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A novel tetrahedral element for static and dynamic analysis of laminated composites

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Abstract. This paper presents a novel approach to formulating finite elements for modelling laminated composites. Most FE systems provide laminated composite elements based upon layered shell elements and some provide layered brick elements, but these require a hexahedral mesh which is difficult to generate for arbitrary geometries. This paper describes a tetrahedral element based on the concept of an equivalent graded material whose properties vary smoothly (rather than stepwise) and whose global behaviour replicates that of the laminate. This allows the possibility of automatically-generated models involving laminated composite behaviour. The element requires a non-standard integration system, and is presented for the cases of elastic, thermoelastic and dynamic behaviour.

1. Introduction
This work is motivated by the desire to undertake finite element analysis of laminated components of non-trivial geometry, specifically where the geometry is defined within a solid modelling system and is difficult to reduce to an essentially prismatic form. Such geometries are often difficult or impractical to mesh with laminated brick or wedge elements, since arbitrary geometries can only be meshed fully automatically via tetrahedral elements.

No evidence could be found of existing laminated tetrahedral elements, so there appears to be little to start from. There have, of course, been element formulations published relating to functionally graded materials (e.g. Panda et al. [1]). Conversely, laminated elements have been used to analyse functionally graded materials (e.g. Lee [2]). However, we are not aware of any existing formulation which approximates a laminate as an equivalent graded material.

The present paper describes a novel approach to this problem, in which a conventional laminated formulation is abandoned in favour of one based on a fictitious graded material, which, when subjected to typical loads (mechanical, thermoelastic and dynamic), exhibit equivalent behaviour to the genuine laminated composite material. Integrations within the element can then be carried out over the continuum of the element without any need to discretise the element into awkward sub-regions and (unlike laminated bricks and wedges) with no requirement for element to be aligned with the planes of the true laminate.

2. Elastically-equivalent graded material
The formulation of the element follows a number of discrete stages.
(i) In order to formulate the element, an assumption must be made on the manner in which displacements (and other quantities) vary within the element. Here, a quadratic variation is assumed within a given element and hence within a given region of the laminated material.

(ii) The concept is then established of an “representative sub-laminate” of the true (laminated) material, spanning the extremities of the through-thickness nodal positions of the element whose stiffness is to be evaluated. A unit projected area of this sub-laminate is assumed to be subjected to a strain field varying in the manner defined in item i, analogous to (but with a higher order than) the linear variation of in-plane displacement assumed within classical laminate theory (CLT) \[3\]:

\[
\epsilon(z) = \begin{bmatrix} \epsilon_x & \epsilon_y & \epsilon_z & \gamma_{yz} & \gamma_{zx} \end{bmatrix}^T = \epsilon^0 + \kappa z + \xi z^2 \tag{1}
\]

where \(\epsilon^0\), \(\kappa z\) and \(\xi z^2\) are vectors of coefficients, of which the first two are respectively analogous to the well-known mid-plane strain and curvature terms in CLT.

(iii) The strain energy in the true sub-laminate is evaluated in terms of the stiffness coefficients within each of its \(N\) laminae or plies, and in terms of the assumed displacement field. It will be assumed that the elastic behaviour at a given through-thickness position \(z\) is defined by the \(6 \times 6\) stiffness matrix \(C(z)\), which for a real laminate will take vary in a piecewise-constant manner, having the value \(C_k\) within the \(k\)th ply. In general, each ply will be orientated arbitrarily, with its off-axis behaviour implicit in the definition of \(C\), without the use of an overbar as is sometimes used in CLT \[3\]. This leads to an expression of the form:

\[
\text{S.E.} = \frac{1}{2} \int_{-h/2}^{h/2} (\epsilon^0 + \kappa z + \xi z^2) C(z) (\epsilon^0 + \kappa z + \xi z^2) \, dz \tag{2}
\]

where \(h\) is the thickness of the sub-laminate (numerically equal to the difference in normal through-thickness position of the uppermost and lowermost nodes of the tetrahedron) and where:

\[
\begin{align*}
A' &= \sum_{k=1}^{N} C_k (z_k - z_{k-1}), \quad B' = \frac{1}{2} \sum_{k=1}^{N} C_k \left( z_k^2 - z_{k-1}^2 \right), \quad D' = \frac{1}{3} \sum_{k=1}^{N} C_k \left( z_k^3 - z_{k-1}^3 \right), \\
F' &= \frac{1}{4} \sum_{k=1}^{N} C_k \left( z_k^4 - z_{k-1}^4 \right), \quad G' = \frac{1}{5} \sum_{k=1}^{N} C_k \left( z_k^5 - z_{k-1}^5 \right) \tag{3a-e}
\end{align*}
\]

(iv) The properties (e.g. the coefficients of the elastic stiffness matrix) of the equivalent material are now assumed to be constant within a plane parallel to the planes of the laminate, but to vary with through-thickness position according to some order of polynomial. This polynomial order is chosen so that the number of coefficients (five in this case, defining a quartic) equals the number of independent constants \(A' - G'\) appearing in equation (2):

\[
C(z) = C^{(0)} + C^{(1)} z + C^{(2)} z^2 + C^{(3)} z^3 + C^{(4)} z^4 \tag{4}
\]

where \(C^{(0)}\) etc. are \(6 \times 6\) matrices of coefficients.

(v) The concept is then established of an “equivalent block” of the fictitious graded material, behaving elastically in the same manner as the representative sub-laminate of the material
spanning the through-thickness extents of the element. This is then subjected to the various
kinematic deformation modes described under section i, and the strain energy determined
in terms of the unknown matrix coefficients in equation (4).

\[
\text{S.E.} = \frac{1}{2} \int_{-h/2}^{h/2} \left[ e^0 \kappa \xi \right] \begin{bmatrix}
C(z) & C(z)z & C(z)z^2 \\
C(z)z & C(z)z^2 & C(z)z^3 \\
C(z)z^2 & C(z)z^3 & C(z)z^4
\end{bmatrix} \left\{ \begin{bmatrix}
e^0 \\
\kappa \\
\xi
\end{bmatrix} \right\} dz
\]

(5)

which can be represented in the form:

\[
\text{S.E.} = \frac{1}{2} \left[ e^0 \kappa \xi \right] \begin{bmatrix}
A'' & B'' & D'' \\
B'' & D'' & F'' \\
D'' & F'' & G''
\end{bmatrix} \left\{ \begin{bmatrix}
e^0 \\
\kappa \\
\xi
\end{bmatrix} \right\}
\]

(6)

After evaluation of the integrals, and representing the summations in matrix form, the
unknown matrix coefficients can be expressed via the following equation system:

\[
\begin{bmatrix}
A'' \\
B'' \\
D'' \\
F'' \\
G''
\end{bmatrix} = \begin{bmatrix}
hI & 0 & h^2 h& 0 & h^3 h& 0 \\
0 & h^4 h& 0 & h^5 h& 0 \\
h^4 h& 0 & h^5 h& 0 & h^6 h& 0 \\
0 & h^6 h& 0 & h^7 h& 0 \\
0 & h^6 h& 0 & h^7 h& 0
\end{bmatrix} \begin{bmatrix}
C(0) \\
C(1) \\
C(2) \\
C(3) \\
C(4)
\end{bmatrix}
\]

(7)

where the identity and zero matrices are all 6 × 6. The value of strain energy from equation
(6) is then equated to that obtained in equation (2), resulting in the observation that
\( A'' = A' \) etc. Inserting this condition into equation (7) and performing the inversion
analytically yields the following expression defining the matrix coefficients in the polynomial
expression for elastic stiffness:

\[
\begin{bmatrix}
C(0) \\
C(1) \\
C(2) \\
C(3) \\
C(4)
\end{bmatrix} = \begin{bmatrix}
3.515625 h^6 & 0 & -65.625 h^4 & 0 & 236.25 h^2 \\
0 & 75 h^4 & 0 & -420 h^2 & 0 \\
-65.625 h^4 & 0 & 420 h^2 & 0 & -9450 h^0 \\
0 & 420 h^2 & 0 & 800 h^0 & 0 \\
236.25 h^2 & 0 & -9450 h^0 & 0 & 44100 h^0
\end{bmatrix} \begin{bmatrix}
A' \\
B' \\
D' \\
F' \\
G'
\end{bmatrix}
\]

(8)

(vi) The modulus matrix for the material can now be reconstructed at any chosen point by
evaluating the polynomial expression for \( C(z) \) at the chosen value of \( z \).

3. Thermoelastic behaviour
A broadly similar approach is taken for defining equivalent graded thermoelastic behaviour,
though the approach here is to obtain graded thermoelastic properties which result in equivalent
distributions of thermal stress for a given temperature distribution.

(i) An analogy is made with the use of “unit force” \( N \) and “unit moment” \( M \) vectors in
classical laminate theory [3] (corresponding to values of direct and shear force and bending
and twisting moment per unit plate width), where the constraint \( \sigma_z = \tau_{xz} = \tau_{yz} \) is implied.
These quantities are simply the integrals of the stress state \( \sigma \) and its product with \( z \) through
the thickness of the laminate (where, for these equations (9a) and (9b) only, \( h \) refers to the
thickness of the whole laminate, in accordance with conventional CLT notation):

\[
N = \int_{-h/2}^{h/2} \sigma(z)dz = \int_{-h/2}^{h/2} \bar{\sigma}(z)\epsilon(z)dz
\]

(9a)

\[
M = \int_{-h/2}^{h/2} z\sigma(z)dz = \int_{-h/2}^{h/2} z\bar{\sigma}(z)\epsilon(z)dz
\]

(9b)
(ii) In the present context we define broadly similar vectors of thermal “forces” etc. which will be denoted $\mathbf{N}'$, $\mathbf{M}'$ etc. which are now integrals of the (3D) thermally-induced stress state $\sigma^{(T)}$ and its product with $z$, and additionally $z^2$. Some of the terms which result have the physical meaning of the forces and moments which arise when strains $e^{(T)}$ due to thermal expansion are prohibited, and correspond to the resultant forces and moments required to force the expanded element of material back to its original shape. The terms involving integrals of $z^2\sigma_z$ etc. should be regarded as the summary of the effect of stress variations in a constrained, thermally-loaded material. Specifically, three sets of integrals are defined:

\[
\mathbf{N}' = \int_{-h/2}^{h/2} \sigma(z) dz = \int_{-h/2}^{h/2} C(z)e^T(z)dz = \int_{-h/2}^{h/2} C(z)\alpha(z)\Delta T(z)dz \quad (10a)
\]

\[
\mathbf{M}' = \int_{-h/2}^{h/2} z\sigma(z) dz = \int_{-h/2}^{h/2} zC(z)e^T(z)dz = \int_{-h/2}^{h/2} zC(z)\alpha(z)\Delta T(z)dz \quad (10b)
\]

\[
\mathbf{P}' = \int_{-h/2}^{h/2} z^2\sigma(z) dz = \int_{-h/2}^{h/2} z^2C(z)e^T(z)dz = \int_{-h/2}^{h/2} z^2C(z)\alpha(z)\Delta T(z)dz \quad (10c)
\]

(iii) A temperature distribution is assumed (in the present case, a linear distribution), under which the real and equivalent materials are to be forced to behave in an equivalent manner:

\[
\Delta T(z) = \Delta T^{(0)} + z\Delta T^{(1)} \quad (11)
\]

(iv) For the true laminate, inserting the assumed temperature distribution into equations (10a)–(10c) results in the following expressions:

\[
\mathbf{N}' = \sum_{k=1}^{N} C_k\alpha_k \int_{z_{k-1}}^{z_k} \left( \Delta T^{(0)} + z\Delta T^{(1)} \right) dz = \begin{bmatrix} \Delta T^{(0)} & \Delta T^{(1)} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \quad (12a)
\]

\[
\mathbf{M}' = \sum_{k=1}^{N} C_k\alpha_k \int_{z_{k-1}}^{z_k} z \left( \Delta T^{(0)} + z\Delta T^{(1)} \right) dz = \begin{bmatrix} \Delta T^{(0)} & \Delta T^{(1)} \end{bmatrix} \begin{bmatrix} L_2 \\ L_3 \end{bmatrix} \quad (12b)
\]

\[
\mathbf{P}' = \sum_{k=1}^{N} C_k\alpha_k \int_{z_{k-1}}^{z_k} z^2 \left( \Delta T^{(0)} + z\Delta T^{(1)} \right) dz = \begin{bmatrix} \Delta T^{(0)} & \Delta T^{(1)} \end{bmatrix} \begin{bmatrix} L_3 \\ L_4 \end{bmatrix} \quad (12c)
\]

where:

\[
L_1 = \sum_{k=1}^{N} C_k\alpha_k (z_k - z_{k-1}) \quad L_2 = \frac{1}{2} \sum_{k=1}^{N} C_k\alpha_k \left( z_k^2 - z_{k-1}^2 \right)
\]

\[
L_3 = \frac{1}{3} \sum_{k=1}^{N} C_k\alpha_k \left( z_k^3 - z_{k-1}^3 \right) \quad L_4 = \frac{1}{4} \sum_{k=1}^{N} C_k\alpha_k \left( z_k^4 - z_{k-1}^4 \right) \quad (13a-d)
\]

(v) Considering now the fictitious functionally graded material, the coefficient of thermal expansion is henceforward assumed to vary as a polynomial in $z$, involving as many coefficients as there are independent coefficients $L_1$...$L_4$ in equation (13a-d):

\[
\alpha(z) = \alpha^{(0)} + \alpha^{(1)}z + \alpha^{(2)}z^2 + \alpha^{(3)}z^3 \quad (14)
\]

It should also be recalled that $C(z)$ is expressed as a polynomial in $z$ via equation (4). These various expressions for $\Delta T(z)$, $C(z)$ and $\alpha(z)$ are into the following expressions for
thermal forces, moments etc. \( \mathbf{N''}, \mathbf{M''} \) and \( \mathbf{P''} \) in the graded material:

\[
\mathbf{N''} = \int_{-h/2}^{h/2} C(z)\alpha(z)\Delta T(z)\,dz, \quad \mathbf{M''} = \int_{-h/2}^{h/2} zC(z)\alpha(z)\Delta T(z)\,dz, \\
\mathbf{P''} = \int_{-h/2}^{h/2} z^2C(z)\alpha(z)\Delta T(z)\,dz
\]

After some manipulation including rearrangement into matrix form, the following expressions are obtained:

\[
\mathbf{N''} = \begin{bmatrix} \Delta T^{(0)} & \Delta T^{(1)} \end{bmatrix} \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \\ K_2 & K_3 & K_4 & K_5 \end{bmatrix} \begin{bmatrix} \alpha^{(0)} & \alpha^{(1)} & \alpha^{(2)} & \alpha^{(3)} \end{bmatrix}^T \quad (16a)
\]

\[
\mathbf{M''} = \begin{bmatrix} \Delta T^{(0)} & \Delta T^{(1)} \end{bmatrix} \begin{bmatrix} K_2 & K_3 & K_4 & K_5 \\ K_3 & K_4 & K_5 & K_6 \end{bmatrix} \begin{bmatrix} \alpha^{(0)} & \alpha^{(1)} & \alpha^{(2)} & \alpha^{(3)} \end{bmatrix}^T \quad (16b)
\]

\[
\mathbf{P''} = \begin{bmatrix} \Delta T^{(0)} & \Delta T^{(1)} \end{bmatrix}^T \begin{bmatrix} K_3 & K_4 & K_5 & K_6 \\ K_4 & K_5 & K_6 & K_7 \end{bmatrix} \begin{bmatrix} \alpha^{(0)} & \alpha^{(1)} & \alpha^{(2)} & \alpha^{(3)} \end{bmatrix}^T \quad (16c)
\]

where the 6 \times 6 submatrices \( K_1 \ldots K_7 \) are as follows:

\[
K_1 = C^{(0)} + C^{(2)} \frac{h^3}{12} + C^{(4)} \frac{h^5}{80}, \quad K_2 = C^{(1)} \frac{h^3}{12} + C^{(3)} \frac{h^5}{80}, \\
K_3 = C^{(0)} \frac{h^3}{12} + C^{(2)} \frac{h^5}{80} + C^{(4)} \frac{h^7}{448}, \quad K_4 = C^{(1)} \frac{h^3}{12} + C^{(3)} \frac{h^5}{80}, \\
K_5 = C^{(0)} \frac{h^5}{80} + C^{(2)} \frac{h^7}{448} + C^{(4)} \frac{h^9}{2304}, \quad K_6 = C^{(1)} \frac{h^5}{80} + C^{(3)} \frac{h^7}{448}, \\
K_7 = C^{(0)} \frac{h^7}{448} + C^{(2)} \frac{h^9}{2304} + C^{(4)} \frac{h^{11}}{11264}, \quad (17a-g)
\]

(vi) Setting \( \mathbf{N'} = \mathbf{N''} \) etc., it is apparent from equations (12) and (16a) that the following set of equations can be constructed:

\[
\begin{aligned}
\begin{cases}
L_1 \\
L_2 \\
L_3 \\
L_4 
\end{cases} &= \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \\ K_2 & K_3 & K_4 & K_5 \\ K_3 & K_4 & K_5 & K_6 \\ K_4 & K_5 & K_6 & K_7 \end{bmatrix} \begin{bmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \end{bmatrix} \\
&= \begin{bmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \end{bmatrix}
\end{aligned} \quad (18)
\]

This equation system is straightforward to solve in order to find \( \alpha^{(0)} \ldots \alpha^{(3)} \).

4. Equivalent dynamic behaviour

A similar approach is can again be taken for the formulation of the equivalent density of the graded material for use in dynamic calculations, though it is emphasised that this remains untested at the time of writing. The velocity \( \dot{u} \) at a point may be expressed in terms of through-thickness position \( z \):

\[
\dot{u}(z) = \dot{u}^{(0)} + \dot{u}^{(1)}z + \dot{u}^{(2)}z^2
\]

The kinetic energy per unit area is as follows:

\[
\text{K.E.} = \frac{1}{2} \int_{-h/2}^{h/2} \rho(z)(\dot{u}(z))^T\dot{u}(z)\,dz \\
= \frac{1}{2} \int_{-h/2}^{h/2} \rho(z) \begin{bmatrix} \dot{u}^{(0)} & \dot{u}^{(1)} & \dot{u}^{(2)} \end{bmatrix} \begin{bmatrix} I & z & z^2 \\ z & z^2 & z^3 \\ z^2 & z^3 & z^4 \end{bmatrix} \begin{bmatrix} \dot{u}^{(0)} \\ \dot{u}^{(1)} \\ \dot{u}^{(2)} \end{bmatrix}
\]
For the true laminate, this evaluates to:

$$\text{K.E.} = \frac{1}{2} \left[ \dot{\mathbf{u}}^{(0)} \quad \dot{\mathbf{u}}^{(1)} \quad \dot{\mathbf{u}}^{(2)} \right] \left[ \begin{array}{ccc}
M_1 & M_2 & M_3 \\
M_4 & M_5 & \mathbf{I} \\
M_6 & M_7 & \mathbf{I} \\
\end{array} \right] \left\{ \begin{array}{c}
\dot{\mathbf{u}}^{(0)} \\
\dot{\mathbf{u}}^{(1)} \\
\dot{\mathbf{u}}^{(2)} \\
\end{array} \right\} \quad (21)$$

where:

$$M_1 = \sum_{k=1}^{N} \rho_k (z_k - z_{k-1}) \quad \ldots \quad M_5 = \frac{1}{5} \sum_{k=1}^{N} \rho_k (z_k^5 - z_{k-1}^5) \quad (22a-e)$$

For the graded material, the following variation of density $\rho$ with position $z$ is assumed in order to give an equal number of unknown coefficients as independent terms $M_1 \ldots M_5$:

$$\rho(z) = \rho^{(0)} + \rho^{(1)} z + \rho^{(2)} z^2 + \rho^{(3)} z^3 + \rho^{(4)} z^4 \quad (23)$$

The expression for kinetic energy can now be evaluated as

$$\text{K.E.} = \frac{1}{2} \left[ \dot{\mathbf{u}}^{(0)} \quad \dot{\mathbf{u}}^{(1)} \quad \dot{\mathbf{u}}^{(2)} \right] \left[ \begin{array}{ccc}
M'_1 & M'_2 & M'_3 \\
M'_4 & M'_5 & \mathbf{I} \\
M'_6 & M'_7 & \mathbf{I} \\
\end{array} \right] \left\{ \begin{array}{c}
\dot{\mathbf{u}}^{(0)} \\
\dot{\mathbf{u}}^{(1)} \\
\dot{\mathbf{u}}^{(2)} \\
\end{array} \right\} \quad (24)$$

where, in a similar manner to equations (6) and (7), $M'_1 = \rho^{(0)} h + \rho^{(2)} \frac{h^3}{12} + \rho^{(4)} \frac{h^5}{360}$ etc. By equating $M_1 = M'_1$ etc., expressing the equations in matrix form and inverting analytically following a very similar procedure to that outlined in section 2, the coefficients $\rho^{(0)} \ldots \rho^{(4)}$ can be found.

5. Non-standard integration scheme

It is apparent from the above process that, as the material properties vary within the element, the order of variation of strain energy density will be higher than that encountered within conventionally-formulated elements. This implies that, in order to capture the element stiffness accurately, a high integration order must be used within the element. However, this high-order variation occurs only in the through-thickness direction, so it would be wasteful to employ high-order integration in the in-surface directions. A novel integration scheme was therefore developed involving different integration orders in different directions. It was extended from a concept by Rathod et al., [4] who developed integration schemes of arbitrary order by degenerating straightforward hexahedral Gaussian integration schemes. The approach taken here is to degenerate a pentahedral (wedge) integration scheme so that triangular integration is used within planes parallel to the base of the tetrahedron, but an arbitrary order of conventional one-dimensional Gaussian integration is performed in the “height” direction. Full integration can therefore be retained for the direction closest to through-thickness while reduced integration similar to that used in reduced-integration wedge elements is used for integration within planes approximately parallel to the planes of the plies. In practice, the orientation of the element is unknown a priori so the integration scheme was further modified by permuting the three local axes. A fourth case, relating to the situation where the “base” of the element maps to the oblique face of the local element, is also obtained. The four possible integration schemes are illustrated in Figure 1. If none of the faces is close to being parallel to the plane of the plies, full integration can of course be used.

6. Trial FE implementation

A full trial implementation of the concept for static mechanical loads has taken place within the LibMesh open-source programming framework [5]. The program makes it possible to compare the results of the present element with those of an existing (hexahedral) element.
Formal verification of the element has not yet taken place, but preliminary verification of the static elastic and thermoelastic behaviour demonstrates the potential usefulness of the element. The test case consists of the unit cube, restrained against normal movement on the surfaces \( x = 0, \, y = 0 \) and \( y = 1 \), and with the node at \((0,0,0)\) restrained against movement in the \( z\)-direction. Three sets of properties (given in consistent units) have been chosen:

(i) Isotropic: \( E = 100000, \, \nu = 0.3, \, G = 38461, \, \alpha = 13 \times 10^{-6} \, ^\circ\text{C} \)

(ii) Cross-ply asymmetric: \( E_1 = 50000, \, E_2 = E_3 = 10000, \, \nu_{12} = 0.3, \, \nu_{23} = 0.35, \, G_{12} = G_{13} = 5000, \, G_{23} = 4000, \, \alpha_1 = 5 \times 10^{-6} \, ^\circ\text{C}, \, \alpha_2 = \alpha_3 = 15 \times 10^{-6} \, ^\circ\text{C} \) 4 plies of 0.25 thickness each, with the ply sequence \( 0^\circ/90^\circ/0^\circ/90^\circ \)

(iii) Angle-ply asymmetric: as for cross-ply, but with ply sequence \( 0^\circ/45^\circ/-45^\circ/90^\circ \)

In each case the problem is modelled in two ways: (a) via meshes of hexahedral (brick) elements with \( 2 \times 2 \times 2 \) reduced integration, with separate elements assigned to each layer (b) via a meshes of graded tetrahedral elements, assuming nine layers of 3 Gauss points.

In each case two meshes have been used: one coarse (Hex1, 32 elements, 261 nodes, 783 dof; and Tet1, 6 elements, 27 nodes, 81 dof), the other somewhat finer (Hex2, 128 elements, 785 nodes, 2355 dof; and Tet2, 227 elements, 462 nodes, 1386 dof). The solution noted as “CLT” relates (in the case of the elastic stiffness) simply to the corresponding direct \((A)\) and bending \((D)\) stiffnesses from classical laminate theory, and (in the case of thermal forces and moments) the numerical integration of thermoelastic stresses following a CLT approach. The true angle-ply laminate situation involves significant interlaminar shearing so the CLT solution was of limited applicability. For reasons of space, only the results relating to the cross-ply laminate are tabulated in detail.

For the mechanical test cases, the surface \( x = 1 \) is subjected to a prescribed tensile displacement of 1 unit or a bending displacement of 1 radian as appropriate. The reaction forces (in the tension cases) and moments (in the bending cases) are presented in Table 1. Results for the isotropic and cross-ply test cases showed excellent agreement with the reference solutions; for the angle-ply laminate (not shown here) results were still good, with errors of around 1% compared with the Hex2 solution.

Informal verification of the thermoelastic analysis took place by again considering a unit cube, but this time with in-plane displacements constrained to zero, and with a thermal gradient of \( \Delta T = 100 + 200z \, ^\circ\text{C} \) imposed. The reaction forces and moments are again presented in Table 1.

The agreement with the CLT test case is good (typical error up to 1-2%, not quite as good as in the mechanical load examples) except for the angle ply example. Comparison in that case is again more meaningful against the Hex2 example, showing agreement to within a few percent.

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**Figure 1.** Four variants on the non-standard integration scheme.
Table 1. Static stiffness results: reactions for tension are units of force, reactions for bending are units of moment.

| Load case | Solution/mesh | CLT | Hex1 | Hex2 | Tet1 | Tet2 |
|-----------|---------------|-----|------|------|------|------|
|           | Integration scheme | 2 × 2 × 2 | 2 × 2 × 2 | 9 × 3 | 9 × 3 |
| Mechanical | Tension       | 30549.89817 | 30549.8 | 30549.8 | 30383.2 | 30552 |
|           | diff. from CLT | 0.000% | 0.000% | -0.546% | 0.007% |
|           | diff. from Hex2 | 0.000% | -0.545% | 0.007% |
|           | Bending       | 2545.824847 | 2545.81 | 2545.82 | 2522.9 | 2544.97 |
|           | diff. from CLT | -0.001% | 0.000% | -0.900% | -0.034% |
|           | diff. from Hex2 | 0.000% | -0.900% | -0.033% |
| Thermal   | Thermal force | -21.7668 | -21.7667 | -21.7668 | -22.2214 | -22.0095 |
|           | diff. from CLT | 0.000% | 0.000% | 2.089% | 1.115% |
|           | diff. from Hex2 | 0.000% | 2.089% | 1.115% |
|           | Thermal moment | -3.076205 | -3.0762 | -3.0762 | -3.13285 | -3.10479 |
|           | diff. from CLT | 0.000% | 0.000% | 1.841% | 0.929% |
|           | diff. from Hex2 | 0.000% | 1.842% | 0.929% |

7. Conclusions
(i) A novel approach for formulating a tetrahedral element for the analysis of laminates for mechanical and thermoelastic loading is presented. The approach taken is to create a fictitious functionally-graded material whose elastic and thermoelastic behaviour at structural level is the same as that of the true laminate.

(ii) The element has been implemented in trial form using the LibMesh finite element library. The underlying concept has also been implemented in non-FE form to permit closer examination of the algorithms.

(iii) Some informal verification results have been presented; these are not intended to be complete. In particular, the dynamic aspects still require implementation and testing.

(iv) Results are very encouraging, with errors ranging from negligible to the order of a few percent. No shear locking seems to be apparent.

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