A systematic study of finite field dependent BRST-BV transformations in $Sp(2)$ extended field-antifield formalism

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Abstract

In the framework of $Sp(2)$ extended Lagrangian field-antifield BV formalism we study systematically the role of finite field-dependent BRST-BV transformations. We have proved that the Jacobian of a finite BRST-BV transformation is capable of generating arbitrary finite change of the gauge-fixing function in the path integral.

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1 INTRODUCTION AND SUMMARY

In the previous articles [1, 2, 3] systematic studies of the role of finite field dependent BRST transformations have been conducted in various types of quantization formalisms. Firstly, this was done in the framework of the standard generalized Hamiltonian BFV formalism [1], as well as in its Sp(2) extended counterpart [3]. Secondly, a similar study was performed in the framework of the standard Lagrangian field-antifield BV formalism [2]. The main result in all cases is that the Jacobian of a finite field dependent BRST transformation is capable of reproducing an arbitrary finite change of the gauge-fixing function in the corresponding path integral.

In the present article, we will develop a similar study in the Sp(2) extended Lagrangian field-antifield formalism [4, 5, 6] (see also an alternative approach to the problem of BRST-antiBRST invariant quantization of general gauge theories in Lagrangian formalism [7]). The main new feature, in the situation at hand, is that the algebra of BRST-BV transformations is open in the antifield sector, while it is Abelian in the sector of fields. It is therefore in general impossible to have integrable partial differential equations in the Sp(2) extended BRST parameters for the total set of field-antifield variables. However, as the parameters are only allowed to depend on fields, the corresponding partial differential equations are integrable. At the same time, for the total set of variables, one can formulate the Lie equations in terms of a Bosonic rescaling variable for the Fermionic BRST parameters. Then, when considering the Jacobian of the transformation, we can split the Jacobian into two pieces: the first one is calculated for constant BRST parameters; the second piece comes from the field dependence of the BRST parameters. The second piece is just responsible for generating the required change of the gauge-fixing function in the path integral. It appears that integrable Lie equations in the field sector are consistent with the compensation equation, which determines the necessary field dependence for the BRST parameters.

2 STANDARD FORMULATION AND Sp(2) DIFFERENTIAL FOR FIELDS

Let
\[ \Phi^A, \pi^{Aa}, \Phi^{\ast A}, \Phi^{\ast \ast A} \]  
be a set of variables necessary to describe an arbitrary Lagrangian gauge field theory within the framework of Sp(2) extended field-antifield formalism [4, 5, 6]. Here in (2.1) a capital Latin index “A”, “B”, . . . , from the beginning of the alphabet, should be understood in the sense of DeWitt’s condensed notation [8] as a condensed index of fields and antifields, while a small Latin index “a”, “b”, . . . , from the beginning of the alphabet, is a vector index of the Sp(2)
The variables (2.1) have Grassmann parities

$$\varepsilon_A, \varepsilon_A + 1, \varepsilon_A + 1, \varepsilon_A,$$

respectively. We will denote partial derivatives with respect to the variables (2.1) as

$$\partial_A, \partial_{Aa}, \partial^{Aa}, \partial^A.$$

In terms of the variables (2.1), the path integral for the partition function reads

$$Z_F = \int D\Phi D\pi D\Phi^* D\Phi^{**} D\lambda \exp\{(i/\hbar)W_F\},$$

where

$$W_F = W + \Phi^*_A \pi^A + (\Phi^{**}_A - F \partial_A)\lambda^A + (1/2)F \partial_A \pi^A \partial_B \pi^B \varepsilon_{ab},$$

and where $\varepsilon_{ab} = -\varepsilon_{ba}$ is the constant $Sp(2)$ invariant tensor, while $\varepsilon^{ab} = -\varepsilon^{ba}$ stands for its inverse. In (2.4) we have also integrated over dynamically passive Lagrange multipliers $\lambda^A$ with Grassmann parity $\varepsilon_A$. The gauge-fixing function $F(\Phi)$ in (2.5) is a Boson. The quantum master action $W = W(\Phi, \Phi^*, \Phi^{**})$ satisfies the $Sp(2)$ extended quantum master equation,

$$(\Delta^a + (i/\hbar)V^a) \exp\{(i/\hbar)W\} = 0,$$

where

$$\Delta^a = (-1)^{\varepsilon_A} \partial_A \partial^{Aa},$$

$$V^a = \varepsilon^{ab} \Phi^*_A \partial^A.$$

Fermionic operators (2.7), (2.8) satisfy the algebra

$$[\Delta^a, \Delta^b] = \Delta^b \{a\Delta^b\} = 0,$$

$$[V^a, V^b] = V^b \{aV^b\} = 0,$$

$$[\Delta^a, V^b] = 0.$$

Here and below ...{a...b}... means symmetrization in $a$ and $b$ indices. The quantum master equation (2.6) can equivalently be rewritten in its quadratic form, which is convenient for $\hbar$ series expansion,

$$(1/2)(W, W)^a + V^a W = i\hbar \Delta^a W,$$

where in the l. h. s., we have denoted the $Sp(2)$ extended antibracket

$$(F, G)^a = F(\partial_A \partial^A - \partial^{Aa} \partial_A)G = -(G, F)^a (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}.$$

Here $[,]$ means the supercommutator, which is defined for any quantities $G, H$ as $[G, H] = GH - (-1)^{\varepsilon(G)\varepsilon(H)}HG.$
The antibracket (2.13) is Grassmann odd,
\[ \varepsilon((F, G)^a) = \varepsilon(F) + \varepsilon(G) + 1; \] (2.14)
it satisfies the Leibniz rule,
\[ (F, GH)^a = (F, G)^a H + G(F, H)^a (-1)^{(\varepsilon(F)+1)\varepsilon(G)}; \] (2.15)
it satisfies the Jacobi identity,
\[ ((F, G)^a, H^b)^c (-1)^{(\varepsilon(F)+1)\varepsilon(H)+1} + \text{cycle}(F, G, H) = 0; \] (2.16)
and it is differentiated by both the operators (2.7) and (2.8),
\[ \Delta((a(F, G)^b)^c = (\Delta(aF, G)^b) - (F, \Delta(aG)^b)^c (-1)^\varepsilon(F), \] (2.17)
\[ V((a(F, G)^b)^c = (V(aF, G)^b) - (F, V(aG)^b)^c (-1)^\varepsilon(F). \] (2.18)

Now let us make the essential observation that the gauge-fixed quantum action can be rewritten in the following equivalent form
\[ W_F = W + \Phi^*_A a \pi_A + \Phi^*_a \lambda a + (1/2) F d^a d^b \varepsilon_{ba}, \] (2.19)
where the (right) Fermionic Sp(2) differential is defined by
\[ \overleftrightarrow{d}^a = \overleftrightarrow{\partial}_A \pi_A - \overleftrightarrow{\partial}_{Ab} \lambda^A \varepsilon_{ba}, \] (2.20)
\[ [d^a, d^b] = d^{(a} d^{b)} = 0, \] (2.21)
\[ d^a d^b d^c = 0. \] (2.22)
Its counterpart acting from the left reads
\[ d^a = (-1)^{\varepsilon_A} a \pi_{A} \partial_A + (-1)^{\varepsilon_A} \lambda^A \varepsilon_{ab} \partial_{Ab}, \] (2.23)
\[ [d^a, d^b] = d^{(a} d^{b)} = 0, \] (2.24)
\[ G d^a \mu_a = \mu_a d^a G, \varepsilon(\mu_a) = 1, \] (2.25)
where G is a function of all variables. It is a remarkable feature of (2.19) that the dependence on the gauge-fixing function F is here accumulated in the fourth term alone, which is similar to what happens in the corresponding Hamiltonian formulation [3].

3 Sp(2) EXTENDED BRST-BV GENERATORS AND THEIR ALGEBRA

Now let us define the (left) Sp(2) extended BRST-BV generators by
\[ \mathfrak{D}^a = d^a + (W \partial_A) \partial^A + \varepsilon_{ab} \Phi^* A_a \partial^a. \] (3.1)
Due to the quantum master equation (2.12), it follows from (3.1) that
\[ \mathcal{D}^a W_F = i\hbar \Delta^a W_F, \] (3.2)
and that the generators (3.1) satisfy the algebra
\[ [\mathcal{D}^a, \mathcal{D}^b] = (D^{(a} W^{\partial A}) \partial^{Ab}) + W^{\partial A(a} \partial_B^{\partial A} \partial_B), \] (3.3)
where
\[ D^a = d^a + i\hbar \Delta^a, \] (3.4)
\[ [D^a, D^b] = D^{(a} D^{b)} = 0, \quad [D^a, \partial_A] = 0. \] (3.5)

The (right) \( Sp(2) \) extended BRST-BV generators are written as
\[ \overleftrightarrow{\mathcal{D}}^a = \overleftarrow{d}^a + \overleftrightarrow{\partial}^A a(\partial_A W) + \overleftarrow{\partial}^A \Phi^*_{A b} \varepsilon^b a (-1)^{e_A}, \] (3.6)
so that the relation
\[ G \mathcal{D}^a \mu_a = \mu_a \mathcal{D}^a G, \quad \varepsilon(\mu_a) = 1 \] (3.7)
holds for a function \( G \) of all variables.

Thus, we conclude from (3.3) that in the total field-antifield space, the algebra is open, while it is Abelian in the field sector \( \{ \Phi^A, \pi^{A a} \} \).

4 LIE EQUATIONS AND FINITE \( Sp(2) \) EXTENDED BRST-BV TRANSFORMATIONS

Let us denote the set of field-antifield variables (2.1) by \( z^i \). We next consider the Lie equation for a finite BRST-BV transformation, restricted to a one-parameter subgroup of \( t \)-rescaling of the Fermionic parameter \( \mu_a, \mu_a \to t \mu_a \), where \( t \) is a Bosonic parameter,
\[ \frac{d}{dt} z^i (z, t \mu) = \mu_a \overleftrightarrow{\mathcal{D}}^a z^i = z^i \overleftarrow{\mathcal{D}}^a \mu_a, \] (4.1)
\[ z^i (t = 0) = z^i. \] (4.2)

For constant \( \mu_a \), the formal solution to that equations reads
\[ z^i = \exp\{t \mu_a \mathcal{D}^a\} z^i = z^i \exp\{t \overleftarrow{\mathcal{D}}^a \mu_a\}. \] (4.3)

It is also natural to extend the solution (4.3) to the case of parameters \( \mu_a \) being dependent of the initial data (1.2), \( \mu_a = \mu_a(z) \), as follows: all \( \mu_a \) should stand to the left (right) of all
in the first (second) equality in (4.1). We will mean that extension when considering the Jacobian and the compensation equation in Sec. 5.

As the algebra (3.3) is open in the total field-antifield space, it is impossible to deduce integrable partial differential equations in the parameters \( \mu_a \) themselves for all the variables \( z^i \). However, if one considers only the field sector \( z^\alpha = \{ \Phi^A, \pi^{Aa} \} \), then the integrable partial differential equations

\[
\partial^a z^\alpha(z, t\mu) = \frac{i}{\hbar} z^\alpha, \quad \partial^a = \frac{\partial}{\partial \mu_a},
\]

(4.4)
directly imply the corresponding equations (4.1) in the field sector \( z^\alpha \). These equations (4.4) are integrable due to (2.24). When multiplied by \( \mu_a \) from the left, equations (4.4) in the \( z^\alpha \) sector yield exactly (4.1), cf. eq. (3.1). A version of (4.4) using right differentials reads

\[
z^\alpha(z, t\mu) \overset{\leftarrow}{\partial^a} = \overline{z^\alpha \partial^a} t.
\]

(4.5)

For further convenience, let us also write down a counterpart to (4.5) for inverse transformation,

\[
z^\alpha(\overline{z}, t\mu) \overset{\leftarrow}{\partial^a} = -\overline{z^\alpha \partial^a} t.
\]

(4.6)

It follows from (3.2) and (4.1) that

\[
(\frac{d}{dt}) \ln J_F = \mu_a \Delta^a W_F - \text{tr} [(1 + t\kappa)^{-1} \kappa],
\]

(4.7)

\[\kappa^b_a = (\mu_a \partial^b),\]
and where we have used that the $t$-derivative of the second term in (5.2) equals

$$G^\alpha_\beta(z^\beta \partial^a)(\mu_a \partial_\alpha)(-1)^{\varepsilon_a},$$

(5.5)

with $G^\alpha_\beta$ satisfying the equation

$$G^\alpha_\gamma + t(z^\alpha \partial^a)(\mu_a \partial_\gamma)G^\beta_\gamma = \delta^\alpha_\gamma.$$  

(5.6)

It follows from (5.6) that

$$(\mu_a \partial_\alpha)G^\alpha_\beta = [(1 + t\kappa)^{-1}]_a^b (\mu_b \partial_\beta).$$

(5.7)

By substituting (5.7) into (5.5), we get for the latter exactly the second term in (5.3). Now, it follows from (4.7) and (5.3) that

$$(d/dt)((i/\hbar)WF + \ln J) = - \text{tr} [(1 + t\kappa)^{-1}\kappa],$$

(5.8)

so that

$$(i/\hbar)WF + \ln J)_{t=1} = (i/\hbar)WF - \text{tr} \ln (1 + \kappa).$$

(5.9)

In order to change the gauge-fixing function $F \rightarrow F_1 = F + \delta F$

(5.10)

with a finite amount $\delta F$, we must have

$$- \text{tr} \ln (1 + \kappa) = (i/\hbar)(1/2)\delta F d^a d^b \varepsilon_{ba} = - \text{tr} x,$$

(5.11)

where

$$x^b_a = -(i/\hbar)(1/2)\varepsilon_{ac} \delta F d^c d^b.$$  

(5.12)

We must also require the following condition to hold

$$\ln (1 + \kappa) = x,$$

(5.13)

so that

$$\kappa = \exp(x) - 1,$$

(5.14)

or explicitly

$$\mu_a \partial_\beta = [\exp(x) - 1]_a^b.$$  

(5.15)

We call (5.15) "a compensation equation". This equation determines the necessary dependence of the parameters $\mu_a$ on fields. There is an obvious explicit solution to that equation, namely

$$\mu_a = -(i/\hbar)[f(x)]_a^b(1/2)\varepsilon_{bc} \delta F d^c,$$

(5.16)
where
\[ f(x) = (\exp(x) - 1) x^{-1}. \] (5.17)

If one chooses the parameters \( \mu_a \) in the form (5.16), then it follows due to (5.9) and (5.11), that
\[ Z_{F_1} = Z_F, \] (5.18)
so that the partition function is independent of the gauge-fixing \( F \).

6  \( Sp(2) \) EXTENDED MODIFIED WARD IDENTITIES

Let us define the generating functional for Green's functions by adding the action of external sources to the gauge-fixed quantum action \( W_F \) (2.5) or (2.19),
\[
Z_F(\zeta, \zeta^*, \zeta^{**}) = \int D\Phi D\pi D\Phi^* D\Phi^{**} \exp\{ (i/\hbar) [W_F + \zeta_\alpha z^\alpha + 
+ \zeta^{**}_\alpha z^{\alpha} \overleftarrow{d^a} + \zeta^{*}_\alpha z^{\alpha} \overleftarrow{d^a d^b (1/2) \varepsilon_{ba}}] \};
\] (6.1)

where, again, \( z^\alpha = \{ \Phi^A, \pi^{Aa} \} \) is the field sector.

Let us make in (6.1) an infinitesimal change with constant \( \mu_a \),
\[
\delta z^i = z^i \overleftarrow{\delta^a}_a \mu_a, \quad \mu_a \to 0,
\] (6.2)
of the total set \( z^i \) of the field-antifield variables (2.1). We get then the standard Ward Identities
\[
(\zeta_\alpha \partial^{\alpha} + \varepsilon^{ab} \zeta^{*}_\alpha \overleftarrow{\partial^a}) Z_F = 0,
\] (6.3)
where \( \partial^\alpha \) and \( \partial^{\alpha} \) is the partial derivative with respect to \( \zeta^{**}_\alpha \) and \( \zeta^{*}_\alpha \), respectively.

Now let us perform in (6.1) a finite BRST-BV transformation defined by (4.1) and (4.2), with arbitrary \( \mu_a(z) \); we get then what we call ”a modified Ward identity”,
\[
< \exp\{ (i/\hbar) [\zeta_\alpha (\overleftarrow{\zeta} - z)^\alpha + \zeta^{**}_\alpha (z^{\alpha} \overleftarrow{d^a} - z^{\alpha} \overleftarrow{d^a} + \zeta^{*}_\alpha (z^{\alpha} \overleftarrow{d^a d^b} - z^{\alpha} d^a d^b (1/2) \varepsilon_{ba})] -
- \text{tr} \ln (\delta^b_a + \mu_a d^b) \} >_{F;\zeta,\zeta^*,\zeta^{**}} = 1,
\] (6.4)
where we have defined the source-dependent quantum mean value
\[
< \ldots >_{F;\zeta,\zeta^*,\zeta^{**}} = [Z_F(\zeta, \zeta^*, \zeta^{**})]^{-1} \int D\Phi D\pi D\Phi^* D\Phi^{**} \exp\{ (i/\hbar) [W_F + 
+ \zeta_\alpha z^\alpha + \zeta^{**}_\alpha z^{\alpha} \overleftarrow{d^a} + \zeta^{*}_\alpha z^{\alpha} \overleftarrow{d^a d^b (1/2) \varepsilon_{ba}}] \}. \] (6.5)

Now, let us choose in (6.4) the parameters \( \mu_a \) to coincide with the solution (5.16) to the compensation equation with the inverse sign of \( \delta F \); then we get from (6.4) the relation generalizing the one (5.18) to the presence of external sources.
\[
Z_{F_1} = Z_F < \exp\{ (i/\hbar) [\ldots] \} >_{F;\zeta,\zeta^*,\zeta^{**}}, \] (6.6)
where [\ldots] means the expression in the square brackets in the exponential on the l. h. s. of (6.4).
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