DEPENDENCE RESULT OF THE WEAK SOLUTION OF ROBIN BOUNDARY VALUE PROBLEMS

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Abstract. In this article we establish an approximation result involving the Laplacian with Robin boundary conditions. It informs about the weak solution’s dependence from the input function on the boundary.

1. Introduction

Let $\Omega$ be a bounded domain with Lipschitz boundary. We consider the problem of the Laplacian with Robin boundary conditions,

$$\frac{\partial u}{\partial \nu} + \beta u = 0$$ (1.1)

where $\nu$ is the outward normal vector and $\beta$ is a measurable positive bounded function on the boundary $\partial \Omega$. This kind of problems was extensively studied by many authors, we refer to [1], [2], [3], [7], [5] and references therein for more details.

The aim of this article is to show a dependence result of a sequence of weak solutions $(u_n)_{n \geq 0}$ with a sequence of input functions $(\beta_n)_{n \geq 0}$. The proof is based on a technical Lemma due to Stampaccia [6].

2. Preliminaries and main result

We assume that $\Omega \subset \mathbb{R}^d$ ($d \geq 3$) is a bounded domain with Lipschitz boundary. We denote by $\sigma$ the restriction to $\partial \Omega$ of the $(d-1)$-dimensional Hausdorff measure.

We know that the following continuous embedding holds,

$$H^1(\Omega) \rightarrow L^q(\Omega), \quad q = \frac{2d}{d-2}$$ (2.1)

Moreover each function $u \in H^1(\Omega)$ has a trace which is in $L^s(\partial \Omega)$, where $s = \frac{2(d-1)}{d-2}$; i.e. there is a constant $c > 0$ such that

$$\|u\|_{s, \partial \Omega} \leq c \|u\|_{H^1(\Omega)} \text{ for all } u \in H^1(\Omega)$$ (2.2)

Let $\lambda > 0$ be a real number, $f \in L^p(\Omega) \ (p > d)$ and $\beta$ be a nonnegative bounded measurable function on $\partial \Omega$. We consider the following Robin boundary value problem

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\[
\begin{aligned}
&\begin{cases}
-\Delta u + \lambda u = f \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{in } \partial \Omega
\end{cases} \\
\end{aligned}
\]

(2.3)

The form associated with the Laplacian with Robin boundary condition is

\[ a_\beta(u, v) = \int_{\Omega} \nabla u \nabla v \, dx + \int_{\partial \Omega} \beta uv \, d\sigma \quad \text{for all } u, v \in H^1(\Omega) \]

We start by the definition of the weak solution of the problem (2.3).

**Definition 2.1.** Let \( f \in L^p(\Omega) \). For each \( \lambda > 0 \), a function \( u = G_\lambda(\beta) \in H^1(\Omega) \) is called a weak solution of the Robin boundary value Problem (associated with \( \beta \)) if for every \( v \in H^1(\Omega) \)

\[ a_\lambda(\beta)(u, v) = \int_{\Omega} fv \, dx, \]

where for \( u, v \in H^1(\Omega) \)

\[ a_\lambda(\beta)(u, v) = a_\beta(u, v) + \lambda \int_{\Omega} uv \, dx \]

It is clear that the closed bilinear form \( a_\beta \) is continous on \( H^1(\Omega) \) and also coercive on \( H^1(\Omega) \) in the sens that there exists a constant \( c > 0 \) such that for all \( u \in H^1(\Omega) \)

\[ a_\lambda(\beta)(u, u) \geq \|u\|_{H^1(\Omega)}^2 \]

Let \( L \) be the linear functional on \( H^1(\Omega) \) defined by : for \( v \in H^1(\Omega) \)

\[ Lv := \int_{\Omega} fv \, dx \]

Since \( p \geq 2 \), the functional \( L \) is well defined and continous on \( H^1(\Omega) \). Thus by coerciveness of the bilinear form \( a_\beta \), the Lax-Milgram Lemma (see [4, Corollaire V.8 p:84]) implies that there exists a unique weak solution \( u \in H^1(\Omega) \) of the boundary value problem (2.3).

The following lemma is important in the proof of Theorem 2.4, we can find its proof in [6] Lemma 4.1.

**Lemma 2.2.** Let \( \varphi = \varphi(t) \) be a nonnegative, nonincreasing function on the half line \( t \geq k_0 \geq 0 \) such that there are positive constants \( c, \alpha \) and \( \delta(\delta > 0) \) such that

\[ \varphi(h) \leq c(h - k)^{-\alpha} \varphi(k)^\delta \]

for all \( h > k \geq k_0 \). Then we have

\[ \varphi(k_0 + d) = 0, \quad \text{where } d > 0 \text{ satisfies } d^\alpha = c \varphi(k_0)^{\delta - 1} 2^{\delta - 1} \]

**Theorem 2.3.** Let \( u \) be a weak solution and assume that \( p > d \). Then

1) if \( \lambda = 0 \) and \( \Omega \) is of finite volume, there exists a strictly positive constants \( C_1 = C_1(d, p, |\Omega|) \) such that

\[ |u(x)| \leq C_1 \|f\|_p \quad \text{a.e on } \Omega \]
2) if $\lambda > 0$ and $\Omega$ is an arbitrary domain, there exist a strictly positive constant $C_2 = C_2(d, p, \lambda)$ such that

$$|G_\beta^\lambda f(x)| \leq C_2\|f\|_p \quad \text{a.e on } \overline{\Omega}$$

The proof can be found in [7] and is based on the Maza'ya inequality and a standard argument as in Theorem 4.1 of [6].

Our main result is the following Theorem,

**Theorem 2.4.** Any sequence $(u_n)_{n \geq 0}$ of weak solutions of the Robin boundary value problem associated to the sequence $(\beta_n)_{n \geq 0}$ verify the following inequality:

$$\|u_n - u_m\|_{\infty, \overline{\Omega}} \leq C\|u_n\|_{\infty, \partial \Omega}\|\beta_n - \beta_m\|_{\infty, \partial \Omega} \quad (2.4)$$

for all $n, m \in \mathbb{N}$ and where $C$ may depend of $\lambda$.

### 3. Proof of Theorem 2.4

**Proof.** Let $(u_n)_{n \geq 0}$ be a sequence of weak solutions associated with the sequence $(\beta_n)_{n \geq 0}$. Let $k \geq 0$ be a real number and define $u_{n,m} := u_n - u_m$.

Define $v_{n,m} := (|u_{n,m}| - k)^+ \text{sgn}(u_{n,m})$. Then $v_{n,m} \in H^1(\Omega)$ and

$$\nabla v_{n,m} = \begin{cases} \nabla u_{n,m} & \text{in } A_{n,m}(k); \\ 0 & \text{otherwise} \end{cases}$$

where $A_{n,m}(k) = \{x \in \overline{\Omega} : |u_{n,m}(x)| > k\}$. In the following, we write $u, v, A(k)$ instead of $u_{n,m}, v_{n,m}, A_{n,m}(k)$...

It is clear that $a_\beta^\lambda (u, v) - a_{\beta_n}^\lambda (u_m, v) = 0$. Calculating we obtain:

$$0 = \int_{\Omega} \nabla (u_n - u_m) \nabla v \, dx + \int_{\partial \Omega} (\beta_n u_n - \beta_m u_m) v \, d\sigma + \lambda \int_{\Omega} (u_n - u_m) v \, dx$$

$$= \int_{A(k)} |\nabla v|^2 \, dx + \int_{\partial \Omega \setminus A(k)} (\beta_n \beta_m) u_n \, v \, d\sigma + \beta_m(u_n - u_m) v \, d\sigma$$

$$= \int_{A(k)} |\nabla v|^2 \, dx + \int_{\partial \Omega \setminus A(k)} (\beta_n - \beta_m) u_n v \, d\sigma + \beta_m(u_n - u_m) v \, d\sigma$$

$$+ \lambda \int_{A(k)} (u_n - u_m) v \, dx$$

$$= \int_{A(k)} |\nabla v|^2 \, dx + \int_{\partial \Omega \setminus A(k)} (\beta_n - \beta_m) u_n v \, d\sigma + \int_{\partial \Omega \setminus A(k)} \beta_m v^2 \, d\sigma$$

$$+ k \int_{\partial \Omega \setminus A(k)} \beta_m |v| \, d\sigma + \lambda \int_{A(k)} v^2 \, dx + \lambda k \int_{A(k)} |v| \, dx$$

$$= a_{\beta_n}^\lambda (v, v) + \int_{\partial \Omega \setminus A(k)} (\beta_n - \beta_m) u_n v \, d\sigma + k \int_{\partial \Omega \setminus A(k)} \beta_m |v| \, d\sigma + \lambda k \int_{A(k)} |v| \, dx \quad (3.1)$$
It follows that
\[
a_h^\lambda (v, v) + \int_{\partial \Omega \setminus A(k)} (\beta_n - \beta_m) u_n v d\sigma = -k \int_{\partial A \setminus A(k)} \beta_m |v| d\sigma - \lambda k \int_{A(k)} |v| dx \\
\leq 0
\]  
\tag{3.2}
\]

Which leads to
\[
a_h^\lambda (v, v) \leq \int_{\partial \Omega \setminus A(k)} (\beta_n - \beta_m) u_n v d\sigma
\]

Using the Hölder inequality and (2.2), we obtain the following estimates,
\[
a_h^\lambda (v, v) \leq \int_{\partial \Omega \setminus A(k)} (\beta_n - \beta_m) u_n v d\sigma
\]
\[
\leq \|\beta_n - \beta_m\|_{\infty, \partial \Omega} \int_{\partial \Omega \setminus A(k)} u_n v d\sigma
\]
\[
\leq \|\beta_n - \beta_m\|_{\infty, \partial \Omega} \|u_n\|_{2, \partial \Omega \setminus A(k)} \|v\|_{2, \partial \Omega \setminus A(k)}
\]
\[
\leq \|\beta_n - \beta_m\|_{\infty, \partial \Omega} \|u_n\|_{\infty, \partial \Omega} \|\partial \Omega \cap A(k)\|^{\frac{1}{2}} \|\partial \Omega \cap A(k)\|^{\frac{1}{2}} \|v\|_{s, \partial \Omega}
\]
\[
\leq c \|\beta_n - \beta_m\|_{\infty, \partial \Omega} \|u_n\|_{\infty, \partial \Omega} \|\partial \Omega \cap A(k)\|^{1-\frac{1}{2}} \|v\|_{H^1(\Omega)}
\]

We have then,
\[
\alpha \|v\|^2_{H^1(\Omega)} \leq a_h^\lambda (v, v)
\]
\[
\leq c \|\beta_n - \beta_m\|_{\infty, \partial \Omega} \|u_n\|_{\infty, \partial \Omega} \|\partial \Omega \cap A(k)\|^{1-\frac{1}{2}} \|v\|_{H^1(\Omega)}
\]
\tag{3.4}
\]

It follows that
\[
\|v\|_{H^1(\Omega)} \leq c_1 \|\beta_n - \beta_m\|_{\infty, \partial \Omega} \|u_n\|_{\infty, \partial \Omega} \|\partial \Omega \cap A(k)\|^{1-\frac{1}{2}}
\]
\tag{3.5}
\]

Using the inequalities (2.1) and (2.2), we obtain the following estimates,
\[
\|v\|_{s, \partial \Omega \cap A(k)} \leq c_2 \|\beta_n - \beta_m\|_{\infty, \partial \Omega} \|u_n\|_{\infty, \partial \Omega} \|\partial \Omega \cap A(k)\|^{1-\frac{1}{2}}
\]
\tag{3.6}
\]

and,
\[
\|v\|_{q(A(k))} \leq c_3 \|\beta_n - \beta_m\|_{\infty, \partial \Omega} \|u_n\|_{\infty, \partial \Omega} \|\partial \Omega \cap A(k)\|^{1-\frac{1}{2}}
\]
\tag{3.7}
\]

Let now \(h > k \geq 0\). Then \(A(h) \subset A(k)\) and on \(A(h)\) we have \(|v| \geq h - k\). It follows that
\[
\|v\|_{s, \partial \Omega \cap A(h)} \geq \|v\|_{s, \partial \Omega \cap A(h)}
\]
\[
\geq \|u| - k\|_{s, \partial \Omega \cap A(h)}
\]
\[
\geq (h - k) \|\partial \Omega \cap A(h)\|^\frac{1}{2}
\]
\tag{3.8}
\]

We deduce from (3.6) that
\[
(h - k) \|\partial \Omega \cap A(h)\|^\frac{1}{2} \leq c_2 \|\beta_n - \beta_m\|_{\infty, \partial \Omega} \|u_n\|_{\infty, \partial \Omega} \|\partial \Omega \cap A(k)\|^{1-\frac{1}{2}}
\]

which reduces to,
\[
|\partial \Omega \cap A(h)| \leq c_2^2(h - k)^{-s} \|\beta_n - \beta_m\|_{\infty, \partial \Omega} \|u_n\|_{\infty, \partial \Omega} \|\partial \Omega \cap A(k)\|^{s-1}
Set \( \phi(h) = |\partial \Omega \cap A(h)| \), we obtain,
\[
\phi(h) \leq C(h - k)^{-s}\phi(k)^{s-1}
\]
where \( C = c_2^s\|\beta_n - \beta_m\|_{\infty, \partial \Omega}^s\|u_n\|_{\infty, \partial \Omega}^s \).

As \( s - 1 > 1 \), then the conditions of the Lemma 2.2 are satisfied with \( \delta = s - 1 \) and \( k_0 = 0 \), one obtain \( \phi(d) = 0 \) where \( d > 0 \) satisfies \( d^s = C\phi(0)^{s-2}(s-1)(s-2) \), consequently
\[
d = c_4\|\beta_n - \beta_m\|_{\infty, \partial \Omega}u_n\|_{\infty, \partial \Omega}
\]
and
\[
\|u_n - u_m\|_{\infty, \partial \Omega} \leq c_4\|u_n\|_{\infty, \partial \Omega}\|\beta_n - \beta_m\|_{\infty, \partial \Omega}
\]  
(3.9)

In the same way as in (3.8), we obtain
\[
\|v\|_{A(k)} \geq (h - k)|A(k)|^{\frac{1}{2}}
\]
From (3.7), we deduce
\[
(h - k)|A(h)|^{\frac{1}{2}} \leq c_3\|\beta_n - \beta_m\|_{\infty, \partial \Omega}u_n\|_{\infty, \partial \Omega}\|\partial \Omega \cap A(k)\|^{1-\frac{1}{2}}
\]
We take \( k = d \) and \( h = \gamma d \) with \( \gamma > 1 \), we obtain \( |A(\gamma d)| = 0 \) which leads to
\[
\|u_n - u_m\|_{\infty, \partial \Omega} \leq \gamma d
\]  
\[
\leq c_4\|u_n\|_{\infty, \partial \Omega}\|\beta_n - \beta_m\|_{\infty, \partial \Omega}
\]  
(3.10)

From (3.9) and (3.10) we obtain our Theorem.

**Corollary 3.1.** Let \( (u_n)_{n \geq 0} \) be a sequence weak solutions associated with the sequence \( (\beta_n)_{n \geq 0} \in L^\infty(\partial \Omega) \) such that \( \inf_n \beta_n > 0 \) then if \( (u_n)_{n \geq 0} \) is uniformly bounded we have for \( p > d \)
\[
\|u_n - u_m\|_{\infty, \partial \Omega} \leq C\|f\|_{p}\|\beta_n - \beta_m\|_{\infty, \partial \Omega}
\]  
(3.11)

for all \( n, m \in \mathbb{N} \) and where \( C \) may depend of \( \lambda \).

In the case where the sequence of weak solutions \( (u_n)_{n \geq 0} \) is uniformly bounded with respect to \( n \) we have the following consequence

**Corollary 3.2.** Let \( (u_n)_{n \geq 0} \) be a sequence weak solutions associated with the sequence \( (\beta_n)_{n \geq 0} \in L^\infty(\partial \Omega) \) such that \( \inf_n \beta_n > 0 \) and \( \lim_n \beta_n(x) = \beta(x) \) a.e \( x \in \partial \Omega \) then if \( (u_n)_{n \geq 0} \) is uniformly bounded we have \( \lim_n u_n(x) = u(x) \) a.e \( x \in \mathbb{N} \), where \( u \) is the weak solution associated with \( \beta \).

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