Projector matrix product operators, anyons and higher relative commutants of subfactors

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Abstract
A bi-unitary connection in subfactor theory of Jones producing a subfactor of finite depth gives a 4-tensor appearing in a recent work of Bultinck–Mariën–Williamson– Şahinoğlu-Haegeman–Verstraete on two-dimensional topological order and anyons. In their work, they have a special projection called a projector matrix product operator. We prove that the range of this projection of length \( k \) is naturally identified with the \( k \)th higher relative commutant of the subfactor arising from the bi-unitary connection. This gives a further connection between two-dimensional topological order and subfactor theory.

1 Introduction

The Jones theory of subfactors [5] in operator algebras has found many profound relations to other topics in low-dimensional topology and mathematical physics. Here we present a new connection between subfactor theory and two-dimensional topological order.

Theory of topological phases of matter has recently caught much attention both in mathematics and physics. A recent paper [2] on two-dimensional topological order, tensor networks and anyons attracted much interest of several researchers and this topic is closely related to theory of topological quantum computation [21]. A certain

Dedicated to the memory of Vaughan Jones

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operator $P^k$ on a finite dimensional Hilbert space called a projector matrix product operator (PMPO) [2, Sect. 3], arising from a certain 4-tensor which means a finite family of complex numbers indexed with 4 indices, plays a key role and its range is important in studies of gapped Hamiltonians and projected entangled pair states (PEPS) as in [2, Sections 4, 5] in connection to [6, 11]. The ranges of the projector matrix product operators $P^k$ give an increasing sequence of finite dimensional Hilbert spaces indexed by $k$. Our main result, Theorem 3.3, states that this space has a natural meaning as the $k$th higher relative commutant of the subfactor arising from the 4-tensor in the Jones theory, through repeated basic constructions. (See Fig. 15 for a matrix product operator $O^k_w$ which is used in the definition of $P^k = \sum_a d_a w O^k_a$.)

We note that flatness of a field of strings in the sense of [4, Theorems 11.15] is known to play an important role in subfactor theory and it is also a key notion in our main result. (This flatness was first introduced by Ocneanu [16].) Recall that the tower of higher relative commutants is one of the most important objects in subfactor theory.

We have already seen a connection of the work [2] to subfactor theory and the meaning of anyons there in [9], [10] and we now present a more direct and deeper connection. See [14] for another recent connection to theory of fusion categories, which is also closely related to subfactor theory. See [8] for more general relations among subfactor theory, two-dimensional conformal field theory and tensor categories.

Many researchers work on a formulations based on fusion categories in two-dimensional topological orders. As shown in [4, Chapter 12], a fusion category framework in terms of $6j$-symbols and one based on flat bi-unitary connections (Definition 2.2) are equivalent. A possible advantage of our framework is that the size of numerical data is much smaller for bi-unitary connections than $6j$-symbols and this could be more suited to actual (numerical) computations. We also treat non-flat bi-unitary connections simultaneously as flat bi-unitary connections and this generality could cover a wider class of examples. (See Remark 3.11 on this point).

Recently we have much advance in operator algebraic classification of gapped Hamiltonians on quantum spin chains [17] and we see some formal similarities of mathematical structures there. It would be interesting to exploit this possible connection. For example, the range of a projector matrix product operator should be a space of ground states in some appropriate sense and this viewpoint is to be further explored.

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2 A bi-unitary connection and a subfactor of finite depth

In subfactor theory, finite bipartite graphs play an important role as principal graphs of subfactors. A vertex of a principal graph represents an irreducible object in a certain tensor category and an edge represents the dimension of a certain Hom space in such a category. We treat certain 4-tensors and their 4 wires are labeled with edges of such finite bipartite graphs (and their slight generalizations). That is, a choice of four edges
Fig. 1 Four graphs

\[
\begin{array}{ccc}
V_0 & G & V_3 \\
H & H' & \\
V_1 & G' & V_2
\end{array}
\]

gives a complex number and such an object is known as a bi-unitary connection as in Definition 2.2 in subfactor theory. We first prepare notations and conventions on bi-unitary connections as in [1], [4, Chapter 11], [7, 9, 15, 16].

We have four finite unoriented connected bipartite graphs \( G, G', H, H' \). (These graphs are allowed to have multiple edges between a pair of vertices. The set of vertices of each graph is divided into two classes, even add odd ones.) The even vertices of \( G \) and \( H \) are identified and we write \( V_0 \) for the set of these vertices. The odd vertices of \( H \) and \( G' \) are identified and we write \( V_1 \) for the set of these vertices. The even vertices of \( G' \) and \( H' \) are identified and we write \( V_2 \) for the set of these vertices. The odd vertices of \( G \) and \( H' \) are identified and we write \( V_3 \) for the set of these vertices. They are depicted as in Fig. 1. We assume that all of the four graphs have more than one edges.

Let \( \Delta_{G,xy} \) be the number of edges of \( G \) between \( x \in V_0 \) and \( y \in V_3 \). Let \( \Delta_{G',xy} \) be the number of edges of \( G' \) between \( x \in V_1 \) and \( y \in V_2 \). Let \( \Delta_{H,xy} \) be the number of edges of \( H \) between \( x \in V_0 \) and \( y \in V_1 \). Let \( \Delta_{H',xy} \) be the number of edges of \( H' \) between \( x \in V_3 \) and \( y \in V_2 \). We assume that we have the following identities for some positive numbers \( \gamma_1, \gamma_2 \). For each vertex \( x \), we have a positive number \( \mu_x \). We assume the following identities. That is, for each of \( V_0, V_1, V_2, V_3 \), the vector given by \( \mu_x \) gives a Perron-Frobenius eigenvector for the adjacency matrix of one of the four graphs, and the numbers \( \gamma_1, \gamma_2 \) are the Perron-Frobenius eigenvalues of these matrices. Since all the four graphs have more than one edge, we have \( \gamma_1, \gamma_2 > 1 \).

\[
\sum_x \Delta_{G,xy} \mu_x = \gamma_1 \mu_y, \quad x \in V_0, y \in V_3, \\
\sum_y \Delta_{G,xy} \mu_y = \gamma_1 \mu_x, \quad x \in V_0, y \in V_3, \\
\sum_x \Delta_{G',xy} \mu_x = \gamma_1 \mu_y, \quad x \in V_1, y \in V_2, \\
\sum_y \Delta_{G',xy} \mu_y = \gamma_1 \mu_x, \quad x \in V_1, y \in V_2, \\
\sum_x \Delta_{H,xy} \mu_x = \gamma_2 \mu_y, \quad x \in V_0, y \in V_1, \\
\sum_y \Delta_{H,xy} \mu_y = \gamma_2 \mu_x, \quad x \in V_0, y \in V_1, \\
\sum_x \Delta_{H',xy} \mu_x = \gamma_2 \mu_y, \quad x \in V_3, y \in V_2, \\
\sum_y \Delta_{H',xy} \mu_y = \gamma_2 \mu_x, \quad x \in V_3, y \in V_2,
\]
For an edge $\xi$ of one of the graphs $G, G', H, H'$, we regard it oriented, and write $s(\xi)$ and $r(\xi)$ for the source (starting vertex) and the range (ending vertex). (Each graph is unoriented in the sense that for each edge $\xi$, its reversed edge $\tilde{\xi}$ from $r(\xi)$ to $s(\xi)$ is also an edge of the graph and this reversing map is bijective on the set of edges.) Let $\xi_0, \xi_1, \xi_2, \xi_3$ be oriented edges of $G, H, G', H'$, respectively. If we have $s(\xi_0) = x_0 \in V_0, r(\xi_0) = x_1 \in V_1, s(\xi_1) = x_1 \in V_1, r(\xi_1) = x_2 \in V_2, s(\xi_2) = x_2 \in V_2, r(\xi_2) = x_2 \in V_2, s(\xi_3) = x_0 \in V_0, \text{and } r(\xi_3) = x_3 \in V_3$, then we call a combination of $\xi_i$ a cell, as in Fig. 2.

**Definition 2.1** Assignment of a complex number to each cell is called a connection. We write $W$ for this map and write $W$ within a cell to represent this number as in Fig. 3.

Note that this setting is similar to an interaction-round-a-face (IRF) model in theory of solvable lattice models, where we also assign a complex number to each cell arising from one graph (rather than a combination of four graphs).

The unitarity axiom for $W$ is given in Fig. 4, where the bar on the right cell denotes the complex conjugate of the connection value.

We define a new connection $W'$ as in Fig. 5 on the four graphs $\tilde{G}, \tilde{G}', H', H$, where $\tilde{\xi}$ is the reversed edge of $\xi$ from $r(\xi)$ to $s(\xi)$ and $\tilde{G}$ is the reversed graph of $G$ consisting of such reversed edges as in Fig. 6. We call this rule of giving a new connection Renormalization.

We now have the following definition of a bi-unitary connection.

$$\sum_y \Delta_{H', xy} \mu_y = \gamma_2 \mu_y, \quad x \in V_3, y \in V_2,$$
Definition 2.2 If unitarity holds for $W$ and $W'$, then we say that $W$ is a bi-unitarity connection.

That is, bi-unitarity means that we have unitarity for both the original connection $W$ and the new connection $W'$ defined by Renormalization in Fig. 5. Roughly speaking, bi-unitarity means the connection is “doubly unitary” for the original one and its reflection, but the connection value should be adjusted for the reflection, and this adjustment up to normalization constants is given by Renormalization, Fig. 5.

Ocneanu and Haagerup found that a bi-unitary connection characterizes a non-degenerate commuting squares of finite dimensional $C^*$-algebras with a trace as in [4, Sect. 11.2].

Example 2.3 A typical example of a bi-unitary connection is given as follows. Fix one of the Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$, $E_8$ and let $N$ be its Coxeter number. Set all $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{H}$, $\mathcal{H}'$ to be this bipartite graph so that $V_0$ and $V_2$ [resp. $V_1$ and $V_3$] give the even [resp. odd] vertices of this graph and set both $\gamma_1$, $\gamma_2$ to be $2 \cos \frac{\pi}{N}$. We set $\epsilon = \sqrt{-1} \exp \left( \frac{\pi \sqrt{-1}}{2N} \right)$ We then have a bi-unitary connection as in Fig. 7, [4, Fig. 11.32].

When the graph is $A_n$, this is related to the quantum group $U_q(sl_2)$ with $q$ being a root of unity. Also see [18] for the corresponding IRF models.

We assume this bi-unitarity for $W$ from now on. (We do not assume flatness of $W$ in the sense of [4, Definition 11.16]. If we have flatness with respect to a vertex in $V_0$ and another in $V_2$, then this bi-unitary connection gives a paragroup in the sense of Ocneanu [4, Chapter 10], when we would automatically have $\mathcal{G} = \mathcal{H}$ and $\mathcal{G}' = \mathcal{H}'$. In this sense, a bi-unitary connection is a more general form of a paragroup.)
We also define new connections $\tilde{W}$ and $\tilde{W}'$ as in Figs. 8 and 9, again on (partially) reversed graphs. They are both bi-unitary connections automatically.

**Definition 2.4** We define the value of another diagram in the left hand side of Fig. 10 as in Fig. 10.

Note that we have the identity in Fig. 11 due to Figures 5 and 10.

Let $\tilde{W}$ be the (vertical) product of $W$ and $\tilde{W}'$ as in Fig. 12. That is, we multiply two connection values and make a summation over all possible choices of $\xi_7$, like concatenation of tensors. We make irreducible decomposition of powers of $\tilde{W}$. As
in [1, Sect. 3], this product and irreducible decomposition correspond to the relative tensor product and irreducible decompositions of $A_{0,\infty} - A_{0,\infty}$ bimodules arising from the subfactor $A_{0,\infty} \subset A_{1,\infty}$. (These bimodules are also understood in terms of sectors as in [13].)

Let $\{W_a\}_{a \in V}$ be the set of representative of irreducible bi-unitary connections, up to equivalence, appearing in the irreducible decompositions of the powers of $\tilde{W}$. (See [1, Sect. 3] for the definition of equivalence of connections. This corresponds to an isomorphism of bimodules.) The finite depth assumption exactly means that the set $V$ is finite. Each $a$ corresponds to an irreducible $A_{0,\infty} - A_{0,\infty}$ bimodules arising from the subfactor $A_{0,\infty} \subset A_{1,\infty}$. Each $a$ thus also corresponds to an even vertex of the principal graph of the subfactor $A_{0,\infty} \subset A_{1,\infty}$. Note that the horizontal graph of each $W_a$ is always the original graph $G$.

Let $d_a$ be the Perron-Frobenius eigenvalue of the vertical graph corresponding to the bi-unitary connection $W_a$. This is equal to the dimension of the bimodule corresponding to $W_a$ as in [1, Sect. 3]. We define $w = \sum_{a \in V} d_a^2$, which is sometimes called the global index of the subfactor $A_{0,\infty} \subset A_{1,\infty}$. The original Perron–Frobenius vector $(\mu_x)_{x \in V_0}$ of $\mathcal{H}$ is unique up scalar. We now normalize this vector so that we have $\sum_{x \in V_0} \mu_x^2 = w$. Note that the Perron–Frobenius vector $(\mu_x)_{x \in V_0}$ is also an eigenvector for the vertical graph corresponding to each bi-unitary connection $W_a$.

Let $M_{xy}^a$ be the number of vertical edges with vertex $x \in V_0$ at the upper left corner and $y \in V_0$ at the lower left corner for the connection $W_a$. Note that the Perron-Frobenius eigenvalue property gives $\sum_{y \in V_0} M_{xy}^a \mu_y = d_a \mu_x$. For $a, b, c \in V$, let $N_{ab}^c$ be the multiplicity of $W_c$ in the irreducible decomposition of the product $W_a W_b$. This is also the structure constant of relative tensor products of the corresponding bimodules over $A_{0,\infty}$.

We define $\tilde{a}$ to be $b \in V$ so that $\tilde{W}_a$ is equivalent to $W_b$. We have $M_{xy}^{\tilde{a}} = M_{yx}^a$ by the Frobenius reciprocity, [4, Sect. 9.8]. The $A_{0,\infty} - A_{0,\infty}$ bimodule corresponding to $\tilde{a}$ is contragredient to the one corresponding to $a$ by a result in [1, Page 17].

Finally, we recall the following elementary lemma about a conditional expectation in the string algebra. (See [4, Definitions 11.1, 11.4] for string algebras and a trace there.)

**Lemma 2.5** Let $A = \mathbb{C} \subset B \subset C$ be an increasing sequence of string algebras of length 0, 1, 2 on a Bratteli diagram. We write $*$ for the initial vertex of the Bratteli diagram corresponding to $A = \mathbb{C}$. We fix a faithful trace on $C$. The conditional expectation $E$ from $C$ onto $B' \cap C$ is given as follows.
Let $\xi_1, \xi_2$ be edges of the Bratteli diagram corresponding to $A \subseteq B$, $\eta_1, \eta_2$ be edges of the one corresponding to $B \subseteq C$. Assume $r(\xi_1) = s(\eta_1)$, $r(\xi_2) = s(\eta_2)$, $r(\eta_1) = r(\eta_2)$. We then have

$$E((\xi_1 \cdot \eta_1, \xi_2 \cdot \eta_2)) = \delta_{\xi_1, \xi_2} \frac{1}{K_{r(\xi_1)}} \sum_\xi (\xi \cdot \eta_1 \cdot \xi \cdot \eta_2),$$

where $K_{r(\xi_1)}$ is the number of edges from $*$ to $r(\xi_1)$ on the Bratteli diagram corresponding to $A \subseteq B$.

**Proof** We have this identity by a direct computation. \hfill \square

### 3 A 4-tensor and a projector matrix product operator

We define projector matrix product operators [2, Sect. 3.1], which was originally defined in terms of 4-tensors, with language of bi-unitary connections in the previous Section.

We define a 4-tensor $a$ as in Fig. 13 and [9, Fig. 11]. Note that we have a horizontal concatenation of the connections $W_a$ and $W'_a$ here, since we have considered only symmetric bi-unitary connections in [9, Sect. 2] while we do not assume this symmetric condition here. (If we have $s(\xi_1) \neq s(\xi_6)$, then the value of the 4-tensor is set to be 0. Similarly, if the edges do not make a cell for one of the two squares, the value of the 4-tensor is 0.)

**Remark 3.1** When we concatenate edges $\xi_1, \xi_2, \ldots, \xi_k$ taken from the horizontal graph of $W_a$, we impose the condition $r(\xi_m) = s(\xi_{m+1})$ for $m = 1, 2, \ldots, k - 1$. In the 4-tensor setting, we do not impose such a condition for concatenation of edges, but this difference does not cause any problem here. If we have $r(\xi_m) \neq s(\xi_{m+1})$, the path $\xi_1 \ldots \xi_2 \ldots \xi_k$ is mapped to zero by any matrix product operator and we can ignore this path, since we are interested in the range of a matrix product operator.

Fix a positive integer $k$. Let $\text{Path}^{2k}(G)$ be the $\mathbb{C}$-vector space with a basis consisting of paths of length $2k$ on $G$ starting at an even vertex of $G$. We define a matrix product operator $O^{x,y}_{a,x}$ from $\text{Path}^{2k}_{x,x}(G)$ to $\text{Path}^{2k}_{x,y}(G)$, where $\text{Path}^{2k}_{x,x}(G)$ is a $\mathbb{C}$-linear space spanned by paths of length $2k$ starting from $x$ to $x$ on $G$, as in Fig. 14, where $\xi_1$ and $\xi_2$ have length $k$ each.

**Fig. 13** The 4-tensor $a$ and the connection $W_a$
We next define a matrix product operator $O^k_a$ by

$$O^k_a\left(\bigoplus_x \xi_x\right) = \bigoplus_y \sum_x O^{k, y}_{a, x} \xi_x,$$

where $\xi_x \in \text{Path}^{2k}_{x, x}(G)$. Note that this is the same as the matrix product operator $O^k_a$ defined by Fig. 15 as in [2, Sect. 3.2]. We have different normalization convention for the tensor $a$ and the connection $W_a$ as in Fig. 13, but these coefficients cancel out due to the horizontal periodicity of the picture. (Remark 3.11 again applies here about the domains of the two operators $O^k_a$.)

We then have $O^k_a O^k_b = \sum_c N^c_{ab} O^k_c$. We further define a projector matrix product operator $P^k = \sum_a d^a_w O^k_a$ as in [2, Sect. 3.1]. (This is a projection as shown there.)

For a path $\xi_1 \cdot \xi_2$ with $r(\xi_1) = s(\xi_2)$ and $|\xi_1| = |\xi_2| = k$, we define $\Phi^k(\xi_1 \cdot \xi_2) = \sqrt{\frac{\mu(s(\xi_1))}{\mu(r(\xi_1))}}(\xi_1, \tilde{\xi}_2)$, which is a map from $\text{Path}^{2k}(G)$ to $B_k$, where $\tilde{\xi}_2$ is the reversed path of $\xi_2$, $B_k = \bigoplus_x \text{Str}^k_x(G)$ and $\text{Str}^k_x(G)$ is the string algebra on $G$ with length $k$ starting at a vertex $x \in V_0$ of $G$. (See [4, Definitions 11.1, 11.4] for string algebras.)

We define a matrix product operator $\tilde{O}^{k, y}_{a, x}$ from $\text{Str}^k_x(G)$ to $\text{Str}^k_y(G)$ as in Fig. 16.
We next define a matrix product operator \( \tilde{O}_a^k \) by
\[
\tilde{O}_a^k \left( \bigoplus_x \xi_x \right) = \bigoplus_y \sum_x \tilde{O}_{a,xy}^k \xi_x,
\]
where \( \xi_x \in \text{Str}_x^k(G) \). We again have \( \tilde{O}_a^k \tilde{O}_b^k = \sum_{c \in V} N_{ab}^c \tilde{O}_c^k \). We further define a projector matrix product operator \( \tilde{P}_k = \sum \frac{d_a}{w} \tilde{O}_a^k \) again as in [2, Sect. 3.1].

We then have \( \Phi^k \Theta_a^k = \tilde{O}_a^k \Phi^k \) because of the normalization in Fig. 11 and \( (\tilde{P}_k)^2 = \tilde{P}_k \) for the same reason as \( (P^k)^2 = P_k \).

Each \( \text{Str}_x^k(G) \) has a standard normalized trace \( \text{tr} \) as in [4, page 554]. We set \( \text{tr}(\sigma) = \sum_{x \in V_0} \frac{\mu_x^2}{w} \text{tr}_x(\sigma_x) \) for \( \sigma = \bigoplus_{x \in V_0} \sigma_x \in \bigoplus_{x \in V_0} \text{Str}_x^k(G) \). We let \( \| \sigma \|_{\text{st},2} = \sqrt{\text{tr}(\sigma^* \sigma)} \) for \( \sigma \in B_k \).

Let \( C \) be the maximum of the number of \( x \in V_0 \), the number of \( a \in V \), \( \| \tilde{O}_a^k \| \) over all \( a \in V \) and \( \frac{d_a \mu_x}{w \mu_y} \) over all \( a \in V \), \( x, y \in V_0 \). Here the norm \( \| \tilde{O}_a^k \| \) is the operator norm on \( B_k \) with respect to \( \| \cdot \|_{\text{st},2} \). Note that we have \( C \geq 1 \).

Let \( K_n^x \) be the number of paths from \( x \) to \( x \) on \( H \) of length \( 2n \). Let \( \alpha_n = \sqrt{\sum_{x} (K_n^x)^2} \), and \( \kappa_n^x = K_n^x / \alpha_n \). By the Perron–Frobenius theorem, we have \( \kappa_n^x \rightarrow \mu_x / \sqrt{w} \) as \( n \rightarrow \infty \) for all \( x \in V_0 \).

For a positive integer \( n \), let \( \tilde{W}_n \cong \sum_a L_n^a W_a \), \( \beta_n = \sqrt{\sum_a (L_n^a)^2} \) and \( \lambda_n^a = L_n^a / \beta_n \). By the Perron–Frobenius theorem again, we have \( \lambda_n^a \rightarrow d_a / \sqrt{w} \) as \( n \rightarrow \infty \) for all \( a \in V \).

We recall the following elementary lemma.

**Lemma 3.2** Let \( M \) be a von Neumann algebra with a normalized trace \( \text{tr} \) and \( P \) be its subalgebra. For \( \sigma \in M \) and \( \varepsilon < 1 \), if we have \( \| \sigma \|_2 - \| EP(\sigma) \|_2 < \varepsilon \| \sigma \|_2 \), then we have \( \| \sigma - EP(\sigma) \|_2 < \sqrt{2} \sqrt{\varepsilon} \| \sigma \|_2 \).

**Proof** Since \( \| EP(\sigma) \|_2 > (1 - \varepsilon) \| \sigma \|_2 \) and \( \| \sigma \|_2^2 = \| EP(\sigma) \|_2^2 + \| \sigma - EP(\sigma) \|_2^2 \), we have the conclusion. \( \square \)

With these preparations, we are going to prove the following main result of this paper.

**Theorem 3.3** The range of the projector matrix product operator \( P_k \) of length \( k \) is naturally identified with the \( k \)th higher relative commutant \( A'_\infty,0 \cap A_\infty,k \) for the subfactor \( A_\infty,0 \subset A_\infty,1 \) arising from the original connection \( W \).

**Proof** Note that the map \( \Phi^k \) gives a linear isomorphism from the range of \( P_k \) to that of \( \tilde{P}_k \) in \( B_k \).

We first construct a linear isomorphism \( \Delta \) from \( A'_\infty,0 \cap A_\infty,k \) to the range of \( \tilde{P}_k \). By [4, Theorem 11.15], an arbitrary element in \( A'_\infty,0 \cap A_\infty,k \) is given by a flat field \( \bigoplus_x \sigma_x \in B_k \) and identified with \( \sigma_x \in A_{0,k} \).
Fig. 17 The operator $T^{k,y}_{a,x,\xi_1,\xi_2}$

We define an operator $T^{k,y}_{a,x,\xi_1,\xi_2}$ from $\text{Str}^k_x(G)$ to $\text{Str}^k_y(G)$ as in Fig. 17.

Then flatness of the field [4, Theorems 11.15] gives the equality $T^{k,y}_{a,x,\xi_1,\xi_2}(\sigma_x) = \delta_{\xi_1,\xi_2}\sigma_y$. (This holds as in [4, Fig. 11.19]. Though flatness of the bi-unitary connection is not assumed here, flatness of the fields works instead.) This implies that $	ilde{O}^k_a\sigma_x = \bigoplus_y M^x_{ya}\sigma_y$. Note we have

$$\sum_{x \in V_0, a \in V} d_a \mu_x M^y_{xa} = \sum_{a \in V} \mu_x \sum_{x \in V_0} d_a \mu_y = w \mu_y.$$ 

We then have

$$\tilde{P}^k\left(\bigoplus_{x \in V_0} \mu_x \sigma_x\right) = \sum_{x \in V_0} \frac{d_a}{w} \tilde{O}^k_a \mu_x \sigma_x = \bigoplus_{y \in V_0} \sum_{x \in V_0, a \in V} \frac{d_a}{w} \mu_x M^y_{xa} \sigma_y = \bigoplus_{y \in V_0} \mu_y \sigma_y$$

so the map $\Delta$ assigning $\bigoplus_{x \in V_0} \mu_x \sigma_x$ to $\sigma_\ast$ gives a linear injection from $A'_{\infty,0} \cap A_{\infty,k}$ to the range of $\tilde{P}^k$ in $B_k$.

We next construct an injective linear map for the converse direction. Let $\bigoplus_{x \in V_0} \mu_x \sigma_x$ be in the range of $\tilde{P}^k$ in $B_k$. For a positive integer $n$, we set

$$\sigma^{(n)}(x) = \sum_{x \in V_0} \sum_{\xi} (\xi, \xi) \cdot \sigma_x \in A_{2n,k},$$

where $\xi$ gives all paths from $\ast$ to $x$ on $\mathcal{H}$ with length $2n$. We assume that $n$ is sufficiently large so that the numbers $K^n_x$ are all nonzero.

Suppose that we have the following three estimates for sufficiently small $\varepsilon > 0$.

\begin{align*}
1 - \varepsilon \frac{1}{\sqrt{w}} < a^2_p Y_2^{-2p} < 1 + \varepsilon \frac{1}{\sqrt{w}}, & \quad \text{for all } p \geq n \\
(1 - \varepsilon) \frac{\mu_x}{\sqrt{w}} < \kappa^n_x < (1 + \varepsilon) \frac{\mu_x}{\sqrt{w}}, & \quad \text{for all } x \in V_0 \\
(1 - \varepsilon) \frac{d_a}{\sqrt{w}} < \lambda^m_a < (1 + \varepsilon) \frac{d_a}{\sqrt{w}}, & \quad \text{for all } V \in a
\end{align*}
A computation shows that $E_{A_{2n+2m,0}^n} (\sigma^{(n)})$ is equal to

$$\sum_{y \in V_0} \sum_{\xi} \langle \xi, \xi \rangle \cdot \sum_{a \in V, x \in V_0} \frac{K^n_x}{K^{n+m}_y} L^m_a \tilde{\mathcal{O}}^{k,y}_{a,x} (\sigma_x)$$

by Lemma 2.5, where $\xi$ gives all paths from $*$ to $y$ on $\mathcal{H}$ with length $2(n + m)$, since we have $\tilde{\mathcal{W}} \cong \sum_{a \in V} L^m_a W_a$. Here for large $n$ and $m$, $K^n_x$ is almost equal to $\alpha_n \mu_x \sqrt{w}$, $L^m_a$ is almost equal to $\beta_m d_a \sqrt{w}$, and $K^{n+m}_y$ is almost equal to $\sum_{a \in V, x \in V_0} \alpha_n \mu_x \sqrt{w} \beta_m d_a \sqrt{w}$.

If we had exact equalities for all these three pairs, then we would have

$$\bigoplus_{y \in V_0} \sum_{a \in V, x \in V_0} \frac{K^n_x}{K^{n+m}_y} L^m_a \tilde{\mathcal{O}}^{k,y}_{a,x} (\sigma_x) = \bigoplus_{y \in V_0} \sum_{a \in V, x \in V_0} \mu_x \frac{\beta_m d_a}{\sqrt{w}} \frac{\beta_m d_a}{\sqrt{w}} \tilde{\mathcal{O}}^{k,y}_{a,x} (\sigma_x)$$

$$= \bigoplus_{y \in V_0} \sum_{a \in V, x \in V_0} d_a \mu_x \frac{\beta_m d_a}{\sqrt{w}} \tilde{\mathcal{O}}^{k,y}_{a,x} (\sigma_x)$$

$$= \bigoplus_{y \in V_0} \sigma_y,$$

where the last equality would follow from

$$\sum_{a \in V} d_a \left( \bigoplus_{x \in V_0} \mu_x \sigma_x \right) = \tilde{\mathcal{P}} \left( \bigoplus_{x \in V_0} \mu_x \sigma_x \right) = \bigoplus_{y \in V_0} \mu_y \sigma_y,$$

so $E_{A_{2n+2m,0}^n \cap A_{2n+2m,k}^n} (\sigma^{(n)})$ would be equal to $\sigma^{(n+m)}$. Now we take the approximation errors into account. Suppose we have the estimates (2) and (3). We then have

$$(1 - \varepsilon)^2 \alpha_n \beta_m \mu_y < K^{n+m}_y < (1 + \varepsilon)^2 \alpha_n \beta_m \mu_y,$$ for all $y$

and then

$$\frac{1 - \varepsilon}{1 + \varepsilon} \frac{\mu_x}{\sqrt{w} \beta_m \mu_y} < \frac{K^n_x}{K^{n+m}_y} < \frac{1 + \varepsilon}{1 - \varepsilon} \frac{\mu_x}{\sqrt{w} \beta_m \mu_y},$$ for all $x, y$. 

\quad  \square
We further have
\[
\frac{(1 - \varepsilon)^2 \mu_x d_a}{(1 + \varepsilon)^2 \mu_y w} < \frac{K^n_x}{K^n_{r+m} L^m_a} < \frac{(1 + \varepsilon)^2 \mu_x d_a}{(1 - \varepsilon)^2 \mu_y w},
\]
which means
\[
\frac{(1 - 5\varepsilon) \mu_x d_a}{\mu_y w} < \frac{K^n_x}{K^n_{r+m} L^m_a} < \frac{(1 + 5\varepsilon) \mu_x d_a}{\mu_y w},
\]
This shows
\[
\|E_A^{n+2m,0} - \sigma^{(n+m)}\|_2 \leq 5C^5 \varepsilon \|\sigma^{(n+m)}\|_2 \leq 6C^5 \varepsilon \|\sigma\|_{st,2}.
\]
This is because we have
\[
(1 - \varepsilon) \|\sigma\|_{st,2} < \|\sigma^{(n)}\|_2 < (1 + \varepsilon) \|\sigma\|_{st,2}
\]
and
\[
(1 - \varepsilon) \|\sigma\|_{st,2} < \|\sigma^{(n+m)}\|_2 < (1 + \varepsilon) \|\sigma\|_{st,2},
\]
since the trace value of the minimal central projection corresponding to the vertex \(x\) in \(A_{2n,0}\) is equal to \(\alpha_n \kappa_x^n \gamma_2^{-2n} \mu_x\) while the trace value of the central projection corresponding to the vertex \(x\) in \(\bigoplus_{x \in V_0} \text{Str}^k_x(G)\) is \(\mu_x^2 / w\) and we have (1) and (2). We now have
\[
\|E_A^{n+2m,0} - \sigma^{(n)}\|_2 - \|\sigma^{(n)}\|_2 < 8C^5 \varepsilon \|\sigma\|_{st,2}.
\]
By Lemma 3.2, we then have
\[
\|E_A^{n+2m,0} - \sigma^{(n)}\|_2 < 4C^3 \sqrt{\varepsilon} \|\sigma\|_{st,2}.
\]
We now have
\[
\|\sigma^{(n+m)} - \sigma^{(n)}\|_2 \leq 10C^5 \sqrt{\varepsilon} \|\sigma\|_{st,2}.
\]
We first choose \(n_1\) so that we have (1) and (2) with \(n = n_1\) and \(\varepsilon = \frac{1}{100C^{10} \cdot 4^l}\). Starting with \(l = 1\), we make the following procedure inductively. We choose \(m_l\) so that we have (3) with \(m = m_l\) and \(\varepsilon = \frac{1}{100C^{10} \cdot 4^l}\) and (2) and (2) with \(n = n_l + m_l\) and \(\varepsilon = \frac{1}{100C^{10} \cdot 4^{l+1}}\). (Note that \(\alpha_p \gamma_2^{-2p} \mu_x \rightarrow \frac{\mu_x^2}{w}\) as \(p \rightarrow \infty\) because we have \(\alpha_p \kappa_p^x \gamma_2^{-2p} \mu_x \rightarrow \frac{\mu_x^2}{w}\) and \(\kappa_p \rightarrow \frac{\mu_x}{\sqrt{w}}\) as \(p \rightarrow \infty\).) We next set \(n_{l+1} = n_l + m_l\).
Then we have

\[ \| \sigma^{(n)} - \sigma^{(n+1)} \|_2 \leq \frac{1}{2^l} \| \sigma \|_{st,2} \]

for all \( \sigma \). Because of this estimate, we know that the sequence \( \{ \sigma^{(n)} \}_l \) converges in \( A_{\infty,k} \) in the strong operator topology for all \( \sigma \). We set \( \Gamma(\bigoplus_{x \in V_0} \mu_x \sigma_x) = \lim_{l \to \infty} \sigma^{(n)} \). Since \( \sigma^{(n)} \in A'_{\infty,0} \cap A_{\infty,k} \), we have \( \Gamma(\bigoplus_{x \in V_0} \mu_x \sigma_x) \in A'_{\infty,0} \cap A_{\infty,k} \). This \( \Gamma \) is clearly a linear map. We have \( \| \Gamma(\bigoplus_{x \in V_0} \mu_x \sigma_x) \|_2 = \| \bigoplus_{x \in V_0} \sigma_x \|_{st,2} \), so \( \Gamma \) is injective. This shows the dimension of the range of \( \tilde{P}^k \) is smaller than or equal to \( \dim(A'_{\infty,0} \cap A_{\infty,k}) \). We thus conclude that the map \( \Delta \) constructed above is a linear isomorphism. (This actually shows that \( \bigoplus_{x \in V_0} \sigma_x \) is a flat field and all \( \sigma^{(n)} \) are equal in \( A_{\infty,k} \).)

**Remark 3.4** The range of the projector matrix product operator of length \( k \) has obvious invariance under rotation of \( 2\pi/k \). This passes to invariance of flat fields of strings of length \( k \) under rotation of \( 2\pi/k \). Such invariance was observed by Ocneanu in early days of the theory and this rotation was called a Fourier transform of a flat field of strings. See [12] for a recent progress of this notion of the Fourier transform.

**Remark 3.5** Replace the initial bi-unitary connection \( W \) with \( W' \). The resulting subfactor \( A_{0,\infty} \subset A_{1,\infty} \) does not change, so the set \( \{a, b, \ldots\} \) of labels of the irreducible bi-unitary connections does not change, but the subfactor \( A_{\infty,0} \subset A_{\infty,1} \) changes to its dual subfactor. So the range of the projector matrix product operator also changes from the higher relative commutant to the dual higher relative commutant of a subfactor in this process.

**Remark 3.6** Recall that the Drinfel’d center of the fusion category of \( A_{0,\infty} - A_{0,\infty} \) bimodules arising from the subfactor \( A_{0,\infty} \subset A_{1,\infty} \) is a modular tensor category related to the 2-dimensional topological order appearing in [2, Sect. 5], as shown in [9, Theorem 3.2]. Note that the higher relative commutants of the *other* subfactor \( A_{\infty,0} \subset A_{\infty,1} \) appear here in this paper. Relations between these two subfactors are clarified in [20, Theorem 3.3].

**Remark 3.7** The range of \( P^k \) does not have a natural algebra structure, but we know from the above Theorem that it has a natural structure of a *-algebra.

**Example 3.8** An almost trivial example is given as follows. All the sets \( V_0, V_1, V_2, V_3 \) are one-point sets and identified with \( \{x\} \). All the graphs \( \mathcal{G}, \mathcal{G}', \mathcal{H}, \mathcal{H}' \) consist of \( d \) multiple edges from \( x \) to \( x \) and they are all identified. We have \( \mu_x = 1, \gamma_1 = \gamma_2 = d \) and the connection \( W \) is given as in Fig. 18.

This is a flat connection, and the set \( V \) is identified with \( \{x\} \). We have \( d_x = 1 \) and \( w = 1 \).

In this case, the range of \( \tilde{P}^k \) is \( M_d(\mathbb{C}) \otimes k \), where \( M_d(\mathbb{C}) \) is the \( d \times d \) full matrix algebra with complex entries.

In this example, the natural \( C^* \)-algebra appearing in the inductive limit is a UHF algebra. In the general case, we have an AF algebra instead.
Fig. 18  An almost trivial example

\[ \begin{array}{c}
\xi_0 \\
W \\
\xi_2 = \delta_{\xi_0,\xi_2} \delta_{\xi_1,\xi_3}.
\end{array} \]

**Example 3.9** An easy example of a bi-unitary connection arises from a finite group \( G \) as in [4, Fig. 10.25]. This corresponds to a trivial 3-cocycle case considered in [2, Sect. 6].

We have \( \gamma_1 = \gamma_2 = \sqrt{|G|} \) and this is a flat connection. The sets \( V_0 \) and \( V \) are both identified with \( G \) as sets. All \( d_a \) are 1 and \( w = |G| \).

**Example 3.10** Consider the example arising from the Dynkin diagram \( A_n \) as in Example 2.3. This is a flat connection and the set \( V \) is identified with \( V_0 \) which consists of \( [(n + 1)/2] \) vertices. The value \( w \) is equal to \( n + 1 \) \( 4 \sin^2 \frac{\pi}{n + 1} \) \[3\]. In this case, the range of \( \tilde{P}^k \) is generated by the Jones projections \( e_1, e_2, \ldots, e_{k-1} \), where the Jones projections are given as in [4, Definition 11.5].

**Remark 3.11** Since our treatments include non-flat bi-unitary connections, our setting looks more general than that in [14]. That is, it seems our 4-tensors give a larger class than those covered in [14], but exact relations between ours and theirs are not clearly understood. It would be interesting to clarify this issue.

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