A Liouville comparison principle for solutions of semilinear elliptic second-order partial differential inequalities

Vasilii V. Kurta

June 17, 2021

Abstract

We consider semilinear elliptic second-order partial differential inequalities of the form

\[ Lu + |u|^{q-1}u \leq Lv + |v|^{q-1}v \quad (\ast) \]

in the whole space \( \mathbb{R}^n \), where \( n \geq 2 \), \( q > 0 \) and \( L \) is a linear elliptic second-order partial differential operator in divergence form. We assume that the coefficients of the operator \( L \) are measurable and locally bounded such that the quadratic form associated with the operator \( L \) is symmetric and non-negative definite. We obtain a Liouville comparison principle in terms of capacities associated with the operator \( L \) for solutions of \((\ast)\) which are measurable and belong locally in \( \mathbb{R}^n \) to a Sobolev-type function space also associated with the operator \( L \).

1 Definitions

Let \( L \) be a linear elliptic second-order partial differential operator defined by

\[ Lu := \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} \quad (1) \]

in the whole space \( \mathbb{R}^n \), \( n \geq 2 \). The coefficients \( a_{ij} \) of the operator \( L \) are measurable and locally bounded in \( \mathbb{R}^n \), \( a_{ij} = a_{ji} \), \( i, j = 1, \ldots, n \). We assume that the quadratic form associated with the operator \( L \) is non-negative definite, namely, it satisfies the condition

\[ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0 \quad (2) \]

for all \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \) and almost all \( x \in \mathbb{R}^n \).
We study solutions \((u, v)\) of semilinear elliptic second-order partial differential inequalities of the form
\[
Lu + |u|^{q-1}u \leq Lv + |v|^{q-1}v,
\]
where \(q > 0\). We assume that \((u, v)\) is measurable in \(\mathbb{R}^n\) and belong locally to a Sobolev-type function space associated with the differential operator \(L\). We call such solution the entire solution of the inequality (3) in \(\mathbb{R}^n\).

Note that if \(u\) and \(v\) satisfy, respectively, the semilinear elliptic second-order partial differential inequalities
\[
- Lu \geq |u|^{q-1}u \quad \text{(4)}
\]
and
\[
- Lv \leq |v|^{q-1}v, \quad \text{(5)}
\]
then the pair \((u, v)\) satisfies the inequality (3). Hence, all the results obtained in this paper for solutions of (3) are valid for the corresponding solutions of the system (4)–(5).

**Definition 1.** Let \(n \geq 2\), \(q > 0\), \(\hat{q} = \max\{1, q\}\), \(L\) be a differential operator defined by (1) in \(\mathbb{R}^n\), and \(Q\) be a bounded domain in \(\mathbb{R}^n\). By \(W^{L,\hat{q}}(Q)\) we denote the completion of the function space \(C^\infty(Q)\) with respect to the norm defined by the expression
\[
\|f\|_{W^{L,\hat{q}}(Q)} := \left( \int_Q \sum_{i,j=1}^n a_{ij} f_{x_i} f_{x_j} \, dx \right)^{1/2} + \left( \int_Q |f|^\hat{q} \, dx \right)^{1/\hat{q}}. \quad \text{(6)}
\]

**Definition 2.** Let \(n \geq 2\), \(q > 0\), and \(L\) be a differential operator defined by (1) in \(\mathbb{R}^n\). We say that a function \(f\) belongs to the function space \(W^{L,\hat{q}}_{\text{loc}}(\mathbb{R}^n)\), if it belongs to \(W^{L,\hat{q}}(Q)\) for every bounded domain \(Q\) in \(\mathbb{R}^n\).

**Definition 3.** Let \(n \geq 2\), \(q > 0\), and \(L\) be a differential operator defined by (1) in \(\mathbb{R}^n\). We say that a pair of functions \((u, v)\) is an entire solution of the inequality (3) in \(\mathbb{R}^n\), if \(u\) and \(v\) belong to the function space \(W^{L,\hat{q}}_{\text{loc}}(\mathbb{R}^n)\) and satisfy the integral inequality
\[
\int_{\mathbb{R}^n} \left( \sum_{i,j=1}^n a_{ij} u_{x_i} \zeta_{x_j} - |u|^{q-1} u \zeta \right) \, dx \geq \int_{\mathbb{R}^n} \left( \sum_{i,j=1}^n a_{ij} v_{x_i} \zeta_{x_j} - |v|^{q-1} v \zeta \right) \, dx \quad \text{(7)}
\]
for every non-negative function \(\zeta \in C^\infty(\mathbb{R}^n)\) with compact support.

We understand inequality (7) in the sense that discussed as in [4] or [9]. Definitions of entire solutions to inequalities (4)–(5), are the special cases of Definition 3 with \(v \equiv 0\) or \(u \equiv 0\), respectively.

**Definition 4.** Let \(n \geq 2\). We denote by \(\text{Lip}_{\text{loc}}(\mathbb{R}^n)\) the space of measurable functions \(f\) defined in \(\mathbb{R}^n\) which satisfy the Lipschitz condition
\[
|f(x) - f(y)| \leq K|x - y|
\]
on any compact set \(K\) in \(\mathbb{R}^n\), with \(K\) some positive constant which possibly depends on \(K\).
Definition 5. Let \( n \geq 2 \), \( \mathcal{E} \) be a domain in \( \mathbb{R}^n \), and \( \mathcal{D}, \mathcal{G} \) be two sets in \( \mathcal{E} \), which are non-overlapping and closed with respect to \( \mathcal{E} \). We call such a triple \((\mathcal{D}, \mathcal{G}; \mathcal{E})\) the condenser.

For a given differential operator \( L \) defined by (1) in \( \mathbb{R}^n \), following [7], we use a notion of the \((L,p)\)-capacity of the condenser \((\mathcal{D}, \mathcal{G}; \mathcal{E})\) in the particular case when \( \mathcal{E} \) coincides with the whole space \( \mathbb{R}^n \).

Definition 6. Let \( n \geq 2 \), \( p > 1 \), and \( L \) be a differential operator defined by the relation (1) in \( \mathbb{R}^n \). We call the quantity

\[
\text{cap}_{L,p}(\mathcal{D}, \mathcal{G}; \mathbb{R}^n) = \inf \left\{ \int_{\mathbb{R}^n} \left( \sum_{i,j=1}^{n} a_{ij}(x) \varphi_{x_i} \varphi_{x_j} \right)^{p/2} \, dx \right\}
\]

the \((L,p)\)-capacity of the condenser \((\mathcal{D}, \mathcal{G}; \mathbb{R}^n)\). Here the infimum is taken over the all functions \( \varphi \) in the space \( \text{Lip}_{\text{loc}}(\mathbb{R}^n) \) such that \( \varphi = 1 \) on \( \mathcal{D} \), \( \varphi = 0 \) on \( \mathcal{G} \), and \( 1 \geq \varphi \geq 0 \) in \( \mathbb{R}^n \).

In the case when the coefficients \( a_{ij} \) of a differential operator \( L \) coincide with Kronecker’s symbols

\[
\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}
\]

the \((L,p)\)-capacity \( \text{cap}_{L,p} \) associated with the operator \( L \) is denoted by \( \text{cap}_p \). Notice that if \( p = 2 \), then \( \text{cap}_2 \) is the Wiener capacity.

Now, we give an estimate for the \((L,p)\)-capacity of the condenser \((\overline{B}_r \setminus B_R; \mathbb{R}^n)\) via the well-known nonlinear \( p \)-capacity \( \text{cap}_p \) of the same condenser, and with the coefficients \( a_{ij}(x) \) of the operator \( L \) defined on the set \( B_R \setminus B_r \) for any \( R > r > 1 \), where \( B_r = \{ x \in \mathbb{R}^n : |x| < r \} \) and \( \overline{B}_r = \{ x \in \mathbb{R}^n : |x| \leq r \} \).

Proposition 1. Let \( n \geq 2 \), \( p > 1 \), and let \( L \) be a differential operator defined by (1) in \( \mathbb{R}^n \). Then

\[
\text{cap}_{L,p}(\overline{B}_r \setminus B_R; \mathbb{R}^n) \leq \sup_{x \in B_R \setminus B_r} \left( \sum_{i,j=1}^{n} a_{ij}^2(x) \right)^{p/4} \text{cap}_p(\overline{B}_r \setminus B_R; \mathbb{R}^n)
\]

for all \( R > r > 1 \).

It is well known (see, e.g., [2], p.178 or [8], p.12) that for any \( n \geq 2 \) and \( p > 1 \) the inequality

\[
\text{cap}_p(\overline{B}_{R/2} \setminus B_R; \mathbb{R}^n) \leq C(1 + R^2)^{(n-p)/2}
\]

(10)
holds for all $R > 0$ with $C$ being some positive constant which depends only on $n$ and $p$. Now, based on inequalities (9) and (10), we give an estimate for the $(L,p)$-capacity of the condenser $(\bar{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n)$ in the case when the coefficients $a_{ij}$ of a differential operator (1) satisfy

$$\sup_{x \in \mathbb{B}_R \setminus \mathbb{B}_{R/2}} \sum_{i,j=1}^n a_{ij}^2(x) \leq AR^{-2\sigma}$$

(11)

for all sufficiently large $R$, with some constants $A > 0$ and $\sigma$.

**Proposition 2.** Let $n \geq 2$, $p > 1$, and let $L$ be a differential operator defined by (1) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$, with some constants $A > 0$ and $\sigma$. Then

$$\text{cap}_{L,p}(\bar{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n) \leq \hat{C} R^{2n-p(\sigma+2)/2}$$

(12)

for all sufficiently large $R$, where $\sigma$ is the same constant as in condition (11) and $\hat{C}$ is a positive constant which depends only on $A$, $n$, $p$ and $\sigma$.

## 2 Results

For a given differential operator $L$ defined by (1) in $\mathbb{R}^n$ and $q > 0$, our goal in this paper is to establish a Liouville comparison principle for solutions of the inequality (3) defined in $\mathbb{R}^n$ in terms of the capacity, associated with the operator $L$, of condensers of the form $(\bar{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n)$ as $R \to \infty$. We illustrate our results in the particular case when the coefficients of a differential operator $L$ defined by (1) in $\mathbb{R}^n$ satisfy condition (11). More precisely, we obtain the following results.

**Theorem 1.** Let $n \geq 2$, $q > 0$, let $L$ be a differential operator defined by (1) in $\mathbb{R}^n$, let

$$\liminf_{R \to \infty} \text{cap}_{L,q}(\bar{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n) < \infty,$$

(13)

and let $(u,v)$ be an entire solution of the inequality (3) in $\mathbb{R}^n$ such that $u(x) \geq v(x)$ a.e. in $\mathbb{R}^n$. Then $u = v$ a.e. in $\mathbb{R}^n$.

Next, we give a condition on the coefficients of the operator $L$, which guarantees that the inequality (13) holds.

**Proposition 3.** Let $n \geq 2$, and let $L$ be a differential operator defined by (1) in $\mathbb{R}^n$ and such that its coefficients satisfy condition (11) for all sufficiently large $R$, with some constants $A > 0$ and $\sigma \geq n - 2$. Then the inequality (13) holds.

The following example shows the sharpness of Proposition 3.
Example 1. Let \( n \geq 2 \). Consider the differential operator \( L \) defined by the relation
\[
Lu := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[ \frac{1}{(1 + |x|^2)^{\sigma/2}} \frac{\partial u}{\partial x_i} \right]
\] (14)
in \( \mathbb{R}^n \), with some constant \( \sigma \). Using standard arguments in the capacity theory (see, e.g., [2, p.178] or [8, p.12]), it is not difficult to verify that the \( (L, 2) \)-capacity, associated with the operator \( L \), of the condenser \((B_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n)\) satisfies the two-sided inequality
\[
C_2 R^{n-\sigma-2} \geq \text{cap}_{L, 2}(B_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n) \geq C_1 R^{n-\sigma-2}
\] (15)
for all \( R > 1 \), where \( \sigma \) is the same constant as in the relation (14) and \( C_1, C_2 \) some positive constants which depend only on \( n \) and \( \sigma \). In turn, the inequality (15) yields that the \( (L, 2) \)-capacity of the condenser \((B_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n)\) is bounded above by a constant which depends only on \( n \) and \( \sigma \), for all \( R > 1 \) when \( \sigma \geq n - 2 \), and tends to infinity as \( R \to \infty \) when \( \sigma < n - 2 \).

As a simple corollary of Theorem 1 and Proposition 3, we have the following result.

Theorem 2. Let \( n \geq 2 \), \( q > 0 \). Assume that \( L \) is a differential operator defined by (1) in \( \mathbb{R}^n \) such that its coefficients satisfy condition (11) for all sufficiently large \( R \) with some constants \( A > 0 \) and \( \sigma \geq n - 2 \). Let \((u, v)\) be an entire solution of the inequality (3) in \( \mathbb{R}^n \) such that \( u \geq v \) a.e. in \( \mathbb{R}^n \). Then \( u = v \) a.e. in \( \mathbb{R}^n \).

The following example demonstrates the sharpness of Theorem 2 in the case when \( 1 > q > 0 \).

Example 2. Let \( n \geq 2 \) and \( 1 > q > 0 \). Consider the differential operator \( L \) from Example 1 defined by (14) with \( \sigma < n - 2 \). It is easy to verify that the pair \((u, v)\) of the functions
\[
u(x) = \alpha(1 + |x|^2)^{(2+\sigma)/(2(1-q))} + (1 + |x|^2)^{-\mu}
\]
and
\[
v(x) = \alpha(1 + |x|^2)^{(2+\sigma)/(2(1-q))},
\]
where \( \alpha \) is a suitable sufficiently large positive constant and \( 0 < \mu < (n - 2 - \sigma)/2 \), is an entire solution of inequality (4) such that \( u(x) > v(x) \) a.e. in \( \mathbb{R}^n \).

Combining Examples 1 and 2 gives the sharpness of Theorem 1 in the case when \( 1 > q > 0 \).

In what follows, we consider the case when \( q \geq 1 \).

Theorem 3. Let \( n \geq 2 \), \( q > 1 \), \( \nu \in (0, 1) \cap (0, q - 1) \), \( p = 2(q - \nu)/(q - 1) \), let \( L \) be a differential operator defined by (1) in \( \mathbb{R}^n \), let
\[
\liminf_{R \to \infty} \text{cap}_{L, p}(B_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n) < \infty,
\] (16)
and \((u, v)\) be an entire solution of inequality (3) in \( \mathbb{R}^n \) such that \( u \geq v \) a.e. in \( \mathbb{R}^n \). Then \( u = v \) a.e. in \( \mathbb{R}^n \).
Now, we give conditions on the coefficients of the operator \( L \) and on the parameter \( p \) which guarantee that inequality (16) holds.

**Proposition 4.** Let \( n \geq 2 \), and let \( L \) be a differential operator defined by (11) in \( \mathbb{R}^n \) such that its coefficients satisfy condition (11) for all sufficiently large \( R \) with some constants \( A > 0 \) and \( n - 2 > \sigma > -2 \). Then inequality (16) holds for any \( p \geq 2n/(\sigma + 2) \).

**Proposition 5.** Let \( n \geq 2 \), and let \( L \) be a differential operator defined by (11) in \( \mathbb{R}^n \) such that its coefficients satisfy condition (11) for all sufficiently large \( R \) with some constants \( A > 0 \) and \( n - 2 > \sigma > -2 \). Assume also that \( q > 1 \), \( \nu \in (0,1) \cap (0,q-1) \) and \( q \leq (n-\nu(\sigma+2))/(n-\sigma-2) \). Then inequality (16) holds for \( p = 2(q-\nu)/(q-1) \).

The following example shows the sharpness of Propositions 4 and 5.

**Example 3.** Let \( n \geq 2 \) and \( p > 1 \). Consider the differential operator \( L \) defined by (14) in \( \mathbb{R}^n \), with some constant \( \sigma \). As in Example 1, using the standard arguments in capacity theory (see, e.g., [2, p.178] or [8, p.12]), it is not difficult to verify that the \((L,p)\)-capacity, associated with the operator \( L \), of the condenser \((B_{R/2} \setminus B_R; \mathbb{R}^n)\) satisfies the two-sided inequality

\[
C_4 R^{(2n-p(\sigma+2))/2} \geq \text{cap}_{L,p}(B_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n) \geq C_3 R^{(2n-p(\sigma+2))/2}
\]

(17)

for all \( R > 1 \), with the same constant \( \sigma \) as in (14) and some positive constants \( C_3, C_4 \) which depend only on \( n \), \( p \) and \( \sigma \). Moreover, for any \( \sigma \) such that \( n-2 > \sigma > -2 \), inequality (17) yields that if \( p \geq 2n/(\sigma + 2) \), then for all \( R > 1 \), the \((L,p)\)-capacity of the condenser \((B_{R/2} \setminus B_R; \mathbb{R}^n)\) is bounded above by a constant which depends only on \( n \), \( p \) and \( \sigma \), whereas \((B_{R/2} \setminus B_R; \mathbb{R}^n)\) tends to infinity as \( R \to \infty \) if \( p < 2n/(\sigma + 2) \). Thus, indeed, Proposition 4 is sharp.

Moreover, inequality (22), for any \( q > 1 \) and any \( \nu \in (0,1) \cap (0,q-1) \) with \( p = 2(q-\nu)/(q-1) \) as in Theorem 3 and Proposition 5, can be rewritten in the form

\[
C_4 R^{(n-\sigma-2)(q-(n-\nu(\sigma+2)))/(n-\sigma-2)(q-1)} \geq \text{cap}_{L,p}(B_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n) \geq C_3 R^{(n-\sigma-2)(q-(n-\nu(\sigma+2)))/(n-\sigma-2)(q-1)}
\]

(18)

for all \( R > 1 \), with the same constants \( \sigma, C_3 \) and \( C_4 \) as in inequality (17). It is easy to verify using inequality (18) that for all \( R > 1 \) if \( 1 < q \leq (n-\nu(\sigma+2))/(n-\sigma-2) \), then for any \( \sigma \) such that \( n-2 > \sigma > -2 \), the \((L,p)\)-capacity of the condenser \((B_{R/2} \setminus B_R; \mathbb{R}^n)\) is bounded above by a constant which depends only on \( n \), \( p \) and \( \sigma \). Whereas \((B_{R/2} \setminus B_R; \mathbb{R}^n)\) tends to infinity as \( R \to \infty \) if \( q > n/(n-\sigma-2) \). Thus, indeed, Proposition 5 is sharp.

Since the positive parameter \( \nu \) in Proposition 5 may be chosen arbitrarily small, the next result follows directly from Theorem 3 and Proposition 5.

**Theorem 4.** Let \( n \geq 2 \). Assume that \( L \) is a differential operator defined by (11) in \( \mathbb{R}^n \) such that its coefficients satisfy condition (11) for all sufficiently large \( R \) with some constants \( A > 0 \) and \( n - 2 > \sigma > -2 \). Let \( 1 < q < n/(n-\sigma-2) \), and \((u,v)\) be an entire solution of inequality (3) in \( \mathbb{R}^n \) such that \( u \geq v \) a.e. in \( \mathbb{R}^n \). Then \( u = v \) a.e. in \( \mathbb{R}^n \).
**Remark 1.** The case $\sigma \geq n - 2$ is covered by Theorem 2.

The next example demonstrates the sharpness of the hypothesis $\sigma > -2$ in Theorem 4, as well as in Propositions 4–8 and Theorems 6 and 8 given below.

**Example 4.** Let $n \geq 2$ and $q > 1$. Consider the differential operator $L$ from Example 1 defined by (14) with $\sigma \leq -2$. It is easy to verify that the pair $(u, v)$ of the functions

$$u(x) = \alpha(1 + |x|^2)^{-\mu} \quad \text{and} \quad v(x) = 0,$$

where $(n - \sigma - 2)/2 \geq \mu > -(\sigma + 2)/2$ and $\alpha$ is a sufficiently small positive constant which depends only on $\mu$, is an entire solution of inequality (3) such that $u > v$ a.e. in $\mathbb{R}^n$.

To complete our study in the case $q > 1$, consider $1 < q \leq n/(n - \sigma - 2)$ and introduce the quantity

$$C_{L,p_1,p_2}(R) := \frac{1}{2} \frac{1}{\mathbb{R}^n \backslash B_{2R}} \mathbb{R}^n \backslash B_{2R}) \frac{1}{\mathbb{R}^n \backslash B_{R}} \mathbb{R}^n \backslash B_{R}),$$

which includes both the $(L, p_1)$-capacity of the condenser $(\mathbb{B}_R, \mathbb{R}^n \backslash B_{2R})$ and the $(L, p_2)$-capacity of the condenser $(\mathbb{B}_{R/2}, \mathbb{R}^n \backslash B_{R})$ for all $R > 0$, where $p_1 > 1$ and $p_2 > 1$.

**Theorem 5.** Let $n \geq 2$, $q > 1$, $\nu \in (0, 1) \cap (0, q - 1)$, $p_1 = 2(q - \nu)/(q - 1)$, $p_2 = 2q/(q - 1 - \nu)$. Assume that $L$ is a differential operator defined by (11) in $\mathbb{R}^n$. Let

$$\lim_{R \to \infty} \frac{1}{C_{L,p_1,p_2}(R)} < \infty, \quad (19)$$

and $(u, v)$ be an entire solution of inequality (3) in $\mathbb{R}^n$ such that $u > v$ a.e. in $\mathbb{R}^n$. Then $u = v$ in $\mathbb{R}^n$.

Before imposing conditions on the coefficients of the operator $L$ and on the parameters $p_1$ and $p_2$, which would provide inequality (19), we show by the following example that the parameter $q$ must indeed satisfy the condition $1 < q \leq n/(n - \sigma - 2)$.

**Example 5.** Let $n \geq 2$. Consider the differential operator $L$ from Example 1 defined by (11) with $n - 2 > \sigma > -2$, and $q > n/(n - \sigma - 2)$. It is easy to verify that the pair $(u, v)$ of the functions

$$u(x) = \alpha(1 + |x|^2)^{-\mu} \quad \text{and} \quad v(x) = 0,$$

where $(n - 2 - \sigma)/2 \geq \mu > (\sigma + 2)/(2(q - 1))$ and $\alpha$ is a sufficiently small positive number which depends only on $\mu$ is an entire solution of inequality (3) such that $u > v$ a.e. in $\mathbb{R}^n$.

**Proposition 6.** Let $n \geq 2$ and $L$ be a differential operator defined by (11) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$, with some constants $A > 0$, $n - 2 > \sigma > -2$. Furthermore, assume $\nu \in (0, 1) \cap (0, (\sigma + 2)/(n - \sigma - 2))$. Then inequality (19) holds for any $p_1 \geq 2(n - \nu(n - \sigma - 2))/(\sigma + 2)$ and $p_2 \geq 2n/(\sigma + 2 - \nu(n - \sigma - 2))$. 


Furthermore, we have

**Proposition 7.** Let \( n \geq 2 \) and \( L \) be a differential operator defined by \((11)\) in \( \mathbb{R}^n \) such that its coefficients satisfy condition \((11)\) for all sufficiently large \( R \), with some constants \( A > 0, \sigma \). Furthermore, assume that either \( n - 2 > \sigma > -2 \), \( 1 < q \leq n/(n - \sigma - 2) \) and \( \nu \in (0, 1) \cap (0, q - 1) \), or \( \nu \geq n - 2, q > 1 \) and \( \nu \in (0, 1) \cap (0, q - 1) \). Then inequality \((19)\) holds for \( p_1 = 2(q - \nu)/(q - 1) \) and \( p_2 = 2q/(q - 1 - \nu) \).

The following example shows the sharpness of Propositions 6 and 7.

**Example 6.** Let \( n \geq 2 \) and \( p_2 > p_1 > 2 \). Consider the differential operator \( L \) defined by \((14)\) in \( \mathbb{R}^n \) with some constant \( \sigma \). From \((17)\), it follows that the two-sided inequalities

\[
C_5 R^{(2n-p_1(\sigma+2))/4} \geq \left( \text{cap}_{L,p_1}(B_{R/2}; \mathbb{R}^n \setminus B_R; \mathbb{R}^n) \right)^{1/2} \geq C_5 R^{(2n-p_1(\sigma+2))/4} \tag{20}
\]

and

\[
C_8 R^{(2n-p_2(\sigma+2))/(2p_2)} \geq \left( \text{cap}_{L,p_2}(B_{R/2}; \mathbb{R}^n \setminus B_{2R}; \mathbb{R}^n) \right)^{1/p_2} \geq C_7 R^{(2n-p_2(\sigma+2))/(2p_2)}, \tag{21}
\]

hold for all \( R > 1 \), where \( \sigma \) is the same constant as in \((14)\), whereas \( C_5 \), \( C_6 \), \( C_7 \) and \( C_8 \) are some positive constants which depend, possibly, only on \( n \), \( p_1 \), \( p_2 \) and \( \sigma \).

Furthermore, from inequalities \((20)\) and \((21)\) it follows that the two-sided inequality

\[
C_{10} R^{(2n-p_1(\sigma+2))/4} R^{(2n-p_2(\sigma+2))/(2p_2)} \geq \mathcal{C}_{L,p_1,p_2}(R) \geq C_9 R^{(2n-p_1(\sigma+2))/4} R^{(2n-p_2(\sigma+2))/(2p_2)}, \tag{22}
\]

holds for all \( R > 1 \), where \( \sigma \) is the same constant as in \((14)\), whereas \( C_9 \) and \( C_{10} \) are some positive constants which depend only on \( n \), \( p_1 \), \( p_2 \) and \( \sigma \).

From \((22)\), under the assumptions as in Proposition \(8\) that is for any \( \sigma \) such that \( n - 2 > \sigma > -2 \) and any \( \nu \in (0, 1) \cap (0, (\sigma+2))/(n-\sigma-2) \), choosing \( p_1 \) and \( p_2 \) such that \( p_1 \geq 2(n - \nu(n - \sigma - 2))/(\sigma + 2) \) and \( p_2 \geq 2n/(\sigma + 2 - \nu(n - \sigma - 2)) \), we have

\[
C_{11} \geq \mathcal{C}_{L,p_1,p_2}(R), \quad \forall R > 1.
\]

Here \( C_{11} \) is some positive constant that depends only on \( n \), \( q \), \( p_1 \), \( p_2 \), \( \sigma \) and \( \nu \). Thus, indeed, Proposition \(8\) is sharp.

Let us show the sharpness of Proposition \(7\). From \((22)\), under the assumptions as in Theorem 5 and Proposition \(7\), that is for any \( \sigma \), any \( q > 1 \) and any \( \nu \in (0, 1) \cap (0, q - 1) \), letting \( p_1 = 2(q - \nu)/(q - 1) \) and \( p_2 = 2q/(q - 1 - \nu) \), we have the two-sided inequality

\[
C_{13} R^{(2q-1-\nu)(q(n-\sigma-2)-n)/(2q(q-1))} \geq \mathcal{C}_{L,p_1,p_2}(R) \geq C_{12} R^{(2q-1-\nu)(q(n-\sigma-2)-n)/(2q(q-1))} \tag{23}
\]

which holds for all \( R > 1 \), with the same constant \( \sigma \) as in \((14)\) and with some positive constants \( C_{12}, C_{13} \) which depend on \( n \), \( q \), \( \sigma \) and \( \nu \).
Inequality (23) implies that if $1 < q \leq n/(n - \sigma - 2)$, then for all $R > 1$ and for any $\sigma$ such that $n - 2 > \sigma > -2$, $C_{L,p_1,p_2}(R)$ is bounded above by a constant which depends only on $n$, $q$, $\sigma$ and $\nu$. Furthermore, $C_{L,p_1,p_2}(R)$ tends to infinity as $R \to \infty$ if $q > n/(n - \sigma - 2)$.

Moreover, from (23) it follows that for any $\sigma$ such that $\sigma \geq n - 2$ and for all $R > 1$, $C_{L,p_1,p_2}(R)$ is bounded above by a constant which depends only on $n$, $q$, $\sigma$ and $\nu$.

Combining Examples 5 and 6 shows the sharpness of Theorem 5.

The next result follows directly from Theorem 5 and Proposition 7.

**Theorem 6.** Assume that $n \geq 2$ and $L$ is a differential operator defined by (1) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$, with some constants $A > 0$ and $n - 2 > \sigma > -2$. Let $1 < q \leq n/(n - \sigma - 2)$ and $(u,v)$ be an entire solution of inequality (3) in $\mathbb{R}^n$ such that $u \geq v$ a.e. in $\mathbb{R}^n$. Then $u = v$ a.e. in $\mathbb{R}^n$.

Note that the case when $\sigma \geq n - 2$ is covered by Theorem 2.

**Remark 2.** Example 5 shows that the hypothesis in Theorem 6, which requires that $1 < q \leq n/(n - \sigma - 2)$, is sharp. Moreover, combining Example 3 with Examples 4 and 5 gives that the result in Theorem 3 is sharp, except for the case when $q = n/(n - \sigma - 2)$ which is covered by Theorem 5.

Finally, we complete our study by considering the case when $n \geq 2$ and $q = 1$.

**Theorem 7.** Assume that $n \geq 2$, $q = 1$, and $L$ is a differential operator defined by (1) in $\mathbb{R}^n$. Let

$$
\liminf_{R \to \infty} \text{cap}_{L,2}(\mathbb{R} \setminus B_R; \mathbb{R}^n) R^{-n} = 0, \quad (24)
$$

and $(u,v)$ be an entire solution of inequality (3) in $\mathbb{R}^n$ such that $u \geq v$ a.e. in $\mathbb{R}^n$. Then $u = v$ a.e. in $\mathbb{R}^n$.

Now, we give a condition on the coefficients of the operator $L$ which guarantees that equality (24) holds.

**Proposition 8.** Let $n \geq 2$, and $L$ be a differential operator defined by (1) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$ with some constants $A > 0$ and $\sigma > -2$. Then equality (24) holds.

The following result is a simple corollary of Theorem 7 and Proposition 8.

**Theorem 8.** Assume that $n \geq 2$, $q = 1$, and $L$ is a differential operator defined by (1) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$ with some constants $A > 0$ and $\sigma > -2$. Let $(u,v)$ be an entire solution of inequality (3) in $\mathbb{R}^n$ such that $u \geq v$ a.e. in $\mathbb{R}^n$. Then $u = v$ a.e. in $\mathbb{R}^n$. 

9
Combining Example 1 with the following example gives the sharpness of Theorems 7, 8 and Proposition 8.

**Example 7.** Let \( n \geq 2 \) and \( q = 1 \). Consider the differential operator \( L \) from Example 1 defined by (14) with \( \sigma \leq -2 \). It is easy to verify that the pair \((u, v)\) of the functions

\[
u(x) = (1 + |x|^2)^{-\mu} \quad \text{and} \quad v(x) = 0,
\]

where \( \mu \) is a constant such that \( 1/(2n) \leq \mu \leq -(\sigma + 2)/2 \) for \( \sigma \leq -2 - 1/n \) and \( (n - \sigma - 2 - \sqrt{(n - \sigma - 2)^2 - 4})/4 \leq \mu \leq (n - \sigma - 2 + \sqrt{(n - \sigma - 2)^2 - 4})/4 \) for \( \sigma > -2 - 1/n \), is an entire solution of inequality (3) with the operator \( L \) in \( \mathbb{R}^n \) such that \( u(x) > v(x) \) in \( \mathbb{R}^n \).

The results in Theorems 1, 3, 5 and 7 are new; they are also new in the case when entire solutions of inequalities (3), (4)–(5) in \( \mathbb{R}^n \) belong to the function space \( W_{1,q}^{1,0}(\mathbb{R}^n) \). Theorems 2, 4, 6 and 8 were proved in [6]; in the present paper, we show that these theorems can be obtained as simple corollaries of Theorems 1, 3, 5 and 7. The result in Theorem 5 and its proof generalize and correct the result in Theorem 1 and its proof obtained in [5]. We would like also to note that the results obtained in this paper were motivated by results established in [1] and [4].

### 3 Proofs

**Proof of Theorem** 1 Let \( n \geq 2 \), \( q > 0 \), and \( L \) be a differential operator defined by (1), and let \((u, v)\) be an entire solution of inequality (3) in \( \mathbb{R}^n \) such that \( u(x) > v(x) \) a.e. in \( \mathbb{R}^n \). Then, from (7) we have the inequality

\[
\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij} (u - v)_{x_i} \zeta_{x_j} dx \geq \int_{\mathbb{R}^n} (|u|^{q-1}u - |v|^{q-1}v)\zeta dx, \tag{25}
\]

which holds for every non-negative function \( \zeta \in C^\infty(\mathbb{R}^n) \) with compact support.

Set \( w(x) = u(x) - v(x) \) and let \( R \) and \( \varepsilon \) be positive numbers, and \( \varphi \) be a function such that \( \varphi \in \text{Lip}_{\text{loc}}(\mathbb{R}^n) \), \( \varphi = 1 \) on \( B_{R/2} \), \( \varphi = 0 \) outside \( B_R \), and \( 1 \geq \varphi \geq 0 \) in \( \mathbb{R}^n \). Without loss of generality, we may substitute the test function \( \zeta(x) = (w(x) + \varepsilon)^{-1}\varphi^2(x) \) in (25). Then integrating by parts, we obtain the inequality

\[
2 \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^n a_{ij} w_{x_i} \varphi_{x_j} (w + \varepsilon)^{-1} \varphi dx \geq \int_{B_R} \sum_{i,j=1}^n a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2} \varphi^2 dx + \int_{B_R} (|u|^{q-1}u - |v|^{q-1}v)(w + \varepsilon)^{-1} \varphi^2 dx. \tag{26}
\]
Estimating the integral on the left side of (26) by Hölder’s inequality we obtain

\[
2 \left( \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_x w_x (w + \varepsilon)^{-2} \varphi^2 dx \right)^{1/2} \times \\
\left( \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} \phi_x \phi_x dx \right)^{1/2} \geq \int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_x w_x (w + \varepsilon)^{-2} \varphi^2 dx + \\
\int_{B_R} (|u|^{q-1}u - |v|^{q-1}v)(w + \varepsilon)^{-1} \varphi^2 dx. 
\] (27)

Since both terms on the right side of (27) are non-negative, we have the inequalities

\[
2 \left( \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_x w_x (w + \varepsilon)^{-2} \varphi^2 dx \right)^{1/2} \times \\
\left( \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} \phi_x \phi_x dx \right)^{1/2} \geq \int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_x w_x (w + \varepsilon)^{-2} \varphi^2 dx, 
\]

and

\[
4 \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} \phi_x \phi_x dx \geq \int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_x w_x (w + \varepsilon)^{-2} \varphi^2 dx, 
\]

which then yield the inequalities

\[
2 \left( \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_x w_x (w + \varepsilon)^{-2} dx \right)^{1/2} \times \\
\left( \int \sum_{i,j=1}^{n} a_{ij} \phi_x \phi_x dx \right)^{1/2} \geq \int_{B_{R/2}} (|u|^{q-1}u - |v|^{q-1}v)(w + \varepsilon)^{-2} dx 
\] (28)

and

\[
4 \int \sum_{i,j=1}^{n} a_{ij} \phi_x \phi_x dx \geq \int \sum_{i,j=1}^{n} a_{ij} w_x w_x (w + \varepsilon)^{-2} dx. 
\] (29)
Minimizing the left sides of (28) and (29) over all functions \( \varphi(x) \) admissible in the definition of the \((L, 2)\)-capacity of the condenser \((\overline{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n)\), we obtain the inequalities

\[
2 \left( \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2} dx \right)^{1/2} \times (\text{cap}_{L,2}(\overline{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n))^{1/2} \geq \int_{B_{R/2}} (|u|^{q-1} u - |v|^{q-1} v)(w + \varepsilon)^{-2} dx \tag{30}
\]

and

\[
4 \text{cap}_{L,2}(\overline{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n) \geq \int_{B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2} dx. \tag{31}
\]

Moreover, since by one of the hypotheses of Theorem 1 there exists a non-negative number \( \Gamma \) and an increasing sequence of positive numbers \( R_k \) such that \( R_k \to \infty \) and

\[
\text{cap}_{L,2}(\overline{B}_{R_k/2}, \mathbb{R}^n \setminus B_{R_k}; \mathbb{R}^n) \to \Gamma \tag{32}
\]

as \( R_k \to \infty \), then, from (31) and (32), we have the inequality

\[
\int_{B_{R_k/2}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2} dx \leq 4 \Gamma, \tag{33}
\]

which holds as \( R_k \to \infty \). Due to condition (2), the quantity

\[
H(R) := \int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2} dx
\]

increases monotonically with respect to \( R \). Hence by inequality (33), which holds as \( R_k \to \infty \), we derive the inequality

\[
\int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2} dx \leq 4 \Gamma,
\]

which holds for all \( R > 0 \). Due to the monotonicity of \( H(R) \) with respect to \( R \), \( H(R) \) has a limit as \( R \to \infty \), bounded by the constant \( 4 \Gamma \), namely,

\[
\lim_{R \to \infty} \int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2} dx \leq 4 \Gamma. \tag{34}
\]
From (34), again due to the monotonicity of $H(R)$ with respect to $R$, we obtain the equality

$$\lim_{R \to \infty} \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^n a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2} = 0,$$

and, in particular,

$$\lim_{R_k \to \infty} \int_{B_{R_k} \setminus B_{R_k/2}} \sum_{i,j=1}^n a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2} = 0, \quad (35)$$

where $R_k$ is the same sequence as in (32).

Since $|u|^{q-1} u \geq |v|^{q-1} v$ in $\mathbb{R}^n$, we derive that the quantity

$$I(R) := \int_{B_{R/2}} (|u|^{q-1} u - |v|^{q-1} v)(w + \varepsilon)^{-2} dx,$$

which is the right side of (30), increases monotonically with respect to $R$ and, therefore, has a limit as $R \to \infty$, which, generally speaking, can be equal to infinity.

Now, taking in (30) the same sequence $R = R_k$ as in (32) and then passing to the limit as $R_k \to \infty$, we obtain, due to (32) and (35), the equality

$$\lim_{R_k \to \infty} \int_{B_{R_k/2}} (|u|^{q-1} u - |v|^{q-1} v)(w + \varepsilon)^{-2} dx = 0. \quad (36)$$

Finally, from (36), since $|u|^{q-1} u \geq |v|^{q-1} v$ in $\mathbb{R}^n$, we deduce that $u(x) = v(x)$ almost everywhere in $\mathbb{R}^n$, and this concludes the proof of Theorem 1.

**Proof of Theorem 3** Let $n \geq 2$, $q > 1$, let $L$ be a differential operator defined by (11), and let $(u, v)$ be an entire solution of the inequality (3) in $\mathbb{R}^n$ such that $u(x) \geq v(x)$. Using the algebraic inequality

$$(|u|^{q-1} u - |v|^{q-1} v)(u - v) \geq c_1 |u - v|^{q+1}, \quad (37)$$

which, for every $q \geq 1$ and a some positive constant $c_1$ depending only on $q$, holds for all real numbers $u$ and $v$, we obtain from (7) the inequality

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij} (u - v)_{x_i} \zeta_{x_j} dx \geq c_1 \int_{\mathbb{R}^n} (u - v)^q \zeta dx, \quad (38)$$

which holds for every non-negative function $\zeta \in C^\infty(\mathbb{R}^n)$ with compact support.
Set \( w(x) = u(x) - v(x) \), and let \( R \) and \( \varepsilon \) be positive numbers, \( \nu \) is a positive number such that \( \nu \in (0, 1) \cap (0, q - 1) \). Let \( \varphi \) be a function such that \( \varphi \in \text{Lip}_{\text{loc}}(\mathbb{R}^n) \), \( \varphi = 1 \) on \( B_{R/2} \), \( \varphi = 0 \) outside \( B_R \), and \( 1 \geq \varphi \geq 0 \) in \( \mathbb{R}^n \). Without loss of generality, we may substitute the test function \( \zeta(x) = (w(x) + \varepsilon)^{-\nu} \varphi^s(x) \) in (38). Hence, we get the inequality

\[
\nu \int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} \varphi^s \, dx \geq \int_{B_R} w^q (w + \varepsilon)^{-\nu} \varphi^s \, dx. \tag{39}
\]

Estimating the integrand on the left side of (39) by Cauchy’s inequality, we obtain the inequality

\[
\nu \int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} \varphi^s \, dx \geq \int_{B_R} w^q (w + \varepsilon)^{-\nu} \varphi^s \, dx. \tag{40}
\]

Further estimating the integrand on the left side of (40) by Young’s inequality, we arrive at the inequality

\[
\frac{\nu}{2} \int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} \varphi^s \, dx +

\int_{B_R} \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} (w + \varepsilon)^{1-\nu} \varphi^s \, dx \geq

\nu \int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} \varphi^s \, dx \geq \int_{B_R} w^q (w + \varepsilon)^{-\nu} \varphi^s \, dx. \tag{41}
\]

Here and in what follows in the proof of Theorem 3 we use the symbols \( c_i, i = 2, \ldots, \) to denote positive constants depending possibly on \( n, q \) or \( \nu \) but not on \( R \) or \( \varepsilon \).

Due to condition (2), from (41) we obtain the inequality

\[
c_3 \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} (w + \varepsilon)^{1-\nu} \varphi^s \, dx \geq \int_{B_R} w^q (w + \varepsilon)^{-\nu} \varphi^s \, dx. \tag{42}
\]
Estimating the left side of (42) by H"older’s inequality, we arrive at the inequality
\[
c_3 \left( \int_{B_R \setminus B_{R/2}} \left( \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} \right)^{(q-\nu)/(q-1)} \varphi^{s-2(q-\nu)/(q-1)} dx \right)^{(q-1)/(q-\nu)} \times \left( \int_{B_R \setminus B_{R/2}} (w + \varepsilon)^{q-\nu} \varphi^s dx \right)^{(1-\nu)/(q-\nu)} \geq \int_{B_R} w^q (w + \varepsilon)^{-\nu} \varphi^s dx. \tag{43}\]

Observe that \( s = 2(q - \nu)/(q - 1) \). Hence setting \( p = 2(q - \nu)/(q - 1) \), we obtain the inequality
\[
c_3 \left( \int_{B_R \setminus B_{R/2}} \left( \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} \right)^{p/2} dx \right)^{2/p} \times \left( \int_{B_R \setminus B_{R/2}} (w + \varepsilon)^{p(q-1)/2} \varphi^s dx \right)^{(p-2)/p} \geq \int_{B_R} w^{p(q-1)/2 + \nu} (w + \varepsilon)^{-\nu} \varphi^s dx. \tag{44}\]

Passing to the limit in (44) as \( \varepsilon \to 0 \), we obtain by Lebesgue’s theorem (see, e.g., [3], p.303) the inequality
\[
c_3 \left( \int_{B_R \setminus B_{R/2}} \left( \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} \right)^{p/2} dx \right)^{2/p} \times \left( \int_{B_R \setminus B_{R/2}} w^{p(q-1)/2} \varphi^s dx \right)^{(p-2)/p} \geq \int_{B_{R/2}} w^{p(q-1)/2} \varphi^s dx,
\]
which yields
\[
c_3 \left( \int_{B_R} \left( \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} \right)^{p/2} dx \right)^{2/p} \times \left( \int_{B_R \setminus B_{R/2}} w^{p(q-1)/2} dx \right)^{(p-2)/p} \geq \int_{B_{R/2}} w^{p(q-1)/2} dx,
\]

15
\( c_4 \int_{B_R} \left( \sum_{i,j=1}^{n} a_{ij} \varphi_x \varphi_{x_j} \right)^{p/2} \, dx \geq \int_{B_{R/2}} w^{p(q-1)/2} dx. \)

Minimizing the left sides of these inequalities over all functions \( \varphi(x) \) admissible in the definition of the \( (L,p) \)-capacity of the condenser \((B_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n)\), we obtain the inequalities

\[
c_3 \left( \left( \text{cap}_{L,p}(B_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n) \right)^{p/2} \right)^{2/p} \times \left( \int_{B_R \setminus B_{R/2}} w^{p(q-1)/2} \, dx \right)^{(p-2)/p} \geq \int_{B_{R/2}} w^{p(q-1)/2} \, dx \tag{45}\]

and

\[
c_4 \text{cap}_{L,p}(B_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n) \geq \int_{B_{R/2}} w^{p(q-1)/2} \, dx. \tag{46}\]

Since by the hypotheses of Theorem 3 there exists a non-negative number \( \Lambda \) and an increasing sequence of positive numbers \( R_k \) such that \( R_k \to \infty \) and

\[
\text{cap}_{L,p}(B_{R_k/2}, \mathbb{R}^n \setminus B_{R_k}; \mathbb{R}^n) \to \Lambda \tag{47}\]

as \( R_k \to \infty \), we obtain from \((45)\) the inequality

\[
\int_{B_{R_k/2}} w^{p(q-1)/2} \, dx \leq c_4 \Lambda, \tag{48}\]

which holds as \( R_k \to \infty \). Moreover, the quantity

\[
J(R) := \int_{B_R} w^{p(q-1)/2} \, dx
\]

increases monotonically with respect to \( R \). Hence, from inequality \((48)\), which holds as \( R_k \to \infty \), we arrive at the inequality

\[
\int_{B_R} w^{p(q-1)/2} \, dx \leq c_4 \Lambda,
\]

which holds for all \( R > 0 \) and which, due to the monotonicity of \( J(R) \) with respect to \( R \), yields that \( J(R) \) has a limit, as \( R \to \infty \), bounded by the constant \( c_4 \Lambda \), namely,

\[
\lim_{R \to \infty} \int_{B_R} w^{p(q-1)/2} \, dx \leq c_4 \Lambda. \tag{49}\]
Hence, due to the monotonicity of $J(R)$ with respect to $R$, we have the equality
\[ \lim_{R \to \infty} \int_{B_R \setminus B_{R/2}} w^{p(q-1)/2} dx = 0, \]
and, in particular, the equality
\[ \lim_{R_k \to \infty} \int_{B_{R_k} \setminus B_{R_k/2}} w^{p(q-1)/2} dx = 0, \]
where $R_k$ is the same sequence as in (47).

Moreover, observe that the right side of inequality (45), which is equal to $J(R/2)$, increases monotonically with respect to $R$ and, due to (49), has a limit, as $R \to \infty$, bounded from above by $c_4 \Lambda$. Let $R = R_k$ be the same sequence as in (47). Then passing to the limit as $R_k \to \infty$, due to (47) and (50), we obtain
\[ \lim_{R_k \to \infty} \int_{B_{R_k/2}} w^{p(q-1)/2} dx = 0. \]
This implies that $w(x) = 0$ almost everywhere in $\mathbb{R}^n$, and thus $u(x) = v(x)$ almost everywhere in $\mathbb{R}^n$. This concludes the proof of Theorem 3.

**Proof of Theorem 5** Let $n \geq 2$, $q > 1$, $L$ be a differential operator defined by (1), and $(u, v)$ be an entire solution of the inequality (3) in $\mathbb{R}^n$ such that $u(x) \geq v(x)$. Then, as we show in the proof of Theorem 3, $(u, v)$ satisfies the inequality (38). Let $w(x) = u(x) - v(x)$, and $R$ be a positive number, $\varphi$ be a function such that $\varphi \in \text{Lip}_{\text{loc}}(\mathbb{R}^n)$, $\varphi = 1$ on $B_{R/2}$, $\varphi = 0$ outside $B_R$, and $1 \geq \varphi \geq 0$ in $\mathbb{R}^n$. Substituting the test function $\zeta(x) = \varphi^2(x)$ in (38) we obtain the inequality
\[ 2 \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij} w_{x_i} \varphi_{x_j} \varphi dx \geq c_1 \int_{\mathbb{R}^n} w^q \varphi^2 dx, \]
where $c_1$ is the same constant as in (37).

Let $\varepsilon$ be a positive number and $\nu$ be a positive number such that $\nu \in (0, 1) \cap (q - 1)$. Estimating the left side of (51) by Hölder’s inequality, we obtain the inequality
\[
c_2 \left( \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-1-\nu} dx \right)^{1/2} \times \left( \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} (w + \varepsilon)^{1+\nu} \varphi^2 dx \right)^{1/2} \geq \int_{B_R} w^q \varphi^2 dx. \tag{51}
\]
Here and in what follows in the proof of Theorem 5, we use the symbols $c_i, i = 2, \ldots$, to denote positive constants depending possibly on $n, q$ or $\nu$ but not on $R$ or $\varepsilon$.

Estimating the second term on the left side of (51) by Hölder’s inequality, we arrive at the inequality

$$c_2 \left( \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-1-\nu} dx \right)^{1/2} \times 
left( \int_{B_R \setminus B_{R/2}} \left( \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} \right)^{q/(q-1-\nu)} \varphi^2 dx \right)^{(q-1-\nu)/(2q)} \times 
left( \int_{B_R \setminus B_{R/2}} (w + \varepsilon)^{q} \varphi^2 dx \right)^{(1+\nu)/(2q)} \geq \int_{B_R} w^q \varphi^2 dx,$$

which yields the inequality

$$c_2 \left( \int_{B_R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-1-\nu} dx \right)^{1/2} \times 
left( \int_{B_R} \left( \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} \right)^{q/(q-1-\nu)} dx \right)^{(q-1-\nu)/(2q)} \times 
left( \int_{B_R \setminus B_{R/2}} (w + \varepsilon)^{q} \varphi^2 dx \right)^{(1+\nu)/(2q)} \geq \int_{B_R} w^q \varphi^2 dx. \quad (52)$$

Now, we estimate the first cofactor on the left side of (52). To this end, as above, we use the inequality (38). Let $\psi$ be a function such that $\psi \in \text{Lip}_{\text{loc}}(\mathbb{R}^n)$, $\psi = 1$ on $\overline{B}_R$, $\psi = 0$ outside $B_{2R}$, and $1 \geq \psi \geq 0$ in $\mathbb{R}^n$. Without loss of generality, we may substitute the test function $\zeta(x) = (w(x) + \varepsilon)^{-\nu} \psi^s(x)$ in (38), where $s = 2(q - \nu)/(q - 1)$. Then integrating by parts, we get the inequality

$$s \int_{B_{2R}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} \psi_{x_j} (w + \varepsilon)^{-\nu} \psi^{s-1} dx \geq 
\nu \int_{B_{2R}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} \psi^s dx + c_1 \int_{B_{2R}} w^q (w + \varepsilon)^{-\nu} \psi^s dx. \quad (53)$$
Estimating the integrand on the left side of (53) by Cauchy’s inequality we obtain the inequality

\[
\int_{B_2 R} s \left( \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} \right)^{1/2} \left( \sum_{i,j=1}^{n} a_{ij} \psi_{x_i} \psi_{x_j} \right)^{1/2} (w + \varepsilon)^{-\nu} \psi^{s-1} dx \geq \nu \int_{B_2 R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} \psi^s dx + c_1 \int_{B_2 R} w^q (w + \varepsilon)^{-\nu} \psi^s dx. \quad (54)
\]

Next, estimating the integrand on the left side of (54) by Young’s inequality, we arrive at the inequality

\[
\frac{\nu}{2} \int_{B_2 R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} \psi^s dx + c_3 \int_{B_2 R} \sum_{i,j=1}^{n} a_{ij} \psi_{x_i} \psi_{x_j} (w + \varepsilon)^{1-\nu} \psi^{s-2} dx \geq \nu \int_{B_2 R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} \psi^s dx + c_1 \int_{B_2 R} w^q (w + \varepsilon)^{-\nu} \psi^s dx,
\]

which yields the inequality

\[
c_3 \int_{B_2 R} \sum_{i,j=1}^{n} a_{ij} \psi_{x_i} \psi_{x_j} (w + \varepsilon)^{1-\nu} \psi^{s-2} dx \geq \frac{\nu}{2} \int_{B_2 R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} \psi^s dx + c_1 \int_{B_2 R} w^q (w + \varepsilon)^{-\nu} \psi^s dx. \quad (55)
\]

Estimating the left side of (55) by Young’s inequality we obtain the inequality

\[
c_1 \int_{B_2 R} (w + \varepsilon)^{-\nu} \psi^s dx + c_4 \int_{B_2 R} \left( \sum_{i,j=1}^{n} a_{ij} \psi_{x_i} \psi_{x_j} \right)^{(q-\nu)/(q-1)} \psi^{s-2(q-\nu)/(q-1)} dx \geq \frac{\nu}{2} \int_{B_2 R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} \psi^s dx + c_1 \int_{B_2 R} w^q (w + \varepsilon)^{-\nu} \psi^s dx,
\]

which, by \( s = 2(q - \nu)/(q - 1) \), yields the inequality

\[
c_1 \int_{B_2 R} (w + \varepsilon)^{-\nu} \psi^s dx - c_1 \int_{B_2 R} w^q (w + \varepsilon)^{-\nu} \psi^s dx + c_4 \int_{B_2 R} \left( \sum_{i,j=1}^{n} a_{ij} \psi_{x_i} \psi_{x_j} \right)^{(q-\nu)/(q-1)} dx \geq \frac{\nu}{2} \int_{B_2 R} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-\nu-1} dx. \quad (56)
\]
Now, we return to the inequality \((52)\) and estimate the first cofactor on the left side of this inequality by \((56)\). As a result, we have the inequality

\[
c_2 \left( 2c_1 \nu^{-1} \int_{B_{2R}} (w + \varepsilon)^{q-\nu} \psi^s dx - 2c_1 \nu^{-1} \int_{B_{2R}} w^q (w + \varepsilon)^{-\nu} \psi^s dx + \right.
\]

\[
2c_4 \nu^{-1} \int_{B_{2R}} \left( \sum_{i,j=1}^n a_{ij} \psi_{x_i} \psi_{x_j} \right)^{(q-\nu)/(q-1)} dx \times
\]

\[
\left( \int_{B_R} \left( \sum_{i,j=1}^n a_{ij} \phi_{x_i} \phi_{x_j} \right)^{(q/(q-1)-\nu)/(2q)} dx \right)^{(1+\nu)/(2q)} \times \left( \int_{B_R \setminus B_{R/2}} w^q \varphi^2 dx \right)^{(1+\nu)/(2q)} \geq \int_{B_R} w^q \varphi^2 dx. \tag{57}
\]

Passing to the limit in \((57)\) as \(\varepsilon \to 0\), we obtain by Lebesgue’s theorem (see, e.g., [3], p.303) the inequality

\[
c_5 \left( \int_{B_{2R}} \left( \sum_{i,j=1}^n a_{ij} \psi_{x_i} \psi_{x_j} \right)^{(q-\nu)/(q-1)} dx \right)^{1/2} \times
\]

\[
\left( \int_{B_R} \left( \sum_{i,j=1}^n a_{ij} \phi_{x_i} \phi_{x_j} \right)^{q/(q-1)-\nu} dx \right)^{(q-\nu)/(2q)} \times \left( \int_{B_R \setminus B_{R/2}} w^q \varphi^2 dx \right)^{(1+\nu)/(2q)} \geq \int_{B_R} w^q \varphi^2 dx,
\]

which yields the inequalities

\[
c_5 \left( \int_{B_{2R}} \left( \sum_{i,j=1}^n a_{ij} \psi_{x_i} \psi_{x_j} \right)^{(q-\nu)/(q-1)} dx \right)^{1/2} \times
\]

\[
\left( \int_{B_R} \left( \sum_{i,j=1}^n a_{ij} \phi_{x_i} \phi_{x_j} \right)^{q/(q-1)-\nu} dx \right)^{(q-\nu)/(2q)} \times \left( \int_{B_R \setminus B_{R/2}} w^q dx \right)^{(1+\nu)/(2q)} \geq \int_{B_{R/2}} w^q dx. \tag{58}
\]
and

\[
c_5 \left( \frac{\int_{\mathcal{B}_R} \left( \sum_{i,j=1}^n a_{ij} \psi_{x_i} \psi_{x_j} \right)^{(q-\nu)/(q-1)} \, dx}{\int_{\mathcal{B}_R} \left( \sum_{i,j=1}^n a_{ij} \varphi_{x_i} \varphi_{x_j} \right)^{q/(q-1-\nu)} \, dx} \right)^{(q-1-\nu)/(2q)} \left( \frac{\int_{\mathcal{B}_{R/2}} w^q \, dx}{\int_{\mathcal{B}_R \setminus \mathcal{B}_{R/2}} \varphi_{x_i} \varphi_{x_j} \, dx} \right)^{2/(2q-1)} \geq \int_{\mathcal{B}_{R/2}} w^q \, dx,
\]

(59)

Taking \( p_1 = 2(q-\nu)/(q-1) \), \( p_2 = 2q/(q-1-\nu) \) in (58) and (59), and minimizing the left sides of these inequalities first over all functions \( \psi(x) \) admissible in the definition of the \((L, p_1)\)-capacity of the condenser \((\mathcal{B}_R, \mathbb{R}^n \setminus \mathcal{B}_2R; \mathbb{R}^n)\) and then over all functions \( \varphi(x) \) admissible in the definition of the \((L, p_2)\)-capacity of the condenser \((\overline{\mathcal{B}}_{R/2}, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n)\), we arrive at the inequalities

\[
c_5 \left( \text{cap}_{L,p_1}(\overline{\mathcal{B}}_R, \mathbb{R}^n \setminus \mathcal{B}_2R; \mathbb{R}^n) \right)^{1/2} \left( \text{cap}_{L,p_2}(\overline{\mathcal{B}}_{R/2}, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n) \right)^{1/p_2} \times \left( \int_{\mathcal{B}_R \setminus \mathcal{B}_{R/2}} w^q \, dx \right)^{(1+\nu)/(2q)} \geq \int_{\mathcal{B}_{R/2}} w^q \, dx,
\]

(60)

and thus we get

\[
c_5 \left( \text{cap}_{L,p_1}(\overline{\mathcal{B}}_R, \mathbb{R}^n \setminus \mathcal{B}_2R; \mathbb{R}^n) \right)^{1/2} \times \left( \text{cap}_{L,p_2}(\overline{\mathcal{B}}_{R/2}, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n) \right)^{1/p_2} \geq \left( \int_{\mathcal{B}_{R/2}} w^q \, dx \right)^{(2q-1-\nu)/(2q)}.
\]

(61)

Since by one of the hypotheses of Theorem 5 there exists a non-negative number \( \Upsilon \) and an increasing sequence of positive numbers \( R_k \) such that \( R_k \to \infty \) and

\[
\mathcal{C}_{L,p_1,p_2}(R_k) \to \Upsilon
\]

(62)
as \( R_k \to \infty \), we have from (61) the inequality

\[
\int_{\mathcal{B}_{R_k/2}} w^q \, dx \leq c_6 \Upsilon^{2q/(2q-1-\nu)},
\]

(63)

which holds as \( R_k \to \infty \). Observe that the quantity

\[
Q(R) := \int_{\mathcal{B}_R} w^q \, dx
\]
increases monotonically with respect to $R$. Hence, from inequality (63), which holds as $R_k \to \infty$, we derive the inequality
\[
\int_{B_R} w^q \, dx \leq c_6 \Upsilon^{2q/(2q-1-\nu)},
\]
which holds for all $R > 0$. Due to the monotonicity of $Q(R)$ with respect to $R$, this yields that $Q(R)$ has a limit, as $R \to \infty$, bounded from above by the constant $c_6 \Upsilon^{2q/(2q-1-\nu)}$, namely,
\[
\lim_{R \to \infty} \int_{B_R} w^q \, dx \leq c_6 \Upsilon^{2q/(2q-1-\nu)}. \tag{64}
\]
From (64), again due to the monotonicity of $Q(R)$ with respect to $R$, we have the equality
\[
\lim_{R \to \infty} \int_{B_R \setminus B_{R/2}} w^q \, dx = 0, \tag{65}
\]
and thus we have the equality
\[
\lim_{R_k \to \infty} \int_{B_{R_k} \setminus B_{R_k/2}} w^q \, dx = 0,
\]
where $R_k$ is the same sequence as in (62).

Observe that the right side of (60), which is equal to $Q(R/2)$, increases monotonically with respect to $R$. Thus, due to (64), it has a limit, as $R \to \infty$, bounded from above by $c_6 \Upsilon^{2q/(2q-1-\nu)}$. Hence, taking the same sequence $R = R_k$ as in (62) and passing to the limit as $R_k \to \infty$, we obtain, due to (62) and (95), the equality
\[
\lim_{R_k \to \infty} \int_{B_{R_k} \setminus B_{R_k/2}} w^q \, dx = 0. \tag{66}
\]
Finally, from (66), we deduce that $w(x) = 0$ almost everywhere in $\mathbb{R}^n$, which implies that $u(x) = v(x)$ almost everywhere in $\mathbb{R}^n$, and this concludes the proof of Theorem 5.

**Proof of Theorem 7.** We prove this theorem by contradiction. Let $n \geq 2$, $q = 1$, $L$ be a differential operator defined by (1), and $(u, v)$ be an entire solution of inequality (3) in $\mathbb{R}^n$ such that $u(x) > v(x)$. Then, as it has been shown in Theorem 3, $(u, v)$ satisfies inequality (38) with $q = 1$ and $c_1 = 1$. Further, let $R$ and $\varepsilon$ be positive numbers, and $\varphi$ be a function such that $\varphi \in \text{Lip}_{loc}(\mathbb{R}^n)$, $\varphi = 1$ on $\overline{B}_{R/2}$, $\varphi = 0$ outside $B_R$, and $1 \geq \varphi \geq 0$ in $\mathbb{R}^n$. 

Without loss of generality we may substitute the function \( \zeta(x) = (w(x) + \varepsilon)^{-1}\varphi^2(x) \) in (38), where \( w(x) = u(x) - v(x) \). Then integrating by parts, we have the inequality

\[
2 \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} \varphi_{x_j} (w + \varepsilon)^{-1}\varphi dx \geq 
\int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2}\varphi^2 dx + \int_{B_R} w(w + \varepsilon)^{-1}\varphi^2 dx. \tag{67}
\]

Estimating the integrand on the left side of (67) by Cauchy’s inequality, we obtain the inequality

\[
\int_{B_R \setminus B_{R/2}} 2 \left( \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} \right)^{1/2} \left( \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} \right)^{1/2} (w + \varepsilon)^{-1}\varphi dx \geq 
\int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2}\varphi^2 dx + \int_{B_R} w(w + \varepsilon)^{-1}\varphi^2 dx. \tag{68}
\]

Now estimating the integrand on the left side of (68) by Young’s inequality, we arrive at the inequality

\[
\frac{1}{2} \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2}\varphi^2 dx + c_1 \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} dx \geq 
\int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2}\varphi^2 dx + \int_{B_R} w(w + \varepsilon)^{-1}\varphi^2 dx. \tag{69}
\]

Here and in what follows in the proof of Theorem 7, we use the symbols \( c_i, i = 1, \ldots, \) to denote positive constants depending possibly on \( n \) but not on \( R \) or \( \varepsilon \).

From (69) due to (2) we have the inequality

\[
c_2 \int_{B_R \setminus B_{R/2}} \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} dx \geq \int_{B_R} w(w + \varepsilon)^{-1}\varphi^2 dx,
\]

which yields the inequality

\[
c_2 \int_{B_R} \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} dx \geq \int_{B_R/2} w(w + \varepsilon)^{-1} dx. \tag{70}
\]

Passing to the limit in (70) as \( \varepsilon \to 0 \) by Lebesgue’s theorem (see, e.g., [3], p.303) we derive

\[
c_2 \int_{B_R} \sum_{i,j=1}^{n} a_{ij} \varphi_{x_i} \varphi_{x_j} dx \geq \int_{B_{R/2}} dx. \tag{71}
\]
Minimizing the left side of (71) over all functions $\varphi(x)$ admissible in the definition of the $(L, 2)$-capacity of the condenser $(\mathcal{B}_{R/2}, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n)$, we obtain the inequality

$$c_3 \text{cap}_{L, 2}(\mathcal{B}_{R/2}, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n) \geq R^n. \tag{72}$$

However, by a hypothesis of Theorem 7 there exists an increasing sequence of positive numbers $R_k \to \infty$ such that the equality

$$\lim_{R_k \to \infty} \text{cap}_{L, 2}(\mathcal{B}_{R_k/2}, \mathbb{R}^n \setminus \mathcal{B}_{R_k}; \mathbb{R}^n) R_k^{-n} = 0 \tag{73}$$

holds. This implies the desired contradiction, i.e. inequality (72) and equality (73) contradict each other.

**Proof of Proposition 1.** Let $n \geq 2$, $p > 1$, and $L$ be a differential operator defined by (1) in $\mathbb{R}^n$. Using the algebraic inequality

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \left( \sum_{i,j=1}^n a_{ij}^2(x) \right)^{\frac{1}{2}} \left( \sum_{i=1}^n \xi_i^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^n \xi_j^2 \right)^{\frac{1}{2}},$$

which holds for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and almost all $x \in \mathbb{R}^n$, we obtain the inequality

$$\int_{\mathbb{R}^n} \left( \sum_{i,j=1}^n a_{ij}(x)\varphi_{x_i}\varphi_{x_j} \right)^{\frac{p}{2}} dx \leq \sup_{x \in \mathcal{B}_R \setminus \mathcal{B}_r} \left( \sum_{i,j=1}^n a_{ij}^2(x) \right)^{\frac{p}{4}} \int_{\mathbb{R}^n} |\nabla \varphi|^p dx, \tag{74}$$

which holds for all functions $\varphi(x)$ admissible in the definition of the $(L, p)$-capacity of the condenser $(\mathcal{B}_r, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n)$, with $R > r > 1$.

Minimizing first the left side and then the right side of (74) over all functions $\varphi(x)$ admissible in the definition of the $(L, p)$-capacity of the condenser $(\mathcal{B}_r, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n)$, we obtain inequality (9) for any $R > r > 1$.

**Proof of Proposition 2.** Let $n \geq 2$, $p > 1$, and let $L$ be a differential operator defined by the relation (1) in $\mathbb{R}^n$ and such that its coefficients satisfy condition (11) for all sufficiently large $R$, with $A > 0$ and $\sigma$ some constants. Then, from (9) and (11), we obtain the inequality

$$\text{cap}_{L, p}(\mathcal{B}_{R/2}, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n) \leq A^{n/4} R^{-\sigma p/2} \text{cap}_{p}(\mathcal{B}_{R/2}, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n), \tag{75}$$

which holds for all sufficiently large $R$, with $A > 0$ and $\sigma$ the same constants as in (11). In turn, from (75) and (10), we derive the inequality (12), which holds for all sufficiently large $R$, with $\sigma$ the same constant as in (11) and $\hat{C}$ some positive constant which depends only on $A$, $n$, $p$ and $\sigma$, and this concludes the proof of Proposition 2.

**Proof of Proposition 3.** Let $n \geq 2$, and $L$ be a differential operator defined by (1) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$, with some constants $A > 0$ and $\sigma \geq n - 2$. Then, in (12), letting $p = 2$, we derive the inequality

$$\text{cap}_{L, 2}(\mathcal{B}_{R/2}, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n) \leq \hat{C} R^{n-\sigma-2}, \tag{76}$$

which
which holds for all sufficiently large $R$, with some positive constant $\hat{C}$ which depends only on $A$, $n$ and $\sigma$. From (76), we obtain that for any $\sigma \geq n - 2$ the $(L,2)$-capacity of the condenser $(\overline{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n)$ is bounded above by a constant which depends only on $A$, $n$ and $\sigma$, for all sufficiently large $R$. This concludes the proof of Proposition 3.

**Proof of Proposition 4.** Let $n \geq 2$, and $L$ be a differential operator defined by (1) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$, with some constants $A > 0$ and $n - 2 > \sigma > -2$. Then, from (12) we obtain that for any $p \geq 2n/(\sigma + 2)$, the $(L,p)$-capacity of the condenser $(\overline{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n)$ is bounded above by a constant, which depends only on $A$, $n$, $p$ and $\sigma$, for all sufficiently large $R$.

**Proof of Proposition 5.** Let $n \geq 2$, and $L$ be a differential operator defined by (1) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$, with some constants $A > 0$ and $n - 2 > \sigma > -2$. Then, for any $q > 1$ and any $\nu \in (0, 1) \cap (0, q - 1)$, taking $p = 2(q - \nu)/(q - 1)$ in (12) we obtain the inequality

\[
\text{cap}_{L,p}(\overline{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n) \leq \hat{C} R^{(n-\sigma-2)(q-(n-\nu(\sigma+2)))/(n-\sigma-2)/(q-1)},
\]

which holds for all sufficiently large $R$, with some positive constant $\hat{C}$ which depends only on $A$, $n$, $p$ and $\sigma$. From (77) we obtain that for any $\sigma < n - 2$ and any $1 < q \leq (n - \nu(\sigma + 2))/(n - \sigma - 2)$ the $(L,p)$-capacity of the condenser $(\overline{B}_{R/2}, \mathbb{R}^n \setminus B_R; \mathbb{R}^n)$ is bounded above by a constant which depends only on $A$, $n$, $p$ and $\sigma$, for all sufficiently large $R$.

**Proof of Proposition 6.** Let $n \geq 2$, and $L$ be a differential operator defined by (1) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$, with some constants $A > 0$ and $n - 2 > \sigma > -2$. Then, from (12) we obtain the inequality

\[
\mathcal{C}_{L,p_1,p_2}(R) \leq c R^{(2n-p_1(\sigma+2))/4} R^{(2n-p_2(\sigma+2))/2p_2}
\]

for all sufficiently large $R$, with some positive constant $c$ which depends only on $A$, $n$, $p_1$, $p_2$ and $\sigma$. Hence for any $\nu \in (0, 1) \cap (0, (\sigma + 2)/(n - \sigma - 2))$, choosing $p_1 \geq 2(n - \nu(n - \sigma - 2))/(\sigma + 2)$ and $p_2 \geq 2n/(\sigma + 2 - \nu(n - \sigma - 2))$ in (78) we obtain that the quantity $\mathcal{C}_{L,p_1,p_2}(R)$ is bounded above by a constant, which depends only on $A$, $n$, $p_1$, $p_2$ and $\sigma$, for all sufficiently large $R$, and this concludes the proof of Proposition 6.

**Proof of Proposition 7.** Let $n \geq 2$, and $L$ be a differential operator defined by (1) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$, with some constants $A > 0$ and $n - 2 > \sigma > -2$. Then, for any $q > 1$ and any $\nu \in (0, 1) \cap (0, q - 1)$, letting $p_1 = 2(q - \nu)/(q - 1)$ and $p_2 = 2q/(q - 1 - \nu)$ in (78), we obtain the inequality

\[
\mathcal{C}_{L,p_1,p_2}(R) \leq c R^{(2q-1-\nu)(n-\sigma-2)(q-n)/(n-\sigma-2)/(2q(1-\nu))}
\]

for all sufficiently large $R$, with some positive constant $c$ which depends only on $A$, $n$, $p_1$, $p_2$ and $\sigma$. From (79), we have that for any $q \leq n/(n - \sigma - 2)$ the quantity $\mathcal{C}_{L,p_1,p_2}(R)$ is bounded above by a constant, which depends only on $A$, $n$, $p_1$, $p_2$ and $\sigma$, for all sufficiently large $R$, and this concludes the proof of Proposition 7.
Proof of Proposition 8. Let $n \geq 2$, and $L$ be a differential operator defined by the relation (1) in $\mathbb{R}^n$ such that its coefficients satisfy condition (11) for all sufficiently large $R$, with some constants $A > 0$ and $\sigma > -2$. Then letting $p = 2$ in (12) and multiplying both sides of this inequality by $R^{-n}$ we obtain the inequality
\[
cap_{L,2}(\mathcal{B}_{R/2}, \mathbb{R}^n \setminus \mathcal{B}_R; \mathbb{R}^n) R^{-n} \leq \hat{C} R^{-\sigma - 2},
\] (80)
which holds for all sufficiently large $R$, with some positive constant $\hat{C}$ which depends only on $A$, $n$ and $\sigma$. From (80), for any $\sigma > -2$, we obtain equality (??), and this concludes the proof of Proposition 8.
References

[1] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math., 34 (1981), no. 4, 525-598.

[2] V.M. Gol’dshtein and Yu.G. Reshetnyak, *Introduction to the theory of functions with generalized derivatives and quasiconformal mappings*, (Russian) ”Nauka”, Moscow (1983), 285 pp.

[3] A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1970, 403 pp.

[4] V.A. Kondrat’ev and E.M. Landis, *Semilinear second-order equations with nonnegative characteristic form*, (Russian) Mat. Zametki 44 (1988), no. 4, 457-468; translation in Math. Notes 44 (1988), no. 3-4, 728-735.

[5] V.V. Kurta, *On the absence of positive solutions to semilinear elliptic equations*, (Russian) Tr. Mat. Inst. Steklova 227 (1999), Issled. po Teor. Differ. Funkts. Mnogikh Perem. i ee Prilozh. 18, 162–169; translation in Proc. Steklov Inst. Math. 1999, no. 4 (227), 155–162.

[6] V.V. Kurta, *Liouville comparison principles for solutions of semilinear elliptic second-order partial differential inequalities*, Complex Var. Elliptic Equ. 58 (2013), no. 9, 1299-1319.

[7] W. Littman, G. Stampacchia and H.F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa (3) 17 (1963), 43-77.

[8] V.G. Maz’ya and T.O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Pitman (Advanced Publishing Program), Boston, MA (1985), 344 pp.

[9] O.A. Oleinik and E.V. Radkevich, *Second order equations with nonnegative characteristic form*, Translated from the Russian by Paul C. Fife. Plenum Press, New York-London, 1973, 259 pp.
Author’s address:

Vasilii V. Kurta
Mathematical Reviews
416 Fourth Street, P.O. Box 8604
Ann Arbor, Michigan 48107-8604, USA

e-mail: vkurta@umich.edu, vvk@ams.org