Saari’s homographic conjecture for a planar equal-mass three-body problem under a strong force potential

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Abstract
Saari conjectured that the N-body motion with a constant configurational measure is a motion with fixed shape. Here, the configurational measure $\mu$ is a scale-invariant product of the moment of inertia $I = \sum_k m_k |q_k|^2$ and the potential function $U = \sum_{i<j} m_i m_j |q_i - q_j|^\alpha$, $\alpha > 0$. Namely, $\mu = I^{\alpha/2} U$. We will show that this conjecture is true for a planar equal-mass three-body problem under the strong force potential $\sum_{i<j} 1/|q_i - q_j|^2$.

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1. Saari’s homographic conjecture
In 1969, Saari conjectured that if an N-body system has a constant moment of inertia, then the motion is a rotation with constant mutual distances $r_{ij}$ [8]. Here, the moment of inertia $I$ is defined by

$$I = \sum_k m_k |q_k|^2,$$

with $m_k$ and $q_k$ being the masses and position vectors of the body, $k = 1, 2, 3, \ldots, N$. This is now called Saari’s original conjecture. In the conference ‘Saarifest 2005’ at Guanajuato, Mexico, Moeckel proved that the original conjecture is true for the three-body problem in $\mathbb{R}^d$ for any $d \geq 2$ [4, 5].
In the same conference, Saari extended his conjecture. His new conjecture is ‘if the configurational measure $I^{\alpha/2}U$ is constant, then the $N$-body motion is homographic’ [9, 10], where

$$U = \sum_{i<j} \frac{m_im_j}{r_{ij}^{\alpha}}$$

(2)

is the potential function. This is indeed a natural extension of the original conjecture. Note that a solution $q_k$ of $N$ bodies is called homographic if the configuration formed by the $N$ bodies moves in such a way as to remain similar to itself. For $\alpha \neq 2$, we can show that if the moment of inertia is constant, then $U$ is constant; therefore, the configurational measure is also constant. For $\alpha = 2$, on the other hand, $I = \text{constant}$ does not yield $U = \text{constant}$ [1]. Actually, there are some counter examples for the original conjecture for $\alpha = 2$ [2, 7]. However, the extended conjecture is expected to be true for $\alpha = 2$ and all $\alpha > 0$.

Diacu et al called this conjecture ‘Saari’s homographic conjecture’ and partly proved this conjecture for some cases [3]. No one proved this conjecture completely, as far as we know.

Obviously, Saari’s conjecture is related to the motion in shape. Here, a shape is a configuration of $N$ bodies up to rotation and scaling. To prove Saari’s original and homographic conjecture, it is important to find appropriate variables to describe motion in shape. The moment of inertia $I$ describes the motion in size, and the angular momentum $C$ describes the rotation. What are the appropriate variables to describe the motion in shape?

An answer was given by Moeckel and Montgomery [6]. They used the ratio of the Jacobi coordinates to describe the motion in shape for the planar three-body problem. Let us explain precisely. To avoid non-essential complexity, let us consider the equal-mass case and set $m_k = 1$ in this paper. We take the center of mass frame. So, we have

$$\sum q_k = 0.$$  

(3)

In the three-body problem, we have two Jacobi coordinates

$$z_1 = q_2 - q_1,$$  

(4)

$$z_2 = q_3 - \frac{q_1 + q_2}{2} = \frac{3}{2}q_3.$$  

(5)

Since we are considering planar motions, let us identify $q_k$ and $z_i$ with complex numbers. Then, we can define the ratio of the Jacobi coordinates:

$$\zeta = \frac{z_2}{z_1} = \frac{3}{2} \frac{q_3}{q_2 - q_1}.$$  

(6)

Note that the variable $\zeta$ is invariant under size change and rotation, and $q_k \mapsto \lambda e^{i\theta}q_k$ with $\lambda, \theta \in \mathbb{R}$. Therefore, $\zeta$ depends only on the shape. The great idea by Moeckel and Montgomery is to use the variable $\zeta$ to describe the shape. They actually write down the Lagrangian using the variable $\zeta$, the moment of inertia $I$ and the rotation angle $\theta$. They also write down the equations of motion for these variables.

Using the formulation developed by Moeckel and Montgomery, we will show that Saari’s homographic conjecture is true for the planar equal-mass three-body problem under the strong force potential

$$U = \sum_{i<j} \frac{1}{|q_i - q_j|^\alpha}.$$  

(7)

Namely, we will show that $d\zeta/dt = 0$ if and only if $IU = \text{constant}$. 

[2]

Figure 1. The definition of shape variable \( \zeta \). A transformation keeps the similarity and the orientation transforms the triangle \( q_1 q_2 q_3 \) into the triangle \( ab \zeta \) with \( a = -1/2 \) and \( b = 1/2 \).

In section 2, we derive the Lagrangian in terms of \( I, \theta \) and \( \zeta \) by elementary calculations. The equations of motion and some useful relations are also shown in this section. Every relation in section 2 is valid for \( \alpha \neq 0 \). In section 3, we concentrate on the strong force potential \( \alpha = 2 \). We will prove Saari’s homographic conjecture for this case. Section 4 is devoted to discussions. Details of calculations are given in the appendix.

2. Lagrangian for the planar equal-mass three-body problem in terms of shape, size and rotation angle

In this section, we consider the planar equal-mass three-body problem under the potential function (2) with \( m_1 = m_2 = m_3 = 1 \) for \( \alpha \neq 0 \). Let \( K = \sum_k |dq_k/dt|^2 \) be twice the kinetic energy, and let the Lagrangian and the total energy be \( L = K/2 + U/\alpha \) and \( E = K/2 - U/\alpha \), respectively. At the center of mass frame (3), all quantities \( \xi_k = q_k/(q_2 - q_1) \) are expressed by the shape variable (6) as follows:

\[
\xi_1 = \frac{q_1}{q_2 - q_1} = -\frac{1}{2} - \frac{\zeta}{3},
\]

\[
\xi_2 = \frac{q_2}{q_2 - q_1} = +\frac{1}{2} - \frac{\zeta}{3},
\]

\[
\xi_3 = \frac{q_3}{q_2 - q_1} = \frac{2}{3}\zeta.
\]

Obviously, the triangle made of \( q_1, q_2 \) and \( q_3 \) is similar to the triangle made of \( \xi_1, \xi_2 \) and \( \xi_3 \). Therefore, there are some \( I \geq 0 \) and \( \theta \in \mathbb{R} \), such that

\[
q_k = \sqrt{I} e^{i\theta} \frac{\xi_k}{\sqrt{\sum_j |\xi_j|^2}}.
\]

We treat \( I, \theta \) and \( \zeta \) as independent dynamical variables.

Let us explain a geometrical interpretation of the variable \( \zeta \). For the given triangle \( q_1 q_2 q_3 \), we can transform the points \( q_1 \mapsto a = -1/2 \) and \( q_2 \mapsto b = 1/2 \) keeping the similarity and the orientation of the triangle. The points \( a \) and \( b \) are fixed. Let \( \zeta \) be the image of \( q_1 \) by this transformation. So, the triangles \( q_1 q_2 q_3 \) and \( ab \zeta \) are similar and have same orientation. See figure 1. It is clear that the variable \( \zeta \) describes the shape of the triangle \( q_1 q_2 q_3 \). This gives an alternative definition of the shape variable \( \zeta \). The center of mass of the triangle \( ab \zeta \) is \( \zeta/3 \). To fix the center of mass at the origin, we subtract \( \zeta/3 \) from the three vertices. Thus, we have three vertices: \( \xi_1 = -1/2 - \zeta/3, \xi_2 = 1/2 - \zeta/3 \) and \( \xi_3 = 2\zeta/3 \). These are represented in equations (8)–(10).
2.1. Lagrangian

Direct calculations for $K$ yield

$$K = \frac{i^2}{4I} + I\left(\dot{\theta} + \frac{2}{3} \frac{\dot{\xi} \wedge \dot{\xi}}{\frac{1}{2} + \frac{2}{3} |\dot{\xi}|^2}\right)^2 + \frac{I}{3} \left(\frac{|\dot{\xi}|^2}{\frac{1}{3} + \frac{2}{3} |\dot{\xi}|^2}\right).$$ (12)

Here, the wedge product $\wedge$ represents $(a + ib) \wedge (c + id) = ad - bc$ for $a, b, c, d \in \mathbb{R}$, and the dot $d/dt$. On the other hand, the potential function $U$ is

$$U = \frac{1}{F^{2/2}} \left(\frac{1}{2} + \frac{2}{3} |\dot{\xi}|^2\right)^{2/3} \left(1 + \frac{1}{|\dot{\xi} - 1/2|^\alpha} + \frac{1}{|\dot{\xi} + 1/2|^\alpha}\right).$$ (13)

Therefore, we obtain the Lagrangian

$$\mu(\xi) = F^{2/2} U = \left(\frac{1}{2} + \frac{2}{3} |\dot{\xi}|^2\right)^{\alpha/2} \left(1 + \frac{1}{|\dot{\xi} - 1/2|^\alpha} + \frac{1}{|\dot{\xi} + 1/2|^\alpha}\right).$$ (14)

Thus, we obtain the Lagrangian

$$L = \frac{i^2}{8I^2} + \frac{I}{2} \left(\dot{\theta} + \frac{2}{3} \frac{\dot{\xi} \wedge \dot{\xi}}{\frac{1}{2} + \frac{2}{3} |\dot{\xi}|^2}\right)^2 + \frac{I}{6} \left(\frac{|\dot{\xi}|^2}{\frac{1}{3} + \frac{2}{3} |\dot{\xi}|^2}\right)^2 + \frac{\mu(\xi)}{a F^{\alpha/2}}.$$ (15)

in terms of $I, \theta, \dot{\xi}$ and their velocities, or identifying $\dot{\xi} = x + iy$ with $x, y \in \mathbb{R}$ to a two-dimensional vector $\mathbf{x} = (x, y)$, the Lagrangian is expressed as

$$L = \frac{i^2}{8I^2} + \frac{I}{2} \left(\dot{\theta} + \frac{2}{3} \mathbf{x} \wedge \dot{\mathbf{x}}\right)^2 + \frac{I}{6} \left(\frac{\dot{\mathbf{x}}^2}{\frac{1}{2} + \frac{2}{3} |\mathbf{x}|^2}\right)^2 + \frac{\mu(\mathbf{x})}{a F^{\alpha/2}}.$$ (16)

with $\mathbf{x} \wedge \dot{\mathbf{x}} = xy - yx$ and

$$\mu(\mathbf{x}) = \left(\frac{1}{2} + \frac{2}{3} |\mathbf{x}|^2\right)^{\alpha/2} \left(1 + \frac{1}{(x - 1/2)^2 + y^2}^{\alpha/2} + \frac{1}{(x + 1/2)^2 + y^2}^{\alpha/2}\right).$$ (17)

2.2. Equation of motion for the rotation angle

Obviously, the variable $\dot{\theta}$ is cyclic. Therefore, we obtain the conservation law of the angular momentum:

$$C = \frac{\partial L}{\partial \dot{\theta}} = I \left(\dot{\theta} + \frac{2}{3} \mathbf{x} \wedge \dot{\mathbf{x}}\right) = \text{constant.}$$ (18)

Then, the kinetic energy $K/2$ is given by

$$K = \frac{i^2}{8I} + \frac{C^2}{2I} + \frac{I}{6} \left(\frac{\dot{\mathbf{x}}^2}{\frac{1}{2} + \frac{2}{3} |\mathbf{x}|^2}\right)^2.$$ (19)

The three terms on the right-hand side represent kinetic energies for the scale change, for the rotation and for the shape change, respectively.

2.3. Equation of motion for the moment of inertia

The Euler–Lagrange equation for the moment of inertia $I$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{x}^2}\right) - \frac{\partial L}{\partial \mathbf{x}} = 0$$

yields
\[ \frac{\dot{I}}{4} = \frac{\dot{I}^2}{8I} + \frac{C^2}{2I} + \frac{I}{6} \left( \frac{1}{2} + \frac{2}{3} |x|^2 \right)^2 - \frac{1}{2} \frac{\mu(x)}{J^{n/2}} \] (20)
\[ = E + \left( \frac{1}{\alpha} - \frac{1}{2} \right) U. \] (21)

Multiplying \( \dot{I} \) on both sides of equation (21), we obtain
\[ -\frac{\mu}{\alpha} (1 - \alpha/2) I^{-\alpha/2} \dot{I} = EI - \frac{1}{4} \ddot{I}. \]

This means
\[ -\frac{\mu}{\alpha} \frac{d}{dt} (I^{1-\alpha/2}) = \frac{d}{dt} \left( EI - \frac{I^2}{8} \right) = \frac{d}{dt} \left( \frac{C^2}{2} + \frac{I^2}{6} \left( \frac{1}{2} + \frac{2}{3} |x|^2 \right)^2 - \frac{\mu}{\alpha} I^{1-\alpha/2} \right). \]

Therefore, we obtain
\[ \frac{d}{dt} \left( I \left( \frac{1}{2} + \frac{2}{3} |x|^2 \right) \right) = \frac{1}{\alpha} \frac{d\mu}{dt}. \] (22)

This relation was first derived by Saari [9]. We call this ‘Saari’s relation’. Inspired by this relation, let us introduce new ‘time’ variable \( s \) defined by
\[ ds = \left( \frac{1}{2} + \frac{2}{3} |x|^2 \right) dt. \] (23)

Then, we have
\[ \frac{d}{dt} = \frac{1}{I} \left( \frac{1}{2} + \frac{2}{3} |x|^2 \right) \frac{d}{ds} \] (24)

and Saari’s relation (22) is
\[ \frac{d}{ds} \left( I \left( \frac{1}{2} + \frac{2}{3} |x|^2 \right) \right) = \frac{1}{\alpha} \frac{d\mu}{ds}. \] (25)

2.4. Equation of motion for the shape variables

The Euler–Lagrange equation for \( x \)
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \]
yields
\[ \frac{d}{dt} \left( \frac{I}{3(1/2 + 2|x|^2/3)^2} \frac{dx}{dt} \right) = \frac{2C}{3} \frac{1}{(1/2 + 2|x|^2/3)^3} (y, -x) \]
\[ + \frac{2}{3} \frac{1}{(1/2 + 2|x|^2/3)^3} \left( \frac{\dot{x}}{\dot{y}} \right) + \frac{1}{\alpha J^{n/2}} \frac{\partial \mu}{\partial x}. \]
Using the ‘time’ variable $s$, this equation of motion is

$$\frac{d^2\mathbf{x}}{ds^2} = 2C - \frac{1}{\alpha} \left( \mathbf{x} \wedge \frac{d\mathbf{x}}{ds} \right) \left( \frac{dy}{ds} + \frac{dx}{ds} \right) + \frac{3I^{1-\alpha/2}}{\alpha} \frac{\partial \mu}{\partial \mathbf{x}}. \quad (26)$$

The inner product of $d\mathbf{x}/ds$ and $d^2\mathbf{x}/ds^2$ yields

$$\frac{dx}{ds} \frac{d^2\mathbf{x}}{ds^2} = 3I^{1-\alpha/2} \frac{dx}{ds} \frac{\partial \mu}{\partial \mathbf{x}} = 3I^{1-\alpha/2} \frac{d\mu}{ds}. \quad (27)$$

This is nothing but Saari’s relation (25). However, the wedge product of the same pair yields

$$\frac{dx}{ds} \wedge \frac{d^2\mathbf{x}}{ds^2} = -\frac{2C - \frac{1}{\alpha} \left( \mathbf{x} \wedge \frac{d\mathbf{x}}{ds} \right)}{\frac{1}{2} + \frac{1}{\alpha}} \left( \frac{dx^2}{ds} + 3I^{1-\alpha/2} \frac{dx}{ds} \frac{\partial \mu}{\partial \mathbf{x}}. \right) \quad (28)$$

Every equation is valid for all $\alpha \neq 0$. We will use these expressions later.

3. Proof of Saari’s homographic conjecture under the strong force potential

In this section, we will prove Saari’s homographic conjecture for the case $\alpha = 2$, namely $\zeta = 0$ if and only if $\mu$ = constant.

Let us assume

$$\mu = \mu_0 = \text{constant.}$$

Then, by Saari’s relation (25), we have

$$\left| \frac{dx}{ds} \right|^2 = k^2 \quad (29)$$

with the constant $k \geq 0$.

If $k = 0$, then $dx/ds = 0$, namely $\zeta = 0$.

Let us examine the case $k > 0$. For this case, the point $\mathbf{x}(s)$ moves on the curve $\mu(\mathbf{x}) = \mu_0$ with the constant speed $|dx/ds| = k$. This motion of $\mathbf{x}(s)$ is not able to keep $\partial \mu/\partial \mathbf{x} = 0$, because the points that satisfy $\partial \mu/\partial \mathbf{x} = 0$ are only five central configurations at $\mathbf{x} = (\pm 3/2, 0), (0, 0), (0, \pm \sqrt{3}/2)$. See figure 2.

So, we can take some finite arc of the curve $\mu(\mathbf{x}) = \mu_0$ on which

$$\frac{\partial \mu}{\partial \mathbf{x}} \neq 0.$$

Then, equation (28) and

$$\frac{dx}{ds} \cdot \frac{\partial \mu}{\partial \mathbf{x}} = \frac{d\mu}{ds} = 0$$

yield

$$\frac{dx}{ds} = \frac{\epsilon k}{|\partial \mu/\partial \mathbf{x}|} \left( -\frac{\partial \mu}{\partial y} \frac{\partial \mu}{\partial x} \right). \quad (30)$$

Here, $\epsilon = \pm 1$ determines the direction of the motion on the curve $\mu = \mu_0$. Differentiating this expression by $s$ again, we have

$$\frac{d^2x}{ds^2} = \frac{\epsilon k}{|\partial \mu/\partial \mathbf{x}|} \frac{d}{ds} \left( -\frac{\partial \mu}{\partial y} \frac{\partial \mu}{\partial x} \right) + \left( \frac{\partial \mu}{\partial y} \frac{\partial \mu}{\partial x} \right) \frac{d}{ds} \left( \frac{\epsilon k}{|\partial \mu/\partial \mathbf{x}|} \right). \quad (31)$$

Therefore, we obtain

$$\frac{dx}{ds} \wedge \frac{d^2\mathbf{x}}{ds^2} = \frac{\epsilon k^3}{|\partial \mu/\partial \mathbf{x}|^3} \left( \left( \frac{\partial \mu}{\partial y} \right)^2 \frac{\partial^2 \mu}{\partial x^2} - 2 \frac{\partial \mu}{\partial x} \frac{\partial \mu}{\partial y} \frac{\partial^2 \mu}{\partial x \partial y} + \left( \frac{\partial \mu}{\partial x} \right)^2 \frac{\partial^2 \mu}{\partial y^2} \right). \quad (32)$$
Therefore, the curvature $\rho^{-1}$ of the curve $\mu = \mu_0$ should be

$$\rho^{-1} = \frac{\epsilon}{|\partial \mu / \partial \mathbf{x}|} \left( \left( \frac{\partial \mu}{\partial y} \right)^2 \frac{\partial^2 \mu}{\partial x^2} - 2 \frac{\partial \mu}{\partial x} \frac{\partial \mu}{\partial y} \frac{\partial^2 \mu}{\partial x \partial y} + \left( \frac{\partial \mu}{\partial x} \right)^2 \frac{\partial^2 \mu}{\partial y^2} \right).$$  \hspace{1cm} (31)

On the other hand, by relation (27), which is a result of the equation of motion and the expression for the velocity (29), we have another expression for the curvature

$$\rho^{-1} = \frac{1}{1/2 + 2|\mathbf{x}|^2/3} \left( -\frac{2|\mathbf{x}|^2}{k} + \frac{4\epsilon}{3|\partial \mu / \partial \mathbf{x}|} \frac{\partial \mu}{\partial \mathbf{x}} \right) - \frac{3\epsilon}{2k^2} \left| \frac{\partial \mu}{\partial \mathbf{x}} \right|. \hspace{1cm} (32)$$

Our plan to exclude the case $k > 0$ is the following. Since $|\mathbf{d}x / ds| = k = \text{constant}$, the parameter $s$ is proportional to the arc length of the curve $\mathbf{x}(s)$ on $\mu = \mu_0$. Therefore, if $k > 0$, there must be a finite arc, on which the curvature (31) coincides with (32). We call such an arc the non-Saari arc. In the following, we will show that the non-Saari arc does not exist. Namely, $k > 0$ is impossible.

Let us examine the condition for the two curvatures having the same value. The condition is

$$\frac{1}{|\partial \mu / \partial \mathbf{x}|^3} \left( \left( \frac{\partial \mu}{\partial y} \right)^2 \frac{\partial^2 \mu}{\partial x^2} - 2 \frac{\partial \mu}{\partial x} \frac{\partial \mu}{\partial y} \frac{\partial^2 \mu}{\partial x \partial y} + \left( \frac{\partial \mu}{\partial x} \right)^2 \frac{\partial^2 \mu}{\partial y^2} \right) = \frac{1}{1/2 + 2|\mathbf{x}|^2/3} \left( -\frac{2|\mathbf{x}|^2}{k} + \frac{4\epsilon}{3|\partial \mu / \partial \mathbf{x}|} \frac{\partial \mu}{\partial \mathbf{x}} \right) - \frac{3\epsilon}{2k^2} \left| \frac{\partial \mu}{\partial \mathbf{x}} \right|.$$
Therefore,
\[
\frac{4C^2}{k^2N^2}|\nabla \mu|^6 - \left(\frac{\partial \mu}{\partial y}\right)^2 \frac{\partial^2 \mu}{\partial x^2} - 2 \frac{\partial \mu}{\partial x} \frac{\partial \mu}{\partial y} \frac{\partial^2 \mu}{\partial x \partial y} + \left(\frac{\partial \mu}{\partial x}\right)^2 \frac{\partial^2 \mu}{\partial y^2} - \frac{4}{3N}(x \cdot \nabla \mu)|\nabla \mu|^2
+ \frac{3}{2k^2}|\nabla \mu|^4) = 0.
\]
(33)
where \( N = 1/2 + 2|x|^2/3 \) and \( \nabla \mu = \partial \mu / \partial x \). The left-hand side is a ratio of polynomials of \( x^2, y^2, C^2 \) and \( k^2 \). Let the numerator of this ratio be a polynomial \( P(x^2, y^2, C^2, k^2) \); then, \( x^2 \) and \( y^2 \) must satisfy the following equation:
\[
P(x^2, y^2, C^2, k^2) = 0.
\]
(34)
The maximum power of the variables for \( P \) is \( x^{60}, y^{60}, C^2 \) and \( k^4 \).

On the other hand, the equation \( \mu = \mu_0 \) is also a ratio of polynomials of \( x^2, y^2 \) and \( \mu_0 \).
Let the numerator of this ratio be a polynomial \( Q \); then, we have the following equation:
\[
Q(x^2, y^2, \mu_0) = 27 + 64x^6 + 156y^2 + 208y^4 + 64y^6 + 48x^4(3 + 4y^2)
+ 4x^2(27 + 88y^2 + 48y^4) - 6\mu_0(16x^4 + 8x^2(-1 + 4y^2) + (1 + 4y^2)^2)
= 0.
\]
(35)
The non-Saari arc must satisfy both \( P = 0 \) and \( Q = 0 \) for some values of parameters \( C^2, k^2 \) and \( \mu_0 \).

There is no finite arc with \( x = x_0 \) fixed and \( \mu = \mu_0 \), because, for \( x = x_0, Q = 0 \) is a polynomial of \( y^2 \) of order \( y^6 \) with the coefficient of \( y^6 \) being \( 64 \neq 0 \). Therefore, solutions of \( y \) for \( Q = 0 \) are discrete. Thus, any finite arc must have some finite interval \( x_1 \leq x \leq x_2 \).

There is no finite interval \( x_1 \leq x \leq x_2 \) that every \( x \) in this interval satisfies \( P = Q = 0 \). To show this, we eliminate \( y^2 \) from \( P(x^2, y^2) = Q(x^2, y^2) = 0 \) to obtain the new polynomial \( R(x^2) = 0 \). This polynomial turns out to be of order \( x^{68} \):
\[
R = A x^8(4x^2 - 1)^6 \sum_{0 \leq n \leq 24} c_n(C^2, k^2, \mu_0)x^{2n},
\]
(36)
where \( A \) is a big integer. To have a continuous solution of \( x \) for \( R = 0 \), the polynomial \( R \) must be identically equal to zero. Namely, all coefficients \( c_n, n = 0, 1, 2, \ldots, 24 \) must be zero. Therefore, we have 25 conditions for only three parameters \( C^2, k^2 \) and \( \mu_0 \). Actually, there are no parameters to make all 25 coefficients vanish. See the appendix for details. Therefore, there is no finite interval of \( x \) on which \( P = Q = 0 \) is satisfied. Thus, the \( k > 0 \) case is excluded.

Therefore, we have proved that if \( \mu = \) constant, then \( \dot{\zeta} = 0 \), namely the shape of three-body triangle remains similar.

Inversely, if \( \zeta = 0 \), then obviously \( \mu(\zeta) \) is constant. This completes the proof for Saari’s homographic conjecture for the case \( \alpha = 2 \) planar equal-mass three-body problem.

4. Discussions

Let us explain why we treat only the equal-mass case and \( \alpha = 2 \) in this paper.

For general masses, we observe that the contour lines of \( J^{\alpha/2}U \) in \( \zeta \) are complex. In particular, it is not invariant under the transformation \( \zeta \rightarrow -\zeta \). So, the equation of motion for \( \zeta \) is not invariant under the same transformation. Therefore, the polynomials \( P \) and \( Q \) will be
functions of $x$ and not of $x^2$. This will make the polynomial $R$ much complex. Obviously, we need some method to treat the theory more concisely.

For $\alpha = 2$, equations (26) and (27) are independent of $I$. Therefore, the curvature $\rho^{-1}$ in equation (32) and the condition $P = 0$ do not contain the variable $I$. On the other hand, in the Newtonian case $\alpha = 1$ and other $\alpha$, equations (26) and (27) depend on $I$; thus, the condition $P = 0$ contains $I$. Therefore, we have to perform one more step to eliminate $I$ to have a condition for shape variables. This makes the proof much more complex. This is the reason why we consider $\alpha = 2$ in this paper.

We hope, some day, we will find a method to express $P$ and $Q$ in a concise way and extend our method to the $\alpha = 1$ case and/or general-mass case.

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Appendix. Details of calculations

In this appendix, details of calculations are given. The following calculations were performed using Mathematica 8.0.1.0.

To eliminate the variable $y^2$ from the equation $P = Q = 0$, we calculate the resultant of $P$ and $Q$ with respect to $y^2$:

$$R = \text{Resultant}[P, Q, y^2]. \quad (A.1)$$

In the actual calculations, we replaced $y^2$ with $Y$ and calculated $\text{Resultant}[P, Q, Y]$, because Mathematica does not accept $y^2$ as a variable.

Then, the coefficients $c_n$, $n = 0, 1, 2, \ldots, 24$, in equation (36) are the polynomials of $C^2$, $k^2$ and $\mu_0$. To eliminate $k^2$ from the equation $c_{24} = c_{23} = 0$, we again calculate the resultant of $c_{24}$ and $c_{23}$ with respect to $k^2$:

$$d_1 = \text{Resultant}[c_{24}, c_{23}, k^2]$$

$$= D_1 \mu_0^6 (\mu_0 - 1) (2\mu_0 - 1)^3 ((3\mu_0 - 1)C^2 + 2\mu_0(2\mu_0 - 1)^2)C^2, \quad (A.2)$$

with a big integer $D_1$. The configurational measure $\mu$ is not smaller than 3:

$$\mu = \frac{1}{3} \left( \sum_{i<j} r_{ij}^2 \right) \left( \sum_{i<j} \frac{1}{r_{ij}} \right) \geq 3 \left( \frac{r_{12}^2 r_{23}^2 r_{13}^2}{r_{12}^2 r_{23}^2 r_{13}^2} \right)^{1/3} \left( \frac{1}{r_{12}^2 r_{23}^2 r_{13}^2} \right)^{1/3} = 3.$$

Therefore, $\mu_0 \geq 3$. Then, $d_1 = 0$ yields $C^2 = 0$. For $C^2 = 0$, the resultant of $c_{24}$ and $c_{22}$ with respect to $k^2$ is

$$d_2 = \text{Resultant}[c_{24}, c_{22}, k^2]$$

$$= D_2 \mu_0^2 (\mu_0 - 1)^4 (2\mu_0 - 1)^6 \neq 0, \quad (A.3)$$

where $D_2$ is another big integer. Therefore, it is impossible to make $c_{24} = c_{23} = c_{22} = 0$ simultaneously.

This completes the proof that there is no parameter $C^2$, $k^2$ or $\mu_0$ to make $R = 0$ identically.
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