Vortex-Antivortex Lattice in Ultra-Cold Fermi Gases

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We discuss ultra-cold Fermi gases in two dimensions, which could be realized in a strongly confining one-dimensional optical lattice. We obtain the temperature versus effective interaction phase diagram for an s-wave superfluid and show that, below a certain critical temperature $T_c$, spontaneous vortex-antivortex pairs appear for all coupling strengths. In addition, we show that the evolution from weak to strong coupling is smooth, and that the system forms a square vortex-antivortex lattice at a lower critical temperature $T_M$.

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The presence of quantized vortices is a strong indication of the existence of a superfluid state. The recent experiment from the MIT group [1] marked the first observation of vortices in atomic Fermi gases. This result complements previous experiments involving s-wave ultra-cold Fermi gases [2, 3, 4, 5, 6]. These studies correspond to the experimental realization of the theoretically proposed BCS-to-BEC (weak-to-strong coupling) crossover in three dimensional realization of the theoretically proposed BCS-boson superfluidity of confined 2D [8, 9, 10]. More recent generalizations [11] of these earlier theoretical results indicate that the critical temperature in a trap geometry is higher than in the continuum case throughout the BCS-to-BEC evolution. S-wave Fermi systems have also been investigated in one dimensional (1D) [12] and three-dimensional (3D) [13] optical lattices. Furthermore, the effects of dimensionality have been analyzed in p-wave Fermi [14] and Bose [15] systems. In the particular case of Bose systems, the presence of topological defects associated with the phase of the order parameter has been detected in a nearly two-dimensional geometry [16]. These optical lattice experiments [12, 13, 14, 15] are now allowing the exploration of dimensional (1D, 2D, and 3D) and correlation effects in interacting Fermi gases.

Given all the recent advances in experimental techniques, we discuss the evolution from weak to strong coupling superfluidity of confined 2D s-wave ultra-cold Fermi gases. We show that a Berezinskii-Kosterlitz-Thouless (BKT) transition [14, 17] occurs at finite temperatures, and that the strong coupling produces a critical temperature $T_{BKT} = 0.5\epsilon_F$, where $\epsilon_F$ is the Fermi energy. In the strong coupling limit of a 2D Fermi gas, the superfluid transition is not characterized by Bose-Einstein condensation (BEC) as in 3D, but by the BKT transition. Below $T_{BKT}$, pairs of vortices and antivortices appear spontaneously for all couplings, and eventually condense into a vortex-antivortex (VA) square lattice as the temperature is lowered further. The lattice melting temperature is shown to be $T_M \approx 0.1\epsilon_F$ in the strong coupling limit, and the melting mechanism is controlled by dislocations [17, 18, 19].

The appearance of spontaneously generated VA bound states and the existence of the VA square lattice are major characteristics of the 2D physics proposed here. This spectacular effect could be measured via density or velocity sensitive techniques. Vortices or antivortices could be detected in a density sensitive experiment without stirring the condensate, but the topological charge associated with the sense of rotation should be detectable only in velocity sensitive experiments.

Two-dimensional Fermi systems can be prepared by means of a 1D optical lattice, where tunnelling between lattice sites is suppressed by a large trapping potential. A typical trapping potential is $V_{trap} = -V_0 \exp[-2(x^2 + y^2)/w^2] \cos^2(k_s z)$, where $\lambda_s = 2\pi/k_s$ is the wavelength of the light used in the laser beam. We assume that the width $w$ is such that $w \gg \lambda_F$, where $\lambda_F = 2\pi/k_F$ is proportional to the interparticle spacing of a Fermi gas with Fermi wavevector $k_F$. The atom transfer energy along the $z$ direction can be estimated using a simple WKB expression leading to $t_z = W_0 \exp\left[-\pi\sqrt{(V_0 - W_0)/E_c}\right]$, where $W_0 = \hbar\omega_0/2 = \sqrt{V_0 E_c}/2$. Here, $E_c = \hbar^2 k_F^2/2M$ is the recoil energy of atoms with mass $M$. Thus, in the 2D limit of $t_z \to 0$, there is no finite critical temperature for BEC, and the superfluid phase in both the BEC and BCS regimes is characterized by VA bound pairs.

To explore the physics described above, we consider a 2D continuum model of Fermi atoms of mass $M$ and density $n = k_F^2/2\pi$, with Hamiltonian ($\hbar = k_B = 1$)

$$\mathcal{H} = \sum_{k, \sigma} \xi_k \psi^\dagger_{k\sigma} \psi_{k\sigma} + \sum_{k, k', q} V_{k'k} b^\dagger_{qk} b_{k'q},$$

where $b_{qk} = \psi_{-k+q/2, \uparrow} \psi_{k+q/2, \downarrow}$ and $\xi_k = \epsilon_k - \mu$, with energy dispersion $\epsilon_k = k^2/2m$ and chemical potential $\mu$. Following the procedure discussed in detail in Ref. [20], we obtain the following separable expression for the interparticle potential in $k$-space,

$$V_{kk'} = -\lambda \Gamma(k) \Gamma(k'),$$

where $\lambda$ is the interaction strength and, in the particular case of s-wave symmetry ($\ell = 0$), $\Gamma(k) = (1 + k/k_0)^{-1/2}$, with $R_0 \sim k_0^{-1}$ playing the role of the interaction range.
The partition function $Z$ at temperature $T = \beta^{-1}$ is an imaginary-time functional integral with action

$$S = \int d\tau \left[ \sum_{k,\sigma} \bar{\psi}_{k\sigma}(\tau) \partial_\tau \psi_{k\sigma}(\tau) + \mathcal{H} \right].$$

Introducing the Hubbard-Stratonovich field $\phi_\eta(\tau)$, which couples to $\bar{\psi}\psi$, and integrating out the fermionic degrees of freedom, we obtain

$$Z = \int \mathcal{D}\phi \mathcal{D}\phi^* \exp(-S_{\text{eff}}[\phi, \phi^*]),$$

with the effective action given by

$$S_{\text{eff}} = \int_0^\beta d\tau \sum_k \left( \frac{\phi_k^2(\tau)}{\lambda} + \xi_k \right) - \text{Tr} \left( \ln G^{-1}_{k,k'}(\tau) \right).$$

The symbol $\text{Tr}$ denotes the trace over momentum, imaginary time and Nambu indices, and $G^{-1}_{k,k'}(\tau)$ is the (inverse) Nambu matrix,

$$G^{-1}_{k,k'}(\tau) = \begin{pmatrix} \Lambda_{k,k'}(\tau) & \Lambda^*_{k',k}(\tau) \\ - (\partial_\tau - \xi_k) \delta_{k,k'} & - (\partial_\tau + \xi_k) \delta_{k,k'} \end{pmatrix},$$

with $\Lambda_{k,k'}(\tau) = \phi_k - \phi_{-k'}(\tau) \Gamma((k + k')/2)$.

Assuming $\phi_\eta(\tau) = \Delta_0 \delta_\eta + \eta_0(\tau)$ and performing an expansion in $S_{\text{eff}}$ to quadratic order in $\eta$, one can write the effective action as $S_{\text{eff}} = S_0[\Delta_0] + S_{\text{fluct}}[\eta, \eta^*]$, where $S_0[\Delta_0]$ is the saddle point action, and the fluctuation action $S_{\text{fluct}}[\eta, \eta^*]$ can be expressed in terms of amplitudes and phases via $\eta(q) \equiv |\eta(q)|e^{i\theta(q)}$. Integrating out the amplitudes, and Fourier transforming to position and imaginary time $r = (r, \tau)$, one obtains

$$S_{\text{fluct}} = \frac{1}{2} \int dr \left[ \rho_{ij} \partial_i \theta(r) \partial_j \theta(r) + A(\partial_\tau \theta)^2 - iB \partial_\tau \theta(r) \right].$$

In this expression,

$$A(\mu, \Delta_0, T) = \frac{1}{4L^2} \sum_k \left[ \frac{\Delta_0^2}{E_k^2} \tanh \left( \frac{E_k}{2T} \right) + \frac{\xi_k^2}{2E_k} \right],$$

where $Y_k = (2T)^{-1} \text{sech}^2(E_k/2T)$ is the Yoshida distribution, and

$$B(\mu, \Delta_0, T) = \frac{1}{L^2} \sum_k \frac{\xi_k}{E_k} \tanh \left( \frac{E_k}{2T} \right).$$

The quantity $\rho_{ij}$ represents the superfluid density tensor

$$\rho_{ij}(\mu, \Delta_0, T) = \frac{1}{L^2} \sum_k \left[ 2n_0(k) \partial_\xi \xi_{i\xi_{j\xi_{k\xi_k}}} - Y_k \partial_\xi \xi_k \partial_\xi \xi_k \right],$$

where $\partial_\xi$ denotes the partial derivative with respect to $k_i$, and

$$n_0(k) = \frac{1}{2} \left[ 1 - \frac{\xi_k}{E_k} \tanh \left( \frac{E_k}{2T} \right) \right]$$

is the momentum distribution. Notice that $\rho_{xx} = \rho_{yy} \equiv \rho$, while $\rho_{xy} = \rho_{yx} = 0$.

The decomposition of $\theta(r, \tau) = \theta_s(r, \tau) + \theta_v(r, \tau)$ into a static vortex part $\theta_s(r, \tau)$ and a spin-wave part $\theta_v(r, \tau)$ permits us to rewrite

$$S_{\text{fluct}} = S_v + S_{sw},$$

where $S_v = \frac{1}{2} \int d\tau d\rho \left[ \nabla \theta_v(r) \right]^2$, while $S_{sw} = \frac{1}{2} \int d\tau \left[ \rho_s \left[ \nabla \theta_{sw}(r) \right]^2 + A(\partial_\tau \theta_{sw}(r))^2 - iB \partial_\tau \theta_{sw}(r) \right]$.

The spin-wave part can be integrated out to give $\Omega_{sw} = \int d\tau \ln \left[ 1 - \exp(-\omega_T/T) \right]$, where $\omega_T(q) = c|q|$ is the frequency and $c = \sqrt{\rho_s/\Lambda}$ is the speed of the spin-wave. Here, $\Omega_{sw}$ is the spin-wave contribution to the thermodynamic potential $\Omega = \Omega_0 + \Omega_{sw} + \Omega_v$, where $\Omega_0 = TS_0[\Delta_0]$ and $\Omega_v$ are the saddle-point and vortex parts, respectively.

The self-consistent equations for $\mu$, $\Delta_0$ and $T_c$ can be derived from the effective action $S_{\text{eff}} = S_0 + S_{\text{fluct}} + S_v$ as follows. The order parameter equation is obtained through the stationarity condition $[\delta S_{\text{eff}}/\delta \phi_0^*(\tau')]|_{\Delta_0 = 0}$, leading to

$$\frac{1}{\lambda} = \sum_k \frac{|\Gamma(k)|^2}{2E_k} \tanh \left( \frac{E_k}{2T} \right),$$

where $E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$ is the quasiparticle excitation energy, and $\Delta_k = \Delta_0 \Gamma(k)$ plays the role of the order parameter function. Elimination of the interaction strength $A$ in favor of the two-body bound state energy $E_b$ in vacuum is possible through the relation

$$\frac{1}{\lambda} = \sum_k \frac{|\Gamma(k)|^2}{2E_k} \frac{1}{E_b - E_b}. $$

The number equation is obtained via $N_p = -\partial \Omega_p / \partial \mu$, where $\Omega_p = \Omega_0 + \Omega_{sw}$, leading to

$$N_p = N_0 + N_{sw}. $$

Here, $N_0 = -\partial \Omega_0 / \partial \mu = 2 \sum_k n_0(k)$, and $N_{sw} = -\partial \Omega_{sw} / \partial \mu$.

The equation for the critical temperature $T_c = T_{\text{BKT}}$ is determined by the Kosterlitz-Thouless condition

$$T_{\text{BKT}} = \frac{\pi}{2} \rho_s(\mu, \Delta_0, T_{\text{BKT}}).$$

The self-consistent solutions of Eqs. (3), (5) and (9) determine $\mu$, $\Delta_0$ and $T_{\text{BKT}}$ as functions of the two-body binding energy $E_b$ (or interaction strength $\lambda$). Solutions for $T_{\text{BKT}}$ are shown in Fig. 11. The curves labeled MF represent the mean-field (saddle-point) solution for $T_c = T_{\text{MF}}$, which is obtained by solving only Eqs. (9) and (3) with $\Delta_0 = 0$. The disparity between the BKT and MF solutions is larger with increasing $\lambda$ (or $|E_b|$), indicating that strong coupling results are dramatically affected by phase fluctuations. Furthermore, $T_{\text{BKT}} = 0.5 T_{\text{MF}}$ in the strong coupling limit where $\rho \sim n/M$, with $n = N_0/L^2$. 


only on the vortex action, which becomes the topological charge (vorticity). Thus, we focus next on the vortex contribution to the action via a decreasing (increasing) function of the binding energy $\Delta E_L(b)$ as functions of (a) absolute value $|E_b|$ (main frame) and of the interaction strength $\lambda$ (in units of $g_{2D}$), where $g_{2D}$ is the two-dimensional density of states.

being the fermion density. This shows that although there is no finite $T_c$ for BEC in 2D, there is still a superfluid transition with a high $T_c \propto n$. Finally, $\mu(T_{\text{BKT}})$ and $\Delta \mu(T_{\text{BKT}})$ are shown in Fig. 2 as functions of $|E_b|$ or $\lambda$. Notice that $\mu(T_{\text{BKT}})$ ($\Delta \mu(T_{\text{BKT}})$) is a monotonically decreasing (increasing) function of $|E_b|$ or $\lambda$, and that $\mu$ changes sign at $|E_b| \approx 2.23$ ($\lambda \approx 0.419$).

Since the vortex part is directly related to the transverse velocity $v_i(r) = \nabla \theta_i(r)$, where $\nabla \cdot v_i(r) = 0$, we can express the vortex contribution to the action via $\nabla \times v_i(r) = 2\pi \delta \sum_i n_i \delta(r-r_i)$, where $n_i = \pm 1$ is the vortex topological charge (vorticity). Thus, we focus next only on the vortex action, which becomes

$$S_\text{v} = \frac{H_v}{T} = (2\pi) \frac{D_s}{2} \sum_{i \neq j} n_i n_j G(r_i - r_j) + \sum_i E_v n_i^2,$$

where $H_v$ is the vortex Hamiltonian, $E_v$ is the vortex core energy, and $G(r_i - r_j)$ plays the role of the interaction potential between topological charges $n_i$ and $n_j$, and satisfies $\nabla^2 G(r-r') = \delta(r-r')$. Given that in the vicinity of $T_{\text{BKT}}$ VA pairs are formed, it is possible that they crystallize into a solid at a lower temperature $T_M$. We note that the vortex action just obtained allows for the appearance of a VA solid (with well defined lattice structure) at a stable minimum of the interaction potential. At $T \ll T_M$, the lattice configuration that is compatible with short range core repulsions is such that vortices and antivortices form two square sublattices of side $b = a\sqrt{2}$, where $2a$ is the VA pair size. This VA lattice can be constructed as a superposition $\theta_V(x, y) = \sum_{ij} c_{ij} \theta_{VA}(x - ib, y - jb/2)$, where

$$\theta_{VA}(x, y) = \arctan \left( \frac{2ay}{x^2 + y^2} \right).$$

For convenience, we choose the line connecting the vortex to the antivortex in a pair to be along the $x$ axis. The optimal coefficients correspond to $c_{ij} = (-1)^j$, where the VA pairs orient themselves like dipoles in an antiferroelectric material. All topological dipoles are aligned along the $x$ axis and anti-aligned along the $y$ axis, as shown in Fig. 3. However, there is no easy axis as the system is rotationally invariant. The configuration where VA pairs are aligned as dipoles in a ferroelectric material is higher in energy.

As the temperature is raised towards $T_M$, the importance of vibrations and defects of the VA lattice increases and eventually causes its melting into a VA liquid state. In order to obtain the correct equations of motion, we have to recall that vortices and antivortices move perpendicularly to the applied force $\mathbf{F}$. Thus, their dynamics is Eulerian instead of Newtonian as in a lattice of atoms. This leads to the equation of motion

$$n_i \frac{\partial r_i}{\partial t} = \frac{\hbar}{M} \hat{\mathbf{z}} \times \mathbf{F},$$

where $\mathbf{F}_i = -\nabla \theta_i \sum_{j \neq i} n_i n_j G(r_i - r_j)$ plays the role of the force. We can define displacement fields similar to the phonon problem in solids as $\mathbf{u}_p(R_i) = \mathbf{r}_p(R_i) - \mathbf{R}_i$, with the index $p = V, A$ indicating a vortex (V) or an antivortex (A) sublattice, and $\mathbf{R}_i$ labeling the equilibrium lattice sites. This equation produces two modes in the long wavelength limit: a longitudinal (L) mode with frequency $\omega_L = c_L \omega$, with $c_L = (\hbar/M)(\gamma_L/b)$, and a transverse (T) mode with frequency $\omega_T = c_T \omega$, with $c_T = (\hbar/M)(\gamma_T/b)$, where $\gamma_L < \gamma_T$ are dimensionless constants. The corresponding eigenvectors are $\eta_L = \mathbf{u}_V + \mathbf{u}_A$, and $\eta_T = \mathbf{u}_V - \mathbf{u}_A$. The harmonic motion in diagonal form is

$$S_{oh} = \frac{\pi \rho_s}{b^2} \int d^2 r \left[ \alpha_L I_{ij}^2 + \alpha_L L_{ij}^2 + \alpha_T T_{ij}^2 + \alpha_T T_{ij}^2 \right],$$

where $0 < \alpha_L, \alpha_T < 1$ are dimensionless elastic constants obtained from the interaction potential, and $\Lambda_{ij} = \frac{1}{2}[\partial_i \eta_{L,j} + \partial_j \eta_{L,i}]$, with $\Lambda = L, T$. The VA lattice melting transition is then controlled by the Kosterlitz-Thouless-Halperin-Nelson-Young (KTHNY) mechanism [17, 18, 19]. The melting temperature $T_M$ is determined by the lowest energy mode ($L$) via the relation

$$T_M = \rho_s \left[ \frac{\alpha_L(\alpha_L + 2\alpha_L)}{2\alpha_L} \right],$$

FIG. 1: Mean field (MF), Berezinskii-Kosterlitz-Thouless (BKT) and vortex-antivortex lattice melting (M) critical temperatures (in units of $\epsilon_F$) as functions of (a) absolute value of the binding energy $|E_b|$ (in units of $\epsilon_F$) and (b) interaction strength $\lambda$ (in units of $g_{2D}$).
In the strong coupling limit, the melting temperature without the need of rotating the Fermi superfluid. The measurement of sound velocities can be useful to extract information about the short-range nature of the interaction forces through $\alpha_{L_1}$ and $\alpha_{T_1}$. The values used in Fig. 1 correspond to $T_M = 0.3\rho_0(\mu, \Delta_0, T_M)$.

In summary, we have studied the superfluid state of Fermi gases in a strongly confining one-dimensional optical lattice that suppresses quantum mechanical atom transfer (tunneling), making the Fermi system two-dimensional. In the s-wave channel, we have found that the Fermi gas undergoes a superfluid BKT transition and that vortex-antivortex (VA) bound states appear below $T_{\text{BKT}}$ during the evolution from weak to strong coupling. In the strong coupling limit, $T_{\text{BKT}}$ is large and controlled by the Fermi energy of the atomic system, thus allowing for spontaneous VA pairs to be observed directly (without stirring the condensate). In addition, we have found that the ground state of the system evolves smoothly from weak to strong coupling, but supports a square VA lattice. Unlike the case of a lattice of identical vortices, which is triangular, a square VA lattice is favored, because topological charge frustration pushes the triangular VA configuration to higher energy. In addition, we have shown that the VA lattice melts via the KTHNY mechanism of dislocation mediated melting. Because the symmetry of the VA lattice is square, we expect that the melted lattice will not be hexatic, but quartic. The VA lattice might be detected in a trap release experiment without the need of rotating the Fermi superfluid. Experimental chances of this observation are higher in the strong coupling limit, where the melting temperature $T_M < T_{\text{BKT}}$ is also controlled by the Fermi energy.

Although we have not yet studied in detail the possible quartic liquid VA state, there can be an additional transition into an isotropic VA liquid before the BKT transition is reached. In case experiments can be done using a velocity sensitive probe instead of a density probe as usual, not only the vortex cores would be seen, but also the direction of their rotation (vorticity). Lastly, in addition to the spin wave mode, we have found two VA lattice vibrational modes, one longitudinal and one transverse. All these modes disperse linearly, but the presence of a transverse mode is also a direct indication of the existence of the VA lattice.

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