ON SOVEREIGN, BALANCED AND RIBBON QUASI-HOPF ALGEBRAS

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ABSTRACT. We introduce the notions of sovereign, spherical and balanced quasi-Hopf algebra. We investigate the connections between these, as well as their connections with the class of pivotal, involutory and ribbon quasi-Hopf algebras, respectively. Examples of balanced and ribbon quasi-Hopf algebras are obtained from a sort of double construction which associates to a braided category (resp. rigid braided) a balanced (resp. ribbon) one.

1. Introduction

The theory of monoidal categories plays an important role in (quantum) topology, a domain with applications in knot theory, link theory, the classification of manifolds, algebraic geometry, etc. In some cases, the monoidal categories are identified with categories of (co)representations of a Hopf like algebra; for instance, some modular categories identify with categories of finite-dimensional comodules of a certain weak Hopf algebra (see [30, Theorem 1.1]), while fusion categories for which each simple object has an integer Frobenius-Perron dimension are precisely the categories of representations of a finite-dimensional quasi-Hopf algebra (see [15, Theorem 8.33]). This led to a growing interest for the study of those categories of (co)modules which are monoidal, rigid, sovereign, pivotal, spherical, braided, balanced, ribbon or modular, to name a few. Many such monoidal categories were invented with topological applications in mind. For instance, spherical categories were introduced in [1] in order to generalize the Turaev-Viro state sum model invariant of a closed piecewise-linear 3-manifold, and the main sources of them are categories of representations of involutory Hopf algebras and of quantised enveloping algebras at a root of unity; similarly, ribbon categories give rise to link invariants, and in particular ribbon Hopf algebras give rise to a topological invariant of knots and links is the 3-sphere (see [31]). Monoidal categories drawn also attention in physics, algebra and computer science.

This paper deals with the study of the category of representations of a quasi-Hopf algebra $H$ in the general framework of monoidal categories mentioned above. Otherwise stated, we deal with categories $\mathcal{C}$ of modules over a $k$-algebra $H$ for which the forgetful functor to the category of $k$-vector spaces is quasi-monoidal and, moreover, rigid when it is restricted to the category of finite-dimensional $H$-representations, $H\mathcal{M}^{\text{fd}}$ (see [11]). We study first when $H\mathcal{M}^{\text{fd}}$ is sovereign, that is rigid such that the left and right duality functors coincide as monoidal functors; as sovereign categories identify with the pivotal ones (according to [17], see also Theorem 2.4) it comes up that a quasi-Hopf algebra $H$ is sovereign if and only if it is pivotal, and by [4] the latter is equivalent to the existence of a kind of grouplike element in $H$ that defines the square of the antipode as an inner automorphism of $H$ (see Proposition 3.1). Then we get for free necessary and sufficient conditions for $H\mathcal{M}^{\text{fd}}$ to be spherical, i.e. a pivotal category for which the left and right traces of an endomorphism in $H\mathcal{M}^{\text{fd}}$ are equal. We should point out that, as in the Hopf algebra case, particular examples of spherical quasi-Hopf algebras are obtained from involutory quasi-Hopf algebras (see [31]).

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Theorem 3.9) and their quantum doubles. This leads also to a positive answer for a question raised in [4]: $H$ is involutory if and only if so is its quantum double $D(H)$ (see Corollary 5.10).

Starting with Section 4 we move to the braided case. It is well understood by now that ribbon categories are balanced, spherical, sovereign, etc. categories, so they are good sources for constructing various topological invariants. With these topological applications in mind, Kassel and Turaev [23] associate to any rigid monoidal category $\mathcal{C}$ a ribbon one, denoted by $\mathcal{D}(\mathcal{C})$. Their construction extends the center construction due to Drinfeld (unpublished), Joyal and Street [21] and Majid [26], a construction that associates to a monoidal category a braided one. As observed by Drabant [13], the idea behind of the construction of Kassel and Turaev can be used also to obtain balanced categories (or braided sovereign categories according to Theorem 2.10, a result owing to Deligne [12]), from monoidal ones. In particular, any quasitriangular (QT for short) bialgebra (resp. Hopf algebra) gives rise to a balanced bialgebra (resp. ribbon Hopf algebra).

In Proposition 4.1 we generalize the construction of Drabant, by “replacing” the centre category with an arbitrary braided category $(\mathcal{C}, c)$. The same thing we do in Section 5 where in Theorem 5.2 we generalize the construction of Kassel and Turaev to an arbitrary rigid braided category $(\mathcal{C}, c)$. Our approaches allow to simplify the computations in the case when we apply these constructions to the category of representations of a quasi-bialgebra or a quasi-Hopf algebra $H$ (in general complicated by the apparitions of the reassociator $\Phi$ of $H$ and of the triple that defines the antipode of it). More exactly, to any QT quasi-bialgebra $(H, R)$ we associate a balanced one, denoted by $H[\theta, \theta^{-1}]$, in such a way that $H[\theta, \theta^{-1}]$ and $\mathcal{B}(H, \mathcal{M}, c)$ are isomorphic as balanced categories (see Proposition 4.4), where in general by $\mathcal{B}(\mathcal{C}, c)$ we denote the balanced category associated through our construction to the braided category $(\mathcal{C}, c)$. Similarly, to a QT quasi-Hopf algebra $(H, R)$ we can associate a ribbon quasi-Hopf algebra $(H(\theta), R)$ is such a way that $\mathcal{R}(H, \mathcal{M}^{fd})$ identifies as a ribbon category with $H[\theta, \theta^{-1}]M^{fd}$ (see Theorem 5.10), where in general $\mathcal{R}(\mathcal{C}, c)$ stands for the ribbon category associated through our construction to the rigid braided category $(\mathcal{C}, c)$. Furthermore, when we apply this to the category of finite dimensional Yetter-Drinfeld modules over an arbitrary finite-dimensional quasi-Hopf algebra $H$ we get for free that $\mathcal{R}(H, \mathcal{M}^{fd}, c)$ and $D(H)/\theta, \mathcal{M}^{fd}$ are isomorphic as ribbon categories, where $D(H)$ is the quantum double of $H$ (see Corollary 5.12). We conclude by mentioning, one more time, that our constructions apply to any (rigid) braided category (so for instance to $H, \mathcal{M}$ with $(H, R)$ a QT quasi-bialgebra or quasi-Hopf algebra), hence not necessarily equals a centre of a (rigid) monoidal category, leading thus to new link invariants.

2. Preliminaries

2.1. Quasi-bialgebras and quasi-Hopf algebras. We work over a field $k$. All algebras, linear spaces, etc. will be over $k$; unadorned $\otimes$ means $\otimes_k$. Following Drinfeld [14], a quasi-bialgebra is a four-tuple $(H, \Delta, \varepsilon, \Phi)$ where $H$ is an associative algebra with unit, $\Phi$ is an invertible element in $H \otimes H \otimes H$, and $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ are algebra homomorphisms satisfying the identities

\begin{align}
(\text{Id}_H \otimes \Delta)(\Delta(h)) &= \Phi(\Delta \otimes \text{Id}_H)(\Delta(h))\Phi^{-1}, \\
(\text{Id}_H \otimes \varepsilon)(\Delta(h)) &= h, \quad (\varepsilon \otimes \text{Id}_H)(\Delta(h)) = h,
\end{align}

for all $h \in H$, where $\Phi$ is a 3-cocycle, in the sense that

\begin{align}
(1 \otimes \Phi)(\text{Id}_H \otimes \Delta \otimes \text{Id}_H)(\Phi)(1) &= (\text{Id}_H \otimes \Delta)(\Phi)(\Delta \otimes \text{Id}_H \otimes \text{Id}_H)(\Phi), \\
(\text{Id} \otimes \varepsilon \otimes \text{Id}_H)(\Phi) &= 1 \otimes 1.
\end{align}

The map $\Delta$ is called the coproduct or the comultiplication, $\varepsilon$ is the counit, and $\Phi$ is the reassociator. As for Hopf algebras we denote $\Delta(h) = h_1 \otimes h_2$, but since $\Delta$ is only quasi-coassociative we adopt
the further convention (summation understood):
\[
(\Delta \otimes \text{Id}_H)(\Delta(h)) = h_{(1.1)} \otimes h_{(1.2)} \otimes h_{2}, \quad (\text{Id}_H \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2.1)} \otimes h_{(2.2)},
\]
for all \( h \in H \). We will denote the tensor components of \( \Phi \) by capital letters, and the ones of \( \Phi^{-1} \) by lower case letters, namely
\[
\Phi = X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = V^1 \otimes V^2 \otimes V^3 = \cdots
\]
\[
\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = v^1 \otimes v^2 \otimes v^3 = \cdots
\]

\( H \) is called a quasi-Hopf algebra if, moreover, there exists an anti-morphism \( S \) of the algebra \( H \) and elements \( \alpha, \beta \in H \) such that, for all \( h \in H \), we have:

\[
S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad S(h_1)\beta S(h_2) = \varepsilon(h)\beta,
\]
\[
X^1 \beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad S(x^1)\alpha x^2 \beta S(x^3) = 1.
\]

Our definition of a quasi-Hopf algebra is different from the one given by Drinfeld \([14]\) in the sense that we do not require the antipode to be bijective. In the case where \( H \) is finite dimensional or quasitriangular, bijectivity of the antipode follows from the other axioms, see \([2]\) and \([6]\), so the two definitions are equivalent.

It is well-known that the antipode of a Hopf algebra is an anti-morphism of coalgebras. For a quasi-Hopf algebra, we have something close that follows from the following general result due to Drinfeld, see \([14]\, Lemma 2]\).

**Lemma 2.1.** Let \( H \) be a quasi-Hopf algebra and \( A \) a \( k \)-algebra. Suppose that there exist an algebra map \( f : H \to A \), an anti-algebra map \( g : H \to A \), and elements \( \rho, \sigma \in A \) such that

\[
g(h_1)\rho f(h_2) = \varepsilon(h)\rho, \quad f(h_1)\sigma g(h_2) = \varepsilon(h)\sigma, \quad \forall h \in H,
\]
\[
f(X^1)\sigma g(X^2)\rho f(X^3) = 1_A, \quad g(x^1)\rho f(x^2)\sigma g(x^3) = 1_A.
\]

If \( \overline{f} : H \to A \) is another anti-algebra map and \( \overline{\sigma}, \overline{\rho} \in A \) are such that \((2.7)\) and \((2.8)\) hold for \( f, \overline{f}, \sigma, \overline{\sigma} \) as well, then there exists a unique invertible element \( F \in A \) such that \( \overline{f} = F \rho, \overline{\sigma} = \sigma F^{-1} \) and \( \overline{f}(h) = F g(h) F^{-1} \), for all \( h \in H \). Furthermore,

\[
F = \overline{f}(x^1)\overline{\rho} f(x^2)\sigma g(x^3) \quad \text{with} \quad F^{-1} = g(x^1)\rho f(x^2)\overline{\sigma} \overline{f}(x^3).
\]

If we define \( \gamma, \delta \in H \otimes H \) by

\[
\gamma = S(x^1 x^2)\alpha x^3 \otimes S(X^1)\alpha x^2 X^3 & \text{[2.9]} S(X^2 x^1)\alpha X^3 x^2 \otimes S(X^1 x^1)\alpha x^3,
\]
\[
\delta = X^1 x^2 \beta S(X^3) \otimes X^2 x^3 \beta S(x^3 x^1) = \text{[2.10]} x^1 \beta S(x^2 x^3) \otimes x^2 x^1 \beta S(x^1 x^2)
\]

and apply Lemma 2.1 to \( A = H \otimes H \), \( f = \Delta \), \( g = \Delta \circ S : H \to H \otimes H \), \( \rho = \Delta(\alpha) \), \( \sigma = \Delta(\beta) \), \( \overline{f} = (S \otimes S) \circ \Delta^{\text{cop}} : H \to H \otimes H \), \( \overline{\sigma} = \gamma \) and \( \overline{\rho} = \delta \), then there exists a unique invertible element \( f = f^1 \otimes f^2 \in H \otimes H \), called the Drinfeld twist or the gauge transformation,

\[
f = (S \otimes S)(\Delta^{\text{cop}}(x^1))\gamma \Delta(x^2 \beta S(x^3)) \quad \text{with} \quad f^{-1} = \Delta(S(x^1)\alpha x^2 \beta)(S \otimes S)(\Delta^{\text{cop}}(x^3)),
\]

such that \( \varepsilon(f^1)f^2 = \varepsilon(f^2)f^1 = 1 \) and

\[
f \Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\text{cop}}(h)), \quad \forall h \in H,
\]
\[
f \Delta(\alpha) = \varepsilon, \quad \Delta(\beta) f^{-1} = \delta.
\]

Furthermore, \((2.13)\) is part of the fact that \( S : H^{\text{op.cop}} \to H_f \) is a quasi-Hopf algebra morphism, where \( H_f \) is the twisting of the quasi-Hopf algebra \( H \) by the Drinfeld twist \( f \). We refer to \([14]\) for more details; as far as we are concerned, record only that this property of \( S \) implies

\[
X^1 g_1^1 G^1 \otimes X^2 g_2^1 G^2 \otimes X^3 g_2^2 = g^1 S(X^3) \otimes g_2^1 G^1 S(X^2) \otimes g_2^2 G^2 S(X^1),
\]
\[
f^1 \beta S(f^2) = S(\alpha) \quad \text{and} \quad S(g^1) \alpha g^2 = S(\beta).
\]
where \( f = f^1 \otimes f^2 \) and \( f^{-1} := g^1 \otimes g^2 = G^1 \otimes G^2 \).

2.2. Sovereign and spherical categories. For the definition of a monoidal category \( C \) and related topics we refer to [24] [27]. Usually, for a monoidal category \( C \), we denote by \( \otimes \) the tensor product, by \( \mathbb{1} \) the unit object, and by \( a, l, r \) the associativity constraint and the left and right unit constraints, respectively. The monoidal category \((C, \otimes, \mathbb{1}, a, l, r)\) is called strict if all the natural isomorphisms \( a, l \) and \( r \) are defined by identity morphisms in \( C \).

If \((C, \otimes, \mathbb{1}, a, l, r)\) is a monoidal category, by

\[
\overline{C} := (C, \overline{\otimes} := \otimes \circ \tau, \overline{a} := (a_{X,Y,Z}^{-1})_{X,Y,Z \in C}, \overline{1} := \tau, \overline{r} := \lambda)
\]

we denote the reverse monoidal category associated to \( C \), where \( \tau : C \times C \rightarrow C \times C \) stands for the twist functor, that is \( \tau(X, Y) = (Y, X) \), for any \( X, Y \in C \), and \( \tau(f, g) = (g, f) \), for any morphisms \( X \xrightarrow{f} X' \) and \( Y \xrightarrow{g} Y' \) in \( C \).

When \( C \) is, moreover, left rigid we denote by \( X^* \) the left dual of an object \( X \) of \( C \) and by \( \coev_X = \sum_{X}^{X^*} : \mathbb{1} \rightarrow X \otimes X^* \) and \( \ev_X = \sum_{X^*}^{X} : X^* \otimes X \rightarrow \mathbb{1} \) the corresponding coevaluation and evaluation morphisms. When \( C \) is strict we have that

\[
\lambda_{X,Y} = \coev_{X \otimes Y}, \quad \lambda_{X,Y}^{-1} = \ev_{X \otimes Y}.
\]

And if \( C \) is left rigid we have a well defined functor \((-)^* : C \rightarrow C^{\text{op}}\) that maps \( X \) to \( X^* \) and a morphism \( f \) to \( f^* \), the transpose of \( f \). Here \( C^{\text{op}} \) is the opposite category associated to \( C \). Furthermore, \((-)^* \) is a strong monoidal functor if it is regarded as a functor from \( C \) to \( C^{\text{op}} := (C^{\text{op}}, \otimes^{\text{op}} := \otimes \circ \tau, \mathbb{1}^{\text{op}} := \mathbb{1}^{-1}, a^{\text{op}} := a_{X,Y,Z}^{-1}) \), where \( \tau : C \times C \rightarrow C \times C \) is the switch functor. The functor \((-)^* \) is called the left dual functor, and its strong monoidal structure is mostly determined by

\[
\lambda_{X,Y} = \coev_{X \otimes Y}, \quad \lambda_{X,Y}^{-1} = \ev_{X \otimes Y}.
\]

in the sense that \( \lambda := (\lambda_{X,Y} : (X \otimes Y)^* \rightarrow Y^* \otimes X^*)_{X,Y \in C} \) is a natural isomorphism from \((-)^* \circ \otimes \rightarrow \otimes \circ (-)^* \circ \otimes^{\text{op}}\) such that its inverse \( \lambda^{-1} := (\lambda_{X,Y}^{-1})_{X,Y \in C} \) satisfies the conditions in [24] Definition XI.4.1]. Here, and in what follows,

\[
\coev_{X \otimes Y} = \frac{1}{\otimes} \quad \text{and} \quad \ev_{X \otimes Y} = \frac{1}{\otimes}.
\]

is the diagrammatic notation for the coevaluation and evaluation morphisms of \( X \otimes Y \).

A monoidal category \( C \) is right rigid if \( C \) is left rigid. If this is the case, the right dual of an object \( X \) of \( C \) is denoted by \( X^* \), and by \( \coev'_X := \sum_{X}^{X^*} : \mathbb{1} \rightarrow X \otimes X^* \) and \( \ev'_X := \sum_{X^*}^{X} : X^* \otimes X \rightarrow \mathbb{1} \)
we denote the corresponding coevaluation and evaluation morphisms. Then, if \( C \) is strict,

\[
\begin{align*}
\begin{array}{c}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \uparrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  =
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \downarrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}

  \text{and}

  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \uparrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  =
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \downarrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}.
\end{array}
\end{align*}
\]

As in the left rigid case, we have a strong monoidal functor \( *(-) : C \rightarrow \mathcal{C}_{\text{op}} \), called the right dual functor. Note that \( *(-) \) is just \((-)^*\) considered for \( \mathcal{C} \) instead of \( \mathcal{C} \).

**Definition 2.2.** A monoidal category is called sovereign if it is left and right rigid such that the corresponding left and right dual functors \((-)^* \), \(*(-) : C \rightarrow \mathcal{C}_{\text{op}} \) are equal as strong monoidal functors.

If \( C \) is left rigid, the double dual functor \((-)^{**} := ((-)^*)^* : C \rightarrow C \) is strong monoidal, too.

**Definition 2.3.** Let \( C \) be a left rigid monoidal category and denote by \( \text{Id}_C \) the identity functor of \( C \). A pivotal structure on \( C \) is a monoidal natural isomorphism \( i \) between the strong monoidal functors \( \text{Id}_C \) and \((-)^{**} \). Such a pair \((C, i)\) is called pivotal category.

It seems that the equivalence between sovereign and pivotal notions goes back to Freyd and Yetter [17]. For the sake of completeness and also for further use we sketch below this equivalence.

**Theorem 2.4.** Let \( C \) be a left rigid monoidal category. Then \( C \) admits a pivotal structure if and only if it is sovereign.

**Proof.** Let \( C \) be a sovereign category (in particular, it is also right rigid). For all \( X \in C \) we have an isomorphism \( \theta_X : X \rightarrow (*X)^* \),

\[
\theta_X : \begin{align*}
\begin{array}{c}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \uparrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \end{array}
  & \xrightarrow{r_X^i} \begin{array}{c}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \downarrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \end{array} \\
  \begin{array}{c}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \uparrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \end{array}
  & \xrightarrow{\text{Id}_X \otimes \text{coev}^* X} \begin{array}{c}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \downarrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \end{array} \\
  \begin{array}{c}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \uparrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \end{array} \\
  \begin{array}{c}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \uparrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \end{array} = \begin{array}{c}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \begin{array}{c}
    \downarrow
  \end{array}
  \begin{array}{c}
    \mathcal{X}
  \end{array}
  \end{array}.
\end{align*}
\]

(2.20) Since \( C \) is sovereign we have \((-)^* = *(-) \), so \((X)^* = X^{**} \). Thus we have \( \theta_X : X \rightarrow X^{**} \). One can easily see that \( \theta = (\theta_X)_{X \in \text{Ob}(C)} : \text{Id}_C \rightarrow (-)^{**} \) is a natural isomorphism. It is, moreover, a natural monoidal transformation, and so provides a pivotal structure on \( C \).

Conversely, let \( i \) be a pivotal structure on \( C \), and for all \( V \in C \) define

\[
\text{ev}_{V'} : V \otimes V^* \xrightarrow{i_{V} \otimes \text{Id}_{V^*}} V^{**} \otimes V^* \xrightarrow{\text{ev}^*_{V'}} 1 \quad \text{and} \quad \text{coev}_{V'} : 1 \xrightarrow{\text{coev}^*_{V'}} V^* \otimes V^{**} \xrightarrow{\text{Id}_{V^*} \otimes i_{V}^{-1}} V^* \otimes V.
\]

It is immediate that \((V^*, \text{ev}_{V'}, \text{coev}_{V'})\) is a right dual for \( V \in C \), and so \( C \) is right rigid, too. With respect to this right duality we have that the left and right duality functors coincide as monoidal functors, and therefore \( C \) is sovereign.

For a sovereign (or, equivalently, for a pivotal) category \( C \) one can define the left and right traces \( \text{tr}_l(f), \text{tr}_r(f) \in \text{End}_C(1) \) of an endomorphism \( f \in \text{End}_C(V) \) by

\[
\text{tr}_l(f) := \begin{array}{c}
  \begin{array}{c}
    \mathcal{1}
  \end{array}
  \begin{array}{c}
    \uparrow
  \end{array}
  \begin{array}{c}
    \mathcal{1}
  \end{array}\end{array} \quad \text{and} \quad \text{tr}_r(f) := \begin{array}{c}
  \begin{array}{c}
    \mathcal{1}
  \end{array}
  \begin{array}{c}
    \uparrow
  \end{array}
  \begin{array}{c}
    \mathcal{1}
  \end{array}\end{array}.
\]

The definition of a spherical category in terms of a pivotal structure was given in [17, Definition 2.4]. If we deal with the sovereign property instead, it reduces to the following.
Definition 2.5. A sovereign category is called spherical if \( \text{tr}_l(f) = \text{tr}_r(f) \), for all \( V \in \mathcal{C} \) and \( f \in \text{End}_\mathcal{C}(V) \).

Note that \( \text{tr}_l(f) = \text{tr}_r(f^*) \), for all \( f \in \text{End}_\mathcal{C}(V) \), and therefore a sovereign (≡ pivotal) category is spherical if and only if \( \text{tr}_l(f) = \text{tr}_r(f^*) \) or \( \text{tr}_r(f) = \text{tr}_r(f^*) \), for all \( V \in \mathcal{C} \) and \( f \in \text{End}_\mathcal{C}(V) \). This is possible since in any sovereign category we have, for all \( V \in \mathcal{C} \),

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{V} \\
\text{V}^* \\
\end{array}
\end{array}
&= \\
\begin{array}{c}
\begin{array}{c}
\text{V} \\
\text{V} \\
\end{array}
\end{array}
\quad \text{and} \quad \\
\begin{array}{c}
\begin{array}{c}
\text{V} \\
\text{V}^* \\
\end{array}
\end{array}
&= \\
\begin{array}{c}
\begin{array}{c}
\text{V} \\
\text{V}^* \\
\end{array}
\end{array}
\end{align*}
\]

2.3. Balanced and ribbon categories. Roughly speaking, a monoidal category \( \mathcal{C} \) is braided if it has a braiding \( c \), i.e. a family of natural isomorphisms \( c_{XY} : X \otimes Y \rightarrow Y \otimes X \) satisfying two hexagon axioms, see [24, XIII.1]. Any braiding \( c \) obeys the categorical version of the Yang-Baxter equation. Namely, for any objects \( X, Y \) and \( Z \) of \( \mathcal{C} \), we have

\[
\begin{equation}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{X} \\
\text{Y} \\
\text{Z}
\end{array}
\end{array}
\end{array}
&= \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{X} \\
\text{Y} \\
\text{Z}
\end{array}
\end{array}
\end{array}
\quad \text{and} \quad \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{X} \\
\text{Y} \\
\text{Z}
\end{array}
\end{array}
\end{array}
&= \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{X} \\
\text{Y} \\
\text{Z}
\end{array}
\end{array}
\end{array}
\end{equation}
\]

Here \( c_{XY} = \begin{array}{c} X \ Y \\ Y \ X \end{array} \) is the notation for the braiding \( c \) of \( \mathcal{C} \); similarly, we denote by \( \begin{array}{c} Y \ X \\ X \ Y \end{array} \) the inverse morphism of \( c_{XY} \) in \( \mathcal{C} \).

Let \( H \) be a quasi-bialgebra. It is well known that \( H \mathcal{M} \) is braided if and only if there exists an invertible element \( R = R^1 \otimes R^2 = r^1 \otimes r^2 \in H \otimes H \) (formal notation, summation implicitly understood) such that the following relations hold:

\[
\begin{align*}
(\Delta \otimes \text{Id}_H)(R) &= X^{2}R^{1}x^{1}Y^{1} \otimes X^{3}x^{3}r^{1}Y^{2} \otimes X^{1}R^{2}x^{2}r^{2}Y^{3}, \\
(\text{Id}_H \otimes \Delta)(R) &= x^{3}R^{1}X^{2}r^{1}y^{1} \otimes x^{1}X^{1}r^{2}y^{2} \otimes x^{2}R^{2}X^{3}y^{3}, \tag{2.23} \\
\Delta^{\text{cop}}(h)R &= R\Delta(h), \ \forall h \in H. \tag{2.24} \\
\end{align*}
\]

We say in this case that \( H \) is a quasitriangular, QT for short, quasi-bialgebra. Note that the braiding \( c \) on \( H \mathcal{M} \) defined by \( R \) as above is given by

\[
\begin{equation}
\begin{array}{c}
\begin{array}{c}
\text{X} \ Y \\
\text{Y} \ X
\end{array}
\end{array}
\text{cop}(h)\text{cop}(c) = X \ Y \\
\text{Y} \ X
\end{equation}
\]

When we refer to a QT quasi-bialgebra or quasi-Hopf algebra we always indicate the \( R \)-matrix \( R \) that produces the QT structure by pointing out the couple \( (H, R) \). Also, if \( t \) denotes a permutation of \( \{1, 2, 3\} \), then we set \( \Phi_{(1)(2)(3)} = X^{t^{-1}(1)} \otimes X^{t^{-1}(2)} \otimes X^{t^{-1}(3)} \), and by \( R_{ij} \) we denote the element obtained by acting with \( R \) non-trivially in the \( i \)th and \( j \)th positions of \( H \otimes H \otimes H \).

For \( (H, R) \) a QT quasi-Hopf algebra, \( u \) is the element of \( H \) defined by

\[
u = S(R^{2}x^{2}\beta S(x^{3}))\alpha R^{1}x^{1}, \tag{2.26}\]

By \( R \), \( u \) is an invertible element of \( H \) and the following equalities hold:

\[
\begin{align*}
S^{2}(h) &= uhu^{-1}, \ \forall h \in H, \tag{2.27} \\
S(\alpha)u &= S(R^{2})\alpha R^{1}, \tag{2.28} \\
R^{1}\beta S(R^{2}) &= S(\beta u), \tag{2.29} \\
\Delta(u) &= (R_{21}R)^{-1}f^{-1}(S \otimes S)(f_{21})(u \otimes u), \tag{2.30} \\
\Delta(S(u)) &= (R_{21}R)^{-1}(S(u) \otimes S(u))(S \otimes S)(f_{21})f. \tag{2.31}
\end{align*}
\]
We move now to the balanced/ribbon case.

**Definition 2.6.** (i) A braided category \((\mathcal{C}, c)\) is called balanced if there exists a natural isomorphism \(\eta = (\eta_V : V \to V)_{V \in \text{Ob}(\mathcal{C})}\), called twist on \(\mathcal{C}\), such that for all \(V, W \in \mathcal{C}\),

\[
\eta_V \otimes W = (\eta_V \otimes \eta_W) \circ c_{W,V} \circ c_{V,W}.
\]

(ii) A balanced category \((\mathcal{C}, c, \eta)\) is called ribbon if, in addition, \(\mathcal{C}\) is left rigid and

\[
\eta_V^{-1} = (\eta_V)^*\]

for all \(V \in \mathcal{C}\). If this is the case then \(\eta\) is called a ribbon twist on \(\mathcal{C}\).

For \(H\) a quasi-bialgebra, when \(_H\mathcal{M}\) is a balanced/ribbon category was studied in [7].

**Proposition 2.7.** Let \((H, R)\) be a finite dimensional QT quasi-Hopf algebra, so \(_H\mathcal{M}\) is a rigid braided category. Set \(R = R^1 \otimes R^2\) and \(R_{21} = R^2 \otimes R^1\). Then \(_H\mathcal{M}\) is, moreover,

(i) a balanced category if and only if there exists an invertible central element \(\eta \in H\) such that

\[
\Delta(\eta) = (\eta \otimes \eta)R_{21}R;
\]

(ii) a ribbon category if and only if there is an element \(\eta \in H\) as in (i) satisfying also the condition

\[
S(\eta) = \eta.
\]

**Proof.** Since \(H\) is finite dimensional we can regard \(H \in _H\mathcal{M}\) via its multiplication.

(i) Suppose that \((\eta_V : V \to V)\), indexed by \(V \in _H\mathcal{M}\), defines a balanced structure on \(_H\mathcal{M}\).

Then \(\eta_H\) is completely determined by \(\eta := \eta(1_H)\). Actually,

\[
\eta_V(v) = \eta \cdot v, \quad \forall v \in V.
\]

Thus (2.33) is satisfied for all \(V, W \in _H\mathcal{M}\) if and only if (2.35) holds. Furthermore, \(\eta_V\) is an isomorphism for any \(V \in _H\mathcal{M}\) if and only if \(\eta\) is invertible, and \(\eta_V\) is left \(H\)-linear, for all \(V \in _H\mathcal{M}\), if and only if \(\eta\) is a central element of \(H\).

(ii) For the ribbon case, (2.34) is equivalent to (2.36). \(\square\)

The next definitions are imposed by the characterization in Proposition 2.7.

**Definition 2.8.** (i) We call a QT quasi-bialgebra or quasi-Hopf algebra \((H, R)\) balanced if there exists an invertible central element \(\eta \in H\) satisfying (2.35).

(ii) A QT quasi-Hopf algebra \((H, R)\) is called a ribbon quasi-Hopf algebra if there exists an invertible central element \(\eta \in H\) satisfying (2.35) and (2.36).

The following result is [13, Proposition 2.5]. It says that if \(\mathcal{C}\) is left rigid then \(\mathcal{C}\) is balanced if and only if there is a natural transformation \(\eta : \text{Id}_\mathcal{C} \to \text{Id}_\mathcal{C}\) a natural transformation satisfying (2.33), i.e. the fact that \(\eta\) is as well a natural isomorphism is automatic; furthermore, the inverse of the square of the twist \(\eta\) is completely determined by the left rigid braided structure of the category.

**Proposition 2.9.** Let \((\mathcal{C}, c)\) be a left rigid category and \(\eta : \text{Id}_\mathcal{C} \to \text{Id}_\mathcal{C}\) a natural transformation satisfying (2.33). Then, for all \(V \in \mathcal{C}\), \(\eta_V\) is an isomorphism in \(\mathcal{C}\) with inverse given by

\[
\eta_V^{-1} = (\text{Id}_V \otimes \text{Id}_V)(\text{Id}_V \otimes \text{c}_{V,V}^{-1})(\text{c}_{V,V} \circ \text{coev}_V \otimes \text{Id}_V).
\]

and therefore \((\mathcal{C}, c, \eta)\) is balanced. Consequently, \((\mathcal{C}, c, \eta)\) is ribbon if and only if, for all \(V \in \mathcal{C}\),

\[
\eta_V^{-2} = (\text{ev}_V \otimes \text{Id}_V)(\text{Id}_V \otimes \text{c}_{V,V}^{-1})(\text{c}_{V,V} \circ \text{coev}_V \otimes \text{Id}_V).
\]
Any left/rigid braided category is braided, so any ribbon category is rigid. Furthermore, for a ribbon category the right rigid structure can be constructed from the left rigid structure, braiding and twist in such a way that the left and right dual functors coincide. Thus the choice of the left duals in the definition of a ribbon category is irrelevant.

More generally, if \((\mathcal{C}, c, \eta)\) is a left rigid balanced category, \(V\) is an object of \(\mathcal{C}\) and \(V^*\) is the left dual of \(V\) in \(\mathcal{C}\) with evaluation and coevaluation morphisms \(\text{ev}_V : V^* \otimes V \to 1\) and \(\text{coev}_V : 1 \to V \otimes V^*\), then

\[
(V^*, \text{ev}'_V := \text{ev}_V c_{V, V^*}(\eta \otimes \text{Id}_{V^*}), \text{coev}'_V := (\eta^* \otimes \text{Id}_V)c_{V^*, V} \text{coev}_V)
\]

is a right dual for \(V\) in \(\mathcal{C}\). Furthermore, with respect to it the left and right dual functors coincide as strong monoidal functors, and therefore any left rigid balanced category is sovereign. In particular, any ribbon category is sovereign; note that in this case, from \(\eta^*_V = (\eta_V)^*\) it follows that \(\text{coev}'_V\) in (2.40) can be restated as

\[
\text{coev}'_V = (\text{Id}_{V^*} \otimes \eta_V)c_{V^*, V} \text{coev}_V.
\]

Actually, we can characterize ribbon categories in terms of sovereign categories. That rigid braided balanced categories are actually braided sovereign categories was proved by Deligne [12]. The other statements of the theorem below were taken from [20, Proposition A.4].

**Theorem 2.10.** Let \((\mathcal{C}, c)\) be a left rigid braided category. Then

(i) \((\mathcal{C}, c)\) is balanced if and only if \(\mathcal{C}\) is sovereign;

(ii) \((\mathcal{C}, c)\) is ribbon if and only if it is sovereign and with respect to the rigid structure given by the fact that \(\mathcal{C}\) is sovereign we either have

\[
\begin{align*}
V & \quad V^* = \quad V & \quad \text{or} & \quad V^* \quad V = \quad V, \\
V & \quad V & \quad V^* & \quad V^*
\end{align*}
\]

\[
\forall V \in \mathcal{C}.
\]

**Proof.** (i) We have just seen that a left rigid balanced category is sovereign. For the converse, if \((\mathcal{C}, c)\) is a braided sovereign category define, for all \(V \in \mathcal{C}\),

\[
\eta_V := \begin{array}{c}
V \\
V
\end{array}, \quad \eta^{-1}_V := \begin{array}{c}
V^* \\
V
\end{array}, \quad \theta_V := \begin{array}{c}
V^* \\
V
\end{array}, \quad \theta^{-1}_V := \begin{array}{c}
V \\
V^*
\end{array}.
\]

Then \(\eta := (\eta_V)_{V \in \text{Ob}(\mathcal{C})}\) and \(\theta := (\theta_V)_{V \in \text{Ob}(\mathcal{C})}\) are twists on \(\mathcal{C}\) with inverses defined by \(\eta^{-1} := (\eta^{-1}_V)_{V \in \text{Ob}(\mathcal{C})}\) and \(\theta^{-1} := (\theta^{-1}_V)_{V \in \text{Ob}(\mathcal{C})}\), respectively. Hence \((\mathcal{C}, c, \eta)\) and \((\mathcal{C}, c, \theta)\) are braided balanced categories.

(ii) By [20, Proposition A.4] we know that (ii) is equivalent to the first equality in (2.42). So we only have to show that, for all \(V \in \mathcal{C}\), \(\eta_{V^*} = (\eta_V)^*\) if and only if the second equality in (2.42) holds. To this end, note that (2.40) and the naturality of \(c\) imply

\[
\begin{align*}
V^* & \quad \eta_{V^*} := \begin{array}{c}
V^* \\
V^*
\end{array} = \begin{array}{c}
V^* \\
V^*
\end{array} = \text{Id}_{V^*},
\end{align*}
\]
and therefore

\[
\begin{align*}
V^* & \quad \eta_V \circ V^* \\
\eta_V & \quad \eta_V \circ V^* \\
V^* & \quad \eta_V \circ V^* \\
\eta_V & \quad \eta_V \circ V^*
\end{align*}
\]

and this finishes the proof. \hfill \Box

Hence, for a braided sovereign category \( C \) we have two twists \( \eta \) and \( \theta \) which are equal if and only if one of them provides a ribbon structure on \( C \). If this is the case then \( C \) endowed with the pivotal structure produced by the ribbon twist \( \eta = \theta \) is a spherical category. Note that the converse is also true, provided that \( C \) is semisimple; see [20, Proposition A.4].

3. SOVEREIGN AND SPHERICAL QUASI-HOPF ALGEBRAS

If \( H \) is a quasi-bialgebra, the category \( H \mathcal{M} \) of left \( H \)-representations is monoidal. If \( U, V \) are left \( H \)-modules then the tensor product between \( U \) and \( V \) is the tensor product over \( k \) equipped with the left \( H \)-module structure given by \( \Delta \), i.e. \( h \cdot (u \otimes v) = h_1 \cdot u \otimes h_2 \cdot v \), for all \( h \in H \), \( u \in U \) and \( v \in V \). The associativity constraint on \( H \mathcal{M} \) is the following: for \( U, V, W \in H \mathcal{M} \), \( a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W) \) is given by

\[
a_{U,V,W}((u \otimes v) \otimes w) = X_1 \cdot u \otimes (X^2 \cdot v \otimes X^3 \cdot w).
\]

The unit object is \( k \) considered as a left \( H \)-module via \( \varepsilon \), the counit of \( H \). The left and right unit constraints are the same as for the category \( \mathcal{M} \) of \( k \)-vector space.

Let \( H \mathcal{M}^{fd} \) be the full subcategory of \( H \mathcal{M} \) consisting of finite dimensional vector spaces. Then \( H \mathcal{M}^{fd} \) is a category with left duality, see Remark 3.5 below. Our goal is to see when \( H \mathcal{M}^{fd} \) is a sovereign (resp. spherical) category, in the sense of Definition 2.2. By the general results presented in Subsection 2.2 it comes up that sovereign structures on \( H \mathcal{M}^{fd} \) are in a one to one correspondence with the pivotal ones, and that the latter are completely determined by certain elements of \( H \).

**Proposition 3.1.** Let \( H \) be a finite dimensional quasi-Hopf algebra over a field \( k \). Then we have a bijective correspondence between

(i) pivotal structures \( i \) on \( C = H \mathcal{M}^{fd} \);
(ii) sovereign structures on \( C = H \mathcal{M}^{fd} \);
(iii) invertible elements \( g_i \in H \) satisfying

\[
S^2(h) = g_i^{-1} h g_i, \quad \forall \ h \in H,
\]

\[
\Delta(g_i) = (g_i \otimes g_i)(S \otimes S)(f_{21}^{-1}) f,
\]

where \( f = f_1 \otimes f_2 \) is the Drinfeld twist defined in (2.11) and \( f_{21} = f_2 \otimes f_1 \).

**Proof.** The bijection between (i) and (ii) is established by Theorem 2.4 while the equivalence between (i) and (iii) was established in [3, Proposition 4.2]. \hfill \Box
Definition 3.2. A sovereign quasi-Hopf algebra is a quasi-Hopf algebra (not necessarily finite dimensional) $H$ for which there exists an invertible element $g \in H$ satisfying (3.1) and (3.2).

Thus the definition of a sovereign quasi-Hopf algebra is designed in such a way that its category of finite-dimensional left representations is sovereign or, equivalently, pivotal.

Under some conditions imposed to the field $k$, a family of sovereign quasi-Hopf algebras $H$ is given by the semisimple ones. Recall that $H$ is semisimple if it is semisimple as an algebra, and that this is equivalent to the existence of a left integral $t$ in $H$ (i.e. of an element $t \in H$ obeying $ht = \varepsilon(h)t$, for all $h \in H$) such that $\varepsilon(t) = 1$. If this is the case then $t$ is also a right integral in $H$, that is $th = \varepsilon(h)t$, for all $h \in H$. We refer to [29] for more details related to this topic.

Example 3.3. A finite dimensional semisimple quasi-Hopf algebra over an algebraic closed field of characteristic zero is sovereign via the element $g := q^2t_2p^2q(q^1t_1p^1)$, where $p_R, q_R$ are the elements defined by

$$
\begin{align*}
p_R &= p^1 \otimes p^2 = x^1 \otimes x^2 \beta(x^3), \\
q_R &= q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3)X^2,
\end{align*}
$$

and $t$ is a left (and so right as well) integral in $H$ satisfying $\varepsilon(t) = 1$.

Proof. It follows from [15] Propositions 8.24 & 8.23] that $H_{\text{fd}}$ has a unique pivotal (= sovereign) structure such that $\text{dim}(V) := \text{tr}_r(\text{Id}_V)$ equals $\dim_k(V)$, for any simple object of $H_{\text{fd}}$. Furthermore, by [28] the element $g \in H$ that produces this pivotal structure on $H_{\text{fd}}$ is just the one mentioned in the statement.

We refer to [19][11] for the explicit quasi-Hopf algebra structure of the quantum double $D(H)$ of a finite dimensional quasi-Hopf algebra $H$.

Example 3.4. If $H$ is a finite-dimensional sovereign quasi-Hopf algebra then so is its quantum double $D(H)$.

Proof. The category $D(H)_{\text{fd}}$ identifies to the right centre of the monoidal category $H_{\text{fd}}$, see [3][8]. Thus our result follows from [16] Exercise 7.13.6].

Remark 3.5. If $H$ is a sovereign quasi-Hopf algebra via an invertible element $g$ of $H$ obeying (3.1) and (3.2), then $H_{\text{fd}}$ is sovereign with the following rigid structure.

If $V \in H_{\text{fd}}$, with $H$-action denoted by $H \otimes V \ni h \otimes v \mapsto h \cdot v \in V$, then the left dual of $V$ is $V^*$ with $H$-module structure $(h \cdot v^*)(v) = v^*(S(h) \cdot v)$, for all $v^* \in V^*$, $v \in V$ and $h \in H$, and $ev_V$ and $\text{coev}_V$ given by

$$
\begin{align*}
ev_V : V^* \otimes V \ni v^* \otimes v &\mapsto v^*(\alpha \cdot v) \in k, \\
\text{coev}_V : k \ni \beta \cdot v_i \otimes v^i &\mapsto V \otimes V^*,
\end{align*}
$$

where $\{v_i, v^i\}_i$ are dual bases in $V$ and $V^*$ (summation implicitly understood). The right dual of $V$ is again $V^*$ considered as a left $H$-module via the antipode $S$ of $H$ as above, and with the evaluation and coevaluation morphisms given by

$$
\begin{align*}
ev'_V : V \otimes V^* \ni v \otimes v^* &\mapsto v^*(g^{-1}S^{-1}(\alpha) \cdot v) = v^*(S(\alpha)g^{-1} \cdot v) \in k, \\
\text{coev}'_V : k \ni 1_k \mapsto v^i \otimes S^{-1}(\beta)g \cdot v_i = v^i \otimes gS(\beta) \cdot v_i \in V^* \otimes V,
\end{align*}
$$

summation implicitly understood, where $\{v_i, v^i\}_i$ are dual bases in $V$ and $V^*$.

When we refer to a sovereign quasi-Hopf algebra we always indicate the element that produces the sovereign structure, by pointing out the couple $(H, g)$. Also, for $V$ a left $H$-module we denote by $\text{End}_H(V)$ the set of $H$-endomorphisms of $V$.  

Corollary 3.6. If \((H, \varrho)\) is a sovereign quasi-Hopf algebra and \(V\) is a finite dimensional left \(H\)-module then for all \(f \in \text{End}_H(V)\) we have

\[
\text{tr}_r(f) = \text{tr}(V \ni v \mapsto f(\varrho S(\beta) \alpha \cdot v) \in V) \quad \text{and} \quad \text{tr}_r(f) = \text{tr}(V \ni v \mapsto f(g^{-1} \beta S(\alpha) \cdot v) \in V),
\]

where by \(\text{tr}(\chi)\) we denote the usual trace of a \(k\)-linear endomorphism \(\chi\).

Proof. By Remark 3.5 and (2.21), for all \(f \in \text{End}_H(V)\) we have

\[
\text{tr}_r(f) = \langle v', \alpha \cdot f(\varrho S(\beta) \cdot v) \rangle = \langle S^{-1}(\alpha) \cdot v', f(\varrho S(\beta) \cdot v) \rangle = \langle v', f(\varrho S(\beta) \alpha \cdot v) \rangle = \text{tr}(V \ni v \mapsto f(\varrho S(\beta) \alpha \cdot v) \in V),
\]

as stated. The formula for \(\text{tr}_r(f)\) can be computed in a similar manner, so we are done. \(\square\)

By analogy with the Hopf case [11], we call a quasi-Hopf algebra spherical if its category of left finite dimensional representations is spherical. By the above results we have the following.

**Proposition 3.7.** A quasi-Hopf algebra \((H, \varrho)\) is spherical if and only if there exists an invertible element \(g \in H\) satisfying (3.2) and (3.3), and such that

\[
\text{tr}(V \ni v \mapsto f(\varrho S(\beta) \alpha \cdot v) \in V) = \text{tr}(V \ni v \mapsto f(g^{-1} \beta S(\alpha) \cdot v) \in V),
\]

for any finite dimensional left \(H\)-module \(V\) and any \(f \in \text{End}_H(V)\).

As any ribbon category is spherical we get the following.

**Example 3.8.** If \((H, R, \eta)\) is a ribbon quasi-Hopf algebra as in Definition 2.28 then \(H\mathcal{M}^{fd}\) is spherical via the element \(g = (u\eta)^{-1}\), where \(u\) is as in (2.27).

An important class of spherical categories is defined by involutory Hopf algebras, which coincides to the class of semisimple Hopf algebras provided that \(k\) has characteristic zero (see [25]). As we will see a similar result holds for quasi-Hopf algebras.

Recall from [11] that a quasi-Hopf algebra is called involutory if

\[
S^2(h) = S(\beta) \alpha h S(\alpha), \quad \forall h \in H.
\]

If \(H\) is involutory then \(\alpha, \beta \in H\) are invertible and \((\beta S(\alpha))^{-1} = S(\beta)\alpha\), cf. [11] Lemma 3.2.

**Theorem 3.9.** Any involutory quasi-Hopf algebra is spherical, and therefore sovereign, too.

Proof. We show that \(g := \beta S(\alpha)\) has all the required properties. In fact, by the above comments we have \(S^2(h) = g^{-1} h g\), for all \(h \in H\), and the equality in (3.3) is fulfilled since the both sides of it are equal to the usual trace of \(f\). So it remains to show that \(g = \beta S(\alpha)\) satisfies (3.2). If we assume \(k\) algebraic closed of characteristic zero then this follows from [11] Remarks 3.5 1). As we work over an arbitrary field, a proof for the fact that \(g = \beta S(\alpha)\) satisfies (3.2) should be given. Towards this end, for the quasi-Hopf algebra \((H^{\text{cop}}, \Delta^{\text{cop}}, \varrho, \varrho^{-1}) := x^3 \otimes x^2 \otimes x^1, S^{-1}, S^{-1}(\alpha), S^{-1}(\beta))\) and the tensor product algebra \(A = H \otimes H\) consider \(f, g, \rho, \sigma\) as in Lemma 2.21 i.e.

\[
f = \Delta^{\text{cop}}, \quad g = \Delta^{\text{cop}} \circ S^{-1}, \quad \rho = \Delta(S^{-1}(\alpha)), \quad \sigma = \Delta^{\text{cop}}(S^{-1}(\beta)).
\]

Define the anti-algebra map \(\varpi = \Delta^{\text{cop}} \circ S : H^{\text{cop}} \to H \otimes H\) and the elements \(\varpi = \Delta^{\text{cop}}(\beta^{-1})\), \(\tau = \Delta^{\text{cop}}(\alpha^{-1})\) \(\in H\). By [11] Proposition 3.4 we have \(S(h_2)\beta^{-1} h_1 = \varpi(h)\beta^{-1} \quad \text{and} \quad h_2 \alpha^{-1} S(h_1) = \varpi(h)\alpha^{-1}, \quad \forall h \in H\), and therefore \((f, \varrho, \varpi, \tau)\) satisfies for \(H^{\text{cop}}\) and \(A = H \otimes H\) the relations in (2.27). The ones in (2.28) are satisfied as well since, according to [11] Lemma 3.2, \(\beta S(\beta)\alpha = \alpha^{-1}\) and \(S(\beta)\alpha = \beta^{-1}\). Therefore

\[
x^3 \alpha^{-1} S(x^2) \beta^{-1} x^1 = x^3 \beta S(\alpha) S(\alpha x^2 \beta) S(\beta) \alpha x^1 = x^3 S^{-1}(\alpha x^2 \beta) x^1 = S^{-1}(S(x^1) \alpha x^2 \beta S(x^3)) = 1,
\]
and similarly
\[ S(X^3)\beta^{-1}X^2\alpha^{-1}S(X^1) = \frac{3\beta}{4} S(\alpha X^3) S(\beta)\alpha X^2 \beta S(\alpha) S(X^1) \]

Hence, Lemma 3.4 guarantees the existence and the uniqueness of an invertible element \( F \in H \otimes H \) satisfying, for all \( h \in H \), the relations
\[ \Delta^{\text{cop}}(S(h)) = F \Delta^{\text{cop}}(S^{-1}(h)) F^{-1}, \quad \Delta^{\text{cop}}(\alpha^{-1}) = \Delta^{\text{cop}}(S^{-1}(\beta)) F, \quad \Delta^{\text{cop}}(\beta^{-1}) = F \Delta^{\text{cop}}(S^{-1}(\alpha)). \]

Now, from \( 3.5 \) and \( 2.13 \) we see that
\[
\begin{align*}
f^{i-1}_2 (S \otimes S)(f)(g^{-1} \otimes g^{-1}) & \Delta^{\text{cop}}(S^{-1}(h))(g \otimes g)(S \otimes S)(f^{-1}) f^{i-1}_2 \\
& = f^{i-1}_2 (g^{-1} \otimes g^{-1})(S^{-1} \otimes S^{-1})(f) \Delta^{\text{cop}}(S^{-1}(h))(S^{-1} \otimes S^{-1})(f^{-1})(g \otimes g) f^{i-1}_2 \\
& = f^{i-1}_2 (g^{-1} \otimes g^{-1})(S^{-1} \otimes S^{-1})(\Delta(h))(g \otimes g) f^{i-1}_2 \\
& = f^{i-1}_2 (S \otimes S)(\Delta(h)) f^{i-1}_2 = \Delta^{\text{cop}}(S(h)),
\end{align*}
\]
for all \( h \in H \). We get from here that \( F = f^{i-1}_2 (S \otimes S)(f)(g^{-1} \otimes g^{-1}) \), which together with \( \Delta^{\text{cop}}(\alpha^{-1}) = \Delta^{\text{cop}}(S^{-1}(\beta)) F^{-1} \) and \( \alpha^{-1} = S^{-1}(\alpha) \beta \) leads to
\[ \Delta(S^{-1}(\alpha)\beta) = (g \otimes g)(S \otimes S)(f^{i-1}_2) f. \]

But \( S(\beta S(\alpha)) = S^2(\alpha)S(\beta) = S(\beta)\alpha^2 \beta S(\beta \alpha) = S(\beta) \alpha \) because, once more, \( \alpha^{-1} = \beta S(\alpha) \). Thus \( g = \beta S(\alpha) = S^{-1}(\beta) \alpha = S^{-1}(\alpha) \beta \), and this finishes the proof. \( \square \)

It was proved in [4, Proposition 4.4] that \( D(H) \) is involutory if \( H \) is so, provided that the relation
\[ \Delta(S(\beta)\alpha) = f^{-1}(S \otimes S)(f^{i-1}_2)(S(\beta) \alpha \otimes S(\beta) \alpha) \]
holds. As \( g^{-1} = S(\beta) \alpha \) it follows from the proof of Theorem 3.9 that \( 3.5 \) is valid for any involutory quasi-Hopf algebra. So we get the following result, which in particular produces a new class of spherical quasi-Hopf algebras. It answer also in positive to a question raised in [4].

**Corollary 3.10.** A quasi-Hopf algebra \( H \) is involutory if and only if so is its quantum double \( D(H) \).

**Proof.** The direct implication follows from the above comments, while the converse follows since \( H \) is a quasi-Hopf subalgebra of \( D(H) \). \( \square \)

Hence all the examples of involutory quasi-Hopf algebras presented in [4] are as well examples of spherical (and implicitly sovereign) quasi-Hopf algebras. Another one is the following.

**Example 3.11.** Let \( k \) be a field containing a primitive root of unity \( q \) of degree \( n \) so in particular \( n \neq 0 \) in \( k \) and \( C_n \) the cyclic group of order \( n \), say generated by \( g \). We endow the group algebra \( k[C_n] \) of \( k \) and \( C_n \) with the comultiplication given by \( \Delta(g) = g \otimes g \) and \( \varepsilon(g) = 1 \), for all \( g \in G \), extended to the whole \( k[C_n] \) as algebra morphisms. In this way we can see \( k[C_n] \) as a quasi-bialgebra with reassocciator \( \Phi_q = \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-[i-j]} q^{i(j+1)} 1_i 1_j 1_l \), where \( [a] \) stands for the integer part of the rational number \( a \) and \( 1_i := \frac{1}{n} \sum_{a=0}^{n-1} q^{i-a} q^a \), for all \( 1 \leq i \leq n - 1 \). Furthermore, this structure of \( k[C_n] \) can be completed up to a quasi-Hopf algebra one by defining \( S(g) = g^{-1} \), extended to the whole \( k[C_n] \) as an anti-morphisms of algebras, \( \alpha = g^{-1} \) and \( \beta = 1 \). We denote by \( H_q(n) \) this quasi-Hopf algebra structure of \( k[C_n] \).

\( H_q(n) \) is an involutory quasi-Hopf algebra, and so \( (H_q(n), g) \) is a spherical quasi-Hopf algebra.

**Proof.** That \( H_q(n) \) is a quasi-Hopf algebra is a well known fact, see for instance [18]. Actually, since \( H_q(n) \) is commutative, the only thing we must check is that \( \Phi_q \) is a normalized 3-cocycle, that is \( \Phi_q \) satisfies the relation \( 2.4 \). This follows from the fact that \( \Phi_q \) is the 3-cocycle (in the Harrison cohomology, see [5]) dual to the 3-cocycle of the group \( C_n \) corresponding to \( q \).
We have $S^2 = \text{Id}$ and because $H_n(q)$ is commutative we can also see $S^2(h) = g^{-1}hg$, for all $h \in H(n)$, with $g = \beta S(\alpha) = g$. Thus $H_n(q)$ is involutory, and so Theorem 3.11 applies.

It was proved in [28, Theorem 7.2] that for a finite dimensional semisimple quasi-Hopf algebra $H$ over an algebraic closed field of characteristic zero one has $g^{-1} = S(g)$, where $g$ is the element of $H$ responsible for the sovereign structure of $H$ as in Example 3.3. As we will see, this is a consequence of a more general result valid for sovereign quasi-Hopf algebras (over arbitrary fields).

**Proposition 3.12.** If $(H, g)$ is a sovereign quasi-Hopf algebra then $g^{-1} = S(g)$.

**Proof.** As $g$ is invertible, it suffices to show that $gS(g) = 1$. Indeed, by applying $\varepsilon \otimes \varepsilon$ to the both sides of (3.2) we get $\varepsilon(g) = 1$. Thus, by using again (3.2) we see that

$$
\beta = \varepsilon(g)\beta = g_1\beta S(g_2)
= gS(g^2)f_1^1\beta S(gS(g^1)f_2^2) = gS(g^2)S(\alpha)S(gS(g^1))
= gS^2(\beta)S(g) = \beta gS(g).
$$

On the other hand, (3.1) allows to compute that

$$
A\beta B = A\beta gS(g)B = A\beta gS(S^{-1}(B)g) = A\beta gS(gS(B)) = A\beta gS^2(B)S(g) = A\beta BgS(g),
$$

for all $A \otimes B \in H \otimes H$. By taking $A \otimes B = X^1 \otimes S(X^2)\alpha X^3$ in the above equality, by (2.16) we conclude that $1 = gS(g)$, as needed. \qed

## 4. Balanced Categories and Balanced Quasi-bialgebras

Kassel and Turaev [29] associate to any rigid monoidal category $C$ a ribbon category $D(C)$ and specialize this construction to the category of representations of a Hopf algebra. The importance of their construction resides in the fact that any ribbon category produces link invariants (see [34]), and so out of any rigid monoidal category $C$ we can construct link invariants taking values in the semigroup of the endomorphisms of the unit object $1$ of $C$.

In this section we present a more general construction, in the sense that in place of the centre we consider an arbitrary braided category. Our construction has the advantage that in the quasi-Hopf case it leads to a more conceptual and less computational proof for certain balanced/ribbon isomorphisms of categories.

### 4.1. Balanced categories obtained from braided categories

Following the idea of Kassel and Turaev [29], balanced categories from monoidal ones were obtained by Drabant in [13]. The construction of Drabant generalizes as follows.

**Proposition 4.1.** Let $(C, c)$ be a braided category. Denote by $B(C, c)$ the category whose

- objects are pairs $(V, \eta_V)$ consisting of an object $V \in C$ and an automorphism $\eta_V : V \to V$ of $V$ in $C$;

- morphisms between $(V, \eta_V)$ and $(W, \eta_W)$ are morphisms $f : V \to W$ in $C$ fulfilling $\eta_W f = f \eta_V$.

Then $B(C, c)$ is a balanced category via the following structure:

(i) The tensor product on $B(C, c)$ is given by $(V, \eta_V) \otimes (W, \eta_W) = (V \otimes W, \eta_V \otimes \eta_W)$, with $\eta_V \otimes \eta_W = (\eta_V \otimes \eta_W) c_{W,V,V,W}$, and on morphisms it acts as the tensor product of $C$. The unit object in $B(C, c)$ is $(1, \eta_1 := \text{Id}_1)$.

Together with the associativity and the left and right unit constraints of $C$ these give the monoidal structure on $B(C, c)$;

(ii) The braiding on $B(C, c)$ is determined by the braiding $c$ of $C$;

(iii) The balancing on $B(C, c)$ is produced by $\eta := (\eta_V)(V,\eta_V) \in B(C, c)$.
Proof. We check that the associativity constraint of $\mathcal{C}$ is a morphism in $\mathcal{B}(\mathcal{C}, c)$; the remaining details are trivial. Assuming $\mathcal{C}$ strict monoidal, this reduces to the equality $\eta_{V \otimes W} \otimes T = \eta_V \otimes (W \otimes T)$, for all $V, W, T \in \mathcal{C}$. In diagrammatic notation, the latter comes out as

$$
\begin{array}{ccc}
V & W & T \\
\text{a} & \text{b} & \text{c} \\
\text{d} & \text{e} & \text{f}
\end{array} =
\begin{array}{ccc}
V & W & T \\
\text{g} & \text{h} & \text{i} \\
\text{j} & \text{k} & \text{l}
\end{array}.
$$

To prove the above equality we compute the right hand side of it as follows: apply twice $\circlearrowright 222$, and then apply the naturality of the braiding $c$ to the morphism $c_{W, V} c_{W, W} : V \otimes W \rightarrow V \otimes W$. In this way we get the left hand side of the equality. \qed

Remark 4.2. By Proposition 4.1 we can associate to any monoidal category $\mathcal{C}$ two balanced ones. Namely, $\mathcal{B}_l(\mathcal{C}) := \mathcal{B}(\mathcal{Z}_l(\mathcal{C}), c)$ and $\mathcal{B}_r(\mathcal{C}) := \mathcal{B}(\mathcal{Z}_r(\mathcal{C}), c)$, where $\mathcal{Z}_l/r(\mathcal{C})$ is the left/right center of $\mathcal{C}$.

As a concrete example, we can take $\mathcal{C} = \h \mathcal{M}$, $H$ a quasi-Hopf algebra with bijective antipode, in which case we have $\mathcal{B}_l(\mathcal{C}) = \mathcal{B}(\h \mathcal{D}, c)$. Here $\h \mathcal{D}$ is the category of left-right Yetter-Drinfeld modules over $H$, a braided category with braiding $c$ given by $c_{V, W} : V \otimes W \ni v \otimes w \mapsto v_{(-1)} w \otimes v_{(0)} \in W \otimes V$, for all $V, W \in \h \mathcal{D}$, where $V \ni v \mapsto v_{(-1)} \otimes v_{(0)} \in H \otimes H$ is our notation for the right $H$-action on $V$ (more details can be found in [3]). Analogously, $\mathcal{B}_r(\mathcal{C}) = \mathcal{B}(\h \mathcal{D}^H, c)$, where $\h \mathcal{D}^H$ is the category of left-right Yetter-Drinfeld modules over $H$ endowed with the braiding $c$ defined by

$$
(4.1) \quad c_{V, W} : V \otimes W \ni v \otimes w \mapsto w_{(0)} \otimes w_{(1)} \cdot v \in W \otimes V,
$$

for all $V, W \in \h \mathcal{D}^H$ (this time $W \ni w \mapsto w_{(0)} \otimes w_{(1)} \in W \otimes H$ is the notation for the right coaction of $H$ on $W$).

Proposition 4.1 can be applied also to the category $\mathcal{H} \mathcal{M}$, where $(H, R)$ is a QT quasi-Hopf algebra. We will exploit this fact in what follows.

4.2. A class of balanced quasi-Hopf algebras. To a finite dimensional quasi-Hopf algebra $H$ we can associate a QT one, its quantum double $D(H)$. Furthermore, the category of $D(H)$-representations is braided isomorphic to the category of Yetter-Drinfeld modules. In this subsection we will go further by showing that to any QT quasi-bialgebra $(H, R)$ we can associate a balanced one, denoted by $\h[H, \theta^{-1}]$, and that the category of $\h[H, \theta^{-1}]$-representations is balanced isomorphic to $\mathcal{B}(\h \mathcal{M}, c)$, where $c$ is the braiding of $\h \mathcal{M}$ defined by the $R$-matrix $R$.

More exactly, for $H$ a quasi-bialgebra we denote by $\h[H, \theta^{-1}]$ the free $k$-algebra generated by $H$ and $\theta$, with relations $h \theta = \theta h^2$, for all $h \in H$, and $\theta \theta^{-1} = \theta^{-1} \theta = 1$; by analogy with the commutative case and the terminology used in [13], we call $\h[H, \theta^{-1}]$ the Laurent polynomial algebra over $H$.

Proposition 4.3. Let $(H, R)$ be a QT quasi-bialgebra. Then $\h[H, \theta^{-1}]$ is a balanced quasi-bialgebra with structure given by

$$
\Delta |_{\h[H, \theta^{-1}]} = \Delta_H \cdot \Delta(\theta^\pm) = (R_{21} R)^\mp (\theta^\pm \otimes \theta^\pm), \quad \varepsilon |_{\h[H, \theta^{-1}]} = \varepsilon_H \cdot \varepsilon(\theta^\pm) = 1,
$$

where, for simplicity, we denote $\theta = \theta^+$ and $\theta^{-} = \theta^{-}$, and similarly for $(R_{21} R)^\pm$.

Proof. The only thing we have to prove is the quasi-coassociativity of $\Delta$ on $\theta$. It will follow then that the natural inclusion of $H$ into $\h[H, \theta^{-1}]$ is a quasi-bialgebra morphism, and that $\h[H, \theta^{-1}]$ is balanced via $\theta$.\qed
Now, since $\Delta(h) R_{21} R = R_{21} R \Delta(h)$, for all $h \in H$, the quasi-coassociativity of $\Delta$ on $\theta$ reduces to

$$\Phi(R_{21} R \otimes 1_H)(\Delta_H \otimes \text{Id}_H)(R_{21} R) \Phi^{-1} = (1_H \otimes R_{21} R)(\text{Id}_H \otimes \Delta_H)(R_{21} R).$$

This can be restated as

$$\Phi(\Delta_H \otimes \text{Id}_H)(R_{21} R)R_{21}R_{12} \Phi^{-1} = R_{32}R_{23}(\text{Id}_H \otimes \Delta_H)(R_{21} R).$$

By using (2.23) and (2.24), the left hand side of the above equality equals

$$\Phi(\Delta(R^2) \otimes R^1)(\Delta(r^1) \otimes r^2)R_{21}R_{12} \Phi^{-1} = R_{32}R_{132}R_{313}R_{132}R_{23}\Phi R_{21}R_{12} \Phi^{-1},$$

while its right hand side is equal to

$$R_{32}R_{23}(R^2 \otimes \Delta_H(R^1))(r^1 \otimes \Delta_H(r^2)) = R_{32}R_{23}\Phi R_{21}R_{213}R_{313}R_{213}R_{12} \Phi^{-1}.$$ 

Thus we must show that

$$\Phi_{132}R_{313}R_{132}R_{213}\Phi R_{21} = R_{23}\Phi R_{21}R_{213}R_{313}R_{213}$$

If $\tau$ is the usual switch for the category of $k$-vector spaces, the latter follows from

$$R_{23}\Phi R_{21}R_{213}R_{313}R_{213} \Phi \tau = R_{32}R_{132}R_{313}R_{132}R_{23}\Phi R_{21}R_{12} \Phi^{-1},$$

as desired.  

**Proposition 4.4.** Let $(H, R)$ be a QT quasi-bialgebra and $C = H \mathcal{M}$. Then $\mathcal{B}(C, c)$ and $H[\theta, \theta^{-1}] \mathcal{M}$ are isomorphic as balanced categories, where $c$ is as in (2.29) and $\mathcal{B}(C, c)$ is as in the last part of Remark 4.2.

**Proof.** Indeed, the desired isomorphism is produced by the following correspondence. To $(V, \eta_V)$ in $\mathcal{B}(C, c)$ we associate $V$ regarded as a left $H[\theta, \theta^{-1}]$-module via the $H$-module structure of it and $\theta \cdot v = \eta_V(v)$, for all $v \in V$. In this way a morphism in $\mathcal{B}(C, c)$ becomes a morphism in $H[\theta, \theta^{-1}] \mathcal{M}$.  

5. **Ribbon categories and ribbon quasi-Hopf algebras**

To any left rigid braided category $(C, c)$ (which is consequently rigid) we assign a ribbon category $\mathcal{R}(C, c)$. Thus, to any left (resp. right) rigid monoidal category $C$ we can associate a ribbon monoidal one, that will be denoted by $\mathcal{R}_l(C)$ (resp. $\mathcal{R}_r(C)$). The latter are possible due to the left and right center constructions and coincide to the ones defined in [23]. Our construction allows to associate to any QT quasi-Hopf algebra a ribbon one, and suggests also how to construct a class of ribbon quasi-Hopf algebras.

5.1. **Ribbon categories obtained from rigid monoidal categories.** Inspired by the formula in (2.39) we introduce the following category, that will turn out to be a ribbon category.

**Definition 5.1.** If $(C, c)$ is a left rigid braided (strict) monoidal category then $\mathcal{R}(C, c)$ is the category whose
• objects are pairs \((V, \eta_V)\) consisting of an object \(V\) of \(\mathcal{C}\) and an automorphism \(\eta_V\) of \(V\) in \(\mathcal{C}\) satisfying

\[
\eta_V^2 = \begin{array}{c}
\begin{array}{c}
V \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
V \\
\end{array}
\end{array} ;
\]

\[
(5.1)
\]

• morphisms \(f : (V, \eta_V) \rightarrow (W, \eta_W)\) are morphisms \(f : V \rightarrow W\) in \(\mathcal{C}\) such that \(\eta_W f = f \eta_V\).

The composition in \(\mathcal{R}(\mathcal{C})\) is given by the composition in \(\mathcal{C}\), and the identity morphism of an object \((V, \eta_V)\) is \(\text{Id}_V\).

Otherwise stated, \(\mathcal{R}(\mathcal{C}, c)\) is the full subcategory of \(\mathcal{B}(\mathcal{C}, c)\) considered in Proposition 4.1 determined by those objects \((V, \eta_V)\) of \(\mathcal{B}(\mathcal{C}, c)\) for which \(\eta_V\) obeys \((5.1)\). As we pointed out in the second part of Proposition 2.9, this is the necessary and sufficient condition that turns \(\mathcal{B}(\mathcal{C}, c)\) into a ribbon category. Actually, the ribbon structure of \(\mathcal{R}(\mathcal{C}, c)\) is encoded in the following result.

**Theorem 5.2.** Let \((\mathcal{C}, c)\) be a left rigid braided (strict) monoidal category. Then \(\mathcal{R}(\mathcal{C}, c)\) is a ribbon monoidal category as follows:

- If \((V, \eta_V)\), \((W, \eta_W)\) \(\in \mathcal{R}(\mathcal{C}, c)\) then \((V, \eta_V) \otimes (W, \eta_W) = (V \otimes W, \eta_V \otimes \eta_W)\), where

\[
\eta_{V \otimes W} = (\eta_V \otimes \eta_W) c_{W,V} c_{V,W};
\]

- The unit object is \((1_c, \eta_1) = \text{Id}_1\), and the associativity and the left and right unit constraints are the same as those of \(\mathcal{C}\);

- The braiding equals \(c\), regarded as an isomorphism in \(\mathcal{R}(\mathcal{C}, c)\);

- For \((V, \eta_V)\) an object in \(\mathcal{R}(\mathcal{C}, c)\), a left dual object for it is \((V^*, \eta_{V^*})\), where

\[
\eta_{V^*} = (\eta_V)^* ,
\]

with evaluation and coevaluation morphisms equal to \(\text{ev}_V\) and \(\text{coev}_V\), viewed now as morphisms in \(\mathcal{R}(\mathcal{C}, c)\);

- The twist is given by

\[
\eta_V : (V, \eta_V) \rightarrow (V, \eta_V),
\]

an automorphism in \(\mathcal{R}(\mathcal{C}, c)\).

**Proof.** We start by proving that \((V \otimes W, \eta_{V \otimes W})\) is an object of \(\mathcal{R}(\mathcal{C}, c)\), i.e. \(\eta_{V \otimes W}\) in \((5.2)\) obeys \((5.1)\). To this end, we need the equalities

\[
(5.5)
\]

\[
(a) \quad \begin{array}{c}
V W X Y \\
\end{array} = \begin{array}{c}
V W X Y \\
\end{array}, \quad (b) \quad \begin{array}{c}
V X \\
\end{array} = \begin{array}{c}
V X \\
\end{array}, \quad \begin{array}{c}
Y Z V T \\
\end{array} = \begin{array}{c}
Y Z V T \\
\end{array},
\]

for any morphisms \(f : X \otimes Y \rightarrow Z\) and \(g : X \rightarrow Y \otimes Z \otimes T\) in \(\mathcal{C}\), and respectively

\[
(5.6)
\]

\[
\begin{array}{c}
X Y Z \\
\end{array} = \begin{array}{c}
X Y Z \\
\end{array}, \quad \begin{array}{c}
Z Y X \\
\end{array} = \begin{array}{c}
Z Y X \\
\end{array},
\]

for any morphisms \(f : X \otimes Y \rightarrow Z\) and \(g : X \rightarrow Y \otimes Z \otimes T\) in \(\mathcal{C}\), and respectively.
valid for all $X, Y, Z \in \mathcal{C}$, which follow from the fact that $c_{\cdot, \cdot}$ is a natural isomorphism. Note that (5.10) is nothing but an equivalent form of the categorical version of the Yang-Baxter equation (2.22).

Now, if $\lambda_{V,W} : (V \otimes W)^* \to W^* \otimes V^*$ is the isomorphism in $\mathcal{C}$ defined in (2.18) and $\lambda_{V,W}^{-1}$ is its inverse, then we compute

$$V W \xrightarrow{(5.5) a)} V W \xrightarrow{(5.5) b)} V W \xrightarrow{\lambda_{V,W}} V W \xrightarrow{\lambda_{V,W}^{-1}} V W = V W,$$

where $(\cdot)$ is

This ends the proof of the fact that the tensor product of $\mathcal{R}(\mathcal{C}, c)$ is well defined at the level of objects. It is easy to see that for any two morphisms $f, f'$ in $\mathcal{R}(\mathcal{C}, c)$ their tensor product $f \otimes f'$ in $\mathcal{C}$ is actually a morphism in $\mathcal{R}(\mathcal{C}, c)$. Therefore, we have a tensor product functor on $\mathcal{R}(\mathcal{C}, c)$
that together with the associativity and the left and right unit constraints of $\mathcal{C}$ defines on $\mathcal{R}(\mathcal{C}, c)$ a monoidal structure; the unchecked details are left to the reader.

We next show that $c$ provides a braiding on $\mathcal{R}(\mathcal{C}, c)$. The only thing that must be verified is that $c_{V,W}$ is a morphism in $\mathcal{R}(\mathcal{C}, c)$, for any objects $(V, \eta_V)$, $(W, \eta_W)$ of $\mathcal{R}(\mathcal{C}, c)$. This follows directly from the definitions and from the naturality of $c$.

Let $(V^*, \eta_{V^*})$ be as in (5.3). By using the naturality of $c$ one sees that

\begin{align*}
\eta_{V^*}^{-2} &= V^* \Rightarrow V \Leftrightarrow V \Rightarrow V^* \Leftrightarrow V^* = (\eta_{V^*})^{-2}.
\end{align*}

where in the last equality of the second computation we applied the naturality of $c_{V^*,-}$ to the morphism $ev_{V^*} : V \otimes V^* \rightarrow 1$. These formulas allow to prove that

\begin{align*}
V^* \Rightarrow V \Leftrightarrow V \Rightarrow V^* \Leftrightarrow V^* = (\eta_{V^*})^{-2}.
\end{align*}

We used in the first equality an equivalent form of the naturality of $c_{V,-}^{-1}$ applied to $ev_V$, the relation (2.17) in the second equality, (5.7) in the third equality and in the fourth equality (2.17) and (5.8), respectively. In other words, the relation in (5.1) is satisfied by $\eta_{V^*}$, and thus $(V^*, \eta_{V^*})$ is an object of $\mathcal{R}(\mathcal{C}, c)$. That it is a left dual of $(V, \eta_V)$ in $\mathcal{R}(\mathcal{C}, c)$ reduces to the fact that $ev_V$ and
coev_V are morphisms in R(C, c). Towards this end, we need the equivalence

\[
\text{(5.9)} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{W}^* V
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array}
\text{=} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{W}^* V
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{W}^* V
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{V}^*
\end{array}
\end{array}
\end{array}
\end{array}
\text{=} \quad f^*,
\end{array}
\]

true for any morphism \( f : V \rightarrow W \) in \( C \) with \((V, \eta_V) \in \mathcal{R}(C, c)\), which follows from the naturality of \( c_{V,-} \) applied to \( ev_V(\text{Id}_{V^*} \otimes f \eta_W^2)_{C,V,W} : V \otimes W^* \rightarrow 1 \) and the definition of \( f^* \).

Now, that \( ev_V \) is a morphism in \( \mathcal{R}(C, c) \) is a consequence of the computation

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{V}^* V
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{V}^* V
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{V}^* V
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{V}^* V
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
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\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\text{=} \quad f^*,
\end{array}
\]

Likewise, we compute that

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{V}^* V
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{V}^* V
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
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\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\text{=} \quad \text{coev}_V,
\end{array}
\]

and so \( \text{coev}_V \) is a morphism in \( \mathcal{R}(C, c) \), as stated.

Finally, from the left rigid monoidal structure of \( \mathcal{R}(C, c) \) we get that \( \eta := (\eta_V)_V \) is a twist on \( \mathcal{R}(C, c) \), and so the proof is finished. \( \square \)

Theorem \ref{thm:ribbon-monoidal} allows to construct ribbon categories from left or right rigid monoidal categories.

**Proposition 5.3.** Let \( C \) be a left rigid (strict) monoidal category. Then \( \mathcal{R}_l(C) \) is a category whose

- objects are triples \((V, c_{V,-}, \eta_V)\) consisting of an object \( V \) of \( C \), a natural isomorphism \( c_{V,-} = (c_{V,X} : V \otimes X \rightarrow X \otimes V)_{X \in \text{Ob}(C)} \) and an automorphism \( \eta_V \) of \( V \) in \( C \) such that

\[
\text{(5.10)} \quad c_{V,X \otimes Y} = (\text{Id}_X \otimes c_{V,Y})(c_{V,X} \otimes \text{Id}_Y), \quad \forall X, Y \in C,
\]

\[
\text{(5.11)} \quad (\text{Id}_X \otimes \eta_V)c_{V,X} = c_{V,X}(\eta_V \otimes \text{Id}_X), \quad \forall X \in \text{Ob}(C);
\]

- morphisms \( f : (V, c_{V,-}, \eta_V) \rightarrow (V', c_{V',-}, \eta_{V'}) \) are morphisms \( f : V \rightarrow V' \) in \( C \) for which

\[
(\text{Id}_X \otimes f)c_{V,X} = c_{V',X}(f \otimes \text{Id}_X), \quad \text{for any object } X \text{ of } C, \text{ and } f \eta_V = \eta_{V'}f.
\]

**Proof.** One can easily check that \( \mathcal{R}_l(C) = \mathcal{R}(\mathcal{Z}_l(C), c) \), where \( \mathcal{Z}_l(C) \) is the left center of \( C \) as in \ref{def:quasi-hopf-center}, a braided category. We only notice that \ref{lem:braided} is nothing but the second equality in \ref{eq:ribbon-identity}. \( \square \)
We now uncover the ribbon structure of \( \mathcal{R}_l(\mathcal{C}) \), the category denoted by \( \mathcal{D}(\mathcal{C}) \) in [23]. For the choice of the natural isomorphism \( c_{V,-} \) below see [22] Proposition 7.4 or [35].

**Theorem 5.4.** Let \( \mathcal{C} \) be a left rigid monoidal category. Then \( \mathcal{R}_l(\mathcal{C}) \) is a ribbon category with the following structure:

- If \( (V, c_{V,-}, \eta_V), (W, c_{W,-}, \eta_W) \in \mathcal{R}_l(\mathcal{C}) \) then
  \[
  (V, c_{V,-}, \eta_V) \otimes (W, c_{W,-}, \eta_W) = (V \otimes W, c_{V \otimes W,-}, \eta_{V \otimes W}),
  \]
  where \( c_{V \otimes W,-} \) and \( \eta_{V \otimes W} \) are respectively given by
  \[
  c_{V \otimes W,X} = (c_{V,X} \otimes \text{Id}_W)(\text{Id}_V \otimes c_{W,X}), \quad \forall \ X \in \mathcal{C},
  \]
  \[
  \eta_{V \otimes W} = (\eta_V \otimes \eta_W) c_{W,V};
  \]
- The unit object is \((\textbf{1}, c_{\textbf{1},-}) = (r_X^{-1}l_X)_{X \in \mathcal{C}} \equiv \text{Id}, \eta_\textbf{1} = \text{Id}_\textbf{1}\), and the associativity and the left and right unit constraints are the same as those of \( \mathcal{C} \);
- The braiding \( c \) is determined by
  \[
  c_{V,W} : (V, c_{V,-}, \eta_V) \otimes (W, c_{W,-}, \eta_W) \to (W, c_{W,-}, \eta_W) \otimes (V, c_{V,-}, \eta_V),
  \]
  an isomorphism in \( \mathcal{R}_l(\mathcal{C}) \);
- For \( (V, c_{V,-}, \eta_V) \in \mathcal{R}_l(\mathcal{C}) \), a left dual object for it is \((V^*, c_{V^*,-}, \eta_{V^*})\), where
  \[
  c_{V^*,X} = (e_{V^*} \otimes \text{Id}_{X \otimes V^*})(\text{Id}_{V^*} \otimes c_{V^*,X}^{-1}\otimes \text{Id}_{V^*})(\text{Id}_{V^* \otimes X} \otimes \text{coev}_V),
  \]
  for all \( X \in \text{Ob}(\mathcal{C}) \), and
  \[
  \eta_{V^*} = (\eta_V)^*.
  \]

The evaluation and coevaluation morphisms are \( \text{ev}_V \) and \( \text{coev}_V \), viewed now as morphisms in \( \mathcal{R}_l(\mathcal{C}) \);
- The twist is given by
  \[
  \eta_V : (V, c_{V,-}, \eta_V) \to (V, c_{V,-}, \eta_V),
  \]
  an automorphism in \( \mathcal{R}_l(\mathcal{C}) \).

**Proof.** Since \( \mathcal{R}_l(\mathcal{C}) = \mathcal{R}(\mathcal{Z}_l(\mathcal{C}), c) \), the only thing we must check is the fact \( \mathcal{Z}_l(\mathcal{C}) \) is left rigid, provided that \( \mathcal{C} \) is so. To this end, according to [32] Proposition 3 it suffices to show that \( c_{V,-} \) in [5.16] is an isomorphism in \( \mathcal{C} \), for all \( X \in \text{Ob}(\mathcal{C}) \), since this will imply that \((V^*, c_{V^*,-})\) is a left dual of \((V, c_{V,-})\) in \( \mathcal{Z}_l(\mathcal{C}) \). But this fact was proved in [23] Lemma 4.1], by showing that

\[
\begin{array}{c}
\begin{array}{c}
X \quad V^* \\
\downarrow \quad \downarrow \\
V^* \quad X
\end{array}
\end{array}
\]

is the inverse of \( c_{V,-} \) in \( \mathcal{C} \), for all \( X \in \mathcal{C} \). Here, as the graphical notation suggests, the evaluation and coevaluation morphisms with a black dot are \( \text{ev}_V \) and \( \text{coev}_V \) defined as in (2.40) and (2.41), respectively, of course with \( c_{V,V} \) replaced by the component \( V^* \) of our \( c_{V,-} \). \qed

**Definition 5.5.** If \((\mathcal{C}, c, \eta)\) and \((\mathcal{D}, d, \theta)\) are ribbon categories then a functor \( F : \mathcal{C} \to \mathcal{D} \) is called a ribbon functor if it is a braided monoidal functor such that \( F(\eta_V) = \theta_{F(V)} \), for all \( V \in \text{Ob}(\mathcal{C}) \), and is compatible with the duality.

The left handed version of the universal property of the centre (see [21] Proposition XIII.4.3]) leads to a similar property for \( \mathcal{R}_l(\mathcal{C}) \); see [23] Theorem 2.5]. It can be derived also from the universal property of our \( \mathcal{R}(\mathcal{C}, c) \) that we prove below. Indeed, the universal property of the (left) centre assigns (under some conditions) to a monoidal functor \( F \) from a braided category \( \mathcal{D} \) to a monoidal category \( \mathcal{C} \) a braided monoidal functor \( \mathcal{Z}_l(F) \) from \( \mathcal{D} \) to \( \mathcal{Z}_l(\mathcal{C}) \), the (left) centre of \( \mathcal{C} \). Then [24] Proposition XIII.4.3] follows by applying our universal property to the braided functor \( \mathcal{Z}_l(F) \).
Proposition 5.6. Let $(\mathcal{D}, d, \theta)$ be a ribbon category, $(\mathcal{C}, c)$ a left rigid braided category and $F : (\mathcal{D}, d) \to (\mathcal{C}, c)$ a braided monoidal functor. Then there exists a unique ribbon functor $\mathcal{R}(F) : \mathcal{D} \to \mathcal{R}(\mathcal{C})$ such that $\Pi \circ \mathcal{R}(F) = F$, where $\Pi : \mathcal{R}(\mathcal{C}) \to \mathcal{C}$ is the functor that forgets the ribbon twist.

Proof. If there exists a ribbon functor $\mathcal{R}(F) : \mathcal{D} \to \mathcal{R}(\mathcal{C})$ such that $\Pi \circ \mathcal{R}(F) = F$ it follows that $\mathcal{R}(F)(X) = (F(X), F(\theta_X))$, for any object $X$ of $\mathcal{D}$, and that $\mathcal{R}(F)(f) = F(f)$, for any morphism $f$ in $\mathcal{D}$. This proves the uniqueness of $\mathcal{R}(F)$.

Conversely, define $\mathcal{R}(F)(X) = (F(X), F(\theta_X))$, for any object $X$ of $\mathcal{D}$, and for a morphism $f$ in $\mathcal{D}$ set $\mathcal{R}(F)(f) = F(f)$. Since $F$ is strong monoidal it respects the left dualities on $\mathcal{D}$ and $\mathcal{C}$. By using the uniqueness (up to isomorphism) of a left dual object we can assume without loss of generality that $F(X^\sharp) = F(X)^*$, where $X^\sharp$ is the left dual object of $X$ in $\mathcal{D}$ and $F(X)^*$ is the left dual of $F(X)$ in $\mathcal{C}$. Then the evaluation and coevaluation morphisms for the adjunction $F(X)^* \dashv F(X)$ in $\mathcal{C}$ are obtained from those of $X^\sharp \dashv X$ in $\mathcal{D}$ and the strong monoidal structure of the functor $F$. But $F$ is, moreover, braided monoidal and together with the above arguments and the fact that $\theta$ verifies (2.39) this implies that $\eta_{F(X)} := F(\theta_X)$ verifies (5.1). In other words $\mathcal{R}(F)$ is well defined on objects. It is well defined on morphisms, too, since $\theta$ is a natural isomorphism. \(\square\)

Corollary 5.7. If $(\mathcal{C}, c, \eta)$ is a ribbon category then there exists a unique ribbon functor $\mathcal{R} : \mathcal{C} \to \mathcal{R}(\mathcal{C}, c)$ such that $\Pi \circ \mathcal{R} = \text{Id}_\mathcal{C}$.

Proof. Take $\mathcal{D} = \mathcal{C}$ and $F = \text{Id}_\mathcal{C}$ in Proposition 5.6. Then $\mathcal{R} = \mathcal{R}(\text{Id}_\mathcal{C})$. \(\square\)

In what follows we also need the right handed version of $\mathcal{R}_l(\mathcal{C})$. In fact, if we start with a right rigid monoidal category then $\mathcal{C}$ is a left rigid monoidal category, and so we can consider $\mathcal{R}_l(\mathcal{C})$. Thus $\mathcal{R}_r(\mathcal{C}) := \mathcal{R}_l(\mathcal{C})$ is a ribbon category, too. More precisely, we have the following.

Proposition 5.8. Let $\mathcal{C}$ be a right rigid monoidal category. Then the objects of $\mathcal{R}_r(\mathcal{C})$ are triples $(V, c_{-V}, \theta_V)$ consisting of an object $V$ of $\mathcal{C}$, a natural isomorphism $c_{-V} = (c_{X,V} : X \otimes V \to V \otimes X)_{X \in \text{Ob}(\mathcal{C})}$ and a morphism $\theta_V : V \to V$ in $\mathcal{C}$, subject to the following conditions:

- $c_{X,V} \equiv \text{Id}_V$ and $c_{X\otimes Y,V} = (c_{X,V} \otimes \text{Id}_Y)(\text{Id}_X \otimes c_{Y,V})$, for all $X, Y \in \mathcal{C}$;
- $\theta_V$ is an automorphism of $V$ in $\mathcal{C}$ obeying $c_{X,V}(\text{Id}_X \otimes \theta_V) = (\theta_V \otimes \text{Id}_X)c_{X,V}$, for all $X \in \mathcal{C}$, and

\begin{equation}
\theta_V^{-2} := (\text{Id}_V \otimes \text{ev}'_V)(c_{-V,V}^{-1} \otimes \text{Id}_V)(\text{Id}_V \otimes c_{V,V})(\text{Id}_V \otimes \text{coev}'_V).
\end{equation}

A morphism $f : (V, c_{-V}, \theta_V) \to (W, c_{-W}, \theta_W)$ in $\mathcal{R}_r(\mathcal{C})$ is a morphism $f : V \to W$ in $\mathcal{C}$ such that $c_{X,W}(f \otimes \text{Id}_X) = (\text{Id}_X \otimes f)c_{X,V}$, for all $X \in \mathcal{C}$.

The category $\mathcal{R}_r(\mathcal{C})$ is ribbon via the following structure:

- the tensor product of $(V, c_{-V}, \theta_V)$ and $(W, c_{-W}, \theta_W)$ in $\mathcal{R}_r(\mathcal{C})$ is $(V \otimes W, c_{W,V}, \theta_{V\otimes W})$, where $c_{W,V} = (c_{V,W} \otimes \text{Id}_W)(c_{X,W} \otimes \text{Id}_W)$, for all $X \in \mathcal{C}$, and $\theta_{V\otimes W} = (\theta_V \otimes \theta_W)c_{W,V}c_{V,W}$, and the tensor product of two morphisms in $\mathcal{R}_r(\mathcal{C})$ is their tensor product in $\mathcal{C}$, while the associativity and the left and right unit constraints are the same as those of $\mathcal{C}$;
- the braiding between two objects $(V, c_{-V}, \theta_V)$ and $(W, c_{-W}, \theta_W)$ in $\mathcal{R}_r(\mathcal{C})$ is given by $c_{V,W}$;
- the right dual object of $(V, c_{-V}, \theta_V)$ in $\mathcal{R}_r(\mathcal{C})$ is $(*V, *c_{-V}, *\theta_V)$ determined by

\begin{equation}
c_{*V,V} = (\text{Id}_{V^\otimes V} \otimes \text{ev}'_V)(\text{Id}_{V^\otimes V} \otimes c_{V,V}^{-1})(\text{Id}_{V^\otimes V} \otimes \text{coev}_V)(\text{Id}_{V^\otimes V} \otimes \text{coev}_V),
\end{equation}

for all $X \in \mathcal{C}$, $\theta_V = *\theta_V$, and the evaluation and coevaluation morphisms equal $\text{ev}'_V$ and $\text{coev}'_V$, respectively;
- the twist is given by $\theta_V : (V, c_{-V}, \theta_V) \to (V, c_{-V}, \theta_V)$, for all $V \in \mathcal{C}$.

Proof. As $\mathcal{R}_r(\mathcal{C}) := \mathcal{R}_l(\mathcal{C})$, everything follows from the above comments and results. \(\square\)
5.2. A class of ribbon quasi-Hopf algebras. In general, \( H[\theta, \theta^{-1}] \) considered in Subsection 4.2 is not a quasi-Hopf algebra, and so neither a ribbon quasi-Hopf algebra. To "make" it ribbon, we have to consider a quotient of it. Actually, we have to consider the \( k \)-algebra \( H[\theta] := \frac{H[\theta]}{\langle \theta^2 - uS(u) \rangle} \) instead of \( H[\theta, \theta^{-1}] \), where \( H[\theta] \) is the free \( k \)-algebra generated by \( H \) and \( \theta \) with relations \( h\theta = \theta h \), for all \( h \in H \), and \( u \) is as in (2.27).

In what follows, we still denote by \( \theta \) the class in \( H(\theta) \) of \( \theta \).

Proposition 5.9. If \( (H, R) \) is a QT quasi-Hopf algebra then \( H(\theta) \) is a quasi-Hopf algebra with structure determined by \( \Delta |_H = \Delta_H, \varepsilon |_H = \varepsilon_H, S |_H = S \),

\[
\Delta(\theta) = (\theta \otimes \theta)(R_{21}R)^{-1}, \quad \varepsilon(\theta) = 1, \quad S(\theta) = \theta,
\]

and the reassociator and distinguished elements that define the antipode equal to those of \( H \). Furthermore, \( (H(\theta), R) \) is QT and \( \theta^{-1} \) defines a ribbon twist on \( H(\theta)M^1 \) as in (2.37).

Proof. One can see easily that \( \Delta(h)R_{21}R = R_{21}R\Delta(h) \), for all \( h \in H \). So by (2.31) and (2.32) one compute that

\[
\Delta(uS(u)) = f^{-1}(S \otimes S)(f_{21})(uS(u) \otimes uS(u))(S \otimes S)(f_{21})^{-1}f(R_{21}R)^{-2},
\]

and because \( uS(u) \) is central in \( H \) we get \( \Delta(uS(u)) = (uS(u) \otimes uS(u))(R_{21}R)^{-2} \). This shows that \( \Delta \) is well defined on \( \theta \). Also, it follows from Proposition 4.3 that \( H(\theta) \) is a quasi-bialgebra.

Furthermore, since \( S(\theta^2 - uS(u)) = S(\theta^2) - S^2(u)S(u) = \theta^2 - uS(u) \) we deduce that \( S \) is well defined on \( \theta \), too. So it remains to show the equalities

\[
S(\theta_1)\alpha_2 = \alpha \quad \text{and} \quad \theta_1\beta S(\theta_2) = \beta.
\]

The equality in (2.29) can be rewritten as \( S(R^2)uR^1 = \alpha \) or, equivalently, as \( S(R^1)\alpha R^2 = S^{-1}(\alpha u^{-1}) = S(u^{-1}\alpha) \), cf. (2.28). Hence

\[
S(\theta_1)\alpha_2 = S(R^1)\alpha R^2 = S(u^{-1}\alpha)S(\beta^2) = S(u^{-1}\alpha) = \alpha u^{-1}S(u^{-1})\theta^2 = \alpha,
\]

as required; in the last equality we used that \( uS(u) = S(u)u \).

Notice that the formula in (2.30) is equivalent to \( R \), \( S(R^2)\beta u = \beta \), and therefore with \( R^2 \beta S(R^1) = u^{-1}S(\beta) \), because of (2.28). The latter gives us

\[
\theta_1\beta S(\theta_2) = \theta_1\beta S(R^2) = \theta_1 u^{-1}S(R^2) = u^{-1}(S^2(R^2)\beta S(R^1))\theta^2 = \theta_1 u^{-1}S^2(\beta)\theta = \theta_1 u^{-1}S^2(\beta) = \beta,
\]

since \( uS(u) = S(u)u \). Finally, \( R \) is an \( R \)-matrix for \( H(\theta) \) because so is for \( H \) and

\[
R\Delta(\theta) = R(R_{21}R)^{-1}(\theta \otimes \theta) = R_{21}^{-1}(\theta \otimes \theta) = (RR_{21})^{-1}R(\theta \otimes \theta) = \Delta^{cop}(\theta)R.
\]

Clearly \( \theta^{-1} \) defines a ribbon structure for \( (H(\theta), R) \), and this completes the proof. \( \square \)
Our next goal is to show that the category of finite dimensional left modules over \( H(\theta) \) can be identified with the category \( \mathcal{R}(H, \mathcal{M}_{\text{fd}}^d, c) \), where \( c \) is defined by (2.20).

**Theorem 5.10.** Let \((H,R)\) be a QT quasi-Hopf algebra, so \(H, \mathcal{M}_{\text{fd}}^d\) is a rigid braided category. Then \( \mathcal{R}(H, \mathcal{M}_{\text{fd}}^d, c) \) identifies as a ribbon category with \(H(\theta), \mathcal{M}_{\text{fd}}\).

**Proof.** By taking \(C = H, \mathcal{M}_{\text{fd}}^d\) in Definition 4.1, we deduce that an object of the category \( \mathcal{R}(H, \mathcal{M}_{\text{fd}}^d, c) \) is a pair \((V, \eta_V)\) consisting of a finite dimensional left \( H\)-module \( V \) and an \( H\)-automorphism \( \eta_V \) of \( V \) such that

\[
\eta^{-2}(v) = X^1 r^2 R^1 \beta S(X^2 r^1 R^2) \alpha X^3 \cdot v
\]

for all \( v \in V \). Thus objects of \( \mathcal{R}(H, \mathcal{M}_{\text{fd}}^d, c) \) are pairs \((V, \eta_V)\) consisting of a finite dimensional left \( H\)-module and an \( H\)-automorphism \( \eta_V \) of \( V \) satisfying \( \eta^{-2}(v) = (uS(u))^{-1} \cdot v \), for all \( v \in V \).

Clearly, a morphism \( f : (V, \eta_V) \to (W, \eta_W) \) is a left \( H\)-linear morphism \( f : V \to W \) such that \( \eta_W f = f \eta_V \).

Let \( F : \mathcal{R}(H, \mathcal{M}_{\text{fd}}^d, c) \to H(\theta), \mathcal{M}_{\text{fd}}^d \) be the functor defined as follows: \( F(V, \eta_V) = V \) regarded as \( H(\theta)\)-module via the \( H\)-action on \( V \) and \( \theta \cdot v = \eta^{-1}_V(v) \); \( F \) acts as identity on morphisms.

We can easily see that \( \theta^2 = uS(u) \) together with \( \eta^{-2}(v) = (uS(u))^{-1} \cdot v \), for all \( v \in V \), implies that \( F \) is a well defined functor. It provides an isomorphism of categories, its inverse being the functor \( G : H(\theta), \mathcal{M}_{\text{fd}}^d \to \mathcal{R}(H, \mathcal{M}_{\text{fd}}^d, c) \) given by \( G(V) = (V, \eta_V : V \ni v \mapsto \theta^{-1} \cdot v \in V \) for all \( V \in H(\theta), \mathcal{M}_{\text{fd}}^d \).

The functor \( F \) is monoidal since

\[
\theta \cdot (v \otimes w) = \eta^{-1}_V(v \otimes w)
\]

for all \( V, W \in \mathcal{R}(H, \mathcal{M}_{\text{fd}}^d, c) \), \( v \in V \) and \( w \in W \). Furthermore, \( F \) is braided because for both categories the braiding is defined by \( c \).

The ribbon structure \( \tilde{\eta} \) of \( H(\theta), \mathcal{M}_{\text{fd}}^d \) is induced by the element \( \theta^{-1} \), in the sense that \( \tilde{\eta}_V(v) = \theta^{-1} \cdot v = \eta_V(v) \), for all \( v \in V \in H(\theta), \mathcal{M}_{\text{fd}}^d \). Otherwise stated, the two categories have the same ribbon structure, and therefore \( F \) is a ribbon isomorphism functor. Observe also that \( F \) is compatible with the left rigid monoidal structures on \( \mathcal{R}(H, \mathcal{M}_{\text{fd}}^d, c) \) and \( H(\theta), \mathcal{M}_{\text{fd}}^d \). More precisely, we have \( F \left( V^*, \eta_{V^*} \right) = V^* \) with left \( H(\theta)\)-module structure induced by that of the left \( H\)-module and \( \theta \cdot v^* = \eta^{-1}_{V^*}(v^*) = (\eta^{-1}_V)^*(v^*) = v^* \circ \eta^{-1}_V = v^*(\theta) = v^*(S(\theta)) \), for all \( v^* \in V^* \), as required.

**Remark 5.11.** If \((H,R)\) is QT then \( \tilde{R} = R_{\text{s}}^{-1} = R_{\text{s}}^2 \otimes R_{\text{s}}^{-1} \) is another \( R\)-matrix for \( H \). In addition, if \( \tilde{u} \) is the element (2.23) corresponding to \((H,\tilde{R})\) then by the relation (6.23) in [M] we have that \( \tilde{u} = S(u^{-1}) \), so \( \tilde{u}S(\tilde{u}) = (uS(u))^{-1} \). Therefore, if \( c \) is the braiding on \( H, \mathcal{M}_{\text{fd}}^d \) defined by \( \tilde{R} \) then \( \mathcal{R}(H, \mathcal{M}_{\text{fd}}^d, \tilde{c}) \) is a ribbon category that is isomorphic to \( H(\theta), \mathcal{M}_{\text{fd}}^d \), too. To see this, observe that an object of \( \mathcal{R}(H, \mathcal{M}_{\text{fd}}^d, \tilde{c}) \) is a pair \((V, \eta_V)\) consisting of \( V \in H, \mathcal{M}_{\text{fd}}^d \) and an automorphism \( \eta_V \) of the left \( H\)-module \( V \) such that \( \theta^2_{\text{s}}(v) = uS(u) \cdot v \), for all \( v \in V \). It is clear at this point that \( (V, \eta_V) \mapsto (V, \theta_V := \eta^{-1}_V) \) defines the desired isomorphism of categories.
Corollary 5.12. Let $H$ be a finite dimensional quasi-Hopf algebra and $D(H)$ its quantum double. If $D(H, \theta) := D(H) \theta$ then
\[ R_r(H, M_{fd}) \simeq R(L, D^{H_{fd}}) \simeq D(H, \theta, M_{fd}) \]
as ribbon categories, where $\epsilon$ is the braiding defined in (4.7).

Proof. The first isomorphism can be deduced from the definition $R_r(H, M_{fd}) = R(L, D^{H_{fd}})$ and the braided isomorphism between $Z_r(H, M_{fd})$ and $H, \theta, D^{H_{fd}}$ established by [3, Theorem 2.2]. The second one follows from the braided isomorphism $(H, D^{H_{fd}}, \epsilon) \cong (D(H, M_{fd}, R_D)$ established in [10, Proposition 3.1], and Theorem 5.10.

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