Existence of Brans-Dicke flat universe

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In this paper, we examine the existence of Brans-Dicke flat universe by performing independent/combined test from H(z), SN Ia and BAO (Baryon Acoustic Oscillation) observational data sets. We find the constraints on \( \Omega_m^0 \) and \( H_0 \) from H(z), H(z) + SN Ia and H(z) + BAO data sets as 0.259, 0.224 & 0.279, 0.0713 Gyr^{-1}, 0.0744 Gyr^{-1} & 0.0704 Gyr^{-1} respectively. We have derived the cosmological quantities in term of red-shift and estimates it’s present values. The derived model shows a signature flip in the evolution of deceleration parameter at 0.65 \( \leq z \leq 0.95 \) which clearly indicates that the current universe is in accelerating mode but it was in decelerating mode from its beginning. The present value of deceleration parameter and age of universe of BD flat universe are found in close proximity with corresponding values, obtained by recent Plank collaboration and WMAP observations.

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I. INTRODUCTION

The cosmological constant (\( \Lambda \)) cold dark matter model of universe is credited today as the most simplest and successful cosmological model that describes the dynamics of present universe with acceleration but it suffers many problems on theoretical as well as observational ground, which are reported by numerous cosmologists [1–6]. So, in the literature various dark energy models exist with dynamical behavior of equation of state (EoS) parameter \( \omega^{(\text{eff})} \) [7–10]. It is well known that \( \omega^{(\text{eff})} = -1 \) represents the \( \Lambda \)CDM universe [11, 12]. However, to concur with high precision observational data, we have to have small deviation from \( \Lambda \)CDM model or modification to general theory of relativity but indeed, we still do not have a promising and concrete fundamental theory to handle this issue [13].

In 1961, Brans-Dicke [14] had proposed a scalar-tensor theory in which the average expansion rate is modified due to alignment of scalar field with geometry while the geometrization of tensor field remains alone [15]. So, both the scalar and tensor field have more or less intrinsic geometrical consequences and finally executes a more general method of geometrizing gravitation.

With motivation provided by above, it is worth to investigate the effect of \( \Lambda \) in Brans-Dicke gravity. In 2010, Setare and Jamil [16] have constructed the chameleon model of holographic dark energy in BD theory and compared his findings with general theory of relativity (GTR) for large value of coupling constant \( \omega \). It is to be noted that for large value of \( \omega \), BD theory has nice agreement with GTR. Later on, Karmani et al [17] and Jamil et al [18] have investigated some chameleon cosmological models with different physical contexts in the frame work of BD gravity. In this connection, Yadav [19] and Ali et al [20] have investigated some non-isotropic, non-flat and in-homogeneous model of accelerating universe in Brans-Dike scalar-tensor theory of gravitation. In 1984, Singh and Singh [21] had investigated Brans-Dike cosmological model by choosing \( \Lambda = \Lambda(\phi) \) and this idea was extended by Azad and Islam [22] in an-isotropic and homogeneous Bianchi - I space time. A higher dimensional interacting scalar field and Higgs model is obtained by Qiang et al [23]. Later on, using classical approach, Smolyakov [24] has...
investigated Brans-Dicke cosmological model with effective value of $\Lambda$ in 5 D. Das and Banerjee \cite{24} have investigated model of accelerating universe with variable deceleration parameter in BD theory. Some other useful applications of Brans-Dicke theory are given in Refs. \cite{26–30}. In the recent past, Perivolaropoulos \cite{31} has reported the strongest constraints of Brans-Dicke coupling parameter as $\omega > 40000$ from the solar system scale at 2$\sigma$ confidence level. Recently, Avilez and Skordis \cite{32} have reported cosmological constraints on the mass-less Brans-Dicke theory using CMB data. Also the forecasting of observational future of scalar-tensor theory is reported in the literature \cite{33,34}.

The outlines of the present work is as follows: Section II deals with the model and basic equations. In section III we describe the data and likelihoods, $\chi^2$ test and best fit curve for the model under consideration in detail. The mathematical expressions for cosmological quantities and it’s significance are given in section III, VII. Finally the closing remarks are listed in section VIII.

II. THE MODEL AND BASIC EQUATIONS

The Einstein’s field equations in Brans-Dicke theory is given by

$$R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = \frac{8\pi}{c^4} T_{ij}$$

and

$$-\frac{\omega}{\phi^2} \left( \phi_i \phi_j - \frac{1}{2} g_{ij} \phi \phi \right) - \frac{1}{\phi} (\dot{\phi} - g_{ij} \Box \phi)$$

(1)

where $\omega$ is the Brans-Dicke coupling constant; $\phi$ is Brans-Dicke scalar field and $\Lambda$ is the cosmological constant.

The FRW space-time is read as

$$ds^2 = c^2 dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$$

(3)

where $a(t)$ is scale factor.

The energy momentum tensor of perfect fluid is given by

$$T_{ij} = (p + \rho) u_i u_j - p g_{ij}$$

(4)

Here, $p$ and $\rho$ are the isotropic pressure and energy density of the matter under consideration.

where $u^i u_i = 1$ and $u^i$ is the four velocity vector.

The field equations \cite{1} for space-time \cite{3} are read as

$$\frac{\ddot{a}}{a} + \frac{\dot{a}}{a} - \frac{\omega}{2\phi^2} \frac{\dot{\phi}}{\phi} - \frac{\dot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi} = \frac{8\pi}{c^4} p + \Lambda c^2$$

(5)

The energy momentum tensor of perfect fluid is given by

$$T_{ij} = (p + \rho) u_i u_j - p g_{ij}$$

(4)

$$\frac{\ddot{a}}{a} + \frac{\dot{a}}{a} - \frac{\omega}{2\phi^2} \frac{\dot{\phi}}{\phi} - \frac{\dot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi} = \frac{8\pi}{c^4} p + \Lambda c^2$$

(6)

$$\frac{\ddot{\phi}}{\phi} + \frac{3}{\omega} \frac{\dot{\phi}}{\phi} = \frac{8\pi(\rho - 3p)}{(2\omega + 3)c^2 \phi} + \frac{2\Lambda c^2}{2\omega + 3}$$

(7)

where over dot denotes derivatives with respect to time $t$.

A. The model: Brans-Dicke flat universe

The density parameters are read as

$$\Omega_m = \frac{8\pi \rho_m}{3c^2 H^2 \phi}$$

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$$

(8)

where $\rho_m = (\rho_m)_0 a^{-3}$ and $\Omega_m$ and $\Omega_\Lambda$ represent the dimensionless density parameters for matter and $\Lambda$-energy respectively.

For flat FRW model, we know that

$$\Omega_m + \Omega_\Lambda = 1$$

(9)

The scale factor $a$ and $\phi$ in connection with $z$ are read as

$$a = \frac{a_0}{1 + \frac{1}{(1 + z)\beta \Lambda}}$$

(10)

where $a_0$ is the present value of scale factor.

Equations \cite{8}, \cite{9} and \cite{10} leads to

$$H_{BD} = \frac{H_0}{\left(1 + \frac{5\omega + 6}{6(\omega + 1)}\right)^\frac{1}{2}} \left[ (\Omega_m)_0 (1 + z)^{\frac{3\omega + 4}{2\omega + 3}} + (\Omega_\Lambda)_0 \right]^\frac{1}{2}$$

(11)

where $H_0$ is the present value of Hubble’s parameter and in this paper it is obtained as $H_0 = 0.0713$ $Gyr^{-1}$ $\sim 69.76$ $km s^{-1} Mpc^{-1}$ by bounding the model under consideration with 38 OHD data points (see Fig. 1).

B. $\Lambda$CDM model

For $\Lambda$CDM model, the field equations of flat universe are given by

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}}{a} - \frac{\omega}{2\phi^2} \frac{\dot{\phi}}{\phi} - \frac{\dot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi} = \frac{8\pi}{c^4} p + \Lambda c^2$$

(12)

$$3\frac{\dot{\phi}}{a} + \frac{\dot{\phi}}{\phi} = \frac{8\pi}{c^4} p + \Lambda c^2$$

(13)

In this case the expression for Hubble parameter for the universe filled with pressure-less matter and $\Lambda$ energy is read as

$$H_{\Lambda CDM} = H_0[(\Omega_m)_0 (1 + z)^3 + (\Omega_\Lambda)_0]^{\frac{1}{2}}$$

(14)
III. DATA AND LIKELIHOODS

We consider 38 OHD points given in the table 1 of the paper of Farooq et al. [36] to estimate the present values of Hubble’s constant and energy densities parameter of derived model. For this sake, we define $\chi^2$ as following:

$$\chi^2_{OHD}(H_o, (\Omega_m)_0) = \sum_{i=1}^{38} \frac{[H(z_i, H_o, (\Omega_m)_0) - H_{obs}(z_i)]^2}{\sigma_i}$$

(15)

where $H_{obs}(z_i)$ is the observed value of Hubble’s parameter with standard deviation $\sigma_i$ and $H(z_i, H_o, (\Omega_m)_0)$ is the theoretical values obtained from bounding equation (14) with 38 OHD points. We find that the best fit values of the parameters are $H_0 = 0.0713 Gy^{-1} \sim 69.76 km s^{-1} Mpc^{-1}$ & $(\Omega_m)_0 = 0.259$ together with reduced $\chi^2_{OHD} = 0.91$. 

The distance modulus is given by

$$\mu(z) = m_b - M = 5 \log_{10}D_L(z) + \mu_0$$

(16)

where $m_b$, $M$ are the apparent magnitude and absolute magnitude of a standard candle respectively. The luminosity distance $D_L$ and nuisance parameter ($\mu_0$) are read as

$$D_L = \frac{c(1+z)}{H_0} \int_0^z \frac{dz}{h(z)}; \quad h(z) = \frac{H(z)}{H_0}$$

(17)

and

$$\mu_0 = 5 \log_{10} \left( \frac{H_0^{-1}}{Mpc} \right) + 25$$

(18)

Similarly, joint $\chi^2$ test for combined observational $H(z)$ and SN Ia data is obtained as

$$\chi^2_{joint} = \chi^2_{OHD} + \chi^2_{SN}$$

From $\chi^2_{joint}$ test along with $H(z)$ and SN Ia, we obtain that the best fit values of the parameters are $H_0 = 0.0744 Gy^{-1} \sim 72.80 km s^{-1} Mpc^{-1}$ & $(\Omega_m)_0 = 0.224$ together with reduced $\chi^2_{OHD} = 0.86$. 

Also, the joint $\chi^2$ test for combined observational $H(z)$ and BAO data is obtained as

$$\chi^2_{joint} = \chi^2_{OHD} + \chi^2_{BAO}$$

From $\chi^2_{joint}$ test with $H(z)$ and BAO observational data, we find that the best fit values of the parameters are $H_0 = 0.0704 Gy^{-1} \sim 68.89 km s^{-1} Mpc^{-1}$ & $(\Omega_m)_0 = 0.279$ together with reduced $\chi^2_{OHD} = 0.62$. The summary of numerical result is listed in table 1.

IV. OM(Z) Diagnostics

The Om(z) parameter of BD flat universe is read as

$$Om(z)_{BD} = \frac{[(\Omega)_0(1+z)^{\frac{3\omega+4}{\omega+1}} + (\Omega_\Lambda)_0]^\frac{2}{3} - 1}{(1+z)^3 - 1}$$

(19)

The behavior of Om(z) parameter for BD flat universe is graphed in Fig. 7. The Om(z) parameter of BD flat universe have positive values and shows a small variation with it’s value in the case of ΛCDM universe. The negative, zero and positive values of Om(z) represents the quintessence, ΛCDM and phantom dark energy models respectively [39]. Thus the derived BD flat universe is equivalent to the phantom dark energy universe.

V. Deceleration Parameter

The differential age of the galaxies are obtained by following equation [37, 38]

$$H(z) = - \frac{1}{1+z} \frac{dz}{dt}$$

(20)

The deceleration parameter (DP) in terms of H(z) is given by

$$q = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{(1+z)H'(z)}{H}$$

(21)

where $H'(z) = \frac{dH(z)}{dz}$.

Equations (11) and (21) lead to

$$q = -1 + \frac{\frac{3\omega+4}{\omega+1}(\Omega_m)_0(1+z)^{\frac{3\omega+4}{\omega+1}}}{2[(\Omega_m)_0(1+z)^{\frac{3\omega+4}{\omega+1}} + (\Omega_\Lambda)_0]}$$

(22)

The present value of DP ($q_0$) is obtained by putting $z = 0$ in equation (22) i. e.

$$q_0 = -1 + \frac{\frac{3\omega+4}{\omega+1}(\Omega_m)_0}{2[(\Omega_m)_0 + (\Omega_\Lambda)_0]}$$

(23)

The dynamics of DP versus $z$ is shown in Fig. 8. For large value of $z$, $q$ varies with positive sign which confirms that the early universe was in decelerating mode which turns into accelerating mode at $0.65 \leq z \leq 0.95$ (see Fig. 8).

VI. Present Age of BD Flat Universe

The age of BD flat universe is computed with following integral

$$t = \int dt = - \int \frac{dz}{(1+z)H(z)}$$

(24)
TABLE I: Summary of the numerical result.

| Source/Data | Model parameters | BD flat universe | ΛCDM flat universe |
|-------------|------------------|------------------|--------------------|
| H(z)        | (Ω_m)_0          | 0.259            | 0.302              |
| H(z)        | (Ω_Λ)_0          | 0.741            | 0.698              |
| H(z)        | ω                | 4.5 × 10^4       | -                  |
| H(z)        | H_0              | 0.0713 (Gyr^{-1})| 0.0701 (Gyr^{-1})  |
| H(z)        | χ_2_{min}/dof    | 0.91             | 0.895              |
| H(z)+SN Ia  | (Ω_m)_0          | 0.224            | 0.257              |
| H(z)+SN Ia  | (Ω_Λ)_0          | 0.776            | 0.743              |
| H(z)+SN Ia  | H_0              | 0.0744 (Gyr^{-1})| 0.0735 (Gyr^{-1})  |
| H(z)+SN Ia  | χ_2_{min}/dof    | 0.86             | 0.89               |
| H(z)+BAO    | (Ω_m)_0          | 0.279            | 0.317              |
| H(z)+BAO    | (Ω_Λ)_0          | 0.721            | 0.683              |
| H(z)+BAO    | H_0              | 0.0704 (Gyr^{-1})| 0.0696 (Gyr^{-1})  |
| H(z)+BAO    | χ_2_{min}/dof    | 0.62             | 0.55               |

FIG. 1: (Left panel) The likelihood contour at 1σ and 2σ confidence level around the best fit values as $H_0 = 0.0713 \text{ Gyr}^{-1}$ and $(\Omega_m)_0 = 0.259$ in $H_0 - \Omega_m$ plane obtained by bounding the model under consideration with 38 OHD points. (Right panel) The likelihood contour at 1σ and 2σ confidence level around the best fit values as $H_0 = 0.0744 \text{ Gyr}^{-1}$ and $(\Omega_m)_0 = 0.224$ in $H_0 - \Omega_m$ plane obtained by bounding the ΛCDM universe with 38 OHD points.

FIG. 2: (Left panel) The likelihood contour at 1σ and 2σ confidence level around the best fit values as $H_0 = 0.0701 \text{ Gyr}^{-1}$ and $(\Omega_m)_0 = 0.302$ in $H_0 - \Omega_m$ plane obtained by bounding the model under consideration with H(z) + SN Ia data. (Right panel) The likelihood contour at 1σ and 2σ confidence level around the best fit values as $H_0 = 0.0735 \text{ Gyr}^{-1}$ and $(\Omega_m)_0 = 0.257$ in $H_0 - \Omega_m$ plane obtained by bounding the ΛCDM universe with H(z) + SN Ia data.
FIG. 3: (Left panel) The likelihood contour at 1σ and 2σ confidence level around the best fit values as $H_0 = 0.0704 \, Gyr^{-1}$ and $(\Omega_m)_0 = 0.279$ in $H_0 - \Omega_{m0}$ plane obtained by bounding the BD flat universe with observational $H(z) +$ BAO. (Right panel) The likelihood contour at 1σ and 2σ confidence level around the best fit values as $H_0 = 0.0696 \, Gyr^{-1}$ and $(\Omega_m)_0 = 0.317$ in $H_0 - \Omega_{m0}$ plane obtained by bounding the $\Lambda$CDM universe with $H(z) +$ BAO data.

FIG. 4: The observational 38 $H(z)$ points are shown with error bar (red colour). The best fit Hubble’s parameter ($H(z)$) curve (solid black line) based on theoretical values is shown.

FIG. 5: The observational 580 SN Ia points are shown with error bar (red colour). The best fit distance modulus $\mu(z)$ curve (solid black line) based on theoretical values is shown.

TABLE II: Numerical values of $q_0$, $j_0$ and $t_0$ for BD flat universe.

| Source/Data       | $q_0$      | $j_0$       | $t_0$   |
|-------------------|------------|-------------|---------|
| $H(z)$            | $-0.611$   | $1.000007$  | $14.081$ (Gyr) |
| $H(z) +$ SN Ia    | $-0.663$   | $1.000006$  | $14.032$ (Gyr) |
| $H(z) +$ BAO      | $-0.581$   | $1.000008$  | $13.973$ (Gyr) |
FIG. 6: The observational $H(z)$ and BAO data points are shown with error bar (red colour). The best fit Hubble’s parameter ($H(z)$) curve (solid black line) based on theoretical values is shown.

FIG. 7: The behavior of $\Omega_m(z)$ parameter versus $z$ for BD flat universe.

FIG. 8: The dynamics of deceleration parameter for BD flat universe.

FIG. 9: The dynamics of jerk parameter for BD flat universe.
Using equation [11] in equation [24], we obtain
\[ t = -\int \frac{\left(1 + \frac{5\omega + 6}{6(\omega + 1)^2}\right)^{\frac{3}{2}} dz}{(1 + z)H_0(\Omega_m)_0(1 + z)^{\frac{3\omega + 4}{\omega + 1}} + (\Omega_L)_0^{\frac{3}{2}}} \] (25)

The present age of derived universe is computed as following
\[ t_0 = \lim_{z \to 0} \left[ -\int (1 + z)^{\frac{3\omega + 4}{\omega + 1}} dz \right] \frac{2(1 + \omega)}{(4 + 3\omega)H_0 \sqrt{\Omega_0}} \] (26)

The numerical values of DP and age of BD flat universe at present, obtained by bounding the derived model with H(z), H(z) + SN Ia and H(z) + BAO observations are tabulated in Table II. The recent observations of WMAP [10] and PLANK [3] confirm that the present age of universe is 13.73 Gyr and 13.799 Gyr which are close to the computed age of derived BD flat universe. Thus the derived model have pretty consistency with observations.

VII. JERK PARAMETER

The jerk parameter (j) is connected to the third order derivatives of scale factor and hence play a vital role in cosmology [11]. Mathematically, it is defined as
\[ j = \frac{\dot{a}}{aH^3} = 1 - (1 + z)H'(z) + \frac{1}{2} (1 + z)^2 \frac{H''(z)}{H^2(z)} \] (27)

Equation [11] and equation [27] lead to
\[ j = 1 - \frac{(\Omega_m)_0 \frac{3\omega + 4}{\omega + 1}(1 + z)^{\frac{3\omega + 4}{\omega + 1}} + (\Omega_L)_0}{(\Omega_m)_0(1 + z)^{\frac{3\omega + 4}{\omega + 1}} + (\Omega_L)_0^{\frac{3}{2}}} - \frac{(\Omega_m)_0 \left[ \frac{3\omega + 4}{\omega + 1} + (\Omega_L)_0 \right]}{2[(\Omega_m)_0(1 + z)^{\frac{3\omega + 4}{\omega + 1}} + (\Omega_L)_0^{\frac{3}{2}}]} \] (28)

The graphical behavior of jerk parameter is plotted in Fig. 9 and its present value, listed in Table II, is obtained by bounding the derived model with H(z), H(z)+SN Ia and H(z) + BAO data sets. The observed value of jerk parameter for ΛCDM universe is 1 at present epoch. It is worth to note that the present value of jerk parameter for BD flat universe is very close to 1. We can therefore note that BD flat universe is well consistent with recent astrophysical observations.

VIII. CONCLUDING REMARKS

Brans-Dicke theory of gravitation successfully pass the Mach’s principle and has the elegance of describing both the inflation as well as late time acceleration in our universe without aid of any exotic fluids. The present work deals with the modeling of flat universe in BD theory of gravitation and it’s independent/combined analysis with different observational data sets. In general, the derived BD flat universe has nice agreement with recent astrophysical observations. Some key observations are as following:

- The estimation for numerical value of DP is done by bounding the model under consideration with recent H(z), H(z)+SN Ia and H(z) + BAO observational data. The value of DP at present is in close agreement with it’s observed value obtained by Plank collaboration [42]. The behavior of q(z) is evident in Fig. 8. We observe that BD flat universe shows a transition in the signature of q(z) and the transition red-shift is pretty consistent with Plank collaborations [3, 42].

- We estimate the present age of derived universe as 14.081 Gyr, 14.032 Gyr, 13.973 Gyr from H(z), H(z) + SN Ia and H(z) + BAO respectively.

- We have carried out Om(z) and jerk parameter diagnostics for BD flat universe and it’s behaviors are graphed in Fig. 7 and Fig. 10 respectively.

- We find the constraints on (Ω_m)_0 and H_0 of both BD flat universe as well ΛCDM universe from independent/combined observational data sets (see Figs. 1, 2, 3). The H(z) data constrain (Ω_m)_0 as (Ω_m)_0 = 0.259 and H_0 = 0.0714 Gyr\(^{-1}\) while the joint test of H(z) & SN Ia estimates (Ω_m)_0 = 0.224 and H_0 = 0.0744 Gyr\(^{-1}\). The joint test of H(z) & BAO gives (Ω_m)_0 = 0.224 and H_0 = 0.0744 Gyr\(^{-1}\). The result of numerical analysis is summarized in Table I.

- The best fitting curve of derived BD flat universe fits pretty well with H(z), H(z) + SN Ia and H(z) + BAO observational data points (See Figs. 4, 5, 6).

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