Suppression of the critical collapse for one-dimensional solitons by saturable quintic nonlinear lattices

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The stabilization of one-dimensional solitons by a nonlinear lattice against the critical collapse in the focusing quintic medium is a challenging issue. We demonstrate that this purpose can be achieved by combining a nonlinear lattice and saturation of the quintic nonlinearity. The system supports three species of solitons, namely, fundamental (even-parity) ones and dipole (odd-parity) modes of on- and off-site-centered types. Very narrow fundamental solitons are found in an approximate analytical form, and systematic results for very broad unstable and moderately broad partly stable solitons, including their existence and stability areas, are produced by means of numerical methods. Stability regions of the solitons are identified by means of systematic simulations. The stability of all the soliton species obeys the Vakhitov-Kolokolov criterion.

I. INTRODUCTION

Although the possibility of the self-trapping of two- and three-dimensional (2D and 3D) optical solitons was first predicted half a century ago [1–3], the creation of such multi-dimensional localized modes remains a challenging topic [4–6]. In particular, a major problem is the fact that the 2D and 3D solitons in media with the generic Kerr (cubic) nonlinearity are vulnerable to instability against the critical and supercritical collapse, respectively [7–9]. As a result, multidimensional solitons are not routinely observed in experiments, except for spatial [10] and spatiotemporal [11] quasi-2D beams and pulses in waveguides with the quadratic nonlinearity, and spatial 2D solitons in bulk media with a competing nonlinear response of cubic-quintic [12] and quintic-septimal [13] types. Very recently, the creation of 3D matter-wave solitons in the form of “quantum droplets”, stabilized by the Lee-Huang-Yang corrections to the mean-field approximation [14–16], has been reported in Bose-Einstein condensates (BECs) with long-range dipole-dipole interactions [17–19], as well as in binary condensates with local attraction between the components [20–22].

Diverse strategies have been theoretically elaborated to stabilize solitons in 2D and 3D geometries [23]. A versatile
stabilization mechanism relies on the use of periodic (lattice) potentials, which allow one to predict not only robust multidimensional fundamental solitons, but also ones with embedded vorticity (alias solitary vortices) [24, 25]. Available experimental techniques make it possible to readily create potential lattices in optical media and in BECs, in the form of photonic crystals [26] and optical lattices [5, 27, 28], respectively.

Recently, this technique was extended (still, chiefly in the theoretical form) to the use of nonlinear lattices (NLs), i.e., periodic modulations of the local nonlinearity strength, as well as their combinations with linear-lattice potentials [5]. NLs can be realized in optics and BECs by means of diverse techniques, e.g., filling voids of photonic crystals with index-matching materials [5], or subjecting BEC to the action of the Feshbach resonance controlled by a spatially periodic field [29, 30]. In 1D settings, NLs have been used to predict stable single-hump solitons and various soliton complexes, both single- and two-component ones [31–36]. In the 2D geometry, stable solitons, including fundamental, dipole, and vortex modes, may only be supported by NLs with sharp edges, such as a period array of cylinders [37]. Stable solitons supported by NLs with competing nonlinearities [38] and combined linear-nonlinear lattices [39–42], have been predicted too.

Similar to the challenging problem of stabilizing 2D solitons in focusing Kerr media, the occurrence of the critical collapse makes the stabilization of 1D solitons in self-focusing quintic media a nontrivial problem too [43–45]. This attractive quintic nonlinearity is physically relevant, as it was predicted [46] and experimentally demonstrated [47] that it may be realized as a super-Tonks-Girardeau gas (a gas of hard-core bosons in a strongly excited state). On the other hand, the quintic self-focusing in a nearly “pure” form can be experimentally realized for the light propagation in colloidal waveguides with suspensions of metallic nanoparticles [48].

Periodic lattices acting on the quintic-only self-focusing in 1D cannot readily stabilize solitons, similar to the above-mentioned situation with the 2D NLs applied to the cubic self-focusing. As demonstrated in Ref. [38], in this case the stabilization is possible in a very narrow region of the solitons’ existence area. In this work, we address the existence and stability of 1D solitons under the action of a NL in a saturable self-focusing quintic medium. Numerical analysis of the model gives rise to two species of solitons, fundamental and dipole ones (with even and odd shapes, respectively). In particular, the saturation makes the stability region of the solitons much broader than it was provided by the NL acting on the unsaturated quintic term [38].

The model is formulated in Section II. The same section also reports some approximate analytical results for narrow fundamental solitons. Systematic numerical findings for very broad (unstable) and moderately broad (partly stable) fundamental solitons, as well as for two types of dipole modes, on- and off-center-cited ones, are presented in Section III. The paper is concluded by Section IV.

II. THE MODEL AND ANALYTICAL APPROXIMATIONS

We consider the propagation of light beams in a planar waveguide governed by the nonlinear Schrödinger equation for the field amplitude, \( \psi(x, z) \):

\[
 i \frac{\partial \psi}{\partial z} = \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} - g \cos(2x) \left| \psi \right|^4 \psi \left/ \left(1 + S |\psi|^4 \right) \right.,
\]

where \( S > 0 \) determines the saturation of the quintic nonlinearity, and \( g > 0 \) is the strength of the NL, whose period is fixed by scaling to be \( \pi \). Although the model seems quite specific, it may be realized as a solid-phase sol (similar to “cranberry glass”) with a periodic modulation of parameters of the dispersed nanoparticles. In this case, the variation of the parameters, such as the concentration of the nanoparticles and their size, may change the sign of the nonlinearity [48, 49], which also features saturation. In fact, a broader class of models similar to the one based on Eq. (1) will produce results similar to those reported below. The nonlinearity is considered here as self-focusing because we consider solitons with centers placed at \( x = 0 \), around which the sign indeed corresponds to focusing.

We aim to find stationary solutions of Eq. (1) with propagation constant \( b \), as \( \psi(x, z) = \phi(x) \exp(i bz) \), where real stationary field \( \phi(x) \) is determined by the equation

\[
 b \phi = \frac{1}{2} \frac{d^2 \phi}{dx^2} + g \cos(2x) \phi^5 \left/ \left(1 + S |\phi|^4 \right) \right..
\]

Solitons produced by Eq. (2) are characterized by the total power (alias norm),

\[
 P = \int_{-\infty}^{+\infty} \phi^2(x)\,dx.
\]

In the case of small \( S \), when the saturation is negligible, the present model is tantamount to the one considered in Ref. [38], where the above-mentioned narrow stability region for bright solitons was found in a numerical form. On the other hand, for narrow solitons in the present model, with width

\[
 w \ll \pi/4
\]

and centered at \( x = 0 \), one may replace \( \cos(2x) \) by 1 in Eq. (2). In this case, it is easy to integrate the stationary equation to obtain

\[
 b \phi^2 = \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{g}{S} \left[ \phi^2 - \frac{1}{2} \sqrt{S} \arctan \left( \sqrt{S} \phi^2 \right) \right].
\]

It immediately follows from Eq. (5) that the narrow solitons exist in the interval of

\[
 0 < b < b_{\max} \equiv g/S.
\]

Close to the largest possible propagation constant, viz., at \( 0 < \delta b \equiv b_{\max} - b < b_{\max} \), it follows from Eq. (6) that the soliton’s peak power grows as

\[
 \phi^2(x = 0) \approx \left( \pi/2S^{1/2} \right) (\delta b)^{-1},
\]

where

\[
 \phi^2(x = 0) \approx \left( \pi/2S^{1/2} \right) (\delta b)^{-1},
\]
while its width remains finite, \( w \sim 1/\sqrt{2b_{\text{max}}} \equiv \sqrt{S/(2g)} \), hence the total power \( P \) diverges at \( \delta b \to 0 \) as
\[
P \sim (S\sqrt{\delta b})^{-1}.
\]
Eq. (8)

Lastly, the above-mentioned condition, \( w \ll \pi/4 \), which allows one to replace \( \cos(2x) \) by 1, amounts to \( S \ll g \).

Under more restrictive conditions, \( 1 \ll \delta b \ll b_{\text{max}} \) (rather than simply \( \delta b/b_{\text{max}} \to 0 \), as considered above), Eq. (2) gives rise to an explicit approximate solution in the form of a compacton [50]:
\[
\phi(x) = \sqrt{\frac{\pi g}{4S^{3/2}\delta b}} \left\{ \begin{array}{ll}
\cos \left( \frac{\sqrt{2\delta b}x}{S} \right), & \text{at } |x| < \pi/\left(2\sqrt{2\delta b}\right), \\
0, & \text{at } |x| > \pi/\left(2\sqrt{2\delta b}\right),
\end{array} \right.
\]
Eq. (9)

with integral power
\[
P = \frac{\pi^2 g}{4(2\sqrt{2\delta b})^{3/2}}.
\]
Eq. (10)

Note that both dependences \( P(b) \) given by Eqs. (8) and (10) satisfy the Vakhitov-Kolokolov (VK) criterion [51], \( dP/db > 0 \), which is a well-known necessary (but, generally speaking, not sufficient) stability condition for any soliton family supported by self-focusing nonlinearity.

### III. NUMERICAL RESULTS

#### A. Fundamental solitons

In the numerical form, soliton solutions to Eq. (2), both fundamental (spatially even) and dipole (odd) ones, were produced by means of the Newton-Raphson iteration method, that provides fast convergence with a properly selected input. The stability of the solitons was then tested by means of direct simulations of their perturbed evolution in the framework of Eq. (1), using the finite-difference method.

An example of a stable narrow soliton supported by the saturable quintic NL is shown in Fig. 1(a). The soliton completely localizes itself within a single cell of the NL, resembling a similar soliton found in the unsaturated quintic NL model [38], i.e., its width \( w \) indeed satisfies condition (4). As predicted above, the stability region of the narrow solitons, displayed in Fig. 1(b), is close enough to the largest possible propagation constant \( b_{\text{max}} \equiv g/S \), where the soliton’s power may be very large, according to Eq. (8). This result is drastically different from the situation in the unsaturated model, where the power of stable solitons is restricted to small values [38]. Because the NL does not essentially affect the structure and stability of the narrow solitons, the main objective of this work is, instead, to focus on the consideration of broader localized states, whose width is comparable to or larger than the period of the NL.

Figures 2(a,c) and (b,d) demonstrate, respectively, that the NL supports both very broad unstable fundamental solitons, which spread over several NL periods, and stable moderately broad ones, which are trapped, essentially, in a single self-focusing stripe. Nevertheless, the latter type is clearly different from the narrow solitons displayed above in Fig. 1. It is relevant to mention that the stability of the solitons was tested by direct simulations run in a sufficiently large domain, with the Neumann boundary conditions [see Figs. 2(c,d) and Fig. 6 below]. Although radiation emitted by unstable solitons may partly bounce back from the boundaries, the size of the integration domain is large enough to make it sure that the instability of the solitons, if any, is driven by their internal dynamics, rather than by perturbations induced by reflected radiation waves.

The difference between the very broad and moderately broad solitons is natural, taking into account the above-mentioned general fact that NLs may stabilize solitons under the action of the critical nonlinearity (cubic and quintic in the 2D and 1D settings, respectively), provided that the modulation profile is sharp enough: this condition holds for the moderately broad solitons, while very broad ones feel the action of the underlying lattice in an averaged, i.e., effectively smooth form [38].

The results are summarized in Fig. 3, by dint of plots showing dependence \( P(b) \) for the numerically generated families of the fundamental solitons. In the same figure, stability regions of the fundamental solitons are shown too, showing that the stability agrees with the VK criterion, as stable solitons always obey condition \( dP/db > 0 \). In addition, comparison of Figs. 3(a) and (b), whose stability regions are, respectively, \( b \in [0.38, 2.95] \) and \( b \in [0.146, 1.12] \), shows that the increase of saturation \( S \) leads to decrease of the stability threshold, and the corresponding cutoff, \( b_{\text{max}} \), rapidly decreases too, similarly to what is predicted for narrow solitons by Eq. (6). By increasing the nonlinearity strength \( g \), while keeping \( S \) fixed, the stability region broadens, as seen from the comparison of Figs. 3(a,c) and (b,d). A noteworthy feature of the present model is that, once power \( P \) or propagation constant \( b \) become

![Fig. 1.](image-url) (Color online) (a) A typical shape of a stable narrow soliton. (b) Total power \( P \) of the narrow solitons vs. their propagation constant \( b \), for nonlinear strength \( g = 1 \) and saturation \( S = 0.01 \) in Eq. (1). The soliton displayed in (a) corresponds to the bold dot in (b). Shaded regions in (a), as well as in Figs. 2(a,b) and 5 below correspond to maxima of the local nonlinearity. In panel (b), as well in Figs. 3, 4, 7, and 8 below, the stability region is shaded. The inset in (b) and the ones in Figs. 3 and 7 below show curve \( P(b) \) at small values of \( b \).
large enough to support stable solitons, e.g., one reaches the corresponding stability threshold, \( b = b_{\text{min}} \) [which is, for instance, 0.38 in Fig. 3(a)], the fundamental solitons remain stable up to arbitrarily large values of \( P \), i.e., up to the cutoff propagation constant, \( b = b_{\text{max}} \), at which \( P \) diverges. This feature is a natural consequence of the nonlinearity saturation in Eq. (1). The value of \( P \) diverges too at \( b \to 0 \), as seen from the inside subplot for \( P(b) \) relation in Fig. 3 (and the following Fig. 6). The explanation is that, the corresponding soliton’s amplitude decreases at \( b \to 0 \), the soliton quickly broadens, while the amplitude decreases at a smaller rate.

The results are further collected in Fig. 4, which shows stability areas for the fundamental solitons in the planes of \((S, b)\) and \((g, b)\). In particular, the trend to the decrease of the stability threshold, \( b_{\text{min}} \), with the increase of \( S \), observed in Figs. 4(a,c,e), is quite natural, as larger values of \( S \) make smaller amplitudes sufficient for affecting the existence and stability of the solitons through the saturation of the nonlinearity. On the other hand, the same figures demonstrate that the entire stability areas shrinks with the increase of \( S \) (the same conclusion follows from the comparison of the stability areas displayed in Figs. 4(b,d,f) for different fixed values of \( S \)).

Comparing Figs. 4(b,d,f), one can see that the stability area for the fundamental solitons shrinks at both very small and very large values of nonlinearity strength \( g \). Further, reducing \( S \) leads to an increase of possible stable region, as can be seen from the planes of \((g, b)\) in the right column [Fig. 4(b,d,f)].

**B. Dipole solitons**

Besides the fundamental solitons, the NL in the present model also supports two species of localized dipole states, with the spacing between local power maxima \( \pi \) and \( 2\pi \), which correspond to the single and double lattice period, severally, see examples in Fig. 5 and 6. We call these soliton modes the off-site-centered and on-site-centered dipole ones, respectively. By means of the systematic simulations, we have verified that all dipole modes are unstable (or do not exist) both in the case of the saturable quintic nonlinearity without the NL [with \( \cos(2x) \) replaced by 1 in Eq. (1)], and in the presence of the NL without the saturation \((S = 0)\), while stabilization can be readily achieved in the present model, which combines the NL and saturation. Similarly to the fundamental solitons, the power peaks of unstable dipole modes, of both off- and on-site-centered types, expand across several cells of the lattice, while the stable ones are nested in a single well, as seen in Figs. 5 and 6.

Dependences \( P(b) \) and respective stability regions for the dipole solitons are displayed in Fig. 7, and findings for the stability are summarized in Fig. 8, which displays stability areas in the \((S, b)\) and \((g, b)\) parameter planes, cf. similar diagrams for the fundamental solitons presented above in Fig. 3. The first noteworthy conclusion is that the cutoff value of the propagation constant, \( b_{\text{max}} \), above which the stationary solu-
FIG. 4. (Color online) Stability areas (shaded) for the fundamental solitons in the \((S, b)\) plane for different fixed values of the nonlinearity strength: \(g = 2\) (a), \(g = 1\) (c), \(g = 0.5\) (e), and in the \((g, b)\) plane for different values of the saturation parameter: \(S = 1\) (b), \(S = 0.5\) (d), \(S = 0.25\) (f). The fundamental solitons are unstable below bottom boundaries of the shaded areas, and solitons do not exist above the top boundaries.

It is seen that, as in the case of the fundamental solitons (cf. Fig. 4), the increase of saturation \(S\) leads to the decrease of essential values of \(b\), the effect of the variation of \(g\) on the stability being also similar to what was shown above for the fundamental solitons.

On the other hand, Figs. 7 and 8 clearly show that the stability threshold (i.e., the bottom boundaries of the stability areas in Fig. 8) are much lower for the on-site-centered dipole solitons than for their off-site-centered counterparts, which is explained by the fact that the modes of the latter type are easier destabilized by the stronger interaction between the two power peaks, which are separated by a smaller distance.

Finally, the comparison of Figs. 4 and 8 demonstrates that the dependence of the stability areas of the dipoles of both types on the saturation and nonlinearity coefficients, \(S\) and \(g\), is qualitatively similar to that for fundamental solitons. This conclusion is quite natural too, as the effect of these parameters on the stability is not essentially different for the solitons of different types.
FIG. 7. (Color online) Power $P$ vs. propagation constant $b$ for the off-site- (a,b) and on-site- (c,d) centered dipole solitons, respectively. The saturation coefficient is $S = 0.2$ in (a,c), and $S = 0.4$ in (b,d), while $g = 2$ is fixed in all the cases. The dipole solitons are stable in shaded regions. Points (C,D) in (a) correspond to the dipole modes displayed in Figs. 5(a) and 5(b), respectively, while points (E,F) in (c) correspond to Figs. 5(c) and 5(d).

FIG. 8. (Color online) Stability areas for dipole solitons of the off-site-centered (a,c) and on-site-centered (b,d) types. Panels (a,b) display the stability in the $(S,b)$ plane at fixed $g = 2$, while (c,d) show it in the $(g,b)$ plane at $S = 0.5$.

IV. CONCLUSION

In this work, we have investigated a possible way to stabilize three types of bright solitons, viz., the fundamental ones and dipoles with on- and off-site-centered structure, against the critical quasi-collapse in the 1D medium with the focusing saturable quintic nonlinearity by embedding the NL (nonlinear lattice) into it. The stabilization is provided by the interplay between the NL and nonlinearity saturation. For very narrow fundamental solitons, approximate analytical results were reported, while the systematic analysis for very broad unstable solitons and moderately broad partly stable ones was performed by means of numerical methods. In particular, it was found that the stability region is much broader for the on-site-centered dipole modes than for their off-site-centered counterparts. The stability of both the fundamental solitons and both types of the dipoles conforms to the VK (Vakhitov-Kolokolov) criterion. The predicted self-trapped states may be realized in nonlinear optical waveguides built of solid colloidal materials with metallic nanoparticles.

As an extension of the present analysis, it may be interesting to study possible mobility of solitons in this and similar models.

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