SUPPORT VARIETIES AND COHOMOLOGY OF VERDIER QUOTIENTS OF STABLE CATEGORY OF COMPLETE INTERSECTION RINGS

TONY J. PUTHENPURAKAL

Abstract. Let \((A, m)\) be a complete intersection with \(k = A/m\) algebraically closed. Let \(\text{CM}(A)\) be the stable category of maximal Cohen-Macaulay \(A\)-modules. For a large class of thick subcategories \(S\) of \(\text{CM}(A)\) we show that there is a theory of support varieties for the Verdier quotient \(T = \text{CM}(A)/S\). As an application we show that the analogous version of Auslander-Reiten conjecture, Murthy's result, Avramov-Buchweitz result on symmetry of vanishing of cohomology holds for \(T\).

1. Introduction

Quillen's geometric methods to study cohomology of finite groups is an important contribution in modular representation theory, see [21]. The techniques involved have been generalized and extended to representations of various Hopf algebras, for instance see [13]. In commutative algebra Avramov and Buchweitz introduced the notion of support varieties of a pair of modules over local complete intersections and as an application proved the symmetry of vanishing of Ext over such rings; see [5]. In [7], [8] the notion of support varieties was extended to certain class of triangulated categories

Let \((A, m)\) be a commutative Gorenstein local ring with residue field \(k\). Let \(\text{CM}(A)\) denote the full subcategory of maximal Cohen-Macaulay (= MCM) \(A\)-modules and let \(\text{CM}(A)\) denote the stable category of MCM \(A\)-modules. It is well-known that \(\text{CM}(A)\) is a triangulated category with translation functor \(\Omega^{-1}\), (see [10]; cf. [25]).

We use Neeman’s book [19] for notation on triangulated categories. However we will assume that if \(C\) is a triangulated category then \(\text{Hom}_C(X, Y)\) is a set for any objects \(X, Y\) of \(C\).

1.1. For the rest of the paper let us assume that \((A, m)\) is a complete complete intersection ring of dimension \(d\) and codimension \(c\). Assume \(k = A/m\) is algebraically closed. Some of our results are applicable more generally. However for simplicity we will make this hypothesis throughout this paper.

There is a theory of support varieties for modules over \(A\). Essentially for every finitely generated module \(E\) over \(A\) an algebraic cone \(V(E)\) in \(k^c\) is attached, see [3, 6.2]. Conversely it is known that if \(V\) is an algebraic cone in \(k^c\) then there exists a finitely generated module \(E\) with \(V(E) = V\), see [6, 2.3]. It is known...
that $\mathcal{V}(\mathcal{P}^n(E)) = \mathcal{V}(E)$ for any $n \geq 0$. Thus we can assume $E$ is maximal Cohen-Macaulay.

**Remark 1.2.** It is easier to work with varieties in $\mathbb{P}^{c-1}$ than with algebraic cones in $k^c$. So we let $\mathcal{V}^*(E)$ be the algebraic set in $\mathbb{P}^{c-1}$ corresponding to $\mathcal{V}(E)$.

We now give the class of thick subcategories of $\text{CM}(A)$ that are of interest to us.

**Definition 1.3.** Let $X$ be an algebraic set in $\mathbb{P}^{c-1}$. Set

$$S_X = \{M \mid M \text{ is MCM and } \mathcal{V}^*(M) \subseteq X\}$$

It is easy to verify that $S_X$ is a thick subcategory of $\text{CM}(A)$; see Lemma 3.2. We will be interested in the Verdier quotient

$$T_X = \text{CM}(A)/S_X.$$ 

Our definition of support variety for objects in $T_X$ is simply

$$\mathcal{V}_X(M) = \mathcal{V}^*(M) \setminus X.$$ 

In Corollary 3.5 we show that this is a well-defined notion in $T_X$.

Although our definition of support varieties in $T_X$ is very simple, it has significant consequences. We now describe our:

**Applications:**

(1) Auslander and Reiten conjectured that if $R$ is an Artin ring, $\Lambda$ is an Artin $R$-algebra and $M$ is a finitely generated $\Gamma$-module then

$$\text{Ext}^i_{\Lambda}(M, M \oplus \Lambda) = 0 \quad \text{for all } i \geq 1 \implies M \text{ is projective}.$$ 

We note that this conjecture makes sense for any ring $\Lambda$. In [2, 1.9] it is shown that AR-conjecture holds for complete intersection rings. Also see [17, Main Theorem] and [11, Corollary 4] where it is shown that AR-conjecture holds for normal local Gorenstein rings. The generalized Auslander Reiten (GAR) conjecture states that if $\Lambda$ is any ring then

$$\text{Ext}^i_{\Lambda}(M, M \oplus \Lambda) = 0 \quad \text{for all } i \geq m \implies \text{projdim}_\Lambda M < m.$$ 

GAR has been verified for complete intersections, see [5, 4.2]. In this case we just require $\text{Ext}^i_R(M, M) = 0$ for some $i \geq 1$. We note that if $R$ is Gorenstein local and $M, N$ are MCM $R$-modules then $\text{Ext}^j_R(M, N) = \text{Hom}_R(M, \Omega^{-j}(N))$. We say a triangulated category $\mathcal{T}$ with shift operator $\Sigma$ satisfies generalized Auslander-Reiten property (GAR) if for $U \in \mathcal{T}$

$$\text{Hom}_\mathcal{T}(U, \Sigma^n U) = 0 \quad \text{for all } n \gg 0 \implies U = 0.$$ 

Our first application is

**Theorem 1.4.** (with hypotheses as in 1.1 and notation as in 1.3) The triangulated category $T_X$ satisfies GAR for any algebraic set $X \subseteq \mathbb{P}^{c-1}$.

(2) Murthy, [18, 1.6] proved that if $A$ is a complete intersection of codimension $c$ and for some $m > 0$

$$\text{Tor}^A_i(M, N) = 0 \quad \text{for } i = m, m+1, \ldots, m+c;$$
then Tor^i_A(M, N) = 0 for all i ≥ m. The corresponding property with Ext^*(−, −) is also true. We say a triangulated category \( T \) with shift operator \( \Sigma \) satisfies Murthy’s property with order \( r \) ≥ 1 if for some \( m \) and \( U, V \in T \)

\[ \text{Hom}_T(U, \Sigma^nV) = 0 \quad \text{for all} \quad n = m, m + 1, \cdots, m + r \]

\[ \implies \text{Hom}_T(U, \Sigma^nV) = 0 \quad \text{for all} \quad n \geq m. \]

Clearly CM\((A)\) satisfies Murthy’s property with order \( c \) if \( A \) is a local complete intersection of codimension \( c \). See [7] for some examples of triangulated categories satisfying Murthy’s property. Our second application is

**Theorem 1.5.** (with hypotheses as in 1.1 and notation as in 1.3.) The triangulated category \( T_X \) satisfies Murhty’s property with order \( c \) for any algebraic set \( X \subseteq \mathbb{P}^{c-1} \).

(3) A spectacular application of Avramov and Buchweitz’s definition of support variety of a pair of modules over a complete intersection \( A \) is the following:

\[ \text{Ext}^i_A(M, N) = 0 \quad \text{for all} \quad i \gg 0 \quad \text{if and only if} \quad \text{Ext}^i_A(N, M) = 0 \quad \text{for all} \quad i \gg 0 \]

We say a triangulated category \( T \) with shift operator \( \Sigma \) satisfies symmetry in vanishing of cohomology if for \( U, V \in T \)

\[ \text{Hom}_T(U, \Sigma^nV) = 0 \quad \text{for all} \quad n \gg 0 \quad \implies \quad \text{Hom}_T(V, \Sigma^nU) = 0 \quad \text{for all} \quad n \gg 0. \]

Clearly CM\((A)\) satisfies symmetry in vanishing of cohomology if \( A \) is a local complete intersection of codimension \( c \). See [8] for some examples of triangulated categories satisfying symmetry in vanishing of cohomology.

Before we state our next result we need the following:

**Definition 1.6.** We say a module \( M \) is essentially disjoint from \( X \) if \( M \cong M_1 \oplus M_2 \) where \( V^*(M_1) \subseteq X \) and \( V^*(M_2) \subseteq \mathbb{P}^{c-1} \setminus X \).

**Remark 1.7.** (1) Note \( M \cong M_2 \) in \( T_X \).

(2) In [10] we show that if \( M \cong N \) in \( T_X \) and \( M \) is essentially disjoint from \( X \) then so is \( N \).

Our third application is

**Theorem 1.8.** (with hypotheses as in 1.1 and notation as in 1.3.) Let \( X \) be any algebraic set in \( \mathbb{P}^{c-1}(k) \). Let \( M, N \) be any two MCM \( A \)-modules. Consider the following two statements:

1. \( \text{Hom}_{T_X}(M, \Omega^{-n}(N)) = 0 \quad \text{for all} \quad n \gg 0. \)
2. \( V_X(M) \cap V_X(N) = \emptyset. \)

Then (1) \( \implies \) (2). If \( M \) or \( N \) is essentially disjoint from \( X \) then (2) \( \implies \) (1).

We make the following:

**Conjecture 1.9.** (with hypotheses as in Theorem 1.8) The assertions (2) \( \implies \) (1) hold in general.

**Remark 1.10.** The main reason we are unable to prove Conjecture 1.9 is that we do not have a good notion of support variety for a pair of objects \( M, N \) in \( T_X \). Philosophically it should be \( V_X(M) \cap V_X(N) \) but we do not have a cohomological criterion for support varieties of the pair \( M, N \).
1.11. We give another class of Verdier-quotient’s of $\text{CM}(A)$ for which GAR and Muthy’s property holds. Let $\text{cx}_A M$ denote the complexity of a module $M$. For $i = 1, \ldots, c - 1$ let

$$\text{CM}_{\leq i}(A) = \{M \mid M \text{ is MCM and } \text{cx}_A M \leq i\}.$$ 

Then it is easy to check that $\text{CM}_{\leq i}(A)$ is a thick subcategory of $\text{CM}(A)$. Set $T_i = \text{CM}(A)/\text{CM}_{\leq i}(A)$.

If $Y$ is a variety in $\mathbb{P}^{c-1}$, write $Y = \bigcup_{j=1}^m Y_j$ where $Y_j$ is irreducible.

Assume $\dim Y_j = \dim Y$ for $1 \leq j \leq r$ and $\dim Y_j < \dim Y$ for $j > r$. Set $\text{top}(Y) = \bigcup_{i=1}^r Y_i$.

It is easy to prove that $\text{top}(Y)$ is an invariant of $Y$. We define support variety of an object $M$ in $T_i$ as follows

$$V_i(M) = \begin{cases} \text{top}(V^*(M)), & \text{if } \text{cx}_A M > i; \\ \emptyset, & \text{otherwise}. \end{cases}$$

In Corollary 3.10 we show that this is a well-defined notion in $T_i$. We show

**Theorem 1.12.** *(with hypotheses as in 1.1 and notation as in 1.11)* For $i = 1, \ldots, c - 1$ the triangulated category $T_i$ satisfies GAR.

Next we show:

**Theorem 1.13.** *(with hypotheses as in 1.1 and notation as in 1.11)* For $i = 1, \ldots, c - 1$ the triangulated category $T_i$ satisfies Murthy’s property with order $c - i$.

Huneke and Wiegand, [16, 1.9] showed that if $A$ is a hypersurface ring and $M, N$ are MCM $A$-modules then for some $i \geq 1$ we have

$$\text{Tor}_i^A(M, N) = \text{Tor}_{i+1}^A(M, N) = 0 \quad \text{then either } M \text{ or } N \text{ is free.}$$

Similar result holds for $\text{Ext}(-, -)$. Note that any MCM module over a hypersurface is two periodic, i.e., $\Omega^2(M) = M$ for any MCM $A$-module $M$ with no free summands. We say a triangulated category $\mathcal{T}$ is two periodic if $\Sigma^2(U) = U$ for any object $U$ in $\mathcal{T}$. We say a two periodic triangulated category $\mathcal{T}$ has the Huneke-Wiegand property if

$$\text{Hom}_\mathcal{T}(U, V) = \text{Hom}_\mathcal{T}(U, \Sigma(V)) = 0 \quad \Longrightarrow \quad U = 0 \text{ or } V = 0.$$

In an earlier paper we showed that $T_{c-1}$ is 2-periodic, see [20, 3.1] In this paper we show:

**Theorem 1.14.** *(with hypotheses as in 1.1 and notation as in 1.11)* The triangulated category $T_{c-1}$ has the Huneke-Wiegand property.
2. Preliminaries

In this paper all rings will be Noetherian local. All modules considered are finitely generated unless otherwise stated. However note that Home-sets in Verdier quotients need not be finitely generated.

Let $(A, \mathfrak{m})$ be a local ring and let $k = A/\mathfrak{m}$ be its residue field. Let $\dim A = d$. If $M$ is an $A$-module then $\mu(M) = \dim_k M/\mathfrak{m}M$ is the number of a minimal generating set of $M$. Also let $\ell(M)$ denote its length. In this section we discuss a few preliminary results that we need.

2.1. Let $M$ be an $A$-module. For $i \geq 0$ let $b_i(M) = \dim_k \text{Tor}_i^A(M, k)$ be its $i^{th}$ betti-number. Let $P_M(z) = \sum_{n \geq 0} b_n(M)z^n$, the Poincare series of $M$. Set

$$cx(M) = \inf \{ r \mid \limsup \frac{b_n(M)}{n^{r-1}} < \infty \}$$

It is possible that $cx(M) = \infty$, see [4, 4.2.2].

2.2. It can be shown that for any $A$-module $M$ we have

$$cx(M) \leq cx(k)$$

see [4, 4.2.4].

2.3. If $A$ is a complete intersection of co-dimension $c$ then for any $A$-module $M$ we have $cx(M) \leq c$. Furthermore for each $i = 0, \ldots, c$ there exists an $A$-module $M_i$ with $cx(M_i) = i$. Also note that $cx(k) = c$. [4] 8.1.1(2).

2.4. The stable category of a Gorenstein local ring:

Let $(A, \mathfrak{m})$ be a commutative Gorenstein local ring with residue field $k$. Let $\text{CM}(A)$ denote the full subcategory of maximal Cohen-Macaulay (= MCM) $A$-modules and let $\text{CM}(A)$ denote the stable category of MCM $A$-modules. Recall that objects in $\text{CM}(A)$ are same as objects in $\text{CM}(A)$. However the set of morphisms $\text{Hom}_A(M, N)$ between $M$ and $N$ is $= \text{Hom}_A(M, N)/P(M, N)$ where $P(M, N)$ is the set of $A$-linear maps from $M$ to $N$ which factor through a free module. It is well-known that $\text{CM}(A)$ is a triangulated category with translation functor $\Omega^{-1}$, (see [10, 4.7]; cf. [2.5]). Here $\Omega(M)$ denotes the syzygy of $M$ and $\Omega^{-1}(M)$ denotes the co-syzygy of $M$. Also recall that an object $M$ is zero in $\text{CM}(A)$ if and only if it is free considered as an $A$-module. Furthermore $M \cong N$ in $\text{CM}(A)$ if and only if there exists finitely generated free modules $F, G$ with $M \oplus F \cong N \oplus G$ as $A$-modules.

2.5. Triangulated category structure on $\text{CM}(A)$.

We first describe basic exact triangle. Let $f: M \to N$ be a morphism in $\text{CM}(A)$. Note we have an exact sequence $0 \to M \xrightarrow{i} Q \xrightarrow{p} \Omega^{-1}(M) \to 0$, with $Q$-free. Let $C(f)$ be the pushout of $f$ and $i$. Thus we have a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & M & \xrightarrow{i} & Q & \xrightarrow{p} & \Omega^{-1}(M) & \rightarrow & 0 \\
\downarrow{f} & & \downarrow & & \downarrow{j} & & \downarrow & & \\
0 & \rightarrow & N & \xrightarrow{i'} & C(f) & \xrightarrow{\bar{p}'} & \Omega^{-1}(M) & \rightarrow & 0
\end{array}
\]

Here $j$ is the identity map on $\Omega^{-1}(M)$. As $N, \Omega^{-1}(M) \in \text{CM}(A)$ it follows that $C(f) \in \text{CM}(A)$. Then the projection of the sequence

$$M \xrightarrow{f} N \xrightarrow{i'} C(f) \xrightarrow{\bar{p}'} \Omega^{-1}(M)$$
in \( \text{CM}(A) \) is a basic exact triangle. Exact triangles in \( \text{CM}(A) \) are triangles isomorphic to a basic exact triangle.

2.6. By construction of exact triangles in \( \text{CM}(A) \) it follows that if 

\[
M \to N \to L \to \Omega^{-1}(M)
\]

is an exact triangle in \( \text{CM}(A) \) then we have a short exact sequence 

\[
0 \to N \to E \to \Omega^{-1}(M) \to 0,
\]

where \( E \cong L \) in \( \text{CM}(A) \).

2.7. Support varieties of modules over local complete intersections:

This is relatively simple in our case since \( A \) is complete with algebraically closed residue field.

2.8. Let \( A = Q/(\mathbf{u}) \) where \( (Q, \mathfrak{n}) \) is a complete regular local ring and \( \mathbf{u} = u_1, \ldots, u_c \in \mathfrak{n}^2 \) is a regular sequence. We need the notion of cohomological operators over a complete intersection ring.

The Eisenbud operators, \([12]\) are constructed as follows:

Let \( F: \cdots \to F_{i+2} \xrightarrow{\partial} F_{i+1} \xrightarrow{\partial} F_i \to \cdots \) be a complex of free \( A \)-modules.

Step 1: Choose a sequence of free \( Q \)-modules \( \tilde{F}_i \) and maps \( \tilde{\partial} \) between them:

\[
\tilde{F}: \cdots \to \tilde{F}_{i+2} \xrightarrow{\tilde{\partial}} \tilde{F}_{i+1} \xrightarrow{\tilde{\partial}} \tilde{F}_i \to \cdots
\]

so that \( F = A \otimes \tilde{F} \).

Step 2: Since \( \tilde{\partial}^2 \equiv 0 \) modulo \( (\mathbf{u}) \), we may write \( \tilde{\partial}^2 = \sum_{j=1}^c u_j \tilde{t}_j \) where \( \tilde{t}_j: \tilde{F}_i \to \tilde{F}_{i-2} \) are linear maps for every \( i \).

Step 3: Define, for \( j = 1, \ldots, c \) the map \( t_j = t_j(Q, f, \mathbf{u}): F \to F(-2) \) by \( t_j = A \otimes \tilde{t}_j \).

2.9. The operators \( t_1, \ldots, t_c \) are called Eisenbud’s operator’s (associated to \( \mathbf{u} \)). It can be shown that

1. \( t_i \) are uniquely determined up to homotopy.
2. \( t_i, t_j \) commute up to homotopy.

2.10. Let \( R = A[t_1, \ldots, t_c] \) be a polynomial ring over \( A \) with variables \( t_1, \ldots, t_c \) of degree \( 2 \). Let \( M, N \) be finitely generated \( A \)-modules. By considering a free resolution \( F \) of \( M \) we get well defined maps

\[
t_j: \text{Ext}_A^n(M, N) \to \text{Ext}_A^{n+2}(M, N) \quad \text{for} \quad 1 \leq j \leq c \quad \text{and} \quad \text{all} \ n,
\]

which turn \( \text{Ext}_A^n(M, N) = \bigoplus_{i \geq 0} \text{Ext}_A^i(M, N) \) into a module over \( R \). Furthermore these structure depend on \( \mathbf{u} \), are natural in both module arguments and commute with the connecting maps induced by short exact sequences.

2.11. Gulliksen, \([14]\ 3.1\), proved that \( \text{Ext}_A^n(M, N) \) is a finitely generated \( R \)-module. We note that \( \text{Ext}^n(M, k) \) is a finitely generated graded module over \( T = k[t_1, \ldots, t_c] \). Define \( V^* = \text{Var} \text{ann}_R(\text{Ext}^*(M, k)) \) in the projective space \( \mathbb{P}^{c-1} \). We call \( V^* \) the support variety of a module \( M \). Note that in \([5]\) support varieties are defined as \( \text{Var} \text{ann}_R(\text{Ext}^*(M, k)) \) in \( k^c \). However as the ideal involved is homogeneous we get a similar notion. Set \( \mathcal{V}^*(M, N) = \text{Var} \text{ann}_R(\text{Ext}^*(M, N) \otimes k) \) in \( \mathbb{P}^{c-1} \). Then in \([5]\ 5.6\) it is proved that \( \mathcal{V}^*(M, N) = \mathcal{V}^*(M) \cap \mathcal{V}^*(N) \).

2.12. Set \( \dim \emptyset = -1 \). For any \( A \)-module \( M \) it is clear that \( \dim \mathcal{V}^*(M) = c_{\text{ch}} A M - 1 \).
2.13. [5 5.6]: If \( 0 \to M_1 \to M_2 \to M_3 \to 0 \) is a short exact sequence and \( N \) is any \( A \)-module then for \( \{i, j, h\} = \{1, 2, 3\} \) we have
\[
\mathcal{V}^*(M_i, N) \subseteq \mathcal{V}^*(M_j, N) \cup \mathcal{V}^*(M_h, N).
\]

2.14. Let \( a \in \mathbb{P}^{c-1} \). Define
\[
\mathcal{I}_a = \{ M \mid M \text{ is MCM and } \mathcal{V}^*(M) = \{a\} \}.
\]
By [6, 2.3] there exists \( N \) with \( \mathcal{V}^*(N) = \{a\} \). Then \( \Omega^d(N) \in \mathcal{I}_a \). Thus \( \mathcal{I}_a \neq \emptyset \) for all \( a \in \mathbb{P}^{c-1} \). We call \( \mathcal{I}_a \) the class of indicator MCM’s for \( a \in \mathbb{P}^{c-1} \). The utility of \( \mathcal{I}_a \) will be apparent in the next section. We will need the following result:

**Lemma 2.15.** (with notation as in 2.14). For each \( a \in \mathbb{P}^{c-1} \) choose any \( N_a \in \mathcal{I}_a \).

Then for an \( A \)-module \( M \) the following conditions are equivalent:

(i) \( a \in \mathcal{V}^*(M) \).

(ii) \( \text{Ext}^*_A(M, N_a) \neq 0 \).

(iii) \( \text{Ext}^*_A(N_a, M) \neq 0 \).

**Proof.** The result follows since for any \( b \in \mathbb{P}^{c-1} \) we have
\[
\mathcal{V}^*(M, N_b) = \mathcal{V}^*(N_b, M) = \mathcal{V}^*(M) \cap \mathcal{V}^*(N_b) = \mathcal{V}^*(M) \cap \{b\}.
\]
\[\square\]

2.16. Let \( \mathcal{T} \) be a triangulated category and let \( \mathcal{S} \) be a thick subcategory of \( \mathcal{T} \). Let \( \text{Mor}_\mathcal{S} \) denote the collections of morphisms \( f: X \to Y \) such that the cone of \( f \) is in \( \mathcal{S} \). Recall the Verdier quotient \( \mathcal{T}/\mathcal{S} \) is obtained by formally inverting all morphisms in \( \text{Mor}_\mathcal{S} \); see [19, Chapter 2]. A morphism \( \phi: X \to Y \) can be written as a left fraction
\[
\begin{array}{ccc}
X & \xrightarrow{\phi=fu^{-1}} & Y \\
\downarrow{u} & & \downarrow{f} \\
U & & \\
& \xrightarrow{f} & \\
& \downarrow{u} & \\
\end{array}
\]
with \( u \in \text{Mor}_\mathcal{S} \); or as a right fraction
\[
\begin{array}{ccc}
X & \xrightarrow{\phi=v^{-1}g} & Y \\
\downarrow{g} & & \downarrow{v} \\
V & & \\
& \xrightarrow{v} & \\
& \downarrow{v} & \\
& & \end{array}
\]
with \( v \in \text{Mor}_\mathcal{S} \).

3. Support Varieties for some Verdier Quotients

In this section we show that in the Verdier quotients that we consider objects have a well-defined notion of support variety. Throughout our assumptions will be as in [11]. We will also consider support varieties of modules over complete intersections as defined in 2.11.

3.1. Let \( X \) be an algebraic set in \( \mathbb{P}^{c-1} \). Set
\[
\mathcal{S}_X = \{ M \mid M \text{ is MCM and } \mathcal{V}^*(M) \subseteq X \}
\]

**Lemma 3.2.** \( \mathcal{S}_X \) is a thick subcategory of \( \text{CM}(A) \).
Proof. Clearly $S_X$ is closed under isomorphisms. If $M \in S_X$ then as $V^*(\Omega^{-1}(M)) = V^*(M)$ we get $\Omega^{-1}(M) \in S_X$. Let

$$M \to N \to L \to \Omega^{-1}(M)$$

be an exact triangle in $\text{CM}(A)$ with $M, N \in S_X$. We then have a short exact sequence

$$0 \to N \to L' \to \Omega^{-1}(M) \to 0$$

with $L' \cong L$ in $\text{CM}(A)$. By 2.13 we get $V^*(L') \subseteq X$. Note $V^*(L) = V^*(L')$. Thus $S_X$ is a triangulated subcategory of $\text{CM}(A)$. Let $M \oplus N \in S_X$. As

$$V^*(M \oplus N) = V^*(M) \cup V^*(N)$$

we get $M, N \in S_X$. Thus $S_X$ is a thick subcategory of $\text{CM}(A)$.

3.3. Let $T_X = \text{CM}(A)/S_X$ denote the Verdier quotient of $\text{CM}(A)$ by $S_X$.

Lemma 3.4. (with hypotheses as in 3.2) Let $f: M \to N$ be in $\text{Mor}_{S_X}$. Then

$$V^*(M) \setminus X = V^*(N) \setminus X.$$

Proof. We have a triangle

$$\Omega(L) \to M \to N \to L$$

with $L \in S_X$. Let $a \in \mathbb{P}^{c-1} \setminus X$ and let $U \in T_a$. Then notice $\text{Hom}_A(U, L) = \text{Hom}_A(U, \Omega(L)) = 0$. So we have an isomorphism:

$$\text{Hom}_A(U, M) \cong \text{Hom}_A(U, N).$$

Similarly we have an isomorphism:

$$\text{Hom}_A(\Omega U, M) \cong \text{Hom}_A(\Omega U, N).$$

The result follows from 2.15.

As a consequence we have:

Corollary 3.5. (with hypotheses as in 3.2) Let $\phi: M \to N$ be an isomorphism in $T_X$. Then

$$V^*(M) \setminus X = V^*(N) \setminus X.$$

Proof. We write $\phi$ as a left fraction $f u^{-1}$

$$\begin{array}{ccc}
U & \xrightarrow{u} & M \\
\downarrow{f} & & \downarrow{\phi = fu^{-1}} \\
N & & \\
\end{array}$$

with $u \in \text{Mor}_{S_X}$. By Lemma 3.4 we get

$$V^*(M) \setminus X = V^*(U) \setminus X.$$

As $\phi$ is an isomorphism we get that $f \in \text{Mor}_{S_X}$ also. Again by Lemma 3.4 we get

$$V^*(N) \setminus X = V^*(U) \setminus X.$$

The result follows.

Definition 3.6. Let $M \in T_X$. Set

$$V_X(M) = V^*(M) \setminus X.$$

By Corollary 3.5 this is a well-defined invariant of $M$ in $T_X$. 

3.7. For \( i = 1, \ldots, c - 1 \) let
\[
CM_{\leq i}(A) = \{ M \mid M \text{ is MCM and } cx_A M \leq i \}.
\]
Then by a proof similar to 3.2 we get that \( CM_{\leq i}(A) \) is a thick subcategory of \( CM(A) \). Set
\[
T_i = CM(A)/CM_{\leq i}(A).
\]
3.8. If \( Y \) is a variety in \( \mathbb{P}^{c-1} \), write
\[
Y = \bigcup_{j=1}^{m} Y_j \text{ where } Y_j \text{ is irreducible.}
\]
Assume \( \dim Y_j = \dim Y \) for \( 1 \leq j \leq r \) and \( \dim Y_j < \dim Y \) for \( i > r \). Set
\[
\text{top}(Y) = \bigcup_{j=1}^{r} Y_j.
\]
It is easy to prove that \( \text{top}(Y) \) is an invariant of \( Y \).

**Lemma 3.9.** (with hypotheses as in 3.7) Let \( f: M \rightarrow N \) be in \( \text{Mor}_{CM_{\leq i}} \). Then
\[
\text{top}(V^*(M)) = \text{top}(V^*(N)).
\]

**Proof.** We write \( V^*(M) \) and \( V^*(N) \) as a union of irreducible subvarieties:
\[
V^*(M) = \bigcup_{j=1}^{m} Y_j \quad \text{and} \quad V^*(N) = \bigcup_{j=1}^{m} Z_j
\]
Assume \( \dim Y_j = \dim V^*(M) \) for \( 1 \leq j \leq r \) and \( \dim Y_j < V^*(M) \)dim for \( j > r \). Also assume \( \dim Z_j = \dim V^*(N) \) for \( 1 \leq j \leq s \) and \( \dim Z_j < V^*(N) \) for \( j > s \).

We have a triangle
\[
\Omega(L) \rightarrow M \rightarrow N \rightarrow L
\]
with \( L \in CM_{\leq i} \). Fix \( j \) with \( 1 \leq j \leq r \). As \( \dim V^*(L) \leq i - 1 \) and \( \dim Y_j \geq i \) we get that \( Y_j \setminus V^*(L) \neq \emptyset \). Let \( a \in Y_j \setminus V^*(L) \) and let \( U \in I_a \). By an argument similar to proof of Lemma 3.4 we get that \( \text{Ext}^*(U, N) \neq 0 \). So \( a \in V^*(N) \). Thus
\[
Y_j \setminus V^*(L) \subseteq V^*(N)
\]
It follows that \( Y_j \subseteq V^*(N) \). An elementary argument yields that \( Y_j \subseteq \text{top}(V^*(N)) \).

Corollary 3.10. (with hypotheses as in 3.7) Let \( \phi: M \rightarrow N \) be an isomorphism in \( T_i \). Then
\[
\text{top}(V^*(M)) \cong \text{top}(V^*(N)).
\]

The proof of the above Corollary uses 3.9 and is analogous to proof of Corollary 3.5.
4. Proof of Theorems 1.8 (partly) and 1.4

In this section we give proof of part of Theorem 1.8. As a consequence we give a proof of Theorem 1.4. We restate the part of Theorem 1.8 we will prove:

**Theorem 4.1.** (with hypotheses as in 1.1 and notation as in 1.3) Let $X$ be any algebraic set in $\mathbb{P}^{c-1}(k)$. Let $M,N$ be any two MCM $A$-modules. Consider the following two statements:

1. $\text{Hom}_{T_X}(M, \Omega^{-n}(N)) = 0$ for all $n \gg 0$.
2. $\mathcal{V}_X(M) \cap \mathcal{V}_X(N) = \emptyset$.

Then $(1) \implies (2)$.

As a corollary we prove

**Proof of Theorem 1.4.** By Theorem 4.1 we get $\mathcal{V}_X(M) = \emptyset$. So $\mathcal{V}^*(M) \subseteq X$. This implies $M = 0$ in $T_X$. □

We now give

**Proof of Theorem 4.1.** Assume $\text{Hom}_T(M, \Omega^{-n}(N)) = 0$ for all $n \geq n_0$. We know that $E(M,N) = \text{Ext}^*_A(M,N)$ is finitely generated over ring of cohomology operators $R = A[t_1, \ldots, t_c]$. Set

$$E^{ev}(M,N) = \bigoplus_{n \geq 0} \text{Ext}^{2n}(M,N) \quad \text{and} \quad E^{odd}(M,N) = \bigoplus_{n \geq 0} \text{Ext}^{2n+1}(M,N).$$

Then $E(M,N) = E^{ev}(M,N) \oplus E^{odd}(M,N)$ as $R$-modules. We may assume that for some $m \geq n_0$ the modules $E^{ev}(M,N)_{\geq m}$ and $E^{odd}(M,N)_{\geq m}$ are generated by $\text{Ext}^{2m}_A(M,N)$ and $\text{Ext}^{2m+1}_A(M,N)$ respectively as $R$-modules.

Let $u_1, \ldots, u_r \in \text{Ext}^{2m}_A(M,N)$ generate $E^{ev}(M,N)_{\geq m}$ as an $R$-module. Note

$$\text{Ext}^{2m}_A(M,N) = \text{Hom}_A(M, \Omega^{-2m}(N)) \to \text{Hom}_{T_X}(M, \Omega^{-2m}(N)) = 0.$$

So by [19, 2.1.26] there exits $W_j$ with $\mathcal{V}^*(W_j) \subseteq X$ such that the map $u_j \in \text{Hom}_A(M, \Omega^{-2m}(N))$ factors through $W_j$. We have a commutative diagram

$$\begin{array}{ccc}
W_j & \to & \Omega^{-2m}(N) \\
\downarrow w_j & & \downarrow \circlearrowleft \\
M & \to & \Omega^{-2m}(N)
\end{array}$$

So we have map $w_j^*: \text{Ext}^*(M,W_j) \to \text{Ext}^*_{\geq 2m}(M,N)$ of $R$-modules. Note $u_j \in \text{image } w_j^*$. Thus we have a surjective map

$$\bigoplus_{j=1}^r \text{Ext}^*(M,W_j) \xrightarrow{\bigoplus w_j} E^{ev}(M,N)_{\geq m} \to 0.$$ 

Set $W = \oplus_j W_j$. Note $W \in S_X$. Set $T = k[t_1, \ldots, t_c] = R \otimes k$. Tensoring with $k$ and taking annihilators (and their varieties) we get

$$\text{Var}(\text{ann}_T E^{ev}(M,N)_{\geq m} \otimes k) \subseteq X.$$ 

Similarly we obtain

$$\text{Var}(\text{ann}_T E^{odd}(M,N)_{\geq m} \otimes k) \subseteq X.$$
Notice
\[ \text{ann}_T E^{\text{odd}}(M, N)_{\geq m} \otimes k \cap \text{ann}_T E^{ev}(M, N)_{\geq m} \otimes k = \text{ann}_T E(M, N)_{\geq 2m} = K \text{(say)}. \]

Clearly \( \text{Var}(K) \subseteq X \). Set \( q \) to be the unique graded maximal ideal of \( T \). Then \( q^l K \subseteq \text{ann}_T E(M, N) \otimes k \) for some \( l \geq 1 \). It follows that
\[ \mathcal{V}^*(M) \cap \mathcal{V}^*(N) \subseteq X. \]

The result follows. \( \Box \)

5. Proof of Theorem 1.8

In this section we give a proof of Theorem 1.8. We first need:

**Lemma 5.1.** [with hypotheses as in 1.8] Let \( M \in \mathcal{CM}(A) \) be essentially disjoint from \( X \). Let \( f : N \to M \) and \( g : M \to L \) be in \( \text{Mor}_{S_X} \). Then \( N, L \) are essentially disjoint from \( X \). Further let \( M = M_1 \oplus M_2 \), \( N = N_1 \oplus N_2 \) and \( L = L_1 \oplus L_2 \) where \( \mathcal{V}^*(M_1 \oplus N_1 \oplus L_1) \subseteq X \) and \( \mathcal{V}^*(M_2 \oplus N_2 \oplus L_2) \subseteq \mathbb{P}^{c-1} \setminus X \). Consider the natural maps \( i_M : M_2 \to M \), \( \pi_M : M \to M_2 \), \( i_N : N_2 \to N \) and \( \pi_L : L \to L_2 \). Then \( \pi_M \circ f \circ i_N : N_2 \to M_2 \) and \( \pi_L \circ g \circ i_M : M_2 \to L_2 \) are isomorphisms.

**Proof.** Note that \( i_M, \pi_M \in \text{Mor}_{S_X} \). So \( \pi_M \circ f : N \to M_2 \) and \( g \circ i_M : M_2 \to L \) are in \( \text{Mor}_{S_X} \). By 3.4 we get that
\[ \mathcal{V}^*(N) \setminus X = \mathcal{V}^*(M_2) \setminus X = \mathcal{V}^*(M_2). \]

So we have a disjoint union
\[ \mathcal{V}^*(N) = (\mathcal{V}^*(N) \cap X) \sqcup \mathcal{V}^*(M_2). \]

By 3.1 we get that \( N = N_1 \oplus N_2 \) where \( \mathcal{V}^*(N_1) = \mathcal{V}^*(N) \cap X \) and \( \mathcal{V}^*(N_2) = \mathcal{V}^*(M_2) \). A similar assertion holds for \( L \).

As \( i_N : N_2 \to N \) is in \( \text{Mor}_{S_X} \) we get that \( h = \pi_M \circ f \circ i_N : N_2 \to M_2 \) is in \( \text{Mor}_{S_X} \). So we have a triangle
\[ W \xrightarrow{\phi} N_2 \xrightarrow{h} M \xrightarrow{\psi} \Omega^{-1}(W); \]

where \( W \in S_X \). As
\[ \mathcal{V}^*(M_2) \cap X = \mathcal{V}^*(N_2) \cap X = \emptyset; \]

we get \( \phi = 0 \) and \( \psi = 0 \). As \( \psi = 0 \) we get by 13.4 that \( \phi \) is a section. But \( \phi = 0 \). So \( W = 0 \). Thus \( h \) is an isomorphism. An analogous proof shows that \( \pi_L \circ g \circ i_M \) is an isomorphism. \( \Box \)

We also need:

**Proposition 5.2.** [with hypotheses as in 1.8] Let \( M \in \mathcal{CM}(A) \) be such that \( \mathcal{V}^*(M) \subseteq \mathbb{P}^{c-1} \setminus X \). Let \( N, L \in \mathcal{CM}(A) \) be any. Then the natural maps
\[ \Psi : \text{Hom}_A(N, M) \to \text{Hom}_{\mathcal{T}_X}(N, M) \quad \text{and} \quad \Phi : \text{Hom}_A(M, L) \to \text{Hom}_{\mathcal{T}_X}(M, L); \]

are isomorphisms.

**Proof.** Suppose \( \Psi(f) = 0 \). Then factors through some \( W \in S_X \). So we have a commutative diagram

\[ \begin{array}{ccc}
W & \xrightarrow{\psi} & M \\
\downarrow{\phi} & & \downarrow{\psi} \\
N & \xrightarrow{f} & M
\end{array} \]
As $\mathcal{V}^*(M) \subseteq \mathbb{P}^{c-1} \setminus X$ we get that $w = 0$. So $f = 0$. Thus $\Psi$ is injective. A similar argument shows that $\Phi$ is injective.

Let $\xi \in \text{Hom}_{T_X}(M,L)$. We write $\xi$ as a left fraction:

$$
\begin{array}{ccc}
U & \xrightarrow{f} & L \\
\downarrow{u} & & \\
M & \xrightarrow{\xi = fu^{-1}} & L
\end{array}
$$

where $u \in \text{Mor}_S$. By 5.1 $U$ is essentially disjoint from $X$. Say $U = U_1 \oplus U_2$ with $\mathcal{V}^*(U_1) \subseteq X$ and $\mathcal{V}^*(U_2) \subseteq \mathbb{P}^{c-1} \setminus X$. Let $i_U : U_2 \to U$ be the natural map. Then $i_U \in \text{Mor}_S$. Set $u' = u \circ i_U$ and $f' = f \circ i_U$. Then notice $\xi = f'(u')^{-1}$. So we can write $\xi$ as an equivalent left fraction:

$$
\begin{array}{ccc}
U_2 & \xrightarrow{f'} & L \\
\downarrow{u'} & & \\
M & \xrightarrow{\xi = f'(u')^{-1}} & L
\end{array}
$$

By 5.1 we get that $u'$ is an isomorphism in $\text{CM}(A)$. It follows that $\xi \in \text{image } \Phi$. Thus $\Phi$ is surjective. The argument to show $\Psi$ is surjective is similar. However to prove this it is convenient to express elements of $\text{Hom}_{T_X}(N,M)$ as right fractions.

Note that we proved half of Theorem 1.8 in the previous section. Here we prove a stronger result which implies the rest of Theorem 1.8 that we wish to prove.

**Theorem 5.3.** (with hypotheses as in 1.1 and notation as in 1.3) Let $X$ be any algebraic set in $\mathbb{P}^{c-1}(k)$. Let $M,N$ be any two MCM $A$-modules. Assume $M$ or $N$ is essentially disjoint from $X$. If $\mathcal{V}_X(M) \cap \mathcal{V}_X(N) = \emptyset$ then $\text{Hom}_{T_X}(M,\Omega^{-n}(N)) = 0$ for all $n \in \mathbb{Z}$.

**Proof.** Fix $n \in \mathbb{Z}$. First assume that $M$ is essentially disjoint from $X$. Then $M = M_1 \oplus M_2$ where $\mathcal{V}^*(M_1) \subseteq X$ and $\mathcal{V}^*(M_2) \subseteq \mathbb{P}^{c-1} \setminus X$. It follows that $\mathcal{V}^*(M_2) \cap \mathcal{V}^*(N) = \emptyset$. It follows that $\text{Hom}_A(M_2,N) = 0$. Notice $M_1 = 0$ in $T_X$. So we have,

$$
\text{Hom}_{T_X}(M,\Omega^{-n}(N)) = \text{Hom}_{T_X}(M_2,\Omega^{-n}(N))
$$

$$
\cong \text{Hom}_A(M_2,\Omega^{-n}(N)), \text{ by 5.2}
$$

$$
= 0
$$

The case when $N$ is essentially disjoint from $X$ is similar.

\qed

6. PROOF OF THEOREM 1.5

In this section we give a proof of Theorem 1.5. We first need the following result from 20.3.7:

**Lemma 6.1.** (with hypotheses as in 1.1) Let $M \in \text{CM}(A)$ with $cx_A M \geq 2$. Then there exists for some $n \geq 0$ a short exact sequence

$$
0 \to K \to \Omega^{n+2}(M) \to \Omega^n(M) \to 0
$$

where $cx_K K = cx_A M - 1$. 

We will also need the following:

**Definition 6.2.** Let $M \in \mathcal{T}_X$. Set 
\[
\text{cx} M = \min \{ \text{cx}_A L \mid L \cong M \text{ in } \mathcal{T}_X \}.
\]

The proof of Theorem 6.3 is by induction. It is convenient to prove a stronger form of the result in a special case:

**Theorem 6.3.** (with hypotheses as in [L.7] and notation as in [L.3]) Let $M, N$ be objects in $\mathcal{T}_X$. Assume $\mathcal{V}_X(M) \cap \mathcal{V}_X(N) = \emptyset$. Let $r = \text{cx} N$. Then 
\[
\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-n}(N)) = 0 \quad \text{for } n = m, \cdots, m + r - 2
\]
\[
\implies \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-n}(N)) = 0 \quad \text{for all } n \geq m.
\]

**Remark 6.4.** Our Conjecture 1.9 asserts that in the case above $\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-n}(N)) = 0$ for $n \gg 0$ without any initial condition.

We now give:

**Proof of Theorem 6.3.** we use induction on $r = \text{cx}_A N$ to prove the result. We may assume $r = \text{cx}_A N$. We first consider the case when $r = 0$. Then $N = 0$ and so we have nothing to prove. Next we consider the case when $r = 1$. Then note that $\mathcal{V}^1(N) = \{a\}$ for some $a \in \mathbb{P}^{n-1} \setminus X$. The result follows from [L.6].

Next we consider the case when $r = \text{cx}_A N \geq 2$ and assume the results are proved for MCM modules $E$ with $\text{cx}_A E < r$. By [L.1] we have a short exact sequence, 
\[
0 \to K \to \Omega^{n+2}(N) \to \Omega^n(N) \to 0;
\]
for some $n \geq 0$ with $\text{cx}_A K = r - 1$. So we have a triangle in $\text{CM}(A)$ 
\[
K \to \Omega^{n+2}(N) \to \Omega^n(N) \to \Omega^{-1}(K)
\]
Rotating it several times we get the following triangle in $\text{CM}(A)$ 
\[
\Omega^2(N) \to N \to E \to \Omega(N)
\]
where $\text{cx} E = r - 1$. Also note $\mathcal{V}^s(E) \subseteq \mathcal{V}^s(N)$. We takes its image in $\mathcal{T}_X$ 
\[
s : \Omega^2(N) \to N \to E \to \Omega(N).
\]
Note $\mathcal{V}_X(M) \cap \mathcal{V}_X(E) = \emptyset$. By applying $\text{Hom}_{\mathcal{T}_X}(M, -)$ to $s$ we obtain long exact sequence:

(†) \[
\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(N)) \to \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(E)) \to \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j+1}(N)) \to \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j-1}(N)) \to \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j-1}(E)) \to \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(N)) \to .
\]
It follows that 
\[
\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(E)) = 0 \quad \text{for } j = m + 1, \ldots, m + r - 2.
\]
As $\text{cx}_A E \leq \text{cx}_A E = r - 1$ we get by induction hypotheses that 
\[
\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(E)) = 0 \quad \text{for all } j \geq m + 1.
\]
Putting $j = m + r - 2$ in (†) we get 
\[
\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-m-r+1}(N)) = 0.
\]
Iterating we obtain $\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(N)) = 0$ for all $j \geq m$. \hfill \Box

We now state and prove a result which implies Theorem 1.5.
Theorem 6.5. (with hypotheses as in [1,3] and notation as in [1,3]) Let $M, N$ be objects in $\mathcal{T}_X$. Let $r = \text{ecx} N$. Then
\[
\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-n}(N)) = 0 \text{ for } n = m, m+1, \cdots, m+r
\]
\[
\implies \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-n}(N)) = 0 \text{ for all } n \geq m.
\]

Proof. Set $a = \dim \mathcal{V}_X(M) \cap \mathcal{V}_X(N)$. We prove the result by induction on $a$. We first consider the case when $a = -1$; i.e., $\mathcal{V}_X(M) \cap \mathcal{V}_X(N) = \emptyset$. In this case the result follows from Theorem 6.3.

Let $a = t \geq 0$ and assume the result is proved when $a < t$. Next we consider the case when $r = \text{ecx}_A N \geq 2$ and assume the results are proved for MCM modules $E$ with $\text{ecx}_A E < r$. By [1,3] we have a short exact sequence,
\[
0 \to K \to \Omega^{n+2}(N) \to \Omega^n(N) \to 0;
\]
for some $n \geq 0$ with $\text{ecx}_A K = r - 1$. So we have a triangle in $\text{CM}(A)$
\[
K \to \Omega^{n+2}(N) \to \Omega^n(N) \to \Omega^1(K)
\]
Rotating it several times we get the following triangle in $\text{CM}(A)$
\[
\Omega^2(N) \to N \to E \to \Omega(N)
\]
where $\text{cx} E = r - 1$. Also note $\mathcal{V}^*(E) \subseteq \mathcal{V}^*(N)$. We takes its image in $\mathcal{T}_X$
\[
s : \Omega^2(N) \to N \to E \to \Omega(N).
\]
Note $\mathcal{V}_X(M) \cap \mathcal{V}_X(E) \subseteq \mathcal{V}_X(M) \cap \mathcal{V}_X(E)$. So $\dim \mathcal{V}_X(M) \cap \mathcal{V}_X(E) \leq t$. By applying $\text{Hom}_{\mathcal{T}_X}(M, -)$ to $s$ we obtain long exact sequence:
\[
\vdash \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(N)) \to \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(E)) \to \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j+1}(N)) \to \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j+1}(E)) \to \text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(N)) \to \cdot.
\]
It follows that
\[
\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(E)) = 0 \text{ for } j = m+1, \ldots, r.
\]
As $\text{ecx}_A E \leq \text{cx}_A E = r - 1$ we get by induction hypotheses that
\[
\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(E)) = 0 \text{ for all } j \geq m + 1.
\]
Putting $j = m + r$ in $(\vdash)$ we get $\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-m-r+1}(N)) = 0$. Iterating we obtain $\text{Hom}_{\mathcal{T}_X}(M, \Omega^{-j}(N)) = 0$ for all $j \geq m$. \hfill \Box

7. Proof of Theorems 1.12, 1.13 and 1.14

In this section we give proofs of Theorems 1.12, 1.13 and 1.14. We first prove:

Theorem 7.1. (with hypotheses as in [1,3] and notation as in [1,11]) Fix $i \in \{1, \ldots, c-1\}$. Let $M, N$ be any two MCM $A$-modules. Consider the following two statements:
(1) $\text{Hom}_{\mathcal{T}}(M, \Omega^{-n}(N)) = 0$ for all $n \gg 0$.
(2) $\dim(\mathcal{V}_i(M) \cap \mathcal{V}_i(N)) \leq i - 1$.

Then (1) $\implies$ (2).

Proof. The proof of this result is completely analogous to proof of Theorem 1.1.
One simply replaces $W_j \in \mathcal{S}_X$ with $W_j \in \text{CM}_{\leq i}(A)$.

As a consequence we give
Proof of Theorem 1.12. We get \( \dim \text{top}(M) = \dim V_i(M) \leq i - 1 \). This forces \( \dim V^i(M) \leq i - 1 \). So \( \text{cx}_A M \leq i \). Therefore \( M = 0 \) in \( T_i \). □

As we observed in the proof of Theorem 1.5 it is convenient to prove a slightly more general result:

**Theorem 7.2.** (with hypotheses as in 1.1 and notation as in 1.11) Let \( M, N \) be objects in \( T_X \). Let \( r = \text{cx}_A N \). Then

\[
\text{Hom}_{T_X}(M, \Omega^{-n}(N)) = 0 \quad \text{for} \quad n = m, m + 1, \ldots, m + r - i
\]

\[
\implies \text{Hom}_{T_X}(M, \Omega^{-n}(N)) = 0 \quad \text{for all} \quad n \geq m.
\]

**Proof.** If \( r \leq i \) then \( N = 0 \) in \( T_i \) and so we have nothing to prove. Next we consider the case when \( \text{cx}_A N = i + 1 \). By 6.1 we have a short exact sequence:

\[
0 \to K \to \Omega^{n+2}(N) \to \Omega^n(N) \to 0;
\]

for some \( n \geq 0 \) and \( \text{cx}_A K = i \). This yields an exact triangle in \( \text{CM}(A) \)

\[
K \to \Omega^{n+2}(N) \to \Omega^n(N) \to \Omega^{-1}(K).
\]

Taking the image of the above exact triangle in \( T_i \) and noting \( K = 0 \) in \( T_i \) we get \( \Omega^{n+2}(N) \cong \Omega^n(N) \) in \( T_i \). It follows that \( N \) is two periodic in \( T_i \). The result trivially follows.

Rest of the proof is similar to proof of Theorem 6.5. □

Finally we give:

Proof of Theorem 1.14. Suppose \( U, V \) are MCM \( A \)-modules such that

\[
\text{Hom}_{T_{c-1}}(U, V) = 0 = \text{Hom}_{T_{c-1}}(U, \Omega^{-1}(V)).
\]

Suppose if possible \( U, V \neq 0 \) in \( T_{c-1} \). Thus \( \text{cx}_A U = \text{cx}_A V = c \). We have \( V^*(U) = V^*(V) = \mathbb{P}^{c-1} \). Thus \( V_{c-1}(U) = V_{c-1}(V) = \mathbb{P}^{c-1} \).

Note \( V \) is two periodic in \( T_{c-1} \). So we have

\[
\text{Hom}_{T_{c-1}}(U, \Omega^{-j}(V)) = 0 \quad \text{for all} \quad j \geq 0.
\]

By Theorem 7.1 we get \( \dim(V_{c-1}(U) \cap V_{c-1}(V)) \leq c - 2 \). We get \( \dim \mathbb{P}^{c-1} \leq c - 2 \) which is a contradiction. □

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Department of Mathematics, IIT Bombay, Powai, Mumbai 400 076

Email address: tputhen@math.iitb.ac.in