Remark on atomic decompositions
for the Hardy space $H^1$ in the rational Dunkl setting

by

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Abstract. Let $\Delta$ be the Dunkl Laplacian on $\mathbb{R}^N$ associated with a normalized root system $R$ and a multiplicity function $k(\alpha) \geq 0$. We say that a function $f$ belongs to the Hardy space $H^1_\Delta$ if the nontangential maximal function defined by $M_{H^1_\Delta} f(x) = \sup_{\|x-y\| < t} |\exp(t^2 \Delta) f(x)\|$ belongs to $L^1(w(x) \, dx)$, where $w(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k(\alpha)}$. We prove that $H^1_\Delta$ admits atomic decompositions into atoms in the sense of Coifman–Weiss on the space of homogeneous type $\mathbb{R}^N$ equipped with the Euclidean distance $\|x-y\|$ and the measure $w(x) \, dx$. To this end we improve estimates for the heat kernel of $e^{t \Delta}$.

1. Introduction. Let $\Delta$ be the Dunkl Laplacian on $\mathbb{R}^N$ associated with a reduced normalized root system $R$ and a multiplicity function $k(\alpha) \geq 0$. Let $dw(x) = w(x) \, dx$, where

\begin{equation}
(1.1) \quad w(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k(\alpha)},
\end{equation}

be the associated measure on $\mathbb{R}^N$. Let $H_t = e^{t \Delta}$ be the Dunkl heat semigroup. The operators $H_t$ form strongly continuous semigroups of linear contractions on $L^p(dw)$ for $1 \leq p < \infty$, which are self-adjoint operators on $L^2(dw)$. Moreover, the maximal operator $\sup_{t>0} |\exp(t \Delta) f(x)|$ is bounded from $L^1(dw)$ into $L^{1,\infty}(dw)$ (see [21] Theorems 6.1 and 6.2).

We say that an $L^1(dw)$-function $f$ belongs to the real Hardy space $H^1_\Delta$ if the nontangential maximal function

$$M_{H^1_\Delta} f(x) = \sup_{\|x-y\| < t} |\exp(t^2 \Delta) f(y)|$$

2010 Mathematics Subject Classification: Primary 42B30; Secondary 42B25, 35K08, 42B35, 42B37.

Key words and phrases: Dunkl operator, Hardy spaces, maximal operator, atomic decomposition.

Received 18 June 2018; revised 2 November 2018.

Published online 12 August 2019.

DOI: 10.4064/sm180618-25-11
belongs to $L^1(dw)$. The space $H^1_\Delta$ is a Banach space with the norm
\[ \|f\|_{H^1_{\max,H}} = \|M_H f\|_{L^1(dw)}. \]

In [2] characterizations of $H^1_\Delta$ by relevant Riesz transforms, Littlewood–Paley square functions, and atomic decompositions were proved. Let us recall the notions of atoms considered in [2]. For a positive integer $M$, let $\mathcal{D}(\Delta^M)$ denote the domain of $\Delta^M$ as an (unbounded) operator on $L^2(dw)$. Let $G$ be the Weyl group of the root system $R$. Set $\mathcal{O}(x) = \bigcup_{\sigma \in G} \{a(x)\}$. Similarly, if $B$ is a Euclidean ball then $\mathcal{O}(B) = \bigcup_{\sigma \in G} \sigma(B)$ is the $G$-orbit of $B$.

**Definition 1.1.** Let $1 < q \leq \infty$ and $M$ be a positive integer. A function $a(x)$ is said to be a $(1, q, \Delta, M)$-atom if $a \in L^2(dw)$ and there is $b \in \mathcal{D}(\Delta^M)$ and a Euclidean ball $B = B(y_0, r)$ such that

- $a = \Delta^M b$;
- $\text{supp} \Delta^\ell b \subset \mathcal{O}(B)$ for $\ell = 0, 1, \ldots, M$;
- $\|(r^2 \Delta)\ell b\|_{L^q(dw)} \leq r^{2M} w(B)^{1/q-1}$, $\ell = 0, 1, \ldots, M$.

**Definition 1.2.** A function $f$ is in $H^1_{(1,q,\Delta,M)}$ if there are $(1, q, \Delta, M)$-atoms $a_j$ and $\lambda_j \in \mathbb{C}$ such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. Then we set $\|f\|_{H^1_{(1,q,\Delta,M)}} = \inf \{ \sum_{j=1}^{\infty} |\lambda_j| \}$, where the infimum is taken over all representations of $f$ as above.

It was proved in [2] that the spaces $H^1_\Delta$ and $H^1_{(1,q,\Delta,M)}$ coincide and the corresponding norms are equivalent.

Let us note that the atoms considered in [2] (see Definition 1.1) are in the spirit of [10], which means that they are of the form $a = \Delta^M b$ for appropriate functions $b$. Our aim is to prove that the Hardy space $H^1_\Delta$ admits other atomic decompositions, namely into atoms in the sense of Coifman–Weiss [5] on the space of homogeneous type $(\mathbb{R}^N, \|x - y\|, dw)$.

**Definition 1.3.** Fix $1 < q \leq \infty$. A function $a(x)$ is a $(1, q)$-atom if there is a Euclidean ball $B$ such that

(A) $\text{supp} \ a \subset B$;
(B) $\|a\|_{L^q(dw)} \leq w(B)^{1/q-1}$;
(C) $\int a(x) \ dw(x) = 0$.

**Definition 1.4.** A function $f$ belongs to $H^1_{(1,q)}$ if there are $\lambda_j \in \mathbb{C}$ and $(1, q)$-atoms $a_j$ such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. Then
\[ \|f\|_{H^1_{(1,q)}} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \right\}, \]
where the infimum is taken over all representations of $f$ as above.

We are now in a position to state our main result.
Theorem 1.5. For every $1 < q \leq \infty$ the spaces $H^1_\Delta$ and $H^1_{(1,q)}$ coincide and the corresponding norms are equivalent, that is, there is a constant $C > 0$ such that

$$C^{-1} \|f\|_{H^1_{(1,q)}} \leq \|f\|_{H^1_{\text{max},H}} \leq C \|f\|_{H^1_{(1,q)}}. \tag{1.2}$$

Remark 1.6. Since every $(1,\infty)$-atom is a $(1,q)$-atom, it suffices to prove the first inequality in (1.2) for $q = \infty$.

In order to prove the theorem we first derive improvements of the estimates obtained in [2] of the heat kernel of the semigroup $e^{t\Delta}$, and consequently of other kernels associated with translations of radial functions. This is presented in Section 3. Then, in Section 4 we use a characterization of $H^1_\Delta$ by Littlewood–Paley square functions to obtain decomposition into $(1,2)$-atoms. Finally, $(1,\infty)$-atomic decomposition is achieved by a standard decomposition of $(1,2)$-atoms into $(1,\infty)$-atoms.

Let us remark that if $k \equiv 0$, then the Hardy space $H^1_\Delta$ coincides with the classical real Hardy space $H^1$ on the Euclidean space $\mathbb{R}^N$ studied originally by Stein and Weiss [20], Fefferman and Stein [9], and Coifman [3]. More information concerning the classical theory of $H^p$ spaces can be found in the book [19] and references therein.

2. Preliminaries. In this section we present basic facts concerning Dunkl operators. For details we refer the reader to [6], [11], [15], and [17].

We consider the Euclidean space $\mathbb{R}^N$ with the scalar product $\langle x,y \rangle = \sum_{j=1}^N x_j y_j$, $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N)$. For a nonzero vector $\alpha \in \mathbb{R}^N$ the reflection $\sigma_\alpha$ with respect to the hyperplane $\alpha^\perp$ orthogonal to $\alpha$ is given by

$$\sigma_\alpha(x) = x - 2\frac{\langle x, \alpha \rangle}{\|\alpha\|^2}\alpha. \tag{2.1}$$

In this paper we fix a normalized root system in $\mathbb{R}^N$, that is, a finite set $R \subset \mathbb{R}^N \setminus \{0\}$ such that $\sigma_\alpha(R) = R$ and $\|\alpha\| = \sqrt{2}$ for every $\alpha \in R$. The finite group $G$ generated by the reflections $\sigma_\alpha$, $\alpha \in R$, is called the Weyl group (reflection group) of the root system. A multiplicity function is a $G$-invariant function $k : R \to \mathbb{C}$ which will be fixed and $\geq 0$ throughout this paper.

Let

$$\gamma = \frac{1}{2} \sum_{\alpha \in R} k(\alpha) \quad \text{and} \quad N = 2\gamma + N. \tag{2.2}$$

The number $N$ is called the homogeneous dimension of the system, since

$$w(B(tx, tr)) = t^N w(B(x, r)) \quad \text{for} \quad x \in \mathbb{R}^N, t, r > 0,$$
where $B(x, r)$ denotes the Euclidean ball centered at $x$ with radius $r > 0$. Observe that
\begin{equation}
(2.3) \quad w(B(x, r)) \sim r^N \prod_{\alpha \in R} (|\langle x, \alpha \rangle| + r)^{k(\alpha)},
\end{equation}
so $dw(x)$ is doubling, that is, there is a constant $C > 0$ such that
\begin{equation}
(2.4) \quad w(B(x, 2r)) \leq Cw(B(x, r)) \quad \text{for } x \in \mathbb{R}^N, \ r > 0.
\end{equation}
Moreover, by (2.3),
\begin{equation}
(2.5) \quad C^{-1} \left( \frac{r_2}{r_1} \right)^N \leq w(B(x, r_2)) \leq C \left( \frac{r_2}{r_1} \right)^N \quad \text{for } 0 < r_1 < r_2.
\end{equation}

Given a (normalized) root system $R$ and a multiplicity function $k(\alpha)$, the Dunkl operator $T_\xi$ is the following $k$-deformation of the directional derivative $\partial_\xi$ by a difference operator:
\begin{equation}
T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_{\alpha}(x))}{\langle \alpha, x \rangle}.
\end{equation}
The Dunkl operators $T_\xi$ were introduced in [6]. They pairwise commute and are skew-symmetric in $L^2(\mathbb{R}^N, dw)$. Moreover, if $f, g \in C^1(\mathbb{R}^N)$ and at least one of them is $G$-invariant, then
\begin{equation}
(2.6) \quad T_\xi (fg) = (T_\xi f) \cdot g + f \cdot (T_\xi g).
\end{equation}
Let $e_j, j = 1, \ldots, N,$ denote the canonical orthonormal basis in $\mathbb{R}^N$ and let $T_j = T_{e_j}$.

**Dunkl kernel and Dunkl transform.** For fixed $y \in \mathbb{R}^N$ the Dunkl kernel $E(x, y)$ is a unique solution of the system
\begin{equation}
T_\xi f = \langle \xi, y \rangle f, \quad f(0) = 1.
\end{equation}
In particular
\begin{equation}
(2.7) \quad T_{j,x} E(x, y) = y_j E(x, y),
\end{equation}
where $T_{j,x}$ denotes the action of $T_j$ with respect to the variable $x$.

The function $E(x, y)$ was introduced in [7]. It generalizes the exponential function $e^{\langle x, y \rangle}$ and has a unique extension to a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$. We have
\begin{enumerate}
\item[(a)] $E(\lambda x, y) = E(x, \lambda y) = E(\lambda y, x) = E(\lambda \sigma(x), \sigma(y))$ for all $x, y \in \mathbb{C}^N$, $\sigma \in G, \lambda \in \mathbb{C}$;
\item[(b)] $E(x, y) > 0$ for all $x, y \in \mathbb{R}^N$;
\item[(c)] $|E(-ix, y)| \leq 1$ for all $x, y \in \mathbb{R}^N$;
\item[(d)] $E(0, y) = 1$ for all $y \in \mathbb{C}^N$.
\end{enumerate}
The proof of (a) can be found in [7]; the other properties are direct consequences of [13, Proposition 5.1]. More details concerning the Dunkl kernel \( E(x, y) \) can be found in the lecture notes [15], [17] and references therein.

The Dunkl transform, which generalizes the classical Fourier transform, is defined on \( L^1(dw) \) by (see [11], [17])
\[
\mathcal{F} f(\xi) = c_k^{-1} \int_{\mathbb{R}^N} f(x)E(x, -i\xi) \, dw(x),
\]
where
\[
c_k = \int_{\mathbb{R}^N} e^{-\|x\|^2/2} \, dw(x).
\]
The Dunkl transform is a topological automorphism of the Schwartz space \( \mathcal{S}(\mathbb{R}^N) \), has a unique extension to an isometric automorphism of \( L^2(dw) \) and satisfies the following inversion formula (see [11]): for every \( f \in L^1(dw) \) such that \( \mathcal{F} f \in L^1(dw) \), we have
\[
f(x) = \mathcal{F}^2 f(-x) \quad \text{for all } x \in \mathbb{R}^N.
\]
For \( \lambda > 0 \), we have \( \mathcal{F}(f_\lambda)(\xi) = \mathcal{F} f(\lambda \xi) \), where \( f_\lambda(x) = \lambda^{-N} f(\lambda^{-1} x) \).

**Dunkl translations and Dunkl convolution.** The Dunkl translation \( \tau_x f \) of a function \( f \in \mathcal{S}(\mathbb{R}^N) \) by \( x \in \mathbb{R}^N \) is defined by
\[
\tau_x f(y) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, x)E(i\xi, y) \mathcal{F} f(\xi) \, dw(\xi).
\]
We list some properties of Dunkl translations:
- each translation \( \tau_x \) is a continuous linear map of \( \mathcal{S}(\mathbb{R}^N) \) into itself, which extends to a contraction on \( L^2(dw) \);
- (identity) \( \tau_0 = I \);
- (symmetry) \( \tau_x f(y) = \tau_y f(x) \) for all \( x, y \in \mathbb{R}^N, f \in \mathcal{S}(\mathbb{R}^N) \);
- (scaling) \( \tau_x(f_\lambda) = (\tau_{\lambda^{-1} x}f)_\lambda \) for all \( \lambda > 0, x \in \mathbb{R}^N, f \in \mathcal{S}(\mathbb{R}^N) \);
- \( T_\xi(\tau_x f) = \tau_x(T_\xi f) \) for all \( x, \xi \in \mathbb{R}^N \);
- (skew-symmetry) for all \( x \in \mathbb{R}^N \) and \( f, g \in \mathcal{S}(\mathbb{R}^N) \) we have
\[
\int_{\mathbb{R}^N} \tau_x f(y) g(y) \, dw(y) = \int_{\mathbb{R}^N} f(y) \tau_{-x} g(y) \, dw(y).
\]
The latter formula allows us to define the Dunkl translations \( \tau_x f \) in the distributional sense for \( f \in L^p(dw) \) with \( 1 \leq p \leq \infty \). Further,
\[
\int_{\mathbb{R}^N} \tau_x f(y) \, dw(y) = \int_{\mathbb{R}^N} f(y) \, dw(y) \quad \text{for all } x \in \mathbb{R}^N, f \in \mathcal{S}(\mathbb{R}^N).
\]

The **Dunkl convolution** of two reasonable functions (for instance Schwartz functions) is defined by
\[
(f \ast g)(x) = c_k \mathcal{F}^{-1}[(\mathcal{F} f)(\mathcal{F} g)](x) = \int_{\mathbb{R}^N} (\mathcal{F} f)(\xi)(\mathcal{F} g)(\xi)E(x, i\xi) \, dw(\xi)
\]
for all \( x \in \mathbb{R}^N \), or equivalently by
\[
(f * g)(x) = \int_{\mathbb{R}^N} f(y) \tau_x g(-y) \, dw(y) = \int f(y)g(x, y) \, dw(y);
\]
here and subsequently,
\[
g(x, y) = \tau_x g(-y)
\]
for a reasonable function \( g(x) \) on \( \mathbb{R}^N \).

**Dunkl heat semigroup.** The *Dunkl Laplacian* associated with \( G \) and \( k \) is the differential-difference operator
\[
\Delta = \sum_{j=1}^{N} T_j^2.
\]
It acts on \( C^2(\mathbb{R}^N) \) functions by
\[
\Delta f(x) = \Delta_{\text{eucl}} f(x) + \sum_{\alpha \in \mathbb{R}^N} k(\alpha) \delta_{\alpha} f(x),
\]
\[
\delta_{\alpha} f(x) = \frac{\partial_{\alpha} f(x)}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma(\alpha))}{\langle \alpha, x \rangle^2}.
\]
The operator \( \Delta \) is essentially self-adjoint on \( L^2(dw) \) and generates the semigroup \( H_t = e^{t\Delta} \) of linear self-adjoint contractions on \( L^2(dw) \). The semigroup has the form
\[
e^{t\Delta} f(x) = \int_{\mathbb{R}^N} h_t(x, y) f(y) \, dw(y),
\]
where
\[
h_t(x, y) = c_k^{-1}(2t)^{-N/2} e^{-\frac{\|x\|^2 + \|y\|^2}{4t}} E\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right)
\]
(see [12] Section 4). The heat kernel \( h_t(x, y) \) is a \( C^\infty \) function of all variables \( x, y \in \mathbb{R}^N, t > 0 \) and satisfies
\[
0 < h_t(x, y) = h_t(y, x),
\]
\[
\int_{\mathbb{R}^N} h_t(x, y) \, dw(y) = 1.
\]
In particular (see [12]) for every \( t > 0 \) and all \( x, y \in \mathbb{R}^N \),
\[
h_t(x, y) = \tau_x h_t(-y), \quad \text{where}
\]
\[
h_t(x) = \tilde{h}_t(|x|) = c_k^{-1} (2t)^{-N/2} e^{-|x|^2/(4t)}.
\]

**Dunkl translations of radial functions.** The following specific formula was obtained by Rösler [14] for the Dunkl translations of (reasonable) radial functions \( f(x) = \tilde{f}(|x|) \):
\[
\tau_x f(-y) = \int_{\mathbb{R}^N} (\tilde{f} \circ A)(x, y, \eta) \, d\mu_x(\eta) \quad \text{for all } x, y \in \mathbb{R}^N.
\]
Here
\[ A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2\langle y, \eta \rangle} = \sqrt{\|x\|^2 - \|\eta\|^2 + \|y - \eta\|^2} \]
and \( \mu_x \) is a probability measure supported in \( \text{conv } O(x) \).

Let
\[ d(x, y) = \min_{\sigma \in G} \|\sigma(x) - y\| \]
denote the distance between the orbits \( O(x) \) and \( O(y) \). Since
(2.15) \[ A(x, y, \eta) \geq d(x, y) \quad \text{for } \eta \in \text{conv } O(x), \]
the formulas (2.13) and (2.14) imply (see e.g. [17])
(2.16) \[ h_t(x, y) \leq c_k^{-1} (2t)^{-N/2} e^{-d(x,y)^2/(4t)}. \]

For \( x, y \in \mathbb{R}^N \) and \( t > 0 \) we set
\[ V(x, y, t) = \max \{ w(B(x, t)), w(B(y, t)) \}. \]
It was proved in [2, Theorem 4.1] that the factor \( t^{N/2} \) in (2.16) can be replaced by \( V(x, y, \sqrt{t}) \), which gives the following estimates on the heat kernel in the spirit of analysis on spaces of homogeneous type. Let
\[ G_t(x, y) = \frac{1}{V(x, y, \sqrt{t})} \sum_{\sigma \in G} \exp \left( -\frac{\|x - \sigma(y)\|^2}{t} \right) \]
\[ \sim \frac{1}{V(x, y, \sqrt{t})} \exp \left( -\frac{d(x, y)^2}{t} \right). \]

There are constants \( C, c > 0 \) such that
(2.17) \[ h_t(x, y) \leq C G_{t/c}(x, y). \]
Note that \( V(x, y, t) \) and \( G_t(x, y) \) are \( G \)-invariant in \( x \) and \( y \). Below we list further inequalities for the kernel \( h_t \) proved in [2, Section 4]. For every nonnegative integer \( m \) and for any multi-indices \( \alpha, \beta \), there are constants \( C, c > 0 \) such
(2.18) \[ |\partial_t^m \partial_x^\alpha \partial_y^\beta h_t(x, y)| \leq C t^{-m-|\alpha|/2-|\beta|/2} h_{2t}(x, y). \]
Hence,
(2.19) \[ |\partial_t^m \partial_x^\alpha \partial_y^\beta h_t(x, y)| \leq C t^{-m-|\alpha|/2-|\beta|/2} G_{t/c}(x, y). \]
Moreover, if \( \|y - y'\| \leq \sqrt{t} \), then
(2.20) \[ |\partial_t^m h_t(x, y) - \partial_t^m h_t(x, y')| \leq C t^{-m} \frac{\|y - y'\|}{\sqrt{t}} G_{t/c}(x, y). \]

3. Estimates of the Dunkl heat kernel. The main goal of this section is to improve the estimates (2.19) (see Theorem [3.1]). Then, using Theorem [3.1] we deduce bounds for the Poisson kernel and for the Dunkl translations of radial compactly supported continuous functions.
Theorem 3.1. For every nonnegative integer $m$ and any multi-indices $\alpha, \beta$ there are constants $C_{m, \alpha, \beta}, c > 0$ such that

\begin{equation}
|\partial_t^m \partial_\alpha \partial_\beta h_t(x, y)| \leq C_{m, \alpha, \beta} t^{-m-|\alpha|/2-|\beta|/2} \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{-2} G_t/c(x, y).
\end{equation}

Moreover, if $\|y - y'\| \leq \sqrt{t}$, then

\begin{equation}
|\partial_t^m h_t(x, y) - \partial_t^m h_t(x, y')| \\
\leq C_m t^{-m} \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{-2} G_t/c(x, y).
\end{equation}

Remark 3.2. Observe that the estimates (2.19) and (3.1) differ by the factor $(1 + \|x - y\|/\sqrt{t})^{-2}$. We want to emphasize that the presence of this factor is crucial to the proof of the atomic decomposition stated in Theorem 1.5.

Lemma 3.3. For all $x, y \in \mathbb{R}^N$ and for any $t > 0$ we have

\begin{align}
T_{j, x} h_t(x, y) &= \frac{y_j - x_j}{2t} h_t(x, y), \\
T_{j, x}^2 h_t(x, y) &= \frac{(y_j - x_j)^2}{(2t)^2} h_t(x, y) - \frac{1}{2t} h_t(x, y) - \frac{1}{2t} \sum_{\alpha \in H} k(\alpha) \alpha_j^2 h_t(\sigma_\alpha(x), y).
\end{align}

Proof. The function $x \mapsto \exp(-\frac{\|x\|^2 + \|y\|^2}{4t})$ is $G$-invariant, so, by (2.6),

\begin{align}
c_k(2t)^{N/2} T_{j, x} h_t(x, y) &= \partial_{x_j} \left(\exp\left(-\frac{\|x\|^2 + \|y\|^2}{4t}\right)\right) E\left(x, \frac{y}{2t}\right) \\
&\quad + T_{j, x} \left(E\left(x, \frac{y}{2t}\right)\right) \exp\left(-\frac{\|x\|^2 + \|y\|^2}{4t}\right) \\
&= c_k(2t)^{N/2} \left(-\frac{x_j}{2t} h_t(x, y) + \frac{y_j}{2t} h_t(x, y)\right),
\end{align}

where in the last equality we have used (2.7). Thus (3.3) is established.

To prove (3.4) we utilize (3.3) and get

\begin{align}
T_{j, x}^2 h_t(x, y) &= T_{j, x} \left(\frac{y_j - x_j}{2t} h_t(x, y)\right) \\
&= \frac{y_j}{2t} T_{j, x} h_t(x, y) - T_{j, x} h_t(x, y) - S_j(x, y, t).
\end{align}
Let \( (\sigma_\alpha(x))_j \) denote the \( j \)th coordinate of \( \sigma_\alpha(x) \). Further,

\[
S_j(x, y, t) = \partial_{x_j} \left( \frac{x_j}{2t} h(t, x, y) \right) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \alpha_j \frac{x_j}{2t} h(t, x, y) - \frac{(\sigma_\alpha(x))_j}{2t} h(t, \sigma_\alpha(x), y) \\
= \frac{1}{2t} h(t, x, y) + \frac{x_j}{2t} \partial_{x_j} h(t, x, y) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \alpha_j \frac{x_j}{2t} h(t, \sigma_\alpha(x), y) - \frac{(\sigma_\alpha(x))_j}{2t} h(t, \sigma_\alpha(x), y) \\
+ \sum_{\alpha \in R} \frac{k(\alpha)}{2} \alpha_j \frac{x_j}{2t} h(t, \sigma_\alpha(x), y) - \frac{(\sigma_\alpha(x))_j}{2t} h(t, \sigma_\alpha(x), y).
\]

Note that \( x_j - (\sigma_\alpha(x))_j = \langle x, \alpha \rangle \alpha_j \). Therefore

\[
S_j(x, y, t) = \frac{1}{2t} h(t, x, y) + \frac{x_j}{2t} y_j - \frac{x_j}{2t} h(t, x, y) + \frac{1}{2t} \sum_{\alpha \in R} \frac{k(\alpha)}{2} \alpha_j^2 h(t, \sigma_\alpha(x), y).
\]

Finally,

\[
T_{j,x}^2 h(t, x, y) = \frac{y_j - x_j}{2t} h(t, x, y) - \frac{1}{2t} h(t, x, y) \\
- \frac{x_j}{2t} \frac{y_j - x_j}{2t} h(t, x, y) - \frac{1}{2t} \sum_{\alpha \in R} \frac{k(\alpha)}{2} \alpha_j^2 h(t, \sigma_\alpha(x), y).
\]

Proof of Theorem 3.1 Clearly, \( \Delta x h(t, x, y) = \partial_t h(t, x, y) \). Hence, summing (3.4) over \( j = 1, \ldots, N \), we obtain

\[
(3.5) \quad \partial_t h(t, x, y) = \frac{\|x - y\|^2}{(2t)^2} h(t, x, y) - \frac{N}{2t} h(t, x, y) - \frac{1}{2t} \sum_{\alpha \in R} k(\alpha) h(t, \sigma_\alpha(x), y).
\]

Applying (3.5) together with (2.18) we get

\[
(3.6) \quad \left(1 + \frac{\|x - y\|}{\sqrt{t}}\right)^2 h(t, x, y) \lesssim h_{2t}(x, y) + \sum_{\alpha \in R} k(\alpha) h(t, \sigma_\alpha(x), y).
\]

Using (3.6) and (2.17) we obtain

\[
(3.7) \quad h(t, x, y) \lesssim \left(1 + \frac{\|x - y\|}{\sqrt{t}}\right)^{-2} G_{t/c}(x, y),
\]

which completes the proof of (3.1) for \( m = 0, \alpha = \beta = 0 \). Now (3.1) in its general form is a direct consequence of (2.18) and (3.7).
The inequality (3.2) can be proved in a similar way. To this end we repeat
the arguments from the proof of (2.20) presented in [2, Theorem 4.1(b)] and
apply (at the very end) (3.7) to get the additional factor \((1 + \|x - y\|/\sqrt{t})^{-2}\).
We omit the details.

**Remark 3.4.** Let us note that iteration of the procedure presented in
the proof of Theorem 3.1 may lead to an improvement of (3.1). Indeed, if
we use (3.6) twice, then

\[
\begin{align*}
\eta_t(x, y) &\lesssim \left(1 + \frac{\|x - y\|}{\sqrt{t}}\right)^{-4} \left(\eta_{4t}(x, y) + \sum_{\alpha \in R} \eta_{2t}(\sigma_{\alpha}(x), y)\right) \\
&\quad + \left(1 + \frac{\|x - y\|}{\sqrt{t}}\right)^{-2} \sum_{\alpha \in R} k(\alpha) \left(1 + \frac{\|\sigma_{\alpha}(x) - y\|}{\sqrt{t}}\right)^{-2} \eta_{2t}(\sigma_{\alpha}(x), y) \\
&\quad + \left(1 + \frac{\|x - y\|}{\sqrt{t}}\right)^{-2} \sum_{\alpha \in R} k(\alpha) \left(1 + \frac{\|\sigma_{\alpha}(x) - y\|}{\sqrt{t}}\right)^{-2} \sum_{\beta \in R} \eta_{t}(\sigma_{\beta}(\sigma_{\alpha}(x)), y) \\
&\lesssim \left(1 + \frac{\|x - y\|}{\sqrt{t}}\right)^{-4} g_{t/c}(x, y) \\
&\quad + \left\{ \left(1 + \frac{\|x - y\|}{\sqrt{t}}\right)^{-2} \sum_{\alpha \in R} k(\alpha) \left(1 + \frac{\|\sigma_{\alpha}(x) - y\|}{\sqrt{t}}\right)^{-2} \right\} g_{t/c}(x, y).
\end{align*}
\]

In particular, for the root system \(A_2\), if \(x_1, \ldots, x_5\) and \(y\) are located as in
Figure 1 then

\[
\begin{align*}
\eta_t(x_j, y) &\lesssim \begin{cases} 
  w(B(y, \sqrt{t}))^{-1} \left(1 + \frac{\|x_j - y\|}{\sqrt{t}}\right)^{-2} & \text{for } j = 1, 2, 3, \\
  w(B(y, \sqrt{t}))^{-1} \left(1 + \frac{\|x_j - y\|}{\sqrt{t}}\right)^{-4} & \text{for } j = 4, 5.
\end{cases}
\end{align*}
\]

![Fig. 1. The points \(y, x_1, \ldots, x_5\) and the root system \(A_2\)]
Let us also elaborate on the product case \( A_1^N \), where \( R = \{ \pm \sqrt{2} e_j : j = 1, \ldots, N \} \) and \( e_j \) is the canonical orthonormal basis in \( \mathbb{R}^N \). If \( y = (1, \ldots, 1) \) and \( x = (\varepsilon_1, \ldots, \varepsilon_N) \), \( \varepsilon_j \in \{ -1, 1 \} \), then iteration of (3.6) leads to

\[
(3.8) \quad h_t(x, y) \lesssim w(B(y, \sqrt{t}))^{-1} \left( 1 + \left\| \frac{x - y}{\sqrt{t}} \right\| \right)^{-4\ell},
\]

where \( \ell = \# \{ j : \varepsilon_j = -1 \} \), which is exactly the smallest number of reflections \( \sigma_{\sqrt{2}e_j} \) that are needed to pass from \( y \) to \( x \). If \( k > 0 \), then (3.8) is sharp, since the heat kernel \( h_t(x, y) \) is the product of one-dimensional heat kernels, whose behavior, in this case, is well understood (see e.g. [1, Proposition 2.3]).

**Corollary 3.5.** Assume that \( \Phi(x) \) is a radial continuous function which is supported in \( B(0, 1) \). Let \( \Phi_t(x) = t^{-N} \Phi(x/t) \). There is a constant \( C = C(\Phi) > 0 \) such that

\[
(3.9) \quad |\Phi_t(x, y)| \leq CV(x, y, t)^{-1} \left( 1 + \frac{||x - y||}{t} \right)^{-2} \chi_{[0,1]}(d(x, y)/t).
\]

**Proof.** There is a constant \( C > 0 \) such that

\[
|\Phi_t(x)| \leq Ch_{t^2}(x),
\]

where \( h_t(x) = c_k^{-1} (2t)^{-N/2} e^{-\|x\|^2/(4t)} \). Applying (2.14) we obtain

\[
|\Phi_t(x, y)| \leq Ch_{t^2}(x, y) \leq C'V(x, y, t)^{-1} \left( 1 + \frac{||x - y||}{t} \right)^{-2}.
\]

By (2.14) and (2.15) we have \( \Phi_t(x, y) = 0 \) if \( d(x, y) > t \), so the proof of (3.9) is complete. \( \blacksquare \)

**Estimates for the Poisson kernel.** Let \( p_t(x, y) \) denote the integral kernel of the operator \( P_t = e^{-t\sqrt{-\Delta}} \). It is related to the heat semigroup by the subordination formula

\[
(3.10) \quad p_t(x, y) = \pi^{-1/2} \int_0^\infty e^{-u} h_{t^2/(4u)}(x, y) \frac{du}{\sqrt{u}}.
\]

The kernel \( p_t(x, y) \) was introduced and studied in [16]. It was called the \( k\)-Cauchy kernel there. For a continuous bounded function \( f \) defined on \( \mathbb{R}^N \), the function \( v(t, x) = P_t f(x) \), \( v(0, x) = f(x) \), solves the Cauchy problem \((\partial^2_t + \Delta)u = 0\), and \( v \) is continuous and bounded on \([0, \infty) \times \mathbb{R}^N \) (see [16, Theorem 5.6]).

Proposition 5.1 of [2] asserts that there is a constant \( C > 0 \) such that

\[
(3.11) \quad p_t(x, y) \leq \frac{C}{V(x, y, t + d(x, y)) \ t + d(x, y)}
\]

for every \( t > 0 \) and all \( x, y \in \mathbb{R}^N \). Moreover, for any nonnegative integer \( m \) and for any multi-index \( \beta \), there is a constant \( C \geq 0 \) such that, for every
Using Theorem 3.1, we have
\begin{equation}
|\partial^m \partial^\beta_p(t, x, y)| \leq C \rho(t, x, y)(t + d(x, y))^{-m-|\beta|} \times \begin{cases} 1 & \text{if } m = 0, \\ 1 + d(x, y)/t & \text{if } m > 0. \end{cases}
\end{equation}

The following proposition improves (3.11).

**Proposition 3.6.** If \( N \geq 2 \), then
\begin{equation}
p(t, x, y) \lesssim \frac{t}{V(x, y, d(x, y) + t)} \cdot \frac{d(x, y) + t}{\|x - y\|^2 + t^2}.
\end{equation}

If \( N = 1 \), then
\begin{equation}
p(t, x, y) \lesssim \frac{t}{V(x, y, d(x, y) + t)} \cdot \frac{d(x, y) + t}{\|x - y\|^2 + t^2} \cdot \ln \left( 1 + \frac{\|x - y\| + t}{d(x, y) + t} \right).
\end{equation}

Proof. The proof is similar to the proof of [8, Proposition 6] and uses (3.10) together with (3.1). We present the details. In order to prove (3.13), we first consider the case \( d(x, y) \leq t \). In this case \( d(x, y) + t \simeq t \). If \( \|x - y\| < t \) then (3.13) reduces to (3.11). If \( \|x - y\| \geq t \) then by (3.10) and (3.1),

\[ p(t, x, y) \lesssim w(B(x, t))^{-1} \int_0^\infty e^{-u} \frac{t^2/(4u)}{\|x - y\|^2 + t^2/(4u)} \frac{w(B(x, t))}{w(B(x, t^{1/2}))} \frac{du}{\sqrt{u}} \]

\[ \lesssim w(B(x, t))^{-1} \int_0^{1/4} \frac{t^2/(4u)}{\|x - y\|^2 + t^2/(4u)} \frac{du}{\sqrt{u}} + w(B(x, t))^{-1} \frac{t^2}{\|x - y\|^2} \]

\[ \lesssim w(B(x, t))^{-1} \frac{t^2}{\|x - y\|^2}. \]

Now we turn to the case \( \|x - y\| \geq d(x, y) \geq t \). Then \( d(x, y) + t \simeq d(x, y) \). Using Theorem 3.1, we have

\[ p(t, x, y) \lesssim \int_0^\infty e^{-u} \exp(-4cud(x, y)^2/t^2) \frac{t^2/(4u)}{\|x - y\|^2 + t^2/(4u)} \frac{du}{\sqrt{u}} \]

\[ = \int_0^{t^2/d(x, y)^2} + \int_{t^2/d(x, y)^2}^\infty = J_1 + J_2. \]
Further, since \( N \geq 2 \),
\[
J_1 \lesssim w(B(x, d(x, y)))^{-1} \int_0^{t^2/d(x,y)^2} \frac{w(B(x, d(x, y)))}{w(B(x, t/2u^{1/4}))} \left( \frac{\sqrt{u} d(x, y)}{t} \right)^N \frac{t^2/(4u)}{\|x - y\|^2 + t^2/(4u)} \frac{du}{\sqrt{u}}
\]
\[
\lesssim w(B(x, d(x, y)))^{-1} \int_0^{t^2/d(x,y)^2} \left( \frac{\sqrt{u} d(x, y)}{t} \right)^2 \frac{t^2/(4u)}{\|x - y\|^2} \frac{du}{\sqrt{u}}
\]
\[
\lesssim w(B(x, d(x, y)))^{-1} \int_0^{t^2/d(x,y)^2} \frac{d(x, y)^2}{\|x - y\|^2} \frac{du}{\sqrt{u}}
\]
\[
\lesssim w(B(x, d(x, y)))^{-1} \frac{td(x, y)}{\|x - y\|^2}.
\]

For \( J_2 \) we obtain
\[
J_2 \lesssim \frac{1}{w(B(x, d(x, y)))} \int_0^{\frac{t^2}{d(x,y)^2}} e^{-\frac{4cud(x,y)^2}{t^2}} \left( \frac{2d(x, y) \sqrt{u}}{t} \right)^N \frac{t^2 du}{4\|x - y\|^2 u^{3/2}}
\]
\[
\lesssim \frac{1}{w(B(x, d(x, y)))} \int_0^{\frac{t^2}{d(x,y)^2}} \frac{t^2}{\|x - y\|^2} \frac{du}{u^{3/2}} \lesssim w(B(x, d(x, y)))^{-1} \frac{td(x, y)}{\|x - y\|^2}.
\]

The proof of (3.14) goes in a similar way. We omit the details. ■

4. Atomic decompositions: proof of Theorem 1.5

**Inclusion** \( H^1_{(1, q)} \subseteq H^1_{\Delta} \). Note that the inclusion \( H^1_{(1, q)} \subseteq H^1_{\Delta} \) and the second inequality in (1.2) are easy consequences of (2.19) and (2.20). The proof is standard (see e.g. [19]). To this end, it is enough to prove that there is \( C > 0 \) such that
\[
\| \mathcal{M}_H a \|_{L^1(dw)} \leq C
\]
for any \((1, q)\)-atom. Let \( a \) be a \((1, q)\)-atom associated with \( B(x_0, r) \). It follows from (2.17) that \( \mathcal{M}_H \) is bounded on \( L^q(dw) \), hence by Hölder’s inequality and conditions [A] and [B] of Definition 1.3 we have
\[
\| \mathcal{M}_H a \|_{L^1(O(B(x_0, 2r), dw))} \leq \| \mathcal{M}_H a \|_{L^q(dw)} w(O(B(x_0, 2r)))^{1-1/q}
\]
\[
\leq C \| a \|_{L^q(dw)} w(O(B(x_0, 2r)))^{1-1/q} \leq C'.
\]
We now estimate \( \mathcal{M}_H a \) on \( O(B(x_0, 2r))^c \). Using condition [C] of Defini-
where $q$ the operators $\big| f \big|$ and the space $H$ which implies not difficult to check that by (2.5) and (2.17) we have

$$\sup_{d(x,y)<t} \frac{r}{t} G_{t/2/c}(y,x_0) \leq C \frac{r}{d(x,x_0)} w(B(x_0, d(x, x_0)))^{-1},$$

where in the last inequality we have used the fact that $\|a\|_{L^1(dw)} \leq 1$. It is not difficult to check that by (2.5) and (2.17) we have

$$\sup_{d(x,y)<t} \frac{r}{t} G_{t/2/c}(y,x_0) \leq C \frac{r}{d(x,x_0)} w(B(x_0, d(x, x_0)))^{-1},$$

which implies $\|M_H a\|_{L^1(O(B(x_0,2r))^c, dw)} \leq C$. 

**Square function characterization of $H^1_\Delta$ and tent spaces.** Let

$$Q_t f = t \sqrt{-\Delta} e^{-t\Delta} f = t \frac{d}{dt} P_t f = \mathcal{F}^{-1}(t||\xi||e^{-t||\xi||} \mathcal{F} f(\xi)).$$

The operators $Q_t$, initially defined on $L^1(dw) \cup L^2(dw)$, have the form

$$Q_t f(x) = \int_{\mathbb{R}^N} q_t(x, y) f(y) dw(y),$$

where $q_t(x, y) = t \frac{d}{dt} p_t(x, y)$. It can be easily deduced from (3.12) that $|q_t(x, y)| \leq C p_t(x, y)$. Thus for every $1 \leq p < \infty$, the operators $Q_t$ are uniformly bounded on $L^p(dw)$.

Consider the square function

$$(4.1) \quad Sf(x) = \left( \int \int |Q_t f(y)|^2 \frac{dt \, dw(y)}{tw(B(x,t))} \right)^{1/2}$$

and the space $H^1_{\text{square}} = \{ f \in L^1(dw) | \|Sf\|_{L^1(dw)} < \infty \}$. The following theorem was proved in [2].

**Theorem 4.1.** The spaces $H^1_\Delta$ and $H^1_{\text{square}}$ coincide and the corresponding norms $\|f\|_{H^1_{\text{max},H}}$ and $\|Sf\|_{L^1(dw)}$ are equivalent.

In order to prove our main result about atomic decomposition we use the relation between $H^1_{\text{square}}$ and the tent space $T^1_2$. The tent spaces were introduced by Coifman, Meyer, and Stein [4]. Recall that a function $F$ defined on $\mathbb{R}_+ \times \mathbb{R}^N$ is in the tent space $T^p_2$ if $\|F\|_{T^p_2} := \|A F(x)\|_{L^p(dw)} < \infty$, where

$$AF(x) := \left( \int_0^\infty \int_{\|y-x\|<t} |F(t, y)|^2 \frac{dw(y)}{w(B(x,t))} \frac{dt}{t} \right)^{1/2}.$$ 

So, $f \in H^1_{\text{square}}$ if and only if $F(t, x) = Q_t f(x)$ belongs to the tent space $T^1_2$. 

A measurable function $A(t, x)$ is said to be a $T^1_2$-atom if there is a Euclidean ball $B = B(y_0, r)$ such that

1. $\operatorname{supp} A \subset \hat{B} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \mid B(x, t) \subset B(y_0, r)\}$;
2. $\int_0^\infty \int_{\mathbb{R}^N} |A(t, x)|^2 \, dw(x) \frac{dt}{t} \leq w(B)^{-1}$.

It is well known that $F(t, x)$ belongs to $T^1_2$ if and only if there are sequences $A_j$ of $T^1_2$-atoms and $\lambda_j \in \mathbb{C}$ such that

$$\sum_{j=1}^\infty \lambda_j A_j = F, \quad \sum_{j=1}^\infty |\lambda_j| \sim \|F\|_{T^1_2},$$

where the convergence is in $T^1_2$ norm and a.e. (see [4] and [18]).

**Remark 4.2.** The functions $\lambda_j A_j(t, x)$ can be taken of the form

$$\lambda_j A_j(t, x) = F(t, x) \chi_{S_j}(t, x),$$

where $S_j \subset \mathbb{R}_+ \times \mathbb{R}^N$ are mutually disjoint (see [18]).

**Calderón reproducing formula.** From now on we choose a radial function $\Psi \in C_c^\infty(B(0, 1/4))$ satisfying $\int \Psi(x) \, dw(x) = 0$ such that the Calderón reproducing formula

$$f(x) = \int_0^\infty \Psi_t(x, y) Q_t f(y) \, dw(y) \frac{dt}{t}$$

holds for all $f \in L^2(dw)$, with convergence in $L^2(dw)$ (see [2]). Let us recall that $\Psi_t(x, y) = \tau_x \Psi_t(-y)$, $\Psi_t(x) = t^{-N} \Psi(x/t)$, $\int \Psi_t(x, y) \, dw(y) = 0$, and $\Psi_t(x, y) = 0$ if $d(x, y) > t$.

**Atomic decomposition of $H^1_{\text{square}}$ into $(1, 2)$-atoms.** We are now in a position to prove decomposition of $f \in H^1_{\text{square}} = H^1_\Delta$ into $(1, 2)$-atoms. We start by assuming additionally that $f \in L^2(dw)$; later this assumption is easily removed by the approximation argument presented in [2] Theorem 11.4, Lemma 13.6, Proposition 13.8]. Set

$$\pi_{\Psi} F(x) = \int_0^\infty \int_{\mathbb{R}^N} \Psi_t(x, y) F(t, y) \, dw(y) \frac{dt}{t}.$$

Then $\|\pi_{\Psi} F\|_{L^2(dw)} \leq C \|F\|_{T^1_2}$. Let $F(t, x) = Q_t f(x)$. Note that $F \in T^1_2 \cap T^2_2$. Applying atomic decomposition of $F$ as a function in $T^1_2$ combined with Remark 4.2 we get

$$f(x) = \pi_{\Psi} F(x) = \sum_{j=1}^\infty \lambda_j \pi_{\Psi} A_j(x),$$
where the convergence is in $L^2(dw)$. Hence, it suffices to show that there is a constant $C > 0$ such that

(4.2) \[ \| \pi \Psi A \|_{H^1_{(1,2)}} \leq C \]

for any $T_2^1$-atom $A(t, x)$. To this end assume that $A$ is associated with $\hat{B}$, where $B = B(y_0, r)$. Set

\[ a(t, x) = \int_{\mathbb{R}^N} \Psi_t(x, y) A(t, y) \, dw(y), \]
\[ g(x) = \pi \Psi A(x) = \int_0^\infty \int_{B(y_0, r)} \Psi_t(x, y) A(t, y) \, dw(y) \, \frac{dt}{t} = a(t, x) \frac{dt}{t}. \]

Then

\[ \| g \|_{L^2(dw)} \leq C \| A \|_{T^1_2} \leq C'/w(B)^{1/2}, \]
\[ \text{supp } g \subseteq O(B) = \bigcup_{\sigma \in G} B(\sigma(y_0), r), \text{ and } \int g(x) \, dw(x) = 0. \]

We denote by $M_{HL}$ the Hardy–Littlewood maximal function

\[ M_{HL} f(x) = \sup_{B(y, R) \ni x} \frac{1}{w(B(y, R))} \int_{B(y, R)} |f(y')| \, dw(y'). \]

**Lemma 4.3.** Assume that $\sigma \in G$ is such that $\| \sigma(y_0) - y_0 \| > 4r$. Then for $x \in B(\sigma(y_0), r)$ we have

\[ |a(t, x)| \leq C \frac{t^2}{\| \sigma(y_0) - y_0 \|^2} \sum_{\sigma' \in G} M_{HL}(A(t, \cdot))(\sigma'(x)). \]

The constant $C > 0$ is independent of $A(t, x)$, $\sigma \in G$, $x \in \mathbb{R}^N$ and $t > 0$.

**Proof.** For $(t, y) \in \text{supp } A \subset \hat{B}$ and $x \in B(\sigma(y_0), r)$ we have $\|x - y\| \sim \|y_0 - \sigma(y_0)\|$. Therefore, by Corollary 3.5

\[ |a(t, x)| \lesssim \int_{\mathbb{R}^N} \frac{t^2}{\|x - y\|^2 V(x, y, t)^\lambda_{[0,1]}} \left( \frac{d(x, y)}{t} \right) |A(t, y)| \, dw(y) \]
\[ \lesssim \sum_{\sigma' \in G} \int_{B(\sigma'(x), t)} V(\sigma'(x), y, t)^{-1} \frac{t^2}{\|\sigma(y_0) - y_0\|^2} |A(t, y)| \, dw(y) \]
\[ \lesssim \left( \sum_{\sigma' \in G} M_{HL}(A(t, \cdot))(\sigma'(x)) \right) \frac{t^2}{\|\sigma(y_0) - y_0\|^2}. \]

**Lemma 4.4.** If $\| \sigma(y_0) - y_0 \| > 4r$, then

\[ \| g \|_{L^2(B(\sigma(y_0), r), dw)} \leq \frac{C}{w(B(y_0, r))^{1/2}} \frac{r^2}{\| \sigma(y_0) - y_0 \|^2}. \]

The constant $C > 0$ is independent of $A(t, x)$ and $\sigma \in G$. 
Proof. By the Minkowski integral inequality, the Cauchy–Schwarz inequality and Lemma 4.3 we have

\[
\left( \int_{B(\sigma(y_0), r)} |g(x)|^2 \, dw(x) \right)^{1/2}
\leq \int_{0}^{r} \left( \int_{B(\sigma(y_0), r)} |a(t, x)|^2 \, dw(x) \right)^{1/2} \frac{dt}{t}
\leq \int_{0}^{r} \left( \int_{\mathbb{R}^N} \left( \sum_{\sigma' \in G} M_{HL}(A(t, \cdot))(\sigma'(x)) \right)^2 \frac{t^4}{\|\sigma(y_0) - y_0\|^4} \, dw(x) \right)^{1/2} \frac{dt}{t}
\leq \int_{0}^{r} \frac{t^2}{\|\sigma(y_0) - y_0\|^2} \sum_{\sigma' \in G} \left( \int_{\mathbb{R}^N} |A(t, x)|^2 \, dw(x) \right)^{1/2} \frac{dt}{t}
\leq \int_{0}^{r} \frac{t^2}{\|\sigma(y_0) - y_0\|^2} \left( \int_{\mathbb{R}^N} |A(t, x)|^2 \, dw(x) \right)^{1/2} \frac{dt}{t}
\leq \frac{r^2}{\|\sigma(y_0) - y_0\|^2} \frac{1}{w(B(y_0, r))^{1/2}}.
\]

**Proposition 4.5.** There exists \( C > 0 \) independent of \( A(t, x) \) such that \( g = \pi_{\psi} A \) can be written as

\[
g = \sum_{j=1}^{\infty} \lambda_j a_j,
\]

where \( a_j \) are \((1, 2)\)-atoms and \( \sum_{j=1}^{\infty} |\lambda_j| \leq C \).

Proof. Let \( \sigma_0 = e \) and \( G = \{ \sigma_0, \sigma_1, \ldots, \sigma_{|G|-1} \} \). We know that \( g = \pi_{\psi} A \) is supported by

\[
O(B) = \bigcup_{j=0}^{|G|-1} B(\sigma_j(y_0), r).
\]

Set \( E_0 = B(y_0, r) \),

\[
E_j = B(\sigma_j(y_0), r) \setminus \bigcup_{i=0}^{j-1} B(\sigma_i(y_0), r) \quad \text{for } j = 1, \ldots, |G| - 1,
\]

and \( g_j = g \chi_{E_j} \). Then \( g = \sum_{j=0}^{|G|-1} g_j \), \( \text{supp } g_j \subset E_j \subseteq B(\sigma_j(y_0), r) \). Define \( \mathcal{I} = \{ j \in \{1, \ldots, |G|-1 \} \mid \|\sigma_j(y_0) - y_0\| \geq 4r \} \), \( \mathcal{J} = \{0, 1, \ldots, |G|-1\} \setminus \mathcal{I} \).
For $j \in I$ let $m_j = \lfloor (\|\sigma_j(y_0) - y_0\|/r) \rfloor$. Set
\[
x^{\{j\}}_n = \sigma_j(y_0) + \frac{n y_0 - \sigma_j(y_0)}{m_j}
\]
for $n = 0, 1, \ldots, m_j$.

Then $r \leq \|x^{\{j\}}_n - x^{\{j\}}_{n+1}\| \leq 2r$. Let $c_j = \int_{\mathbb{R}^N} g_j(x) \, dw(x)$. By Lemma 4.4 and the Cauchy–Schwarz inequality we have
\[
|c_j| \leq C_1 \frac{r^2}{\|\sigma_j(y_0) - y_0\|^2}.
\]
Set
\[
a_0^{\{j\}} = \frac{\|\sigma_j(y_0) - y_0\|^2}{r^2} \left( g_j - c_j \frac{1}{w(B(x^{\{j\}}_1, r))} \chi_{B(x^{\{j\}}_1, r)} \right),
\]
\[
a_n^{\{j\}} = c_j \frac{\|\sigma_j(y_0) - y_0\|^2}{r^2} \frac{1}{w(B(x^{\{j\}}_n, r))} \chi_{B(x^{\{j\}}_n, r)}
- c_j \frac{\|\sigma_j(y_0) - y_0\|^2}{r^2} \frac{1}{w(B(x^{\{j\}}_{n+1}, r))} \chi_{B(x^{\{j\}}_{n+1}, r)}
\]
for $n = 1, \ldots, m_j - 1$, and
\[
b_j = c_j \frac{1}{w(B(x_{m_j}, r))} \chi_{B(x_{m_j}, r)} = c_j \frac{1}{w(B(y_0, r))} \chi_{B(y_0, r)}.
\]

The positions of $B(x^{\{j\}}_n, r)$ for the root system $A_2$ are schematized in Figure 2.

![Figure 2. The balls $B(x^{\{j\}}_n, r)$ for the root system $A_2$](image)

Clearly,
\[
g_j = \sum_{n=0}^{m_j-1} \frac{r^2}{\|\sigma_j(y_0) - y_0\|^2} a_n^{\{j\}} + b_j.
\]
It follows from Lemma 4.4 and the doubling property that each function $a_n^{(j)}$ (for $j = 0, 1, \ldots, m_j - 1$) is a multiple of a $(1,2)$-atom associated with the ball $B(x_n^{(j)}, 4r)$. The multiplicity constant depends on $\Psi$ and the doubling constant but it is independent of $A(t,x)$. Write

$$a = \sum_{j \in J} g_j + \sum_{j \in I} b_j.$$  

Then

$$g = \sum_{j=0}^{\lfloor G \rfloor - 1} g_j = a + \sum_{j \in I} \sum_{i=0}^{m_j - 1} \frac{r^2}{\|\sigma_j(y_0) - y_0\|^2} a_i^{(j)}.$$  

Note that by the construction above $\text{supp}\ a \subset B(y_0, 16r)$ and, by Lemma 4.4 and (4.3), we have

$$\|a\|_{L^2(dw)} \leq \sum_{j \in J} \|g_j\|_{L^2(dw)} + \sum_{j \in I} \|b_j\|_{L^2(dw)} \leq C_2 \frac{1}{w(B(y_0, 16r))^{1/2}}.$$  

Moreover, $\int a(x) \, dw(x) = 0$, since $\int g(x) \, dw(x) = 0$ and $\int a_i^{(j)}(x) \, dw(x) = 0$. So the function $a$ is a multiple of a $(1,2)$-atom. Further,

$$\sum_{j \in I} \sum_{i=0}^{m_j - 1} \frac{r^2}{\|\sigma_j(y_0) - y_0\|^2} \leq \sum_{j \in I} \frac{r^2}{\|\sigma_j(y_0) - y_0\|^2} m_j \leq |G|/4.$$  

Therefore (4.4) is the desired atomic decomposition. $lacksquare$

Thus we have proved Theorem 1.5 for $q = 2$.

**Decomposition into $(1, \infty)$ atoms.** To finish the proof of Theorem 1.5 it suffices to refer to the following known proposition. For the convenience of the reader we provide a short proof.

**Proposition 4.6.** There is a constant $C > 0$ such that any $(1,2)$-atom $a(x)$ can be written as

$$a(x) = \sum_{j=1}^{\infty} \lambda_j b_j(x),$$

where $b_j$ are $(1,\infty)$-atoms and $\sum_{j=1}^{\infty} |\lambda_j| \leq C$.

**Proof.** Fix a $(1,2)$-atom $a(x)$. Since the measure $dw$ is doubling, without loss of generality we can assume that $a(x)$ is associated with a cube $Q$, i.e.

$$\text{supp}\ a \subset Q, \quad \|a\|_{L^2(dw)} \leq w(Q)^{-1/2}, \quad \int_{\mathbb{R}^N} a \, dw = 0.$$

Clearly, there is a constant $C_1 > 1$, which depends on the doubling constant and $N$, such that $w(Q) \leq C_1 w(Q')$, where $Q'$ is any subcube of $Q$, and $\ell(Q') = \ell(Q)/2$, where $\ell(Q)$ denote the side length of $Q$. Form the
Calderón–Zygmund decomposition of $|a|^2$ at height $\lambda = \varepsilon^{-2}w(Q)^{-2}$, where $\varepsilon = 4^{-1/2}C_1$. This yields a sequence of disjoint cubes $Q_j \subseteq Q$ such that

\begin{equation}
(4.6) \quad w(Q_j) \leq \lambda^{-1} \int_{Q_j} |a(x)|^2 \, dw(x) < C_1 w(Q_j),
\end{equation}

\begin{equation}
(4.7) \quad |a(x)|^2 \leq \lambda \quad \text{for } x \in \Omega = Q \setminus \bigcup_{j=1}^{\infty} Q_j.
\end{equation}

Set $a_{Q_j} = \frac{1}{w(Q_j)} \int_{Q_j} a(x) \, dw(x)$. We write

$$a = \sum_{j=1}^{\infty} (a - a_{Q_j}) \chi_{Q_j} + \left( a \chi_{\Omega} + \sum_{j=1}^{\infty} a_{Q_j} \chi_{Q_j} \right) = \sum_{j=1}^{\infty} a_j + b_1.$$ 

We first prove that $b_1$ is a multiple of a $(1, \infty)$-atom associated with $Q$. Clearly, $\text{supp} \ b_1 \subset Q$ and $\int_{Q} b_1 = 0$. Moreover,

$$|a_{Q_j}| = \left| \frac{1}{w(Q_j)} \int_{Q_j} a(x) \, dw(x) \right| \leq C_1^{1/2} \lambda^{1/2} = C_1^{1/2} \varepsilon^{-1} w(Q)^{-1}.$$ 

Therefore, by $(4.7)$ and $(4.6)$,

$$|b_1(x)| \leq \varepsilon^{-1} w(Q)^{-1} + C_1^{1/2} \frac{1}{\varepsilon w(Q)} \leq (1 + C_1^{1/2}) \varepsilon^{-1} w(Q)^{-1} = C_2 w(Q)^{-1}.$$ 

Next, we show that $a_j$ is a multiple of a $(1, 2)$-atom associated with $Q_j$. Obviously, $\text{supp} \ a_j \subset Q_j$ and $\int_{Q_j} a_j = 0$. Furthermore, by $(4.6)$,

$$\|a_j\|_{L^2(dw)} \leq 2C_1^{1/2} \lambda^{1/2} w(Q_j)^{1/2} = 2C_1^{1/2} \varepsilon^{-1} w(Q)^{-1} w(Q_j)^{1/2},$$

which implies

$$\|a_j\|_{H^1_{(1,2)}} \leq 2C_1^{1/2} \varepsilon^{-1} w(Q)^{-1} w(Q_j).$$

Finally, note that by $(4.5)$ and $(4.6)$,

$$\sum_{j=1}^{\infty} w(Q_j) \leq \lambda^{-1} \sum_{j=1}^{\infty} \int_{Q_j} |a(x)|^2 \, dw(x) \leq \lambda^{-1} \|a\|_{L^2(dw)}^2 \leq \frac{1}{\lambda w(Q)} = \varepsilon^2 w(Q),$$

which implies

$$\sum_{j=1}^{\infty} \|a_j\|_{H^1_{(1,2)}} \leq 2C_1^{1/2} \varepsilon^{-1} w(Q)^{-1} \sum_{j=1}^{\infty} w(Q_j) \leq 2C_1^{1/2} \varepsilon = 1/2.$$ 

We repeat the argument above with $a_j$ in place of $a$. Iterating this procedure, we obtain a representation $a(x) = \sum_{j=1}^{\infty} \lambda_j b_j(x)$, where $b_j$ are $(1, \infty)$-atoms and $\sum_{j=1}^{\infty} |\lambda_j| \leq 2C_2$. \hfill \blacksquare
Acknowledgments. The authors want to thank Jean-Philippe Anker, Detlef Müller, Margit Rösler, and Michael Voit for conversations on the subject of the paper. The authors are also greatly indebted to the referees for helpful comments which have improved the presentation of the paper.

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