On weak convergence of stochastic heat equation with colored noise

Pavel Bezdek

Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090

Abstract

In this work we are going to show weak convergence of probability measures. The measure corresponding to the solution of the following one dimensional nonlinear stochastic heat equation

$$\frac{\partial}{\partial t}u_t(x) = \kappa \frac{\partial^2}{\partial x^2}u_t(x) + \sigma(u_t(x))\eta_\alpha$$

with colored noise $\eta_\alpha$ will converge to the measure corresponding to the solution of the same equation but with white noise $\eta$, as $\alpha \uparrow 1$. Function $\sigma$ is taken to be Lipschitz and the Gaussian noise $\eta_\alpha$ is assumed to be colored in space and its covariance is given by $E[\eta_\alpha(t,x)\eta_\alpha(s,y)] = \delta(t-s)f_\alpha(x-y)$ where $f_\alpha$ is the Riesz kernel $f_\alpha(x) \propto 1/|x|^{\alpha}$. We will work with the classical notion of weak convergence of measures, that is convergence of probability measures on a space of continuous function with compact domain and sup-norm topology. We will also state a result about continuity of measures in $\alpha$, for $\alpha \in (0,1)$.

Keywords: The Stochastic heat equation, colored noise, Riesz kernel

1. Introduction

Throughout this work we will consider the following one-dimensional heat equation

$$\frac{\partial}{\partial t}u_{x,t}(x) = \kappa \frac{\partial^2}{\partial x^2}u_{x,t}(x) + \sigma(u_{x,t}(x))\eta_\alpha, \quad x \in \mathbb{R}, \quad t \geq 0,$$

$$u_{x,0} = w(x),$$

with $\kappa > 0$ and Gaussian space time colored noise $\eta_\alpha$ [7]. The noise $\eta_\alpha$ is assumed to have a particular covariance structure

$$E[\eta_\alpha(t,x)\eta_\alpha(s,y)] = \delta(t-s)f_\alpha(x-y),$$

This research was supported in part by the NSFs grant DMS-1307470

Email address: bezdek@math.utah.edu (Pavel Bezdek)

Preprint submitted to Elsevier Thursday 14th January, 2016

© 2016. This manuscript version is made available under the Elsevier user license
http://www elsevier com/open-access/userlicense/1.0/
where [7, Ex. 1]
\[ f_\alpha(x) = c_{1-\alpha}g_\alpha(x) = \hat{g}_{1-\alpha}(x) , \quad g_\alpha(x) = \frac{1}{|x|^{\alpha}} \quad \text{for } \alpha \in (0,1), \]
and the constant \( c_\alpha \) is [9, (12) on pg. 173]
\[ c_\alpha = 2\frac{\sin\left(\frac{\alpha \pi}{2}\right) \Gamma(1-\alpha)}{(2\pi)^{1-\alpha}}. \]

The function \( \hat{g}_{1-\alpha} \) denotes the Fourier of function \( g_{1-\alpha} \). For \( F \in L^1(\mathbb{R}) \), we will take \( \hat{F}(\xi) = \int_{\mathbb{R}} e^{-2\pi \xi x} F(x) dx \). The initial condition, \( w(x) \) is taken to be bounded and \( \varrho \)-Hölder continuous. We will also assume \( \sigma \) to be Lipschitz continuous, there exists \( K \geq 0 \) such that \( |\sigma(x) - \sigma(y)| \leq K |x-y| \) and \( |\sigma(x)| \leq K(1+|x|) \). Stochastic PDEs such as (1) have been studied in [7, 14, 2, 13, 5] and others.

The function \( f_\alpha \) can be thought of as an ‘approximation’ to the delta function in the following special sense, we know that one-dimensional Fourier transform of \( g_{1-\alpha} \), denoted by \( \hat{g}_{1-\alpha} \), is equal to \( f_\alpha \). We also know that the Fourier transform of a constant is \( \delta \) distribution. Observe that \( g_{1-\alpha} \) converges pointwise to 1 as \( \alpha \uparrow 1 \). We will study the solution of (1) as a function of \( \alpha \). This arises noticeably in [1, Sec. 7] where the authors have shown that \( L^2(P) \) norm of \( u_{\alpha,t}(x) \) converges to \( L^2(P) \) norm of the solution to (5) as \( \alpha \uparrow 1 \) for every \( t > 0, x \in \mathbb{R} \) and \( \sigma(x) = x \).

The main question that has motivated this work, is whether the solution of (1) converges in the appropriate sense to the solution of the same equation, but with white noise \( \eta \) instead of colored noise \( \eta_\alpha \) as \( \alpha \uparrow 1 \). By that we mean, the solution to
\begin{align*}
\frac{\partial}{\partial t} u_t(x) &= \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} u_t(x) + \sigma(u_t(x))\eta \quad x \in \mathbb{R}, \quad t \geq 0, \\
u_0(x) &= w(x),
\end{align*}
where \( \eta \) denotes white noise. We will state the main theorem in terms of measures corresponding to solutions. Let \( \mathcal{C} = C([0,T] \times [-N,N]) \) be the space of continuous functions on \([0,T] \times [-N,N] \subset \mathbb{R}^+ \times \mathbb{R} \) with supremum norm. Denote by \( P_\alpha \), the measure corresponding to \( u_\alpha \) restricted to \( D = [0,T] \times [-N,N] \),
\[ P_\alpha(A) := \begin{cases} P \{ u_\alpha \in A^\circ \} & \text{for } \alpha \in (0,1), \\
\{ u \in A^\circ \} & \text{for } \alpha = 1,
\end{cases} \]
for any Borel set \( A \) of space \( \mathcal{C} \). By \( A^\circ \), we denote the embedding of the set \( A \) in a larger space \( C(\mathbb{R}_+ \times \mathbb{R}) \), that is
\[ A^\circ = \{ f \in C(\mathbb{R}_+ \times \mathbb{R}) : f \text{ restricted to } [0,T] \times [-N,N] \text{ is in } A \}.
\]
Here is the main theorem:

**Theorem 1.** Measure $P_{\alpha}$ is continuous in $\alpha$, for $\alpha \in (0, 1]$. We precisely mean that $P_{\alpha}$ converges weakly to $P_1$ as $\alpha \uparrow 1$ and $P_{\alpha}$ converges weakly to $P_{\alpha_0}$ as $\alpha \to \alpha_0$ for any $\alpha_0 \in (0, 1]$.

The notion of weak convergence in Theorem 1 is the classical one [4]. Theorem 1 gives us a new way of thinking about the stochastic heat equation with white noise. Instead of studying the solution to (5) we can study the solution to (1) for $\alpha \approx 1$. Also note, that the noise with Riesz kernel spatial covariance produces noise which is less regular. We like to think that this ‘roughness’ better captures properties of the stochastic heat equation with white noise.

Before we begin the proof of Theorem 1, let us recall the rigorous definition of the noises $\eta_{\alpha}, \eta$ as well as the form of [mild] solutions to (1) and (5). The noises $\eta_{\alpha}$ and $\eta$ are $L^2(P)$ (random variables with finite second moment) valued set functions such that [7, 6, 12]

$$
\eta_{\alpha}([0,t] \times B) \text{ and } \eta_{\alpha}([0,t] \times B) \text{ for } t \geq 0, B \text{ bounded Borel set},
$$

are mean zero Gaussian random variables. If $A$ is another bounded Borel set, then the covariance of noises $\eta_{\alpha}$ and $\eta$ will be

$$
E[\eta_{\alpha}([0,t] \times A) \eta_{\alpha}([0,t] \times B)] = t \int \int \mathbb{1}_{A}(x)f_{\alpha}(x-y)\mathbb{1}_{B}(y)dxdy,
$$

$$
E[\eta([0,t] \times A) \eta([0,t] \times B)] = t \int \int \mathbb{1}_{A}(x)\mathbb{1}_{B}(x)dx.
$$

We often talk about Martingale valued measure, since $\eta_{\alpha}([0,t] \times A)$ and $\eta_{\alpha}([0,t] \times A)$ are set functions, which are $L^2(P)$ martingales in $t$. We refer the reader to [7, 6, 12] for more details.

The mild solutions are interpreted as solutions of the following integral equations [7, 6, 12]

$$
u_{\alpha,t}(y) = (u_{\alpha,0} * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_{\alpha,s}(x))\eta_{\alpha}(ds,dx),
$$

$$
u_{t}(y) = (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_s(x))\eta(ds,dx),
$$
where \( p_t \) is the heat kernel

\[
p_t(x) = \frac{1}{\sqrt{2\pi kt}} \exp\left(-\frac{x^2}{2kt}\right),
\]

and \( * \) denotes the convolution of two function \((f * g(x) = \int_{\mathbb{R}} f(y)g(x - y)dy)\).

### 2. Proof of Theorem 1

We will only show the first part of the theorem, \( P_\alpha \) converges weakly to \( P_1 \) as \( \alpha \to 1 \). This is the worst case scenario. The second statement of Theorem 1 follows almost directly from the proof in this section.

The proof of the upcoming Theorem 2 uses coupling, which allows us to put both noises \( \eta_\alpha \) and \( \eta \) on the same probability space. This idea was introduced in [5] and lets us write our noise \( \eta_\alpha \), for every \( \alpha \in (0,1) \) in terms of one white noise \( \eta \) with covariance

\[
E[\eta(t,x)\eta(s,y)] = \delta(t-s)\delta(x-y).
\]

The idea of coupling, or smoothing the noise in the spatial variable is not new. Authors in [3] smoothed the noise in the spatial variable by an infinitely differentiable function with compact support. They have showed that this kind of smoothing converges to the heat equation with white noise as our smoothing function converges to \( \delta \) distribution.

By coupling we mean that the martingale measure \( \eta_\alpha \) will be defined as

\[
\eta_\alpha([0,t] \times A) = \int_0^t \int_{\mathbb{R}} (1_A \ast h_\alpha)(x)\eta(ds,dx),
\]

where

\[
h_\alpha(x) = c_{\frac{1}{2} - \alpha} g_{\frac{1}{2} + \alpha}(x) = \hat{g}_{\frac{1}{2} - \alpha}(x).
\]

This choice of \( h_\alpha \) produces correct \( f_\alpha \) in (2), that is

\[
f_\alpha(x) = (h_\alpha \ast h_\alpha)(x),
\]

since

\[
g_{1-\alpha}(\xi) = g_{\frac{1}{2} - \alpha}(\xi) \cdot g_{\frac{1}{2} - \alpha}(\xi).
\]
The typical stochastic Fubini theorem (see [5, pg. 492] or [12, pg. 14, pg. 50]) which would allow us to write (6), requires that \( h_\alpha \in L^2(\mathbb{R}) \). One might notice that \( h_\alpha \notin L^2(\mathbb{R}) \), but \( \eta_\alpha \) is a well defined martingale measure, we refer the reader to [5] for more details. Before we state the main theorem of this section, let us state a technical lemma and define the following norms [12]

\[
\mathcal{N}_{\gamma,k}(u) = \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} \left( e^{-\gamma t} \| u_t(x) \|_{L^k(P)} \right), \quad \gamma > 1, \quad k \geq 2.
\]

The norm \( \| \cdot \|_{L^k(P)} \) in the definition of \( \mathcal{N}_{\gamma,k} \) stands for \( L^k \) norm on a probability space. For a random variable \( X \) it is defined as \( \| X \|_{L^k(P)} = E[|X|^k]^{1/k} \). We will often write \( \| \cdot \|_{k} \) instead of \( \| \cdot \|_{L^k(P)} \).

**Lemma 1.** [11, 3.478] The following equality holds for \( s > 0 \) and \( \beta \in [0,1) \)

\[
\int_{\mathbb{R}} |x|^{-\beta} e^{-s4\pi^2 x^2} dx = \left( \frac{1}{s4\pi^2} \right)^{-\beta/2} \Gamma(-\beta/2 + 1/2).
\] (8)

In the rest of this section, we will prove the following main theorem.

**Theorem 2.** For every \( k \geq 2 \) we can find \( \gamma \) such that

\[
\lim_{\alpha \uparrow 1} \mathcal{N}_{\gamma,k}(u_\alpha - u) = 0.
\]

Take the constant \( T \) that appears in Theorem 1 and the definition of the norm \( \mathcal{N}_{\gamma,k} \) to be fixed throughout the whole proof. We will start our proof with Picard iterations for both noises \( \eta_\alpha \) and \( \eta \)

\[
u_t^{(n+1)}(y) = (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u_s^{(n)}(x)) \eta(ds, dx)
\]

\[
u_{\alpha,t}^{(n+1)}(y) = (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \sigma(u_{\alpha,s}^{(n)}(x)) \eta_\alpha(ds, dx),
\]

which is equivalent to the following, thanks to [5, Sec. 3.2]:

\[
u_t^{(n+1)}(y) = (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} (p_{t-s}(\cdot - y) \sigma(u_s^{(n)}(\cdot)) * \delta)(x) \eta(ds, dx)
\]

\[
u_{\alpha,t}^{(n+1)}(y) = (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} (p_{t-s}(\cdot - y) \sigma(u_{\alpha,s}^{(n)}(\cdot)) * \delta)(x) \eta_\alpha(ds, dx).
\]
First, let us estimate the $L^k(P)$ norm of the difference of Picard iterates $u^{(n+1)}_{\alpha,t}(y) - u^{(n+1)}_t(y)$,

$$E\left[\left|u^{(n+1)}_{\alpha,t}(y) - u^{(n+1)}_t(y)\right|^k\right]$$

Adding and subtracting the following term, $(p_{t-s}(\cdot - y)\sigma(u^{(n)}_{\alpha,s}) * h_\alpha)(x) - (p_{t-s}(\cdot - y)\sigma(u^{(n)}_{s}) * \delta)(x)) \eta(ds, dx)$ inside the integral and using the inequality $|a - b|^k \leq 2^k|a|^k + 2^k|b|^k$ yields

$$E\left[\left|\int_0^t \int_\mathbb{R} (p_{t-s}(\cdot - y)\sigma(u^{(n)}_{\alpha,s}) - \sigma(u^{(n)}_{s}) * \delta)(x)) \eta(ds, dx)\right|^k\right].$$

The next series of steps will be used multiple times throughout this work. First we will use Burkholder-Davis-Gundy inequality and Minkowski integral inequality. Burkholder-Davis-Gundy (BDG) inequality (see for example [12, Thm. B.1]) states that for any continuous $L^2$ martingale $M_t$ and $k \geq 2$ we have $\|M_t\|_k \leq 4\|\langle M \rangle_t\|_{k/2}$, where $\langle M \rangle_t$ denotes the quadratic variation of $M$. Applying this inequality and evaluating the quadratic variation term [6, Thm. 5.26] on both terms gives us
Minkowski integral inequality states that \((\int f d\mu)^{k/2} \leq \int (f^k d\nu)^{1/k} d\mu\) for any \(\sigma\)-finite measures \(\mu, \nu\) and jointly measurable \((\mu \times \nu)\) positive function \(f\). We use this inequality on the first term in order to obtain

\[
E \left[ \left| u_{\alpha,t}^{(n+1)}(y) - u_{t}^{(n+1)}(y) \right|^k \right] 
\leq \text{const} \cdot E \left[ \left( \int_{[0,t] \times \mathbb{R}^2} p_{t-s}(x-y) v_s^{(n)}(x,z) f_{\alpha}(x-z) p_{t-s}(z-y) dxdzd\bar{s} \right)^{k/2} \right]
\]

\[
+ \text{const} \cdot E \left[ \left( \int_{[0,t] \times \mathbb{R}^2} p_{t-s}(x-y) (z) \sigma(u_s^{(n)}(x)) (f_{\alpha} - 2h_{\alpha} + \delta)(x-z) \right. \right.
\]

\[
\left. \left. \left. \left. p_{t-s}(z-y) \sigma(u_s^{(n)}(z)) dxdzd\bar{s} \right)^{k/2} \right] \right],
\]

where

\[
v_s^{(n)}(x,z) = (\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(x)(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(z).
\]

and \(v_s^{(n)}\) denotes

\[
v_s^{(n)}(x,z) = E \left[ \left| (\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(x) \right|^{k/2} \left| (\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(z) \right|^{k/2} \right]^{2/k}.
\]
Ultimately, we would like to show that $u_\alpha$ is close to $u$ as $\alpha \uparrow 1$. The original estimate for Picard iterates splits into two terms $\mathfrak{A}_{n,\alpha}$ and $\mathfrak{B}_{n,\alpha}$ where

For term $\mathfrak{A}_{n,\alpha}$ we use Cauchy-Schwarz inequality and take supremum over the term involving expectation, which yields

$$\mathfrak{A}_{n,\alpha} \leq \left( \int_0^t \sup_{x \in \mathbb{R}} E \left[ \left| (\sigma(u^{(n)}_{\alpha,s}) - \sigma(u^{(n)}_s))(x) \right|^k \right] \int_{\mathbb{R}^2} p_{t-s}(x-y)f_\alpha(x-z)p_{t-s}(z-y)dxdzds \right)^{k/2}.$$ 

The following identity holds

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x)f_\alpha(x-y)\varphi(y)dxdy = \int_{\mathbb{R}} f_\alpha(x)(\varphi \ast \hat{\varphi})(x)dx = \int_{\mathbb{R}} g_{1-\alpha}(\xi)\left| \hat{\varphi}(\xi) \right|^2 d\xi, \quad (10)$$

for any $\varphi$ from Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing test functions, where $\hat{\varphi}(x) = \varphi(-x)$. This is a consequence of elementary properties of Fourier transform [7, pg. 6], [10, pg. 151,152]. We can further rewrite $\mathfrak{A}_{n,\alpha}$, using identity (10) and the assumption that $\sigma$ is Lipschitz continuous (there exists $K \geq 0$, such that $|\sigma(x) - \sigma(y)| \leq K |x-y|$ and $|\sigma(x)| \leq K(1+|x|)$), as

$$\mathfrak{A}_{n,\alpha} \leq \left( \int_0^t \sup_{x \in \mathbb{R}} E \left[ \left| (\sigma(u^{(n)}_{\alpha,s}) - \sigma(u^{(n)}_s))(x) \right|^k \right] \int_{\mathbb{R}} g_{1-\alpha}(\xi)\left| \hat{\varphi}_{t-s}(\xi) \right|^2 d\xi ds \right)^{k/2} \leq K^k \left( \int_0^t \sup_{x \in \mathbb{R}} E \left[ \left| (u^{(n)}_{\alpha,s} - u^{(n)}_s)(x) \right|^k \right] \int_{\mathbb{R}} g_{1-\alpha}(\xi)\left| \hat{\varphi}_{t-s}(\xi) \right|^2 d\xi ds \right)^{k/2}.$$ 

Multiply by term $e^{-k\gamma t}$ and obtain $\mathcal{N}_{\gamma,k}$ norm in the estimate

$$e^{-k\gamma t} \mathfrak{A}_{n,\alpha} \leq K^k \left( \int_0^t e^{-2\gamma s} \sup_{x \in \mathbb{R}} E \left[ \left| (u^{(n)}_{\alpha,s} - u^{(n)}_s)(x) \right|^k \right] e^{-2\gamma(t-s)} \int_{\mathbb{R}} g_{1-\alpha}(\xi)\left| \hat{\varphi}_{t-s}(\xi) \right|^2 d\xi ds \right)^{k/2}$$

$$\leq K^k \mathcal{N}_{\gamma,k}(u^{(n)}_\alpha - u^{(n)})^k \left( \int_0^t e^{-2\gamma(t-s)} \int_{\mathbb{R}} g_{1-\alpha}(\xi)\left| \hat{\varphi}_{t-s}(\xi) \right|^2 d\xi ds \right)^{k/2} \leq K^k \mathcal{N}_{\gamma,k}(u^{(n)}_\alpha - u^{(n)})^k \left( \int_0^t e^{-2\gamma(t-s)} \int_{\mathbb{R}} \frac{1}{|\xi|^{1-\alpha}} e^{-(t-s)\kappa_4\pi^2\xi^2} d\xi ds \right)^{k/2}. \quad (11)$$

Later on, we will see that we can make the integral on right hand side arbitrarily small. The estimate for $\mathfrak{B}_{n,\alpha}$ uses a similar technique as the estimate for $\mathfrak{A}_{n,\alpha}$, but
some extra work is required because of the term \((f_\alpha - 2h_\alpha + \delta)\) inside the integral is not a positive function. Thanks to [7],[8, Cor. 3.4] identity (10) extends to a much broader class of functions. We will use this identity to bound term \(B_{n,\alpha}\). Quantity \(\sigma(u_s^{(n)}(\cdot))p_{t-s}(\cdot - y) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})\) almost surely, because

\[
E \left[ \|\sigma(u_s^{(n)}(\cdot))p_{t-s}(\cdot - y)\|_{L^2(\mathbb{R})}^2 \right] \leq 2K^2(1 + \mathcal{N}_{\gamma,2}(u^{(n)}))\|p_{t-s}(\cdot)\|_{L^2(\mathbb{R})}^2,
\]

and \(\mathcal{N}_{\gamma,2}(u^{(n)})\) is bounded uniformly (in \(n, \gamma\)) for every \(n \in \mathbb{N}\) and all \(\gamma > \gamma_1\) [12, proof of Thm. 5.5]. The constant \(\gamma_1\) depends on \(K, \kappa\) and \(\sup_x w(x)\) [12, Thm. 5.5]. A similar reasoning applies for \(\|\cdot\|_{L^1(\mathbb{R})}\). We can write

\[
E \left[ \left( \int_{[0,t] \times \mathbb{R}^2} p_{t-s}(x - y)\sigma(u_s^{(n)}(x)) (f_\alpha - 2h_\alpha + \delta) (x - z)p_{t-s}(z - y)\sigma(u_s^{(n)}(z))dx dz ds \right)^{k/2} \right]
\]

\[
= E \left[ \left( \int_{[0,t] \times \mathbb{R}} (g_{1-\alpha} - 2g_{1-\alpha}/2 + 1)(\xi) |\mathcal{F}(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot)))\,(\xi)|^2 d\xi ds \right)^{k/2} \right].
\]

Split this integral into two parts, and use inequality \(|a - b|^{k/2} \leq 2^{k/2} |a|^{k/2} + 2^{k/2} |b|^{k/2}\) to get

\[
E \left[ \left( \int_{[0,t] \times \mathbb{R}} (g_{1-\alpha} - 2g_{1-\alpha}/2 + 1)(\xi) |\mathcal{F}(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot)))\,(\xi)|^2 d\xi ds \right)^{k/2} \right]
\]

\[
\leq \text{const} \cdot (\mathcal{C}_{n,\alpha} + \mathcal{D}_{n,\alpha}),
\]

where

\[
\mathcal{C}_{n,\alpha} = E \left[ \left( \int_{0}^{t} \int_{-1}^{1} (g_{1-\alpha} - 2g_{1-\alpha}/2 + 1)(\xi) |\mathcal{F}(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot)))\,(\xi)|^2 d\xi ds \right)^{k/2} \right]
\]

\[
\mathcal{D}_{n,\alpha} = E \left[ \left( \int_{0}^{t} \int_{\mathbb{R}\setminus[-1,1]} (g_{1-\alpha} - 2g_{1-\alpha}/2 + 1)(\xi) |\mathcal{F}(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot)))\,(\xi)|^2 d\xi ds \right)^{k/2} \right].
\]

Properties of Fourier transform and Lipschitz continuity of \(\sigma(x)\) give us

\[
|\mathcal{F}(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot)))\,(\xi)|^2 \leq \|p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\|^2_{L^1(\mathbb{R})}
\]

\[
\leq K^2\|p_{t-s}(\cdot - y)(1 + |u_s^{(n)}(\cdot)|)\|^2_{L^1(\mathbb{R})} \leq K^2(2 + 2\|p_{t-s}(\cdot - y)u_s^{(n)}(\cdot)\|^2_{L^1(\mathbb{R})} ).
\]

(12)
for the term inside of $\mathcal{C}_{n,\alpha}$. Splitting the term (12) inside of the integral into two yields
\[
\mathcal{C}_{n,\alpha} \leq E \left[ \left( \int_0^t \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) K^2 (2 + 2\|p_{t-s}(\cdot - y)u_s^{(n)}(\cdot)\|^2_{L^1(\mathbb{R})}) d\xi ds \right)^{k/2} \right] \\
\leq C_\alpha \cdot \text{const} \cdot E \left[ \left( \int_0^t \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) \|p_{t-s}(\cdot - y)u_s^{(n)}(\cdot)\|^2_{L^1(\mathbb{R})} d\xi ds \right)^{k/2} \right],
\]
where $C_\alpha$ denotes
\[
C_\alpha = \text{const} \left( \int_0^t \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds \right)^{k/2}.
\]
Term $C_\alpha$ can be made as small as we like, due to the dominated convergence theorem. We use Minkowski integral inequality and get
\[
\mathcal{C}_{n,\alpha} \leq C_\alpha + \text{const} \cdot \text{sup}_{x \in \mathbb{R}} E \left( \left| u_s^{(n)}(x) \right|^k \right)^{2/k} \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds \\
\leq C_\alpha + \text{const} \left( \int_0^t e^{-k\gamma s} \text{sup}_{x \in \mathbb{R}} E \left( \left| u_s^{(n)}(x) \right|^k \right)^{2/k} e^{k\gamma s} \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds \right)^{k/2} \\
\leq C_\alpha + \text{const} \cdot \mathcal{N}_{\gamma,k}(u^{(n)})^k \left( \int_0^t e^{k\gamma s} \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds \right)^{k/2}.
\]
From the general theory of stochastic partial differential equations [12, proof of Thm. 5.5] we know that the term $\mathcal{N}_{\gamma,k}(u^{(n)})$ is bounded uniformly in $n$ and $\gamma$ for every $n \in \mathbb{N}$ and $\gamma > \gamma_2$. Where $\gamma_2$ again depends on $K, \kappa$ and $\sup_{x \in \mathbb{R}} w(x)$. The integral term bounding $\mathcal{C}_{n,\alpha}$ can be made arbitrarily small, again from the dominated convergence theorem. Overall, we get that $\mathcal{C}_{n,\alpha}$ converges uniformly in $n$ to zero as $\alpha \uparrow 1$.

All we have left to do is find the estimate for $\mathcal{D}_{n,\alpha}$. Add and subtract the term $\sigma(u_s(x))$ inside the Fourier transform, split into two integrals and obtain
\[
\mathcal{D}_{n,\alpha} \leq \text{const} \cdot \mathcal{D}_{n,\alpha}^{(1)} + \text{const} \cdot \mathcal{D}_{n,\alpha}^{(2)},
\]
where
\[
\mathcal{D}_{n,\alpha}^{(1)} = E \left[ \left( \int_0^t \int_{\mathbb{R} \setminus [-1,1]} (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) \cdot \mathcal{F} \left( p_{t-s}(\cdot - y) \left( \sigma(u_s^{(n)}(\cdot)) - \sigma(u_s(\cdot)) \right) \right) (\xi)^2 d\xi ds \right)^{k/2} \right],
\]
\[
\mathcal{D}_{n,\alpha}^{(2)} = E \left[ \left( \int_0^t \int_{\mathbb{R} \setminus [-1,1]} (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) \cdot \mathcal{F} \left( p_{t-s}(\cdot - y) \left( \sigma(u_s^{(n)}(\cdot)) - \sigma(u_s(\cdot)) \right) \right) (\xi)^2 d\xi ds \right)^{k/2} \right].
\]
\[ \mathcal{D}_{n,\alpha}^{(2)} = E \left( \int_0^t \int_{\mathbb{R} \setminus [-1,1]} (g_{1-\alpha} - 2g_{1-\alpha}^2 + 1)(\xi) \left| \mathcal{F}(p_{t-s}(\cdot - y) \sigma(u_s(\cdot)) (\xi)) \right|^2 d\xi ds \right)^{k/2} \].

Term \( \mathcal{D}_{n,\alpha}^{(1)} \) converges to zero as \( n \uparrow \infty \), uniformly in \( \alpha \in (0,1) \). Use Plancherel’s theorem and the fact that \((g_{1-\alpha} - 2g_{1-\alpha}^2 + 1)\) is bounded by a constant on \( \mathbb{R} \setminus [-1,1] \), uniformly for all \( \alpha \in (0,1) \) to write

\[ \mathcal{D}_{n,\alpha}^{(1)} \leq E \left( \left( \int_0^t \int_{\mathbb{R}} (p_{t-s}(x - y) (\sigma(u_s^{(n)}(x)) - \sigma(u_s(x)))^2 dx dt \right)^{k/2} \right). \]

From the convergence of Pickard’s iterations and theory of SPDEs [12, proof of Thm. 5.5] we get convergence of \( \mathcal{D}_{n,\alpha}^{(1)} \) to zero as \( n \to \infty \), uniformly in \( \alpha \in (0,1) \). For \( \epsilon > 0 \), we can find \( n_0(\epsilon) \) such that for every \( n > n_0 \) we have \( \mathcal{D}_{n,\alpha}^{(1)} < \epsilon/2 \). If \( n \leq n_0 \), we can find \( \alpha_n \) such that \( \mathcal{D}_{n,\alpha}^{(1)} < \epsilon/2 \) for \( \alpha \in (\alpha_n, 1) \), by dominated convergence theorem. Do not forget that \((g_{1-\alpha} - 2g_{1-\alpha}^2 + 1)(\xi)\) converges pointwise to zero. If we apply dominated convergence theorem to \( \mathcal{D}_{n,\alpha}^{(2)} \), we get that for \( \alpha \in (\alpha_n, 1) \), \( \mathcal{D}_{n,\alpha}^{(2)} < \epsilon/2 \). Altogether we have that for every \( \epsilon > 0 \) there is \( \alpha_\epsilon := \max_{0 \leq n \leq n_0} \alpha_n \) such that \( \mathcal{D}_{n,\alpha} < \epsilon \) for \( \alpha \in (\alpha_\epsilon, 1) \) and every \( n \in \mathbb{N} \). Therefore \( \mathcal{D}_{n,\alpha} \) converges uniformly in \( n \) to zero as \( \alpha \uparrow 1 \).

We have shown that

\[ E \left[ \left| u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y) \right|^k \right] \leq \text{const} \cdot \mathfrak{A}_{n,\alpha} + \text{const} \cdot \mathfrak{C}_{n,\alpha} + \text{const} \cdot \mathcal{D}_{n,\alpha}, \]

where \( \mathfrak{C}_{n,\alpha} + \mathcal{D}_{n,\alpha} \) converges to zero as \( \alpha \uparrow 1 \), uniformly in \( n \). For \( \epsilon > 0 \), we can pick \( \alpha_\epsilon \) such that

\[ E \left[ \left| u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y) \right|^k \right] \leq \text{const} \cdot \mathfrak{A}_{n,\alpha} + \epsilon, \]

for \( \alpha \in (\alpha_\epsilon, 1) \). Multiply the previous line by \( e^{-k\gamma t} \) and use (11) to arrive at

\[ e^{-k\gamma t} E \left[ \left| u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y) \right|^k \right] \]

\[ \leq \text{const} \cdot \mathcal{N}_{\gamma,k}(u_{\alpha}^{(n)} - u^{(n)})^k \left( \int_0^t e^{-2\gamma(t-s)} \int_{\mathbb{R}} \frac{1}{|\xi|^{1-\alpha}} e^{-(t-s)k\pi^2 \xi^2} d\xi ds \right)^{k/2} + \epsilon, \]
where \( \gamma \geq 1 \). We use Lemma 1 to evaluate the term inside the integral. A straightforward calculation yields, for \( 1 > \alpha > \alpha_\epsilon > 0 \),

\[
e^{-k\gamma t} E \left[ \left| u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y) \right|^k \right]
\]

\[
\leq \text{const} \cdot N_{\gamma,k}(u_{\alpha}^{(n)} - u^{(n)})^k \left( \int_0^t e^{-2\gamma(t-s)}(t-s)^{-\alpha/2} ds \right)^{k/2} + \epsilon
\]

\[
\leq \text{const} \cdot N_{\gamma,k}(u_{\alpha}^{(n)} - u^{(n)})^k \left( \frac{1}{\gamma} \right)^{1-\alpha/2} \left( \frac{1}{\gamma} \right)^{k/4} + \epsilon .
\]

We can take supremum over \( y \in \mathbb{R} \) and \( t \in [0, T] \) to get

\[
N_{\gamma,k}(u_{\alpha}^{(n+1)} - u^{(n+1)})^k \leq a \cdot N_{\gamma,k}(u_{\alpha}^{(n)} - u^{(n)})^k + \epsilon . \tag{13}
\]

where \( a = \text{const} \left( \frac{1}{\gamma} \right)^{k/4} \). The constant in \( a \) depends only on \( K,k \) and choice of \( \alpha_\epsilon \). It can be made explicit by tracking constants in front of \( A_n,\alpha \) together with a constant dependent on \( \alpha_\epsilon \), which comes from Lemma 1. The dependency on \( \alpha_\epsilon \) comes only from constant that appears in Lemma 1 and can be bounded from above as long as \( \alpha_\epsilon \) is bounded away from zero. We can always pick \( \alpha_\epsilon > 1/2 \) and get rid of dependency in \( \alpha_\epsilon \). Equation (13) defines a convergent geometric series assuming that the coefficient \( a < 1 \) and \( \gamma > \max(\gamma_1, \gamma_2) \). We have

\[
N_{\gamma,k}(u_{\alpha}^{(n+1)} - u^{(n+1)})^k \leq a^nN_{\gamma,k}(u_{\alpha}^{(1)} - u^{(1)})^k + \sum_{i=1}^{n-1} a^i\epsilon,
\]

and

\[
N_{\gamma,k}(u_{\alpha}^{(n+1)} - u^{(n+1)})^k \leq \frac{\epsilon}{1 - a}.
\]

Let \( n \) go to infinity and conclude the proof.

2.1. Continuity in \( N_{\gamma,k} \) norm

Our proof of Theorem 2 also implies continuity in \( \alpha \) for \( \alpha \in (0, 1) \). We will only comment on how the proof would change in section 2.

**Theorem 3.** For every \( k \geq 2 \) and \( \alpha_0 \in (0, 1) \) we can find \( \gamma \) such that

\[
\lim_{\alpha \to \alpha_0} N_{\gamma,k}(u_{\alpha} - u_{\alpha_0}) = 0.
\]

12
The proof of Theorem 3 follows the same general direction of the proof of Theorem 2 with the following changes. We need to replace \( u_t(x) \) with \( u_{\alpha,t}(x) \) and change \((f_\alpha - 2h_\alpha + \delta)\) in the estimate for \( \mathfrak{B}_{n,\alpha} \) to \((f_\alpha - 2\frac{f_\alpha + a_\alpha}{2} + f_{a_\alpha})\) and change \((g_{1-\alpha} - 2g_{1-\alpha} + 1)\) to \((g_{1-\alpha} - 2g_{1-\alpha} + a_\alpha)\). We will also need an existence of \( \gamma_0 \) such that for every \( \gamma > \gamma_0 \) norms \( \mathcal{N}_{\gamma,k}(u_{a_\alpha}), \mathcal{N}_{\gamma,k}(u_{a_\alpha}^{(n)}), \mathcal{N}_{\gamma,2}(u_{a_\alpha}), \mathcal{N}_{\gamma,2}(u_{a_\alpha}^{(n)}) \) are finite, uniformly in \( n \in \mathbb{N} \) and \( \gamma \). This can be obtained from [5, Prep. 9.1].

One can notice in both Theorem 2 and 3, the coefficient \( e^{-\gamma t} \) in \( \mathcal{N}_{\gamma,k} \) norm served only as a helping hand to make part of the term \( \mathfrak{B}_{n,\alpha} \) in (11) small. Let us summarize our effort in the following corollary.

**Corollary 1.** Define \( u_{1,t}(x) \equiv u_t(x) \), then for all \( \alpha_0 \in (0, 1) \) we have

\[
\lim_{\alpha \to \alpha_0} \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} E \left[ |u_{\alpha,t}(x) - u_{\alpha_0,t}(x)|^k \right].
\]

### 2.2. Convergence of finite dimensional distributions

Theorem 2 also states that the solution \( u_\alpha \) converges to \( u \) in \( L^2(P) \) norm for every \( t \in [0,T] \) and \( x \in \mathbb{R} \). This implies weak convergence of finite dimensional distributions of \( u_\alpha \) to finite dimensional distributions of \( u \). The easiest way to see that is to show convergence in probability for a finite number of pairs \((t_i, x_i) \in [0, T] \times [-N, N]\), which implies weak convergence of finite dimensional distribution. By Chebyshev's inequality, we have for \( x_i \in [-N, N] \) and \( t_i \in [0, T] \)

\[
P\left( \sum_{i=1}^{l}(u_{\alpha,t_i}(x_i) - u_{t_i}(x_i))^2 > \varepsilon^2 \right) \leq \frac{\sum_{i=1}^{l} E[(u_{\alpha,t_i}(x_i) - u_{t_i}(x_i))^2]}{\varepsilon^2}.
\]  

(14)

Right hand side of (14) converges to zero by Theorem 2, which also means that we have a convergence in probability of random vectors

\[
(u_{\alpha,t_1}(x_1), \ldots, u_{\alpha,t_l}(x_l)) \to (u_{t_1}(x_1), \ldots, u_{t_l}(x_l)).
\]

From convergence in probability of finite dimensional distributions, we can conclude the weak convergence of finite dimensional distributions [4, pg. 27, pg. 207].

### 2.3. Estimates for Kolmogorov’s continuity theorem and Tightness

We will prove tightness (and thus weak convergence) from Kolmogorov’s continuity theorem [12, pg. 107]. Before we begin the proof, we will need the following two lemmas.
Lemma 2. [5, Lemma 6.4] For all \( t > 0 \) and \( x \in \mathbb{R} \)

\[
\int_{\mathbb{R}} |p_t(y - x) - p_t(y)| \, dy \leq \text{const} \cdot \left( \frac{|x|}{\sqrt{\kappa t}} \wedge 1 \right),
\]

where the implied constant does not depend on \((t, x)\).

Lemma 3. For all \( t, \epsilon > 0 \) we have

\[
\int_{\mathbb{R}} |p_{t+\epsilon}(y) - p_t(y)| \, dy \leq \text{const} \cdot ((\log(t + \epsilon) - \log(t)) \wedge 1).
\]

Proof. Direct computation gives us

\[
\int_{\mathbb{R}} |p_{t+\epsilon}(y) - p_t(y)| \, dy = \int_{\mathbb{R}} \int_{t}^{t+\epsilon} \dot{p}_s(y) ds \, dy = \int_{\mathbb{R}} \int_{t}^{t+\epsilon} \left( -\frac{1}{2s} + \frac{y^2}{2s^2 \kappa} \right) p_s(y) ds \, dy \\
\leq \int_{t}^{t+\epsilon} \int_{\mathbb{R}} \left( \frac{1}{2s} + \frac{y^2}{2s^2 \kappa} \right) p_s(y) ds dy = \int_{t}^{t+\epsilon} \frac{1}{s} ds = (\log(t + \epsilon) - \log(t)).
\]

In addition, we have that

\[
\int_{\mathbb{R}} |p_{t+\epsilon}(y) - p_t(y)| \, dy \leq 2.
\]

Lemma 4. [15, pg. 314] Let \( w \) be a bounded \( \varrho \)-Hölder continuous function, then there exists \( C > 0 \) such that for every \( t > 0, \delta > 0, x, z \in \mathbb{R} \) we have

\[
\int_{\mathbb{R}} (p_t(x - y) - p_t(z - y)) w(y) \, dy \leq C \cdot |x - z|^{\varrho},
\]

\[
\int_{\mathbb{R}} (p_{t+\delta}(x - y) - p_t(x - y)) w(y) \, dy \leq C \cdot \delta^{\varrho}.
\]

2.3.1. Difference in the spatial variable

Denote

\[
I_{\alpha,t}(x) = \int_{0}^{t} \int_{\mathbb{R}} (p_{t-s}(x - z) - p_{t-s}(y - z)) \sigma(u_{\alpha,s}(z)) \eta_{\alpha}(ds, dz),
\]
which is the stochastic integral for the mild solution. We will estimate the spatial and the time difference of the stochastic integral $I$ in this and next subsection. The estimates of differences of the solution $u_\alpha$ will be obtained by combining Lemma 4 and estimates on $I$.

Let us estimate the difference in the spatial variable is

$$E \left[ |I_{\alpha,t}(x) - I_{\alpha,t}(y)|^k \right] = E \left[ \left| \int_0^t \int_\mathbb{R} \left( p_t(x - z) - p_t(y - z) \right) \sigma(u_{\alpha,s}(z)) \eta_\alpha(ds, dz) \right|^k \right],$$

and denote

$$B_s(z) = (p_t(x - z) - p_t(y - z)),$$

$$A_s(x, y) = \sigma(u_{\alpha,s}(x)) \sigma(u_{\alpha,s}(y)).$$

We will proceed just as in Section 2. We use Burkholder-Davis-Gundy inequality, Minkowski integral inequality, Cauchy-Schwarz inequality and take the absolute value inside the integral and get

$$E \left[ |I_{\alpha,t}(x) - I_{\alpha,t}(y)|^k \right]$$

$$\leq \text{const} \cdot E \left[ \left( \int_0^t \int_\mathbb{R} \int_\mathbb{R} f_\alpha(z - w) B_s(z) B_s(w) A_s(x, y) dz dw ds \right)^{k/2} \right]$$

$$\leq \text{const} \cdot \left( \int_0^t \sup_{x \in \mathbb{R}} \|\sigma(u_{\alpha,s}(x))\|^2_k \left( \int_\mathbb{R} \left( \int_\mathbb{R} f_\alpha(z - w) B_s(z) B_s(w) dz \right)^{1/2} \right)^2 \right)^{k/2}$$

$$\leq \text{const} \cdot (1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k) \left( \int_0^t \left( \int_\mathbb{R} \left( f_\alpha * p_t(s) \right) \left| p_t(x - z) - p_t(y - z) \right| dz \right)^{1/2} \right)^{k/2}$$

$$\leq \text{const} \cdot (1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k) \left( \int_0^t \left( \int_\mathbb{R} \left| f_\alpha * p_t(s) \right| \left( \frac{|x - y|}{\sqrt{\kappa(t - s)}} \right) dz \right)^{1/2} \right)^{k/2},$$

where the last inequality is due to Lemma 2. We also used the fact that

$$\int_\mathbb{R} f_\alpha(z - w) \left| p_t(x - w) - p_t(y - w) \right| dw \leq 2 \left( f_\alpha * p_t \right)(0). \quad (16)$$
The previous line (16) can be easily checked by the use of the triangle inequality and maximization over variables. The inequality $r \wedge 1 \leq r^{2a}$ for $a \in (0, 1/2)$ gives us

$$E \left[ |I_{\alpha,t}(x) - I_{\alpha,t}(y)|^k \right] \leq \text{const}(1 + N_{\gamma,k}(u_{\alpha})^k) |x - y|^{ak} \left| \int_0^t (f_\alpha * p_{t-s})(0) \cdot (t - s)^{-a} ds \right|^{k/2} \quad (17)$$

It remains to show that the integral on the right hand side is bounded for all $\alpha \in (\alpha_0, 1)$, $\alpha_0 > 0$. To show this, we will need an explicit form of $f_\alpha$. The result is stated in the next lemma.

**Lemma 5.** For every $1 > \alpha > \alpha_0 > 0$ we have

$$f_\alpha * p_s(0) \leq \text{const} \cdot s^{-\alpha/2},$$

where the constant depends only on our choice of $\alpha_0$.

**Proof.** By direct computation and (8) we get

$$(f_\alpha * p_s)(0) = c_{1-\alpha} \int_{\mathbb{R}} \frac{1}{|x|^{\alpha}} p_s(x) dx = 2^{1-\alpha} (\frac{1-\alpha}{2})^{\alpha} \Gamma(\alpha) \Gamma(1 - \alpha) \pi s^{-\alpha/2} \pi^{-1/2}.$$  

The boundedness of constant
d

$$\frac{\sin \left( \frac{(1-\alpha)\pi}{2} \right) \Gamma(\alpha) \Gamma(1 - \alpha)}{(2\pi)^{\alpha}} 2^{-\alpha/2} \pi s^{-\alpha/2} \pi^{-1/2}$$

can be concluded from Euler’s reflection formula $(\Gamma(1 - z)\Gamma(z) = \pi / \sin(\pi z))$ for $z = (1 - \alpha)/2$. $\square$

Because of the lemma above, the integral on the right hand side of (17) is finite as long as $\alpha/2 + a < 1$. Since $\alpha \in (0, 1)$, we can always take $a \in (0, 1/2)$. In [5, proof of Prep. 6.5], authors also get (17), but some extra effort is required to show that (17) holds with one constant on right hand side for $\alpha \in (\alpha_0, 1), \alpha_0 > 0$. 

16
2.3.2. Difference in the time variable

The difference in the time variable is going to be, for $\delta > 0$

\[
E \left[ |I_{x,t+\delta}(x) - I_{x,t}(x)|^k \right] \\
= \text{const} \cdot E \left[ \left| \int_0^t \int_\mathbb{R} \left( p_{t+\delta-s}(x-z) - p_{t-s}(x-z) \right) \sigma(u_{\alpha,s}(z)) \eta_\alpha(ds,dz) \right|^k \right] \\
+ \text{const} \cdot E \left[ \left| \int_t^{t+\delta} \int_\mathbb{R} p_{t+\delta-s}(x-z) \sigma(u_{\alpha,s}(z)) \eta_\alpha(ds,dz) \right|^k \right].
\]

Let us estimate the second integral, we can use the same technique as in the case of the spatial variable and write

\[
E \left[ \left| \int_t^{t+\delta} \int_\mathbb{R} p_{t+\delta-s}(x-z) \sigma(u_{\alpha,s}(z)) \eta_\alpha(ds,dz) \right|^k \right] \\
\leq \text{const} \left( \int_t^{t+\delta} \sup_x E \left[ |\sigma(u_{\alpha,s}(x))|^k \right] \right)^{2/k} \left( \int_\mathbb{R} |\xi|^{1-\alpha} \left| \hat{p}_{t+\delta-s}(\xi) \right|^2 d\xi ds \right)^{k/2} \\
\leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k) \left( \int_t^{t+\delta} (t + \delta - s)^{-\alpha/2} ds \right)^{k/2} \\
\leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k) |\delta|^{(2-\alpha)/4} \leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k) |\delta|^{k/4}.
\]

The estimate for the second integral will be

\[
E \left[ \left| \int_t^t \int_\mathbb{R} \left( p_{t+\delta-s}(x-z) - p_{t-s}(x-z) \right) \sigma(u_{\alpha,s}(z)) \eta_\alpha(ds,dz) \right|^k \right] \\
\leq \text{const} \left( \int_0^t \sup_{x \in \mathbb{R}} E \left[ |\sigma(u_{\alpha,s}(x))|^k \right] \right)^{2/k} \left( f_\alpha * p_{t-s}(0) \right) \left( \int_\mathbb{R} \left| p_{t+\delta-s}(z) - p_{t-s}(z) \right| dz ds \right)^{k/2} \\
\leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k) \left( \int_0^t \left( f_\alpha * p_{t-s}(0) \right) \left( \int_\mathbb{R} \left| p_{t+\delta-s}(z) - p_{t-s}(z) \right| dz ds \right)^{k/2} \\
\leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k) \left( \int_0^t s^{-1/2} (\log(s + \delta) - \log(s)) ds \right)^{k/2} \\
\leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k) \left( 4\sqrt{\delta} \arctan \left( \sqrt{\frac{t}{\delta}} \right) + 2\sqrt{t} \log(1 + \delta/t) \right)^{k/2}, \quad (18)
\]
by using a similar technique as in the case for the spatial variable and Lemma 3. The first step in (18) uses that \((f_\alpha * |p_{t+\delta-s} - p_{t-s}|)(z) \leq f_\alpha * p_{t+\delta-s}(0) + f_\alpha * p_{t-s}(0) \leq 2f_\alpha * p_{t-s}(0)\). The last step in (18) can be verified by differentiating function \(4\sqrt{\delta} \tan(\sqrt{t/\delta}) + 2\sqrt{t} \log(1 + \delta/t)\). The inequality \(\log(1 + \zeta) < \sqrt{\zeta}\) for all \(\zeta > 0\) gives us

\[
E \left[ \left| \int_0^t \int_\mathbb{R} (p_{t+\delta-s}(x-z) - p_{t-s}(x-z)) \sigma(u_{\alpha,s}(z)) \eta(ds,dz) \right|^k \right] \leq \text{const} \cdot (1 + \mathcal{N}_{\gamma,k}(u_\alpha))^k \delta^{k/4}. \tag{19}
\]

We can combine both estimates (18) and (19) to finally get

\[
E \left[ |I_{\alpha,t+\delta}(x) - I_{\alpha,t}(x)|^k \right] = \text{const} \cdot (1 + \mathcal{N}_{\gamma,k}(u_\alpha))^k \delta^{k/4}. \tag{20}
\]

Estimates similar to 2.3.2 and 2.3.1 can be found in numerous places in the literature, for example [16]. Authors in [16] use a different technique and investigate noise with more general covariance structure. We do not know of continuity estimates which take into account \(\alpha\) as a variable, thus our estimates in subsections 2.3.2 and 2.3.1 are novel in that sense.

### 2.4. Kolmogorov’s continuity theorem and tightness

Let us mention that \(\mathcal{N}_{\gamma}(u_\alpha)\) is uniformly bounded in \(\gamma, \gamma > 0\) and \(\alpha \in (\alpha_0, 1)\) where \(\alpha_0 > 0\). This follows from Corollary 1 and it is important for bounds on differences in sections 2.3.2 and 2.3.1. We have that for every \(1 > \alpha > \alpha_0 > 0\) and \((s,x), (t,y)\) from \(D := [0,T] \times [-N,N] \subset \mathbb{R}_0^+ \times \mathbb{R}\) the following holds for \(k \geq 2\)

\[
E \left[ |u_{\alpha,s}(x) - u_{\alpha,t}(y)|^k \right] \leq \text{const} \cdot |x-y|^{ka} + \text{const} \cdot |t-s|^{kb},
\]

where \(a \in (0, \frac{1}{2} \land g)\) and \(b \in (0, \frac{1}{2} \land g)\), thanks to our estimates from sections 2.3.1,2.3.2 and Lemma 4. Denote \(\rho(t,x) = |x|^a + |t|^b\), then Kolmogorov’s continuity theorem states that there is a modification of \(u_{\alpha,s}(x)\) such that (see for example [12, pg. 107])

\[
E \left[ \sup_{(s,x),(t,y) \in D} \frac{|u_{\alpha,s}(x) - u_{\alpha,t}(y)|^k}{\rho(s-t, x-y)^q} \right] < \Lambda < +\infty \tag{21}
\]

for every \(\alpha \in (\alpha_0, 1)\) and \(q \in (0, 1 - H/k)\) where \(H = 1/a + 1/b\). By Markov’s inequality and (21), we can write
\[
\begin{align*}
\mathbb{P} \left\{ \sup_{(s,x),(t,y) \in D} |u_{\alpha,s}(x) - u_{\alpha,t}(y)| > \epsilon \right\} < \frac{\Lambda}{k^{q_k}},
\end{align*}
\]

which implies

\[
\lim_{\delta \to 0} \sup_{\alpha \in (\alpha_0,1)} \mathbb{P} \left\{ \sup_{(s,x),(t,y) \in D} |u_{\alpha,s}(x) - u_{\alpha,t}(y)| > \epsilon \right\} = 0, \tag{22}
\]

for every \( \epsilon > 0 \). We established both convergence of finite dimensional distributions [17, Thm. 2 (i)] and tightness (22) for measures \( P_\alpha \) on \( C \) [17, Thm. 2 (ii)]. The tightness of \( P_\alpha \) can be seen by adapting [4, Thm. 7.3] to a setting of two dimensional continuous functions with compact domain and supremum norm. We can conclude [17, Thm. 2] that the measure \( P_\alpha \) corresponding to \( u_\alpha \) restricted to \( D \) converges weakly as \( \alpha \uparrow 1 \) to a measure \( P_1 \) corresponding to \( u \) restricted to \( D \). We can also conclude weak convergence of \( P_\alpha \) to \( P_1 \) as \( \alpha \uparrow 1 \) by adapting [4, Chapter 2] to the two dimensional setting.

The second part of the main theorem, that is weak convergence of \( P_\alpha \) to \( P_{\alpha_0} \) as \( \alpha \to \alpha_0 \in (0,1) \) also follows from the current section. We have tightness for this claim, but we are missing convergence of finite dimensional distributions. Little modification of section 2.2 would give us that.

Acknowledgement

This article was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics at the University of Utah under the supervision of Professor Davar Khoshnevisan. This research was supported in part by the NSFs grant DMS-1307470.

References

[1] R. M. Balan and D. Conus. Intermittency for the wave and heat equations with fractional noise in time. To be published in the Annals of Probability.

[2] Raluca M. Balan and Ciprian A. Tudor. Stochastic heat equation with multiplicative fractional-colored noise. Journal of Theoretical Probability, 23(3):834–870, 2010.
[3] Lorenzo Bertini and Nicoletta Cancrini. The stochastic heat equation: Feynman-kac formula and intermittence. *Journal of Statistical Physics*, 78(5-6):1377–1401, 1995.

[4] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons, 1999.

[5] Daniel Conus, Mathew Joseph, Davar Khoshnevisan, and Shang-Yuan Shiu. On the chaotic character of the stochastic heat equation, ii. *Probability Theory and Related Fields*, 156(3-4):483–533, 2013.

[6] R.C. Dalang, D. Khoshnevisan, and F. Rassoul-Agha. A *Minicourse on Stochastic Partial Differential Equations*. Number no. 1962 in A Minicourse on Stochastic Partial Differential Equations. Springer, 2009.

[7] Robert Dalang. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.’s. *Electron. J. Probab.*, 4:no. 6, 1–29, 1999.

[8] Mohammud Foondun and Davar Khoshnevisan. On the stochastic heat equation with spatially-colored random forcing. *Transactions of the American Mathematical Society*, 365(1):409–458, 2013.

[9] I.M. Gelfand and G.E. Chilov. *Generalized Functions: Properties and operations*. Generalized functions. Academic Press, 1964.

[10] Izrail M Gelfand and N Yao Vilenkin. Generalized functions: Vol.: 4. : Applications of harmonic analysis. 1964.

[11] I.S. Gradshteyn and I.M. Ryzhik. *Table of integrals, series, and products*. Academic Press, 2007.

[12] Davar Khoshnevisan. *Analysis of Stochastic Partial Differential Equations*, volume 119. American Mathematical Soc., 2014.

[13] Carl Mueller and Roger Tribe. A singular parabolic anderson model. *Electron. J. Probab.*, 9:no. 5, 98–144, 2004.

[14] Thomas Rippl and Anja Sturm. New results on pathwise uniqueness for the heat equation with colored noise. *Electron. J. Probab.*, 18:no. 77, 1–46, 2013.

[15] Marta Sanz-Sole and Mònica Sarrà. Path properties of a class of gaussian processes with applications to spdes. *Canadian Mathematical Society, Conference Proceedings*, 28:303–316, 2000.
[16] Marta Sanz-Solé and Mònica Sarrà. Hölder continuity for the stochastic heat equation with spatially correlated noise. *Seminar on Stochastic Analysis, Random Fields and Applications III*, pages 259–268, 2002.

[17] Michael J. Wichura. Inequalities with applications to the weak convergence of random processes with multi-dimensional time parameters. *The Annals of Mathematical Statistics*, 40(2):681–687, 04 1969.