POISSON LAW FOR RETURNS OF MAPS ON COMPACT MANIFOLDS

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Abstract. We consider invariant measures of maps on manifolds whose correlations decay at a sufficient rate and which satisfy a geometric contraction property. We then prove the that the limiting distribution of returns to geometric balls is Poissonian. This does not assume an tower construction. The decay of correlations is used to show that the independence generated results in the Poisson distribution for returns that are sufficiently separated. A geometric contraction property is then used to show that short return times have a vanishing contribution to the return times distribution. We then also show that the set of very short returns which are of a small linear order of the logarithm of the radius of the balls has a vanishing measure. We obtain error terms which decay polynomially in the logarithm of the radius. We also obtain a extreme value law for such systems.

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1. Introduction

The limiting distribution of higher order return times in dynamics goes back to Doeblin [8] who established the Poisson distribution in the limit for the Gauss map at the origin. In more recent times there has been a large number of results for returns to cylinder sets where certain mixing properties are assumed. Pitskel [19] proved for Axiom A maps and equilibrium measures for Hölder continuous potentials that unions of cylinder sets have in the limit Poisson distributed return times. An approximation argument allowed him then to also deduce the same result for balls in the case of an Axiom A map on the two-dimensional torus. He showed that the moments converge and then invoked a theorem of Sevast’yanov to conclude that the return times are Poissonian in the limit. Other subsequent results like by Denker [7] which is along similar lines and in [15] which considers parabolic maps on the interval extended those results to more general settings.

For rational maps this was done in [10] which allowed approximations for balls if the dimension of the measure was not too large. More recently in [12] results were obtain for $\phi$-mixing and $\alpha$-mixing systems along a nested sequence of cylinder sets. Using the method of Chen and Stein the Poisson distribution was established in [13] along sequences of unions of cylinders for $\phi$-mixing systems over countable alphabets. This also allowed for approximations for balls under some favourable conditions. In [12] this was extended to $\alpha$-mixing systems and applied to the returns to Bowen balls. For a review see e.g. [11].

For returns to geometric balls apart from Pitskel’s result [19] from 1990 all results are quite recent except for the one by Pitskel and on intervals where approximations by cylinder sets can easily be used. For systems that can be modelled by Young towers, Chazottes and Collet [6] proved the limiting distribution to be Poissonian if the decay of correlations are exponential and the unstable manifold is one dimensional. This was generalised to polynomially decaying correlations and arbitrary dimensions in [14]. In this case the speed of convergence is polynomial in the logarithm of the radius of the return ball. A similar result without speed of convergence was obtained in [18] by using the Lebesgue density theorem. In this paper we provide a limiting result for maps on manifolds whose correlations decay and which satisfy a certain uniform contraction property (Assumption (V) below).

2. Assumptions and main results

Let $M$ be a manifold and $T : M \to M$ a map with the properties described below in the assumptions. Let $\mu$ be a $T$-invariant probability measure on $M$.

For a ball $B_\rho(x) \subset M$ we define the counting function

\[
\xi_{\rho,x}^{t}(x) = \sum_{n=0}^{\lfloor t/\mu(B_\rho(x)) \rfloor - 1} 1_{B_\rho(x)} \circ T^n(x).
\]

which tracks the number of visits a trajectory of the point $x \in M$ makes to the ball $B_\rho(x)$ on an orbit segment of length $N = \lfloor t/\mu(B_\rho(x)) \rfloor$, where $t$ is a positive parameter. (We often omit the sub- and superscripts and simply use $\xi(x)$.)

Let $\Gamma^u$ be a collection of unstable leaves $\gamma^u$ and $\Gamma^s$ a collection of stable leaves $\gamma^s$. We assume that $\gamma^u \cap \gamma^s$ consists of a single point for all $(\gamma^u, \gamma^s) \in \Gamma^u \times \Gamma^s$. The map $T$ contracts along the stable leaves and similarly $T^{-1}$ contracts along the unstable leaves.
For an unstable leaf $\gamma^u$ denote by $\mu_{\gamma^u}$ the disintegration of $\mu$ to the $\gamma^u$. We assume that $\mu$ has a product like decomposition $d\mu = d\mu_{\gamma^u} dv(\gamma^u)$, where $v$ is a transversal measure. That is, if $f$ is a function on $M$ then

$$\int f(x) \, d\mu(x) = \int_{\gamma^u} \int_{\gamma^u} f(x) \, d\mu_{\gamma^u}(x) \, dv(\gamma^u)$$

If $\gamma^u, \hat{\gamma}^u \in \Gamma^u$ are two unstable leaves then the holonomy map $\Theta : \gamma^u \cap \Lambda \to \hat{\gamma}^u \cap \Lambda$ is defined by $\Theta(x) = \hat{\gamma}^u \cap \gamma^u(x)$ for $x \in \gamma^u \cap \Lambda$, where $\gamma^u(x)$ be the local unstable leaf through $x$.

Let us denote by $J_n = \frac{dT^n_{\gamma^u}}{d\mu_{\gamma^u}}$ the Jacobian of the map $T^n$ with respect to the measure $\mu$ in the unstable direction.

Let $\gamma^u$ be a local unstable leaf. Assume there exists $R > 0$ and for every $n \in \mathbb{N}$ finitely many $y_k \in T^n \gamma^u$ so that $T^n \gamma^u \subset \bigcup_k B_{R,\gamma^u}(y_k)$, where $B_{R,\gamma^u}(y)$ is the embedded $R$-disk centered at $y$ in the unstable leaf $\gamma^u$. Denote by $\zeta_{\varphi,k} = \varphi(B_{R,\gamma^u}(y_k))$ where $\varphi \in \mathcal{A}_n$ and $\mathcal{A}_n$ denotes the inverse branches of $T^n$. We call $\zeta$ an $n$-cylinder. Then there exists a constant $L$ so that the number of overlaps $N_{\varphi,k} = |\{\zeta_{\varphi',k'} : \zeta_{\varphi,k} \cap \zeta_{\varphi',k'} \neq \emptyset, \varphi' \in \mathcal{A}_n\}|$ is bounded by $L$ for all $\varphi \in \mathcal{A}_n$ and for all $k$ and $n$. This follows from the fact that $N_{\varphi,k}$ equals $|\{k' : B_{R,\gamma^u}(y_k) \cap B_{R,\gamma^u}(y_{k'}) \neq \emptyset\}|$ which is uniformly bounded by some constant $L$.

We make the following assumptions:

(I) Decaying of correlations: There exists a decay function $\lambda(k)$ so that

$$\left| \int_M G(H \circ T^k) \, d\mu - \mu(G) \mu(H) \right| \leq \lambda(k) \|G\|_{Lip} \|H\|_{\infty} \quad \forall k \in \mathbb{N},$$

for functions $H$ which are constant on local stable leaves $\gamma^s$ of $T$.

(II) Dimension: There exist $0 < d_0 < d_1$ such that $\rho^{d_0} \geq \mu(B_\rho) \geq \rho^{d_1}$.

(III) Unstable dimension: There exists a $u_0$ so that $\mu_{\gamma^u}(B_{\rho}(x)) \leq C_1 \rho^{u_0}$ for all $\rho > 0$ small enough and for almost all $x \in \gamma^u$, every unstable leaf $\gamma^u$.

(IV) Distortion: We require that

$$\frac{J_n(x)}{J_n(y)} = \mathcal{O}(\omega(n))$$

for all $x, y \in \zeta$ and $n$, where $\zeta$ are $n$-cylinders in unstable leaves $\gamma^u$ and $\omega(n)$ is a non-decreasing sequence.

(V) Contraction: We require that there exists a function $\delta(n) \to 0$ which decays at least summably polynomially, i.e. $\delta(n) = \mathcal{O}(n^{-\kappa})$ with $\kappa > 1$, so that $\text{diam} \zeta \leq \delta(n)$ for all $n$-cylinder $\zeta$ and all $n$.

(VI) Annulus condition: Assume that for some $\eta, \beta > 0$:

$$\frac{\mu(B_{\rho+r} \setminus B_{\rho-r})}{\mu(B_\rho)} = \mathcal{O}(r^\eta \rho^{-\beta})$$

for every $r < \rho_0$ for some $\rho_0 < \rho$ (see remark below).

For a positive parameter $a$ define the set

$$V_\rho(a) = \{x \in M : B_\rho(x) \cap T^n B_\rho(x) \neq \emptyset \text{ for some } 1 \leq n < a|\log \rho|\},$$

where $\rho > 0$. The set $V_\rho$ represents the points within $M$ with very short return times.
2.1. Return times are Poisson distributed.

**Theorem 1.** Assume that the map $T : M \rightarrow M$ satisfies the assumptions (I)-(VI) where $\lambda(k)$ decays at least polynomially with power $p > \frac{\beta + d_1}{d_0}$. Moreover we assume that $d_0(\kappa \eta - 1) > \beta$ and $\kappa u_0 > 1$. If $\delta(j)$ decays polynomially with power $\kappa$ and $\omega(j) \sim j^{\kappa'}$ for some $\kappa' \in [0, \kappa u_0 - 1)$.

Then

$$P(\xi_{\rho, x} = r) = e^{-t} \frac{r^r}{r!} + O(|\log \rho|^{-\sigma})$$

for all $x \not\in V_\rho(a)$ for some positive $a$, where $\sigma = \kappa u_0 - \kappa' - 1$ is positive. Moreover, there exists an $a > 0$ so that

$$\mu(V_\rho(a)) = O(|\log \rho|^{-\sigma}).$$

If $\delta(n) = O(\vartheta^n)$, $\vartheta < 1$ is exponential then

$$P(\xi_{\rho, x} = r) = e^{-t} \frac{r^r}{r!} + O(t \rho^{u_0 a |\log \vartheta|}).$$

The proof of the first part of the theorem is given in the next section. The bound on the size of the very short return set is given in Section 4.2.

**Remark 1:** The standard case of bounded distortion corresponds to the value $\kappa' = 0$. Then the rate of convergence is $\sigma = \kappa u_0 - 1$.

**Remark 2:** If $\mu$ has dimension $d$ then the condition on the decay of correlation is $p > 1 + \frac{\beta}{d \eta}$.

**Remark 3:** For an absolutely continuous measure $\beta$ and $\beta$ can be chosen arbitrarily close to 1, $d_0$ and $d_1$ are arbitrarily close to $D = \dim M$; thus the requirement for $p$ is to be larger than $\frac{1}{D} + 1$ and $\kappa > \frac{1}{D} + 1$ as $u_0$ is arbitrarily close to $D$.

**Remark 4:** In the annulus condition (IV) we require that $r < \rho_0$ where according to Section 3.4 $\rho_0 = O(\rho^{v \kappa})$ and $v < d_0$ can be arbitrarily close to $d_0$.

2.2. Extremal Values Distribution. We take a point $z \in M$ and define

$$\varphi(x) = g(\mu(B_d(x, z)(z)))$$

where $g$ is a function from $M$ to $\mathbb{R} \cup \{+\infty\}$ with the following properties: $g$ is strictly decreasing in a neighborhood of 0; 0 is a global maximum for $g$; $g$ satisfies one of the following three properties:

Type 1. There exist some strictly positive function $p$ such that

$$g^{-1}\left(g\left(\frac{1}{n}\right) + yp(g\left(\frac{1}{n}\right))\right) = (1 + \varepsilon_n) \frac{e^{-y}}{n}.$$

with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ for all $y \in \mathbb{R}$.

Type 2. $g(0) = +\infty$ and there exist $\beta > 0$ such that

$$g^{-1}\left(g\left(\frac{1}{n}\right)y\right) = (1 + \varepsilon_n) \frac{y^{-\beta}}{n}$$

for all $y > 0$. 
Type 3. \( g(0) = D < +\infty \) and there exist \( \gamma > 0 \) such that
\[
g^{-1} \left( D - g \left( \frac{1}{n} \right) y \right) = (1 + \varepsilon_n) g^{-1} \left( D - g \left( \frac{1}{n} \right) \right) y^{\gamma} / n
\]
for all \( y > 0 \).

Examples of functions satisfying the three types are \( g_1(x) = -\log x \), \( g_2(x) = x^{1-\beta} \) and \( g_3(x) = D - x \Gamma \).

We put \( X_n = \varphi \circ T^n \) and \( M_n = \max\{X_k : 0 \leq k \leq n - 1\} \). We also write
\[
M_{j,n} = \max\{X_j, \ldots, X_{j+n-1}\}.
\]

Let \( \{\hat{X}_n\} \) be the a stationary, independent process such that \( \hat{X}_0 \) has the same distribution as \( X_0 \). Denote by \( \hat{M}_n \) the corresponding maxima of \( \{\hat{X}_n\} \).

From the extreme value theory of stationary, independent processes we know that under proper linear normalization, \( a_n(\hat{M}_n - b_n) \) converges to one of the following three limits:

Type 1.
\[
G(x) = e^{-e^{-x}}, \quad -\infty < x < \infty.
\]

Type 2.
\[
G(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
 e^{-x-\beta} & \text{if } x > 0, \text{ for some } \beta > 0.
\end{cases}
\]

Type 3.
\[
G(x) = \begin{cases} 
e^{-(-x)\gamma} & \text{if } x \leq 0 \text{ for some } \gamma > 0 \\
1 & \text{if } x > 0.
\end{cases}
\]

Now we state the Theorem.

**Theorem 2.** Assume that the map \( T : M \to M \) satisfies the assumptions (I)–(VI) where \( \lambda(k) \) decays at least polynomially with power \( p > \beta \eta d + 1 \). Moreover we assume that \( \kappa \eta - 1 > \beta \) and \( \kappa u_0 > 2 \) if \( \delta(j) \) decays polynomially with power \( \kappa \) and \( \omega(j) \sim j^{\kappa'} \) for some \( \kappa' \in [0, \kappa u_0 - 3] \). Then we have Type \( i \) extreme value law for observables \( g_i \) with type \( i \), \( i = 1, 2, 3 \).

3. Proof of Theorem 1

3.1. Poisson approximation of the return times distribution. To prove Theorem 1 we will employ the Poisson approximation theorem from Section 5. Let \( x \) be a point in the phase space and \( B_\rho := B_\rho(x) \) for \( \rho > 0 \). Let \( X_n = 1_{B_\rho} \circ T^{n-1} \), then we put \( N = \lceil t / \mu(B_\rho) \rceil \), where \( t \) is a positive parameter. We write \( S_0 = \sum_{n=a}^{b} X_n \) (and \( S = S_1^N \)). Then for any \( 2 \leq \Delta \leq N \) (\( C_3 \) from Section 5)
\[
\left| \mathbb{P}(S = k) - \frac{t^k}{k!} e^{-t} \right| \leq C_3(N(\mathcal{R}_1 + \mathcal{R}_2) + \Delta \mu(B_\rho)),
\]

(2)
where

\[
R_1 = \sup_{0 < j < N - \Delta} \sup_{0 < q < N - \Delta - j} \left| E(1_{B_\rho} 1_{S_{\Delta + j} = q}) - \mu(B_\rho) E(1_{S_{\Delta + j} = q}) \right|
\]

\[
R_2 = \sum_{n=1}^{\Delta-1} E(1_{B_\rho} 1_{B_\rho \circ T_n}).
\]

Since we restrict to the complement of the set \( V_\rho \) (cf. (1)) we have from now on

\[
R_2 = \sum_{n=J}^{\Delta-1} \mu(B_\rho \cap T^{-n} B_\rho),
\]

where \( J = \lfloor a \log \rho \rfloor \). Since \( P(S = k) = 0 \) for \( k > N \) we obtain

\[
|P(S = k) - \frac{t^k}{k!} e^{-t}| = \frac{t^k}{k!} e^{-t} \leq c_1 |\log \rho|^{-\kappa u_0} \quad \forall k > N
\]

using the fact that \( \mu(B_\rho) \lesssim \rho^{d_0} \) and for \( \rho \) sufficiently small.

We now proceed to estimate the error between the distribution of \( S \) and a Poissonian for \( k \leq N \) based on Theorem 3.

3.2. Estimating \( R_1 \). By invariance of the measure \( \mu \) we can also write

\[
R_1 = \sup_{0 < j < N - \Delta} \left| \mu(B_\rho \cap T^{-\Delta} \{ S_{1}^{N-j-\Delta} = q \}) - \mu(B_\rho) \mu(\{ S_{1}^{N-j-\Delta} = q \}) \right|.
\]

We now use the decay of correlations (I) to obtain an estimate for \( R_1 \). Approximate \( 1_{B_\rho} \) by Lipschitz functions from above and below as follows:

\[
\phi(x) = \begin{cases} 
1 & \text{on } B_\rho \\
0 & \text{outside } B_{\rho+\delta}\rho
\end{cases}
\text{ and } \quad \tilde{\phi}(x) = \begin{cases} 
1 & \text{on } B_{\rho-\delta}\rho \\
0 & \text{outside } B_\rho
\end{cases}
\]

with both functions linear within the annuli. The Lipschitz norms of both \( \phi \) and \( \tilde{\phi} \) are equal to \( 1/\delta \rho \) and \( \phi \leq 1_{B_\rho} \leq \phi \).

We obtain

\[
\mu(B_\rho \cap \{ S_{\Delta}^{N-j} = q \}) - \mu(B_\rho) \mu(\{ S_{\Delta}^{N-j-\Delta} = q \})
\]

\[
\leq \int_M \phi \cdot 1_{S_{\Delta}^{N-j} = q} d\mu - \int_M 1_{B_\rho} d\mu \int_M 1_{S_{1}^{N-j-\Delta} = q} d\mu
\]

\[
= X + Y
\]

where

\[
X = \left( \int_M \phi d\mu - \int_M 1_{B_\rho} d\mu \right) \int_M 1_{S_{1}^{N-j-\Delta} = q} d\mu
\]

\[
Y = \int_M \phi \left( 1_{S_{\Delta}^{N-j} = q} \right) d\mu - \int_M \phi d\mu \int_M 1_{S_{1}^{N-j-\Delta} = q} d\mu.
\]
The two terms $X$ and $Y$ are estimated separately. The first term is estimated by:

$$X \leq \int_M \mathbb{1}_{S_1^{N-j-\Delta} = q} d\mu \int_M (\phi - \mathbb{1}_{B_\rho}) d\mu \leq \mu(B_{\rho + \delta} \setminus B_\rho).$$

In order to estimate the second term $Y$ we use the decay of correlations and have to approximate $\mathbb{1}_{S_1^{N-j-\Delta} = q}$ by a function which is constant on local stable leaves. For that purpose put

$$S_n = \bigcup_{T^n \gamma^s \subset B_\rho} T^n \gamma^s, \quad \partial S_n = \bigcup_{T^n \gamma^s \cap B_\rho \neq \emptyset} T^n \gamma^s$$

and

$$\mathcal{F}_\Delta^{N-j} = \bigcup_{n=\Delta} S_n, \quad \partial \mathcal{F}_\Delta^{N-j} = \bigcup_{n=\Delta} \partial S_n.$$

The set

$$\mathcal{F}_\Delta^{N-j}(q) = \{S_1^{N-j-\Delta} = q\} \cap \mathcal{F}_\Delta^{N-j}$$

is then a union of local stable leaves. This follows from the fact that by construction $T^n y \in B_\rho$ if and only if $T^n \gamma^s(y) \subset B_\rho$. We also have $\{S_1^{N-j} = q\} \subset \mathcal{F}_\Delta^{N-j}(q)$ where the set $\mathcal{F}_\Delta^{N-j}(k) = \mathcal{F}_\Delta^{N-j}(q) \cup \partial \mathcal{F}_\Delta^{N-j}$ is a union of local stable leaves.

Denote by $\psi_\Delta^{N-j}$ the characteristic function of $\mathcal{F}_\Delta^{N-j}(k)$ and by $\tilde{\psi}_\Delta^{N-j}$ the characteristic function of $\tilde{\mathcal{F}}_\Delta^{N-j}(k)$. Then $\psi_\Delta^{N-j}$ and $\tilde{\psi}_\Delta^{N-j}$ are constant on local stable leaves and satisfy

$$\psi_\Delta^{N-j} \leq \mathbb{1}_{S_1^{N-j-\Delta} = q} \leq \tilde{\psi}_\Delta^{N-j}.$$

Since $\{y : \psi_\Delta^{N-j}(y) \neq \tilde{\psi}_\Delta^{N-j}(y)\} \subset \partial \mathcal{F}_\Delta^{N-j}$ we need to estimate the measure of $\partial \mathcal{F}_\Delta^{N-j}$.

By the contraction property $\text{diam}(T^n \gamma^s(y)) \leq \delta(n)$ and consequently

$$\mu(\partial \mathcal{F}_\Delta^{N-j}) \leq \mu\left(\bigcup_{n=\Delta} T^n \gamma^s \subset B_{\rho + \delta(n)} \setminus B_{\rho - \delta(n)}\right) \leq \sum_{n=\Delta} \mu(B_{\rho + \delta(n)} \setminus B_{\rho - \delta(n)}).$$

and therefore

$$\mu(\partial \mathcal{F}_\Delta^{N-j}) = \mathcal{O}(1) \sum_{n=\Delta}^{-\kappa \eta} \mu(B_\rho) = \mathcal{O}(\rho^{(\kappa \eta - 1) - \beta}) \mu(B_\rho)$$

provided $\delta(n) = \mathcal{O}(n^{-\kappa})$ and $\Delta \sim \rho^{-v}$ for some positive $v > \frac{\beta}{\kappa \eta - 1}$ which is determined in Section 3.4 below. If we split $\Delta = \Delta' + \Delta''$ then we can estimate as follows:

$$Y = \int_M \phi T^{-\Delta'}(\mathbb{1}_{S_1^{N-j-\Delta'} = q}) d\mu - \int_M \phi d\mu \int_M \mathbb{1}_{S_1^{N-j-\Delta} = q} d\mu \leq \lambda(\Delta') ||\phi||_{Lip} ||\mathbb{1}_{\mathcal{F}_\Delta^{N-j-\Delta'}}||_{L^\infty} + 2 \mu(\partial \mathcal{F}_\Delta^{N-j}).$$
Hence
\[\mu(B_\rho \cap T^{-\Delta} \{ S_1^{N-j-\Delta} = q \}) - \mu(B_\rho) \mu(\{ S_1^{N-j-\Delta} = q \}) \leq \frac{\lambda(\Delta/2)}{\delta \rho} + \mu(B_{\rho+\delta \rho} \setminus B_\rho) + O(\rho^{v(\kappa_1) - \beta}) \mu(B_\rho)\]
by taking \(\Delta' = \Delta'' = \frac{\Delta}{2}\). A similar estimate from below can be done using \(\tilde{\phi}\). Hence
\[(4) \quad \mathcal{R}_1 \leq c_2 \left( \frac{\lambda(\Delta/2)}{\delta \rho} + \mu(B_{\rho+\delta \rho} \setminus B_{\rho-\delta \rho}) \right) + O(\rho^{v(\kappa_1) - \beta}) \mu(B_\rho).\]

In the exponential case when \(\delta(n) = O(\theta^n)\) we choose \(\Delta = s|\log \rho|\) for some \(s > 0\) and obtain the estimate
\[\mathcal{R}_1 \leq c_2 \left( \frac{\lambda(\Delta/2)}{\delta \rho} + \mu(B_{\rho+\delta \rho} \setminus B_{\rho-\delta \rho}) \right) + O(\rho^{s|\log \theta| - \beta}) \mu(B_\rho).\]

3.3. Estimating the terms \(\mathcal{R}_2\). We will estimate the measure of each of the summands comprising \(\mathcal{R}_2\) individually. We use the product form of the measures \(\mu\). For that purpose fix \(j\) and let \(\gamma^u\) be an unstable local leaf through \(B\). Then we put
\[\mathcal{E}_j(B, \gamma^u) = \{ \zeta_{\varphi, j} : \zeta_{\varphi, j} \cap B \neq \emptyset, \varphi \in \mathcal{I}_j \}\]
for the cluster of \(j\)-cylinders that covers the set \(B\), where the sets \(\zeta_{\varphi, j}\) are the pre-images of embedded \(R\)-balls in \(T^j\gamma^u\). Then
\[\mu_{\gamma^u}(T^{-j}B_\rho \cap B_\rho) \leq \sum_{\zeta \in \mathcal{E}_j(B_\rho, \gamma^u)} \frac{\mu_{\gamma^u}(T^{-j}B_\rho \cap \zeta)}{\mu_{\gamma^u}(\zeta)} \mu_{\gamma^u}(\zeta) \leq \sum_{\zeta \in \mathcal{E}_j(B_\rho, \gamma^u)} c_3 \omega(j) \mu_{T^j\gamma^u}(B_\rho \cap T^j\zeta) \mu_{T^j\gamma^u}(T^j\zeta) \mu_{\gamma^u}(\zeta)\]
Since \(\mu_{T^j\gamma^u}(T^j\zeta) = \mu_{T^j\gamma^u}(B_{R, \gamma^u}(y_k))\) (for some \(y_k\)) is uniformly bounded from below, we obtain
\[\mu_{\gamma^u}(T^{-j}B_\rho \cap B_\rho) \leq c_3 \omega(j) \mu_{T^j\gamma^u}(B_\rho) \sum_{\zeta \in \mathcal{E}_j(B_\rho, \gamma^u)} \mu_{\gamma^u}(\zeta) \leq c_3 \omega(j) \mu_{T^j\gamma^u}(B_\rho) L \mu_{\gamma^u} \left( \bigcup_{\zeta \in \mathcal{E}_j(B_\rho, \gamma^u)} \zeta \right)\]
Now, since \(\text{diam } \bigcup_{\zeta \in \mathcal{E}_j(B_\rho, \gamma^u)} \zeta \leq \delta(j) + \text{diam } B_\rho \leq c_1 \delta(j)\) (as we can assume that \(\rho < \delta(j)\)) we obtain
\[\mu_{\gamma^u}(T^{-j}B_\rho \cap B_\rho) \leq c_4 \omega(j) \mu_{T^j\gamma^u}(B_\rho) \delta(j)^{u_0}\]
Since \(d\mu = d\mu_{\gamma^u} \cdot d\nu(\gamma^u)\) we obtain
\[\mu(T^{-j}B_\rho \cap B_\rho) \leq c_5 \omega(j) \mu(B_\rho) \delta(j)^{u_0}\].
Summing up the $\mu(T^{-j}B_\rho \cap B_\rho)$ over $j = J, \ldots, \Delta - 1$, we see that outside the set of forbidden ball centers $V_\rho$ we get

$$R_2 = \sum_{j=J}^{\Delta-1} \mu(T^{-j}B_\rho \cap B_\rho) \leq c_5 \sum_{j=J}^{\Delta-1} \omega(j)\delta(j)\mu(B_\rho)$$

for some $c_5$. If we assume that $\omega(j) \sim j^{\kappa'}$ and $\delta(j)$ decays polynomially with power $\kappa$ then

$$R_2 \leq c_6 \mu(B_\rho) \sum_{j=J}^{\Delta-1} j^{\kappa'} j^{-\kappa u_0} \leq c_7 J^{-\sigma} \mu(B_\rho)$$

for $\rho$ small enough, where $\sigma = \kappa u_0 - \kappa' - 1$ is positive by assumption. If $\delta(j)$ decays super polynomially then

$$R_2 \leq c_7 J^{\kappa'} \delta(J)^u_0 \mu(B_\rho).$$

In the exponential case ($\delta(n) = O(\theta^n)$) we obtain

$$R_2 = O\left(\rho^{u_0|\log \theta|}\mu(B_\rho)\right)$$

as $J = a|\log \rho|$. 

3.4. **The total error.** For the total error we now put $\delta \rho = \rho^w$ and $\Delta = \rho^{-v}$ where $v < d_0$ since $\Delta \ll N$ and $N \geq \rho^{-d_0}$. Then $\lambda(\Delta) = O(\Delta^{-p}) = O(\rho^{\nu p})$ and thus (in the polynomial case)

$$\left| P(S = k) - \frac{t^k}{k!} e^{-t} \right| \leq N c_1 \left( \frac{\lambda(\Delta)}{\delta \rho} + \mu(B_{\rho+\delta \rho} \setminus B_{\rho-\delta \rho}) \right) + c_7 J^{-\sigma} + O(\rho^{v(\kappa \eta - 1) - \beta})$$

$$\leq c_8 \left( \rho^{v p w - d_1} + \rho^{\nu \eta - \beta} + t J^{-\sigma} + \rho^{v(\kappa \eta - 1) - \beta} \right).$$

We can choose $v < d_0$ arbitrarily close to $d_0$ and then require $d_0 p - w - d_1 > 0$, $w \eta - \beta > 0$ and $d_0 (\kappa \eta - 1) - \beta > 0$. We can choose $w > \frac{\beta}{\eta}$ arbitrarily close to $\frac{\beta}{\eta}$ and can satisfy all requirements if $p > \frac{\beta + d_1}{d_0}$ in the case when $\lambda$ decays polynomially with power $p$, i.e. $\lambda(k) \sim k^{-p}$. Hence

$$\left| P(S = k) - \frac{t^k}{k!} e^{-t} \right| \leq C_1 J^{-\sigma}$$

for some $C_1$.

In the exponential case ($\delta(n) = O(\theta^n)$) we obtain

$$P(S = k) - \frac{t^k}{k!} e^{-t} = O(1) \left( \rho^{s|\log \theta| - w - d_1} + \rho^{w \eta - \beta} + t \rho^{u_0|\log \theta|} + \rho^{\kappa \eta - \beta} \right).$$

Choosing $s$ and $w$ large enough, we obtain that the RHS is of order $O(t \rho^{u_0|\log \theta|}).$  \qed
4. Very Short Returns

4.1. Assumptions. Let \((M, T)\) be a dynamical system equipped with a metric \(d\). Assume that the map \(T : M \to M\) is a \(C^2\)-diffeomorphism. As at the start of the paper the set \(V_\rho \subset M\) is given by

\[V_\rho = \{x \in M : B_\rho(x) \cap T^n B_\rho(x) \neq \emptyset \text{ for some } 1 \leq n < J\},\]

where \(J = \lfloor a \log \rho \rfloor\) and \(a = (4 \log A)^{-1}\) with \(A = \sup_\omega (\|DT\|_{L^\infty} + \|DT^{-1}\|_{L^\infty})\) \((A \geq 2)\). We will need the following assumptions:

(V1) Distortion: We require that \(J_n(x) = \mathcal{O}(\omega(n))\) for all \(x, y \in \zeta\) and \(n\), where \(\zeta\) are \(n\)-cylinders on unstable leaves \(\gamma^u\) and \(\omega(n)\) is a (non-decreasing) sequence.

(V2) Contraction: There exists a function \(\delta(n) \to 0\) so that \(\text{diam } \zeta \leq \delta(n)\) for all \(n\)-cylinder \(\zeta\) and all \(n\) and \(\omega\).

(V3) Geometric regularity of the measure on the unstable leaves: Assume there exists \(u_0 > 0\) such that

\[\mu_{\gamma^u}(B_\rho(x)) \leq \rho^{u_0}\]

for all \(x\), unstable leaves \(\gamma^u\) and \(\rho\) small enough.

4.2. Estimate on the measure of \(V_\rho\). Now we can show that the set of centres where small balls have very short returns is small. To be precise we have the following result:

Proposition 1. Assume that the map \(T : M \to M\) satisfies the assumptions (V1)–(V3). Then there exist constants \(C_2 > 0\) such that for all \(\rho\) small enough

\[\mu(V_\rho) \leq \frac{C_2}{|\log \rho|^\sigma}\]

where \(\sigma = \kappa u_0 - \kappa' - 1\) if \(\delta\) decays polynomially with power \(\kappa > 1\) and \(\omega\) grows polynomially with power \(\kappa' \geq 0\) assuming \(\sigma > 0\).

If \(\delta\) decays exponentially and \(\limsup_{n \to \infty} \frac{\log \log \omega(n)}{\log n} < \frac{1}{2}\) then the error term on the RHS is \(\mathcal{O}(\delta(\log \rho)^{u_0})\).

Proof. We follow the proof of Proposition 5.1 of \([14]\) which modelled after Lemma 4.1 of \([6]\). Let us note that since \(T\) is a diffeomorphism one has

\[B_\rho(x) \cap T^n B_\rho(x) \neq \emptyset \iff B_\rho(x) \cap T^{-n} B_\rho(x) \neq \emptyset.\]

We partition \(V_\rho\) into level sets \(N_\rho(n)\) as follows

\[V_\rho = \{x \in M : B_\rho(x) \cap T^{-n} B_\rho(x) \neq \emptyset \text{ for some } 1 \leq n < J\} = \bigcup_{n=1}^{J-1} N_\rho(n)\]

where

\[N_\rho(n) = \{x \in M : B_\rho(x) \cap T^{-n} B_\rho(x) \neq \emptyset\}.\]
The above union is split into two collections \( V^1_\rho \) and \( V^2_\rho \), where

\[
V^1_\rho = \bigcup_{n=1}^{bJ} \mathcal{N}_\rho(n) \quad \text{and} \quad V^2_\rho = \bigcup_{n=bJ+1}^{J} \mathcal{N}_\rho(n).
\]

and where the constant \( b \in (0,1) \) will be chosen below. In order to find the measure of the total set we will estimate the measures of the two parts separately.

**I. Estimate of \( V^2_\rho \)**

We will derive a uniform estimate for the measure of the level sets \( \mathcal{N}_\rho(n) \) when \( n > bJ \).

Then

\[
\mu(\mathcal{N}_\rho(n)) = \mu(T^{-n}\mathcal{N}_\rho(n)) \leq \sum_{\zeta} \mu(T^{-n}\mathcal{N}_\rho(n) \cap \zeta)
\]

We will consider each of the measures \( \mu(T^{-n}\mathcal{N}_\rho(n) \cap \zeta) \) separately by using the product form of the measures \( \mu \). By distortion of the Jacobian we obtain

\[
\mu_{\gamma^n}(T^{-n}\mathcal{N}_\rho(n) \cap \zeta) = \frac{\mu_{\hat{\gamma}^n}(T^{-n}\mathcal{N}_\rho(n) \cap \zeta)}{\mu_{\gamma^n}(\zeta)} \mu_{\gamma^n}(\zeta)
\]

where, as before, \( \hat{\gamma}^n = \gamma^n(T^nx) \) for \( x \in \zeta \cap \gamma^n \). We estimate the numerator by finding a bound for the diameter of the set. Let the points \( x \) and \( z \) in \( T^{-n}\mathcal{N}_\rho(n) \) be such that \( x, z \in T^{-n}\mathcal{N}_\rho(n) \cap \zeta \cap \gamma^n \) for an unstable leaf \( \gamma^n \).

Note that \( T^nx, T^nz \in \mathcal{N}_\rho(n) \), there exists \( y \in B_\rho(T^nx) \) such that \( T^ny \in B_\rho(T^nx) \), thus

\[
d(T^n x, x) \leq d(T^n x, T^ny) + d(T^ny, y) + d(y, x) \leq \rho + \rho + A^nd(T^n x, T^ny) \leq (2 + A^n)\rho.
\]

Hence

\[
d(T^n x, T^n z) \leq d(T^n x, x) + d(x, z) + d(z, T^n z) \leq 4A^n\rho + d(x, z).
\]

We have

\[
dx(z) \leq \text{diam } \zeta < \delta(n)
\]

by assumption. Therefore

\[
d(T^n x, T^n z) \leq 4A^n\rho + d(x, z) \leq 4A^n\rho + \delta(n)
\]

If we choose \( a > 0 \) so that \( a < \frac{1}{2\log A} \) then \( A^n\rho < e^{-\frac{1}{4}\log \rho |\log \rho|^1/2} \). If \( n \geq b|\log \rho| \) for some \( b \in (0, a) \) then

\[
d(T^n x, T^n z) \leq c_2(e^{-c'|\log \rho|^1/2} + \delta(n))
\]

for some constant \( c_2 \) where \( c' = \min(\frac{1}{2}, \sqrt{b}) \). Taking the supremum over all points \( x \) and \( z \) yields

\[
|T^n(T^{-n}\mathcal{N}_\rho(n) \cap \zeta \cap \gamma^n)| \leq c_2(e^{-c'|\log \rho|^1/2} + \delta(n)).
\]

By assumption (V3) on the relationship between the measure and the metric

\[
\mu_{\hat{\gamma}^n}(T^n(T^{-n}\mathcal{N}_\rho(n) \cap \zeta)) \leq c_3(e^{-u_0c'|\log \rho|^1/2} + \delta(n)^{u_0})
\]

Incorporating the estimate into (i) yields

\[
\mu_{\gamma^n}(T^{-n}\mathcal{N}_\rho(n) \cap \zeta) \leq c_4\omega(n)(e^{-u_0c'|\log \rho|^1/2} + \delta(n)^{u_0})\mu(\zeta),
\]
for some $c_4$. Integrating over $dv(\gamma^u)$ and summing over $\zeta$ yields
\[
\mu(\mathcal{N}_\rho(n)) \leq c_4\omega(n)(e^{-u\rho'\log \rho} \log |\log \rho|^{1/2} + (\delta(n)u o) \sum \mu(\zeta) \leq c_5\omega(n)(e^{-u\rho'\log \rho} \log |\log \rho|^{1/2} + \delta(n)u o)
\]
as $\sum \mu(\zeta) = O(1)$. Consequently, if $\omega(n)$ is so that $\limsup_{n \to \infty} \frac{\log \omega(n)}{\log n} < \frac{1}{2}$ (as can be seen from the estimates above), the value $\frac{1}{2}$ can be replaced by 1) then
\[
(7) \quad \mu(\mathcal{V}_\rho^2) \leq \sum_{n=\lfloor bJ \rfloor}^J \mu(\mathcal{N}_\rho(n))
\]
\[
\leq c_5e^{-u\rho'\log \rho} \log |\log \rho|^{1/2} + \omega(n)\delta(n)u o
\]
\[
\leq c_6(e^{-u\rho'\log \rho} \log |\log \rho|^{1/2} + (ab|\log \rho|^{-\sigma})
\]
for some constant $c'' > 0$ (and $\rho$ small enough) as $J = \lfloor a|\log \rho| \rfloor$. As before, $\sigma = \kappa u_0 - \kappa' - 1$ if $\delta(n) \sim n^{-\kappa}$ and $\omega(n) \sim n^{\kappa'}$.

(II) Estimate of $\mathcal{V}_\rho^1$

We will need the following version of Lemma B.3 from [6].

**Lemma 1.** Put $s_p = 2^p \frac{A^{n^p-1}}{A^n-1}$. Then for every $p, k$ integers, $\rho > 0$
\[
\{x \in M : B_\rho(x) \cap T^k B_\rho(x) \neq \emptyset\} \subset \{x \in M : B_{sp}(x) \cap T^{2p} B_{sp}(x) \neq \emptyset\}.
\]

**Proof.** Consider the case $p = 1$. Let $x$ such that $B_\rho(x) \cap T^k B_\rho(x) \neq \emptyset$. This implies that there exist $z \in B_\rho(x) \cap T^{-k} B_\rho(x)$. For any $u \in T^k B_\rho(x)$, there exist $v \in B_\rho(x)$ such that $T^k v = u$, thus
\[
d(u, x) \leq d(u, T^k z) + d(T^k z, x) \leq d(T^k v, T^k z) + 2\rho \leq (2A^k + 2)\rho.
\]
Therefore, $T^k B_\rho(x) \subset B_{(2A^k+2)\rho}(x)$.

One can observe that if $B_\rho(x) \cap T^k B_\rho(x) \neq \emptyset$ then $T^k (B_\rho(x) \cap T^k B_\rho(x)) \neq \emptyset$ thus $T^k B_\rho(x) \cap T^k (T^k B_\rho(x)) \neq \emptyset$ and therefore $B_{(2A^k+2)\rho}(x) \cap T^k (T^k B_{(2A^k+2)\rho}(x)) \neq \emptyset$. Finally, this gives us
\[
\{x \in M : B_\rho(x) \cap T^k B_\rho(x) \neq \emptyset\} \subset \{x \in M : B_{(2A^k+2)\rho}(x) \cap T^k (T^k B_{(2A^k+2)\rho}(x)) \neq \emptyset\}.
\]
The general case is shown similarly. \hfill $\Box$

The lemma thus shows that $\mathcal{N}_\rho(n) \subset \mathcal{N}_{sp}(2^p n)$ and consequently we only need to estimate $\mu(\mathcal{N}_{sp}(2^p n))$.

Let us now consider the case $1 \leq n \leq \lfloor bJ \rfloor$ and let as in Lemma 1 $s_p = 2^p \frac{A^{n^p-1}}{A^n-1}$. Hence by Lemma 1 one has $\mathcal{N}_\rho(n) \subset \mathcal{N}_{sp}(2^p n)$. for any $p \geq 1$, and in particular for $p(n) = \lfloor \log bJ - \log n \rfloor + 1$. Therefore
\[
\bigcup_{n=1}^{\lfloor bJ \rfloor} \mathcal{N}_\rho(n) \subset \bigcup_{n=1}^{\lfloor bJ \rfloor} \mathcal{N}_{sp(n)}(2^{p(n)} n).
\]
Now define
\[ n' = n2^p(n) \quad \text{and} \quad \rho' = s_{p(n)}\rho. \]
A direct computation shows that \( 1 \leq n \leq \lceil bJ \rceil \) implies \( \lceil bJ \rceil \leq n' \leq 2bJ \) and so
\[
\mathcal{V}_{\rho}^1 = \bigcup_{n=1}^{\lceil bJ \rceil} \mathcal{N}_\rho(n) \subset \bigcup_{n=1}^{\lceil bJ \rceil} \mathcal{N}_{s_{p(n)}\rho}(2^p(n)n) \subset \bigcup_{n'=\lceil bJ \rceil} \mathcal{N}_{\rho'}(n').
\]
Therefore to estimate the measure of \( \mathcal{V}_{\rho}^1 \) it suffices to find a bound for \( \mathcal{N}_{\rho'}(n') \) when \( n' \geq bJ \). This is accomplished by using an argument analogous to the first part of the proof. We replace all the \( n \) with \( n' \) and \( \rho \) with \( \rho' \). We get for \( b < 1/3 \)
\[
\mu(\mathcal{N}_{\rho'}(n')) \leq c_5\xi(n')(e^{-u_0|\log \rho'|^1/2} + \delta(n')^{u_0})
\]
and thus obtain an estimate similar to (7):
\[
\mu(\mathcal{V}_{\rho}^1) \leq \sum_{n'=\lceil bJ \rceil}^{2bJ} \mu(\mathcal{N}_{\rho'}(n')) \leq c_8(e^{-c|\log \rho'|^1/2} + (ab|\log \rho|)^{-\sigma}).
\]
for some \( c \in (0, u_0) \).

(III) Final estimate
Overall we obtain for all \( \rho \) sufficiently small
\[
\mu(\mathcal{V}_\rho) \leq \mu(\mathcal{V}_{\rho}^1) + \mu(\mathcal{V}_{\rho}^2) \leq c_9(e^{-c|\log \rho'|^1/2} + (ab|\log \rho|)^{-\sigma}) \leq C_2|\log \rho|^{-\sigma},
\]
for some \( C_2 \).

5. Poisson Approximation Theorem

This section contains the abstract Poisson approximation theorem which establishes the distance between sums of \{0,1\}-valued dependent random variables \( X_n \) and a random variable that is Poisson distributed. It is used in Section 3.1 in the proof of Theorem 1 and compares the number of occurrences in a finite time interval with the number of occurrences in the same interval for a Bernoulli process \( \{\tilde{X}_n : n \} \).

**Theorem 3.** Let \( (X_n)_{n \in \mathbb{N}} \) be a stationary \{0,1\}-valued process and \( t \) a positive parameter. Let \( S_a^b = \sum_{n=a}^{b} X_n \) and define \( S := S_1^N \) for convenience’s sake where \( N = \lfloor t/\epsilon \rfloor \) and \( \epsilon = \mathbb{P}(X_1 = 1) \). Additionally, let \( \nu \) be the Poisson distribution measure with mean \( t > 0 \). Finally, assume that \( \epsilon < \frac{t}{2} \). Then there exists a constant \( C_3 \) such that for any \( E \subset \mathbb{N}_0 \), and \( 2 \leq \Delta < N \) we have
\[
|\mathbb{P}(S \in E) - \nu(E) | \leq C_3\#\{E \cap [0,N]\} (N(\mathcal{R}_1 + \mathcal{R}_2) + \Delta \epsilon)
\]
where,
\[
\mathcal{R}_1 = \sup_{0 < j < N - \Delta} \{ \mathbb{P}(X_1 = 1 \land S_{\Delta+1}^{N-j} = q) - \epsilon \mathbb{P}(S_{\Delta+1}^{N-j} = q) \}
\]
\[
\mathcal{R}_2 = \sum_{n=2}^{\Delta} \mathbb{P}(X_1 = 1 \land X_n = 1).
\]
Proof. Let \((\tilde{X}_n)_{n \in \mathbb{N}}\) be a sequence of independent, identically distributed random variables taking values in \(\{0, 1\}\), constructed so that \(\mathbb{P}(\tilde{X}_1 = 1) = \epsilon\). Further assume that the \(\tilde{X}_n\)'s are independent of the \(X_n\)'s. Let \(S = \sum_{n=1}^{N} \tilde{X}_n\). Then

\[
|\mathbb{P}(S \in E) - \nu(E)| \leq |\mathbb{P}(S \in E) - \mathbb{P}(\tilde{S} \in E)| + |\mathbb{P}(\tilde{S} \in E) - \nu(E)|
\]

Thanks to [3] we can bound the second sum using the estimate

\[
(8) \quad \sum_{k=0}^{\infty} \left| \mathbb{P}(\tilde{S} = k) - \frac{t^k}{k!} e^{-t} \right| \leq \frac{2t^2}{N}.
\]

For summands of the remaining term we utilize the proof of Theorem 2.1 from [6] according to which for every \(k \leq N\),

\[
|\mathbb{P}(S = k) - \mathbb{P}(\tilde{S} = k)| \leq 2N(\mathcal{R}_1 + \mathcal{R}_2 + \Delta \epsilon^2) + 4\Delta \epsilon.
\]

As \(N \leq t/\epsilon\) this becomes

\[
(9) \quad |\mathbb{P}(S = k) - \mathbb{P}(\tilde{S} = k)| \leq 6t (N(\mathcal{R}_1 + \mathcal{R}_2) + \Delta \epsilon).
\]

Combining (8) and (9) yields

\[
|\mathbb{P}(S \in E) - \nu(E)| \leq \sum_{k \in \mathbb{E} \cap [0, N]} |\mathbb{P}(S = k) - \mathbb{P}(\tilde{S} = k)| + \frac{2t^2}{N}
\]

\[
\leq \sum_{k \in \mathbb{E} \cap [0, N]} 6t (N(\mathcal{R}_1 + \mathcal{R}_2) + \Delta \epsilon) + \frac{2t^2}{t/\epsilon - 1}
\]

\[
\leq 6t \#\{E \cap [0, N]\} (N(\mathcal{R}_1 + \mathcal{R}_2) + \Delta \epsilon) + 4t \epsilon
\]

\[
\leq C_3 \#\{E \cap [0, N]\} (N(\mathcal{R}_1 + \mathcal{R}_2) + \Delta \epsilon)
\]

for some \(C_3 < \infty\). \(\square\)

6. Proof of Theorem [2]

In [17] Leadbetter et al gave two conditions called \(D\) and \(D'\), under which \(a_n(M_n - b_n) \to G\) is equivalent to \(a_n(\tilde{M}_n - b_n) \to G\). Recall that \(\tilde{M}_n\) is the maxima of the independent, stationary process \(\{\tilde{X}_n\}\). Later Freitas et al [9] replaced condition \(D\) by \(D_2\) and obtained the same result. To state the conditions we put \(u_n = v/a_n + b_n\) for \(v \in \mathbb{R}\) and sequences \(a_n, b_n\).

**Condition.** \(D_2(u_n)\) [9] We say condition \(D_2(u_n)\) holds if for any integers \(l, t\) and \(n\)

\[
|\mu(X_0 > u_n, M_{l,t} < u_n) - \mu(X_0 > u_n)\mu(M_t < u_n)| \leq \gamma(n, t)
\]

where \(\gamma(n, t)\) is a non-increasing sequence in \(t\) for every \(n\) and satisfies \(\gamma(n, t_n) = o\left(\frac{1}{n}\right)\) for some sequence \(t_n = o(n), t_n \to \infty\).
Condition. $D'(u_n)$ \[9\] We say condition $D'(u_n)$ holds if

$$\lim_{k \to \infty} \limsup_n n \cdot \sum_{j=1}^{\lfloor n/k \rfloor} \mu(X_0 > u_n, X_j > u_n) = 0$$

Below we will verify both conditions for Type I observable, i.e. $g$ with $g^{-1}\left(g\left(\frac{1}{n}\right) + y\phi\left(g\left(\frac{1}{n}\right)\right)\right) = (1 + \varepsilon_n) e^{-u_n}$. The other two cases follow similarly.

6.1. Condition $D_2(u_n)$. First we show $D_2(u_n)$. Put $u_n(y) = g\left(\frac{1}{n}\right) + y\phi\left(g\left(\frac{1}{n}\right)\right)$, then

$$\{X_0 > u_n\} = B_t(g^{-1}(u_n))(z);$$

here $l(y) = \inf\{r > 0 : \mu(B_r(z)) \geq y\}$. Since $\mu(B_l(y))(z) = y$, by Assumption (II) we get

$$C y^{1/d_0} \leq l(y) \leq C' y^{1/d_1}$$

for some constant $C$ and $C'$. In particular we have

$$\sum_{j=0}^{\infty} \sum_{t=1}^{\infty} \int_B \left| 1_{B_t} 1_{\{M_t > u_n\}} d\mu - \left( \int 1_B d\mu \right) \left( \int 1_{M_t > u_n} d\mu \right) \right| \leq 2 \left( \frac{\lambda t/2}{\delta r_n} + \mu(B_{r_n + \delta r_n} \setminus B_{r_n - \delta r_n}) \right) + O\left(r_n^{v(\kappa - 1 - \beta)}\right)$$

Putting $\delta r_n = \frac{r_n^w}{n}$, $w > 1$, and $t = n^\nu$ with $0 < \nu < 1$ gives

$$\gamma(n, t) = \mu(Y_0 > u_n, M_t < u_n) - \mu(Y_0 > u_n) \mu(M_t < u_n)$$

$$\leq O(1) \left(n^{-vp + w/d_1} + n^{-\nu(\kappa - 1 - \beta)/d_1}\right) + O\left(r_n^{v(\kappa - 1 - \beta)}\right)$$

as $\mu(Y_0 > u_n) = \mu(B_{r_n}) = O(1/n)$. In order that $n\gamma(n, t) \to 0$ we need $1 - vp + w/d_1 < 0$, $-(\nu - \beta)/d_1 < 0$ and $v(\kappa - 1 - \beta) > 0$. We choose $w > \beta/\nu$ and $\nu$ close to 1. In the case when $\lambda$ decays polynomially with power $p$, $1 - vp + w/d_1 < 0$ is satisfied if $p > \frac{\beta}{\nu d_1} + 1$.

6.2. Condition $D'(u_n)$. Notice that

$$\sum_{j=1}^{\lfloor n/k \rfloor} \mu(X_0 > u_n, X_j > u_n) = \sum_{j=1}^{\lfloor n/k \rfloor} \mu(B_{r_n} \cap T^{-j}B_{r_n}).$$

This is exactly $R_2$ in Section 3.1 with $\Delta = \lfloor n/k \rfloor$. We split the sum into two parts as follows:

$$\sum_{j=1}^{\lfloor n/k \rfloor} \mu(B_{r_n} \cap T^{-j}B_{r_n}) = \sum_{j=1}^{\lfloor n/k \rfloor} \mu(B_{r_n} \cap T^{-j}B_{r_n}) + \sum_{j=J}^{\lfloor n/k \rfloor} \mu(B_{r_n} \cap T^{-j}B_{r_n})$$
with \( J = \lfloor a \log r_n \rfloor \). By (5) we get
\[
\sum_{j=J}^{\Delta} \mu(B_{r_n} \cap T^{-j} B_{r_n}) \leq c_5 \sum_{j=J}^{\Delta-1} \omega(j) \delta(j) u_0 \mu(B_{r_n}),
\]
where \( \mu(B_{r_n}) = \mathcal{O}(\frac{1}{n^s}) \). For the second term we restrict to points \( z \not\in V_{r_n} \) which implies that \( B_{r_n} \cap T^{-j} B_{r_n} = \emptyset \) for \( j = 1, \ldots, J - 1 \), where by Section 4
\[
\mu(V_{r_n}) \leq C_2 |\log r_n|^{-\sigma},
\]
with \( \sigma = \kappa u_0 - \kappa' - 1 > 0 \).

To finish the proof we use the maximal function technique by Collet in [5]. For this purpose we fix some \( 0 < \xi < \theta < 1 \) and define the set
\[
F_k = \left\{ \mu(B_{r_n}^{\exp(\xi)} \cap V_{r_n}^{\exp(\xi)}) \geq \mu(B_{r_n}^{\exp(\xi)}) \cdot k^{-\theta} \right\}
\]
and
\[
M_r(x) = \sup_{s>0} \frac{1}{\mu(B_s(x))} \int_{B_s(x)} 1_{V_r}(y) d\mu(y)
\]
where \( V_r = \{ x \in M : B_r \cap T_n B_r \neq \emptyset \text{ for some } 1 \leq n < J \} \) as before. Since
\[
F_k \subset \left\{ M_{r_n}^{\exp(\xi)} \geq k^{-\theta} \right\},
\]
we conclude that
\[
\mu(F_k) \leq \mu(M_{r_n}^{\exp(\xi)} \geq k^{-\theta}) \leq \frac{\mu(1_{V_r}^{\exp(\xi)}(y))}{k^{-\theta}} \leq k^{-((\xi - \theta) \sigma)}.
\]
If \( \xi - \theta > 1 \) we get \( \sum_k \mu(F_k) < \infty \) and thus by Borel-Cantelli there exist \( N(x) \) for almost every \( x \) such that \( x \not\in F_k \) for all \( k > N(x) \). For every \( n \), we choose \( k \) such that
\[
\exp(k^\xi) \leq n < \exp((k+1)^\xi),
\]
we have \( r_n^{\exp((k+1)^\xi)} \leq r_n \leq r_n^{\exp(k^\xi)} \). As a result
\[
B_{r_n} \cap T^{-j} B_{r_n} \subset B_{r_n}^{\exp(k^\xi)} \cap T^{-j} B_{r_n}^{\exp(k^\xi)} \subset B_{r_n}^{\exp(k^\xi)} \cap V_{r_n}^{\exp(k^\xi)}
\]
for every \( j < J \). Therefore
\[
n \cdot \sum_{j=1}^{J} \mu(B_{r_n} \cap T^{-j} B_{r_n}) \leq n \cdot \sum_{j=1}^{J} \mu(B_{r_n}^{\exp(k^\xi)} \cap V_{r_n}^{\exp(k^\xi)}) \leq C \exp((k+1)^\xi) \cdot (k+1)^\xi \mu(B_{r_n}^{\exp(k^\xi)}) \cdot k^{-\theta} \leq C \frac{\exp((k+1)^\xi)}{\exp(k^\xi)} \cdot k^\xi \cdot k^{-\theta} \to 0
\]
since \( \xi - \theta < 0 \). If \( \sigma = \kappa u_0 - \kappa' - 1 > 2 \) then we can choose \( 0 < \xi < \theta < 1 \) both close to 1 so that \( \sigma \xi - \theta > 1 \).
7. Example

As an example we consider the Manville-Pommeau map on the unit interval. It is given by

\[ Tx = \begin{cases} 
  x + 2^{1+\alpha} x^{1+\alpha} & \text{for } 0 \leq x \leq \frac{1}{2} \\
  2x - 1 & \text{for } \frac{1}{2} < x \leq 1
\end{cases} \]

where \( \alpha \in (0, 1) \) is a parameter. In this case \( T \) has an absolutely continuous invariant measure \( \mu \) whose density is \( h(x) \sim x^{-\alpha} \). The return times distribution has previously been shown to be Poissonian in [15]. Also, an inducing argument was used in [4] to show that the transfer operator converges at a polynomial rate and thus that the decay of correlations (as in Assumption (I)) is polynomial.

There is a sequence of points \( a_n, n = 0, 1, \ldots \) which decreases to 0 so that \( T_0 = \frac{1}{2} \) and \( T a_{n+1} = a_n \) for all \( n \). If we put \( I_n = (a_{n+1}, a_n) \), then all the intervals \( I_n \) are pairwise disjoint and satisfy \( TI_{n+1} = I_n \) for all \( n \) and \( \bigcup_n I_n = (0, \frac{1}{2}) \). Moreover \( a_n \sim n^{-\gamma} \), where \( \gamma = \frac{1}{\alpha} > 1 \). Since \( h(x) \sim x^{-\alpha} \) one has \( \mu(I_n) \sim n^{-\gamma} \) and \( \mu(J_n) = n^{-1-\gamma} \) where \( J_n = \bigcup_{j=n}^{\infty} I_j \) is a punctured neighbourhood of 0.

The two elements \( (0, \frac{1}{2}) \) and \( (\frac{1}{2}, 1] \) cover the entire unit interval and denote by \( \mathcal{I}_n \) the inverse branches of \( T^n \).

Put \( A_k = (a_{k+1}, a_0) = \bigcup_{j=0}^k I_k \) and let \( \mathcal{I}_n \) be the inverse branches of \( T^n \). If \( \tilde{\zeta}_\varphi \) is an \( n \)-cylinder, that is a preimage of either \( A_k \) or \( (\frac{1}{2}, 1] \) under the inverse branch \( \varphi \in \mathcal{I}_n \) then the distortion of \( DT^n \) on \( \tilde{\zeta}_\varphi \) is bounded by \( c_1 (\frac{1}{k} + \frac{1}{n})^{-\gamma(\gamma+1)} \) for some constant \( c_1 \). In particular if we choose \( k = n^\theta \) for some \( \theta \in [0, 1) \) then we can put \( \omega(n) = c_2 n^{\theta \gamma(\gamma+1)} \) for some \( c_2 \).

We now can nearly use the theorem for higher order returns, but let us remark that if \( x \in (0, 1) \) then for \( n \) large enough we have that \( x \in A_n \cup (\frac{1}{2}, 1] \). If we proceed as in the estimate of the term \( R_2 \) we obtain

\[ T^{-j} B_\rho \cap B_\rho \subset \bigcup_{\zeta : \zeta \cap B_\rho \neq \emptyset} \zeta = \mathcal{P}_1 \cup \mathcal{P}_2 \]

where the union is over \( j \)-cylinders \( \zeta \) and

\[ \mathcal{P}_1 = \bigcup_{\zeta : \zeta \cap B_\rho \neq \emptyset} T^{-j} B_\rho \cap \tilde{\zeta}, \quad \mathcal{P}_2 = \bigcup_{\zeta : \zeta \cap B_\rho \neq \emptyset} T^{-j} B_\rho \cap \zeta \setminus \tilde{\zeta}. \]

The first set is estimated as before in the main theorem. For the second term notice that

\[ \mathcal{P}_2 = \bigcup_{A \in \mathcal{A}, \varphi \in \mathcal{I}_j, \varphi(A) \cap B_\rho \neq \emptyset} T^{-j} B_\rho \cap \varphi(A) = \bigcup_{A \in \mathcal{A}, \varphi \in \mathcal{I}_j, \varphi(A_0) \cap B_\rho \neq \emptyset} T^{-j} B_\rho \cap \varphi(J_n^\rho) \]

where \( \mathcal{A} = \{ (0, \frac{1}{2}), (\frac{1}{2}, 1] \} \) and \( A = A^\rho \) if \( A = (0, \frac{1}{2}) \) and otherwise \( \hat{A} = A \). Hence

\[ \mathcal{P}_2 = \bigcup_{\varphi \in \mathcal{I}_j, \varphi(A_0) \cap B_\rho \neq \emptyset} \varphi(B_\rho \cap J_n^\rho) \]
which is empty for $n$ large enough, i.e. so that $a_n^\theta < x$.

By [16] Proposition 5.2 one has that the correlations decay polynomially at the rate of $\gamma - 1$, that is $\lambda(k) = c_3 k^{1-\gamma}$ ($p = \gamma - 1$) for some $c_3$. The dimensions here are $d_0 = d_1 = 1$ and the annulus condition is satisfied with $\eta = 1$ and $\beta = 0$. Similarly, $u_0 = 1$. In order to get the contraction rate consider the ‘worst’ case for the contraction, when the partition element $(0, \frac{1}{\gamma}]$ is $n$ times mapped by the inverse branch that contains the parabolic branch. Its image is then $(0, a_n)$ and therefore $\delta(n) = a_n \sim n^{-\gamma}$. Hence $\kappa = \gamma$. Since $\gamma > 1$ the conditions of the theorem are satisfied since we can choose $\theta > 0$ arbitrarily close to $0$.

For any $\sigma < \gamma$ one can choose $\theta > 0$ so that $\sigma \leq \gamma - \theta \gamma (\gamma - 1)$ and therefore we obtain the following result.

**Corollary.** Let $T$ be the Manneville-Pommeau map for the parameter $\alpha \in (0, 1)$. Let $\mu$ be the invariant absolutely continuous probability measure. Then for any $\sigma < \gamma = \frac{1}{\alpha}$ one has

$$
P(\xi_{\rho, x} = r) = e^{-\tau r} \frac{r^\sigma}{r!} + O(|\log \rho|^{-\sigma})$$

for all $x \notin V_\rho(a)$ for some positive $a$. Moreover $\mu(V_\rho(a)) = O(|\log \rho|^{-\sigma})$.

**References**

[1] M Abadi: Poisson approximations via Chen-Stein for non-Markov processes; *In and Out of Equilibrium 2* V Sidoravicius and M E Vares (editors), 2008, pp1–19.
[2] M Abadi and N Vergne: Sharp errors for point-wise Poisson approximations in mixing processes; *Nonlinearity* 21 (2008), 2871–2885.
[3] R. Arratia, L. Goldstein and L. Gordon: Two moments suffice for Poisson approximations: the Chen-Stein method; *Ann. Probab.* 17(1) (1989), 9–25.
[4] H. Bruin, B. Saussol, S. Troubetzkoy and S. Vaienti. Return time statistics via inducing, 2003, *Ergodic Theory Dynam. Systems* 23: 991–1013.
[5] P Collet: Statistics of closest return for some non-uniformly hyperbolic systems; 2001, *Ergod. Th. & Dynam. Sys.* 21 (2001), 410–420.
[6] J-R Chazottes and P Collet: Poisson approximation for the number of visits to balls in nonuniformly hyperbolic dynamical systems; *Ergod. Th. & Dynam. Sys.* 33 (2013), 49–80.
[7] M Denker: Remarks on weak limit laws for fractal sets; *Progress in Probability* Vol. 37, Birkhäuser 1995, 167–178.
[8] W Doeblin: Remarques sur la théorie métrique des fraction continues; *Compositio Mathematica* 7 (1940), 353–371.
[9] A C M Freitas and J M Freitas: On the link between dependence and independence in extreme value theory for dynamical systems, *Stat. Probab. Lett.* 78 (2008), 1088–1093.
[10] N T A Haydn: Statistical properties of equilibrium states for rational maps; *Ergod. Th. & Dynam. Sys.* 20 (2000), 1371–1390.
[11] N T A Haydn: Entry and return times distribution; *Dynamical Systems: An International Journal* 28(3) (2013), 333–353.
[12] N T A Haydn and F Yang: Entry times distribution for mixing systems; preprint available at [http://arxiv.org/abs/1412.4197](http://arxiv.org/abs/1412.4197)
[13] N T A Haydn and Y Psiloyenis: Return times distribution for Markov towers with decay of correlations, 2010, preprint available at [http://arxiv.org/abs/1010.1029](http://arxiv.org/abs/1010.1029)
[14] N T A Haydn and K Wasilewska: Limiting distribution and error terms for the number of visits to balls in non-uniformly hyperbolic dynamical systems; preprint available at [http://arxiv.org/abs/1402.2990](http://arxiv.org/abs/1402.2990)
[15] M Hirata, B Saussol and S Vaienti: Statistics of return times: a general framework and new applications; *Comm. Math. Phys.* 206 (1999), 33–55.
[16] H Hu: Decay of correlations for piecewise smooth maps with indifferent fixed points; Ergod. Theory & Dynam. Sys. 24 (2004), 495–524.
Israel J. Math. 132 (2002), 253–263.
[17] M R Leadbetter, G Lindgren and H Rootzén: Extremes and Related Properties of Random Sequences and Processes, Springer–Verlag, 1980.
[18] F Pène and B Saussol: Poisson law for some nonuniformly hyperbolic dynamical systems with polynomial rate of mixing; preprint Université de Bretagne Occidentale
[19] B Pitskel: Poisson law for Markov chains; Ergod. Th. & Dynam. Syst. 11 (1991), 501–513.
vol. 3, Gauthiers-Villars, Paris 1899.