Generalised dihedral subgroups of $\text{SO}(3, \mathbb{Q})$

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Abstract. Subgroups of the special orthogonal group $\text{SO}(3, \mathbb{R})$ that are generated by two rotations of finite order about axes that are also separated by an angle of finite order are sometimes called generalised dihedral groups. They appear as orientation groups in the theory of tilings of Euclidean 3-space. As our main result, we classify the generalised dihedral subgroups of $\text{SO}(3, \mathbb{Q})$ as the finite subgroups of $\text{SO}(3, \mathbb{Q})$ except the alternating group $A_4$.

Introduction

We consider those subgroups of $\text{SO}(3, \mathbb{R})$ that are generated by two rotations of finite order about rotation axes that are themselves separated by an angle of finite order, i.e. an angle that is a rational multiple of $\pi$. These groups are called generalised dihedral groups in [9]. They appear as orientation groups in the theory of tilings of Euclidean 3-space. For example, the congruent triangular prisms in the “quaquaversal tiling” constructed in [3] appear in an infinite number of orientations. There exists a generalised dihedral group such that the orientations of any two prisms in that tiling are related by an element of this group.

The generalised dihedral subgroups of $\text{SO}(3, \mathbb{R})$ are mostly free products or amalgamated free products of cyclic or dihedral groups and thus generally infinite, the only exceptions being the rotation symmetry groups of certain polyhedra. We use Cayley’s parametrisation of $\text{SO}(3, \mathbb{R})$ to represent rotations by quaternions and combine group theoretic results on amalgamated free products of groups with previous results on generalised dihedral subgroups of $\text{SO}(3, \mathbb{R})$ to obtain our main result. It classifies the generalised dihedral subgroups of $\text{SO}(3, \mathbb{Q})$ as the finite subgroups of $\text{SO}(3, \mathbb{Q})$ except the alternating group $A_4$.

$\text{SO}(3, \mathbb{Q})$ is the group of coincidence rotations of the cubic lattice $\mathbb{Z}^3$ and is thus of basic interest in crystallography for example when studying grain boundaries. The situation here is more complex than for the group $\text{SO}(2, \mathbb{Q})$ of coincidence rotations of the square lattice. This is because $\text{SO}(3, \mathbb{Q})$ is non-Abelian. Generally, the 2-dimensional case is, in contrast to the 3-dimensional one, rather well-understood [1, 6, 2]. Studying the structure of $\text{SO}(3, \mathbb{Q})$ leads to the task of determining its subgroups. The classification of finite subgroups of $\text{SO}(3, \mathbb{Q})$ is well known and this text focuses on a special class of 2-generator subgroups.

This article summarises results of [4] with an emphasis on presenting the main ideas rather than providing all the details. For those, the reader is referred to the original source.
1. Amalgamated free products of groups

Let $G_1$ and $G_2$ be groups. If $G = G_1 \star G_2$ is the free product of $G_1$ and $G_2$, then every element $g \in G$ can be uniquely written in the form

$$g = g_1 g_2 \cdots g_\ell \quad (\ell \geq 0),$$

where $e \neq g_i \in G_{\lambda_i}$ and $\lambda_i \neq \lambda_{i+1}$ for $1 \leq i \leq \ell - 1$. Here, $\ell = 0$ is understood as $g = e$, the identity element of $G$. This is called the normal form of $g$. Note that the product of two elements of $G$ is given by juxtaposition followed by the reduction to normal form; cf. [7, Ch. 6.2] for details.

In turn, the free product of groups is characterised by the existence of a normal form. Namely, if $G$ is a group generated by subgroups $G_\lambda$, $\lambda \in \Lambda$, and every element has a unique expression of the form (1), then $G$ is the free product of the $G_\lambda$’s (cf. [7, Thm. 6.2.4]).

Given the free product $G_1 \star G_2$ of two groups $G_1$ and $G_2$, there are natural monomorphisms $\iota_i: G_i \to G_1 \star G_2$ for $i \in \{1, 2\}$. Now let $H$ be a group that is isomorphic to subgroups of $G_1$ and $G_2$ via monomorphisms $\varphi_i: H \to G_i$, $(i \in \{1, 2\})$. Furthermore let $N$ be the normal closure in $G_1 \star G_2$ of the set

$$\{ \varphi_1(h) (\varphi_2(h))^{-1} \mid h \in H \}.$$

Then the factor group

$$G = (G_1 \star G_2)/N$$

is called the free product of $G_1$ and $G_2$ with amalgamated subgroup $H$ (with respect to $\varphi_1$ and $\varphi_2$). Note that $G$ depends on the choice of the monomorphisms $\varphi_1$ and $\varphi_2$. Nevertheless, we denote $G$ by $G_1 \star_H G_2$.

Since $\varphi_1(h) \equiv \varphi_2(h) \mod N$ for all $h \in H$, the subgroups $(\varphi_1(H)N)/N$ and $(\varphi_2(H)N)/N$ are identified in $G_1 \star_H G_2$.

**Theorem 1.** [8, Thm. 11.58] $G_1 \star_H G_2$ satisfies the following universal property. Let $\psi_i: G_i \to T$ $(i \in \{1, 2\})$ be homomorphisms into some group $T$ with $\psi_1|_H = \psi_2|_H$. Then there exists a unique homomorphism $\psi: G_1 \star_H G_2 \to T$ that makes the diagram below commutative.

$$
\begin{array}{ccc}
H & \xrightarrow{\varphi_1} & G_1 \\
\downarrow{\varphi_2} & & \downarrow{\tau_1} \\
G_2 & \xrightarrow{\tau_2} & G_1 \star_H G_2 \\
\downarrow{\psi_2} & & \downarrow{\psi_1} \\
T & & T
\end{array}
$$

Here, $\tau_i := \text{pr} \circ \iota_i$, where pr is the canonical projection from $G_1 \star G_2$ onto $G_1 \star_H G_2$.

In other words, $G_1 \star_H G_2$ together with $\tau_1$ and $\tau_2$ is the pushout of $\varphi_1: H \to G_1$ and $\varphi_2: H \to G_2$ in the category of groups.

**Corollary 1.** If $G'$ is any other group with group homomorphisms $\tau'_i: G_i \to G'$ $(i \in \{1, 2\})$ satisfying the universal property of Theorem 1, then $G' \simeq G_1 \star_H G_2$. Thus, the universal property characterises free products with amalgamated subgroup.

Taking a closer look at the involved homomorphisms in the diagram above, one finds that the homomorphisms $\tau_i: G_i \to G_1 \star_H G_2$ are injective. Thus, one can identify $G_i$ with its image $\tau_i(G_i)$, making $G_i$ a subgroup of $G_1 \star_H G_2$. Moreover, $G_1 \star_H G_2$ is generated by $\tau_1(G_1)$ and $\tau_2(G_2)$, and their intersection is $\tau_1(H) = \tau_2(H)$, the latter being isomorphic to $H$; cf. [8, Thm. 10.67] for details. Therefore $H$ can as well be viewed as a subgroup of $G_1 \star_H G_2$. 

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Theorem 2. [8, Thm. 11.58] If $G_1$ has a presentation $\langle Y_1 \mid R_1 \rangle$ and $G_2$ has a presentation $\langle Y_2 \mid R_2 \rangle$ such that $Y_1 \cap Y_2 = \emptyset$, then the free product of $G_1$ and $G_2$ with amalgamated subgroup $H$ has a presentation

$$\langle Y_1 \cup Y_2 \mid R_1 \cup R_2 \cup \{ \varphi_1(h)(\varphi_2(h))^{-1} \mid h \in H \} \rangle.$$ 

Example 1. Let $C_4 = \langle x \mid x^4 \rangle$ and $C_6 = \langle y \mid y^6 \rangle$ be cyclic groups of order 4 and 6 respectively. Since $x^2$ and $y^3$ both have order 2, $C_4$ and $C_6$ have a common subgroup $C_2$. More precisely, let $C_2 = \langle z \mid z^2 \rangle$. The homomorphisms $\varphi_1: C_2 \to C_4$ defined by $\varphi_1(z) = x^2$ and $\varphi_2: C_2 \to C_6$ defined by $\varphi_2(z) = y^3$ are injective and the subgroups $\varphi_1(C_2)$ of $C_4$ and $\varphi_2(C_2)$ of $C_6$ are isomorphic. By Theorem 2, the free product of $C_4$ and $C_6$ with amalgamated subgroup $C_2$ has a presentation

$$C_4 \star_{C_2} C_6 = \langle x, y \mid x^4, y^6, \{ \varphi_1(h)(\varphi_2(h))^{-1} \mid h \in H \} \rangle = \langle x, y \mid x^4, y^6, \varphi_1(z)(\varphi_2(z))^{-1} \rangle = \langle x, y \mid x^4, y^6, x^2y^{-3} \rangle = \langle x, y \mid x^4, x^2y^{-3} \rangle.$$ 

Thus $x^2$ and $y^3$ are identified in $C_4 \star_{C_2} C_6$.

Hence it is easily possible to achieve a presentation of the free product $G_1 \star_H G_2$ with amalgamated subgroup $H$ from presentations of $G_1$ and $G_2$. Given a presentation, it may be very difficult though (if not impossible) to determine the order of the group being presented; cf. [8, Ch. 12]. Without taking a presentation into account, there is a useful theorem to determine whether a free product with amalgamated subgroup has infinite order.

Theorem 3. [7, Thm. 6.4.3] If both $G_1$ and $G_2$ are different from $H$, then $G_1 \star_H G_2$ has an element of infinite order. Moreover, any element of finite order is conjugate to an element of $G_1$ or $G_2$.

Thus, the group $C_4 \star_{C_2} C_6$ of Example 1 is infinite. In the next lemma, we consider the case where one of the $G_i$’s coincides with the amalgamated subgroup $H$.

Lemma 1. One has $H \star_{H} G_2 \simeq G_2$.

Proof. This follows easily using Corollary 1. 

2. Generalised dihedral groups

We review some results of [9], where all generalised dihedral groups are classified. To be more concrete, we adopt the notation of [9] and denote by $\ell$ the line in the $x$-$y$-plane of $\mathbb{R}^3$ through the origin that makes an angle of $2\pi n/m$ with the $x$-axis (where $n, m \in \mathbb{N}$ with $\gcd(n, m) = 1$ and $m > 2$). For $p \in \mathbb{N}$ let $R_{x}^{2\pi/p}$ be the rotation by $2\pi/p$ about the $x$-axis and define $R_{x}^{2\pi/q}$ for $q \in \mathbb{N}$ accordingly, see Figure 1 for an illustration.

Define $G_{n/m}(p, q)$ to be the subgroup of $\text{SO}(3, \mathbb{R})$ generated by $R_{x}^{2\pi/p}$ and $R_{x}^{2\pi/q}$. From three simple facts on rotations by multiples of $\pi/2$ all relations between the two generators $R_{x}^{2\pi/p}$ and $R_{x}^{2\pi/q}$ are deduced, and thereby a group presentation for $G_{n/m}(p, q)$ is obtained. These simple facts are the following.

\begin{align*}
R_{x}^{\pi}R_{y}^{\theta}R_{x}^{\pi} &= R_{y}^{-\theta}, & \text{where } 0 \leq \theta < 2\pi \\
R_{y}^{\pi/2}R_{x}^{\pi/2}R_{y}^{\pi/2} &= R_{x}^{\pi/2}R_{y}^{\pi/2}R_{x}^{\pi/2}
\end{align*}

(2) (3)
Figure 1. The generators of $G_{n/m}(p, q)$.

Many finite groups are generalised dihedral groups. For example, all finite cyclic groups $C_p$, $p \in \mathbb{N}$, can be viewed as generalised dihedral groups by adding to $R_x^{2\pi/p}$ a second redundant generator, say $R_y^{2\pi}$. One obviously has $C_p \cong G_{1/4}(p, 1)$. The dihedral group $D_p$ of order $2p$, $p \in \mathbb{N}$, has the group presentation $D_p = \langle \alpha, \beta \mid \alpha^p = e, \beta^2 = e, (\alpha \beta)^2 = e \rangle$. We use the simple fact (2) from Euclidean geometry and set $\theta = 2\pi/p$. This implies that $R_x^{\pi} = \alpha$ and $R_y^{\pi} = \beta$ generate $D_m$ and hence $D_m \cong G_{1/4}(p, 2)$. Furthermore, one has $S_4 \cong G_{1/4}(4, 4)$. One can show that $S_4$ has the group presentation $S_4 = \langle u, v \mid u^4, v^4, (u^2v)^2, (uv^2)^2, (uv)^3 \rangle$. Here $(u^2v)^2$ and $(uv^2)^2$, respectively, are consequences of (2), whereas the last relation $(uv)^3$ is a consequence of (3).

But not all finite subgroups of $SO(3, \mathbb{R})$ are generalised dihedral groups.

**Lemma 2.** The rotation symmetry group of the tetrahedron $A_4$ and the rotation symmetry group of the icosahedron $A_5$ can each be generated by two rotations of finite order, but only with axes separated by an angle of infinite order. Thus, neither $A_4$ nor $A_5$ is a generalised dihedral group.

**Proof.** We show that any two rotations of $A_4$ whose rotation axes are separated by an angle of finite order do not generate $A_4$. Consider the angles between the rotation axes of elements of the tetrahedral group $A_4$. The only angles of finite order that appear are the ones between two axes of 2-fold symmetry passing from the midpoint of an edge of the tetrahedron to the midpoint of the opposite edge. Two such axes form an orthogonal angle. The corresponding two rotations by $\pi$ generate a group isomorphic to $G_{1/4}(2, 2) \cong D_2$. Thus $A_4$ is not a generalised dihedral group. One proceeds similarly for the group $A_5$. \qed
Remark 4. In view of our main objective we can already state that all finite subgroups of $SO(3,\mathbb{Q})$ except the alternating group $A_4$ are generalised dihedral subgroups of $SO(3,\mathbb{Q})$. This leaves the task to classify the infinite generalised dihedral subgroups of $SO(3,\mathbb{Q})$, if there are any.

The classification of the generalised dihedral groups of $SO(3,\mathbb{R})$ obtained in [9] can be summarised as follows. Mostly, these groups are free products, or amalgamated free products of cyclic or dihedral groups. In most cases, $G_{n/m}(p,q)$ is thus infinite (cf. Section 1), the only exceptions being the groups $S_4$, $C_p$ and $D_p$, where $p \in \mathbb{N}$. Together with Lemma 2 and the classification of finite subgroups of $SO(3,\mathbb{R})$, this implies the following.

Lemma 3. The only finite subgroups of $SO(3,\mathbb{R})$ are the alternating groups $A_4$ and $A_5$ and the finite generalised dihedral groups.

The generalised dihedral groups play a role in the theory of tilings of Euclidean 3-space. An example called “quaquaversal tilings” was constructed in [3], see [10] for an image. A quaquaversal tiling of $\mathbb{R}^3$ consists of congruent copies of a single triangular prism that appears in an infinite number of orientations. This set of orientations forms a dense subgroup of $SO(3,\mathbb{R})$ that, in fact, is the generalised dihedral group $G_{1/4}(6,4) \cong D_6 \ast D_2, D_4$ [11]. (By Lemma 3, this group is infinite.) The quaquaversal tiling can be viewed as a 3-dimensional extension of the pinwheel tiling of the plane. Another example is a 3-dimensional “dite and kart tiling” constructed in [11], which is based on a version of the 2-dimensional kite and dart tiling and has $G_{1/4}(10,4)$ as orientation group.

3. Cayley’s parametrisation

We recall some results on quaternions which can be found in [5, 1]. The Hamiltonian quaternion algebra is

$$\mathbb{H}(K) = \mathbb{R}e + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$$  \hspace{1cm} (5)

where the defining relations for the generating elements $e, i, j, k$ are given by

$$i^2 = j^2 = k^2 = -e \quad \text{and} \quad ij = -ji = k. \hspace{1cm} (6)$$

$\mathbb{H}(\mathbb{R})$ is a skew field with unit element $e$ and its elements are called quaternions. For a quaternion $q$ it is customary to write $q = (x_0, x_1, x_2, x_3)$ instead of $q = x_0 e + x_1 i + x_2 j + x_3 k$. Assign to a nonzero quaternion $\rho = (\kappa, \lambda, \mu, \nu)$ the following matrix $R(\rho)$ in $SO(3,\mathbb{R})$:

$$R(\rho) = \frac{1}{|\rho|^2} \begin{pmatrix}
\kappa^2 + \lambda^2 - \mu^2 - \nu^2 & -2\kappa\nu + 2\lambda\mu & 2\kappa\mu + 2\lambda\nu \\
2\kappa\nu + 2\lambda\mu & \kappa^2 - \lambda^2 + \mu^2 - \nu^2 & -2\kappa\lambda + 2\mu\nu \\
-2\kappa\mu + 2\lambda\nu & 2\kappa\lambda + 2\mu\nu & \kappa^2 - \lambda^2 - \mu^2 + \nu^2
\end{pmatrix}, \hspace{1cm} (7)$$

where $|\rho|^2 = \kappa^2 + \lambda^2 + \mu^2 + \nu^2$. The next result goes back to Euler in 1770.

Theorem 5. [5, Ch. 3.6] Every rotation matrix $M \in SO(3,\mathbb{R})$ is of the form $M = R(\rho)$ for some nonzero $\rho \in \mathbb{H}(\mathbb{R})$, where $R(\rho)$ is defined as in (7). The mapping $\rho \mapsto R(\rho)$ is called Cayley’s parametrisation of $SO(3,\mathbb{R})$.

Thus, in order to obtain all rotation matrices of $SO(3,\mathbb{R})$, it suffices to consider all nonzero quaternions $\rho = (\kappa, \lambda, \mu, \nu)$ such that $\kappa, \lambda, \mu, \nu \in \mathbb{R}$ and apply the mapping given by (7). But like this, any matrix in $SO(3,\mathbb{R})$ is obtained multiple times since $R(t\rho) = R(\rho)$ for all nonzero $t \in \mathbb{R}$. This immediately suggests the question whether it is possible to further restrict the entries of $\rho$ and still obtain all elements of $SO(3,\mathbb{R})$. Since our aim is to classify the generalised dihedral subgroups of $SO(3,\mathbb{Q})$, we shall restrict the entries of $\rho$ even further. With this in mind, we call a quaternion $\rho = (\kappa, \lambda, \mu, \nu)$ primitive, if $\kappa, \lambda, \mu, \nu \in \mathbb{Z}$ and gcd($\kappa, \lambda, \mu, \nu) = 1$. 

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Lemma 4. Let \( \kappa, \lambda, \mu, \nu \) of \( \langle \kappa, 0, 0, 0 \rangle \in \mathbb{H}(\mathbb{R}) \) be an element of \( \mathbb{R} \), and \( \phi \) the rotation angle \( \phi = 2\pi n/m \) between those two generators satisfies

\[
\cos \left( \frac{2\pi n}{m} \right) = \frac{\langle (\lambda, \mu, \nu)^t, (\lambda', \mu', \nu')^t \rangle}{\sqrt{\lambda^2 + \mu^2 + \nu^2} \cdot \sqrt{\lambda'^2 + \mu'^2 + \nu'^2}}
\]

- The rotation angle \( \phi \) of \( R(\rho) \) is given by

\[
\cos(\phi) = \frac{\kappa^2 - \lambda^2 - \mu^2 - \nu^2}{\kappa^2 + \lambda^2 + \mu^2 + \nu^2} \in \mathbb{Q},
\]

due to \( \text{tr}(R(\rho)) = 1 + 2\cos(\phi) \).

This enables us to “translate” the information given on a generalised dihedral group \( G_{n/m}(p, q) \) (i.e. the order of the generators and the enclosed angle between their rotation axes) into equations of \( \kappa, \lambda, \mu, \nu \) over \( \mathbb{Z} \).

Lemma 4. Let \( R \) be an element of \( \text{SO}(3, \mathbb{Q}) \) of finite order. Then \( R \) is a rotation of order 1, 2, 3, 4 or 6.

Proof. One easily sees that there are rotations of order 1 and 2 in \( \text{SO}(3, \mathbb{Q}) \). We use Cayley’s parametrisation of \( \text{SO}(3, \mathbb{Q}) \) with primitive quaternions. Let \( \rho = (\kappa, \lambda, \mu, \nu) \in \mathbb{H}(\mathbb{Q}) \) be a primitive quaternion such that \( R(\rho) = R \). Since \( R \) has finite order, its rotation angle \( \phi \in [0, 2\pi) \) can be expressed as \( \phi = 2\pi n/m \), where \( \gcd(n, m) = 1 \). Equation (8) implies

\[
\cos(\phi) = \frac{\kappa^2 - \lambda^2 - \mu^2 - \nu^2}{\kappa^2 + \lambda^2 + \mu^2 + \nu^2} \in \mathbb{Q}.
\]

Denote by \( \varphi \) Euler’s totient function. One has \( \mathbb{Q} = \mathbb{Q}(\cos(2\pi n/m)) = \mathbb{Q}(2\cos(2\pi/m)) \), the maximal real subfield of \( \mathbb{Q}(\zeta_m) \). If \( m \geq 3 \), its field degree over \( \mathbb{Q} \) is \( \varphi(m)/2 \). Therefore, one has \( \varphi(m)/2 = 1 \) yielding \( m \in \{3, 4, 6\} \).

4. Generalised dihedral subgroups of \( \text{SO}(3, \mathbb{Q}) \)

We have already encountered some examples of generalised dihedral subgroups of \( \text{SO}(3, \mathbb{Q}) \). The cyclic group \( C_k, \) where \( k \in \{1, 2, 3, 4, 6\} \), is a subgroup of \( \text{SO}(3, \mathbb{Q}) \) generated by a rotation by \( 2\pi/k \). It can be viewed as a generalised dihedral group by adding to this generator a second redundant one, given by the rotation by \( 2\pi \) about an axis orthogonal to the rotation axis of the first generator. The dihedral groups \( D_r \) with \( r \in \{1, 2, 3, 4, 6\} \) are also generalised dihedral subgroups of \( \text{SO}(3, \mathbb{Q}) \). They are generated by a rotation of order \( r \) and one of order 2 whose rotation axes are separated by an orthogonal angle, cf. Section 2. Furthermore, it was shown that the rotation symmetry group of the cube, \( S_4 \), is the generalised dihedral group \( G_{1/4}(4, 4) \).
It is generated by the two rotations by $\pi/2$ about the $x$-axis and $y$-axis, respectively. In matrix form, with respect to the canonical basis, these two generators are

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]

Therefore, $S_4$ is a generalised dihedral subgroup of $SO(3, \mathbb{Q})$. All these examples are finite groups. Since $A_4$ is not a generalised dihedral group, this leaves no finite groups to consider.

What about other generalised dihedral groups, which are in most cases free products or amalgamated free products of cyclic or dihedral groups? It turns out that, in the rational case, no infinite generalised dihedral groups occur.

**Theorem 7.** Let $G$ be a nontrivial generalised dihedral subgroup of $SO(3, \mathbb{Q})$, i.e., let $G$ be a nontrivial subgroup of $SO(3, \mathbb{Q})$ that is generated by two rotations of finite order about axes that are separated by an angle of finite order. Then $G$ is one of the following groups:

- the cyclic group $C_k$ with $k \in \{2, 3, 4, 6\}$,
- the dihedral group $D_\ell$ of order $2\ell$ with $\ell \in \{2, 3, 4, 6\}$,
- or the symmetric group $S_4$.

In particular, all generalised dihedral subgroups of $SO(3, \mathbb{Q})$ are finite.

The proof splits into numerous cases depending on the order of the two generators of the generalised dihedral group considered. For a complete proof, the reader is referred to [4, Thm. 2.42]. Here, we shall merely work through a single case that will nonetheless give an idea how the previous results on amalgamated free products, Cayley’s parametrisation of $SO(3, \mathbb{Q})$ and the classification of generalised dihedral groups of $SO(3, \mathbb{R})$ are combined.

Prior to that, let us formulate one consequence of the main theorem with regard to the classification of finite subgroups of $SO(3, \mathbb{Q})$.

**Corollary 2.** The finite subgroups of $SO(3, \mathbb{Q})$ are exactly the generalised dihedral subgroups of $SO(3, \mathbb{Q})$ and the alternating group $A_4$. 

4.1. Proof of Theorem 7

We need some further preliminaries. Consider the following equation.

\[ \kappa^2 = \lambda^2 + \mu^2 + \nu^2 \]  

\[(I)\]

**Lemma 5.** If $\rho = (\kappa, \lambda, \mu, \nu)$ is a primitive quaternion, then the following statements hold.

(i) $R(\rho)$ is a rotation of order 4 if and only if $\rho$ satisfies Equation \((I)\).

(ii) If Equation \((I)\) holds, then $\kappa$ is odd. Moreover, exactly one of $\lambda, \mu, \nu$ is odd as well.

**Proof.** (i) If $R(\rho)$ has order 4, then straightforward calculations for its rotation angle $\phi$ show that $\phi = \pi/2$ or $\phi = 3\pi/2$. In both cases $\cos(\phi) = 0$ and, again by (8), Equation (I) follows. If conversely $\rho$ fulfills (I), let $\varphi \in [0, 2\pi)$ be the rotation angle of $R(\rho)$. We deduce from Equation (I) that $k^2 - \lambda^2 - \mu^2 - \nu^2 = 0$ and one therefore has $\cos(\varphi) = 0$ by (8). Thus $\varphi = \pi/2$ or $\varphi = 3\pi/2$ which implies that $R(\rho)$ has order 4.

(ii) The squares of integers are congruent to 0 or to 1 mod 4. If $\kappa^2 \equiv 0$ mod 4, then $\lambda^2 + \mu^2 + \nu^2 \equiv 0$ mod 4 and hence, $\lambda^2 \equiv \mu^2 \equiv \nu^2 \equiv 0$ mod 4. This implies that all $\lambda, \mu, \nu$ are even. But $\kappa$ is even as well, contradicting the fact that $\rho$ is primitive. Thus $\kappa^2 \equiv 1$ mod 4, which shows that $\kappa$ is odd. Then one furthermore has $1 \equiv \lambda^2 + \mu^2 + \nu^2$ mod 4. This implies that exactly one of $\lambda^2, \mu^2, \nu^2$ is congruent to 1 mod 4, and therefore exactly one of $\lambda, \mu, \nu$ is odd.
Now let \( G \) be a nontrivial generalised dihedral subgroup of \( \text{SO}(3, \mathbb{Q}) \) that is generated by a rotation of order \( p \) and a rotation of order \( q \) about rotation axes that enclose an angle of \( 2\pi n/m \).

To prove the main theorem, it suffices to show that \( G \) is finite. Due to Lemma 4, only rotations of order 1, 2, 3, 4 and 6 are possible. In other words, we only have to check for \( p, q \in \{1, 2, 3, 4, 6\} \).

Here, we merely consider the case \( p = q = 4 \).

We employ Cayley’s parametrisation of \( \text{SO}(3, \mathbb{Q}) \) with primitive quaternions. Let \( \rho = (\kappa, \lambda, \mu, \nu) \) and \( \rho' = (\kappa', \lambda', \mu', \nu') \) be two primitive quaternions such that \( R(\rho) \) and \( R(\rho') \) generate \( G \) and such that \( R(\rho) \) and \( R(\rho') \) are both rotations of finite order \( p \). Let their rotation axes be separated by an angle of \( 2\pi n/m \) with \( n, m \in \mathbb{N} \), \( n < m \), \( \gcd(n,m) = 1 \) and \( m > 2 \).

\( R(\rho) \) and \( R(\rho') \) are both rotations of order 4. We translate the information given on \( G_{n/m}(4,4) \) into equations over the integers. By Lemma 5(i), one has

\[
\begin{align*}
\kappa^2 &= \lambda^2 + \mu^2 + \nu^2 \quad &\text{(I)} \\
(\kappa')^2 &= (\lambda')^2 + (\mu')^2 + (\nu')^2 \quad &\text{(I')} \\
\cos \left( \frac{2\pi n}{m} \right) &= \frac{\lambda \lambda' + \mu \mu' + \nu \nu'}{\sqrt{\lambda^2 + \mu^2 + \nu^2} \cdot \sqrt{(\lambda')^2 + (\mu')^2 + (\nu')^2}}. \quad &\text{(II)}
\end{align*}
\]

\( G_{n/m}(4,4) \) is a subgroup of \( \text{SO}(3, \mathbb{Q}) \) if and only if the three equations above have an integer solution \( \{\kappa, \lambda, \mu, \nu, \kappa', \lambda', \mu', \nu'\} \) that is subject to the conditions \( \gcd(\kappa, \lambda, \mu, \nu) = 1 \) and \( \gcd(\kappa', \lambda', \mu', \nu') = 1 \).

We confine ourselves to the case where \( m \) is odd. But first consider the equations at hand. By Lemma 5(ii), Equations (I) and (I') imply that \( \kappa \) and \( \kappa' \) are odd. Furthermore, inserting these two equations into Equation (II) yields

\[
\cos \left( \frac{2\pi n}{m} \right) = \frac{\lambda \lambda' + \mu \mu' + \nu \nu'}{\kappa \kappa'}. \quad &\text{(9)}
\]

If \( m \) is odd, the classification of generalised dihedral groups of \( \text{SO}(3, \mathbb{R}) \) [9] shows that

\[
G_{n/m}(4,4) \simeq C_4 \ast_{C_2} D_m,
\]

where \( C_4 \ast_{C_2} D_m \) denotes the free product of \( D_m \) and \( C_4 \) with amalgamated subgroup \( C_2 \). This group contains \( D_m \) as a subgroup as seen in Section 1. Assuming that \( G_{n/m}(4,4) \) is a subgroup of \( \text{SO}(3, \mathbb{Q}) \), one has

\[
D_m \subset G_{n/m}(4,4) \subset \text{SO}(3, \mathbb{Q}).
\]

Now Corollary 4 yields \( m \in \{1, 2, 3, 4, 6\} \) as a necessary condition. But \( m \) is odd and \( m > 2 \), hence \( m = 3 \). Using \( \cos(2\pi n/3) = -1/2 \) in Equation (9) yields

\[
-\frac{1}{2} = \frac{\lambda \lambda' + \mu \mu' + \nu \nu'}{\kappa \kappa'},
\]

and thus

\[
-\kappa \kappa' = 2(\lambda \lambda' + \mu \mu' + \nu \nu').
\]

The left hand side of this equation is odd, because \( \kappa \) and \( \kappa' \) are both odd. But the right hand side is even. This shows that the system of equations consisting of (I), (I') and (II) does not have a solution over the integers if \( m \) is odd. Hence \( G_{n/m}(4,4) \) is not a subgroup of \( \text{SO}(3, \mathbb{Q}) \) for \( m \) odd.
5. Generalised dihedral subgroups of $SO(3, \mathbb{Q}(\tau))$

Classifications of all generalised dihedral groups of $SO(3, \mathbb{R})$ and of the generalised dihedral subgroups of $SO(3, \mathbb{Q})$ have been accomplished. Other real algebraic number fields $K$ are of interest as well. For example, $K = \mathbb{Q}(\tau)$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio, is an important object of study in quasicrystallography when dealing with modules that display icosahedral symmetry. It is possible to parametrise also the group $SO(3, \mathbb{Q}(\tau))$ via a certain type of quaternions. Not surprisingly, the finite generalised dihedral subgroups of $SO(3, \mathbb{Q}(\tau))$ are precisely the cyclic groups $C_k$ with $k \in \{1, 2, 3, 4, 5, 6, 10\}$, the dihedral groups $D_{2\ell}$ with $\ell \in \{2, 3, 4, 5, 6, 10\}$ and the symmetric group $S_4$; cf. [4, Thm. 2.48].

But in contrast to the rational case, $SO(3, \mathbb{Q}(\tau))$ contains infinite generalised dihedral groups. For example, the group of orientations of the quaquaversal tiling $G_{1/4}(6, 4) \cong D_6 \ast D_2 D_4$ is infinite by Lemma 3. It is generated by $R(\rho)$ and $R(\rho')$, where $\rho = (2\tau, -1, \tau, \tau + 1)$ and $\rho' = (3\tau, \tau + 1, 0, 1)$ [4, Rem. 2.49]. One easily checks that $R(\rho)$ and $R(\rho')$ are elements of $SO(3, \mathbb{Q}(\tau))$.

However, the complete classification of all generalised dihedral subgroups of $SO(3, \mathbb{Q}(\tau))$ remains an open problem so far.

Acknowledgments

This work was supported by the German Research Council (DFG), within the CRC 701.

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