A VARIATION EMBEDDING THEOREM AND APPLICATIONS

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Abstract. Fractional Sobolev spaces, also known as Besov or Slobodetzki spaces, arise in many areas of analysis, stochastic analysis in particular. We prove an embedding into certain $q$-variation spaces and discuss a few applications. First we show $q$-variation regularity of Cameron-Martin paths associated to fractional Brownian motion and other Volterra processes. This is useful, for instance, to establish large deviations for enhanced fractional Brownian motion. Second, the $q$-variation embedding, combined with results of rough path theory, provides a different route to a regularity result for stochastic differential equations by Kusuoka. Third, the embedding theorem works in a non-commutative setting and can be used to establish Hölder/variation regularity of rough paths.

1. Fractional Sobolev Spaces

For a real valued measurable path $h : [0, 1] \to \mathbb{R}$ and $\delta \in (0, 1)$ and $p \in (1, \infty)$ we define the fractional Sobolev (semi-)norm

$$|h|_{W^{\delta, p}} = \left( \int \int_{[0,1]^2} \frac{|h_t - h_s|^p}{|t - s|^{1+\delta p}} ds dt \right)^{1/p} \in [0, +\infty]$$

For $\delta = 1$ and $p \in (1, \infty)$, writing $\dot{h}$ for the weak derivative, we set

$$|h|_{W^{1, p}} = \left( \int_0^1 |\dot{h}_t|^p dt \right)^{1/p} \in [0, +\infty].$$

Define $W^{\delta, p}$ as the set of $h$ for which $|h|_{L^p} + |h|_{W^{\delta, p}} < \infty$. They are known to be Banach-spaces. For $1 \geq \delta > 1/p > 0$ one can assume that $h$ is continuous; compare with the embedding theorems below. It then makes sense to consider the closed subspace

$$W^{\delta, p}_0 = \{h \in W^{\delta, p} : h(0) = 0\}$$

which is Banach under $|h|_{W^{\delta, p}}$. We finally remark that the space $W^{1, p}$ is precisely the set of absolutely continuous paths on $[0, 1]$ with (a.e. defined) derivative in $L^p[0, 1]$. The space $W^{1, 2}_0$ is the usual Cameron-Martin space for Brownian motion. We recall some well-known continuous resp. compact embeddings\(^1\) [1],[3],[2],

\begin{align*}
(1.1) \quad & p \in (1, \infty), \ 1 \geq \delta > \delta \geq 0 \implies W^{\delta, p} \subset W^{\delta, p}, \\
(1.2) \quad & 1 < p \leq q < \infty, \ \delta \equiv 1 - 1/p + 1/q > 0 \implies W^{1, p} \subset W^{\delta, q}.
\end{align*}

\(^1\)The symbol $\subset\subset$ means compact embedding.
Theorem 1. Let $p \in (1, \infty)$ and $\alpha = 1 - 1/p > 0$. Then the variation of any $h \in W^{1,p}$ is controlled by the control function\(^2\)

$$\omega(s, t) = |h|_{W^{1,p};[s,t]}(t - s)^\alpha, \quad 0 \leq s \leq t \leq 1$$

and we have the continuous embeddings

$$W^{1,p} \subset C^{\alpha-Hölder} \quad \text{and} \quad W^{1,p} \subset C^{1-var}.$$  

Proof. By absolute continuity and Hölder’s inequality with conjugate exponents $p$ and $1/\alpha$

$$|h_{s,t}| = \int_s^t |h_r| dr \leq (t - s)^\alpha \left( \int_s^t |h_r|^p dr \right)^{1/p}$$

$$= |h|_{W^{1,p};[s,t]}(t - s)^\alpha.$$

We now show that the variation of $h$ is controlled by the control function

$$\omega(s, t) = |h|_{W^{1,p};[s,t]}(t - s)^\alpha, \quad t \geq s.$$  

Only super-additivity, $\omega(s, t) + \omega(t, u) \leq \omega(s, u)$ with $s \leq t \leq u$, is non-trivial. Note $p \in (1, \infty)$. From Hölder’s inequality with conjugate exponents $p$ and $p/(p-1) = 1/\alpha$ we obtain

$$|h|_{W^{1,p};[s,t]}(t - s)^\alpha + |h|_{W^{1,p};[t,u]}(u - t)^\alpha$$

$$\leq \left( |h|_{W^{1,p};[s,t]}^p + |h|_{W^{1,p};[t,u]}^p \right)^{1/p} \left[ (t - s)^{\frac{p}{p-1}} + (u - t)^{\frac{p}{p-1}} \right]^{(p-1)/p}$$

$$= |h|_{W^{1,p};[s,u]}(u - t)^\alpha.$$  

This shows that $\omega$ is super-additive and we conclude that for any $0 < a < b \leq 1$,

$$|h|_{1-var;[a,b]} \leq \omega(a, b) = (b - a)^\alpha |h|_{W^{1,p};[a,b]}.$$  

In particular, we established $W^{1,p} \subset C^{\alpha-Hölder} \quad \text{and} \quad W^{1,p} \subset C^{1-var}. \quad \Box$

Theorem 2. Let $0 < \delta < 1$ and $p \geq 1$ such that

$$\alpha = \delta - 1/p > 0.$$  

Set $q = 1/\delta$. Then the $q$-variation of any $h \in W^{\delta,p}$ is controlled by a constant multiple of the control function

$$\omega(s, t) = |h|_{W^{\delta,p};[s,t]}^q(t - s)^\alpha, \quad 0 \leq s \leq t \leq 1.$$  

and we have the continuous embeddings

$$W^{\delta,p} \subset C^{\alpha-Hölder} \quad \text{and} \quad W^{\delta,p} \subset C^{q-var}.$$  

Proof. We have

$$|h|_{W^{\delta,p};[s,t]}^p \equiv F_{s,t} = \int_{[s,t]^2} \frac{|h_{u,v}|^p}{(u-v)^{1+\delta p}} dvdu = \int \int_{[s,t]^2} \left( \frac{|h_{u,v}|}{(u-v)^{1/\delta+\delta}} \right)^p dvdu.$$  

\(^2\)A continuous, super-additive map $(s, t) \mapsto \omega(s, t) \in [0, \infty)$, defined for $0 \leq s \leq t \leq 1$.  


The Garsia-Rodemich-Rumsey lemma with \( \Psi(\cdot) = (\cdot)^p \) and \( p(\cdot) = (\cdot)^{1/p+\delta} \) yields

\[
|h_{s,t}| \leq C \int_0^{t-s} \left( \frac{F_{s,t}}{u^2} \right)^{1/p} dp(u) = C |h|_{W^{s,p};[s,t]} \int_0^{t-s} u^{-2/p} dp(u) \\
= C |h|_{W^{s,p};[s,t]} \int_0^{t-s} u^{-1/p+\delta-1} du = C |h|_{W^{s,p};[s,t]} (t-s)^{\delta-1/p},
\]

using \( \alpha = \delta - 1/p > 0 \). We now show that the \( q \)-variation of \( h \) is controlled by the control function

\[
\omega(s,t) := |h|_{W^{s,p};[s,t]}^q (t-s)^{\alpha q}, \quad t \geq s.
\]

Only super-additivity, \( \omega(s,t) + \omega(t,u) \leq \omega(s,u) \) with \( s \leq t \leq u \), is non-trivial. Note that \( p/q = 1/(p\alpha + 1) \in (1,\infty) \). From Hölder’s inequality with conjugate exponents \( p/q \) and \( p/(p-q) \) we obtain

\[
|h|_{W^{s,p};[s,t]}^q (t-s)^{\alpha q} + |h|_{W^{s,p};[t,u]}^q (u-t)^{\alpha q} \\
\leq \left( |h|_{W^{s,p};[s,t]}^p + |h|_{W^{s,p};[t,u]}^p \right)^{q/p} \left( (t-s)^{\alpha q - q\alpha/p - \alpha q/p} + (u-t)^{\alpha q - q\alpha/p - \alpha q/p} \right)^{(p-q)/p}
\]

The first factor is easily estimated

\[
\left( |h|_{W^{s,p};[s,t]}^p + |h|_{W^{s,p};[t,u]}^p \right)^{q/p} \leq |h|_{W^{s,p};[s,u]}^q.
\]

To estimate the second factor note that the exponent of \( (t-s) \) resp. \( (u-t) \) equals one, indeed

\[
\frac{q\alpha p}{p-q} = 1 \iff q = \frac{p}{p\alpha + 1}
\]

and the second factor equals

\[
(u-s)^{(p-q)/p} = (u-t)^{\alpha q}.
\]

This shows that \( \omega \) is super-additive and we conclude that for any \( 0 \leq a < b \leq 1 \),

\[
|h|_{q\text{-var};[a,b]} \leq C \omega(a,b)^{1/q} = C |h-a|^\alpha |h|_{W^{s,p};[a,b]}.
\]

In particular, we have established continuity of the embeddings

\[
W^{\delta,p} \subset C^{\alpha\text{-Hölder}} \quad \text{and} \quad W^{\delta,p} \subset C^{q\text{-var}}.
\]

\[ \square \]

The case \( p = 2 \) deserves special attention. The assumptions of Theorem 2 are then satisfied for any \( \delta \in (1/2,1) \).

**Remark 1.** In [6], Kusuoka discusses differentiability of SDE solution beyond the usual Malliavin sense. In particular, he shows the existence of a nice version of the Itô-map which has derivatives in directions \( W^{\delta,2}_0 \subset W^{1,2}_0 \) for \( \delta \in (1/2,1) \). Since \( W^{\delta,2}_0 \subset C^{q\text{-var}} \) with \( q = 1/\delta < 2 \) this result is now explained by Lyons’ theory of rough paths [8, 7]. Note that in Lyons’ continuity statements the modulus \( \omega \) is preserved. This implies that after perturbing a Brownian path in a \( W^{\delta,2}_0 \)-direction the solution maintains \( \alpha \)-Hölder regularity with \( \alpha = \delta - 1/2 \). (Clearly, this is not true for an arbitrary perturbation in \( C^{q\text{-var}} \).) For what it’s worth, we can extend Gateaux-differentiability to suited \( W^{\delta,p} \)-spaces as long as \( \delta - 1/p > 0 \) and even apply this to rough path differential equations driven by enhanced fBM. On the other hand, we do not attempt to recover Kusuoka’s full statement (Fréchet in both
starting point and $W^{5,2}_0$). This requires a careful formulation of Lyons’ universal limit theorem and will be addressed in a forthcoming monograph.

**Remark 2.** Integrals of form $\int f dg$ for $f, g \in W^{5,2}$ are discussed in [11]. Theorem 2 reveals them as normal Young-interval. Following [9] its continuity properties are conveniently expressed in terms of the modulus $\omega$. In particular, the modulus of continuity of $\int f dg$ is immediately controlled by the $W^{5,2}$-Sobolev-norms of $f$ and $g$ and we can easily extend this to $W^{5,p}$ provided $\delta - 1/p > 0$. On the other hand, our approach does not allow us to control the $W^{5,2}$-norm of the indefinite integral $\int f dg$.

**Remark 3.** When $\delta < 1$ the notion of $W^{5,p}$ makes perfect sense for paths with values in a metric space $(E, d)$. Theorem 2 still holds with the same proof. The case of the free step-N nilpotent group $(G^N(\mathbb{R}^d), \otimes)$ with Carnot-Caratheodory norm $\|\cdot\|$ and distance $d(x, y) = \|x^{-1} \otimes y\|$ is of particular importance: Theorem 2 is a criterion for variation and Hölder regularity of a $G^N(\mathbb{R}^d)$-valued path, a fundamental aspect in Lyons’ theory of rough paths, [7]. To illustrate the idea we give a simple application to enhanced Brownian motion $B$, see [5, 4]. Then\(^3\)

$$E \|B\|_{W^{5,p};[0,1]}^p = \int \int_{[0,1]^2} \frac{E \|B_{s,t}\|^p}{|t-s|^{1+\delta} p} dsdt = E \|B_{0,1}\|^p \int \int_{[0,1]^2} |t-s|^{p/2-1-\delta} dsdt.$$  

For every $\alpha < 1/2$ and $\delta \in (\alpha, 1/2)$ there exists $p_0(\delta)$ such that for all $p \geq p_0$ the double integral is bounded by 1. Thus for all $p$ large enough,

$$E \|B\|_{W^{5,p};[0,1]}^p \leq E \|B_{0,1}\|^p.$$  

It is well-known, [5], that $\|B_{0,1}\|$ has a Gaussian tail and it follows that $\|B\|_{W^{5,p}}$ has a Gaussian tail, provided $p \geq p_0(\delta)$. For $p$ large enough we have $\alpha \leq \delta - 1/p$ and we conclude that $\|B\|_{\alpha}$-Hölder has a Gaussian tail, too. For a direct proof see [4]. Note that the law of $B$ is not Gaussian and there are no Fernique-type results.

**Remark 4.** Potential spaces, see [2] and the references therein, are a popular alternative to fractional Sobolev spaces. But only the latter adapt easily to $(E,d)$-valued paths as required in rough path analysis.

**Remark 5.** The $W^{5,p}$-embedding of Theorem 2 has two different regimes:

1. For $p$ large one has $q = 1/\delta \sim 1/\alpha$. Since every $\alpha$-Hölder path has finite $1/\alpha$-variation (the converse not being true) one can forget about $q$-variation.

2. When $p$ is small, the variation parameter $q = 1/\delta$ can be considerably smaller than $1/\alpha$ and $q$-variation is an essential part of the regularity. Elementary examples show that $q$-variation does not imply any Hölder regularity and therefore one should not forget about $\alpha$-Hölder regularity. The fractional Sobolev space $W^{5,p}$ resp. the modulus $\omega$ are tailor-made to keep track of both regularity aspects.

3. **Cameron Martin space of fBM**

We consider fractional Brownian motion with $H \in (0, 1/2)$. Call $\mathcal{H}^H$ the associated Cameron-Martin space.

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\(^3\)Simply write $h_{s,t} \equiv d(h_s, h_t)$ and note that the Garsia-Rodemich-Rumsey lemma works for $(E, d)$-valued continuous functions.

\(^4\)Note $\|B_{s,t}\| \overset{D}{=} |t-s|^{1/2} \|B_{0,1}\|$.
Theorem 3. Let $1/2 < \delta < H + 1/2$. Then $\mathcal{H}^H \subset W^\delta$.  

Proof. From [2] and the references therein we know that $\mathcal{H}^H$ is continuously embedded in the potential space $I_{H+1/2,2}^+$ which we need not define here. Then, [3, 2], $I_{H+1/2,2}^+ \subset W^\delta$, so that  

$$ (3.1) \quad \mathcal{H}^H \subset W^\delta. $$  

The compact embeddings is obtained by a standard squeezing argument: Replace $\delta$ by $\delta \in (\delta, H + 1/2)$, repeat the argument for $\delta$ and then use (1.1). \hfill $\square$

Corollary 1. For $\alpha \in (0, H)$ and $1/(H + 1/2) < q < \infty$ we have 

$$ \mathcal{H}^H \subset C^\alpha, \quad \mathcal{H}^H \subset C^\alpha_{\text{var}}. $$

Remark 6. From $\mathcal{H}^H \subset I_{H+1/2,2}^+$ it follows that $\mathcal{H}^H \subset C^{H, \text{Hölder}}$, this is well-known, [2].

Remark 7. For any $H \in (0, 1/2)$ we can find $1/(H + 1/2) < q < 2$. This has useful consequences. For instance, for $h, g \in \mathcal{H}^H$ that integral $\int h dg$ makes sense as classical Young integral with all its continuity properties. In particular, the lift of $h \in \mathcal{H}^H$ to a geometric $p$-rough paths $p > 1/H$, see [10], is well-defined and convergence of piecewise linear approximations, uniformly over bounded sets in $\mathcal{H}^H$, is an easy consequence. Such results are useful to establish large deviations principles for enhanced Gaussian processes, enhanced fBM being a particular example. We will discuss this in forthcoming work.

4. APPENDIX

The proof of (3.1) appears somewhat spread out in the references. We present a direct argument which avoids potential spaces and fractional calculus and extends to other Volterra kernels.  

Step 1: $\mathcal{H}^H$ is the image of $L^2[0,1]$ under the integral operator $K = K_1 + K_2$ where 

$$ K_1(t,s) = (t-s)^{H-1/2}, \quad K_2(t,s) = s^{H-1/2}F_1(t/s); \quad F_1 = \int_0^{t-1} u^{H-3/2} \left( 1 - (u+1)^{H-1/2} \right). $$  

for $s < t$. Set $h_i = K_i g \equiv \int_0^s K_i (\cdot, s) g(s) \, ds$ with $g \in L^2[0,1]$, $i = 1, 2$.

Step 2: An elementary computation shows 

$$ \sup_{u \in [0,1]} \int_0^{1-t} |K_1(s+t, u) - K_1(s,u)| \, ds = O \left( t^{H+1/2} \right), $$

$$ \sup_{s \in [0,1-t]} \int_0^1 |K_1(s+t, u) - K_1(s,u)| \, du = O \left( t^{H+1/2} \right). $$

From Cauchy-Schwartz and trivial sup-estimates, 

$$ (\ast) = \int_{s=0}^{1-t} |h_1(s+t) - h(s)|^2 \, ds = |g|_{L^2}^2 \times O \left( t^{1+2H} \right). $$

5For instance, every kernel for which one can get estimates as those in Step 2 will lead to a fractional Sobolev embedding.
The $W^{\delta,2}$-norm of $h_1$ is equivalent to $\int dt \left( t^{1+2\delta} \right) / t^{1+2\delta}$ which is less than $C|g|_{L^2}^2$ provided $1 + 2H - (1 + 2\delta) > -1$ and this happens precisely for $\delta < H + 1/2$.

Step 3: A straight-forward computation shows (one can assume $g \in C^1 \cap L^2$ for the computation) that $|\hat{h}_2| < C|g|_{L^2}^2$ provided $p < 1/(1-H)$ and hence $h_2 \in W^{1,p}$.

From (1.2), $W^{1,1/(1-H)} \subset W^{H+1/2,2}$. Similarly, given $\delta < H + 1/2$ we can find $p < 1/(1-H)$, close enough to $1/(1-H)$ so that $W^{1,p} \subset W^{\delta,2}$.

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