THE DARK GEOMETRY OF A NULL EXTRA DIMENSION

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Abstract

We introduce a dynamical formulation of vacuum gravity based on an extra dimension of vanishing proper length. The most general solution to the field equations are presented, leading to an emergent Einsteinian theory associated with a stress-tensor. As an ideal fluid composite, the latter admits a generic decomposition into a unique scalar resembling a running vacuum energy, pure radiation and a vector-tensor multiplet exhibiting a bounded equation of state. As one of the most important phenomenological imports of the null extra dimension, this multiplet is shown to provide a potential resolution to the ‘dark matter’ problem. Based on these nonpropagating geometric fields, we suggest an explanation of the stability of the galactic halo and of the flat rotation curves. The proposed theory also predicts the purely axionic and tensorial fraction of the halo mass density, and is open to experimental tests.

1 Introduction

The physics of extra dimensions, apart from being a fascinating idea in itself, has often provided uncanny insights into some of the outstanding problems in general relativity and particle physics. One of its most remarkable applications was due to Kaluza in his endeavour to unify gravitational and electromagnetic interactions [1], along with Klein who proposed that the extra spatial dimension is curled up in a small circle to justify why it is not observable [2]. Another conceptual breakthrough emerged through the notion of warped extra dimension in a Universe with 3-branes, introduced as a potential explanation to the hierarchy between the weak and Planck scale [3, 4, 5]. However, the fact that there are no signs of Kaluza-Klein particles or of small or large extra dimensions as yet makes it worthwhile to explore alternative ideas.
Here we propose a different scenario altogether, based on a dynamical theory of vacuum gravity with an extra dimension of vanishing metrical length. The five dimensional action principle is based on the first order formulation. First, we find the most general solution to the field equations resulting from the variation of the Lagrangian. Subsequently, we provide an emergent Einsteinian description characterized by an invertible metric and an additional set of (geometric) fields in four dimensions. The associated phenomenological prospects are rich and intriguing, even though the fifth direction itself can never be detected directly.

While this theory exhibits a unique scalar component that provides a purely geometric realization of a variable vacuum energy, the vector-tensor field multiplet emerges as the natural candidate to supersede the ‘invisible’ matter which has long been conjectured to be one of the missing elements of standard Einsteinian gravity. The presence of the null dimension is reflected through the nonpropagating nature of these geometric fields. In the four dimensional effective action, these do not generate any interaction other than gravitational. In the context of galactic physics, this is precisely the characteristic which is required to explain the stability of the (invisible) halo against nongravitational interactions in general. Building upon this observation followed by a general analysis of the field composite as an ideal fluid, we show that this formulation of gravity with null extra dimension(s) provides a potential resolution to the ‘dark matter’ problem [6, 7, 8, 9, 11, 10, 12].

In this context, we also analyze the implications of this theory on the galactic rotation curves, and provide an explanation to their flatness at the outermost regions [8, 9]. The circular velocity formula so obtained could be used to predict the equation of state of the vector-tensor multiplet. In addition, it also leads to numerical constraints on the axionic structure underlying this theory. Such theoretical estimates could be subject to direct experimental verification, either in colliders or through large scale observations in the current and near future.

The idea of dynamically embedding one (or more) extra dimension of zero length in vacuum gravity within an appropriate first order Lagrangian formulation and studying the emergent lower dimensional structure has not been explored before. However, degenerate spacetime solutions in four dimensional gravity theory have been considered earlier [13, 14, 15]. The geometric and topological properties (e.g. geodesic completeness, space or time orientability etc.) of such solutions are fundamentally distinct from those introduced here [17, 18, 19, 20].
2 The fundamental theory

The independent fields in a five dimensional first-order formulation are the vielbein \( \hat{e}_I^\mu(x) \) and super-connection \( \hat{w}_{IJ}^\mu \) \((\mu \equiv (t, x, y, z, w), I \equiv (0, 1, 2, 3, 4))\). The internal metric is defined as \( \eta_{IJ} = \text{diag}[-1, 1, 1, 1, \sigma] \), where \( \sigma = \pm 1 \) corresponds to either of the two possible internal gauge groups \( \text{SO}(4, 1) \) or \( \text{SO}(3, 2) \), respectively. We define the vacuum Lagrangian density to be the five-dimensional Hilbert-Palatini functional:

\[
\mathcal{L} = \frac{1}{\lambda} \epsilon^{\mu \alpha \beta \rho} \epsilon_{IJKLM} \hat{e}_I^\mu \hat{e}_J^\nu \hat{e}_K^\sigma \hat{R}_{\beta \rho}^{LM}(w),
\]

where \( |\lambda| \sim L^3 \) is the five-dimensional gravitational coupling. The field equations are obtained directly from the variation of the above:

\[
\epsilon^{\mu \alpha \beta \rho} \epsilon_{IJKLM} \hat{e}_I^\mu \hat{e}_J^\nu \hat{D}_\alpha(\hat{w}) \hat{e}_K^\sigma = 0, \tag{1a}
\]

\[
\epsilon^{\mu \alpha \beta \rho} \epsilon_{IJKLM} \hat{e}_I^\mu \hat{e}_J^\nu \hat{\bar{R}}_{\alpha \rho}^{KL}(\hat{w}) = 0, \tag{1b}
\]

Here we have defined \( \hat{D}_\mu \) as the gauge-covariant derivative with respect to the super-connection \( \hat{w}_{IJ}^\mu \). In general, these equations admit solutions which could have either vanishing or nonvanishing determinant of the vielbein. Since our goal here is to explore the consequence of an extra dimension of zero metrical length in vacuum gravity, we need to find the most general solution to the above equations where the vielbein has a zero eigenvalue. For definiteness, the zero may be chosen to lie along a particular direction \( (v) \):

\[
\hat{e}_v^I = 0. \tag{2}
\]

This implies:

\[
\hat{e}_I^\mu = \begin{bmatrix} e_a^i & 0 \\ 0 & 0 \end{bmatrix}
\]

Since the (emergent) tetrad fields \( e_a^i \) are invertible (\( \det e_a^i \neq 0 \)), we may define their inverse as \( e^a_i \) \((\neq \hat{e}_i^a \), since \( \hat{e}_I^\mu \) do not exist):

\[
e^a_i e_b^b = \delta_a^b, \quad e^a_i e^b_j = \delta^b_j.
\]

2.1 General solution to the connection equations of motion

Let us first attempt to solve the connection equations, which are linear in the connection fields. These decouple into two sets:

\[
\epsilon^{abcd} \epsilon_{IJKLM} e_a^I e_b^J \hat{D}_c e_d^K = 0, \quad \epsilon^{abcd} \epsilon_{IJKLM} e_a^I e_b^J \hat{\bar{R}} e_d^K = 0. \tag{3}
\]

The first set above may be solved for the connection components \( \hat{w}_{IJ}^\mu \) as:

\[
\hat{w}_{IJ}^\mu = -\epsilon^{\mu IJ} \partial_\alpha e_a^\alpha, \quad \hat{w}_v^{4j} = 0. \tag{4}
\]
In the second set of equations, the $M \neq 4, L \neq 4$ component leads to six equations:

$$e^{c[i} \hat{w}^{j]} = 0 \implies \hat{w}^{4i} = M^{ij} e_{aj}$$

where the arbitrary spacetime field $M^{ij}(t, x, y, z) = M^{ji}(t, x, y, z)$ represents a $4 \times 4$ matrix. In other words, these six equations fix the six antisymmetric components of $\hat{w}^{4i}$ to be zero, leaving the remaining ten as arbitrary. Next, the $M = 4$ component of the same set implies:

$$e^a_k \hat{D}_c e^k_d = 0,$$

which provides a general solution for the connection components $\hat{w}^{ij}$ as:

$$\hat{w}^{ij} = w^{ij}(e) + K^{ij}.$$  

In the above, $w^{ij}(e) = \frac{1}{2} e_i^d \partial_a e^j_b - e_l^d \partial_a e^b_l - e_i^d e_j^l \partial_a e^{pl}$ is completely determined by the tetrads through the torsionfree condition $D_{[a}(w)e^c_b] = 0$ and the contortion tensor $K^{ij} = -K^{ji}$ satisfies:

$$e^a_j K^{ij} = K^{ia}_a = 0.$$  

The remaining twenty (arbitrary) components of the contortion can be parametrized in terms of an axial vector field $L_i(t, x, y, z)$ and a tensor field $N^{ijk}(t, x, y, z) = -N^{ikj}(t, x, y, z)$ with $N^{iik} = 0, \epsilon_{ijkl} N^{ijkl} = 0$:

$$K^{ij} = \epsilon^{ijkl} e_{ak} L_d + 2 e_{al} N^{ild}. $$

To summarize, the most general solution to the connection equations (5) is given by:

$$\hat{w}^{ij} = -e^{ij} \partial_v e^i_c, \hat{w}_v^{4i} = 0, \hat{w}_v^{4i} = M^{ij} e_{aj}$$ (with $M_{ij} = M_{ji}$),

$$\hat{w}^{ij}_a = w^{ij}(e) + K^{ij}_a$$ (with $e^{a}_d K^{ij} = 0$).

Note that since $\hat{w}_v^{kl}$ is a pure gauge, it is always possible to choose a gauge where $e^{k}_c$ is $\nu$-independent and $\hat{w}_v^{kl} = 0$. We shall adopt this choice hereon. This also implies that $e^{a}_d \partial_v e^i_a = 0$, reflecting the fact that the determinant of the tetrad $e^i_a$ is then independent of the fifth coordinate.

### 2.2 General solution to vielbein equations of motion

The vielbein equations (1a) also consist of two sets:

$$\epsilon^{abcd} e_{[a}^{IJKL} e_{b}^{J} \hat{R}_{c}^{K} L^L(w) = 0, \epsilon^{abcd} e_{[a}^{IJKL} e_{b}^{J} \hat{K}_{c}^{K} L^L(w) = 0.$$  

The $I \neq 4$ component of the first set above implies:

$$R^{i4} = 0 \implies \hat{D}_v M^{kl} = 0,$$
In the gauge where $e^a_i$ are $v$-independent, this equation implies that $M^{kl}$ are also $v$-independent. The $I = 4$ component, on the other hand, is identically satisfied. The second set leads to further nontrivial constraints. The one for $I \neq 4$ implies:

$$e^a_1 e^b_2 \hat{R}_{ab}^{ij} = 0 \implies [\delta^a_i \delta_{kl} - e^a_k e_{il}] D_a M^{kl} = 0 \quad (12)$$

Note that the covariant derivative $D_a$ is defined with respect to the torsion-free connection $\omega_{ij}^a(e)$. To find a solution of the above, let us decompose $M_{kl}$ in terms of the trace $M \equiv \eta_{ij} M^{ij}$ $(\eta_{ij} = diag[-1, 1, 1, 1])$ and the tracefree part as:

$$M_{ij} = \frac{1}{4} M \eta_{ij} + \sqrt{2} S_{ij}, \quad (13)$$

Eq. (12) then reduces to a condition that may be solved for $M$:

$$\partial_a M = \frac{4\sqrt{2}}{3} e^b_k e_{il} D_a S^{kl} \quad (14)$$

In the above, of special interest is the solution where the scalar component is a spacetime constant (to be interpreted later):

$$S_{kl} = 0 \implies M = \text{const.} \quad (15)$$

Finally, the $I = 4$ component of the same set of field equations reads:

$$e^a_i e^b_j \hat{R}_{ab}^{ij} = 0 \implies e^a_i e^b_j \left[ R_{ab}^{ij}(w(e)) + K_{ik} K_{bj}^j - \sigma M_{ik} M_{bj} e^a_i e^b_j \right] = 0, \quad (16)$$

where $R_{ab}^{ij}(w(e)) = \partial_{[a} w_{b]}^{ij} + w_{[a}^{ik} w_{b]}^{jq} - \sigma M_{ik} M_{bj} e^a_i e^b_j$ is completely determined by the tetrads through the torsionless connection components $w_{a}^{ij}(e)$.

### 2.3 The emergent Einstein equation

The scalar equation (16) has the general solution:

$$\hat{R}_{ab} = \hat{t}_{ab},$$

where $\hat{t}_{ab}$ is an arbitrary tracefree tensor field. The symmetric part of the above solution involves the four-dimensional Ricci tensor $R_{ab} \equiv \frac{1}{2} R_{(a}^i e_{b)i}$, that depends only on $e^a_i$, leading to an effective Einstein equation:

$$R_{ab} = \hat{t}_{(ab)} - \left[ 2(L_a L^c g_{ab} - L_a L_b) + \frac{1}{2} N_a^{jk} N_{bijk} - \sigma(M_c^e M_{ab} - M_a^e M_{cb}) \right]$$

$$- \frac{1}{2} \epsilon^r_{(a} e^s_{b)} e^t_j D_r N_{kij} - \left[ \left( \epsilon^{jkn} N_{jk}^q + \epsilon^{jkn} N_{jk}^p \right) L_n e_{a_p e_{b_q} + (a \leftrightarrow b)} \right]$$
In writing the above, some of the internal indices in the contortion fields have been converted to four-spacetime indices through the tetrads and their inverses, i.e. \( L^a_a = L^k e_{ak}, M^k_j = M^{jk} e_{aj}, N^j_k = N^{ijk} e_{ai} \). The antisymmetric part, on the other hand, implies three constraints relating the torsion to \( \hat{t}_{ab} = \frac{1}{2} (t_{ab} - t_{ba}) \), and has no dynamical content. Next, using the decomposition (13) and then redefining the symmetric traceless piece as \( t_{ab} = \hat{t}_{ab} + \frac{1}{\sqrt{2}} M S_{ab} - e^k_i e^j_b e^a_j D_i N^{kij} \), we finally obtain the (effective) Einstein tensor to be:

\[
R_{ab} - \frac{1}{2} g_{ab} R = t_{ab} - \frac{3\sigma}{8} M^2 g_{ab} + \psi^L_{ab} + \psi^S_{ab} + \psi^N_{ab} \tag{17}
\]

where the LHS depends only upon the emergent metric \( g_{ab} = e^i_a e^j_b \) and the last three pieces depend on the axial vector \( L^i \), the two-tensor \( S^{ij} \) and the three-tensor \( N^{ijk} \) only, respectively:

\[
\begin{align*}
\psi^L_{ab} &= 2 L^a_a L_b + L_c L^c g_{ab}, \\
\psi^S_{ab} &= -\sigma (2 S^c_a S^i_b - S_{cd} S^{cd} g_{ab}), \\
\psi^N_{ab} &= -\left[ 2 N^j_k N_{bjk} - N_{cde} N^{cde} g_{ab} \right] \tag{18}
\end{align*}
\]

The effective Einstein equation in the above form is particularly illuminating. The geometrical field content originating due to the extra dimension is composed of a (traceless) radiation term, a unique non-derivative scalar mode reflecting a spacetime-dependent vacuum energy (positive or negative) and a vector-tensor composite \( \psi_{ab} \equiv (\psi^L_{ab} + \psi^S_{ab} + \psi^N_{ab}) \) which is to be interpreted next.

For the particular solution (15), the vacuum energy reduces to an emergent cosmological constant.

3 Four dimensional effective action

Due to the vanishing proper length of the extra dimension, a naive dimensional reduction by integrating over the fifth direction does not lead to a four-dimensional effective action. Rather, the appropriate action is generated by the four-dimensional Hilbert-Palatini term, along with a Lagrange multiplier field to implement the associated constraints and a source term depending on \( M^i_a \):

\[
S^{eff}_{ij} (e^i_a, \hat{w}_{ab}^i, \zeta^a) = \frac{1}{2 l_p^2} \int d^4 x \ e^i_a e^b_j \hat{R}_{ab} \hat{w}_{ab}^i (\hat{w}) + 2 \zeta^a e^i_a \hat{D}_{ia} (\hat{w}) e^b_j - e^i_a e^b_j J_{ab}^{ij} (M) \]

Here the source is defined as \( J_{ab}^{ij} (M) = \sigma M^i_a M^j_b \) and \( l_p \) is the four dimensional Planck length. Also, the covariant derivative and the field-strength here are defined with respect to the connection with contortion: \( \hat{w}_{ab}^i = \)
$w_{ij} + K_{ij}$. The action is manifestly invariant under the four dimensional general coordinate transformations as well as the $SO(3,1)$ gauge rotations, since $J_{ab}^{ij}(M)$ transforms homogeneously under both.

As is straightforward to verify, the variation of this action with respect to $\zeta^a$ yields:

$$e^b_i \hat{D}_a (\hat{w}) e^i_j \approx 0,$$

which are equivalent to the field equations (6). The weak equality $\approx$ implies that these equations of motion are to be implemented only after all the variations have been performed. Next, the connection equations imply:

$$\zeta^a \approx 0.$$

Finally, a variation with respect to the tetrad, upon using the above two field equations, leads to exactly the same effective Einstein equation as given by (17).

4 A perfect fluid model

The traceless tensor $t_{ab}$ corresponds to the radiation equation of state $\omega = \frac{1}{3}$ and the pure trace corresponds to $\omega = -1$. Let us now analyze the potential role of the remaining part of the torsional composite in (17), assuming it to be an ideal fluid:

$$\psi_L^b + \psi_S^b + \psi_N^b = \left[ (\rho_\psi + P_\psi) u_a u_b + P_\psi g_{ab} \right],$$

where $u^a$ is the four-velocity of the fluid in the comoving frame. If the fields $L^i, S^{jk}, N^{jk}$ are fundamental (and hence cannot be decomposed any further into other elementary fields already present in this theory), then the coefficients of $g_{ab}$ at both sides must be equal since $g_{ab}$ is independent of the contortion fields. Using this fact and then taking the trace, the effective density and pressure are obtained as:

$$\rho_\psi = -3L_i L^i + \sigma S_{kl} S^{kl} + N_{ijk} N^{ijk},$$

$$P_\psi = L_i L^i + \sigma S_{kl} S^{kl} + N_{ijk} N^{ijk}.$$  

Demanding nonnegative effective energy densities for each of the three independent components $\rho_L, \rho_S, \rho_N$ in the total energy density (the axial vector $L^i$ is necessarily non-spacelike: $L_i L_i \leq 0$), the respective equation of state parameters are found to be:

$$\omega_L = -\frac{1}{3}, \quad \omega_S = 1 = \omega_N.$$  

In other words, while $L_i$ mimics an extremely exotic (in the sense of negative pressure) fluid, the tensorial multiplet resembles an ultra-relativistic fluid.
A purely non-relativistic (pressureless) composite would involve both vector and tensor components with $-L_i L_i = \sigma S_{kl} S^{kl} + N_{ijk} N^{ijk} \neq 0$. Altogether, equation of state of the torsional composite is bounded both above and below:

$$-\frac{1}{3} \leq \omega_\psi \leq 1.$$  \hfill (22)

The above analysis indicates a rich phenomenology in general. The most intriguing but seemingly natural prospect of them all is, if the multiplet $(L^i, S^{ij}, N^{ijk})$ could supercede the hypothetical ‘dark matter’.

## 5 Axionic structure and theoretical predictions

Before going on to explore the possibility just mentioned, let us analyze the axial vector field $L_i$ further. We show that it underlies an axionic structure, which paves the way for a direct comparison of this framework with the phenomenology of the ‘axionic dark matter’ scenarios [24].

The four components of $L_i$ admit a general parametrization in terms of a pseudoscalar $\phi$ and a three-vector $n^i$, which are mutually orthogonal ($e^a_i n^i \partial_a \phi = 0$):

$$L_i = e^a_i \partial_a \phi + n^i.$$  \hfill (23)

The effective energy density due to $L_i$ becomes (upto a tracefree term which could be absorbed away into the radiation $t_{ab}$ in the effective Einstein equation):

$$\rho_L = -3L_i L^i = \rho_\phi + \rho_n$$  \hfill (24)

where $\rho_\phi = -3\partial_a \phi \partial^a \phi$, $\rho_n = -3n_i n^i$. Further, this leads to the following expressions for the total effective density and pressure:

$$\rho_\psi = -3(\partial_a \phi \partial^a \phi + n_i n^i) + (\sigma S_{kl} S^{kl} + N_{ijk} N^{ijk})$$

$$P_\psi = (\partial_a \phi \partial^a \phi + n_i n^i) + (\sigma S_{kl} S^{kl} + N_{ijk} N^{ijk})$$  \hfill (25)

Notably, this leads to a theoretical bound on the axionic mass density $\rho_\phi$ as follows. If we assume the torsional composite $(\psi \equiv (\phi, n^i, S^{ij}, N^{ijk}))$ to be characterized by the equation of state $P_\psi = \omega_\psi \rho_\psi$, then we obtain:

$$\frac{\rho_\phi}{\rho_\psi} \leq \frac{3}{4} (1 - \omega_\psi).$$  \hfill (26)

On the other hand, the fractional contribution to the total density due to the tensorial components are predicted to be:

$$\frac{\rho_{S,N}}{\rho_\psi} \leq \frac{1}{4} (1 + 3\omega_\psi).$$  \hfill (27)
Next, we demonstrate explicitly that the geometric field multiplet \((L^i, S^{ij}, N^{ijk})\) does provide a consistent model of the (spherical) halo for any individual galaxy. For the sake of simplicity, we ignore the physics of baryonic mass and vacuum energy (in this particular context), which lead to subdominant effects far away from the galactic disc.

6 Implications for the galactic rotation curve

Let us begin with the assumption that the spacetime outside the galactic disc is modelled by a spherically symmetric halo, represented by the static metric \([20, 21, 22]\):

\[
\text{ds}^2 = -e^{\mu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]  

(28)

The squared circular velocity of the test particles at a radial distance \(r\) is given by \([20]\):

\[
v^2(r) = e^{-\mu} r^2 \left( \frac{d\phi}{dt} \right)^2 = \frac{1}{2} r \mu'(r).
\]  

(29)

The three independent diagonal components of the effective Einstein’s equations \([17]\) lead to (assuming isotropic pressure):

\[
\rho_\psi e^\lambda = \frac{\lambda'}{r} - \frac{1}{r^2} \left( 1 - e^\lambda \right)
\]

\[
P_\psi e^\lambda = \frac{\mu'}{r} + \frac{1}{r^2} \left( 1 - e^\lambda \right)
\]

\[
P_\psi e^\lambda = \mu'' + \left( \frac{\mu'}{2} + \frac{1}{r} \right) \left( \mu' - \lambda' \right)
\]  

(30)

where \(\rho_\psi\) and \(P_\psi\) are defined in eq.\([20]\) in terms of the torsional fields. Using the fact that the velocities of gas clouds and stars within the halo are nonrelativistic \((v \ll 1)\), the above equations in the weak-field limit \((e^\lambda \approx 1)\) can be translated into an equation for the circular velocity within the halo:

\[
[r v^2(r)]' \approx 4 \pi r^2 (1 + 3w_\psi) \rho_\psi
\]

\[
= 16 \pi r^2 \left( \sigma S_{kl} S^{kl} + N_{ijk} N^{ijk} \right)
\]  

(31)

Thus, given the total halo mass and the terminal velocity at the flat region \((v(r) = \text{constant})\), we can find the equation of state parameter of the effective fluid making up the halo as well as the total tensorial contribution \((\rho_S + \rho_N) = (\sigma S_{kl} S^{kl} + N_{ijk} N^{ijk})\).

Based on these general considerations, let us now find an order of magnitude estimate for the equation of state in the vicinity where the rotation curve is flat. Assuming some typical numbers such as \(v \approx 238\) km/s, \(r \approx 10\)
kpc, \( \rho_\psi \approx 0.01M_\odot/pc^3 \), we obtain \( w_\psi \approx 6 \times 10^{-3} \). In terms of the underlying axionic structure, the associated pure axionic contribution is found to be bounded as: \( \rho_\phi \lesssim 0.74\rho_\psi \) where \( \rho_\psi \) now represents the total effective mass density at the halo (ignoring baryons). On the other hand, for the Milky Way halo, the KSVZ axions (which couple only to quarks but not leptons) are thought to contribute less or equal to \( 0.45\text{Gev/cm}^3 = 0.012M_\odot/pc^3 \) of mass density to the local halo mass density \( \rho_H \approx 0.029M_\odot/pc^3 \), implying that \( \rho_\phi \lesssim 0.41\rho_\psi \).\[24\]. It seems remarkable that this observational bound is consistent with the crude theoretical prediction obtained above.

We note that there is also a new prediction now, regarding the (non-axial) tensorial contribution to the halo density, which could be due to either \( S_{ij} \) or \( N_{ijk} \) or both. For the typical numbers used above, the associated bound turns out to be: \( \rho_{S,N} \lesssim 0.26\rho_\psi \). This may be compared with an observational estimate, once that becomes available in the near future.

### 6.1 Region of flat rotation profile

The previous discussion leading to eq. \(31\) applies to any region within the halo provided that is far way from the disc. Let us now apply these considerations to the specific region where the rotation profile becomes flat (\( v \to v_0 \)) and baryonic disc has the least dominant effect. The approximate solution for \( \mu(r) \) there reads:

\[ \mu(r) = 2v_0^2\ln \left[ \frac{r}{r_0} \right], \tag{32} \]

where \( r_0 \) is an integration constant. The \( G_{rr} \) and \( G_{\theta\theta} \) equations can be solved for the other metric function up to an integration constant \( C \) as \[20\] \[23\]:

\[ e^{-\lambda(r)} = \frac{1}{1 + 2v_0^2 - v_0^4} + Cr^\frac{2(v_0^2 - v_0^4)}{1 + v_0^4} \tag{33} \]

\( C \) may be fixed by joining the halo metric to the exterior vacuum (which may be taken to be the Schwarzschild metric) at the boundary of the halo. This leads to the following expressions for the effective density and pressure, respectively:

\[ \rho_\psi = \left[ \frac{v_0^4(2 - v_0^2)}{1 + 2v_0^2 - v_0^4} \right] \frac{1}{r^2} - \frac{3 + 5v_0^2 - 4v_0^4}{1 + v_0^2}Cr\left[ \frac{2v_0^2(1 - v_0^2)}{1 + v_0^4} \right] \]

\[ P_\psi = \left[ \frac{v_0^4}{1 + 2v_0^2 - v_0^4} \right] \frac{1}{r^2} + (1 + 2v_0^2)Cr\left[ \frac{2v_0^2(1 - v_0^2)}{1 + v_0^4} \right] \tag{34} \]

In the weak field approximation the constant \( C \) represents a small correction \( (C \sim o(v_0^4)) \) and may be dropped. This also implies: \( w \approx \frac{v_0^2}{2} \). We
may now relate the condensate density to the radial variation of the halo mass $M_H(r)$ as:

$$\partial_r M_H(r) = 4\pi r^2 \rho_\psi(r) = \frac{16\pi r^2}{3(1 - w)} \rho_L(r) \quad (35)$$

Hence, given the terminal velocity $v_0$ and $\partial_r M_H(r)$, which may be determined from the study of gravitational lensing within or outside the halo [25], one can determine the total halo density $\rho_\psi$ or the axial vector contribution $\rho_L$. This in turn may be translated to an upper bound on the axionic density at the halo:

$$\rho_\psi(r) \leq \frac{3(1 - w)}{16\pi r^2} \partial_r M_H(r) \quad (36)$$

That would provide another independent test of the halo model proposed here.

7 Conclusions

Within a five dimensional Lagrangian formulation of vacuum gravity, we have explored the dynamical consequences of an extra dimension of vanishing proper length. From the most general solution, we extract an effective four dimensional Einstein equation where the emergent stress-energy tensor is composed of pure radiation, a running vacuum energy and a nonpropagating field multiplet. The latter set of fields are composed of a pseudovector, a rank two symmetric tensor and a three-index traceless tensor. These are analyzed within a perfect fluid model. Based on their distinctive features, these are proposed to supercede the ‘invisible matter’ invoked to explain various large scale observations. The emergent four dimensional theory is fully relativistic.

Note that as an infrared modification of gravity, our formulation is different from modified theories of gravity in general (such as higher curvature gravity or the MOND programme or conformal gravity etc. [26, 27]). As an extra dimensional formulation also, this theory is distinct from the earlier ones based on compact or large or warped dimensions [1, 2, 3, 4, 5].

The fifth dimension here, as it is, does not introduce a metrical scale unlike in other scenarios based on factorizable (Kaluza-Klein) or non-factorizable (Randall-Sundrum) higher dimensional geometries. However, there does exist a characteristic mass associated with the torsional composite in the emergent stress-tensor. This scale is equivalent to the vev of the scalar condensate (the emergent cosmological constant). This also could have nontrivial imports in the weak or Planck scale physics, something which deserves an elaboration elsewhere. So does the implications of this general formulation on cosmological dynamics, which does seem rich in prospects and possible surprises.
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