Shadowing for differential equations with grow-up.

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Abstract

We consider the problem of shadowing for differential equations with grow-up. We introduce so-called nonuniform shadowing properties (in which size of the error depends on the point of the phase space) and prove for them analogs of shadowing lemma. Besides, we prove a theorem about weighted shadowing for flows. We compactify the system (using Poincare compactification, for example), apply the results about nonuniform or weighted shadowing to the compactified system, and then transfer the results back to the initial system using the decompactification procedure.

Keywords. Shadowing, grow-up, hyperbolicity, Poincare compactification, time change.

1 Introduction and main definitions.

Consider a system of ODEs

$$\dot{x} = X(x), \quad x \in \mathbb{R}^N. \quad (1)$$

We say that it has grow-up if it has a solution $|x(t, x_0)| \to \infty$ as $t \to +\infty$.

In the modern literature there are a lot of works devoted to study of grow-up and blow-up (a solution "reaches" infinity within a finite time) both for ODEs and PDEs (see, e.g., [2,6,15]). Developing theory of shadowing for such equations seems to be an interesting and challenging problem.

Theory of shadowing studies the problem of closeness of approximate and exact trajectories of dynamical systems. Roughly speaking, a dynamical system has a shadowing property if any sufficiently precise approximate trajectory is close to some exact trajectory. We are interested in introducing shadowing properties for differential equations with grow-up and in obtaining relevant criteria. Thus we want to answer the following question (under reasonable assumptions): suppose we have a reasonable approximate solution going to

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infinity for infinite time; is it true that there exists an exact solution that is in some sense close to our approximate solution?

Usually theory of shadowing (see [7, 11] for review of classical results and [12] for review of modern results) establishes shadowing properties for dynamical systems on a compact phase space or establishes shadowing properties in a small neighborhood of a compact invariant set (e.g., shadowing near a hyperbolic set). Note that we deal with a dynamical system on a noncompact phase space (i.e. on \(\mathbb{R}^N\)).

It is reasonable to act according to the following plan:

1. to compactify our system (using, e.g., Poincare compactification),
2. to establish some shadowing property for the compactified system,
3. to transfer the property back to the original system.

It is relatively easy to understand that the standard shadowing property for flows (we will remind the definition below in the paper) is bad for this scheme. In order to act according to this scheme, one should consider shadowing properties with errors decreasing to zero sufficiently fast (weighted shadowing) or shadowing properties with errors depending on the point of the phase space (we call it nonuniform shadowing).

The rest of the paper is organized as follows: in Section 2 we discuss compactifications (Step 1 of the plan), Section 3 is a brief introduction to classical theory of shadowing, in Section 4 we define and study nonuniform shadowing properties, in Section 5 we study weighted shadowing properties, and in Section 6 we discuss plans for further research.

Main results of the paper are Theorems 4.10 and 5.3 and their compactified versions Theorems 4.6 and 5.1.

2 Poincare compactification.

It is possible to compactify our system (1) in various ways (see [5] for excellent description of compactifications). The most obvious way is just to add one point as infinity. If we do it, we will get a system (or vector field) on the \(S^N\), without of one point. But, of course, in general, in order to get a vector field on \(S^N\), (consider, e.g., any system with blow-up) we should apply a time change of a certain type. This procedure (the compactification of space and the time change) is called Bendixon compactification. It works not for an arbitrary vector field, but only for so-called normalizable vector fields. Any polynomial vector field belongs to the class of normalizable vector fields.

However we are not going to apply Bendixon compactification by the following reasons:

1. it is very likely that the point on \(S^N\) corresponding to infinity will be a degenerate point of very high order;
2. Bendixon compactification does not allow to distinguish ”convergence to infinity by different directions”.

Instead we will use the procedure called Poincare compactification. Similarly with Bendixon compactification it consists of two phases: a compactification of the phase space and a change of time.

We compactify the phase space in the following way: we consider the map \(\Theta : \mathbb{R}^N \rightarrow B_N\) defined by the formula

\[
\Theta(x) := \frac{x}{\sqrt{(|x|^2 + 1)}},
\]

(2)
where the coordinates in the $N$-dimensional ball, $B_N$, are chosen like on Fig. 1.

If we apply the compactification (2) to the system (1), we will get a system $\dot{\bar{x}} = \bar{X}(\bar{x})$ on $B_N \setminus \partial B_N$ (hereinafter $\partial B_N$ denotes the boundary of the $N$-dimensional ball $B_N$). It is easy to understand that, in general, we will not get a system on $B_N$ (consider, e.g., any system with blow-up). Similarly with Bendixson compactification in order to get a system on $B_N$ we should apply a time change of a certain type. This procedure (the change of phase space and the change of time) is called Poincare compactification. Poincare compactification is defined not for an arbitrary vector field (1), but only for the class of so-called normalizable vector fields. Any polynomial vector field belongs to the class of normalizable vector fields. Hereinafter we assume that we deal with normalizable vector fields.

Consider polar coordinates $x = (z, \phi_1, \ldots, \phi_{N-1})$ in $\mathbb{R}^N$. Consider polar coordinates in $B_N$: $\bar{x} = (\bar{z}, \phi_1, \ldots, \phi_{N-1})$. Naturally the compactification map (2) can be rewritten in the following way:

$$\bar{z} = \sqrt{\frac{z^2}{1+z^2}} = \sqrt{\frac{1}{z^2} - 1}$$

and angles $\phi_1, \ldots, \phi_{N-1}$ do not change.

Consider a ball $U(\bar{R}, \bar{x}) \subset B_N$ of radius $\bar{R}$. We assume that $U(\bar{R}, \bar{x})$ does not intersect the boundary. We want to find (a reasonably small) $R$ such that

$$\Theta^{-1}(U(\bar{R}, \bar{x})) \subset U(R, x),$$

i.e. we want to understand how the ball $U(\bar{R}, \bar{x})$ can be expanded via the decompactification procedure.

Note that, since we are interested only in getting a qualitative estimate, and polar coordinates and Cartesian coordinates generate equivalent topologies, it is enough to consider only the change of radial coordinates. Put $\bar{y} = 1 - \bar{z}$ (i.e. $\bar{y}$ is radial distance to the boundary). Then it easy to compute that

$$z = \sqrt{\frac{1}{2\bar{y} - \bar{y}^2} - 1}.$$

Assuming that the ball does not intersect the boundary, the points $(\bar{y} - \bar{R}, \ldots)$ and $(\bar{y} + \bar{R}, \ldots)$ are mapped to the points $(\sqrt{-1 + 1/(2(\bar{y} - \bar{R}) - (\bar{y} - \bar{R})^2)}, \ldots)$
and \((\sqrt{-1+1/(2(\bar{y}+R)) - (\bar{y}+R)^2}), \ldots\). After careful calculations we see that
\[
\Theta^{-1}(U(\bar{R}, \bar{x})) \subset U(R, \Theta^{-1}(x)), \quad \text{where } R = O(\bar{R}/(\bar{y}^{3/2})).
\]
and taking into consideration that
\[
\bar{y} = 1 - \sqrt{1 - 1/(z^2 + 1)} = O\left(\frac{1}{z^2}\right) = O\left(\frac{1}{|x|^2}\right).
\]
we get
\[
\Theta^{-1}(U(\bar{R}, \bar{x})) \subset U(R, \Theta^{-1}(x)), \quad \text{where } R = O(\bar{R}|x|^3).
\]

Now we consider the inverse problem. Consider a ball \(U(R, x) \subset \mathbb{R}^N\). We want to find (a reasonably small) \(\bar{R}\) such that
\[
\Theta(U(R, x)) \subset U(\bar{R}, \bar{x}).
\]

By (3), the points \((z - R, \ldots)\) and \((z + R, \ldots)\) are mapped to the points \((1 - \sqrt{1 - 1/((z - R)^2 + 1)), \ldots)\) and \((1 - \sqrt{1 - 1/((z + R)^2 + 1)), \ldots)\). After careful calculations and using (4), we observe that
\[
\Theta(U(R, x)) \subset U(\bar{R}, \Theta(x)), \quad \text{where } \bar{R} = O(R/|x|^3) = O(R|\bar{y}|^{3/2}).
\]

3 Standard shadowing properties.

Consider a diffeomorphism \(f\) of a compact smooth Riemannian manifold \(M\) with Riemannian metric \(\text{dist}\). A trajectory of a point \(q\) of the diffeomorphism \(f\) is the sequence
\[
O(q, f) = \{f^k(q)\}_{k \in \mathbb{Z}}.
\]
A sequence \(\{x_k\}_{k \in \mathbb{Z}}\) of points of \(M\) is a \(d\)-pseudotrajectory if
\[
\text{dist}(x_{k+1}, f(x_k)) \leq d \quad \forall k \in \mathbb{Z}.
\]
Clearly the notion of a pseudotrajectory is one of possible formalizations of the notion of an approximate trajectory.

A diffeomorphism \(f\) has standard shadowing property if for any \(\epsilon > 0\) there exists \(d_0\) such that for any \(d\)-pseudotrajectory \(\{x_k\}_{k \in \mathbb{Z}}\) with \(d \leq d_0\) there exists a point \(q\) such that
\[
\text{dist}(x_k, f^k(q)) \leq \epsilon \quad \forall k \in \mathbb{Z}.
\]
Thus standard shadowing means that any sufficiently precise pseudotrajectory is pointwisely close to some exact trajectory.

This property is also called two-sided standard shadowing property, because biinfinite trajectories and pseudotrajectories are considered. Also so-called one-sided standard shadowing property is considered, in which pseudotrajectories and trajectories are indexed by natural numbers (clearly this property is weaker for diffeomorphisms than the two-sided version). Moreover, so-called Lipschitz standard shadowing property is considered (in which \(d = \epsilon/L\), where \(L\) is a global constant).

One of the main results of theory of shadowing is so-called shadowing lemma (see [1, 3]):

3.1. Theorem (Anosov, Bowen). A diffeomorphism has Lipshitz standard shadowing property in a small neighborhood of a hyperbolic set.
Recently the following result was obtained (see [13]):

3.2. Theorem (Pilyugin, Tikhomirov). Lipschitz standard shadowing property is equivalent to structural stability.

For flows the situation with shadowing properties is more difficult. First of all, there is no canonical way to formalize the notion of a pseudotrajectory for a flow. We will use here the definitions offered by S.Yu. Pilyugin (see [11]).

Let $\Phi$ be a flow on a compact smooth Riemannian manifold $M$. A $(d,T)$-pseudotrajectory of a flow $\Phi$ is a function $\Psi : M \to \mathbb{R}$ such that

$$\text{dist}(\Psi(t + \tau), \Phi(\tau, \Psi(t))) \leq d \quad \forall|\tau| \leq T, \forall t \in \mathbb{R}.$$ 

Note that a function $\Psi$ is not assumed to be continuous.

Let $\text{Rep}$ be the class of all increasing homeomorphisms of $\mathbb{R}$. Put

$$\text{Rep}(\epsilon) = \{ \alpha \in \text{Rep} \mid \frac{|\alpha(t) - \alpha(s)|}{t - s} \leq \epsilon \quad \forall t \neq s \}.$$ 

A flow $\Phi$ has oriented shadowing property if for any $\epsilon > 0$ there exists $d_0$ such that for any $(d,1)$-pseudotrajectory with $d \leq d_0$ there exist a point $q$ and a reparametrization $\alpha \in \text{Rep}$ such that

$$\text{dist}(\Psi(t), \Phi(\alpha(t), q)) \leq \epsilon \quad \forall t \in \mathbb{R}.$$ 

It is necessary to use time reparametrizations because of possible existence of periodic trajectories. However if a flow has no periodic trajectories, but is good (e.g., is a Smale flow), then no reparametrizations are required.

A flow $\Phi$ has standard shadowing property if for any $\epsilon > 0$ there exists $d_0$ such that for any $(d,1)$-pseudotrajectory with $d \leq d_0$ there exist a point $q$ and a reparametrization $\alpha \in \text{Rep}(\epsilon)$ such that

$$\text{dist}(\Psi(t), \Phi(\alpha(t), q)) \leq \epsilon \quad \forall t \in \mathbb{R}.$$ 

Standard shadowing property is not preserved via time changes. Similarly with the case of discrete time systems, Lipschitz version of standard shadowing property can be defined (when $d = \epsilon/L$, where $L$ is a global constant).

Shadowing lemma for flows was proved by S.Yu. Pilyugin and K. Palmer:

3.3. Theorem (Pilyugin). A flow has Lipschitz shadowing property in a small neighborhood of a hyperbolic set.

Palmer, Pilyugin, and Tikhomirov obtained the following result (see [8]):

3.4. Theorem (Palmer, Pilyugin, Tikhomirov). Structural stability for flows is equivalent to Lipschitz shadowing property.

4 Nonuniform shadowing.

4.1 Definitions and basic results.

Let $M$ be a smooth compact $N$-dimensional Riemannian manifold with boundary $\partial M$. By Whitney theorem, we assume that $M$ is embedded into an Euclidean space of a sufficiently large dimension. For any $x \in M$ define

$$r(x) = \text{dist}(x, \partial M) = \min_{y \in \partial M} |x - y|.$$
Without loss of generality, we assume that $M$ has diameter less than 1. Consequently, $r(M) \subset [0,1]$. Denote $\text{Int}(M) = M \setminus \partial M$.

A sequence $\{x_k\}_k \subset \text{Int}(M)$ is a nonuniform $(n, \delta)$-pseudotrajectory ($n \geq 1$) if
\[ |x_{k+1} - f(x_k)| \leq d(r(f(x_k))), \quad \forall k \geq 0, \quad (8) \]
where
\[ d(z) = \delta z^n, \quad \forall z \in \mathbb{R}_{>0}. \quad (9) \]

**4.1. Remark.**

1) Any $(n, \delta)$-pseudotrajectory is a $\delta$-pseudotrajectory (in the classical sense).

2) If we put $d(r(f(x_{k+1})))$ in (8) in the definition of an $(n, \delta)$-pseudotrajectory, then we obtain an equivalent definition. Note that $n \geq 1$.

We say that a diffeomorphism $f$ of $M$ has nonuniform shadowing property with exponent $m \geq 0$ if for any number $\Delta$ and the function $\epsilon(z) = \Delta z^m$, $\forall z \in \mathbb{R}$ there exist numbers $\delta_0$ and $n_0$ such that for any nonuniform $(n, \delta)$-pseudotrajectory with $\delta \leq \delta_0$ and $n \geq n_0$ there exists a point $q$ such that
\[ |x_k - f^k(q)| \leq \epsilon(r(f^k(q))), \quad \forall k \geq 0. \quad (11) \]

**4.2. Remark.**

1) For $m = 0$ and $n_0 = 0$ this property is standard shadowing property.

2) It is possible to put $\epsilon(r(x_k))$ instead of $\epsilon(r(f^k(q)))$ in (11) in the previous definition, but it does not lead to an equivalent definition, generally speaking. However the previous definition seems more natural to us, and the definition remains equivalent if $m \geq 1$.

For flows on $M$ this concept can be defined in the following way. A (not necessarily continuous) function $\Phi : \mathbb{R} \mapsto \text{Int}(M)$ is a nonuniform $(n, \delta, T)$-pseudotrajectory if for the function
\[ \psi(t) := \max_{|\tau| \leq T} |\Psi(t + \tau) - \Phi(\tau, \Psi(t))|, \]
and the function $d(\cdot)$ defined by (9) the following holds:
\[ \psi(t) \leq \min_{|\tau| \leq T} d(r(\Phi(\tau, \Psi(t)))) \quad \forall t \in \mathbb{R}_{>0}. \quad (12) \]

A flow $\Phi$ has nonuniform shadowing property with exponent $m$ if for any number $\Delta$ and the function $\epsilon(\cdot)$ defined by (10) there exist numbers $\delta_0$ and $n_0$ such that for any nonuniform $(n, \delta, 1)$-pseudotrajectory with $\delta \leq \delta_0$ and $n \geq n_0$ there exists a point $q$ such that
\[ |\Psi(t) - \Phi(t, q)| \leq \max_{|t| \leq \tau \leq |t|+1} \epsilon(r(\Phi(\tau, q))), \quad \forall t \geq 0, \]
where $\lfloor t \rfloor$ is the maximal integer number no more than $t$.

Note that oriented shadowing property is a nonuniform oriented shadowing property with exponent 0. We do not use reparametrizations in this property since in (very specific) situations that we will consider it is possible to choose the identity map as the reparametrization.

We use the following proposition:
4.3. Proposition. Consider the time-one map $f$ for a flow $\Phi$. Suppose that $f$ has a nonuniform shadowing property with exponent $m$; then $\Phi$ has nonuniform shadowing property with exponent $m$.

Proof. Let $\Psi$ be a nonuniform $(\delta, n, 1)$-pseudotrajectory. Consider the sequence $\xi = \{x_k\}_{k \in \mathbb{Z}} = \{\Psi(k)\}_{k \in \mathbb{Z}}$. We claim that $\xi$ is a $(\delta, n)$-pseudotrajectory. Indeed,\

$$|x_{k+1} - f(x_k)| \leq \psi(k) \leq d(r(f(x_k))).$$\

Choose a point $q$ such that (11) holds. Fix any $t \in [k, k+1]$. Define\

$$H = \max_{x \in M, 0 \leq \tau \leq 1} \left\| \frac{\partial \Phi}{\partial x}(\tau, x) \right\|.$$\

Then, by (12),\

$$|\Psi(t) - \Phi(t, q)| \leq |\Psi(t) - \Phi(t - k, \Psi(k))| + |\Phi(t - k, \Psi(k)) - \Phi(t, q)| \leq$$\

$$\leq d(r(\Phi(t - k, \Psi(k)))) + H|\Psi(k) - \Phi(k, q)| \leq d(r(\Phi(t - k, \Psi(k)))) +$$\

$$+ H\epsilon(r(\Phi(k, q))) \leq (1 + H) \max_{k \leq \tau \leq k+1} \epsilon(r(\Phi(\tau, q))).$$

Now let us investigate how nonuniform shadowing property is preserved via the decompactification procedure.

4.4. Proposition. Suppose that the compactified flow has nonuniform shadowing property with exponent $\hat{m}$ and numbers $\hat{\delta}$ and $\hat{n}_0$.

Then the initial flow has the following analog of nonuniform shadowing property, which we call noncompact nonuniform oriented shadowing property:

There exists a time change $\alpha : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ such that for any function $\epsilon(t) = \Delta |t|^{-2\hat{m}+3}$ there exist numbers $\delta_0$ and $n_0 = 2\hat{n}_0 - 3$ such that for any function $d(t) = \delta |t|^{-n}$ with $\delta \leq \delta_0$ and $n \geq n_0$ and any function $\Psi$ (which we call a noncompact nonuniform $(\delta, n, 1)$-pseudotrajectory) such that\

$$\max_{|\tau| \leq 1} |\Psi(t + \tau) - \Phi(\alpha(\tau, \Psi(t)), \Psi(t))| \leq \min_{|\tau| \leq 1} d(|\Phi(\alpha(\tau, \Psi(t)), \Psi(t))|)$$\

there exists a point $q \in \mathbb{R}^N$ and a reparametrization $\alpha \in \text{Rep}$ such that\

$$|\Psi(t) - \Phi(\alpha(t, q), q)| \leq \max_{\alpha(t, q) \leq \tau \leq \alpha(t+1, q)} \epsilon(\Phi(\tau, q)).$$

In particular, if $\hat{m} \geq 3/2$, then the initial noncompactified flow has oriented shadowing property. Even for $\hat{m} < 3/2$ we still have some sort of shadowing (despite our errors grow as the pseudotrajectory goes to infinity). Thus these shadowing properties even for $\hat{m} < 3/2$ can be used to determine grow-up.

Proof. Let $\alpha$ be the inverse map to the time change used in the compactification procedure. Note that, by (4) and (6),\

$$|\Theta(\Psi(t+\tau)) - \Phi(\tau, \Theta(\Psi(t)))| \leq |\Psi(t+\tau) - \Phi(\alpha(\tau, \Psi(t)), \Psi(t))||r(\Phi(\tau, \Theta(\Psi(t))))|^{3/2} \leq$$\

$$\leq \delta|\Phi(t, q)|^{-n}|r(\Phi(\tau, \Psi(t)))|^{3/2} \leq \delta|r(\Phi(\tau, \Psi(t)))|^{n/2+3/2}$$

(strictly speaking, since we have used asymptotic inequalities, we should have written a multiplicative constant $C_0$ in the right side of the previous equation; however, for simplicity, in such cases we omit such multiplicative constants).
Since \( n/2 + 3/2 \geq \bar{n}_0 \), by assumption of the proposition, there exists a point \( \Theta(q) \) such that
\[
|\Theta(\Psi(t)) - \Phi(t, \Theta(q))| \leq \Delta |r(\Phi(t, \Theta(q)))|^{\bar{m}}.
\]

Note that, by \([4]\) and \([5]\),
\[
|\Psi(t) - \Phi(\alpha(t, q), q)| \leq |\Theta(\Psi(t)) - \Phi(\alpha(t, q), \Theta(q))||r(\Phi(t, \Theta(q)))|^{-3/2} \leq \Delta \max_{|t| \leq \tau \leq |t| + 1} |r(\Phi(t, \Theta(q)))|^{\bar{m} - 3/2} \leq \Delta \max_{\alpha(t, q) \leq \tau \leq \alpha(t, q) + 1, q} |\Phi(t, q)|^{-2\bar{m} + 3}.
\]

\[\square\]

### 4.2 Reasoning of Conley.

Let \( \{g_k : \mathbb{R}^N \to \mathbb{R}^N\} \) be a sequence of diffeomorphisms. It is possible to use a manifold instead of \( \mathbb{R}^N \), since the reasoning is local. We assume that this sequence of maps is hyperbolic on some compact locally maximal invariant set \( \Lambda \subset \mathbb{R}^N \), and they locally preserve the foliation of the phase space on \( s \)-dimensional stable and \( u \)-dimensional unstable manifolds.

Let \( U(\varepsilon_0, \Lambda) \) be a small neighborhood of \( \Lambda \). Consider a continuous function \( \delta(p) \) (later we impose additional restrictions on it).

Let \( W_{2\delta}(p) \) and \( W_{2\delta}^u(p) \) be \( s \)-dimensional and \( u \)-dimensional submanifolds in \( \mathbb{R}^N \) of size \( \delta(p) \) respectively (corresponding to stable and unstable manifolds at \( p \)). Consider the set \( W_{2\delta}(p) \), the neighborhood of the point \( p \), and a map \( \chi_p : W_{2\delta}(p) \to E_{2\delta}(p) = E_{2\delta}^s(p) \times E_{2\delta}^u(p) \), where \( E_{2\delta}(p) \) is the standard cube, and stable manifolds are mapped to \( s \)-components and unstable manifolds are mapped to \( u \)-components. Denote by \( pr_s \) and \( pr_u \) natural projections on \( E_s \) along \( E_s \) and on \( E_u \) along \( E_u \) respectively. Fix \( \{x_k\}_k \subset \mathbb{R}^N \) (in the applications \( \{x_k\}_k \) will be a nonuniform pseudotrajectory). Consider a sequence of neighborhoods \( \{U(\delta(x_k), x_k)\}_k \) Assume this neighborhoods are so small that
\[
U(\delta(x_k), x_k) \subset W_{2\delta(x_k)}(x_k), \quad g_k(U(\delta(x_k), x_k)) \subset W_{2\delta(x_k+1)}(x_k+1) \quad \forall k \geq 0,
\]
\[
pr_s \chi_{x_k+1} g_k(U(\delta(x_k), x_k)) \subset pr_s \chi_{x_k+1} U(\delta(x_k+1), x_k+1) \quad \forall k \geq 0,
\]
\[
pr_u \chi_{x_k+1} g_k(U(\delta(x_k), x_k)) \cup pr_u \chi_{x_k+1} U(\delta(x_k+1), x_k+1) \quad \forall k \geq 0.
\]

Clearly estimates \([14]\) and \([15]\) hold, since \( \delta(x_k) \) can be chosen (uniformly) sufficiently small, \( \Lambda \) is a hyperbolic set, and it is possible to consider as a new sequence of \( \{g_k\} \) finite compositions \( \{g_T \circ \cdots \circ g_1 \circ g_T \circ \cdots \circ g_T \circ \cdots\} \), where \( T \) is a sufficiently large number. Note that passage to such finite compositions preserves shadowing.

The dynamics of \( \{g_k\} \) is depicted on Fig. 2 (vertical direction corresponds to contraction, and horizontal direction corresponds to expansion).

Consider the map
\[
h_k = \chi_{x_k+1} \circ g_k \circ \chi_{x_k}^{-1}.
\]

Clearly the analogs of estimates \([14]\) and \([15]\) hold for the maps \( h_k \).

Consider the cube \( E_{2\delta(p)} \). A horizontal \( u \)-dimensional surface is a surface \( S \subset E_{2\delta} \) such that \( pr_u S = E_u \). A vertical \( s \)-dimensional surface is a surface \( S \subset E_{2\delta(p)} \) such that \( pr_v S = E_s \).

We will need the following lemma. In essence, it was used without of proof in paper \([4]\). Paper \([4]\) contains the proof of this lemma for two-dimensional case. In essence, we generalize this proof to the case of higher dimensions.
4.5. Lemma. Let \( g_k : M \mapsto M \) be a sequence of smooth maps, let \( \{ x_k \}_k \) and \( \{ \delta(x_k) \}_k \) be such that relations \([13],[14],[15]\) hold. Put

\[
\text{Inv}_+^s(\{x_k\}_{k \geq 0}, \{g_k\}) := \{ q \mid g_k \circ \ldots \circ g_0(q) \in U(\delta(x_{k+1}), x_{k+1}) \quad \forall k \geq 0 \}.
\]

Then the set \( \chi_{x_0} \text{Inv}_+^s(\{x_k\}_{k \geq 0}, \{g_k\}) \) contains a unique vertical \( s \)-dimensional surface.

Proof. 1) Consider the maps \( h_k : E_{\delta(x_k)} \mapsto E_{\delta(x_{k+1})} \) defined by \([16]\). Note that \( \text{Inv}_+^s(\{0\}_{k \geq 0}, \{h_k\}_{k \geq 0}) = \chi_{x_0} \text{Inv}_+^s(\{x_k\}_{k \geq 0}, \{g_k\}_{k \geq 0}) \), where

\[
\text{Inv}_+^s(\{0\}_{k \geq 0}, \{h_k\}_{k \geq 0}) = \{ q \mid h_k \circ \ldots \circ h_0(q) \in E_{\delta(x_{k+1})}(x_{k+1}) \}.
\]

In order to prove Lemma 4.5, it is sufficient to prove that the set \( \text{Inv}_+^s(\{0\}_{k \geq 0}, \{h_k\}_{k \geq 0}) \) contains a vertical \( s \)-dimensional surface.

2) The case of \( s = N \) is trivial; that is why we do not consider it in details.

In this case \( \text{Inv}_+^s(\{0\}_{k \geq 0}, \{h_k\}_{k \geq 0}) = E_{\delta(x_k)}(x_0) \).

3) We divide \( E_{\delta(x_0)}(x_0) \) on cubes of size \( 1/2^n \). Denote by \( \text{Inv}_n \) the set of all cubes that intersect \( \text{Inv}_+^s(\{0\}_{k \geq 0}, \{h_k\}_{k \geq 0}) \). By \([15]\), it is sufficient to prove that for any \( n \) the set \( \text{Inv}_n \) contains a vertical \( s \)-dimensional surface (since a limit of vertical \( s \)-dimensional surfaces as \( n \to +\infty \) is a vertical \( s \)-dimensional surface).

4) Consider the set \( B = E_{\delta(x_0)}^u(x_0) \times \partial E_{\delta(x_0)}^u(x_0) \). Let \( J_n \) be the set of all cubes \( C \) such that the following holds:

- any cube \( C \subset J_n \) can be connected with \( B \) by a horizontal \( u \)-dimensional surface contained in \( J_n \),
- any cube \( C \subset J_n \) does not intersect \( \text{Inv}_+^s(\{0\}_{k \geq 0}, \{h_k\}_{k \geq 0}) \).

Due to \([14]\), \( J_n \) is not empty if \( n \) is sufficiently large (and contains all cubes adjacent to \( B \)).

5) Note that the set \( \partial J_n \) contains \( E_{\delta(x_0)}^u(x_0) \times \partial E_{\delta(x_0)}^u(x_0) \). Moreover, \( J_n \) is homeomorphic to the direct product of \( E_{\delta(x_0)}^u(x_0) \times \partial E_{\delta(x_0)}^u(x_0) \) and some other set. Consequently, \( B \) contains a vertical \( s \)-dimensional surface. Its uniqueness follows from \([15]\). We have proved our lemma. \( \square \)

4.3 Shadowing lemma for nonuniform shadowing, case of diffeomorphisms.

Let \( f \) be a diffeomorphism of an \( N \)-dimensional manifold \( M \) with the boundary \( \partial M \). Note that, since we are interested in criteria for shadowing, it is possible to consider \( f^T \) instead of \( f \), where \( T \) is a sufficiently large number. It allows to simplify hyperbolicity estimates.
Setting. Assume that $f$ has a locally maximal compact invariant set $\Lambda \subset \partial M$ and the following holds (we assume either 4.a) or 4.b)):

Main assumption. For any point $p \in \Lambda$ there exists a one-dimensional subspace $\ell(p) \subset T_pM$ such that:

- $\ell(p) \notin T_p\partial M$;
- $\ell(f(p)) = Df(p)\ell(p)$;
- $\ell(p)$ continuously depends on $p$;
- for any $v \in \ell(p)$ $\mu_1|v| \leq |Df(p)v| \leq \mu_2|v|$, and either $\mu_2 < 1$ or $\mu_1 > 1$ (hereinafter we consider only the case $\mu_2 < 1$, since the other case is completely similar, and we just get shadowing for negative indices instead of positive indices)

4.a) (if $\mu_2 < 1$) we choose $\lambda_{\text{min}}^s$ such that $|Df(p)v| \geq \lambda_{\text{min}}^s|v|$ for any $v \in T_pM$

OR

4.b) the set $\Lambda$ is hyperbolic for $f$, hyperbolicity is controlled by constants $\lambda_{\text{min}}^s < \lambda_{\text{max}}^s < 1$ for the case of the stable space and $1 < \lambda_{\text{min}}^u < \lambda_{\text{max}}^u$ for the case of the unstable space.

4.6. Theorem. Suppose that Main assumption holds. Let $U$ be a sufficiently small neighborhood of $\Lambda$ such that the analogues of estimates from Main Assumption hold in it.

1) Assume that $\mu_2 < 1$ and Item 4.a holds. Suppose that $\mu_2^m < \lambda_{\text{min}}^s$, $m > \ln \lambda_{\text{min}}^s / \ln \mu_2$, then $f$ has one-sided nonuniform shadowing property in $U$ with the exponent $m$ (which is Lipshitz if $m \geq 1$). If a pseudotrajectory is fully contained in $U$, then the point $q$ from the definition of nonuniform shadowing property is unique. If only a finite part of a pseudotrajectory is contained in $U$, then the set of points $q$ such that the analog of (11) holds is a small ball.

2) Suppose that $\mu_2 < 1$, Item 4.b) holds, and $\mu_1^m > \lambda_{\text{max}}^s$, $m < \ln \lambda_{\text{max}}^s / \ln \mu_1$, then $f$ has one-sided nonuniform shadowing property in $U$ with the exponent $m$ (which is Lipshitz if $m \geq 1$). If a pseudotrajectory is fully contained in $U$, then the set of points $q$ such that the analog of (11) holds is an $s$-dimensional disk $D_s$. If only finite part of a pseudotrajectory is contained in $U$, then the set of points $q$ such that the analog of (11) holds is a small neighborhood of an $s$-dimensional disk $D_s$.

4.7. Remark. 1) The simplest application of the theorem is when $\Lambda$ is a hyperbolic fixed point. In this case Main Assumption holds and the conditions of Theorem are naturally formulated in terms of eigenvalues of the corresponding matrix.

2) Note that condition (17) implies that $m > 1$, and condition (18) implies that $0 < m < 1$.

3) It is possible to give a more refined, stronger, and more technical version of the theorem using the theorem about filtrations (see [9]).

Proof. Step 1. Introduction of the coordinates. Here we consider only case 4.b) (case 4.a) is easier and is treated similarly).

We will define the coordinates in some fixed small neighborhood of the boundary $U(\epsilon_0, \partial M)$. We introduce a finite (but large number) of coordinate
charts. We denote by $\epsilon_1$ maximal diameter of the charts. In any of the coordinate charts the boundary is mapped to a hyperplane. This hyperplane contains center of the coordinate chart. We denote by $\theta(x)$ the coordinate of $x \in M$ in one of the charts. Any chart is the set of points $\{ |\theta(x)| \leq 1 \}$. We represent $\theta(x)$ as $(\theta(x)^{(1)}, \ldots, \theta(x)^{(s+u)})$, where $\theta(x)^{(s)} = (\theta(x)^{(1)}, \ldots, \theta(x)^{(s)})$ correspond to the stable space coordinates and $\theta(x)^{(w)} = (\theta(x)^{(s+1)}, \ldots, \theta(x)^{(u)})$ correspond to the unstable space coordinates, and $\theta(x)^{(1)}$ corresponds to the direction orthogonal to the boundary (we assume that $\epsilon_1$ is small, hence, it is possible to choose the stable and unstable coordinates and the transversal to the boundary coordinate uniformly in the coordinate system centered at $x$). Denote by $C$ the Lipschitz constant of the coordinate maps $\theta$ and their inverses $\theta^{-1}$. Note that since we can assume that $C$ is sufficiently close to 1,

$$C^{2m} \mu_2^m < \lambda_{\min}$$

in case 1) and similar inequalities in other cases. Besides, we assume that the coordinates are chosen such that if two points $z_1$ and $z_2$ are contained in one chart, then

$$g(\theta(z_2)) = g(\theta(z_1)) + A(z_1)(\theta(z_2) - \theta(z_1)) + \phi(z_1)(\theta(z_2) - \theta(z_1)), \quad \phi(z_1)(z) = o(z)$$

(where $g$ describes the dynamics in the coordinates).

In particular, the coordinates are constructed in such a way that the transversal to the boundary tangent direction is orthogonal to the boundary. Besides, we assume that the coordinate charts are monotonous in the following sense. Suppose that two points $z_1$ and $z_2$ are contained in an image of some coordinate chart; then

$$\text{dist}(z_1, \partial M) < \text{dist}(z_2, \partial M) \iff \theta(z_1)^{(1)} < \theta(z_2)^{(1)}.$$

Let $\{x_k\}_{k \geq 0}$ be a nonuniform pseudotrajectory with $n \geq 1$. Suppose that a point $p$ is $\epsilon_1$-close to $x_k$ and a point $f(p)$ is $\epsilon_1$-close to $f(x_k)$; then using Main assumption we conclude that in the coordinates that contain both $f(p)$ and $f(x_k)$

$$\theta(f(x_k))^{(1)} = \theta(f(p))^{(1)} + \mu(x_k)\theta(p)^{(1)} + \phi(x_k)^{(1)}(\theta(p)),$$

$$\theta(f(x_k))^{(s)} = \theta(f(p))^{(s)} + A(x_k)^s\theta(p)^{(s)} + \phi(x_k)^s(\theta(p)),$$

$$\theta(f(x_k))^{(u)} = \theta(f(p))^{(u)} + A(x_k)^u\theta(p)^{(u)} + \phi(x_k)^u(\theta(p)).$$

Assumptions of the theorem imply the corresponding estimates on the products of $|\mu(x_k)|$, $|A(x_k)^s|$, and $|A(x_k)^u|$.

By invariance of the direction towards the boundary and of the stable and unstable spaces, $\phi(x_k)^{(1)}(\theta(p)) = o(\theta(p)^{(1)}), \phi(x_k)^s(\theta(p)) = o(\theta(p)^{(s)}), \phi(x_k)^u(\theta(p)) = o(\theta(p)^{(u)})$.

Let $\Delta_0$ be a sufficiently small number (if necessary we decrease $\epsilon_1$) such that locally in each chart for $\phi(x_k)(\theta(p))$ we have

$$|\phi(x_k)(\theta(p))| \leq \Delta_0|\theta(p)|, \quad |\phi(x_k)^{(1)}(\theta(p))| \leq \Delta_0|\theta(p)|^{(1)}|,$$

$$|\phi(x_k)^s(\theta(p))| \leq \Delta_0|\theta(p)^{(s)}|, \quad |\phi(x_k)^u(\theta(p))| \leq \Delta_0|\theta(p)^{(u)}|.$$  

Note that these formulas imply monotonicity of sufficiently precise nonuniform pseudotrajectories with respect to the boundary (for $n \geq 1$). We consider only the case of $\mu_2 < 1$ (the case of $\mu_1 > 1$ is similar). Observe that

$$|\theta(x_{k+1})^{(1)}| \leq \mu_2|\theta(x_k)^{(1)}| + o(|\theta(x_k)^{(1)}|) + d((f(x_k))^{(1)}C) \leq$$
\[ \leq \mu_2|\theta(x_k)| + o(|\theta(x_k)|) + \delta(\mu_2c|\theta(x_k)| + o(|\theta(x_k)|))^n, \]
\[ |\theta(x_{k+1})| \geq \mu_1|\theta(x_k)| + o(|\theta(x_k)|) - \delta(\mu_1c|\theta(x_k)| + o(|\theta(x_k)|))^n \]

(we suppose that \(d(\cdot)\) satisfies (9)). Moreover, (since, by (19), \(C\mu_2 < 1\), and \(\delta\) can be chosen so small that \(2\delta\mu_2c \leq \Delta_0\)) it follows from (22) that
\[ (\mu_1 - 2\Delta_0)|\theta(x_k)| \leq |\theta(x_{k+1})| \leq (\mu_2 + 2\Delta_0)|\theta(x_k)|. \]

By decreasing (if necessary) \(\epsilon_1\) (and, consequently, \(\Delta_0\) too), we assume that
\[ \Delta_0 < \min(|1 - \mu_1/4|, |1 - \mu_2/4|). \]

By the choice of coordinate charts, estimates (24) imply that
\[ \text{dist}(x_{k+1}, \partial B_N) < \text{dist}(x_k, \partial B_N) \]
if \(\mu_2 < 1\) and
\[ \text{dist}(x_{k+1}, \partial B_N) > \text{dist}(x_k, \partial B_N) \]
if \(\mu_1 > 1\).

Note that we got monotonicity for \(n \geq 1\). Generally speaking, for \(0 < n < 1\) monotonicity does not hold. That is why we require \(n \geq 1\) even if \(0 < m < 1\).

Hereinafter, we assume that \(d(\cdot) \leq \epsilon(\cdot)/L\) for sufficiently large \(L\) (if \(m \geq 1\) it is sufficient to take \(d(\cdot) = \epsilon(\cdot)/L\).

Without loss of generality we assume that \(\delta\) is sufficiently small such that for any two points \(q_1\) and \(q_2\) (in one of the coordinate charts) that are \(\delta\)-close
\[ d_H(\text{pr}_u\chi_{q_1}U(q_1, \epsilon_1), \text{pr}_u\chi_{q_2}U(q_2, \epsilon_1)) \leq 1/L, \]
\[ d_H(\text{pr}_s\chi_{q_1}U(q_1, \epsilon_1), \text{pr}_s\chi_{q_2}U(q_2, \epsilon_1)) \leq 1/L, \]

where \(\chi_q\) is the analog of the map \(\chi_q\) defined at the beginning of Section 4.2.

**Step 2. The method of Conley.**

Let \(\{x_k\}\) be a sufficiently precise nonuniform pseudotrajectory contained in \(U(\epsilon_1, \partial M)\). We use notations from Section 4.2. Put \(\delta(x_k) = \epsilon(\theta(x_k)^{(1)}C)\).

Without loss of generality we assume the analogs of (13).

**Proof of Item 1** (Case 1). Suppose that inequality (17) holds. Without loss of generality by (19), assume that \(L\) is so large that it satisfies the following inequality
\[ C^{2m}(1 + 1/L)(1 + \Delta/L^m)\mu_2^m < \lambda_{min}^s. \]

The dynamics of \(f\) in Case 1 is depicted on Fig. 3

![Figure 3: Mappings of squares in Case 1.](image-url)
We need to prove the following inequality

\[ U(θ(x_{k+1}), anything) \subset g_k(U(θ(x_k), anything)) \]  

(28)

(where \( g_k \) is the corresponding coordinate representation of \( f \)). Let us remind the reader that \( anything \) satisfies 10. Since the Hausdorff distance between \( U(θ(x_{k+1}), anything) \) and \( U(θ(x_{k+1}), anything) \) is no more than \( \Delta(\delta(θ(f(x_{k+1}))))mC^m \leq \Delta(\delta(θ(f(x_{k+1}))))/L)mC^m \leq \Delta C^m/(L^m) \), inequality 28 would follow from

\[ U(θ(x_{k+1}), anything) \subset g_k(U(θ(x_k), anything)/C^m)) \]  

Note that inequality (32) for any sufficiently small \( \Delta \).

\[ U(θ(x_{k+1}), r) \subset U(θ(f(x_k)), (1+1/L)r). \]  

(30)

It follows from 30 that

\[ U(θ(x_{k+1}), (1+\Delta/L^m)anything(θ(f(x_k)))C^m) \subset \]  

\[ U(θ(f(x_k)), (1+1/L)(1+\Delta/L^m)anything(θ(f(x_k)))C^m). \]

Note that (since we may assume that \( \epsilon_1 \) (and, hence, \( \Delta_0 \)) is sufficiently small)

\[ U(θ(f(x_k)), (\lambda_{min} - \Delta_0)anything(θ(x_k)))C^m \) \( g_k(U(θ(x_k), anything(θ(x_k)))C^m). \)

Thus in order to get 29 it is enough to prove the following inequality

\[ (1+1/L)(1+\Delta/L^m)anything(θ(f(x_k)))C^m < (\lambda_{min} - \Delta_0)anything(θ(x_k))C^m. \]  

(31)

Inequality 31 would follow from 24 and

\[ (1+1/L)(1+\Delta/L^m)\delta(\mu_2_+\Delta_0)manything(θ(x_k))C^m \leq \delta(\lambda_{min} - \Delta_0)(anything(θ(x_k))C^m \]  

(32)

Note that inequality 32 for any sufficiently small \( \Delta_0 \) follows from 17 (one of conditions of the theorem) and 27.

Inequality 28 implies the analog of relation 15 (where the quasi-unstable space is \( \mathbb{R}^n \), i.e. \( s \) is 0 and \( u \) is changed to \( s + u \)). Thus it follows from Lemma 4.5 (applied to the sequence of maps \( \{g_k\}_k \)) that there exists a unique point \( q \) such that

\[ f^k(q) \in U(x, \epsilon(r(x))) \quad \forall k \geq 0. \]

Item 1 is proved.

The case when only a finite part of a nonuniform pseudotrajecory is contained in \( U(\epsilon_1, \Lambda) \) is treated similarly.

**Proof of Item 2** (Case 2). Suppose that inequality 15 holds.

Assume that a nonuniform pseudotrajecory \( \{x_k\}_{k \geq 0} \) is fully contained in \( U(\epsilon_1, \partial M) \). The dynamics of \( f \) in Case 2 is depicted on Fig. 2.

We will establish nonuniform shadowing for the sequence of maps \( \{g_k\}_k \), and then transfer the property to the map \( f \) (it will change only the constant but not the exponent).

We use notations from Section 4.2. As before, let \( pr_s \) and \( pr_u \) be natural projections on stable and unstable manifolds along unstable and stable manifolds, respectively, (the stable manifold corresponds to \( \theta(x)^u = 0 \), and the
Thus, in order to obtain (37), it is sufficient to prove that

\[ L \text{ is sufficiently large} \]

However, the last inequality holds trivially for any sufficiently small \( \Delta \)

To obtain the following inclusion:

\[ \text{Fix } z \in U(\theta(x_k), \epsilon(\theta(x_k)^{(1)})) \]  

Then, in order to obtain (37), it is sufficient to prove that

\[ (1 + 1/L)pr_u \chi(\theta(f(x_k)))U(\theta(f(x_k)), (1 + \Delta/L)\epsilon(\theta(f(x_k))^{(1)})C^m) \subset \]

(1 - 1/L)pr_u \chi(\theta(f(x_k)))U(\theta(x_k), \epsilon(\theta(x_k)^{(1)})C^m)).

Fix \( z \in U(\theta(x_k), \epsilon(\theta(x_k)^{(1)})) \) By (21) applied to \( x_k \) and \( z \), and by (23),

\[ |pr_u \theta(f(x_k)) - pr_u \theta(f(z))| \leq (\lambda_{\min}^u - \Delta_0)|\theta(x_k)^{(u)} - \theta(z)^{(u)}| \]

Thus, in order to obtain (37), it is sufficient to prove that

\[ (1 + 1/L)(1 + \Delta/L^m)\epsilon(\theta(f(x_k))^{(1)})C^m \leq (1 + 1/L)(\lambda_{\min}^u - \Delta_0)\epsilon(\theta(x_k)^{(1)})/C^m. \]

We obtain this inequality as soon as we prove that

\[ (1 + 1/L)(1 + \Delta/L^m)(\mu_2 + \Delta_0)^m|\theta(x_k)^{(1)}|C^m \leq (1 + 1/L)(\lambda_{\min}^u - \Delta_0)|\theta(x_k)^{(1)}|C^m. \]

However, the last inequality holds trivially for any sufficiently small \( \Delta_0 \), \( \Delta \), any sufficiently large \( L \), and any \( C \) that is sufficiently close to 1, since \( \mu_2 < 1 \) and \( \lambda_{\min}^u > 1 \). Inclusion (36) (and, hence, inclusion (34)) is proved.

Let us prove inclusion (35).

It follows from (20) that, in order to get inclusion (35), it is sufficient to obtain the following inclusion:

\[ (1 + 1/L)pr_u \chi(\theta(f(x_k)))U(\theta(f(x_k)), (1 + \Delta/L^m)\epsilon(\theta(f(x_k))^{(1)})C^m) \supset \]

(1 - 1/L)pr_u \chi(\theta(f(x_k)))U(\theta(x_k), \epsilon(\theta(x_k)^{(1)})C^m)).

Fix \( z \in U(\theta(x_k), \epsilon(\theta(x_k)^{(1)})) \) By (20) applied to \( x_k \) and \( z \), and by (23),

\[ |pr_u \theta(f(x_k)) - pr_u \theta(f(z))| \leq (\lambda_{\max}^s + \Delta_0)|\theta(x_k)^{(s)} - \theta(z)^{(s)}|. \]

Thus, in order to prove (38), it is sufficient to get the inequality

\[ (1 - 1/L)(1 + \Delta/L^m)\epsilon(\theta(f(x_k))^{(1)})C^m \geq (1 + 1/L)(\lambda_{\max}^s + \Delta_0)\epsilon(\theta(x_k)^{(1)})C^m. \]
We will prove this inequality as soon as we prove that
\[ (1-1/L)(1-\Delta/L^m)(\mu_1-\Delta_0)^m|\theta(x_k)^{(1)}|^m/C^m \geq (1+1/L)(\lambda^s_{\max}+\Delta_0)|\theta(x_k)^{(1)}|^mC^m. \]
However it follows from [18] (one of the conditions of the theorem) that the last inequality holds for any sufficiently small \( \Delta_0, \Delta, \) any sufficiently large \( L, \) and any \( C \) that is sufficiently close to 1. Inclusion (35) (and, consequently, inclusion (33)) is proved.

Relations (33) and (34) are analogs of relations (14) and (15). Thus we can apply Lemma 4.5. It follows from Lemma 4.5 that there exists an \( s \)-dimensional disk \( D_s \) such that for any point \( q \in D_s \)
\[ f^k(q) \in U(x_k, \epsilon(r(x_k))) \quad \forall k \geq 0. \]
The case when only finite part of a pseudotrajectory is contained in \( U(\epsilon_1, \Lambda) \) can be treated similarly.

\[ \square \]

4.4 Nonuniform shadowing for flows.

In these section we formulate the analogs of Theorem 4.6 for flows.

Let \( \Phi : \mathbb{R} \times M \mapsto M \) be a flow on a smooth compact Riemannian manifold \( M \) with boundary. Assume that \( M \) is embedded in an Euclidean space of sufficiently large dimension. We assume that \( \Phi \) satisfies the following:

**Main Assumption for flows.** There exists a compact locally maximal invariant set \( \Lambda \subset \partial M \) such that for any point \( p \in \Lambda \) there exists a one-dimensional subspace \( \ell(p) \subset T_p M \) such that for some sufficiently large constant \( C_0 \)
0) \( \ell(p) \notin T_p \partial M; \)
1) \( \ell(\Phi(t, p)) = D\Phi(t, p)\ell(p) \) for any \( t \in \mathbb{R}; \)
2) \( \ell(p) \) continuously depends on \( p; \)
3) \( (\exp(\mu_1 t)C_0)|v| \leq |D\Phi(t, p)v| \leq C_0 \exp(\mu_2 t)|v| \) for any \( v \in \ell(p), \) \( t \in \mathbb{R}; \)
4.a) (if \( \mu_2 < 0 \)) we choose \( \lambda^s_{\min} \) such that \( |D\Phi(t, p)v| \geq \exp(\lambda^s_{\min} t)|v|/C_0 \)
for any \( v \in T_p M, \) \( t \in \mathbb{R} \)

OR

4.b) \( \Lambda \) is a hyperbolic set for \( \Phi \) in the sense of [11], i.e. there exist invariant subspaces \( S(p), U(p) \subset T_p M \) such that \( S(p) \oplus U(p) = T_p M \) if \( p \) is a fixed point of \( \Phi \) or \( S(p) \oplus U(p) \) has codimension 1 in \( T_p M \) (and is transversal to the vector field) if \( p \) is not a fixed point, and there exist numbers \( \lambda^s_{\min} \leq \lambda^s_{\max} < 0, \)
0 < \( \lambda^s_{\min} \leq \lambda^s_{\max} \) such that
\[ e^{\lambda^s_{\min} t}|v|/C_0 \leq |D\Phi(t, p)v| \leq C_0 e^{\lambda^s_{\max} t}|v|, \quad \forall v \in S(p), t \in \mathbb{R}, \]
\[ e^{\lambda^s_{\min} t}|v|/C_0 \leq |D\Phi(t, p)v| \leq C_0 e^{\lambda^s_{\max} t}|v|, \quad \forall v \in U(p), t \in \mathbb{R}. \]

For simplicity we assume that \( \mu_2 < 0 \) (case \( \mu_2 > 0 \) is similar).

Theorem 4.6 can be generalized for flows on \( M \) in the following way (for simplicity, we treat only the case \( \mu_2 < 0 \), and naturally the corresponding analogs for finite shadowing also hold):

**4.8. Theorem.** Let \( \Phi : \mathbb{R} \times M \mapsto M \) be a flow, and let \( \Lambda \subset \partial M \) be a compact locally maximal invariant set. Let \( U \) be a sufficiently small neighborhood of \( \Lambda \) (such that the analogs of estimates from Main Assumption hold in \( U \)).

1) Suppose that Main Assumption for flows with Item 4.a) holds, and \( m \) is a sufficiently large number such that
\[ e^{\mu_2 m} < e^{\lambda^s_{\max}}, \quad m > \lambda^s_{\max}/\mu_2; \]
then $\Phi$ has nonuniform shadowing with exponent $m$ by a unique point.

2) Suppose that Main Assumption for flows with Item 4.b) holds, and $m$ is such that

$$e^{\mu_1 m} > e^{\lambda_{\text{min}}}, \quad m < \lambda_{\text{min}} / \mu_1; \quad (40)$$

then $\Phi$ has nonuniform shadowing with exponent $m$.

4.9. Remark. Note that, since $\mu_2 \leq \lambda_{\text{max}}$, in Item 1) $m > 1$, and, since $\mu_1 \geq \lambda_{\text{min}}$, in Item 2) $0 < m < 1$.

Proof. We start from Item 1). Consider the time-$T$ map $f$ for $\Phi$, where $T$ is a sufficiently large number (such that $(\lambda_{\text{max}})^T C < 1$, $(\lambda_{\text{min}})^T / C > 1$). Note that it satisfies Main Assumption for diffeomorphisms with Item 4.a) (clearly (39) implies (17)). Thus by Theorem 4.6 $f$ has nonuniform shadowing with exponent $m$ (by a unique point). Next we apply Proposition 4.3 and observe that $\Phi$ has nonuniform shadowing with exponent $m$ (by a unique point).

Item 2) is much more technical. That is why in this case we give just a brief outline of the proof. Using reasoning required to prove the standard shadowing lemma for flows (see [11]), we conclude that in order to prove nonuniform shadowing for $\Phi$ it is sufficient to prove the analog of Item 2) of Theorem 4.6 for a sequence of diffeomorphisms. This sequence of diffeomorphisms satisfies the analog of Main Assumption for diffeomorphisms with Item 4.b) (the analog of (18) follows from (40)). The shadowing lemma for a sequence of diffeomorphisms can be proved similarly with Item 2) of Theorem 4.6, but is much more technical. That is why we do not give a detailed proof here. \[\square\]

4.10. Theorem. Let $\Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a flow and $U$ be a sufficiently small neighborhood of infinity.

1) Suppose that after the compactification procedure (which includes applying a time change) the compactified flow $\tilde{\Phi} : \mathbb{R} \times B_N \to B_N$ satisfies Main Assumption for flows with Item 4.a), $\mu_2 < 0$ and $\Lambda = \partial B_N$. Let $m$ be such that (39) holds. Then $\Phi$ has noncompact oriented nonuniform shadowing from Proposition 4.4 in $U$ with exponent $3 - 2m$.

2) Suppose that after the compactification procedure the compactified flow $\tilde{\Phi} : \mathbb{R} \times B_N \to B_N$ satisfies Main Assumption for flows with Item 4.b), $\mu_2 < 0$ and $\Lambda = \partial B_N$. Let $m$ be such that (40) holds. Then $\Phi$ has noncompact oriented nonuniform shadowing from Proposition 4.4 in $U$ with exponent $3 - 2m$.

Proof. The theorem follows from Theorem 4.8 and Proposition 4.4 \[\square\]

4.11. Remark. Note that if $3 - 2m > 0$, i.e. $0 < m < 3/2$, we get a shadowing property with errors that are not bounded, but their growth is controlled. Whereas for $m \geq 3/2$ the errors are bounded.

5 Weighted shadowing.

5.1 Weighted shadowing for flows on compact manifolds.

We use notations from Section 3. As before

$$\psi(t) := \sup_{|\tau| \leq 1} |\Psi(t + \tau) - \Phi(\tau, \Psi(t))|.$$
5.1. Theorem. Let $\Phi$ be a flow on a compact smooth Riemannian manifold $M$ with the boundary $\partial M$ (e.g., $M = B_N$). Let $U$ be a small neighborhood of $\partial M$. There exist constants $C > 1$ and $L$ such that for any sufficiently small number $d$ and any $(d, 1)$-pseudotrajectory such that
\[
\int_{t \geq 0} C^t \psi(t) dt \leq d \tag{41}
\]
there exists a point $p$ such that
\[
\int_{t \geq 0} C^t |\Phi(t, p) - \Psi(t)| dt \leq Ld. \tag{42}
\]

5.2. Remark. The analog of Theorem 5.1 for discrete dynamical systems was formulated and proved in the book [11].

Proof. Choose
\[
C \geq \max_{|\tau| \leq 1, p \in M} ||D\Phi(\tau, p)|| \tag{43}
\]
Let $\Psi(t)$ be a function that satisfies relations (41). Put $x_k = \Psi(k)$ for $k \in \mathbb{N}$. Since an integral is a limit of Darboux sums, the sequence $x_k$ satisfies the following discrete analog of (41)
\[
\sum_{k} C^k |x_{k+1} - \Phi(1, x_k)| \leq d
\]
(i.e. it is a weighted pseudotrajectory for $\Phi(1, \cdot)$). Next due to (43) we apply for $\Phi(1, \cdot)$ the result of Pilyugin for discrete time systems (see [11]). Thus there exist a global constant $L$ and a point $q$ such that
\[
\sum_{k} C^k |\Psi(k) - \Phi(k, q)| \leq Ld.
\]
Let $L_0$ be such that $|D\Phi(\tau, \cdot)| \leq L_0$ for all $|\tau| \leq 1$. Since for any $|\tau| \leq 1$
\[
|\Psi(k + \tau) - \Phi(k + \tau, q)| \leq |\Psi(k + \tau) - \Phi(\tau, \Psi(k))| + |\Phi(\tau, \Psi(k)) - \Phi(k + \tau, q)| \leq \psi(t) + C|\Psi(k) - \Phi(k, q)|,
\]
and the following holds
\[
\sum_{k} C^k |\Psi(k + \tau) - \Phi(k + \tau, q)| \leq d + CLd,
\]
which gives desired estimate (42) for the integral.

5.2 Weighted shadowing for flows on $\mathbb{R}^N$.

In this section we formulate a noncompact version of Theorem 5.1

5.3. Theorem. Let $C$ be a sufficiently large number. There exists a map $\alpha : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ such that if $\Psi(t)$ satisfies the following analog of (41)
\[
\int_{t \geq 0} C^{5t/2} \psi_\alpha(t) dt \leq d, \tag{44}
\]
where
\[
\psi_\alpha(t) = \max_{|\tau| \leq 1} |\Psi(t + \tau) - \Phi(\alpha(\tau), \Psi(t))|,
\]
then there exists a point $q$ such that the following analog of (42) holds:
\[
\int_{t \geq 0} C^t |\Phi(\alpha(t, q), q) - \Psi(t)| dt \leq Ld. \tag{45}
\]
Proof. Denote by $\Phi$ the compactified flow. We assume that the number $C$ is such that

$$C \geq \max_{|t| \leq 1, x \in B_N} |D\Phi(t, x)|.$$

Let $\alpha$ be the inverse to time change used in the compactification. Note that, by (4) and (6), and since $r([R_0 \geq 0] \subset [0, 1],

$$|\Theta(\Psi(t + \tau)) - \Phi(\tau, \Theta(\Psi(t)))| \leq |\Psi(t + \tau) - \Phi(\alpha(\tau, \Psi(t)), \Psi(t))|.$$

Thus, by (44),

$$\int_{t \geq 0} C^{5t/2} \max_{|r| \leq 1} |\Theta(\Psi(t + \tau)) - \Phi(\tau, \Theta(\Psi(t)))|dt \leq \int_{t \geq 0} C^{5t/2} \psi_\alpha(t)dt \leq d,$$

which allows us to apply Theorem 5.1 and (by increasing $C$ even more if necessary) to get for some point $\Theta(q)$ the following analog of (42)

$$\int_{t \geq 0} C^{5t/2} |\Phi(t, \Theta(q)) - \Psi(t)| \leq Ld.$$  \hfill (46)

Similarly, by (4) and (5),

$$|\Phi(\alpha(t, q), q) - \Psi(t)| \leq \frac{|\Phi(t, \Theta(q)) - \Theta(\Psi(t))|}{|r(\Phi(t, \Theta(q)))|^{3/2}} \leq |\Phi(t, \Theta(q)) - \Theta(\Psi(t))|C^{3t/2}.$$

Thus, we derive (45) from (46):

$$\int_{t \geq 0} C^{t} |\Phi(\alpha(t, q), q) - \Psi(t)|dt \leq \int_{t \geq 0} C^{5t/2} |\Phi(t, \Theta(q)) - \Theta(\Psi(t))|dt \leq Ld.$$

6 Plans for further research.

1. Analogs of theorems about structural stability and $\Omega$-stability for systems with nonuniform shadowing.
2. Quantitative study of transfer of nonuniform shadowing via time reparametrizations.
3. Nonuniform shadowing near nonhyperbolic fixed points (analogs of results from [10]).
4. Study of shadowing properties of polynomial ODEs.

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References

[1] D.V. Anosov, *On a class of invariant sets of smooth dynamical systems (in Russian)*, Proc. 5th Int. Conf. on Nonl. Oscill., 2, Kiev, 1970, 39–45.

[2] N. Ben-Gal, *Grow-up solutions and heteroclinics to infinity for scalar parabolic PDEs*, text of the Ph.D. thesis.

[3] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*. Lect. Notes in Math., 470, Springer-Verlag, 1975.

[4] C.C. Conley, *Hyperbolic sets and shift automorphisms*, Dynamical systems, theory and applications, Lecture notes in physics, 1975, Volume 38, 539 – 549.

[5] J. Hell, *Conley Index at Infinity*, text of the Ph.D. thesis.

[6] B. Fiedler, H. Matano, *Global dynamics of blow-up profiles in one-dimensional reaction diffusion equations*, J. Dyn. Diff. Eqs., Vol. 19, 2007, 4, pp. 867–893.

[7] K. Palmer, *Shadowing in Dynamical Systems. Theory and Applications*, Kluwer Academic Publishers, 2000.

[8] K. Palmer, S.Yu. Pilyugin, S.B. Tikhomirov, *Lipschitz Shadowing for Flows*, J. Diff. Eqs, 2012, Vol. 252, 1723–1747.

[9] Y.B. Pesin, *Lectures on partial hyperbolicity and stable ergodicity*, Zurich Lect. in Adv. Math, Eur. Math. Soc., 2004.

[10] A. A. Petrov, S. Yu. Pilyugin, *Shadowing near nonhyperbolic fixed points*, Discrete Contin. Dyn. Syst., Vol. 34, 9, 2014, pp. 3761–3772.

[11] S. Yu. Pilyugin, *Shadowing in Dynamical Systems*. Lecture Notes in Math., Springer-Verlag, Berlin, 1706, 1999.

[12] S. Yu. Pilyugin, *Theory of pseudo-orbit shadowing in dynamical systems*, Diff. Eqs., 2011, Vol. 47, 13, 1929–1938.

[13] S. Yu. Pilyugin, S. B. Tikhomirov, *On Lipschitz Shadowing and structural stability*, Nonlinearity, 23, 2010, 2509-2515.

[14] S.Yu. Pilyugin, K. Sakai, *C0-transversality and shadowing properties*, Proceedings of the Steklov Institute of Mathematics, 2007, Vol. 256, 290–305.

[15] A. Vanderbauwhede, B. Fiedler, *Homoclinic period blow-up in reversible and conservative systems*, J. ZAMP, Vol. 43, 2, 1992, 292–318.