Admissible pairs vs Gieseker-Maruyama

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Abstract. Morphisms between the moduli functor of admissible semi-stable pairs and the Gieseker-Maruyama moduli functor (of semistable coherent torsion-free sheaves) with the same Hilbert polynomial on the surface are constructed. It is shown that these functors are isomorphic, and the moduli scheme for semistable admissible pairs \((\bar{S}, \bar{L}, \bar{E})\) is isomorphic to the Gieseker-Maruyama moduli scheme. All the components of moduli functors and corresponding moduli schemes which exist are looked at here.

Bibliography: 16 titles.

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§ 1. Introduction

In this article we complete our investigation of the compactification of the moduli of stable vector bundles on a surface by locally free sheaves by examining the whole of the moduli space of admissible semistable pairs. Various aspects of the construction of the main components (constituting the compactification of the moduli of vector bundles) and basic properties were given in preceding papers by the author [1]–[9]. In these articles the key restriction was that all the families under consideration include so-called S-pairs. In this article we eliminate this restriction.

Let \(S\) be a smooth irreducible projective algebraic surface over a field \(k = \bar{k}\) of zero characteristic, \(\mathcal{O}_S\) its structure sheaf, \(E\) a coherent torsion-free \(\mathcal{O}_S\)-module and \(E^\vee := \mathcal{H}om_{\mathcal{O}_S}(E, \mathcal{O}_S)\) its dual \(\mathcal{O}_S\)-module. \(E^\vee\) is reflexive and hence locally free. A locally free sheaf and the vector bundle corresponding to it are canonically identified and both terms are used synonymously. Let \(L\) be a very ample invertible sheaf on \(S\); it is fixed and used as a polarization. The symbol \(\chi(\cdot)\) denotes the Euler-Poincaré characteristic and \(c_i(\cdot)\) is the \(i\)th Chern class.

Definition 1 (see [4] and [5]). A polarized algebraic scheme \((\bar{S}, \bar{L})\) is called admissible if it satisfies one of the following conditions:

(i) \((\bar{S}, \bar{L}) \cong (S, L)\);

(ii) \(\bar{S} \cong \text{Proj} \bigoplus_{s \geq 0}(I(t) + (t))^s/(t^{s+1})\) where \(I = \mathcal{Fitt}^0 \mathcal{E}xt^2(\mathcal{O}_S, \mathcal{O}_S)\) for an Artinian quotient sheaf \(q_0: \bigoplus^r \mathcal{O}_S \to \mathcal{O}_S\) of length \(l(\mathcal{O}_S) \leq c_2\), and \(\bar{L} = L \otimes (\sigma^{-1}I \cdot \mathcal{O}_S)\)

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is a very ample invertible sheaf on the scheme $\tilde{S}$; this polarization $\tilde{L}$ is called the distinguished polarization.

Now we discuss the concepts and objects involved in this definition.

Recall the definition of a sheaf of 0th Fitting ideals, known from commutative algebra. Let $X$ be a scheme and $F$ an $\mathcal{O}_X$-module of finite presentation $F_1 \xrightarrow{\varphi} F_0 \to F$. Without loss of generality we assume that $\text{rank } F_1 \geq \text{rank } F_0$.

**Definition 2.** The sheaf of 0th Fitting ideals of the $\mathcal{O}_X$-module $F$ is defined by $\mathcal{F}itt^0 F = \operatorname{im} (\bigwedge^{\text{rank } F_0} F_1 \otimes \bigwedge^{\text{rank } F_0} F_0^\vee \xrightarrow{\varphi'} \mathcal{O}_X)$, where $\varphi'$ is the morphism of $\mathcal{O}_X$-modules induced by $\varphi$.

**Remark 1.** In what follows, we replace $L$ by a big enough tensor power of it, if this is necessary for $\tilde{L}$ to be very ample. This power can be chosen to be finite, uniform and fixed, as shown in [5]. All Hilbert polynomials are computed according to the new polarizations $L$ and $\tilde{L}$, respectively.

It was shown in [4] that if $\tilde{S}$ satisfies condition (ii) in Definition 1, then it can be decomposed into a union of several components $\tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i$. It has a morphism $\sigma: \tilde{S} \to S$ which is induced by the structure of an $\mathcal{O}_S$-algebra on the graded object $\bigoplus_{s \geq 0} (I[t] + (t))^s/(t^{s+1})$. The scheme $\tilde{S}$ can be produced as follows. Take a product $(\text{Spec } k[t]) \times S$ and a blowup of it $\text{Bl}_\mathcal{I}(\text{Spec } k[t]) \times S$ in the sheaf of ideals $\mathcal{I} = (t) + I[t]$ corresponding to the subscheme with ideal $I$ in the zero-fibre $0 \times S$.

If $\sigma: \text{Bl}_\mathcal{I}(\text{Spec } k[t]) \times S \to (\text{Spec } k[t]) \times S$ is the blowup morphism then $\tilde{S}$ is a zero-fibre of the composite

$$\text{pr}_1 \circ \sigma: \text{Bl}_\mathcal{I}(\text{Spec } k[t]) \times S \to (\text{Spec } k[t]) \times S \to \text{Spec } k[t].$$

**Definition 3** (see [5]). An $S$-stable ($S$-semistable, respectively) pair $((\tilde{S}, \tilde{L}), \tilde{E})$ is given by the following data:

- $\tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i$ is an admissible scheme, $\sigma: \tilde{S} \to S$ is a morphism, which is called canonical, and $\sigma_i: \tilde{S}_i \to S$ is its restriction to the component $\tilde{S}_i$, $i \geq 0$;
- $\tilde{E}$ is a vector bundle on the scheme $\tilde{S}$;
- $\tilde{L} \in \text{Pic } \tilde{S}$ is the distinguished polarization;

such that

- $\chi(\tilde{E} \otimes \tilde{L}^n) = rp(n)$, the polynomial $p(n)$ and the rank $r$ of the sheaf $\tilde{E}$ are fixed;
- the sheaf $\tilde{E}$ on the scheme $\tilde{S}$ is stable (semistable, respectively) in the sense of Gieseker (cf. Definition 4 below);
- on each additional component $\tilde{S}_i$, $i > 0$, the sheaf $\tilde{E}_i := \tilde{E}|_{\tilde{S}_i}$ is quasi-ideal, that is, it admits a description of the form

$$\tilde{E}_i = \sigma_i^* \ker q_0/\text{tors}_i$$

(1.1)

for some $q_0 \in \bigsqcup_{t \leq c_2} \operatorname{Quot}^t \bigoplus^r \mathcal{O}_S$.

The definition of the subsheaf $\text{tors}_i$ will be given below. The coefficients of the Hilbert polynomial $rp(n)$ depend on Chern classes. In particular, $c_2$ is the 2nd Chern class of a sheaf with Hilbert polynomial equal to $rp(n)$. 
Pairs \(((\tilde{S}, \tilde{L}), \tilde{E})\) such that \((\tilde{S}, \tilde{L}) \cong (S, L)\) will be called \textit{S-pairs}.

In the series of articles [1]–[5] by the author, a projective algebraic scheme \(\tilde{M}\) was built up as the reduced moduli scheme of \(S\)-semistable admissible pairs and in [6] it was constructed as a possibly nonreduced moduli space.

The scheme \(\tilde{M}\) contains an open subscheme \(\tilde{M}_0\) which is isomorphic to the subscheme \(M_0\) of Gieseker-semistable vector bundles in the Gieseker-Maruyama moduli scheme \(\tilde{M}\) of torsion-free semistable sheaves whose Hilbert polynomial is equal to \(\chi(E \otimes L^n) = r p(n)\). We use the following definition of Gieseker-semistability.

\textbf{Definition 4} (see [10]). The coherent \(\mathcal{O}_S\)-sheaf \(E\) is \textit{stable} (semistable) if for any proper subsheaf \(F \subset E\) of rank \(r' = \text{rank} F\) for \(n \gg 0\)

\[
\frac{\chi(F \otimes L^n)}{r'} < \frac{\chi(E \otimes L^n)}{r} \quad \left( \frac{\chi(F \otimes L^n)}{r'} \leq \frac{\chi(E \otimes L^n)}{r}, \text{ respectively} \right).
\]

Let \(E\) be a semistable locally free sheaf. Then it is obvious that the sheaf \(I = \mathcal{F}itt^0 \mathcal{E}xt^1(E, \mathcal{O}_S)\) is trivial and \(\tilde{S} \cong S\). In this case \(((\tilde{S}, \tilde{L}), \tilde{E}) \cong ((S, L), E)\) and we have a bijective correspondence \(\tilde{M}_0 \cong M_0\).

Let \(E\) be a semistable coherent sheaf that is not locally free; then the scheme \(\tilde{S}\) contains a reduced irreducible component \(\tilde{S}_i\) such that the morphism \(\sigma_0 := \sigma|_{\tilde{S}_0}: \tilde{S}_0 \to S\) is a blowup morphism of the scheme \(S\) in the sheaf of ideals \(I = \mathcal{F}itt^0 \mathcal{E}xt^1(E, \mathcal{O}_S)\). Formation of a sheaf \(I\) is a way of characterizing the singularities of the sheaf \(E\), that is, showing how it differs from a locally free sheaf. Indeed, the quotient sheaf \(\mathcal{E}xt^1(E, \mathcal{O}_S) \cong \mathcal{E}xt^2(\mathcal{E}, \mathcal{O}_S)\). Then \(\mathcal{F}itt^0 \mathcal{E}xt^2(\mathcal{E}, \mathcal{O}_S)\) is a sheaf of ideals of an (in general case nonreduced) subsheaf \(Z\) of bounded length [6] supported at a finite set of points on the surface \(S\). As was shown in [4], the other components \(\tilde{S}_i, i > 0\), of the scheme \(\tilde{S}\) can in general carry nonreduced scheme structures.

Each semistable coherent torsion-free sheaf \(E\) corresponds to a pair \(((\tilde{S}, \tilde{L}), \tilde{E})\) where \((\tilde{S}, \tilde{L})\) is defined as described.

Now we describe the construction of the subsheaf tors in (1.1). Let \(U\) be a Zariski-open subset in some component \(\tilde{S}_i, i \geq 0\), and let \(\sigma^*E|_{\tilde{S}_i}(U)\) be the corresponding group of sections. This group is an \(\mathcal{O}_{\tilde{S}_i}(U)\)-module. The sections \(s \in \sigma^*E|_{\tilde{S}_i}(U)\) annihilated by prime ideals of positive codimension in \(\mathcal{O}_{\tilde{S}_i}(U)\) form a submodule in \(\sigma^*E|_{\tilde{S}_i}(U)\). This submodule is denoted by \(\text{tors}_i(U)\). The correspondence \(U \mapsto \text{tors}_i(U)\) defines a subsheaf \(\text{tors}_i \subset \sigma^*E|_{\tilde{S}_i}\). Note that the associated prime ideals of positive codimension which annihilate sections \(s \in \sigma^*E|_{\tilde{S}_i}(U)\) correspond to subschemes supported in the preimage \(\sigma^{-1}(\text{Supp } \mathcal{E}) = \bigcup_{i > 0} \tilde{S}_i\). Since, by construction, the scheme \(\tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i\) is connected [4], the subsheaves \(\text{tors}_i, i \geq 0\), allow us to construct a subsheaf tors \(\subset \sigma^*E\). It is defined as follows.

A section \(s \in \sigma^*E|_{\tilde{S}_i}(U)\) satisfies the condition \(s \in \text{tors}_i|_{\tilde{S}_i}(U)\) if and only if

\begin{itemize}
  \item there exist a section \(y \in \mathcal{O}_{\tilde{S}_i}(U)\) such that \(ys = 0\);
  \item at least one of the following two conditions is satisfied: either \(y \in p\), where \(p\) is a prime ideal of positive codimension or there exist a Zariski-open subset \(V \subset \tilde{S}\) and a section \(s' \in \sigma^*E(V)\) such that \(V \supset U\), \(s'|_{U} = s\), and \(s'|_{V \cap \tilde{S}_0} \in \text{tors}(\sigma^*E|_{\tilde{S}_0})(V \cap \tilde{S}_0)\).
\end{itemize}
(In the last expression the torsion subsheaf \(\text{tors}(\sigma^*E|_{\bar{S}_0})\) is understood in the usual sense.)

The role of the subsheaf \(\text{tors} \subset \sigma^*E\) in our construction is analogous to the role of the torsion subsheaf in the case of a reduced and irreducible base scheme. Since no confusion will occur, the symbol \(\text{tors}\) is understood throughout in the sense described. The subsheaf \(\text{tors}\) is called the \textit{torsion subsheaf}.

In [5] we proved that the sheaves \(\sigma^*E/\text{tors}\) are locally free. The sheaf \(\bar{E}\) included in the pair \(((\bar{S}, \bar{L}), \bar{E})\) is defined by the formula \(\bar{E} = \sigma^*E/\text{tors}\). In these circumstances there is an isomorphism \(H^0(\bar{S}, \bar{E} \otimes \bar{L}) \cong H^0(S, E \otimes L)\).

We proved in [5] that the restriction of the sheaf \(\bar{E}\) to each component \(\bar{S}_i, i > 0\), is given by the quasi-ideality relation (1.1) where \(q_0: \mathcal{O}_S^{\oplus r} \rightarrow \mathcal{E}\) is an epimorphism defined by the exact triple \(0 \rightarrow E \rightarrow E^\vee \vee \rightarrow \mathcal{E} \rightarrow 0\) as the sheaf \(E^\vee \vee\) is locally free.

Resolution of singularities of a semistable sheaf \(E\) can be globalized in a flat family by means of the construction developed in various versions in [2], [3], [5] and [8].

Let \(T\) be a scheme, \(\mathcal{E}\) a sheaf of \(\mathcal{O}_{T \times S}\)-modules, \(\mathcal{L}\) an invertible \(\mathcal{O}_{T \times S}\)-sheaf which is very ample relative to \(T\) and such that \(\mathcal{L}|_{t \times S} = L\), and \(\chi(\mathcal{E} \otimes \mathcal{L}^n|_{t \times S}) = rp(n)\) for all closed points \(t \in T\). Under the assumption that \(\mathcal{E}\) and \(\mathcal{L}\) are flat relative to \(T\) and that \(T\) contains a nonempty open subset \(T_0\) such that \(\mathcal{E}|_{T_0 \times S}\) is a locally free \(\mathcal{O}_{T_0 \times S}\)-module the following objects were defined:

- \(\pi: \Sigma \rightarrow \tilde{T}\), which is a flat family of admissible schemes with invertible \(\mathcal{O}_\Sigma\)-module \(\mathcal{L}\) such that \(\mathcal{L}|_{t \times S}\) is the distinguished polarization of the scheme \(\pi^{-1}(t)\);
- \(\tilde{E}\), which is a locally free \(\mathcal{O}_\Sigma\)-module; in addition, \(((\pi^{-1}(t), \mathcal{L}|_{\pi^{-1}(t)}), \mathcal{E}|_{\pi^{-1}(t)})\) is an \(S\)-semistable admissible pair.

In this situation there is a blowup morphism \(\Phi: \Sigma \rightarrow \tilde{T} \times S\) and the scheme \(\tilde{T}\) is birational to the initial base scheme \(T\). The mechanism described was called a \textit{standard resolution}. In [8] the procedure of standard resolution was modified so that \(\tilde{T} \cong T\).

In this paper the open subset \(T_0\) such that \(\mathcal{E}|_{T_0 \times S}\) is locally free, can be empty. This makes some of the arguments in our previous papers invalid.

In this article we prove the following results.

**Theorem 1.** (i) There is a natural transformation \(\kappa: \mathcal{f}^{\text{GM}} \rightarrow \mathcal{f}\) taking the Gieseker-Maruyama moduli functor to the moduli functor of admissible semistable pairs with the same rank and Hilbert polynomial.

(ii) There is a natural transformation \(\tau: \mathcal{f} \rightarrow \mathcal{f}^{\text{GM}}\) of the moduli functor of admissible semistable pairs to the Gieseker-Maruyama moduli functor for sheaves with the same rank and Hilbert polynomial.

(iii) The natural transformations \(\kappa\) and \(\tau\) are mutually inverse. Hence both morphisms of nonreduced moduli functors \(\kappa: \mathcal{f}^{\text{GM}} \rightarrow \mathcal{f}\) and \(\tau: \mathcal{f} \rightarrow \mathcal{f}^{\text{GM}}\) are isomorphisms.

**Corollary 1.** The nonreduced moduli scheme \(\widetilde{M}\) for \(\mathcal{f}\) is isomorphic to the nonreduced Gieseker-Maruyama scheme \(\bar{M}\) for sheaves with same rank and Hilbert polynomial.
In §2 we recall the definitions of the functor \( f^{GM} \) of moduli of semistable coherent torsion-free sheaves (the ‘Gieseker-Maruyama functor’) given in (2.3), (2.4) and the functor \( f \) of moduli of admissible semistable pairs (2.2), (2.1). The rank \( r \) and polynomial \( p(n) \) are fixed and equal for both moduli functors.

Then in §3 we give the transformation of the family of coherent torsion-free sheaves \((T, L, E)\) with base scheme \( T \) into a family of admissible semistable pairs \((\pi: \Sigma \to T, \Sigma, \Sigma)\). This transformation generalizes the procedure of standard resolution to the case when the initial family does not necessarily contain locally free sheaves. It leads to the functorial morphism \( \kappa: f^{GM} \to f \) and proves part (i) of Theorem 1.

In §4 we give a description of the transformation of a family of semistable admissible pairs \((\pi: \Sigma \to T, \Sigma, \Sigma)\) with (possibly, nonreduced) base scheme \( T \) to a family \( E \) of coherent torsion-free semistable sheaves with the same base \( T \). This transformation provides a morphism of the functor \( \tau \) of admissible semistable pairs \( f \) to the Gieseker-Maruyama functor \( f^{GM} \) and proves part (ii) of Theorem 1.

In §5 we show that the morphisms of functors \( \kappa: f^{GM} \to f \) and \( \tau: f \to f^{GM} \) we have constructed are mutually inverse. In this way the functors under consideration are isomorphic and this completes the proof of Theorem 1.

**§2. Moduli functors**

Following [11], Ch. 2, §2.2, we recall some definitions. Let \( \mathcal{C} \) be a category, \( \mathcal{C}^o \) its dual and \( \mathcal{C}' = \text{Funct}(\mathcal{C}^o, \text{Sets}) \) the category of functors to the category of sets. By Yoneda’s lemma, the functor

\[
\mathcal{C} \to \mathcal{C}' : F \mapsto (F: X \mapsto \text{Hom}_\mathcal{C}(X, F))
\]

includes \( \mathcal{C} \) into \( \mathcal{C}' \) as a full subcategory.

**Definition 5** (see [11], Ch. 2, Definition 2.2.1). The functor \( f \in \mathcal{Ob}' \mathcal{C}' \) is corepresented by an object \( M \in \mathcal{Ob} \mathcal{C} \) if there exists a \( \mathcal{C}' \)-morphism \( \psi: f \to M \) such that any morphism \( \psi': f \to F' \) factors through the unique morphism \( \omega: M \to F' \).

**Definition 6.** The scheme \( \widetilde{M} \) is a coarse moduli space for the functor \( f \) if \( f \) is corepresented by the scheme \( \widetilde{M} \).

Let \( T \) and \( S \) be schemes over a field \( k \) and let \( \pi: \Sigma \to T \) be a morphism of \( k \)-schemes.

**Definition 7** (see [8], Definition 5). The family of schemes \( \pi: \Sigma \to T \) is birationally \( S \)-trivial if there exist isomorphic open subschemes \( \Sigma_0 \subset \Sigma \) and \( \Sigma_0 \subset T \times S \) such that there is a scheme equality \( \pi(\Sigma_0) = T \).

The last equality means in particular that all fibres of the morphism \( \pi \) have nonempty intersections with the open subscheme \( \Sigma_0 \). In particular, if \( T = \text{Spec} k \) then \( \pi \) is a constant morphism and \( \Sigma_0 \cong \Sigma_0 \) is an open subscheme in \( S \).

Since we only consider birationally \( S \)-trivial families here, we will refer to them as birationally trivial families.
We consider sets of families of semistable pairs

\[
\mathcal{F}_T = \left\{ \begin{array}{l}
\pi : \Sigma \to T \text{ is birationally } S\text{-trivial;}

\tilde{L} \in \text{Pic} \tilde{\Sigma} \text{ is a sheaf which is flat over } T;

\text{for } m \gg 0 \quad \tilde{L}^m \text{ is a sheaf which is very ample relative to } T;

\forall t \in T \quad \tilde{L}_t = \tilde{L}|_{\pi^{-1}(t)} \text{ is an ample sheaf;}

\chi(\tilde{E}_t^n)|_{\pi^{-1}(t)} = \text{rp}(n);

((\pi^{-1}(t), \tilde{L}_t), \tilde{E}|_{\pi^{-1}(t)}) \text{ is a semistable pair}\end{array} \right. \right\}
\]

(2.1)

and a functor

\[
f : (\text{Schemes}_k)^o \to (\text{Sets})
\]

(2.2)

from the category of \(k\)-schemes to the category of sets. It attaches to each scheme \(T\) the set of equivalence classes of families of the form \((\mathcal{F}_T/\sim)\).

The equivalence relation \(\sim\) is defined as follows. Families \(((\pi : \Sigma \to T, \tilde{L}, \tilde{E}))\) and \(((\pi' : \Sigma \to T, \tilde{L}', \tilde{E}'))\) in the class \(\mathcal{F}_T\) are said to be equivalent (notation: \((\pi : \Sigma \to T, \tilde{L}, \tilde{E}) \sim (\pi' : \Sigma \to T, \tilde{L}', \tilde{E}'))\) if

1) there exist an isomorphism \(\iota : \Sigma \sim \tilde{\Sigma}'\) such that the diagram

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\sim} & \tilde{\Sigma}' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
T & & T
\end{array}
\]

commutes;

2) there exist line bundles \(L'\) and \(L''\) on the scheme \(T\) such that \(\iota^*\tilde{E}' = \tilde{E} \otimes \pi^*L'\) and \(\iota^*\tilde{L}' = \tilde{L} \otimes \pi^*L''\).

Now we discuss what is the ‘size’ of the maximal object \(\tilde{\Sigma}_0\) (under inclusion) among the open subschemes in the family of admissible schemes \(\tilde{\Sigma}\) that are isomorphic to appropriate open subschemes in \(T \times S\) in Definition 7. The set \(F = \tilde{\Sigma} \setminus \tilde{\Sigma}_0\) is closed. If \(T_0\) is an open subscheme in \(T\) whose points carry fibres isomorphic to \(S\), then \(\tilde{\Sigma}_0 \supseteq \pi^{-1}T_0\) (we do not have equality because \(\pi(\tilde{\Sigma}_0) = T\) in Definition 7).

The subscheme \(\Sigma_0\) that is open in \(T \times S\) and isomorphic to \(\tilde{\Sigma}_0\) has the property \(\Sigma_0 \supseteq T_0 \times S\). If \(\pi : \tilde{\Sigma} \to T\) is a family of admissible schemes then \(\tilde{\Sigma}_0 \cong \Sigma \setminus F\), and \(F\) is (set-theoretically) the union of the additional components of fibres that are nonisomorphic to \(S\). In particular, this means that \(\text{codim}_{T \times S}(T \times S) \setminus \Sigma_0 \geq 2\).

The Gieseker-Maruyama functor

\[
f^{\text{GM}} : (\text{Schemes}_k)^o \to \text{Sets}
\]

(2.3)
attaches to each scheme $T$ the set of equivalence classes of families of the form $\tilde{\mathcal{C}}_{T}^{GM}/\sim$, where

$$\tilde{\mathcal{C}}_{T}^{GM} = \left\{ \begin{array}{l}
\mathbb{E} \text{ is a sheaf of } \mathcal{O}_{T\times S}\text{-modules that are flat over } T; \\
\mathbb{L} \text{ is an invertible sheaf of } \mathcal{O}_{T\times S}\text{-modules which is ample relative to } T \text{ and such that } L_{t} := \mathbb{L}|_{t\times S} \cong L \text{ for any point } t \in T; \\
E_{t} := \mathbb{E}|_{t\times S} \text{ is torsion-free and Gieseker-semistable}; \\
\chi(E_{t} \otimes L_{t}^{n}) = rp(n)
\end{array} \right\}. \tag{2.4}$$

Families $(\mathbb{E}, \mathbb{L})$ and $(\mathbb{E}', \mathbb{L}')$ in the class $\tilde{\mathcal{C}}_{T}^{GM}$ are said to be equivalent (notation: $(\mathbb{E}, \mathbb{L}) \sim (\mathbb{E}', \mathbb{L}')$) if there exist line bundles $L'$ and $L''$ on the scheme $T$ such that $E' = \mathbb{E} \otimes p^{*}L'$ and $L' = \mathbb{L} \otimes p^{*}L''$, where $p: T \times S \to T$ is the projection onto the first factor.

**Remark 2.** Since $\text{Pic}(T \times S) = \text{Pic}T \times \text{Pic}S$, our definition of the moduli functor $\mathcal{f}^{GM}$ is equivalent to the standard definition which can be found, for example, in [11]: the difference in the choice of polarizations $\mathbb{L}$ and $\mathbb{L}'$ having isomorphic restrictions to fibres over the base $T$ is avoided by the equivalence induced by tensoring by the inverse image of an invertible sheaf $L''$ from the base $T$.

### § 3. GM-to-Pairs transformation (standard resolution)

The morphism of functors $\kappa: \mathcal{f}^{GM} \to \mathcal{f}$ is defined by the commutative diagrams

$$T \xrightarrow{\mathcal{f}^{GM}} \tilde{\mathcal{C}}_{T}^{GM}/\sim \xrightarrow{\kappa(T)} \tilde{\mathcal{C}}_{T}/\sim \tag{3.1}$$

where $T \in \text{Ob}($Schemes$)_{k}$ and $\kappa(T): (\tilde{\mathcal{C}}_{T}^{GM}/\sim) \to (\tilde{\mathcal{C}}_{T}/\sim)$ is a morphism in the category of sets (a mapping).

The aim of this section is to build up a transformation of the family $(T, \mathbb{L}, \mathbb{E})$ of semistable coherent torsion-free sheaves to the family $((T, \pi): \tilde{\Sigma} \to T, \tilde{\mathbb{L}}), \tilde{\mathbb{E}})$ of admissible semistable pairs. Since the initial family of sheaves $\mathbb{E}$ does not necessarily contain at least one locally free sheaf, the codimension of the singular locus $\text{Sing}\mathbb{E}$ in $\Sigma = T \times S$ can be equal to 2. Hence, if the blowup $\sigma: \tilde{\Sigma} \to \Sigma$ of the sheaf of ideals $\mathcal{F}itt^{0}\mathcal{O}_{xt}^{\mathbb{E}}(\mathbb{E}, \mathcal{O}_{\Sigma})$ is considered then the fibres of the composite $p \circ \sigma$ are not necessarily equidimensional. Such a blowup cannot produce a family of admissible schemes.

We use the following artificial but obvious trick to overcome this difficulty. Consider the product $\Sigma' = \Sigma \times \mathbb{A}^{1}$ and fix a closed immersion $i_{0}: \Sigma \hookrightarrow \Sigma'$ which identifies $\Sigma$ with the zero fibre $\Sigma \times 0$. Now let $Z \subset \Sigma$ be a subscheme defined by the sheaf of ideals $\mathbb{I} = \mathcal{F}itt^{0}\mathcal{O}_{xt}^{\mathbb{E}}(\mathbb{E}, \mathcal{O}_{\Sigma})$. Then consider the sheaf of ideals $\mathbb{I}' := \ker(\mathcal{O}_{\Sigma'} \to i_{0*}\mathcal{O}_{Z})$ and the blowup morphism $\sigma': \tilde{\Sigma}' \to \Sigma'$ defined by the sheaf $\mathbb{I}'$. Denote the projection onto the product of factors $\Sigma' \to T \times \mathbb{A}^{1} =: T'$ by $p'$ and the composite $p' \circ \sigma'$ by $\tilde{p}'$. We are interested in the induced morphism $\pi: \Sigma := i_{0}(\Sigma) \times_{\Sigma'} \tilde{\Sigma}' \to T$. Under the identification $\Sigma \cong i_{0}(\Sigma)$ we denote the
induced morphism $\Sigma \to \Sigma$ by $\sigma$. Set $L' := L \otimes O_{A^1}$. Obviously, there exists $m \geq 0$ such that the invertible sheaf $L' := \sigma^*L^m \otimes (\sigma^{-1})' \cdot O_{\Sigma'}$ is ample relative to $\Sigma'$. For brevity of notation we fix this $m$ and replace $L$ by its $m$th tensor product throughout the text that follows. Set $\tilde{L} := L'|_{\Sigma'}$.

**Proposition 1.** The morphism $\pi: \Sigma \to T$ is flat and the fibrewise Hilbert polynomial computed with respect to $\tilde{L}$, that is, $\chi(\tilde{L}^n|_{\pi^{-1}(t)})$ is uniform over $t \in T$.

**Proof.** First recall the following definition from [12], Ch. 0, §9, Definition 9.1.1.

**Definition 8.** A continuous mapping $f: X \to Y$ is called quasi-compact if any open quasi-compact subset $U \subset Y$ has a quasi-compact preimage $f^{-1}(U)$. A subset $Z$ is called retro-compact in $X$ if the canonical injection $Z \to X$ is quasi-compact and for any open quasi-compact subset $U \subset X$ the intersection $U \cap Z$ is quasi-compact.

Let $f: X \to S$ be a scheme morphism of finite presentation and let $\mathcal{M}$ be a quasi-coherent $O_X$-module of finite type.

**Definition 9** (see [13], Part 1, Definition 5.2.1). The $O_X$-module $\mathcal{M}$ is $S$-flat in dimension $\geq n$ if there exists a retro-compact open subset $V \subset X$ such that $\dim(X \setminus V)/S < n$ and if $\mathcal{M}|_V$ is an $S$-flat module of finite presentation.

If $\mathcal{M}$ is an $S$-flat module of finite presentation and schemes $X$ and $S$ are of finite type over the field, then any open subset $V \subset X$ can be used in the definition. Setting $V = X$ we have $X \setminus V = \emptyset$ and $\dim(X \setminus V)/S = -1 - \dim S$. Consequently, an $S$-flat module of finite presentation is flat in dimension $\geq -\dim S$.

Conversely, let an $O_X$-module $\mathcal{M}$ be $S$-flat in dimension $\geq -\dim S$. Then there is an open retro-compact subset $V \subset X$ such that $\dim(X \setminus V)/S < -\dim S$ and $\mathcal{M}|_V$ is an $S$-flat module. By the above inequality between dimensions we have $\dim(X \setminus V) < 0$, which implies that $X = V$, and $\mathcal{M}|_V = \mathcal{M}$ is $S$-flat.

**Definition 10** (see [13], Part 1, Definition 5.1.3). Let $f: S' \to S$ be a morphism of finite type and let $U$ be an open subset in $S$. The morphism $f$ is called a $U$-admissible blowup if there exist a closed subscheme $Y \subset S$ of finite presentation that is disjoint from $U$ and such that $f$ is isomorphic to the blowup of $S$ in $Y$.

**Proposition 2** (see [13], Theorem 5.2.2). Let $S$ be a quasi-compact quasi-separated scheme, $U$ an open quasi-compact subscheme in $S$, let $f: X \to S$ be a morphism of finite presentation, $\mathcal{M}$ an $O_X$-module of finite type and $n$ an integer. Assume that $\mathcal{M}|_{f^{-1}(U)}$ is flat over $U$ in dimension $\geq n$. Then there exists a $U$-admissible blowup $g: S' \to S$ such that $g^*\mathcal{M}$ is $S'$-flat in dimension $\geq n$.

Recall the following.

**Definition 11** (see [14], Definition 6.1.3). A scheme morphism $f: X \to Y$ is quasi-separated if the diagonal morphism $\Delta_f: X \to X \times_Y X$ is quasi-compact. A scheme $X$ is quasi-separated if it is quasi-separated over Spec $\mathbb{Z}$.

If the scheme $X$ is Noetherian, then any morphism $f: X \to Y$ is quasi-compact. Since we are working in the category of Noetherian schemes, all morphisms of interest to us and all the schemes which arise are quasi-compact.
Under the assumptions of Proposition 1 set \( f = \tilde{\pi}', \mathcal{M} = \mathcal{O}_{\tilde{\Sigma}'}, \) and \( U = T' \setminus T \times 0. \) Then by Proposition 2 there exists a \( T' \setminus T \times 0 \)-admissible blowup \( g : \tilde{T}' \rightarrow T' \) such that in the fibred square

\[
\begin{array}{ccc}
\tilde{\Sigma}' & \xrightarrow{\tilde{g}} & \tilde{\Sigma}' \\
\downarrow{\tilde{\pi}'} & & \downarrow{\tilde{\pi}'} \\
\tilde{T}' & \xrightarrow{g} & T'
\end{array}
\]

\( \mathcal{O}_{\tilde{T}'} = \tilde{g}^* \mathcal{O}_{\tilde{\Sigma}'} \) is a flat \( \mathcal{O}_{\tilde{T}'} \)-module.

By the arguments and results in [8], §3, and by [8], Proposition 3 (the proof is applicable to the invertible sheaves \( \tilde{\mathbb{L}}' \) and \( \mathbb{L}' \) instead of \( \mathbb{L} \) and \( \mathbb{L} \), respectively), the morphism \( \tilde{\pi}' \) is flat and a computation of fibrewise Hilbert polynomials with respect to \( \tilde{\mathbb{L}}' \) leads to polynomials which are uniform over the base \( T' \). Then we set \( g = \text{id}_{T'} \), \( \tilde{g} = \text{id}_{\tilde{\Sigma}} \), and \( \tilde{\Sigma}' = \tilde{\Sigma}' \).

Now \( \pi : \tilde{\Sigma} \rightarrow T \) is flat since it is obtained from the flat morphism \( \tilde{\pi}' \) by a change of base. This completes the proof of Proposition 1.

We set \( \sigma := \sigma'|_{\tilde{\Sigma}} : \tilde{\Sigma} \rightarrow \Sigma \).

To resolve the singularities of the sheaf \( E \) we repeat all the manipulations from [8] for the morphism \( \sigma \) which has just been defined.

Let \( T \) be an arbitrary (possibly nonreduced) \( k \)-scheme of finite type. We assume that its reduction \( T_{\text{red}} \) is irreducible. If \( E \) is a family of coherent torsion-free sheaves on the surface \( S \) having reduced base \( T \), then the homological dimension of \( E \) as a \( \mathcal{O}_{T \times S} \)-module is not greater than 1. The proof of this fact for a reduced equidimensional base can be found in [1], Proposition 1, for example.

Now we need the following simple lemma, which concerns the homological dimension of the family \( E \) with nonreduced base and was proved in [8], Lemma 1.

**Lemma 1.** Let the coherent \( \mathcal{O}_{T \times S} \)-module \( E \) of finite type be \( T \)-flat and let its reduction \( E_{\text{red}} := E \otimes_{\mathcal{O}_T} \mathcal{O}_{T_{\text{red}}} \) have homological dimension no greater than 1: \( \text{hd}_{T_{\text{red}} \times S} E_{\text{red}} \leq 1. \) Then \( \text{hd}_{T \times S} E \leq 1 \).

We will carry out some computations as in [2] and [8] but here the morphism \( \sigma \) is defined differently. We choose and fix a locally free \( \mathcal{O}_{T \times S} \)-resolution of the sheaf \( E \):

\[
0 \rightarrow E_1 \rightarrow E_0 \rightarrow E \rightarrow 0. \tag{3.2}
\]

Then we apply the inverse image \( \sigma^* \) to the dual sequence of (3.2):

\[
\sigma^* E^\vee \rightarrow \sigma^* E_0^\vee \rightarrow \sigma^* \mathcal{W} \rightarrow 0, \tag{3.3}
\]

The symbol \( \mathcal{W} \) stands for the sheaf

\[
\ker (E_1^\vee \rightarrow \mathcal{E}xt^1(E, \mathcal{O}_\Sigma)) = \text{coker}(E^\vee \rightarrow E_0^\vee).
\]

In (3.3) set \( N := \ker (\sigma^* E_1^\vee \rightarrow \sigma^* \mathcal{E}xt^1(E, \mathcal{O}_\Sigma)) \). The sheaf \( \mathcal{F}itt^0(\sigma^* \mathcal{E}xt^1(E, \mathcal{O}_\Sigma)) \) is invertible by the functorial property of \( \mathcal{F}itt \):

\[
\mathcal{F}itt^0(\sigma^* \mathcal{E}xt^1_{\mathcal{O}_\Sigma}(E, \mathcal{O}_\Sigma)) = (\sigma^{-1} \mathcal{F}itt^0(\mathcal{E}xt^1_{\mathcal{O}_{\Sigma}}(E, \mathcal{O}_\Sigma))) \cdot \mathcal{O}_{\Sigma} = (\sigma^{-1} \mathcal{I}) \cdot \mathcal{O}_{\Sigma} = (\sigma^{-1} \mathcal{I}_0) \cdot \mathcal{O}_{\Sigma} = (\sigma^{-1} \mathcal{I}') \cdot \mathcal{O}_{\Sigma} = (\tilde{\sigma}')^{-1} \mathcal{I}' \cdot \mathcal{O}_{\Sigma} = (\tilde{\sigma}')^{-1} \mathcal{I}' \cdot \mathcal{O}_{\Sigma}.
\]
Here we have used the fact that $\mathcal{I}' = \text{pr}_1^* \mathcal{I} + (t)$ where $\mathbb{A}^1 = \text{Spec } k[t]$, $\text{pr}_1 : \Sigma \times \mathbb{A}^1 \to \Sigma$ is the natural projection and the closed immersion $\tilde{i}_0 : \tilde{\Sigma} \hookrightarrow \tilde{\Sigma}'$ is fixed by the fibred square

$$\begin{array}{ccc}
\tilde{\Sigma}' & \xrightarrow{\sigma'} & \Sigma \times \mathbb{A}^1 \\
\downarrow & & \downarrow \\
\tilde{\Sigma} & \xrightarrow{\sigma} & \Sigma
\end{array}$$

**Lemma 2** (see [8], Lemma 2). Let $X$ be a Noetherian scheme such that its reduction $X_{\text{red}}$ is irreducible and let $\mathcal{F}$ be a nonzero coherent $\mathcal{O}_X$-sheaf supported on a subscheme of codimension $\geq 1$. Then the sheaf of 0th Fitting ideals $\mathcal{Fitt}^0(\mathcal{F})$ is an invertible $\mathcal{O}_X$-sheaf if and only if $\mathcal{F}$ has homological dimension 1: $\text{hd}_X \mathcal{F} = 1$.

**Remark 3.** If the scheme $\tilde{\Sigma}$ has an irreducible reduction then Lemma 2 can be applied directly and we conclude that $\text{hd} \mathcal{O}^* \mathcal{E}xt^0_{\mathcal{O}_{\Sigma}}(E, \mathcal{O}_\Sigma) = 1$. If $\tilde{\Sigma}$ has a reducible reduction then there is a natural decomposition $\tilde{\Sigma} = \tilde{\Sigma}_0 \cup \mathcal{D}'$, where $\tilde{\Sigma}_0 = \text{Proj} \bigoplus_{s \geq 0} \mathcal{I}^s$ is the scheme obtained by blowing up $\Sigma$ in the sheaf of ideals $\mathcal{I}$ while $\mathcal{D}'$ is an exceptional divisor of the blowup morphism $\mathfrak{s}' : \mathcal{D}' \to \Sigma \times \mathbb{A}^1$.

Their scheme-theoretic intersection equals the exceptional divisor $\mathcal{D}$ of the blowup morphism $\mathfrak{s}_0 : \tilde{\Sigma}_0 \to \Sigma$:

$$\tilde{\Sigma}_0 \cap \mathcal{D}' = \mathcal{D},$$

and we arrive at the decomposition of the morphism $\mathfrak{s}$:

$$\tilde{\Sigma} \xrightarrow{\mathfrak{s}} \tilde{\Sigma}_0 \xrightarrow{\mathfrak{s}_0} \Sigma,$$

where $\mathfrak{s}$ acts identically on $\tilde{\Sigma}_0$ and its action on $\mathcal{D}'$

$$\mathfrak{s}|_{\mathcal{D}'} : \mathcal{D}' \to \tilde{\Sigma}_0$$

factors through the exceptional divisor $\mathcal{D} = \text{Proj} \bigoplus_{s \geq 0} \mathcal{I}^s / \mathcal{I}^{s+1}$ of the morphism $\mathfrak{s}_0$ and is defined by the structure of a $\bigoplus_{s \geq 0} \mathcal{I}^s / \mathcal{I}^{s+1}$-algebra on the graded ring $\bigoplus_{s \geq 0} \mathcal{I}^s / \mathcal{I}^{s+1}$.

In this case the morphism $\mathfrak{s}$ can be replaced by $\mathfrak{s}_0$ and the sheaf $\mathcal{Fitt}^0(\mathfrak{s}_0^* \mathcal{E}xt^1_{\mathcal{O}_{\Sigma}}(E, \mathcal{O}_\Sigma))$ is also invertible. Applying Lemma 2 we conclude that $\text{hd} \mathfrak{s}_0^* \mathcal{E}xt^1_{\mathcal{O}_{\Sigma}}(E, \mathcal{O}_\Sigma) = 1$ and hence $\text{hd} \mathfrak{s}^* \mathcal{E}xt^1_{\mathcal{O}_{\Sigma}}(E, \mathcal{O}_\Sigma) = 1$ as well.

Hence the sheaf $\mathcal{N} = \ker(\mathfrak{s}^* E_1^\vee \to \mathfrak{s}^* \mathcal{E}xt^1_{\mathcal{O}_{\Sigma}}(E, \mathcal{O}_\Sigma))$ is locally free. Then there is a morphism of locally free sheaves $\mathfrak{s}^* E_0^\vee \to \mathcal{N}$. Let $Q$ be a sheaf of $\mathcal{O}_{\Sigma}$-modules which factors the morphism $E_0^\vee \to E_1^\vee$ into a composite of an epimorphism and a monomorphism. By the definition of the sheaf $\mathcal{N}$ it also factors the morphism $\mathfrak{s}^* Q \to \mathfrak{s}^* E_1^\vee$ into a composite of an epimorphism and a monomorphism and $\mathfrak{s}^* E_0^\vee \to \mathfrak{s}^* Q$ is an epimorphism. From this we conclude that the composite $\mathfrak{s}^* E_0^\vee \to \mathfrak{s}^* Q \to \mathcal{N}$ is an epimorphism of locally free sheaves. Then its kernel is also a locally free sheaf. Now set $\tilde{E} := \ker(\mathfrak{s}^* E_0^\vee \to \mathcal{N})^\vee$. Consequently, we have an exact triple of locally free $\mathcal{O}_{\Sigma}$-modules:

$$0 \to \tilde{E}^\vee \to \mathfrak{s}^* E_0^\vee \to \mathcal{N} \to 0.$$

Its dual is also exact.
Now there is a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N} & \longrightarrow & \sigma^*E_0 & \longrightarrow & \tilde{E} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\sigma^*E_1 & \longrightarrow & \sigma^*E_0 & \longrightarrow & \sigma^*E & \longrightarrow & 0
\end{array}
$$

(3.4)

where the right-hand vertical arrow is an epimorphism.

Remark 4. Since $\tilde{E}$ is locally free as an $\mathcal{O}_{\Sigma}$-module and $\mathcal{O}_{\Sigma}$ is $\mathcal{O}_T$-flat, then $\tilde{E}$ is also flat over $T$.

The epimorphism

$$\sigma^*E \twoheadrightarrow \tilde{E}$$

induced by the right-hand vertical arrow in (3.4), provides quasi-ideality on closed fibres of the morphism $\pi$.

The transformation of families that we have constructed has the form

$$(T, L, E) \longmapsto (\pi: \Sigma \rightarrow T, \Sigma, \tilde{E})$$

and is defined by the commutative diagram

$$\begin{array}{ccc}
T & \longrightarrow & \{(T, L, E)\} \\
\uparrow & & \uparrow \\
T & \longleftarrow & \{(\pi: \Sigma \rightarrow T, \Sigma, \tilde{E})\}
\end{array}$$

The right-hand vertical arrow is a map of sets. Their elements are families of objects to be parametrized. This map is determined by the procedure of resolution as it has been developed in this section.

Remark 5. The transformation we have constructed now defines a morphism of functors $\kappa: \mathcal{F}_{GM} \rightarrow \mathcal{F}$.

§ 4. Pairs-to-GM transformation

In what follows we show that there is a morphism of the nonreduced moduli functor of admissible semistable pairs to the nonreduced Gieseker-Maruyama moduli functor. Given a scheme $T$ we build up a correspondence $((\pi: \Sigma \rightarrow T, \Sigma, \tilde{E}) \mapsto ( L, E)$. It leads to a set mapping $\{(\pi: \Sigma \rightarrow T, \Sigma, \tilde{E})\}/\sim \mapsto \{( L, E)\}/\sim$.

This means that the family of semistable coherent torsion-free sheaves $E$ with the same base $T$ can be constructed using any family $((\pi: \Sigma \rightarrow T, \Sigma, \tilde{E})$ of admissible semistable pairs that is birationally trivial and flat over $T$.

First we construct a $T$-morphism $\phi: \Sigma \rightarrow T \times S$. Since the family $\pi: \Sigma \rightarrow T$ is birationally trivial, there is a fixed isomorphism $\phi_0: \Sigma_0 \cong \Sigma$ of maximal open subschemes $\Sigma_0 \subset \Sigma$ and $\Sigma_0 \subset T \times S$. We define an invertible $\mathcal{O}_{T \times S}$-sheaf $L$ by the equality

$$L(U) := \tilde{L}(\phi_0^{-1}(U \cap \Sigma_0))$$
for all open \( U \subset T \times S \). Identifying \( \tilde{\Sigma}_0 \) with \( \Sigma_0 \) by the isomorphism \( \phi_0 \) we conclude that the sheaves \( L|_{\Sigma_0} \) and \( \tilde{L}|_{\tilde{\Sigma}_0} \) are also isomorphic.

For each closed point \( t \in T \) there is a canonical morphism of the fibre \( \sigma_t: \tilde{S}_t \to S \) where \( \tilde{S}_t = \pi^{-1}(t) \).

**Proposition 3.** For any closed point \( t \in T \) and any open \( V \subset S \)

\[
L \otimes (k_t \boxtimes \mathcal{O}_S)(V) = \tilde{L}_t(\sigma_t^{-1}(V) \cap \tilde{\Sigma}_0).
\]

In particular, \( L \otimes (k_t \boxtimes \mathcal{O}_S) = L \).

**Proof.** The restriction \( L|_{t \times S} \) is the sheaf associated to the presheaf

\[
V \mapsto L(U) \otimes \mathcal{O}_T(U) \boxtimes \mathcal{O}_S(U \cap t \times S)
\]

for any open \( U \subset T \times S \) such that \( U \cap (t \times S) = V \). Since \( \text{codim } T \times S \setminus \Sigma_0 \geq 2 \), it follows that

\[
\mathcal{O}_{T \times S}(U) = \mathcal{O}_{T \times S}(U \cap \Sigma_0)
\]

and

\[
(k_t \boxtimes \mathcal{O}_S)(U \cap t \times S) = (k_t \boxtimes \mathcal{O}_S)(U \cap \Sigma_0 \cap t \times S) = \mathcal{O}_{\tilde{S}_t}(\phi_0^{-1}(U \cap \Sigma_0) \cap \pi^{-1}(t)).
\]

Hence \( L|_{t \times S} \) is associated to the presheaf

\[
V \mapsto \tilde{L}(\phi_0^{-1}(U \cap \Sigma_0)) \boxtimes \mathcal{O}_{\tilde{S}_t}(\phi_0^{-1}(U \cap \Sigma_0)) \mathcal{O}_{\tilde{S}_t}(\phi_0^{-1}(U \cap \Sigma_0) \cap \pi^{-1}(t)),
\]

or, equivalently,

\[
V \mapsto \tilde{L}_t(\phi_0^{-1}(U \cap \Sigma_0) \cap \tilde{S}_t) = L(U \cap \Sigma_0 \cap t \times S) = L(U \cap t \times S).
\]

We bear in mind that \( \phi_0^{-1}(U \cap \Sigma_0) \cap \tilde{S}_t = \sigma_t^{-1}(V) \cap \tilde{\Sigma}_0 \); this completes the proof.

Define a sheaf \( L' \) by the correspondence \( U \mapsto \tilde{L}(U \cap \tilde{\Sigma}_0) \) for any open \( U \subset \tilde{\Sigma}_0 \). It carries the natural structure of an invertible \( \mathcal{O}_{\tilde{\Sigma}} \)-module. This structure is induced by the commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_{\tilde{\Sigma}}(U) \times L'(U) & \longrightarrow & L'(U) \\
\text{res} \downarrow & & \downarrow \\
\mathcal{O}_{\tilde{\Sigma}}(U \cap \tilde{\Sigma}_0) \times \tilde{L}(U \cap \tilde{\Sigma}_0) & \longrightarrow & \tilde{L}(U \cap \tilde{\Sigma}_0)
\end{array}
\]

where the vertical arrow is induced by the natural restriction map in \( \mathcal{O}_{\tilde{\Sigma}} \). We compare the direct images \( p_* L \) and \( \pi_* L' \); for any open \( V \subset T \),

\[
p_* L(V) = \tilde{L}(p^{-1} V) = \tilde{L}(p^{-1} V \cap \Sigma_0).
\]

By the definition of \( L' \)

\[
L(p^{-1} V \cap \Sigma_0) = \tilde{L}(\pi^{-1} V \cap \tilde{\Sigma}_0) = L'(\pi^{-1} V) = \pi_* L'(V).
\]

Thus \( \pi_* L' = p_* L \).
The invertible sheaf $\mathbb{L}'$ induces a morphism $\phi': \tilde{\Sigma} \to \mathbb{P}(\pi_*\mathbb{L}')^\vee$, which is included in the commutative diagram of $T$-schemes

\[
\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\phi'} & \mathbb{P}(\pi_*\mathbb{L}')^\vee \\
\downarrow \cong & & \downarrow \cong \\
\tilde{\Sigma}_0 = \Sigma_0 & \xrightarrow{i_L} & \mathbb{P}(p_*\mathbb{L})^\vee \\
\end{array}
\]

where $i_L$ is a closed immersion induced by $\mathbb{L}$ and $\phi'|_{\tilde{\Sigma}_0}$ is also an immersion. From now on we identify $\mathbb{P}(\pi_*\mathbb{L}')^\vee$ and $\mathbb{P}(p_*\mathbb{L})^\vee$ and use the common notation $\mathbb{P}$ for these projective bundles. The formation of scheme closures of images of $\tilde{\Sigma}_0$ and $\Sigma_0$ in $\mathbb{P}$ leads to $\phi'(\tilde{\Sigma}_0) = \tilde{i}_L(T \times S) = T \times S$. Also by the definition of the sheaf $\mathbb{L}'$ for any open $U \subset \tilde{\Sigma}$ and $V \subset T \times S$ such then $U \cap \tilde{\Sigma}_0 \cong V \cap \Sigma_0$ the following chain of equalities holds:

$$\mathbb{L}'(U) = \mathbb{L}'(U \cap \tilde{\Sigma}_0) = \mathbb{L}(V \cap \Sigma_0) = \mathbb{L}(V). \quad (4.1)$$

Now we suppose for a moment that $T$ is affine: $T = \text{Spec } A$ for some commutative algebra $A$ and $\mathbb{P} = \text{Proj } A[x_0 : \cdots : x_N]$ where $x_0, \ldots, x_N \in H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1))$ generate $\mathcal{O}_\mathbb{P}(1)$. The images $\phi'^* x_i = s'_i$, $i = 0, \ldots, N$, generate $\mathbb{L}'$ along $\tilde{\Sigma}_0$ and they need not generate $\mathbb{L}'$ along the whole of $\tilde{\Sigma}$. The images $i_L^* x_i = s_i$, $i = 0, \ldots, N$, generate $\mathbb{L}$ along the whole of $T \times S$ and so $i_L$ is a closed immersion.

We pass to the standard affine covering by

$$\mathbb{P}_i = \text{Spec } A[x_0, \ldots, \hat{x}_i, \ldots, x_N], \quad i = 0, \ldots, N,$$

where $\hat{}$ means the symbol below is omitted. Set $(i_L(T \times S))_i := i_L(T \times S) \cap \mathbb{P}_i$ and $(\phi'(\tilde{\Sigma}))_i := \phi'(\tilde{\Sigma}) \cap \mathbb{P}_i$. Also set $(T \times S)_i := i_L^{-1}(i_L(T \times S))_i$ and $\tilde{\Sigma}_i := \phi'^{-1}(\phi'(\tilde{\Sigma}))_i$.

Now we have mappings

$$A[x_0, \ldots, \hat{x}_i, \ldots, x_N] \to \Gamma(\tilde{\Sigma}_i, \mathbb{L}') : x_j \mapsto s'_j$$

and

$$A[x_0, \ldots, \hat{x}_i, \ldots, x_N] \to \Gamma((T \times S)_i, \mathbb{L}) : x_j \mapsto s_j,$$

which fit into the triangular diagram

\[
\begin{array}{ccc}
A[x_0, \ldots, \hat{x}_i, \ldots, x_N] & \xrightarrow{\phi'^*} & \Gamma(\tilde{\Sigma}_i, \mathbb{L}') \\
& & \downarrow \cong \\
& & \Gamma((T \times S)_i, \mathbb{L})
\end{array}
\]

(4.2)

where the vertical equality sign means the bijection (4.1) rewritten for the covering under consideration. Since the diagram (4.2) commutes, thus $\phi'$ factors through $i_L(T \times S)$, that is, $\phi'(\tilde{\Sigma}) = i_L(T \times S)$.

Now identifying $i_{\mathbb{L}}(T \times S)$ with $T \times S$ by means of an obvious isomorphism we arrive at a $T$-morphism

$$\phi: \tilde{\Sigma} \to T \times S.$$  

It coincides with $\phi_0: \tilde{\Sigma}_0 \sim \Sigma_0$ when restricted to $\tilde{\Sigma}_0$.

For $n > 0$ consider an invertible $\mathcal{O}_S \times T$-sheaf $U \mapsto \mathbb{L}^n(\phi_0^{-1}(U \cap \Sigma_0))$. It coincides with $\mathbb{L}^n$ on $\Sigma_0$ and hence on the whole of $T \times S$.

Now there is a commutative triangle

$$\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\phi} & T \times S \\
\downarrow{\pi} & & \downarrow{p} \\
T & & T
\end{array}$$

First, note that $T$ contains at least one closed point, say $t \in T$; let $\tilde{S}_t = \pi^{-1}(t)$ be the corresponding closed fibre and let $\tilde{L}_t = \mathbb{L}|_{\tilde{S}_t}$ and $\tilde{E}_t = \mathbb{E}|_{\tilde{S}_t}$ be the restrictions of sheaves to it. By the definition of an admissible scheme there is a canonical morphism $\sigma_t: \tilde{S}_t \to S$. Then $(\sigma_t^* \tilde{L}_t)^{\vee\vee} = L$.

Second, the family $\pi: \tilde{\Sigma} \to T$ is birationally trivial, that is, there exist isomorphic open subschemes $\tilde{\Sigma}_0 \subset \tilde{\Sigma}$ and $\Sigma_0 \subset T \times S$. Note that the ‘boundary’ $\Delta = S \times T \setminus \Sigma_0$ has codimension $\geq 2$ and that, for any closed point $t \in T$, codim $\Delta \cap (t \times S) \geq 2$.

Third, the morphism of multiplication of sections

$$(\sigma_t^* \tilde{L}_t)^n \to \sigma_t^* \tilde{L}_t^n$$

induces the morphism of reflexive hulls

$$( (\sigma_t^* \tilde{L}_t)^n )^{\vee\vee} \to ( \sigma_t^* \tilde{L}_t^n )^{\vee\vee},$$

which are locally free sheaves on a surface and coincide apart from a collection of points. Hence they are equal. Also the sheaf $((\sigma_t^* \tilde{L}_t)^{\vee\vee})^n = L^n$ coincides with them for the analogous reason. Then for all $n > 0$

$$( (\sigma_t^* \tilde{L}_t)^n )^{\vee\vee} = L^n.$$ 

Now take a product $\mathbb{A}^1 \times S$, $\mathbb{A}^1 = \text{Spec } k[u]$. Let $I \subset \mathcal{O}_S$ be the sheaf of ideals such that $\tilde{S}_t = \text{Proj } \bigoplus_{s \geq 0} (I[u] + (u))^n/(u^{s+1})$ and consider the blowup $\text{Bl}_{\mathcal{I}} \mathbb{A}^1 \times S$ in the sheaf of ideals $\mathcal{I} = (u) + I[u]$ corresponding to the subscheme with ideal $I$ in the zero-fibre $0 \times S$. If $\sigma: \text{Bl}_{\mathcal{I}}(\mathbb{A}^1 \times S) \to \mathbb{A}^1 \times S$ is the blowup morphism, then a zero-fibre $\tilde{S}_0$ of the composite

$$\text{pr}_1 \circ \sigma: \text{Bl}_{\mathcal{I}}(\mathbb{A}^1 \times S) \to \text{Spec } k[u] \times S \to \text{Spec } k[u]$$

is isomorphic to $\tilde{S}_t$. The other closed fibres are isomorphic to $S$. Since this composite is a flat morphism, the invertible sheaf

$$\mathbb{L}_t = \sigma^*(\mathcal{O}_{\mathbb{A}^1} \boxtimes L) \otimes \mathcal{O}_S \cdot \mathcal{O}_{\text{Bl}_{\mathcal{I}}(\mathbb{A}^1 \times S)}$$

is flat over $\mathbb{A}^1$. Now $\tilde{L}_t = \mathbb{L}|_{\tilde{S}_0}$, and for $n \gg 0$ we have

$$h^0(\tilde{S}_t, \tilde{L}_t^n) = h^0(S, L^n).$$
Proposition 4. There are morphisms of $\mathcal{O}_T \times S$-sheaves

$$\phi_* \tilde{\mathbb{L}}^n \to \mathbb{L}^n$$

for all $n > 0$.

**Proof.** For any open $U \in T \times S$ and any $n > 0$ there is a restriction map of sections $\text{res}: (\phi_* \tilde{\mathbb{L}}^n)(U) \to (\phi_* \tilde{\mathbb{L}}^n)(U \cap \Sigma_0)$. Denoting the preimage $\phi^{-1}(\Sigma_0)$ by $\Sigma_0$ as usual (recall that $\phi|_{\Sigma_0} = \phi_0$ is an isomorphism) we arrive at the chain of equalities:

$$(\phi_* \tilde{\mathbb{L}}^n)(U \cap \Sigma_0) = \tilde{\mathbb{L}}^n(\phi^{-1}(U \cap \Sigma_0)) = \mathbb{L}^n(U).$$

Applying $p_*$ to the above morphisms yields the following.

Corollary 2. For $n > 0$ the morphisms $\phi_* \tilde{\mathbb{L}}^n \to \mathbb{L}^n$ induce isomorphisms of $\mathcal{O}_T$-sheaves

$$\pi_* \mathbb{L}^n \sim p_* \mathbb{L}^n.$$

**Proof.** Both the sheaves $\pi_* \tilde{\mathbb{L}}^n$ and $p_* \mathbb{L}^n$ are locally free and have equal ranks. Passing to fibrewise consideration we get $\pi_* \tilde{\mathbb{L}}^n \otimes k_t \to p_* \mathbb{L}^n \otimes k_t$ or, equivalently, $H^0(S_t, \tilde{\mathbb{L}}^n_t) \to H^0(t \times S, \mathbb{L}^n)$. This map is an isomorphism and hence $\pi_* \tilde{\mathbb{L}}^n \sim p_* \mathbb{L}^n$.

We will need sheaves

$$\tilde{\mathbb{V}}_m = \pi_*(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m)$$

for $m \gg 0$ such that the $\tilde{\mathbb{V}}_m$ are locally free of rank $rp(m)$ and the $\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m$ are fibrewise globally generated in the following sense: the canonical morphisms

$$\pi^* \tilde{\mathbb{V}}_m \to \tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m$$

are surjective for these $m$.

Also let

$$\mathcal{E}_m = \phi_*(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m)$$

for $m \gg 0$; then

$$p_* \mathcal{E}_m = p_* \phi_*(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m) = \pi_*(\tilde{\mathbb{E}} \otimes \tilde{\mathbb{L}}^m) = \tilde{\mathbb{V}}_m.$$

We intend to verify that the sheaves $p_* (\mathcal{E}_m \otimes \mathbb{L}^n)$ are locally free of rank $rp(m+n)$ for all $m, n > 0$. This implies that $\mathcal{E}_m$ is $T$-flat.

To proceed further we need morphisms $\tilde{\mathbb{L}}^n \to \phi^* \mathbb{L}^n, n > 0$.

Proposition 5. For all $n > 0$ there are injective morphisms $\iota_n: \tilde{\mathbb{L}}^n \to \phi^* \mathbb{L}^n$ of invertible $\mathcal{O}_{\tilde{\Sigma}}$-sheaves.

**Proof.** For $n > 0$ and for any open $U \subset \tilde{\Sigma}$ there is a restriction map on sections

$$\tilde{\mathbb{L}}^n(U) \xrightarrow{\text{res}} \tilde{\mathbb{L}}^n(U \cap \tilde{\Sigma}_0) = \mathbb{L}^n(\phi_0(U \cap \Sigma_0)) = \mathbb{L}^n(\phi_0(U) \cap \Sigma_0) = \mathbb{L}^n(\phi(U)).$$

Since $\phi$ is projective and hence takes closed subsets to closed subsets (open to open ones, respectively), this implies a sheaf morphism $\tilde{\mathbb{L}}^n \to \phi^{-1} \mathbb{L}^n$. Combining it with multiplication by the unity section $1 \in \mathcal{O}_{\tilde{\Sigma}}(U)$ leads to a morphism $\iota_n: \tilde{\mathbb{L}}^n \to \phi^* \mathbb{L}^n$ of invertible $\mathcal{O}_{\tilde{\Sigma}}$-modules.
Remark 6. By the definition of the invertible $\mathcal{O}_{\Sigma}$-sheaf $L'$ it follows from the above proof that there are injective morphisms of invertible $\mathcal{O}_{\Sigma}$-modules $\tilde{L}^n \rightarrow L^m$.

**Proposition 6.** The sheaf $\mathcal{E}_m$ is $T$-flat for $m \gg 0$.

**Proof.** Consider the morphism of multiplication of sections

$$p_\ast \phi_\ast (\tilde{E} \otimes \tilde{L}^m) \otimes p_\ast L^n \rightarrow p_\ast (\phi_\ast (\tilde{E} \otimes \tilde{L}^m) \otimes L^n),$$

which is surjective for $m, n \gg 0$. By the projection formula

$$p_\ast (\phi_\ast (\tilde{E} \otimes \tilde{L}^m) \otimes L^n) = p_\ast (\phi_\ast (\tilde{E} \otimes \tilde{L}^m \otimes \phi^* L^n)).$$

Also there is another morphism of multiplication of sections for the projection $\pi$,

$$\pi_\ast (\tilde{E} \otimes \tilde{L}^m) \otimes \pi_\ast L^n \rightarrow \pi_\ast (\tilde{E} \otimes \tilde{L}^{m+n}).$$

The injective $\mathcal{O}_{\Sigma}$-morphism $\tilde{L}^n \hookrightarrow \phi^* L^n$, after tensoring by $\tilde{E} \otimes \tilde{L}^m$ and applying $\pi_\ast$, leads to

$$\pi_\ast (\tilde{E} \otimes \tilde{L}^{m+n}) \hookrightarrow \pi_\ast (\tilde{E} \otimes \tilde{L}^m \otimes \phi^* L^n).$$

Taking the isomorphism $p_\ast L^n = \pi_\ast \tilde{L}^n$ and Proposition 5 into account we collect these mappings into the commutative diagram

\[
\begin{array}{ccc}
p_\ast \phi_\ast (\tilde{E} \otimes \tilde{L}^m) \otimes p_\ast L^n & \longrightarrow & p_\ast (\phi_\ast (\tilde{E} \otimes \tilde{L}^m \otimes \phi^* L^n)) \\
\| & & \\
\pi_\ast (\tilde{E} \otimes \tilde{L}^m) \otimes \pi_\ast L^n & \longrightarrow & \pi_\ast (\tilde{E} \otimes \tilde{L}^{m+n})
\end{array}
\] (4.3)

As the diagram (4.3) commutes, we conclude that

$$\pi_\ast (\tilde{E} \otimes \tilde{L}^{m+n}) = p_\ast (\phi_\ast (\tilde{E} \otimes \tilde{L}^m) \otimes L^n),$$

or, in our notation, $p_\ast (\mathcal{E}_m \otimes L^n) = \tilde{V}_{m+n}$ for $m, n \gg 0$. This guarantees that the $\mathcal{E}_m$ are $T$-flat for $m \gg 0$.

We intend to confirm that the $\mathcal{E}_m \otimes L^{-m}$ are families of semistable sheaves on $S$ as required. First we prove the following.

**Proposition 7.** For any $m \gg 0$ and $n > 0$, $\mathcal{E}_{m+n} = \mathcal{E}_m \otimes L^n$.

**Proof.** By the definition of the sheaves $\mathcal{E}_m = \phi_\ast (\tilde{E} \otimes \tilde{L}^m)$, for any $m > 0$ there is an injective $\mathcal{O}_{T \times S}$-morphism

$$\varepsilon_{m+n} : \mathcal{E}_m \hookrightarrow \mathcal{E}_{m+n}$$

induced locally by multiplication by the generator of $\tilde{L}^n$.

Consider the sheaf inclusion $i_n : \tilde{L}^n \hookrightarrow \phi^* L^n$, which is valid for any $n > 0$. Tensoring by the locally free $\mathcal{O}_{\Sigma}$-module $\tilde{E} \otimes \tilde{L}^m$, the formation of the direct image under $\phi$ and the projection formula yield the inclusion

$$i_{m,n} : \mathcal{E}_{m+n} \hookrightarrow \mathcal{E}_m \otimes L^n.$$
Both sheaves $E_m \otimes L^n$ and $E_{m+n}$ become normal if restricted to $T_{\text{red}} \times S$ and coincide apart from their singular locus $T_{\text{red}} \times S \setminus \Sigma_{0\text{red}}$, which has codimension $\geq 2$. Hence they coincide along the whole of $T_{\text{red}} \times S$. Let $T$ be the cokernel of $i_{m,n}$: since $E_{m+n}|_{\Sigma_0} = E_m \otimes L^n|_{\Sigma_0}$, it follows that $\text{Supp} \ T \subset T \times S \setminus \Sigma_0$. If $\text{Supp} \ T \neq \emptyset$, it contains at least one closed point $t \times s$ and $T_{t \times s} \neq 0$. Now $t \times s \in T_{\text{red}}$ but $T \otimes T_{\text{red}} = 0$. This contradiction leads us to conclude that $\text{Supp} \ T = \emptyset$ and $T = 0$, which proves the proposition.

We can introduce the sheaf which is the goal of our construction

$$E := E_m \otimes L^{-m}.$$  

By the proposition proved above this definition is independent of $m$ at least in the case when $m \gg 0$. The sheaf $E$ is $T$-flat.

**Proposition 8.** With respect to the invertible sheaf $\mathbb{L}$ the sheaf $E$ has a fibrewise Hilbert polynomial equal to $rp(n)$, that is, for $n \gg 0$

$$\text{rank } p_*(E \otimes \mathbb{L}^n) = rp(n).$$  

**Proof.** For $n \gg m \gg 0$ by (4.4) we have the chain of equalities $p_*(E \otimes \mathbb{L}^n) = p_*(E_m \otimes \mathbb{L}^{n-m}) = p_*(\phi_*(E \otimes \mathbb{L}^n) \otimes \mathbb{L}^{n-m}) = \pi_*(E \otimes \mathbb{L}_n)$. The last sheaf in this chain has rank $rp(n)$.

**Proposition 9.** For any closed point $t \in T$ the sheaf

$$E_t := E|_{t \times S}$$  

is torsion-free and Gieseker-semistable with respect to the polarization

$$L_t := \mathbb{L}|_{t \times S} \cong L.$$  

**Proof.** The isomorphism $\mathbb{L}|_{t \times S} \cong L$ was investigated in Proposition 3. Now, for $E_t$ we have

$$E_t = E|_{t \times S} = (E_m \otimes L^{-m})|_{t \times S} = E_m|_{t \times S} \otimes L^{-m} = \phi_*(E \otimes L^n)|_{t \times S} \otimes L^{-m}.$$  

Letting $i_t: t \times S \hookrightarrow T \times S$ and $\tilde{i}_t: \tilde{S}_t \hookrightarrow \tilde{\Sigma}$ denote the morphisms of closed immersions of fibres we obtain $\phi_*(E \otimes L^n)|_{t \times S} \otimes L^{-m} = (i_t^*\phi_*(E \otimes L^n)) \otimes L^{-m}$ and the base change morphism

$$\beta_t: i_t^*\phi_*(E \otimes L^n) \rightarrow \sigma_t \tilde{i}_t^*\phi_*(E \otimes L^n) \quad (4.5)$$  

in the fibred square

$$\begin{array}{ccc}
\tilde{S}_t & \xrightarrow{\tilde{i}_t} & \tilde{\Sigma} \\
\downarrow \sigma_t & & \downarrow \phi \\
t \times S & \xrightarrow{i_t} & T \times S
\end{array}$$  

The quasi-ideality of the sheaf $\tilde{E}_t = \tilde{E}|_{\tilde{S}_t}$ validates the following lemma, which we prove later.
Lemma 3. The sheaf $\sigma_{ts} \bar{E}_t$ is torsion-free.

Both sheaves in (4.5) coincide along $(t \times S) \cap \Sigma_0$. Now consider the corresponding map of global sections:

$$H^0(\beta_t) : H^0(t \times S, i_t^* \phi_*(\bar{E} \otimes \bar{L}^m)) \to H^0(t \times S, \sigma_{ts} i_t^* (\bar{E} \otimes \bar{L}^m)).$$

It is injective. The left-hand side takes the form

$$H^0(t \times S, i_t^* \phi_*(\bar{E} \otimes \bar{L}^m)) \otimes k_t = i_t^* p_* \phi_*(\bar{E} \otimes \bar{L}^m) = k_t^{\oplus rp(m)}.$$

On the right-hand side we have

$$H^0(t \times S, \sigma_{ts} i_t^* (\bar{E} \otimes \bar{L}^m)) \otimes k_t = H^0(S_t, \bar{E}_t \otimes \bar{L}_t^m) \otimes k_t = k_t^{\oplus rp(m)}.$$

This implies that $H^0(\beta_t)$ is bijective and there is a commutative diagram

$$\begin{array}{ccc}
H^0(t \times S, \sigma_{ts} (\bar{E}_t \otimes \bar{L}_t^m)) \otimes \mathcal{O}_S & \longrightarrow & \sigma_{ts} (\bar{E}_t \otimes \bar{L}_t^m) \\
\uparrow H^0(\beta_t) & & \uparrow \beta_t \\
i_t^* p^* \bar{V}_m & \longrightarrow & i_t^* \mathcal{E}_m \end{array} \quad (4.6)
$$

We observe that

$$\sigma_{ts} (\bar{E}_t \otimes \bar{L}_t^m) = \sigma_{ts} (\bar{E}_t \otimes \sigma_t^* L^m \otimes (\sigma_t^{-1} I \cdot \mathcal{O}_{S_t})^m)$$

$$= \sigma_{ts} (\bar{E}_t \otimes (\sigma_t^{-1} I \cdot \mathcal{O}_{S_t})^m) \otimes L^m,$$

and for $m \gg 0$ the latter sheaf is globally generated. This implies that the upper horizontal arrow in (4.6) is surjective. It follows from (4.6) that $\beta_t$ is surjective. Since $\ker H^0(\beta_t) = H^0(t \times S, \ker \beta_t) = 0$, $\beta_t$ is an isomorphism.

Now take a subsheaf $F_t \subset E_t$. For $m \gg 0$ there is a commutative diagram

$$\begin{array}{ccc}
H^0(t \times S, E_t \otimes L^m) \otimes \mathcal{O}_S & \longrightarrow & E_t \otimes L^m \\
\uparrow & & \uparrow \\
H^0(t \times S, F_t \otimes L^m) \otimes \mathcal{O}_S & \longrightarrow & F_t \otimes L^m \\
\end{array}
$$

The isomorphism $E_t \otimes L^m = \sigma_{ts} (\bar{E}_t \otimes \bar{L}_t^m)$, proved above, fixes a bijection on global sections

$$H^0(t \times S, E_t \otimes L^m) \simeq H^0(t \times S, \sigma_{ts} (\bar{E}_t \otimes \bar{L}_t^m)) = H^0(S_t, \bar{E}_t \otimes \bar{L}_t^m).$$

Let $\bar{V}_t \subset H^0(S_t, \bar{E}_t \otimes \bar{L}_t^m)$ be the subspace corresponding to $H^0(t \times S, F_t \otimes L^m) \subset H^0(t \times S, E_t \otimes L^m)$ under this bijection. Now we have a commutative diagram

$$\begin{array}{ccc}
H^0(S_t, \bar{E}_t \otimes \bar{L}_t^m) \otimes \mathcal{O}_{S_t} & \longrightarrow & \bar{E}_t \otimes \bar{L}_t^m \\
\uparrow \gamma & & \uparrow \\
\bar{V}_t \otimes \mathcal{O}_{S_t} & \longrightarrow & \bar{F}_t \otimes \bar{L}_t \\
\end{array}$$
where $\tilde{F}_t \otimes \tilde{L}_t^m \subset \tilde{E}_t \otimes \tilde{L}_t^m$ is defined as its subsheaf generated by the subspace $\tilde{V}_t$ by means of the morphism $\varepsilon \circ \Upsilon$. The associated map of global sections

$$H^0(\varepsilon') : \tilde{V}_t \to H^0(\tilde{S}_t, \tilde{F}_t \otimes \tilde{L}_t^m)$$

fits in the commutative triangle

$$\begin{array}{ccc}
\tilde{V}_t & \xrightarrow{H^0(\varepsilon')} & H^0(\tilde{S}_t, \tilde{F}_t \otimes \tilde{L}_t^m) \\
\downarrow & & \downarrow \\
H^0(\tilde{S}_t, \tilde{E}_t \otimes \tilde{L}_t^m) & & \\
\end{array}$$

which implies that $H^0(\varepsilon')$ is injective. Since each section in $H^0(\tilde{S}_t, \tilde{F}_t \otimes \tilde{L}_t^m)$ corresponds to a section in $H^0(t \times S, F_t \otimes L^m) \subset H^0(t \times S, E_t \otimes L^m)$, $H^0(\varepsilon')$ is surjective. Hence $h^0(t \times S, F_t \otimes L^m) = h^0(\tilde{S}_t, \tilde{F}_t \otimes \tilde{L}_t^m)$ for all $m \gg 0$ and stability (semistability) for $\tilde{E}_t$ implies stability (semistability, respectively) for $E_t$.

Proposition 9 is proved.

Proof of Lemma 3. Since the sheaves $\tilde{E}_t$ and $\sigma_t^* \tilde{E}_t$ coincide along the identified open subschemes $\tilde{S}_t \cap \Sigma_0 \simeq t \times S \cap \Sigma_0$, it is enough to verify that there is no torsion subsheaf in $\sigma_t^* \tilde{E}_t$ concentrated on $t \times S \cap (T \times S \setminus \Sigma_0)$. Assume that $T \subset \sigma_t^* \tilde{E}_t$ is such a torsion subsheaf, that is, $T \neq 0$ and $T(U) = 0$ for any open $U \subset t \times S \cap \Sigma_0$. Let $A = \text{Supp} \, T \subset t \times S$ and let $U \subset t \times S$ be an open subset such that $T(U) \neq 0$, that is, $U \cap A \neq \emptyset$. Now

$$T(U) \subset \sigma_t^* \tilde{E}_t(U) = \tilde{E}_t(\sigma^{-1}(U))$$

and any nonzero section $s \in T(U)$ is supported in $U \cap A$ and comes from a section $\tilde{s} \in \tilde{E}_t(\sigma^{-1}(U))$ with support in $\sigma^{-1}(U \cap A)$. This means that $\tilde{s}$ is supported in some additional component $\tilde{S}_{t,j}$ of the admissible scheme $\tilde{S}_t$. Hence $\tilde{s} \in \text{tors}_j$. But as $\tilde{E}_t$ is quasi-ideal on the additional components $\tilde{S}_{t,j}$ of $\tilde{S}_t$, we have $\text{tors}_j = 0$. This implies that $T = 0$.

§ 5. Functor isomorphism

In this section we prove that the natural transformations $\kappa : f^\text{GM} \to f$ and $\tau : f \to f^\text{GM}$ are mutually inverse and hence provide an isomorphism between the functor of moduli of admissible semistable pairs and the functor of moduli in the sense of Gieseker and Maruyama. As a corollary, we obtain an isomorphism of moduli schemes for these moduli functors which is independent of the number and geometry of their connected components, of the scheme structure being reduced and of the presence of locally free sheaves (respectively, $S$-pairs) in each component.

The proof has two aspects.

1. Pointwise. a) For any torsion-free semistable $\mathcal{O}_S$-sheaf the composite of transformations

$$E \mapsto ((\tilde{S}, \tilde{L}), \tilde{E}) \mapsto E'$$

returns $E' = E$. 
b) Conversely, for any admissible semistable pair \(((\tilde{S}, \tilde{L}), \tilde{E})\) the composite of transformations 
\[
((\tilde{S}, \tilde{L}), \tilde{E}) \mapsto E \mapsto ((\tilde{S}', \tilde{L}'), \tilde{E}')
\]
returns \(((\tilde{S}', \tilde{L}'), \tilde{E}') = ((\tilde{S}, \tilde{L}), \tilde{E}).

2. Global. a) For any family of semistable torsion-free sheaves \(E\) with base scheme \(T\) the composite 
\[
(E, L) \mapsto ((\pi: \Sigma \rightarrow T, \tilde{L}), \tilde{E}) \mapsto (E', L')
\]
returns a family \((E', L') \sim (E, L)\) in the sense of the description of the functor \(f^{\text{GM}}\) in (2.3), (2.4).

b) Conversely, for any family \(((\pi: \Sigma \rightarrow T, \tilde{L}), \tilde{E})\) of admissible semistable pairs with base scheme \(T\) the composite of transformations 
\[
((\pi: \Sigma \rightarrow T, \tilde{L}), \tilde{E}) \mapsto (E, L) \mapsto ((\pi': \Sigma' \rightarrow T, \tilde{L}'), \tilde{E}')
\]
returns a family \(((\pi': \Sigma' \rightarrow T, \tilde{L}'), \tilde{E}') \sim ((\pi: \Sigma \rightarrow T, \tilde{L}), \tilde{E})\) in the sense of the description of the functor \(f\) (2.2), (2.1).

We begin with 2.a); it will be specialized to the pointwise version 1.a) when 
\(T = \text{Spec } k\).

The families of polarizations \(L\) and \(L'\) coincide along the open subset \(\Sigma_0\) (the locally free locus for the sheaves \(E\) and \(E'\)) where \(\text{codim}_{T \times S} T \times S \setminus \Sigma_0 \geq 2\). Since \(L\) and \(L'\) are locally free, this implies that 
\(L = L'\).

Now consider three locally free \(\mathcal{O}_T\)-sheaves of rank \(rp(m)\):
\[
\mathbb{V}_m = p_*(E \otimes L^m), \quad \mathbb{V}_m = \pi_*(\tilde{E} \otimes \tilde{L}^m), \quad \mathbb{V}_m = p_*(E' \otimes L'^m).
\]

Lemma 4. There is an isomorphism of sheaves: \(\mathbb{V}_m \cong \mathbb{V}_m \cong \mathbb{V}_m'\).

Proof. We start with the epimorphism \(\sigma^*E \rightarrow \tilde{E}\). Tensoring it by \(\tilde{L}^m\) and the direct image \(\sigma_*\) yield a morphism of \(\mathcal{O}_T\)-sheaves
\[
\sigma_*(\sigma^*E \otimes \tilde{L}^m) \rightarrow \sigma_*(\tilde{E} \otimes \tilde{L}^m).
\]

We turn our attention to the following result.

Lemma 5 (see [9], Lemma 2.2). Let \(f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)\) be a morphism of locally ringed spaces such that \(f_*\mathcal{O}_X = \mathcal{O}_Y\), let \(\mathcal{E}\) be an \(\mathcal{O}_Y\)-module of finite presentation, and \(\mathcal{F}\) an \(\mathcal{O}_X\)-module. Then there is a monomorphism
\[
\mathcal{E} \otimes f_*\mathcal{F} \hookrightarrow f_*[f^*\mathcal{E} \otimes \mathcal{F}].
\]

Remark 7. In the general case we only have the morphism \(\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X\) at our disposal. Applying the inverse image \(f^*\), the tensor product \(\otimes \mathcal{F}\) and the direct image \(f_*\) to a finite presentation \(E_1 \rightarrow E_0 \rightarrow \mathcal{E} \rightarrow 0\) we obtain the commutative diagram
\[
\begin{array}{c}
 f_*[f^*E_1 \otimes \mathcal{F}] \longrightarrow f_*[f^*E_0 \otimes \mathcal{F}] \longrightarrow f_*[f^*\mathcal{E} \otimes \mathcal{F}] \\
 E_1 \otimes f_*\mathcal{F} \longrightarrow E_0 \otimes f_*\mathcal{F} \longrightarrow \mathcal{E} \otimes f_*\mathcal{F} \longrightarrow 0
\end{array}
\]
where the right-hand side is the morphism in question

\[ \mathcal{E} \otimes f_* \mathcal{F} \to f_! [f^* \mathcal{E} \otimes \mathcal{F}] \]

Setting \( \mathcal{E} = \mathcal{E}, \mathcal{F} = \widehat{L}^m \) and \( f = \sigma \) we obtain

\[ \mathcal{E} \otimes \widehat{L}^m \otimes \sigma_*(\sigma^{-1} I \cdot \mathcal{O}_{\widehat{S}})^m \]

and after taking the direct image \( p_\ast \)

\[ p_\ast (\mathcal{E} \otimes \widehat{L}^m \otimes \sigma_*(\sigma^{-1} I \cdot \mathcal{O}_{\widehat{S}})^m) \to p_\ast (\mathcal{E} \otimes \widehat{L}^m) \]

where the upper horizontal arrow is the natural immersion into the reflexive hull. Since the target sheaf of the composite map \( \eta \) is reflexive, \( \eta \) factors through \( p_\ast (\mathcal{E} \otimes \widehat{L}^m) \) as the reflexive hull of the source. This yields the existence of a morphism of locally free sheaves of equal rank

\[ \widetilde{\eta}: \pi_\ast (\mathcal{E} \otimes \widehat{L}^m) \to \sigma_\ast (\mathcal{E} \otimes \widehat{L}^m) \]

A morphism of sheaves is an isomorphism if and only if it is stalkwise isomorphic. Fix an arbitrary closed point \( t \in T \); up to the end of this proof we omit the subscript \( t \) in the notation related to the sheaves corresponding to \( t \): \( \mathcal{E}_t =: \mathcal{E}, \widetilde{E}_t =: \widetilde{E}, \widetilde{S}_t =: \widetilde{S} \) and \( \sigma_t =: \sigma: \widetilde{S} \to S \). There is an epimorphism of \( \mathcal{O}_{\widetilde{S}} \)-modules

\[ \sigma^* \mathcal{E} \otimes \widehat{L}^m \to \widetilde{E} \otimes \widehat{L}^m \]

As in the global case, the analogue of the projection formula leads to

\[ \mathcal{E} \otimes L^m \otimes \sigma_*(\sigma^{-1} I \cdot \mathcal{O}_{\widehat{S}})^m \]

and taking global sections we obtain

\[ H^0(S, \mathcal{E} \otimes L^m \otimes \sigma_*(\sigma^{-1} I \cdot \mathcal{O}_{\widehat{S}})^m) \to H^0(S, \mathcal{E} \otimes L^m) \]

\[ \eta \]

\[ \widetilde{\eta} \]

\[ \mathcal{E} \otimes \sigma_*(\sigma^* \mathcal{E} \otimes \widehat{L}^m) \to \sigma_\ast (\mathcal{E} \otimes \widehat{L}^m) \]

\[ \mathcal{E} \otimes \sigma_*(\sigma^* \mathcal{E} \otimes \widehat{L}^m) \to \sigma_\ast (\mathcal{E} \otimes \widehat{L}^m) \]
These blowup morphisms are defined by the same sheaf of ideals $H^0$, hence the images of both schemes coincide off their additional components, that is, $r: (S, E \otimes L^m) \esarrow (S \cap S_0, E \otimes L^m)$, where both the horizontal arrows are restriction maps and the upper restriction map is injective. Hence $\tilde{\eta}_t$ is also injective. Since it is a monomorphism of vector spaces of equal dimension, it is an isomorphism. Then $\eta: \mathbb{V}_m \rightarrow \mathbb{V}_m'$ is also an isomorphism. The isomorphism $\mathbb{V}_m \cong \mathbb{V}_m'$ was established in the previous section (the proof of Proposition 8).

Lemma 4 is proved.

Identifying locally free sheaves $\mathbb{V}_m = \mathbb{V}'_m$ consider the relative Grothendieck scheme $\text{Quot}^{rp(n)}(p^*\mathbb{V}_m \otimes \mathbb{L}^{-m})$ and two $T$-morphisms of closed immersion

$$T \times S \overset{\tilde{\nu}}{\rightarrow} \text{Quot}^{rp(n)}(p^*\mathbb{V}_m \otimes \mathbb{L}^{-m}) \times S \overset{\tilde{\nu}'}{\rightarrow} T \times S.$$ 

The morphism $\tilde{\nu}$ is induced by the morphism $\nu: p^*\mathbb{V}_m \otimes \mathbb{L}^{-m} \rightarrow E$ and $\tilde{\nu}'$ is induced by the morphism $\nu': p^*\mathbb{V}_m \otimes \mathbb{L}^{-m} \rightarrow E'$.

Since both morphisms $\tilde{\nu}$ and $\tilde{\nu}'$ are proper and coincide along $\Sigma_0$ such that $\text{codim}_{T \times S}(T \times S) \setminus \Sigma_0 \geq 2$, we have $\tilde{\nu} = \tilde{\nu}'$ and $\tilde{\nu}(T \times S) = \tilde{\nu}'(T \times S)$ in the sense of schemes. Hence by the universality of the Quot scheme, $E = E'$ as the inverse images of the universal quotient sheaf over $\text{Quot}^{rp(n)}(p^*\mathbb{V}_m \otimes \mathbb{L}^{-m})$ under the morphisms $\tilde{\nu} = \tilde{\nu}'$.

Now we turn to 1, b). For $m \gg 0$ there are surjective morphisms

$$H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \otimes \mathcal{O}_{\tilde{S}} \rightarrow \tilde{E} \otimes \tilde{L}^m$$

and

$$H^0(\tilde{S}', \tilde{E}' \otimes \tilde{L}^m) \otimes \mathcal{O}_{\tilde{S}'} \rightarrow \tilde{E}' \otimes \tilde{L}^m,$$

where $H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \cong H^0(\tilde{S}', \tilde{E}' \otimes \tilde{L}^m) = V_m$, $\dim V_m = rp(m)$. There are two induced closed immersions of schemes $\tilde{S}$ and $\tilde{S}'$ into the Grassmann variety of $r$-quotient spaces of a vector space of dimension $rp(m)$:

$$\tilde{S} \overset{i}{\rightarrow} G(rp(m), r) \overset{j'}{\rightarrow} \tilde{S}' .$$

The images of both schemes coincide off their additional components, that is,

$$j\left(\tilde{S} \setminus \bigcup_{i > 0} \tilde{S}_i\right) = j'\left(\tilde{S}' \setminus \bigcup_{i > 0} \tilde{S}'_i\right).$$

Hence $j(\tilde{S}_0) = j'(\tilde{S}'_0)$. Since $\sigma_0 = \sigma|_{\tilde{S}_0}$ is a blowup morphism, as is $\sigma' = \sigma'|_{\tilde{S}'_0}$, these blowup morphisms are defined by the same sheaf of ideals $I \subset \mathcal{O}_S$. This leads
to the conclusion that the schemes $\widetilde{S}$ and $\widetilde{S}'$ as a whole are defined by the same sheaf of ideals $I$ and hence $\widetilde{S} \cong \widetilde{S}'$.

Since $\widetilde{L} = L \otimes \sigma^{-1} I \cdot \mathcal{O}_{\widetilde{S}}$ and $\widetilde{L}' = L \otimes \sigma'^{-1} I \cdot \mathcal{O}_{\widetilde{S}}$, where $\widetilde{S} \cong \widetilde{S}'$, $\sigma = \sigma'$, it follows that $\widetilde{L} \cong \widetilde{L}'$.

Now it remains to verify that $\widetilde{E} \cong \widetilde{E}'$. This will follow from global considerations for $T = \text{Spec } k$. We turn to the global case.

For the global version we consider families $(\pi: \Sigma \to T, L)$ and $(\pi': \Sigma' \to T, L')$ and epimorphisms $\pi^* \pi_* L \to L$ and $\pi'^* \pi'_* L' \to L'$. We can assume that $\pi_* L = \pi'_* L'$ and then identify the projective bundles $\mathbb{P}(\pi_* L)^\vee = \mathbb{P}(\pi'_* L')^\vee$. The closed immersions of $T$-schemes

$$j: \Sigma \hookrightarrow \mathbb{P}(\pi_* L)^\vee, \quad j': \Sigma' \hookrightarrow \mathbb{P}(\pi'_* L')^\vee$$

and the diagonal immersion $\mathbb{P}(\Delta) \hookrightarrow \mathbb{P}(\pi_* L)^\vee \times_T \mathbb{P}(\pi'_* L')^\vee$ lead to the commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}(\pi_* L)^\vee & \sim & \mathbb{P}(\pi'_* L')^\vee \\
\downarrow j & & \downarrow j' \\
\Sigma & \sim & \Sigma' \\
\end{array}
$$

The fibred product $\Sigma \times_T \Sigma' \hookrightarrow \mathbb{P}(\pi_* L)^\vee \times_T \mathbb{P}(\pi'_* L')^\vee$ gives rise to the intersection subscheme $\Sigma_\Delta = (\Sigma \times_T \Sigma') \cap \mathbb{P}_\Delta$. Now we observe that there is a commutative square

$$
\begin{array}{ccc}
\Sigma & \sim & \Sigma_\Delta \\
\downarrow \pi & & \downarrow \\
T & \sim & \Sigma' \\
\end{array}
$$

where, as we have seen before, for any closed point $t \in T$ the corresponding fibres of the schemes $\Sigma$, $\Sigma'$ and $\Sigma_\Delta$ are identified isomorphically by the arrows in the diagram.

Also, from the commutative diagram

$$
\begin{array}{ccc}
\Sigma_\Delta & \sim & \mathbb{P}(\pi_* L)^\vee \\
\downarrow j & & \downarrow \\
\Sigma & \sim & \mathbb{P}(\pi_* L)^\vee \\
\end{array}
$$

we conclude that the left-hand vertical arrow is a closed immersion. There is a closed immersion $\Sigma_\Delta \hookrightarrow \Sigma'$ for the same reason.

Now we make use of the following algebraic result.
Proposition 10 (see [15], Ch. 1, Proposition 2.5). Let $B$ be a flat $A$-algebra and let $b \in B$. If the image of $b$ in $B/mB$ is not a zero divisor for any maximal ideal $m$ in $A$, then $B/\langle b \rangle$ is a flat $A$-algebra.

Take a section $(s,s')$ of $\mathcal{O}_P(\pi_\Sigma(\pi_\Delta)^\vee)\otimes_{\mathcal{O}_T} \mathcal{O}_{\Delta}$ and let $b = (s',s'')$ be its image in $\mathcal{O}_\Sigma(\pi_\Delta)^\vee$. In our situation $m = m_t$, and $b$ has an image in $\mathcal{O}_{\pi^{-1}(t)}(\pi_\Delta)^\vee$ which is not a zero divisor. Iterating, using Proposition 10 repeatedly in a regular $O$-composite map in (5.1). Since $\Sigma$ is flat over $T$. It remains to compare the Hilbert polynomials of the fibres over $t$ in the exact triple

$$0 \to \mathcal{I}_{\Sigma_\Delta, \Sigma} \to \mathcal{O}_{j(\Sigma)} \to \mathcal{O}_{j(\Sigma_\Delta)} \to 0$$

when the isomorphic projective bundles $P_\Delta \cong \mathbb{P}(\pi_\Sigma)^\vee$ are identified under the composite map in (5.1). Since $\mathcal{O}_\Sigma$ and $\mathcal{O}_{\Sigma_\Delta}$ are $T$-flat, $\mathcal{I}_{\Sigma_\Delta, \Sigma}$ is also $T$-flat. By the infinitesimal criterion for flatness [16], the fibrewise Hilbert polynomials

$$\chi(\mathcal{I}_{\Sigma_\Delta, \Sigma} \otimes \mathbb{L}^n|_{j(\pi^{-1}(t))})$$

do not depend on the closed point $t \in T$.

We have $\chi(\mathcal{O}(\pi)|_{j(\pi^{-1}(t)))}) = \chi(\mathcal{O}(\pi)|_{j(\Sigma_\Delta)})$ and hence we conclude that $\chi(\mathcal{I}_{\Sigma_\Delta, \Sigma} \otimes \mathbb{L}^n|_{j(\pi^{-1}(t))}) = 0$. Now $j$ and $j_\Delta$ are identified under the isomorphism $\mathbb{P}(\pi_\Sigma)^\vee = P_\Delta$. For the same reason $j'$ and $j_\Delta$ are identified under the isomorphism $\mathbb{P}((\pi_\Sigma)^\vee) = \mathbb{P}((\pi_\Sigma)^\vee)$ and hence $\Sigma \cong \Sigma'$ and under this identification $\mathbb{L} = j^*\mathcal{O}(1) = j'^*\mathcal{O}(1) = \mathbb{L}'$ too.

To verify that $\widetilde{E} = \widetilde{E}'$ as well, we argue in a similar way and consider closed immersions of the $T$-schemes $\Sigma$ and $\Sigma'$ into Grassmann varieties:

$$\Sigma \xrightarrow{j} \text{Grass}(\mathbb{V}_m, r) = \text{Grass}(\mathbb{V}'_m, r) \xrightarrow{j'} \Sigma'.$$

We introduce the shorthand notation $G := \text{Grass}(\mathbb{V}_m, r)$ and $G' := \text{Grass}(\mathbb{V}'_m, r)$ and form a fibred product $G \times_T G'$ together with the diagonal $G_\Delta \hookrightarrow G \times_T G'$; and we also form the subscheme $\Sigma_\Delta = (\Sigma \times_T \Sigma') \cap G_\Delta$. As previously, there is a commutative square

$$\begin{array}{ccc}
\widetilde{\Sigma}_\Delta & \xrightarrow{j_\Delta} & G_\Delta \\
\downarrow & & \downarrow \sim \\
\widetilde{\Sigma}' & \xrightarrow{j} & G
\end{array}$$

from which we see that $\widetilde{\Sigma}_\Delta \hookrightarrow \widetilde{\Sigma}$ as closed subscheme. Applying Proposition 10 we conclude that $\widetilde{\Sigma}_\Delta$ is flat over $T$. Since fibres of the schemes $\widetilde{\Sigma}_\Delta$ and $\widetilde{\Sigma}$ over same closed point $t \in T$ coincide, they have equal Hilbert polynomials as subschemes in the Grassmann variety $G_t \cong G(rp(m), r)$. Hence $j_\Delta(\Sigma_\Delta) = j(\Sigma)$ under the identification $G_\Delta = G$ and also $j_\Delta(\Sigma_\Delta) = j'(\Sigma')$ under the identification $G_\Delta = G'$. Now let $\pi^*\mathbb{V}_m \to \mathcal{O}$ be the universal quotient bundle on $G = G'$. Then $\mathbb{E} \otimes \mathbb{L} = j^*\mathcal{O} = j'^*\mathcal{O} = \mathbb{E}' \otimes \mathbb{L}'$ and hence $\mathbb{E} = \mathbb{E}'$.

This is dedicated to the blessed memory of my Mom.
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