Embedded hypersurfaces with constant $m^{\text{th}}$ mean curvature in a unit sphere

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Abstract

In this paper, we study $n$-dimensional hypersurfaces with constant $m^{\text{th}}$ mean curvature in a unit sphere $S^{n+1}(1)$ and construct many compact nontrivial embedded hypersurfaces with constant $m^{\text{th}}$ mean curvature $H_m > 0$ in $S^{n+1}(1)$, for $1 \leq m \leq n - 1$. In particular, if the $4^{\text{th}}$ mean curvature $H_4$ takes value between $\frac{1}{(\tan \frac{\pi}{k})^4}$ and $\frac{k^4 - 4}{n(n - 4)}$ for any integer $k \geq 3$, then there exists an $n$-dimensional $(n \geq 5)$ compact nontrivial embedded hypersurface with constant $H_4$ in $S^{n+1}(1)$.

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1. Introduction

It is well known that Alexandrov [1] and Montiel-Ros [11] proved that the standard round spheres are the only possible oriented compact embedded hypersurfaces with constant $m^{\text{th}}$ mean curvature $H_m$ in a Euclidean space $\mathbb{R}^{n+1}$, for $m \geq 1$. On the other hand, one knows that standard round spheres and Clifford hypersurfaces $S^l(a) \times S^{n-l}(b)$, $1 \leq l \leq n - 1$ are compact embedded hypersurfaces in a unit sphere $S^{n+1}(1)$. Hence, it is natural to ask the following:

Question: Do there exist compact embedded hypersurfaces with constant $m^{\text{th}}$ mean curvature $H_m$ in $S^{n+1}(1)$ other than the standard round spheres and Clifford hypersurfaces?

When $m = 1$, namely, when the mean curvature is constant, Brito-Leite [2] and Perdomo [13] have proved that there exist compact embedded hypersurfaces with constant mean curvature $H$ in $S^{n+1}(1)$, which are not isometric to the standard round spheres and the Clifford hypersurfaces.

For $m = 2$, that is, when the scalar curvature is constant, Leite [7] has proved that there exist compact nontrivial embedded hypersurfaces with constant scalar curvature $R$ satisfying $(n - 1)(n - 2) < R < n(n - 1)$ in $S^{n+1}(1)$. Furthermore, Li-Wei [9] have proved that there exist many compact nontrivial embedded hypersurfaces with constant scalar curvature $R$ satisfying $R > n(n - 1)$ in $S^{n+1}(1)$, recently. But for $m > 2$, one knows little about existence of compact embedded hypersurfaces with constant $m^{\text{th}}$ mean curvature.
curvature $H_m$ in $S^{n+1}(1)$. In this paper, we prove that there exist many compact nontrivial embedded hypersurfaces with constant $m$th mean curvature $H_m > 0$ in $S^{n+1}(1)$, for $1 \leq m \leq n - 1$. In particular, for $m = 4$, we prove that there exist a lot of compact embedded hypersurfaces with constant 4th mean curvature $H_4 > 0$ in $S^{n+1}(1)$ if it takes value between $\frac{1}{(\tan \frac{\pi}{k})^4}$ and $\frac{k^4 - 4}{n(n - 4)}$ for any integer $k \geq 3$. Furthermore, for $m = 1$, our results reduce to the conclusion of Brito-Leite [2] and Perdomo [14]. For $m = 2$, we prove that there are many new compact embedded hypersurfaces with constant scalar curvature satisfying $R > n(n - 1)$ in $S^{n+1}(1)$, other than ones of Li-Wei [9].

2. Preliminaries

Let $M$ be an $n$-dimensional hypersurface of a unit sphere $S^{n+1}(1)$ with constant $m$th mean curvature $H_m$. We choose a local orthonormal frame $\{e_A\}_{1 \leq A \leq n+1}$ in $S^{n+1}$, with dual coframe $\{\omega_A\}_{1 \leq A \leq n+1}$, such that, at each point of $M$, $e_1, \ldots, e_n$ are tangent to $M$ and $e_{n+1}$ is the positively oriented unit normal vector. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \ldots, \leq n + 1; \quad 1 \leq i, j, k, \ldots, \leq n.$$ 

Then the structure equations of $S^{n+1}$ are given by

$$d\omega_A = \sum_{B=1}^{n+1} \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad (2.1)$$

$$d\omega_{AB} = \sum_{C=1}^{n+1} \omega_{AC} \wedge \omega_{CB} - \omega_A \wedge \omega_B. \quad (2.2)$$

When restricted to $M$, we have $\omega_{n+1} = 0$ and

$$0 = d\omega_{n+1} = \sum_{i=1}^{n} \omega_{n+1i} \wedge \omega_i. \quad (2.3)$$

By Cartan’s lemma, there exist functions $h_{ij}$ such that

$$\omega_{n+1} = \sum_{j=1}^{n} h_{ij} \omega_j, \quad h_{ij} = h_{ji}. \quad (2.4)$$

This gives the second fundamental form of $M$, $B = \sum_{i,j} h_{ij} \omega_i \omega_j e_{n+1}$. The mean curvature $H$ is defined by $H = \frac{1}{n} \sum_i h_{ii}$. From (2.1)-(2.4) we obtain the structure equations of $M$ (see [8])

$$d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \quad (2.5)$$

$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l=1}^{n} R_{ijkl} \omega_k \wedge \omega_l. \quad (2.6)$$
and the Gauss equations
\[ R_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} + (h_{ik} h_{jl} - h_{il} h_{jk}). \]  
(2.7)

\[ R - n(n-1) = n(n-1)(r-1) = n^2 H^2 - S. \]  
(2.8)

where \( R_{ijkl} \) denotes the components of the Riemannian curvature tensor of \( M \), \( R = n(n-1)r \) is the scalar curvature of \( M \) and \( S = \sum_{i,j=1}^{n} h_{ij}^2 \) is the square norm of the second fundamental form of \( M \).

Let \( h_{ijkl} \) denote the covariant derivative of \( h_{ij} \). We then have (see [8])
\[ \sum_{k} h_{ijkl} \omega_{k} = dh_{ij} + \sum_{k} h_{kj} \omega_{kl} + \sum_{k} h_{ik} \omega_{kj}. \]  
(2.9)

Thus, by exterior differentiation of (2.4), we obtain the Codazzi equation (see [8])
\[ h_{ijk} = h_{ikj}. \]  
(2.10)

We choose \( e_1, \ldots, e_n \) such that
\[ h_{ij} = \lambda_i \delta_{ij}. \]  
(2.11)

Let \( H_m \) be \( m^{th} \) mean curvature of \( M \), then we have
\[ C_n^m H_m = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \lambda_{i_1} \cdots \lambda_{i_m}, \]  
(2.12)

where \( C_n^m = \frac{n!}{m!(n-m)!} \).

In [12], Otsuki proved the following

**Lemma 2.1 ([12])**. Let \( M \) be an \( n \)-dimensional hypersurface in a unit sphere \( S^{n+1} \) such that the multiplicities of principal curvatures are all constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.

From Lemma 2.1, we can easily obtain the following theorem.

**Theorem 2.1.** Let \( M \) be an \( n \)-dimensional oriented complete hypersurface in a unit sphere \( S^{n+1} \) with constant \( m^{th} \) mean curvature \( H_m \) and with two distinct principal curvatures. If the multiplicities of these two distinct principal curvatures are greater than 1, then \( M \) is isometric to Riemannian product \( S^k(a) \times S^{n-k}(b), 2 \leq k \leq n-2 \).
3. A representation formula of principal curvatures

Now, let us consider that $M$ is an $n$-dimensional oriented hypersurface with constant $m^\text{th}$ mean curvature $H_m$ and with two distinct principal curvatures in $S^{n+1}$. If multiplicities of these two distinct principal curvatures are all great than 1, then we can deduce from Theorem 2.1 that $M$ is isometric to $S^k(a) \times S^{n-k}(b)$, $2 \leq k \leq n - 2$. Hence, we shall assume that one of these two distinct principal curvatures is simple, that is, we assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu. \quad (3.1)$$

Since $H_m$ is constant, we obtain from (2.12) that

$$C_n^m H_m = C_{n-1}^m \lambda^m + C_{n-1}^{m-1} \lambda^{m-1} \mu. \quad (3.2)$$

By Lemma 2.1, let us denote the integral submanifold through $x \in M$, corresponding to $\lambda$ by $M^{n-1}_i(x)$. We write

$$d\lambda = \sum_i \lambda_i \omega_i, \quad d\mu = \sum_j \mu_j \omega_j. \quad (3.3)$$

We assume that $\lambda > 0$ on $M$. Then Lemma 2.1 implies

$$\lambda_1 = \cdots = \lambda_{n-1} = 0. \quad (3.4)$$

Then (3.2) yields

$$\mu = \frac{C_n^m H_m - C_{n-1}^m \lambda^m}{C_{n-1}^{m-1} \lambda^{m-1}} = \frac{nH_m - (n - m)\lambda^m}{m\lambda^{m-1}}, \quad (3.5)$$

and from the formula

$$\lambda - \mu = \frac{n(\lambda^m - H_m)}{m\lambda^{m-1}}, \quad (3.6)$$

we obtain that

$$\lambda^m - H_m \neq 0. \quad (3.7)$$

By means of (2.9) and (2.11), we obtain

$$\sum_k h_{ijk} \omega_k = \delta_{ij} d\lambda_j + (\lambda_i - \lambda_j) \omega_{ij}. \quad (3.8)$$

We adopt the notational convention that

$$1 \leq a, b, c, \cdots \leq n - 1.$$

From (3.1), (3.2) and (3.8), we have

$$h_{ijk} = 0, \quad \text{if } i \neq j, \quad \lambda_i = \lambda_j, \quad (3.9)$$

$$h_{aab} = 0, \quad h_{aan} = \lambda_n, \quad (3.10)$$

$$h_{nna} = 0, \quad h_{nnn} = \mu_n. \quad (3.11)$$
Combining this with (2.10) and the formula
\[
\sum_i h_{ani} \omega_i = dh_{an} + \sum_i h_{in} \omega_{ia} + \sum_i h_{ai} \omega_{im} = (\lambda - \mu) \omega_{an},
\]
we obtain from (3.10) and (3.6)
\[
\omega_{an} = \frac{\lambda_n}{\lambda - \mu} \omega_a = \frac{m \lambda^{m-1} \lambda_n}{n(\lambda^m - H_m)} \omega_a.
\]
Therefore we have
\[
d\omega_n = \sum_a \omega_{na} \wedge \omega_a = 0.
\]

Notice that we may consider \( \lambda \) to be locally a function of the parameter \( s \), where \( s \) is the arc length of an orthogonal trajectory of the family of the integral submanifolds corresponding to \( \lambda \). We may put
\[
\omega_n = ds.
\]
Thus, for \( \lambda = \lambda(s) \), we have
\[
d\lambda = \lambda_n ds, \quad \lambda_n = \lambda'(s).
\]
From (3.6) and (3.13), we get
\[
\omega_{an} = \frac{m \lambda^{m-1} \lambda_n}{n(\lambda^m - H_m)} \omega_a = \frac{m \lambda^{m-1} \lambda'(s)}{n(\lambda^m - H_m)} \omega_a = \{ \log |\lambda^m - H_m|^{1/n} \} \omega_a,
\]
which shows that the integral submanifolds \( M_i^{n-1}(x) \) corresponding to \( \lambda \) is umbilical in \( M \) and \( S^{n+1} \).

On the other hand, we can deduce from (3.16) that
\[
\nabla_{e_n} e_n = \sum_{k=1}^n \omega_{ni}(e_n) e_i = 0.
\]

According to the definition of geodesic, we know that the integral curve of the principal vector field \( e_n \) corresponding to the principal curvature \( \mu \) is a geodesic.

This proves the following result:

**Lemma 3.1.** If \( M \) is an \( n \)-dimensional oriented complete hypersurface \( (n \geq 3) \) in \( S^{n+1} \) with constant \( m^{th} \) mean curvature \( H_m \) and with two distinct principal curvatures, one of which is simple, then

1. the integral submanifold \( M_i^{n-1}(x) \) through \( x \in M \) corresponding to \( \lambda \) is umbilical in \( M \) and \( S^{n+1} \),
2. the integral curve of the principal vector field \( e_n \) corresponding to the principal curvature \( \mu \) is a geodesic.

Now we state our Theorem 3.1 as follows:
Theorem 3.1. If $M$ is an $n$-dimensional oriented complete hypersurface $(n \geq 3)$ in $S^{n+1}$ with constant $m^{th}$ mean curvature $H_m$ and with two distinct principal curvatures one of which is simple, then $M$ is isometric to a complete hypersurface of $S^{n-1}(c(s)) \times M^1$, where $S^{n-1}(c(s))$ is of constant curvature $[(\log |\lambda^n - H_m|^{1/n})^2 + \lambda^2 + 1$. And $w = |\lambda^n - H_m|^{-1/n}$ satisfies the following ordinary differential equation of order 2:

$$
\frac{d^2w}{ds^2} - w \left\{ \frac{(n-m)(w^{-n} + H_m)(2-m)/m}{mw^n} - H_m(w^{-n} + H_m)(2-m)/m - 1 \right\} = 0. \quad (3.17)
$$

The proof of Theorem 3.1. According to the structure equations of $S^{n+1}$ and (3.16), we may compute

$$
d\omega_{an} = \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_{bn} + \omega_{an+1} \wedge \omega_{n+1n} - \omega_a \wedge \omega_n
$$

$$
= (\log |\lambda^n - H_m|^{1/n})' \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_{b} - \lambda \omega_a \wedge ds - \omega_a \wedge ds,
$$

$$
d\omega_{an} = d[(\log |\lambda^n - H_m|^{1/n})' \omega_a]
$$

$$
= \{ (\log |\lambda^n - H_m|^{1/n})'' ds \wedge \omega_a + (\log |\lambda^n - H_m|^{1/n})' d\omega_a
$$

$$
= \left\{ -(\log |\lambda^n - H_m|^{1/n})'' + \left[ (\log |\lambda^n - H_m|^{1/n})' \right]^2 \right\} \omega_a \wedge ds
$$

$$
+ (\log |\lambda^n - H_m|^{1/n})' \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_b.
$$

Then we obtain from two equalities above that

$$
\{ (\log |\lambda^n - H_m|^{1/n})'' \}'' - \left[ (\log |\lambda^n - H_m|^{1/n})' \right]^2 - \lambda \mu - 1 = 0. \quad (3.18)
$$

Combining (3.18) with (3.6), we have

$$
\{ (\log |\lambda^n - H_m|^{1/n})'' \}'' - \left[ (\log |\lambda^n - H_m|^{1/n})' \right]^2 + \frac{(n-m)\lambda^n - nH_m}{m\lambda^{m-2}} - 1 = 0. \quad (3.19)
$$

We know that $\lambda^n - H_m \neq 0$. If $\lambda^n - H_m < 0$, from (3.6), we have

$$
\lambda^2 - \lambda \mu = \frac{n(\lambda^n - H_m)}{m\lambda^{m-2}} < 0, \quad H_m > 0. \quad (3.20)
$$

According to the Gauss equation (2.7), we know that the sectional curvature of $M$ is not less than 1 and $H_m > 0$. By a direct calculation, we know that $M$ is isometric to a totally umbilical hypersurface. This is impossible because $M$ has two distinct principal curvatures. Hence, $\lambda^n - H_m > 0$. Let us define a positive function $w(s)$ over $s \in (-\infty, +\infty)$ by

$$
w = (\lambda^n - H_m)^{-1/n}, \quad (3.21)$$

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then (3.19) reduces to
\[ \frac{d^2 w}{ds^2} - w \left\{ \frac{(n - m)(w^{-n} + H_m)^{(2-m)/m}}{m w^n} - H_m(w^{-n} + H_m)^{(2-m)/m} - 1 \right\} = 0. \] (3.22)

Integrating (3.22), we obtain
\[ \left( \frac{dw}{ds} \right)^2 = C - w^2 (w^{-n} + H_m)^{\frac{n}{m}}, \] (3.23)
where \( C \) is the constant of integration.

We consider the frame \{\( x, e_1, e_2, \cdots, e_n, e_{n+1} \)\} in the Euclidean space \( \mathbb{R}^{n+2} \). Then, by (2.4), (3.13) and (3.18), we obtain
\[
d e_a = \sum_{b=1}^{n-1} \omega_{ab} e_b + \omega_{an} e_n + \omega_{an+1} e_{n+1} - \omega_a e_{n+2} \\
= \sum_{b=1}^{n-1} \omega_{ab} e_b + (\log | \lambda^m - H_m |^{1/n})' \omega_a e_n - \lambda \omega_a e_{n+1} - \omega_a e_{n+2} \\
= \sum_{b=1}^{n-1} \omega_{ab} e_b + \left\{ \left( \log | \lambda^m - H_m |^{1/n} \right)' - \lambda e_{n+1} - e_{n+2 \text{ mod } \{e_1, \cdots, e_{n-1}\}} \right\} \omega_a \\
\]
\[ d \left\{ \left( \log | \lambda^m - H_m |^{1/n} \right)' - \lambda e_{n+1} - e_{n+2 \text{ mod } \{e_1, \cdots, e_{n-1}\}} \right\} \]
\[ = \left\{ \left( \log | \lambda^m - H_m |^{1/n} \right)^{''} - \lambda \mu - 1 \right\} e_n \omega_n \\
= \left\{ \lambda' + (\log | \lambda^m - H_m |^{1/n})' \mu \right\} e_{n+1} \omega_n \ (\text{mod } \{e_1, \cdots, e_{n-1}\}) \\
= \left( \log | \lambda^m - H_m |^{1/n} \right)' \left\{ \left( \log | \lambda^m - H_m |^{1/n} \right)' e_n - \lambda e_{n+1} - e_{n+2 \text{ mod } \{e_1, \cdots, e_{n-1}\}} \right\} ds \\
\]

By putting
\[ W = e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1} \wedge \left\{ \left( \log | \lambda^m - H_m |^{1/n} \right)' e_n - \lambda e_{n+1} - e_{n+2 \text{ mod } \{e_1, \cdots, e_{n-1}\}} \right\}, \] (3.24)
we can show that
\[ dW = (\log | \lambda^m - H_m |^{1/n} \right)' W ds. \] (3.25)
(3.25) shows that $n$-vector $W$ in $\mathbb{R}^{n+2}$ is constant along $M_1^{n-1}(x)$. Hence there exists an $n$-dimensional linear subspace $E^n(s)$ in $\mathbb{R}^{n+2}$ containing $M_1^{n-1}(x)$. (3.25) also implies that the $n$-vector field $W$ only depends on $s$ and by integrating it, we get

$$W = \left\{ \frac{\lambda^m(s) - H_m}{\lambda^m(s_0) - H_m} \right\}^{1/n} W(s_0).$$

Therefore, we have that $E^n(s)$ is parallel to $E^n(s_0)$ in $\mathbb{R}^{n+2}$ for every $s$.

From the calculation

$$d\omega_{ab} - \sum_{c=1}^{n-1} \omega_{ac} \wedge \omega_{cb} = \omega_{an} \wedge \omega_{nb} + \omega_{an+1} \wedge \omega_{n+1b} - \omega_a \wedge \omega_b \Rightarrow -\left\{ [(\log |\lambda^m - H_m|^{1/n})]^2 + \lambda^2 + 1 \right\} \omega_a \wedge \omega_b,$$

we see that the curvature of $M_1^{n-1}(x)$ is $[(\log |\lambda^m - H_m|^{1/n})]^2 + \lambda^2 + 1$ and $M_1^{n-1}(x)$ is locally isometric to $S^{n-1}(c(s))$. Therefore, $M$ is isometric to a complete hypersurface of revolution $S^{n-1}(c(s)) \times M^1$.

This proves Theorem 3.1.

4. A representation formula of radius

One knows that the following immersion:

$$x : M^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2},$$

$$(s, t_1, \cdots, t_{n-1}) \mapsto (y_1(s) \varphi_1, \cdots, y_1(s) \varphi_n, y_{n+1}(s), y_{n+2}(s)).$$

$$\varphi_i = \varphi_i(t_1, \cdots, t_{n-1}), \quad \varphi_1^2 + \cdots + \varphi_n^2 = 1$$

is a parametrization of a rotational hypersurface generated by a curve $(y_1(s), y_{n+1}(s), y_{n+2}(s))$. Since the curve $(y_1(s), y_{n+1}(s), y_{n+2}(s))$ belongs to $S^2(1)$ and the parameter $s$ can be chosen as its arc length, we have

$$y_1^2(s) + y_{n+1}^2(s) + y_{n+2}^2(s) = 1, \quad \dot{y}_1^2(s) + \dot{y}_{n+1}^2(s) + \dot{y}_{n+2}^2(s) = 1$$

where the dot denotes the derivative with respect to $s$ and from (4.3) we can obtain $y_{n+1}(s)$ and $y_{n+2}(s)$ as functions of $y_1(s)$. In fact, we can write

$$y_1(s) = \cos \vartheta(s), \quad y_{n+1}(s) = \sin \vartheta(s) \cos \theta(s), \quad y_{n+2}(s) = \sin \vartheta(s) \sin \theta(s).$$

We can deduce from (4.3) that

$$\dot{\vartheta}^2 + \dot{\theta}^2 \sin^2 \vartheta = 1.$$

It follows from equation (4.5) that $\dot{\vartheta}^2 \leq 1$. Combining these with $\dot{\vartheta}^2 = \frac{\dot{y}_1^2}{1-y_1^2}$, we have

$$\dot{y}_1^2 + y_1^2 \leq 1.$$
We can get the plane curve $\zeta$ from $\alpha$ by projection of $S^2_+ = \{(y_1, y_{n+1}, y_{n+2}) \mid y_1 \geq 0, y_1^2 + y_{n+1}^2 + y_{n+2}^2 = 1\}$ onto the unit disk $E = \{(y_{n+1}, y_{n+2}) \mid y_{n+1}^2 + y_{n+2}^2 \leq 1\}$. Then the plane curve $\zeta$ can be written as
\[
y_{n+1}(s) = \sin \vartheta(s) \cos \theta(s), \quad y_{n+2}(s) = \sin \vartheta(s) \sin \theta(s).
\] (4.7)

Writing $r(s) = y_1(s)$, (4.5) can be written as
\[
\dot{\vartheta}^2 = \frac{1 - \dot{\vartheta}^2}{\sin^2 \vartheta} = \frac{1 - r^2 - \dot{r}^2}{(1 - r^2)^2}.
\] (4.8)

Do Carmo and Dajczer proved the following

**Lemma 4.1** ([3]). Let $M^n$ be a rotational hypersurface of $S^{n+1}(1)$. Then the principal curvatures $\lambda_i$ of $M^n$ are
\[
\lambda_i = \lambda = -\frac{\sqrt{1 - r^2 - \dot{r}^2}}{r}
\] for $i = 1, \cdots, n - 1$, and
\[
\lambda_n = \mu = \frac{\dot{r} + r}{\sqrt{1 - r^2 - \dot{r}^2}}
\] (4.9) (4.10)

On the other hand, let us fix a point $p_0 \in M$, let $\gamma(u)$ be the only geodesic in $M$ such that $\gamma(0) = p_0$ and $\gamma'(0) = e_n(p_0)$. From (3.16) of Section 3, we know that $\gamma(u) = e_n(\gamma(u))$. Note that $\gamma(u)$ is also a line of curvature. Let us denote by $g(u) = w(\gamma(u))$. Since $H_m$ is constant, we know from (3.23) that
\[
(g')^2 + g^2 (g^{-n} + H_m)^\frac{n}{m} + g^2 = C.
\] (4.11)

From (4.11), we have $C > 0$. Moreover, by a direct calculation, we get
\[
q(x) = C - x^2 (x^{-n} + H_m)^\frac{2}{m} - x^2
\] is positive on a interval $(t_1, t_2)$ with $0 < t_1 < t_2$ and $q(t_1) = q(t_2) = 0$. From (4.11), we know that $g(u)$ is periodic. And the period is the following
\[
T = 2 \int_{t_1}^{t_2} \frac{dt}{\sqrt{C - t^2 (t^{-n} + H_m)^\frac{2}{m} - t^2}}.
\] (4.13)

From (4.1) and Theorem 3.1, we have
\[
\frac{1}{r^2} = [(\log | \lambda^n - H_m |^{1/n})']^2 + \lambda^2 + 1.
\]
Then we know form (3.23), (4.11) that
\[
r(u) = \frac{g(u)}{\sqrt{C}}, \quad g(u) = (\lambda^n - H_m)^{\frac{1}{n}}.
\] (4.14)

From (4.8), (4.9) and (4.11), we obtain the period $P(H_m, n, c)$ of hypersurfaces
\[ P(H_m, n, C) = \theta(T) = \int_{0}^{T} \frac{\sqrt{1 - r^2 - \dot{r}^2}}{1 - r^2} \, ds \]
\[ = \int_{0}^{T} \frac{r(s)\lambda(s)}{1 - r^2(s)} \, ds \]
\[ = 2 \int_{0}^{T} \frac{r(s)\lambda(s)}{1 - r^2(s)} \, ds. \]  

(4.15)

5. Embedded hypersurfaces with constant \( H_m > 0 \)

At first, we give the following Lemma due to Perdomo [14]

**Lemma 5.1.** Let \( \epsilon \) and \( \delta \) be positive numbers and \( f : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R} \) and \( y : (-\delta, \delta) \times (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R} \) be two smooth functions such that \( f(t_0) = f'(t_0) = 0 \) and \( f''(t_0) = -2a < 0 \). If for any small \( c > 0 \), \( t_1(c) < t_0 < t_2(c) \) are such that \( f(t_1(c)) + c = 0 = f(t_2(c)) + c \), then

\[ \lim_{c \to 0^+} \int_{t_1(c)}^{t_2(c)} \frac{y(c, t) \, dt}{\sqrt{f(t) + c}} = \frac{y(0, t_0)\pi}{\sqrt{a}}. \]

Now we state our main theorem.

**Theorem 5.1.** For any \( n \geq 5 \) and any integer \( k \geq 3 \), if 4th mean curvature \( H_4 \) takes value between \( \frac{1}{(\tan \frac{\pi}{k})^4} \) and \( \frac{k^4 - 4}{n(n - 4)} \), then there exists an \( n \)-dimensional compact nontrivial embedded hypersurface with constant \( H_4 > 0 \) in \( S^{n+1}(1) \).

**Proof.** Let us rewrite (4.11) as

\[ (g')^2 = q(g), \quad \text{where} \quad q(v) = C - v^2(v^{-n} + H_m)^{\frac{2}{m}} - v^2. \]  

(5.1)

We know that for some value of \( C \), the function \( q \) has positive values between two positive roots of \( q \), denoted by \( t_1 \) and \( t_2 \). A direct calculation shows that

\[ q'(v) = 2v \left\{ -(v^{-n} + H_m)^{\frac{2}{m}} + \frac{n}{m}v^{-n}(v^{-n} + H_m)^{\frac{2}{m}} - 1 \right\}. \]  

(5.2)

\[ q''(v) = -\frac{2(v^{-n} + H_m)^{\frac{2-2m}{m}}}{m^2} \left( (2n^2 - 3nm + m^2)v^{-2n} + m(n^2 - 3n + 2m)H_m v^{-n} + m^2 H_m^2 \right) - 2 < -2. \]  

(5.3)

If \( m = 4 \) and \( H_4 = 1 \), we have the only positive root of \( q' \) is

\[ v_0 = \left( \frac{(n - 4)^2}{8n - 16} \right)^{\frac{1}{n}}. \]  

(5.4)
Therefore, for positive values of $v$, the function $q$ increase from 0 to $v_0$ and decrease for values greater than $v_0$. Then we obtain that $q(v_0) = C - c_0$, where

$$c_0 = v_0^2((v_0^n + 1)\frac{1}{2} + 1) = \left(\frac{(n-4)^2}{8n-16}\right)^{\frac{1}{2}} \times \left(\frac{n}{n-4} + 1\right). \tag{5.5}$$

Therefore, whenever $C > c_0$, we will have the two positive roots of the function $q(v)$ that we will denote by $t_1(C)$ and $t_2(C)$. By computing, we have $q''(v_0) = -2a$, where

$$a = \frac{2(n-2)^2}{n}. \tag{5.6}$$

Hence, we get from (4.15) that

$$P(H_4, n, C) = 2 \int_0^{\frac{\pi}{2}} r(s)\lambda(s) ds. \tag{5.7}$$

Since $r(s) = g(s)$ and $\lambda(s) = (g^{-n} + 1)\frac{1}{2}$, we have

$$P(H_4, n, C) = 2 \int_0^{\pi} \sqrt{C}g(s)(g^{-n}(s) + 1)\frac{1}{2} ds. \tag{5.8}$$

Since $g(0) = t_1(C)$ and $g(T) = t_2(C)$, by doing the substitutions $t = g(s)$, we have

$$P(H_4, n, C) = 2 \int_{t_1(C)}^{t_2(C)} \sqrt{C}(t^{-n} + 1)\frac{1}{2} \frac{1}{\sqrt{q(t)}} ds. \tag{5.9}$$

Using Lemma 5.1, we have

$$\lim_{C \to c_0^+} P(H_4 = 1, n, C) = \frac{2\pi}{\sqrt{a}} \frac{\sqrt{c_0}v_0(v_0^n + 1)\frac{1}{2}}{c_0 - v_0^2} = \frac{2\pi\sqrt{n-2}}{n-2}. \tag{5.10}$$

On the other hand, we will estimate $P(H_m, n, C)$ when $C \to \infty$, we make the substitution $t = r(s)$ and obtain

$$P(H_m, n, C) = 2 \int_{t_1(C)}^{t_2(C)} \frac{t((\sqrt{C}t)^{-n} + H_m)^{\frac{1}{m}}}{(1-t^2)\sqrt{1-t^2(1 + \frac{H_m + (\sqrt{C}t)^{-n})^{\frac{2}{m}}}}). \tag{5.11}$$

Since

$$\tilde{q} = 1 - t^2(1 + (H_m + (\sqrt{C}t)^{-n})^{\frac{2}{m}}) \tag{5.12}$$

have two positive roots converge to 0 and $\frac{1}{\sqrt{1+H_m}}$, we obtain

$$\lim_{C \to \infty} P(H_m, n, C) = 2 \int_0^{\frac{1}{\sqrt{1+H_m}}} tH_m^{\frac{1}{m}} dt = 2 \arctan \frac{1}{H_m}. \tag{5.13}$$
If \( m = 4 \) and \( H_4 = 1 \), we have that

\[
\lim_{C \to \infty} P(H_4 = 1, n, C) = 2 \arctan \frac{1}{(H_4)^{\frac{1}{4}}} = \frac{\pi}{2}.
\]  

(5.14)

Next, we consider the case \( m = 4 \) and \( 0 < H_4 \neq 1 \).

In this case, we have the only positive root of \( q' \) is

\[
v_0 = \left( \frac{\sqrt{n(n - 4)H_4 + 4 - nH_4 + 4H_4 - 2}}{4H_4(1 - H_4)} \right)^\frac{1}{n}.
\]  

(5.15)

A direct calculation shows that \( q(v_0) = C - c_0 \), where

\[
c_0 = v_0^2(v_0^{-n} + H_4)^\frac{1}{2} + v_0^2
\]

\[
= \left( \frac{\sqrt{n(n - 4)H_4 + 4 - nH_4 + 4H_4 - 2}}{4H_4(1 - H_4)} \right)^\frac{1}{n}
\]

\[
\times \left\{ \frac{H_4(\sqrt{n(n - 4)H_4 + 4 - nH_4 + 2})}{\sqrt{n(n - 4)H_4 + 4 - nH_4 + 4H_4 - 2}} \right\}^\frac{1}{2} + 1 \right].
\]  

(5.16)

\[
q''(v_0) = -2a
\]

\[
= \frac{-2H_4^\frac{1}{n}}{(\sqrt{n(n - 4)H_4 + 4 - nH_4 + 2})^\frac{1}{n}} \times \left\{ n^2(n - 4)H_4^2 + n(-n^2 + 4n + 4)H_4 - 4n + [n^2 - 2n + (-n^2 + 2n)H_4]\sqrt{n(n - 4)H_4 + 4} \right\}.
\]  

(5.17)

Therefore, whenever \( C > c_0 \), we will have two positive roots of the function \( q(v) \) that we will denote by \( t_1 \) and \( t_2 \).

Using the results of section 4, we have from (4.15) that

\[
P(H_4, n, C) = 2 \int_0^T \frac{r(s)\lambda(s)}{1 - r^2(s)} ds.
\]  

(5.18)

From (4.14), we have \( r(s) = g(s)^{-n} \sqrt{\frac{C}{C - g^2(s)}} \) and \( \lambda(s) = (g(s)^{-n} + H_m)^{\frac{1}{n}} \), then we get that

\[
P(H_4, n, C) = 2 \int_0^T \sqrt{C}g(s)(g(s)^{-n} + H_4)^{\frac{1}{n}} ds.
\]  

(5.19)

Since \( g(0) = t_1 \) and \( g(\frac{T}{2}) = t_2 \), by doing the substitution \( t = g(s) \) and using Lemma 5.1, we obtain
\[
\lim_{c \to c_0} P(H_4, n, C) = \frac{2\pi \sqrt{c_0}}{\sqrt{a \sqrt{c_0 - v_0^2}}}
\]

\[
= 2\pi \frac{\sqrt{(n - 2)(n - nH_4) + (nH_4 - n)\sqrt{n(n - 4)H_4 + 4}}}{\sqrt{n^2(n - 4)H_4^2 + n(-n^2 + 4n + 4)H_4 - 4n + [n^2 - 2n + (-n^2 + 2n)H_4]\sqrt{n(n - 4)H_4 + 4}}^{1/2}}
\]

\[
= 2\pi \frac{|(n - 2) - \sqrt{n(n - 4)H_4 + 4}|^{1/2}}{|(n(n - 4)H_4 - 4) + (n - 2)\sqrt{n(n - 4)H_4 + 4}|^{1/2}}
\]

\[
= \frac{2\pi}{|n(n - 4)H_4 + 4|^{1/2}}.
\]

On the other hand, we know that

\[
\lim_{c \to \infty} P(H_4, n, C) = 2 \arctan \frac{1}{H_4^{1/4}}.
\]  

(5.20)

Therefore, for any fixed \(H_4 > 0\), the function \(P(H_4, n, C)\) takes all the values between

\[
A(H_4) = 2 \arctan \frac{1}{H_4^{1/4}}, \quad B(H_4) = \frac{2\pi}{|n(n - 4)H_4 + 4|^{1/4}}.
\]

(5.22)

By a direct computation, we know that \(A(H_4)\) and \(B(H_4)\) are decreasing functions. Since

\[
A\left(\frac{1}{(\tan \frac{k}{2})^4}\right) = B\left(\frac{k^4 - 4}{n(n - 4)}\right) = \frac{2\pi}{k},
\]

(5.23)

where \(k \geq 3\) is any integer, we deduce that the number \(\frac{2\pi}{k}\) lies between \(A(H_4)\) and \(B(H_4)\) since they are decreasing functions, hence, for some constant \(C_1\), we have that \(P(H_4, n, C_1) = \frac{2\pi}{k}\). If the period is \(\frac{2\pi}{k}\), then there exists a compact embedded hypersurfaces with constant \(H_4\) which is not isometric to a round sphere or a Clifford hypersurface.

We complete the proof of Theorem 5.1.

For constant \(H_m > 0\), we can prove the following

**Theorem 5.2.** For any integer \(1 \leq m \leq n - 1\), there exist many nontrivial embedded hypersurfaces with constant \(H_m > 0\) in \(S^{n+1}(1)\).

**Proof.** By using the similar arguments with the proof of Theorem 5.1, we have that

\[
v_0 = \left(\frac{n - m}{m}\right)^{\frac{m}{n}}, \quad c_0 = \left(\frac{n - m}{m}\right)^{\frac{m}{n}} \times \frac{n}{n - m},
\]

\[
q''(v_0) = -2a = -\frac{4n}{m}.
\]

From (4.14), we have \(r = \frac{g}{\sqrt{C}}\) and \(\lambda = g^{-\frac{n}{m}}\), then we obtain
\[
\lim_{C \to c_0^+} P(H_m = 0, n, C) = \frac{2\pi \sqrt{c_0}}{\sqrt{a} \sqrt{c_0 - v_0^2}} = \sqrt{2\pi},
\]

by continuity arguments, we can fix \(H_m\) sufficiently small such that
\[
\lim_{C \to c_0^+} P(H_m, n, C) > \pi.
\]

On the other hand, we deduce from \(H_m > 0\) that
\[
\lim_{C \to \infty} P(H_m, n, C) = 2 \arctan \frac{1}{H_m^2} < \pi.
\]

Hence, there exists \(C_2\), such that \(P(H_m, n, C_2) = \pi\).

We complete the proof of Theorem 5.2.

**Remark 5.1.** When \(m = 1\), Theorem 5.2 reduces to the results of Brito and Leite [2].

Using the similar arguments as above, we can obtain the following:

When \(m = 1\), we have

**Proposition 5.1 ([14]).** For any \(n \geq 2\) and any integer \(k \geq 2\), if mean curvature \(H\) takes value between \(\frac{1}{\left(\tan \frac{\pi}{k}\right)^2}\) and \(\frac{(k^2 - 2)\sqrt{n - 1}}{n\sqrt{k^2 - 1}}\), then there exists an \(n\)-dimensional compact nontrivial embedded hypersurface with constant mean curvature \(H > 0\) in \(S^{n+1}(1)\).

**Remark 5.2.** Proposition 5.1 is also proved by Perdomo [14].

When \(m = 2\), we have

**Proposition 5.2.** For any \(n \geq 3\) and any integer \(k \geq 2\), if \(H_2 = \frac{R-n(n-1)}{n(n-1)}\) takes value between \(\frac{1}{(\tan \frac{\pi}{k})^2}\) and \(\frac{k^2 - 2}{n}\), then there exists an \(n\)-dimensional compact nontrivial embedded hypersurface \(M\) with constant 2-th mean curvature \(H_2 > 0\) (i.e. scalar curvature \(R > n(n-1)\)) in \(S^{n+1}(1)\), where \(R\) is the scalar curvature of \(M\).

**Proof.** By using the similar arguments with the proof of Theorem 5.1, we have that
\[
q''(v_0) = -2a = -2n(H_2 + 1),
\]

\[
\lim_{C \to c_0^+} P(H_2, n, C) = \frac{2\pi \sqrt{c_0}}{\sqrt{a} \sqrt{c_0 - v_0^2}} = \frac{2\pi}{\sqrt{nH_2 + 2}}.
\]

On the other hand,
\[
\lim_{C \to \infty} P(H_2, n, C) = 2 \arctan \frac{1}{H_2^2}.
\]
Therefore, for any fixed $H_2 > 0$, the function $P(H_2, n, C)$ takes all the values between

$$E(H_2) = 2 \arctan \frac{1}{H_2^2}, \quad F(H_2) = \frac{2\pi}{\sqrt{nH_2 + 2}}$$

By a direct computation, we know that $E(H_2)$ and $F(H_2)$ are decreasing functions. Since

$$E\left( \frac{1}{(\tan \frac{\pi}{k})^2} \right) = F\left( \frac{k^2 - 2}{n} \right) = \frac{2\pi}{k},$$

where $k \geq 2$ is any integer, we deduce that the number $\frac{2\pi}{k}$ lies between $E(H_4)$ and $F(H_4)$ since they are decreasing functions, hence, for some constant $C_3$, we have that $P(H_2, n, C_3) = \frac{2\pi}{k}$. If the period is $\frac{2\pi}{k}$, then there exists a compact embedded hypersurfaces with constant $H_2$ (i.e. constant scalar curvature) which is not isometric to a round sphere or a Clifford hypersurface.

We complete the proof of Proposition 5.2.

**Remark 5.3.** Since $H_2 = \frac{R-n(n-1)}{n(n-1)}$, by a direct calculation, we know that when $3 \leq n \leq 6$, Proposition 5.2 reduces to Theorem 1.1 and Theorem 1.2 due to Li-Wei [9]; when $n > 6$ and $k = 2$, Proposition 5.2 reduces to Theorem 1.3 due to Li-Wei [9]. In Proposition 5.2, we find there exist a lot of new examples satisfying $R > n(n - 1)$. Hence, Proposition 5.2 is the generalization of Li-Wei’s results [9].

**Remark 5.4.** For some special $4 \neq m > 3$, we can also obtain some nontrivial embedded hypersurfaces with $H_m = \text{constant}$ in $S^{n+1}(1)$ using the same methods.
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