Self-Intersection Local Times of Generalized Mixed Fractional Brownian Motion as White Noise Distributions

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Abstract. The generalized mixed fractional Brownian motion is defined by taking linear combinations of a finite number of independent fractional Brownian motions with different Hurst parameters. It is a Gaussian process with stationary increments, possesses self-similarity property, and, in general, is neither a Markov process nor a martingale. In this paper we study the generalized mixed fractional Brownian motion within white noise analysis framework. As a main result, we prove that for any spatial dimension and for arbitrary Hurst parameter the self-intersection local times of the generalized mixed fractional Brownian motions, after a suitable renormalization, are well-defined as Hida white noise distributions. The chaos expansions of the self-intersection local times in the terms of Wick powers of white noises are also presented.

1. Introduction
The notion of mixed fractional Brownian motion was introduced by Cheridito in [2]. Let $a$ and $b$ be two real numbers such that $(a, b) \neq (0, 0)$. A mixed fractional Brownian motion (MFBM) $M^H := (M^H_t)_{t \geq 0} := (M^H_{a,b})_{t \geq 0}$ of parameter $H$, $a$, and $b$ is a stochastic process defined on some probability space by $M^H_t = M^H_{a,b} := aB^H_t + bB^H_t$, where $(B^H_t)_{t \geq 0}$ is a Brownian motion and $(B^H_t)_{t \geq 0}$ is an independent fractional Brownian motion of Hurst parameter $H \in (0, 1)$. This process was introduced to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. Some stochastic and analytic properties of MFBM was investigated by Zili in [14].

MFBM has been further generalized by Thäle in [13] to an arbitrary finite linear combinations of independent fractional Brownian motions. In this paper we shall name this process generalized mixed fractional Brownian motion. Let $\alpha_1, \ldots, \alpha_n$, $n \in \mathbb{N}$ be real numbers and not all $\alpha_k$ equals zero. A generalized mixed fractional Brownian motion (GMFBM) of parameter $H = (H_1, \ldots, H_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a stochastic process $Z^H := (Z^H_t)_{t \geq 0} := (Z^H_{\alpha})_{t \geq 0}$ defined on some probability space by

$$Z^H_t = Z^H_{\alpha} := \sum_{k=1}^{n} \alpha_k B^H_k,$$

where $(B^H_k)_{t \geq 0}$, $k = 1, \ldots, n$ are independent fractional Brownian motions of Hurst parameter $H_k \in (0, 1)$. Research in the internet traffic modeling using self-similar process shows the need
of study of GMFBM, see e.g. [4]. Below we collect some basic properties of the GMFBM. For proofs and further comments on the importance of this process see [13] and references therein.

**Proposition 1.1.** The GMFBM $Z^H = (Z_t^{H, \alpha})_{t \geq 0}$ is a centered Gaussian process with variance $\sum_{k=1}^{n} \alpha_k^2 t^{2H_k}$ and covariance function

$$
\mathbb{E}(Z_t^{H, \alpha} Z_s^{H, \alpha}) = \frac{1}{2} \sum_{k=1}^{n} \alpha_k^2 (t^{2H_k} + s^{2H_k} - |t-s|^{2H_k}).
$$

$Z^H$ has stationary increments and they are uncorrelated if and only if $H_k = \frac{1}{2}$ for all k. $Z^H$ is also $(c_1, \ldots, c_n; H_1, \ldots, H_n)$-self-similar in the sense that $\sum_{k=1}^{n} \alpha_k c_k^{H_k} B_{c_k t} = \sum_{k=1}^{n} \alpha_k B_k^{H_k}$ in law. $Z^H$ is neither a Markov process nor a semimartingale, unless $H_k = \frac{1}{2}$ for all k. $Z^H$ exhibits a long-range dependence if and only if there exists $k$ with $H_k > \frac{1}{2}$. For all $T > 0$, with probability one $Z^H$ has a version, the sample path of which are Hölder continuous of order $\gamma < \min_{1 \leq k \leq n} H_k$ on the interval $[0, T]$. Every sample path of $Z^H$ is almost surely nowhere differentiable.

In this paper we focus on analyzing self-intersection local time of GMFBM using white noise theory. More precisely, we will study GMFBM on a specific probability space namely white noise space. The self-intersection local time of GMFBM $Z^H$ is formally defined by the Tanaka formula

$$
L(T) := \int_0^T \int_0^T \delta(Z_t^H - Z_s^H) \, ds \, dt,
$$

where $\delta$ is the Dirac delta function. Intuitively it measures the amount of time that the process spends intersecting itself on the time interval $[0, T]$. A rigorous definition of this object may be obtained by approximating the Dirac delta function by the heat kernel

$$
p_{\varepsilon}(x) := \frac{1}{(2\pi \varepsilon)^{1/2}} e^{-\frac{1}{2\varepsilon} |x|^2},
$$
as $\varepsilon > 0$ tends to zero. The first investigation related to this topic within white noise analysis framework was done by Faria et al [3] for the case $\alpha_1 = 1, \alpha_2 = \cdots = \alpha_n = 0$, and $H = \frac{1}{2}$ (Brownian motion). Drumond et al in [5] have extended this approach for the case $\alpha_1 = 1, \alpha_2 = \cdots = \alpha_n = 0$, and $H \in (0, 1)$ (fractional Brownian motion). In this present work we are able to give meaning to $L^{(N)}$, i.e. a renormalized counterpart of $L(T)$, as a well-defined object in some white noise distribution space. Moreover, we find the chaos expansion of $L^{(N)}$ in terms of Wick powers of white noise. This allows us to define the kernel for $L^{(N)}$.

2. **White Noise Analysis of GMFBM**

We begin with some basic concepts of white noise theory used throughout this paper. For a more comprehensive explanation including various applications of white noise analysis, see [7, 8, 9]. In particular, for a survey on Gaussian white noise analysis and its application to Feynman’s approach to quantum mechanics we refer to [12].

In the first part of this section we will recall a way to represent GMFBM as a generalized random variable on the white noise space. Firstly, we summarize the construction of a fractional Brownian motion in the white noise space. This approach was first introduced by Bender in [1]. Since GMFBM is a linear combination of independent fractional Brownian motions, its realization in the white noise space can be easily derived. For this purpose we follow closely the approach developed by Drumond et al in [5]. For a more detail on white noise approach of GMFBM see [10].
Let \((S'_d(\mathbb{R}), C, \mu)\) be the vector-valued white noise space, i.e., \(S'_d(\mathbb{R})\) is the space of vector-valued tempered distribution, \(C\) is the Borel \(\sigma\)-algebra generated by cylinder sets in \(S'_d(\mathbb{R})\), and the probability measure \(\mu\) is uniquely determined by the Bochner-Minlos theorem such that its characteristic function given by the probability measure \(\mu\) valued tempered distribution, \(\mathcal{C}\)

\[
\langle \cdot, \cdot \rangle_{\text{pairing}} \text{ real Hilbert space of all vector-valued Lebesgue square-integrable functions} \]

\[
\langle \vec{g}, \vec{f} \rangle = \sum_{j=1}^{d} \int_{\mathbb{R}} g_j(x)f_j(x) \, dx,
\]

for all \(\vec{g} = (g_1, \ldots, g_d) \in L^2_d(\mathbb{R})\) and \(\vec{f} = (f_1, \ldots, f_d) \in S_d(\mathbb{R})\). We should remark that we have the following Gel'fand triple

\[
S_d(\mathbb{R}) \subset L^2_d(\mathbb{R}) \subset S'_d(\mathbb{R}).
\]

It is well known that in the white noise space a version of the \(d\)-dimensional Brownian motion is given by the stochastic process \((B_t)_{t \geq 0}\) with

\[
B_t := \left( \langle \cdot, 1_{[0,t]} \rangle, \cdots, \langle \cdot, 1_{[0,t]} \rangle \right),
\]

such that

\[
B_t(\bar{\omega}) := \left( \langle \omega_1, 1_{[0,t]} \rangle, \cdots, \langle \omega_d, 1_{[0,t]} \rangle \right), \quad \bar{\omega} = (\omega_1, \ldots, \omega_d) \in S'_d(\mathbb{R})
\]

where \(1_A\) denotes the indicator function on a set \(A\).

A representation of fractional Brownian motion in term of indicator function is constructed using the concept of Weyl's fractional integral and Marchaud's fractional derivative. Precisely speaking, for an arbitrary Hurst parameter \(H \in (0, 1)\), a version of a \(d\)-dimensional fractional Brownian motion is given by

\[
B^H_t(\bar{\omega}) := \left( \langle \omega_1, N^H(t1_{[0,t]}) \rangle, \cdots, \langle \omega_d, N^H(t1_{[0,t]}) \rangle \right), \quad \bar{\omega} = (\omega_1, \ldots, \omega_d) \in S'_d(\mathbb{R})
\]

where for a real-valued function \(f\) the operator \(N^H\) is defined by

\[
N^H f := \begin{cases} 
\left( \frac{1}{\Gamma(H+\frac{1}{2})} \right)^H \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{f(x)-f(x+y)}{y^{\frac{1}{2}-H}} \, dy, & \text{if } H \in (0, \frac{1}{2}) \\
\frac{K_H}{\Gamma(H+\frac{3}{2})} \int_{-\infty}^{\infty} f(y)(y-x)^{H-\frac{3}{2}} \, dy, & \text{if } H = \frac{1}{2} \\
\frac{K_H}{\Gamma(H+\frac{3}{2})} \int_{-\infty}^{\infty} f(y)(y-x)^{H-\frac{3}{2}} \, dy, & \text{if } H \in (\frac{1}{2}, 1)
\end{cases}
\]

provided the integrals exist for almost all \(x \in \mathbb{R}\). Here \(\Gamma\) denotes Gamma function and \(K_H\) is the normalizing constant. We also consider the operator \(N^H_+\) defined by

\[
N^H_+ f := \begin{cases} 
\left( \frac{1}{\Gamma(H+\frac{1}{2})} \right)^H \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{f(x)-f(x-y)}{y^{\frac{1}{2}-H}} \, dy, & \text{if } H \in (0, \frac{1}{2}) \\
\frac{K_H}{\Gamma(H-\frac{1}{2})} \int_{-\infty}^{x} f(y)(x-y)^{H-\frac{3}{2}} \, dy, & \text{if } H = \frac{1}{2} \\
\frac{K_H}{\Gamma(H-\frac{1}{2})} \int_{-\infty}^{x} f(y)(x-y)^{H-\frac{3}{2}} \, dy, & \text{if } H \in (\frac{1}{2}, 1)
\end{cases}
\]
For \( H = (H_1, \ldots, H_n) \), and \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( H_k \in (0, 1) \), \( \alpha_k \in \mathbb{R} \), \( n \in \mathbb{N} \) a version of a \( d \)-dimensional GMFBM of parameter \( H \) and \( \alpha \) in the white noise analysis setting is given by

\[
Z_t^{H, \alpha}(\vec{\omega}) := \left( \left\langle \omega_1, \sum_{k=1}^n \alpha_k N^{-H_k}_t 1_{[0,t]} \right\rangle, \ldots, \left\langle \omega_d, \sum_{k=1}^n \alpha_k N^{H_k}_t 1_{[0,t]} \right\rangle \right),
\]

for \( \vec{\omega} = (\omega_1, \ldots, \omega_d) \in S'_d(\mathbb{R}) \). Within this formalism we can choose a continuous version of the process according to Kolmogorov-Centsov theorem. At this point we should also emphasize that the white noise analysis approach in defining GMFBM has the advantage that the underlying probability space does not depend on the Hurst parameter under consideration.

It is known that for some type of functions, for example \( f = 1_{[0,t]}t > 0 \) or \( f \in S_1(\mathbb{R}) \), \( N^H_- \) and \( N^H_+ \) are dual operators in the sense that the following integration by parts formula holds

\[
\int_{\mathbb{R}} f(x) N^H_- g(x) \, dx = \int_{\mathbb{R}} (N^H_+ f)(x) g(x) \, dx.
\]

Moreover, the following estimation was proved in [5] and will be very useful for our purpose.

**Lemma 2.1.** If \( H \in (0, 1) \) and \( f \in S_1(\mathbb{R}) \), then there exists a nonnegative constant \( C_H \), independent of \( f \), such that

\[
\left| \int_{\mathbb{R}} f(x) N^H_1 1_{[s,t]}(x) \, dx \right| \leq C_H (t-s) \left( \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \in \mathbb{R}} |f'(x)| + |f|_0 \right)
\]

for all \( s < t \).

In the second part of this section we will review the concept of Hida distribution. Consider \( L^2(\mu) := L^2(S'_d(\mathbb{R}), C, \mu) \) as a complex Hilbert space. This space is unitary isomorphic to the Fock space of symmetric square-integrable function, i.e.

\[
L^2(\mu) \cong \left( \bigoplus_{k=0}^{\infty} \text{sym} L^2(\mathbb{R}^k, k! dx^k) \right) ^{\otimes d}.
\]

This will lead to the chaos expansion of an element \( F \in L^2(\mu) \), i.e.

\[
F(\omega_1, \ldots, \omega_d) = \sum_{(m_1, \ldots, m_d) \in \mathbb{N}^d_{\geq 0}} \left\langle : \omega_1^{\otimes m_1} : \otimes \ldots \otimes : \omega_d^{\otimes m_d} : , \vec{f}_{(m_1, \ldots, m_d)} \right\rangle,
\]

with kernel functions \( \vec{f}_{(m_1, \ldots, m_d)} \) of the \( m \)-th chaos are in the Fock space. Here \( : \omega^{\otimes n} : \) denotes the \( m \)-th Wick power of \( \omega \in S'_1(\mathbb{R}) \), see e.g. [8] for details. For simplicity we introduce the following notations

\[
m = (m_1, \ldots, m_d) \in \mathbb{N}^d_{\geq 0}, \quad m = \sum_{j=1}^d m_j, \quad m! = \prod_{j=1}^d m_j!,
\]

which reduce (1) to

\[
F(\vec{\omega}) = \sum_{m \in \mathbb{N}^d_{\geq 0}} \left\langle : \vec{\omega}^{\otimes m} : , \vec{f}_m \right\rangle, \quad \vec{\omega} \in S'_d(\mathbb{R}).
\]

Using, for example, the Wiener-Ito chaos expansion theorem and the second quantization operator of the Hamiltonian of harmonic oscillator, we can construct a Gel’fand triple

\[
(S) \subset L^2(\mu) \subset (S)^*
\]
where \((S)\) is the space of white noise test functions such that its topological dual space is the space \((S)^*\), the space of generalized white noise functionals. Elements of \((S)\) and \((S)^*\) are also known as Hida test functions and Hida distributions, respectively. The reader not familiar with this subject may consult [7] or [8]. As we know with probability one sample paths of a GMFMB \(Z^H\) are nowhere differentiable. However, it is possible to show that \(Z^H\) is differentiable as a mapping from \(\mathbb{R}\) into \((S)^*\), and the distributional derivative of \(Z^H = Z^H_t\) in \((S)^*\) is given by

\[
W_t^{H,\alpha}(\varpi) := \left( \left\langle \omega_1, \sum_{k=1}^{n} \alpha_k \delta_t \circ N^H_k \right\rangle, \cdots, \left\langle \omega_d, \sum_{k=1}^{n} \alpha_k \delta_t \circ N^H_k \right\rangle \right),
\]

where \(\varpi = (\omega_1, \ldots, \omega_d) \in S'_d(\mathbb{R})\) and \(\delta_t\) denotes the Dirac delta function at \(t\). The Hida distribution \(W_t^{H,\alpha}\) is called generalized mixed fractional white noise. See [10] for details.

In the rest of this section we state an integrability criterion of a Hida distribution via the so-called S-transform. For a given \(\vec{f} \in S_d(\mathbb{R})\) and the corresponding Wick exponential

\[
e(\varpi, \lambda) := \sum_{n \in \mathbb{N}^d} \left\langle \varpi^{\otimes n}, \vec{f}^{\otimes n} \right\rangle = e^{\langle \varpi, \lambda \rangle} - \frac{1}{2} ||\vec{f}||^2_0,
\]

we define the S-transform of an element \(\Phi \in (S)^*\) by

\[
(S\Phi)(\vec{f}) := \left( \left\langle \Phi, e^{\langle \cdot, \cdot \rangle} \right\rangle \right), \quad \text{for all } \vec{f} \in S_d(\mathbb{R}). \tag{2}
\]

Here \(\langle \cdot, \cdot \rangle\) denotes the dual pairing between \((S)\) and \((S)^*\) which is defined as the bilinear extension of the sesquilinear inner product on \(L^2(\mu)\). We observe that the multilinear expansion of (2) \(S\Phi(\vec{f}) = \sum_{m \in \mathbb{N}^d} \left\langle F_m, \vec{f}^{\otimes m} \right\rangle\), extends the chaos expansion to \(\Phi \in (S)^*\) with distribution-valued kernels \(F_m\) such that \(\langle \Phi, \varphi \rangle = \sum_{m \in \mathbb{N}^d} m! \langle F_m, \varphi_m \rangle\), for every Hida test function \(\varphi \in (S)\) with kernel functions \(\varphi_m\). The S-transform provides a quite useful way to study the Bochner integrability of a family of Hida distributions which depends on an additional parameter. For details and proofs see [6].

**Theorem 2.2.** Let \((\Omega, \mathcal{F}, \nu)\) be a measure space and \(\lambda \mapsto \Phi_\lambda\) be a mapping from \(\Omega\) to \((S)^*\). If the S-transform of \(\Phi_\lambda\) fulfils the following two conditions:

1. the mapping \(\lambda \mapsto S(\Phi_\lambda)(\vec{f})\) is measurable for all \(\vec{f} \in S_d(\mathbb{R})\), and
2. there exist \(C_1(\lambda) \in L^1(\Omega, \nu)\), \(C_2(\lambda) \in L^\infty(\Omega, \nu)\) and a continuous norm \(\| \cdot \|\) on \(S_d(\mathbb{R})\) such that

\[
\left| (S\Phi_\lambda)(z\vec{f}) \right| \leq C_1(\lambda)e^{C_2(\lambda)|z|^2\|\vec{f}\|^2}, \quad \text{for all } z \in \mathbb{C}, \; \vec{f} \in S_d(\mathbb{R}),
\]

then \(\Phi_\lambda\) is Bochner integrable with respect to some Hilbertian norm which topologizing \((S)^*\). Hence \(\int_\Omega \Phi_\lambda d\nu(\lambda) \in (S)^*\), and furthermore

\[
S \left( \int_\Omega \Phi_\lambda d\nu(\lambda) \right)(\vec{f}) = \int_\Omega (S\Phi_\lambda)(\vec{f}) d\nu(\lambda).
\]

3. Self-Intersection Local Times of GMFBM

In several applications, for example in the context of polymer physics, Feynman’s path integral, and financial mathematics, we need to pin a stochastic process at a particular spatial coordinate. Motivated by this fact we study the so-called Donsker delta function of the generalized mixed fractional Brownian motion. Therefore, in order to pin GMFBM at a point \(c \in \mathbb{R}^d\) we consider the Donsker’s delta function of GMFBM which is defined as the formal composition of the Dirac delta distribution \(\delta_d \in S'(\mathbb{R}^d)\) with a \(d\)-dimensional GMFBM \((Z^H_t)_{t \geq 0}\), i.e., \(\delta_d(Z^H_t - c)\). We
can give a precise meaning to the Donsker’s delta function as a Hida distribution. First, we recall the Fourier-transform representation of Dirac delta function which is given by

\[ \delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda x} d\lambda. \]

The following result has been proven in [11] and here we present a sketch of the proof for the sake of completeness.

**Proposition 3.1.** The Bochner integral

\[ \delta_d \left( Z^H_t - Z^H_s \right) := \left( \frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} e^{i\lambda (Z^H_t - Z^H_s)} d\lambda, \ t \neq s \]

is a Hida distribution with S-transform given by

\[
\begin{align*}
(S\delta_d \left( Z^H_t - Z^H_s \right))(\tilde{f}) &= \frac{1}{(2\pi \sum_{k=1}^{n} \alpha_k^2 |t - s|^{2H_k})^{d/2}} \times \exp \left( -\frac{1}{2 \sum_{k=1}^{n} \alpha_k^2 |t - s|^{2H_k}} \sum_{j=1}^{d} \left( \int_{\mathbb{R}} \sum_{k=1}^{n} \alpha_k f_j(x) N_{-H_k} 1_{[s,t]}(x) dx \right)^2 \right),
\end{align*}
\]

for all \( \tilde{f} = (f_1, \ldots, f_d) \in S_d(\mathbb{R}) \). In particular, its generalized expectation is given by

\[ \mathbb{E}_\mu \left( \delta_d \left( Z^H_t - Z^H_s \right) \right) = \frac{1}{(2\pi \sum_{k=1}^{n} \alpha_k^2 |t^{2H_k}|)^{d/2}}. \]

**Proof.** Without loss of generality we may assume \( t > s \). For any \( \tilde{f} \in S_d(\mathbb{R}) \) we have

\[ F_\lambda(\tilde{f}) = S e^{i\lambda (Z^H_t - Z^H_s)}(\tilde{f}) \]

\[ = \left\langle \left\langle \exp \left( i\lambda \left( \sum_{k=1}^{n} \alpha_k N_{-H_k} 1_{[0,t]} \right) - \sum_{k=1}^{n} \alpha_k N_{-H_k} 1_{[s,t]} \right) \right), e^{\langle \cdot, \tilde{f} \rangle} \right\rangle \]

\[ = \exp \left( -\frac{1}{2} \left\| \tilde{f} \right\|_0^2 \right) \int_{S_d(\mathbb{R})} \exp \left( \left\langle \omega, i\lambda \sum_{k=1}^{n} \alpha_k N_{-H_k} 1_{[s,t]} + \tilde{f} \right\rangle \right) d\mu(\omega) \]

\[ = \exp \left( -\frac{1}{2} |\lambda|^2 \sum_{k=1}^{n} \alpha_k^2 (t - s)^{2H_k} \right) \exp \left( i\lambda \int_{\mathbb{R}} \tilde{f}(x) \sum_{k=1}^{n} \alpha_k N_{-H_k} 1_{[s,t]}(x) dx \right). \]

The mapping \( \lambda \mapsto F_\lambda(\tilde{f}) \) is measurable for all \( \tilde{f} \in S_d(\mathbb{R}) \) and \( \lambda \in \mathbb{R}^d \). To apply Theorem 2.2, it remains to show the boundedness. Let \( z \in \mathbb{C} \) and by an application of Lemma 2.1 we obtain

\[ |F_\lambda(z\tilde{f})| \leq \exp \left( -\frac{1}{2} |\lambda|^2 \sum_{k=1}^{n} \alpha_k^2 (t - s)^{2H_k} \right) \times \exp \left( |z||\lambda| \sum_{j=1}^{d} \sum_{k=1}^{n} |\alpha_k| C_{H_k} (t - s) \left( \sup_{x \in \mathbb{R}} |f_j(x)| + \sup_{x \in \mathbb{R}} |f'_j(x)| + |f_{j0}| \right) \right). \]
Now define a continuous norm on $S_d(\mathbb{R})$ as follows
\[
\| \tilde{f} \|_* := \sum_{j=1}^{d} \left( \sup_{x \in \mathbb{R}} |f_j(x)| + \sup_{x \in \mathbb{R}} |f_j'(x)| + |f_j(0)| \right).
\]

Thus,
\[
F_\lambda(z\tilde{f}) \leq \exp\left( -\frac{1}{4} |\lambda|^2 \sum_{k=1}^{n} \alpha_k^2 (t-s)^{2H_k} \right) \exp\left( \frac{(\sum_{k=1}^{n} K_{H_k})^2}{\sum_{k=1}^{n} \alpha_k^2 (t-s)^{2H_k}} |z|^2 \| \tilde{f} \|_*^2 \right),
\]

where $K_{H_k} := |\alpha_k| C_{H_k} (t-s)$. The first factor is an integrable function of $\lambda$, and the second factor is constant with respect to $\lambda$. Therefore, according to the Theorem 2.2, $\delta_d (Z_t^H - Z_s^H) \in (S)^*$. To obtain an explicit expression for the S-transform of $\delta_d (Z_t^H - Z_s^H)$ we calculate as follow.

\[
S \left( \delta_d (Z_t^H - Z_s^H) \right) (\tilde{f}) = \left( \frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} S e^{i\lambda(z\tilde{f})} d\lambda = \left( \frac{1}{2\pi} \right)^d \left( \frac{\pi}{2 \sum_{k=1}^{n} \alpha_k^2 (t-s)^{2H_k}} \right)^{d/2} \exp\left( \frac{i \int_{\mathbb{R}} \tilde{f}(x) \sum_{k=1}^{n} \alpha_k N_{H_k} 1_{[s,t]}(x) dx}{2 \sum_{k=1}^{n} \alpha_k^2 (t-s)^{2H_k}} \right)^2.
\]

The generalized expectation is easily obtained by choosing $\tilde{f} = 0$. \hfill \Box

Now we are ready to prove our main result on self-intersection local times $L(T)$ as well as their subtracted counterparts $L^{(N)}(T)$. For simplicity we assume $T = 1$ and write $L(T)$ and $L^{(N)}(T)$ simply by $L$ and $L^{(N)}$, respectively. In the sequel we fix the following notations $H_* := \max_{1 \leq k \leq n} H_k$, $\Delta := \{(s,t) \in \mathbb{R}^2 : 0 < s < t < 1\}$, $d^2(s,t)$ is the Lebesgue measure on $\Delta$, $\| \cdot \|_*$ is the norm on $S_d(\mathbb{R})$ defined by Equation (3), and $\exp^{(N)}(x) := \sum_{m=0}^{\infty} \frac{x^m}{m!}$. We also define the truncated Donsker’s delta function $\delta^{(N)}_d (Z_t^H - Z_s^H)$ via its S-transform, i.e.

\[
S \left( \delta^{(N)}_d (Z_t^H - Z_s^H) \right) (\tilde{f}) = \left( \frac{1}{2\pi} \right)^d \left( \frac{\pi}{2 \sum_{k=1}^{n} \alpha_k^2 (t-s)^{2H_k}} \right)^{d/2} \exp^{(N)}\left( -\frac{1}{2 \sum_{k=1}^{n} \alpha_k^2 (t-s)^{2H_k}} \sum_{j=1}^{d} \left( \int_{\mathbb{R}} \sum_{k=1}^{n} \alpha_k f_j(x) N_{H_k} 1_{[s,t]}(x) dx \right)^2 \right),
\]

This is a unique Hida distribution whose existence is given via the characterization theorem of white noise distributions, see e.g. [6].
Theorem 3.2. Let \( \left( Z_{t}^{H,\alpha} \right)_{t \geq 0} \) be a GMFBM with parameter \( H = (H_1, \ldots, H_n) \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( H_k \in (0, 1) \), \( \alpha_k \in \mathbb{R} \), \( n \in \mathbb{N} \). For any pair of integer \( d \geq 1 \) and \( N \geq 0 \) such that \( 2N(H_s - 1) + dH_s < 1 \), the Bochner integral

\[
L(N) := \int_{\Delta} \delta_d^{(N)} (Z_t^H - Z_s^H) \, d^2(s, t)
\]

is a Hida distribution.

Proof. Note that the S-transform of \( \delta_d^{(N)} (Z_t^H - Z_s^H) \) is a measurable function. Furthermore, for every \( z \in \mathbb{C} \)

\[
|S \left( \delta_d^{(N)} (Z_t^H - Z_s^H) \right) (z\tilde{f})| \leq \frac{1}{(2\pi \sum_{k=1}^{n} \alpha_k^2 (t-s)^{2H_k})^{d/2}} \exp \left( \frac{(t-s)^2}{2} \sum_{k=1}^{n} \alpha_k |C_{H_k}|^2 \right) \exp \left( \frac{(t-s)^2}{2} \sum_{k=1}^{n} \alpha_k |C_{H_k}|^2 \right) \exp \left( \frac{(t-s)^2}{2} \sum_{k=1}^{n} \alpha_k |C_{H_k}|^2 \right),
\]

where \( t-s)^2(1-H_s-dH_s \) is integrable with respect to the Lebesgue measure on \( \Delta \) if and only if \( 2N(1-H_s) - dH_s > 1 \). Therefore we can conclude using Theorem 2.2 that \( L(N) \in (S)^* \).

We are also able to derive the chaos expansion for the (truncated) self-intersection local times \( L(N) \).

Theorem 3.3. Let \( \left( Z_{t}^{H,\alpha} \right)_{t \geq 0} \) be a GMFBM with parameter \( H = (H_1, \ldots, H_n) \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( H_k \in (0, 1) \), \( \alpha_k \in \mathbb{R} \), \( n \in \mathbb{N} \). For any pair of integer \( d \geq 1 \) and \( N \geq 0 \) such that \( 2N(H_s - 1) + dH_s < 1 \), the kernel functions \( F_{2m} \) of \( L(N) \) are given by

\[
F_{2m}(u_1, \ldots, u_{2m}) = \left( -\frac{1}{2} \right)^m \frac{m!}{m!} \left( \frac{1}{2\pi} \right)^{d/2} \int_{\Delta} \prod_{l=1}^{2m} \left( \sum_{k=1}^{n} \alpha_k N_{H_k} 1_{[s,t]} (u_l) \right) \left( \sum_{k=1}^{n} \alpha_k^2 (t-s)^{2H_k} \right)^{m+d/2} \, d^2(s, t)
\]

for each \( m \in \mathbb{N}^0 \) such that \( m \geq N \). All other kernel functions \( F_m \) are identically equal to zero.

Proof. By Theorem 2.2, the S-transform of self-intersection local times \( L(N) \) is obtained by integrating (4) over \( \Delta \). Hence, given an element \( f = (f_1, \ldots, f_d) \in S_d(\mathbb{R}) \),

\[
SL^{(N)}(\tilde{f}) = S \left( \int_{\Delta} \delta_d^{(N)} (Z_t^H - Z_s^H) \, d^2(s, t) \right) (\tilde{f}) = \int_{\Delta} S \left( \delta_d^{(N)} (Z_t^H - Z_s^H) \right) (\tilde{f}) \, d^2(s, t) = \int_{\Delta} \frac{1}{(2\pi \sum_{k=1}^{n} \alpha_k^2 (t-s)^{2H_k})^{d/2}} \exp \left( \frac{-1}{2} \sum_{k=1}^{n} \alpha_k^2 (t-s)^{2H_k} \right) \, d^2(s, t)
\]

\[
\times \exp \left( \frac{-1}{2} \sum_{k=1}^{n} \frac{1}{\alpha_k^2 (t-s)^{2H_k}} \sum_{j=1}^{d} \left( \int_{\mathbb{R}} \sum_{k=1}^{n} \alpha_k f_j(x) N_{H_k} 1_{[s,t]} (x) \, dx \right)^2 \right) \, d^2(s, t)
\]
\[
\frac{1}{2\pi} d/2 \int_{\Delta} \sum_{m=N}^{\infty} \left( \frac{1}{2} \right)^m \frac{1}{(\sum_{k=1}^{n} \alpha_k^2(t-s)^{2H_k})^{m+d/2}} \times \sum_{m_1+\ldots+m_d=m} \frac{1}{m!} \prod_{j=1}^{d} \left( \int_{\mathbb{R}} f_j(x) \sum_{k=1}^{n} \alpha_k N^{-H_k} 1_{[s,t]}(x) dx \right)^{2m_j} d^2(s,t).
\]

Remember the general form of the chaos expansion
\[
L^{(N)} = \sum_{m \in \mathbb{N}_0} \langle \mathcal{N}^\otimes m : F_m \rangle \quad \text{and} \quad SL^{(N)}(\vec{f}) = \sum_{m \in \mathbb{N}_0} \langle F_m, \vec{f}^\otimes m \rangle.
\]

Thus, we conclude that
\[
F_m = \left( -\frac{1}{2} \right)^m \frac{1}{m!} \left( \frac{1}{2\pi} \right)^{d/2} \int_{\Delta} \frac{\left( \sum_{k=1}^{n} \alpha_k N^{-H_k} \right)^{\otimes 2m}}{\left( \sum_{k=1}^{n} \alpha_k^2(t-s)^{2H_k} \right)^{m+d/2}} d^2(s,t).
\]

More precisely, for every \( m \in \mathbb{N}_0 \) such that \( m \geq N \) and \( u_1, \ldots, u_{2m} \in \mathbb{R} \) we have
\[
F_{2m}(u_1, \ldots, u_{2m}) = \left( -\frac{1}{2} \right)^m \frac{1}{m!} \left( \frac{1}{2\pi} \right)^{d/2} \int_{\Delta} \frac{\prod_{l=1}^{2m} \left( \sum_{k=1}^{n} \alpha_k N^{-H_k} 1_{[s,t]} \right) (u_l)}{\left( \sum_{k=1}^{n} \alpha_k^2(t-s)^{2H_k} \right)^{m+d/2}} d^2(s,t),
\]

while all other odd kernels \( F_m \) vanish. In particular, we obtain the generalized expectation of (truncated) self-intersection local times \( L^{(N)} \), that is
\[
\mathbb{E}_\mu(L^{(N)}) = F_0 = \left( \frac{1}{2\pi} \right)^{d/2} \int_{\Delta} \frac{1}{\left( \sum_{k=1}^{n} \alpha_k^2(t-s)^{2H_k} \right)^{d/2}} d^2(s,t). \]

Theorem 3.2 shows that for one-dimensional GMFBM all self-intersection local times are well-defined as Hida distributions for all possible Hurst parameter \( H_k \in (0,1) \), \( k = 1, \ldots, n \). For \( d \geq 2 \), self-intersection local times are well-defined only for \( \max_{1 \leq k \leq n} H_k \leq \frac{1}{d} \). Informally speaking, if \( H_k \geq \frac{1}{d} \) for some \( k \) with \( d \geq 2 \), the local times only become well-defined once the divergent terms are removed.

4. Concluding Remarks
We proved under some conditions on the number of truncation of divergent terms, Hurst parameters, and the spatial dimension, that self-intersection local times of GMFBM are Hida distribution. Explicit expressions for the chaos expansions of the self-intersection local times are also presented. All of these results are obtained in the white noise analysis framework.

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