De Sitter vacua in ghost-free massive gravity theory

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We present a simple procedure to obtain all de Sitter solutions in the ghost-free massive gravity theory by using the Gordon ansatz. For these solutions the physical metric can be conveniently viewed as describing a hyperboloid in 5D Minkowski space, while the flat reference metric depends on the Stuckelberg field $T(t, r)$ that satisfies the equation $(\partial_t T)^2 - (\partial_r T)^2 = 1$. This equation has infinitely many solutions, hence there are infinitely many de Sitter vacua with different physical properties. Only the simplest solution with $T = t$ has been previously studied since it is manifestly homogeneous and isotropic, but it is unstable. However, other solutions could be stable. We require the timelike isometry to be common for both metrics, and this gives physically distinguished solutions since only for them the canonical energy is time-independent. We conjecture that these solutions minimize the energy and are therefore stable. We also show that in some cases solutions can be homogeneous and isotropic in a non-manifest way such that their symmetries are not obvious. All of this suggests that the theory may admit viable cosmologies.

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I. INTRODUCTION

The discovery of the ghost-free massive gravity theory by de Rham, Gabadadze, and Tolley (dRGT) [1] (see [2] for a review) opens up the possibility to explain the dark energy and the cosmic acceleration [3] by a tiny mass of the gravitons. The dRGT field equations admit the de Sitter solution with the cosmological constant mimicked by the graviton mass. This solution can describe the late time acceleration, but a special analysis is needed to decide whether its other properties are physically acceptable.

A closer look reveals that the de Sitter solution in the dRGT theory is actually not unique, and a number of its versions have been found [4],[5],[6],[7],[8],[9]. A special attention was received by one solution whose physical and reference metrics are of the manifestly homogeneous and isotropic Freedman-Lemaître-Robertson-Walker (FLRW) form [7]. However, a detailed analysis revealed that this solution is unstable [10]. For other known solutions only the physical metric is manifestly FLRW while the reference metric looks inhomogeneous, for which reason they are considered to be less interesting [6]. All of this has reduced the interest towards the dRGT theory, the focus shifting towards its extensions, as for example the bigravity [11],[12],[13] and other generalizations admitting FLRW solutions [14].

However, we would like to argue that it may be premature to abandon the dRGT theory on the basis of negative evidence obtained from just one solution, because the theory admits infinitely many other solutions that could be physically interesting. They all have the same physical (de Sitter) metric but different values of the reference metric depending on the Stuckelberg field \( T(t, r) \) subject to a complicated differential equation [6],[8],[9]. Below we shall describe a simple way to obtain these solutions by applying the Gordon ansatz [15] and using the global embedding coordinates. The \( T \)-equation then assumes a simple form, \((\partial_t T)^2 - (\partial_r T)^2 = 1\), whose essentially general solution is known. The simplest solution \( T = t \) [7] is unstable [10] but other solutions could be stable. One can choose \( T \) such that both metrics are invariant under the timelike isometry, which gives special solutions since only for them the canonical energy is time independent. We conjecture that their energy is minimal and hence these solutions are stable. We also give explicit examples where the reference metric looks inhomogeneous but shares with the physical metric the same translational and rotational isometries. Hence, solutions considered to be non-FLRW can actually be homogeneous and isotropic. All of this suggests that viable dRGT cosmologies may exist.
II. THE DRGT MASSIVE GRAVITY

The theory is defined on a four-dimensional spacetime manifold endowed with two Lorentzian metrics, the physical metric \( g_{\mu\nu} \) and the flat reference metric \( f_{\mu\nu} = \eta_{AB} \partial_\mu \Phi^A \partial_\nu \Phi^B \) with \( \eta_{AB} = \text{diag}[-1, 1, 1, 1] \). The fields \( \Phi^A(x) \) are sometimes called Stuckelberg scalars. The theory is defined by the action

\[
S = \frac{M_{Pl}^2}{m^2} \int \left( \frac{1}{2} R(g) - U \right) \sqrt{-g} \, d^4x. \tag{2.1}
\]

The metrics and all coordinates are assumed to be dimensionless, the length scale being the inverse graviton mass \( 1/m \). The interaction between the two metrics is expressed by a scalar function of the tensor \( \gamma^\mu_\nu = \sqrt{g^{\mu\sigma} f_{\nu\sigma}} \) where \( g^{\mu\nu} \) is the inverse of \( g_{\mu\nu} \) and the square root is understood in the matrix sense, i.e. \( (\gamma^2)^{\mu_\nu} \equiv \gamma^\mu_\alpha \gamma^\alpha_\nu = g^{\mu\alpha} f_{\nu\alpha} \). If \( \lambda_A \ (A = 0, 1, 2, 3) \) are the eigenvalues of \( \gamma^\mu_\nu \), then the interaction potential is given by

\[
U = \sum_{n=0}^{4} b_k U_k \quad \text{where} \quad b_k \text{ are parameters}
\]

and \( U_k \) are defined by the relations

\[
U_0 = 1, \quad U_1 = \sum_A \lambda_A = [\gamma], \quad U_2 = \sum_{A < B} \lambda_A \lambda_B = \frac{1}{2!} ([\gamma]^2 - [\gamma^2]),
\]

\[
U_3 = \sum_{A < B < C} \lambda_A \lambda_B \lambda_C = \frac{1}{3!} ([\gamma]^3 - 3[\gamma][\gamma^2] + 2[\gamma^3]),
\]

\[
U_4 = \lambda_0 \lambda_1 \lambda_2 \lambda_3 = \frac{1}{4!} ([\gamma]^4 - 6[\gamma]^2[\gamma^2] + 8[\gamma][\gamma^3] + 3[\gamma^2]^2 - 6[\gamma^4]).
\]

Here, using the hat to denote matrices, one defines \( [\gamma] \equiv \text{tr}(\hat{\gamma}) = \gamma^{\mu}_\mu, \quad [\gamma^k] \equiv \text{tr}(\hat{\gamma}^k) = (\gamma^k)^{\mu}_\mu \).

The parameters \( b_k \) can apriori be arbitrary, but if one requires the flat space to be a solution of the theory and \( m \) to be the Fierz-Pauli mass of the gravitons in flat space, then the five \( b_k \) are expressed in terms of two arbitrary parameters, sometimes called \( c_3 \) and \( c_4 \), as

\[
b_0 = 4c_3 + c_4 - 6, \quad b_1 = 3 - 3c_3 - c_4, \quad b_2 = 2c_3 + c_4 - 1, \quad b_3 = -(c_3 + c_4), \quad b_4 = c_4. \quad \tag{2.2}
\]

The metric \( g_{\mu\nu} \) and the scalars \( \Phi^A \) are the variables of the theory. Varying the action (2.1) with respect to \( g_{\mu\nu} \) gives the Einstein equations \( G_{\mu\nu} = T_{\mu\nu} \) with

\[
T^\mu_\nu = \{ b_1 U_0 + b_2 U_1 + b_3 U_2 + b_4 U_3 \} \gamma^\mu_\nu - \{ b_2 U_0 + b_3 U_1 + b_4 U_2 \} (\gamma^2)^{\mu}_\nu
\]

\[
+ \{ b_3 U_0 + b_4 U_1 \} (\gamma^3)^{\mu}_\nu - b_4 U_0 (\gamma^4)^{\mu}_\nu - U \delta^\mu_\nu. \tag{2.3}
\]

Varying the action with respect to \( \Phi^A \) gives the conservation conditions \( \nabla_\mu T^\mu_\nu = 0 \). These are equations for the Stuckelberg scalars, but they are actually not independent and follow from the Bianchi identities for the Einstein equations.
III. DE SITTER SPACE

The above field equations admit solutions for which the physical metric is de Sitter. Specifically, the de Sitter space can be globally visualized as the hyperboloid

\[-X_0^2 + \sum_i X_i^2 + X_4^2 = \alpha^2\]  

is the 5D Minkowski space with the metric

\[ds^2 = -dX_0^2 + \sum_i dX_i^2 + dX_4^2.\]  

Rescaling the coordinates, \(X_0 = \alpha t, X_i = \alpha x_i, X_4 = \alpha r\) with \(x_i \equiv (x, y, z)\), the metric reads

\[ds_g^2 = \alpha^2 \left\{ -dt^2 + dr^2 + dx^2 + dy^2 + dz^2 \right\}
= \alpha^2 \left\{ -dt^2 + dr^2 + dR^2 + R^2 d\Omega^2 \right\}\]  

where \(d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\phi^2\) and

\[R^2 \equiv x^2 + y^2 + z^2 = 1 + t^2 - r^2.\]  

Let us choose the flat reference metric as

\[ds_f^2 = \alpha^2 u^2 \left\{ -dT^2 + dX^2 + dY^2 + dZ^2 \right\},\]  

where \(u\) is a constant and \(T, X, Y, Z\) are the Stuckelberg fields.

It turns out that \(\alpha, u\) and \(T, X, Y, Z\) can be chosen such that the two metrics fulfill the field equations. Specifically, it is sufficient to make sure that they fulfill the following relation (the Gordon ansatz) \([15]\),

\[f_{\mu\nu} = \omega^2 \left( g_{\mu\nu} + (1 - \zeta^2) V^\mu V_\nu \right),\]  

where \(\omega, \zeta\) are functions and

\[g^{\mu\nu} V_\mu V_\nu \equiv V^\mu V_\mu = -1.\]  

If Eq.(3.6) is fulfilled, then one can see at once that

\[\gamma^\mu_\nu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}} = \omega \left( \delta^\mu_\nu + (1 - \zeta) V^\mu V_\nu \right),\]  

since \(\gamma^\mu_\nu \gamma_\nu^\alpha = g^{\mu\alpha} f_{\alpha\nu}\). One has \((\gamma^n)^\mu_\nu = \omega^n \left( \delta^\mu_\nu + (1 - \zeta^n) V^\mu V_\nu \right)\) and so the energy-momentum tensor (2.3) is

\[T^\mu_\nu = - \left\{ P_0(\omega) - \zeta P_1(\omega) \right\} \delta^\mu_\nu + \omega (\zeta - 1) P_1(\omega) V^\mu V_\nu\]  

with
\[ P_m(\omega) = b_m + 2b_{m+1} \omega + b_{m+2} \omega^2; \quad m = 0, 1, 2. \] (3.10)

Let us set \( \omega = u \) where \( u \) is a constant chosen such that \( P_1(u) = 0. \) (3.11)

Then the energy-momentum tensor (3.9) reduces to \( T_\mu^\nu = -P_0(u)\delta_\mu^\nu \) and the Einstein equations become
\[ G_\mu^\nu + \Lambda \delta_\mu^\nu = 0 \] (3.12)

with \( \Lambda = P_0(u) \). The de Sitter metric (3.3) is a solution of these equations provided that
\[ \frac{1}{\alpha^2} = \frac{\Lambda}{3} = \frac{P_0(u)}{3}. \] (3.13)

Therefore, the metrics (3.3) and (3.5) will indeed fulfill the field equations if \( u \) and \( \alpha \) are defined by (3.11),(3.13), provided that one can adjust the functions \( T, X, Y, Z \) such that the Gordon relation (3.6) is fulfilled.

Let us choose in (3.5) \( T = T(t, r), X = x, Y = y, Z = z \) so that the f-metric becomes
\[ ds_f^2 = \alpha^2 u^2 \left\{-dT^2 + dx^2 + dy^2 + dz^2\right\} = \alpha^2 u^2 \left\{-dT^2 + dR^2 + R^2d\Omega^2\right\}. \] (3.14)

The two metrics (3.3) and (3.14) are related to each other as
\[ ds_f^2 = u^2 (ds_g^2 + dt^2 - dr^2 - dT^2). \] (3.15)

This will be compatible with the Gordon relation (3.6) if
\[ \partial_\mu t \partial_\nu t - \partial_\mu r \partial_\nu r - \partial_\mu T \partial_\nu T = (1 - \zeta^2)V_\mu V_\nu. \] (3.16)

Assuming that the indices \( \mu, \nu \) correspond to \( (t, r, \vartheta, \varphi) \) yields \( V_\theta = V_\varphi = 0 \) and
\[ (\partial_t T)^2 - 1 = (\zeta^2 - 1)V_t^2, \]
\[ (\partial_r T)^2 + 1 = (\zeta^2 - 1)V_r^2, \]
\[ \partial_t T \partial_r T = (\zeta^2 - 1)V_t V_r. \] (3.17)

From the first two of these relations one obtains
\[ V_t^2 = \frac{(\partial_t T)^2 - 1}{\zeta^2 - 1}, \quad V_r^2 = \frac{(\partial_r T)^2 + 1}{\zeta^2 - 1}, \] (3.18)
while the normalization condition (3.7) determines $\zeta$. Finally, inserting (3.18) to the third relation in (3.17) yields

\[(\partial_t T)^2(\partial_r T)^2 = ((\partial_t T)^2 - 1)((\partial_r T)^2 + 1)\]  
(3.19)

and therefore

\[(\partial_t T)^2 - (\partial_r T)^2 = 1.\]  
(3.20)

This completes the procedure, because $V_\mu$ and $\zeta$ are determined by the above formulas and the Gordon relation is fulfilled.

Summarizing, the de Sitter solution in the theory is described by Eqs.(3.3),(3.14) where $u, \alpha$ are defined by (3.11),(3.13) and $T$ is a solution of the differential equation (3.20). Since there are infinitely many $T$’s subject to (3.20), there are infinitely many de Sitter solutions. They all have the same physical metric (3.3) but differ one from the other by the choice of $T$ in the reference metric (3.14). The physical properties of solutions with different $T$’s, as for example their stability, can be different.

These solutions were actually obtained previously [6],[8],[9], but within a different computation scheme yielding the $T$-equation in a form that gives little hope to solve it (see Eq.(A.3) in the Appendix). Our procedure yields its equivalent form (3.20), which is simple and admits a general solution. In addition, by slightly modifying our procedure, we can obtain new things. Specifically, it was assumed in the above derivation that both metrics have the same spatial $SO(3)$ symmetry. However, let us rather choose

\[ds_f^2 = \alpha^2 u^2 \{-dt^2 + dx^2 + dy^2 + dZ^2\}\]  
(3.21)

with $Z = Z(r, z)$, so that the two metrics share the same $SO(1,2)$ symmetry in the $t, x, y$ subspace. Repeating the above analysis one obtains

\[(\partial_r Z)^2 + (\partial_z Z)^2 = 1,\]  
(3.22)

and this gives new solutions. When expressed in the standard spherical coordinates, their $f$-metric will not look spherically symmetric, since for generic $Z$ it has no common with the $g$-metric $SO(3)$ symmetry, although it has its own $SO(3)$ in the $x, y, Z$ space. Below we shall mainly be discussing equation (3.20) since the analysis of (3.22) is similar.
IV. THE SIMPLEST SOLUTION

Even though there are infinitely many solutions of Eq. (3.20), almost all known dRGT cosmologies reported in the literature correspond to just one simplest solution,

\[ T = t. \] (4.1)

A slightly more general choice is

\[ T = \cosh(\xi) t + \sinh(\xi) r \] (4.2)

with a constant \( \xi \). However, the value of \( \xi \) can be changed by a boost in the \( t, r \) plane of the ambient 5D Minkowski space, which does not affect the g-metric (3.3), hence one can set \( \xi = 0 \) without loss of generality. Rewriting (4.1) in different coordinates gives results which look very different, and it has not been recognized that they actually describe the same solution. Let us see what happens when this solution is expressed in the standard spatially flat, closed, and open coordinate systems.

1. Flat slicing

Let us express \( t, r, R \) in (3.3) in terms of two new coordinates \( \tau \) and \( \rho \) as

\[ t = \sinh \tau + \frac{\rho^2}{2} e^\tau, \quad r = \cosh \tau - \frac{\rho^2}{2} e^\tau, \quad R = e^\tau \rho. \] (4.3)

This solves the constraint (3.4) and transforms the de Sitter metric (3.3) to the standard FLRW form with flat spatial sections,

\[ ds^2_g = \alpha^2 \{-d\tau^2 + a^2(\tau)(d\rho^2 + \rho^2 d\Omega^2)\}, \] (4.4)

where \( a(\tau) = e^\tau \). The function \( T = t \) can be represented as

\[ T = \frac{1}{2} \int \frac{d\tau}{a(\tau)} + \frac{1}{2} (1 + \rho^2) a(\tau). \] (4.5)

This solution was found in Ref. [6] for \( b_k \) given by (2.2) with \( c_3 = c_4 = 0 \), and later for arbitrary \( b_k \) [9] (solution in [6], [9] contains an integration constant that can be obtained by using (4.2) instead of (4.1)). Although the g-metric (4.4) is manifestly homogeneous and isotropic, the f-metric (3.14), when expressed in the \( \tau, \rho \) coordinates, becomes non-diagonal and \( \rho \)-dependent, which suggests that it is inhomogeneous. For this reason it is sometimes said that the dRGT theory does not admit genuinely homogeneous and isotropic cosmologies with flat spatial sections [6]. However, we shall shortly comment on this.
2. Closed slicing

If one chooses in (3.3)
\[ t = \sinh(\tau), \quad r = \cosh(\tau) \cos(\rho), \quad R = \cosh(\tau) \sin(\rho), \]
this solves the constraint (3.4) and the de Sitter metric (3.3) assumes the FLRW form with closed spatial sections,
\[ ds_g^2 = \alpha^2 \{-d\tau^2 + a^2(\tau)(d\rho^2 + \sin^2(\rho)d\Omega^2)\} \]
with \( a(\tau) = \cosh(\tau) \). These coordinates cover the whole of de Sitter space. The Stuckelberg field is \( T = \sinh(\tau) \), and the f-metric (3.14) expressed in the \( \tau, \rho \) coordinates is again non-diagonal and \( \rho \)-dependent, which suggests that there are no genuinely homogeneous and isotropic cosmologies with closed spatial sections either.

3. Open slicing

For the open slicing one has
\[ t = \sinh(\tau) \cosh(\rho), \quad r = \cosh(\tau), \quad R = \sinh(\tau) \sinh(\rho), \]
and the g-metric becomes
\[ ds_g^2 = \alpha^2 \{-d\tau^2 + a^2(\tau)(d\rho^2 + \sinh^2(\rho)d\Omega^2)\} \]
with \( a(\tau) = \sinh(\tau) \). The Stuckelberg field is \( T = \sinh(\tau) \cosh(\rho) \) and the specialty now is that the f-metric (3.14) becomes diagonal in the \( \tau, \rho \) coordinates,
\[ ds_f^2 = \alpha^2 u^2 \{-\cosh(\tau)^2 d\tau^2 + a^2(\tau)(d\rho^2 + \sinh^2(\rho)d\Omega^2)\}. \]
This solution, discovered in Ref.[7], is broadly viewed as the only homogeneous and isotropic dRGT cosmology, because both metrics are manifestly homogeneous and isotropic, so that they share the same rotational and translational Killing symmetries. However, this solution is completely equivalent to its flat and closed versions. Therefore, the latter also have the same common isometries, hence they are all homogeneous and isotropic, although their symmetries are not manifest. The conclusion is that sometimes solutions can be FLRW in a non-manifest way.

At the same time, although homogeneous and isotropic, the solution \( T = t \) is not static whereas the de Sitter space is. Specifically, let us consider the
4. Static slicing

Setting
\[ t = \sqrt{1 - \rho^2} \sinh(\tau), \quad r = \sqrt{1 - \rho^2} \cosh(\tau), \quad R = \rho \]  
(4.11)
solves the condition (3.4) and reduces the de Sitter metric (3.3) to the static form
\[ ds_g^2 = \alpha^2 \left\{ -\Sigma d\tau^2 + \frac{d\rho^2}{\Sigma} + \rho^2 d\Omega^2 \right\} \]  
(4.12)
with \( \Sigma = 1 - \rho^2 \). The \( T = t \) solution then becomes
\[ T(\tau, \rho) = \sqrt{1 - \rho^2} \sinh(\tau), \]  
(4.13)
and it is non-static even in static coordinates. Therefore, the g-metric is invariant under the action of the (locally) timelike Killing vector \( \partial/\partial \tau \), but the Stuckelberg field \( T \) and the f-metric are not invariant. As a result, the timelike isometry is not shared by both metrics.

As the solution \( T = t \) is not static, it is unlikely to describe the “ground state” of the theory. This is probably the reason why this solution was found to be unstable [10]. Therefore, we need to consider other solutions for \( T \).

V. OTHER SOLUTIONS

Solutions of the \( T \)-equation \((\partial T)^2 - (\partial_r T)^2 = 1\) can be constructed in different ways. A fairly general solution containing an arbitrary function \( W(\xi) \) is given by [16],
\[ T = \cosh(\xi) t + \sinh(\xi) r + W(\xi), \]  
\[ 0 = \sinh(\xi) t + \cosh(\xi) r + \frac{dW(\xi)}{d\xi}, \]  
(5.1)
where the second line implicitly determines the dependence of \( \xi \) on \( t, r \). Together with (4.2), this gives if not all but probably almost all solutions. However, this formula is difficult to use since one cannot explicitly determine \( \xi(t, r) \) for a generic \( W(\xi) \).

The \( T \)-equation can also be integrated by applying the method of characteristics [16], which has a simple geometric interpretation. Let us consider the 2D Minkowski space spanned by \( x^a \equiv \{ t, r \} \) with the metric \( g_{ab} = \text{diag}[1, -1] \). The \( T \)-equation reads \( g^{ab} \partial_a T \partial_b T \equiv \langle \nabla T, \nabla T \rangle = 1 \). Let \( \gamma = x^a(s) \) be a spacelike curve and \( T \) is constant along it. At every point of \( \gamma \) there is a unit timelike normal \( n \) such that \( \langle n, n \rangle = 1 \) and \( \langle n, \partial/\partial s \rangle = 0 \). The \( T \)-equation is equivalent to \( \partial T/\partial n = 1 \) [16].
This allows one to pass from $\gamma$ where $T = T(\gamma)$ to a neighboring curve $\tilde{\gamma}$ where $T = T(\tilde{\gamma})$ (see Fig. 1) and so on, thereby extending $T$ to the whole of the space. The solution is therefore defined, up to an additive constant, by the choice of the initial curve $\gamma$. For example, the solution $(4.2)$ can be obtained by choosing $\gamma$ to be a straight line.

In practice solutions of $(\partial_t T)^2 - (\partial_r T)^2 = 1$ can be obtained by changing the variables and then separating them [17]. Let us illustrate the method by passing to the static coordinates $(4.11)$, in which case the $T$-equation becomes

$$\frac{1}{\Xi} \left( \frac{\partial T}{\partial \tau} \right)^2 - \frac{\Xi}{1 - \Xi} \left( \frac{\partial T}{\partial \rho} \right)^2 = 1. \quad (5.2)$$

It is easy to see that $T(\tau, \rho)$ given by $(4.13)$ fulfills this equation, but now we can obtain also other solutions, in particular those for which $dT$ does not depend on time and the f-metric is static. The most general solution of $(5.2)$ of this type is obtained by separating the variables,

$$T = \sqrt{1 + q^2} \tau + \int \frac{\rho \, d\rho}{\Sigma} \sqrt{q^2 + \rho^2}, \quad (5.3)$$

where $q$ is an integration constant. If $q = 0$ then the solution becomes especially simple,

$$T = \tau + \int \frac{d\rho}{\Sigma} - \rho \equiv V - \rho, \quad (5.4)$$

and choosing $V$ and $\rho$ as coordinates, the two metrics become

$$ds_g^2 = \alpha^2 \{ -\Sigma \, dV^2 + 2dV \, d\rho + \rho^2 d\Omega^2 \},$$
$$ds_f^2 = u^2 \alpha^2 \{ -dV^2 + 2dV \, d\rho + \rho^2 d\Omega^2 \}. \quad (5.5)$$
VI. ENERGY

One can compute the canonical energy for systems with non-trivial Stuckelberg fields in the same way as this is done in the unitary gauge \[18\]. The computation will be presented separately \[19\] but its result is as follows. For a solution expressed in the static coordinates \(\tau, \rho\) the energy on a hypersurface of constant \(\tau\) is

\[ E = \int E \, d\rho \quad (6.6) \]

with the radial energy density

\[ \mathcal{E} = u^2 P_2(u) \rho^2 \partial_\tau T. \quad (6.7) \]

Applying this formula to the \(T = t\) solution \((4.13)\) gives the time-dependent value,

\[ \mathcal{E} = u^2 P_2(u) \rho^2 \sqrt{1 - \rho^2 \cosh(\tau)}, \quad (6.8) \]

which indicates once again that this solution cannot describe the ground state. On the other hand, the energy will be time independent if \(\partial_\tau T\) is time independent, but all such solutions are given by \((5.3)\), in which case

\[ \mathcal{E} = u^2 P_2(u) \sqrt{1 + q^2 \rho^2}. \quad (6.9) \]

This corresponds to the constant volume energy density

\[ \epsilon = u^2 P_2(u) \sqrt{1 + q^2}, \quad (6.10) \]

and the total energy is \(E = \epsilon V\) where \(V\) is the (infinite) volume of the 3-space. We remember that \(u\) is a solution of the algebraic equation \(P_1(u) = 0\), therefore, depending on choice of \(u\) and also on values of the parameters \(b_k\), the energy can be positive, negative, or zero.

The actual value of the background energy is probably not so important, but it is important to know if the energy is minimal or not. We conjecture that the static solutions \((5.3)\) correspond to the energy minima and are therefore stable. Therefore, they are candidates for describing the de Sitter ground state in the theory. To prove the conjecture will require to resolve the constraints and to compute the energy for deformations of the background \[18\]. We presently have partial results supporting our conjecture, but the detailed analysis will be presented elsewhere \[19\].
VII. CONCLUSIONS

We have shown that the de Sitter vacua in the dRGT theory are labeled by solutions of 
\[(\partial_t T)^2 - (\partial_r T)^2 = 1.\] 
The simplest solution \(T = t\) is manifestly homogeneous and isotropic when written in the open chart, but it is unstable. Therefore, one should study other solutions. One could worry that other solutions will not be FLRW because their reference metric is inhomogeneous. However, as we have seen, this is not necessarily the case, as the reference metric can look inhomogeneous in some coordinates while sharing common translational isometries with the physical metric.

The important issue is the number of common isometries of the two metrics. Since each of them describes a maximal symmetry space, each metric has ten isometries, some of which can be common, as for example the SO(3) rotational isometries. The number of common isometries depends on choice of \(T\), for example for \(T = t\) this number is six, but the same can be true for other choices of \(T\) as well.

Requiring the timelike isometry to be common for both metrics reduces the set of solutions to a one-parameter family (5.3). These solutions are physically distinguished since only for them the energy is time-independent. We conjecture that these solutions are stable and describe therefore the de Sitter ground state of the theory. The stability will follow if one shows that the energy increases for deformations of the de Sitter background, but such an analysis goes beyond the scope of the present paper and will be reported separately [19].

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Appendix A: \(T\)-equation in general coordinates

The most general spherically symmetric g-metric can be represented as

\[ds_g^2 = \alpha^2 \left\{-N^2 dt^2 + \frac{1}{\Delta^2} (dr + \beta dt)^2 + R^2 d\Omega^2\right\},\]  
(A.1)
where $N, \beta, \Delta, R$ are functions of two coordinates $t, r$. Choosing the f-metric to be

$$ds_f^2 = \alpha^2 u^2 \left\{ -d T^2(t,r) + dR^2 + R^2 d\Omega^2 \right\}$$  \hspace{1cm} (A.2)$$
and analyzing the field equation $G_{\mu\nu} = T_{\mu\nu}$ component by component, one finds [9] that they reduce to $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ with $\Lambda = P_0(u)$ provided that $P_1(u) = 0$ and $\alpha^2 = 3/\Lambda$, and if $(b_2 + b_3u) Y = 0$ hence either $b_2 + b_3u = 0$ or $Y = 0$. Here

$$Y \equiv \left( \dot{T} - \beta T' + N\Delta R' \right)^2 - \left( \dot{R} - \beta R' + N\Delta T' \right)^2 - \left( \Delta(\dot{R}R' - \dot{T}T') + N \right)^2,$$  \hspace{1cm} (A.3)$$where the dot and the prime denote the derivatives with respect to $t$ and $r$, respectively.

If the g-metric is de Sitter and $t, r$ coincide with the $t, r$ coordinates of the ambient 5D Minkowski space used in (3.3), then one has

$$N = \frac{1}{\sqrt{1 + t^2}}, \quad R = \sqrt{1 + t^2 - r^2}, \quad \Delta = NR, \quad \beta = -\frac{tr}{1 + t^2}.$$  \hspace{1cm} (A.4)$$
Inserting this to the condition $Y = 0$ gives equation (3.20) for $T(t,r)$ in the main text. The other possibility is to set $b_2 + b_3u = 0$ [5],[20], which restricts the values of the parameters $b_k$ but the function $T(t,r)$ then remains arbitrary, which presumably indicates some hidden gauge invariance.

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