On Darboux-Bäcklund Transformations for the Q-Deformed Korteweg-de Vries Hierarchy

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Abstract

We study Darboux-Bäcklund transformations (DBTs) for the q-deformed Korteweg-de Vries hierarchy by using the q-deformed pseudodifferential operators. We identify the elementary DBTs which are triggered by the gauge operators constructed from the (adjoint) wave functions of the hierarchy. Iterating these elementary DBTs we obtain not only q-deformed Wronskian-type but also binary-type representations of the tau-function to the hierarchy.

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I. INTRODUCTION

There has been remarkable interest in the theory of soliton ever since the discovery of the Inverse Scattering Method (ISM) for the Korteweg-de Vries (KdV) equation (see, for example, [1]). Although the ISM has been extended to many nonlinear systems which describe phenomena in many branches of science, the KdV equation still plays an important role in the development of modern soliton theory (see, for example [2]). In particular, many concepts were established first for the KdV equation and then generalized to other systems in different ways.

A convenient approach to formulate the KdV hierarchy relies on the use of fractional-power pseudo-differential operators associated with the scalar Lax operator $L = \partial^2 + u$ which was further generalized by Gelfand and Dickey [3] to the $N$-th KdV hierarchy which has Lax operator of the form $L_N = \partial^N + u_{N-1}\partial^{N-1} + \cdots + u_0$. In the past few years, there are several works concerning the extensions of the KdV hierarchy in Lax formulation, such as Drinfeld-Skolev theory [4] and supersymmetric generalizations [5], etc. The common features of these extensions show that they preserve the integrable structure of the KdV hierarchy and contain the KdV as reductions in some limiting cases.

Recently, a new kind of extension called $q$-deformed KdV ($q$-KdV) hierarchy has been proposed and studied [6–14]. In this extension the partial derivative $\partial$ is replaced by the $q$-deformed differential operators ($q$-DO) $\partial_q$ such that

$$\left(\partial_q f(x)\right) = \frac{f(qx) - f(x)}{x(q - 1)}$$

which recovers the ordinary differentiation $(\partial_x f(x))$ as $q$ goes to 1. Even though many structures of the $q$-KdV hierarchy such as infinite conservation laws [6], bi-Hamiltonian structure [7], tau-function [11–14], Vertex operators [14], Virasoro and $W$-algebras [9], etc. have been studied however the $q$-version of Darboux-Bäcklund transformations (DBTs) for this system are still unexplored. It is well known that the DBT is an important property to characterize the integrability of the hierarchy [15]. Thus it is worthwhile to investigate the DBTs associated with the $q$-KdV hierarchy. Once this goal can be achieved, it will deepen our understanding on the soliton solutions of the hierarchy.

Our paper is organized as follows: In Sec. II, we recall the basic facts concerning the $q$-deformed pseudodifferential operators ($q$-PDO) and define the $N$th $q$-deformed Korteweg-de Vries ($q$-KdV) hierarchy. In Sec. III, we construct the Darboux-Bäcklund transformations (DBTs) for the $N$th $q$-KdV hierarchy, which preserve the form of the Lax operator and the hierarchy flows. Iteration of these DBTs generates the $q$-analogue of soliton solutions of the hierarchy. In Sec. IV, the case for $N = 2$ ($q$-KdV hierarchy) is studied in detail to illustrate the $q$-deformed formulation. Concluding remarks are presented in Sec. V.

II. $Q$-DEFORMED PSEUDODIFFERENTIAL OPERATORS

Let us define the $q$-shift operator $\theta$ as

$$\theta(f(x)) = f(qx)$$

(2.1)
then it is easy to show that $\theta$ and $\partial_q$ do not commute but satisfy

$$(\partial_q \theta^k(f)) = q^k \theta^k(\partial_q f), \quad k \in \mathbb{Z} \tag{2.2}$$

Using (1.1) and (2.1) we have $(\partial_q f g) = (\partial_q f)g + \theta(f)(\partial_q g)$ which implies that

$$\partial_q f = (\partial_q f) + \theta(f)\partial_q \tag{2.3}$$

We also denote $\partial_q^{-1}$ as the formal inverse of $\partial_q$ such that

$$\partial_q \partial_q^{-1} f = \partial_q^{-1} \partial_q f = f \quad \text{and hence} \quad \partial_q^{-1} f = \sum_{k \geq 0} (-1)^k q^{-(k+1)/2} \theta^{-k} (\partial_q^k f) \partial_q^{-k}. \tag{2.4}$$

In general one can justify the following $q$-deformed Leibnitz rule:

$$\partial_q^n f = \sum_{k \geq 0} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \theta^{n-k}(\partial_q^k f) \partial_q^{n-k} \quad n \in \mathbb{Z} \tag{2.5}$$

where we introduce the $q$-number and the $q$-binomial as follows

$$[n]_q = \frac{q^n - 1}{q - 1} \tag{2.6}$$

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[1]_q [2]_q \cdots [k]_q}, \quad \left[ \begin{array}{c} n \\ 0 \end{array} \right]_q = 1 \tag{2.7}$$

For a $q$-PDO of the form

$$P = \sum_{i=-\infty}^{\infty} u_i \partial_q^i \tag{2.8}$$

it is convenient to separate $P$ into the differential and integral parts as follows

$$P_+ = \sum_{i \geq 0} u_i \partial_q^i \quad P_- = \sum_{i \leq -1} u_i \partial_q^i \tag{2.9}$$

and denote $(P)_0$ as the zero-th order term of $P$. The residue of $P$ is defined by $\text{res}(P) = u_{-1}$ and the conjugate operation “$*$” for $P$ is defined by $P^* = \sum_i (\partial_q^*)^i u_i$ with

$$\partial_q^* = -\partial_q \theta^{-1} \tag{2.10}$$

Then a straightforward calculation shows that

$$(PQ)^* = Q^* P^* \tag{2.11}$$

where $P$ and $Q$ are any two $q$-PDOs. Finally, for a set of functions $f_1, f_2, \cdots, f_n$, we define $q$-deformed Wronskian determinant $W_q[f_1, f_2, \cdots, f_n]$ as

$$W_q[f_1, f_2, \cdots, f_n] = \begin{vmatrix} f_1 & \cdots & f_n \\ (\partial_q f_1) & \cdots & (\partial_q f_n) \\ \vdots & \ddots & \vdots \\ (\partial_q^{n-1} f_1) & \cdots & (\partial_q^{n-1} f_n) \end{vmatrix} \tag{2.12}$$
With these definitions in hand, we have the following identities which will simplify the computations involving compositions of \( q \)-PDOs.

**Proposition 1:**

\[
(P^*)_+ = (P_+)^*, \quad (P^*)_\_ = (P_\_)^* \tag{2.13}
\]

\[
(\partial_q^{-1} P)_\_ = \partial_q^{-1} (P^*)_0 + \partial_q^{-1} P - P \partial_q^{-1} \tag{2.14}
\]

\[
(P \partial_q^{-1})_\_ = (P_0) \partial_q^{-1} + P \partial_q^{-1} \tag{2.15}
\]

\[
\text{res}(P^*) = -\theta^{-1}(\text{res}(P)), \quad (\partial_q \text{res}(P)) = \text{res}(\partial_q P) - \theta(\text{res}(P \partial_q)) \tag{2.16}
\]

\[
\text{res}(P \partial_q^{-1}) = (P)_0 \quad \text{res}(\partial_q^{-1} P) = \theta^{-1}((P^*)_0) \tag{2.17}
\]

\[
\text{res}(\partial_q^{-1} P_1 P_2 \partial_q^{-1}) = \text{res}(\partial_q^{-1} (P_1^*)_0 P_2 \partial_q^{-1}) + \text{res}(\partial_q^{-1} P_1 (P_2)_0 \partial_q^{-1}) \tag{2.18}
\]

where \( P_1 = (P_1)_+ \) and \( P_2 = (P_2)_+ \).

The \( N \)-th \( q \)-KdV hierarchy is defined, in Lax form, as

\[
\partial_{t_n} L = [B_n, L], \quad n = 1, 2, 3, \ldots \tag{2.19}
\]

with

\[
L = \partial_q^N + u_{N-1} \partial_q^{N-1} + \cdots + u_0, \quad B_n \equiv L_n^{n/(N)} \tag{2.20}
\]

where the coefficients \( u_i \) are functions of the variables \((x, t_1, t_2, \cdots)\) but do not depend on \((t_N, t_{2N}, t_{3N}, \cdots)\).

In fact, we can rewrite the hierarchy equations (2.19) as follows:

\[
\partial_{t_m} B_n - \partial_{t_n} B_m - [B_n, B_m] = 0 \tag{2.21}
\]

which is called the zero-curvature condition and is equivalent to the whole set of equations of (2.19). If we can find a set of functions \( \{u_i, i = 0, 1, \ldots, N-1\} \) and hence a corresponding Lax operator \( L \) (or \( B_n \)) satisfying (2.19) (or (2.21)), then we have a solution to the \( N \)-th \( q \)-KdV hierarchy.

For the Lax operator (2.20), we can formally expand \( L^{1/(N)} \) in powers of \( \partial_q \) as follows

\[
L^{1/(N)} = \partial_q + s_0 + s_1 \partial_q^{-1} + \cdots \tag{2.22}
\]

such that \((L^{1/(N)})^N = L\) which gives all the \( s_i \) being \( q \)-deformed differential polynomials in \( \{u_i\} \). Especially, for the coefficient of \( \partial_q^{N-1} \) we have

\[
u_{N-1} = s_0 + \theta(s_0) + \cdots + \theta^{N-1}(s_0). \tag{2.23}
\]

The Lax equations (2.19) can be viewed as the compatibility condition of the linear system

\[
L \phi = \lambda \phi, \quad \partial_{t_n} \phi = (B_n \phi) \tag{2.24}
\]

where \( \phi \) and \( \lambda \) are called wave function and eigenvalue of the linear system, respectively. On the other hand, we can also introduce adjoint wave function \( \psi \) which satisfies the adjoint linear system

\[
L^* \psi = \mu \psi, \quad \partial_{t_n} \psi = -(B_n^* \psi) \tag{2.25}
\]

For convenience, throughout this paper, \( \phi_i \) (\( \psi_i \)) will stand for (adjoint) wave functions with eigenvalues \( \lambda_i(\mu_i) \), respectively without further mention.
III. ELEMENTARY DARBOUX-BÄCKLUND TRANSFORMATIONS

In this section we would like to construct DBTs for the $N$-th $q$-KdV hierarchy. To attain this purpose, let us consider the following transformation

\[ L \rightarrow L^{(1)} = TLT^{-1} \]  \hspace{1cm} (3.1)

where $T$ is any reasonable $q$-PDO and $T^{-1}$ denotes its inverse. In order to obtain the new solution $(L^{(1)})$ from the old one $(L)$ the gauge operator $T$ can not be arbitrarily chosen. It should be constructed in such a way that the transformed Lax operator $L^{(1)}$ preserves the form of $L$ and satisfies the Lax equation (2.19). From the zero-curvature condition (2.21) point of view, the operator $B_n$ should be transformed according to

\[ B_n \rightarrow B_n^{(1)} = T B_n T^{-1} + \partial_n T T^{-1} \]  \hspace{1cm} (3.2)

which will, in general, not be a pure $q$-DO although the $B_n$ does. However if we suitable choose the gauge operator $T$ such that $B_n^{(1)}$, as defined by (3.2), is also a purely $q$-PDO, then $L_n^{(1)}$ represents a valid new solution to the $N$-th $q$-KdV hierarchy. This is the goal we want to achieve in this letter.

To formulate the DBTs, we follow [16] to introduce a $q$-version of the bilinear potential $\Omega(\phi, \psi)$ which is constructed from a wave function $\phi$ and an adjoint wave function $\psi$. The usefulness of this bilinear potential will be clear when we use it to construct DBTs (see below).

**Proposition 2:** For any pair of $\phi$ and $\psi$, there exists a bilinear potential $\Omega(\phi, \psi)$ satisfying

\[ (\partial_q \Omega(\phi, \psi)) = \phi \psi \]  \hspace{1cm} (3.3)

\[ \partial_n \Omega(\phi, \psi) = \text{res} (\partial_q^{-1} \psi B_n \phi \partial_q^{-1}) \]  \hspace{1cm} (3.4)

In fact the bilinear potential $\Omega(\phi, \psi)$ can be formally represented by a $q$-integration as $\Omega(\phi, \psi) = (\partial_q^{-1} \phi \psi)$.

Motivated by the DBTs for the ordinary KdV [15] (or Kadomtsev-Petviashvili (KP) [16-18]) hierarchy, we can construct a qualified gauge operator $T$ as follows

\[ T_1 = \theta(\phi_1) \partial_q \phi_1^{-1} = \partial_q - \alpha_1, \quad \alpha_1 \equiv \frac{(\partial_q \phi_1)}{\phi_1} \]  \hspace{1cm} (3.5)

where $\phi_1$ is a wave function associated with the linear system (2.24). It is not hard to show that the transformed Lax operator $L^{(1)}$ is a purely $q$-PDO with order $N$ and the Lax equation (2.19) transforms covariantly, i.e. $\partial_n L^{(1)} = [(L^{(1)})_{n/N}, L^{(1)}]$. The transformed coefficients $\{u_i^{(1)}\}$ then can be expressed in terms of $\{u_i\}$ and $\phi_1$. On the other hand, for a given generic wave function $\phi \neq \phi_1$ (or adjoint function $\psi$) its transformed result can be expressed in terms of $\phi_1$ and itself.

**Proposition 3:** Suppose $\phi_1$ is a wave function of the linear system (2.24). Then the gauge operator $T_1$ triggers the following DBT:
\[
L^{(1)} = T_1LT_1^{-1} = \partial_q^N + u_1^{(1)} \partial_q^{N-1} + \cdots + u_0^{(1)}
\]
(3.6)

\[
\phi^{(1)} = (T_1 \phi) = \frac{W_q[\phi_1, \phi]}{\phi_1}, \quad \phi \neq \phi_1
\]
(3.7)

\[
\psi^{(1)} = ((T_1^{-1})^* \psi) = -\frac{\theta(\Omega(\phi_1, \psi))}{\theta(\phi_1)}
\]
(3.8)

where \(L^{(1)}, \phi^{(1)}\) and \(\psi^{(1)}\) satisfy (2.19), (2.24) and (2.25) respectively.

Just like the DBTs for the ordinary KdV hierarchy, the DBT triggered by the gauge operator \(T_1\) is by no means the only transformation in this \(q\)-analogue framework. We have another construction of DBT by using the adjoint wave function associated with the adjoint linear system (2.23). Indeed, for a given adjoint wave function \(\psi_1\) we can construct a gauge operator

\[
S_1 = \theta^{-1}(\psi_1^{-1}) \partial_q^{-1} \psi_1 = (\partial_q + \beta_1)^{-1}, \quad \beta_1 \equiv \frac{\partial_q \theta^{-1}(\psi_1)}{\psi_1}
\]
(3.9)

which preserves the form of the Lax operator and the Lax equation.

**Proposition 4:** Suppose \(\psi_1\) is an adjoint wave function of the adjoint linear system (2.25). Then the gauge operator \(S_1\) triggers the following adjoint DBT:

\[
L^{(1)} = S_1LS_1^{-1}
\]
(3.10)

\[
\phi^{(1)} = (S_1 \phi) = \frac{\Omega(\phi, \psi_1)}{\theta^{-1}(\psi_1)}
\]
(3.11)

\[
\psi^{(1)} = ((S_1^{-1})^* \psi) = \frac{\tilde{W}_q[\psi_1, \psi]}{\psi_1}, \quad \psi \neq \psi_1
\]
(3.12)

where \(L^{(1)}, \phi^{(1)}\) and \(\psi^{(1)}\) satisfy (2.19), (2.24) and (2.25), respectively and \(\tilde{W}_q\) is obtained from \(W_q\) by replacing \(\partial_q\) with \(\partial_q^*\).

So far, we have constructed two elementary DBTs triggered by the gauge operators \(T_1\) and \(S_1\). Regarding them as the building blocks, we can construct more complicated transformations from the compositions of these elementary DBTs. However, we will see that it is convenient to consider a DBT followed by an adjoint DBT and vice versa because such combination will frequently appear in more complicated DBTs. So let us compose them to form a single operator \(R_1\) which we call binary gauge operator. The construction of the binary gauge operator \(R_1\) can be realized as follows: first we perform a DBT triggered by the gauge operator \(T_1 = \theta(\phi_1)\partial_q \phi_1^{-1}\) and the adjoint wave function \(\psi_1\) is thus transformed to \(\psi_1^{(1)} = ((T_1^{-1})^* \psi_1) = -\theta(\phi_1^{-1})\theta(\Omega(\phi_1, \psi_1))\). Then a subsequent adjoint DBT triggered by \(S_1^{(1)} = \theta^{-1}((\psi_1^{(1)})^{-1}) \partial_q^{-1}(\psi_1^{(1)})\) is performed and the composition of these two transformations gives

\[
R_1 = S_1^{(1)}T_1 = 1 - \phi_1 \Omega(\phi_1, \psi_1)^{-1} \partial_q^{-1} \psi_1
\]
(3.13)

**Proposition 5:** Let \(\phi_1\) and \(\psi_1\) be wave function and adjoint wave function associated with the linear systems (2.24) and (2.25), respectively. Then the gauge operator \(R_1\) triggers the following binary DBT:
\[ L^{(1)} = R_1 L R_1^{-1} \]
\[ \phi^{(1)} = (R \phi) = \phi - \Omega(\phi_1, \psi_1)^{-1} \phi_1 \Omega(\phi, \psi_1), \quad \phi \neq \phi_1 \]
\[ \psi^{(1)} = ((R^{-1})^* \psi) = \psi - \theta(\Omega(\phi_1, \psi_1)^{-1}) \psi_1 \theta(\Omega(\phi_1, \psi)), \quad \psi \neq \psi_1 \]

where \( L^{(1)} \), \( \phi^{(1)} \) and \( \psi^{(1)} \) satisfy (3.13), (3.24) and (3.25) respectively.

We would like to remark that the construction of the binary gauge operator \( R_1 \) is independent of the order of the gauge operators \( T \) and \( S \). If we apply the gauge operator \( S_1 \) followed by \( T^{(1)}_1 \), then a direct calculation shows that \( R = T^{(1)}_1 S_1 \) has the same form as (3.13).

The remaining part of this section is to consider the iteration of the DBTs by using the DBT, the adjoint DBT, and the binary DBT triggered by the gauge operators \( T \), \( S \), and \( R \), respectively. For example, by iterating the DBT triggered by the gauge operator \( T \), we can express the solution of the \( N \)-th \( q \)-KdV hierarchy through the \( q \)-deformed Wronskian representation. This construction starts with \( n \) wave functions \( \phi_1, \phi_2, \ldots, \phi_n \) of the linear system (2.24). Using \( \phi_1 \), say, to perform the first DBT of Proposition 1, then all \( \phi_i \) are transformed to \( \phi^{(1)}_i = (T_1 \phi_i) \). Obviously, we have \( \phi^{(1)}_1 = 0 \). The next step is to perform a subsequent DBT triggered by \( \phi^{(1)}_2 \), which leads to the new wave functions \( \phi^{(2)}_i \) with \( \phi^{(2)}_2 = 0 \). Iterating this process such that all the wave functions are exhausted, then an \( n \)-step DBT triggered by the gauge operator \( T_n = (\partial_q - \alpha^{(n-1)}_n)(\partial_q - \alpha^{(n-2)}_{n-1}) \cdots (\partial_q - \alpha_1) \) is obtained, where \( \alpha^{(j)}_i \equiv (\partial_q \phi^{(j)}_i)/\phi^{(j)}_i \). It is easy to see that \( T_n \) is an \( n \)-th-order \( q \)-DO of the form \( T_n = \partial_q^n + a_{n-1} \partial_q^{n-1} + \cdots + a_0 \) with \( a_i \) defined by the conditions \( (T_n \phi_j) = 0, j = 1, 2, \ldots, n \).

Following the Cramer’s formula it turns out that \( a_i = -W_q(\phi_1, \ldots, \phi_i, \phi)/W_q(\phi_1, \ldots, \phi_n) \) where \( W_q(\phi_i) \) is obtained from \( W_q \) with its \( i \)-th row replaced by \( (\partial_q^n \phi_1, \ldots, \partial_q^n \phi_n) \). This implies that the \( n \)-step transformed wave function \( \phi^{(n)}(\phi \neq \phi_i) \) is given by

\[ \phi^{(n)} = (T_n \phi) = \frac{W_q(\phi_1, \ldots, \phi_n, \phi)}{W_q(\phi_1, \ldots, \phi_n)} \]

and the \( n \)-step gauge operator \( T_n \) can be expressed as

\[ T_n = \frac{1}{W_q(\phi_1, \ldots, \phi_n)} \left| \begin{array}{cccc} \phi_1 & \cdots & \phi_n & 1 \\ (\partial_q \phi_1) & \cdots & (\partial_q \phi_n) & \partial_q \\ \vdots & \ddots & \vdots & \vdots \\ (\partial_q^n \phi_1) & \cdots & (\partial_q^n \phi_n) & \partial_q^n \end{array} \right| \]

where it should be realized that in the expansion of the determinant by the elements of the last column, \( \partial_q^n \) have to be written to the right of the minors.

Next let us turn to the iteration of the adjoint DBT. In this case, the \( n \)-step gauge operator can be constructed in a similar manner by preparing \( n \) initial adjoint wave functions \( \psi_1, \ldots, \psi_n \) such that \( S_n^{-1} = (\partial_q + \beta^{(n-1)}_n)(\partial_q + \beta^{(n-2)}_{n-1}) \cdots (\partial_q + \beta_1) = \partial_q^n + \sum_{i=1}^{n-1} \partial_q^i b_i \). Using the required conditions \( ((S_n^{-1})^* \psi_i) = 0, j = 1, \ldots, n \) we obtain \( b_i = -W_q(\psi_1, \ldots, \psi_n)/W_q(\psi_1, \ldots, \psi_n) \) which give the \( n \)-step transformed adjoint wave function

\[ \psi^{(n)} = ((S_n^{-1})^* \psi) = \frac{W_q(\psi_1, \ldots, \psi_n, \psi)}{W_q(\psi_1, \ldots, \psi_n)} \]
and the gauge operator

\[ (S_n^{-1})^* = \frac{1}{W_q[\psi_1, \ldots, \psi_n]} \begin{vmatrix} \psi_1 & \cdots & \psi_n & 1 \\ (\partial_q^* \psi_1) & \cdots & (\partial_q^* \psi_n) & \partial_q^* \\ \vdots & \vdots & \vdots & \vdots \\ ((\partial_q^*)^n \psi_1) & \cdots & ((\partial_q^*)^n \psi_n) & (\partial_q^*)^n \end{vmatrix} \] (3.20)

Finally, we shall construct \( n \)-step binary DBT by preparing \( n \) wave functions \( \phi_i \) and \( n \) adjoint wave functions \( \psi_i \) at the beginning. Then we perform the first binary DBT by using the gauge operator \( R_1 \) which is constructed from \( \phi_1 \) and \( \psi_1 \) as in (3.13). At the same time, all \( \phi_i \) and \( \psi_i \) are transformed to \( \phi_i^{(1)} = (R_1 \phi_i) \) and \( \psi_i^{(1)} = ((R_1)^{-1} \psi_i) \), respectively except \( \phi_1^{(1)} = \psi_1^{(1)} = 0 \).

We then use the pair \( \{\phi_2^{(1)}, \psi_2^{(1)}\} \) to perform the next binary DBT. Iterating this process until all the pairs \( \{\phi_i, \psi_i\} \) are exhausted, then we obtain an \( n \)-step gauge operator of the form \( R_n = 1 - \sum_{j=1}^{n-1} c_j \partial_q^{-1} \psi_j \). Solving the conditions \( (R_n \phi_j) = 0, j = 1, \ldots, n \), we obtain the coefficients \( c_j = \det \Omega^{(i)}/\det \Omega \) where \( \Omega_{ij} \equiv \Omega(\phi_i, \psi_j) \) and \( \Omega^{(i)} \) is obtained from \( \Omega \) with its \( i \)-th column replaced by \( (\phi_i, \ldots, \phi_n)^t \). This leads to the following representations for \( R_n \):

\[ R_n = \frac{1}{\det \Omega} \begin{vmatrix} \Omega_{11} & \cdots & \Omega_{1n} & \phi_1 \\ \vdots & \vdots & \vdots & \vdots \\ \Omega_{n1} & \cdots & \Omega_{nn} & \phi_n \\ \partial_q^{-1} \psi_1 & \cdots & \partial_q^{-1} \psi_n & 1 \end{vmatrix} \] (3.21)

Moreover, the \( n \)-step transformed bilinear potential \( \Omega(\phi^{(n)}, \psi^{(n)}) \) can be expressed in terms of binary-type determinant as

\[ \Omega(\phi^{(n)}, \psi^{(n)}) = \frac{1}{\det \Omega} \begin{vmatrix} \Omega_{11} & \cdots & \Omega_{1n} & \Omega(\phi_1, \psi) \\ \vdots & \vdots & \vdots & \vdots \\ \Omega_{n1} & \cdots & \Omega_{nn} & \Omega(\phi_n, \psi) \\ \Omega(\phi, \psi_1) & \cdots & \Omega(\phi, \psi_n) & \Omega(\phi, \psi) \end{vmatrix} \] (3.22)

where \( \phi^{(n)} = (R_n \phi) \) and \( \psi^{(n)} = ((R_n)^{-1} \psi) \).

**IV. Q-DEFORMED KDV HIERARCHY (N = 2)**

This section is devoted to illustrate the DBTs for the simplest nontrivial example: \( q \)-deformed KdV hierarchy \((N = 2)\) case. Let

\[ L = \partial_q^2 + u_1 \partial_q + u_0 \] (4.1)

then the Lax equations

\[ \partial_{t_n} L = [L^{n/2}, L], \quad n = 1, 3, 5, \ldots \] (4.2)

define the evolution equations for \( u_1 \) and \( u_0 \). In particular, for the \( t_1 \)-flow, we have

\[ \partial_{t_1} u_1 = x(q - 1) \partial_{t_1} u_0 \] (4.3)
\[ \partial_{t_1} u_0 = (\partial_q u_0) - (\partial_q^2 s_0) - (\partial_q s_0^2) \] (4.4)
which is nontrivial and recovers the ordinary case as $q$ goes to 1. For higher hierarchy flows
the evolution equations for $u_1$ and $u_0$ become more complicated due to the non-commutative
nature of the $q$-deformed formulation.

We now perform the DBT of Proposition 3 to the Lax operator (4.1), then the transformed coefficients become

$$
u_1^{(1)} = \theta(u_1) - \alpha_1 + \theta^2(\alpha_1) \quad (4.5)$$

$$
u_0^{(1)} = \theta(u_0) + (\partial_q u_1) + (q + 1)\theta(\partial_q \alpha_1) - \alpha_1 u_1 + \theta(\alpha_1) u_1^{(1)} \quad (4.6)$$

Since $\phi_1$ is a wave function associated with the Lax operator (4.1), i.e. $L\phi_1 = \lambda_1 \phi_1$, one can
easily verify that $(\partial_q \alpha_1) + \theta(\alpha_1) + \alpha_1 u_1 + u_0 = \lambda_1$ and hence Eqs.(4.5) and (4.6) can be
simplified as

$$
u_1^{(1)} - u_1 = x(q - 1)(\nu_0^{(1)} - u_0) \quad (4.7)$$

$$
u_0^{(1)} - u_0 = \partial_q(u_1 + \alpha_1 + \theta(\alpha_1)) \quad (4.8)$$

Furthermore, using the facts that $\partial_t \phi_1 = (L_+^{1/2} \phi_1) = (\partial_q \phi_1) + s_0 \phi_1$ and $u_1 = \theta(s_0) + s_0$, we
can rewrite (4.8) as

$$
u_0^{(1)} = u_0 + \partial_q \partial_t_1 \ln \phi_1 \theta(\phi_1) \quad (4.9)$$

Eqs.(4.7) and (4.9) are just the desired result announced in Sec. III.

Similarly, by applying the above analysis to the adjoint and binary DBTs, we obtain the
following result:

**Proposition 6:** Let $\phi_1$ and $\psi_1$ be (adjoint) wave function associated with the Lax
operator (4.1). Then under the DBT, adjoint DBT, and binary DBT, the transformed coefficients are given by

$$
u_1^{(n)} - u_1 = x(q - 1)(\nu_0^{(n)} - u_0) \quad (4.10)$$

with

$$
u_0^{(1)} = u_0 + \partial_q \partial_t_1 \ln \phi_1 \theta(\phi_1), \quad (DBT) \quad (4.11)$$

$$
u_0^{(1)} = u_0 + \partial_q \partial_t_1 \ln \psi_1 \theta^{-1}(\psi_1), \quad (adjoint DBT) \quad (4.12)$$

$$
u_0^{(1)} = u_0 + \partial_q \partial_t_1 \ln \Omega_{11} \theta(\Omega_{11}), \quad (binary DBT) \quad (4.13)$$

Eqs.(4.11)-(4.13) effectively represent the 1-step transformations. To obtain the $n$-step DBT,
adjoint DBT and binary DBT we just need to iterate the corresponding 1-step transformations successively by inserting the triggered wave function (3.17), adjoint wave function (3.19) and bilinear potential (3.22) into the logarithm in Eqs. (4.11), (4.12) and (4.13), respectively.

**Proposition 7:** Let $\phi_i$ and $\psi_i$ ($i = 1, 2, \cdots, n$) be (adjoint) wave functions associated
with the Lax operator (4.1). Then under the successive DBT, adjoint DBT and binary DBT
of Proposition 6, the $n$-step transformed coefficients are given by

$$u_1^{(n)} - u_1 = x(q - 1)(\nu_0^{(n)} - u_0) \quad (4.14)$$
with

\[ u_0^{(n)} = u_0 + \partial_q \partial_{t_1} \ln W_q[\phi_1, \cdots, \phi_n] \theta(W_q[\phi_1, \cdots, \phi_n]), \quad \text{(DBT)} \tag{4.15} \]

\[ u_0^{(n)} = u_0 + \partial_q \partial_{t_1} \ln \tilde{W}_q[\psi_1, \cdots, \psi_n] \theta^{-1}(\tilde{W}_q[\psi_1, \cdots, \psi_n]), \quad \text{(adjoint DBT)} \tag{4.16} \]

\[ u_0^{(n)} = u_0 + \partial_q \partial_{t_1} \ln \det \Omega \theta(\det \Omega), \quad \text{(binary DBT)} \tag{4.17} \]

Eqs. (4.15)-(4.17) provide us a convenient way to construct new solutions from the old ones. Especially, starting from the trivial solution \((u_1 = u_0 = 0)\) we can obtain nontrivial multi-soliton solutions just by putting the (adjoint) wave functions into the formulas (4.15)-(4.17).

For example, the wave functions \(\phi_i (i = 1, \cdots, n)\) associated with the trivial Lax operator \(L = \partial_q^2\) satisfy

\[ \partial_q^2 \phi_i = p_i^2 \phi_i, \quad \partial_q \phi_i = (\partial_q^n \phi_i) \quad p_i \neq p_j \tag{4.18} \]

which give the following solutions

\[ \phi_i(x, t) = E_q(p_i x) \exp\left(\sum_{k=0}^{\infty} p_i^{2k+1} t_{2k+1}\right) + \gamma_i E_q(-p_i x) \exp\left(-\sum_{k=0}^{\infty} p_i^{2k+1} t_{2k+1}\right) \tag{4.19} \]

where \(\gamma_i\) are constants and \(E_q(x)\) denotes the \(q\)-exponential function which satisfies \(\partial_q E_q(px) = pE_q(px)\) and has the following representation:

\[ E_q(x) = \exp\left[\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k\right] \tag{4.20} \]

Substituting (4.19) into (4.15)-(4.17) gives us \(n\)-solution solutions. Note that for the soliton solutions constructed out from the trivial one \((u_1 = u_0 = 0)\) as described above, they satisfy a simple relation: \(u_1^{(n)} = x(q-1)u_0^{(n)}\). This is just the case considered by Haine and Iliev in [11,13]. In general, it can be shown [14] that the solutions of the \(q\)-KdV hierarchy can be expressed in terms of a single function \(\tau_q\) called tau-function such that

\[ u_1(x, t) = \partial_{t_1} \ln \frac{\theta^2(\tau_q)}{\tau_q} = x(q-1)\partial_q \partial_{t_1} \ln \tau_q(x, t) \theta(\tau_q(x, t)) \tag{4.21} \]

Eq. (4.21) together with Proposition 7 shows that for a given solution (or \(\tau_q\)) of the \(q\)-KdV hierarchy, the transformation properties of \(\tau_q\) can be summarized as follows

\[ \tau_q \rightarrow \tau_q^{(n)} = W_q[\phi_1, \cdots, \phi_n] \cdot \tau_q \quad \text{(DBT)} \]

\[ \tau_q \rightarrow \tau_q^{(n)} = \theta^{-1}(\tilde{W}_q[\psi_1, \cdots, \psi_n]) \cdot \tau_q \quad \text{(adjoint DBT)} \tag{4.22} \]

\[ \tau_q \rightarrow \tau_q^{(n)} = \det \Omega \cdot \tau_q \quad \text{(binary DBT)} \]

which implies that the Wronskian-type (or binary-type) tau-function can be viewed as the \(n\)-step transformed tau-function constructed from the trivial solution \((\tau_q = 1)\).
V. CONCLUDING REMARKS

We have constructed the elementary DBTs for the $q$-KdV hierarchy, which preserve the form of the Lax operator and are compatible with the Lax equations. Iterated application of these elementary DBTs produces new soliton solutions (tau-functions) of the $q$-KdV hierarchy out of given ones. Following the similar treatment and using the tau-function representation $u_{N-1} = \partial_{t_1} \ln \theta^N(\tau_q)/\tau_q$ [14] for $N > 2$ we can reach the same result as (4.22) except now $\partial \tau_q/\partial t_{iN} = 0$.

In fact these DBTs can be applied to the $q$-deformed KP hierarchy without difficulty by considering the Lax operator of the form $L = \partial_q + \sum_{i=0}^{\infty} u_i \partial_{q^{-i}}$. The $q$-KdV is just a reduction of the $q$-KP by imposing the condition $(L^N)_+ = L^N$. Since the ordinary KP hierarchy admits other reductions which are also integrable. Hence it is quite natural to ask whether there exist $q$-analogue of these reductions and what are the DBTs associated with them? We hope we can answer this question in the near future.

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