (2) with the initial data satisfying

$$
\|\nabla u_0\|_{L^\infty} < \min\{\frac{\nu}{2C_0}, \frac{\eta}{C_0}\}, \quad \|\nabla \Psi_0\|_{L^\infty} < \frac{\nu}{2C_0} \|\nabla u_0\|_{L^\infty},
$$

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where $B^{0}_{\infty,1}$ is the Besov space (see [2] for the definition), while they proposed that global well-posedness for (1) is an open question (see P. 3597 in that paper). Later, Wan [13] obtained global solutions to (2) with large initial velocity data by exploiting a new decomposition technique which is splitting the damped Navier-Stokes equations from (2). In fact, in [13], the initial data satisfies
\[
\frac{C_0}{\sqrt{|D|}} \|P_0\|_{\dot{H}^{m-1}} \exp\left\{C(\frac{1}{\nu} + \eta)\|u_0\|_{\dot{H}^m} \exp\left\{\frac{C}{\nu} \|\Omega_0\|_{L^2 \cap L^\infty} A(\nu, u_0, \Omega_0)\right\}\right\} < \min\{\nu, \frac{1}{\nu}\} (m > 3)
\]
for some constants $C > 0$ and $C_0 > 0$, where $\Omega_0 \equiv \nabla \times u_0$ and $A(\nu, u_0, \Omega_0) \equiv \ln(\epsilon + \|u_0\|_{H^m}) \exp\left\{\frac{C}{\nu} \|\Omega_0\|_{L^2 \cap L^\infty}\right\}$. We refer [17] and references therein for the related works.

Inspired by the studies of the MHD equations (see, e.g., [11, 15, 16]), the contribution of this article is the global existence and uniqueness of solutions of (1) with sufficiently smooth initial data $(u_0, P_0)$ close to the equilibrium state $(0, x_2)$. In fact, using $\Psi = \theta + x_2$, we return to seek the solutions of the following system
\[
\begin{cases}
\partial_t \omega + u \cdot \nabla \omega + \nu \omega = \partial_1 \theta, \\
\partial_t \theta + u \cdot \nabla \theta = -u_2, \\
u = \nabla^\perp \Lambda^{-2} \omega, \\
\omega|_{t=0} = \omega_0(x), \quad \theta|_{t=0} = \theta_0(x),
\end{cases}
\]
(3)
where $\nabla^\perp \equiv (\partial_2, -\partial_1)$ and $\omega \equiv \partial_1 u_2 - \partial_2 u_1$ is the vorticity of the velocity $u$. When $\nu = 0$, Elgindi-Widmayer [4] proved the long time existence of (3), that is, the lifespan of the associated solutions is $\epsilon^{-\frac{3}{2}}$ if the initial data is of size $\epsilon$. We point out that the initial data in $W^{s,1}(\mathbb{R}^2)$ is very crucial in the decay estimate of the semi-group $e^{R\cdot t}$. Later, Wan-Chen [12] obtained global well-posedness under the initial data near a nontrivial equilibrium $(0, \kappa x_2)$ ($\kappa$ large enough), which is consistent with the corresponding work of [4] if $\kappa = 1$. Recently, by establishing some Strichartz type estimates of the semi-group $e^{R\cdot t}$, Wan [14] obtained the same result as [4] without assuming that the initial data is in $W^{s,1}(\mathbb{R}^2)$.

For some convenience, we set $\nu = 1$, that is, we consider
\[
\begin{cases}
\partial_t \omega + u \cdot \nabla \omega + \omega = \partial_1 \theta, \\
\partial_t \theta + u \cdot \nabla \theta = -u_2, \\
u = \nabla^\perp \Lambda^{-2} \omega, \\
\omega|_{t=0} = \omega_0(x), \quad \theta|_{t=0} = \theta_0(x),
\end{cases}
\]
(4)
in the present work and use the denotations below:
\[
A(T) \equiv \|\omega\|^2_{L^\infty(\mathbb{R}^+)} + \|\theta\|^2_{L^\infty(\mathcal{H}^{s+1})}, \\
A_1(T) \equiv \|\omega\|^2_{L^2(\mathbb{R}^+)} + \|\partial_1 \theta\|^2_{L^2(\mathbb{H}^{s})}, \\
A(0) \equiv \|\omega_0\|^2_{\mathbb{H}^s} + \|\theta_0\|^2_{\mathcal{H}^{s+1}},
\]
where the definitions of the function spaces $\mathcal{H}^s(\mathbb{R}^2)$ and $\mathbb{H}^s(\mathbb{R}^2)$ are given in section 2. Now, we state the main result.

**Theorem 1.1.** Let $s \geq 5$. Assume that
\[
\omega_0 \in \mathbb{H}^s(\mathbb{R}^2), \quad \theta_0 \in \mathcal{H}^{s+1}(\mathbb{R}^2).
\]
There exists a positive constant $C'$ such that if
\[ C'A(0) < 1 \] (5)
then system (4) has a unique global solution $(\omega, \theta)$ satisfying
\[ A(T) + A_1(T) \leq C'A(0), \]
for all $T > 0$.

Remark 1.2. We shall point out that the solutions in Sobolev space do not grow over time, which is very different from the work [15] on the 2D MHD system with a velocity damping term, where the Sobolev norm of the solutions may grow over time, see P.2631 in that paper. Furthermore, our result gives a positive answer to the question proposed in [1] under the initial data near equilibrium state $(0, x_2)$.

Let us making the following comments concerning this theorem:

- By the standard energy method, we need to bound the integral $\int_0^T \|\partial_t u_2\|_{L^\infty} dt$, which is very difficult, even if one applies the techniques in the following sections. By utilising $-u_2 = \partial_t \theta + u \cdot \nabla \theta$ and integrating by parts, the estimate of this integral is reduced to the estimate of $R(T)$ (see (15) for the definition).
- To get the estimate of $R(T)$, we shall estimate the integral $\int_0^T \|u_2\|_{L^\infty} \frac{4}{3} dt$, which strongly relies on the diagonalization process (section 5) and the energy estimate II (section 6). In addition, we shall use some unexpected techniques like (32).
- The unnatural condition in (5) is that the initial data satisfies $\omega_0 \in \dot{H}^{-2}(\mathbb{R}^2)$ and $\theta_0 \in \dot{H}^{-1}(\mathbb{R}^2)$. In fact, these conditions play a very important role in the energy estimate II and the estimate of $\int_0^T \|u_2\|_{L^\infty} \frac{4}{3} dt$.

The present paper is structured as follows:

In section 2, we provide some definitions of spaces and several lemmas. Section 3 devotes to obtaining the estimate of $E(T) + E_1(T)$ (see Lemma 3.1 for the definition). Section 4 bounds the estimate of $\int_0^T \int \partial_2 u_2 (\partial_2^{m+1} \theta)^2 dx dt$. Section 5 provides the estimate of $\int_0^T \|u_2\|_{L^\infty} \frac{4}{3} dt$. In section 6, we give the estimate of $E_2(T)$ (see (21) for the definition). In the last section, we prove Theorem 1.1.

Let us complete this section by describing the notations we shall use in this paper.

Notations. For two operators $A$ and $B$, we denote by $[A, B] = AB - BA$ the commutator between $A$ and $B$. In some places of this paper, we may use $L^p$, $H^s$ and $H^s$ to stand for $L^p(\mathbb{R}^2)$, $H^s(\mathbb{R}^2)$ and $H^s(\mathbb{R}^2)$, respectively. $a \approx b$ means $\mathcal{C}^{-1}b \leq a \leq \mathcal{C}b$ for some positive constant $\mathcal{C}$. $\langle t \rangle$ means $1 + t$. The uniform constant $C$ may be different on different lines. We use $\|f\|_{L^p}$ to denote the $L^p(\mathbb{R}^2)$ norm of $f$, and use $L^p_T(X) = L^p([0, T]; X)$. We shall denote by $(a|b)$ the $L^2$ inner product of $a$ and $b$, and
\[ (a|b)_{H^s} \overset{\text{def}}{=} (\Lambda^s a|\Lambda^s b), \quad (a|b)_{H^m} \overset{\text{def}}{=} (\partial^m a|\partial^m b) \quad (m \text{ is an integer}), \]
\[ (a|b)_{H^s} \overset{\text{def}}{=} (a|b)_{H^s}. \]

2. Preliminaries. In this section, we provide some essential definitions, propositions and lemmas.
The fractional Laplacian operator $\Lambda^{\alpha} = (-\Delta)^{\frac{\alpha}{2}}$ is defined through the Fourier transform, namely,
\[ \hat{\Lambda^{\alpha} f}(\xi) \overset{\text{def}}{=} |\xi|^\alpha \hat{f}(\xi), \]
where the Fourier transform is given by
\[ \hat{f}(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx. \]
Sometimes, we also use $\mathcal{F}[f]$ to stand for the Fourier transform of $f$.

The norms $\|f\|_{\dot{H}^{\alpha_1}(\mathbb{R}^2)}$ ($\alpha_1 \in \mathbb{R}$) and $\|f\|_{\dot{H}^{\alpha_2}(\mathbb{R}^2)}$ ($\alpha_2 > 0$) can be defined by
\[ \|f\|_{\dot{H}^{\alpha_1}(\mathbb{R}^2)} \overset{\text{def}}{=} \|\Lambda^{\alpha_1} f\|_{L^2(\mathbb{R}^2)} \]
and
\[ \|f\|_{\dot{H}^{\alpha_2}(\mathbb{R}^2)} \overset{\text{def}}{=} \|f\|_{L^2(\mathbb{R}^2)} + \|\Lambda^{\alpha_2} f\|_{L^2(\mathbb{R}^2)}. \]
Especially, provided that $\alpha_1 \in \mathbb{R}$ and $\alpha_2 > 0$ are integers, we also have
\[ \|f\|_{\dot{H}^{\alpha_1}(\mathbb{R}^2)} \overset{\text{def}}{=} \|\partial^{\alpha_1} f\|_{L^2(\mathbb{R}^2)} \]
\[ \|f\|_{\dot{H}^{\alpha_2}(\mathbb{R}^2)} \overset{\text{def}}{=} \|f\|_{L^2(\mathbb{R}^2)} + \|\partial^{\alpha_2} f\|_{L^2(\mathbb{R}^2)}. \]

For some convenience, we introduce two new functional spaces as follows:
\[ \mathcal{H}^s(\mathbb{R}^2) \overset{\text{def}}{=} \dot{H}^{-1}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2), \]
\[ \mathbb{H}^s(\mathbb{R}^2) \overset{\text{def}}{=} \dot{H}^{-2}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2). \]

**Lemma 2.1.** (i)\[6\] Let $s > 0$, $1 \leq p, r \leq \infty$, then
\[ \|\Lambda^s(fg)\|_{L^p(\mathbb{R}^2)} \leq C \left\{ \|f\|_{L^{p_1}(\mathbb{R}^2)} \|\Lambda^s g\|_{L^{p_2}(\mathbb{R}^2)} + \|g\|_{L^{r_1}(\mathbb{R}^2)} \|\Lambda^s f\|_{L^{r_2}(\mathbb{R}^2)} \right\}, \]
where $1 \leq p_1, r_1 \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$.

(ii)\[7\] Let $s > 0$, and $1 < p < \infty$, then
\[ \|[\Lambda^s, f]g\|_{L^p(\mathbb{R}^2)} \leq C \left\{ \|\nabla f\|_{L^{p_1}(\mathbb{R}^2)} \|\Lambda^{s-1} g\|_{L^{p_2}(\mathbb{R}^2)} + \|\Lambda^s f\|_{L^{p_3}(\mathbb{R}^2)} \|g\|_{L^{p_4}(\mathbb{R}^2)} \right\}, \]
where $1 < p_2, p_3 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

**Lemma 2.2.** If the functions $f_i$ $(i = 1, 2, 3, 4, 5)$ satisfy that
\[ \partial_t f_j \in L^p_T(\mathcal{H}^1(\mathbb{R}^2)), \quad f_j \in L^\infty_T(H^1(\mathbb{R}^2)), \quad j = 1, 2, \]
\[ f_3 \in L^p_T(\mathcal{H}^\infty(\mathbb{R}^2)), \quad f_4 \in L^q_T(\mathcal{L}^2(\mathbb{R}^2)), \quad f_5 \in L^\infty_T(\mathcal{L}^2(\mathbb{R}^2)), \]
where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $(p, q) \in [2, \infty]$, then there holds
\[ \left| \int_{\mathbb{R}^2} \int_0^T f_1(x) f_2(x) f_3(x) f_4(x) f_5(x) dx dt \right| \leq C \left( \|\partial_t f_1, \partial_t f_2\|_{L^2_T(\mathcal{H}^1)} \right) \left( \|f_1, f_2\|_{L^p_T(\mathcal{H}^1)} \|f_3\|_{L^\infty_T(L^2)} \|f_4\|_{L^q_T(L^2)} \|f_5\|_{L^\infty_T(L^2)} \right). \]

**Proof.** By using
\[ \|f\| \leq C \|\partial_t f\|, \]

\[ \|f\| \leq C \|\partial_t f\|, \]

Minkowski’s inequality for integrals and the embedding relation $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we have
\[
\left| \int_{\mathbb{R}^2} f_1(x)f_2(x)f_3(x)f_4(x)f_5(x)dx \right| \\
\leq \left| \int_{\mathbb{R}^2} \|f_1\|_{L^2_1} \|f_5\|_{L^2_2} \prod_{i=1}^3 \|f_i\|_{L^\infty_2} dx \right| \\
\leq C \|f_3\|_{L^\infty(\mathbb{R}^2)} \|f_4\|_{L^2(\mathbb{R}^2)} \|f_5\|_{L^2(\mathbb{R}^2)} \left\| \prod_{i=1}^3 \|f_i\|_{L^2_2} \|\partial_1 f_i\|_{L^2_1} \right\|_{L^\infty_2} \\
\leq C \|f_3\|_{L^\infty(\mathbb{R}^2)} \|f_4\|_{L^2(\mathbb{R}^2)} \|f_5\|_{L^2(\mathbb{R}^2)} \\
\times (\|f_1\|_{H^1(\mathbb{R}^2)} + \|f_2\|_{H^1(\mathbb{R}^2)})(\|\partial_1 f_1\|_{H^1(\mathbb{R}^2)} + \|\partial_1 f_2\|_{H^1(\mathbb{R}^2)}).
\]
Integrating the inequality above in time, and then using Hölder’s inequality can yield the desired result. \hfill \Box

3. **Energy estimate I.** In this section, we prove some a priori estimates, which are given by the following lemma:

**Lemma 3.1.** Let $(\omega, \theta)$ be sufficiently smooth functions which solves (4) and satisfy $(\omega_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$, then there holds
\[
E(T) + E_1(T) \leq C \left( E(0) + E(T) \frac{3}{2} E_1(T) + \left| \int_0^T I_1(t) dt \right| \right),
\]
where
\[
E(T) \overset{\text{def}}{=} \|\omega\|_{L^2(\mathbb{H}^s)} + \|\theta\|_{L^2(\mathbb{H}^{s+1})}, \\
E_1(T) \overset{\text{def}}{=} \|\omega\|_{L^2(\mathbb{H}^s)} + \|\partial_1 \theta\|_{L^2(\mathbb{H}^s)}, \\
E(0) \overset{\text{def}}{=} \|\omega_0\|_{\mathbb{H}^s} + \|\theta_0\|_{\mathbb{H}^{s+1}}, \\
I_1 \overset{\text{def}}{=} \int \partial_2 u_2 (\partial_2^{-1} \theta)^2 dx.
\]
and $C$ is a positive constant independent of $T$.

**Remark 3.2.** $\omega_0 \in \dot{H}^{-1}$ is a natural condition, which is equal to $u_0 \in L^2$. Thanks to this condition, together with $\theta_0 \in L^2$, we can get the global kinetic energy.

**Proof.** Using the cancelation relations
\[
(u \cdot \nabla \theta | \theta) = (u \cdot \nabla \omega | \omega)_{\dot{H}^{-1}} = 0
\]
and
\[
(\partial_1 \theta | \omega)_{\dot{H}^s} + (\partial_1 \Lambda^{-2} \omega | \theta)_{\dot{H}^{s+1}} = 0,
\]
we can get
\[
\frac{d}{dt} (\|\omega\|_{\dot{H}^s}^2 + \|\theta\|_{\dot{H}^{s+1}}^2) + \|\omega\|_{\dot{H}^s}^2 = -(u \cdot \nabla \omega | \omega)_{\dot{H}^s} - (u \cdot \nabla \theta | \theta)_{\dot{H}^{s+1}}, \]
\[
= -(u \cdot \nabla \omega | \omega)_{\dot{H}^s} - (u \cdot \nabla \theta | \theta)_{\dot{H}^{s+1}}. \quad (9)
\]
Using \((u \cdot \nabla \omega | \Lambda^\omega) = 0\) and (6), we have
\[-(u \cdot \nabla \omega | \Lambda^\omega)|_{t, r} = - \int [\Lambda^\omega, u \cdot \nabla] \omega \Lambda^\omega dx\]
\[\leq ||[\Lambda^\omega, u \cdot \nabla]|_{L^2}||\Lambda^\omega|_{L^2}\]
\[\leq C||\nabla u||_{L^\infty}||\Lambda^\omega|_{L^2}^2 + C||\nabla \omega||_{L^4}||\Lambda^\omega|_{L^4}||\Lambda^\omega|_{L^2}\]
\[\leq C \|\omega\|_{H^s}^3\]

For the estimate of \(I\), using \((u \cdot \nabla \vartheta_{s+1}\partial^s\theta| \partial^s\theta) = 0\), we obtain
\[I = - \int \partial^{s+1}(u \cdot \nabla \theta) \partial^{s+1}\theta dx = - \sum_{1 \leq \alpha \leq s+1} C_{s+1}^{\alpha} \int \partial^{\alpha}u \cdot \nabla \partial^{s+1-\alpha}\theta \partial^{s+1}\theta dx.\]

Depending on the derivatives \(\partial^{s+1}\), we will split the estimate into two cases: (1) \(\partial^{s+1}\) including at least one derivative on \(x_1\) and (2) \(\partial^{s+1} = \partial^2_2\). For the case (1), it is easy to get
\[I \leq C\|\omega\|_{H^s} ||\partial_1\theta||_{H^s} \|\theta\|_{H^{s+1}}.\]

For the case (2), thanks to
\[\sum_{1 \leq \alpha \leq s+1} C_{s+1}^{\alpha} \int \partial^{\alpha}_1u_1 \partial^{s+1}_1\theta \partial^{s+1}\theta dx \leq C\|\omega\|_{H^s} ||\partial_1\theta||_{H^s} \|\theta\|_{H^{s+1}}\]
and
\[\sum_{2 \leq \alpha \leq s+1} C_{s+1}^{\alpha} \int \partial^{\alpha}_2u_2 \partial^{s+2}_2\theta \partial^{s+1}\theta dx\]
\[-= - \sum_{2 \leq \alpha \leq s+1} C_{s+1}^{\alpha} \int \partial_1 \partial^{\alpha-1}_2 \partial_1 \partial^{s+2-\alpha}_2\theta \partial^{s+1}\theta dx\]
\[= \sum_{2 \leq \alpha \leq s+1} C_{s+1}^{\alpha} \int \partial^{\alpha-1}_2 \partial_1 \partial^{s+2-\alpha}_2\theta \partial^{s+1}\theta dx + \int \partial^{\alpha-1}_2 \partial_1 \partial^{s+2-\alpha}_2\theta \partial_1 \partial^{s+1}\theta dx\]
\[\leq C\|\omega\|_{H^s} ||\partial_1\theta||_{H^s} \|\theta\|_{H^{s+1}},\]

where we have used \(\partial_2u_2 = -\partial_1u_1\) and integration by parts two times, then
\[I = - \sum_{1 \leq \alpha \leq s+1} C_{s+1}^{\alpha} \left( \int \partial^{\alpha}_2u_2 \partial_1 \partial^{s+1-\alpha}_2\theta \partial^{s+1}\theta dx + \int \partial^{\alpha}_2u_2 \partial^{s+2-\alpha}_2\partial^{s+1}\theta dx \right)\]
\[\leq C\|\omega\|_{H^s} ||\partial_1\theta||_{H^s} \|\theta\|_{H^{s+1}} + I_1.\]

The estimate of \(I_1\) will be given in section 4. So (9) reduces to
\[\frac{1}{2} \frac{d}{dt} \left( \|\omega\|_{H^r}^2 + ||\theta||_{H^{s+1}}^2 \right) \leq C\|\omega\|_{H^s} ||\partial_1\theta||_{H^s} \|\theta\|_{H^{s+1}} + ||\omega\|_{H^s}^3 + I_1.\]

Next, we will find the dissipation of \(\theta\). As a matter of fact, we have
\[||\partial_1\theta||_{H^s}^2 = (\partial_1\omega|\partial_1\theta)_{H^s} + (u \cdot \nabla \omega|\partial_1\theta)_{H^s} + (\omega|\partial_1\theta)_{H^s}.\]
Using (4), we have
\[
(\partial_t \omega | \partial_t \theta)_H^s = \frac{d}{dt}(\omega | \partial_t \theta)_H^s - (\omega | \partial_t \partial_t \theta)_H^s
\]
\[
= \frac{d}{dt}(\omega | \partial_t \theta)_H^s - (\omega | \partial_t^2 \Lambda^{-2} \omega)_H^s + (\partial_t (u \cdot \nabla) \omega)_H^s
\]
\[
= \frac{d}{dt}(\omega | \partial_t \theta)_H^s + \|\partial_t \Lambda^{-1} \omega \|^2_{H^s} + (\partial_t (u \cdot \nabla) \omega)_H^s.
\]
So there exists a positive $C_0$ such that
\[
\|\partial_t \theta\|^2_{H^s} - \frac{d}{dt}(\omega | \partial_t \theta)_H^s = (\omega | \partial_t \theta)_H^s + \|\partial_t \Lambda^{-1} \omega \|^2_{H^s}
\]
\[
+ (\partial_t (u \cdot \nabla) \omega)_H^s + (u \cdot \nabla \omega | \partial_t \theta)_H^s
\]
\[
\leq C_0 \|\omega\|^2_{H^s} + \frac{1}{2} \|\partial_t \theta\|^2_{H^s} + (\partial_t (u \cdot \nabla) \omega)_H^s
\]
\[
+ (u \cdot \nabla \partial_t \theta)_H^s + (u \cdot \nabla \omega | \partial_t \theta)_H^s,
\]
where we have used
\[
(u \cdot \nabla \omega | \partial_t \theta) + (u \cdot \nabla \partial_t \theta | \omega) = 0.
\]
It is easy to get
\[
(\partial_t u \cdot \nabla) \theta)_H^s \leq C \|\omega\|^2_{H^s} \|\theta\|_{H^{s+1}}.
\]
Using the cancelation property
\[
\int u \cdot \nabla \Lambda^s \omega \partial_t \Lambda^s \theta dx + \int u \cdot \nabla \partial_t \Lambda^s \theta \Lambda^s \omega dx = 0,
\]
then
\[
(u \cdot \nabla \partial_t \theta | \omega)_H^s + (u \cdot \nabla \omega | \partial_t \theta)_H^s
\]
\[
= \int [\Lambda^s, u \cdot \nabla] \partial_t \Lambda^s \theta dx + \int [\Lambda^s, u \cdot \nabla] \omega \partial_t \Lambda^s \theta dx
\]
\[
\leq \|[\Lambda^s, u \cdot \nabla] \partial_t \theta\|_{L^2} \|\omega\|_{H^s} + \|[\Lambda^s, u \cdot \nabla] \omega\|_{L^2} \|\partial_t \theta\|_{H^s}.
\]
Thanks to (7) and interpolation inequalities, we have
\[
\|[\Lambda^s, u \cdot \nabla] \partial_t \theta\|_{L^2} \leq C(\|\nabla u\|_{L^\infty} \|\partial_t \theta\|_{H^s} + \|\nabla \partial_t \theta\|_{L^4} \|\Lambda^s u\|_{L^4})
\]
\[
\leq C(\|\nabla u\|_{L^\infty} \|\partial_t \theta\|_{H^s} + \|\nabla \partial_t \theta\|_{L^2} \|\Delta \partial_t \theta\|_{L^2} \|\Lambda^s u\|_{L^4} \|\Lambda^s \omega\|_{L^2})
\]
\[
\leq C \|\omega\|_{H^s} \|\partial_t \theta\|_{H^s}.
\]
Similarly, we can obtain
\[
\|[\Lambda^s, u \cdot \nabla] \omega\|_{L^2} \leq C \|\omega\|^2_{H^s}.
\]
Thus,
\[
(u \cdot \nabla \partial_t \theta | \omega)_H^s + (u \cdot \nabla \omega | \partial_t \theta)_H^s \leq C \|\omega\|^2_{H^s} \|\partial_t \theta\|_{H^s}.
\]
Plugging the estimates above into (11) yields
\[
\frac{1}{2} \|\partial_t \theta\|^2_{H^s} - \frac{d}{dt}(\omega | \partial_t \theta)_H^s \leq C_0 \|\omega\|^2_{H^s} + C \|\omega\|^2_{H^{s+1}} \|\partial_t \theta\|_{H^s}.
\]
Multiplying (10) by $2C_0$, and adding the resulting inequality to (12), we can get
\[
\frac{d}{dt} \{C_0 (\|\omega\|^2_{H^s} + \|\theta\|^2_{H^{s+1}}) - (\omega | \partial_t \theta)_H^s\}
\]
\[
+ C_0 \|\omega\|^2_{H^s} + \frac{1}{2} \|\partial_t \theta\|^2_{H^s}
\]
\[
\leq C(\|\omega\|_{H^s} + \|\theta\|_{H^{s+1}})(\|\omega\|^2_{H^s} + \|\partial_t \theta\|^2_{H^s}) + 2C_0 I_1.
\]
Using
\[ C_0(\|\omega\|_{H^s}^2 + \|\theta\|_{H^{s+1}}^2) - (\omega|\partial_t \theta)_{H^s} \approx \|\omega\|_{H^s}^2 + \|\theta\|_{H^{s+1}}^2, \]
and integrating (13) in time can lead to the desired estimate (8).
\[ \square \]

4. The estimate of \( \int_0^T \int \partial_2 u_2 (\partial_2^{s+1} \theta)^2 \, dx \, dt \).

Lemma 4.1. Under the conditions in Lemma 3.1, then there holds
\[ \int_0^T \int \partial_2 u_2 (\partial_2^{s+1} \theta)^2 \, dx \, dt \leq C(M(T) + R(T)), \tag{14} \]
where
\[ R(T) \overset{\text{def}}{=} \left| \int_0^T \int u_2 \partial_2 \theta (\partial_2^{s+1} \theta)^2 \, dx \, dt \right|. \tag{15} \]

Moreover, we have
\[ E(T) + \frac{19}{20} E_1(T) \leq C(E(0) + M(T)) + CE(T) \frac{3}{2} \int_0^T \|u_2\|_{L^\infty}^3 \, dt. \tag{16} \]
Here \( C \) is a positive constant independent of \( T \).

Remark 4.2. The quantity \( \int_0^T \|\partial_2 u_2\|_{L^\infty} \, dt \) seems difficult to be bounded, but instead section 5 gives a bound for \( \int_0^T \|\partial_2 u_2\|_{L^\infty}^{\frac{3}{2}} \, dt \). This motivates the setting and results of section 4.

Proof. We shall find a new way to bound this integral. In fact, using \( -u_2 = \partial_t \theta + u \cdot \nabla \theta \), we have
\[ \int_0^T \int \partial_2 u_2 (\partial_2^{s+1} \theta)^2 \, dx \, dt \]
\[ = -\int_0^T \int \partial_2 \partial_t \theta (\partial_2^{s+1} \theta)^2 \, dx \, dt - \int_0^T \int \partial_2 (u \cdot \nabla \theta) (\partial_2^{s+1} \theta)^2 \, dx \, dt \]
\[ \overset{\text{def}}{=} J_1 + J_2. \]

• The estimate of \( J_1 \)

Using \( \partial_t \theta = -u \cdot \nabla \theta - u_2 \), we can obtain
\[ J_1 = -\int_0^T \int \partial_2 \theta (\partial_2^{s+1} \theta)^2 \, dx \bigg|_{t=0}^T + 2 \int_0^T \int \partial_2 \theta \partial_2^{s+1} \theta \partial_2^{s+1} \theta \, dx \, dt \]
\[ \leq \|\theta\|_{L_T^\infty(H^{s+1})}^3 + 2(-J_{11} + J_{12}) \]
\[ \leq E(T)^\frac{3}{2} + 2(-J_{11} + J_{12}), \]
where
\[ J_{11} \overset{\text{def}}{=} \int_0^T \int \partial_2 \theta \partial_2^{s+1} u_2 \partial_2^{s+1} \theta \, dx \, dt, \quad J_{12} \overset{\text{def}}{=} -\int_0^T \int \partial_2 \theta \partial_2^{s+1} (u \cdot \nabla \theta) \partial_2^{s+1} \theta \, dx \, dt. \]
For $J_{11}$, using $\partial_2 u_2 = -\partial_1 u_1$ and integrating by parts two times lead to

$$J_{11} = - \int_0^T \int \partial_2 \theta \partial_2^* \partial_1 u_1 \partial_2^{*+1} \theta dxdt$$

$$= \int_0^T \int \partial_2 \partial_2^* \partial_2 u_1 \partial_1 \partial_2^{*+1} \theta dxdt + \int_0^T \int \partial_2 \partial_2^* \partial_1 u_1 \partial_1 \partial_2^{*+1} \theta dxdt$$

$$= \int_0^T \int \partial_1 \partial_2 \partial_2^* \partial_2 u_1 \partial_2^{*+1} \theta dxdt - \int_0^T \int \partial_2 \partial_2^* \partial_1 u_1 \partial_1 \partial_2^{*+1} \theta dxdt$$

$$- \int_0^T \int \partial_2 \partial_2^* \partial_2^{*+1} u_1 \partial_1 \partial_2^{*+1} \theta dxdt$$

$$\leq C \|\partial_1 \theta\|_{L^2_s(H^s)} \|\omega\|_{L^2_s(H^{s+1})} \leq C E(T)^{1/2} E_1(T).$$

For $J_{12}$, we have

$$J_{12} = - \sum_{0 \leq \alpha \leq s+1} C_{s+1}^\alpha \int_0^T \int \partial_2 \theta \partial_2^* u \cdot \nabla \partial_2^{*+1} \partial_2^{*+1} \theta dxdt$$

$$= - \sum_{1 \leq \alpha \leq s+1} C_{s+1}^\alpha \int_0^T \int \partial_2 \theta \partial_2^* u \cdot \nabla \partial_2^{*+1} \partial_2^{*+1} \theta dxdt$$

$$- \frac{1}{2} \int_0^T \int \partial_2 \theta u \cdot \nabla (\partial_2^{*+1} \theta)^2 dxdt$$

$$= - \sum_{1 \leq \alpha \leq s+1} C_{s+1}^\alpha \int_0^T \int \partial_2 \partial_2^* u \cdot \nabla \partial_2^{*+1} \partial_2^{*+1} \theta dxdt$$

$$+ \frac{1}{2} \int_0^T \int u \cdot \nabla \partial_2 \theta (\partial_2^{*+1} \theta)^2 dxdt$$

$$\overset{\text{def}}{=} J_{121} + J_{122}.$$
By using the previous approach, one can get the estimate as follows:

\[ K_1 \leq CE(T)E_1(T), \quad K_2 \leq CE(T)E_1(T), \]

whose proof is given in the appendix. As for the estimate of \( K_3 \), using the equation of \( \theta \) two times and \( \partial_2 u_2 = -\partial_1 u_1 \), we have

\[
K_3 \leq (s + 1) \int_0^T \int \partial_2 \partial_2 \partial_1 \partial_1 (\partial_2^{s+1} \theta)^2 \, dx \, dt \\
+ (s + 1) \int_0^T \int \partial_2 \partial_2 \partial_2 (\partial_2^{s+1} \theta)^2 \, dx \, dt \\
= \frac{s + 1}{2} \int \underbrace{(\partial_2 \theta)^2}_{\text{Terms involving } \theta} (\partial_2^{s+1} \theta)^2 \, dx \bigg|_0^T \\
- (s + 1) \int_0^T \int \partial_2 \partial_2 (u \cdot \nabla \theta)(\partial_2^{s+1} \theta)^2 \, dx \, dt \\
+ (s + 1) \int_0^T \int \partial_2 \partial_2 \partial_2 (u \cdot \nabla \theta)(\partial_2^{s+1} \theta)^2 \, dx \, dt \\
\leq CE(T)^2 + (s + 1) \int_0^T \int (\partial_2 \theta)^2 \partial_2^{s+1} \theta \partial_2 u_1 \, dx \, dt \\
+ (s + 1) \int_0^T \int (\partial_2 \theta)^2 \partial_2^{s+1} \theta (u \cdot \nabla \theta) \, dx \, dt \\
- (s + 1) \int_0^T \int \partial_2 \partial_2 \partial_2 (u \cdot \nabla \theta)(\partial_2^{s+1} \theta)^2 \, dx \, dt \\
\overset{\text{def}}{=} CE(T)^2 + \sum_{i=1}^3 K_{3i}.
\]

Integrating by parts two times, one can get

\[ K_{31} \leq CE(T)E_1(T). \]

Integrating by parts and using Lemma 2.2 with \((p, q) = (2, \infty)\) and \((p, q) = (\infty, 2)\), we have

\[
K_{32} \leq C \left| \int_0^T \int (\partial_2 \theta)^2 u \cdot \nabla (\partial_2^{s+1} \theta)^2 \, dx \, dt \right| \\
+ C \sum_{1 \leq \alpha \leq s+1} \left| \int_0^T \int (\partial_2 \theta)^2 \partial_\alpha u \cdot \nabla \partial^{s+1-\alpha} \theta \partial_2^{s+1} \theta \, dx \, dt \right| \\
\leq C \left| \int_0^T \int u \cdot \nabla (\partial_2 \theta)^2 (\partial_2^{s+1} \theta)^2 \, dx \, dt \right| \\
+ C \sum_{1 \leq \alpha \leq s+1} \left| \int_0^T \int (\partial_2 \theta)^2 \partial_\alpha u \cdot \nabla \partial^{s+1-\alpha} \theta \partial_2^{s+1} \theta \, dx \, dt \right| \\
+ C \sum_{s \leq \alpha \leq s+1} \left| \int_0^T \int (\partial_2 \theta)^2 \partial_\alpha u \cdot \nabla \partial^{s+1-\alpha} \theta \partial_2^{s+1} \theta \, dx \, dt \right| \\
\leq C \left| \partial_1 \theta \right|_{L_2^2(H^{s+1})} \left| \theta \right|_{L_2^2(H^{s+1})} \left| u \right|_{L_2^2(H^{s+1})} \left| \theta \right|_{L_2^2(H^{s+1})} \\
\leq CE(T)^{\frac{3}{2}} E_1(T).
\]
Similarly, using Lemma 2.2 with \((p, q) = (2, \infty)\), we can get
\[
K_{33} \leq C \left[ \int_0^T \int \partial_2 \partial_2 u \cdot \nabla \theta (\partial_2^{\ast + 1} \theta) \, dx \, dt \right] + C \left[ \int_0^T \int \partial_2 \theta \cdot \nabla \theta (\partial_2^{\ast + 1} \theta) \, dx \, dt \right] \\
\leq C \| \partial \theta \|_{L_2^\infty (H^1)} \| \theta \|_{L_2^\infty (H^3)} \| u \|_{L_2^2 (H^3)} \| \theta \|_{L_2^\infty (H^{\ast + 1})}^2 \\
\leq C E(T) \frac{2}{2} E_1(T). 
\]
So we can get the estimate of \(K_3\), and combining with the estimates of \(K_1\) and \(K_2\) yields
\[
J_{121} \leq C \left( E(T)^2 + E(T)E_1(T) + E(T)^{\frac{2}{2}} E_1(T) \right). 
\]
For the estimate of \(J_{122}\), we have
\[
J_{122} \leq C \left[ \int_0^T \int u \partial_1 \partial_2 \theta (\partial_2^{\ast + 1} \theta) \, dx \, dt \right] + C \left[ \int_0^T \int u \partial_2 \partial_2 \theta (\partial_2^{\ast + 1} \theta) \, dx \, dt \right] \\
\leq C \| u \|_{L_2^2 (L^\infty)} \| \partial \theta \|_{L_2^2 (H^\ast)} \| \theta \|_{L_2^\infty (H^{\ast + 1})}^2 + C \mathcal{R}(T) \\
\leq C E(T) E_1(T) + C \mathcal{R}(T). 
\]
Hence,
\[
J_1 \leq C E(T) \frac{1}{2} \left( E(T) + E(T) \frac{1}{2} E_1(T) + E(T)^{\frac{2}{2}} E_1(T) \right) + C \mathcal{R}(T). 
\]  
\[
(18)
\]
- The estimate of \(J_2\) We have
\[
J_2 \leq \left[ \int_0^T \int \partial_2 u \partial_1 \partial_1 \theta (\partial_2^{\ast + 1} \theta) \, dx \, dt \right] + \left[ \int_0^T \int \partial_2 u \partial_2 \partial_2 \theta (\partial_2^{\ast + 1} \theta) \, dx \, dt \right]. 
\]
It is easy to get
\[
\left[ \int_0^T \int \partial_2 u \partial_1 \partial_1 \theta (\partial_2^{\ast + 1} \theta) \, dx \, dt \right] \leq E(T) E_1(T), 
\]
while the second term can be bounded as the estimate of \(K_3\). So we can get
\[
J_2 \leq C E(T) \left( E_1(T) + E(T)^{\frac{1}{2}} E_1(T) + E(T) \right). 
\]
We can get the desired result (14) by combining with the estimates of \(J_1\) and \(J_2\).

Next, we show (16). One can deduce from Lemma 3.1 and Lemma 4.1 that
\[
E(T) + E_1(T) \leq C \left( E(0) + M(T) + \mathcal{R}(T) \right). 
\]
It is easy to see that \(M(T)\) is a good term, since \(M(T)\) can be bounded by \(E(T)^{\alpha} E_1(T)^{\beta}\) for \(\alpha + \beta > 1\) and \((\alpha, \beta) \in [0, \infty)^2\). Next, it suffices to show the estimate of \(\mathcal{R}(T)\). Thanks to
\[
\| \partial_2 \theta \|_{L^\infty} \leq C \| \partial_2 \theta \|_{H^1} \| \partial_1 \partial_2 \theta \|_{H^1}, 
\]
we have

\[ \mathcal{R}(T) \leq \int_0^T \|u_2\|_{L^\infty} \|\partial_2^2 \theta\|_{L^\infty} \|\theta\|_{H^{s+1}}^2 \, dt \]
\[ \leq C \int_0^T \|u_2\|_{L^\infty} \|\partial_2^2 \theta\|_{H^1}^{\frac{3}{2}} \|\partial_1 \partial_2^2 \theta\|_{H^1} \|\theta\|_{H^{s+1}}^2 \, dt \]
\[ \leq CE(T)^{\frac{5}{2}} \|\partial_2 \theta\|_{L^2_t\left(H^s\right)} \|u_2\|_{L^2_t(L^\infty)}^{\frac{1}{2}} \]
\[ \leq CE(T)^{\frac{5}{2}} \|u_2\|_{L^2_t(L^\infty)}^{\frac{1}{2}} + \frac{1}{20} E_1(T), \]

which yields (16).

To close the estimate of (16), we shall bound \( \int_0^T \|u_2\|_{L^\infty}^{\frac{4}{3}} \, dt \), which is the main goal in the following section.

5. Spectral analysis and the estimate of \( \int_0^T \|u_2\|_{L^\infty}^{\frac{4}{3}} \, dt \). In this section, we get the expression of solution and then show the estimate of \( \int_0^T \|u_2\|_{L^\infty}^{\frac{4}{3}} \, dt \).

**Lemma 5.1.** Under the conditions in Lemma 3.1, then there holds

\[ \int_0^T \|u_2\|_{L^\infty}^{\frac{4}{3}} \, dt \leq C \mathfrak{M}(T), \]  

where

\[ \mathfrak{M}(T) = CE(T)^{\frac{1}{2}} + C E_1(T) + C \left(E_1(T)^{\frac{1}{2}} + E_2(T)^{\frac{1}{2}}\right) E_1(T)^{\frac{1}{2}} \]
\[ + C \left(E_1(T)^{\frac{1}{2}} + E_2(T)^{\frac{1}{2}}\right) E(T)^{\frac{1}{2}} E_1(T)^{\frac{1}{2}}, \]
\[ E_2(T) = \|\Lambda^{-2} \omega\|_{L^2_t(L^2)}. \]  

In addition, we have

\[ E(T) + \frac{19}{20} E_1(T) \leq C (E(0) + M(T)) + CE(T)^{\frac{5}{4}} \mathfrak{M}(T). \]  

Here \( C \) is a positive constant independent of \( T \).

**Proof.** The proof consists of the following two subsections. We need first a diagonalization process.

5.1. Spectral analysis. To obtain the estimate of \( \int_0^T \|u_2\|_{L^\infty}^{\frac{4}{3}} \, dt \), we shall first investigate the spectrum properties to the following system:

\[ \begin{align*}
\partial_t \omega + \omega &= \partial_1 \theta + G, \\
\partial_t \theta &= \partial_1 \Lambda^{-2} \omega + H, \\
G &= -u \cdot \nabla \omega, \quad H = -u \cdot \nabla \theta.
\end{align*} \]  

Denote

\[ A \overset{\text{def}}{=} \begin{pmatrix}
-1 & -i\xi_1 \\
\xi_1 & 0
\end{pmatrix}, \]

then we can get from (23) that

\[ \partial_t \left( \begin{pmatrix} \omega \\ \theta \end{pmatrix} \right)(\xi) = A \left( \begin{pmatrix} \omega \\ \theta \end{pmatrix} \right)(\xi) + \begin{pmatrix} G \\ H \end{pmatrix}(\xi). \]
One can get the eigenvalues of the matrix $A$ as follows:

$$
\lambda_{\pm} = \begin{cases} 
-\frac{1+i|\xi|^{-1}\sqrt{|\xi|^2 - 4\xi_1^2}}{2}, & \text{when } |\xi| \geq 2|\xi_1|, \\
-\frac{1+i|\xi|^{-1}\sqrt{4\xi_1^2 - |\xi|^2}}{2}, & \text{when } |\xi| < 2|\xi_1| 
\end{cases}
$$

(25)

and

$$
P^{-1}AP = \begin{pmatrix} 
\lambda_+ & 0 \\
0 & \lambda_- 
\end{pmatrix},
$$

(26)

where the matrices $P$ and $P^{-1}$ are given by

$$
P = \begin{pmatrix} 
\lambda_+|\xi|^2 & \lambda_-|\xi|^2 \\
-i\xi_1 & -i\xi_1 
\end{pmatrix}, \quad P^{-1} = \frac{1}{|A|} \begin{pmatrix} 
-i\xi_1 & -\lambda_-|\xi|^2 \\
i\xi_1 & \lambda_+|\xi|^2 
\end{pmatrix}.
$$

Thanks to (24) and (26), denote

$$
W \overset{\text{def}}{=} -i\xi_1\hat{\omega} - \lambda_-|\xi|^2\hat{\theta}, \quad V \overset{\text{def}}{=} i\xi_1\hat{\omega} + \lambda_+|\xi|^2\hat{\theta},
$$

we have

$$
\begin{cases} 
\partial_t W = \lambda_+ W + (-i\xi_1\hat{G} - \lambda_-|\xi|^2\hat{H}), \\
\partial_t V = \lambda_- V + i\xi_1\hat{G} + \lambda_+|\xi|^2\hat{H}, 
\end{cases}
$$

which is equal to

$$
W = e^{\lambda_+t}W_0 + \int_0^t e^{\lambda_+(t-\tau)}(-i\xi_1\hat{G} - \lambda_-|\xi|^2\hat{H})d\tau,
$$

$$
V = e^{\lambda_-t}V_0 + \int_0^t e^{\lambda_-(t-\tau)}(i\xi_1\hat{G} + \lambda_+|\xi|^2\hat{H})d\tau.
$$

So we can get

$$
\hat{\omega}(\xi, t) = M_1(t)\hat{\omega}_0(\xi) + M_2(t)\hat{\theta}_0(\xi)
\begin{align*} 
&+ \int_0^t M_1(t-\tau)\hat{G}(\xi)d\tau + \int_0^t M_2(t-\tau)\hat{H}(\xi)d\tau, 
\end{align*}
$$

(27)

where

$$
M_1(t) \overset{\text{def}}{=} \begin{cases} 
\frac{-|\xi|(\lambda_+ e^{\lambda_+t} - \lambda_- e^{\lambda_-t})}{\sqrt{|\xi|^2 - 4\xi_1^2}}, & \text{when } |\xi| \geq 2|\xi_1|, \\
\frac{-|\xi|(\lambda_- e^{\lambda_-t} - \lambda_+ e^{\lambda_+t})}{i\sqrt{4\xi_1^2 - |\xi|^2}}, & \text{when } |\xi| < 2|\xi_1|, 
\end{cases}
$$

(28)

and

$$
M_2(t) \overset{\text{def}}{=} \begin{cases} 
\frac{\xi_1|\xi|(e^{\lambda_+t} - e^{\lambda_-t})}{\sqrt{|\xi|^2 - 4\xi_1^2}}, & \text{when } |\xi| \geq 2|\xi_1|, \\
\frac{\xi_1|\xi|(e^{\lambda_-t} - e^{\lambda_+t})}{\sqrt{4\xi_1^2 - |\xi|^2}}, & \text{when } |\xi| < 2|\xi_1|.
\end{cases}
$$

(29)

5.2. The estimate of $\int_0^T \|u_2\|^4_{L^4}dt$. Using $\|f\|_{L^\infty} \leq \|\hat{f}(\xi)\|_{L^1}$, we have

$$
\begin{align*} 
\int_0^T \|u_2\|^4_{L^4}dt &\leq \int_0^T \|\hat{u}_2(\xi)\|^4_{L_4^1}dt \\
&\leq \int_0^T \|\hat{u}_2(\xi)\|^4_{L_1^1(D_1)}dt \\
&+ \int_0^T \|\hat{u}_2(\xi)\|^4_{L_1^1(D_2)}dt \\
&+ \int_0^T \|\hat{u}_2(\xi)\|^4_{L_1^1(D_3)}dt,
\end{align*}
$$

(30)
where

\[ D_1 \triangleq \{ \xi \in \mathbb{R}^2 : |\xi| \geq 3|\xi_1| \}, \]

\[ D_2 \triangleq \{ \xi \in \mathbb{R}^2 : 2|\xi_1| \leq |\xi| < 3|\xi_1| \}, \]

\[ D_3 \triangleq \{ \xi \in \mathbb{R}^2 : |\xi| < 2|\xi_1| \}. \]

**Case 1.** \( \xi \in D_1 \)

In this case, one can get from (25), (28) and (29) that

\[
\begin{align*}
\lambda_+ & \leq -\frac{\xi_1^2}{|\xi|^2}, \quad \lambda_- \leq -\frac{1}{2}, \\
M_1(t) & \leq C(e^{-\frac{1}{2}t} + \frac{\xi_1^2}{|\xi|^2}e^{-\frac{\xi_1^2}{|\xi|^2}t}), \\
M_2(t) & \leq C|\xi_1|(e^{-t} + e^{-\frac{\xi_1^2}{|\xi|^2}t}).
\end{align*}
\]

Consequently, using (27), we can obtain

\[
\begin{align*}
\left\| \frac{\xi_1}{|\xi|} \hat{\omega} \right\|_{L^1(D_1)} & \leq \left\| \frac{\xi_1}{|\xi|^2} M_1(t) \hat{\omega}_0 \right\|_{L^1(D_1)} + \left\| \frac{\xi_1}{|\xi|^2} M_2(t) \hat{\rho}_0 \right\|_{L^1(D_1)} + \left( \int_0^t \left\| \frac{\xi_1}{|\xi|^2} M_1(t-\tau) \hat{G} \right\|_{L^1(D_1)} d\tau \right)_{L^\frac{4}{3}} \\
& \quad + \left( \int_0^t \left\| \frac{\xi_1}{|\xi|^2} M_2(t-\tau) \hat{H} \right\|_{L^1(D_1)} d\tau \right)_{L^\frac{4}{3}} \\
& \defeq L_1 + L_2 + L_3 + L_4.
\end{align*}
\]

Using (31), we have

\[
\left\| \frac{\xi_1}{|\xi|} M_1(t) \right\|_{L^\infty} \leq C(t)^{-\frac{3}{2}}, \quad \left\| \frac{\xi_1}{|\xi|^2} M_2(t) \right\|_{L^\infty} \leq C(t)^{-1},
\]

which yields

\[
L_1 \leq C \left\| (t)^{-\frac{3}{2}} \hat{\lambda}^{-1} \hat{\omega} \right\|_{L^\frac{4}{3}} \leq C \left\| (t)^{-\frac{3}{2}} \|u\|_{H^{1+\eta}} \right\|_{L^\frac{4}{3}} \leq C \|u\|_{L^p(H^{1+\eta})} \leq CE(T)^\frac{3}{2},
\]

\[
L_2 \leq C \left\| (t)^{-1} \|\hat{\theta}\|_{L^1} \right\|_{L^\frac{4}{3}} \leq C \left\| (t)^{-1} \|\theta\|_{H^{1+\eta}} \right\|_{L^\frac{4}{3}} \leq C \|\theta\|_{L^p(H^{1+\eta})} \leq CE(T)^{\frac{1}{2}},
\]

where \( 0 < \eta \ll 1 \), and

\[
L_3 \leq C \left\| \int_0^t (t-\tau)^{-\frac{3}{2}} \|u \otimes \omega\|_{H^{1+\eta}} d\tau \right\|_{L^\frac{4}{3}}.
\]

Let

\[
f_1(s) = (s)^{-\frac{3}{2}} \chi_{(0,\infty)}(s) \in L^p \quad (p \geq 1), \quad g_1(s) = \|u \otimes \omega\|_{H^{1+\eta}} \chi_{(0,\infty)},
\]

by Young’s inequality, (6) and \( H^{1+\eta}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \), then we have

\[
L_3 \leq \|f_1 * g_1\|_{L^\frac{4}{3}(\mathbb{R})} \leq C \|f_1\|_{L^4(\mathbb{R})} \|g_1\|_{L^1(\mathbb{R})} \leq C \|u\|_{L^\infty} \|\omega\|_{H^{1+\eta}} + \|\omega\|_{L^\infty} \|u\|_{H^{1+\eta}} \leq C \|u\|_{L^\frac{4}{3}(H^{1+\eta})} \|\omega\|_{L^\frac{4}{3}(H^{1+\eta})} \leq CE_1(T).
\]
For the last term $L_4$, we shall first give some analysis on $\tilde{u} \cdot \nabla \theta$. Applying $u \cdot \nabla = u_1 \partial_1 + u_2 \partial_2$, we can get

$$L_4 \leq C \left\| \int_0^t (t - \tau)^{-1} \| u_1 \partial_1 \theta \|_{L^1(D_1)} d\tau \right\|_{L^4_T} + C \left\| \int_0^t \| M_2(t - \tau) u_2 \partial_2 \theta \|_{L^1(D_1)} d\tau \right\|_{L^4_T} \defeq L_{41} + L_{42},$$

where $L_{41}$ can be bounded like $L_3$, indeed, denote

$$f_2(s) = \langle s \rangle^{-1} \chi_{(0,\infty)}(s) \in L^p \quad (p > 1), \quad g_2(s) = \| u_1 \partial_1 \theta \|_{H^{1+\alpha}},$$

similarly, we also have

$$L_{41} \leq C \| f_2 \ast g_2 \|_{L^\frac{4}{3}(\mathbb{R})} \leq C \| f_2 \|_{L^\frac{4}{3}(\mathbb{R})} \| g_2 \|_{L^1(\mathbb{R})} \leq C \| u_1 \|_{L^2_{t,h}(H^{1+\alpha})} \| \partial_1 \theta \|_{L^2_{t,h}(H^{1+\alpha})} \leq CE_1(T),$$

However, this strategy cannot be used to the estimate of $L_{42}$, due to the fact that

$$\| \partial_2 \theta \|_{L^2_{t,h}(H^{1+\alpha})}$$

cannot be bounded by $E_1(T)$. In order to avoid this bad term, we shall seek a different approach. We begin with the analysis of $\tilde{u}_2 \partial_2 \theta$. Using $u_2 = -\partial_1 \Lambda^{-2} \omega$, one has

$$\tilde{u}_2 \partial_2 \theta = \int (\xi_1 - \eta_1) [\xi - \eta]^{-2} \tilde{\omega}(\xi - \eta) \eta \partial_2 \theta(\eta) d\eta$$

$$= \xi_1 \int |\xi - \eta|^{-2} \tilde{\omega}(\xi - \eta) \eta \partial_2 \theta(\eta) d\eta$$

$$- \int |\xi - \eta|^{-2} \tilde{\omega}(\xi - \eta) \eta \partial_2 \theta(\eta) d\eta$$

$$= i \xi_1 \int \Lambda^{-2} \omega(\xi - \eta) \partial_2 \theta(\eta) d\eta$$

$$+ i \int \Lambda^{-2} \omega(\xi - \eta) \partial_1 \partial_2 \theta(\eta) d\eta$$

$$= i \xi_1 \partial_{\xi} \mathcal{F}[\Lambda^{-2} \omega \partial_2 \theta](\xi) + \mathcal{F}[\Lambda^{-2} \omega \partial_1 \partial_2 \theta](\xi),$$

with (29) leads to

$$\left\| \frac{\xi_1}{|\xi|^2} M_2(t - \tau) \tilde{u}_2 \partial_2 \theta \right\|_{L^1(D_1)} \leq C \| t - \tau \|^{-\frac{3}{2}} \left\| \mathcal{F}[\Lambda^{-2} \omega \partial_2 \theta] \right\|_{L^1}$$

$$+ C \| t - \tau \|^{-1} \left\| \mathcal{F}[\Lambda^{-2} \omega \partial_1 \partial_2 \theta] \right\|_{L^1}$$

$$\leq C \| t - \tau \|^{-\frac{3}{2}} \left\| \Lambda^{-2} \omega \partial_2 \theta \right\|_{H^{2+\eta}}$$

$$+ C \| t - \tau \|^{-1} \left\| \Lambda^{-2} \omega \partial_1 \partial_2 \theta \right\|_{H^{1+\eta}}$$

$$\leq C \| t - \tau \|^{-\frac{3}{2}} \left\| \Lambda^{-2} \omega \partial_2 \theta \right\|_{H^{2+\eta}}$$

$$+ C \| t - \tau \|^{-1} \left\| \Lambda^{-2} \omega \|_{H^{1+\eta}} \left\| \partial_1 \partial_2 \theta \right\|_{H^{1+\eta}}$$

$$\defeq N_1(t) + N_2(t).$$

So we have

$$L_{42} \leq \| N_1(t) \|_{L^\frac{4}{3}_T} + \| N_2(t) \|_{L^\frac{4}{3}_T}. \tag{33}$$
Denote $g_3(s) = \|\Lambda^{-2}\omega\|_{H^{2+n}}\|\partial_t \partial_2 \theta\|_{H^{2+n}}\chi(0,T)$, by Young’s inequality and (6), we have
\[
\|N_2(t)\|_{L_t^2}^\frac{4}{3} \leq C\|f_2 * g_3\|_{L_t^\frac{4}{3}}^\frac{4}{3} \\
\leq C\|f_2\|_{L_t^4(R)}\|g_3\|_{L_t^1(R)} \\
\leq C\|\Lambda^{-2}\omega\|_{L_t^3(H^{2+n})}\|\partial_t \partial_2 \theta\|_{L_t^3(H^{2+n})} \\
\leq C\|\Lambda^{-2}\omega\|_{L_t^3(H^{2+n})} E_1(T)^\frac{1}{2} \\
\leq C(E_2(T)^\frac{1}{2} + E_1(T)^\frac{1}{2}) E_1(T)^\frac{1}{2}.
\]

**Remark 5.2.** $E_2(T)$ plays an essential role in the estimate of $L_{42}$. The regularity index -2 in (21) may be slightly improved to $-1 - \delta$ for some $\delta \in (0,1)$, but it seems difficult to obtain the case $\delta = 0$.

Denote $g_4(s) = \|\Lambda^{-2}\omega \partial_2 \theta\|_{H^{2+n}}\chi(0,T)$, by Young’s inequality, (6) and
\[
\|h\|_{L_t^\infty} \leq C\|h\|_{L_t^\frac{2}{3}} \|\partial_1 h\|_{L_t^\frac{2}{3}} \|L_t^\infty \leq C\|h\|_{H^1} \|\partial_1 h\|_{H^1},
\]
we can get
\[
\|N_1(t)\|_{L_t^\frac{4}{3}} \leq \|f_1 * g_4\|_{L_t^\frac{4}{3}} \\
\leq C\|f_1\|_{L_t^1} \|\Lambda^{-2}\omega \partial_2 \theta\|_{H^{2+n}} \|L_t^\frac{4}{3}} \\
\leq C\|f_1\|_{L_t^1} \|\Lambda^{-2}\omega \partial_2 \theta\|_{H^*) \|L_t^\frac{4}{3}} \\
\leq C\|\Lambda^{-2}\omega\|_{L_t^2} \|\partial_2 \theta\|_{H^{3,\infty}} \|L_t^\frac{4}{3}} \\
+ \|\Lambda^{-2}\omega\|_{H^*) \|\partial_2 \theta\|_{H^{1}} \|L_t^\frac{4}{3}} \\
\leq C\|\Lambda^{-2}\omega\|_{L_t^3(L^2)} \|\partial_2 \theta\|_{H^*)} \|\partial_1 \partial_2 \theta\|_{H^1} \|L_t^\frac{4}{3}} \\
+ C\|\Lambda^{-2}\omega\|_{L_t^3(H^2)} \|\partial_2 \theta\|_{H^*)} \|\partial_1 \partial_2 \theta\|_{H^1} \|L_t^\frac{4}{3}} \\
\leq C \left( E_1(T)^\frac{1}{2} + E_2(T)^\frac{1}{2} \right) E(T)^\frac{1}{2} E_1(T)^\frac{1}{2}.
\]
Thus we have
\[
L_{42} \leq C \left( E_1(T)^\frac{1}{2} + E_2(T)^\frac{1}{2} \right) E_1(T)^\frac{1}{2} \\
+ C \left( E_1(T)^\frac{1}{2} + E_2(T)^\frac{1}{2} \right) E(T)^\frac{1}{2} E_1(T)^\frac{1}{2},
\]
with the estimate of $L_{41}$ leads to
\[
L_4 \leq C \left( E_1(T)^\frac{1}{2} + E_2(T)^\frac{1}{2} \right) E_1(T)^\frac{1}{2} \\
+ C \left( E_1(T)^\frac{1}{2} + E_2(T)^\frac{1}{2} \right) E(T)^\frac{1}{2} E_1(T)^\frac{1}{2} + CE_1(T).
\]
Therefore, we can get
\[
\int_0^T \|\bar{u}_2(\xi)\|_{L_t^1(D_1)}^\frac{4}{3} dt \leq 2\mathcal{M}(T).
\]
Case 2. $\xi \in D_2$

In this case, we have

$$\frac{1}{9} < \frac{\xi_1^2}{|\xi|^2} \leq \frac{1}{4},$$

and then

$$|M_1(t)| \leq \frac{(\lambda_- - \lambda_+)e^{\lambda_- t}}{|\xi|^{-1} \sqrt{|\xi|^2 - 4\xi_1^2}} + \frac{|\lambda_+(e^{\lambda_+ t} - e^{\lambda_- t})|}{|\xi|^{-1} \sqrt{|\xi|^2 - 4\xi_1^2}}$$

$$\leq Ce^{-\frac{1}{4}t} + C e^{\frac{\xi_1^2}{|\xi|^2} \sqrt{4\xi_1^2 - |\xi|^2}}$$

$$\leq Ce^{-\frac{1}{4}t} + C e^{\frac{\xi_1^2}{|\xi|^2} \sqrt{4\xi_1^2 - |\xi|^2}}$$

$$\leq Ce^{-\frac{1}{4}t},$$

and

$$|M_2(t)| \leq C |\xi_1| \frac{\lambda_+ - \lambda_-}{|\xi|^{-1} \sqrt{|\xi|^2 - 4\xi_1^2}}$$

$$\leq C |\xi_1| e^{-\frac{1}{4}t}$$

$$\leq C |\xi_1| e^{-\frac{1}{2}t}.$$

It means $M_i(t) \ (i = 1, 2)$ admits a faster decay than the Case 1. So one can obtain by following the previous procedure that

$$\int_0^T \| \mathfrak{u}_2(\xi) \|_{L^1(D_2)}^\frac{1}{2} dt \leq \mathfrak{M}(T).$$

Case 3. $\xi \in D_3$

We have

$$|M_1(t)| \leq \left| \frac{\xi_1 (\lambda_- - \lambda_+) e^{\lambda_- t}}{\sqrt{4\xi_1^2 - |\xi|^2}} \right| + \left| \frac{|\xi_1| \lambda_+ (e^{\lambda_+ t} - e^{\lambda_- t})}{\sqrt{4\xi_1^2 - |\xi|^2}} \right|$$

$$\leq Ce^{-\frac{1}{2}t} + Ce^{-\frac{1}{4}t}$$

$$\leq Ce^{-\frac{1}{2}t} + Ce^{-\frac{1}{4}t}$$

$$\leq Ce^{-\frac{1}{2}t},$$

where we have used

$$\left| \frac{\sin x}{x} \right| \leq 1.$$
Similarly, we also have
\[
|M_2(t)| \leq C|\xi_1|e^{-\frac{1}{2}t}\frac{e^{\frac{1}{2}t\sqrt{|\xi|^2 - |\xi_1|^2}} - e^{-\frac{1}{2}t\sqrt{|\xi|^2 - |\xi_1|^2}}}{|\xi|^{-1}\sqrt{4|\xi|^2 - |\xi_1|^2}}
\]
\[
\leq C|\xi_1|e^{-\frac{1}{2}t}t
\]
\[
\leq C|\xi_1|e^{-\frac{1}{2}t}.
\]
Then one can get by following the previous procedure line by line that
\[
\int_0^T \|\hat{w}_2(\xi)\|^\frac{4}{3}_{L^1(D_3)}dt \leq \mathcal{M}(T).
\]
So
\[
\int_0^T \|\hat{w}_2(\xi)\|^\frac{4}{3}_{L^1}dt \leq \mathcal{M}(T).
\]
Inserting the estimate above in (16), we can get (22).

6. Energy estimate II. Due to the appearance of $E_2(T)$, the estimate (22) is not closed. The following lemma will help us close this estimate.

**Lemma 6.1.** Let $(\omega, \theta)$ be sufficiently smooth functions which solves (4) and satisfy $(\omega_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)$, then there holds
\[
\|\omega\|^2_{L^\infty_T(H^{-2})} + \|\theta\|^2_{L^\infty_T(H^{-1})} + \int_0^T (\|\omega\|^2_{H^{-2}} + \|\partial_1\theta\|^2_{H^{-2}})dt \leq C(\|\omega_0\|^2_{H^{-2}} + \|\theta_0\|^2_{H^{-1}}) + (A(T)^\frac{1}{2} + A(T))A_1(T),
\]
where $C$ is a positive constant independent of $T$.

**Proof.** Using energy method, we have
\[
\frac{1}{2} \frac{d}{dt}\|\omega\|^2_{H^{-2}} + \|\omega\|^2_{H^{-2}} = -(u \cdot \nabla \omega)|\omega|_{H^{-2}} + (\partial_1 \theta|\omega|)_{H^{-2}}
\]
and
\[
\frac{1}{2} \frac{d}{dt}\|\theta\|^2_{H^{-1}} = -(u \cdot \nabla \theta)|\theta|_{H^{-1}} + (\partial_1 A^{-2}\omega|\theta|)_{H^{-1}}.
\]
Thanks to the cancelation property
\[
(\partial_1 \theta|\omega)_{H^{-2}} + (\partial_1 A^{-2}\omega|\theta)_{H^{-1}} = 0,
\]
and
\[
|(u \cdot \nabla \omega)_{H^{-2}}| \leq C\|\nabla \times (u \cdot \nabla u)|_{H^{-2}}\|\omega\|_{H^{-2}} \leq C\|u \otimes u\|_{L^2}\|\omega\|_{H^{-2}} \leq C\|u\|_{L^4}\|\omega\|_{H^{-2}} \leq C\|u\|_{L^2}\|\omega\|_{L^2}\|\omega\|_{H^{-2}},
\]
we suffice to show the estimate of $(u \cdot \nabla \theta|\theta)_{H^{-1}}$. Denote $-\Delta f = \theta$, using integration by parts many times, we have
\[
(u \cdot \nabla \theta|\theta)_{H^{-1}} = \int u \cdot \nabla \theta(-\Delta)^{-1}\theta = -\int u \cdot \nabla \Delta f f
\]
\[
= -\int \partial_i(u \cdot \nabla \partial_i f)f + \int \partial_i u \cdot \nabla \partial_i f f
\]
then by interpolation inequality,
\[
\|(u \cdot \nabla \theta)_{H^{-1}}\|
\leq C\|\omega\|_{L^4}\|\partial_1 f\|_{L^2}\|\nabla f\|_{L^4} + C\|u_1\|_{L^\infty}\|\partial_1 \partial_2 f\|_{L^2}\|\partial_2 f\|_{L^2}
\]
\[
= C\|\omega\|_{L^4}\|\partial_1 \Lambda^{-2} \theta\|_{L^2}\|\nabla \Lambda^{-2} \theta\|_{L^4} + C\|u_1\|_{L^\infty}\|\partial_1 \partial_2 \Lambda^{-2} \theta\|_{L^2}\|\partial_2 \Lambda^{-2} \theta\|_{L^2}
\]
\[
\leq C\|u\|_{H^2}\|\partial_1 \Lambda^{-2} \theta\|_{H^1}\|\Lambda^{-1} \theta\|_{H^2}.
\]
By interpolation inequality, thus we can get
\[
\frac{1}{2} \frac{d}{dt}(\|\omega\|^2_{H^{-2}} + \|\theta\|^2_{H^{-2}}) + \|\omega\|_{H^{-2}}^2 \leq C\|\partial_1 \Lambda^{-2} \theta\|_{H^1}\|\Lambda^{-1} \theta\|_{H^2}
\]
\[
\leq C(\|\omega\|^2_{H^{-2}} + \|\Lambda^{-1} \theta\|^2_{H^2})
\]
\[
\times (\|u\|^2_{H^2} + \|\partial_1 \theta\|^2_{H^{-2}} + \|\partial_1 \theta\|^2_{L^2}).
\]
Then there exists a positive constant \(C_1\) such that
\[
\|\partial_1 \theta\|^2_{H^{-2}} - \frac{d}{dt}(\|\omega\|_{H^{-2}}^2) = (\omega|\partial_1 \theta)_{H^{-2}} - (\partial_1 \Lambda^{-2} \omega|\omega)_{H^{-2}}
\]
\[
+ (u \cdot \nabla \omega|\partial_1 \theta)_{H^{-2}} + (\partial_1 (u \cdot \nabla \theta)|\omega)_{H^{-2}}
\]
\[
\leq C_1\|\omega\|^2_{H^{-2}} + \frac{1}{2} \|\partial_1 \theta\|^2_{H^{-2}}
\]
\[
+ \|u \cdot \nabla \omega\|^2_{H^{-2}} + \|\partial_1 \text{div}(u \otimes \theta)\|^2_{H^{-2}}
\]
\[
\leq C_1\|\omega\|^2_{H^{-2}} + \frac{1}{2} \|\partial_1 \theta\|^2_{H^{-2}}
\]
\[
+ \|u \cdot \nabla u\|^2_{H^{-1}} + \|u \otimes \theta\|^2_{L^2}
\]
\[
\leq C_1\|\omega\|^2_{H^{-2}} + \frac{1}{2} \|\partial_1 \theta\|^2_{H^{-2}}
\]
\[
+ \|u \otimes \theta\|^2_{L^2} + \|u\|_{L^\infty}^2 \|\theta\|^2_{L^2}
\]
\[
\leq C_1\|\omega\|^2_{H^{-2}} + \frac{1}{2} \|\partial_1 \theta\|^2_{H^{-2}}
\]
\[
+ \|u\|_{L^2}^2 \|\omega\|^2_{L^2} + \|u\|_{L^\infty}^2 \|\theta\|^2_{L^2}.
\]
Multiplying (36) by 2\(C_1\), and adding to (37), we can get
\[
\frac{d}{dt}\{C_1(\|\omega\|^2_{H^{-2}} + \|\theta\|^2_{H^{-2}}) - (\omega|\partial_1 \theta)_{H^{-2}}\}
\]
\[
+ C_1\|\omega\|^2_{H^{-2}} + \frac{1}{2} \|\partial_1 \theta\|^2_{H^{-2}}
\]
\[
\leq (\|\omega\|_{\dot{H}^{-2}} + \|\Lambda^{-1}\theta\|_{H^2} + \|\theta, \omega\|_{L^2}^2 )
\times (\|u\|_{H^2}^2 + \|\partial_t \theta\|_{\dot{H}^{-2}}^2 + \|\partial_t \theta\|_{L^2}^2 ).
\]

Integrating (38) in time, and using
\[
2C_1(\|\omega\|_{\dot{H}^{-2}}^2 + \|\theta\|_{\dot{H}^{-2}}^2) - (\omega|\partial_t \theta|)_{\dot{H}^{-2}} \approx \|\omega\|_{\dot{H}^{-2}}^2 + \|\theta\|_{\dot{H}^{-2}}^2,
\]
we have
\[
\|\omega\|_{L^p{T}\dot{H}^{-2}}^2 + \|\theta\|_{L^p{T}\dot{H}^{-1}}^2
\leq C(\|\omega\|_{\dot{H}^{-2}}^2 + \|\theta_0\|_{\dot{H}^{-1}}^2) + \int_0^T (\|\omega\|_{\dot{H}^{-2}} + \|\Lambda^{-1}\theta\|_{H^2} + \|\theta, \omega\|_{L^2}^2 )
\times (\|u\|_{H^2}^2 + \|\partial_t \theta\|_{\dot{H}^{-2}}^2 + \|\partial_t \theta\|_{L^2}^2 )dt.
\]

Using Hölder’s inequality, we complete the proof of Lemma 6.1.

7. Proof of Theorem 1.1. In this section, we give the proof of Theorem 1.1. Local well-posedness can be proved by using standard method. Here we only show the global a priori bound. Combining with (22) and Lemma 6.1, we can get
\[
A(T) + A_1(T) \leq C_2 A(0) + C_2 A(T) \{ A(T) + A_1(T) + A(T) \frac{3}{2} + A(T) A_1(T) \} + C_2 (A(T) \frac{3}{2} + A_1(T) \} \{ A(T) \frac{3}{2} + A_1(T) + A(T) \frac{3}{4} A_1(T) \frac{3}{4} \}.
\]

Denote
\[
\tilde{T} \stackrel{\text{def}}{=} \{ T \in (0, T^*) : A(T) + A_1(T) \leq 4C_2 A(0) \},
\]
where \( T^* > 0 \) is the maximal existence time of the local solution. Assume \( \tilde{T} < T^* \). Thanks to (5), we can get from (40) that
\[
A(T) + A_1(T) \leq 2C_2 A(0),
\]
which yields a contradiction with \( \tilde{T} < T^* \) by the continuous arguments. Thus we can get \( \tilde{T} = T^* \), which leads to the desired result.

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Appendix.

Proof of (17). We only give the estimate of \( K_2 \), since the estimate of \( K_1 \) is similar. Integrating by parts, we have
\[
K_2 = \sum_{2 \leq \alpha \leq \alpha+1} C_{\alpha+1}^\alpha \int_0^T \int \partial_\theta \partial_\theta^{-\alpha} \partial_1 u_1 \partial_2^{\alpha+2-\alpha} \theta \partial_2^{\alpha+1} \theta dx dt
\]
\[
= - \sum_{2 \leq \alpha \leq \alpha+1} C_{\alpha+1}^\alpha \int_0^T \int \partial_1 \partial_\theta \partial_\theta^{-\alpha} \partial_1 u_1 \partial_2^{\alpha+2-\alpha} \theta \partial_2^{\alpha+1} \theta dx dt
\]
\[
- \sum_{2 \leq \alpha \leq \alpha+1} C_{\alpha+1}^\alpha \int_0^T \int \partial_\theta \partial_\theta^{-\alpha} u_1 \partial_1 \partial_2^{\alpha+2-\alpha} \theta \partial_2^{\alpha+1} \theta dx dt.
\]
Thus, we get
\[
- \sum_{2 \leq \alpha \leq s+1} C_{s+1}^\alpha \int_0^T \int \partial_3 \theta \partial_2^{s+1-\alpha} u_1 \partial_2^{s+2-\alpha} \theta \partial_1 \partial_2^{s+1} \theta dxdt = \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3.
\]

For \( \tilde{K}_1 \), we have
\[
\tilde{K}_1 = - \left( \sum_{2 \leq \alpha \leq 3} \sum_{4 \leq \alpha \leq s+1} C_{s+1}^\alpha \right) \int_0^T \int \partial_1 \partial_2 \partial_2^{\alpha-1} u_1 \partial_2^{s+2-\alpha} \theta \partial_1 \partial_2^{s+1} \theta dxdt
\]
\[
= \tilde{K}_{11} + \tilde{K}_{12}.
\]

It follows by Hölder’s inequality and \( \|f\|_{L^\infty} \leq C\|f\|_{H^2} \) that
\[
\tilde{K}_{11} \leq C \sum_{2 \leq \alpha \leq 3} \|\partial_1 \partial_2 \theta\|_{L_2^2(L^\infty)} \|\partial_2^{\alpha-1} u_1\|_{L_2^3(L^\infty)} \|\partial_2^{s+2-\alpha} \theta\|_{L_2^2(L^2)} \|\partial_1 \partial_2^{s+1} \theta\|_{L_2^2(L^2)}
\]
\[
\leq C \|\partial_1 \partial_2 \theta\|_{L_2^2(H^3)} \|u_1\|_{L_2^3(H^4)} \|\theta\|_{L_2^2(H^{s+1})} ^2
\]
\[
\leq CE(T)E_1(T)
\]

and
\[
\tilde{K}_{12} \leq C \sum_{4 \leq \alpha \leq s+1} \|\partial_1 \partial_2 \theta\|_{L_2^2(L^\infty)} \|\partial_2^{\alpha-1} u_1\|_{L_2^3(L^\infty)} \|\partial_2^{s+2-\alpha} \theta\|_{L_2^2(L^2)} \|\partial_1 \partial_2^{s+1} \theta\|_{L_2^2(L^2)}
\]
\[
\leq C \|\partial_1 \partial_2 \theta\|_{L_2^2(H^3)} \|u_1\|_{L_2^3(H^4)} \|\theta\|_{L_2^2(H^{s+1})} ^2
\]
\[
\leq CE(T)E_1(T).
\]

Thus, we get
\[
\tilde{K}_1 \leq CE(T)E_1(T).
\]

Along the similar arguments can yield the estimate of \( \tilde{K}_2 \):
\[
\tilde{K}_2 \leq CE(T)E_1(T).
\]

For \( \tilde{K}_3 \), integrating by parts again, one can deduce that
\[
\tilde{K}_3 = \sum_{2 \leq \alpha \leq s+1} C_{s+1}^\alpha \int_0^T \int \partial_2 \partial_2^{\alpha-1} u_1 \partial_2^{s+2-\alpha} \theta \partial_1 \partial_2^{s+1} \theta dxdt
\]
\[
+ \sum_{2 \leq \alpha \leq s+1} C_{s+1}^\alpha \int_0^T \int \partial_2 \partial_2^{\alpha-1} u_1 \partial_2^{s+2-\alpha} \theta \partial_1 \partial_2^{s+1} \theta dxdt
\]
\[
+ \sum_{2 \leq \alpha \leq s+1} C_{s+1}^\alpha \int_0^T \int \partial_2 \partial_2^{\alpha-1} u_1 \partial_2^{s+3-\alpha} \theta \partial_1 \partial_2^{s+1} \theta dxdt
\]

Like the previous way yielding (41), the three parts on the right hand side of (42) can also be bounded by \( CE(T)E_1(T) \). Collecting the above estimates can yield the desired bound of \( K_2 \).

\[ \square \]

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E-mail address: rhwanmath@163.com, wrh@njnu.edu.cn