CALABI-YAU QUOTIENTS WITH TERMINAL SINGULARITIES

FILIPPO F. FAVALE

ABSTRACT. In this paper we are interested in quotients of Calabi-Yau threefolds with isolated singularities. In particular, we analyze the case when $X/G$ has terminal singularities. A complete classification for the case of a cyclic group $G$ of prime order is given. We also investigate some necessary conditions for the order of $G$ when a quotient $X/G$ has terminal singularities.

CONTENTS

1. Introduction 1
2. General facts 2
2.1. Quotients with terminal singularities 4
2.2. Small involutions 7
2.3. Automorphism of order three with isolated fixed points 8
2.4. Automorphism of order five with isolated fixed points 8
3. Some examples 10
3.1. Quotient with terminal singularities 10
3.2. A quotient with non-isolated Gorenstein singularities 12
References 13

1. INTRODUCTION

A Calabi-Yau variety is a projective variety $X$ that has trivial canonical bundle and no non-zero holomorphic $p$–forms for $1 \leq p \leq \dim(X) - 1$. Even for threefolds, the ones concerning us in this paper, very little is known about many geometric aspects. For example, a topological classification is very far to be understood and all possible sets of Betti numbers of a Calabi-Yau threefold are not known. Here we are interested in automorphisms of a Calabi-Yau variety. Most of the interesting results in this area are for particular families of Calabi-Yau varieties. For example, Wilson has proved that a Calabi-Yau manifold whose second Chern class is positive on the Kähler cone has a finite automorphism group (see [Wil11]). In [Ogu13], it is shown that the same is true if $X$ is a Calabi-Yau threefold with Picard number 2. In this paper, we would like to say something about groups $G$ of automorphism of Calabi-Yau threefold that give quotients with terminal singularities.

The main tool we will use is the holomorphic Lefschetz fixed point formula. If $g$ is an automorphism of a complex threefold and if the fixed points of $g$ are isolated, this
formula, in its basic form, gives a relation between the traces of \( g^* \) restricted on \( H^{0,k}(X) \) and some contributions that depend only on the fixed points and the local actions in a neighbourhood of the fixed points of \( g \). First of all, we will use the Lefschetz formula to find conditions that have to be satisfied by prime-order automorphisms with isolated fixed points. In Theorem 2.3 we will see that, if we allow only terminal singularities for the quotient \( X/\mathbb{Z}_p \), then \( p \) can be equal to 2, 3 or 5. Also, we are able to tell precisely the number of the fixed points for each case.

We will then focus on small automorphisms of order 2, 3 or 5. When \( g \) has order 2, a description of the quotient \( X/\mathbb{Z}_2 \) for all the possible values of \( \dim(\text{Fix}(g)) \) is given. It is shown that \( \dim(\text{Fix}(g)) \) has always pure dimension and that the quotient with terminal singularities are also the ones with isolated singularities. When the order of \( g \) is 3, a discussion on the number of fixed points for \( \text{Fix}(g) \) of dimension 0 is given. Basically, one can only have 9 fixed points if \( X/\mathbb{Z}_3 \) has terminal singularities or an even number of fixed points otherwise, i.e. when \( X/\mathbb{Z}_3 \) has Gorenstein singularities and it is a singular Calabi-Yau threefold. If \( g \) has order 5, the number of fixed points for the case \( \dim(\text{Fix}(g)) = 0 \) is studied when \( g \) is not symplectic, i.e., \( g^*|_{H^{0,3}(X)} \neq \text{Id} \). It is shown that the minimal number of fixed points is achieved only if \( X/\mathbb{Z}_5 \) is terminal.

Finally, we will present some examples of automorphisms giving cyclic quotients with terminal singularities to show that each \( p \in \{2, 3, 5\} \) can, in fact, occur. An example of a terminal quotient \( X/(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \) is given. Two quotient \( X/G \) with Gorenstein non-isolated singularities (one with \( G \) of order 32 and the other with respect to \( \mathbb{Z}_2 \)) are also investigated.

2. General facts

Let \( X \) be a smooth Calabi-Yau threefold and let \( g \) be an automorphism of \( X \) such that \( o(g) = p \) is prime. Denote by \( G \) the group generated by \( g \) and assume that \( \dim(\text{Fix}(g)) = 0 \), i.e., \( g \) has only isolated fixed points. Under this assumption, we can apply the holomorphic Lefschetz fixed point formula to \( g \), namely:

\[
\sum_{P \in \text{Fix}g} \frac{1}{\det(\text{Id} - d_Pg)} = \sum_{k=0}^{3} (-1)^k \text{Tr} \left( g^*|_{H^{0,k}(X)} \right),
\]

where \( d_Pg \) is the differential map induced by \( g \) on \( T_PX \). We will investigate the restriction given by this identity under some assumptions on the types of singularities of \( X/G \).

First of all, let’s try to understand the contribution given by one point to the right hand side of (1). In a neighbourhood of a fixed point \( P \), the action of \( g \) on \( X \) can be described in terms of a linearization of \( g \), i.e., the action of \( d_Pg : T_PX \to T_{g(P)}X = T_PX \). The automorphism \( g \) has finite order, so we can diagonalize \( d_Pg \), thus obtaining

\[
d_Pg \leftrightarrow \begin{bmatrix}
\omega^{a_1(P)} & 0 & 0 \\
0 & \omega^{a_2(P)} & 0 \\
0 & 0 & \omega^{a_3(P)}
\end{bmatrix},
\]
where \( \omega = e^{2\pi i/p} \) and \( 0 \leq a_i(P) \leq p-1 \) are the exponents determined up to permutation. The local equations for the fixed locus are \( z_i(1 - \omega^{a_i(P)}) = 0 \); if the fixed point is isolated \( a_i(P) > 0 \). Call \( s(P) = a_1(P) + a_2(P) + a_3(P) \).

Recall that a Calabi-Yau threefold has \( h^{1,1}(X) = h^{0,2}(X) = 0 \) and trivial canonical bundle. Hence there exists an everywhere non vanishing holomorphic 3–form which will be denoted by \( \eta \). Moreover, \( H^{0,3}(X) = \eta \cdot \mathbb{C} \), so the action of an automorphism \( g \) on \( H^{0,3}(X) \) is simply the multiplication by an element of \( \mathbb{C}^* \) that is a root of unity of order \( o(g) \). Therefore, the right-hand side of (1) is simply \( 1 - \text{Tr} (g^*|_{H^{0,3}(X)}) = 1 - \omega^r \) for some \( r \). We will call an automorphism of a Calabi-Yau threefold such that \( g^*|_{H^{0,3}(X)} = \text{Id} \) a symplectic automorphism. The set \( S(X) \) of such automorphisms is easily proven to be a normal subgroup of \( \text{Aut}(X) \).

As the following lemma shows, if we know the local action around a fixed point we can obtain information about the action of \( g \) on \( H^{0,3}(X) \).

**Lemma 2.1.** If \( X \) is a Calabi-Yau threefold and \( g \in \text{Aut}(X) \), then the following holds:

1. \( \text{Fix}(g) \) is empty and \( g \in S(X) \);
2. \( P \in \text{Fix}(g) \) and there exists an action of \( g \) on the stalk \( \Omega^3_{X,P} \); the action of \( g \) is the multiplication by \( \det d_P g = \omega^{s(P)} \).
3. One has \( \text{Tr} (g^*|_{H^{0,3}(X)}) = \omega^{-s(P)} \) for every \( P \in \text{Fix}(g) \).

**Proof.** If \( \text{Fix}(g) \) is empty, the holomorphic Lefschetz fixed point formula gives the equation \( 0 = 1 - \omega^r \), so \( r = 0 \), and this is equivalent to asking that \( g \in S(X) \). Now, if there is a fixed point \( P \), let’s prove that the action of \( g \) on the stalk of the canonical sheaf is given by \( \det(d_P g) \). If \( \rho_P(g^*) \) is the map given by \( g \) on the stalk over \( P \) of the sheaf \( \Omega^3_X \), we have the relation \( \wedge^3(d_P g^*) = \rho_P(g^*) \). But \( d_P g \) is a linear automorphism of \( T_P X \) whose dimension is 3. This implies that \( \wedge^3(d_P g) \) is the multiplication by \( \det(d_P g) \), thus proving the claim. The last part is an easy consequence of the second. \( \square \)

Let \( X \) be a complex threefold and consider a fixed point \( P \) of \( g \in \text{Aut}(X) \). The **age** of \( P \) with respect to the primitive root of order \( o(g) \) \( \lambda \) is

\[
\text{age}(P, \lambda) := (a_1 + a_2 + a_3)/o(g),
\]

where \( \lambda^{a_i} \) are the eigenvalues of \( d_P g \) and \( 0 \leq a_i \leq o(g) - 1 \). Recall that if \( V \) is a vector space and \( f : V \to V \) is linear, \( f \) is a quasi-reflection if \( \text{Rk}(f - \text{Id}) = 1 \). A group \( G \) acting on a complex manifold is said to be **small** if for every \( g \in G \) and every \( P \in \text{Fix}(g) \) one has that \( d_P g : T_P X \to T_P X \) is not a quasi-reflection. This condition is equivalent to asking that \( \text{Fix}(G) \) has codimension at least 2. The following theorem recalls some well known facts about some types of singularities (see, for example, [MGS4]).

**Theorem 2.2.** Let \( X \) be a complex threefold and consider a small group \( G \) that acts on \( X \). Call \( \pi : X \to X/G \) the projection on the quotient and \( G_P = \text{Stab}_G\{P\} \) the isotropy of \( P \). Then \( \text{Sing}(X/G) = \text{Fix}(G)/G \) and

- \( \pi(P) \) is a Gorenstein singularity if and only if \( d_P g \leq \text{SL}(T_P X) \) for each \( g \in G_P \);
- \( \pi(P) \) is a canonical singularity if and only if \( \text{age}(g, \lambda) \geq 1 \) for each primitive \( \lambda \) and for each \( g \in G_P \);
• \( \pi(P) \) is a terminal singularity if and only if \( \text{age}(g, \lambda) > 1 \) for each primitive \( \lambda \) and for each \( g \in G_P \);
• \( \pi(P) \) is a terminal singularity if and only if \( \det(d_pg) \) is an eigenvalue of \( d_pg \) for each \( g \in G_P \).

2.1. Quotients with terminal singularities. From now on, \( X \) is a Calabi-Yau threefold. Here we are interested in groups which give a quotient with terminal singularities. We are interested in the cyclic groups of prime order. The following calculation follows the one in [Sob00] for a Fano threefold.

Assume that \( g \) is an automorphism of a Calabi-Yau threefold with a finite number of fixed point and with prime order \( p \). If \( P \in \text{Fix}(g) \), define \( S_n(P) \) to be

\[
\sum_{0 \leq k_1, k_2, k_3 \leq p-1 \atop a_1(P)k_1 + a_2(P)k_2 + a_3(P)k_3 \equiv n \mod p} k_1k_2k_3,
\]

where \( \omega^{a_i}(P) \) are the eigenvalues of \( d_pg \) and \( \omega \) is a \( p \)-th primitive root of unity.

**Theorem 2.3.** Let \( X \) be a Calabi-Yau threefold and \( g \) an automorphism of prime order \( p \) with only isolated fixed point. Let \( 0 \leq r \leq p - 1 \) such that \( \text{Tr} (g^* |_{H^{0,3}(X)}) = \omega^r \). Then the Lefschetz formula implies

\[
\sum_{x \in \text{Fix}(g)} \left( \frac{p^3(p-1)^3}{8} - pS_0(x) \right) = \begin{cases} p^4 & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}
\]

**Proof.** Starting from the Lefschetz fixed point formula, we have

\[
\sum_{P \in \text{Fix}(G)} \frac{1}{\det(I - d_pg)} =: \Lambda(X, g) = 1 - \omega^r
\]

If, as before, we call \( \omega \) a root of unity and we let \( \omega^{a_i}(P) \) be the eigenvalues of \( d_pg \) one has

\[
\sum_{P \in \text{Fix}(G)} \frac{1}{\det(I - d_pg)} = \sum_{P \in \text{Fix}(G)} \frac{1}{(1 - \omega^{a_1}(P))(1 - \omega^{a_2}(P))(1 - \omega^{a_3}(P))}.
\]

If \( \lambda \) is a primitive root of unity, the following relation holds

\[
\frac{1}{1 - \lambda} = -\frac{1}{p} \sum_{k=1} \lambda^k.
\]

Every \( \omega^{a_i}(P) \) is a primitive root. Indeed, the fixed locus has dimension 0, hence \( a_i \neq 0 \). We can then write

\[
\sum_{P \in \text{Fix}(G)} \frac{1}{\det(I - d_pg)} = -\frac{1}{p^3} \sum_{P \in \text{Fix}(G)} \prod_{i=1}^3 \left( \sum_{k_i} k_i \omega^{k_1a_1(P)} \right) =
\]

\[
= -\frac{1}{p^3} \sum_{P \in \text{Fix}(G)} \sum_{0 \leq k_1, k_2, k_3 \leq p-1} k_1k_2k_3\omega^{k_1a_1(P)+k_2a_2(P)+k_3a_3(P)}.
\]
The coefficient of $\omega^n$ (seeing $\mathbb{Q}(\omega)$ as a $\mathbb{Q}$ vector space with the standard basis) given by $P$ is precisely $-\frac{1}{p^3}S_n(P)$; so we can rewrite the left hand side of the holomorphic Lefschetz formula in the following way:

$$-\frac{1}{p^3} \sum_{P \in \text{Fix}(G)} \sum_{n=0}^{p-1} \left( \sum_{0 \leq k_1, k_2, k_3 \leq p-1 \atop \alpha_1(P)k_1 + \alpha_2(P)k_2 + \alpha_3(P)k_3 \equiv n} k_1k_2k_3 \right) \omega^n =$$

$$= -\frac{1}{p^3} \sum_{P \in \text{Fix}(G)} \sum_{n=0}^{p-1} S_n(x)\omega^n.$$

We end up with the following formula:

$$\Lambda(X, g) = \sum_{x \in \text{Fix}(G)} \frac{1}{\det(I - d_x g)} = -\frac{1}{p^3} \sum_{P \in \text{Fix}(G)} \sum_{n=0}^{p-1} S_n(P)\omega^n.$$  

From equations (2) and (3) we obtain

$$\sum_{P \in \text{Fix}(G)} \sum_{n=0}^{p-1} S_n(P)\omega^n = p^3\omega^r - p^3$$

$$\sum_{n=0}^{p-1} \left( \sum_{P \in \text{Fix}(G)} S_n(P) \right) \omega^n + p^3 - p^3\omega^r = 0.$$

It is useful to separate the case $r = 0$ and $r \neq 0$. Call $B_n$ the coefficient of $\omega^n$ in the left hand side of the last equation, namely

$$r \neq 0$$

$$B_n := \sum_{P \in \text{Fix}(G)} S_n(P) + \begin{cases} \hfill 0 \hfill & \text{se } n \neq 0, r \\ \hfill p^3 \hfill & \text{se } n = 0 \\ \hfill -p^3 \hfill & \text{se } n = r \end{cases} \quad B_n := \sum_{P \in \text{Fix}(G)} S_n(P)$$

This relation

$$\sum_{n=0}^{p-1} B_n\omega^n = 0$$

is true if and only if the coefficients $B_n$ are all equal. From $B_0 = B_1 = \cdots = B_{p-1}$ one has $B_n - B_0 = 0$ for all $1 \leq n \leq p - 1$. Hence

$$\sum_{n=1}^{p-1} (B_n - B_0) = 0 \iff \sum_{n=1}^{p-1} (B_n) - (p - 1)B_0 = 0 \iff \sum_{n=0}^{p-1} (B_n) - pB_0 = 0$$

If we solve for $B_n$, we have

$$r \neq 0$$

$$\sum_{P} \left( (\sum_{n=0}^{p-1} S_n(P)) - pS_0(P) \right) = p^4 \quad r = 0$$

$$\sum_{P} \left( (\sum_{n=0}^{p-1} S_n(P)) - pS_0(P) \right) = 0.$$
The quantity $S_n(x)$ is the sum of the product $k_1k_2k_3$ for which $a_1(P)k_1 + a_2(P)k_2 + a_3(P)k_3 \equiv n$; hence the sum of $S_n(x)$ for $0 \leq n \leq p - 1$ is simply the sum of $k_1k_2k_3$ for $0 \leq k_i \leq n - 1$. This is equivalent to the third power of the sum of the first $p - 1$ integers. In the end, this can be written as

$$\sum_{P} \left( \sum_{n=0}^{p-1} S_n(P) - pS_0(P) \right) = \sum_{P} \left( \sum_{0 \leq k_i \leq p-1} k_1k_2k_3 - pS_0(P) \right) = \sum_{P} \left( \frac{p^3(p-1)^3}{8} - pS_0(P) \right).$$

This is the statement of the Theorem. □

**Lemma 2.4.** Let $p$ be a prime and let $0 \leq a, b, c \leq p - 1$ such that

$$((a + b + c) \mod p) \in \{a, b, c\}.$$

Then

$$\sum_{0 \leq k_1, k_2, k_3 \leq p-1 \atop ak_1 + bk_2 + ck_3 \equiv 0} k_1k_2k_3 = \frac{p}{2} \left[ \frac{p^2(p-1)^2}{4} - \frac{p(p-1)(2p-1)}{6} \right].$$

**Proof.** See [Sob00]. □

By Theorem (2.2), a fixed point $P \in X$ of $g$ satisfies $((a_1(P) + a_2(P) + a_3(P)) \mod p) \in \{a_i(P)\}$ if and only if $P$, on $X/ \langle g \rangle$, gives a terminal singularity. Hence Lemma (2.4) tells us that $S_0(P)$ doesn’t depend on $P$ if the singularity is terminal but only on the order of $g$.

**Theorem 2.5.** Assume that $X$ is a Calabi-Yau threefold and $g$ is an automorphism of prime order $p$ with at most isolated fixed points. Call $G := \langle g \rangle$ and $q$ the number of fixed point of $g$. If $X/G$ has at most terminal singularities, then one of the following holds:

- $G$ acts freely on $X$ ($q = 0$) and the action is symplectic;
- $G$ has fixed points and the action of $G$ is not symplectic; Moreover we have $p \in \{2, 3, 5\}$.

If the second case occurs, $g$ has 16, 9 or 5 fixed point if $p = 2, 3$ or 5, respectively.

**Proof.** The quotient $X/G$ has at most terminal singularities, so $G$ is small because its fixed locus is either empty or it has dimension 0. If $q = 0$ then, by Lemma (2.1), $G$ is symplectic and, by definition it acts freely on $X$. Assume that there are fixed points. Using Lemma (2.4) one has

$$\sum_{x \in \text{Fix}(g)} \left( \frac{p^3(p-1)^3}{8} - pS_0(x) \right) = \sum_{x \in \text{Fix}(g)} \left( \frac{p^3(p-1)^3}{8} - \frac{p}{2} \left[ \frac{p^2(p-1)^2}{4} - \frac{p(p-1)(2p-1)}{6} \right] \right) =$$
\[ = q \left( \frac{p^3(p-1)^3}{8} - p^2 \left[ \frac{p^2(p-1)^2}{4} - \frac{p(p-1)(2p-1)}{6} \right] \right) = q \frac{p^3(p^2-1)}{24} \]

which is equal to
\[
\begin{cases} 
p^4 & \text{if } r \neq 0 \\
0 & \text{if } r = 0,
\end{cases}
\]

by Theorem (2.3). In the end
\[
q \frac{(p^2-1)}{24} = \begin{cases} 
p & \text{if } r \neq 0 \\
0 & \text{if } r = 0
\end{cases}
\]

\(G\) is a subgroup of \(S(X)\) if and only if \(r = 0\) but this implies \(q = 0\). So, if there are fixed points, the action of \(G\) is not symplectic. We can write \(q\) as depending on \(p\) in the following way \(q = 24p/(p^2 - 1)\). The only values of \(p\) for which \(q\) is a positive integer are 2, 3 and 5, for which \(q\) is 16, 9 and 5, respectively. \(\square\)

As an easy consequence of the last theorem we have the following

**Corollary 2.6.** Assume that \(X\) is a Calabi-Yau threefold and that \(G \leq \text{Aut}(X)\) is a small group such that \(X/G\) has at most terminal singularities. Then \(|G| = 2^a3^b5^c\).

In section (3) we will give some examples of Calabi-Yau threefolds with quotients having terminal singularities. Now, we will analyse the cases for which \(p \in \{2, 3, 5\}\).

### 2.2. Small involutions

Here we are interested in small involutions. For this case, we will not restrict to the case of fixed locus of dimension 0.

**Proposition 2.7.** Let \(X\) be a Calabi-Yau threefold and let \(g\) be a small involution with fixed points. The following are equivalent:

1. \(g\) is symplectic;
2. \(\text{Fix}(g)\) contains a curve;
3. \(\text{Fix}(g)\) is smooth of pure dimension 1.

**Proof.** If \(g \in S(X)\), then there exist local coordinates around a fixed points \(P\) such that \(d_P g\) acts as
\[
(z_1, z_2, z_3) \mapsto (z_1, -z_2, -z_3).
\]
The fixed locus has local equation \(z_2 = z_3 = 0\), and so is smooth at \(P\) and the component containing \(P\) has dimension 1. To complete the proof, one can use Lemma 2.1 and see that the same description is true near all the fixed points of \(g\). \(\square\)

Quotients of the form \(X/G\), where \(G = \langle g \rangle\) with \(g\) a small involution, are of three types and they are described by their fixed locus.

**Proposition 2.8.** Let \(g\) be a small involution on a Calabi-Yau threefold \(X\). Call \(G\) the cyclic group generated by \(g\). Then one of the following holds:

- \(\text{Fix}(g)\) is empty, \(g\) is symplectic and \(X/G\) is a smooth Calabi-Yau threefold.
- \(\dim(\text{Fix}(g)) = 0\), \(g\) is not symplectic and \(X/G\) has precisely 16 singular points that are all terminal.
\[ \text{Dim}(\text{Fix}(g)) = 1, g \text{ is symplectic, } X/G \text{ is a singular Calabi-Yau threefold whose singular locus has pure dimension 1.} \]

**Proof.** If \( g \) has empty fixed locus, the action of \( g \) is free, proving the first part. If the fixed locus has dimension 0, the eigenvalues of \( d_P g \) are all equal to \(-1\) for each \( P \in \text{Fix}(g) \). This implies that \( g \) isn’t symplectic. Moreover, because \((1 + 1 + 1) \equiv 2 \mod 1\), all the fixed points of \( g \) gives terminal singularities on the quotients. By Theorem (2.5) the fixed points are 16 and each of them corresponds to a singular point. Finally, if \( \text{Fix}(g) \) has dimension 1, \( X/G \) is a normal projective threefold with canonical and Gorenstein singularities, i.e., it is a singular Calabi-Yau threefold\(^1\). \( \square \)

2.3. **Automorphism of order three with isolated fixed points.** Assume that \( g \) is an automorphism of order 3 with isolated fixed points on a Calabi-Yau threefold. Call \( \lambda \) a primitive root of unity of order 3. By Lemma (2.1), for each fixed point \( P \), \( d_P g \) has the same determinant. There are three possible case, namely \( \det(d_P g) = 1, \lambda \) and \( \lambda^2 \).

If \( \det(d_P g) = 1 \), i.e., if \( g \) is symplectic, then \((\lambda^a(P), \lambda^b(P), \lambda^c(P)) \in \{(\lambda, \lambda, \lambda), (\lambda^2, \lambda^2, \lambda^2)\}\). Denote by \( x_1 \) the contribution \( C(P) \) of a point such that \((\lambda^a(P), \lambda^b(P), \lambda^c(P)) = (\lambda, \lambda, \lambda) \) and \( x_2 \) the contribution of a point of the other type. Call \( n_1 \) and \( n_2 \) the number of such points. Is easy to see that

\[
 x_1 = \frac{1}{(1 - \lambda)^3} = \pm \frac{i\sqrt{3}}{9}
\]

and that \( x_1 = x_2 \). The Lefschetz fixed point formula is then \( 0 = n_1 x_1 + n_2 x_1 \). Being \( x_1 \) pure imaginary, one has \( n_1 = n_2 \). In particular, \( |\text{Fix}(g)| \) is even.

If \( \det(d_P g) = \lambda \) then, for each \( P \in \text{Fix}(g) \), \((\lambda^a(P), \lambda^b(P), \lambda^c(P)) = (\lambda, \lambda, \lambda^2) \) up to permutations. This implies that every point will give a terminal point on the quotient. So, by Theorem 2.5 the fixed points are 9. The case for which \( \det(d_P g) \) is similar.

We have proved the following Proposition:

**Proposition 2.9.** Let \( g \) be an automorphism of order 3 on a Calabi-Yau threefold \( X \). Call \( G \) the cyclic group generated by \( g \) and assume that it has a finite number of fixed points. Then one of the following holds:

- \( g \) is symplectic and \( X/G \) is a singular Calabi-Yau threefold with an even number of singular points.
- \( g \) isn’t symplectic, \( |\text{Fix}(g)| = 9 \) and \( X/G \) has exactly 9 singular points. All of them are terminal.

2.4. **Automorphism of order five with isolated fixed points.** Consider now an automorphism \( g \) of order 5 such that \( \text{Fix}(g) \) has dimension 0. We have seen that if \( g \) is such that \( X/\langle g \rangle \) has terminal singularities, then \( |\text{Fix}(G)| = 5 \). Now we will show that if \( g \) is not symplectic (and it has isolated fixed points) then it has at least 5 fixed points and the minimum is achieved if and only if \( X/\langle g \rangle \) has terminal singularities.

\(^1\)They are interesting because a crepant resolution always exists and it is a smooth Calabi-Yau.
Recall that, given an isolated fixed point $P$, we have defined

$$S_n(P) = \sum_{0 \leq k_1, k_2, k_3 \leq p-1} \sum_{a_1(P)k_1 + a_2(P)k_2 + a_3(P)k_3 \equiv p \mod n} k_1k_2k_3$$

and that, if $g \not\in S(X)$, we have

$$\sum_{x \in \text{Fix}(g)} \left( \frac{p^3(p-1)^3}{8} - pS_0(x) \right) = p^4$$

by Theorem (2.3). If we define

$$A := \{(4, 1, 1), (3, 2, 1), (4, 2, 1), (3, 2, 2), (4, 3, 1), (3, 3, 2), (4, 4, 1), (4, 3, 2)\}$$

and

$$B := \{(2, 2, 2), (4, 4, 3), (3, 3, 1), (4, 4, 1), (1, 1, 1), (4, 2, 2), (2, 1, 1), (3, 3, 3)\},$$

these sets correspond to all the possible values for $(a(P), b(P), c(P))$ for the case $g \not\in S(X)$. $A$ is precisely the set for $(a(P) + b(P) + c(P) \mod 5) \in \{a(P), b(P), c(P)\}$, i.e. the set corresponding to $P$ that gives terminal singularities on $X/ < g >$. By direct inspection, we see that $S_0(P)$ is equal to 175 if and only if $(a(P), b(P), c(P)) \in A$. The values that $S_0(P)$ can assume in the other case are 200 and 225. Call $n, q_1, q_2$ the number of points for which $S_0(P)$ is equal respectively to 175, 200 and 225. In particular, $X/ < g >$ has $n + q_1 + q_2$ singular points and exactly $n$ are terminal. The relation in Theorem (2.3) is then

$$p^4 = \sum_{x \in \text{Fix}(g)} \left( \frac{p^3(p-1)^3}{8} - pS_0(x) \right) =$$

$$= n \left( \frac{p^3(p-1)^3}{8} - 175p \right) + q_1 \left( \frac{p^3(p-1)^3}{8} - 200p \right) + q_2 \left( \frac{p^3(p-1)^3}{8} - 225p \right) =$$

$$= (n + q_1 + q_2) \left( \frac{p^3(p-1)^3}{8} - 175p \right) - 25pq_1 - 50pq_2 = (n + q_1 + q_2)5^3 - 5^3q_1 - 2 \cdot 5^3q_2$$

that is

$$5 = (n + q_1 + q_2) - q_1 - 2q_2 = n - q_2 \implies n = 5 + q_2.$$

This implies that the number of fixed points is $5 + q_1 + 2q_2$. In particular it is at least 5 and, moreover, it is equal to 5 if and only if $q_1 = q_2 = 0$.

**Proposition 2.10.** Let $g$ be a non symplectic automorphism of order 5 on a Calabi-Yau threefold $X$. Assume that it has a finite number of fixed points and call $G$ the cyclic group generated by $g$. Then one of the following holds:

- $|\text{Fix}(G)| = 5$ and $X/G$ has only terminal singularities.
- $|\text{Fix}(G)| > 5$, $X/G$ has 5 or more terminal singularities and at least another fixed point.
3. Some examples

3.1. Quotient with terminal singularities. Here we will construct quotients of Calabi-Yau threefolds with only terminal singularities with respect to $\mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Example 1
Let $X$ be $\mathbb{P}^2 \times \mathbb{P}^2$ with projective coordinates $x_i$ and $y_i$ on the two factors of $X$. Consider the automorphism of $X$ defined by

$$g((x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2)) := (x_0 : x_1 : \lambda x_2) \times (y_0 : \lambda y_1 : \lambda^2 y_2)$$

with $\lambda$ a primitive root of unity of order 3. It is easy to see that $g$ has order 3 and that its fixed locus has 6 irreducible components, three of which are rational curves. More precisely, if $S := \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$, then

$$\text{Fix}(g) = \{(x_0 : x_1 : 0) | (x_0 : x_1) \in \mathbb{P}^1\} \cup \{(0 : 0 : 1)\} \times S.$$ 

If $G := \langle g \rangle$, define $V$ to be the vector space $H^0(X, -K_X)^G$ of all $G$–invariants anticanonical sections. By direct computations, we see that the generic element of $V$ is smooth (because $V$ has empty base locus) and doesn’t vanish on any of isolated points of Fix($g$). The generic element gives an invariant Calabi-Yau 3fold, we can conclude that $Y/G = Y/\mathbb{Z}_3$ has exactly 9 singular points and that each of them are terminal.

Example 2
Let $X$ be $\mathbb{P}^4$ with projective coordinates $x_i$. Take $g$ to be the automorphism of $X$ defined by

$$g((x_0 : x_1 : x_2 : x_3 : x_4) := (x_0 : x_1 : \lambda x_2 : \lambda^2 x_3 : \lambda^3 x_4),$$

with $\lambda^5 = 1$ and primitive. Call $G$ the group generated by $g$ and $Y$ the Fermat hypersurface of degree five in $\mathbb{P}^4$. $Y$ is a smooth Calabi-Yau threefold and is easily seen to be invariant with respect to $G$. The fixed locus of $g$ on $X$ is

$$\text{Fix}(g) = \{(x_0 : x_1 : 0 : 0 : 0) | (x_0 : x_1) \in \mathbb{P}^1\} \cup \{(0 : 0 : 1 : 0 : 0), (0 : 0 : 0 : 1 : 0), (0 : 0 : 0 : 0 : 1)\}$$

and $Y$ doesn’t meet the isolated points of Fix($g$). The intersection of $Y$ with Fix($g$) are 5 isolated points. If one consider the quotient $Y/G$, because $G$ is small, one has that $\text{Sing}(Y/G) = \text{Fix}(g)$. In particular $Y/G = Y/\mathbb{Z}_5$ has exactly 5 isolated fixed points and each of them is terminal.

Example 3
Set $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with $(x_i : y_i)$ that are projective coordinates on the $i$–th $\mathbb{P}^1$. In [BF11] and [BFNP13] the authors study the automorphisms of Calabi-Yau manifolds embedded in $X$ that have empty fixed locus. The authors produce a classification of all the admissible pairs in $X$, i.e. the pairs $(Y, G)$ where $Y$ is a Calabi-Yau threefold and $G$ is a finite group of automorphisms of $X$ that stabilizes $Y$ and acts freely on $Y$. Here
we will show that one can easily construct examples with a different kind of fixed points.

Every \( g \in \text{Aut}(X) \) acts on the 4 factors (see, for instance, [BF11]) giving a surjective homomorphism \( \pi : \text{Aut}(X) \to S_4 \) with kernel \( \text{PGL}(2)^4 \). On the other hand the permutations of the factors give an inclusion \( S_4 \hookrightarrow \text{Aut}(X) \) splitting \( \pi \) and therefore giving a structure of semidirect product

\[
\text{Aut}(X) \cong S_4 \rtimes \text{PGL}(2)^4
\]

Concretely this gives, \( \forall g \in \text{Aut} X \), a unique decomposition \( g = (A_i) \circ \sigma \) where \( \sigma = \pi(g) \) and \( (A_i) = (A_1, A_2, A_3, A_4) \in \text{PGL}(2)^4 \). Denote with \( A \) and \( B \) the automorphisms of \( \mathbb{P}^1 \) that send \( (x_1 : y_1) \) respectively in \( (x_1 : -y_1) \) and \( (y_1 : x_1) \).

Call \( g := (\text{Id}, \text{Id}, A, A) \circ (12) \) and \( h := (A, A, \text{Id}, \text{Id}) \circ (34) \). It is easy to see that \( G := < g, h > \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). As an automorphism of \( X \), \( g \) have fixed locus composed of 4 rational curves. These are

\[
\{(P, P, Q_1, Q_2) \mid P \in \mathbb{P}^1, Q_1, Q_2 \in \{(1 : 0), (0 : 1)\}\}.
\]

Call \( C \) a component of \( \text{Fix}(g) \) and consider a generic element \( Y \in |-K_X| \). It is easy to see that \( Y \cdot C = 4 \) so generically one expect that \( Y \cap C \) has 4 fixed points. A similar result holds for \( h \). For \( gh = (A, A, A, A) \circ (12)(34) \) one can see that \( \text{Fix}(gh) \) is composed of isolated points so generically one expect that the general member of \( |-K_X| \) doesn’t meet \( \text{Fix}(gh) \). Call \( V \) the vector space \( H^0(X, -K_X)^G \) of all \( G \)-invariants anticanonical sections. By direct computations one see that \( V \) has empty base locus so the generic element is a smooth Calabi-Yau that admits an action of \( G \).

We point out that the Calabi-Yau \( Y \) constructed as zero locus of the generic section of \( V \) doesn’t contain \( \text{Fix}(g) \). Then \( \text{Fix}(g) \) meets \( Y \) at isolated points. By Theorem 2.5, \( g \) is an involution with exactly 16 fixed points and acts as \(-1\) on \( H^{0,3}(X) \). The same is true for \( h \) and \( gh \) which have to act as \( \text{Id} \) on \( H^{0,3}(X) \). Because \( gh \) is an involution and \( (gh)^*|_{H^{0,3}(X)} = \text{Id} \) one has that \( \text{Fix}(gh) \) is either empty or it has pure dimension 1. The latter cannot occur because \( gh \) has a finite number of fixed points on \( X \) so \( gh \) acts freely on \( Y \).

By direct computation the common fixed points of \( g \) and \( h \) on \( X \) are 4, namely:

\[
\{(P, P, Q, Q) \mid P, Q \in \{(1 : 0), (0 : 1)\}\}.
\]

The generic invariant section isn’t zero on any of these point so, for the generic invariant Calabi-Yau \( Y \), the fixed locus of \( g \) and the one of \( h \) are disjoint.

To conclude, for \( Y \) generic, we have given four quotients, namely: \( Z_1 = Y/ < g >, Z_2 = Y/ < h >, Z_3 = Y/ < gh > \) and \( Z_4 = Y/G = Y/(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \). \( Z_3 \) is smooth because \( gh \) acts freely on \( Y \); thus the map \( Y \to Y/ < gh > \) is an étale cover of degree 2. \( Z_1, Z_2 \) and \( Z_4 \) have only terminal singularities by construction. \( Z_1 \) and \( Z_2 \) have exactly 16 singular points. To compute the number of singular points of \( Z_4 \) we can use Burnside’s Lemma on the action of \( G \) restricted on \( \text{Fix}(G) \). In fact \( \text{Sing}(Y/G) = \text{Fix}(G)/G \) because
Another way to see this fact is to note that \( g \) generated by an involution which we will denote \( \hat{g} \).

In conclusion \( Z_4 \), like \( Z_1 \) and \( Z_2 \), has exactly 16 isolated and terminal singularities.

Another way to see this fact is to note that \( < gh > \) is normal in \( G \) with quotient generated by an involution which we will denote \( \hat{g} \). We can write the quotient \( Z_4 = Y/G \) as \( (Y/ < gh >)/(G/ < gh >) = Z_3/(G/ < gh >) \). Hence, \( Y/G \) can be seen as the quotient of a smooth Calabi-Yau threefold by the single involution \( \hat{g} \), an involution that has isolated fixed points. By our classification of fixed locus of involutions on Calabi-Yau manifolds, \( \hat{g} \) has exactly 16 fixed points. Since we have

\[ \text{Sing}(Y/G) = \text{Fix}(\hat{g})/ < \hat{g} > = \text{Fix}(\hat{k}), \]

we obtain again that \( Z_4 \) has 16 fixed points.

### 3.2. A quotient with non-isolated Gorenstein singularities.

Here we will construct a quotient of a Calabi-Yau by a group of order 32 that is contained in \( S(X) \).

**Example 4**

Use the same notation introduced in the last example. Call

\[
g := (\text{Id}, \text{Id}, \text{Id}, A) \circ (1324), h := (B, B, B, B), k := (14)(23)\]

and consider the group \( G := < g, h, k > \). \( g, h \) and \( k \) satisfy the following relations

\[
g^8 = h^2 = k^2 = 1 \quad gh = hg \quad kh = hk \quad gk = kg^{-1}.\]

Using these relations it is easy to see that \( G \) has 32 elements and that \( G \) is isomorphic to \( D_{16} \times \mathbb{Z}_2 \) where \( D_{16} = < g, k > \) is the dihedral group with 16 elements and \( \mathbb{Z}_2 = < h > \). The elements of \( G \) can be written uniquely as \( g^a h^b k^c \) with \( 0 \leq a \leq 7 \) and \( 0 \leq b, c \leq 1 \). All the elements with \( c = 1 \) are involutions. Considering all the elements as automorphisms acting on \( X \), \( \text{Fix}(g^a h^b) \) has a finite number of elements whereas \( \text{Fix}(k) \) has pure dimension 2 and the same is true for \( \text{Fix}(g^a h^b k) \).

If we denote by

\[
V := H^0(X, K_X)^G
\]

it can be seen that the Calabi-Yau \( Y \) given by the generic \( s \in V \), i.e., the generic \( G \)--invariant Calabi-Yau is smooth and satisfies

\[
\text{Fix}(g^a h^b) \cap Y = \emptyset \quad \text{Dim(} \text{Fix}(k) \cap Y) = 1
\]

for all \( a, b \). the involution \( k \) acts on \( Y \) with a fixed locus of dimension 1. By Proposition 2.8 this implies that \( k \in S(Y) \). The two elements \( g \) and \( h \) do not have fixed points on \( Y \) so they are elements of \( S(Y) \). This is enough to conclude that \( G \leq \text{Gor}(Y) \). It can be shown that \( \text{Aut}(Y) = S(Y) \) for the generic \( G \)--invariant Calabi-Yau threefold.

By the Moishezon-Nakai criterion, being \( Y \subseteq | - K_X | \) ample, \( Y \) cannot be disjoint from \( \text{Fix}(g^a h^b k) \) that has dimension 2 on \( X \). We know that \( g^a h^b k \) is a symplectic involution so we conclude that \( \text{Fix}(g^a h^b k) \cap Y \) is smooth of pure dimension 1 for each \( a, b \).

All the irreducible components of \( \text{Fix}(G) \) are obtained from the irreducible components of \( \text{Fix}(k), \text{Fix}(gk), \text{Fix}(hk) \) and \( \text{Fix}(ghk) \) using \( \text{Fix}(b^{-1}ab) = b(\text{Fix}(a)) \) and \( x^{-1}(g^a h^b k)x = \)
$g^{a+2d}h^bk$ for $x \in G$. Using MAGMA, it is possible to check that there are at least 32 components and some of them are rational curves. Starting from these remarks we can argue that there are at least 4 orbits for the action of $G$ on $\text{Fix}(G)$.

It is interesting to note that $H := \langle g, h \rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$ is a subgroup of $G$ of index 2 and thus is a normal subgroup of $G$. The quotient $Y/G$ can be viewed as $(Y/H)/(G/H)$. It is interesting to write the quotient like this because $G/H \cong \mathbb{Z}_2$ and it is generated by an involution which we will denote $k$. Moreover $\text{Fix}(H)$ is empty (the pair $(Y, H)$ is one of the admissible pairs studied in [BFNP13]) so $Y \to Y/H$ is an étale cover of degree 16 and $Y/H$ is again a smooth Calabi-Yau threefold. Hence $Y/G$ may be viewed as the quotient of a smooth Calabi-Yau threefold by the single involution $k$. The fixed locus of $k$ satisfies $\text{Fix}(k) = \text{Fix}(k)/\langle k \rangle = \text{Sing}(Y/G) = \text{Fix}(G)/G$, so the singular locus of $Y/G$ has at least 4 irreducible components and at least one of them is a rational curve.

**References**

[BF11] G. Bini, F.F. Favale, *Groups Acting Freely on Calabi–Yau Threefolds Embedded in a Product of del Pezzo Surfaces*. Advances in Theoretical and Mathematical Physics **16** (2012), no. 3, 887–933.

[BFNP13] G. Bini, F.F. Favale, J. Neves, R. Pignatelli, *New examples of Calabi–Yau threefolds and genus zero surfaces*. [arXiv:1211.2390](http://arxiv.org/abs/1211.2390) to appear in CCM Journal (2013 issue).

[Wil11] Wilson, P. M. H., *The role of $c_2$ in Calabi-Yau classification: a preliminary survey*. Mirror symmetry, II, 381–392, AMS/IP Stud. Adv. Math., 1, Amer. Math. Soc., Providence, RI, 1997.

[Ogu13] Keiji Oguiso, *Automorphism groups of Calabi-Yau manifolds of Picard number two*. [arXiv:1206.1549](http://arxiv.org/abs/1206.1549)

[MG84] Morrison, David R. and Stevens, Glenn *Terminal quotient singularities in dimensions three and four*. Proc. Amer. Math. Soc. **90** (1984), no. 1, 15–20.

[Gar13] Garbagnati, Alice; *New families of Calabi-Yau threefolds without maximal unipotent monodromy*. Manuscripta Math. **140** (2013), no. 3–4, 273–294.

[Cam12] Camere, Chiara; *Symplectic involutions of holomorphic symplectic four-folds*. Bull. Lond. Math. Soc. **44** (2012), no. 4, 687–702.

[Sob00] Sobolev, I. V.; *The action of cyclic groups on Fano 3-folds*. Math. Notes **68** (2000), no. 5-6, 672–674.

[AS68] Atiyah, M. F., Singer, I. M.; *The index of elliptic operators. III*. Ann. of Math. (2) **87** (1968), 546–604.

(Filippo F. Favale) **DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PAVIA,**

**VIA FERRATA 1, I-27100 PAVIA, ITALY**

*E-mail address: filippo.favale@unipv.it*