DUAL FIBONACCI QUATERNIONS

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Abstract. In this study, we define the dual Fibonacci quaternion and the dual Lucas quaternion. We derive the relations between the dual Fibonacci and the dual Lucas quaternion which connected the Fibonacci and the Lucas numbers. Furthermore, we give the Binet and Cassini formulas for these quaternions.

1. Introduction

The quaternions are a number system which extends to the complex numbers. They are members of noncommutative algebra, first invented by William Rowan Hamilton in 1843. Hamilton defined a quaternion as the quotient of two vectors. The algebra of quaternions is denoted by $\mathbb{H}$, also by the Clifford algebra classifications $Cl_{0,2}(\mathbb{R}) \cong Cl_{3,0}(\mathbb{R})$. A quaternion is defined in the form

$$q = q_0 + iq_1 + jq_2 + kq_3$$

where $q_0$, $q_1$, $q_2$, $q_3$ are real numbers and $i, j, k$ are standard orthonormal basis in $\mathbb{R}^3$ which satisfy the quaternion multiplication rules as

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$ 

The conjugate of the quaternion $q = q_0 + iq_1 + jq_2 + kq_3$ is given by $\overline{q} = q_0 - iq_1 - jq_2 - kq_3$.

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Also if two quaternions are $q$ and $p$, then

$$
\overline{q} = q, \quad (qp) = \overline{p} \overline{q}.
$$

The norm of $q$ is defined by $N_q$ and

$$
N_q = ||q|| = q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2.
$$

The quaternion $q$ is called a unit quaternion if $N_q = 1$.

The inverse of $q$ is denoted by $q^{-1}$ as

$$
q^{-1} = \frac{\overline{q}}{N_q} = \frac{\overline{q}}{q\overline{q}}.
$$

Since we will clarify the dual Fibonacci quaternions, we now give the basic notations of dual quaternions.

Clifford [2] published his work on dual numbers in 1873 and provided us with a powerful tool for the analysis of complex numbers. The dual numbers extend to the real numbers has the form

$$
d = a + \varepsilon a^*
$$

where $\varepsilon$ is the dual unit and $\varepsilon^2 = 0$, $\varepsilon \neq 0$. Dual numbers form two dimensional commutative associative algebra over the real numbers. Also the algebra of dual numbers is a ring.

A dual quaternion is an extension of dual numbers whereby the elements of that quaternion are dual numbers. Dual quaternions are used as an appliance for expressing and analyzing the physical properties of rigid bodies. They are computationally efficient approach of representing rigid transforms like translation and rotation.

The dual quaternion is represented in the form

$$
Q = q + \varepsilon q^*
$$

where $q$ and $q^*$ are quaternions and $\varepsilon$ is the dual unit.

If $q = q_0 + iq_1 + jq_2 + kq_3$ and $q^* = q_0^* + iq_1^* + jq_2^* + kq_3^*$, then the dual quaternion $Q$ can be denoted as;

$$
Q = q + \varepsilon q^* \\
= (q_0 + iq_1 + jq_2 + kq_3) + \varepsilon(q_0^* + iq_1^* + jq_2^* + kq_3^*) \\
= (q_0 + \varepsilon q_0^*) + i(q_1 + \varepsilon q_1^*) + j(q_2 + \varepsilon q_2^*) + k(q_3 + \varepsilon q_3^*).
$$

So the dual quaternion $Q$ is constructed from eight real parameters. Also $Q$ can be written as

$$
Q = S_Q + V_Q
$$

where

$$
S_Q = q_0 + \varepsilon q_0^* = S_q + \varepsilon S_{q^*}, \\
V_Q = i(q_1 + \varepsilon q_1^*) + j(q_2 + \varepsilon q_2^*) + k(q_3 + \varepsilon q_3^*) = V_q + \varepsilon V_{q^*}.
$$

Similarly in quaternions, $S_Q$ and $V_Q$ are called scalar part and vector part of the dual quaternion $Q$, respectively.

If two dual quaternions are $Q = q + \varepsilon q^*$ and $P = p + \varepsilon p^*$, then the addition and subtraction is given by

$$
Q \mp P = (q \mp p) + \varepsilon(q^* \mp p^*)
$$
and multiplication is
\[ Q.P = q.p + \varepsilon(q.p^* + q^*.p) \]
where \( q = q_0 + iq_1 + jq_2 + kq_3 \), \( q^* = q_0^* + iq_1^* + jq_2^* + kq_3^* \), and \( p = p_0 + ip_1 + jp_2 + kp_3 \), \( p^* = p_0^* + ip_1^* + jp^* + kp_3^* \).

The conjugate of the dual quaternion \( Q = q + \varepsilon q^* \) is defined as;
\[
\overline{Q} = q + \varepsilon q^* = (q_0 + \varepsilon q_0^*) - i(q_1 + \varepsilon q_1^*) - j(q_2 + \varepsilon q_2^*) - k(q_3 + \varepsilon q_3^*).
\]

If two quaternions are \( Q \) and \( P \), then
\[
\overline{Q} = \frac{Q}{P} = \overline{P} \overline{Q}.
\]

\( N_Q \) is called the norm of \( Q \) and given by
\[
N_Q = ||Q|| = Q\overline{Q} = A^2 + B^2 + C^2 + D^2
\]
where \( A = q_0 + \varepsilon q_0^* \), \( B = q_1 + \varepsilon q_1^* \), \( C = q_2 + \varepsilon q_2^* \), \( D = q_3 + \varepsilon q_3^* \). Also the dual number \( N_Q \) is called the magnitude of the dual quaternion \( Q \).

The dual quaternion with \( N_Q = 1 \) is called a unit dual quaternion.

The inverse of \( Q \) is
\[
Q^{-1} = \frac{\overline{Q}}{N_Q} = \frac{\overline{Q}}{QQ}.
\]

From these notations it can be said that the above properties are dual version of quaternions.

There are many works on Fibonacci and Lucas numbers. Dunlap [3], Vajda [13], Verner [14] and Hoggatt [14] explained the properties of Fibonacci and Lucas numbers and computed the relations between them.

Horadam defined the generalized Fibonacci sequences in [6]. Then the \( n^{th} \) Fibonacci and \( n^{th} \) Lucas quaternions were described by Horadam in [7] as
\begin{align}
Q_n &= F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \\
K_n &= L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}
\end{align}
respectively, where \( F_n \) is Fibonacci number and \( L_n \) is Lucas number. Also \( i, j, k \) are standard orthonormal basis in \( \mathbb{R}^3 \).

Swamy [12] gave the relations of generalized Fibonacci quaternions. Iyer studied Fibonacci quaternions in [8] and obtained some other relations about Fibonacci and Lucas quaternions.

In [4], Halıcı expressed the generating function and Binet formulas for these quaternions. Akyı órgão, Kösäl and Tosun [1] defined the split Fibonacci and split Lucas quaternions. They also gave Binet formulas and Cassini identities for these quaternions.

In this paper, we define the dual Fibonacci quaternion and the dual Lucas quaternion by combining Fibonacci, Lucas quaternions and dual quaternions. We find the equations between the given quaternions and give the Binet and Cassini formulas for them.
2. Dual Fibonacci Quaternions

Complex Fibonacci numbers are given in [7] by Horadam as:
\[ C_n = F_n + iF_{n+1}, \quad i^2 = -1 \]
where \( F_n \) is the \( n \)th Fibonacci number.

Also Halıcı [5] described the \( n \)th complex Fibonacci quaternion as follows:
\[ R_n = Q_n + iQ_{n+1}, \quad i^2 = -1 \]
where \( Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \) is the \( n \)th Fibonacci quaternion.

With the same logic we can define dual Fibonacci, Lucas numbers and dual Fibonacci, Lucas quaternions.

The \( n \)th dual Fibonacci and \( n \)th dual Lucas numbers are defined by
\[
\begin{align*}
F_n & = F_n + \varepsilon F_{n+1} \\
L_n & = L_n + \varepsilon L_{n+1}
\end{align*}
\]
respectively, where \( \varepsilon \) is the dual unit and \( \varepsilon^2 = 0, \varepsilon \neq 0 \). Here \( F_n \) is the \( n \)th Fibonacci number and \( L_n \) is the \( n \)th Lucas number.

The dual Fibonacci quaternion and \( n \)th dual Lucas quaternions are defined as:
\[
\begin{align*}
\tilde{Q}_n & = Q_n + \varepsilon Q_{n+1} \\
\tilde{K}_n & = K_n + \varepsilon K_{n+1}
\end{align*}
\]
respectively. Here \( Q_n = F_n + F_{n+1} + jF_{n+2} + kF_{n+3} \) is the \( n \)th Fibonacci quaternion and \( K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3} \) is the \( n \)th Lucas quaternion. \( i, j, k \) are quaternion units or standard orthonormal basis in \( \mathbb{R}^3 \) which satisfy the following rules:
\[
\begin{align*}
i^2 & = j^2 = k^2 = -1 \\
i j & = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\end{align*}
\]

The dual Fibonacci quaternion \( \tilde{Q}_n \) consists four dual elements and can be represented as
\[
\tilde{Q}_n = (F_n + \varepsilon F_{n+1}) + i(F_{n+1} + \varepsilon F_{n+2}) + j(F_{n+2} + \varepsilon F_{n+3}) + k(F_{n+3} + \varepsilon F_{n+4})
\]
By using dual Fibonacci numbers we can get
\[
\tilde{Q}_n = \tilde{F}_n + i\tilde{F}_{n+1} + j\tilde{F}_{n+2} + k\tilde{F}_{n+3}.
\]
The scalar part and vector part of the dual Fibonacci quaternion \( \tilde{Q}_n \) are given by
\[
\begin{align*}
S_{\tilde{Q}_n} & = F_n + \varepsilon F_{n+1} \\
V_{\tilde{Q}_n} & = i(F_{n+1} + \varepsilon F_{n+2}) + j(F_{n+2} + \varepsilon F_{n+3}) + k(F_{n+3} + \varepsilon F_{n+4})
\end{align*}
\]
respectively.

Let \( \tilde{Q}_n = Q_n + \varepsilon Q_{n+1} \) and \( \tilde{P}_n = P_n + \varepsilon P_{n+1} \) be two dual Fibonacci quaternions.
The addition and subtraction of them is given by
\[
\begin{align*}
\tilde{Q}_n \mp \tilde{P}_n & = (Q_n \mp P_n) + \varepsilon(Q_{n+1} \mp P_{n+1}) \tag{2.5}
\end{align*}
\]
and multiplication is
\[
\begin{align*}
\tilde{Q}_n \tilde{P}_n & = Q_n P_n + \varepsilon(Q_{n+1} P_{n+1} + Q_{n+1} P_n) \tag{2.6}
\end{align*}
\]
where \( Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \), \( Q_{n+1} = F_{n+1} + iF_{n+2} + jF_{n+3} + kF_{n+4} \) and \( P_n = X_n + iX_{n+1} + jX_{n+2} + kX_{n+3} \), \( P_{n+1} = X_{n+1} + iX_{n+2} + jX_{n+3} + kX_{n+4} \) are Fibonacci quaternions.

The conjugate of the dual Fibonacci quaternion \( \tilde{Q}_n \) is defined by
\[
\overline{\tilde{Q}_n} = S_{\tilde{Q}_n} - V_{\tilde{Q}_n} = F_n + \varepsilon F_{n+1} - i(F_{n+1} + \varepsilon F_{n+2}) - j(F_{n+2} + \varepsilon F_{n+3}) - k(F_{n+3} + \varepsilon F_{n+4}).
\]

Also the norm of \( \tilde{Q}_n \) can be given as
\[
N_{\tilde{Q}_n} = \|\tilde{Q}_n\| = \overline{\tilde{Q}_n}Q_n = A^2 + B^2 + C^2 + D^2
\]
where \( A = F_n + \varepsilon F_{n+1} \), \( B = F_{n+1} + \varepsilon F_{n+2} \), \( C = F_{n+2} + \varepsilon F_{n+3} \), \( D = F_{n+3} + \varepsilon F_{n+4} \).

Additionally, after using the properties of Fibonacci numbers we can write the norm
\[
(2.7) \quad N_{\tilde{Q}_n} = F_{2n+1} + F_{2n+5} + 2\varepsilon(F_{2n+2} + F_{2n+6}).
\]

If \( N_{\tilde{Q}_n} = 1 \), then \( \tilde{Q}_n \) is called unit dual Fibonacci quaternion.

The inverse of \( \tilde{Q}_n \) can be computed by
\[
\tilde{Q}_n^{-1} = \frac{\tilde{Q}_n}{N_{\tilde{Q}_n}} = \frac{\tilde{Q}_n}{\overline{\tilde{Q}_n}Q_n}.
\]

After above notations now we will state the theorems.

**Theorem 1.** Let \( \tilde{L}_n \) and \( \tilde{Q}_n \) be a dual Lucas number and a dual Fibonacci quaternion, respectively. For \( n \geq 1 \), the following relations hold:

1. \( \tilde{L}_n + \tilde{Q}_{n+1} = \tilde{Q}_{n+2} \)
2. \( \tilde{Q}_{n+1} - i\tilde{Q}_{n+1} - j\tilde{Q}_{n+2} - k\tilde{Q}_{n+3} = \tilde{L}_{n+3} \)
3. \( \tilde{Q}_n\tilde{Q}_m + \tilde{Q}_{n+1}\tilde{Q}_{m+1} = -\left(\tilde{L}_{n+m+2} + \tilde{L}_{n+m+6}\right) + 2\tilde{Q}_{n+m+1} + \varepsilon(-\tilde{L}_{n+m+3} - \tilde{L}_{n+m+7} + 2Q_{n+m+2}) \)

**Proof.** 1) By using the equation (2.3) and (2.5), we get
\[
\tilde{Q}_n + \tilde{Q}_{n+1} = (Q_n + \varepsilon Q_{n+1}) + (Q_{n+1} + \varepsilon Q_{n+2})
= (Q_n + Q_{n+1}) + \varepsilon(Q_{n+1} + Q_{n+2}).
\]

If we use the equation (1.2) and the identity of Fibonacci numbers that is \( F_n = F_{n-1} + F_{n-2} \), then the above equation becomes as
\[
\tilde{Q}_n + \tilde{Q}_{n+1} = F_n + F_{n+1} + i(F_{n+1} + F_{n+2}) + j(F_{n+2} + F_{n+3}) + k(F_{n+3} + F_{n+4})
+ \varepsilon(F_{n+1} + F_{n+2} + i(F_{n+2} + F_{n+3}) + j(F_{n+3} + F_{n+4}) + k(F_{n+4} + F_{n+5}))
= F_{n+2} + iF_{n+3} + jF_{n+4} + kF_{n+5}
+ \varepsilon(F_{n+3} + iF_{n+4} + jF_{n+5} + kF_{n+6})
= Q_{n+2} + \varepsilon Q_{n+3}.
\]

From the definition of dual Fibonacci quaternions it results that
\[
\tilde{Q}_n + \tilde{Q}_{n+1} = \tilde{Q}_{n+2}.
\]
2) From the equation (2.3) and (2.4) , and the identity of Fibonacci quaternions
\[ Q_n - \imath Q_{n+1} - j Q_{n+2} - k Q_{n+3} = L_{n+3} \] which is given in Iyer [8], we find that
\[ \tilde{Q}_n - \imath \tilde{Q}_{n+1} - j \tilde{Q}_{n+2} - k \tilde{Q}_{n+3} = (Q_n + \varepsilon Q_{n+1}) - \imath (Q_{n+1} + \varepsilon Q_{n+2}) - j (Q_{n+2} + \varepsilon Q_{n+3}) - k (Q_{n+3} + \varepsilon Q_{n+4}) \]
\[ = (Q_n - j Q_{n+1} - j Q_{n+2} - k Q_{n+3}) \]
\[ + \varepsilon (Q_{n+1} - j Q_{n+2} - j Q_{n+3} - k Q_{n+4}) \]
\[ = L_{n+3} + \varepsilon L_{n+4} \]
\[ = \tilde{L}_{n+3}. \]

3) By using the equation (2.3) and (2.5), we simply get;
\[ \tilde{Q}_n \tilde{Q}_m + \tilde{Q}_{n+1} \tilde{Q}_{m+1} = (Q_n + \varepsilon Q_{n+1}) (Q_m + \varepsilon Q_{m+1}) + (Q_{n+1} + \varepsilon Q_{n+2}) (Q_{m+1} + \varepsilon Q_{m+2}) \]
\[ = Q_n Q_m + Q_{n+1} Q_{m+1} \]
\[ + \varepsilon [(Q_n Q_{m+1} + Q_{n+1} Q_m) + (Q_{n+1} Q_m + Q_n Q_{m+1})]. \quad (2.8) \]
Let us compute \( Q_n Q_m + Q_{n+1} Q_{m+1} \) for Fibonacci quaternions to use it in the equation by (1.2).
\[ Q_n Q_m + Q_{n+1} Q_{m+1} = (F_n + i F_{n+1} + j F_{n+2} + k F_{n+3}) (F_m + i F_{m+1} + j F_{m+2} + k F_{m+3}) \]
\[ + (F_n + i F_{n+1} + j F_{n+2} + k F_{n+3}) (F_{m+1} + i F_{m+2} + j F_{m+3} + k F_{m+4}) \]
\[ = F_{n+m+1} - F_{n+m+3} - F_{n+m+5} - F_{n+m+7} \]
\[ + (2 F_{n+m+2}) i + (2 F_{n+m+3}) j + (2 F_{n+m+4}) k \]
\[ = F_{n+m+1} - F_{n+m+3} - F_{n+m+5} - F_{n+m+7} - 2 F_{n+m+1} \]
\[ + (2 F_{n+m+1}) + (2 F_{n+m+2}) i + (2 F_{n+m+3}) j + (2 F_{n+m+4}) k \]
\[ = - (F_{n+m+1} + F_{n+m+3} + F_{n+m+5} + F_{n+m+7}) + 2 Q_{n+m+1} \]
\[ = - (L_{n+m+2} + L_{n+m+6}) + 2 Q_{n+m+1}. \]
Thus we can write
\[ (2.9) \quad Q_n Q_{m+1} + Q_{n+1} Q_m = - (L_{n+m+3} + L_{n+m+7}) + 2 Q_{n+m+2} \]
and
\[ (2.10) \quad Q_{n+1} Q_m + Q_{n+2} Q_{m+1} = - (L_{n+m+3} + L_{n+m+7}) + 2 Q_{n+m+2}. \]
Putting the equations (2.9) and (2.10) in (2.8), we obtain the result as;
\[ \tilde{Q}_n \tilde{Q}_m + \tilde{Q}_{n+1} \tilde{Q}_{m+1} = - \left( L_{n+m+2} + L_{n+m+6} \right) + 2 Q_{n+m+1} + \varepsilon \left[ - \left( L_{n+m+3} + L_{n+m+7} \right) + 2 Q_{n+m+2} \right] \]
\[ = - \left( \tilde{L}_{n+m+2} + \tilde{L}_{n+m+6} \right) + 2 \tilde{Q}_{n+m+1} \]
\[ + \varepsilon \left( - L_{n+m+3} - L_{n+m+7} + 2 Q_{n+m+2} \right). \]

\[ \square \]

Theorem 2. Let \( \tilde{Q}_n \) and \( \tilde{K}_n \) be a dual Fibonacci quaternion and a dual Lucas quaternion, respectively. For \( n \geq 1 \), the following relations hold:

1) \( \tilde{Q}_{n-1} + \tilde{Q}_{n+1} = \tilde{K}_n \)
2) \( \tilde{Q}_{n+2} - \tilde{Q}_{n-2} = \tilde{K}_n \)
Proof. 1) From the equations (2.3) and (2.5), we get

\[
\bar{Q}_{n-1} + \bar{Q}_{n+1} = (Q_{n-1} + \varepsilon Q_n) + (Q_{n+1} + \varepsilon Q_{n+2})
\]

\[
= Q_{n-1} + Q_{n+1} + \varepsilon (Q_n + Q_{n+2}).
\]

By using the equations (1.2), (1.3) and the relation between Fibonacci numbers and Lucas numbers \(L_n = F_{n-1} + F_{n+1}\) (see Vajda [13]), the equation becomes

\[
\bar{Q}_{n-1} + \bar{Q}_{n+1} = F_{n-1} + F_{n+1} + i(F_n + F_{n+2}) + j(F_{n+1} + F_{n+3}) + k(F_{n+2} + F_{n+4})
\]

\[
+ \varepsilon (F_n + F_{n+2} + i(F_{n+1} + F_{n+3}) + j(F_{n+2} + F_{n+4}) + k(F_{n+3} + F_{n+5}))
\]

\[
= L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}
\]

\[
+ \varepsilon (L_{n+1} + iL_{n+2} + jL_{n+3} + kL_{n+4})
\]

\[
= K_n + \varepsilon K_{n+1}
\]

If we use the definition of dual Lucas quaternion, the last equation completed the proof as

\[
\bar{Q}_{n-1} + \bar{Q}_{n+1} = \bar{K}_n.
\]

2) Similarly in proof 1), we can find that

\[
\bar{Q}_{n+2} - \bar{Q}_{n-2} = (Q_{n+2} + \varepsilon Q_{n+3}) - (Q_{n-2} + \varepsilon Q_{n-1})
\]

\[
= Q_{n+2} - Q_{n-2} + \varepsilon (Q_{n+3} - Q_{n-1})
\]

\[
= F_{n+2} - F_{n-2} + i(F_{n+3} - F_{n-1}) + j(F_{n+4} - F_n) + k(F_{n+5} - F_{n+1})
\]

\[
+ \varepsilon (F_{n+3} - F_{n-1} + i(F_{n+4} - F_n) + j(F_{n+5} - F_{n+1}) + k(F_{n+6} - F_{n+2})).
\]

Using the identity \(L_n = F_{n+2} - F_{n-2}\) (see Vajda [13]) and the equations (1.3), (2.4), it results as

\[
\bar{Q}_{n+2} - \bar{Q}_{n-2} = \bar{K}_n
\]

\[
\square
\]

Theorem 3. Let \(\bar{Q}_n\) be a dual Fibonacci quaternion, \(\bar{Q}_n\) be conjugate of \(\bar{Q}_n\), \(\bar{F}_n\) be a dual Fibonacci number and \(\bar{L}_n\) be a dual Lucas number. Then the following equations can be given:

1) \(\bar{Q}_n \bar{Q}_n = 3(\bar{F}_{2n+3} + \varepsilon \bar{F}_{2n+4})\)

2) \(\bar{Q}_n + \bar{Q}_n = 2\bar{F}_n\)

3) \(\bar{Q}_n^2 = 2\bar{Q}_n \bar{F}_n - 3(\bar{F}_{2n+3} + \varepsilon \bar{F}_{2n+4})\)

4) \(\bar{Q}_n \bar{Q}_n + \bar{Q}_{n-1} \bar{Q}_{n-1} = 3(\bar{L}_{2n+2} + \varepsilon \bar{L}_{2n+3})\)

5) \(\bar{Q}_n^2 + \bar{Q}_{n-1}^2 = 2\bar{Q}_{2n-1} - 3\bar{L}_{2n+2} + \varepsilon (2\bar{Q}_{2n} - 3\bar{L}_{2n+3})\)
Proof. 1) From the equation (2.1) , (2.7) and the identity \( F_n = F_{n-1} + F_{n-2} \), we clearly get

\[
\tilde{Q}_n \tilde{Q}_n = F_{2n+1} + F_{2n+5} + 2\varepsilon(F_{2n+2} + F_{2n+6}) \\
= 3(F_{2n+3} + 2\varepsilon F_{2n+4}) \\
= 3(F_{2n+3} + \varepsilon F_{2n+4}) \\
= 3(F_{2n+3} + \varepsilon F_{2n+4}).
\]

2) By using the equations (2.3), (2.5) and (2.7) we can compute;

\[
\tilde{Q}_n + \tilde{Q}_n = 2(F_n + \varepsilon F_{n+1}) \\
= 2\tilde{F}_n.
\]

3) We obtain the result by using the equations given in theorem 3 (identity 1) and 2)) as;

\[
\tilde{Q}_n^2 = \tilde{Q}_n \tilde{Q}_n \\
= \tilde{Q}_n(2\tilde{F}_n - \tilde{Q}_n) \\
= 2\tilde{Q}_n \tilde{F}_n - \tilde{Q}_n^2 \\
= 2\tilde{Q}_n \tilde{F}_n - 3(F_{2n+3} + \varepsilon F_{2n+4}).
\]

4) If we consider identity 1) in this theorem and \( L_n = F_{n-1} + F_{n+1} \) (see Vajda [13]), we can obtain the result;

\[
\tilde{Q}_n \tilde{Q}_n + \tilde{Q}_{n-1} \tilde{Q}_{n-1} = 3(\tilde{F}_{2n+3} + \varepsilon F_{2n+4}) + 3(\tilde{F}_{2n+1} + \varepsilon F_{2n+2}) \\
= 3(\tilde{F}_{2n+1} + \tilde{F}_{2n+3} + \varepsilon(F_{2n+2} + F_{2n+4})) \\
= 3(L_{2n+2} + \varepsilon L_{2n+3}).
\]

5) From the equations (2.3) and (2.5), we have;

\[
(2.11) \quad \tilde{Q}_n^2 + \tilde{Q}_{n-1}^2 = Q_n^2 + Q_{n-1}^2 + 2\varepsilon (Q_{n-1}Q_n + Q_nQ_{n+1}).
\]

Here if we use the identity \( Q_n^2 + Q_{n-1}^2 = 2Q_{2n-1} - 3L_{2n+2} \) (see Swamy [12]), we find;

\[
\tilde{Q}_n^2 + \tilde{Q}_{n-1}^2 = 2Q_{2n-1} - 3L_{2n+2} + 2\varepsilon (Q_{n-1}Q_n + Q_nQ_{n+1}).
\]

Now by using \( F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1} \) (Vajda [13]) and \( F_n^2 + F_{n+1}^2 = F_{2n+1} \) (Vajda [13]), we get the following equation;

\[
Q_{n-1}Q_n + Q_nQ_{n+1} = F_{n-1}F_n - F_{n+3}F_{n+4} - 2F_{2n+4} + i(2F_{2n+1}) + j(2F_{2n+2}) + k(2F_{2n+3}).
\]
In this equation, we take into account that \( F_n = F_{n-1} + F_{n-2} \) and (1.2), thus we have:

\[
Q_{n-1}Q_n + Q_nQ_{n+1} = F_n - F_{n-1} + i(2F_{2n+1}) + j(2F_{2n+2}) + k(2F_{2n+3})
\]

\[
= F_n - F_{n-1} - F_{n-3}F_{n+2} - 2F_{2n+2} + 2F_{n+2} + kF_{2n+3}
\]

\[
= -2(F_{2n+4} + F_{2n}) + 2(F_{2n} + iF_{2n+1} + jF_{2n+2} + kF_{2n+3})
\]

\[
= F_n - F_{n-1} - F_{n+2}^2 - F_{n-1}^2 - F_{n-1}F_n - 2(3F_{2n+2}) + 2(2F_{2n+1} + 2F_{2n+2} + 3F_{2n+3}) + 2(2F_{2n+1} + 2F_{2n+2} + 3F_{2n+3})
\]

Putting this equation in (2.11), using the identity \( L_n = F_{n-1} + F_{n+1} \), the equations (2.3), (2.4) and composing the expression, the result is obtained as:

\[
\widetilde{Q}_n^2 + \tilde{Q}_{n-1}^2 = 2Q_{n-1} - 3L_{2n+2} + 2\varepsilon(-3F_{2n+3} - 6F_{2n+2} + 2Q_{2n})
\]

\[
= 2(Q_{2n-1} + \varepsilon Q_{2n}) + \varepsilon Q_{2n} - 3L_{2n+2} + 2\varepsilon[-3(F_{2n+3} + 2F_{2n+2}) + Q_{2n}]
\]

\[
= 2\tilde{Q}_{2n-1} - 3(L_{2n+2} + \varepsilon L_{2n+3}) + \varepsilon(2Q_{2n} - 3L_{2n+3})
\]

\[
= \tilde{Q}_{2n-1} - 3\tilde{L}_{2n+2} + \varepsilon(2Q_{2n} - 3L_{2n+3}).
\]

\[\square\]

**Theorem 4.** Let \( \tilde{Q}_n \) be a dual Fibonacci quaternion. Then the following summation formulas hold:

1) \( \sum_{s=1}^{n} \tilde{Q}_s = \tilde{Q}_{s+2} - \tilde{Q}_2 \)

2) \( \left( \sum_{s=0}^{n} \tilde{Q}_{n+s} \right) + \tilde{Q}_{n+1} = \tilde{Q}_{n+p+2} \)

3) \( \sum_{s=1}^{n} \tilde{Q}_{2s-1} = \tilde{Q}_{2n} - \tilde{Q}_0 \)

4) \( \sum_{s=1}^{n} \tilde{Q}_{2s} = \tilde{Q}_{2n+1} - \tilde{Q}_1 \)
Proof. 1) If we use the equation (2.3) and 
\[ \sum_{s=1}^{n} Q_s = Q_{n+2} - Q_2 \] (see Halıcı, [4]),
we obtain the result:

\[
\sum_{s=1}^{n} \tilde{Q}_s = \sum_{s=1}^{n} (Q_s + \varepsilon Q_{s+1}) \\
= \sum_{s=1}^{n} Q_s + \varepsilon \sum_{s=1}^{n} Q_{s+1} \\
= Q_{n+2} - Q_2 + \varepsilon (Q_{n+3} - Q_3) \\
= Q_{n+2} - \tilde{Q}_2.
\]

2) Since

\[
\sum_{s=0}^{n} Q_{n+s} = \sum_{r=1}^{n+p} Q_r - \sum_{r=1}^{n-1} Q_r
\]
then we can write:

\[
\left( \sum_{s=0}^{n} \tilde{Q}_{n+s} \right) + \tilde{Q}_{n+1} = \left( \sum_{s=0}^{n} Q_{n+s} + \varepsilon Q_{n+s+1} \right) + (Q_{n+1} + \varepsilon Q_{n+2}) \\
= \left[ \left( \sum_{s=0}^{n} Q_{n+s} \right) + Q_{n+1} \right] + \varepsilon \left[ \left( \sum_{s=0}^{n} Q_{n+s+1} \right) + Q_{n+2} \right] \\
= \left[ \left( \sum_{r=1}^{n+p} Q_r - \sum_{r=1}^{n-1} Q_r \right) + Q_{n+1} \right] \\
+ \varepsilon \left[ \left( \sum_{r=1}^{n+p+1} Q_r - \sum_{r=1}^{n} Q_r \right) + Q_{n+2} \right] \\
= Q_{n+p+2} + \varepsilon Q_{n+p+3} \\
= \tilde{Q}_{n+p+2}.
\]

3) Using (2.3), we obtain the result clearly:

\[
\sum_{s=1}^{n} \tilde{Q}_{2s-1} = \sum_{s=1}^{n} (Q_{2s-1} + \varepsilon Q_{2s}) \\
= \sum_{s=1}^{n} (Q_{2s} - Q_{2s-2}) + \varepsilon \sum_{s=1}^{n} (Q_{2s+1} - Q_{2s-1}) \\
= Q_{2n} - Q_0 + \varepsilon (Q_{2n+1} - Q_1) \\
= \tilde{Q}_{2n} - \tilde{Q}_0.
\]
4) Similarly in proof 3):
\[
\sum_{s=1}^{n} \tilde{Q}_{2s} = \sum_{s=1}^{n} (Q_{2s} + \varepsilon Q_{2s+1}) \\
= \sum_{s=1}^{n} (Q_{2s+1} - Q_{2s-1}) + \varepsilon \sum_{s=1}^{n} (Q_{2s+2} - Q_{2s}) \\
= Q_{2n+1} - Q_1 + \varepsilon (Q_{2n+2} - Q_2) \\
= \tilde{Q}_{2n+1} - \tilde{Q}_1 \\
\]
\[\square\]

The explicit formulas for Fibonacci and Lucas numbers were given by Jacques-Phillipe-Marie Binet in 1843, which are called Binet formulas. The positive and negative roots of the quadratic equation \(x^2 - x - 1 = 0\) are \(\alpha\) and \(\beta\), respectively. They are:
\[
\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2}.
\]

Then the Binet formulas for Fibonacci and Lucas numbers are given by (see Koshy, [11])
\[
F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n.
\]

Now we will express the following theorems which are about Binet formula and Cassini identity.

**Theorem 5.** Let \(\tilde{Q}_n\) and \(\tilde{K}_n\) be dual Fibonacci and dual Lucas quaternions, respectively. For \(n \geq 0\), the Binet formulas for these quaternions are given as:
\[
\tilde{Q}_n = \frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta}
\]

and
\[
\tilde{K}_n = \alpha^* \alpha^n + \beta^* \beta^n
\]

where \(\alpha^* = \alpha (1 + \varepsilon \alpha)\), \(\beta^* = \beta (1 + \varepsilon \beta)\) and \(\alpha = 1 + i \alpha + j \alpha^2 + k \alpha^3\), \(\beta = 1 + i \beta + j \beta^2 + k \beta^3\).

**Proof.** In [4], Halıcı gave the Binet formula for Fibonacci quaternion by
\[
Q_n = \frac{\alpha \alpha^n - \beta \beta^n}{\alpha - \beta}
\]

where \(\alpha = \frac{1 + \sqrt{5}}{2}\), \(\beta = \frac{1 - \sqrt{5}}{2}\) and \(\alpha = 1 + i \alpha + j \alpha^2 + k \alpha^3\), \(\beta = 1 + i \beta + j \beta^2 + k \beta^3\). Thus it can be written;
\[
Q_{n+1} = \frac{\alpha \alpha^{n+1} - \beta \beta^{n+1}}{\alpha - \beta}.
\]
So by using the equations (2.3), (2.12) and (2.13), we have:

\[
\tilde{Q}_n = Q_n + \varepsilon Q_{n+1}
\]

\[
= \frac{\alpha \alpha^n - \beta \beta^n}{\alpha - \beta} + \varepsilon \frac{\alpha \alpha^{n+1} - \beta \beta^{n+1}}{\alpha - \beta}
\]

\[
= \frac{\alpha \alpha^n (1 + \varepsilon \alpha) - \beta \beta^n (1 + \varepsilon \beta)}{\alpha - \beta}.
\]

Taking \(\alpha (1 + \varepsilon \alpha) = \alpha^*\) and \(\beta (1 + \varepsilon \beta) = \beta^*\) in last equation, then the proof completed as:

\[
\tilde{Q}_n = \frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta}.
\]

Similarly in [4], the Binet formula for Lucas quaternion is given by

\[
T_n = \alpha \alpha^n + \beta \beta^n.
\]

We can clearly compute that:

\[
\tilde{K}_n = K_n + \varepsilon K_{n+1}
\]

\[
= \frac{\alpha \alpha^n + \beta \beta^n + \varepsilon (\alpha \alpha^{n+1} + \beta \beta^{n+1})}{\alpha - \beta}
\]

\[
= \frac{\alpha^* \alpha^n + \beta^* \beta^n}{\alpha - \beta}.
\]

\[\square\]

**Theorem 6.** Let \(\tilde{Q}_n\) and \(\tilde{K}_n\) be dual Fibonacci and dual Lucas quaternions, respectively. For \(n \geq 1\), the Cassini identities for these quaternions are given as:

\[
\tilde{Q}_{n-1} \tilde{Q}_{n+1} - \tilde{Q}_n^2 = (-1)^n \left[ 2\tilde{Q}_1 - 3k - \varepsilon 9k \right]
\]

and

\[
\tilde{K}_{n-1} \tilde{K}_{n+1} - \tilde{K}_n^2 = 5(-1)^n \left[ 2\tilde{K}_1 - 4k + \varepsilon (-2i - 17k) \right]
\]

**Proof.** From the equations (2.3), (2.5) and (2.6), we have;

(2.14) \(\tilde{Q}_{n-1} \tilde{Q}_{n+1} - \tilde{Q}_n^2 = (Q_{n-1}Q_{n+1} - Q_n^2) + \varepsilon (Q_{n-1}Q_{n+2} - Q_nQ_{n+1})\).

If we use the equation for Fibonacci numbers called D’ocagne’s identity which is \(F_mF_{n+1} - F_m+1F_n = (-1)^n F_{m-n}\) (see Weisstein, [15]) and the identity of negafibonacci numbers \(F_n = (-1)^{n+1} F_n\) (see Knuth, [10]), we get

\[
Q_{n-1}Q_{n+2} - Q_nQ_{n+1} = (-1)^n (2 + 4i + 6j + k).
\]

Also Halıcı denoted the equation in [4] as:

\[
Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n (2Q_1 - 3k).
\]

Putting these last two equations in (2.14) and using the definition of Fibonacci quaternion, we reach the result;

\[
\tilde{Q}_{n-1} \tilde{Q}_{n+1} - \tilde{Q}_n^2 = (Q_{n-1}Q_{n+1} - Q_n^2) + \varepsilon (Q_{n-1}Q_{n+2} - Q_nQ_{n+1})
\]

\[
= (-1)^n (2Q_1 - 3k) + \varepsilon((-1)^n (2 + 4i + 6j + k))
\]

\[
= (-1)^n (2\tilde{Q}_1 - 3k - \varepsilon 9k).
\]
By the equation (2.4), the identity of Lucas numbers which are $L_{n-1}L_{n+1} - L_n^2 = 5(-1)^{n-1}$ (see Koshy, [11]) and $L_{n+2} = L_{n+1} + L_n$ (see Dunlap, [3]), the proof completed as:

$$\tilde{K}_{n-1}\tilde{K}_{n+1} - \tilde{K}_n^2 = 5(-1)^n \left[ 2\tilde{K}_1 - 4k + \varepsilon(-2i - 17k) \right].$$

□

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