A Group-Theoretic Approach to Abstraction: Hierarchical, Interpretable, and Task-Free Clustering

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Abstract
Abstraction plays a key role in concept learning and knowledge discovery. While pervasive in both human and artificial intelligence, it remains mysterious how concepts are abstracted in the first place. We study the nature of abstraction through a group-theoretic approach, formalizing it as a hierarchical, interpretable, and task-free clustering problem. This clustering framework is data-free, feature-free, similarity-free, and globally hierarchical—the four key features that distinguish it from common clustering models. Beyond a theoretical foundation for abstraction, we also present a top-down and a bottom-up approach to establish an algorithmic foundation for practical abstraction-generating methods. Lastly, using both a theoretical explanation and a real-world application, we show that the coupling of our abstraction framework with statistics realizes Shannon’s information lattice and even further, brings learning into the picture. This gives a first step towards a principled and cognitive way of automatic concept learning and knowledge discovery.

Keywords: Abstraction, Partition (Clustering), Group, Symmetry, Lattice

1. Introduction
Abstraction describes the process of generalizing high-level concepts from specific data samples by “forgetting the details” (Weinberg, 1968; Giunchiglia and Walsh, 1992; Saitta and Zucker, 1998). This conceptual process is pervasive in human reasoning, and it is evident that more advanced concepts can be abstracted once a “conceptual base” is established (Mandler, 2000). However, it remains mysterious how concepts are abstracted in the first place, which is generally attributed to innate biology (Mandler, 2000; Gómez and Lakusta, 2004; Biederman, 1987).
Considering artificial intelligence rather than biological minds, there are now algorithms to automate abstraction in various concept learning tasks (Saitta and Zucker, 2013; LeCun et al., 2015; Bredeche et al., 2006; Yu et al., 2016). However, almost all require handcrafted priors—the counterpart to innate biology (Marcus, 2018; Dietterich, 2018). While a prior can take many forms such as rules in automatic reasoning, distributions in Bayesian inference, features in classifiers, or architectures in neural networks, it is typically task-specific and/or domain-specific (Raina et al., 2006; Yu et al., 2007; Krupka and Tishby, 2007). Therefore, extensive hand-design from domain knowledge is sometimes considered “cheating” if one hard codes all known high-level abstractions as priors (Ram and Jones, 1994). This motivates us to consider only universal priors as innate knowledge for abstraction.

This paper establishes both a theoretical and an algorithmic foundation for abstraction. It is worth noting that our abstraction framework is universal in the following two senses. First, we consider the general question of conceptualizing a domain—a task-free preparation phase before specific problem solving (Zucker, 2003). Second, we consider symmetries in nature (or groups in mathematics)—a universal prior that encodes no domain knowledge. The ultimate goal is to learn domain concepts/knowledge when our group-theoretic abstraction framework is connected to statistical learning. This is contrary to much prior work at the intersection of group theory and learning (Kondor, 2008) that often encodes domain-relevant symmetries in features or kernels rather than learns them as new findings.

### 1.1 Theoretical Foundation for Abstraction

Existing formalizations of abstraction form at least two camps. One uses mathematical logic where abstraction is explicitly constructed from abstraction operators and formal languages (Saitta and Zucker, 2013; Zucker, 2003; Bundy et al., 1990); another uses deep learning where abstraction is hinted at by the layered architectures of neural networks (LeCun et al., 2015; Bengio, 2009). Their key characteristics—commonly known as rule-based (deductive) and data-driven (inductive)—are quite complimentary. The former enjoys model interpretability, but requires explicit handcrafting of complicated logic with massive domain expertise; the latter shifts the burden of model crafting to data, but makes the model less transparent.

This paper takes a new viewpoint, aiming for a middle ground between the two camps. We formalize abstraction as a symmetry-generated clustering, or more precisely a group-generated partition, which admits statistical learning. By clustering, we forget within-cluster variations and discern only between-cluster distinctions (Bélaï and Jaoua, 1998; Sheikhhalishahi et al., 2016), revealing the nature of abstraction (Livingston, 1998). While clustering is common in machine learning (Duda et al., 2012, chap. 10), our clustering model is in stark contrast with the common settings, as follows.

1. **Data-free.** Our clustering model considers partitioning an input space rather than data samples. It is treated more as conceptual clustering than data clustering like k-means (Michalski and Stepp, 1983; Fisher, 1987): clusters are formed in a mechanism-driven, not data-driven, fashion; and the mechanisms considered here are symmetries. The process is causal, and the results are interpretable. More importantly, a single clustering mechanism transfers to multiple domains, and a single clustering result transfers to various datasets.
2. Feature-free. Our clustering model involves no feature engineering, so no domain expertise. This particularly means three things. First, no feature design for inputs: we directly deal with mathematical spaces, e.g. vector spaces or manifolds. Second, no feature/assignment function for cluster designation: this differs from algorithms that hand-design abstraction operators (Zucker, 2003), arithmetic descriptors (Yu et al., 2016), or decision-tree-like feature thresholding (Sheikhalishahi et al., 2016). Third, no meta-feature tuning such as pre-specifying the number of clusters.

3. Similarity-free. Our clustering model does not depend on a predefined notion of similarity. This differs from most clustering algorithms where much effort has been expended in defining “closeness” (Raman and Varshney, 2018; Rand, 1971). Instead, pairwise similarity is replaced by an equivalence relation induced from symmetry. Note that the definitions of certain symmetries may require additional structure of the input space, e.g. topology or metric, but this is not used as a direct measurement for inverse similarity. Therefore, points that are far apart (in terms of metric distance) in a metric space can be grouped together (in terms of equivalence) under certain symmetries, resulting in a “discontinuous” cluster comprising disconnected regions in the input space. This is not likely to happen for algorithms such as $k$-means.

It is noteworthy that being feature-free and similarity-free makes a clustering model universal (Raman and Varshney, 2018), becoming more of a science than an art (Von Luxburg et al., 2012). Besides the above three distinguishing features, our clustering model exhibits one more distinction regarding hierarchical clustering for multi-level abstractions:

4. Global hierarchy. Like many hierarchical clusterings (Jain and Dubes, 1988; Rokach and Maimon, 2005), our clustering model outputs a family of multi-level partitions and a hierarchy showing their interrelations. However, here we have a global hierarchy formalized as a partition (semi)lattice, which is generated from another hierarchy of symmetries represented by a subgroup lattice. This is in contrast with greedy hierarchical clusterings such as agglomerative/divisive clustering (Cormack, 1971; Kaufman and Rousseeuw, 2009) or topological clustering via persistent homology (Oudot, 2015). These greedy algorithms lose many possibilities for clusterings since the hierarchy is constructed by local merges/splits made in a one-directional procedure, e.g. growing a dendrogram or a filtration. In particular, greedy hierarchical clustering is oft-criticized since it is hard to recover from bad clusterings in early stages of construction (Oudot, 2015). Lastly, our global hierarchy is represented by a directed acyclic graph rather than tree-like charts (essentially a linear structure) such as dendrograms or barcodes.

1.2 Algorithmic Foundation for Abstraction

Besides a group-theoretic formalism of hierarchical abstractions as clusterings induced from hierarchical symmetries, we introduce two general principles, a top-down approach and a bottom-up approach, which systematically enumerate symmetries to construct hierarchical abstraction families. Each principle leverages a different duality developed in the formalism, and leads to practical algorithms that realize the abstraction generating process.

1. A top-down approach. Starting from all possible symmetries, we gradually restrict our attention to certain types of symmetries which can lead to practical abstraction-
construction algorithms. In general, the choices for restrictions can be made arbitrarily. However, it turns out that we can find a complete identification of all symmetries induced from affine transformations, where we explicitly give a full parametrization of affine symmetries. This complete identification not only decomposes a large symmetry-enumeration problem into smaller enumeration subproblems, but also suggests ways of adding restrictions to obtain desired symmetries. This approach from general symmetries to more restrictive ones corresponds to top-down paths in the symmetry hierarchy, which explains where the name comes from.

2. A bottom-up approach. Starting from a set of atomic symmetries, we generate all symmetries that are seeded from the given set. Based on a strong duality result developed in the formalism, we introduce an induction algorithm which directly computes a hierarchical family of abstractions without explicitly enumerating the corresponding symmetries. This induction algorithm is much more efficient than generating all abstractions from scratch, i.e. from symmetries. So, it is a good choice to quickly build an abstraction family in the first place, after which one can fine tune the generating set to balance the trade-off between efficiency and expressiveness. This approach from atomic symmetries to more complicated ones corresponds to bottom-up paths in the symmetry hierarchy, which explains where the name comes from.

The rest of the paper is organized as follows. In Section 2, we describe abstraction informally to give intuition on its nature and key properties via everyday examples; all points made in this section are formalized and algorithmically realized in later sections. Section 3 sets up the theoretical foundation for abstraction, where we formalize abstraction, symmetry, and their hierarchies; and cast symmetry-generated abstractions in a primal-dual viewpoint. Sections 4 and 5 set up the algorithmic foundation for abstraction, where we introduce the top-down approach and the bottom-up approach, respectively. In Section 6, we describe tricks and cautions in real implementations where abstractions have to be restricted to finite subspaces of an input space. In Section 7, we discuss connections to Shannon’s information lattice—a special case under our abstraction formalism—and a real application that realizes learning in an information lattice for automatic concept learning.

2. Abstraction: Informal Description

We informally discuss abstraction by drawing examples from different domains and summarizing their commonalities. Although expressed in everyday terms from specific domains, the conclusions from this section cover all key properties of abstraction that the remainder of the paper aims to capture formally. In particular, the rest of the paper formalizes the ideas from this section in a precise and general manner that, importantly, leads to principled algorithmic approaches for automatic concept learning and knowledge discovery.

2.1 Everyday Abstraction

Whether aware or not, abstraction is everywhere in our daily behaviors. It is in the nature of abstraction that it treats the set of instances that it subsumes as if they were qualitatively identical, although in fact they are not (Livingston, 1998). Examples of people making abstractions can be as simple as observing ourselves through social categories such as race
or gender (Macrae and Bodenhausen, 2000); or as complicated as a systematic taxonomy of a subject domain. Here, we present examples of two systematic abstractions (Figure 1): one is from a taxonomy of animals; the other is from a classification of music chords.

### 2.2 Abstraction in Common

There are many commonalities in examples from Section 2.1 as well as in many other real-life examples of abstraction. We summarize the key properties shared in these abstraction examples, which will be formalized in the following sections.

**Nature of abstraction: clustering or classification?** One shared property among many examples of abstraction is the idea of *clustering* and then forgetting within-cluster variations. For instance, we cluster people into \{men, women\}, forgetting the difference between John and David, Mary and Rachel; we cluster animals with a backbone into \{fish, amphibians, reptiles, birds, mammals\}, forgetting the difference between penguins and eagles, bats and dogs; we cluster music triads into \{major, minor, augmented, diminished, …\}, forgetting the difference between C-E-G and F-A-C, C-E♭-G and A-C-E. This idea of clustering is pervasive in various definitions of abstraction, but more often termed as classification (or categorization, taxonomy). Although clustering and classification (likewise clusters and classes) are more or less synonyms in everyday life, there is a clear difference between the two in machine learning. The former generally falls under the realm of unsupervised learning, whereas the latter falls under supervised learning. The difference is merely whether or not there is a label for each cluster. Note that labels are important in supervised learning, since a perfect binary classifier with a 100% accuracy is clearly different from a bad one with a 0% accuracy. However, in light of clustering, the two classifiers are identical: the “bad” one, for instance, simply calls all men as women and all women as men, but still accurately captures the concept of gender. Consequently in this paper, we treat the nature
of abstraction as clustering rather than classification, further formalized as a partition or equivalence relation. So, men and women are two equivalence classes of people: all men are equivalent, so are all women. An extended discussion on clustering and classification can be found in Section 7.1, relating to information elements and random variables.

**Hierarchy.** Another shared property among many examples of abstraction is the presence of a hierarchy, where “later” abstractions can be made recursively from “earlier” ones. For instance, we cluster animals into \{fish, birds, mammals, annelids, mollusks, \ldots\}, and further cluster these abstracted terms into \{vertebrates, invertebrates\}; we cluster music chords into \{major, minor, dominant, German, \ldots\}, and further cluster these abstracted terms into \{triads, seventh chords, sixth chords, \ldots\}, and even further into \{trichords, tetrachords, \ldots\}. Hierarchy, being either explicit or implicit, brings the notion of level of an abstraction. For instance, biological taxonomy gives an explicit description of abstraction levels: kindom \rightarrow phylum \rightarrow class \rightarrow order \rightarrow family \rightarrow genus \rightarrow species; whereas the abstraction levels of music chords are relatively implicit but still present. In general, an abstraction hierarchy can be more complicated than simply linear due to various clustering possibilities. In this paper, a general hierarchy is formalized by a mathematical lattice.

**Mechanism.** A third shared property among many examples of abstraction is the existence of a mechanism—a driving force that causes the resulting abstraction. For instance, the presence or absence of a backbone is the underlying mechanism that results in the abstraction of animals into vertebrates and invertebrates; the intervalic quality is the underlying mechanism that results in the abstraction of music chords. Having a mechanism is important for at least three reasons. First, it makes the abstraction process logical, so that every abstraction is made for a reason. This is a distinguishing feature in human intelligence, which is further key to the development of concepts and knowledge. Second, different mechanisms yield different abstractions, which further yield different attributes of an object. For instance, a bat can be abstracted as a mammal since, among many other reasons, it nurses its pups with milk; a bat can also be abstracted as a flying animal based on its capability of flying. In comparison, under the same two mechanisms, a penguin is abstracted as a bird but flightless. Third, perhaps most importantly, having a mechanism allows generalization, i.e. we can transfer a mechanism from one domain to another. For instance, generalizing the same mechanism under which we abstract people into men and women to other species, we get roosters and hens, bulls and cows, etc. As a result, we emphasize the generating mechanisms for abstractions. In this paper, we focus on symmetries—a type of domain-independent mechanism—and symmetry-generated abstractions.

**Towards laws of nature.** Lastly, abstraction is a very important stage towards laws—or less seriously, rules or patterns—of nature (Schmidt and Lipson, 2009). An abstraction itself is not a rule, but an abstraction paired with a property describing that abstraction can be treated as a rule. For instance, the abstraction \{fish, amphibians, reptiles, birds, mammals\} of animals is not a rule, but a statement like “Most of the birds fly, whereas only a few fish, amphibians, reptiles, or mammals fly” is a rule which indicates what is special about this abstraction. While this paper focuses on abstractions only rather than rules, we discuss probabilistic rules made out of abstractions and their statistical properties in Section 7. There, we introduce the information lattice and a real implementation of probabilistic rule learning from our earlier work (Yu et al., 2016; Yu and Varshney, 2017; Yu et al., 2017).
3. Abstraction: Mathematical Formalism

We formalize an abstraction process on an underlying space as a clustering problem. In this process, elements of the space are grouped into clusters, abstracting away within-cluster variations. The outcome is a coarse-grained abstraction space whose elements are the clusters. Clustering is performed based on certain symmetries such that the resulting clusters are invariant with respect to the symmetries.

3.1 Abstraction as Partition (Clustering)

We formalize an abstraction of a set as a partition of the set, which is a mathematical representation of the outcome of a clustering process. Throughout this paper, we reserve $X$ to exclusively denote a set which we make abstractions of. The set $X$ can be as intangible as a mathematical space, e.g. $\mathbb{R}^n$, $\mathbb{Z}^n$, a general manifold; or as concrete as a collection of items, e.g. \{rat, ox, tiger, rabbit, dragon, snake, horse, sheep, monkey, rooster, dog, pig\}.

Preliminaries (Appendix A.1): partition of a set ($\mathcal{P}$), partition cell ($P \in \mathcal{P}$); equivalence relation on a set ($\sim$), quotient ($X/\sim$).

Remark 1 An abstraction is a partition, and vice versa. The two terms refer to the same thing, with the only nuance being that one is used less formally, whereas the other is used in the mathematical language. When used as a single noun, these two terms are interchangeable in this paper.

Remark 2 A partition is not an equivalence relation. The two terms do not refer to the same thing (one is a set, the other is a binary relation), but convey equivalent ideas since they induce each other bijectively (Appendix A.1). In this paper, we use an equivalence relation to explain a partition: elements of a set $X$ are put in the same cell because they are equivalent. Based on this reason, abstracting the set $X$ is about treating equivalent elements as the same, i.e. collapsing equivalent elements in $X$ into a single entity (namely, an equivalence class or a cell) where collapsing is formalized by taking the quotient.

3.2 Abstraction Universe as Partition Lattice (Hierarchical Clustering)

A set $X$ can have multiple partitions, provided that $|X| > 1$. The number of all possible partitions of a set $X$ is called the Bell number $B_{|X|}$. Bell numbers grow extremely fast with the size of the set: starting from $B_0 = B_1 = 1$, the first few Bell numbers are:

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, \ldots$$

We use $\mathcal{P}^*_X$ to denote the family of all partitions of a set $X$, so $|\mathcal{P}^*_X| = B_{|X|}$. We can compare partitions of a set in two ways. One simple way is to compare by size: given two partitions $\mathcal{P}, \mathcal{Q}$ of a set, we say that $\mathcal{P}$ is no larger than (resp. no smaller than) $\mathcal{Q}$ if $|\mathcal{P}| \leq |\mathcal{Q}|$ (resp. $|\mathcal{P}| \geq |\mathcal{Q}|$). Another way of comparison considers the structure of partitions via a partial order on $\mathcal{P}^*_X$. The partial order further yields a partition lattice, a hierarchical representation of a family of partitions.
Preliminaries (Appendix A.2): partial order, poset; lattice, join (\(\lor\)), meet (\(\land\)), sub-lattice, join-semilattice, meet-semilattice, bounded lattice.

**Definition 3** Let \(P\) and \(Q\) be two abstractions of a set \(X\). We say that \(P\) is at a higher level than \(Q\), denoted \(P \preceq Q\), if as partitions, \(P\) is coarser than \(Q\). For ease of description, we expand the vocabulary for this definition, so the following are all equivalent:

1. \(P \preceq Q\), or equivalently \(Q \succeq P\) (Figure 2).
2. As abstractions, \(P\) is at a higher level than \(Q\) (or \(P\) is an abstraction of \(Q\)).
3. As partitions, \(P\) is coarser than \(Q\) (or \(P\) is a coarsening of \(Q\)).
4. As abstractions, \(Q\) is at a lower level than \(P\) (or \(Q\) is a realization of \(P\)).
5. As partitions, \(Q\) is finer than \(P\) (or \(Q\) is a refinement of \(P\)).
6. Any \(x, x' \in X\) in the same cell in \(Q\) are also in the same cell in \(P\).
7. Any \(x, x' \in X\) in different cells in \(P\) are also in different cells in \(Q\).

\[\begin{array}{ccc}
P & \overset{\text{higher-level abstraction}}{\leftrightarrow} & Q \\
\end{array}\]

Figure 2: The partial order \(\preceq\) compares the levels of abstractions.

It is known that the binary relation “coarser than” on the family \(\mathcal{P}_X^*\) of all partitions of a set \(X\) is a partial order, so is the binary relation “at a higher level than” on abstractions. Given two partitions \(P, Q\) of a set, we can have \(P \preceq Q\), \(Q \preceq P\), or they are incomparable. Further, \((\mathcal{P}_X^*, \preceq)\) is a bounded lattice, in which the greatest element is the finest partition \(\{\{x\} \mid x \in X\}\) and the least element is the coarsest partition \(\{X\}\). For any pair of partitions \(P, Q \in \mathcal{P}_X^*\), their join \(P \lor Q\) is the coarsest common refinement of \(P\) and \(Q\); their meet \(P \land Q\) is the finest common coarsening of \(P\) and \(Q\) (Figure 3).

\[\begin{array}{ccc}
P & \lor & Q \\
\end{array}\]

Figure 3: Two abstractions \(P, Q\) and their join \(P \lor Q\) and meet \(P \land Q\).

**Definition 4** An abstraction universe for a set \(X\) is a sublattice of \(\mathcal{P}_X^*\), or a partition (sub)lattice in short. In particular, we call the partition lattice \(\mathcal{P}_X^*\) itself the complete abstraction universe for \(X\). An abstraction join-semiuniverse (resp. meet-semiuniverse) for a set \(X\) is a join-semilattice (resp. meet-semilattice) of \(\mathcal{P}_X^*\). An abstraction family for a set \(X\), an even weaker notion, is simply a subset of \(\mathcal{P}_X^*\).
If the complete abstraction universe \( (\mathcal{P}_X^*, \preceq) \) is finite, we can visualize its hierarchy as a directed acyclic graph where vertices denote partitions and edges denote the partial order. The graph is constructed as follows: plot all distinct partitions of \( X \) starting at the bottom with the finest partition \( \{x\} | x \in X \), ending at the top with the coarsest partition \( \{X\} \) and, roughly speaking, with coarser partitions positioned higher than finer ones. Draw edges downwards between partitions using the rule that there will be an edge downwards from \( P \) to \( Q \) passing through a chain of intermediate partitions (and a path upward from \( Q \) to \( P \) if \( Q \succeq P \)). For any pair of partitions \( P, Q \in \mathcal{P}_X^* \), the join \( P \lor Q \) can be read from the graph as follows: trace paths downwards from \( P \) and \( Q \) respectively until a common partition \( R \) is reached (note that the finest partition \( \{x\} | x \in X \) at the bottom is always the end of all downward paths in the graph, so it is guaranteed that \( R \) always exists). To ensure that \( R = P \lor Q \), make sure there is no \( R' \preceq R \) (indicated by an upward path from \( R \) to \( R' \)) with upward paths towards both \( P \) and \( Q \) (otherwise replace \( R \) with \( R' \) and repeat the process). Symmetrically, one can read the meet \( P \land Q \) from the graph.

There are limitations to this process, especially if the set \( X \) is infinite. Even for a finite set \( X \) of relatively small size, the complete abstraction universe \( \mathcal{P}_X^* \) can be quite complicated to visualize (recall that we have to draw \( |\mathcal{P}_X^*| = B_{|X|} \) vertices where \( B_{|X|} \) grows extremely fast with \( |X| \), let alone the edges). However, not all arbitrary partitions are of interest to us. In the following subsections, we study symmetry-generated abstractions and abstraction universes. So, later we can focus on certain partitions by considering certain symmetries.

### 3.3 Symmetry-Generated Abstraction

Recall that we explain an abstraction of a set by its inducing equivalence relation, where equivalent elements are treated as the same. Instead of considering arbitrary equivalence relations or arbitrary partitions, we construct every abstraction from an explicit mechanism—a symmetry—so the resulting equivalence classes or partition cells are invariant under this symmetry. To capture various symmetries, we consider groups and group actions.

**Preliminaries (Appendix A.3):** group \((G, \cdot) \) or \( G \), subgroup \( (\leq) \), trivial subgroup \( \{e\} \), subgroup generated by a set \((S)\), cyclic subgroup \((s)\); group action, \( G \)-action on \( X \) \((\cdot: G \times X \rightarrow X)\), orbit of \( x \in X \) \((Gx)\), set of all orbits \((X/G)\).

Consider a special type of group, namely the symmetric group \( (S_X, \circ) \) defined over a set \( X \), whose group elements are all the bijections from \( X \) to \( X \) and whose group operation is (function) composition. The identity element of \( S_X \) is the identity function, denoted \( \text{id} \). A bijection from \( X \) to \( X \) is also called a *transformation* of \( X \). Therefore, the symmetric group \( S_X \) comprises all transformations of \( X \), and is also called the transformation group of \( X \), denoted \( \text{F}(X) \). We use these two terms and notations interchangeably in this paper, with a preference for \( \text{F}(X) \) in general, while reserving \( S_X \) mostly for a finite \( X \).

Given a set \( X \) and a subgroup \( H \leq \text{F}(X) \), we define an \( H \)-action on \( X \) by \( h \cdot x := h(x) \) for any \( h \in H, x \in X \); the *orbit* of \( x \in X \) under \( H \) is the set \( Hx := \{h(x) | h \in H\} \). Orbits in \( X \) under \( H \) define an equivalence relation: \( x \sim x' \) if and only if \( x, x' \) are in the same orbit, and each orbit is an equivalence class. Thus, the quotient \( X/H = X/\sim \) is a...
partition of $X$. It is known that every cell (or orbit) in the abstraction (or quotient) $X/H$ is a minimal non-empty invariant subset of $X$ under transformations in $H$. Therefore, we say this abstraction respects the so-called $H$-symmetry or $H$-invariance.

We succinctly record the above process of constructing an abstraction $X/H$ (of $X$) from a given subgroup $H \leq F(X)$ in the following abstraction generating chain:

$$
\text{a subgroup of } F(X) \overset{\text{group action}}{\rightarrow} \text{orbits} \overset{\text{equiv. rel.}}{\rightarrow} \text{a partition} \overset{\rightarrow}{\rightarrow} \text{an abstraction of } X,
$$

which can be further encapsulated by the abstraction generating function defined as follows.

**Definition 5** The abstraction generating function is the mapping $\pi : \mathcal{H}_{F(X)}^* \rightarrow \mathcal{P}_X^*$ where $\mathcal{H}_{F(X)}^*$ is the collection of all subgroups of $F(X)$, $\mathcal{P}_X^*$ is the family of all partitions of $X$, and for any $H \in \mathcal{H}_{F(X)}^*$, $\pi(H) := X/H := \{Hx \mid x \in X\}$, where $Hx := \{h(x) \mid h \in H\}$.

**Theorem 6** The abstraction generating function $\pi : \mathcal{H}_{F(X)}^* \rightarrow \mathcal{P}_X^*$ is not necessarily injective.

**Proof** Let $X = \{1, 2, 3, 4\}$ and $h = (1234), g = (1324) \in S_4 = F(X)$ be two transformations (also known as permutations, in the cycle notation) of $X$; consider the cyclic groups:

$$
H = \langle h \rangle := \{h^n \mid n \in \mathbb{Z}\} = \{\text{id}, h, h^2, h^3\} = \{\text{id}, (1234), (13)(24), (1423)\};
$$

$$
G = \langle g \rangle := \{g^n \mid n \in \mathbb{Z}\} = \{\text{id}, g, g^2, g^3\} = \{\text{id}, (1234), (12)(34), (1423)\}.
$$

It is clear that $H \neq G$ but $\pi(H) = \pi(G) = \{\{1, 2, 3, 4\}\}$, the coarsest partition of $X$.

**Theorem 7** The abstraction generating function $\pi : \mathcal{H}_{F(X)}^* \rightarrow \mathcal{P}_X^*$ is surjective.

**Proof** For any $a, b \in X$, let $f_{a,b} : X \rightarrow X$ be the bijective function of the form

$$
f_{a,b}(x) = \begin{cases} 
  a & x = b, \\
  b & x = a, \\
  x & \text{otherwise}.
\end{cases}
$$

Pick any partition $P \in \mathcal{P}_X^*$. For any cell $P \in P$, define

$$
S_P := \{f_{a,b} \mid a, b \in P, a \neq b\}, \quad \text{and let } H := \left( \bigcup_{P \in \mathcal{P}} S_P \right).
$$

We claim $\pi(H) = P$. To see this, for any distinct $x, x' \in X$ that are in the same cell in $P$, $f_{x,x'} \in S_P$ for some $P \in \mathcal{P}$, so $f_{x,x'} \in H$. This implies that $x$ and $x'$ are in the same orbit in $\pi(H)$, since $x' = f_{x,x'}(x)$. Therefore, $\pi(H) \supseteq P$. Conversely, for any distinct $x, x' \in X$ that are in the same orbit in $\pi(H)$, there exists an $h \in H$ such that $x' = h(x)$. By definition, $h = h_k \circ \cdots \circ h_1$ for some finite integer $k > 0$ where $h_k, \ldots, h_1 \in \bigcup_{P \in \mathcal{P}} S_P$. Suppose $P' \in \mathcal{P}$ is the cell that $x$ is in, i.e. $x \in P'$, then $h_1(x) \in P'$, since $h_1(x) \in P'$ if $h_1 \in S_{P'}$ and $h_1(x) = x$ otherwise. Likewise, we have $h_2 \circ h_1(x), h_3 \circ h_2 \circ h_1(x), \ldots, h_k \circ \cdots \circ h_1(x) \in P'$. This implies that $x' = h(x) = h_k \circ \cdots \circ h_1(x) \in P'$, i.e. $x$ and $x'$ are in the same cell in $P$. Therefore, $P \supseteq \pi(H)$. Combining both directions yields $\pi(H) = P$, so $\pi$ is surjective.
3.4 Duality: from Subgroup Lattice to Abstraction (Semi)Universe

Given a subgroup of $F(X)$, we can generate an abstraction of $X$ via the abstraction generating function $\pi$. Thus, given a collection of subgroups of $F(X)$, we can generate a family of abstractions of $X$. Further, given a collection of subgroups of $F(X)$ with a hierarchy, we can generate a family of abstractions of $X$ with an induced hierarchy. This leads us to a subgroup lattice generating a partition (semi)lattice, where the latter is dual to the former via the abstraction generating function $\pi$.

Preliminaries (Appendix A.4): the (complete) subgroup lattice for a group $(\mathcal{H}_G^*, \leq)$, join $(A \lor B = (A \cup B))$, meet $(A \land B = A \cap B)$.

We consider the subgroup lattice for $F(X)$, denoted $(\mathcal{H}_{F(X)}^*, \leq)$. Similar to the complete abstraction universe $(\mathcal{B}_X^*, \leq)$, we can draw a directed acyclic graph to visualize $(\mathcal{H}_{F(X)}^*, \leq)$ if it is finite, where vertices denote subgroups and edges denote the partial order. The graph is similarly constructed by plotting all distinct subgroups of $F(X)$ starting at the bottom with $\{\text{id}\}$, ending at the top with $F(X)$ and, roughly speaking, with larger subgroups positioned higher than smaller ones. Draw an upward edge from $A$ to $B$ if $A \leq B$ and there are no subgroups properly between $A$ and $B$. For any pair of subgroups $A, B \in \mathcal{H}_{F(X)}^*$, the join $A \lor B$ can be read from the graph by tracing paths upwards from $A$ and $B$ respectively until a common subgroup containing both is reached, and making sure there are no smaller subgroups; the meet $A \land B$ can be read from the graph in a symmetric manner. For any subgroup $C \in \mathcal{H}_{F(X)}^*$, the subgroup sublattice $(\mathcal{H}_C^*, \leq)$ for $C$ is part of the subgroup lattice $(\mathcal{H}_{F(X)}^*, \leq)$ for $F(X)$, which can be read from the graph for $(\mathcal{H}_C^*, \leq)$ by extracting the part below $C$ and above $\{\text{id}\}$.

**Theorem 8 (Duality)** Let $(\mathcal{H}_{F(X)}^*, \leq)$ be the subgroup lattice for $F(X)$, and $\pi$ be the abstraction generating function. Then $(\pi(\mathcal{H}_{F(X)}^*), \leq)$ is an abstraction meet-semiuniverse for $X$. More specifically, for any $A, B \in \mathcal{H}_{F(X)}^*$, the following hold:

1. partial-order reversal: if $A \leq B$, then $\pi(A) \succeq \pi(B)$;
2. strong duality: $\pi(A \lor B) = \pi(A) \land \pi(B)$ (Figure 4a);
3. weak duality: $\pi(A \land B) \succeq \pi(A) \lor \pi(B)$ (Figure 4b).

**Proof** (Partial-order reversal) Pick any $A, B \in \mathcal{H}_G^*$ and $A \leq B$. For any $x, x' \in X$ that are in the same cell in partition $\pi(A) = X/A = \{Ax \mid x \in X\}$, $x' \in Ax = \{a(x) \mid a \in A\}$. Since $A \leq B$, then $Ax \subseteq Bx$, which further implies that $x' \in Bx$. So, $x$ and $x'$ are in the same cell in partition $\pi(B)$. Therefore, $\pi(A) \succeq \pi(B)$.

(Strong duality) Pick any $A, B \in \mathcal{H}_G^*$. By the definition of join, $A, B \leq A \lor B$, so from what we have shown at the beginning, $\pi(A), \pi(B) \succeq \pi(A \lor B)$, i.e. $\pi(A \lor B)$ is a common coarsening of $\pi(A)$ and $\pi(B)$. Since $\pi(A) \land \pi(B)$ is the finest common coarsening of $\pi(A)$ and $\pi(B)$, then $\pi(A \lor B) \succeq \pi(A) \land \pi(B)$. Conversely, for any $x, x' \in X$ that are in the same cell in partition $\pi(A \lor B) = \pi((A \cup B)) = X/(A \cup B) = \{(A \cup B)x \mid x \in X\}$, $x$ and $x'$ must be in the same orbit under $(A \cup B)$-action on $X$, i.e. $x' \in (A \cup B)x$ which means $x' = f_{k} \circ \cdots \circ f_1(x)$ for some finite integer $k$ where $f_1, \ldots, f_k \in A \cup B$ (note: the fact that $A, B$ are both subgroups ensures that $A \cup B$ is closed under inverses). This implies that $x$
and \( f_1(x) \) are either in the same cell in partition \( \pi(A) \) or in the same cell in partition \( \pi(B) \) depending on whether \( f_1 \in A \) or \( f_1 \in B \), but in either event, \( x \) and \( f_1(x) \) must be in the same cell in any common coarsening of \( \pi(A) \) and \( \pi(B) \). Note that \( \pi(A) \wedge \pi(B) \) is a common coarsening of \( \pi(A) \) and \( \pi(B) \) (regardless of the fact that it is the finest), so \( x \) and \( f_1(x) \) are in the same cell in partition \( \pi(A) \wedge \pi(B) \). Likewise, \( f_1(x) \) and \( f_2 \circ f_1(x) \), \( f_3 \circ f_2 \circ f_1(x) \) and \( f_2 \circ f_1(x) \), \( \ldots \), \( f_{k-1} \circ \cdots \circ f_1(x) \) and \( x' \) are all in the same cell in partition \( \pi(A) \wedge \pi(B) \). Therefore, \( x \) and \( x' \) are in the same cell in partition \( \pi(A) \wedge \pi(B) \). So, \( \pi(A \lor B) \succeq \pi(A) \wedge \pi(B) \). Combining both directions yields \( \pi(A \lor B) = \pi(A) \wedge \pi(B) \).

(Weak duality) Pick any \( A, B \in \mathcal{H}^*_G \). By the definition of meet, \( A, B \succeq A \land B \), so from what have shown at the beginning, \( \pi(A), \pi(B) \preceq \pi(A \land B) \), i.e. \( \pi(A \land B) \) is a common refinement of \( \pi(A) \) and \( \pi(B) \). Since \( \pi(A) \lor \pi(B) \) is the coarsest common refinement of \( \pi(A) \) and \( \pi(B) \), then \( \pi(A \land B) \succeq \pi(A) \lor \pi(B) \). We cannot obtain equality in general. For example, let \( X = \mathbb{Z} \) and \( A = \{ r : \mathbb{Z} \to \mathbb{Z} \mid r(x) = kx, k \in \{-1, 1\} \} \), \( B = \{ t : \mathbb{Z} \to \mathbb{Z} \mid t(x) = x + k, k \in \mathbb{Z} \} \). It is clear that \( A, B \subseteq F(X) \) and \( A \land B = A \cap B = \{ \text{id} \} \), so \( \pi(A \land B) = X/\{ \text{id} \} = \{ \{ x \} \mid x \in \mathbb{Z} \} \), i.e. the finest partition of \( \mathbb{Z} \). However, \( \pi(A) = \{ \{ x, -x \} \mid x \in \mathbb{Z} \} \) and \( \pi(B) = \{ \mathbb{Z} \} \), i.e. the coarsest partition of \( \mathbb{Z} \), so \( \pi(A) \lor \pi(B) = \pi(A) = \{ \{ x, -x \} \mid x \in \mathbb{Z} \} \). In this example, we see that \( \pi(A \land B) \succeq \pi(A) \lor \pi(B) \) but \( \pi(A \land B) \neq \pi(A) \lor \pi(B) \).

**Remark 9 (Practical implication)** The strong duality in Theorem 8 suggests a quick way of computing abstractions. If one has already computed abstractions \( \pi(A) \) and \( \pi(B) \), then instead of computing \( \pi(A \lor B) \) from \( A \lor B \), one can compute the meet \( \pi(A) \land \pi(B) \), which is generally a less expensive operation than computing \( A \lor B \) and identifying all orbits in \( \pi(A \lor B) \).

Theorem 8 further allows us to build an abstraction semiuniverse with a partial hierarchy directly inherited from the hierarchy of the subgroup lattice. Nevertheless, there are cases where \( \pi(A) \preceq \pi(B) \) with incomparable \( A \) and \( B \) since the abstraction generating function \( \pi \) is not injective (Theorem 6). If desired, one needs additional steps to complete the hierarchy or even to complete the abstraction semiuniverse into an abstraction universe.
3.5 More on Duality: from Conjugation to Group Action

Partitions of a set $X$ generated from two conjugate subgroups of $F(X)$ can be related by a group action. We present this relation as another duality between subgroups and abstractions, which can also simplify the computation of abstractions.

**Preliminaries (Appendix A.5):** conjugate, conjugacy class.

**Theorem 10** Let $G$ be a group, $X$ be a set, and $\cdot : G \times X \to X$ be a $G$-action on $X$.

1. for any $g \in G$, $Y \in 2^X$, $g \cdot Y := \{g \cdot y \mid y \in Y\} \in 2^X$, and the corresponding function $\cdot : G \times 2^X \to 2^X$ defined by $g \cdot Y$ is a $G$-action on $2^X$;
2. for any $g \in G$, $P \in \mathcal{P}_X^*$, $g \cdot P := \{g \cdot P \mid P \in \mathcal{P}\} \in \mathcal{P}_X^*$, and the corresponding function $\cdot : G \times \mathcal{P}_X^* \to \mathcal{P}_X^*$ defined by $g \cdot P$ is a $G$-action on $\mathcal{P}_X^*$.

**Proof** See Appendix B.1.

**Theorem 11 (Duality)** Let $X$ be a set, $F(X)$ be the transformation group of $X$, and $\pi$ be the abstraction generating function. Then for any $H \leq F(X)$ and $g \in F(X)$,

$$\pi(g \circ H \circ g^{-1}) = g \cdot \pi(H),$$

where $\cdot$ refers to the group action defined in Statement 2 in Theorem 10.

**Proof** For any $Y \in \pi(g \circ H \circ g^{-1})$, $Y$ is an orbit in $X$ under $g \circ H \circ g^{-1}$, then $Y = (g \circ H \circ g^{-1})X = \{(g \circ h \circ g^{-1})x \mid h \in H\} = \{g \circ h \circ g^{-1}(x) \mid h \in H\} = \{g \circ h((g^{-1})(x)) \mid h \in H\}$ for some $x \in X$. Note that in the above derivation, $g^{-1}(x) \in X$ since $g \in F(X)$. So, $Hg^{-1}(x)$ is the orbit of $g^{-1}(x)$ under $H$, i.e. $Hg^{-1}(x) \in \pi(H)$. This implies that $Y \in g \cdot \pi(H)$. Therefore, $\pi(g \circ H \circ g^{-1}) \subseteq g \cdot \pi(H)$.

Conversely, for any $Y \in g \cdot \pi(H)$, $Y = g \cdot P$ for some $P \in \pi(H)$. Note that $P$ is an orbit in $X$ under $H$, i.e. $P = P \subseteq \{h \cdot x \mid h \in H\}$ for some $x \in X$, then $Y = g \cdot P = \{g \cdot y \mid y \in P\} = \{g \cdot (h \cdot x) \mid h \in H\} = \{(g \circ h(x)) \mid h \in H\} = \{g \circ h \circ g^{-1} \circ g(x) \mid h \in H\} = \{(g \circ h \circ g^{-1})(g(x)) \mid h \in H\} = \{(g \circ h \circ g^{-1}) \cdot g(x) \mid h \in H\} = (g \circ H \circ g^{-1})g(x)$ for some $x \in X$. Note that in the above derivation, $g(x) \in X$ since $g \in F(X)$. Therefore, $(g \circ H \circ g^{-1})g(x)$ is the orbit of $g(x)$ under $g \circ H \circ g^{-1}$, i.e. $(g \circ H \circ g^{-1})g(x) \in \pi(g \circ H \circ g^{-1})$. This implies that $Y \in \pi(g \circ H \circ g^{-1})$. So, $g \cdot \pi(H) \subseteq \pi(g \circ H \circ g^{-1})$. 

**Remark 12 (Practical implication)** Theorem 11 relates conjugation in the subgroup lattice $\mathcal{H}^*_X$ to group action on the partition lattice $\mathcal{P}^*_X$. In other words, the group action on the partition lattice is dual to the conjugation in the subgroup lattice. This duality suggests a quick way of computing abstractions. If one has already computed abstraction $\pi(H)$, then instead of computing $\pi(g \circ H \circ g^{-1})$ from $g \circ H \circ g^{-1}$, one can compute $g \cdot \pi(H)$, which is generally a less expensive operation than computing $g \circ H \circ g^{-1}$ and identifying all orbits in $\pi(g \circ H \circ g^{-1})$. 

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3.6 Partial Subgroup Lattice

Theoretically, through the abstraction generating function $\pi$ and necessary hierarchy completions, we can construct the complete abstraction universe $\mathcal{P}_X^*$ from the complete subgroup lattice $\mathcal{H}_{F(X)}^*$. This is because the subgroup lattice is a larger space that “embeds” the partition lattice (more precisely, Theorem 6 and 7). However, as we mentioned earlier, it is not practical to even store $\mathcal{P}_X^*$ for small $X$, and not all arbitrary partitions of $X$ are equally useful. Instead of considering all subgroups of $F(X)$, we draw our attention to certain parts of the complete subgroup lattice $\mathcal{H}_{F(X)}^*$. We introduce two general principles in extracting partial subgroup lattices: the top-down approach and the bottom-up approach.

The Top-Down Approach. We consider the subgroup sublattice $(\mathcal{H}_G^*, \leq)$ for some subgroup $G \leq F(X)$. If $X$ is finite, this is the part below $G$ and above $\{\text{id}\}$ in the directed acyclic graph for the complete subgroup lattice $(\mathcal{H}_{F(X)}^*, \leq)$. As the name suggests, the top-down approach first specifies a “top” in $\mathcal{H}_{F(X)}^*$ (i.e. a subgroup $G \leq F(X)$), and then extract everything below the “top” (i.e. the subgroup lattice $\mathcal{H}_G^*$). The computer algebra system GAP (The GAP Group, 2018) provides efficient algorithmic methods to construct the subgroup lattice for a given group, and even maintains several data libraries for special groups and their subgroup lattices. In general, enumerating all subgroups of a group can be computationally intense, and therefore, is applied primarily to small groups. When computationally prohibited, a general trick is to enumerate subgroups up to conjugacy (which is also supported by the GAP system). Computing abstractions within the conjugacy class of any subgroup is then easy by the duality in Theorem 11, once the abstraction generated by a representative is computed. More details on picking a special subgroup (as the “top”) of $F(X)$ are discussed in Section 4.

The Bottom-Up Approach. We first pick some finite subset $S \subseteq F(X)$, and then generate a partial subgroup lattice for $\langle S \rangle$ by computing $\langle S' \rangle$ for every $S' \subseteq S$, starting from smaller subgroups. As the name suggests, the bottom-up approach first constructs the trivial subgroup $\langle \emptyset \rangle = \{\text{id}\}$, i.e. the bottom vertex in the direct acyclic graph for $\mathcal{H}_{F(X)}^*$ if $X$ is finite, and then cyclic subgroups $\langle s \rangle$ for every $s \in S$. We continue to construct larger subgroups from smaller ones by taking the join, which corresponds to gradually moving upwards in the graph for $\mathcal{H}_{F(X)}^*$ when $X$ is finite. In general, this approach will produce at most $2^{|S|}$ subgroups for a given subset $S \subseteq F(X)$, and will not produce the complete subgroup sublattice $\mathcal{H}_S^*$ unless $S = \langle S \rangle$. Computing abstractions using this bottom-up approach is easy by the strong duality in Theorem 8, once the abstractions generated by all cyclic subgroups are computed. More details on this abstraction generating process and picking a generating set (as the “bottom”) are discussed in Section 5.

4. The Top-Down Approach: Special Subgroups

We follow a top-down approach to discuss subgroup enumeration problems. The plan is to start with the transformation group of $X = \mathbb{R}^n$, and then to consider special subgroups of $F(\mathbb{R}^n)$ and special subspaces of $\mathbb{R}^n$. To do this systematically, we derive a principle that allows us to hierarchically break the enumeration problem into smaller and smaller enumeration subproblems. This hierarchical breakdown can guide us in restricting both the
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Figure 5: Special subgroups and spaces as well as their hierarchies. (a) presents a backbone of the complete subgroup lattice $\mathcal{H}_F(\mathbb{R}^n)$, including important subgroups and their breakdowns. One can check the above directed acyclic graph indeed represents a sublattice: it is closed under both join and meet. (b) presents important subspaces of $\mathbb{R}^n$, where restrictions are gradually added to eventually lead to practical abstraction-construction algorithms.

type of subgroups and the type of subspaces, so that the resulting abstraction (semi)universe fits our desiderata, and more importantly can be computed in practice. Figure 5 presents an outline consisting of special subgroups and subspaces considered in this section as well as their hierarchies.

Note: we do not claim the originality of the content in this section. Indeed, many parts have been studied in various contexts. Our work is to extend existing results from specific context to a general setting. This generalization coherently puts different pieces of context-specific knowledge under one umbrella, forming the guiding principle of the top-down approach.

Preliminaries (Appendix A.6): group homomorphism, isomorphism ($\cong$); normalizer of a set in a group ($N_G(S) := \{g \in G \mid gSg^{-1} = S\}$), normal subgroup ($\triangleleft$); group decomposition, inner semi-direct product, outer semi-direct product ($\rtimes$).

4.1 The Affine Transformation Group $\text{AFF}(\mathbb{R}^n)$

An affine transformation of $\mathbb{R}^n$ is a function $f_{A,u} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form

$$f_{A,u}(x) = Ax + u$$

for any $x \in \mathbb{R}^n$,

where $A \in \text{GL}_n(\mathbb{R})$ is an $n \times n$ real invertible matrix and $u \in \mathbb{R}^n$ is an $n$-dimensional real vector. We use $\text{AFF}(\mathbb{R}^n)$ to denote the set of all affine transformations of $\mathbb{R}^n$. There are two special cases:

1. A translation of $\mathbb{R}^n$ is a function $t_u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form $x \mapsto x + u$ where $u \in \mathbb{R}^n$; we use $T(\mathbb{R}^n)$ to denote the set of all translations of $\mathbb{R}^n$.

2. A linear transformation of $\mathbb{R}^n$ is a function $r_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form $x \mapsto Ax$ where $A \in \text{GL}_n(\mathbb{R})$; we use $L(\mathbb{R}^n)$ to denote the set of all linear transformations of $\mathbb{R}^n$. 

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It is easy to check that \( T(\mathbb{R}^n), L(\mathbb{R}^n) \leq \text{AFF}(\mathbb{R}^n) \leq F(\mathbb{R}^n) \); further, \((T(\mathbb{R}), \circ)\) and \((L(\mathbb{R}^n), \circ)\) are isomorphic to \((\mathbb{R}^n, +)\) and \((\text{GL}_n(\mathbb{R}), \cdot)\), respectively. It is known that

\[
\text{AFF}(\mathbb{R}^n) = T(\mathbb{R}^n) \circ L(\mathbb{R}^n) \cong T(\mathbb{R}^n) \times L(\mathbb{R}^n) \cong \mathbb{R}^n \times \text{GL}_n(\mathbb{R}).
\]

So every affine transformation can be uniquely identified with a pair \((u, A) \in \mathbb{R}^n \times \text{GL}_n(\mathbb{R})\). In particular, the identity transformation is identified with \((0, I)\), the translation group \(T(\mathbb{R}^n)\) is identified with \(\{(u, I) \mid u \in \mathbb{R}^n\}\), and the linear transformation group \(L(\mathbb{R}^n)\) is identified with \(\{(0, A) \mid A \in \text{GL}_n(\mathbb{R})\}\). Under this identification, compositions and inverses of affine transformations become

\[
(u, A)(u', A') = (u + Au', AA') \quad \text{and} \quad (u, A)^{-1} = (-A^{-1}u, A^{-1}). \tag{1}
\]

The above identification further allows us to introduce two functions \(\ell : \text{AFF}(\mathbb{R}^n) \to \text{GL}_n(\mathbb{R})\) and \(\tau : \text{AFF}(\mathbb{R}^n) \to \mathbb{R}^n\) to extract the linear and translation part of an affine transformation, respectively, where

\[
\ell(f_{A,u}) = A, \quad \tau(f_{A,u}) = u \quad \text{for any} \quad f_{A,u} \in \text{AFF}(\mathbb{R}^n).
\]

Now we can start our journey towards a complete identification of every subgroup \(H\) of \(\text{AFF}(\mathbb{R}^n)\). We introduce the first foundational quantity \(T := T(\mathbb{R}^n) \cap H\), which is the set of pure translations in \(H\), called the translation subgroup of \(H\). It is easy to check that \(T \leq H\) since translations are normal in affine transformations. Therefore, the quotient group \(H/T = \{T \circ h \mid h \in H\}\) is well-defined. The elements in \(H/T\) are called cosets. The following theorems reveal more structures of \(H/T\), the second foundational quantity.

**Lemma 13** \(\ell : \text{AFF}(\mathbb{R}^n) \to \text{GL}_n(\mathbb{R})\) is a homomorphism.

**Proof** For any \(f_{A,u}, f'_{A',u'} \in \text{AFF}(\mathbb{R}^n)\), we have \(\ell(f_{A,u} \circ f'_{A',u'}) = \ell(f_{AA',Au'+u}) = AA' = \ell(f_{A,u})\ell(f'_{A',u})\), which implies that \(\ell\) is a homomorphism. ■

**Theorem 14** Let \(H \leq \text{AFF}(\mathbb{R}^n), \; T = T(\mathbb{R}^n) \cap H\). Then \(h, h' \in H\) are in the same coset in \(H/T\) if and only if they have the same linear part, i.e. \(\ell(h) = \ell(h')\).

**Proof** See Appendix B.2. ■

**Theorem 15** Let \(H \leq \text{AFF}(\mathbb{R}^n), \; T = T(\mathbb{R}^n) \cap H\). If \(h, h' \in H\) are in the same coset in \(H/T\), then \(\tau(h') - \tau(h) \in \tau(T) := \{u \mid t_u \in T\}\).

**Proof** See Appendix B.3. ■

**Remark 16** Theorems 14 and 15 present two characterizations of elements in the same coset in \(H/T\), respectively. The former, through the linear part, is an if-and-only-if characterization; while the latter, through the translation part, is a necessary but not sufficient characterization.
Theorem 17 Let $H \leq \text{AFF}(\mathbb{R}^n)$, $T = \text{T}(\mathbb{R}^n) \cap H$. Then $H/T \cong \ell(H)$.

Proof It is clear that $\ell(H) \leq \text{GL}_n(\mathbb{R})$, since $H \leq \text{AFF}(\mathbb{R}^n)$ and $\ell$ is a homomorphism (Lemma 13) which preserves subgroups. Let $\bar{\ell} : H/T \to \ell(H)$ be the function of the form $\bar{\ell}(T \circ h) = \ell(h)$, we claim that $\bar{\ell}$ is an isomorphism. To see this, for any $T \circ h, T \circ h' \in H/T$,

$$\bar{\ell}((T \circ h)(T \circ h')) = \bar{\ell}(T \circ (h \circ h')) = \ell(h \circ h') = \ell(h)\ell(h') = \bar{\ell}(T \circ h)\bar{\ell}(T \circ h'),$$

which implies $\bar{\ell}$ is a homomorphism. Further, for any $T \circ h, T \circ h' \in H/T$, if $\bar{\ell}(T \circ h) = \bar{\ell}(T \circ h')$, then $\ell(h) = \ell(h')$. By Theorem 14, this implies that $T \circ h = T \circ h'$, so $\bar{\ell}$ is injective. Lastly, for any $A \in \ell(H)$, there exists an $h \in H$ such that $\ell(h) = A$. For this particular $h$, $T \circ h \in H/T$, and $\bar{\ell}(T \circ h) = \ell(h) = A$. This implies that $\bar{\ell}$ is surjective. □

Remark 18 Theorem 17 can be proved directly from the first isomorphism theorem, by recognizing $\ell|_H$ is a homomorphism whose kernel and image are $T$ and $\ell(H)$, respectively. However, the above proof explicitly gives the isomorphism $\ell$ which is useful in the sequel.

Theorem 19 (Compatibility) Let $H \leq \text{AFF}(\mathbb{R}^n)$, $T = \text{T}(\mathbb{R}^n) \cap H$. For any $A \in \ell(H)$ and $v \in \tau(T)$, we have $Av \in \tau(T)$. Further, if we define a function $\cdot : \ell(H) \times \tau(T) \to \tau(T)$ of the form $(A, v) \mapsto Av$, then $\cdot$ is a group action of $\ell(H)$ on $\tau(T)$.

Proof See Appendix B.4. □

So far, we have seen that for any subgroup $H \leq \text{AFF}(\mathbb{R}^n)$, its subset of pure translations $T := \text{T}(\mathbb{R}^n) \cap H$ is a normal subgroup of $H$; $T$ is also a normal subgroup of $\text{T}(\mathbb{R}^n)$, since $\text{T}(\mathbb{R}^n)$ is a commutative group. As a result, both quotient groups $H/T$ and $\text{T}(\mathbb{R}^n)/T$ are well-defined. We next introduce a function, called a vector system, which connects the two quotient groups. It turns out that vector systems comprise the last piece of information that leads to a complete identification of every subgroup of $\text{AFF}(\mathbb{R}^n)$. Note that $H/T \cong \ell(H)$ (Theorem 17) and $\text{T}(\mathbb{R}^n)/T \cong \mathbb{R}^n/\tau(T)$; thus for conceptual ease (think in terms of matrices and vectors), we introduce vector systems connecting $\ell(H)$ and $\mathbb{R}^n/\tau(T)$ instead.

Definition 20 (Vector system) For any $L \leq \text{GL}_n(\mathbb{R})$ and $V \leq \mathbb{R}^n$, an $(L, V)$-vector system is a function $\xi : L \to \mathbb{R}^n/V$, which in addition satisfies the following two conditions:

1. compatibility condition: for any $A \in L$, $AV = \{Av \mid v \in V\} = V$;
2. cocycle condition: for any $A, A' \in L$, $\xi(AA') = \xi(A) + A\xi(A')$.

Note: elements in $\mathbb{R}^n/V$ are cosets of the form $V + u$ for $u \in \mathbb{R}^n$. It is easy to check: for any two cosets in $\mathbb{R}^n/V$, the sum

$$(V + u) + (V + u') = \{v + u + v' + u' \mid v, v' \in V\} = V + (u + u');$$

for any $A \in L$ and any coset in $\mathbb{R}^n/V$, the product

$$A(V + u) = \{A(v + u) \mid v \in V\} = V + Au.$$

So, the sum and product in the cocycle condition are defined in the above sense.
We use $\Xi_{L,V}$ to denote the family of all $(L,V)$-vector systems. One can check that $\Xi_{L,V} \neq \emptyset$ if and only if $L,V$ are compatible (consider the trivial vector system $\xi_{L,V}^0$ given by $\xi_{L,V}^0(A) = V$ for all $A \in L$). We use $\Xi^* := \{\Xi_{L,V} \mid L \leq \text{GL}_n(\mathbb{R}), V \leq \mathbb{R}^n \text{ compatible}\}$ to denote the universe of all vector systems.

Remark 21 The universe of all vector systems $\Xi^*$ can be parameterized by the set of compatible pairs $(L,V) \in H^*_{\text{GL}_n(\mathbb{R})} \times H^*_{\mathbb{R}^n}$. The reason is straightforward: $L$ and $V$ respectively define the domain and codomain of a function, and two functions are different if either their domains or their codomains are different.

Lemma 22 Let $L \leq \text{GL}_n(\mathbb{R})$, $V \leq \mathbb{R}^n$, and $\xi \in \Xi_{L,V}$, then

1. for the identity matrix $I \in L$, $\xi(I) = V$;
2. for any $A \in L$, $\xi(A^{-1}) = -A^{-1}\xi(A)$.

Proof See Appendix B.5. ■

Theorem 23 (Affine subgroup identification) Let

$$\Sigma := \{(L,V,\xi) \mid L \leq \text{GL}_n(\mathbb{R}), V \leq \mathbb{R}^n, \xi \in \Xi_{L,V}\},$$

then there is a bijection between $H^*_{\text{AFF}(\mathbb{R}^n)}$ and $\Sigma$.

Proof (Outline) Let $\Psi : H^*_{\text{AFF}(\mathbb{R}^n)} \rightarrow \Sigma$ be the function defined by

$$\Psi(H) := (\ell(H), \tau(T), \xi_H) \quad \text{for any } H \in H^*_{\text{AFF}(\mathbb{R}^n)},$$

where $T := T(\mathbb{R}^n) \cap H$, and $\xi_H : \ell(H) \rightarrow \mathbb{R}^n/\tau(T)$ is given by $\xi_H(A) = \tau(\ell^{-1}(A))$ with $\ell : H/T \rightarrow \ell(H)$ being the isomorphism defined in the proof of Theorem 17. The plan is to first show that $\Psi$ is well-defined, and then to show that it is bijective; in particular, we will show that the inverse function

$$\Psi^{-1}((L,V,\xi)) = \{f_{A,u} \in \text{AFF}(\mathbb{R}^n) \mid A \in L, u \in \xi(A)\} \quad \text{for any } (L,V,\xi) \in \Sigma.$$ 

The entire proof is divided into four parts. We relegate the full proof to Appendix B.6. ■

Remark 24 The bijection $\Psi$ from $H^*_{\text{AFF}(\mathbb{R}^n)}$ to $\Sigma$ allows us to use the latter to parameterize the former. Further, through the inverse function $\Psi^{-1}$, we can enumerate affine subgroups by enumerating triplets $(L,V,\xi) \in \Sigma$, or more specifically, by enumerating matrix subgroups of $\text{GL}_n(\mathbb{R})$, vector subgroups of $\mathbb{R}^n$, and then vector systems for every compatible pair of a matrix subgroup and a vector subgroup. Note that enumeration for each element in the triplet is still not practical if no restriction is imposed. Nevertheless, we have broken the original subgroup enumeration problem into three smaller enumeration problems. More importantly, we are now more directed in imposing restrictions on both subgroups and spaces, under which the three smaller enumerations become practical. We will discuss these restrictions (e.g. being isometric, finite, discrete, compact) in more details in the sequel.
4.2 The Isometry Group $\text{ISO}(\mathbb{R}^n)$

One way to restrict $\Sigma := \{(L, V, \xi) \mid L \leq \text{GL}_n(\mathbb{R}), V \leq \mathbb{R}^n, \xi \in \Xi_{L, V}\}$ is to consider a special subgroup of $\text{GL}_n(\mathbb{R})$. Instead of all subgroups of $\text{GL}_n(\mathbb{R})$, we consider only subgroups consisting of orthogonal matrices. This restriction gives rise to the subgroup lattice $\mathcal{H}_{\text{ISO}(\mathbb{R}^n)}$ where $\text{ISO}(\mathbb{R}^n)$ denotes the group of isometries of $\mathbb{R}^n$. In this subsection, we first give an overview of $\text{ISO}(\mathbb{R}^n)$, and then cast $\mathcal{H}_{\text{ISO}(\mathbb{R}^n)}$ in the big picture of $\mathcal{H}_{\text{AFF}(\mathbb{R}^n)}$ and $\Sigma$.

An isometry of $\mathbb{R}^n$, with respect to the Euclidean distance $d : \mathbb{R}^n \to \mathbb{R}^n$ which preserves distances: $d(h(x), h(x')) = d(x, x')$, for all $x, x' \in \mathbb{R}^n$. We use $\text{ISO}(\mathbb{R}^n)$ to denote the set of all isometries of $\mathbb{R}^n$, which is a subgroup of the transformation group $F(\mathbb{R}^n)$. So, we call $\text{ISO}(\mathbb{R}^n)$ the isometry group of $\mathbb{R}^n$.

A (generalized) rotation of $\mathbb{R}^n$ is a linear transformation $r_A : \mathbb{R}^n \to \mathbb{R}^n$ given by $x \mapsto Ax$, for some orthogonal matrix $A \in \text{O}_n(\mathbb{R}) := \{A \in \mathbb{R}^{n \times n} \mid A^T = A^{-1}\} \leq \text{GL}_n(\mathbb{R})$. We use $R(\mathbb{R}^n)$ to denote the set of all rotations of $\mathbb{R}^n$, which is a subgroup of the linear transformation group $L(\mathbb{R}^n)$. So, we call $R(\mathbb{R}^n)$ the rotation group of $\mathbb{R}^n$.

There are two key characterizations of $\text{ISO}(\mathbb{R}^n)$. The first one regards its components:

$$\text{ISO}(\mathbb{R}^n) = (T(\mathbb{R}^n) \cup R(\mathbb{R}^n)) \quad \text{where} \quad T(\mathbb{R}^n) \cap R(\mathbb{R}^n) = \{\text{id}\}.$$  

This characterization says that $\text{ISO}(\mathbb{R}^n)$ comprises exclusively translations, rotations, and their finite compositions. Note that we can rewrite the above characterization as $T(\mathbb{R}^n) \cup R(\mathbb{R}^n) = \text{ISO}(\mathbb{R}^n)$ and $T(\mathbb{R}^n) \cap R(\mathbb{R}^n) = \{\text{id}\}$. This determines the positions of the four subgroups $\{\text{id}\}$, $T(\mathbb{R}^n)$, $R(\mathbb{R}^n)$, and $\text{ISO}(\mathbb{R}^n)$ in the subgroup lattice $\mathcal{H}_{\text{AFF}(\mathbb{R}^n)} \leq$, which forms a diamond shape in the direct acyclic graph in Figure 5a. The second characterization of $\text{ISO}(\mathbb{R}^n)$ regards a unique representation for every isometry of $\mathbb{R}^n$, which is done by a group decomposition of $\text{ISO}(\mathbb{R}^n)$ as semi-direct products:

$$\text{ISO}(\mathbb{R}^n) = T(\mathbb{R}^n) \circ R(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes \text{O}_n(\mathbb{R}).$$

This characterization says that every isometry of $\mathbb{R}^n$ can be uniquely represented as an affine transformation $f_{A, u} \in \text{AFF}(\mathbb{R}^n)$ where $A \in \text{O}_n(\mathbb{R})$ and $u \in \mathbb{R}^n$. This further implies that $\text{ISO}(\mathbb{R}^n)$ is a special subgroup of $\text{AFF}(\mathbb{R}^n)$.

Let $\Psi : \mathcal{H}_{\text{AFF}(\mathbb{R}^n)} \to \Sigma$ be the bijection defined in the proof of Theorem 23, and let

$$\Sigma' := \{(L, V, \xi) \mid L \leq \text{O}_n(\mathbb{R}), V \leq \mathbb{R}^n, \xi \in \Xi_{L, V}\} \subseteq \Sigma.$$ 

One can check: $\Psi^{-1}(\Sigma') = \mathcal{H}_{\text{ISO}(\mathbb{R}^n)}$. This means $\Psi|_{\mathcal{H}_{\text{ISO}(\mathbb{R}^n)}} : \mathcal{H}_{\text{ISO}(\mathbb{R}^n)} \to \Sigma'$ is well-defined and bijective. Therefore, the subgroups of $\text{ISO}(\mathbb{R}^n)$ can be enumerated by the triplets in $\Sigma'$ in a similar manner as in Remark 24. The only difference is that we now enumerate subgroups of $\text{O}_n(\mathbb{R})$ instead of the entire $\text{GL}_n(\mathbb{R})$.

Note that restricting to subgroups of $\text{O}_n(\mathbb{R})$ does not really make the enumeration problem practical. However, there are many ways of imposing additional restrictions on $\text{ISO}(\mathbb{R}^n)$ to eventually achieve practical enumerations. We want to point out that there is no universal way of constraining the infinite enumeration problem into a practical one: the design of restrictions is most effective if it is consistent with the underlying topic domain. So, for instance, one can start with his/her intuition to try out some restrictions whose effectiveness can be verified via a subsequent learning process (cf. Section 7). In the next subsection, we give two examples to illustrate some of the existing design choices that have been made in two different domains.
4.3 Special subgroups of $\text{ISO}(\mathbb{R}^n)$ used in Chemistry and Music

From two examples, we show how additional restrictions can be imposed to yield a finite collection of subgroups of $\text{ISO}(\mathbb{R}^n)$, capturing different parts of the infinite subgroup lattice $\mathcal{H}_{\text{ISO}(\mathbb{R}^n)}$. The two examples are from two different topic domains: one is from chemistry (or more precisely, crystallography), the other is from music. The ways of adding restrictions in these two examples are quite different: one introduces conjugacy relations to obtain a finite collection of subgroup types; the other restricts the space to be discrete or even finite.

4.3.1 Crystallographic Space Groups

In crystallography, symmetry is used to characterize crystals, to identify repeating parts of molecules, and to simplify both data collection and subsequent calculations. Further, the symmetry of physical properties of a crystal such as thermal conductivity and optical activity has a strong connection with the symmetry of the crystal. So, a thorough knowledge of symmetry is crucial to a crystallographer. A complete set of symmetry classes is captured by a collection of 230 unique 3-dimensional space groups. However, space groups represent a special type of subgroups of $\text{ISO}(\mathbb{R}^n)$ which can be defined in general for any dimension.

We give a short review of known results from crystallography, and then identify space groups in the parametrization set $\Sigma$ that we derived earlier. A crystallographic space group or space group $\Gamma$ is a discrete (with respect to the subset topology) and cocompact (i.e. the abstraction space $\pi(\Gamma) := \mathbb{R}^n/\Gamma$ is compact with respect to the quotient topology) subgroup of $\text{ISO}(\mathbb{R}^n)$. So, if the underlying topic domain indeed considers only compact abstractions, space groups are good candidates. A major reason is that for a given dimension, there exist only finitely many space groups (up to isomorphism or affine conjugacy) by Bieberbach’s second and third theorems (Bieberbach, 1911; Charlap, 2012).

Bieberbach’s first theorem (Bieberbach, 1911; Charlap, 2012) gives an equivalent characterization of space groups: a subgroup $\Gamma$ of $\text{ISO}(\mathbb{R}^n)$ is a space group if $T := T(\mathbb{R}^n) \cap \Gamma$ is isomorphic to $\mathbb{Z}^n$ and $\tau(T)$ spans $\mathbb{R}^n$. In particular, for a space group $\Gamma$ in standard form, we have $\ell(\Gamma) \leq \text{O}_n(\mathbb{Z})$, $\tau(T) = \mathbb{Z}^n$ (Eick and Souvignier, 2006). Therefore, we can use

$$\Sigma_{\text{cryst}}' := \{(L, V, \xi) \mid L \leq \text{O}_n(\mathbb{Z}), V = \mathbb{Z}^n, \xi \in \Xi_{L,V}\} \subseteq \Sigma' \subseteq \Sigma$$

to parameterize the set of all space groups in standard form. We will soon (in Section 4.3.2) see that $|\text{O}_n(\mathbb{Z})| = n!2^n$ which is finite. For every $L \leq \text{O}_n(\mathbb{Z})$, the enumeration of vector systems $\xi \in \Xi_{L,\mathbb{Z}^n}$ is also made feasible in Zassenhaus (1948) by identifying orbits in $H^1(L, \mathbb{R}^n/\mathbb{Z}^n)$ under the group action of $\text{NGL}_n(\mathbb{Z})(L)$ on $H^1(L, \mathbb{R}^n/\mathbb{Z}^n)$, where $H^1(L, \mathbb{R}^n/\mathbb{Z}^n)$ is the first cohomology group of $L$ with values in $\mathbb{R}^n/\mathbb{Z}^n$ and $\text{NGL}_n(\mathbb{Z})(L)$ is the integral normalizer of $L$. We refer interested readers to the original Zassenhaus algorithm (Zassenhaus, 1948) and the GAP package CrystCat (Felsch and Gähler, 2000) for more details on the algorithmic implementation of space groups.

4.3.2 Isometries of $\mathbb{Z}^n$ in Music

Another example of obtaining a finite collection of subgroups of $\text{ISO}(\mathbb{R}^n)$ comes from computational music theory. This is an extension to our earlier work on building an automatic music theorist (Yu et al., 2016; Yu and Varshney, 2017; Yu et al., 2017). In this example,
we impose restrictions on the space, focusing on discrete subsets of $\mathbb{R}^n$ that represent music pitches from equal temperament. Restrictions on the space further result in restrictions on the subgroups under consideration, namely only those subgroups that stabilize the restricted subsets of $\mathbb{R}^n$. We start our discussion on isometries of $\mathbb{Z}^n$, while further restrictions for a finite discrete subspace such as $\mathbb{Z}_{[a,b]}^n$ or $\mathbb{Z}_{[0,b]}^n$ (Figure 5b) will be presented in Section 6. We first introduce a few definitions regarding the space $\mathbb{Z}^n$ in parallel with their counterparts regarding $\mathbb{R}^n$, and then establish their equivalences under restricted setwise stabilizers.

**Definition 25** An isometry of $\mathbb{Z}^n$, with respect to the Euclidean distance $d$ (or more precisely $d|_{\mathbb{Z}^n}$) is a function $h' : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ which preserves distances: $d(h'(x), h'(x')) = d(x, x')$, for all $x, x' \in \mathbb{Z}^n$. We use $\text{ISO}(\mathbb{Z}^n)$ to denote the set of all isometries of $\mathbb{Z}^n$.

**Definition 26** A translation of $\mathbb{Z}^n$ is a function $t'_u : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ of the form $x \mapsto x + u$, where $u \in \mathbb{Z}^n$. We use $T(\mathbb{Z}^n)$ to denote the set of all translations of $\mathbb{Z}^n$.

**Definition 27** A (generalized) rotation of $\mathbb{Z}^n$ is a function $r'_A : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ of the form $x \mapsto Ax$, where $A \in O_n(\mathbb{Z}) := \{A \in \mathbb{Z}^{n \times n} \mid A^T = A^{-1}\}$. We use $R(\mathbb{Z}^n)$ to denote the set of all rotations of $\mathbb{Z}^n$.

It is easy to check that $(T(\mathbb{Z}^n), \circ)$ is isomorphic to $(\mathbb{Z}^n, +)$, and $(R(\mathbb{Z}^n), \circ)$ is isomorphic to $(O_n(\mathbb{Z}), \cdot)$; further, $T(\mathbb{Z}^n), R(\mathbb{Z}^n) \leq F(\mathbb{Z}^n)$, and $T(\mathbb{Z}^n), R(\mathbb{Z}^n) \subseteq \text{ISO}(\mathbb{Z}^n)$, so translations and rotations of $\mathbb{Z}^n$ are transformations and are also isometries. However, we do not know yet whether $(\text{ISO}(\mathbb{Z}^n), \circ)$ is a group or whether $\text{ISO}(\mathbb{Z}^n) \subseteq F(\mathbb{Z}^n)$. It turns out that the results are indeed positive, i.e. $\text{ISO}(\mathbb{Z}^n) \leq F(\mathbb{Z}^n)$, but we need more steps to see this.

**Definition 28** Let $G \leq F(X), Y \subseteq X$, and $G_Y := \{g \in G \mid g(Y) = Y\}$ be the setwise stabilizer of $Y$ under $G$. The restricted setwise stabilizer of $Y$ under $G$ is the set

$$G_Y|_Y := \{g|_Y \mid g \in G_Y\},$$

where $g|_Y : Y \rightarrow Y$ is the (surjective) restriction of the function $g$ to $Y$.

**Theorem 29** For any $Y \subseteq X$, $F(Y) = F(X)|_Y$.

**Proof** See Appendix B.7.

**Corollary 30** $F(\mathbb{Z}^n) = F(\mathbb{R}^n)|_{\mathbb{Z}^n}$.

**Theorem 31** $T(\mathbb{Z}^n) = T(\mathbb{R}^n)|_{\mathbb{Z}^n}$, and $R(\mathbb{Z}^n) = R(\mathbb{R}^n)|_{\mathbb{Z}^n}$.

**Proof** See Appendix B.8.

**Theorem 32** $\text{ISO}(\mathbb{Z}^n) = \text{ISO}(\mathbb{R}^n)|_{\mathbb{Z}^n}$.

**Proof** See Appendix B.9.
Remark 33 Through restricted setwise stabilizers, Corollary 30 as well as Theorems 31 and 32 collectively verify that transformations, translations, rotations, and isometries of \( \mathbb{Z}^n \) are precisely those transformations, translations, rotations, and isometries of \( \mathbb{R}^n \) that stabilize \( \mathbb{Z}^n \), respectively. In particular, it is now clear that \( (\text{ISO}(\mathbb{Z}^n), \circ) \) is indeed a group, and moreover \( T(\mathbb{Z}^n), R(\mathbb{Z}^n) \leq \text{ISO}(\mathbb{Z}^n) \leq F(\mathbb{Z}^n) \).

The parallels between translations, rotations, isometries of \( \mathbb{Z}^n \) and \( \mathbb{R}^n \) yield the two characterizations of \( \text{ISO}(\mathbb{Z}^n) \) which are parallel to the those of \( \text{ISO}(\mathbb{R}^n) \):

\[
\text{ISO}(\mathbb{Z}^n) = \langle T(\mathbb{Z}^n) \cup R(\mathbb{Z}^n) \rangle \quad \text{where} \quad T(\mathbb{Z}^n) \cap R(\mathbb{Z}^n) = \{\text{id}\};
\]

\[
\text{ISO}(\mathbb{Z}^n) = T(\mathbb{Z}^n) \circ R(\mathbb{Z}^n) \cong \mathbb{Z}^n \times O_n(\mathbb{Z}).
\]

This further yields the parametrization of \( \mathcal{H}_{\text{ISO}(\mathbb{Z}^n)}^* \) by

\[
\Sigma_{\text{iso}zn}'' := \{(L, V, \xi) \mid L \leq O_n(\mathbb{Z}), V \leq \mathbb{Z}^n, \xi \in \Xi_{L, V}' \} \subseteq \Sigma' \subseteq \Sigma,
\]

where \( \Xi_{L, V}' := \{\xi \in \Xi_{L, V} \mid \xi(L) \subseteq \mathbb{Z}^n/V\} \subseteq \Xi_{L, V} \). Note that \( \text{ISO}(\mathbb{Z}^n) \) still have infinitely many subgroups, since the choices for \( V \) and \( \xi \) are still unlimited. Next we will show how to enumerate a finite subset from \( \mathcal{H}_{\text{ISO}(\mathbb{Z}^n)}^* \) when considering the music domain.

The space of music pitches from equal temperament can be denoted by \( \mathbb{Z} \). Every adjacent pitch is separated by a half-step (or semi-tone) denoted by the integer 1, which is also the distance between every adjacent keys (regardless of black or white) in a piano keyboard. While the absolute integer assigned to each music pitch is not essential, in the standard MIDI convention, C4 (the middle C) is 60, C♯4 is 61, and so forth. Therefore, the space \( \mathbb{Z}^n \) represents the space of chords consisting of \( n \) pitches. For instance, \( \mathbb{Z}^3 \) denotes the space of trichords, \( \mathbb{Z}^4 \) denotes the space of tetrachords, and so forth. Known music transformations of fixed-size chords (Tymoczko, 2010; Lewin, 2010) can be summarized as a subset of the following parametrization set

\[
\Sigma_{\text{music}}'' := \{(L, V, \xi) \mid L \leq O_n(\mathbb{Z}), V \in \mathcal{H}_{\mathbb{Z}^n}^M, \xi = \xi_{L, V}^0 \in \Xi_{L, V}' \} \subseteq \Sigma_{\text{iso}zn}'',
\]

where \( \mathcal{H}_{\mathbb{Z}^n}^M := \{(1), (12\mathbb{Z})^n, (1) \vee (12\mathbb{Z})^n\} \) is a finite collection of music translation subgroups including music transpositions, octave shifts, and their combinations; \( \xi_{L, V}^0 \) is the trivial vector system given by \( \xi_{L, V}^0(A) = V \) for any \( A \in L \) requiring the inclusion of all rotations to include music permutations and inversions. Together with the fact that \( O_n(\mathbb{Z}) \) is finite, the enumeration of each element in the triplet \( (L, V, \xi) \) is finite, yielding a finite \( \Sigma_{\text{music}}'' \).

It is important to recognize that the significance of the parametrization set \( \Sigma_{\text{music}}'' \) is not limited to recover known music-theoretic concepts but to complete existing knowledge by forming a music “closure” \( \Sigma_{\text{music}}'' \). Such a “closure” can be further fine-tuned to be either more efficient (e.g. by removing uninteresting rotation subgroups) or more expressive (e.g. by adding more translation subgroups).

4.4 Section Summary

In this section, we first moved down from the full transformation group of \( \mathbb{R}^n \)—the top vertex in the subgroup lattice \( (\mathcal{H}^*_{\mathbb{R}^n}, \leq) \)—to the affine group of \( \mathbb{R}^n \). Focusing on \( \text{AFF}(\mathbb{R}^n) \), we derived a complete identification of its subgroups by constructing a parametrization set \( \Sigma \).
Figure 6: Roadmap of the top-down paths in terms of collection of subgroups (left) and the corresponding parametrization paths (right) under the parametrization map $\Psi$.

and a bijection $\Psi : \mathcal{H}^*_{\text{AFF}(\mathbb{R}^n)} \to \Sigma$. So, every subgroup of $\text{AFF}(\mathbb{R}^n)$ bijectively corresponds to a unique triplet in $\Sigma$. Towards the goal of a finite collection of affine subgroups, we further moved down in the subgroup lattice $(\mathcal{H}^*_{\text{AFF}(\mathbb{R}^n)}, \leq)$ from the affine group of $\mathbb{R}^n$ to the isometry group of $\mathbb{R}^n$. Focusing on $\text{ISO}(\mathbb{R}^n)$, we identified the parametrization of $\mathcal{H}^*_{\text{ISO}(\mathbb{R}^n)}$ by a subset $\Sigma' \subseteq \Sigma$. From there, we made a dichotomy in our top-down path, and presented two examples to obtain two collections of subgroups used in two different topic domains. One is a finite collection of space groups (in standard form and up to affine conjugacy) used in crystallography, which is parameterized by $\Sigma''_{\text{cryst}} \subseteq \Sigma'$; the other is a finite completion of existing music concepts, which is parameterized by $\Sigma''_{\text{music}} \subseteq \Sigma''_{\text{isozn}} \subseteq \Sigma'$. A complete roadmap that we have gone through is summarized in Figure 6.

We finally reiterate that the selection of top-down paths is one’s design choice. Whenever necessary, one should make his/her own decision on creating a new branch or even trying out several branches along major downward paths. The top-down path with two branches introduced in this section serve for illustration purposes.

5. The Bottom-Up Approach: Generating Set

We follow a bottom-up approach to extract a partial subgroup lattice $\mathcal{H}_{(S)}$ from a generating set $S$. This is done by an induction procedure which first extracts cyclic subgroups $\{\langle s \rangle \mid s \in S\}$ as base cases, and then inductively extracts other subgroups via the join of the extracted ones. The resulting collection of subgroups $\mathcal{H}_{(S)}$ is generally not the complete subgroup lattice $\mathcal{H}^*_{(S)}$ since some of its subgroups are missing. The dual of this induction procedure gives a mirrored induction algorithm that computes the corresponding abstraction semiuniverse in an efficient way. The missing subgroups can be made up by adding more generators, but this hinders the efficiency. At the end of this section, we will discuss the trade-off between expressiveness and efficiency when designing a generating set in practice.

5.1 From Generating Set to Subgroup (Semi)Lattice

Let $S \subseteq F(X)$ be a finite subset consisting of transformations of a set $X$. We construct a collection $\mathcal{H}_{(S)}$ consisting of subgroups of $(S)$ where every subgroup is generated by a subset of $S$. To succinctly record this process and concatenate it with the abstraction generating
chain, we introduce the following one-step subgroup generating chain:

\[ \text{a subset of } F(X) \xrightarrow{\text{group generation}} \text{a subgroup of } F(X), \]

which can be further encapsulated by the subgroup generating function defined as follows.

**Definition 34** The subgroup generating function is the mapping \( \pi' : 2^{F(X)} \to \mathcal{H}_{F(X)}^* \) where \( 2^{F(X)} \) is the power set of \( F(X) \), \( \mathcal{H}_{F(X)}^* \) is the collection of all subgroups of \( F(X) \), and for any \( S \in 2^{F(X)} \), \( \pi'(S) := \{ S_k \circ \cdots \circ s_1 \mid s_i \in S \cup S^{-1}, i = 1, \ldots, k, k \in \mathbb{Z}_{\geq 0} \} \) where \( S^{-1} := \{ s^{-1} \mid s \in S \} \). By convention, \( s_k \circ \cdots \circ s_1 = \text{id} \) for \( k = 0 \), and \( \pi'(\emptyset) = \langle \emptyset \rangle = \{ \text{id} \} \).

**Remark 35** The subgroup generating function in Definition 34 is nothing but generating a subgroup from its given generating set. However, we can now write the procedure at the beginning of this subsection succinctly as \( \mathcal{H} = \pi'(2^S) \) for any finite subset \( S \subseteq F(X) \); further, the subgroup generating chain and the abstraction generating chain can now be concatenated, which is denoted by the composition \( \Pi := \pi \circ \pi' \).

Like the abstraction generating function \( \pi \), the subgroup generating function \( \pi' \) is not necessarily injective, since a generating set of a group is generally not unique; \( \pi' \) is surjective, since every subgroup per se is also its own generating set. The following theorem captures the structure of \( \pi'(2^S) \) for a finite subset \( S \subseteq F(X) \).

**Theorem 36** Let \( S \subseteq F(X) \) be a finite subset, and \( \pi' \) be the subgroup generating function. Then \( (\pi'(2^S), \leq) \) is a join-semilattice, but not necessarily a meet-semilattice. In particular,

\[ \pi'(A \cup B) = \pi'(A) \lor \pi'(B) \quad \text{for any } A, B \subseteq S. \]

**Proof** For any \( A, B \subseteq S \), we have

\[ \pi'(A \cup B) = \langle A \cup B \rangle = \langle \langle A \rangle \cup \langle B \rangle \rangle = \langle \pi'(A) \cup \pi'(B) \rangle = \pi'(A) \lor \pi'(B). \]

Then for any \( \pi'(A), \pi'(B) \in \pi'(2^S) \) where \( A, B \subseteq S \), the join \( \pi'(A) \lor \pi'(B) = \pi'(A \cup B) \in \pi'(2^S) \), since \( A \cup B \subseteq S \). So, \( (\pi'(2^S), \leq) \) is a join-semilattice.

We give an example in which \( (\pi'(2^S), \leq) \) is not a meet-semilattice. Let \( X = \mathbb{R}^n \) and \( S = \{ t_{e_1}, t_{e_2}, t_{(3/2)1} \} \) be a set consisting of three translations where \( e_1 = (1, 0), e_2 = (0, 1), 1 = (1, 1) \). Further, let \( A = \{ t_{e_1}, t_{e_2} \} \) and \( B = \{ t_{(3/2)1} \} \). The meet \( \pi'(A) \land \pi'(B) = \langle A \rangle \cap \langle B \rangle = \langle t_{3,1} \rangle \not\in \pi'(2^S) \).

**Remark 37** Although the collection of subgroups generated by the subgroup generating function \( \pi' \) is not a lattice in general, it is sufficient that it is a join-semilattice. This is because the family of abstractions generated by the abstraction generating function \( \pi \) is a meet-semiuniverse (recall the strong and weak dualities in Theorem 8). As a result, the closedness of \( \pi'(2^S) \) under join is carried over through the strong duality to preserve the closedness of \( \pi(\pi'(2^S)) \) under meet. This preservation of closednesses under join and meet has a significant practical implication: it directly yields an induction algorithm that implements \( \Pi(2^S) := \pi \circ \pi'(2^S) \) from a finite subset \( S \).
5.2 An Induction Algorithm

We describe an algorithmic implementation of $\Pi(2^S) := \pi \circ \pi'(2^S)$, where $S \subseteq F(X)$ as the input is a finite subset, and $\Pi(2^S)$ as the output is the resulting abstraction semigroup. Here we assume $X$ is a finite space for computational feasibility. A naive implementation will first compute the subgroup join-semilattice $\pi'(2^S)$ as an intermediate step, and then compute the abstraction meet-semiuniverse $\pi(\pi'(2^S))$ as the second step. However, as mentioned in Remark 9, computing every abstraction of $X$ by identifying orbits from a subgroup action can be expensive, and even computationally prohibitive. In this subsection, we first analyze the naive two-step implementation, and then introduce an induction algorithm that efficiently computes $\Pi(2^S)$ without the intermediate step, avoiding expensive computations for all abstractions from orbits identifications.

A Naive Two-Step Implementation. For a given input $S \subseteq F(X)$ where both $S$ and $X$ are finite, we first compute $\pi'(2^S)$ which is straightforward, since we can simply index (possibly with duplication) every subgroup in $\pi'(2^S)$ by its generating set $S' \subseteq S$. Now consider the second step: for every $S' \subseteq S$ and its corresponding $\pi'(S') = \langle S' \rangle$, we compute $\pi(\pi'(S')) = X/\langle S' \rangle$ by identifying the set of orbits $\{\langle S' \rangle x \mid x \in X\}$. More specifically, for every pair $x, x' \in X$, we need to check whether or not they are in the same orbit. The number of checks needed is $O(|X|^2)$ which can be computationally prohibitive if $|X|$ is large. Nevertheless, what really makes this naive thought fail is that most checks cannot finish in finite time. Take $S' = \{s_1, s_2\}$ for example, without additional properties to leverage, there are infinitely many ways of causing $x, x'$ to be in the same orbit, e.g. $x' = s_1(x)$, $x' = s_1^{-1}(x)$, $x' = s_2(x)$, $x' = s_1 \circ s_2 \circ s_1^{-1}(x)$ and so forth.

An Induction Algorithm. Instead, we give an algorithm based on induction on $|S'|$ for all nonempty subsets $S' \subseteq S$.

Base case: compute $\Pi(S')$ for $|S'| = 1$ (Algorithm 1) as orbits under a cyclic subgroup:

$$\Pi(S') = \{\langle S' \rangle x \mid x \in X\}.$$

Induction step: compute $\Pi(S')$ for $|S'| > 1$ (Algorithm 2) as the meet of two partitions:

$$\Pi(S') = \Pi(S'') \land \Pi(S' \setminus S'') \quad \text{for any } S'' \subset S'.$$

In the base case, every base partition is generated through orbits identification (or more precisely, orbits tracing), which however does not require any endless checks since there is only one generator. As a result, the computational complexity of Algorithm 1 is linear rather than quadratic in the size of the set $X$. The correctness of the induction step is backed by Theorem 36 and the strong duality in Theorem 8, or more explicitly,

$$\Pi(S') = \pi \circ \pi'(S'' \cup (S' \setminus S''))$$
$$= \pi(\pi'(S'') \lor \pi'(S' \setminus S''))$$
$$= \pi \circ \pi'(S'') \land \pi \circ \pi'(S' \setminus S'')$$
$$= \Pi(S'') \land \Pi(S' \setminus S'').$$

It is the meet operation that successfully bypasses the endless checks in the naive implementation.
Input: a generator \( s \) and a set \( X \)
Output: the base partition \( \Pi(\{s\}) \)

Function \( \text{BasePartn}(s) \):
- Initialize label id: \( l = 0 \);
- For each point \( x \in X \) do
  - If \( x \) is not labelled then
    - Initialize a new orbit: \( O = \{x\} \);
    - Transform: \( y = s(x) \);
    - While \( y \in X \) and \( y \neq x \) and \( y \) is not labelled do
      - Enlarge the orbit: \( O = O \cup \{y\} \);
      - Transform: \( y = s(y) \);
    end
  - If \( y \notin X \) or \( y = x \) then
    - Create a new label: \( l = l + 1 \);
  end
- If \( y \) is labelled then
  - Use \( y \)'s label: \( l = y \)'s label;
end
- Label every point in the orbit \( O \) by \( l \);
return the partition according to the labels;

Algorithm 1: Computing base partitions by identifying orbits: \( O(|X|) \).

Input: two partitions \( P \) and \( Q \) of a set \( X \)
Output: the meet \( P \land Q \), i.e. finest common coarsening of \( P \) and \( Q \)

Function \( \text{Meet}(P, Q) \):
- For each cell \( Q \in Q \) do
  - Initialize a new cell: \( P_{\text{merge}} = \emptyset \);
  - For each cell \( P \in P \) do
    - If \( P \cap Q \neq \emptyset \) then
      - Merge: \( P_{\text{merge}} = P_{\text{merge}} \cup P \);
      - Remove: \( P = P \setminus \{P\} \);
    end
  end
- Insert: \( P = P \cup \{P_{\text{merge}}\} \);
return \( P \);

Algorithm 2: Computing partitions generated from more than one generators inductively by taking the meet of two partitions computed earlier: \( O(|P||Q|) \). Normally, all base partitions should be already computed and cached before running the induction steps.

5.3 Finding a Generating Set of \( \text{ISO}(\mathbb{Z}^n) \)

We give an example of finding a finite generating set. The key idea is based on recursive group decompositions. In light of storing abstractions of a set \( X \) in digital computers, we consider the discrete space \( X = \mathbb{Z}^n(\subseteq \mathbb{R}^n) \). Further, we restrict our attention to generators that are isometries of \( \mathbb{Z}^n \), since \( \text{ISO}(\mathbb{Z}^n) \) is finitely generated. We show this by explicitly finding a finite generating set of \( \text{ISO}(\mathbb{Z}^n) \).

Recall that (in Section 4.3.2) we presented one of the characterizations of \( \text{ISO}(\mathbb{Z}^n) \) as

\[
\text{ISO}(\mathbb{Z}^n) = \langle T(\mathbb{Z}^n) \cup R(\mathbb{Z}^n) \rangle \quad \text{where} \quad T(\mathbb{Z}^n) \cap R(\mathbb{Z}^n) = \{\text{id}\}. \tag{2}
\]
We start from this characterization, and seek a generating set of $T(\mathbb{Z}^n)$ and a generating set of $R(\mathbb{Z}^n)$. Finding generators of $T(\mathbb{Z}^n)$ is easy: $T(\mathbb{Z}^n) = \langle t'_e \cup \cdots \cup t'_{e_n} \rangle$. However, finding generators of $R(\mathbb{Z}^n)$ requires more structural inspections. The strategy is to first study the matrix group $O_n(\mathbb{Z})$ which is isomorphic to $R(\mathbb{Z}^n)$, and then transfer results to $R(\mathbb{Z}^n)$. Interestingly, $O_n(\mathbb{Z})$ has a decomposition similar to what $\text{ISO}(\mathbb{Z}^n)$ has in Equation (2). By definition, $O_n(\mathbb{Z})$ consists of all orthogonal matrices with integer entries. For any $A \in O_n(\mathbb{Z})$, the orthogonality and integer-entry constraints restrict every column vector of $A$ to be a unique standard basis vector or its negation. This will lead to the decomposition of $R(\mathbb{Z}^n)$.

Notations. $1 = (1, \ldots, 1) \in \mathbb{R}^n$ is the all-ones vector; $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$ where $e_i \in \{0, 1\}^n$ has a 1 in the $i$th coordinate and 0s elsewhere; $\nu_1, \ldots, \nu_n$ are the so-called unit negation vectors of $\mathbb{R}^n$ where $\nu_i \in \{-1, 1\}^n$ has a $-1$ in the $i$th coordinate and 1s elsewhere.

Definition 38 (Permutation) A permutation matrix is a matrix obtained by permuting the rows of an identity matrix; we denote the set of all permutation matrices by $P_n$. A permutation of an index set is a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$; the set of all permutations of the size-$n$ index set is known as the symmetric group $S_n$. A permutation of (integer-valued) vectors is a rotation $t'_P : \mathbb{Z}^n \to \mathbb{Z}^n$ for some $P \in P_n$; we denote the set of all permutations of $n$-dimensional vectors by $R_P(\mathbb{Z}^n) \subseteq R(\mathbb{Z}^n)$.

Definition 39 (Negation) A (partial) negation matrix is a diagonal matrix whose diagonal entries are drawn from $\{-1, 1\}$; we denote the set of all $n \times n$ negation matrices by $N_n$. A (partial) negation of (integer-valued) vectors is a rotation $t'_N : \mathbb{Z}^n \to \mathbb{Z}^n$ for some $N \in N_n$; we denote the set of all negations of $n$-dimensional vectors by $R_N(\mathbb{Z}^n) \subseteq R(\mathbb{Z}^n)$.

Remark 40 Under Definitions 38 and 39, one can verify that a permutation (of vectors) maps $x$ to $Px$ by permuting $x$’s coordinates according to $P \in P_n$; likewise, a negation (of vectors) maps $x$ to $Nx$ by negating $x$’s coordinates according to $N \in N_n$.

Theorem 41 We have the following characterizations of permutations and negations:

$$(R_P(\mathbb{Z}^n), \circ) \cong (P_n, \cdot) \cong (S_n, \circ) \quad \text{and} \quad (R_N(\mathbb{Z}^n), \circ) \cong (N_n, \cdot).$$

In particular, these imply that $|R_P(\mathbb{Z}^n)| = |P_n| = |S_n| = n!$ and $|R_N(\mathbb{Z}^n)| = |N_n| = 2^n$.

Proof It is an exercise to check that all entities in the theorem are indeed groups.

Let $\phi_P : R_P(\mathbb{Z}^n) \to P_n$ be the function given by $\phi_P(t'_P) = P$, for any $t'_P \in R_P(\mathbb{Z}^n)$. For any $r'_P, r'_Q \in R_P(\mathbb{Z}^n)$, if $\phi_P(r'_P) = \phi_P(r'_Q)$, i.e. $P = Q$, then $r'_P = r'_Q$, so $\phi_P$ is injective. For any $P \in P_n$, $t'_P \in R_P(\mathbb{Z}^n)$ and $\phi_P(t'_P) = P$, so $\phi_P$ is surjective. Further, for any $r'_P, r'_Q \in R_P(\mathbb{Z}^n)$, $\phi_P(r'_P \circ r'_Q) = \phi_P(r'_P) \cdot \phi_P(r'_Q) = P \cdot Q = \phi_P(r'_P) \cdot \phi_P(r'_Q)$, so $\phi_P$ is a homomorphism. Now we see that $\phi_P$ is an isomorphism. So, $(R_P(\mathbb{Z}^n), \circ) \cong (P_n, \cdot)$.

Let $\phi_S : S_n \to P_n$ be the function given by $\sigma \mapsto P^\sigma$, where $P^\sigma$ is an $n \times n$ permutation matrix obtained by permuting the rows of the identity matrix according to $\sigma$, i.e.

$$P^\sigma_{ij} = \begin{cases} 1 & i = \sigma(j) \\ 0 & i \neq \sigma(j) \end{cases} \quad \text{for any } i,j \in \{1, \ldots, n\}.$$
For any \( \sigma, \mu \in S_n \), if \( \phi_S(\sigma) = \phi_S(\mu) \), i.e. \( P^\sigma = P^\mu \), then \( \sigma(j) = \mu(j) \) for all \( j \in \{1, \ldots, n\} \), i.e. \( \sigma = \mu \), so \( \phi_S \) is injective. For any \( P \in P_n \), let \( \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \) be the function given by \( \sigma(j) \in \{i | P_{ij} = 1\} \), which is well-defined since \( \{i | P_{ij} = 1\} \) is a singleton for all \( j \in \{1, \ldots, n\} \) given that \( P \in P_n \) is a permutation matrix. It is clear that \( \sigma \in S_n \), and \( \phi_S(\sigma) = P \). So, \( \phi_S \) is surjective. Further, for any \( \sigma, \mu \in S_n \), \( \phi_S(\sigma \circ \mu) = P^\sigma \cdot P^\mu = \phi_S(\sigma) \cdot \phi_S(\mu) \) where the second equality holds because for all \( i, j \in \{1, \ldots, n\} \),

\[
(P^\sigma \cdot P^\mu)_{ij} = \sum_{k=1}^{n} P^\sigma_{ik} \cdot P^\mu_{kj} = \begin{cases} 1 & i = \sigma \circ \mu(j) \\ 0 & i \neq \sigma \circ \mu(j) \end{cases} = P^{\sigma \circ \mu}_{ij},
\]

so \( \phi_S \) is a homomorphism. Now we see that \( \phi_S \) is an isomorphism. So, \((S_n, \circ) \cong (P_n, \cdot)\).

Let \( \phi_N : R_N(Z^n) \to N_n \) be the function given by \( \phi_N(r'_N) = N \), for any \( r'_N \in R_N(Z^n) \). For any \( r'_N, r'_M \in R_N(Z^n) \), if \( \phi_N(r'_N) = \phi_N(r'_M) \), i.e. \( N = M \), then \( r'_N = r'_M \), so \( \phi_N \) is injective. For any \( N \in R_N(Z^n) \), \( r'_N \in R_N(Z^n) \) and \( \phi_N(r'_N) = N \), so \( \phi_N \) is surjective. Further, for any \( r'_N, r'_M \in R_N(Z^n) \), \( \phi_N(r'_N \cdot r'_M) = \phi_N(r'_N) \cdot \phi_N(r'_M) \), so \( \phi_N \) is a homomorphism. Now we see that \( \phi_N \) is an isomorphism. So, \((R_N(Z^n), \circ) \cong (N_n, \cdot)\).

**Theorem 42** We have the following characterization of \( O_n(Z) \):

\[
O_n(Z) = \langle N_n \cup P_n \rangle \quad \text{where} \quad N_n \cap P_n = \{I\}.
\]

**Proof** We first show that \( N_n, P_n \leq O_n(Z) \). \((O_n(Z), \cdot)\) is a group since matrix multiplication \( \cdot \) is associative, \( I \in O_n(Z) \) is the identity element, and for any \( A \in O_n(Z) \), \( A^T \in O_n(Z) \) is its inverse. Pick any \( N \in N_n \), then \( N \in Z^{n \times n} \) and \( N^T N = N N = I \), so \( N \in O_n(Z) \), which implies that \( N_n \subseteq O_n(Z) \). Pick any \( P \in P_n \), then \( P = [e_\sigma(1), \ldots, e_\sigma(n)] \in Z^{n \times n} \) for some \( \sigma \in S_n \) and \( (P^T P)_{ij} = e_\sigma^T(i) e_\sigma(j) = \delta_{ij} \), i.e. \( P^T P = I \), so \( P \in O_n(Z) \), which implies that \( P_n \subseteq O_n(Z) \). Now we perform subgroup tests to show that \( N_n, P_n \leq O_n(Z) \). First, we check that \( 1 \) \( \in N_n \), \( 2 \) for any \( N' \in N_n \), \( NN' = N_n \), \( 3 \) for any \( N \in N_n \), \( N^{-1} = N = N_n \); therefore, \( N_n \leq O_n(Z) \). Second, we check that \( 1 \) \( \in P_n \), \( 2 \) for any \( P, P' \in P_n \), \( PP' = P_n \), \( 3 \) for any \( P \in P_n \), \( P^{-1} = P \) \( \subseteq P_n \); therefore, \( P_n \leq O_n(Z) \).

Now we show that \( N_n \cap P_n = \{I\} \). Pick any \( N \in N_n \setminus \{I\} \) and any \( P \in P_n \). It is clear that \( N \neq P \) since \( N \) has at least one \(-1\) entries while \( P \) has no \(-1\) entries. This implies that \((N_n \setminus \{I\}) \cap P_n = \emptyset \). Further, \( I \in N_n \cap P_n \). Therefore, \( N_n \cap P_n = \{I\} \).

Lastly, we show that \( O_n(Z) = \langle N_n \cup P_n \rangle \). It is clear that \((N_n \cup P_n) \subseteq O_n(Z) \), since \( N_n, P_n \leq O_n(Z) \). Conversely, pick any \( A = [a_1, \ldots, a_n] \in O_n(Z) \) where \( a_i \) denotes the \( i \)th column of \( A \). By definition, \( A^T A = I \), so \( \langle a_i, a_i \rangle = \|a_i\|^2 = 1 \) for \( i \in \{1, \ldots, n\} \), and \( \langle a_i, a_j \rangle = 0 \) for \( i, j \in \{1, \ldots, n\} \) and \( i \neq j \). On the one hand, given \( A \in Z^{n \times n} \), the unit-norm property \( \|a_i\|^2 = 1 \) implies that \( a_i \) is a standard basis vector or its negation, i.e. \( a_i = \pm e_k \) for some \( k \in \{1, \ldots, n\} \). On the other hand, for \( i \neq j \), the orthogonality property \( \langle a_i, a_j \rangle = 0 \) implies that for some \( k \neq k' \), \( a_i = \pm e_k \) and \( a_j = \pm e_{k'} \). Thus, there exist some vector alpha \( \alpha = (a_1, \ldots, a_n) \in \{1, -1\}^n \) and some permutation \( \sigma \in S_n \) such that \( A = [a_1 e_\sigma(1), \ldots, a_n e_\sigma(n)] = \text{diag}(\alpha) [e_\sigma(1), \ldots, e_\sigma(n)] \) \( \in NP \), where \( N = \text{diag}(\alpha) \in N_n \) and \( P = [e_\sigma(1), \ldots, e_\sigma(n)] \in P_n \). This implies \( A \in \langle N_n \cup P_n \rangle \). So, \((O_n(Z), \cdot) \subseteq \langle N_n \cup P_n \rangle \).
Corollary 43 We have the following characterization of $\mathbb{R}(\mathbb{Z}^n)$:

$$\mathbb{R}(\mathbb{Z}^n) = \langle \mathbb{R}_N(\mathbb{Z}^n) \cup \mathbb{R}_P(\mathbb{Z}^n) \rangle$$

where $\mathbb{R}_N(\mathbb{Z}^n) \cap \mathbb{R}_P(\mathbb{Z}^n) = \{\text{id}\}$. \hspace{1cm} (3)

Remark 44 The decomposition of the rotation group $\mathbb{R}(\mathbb{Z}^n)$ in Equation (3) has a similar form compared to the decomposition of the isometry group $\text{ISO}(\mathbb{Z}^n)$ in Equation (2). One can show that $\mathbb{R}(\mathbb{Z}^n)$ has a second characterization that is similar to the second characterization of $\text{ISO}(\mathbb{Z}^n)$, where $\mathbb{R}(\mathbb{Z}^n)$ can also be decomposed as semi-direct products:

$$\mathbb{R}(\mathbb{Z}^n) = \mathbb{R}_N(\mathbb{Z}^n) \circ \mathbb{R}_P(\mathbb{Z}^n) \cong \mathbb{N}_n \times \mathbb{P}_n.$$  

However, this characterization is not used in this paper, so we omit its proof.

By Corollary 43, finding generators of $\mathbb{R}(\mathbb{Z}^n)$ breaks down into finding those of $\mathbb{R}_N(\mathbb{Z}^n)$ and $\mathbb{R}_P(\mathbb{Z}^n)$, respectively. First, from unit negation vectors, we can find generators for negations:

$$\mathbb{R}_N(\mathbb{Z}^n) = \langle \{r'_{\text{diag}(\nu_1)}, \ldots, r'_{\text{diag}(\nu_n)}\} \rangle.$$  

Second, from the fact that the symmetric group is generated by 2-cycles of the form $(i, i+1)$: $S_n = \langle \{(1, 2), \ldots, (n-1, n)\}\rangle$ (Conrad, 2016), we can find generators for permutations:

$$\mathbb{R}_P(\mathbb{Z}^n) = \langle \{r'_{p(1,2)}, \ldots, r'_{p(n-1,n)}\} \rangle,$$

where $P^{(i,i+1)} \in \mathbb{P}_n$ is obtained by swapping the $i$th and $(i+1)$th rows of $I$. Finally,

$$\text{ISO}(\mathbb{Z}^n) = \langle T(\mathbb{Z}^n) \cup \mathbb{R}(\mathbb{Z}^n) \rangle$$

$$= \langle T(\mathbb{Z}^n) \cup \mathbb{R}_N(\mathbb{Z}^n) \cup \mathbb{R}_P(\mathbb{Z}^n) \rangle$$

$$= \langle T_0 \cup \mathbb{R}_N^0 \cup \mathbb{R}_P^0 \rangle,$$  \hspace{1cm} (4)

where $T_0 := \{r'_{e_i}\}_{i=1}^n$, $R_{N0} := \{r'_{\text{diag}(\nu_i)}\}_{i=1}^n$, and $R_{P0} := \{r'_{P(i,i+1)}\}_{i=1}^{n-1}$. Here, we performed recursive group decompositions to yield the generating set $T_0 \cup R_{N0} \cup R_{P0}$ with finite size $n + n + (n - 1) = 3n - 1$. This verifies that $\text{ISO}(\mathbb{Z}^n)$ is indeed finitely generated.

5.4 Trade-off: Minimality or Diversity (Efficiency or Expressiveness)

A generating set of a group is not unique. There are two extremes when considering the size of a generating set. One considers the largest generating set of a group which is the group itself; the other considers a minimal generating set which is not unique either.

Definition 45 Let $G$ be a group, $S \subseteq G$, and $\langle S \rangle$ be the subgroup of $G$ generated by $S$. We say that $S$ is a minimal generating set (of $\langle S \rangle$) if for any $s \in S$, $\langle S \setminus \{s\} \rangle \neq \langle S \rangle$.

Theorem 46 Let $G$ be a group and $S \subseteq G$, then $S$ is a minimal generating set if and only if for any $s \in S$, $s \notin \langle S \setminus \{s\} \rangle$.

Proof See Appendix B.10 \hspace{1cm} ■
According to $\text{ISO}(\mathbb{Z}^n) = \langle T_0 \cup R_{\mathbb{N}_0} \cup R_{\mathbb{P}_0} \rangle$, it is easy to check that $T_0$, $R_{\mathbb{N}_0}$, and $R_{\mathbb{P}_0}$ are minimal individually; whereas their union is not. Nevertheless, it is not hard to show that $S^* := \{t'_{e_1}, r'_{\mathrm{diag}(\nu_1)} \} \cup R_{\mathbb{P}_0}$ is a minimal generating set of $\text{ISO}(\mathbb{Z}^n)$ with size $n + 1$.

There is a trade-off between minimality and diversity, which further leads to the trade-off between efficiency and expressiveness. Again, we use $\text{ISO}(\mathbb{Z}^n)$ as an example. From one extreme, a minimal generating set is most efficient in the following sense: $S \subseteq \text{ISO}(\mathbb{Z}^n)$ is a minimal generating set (of $\langle S \rangle$) if and only if $\pi'|_S$ is injective, i.e. for any $S', S'' \subseteq S$, if $S' \neq S''$, then $\pi'(S') \neq \pi'(S'') = \pi'(S'')$ (an easy check). Therefore, whenever $S$ is not minimal, there are duplicates in the generated subgroups, and thus duplicates in the subsequent abstraction generations. Every occurrence of a duplicate is a waste of computing power since it does not produce a new abstraction in the end. Intuitively, if a generating set is further away from being minimal, then more duplicates tend to occur and the abstraction generating process is less efficient. To the other extreme, the largest generating set is most expressive in the following sense: if $S = \text{ISO}(\mathbb{Z}^n)$, then $\pi'(S^*) = H_{\text{ISO}(\mathbb{Z}^n)}$ i.e. the collection of all subgroups of $\text{ISO}(\mathbb{Z}^n)$; and in general, for any $S \subset S_+ \subseteq \text{ISO}(\mathbb{Z}^n)$, the monotonicity property $\pi'(S^*) \subseteq \pi'(S^+)$ holds (an easy check). However, the largest generating set is also the least efficient not only because it has the largest number of duplicates, but in this case, it is infinite. Thus, to respect the trade-off between efficiency and expressiveness, we need to find a balance between the two extremes.

Our plan is to start from a minimal generating set $S^*$ of $\text{ISO}(\mathbb{Z}^n)$ and then gradually enlarge it by adding the so-called derived generators. In other words, we aim for a filtration: $S^* \subseteq S^+_{1} \subseteq S^+_{2} \subseteq S^+_{3} \subseteq \cdots$ such that the corresponding collections of subgroups satisfy

$$
\pi'(2^{S^*}) \subseteq \pi'(2^{S^+_{1}}) \subseteq \pi'(2^{S^+_{2}}) \subseteq \pi'(2^{S^+_{3}}) \subseteq \cdots \quad \text{and} \quad \bigcup_{m=1}^{\infty} \pi'(2^{S^+_{m}}) = H_{\text{ISO}(\mathbb{Z}^n)}.
$$

**Definition 47** Let $S^*$ be a minimal generating set of $\text{ISO}(\mathbb{Z}^n)$, and define

$$
S^+_{m} := \{s^\alpha_{k_1} \circ \cdots \circ s^\alpha_{1} \mid k \in \mathbb{Z}_{\geq 0}, s_k, \ldots, s_1 \in S^*, \alpha_k, \ldots, \alpha_1 \in \mathbb{Z}, \sum_{i=1}^{k} |\alpha_i| \leq m \}.
$$

A derived generator of length $m$ is an $s \in S^+_{m+1} \setminus S^+_{m}$.

**Remark 48** In Definition 47, $S^+_{m}$ is the “ball” with center $id$ and radius $m$ in the Cayley graph of $S^* \cup (S^*)^{-1}$. It is an easy check that $S^* \cup (S^*)^{-1} = S^+_{1} \subseteq S^+_{2} \subseteq S^+_{3} \subseteq \cdots$.

Note that $\bigcup_{m=1}^{\infty} S^+_{m} = \langle S^* \rangle = \text{ISO}(\mathbb{Z}^n)$, since the growing “ball” will eventually cover the whole Cayley graph. Therefore, $\bigcup_{m=1}^{\infty} \pi'(2^{S^+_{m}}) = H_{\text{ISO}(\mathbb{Z}^n)}$. This suggests we gradually add derived generators of increasing length to $S^*$, and approximate $H_{\text{ISO}(\mathbb{Z}^n)}$ by $\pi'(2^{S^+_{m}})$ for some large $m$. Without any prior preference, one must go through this full procedure to grow the ball $S^+_{m}$ from radius $m = 1$. Although computationally intense, it is incremental. More importantly, this is a one-time procedure, but the resulting abstraction (semi)universe is universal: computed abstractions can be used in different topic domains.

However, just like biological perception systems which have innate preference for certain stimuli, having prior preference for certain derived generators can make the abstraction (semi)universe grow more efficiently. As an illustrative example and a design choice, we
Adding these circulators and synchronizers to \( R \). Inspecting circulators of the orbit \( \langle s^\alpha \rangle \) consists of periodic points from \( \langle s \rangle x \). If \( \langle S \rangle \) is Abelian, the synchronizer of \( S \) is: \( s_k \circ \cdots \circ s_1 \).

We denote the set of all circulators of \( T_0 \) with a fixed period \( \alpha \) by \( T_0^\alpha \). Inspecting circulators of \( R_{N0} \) and \( R_{P0} \) does not yield new generators, since for any \( s \in R_{N0} \cup R_{P0} \), \( s^2 = \text{id} \). The synchronizers of \( T_0 \) and \( R_{N0} \) are \( t' \) and \( t'_1 \) (\( (R_{P0}) \) is not Abelian). Adding these circulators and synchronizers to \( T_0 \cup R_{N0} \cup R_{P0} \) yields the generating set:

\[
S^*_+ := T_0 \cup T_0^2 \cup \cdots \cup T_0^n \cup \{t'_1\} \cup R_{N0} \cup \{t'_1\} \cup R_{P0},
\]

where \( \tau \) denotes an upper bound on period exploration. Note that \( |S^*_+| = \tau n + 1 + n + 1 + (n - 1) = (\tau + 2)n + 1 \). In light of real applications, we can use this generating set to generate an abstraction semiuniverse for automatic music concept learning.

6. Implementation Heads-Up: Restriction to Finite Subspaces

Computers have to work with finite spaces for finite execution time. If the underlying space \( X \) is finite, then there is no issue. However, if \( X \) is infinite (but still discrete) like \( \mathbb{Z}^n \), we have to consider a finite subspace of \( X \) in practice. In this case, we must be careful about both what an abstraction of a subspace means and what potential problems might occur.

**Definition 50** Let \( X \) be a set and \( \mathcal{P} \) be an abstraction of \( X \). For any \( Y \subseteq X \), the restriction of \( \mathcal{P} \) to \( Y \) is an abstraction of \( Y \) given by \( \mathcal{P}|_Y := \{P \cap Y \mid P \in \mathcal{P}\} \setminus \{\emptyset\} \).

**Remark 51** Unless otherwise stated, the term “an abstraction of a subspace” means an abstraction of the ambient space restricted to that subspace. Under this definition, we need extra caution when computing an abstraction of a subspace.

Let \( X \) be a set, and \( H \leq F(X) \) be a subgroup of the transformation group of \( X \). For any \( Y \subseteq X \), according to Definition 50, the correct way of generating the abstraction of the subspace \( Y \) from \( H \) is:

\[
\pi(H)|_Y = (X/H)|_Y = \{hx \cap Y \mid x \in X \}\setminus\{\emptyset\}.
\]

A risky way of computing the abstraction of the subspace \( Y \) is by thinking only in \( Y \) while forgetting the ambient space \( X \). The risk here is to get a partition of \( Y \), denoted \( R^H_Y \), which is strictly finer than \( \pi(H)|_Y = R^H_X|_Y \). In other words, there are possibly cells in \( R^H_Y \).
that should be merged but are not if they are connected via points outside the subspace $Y$. For instance, consider $X = \mathbb{Z}^2$, $Y = \{(0,0), (1,0), (0,1), (1,1)\} \subseteq X$, and the subgroup $H = \langle \langle t_1', r_{-1}' \rangle \rangle \leq \text{ISO}(\mathbb{Z}^2) \leq \mathbb{F}(\mathbb{Z}^2)$. Let $\mathcal{R}_Y^H$ be the abstraction of $Y$ obtained by running the induction algorithm on $Y$ (instead of $X$) in the bottom-up approach. One can check:

$$\mathcal{R}_Y^H = \{ \{(0,0),(1,1)\}, \{(0,1)\}, \{(1,0)\} \};$$

$$\pi(H)|_Y = \{ \{(0,0),(1,1)\}, \{(0,1),(1,0)\} \}.$$ 

The two points $(1,0)$ and $(0,1)$ should be in one cell since $(1,0) \xrightarrow{r_{-1}' \cdot} (-1,0) \xrightarrow{t_1'} (0,1)$, but are not in $\mathcal{R}_Y^H$ since the via-point $(-1,0) \notin Y$. In general, the risk is present if we compute an abstraction of a subspace $Y$ from other abstractions of $Y$ or from orbit tracing.

However, for computational reasons, we want to forget the ambient space $X$! In particular, the risky way is the only practical way if $X$ is infinite and $Y$ is finite, since it is not realistic to identify all orbits in an infinite space. This suggests that we take the risk to generate $\mathcal{R}_Y^H$ as the first step, and rectify the result in a second step to merge cells that are missed in the first step. As a result, we introduce a technique called “expand-and-restrict”.

### 6.1 Expand-and-Restrict

“Expand-and-restrict” is an empirical technique which first expands the subspace and then restricts it back, i.e. to compute $\mathcal{R}_Y^H|_Y$ for some finite subspace $Y_+$ such that $Y \subset Y_+ \subset X$. The expansion $Y_+$ takes more via-points into consideration, so it helps merge cells that are missed in $\mathcal{R}_Y^H$. In practice, we carry out this technique gradually in a sequential manner, which is similar to what we did in enlarging a minimal generating set (cf. Section 5.4).

Given an infinite space $X$ and a finite subspace $Y \subset X$, we first construct a filtration

$$Y = Y_{+0} \subset Y_{+1} \subset Y_{+2} \subset \cdots \subset X \quad \text{where } Y_{+k} \text{ is finite } \forall k \in \mathbb{Z}_{\geq 0} \text{ and } \bigcup_{k=0}^{\infty} Y_{+k} = X.$$ 

We then start a search process for a good expansion $Y_{+k}$. More specifically, we iteratively compute $\mathcal{R}_{Y_{+k}}^H|_Y$ for expansion factors $k = 0, 1, 2, \ldots$ until the results reach a consensus among consecutive iterations. To determine a consensus, theoretically, we need to find the smallest $k$ such that $\mathcal{R}_{Y_{+k}}^H|_Y = \mathcal{R}_{Y_{+k+1}}^H|_Y$ for all $k' > k$, which requires an endless search. In practice, we stop the search whenever $\mathcal{R}_{Y_{+k}}^H|_Y = \mathcal{R}_{Y_{+k+1}}^H|_Y = \cdots = \mathcal{R}_{Y_{+(k+\Delta k)}}^H|_Y$ for some positive integer $\Delta k$. We call this an early stop, whose resulting abstraction $\mathcal{R}_{Y_{+k}}^H|_Y$ is an empirical approximation of the true abstraction $\pi(H)|_Y$. Note that without early stopping, we will have the correct result $\pi(H)|_Y = \mathcal{R}_X^H|_Y$ in the limit of this infinite search process. Therefore, even in cases where the space $X$ is finite, if $X$ is much larger than the subspace $Y$, this empirical search can be more efficient than computing $\pi(H)|_Y$ directly, since earlier search iterations will be extremely cheap and if an early stop happens early there is a win.

### 6.2 An Implementation Example

We give an example to illustrate some implementation details on generating abstractions of a finite subspace. In this example, we consider finite subspaces of $X = \mathbb{Z}^n$ to be the centered hypercubes of the form $Y = \mathbb{Z}_{[-b,b]}^n$ where $\mathbb{Z}_{[-b,b]} := \mathbb{Z} \cap [-b,b]$ and $b > 0$ is finite.
To construct an abstraction semiuniverse for such a finite hypercube, we adopt the bottom-up approach and pick the generating set to be $S^*_n$ defined in Equation (5). Taking $S^*_n$ and $\mathbb{Z}_n^{[−b,b]}$ as inputs, we run the induction algorithm, where both Algorithm 1 (for base cases) and Algorithm 2 (for the induction step) are run on the finite subspace $\mathbb{Z}_n^{[−b,b]}$ instead of the infinite space $\mathbb{Z}^n$. This is the first step which gives abstractions $\mathcal{R}^{(S)}_{\mathbb{Z}^n_{[−b,b]}}$ for $S \subseteq S^*_n$.

As mentioned earlier, for every $S \subseteq S^*_n$, the correct abstraction should be $\mathcal{R}^{(S)}_{\mathbb{Z}^n_{[−b,b]}}$, which is generally not equal to $\mathcal{R}^{(S)}_{\mathbb{Z}^n_{[−b,b]}}$. So, we run the “expand-and-restrict” technique as the second step. We first construct a filtration: let $Y_{+k} = \mathbb{Z}_n^{[−b−k,b+k]}$ be a finite expansion of $Y = \mathbb{Z}_n^{[−b,b]}$, then it is clear that $Y = Y_{+0} \subset Y_{+1} \subset Y_{+2} \subset \cdots \subset X$ and $\cup_{k=0}^{\infty} Y_{+k} = X = \mathbb{Z}^n$. We then start the empirical search process and set $\Delta k = 1$ (the most greedy search). This means we will stop the search whenever $\mathcal{R}^{(S)}_{Y_{+k}}|_Y = \mathcal{R}^{(S)}_{Y_{+(k+1)}}|_Y$, and return the abstraction $\mathcal{R}^{(S)}_{Y_{+k}}|_Y = \mathcal{R}^{(S)}_{\mathbb{Z}_n^{[−b−k,b+k]}}|_{\mathbb{Z}_n^{[−b,b]}}$ as the final result to approximate $\mathcal{R}^{(S)}_{\mathbb{Z}^n_{[−b,b]}} = \Pi(S)|_{\mathbb{Z}^n_{[−b,b]}}$.

There are three additional implementation tricks that are special to this example. The first trick applies to cases where the subspace $\mathbb{Z}_n^{[−b,b]}$ is large, i.e. a large $b$. In this case, every search iteration in the “expand-and-restrict” technique is expensive and gets more expensive as the search goes. However, for the generating set $S^*_n$ specifically, it is typical to have $b \gg \tau$ so as to reveal strong periodic patterns. Thus, we run the entire two-step abstraction generating process for $\mathbb{Z}_n^{[−\tau,\tau]}$ instead of $\mathbb{Z}_n^{[−b,b]}$, pretending $\mathbb{Z}_n^{[−\tau,\tau]}$ is the subspace that we want to abstract. This yields a much faster abstraction process since $\mathbb{Z}_n^{[−\tau,\tau]}$ is much smaller than $\mathbb{Z}_n^{[−b,b]}$. The result is an abstraction $\mathcal{R}^{(S)}_{\mathbb{Z}_n^{[−\tau,\tau]}}|_{\mathbb{Z}_n^{[−b,b]}}$ for some expansion factor $k$.

We reuse this same $k$ and compute $\mathcal{R}^{(S)}_{\mathbb{Z}_n^{[−\tau−k,\tau+k]}}|_{\mathbb{Z}_n^{[−b,b]}}$ as the final result, which is the only expensive computation. Note that this trick adds an additional empirical approximation, assuming that the same expansion factor $k$ works for both small and large subspaces. While we have not yet found a theoretical guarantee for this assumption, this trick works well in practice, and provides huge computational savings.

Note: for some generating subsets $S \subseteq S^*_n$, we can prove (so no approximations) that the expansion factor $k = 0$ (no need to expand) or 1. Although this provides theoretical guarantees in certain cases, the tricks used in the current proofs are case-by-case depending on the chosen generators. Thus, before we find a universal way of proving things, we prefer empirical strategies—like the above search process—which work universally in any event.

The second trick considers the subspace to be any general hypercube in $\mathbb{Z}^n$, which is not necessarily square or centered. The trick here is simply to find a minimum centered square hypercube containing the subspace. If the ambient space $X$ happens to be “spatially stationary”—the absolute location of each element in the space is not important but only their relative position matters (e.g. the space of music pitches)—then we find a minimum square hypercube containing the subspace and center it via a translation. Centering is very important and specific to the chosen generating set $S^*_n$. This is because $S^*_n$ contains only pure translations and pure rotations; and centering square hypercubes makes pure rotations safe: no rotation maps a point in $\mathbb{Z}_n^{[−b,b]}$ outside (one can check that for any $r_A' \in R(\mathbb{Z}^n)$, $r_A'(\mathbb{Z}_n^{[−b,b]}) = \mathbb{Z}_n^{[−b,b]}$). In practice, centering dramatically decreases the number of miss-
merged cells, and makes it safe to choose small $\Delta k$ for early stopping. This explains why we only consider subspaces of the form $\mathbb{Z}_n^{[-b,b]}$ in the first place, and boldly choose $\Delta k = 1$.

The third trick considers a quick-and-dirty pruning of duplicates in generating a family of partitions, leaving room for larger-period explorations. Without this trick, to generate the partition family $\Pi(2^{S^*_+})$, we need $|2^{S^*_+}| = 2^{(r+2)n+1} = O(2^r)$ computations, which hinders exploration on period $\tau$. However, $S^*_+$ is not minimal, so $|\Pi(2^{S^*_+})| < 2^{S^*_+}$, suggesting many computations are not needed since they yield the same abstraction. We focus on circulators, where we exclude computations on those $S \in 2^{S^*_+}$ containing multiple periods. This reduces the number of computations to $(2^{3n+1} - 2^{2n+1})\tau + 2^{2n+1} = O(\tau)$.

A real run on the subspace $\mathbb{Z}_4^{[-12,12]}$ and $\tau = 4$ computes 31232 partitions, during which all search processes in “expand-and-restrict” end in at most three iterations. This means in this experiment we only need to expand the subspace by $k = 0$ or 1 for all abstractions in the family.

Lastly, we briefly mention the task of completing a global hierarchy on an abstraction family $\mathcal{Y}$. A brute-force algorithm makes $O(|\mathcal{Y}|^2)$ comparisons, determining the relation ($\preceq$ or incomparable) for every unordered pair of partitions $P, Q \in \mathcal{Y}$. Locally, we run a subroutine $GetRelation(P,Q)$ implemented via the contingency table (Hubert and Arabie, 1985) whenever we want to query a pair of partitions. Globally, we use two properties to reduce the number of calls to $GetRelation(P,Q)$: 1) transitivity: for any $P, P', P'' \in \mathcal{Y}$, $P \preceq P'$ and $P' \preceq P''$ implies $P \preceq P''$; 2) dualities in Theorem 8. The final output of our abstraction process is a directed acyclic graph of the abstraction (semi)universe $\Pi(2^{S^*_+})$. Similar to the first trick above, in practice it suffices to complete the hierarchy for smaller subspaces like $\mathbb{Z}_n^{[-\tau,\tau]}$, assuming the same hierarchy holds for the actual subspace under consideration.

7. Discussion: Information Lattice and Learning

In his 1953 work, Claude E. Shannon attempted to unravel the nature of information beyond just quantifying its amount (Shannon, 1953). In the specific context of communication problems, he coined the term information element to denote the nature of information which is invariant under “(language) translations” or different encoding-decoding schemes. He further introduced a partial order between a pair of information elements, eventually yielding a lattice of information elements, or information lattice in short.

In this section, we first present a brief overview of Shannon’s original work and then cast the information lattice in our abstraction-generation framework without needing to introduce information-theoretic functionals. Our abstraction-generation framework not only generalizes Shannon’s information lattice, but more importantly presents a generating chain that brings learning into the picture. This eventually opens the opportunity for data-driven concept learning which aims to discover human-interpretable rules from sensory data. After the theoretical connections, we present a real implementation of an information lattice from a music application. In this application, we build an automatic music theorist and pedagogue that self-learns music compositional rules and provides people with personalized music composition lessons (Yu et al., 2016; Yu and Varshney, 2017; Yu et al., 2017).
7.1 Theoretical Generalization: a Separation of Clustering and Statistics

We present an overview of Shannon’s original work and a follow-up work by Li and Chong (2011) which formalizes Shannon’s idea in a more principled way.

Nature of information:

information element or random variable?

An information element is an equivalence class of random variables (of a common sample space) with respect to the “being-informationally-equivalent” relation, where two random variables are informationally equivalent if they induce the same \( \sigma \)-algebra (of the sample space). Under this definition, the notion of an information element—essentially a probability space—is more abstract than that of a random variable: an information element can be realized by different random variables. The relationship between different but informationally-equivalent random variables and their corresponding information element is analogous to the relationship between different translations (say, English, French, or a code) of a message and the actual content of the message. Since different but faithful translations are viewed as different ways of describing the same information, the information itself is then regarded as the equivalence class of all translations or ways of describing the same information. Therefore, the notion of information element reveals the nature of information.

Group-theoretic interpretation:

information lattice \( \rightarrow \) partition lattice \( \rightarrow \) subgroup lattice \( (\rightarrow \) interpretation). \( \text{(6)} \)

An information lattice is a lattice of information elements, where the partial order is defined by \( x \leq y \iff H(x|y) = 0 \) where \( H \) denotes the information entropy. The join of two information elements \( x \lor y = x + y \) is called the total information of \( x \) and \( y \); the meet of two information elements \( x \land y = xy \) is called the common information of \( x \) and \( y \). By definition, every information element can be uniquely determined by its induced \( \sigma \)-algebra. Also, it is known that every \( \sigma \)-algebra of a countable sample space can be uniquely determined by its generating (via union operation) sample-space-partition. Thus, an information lattice has a one-to-one correspondence to a partition lattice. Further, given a partition of a sample space, Li and Chong (2011) constructed a unique permutation subgroup whose group action on the sample space produces orbits that coincide with the given partition. Therefore, under this specific construction (which also validates our Theorem 7), any partition lattice has a one-to-one correspondence to the constructed subgroup lattice (see Li and Chong, 2011, General Isomorphism Theorem). This yields the above Chain (6) which further achieves group-theoretic interpretations of various information-theoretic results, bringing together information theory and group theory (Chan and Yeung, 2002; Yeung, 2008, chap. 16).

Now we cast the above results into our framework, and reveal the key differences.

Nature of abstraction:

clustering or classification?

Generalizing Shannon’s insight on the nature of information essentially reveals the difference between clustering and classification in machine learning. We can similarly define an
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| element          | Partition lattice                          | Information lattice                          |
|------------------|--------------------------------------------|---------------------------------------------|
| partition (P)    | information element (x)                    | probability space (X, Σ, P)                  |
| clustering (X, P) | equival. class of classifications          | equival. class of random variables          |
| equiv. class of classifications |                                |                                             |
| partial order    | P ⪯ Q                                      | x ≤ y =⇒ H(x|y) = 0                         |
| meet             | P ∨ Q                                      | x + y                                       |
| metric           | undefined                                  | ρ(x, y) = H(x|y) + H(y|x)                    |

Table 1: Partition lattice and information lattice: the main difference comes from the fact that a partition lattice is not coupled with a measure; whereas an information lattice is coupled with a probability measure, so both the partial order and the metric can be defined in terms of entropies.

equivalence relation on the set of all classifications where two classifications are equivalent if they yield the same set of classes and only differ by class labels. For instance, given a set of animals, classifying them into \{fish, amphibians, reptiles, birds, mammals\} is equivalent to classifying them into \{poisson, amphibiens, reptiles, des oiseaux, mammiferes\}, where the different class labels are only English and French translations of the same set of animal classes. So, clustering to classification is analogous to information element to random variable; and it is clustering rather than classification that captures the nature of abstraction. This again explains why abstraction is formalized as a clustering problem in this paper.

**Partition lattice and information lattice:** We summarize major connections between a partition lattice and an information lattice in Table 1. The differences are rooted in the separation of clustering from statistics, so roughly speaking, a partition lattice—which is measure-free—can be thought as an information lattice without probability measure. Therefore, abstraction is a more general concept than information, which is not specific to communication problems, and in particular, is not attached to any stochastic processes or information-theoretic functionals such as entropy.

**Group-theoretic learning:**

- subgroup lattice → partition lattice → information lattice (→ learning).  

The separation of clustering and statistics is important since it opens the opportunity for interpretable statistical learning, where interpretability is achieved by the explicit construction of a partition lattice (symmetry-generated hierarchical clustering), and learning is achieved by subsequent statistical inference on this lattice. This is more precisely presented in Chain (7) aiming for learning, which at first glance, is merely a reverse process of Chain (6) aiming for re-interpretation. However, the subgroup lattices in both chains are in stark contrast: the subgroups considered in Chain (7) are based on certain symmetries—the underlying mechanism of abstraction—whereas the subgroups considered in Chain (6) are merely (isomorphic) re-statements of the given partitions. In other words, among possibly many subgroups (recall Theorem 6 and 7: π is surjective but not necessarily injective) that generate the same partition, we only pick the one that explains to us the types of symmetries under consideration. The preservation of interpretable symmetries through Chain (7)
Figure 7: MUS-ROVER’s self-learning loop (the kth iteration). The teacher (discriminator) takes as inputs the student’s latest style $p^{(k-1)}_{stu}$ and the input style $\hat{p}$, and identifies an abstraction $\phi$ under which the two styles manifest the largest statistical gap $D(\cdot || \cdot)$. The identified abstraction is then made into a rule (a constraint set $\Gamma_k$), and augments the ruleset $\{\Gamma_i\}_{i=1}^k$. The student (generator) takes as input the augmented ruleset to update its writing style into $p^{(k)}_{stu}$, meanwhile favors novelty, i.e. more possibilities, by maximizing the Tsallis entropy $S_q$ subject to the rule constraints. In short, the teacher extracts rules while the student applies rules; both perform their tasks by solving optimization problems.

makes the subsequent learning transparent. Therefore, when abstraction does meet statistics, it will yield interpretable machine learning and knowledge discovery, which is beyond simply a re-interpretation of known results.

7.2 A Real Application: Automatic Concept Learning in Music

We present a music application called MUS-ROVER from our earlier and ongoing work (Yu et al., 2016; Yu and Varshney, 2017; Yu et al., 2017), where we show how automatic concept learning is achieved in a real information lattice. In MUS-ROVER, each learned music concept takes the form of a compositional rule, which is represented by a partition of the chord space and a probability measure over the partition cells. Therefore, every rule is an instance of an information element, and rules are extracted as learning in a symmetry-generated information lattice.

MUS-ROVER is an automatic music theorist and pedagogue, which self-learns compositional rules from symbolic music datasets and provides personalized composition lessons. Rules are learned from a “teacher ⇌ student” model (Figure 7). This model is implemented by a self-learning loop between a discriminative component (teacher) and a generative component (student), where both entities cooperate to iterate through the rule-learning process. The student starts as a tabula rasa that picks pitches uniformly at random to form chords and chord progressions. In each iteration, the teacher compares the student’s writing style (represented by a probabilistic model) with the input style (represented by empirical statistics) to identify one music abstraction (represented by a partition of the chord space) that best reveals the gap between the two styles; and then make it a rule for the student. Consequently, the student becomes less and less random by obeying more and more rules, and thus, approaches the input style. From its rule-learning process on a dataset consisting of
Bach’s chorales, MUS-ROVER successfully recovered many known rules, such as “Parallel perfect octaves/fifths are rare” and “Tritons are often resolved either inward or outward”.

In MUS-ROVER’s self-learning loop, both the teacher and the student perform their tasks, namely rule extraction and rule realization respectively, by solving optimization problems. For the student, the rule realization problem is about finding the most random probability distribution over the chord space (i.e. maximizing novelty) as long as it satisfies all the probabilistic rules. This is formalized as the following optimization problem:

$$\text{maximize } S_q(x)$$
$$\text{subject to } x \in \Gamma_k, \ k = 1, \ldots, K,$$

where the optimization variable $x$ denotes the probability distribution over the chord space (note: we pre-specify a finite range of the pitches under consideration, e.g. piano range, vocal range, so the chord space is finite and $x$ is a vector), the objective $S_q(x)$ is the Tsallis entropy of $x$ measuring the randomness of $x$, and the constraint sets $\Gamma_1, \ldots, \Gamma_K$ denote $K$ rules learned thus far. We mention two facts here. First, in the limit as $q \to 1$, $S_q(x) \to H(x)$ which is the Shannon entropy. Second, $x \in \Gamma_k$ is more explicitly represented as a linear equation $A_k^k x = y_k$ where the pair $(A_k^k, y_k)$ denotes the $k$th rule. More specifically, $A_k^k$ is a boolean matrix (called a partition matrix) which stores the full information of a partition: $A_{ij}^k = 1$ if and only if the $j$th chord belongs to the $i$th partition cell; $y_k$ is a probability distribution over the partition cells. We slightly overload the notations and let $x$ and $y_k$ be the information elements that represent the probability space $(X, \Sigma(I), x)$ and $(X, \Sigma(A_k), y_k)$, respectively. In this notation, the sample space $X$ is the chord space and, for a partition matrix $P$, $\Sigma(P)$ denotes the $\sigma$-algebra generated by the partition represented by $P$ (so $\Sigma(I)$ denotes the $\sigma$-algebra generated by the finest partition). Under this setting, the equality constraint $A_k^k x = y_k$ becomes $H(y_k|x) = 0$, i.e. $y_k$ is an abstraction of $x$ as information elements or $y_k \leq x$ by Shannon’s definition. Therefore, the student’s optimization problem for rule realization can be rewritten as follows:

$$\text{maximize } H(x)$$
$$\text{subject to } H(y_k|x) = 0, \ k = 1, \ldots, K,$$

which describes an inference problem in an information lattice. In words, this means that we want to find an information lattice for the student (used as its mental model) such that it agrees on all $K$ abstractions from the information lattice for the dataset and meanwhile achieves the largest randomness in the chord space. As a result, a good student memorizes high-level principles—rules in terms of high-level abstractions and their statistical patterns—in music composition rather than the actual pieces. Indeed the student is encouraged to be as creative as possible as long as the high-level principles are satisfied.

Since learning an information lattice requires both the explicit construction of an abstraction (semi)universe and statistical inference from data, the learning paradigm in MUS-ROVER differs from pure rule-based systems or pure data-driven methods in artificial intelligence, creating a middle ground between the two extremes. Developmental cognitive scientists say this resembles the way babies learn from both empirical experience and biological instinct (Hutson, 2018). In this example, the constructed abstraction semiuniverse
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Figure 8: Visualization of Bach’s music mind for writing chorales. The underlying directed acyclic graph signifies an upside-down information lattice. (Note: edges are oriented upwards according to the convention of a partition lattice; the coarsest partition at the bottom is omitted.) Colors are used to differentiate rule activations from different n-gram settings.

resembles biological instincts on perceiving sound, and the statistical inference in this semiuniverse resembles experiential learning driven by the instincts. Therefore, the entire concept learning process is transparent, in contrast with black-box algorithms. In particular, when we run MUS-ROVER on Bach’s four-part chorales, we can actually visualize the learning process in the information lattice as a process that mimics Bach’s mental activities during chorale composition (Figure 8).

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References

Henda Bélaï and Ali Jaoua. Abstraction of objects by conceptual clustering. *Inf. Sci.*, 109 (1-4):79–94, 1998.

Yoshua Bengio. Learning deep architectures for AI. *Found. Trends Mach. Learn.*, 2(1):1–127, 2009.
Ludwig Bieberbach. Über die bewegungsgruppen der euklidischen räume. Math. Ann., 70 (3):297–336, 1911.

Irving Biederman. Recognition-by-components: a theory of human image understanding. Psychol. Rev., 94(2):115, 1987.

Nicolas Bredeche, Zhongzhi Shi, and Jean-Daniel Zucker. Perceptual learning and abstraction in machine learning: an application to autonomous robotics. IEEE Trans. Syst., Man, Cybern. C, 36(2):172–181, 2006.

Alan Bundy, Fausto Giunchiglia, and Toby Walsh. Building Abstractions. University of Edinburgh, Department of Artificial Intelligence, 1990.

Terence Ho Leung Chan and Raymond Yeung. On a relation between information inequalities and group theory. IEEE Trans. Inf. Theory, 48(7):1992–1995, 2002.

Leonard S Charlap. Bieberbach Groups and Flat Manifolds. Springer Science & Business Media, 2012.

Keith Conrad. Generating sets. http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/genset.pdf, 2016.

Richard M Cormack. A review of classification. J. R. Stat. Soc. Ser. A. Gen., 134(3):321–367, 1971.

Thomas G Dietterich. Reflections on innateness in machine learning. https://medium.com/@tdietterich/reflections-on-innateness-in-machine-learning-4eebefa3e1af, 2018.

Richard O Duda, Peter E Hart, and David G Stork. Pattern Classification. John Wiley & Sons, 2012.

Bettina Eick and Bernd Souvignier. Algorithms for crystallographic groups. Int. J. Quantum Chem., 106(1):316–343, 2006.

Volkmar Felsch and Franz Gähler. CrystCat-a library of crystallographic groups. A refereed Gap, 4, 2000.

Douglas H Fisher. Knowledge acquisition via incremental conceptual clustering. Mach. Learn., 2(2):139–172, 1987.

Fausto Giunchiglia and Toby Walsh. A theory of abstraction. Artif. Intell., 57(2-3):323–389, 1992.

Rebecca L Gómez and Laura Lakusta. A first step in form-based category abstraction by 12-month-old infants. Developmental Sci., 7(5):567–580, 2004.

Lawrence Hubert and Phipps Arabie. Comparing partitions. J. Classif., 2(1):193–218, 1985.

Matthew Hutson. How researchers are teaching AI to learn like a child. http://www.sciencemag.org/news/2018/05/how-researchers-are-teaching-ai-learn-child, 2018.
Anil K Jain and Richard C Dubes. *Algorithms for Clustering Data*. Prentice-Hall, Inc., 1988.

Leonard Kaufman and Peter J Rousseeuw. *Finding Groups in Data: An Introduction to Cluster Analysis*, volume 344. John Wiley & Sons, 2009.

Risi Kondor. *Group Theoretical Methods in Machine Learning*. Columbia University, 2008.

Eyal Krupka and Naftali Tishby. Incorporating prior knowledge on features into learning. In *Proc. 10th Int. Conf. Artif. Intell. Stat. (AISTATS 2007)*, pages 227–234, 2007.

Yann LeCun, Yoshua Bengio, and Geoffrey Hinton. Deep learning. *Nature*, 521(7553):436–444, 2015.

David Lewin. *Generalized Musical Intervals and Transformations*. Oxford University Press, 2010.

Hua Li and Edwin KP Chong. On a connection between information and group lattices. *Entropy*, 13(3):683–708, 2011.

Kenneth R Livingston. *Rationality and the Psychology of Abstraction*. Institute for Objectivist Studies, 1998.

C Neil Macrae and Galen V Bodenhausen. Social cognition: Thinking categorically about others. *Annu. Rev. Psychol.*, 51(1):93–120, 2000.

Jean M Mandler. Perceptual and conceptual processes in infancy. *J. Cogn. Develop.*, 1(1):3–36, 2000.

Gary Marcus. Innateness, AlphaZero, and artificial intelligence. arXiv:1801.05667 [cs.AI], 2018.

Ryszard S Michalski and Robert E Stepp. Learning from observation: Conceptual clustering. *Mach. Learn.*, 1:331–363, 1983.

Steve Y Oudot. *Persistence Theory: From Quiver Representations to Data Analysis*, volume 209. American Mathematical Society Providence, RI, 2015.

Rajat Raina, Andrew Y Ng, and Daphne Koller. Constructing informative priors using transfer learning. In *Proc. 23rd Int. Conf. Mach. Learn. (ICML 2006)*, pages 713–720, 2006.

Ashwin Ram and Eric K Jones. *Foundations of Foundations of Artificial Intelligence*. Department of Computer Science, Victoria University of Wellington, 1994.

Ravi K Raman and Lav R Varshney. Universal clustering. In Yonina Eldar and Miguel Rodrigues, editors, *Information-Theoretic Methods in Data Science*. Cambridge University Press, 2018. in preparation.

William M Rand. Objective criteria for the evaluation of clustering methods. *J. Am. Stat. Assoc.*, 66(336):846–850, 1971.
Lior Rokach and Oded Maimon. Clustering methods. In Lior Rokach and Oded Maimon, editors, *Data Mining and Knowledge Discovery Handbook*, pages 321–352. Springer, 2005.

Lorenza Saitta and Jean-Daniel Zucker. Semantic abstraction for concept representation and learning. In *Proc. Symp. Abstr., Reformul. and Approx.*, pages 103–120, 1998.

Lorenza Saitta and Jean-Daniel Zucker. *Abstraction in Artificial Intelligence and Complex Systems*. Springer, 2013.

Michael Schmidt and Hod Lipson. Distilling free-form natural laws from experimental data. *Science*, 324(5923):81–85, 2009.

Claude Shannon. The lattice theory of information. *Trans. IRE Prof. Group Inf. Theory*, 1(1):105–107, 1953.

Mina Sheikhalishahi, Mohamed Mejri, and Nadia Tawbi. On the abstraction of a categorical clustering algorithm. In *Proc. 12th Int. Conf. Mach. Learn. Data Min. (MLDM 2016)*, pages 659–675, 2016.

The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.9.1. https://www.gap-system.org, 2018.

Dmitri Tymoczko. *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice*. Oxford University Press, 2010.

Rufin VanRullen, Benedikt Zoefel, and Barkin Ilhan. On the cyclic nature of perception in vision versus audition. *Phil. Trans. R. Soc. B*, 369(1641):20130214, 2014.

Rüdiger Von Der Heydt, Esther Peterhans, and MR Dursteler. Periodic-pattern-selective cells in monkey visual cortex. *J. Neurosci.*, 12(4):1416–1434, 1992.

Ulrike Von Luxburg, Robert C Williamson, and Isabelle Guyon. Clustering: Science or art? In *Proc. 2012 ICML Workshop Unsuperv. Transf. Learn. (UTL 2012)*, pages 65–79, 2012.

Julius R Weinberg. *Abstraction, Relation, and Induction: Three Essays in the History of Thought*. University of Wisconsin Press, 1968.

Raymond W Yeung. *Information Theory and Network Coding*. Springer Science & Business Media, 2008.

Haizi Yu and Lav R Varshney. Towards deep interpretability (MUS-ROVER II): Learning hierarchical representations of tonal music. In *Proc. 5th Int. Conf. Learn. Represent. (ICLR 2017)*, 2017.

Haizi Yu, Lav R Varshney, Guy E Garnett, and Ranjitha Kumar. MUS-ROVER: A self-learning system for musical compositional rules. In *Proc. 4th Int. Workshop Music. Metacreation (MUME 2016)*, 2016.

Haizi Yu, Tianxi Li, and Lav R Varshney. Probabilistic rule realization and selection. In *Proc. 31th Annu. Conf. Neural Inf. Process. Syst. (NIPS 2017)*, pages 1561–1571, 2017.
Appendix A. Mathematical Preliminaries

A.1 For Section 3.1

A partition $P$ of a set $X$ is a collection of mutually disjoint non-empty subsets of $X$ whose union is $X$. Elements in $P$ are called cells (or less formally, clusters); the size of $P$ is $|P|$, i.e. the number of cells in $P$. An equivalence relation on a set $X$, denoted $\sim$, is a binary relation satisfying reflexivity, symmetry, and transitivity. An equivalence relation $\sim$ on $X$ induces a partition of $X$: $P = X/\sim := \{[x] \mid x \in X\}$, where the quotient $X/\sim$ is the set of equivalence classes $[x] := \{x' \in X \mid x \sim x'\}$. Conversely, a partition $P$ of $X$ also induces an equivalence relation $\sim$ on $X$: $x \sim x'$ if and only if $x, x'$ are in the same cell in $P$.

A.2 For Section 3.2

A partial order is a binary relation that satisfies reflexivity, antisymmetry, and transitivity. A lattice is a partially ordered set (or a poset) in which every pair of elements has a unique supremum (i.e. least upper bound) called the join and a unique infimum (i.e. greatest lower bound) called the meet. For any pair of elements $p, q$ in a lattice, we denote their join and meet by $p \lor q$ and $p \land q$, respectively. A sublattice is a nonempty subset of a lattice, which is closed under join and meet. A join-semilattice (resp. meet-semilattice) is a poset in which every pair of elements has a join (resp. meet). So, a lattice is both a join-semilattice and a meet-semilattice. A lattice is bounded if it has a greatest element and a least element.

A.3 For Section 3.3

A group is a pair $(G, \ast)$ where $G$ is a set and $\ast : G \times G \to G$ is a binary operation satisfying the group axioms: associativity, the existence of identity (denoted $e$), and the existence of inverse. We also directly say that $G$ is a group, whenever the group operation is understood. Given a group $(G, \ast)$, a subset $H \subseteq G$ is a subgroup, denoted $H \leq G$, if $(H, \ast)$ is a group. The singleton $\{e\}$ is a subgroup of any group, called the trivial group. Given a group $G$ and a subset $S \subseteq G$, the subgroup (of $G$) generated by $S$, denoted $\langle S \rangle$, is the smallest subgroup of $G$ containing $S$; equivalently, $\langle S \rangle$ is the set of all finite products of elements in $S \cup S^{-1}$ where $S^{-1} := \{s^{-1} \mid s \in S\}$. The subgroup generated by a singleton $S = \{s\}$ is called a cyclic group; for simplicity, we also call it the subgroup generated by $s$ (an element), denoted $\langle s \rangle$. Let $G$ be a group and $X$ be a set, then a group action of $G$ on $X$ (or $G$-action on $X$) is a function $\cdot : G \times X \to X$ that satisfies identity ($e \cdot x = x, \forall x \in X$) and compatibility ($g \cdot (h \cdot x) = (g \ast h) \cdot x, \forall g, h \in G, \forall x \in X$). In this paper, we adopt by default the multiplicative notation for group operations and actions, in which $\cdot$ or $\ast$ or both may
be omitted. For any $G$-action on $X$, the orbit of $x$ under $G$ is the set $Gx := \{g \cdot x \mid g \in G\}$, and the quotient of $X$ by $G$-action is the set consisting of all orbits $X/G := \{Gx \mid x \in X\}$.

A.4 For Section 3.4

Let $G$ be a group. We use $\mathcal{H}_G^*$ to denote the collection of all subgroups of $G$. The binary relation “a subgroup of” on $\mathcal{H}_G^*$, denoted $\leq$, is a partial order. $(\mathcal{H}_G^*, \leq)$ is a lattice, called the lattice of all subgroup of $G$, or the (complete) subgroup lattice for $G$ in short. For any pair of subgroups $A, B \subseteq \mathcal{H}_G^*$, the join $A \vee B = \langle A \cup B \rangle$ is the smallest subgroup containing $A$ and $B$; the meet $A \wedge B = A \cap B$ is the largest subgroup contained in $A$ and $B$.

A.5 For Section 3.5

Let $G$ be a group. We say that two elements $a, b \in G$ are conjugate to each other, if there exists a $g \in G$ such that $b = gag^{-1}$, and two subsets $A, B \subseteq G$ are conjugate to each other, if there exists a $g \in G$ such that $B = gAg^{-1}$. In either case, conjugacy is an equivalence relation on $G$ (resp. $2^G$, i.e. the power set of $G$), where the equivalence class of $a \in G$ (resp. $A \subseteq G$) is called the conjugacy class of $a$ (resp. $A$). In particular, we can restrict the above equivalence relation on $2^G$ to the collection of all subgroups $\mathcal{H}_G^*$ which is a subset of $2^G$.

A.6 For Section 4

Let $(G, \ast)$ and $(H, \cdot)$ be two groups. A function $\phi : G \to H$ is called a homomorphism if $\phi(a \ast b) = \phi(a) \cdot \phi(b)$ for all $a, b \in G$. An isomorphism is a bijective homomorphism. We say two groups $G$ and $H$ are homomorphic if there exists a homomorphism $\phi : G \to H$, and say they are isomorphic, denoted $G \cong H$, if there exists an isomorphism $\phi : G \to H$. Let $S$ be a subset of a group $G$; then $N_G(S) := \{g \in G \mid gSg^{-1} = S\}$ is called the normalizer of $S$ in $G$, which is a subgroup of $G$. We say a subset $T \subseteq G$ normalizes another subset $S \subseteq G$ if $T \subseteq N_G(S)$. We say a subgroup $N$ of a group $G$ is a normal subgroup of $G$, denoted $N \unlhd G$, if $G$ normalizes $N$, i.e. $G = N_G(N)$. Let $G$ be a group, $N \subseteq G$, $H \leq G$, $N \cap H = \{e\}$, and $G = NH$; then $NH$ is the inner semi-direct product of $N$ and $H$, and $N \rtimes H$ is the outer semi-direct product of $N$ and $H$. The outer semi-direct product $N \rtimes H$ is the group of all ordered pairs $(n, h) \in N \times H$ with group operation defined by $(n, h)(n', h') = (nn'h^{-1}, hh')$. The inner and outer semi-direct products are isomorphic, i.e. $NH \cong N \rtimes H$. The semi-direct product equation $G = NH$ gives a decomposition of $G$ into “nearly non-overlapping” (i.e. with trivial intersection) subgroups; moreover, for any $g \in G$, these exist a unique $n \in N$ and $h \in H$ such that $g = nh$.

Appendix B. Mathematical Proofs

B.1 Theorem 10

**Proof** Pick any $g \in G$ and $Y \subseteq 2^X$. For any $x \in g \cdot Y$, we have $x = g \cdot y$ for some $y \in Y$. Since $Y \subseteq 2^X$, i.e. $Y \subseteq X$, then $y \in X$. This implies that $x = g \cdot y \in X$. Therefore, $g \cdot Y \subseteq X$, i.e. $g \cdot Y \subseteq 2^X$. To see the corresponding function $\cdot : G \times 2^X \to 2^X$ is a $G$-action on $2^X$, we first check that the identity element $e \in G$ satisfies $e \cdot Y = \{e \cdot y \mid y \in Y\} = \{y \mid y \in Y\} = Y$;
then check that for any \( g, h \in G \), \( g \cdot (h \cdot Y) = \{ g \cdot z \mid z \in \{ h \cdot y \mid y \in Y \} \} = \{ g \cdot (h \cdot y) \mid y \in Y \} = ((gh) \cdot y \mid y \in Y) = (gh) \cdot Y \).

Pick any \( g \in G \) and \( P \in \mathcal{P}_X^* \). For any distinct elements \( Q, Q' \in g \cdot P \), we have \( Q = g \cdot P \) and \( Q' = g \cdot P' \) for some distinct \( P, P' \in \mathcal{P} \), respectively. Since \( P, P' \) are two distinct cells in partition \( \mathcal{P} \), \( P \cap P' = \emptyset \). We claim that \( Q \cap Q' = \emptyset \). Assume otherwise, then there exists a \( q \in Q \cap Q' \) and \( q = g \cdot p = g \cdot p' \) for some \( p \in P, p' \in P' \). This implies that \( p = (g^{-1}g) \cdot p = g^{-1} \cdot (g \cdot p) = g^{-1} \cdot g' \cdot p' = g' = p' \in P \cap P' \), which contradicts the fact that \( P \cap P' = \emptyset \). For any \( x \in X \), \( g^{-1} \cdot x \in X \), then there exists a cell \( P \in \mathcal{P} \) such that \( g^{-1} \cdot x \in P \). This implies that \( x = (gg^{-1})x = g \cdot (g^{-1} \cdot x) \in g \cdot P \) which is an element in \( g \cdot P \).

Therefore, the union of all elements in \( g \cdot \mathcal{P} \) covers \( X \), or more precisely, equals \( X \), since every element in \( g \cdot \mathcal{P} \) is a subset of \( X \). Hence, \( g \cdot \mathcal{P} \) is indeed a partition of \( X \), i.e. \( g \cdot \mathcal{P} \in \mathcal{P}_X^* \).

To see the corresponding function \( \cdot : G \times \mathcal{P}_X^* \to \mathcal{P}_X^* \) is a \( G \)-action on \( \mathcal{P}_X^* \), we first check that the identity element \( e \in G \) satisfies \( e \cdot \mathcal{P} = \{ e \cdot P \mid P \in \mathcal{P} \} = \{ P \mid P \in \mathcal{P} \} = \mathcal{P} \); then check that for any \( g, h \in G \), \( g \cdot (h \cdot \mathcal{P}) = \{ g \cdot Q \mid Q \in \{ h \cdot P \mid P \in \mathcal{P} \} \} = \{ g \cdot (h \cdot P) \mid P \in \mathcal{P} \} = \{(gh) \cdot P \mid P \in \mathcal{P} \} = (gh) \cdot \mathcal{P} \).

\[ T \circ h = T \circ h' \iff h' \circ h^{-1} \in T \iff \ell(h') = \ell(h). \]

The last if-and-only-if condition holds because \( \ell(h' \circ h^{-1}) = \ell(h')\ell(h)^{-1} \) by Lemma 13.

\[ \tau(h') = \tau(h) \iff \tau(h' \circ h^{-1}) = \tau(h') + \ell(h')(-\ell(h)^{-1}\tau(h)) = \tau(h') - \tau(h), \]

where the last equality holds by Theorem 14. Therefore, \( \tau(h') - \tau(h) \in \tau(T) \).

\[ (A,B \cdot v) = (AB) \cdot v \]

\[ B.4 \text{ Theorem 19} \]

\[ \textbf{Proof} \] For any \( A \in \ell(H) \) and \( v \in \tau(T) \), there exists an \( f_{A,u} \in H \) and an \( f_{I,v} \in T \). Since \( T \leq H \), then \( f_{A,u} \circ f_{I,v} \circ f_{A,u}^{-1} \in T \). By Equation (1), we have that \( f_{A,u} \circ f_{I,v} \circ f_{A,u}^{-1} = f_{I,u,v} \), so \( f_{I,u,v} \in T \), i.e. \( Av = \tau(f_{I,u,v}) \in \tau(T) \). To see \( \cdot : \ell(H) \times \tau(T) \to \tau(T) \) defines a group action of \( \ell(H) \) on \( \tau(T) \) is then easy, since it is a matrix-vector multiplication. A quick check shows that for any \( v \in \tau(T) \), \( I \cdot v = v \); for any \( v \in \tau(T) \) and \( A, B \in \ell(H) \), \( A \cdot (B \cdot v) = (AB) \cdot v \).
B.5 Lemma 22

**Proof** For any $A \in L$, $\xi(A) = \xi(IA) = \xi(I) + I\xi(A) = \xi(I) + \xi(A)$. Note that $\xi(A), \xi(I) \in \mathbb{R}^n/V$, so $\xi(A) = V + a$ and $\xi(I) = V + b$ for some $a, b \in \mathbb{R}^n$. Thus,

$$\xi(A) = \xi(I) + \xi(A) \iff V + a = V + (b + a).$$

This further implies that $b \in V$ and $\xi(I) = V + b = V$.

For any $A \in L$, $V = \xi(A^{-1}A) = \xi(A^{-1}) + A^{-1}\xi(A)$. Note that $\xi(A), \xi(A^{-1}) \in \mathbb{R}^n/V$, so $\xi(A) = V + a$ and $\xi(A^{-1}) = V + c$ for some $a, c \in \mathbb{R}^n$. Thus,

$$V = \xi(A^{-1}) + A^{-1}\xi(A) \iff V = V + (c + A^{-1}a).$$

This further implies that $c + A^{-1}a \in V$, or equivalently, $c \in V + (-A^{-1}a)$. Therefore, $c$ and $-A^{-1}a$ are in the same coset and $\xi(A^{-1}) = V + c = V + (-A^{-1}a) = -A^{-1}\xi(A)$. ■

B.6 Theorem 23

**Proof** Let $\Psi : \mathcal{H}_{\mathbb{R}}^\ast \to \Sigma$ be the function defined by

$$\Psi(H) := (\ell(H), \tau(T), \xi_H) \quad \text{for any } H \in \mathcal{H}_{\mathbb{R}}^\ast,$$

where $T := T(\mathbb{R}^n) \cap H$, and $\xi_H : \ell(H) \to \mathbb{R}^n/\tau(T)$ is given by $\xi_H(A) = \tau(\bar{\ell}^{-1}(A))$ with $\bar{\ell} : H/T \to \ell(H)$ being the isomorphism defined in the proof of Theorem 17. We first show $\Psi$ is well-defined, and then show it is bijective. The entire proof is divided into four parts.

1. **Check that $\xi_H$ is well-defined.** More specifically, we want to show that

$$\xi_H(A) \in \mathbb{R}^n/\tau(T) \quad \text{for any } H \in \mathcal{H}_{\mathbb{R}}^\ast \text{ and } A \in \ell(H).$$

For any $A \in \ell(H)$, $\bar{\ell}^{-1}(A)$ is the coset $T \circ h$ in $H/T$ such that $\ell(h) = A$. Pick any $h \in \bar{\ell}^{-1}(A)$ which is possible since as a coset $\bar{\ell}^{-1}(A) \neq \emptyset$. For any $h' \in \bar{\ell}^{-1}(A)$, by Theorem 15, $\tau(h') - \tau(h) \in \tau(T)$, i.e. $\tau(h') \in \tau(T) + \tau(h)$, so $\tau(\bar{\ell}^{-1}(A)) \subseteq \tau(T) + \tau(h)$. Conversely, for any $w \in \tau(T) + \tau(h)$, there exists a $v \in \tau(T)$ such that $w = v + \tau(h)$. Note that the pure translation $t_v \in T \leq H$ and $h \in \bar{\ell}^{-1}(A) \subseteq H$, so their composition $t_v \circ h \in H$. Further, it is an easy check that $\ell(t_v \circ h) = A$ and $\tau(t_v \circ h) = v + \tau(h) = w$. This implies that we have found $h' := t_v \circ h \in \bar{\ell}^{-1}(A)$ and $\tau(h') = w$, thus, $w \in \tau(\bar{\ell}^{-1}(A))$. This finally yields that $\tau(T) + \tau(h) \subseteq \tau(\bar{\ell}^{-1}(A))$. Combining the two directions, we have $\tau(\bar{\ell}^{-1}(A)) = \tau(T) + \tau(h)$; so, $\xi_H(A) = \tau(\bar{\ell}^{-1}(A)) \in \mathbb{R}^n/\tau(T)$. This implies $\xi_H$ is well-defined.

2. **Check that $\Psi$ is well-defined.** More specifically, we want to show that

$$\Psi(H) \in \Sigma \quad \text{for any } H \in \mathcal{H}_{\mathbb{R}}^\ast.$$ 

For any $H \in \mathcal{H}_{\mathbb{R}}^\ast$, it is clear that $\ell(H) \leq \text{GL}_n(\mathbb{R})$, $\tau(T) \leq \mathbb{R}^n$, and they are compatible (Theorem 19); therefore, it suffices to show that $\xi_H \in \Xi_{\ell(H), \tau(T)}$. Note that, for any $A, A' \in \mathbb{R}^n$, it is clear that $\xi_H(A) = \xi_H(A')$ for any $H \in \mathcal{H}_{\mathbb{R}}^\ast$. Therefore, $\Psi$ is well-defined.
\( \ell(H) \), the product of two cosets \( \tilde{\ell}^{-1}(A) \tilde{\ell}^{-1}(A') = (T \circ f_{A,u})(T \circ f_{A',u'}) = T \circ (f_{A,u} \circ f_{A',u'}) = T \circ f_{AA',u+Au'} \), for some \( f_{A,u}, f_{A',u'} \in H \). Therefore,

\[
\xi_H(\{A'\}) = \tilde{\tau}(\tilde{\ell}^{-1}(AA')) = \tau(\tilde{\ell}^{-1}(A)\tilde{\ell}^{-1}(A')) = \tau(T \circ f_{AA',u+Au'}) = \tau(T) + u + Au'.
\]

On the other hand,

\[
\xi_H(A) + A\xi_H(A') = (\tau(T) + u) + A(\tau(T) + u') = \tau(T) + u + Au'.
\]

Therefore, \( \xi_H(\{A'\}) = \xi_H(A) + A\xi_H(A') \) and \( \xi_H \in \Xi(\ell(H),\tau(T)) \). This implies that for any \( H \in H^{*}_{AFF(\mathbb{R}^n)} \), \( \Psi(H) \in \Sigma \), so \( \Psi \) is well-defined.

3. Check that \( \Psi \) is injective. Pick any \( H, H' \in H^{*}_{AFF(\mathbb{R}^n)} \) and suppose \( \Psi(H) = \Psi(H') \), i.e. \( (\ell(H), \tau(T), \xi_H) = (\ell(H'), \tau(T'), \xi_{H'}) \), where \( T := T(\mathbb{R}^n) \cap H \) and \( T' := T(\mathbb{R}^n) \cap H' \). For any \( f_{A,u} \in H \), \( A = \ell(f_{A,u}) \in \ell(H) = \ell(H') \); thus, there exists some \( f_{A,u'} \in H' \). Let \( \tilde{\ell} : H/T \to \ell(H) \) and \( \tilde{\ell}' : H'/T' \to \ell(H') \) be the isomorphisms similarly defined as in Theorem 17. As proved earlier, we have

\[
\xi_H(A) = \tau(\tilde{\ell}^{-1}(A)) = \tau(f_{A,u}) = \tau(T) + u,
\]

\[
\xi_{H'}(A) = \tau(\tilde{\ell}'^{-1}(A)) = \tau(f_{A',u'}) = \tau(T') + u'.
\]

Therefore, \( \tau(T) + u = \tau(T) + u' \). This implies that \( \tau(f_{A,u}) = u \in T(T) + u' = \tau(\tilde{\ell}^{-1}(A)) \).

So, \( f_{A,u} = \tilde{\ell}^{-1}(A) \subseteq H' \), and \( H \subseteq H' \). By a completely symmetrical process, \( H' \subseteq H \). Therefore, \( H = H' \), which implies that \( \Psi \) is injective.

4. Check that \( \Psi \) is surjective. Pick any \( (L, V, \xi) \in \Sigma \) and let

\[
H := \{ f_{A,u} \in AFF(\mathbb{R}^n) | A \in L, u \in \xi(A) \}.
\]

We first show that \( H \subseteq AFF(\mathbb{R}^n) \) by a subgroup test. It is clear \( H \subseteq AFF(\mathbb{R}^n) \). The identity matrix \( I \in L \) and \( 0 \in V = \xi(I) \), so the identity transformation \( \id = f_{I,0} \in H \). For any \( f_{A,u}, f_{A',u'} \in H \), we have \( A, A' \in L \) and \( u \in \xi(A), u' \in \xi(A') \), which respectively implies that \( AA' \in L \) and \( u + Au' \in \xi(A) + A\xi(A') = \xi(AA') \). So, \( f_{A,u} \circ f_{A',u'} = f_{AA',u+Au'} \in H \). For any \( f_{A,u} \in H \), we have \( A \in L \) and \( u \in \xi(A) \), which respectively implies that \( A^{-1} \in L \) and \( -A^{-1}u \in -A^{-1}\xi(A) = \xi(A^{-1}) \). So, \( f_{A,u}^{-1} = f_{A^{-1},-A^{-1}u} \in H \). Therefore, \( H \subseteq AFF(\mathbb{R}^n) \).

Now we show that \( \Psi(H) = (\ell(H), \tau(T), \xi_H) = (L, V, \xi) \). First, for any \( A \in \ell(H) \), there exists an \( f_{A,u} \in H \), so \( A \in L \) which implies that \( \ell(H) \subseteq L \). Conversely, for any \( A \in L, \xi(A) \) is a coset in \( \mathbb{R}^n/V \), so \( \xi(A) \neq \emptyset \). Pick any \( u \in \xi(A) \), then \( f_{A,u} \in H \), so \( A = \ell(f_{A,u}) \in \ell(H) \) which implies \( L \subseteq \ell(H) \). Combining both directions yields \( \ell(H) = L \). Second, note that \( T = T(\mathbb{R}^n) \cap H = \{ f_{I,u} \in AFF(\mathbb{R}^n) | u \in \xi(I) = V \} \), so \( \tau(T) = \{ u | u \in V \} = V \). Third, note that \( \xi_H : \ell(H) \to \mathbb{R}^n/\tau(T) \) and \( \xi : L \to \mathbb{R}^n/V \). We have shown that \( \ell(H) = L \) and \( \tau(T) = V \), so \( \xi_H \) and \( \xi \) have the same domain and codomain. Further, for any \( A \in L, \xi_H(A) = \tau(\tilde{\ell}^{-1}(A)) = \tau(\{ f_{A,u} \in AFF(\mathbb{R}^n) | u \in \xi(A) \}) = \{ u | u \in \xi(A) \} = \xi(A) \). So, \( \xi_H = \xi \). Now we have \( \Psi(H) = (L, V, \xi) \). Therefore, \( \Psi \) is surjective.

\[ \square \]
B.7 Theorem 29

**Proof** Pick any \( f' \in F(Y) \), and let \( f : X \to X \) be the function given by

\[
f(x) = \begin{cases} f'(x), & x \in Y; \\ x, & x \in X \setminus Y. \end{cases}
\]

Then it is clear that \( f(Y) = f'(Y) = Y \) and \( f(X \setminus Y) = X \setminus Y \). For any \( x, x' \in X \) and \( f(x) = f(x') \): if \( f(x) \in X \setminus Y \) then \( x = x' \); otherwise \( x, x' \in Y \) and \( f(x) = f'(x) \) which yields \( x = x' \) since \( f' \) is injective. This implies that \( f \) is injective. \( f \) is also surjective, since \( f(X) = f(Y \cup (X \setminus Y)) = f(Y) \cup f(X \setminus Y) = Y \cup (X \setminus Y) = X \). So \( f \in F(X) \). Further, the fact that \( f(Y) = f'(Y) = Y \) implies that \( f \in F(X)_Y \). 

Therefore, \( f(Y) \subseteq F(X)_Y \). Conversely, pick any \( f'_Y \in F(X)_Y \). \( f'_Y \) is injective since \( f \in F(X)_Y \subseteq F(X) \) is injective; \( f'_Y \) is surjective since \( f'_Y(Y) = f(Y) = Y \). So \( f_Y \in F(Y) \). This implies that \( F(X)_Y \subseteq F(Y) \). 

\[\square\]

B.8 Theorem 31

**Proof** Pick any \( t'_u \in T(Z^n) \), then by definition, \( u \in Z^n \), and \( t'_u(x) = x + u \), for any \( x \in Z^n \). Let \( t : \mathbb{R}^n \to \mathbb{R}^n \) be the function given by \( t(x) = x + u \). Since \( u \in Z^n \), then \( u \in \mathbb{R}^n \), so \( t \in T(\mathbb{R}^n) \). Further, note that \( t(Z^n) = Z^n \); therefore, \( t \in T(\mathbb{R}^n|Z^n) \). It follows that \( t'_u \in T(\mathbb{R}^n|Z^n) \) where \( u \in \mathbb{R}^n \).

Pick any \( t' \in T(\mathbb{R}^n|Z^n) \), then by definition, there exists a \( t_u \in T(\mathbb{R}^n) \) where \( u \in \mathbb{R}^n \) such that \( t_u(Z^n) = Z^n \) and \( t' = t_u|Z^n \), i.e. \( t'(x) = x + u \), for any \( x \in Z^n \). The condition \( t_u(Z^n) = Z^n \) implies in particular \( t_u(0) = u \in Z^n \). It follows that \( t' \in T(Z^n) \), so \( T(\mathbb{R}^n|Z^n) \subseteq T(Z^n) \).

Pick any \( r'_A \in R(Z^n) \), then by definition, \( A \in O_n(Z) \), and \( r'_A(x) = Ax \), for any \( x \in Z^n \). Let \( r : \mathbb{R}^n \to \mathbb{R}^n \) be the function given by \( r(x) = Ax \). Since \( A \in O_n(Z) \), then \( A \in O_n(\mathbb{R}) \), so \( r \in R(\mathbb{R}^n) \). Further, note that \( r(Z^n) = Z^n \); therefore, \( r \in R(\mathbb{R}^n|Z^n) \). It follows that \( r'_A \in R(\mathbb{R}^n|Z^n) \) where \( A \in O_n(\mathbb{R}) \) such that \( r_A(Z^n) = Z^n \) and \( r' = r_A|Z^n \), i.e. \( r'(x) = Ax \), for any \( x \in Z^n \). The condition \( r_A(Z^n) = Z^n \) implies in particular \( Ae_i \in Z^n \) for all \( i \), i.e. the columns of \( A \) are from \( Z^n \). So \( A \in O_n(Z) \). It follows that \( r' \in R(Z^n) \), so \( R(\mathbb{R}^n|Z^n) \subseteq R(Z^n) \). 

\[\square\]

B.9 Theorem 32

**Proof** Pick any \( h' \in ISO(\mathbb{R}^n|Z^n) \), then by definition, there exists an \( h \in ISO(\mathbb{R}^n) \) such that \( h(Z^n) = Z^n \) and \( h' = h|Z^n \). For any \( x, y \in Z^n \),

\[
d(h'(x), h'(y)) = d(h|Z^n(x), h|Z^n(y)) = d(hx, hy) = d(x, y).
\]

This implies that \( h' \in ISO(Z^n) \). So, \( ISO(\mathbb{R}^n|Z^n) \subseteq ISO(Z^n) \).

Conversely, pick any \( h' \in ISO(Z^n) \) and let \( h'_0 = h' - h'(0) \). Note that \( h'_0 \in ISO(Z^n) \) and \( h'_0(0) = 0 \). This implies that \( \|h'_0(x)\|_2 = d(h'_0(x), h'_0(0)) = d(x, 0) = \|x\|_2 \), for any \( x \in Z^n \).
Further, for any \( x, y \in \mathbb{Z}^n \), expanding the distance-preserving equation \( ||h'_0(x) - h'_0(y)||_2^2 = ||x - y||_2^2 \) and cancelling equal terms (i.e. \( ||h'_0(x)||_2^2 = ||x||_2^2 \) and \( ||h'_0(y)||_2^2 = ||y||_2^2 \) yields

\[
\langle h'_0(x), h'_0(y) \rangle = \langle x, y \rangle \quad \text{for all } x, y \in \mathbb{Z}^n.
\]

Now let \( h : \mathbb{R}^n \to \mathbb{R}^n \) be the function given by \( h(x) = Ax + u \), where \( A = [h'_0(e_1), \ldots, h'_0(e_n)] \in \mathbb{Z}^{n \times n} \) and \( u = h'(0) \in \mathbb{Z}^n \). Moreover, \( \langle h'_0(e_i), h'_0(e_j) \rangle = \langle e_i, e_j \rangle = \delta_{ij} \). So, \( A \) is orthogonal, i.e. \( A \in \text{O}_n(\mathbb{Z}) \subseteq \text{O}_n(\mathbb{R}) \). This implies \( h \in \text{ISO}(\mathbb{R}^n) \). We claim \( h(\mathbb{Z}^n) = \mathbb{Z}^n \) and \( h' = h|_{\mathbb{Z}^n} \).

To see \( h(\mathbb{Z}^n) = \mathbb{Z}^n \), first pick any \( x \in h(\mathbb{Z}^n) \), then there exists \( y \in \mathbb{Z}^n \) such that \( x = h(y) = Ay + u \). Since \( A \in \mathbb{Z}^{n \times n} \) and \( u \in \mathbb{Z}^n \), \( x \in \mathbb{Z}^n \) which implies \( h(\mathbb{Z}^n) \subseteq \mathbb{Z}^n \).

Conversely, pick any \( x \in \mathbb{Z}^n \). Let \( y = A^\top (x - u) \), then \( y \in \mathbb{Z}^n \) and \( h(y) = Ay + u = x \). So \( x \in h(\mathbb{Z}^n) \) which implies \( \mathbb{Z}^n \subseteq h(\mathbb{Z}^n) \).

To see \( h' = h|_{\mathbb{Z}^n} \), pick any \( x \in \mathbb{Z}^n \), then \( \langle h'_0(e_i), h'_0(x) \rangle = \langle e_i, x \rangle = \langle Ae_i, Ax \rangle = \langle h'_0(e_i), Ax \rangle \), for all \( i = 1, \ldots, n \). So, \( \langle h'_0(e_i), h'_0(x) - Ax \rangle = 0 \), for all \( i = 1, \ldots, n \), that is

\[
A^\top(h'_0(x) - Ax) = 0.
\]

Multiplying both sides by \( A \) yields \( h'_0(x) = Ax \), for all \( x \in \mathbb{Z}^n \). Hence,

\[
h(x) = Ax + u = h'_0(x) + h'(0) = h'(x) \quad \text{for all } x \in \mathbb{Z}^n.
\]

That is, \( h' = h|_{\mathbb{Z}^n} \). It follows that \( h' \in \text{ISO}(\mathbb{R}^n)_{\mathbb{Z}^n} |_{\mathbb{Z}^n} \). So, \( \text{ISO}(\mathbb{Z}^n) \subseteq \text{ISO}(\mathbb{R}^n)_{\mathbb{Z}^n} |_{\mathbb{Z}^n} \). \( \square \)

### B.10 Theorem 46

**Proof** Suppose for any \( s \in S \), \( s \notin \langle S \setminus \{s\} \rangle \). However, \( s \in \langle S \rangle \); so, \( \langle S \setminus \{s\} \rangle \neq \langle S \rangle \). By definition, \( S \) is a minimal generating set. On the other hand, suppose there exists an \( s \in S \) such that \( s \in \langle S \setminus \{s\} \rangle \), i.e. \( s = s_k \circ \cdots \circ s_1 \) for some \( k \) where \( s_k, \ldots, s_1 \in \langle S \setminus \{s\} \rangle \cup \langle S \setminus \{s\} \rangle^{-1} \). Pick any \( s' \in \langle S \rangle \), \( s' = s'_{k'} \circ \cdots \circ s'_1 \) for some \( k' \) where \( s'_{k'}, \ldots, s'_1 \in S \cup S^{-1} \). For any \( i \in \{1, \ldots, k' \} \), if \( s'_i = s \), replace it with \( s_k \circ \cdots \circ s_1 \); if \( s'_i = s^{-1} \), replace it with \( s_1^{-1} \circ \cdots \circ s_k^{-1} \); otherwise \( s'_i \in \langle S \setminus \{s\} \rangle \cup \langle S \setminus \{s\} \rangle^{-1} \), do nothing. This results in an expression of \( s' \) as the composition of finitely many elements in \( \langle S \setminus \{s\} \rangle \cup \langle S \setminus \{s\} \rangle^{-1} \), i.e. \( s' \in \langle S \setminus \{s\} \rangle \). So, \( \langle S \setminus \{s\} \rangle \subseteq \langle S \setminus \{s\} \rangle \). It is trivial to see that \( \langle S \setminus \{s\} \rangle \subseteq \langle S \rangle \) since \( S \setminus \{s\} \subseteq S \). Therefore, \( \langle S \setminus \{s\} \rangle = \langle S \rangle \). By definition, \( S \) is not a minimal generating set. \( \square \)