DILATION IN FUNCTION SPACES WITH GENERAL WEIGHTS

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Abstract. In this paper, we present more regularity conditions which ensure the boundedness of dilation operators on Besov and Triebel-Lizorkin spaces equipped with general weights.

1. Introduction

Function spaces play a central role in mathematical analysis especially in partial differential equations. Some examples of these spaces can be mentioned such as: Besov and Triebel-Lizorkin spaces. The theory of these spaces had a remarkable development in part due to its usefulness in applications. We refer the reader to the monographs [28] and [29] for further details, historical remarks and more references on these spaces.

In recent years many researchers have modified the classical spaces and have generalized the classical results to these modified ones. For example: function spaces of generalized smoothness. These type of function spaces have been introduced by several authors. We refer, for instance, to Cobos and Fernandez [5], Goldman [16] and [17], and Kalyabin [18]; see also Besov [1] and [2], and Kalyabin and Lizorkin [19].

These type of function spaces appear in the study of trace spaces on fractals, see Edmunds and Triebel [9] and [10], where they introduced the spaces $B_{s,\Psi}^{p,q}$, where $\Psi$ is a so-called admissible function, typically of log-type near 0. For a complete treatment of these spaces we refer the reader the work of Moura [20]. More general function spaces of generalized smoothness can be found in Farkas and Leopold [11], and reference therein.

A. Tyulenev has been introduced in [32] a new family of Besov space of variables smoothness which cover many classes of Besov spaces. Based on this new weighted class and the Littlewood-Paley theory the author in [6] and [7] has been introduced the function spaces $B_{p,q}^{s}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$ and $F_{p,q}^{s}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$ which cover weighted Besov and Triebel-Lizorkin spaces, respectively. Several results, concerning, for instance, Sobolev embeddings, atomic, molecular and wavelet decompositions are presented.

The purpose of the present paper is to study the dilation operators in $A_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$ spaces, where we use this notation to denote either $B_{p,q}^{s}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$ or $F_{p,q}^{s}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$. Their behavior is well known in Besov and Triebel-Lizorkin spaces, see [28] 3.4.1. Further results can be found in [26], [27] and [35]. Allowing $\{t_k\}_{k \in \mathbb{N}_0}$ to vary from point to point will raise extra difficulties which, in general, are overcome by imposing some regularity assumptions on this smoothness. By these additional assumptions we ensure the boundedness of these operators on $A_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$ spaces, but with some appropriate assumptions. More precisely we shall show the following result:
Given a measurable set $E$ almost everywhere. For measurable set $E$, let $f$ be a function where

$$f \in A, \sigma = (\sigma_1 \in \Theta (\mathbb{R}^n), \sigma_2 \in \Theta (\mathbb{R}^n))$$

with $\sigma = (\sigma_1 = \theta (\mathbb{R}^n), \sigma_2 > p)$ and all $f \in A_{p,q} (\mathbb{R}^n)$. Then $f$ belongs to $A_{p,q} (\mathbb{R}^n)$. Theorem 1.1.

Let $1 \leq p < \infty, 1 \leq q < \infty, \alpha_1, \alpha_2 \in \mathbb{R}, \alpha = (\alpha_1, \alpha_2), \lambda \geq 1$. Let $\alpha_2 \geq \alpha_1 > 0$. There exists $1 \leq \theta < p < \infty$ such that for all $\{t_k\}_{k \in \mathbb{N}_0} \in X_{\alpha, \sigma, p}$, with $\sigma = (\sigma_1 = \theta (\mathbb{R}^n), \sigma_2 \geq p)$ and all $f \in A_{p,q} (\mathbb{R}^n), \{t_k\}_{k \in \mathbb{N}_0}$

$$\|f(\lambda)\|_{A_{p,q} (\mathbb{R}^n), \{t_k\}_{k \in \mathbb{N}_0}} \leq c \lambda^{\alpha_2 - \frac{\alpha_1}{p}} H \|f\|_{A_{p,q} (\mathbb{R}^n), \{t_k\}_{k \in \mathbb{N}_0}}}$$

where

$$H = \sup_{k \geq 1, m \in \mathbb{Z}^n} \|t_{k-1} (\lambda^{-\frac{1}{p}})\|_{L_p (Q_{k-1, m})},$$

the constant $c > 0$ independent of $\lambda$ and $\lambda < 2^i \leq 2^\lambda$.

As a consequence, our result cover the classical case, see [28, 3.4.1], also for Besov and Triebel-Lizorkin spaces equipped with power weights. Concerning Sobolev spaces $W^k_p (\mathbb{R}^n, w)$ it holds

$$W^k_p (\mathbb{R}^n, w) = F^k_p (\mathbb{R}^n, w), \quad 1 < p < \infty, k \in \mathbb{N}_0, w \in A_p (\mathbb{R}^n),$$

see [3] Theorem 2.8], where $A_p (\mathbb{R}^n)$ are the Muckenhoupt classes, see Section 2. We can easily prove that

$$\|f(\lambda)\|_{W^k_p (\mathbb{R}^n, w)} \leq \lambda^{k-\frac{\alpha_2}{p}} H \sup_{x \in \mathbb{R}^n} \frac{\omega^{-1}(x)}{\omega(x)} \|f\|_{W^k_p (\mathbb{R}^n, w)}$$

for all $f \in W^k_p (\mathbb{R}^n, w)$. In Section 4 we prove that our estimate (1.2) is better than (1.3).

We mention that the boundedness of these operators in function spaces play an important role in mathematical analysis. They appear in the Gagliardo-Nirenberg inequalities [30, Chapter 4] in the boundedness properties of pseudodifferential operators on Besov spaces and Triebel-Lizorkin [24], and in the spline representations of Besov and Triebel-Lizorkin spaces [25].

2. Basic Tools

Throughout this paper, we denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leq c g$ for some independent constant $c$ (and non-negative functions $f$ and $g$). As usual for any $x \in \mathbb{R}$, $\lfloor x \rceil$ stands for the largest integer smaller than or equal to $x$.

By supp $f$ we denote the support of the function $f$, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of $E$ and $\chi_E$ denotes its characteristic function. By $c$ we denote generic positive constants, which may have different values at different occurrences.

A weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere. For measurable set $E \subset \mathbb{R}^n$ and a weight $\gamma$, $\gamma (E)$ denotes

$$\int_E \gamma (x) dx.$$

Given a measurable set $E \subset \mathbb{R}^n$ and $0 < p \leq \infty$, we denote by $L_p (E)$ the space of all functions $f : E \rightarrow \mathbb{C}$ equipped with the finite quasi-norm

$$\|f\|_{L_p (E)} := \left( \int_E |f(x)|^p dx \right)^{1/p} < \infty, \quad 0 < p < \infty,$$
For a function \( f \) in \( L^1_{\text{loc}} \), we set
\[
M_A(f) := \frac{1}{|A|} \int_A |f(x)| \, dx
\]
for any \( A \subset \mathbb{R}^n \). Furthermore, we put
\[
M_{A,p}(f) := \left( \frac{1}{|A|} \int_A |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 0 < p < \infty.
\]
Further, given a measurable set \( E \subset \mathbb{R}^n \) and a weight \( \gamma \), we denote the space of all functions \( f : \mathbb{R}^n \to \mathbb{C} \) with finite quasi-norm
\[
\|f\|_{L_p(\mathbb{R}^n; \gamma)} = \|f\gamma\|_{L_p(\mathbb{R}^n)}
\]
by \( L_p(\mathbb{R}^n, \gamma) \).

Let \( 0 < p \leq \infty \) and \( 0 < q \leq \infty \). The space \( L_p(\ell_q) \) is defined to be the set of all sequences \( \{f_k\}_{k \in \mathbb{Z}} \) of functions such that
\[
\left\| \{f_k\}_{k \in \mathbb{N}_0} \right\|_{L_p(\ell_q)} := \left\| \left\| f_k \right\|_{\ell_q} \right\|_{L_p(\mathbb{R}^n)} < \infty.
\]
In the limiting case \( q = \infty \) the usual modification is required. If \( 1 \leq p < \infty \) and \( \frac{1}{p} + \frac{1}{q'} = 1 \), then \( p' \) is called the conjugate exponent of \( p \).

The symbol \( S(\mathbb{R}^n) \) is used in place of the set of all Schwartz functions on \( \mathbb{R}^n \). We define the Fourier transform of a function \( f \in S(\mathbb{R}^n) \) by
\[
\mathcal{F}(f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx, \quad \xi \in \mathbb{R}^n.
\]
In what follows, \( Q \) will denote an cube in the space \( \mathbb{R}^n \) with sides parallel to the coordinate axes and \( l(Q) \) will denote the side length of the cube \( Q \). For \( k \in \mathbb{N}_0 \) and \( m \in \mathbb{Z}^n \), denote by \( Q_{k,m} \) the dyadic cube,
\[
Q_{k,m} := 2^{-k}(0,1)^n + m.
\]
For the collection of all such cubes we use \( Q := \{Q_{k,m} : k \in \mathbb{N}_0, m \in \mathbb{Z}^n\} \). For each cube \( Q \), we denote by \( x_{k,m} \) the lower left-corner \( 2^{-k}m \) of \( Q = Q_{k,m} \).

### 2.1. Muckenhoupt weights.

The purpose of this subsection is to review some known properties of Muckenhoupt classes.

**Definition 2.1.** Let \( 1 < p < \infty \). We say that a weight \( \gamma \) belongs to the Muckenhoupt class \( A_p(\mathbb{R}^n) \) if there exists a constant \( C > 0 \) such that for every cube \( Q \) the following inequality holds
\[
M_Q(\gamma)M_{Q,\gamma}(\gamma^{-1}) \leq C. \tag{2.2}
\]

The smallest constant \( C \) for which (2.2) holds, denoted by \( A_p(\gamma) \). As an example, we can take
\[
\gamma(x) = |x|^\alpha, \quad \alpha \in \mathbb{R}.
\]
Then \( \gamma \in A_p(\mathbb{R}^n), 1 < p < \infty, \) if and only if \( -n < \alpha < n(p-1) \).

For \( p = 1 \) we rewrite the above definition in the following way.

**Definition 2.3.** We say that a weight \( \gamma \) belongs to the Muckenhoupt class \( A_1(\mathbb{R}^n) \) if there exists a constant \( C > 0 \) such that for every cube \( Q \) and for a.e. \( y \in Q \) the following inequality holds
\[
M_Q(\gamma) \leq C\gamma(y). \tag{2.4}
\]
The smallest constant $C$ for which \((2.4)\) holds, denoted by $A_1(\gamma)$. The above classes have been first studied by Muckenhoupt [21] and used to characterize the boundedness of the Hardy-Littlewood maximal function on $L^p(\gamma)$, see the monographs [14] and [15] for a complete account on the theory of Muckenhoupt weights.

We recall a few basic properties of the class $A_p(\mathbb{R}^n)$ weights, see [15].

**Lemma 2.5.** Let $1 \leq p < \infty$.

(i) If $\gamma \in A_p(\mathbb{R}^n)$, then for any $1 \leq p < q$, $\gamma \in A_q(\mathbb{R}^n)$.

(ii) Let $1 < p < \infty$, $\gamma \in A_p(\mathbb{R}^n)$ if and only if $\gamma^{1-p'} \in A_p(\mathbb{R}^n)$.

(iii) Let $1 \leq p < \infty$ and $\gamma \in A_p(\mathbb{R}^n)$. There is $C > 0$ such that for any cube $Q$ and a measurable subset $E \subset Q$

\[
\left(\frac{|E|}{|Q|}\right)^{p-1} M_Q(\gamma) \leq CM_E(\gamma).
\]

(iv) Suppose that $\gamma \in A_p(\mathbb{R}^n)$ for some $1 < p < \infty$. Then there exists a $1 < p_1 < p < \infty$ such that $\gamma \in A_{p_1}(\mathbb{R}^n)$.

(v) Let $\gamma \in A_p(\mathbb{R}^n)$. Then $\gamma(\lambda \cdot) \in A_p(\mathbb{R}^n)$ for any $\lambda > 0$.

2.2. The weight class $X_{\alpha,\sigma,p}$. If given $0 < p \leq \infty$ a sequence of weights $\{t_k\}_{k \in \mathbb{N}_0}$ is such that $t_k \in L_p^{\loc}$ for $k \in \mathbb{N}_0$, then the weight sequence $\{t_k\}_{k \in \mathbb{N}_0}$ will be called a $p$-admissible weight sequence. For a $p$-admissible weight sequence $\{t_k\}_{k \in \mathbb{N}_0}$ we set

\[ t_{k,m} := \|t_k\|_{L_p(Q_{k,m})}, \quad k \in \mathbb{N}_0, m \in \mathbb{Z}^n. \]

Tyulenev [31] introduce the following new weighted class and use to study Besov spaces of variable smoothness.

**Definition 2.6.** Let $0 < p \leq \infty, \alpha_1, \alpha_2 \in \mathbb{R}, \sigma_1, \sigma_2 \in (0, +\infty)$, $\alpha = (\alpha_1, \alpha_2)$ and let $\sigma = (\sigma_1, \sigma_2)$. We let $X_{\alpha,\sigma,p} = X_{\alpha,\sigma,p}(\mathbb{R}^n)$ denote the set of $p$-admissible weight sequences $\{t_k\}_{k \in \mathbb{N}_0}$ satisfying the following conditions. There exist numbers $C_1, C_2 > 0$ such that for any $0 \leq k \leq j$ and every cube $Q$,

\[
M_{Q,p}(t_k)M_{Q,\sigma_1}(t_j^{-1}) \leq C_1 2^{\alpha_1(k-j)},
\]

\[
M_{Q,p}^{-1}(t_k)M_{Q,\sigma_2}(t_j) \leq C_2 2^{\alpha_2(j-k)}. \tag{2.7}
\]

The constants $C_1, C_2 > 0$ are independent of both the indexes $k$ and $j$.

**Remark 2.9.** We would like to mention that if $\{t_k\}_{k \in \mathbb{N}_0}$ satisfying \((2.7)\) with $\sigma_1 = r \left(\frac{p}{p}\right)'$ and $0 < r < p \leq \infty$, then $t_k^p \in A_\infty(\mathbb{R}^n)$ for any $k \in \mathbb{N}_0$ with $0 < r < p < \infty$ and $t_k^{-r} \in A_1(\mathbb{R}^n)$ for any $k \in \mathbb{N}_0$ with $p = \infty$.

We say that $t_k \in A_p(\mathbb{R}^n)$, $k \in \mathbb{N}_0$, $1 < p < \infty$ have the same Muckenhoupt constant if

\[ A_p(t_k) = c, \quad k \in \mathbb{N}_0, \]

where $c$ is independent of $k$.

**Example 2.10.** Let $0 < p < \infty$, a weight $\omega^p \in A_\infty(\mathbb{R}^n)$ and $\{s_k\}_{k \in \mathbb{N}_0} = \{2^{ks} \omega^p(2^{-k})\}_{k \in \mathbb{N}_0}$, $s \in \mathbb{R}$. Clearly, $\{s_k\}_{k \in \mathbb{N}_0}$ lies in $X_{\alpha,\sigma,p}$ for $\alpha_2 = \alpha_2 = s$, $\sigma = (r \left(\frac{p}{p}\right)', p)$. An example illustrating the advantage of Definition 2.3 is given in [32], [33] and [34].

**Remark 2.11.** Let $0 < \theta \leq p \leq \infty$. Let $\alpha_1, \alpha_2 \in \mathbb{R}, \sigma_1, \sigma_2 \in (0, +\infty)$, $\sigma_2 \geq p$, $\alpha = (\alpha_1, \alpha_2)$ and let $\sigma = (\sigma_1 = \sigma = (\frac{p}{p})', \sigma_2)$. Let a $p$-admissible weight sequence $\{t_k\}_{k \in \mathbb{N}_0} \in X_{\alpha,\sigma,p}$. Then

\[ \alpha_2 \geq \alpha_1. \]
Also we set

\[ \text{Lemma 2.14.} \]

Let \( f \) holds for all sequence of functions usual, we put \( f \in L^1_\text{loc} \), where the supremum is taken over all cubes with sides parallel to the axis and \( x \in Q \).

Also we set

\[ \mathcal{M}_\sigma(f) := (\mathcal{M}(|f|^\sigma))^{\frac{1}{\sigma}}, \quad 0 < \sigma < \infty. \]

\[ \text{Theorem 2.12.} \]

Let \( 0 < p < \infty, 0 < q \leq \infty \) and \( 0 < \sigma < \min(1, p, q) \). Then

\[ \left\| \left( \sum_{k=0}^{\infty} (\mathcal{M}_\sigma(f_k))^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| \left( \sum_{k=0}^{\infty} |f_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \]  \quad (2.13)

holds for all sequence of functions \( \{f_k\}_{k \in \mathbb{N}_0} \in L_p(\ell_q) \).

We will make use of the following statement, see [12].

\[ \text{Lemma 2.14.} \]

Let \( 1 < \theta \leq p < \infty \) and \( 1 < q < \infty \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \) be a \( p \)-admissible weight sequence such that \( t_k^p \in A^p_\#(\mathbb{R}^n) \), \( k \in \mathbb{N}_0 \). Assume that \( t_k^p, k \in \mathbb{N}_0 \) have the same Muckenhoupt constant, \( A_\#(t_k) = c, k \in \mathbb{N}_0 \). Then

\[ \left\| \left( \sum_{k=0}^{\infty} t_k^q (\mathcal{M}(f_k))^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| \left( \sum_{k=0}^{\infty} t_k^q |f_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \]

holds for all sequence of functions \( \{f_k\}_{k \in \mathbb{N}_0} \in L_p(\ell_q) \). In particular

\[ \left\| \mathcal{M}(f_k) \right\|_{L^p(\mathbb{R}^n, t_k)} \leq c \left\| f_k \right\|_{L^p(\mathbb{R}^n, t_k)} \]

holds for all sequence of functions \( f_k \in L^p(\mathbb{R}^n, t_k), k \in \mathbb{N}_0 \), where \( c > 0 \) is independent of \( k \).

\[ \text{Remark 2.15.} \]

We would like to mention that the result of this lemma is true if we assume that \( t_k \in A^p_\#(\mathbb{R}^n) \), \( k \in \mathbb{N}_0 \), \( 1 < \theta \leq p < \infty \) with

\[ A_\#(t_k) \leq c, \quad k \in \mathbb{N}_0, \]

where \( c > 0 \) independent of \( k \).

3. Function spaces

In this section we present the Fourier analytical definition of Besov and Triebel-Lizorkin spaces of variable smoothness and we recall their basic properties in analogy to the classical Besov and Triebel-Lizorkin spaces. We first need the concept of a smooth dyadic resolution of unity. Let \( \varphi_0 \) be a function in \( \mathcal{S}(\mathbb{R}^n) \) satisfying \( \varphi_0(x) = 1 \) for \( |x| \leq 1 \) and \( \varphi_0(x) = 0 \) for \( |x| > \frac{3}{2} \). We put \( \varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{1-k}x) \) for \( k = 1, 2, 3, \ldots \).

Then \( \{\varphi_k\}_{k \in \mathbb{N}_0} \) is a resolution of unity, \( \sum_{k=0}^{\infty} \varphi_k(x) = 1 \) for all \( x \in \mathbb{R}^n \). Thus we obtain the Littlewood-Paley decomposition

\[ f = \sum_{k=0}^{\infty} \mathcal{F}^{-1} \varphi_k \ast f \]

of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) (convergence in \( \mathcal{S}'(\mathbb{R}^n) \)).

Now, we define the spaces under consideration.
Definition 3.1. Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $\{t_k\}_{k \in \mathbb{N}_0}$ be a $p$-admissible weight sequence.

(i) The Besov space $B_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that
$$
\|f\|_{B_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})} := \left( \sum_{k=0}^{\infty} \| t_k (\mathcal{F}^{-1}\varphi_k * f) \|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} < \infty
$$
with the usual modifications if $q = \infty$.

(ii) Let $0 < p < \infty$. The Triebel-Lizorkin space $F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that
$$
\|f\|_{F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})} := \left( \sum_{k=0}^{\infty} t_k^q |\mathcal{F}^{-1}\varphi_k * f|^q \right)^{\frac{1}{q}} < \infty
$$
with the usual modifications if $q = \infty$.

Let $0 < \theta < p < \infty$ and $0 < q < \infty$. Let $\{t_k\} \in X_{\alpha,\sigma,p}$ be a $p$-admissible weight sequence with $\sigma = (\sigma_1 = \theta \left(\frac{2}{p}\right), \sigma_2 \geq p)$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The definition of the spaces $A_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$ is independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_k\}_{k \in \mathbb{N}_0}$. They are quasi-Banach spaces. They are Banach spaces if $1 \leq p < \infty$ and $1 \leq q < \infty$. We have the embedding
$$
\mathcal{S}(\mathbb{R}^n) \hookrightarrow A_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).
$$

Further results such as the $\varphi$-transform characterization in the sense of Frazier and Jawerth, the case $p = \infty$, duality, complex interpolation, the smooth atomic, molecular and wavelet decomposition and the characterization of these function spaces in terms of the difference relations are given in [6] and [7]. The above function spaces whose elements are not distributions, but rather functions that are locally integrable in some power are studied by Tyulenev [31], [32] and [33] for the Besov case and by the author [8] for Triebel-Lizorkin case.

Using the system $\{\varphi_k\}_{k \in \mathbb{N}_0}$ we can define the quasi-norms
$$
\|f\|_{B_{p,q}^s(\mathbb{R}^n)} := \left( \sum_{k=0}^{\infty} 2^{ksq} \| \mathcal{F}^{-1}\varphi_k * f \|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}}
$$
and
$$
\|f\|_{F_{p,q}^s(\mathbb{R}^n)} := \left( \sum_{k=0}^{\infty} 2^{ksq} |\mathcal{F}^{-1}\varphi_k * f|^q \right)^{\frac{1}{q}}
$$
for constants $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ with $0 < p < \infty$ in the $F$-case. The Besov space $B_{p,q}^s(\mathbb{R}^n)$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{B_{p,q}^s(\mathbb{R}^n)} < \infty$. The Triebel-Lizorkin space $F_{p,q}^s(\mathbb{R}^n)$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{F_{p,q}^s(\mathbb{R}^n)} < \infty$.

It is well-known that these spaces do not depend on the choice of the system $\{\varphi_k\}_{k \in \mathbb{N}_0}$ (up to equivalence of quasi-norms). Further details on the classical theory of these spaces, included the homogeneous case, can be found [13, 28] and [29].

One recognizes immediately that if $\{t_k\}_{k \in \mathbb{N}_0} = \{2^{sk}\}_{k \in \mathbb{N}_0}$, $s \in \mathbb{R}$, then we have
$$
B_{p,q}(\mathbb{R}^n, \{2^{sk}\}_{k \in \mathbb{N}_0}) = B_{p,q}^s(\mathbb{R}^n)
$$
and
$$
F_{p,q}(\mathbb{R}^n, \{2^{sk}\}_{k \in \mathbb{N}_0}) = F_{p,q}^s(\mathbb{R}^n).
$$
Moreover, for \( \{ t_k \}_{k \in \mathbb{N}_0} = \{ 2^{sk} w \}_{k \in \mathbb{N}_0} \), \( s \in \mathbb{R} \) with a weight \( w \) we re-obtain the weighted Besov and Triebel-Lizorkin spaces; we refer, in particular, to the papers [3], [4] and [23] for a comprehensive treatment of the weighted spaces.

**Example 3.2.** A sequence \( \{ \gamma_j \}_{j \in \mathbb{N}_0} \) of positive real numbers is said to be admissible if there exist two positive constants \( d_0 \) and \( d_1 \) such that

\[
d_0 \gamma_j \leq \gamma_{j+1} \leq d_1 \gamma_j, \quad j \in \mathbb{N}_0.
\]

For an admissible sequence \( \{ \gamma_j \}_{j \in \mathbb{N}_0} \), let

\[
\underline{\gamma}_j = \inf_{k \geq 0} \frac{\gamma_{j+k}}{\gamma_k} \quad \text{and} \quad \overline{\gamma}_j = \sup_{k \geq 0} \frac{\gamma_{j+k}}{\gamma_k}, \quad j \in \mathbb{N}_0.
\]

and

\[
\alpha_\gamma = \lim_{j \to \infty} \frac{\log \overline{\gamma}_j}{j} \quad \text{and} \quad \beta_\gamma = \lim_{j \to \infty} \frac{\log \underline{\gamma}_j}{j},
\]

be the upper and lower Boyd index of the given sequence \( \{ \gamma_j \}_{j \in \mathbb{N}_0} \), respectively. Then

\[
\underline{\gamma}_j \gamma_k \leq \gamma_{j+k} \leq \overline{\gamma}_j \gamma_k, \quad j, k \in \mathbb{N}_0
\]

and for each \( \varepsilon > 0 \),

\[
c_1 2^{(\beta_\gamma - \varepsilon)j} \leq \underline{\gamma}_j \leq \overline{\gamma}_j \leq c_2 2^{(\alpha_\gamma + \varepsilon)j}, \quad j \in \mathbb{N}_0
\]

for some constants \( c_1 = c_1(\varepsilon) > 0 \) and \( c_2 = c_2(\varepsilon) > 0 \).

Clearly the sequence \( \{ \gamma_j \}_{j \in \mathbb{N}_0} \) lies in \( X_{\alpha, \sigma, p} \) for \( \alpha_1 = \beta_\gamma - \varepsilon, \alpha_2 = \alpha_\gamma + \varepsilon \) and \( 0 < p, \sigma_1, \sigma_2 \leq \infty \).

These type of admissible sequences are used in [11] to study Besov and Lizorkin-Triebel spaces in terms of a generalized smoothness, see also [22].

Let us consider some examples of admissible sequences. The sequence \( \{ \gamma_j \}_{j \in \mathbb{N}_0} \),

\[
\gamma_j = 2^{sj}(1 + j)^b(1 + \log(1 + j))^c, \quad j \in \mathbb{N}_0
\]

with arbitrary fixed real numbers \( s, b \) and \( c \) is a an admissible sequence with

\[
\beta_\gamma = \alpha_\gamma = s.
\]

**Example 3.3.** Let \( 0 < r < p < \infty \), a weight \( \omega^p \in A_r(\mathbb{R}^n) \) and \( \{ s_k \}_{k \in \mathbb{N}_0} = \{ 2^{ks} \omega^p(2^{-k}) \}_{k \in \mathbb{N}_0} \), \( s \in \mathbb{R} \). Obviously, \( \{ s_k \}_{k \in \mathbb{N}_0} \) lies in \( X_{\alpha, \sigma, p} \) for \( \alpha_1 = \alpha_2 = s, \sigma = r(\frac{p}{r})', p \).

Let \( f \) be an arbitrary function on \( \mathbb{R}^n \) and \( x, h \in \mathbb{R}^n \). Then

\[
\Delta_h f(x) := f(x + h) - f(x), \quad \Delta_h^{M+1} f(x) := \Delta_h(\Delta_h^M f)(x), \quad M \in \mathbb{N}.
\]

These are the well-known differences of functions which play an important role in the theory of function spaces. Using mathematical induction one can show the explicit formula

\[
\Delta_h^M f(x) = \sum_{j=0}^{M} (-1)^j C_j^M f(x + (M - j)h),
\]

where \( C_j^M \) are the binomial coefficients.

Let \( M \in \mathbb{N} \). For \( f \in L^1_{\text{loc}} \), \( x \in \mathbb{R}^n \) and a cube \( Q \), we put

\[
\delta^M(Q) f := \frac{1}{|l(Q)|^{2n}} \int_{l(Q)} \int_Q |\Delta_h^M f(x)| \, dx \, dh.
\]
weight sequence. We set Definition 3.4. Also we set, 0 < p ≤ ∞, and let \( \{ t_k \}_{k \in \mathbb{N}_0} \) be a p-admissible weight sequence. We set

\[
\mathcal{F}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0}) := \left\{ f : f \in L^1_{loc}, \|f\|_{\mathcal{F}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})} < \infty \right\},
\]

where

\[
\|f\|_{\mathcal{F}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})} := \|f\|_{\mathcal{F}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})}^* + \|f\|_{L^p(\mathbb{R}^n, t_0)},
\]

making the obvious modifications for q = ∞, with

\[
\|f\|_{\mathcal{F}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})}^* := \left\{ \left( \sum_{k=1}^{\infty} t_k^q (\delta^M (\cdot + 2^{-k} I^n) f)^q \right)^{\frac{1}{q}} \right\}_{L^p(\mathbb{R}^n)}.
\]

Now we present the definition of Besov spaces of variable smoothness \( \bar{B}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0}) \) as introduced recently in [32].

**Definition 3.5.** Let \( M \in \mathbb{N}, 0 < p, q \leq \infty \), and let \( \{ t_k \}_{k \in \mathbb{N}_0} \) be a p-admissible weight sequence. We set

\[
\bar{B}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0}) := \left\{ f : f \in L^1_{loc}, \|f\|_{\bar{B}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})} < \infty \right\},
\]

where

\[
\|f\|_{\bar{B}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})} := \left( \sum_{k=1}^{\infty} \|t_k\|_{\mathbb{R}^n} \|\delta^M (\cdot + 2^{-k} I^n) f\|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} + \|f\|_{L^p(\mathbb{R}^n, t_0)},
\]

making the obvious modifications for p = ∞ and/or q = ∞.

Let \( M \in \mathbb{N}, 0 < p \leq \infty, 0 < q \leq \infty \), and let \( \{ t_k \}_{k \in \mathbb{N}_0} \) be a p-admissible weight sequence. We set

\[
\|f\|_{\bar{B}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})}^{s,1} := \|f\|_{\bar{B}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})} + \left( \sum_{m \in \mathbb{Z}^n} t_{0,m} \|f\|_{L^1(Q_{0,m})} \right)^{\frac{1}{p}},
\]

where

\[
\|f\|_{\bar{B}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})}^{s,1} := \left\{ \left( \sum_{k=1}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{kn} \|t_k\|_{\mathbb{R}^n} \|\delta^M (Q_{k,m}) f\|_{L^p(\mathbb{R}^n)}^{q} \right)^{\frac{1}{q}} \right\}_{L^p(\mathbb{R}^n)},
\]

Also we set, 0 < p ≤ ∞,

\[
\|f\|_{\bar{B}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})}^{s,1} := \|f\|_{\bar{B}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})} + \left( \sum_{m \in \mathbb{Z}^n} t_{0,m} \|f\|_{L^1(Q_{0,m})} \right)^{\frac{1}{p}},
\]

where

\[
\|f\|_{\bar{B}^M_{p,q}(\mathbb{R}^n, \{ t_k \}_{k \in \mathbb{N}_0})}^{s,1} := \left( \sum_{k=1}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} t_{k,m} \|\delta^M (Q_{k,m}) f\|_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{p}} \right)^{\frac{1}{q}},
\]
with
\[ Q_{k,m} := \prod_{i=1}^{n} \left( \frac{m_i - 2}{2^k}, \frac{m_i + 3}{2^k} \right), \quad \chi_{k,m} := \chi_{Q_{k,m}}, \quad m \in \mathbb{Z}^n \]

and
\[ \delta^M(Q_{k,m}) f := \frac{1}{|l(Q_{k,m})|^{2n}} \int_{\mathbb{R}^n} \int_{Q_{k,m}} |\Lambda_h^M f(z)| \, dz \, dh. \]

For simplicity, in what follows, we use the notation \( \tilde{A}^M_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) to denote either \( \tilde{B}^M_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) or \( \tilde{F}^M_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \). The following theorems are useful for us.

**Theorem 3.6.** Let \( \alpha_1, \alpha_2 \in \mathbb{R} \) and \( \alpha = (\alpha_1, \alpha_2) \). Let \( M \in \mathbb{N}, 0 < \theta < p < \infty \) and \( 0 < q < \infty \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \subset X_{\alpha, \sigma, p} \) be a \( p \)-admissible sequence with \( \sigma = (\sigma_1, \sigma_2), \sigma_1 = \theta \left( \frac{p}{\theta} \right)' \) and \( \sigma_2 \geq p \). Then
\[
\| \| \cdot \|_{\tilde{A}^M_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})}
\]
is an equivalent quasi-norm in \( \tilde{A}^M_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \).

This theorem for Besov case is given in [32], while the Triebel-Lizorkin case can be proved as in [3].

In the following theorem we present the characterizations of \( A_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \) in terms of the difference relations, see [30] for Besov case, while for Triebel-Lizorkin spaces can be obtained by the similar arguments.

**Theorem 3.7.** Let \( 1 \leq \theta < p < \infty, 1 \leq q < \infty, \alpha_1, \alpha_2 \in \mathbb{R}, \alpha = (\alpha_1, \alpha_2) \) and \( M \in \mathbb{N} \). Let \( \{t_k\}_{k \in \mathbb{N}_0} \subset X_{\alpha, \sigma, p} \) be a \( p \)-admissible weight sequence with \( \sigma = (\sigma_1, \sigma_2), \sigma_1 = \theta \left( \frac{p}{\theta} \right)', \sigma_2 \geq p \). Assume that
\[
0 < \alpha_1 \leq \alpha_2 < M.
\]

Then
\[
B_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) = \tilde{B}^M_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})
\]
and
\[
F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) = \tilde{F}^M_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}),
\]
in the sense of equivalent norm.

4. Proof of the main result

First, we prove that our estimate (1.2) is better than (1.3). Obviously
\[
H \leq \sup_{t_{k-1}(x) \in \mathbb{R}^n} \frac{t_{k-1}(\lambda^{-1}x)}{t_{k-1}(x)}.
\]

Let \( \{t_k\}_{k \in \mathbb{N}_0} = \{2^k \omega\}_{k \in \mathbb{N}_0} \), with \( \omega(x) = |x - 1|^s, s > 0 \) and \( \delta, x \in \mathbb{R} \). Then \( t_{k_0} \in A_\mathbb{R}^q(\mathbb{R}) \), \( 1 < \frac{q}{\theta} < \infty \), if and only if \( \frac{1}{p} - \delta < \frac{1}{\theta} - \frac{1}{p} \). We have
\[
\sup_{x \in \mathbb{R}^n} \frac{\omega(\lambda^{-1}x)}{\omega(x)} \geq \lambda^{-\delta} \sup_{|x| = \lambda} \frac{|x - \lambda|^{\delta}}{|x - 1|^{\delta}} \geq \lambda^{-\delta} \sup_{|x| = \lambda} |x - \lambda|^{\delta} = \infty
\]
for any \( \frac{1}{p} < \delta < 0 \) and any \( \lambda > 1 \). Let \( p_0 = \frac{q}{\theta} \). From Lemma 2.5(iv) we conclude
\[
\frac{M_{Q_{k-1,m,p}}(\omega(\lambda^{-1} \cdot))}{M_{Q_{k-1,m,p}}(\omega)} \leq \left( \frac{M_{Q_{k-1,m,p_0}}(\omega)}{M_{Q_{k-1,m,p}}(\omega)} \right)^{-1}.
\]
Let $M$ sense of distributions. On the other hand by the embedding

$$\omega$$

is an equivalent norm in $F$, which can be estimated by

$$\text{Theorem 3.7}$$

$$f$$ and $g$ makes also sense as a locally integrable function. Since $0 < \alpha_1 \leq \alpha_2 < M$. Of course, $f(\lambda x)$ must be interpreted in the sense of distributions. On the other hand by the embedding

$$F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0}) \hookrightarrow L^1_{\text{loc}},$$

if $\alpha_2 \geq \alpha_1 > 0$, and $1 \leq p, q < \infty$, see [6], it follows that $f(x)$ is a regular distribution and $f(\lambda x)$ makes also sense as a locally integrable function. Since $0 < \alpha_1 \leq \alpha_2 < M$, by Theorem 3.7

$$\|f\|_{F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})}$$

is an equivalent norm in $F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})$. We will prove

$$\|f(\lambda \cdot)\|_{F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})} \leq c\lambda^{\alpha_2 - \frac{p}{p}} H \|f\|_{F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})}$$

for all $0 < p, q < \infty$ and

$$\sigma_p = \max \left(0, \frac{n}{p} - n \right) < \alpha_1 \leq \alpha_2 < M.$$

**Step 1.** After a simple change of variable

$$\|f(\lambda \cdot)\|_{L_p(\mathbb{R}^n, t_0)} \leq \lambda^{-\frac{p}{p}} \left(\lambda^{-np} \int_{\mathbb{R}^n} t_0^p(\lambda^{-1} x) \|f\|_{L^1(\mathbb{R}^n)}^p \, dx\right)^{\frac{1}{p}}.$$

The above expression in the brackets can be rewritten as follows

$$\lambda^{-np} \sum_{m \in \mathbb{Z}^n} \int_{Q_{i,m}} t_0^p(\lambda^{-1} x) \|f\|_{L^1(\mathbb{R}^n)}^p \, dx,$$

which can be estimated by

$$\lambda^{-np} \sum_{m \in \mathbb{Z}^n} \|f\|_{L^p(\mathbb{R}^n)}^p \|t_0(\lambda^{-1} \cdot)\|_{L_p(\mathbb{R}^n)}^p,$$

where $Q_{i,m} = \cup_{j=1}^n Q_{i-z_j(m)}$. Therefore (4.1) can be rewritten in the following form

$$\lambda^{-np} \sum_{j=1}^n \sum_{m \in \mathbb{Z}^n} \|t_0(\lambda^{-1} \cdot)\|_{L_p(\mathbb{R}^n)}^p \|f\|_{L^1(\mathbb{R}^n)}^p,$$

which is bounded by

$$c\lambda^{np} H \sum_{j=1}^n \sum_{m \in \mathbb{Z}^n} \left( \sum_{h \in \mathbb{Z}^n, Q_{0,h} \subset Q_{i-z_j(m)}} t_0 \|f\|_{L^1(\mathbb{R}^n)}^p \right)^p,$$

(4.2)
where we have used Lemma 2.3(iii) and the constant $c$ is independent of $\lambda$. We distinguish two cases.

**Case 1.** $1 < p < \infty$. Observe that $|h - 2^i z_j(m)| \leq c2^i$ for any $h \in \mathbb{Z}^n$, such that $Q_{0,h} \subset Q_{k-i,z_j(m)}$, $j = 1, ..., 3^n$. Therefore the number of terms in the sum $\sum_{h \in \mathbb{Z}^n, Q_{0,h} \subset Q_{k-i,z_j(m)}} ...$ does not exceed $c2^n$. By Hölder’s inequality (4.2) is bounded by

$$c\lambda^{\left(2^{-1}\right)} H^p \sum_{j=1}^{3^n} \sum_{m \in \mathbb{Z}^n} \sum_{h \in \mathbb{Z}^n, Q_{0,h} \subset Q_{k-i,z_j(m)}} t_{0,h}^p \|f\|_{L^p(Q_{0,h})}^p,$$

which is bounded by

$$c\lambda^{\left(2^{-1}\right)} H^p \sum_{j=1}^{3^n} \sum_{m \in \mathbb{Z}^n} \sum_{h \in \mathbb{Z}^n} t_{0,h}^p \|f\|_{L^p(Q_{0,h})}^p \sum_{h \in \mathbb{Z}^n} \sum_{Q_{0,h} \subset Q_{k-i,z_j(m)}} t_{0,h}^p \|f\|_{L^p(Q_{0,h})} \chi_{Q_{0,h}}(x)dx \leq \lambda^{\left(2^{-1}\right)} H^p \int_{\mathbb{R}^n} \sum_{h \in \mathbb{Z}^n} \sum_{Q_{0,h} \subset Q_{k-i,m}} t_{0,h}^p \|f\|_{L^p(Q_{0,h})} \chi_{Q_{0,h}}(x)dx \leq \lambda^{\left(2^{-1}\right)} H^p \sum_{h \in \mathbb{Z}^n} t_{0,h}^p \|f\|_{L^p(Q_{0,h})}.$$

**Case 2.** $0 < p \leq 1$. We have (4.2) is bounded by

$$c\lambda^{\left(p^{-1}\right)} H^p \sum_{m \in \mathbb{Z}^n} \sum_{Q_{0,h} \subset Q_{k-i,m}} \int_{Q_{0,h}} t_{0,h}^p \chi_{Q_{0,h}}(x)dx \leq \lambda^{\left(p^{-1}\right)} H^p \sum_{h \in \mathbb{Z}^n} t_{0,h}^p \|f\|_{L^p(Q_{0,h})}.$$

Taking $0 < \theta < p < \infty$ be such that $\theta > \frac{n}{\alpha_2 + \max(1,p)}$ which is possible because of $\alpha_2 > \sigma_p$. Therefore

$$\|f(\cdot)\|_{L^p(\mathbb{R}^n, t_0)} \leq c\lambda^{\theta - \frac{n}{\alpha_2 + \min(1,p)}} H^p \|f\|_{F_{p,q}(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{Z}})}$$

for some positive constant $c$ independent of $\lambda$.

**Step 2.** We will estimate

$$\left\|\left(\sum_{k=1}^{\infty} t_{k}^q (\delta^M(x + 2^{-k} I^n) f(\cdot)) \right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}.$$  (4.3)

It is easily seen that

$$\delta^M(x + 2^{-k} I^n) f(\cdot) \leq 2^{ik} \delta^M(\lambda x + 2^{i-k} I^n) f$$

for any $x \in \mathbb{R}^n$. Therefore (4.3) can be estimated by

$$\lambda^{-\frac{n}{p}} \left\|\left(\sum_{k=1}^{\infty} \left( t_{k} (\lambda^{i-k} \cdot + 2^{i-k} I^n) f \right)^q \right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}.$$

Obviously

$$\sum_{k=1}^{\infty} \left( t_{k} (\lambda^{i-k} x) \delta^M(x + 2^{i-k} I^n) f \right)^q$$

$$= \sum_{k=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \left( t_{k} (\lambda^{i-k} x) \delta^M(x + 2^{i-k} I^n) f \right)^q \chi_{Q_{m,i-k}(x)}.$$

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Let $x \in Q_{k-1,m}$ with $k \in \mathbb{N}$ and $m \in \mathbb{Z}^n$. We find that
\[
\delta^M(x + 2^{-k} I^n)(f) \leq \delta^M(Q_{k-1,m} + 2^{-k} I^n) f
= \left( \frac{1}{|Q_{k-1,m}|} \int_{Q_{k-1,m}} [\delta^M(Q_{k-1,m} + 2^{-k} I^n) f] \, dy \right)^{\frac{1}{\delta}},
\]
where $0 < \delta < \min(1, \theta, q)$. Observing that
\[
Q_{k-1,m} + 2^{-k} I^n = y + Q_{k-1,m} - y + 2^{-k} I^n, \quad y \in Q_{k-1,m}.
\]
We have $Q_{k-1,m} - y \subset 2^{-k} I^n$ for all $y \in Q_{k-1,m}$ and this implies that
\[
Q_{k-1,m} + 2^{-k} I^n \subset y + 2^{-k-1} I^n, \quad y \in Q_{k-1,m}.
\]
Therefore, for any $x \in Q_{k-1,m}$,
\[
\delta^M(x + 2^{-k} I^n)(f) \leq c(M_{Q_{k-1,m}}([\tilde{\delta}^M(\cdot + 2^{-k-1} I^n) f])^\delta)^{\frac{1}{\delta}},
\]
where $c > 0$ is independent of $k, m$ and $x$, with
\[
\tilde{\delta}^M(y + 2^{-k-1} I^n) f = 2^{2(k-1)} \int_{2^{-k} I^n} \int_{y + 2^{-k-1} I^n} |\Delta^n f(\nu)| \, d\nu dh.
\]
Hence
\[
\sum_{k=1}^\infty \left( t_k(\lambda^{-1} x) \delta^M(x + 2^{-k} I^n) f \right)^q
\]
can be estimated by
\[
c \sum_{k=1}^\infty \sum_{m \in \mathbb{Z}^n} t^q_k(\lambda^{-1} x) (M_{Q_{k-1,m}}([\tilde{\delta}^M(\cdot + 2^{-k-1} I^n) f])^\delta)^{\frac{q}{\delta}} \chi_{k-1,m}(x)
= c \sum_{k=1}^\infty \left( \sum_{m \in \mathbb{Z}^n} t^\delta_k(\lambda^{-1} x) M_{Q_{k-1,m}}([\tilde{\delta}^M(\cdot + 2^{-k-1} I^n) f])^\delta \chi_{k-1,m}(x) \right)^{\frac{q}{\delta}}
\]
for all $x \in \mathbb{R}^n$, where $t^\delta_k(\lambda^{-1} \cdot) = (t_k(\lambda^{-1} \cdot))^\delta$. Using this estimate, the quantity
\[
\left\| \left( \sum_{k=1}^\infty \sum_{m \in \mathbb{Z}^n} t^q_k(\lambda^{-1} \cdot) M_{Q_{k-1,m}}([\tilde{\delta}^M(\cdot + 2^{-k-1} I^n) f])^\delta \chi_{k-1,m} \right)^{\frac{1}{\delta}} \right\|_{L_p(\mathbb{R}^n)}
\]
can be estimated from above by
\[
c \left\| \left( \sum_{k=1}^\infty \left( \sum_{m \in \mathbb{Z}^n} t^\delta_k(\lambda^{-1} \cdot) M_{Q_{k-1,m}}([\tilde{\delta}^M(\cdot + 2^{-k-1} I^n) f])^\delta \chi_{k-1,m} \right)^{\frac{1}{\delta}} \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^n)}^{\frac{1}{q}}
\]
By duality the last term with power $\delta$ is bounded by
\[
\sup_{k=1}^\infty \sum_{m \in \mathbb{Z}^n} \int_{Q_{k-1,m}} t^\delta_k(\lambda^{-1} x) M_{Q_{k-1,m}}([\tilde{\delta}^M(\cdot + 2^{-k-1} I^n) f])^\delta |g_k(x)| \, dx,
\]
where the supremum is taking over all sequence of functions $\{g_k\}_{k \in \mathbb{N}} \in L(\tilde{\delta}^M(\theta, q))$ with
\[
\| \{g_k\}_{k \in \mathbb{N}} \|_{L(\tilde{\delta}^M(\theta, q))} \leq 1.
\]
By Hölder’s inequality,
\[
1 = M_{Q_{k-1,m},h}(t^\delta_k(\lambda^{-1} \cdot) t^\delta_k(\lambda^{-1} \cdot)) \leq M_{Q_{k-1,m},r}(t^\delta_k(\lambda^{-1} \cdot)) (M_{Q_{k-1,m},p}(t^\delta_k(\lambda^{-1} \cdot)))
\]
For any $h, \tau > 0$ with $\frac{1}{\tau} = \frac{1}{p} + \frac{1}{r}$ and $m \in \mathbb{Z}^n, k \in \mathbb{N}$. By Jensen’s inequality, the second term is bounded by

$$
\left(M_{Q_{k-i,m}}(t_k(\lambda^{-1}))\right)^{\delta}.
$$

Using the fact that

$$
M_{Q_{k-i,m}}(t_k^{\delta}(\lambda^{-1}))(M_{Q_{k-i,m}}(t_k^{\delta}(\lambda^{-1})g_k) \lesssim (M_{Q_{k-i,m}}((t_k^{-\delta}(\lambda^{-1})\mathcal{M}(t_k^{\delta}(\lambda^{-1})g_k))^\tau)^{\frac{1}{\tau}},
$$

we find that

$$
M_{Q_{k-i,m}}\left([\tilde{\delta}^M(\cdot + 2^{i-k+1}l^n)f]^{\delta}\right) \int_{Q_{k-i,m}} t_k^{\delta}(\lambda^{-1}x)|g_k(x)|dx
$$

can be estimated by

$$
c|Q_{k-i,m}|^{-\frac{\delta}{\tau}} \int_{Q_{k-i,m}} M_{Q_{k-i,m}}(\omega_{k,i,m})(M_{Q_{k-i,m}}((t_k^{-\delta}(\lambda^{-1})\mathcal{M}(t_k^{\delta}(\lambda^{-1})g_k))^\tau)^{\frac{1}{\tau}} dx,
$$

where

$$
\omega_{k,i,m} = (\hat{t}_{k-i,m}\tilde{\delta}^M(\cdot + 2^{i-k+1}l^n)f)^{\delta} \quad \text{and} \quad \hat{t}_{k-i,m} = \|t_k(\lambda^{-1})\|_{L_p(Q_{k-i,m})},
$$

which is bounded by

$$
c|Q_{k-i,m}|^{-\frac{\delta}{\tau}} \int_{Q_{k-i,m}} \mathcal{M}(\omega_{k,i,m}\chi_{k-i,m})(x)\mathcal{M}_{\tau}(t_k^{-\delta}(\lambda^{-1})\mathcal{M}(t_k^{\delta}(\lambda^{-1})g_k))(x)dx.
$$

Therefore,

$$
\sum_{k=1}^{\infty} \sum_{m \in \mathbb{Z}^n} M_{Q_{k-i,m}}((\tilde{\delta}^M(\cdot + 2^{i-k+1}l^n)f)^{\delta}) t_k^{\delta}(\lambda^{-1}x)|g_k(x)|\chi_{k-i,m}(x)dx 
\lesssim \int \sum_{k=1}^{\infty} \mathcal{M}\left(\sum_{z \in \mathbb{Z}^n} 2^{(k-i)\frac{\delta}{p}}\omega_{k,i,z}\chi_{k-i,z}\right)(x)\mathcal{M}_{\tau}(t_k^{-\delta}(\lambda^{-1})\mathcal{M}(t_k^{\delta}(\lambda^{-1})g_k))(x)dx.
$$

By Hölder’s inequality the term inside the integral is bounded by

$$
\left(\sum_{k=1}^{\infty} \left(\mathcal{M}\left(\sum_{z \in \mathbb{Z}^n} 2^{(k-i)\frac{\delta}{p}}\omega_{k,i,z}\chi_{k-i,z}\right)\right)^{\frac{q}{n}}\right)^{\frac{n}{q}} \left(\sum_{k=1}^{\infty} \left(\mathcal{M}_{\tau}(t_k^{-\delta}(\lambda^{-1})\mathcal{M}(t_k^{\delta}(\lambda^{-1})g_k))\right)^{\frac{1}{\tau'}}\right)^{\frac{1}{\tau'}}.
$$

Since $t_k^p \in A_{\frac{p}{p'}}(\mathbb{R}^n) \subset A_{\frac{q}{q'}}(\mathbb{R}^n)$, using Lemma $2.5/(ii), (iv)$ to obtain

$$
t_k^{-\delta(\frac{q}{p})'} \in A_{\frac{q'}{q}}(\mathbb{R}^n), \quad k \in \mathbb{N}_0
$$

and there exist a $1 < \varrho < (\frac{q}{p})'$ such that

$$
t_k^{-\delta(\frac{q}{p})'} \in A_{\frac{q'}{q'}}(\mathbb{R}^n), \quad k \in \mathbb{N}_0.
$$

Taking any $0 < \tau < \min(1, (\frac{q}{p})', (\frac{q}{q'})')$, using the vector-valued maximal inequality of Fefferman and Stein $[2.13]$, and Lemma $2.4$ we find that the second term of the last
expression in $L_{(\frac{p}{q},(\mathbb{R}^n))}$-norm is bounded, while the first term in $L_{\frac{p}{q}}(\mathbb{R}^n)$-norm is bounded by
\[
c(\sum_{k=1}^{\infty} \left( \mathcal{M}(2^{(k-i)}\frac{p}{q}\delta \sum_{\xi \in \mathbb{Z}^n} \omega_{k,i,z} \chi_{k-i,z}) \right)^{\frac{p}{q}}) \| L_{\frac{p}{q}}(\mathbb{R}^n)
\]
\[
\leq \| \left( \sum_{k=1}^{\infty} \sum_{\xi \in \mathbb{Z}^n} 2^{(k-i)}\frac{p}{q}(\hat{t}_{k-i,z} \delta^M(\cdot + 2^{1-k}I_n)f)^q \chi_{k-i,z} \right)^{\frac{1}{q}} \|_{L_p(\mathbb{R}^n)},
\]
where we used the vector-valued maximal inequality of Fefferman and Stein (2.13) since $0 < \delta < \min(1,\theta,q)$. Now by (2.8) we deduce that
\[
\hat{t}_{k-i,m} \lesssim 2^{\alpha_i^2} \| t_{k-i}(\lambda^{-1}) \|_{L_p(Q_{k-i,m})}, \quad m \in \mathbb{Z}^n, k \geq i.
\]
Therefore (4.4), with $\sum_{k=1}^{\infty}$ in place of $\sum_{k=1}^{\infty}$ is bounded by
\[
c\lambda^{\alpha_i^2} H^\delta \left( \sum_{j=1}^{\infty} \sum_{\xi \in \mathbb{Z}^n} 2^{j\frac{p}{q}}(t_{j,z} \delta^M(\cdot + 2^{1-j}I_n)f)^q \chi_{j,z} \right)^{\frac{1}{q}} \|_{L_p(\mathbb{R}^n)}.
\]
Using the fact that
\[
\tilde{\delta}^M(\cdot + 2^{1-j}I_n)f \leq \delta^M(Q_{j,z})f
\]
and Theorem 3.6 we easily estimate (4.3) by
\[
c\lambda^{\alpha_i^2} H^\delta \| f \|_{F_{\frac{p}{q}}(\mathbb{R}^n),\{t_k\}_{k \in \mathbb{Z}^n}}^\delta,
\]
where the constant $c$ independent of $\lambda$.

Now for any $x \in Q_{k-i,z}$, with $k \in \{1, \ldots, i\}$ and $z \in \mathbb{Z}^n$ we have
\[
\delta^M(x + 2^{i-k+1}I_n)f \leq c2^{(k-i)n} \int_{Q_{k,z}} |f(\nu)| d\nu,
\]
where $c$ is independent of $\lambda$ and $\bar{Q}_{k-i,z} = \bigcap_{j=1}^{n} I_j$ with
\[
I_j = [2^{i-j}(z_j - M - 2), 2^{i-k}(z_j + M + 3)), \quad j = 1, \ldots, n.
\]
Hence
\[
\tilde{\delta}^M(x + 2^{i-k+1}I_n)f \lesssim 2^{(k-i)n} \| f \|_{L_1(\bigcup_{k=1}^{T} Q_{k-i,z})}
\]
for some $T \in \mathbb{N}$. Therefore
\[
\hat{t}_{k-i,z} \delta^M(x + 2^{i-k+1}I_n)f
\]
\[
\lesssim 2^{(k-i)n} \sum_{l=1}^{T_n} \sum_{h \in \mathbb{Z}^n:Q_{0,h} \subset Q_{k-i,z}} \| t_k(\lambda^{-1}) \|_{L_p(Q_{k-i,z})} \| f \|_{L_1(Q_{0,h})}
\]
for any $x \in Q_{k-i,z}$, with $k \in \{1, \ldots, i\}$ and $z \in \mathbb{Z}^n$. Using the fact that $Q_{k-i,z} \subset \bigcup_{l=1}^{T} Q_{k-i,z}$ and Lemma 2.5 (iii) to estimate this expression by
\[
c2^{(k-i)(n - \frac{p}{q})} \sum_{l=1}^{T} \sum_{h \in \mathbb{Z}^n:Q_{0,h} \subset Q_{k-i,z}} \hat{t}_{k,h} \| f \|_{L_1(Q_{0,h})}
\]
\[
\lesssim 2^{(k-i)(n - \frac{p}{q})+\alpha_2} \sum_{l=1}^{T} \sum_{h \in \mathbb{Z}^n:Q_{0,h} \subset Q_{k-i,z}} \tilde{t}_{0,h} \| f \|_{L_1(Q_{0,h})},
\]
by (2.8) with \( \tilde{t}_{k,h} = \| t_k(\lambda^{-1}) \|_{L^p(Q_{0,h})} \). Observe that \((1 + |x - h|)^d \lesssim 2^{(i-k)d}\) for any \( x \in Q_{k-i,z} \) and any \( d \in \mathbb{N} \), such that \( Q_{0,h} \subset Q_{k-i,z_l} \), \( l \in \{1, \ldots, T\} \). Therefore

\[
\tilde{t}_{k-i,z}^M(x + 2^{i-k+1}I^n) f
\]

can be estimated by

\[
c^2(2^{i-k}(n - \frac{d}{\alpha} - d) + \kappa) a_2 \sum_{h \in \mathbb{Z}^n} (1 + |x - h|)^{-d} \tilde{t}_{0,h} \| f \|_{L^1(Q_{0,h})}
\]

for some positive constant \( c \) independent of \( i, x \) and \( k \). Our estimate partially some decomposition techniques already used in [13]. For any \( j \in \mathbb{N} \) and any \( x \in \mathbb{R}^n \), we define

\[
\Omega_j = \{ h \in \mathbb{Z}^n : 2^{j-1} < |x - h| \leq 2^j \} \text{ and } \Omega_0 = \{ h \in \mathbb{Z}^n : |x - h| \leq 1 \}.
\]

Then,

\[
\sum_{h \in \mathbb{Z}^n} \tilde{t}_{0,h} (1 + |x - h|)^{-d} \| f \|_{L^1(Q_{0,h})} = \sum_{j=0}^{\infty} \sum_{h \in \Omega_j} \tilde{t}_{0,h} (1 + |x - h|)^{-d} \| f \|_{L^1(Q_{0,h})}
\]

\[
\lesssim \sum_{j=0}^{\infty} 2^{-dj} \| f \|_{L^1(Q_{0,h})}.
\]

For any \( 0 < \kappa < \min(1, p) \), the last expression is bounded by

\[
c \sum_{j=0}^{\infty} 2^{-dj} \left( \sum_{h \in \Omega_j} \tilde{t}_{0,h} \| f \|_{L^1(Q_{0,h})}^\kappa \right)^{1/\kappa},
\]

which can be rewritten as

\[
c \sum_{j=0}^{\infty} 2^{(n-d)j} \left( 2^{-jn} \int_{\cup_{h \in \Omega_j} Q_{0,h}} \sum_{m \in \Omega_j} \tilde{t}_{0,m} \| f \|_{L^1(Q_{0,m})}^\kappa \chi_{0,m}(y) dy \right)^{1/\kappa}.
\]

(4.6)

If \( y \in \cup_{h \in \Omega_j} Q_{0,h} \), then \( y \in Q_{0,h} \) for some \( h \in \Omega_j \) and \( 2^{j-1} < |x - h| \leq 2^j \). From this it follows that

\[
|y - x| < |y - h| + |x - h| \leq |y - h| + 2^j \lesssim \sqrt{n} + 2^j < 2^{j+\kappa}, \quad \kappa \in \mathbb{N},
\]

which implies that \( y \) is located in some ball \( B(x, 2^{j+\kappa}) \). By taking \( d \) large enough, such that \( d > \frac{\kappa}{\alpha} \), (4.4) does not exceed

\[
c \mathcal{M}_\kappa \left( \sum_{m \in \mathbb{Z}^n} \tilde{t}_{0,m} \| f \|_{L^1(Q_{0,m})}^\kappa \chi_{0,m}(x) \right), \quad x \in \mathbb{R}^n.
\]

Clearly and since \( 0 < \kappa < \min(1, p) \), the last term in \( L_p(\mathbb{R}^n) \)-quasi-norm is bounded by

\[
c^2(2^{i-k}(n - \frac{d}{\alpha} - d) + \kappa) a_2 \left( \sum_{m \in \mathbb{Z}^n} \tilde{t}_{0,m} \| f \|_{L^1(Q_{0,m})}^p \right)^{\frac{1}{p}} \lesssim 2^{(k-i)(n - \frac{d}{\alpha} - d) + \kappa} H \| f \|_{F_{p,q}^M(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})}^p.
\]

Therefore (4.4), with \( \sum_{k=1}^i \) in place of \( \sum_{k=1}^{\infty} \) is bounded by

\[
C \lambda_{\alpha^2} H \| f \|_{F_{p,q}^M(\mathbb{R}^n, \{t_k\}_{k \in \mathbb{N}_0})}^p.
\]
since $\alpha_2 > \max \left( \frac{\alpha}{\beta} - n, \frac{\alpha}{\beta} - \frac{n}{p} \right)$ and for some positive constant $C$ independent of $\lambda$. The proof is complete.

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