Compact Homogeneous Universes

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ABSTRACT

A thorough classification of the topologies of compact homogeneous universes is given except for the hyperbolic spaces, and their global degrees of freedom are completely worked out. To obtain compact universes, spatial points are identified by discrete subgroups of the isometry group of the generalized Thurston geometries, which are related to the Bianchi and the Kantowski-Sachs-Nariai universes. Corresponding to this procedure their total degrees of freedom are shown to be categorised into those of the universal covering space and the Teichmüller parameters. The former are given by constructing homogeneous metrics on simply connected manifold. The Teichmüller spaces are also given by explicitly constructing expressions for the discrete subgroups of the isometry group.

PACS number(s): 04.20.-q, 04.20.Gz, 02.40.-k.

to appear in Journal of Mathematical Physics
1 Introduction

In the standard cosmology we normally assume homogeneity and isotropy of the universe. On the basis of this cosmological principle the Friedmann-Robertson-Walker metric is constructed and Big-Bang Cosmology has been successfully developed in the theory of Einstein gravity [1]. The homogeneous but anisotropic universe models, the Bianchi [2, 3] and the Kantowski-Sachs-Nariai universes [4, 5], have been studied from various motivations [6], many of which contribute to grasp aspects of more general and more complicated situations in general relativity or in cosmology. For example, people have attempted to clarify the properties of the initial singularity and also attacked the basic problem why the present universe looks so isotropic by studying the time evolution of the homogeneous anisotropic models [7]. The investigation of compact universes may be important when we are interested in the possibility of our universe to have a non-trivial topology or its topology change. The former aspect was studied by Fang and Sato [8] and by Ellis [9] in a special model of torus universe. For the latter we note that the famous theorem by Geroch [10] on topology change assumes the compactness of spatial manifold. It will be more intriguing to study the geometries of compact (locally) homogeneous universes when we investigate minisuperspace models in quantum gravity. There it is necessary to make explicit the global degrees of freedom corresponding to deformation of the compact universes.

For such investigations, we need a classification of compact homogeneous spaces having nontrivial topologies, which are not considered in the traditional Bianchi classification. Ashtekar and Samuel [12] investigated global degrees of freedom of compact locally homogeneous universes. However, it seems to us that their definition of locally homogeneous spaces is so restrictive that many possible types of compact universes are excluded. For example, in the Bianchi V model, they have excluded the compact hyperbolic spaces by restricting the identifications used in the compactification to those in the 3-dimensional Bianchi V group. In fact, the Bianchi V model can be compactified by using the largest 6-dimensional isometry group PSL(2, C) ⊃ Bianchi V group. Our definition and treatment of locally homogeneous spaces, which is overviewed below, may be more natural and
general and coincides with the one adopted by mathematicians.

In a modern point of view, the term “geometry” is used in a specific sense. It is a pair of a manifold $X$ and a group $G$ acting on $X$ transitively with compact point stabilizer (i.e., the isotropy subgroup of $G$ at any point of $X$ is compact). A geometry $(X, G)$ is an alternative expression of a homogeneous Riemannian manifold $(X, g_{ab})$, since we can always construct the $G$-invariant metric $g_{ab}$ in an appropriate manner. Thurston \cite{19} (see also \cite{11}) proved that any maximal and simply connected 3-dimensional geometry which admits a compact quotient is equivalent to one of the so-called Thurston eight geometries. Here, the term “maximal” is defined with respect to inclusion of the group $G$. In other words, maximal geometries are the spaces having such a large symmetry that no extra symmetry can be added. The maximality, however, restricts possible universal covers of compact locally homogeneous spaces. By extending the Thurston’s results to nonmaximal geometries, we shall show a physically relevant theorem, and give the explicit forms of the universal cover metrics. We emphasize that all three-dimensional compact homogeneous manifolds can be obtained from the universal covers with the metrics given in the generalized theorem. We call the parameters in the metrics the universal cover parameters.

Compact homogeneous spaces can be obtained, as already inferred, by identifying the points of the universal covers which are mapped to one another by the action of an appropriate discrete subgroup of the isometry group. This may be well described by the following projection

$$\pi : \tilde{M} \to M = \tilde{M}/\Gamma,$$

where $\tilde{M}$ is a universal covering space, $\Gamma$ the discrete isometry group and $M$ the compact homogeneous space expressed by the projection $\pi$. It is worth noting that $\Gamma$ is isomorphic to the fundamental group of $M$, $\pi_1(M)$. The Teichmüller deformations are expressed by smooth deformations of $\Gamma$. Hence, it is evident that the Teichmüller deformations preserve local quantities such as curvature, whereas the deformations of a universal cover metric alter them. We shall present the possible expressions of $\Gamma$ explicitly. We call the parameters in $\Gamma$ the Teichmüller parameters.

The total (geometrical) degrees of freedom of a compact locally homogeneous space
is defined as the number of the universal cover parameters plus the number of the Teichmüller parameters. For example, we shall see that Bianchi IX universe has two degrees of freedom of the universal cover and no Teichmüller deformations, and that Bianchi VI(0) universe has one universal cover parameter and one Teichmüller parameter.

In this paper, we will not discuss the dynamics of the compact spaces. The Einstein gravity will be investigated for the compact locally homogeneous universes in a separate paper.

The organization of the present paper is the following. In section 2, some mathematical preliminaries are presented to clearly define compact locally homogeneous spaces and the Teichmüller deformation. We also explain Thurston’s theorem which plays a central role in the present work and the eight geometries which appear in the theorem. The homogeneity preserving diffeomorphism is also defined. In section 3, we describe the definition of the compact locally homogeneous cosmological models in terms of the mathematical terminologies explained in section 2. In section 4, we give a complete classification of the types of universal covers $\tilde{M}$. The compact quotients $M = \tilde{M}/\Gamma$ are explicitly constructed in section 5, except for the compact hyperbolic 3-geometries. The final section is devoted to conclusion and discussions.

In the Appendix we present a proof that shows why Bianchi Class B geometries and $(\mathbb{R}^3, \widetilde{SL}(2,\mathbb{R}))$ do not admit compact quotients if one uses the Bianchi groups to find the discrete subgroup $\Gamma$.

2 Mathematical preliminaries

In this section we explain briefly and without proofs the mathematical terms and facts which we will use in what follows. In Sec.2.1 we give the definition and the feature of locally homogeneous manifolds. In Sec.2.2 we explain the theorem of Thurston which will play a great role in this paper. In Sec.2.3 we give the definition of the Teichmüller space. In Sec.2.4 we briefly explain each of the Thurston geometries. The proofs and more details may be found in Wolf [14] and Scott [11].
2.1 Locally homogeneous spaces

A metric on a manifold $M$ is said to be (globally) homogeneous if the isometry group acts transitively on $M$, i.e. for any points $p, q \in M$ there exists an isometry which takes $p$ into $q$. A metric on a manifold $M$ is said to be locally homogeneous if for any points $p, q \in M$ there exist neighborhoods $U, V$ and an isometry $(U, p) \rightarrow (V, q)$. The difference between global and local homogeneity is that in the latter case the isometry may not be globally defined.

Any locally homogeneous manifold is diffeomorphic to the quotient of a simply connected homogeneous manifold by a discrete subgroup of the isometry group. Let $M$ be an arcwise connected compact manifold with a complete locally homogeneous metric. Any arcwise connected manifold $M$ has a unique universal covering manifold $\tilde{M}$ up to diffeomorphisms. $\tilde{M}$ has a natural metric inherited from $M$ which is the pullback of the metric on $M$ by the projection $\tilde{M} \rightarrow M$. This defines a complete locally homogeneous metric on $\tilde{M}$. It is a theorem of Singer \[15\] that a complete locally homogeneous metric on a simply connected manifold is homogeneous. Therefore the metric on $\tilde{M}$ is homogeneous and $\tilde{M}$ is diffeomorphic to $\text{Isom}\tilde{M}/K$ where $\text{Isom}\tilde{M}$ is the isometry group of $\tilde{M}$ and $K$ is the stabiliser of a point in $\tilde{M}$. $K$ is a compact subgroup of $\text{Isom}\tilde{M}$. Let $\Gamma$ be the group of covering transformations, which is isomorphic to the fundamental group $\pi_1(M)$ of $M$. An element of $\Gamma$ is obviously an isometry. It follows that $\Gamma$ is a subgroup of $\text{Isom}\tilde{M}$. Thus $M$ is realized as $\tilde{M}/\Gamma$. We restrict our attention to orientable manifolds. Then each element of $\Gamma$ must preserve the orientation of $\tilde{M}$. This means that in order to get orientable manifolds one has only to replace $\text{Isom}\tilde{M}$ in the above discussion by $\text{Isom}^+\tilde{M}$, the orientation preserving subgroup of $\text{Isom}\tilde{M}$.

Conversely, a quotient space of a homogeneous Riemannian manifold by an appropriate subgroup of the isometry group is locally homogeneous. Let $X$ be a manifold with a complete homogeneous metric, $\text{Isom}X$ be its isometry group, $K$ be a stabiliser of a point, and $\Gamma$ be a subgroup of $\text{Isom}X$. $X/\Gamma$ is Hausdorff if $\Gamma$ acts properly discontinuously on $X$, i.e. for any compact subset $C$ of $X$, $\{\gamma \in \Gamma \mid \gamma C \cap C \neq \emptyset\}$ is finite. $\Gamma$ acts properly discontinuously on $X$ if and only if $\Gamma$ is a discrete subgroup of $\text{Isom}X$. $X/\Gamma$ is a
Riemannian manifold if and only if $\Gamma$ acts \textit{freely} on $X$, i.e. the stabilizer of $\Gamma$ is trivial. If $\Gamma$ does not necessarily act freely then $X/\Gamma$ is an \textit{orbifold}, i.e. a Hausdorff, paracompact space which is locally homeomorphic to the quotient space of $\mathbb{R}^n$ by a finite group action. Evidently $X/\Gamma$ is locally homogeneous if it is a Riemannian manifold. As a result, $X/\Gamma$ is a locally homogeneous Riemannian manifold if $\Gamma$ is a discrete subgroup of $\text{Isom}X$ and acts freely on $X$. So our task is to find out all possible discrete subgroups $\Gamma$ of $\text{Isom}X$ which acts freely on $X$ and makes $X/\Gamma$ compact.

## 2.2 Geometries and Thurston’s theorem

In order to state precisely and take advantage of the result obtained in the recent study in 3-manifolds, we use the term ‘geometry’ in a specific sense. A \textit{geometry} is a pair $(X, G)$ where $X$ is a manifold and $G$ is a group acting transitively on $X$ with compact point stabilizers. Geometries $(X, G)$ and $(X', G')$ are \textit{equivalent} if there is a diffeomorphism $\phi : X \rightarrow X'$ with $\bar{\phi} : g \mapsto \phi \circ g \circ \phi^{-1}$ being an isomorphism from $G$ onto $G'$. Let us call $\phi$ an \textit{equivalence map}. A geometry $(X, G')$ is a \textit{subgeometry} of $(X, G)$ if $G'$ is a subgroup of $G$. A geometry is \textit{maximal} if it is not a proper subgeometry of any geometries.

Although a geometry $(X, G)$ is merely a pair of a manifold $X$ and a group $G$ and has nothing to do with Riemannian metrics on $X$ itself, one can discuss homogeneous Riemannian metrics from the viewpoint of geometries. This is because one can construct a complete homogeneous metric from a geometry $(X, G)$ as follows. Give a random positive definite quadratic form on the tangent space $T_p$ of a point $p$ of $X$. Average it on $p$ by the invariant volume form of the compact stabilizer $G_p$, having a $G_p$-invariant inner product on $T_p$. Carry the quadratic form to each point in $X$ by the action of $G$ which acts transitively on $X$, having the $G$-invariant metric on $X$. Note that there may be many possible Riemannian metrics on $X$ which correspond to a geometry $(X, G)$. Note also that $G$ is not necessarily the isometry group itself but a subgroup of it.

We now quote the theorem of Thurston which is of great use in classifying all compact, locally homogeneous spaces.
THEOREM 1 (Thurston [19]) Any maximal, simply connected 3-dimensional geometry which admits a compact quotient is equivalent to the geometry \((X, \text{Isom}X)\) where \(X\) is one of \(E^3, H^3, S^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \tilde{\text{SL}}(2, \mathbb{R}), \text{Nil}, \text{ or } \text{Sol}\).

A brief proof can be found in Scott [11]. Each \(X\) in the theorem is a manifold with a certain ‘standard’ Riemannian metric on it. Note that this Riemannian metric on \(X\) is not of essential importance in the theorem but is simply useful to express a group \(G\) in a geometry \((X, G)\) as \(\text{Isom}X\) and the action of \(G\) on \(X\). For example, \((E^3, \text{Isom}E^3)\) means the same as \((\mathbb{R}^3, \text{IO}(3))\). A brief explanation of all geometries is given in Sec.2.4.

One important feature of the Thurston geometries is that the compact quotients modeled on the six of them apart from \(H^3\) and \(\text{Sol}\) admit Seifert bundle structures. A Seifert bundle structure is an extended idea of an \(S^1\)-bundle structure over a 2-surface. Though Seifert bundle itself is a 3-manifold, the ‘base space’ may be an orbifold and the fiber at a singular point of the base (critical fiber) does not have to continue smoothly to those of neighboring points (regular fibers). The singular points of the base are always cone points if the total space is orientable. An orientable Seifert bundle is specified by the base orbifold, \(S^1\)-bundle over a 2-manifold obtained by removing cone points from the base orbifold, Seifert invariants \((\alpha, \beta)\) of each critical fiber, and the “Euler number” \(e\). The Seifert invariants \((\alpha, \beta)\) are integers and determine the relation of a critical fiber and its neighboring fibers; \(\alpha\) turns along the critical fiber is equal to \(-\beta\) turns along the regular fiber. The Euler number \(e\) is an integer which tells the nontriviality of the bundle. Regarding a manifold as a Seifert bundle makes it easier to investigate it geometrically, especially in considering possible deformations of it.

2.3 The Teichmüller space

We give the definition of the Teichmüller space of a locally homogeneous manifold. Let \(\tilde{M}\) be a simply connected homogeneous Riemannian manifold with its metric \(\tilde{h}_{ab}\) fixed. Let \(M\) be a quotient of \(\tilde{M}\) by a discrete subgroup \(\Gamma\) of \(\text{Isom}\tilde{M}\). The Teichmüller space of \(M\) is the space of all smooth deformations of \(M\) irrespective of its size, keeping the condition that \(M\) is a quotient of \((\tilde{M}, \tilde{h}_{ab})\).
Let us put it more precisely. Let \( \text{Rep}(M) \) denote the space of all discrete and faithful representations \( \rho : \pi_1(M) \to \text{Isom}^+ \tilde{M} \). Let us call a diffeomorphism \( \phi : \tilde{M} \to \tilde{M} \) a global conformal isometry if \( \phi_* \tilde{h}_{ab} = \text{const.} \cdot \tilde{h}_{ab} \). Let us define a relation \( \sim \) in \( \text{Rep}(M) \) such that \( \rho \sim \rho' \) holds if there exists a global conformal isometry \( \phi \) of \( \tilde{M} \) connected to the identity which gives \( \rho'(a) = \phi \circ \rho(a) \circ \phi^{-1} \) for any \( a \in \pi_1(M) \). It defines an equivalence relation in \( \text{Rep}(M) \). We define the Teichmüller space as

\[
\text{Teich}(M) = \text{Rep}(M)/ \sim .
\] (2)

The Teichmüller space is a manifold. The numbers used to parametrize the Teichmüller space are called the Teichmüller parameters.

If two representations \( \rho \) and \( \rho' \) correspond to the same point in \( \text{Teich}(M) \), the above global conformal isometry \( \phi \) on \( \tilde{M} \) induces a well-defined global conformal isometry from \( \tilde{M}/\Gamma \) onto \( \tilde{M}/\Gamma' \), where \( \Gamma = \rho(\pi_1(M)) \) and \( \Gamma' = \rho'(\pi_1(M)) \). This is because for any \( \gamma \in \Gamma \) there exists unique \( \gamma' \in \Gamma' \) such that \( \phi \circ \gamma = \gamma' \circ \phi \), which guarantees that \( \phi(\tilde{M}/\Gamma) = \tilde{M}/\Gamma' \).

We will see in Sec.4.2 that homogeneous manifolds \( \tilde{M} \) except for the flat space have no global conformal isometries other than isometries.

In this case, the Teichmüller space is \( \text{Rep}(M) \) modulo conjugation by \( \text{Isom}_0 \tilde{M} \). In the case of the flat space, the Teichmüller space is \( \text{Rep}(M) \) modulo conjugation by \( \text{Isom}_0 \tilde{M} \) and modulo total size.

The Teichmüller parameters constitute a subset of dynamical variables of locally homogeneous universes.

### 2.4 The Thurston geometries

In this section we explain the metric and the isometry group of the eight Thurston geometries.

#### 2.4.1 \( E^3 \)

\( E^3 \) is the 3-dimensional flat Riemannian manifold. The standard metric is

\[
ds^2 = dx^2 + dy^2 + dz^2
\] (3)
and an isometry is expressed as
\[ g(x) = Rx + a \] (4)
where \( R \) is a 3×3 orthogonal matrix and \( a \) is a constant vector. The above transformations form a group and we denote it by IO(3). To preserve the orientation \( R \) must be a rotation matrix, and the corresponding transformations form a subgroup, ISO(3), of IO(3).

The generators of IO(3) or ISO(3), i.e. Killing vectors of \( E^3 \), are the following:

\[
\begin{align*}
K_1 &= \frac{\partial}{\partial x}, \\
K_2 &= \frac{\partial}{\partial y}, \\
K_3 &= \frac{\partial}{\partial z}, \\
K_4 &= -z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}, \\
K_5 &= -x \frac{\partial}{\partial z} + z \frac{\partial}{\partial x}, \\
K_6 &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.
\end{align*}
\] (5)

The first three vector fields represent infinitesimal translations while the last three represent infinitesimal rotations.

2.4.2 Nil

Nil is the group of the matrices of the form
\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]
with ordinary matrix multiplication. In other words, the multiplication is defined by
\[
(a, b, c)(x, y, z) = (a + x, b + y, c + z + ay).
\] (6)

The standard metric is one of left-invariant metrics. A basis of left-invariant 1-forms is
\[
\sigma^1 = dz - xdy, \sigma^2 = dx, \sigma^3 = dy.
\] (7)

The standard metric is
\[
\begin{align*}
\text{ds}^2 &= (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \\
&= dx^2 + dy^2 + (dz - xdy)^2.
\end{align*}
\] (8)
The isometry group of course has Nil as its subgroup. There is an additional 1-parameter family of isometries isomorphic to U(1). It is written as

\[
   s_\theta : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ R_\theta \left( \begin{pmatrix} z + \frac{1}{2}\left((x^2 - y^2)\cos \theta - 2xy\sin \theta \right) \sin \theta \right) \end{pmatrix} \end{pmatrix} \quad (0 \leq \theta < 2\pi)
\]

where \( R_\theta \) is the 2-dimensional rotation matrix of angle \( \theta \). Therefore the isometry group is 4-dimensional. The isometry group has two components. A discrete isometry is given by

\[
   (x, y, z) \mapsto (x, -y, -z).
\]

All isometries preserve the orientation of Nil.

**2.4.3 \( H^2 \times \mathbb{R} \)**

The standard metric is

\[
   ds^2 = \frac{1}{y^2}(dx^2 + dy^2) + dz^2
\]

where \( x, y, z \in \mathbb{R} \) and \( y > 0 \). The isometry group is Isom\( H^2 \times \) Isom\( \mathbb{R} \) hence an isometry is written as \((\alpha, \beta)\) where \( \alpha \in \text{Isom}H^2, \beta \in \text{Isom}\mathbb{R} \).

**2.4.4 \( \widetilde{SL}(2, \mathbb{R}) \)**

\( \widetilde{SL}(2, \mathbb{R}) \) is the universal covering group of SL(2,\( \mathbb{R} \)). The standard metric on \( \widetilde{SL}(2, \mathbb{R}) \) is one of left invariant metrics.

This Riemannian manifold can be regarded as the universal cover of the unit tangent bundle \( UH^2 \) of the hyperbolic 2-surface \( H^2 \). A tangent bundle of a Riemannian manifold has a natural metric induced from the base manifold. The induced metric is completely determined by the metric on the base space. So an isometry on the base space induces an isometry on the tangent bundle. We do not explain how to construct the metric on the tangent bundle in general and simply show the explicit form in the case of \( H^2 \).

Let us parametrize \( UH^2 \) by \((x, y, z)\) where \((x, y) \in H^2 \) and \( 0 \leq z < 2\pi \). In terms of the tangent bundle \( TH^2 \), the point \((x, y, z)\) in \( UH^2 \) denotes the tangent vector \((X, Y) = \)
\((y \cos z, y \sin z)\) at \((x, y) \in H^2\). Let us denote \((x, y) = (x^1, x^2), (X, Y) = (X^1, X^2)\) and the metric on \(H^2\) by \(^{(2)}h_{\mu\nu}\). We define the covariant total derivative of a tangent vector \((X^1, X^2)\) by

\[
DX^\mu = dx^\nu \nabla_\nu X^\mu + dz \frac{\partial X^\mu}{\partial z}
\]

where \(^{(2)}\nabla_\nu\) is the covariant derivative associated with \(^{(2)}h_{\mu\nu}\) and Greek indices run from 1 to 2. The metric on \(UH^2\) induced from \(H^2\) is

\[
ds^2 = \frac{1}{y^2}(dx^2 + dy^2 + (dx + ydz)^2).
\]

We get the universal cover \(\tilde{UH}^2\) by simply replacing the condition \(0 \leq z < 2\pi\) by the condition \(z \in \mathbb{R}\). The isometry group of \(\tilde{SL}(2, \mathbb{R})\) is 4-dimensional which includes \(\tilde{SL}(2, \mathbb{R})\). \(\tilde{SL}(2, \mathbb{R})\) is induced from the isometry group \(PSL(2, \mathbb{R})\) of the base. The additional isometries come from the maps which rotate the unit tangent vectors by the same angle, i.e. send \((x, y, z)\) to \((x, y, z + c)\) with some \(c \in \mathbb{R}\).

### 2.4.5 \(H^3\)

\(H^3\) is the 3-dimensional hyperbolic space which has the standard metric

\[
ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2).
\]

An isometry is well expressed in terms of quaternion. A quaternion \(q\) is a number which can be written as \(q = a + bi + cj + dk(a, b, c, d \in \mathbb{R})\). All quaternions form a non-Abelian field. The multiplication rules for \(i, j, k\) are \(ij = -ji = k, jk = -kj = i, ki = -ik = j\). An isometry of \(H^3\) is expressed as

\[
q \mapsto (aq + b)(cq + d)^{-1}
\]

where

\[
q = x + yi + zj, z > 0
\]

and

\[
a, b, c, d \in \mathbb{C}, ad - bc = 1.
\]

These transformations form a Lie group \(PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm1\}\). All isometries preserve the orientation of \(H^3\).
2.4.6 Sol

Sol is the 3-dimensional group with the following multiplication rule:

\[
\begin{pmatrix}
  a & x \\
  b & y \\
  c & z
\end{pmatrix} \longrightarrow
\begin{pmatrix}
  a + e^{-c}x \\
  b + e^c y \\
  c + z
\end{pmatrix}.
\]  

(18)

A basis of left invariant 1-forms are

\[
\sigma_1 = e^z dx, \sigma_2 = e^{-z} dy, \sigma_3 = dz.
\]  

(19)

The standard metric is one of left-invariant metrics and is given by

\[
ds^2 = (\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2 \\
= e^{2z} dx^2 + e^{-2z} dy^2 + dz^2.
\]  

(20)

The identity component of the isometry group is Sol itself. The discrete isometries are

\[
(x, y, z) \mapsto (\pm x, \pm y, z),
\]  

(21)

\[
(x, y, z) \mapsto (\pm y, \pm x, -z).
\]  

(22)

Therefore IsomSol has eight components. Four of them connected to the following elements are orientation preserving:

\[
(x, y, z) \mapsto (x, y, z),
\]  

(23)

\[
(x, y, z) \mapsto (-x, -y, z),
\]  

(24)

\[
(x, y, z) \mapsto (y, x, -z),
\]  

(25)

\[
(x, y, z) \mapsto (-y, -x, -z).
\]  

(26)

2.4.7 $S^3$

$S^3$ is the unit 3-sphere, the unit sphere embedded in $E^4$, and has a metric induced from $E^4$. So the isometry group is $O(4)$ and its orientation preserving subgroup is $SO(4)$. $S^3$ is diffeomorphic to $SU(2)$. It is known that $SO(4)$ is isomorphic to $(SU(2) \times SU(2))/\mathbb{Z}_2$, i.e. there is a homomorphism $\Phi$ from $SU(2) \times SU(2)$ onto $SO(4)$ and the kernel is $\{ \pm 1 \}$. 


The action of \((q_1, q_2) \in \text{SU}(2) \times \text{SU}(2)\) on \(p \in S^3\) is \(p \mapsto q_1 pq_2^{-1}\) where multiplications are those of \(\text{SU}(2)\). Indeed the action of \((q_1, q_2)\) and \((-q_1, -q_2)\) on \(\text{SU}(2)\) are the same.

Let us introduce a coordinate system. An element of \(\text{SU}(2)\) is written as

\[
p = e^{\alpha \chi_3/2} e^{\beta \chi_2/2} e^{\gamma \chi_3/2}
\]  

(27)

where

\[
\begin{pmatrix}
0 & -i \\
-i & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
-i & 0 \\
0 & i
\end{pmatrix},
\]

(28)

and

\[
0 \leq \alpha < 4\pi, \quad 0 \leq \beta < 2\pi, \quad 0 \leq \gamma < \pi.
\]

(29)

A basis of left-invariant 1-forms \(\{\sigma^i\}\) is defined as the component of the matrix \(p^{-1} dp\) of left-invariant 1-forms:

\[
p^{-1} dp = \sigma^i \chi^i_2.
\]

(30)

The basis \(\{\sigma_i\}\) of left-invariant 1-forms is

\[
\sigma_1 = -\sin \beta \cos \gamma d\alpha + \sin \gamma d\beta,
\]

\[
\sigma_2 = \sin \beta \sin \gamma d\alpha + \cos \gamma d\beta,
\]

\[
\sigma_3 = \cos \beta d\alpha + d\gamma.
\]

(31)

The standard metric is bi-invariant in the point of view that \(S^3\) is identified with \(\text{SU}(2)\). It can be written as

\[
ds^2 = (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^2)^2 = d\alpha^2 + d\beta^2 + d\gamma^2 + 2 \cos \beta d\alpha d\gamma.
\]

(32)

2.4.8 \(S^2 \times \mathbb{R}\)

\(S^2 \times \mathbb{R}\) is a direct product of a 2-sphere \(S^2\) and a line \(\mathbb{R}\). \(S^2\) can be considered as a unit sphere in \(E^3\) which has the metric \(dl^2\) induced from \(E^3\). The standard metric of \(S^2 \times \mathbb{R}\) is

\[
ds^2 = dl^2 + dz^2.
\]

(33)
The isometry group of $S^2$ is $O(3)$ and its orientation preserving subgroup is $SO(3)$. An isometry of $\mathbb{R}$ is a translation or a reflection at a point. A translation preserves the orientation of $\mathbb{R}$ while a reflection reverses it. The isometry group of $S^2 \times \mathbb{R}$ is $\text{Isom}\, S^2 \times \text{Isom}\, \mathbb{R}$, i.e. an isometry is expressed as $(\alpha, \beta)$ where $\alpha$ is an element of $O(3)$ and $\beta$ is a translation or a reflection of $\mathbb{R}$. Since both $\text{Isom}\, S^2$ and $\text{Isom}\, \mathbb{R}$ have two components, $\text{Isom}(S^2 \times \mathbb{R})$ has four components. The identity component is $SO(3) \times \mathbb{R}$. An isometry $(\alpha, \beta)$ preserves the orientation of $S^2 \times \mathbb{R}$ when both $\alpha$ and $\beta$ are either orientation preserving or orientation reversing. Therefore $\text{Isom}^+(S^2 \times \mathbb{R})$ has two components.

### 2.5 Homogeneity preserving diffeomorphisms

Let us consider an especially important class of diffeomorphisms, homogeneity preserving diffeomorphisms [12].

Let $(X, G)$ be a geometry. A diffeomorphism $\phi$ on $X$ is a homogeneity preserving diffeomorphism (HPD) if it preserves the action of $G$ on $X$, i.e. it is the self-equivalence map of a geometry $(X, G)$. If a diffeomorphism $\phi$ is an HPD then $\bar{\phi} : g \mapsto \phi \circ g \circ \phi^{-1}$ is an automorphism of $G$, i.e.

$$\bar{\phi} \in \text{Aut}(G).$$

(34)

Group theoretically, an HPD is an element of the normalizer of $G$ in $\text{Diff}X$, the group of all diffeomorphisms.

A diffeomorphism $\phi$ induces a map $\phi_* : TX \to TX$. We show that $\phi_*$ gives an automorphism of Lie algebra of $G$. Let $\text{Expt}\xi$ denote the 1-parameter group of transformations generated by a vector field $\xi$ on $X$. Let $\text{Alg}(G)$ denote the Lie algebra of $G$, the space of vector fields on $X$ which generate transformations in $G$. By definition the condition for $\phi$ to be an HPD is that $\bar{\phi} : \text{Expt}A \mapsto \phi \circ \text{Expt}A \circ \phi^{-1}$ is an automorphism of $G$, where $A \in \text{Alg}(G)$. For any vector field $\xi$ on $X$ and a diffeomorphism $\phi$,

$$\phi \circ \text{Expt}\xi \circ \phi^{-1} = \text{Expt}\phi_*\xi$$

(35)

holds, i.e. the generator of the 1-parameter family of diffeomorphisms on the left hand
side is $\phi_\ast \xi$. So in particular for any $A \in \text{Alg}(G)$ we have

$$\phi \circ \text{Exp} t A \circ \phi^{-1} = \text{Exp} t \phi_\ast A.$$  

Therefore $\phi$ is an HPD if and only if

$$\phi_\ast \text{Alg}(G) = \text{Alg}(G).$$

Since the restriction of $\phi_\ast$ on $\text{Alg}(G)$ is a linear map and preserves the commutator of vector fields, it is a Lie algebra automorphism, i.e.

$$\phi_\ast \in \text{Aut}(\text{Alg}(G))$$

where $\text{Aut}(\text{Alg}(G))$ is the group of all Lie algebra automorphisms of $\text{Alg}(G)$.

Note that the converse is not true. A Lie algebra automorphism is not always induced from an HPD because an HPD has to be globally defined on $X$.

3 Compact locally homogeneous models

In this section we define the compact locally homogeneous universes and their physical degrees of freedom.

A spacetime is represented by a pair $((^{(4)}M, g_{ab}))$, where $^{(4)}M$ is a 4-manifold and $g_{ab}$ is a Lorentzian 4-metric. We 3+1 decompose the spacetime, i.e. we think of a spacetime $((^{(4)}M, g_{ab})$ as the time evolution of a Riemannian 3-metric $h_{ab}(t)$ on a 3-manifold $M$. We require that this decomposition is synchronous; the lapse function is unity and the shift vector vanishes. Then the 4-metric can be written as

$$g_{ab} = -\nabla_a t \nabla_b t + h_{ab}.$$  

We say a spacetime $(M, h_{ab}(t))$ to be spatially locally homogeneous if $h_{ab}(t)$ for every $t$ is locally homogeneous. We say $(M, h_{ab}(t))$ to be spatially compact if $M$ is compact.

Let $(M, h_{ab}(t))$ be a spatially compact locally homogeneous spacetime. Let us separate the conformal factor $a(t)$ from $h_{ab}(t)$ as

$$h_{ab}(t) = a^2(t)\bar{h}_{ab}(t)$$

(40)
and think of $\bar{h}_{ab}$ as the dynamical variables. Therefore, the physical degrees of freedom are the numbers needed to parametrize $\bar{h}_{ab}$. Hereafter we omit the bar of $\bar{h}_{ab}$ and simply denote it by $h_{ab}$.

As mentioned in the previous section, $M$ has a unique universal cover $\tilde{M}$ up to diffeomorphisms, and $(M, h_{ab}(t))$ is metrically diffeomorphic to $(\tilde{M}, \tilde{h}_{ab}(t))/\Gamma(t)$ where $\tilde{h}_{ab}$ is homogeneous and $\Gamma$ is a discrete subgroup of $\text{Isom}(\tilde{M}, \tilde{h}_{ab})$. All $\Gamma(t)$ must be isomorphic for all $\tilde{M}/\Gamma(t)$ to have the same topology.

Thinking of a manifold $M$ as a quotient $\tilde{M}/\Gamma$ of a simply connected homogeneous manifold $\tilde{M}$ naturally leads us to separating the degrees of freedom of $h_{ab}$ into those of $\tilde{h}_{ab}$ and those of $\Gamma$, i.e. those of the universal cover $(\tilde{M}, \tilde{h}_{ab})$ and the Teichmüller deformations. Here we should recall that there is freedom of diffeomorphisms in giving a universal cover $\tilde{M}$ of $M$. If $\phi$ is an orientation preserving diffeomorphism from $\tilde{M}$ onto itself, $(\tilde{M}, \tilde{h}_{ab})/\Gamma$ and $(\tilde{M}, \phi \circ \tilde{h}_{ab})/\phi \circ \Gamma \circ \phi^{-1}$ are diffeomorphic metrically and have the same orientation. This means that different pairs of $\tilde{h}_{ab}$ and $\Gamma$ may give the same manifold $(M, h_{ab})$. To avoid this uninteresting freedom emerged from diffeomorphisms, and recalling that we have already put aside the conformal factor, we define the degrees of freedom of the universal covering manifold as those of

\[
\{\text{homogeneous metrics on a simply connected manifold which admit compact quotients}\} / \{\text{orientation preserving diffeomorphisms}\} / \{\text{global conformal transformations}\}.
\]

We say that there are degrees of freedom of the universal cover if these equivalence classes can be smoothly deformed.

Once a metric on the universal cover $\tilde{M}$ is fixed, the isometry group of $\tilde{M}$ is determined. Remaining deformations of the manifold $M$ are the Teichmüller deformations.

We investigate all homogeneous universal covers in Sec.4 and all compact quotients and their Teichmüller spaces in Sec.5.
4 Types of universal covers

In this section we give all equivalence classes of homogeneous metrics mentioned at the end of Sec.3. We explain the method of exhausting all of them in Sec.4.1, and we give them explicitly in Sec.4.2.

4.1 The method for obtaining nonmaximal universal covers

Theorem 1 tells us all simply connected maximal geometries which admits compact quotients. But we want to consider a general locally homogeneous metric on a compact manifold, so the requirement that the geometry be maximal is too strong for our purpose. For example, Bianchi IX space does not satisfy this requirement; the geometry \((S^3, \text{topologically}, \text{SU}(2))\) is a subgeometry of the isotropic 3-sphere \((S^3, \text{SO}(4))\). In general, if \((X, G')\) is a subgeometry of \((X, G)\) then \(G'\) is a subgroup of \(G\) and acts transitively on \(X\). It follows that the the stabilizer of \(G'\) is a subgroup of that of \(G\). In other words, a metric on a maximal geometry is more isotropic than those on any of its subgeometries.

Theorem 1 says nothing about nonmaximal geometries which have compact quotients. Nevertheless we can show that we have only to consider subgeometries of the above eight geometries. Let \((X, G')\) be a simply connected 3-geometry which admits a compact quotient. Then there is a discrete freely acting subgroup \(\Gamma'\) of \(G'\) which makes \(X/\Gamma'\) compact. Let \((X, G)\) be a maximal geometry which has \((X, G')\) as its subgeometry. Then \(\Gamma'\) is a subgroup of \(G\) which makes \(X/\Gamma'\) compact, so our maximal, simply connected geometry \((X, G)\) admits compact quotients. By Theorem 1 our maximal geometry \((X, G)\) is one of the eight Thurston geometries, which proves that \((X, G')\) is a subgeometry of one of the eight Thurston geometries.

Let us go back to Riemannian manifolds. If a Riemannian manifold \((\tilde{M}, \tilde{h}_{ab})\) admits compact quotients then the geometry \((\tilde{M}, \text{Isom}^+ \tilde{M})\) is a subgeometry of the eight Thurston maximal geometries. Our strategy is to consider all subgeometries \((\tilde{M}, G)\) of the Thurston geometries and \(G\)-invariant metrics on \(\tilde{M}\). Here we must recall that \(G\) may not be the isometry group itself, i.e. it may be the case that the \(G\)-invariant metric on \(\tilde{M}\) admits a larger isometry group than \(G\). So it is necessary to find the whole \(\text{Isom}^+ \tilde{M}\).
in each case. Of course, if \((M, \tilde{h}_{ab})\) and \((M, \tilde{h}'_{ab})\) are in the same equivalence class then \(\text{Isom}^+(\tilde{M}, \tilde{h}_{ab})\) and \(\text{Isom}^+(\tilde{M}, \tilde{h}'_{ab})\) are isomorphic.

We adopt the following procedure to find all equivalence classes of homogeneous metrics on a simply connected manifold which admit compact quotients and their isometry groups:

1) List up all subgeometries \((\tilde{M}, G)\) of each of the Thurston geometries.
2) For each subgeometry \((\tilde{M}, G)\), form a general \(G\)-invariant Riemannian metric \(\tilde{h}'_{ab}\) on \(\tilde{M}\).
3) Transform the metric into a certain simple ‘representative metric’ \(\tilde{h}_{ab}\) by diffeomorphisms together with global conformal transformations and find the equivalence classes.
4) Find the isometry group of \((\tilde{M}, \tilde{h}_{ab})\).

We thus get all possible universal covers of locally homogeneous Riemannian manifolds. Logically speaking, equivalence classes thus found do not necessarily have to admit compact quotients. So we have another task.

5) Make sure whether the equivalence class really admits compact quotients.

The Bianchi classification \([2, 3]\) is of great use in carrying out the procedure given above, for it is the classification of all simply connected 3-dimensional Lie groups up to isomorphisms. A correspondence of the Bianchi classification to the eight Thurston geometries was first pointed out by Fagundes \([13]\). We elaborate this correspondence precisely in terms of geometry in order to make it applicable for our program.

Let us say a geometry \((X, G)\) to be minimal if it does not have a proper subgeometry. We can say that the Bianchi classification is the classification of 3-dimensional minimal geometries, for the following fact is well known:

**FACT 1** Any minimal, simply connected 3-dimensional geometry is equivalent to \((X, G)\) where \(X = \mathbb{R}^3, G = \text{one of Bianchi I to Bianchi VIII groups}; X = S^3, G = \text{Bianchi IX group};\) or \(X = S^2 \times \mathbb{R}, G = \text{SO}(3) \times \mathbb{R}\).

Let us call the minimal geometries with Bianchi groups the **Bianchi minimal geometries**, the last one the **Kantowski-Sachs-Nariai (KSN) minimal geometry**, and in all the **Bianchi-Kantowski-Sachs-Nariai (BKSN) minimal geometries**.
We emphasize that the Thurston geometries are the maximal 3-dimensional geometries which give compact quotients while the BKSNS geometries are the minimal 3-dimensional geometries.

Let us categorize Thurston’s geometries \((X, \text{Isom} X)\) by the topology of \(X\): the one with \(X \cong \mathbb{R}^3\) i.e. \(X = E^3, H^3, H^2 \times \mathbb{R}, \widetilde{\text{SL}(2, \mathbb{R})}, \text{Nil, or Sol}\); the one with \(X = S^3\); and the one with \(X = S^2 \times \mathbb{R}\). A Thurston geometry in the first case has at least one of Bianchi I to Bianchi VIII geometries as its subgeometry. The geometry \((S^3, \text{Isom}^+ S^3)\) has Bianchi IX geometry as its subgeometry. The geometry \((S^2 \times \mathbb{R}, \text{Isom}(S^2 \times \mathbb{R}))\) has the KSN geometry as its subgeometry.

We explain the procedure written above in the case of the Bianchi minimal geometries. The case of the KSN geometry is rather easy and is discussed separately in Sec.4.2. There are two convenient features when \((X, G)\) is a Bianchi geometry. First, we can deal with the \(G\)-invariant 1-forms \((\sigma^i)_a (i = 1, 2, 3)\). Any \(G\)-invariant metric \(\tilde{h}'_{ab}\) can be written as

\[
\tilde{h}'_{ab} = \tilde{h}_{ij} (\sigma^i)_a (\sigma^j)_b
\]

with \(\tilde{h}_{ij}\) being a constant matrix. Second, a Lie algebra automorphism is always induced from a diffeomorphism which is globally defined.

We discuss homogeneity preserving diffeomorphisms (HPDs), because they play an essential role in reducing the redundant degrees of freedom of the metric. We have already seen in Sec.2.5 that an HPD \(\phi\) on a geometry \((\tilde{M}, G)\) induces a Lie algebra automorphism of \(\text{Alg}(G)\), by which the invariant 1-forms transform as

\[
(\sigma'^i)_a = f^i_j (\sigma^j)_a \mapsto (\sigma^i)_a
\]

where \(f^i_j\) is a constant \(3 \times 3\) matrix and \(\sigma^i\) and \(\sigma'^i\) must satisfy the Maurer-Cartan equations,

\[
d\sigma^i = -\frac{1}{2} C^i_{jk} \sigma^j \wedge \sigma^k, \quad (44)
\]

\[
d\sigma'^i = -\frac{1}{2} C^i_{jk} \sigma'^j \wedge \sigma'^k. \quad (45)
\]

Substituting (43) into (45) and using (44), we have

\[
f^i_j C^j_{lm} = C^i_{jk} f^j_l f^k_m. \quad (46)
\]
Conversely, given a Lie algebra automorphism $f^i_j$ which satisfy (46) the corresponding HPD is obtained by solving a differential equation. We must introduce a coordinate system $x = \{x^i\}$ ($i = 1, 2, 3$) to denote a point $p$ in $\tilde{M}$. Let $\phi^i(x)$ denote the $i$-th coordinate of $\phi(p)$. Then we can expand $\sigma^i$ by $dx^i$ as

\[ \sigma^i = \sigma^i_j(x) dx^j. \] (47)

The primed one is the right hand side with $\{x^i\}$ replaced by $\{\phi^i\}$:

\[ \sigma'^i = \sigma'^i_j(\phi(x)) d\phi^j = \sigma^i_j(\phi(x)) \frac{\partial \phi^j}{\partial x^k} dx^k. \] (48)

It follows from $\sigma'^i = f^i_j \sigma^j$ that

\[ \sigma^i_j(\phi(x)) \frac{\partial \phi^j}{\partial x^k} = f^i_j \sigma^j_k(x). \] (49)

So the Jacobi matrix is

\[ \frac{\partial \phi^i}{\partial x^j} = (\sigma^{-1})^i_k(\phi(x)) f^k_\ell \sigma^\ell_j(x). \] (50)

The HPD is the solution to this differential equation.

Analytically, eq.(46) is the local integrability condition for eq.(50). If $G$ is a Bianchi group an HPD $\phi$ can be identified with a Lie group automorphism $\tilde{\phi}$ since the manifold $\tilde{M}$ is diffeomorphic to $G$. Moreover, there is a correspondence between Lie group automorphisms $\tilde{\phi}$ and Lie algebra automorphisms $f^i_j$, for there is a one-to-one correspondence between simply connected Lie groups and Lie algebras. Therefore, in the case of the Bianchi geometries, the necessary and sufficient condition for $f^i_j$ to be induced from an HPD on $\tilde{M}$ is that eq.(46) holds.

Let us reduce the redundant degrees of freedom of a general $G$-invariant metric by the degrees of freedom of HPDs and global conformal transformations. An HPD $\phi$ transforms the invariant 1-forms $(\sigma'^i)_a$ as (43). By (42) the metric transforms as

\[ \tilde{h}'_{kl}(\sigma^i)_a(\sigma^j)_b = f^k_i f^l_j \tilde{h}_{kl}(\sigma^i)_a(\sigma^j)_b \]

\[ = \tilde{h}_{ij}(\sigma'^i)_a(\sigma'^j)_b \]

\[ \rightarrow \tilde{h}_{ij}(\sigma^i)_a(\sigma^j)_b. \] (51)
For any metric $\tilde{h}_{ij}'$ we find a transformation $\tilde{h}_{ij}'$ as (51) by diffeomorphisms to get as simple a metric $\tilde{h}_{ij}$ as possible.

It is more convenient to deal not with $\tilde{h}_{ij}'$ itself but with triad 1-form $(e^i)_a$. A triad 1-form of the metric $\tilde{h}_{ab}'$ is

$$(e^i)_a = b^i_j (\sigma^i)_a$$

where

$$\tilde{h}_{ij}' = \delta_{kl} b^k_i b^l_j.$$  

(53)

We are going to reduce the degrees of freedom of $b^i_j$ by using the degrees of freedom of $f^i_j$ together with a global conformal transformation $b^i_j = a b^i_j \mapsto b^i_j$ to obtain a simple $b^i_j$. Note that infinitely many $b^i_j$ correspond to the same $\tilde{h}_{ij}'$; in fact $b^i_j$ and $r^i_k b^k_j$ correspond to the same $\tilde{h}_{ij}' = \delta_{kl} b^k_i b^l_j$ if $r^i_j \in O(3)$. Therefore the condition for $\tilde{h}_{ij}' = \delta_{kl} b^k_i b^l_j$ to be any positive definite symmetric matrix is for $b^i_j$ to be any element of $O(3) \setminus GL(3, \mathbb{R})$.

The transformation law for $b^i_j$ is

$$b^i_j = b^i_k f^k_j \mapsto b^i_j.$$  

(54)

We must find the matrix $b^i_j$ such that

$$b^i_j = a b^i_k f^k_j$$

(55)

expresses any element of $O(3) \setminus GL(3, \mathbb{R})$, i.e.

$$r^i_k b^k_j = a r^i_k b^k_j f^l_j$$

(56)

expresses any element of $GL(3, \mathbb{R})$, where $a > 0$, $r^i_j \in O(3)$ and $f^i_j$ satisfies (46).

The number $U$ of degrees of freedom of the universal cover is

$$U = \left( \begin{array}{c} \text{degrees of freedom of} \\ \text{GL}(3, \mathbb{R}) \end{array} \right) - \left( \begin{array}{c} \text{degrees of freedom of} \\ r^i_k \end{array} \right) - \left( \begin{array}{c} \text{degrees of freedom of} \\ \text{Lie algebra automorphisms} \end{array} \right)$$

$$= 9 - 3 - 1 - \dim(\text{Aut}(\text{Alg}G))$$

$$= 5 - \dim(\text{Aut}(\text{Alg}G))$$

(57)
for the Bianchi geometries except for Bianchi I geometry. A Bianchi I group-invariant metric has global conformal transformations. The degrees of freedom of conformal transformations (=1) is included in those of diffeomorphisms so that

\[ U = 6 - \text{dim}(\text{Aut}(\text{Alg}G)). \]  

(58)

It follows from \( \text{dim}(\text{Aut}(\text{Alg}G)) = 9 \) that \( U = 0 \).

### 4.2 Types of universal covers

Now we show the procedure shown in Sec.4.1 in detail for a few of the Thurston maximal geometries and show the results for the others. In this section we find the equivalence classes which are slightly different from (I1), where ‘orientation preserving diffeomorphisms’ is replaced by ‘diffeomorphisms’. In Sec.4.3 we find the equivalence classes (I1) as a corollary.

#### 4.2.1 \((E^3, \text{Isom}E^3)\)

We have seen in Sec.2.4 that \( \text{Isom}E^3 = \text{IO}(3) \). A subgroup of \( \text{IO}(3) \) which acts transitively on \( \mathbb{R}^3 \) must contain the translations in (I), i.e. the corresponding Lie algebra must contain \( K_1, K_2, \) and \( K_3 \). The basis of possible Lie subalgebras up to isomorphisms are the following:

1) \( A_1 = K_1, A_2 = K_2, A_3 = K_3 \); (59)

2) \( A_1 = K_1, A_2 = K_2, A_3 = K_3 + K_6 \); (60)

3) \( A_1 = K_1, A_2 = K_2, A_3 = K_3, A_4 = K_6 \); (61)

4) \( A_i = K_i \quad (1 \leq i \leq 6.) \) (62)

The connected Lie groups corresponding to above Lie algebras are 1)\( \mathbb{R}^3 \) (all translations) i.e. Bianchi I group, 2)Bianchi VII(0) group, 3)\( \text{SO}(2) \times \mathbb{R}^3 \), and 4)\( \text{ISO}(3) \). The groups corresponding to minimal geometries are \( \mathbb{R}^3 \) and Bianchi VII(0) group. \( \text{SO}(2) \times \mathbb{R}^3 \) contains \( \mathbb{R}^3 \) as its subgroup and \( \text{ISO}(3) \) contains both \( \mathbb{R}^3 \) and Bianchi VII(0) group. Let us consider each of them in detail.
1) $\mathbb{R}^3$.

The invariant 1-forms of $\mathbb{R}^3$ are

$$\sigma^i = dx^i \quad (i = 1, 2, 3)$$

(63)

where $x^1 = x, x^2 = y, x^3 = z$. Any $\mathbb{R}^3$-invariant metric can be written as

$$\tilde{h}_{ab} = \tilde{h}_{ij}(\sigma^i)_a(\sigma^j)_b.$$  

(64)

Since all structure constants of $\mathbb{R}^3$ vanish, (64) is trivial. This means that $f^{i}_{j}$ can be any element of $\text{GL}(3, \mathbb{R})$. In fact, by (43) and (63) the matrix $f^{i}_{j}$ is the Jacobi matrix itself of an HPD so that any HPD is given by the solution of (50):

$$x^i \mapsto f^{i}_{j} x^j + a^i,$$

(65)

where $f^{i}_{j} \in \text{GL}(3, \mathbb{R})$ and $a^i$ is a constant vector. We can make $b^i$ to be an identity matrix, which is enough for (56) to express a general element of $\text{GL}(3, \mathbb{R})$. Any $\mathbb{R}^3$-invariant metric is in the same equivalence class as the representative metric

$$ds^2 = dx^2 + dy^2 + dz^2.$$  

(66)

The isometry group of this metric is $\text{IO}(3)$, as given in Sec. 2.4.

Note especially that any $\mathbb{R}^3$-invariant metric is $\text{IO}(3)$-invariant, i.e. its isometry group is isomorphic to $\text{IO}(3)$. In this case HPDs are translations, rotations, and stretches. Translations of course do not change the metric because they are isometries. Using degrees of freedom of other two types of HPDs we can transform the metric to the standard metric. In fact, there is no need of global conformal transformations in this case. This comes from the fact that $E^3$ has global conformal isometries. ($E^3$ is the only such a manifold.)

2) Bianchi VII(0) group.

The invariant 1-forms of Bianchi VII(0) group are

$$\sigma^1 = \cos zdz + \sin zdy, \quad \sigma^2 = -\sin zdz + \cos zdy, \quad \sigma^3 = dz.$$  

(67)

The nonvanishing structure constants are

$$C^{1}_{23} = -C^{1}_{32} = C^{2}_{31} = -C^{2}_{13} = 1.$$  

(68)
Solving (66) we have either
\[ f_1 = -f_2, f_2 = f_1, f_3 = f_2 = 0, f_3 = 1 \]  
(69)
or
\[ f_1 = f_2, f_2 = -f_1, f_3 = f_2 = 0, f_3 = -1. \]  
(70)
The simplest choice of the components of \( b_{ij} \) for (56) to express a general element of \( \text{GL}(3, \mathbb{R}) \) is
\[ b_1 = \alpha, b_2 = \alpha^{-1}, b_3 = 1, \]
\[ b_{ij} = 0 \text{ (the others)}, \]  
(71)
where \( \alpha \geq 1 \). If we rewrite the parameter \( \alpha \) as \( \alpha = e^\lambda \) the representative metric is
\[ ds^2 = e^{2\lambda}(\cos z dx + \sin z dy)^2 + e^{-2\lambda}(-\sin z dx + \cos z dy)^2 + dz^2 \]  
(72)
where \( \lambda \geq 0 \). The equivalence classes form a 1-parameter family. Note that the metric (66) is the special case (\( \lambda = 0 \)) of the metric (72).

The isometry is either
\[ \phi_1(x) = \pm(x \cos c^3 - y \sin c^3) + c^3, \]
\[ \phi_2(x) = \pm(x \sin c^3 + y \cos c^3) + c^3, \]
\[ \phi_3(x) = z + c^3 \]  
(73)
or
\[ \phi_1(x) = \pm(x \cos c^3 - y \sin c^3) + c^3, \]
\[ \phi_2(x) = \mp(x \sin c^3 + y \cos c^3) + c^3, \]
\[ \phi_3(x) = -z + c^3 \]  
(74)
where \( c^3(i = 1, 2, 3) \) are constants. The isometry group have four components. All isometries are orientation preserving. The identity component of the isometry group is given by taking the positive signs of (73). It is Bianchi VII(0) group.
3) SO(2) × R³.

The group SO(2) × R³ has R³ as its subgroup. So from 1) any SO(2) × R³-invariant metric is in the same equivalence class as the metric $ds^2 = dx^2 + dy^2 + dz^2$, and hence is IO(3)-invariant.

4) ISO(3).

Any ISO(3)-invariant metric is IO(3)-invariant.

4.2.2 (Nil, IsomNil)

Nil is the only minimal subgroup of the group IsomNil. All Nil-invariant metrics are in the same equivalence class as the representative metric

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$  (75)

Its isometry group is already given explicitly in Sec. 2.4.2. The isometry group is 4-dimensional.

4.2.3 (H² × R, Isom(H² × R))

The subgeometries of the geometry (H² × R, Isom(H² × R)) with connected Lie groups are Bianchi III geometry and (H² × R, Isom₀(H² × R)).

The isometry group of a general Bianchi III-invariant metric is 3-dimensional and its identity component is Bianchi III group itself. This group does not admit compact quotients (Appendix) and will not be discussed further.

Any Isom₀(H² × R)-invariant metric is equivalent to

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2) + dz^2,$$  (76)

which is the standard metric appeared in Sec. 2.4.3 where its isometry group Isom(H² × R) is also given.

4.2.4 (SL(2, R), IsomSL(2, R))

Transitively acting, connected subgroups of the group IsomSL(2, R) are Bianchi III group, Bianchi VIII group = SL(2, R), and Isom₀SL(2, R). The first two correspond to minimal subgeometries.
General Bianchi III-invariant metrics do not admit compact quotients; general $\widetilde{\text{SL}}(2, \mathbb{R})$-invariant metrics do not admit compact quotients either (Appendix). Only $\text{Isom}_0\widetilde{\text{SL}}(2, \mathbb{R})$-invariant metrics admit compact quotients.

The representative metric is of the form

$$ds^2 = \frac{1}{y^2}(e^{(2/\sqrt{3})\lambda}(dx^2 + dy^2) + e^{-(4/\sqrt{3})\lambda}(dx + ydz)^2),$$

which forms a one parameter family. The isometry group is found to be $\text{Isom}\widetilde{\text{SL}}(2, \mathbb{R})$ explained in Sec.2.4.4.

4.2.5 $(H^3, \text{PSL}(2, \mathbb{C}))$

The group $\text{PSL}(2, \mathbb{C})$ has two minimal subgroups, Bianchi V group and Bianchi VII(A) group. Any subgroup of $\text{PSL}(2, \mathbb{C})$ has either of them as its subgroup.

The representative metric of the Bianchi V-invariant metrics is

$$ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2).$$

Any Bianchi V-invariant metric is $\text{PSL}(2, \mathbb{C})$-invariant.

The representative metric of the Bianchi VII(A)-invariant metric is

$$ds^2 = \alpha^2 e^{2Az}(\cos zdx + \sin zdy)^2 + \alpha^{-2} e^{2Az}(-\sin zdx + \cos zdy)^2 + dz^2$$

where $\alpha \geq 1$. Its isometry group is 3-dimensional and has two components. All isometries preserve the orientation. The identity component is Bianchi VII(A) group itself. The metric (78) is a special case of the metric (79). However, Bianchi VII(A) geometry does not admit compact quotients (Appendix).

4.2.6 $(\text{Sol}, \text{IsomSol})$

The geometries $(\text{Sol}, \text{Isom}_0\text{Sol}) = (\mathbb{R}^3, \text{Sol})$ are the only subgeometry of $(\text{Sol}, \text{IsomSol})$ with a connected group. The representative metric is

$$ds^2 = \frac{1}{2} e^{2\lambda}(e^zdx + e^{-z}dy)^2 + \frac{1}{2} e^{-2\lambda}(-e^zdx + e^{-z}dy)^2 + dz^2.$$  

The equivalence classes form a one parameter family. The isometry group is $\text{Isom}^+\text{Sol}$ for $\lambda \neq 0$ and $\text{IsomSol}$ for $\lambda = 0$, which are mentioned in Sec.2.4.6.
4.2.7 \quad (S^3, O(4))

The group O(4) has Bianchi IX group i.e. SU(2) as its minimal subgeometry. The representative metric is

\[ ds^2 = e^{2(\lambda_+/\sqrt{3}+\lambda_-)}(\sigma^1)^2 + e^{2(\lambda_+/\sqrt{3}−\lambda_-)}(\sigma^2)^2 + e^{−(4/\sqrt{3})\lambda_+}(\sigma^3)^2 \] (81)

where \( \{\sigma^i\} (i=1,2,3) \) is a basis of left-invariant 1-forms given in Sec.2.4.7. The equivalence classes form a two parameter family. The isometry group is found to have four components; the identity component is SU(2) itself. The discrete isometries are written as

\[ p \mapsto p\chi_i \quad (i = 1, 2, 3). \] (82)

If we put \( \lambda_- = 0 \), then the isometry group becomes larger. Its identity component is \((SU(2) \times U(1))/\mathbb{Z}_2\). It has two components and the discrete isometry is

\[ p \mapsto p\chi_3. \] (83)

Here the action of U(1) is the right multiplication of a matrix of the form

\[
\begin{pmatrix}
  e^{i\theta} & 0 \\
  0 & e^{-i\theta}
\end{pmatrix}
\]

to the element of SU(2). If we further put \( \lambda_+ = 0 \), then the isometry group is O(4).

4.2.8 \quad (S^2 \times \mathbb{R}, \text{Isom}(S^2 \times \mathbb{R}))

The Thurston geometry \((S^2 \times \mathbb{R}, \text{Isom}(S^2 \times \mathbb{R}))\) has the KSN geometry \((S^2 \times \mathbb{R}, SO(3) \times \mathbb{R})\) as its minimal subgeometry. The group \(SO(3) \times \mathbb{R}\) is the identity component of Isom\((S^2 \times \mathbb{R})\).

Let us parametrize points in \( S^2 \times \mathbb{R} \) by \((r, \theta, z)\) where \( 0 \leq r < \pi, -\pi \leq \theta < \pi \) and \( z \in \mathbb{R} \). Any \( SO(3) \times \mathbb{R}\)-invariant metric can be written as

\[ ds^2 = a^2 dl^2 + b^2 dz^2, \] (84)

where \( dl^2 \) is the metric on \( S^2 \):

\[ dl^2 = dr^2 + \sin^2 r d\theta^2. \] (85)
Transforming the metric by the diffeomorphism \((r, \theta, az) \mapsto (r, \theta, bz)\) and by the global conformal transformation \(\hat{h}_{ij}' \mapsto \hat{h}_{ij} = a^{-2} \hat{h}_{ij}'\), we get the representative metric

\[ ds^2 = dl^2 + dz^2. \]  

(86)

This is the same as the standard metric in Sec.2.4.8 and hence the isometry group is \(\text{Isom}(S^2 \times \mathbb{R})\).

### 4.3 Summary of the section

Let us summarize the result in this section as a theorem.

**THEOREM 2** Any simply connected 3-dimensional Riemannian manifold which admits a compact quotient is one of the types in Table 1 up to diffeomorphism and global conformal transformation.

Now we give equivalence classes (41) which give the universal covers of all orientable compact locally homogeneous Riemannian manifolds.

**THEOREM 3** Any simply connected 3-dimensional Riemannian manifold which admits a compact quotient is one of the types in Table 2 up to orientation preserving diffeomorphism and global conformal transformation.

Let us give a brief proof. If a manifold \(\tilde{M}\) has an orientation reversing isometry the result is the same as Theorem 2 above; if not, \((\tilde{M}, \tilde{g}_{ab})\) and \((\tilde{M}, r_\ast \tilde{g}_{ab})\) are not in the same equivalence class but have isomorphic isometry groups, where \(r\) is an orientation reversing diffeomorphism. An orientation reversing diffeomorphism is well expressed by the group multiplication of the minimal subgeometry for types a to g, namely \(r : p \mapsto p^{-1}\). This map carries a left multiplication to a right multiplication. It can be easily verified by exhaustion that there is an orientation reversing isometry if and only if \(r\) is an isometry, i.e. the left-invariant metric is also right-invariant. Types a1, b, f, g1 and g2 are in the case of \(\text{Isom}\tilde{M} = \text{Isom}^+\tilde{M}\). In the case of Types a and f, the range of the parameter \(\lambda\) must be expanded to \(\mathbb{R}\). In the case of Types b and g1 the metric \((\tilde{g}_R)_{ab} \equiv r_\ast (\tilde{g}_L)_{ab}\) has
right invariant with respect to Nil and SU(2) respectively. In the case of Type g2 the
metric \((\tilde{g}_R)_{ab} \equiv r_*(\tilde{g}_L)_{ab}\) is invariant under right action of SU(2) and left action of U(1).

The number of the degrees of freedom is that of parameters in the representative
metric, which we call \textit{universal cover parameters}. They are also characterized by free
parameters appearing in the relative ratios of the nonvanishing principal sectional curva-
tures. These ratios can be chosen to be dynamical variables.

5 \textbf{Compact quotients and their Teichmüller spaces}

We shall give compact quotients modeled on each of the universal covers from Type
a to Type h given in the previous section and give the dimension \(T\) of their Teichmüller
spaces. We make use of the knowledge of compact quotients modeled on the Thurston
maximal geometries and the dimension of their Teichmüller spaces, without proofs. The
Teichmüller spaces of Seifert bundles modeled on maximal geometries are discussed by
Ohshika \[16\] and Kulkarni \textit{et al.} \[17\].

A quotient \(X/\Gamma\) of a subgeometry \((X, G')\) of a geometry \((X, G)\) can also be viewed as
that of \((X, G)\). Since \(\Gamma\) is a subgroup of \(G'\) and \(G'\) is a subgroup of \(G\), \(\Gamma\) is a discrete
subgroup of \(G\), which implies \(X/\Gamma\) is a quotient of \((X, G)\).

Thus, all compact locally homogeneous Riemannian manifold and their Teichmüller
spaces are obtained by the following procedure.

1) Pick up a Thurston geometry \(X\) (say, \(E^3\)) and list up all compact locally homoge-
neous manifolds \(M \cong X/\Gamma\) modeled on \((X, \text{Isom}^+X)\).

2) List up all types (say, Type a1 and Type a2) of universal covers \(\tilde{M}\) which is subge-
ometries of \((X, \text{Isom}^+X)\).

3) For each universal cover type, check whether \(X/\Gamma\) can be modeled on \((\tilde{M}, \text{Isom}\tilde{M})\)
or not by checking whether \(\Gamma\) is a subgroup of Isom\(\tilde{M}\).

4) Find their Teichmüller spaces.

Note that a representation \(\rho : \pi_1(M) \to \text{Isom}^+\tilde{M}\) is determined by the images of the
fixed generators of \(\pi_1(M)\).
For the universal covers which have L and R types, we show the results of L type; the representations in R type can be obtained in the same way.

5.1 Type a

There are following six compact orientable quotients modeled on $E^3$, i.e. Type a2. (Theorem 3.5.5 of [14]).

1) The fundamental group is generated by $a$, $b$, and $c$ and the relations are

$$[a, b] \equiv aba^{-1}b^{-1} = 1, \ [a, c] = 1, \ [b, c] = 1,$$

which will be denoted as

$$\pi_1(M) = \langle a, b, c; [a, b], [a, c], [b, c] \rangle. \quad (87)$$

Here the convention is adopted that a product $ab$ denotes a turn of curve $b$ followed by a turn of curve $a$, so that the same expression can be used for both an element of the fundamental group and an element of the discrete group in the isometry group. When $\pi_1(M)$ is represented in Isom$E^3$, the generators $a$, $b$, and $c$ are translations in different directions. The manifold is a torus, $T^3$.

2) The fundamental group is

$$\pi_1(M) = \langle a, b, c; [a, b], cac^{-1}a, cbc^{-1}b \rangle. \quad (88)$$

In Isom$E^3$, $a$ and $b$ are translations and $c$ is a screw motion with a rotation angle $\pi$. So the direction of $c$ must be orthogonal to those of $a$ and $b$. The manifold is $T^3/\mathbb{Z}_2$.

3) The fundamental group is

$$\pi_1(M) = \langle a, b, c; [a, b], cac^{-1}b^{-1}, cbc^{-1}ba \rangle. \quad (89)$$

In Isom$E^3$, $a$ and $b$ are translations and $c$ is a screw motion with a rotation angle $2\pi/3$. The direction of $c$ must be orthogonal to those of $a$ and $b$. The vectors of translations $a$ and $b$ must have the same length. The manifold is $T^3/\mathbb{Z}_3$.

4) The fundamental group is

$$\pi_1(M) = \langle a, b, c; [a, b], cac^{-1}b^{-1}, cbc^{-1}a \rangle. \quad (90)$$
In $\text{Isom} E^3$, $a$ and $b$ are translations and $c$ is a screw motion with a rotation angle $\pi/2$. The direction of $c$ must be orthogonal to those of $a$ and $b$. The vectors of translations $a$ and $b$ must have the same length. The manifold is $T^3/\mathbb{Z}_4$.

5) The fundamental group is

$$\pi_1(M) = \langle a, b, c; [a, b], cac^{-1}b^{-1}, cbc^{-1}b^{-1}a \rangle.$$  \hspace{1cm} (92)

In $\text{Isom} E^3$, $a$ and $b$ are translations and $c$ is a screw motion with a rotation angle $\pi/3$. The direction of $c$ must be orthogonal to those of $a$ and $b$. The vectors of translations $a$ and $b$ must have the same length. The manifold is $T^3/\mathbb{Z}_6$.

6) The fundamental group is

$$\pi_1(M) = \langle a, b, c; caba^{-1}b, ab^2a^{-1}b^2, ba^2b^{-1}a^2 \rangle = \langle a, c; ca^2c^{-1}a^2, ac^2a^{-1}c^2 \rangle.$$  \hspace{1cm} (93)

In $\text{Isom} E^3$, all generators are screw motions with a rotation angle $\pi/2$.

Let $\Gamma_i (i = 1, 2, \cdots, 6)$ denote the images of the representations of the fundamental groups from 1) to 6). We will write the quotient of the universal cover of Type a2 by $\Gamma_1$ as $\tilde{M}_{a2}/\Gamma_1$ or simply as $a2/1$, and so forth.

All fundamental groups have representations in $\text{Isom}^+ \tilde{M}_{a1}$. The first five ones do because Type a1 has translations in a plane and one screw motion in the orthogonal direction to the plane. The last one also does by virtue of the discrete elements of $\text{Isom}^+ \tilde{M}_{a1}$, as we will see later.

Let us find the representations of $\pi_1(\tilde{M}_{a1}/\Gamma_1)$ in $\text{Isom} \tilde{M}_{a1}$. It must be faithful, discrete in $\text{Isom}^+ \tilde{M}$, and has no fixed points. First we consider the case that $\pi_1(M)$ is represented in $\text{Isom}_0 \tilde{M}_{a1}$. Let us denote an element $a$ of $\text{Isom}_0 \tilde{M}_{a1}$ as

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv \text{Exp} a_2A_2 \circ \text{Exp} a_1A_1 \circ \text{Exp} a_3A_3,$$  \hspace{1cm} (94)

where $A_i$ are vector fields in eq. (60). This is a screw motion along the $z$-axis followed by a translation in the $xy$-plane. The multiplication rule for Bianchi VII(0) group is then
found by explicit calculations,

\[ ab = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 + b_3 \end{pmatrix} + R_{a_3} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} . \]  

(95)

The inverse of \( a \) is

\[ a^{-1} = \begin{pmatrix} -R_{a_3} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ -a_3 \end{pmatrix} . \]  

(96)

The representation of the \( \pi_1(\tilde{M}_{a_1}/\Gamma) \) is obtained by writing generators with components and demanding (87). The solution is

\[ a = \begin{pmatrix} a_1 \\ a_2 \\ 2l\pi \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ 2m\pi \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ 2n\pi \end{pmatrix} \]  

(97)

where \( l, m \) and \( n \) are integers.

Similarly, the representations including elements of \( \text{Isom}\tilde{M} \) not connected to the identity are generated by \( a, b \) and \( c = h \circ c' \) where \( h : (x, y, z) \mapsto (-x, -y, z) \) and

\[ a = \begin{pmatrix} a_1 \\ a_2 \\ 2l\pi \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ 2m\pi \end{pmatrix}, \quad c' = \begin{pmatrix} c_1 \\ c_2 \\ (2n + 1)\pi \end{pmatrix} \]  

(98)

where we have used that \( h \circ c \circ h^{-1} = h(c) = (-c_1, -c_2, c_3) \).

The Teichmüller space is obtained by taking conjugacy class of discrete groups by \( \text{Isom}\tilde{M} = \text{Bianchi VII}(0) \) group, for the universal cover has no global conformal isometries. In the case that the generators of \( \Gamma \) is (97), the conjugate of the generators by \( s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \in \text{Isom}\tilde{M}_{a_1} \) is

\[ sas^{-1} = \begin{pmatrix} R_{s_3} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ 2l\pi \end{pmatrix}, \quad sbs^{-1} = \begin{pmatrix} R_{s_3} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ 2m\pi \end{pmatrix}, \quad scs^{-1} = \begin{pmatrix} R_{s_3} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ 2n\pi \end{pmatrix} . \]  

(99)
We see that the vectors \( \begin{pmatrix} a_1 \\ a_2 \\ 2l \pi \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ 2m \pi \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \) are rotated by the conjugation. So we can choose a representative element of the equivalence class corresponding to a point in the Teichmüller space as

\[
a = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad (100)
\]

where \( a_1 > 0 \). In the case of \((98)\) the representative is

\[
a = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, c = h \circ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \begin{pmatrix} 0 \\ 2n \pi \end{pmatrix}, \quad (101)
\]

It follows from above that the dimension \( T \) of the Teichmüller space is five.

Along the same line of argument, we can see that for the other fundamental groups the representative elements of the equivalence class corresponding to points in the Teichmüller space and its dimension are the following:

\[
a_{1/2} : \quad a = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}, c = \begin{pmatrix} 0 \\ 0 \\ (2n + 1) \pi \end{pmatrix}; \quad (102)
\]

\[
a_2 : \quad a = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}, c = h \circ \begin{pmatrix} 0 \\ 0 \\ 2n \pi \end{pmatrix}; \quad (103)
\]

\[
a_{a_2} : \quad a = \begin{pmatrix} a_2 \\ 2m \pi \\ 2n \pi \end{pmatrix}, b = \begin{pmatrix} 0 \\ b_2 \\ 2n \pi \end{pmatrix}, c = h k \circ \begin{pmatrix} c_1 \\ 0 \end{pmatrix}, (a_2 > 0); \quad (104)
\]

\[
a_{k_2} : \quad a = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ 0 \\ 2n \pi \end{pmatrix}, c = k \circ \begin{pmatrix} 0 \\ c_2 \end{pmatrix}; \quad (105)
\]

\[T = \text{dim}(\text{Teich(\tilde{M}_{a_1}/\Gamma_1)}) = 3.\]  \quad (106)
Here $k : (x, y, z) \mapsto (-x, y, -z)$, $a_1 > 0$ and the double signs are taken in the same order. Most of the other parameters should not vanish for the representations to be faithful. We have used the fact that $h \circ a \circ h = h(a) = (-a_1, -a_2, a_3)$ and $k \circ a \circ k = k(a) = (-a_1, a_2, -a_3)$ holds for any $a \in \text{Isom} \tilde{M}_{a_1}$, which can be easily verified by the explicit calculation of the
action of $h, k$ and $a$ on $\tilde{M}$.

The dimension of the Teichmüller spaces coincide with those of Type a2. One can verify that the Teichmüller parameters of Type a1 has corresponding ones of Type a2.

5.2 Type b

There are seven types of compact orientable quotients modeled on Nil i.e. Type b; each type consists of infinitely many manifolds. Any quotient admits a Seifert bundle structure over a Euclidean orbifold.

1) $S^1$-bundle over $T^2$.

\[ \pi_1(M) = \langle a, b, c; [a, b]c^{-n}, [a, c], [b, c] \rangle. \] (118)

Generators $a$ and $b$ correspond to the nontrivial loops of the base $T^2$ and $c$ corresponds to that of the fiber $S^1$. The first relation means that if one goes through the both nontrivial closed curves of the base one has turned the fiber $n$ times.

2) $S^1$-bundle over a Klein bottle.

\[ \pi_1(M) = \langle a, b, c; abab^{-1}c^{-n}, [a, c], bcbc^{-1}c \rangle. \] (119)

3) $\pi_1(M) = \langle a, b, c; [a, b]c^{-2m}, cac^{-1}a, cbc^{-1}b \rangle$. (120)

4) $\pi_1(M) = \langle a, b, c; [a, b]c^{-3m}, cac^{-1}b^{-1}, cbc^{-1}ba \rangle$. (121)

5) $\pi_1(M) = \langle a, b, c; [a, b]c^{-4m}, cac^{-1}b^{-1}, cbc^{-1}a \rangle$. (122)

6) $\pi_1(M) = \langle a, b, c; [a, b]c^{-6m}, cac^{-1}b^{-1}, cbc^{-1}b^{-1}a \rangle$. (123)

7) $\pi_1(M) = \langle a, c; ac^2b^{-1}c^{-2}, ca^{-1}c^{-1}a^{-1}c^{-1}b^{-1} \rangle$. (124)

Topologies are classified by a positive integer $n$.

The composition of two elements $a, b$ of $\text{Isom}_0\tilde{M}_b$ are

\[
ab = \begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{pmatrix}
\begin{pmatrix}
b_0 \\
b_1 \\
b_2 \\
b_3
\end{pmatrix}
\]
\[
\begin{pmatrix}
    a_0 + b_0 \\
    \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + R_{a_0} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\
    a_3 + b_3 + a_1(b_1 \sin a_0 + b_2 \cos a_0) + \frac{1}{2}(b_1^2 - b_2^2) \cos a_0 - 2b_1b_2 \sin a_0 \sin a_0
\end{pmatrix}
\]

where

\[
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv \text{Exp}_3 A_3 \circ \text{Exp}_2 A_2 \circ \text{Exp}_1 A_1 \circ \text{Exp}_0 A_0,
\]

\[
A_1 \equiv \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, A_2 \equiv \frac{\partial}{\partial y}, A_3 \equiv \frac{\partial}{\partial z},
\]

\[
A_0 \equiv -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{1}{2}(x^2 - y^2) \frac{\partial}{\partial z}.
\]

This can be verified by writing \( a \in \text{Isom}_0 \tilde{M}_b \) as

\[
a = \begin{pmatrix}
    0 \\
    0 \\
    0
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}
\begin{pmatrix} 0 \\ 0 \\ 0 \\
\end{pmatrix}
\end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}
\]

and use the fact that \( \{\text{Exp}_t A_3 \mid t \in \mathbb{R}\} \) is the center of \( \text{Isom}_0 \tilde{M}_b \) and that \( s_\theta \circ q \circ s_\theta^{-1} = s_\theta(q) \) where \( s_\theta \) and \( q \) are isometries and \( s_\theta = \text{Exp}_\theta A_0 \) and \( q \in \text{Nil} \).

Representative elements of equivalence classes corresponding to points in \( \text{Teich}(M) \) are given by

\[
b/1: \quad a = \begin{pmatrix}
    0 \\
    a_1 \\
    0
\end{pmatrix}
, \quad b = \begin{pmatrix} 0 \\ b_1 \\ b_2 \end{pmatrix}
, \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

\( T = 3 \).

\[
\begin{align*}
    \text{b/1:} & \quad a = \begin{pmatrix} 0 \\ a_1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ b_1 \\ b_2 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
    T & = 3.
\end{align*}
\]
\[
\begin{align*}
b/2: & \quad a = \begin{pmatrix} 0 \\ a_1 \\ 0 \\ 0 \end{pmatrix}, \quad b = h \circ \begin{pmatrix} 0 \\ 0 \\ b_2 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{n}a_1b_2 \end{pmatrix}. \\
T &= 2. \\
b/3: & \quad a = \begin{pmatrix} 0 \\ a_1 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ b_1 \\ b_2 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} \pi \\ -b_1 \\ 0 \end{pmatrix}. \\
T &= 3. \\
b/4: & \quad a = \begin{pmatrix} 0 \\ a_1 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ R_{\pm 2\pi/3} a_1 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} \pm \frac{2}{3}\pi \\ \frac{1}{8}a_1 \\ \pm \frac{\sqrt{3}}{8}a_1 \\ \pm \frac{\sqrt{3}}{2} \left( \frac{1}{3n} + \frac{1}{192} \right) a_1^2 \end{pmatrix}. \\
T &= 1. \\
b/5: & \quad a = \begin{pmatrix} 0 \\ a_1 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \pm a_1 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} \pm \frac{\pi}{2} \\ 0 \\ 0 \end{pmatrix}. \\
T &= 1. \\
b/6: & \quad a = \begin{pmatrix} 0 \\ a_1 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ a_1 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} \pm \frac{\pi}{3} \\ -\frac{3}{8}a_1 \\ \pm \frac{\sqrt{3}}{8}a_1 \\ \pm \frac{\sqrt{3}}{2} \left( \frac{1}{6n} - \frac{3}{64} \right) a_1^2 \end{pmatrix}. \\
T &= 1. \\
b/7: & \quad a = h \circ \begin{pmatrix} \pi \\ a_1 \\ a_2 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \\
T &= 2.
\end{align*}
\]
Here \( h : (x, y, z) \mapsto (-x, y, -z) \), \( a_1 > 0 \), and \( a_2, b_2 \neq 0 \) in all cases. We have used the fact that

\[
h \circ \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \circ h = \begin{pmatrix} -a_0 \\ -a_1 \\ a_2 \\ -a_3 \end{pmatrix}
\]  

(144)

holds for any \( a \in \text{Isom}\tilde{M} \), which can be easily verified by the explicit calculation of the action of \( h \) and \( a \) on \( \tilde{M} \).

It can be shown that the Teichmüller space is homeomorphic to that of the base orbifold. So all Teichmüller parameters of quotients of Type b can be considered as those of the base orbifolds.

5.3 Type c

It is known that any compact quotient of \( H^2 \times \mathbb{R} \) i.e. of the universal cover of Type c admits a unique Seifert bundle structure with \( e = 0 \) whose base is a hyperbolic orbifold. It comes from \( e = 0 \) that the quotient is finitely covered by an orientable trivial \( S^1 \)-bundle over an orientable compact hyperbolic 2-manifold. Since each quotient admit a unique Seifert bundle structure, the topologies of the quotients are completely classified by whether the base orbifold is orientable or not, the genus \( g \) of the base, the number \( k \) of cone points, and the Seifert invariants \((\alpha_i, \beta_i) (i = 1, \cdots, k) \) consistent with \( e = 0 \).

The dimension \( T \) of the Teichmüller space is given by \( 8g - 5 + 2k \) if the base orbifold is orientable and \( 4g - 6 + 2k \) if nonorientable. One degree of freedom corresponds to the length of the circle relative to the base. The remainder correspond to the dimension of the Teichmüller space of the base orbifold and the twist of the fiber around each nontrivial closed curve of the base orbifold. In the former case the dimension of the Teichmüller space of the base is \( 6g - 6 + 2k \) where \( 2k \) corresponds to the positions of the cones on the 2-surface, and the base has \( 2g \) independent nontrivial closed curves. In the latter case the dimension of the Teichmüller space of the base is \( 3g - 6 + 2k \). There are \( g \) generators.
of the fundamental group of the base but the twists of the fiber around them have \( g - 1 \) degrees of freedom because of conjugacy.

5.4 Type d

Similar to Type c, any compact quotient of \( \widetilde{\text{SL}}(2, \mathbb{R}) \) i.e. of the universal cover of Type d admits a unique Seifert bundle structure whose base is a hyperbolic orbifold but with \( e \neq 0 \). Each quotient admit a unique Seifert bundle structure. The topologies are completely classified by whether the base orbifold is orientable, the genus \( g \) of the base orbifold, the number \( k \) of cone points, and the Seifert invariants \( (\alpha_i, \beta_i)(i = 1, \cdots, k) \) and the Euler number \( e(\neq 0) \).

The dimension \( T \) of the Teichmüller space is given by \( 8g - 6 + 2k \) if the base orbifold is orientable and \( 4g - 7 + 2k \) if it is nonorientable. This is the same as the case of Type c but the length of the fiber must be fixed because the universal cover Type d is the unit tangent bundle of \( SL(2, \mathbb{R}) \). Note that this degree of freedom exists in the universal cover. In fact, universal covers which have the same isometry group can be obtained if we consider a tangent bundle of the constant length not equal to one; the parameter \( \lambda \) in Sec.4.2.4 is this degree of freedom.

5.5 Type e

The geometry \((H^3, \text{Isom} H^3)\) admits infinite number of compact quotients. It is known that quotients of \((H^3, \text{Isom} H^3)\) can be classified by the fundamental group. It has also been shown by Mostow [18] that each quotient does not admit Teichmüller deformations. There are no degree of freedoms of the universal cover nor the Teichmüller parameters. We do not investigate the variety of quotients of this type further because the problem is not completely solved and we have already known the degrees of freedom of all quotients. The variety of quotients is discussed by Thurston [19].
5.6 Type f

Any compact quotient of Sol i.e. Type f2 is a $T^2$-bundle over $S^1$. Conversely, any $T^2$-bundle over $S^1$ with hyperbolic gluing map admits a Sol-structure. The fundamental group is

$$
\pi_1(M) = \langle a, b, c; [a, b], cac^{-1}b^{-1}, cbc^{-1}ab^{-n} \rangle,
$$

i.e. it is generated by $a, b$ and $c$ which obey relations

$$
[a, b] = 1, \quad cac^{-1} = b, \quad cbc^{-1} = a^{-1}b^n.
$$

Here $c$ is the generator of the fundamental group of the base $S^1$, while $a$ and $b$ are those of the fiber $T^2$. The equations (147) and (148) indicate that when one goes around the base $S^1$ the fiber $T^2$ is identified by a modular transformation. The topologies are classified by an integer $n$ satisfying $|n| > 2$. Note that if $n = -1, 0$ or $1$ the fundamental group can be realized in $\text{Isom} \tilde{M}_{a1}$ ($\text{Isom} \tilde{M}_{a2}$); in fact they coincide with the fundamental groups of quotients $a1/3$, $a1/4$ and $a1/5$ ($a2/3$, $a2/4$ and $a2/5$) in Sec.5.1, respectively.

All representations of $\pi_1\tilde{M}$ in $\text{Isom}^+ \tilde{M}_{f2}$ are in $\text{Isom}^+ \tilde{M}_{f1}$. If $n$ is positive the representations are in $\text{Isom}0 \tilde{M}_{f1} = \text{Sol}$. Expressing generators with their components in Sol as

$$
a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}
$$

and putting them into (147) and (148), we have

$$
a_3 = b_3 = 0, \quad \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} e^{-c_3} & 0 \\ 0 & e^{c_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.
$$

Then (147) is trivial. The equation (151) is an eigenvalue equation. The numbers $e^{-c_3}$
and $e^{c_3}$ are eigenvalues of \[
\begin{pmatrix}
0 & 1 \\
-1 & n
\end{pmatrix},
\] i.e.
\[
c_3 = \ln n + \sqrt{n^2 - 4}
\] (152)
and
\[
\begin{pmatrix}
a_1 \\
b_1
\end{pmatrix} = \alpha \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \beta \begin{pmatrix} u_2 \\ v_2 \end{pmatrix},
\] (153)
where \(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}\) and \(\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\) are the normalized eigenvectors corresponding to the eigenvalues \(e^{-c_3}\) and \(e^{c_3}\), respectively. The generators are
\[
a = \begin{pmatrix} \alpha u_1 \\ \beta u_2 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} \alpha v_1 \\ \beta v_2 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},
\] (154)
where $\alpha$ and $\beta$ are positive parameters.

If $n$ is negative the representations are generated by
\[
a = \begin{pmatrix} \alpha u_1 \\ \beta u_2 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} \alpha v_1 \\ \beta v_2 \\ 0 \end{pmatrix}, \quad c = h \circ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},
\] (155)
where $-e^{-c_3}$ and $-e^{c_3}$ are eigenvalues of \[
\begin{pmatrix}
0 & 1 \\
-1 & n
\end{pmatrix}; \quad \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}
\] are corresponding unit eigenvectors; and $h : (x, y, z) \mapsto (-x, -y, z)$.

Representative elements of the equivalence classes corresponding to points in Teichmüller space are obtained by taking conjugate of representations by $\text{Isom}_0 \tilde{M}_\Omega = \text{Sol}$. One finds that
\[
a = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix},
\] (156)
for $n > 0$ and

$$a = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ 0 \end{pmatrix}, \quad c = h \circ \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix},$$

(157)

for $n < 0$. The Teichmüller spaces are 1-dimensional and is parametrized by $\alpha$.

It can be regarded as the size of the torus relative to the length of the circle.

### 5.7 Type g

There are infinite number of compact quotients of $S^3$ i.e. Type g3, all of which have finite fundamental groups. They do not admit any Teichmüller deformations, i.e. $T = 0$ for all quotients. It is known that any freely acting discrete subgroup of $\text{Isom}\tilde{M}_{g3}$, i.e. $SO(4)$ in $\text{Isom}\tilde{M}_{g2}$, i.e. $(\text{SU}(2) \times \text{U}(1))/\mathbb{Z}_2$ (Theorem 4.10 of [1]).

The quotients $S^3/\Gamma$ which can be modeled on $(\tilde{M}, \text{Isom}\tilde{M}_{g1})$ are as below, where $n$ is a positive integer, $(q_1, q_2)$ denotes an element of $\text{SU}(2) \times \text{SU}(2)$, and $\mathbb{Z}_2$ in the expression of $\Gamma$ is $\{(1, 1), (-1, -1)\}$.

1) Lens spaces $L(4n, 1)$.

$$\Gamma = \{ (e^{2\pi \chi_3(m/4n)}, \pm 1) \mid m = 0, \cdots, 4n - 1 \} / \mathbb{Z}_2$$

$$\simeq \mathbb{Z}_{4n}.$$  \hspace{1cm} (158)

2) Lens spaces $L(2(2n - 1), 1)$.

$$\Gamma = \{ (e^{2\pi \chi_3(m/2(2n - 1))}, \pm 1) \mid m = 0, \cdots, 2(2n - 1) - 1 \} / \mathbb{Z}_2$$

$$\simeq \mathbb{Z}_{2(2n - 1)}.$$ \hspace{1cm} (159)

The manifold $P^3$ corresponds to the case of $n = 1$.

3) Lens spaces $L(2n - 1, 1)$.

$$\Gamma = \{ (e^{2\pi \chi_3(1/2(2n - 1))}, -1)^m \mid m = 0, \cdots, 2(2n - 1) - 1 \} / \mathbb{Z}_2$$

$$\simeq \mathbb{Z}_{2n - 1}.$$ \hspace{1cm} (160)

The manifold $S^3$ corresponds to the case of $n = 1$. 

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4) Lens spaces \( L(4(2n - 1), 2(2n - 1) + 1) \).

\[
\Gamma = \left\{ (e^{2\pi \chi_3(m/2(2n-1))}, q_2) \mid m = 0, \ldots, 2(2n - 1) - 1; \ q_2 \in \pm 1, \pm 3 \right\} / \mathbb{Z}_2
\]

\[
= \left\{ (e^{2\pi \chi_3(1/(2n-1))}, \chi_3)^m, (-e^{2\pi \chi_3(1/(2n-1))}, -\chi_3)^m \mid m = 0, \ldots, 4(2n - 1) - 1; \right\} / \mathbb{Z}_2
\]

\[
\simeq \mathbb{Z}_{4(2n-1)}. \quad (161)
\]

5) Prism manifolds \( L(4(2n - 1), 2(2n - 1) + 1)/\mathbb{Z}_2 \).

\[
\Gamma = \left\{ (e^{2\pi \chi_3(m/2(2n-1))}, q_2) \mid m = 0, \ldots, 2(2n - 1) - 1; \ q_2 \in \pm 1, \pm \chi_i \ (i = 1, 2, 3) \right\} / \mathbb{Z}_2
\]

\[
\simeq \mathbb{Z}_{2n-1} \times D_2^*. \quad (162)
\]

6) \( S^3/\Gamma \) with

\[
\Gamma = (T^* \times \{ \pm 1 \}) / \mathbb{Z}_2 \simeq T^*. \quad (163)
\]

7) \( S^3/\Gamma \) with

\[
\Gamma = (O^* \times \{ \pm 1 \}) / \mathbb{Z}_2 \simeq O^*. \quad (164)
\]

8) \( S^3/\Gamma \) with

\[
\Gamma = (I^* \times \{ \pm 1 \}) / \mathbb{Z}_2 \simeq I^*. \quad (165)
\]

Here \( D_n^*, T^*, O^* \) and \( I^* \) denote the preimages of \( D_n, T, O \) and \( I \) by the homomorphism \( SU(2) \to SO(3) \), where \( D_n, T, O \) and \( I \) denote the dihedral, tetrahedral, octahedral, icosahedral groups, respectively \([14]\).

The lens spaces and the prism manifolds of the above types have two degrees of freedom of the universal cover and zero Teichmüller parameter, so that the total degrees of freedom is two.

The other quotients are modeled on \( (\check{M}, \mathrm{Isom}\check{M}_{g2}) \) but not on \( (\check{M}, \mathrm{Isom}\check{M}_{g1}) \). These have one degree of universal cover and zero Teichmüller parameter, so that the total degrees of freedom is one.

We do not investigate the variety of the finite subgroups of \( \mathrm{Isom}\check{M}_{g2} \) because we have already known the degrees of freedom. The variety of quotients is discussed in \([14]\).
5.8 Type h

It is known that there exist only two compact, orientable manifold modeled on $S^2 \times \mathbb{R}$ i.e. Type h.

1) The fundamental group is isomorphic to $\mathbb{Z}$ i.e.
\[ \pi_1(M) = \langle a \rangle. \]  (166)
The manifold is $S^2 \times S^1$.

2) The fundamental group is
\[ \pi_1(M) = \langle a, b; a^2, b^2 \rangle. \] (167)
The manifold is $P^3 \# P^3$, which can be regarded as an $S^1$-bundle over $P^2$.

The representation of $\pi_1(M_{h/1})$ in $\text{Isom}^+\tilde{M}_h$ is generated by
\[ a = (R, t) \] (168)
where $R \in \text{SO}(3)$ and $t$ is a translation on $\mathbb{R}$. So $\text{Rep}(M)$ is 4-dimensional.

Let $R(\hat{n}, \theta)$ denote a rotation around axis $\hat{n}$ and with rotation angle $\theta$, and $t_a$ denote a translation $z \mapsto z + a$. The conjugate of (168) by an element of $\text{Isom}_0\tilde{M}_h$ is
\[ (R', t_{a'}) (R(\hat{n}'; \theta), t_a) (R'^{-1}, t_{-a'}) = (R(R'\hat{n}'; \theta), t_a), \] (169)
which states that the rotation axis is changed by conjugation but the rotation angle and the length of translation are not. Thus the representative element of the conjugacy class corresponding to a point in $\text{Teich}(M)$ is generated by
\[ a = (R(\hat{n}_0, \theta), t_a) \] (170)
where $\hat{n}_0$ is a fixed axis. The Teichmüller parameters are $a \in \mathbb{R} \setminus \{0\}$ and $0 \leq \theta \leq \pi$ with $\theta = 0$ and $\theta = \pi$ reflector points, which are the length of $S^1$ relative to the size of $S^2$ and the twist angle of $S^2$ when one goes around $S^1$. The Teichmüller space is 2-dimensional.

Let us find representations of $\pi_1M_{h/2}$ in $\text{Isom}^+\tilde{M}_h$. An element $a$ of $\text{Isom}^+\tilde{M}_h$ is written as
\[ a = h^p \circ (R, t_a) \] (171)
where $h$ is a combination of the antipodal map of $S^2$ and a reflection of $\mathbb{R}$ at 0, and $p$ is 0 or 1. The number $p$ must be 1 to satisfy the relation $a^2 = 1$. Otherwise the second entry of $a^2$ would be a translation by $2a$. If $p = 1$ then $a$ is written as

$$a = (-R, r_{-a/2}) (172)$$

where $r_a$ denotes a reflection at a point $a \in \mathbb{R}$. Since $a^2 = (R^2, 1)$, the rotation angle must be 0 or $\pi$ to satisfy the relation $a^2 = 1$. The latter case is excluded by the requirement that $a$ should have no fixed points. So the representation is generated by

$$a = (-1, r_{-a/2}), b = (-1, r_{-b/2}) (173)$$

where $a, b \in \mathbb{R} - \{0\}$ and $a \neq b$.

The representative element of the conjugacy class has generators

$$a = (-1, r_{-a/2}), b = (-1, r_{0}). (174)$$

The Teichmüller parameter is $a$. It is the length of a nontrivial closed curve corresponding to $ab$. The dimension of the Teichmüller space is one.

### 5.9 Summary of the section

The compact quotients and their dynamical degrees of freedom are in Table 3. The total degrees of freedom $F$ is the sum of the degrees of freedom $U$ of the universal cover and the dimension $T$ of the Teichmüller space.

In the cases that the universal covers are Type a1 and Type a2, we only show the cases of Type a1. This is because a quotient of Type a2 is obtained as the special case of a quotient of Type a1; the universal cover parameters are fixed as $\lambda_+ = \lambda_- = 0$.

Similarly, in the cases of Types f and g, we choose the universal cover which have the largest $U$. 

---

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6 Conclusion and discussions

We have worked out the classification of topologies of compact locally homogeneous universes except for hyperbolic 3-manifolds by studying possible fundamental groups for each Thurston geometry.

We have shown that the total global degrees of freedom of compact locally homogeneous spaces consist of two parts. One is the universal cover parameters while the other is the Teichmüller parameters which parametrize the discrete subgroup $\Gamma$ of the isometry group. The Teichmüller parameters are obtained by constructing the representation of the fundamental group in the isometry group.

Intuitively the difference of the two degrees of freedom is the following. Assuming the homogeneity of the Universe one can locally determine the universal cover degrees of freedom by measuring all ratios of the sectional curvatures. In a sense these are interpreted as local anisotropies. In order to know the Teichmüller parameters, on the other hand, we probably have to send many space explorers, who are supposed to measure the times taken by all possible round trips around non-trivial loops, etc.. In short, the Teichmüller parameters are global quantities.

We have not discussed the dynamics of our compact universes in the present work. In a separate paper we will study the Einstein gravity for a compact locally homogeneous universes.

Thurston conjectured that all possible topologies of compact 3-manifolds can be classified to be connected sums and torus sums of the prime manifolds which are given by the compact quotients of the eight geometries [20]. This may lead us to a speculation that at least some aspects of general inhomogeneous universes are understood by studying combinations of the compact locally homogeneous spaces.

It will be also interesting to investigate quantum field theory in the compact locally homogeneous spaces which have non-trivial topologies. We naturally expect the Cashimir effects, etc..

We hope our present work furnishes a mathematical base on which a new field of cosmology develops.
Acknowledgments

We are greatly indebted to Professors K. Ohshika and S. Kojima for explaining us about the Teichmüller parameters of the Seifert manifolds. We also thank to H. Kodama for a nice lecture and Y. Fujiwara for correspondence. T. K. expresses his sincere thanks to S. Higuchi for useful discussions. This work is partially supported by the Grant-in-Aid for Scientific Research of Ministry of Education, Science and Culture of Japan (No.02640232)(A.H.).

Appendix

It is proved that the Bianchi Class B geometries \((\mathbb{R}^3, G)\) do not admit compact quotients \([12, 22]\). It can be shown by the same argument that a geometry \((G, G)\) where \(G\) is a Lie group of any dimension admits compact quotients only if the trace of the structure constants vanishes. Let \(G\) a group manifold which admits a compact quotient \(G/\Gamma\). The invariant 1-forms \(\{\sigma^i\}\) define the volume element

\[\epsilon = \sigma^1 \wedge \cdots \wedge \sigma^N\]  \hspace{1cm} (175)

on \(G\), where \(N = \dim G\). Let \(\{X_i\}\) the dual basis of \(\{\sigma^i\}\). It is easily verified that

\[\mathcal{L}_X \epsilon = g^{ij} C^k_{\ i} C^d_{\ j} \epsilon\]  \hspace{1cm} (176)

and

\[\exp tX \in \text{Aut}(G),\]  \hspace{1cm} (177)

where

\[X = g^{ij} C^k_{\ i} X_j\]  \hspace{1cm} (178)

If \(C^k_{\ ik} \neq 0\), this means that there is a diffeomorphism from \(G/\Gamma\) onto \(G/\Gamma\) which changes its volume. This contradicts the compactness of \(G/\Gamma\).

The case of \(N = 3\) states that the Bianchi Class B geometries do not admit compact quotients. It should be noted that an invariant metric under a Bianchi Class B group may have compact quotients when the isometry group may be larger than that Bianchi
group. For example, a Bianchi V-invariant metric is invariant under $\text{PSL}(2, \mathbb{C})$ and admits compact quotients.

It can be shown by using the case of $N = 2$ of the above theorem that the geometry $(\mathbb{R}^3, \tilde{\text{SL}}(2, \mathbb{R}))$, which belongs to the Bianchi Class A, does not admit compact quotients. The case of $N = 2$ of the above theorem states that the group $H^2$ does not admit compact quotients, where $H^2$ is a simply connected 2-dimensional Lie group defined by the following multiplication rule:

$$
\begin{pmatrix}
a \\
b
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
a + bx \\
by
\end{pmatrix}.
$$

(179)

The group $\tilde{\text{SL}}(2, \mathbb{R})$ is a semidirect product of $\mathbb{R}$ and $H^2$ group. The manifold can be considered as a $\mathbb{R}$-bundle over $\mathbb{R}^2$; the base is considered as a geometry $(\mathbb{R}^2, H^2)$. Since the action of $\tilde{\text{SL}}(2, \mathbb{R})$ on $\mathbb{R}^3$ is fiber preserving, any quotient of the geometry inherits the same bundle structure. This implies that any compact quotient of $(\mathbb{R}^3, \tilde{\text{SL}}(2, \mathbb{R}))$ has a Seifert bundle structure over a compact orbifold modeled on $(\mathbb{R}^2, H^2)$. Since the base orbifold must be finitely covered by a compact surface the necessary condition for $(\mathbb{R}^3, \tilde{\text{SL}}(2, \mathbb{R}))$ to admit compact quotients is that the base $(\mathbb{R}^2, H^2)$ admits compact quotients. However, this is not satisfied as we have already seen.

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Table Captions

**Table 1:** All equivalence classes of

\[
\{\text{homogeneous metrics on a simply connected manifold which admit compact quotients}\},
\{\text{diffeomorphisms}\} \{\text{global conformal transformations}\},
\]

i.e. all universal covers of compact locally homogeneous Riemannian manifolds.

**Table 2:** All equivalence classes of

\[
\{\text{homogeneous metrics on a simply connected manifold which admit compact quotients}\},
\{\text{orientation preserving diffeomorphisms}\} \{\text{global conformal transformations}\}
\]

i.e. all universal covers of orientable, compact locally homogeneous Riemannian manifolds.

**Table 3:** Degrees of freedom of compact locally homogeneous universes. The plus and minus signs in the Types c and d denote that the base orbifold of the Seifert bundle is orientable and nonorientable, respectively.
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