Physics of glassy systems

Giorgio Parisi\textsuperscript{a} ∗

\textsuperscript{a}Dipartimento di Fisica, Università La Sapienza
and INFN Sezione di Roma I
Piazzale Aldo Moro 2, 00185 Roma (Italy)

In this talk I present some of the recent theoretical results that have been obtained on glassy systems like spin glasses or structural glasses. The physical principles at the basis of the theory are explained in a simple language (without using replicas) and the results are compared with large scale numerical simulations. Finally we introduce the generalized fluctuation dissipation relation that can be directly tested in experiments with the present day technology.

1. Introduction

In this talk I will take the point of view that glassiness (roughly speaking the appearance, when decreasing the temperature, of a very large equilibration time, much longer of those that can be observed on human scale) is related to metastability and to the presence of many equilibrium states.

Replica theory \cite{1,2} is the most powerful tool to deal with systems with many equilibrium states. Here I will present the main results of replica theory stressing the basic ideas, which have been recently recognized to be stochastic stability and separability.

In this talk I will first elaborate on the relations among glassiness and metastability. Later on I will introduce some models of glassy systems (spin glasses, tilings and structural glasses). I will then present the general theoretical interpretation of these phenomena, which can become more quantitative using the two principles of stochastic stability and separability and I will report on some numerical simulations which support these ideas. In the next section I will highlight the connection among this approach and the off equilibrium fluctuation dissipation relations. Finally I will present my conclusions.

2. Glassiness and metastability

It is not simple to give a precise definition of glassiness. In this talk I will take the point of view that glassiness corresponds to the presence of metastability in an open region of parameter space, a new and unusual phenomenon.

Let us recall the usual case in which we have metastability. We consider a system that undergoes a first order phase transition when we change a parameter. The simplest example is the ferromagnetic Ising mode: the control parameter is the magnetic field $h$. At low temperature the magnetization $m(h)$ is given by $m(h) = m_s \text{sign}(h)$ for small $h$ ($m_s$ being the spontaneous magnetization): the magnetization changes discontinuously at $h = 0$ in the low temperature phase where $m_s \neq 0$.

If we slowly change the magnetic field from positive to negative $h$, we enter in a metastable region where the magnetization is positive, and the magnetic field is negative. The system remains in this metastable state for a quite large time, given by $\tau(h) \propto \exp(A/|h|^\alpha)$, where $\alpha = d - 1$. When the observation time is of order of $\tau(h)$ the system suddenly jumps into the stable state. This phenomenon is quite common: generally speaking we always enter into a metastable state when we cross a first order phase transition by changing some parameters.

We can also define a linear response suscepti-
bility which is given by
\[ \beta \chi_{LR} = \lim_{h \to 0^+} \frac{1}{\beta} \sum_i \langle \sigma(i) \sigma(0) \rangle_c. \]  

If we start with the the state where \( m > 0 \) at \( h = 0 \) and we add a positive magnetic field \( h \) at time 0, the linear response susceptibility is equal to
\[ \chi_{LR} = \lim_{t \to \infty} \frac{\partial}{\partial h} m(t, h), \]
\( m(t, h) \) being the magnetization at time \( t \). The linear response susceptibility is not equal to the equilibrium susceptibility that in this case is infinite:
\[ \chi_{eq} = \lim_{h \to 0} \frac{\partial}{\partial h} m(t, h) \bigg|_{h=0} = \infty. \]

This is the usual stuff that is described in books. We claim that in glassy system there is an open region in parameter space where, if we change the parameters of the system (e.g. the magnetic field \( h \)) by an amount \( \Delta h \), we have that \( \chi_{LR} \neq \chi_{eq} \). In the case of spin glasses we expect that for \( |h| < h_c(T) \) we stay in the glassy phase \( (h_c(T)) \) increases when we decrease the temperature and there is a temperature \( T_c \) such that \( h_c(T_c) = 0 \).

In this region
\[ \Delta m(t) = \chi_{LR} \Delta h \quad \text{for} \quad 1 << t << \tau(\Delta h), \]
\[ \Delta m(t) = \chi_{eq} \Delta h \quad \text{for} \quad \tau(\Delta h) << t, \]  
where in some cases one finds numerically that \( \tau(\Delta h) \) is has a power like behaviour (e.g. \( \tau(\Delta h) \propto |\Delta h|^{-4} \)). It is convenient to define \( \chi_{irr} \) as \( \chi_{eq} = \chi_{LR} + \chi_{irr} \). The glassy phase is thus characterized by a non zero value of \( \chi_{irr} \). If we observe the system for a time less that \( \tau(\Delta h) \) the behaviour of the system at a given point of the parameter space depend on the previous story of the system and a strong hysteresis effects are present.

The aim of the the theoretical study of glasses is, at least from my point of view, to get a theoretical understanding of these effects and to arrive to a qualitative and quantitative control of these systems. Before presenting these efforts it is convenient to describe some models which have a glassy behavior.

3. Glassy models

Generally speaking glassy systems can be divided into two categories:

- Systems with quenched disorder, e.g. spin glasses.
- Systems without quenched disorder, which are often translational invariant systems, e.g. tilings and glass forming liquids.

This distinction in two categories is important: although the two categories behave in the same way in the mean field approximation, it is quite possible that in finite dimension they have some different features. We will see later that the definition of an ensemble will be slightly different in the two classes of models.

3.1. Spin glasses

The simplest model of spin glasses is described by an Hamiltonian
\[ H_U(\sigma) = - \sum_{i,k} U(i, k) \sigma(i) \sigma(k) - \sum_i h \sigma(i), \]
where \( U(i, k) \) is a quenched \( Z_2 \) gauge field on the lattice at infinite temperature (in plain words the variables \( U \) are defined on the links, take the values \( \pm 1 \) and are uncorrelated quenched random variables), \( \sigma(i) \) is a matter field which also belongs to \( Z_2 \). We are interested to compute the average value of the free energy (and of the correlation functions) at a temperature \( \beta^{-1} \) of the matter field:
\[ F(\beta) = -\beta^{-1} \ln(Z_U(\beta)) \]
\[ Z_U(\beta) = \sum_{\{\sigma\}} \exp(-\beta H_U(\sigma)). \]

A few comments are in order:

- The overbar denote the average over the \( U \); we are in the quenched approximation. The unquenched case in not physically interesting.
- The quantity \( h \) is the magnetic field, which breaks gauge invariance.
• At \( h = 0 \) “to find the minimum of \( H(U(\sigma)) \)” is equal ”to find the Landau gauge”, \( \sigma \) being the gauge fixing. For given \( U \) in more than two dimension the problem of finding the global minimum of \( H(U(\sigma)) \) is an NP complete problem.

• Gribov ambiguity (which follows from the NP completeness of the problem) implies that there are many local minima of \( H(U(\sigma)) \): their number increase exponentially with the volume.

As we have already remarked there is a glassy region for not too large magnetic fields at low temperature.

3.2. Wang Tilings

The definition of Wang tilings is quite simple. In each point of the lattice there are variables \( \sigma(i) \) which can take \( M \) different values and are called tiles. The Hamiltonian is given by

\[
H(\sigma) = \sum_i \sum_{\mu=1,d} E_\mu(\sigma(i), \sigma(i+\mu)).
\] (7)

The model depend on the functions \( E_\mu(\sigma, \tau) \), which we assume can take only the values 0 and 1. For each \( M \) the number of different models is \( 2^{DM^2} \), however some of them are related by symmetries. Particular examples of Wang tilings are the Ising model (\( M = 2 \)), the \( p \)-state Potts models (\( M = p \)), the Baxter vertex models.

If there are configurations such that \( H(\sigma) = 0 \), we say that the model admits a perfectly matched tiling.

In two dimensions for sufficiently large \( M \) weird things may happen:

• For \( M \geq 16 \) there are functions \( E_\mu(\sigma, \tau) \) such that for open boundary there are global minima of \( H \) with \( H = 0 \), but they cannot be periodic. In a finite volume system with periodic boundary conditions there are no perfectly matched tilings.

• For \( M \geq 56 \) there are functions \( E_\mu(\sigma, \tau) \) such that the problem of deciding if a configuration of tiles on a finite volume can be imbedded in an infinite volume ground state with \( H = 0 \) is not decidable. In other words there is no computer program which can tell us in a \textit{a priori} bounded time if a given configuration can be imbedded in an infinite volume perfectly matched tiling. We can trivially construct a computer program that stops only if the configuration cannot be imbedded in an infinite volume ground state, however if \( L \) is the size of the configuration, the maximum time before stopping increases faster than any computable function of \( L \).

Not so much is known on the thermodynamical properties of those models: most of the investigations deal with the structure of the ground state configurations.
3.3. Glasses

Simple models of glasses are particles of different species with Hamiltonian

\[ H = \sum_{a,b=1,M} \sum_{i=1,N(a)} \sum_{k=1,N(b)} V_{a,b}(x_a(i) - x_b(k)), \]

where \( N(a) = Nc(a), \) \( \sum_a c(a) = 1 \) and the quantities \( c \) are the concentrations of each different species. The model depends on the \( M(M+1)/2 \) functions \( V_{a,b}(x) \) and on the \( M \) concentrations.

In order to have a glass transition it is crucial that the model does not crystallize, i.e. that the crystal ground state energy is bigger than that of lowest energy amorphous structure.

A well studied model of glasses is realized for \( M = 2, \) \( c(1) = .8, \) \( c(2) = .2 \) with a potential that has a simple Lennard-Jones form [8].

4. The theoretical interpretation

One way to interpret the presence of metastable states is to assume that for large (but finite) systems the phase space of equilibrium configurations can be decomposed into many finite volume states or valleys (lumps, as suggested Talarand [8]).

In other word we have a breaking of the ergodicity: the Gibbs measure can be approximately written the sum of smaller disconnect pieces. We warn the reader that we have to navigate between Scilla and Cariddi [4]. We cannot directly take the correlation functions in each state go to zero al large distances (cum grano salis).

- Connected correlation functions in each state go to zero al large distances (cum grano salis).

- The time to go from one state to another state is exponentially large with the volume.

In glassy systems one finds that these states at low temperature have the following unusual features

- Chaos: when we change the parameters in the Hamiltonian of any finite amount (typically as soon as we make a change greater than \( V^{-1/2}, \) \( V \) being the volume of the system) we have state crossing: stable states become metastable.

- No Gibbs rule: states coexist in an open region of parameter space.

The crucial step is the decomposition of the Gibbs measure for a finite system into states:

\[ \langle \cdot \rangle = w_\alpha \langle \cdot \rangle_\alpha, \]

where the weights \( w \) satisfy the relation \( \sum_\alpha w_\alpha = 1. \) Of course the previous formula is only approximate for a finite systems (there is a fraction of phase space with a small, but finite probability that cannot be classified into states) and should become more and more exact when the volume goes to infinity.

It is usually assumed that all the states have similar intensive properties, for example the internal energy, the magnetization, the linear response susceptibility do not depend on the state:

\[ N^{-1}\langle M \rangle_\alpha = m + N^{-1/2}\delta_\alpha, \]

\[ \chi_\alpha \equiv N^{-1}\langle M^2 \rangle_\alpha = \chi_{LR}. \]

In this case we can also define the so called Edward Anderson parameter \( q_{EA} = N^{-1} \sum_\alpha \langle \sigma(i) \rangle_\alpha \), which should not depend on the state \( \alpha. \)

However if we compute the full expression for the susceptibility, we find that the \( O(N^{-1/2}) \) variations in the magnetization do matter: the expression for \( \chi_{irr} \) turns out to be:

\[ \chi_{irr} = \sum_{\alpha,\gamma} w_\alpha w_\gamma (\delta_\alpha - \delta_\gamma)^2. \]

The physical origne of this extra term is quite clear: when we increase the magnetic field, the
states with higher magnetization become more likely than the states with lower magnetization and this effect contributes to the increase in the magnetization. However the time to jump to a state to another state is very high (it is strictly infinite in the infinite volume limit if non linear effects are neglected) and this effect produces the separation of time scales relevant for $\chi_{LR}$ and $\chi_{eq}$.

Having understood that the coexistence of many states in an open region of parameter space with the aforementioned properties is a crucial feature of glassy systems, we face the problem of being more quantitative. This will be done in the next section.

5. A quantitative approach

In order to be specific, let us consider the case of spin glasses. A very important object is the overlap among two configurations (e.g. $\sigma$ and $\tau$):

$$q(\sigma, \tau) = N^{-1} \sum_i \sigma(i)\tau(i).$$

(12)

In a similar way we can define the overlap $q_{\alpha,\gamma}$ among two states $\alpha$ and $\gamma$ as the overlap among two generic configurations belonging to the two states. We also have:

$$q_{\alpha,\gamma} = N^{-1} \sum_i \langle \sigma(i) \rangle_\alpha \langle \sigma(i) \rangle_\gamma.$$

(13)

In a similar way we can define a generalized overlap: given a local operator $A(i)$ we have:

$$q^A_{\alpha,\gamma} = N^{-1} \sum_i \langle A(i) \rangle_\alpha \langle A(i) \rangle_\gamma.$$

(14)

If we take $A(i) = \sigma(i)$ we get $q^A = q$ (i.e. the usual overlap): if we take $A(i) = H(i)$ we get $q^A = q^E$ (i.e. the energy overlap).

Let us consider a given finite system characterized by some parameters that we call $J$. In order to describe the structure of its states, we should give all the $w_\alpha$ and all the $q_{\alpha,\gamma}$. Equivalently we could introduce the probability distribution of the overlap given by

$$P_J(q) = Z^{-2} \sum_{\{\sigma\},\{\tau\}} \exp(-\beta(H(\sigma) + H(\tau))\delta(q(\sigma, \tau) - q)).$$

(15)

$$\approx \sum_{\alpha,\gamma} w_\alpha w_\gamma \delta(q - q_{\alpha,\gamma}).$$

The structure of the states depends on the sample: we show in figs. 4 and 5 two $P_J(q)$ for two different samples of three dimensional spin glasses of size $16^3$. 

Figure 2. The function $P_J(q)$ for a given $16^3$ sample.

Figure 3. The function $P_J(q)$ for another $16^3$ sample.
Is is clear that this description (i.e. the function $P_J(q)$) depends on the sample and numerically the dependence does not disappear by increasing the size. In this case the only thing that makes sense is to introduce an ensemble and to describe the statistical properties of that ensemble. At this end there are a few possibilities:

- To average over the quenched disorder at fixed number of spins $N$.
- To average over $N$ in a window.
- To average over small quenched added disorder for a given sample.

It is evident that the first possibility is empty if no quenched disorder is present.

The statistical description would amount to assign the probability $P([w],[q])$ of finding a given configuration of weights $w$ and overlaps $q$. Such a functional is not so easily parametrized, so that we can introduce its moments, e.g.

$$P(q) = P_J(q), \quad P(q_1,q_2) = P_J(q_1)P_J(q_2). \quad (16)$$

It is evident that $P([w],[q])$ contains the same information of the infinite set of functions $(P(q),P(q_1,q_2),P(q_1,q_2,q_3))$.

In order to put order into this mess and to restrict the form of all possible $P([w],[q])$ two principle can be assumed: stochastic stability and separability.

### 5.1. Stochastic stability

Roughly speaking stochastic stability implies that the ensemble we consider is a generic random ensemble and it does not have any peculiar property [9–11]. More technically it implies that if we add a random perturbation, i.e. we consider the Hamiltonian

$$H(\epsilon)(\sigma) = H(\sigma) + \epsilon H_R(\sigma), \quad (17)$$

where $H_R(\sigma)$ is a random perturbation, the expectation value of everything is smooth in $\epsilon$. A typical example of random Hamiltonian we can add is $H_R(\sigma) = N^{-1/2} \sum_{i,k} R_{i,k} \sigma(i) \sigma(k)$, where $N$ is the total number of spin (as usual) and $R_{i,k}$ are random quantities with zero average and variance 1. It is crucial in the argument that infinite range Hamiltonians are allowed as perturbation. Stochastic stability implies very strong constraints on $P([w],[q])$: the simplest one can be written under the form of the so called Guerra relations:

$$P(q_1,q_2) = \frac{2}{3} P(q_1)\delta(q_1 - q_2) + \frac{1}{3} P(q_1)P(q_2). \quad (18)$$
These relations are very well satisfied in short range models of spin glass. Indeed they imply that
\[
\frac{q_{12}q_{34}}{q_{12}^2} = \frac{2}{3} \left( \frac{q_{12}}{q_{12}} \right)^2 - \frac{1}{3} \frac{q_{12}}{q_{12}}. \tag{19}
\]
This relation which is well satisfied numerically (see fig. 5, where \(q_{12}q_{34}\) is a quantity of order one for \(T < .9\)). As we shall see in the next section stochastic stability has important consequences on the dynamics.

Although for the moment we cannot prove that spin glasses and other random systems are stochastically stable, stochastic stability is rather likely to be correct for systems which do not have any symmetry (or if symmetries are present, it should hold if restricted to those observables which are left invariant by the action of the symmetry group).

### 5.2. Separability

Separability is a more subtle property \[12\]. It implies that there is only one kind of significant overlap. For example in spin glasses it amounts to say all overlaps (depending on the operator \(A\)) are given functions of the spin overlap. In other words for each local operator \(A\) there is a function \(f^A(q)\) (which may depend on the magnetic field, temperature . . . ) such that \(q^A_{\alpha,\gamma} = f^A(q_{\alpha,\gamma})\).

In the general case an infinite number of quantities (i.e. \(q^A_{\alpha,\gamma} \forall A\)) characterizes the mutual relations among \(\alpha\) and \(\gamma\). If separability is assumed this infinite number is reduced to one.

There are some arguments \[12\] that suggest that separability implies the ultrametricity property:
\[
q_{\alpha,\gamma} > \min(q_{\alpha,\beta}, q_{\beta,\gamma}) \forall \beta. \tag{20}
\]
If ultrametricity holds, the set of states of a given system may be ordered on a tree (as it is usual in taxonomy) and the form of the probability functional \(P(\{w\}, \{q\})\) is very similar to the one found in the mean field approximation.

Separability and ultrametricity may be less compulsory properties than stochastic stability: this point is at present under investigations, however the validity of ultrametricity seems to be supported by numerical simulations for spin glasses, at least in 4 dimensions \[13\].

6. Off-equilibrium dynamics

A very interesting question is what happens when we cool the system starting from a random (high temperature) configuration at time zero? The question is non trivial if, cooling the system, we cross a phase transition. The behaviour of the system in these conditions tells us something about the order parameter in the low temperature phase (in the following we will discuss only the behaviour of an infinite volume system). One of the most spectacular phenomenon is ageing.

Let us suppose that we cool very fast the system from an high temperature configuration at time 0. We can define a two times correlation function
\[
C(t, t_w) = \frac{1}{N} \sum_{i} \langle \sigma_i(t_w)\sigma_i(t_w + t) \rangle = q(t_w, t_w + t) \tag{21}
\]
In the limit \(t_w \to \infty\) at fixed \(t\) we recover the usual equilibrium correlation function (i.e. \(\lim_{t_w \to \infty} C(t, t_w) = C(t)\)). However in the region where \(t_w\) and \(t\) are both large, the dependance on \(t_w\) does not disappear: for example in fig. 6 we show the dependance of the correlation function on \(t\) in four dimensional spin glasses: as soon as

![Figure 6. The correlation function \(C(t, t_w)\) in 3-d spin glasses in the low temperature phase.](image)
Figure 7. Three different form of the function $P(q)$ and the related function $S(q)$

$t$ is of order $t_w$, the value of $t_w$ matters. If simple aging holds, we have that $C(t, t_w) \approx C(t/t_w)$ in this region.

We notice that $\lim_{t \to \infty} \lim_{t_w \to \infty} C(t, t_w)$ is a non zero quantity, (i.e. it is equal to $q_{EA}$) while $\lim_{t \to \infty} \lim_{t_w \to \infty} C(t, t_w)$ is zero at zero external magnetic field. The non commutativity of the two limits is a very clear signal of existence of more than one equilibrium state.

Aging can be intuitively interpreted in the following way. At large times the system is ordered in one given phase in regions of size $\xi(t)$, where $\xi(t)$ goes to infinity when $t$ goes to infinity. These regions move with time and their coalescence produces the increase in the correlations length. It is natural to suppose that

$$C(t, t_w) = f(\xi(t + t_w)/\xi(t_w))$$

(a similar formula was suggested to be valid also in infinite range systems where no correlation length can be defined \[13\]). The increase of the the dynamical correlation length $\xi(t)$ corresponds to rearrangements of the regions which belong to different phases (some shrink and some others expand) and it affects the two-times correlation. If $\xi(t)$ increases as power of time, the previous arguments suggest the validity of simple ageing. Of course when $\xi(t)$ hits the size of the box ageing ends, however this phenomenon never happens for an infinity volume system if we use a local dynamics.

In a similar way we can consider a time dependent Hamiltonian

$$H = H_0 + \theta(t - t_w) \sum_i h_i \sigma_i ,$$

$h_i$ being a random magnetic field (which in spin glasses is gauge equivalent to a magnetic field with constant sign).

After cooling the system at time zero, we switch on the random field at time $t_w$. In this way we can define a two times response function (the precise name is relaxation function):

$$\beta S(t, t_w) \equiv \frac{1}{N} \sum_i \frac{\partial \sigma_i (t + t_w)}{\partial h_i} ,$$

A great simplification comes out if we close our eyes on the actual time dependence of the correlation function and of the relaxation function and we plot $S$ versus $C$ \[15,16\]. We can expect that such a plot has a well defined limit when $t_w$ goes to infinity. In this plot we must distinguish two regions:

- The region $C > q_{EA}$ is the equilibrium region $t << t_w$. Here the fluctuation dissipation theorem implies that

$$\frac{dS}{dC} = 1 .$$

- The region $C < q_{EA}$ is the aging region where the fluctuation dissipation theorem cannot be anymore applied. Here we can define the function

$$X(C) = -\frac{dS}{dC} .$$
which characterizes the relations among fluctuations and response in the aging regime.

It is rather surprising to note that a dynamical version of stochastic stability [14] implies the relation

\[ X(C) = \int_0^C dq P(q), \quad (27) \]

which was conjectured to be valid in infinite range models [16,17].

There are three main kinds of the dynamical response \( S(C) \), that correspond to different shapes of the static function \( P(q) \) as shown in fig. (6).

- Case A correspond to no replica symmetry breaking and it is the usual case when two phases are present (e.g. ferromagnetism): \( P(q) \) is a delta function.
- Case B correspond to one step replica symmetry breaking and it is supposed to be realized in structural glasses: \( P(q) \) is the sum of two delta functions.
- Case C correspond to breaking the replica symmetry in an continuous way and is supposed to be realized in glasses and maybe in tiling models: \( P(q) \) is a the sum of two delta functions plus a smooth function in between.

There are many numerical results which confirm the validity of this analysis and classification. Some results are presented in the next two sections.

6.1. Numerical results for spin glasses

In spin glasses it is possible using the parallel tempering technique to fully thermalize systems up to \( O(10^4) \). System of this size could not be thermalized in a reasonable time (eg. less than \( 10^{12} \) Monte Carlo sweep) using the conventional Monte Carlo technique. After thermalizing hundreds of these systems one can measure the average function \( P(q) \) and using the relation (27) one can predict the form of \( X(C) \) and consequently the dependence of \( R \) on \( C \).

These predictions can be tested by taking a large sample (e.g. \( O(10^6) \) spins) and by measuring the correlations and the relaxation function in the off-equilibrium aging regime. These results can be seen in fig. (b) for \( D = 3 \) and fig. (b1) for \( D = 4 \). The result are quite impressive if we con-
consider that no free parameter is present and that the two curves are obtained with a very different procedure.

Similar results have been obtained for a tiling model [6].

6.2. Numerical results for structural glasses

In the case of structural glasses analytic computations [19] and phenomenological considerations [20] support the proposal that

\[ X(C) = m \text{ for } C < C^*, \]
\[ X(C) = 1 \text{ for } C < C^*, \] (28)

where

\[ m \approx T/T_c \] (29)

(i.e. \( m = 1 \) at \( T = T_c \), \( m = 0 \) at \( T = 0 \)). The quantity \( m/\beta \) is roughly independent from the temperature and has the meaning of the effective temperature in the space of the valleys. In other words replica symmetry should be broken at one step level and the symmetry breaking parameter \( m \) should be one at the critical temperature and vanish linearly with the temperature.

This kind of behaviour is exactly the one that has been observed in some long range model of spin glasses. Numerical simulations support the correctness of these predictions. For example we see in fig. 6.2 the numerical evaluation of the parameter \( m \) in a glass forming binary mixture.

7. Conclusions

We have argued that glassiness, metastability and the existence of many equilibrium states are phenomena which are strongly linked. The theory for studying these phenomena has been developed in the framework of replica theory, however we can abstract from the usual replica approach two principles, stochastic stability and separability, which are enough to fully characterize the theory.

The most impressive phenomenon is the presence of a new form of fluctuation dissipation relation in the aging regime, where the characteristic function \( X(C) \) is linked to a quantity that can be defined in the static, i.e the function \( P(q) \). Direct measurements of these fluctuation dissipation relations in real (not numerical) experiments both for glasses and spin glasses are actually under the way. The outcome will be a crucial test for this theoretical approach and I hope that some results will be available in the next future.
REFERENCES

1. M. M´ezard, G. Parisi and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
2. E. Marinari, G. Parisi, F. Ricci-Tersenghi, J. Ruiz-Lorenzo and F. Zuliani, *Replica Symmetry Breaking in Short Range Spin Glasses: A Review of the Theoretical Foundations and of the Numerical Evidence*, cond-mat/9906076, submitted to J. Stat. Phys.
3. A. P. Young (ed.), *Spin Glasses and Random Fields* (World Scientific, Singapore, 1997).
4. S. Franz and G. Parisi, *Critical properties of a three dimensional p-spin model*, cond-mat/9805088, submitted to Europhys. J.; M. Campellone, B. Coluzzi B and G. Parisi, Phys. Rev. 58B, 12081 (1998).
5. B. Gruenbaum and G.C. Shepard *Tilings and Patterns* (New York, Freeman, 1987).
6. L. Leuzzi, G. Parisi *Thermodynamics of a tiling model* (in preparation).
7. W. Kob and H.C. Andersen, *Phys. Rev. Lett.* 73, 1376 (1994).
8. M. Talagrand, *Rigorous low temperature results for the p-spin mean field spin glass model*, to appear in Probability Theory and Related Fields.
9. F. Guerra, Int. J. Mod. Phys. B 10 1675 (1997); S. Ghirlanda and F. Guerra, J. Phys. A: Math. Gen. 31 9149 (1998).
10. M. Aizenman and P. Contucci, J. Stat. Phys. 92 765 (1998).
11. G. Parisi, *On the probabilistic formulation of the replica approach to spin glasses*, cond-mat/9801081, submitted to Europhys. J.;
12. G. Parisi and F. Ricci-Tersenghi *On the physical origin of ultrametricity* cond-mat/9905189, submitted to J. Phys. A.
13. A. Cacciuto, E. Marinari and G. Parisi, J. Phys. A, 30, L263 (1997).
14. S. Franz, M. M´ezard, G. Parisi, L. Peliti Phys. Rev. Lett. 81, 1758 (1998); The response of glassy systems to random perturbations: A bridge between equilibrium and off-equilibrium, cond-mat/9903370.
15. L. Cugliandolo and J. Kurchan 1993, Phys. Rev. Lett. 71, 1.
16. L.F. Cugliandolo and J. Kurchan, *J. Phys. A: Math. Gen.*, 27, 5749 (1994).
17. S. Franz and M. M´ezard 1994, Europhys. Lett. 26, 209.
18. E. Marinari, G. Parisi, F. Ricci-Tersenghi and J. Ruiz-Lorenzo, *J. Phys. A: Math. Gen.*, 31, 2611 (1998).
19. M. M´ezard and G. Parisi, Phys. Rev. Lett. 82, 747 (1988); A first principle computation of the thermodynamics of glasses cond-mat/9812181, J. Chem. Phys., in press.
20. T.R. Kirkpatrick and D. Thirumalai 1995, Transp. Theor. Stat. Phys. 24 927 and references therein.