GLOBAL PHASE PORTRAITS OF A DEGENERATE
BOGDANOV-TAKENS SYSTEM WITH SYMMETRY (II)

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Abstract. The degenerate Bogdanov-Takens system \( \dot{x} = y - (a_1 x + a_2 x^3) \), \( \dot{y} = a_3 x + a_4 x^3 \) has two normal forms, one of which is investigated in [Disc. Cont. Dyn. Syst. B (22)2017, 1273-1293] and global behavior is analyzed for general parameters. To continue this work, in this paper we study the other normal form and perform all global phase portraits on the Poincaré disc. Since the parameters are not restricted to be sufficiently small, some classic bifurcation methods for small parameters, such as the Melnikov method, are no longer valid. We find necessary and sufficient conditions for existences of limit cycles and homoclinic loops respectively by constructing a distance function among orbits on the vertical isocline curve and further give the number of limit cycles for parameters in different regions. Finally we not only give the global bifurcation diagram, where global existences and monotonicities of the homoclinic bifurcation curve and the double limit cycle bifurcation curve are proved, but also classify all global phase portraits.

1. Introduction. A planar differential system is often used to describe natural phenomena mathematically. Even sometimes it is not the most appropriate model directly, it is still regarded as a simplification, which is helpful in study. Directly studying and judging properties of solutions, the qualitative theory and the bifurcation theory of differential systems are very efficient to get an overall understanding of plentiful dynamical phenomenon and the change of flows in the sense of topological structure as parameters vary.

As a generalization of the mass-spring-damper equation \( \ddot{x} + k_1 \dot{x} + k_2 x = 0 \), equation \( \ddot{x} + N(x) \dot{x} + M(x) = 0 \) rises from a great number of mathematical models in physics and usually is written as Liénard system

\[
\dot{x} = y - \int_0^x N(s)ds, \quad \dot{y} = -M(x),
\]

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where $N(x)\dot{x}$ and $M(x)$ are the damping term and the spring term respectively. When both $M(x)$ and $N(x)$ are polynomials, (1) is said (see [7]) to be of type $(m, n)$, where $m := \deg M, n := \deg N$. When either $M(x) \equiv 0$ or $N(x) \equiv 0$, the dynamics of (1) is clear because all solutions can be obtained easily. When either $M(x)$ or $N(x)$ is a nonzero constant, the dynamics is also clear because there exist no equilibria in the case that $M(x)$ is a nonzero constant and there exist no periodic orbits and homoclinic loops in the case that $N(x)$ is a nonzero constant. Thus, usually $m, n$ are considered to be nonzero.

Since global phase portraits describe the dynamics of systems completely, giving global phase portraits is important and, usually, difficult because of the lack of efficient methods. By transformation $x \to x, y \to y + \mu_1 x + \mu_2 x^2/2$, the Bogdanov-Takens system

$$\dot{x} = y, \quad \dot{y} = x(x - 1) + (\mu_1 + \mu_2 x)y \tag{2}$$

is changed into an equivalent system having form (1) of type $(2,1)$. In [25] Perko gives global phase portraits of (2) for global parameters $(\mu_1, \mu_2) \in \mathbb{R}^2$ and proves that (2) has the same qualitative behavior concerning limit cycles and homoclinic loops for global parameters as that for small parameters. Moreover, Perko states two conjectures about the function corresponding to the saddle-loop bifurcation curve in the parameter space, which are solved recently by Gasull et al in [15]. The global bifurcation diagram and global phase portraits are investigated in [3] for a Plant-Herbivore model and in [27] for a Predator-Prey system. For $m + n \leq 4$, interested readers can find global results for system (1) of type $(m,n)$ for global parameters in [6, 7, 14, 23, 24, 25]. For $(m,n) = (3, 2)$, in [8, 9, 10, 11] Dumortier and Li investigate (1), which is regarded as a perturbation of a Hamiltonian vector field, by the Abelian integral method. But for general parameters the global dynamics for (1) of type $(3, 2)$ is still unknown. The main difficulty comes from determining the exactly number of limit cycles, global existences and monotonicities of the homoclinic bifurcation curve and the double limit cycle bifurcation curve.

The special system of form (1) of type $(3, 2)$

$$\dot{x} = y - (a_1 x + a_2 x^3), \quad \dot{y} = a_3 x + a_4 x^3, \quad (3)$$

where $a_2 a_4 \neq 0$, is investigated in [18, 19] for sufficiently small $|a_1|, |a_3|$. When $a_4 > 0$, by $x \to a_2^{-1} \sqrt{-a_4} x, \quad y \to a_2 a_4^{3/2} y, \quad t \to -a_4 a_4^{-1} t$, system (3) can be normalized as

$$\dot{x} = y - (bx + x^3), \quad \dot{y} = ax + x^3. \tag{4}$$

When $a$ and $b$ are sufficiently small, the bifurcation diagram and global phase portraits of an equivalent system of (4) are obtained in [26]. Moreover, the expression of the homoclinic bifurcation curve is given up to degree 6. Global dynamical behavior of system (4) is given recently in [4] for general $a, b$. When $a_4 < 0$, by $x \to a_2^{-1} \sqrt{-a_4} x, \quad y \to a_2^{-2}(-a_4)^{3/2} y, \quad t \to -a_4 a_4^{-1} t$, system (3) can be normalized as

$$\dot{x} = y - (bx + x^3) =: y - F(x), \quad \dot{y} = ax - x^3 =: -g(x) \tag{5}$$

for $(a,b) \in \mathbb{R}^2$. Since system (5)|$_{(a,b) = (0,0)}$ is the 3-jet of a doubly degenerate Bogdanov-Takens point (see [1, 13, 20, 22]) with no quadratic terms in the normal form, (5) is usually called a degenerate Bogdanov-Takens system. As indicated in [8] system (5)|$_{(a,b) = (0,0)}$ is of codimension 4 and has a 4-parametric unfolding $\dot{x} = y - (bx + cx^2 + x^3), \dot{y} = d + ax - x^3$. On the other hand, in the class of
all planar vector fields with $\mathbb{Z}_2$-symmetry system \((5)\) is of codimension 2 and has $\mathbb{Z}_2$-symmetric unfolding \((5)\) as indicated in [5, Chapter 4]. That is, \((5)\) is invariant under the change $x \rightarrow -x$, $y \rightarrow -y$, a rotation of angle $\pi$ of the phase portraits. System \((5)\) is investigated in [2, 5, 16, 19, 22] for sufficient small $|a|$, $|b|$. For general $a$ and $b$, in [20] Khibnik et al give some numerical results about bifurcation curves.

In this paper we study the global bifurcation and the classification of global phase portraits for degenerate Bogdanov-Takens system \((5)\) with $a, b \in \mathbb{R}$. In Section 2 we consider the bifurcations of equilibria for general $a, b$. In Section 3 we give necessary and sufficient conditions for global existences of limit cycles and homoclinic loops by constructing Poincaré-Bendixson Annular regions and a distance function among orbits on the vertical isocline curve respectively. Moreover, the change of amplitudes of limit cycles, monotonocities of the homoclinic bifurcation curve and the double limit cycle bifurcation curve globally depending on parameters are also given without evaluating the Melnikov function. In Section 4 we obtain the global bifurcation diagram and the complete classification of global phase portraits, which are consistent with numerical results given in [20]. In Section 5 we give some concluding remarks.

2. The equilibrium bifurcations. In this section we consider all possible equilibrium bifurcations of system \((5)\). To do this, we firstly give the qualitative properties of equilibria.

**Lemma 2.1.** Equilibria of system \((5)\) are given in Table 1, where $E_0, E_L, E_R$ are points \((0, 0), (-\sqrt{a}, -(a+b)\sqrt{a}), (\sqrt{a}, (a+b)\sqrt{a})\) respectively.

| possibilities of $(a, b)$ | location of equilibria | types and stability |
|---------------------------|------------------------|--------------------|
| $a < 0$                   |                        |                    |
| $b < -2\sqrt{-a}$        | $E_0$                  | $E_0$ unstable bidirectional node |
| $b = -2\sqrt{-a}$        | $E_0$                  | $E_0$ unstable proper node |
| $-2\sqrt{-a} < b < 0$    | $E_0$                  | $E_0$ unstable focus |
| $b = 0$                  | $E_0$                  | $E_0$ stable weak focus of order one |
| $0 < b < 2\sqrt{-a}$    | $E_0$                  | $E_0$ stable focus |
| $b = 2\sqrt{-a}$        | $E_0$                  | $E_0$ stable proper node |
| $b > 2\sqrt{-a}$        | $E_0$                  | $E_0$ stable bidirectional node |
| $a = 0$                  |                        |                    |
| $b < 0$                  | $E_0$                  | $E_0$ unstable degenerate node |
| $b < -3a - 2\sqrt{2a}$  | $E_R, E_L, E_{R_{\epsilon}}$ | $E_0$ saddle; $E_R, E_L$ unstable nodes |
| $b = -3a - 2\sqrt{2a}$  | $E_R, E_L, E_{L_{\epsilon}}$ | $E_0$ saddle; $E_R, E_L$ unstable proper nodes |
| $-3a - 2\sqrt{2a} < b < -3a$ | $E_R, E_L$           | $E_0$ saddle; $E_R, E_L$ unstable foci |
| $b = -3a$                | $E_R, E_L, E_{L_{\epsilon}}$ | $E_0$ saddle; $E_R, E_L$ unstable weak foci of order one |
| $-3a < b < -3a + 2\sqrt{2a}$ | $E_R, E_L$           | $E_0$ saddle; $E_R, E_L$ stable foci |
| $b = -3a + 2\sqrt{2a}$  | $E_R, E_L, E_{L_{\epsilon}}$ | $E_0$ saddle; $E_R, E_L$ stable proper nodes |
| $b > -3a + 2\sqrt{2a}$  | $E_R, E_L, E_{L_{\epsilon}}$ | $E_0$ saddle; $E_R, E_L$ stable nodes |

**Table 1.** Equilibria in finite planes

In Table 1 the second row means that “when $a < 0$ and $b < -2\sqrt{-a}$, there is a unique equilibrium, which lies at $E_0$ and is an unstable bidirectional node”. The other rows are understood similarly. In the following we discuss the qualitative properties of equilibria at infinity, which help us investigate the behavior of orbits when $x$ and $y$ are large.
Lemma 2.2. As in Figure 1, at infinity system (5) has two equilibria \( I_A^\pm \) on the line \( y = x \) and two equilibria \( I_B^\pm \) on the \( y \)-axis, where \( I_A^+, I_B^+ \) lie in the upper half-plane and the other two lie in the lower one. Moreover, \( I_A^\pm \) are unstable nodes and \( I_B^\pm \) are saddles.

![Figure 1. Equilibria at infinity](image)

The result given in Lemma 2.1 is obtained by classical analysis methods of the qualitative theory (see, e.g., [16, 29]) and can be proved similarly to [4, Lemma 2.1]. The properties of equilibria at infinity given in Lemma 2.2 are obtained via Poincaré transformations and Briot-Bouquet ones (see, e.g., [28, 29]) and can be proved similarly to [4, Lemma 2.2]. Thus, we omit the proofs of Lemmas 2.1 and 2.2 for the length of this paper.

Proposition 1. The bifurcation diagram of system (5) includes the following bifurcation curves:

1. pitchfork bifurcation curve \( a = 0 \), i.e., \( b \)-axis;
2. Hopf bifurcation curves \( H_1 = \{(a, b) : a < 0, b = 0\} \) for \( E_0 \), \( H_2 = \{(a, b) : a > 0, b = -3a\} \) for \( E_L \) and \( E_R \);
3. focus-node bifurcation curves \( N_1^\pm = \{(a, b) : a < 0, b = \pm 2\sqrt{-a}\} \) for \( E_0 \), \( N_2^\pm = \{(a, b) : a > 0, b = -3a \pm 2\sqrt{2a}\} \) for \( E_L \) and \( E_R \).

Proof. By Lemma 2.1 the number of equilibria in the finite plane changes from 1 to 3 when \( a \) changes from a nonpositive value to a positive one, implying that \( b \)-axis is the pitchfork bifurcation curve. When \( a < 0 \), by Lemma 2.1 stable weak focus \( E_0 \) of order 1 becomes an unstable rough focus as \( b \) changes from 0 to a small negative value, implying Hopf bifurcations, i.e., \( H_1 \) is a Hopf bifurcation curve for \( E_0 \). When \( a > 0 \), by Lemma 2.1 stable weak foci \( E_L \) and \( E_R \) of order 1 become unstable rough foci as \( b \) changes from \( -3a \) to \( -3a + \epsilon (\epsilon > 0) \), implying that Hopf bifurcations happens, i.e., \( H_2 \) is a Hopf bifurcation curve for both \( E_L \) and \( E_R \). Then, (1) and (2) are proved. When \( a < 0 \), \( E_0 \) becomes a proper node from a focus as \( b \) increases to \( 2\sqrt{-a} \) by Lemma 2.1, implying that \( N_1^\pm \) are the focus-node bifurcation curves for \( E_0 \). When \( a > 0 \), from Lemma 2.1 we see that \( E_L \) and \( E_R \) become proper nodes from foci as \( b \) either decreases to \( -3a - 2\sqrt{2a} \) or increases to \( -3a + 2\sqrt{2a} \), implying that \( N_2^\pm \) are the node bifurcation curves. (3) is proved.

By \((x, y, t) \rightarrow (x/3, y/9 + bx/3 + x^3/27, 3t)\), system (5) is equivalently written as

\[
\dot{x} = y, \quad \dot{y} = 9ax - 3by - x^3 - x^2 y.
\]
As to the equilibrium bifurcation, for sufficiently small $|a|, |b|$ it is indicated in [2, 5, 21] that the bifurcation diagram of system (6) includes the pitchfork bifurcation curve $\{(a, b) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) : a = 0\}$ and the Hopf bifurcation curves $\{(a, b) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) : a < 0, b = 0\}$ for $E_0$, $\{(a, b) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) : a > 0, b = -3a\}$ for $E_L$ and $E_R$, where $\epsilon > 0$ is sufficiently small. In Proposition 1, we get rid of the local limitation for $|a|, |b|$.

3. Limit cycles and homoclinic loops. In this section we study existences of limit cycles and homoclinic loops for system (5). Moreover, the exact number of limit cycles is obtained if they exist. For simplicity, let a large limit cycle be a limit cycle surrounding three equilibria and a small limit cycle be a limit cycle surrounding a single equilibrium. In order to consider the whole two-dimensional parameter space $(a, b)$, we split it into 5 regions as follows:

$$(c_1) \quad \begin{cases} a \leq 0, \\ b < 0; \end{cases} \quad (c_2) \quad \begin{cases} a \leq 0, \\ b \geq 0; \end{cases} \quad (c_3) \quad \begin{cases} a > 0, \\ b \geq -a; \end{cases} \quad (c_4) \quad \begin{cases} a > 0, \\ b \leq -3a; \end{cases} \quad (c_5) \quad \begin{cases} a > 0, \\ -3a < b < -a. \end{cases}$$

Lemma 3.1. Assume that condition $(c_1)$ holds. System (5) has a unique limit cycle, which is stable and hyperbolic. Moreover, for fixed $a$ the amplitude of the limit cycle decreases as $b$ increases.

Proof. Let

$$u = w(x) := \text{sgn}(x) \sqrt{2 \int_{0}^{x} g(s) ds}, \quad \frac{d\tau}{dt} := \frac{g(x)}{u} dt.$$ 

System (5) is rewritten as

$$\frac{du}{d\tau} = y - \hat{F}(u), \quad \frac{dy}{d\tau} = -u,$$

where $\hat{F}(u) := F(w^{-1}(u))$. By [29, Chapter 4, Theorem 4.3], system (7) has a unique limit cycle and it is stable. By [29, Chapter 4, Theorem 2.2], this unique limit cycle is hyperbolic. Thus, system (5) has a unique limit cycle, which is stable and hyperbolic.

In order to consider the change of amplitude of $L$ about $b$, we change (5) into

$$\dot{x} = y, \quad \dot{y} = ax - x^3 - (b + 3x^2)y$$

(8)
by $x \to x, y \to y + F(x)$. For sufficiently small $|a|$ and $|b|$, the dynamical behavior and information about bifurcation curves of system (8) can be found in [21]. From (8), we obtain equation
\begin{equation}
\frac{dy}{dx} = \frac{ax - x^3}{y} - 3x^2 - b.
\end{equation}

As indicated in [29, Chapter 4], the phase portraits of (5) are homeomorphic globally with that of (8). By the First Comparison Theorem (see [17, Chapter 1, Corollary 6.3]) for (9) and (9) $|b|_{b \to b+\epsilon}$ we get the orbits of (8) $|b|_{b \to b+\epsilon}$ as shown in Figure 2, where $0 < \epsilon \ll 1$ and the broken curve is the limit cycle $\hat{L}$ of (8) corresponding to $L$ of (5). Thus, intersection points between the stable limit cycle of (8) $|b|_{b \to b+\epsilon}$ and $x$-axis lie on the line segment $B'B$. Since the line segment $B'B$ is surrounded by the broken curve, the amplitude decreases as $b$ increases. \hfill \square

**Lemma 3.2.** System (5) has no limit cycle and homoclinic loop if either condition (c2) or condition (c3) holds.

**Proof.** When $b \geq 0$, $\text{div}(y - F(x), -g(x)) = -b - 3x^2 \leq 0$. Thus, by Bendixson-Dulac Theorem system (5) has no limit cycle when conditions (c2) holds. Moreover, we only need to consider $-a \leq b < 0$ in condition (c3).

![Figure 3](image-url)

**FIGURE 3.** Discussion in the case that $-a \leq b < 0$

Firstly, we claim that there is no large limit cycles. In fact, if there exists such a limit cycle $L_1$, $0 = \int_{L_1} dE = \int_{L_1} g(x)dx + ydy = \int_{L_1} F(x)dy$, where
\begin{equation}
E(x, y) := \int_0^x g(s)ds + y^2/2.
\end{equation}
By the symmetry with respect to $E_0$ of the phase portraits of (5),
\begin{equation}
\int_{AH} F(x)dy = \frac{1}{2} \int_{L_1} F(x)dy = 0,
\end{equation}
where $A, H$ are intersection points between $L_1$ and $y$-axis as shown in Figure 3(a). Denote by $B, G$ the intersection points between $L_1$ and line $x = \sqrt{-b}$, by $C, F$ the intersection points between $L_1$ and line $x = \sqrt{-a}$, by $D$ the intersection point between $L_1$ and line $y = y_B$, by $E$ the intersection point between $L_1$ and line $y = y_G$, where $y_B, y_G$ are ordinates of $B, G$ respectively. Clearly, $C$ is the peak of $BD$ and $F$ is the nadir of $EG$ because $\dot{y} = -g(x)$. In arc segments $\bar{AB}, \bar{GH}$, let $y = y_1(x), y = y_2(x)$ respectively. Then,
\begin{align*}
\int_{\bar{AB}} F(x)dy + \int_{\bar{GH}} F(x)dy &= \int_0^{\sqrt{-b}} -F(x)g(x) \frac{1}{y_1(x) - F(x)} dx + \int_0^{\sqrt{-a}} -F(x)g(x) \frac{1}{y_2(x) - F(x)} dx
\end{align*}
Assume that \( x = x_1(y) \) and \( x = x_2(y) \) on \( BC \) and \( CD \) respectively. It is obvious that \( x_2(y) > x_1(y) \) for all \( y \in [y_B, y_C] \). Thus, \( F(x_1(y)) - F(x_2(y)) < 0 \) for \( y \in [y_B, y_C] \) because \( F(x) \) is increasing when \( x > \sqrt{-b} \). Therefore,

\[
\int_{BC} F(x)dx + \int_{CD} F(x)dy = \int_{y_B}^{y_C} F(x_1(y))dy + \int_{y_B}^{y_C} F(x_2(y))dy = \int_{y_B}^{y_C} \{F(x_1(y)) - F(x_2(y))\}dy < 0. \tag{12}
\]

Similarly to (12), we obtain \( \int_{EG} F(x)dy < 0 \). Clearly, \( F(x) > 0 \) for all abscissas \( x \) of points on \( DE \) because \( DE \) lies on the right-hand side of \( E_R \). Then, \( \int_{DE} F(x)dy = \int_{y_B}^{y_C} F(x)dy < 0 \). Thus, \( \int_{AH} F(x)dy = \int_{AB + BC + CD + DE + EG + GH} F(x)dy < 0 \), which contradicts (11). Therefore, system (5) has no small limit cycle.

Secondly, by the symmetry of the phase portraits of (5) it is impossible for a limit cycle to surround exactly two equilibria among \( E_L, E_0 \) and \( E_R \). Finally, we claim that there is no small limit cycle. In fact, it is impossible for a small limit cycle to surround \( E_0 \) because \( E_0 \) is a saddle. Assume that there is a small limit cycle (denoted by \( L_2 \)) surrounding \( E_R \) as shown in Figure 3(b). By the expression of \( \dot{y} \) in (5) one can see that \( A \) is the highest point and \( C \) is the lowest one on \( L_2 \). Then,

\[
\int_{AB} F(x)dy + \int_{CD} F(x)dy = \int_{L_2} F(x)dy = \oint_{L_2} dE = 0, \tag{13}
\]

where \( E(x, y) \) is given in (10). Let \( x_1(y), x_2(y) \) be the abscissas of the points on the segmental arcs \( CD \) and \( AB \), respectively. It is obvious that \( x_1(y) < x_2(y) \) for all \( y \in (y_C, y_A) \), where \( y_C, y_A \) are the ordinates of the equilibria \( C, A \) respectively. Note that \( \sqrt{a} \) is the abscissa of \( E_R \). Thus, \( F(x_1(y)) < F(\sqrt{a}) < F(x_2(y)) \). Therefore,

\[
\int_{AB} F(x)dy + \int_{CD} F(x)dy = \int_{y_C}^{y_A} \{F(x_1(y)) - F(x_2(y))\}dy < 0, \tag{14}
\]

which contradicts (13). Hence, (5) has no small limit cycle.

Assume that system (5) has a homoclinic loop \( \Gamma \) surrounding \( E_R \). Similarly to (14), we obtain \( \oint_{\Gamma} dE = \oint_{\Gamma} F(x)dy < 0 \), which contradicts \( \oint_{\Gamma} dE = 0 \) by [29, Chapter 4]. Therefore, system (5) has no homoclinic loop. \( \square \)

Note that, from Lemma 2.1, \( E_0 \) is not of saddle type when \( a \leq 0 \). However, when \( a > 0 \), there are three equilibria and \( E_0 \) is a saddle. In the following lemma, the location of the stable manifold and the unstable one of \( E_0 \) is given.

**Lemma 3.3.** Assume that \( a > 0 \) and \( b < 0 \) in system (5). Then for fixed \( a \)

1. \( x_A \) increases continuously and \( x_B \) decreases continuously as \( b \) increases, where \( x_A, x_B \) are the abscissas of the farthest intersection points of the stable manifold and unstable one on the curve \( y = F(x) \) in the right half-plane, respectively;

2. \( x_C \) increases continuously and \( x_D \) decreases continuously as \( b \) increases, where \( x_C \) is the abscissa of the first intersection point on the curve \( y = F(x) \) in the right half-plane of the orbit starting from \( P \) lying on the positive \( y \)-axis as \( t \to +\infty \), \( x_D \) is the abscissa of the first intersection point on the curve...
$y = F(x)$ in the right half-plane of the orbit starting from the symmetric points $Q$ of $P$ about the origin as $t \to -\infty$.

Proof. Conclusion (2) can be proved similarly to conclusion (1) and, hence, we just give the detailed proof for conclusion (1). By transformation

$$x \to \sqrt{a}x, \ y \to ay + \sqrt{abx} + a\sqrt{ax^3}, \ t \to t/\sqrt{a},$$

system (5) is written as

$$\dot{x} = y, \ \dot{y} = x - x^3 - 3\sqrt{a}\left(\frac{b}{3a} + x^2\right)y.$$  \hspace{1cm} (16)

Transformation (15) changes the original $y$-axis to the new $y$-axis, the original curve $y = F(x)$ to the new $x$-axis. As indicated in [29, Chapter 4], the phase portraits of (5) are homeomorphic globally with that of (16). It is easy to check that (16) has three equilibria at $(-1,0), (0,0), (1,0)$ and, moreover, $(0,0)$ is a saddle. Let $\tilde{W}_0^s$ and $\tilde{W}_0^u$ be the stable manifold and the unstable one of system (16) at the saddle point $(0,0)$, respectively; $\tilde{W}_x^s$ and $\tilde{W}_x^u$ the stable manifold and the unstable one of system (16)$|_{\tilde{x} = \tilde{x}_x, \tilde{y} = \tilde{y}_x}$ at the saddle point $(0,0)$, respectively, where $0 < |\epsilon| \ll 1$; $\tilde{x}_A, \tilde{x}_B, \tilde{x}_A(\epsilon), \tilde{x}_B(\epsilon)$ be the intersection points of $\tilde{W}_0^s, \tilde{W}_0^u, \tilde{W}_x^s$ and $\tilde{W}_x^u$ on the positive $x$-axis, respectively. (16)$|_{\tilde{x} = \tilde{x}_x, \tilde{y} = \tilde{y}_x}$ is regarded as a small perturbation of (16). Denote the points on $\tilde{W}_0^s, \tilde{W}_x^s$ for $x \in (0, \delta)$ as $(x, y_0^s(x))$, $(x, y_x^s(x))$ respectively, where $\delta := \min(\tilde{x}_A, \tilde{x}_B(\epsilon))$. Let $z_{1,b}(x) := y_x^s(x) - y_0^s(x)$ be the distance between $y_x^s(x)$ and $y_0^s(x)$. Note that $z_{1,b}(0) = 0$. For (16) and all $x$ in $(0, \delta),$

$$z_{1,b}(x) = z_{1,b}(x) - z_{1,b}(0) = \left\{y_x^s(\tau) - y_0^s(\tau)\right\}_{\tau = 0}^{\tau = x} = \int_0^x \left\{\frac{\tau - \tau^3 - \left(\frac{b}{3a} + 3\sqrt{a}\tau^2\right)y_x^s(\tau)}{y_0^s(\tau)}\right\}d\tau = H_1(x) + \int_0^x z_{1,b}(\tau)H_2(\tau)d\tau,$$  \hspace{1cm} (17)

where $H_1(x) := -ax/\sqrt{a}, \ H_2(\tau) := (\tau^3 - \tau)/(y_0^s(\tau)y_x^s(\tau))$. It follows from (17) that $H_2(x)z_{1,b}(x) = H_1(x)H_2(x) + H_2(x)\int_0^x z_{1,b}(\tau)H_2(\tau)d\tau$. Then,

$$\frac{dH_3(x)}{dx} - H_2(x)H_3(x) = H_1(x)H_2(x),$$  \hspace{1cm} (18)

where $H_3(x) := \int_0^x z_{1,b}(\tau)H_2(\tau)d\tau$. Solving $H_3$ from (18) we obtain

$$H_3(x) = \int_0^x H_1(\tau)H_2(\tau)\exp\left\{\int_\tau^x H_2(\eta)d\eta\right\}d\tau.$$  \hspace{1cm} (19)

Thus, by (17) and (19),

$$z_{1,b}(x) = H_1(x) + \int_0^x H_1(\tau)H_2(\tau)\exp\left\{\int_\tau^x H_2(\eta)d\eta\right\}d\tau = H_1(0)\exp\left\{\int_0^x H_2(\eta)d\eta\right\} + \int_0^x H'_1(\tau)\exp\left\{\int_\tau^x H_2(\eta)d\eta\right\}d\tau = -\frac{\epsilon}{\sqrt{a}}\int_0^x \exp\left\{\int_\tau^x H_2(\eta)d\eta\right\}d\tau = \left\{\begin{array}{ll}
< 0, & \text{if } \epsilon > 0, \\
> 0, & \text{if } \epsilon < 0,
\end{array}\right.$$  \hspace{1cm} (20)
which implies that $\tilde{W}_{\gamma}'$ lies below $\tilde{W}_{\gamma}$, i.e., $\tilde{x}_A(\epsilon) > \tilde{x}_A$ when $\epsilon > 0$. Therefore, for (16) $\tilde{x}_A$ is increasing as $b$ increases. Similarly, we obtain
\[ z_{2,\delta}(x) = \frac{-\epsilon}{\sqrt{a}} \int_0^x \exp \left\{ \int_r^x \tilde{H}_2(y)dy \right\} dr = \begin{cases} < 0, & \text{if } \epsilon > 0, \\ > 0, & \text{if } \epsilon < 0, \end{cases} \tag{21} \]

where $\tilde{H}_2(x) := (x^3 - x)/(y_0^u(x)y_0^u(x))$ i.e., $\tilde{x}_B$ is decreasing as $b$ increases. It is the same for (5) because in the transformation (15) $x$ is independent on $b$, i.e., $x_A$ increases and $x_B$ decreases as $b$ increases.

Actually by the First Comparison Theorem (see [17, Chapter 1, Corollary 6.3]) for equations
\[ \frac{dy}{dx} = \frac{x - x^3}{y} - \left( \frac{b}{\sqrt{a}} + 3\sqrt{a}x^2 \right), \quad \frac{dy}{dx} = \frac{x - x^3}{y} - \left( \frac{b + \epsilon}{\sqrt{a}} + 3\sqrt{a}x^2 \right) \tag{22} \]
one also can obtain the change of orbits about $x$ or condition (15), it is easy to see that for (5) $x_A$ increases as $a$ increases. Similarly, we find that for (16) $\tilde{x}_A$ is increasing as $a$ increases. Similarly we find that for (16) $\tilde{x}_A$ is increasing as $\epsilon$ increases.

\[ \tilde{z}_{1,a}(x) := (\sqrt{a} - \sqrt{a + \gamma}) \int_0^x \left( 3s^2 - \frac{b}{\sqrt{a^2 + a\gamma}} \right) \exp \left\{ \int_s^x \tilde{H}_2(\tau)d\tau \right\} ds \]
\[ = \begin{cases} > 0, & \text{if } \gamma < 0, \\ < 0, & \text{if } \gamma > 0, \end{cases} \tag{23} \]

where $\tilde{H}_2(x) = (x - x^3)/(y_0^u(x)y_0^u(x))$. Thus, for (16) $\tilde{x}_A$ is increasing as $a$ increases. Similarly we find that for (16)
\[ \tilde{z}_{2,a}(x) := (\sqrt{a} - \sqrt{a + \gamma}) \int_0^x \left( 3s^2 - \frac{b}{\sqrt{a^2 + a\gamma}} \right) \exp \left\{ \int_s^x \tilde{H}_2(\tau)d\tau \right\} ds \]
\[ = \begin{cases} > 0, & \text{if } \gamma < 0, \\ < 0, & \text{if } \gamma > 0, \end{cases} \tag{24} \]
i.e., $\tilde{x}_B$ is decreasing as $a$ increases, where $\tilde{H}_2(x) = (x - x^3)/(y_0^u(x)y_0^u(x))$. By (15), it is easy to see that for (5) $x_A = \sqrt{a}x_A$ and $x_B = \sqrt{a}x_B$; for (5) $x_A(\gamma) = \sqrt{a + \gamma}x_A(\gamma)$ and $x_B(\gamma) = \sqrt{a + \gamma}x_B(\gamma)$. So, $x_A(\gamma) > x_A$ because $\gamma > 0$ and $x_A(\gamma) > x_A$, i.e., $x_A$ increases as $a$ increases. However, it is not easy to judge the change of $x_B$ depending on $a$ because we do not know which one is greater between $x_B(\gamma)$ and $x_B$. Note that $x_B(\gamma) = \sqrt{a + \gamma}x_B(\gamma)$, $x_B = \sqrt{a}x_B$ and $\tilde{x}_B(\gamma) < \tilde{x}_B$.

**Lemma 3.4.** If system (5) has at least two large limit cycles and $L_1, L_2$ are the most close ones to $E_0$, where $L_2$ denotes the outer one, then when either condition (4c) or condition (5c) holds,
\[ \oint_{L_1} \text{div}(y - F(x), -g(x))dt > \oint_{L_2} \text{div}(y - F(x), -g(x))dt. \tag{25} \]
Proof. We firstly consider condition (c4). The corresponding phase portrait is shown in Figure 4(a). By Bendixson-Dulac Theorem ([29, Chapter 4]), each \( L_i \) has two intersection points with the line \( x = x_0 \), where \( x_0 \) is the unique positive zero of \( F'(x) \). Denote these two points by \( B_i, C_i \) as shown in Figure 4(a). By the symmetry of the phase portraits

\[
2 \int_{A_iB_i} f(x)dt = \int_{L_i} f(x)dt = -\int_{L_i} \text{div}(y - F(x), -g(x))dt,
\]

where \( i = 1, 2 \). On the segmental arcs \( A_1B_1 \) and \( A_2B_2 \), let \( y = y_{1,2}(x) \), respectively. Since \( f(x) < 0 \) for \( 0 < x < x_0 \) and \( y_i(x) - F(x) > 0 \) \( (i = 1, 2) \),

\[
\int_{A_2B_2} f(x)dt - \int_{A_1B_1} f(x)dt = \int_0^{x_0} \left\{ \frac{f(x)}{y_2(x) - F(x)} - \frac{f(x)}{y_1(x) - F(x)} \right\} dx
\]

\[
= \int_0^{x_0} \frac{f(x)(y_1(x) - y_2(x))}{(y_1(x) - F(x))(y_2(x) - F(x))} dx > 0.
\]

Similarly, we obtain \( \int_{C_2F_2} f(x)dt - \int_{C_1F_1} f(x)dt > 0 \). By [29, Lemma 4.5, Chapter 4] or [14], \( \int_{B_2G_2C_2} f(x)dt - \int_{B_1G_1C_1} f(x)dt > 0 \). Therefore, (25) holds.

![Figure 4](image)

**Figure 4.** Two large limit cycles

Consider (c5). The corresponding phase portraits are shown in Figure 4(b), where \( B_i, C_i \ (i = 1, 2) \) are the intersection points between \( L_i \) and the line \( x = x_R \). (25) can be proved similarly to the case (c4). In the similar proof, the unique difference is the proof of (27). Thus, here we just give a detailed proof of (27) in the case (c5). In fact, for each \( i = 1, 2 \),

\[
\int_{A_iB_i} f(x)dt
\]

\[
= -\int_0^{x_R} F'(x) \frac{F(x) - y(x)}{F(x) - y_i(x)} dx
\]

\[
= -\int_0^{x_R} \frac{1}{F(x) - y_i(x)} \frac{1}{F(x) - y_i(x)} d(F(x) - y_i(x)) - \int_0^{x_R} \frac{y_i(x)}{F(x) - y_i(x)} dx
\]

\[
= -\ln \left| \frac{F(x_R) - y_i(x_R)}{F(0) - y_i(0)} \right| - \int_0^{x_R} \frac{y_i(x)}{F(x) - y_i(x)} dx
\]

\[
= -\ln \left| \frac{F(0) - y_i(x_R)}{F(0) - y_i(0)} \right| - \int_0^{x_R} \frac{F(x) - y_i(x_R)}{F(0) - y_i(0)} \frac{F(x) - y_i(x_R)}{F(x) - y_i(x_R)} dx
\]

\[
= -\ln \left| \frac{F(x_R) - y_i(x_R)}{F(0) - y_i(x_R)} \right| + \int_0^{x_R} \frac{y_i(x)}{F(0) - y_i(x)} dx - \int_0^{x_R} \frac{g(x)}{(F(x) - y_i(x))} dx
\]

\[
= -\ln \left| \frac{F(x_R) - y_i(x_R)}{F(0) - y_i(x_R)} \right| + \int_0^{x_R} \frac{g(x)}{(F(0) - y_i(x))(F(x) - y_i(x))} dx
\]
we consider the case that 3
Lemma 2.1. Assume that there is a small limit cycle surrounding b
Proof. Firstly, we claim that there is no small limit cycle. In fact, it is impossible
is stable and large. Moreover, for fixed
Lemma 3.5.
From the expressions of $\tilde{f}(\tilde{x})$ expression of $\tilde{F}$
other hand, $\tilde{f}(x) := -b - 3(x - \sqrt{a})^2$, implying that in $(-\infty, \sqrt{a})$, $\tilde{f}(x)$ has a unique
zero $x_0 := -\sqrt{-b/3} + \sqrt{a} < 0$. Moreover, $\tilde{f}(x) < 0$ for all $x \in (-\infty, x_0)$ and $\tilde{f}(x) > 0$ for all $x \in (x_0, \sqrt{a})$. Then (i) of [14, Theorem 2.1] holds. It is easy
to check that $\tilde{F}(x_0) = 2b\sqrt{-b/3}/3 < 0$ and $\tilde{F}(x) \to +\infty$ as $x \to -\infty$. On the
other hand, $\tilde{F}'(x) = \tilde{f}(x) < 0$ for all $x \in (-\infty, x_0)$. Thus, in $(-\infty, x_0)$ function $\tilde{F}(x)$ has a unique zero. Therefore, (ii) of [14, Theorem 2.1] holds. From the
expression of $\tilde{g}$ we obtain $x\tilde{g}(x) = x^2(x - 2\sqrt{a})(x - \sqrt{a})$, implying $x\tilde{g}(x) > 0$ for all
x \in (-\infty, 0) \cup (0, \sqrt{a})$. Thus, (iii) of [14, Theorem 2.1] holds.
Assume that there exist $\tilde{x}_1$ and $\tilde{x}_2$ such that $\tilde{x}_1 < x_0 < 0 < \tilde{x}_2 < \sqrt{a}$ and

\[
\tilde{F}(\tilde{x}_1) = \tilde{F}(\tilde{x}_2) = \frac{\tilde{g}(\tilde{x}_1)}{\tilde{f}(\tilde{x}_1)} = \frac{\tilde{g}(\tilde{x}_2)}{\tilde{f}(\tilde{x}_2)}.
\]

(29)

From the expressions of $\tilde{F}, \tilde{g}, \tilde{f}$, (29) is equivalent to the existence of $(x_1, x_2) := (\tilde{x}_1 - \sqrt{a}, \tilde{x}_2 - \sqrt{a})$ such that

\[
F(x_1) = F(x_2), \quad \frac{g(x_1)}{f(x_1)} = \frac{g(x_2)}{f(x_2)}.
\]

(30)

By the first equality of (30),

\[
x_1^2 + x_1x_2 + x_2^2 + b = 0.
\]

(31)

Let $\eta := x_1 + x_2$. Then, from (31) and the fact that $x_1 < x_2$ we have

\[
\eta = - \frac{x_2 - \sqrt{-4b - 3x_2^2}}{2} + x_2 = \frac{x_2 - \sqrt{-4b - 3x_2^2}}{2},
\]

\[
\int_{\mathbb{A}_2 \mathbb{B}_2} f(x)dt = \int_{\mathbb{A}_1 \mathbb{B}_1} f(x)dt = \int_{y_1(x_R) - F(x_R)} y_1(x_R) - F(x_R) \left( \frac{y_2(x_R) - F(x_R)}{x_R} \right) \left( x_R \right) dx_R.
\]

On the other hand, it is easy to see that $y_2(x) - F(x) > y_1(x) - F(x) > 0, y_2(x) > y_1(x) > 0$ and $g(x) < 0, F(x) < 0$ for all $x \in (0, x_1]$. Therefore,

\[
\frac{y_1(x_R) - F(x_R)}{y_2(x_R)} = \frac{y_2(x_R) - F(x_R)}{y_2(x_R)} > 1, \quad \tilde{F}(x) > 0,
\]

which implies that (27) also holds when (c5) holds. 

Lemma 3.5. When condition (c4) holds, system (5) has a unique limit cycle, which
is stable and large. Moreover, for fixed a the amplitude of the unique limit cycle is
decreasing with respect to b.

Proof. Firstly, we claim that there is no small limit cycle. In fact, it is impossible
for a small limit cycle to surround $E_0$ because $E_0$ is a saddle when a > 0 by
Lemma 2.1. Assume that there is a small limit cycle surrounding $E_L$. Firstly
we consider the case that $3a + b < 0$ in condition (c4). Using transformation
$x \to x - \sqrt{a}, y \to -y - (a + b)\sqrt{a}$, we move $E_L$ of (5) to the origin of system

\[
\dot{x} = -y - F(x - \sqrt{a}) - (a + b)\sqrt{a} =: -y + \tilde{F}(x), \quad \dot{y} = g(x - \sqrt{a}) := \tilde{g}(x).
\]

(28)

Then $\tilde{f}(x) := \tilde{F}' = -b - 3(x - \sqrt{a})^2$, implying that in $(-\infty, \sqrt{a})$, $\tilde{f}(x)$ has a unique
zero $x_0 := -\sqrt{-b/3} + \sqrt{a} < 0$. Moreover, $\tilde{f}(x) < 0$ for all $x \in (-\infty, x_0)$ and $\tilde{f}(x) > 0$ for all $x \in (x_0, \sqrt{a})$. Then (i) of [14, Theorem 2.1] holds. It is easy
to check that $\tilde{F}(x_0) = 2b\sqrt{-b/3}/3 < 0$ and $\tilde{F}(x) \to +\infty$ as $x \to -\infty$. On the
other hand, $\tilde{F}'(x) = \tilde{f}(x) < 0$ for all $x \in (-\infty, x_0)$. Thus, in $(-\infty, x_0)$ function $\tilde{F}(x)$ has a unique zero. Therefore, (ii) of [14, Theorem 2.1] holds. From the
expression of $\tilde{g}$ we obtain $x\tilde{g}(x) = x^2(x - 2\sqrt{a})(x - \sqrt{a})$, implying $x\tilde{g}(x) > 0$ for all
x \in (-\infty, 0) \cup (0, \sqrt{a})$. Thus, (iii) of [14, Theorem 2.1] holds.
which implies that
\[ \eta \in \left( -\frac{\sqrt{a} - \sqrt{4b - 3a}}{2}, -\sqrt{-b} \right) \subset \left( -\frac{2\sqrt{3b}}{3}, -\sqrt{-b} \right) \]
because \( x_2 \in (-\sqrt{a}, 0) \) and \( a \in (0, -b/3) \). Thus, \( \eta^2 \in (-b, -4b/3) \). Let \( h(x) := 3x^2 + 3(a + 2b)x + 2ab + 2b^2 \), which is increasing for \( x \in (-b, -4b/3) \). It is easy to compute that \( h(-b) = -b(a + b) < 0 \) and \( h(-4b/3) = -2b(3a + b)/3 < 0 \). Therefore, \( h(\eta^2) < 0 \), i.e.,
\[ 3\eta^4 + 3(a + 2b)\eta^2 + 2ab + 2b^2 < 0. \] (32)
On the other hand, by (31) we obtain \( x_1 x_2 = \eta^2 + b \). Then, from the second equality of (30) \( 3\eta^4 + 3(a + 2b)\eta^2 + 2ab + 2b^2 = 0 \), which contradicts (32). Therefore, (29) has no solution. By [14, Corollary 2.2] system (28) has no limit cycle on the left-hand side of the line \( x = \sqrt{a} \) and, hence, system (5) has no limit cycle on the left-hand side of the line \( x = 0 \), i.e., there is no limit cycle surrounding \( E_L \). Therefore, system (5) has no small limit cycle when condition (c4) holds and \( 3a + b < 0 \).

Now we claim that system (5) has no small limit cycle when condition (c4) holds and \( 3a + b = 0 \). Assume there exists a small limit cycle \( L \) surrounding \( E_R \), i.e., the limit cycle lies in the interior \((0, +\infty)\). Without loss of generality, we assume that there is no other limit cycles between \( L \) and \( E_R \). Since \( E_R \) is a source as given in Lemma 2.1, \( L \) is inner stable. Let \( A, B \) be the highest point and the lowest one on \( L \) and \( x_1(y), x_2(y) \) be the abscissas of the points on the clockwise segmental arcs \( \overline{BA} \) and \( \overline{AB} \), respectively. It is obvious that \( 0 < x_1(y) < x_2(y) \) for all \( y \in (y_B, y_A) \), where \( y_B, y_A \) are the ordinates of \( B, A \) respectively. Thus, When \( 3a + b = 0 \),
\[ \int_L \text{div}(y - F(x), -g(x))dt = -\int_L f(x)dt = \int_L f(x)\frac{dy}{x} = \int_{y_B}^{y_A} \frac{3}{x_1(y) - x_2(y)}dy = 3 \int_{y_B}^{y_A} \frac{x_2(y) - x_1(y)}{x_1(y)x_2(y)}dy \]
\[ > 0. \] (33)
Therefore, by [29, Theorem 2.2, Chapter 4] limit cycle \( L \) is unstable, which contradicts that \( L \) is inner stable. Hence, system (5) has also no limit cycle only surrounding \( E_R \) when condition (c4) holds and \( 3a + b = 0 \).

![Graph](image_url)

(a) Discussion about \( \Gamma \)  
(b) Change of amplitude

**Figure 5.** \( a, b \) satisfy condition \( (c4) \)

From the above, we see that for system (5) the large limit cycle if it exists. In the following we prove the existence of limit cycles. We first claim that \( x_A < x_B \), where \( x_A, x_B \) are abscissas of the intersection points \( A, B \) of the stable manifold and the unstable one on the curve \( y = F(x) \) in the right half-plane, respectively. In fact,
if \( x_A \geq x_B \) and \( 3a + b < 0 \), from Lemma 3.3 we get that \( x_A(\epsilon) > x_B(\epsilon) \) in system (5)\(_{b \to b+\epsilon} \), for sufficiently small \( \epsilon > 0 \). Thus, for system (5)\(_{b \to b+\epsilon} \) there exists a limit cycle in interior of the closed curve \( \Gamma \) composed by the unstable manifold, \( B(\epsilon)A(\epsilon) \) (on curve \( y = F(x) \)) and the stable manifold in the right half-plane because \( E_R \) is a source, which contradicts the non-existence of limit cycles only surrounding one equilibrium proved in the first part of this lemma. If \( x_A > x_B \) and \( 3a + b = 0 \), there exists a limit cycle in interior of the closed curve \( \Gamma \) composed by the unstable manifold, \( BA \) (on curve \( y = F(x) \)) and the stable manifold in the right half-plane because \( E_R \) is a source, which contradicts the non-existence of limit cycles only surrounding one equilibrium proved in the first part of this lemma. If \( x_A = x_B \) and \( 3a + b = 0 \), i.e., there exists a homoclinic loop \( \Gamma \) in the right half-plane. Since \( E_R \) is a source and there is no small limit cycle surrounding \( E_R \), \( \Gamma \) is stable. As shown in Figure 5(a), let \( P_1, P_2 \in \Gamma \) be sufficiently close to \( E_0 \) and \( \Gamma' \) be the close curve \( \hat{P}_1\hat{P}_2 \cup \gamma \) surrounding \( E_R \), where \( \hat{P}_1\hat{P}_2 \) is the segment on \( \Gamma \) and \( \gamma \) is the smooth segment near \( E_0 \). Then, \( \oint_{\Gamma'} \text{div}(y - F(x), -g(x)) \, dx \) is positive because

\[
\oint_{\Gamma} \text{div}(y - F(x), -g(x)) \, dy > 0
\]

as in (33), where \( C, D \) are the highest point and the lowest one on \( \Gamma' \) and \( x_1(y), x_2(y) \) be the abscissas of the points on the clockwise segmental arcs \( DC \) and \( CD \), respectively. Thus, \( \Gamma \) is unstable, which contradicts its stability. Therefore, \( x_A < x_B \) when \( 3a + b \leq 0 \). Therefore, associated with the instability of equilibria at infinity, system (5) has at least one limit cycle, which is large.

Now we prove the uniqueness of limit cycle. Assume that system (5) has at least two limit cycles. Denote the most close ones to \( E_0 \) by \( L_1, L_2 \), where \( L_2 \) is the outer one. Then \( \oint_{L_1} \text{div}(y - F(x), -g(x)) \, dx \leq 0 \) because \( E_L, E_R \) are sources and system (5) has no small limit cycle. When \( \oint_{L_1} \text{div}(y - F(x), -g(x)) \, dx < 0 \), it follows from (25) that \( \oint_{L_2} \text{div}(y - F(x), -g(x)) \, dx < 0 \). In other words, \( L_1 \) and \( L_2 \) are stable. This is a contradiction. When \( \oint_{L_1} \text{div}(y - F(x), -g(x)) \, dx = 0 \), then \( L_1 \) is either stable or semi-stable since both \( E_L \) and \( E_R \) are sources. In the case that \( L_1 \) is a stable limit cycle, \( L_2 \) is also a stable limit cycle simultaneously which implies a contradiction. In the case that \( L_1 \) is a semi-stable limit cycle, \( L_1 \) is internally stable and externally unstable. For system (5)\(_{b \to b+\epsilon} \) (\( \epsilon > 0 \)), as proved in Lemma 3.3 there will occur two limit cycles in the neighborhood of \( L_1 \), where the internally one \( \hat{L}_1 \) is stable and the externally one \( \hat{L}_1 \) is unstable. From Lemma 3.4, for (5)\(_{b \to b+\epsilon} \) we obtain

\[
\oint_{\hat{L}_1} \text{div}(y - F(x), -g(x)) \, dx > \oint_{L_1} \text{div}(y - F(x), -g(x)) \, dx.
\]  (34)

By stability of \( \hat{L}_1 \) and \( \hat{L}_1 \), \( \oint_{\hat{L}_1} \text{div}(y - F(x), -g(x)) \, dx \leq 0 \leq \oint_{L_1} \text{div}(y - F(x), -g(x)) \, dx \), contradicting (34). Therefore, system (5) has at most one limit cycle, which is stable.

By Lemma 3.3, for (5)\(_{b \to b+\epsilon} \) (\( \epsilon > 0 \)) we obtain the change of \( L \) as shown in Figure 5(b), where \( L \) is the unique limit cycle of unperturbed system (5), \( C, D \) are the intersection points of \( L \) on \( y \)-axis, \( A, B \) are the intersection points of orbits passing through \( C, D \) on the curve \( y = F(x) \) in the right half-plane, respectively. Note that here we only consider the change of limit cycles for system (5) satisfying (c4). Thus, we also require (5)\(_{b \to b+\epsilon} \) to satisfy condition (c4). It is easy to see that the unique limit cycle of (5)\(_{b \to b+\epsilon} \) lies in the inner of \( L \) by the Annular of
Poincaré-Bendixson Theorem because \( \tilde{y} := \tilde{C}B \cup \tilde{BA} \cup \tilde{AD} \cup \tilde{DE} \cup \tilde{EF} \cup \tilde{FC} \) is the external boundary curve, \( E_R \) is a source and there is no homoclinic loop, which is proved when we prove the existence of limit cycles. Therefore, the amplitude of the unique limit cycle decreases with respect to \( b \).

\[ \text{Lemma 3.6. When condition (c5) holds, system (5) has at most one small limit cycle. Moreover, the limit cycle is unstable, hyperbolic and for fixed a has an increasing amplitude with respect to b if it exists.} \]

**Proof.** There is no small limit cycle surrounding \( E_0 \) because \( E_0 \) is a saddle when \( a > 0 \) by Lemma 2.1. By transformation \( x \to -x - \sqrt{a}y \to -y - (a + b)\sqrt{a} \), and \( t \to -t \), \( E_L \) of (5) moves to the origin of system

\[ \dot{x} = -y + F(x + \sqrt{a}) - (a + b)\sqrt{a} =: -y + \tilde{F}(x), \quad \dot{y} = g(x + \sqrt{a}) =: \tilde{g}(x). \] (35)

Then \( \tilde{f}(x) := \tilde{F}' + 3(x + \sqrt{a})^2 \), which has a unique zero \( x_0 := \sqrt{-b/3} - \sqrt{a} < 0 \) in \((-\sqrt{a}, +\infty)\). Similarly to the first paragraph of the proof of Lemma 3.5, one can prove that (i), (ii) and (iii) of [14, Theorem 2.1] hold.

Consider equations

\[ \tilde{F}(\tilde{x}_1) = \tilde{F}(\tilde{x}_2), \quad \frac{\tilde{g}(\tilde{x}_1)}{\tilde{f}(\tilde{x}_1)} = \frac{\tilde{g}(\tilde{x}_2)}{\tilde{f}(\tilde{x}_2)}, \] (36)

where \(-\sqrt{a} < \tilde{x}_1 < x_0 < 0 < \tilde{x}_2 \). From the expressions of \( \tilde{F}, \tilde{g}, \tilde{f} \), (36) is equivalent to (30), where \( x_1 := \tilde{x}_1 + \sqrt{a} \) and \( x_2 := \tilde{x}_2 + \sqrt{a} \) satisfy that \( 0 < x_1 < x_0 + \sqrt{a} < \sqrt{a} < x_2 \). As in the proof of Lemma 3.5, let \( \eta := x_1 + x_2 \). Then the first equality of (30) implies that (31) holds. Thus, \( \eta = x_1 + x_2 = x_1 + (-x_1 + \sqrt{-4b - 3x^2})/2 = (x_1 + \sqrt{-4b - 3x^2})/2 \), from which it is easy to check that \( \eta \in (\sqrt{b}, 2\sqrt{-3b/3}) \).

As in the proof of Lemma 3.5, let \( h(x) := 3x^2 + 3(a + 2b)x + 2ab + 2b^2 \), which is increasing for \( x \in (-b, -4b/3) \) and satisfies that \( h(-b) = -b(a + b) < 0, h(-4b/3) = -2b(3a + b)/3 > 0 \). Thus, in \((-b, -4b/3), h(x) \) has a unique zero. On the other hand, the second equality of (30) implies that \( h(\eta^2) = 0 \). Note that \( \eta^2 \in (-b, -4b/3) \). Then \( h(\eta^2) = 0 \) has a unique root, denoted by \( \eta^* \), in \((-b, \sqrt{2b}/3) \). Thus, (30) has at most one solution \((x_1, x_2)\), satisfying that \( x_1 + x_2 = \eta^*, x_1x_2 = \eta^{*2} + b \).

Therefore, (36) has at most one solution, which means that (iv) of [14, Theorem 2.1] holds.

Straight computation shows that

\[ \frac{[F(x) - (a + b)\sqrt{a}]f(x)}{g(x)} = \frac{3x^4 + 3\sqrt{a}x^3 + (3a + 4b)x^2 + \sqrt{ab}x + (a + b)b}{x^2 + \sqrt{a}x}. \]

Since \( 6a^2 - ab - b^2 = (2a - b)(3a + b) > 0 \), for \( x > \sqrt{a} \),

\[ \frac{d[F(x) - (a + b)\sqrt{a}]f(x)/g(x)}{dx} = \frac{6x^5 + 12\sqrt{a}x^4 + 6ax^3 + 3(a + b)\sqrt{a}x^2 - 2(a + b)bx - (a + b)b\sqrt{a}}{(x^2 + \sqrt{a}x)^2} \]
\[ > \frac{6x^5 + 12\sqrt{a}x^4 + 6ax^3 - 6a\sqrt{a}x^2 - 12a^2x - 6a^2\sqrt{a}}{(x^2 + \sqrt{a}x)^2} \]
\[ = \frac{6(x^3 - a\sqrt{a})(x + \sqrt{a})}{(x^2 + \sqrt{a}x)^2} > 0, \]
There exists a strictly decreasing Proposition 2. system (5) when condition (c5) holds. the following proposition, we continue to study limit cycles and homoclinic loops of that system (5) has at most two small limit cycles when condition (c5) holds. In clinic loops when one of conditions (c1)-(c4) holds. By Lemma 3.6, we only obtain the origin and, moreover, the limit cycle is attracting and hyperbolic if it exists. By [14, Theorem 2.1], system (35) has at most one small limit cycle surrounding field (see [29, Chapter 4]) with respect to $b$ implying that $\frac{\dot{F}(x)}{g(x)}$ is increasing for $x \in (\sqrt{a}, +\infty)$. Thus, $\frac{\dot{F}(x)}{g(x)}$, which equals to $F(x + \sqrt{a}) - (a + b)\sqrt{a}$, is increasing for $x \in (0, +\infty)$. On the other hand, one can check that $\lim_{x \to \pm \infty} F(x) = 0$ and $\lim_{x \to +\infty} \dot{F}(x) = +\infty$. Thus, (v) of [14, Theorem 2.1] holds for system (35). By [14, Theorem 2.1], system (35) has at most one small limit cycle surrounding the origin and, moreover, the limit cycle is attracting and hyperbolic if it exists. Therefore, system (5) has at most one limit cycle surrounding $E_L$ and, moreover, the limit cycle is unstable and hyperbolic if it exists. It is the same for $E_R$ by the symmetry of the phase portraits.

For given $a > 0$ the vector field of system (35) is a generalized rotated vector field (see [29, Chapter 4]) with respect to $b$ for $x > -\sqrt{a}$ because

$$
\begin{vmatrix}
-y + [(b_2 + 3a_1 + 3\sqrt{a}x^2 + x^3] & (x + 2\sqrt{a})(x + \sqrt{a})x \\
-y + [(b_1 + 3a_1 + 3\sqrt{a}x^2 + x^3] & (x + 2\sqrt{a})(x + \sqrt{a})x
\end{vmatrix} 
= (b_2 - b_1)(x + 2\sqrt{a})(x + \sqrt{a})x^2 \geq 0,
$$

where $b_1 < b_2$. Thus, by [29, Theorem 3.5, Chapter 4] the amplitude of the unstable limit cycle surrounding the origin of (35) is monotonous with respect to $b$, so is the unstable limit cycle surrounding $E_L$ of (5). From Lemma 2.1, we see that the Hopf bifurcation occurs when $a = 0$ and the value of $b$ changes from $-3a$ to $-3a + \epsilon$, where $0 < \epsilon \ll 1$, and an unstable limit cycle appears in a small neighborhood of $E_L$ of (5). The amplitude of the unstable limit cycle is sufficiently small if $\epsilon$ is sufficiently small. Therefore, the amplitude increases as $b$ increases.

By Lemmas 3.1, 3.2 and 3.5, we obtain information about limit cycles and homoclinic loops when one of conditions (c1)-(c4) holds. By Lemma 3.6, we only obtain that system (5) has at most two small limit cycles when condition (c5) holds. In the following proposition, we continue to study limit cycles and homoclinic loops of system (5) when condition (c5) holds.

**Proposition 2.** There exists a strictly decreasing $C^\infty$ function $\varphi_1(a)$ for $a > 0$ such that

1. system (5) has a homoclinic loop of figure-eight type if and only if $b = \varphi_1(a)$;
2. $-3a < \varphi_1(a) < -a$ for all $a > 0$ and $\varphi_1(a) = -12a/5 + o(a)$ for small $|a|$;
3. when $-3a < b < \varphi_1(a)$, system (5) has exactly three limit cycles, two of which are unstable, small and surround $E_L, E_R$ respectively, the other one is stable and large;
4. when $b = \varphi_1(a)$, system (5) has a unique limit cycle, which is stable and large;

where for fixed $a$ the amplitudes of all stable (resp. unstable) limit cycles are decreasing (resp. increasing) with respect to $b$.

**Proof.** From Lemma 2.1, one can see that (5) has a unique equilibrium, which means that there is no homoclinic loop when either condition (c1) or condition (c2) holds. Lemma 3.2 states that system (5) has no homoclinic loops when condition (c3) holds. In the proof of Lemma 3.5, we prove that $x_A < x_B$ when considering the existence of limit cycles, where $x_A, x_B$ are abscissas of the intersection points $A, B$ of the stable manifold and the unstable one on the curve $y = F(x)$ in the right half-plane, respectively. Thus, system (5) has no homoclinic loops when (c4) holds. Therefore, we only need to consider (c5).
We claim that \( x_A > x_B \) when \( b = -a \), as shown in Figure 6(a). In fact, it is obvious that \( x_A \neq x_B \) because there is no homoclinic loop when condition (c3) holds. If \( x_A < x_B \) when \( b = -a \), then there exists a limit cycle surrounding \( E_R \) by the Annular of Poincaré-Bendixson Theorem because \( E_R \) is stable as given in Lemma 2.1, which contradicts Lemma 3.2. In the proof of Lemma 3.5, we see that \( x_A < x_B \) when \( b = -3a \). On the other hand, \( x_A - x_B \) is continuous and strictly increasing with respect to \( b \in [-3a, -a] \) by Lemma 3.3. Thus, there exists a unique value of \( b \), denoted by \( \varphi_1(a) \), in \((-3a, -a)\) such that \( x_A - x_B = 0 \). Moreover, by the continuous change of \( x_A \) and \( x_B \) function \( \varphi_1(a) \) is continuous. Consequently, system (5) has a homoclinic loop of figure-eight type if and only if \( b = \varphi_1(a) \). Conclusion (1) and the first part of conclusion (2) are proved.

To prove the smoothness of \( \varphi_1(a) \), we consider the equivalent system (16) and let \( b = \varphi_1(a) \), i.e., there is a homoclinic loop of figure-eight type, which intersects the positive x-axis at \((x_1, 0)\). Taking \((a, b) \to (a + \gamma, b)\) in (16), we assume that \( x_1 + \delta_1 \) and \( x_1 - \delta_2 \) are abscissas of the intersection points of the stable manifold and the unstable one on the positive x-axis. Furthermore, taking \((a + \gamma, b) \to (a + \gamma, b - \epsilon)\) we assume that \( x_1 + \delta_1 - \delta_3 \) and \( x_1 - \delta_2 + \delta_4 \) are abscissas of the intersection points of the stable manifold and the unstable one on the positive x-axis, and \( b - \epsilon = \varphi_1(a + \gamma) \).

We firstly consider that \( \gamma > 0 \). Consequently, \( \epsilon > 0 \) by the proof of Lemma 3.3. From (20), (21), (23) and (24), we obtain \( \bar{z}_{1,a}(x) = K_1(x, a, b)\gamma + o(\gamma) \), \( \bar{z}_{2,a}(x) = K_2(x, a, b)\gamma + o(\gamma) \), \( z_{1,b}(x) = K_3(x, a, b)\epsilon + o(\epsilon) \), \( z_{2,b}(x) = K_4(x, a, b)\epsilon + o(\epsilon) \), where

\[
K_1(x, a, b) := \left(-\frac{1}{2\sqrt{a}}\right)\int_0^x \left(3s^2 - \frac{b}{a}\right) \exp\left\{\int_s^x H_2^*(\tau)d\tau\right\}ds,
\]
\[
K_2(x, a, b) := \left(-\frac{1}{2\sqrt{a}}\right)\int_0^x \left(3s^2 - \frac{b}{a}\right) \exp\left\{\int_s^x H_2^*(\tau)d\tau\right\}ds,
\]
\[
K_3(x, a, b) := -\frac{1}{\sqrt{a}}\int_0^x \exp\left(\int_\tau^x H_2^*(\eta)d\eta\right)d\tau,
\]
\[
K_4(x, a, b) := -\frac{1}{\sqrt{a}}\int_0^x \exp\left(\int_\tau^x H_2^*(\eta)d\eta\right)d\tau,
\]
\[
H_2^*(x) := \frac{(x^3 - x)(y_0^*(x))^2}{(y_0^*(x))^2}, H_2^{**}(x) := \frac{(x^3 - x)(y_0^*(x))^2}{(y_0^*(x))^2}
\]

and \(y_0^*, y_0^{**}\) correspond the orbits on the stable manifold and on the unstable one of (16) respectively.

In the following we compute \( \delta_1 \). It is easy to obtain

\[
\delta_1 = \int_{x_1}^{x_1 + \delta_1} dx = \int_{\bar{z}_{1,a}(x_1)}^{\bar{z}_{1,a}(x_1 + \delta_1)} \frac{dy}{y} = \int_{\bar{z}_{1,a}(x_1)}^{\bar{z}_{1,a}(x_1 + \delta_1)} \frac{y}{g(x) - f(x)y}dy,
\]
where \( \tilde{g}(x) = x - x^3 \), \( \tilde{f}(x) = b/\sqrt{a} + 3\sqrt{a}x^2 \). Then

\[
\delta_1 = \int_{x_1, a(x_1)}^{0} \frac{y}{g(x_1 + \delta_1) + O(y) - f(x_1 + \delta_1)y + o(y)} dy = \left[ \frac{y^2}{2g(x_1 + \delta_1)} + \Gamma_1(y) \right]_{y = \tilde{z}_1, a(x_1)},
\]

where \( \Gamma_1(y) = o(y^2) \). Thus,

\[
\delta_1 = -\frac{\tilde{z}_1^2, a(x_1)}{2\tilde{g}(x_1 + \delta_1)} - \Gamma_1(\tilde{z}_1, a(x_1)) = -\frac{K_1^2(x_1, a, b)}{2\tilde{g}(x_1 + \delta_1)} \gamma^2 + o(\gamma^2).
\]  

(38)

Similarly to \( \delta_1 \), we compute \( \delta_i \) (\( i = 2, 3, 4 \)) and obtain

\[
\delta_2 = -\frac{K_2^2(x_1 - \delta_2, a, b)}{2\tilde{g}(x_1)} \gamma^2 + o(\gamma^2),
\]

\[
\delta_3 = -\frac{K_3^2(x_1 + \delta_1 - \delta_3, a + \gamma, b)}{2\tilde{g}(x_1 + \delta_1)} \epsilon^2 + o(\epsilon^2),
\]

\[
\delta_4 = -\frac{K_4^2(x_1 - \delta_2, a + \gamma, b)}{2\tilde{g}(x_1 - \delta_2 + \delta_4)} \epsilon^2 + o(\epsilon^2).
\]  

(39)

From \( \varphi_1(a + \gamma) = b - \epsilon \), we obtain \( \delta_1 = \delta_3 = \delta_2 + \delta_4 \). Then, it follows from (38) and (39) that

\[
\left( \frac{K_1^2(x_1, a, b)}{2\tilde{g}(x_1 + \delta_1)} + \frac{K_2^2(x_1 - \delta_2, a, b)}{2\tilde{g}(x_1)} \right) \gamma^2 + o(\gamma^2)
\]

\[
= \left( \frac{K_3^2(x_1 + \delta_1 - \delta_3, a + \gamma, b)}{2\tilde{g}(x_1 + \delta_1)} + \frac{K_4^2(x_1 - \delta_2, a + \gamma, b)}{2\tilde{g}(x_1 - \delta_2 + \delta_4)} \right) \epsilon^2 + o(\epsilon^2),
\]

which implies that

\[
\lim_{\gamma \to 0^+} \frac{\varphi_1(a + \gamma) - \varphi_1(a)}{\gamma} = \lim_{\gamma \to 0^+} \frac{\gamma}{\gamma} = \frac{\tilde{g}(x_1)}{2\tilde{g}(x_1)} \frac{K_1^2(x_1, a, b) + K_2^2(x_1, a, b)}{K_3^2(x_1, a, b) + K_4^2(x_1, a, b)} + \tilde{\gamma}(\epsilon)
\]

(40)

where \( \tilde{\gamma}(\gamma) \to 0 \) as \( \gamma \to 0 \), \( \tilde{\gamma}(\epsilon) \to \epsilon \) as \( \epsilon \to 0 \) and all \( K_i \)'s are given in (37). Note that \( \tilde{g}(x_1) < 0 \) and \( x_1 \) is analytic in \( a, b \) and \( b = \varphi_1(a) \). It is easy to check that the expression of the right-hand side of (40) is continuous in \( a \). Similarly to (40), one can obtain

\[
\lim_{\gamma \to 0^+} \frac{\varphi_1(a + \gamma) - \varphi_1(a)}{\gamma} = \frac{\tilde{g}(x_2) + K_3^2(x_1, a, b)}{K_1^2(x_1, a, b) + K_4^2(x_1, a, b)} = \lim_{\gamma \to 0^+} \frac{\varphi_1(a + \gamma) - \varphi_1(a)}{\gamma}.
\]

Thus, \( \varphi_1(a) \) is \( C^1 \). The \( C^{1+\infty} \) smoothness of \( \varphi_1(a) \) can be proved by induction from the expression given in (40).

By [5, Theorem 2.1, Chapter 4], there exists \( \varphi_1(a) := -12a/5 + o(a) \) such that system (6) with sufficiently small \( |a|, |b| \) has a homoclinic loop of figure-eight type if and only if \( b = \varphi_1(a) \), so does the equivalent system (16) of (5) with sufficiently small \( a, b \). Thus, the second part of conclusion (2) is proved.

Consider the case that \( -3a < b < \varphi_1(a) \) firstly. Since \( b = \varphi_1(a) \) corresponds a homoclinic loop of figure-eight type, \( x_A < x_B \) as shown in Figure 6(b) when \( b < \varphi_1(a) \) by Lemma 3.3. On the other hand, \( E_R \) is stable as given in Lemma 2.1. Thus, system (5) has at least one small limit cycle surrounding \( E_R \). Associated with the result of Lemma 3.6, system (5) has exactly one small limit cycle surrounding \( E_R \) and the limit cycle is unstable, so for \( E_L \). The existence, uniqueness, stability
of large limit cycle and the monotonicity of its amplitude with respect to \( b \) can be proved similarly as the proof of Lemma 3.5. Thus, conclusion (3) is proved.

\[ (a) \quad b = \varphi_1(a) \quad \quad (b) \quad b = \varphi_1(a) + \epsilon \]

**Figure 7.** Discussion on existence of limit cycles

\[ (a) \quad \text{Change of orbits} \quad \quad (b) \quad \text{Change of amplitudes} \]

**Figure 8.** Discussion about the perturbation system

Consider the case that \( b = \varphi_1(a) \) secondly. Assume that system (5) has one small limit cycle \( L_1 \) surrounding \( E_R \), as shown in Figure 7(a). By Lemma 3.6, \( L_1 \) is unstable and hyperbolic. From Lemma 3.3 we get that \( x_A(\epsilon) > x_B(\epsilon) \) in system (5) for sufficiently small \( \epsilon > 0 \). By [29, Theorem 3.5, Chapter 4] \( L_1 \) does not disappear and its amplitude is monotonous with respect to \( b \) and, hence, we denote it by \( \hat{L}_1 \) for system (5). Thus, \( \hat{L}_1 \) is sufficiently close to the original \( L_1 \) and still unstable by Lemma 3.6. Then, as shown in Figure 7(b), system (5) has another limit cycle \( \hat{L}_1 \), which contradicts the uniqueness of limit cycles given in Lemma 3.6. Therefore, there is no limit cycle only surrounding \( E_R \), so neither \( E_L \). Now we consider the outside of \( \Gamma_0 \). Since \( \text{div}(y - F(x), -g(x))|_{(0,0)} = -f(0) = -b > 0 \), by the continuity there is a neighborhood \( V_0 \) of the saddle \( E_0 \) such that \( \text{div}(y - F(x), -g(x)) > -b/2 \) for \( (x,y) \in V_0 \). Let \( T_1 \) and \( T_2 \) be the times of the corresponding flow of \( \Gamma_0 \) inside and outside \( V_0 \), respectively. Clearly, \( T_1 \to +\infty \) and \( T_2 \) is bounded, which implies that \( \int_{t_0}^t \text{div}(y - F(x), -g(x))dt = \)
Thus, by Proposition 2 system (16) has a homoclinic loop when \( b = \varphi_2(a) \), which implies that \( b/a \to -12/5 \) as \( a \to 0 \). If additionally \( a \) is sufficiently small, system (16) is a near-Hamiltonian system and the homoclinic loop lies near the curve \( 2y^2 + x^4 - 2x^2 = 0 \). By the form of transformation (15), the homoclinic loop of system (5) lies near the curve \( 2(y - bx - x^3)^2 + x^4 - 2ax^2 = 0 \). Obviously, it lies in a small neighborhood of the origin of system (5) when \( a \) is sufficiently small. When \( a,b \) are not limited to be sufficiently small, equilibria \( E_L \) and \( E_R \) can be far from the origin \( E_0 \) and so does the homoclinic loop.

By Proposition 2, system (5) has a unique large limit cycle when condition (c5) holds and \( b \leq \varphi(a) \). In the following proposition, we continue to study limit cycles and homoclinic loops of system (5) when condition (c5) holds and \( b > \varphi(a) \).

**Proposition 3.** For \( b \in (\varphi_1(a), -a) \), there exists a strictly decreasing \( C^0 \) function \( \varphi_2(a) \) for \( a > 0 \) such that

(1): \( \varphi_1(a) < \varphi_2(a) < -a \) for all \( a > 0 \) and \( \varphi_2(a) = -ca + o(a) \) for small \( |a| \), where \( c \approx 2.256 \);

(2): when \( \varphi_1(a) < b < \varphi_2(a) \), system (5) has exactly two limit cycles, where the two limit cycles are large and the inner (resp. outer) one is unstable (resp. stable);

(3): when \( b = \varphi_2(a) \), system (5) has a unique limit cycle, which is internally unstable, externally stable and large;

(4): when \( \varphi_2(a) < b < -a \), system (5) has no limit cycle;

where for fixed \( a \) the amplitudes of all stable (resp. unstable) limit cycles are decreasing (resp. increasing) with respect to \( b \).

**Proof.** This proof consists of six steps.

The first step is to prove that there is no small limit cycle. Otherwise, by Lemma 3.6 there exists a unique small limit cycle surrounding \( E_R \). By \( b > \varphi_1(a) \) and the change of orbits about \( b \) given in Lemma 3.3, the unique limit cycle is
externally stable. Thus, the limit cycle is semi-stable, contradicting the instability given in Lemma 3.6.

The second step is to there exist at most two large limit cycles. By Lemma 3.3 and Proposition 2, \( x_A > x_B \), where \( x_A, x_B \) are given in Lemma 3.3. Assume that there are at least three large limit cycles and \( L_1, L_2, L_3 \) are the most close ones to \( E_0 \) and denote the inner one, middle one, outer one respectively. Then, \( L_1 \) is internally unstable by \( x_A > x_B \), implying that

\[
\oint_{L_1} \text{div}(y - F(x), -g(x))dt \geq 0
\]  

(41)

by Theorem 2.2 of [29, Chapter 4]. It is to note that either all closed orbits are stable or they are unstable when the integral of the divergence along these closed orbits have the same sign. However, the two closed orbits with the same stability cannot be adjacent to each other. It is easy to prove that \( \geq \) should be \( > \) in (41). Otherwise, by Lemma 3.4,

\[
\oint_{L_1} \text{div}(y - F(x), -g(x))dt < \oint_{L_2} \text{div}(y - F(x), -g(x))dt < 0,
\]

implying that \( L_2, L_3 \) are stable. This is impossible because of their locations. Thus,

\[
\oint_{L_1} \text{div}(y - F(x), -g(x))dt > 0 = \oint_{L_2} \text{div}(y - F(x), -g(x))dt
\]

\[
> \oint_{L_3} \text{div}(y - F(x), -g(x))dt,
\]

i.e., \( L_2 \) is internally stable and externally unstable. Taking \( b \to b + \epsilon \) (\( \epsilon > 0 \)), as the proof of Proposition 2 we obtain that near \( L_2 \) there are two limit cycles, denoted by \( L_2 \) as the inner one by \( \hat{L}_2 \) as the outer one, where \( L_2 \) is stable and \( \hat{L}_2 \) is unstable. Hence,

\[
\oint_{L_2} \text{div}(y - F(x), -g(x))dt \leq 0 \leq \oint_{L_2} \text{div}(y - F(x), -g(x))dt,
\]

contradicting

\[
\oint_{L_2} \text{div}(y - F(x), -g(x))dt > \oint_{L_2} \text{div}(y - F(x), -g(x))dt
\]

given by Lemma 3.4. Therefore, (5) has at most two large limit cycles.

The third step is to prove that the inner large limit cycle and outer one are unstable and stable respectively if they exist. Moreover, for fixed \( a \) their amplitudes are monotonic with respect to \( b \). By Proposition 2 system (5) has one homoclinic loop of figure-eight type and one stable large limit cycle when \( b = \varphi_1(a) \). From the change of orbits given in Lemma 3.3, we obtain that (5)\( |_{b \to \varphi_1(a) + \epsilon} \) (\( \epsilon > 0 \)) has at least two large limit cycles. Associated with the above paragraph (5)\( |_{b \to \varphi_1(a) + \epsilon} \) has exactly two large limit cycles. Let \( L_1, L_2 \) denote the inner one and the outer one, respectively. If both \( L_1 \) and \( L_2 \) are internally unstable and externally stable, as in the above to \( L_2 \) taking a small perturbation \( b \to b + \epsilon \) (\( \epsilon < 0 \)) we obtain that near \( L_i \) (\( i = 1, 2 \)) there are two limit cycles for system (5)\( |_{b \to b + \epsilon} \). Thus, there are more than three limit cycles, which contradicts the fact that there are at most two limit cycles. Thus, from \( x_A > x_B \) and the instability of equilibria at infinity \( L_1 \) is unstable and \( L_2 \) is stable. Further, taking \( b \to b + \epsilon \) (\( \epsilon > 0 \)) we obtain the changes of orbits by Lemma 3.3 as shown in Figure 8(b), where \( P \) is the middle point of \( CD \) and \( Q \) is the symmetric point of \( P \) about the origin and the broken curves are
orbits of unperturbed system (5). Thus, the new unstable limit cycle lies outside of the original $L_1$ and the new limit cycle lies inside of the original $L_2$, i.e., for fixed $a$ the amplitude of the unstable (resp. stable) limit cycle are increasing (resp. decreasing) with respect to $b$.

![Figure 9](image)

**Figure 9.** Discussion about $\Pi(\rho, a, b)$

The fourth step is to prove the existence of $\varphi_2(a)$ such that there exists a unique large limit cycle when $b = \varphi_2(a)$. Moreover, by the stability of equilibria the unique large limit cycle is internally unstable and externally stable, i.e., semistable limit cycle. Consider the equivalent system (16) of (5) and let $\Pi(\rho,a,b)$ be the Poincaré return map for $\rho > 1$ and $\varphi_1(a) < b < -a$. We claim that $\Pi(\rho,a,b)$ is decreasing about $b$, i.e.,

$$\Pi(\rho,a,b) > \Pi(\rho,a,b + \epsilon), \quad 0 < \epsilon \ll 1.$$  \hspace{1cm} (42)

In fact, when $\rho \in (1,x_b]$, by the First Comparison Theorem (see [17, Chapter 1, Corollary 6.3]) for equations given in (22) we obtain the orbits of system (16) as shown in Figure 9(a), where $x_b$ is the abscissa of intersection point of the stable manifold on the positive $x$-axis of system (16), the broken curves are orbits of (16). Thus, (42) holds. When $\rho \in (x_b,x_b+\epsilon]$ and $\rho \in (x_b+\epsilon, +\infty)$, by Lemma 3.3 it is easy to obtain the orbits of system (16) as shown in Figure 9(b) and Figure 9(c) respectively, where $x_{b+\epsilon}$ is the abscissa of intersection point of the stable manifold on the positive $x$-axis of system (16). Thus, (42) holds. Then the successor function $h(\rho,a,b) := \Pi(\rho,a,b) - \rho$ depends continuously on $a,b,\rho$ by the continuous dependence of the solution on parameters and initial values and satisfies that $h(\rho,a,b) - h(\rho,a,b+\epsilon) = \Pi(\rho,a,b) - \Pi(\rho,a,b+\epsilon) > 0$, i.e., $h(\rho,a,b)$ is decreasing in $b$. Similarly, one can prove that $h(\rho,a,b)$ is also decreasing about $a$.

Note that, by the proof of Lemma 3.3 and the instability of equilibria at infinity, $h(\rho,a,b)$ is negative when $\rho$ lies either near 1 or near $+\infty$. Thus, for $\rho > 1$ function $h$ has a maximum value, i.e., $f(a,b) := \max_{\rho > 1} h(\rho,a,b)$ is well-defined. Further for
0 < \epsilon \ll 1, f(a,b) - f(a,b+\epsilon) = f(a,b) - h(\rho^*, a,b) + h(\rho^*, a,b) - f(a,b+\epsilon) > 0,\) where \(\rho^*\) satisfies that \(h(\rho^*, a,b) > 0\) for all \(\rho > 1\) for system (16). So, \(f(a,-a) < 0\). Since (5)|_{b \to \varphi_2(a)+\epsilon} has exactly two limit cycles (one is stable, the other one is unstable) surrounding all equilibria, \(f(a,\varphi_1(a) + \epsilon) > 0\). Thus, by the monotonicity of \(f\) in \(b\) there exists a unique \(\varphi_2(a) \in (\varphi_1(a), -a)\) such that \(f(a,\varphi_2(a)) = 0\), i.e., system (16)|_{b \to \varphi_2(a)} has limit cycles. Since there are at most two limit cycles and, moreover, one is unstable and the other one is stable when there are exactly two limit cycles, we obtain that there exists a unique \(\hat{\rho} > 1\) such that \(h(\hat{\rho},a,\varphi_2(a)) = f(a,\varphi_2(a)) = 0\) for all \(\rho \in (1, \hat{\rho}) \cup (\hat{\rho}, +\infty),\) i.e., (16)|_{b \to \varphi_2(a)} has a unique limit cycle, which is internally unstable and externally stable, so does (5)|_{b \to \varphi_2(a)}. Conclusion (3) is proved.

The fifth step is to give the expansion of \(\varphi_2(a)\) for sufficiently small \(a\) and the relation between \(\varphi_1(a)\) and \(\varphi_2(a)\). For \(0 < |\gamma| \ll 1\) we have \(f(a,\varphi_2(a)) = 0 = f(a + \gamma, \varphi_2(a+\gamma))\), which implies that \(\lim_{\gamma \to 0} \varphi_2(a+\gamma) = \varphi_2(a)\) because the continuity of \(f(a,b)\) in \((a,b)\) and the uniqueness of the value of \(b\) such that \(f(a,b) = 0\). Thus, \(\varphi_2(a)\) is continuous. Further, by the monotonicity of \(f\) in \((a,b)\) we obtain that \(\varphi_2(a+\gamma) - \varphi_2(a)\) is negative (resp. positive) if \(\gamma > 0\) (resp. \(\gamma < 0\)), meaning that \(\varphi_2(a)\) is decreasing. By [5, Theorem 2.1, Chapter 4], there exists a function \(\tilde{\varphi}_2(a) := -ca + o(a)\) such that (6) with sufficiently small \(|a|, |b|\) has a semi-stable limit cycle if and only if \(b = \tilde{\varphi}_2(a)\), where \(c \approx 2.256\), so does the equivalent system (16) of (5) with sufficiently small \(|a|, |b|\). Therefore, conclusion (1) is proved.

The sixth step is to prove that \(h(\rho,a,b)\) has exactly exist two zero points when \(b \in (\varphi_1(a), \varphi_2(a))\). That is, there exactly exist two large limit cycles. When \(\varphi_1(a) < b < \varphi_2(a)\), by the monotonicity of \(f\) in \(b\) we can obtain \(f(a,b) > 0\), implying that there are two values of \(\rho\) such that \(h(\rho,a,b) = 0\) because \(h(\rho,a,b) < 0\) when \(\rho\) lies either near 1 or near +\infty. Thus, there are two limit cycles. Conclusion (2) is proved. When \(\varphi_2(a) < b < -a\), by the monotonicity of \(f\) in \(b\) we obtain \(f(a,b) < 0\), implying that there is no limit cycle. Therefore, conclusion (4) is proved.

4. Bifurcation diagram and global phase portraits. In previous lemmas and propositions, we obtain the qualitative properties of equilibria, limit cycles and homoclinic loops for system (5). In this section by summarizing these results, for general parameters \((a,b) \in \mathbb{R}^2\) we give a bifurcation diagram and a complete classification of global phase portraits.

**Theorem 4.1.** As in Figure 10, the global bifurcation diagram of (5) consists of curves:

1. pitchfork bifurcation curve \(a = 0\), i.e., \(b\)-axis;
2. Hopf bifurcation curves \(H_1 = \{(a,b) : a < 0, b = 0\}\) for \(E_0\), \(H_2 = \{(a,b) : a > 0, b = -3a\}\) for \(E_L\) and \(E_R\);
3. focus-node bifurcation curves \(N_1^\pm = \{(a,b) : a < 0, b = \pm 2\sqrt{a}\}\) for \(E_0\), \(N_2^\pm = \{(a,b) : a > 0, b = -3a \pm 2\sqrt{2a}\}\) for \(E_L\) and \(E_R\);
4. homoclinic bifurcation curve \(HL = \{(a,b) : b = \varphi_1(a), a > 0\}\);
5. double limit cycle bifurcation curve \(DL = \{(a,b) : b = \varphi_2(a), a > 0\}\);

where \(\varphi_1, \varphi_2\) are given in Propositions 2 and 3.
Proof. Conclusions (1-3) follow from Proposition 1. Conclusions (4) and (5) are directly from (1) of Proposition 2 and (2-4) of Proposition 3, respectively.

Since parameters $a, b$ are not limited to be sufficiently small in the proofs of Propositions 2.1, 3.1 and 3.2, these bifurcation curves given in Theorem 4.1 are global for $a, b$. That is, these bifurcation curves exist in the whole parameter space, not in a small neighborhood of $(a, b) = (0, 0)$.

Clearly, after passing $(2, -2)$ the curve $N_2^+$ lies between the line $b = -a$ and $H_2$. It is easy to check that the slope of $N_2^+$ tends to $-3$, equaling to the slope of $H_2$, as $a \to +\infty$. However, in Figure 10 we are unable to judge if $N_2^+$ intersects $DL$ and $HL$ at some points respectively. When $N_2^+$ does not intersect $DL$ and $HL$ as shown in Figure 10, we obtain global phase portraits as in the following theorem.

**Theorem 4.2.** All global phase portraits of (5) are given in Figures 11, 12 and all results about limit cycles and homoclinic loops are given in Table 2, where

- $I := \{(a, b) : a \leq 0, b < -2\sqrt{a}\}$,
- $II := \{(a, b) : a < 0, -2\sqrt{a} < b < 0\}$,
- $III := \{(a, b) : a < 0, 0 \leq b < 2\sqrt{-a}\}$,
- $IV := \{(a, b) : a \leq 0, b \geq 2\sqrt{-a}\}$,
- $V := \{(a, b) : a > 0, b > -3a + 2\sqrt{2a}\}$,
- $VI := \{(a, b) : a > 0, \varphi_2(a) < b < -3a + 2\sqrt{2a}\}$,
- $VII := \{(a, b) : a > 0, 0 < b < -3a - 2\sqrt{2a}\}$,
- $VIII := \{(a, b) : a > 0, -3a - 2\sqrt{2a} < b \leq -3a\}$,
- $IX := \{(a, b) : a > 0, b < \varphi_1(a)\}$,
- $X := \{(a, b) : a > 0, \varphi_1(a) < b < \varphi_2(a)\}$.

Proof. All results about limit cycles and homoclinic loops are summarized from Lemmas 3.1, 3.2, 3.5 and Propositions 2, 3. We obtain all global phase portraits in the case $a \leq 0$ by Lemmas 2.1, 2.2, 3.1, 3.2, in the case $a > 0$ by Lemmas 2.1, 2.2, 3.2, 3.5 and Propositions 2, 3, and show them in Figures 11 and 12.

For general $a, b$, we give global phase portraits and show them in Figures 11 and 12. We can see from Theorem 4.2 that the degenerate Bogdanov-Takens system (5) has the same qualitative behavior for all $(a, b) \in \mathbb{R}^2$ as that given in [2, Section 2, Chapter 4], [5, Theorem 2.1, Chapter 4] and [16, Section 3, Chapter 7] for small $|a|, |b|$. The homoclinic bifurcation curve $HL$ and the double limit cycle bifurcation
curves $DL$ exist globally for general $a > 0$. If curve $N_2^+$ intersects $DL$, $HL$ at some points $A_1, A_2$, i.e., the global bifurcation diagram of (5) is shown as in Figure 13, besides the global phase portraits in Figures 11 and 12 we obtain more regions of $(a, b)$ with interesting global phase portraits.

**Theorem 4.3.** Results about limit cycles and homoclinic loops are given in Table 3, where

\[
\begin{align*}
&\hat{V} := \{(a, b) : a > 0, b > \max\{-3a + 2\sqrt{2a}, \varphi_2(a)\}\}, \\
&\hat{IX} := \{(a, b) : a > 0, b < \min\{\varphi_1(a), -3a + 2\sqrt{2a}\}\}, \\
&\hat{X} := \{(a, b) : a > 0, \varphi_1(a) < b < \min\{\varphi_2(a), -3a + 2\sqrt{2a}\}\}, \\
&\hat{XI} := \{(a, b) : a > 0, \max\{\varphi_1(a), -3a + 2\sqrt{2a}\} < b < \varphi_2(a)\}, \\
&\hat{XII} := \{(a, b) : a > 0, -3a + 2\sqrt{2a} < b < \varphi_1(a)\},
\end{align*}
\]

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
$(a, b) \in$ & limit cycles & homoclinic loops \\
\hline
$I, N_1^-, II$ & one stable, small, surrounding $E_0$ & no \\
\hline
$III, N_1^+, IV$ & no & no \\
\hline
$V, N_2^+, VI$ & no & no \\
\hline
$DL$ & one semi-stable, large & no \\
\hline
$X$ & the inner one is unstable, the outer one is stable & no \\
\hline
$HL$ & one stable, large & one unstable figure-eight type \\
\hline
$IX$ & one stable, large; one unstable, small, surrounding $E_L$; one unstable, small, surrounding $E_R$; & no \\
\hline
$VII, VIII, N_2$ & one stable, large & no \\
\hline
\end{tabular}
\caption{Limit cycles and homoclinic loops to Figure 10}
\end{table}
Figure 12. Global phase portraits for $a > 0$

\[
DL_1 := \{(a, b) : a > 0, b = \varphi_2(a) < -3a + 2\sqrt{2a}\},
\]
\[
DL_2 := \{(a, b) : a > 0, b = \varphi_2(a) = -3a + 2\sqrt{2a}\},
\]
\[
DL_3 := \{(a, b) : a > 0, b = \varphi_2(a) > -3a + 2\sqrt{2a}\},
\]
\[
HL_1 := \{(a, b) : a > 0, b = \varphi_1(a) < -3a + 2\sqrt{2a}\},
\]
Figure 13. Bifurcation diagram of (5) if $N_2^+$ intersects $DL, HL$

$$
HL_2 \ := \ \{(a,b) : a > 0, b = \phi_1(a) = -3a + 2\sqrt{2}a\},
$$
$$
HL_3 \ := \ \{(a,b) : a > 0, b = \phi_1(a) > -3a + 2\sqrt{2}a\},
$$
$$
N_{21}^+ \ := \ \{(a,b) : a > 0, b = -3a + 2\sqrt{2}a > \phi_2(a)\},
$$
$$
N_{22}^+ \ := \ \{(a,b) : a > 0, \phi_1(a) < b = -3a + 2\sqrt{2}a < \phi_2(a)\},
$$
$$
N_{23}^+ \ := \ \{(a,b) : a > 0, b = -3a + 2\sqrt{2}a < \phi_1(a)\}.
$$

Moreover, global phase portraits for $\tilde{V}, \tilde{I}X, \tilde{X}, DL_1, HL_1, N_{21}^+$ are the same as in Figure 12 for $V, IX, X, DL, HL, N_2^+$, respectively. Other phase portraits are given in Figure 14.

| $\tilde{V}, N_{21}^+$ | limit cycles | homoclinic loops |
|------------------------|--------------|-----------------|
| no                     | no           | no              |
| $DL_1, DL_2, DL_3$     | one semi-stable, large | no |
| $\tilde{X}, N_{22}^+, X\bar{I}$ | two, large, inner one is unstable, outer one is stable | no |
| $HL_1, HL_2, HL_3$     | one stable, large | one unstable figure-eight type |
| $\tilde{I}X, N_{23}^+, X\bar{II}$ | one stable, large; two unstable, small; surrounding $E_L, E_R$ separately; | no |

TABLE 3. Limit cycles and homoclinic loops

**Proof.** All results about limit cycles and homoclinic loops are summarized from Lemmas 3.1, 3.2, 3.5 and Propositions 2, 3. We obtain all global phase portraits given in Figure 14 from Lemmas 2.1, 2.2, 3.2, 3.5 and Propositions 2, 3. □

5. **Concluding remarks.** From the global phase portraits shown in Figure 14 we see that for general $(a,b)$ there may exist either an unstable large limit cycle or a homoclinic loop surrounding a node. However, these phenomena do not happen when $|a|, |b|$ are sufficiently small because those bifurcation curves do not intersect each other in a small neighborhood of the origin of the parameter space. Thus,
Comparing system (5) with system (4), which is investigated in [4], we find that the dynamical behavior of system (5) is more complicated than that of system (4). Concretely, the cyclicity (the maximum number of limit cycles) of (4) is 1 but, the cyclicity of (5) is 3, whose analysis is more difficult than the former. The proof of the smoothness of the homoclinic loop bifurcation curve (the graph of function $\varphi_1$) given in Proposition 2 is more difficult than the smoothness of the heteroclinic loop bifurcation curve given in [4, Lemma 3.3] because for (5) we need to estimate the intersection of the invariant manifolds not only on a vertical line but also on $x$-axis but, for (4) we need only to estimate the intersection on a vertical line.

In general, in the investigation of global dynamics it is hard to determine the connection of orbits of infinity and finity. As shown in Figure 11, Figure 12(d-j) and Figure 14, these connections are given explicitly because either there is a large limit cycle surrounding all equilibria or there is a unique equilibrium. However, the
connections in the phase portraits in Figure 12(a-c) are unfinished. One can get the connections numerically by the program P4 (see [12]) but, it is very hard to determine their connections qualitatively. For some special cases it is possible to give completely connections qualitatively. For instance, restricting $a + b \geq 1$ in (5) by straight analysis we can get the global phase portraits as shown in Figure 15, where Figure 15(a) is of either Figure 12(a) or Figure 12(b), Figure 15(b) is of Figure 12(a).

![Phase portraits](image)

**Figure 15.** More global phase portraits for special parameters

![Phase portraits](image)

**Figure 15.** More global phase portraits for special parameters

As mentioned in last section, we do not know if $N_2^+$ intersects $DL, HL$. In the following we try to judge this numerically. Taking $a = 1$ and $b = -2.67958$, we can observe the following:

![Phase portraits](image)

**Figure 16.** The numerical phase portraits when $a = 1$
\[ b = -2.69000, \quad b = -2.70200, \quad b = -2.90000 \]
respectively, we obtain the phase portraits shown in Figure 16 respectively. In Figure 16(a) there is one double limit cycle as in Figure 12(g); In Figure 16(b) there are two large limit cycles as in Figure 12(i); In Figure 16(c) there is one large limit cycle and one figure-eight loop as in Figure 12(j); In Figure 16(d) there are two small limit cycles and one large limit cycles as in Figure 12(h). However, we do not find numerical phase portraits as in Figure 14(c) or (g) even if \(|a|, |b|\) are chosen as sufficiently large constants. This means that \(N^+_2\) may not intersect \(DL, HL\). Thus, we guess that \(N^+_2\) does not intersect \(DL, HL\) in the parameter space but, we are unable to prove it theoretically.

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