STANDARD MODEL AND UNIMODULARITY CONDITION

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Abstract

The unimodularity condition in Connes’ formulation of the standard model is rewritten in terms of group representations.

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1 Introduction

Is our universe open or closed? It is likely that we will never have an experimental answer to this old question. Due to the small value of the speed of light the global geometry of spacetime escapes observation. On the other hand the global properties of the rotation group are well established in quantum physics. Neutrons have to be rotated through an angle of $720^\circ$ before interference patterns repeat [1]. Mathematically this means that spin $\frac{1}{2}$ particles are represented only up to a phase under the rotation group $SO(3)$. If we want genuine representations we must use its universal cover $SU(2)$. In the same spirit we may wonder about the global nature of the internal group $SU(2) \times U(1) \times SU(3)$ coding electro-weak and strong forces. It is well known [2] that the representations classifying quarks, leptons, gauge and Higgs bosons in the standard model are already representations of $S(U(2) \times U(3))$. This experimental fact has no explanation in the frame of Yang-Mills-Higgs theories. In the frame of Connes’ noncommutative geometry [3], where internal groups do not fall from heaven but are derived from associative involution algebras, this coincidence is vital.

2 The global nature of the standard model

For every integer greater than one, $n \geq 2$, the maps

$$U(n) \to SU(2) \times U(1)$$

$$u \mapsto (\det u^{-1/n} u, \det u^{1/n}) \quad (1)$$

$$ds \leftarrow (s, d) \quad (2)$$

define the isomorphism

$$U(n) = SU(n) \times U(1) / \mathbb{Z}_n,$$  

where $\mathbb{Z}_n$ permutes the $n$ roots of the determinant. For instance for $n = 2$, we have

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \mapsto \left( \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}, +i \right) \sim \left( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, -i \right). \quad (4)$$

The map $\rho(u) = \det u^z u, u \in U(n), z \in \mathbb{Z}$, defines a representation of $U(n)$. Under the above isomorphism it induces the fundamental representation of $SU(n)$ with $U(1)$ charges $1 + zn$:

$$\det u^z u = d^{zn} d^{1/n} s = d^{1+zn} s. \quad (5)$$

The $U(n)$ representation $\rho(u) = \det u^z = d^{zn}$ induces the $SU(n)$ singlet representation with $U(1)$ charges $zn$ and likewise for the adjoint representation. The general theory [2] tells us
these are the only $U(1)$ charges possible for the fundamental, singlet and adjoint $SU(n)$ repre-
sentations induced from (continuous, unitary) $U(n)$ representations.

Let us recall the representations of the standard model, we denote by $(m_2, 6y, m_3)$ the tensor
product of a $m_2$ dimensional representation under $SU(2)$ and a $m_3$ dimensional representation
under $SU(3)$ with hypercharge $y$. Note that $U(1)$ charges only make sense after multiplication
with the coupling constant $g_1$ and we have multiplied the conventional hypercharges by 6 in
order to make them integers. The fermion representations are:

\[
\begin{align*}
\left( \begin{array}{c} u \\ d \end{array} \right)_L & : (2, 1, 3) \\
\left( \begin{array}{c} e \\ \nu \end{array} \right)_L & : (2, -3, 1) \\
u_R : (1, 4, 3) & \\
d_R : (1, -2, 3) & \\
e_R : (1, -6, 1), &
\end{align*}
\]

the gauge bosons:

\[
\begin{align*}
W^\pm, \cos \theta_w Z + \sin \theta_w \text{photon} : (3, 0, 1) \\
- \sin \theta_w Z + \cos \theta_w \text{photon} : (1, 0, 1) & \\
gluons : (1, 0, 8) &
\end{align*}
\]

and the Higgs scalar:

\[
\varphi : (2, -3, 1).
\]

Surprisingly, nature has chosen these nine irreducible representations (nineteen representations
if we take into account all three generations of quarks and leptons) such that they can all be
induced from $SU(2) \times U(3)$ and simultaneously from $U(2) \times SU(3)$. In other words, the internal
group of the standard model can be reduced to

\[
S(U(2) \times U(3)) = \frac{SU(2) \times U(1) \times SU(3)}{\mathbb{Z}_2 \times \mathbb{Z}_3}. \tag{6}
\]

3 Connes’ point of view

Connes [3] has generalized Riemannian spaces to include an uncertainty principle. As in quan-
tum mechanics this uncertainty is coded in an associative, noncommutative involution algebra
$\mathcal{A}$ and a representation $\rho$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. In quantum mechanics $\mathcal{A}$ is the algebra
of observables and $\mathcal{H} = L^2(\text{configuration space})$. In order to capture the metric and in order
to generalize differentiation and integration to the noncommutative setting Connes introduces
a selfadjoint operator $\mathcal{D}$ on the Hilbert space, the ‘Dirac operator’. In even dimensional spaces
one also needs a ‘chirality’, a unitary operator $\chi$ and in real spaces one needs a ‘real structure’,
an anti-unitary operator \( J \). The five items, \( \mathcal{A}, \mathcal{H}, \mathcal{D}, \chi, J \) are called spectral triple. They are supposed to satisfy axioms, that are calibrated on Riemannian spin manifolds \( M \), with the \textit{commutative} algebra of differentiable functions on \( M \), \( \mathcal{A} = C^\infty(M) \), \( \mathcal{H} \) is the space of square integrable spinors on which a function acts by pointwise multiplication, the Dirac operator is the genuine one, \( \mathcal{D} = \partial / \partial \), the chirality is \( \chi = \gamma_5 \) and the real structure \( J \) is charge conjugation. The axioms are chosen such that there is a one-to-one correspondence between commutative spectral triples and Riemannian spin manifolds.

Let us spell out the spectral triple for the zero dimensional internal space of the standard model with one generation of quarks and leptons:

\[
\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \ni (a, b, c),
\]

\[
\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c.
\]

\[
\mathcal{H}_L = (\mathbb{C}^2 \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}),
\]

\[
\mathcal{H}_R = ((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}).
\]

The first factor denotes isospin, the second colour. We choose the following basis of \( \mathcal{H} = \mathbb{C}^{30} \):

\[
\begin{pmatrix}
  (u \\ d), (\nu_e \\ e), u_R, e_R, (u \\ d)^c, (\nu_e \\ e)^c, u_R^c, e_R^c.
\end{pmatrix}
\]

The representation \( \rho \) is defined by

\[
\rho(a, b, c) := \begin{pmatrix}
  \rho_w(a, b) & 0 \\
  0 & \bar{\rho}_s(b, c)
\end{pmatrix} := \begin{pmatrix}
  \rho_w L(a) & 0 & 0 & 0 \\
  0 & \rho_w R(b) & 0 & 0 \\
  0 & 0 & \bar{\rho}_s L(b, c) & 0 \\
  0 & 0 & 0 & \bar{\rho}_s R(b, c)
\end{pmatrix}
\]

with

\[
\rho_w L(a) := \begin{pmatrix}
  a \otimes 1_3 & 0 \\
  0 & a
\end{pmatrix}, \quad \rho_w R(b) := \begin{pmatrix}
  B \otimes 1_3 & 0 \\
  0 & \bar{b}
\end{pmatrix},
\]

\[
B := \begin{pmatrix}
  b & 0 \\
  0 & \bar{b}
\end{pmatrix},
\]

\[
\rho_s L(b, c) := \begin{pmatrix}
  1_2 \otimes c & 0 \\
  0 & \bar{b} 1_2
\end{pmatrix}, \quad \rho_s R(b, c) := \begin{pmatrix}
  1_2 \otimes c & 0 \\
  0 & \bar{b}
\end{pmatrix}.
\]

The Dirac operator does not occur in the following calculation, we indicate it for completeness,

\[
\mathcal{D} = \begin{pmatrix}
  0 & \mathcal{M} & 0 & 0 \\
  \mathcal{M}^* & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix}
  m_u & 0 \\
  0 & m_d \\
  m_e & 0 \\
  0 & m_e
\end{pmatrix}.
\]
We will need chirality and charge conjugation,

\[ \chi = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \text{complex conjugation.} \tag{16} \]

Gauge invariance can be defined by

- the group of unitaries,

\[ U(A) = \{ u \in A, \; uu^* = u^*u = 1 \} = SU(2) \times U(1) \times U(3), \tag{17} \]

- the automorphism group $\text{Aut}(A)$. As locally every automorphism $\varphi$ of a matrix algebra $A$ is inner, $\varphi(a) = uau^{-1}$, for a unitary $u \in U(A)$, we can obtain the automorphism group as image of the unitary group under the representation $\varphi$ of $U(A)$ on the vector space $A$,

\[ \text{Aut}(A) \sim SU(2) \times SU(3), \tag{18} \]

- the ‘covering’ $\text{Aut}^H(A)$ of the automorphism group on the Hilbert space $H$. The covering is achieved by the physical representation of the group of unitaries on the fermionic Hilbert space $H$, $\rho(u)J\rho(u)J^{-1}$. This representation is physical because it re-establishes invariance under charge conjugation. Note that it is not an algebra representation, it defines a bimodule. Its image is locally

\[ \text{Aut}^H(A) \sim U(A) = SU(2) \times U(1) \times U(3). \tag{19} \]

After these unsuccessful attempts to obtain the internal group of the standard model $G = SU(2) \times U(1) \times SU(3)$, we are reduced to reduce the group of unitaries by an ad hoc condition, the ‘unimodularity’ condition. At least it has the virtue to make sense for general algebras $A$. For the finite dimensional algebra of the standard model, the unimodularity condition can be formulated conveniently on the Lie algebra level:

\[ G \sim \exp \left\{ X \in u(A), \; \text{tr} [P(\rho(X) + J\rho(X)J^{-1})] = 0 \right\}, \tag{20} \]

where $P$ is the projection on the particles, $H_L \oplus H_R$. For the standard model, the unimodularity condition is equivalent to the requirement that the physical representation $\rho(u)J\rho(u)J^{-1}$ be free of gauge and gravitational anomalies:

\[ \text{tr} [P\chi(\rho(X) + J\rho(X)J^{-1})^3] = 0, \tag{21} \]

\[ \text{tr} [P\chi(\rho(X) + J\rho(X)J^{-1})] = 0. \tag{22} \]
A natural question now is: can we modify the physical representation of $U(A)$ by a phase, that is a central element, such that the image of the modified representation is locally isomorphic to $G$? Let us call $\sigma$ the modified representation for which we try the following ansatz:

$$\sigma(u) = \rho(u)J\rho(u)J^{-1} \rho(p(u))J\rho(p(u))J^{-1}, \quad u \in U(A),$$  \hfill (23)

with the phase $p$,

$$p : U(A) = SU(2) \times U(1) \times U(3) \rightarrow U(A) \cap \text{center}(A)$$

$$u = (u_2, u_1, u_3) \mapsto p(u) = (1, u_1^\alpha \det u_3^\mu, u_1^\beta \det u_3^\nu, 1_3).$$  \hfill (24)

$\sigma$ is a representation because $p(u\bar{u}) = p(u)p(\bar{u})$ and

$$\sigma(u) = \rho(up(u))J\rho(up(u))J^{-1}.$$  \hfill (25)

We will need the explicit form of $\rho J\rho J^{-1}$ restricted to the particles,

$$\rho(u_2, u_1, u_3)J\rho(u_2, u_1, u_3)J^{-1} = \begin{pmatrix} u_2 \otimes u_3 & 0 & 0 & 0 \\ 0 & u_2 u_1^{-1} & 0 & 0 \\ 0 & 0 & (u_1 u_3) & 0 \\ 0 & 0 & 0 & u_1^{-2} \end{pmatrix}. \hfill (26)$$

We want that the representation $\sigma$ restricted to the group $G$ of the standard model close to identity coincide with the physical representation,

$$\sigma(u) = \rho(u)J\rho(u)J^{-1}, \quad \text{for all } u \in G, \quad u \sim 1,$$  \hfill (27)

and that the image of $\sigma$ be locally isomorphic to $G$,

$$\sigma(U(A)) = \sigma(G) = \rho(G)J\rho(G)J^{-1}. \hfill (28)$$

Let us write down the infinitesimal form of $\sigma$. It is sufficient to keep track of the two $U(1)$s. Since $\sigma$ is invariant under charge conjugation we restrict the computation to the particles:

$$\sigma (1_2, e^{i\theta_1}, e^{i\theta_3} 1_3) = 1_{15} + i \text{ diag } \begin{pmatrix} [\beta \theta_1 + (1 + 3\nu)\theta_3]1_2 \otimes 1_3 \\ -[(1 + \alpha)\theta_1 + 3\mu\theta_3]1_2 \\ [(1 + \alpha + \beta)\theta_1 + (1 + 3\mu + 3\nu)\theta_3]1_3 \\ -2[(1 + \alpha)\theta_1 + 3\mu\theta_3] \end{pmatrix} + O(\theta^2).$$  \hfill (29)
For all values of $\theta_1$ et $\theta_3$ this expression must be equal to the exponential of hypercharge,

$$1_{15} + i2\theta \, \text{diag} \begin{pmatrix}
y_1 1_2 \otimes 1_3 \\
y_2 1_2 \\
y_3 1_3 \\
y_4 1_3 \\
y_5
\end{pmatrix} + O(\theta^2).$$  \hfill (30)

In our normalization the hypercharge values are,

$\begin{align*}
y_1 &= \frac{1}{6}, & 6y_1 &= 1 \mod 2 \text{ and } 1 \mod 3, \\
y_2 &= -\frac{1}{2}, & 6y_2 &= 1 \mod 2 \text{ and } 0 \mod 3, \\
y_3 &= \frac{2}{3}, & 6y_3 &= 0 \mod 2 \text{ and } 1 \mod 3, \\
y_4 &= -\frac{1}{3}, & 6y_4 &= 0 \mod 2 \text{ and } 1 \mod 3, \\
y_5 &= -1, & 6y_5 &= 0 \mod 2 \text{ and } 3 \mod 3.
\end{align*}$  \hfill (31)

Equating the two expressions (29) and (30) yields five equations. The last three equations are simply combinations of the first two thanks to the three experimental identities, $y_1 - y_2 = y_3$, $y_1 + y_2 = y_4$ and $2y_2 = y_5$. Note that if $y_1$ and $y_2$ satisfy the conditions from $S(U(2) \times U(3))$ recalled in equations (31), then this is also true for $y_3$, $y_4$ and $y_5$ computed with the three experimental identities. We rewrite the first two equations as:

$$\left( \begin{array}{cc} 1 + \alpha & 3\mu \\ \beta & 1 + 3\nu \end{array} \right) \left( \begin{array}{c} \theta_3/\theta \\ \theta_1/\theta \end{array} \right) = \left( \begin{array}{c} -2y_2 \\ 2y_1 \end{array} \right) = \left( \begin{array}{c} 1 \\ 1/3 \end{array} \right).$$  \hfill (32)

In the physical representation $\rho J \rho J^{-1}$, $\alpha = \beta = \mu = \nu = 0$, hypercharge is given by the following linear combination of the two $u(1)$s, $\theta_3 = \frac{1}{3}\theta_1$,

$$u_{\text{hypercharge}} = (1_2, e^{i\theta}, e^{i\theta/3}1_3).$$  \hfill (33)

We want the two representations of hypercharge to coincide,

$$\sigma(u_{\text{hypercharge}}) = \rho(u_{\text{hypercharge}})J \rho(u_{\text{hypercharge}})J^{-1}.$$  \hfill (34)

This is equivalent to $\alpha = -\mu$ and $\beta = -\nu$. Furthermore we want the image of the representation to be locally isomorphic to $G$. This is equivalent to a vanishing determinant of the matrix in equation (32),

$$(1 + \alpha)(1 + 3\nu) = \beta 3\mu.$$  \hfill (35)

This gives:

$$\sigma(u_2, u_1, u_3) = \rho(u_2, \det \bar{u}_3, \bar{u}_3)J \rho(u_2, \det \bar{u}_3, \bar{u}_3)J^{-1}, \quad \bar{u}_3 := u_1^{\beta} \det u_3^{-\beta} u_3.$$  \hfill (36)

As a matter of fact, the modified representation $\sigma$ ignores $u_1$ and it is indeed a representation of $SU(2) \times U(3)$. 

6
4 Conclusion

The unimodularity condition remains a disturbing feature in the geometric formulation of the standard model. This condition is connected to anomaly cancellation, an intriguing feature of the standard model. We remark that it is also connected to another intriguing feature of the standard model, namely that the internal group may be reduced to $SU(2) \times U(3)$. Of course we would like to use the reduction to $U(2) \times SU(3)$ as well. This points towards the algebra $\mathcal{A} = M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_3(\mathbb{C})$, that has been suggested by Connes in the context of quantum groups and that is a major motivation of this workshop. Within noncommutative Yang-Mills theories, this algebra leads to an unacceptable light neutral scalar. The phenomenological analysis of this algebra within the spectral action is in progress.

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