A characterization of simplicial spaces by an extension property

by

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Summary. Let $\mathcal{H}$ be a function space on a compact Hausdorff space $K$. We provide a characterization of the simpliciality of $\mathcal{H}$ via an extension property.

1. Introduction. Let $X$ be a compact convex set in a real locally convex space. We write $\mathfrak{A}^c(X)$ for the space of all real continuous affine functions on $X$ endowed with the supremum norm. The symbol $\mathfrak{A}^c(X)^+$ stands for the set of nonnegative elements of $\mathfrak{A}^c(X)$. In [4], a compact subset of $X$ is called hyper-extremal if it is a union of compact faces (the class of hyper-extremal subsets of $X$ in fact coincides with compact extremal subsets of $X$; see [4, p. 396] or [5, Proposition 2.69]). The set $X$ is said to have the property $(H)$ if

\begin{equation}
\text{(1.1) there exists a constant } C \in \mathbb{R} \text{ such that for each hyper-extremal set } D \subseteq X \text{ and } a \in \mathfrak{A}^c(\overline{D})^+ \text{ there exists } b \in \mathfrak{A}^c(X)^+ \text{ such that } b = a \text{ on } D \text{ and } \|b\|_K \leq C\|a\|_D.\end{equation}

It was shown by Batty [4] that $X$ has the property $(H)$ if and only if $X$ is a Choquet simplex. A similar characterization was proven before by Andersen [2], but it was restricted to metrizable compact convex sets. In the present paper we provide a similar characterization for abstractly defined affine functions.

First we collect several definitions and basic facts of Choquet theory on function spaces. For details we refer the reader to [5]. In what follows, let

2010 Mathematics Subject Classification: Primary 46A55.
Key words and phrases: simplicial space, extension property.
Received 24 October 2019; revised 5 December 2019.
Published online 10 January 2020.

DOI: 10.4064/ba191024-6-12
Let $K$ be a compact Hausdorff space and $\mathcal{H}$ be a function space on $K$, that is, a linear subspace of the space of continuous functions on $K$, endowed with the supremum norm, containing constant functions and separating the points of $K$.

For $x \in K$ we denote by $\mathcal{M}_x(\mathcal{H})$ the set of all $\mathcal{H}$-representing measures for $x$, that is,

$$\mathcal{M}_x(\mathcal{H}) = \left\{ \mu \in \mathcal{M}^1(K) : f(x) = \int_K f \, d\mu \text{ for any } f \in \mathcal{H} \right\}.$$  

(Here $\mathcal{M}^1(K)$ stands for the set of all Radon probability measures on $K$.)

The Choquet boundary $\text{Ch}_\mathcal{H}(K)$ of $\mathcal{H}$ is the set

$$\text{Ch}_\mathcal{H}(K) = \left\{ x \in K : \mathcal{M}_x(\mathcal{H}) = \{ \delta_x \} \right\},$$

where $\delta_x$ stands for the Dirac measure at a point $x \in K$.

A function $f$ on $K$ satisfying

$$f(x) \leq \int_K f \, d\mu, \quad x \in K, \mu \in \mathcal{M}_x(\mathcal{H}),$$

is termed $\mathcal{H}$-convex. A function $f$ on $K$ is $\mathcal{H}$-concave if $-f$ is $\mathcal{H}$-convex. If $f$ is both $\mathcal{H}$-convex and $\mathcal{H}$-concave, then $f$ is called $\mathcal{H}$-affine. The family of all continuous $\mathcal{H}$-affine functions is denoted by $\mathcal{A}^c(\mathcal{H})$. Then $\mathcal{A}^c(\mathcal{H})$ is again a function space. Moreover, it is closed and contains $\mathcal{H}$.

We recall that given a pair of measures $\mu, \nu \in \mathcal{M}^1(K)$, we say that $\mu \prec_\mathcal{H} \nu$ if $\mu(f) \leq \nu(f)$ for any $\mathcal{H}$-convex continuous function $f$ on $K$ (see [5, Definition 3.19]). A measure which is $\prec_\mathcal{H}$-maximal is called $\mathcal{H}$-maximal.

If $K = X$ is a compact convex set in a locally convex space and $\mathcal{H} = \mathfrak{C}(X)$, then $\mathcal{H}$ is a function space with $\mathcal{A}^c(\mathcal{H}) = \mathcal{H}$ and $\text{Ch}_\mathcal{H}(X) = \text{ext} X$, the set of all extreme points of $X$ (see [5, Theorem 2.40]).

Let $A$ be a Borel subset of $K$. Then $A$ is called measure convex if $x \in A$ whenever $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$ with $\mu(K \setminus A) = 0$. If $A$ is moreover closed, then it is called $\mathcal{H}$-convex. For a subset $F$ of $K$, let

$$\overline{\text{co}}^\mathcal{H}(F) = \bigcap \{ C \subseteq K : C \supseteq F, C \text{ is } \mathcal{H}\text{-convex} \}.$$ 

The subset $F$ is $\mathcal{H}$-convex if and only if $F = \overline{\text{co}}^\mathcal{H}(F)$.

A Borel set $A \subseteq K$ is called measure extremal if for any $x \in A$ and any measure $\mu \in \mathcal{M}_x(\mathcal{H})$, $\mu$ is supported by $A$. Closed measure extremal sets are called $\mathcal{H}$-extremal. Finally, we say that $A$ is a Choquet set if it is both measure convex and measure extremal.

The upper envelope of a bounded function $f$ on $K$ is defined as

$$f^*(x) = \inf \{ s(x) : s \text{ is continuous and } \mathcal{H}\text{-concave, } s \geq f \}, \quad x \in K.$$ 

It is always an upper semicontinuous $\mathcal{H}$-concave function on $K$, and it coincides with $f$ on $\text{Ch}_\mathcal{H}(K)$ for $f$ upper semicontinuous.
The state space $S(\mathcal{H})$ of $\mathcal{H}$ is defined as
$$S(\mathcal{H}) = \{ \varphi \in \mathcal{H}^* : \varphi \geq 0, \varphi(1) = 1 \}.$$ It is a $w^*$-compact convex subset of $\mathcal{H}^*$. The function space $\mathcal{H}$ is called simplicial if $S(\mathcal{A}^c(\mathcal{H}))$ is a Choquet simplex (see [5, Theorem 6.54]). As a canonical example of a simplicial function space serves the space $\mathcal{A}^c_c(X)$ on a simplex $X$ (see [5, Theorem 6.54]). A less obvious example is the following. Let $U \subset \mathbb{R}^d$ be an open bounded set. Then the space of all functions continuous on $U$ that are harmonic on $U$ is an example of a simplicial function space (see [5, Theorem 13.35]).

The evaluation mapping $\phi$ from $K$ into $S(\mathcal{H})$ is defined as $\phi : x \mapsto \phi_x$, $x \in K$, where $\phi_x$ maps a function $h \in \mathcal{H}$ to the real number $h(x)$. We further define a mapping $\Phi : \mathcal{H} \to \mathcal{A}^c(S(\mathcal{H}))$ for $h \in \mathcal{H}$ as $\Phi(h) : s \mapsto s(h), s \in S(\mathcal{H})$.

We point out several important properties of the mappings $\phi$ and $\Phi$. For the proofs of these facts see e.g. [5, Proposition 4.26, Lemma 8.10 and Proposition 8.22].

The mapping $\phi$ is a homeomorphism of $K$ into $S(\mathcal{H})$, $S(\mathcal{H}) = \overline{\text{co}}(\phi(K))$ and $\phi(\text{Ch}_\mathcal{H}(K)) = \text{ext} S(\mathcal{H})$. If $H \subseteq S(\mathcal{H})$ is $\mathcal{A}^c(S(\mathcal{H}))$-measure extremal, then $\phi^{-1}(H \cap \phi(K))$ is measure extremal in $K$. Moreover, for each set $F \subseteq K$ we have
$$\overline{\text{co}}^\mathcal{H}(F) = \phi^{-1}\left( \overline{\text{co}}(\phi(F)) \cap \phi(K) \right).$$

The mapping $\Phi$ is positive, linear and norm-preserving. It is surjective if and only if $\mathcal{H}$ is closed, and in this case the inverse mapping is realized by $\Phi^{-1}(F) = F \circ \phi$, $F \in \mathcal{A}^c(S(\mathcal{H}))$.

If $F$ is a closed subset of $K$ then the space $\mathcal{H}|_F$ of all restrictions of functions from $\mathcal{H}$ to the set $F$ is again a function space.

2. Characterization of simpliciality. We note that from the definition of $\mathcal{H}$-affine functions it follows that the sets of representing measures are the same with respect to both function spaces $\mathcal{H}$ and $\mathcal{A}^c(\mathcal{H})$, that is, for each $x \in K$ we have $\mathcal{M}_x(\mathcal{H}) = \mathcal{M}_x(\mathcal{A}^c(\mathcal{H}))$. From this it follows that $\mathcal{A}^c(\mathcal{A}^c(\mathcal{H})) = \mathcal{A}^c(\mathcal{H})$, and also that the classes of measure extremal and measure convex sets are the same with respect to both these function spaces.

We say that the function space $\mathcal{H}$ has the property $(H)$ if

\begin{equation}
(2.1) \quad \text{there exists a constant } C \geq 0 \text{ such that for each } \mathcal{H}\text{-extremal set } D \subseteq K \text{ and } a \in \mathcal{A}^c(\mathcal{H}|_{\overline{\text{co}}^\mathcal{H}(D)})^+ \text{ there exists } b \in \mathcal{A}^c(\mathcal{H})^+ \text{ with } b|_D = a \text{ and } \|b\|_K \leq C\|a\|_D.
\end{equation}
In the rest of the paper we show that this property \((H)\) characterizes the concept of simpliciality of a function space.

First we show the connection between our definition of property \((H)\) and the property \((\underline{H})\) of a compact convex set as defined by Batty.

**Proposition 2.1.** If a function space \(\mathcal{H}\) has the property \((H)\) in the sense of (2.1) then \(S(\mathcal{A}^c(\mathcal{H}))\) has the property \((\underline{H})\) in the sense of (1.1).

**Proof.** Let \(X = S(\mathcal{A}^c(\mathcal{H}))\). Let \(D \subset X\) be a nonempty hyper-extremal set and \(a \in \mathfrak{X}^c(\mathfrak{co}D)^+\). Then \(D\) is \(\mathfrak{A}^c(X)\)-extremal (see [5, Proposition 2.69]). We consider the closed function space \(\mathcal{A}^c(\mathcal{H})\) and the above-defined mappings \(\phi : K \rightarrow S(X)\) and \(\Phi : \mathcal{A}^c(\mathcal{H}) \rightarrow \mathfrak{A}^c(X)\). We denote \(F = \phi^{-1}(D \cap \phi(K))\). Then \(F\) is \(\mathcal{A}^c(\mathcal{H})\)-extremal (see [5, Lemma 8.10]), and thus \(\mathcal{H}\)-extremal in \(K\).

Since the classes of \(\mathcal{H}\)-convex and \(\mathcal{A}^c(\mathcal{H})\)-convex sets coincide, \(\mathfrak{co}^\mathcal{H}(F) = \mathfrak{co}^\mathcal{A}^c(\mathcal{H})(F)\). Let

\[ \tilde{a}(x) = a(\phi(x)), \quad x \in \mathfrak{co}^\mathcal{H}(F). \]

We claim that \(\tilde{a} \in \mathcal{A}^c(\mathcal{H}|_{\mathfrak{co}^\mathcal{H}(F)})^+\). Obviously, \(\tilde{a} \geq 0\). Let \(x \in \mathfrak{co}^\mathcal{H}(F)\) and \(\mu \in \mathcal{M}_x(\mathcal{H}|_{\mathfrak{co}^\mathcal{H}(F)})\). Then \(\mu \in \mathcal{M}^1(\mathfrak{co}^\mathcal{H}(F))\), and since

\[ \mathfrak{co}^\mathcal{H}(F) = \mathfrak{co}^\mathcal{A}^c(\mathcal{H})(F) = \phi^{-1}(\mathfrak{co}(\phi(F)) \cap \phi(K)), \]

the image \(\phi\mu \in \mathcal{M}^1(X)\) under the mapping \(\phi\) has support in \(\mathfrak{co}(\phi(F)) \subset \mathfrak{co}(D)\). Further, the measure \(\mu\) considered as a measure on \(K\) \(\mathcal{H}\)-represents \(x\). Thus \(\phi\mu \mathfrak{A}^c(X)\)-represents the point \(\phi(x)\) (see [5, Proposition 4.26(c)])

\[ \mu(\tilde{a}) = \mu(a \circ \phi) = \phi\mu(a) = a(\phi(x)) = \tilde{a}(x). \]

Hence \(\tilde{a}\) is an \(\mathcal{A}^c(\mathcal{H}|_{\mathfrak{co}^\mathcal{H}(F)})\)-affine function on \(\mathfrak{co}^\mathcal{H}(F)\).

By (2.1), there exists a function \(\tilde{b} \in \mathcal{A}^c(\mathcal{H})^+\) such that \(\|\tilde{b}\|_K \leq C\|\tilde{a}\|_F\) and \(b = \tilde{a}\) on \(F\). Let \(b = \Phi(\tilde{b}) \in \mathfrak{A}^c(X)^+\). Then \(b = a\) on \(D\).

By \(s \in D\) be given. We find an \(\mathfrak{A}^c(X)\)-maximal measure \(\mu \in \mathcal{M}^1(X)\) which \(\mathfrak{A}^c(X)\)-represents \(s\) (see [5, Theorem 3.65]). Since \(D\) is \(\mathfrak{A}^c(X)\)-extremal, \(\mu\) is supported by \(D\) and, by maximality, by \(\phi(K)\) (see [5, Propositions 3.64 and 4.26(d)]). Let \(\tilde{\mu} \in \mathcal{M}^1(K)\) satisfy \(\phi\tilde{\mu} = \mu\). Then \(\tilde{\mu}\) is supported by \(\phi^{-1}(D \cap \phi(K)) = F\). Thus

\[ a(s) = \mu(a) = (\phi\tilde{\mu})(a) = \int_F a \circ \phi d\tilde{\mu} = \int_F \tilde{a} d\tilde{\mu} = \int_F b d\tilde{\mu} \]

\[ = \int_{\phi(F)} \Phi(\tilde{b}) d\phi\tilde{\mu} = \mu(b) = b(s). \]

Obviously we have

\[ \|b\|_X = \|\tilde{b}\|_K \leq C\|\tilde{a}\|_F \leq C\|a\|_D. \]

Thus \(X = S(\mathcal{A}^c(\mathcal{H}))\) satisfies \((\underline{H})\) in the sense of (1.1). □
In [4], the proof that simplices have the property \((H)\) is deduced from the facts that the closed convex hull of a dilated subset of a simplex is a face (see [3, p. 114]), and that affine continuous functions on a face of a simplex may be extended with preservation of norm (the proof is similar to that of [1, Theorem II.5.19]). (We recall that a closed subset \(D\) of a compact convex set \(X\) is said to be dilated if whenever \(\mu\) is a maximal probability measure on \(X\) that \(\mathcal{A}_c(X)\)-represents a point \(x \in D\) then \(\mu\) is supported by \(D\). Thus a closed set \(D \subset X\) is dilated provided it is measure extremal. On the other hand, the set \(\{0, 1/2, 1\}\) is a dilated subset of \([0,1]\) which is not measure extremal.) The following two lemmas are analogous results in the context of function spaces.

**Lemma 2.2.** Let \(\mathcal{H}\) be a simplicial function space and \(F\) be an \(\mathcal{H}\)-extremal subset of \(K\). Then \(\overline{co}^\mathcal{H}(F)\) is a Choquet set.

**Proof.** If \(F \subset K\) is a nonempty \(\mathcal{H}\)-extremal set, the characteristic function \(\chi_F\) is an upper semicontinuous \(\mathcal{H}\)-convex function. We show that \(\chi_F^* = 1\) on \(\overline{co}^\mathcal{H}(F)\).

Indeed, let \(x \in \overline{co}^\mathcal{H}(F)\) be given. By [3, Proposition 8.18], there exists a measure \(\mu \in M_x(\mathcal{H})\) supported by \(F\). Then by [5, Lemma 3.21],

\[
1 = \mu(F) = \mu(\chi_F) \leq \sup\{\nu(\chi_F) : \nu \in M_x(\mathcal{H})\} = \chi_F^*(x) \leq 1,
\]

which implies \(\chi_F^*(x) = 1\).

Now we prove that \(\overline{co}^\mathcal{H}(F)\) is \(\mathcal{H}\)-extremal. To this end, let \(x \in \overline{co}^\mathcal{H}(F)\) and \(\mu \in M_x(\mathcal{H})\) be given. Let \(\nu \in M^1(K)\) be an \(\mathcal{H}\)-maximal measure satisfying \(\mu \prec_\mathcal{H} \nu\) (see [5, Theorem 3.65]). Then \(\nu \in M_x(\mathcal{H})\). Since \(\mathcal{H}\) is simplicial, \(\chi_F^*\) is \(\mathcal{H}\)-affine (see [5, Theorem 6.5]). Thus by [5, Theorem 3.68],

\[
1 = \chi_F^*(x) = \nu(\chi_F^*) = \nu(\chi_F),
\]

hence \(\nu(F) = 1\). By [5, Proposition 8.24],

\[
\text{spt } \mu \subset \overline{co}^\mathcal{H}(\text{spt } \nu) \subset \overline{co}^\mathcal{H}(F).
\]

Thus \(\overline{co}^\mathcal{H}(F)\) is an \(\mathcal{H}\)-extremal set and the proof is complete. ■

**Lemma 2.3.** Let \(\mathcal{H}\) be simplicial, and \(D\) be a closed Choquet subset of \(K\). Then any \(a \in \mathcal{A}_c^+(\mathcal{H}|_D)\) may be extended to a function in \(\mathcal{A}_c^+(\mathcal{H})\) with the same norm.

**Proof.** We define functions

\[
s(x) = \begin{cases} a(x), & x \in D, \\ 0, & x \in K \setminus D, \end{cases} \quad t(x) = \begin{cases} a(x), & x \in D, \\ \|a\|, & x \in K \setminus D. \end{cases}
\]

Then it is easy to verify that \(s\) is \(\mathcal{H}\)-convex and upper semicontinuous, while \(t\) is \(\mathcal{H}\)-concave and lower semicontinuous. We prove the desired properties
for $s$. Concerning the $\mathcal{H}$-convexity, let $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. We want to show that $s(x) \leq \int_K f \, d\mu$. If $x \in D$, then $\text{spt } \mu \subseteq D$, since $D$ is a Choquet set. But $s$ coincides with $a$ on $D$, and so the desired inequality is satisfied due to the fact that $a \in \mathcal{A}^c(\mathcal{H})$. If $x \in K \setminus D$, then $s(x) = 0$, and since $s$ is nonnegative on $K$, we are done.

Now we show that $s$ is upper semicontinuous. Let $x \in K$. If $x \in D$, then for given $\varepsilon > 0$ there exists a neighborhood $U$ of $x$ such that $s = a \leq a(x) + \varepsilon = s(x) + \varepsilon$ on $U \cap D$. Since on $U \setminus D$ we have $s = 0 < s(x) + \varepsilon$, we see that $s \leq s(x) + \varepsilon$ on $U$. On the other hand, if $x \in K \setminus D$, then since $D$ is closed, $s$ is constant on some neighborhood of $x$, and the upper semicontinuity of $s$ is proven.

Now, since $\mathcal{H}$ is simplicial, by the Edwards in-between theorem (see [5, Theorem 6.6]) there exists a function $f \in \mathcal{A}^c(\mathcal{H})$ such that $s \leq f \leq t$. Then $f$ is clearly a nonnegative extension of $a$ with the same norm.

We obtained the following characterization of simpliciality of a function space.

**Theorem 2.4.** Let $\mathcal{H}$ be a function space on a compact Hausdorff space $K$. Then $\mathcal{H}$ is simplicial if and only if $\mathcal{H}$ has the property $(H)$ in the sense of (2.1).

**Proof.** It follows by Lemmas 2.2 and 2.3 that every simplicial space has the property $(H)$.

On the other hand, if $\mathcal{H}$ has the property $(H)$ in the sense of (2.1) then $S(\mathcal{A}^c(\mathcal{H}))$ has the property $(H)$ in the sense of (1.1) by Proposition 2.1. Thus by [4, Theorem 4] the state space $S(\mathcal{A}^c(\mathcal{H}))$ is a Choquet simplex, so $\mathcal{H}$ is simplicial.

**Acknowledgements.** We thank the anonymous referee for helpful suggestions leading to an improvement of the paper.

This research was supported by the research grant GAČR 17-00941S.

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