Probing $F$-theory With Multiple Branes

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We study multiple 3-branes on an F theory orientifold. The world-volume theory of the 3-branes is $d = 4$, $\mathcal{N} = 2$ $Sp(2k)$ gauge theory with an antisymmetric tensor and four flavors of matter in the fundamental. The solution of this gauge theory is found for vanishing bare mass of the antisymmetric tensor matter, and massive fundamental matter. The integrable system underlying this theory is constructed.
1. Introduction

F-theory may be defined as compactifications of Type IIB string theory in which the dilaton and its Ramond-Ramond sector axion partner are allowed to vary over the internal space \([1]\). F-theory compactification becomes conventional perturbative string compactification for certain \(\mathbb{Z}_2\) orbifolds \([2]\). Deformations away from the orbifold limit are described by the solution of \(d = 4, \mathcal{N} = 2\) \(SU(2)\) gauge theory with \(N_f = 4\) flavors of matter. This can be understood physically by introducing a 3-brane probe \([3]\): it is the T-dual of the type I 5-brane with \(SU(2)\) world-volume gauge symmetry, and the matter comes from 3−7 strings. The variable dilaton-axion field \(\tau\) is the low energy gauge coupling on the 3-brane.

It is natural to ask what happens when several parallel 3-branes are introduced \([4]\). As noted in \([3]\), in general adding 3-branes will change the background and thus the result need not have a probe interpretation. On the other hand, a system of multiple parallel 3-branes has a flat metric on moduli space, suggesting that the additional 3-branes might not affect the probe interpretation, and that the matrix of coupling constants \(\tau_{ij}\) might in fact be equal to \(\delta_{ij}\tau(z_i)\).

It turns out to be quite easy to show that this is true, using an observation of Danielsson and Sundborg \([5]\). The theory with \(N_f\) 7-branes and \(k\) 3-branes near an orientifold point is the T-dual of the theory considered in \([6]\): \(Sp(2k)\) gauge theory with an antisymmetric tensor and \(N_f\) flavors of fundamental matter. One can check that for \(N_f = 4\), this is a finite theory for any \(k\), so the basic physics is as in \([2]\) even with multiple branes.

In this note we construct the solution of this theory. It turns out to be convenient to describe this solution using the framework of Donagi and Witten’s solution of \(SU(N)\) gauge theory with massive adjoint matter \([7]\). We obtain in this way an integrable system underlying the solution. An integrable system describing \(SU(2)\) gauge theory with four flavors of massive fundamental matter is a special case of this construction.

2. Solution of the Multiple 3-brane Probe Gauge Theory

The first element of the solution of this theory is the one-loop prepotential. In terms of the roots \(\alpha\) for the gauge group and weights \(\lambda\) for the matter representations which appear, this is

\[
\mathcal{F} \sim \frac{i}{4\pi} \sum_{\alpha} (\Psi \cdot \alpha)^2 \log \left( \frac{(\Psi \cdot \alpha)^2}{\Lambda^2} \right) - \frac{i}{4\pi} \sum_{\lambda,f} (\Psi \cdot \lambda - m_f)^2 \log \left( \frac{(\Psi \cdot \lambda - m_f)^2}{\Lambda^2} \right),
\]

(2.1)
where $Ψ$ is the $\mathcal{N} = 2$ superfield describing the vector multiplet, $Λ$ is the mass scale of the theory, and $m_\ell$ are bare mass terms. Now, the key point is that for $Sp(2k)$, the set of weights for the antisymmetric tensor representation is a subset of the roots. In a basis where the weights of the fundamental are $±e_i$, the antisymmetric tensor weights are $±e_i ± e_j$, while the roots include all of these and $±2e_i$. Thus the antisymmetric tensor contributions simply cancel the off-diagonal roots.

The complete one-loop prepotential then is simply the sum of those for independent $SU(2)$ factors with $N_f$ flavors,

$$F \sim \frac{i}{2\pi} \sum_i (2a_i)^2 \log \left(\frac{2a_i^2}{\Lambda^2}\right) - \frac{i}{4\pi} \sum_{i,f} (a_i - m_f)^2 \log \left(\frac{(a_i - m_f)^2}{\Lambda^2}\right)$$

$$+ (a_i + m_f)^2 \log \left(\frac{(a_i + m_f)^2}{\Lambda^2}\right).$$

Clearly the gauge coupling $τ_{ij} = \partial^2 F/\partial a_i \partial a_j$ computed from this prepotential will be diagonal, and the 3-branes will behave as completely independent probes of the geometry, to this approximation.

Danielsson and Sundborg’s [5] observation is now that, given a theory with a particular weak coupling behavior and corresponding one-loop prepotential, the Seiberg-Witten-type exact solution for the prepotential should be uniquely determined. If this is true, it means that the exact prepotential for this theory must be equal to the sum of exact prepotentials for $SU(2)$ with $N_f$ flavors. Thus the 3-branes behave as independent probes of the exact geometry.

This can also be argued by considering the effects of a vev for the antisymmetric tensor field, as in [8]. This corresponds to separating the 3-branes in dimensions contained in the 7-branes. Clearly, for large separation the probes are independent. Since hypermultiplet vevs cannot affect the vector effective Lagrangian and $τ$, this will remain true as we bring the probes together.

The answer to the original question turned out to be rather simple, and we see that this particular matter content gives a nice generalization of the finite $SU(2)$, $N_f = 4$ gauge theory.

3. A related non-D-brane gauge theory

A more non-trivial generalization of the theory is to allow a mass term for the antisymmetric tensor. Although this has no direct brane interpretation, we are led by the above to suspect that the theory with this matter content might be tractable. This theory
has some similarity to $SU(N)$ with massive adjoint matter, which was solved in [7]. The techniques introduced for that problem are useful here. Donagi and Witten showed that the solution of a $d = 4$, $\mathcal{N} = 2$ gauge theory determines a complex integrable system. In favorable cases, this system can be realized as a Hitchin system [9] as generalized by Markman [10]. This construction automatically satisfies many of the physical consistency conditions on the solution. We expect the solution of the theory with a massive antisymmetric tensor can be formulated in this framework, though we will not obtain this solution in the present work.

As a first step toward this solution, we reformulate the massless antisymmetric tensor theory in this language. Given the solution of the $SU(2)$, $N_f = 4$ theory [11], this solution follows easily, as might be expected from our previous discussion. For any scale-invariant $SU(2)$ theory, the phase space for the integrable system is the elliptic curve $E_\tau$ fibered over the moduli space $\mathbb{P}^1$ of expectation values for $\bar{u} = \text{Tr} \phi^2$. This is equivalent to the $SU(2)$ Hitchin system on the curve $\sigma \cong E_\tau$. Its phase space consists of gauge equivalence classes of solutions of $F = \bar{D} \Phi = 0$ with the symplectic structure $\{A_\alpha^a(x), \Phi^b(y)\} = \delta^{ab} \delta(x - y)$. The natural guess for the integrable system corresponding to a theory with massive matter is a deformation of this.

The theory with massless adjoint matter had $SL(2, \mathbb{Z})$ symmetry and there is a natural way to add a mass parameter preserving this symmetry: add a ‘charged’ source to the Hitchin equations at a single point $\bar{D} \Phi = \mu \delta(x)$. For $SU(2)$ there is a unique way to do this, while for $SU(N)$ with $N > 2$ there is a unique way to do this which preserves the dimension of the phase space. This leads to the solution of [7].

The theory with $N_f = 4$ massless fundamental flavors has $Spin(8) \ltimes SL(2, \mathbb{Z})$ global symmetry [11], but this is broken to $\Gamma(2) \subset SL(2, \mathbb{Z})$ for generic masses. The full $Spin(8) \ltimes SL(2, \mathbb{Z})$ acts on the combined space of moduli and mass parameters, such that two $Spin(8)$ combinations

\[
R = \frac{1}{2} \sum_i m_i^2 \\
N = \frac{3}{16} \sum_{i > j > k} m_i^2 m_j^2 m_k^2 - \frac{1}{96} \sum_{i \neq j} m_i^2 m_j^4 + \frac{1}{96} \sum_i m_i^6,
\]

are invariant, while three other $Spin(8)$ invariants $T_i$ (only two of which are independent)

\[
T_1 = \frac{1}{12} \sum_{i > j} m_i^2 m_j^2 - \frac{1}{24} \sum_i m_i^4 \\
T_2 = -\frac{1}{2} \prod_i m_i - \frac{1}{24} \sum_{i > j} m_i^2 m_j^2 + \frac{1}{48} \sum_i m_i^4 \\
T_3 = -T_1 - T_2,
\]

are invariant.
are permuted.

This suggests that we introduce charged sources proportional to the conjugacy class
\[ \mu = \text{diag}(1, -1) \] in the Hitchin equations at all four Weierstrass points of the elliptic curve, which we can choose to be at positions \( \nu = 0, 1/2, \tau/2 \) and \((1 + \tau)/2\). The coordinate \( \nu \) is defined in terms of the periodic real parameters \( \sigma_1 \) and \( \sigma_2 \), each of period one, as \( \nu = \sigma_1 + \tau \sigma_2 \) and transforms under \( SL(2, \mathbb{Z}) \) as \( (\sigma_1, \sigma_2) \rightarrow (a \sigma_1 + b \sigma_2, c \sigma_1 + d \sigma_2) \), for \( (a \ b \ c \ d) \in SL(2, \mathbb{Z}) \).

The source at the origin should be a singlet under triality of \( Spin(8) \), while sources at \( 1/2, \tau/2 \) and \((1 + \tau)/2\) should be permuted. When the \( T_i \) and \( N \) vanish the system should reduce to that of the massive adjoint case ([11], 16.26). This does not require all \( m_i = 0 \) but rather the four masses can be \((m, m, 0, 0)\). The equivalence of the two theories can again be motivated by comparing the one-loop prepotentials. Define \( L(a) = a^2 \log a^2 \), then they are

\[ N_f = 4 , \quad F \sim 2L(2a) - 4L(a) - 2L(a - m) - 2L(a + m) \]
\[ N = 4 , \quad F \sim 2L(2a') - L(2a' - m) - L(2a' + m) . \]  

Then \( L(2a) = 4L(a) + \text{regular} \), so with \( a = 2a' \) these have the same singularities. (This may be an interesting generalization of Danielsson and Sundborg’s observation.) The sources at \( 1/2, \tau/2 \) and \((1 + \tau)/2\) should vanish in this limit, in which case the integrable system reduces to that considered by Donagi and Witten [7].

The subsequent analysis is facilitated by introducing the spectral curve

\[ F(t, \tau) = \det(t - \Phi) = 0 . \]  

The sources in the Hitchin equations translate into requiring specific poles for \( F \), with residues proportional to the charges. This, together with the requirements that \( F \) be of degree 2 (where we take \( t, x \) and \( y \) to be of degree 1, 2 and 3 respectively) and that there be no additional singularities, uniquely determines the function \( F \) in terms of the four masses and the parameter \( \tilde{u} \).

Let us use the Weierstrass representation for the curve \( E_\tau \),

\[ y^2 = (x - e_1(\tau))(x - e_2(\tau))(x - e_3(\tau)) , \]  

where the \( e_i \) are roots of the polynomial \( 4x^3 - g_2x - g_3 \), which satisfy \( \sum e_i = 0 \). Here \( g_2 = 60\pi^{-4}G_4(\tau) \) and \( g_3 = 140\pi^{-6}G_6(\tau) \) with \( G_4 \) and \( G_6 \) the usual Eisenstein series [12].
The three points $\nu = 1/2$, $\nu = \tau/2$ and $\nu = (1 + \tau)/2$ map into $(x = e_i, y = 0)$, while $\nu = 0$ maps into the point at infinity. The spectral curve is then (generalizing [7], (3.7))

$$F = t^2 - \bar{R}x + \bar{u} - \frac{P(x)}{(x - e_1)(x - e_2)(x - e_3)},$$

(3.7)

where $P(x)$ is a quartic polynomial in $x$ whose coefficients are related to the five complex order parameters of the theory, which we denote by barred variables

$$P = (\bar{R}x - \bar{u})^3(\bar{T}_1(x - e_1) + \bar{T}_2(x - e_2) + \bar{T}_3(x - e_3)) + \bar{N}(\bar{R}x - \bar{u})^4.$$

(3.8)

The system of equations (3.5) and (3.6) actually describes a curve of genus two and thus the Jacobian is two dimensional. However only a one-dimensional subspace of this is relevant for the physics [7]. As in the case considered there, the piece of the Jacobian coming purely from $E_\tau$ must be projected out. This is accomplished by modding out by the symmetry that takes $t \to -t$ and $y \to -y$, which projects out the form $dx/y$. The resulting curve is found by substituting the invariant variables $z = t^2$ and $w = yt$ into (3.5) and (3.6). The first equation allows us to eliminate $z$ while the second gives

$$w^2 = (\bar{R}x - \bar{u})(x - e_1)(x - e_2)(x - e_3) + P(x).$$

(3.9)

The form of this curve is rather different from what was found in [11]. However, their equivalence is established by the following $SL(2, \mathbb{C})$ transformation of $x$

$$x = \frac{ax' + b}{cx' + d},$$

(3.10)

with

$$c = \frac{\bar{R}}{\sqrt{(\bar{u} - Re_1)(\bar{u} - Re_2)(\bar{u} - Re_3)}},$$

$$a = \frac{\bar{uc}}{\bar{R}},$$

$$b = \bar{Re}_1e_2e_3c,$$

$$d = \frac{(2\bar{u}^2 - \bar{R}^2(e_1^2 + e_2^2 + e_3^2))c}{2\bar{R}}.$$

(3.11)

This transformation also allows us to match our five complex order parameters with those of [11]. The relations are

$$\bar{u} = u,$$

$$\bar{R} = R,$$

$$\bar{T}_1 = T_1A(e_2 - e_3)(u - Re_2)(u - Re_3)/B^2,$$

$$\bar{T}_2 = T_2A(e_3 - e_1)(u - Re_1)(u - Re_3)/B^2,$$

$$\bar{T}_3 = T_3A(e_1 - e_2)(u - Re_1)(u - Re_2)/B^2,$$

$$\bar{N} = NA^2/B^2,$$

(3.12)
where we have defined $A = (e_1 - e_2)(e_2 - e_3)(e_3 - e_1)$ and $B = (u - Re_1)(u - Re_2)(u - Re_3)$.

This equivalence and (3.7) define the Hitchin system which would be the definition of the integrable system for $N_f = 4$ in the Donagi-Witten framework. The construction maintained manifest symmetry under $SL(2, \mathbb{Z})$ and triality, but at a price: the sources in the Hitchin system are not linear in the masses. This does not mean that the solution is inconsistent (after all it is equivalent to that of [11]) but rather that consistency is not manifest. We have the freedom to add an exact form to the symplectic structure and it might be possible to use this to turn the description into another with linear sources, though probably losing manifest $SL(2, \mathbb{Z})$.

The solution generalizes in an obvious way to the $Sp(2k)$ gauge theory with a massless antisymmetric tensor and massive fundamental matter. The charged sources are now taken to be proportional to the conjugacy class $\mu = \text{diag}(1, -1, \cdots, 1, -1)$. Following through the above construction, the spectral curve becomes

$$F = \prod_{i=1}^{k} F_2(u_i) = 0 ,$$

where $F_2$ is given by (3.7) and the $u_i = \phi_i^2$. Here we use the fact that the adjoint vev $\phi$ of $Sp(2k)$ can be diagonalized as $\phi = \text{diag}(\phi_1, \cdots, \phi_k, -\phi_1, \cdots, -\phi_k)$. Note the physics is described by the part of the Jacobian of the surface defined by (3.13) and (3.7) which is invariant under Weyl transformations.

4. Massive antisymmetric tensor matter

We conclude with a few further comments on the generalization to massive antisymmetric tensor matter. There exists a value of the parameters which makes the theory $SL(2, \mathbb{Z})$ invariant, and thus the theory should have the same prepotential as $Sp(k)$ with a massive adjoint. This is analogous to the equivalence of $SU(2)$ with four flavors and masses $(m, m, 0, 0)$ and $SU(2)$ with a massive adjoint. Now the prepotentials of the two $Sp(k)$ theories are

$$N_f = 4 , \quad F = \sum_{\alpha} L(\alpha \cdot a) - L(\alpha \cdot a - m_a) + \sum_{\tilde{\alpha}} L(2\tilde{\alpha} \cdot a) - 4L(\tilde{\alpha} \cdot a - m_f)$$

$$N = 4 , \quad F = \sum_{\alpha} L(\alpha \cdot a) - L(\alpha \cdot a - m) + \sum_{\tilde{\alpha}} L(2\tilde{\alpha} \cdot a) - L(2\tilde{\alpha} \cdot a - m) ,$$

where $\alpha$ are the roots of the form $\pm e_i \pm e_j$, and $\tilde{\alpha}$ are the weights $\pm e_i$. Here $m$, $m_f$ and $m_a$ are the masses of adjoint, fundamental and antisymmetric tensor matter, respectively. These prepotentials are the same (up to an irrelevant constant) if $m_a = m$ and $m_f = m/2$. 
We expect that the $Sp(2k)$ theory with massive adjoint can be expressed as a Hitchin system with a single source. If this is true then by including additional charged sources at all four Weierstrass points it should be possible to construct the integrable system describing the $Sp(2k)$ with a massive antisymmetric tensor and massive fundamental matter.

We note that three-dimensional versions of these gauge theories have been studied recently in [13]. In that case, it is possible to construct the moduli space for arbitrary mass parameters by using the hyperkähler properties of these spaces together with mirror symmetry.

Multiple 3-brane theories and their probe interpretation have also been considered in [14].

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References

[1] C. Vafa, “Evidence for F-theory,” Nucl. Phys. B469 (1996) 403; [hep-th/9602022].
[2] A. Sen, “F-theory and Orientifolds,” Nucl. Phys. B475 (1996) 562; [hep-th/9605150].
[3] T. Banks, M. R. Douglas and N. Seiberg, “Probing F-theory with Branes,” Phys. Lett. B387 (1996) 278; [hep-th/9605199].
[4] P. Windey, private communication; C. Vafa, private communication.
[5] U. H. Danielsson and B. Sundborg, “Exceptional Equivalences in N=2 Supersymmetric Yang-Mills Theory,” Phys. Lett. B370 (1996) 83; [hep-th/9511180].
[6] E. Witten, “Small Instantons in String Theory,” Nucl. Phys. B460 (1996) 541; [hep-th/9511030].
[7] R. Donagi and E. Witten, “Supersymmetric Yang–Mills Theory and Integrable Systems,” Nucl. Phys. B460 (1996) 299-334; [hep-th/9510101].
[8] U. H. Danielsson and P. Stjernberg, “Notes on Equivalences and Higgs Branches in N=2 Supersymmetric Yang-Mills Theory,” Phys. Lett. B380 (1996) 68; [hep-th/9603082].
[9] N. Hitchin, Duke Math. J. 54 (1987) 91.
[10] E. Markman, Comp. Math. 93 (1994) 255.
[11] N. Seiberg and E. Witten, “Monopoles, Duality and Chiral Symmetry Breaking in N=2 Supersymmetric QCD,” Nucl. Phys. B431 (1994) 484, [hep-th/9408099].
[12] N. Koblitz, “Introduction to Elliptic Curves and Modular Forms,” Springer-Verlag New York, 1984.
[13] J. de Boer, K. Hori, H. Ooguri and Y. Oz, “Mirror Symmetry in Three-Dimensional Gauge Theories, Quivers and D-branes,” [hep-th/9611063].
[14] O. Aharony, C. Sonnenschein, S. Yankielowicz, and S. Theisen; “Field Theory Questions for String Theory Answers,” [hep-th/9611222].