A SHEAF-THEORETIC CONSTRUCTION OF SHAPE SPACE

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Abstract. We present a sheaf-theoretic construction of shape space—the space of all shapes. We do this by describing a homotopy sheaf on the poset category of constructible sets, where each set is mapped to its Persistent Homology Transforms (PHT). Recent results that build on fundamental work of Schapira have shown that this transform is injective, thus making the PHT a good summary object for each shape. Our homotopy sheaf result allows us to “glue” PHTs of different shapes together to build up the PHT of a larger shape. In the case where our shape is a polyhedron we prove a generalized nerve lemma for the PHT. Finally, by re-examining the sampling result of Smale-Niyogi-Weinberger, we show that we can reliably approximate the PHT of a manifold by a polyhedron up to arbitrary precision.

1. Introduction and Main Results

Shape spaces are intended to provide a single framework for comparing shapes. Different shapes are rendered as different points in shape space and comparisons of shapes can be formalized in terms of distances between points. Often one wants to decorate shape space with extra structure, such as a set of landmarks (as in the Kendall approach \[31, 32\]) or with a choice of parameterization (as in the Grenander approach \[17\]), but this extra structure is regarded as lying orthogonal to the base manifold of shapes; see Figure 1. Fiber bundles provide a language for formalizing this orthogonality \[29\] and can be used to unify Kendall’s and Grenander’s constructions, as reviewed below, but they are limiting as well.

Implicit in both the Kendall and Grenander approach to shape space is the assumption that each pair of shapes can be related to one another via one-to-one correspondences; for Kendall these are correspondences of landmarks, and for Grenander these are smooth diffeomorphisms. These assumptions severely limit the applicability of these approaches to many datasets of interest. For example, in a dataset of fruit fly wings, some mutant flies have extra lobes of veins \[34\]; or, in a dataset of brain arteries, many of the arteries cannot be continuously mapped to each other \[4\]. Indeed, in large databases such as MorphoSource and Phenome10K \[9, 19\], the CT scans of skulls across many clades are not diffeomorphic. Consequently, there is a real need for a shape space that does not require correspondences and diffeomorphisms.

In this paper we introduce a truly general construction of shape space. Topologically different subsets of \(\mathbb{R}^d\) can be viewed simultaneously and compared in our framework. We accomplish this by passing from the land of fiber bundles to the world of sheaves, which replaces the local triviality condition of fiber bundles with the local continuity condition of sheaves. This passage requires two preparatory steps of categorical generalization:

1. Instead of a “base manifold” of shapes we work with a “base poset” of constructible sets \(\mathcal{CS}(\mathbb{R}^d)\) ordered by inclusion. This poset is equipped with a notion of continuity via a Grothendieck topology.
Figure 1. Previous constructions of shape space by Kendall and Grenander implicitly introduce a fiber bundle perspective on shape space. The fibers encode all possible landmarks or parameterizations of a shape and the base space records the shape as an equivalence relation. The PHT-based shape space introduced here uses a slightly more complex construction where the base space is replaced by a base poset and the fibers are unique sheaf-theoretic representations of the shape.

(2) Each shape—that is, each point $M \in \mathcal{CS}(\mathbb{R}^d)$—is equivalently regarded via its persistent homology transform $\text{PHT}(M)$, which is an object in the derived category of sheaves $\mathcal{D}^b(\text{Shv}(S^{d-1} \times \mathbb{R}))$.

With these observations in place, our main result can be summarized as follows.

**Theorem 1.1.** The following assignment is a homotopy sheaf:

$$\mathcal{F} : \mathcal{CS}(\mathbb{R}^d)^{op} \to \mathcal{D}^b(\text{Shv}(S^{d-1} \times \mathbb{R})) \quad M \mapsto \text{PHT}(M).$$

Intuitively, this result allows us to interpolate between shapes in a continuous way via their persistent homology transforms; continuity is mediated via the Grothendieck topology on $\mathcal{CS}(\mathbb{R}^d)$. More precisely, our main result establishes Čech descent for the persistent homology transform, which is a generalization of the sheaf axiom that holds for higher degrees of homology. In one concrete form, our main result implies the following:

**Theorem 1.2** (Nerve Lemma for the PHT). If $M \in \mathcal{CS}(\mathbb{R}^d)$ is a polyhedron, i.e., it can be written as a finite union of closed linear simplices $\mathcal{M} = \{\sigma_i\}_{i \in \Lambda}$, then the persistent homology transform in degree $n$, written $\text{PHT}^n(M)$, is isomorphic to the $n$-th cohomology of the following complex of sheaves:

$$0 \to \bigoplus_{I \subset \Lambda \text{ s.t. } |I|=1} \text{PHT}^0(\mathcal{M}_I) \to \bigoplus_{J \subset \Lambda \text{ s.t. } |J|=2} \text{PHT}^0(\mathcal{M}_J) \to \cdots$$

Here $\mathcal{M}_I$ with $|I| = k$ denotes the disjoint union of depth $k$ intersections of closed simplices appearing in the cover $\mathcal{M}$.

In Theorem 3.19 we interpret the homotopy sheaf axiom for the PHT in terms of a generalized inclusion-exclusion principle for the Euler Characteristic Transform (ECT), which is the decategorification of the PHT.

It should be noted that positive scalar curvature of a constructible set $M$ (when defined) obstructs Theorem 1.2 from being directly applied, as cover elements may necessarily have
higher homology when viewed in a direction normal to that point. See Figure 2 for an example.

As such it is desirable to have an approximation result that is provably stable under the persistent homology transform and allows us to work degree-0 homology along. We do this by proving a general stability theorem for the PHT across shapes i.e. small perturbations of shapes result in at most small changes in the corresponding persistent homology transforms. The result reads:

**Theorem 1.3. (Stability of the PHT)** For any two constructible sets \( M, N \in \mathcal{CS}(\mathbb{R}^d) \) that are homotopy equivalent and are \( \epsilon \)-close, the PHTs of \( M \) and \( N \) are \( \epsilon \)-close.

**Corollary 1.4 (Approximation of the PHT).** For any compact submanifold \( M \) and any \( \epsilon > 0 \) we can construct a polyhedron \( N \) so that with high probability \( \text{PHT}(M) \) and \( \text{PHT}(N) \) are \( \epsilon \)-close. \( \text{PHT}(N) \) can then be computed using Theorem 1.2.

1.1. Prior Work on Shape Space. We briefly outline the main approaches to shape space to better situate the contributions of this paper. Some of these approaches are quite old and date back to Riemann [28] and the interested reader is encouraged to consult the survey articles [3] and [22] for more information. However, at a high-level, there are three main approaches to shape space that have been worked out in some detail: the landmark approach; the diffeomorphism and optimal control approach; and the persistent homology transform and Euler characteristic transform approach endorsed here.

The first commonly accepted shape space was pioneered in the works of Kendall [31, 32] and Bookstein [6] where a shape is defined by a set of \( k \) landmark points in \( \mathbb{R}^d \), where typically \( d = 2, 3 \). Each shape is defined by \( k \) points and the \( i \)-th point in one shape corresponds to the \( i \)-th point in every other shape in the space; this introduces the central notion of correspondences in shape space. The shape space defined by these landmark points is called Kendall’s shape space and is denoted

\[
\Sigma^k_d := \{(\mathbb{R}^d)^{k-1} \setminus \{0\})/\text{Sim},
\]

where Sim is the group of rotations and dilations. Note that in \( \Sigma^k_d \) a shape is reduced to a \( d \times k \) matrix, which is a very convenient representation. However, the downside of this approach is that a user will need to decide on landmarks before analysis can be carried out, and reducing modern databases of 3-dimensional micro-computed tomography (CT) scans [19, 9] to landmarks can result in a great deal of information loss.
The second commonly accepted shape space was pioneered by Grenander [17], although some aspects were anticipated by [10]. In these works, a shape space is specified for each manifold $M$ and dimension $d$. One then considers all possible immersions modulo the group of reparameterizations of $M$, i.e.

$$\text{Shape}(M) := \text{Imm}(M, \mathbb{R}^d)/\text{Diff}(M).$$

Variation in shape is then modeled by the action of the Lie group of diffeomorphisms on $\mathbb{R}^d$. The advantage of Grenander’s approach is that it bypasses the need for landmarks, but the resulting spaces of interest are infinite-dimensional and shapes with different topology cannot be compared. However, many tools have been developed that efficiently compare the similarity between shapes in large databases via algorithms that continuously deform one shape into another [7, 36, 8, 18].

Finally, building on fundamental work of Schapira [38, 39], two topological transforms—the Euler characteristic transform (ECT) and the persistent homology transform (PHT)—were introduced in [41] to allow for comparison of non-diffeomorphic shapes. The ECT and PHT have two useful properties: standard statistical methods can be applied to the transformed shape and the transforms are injective [25, 13], so no information about the shape is lost via the transform. The utility of the transforms for applied problems in evolutionary anthropology, biomedical applications and plant biology were demonstrated in [11, 43, 40, 2]. The shape space we construct in this paper is a dramatic generalization of the sheaf-theoretic formulation of the PHT found in [13].

2. Background on Constructibility, Persistent Homology and Sheaves

In this section we briefly recall background material from [13].

2.1. O-Minimality. Although shapes in the real world can exhibit wonderful complexity, we impose a fairly weak tameness hypothesis that prohibits us from considering infinitely constructed shapes such as fractals and Cantor sets. This tameness hypothesis is best expressed using the language of o-minimal structures [42].

**Definition 2.1.** An o-minimal structure $\mathcal{O} = \{\mathcal{O}_d\}$, is a specification of a boolean algebra of subsets $\mathcal{O}_d$ of $\mathbb{R}^d$ for each natural number $d \geq 0$. In particular, we assume that $\mathcal{O}_1$ contains only finite unions of points and intervals. We further require that $\mathcal{O}$ be closed under certain product and projection operators, i.e. if $A \in \mathcal{O}_d$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ are in $\mathcal{O}_{d+1}$; and if $A \in \mathcal{O}_{d+1}$, then $\pi(A) \in \mathcal{O}_d$ where $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is axis aligned projection. It is a fact that $\mathcal{O}$ contains all semi-algebraic sets, but may contain certain regular expansions of these sets. Elements of $\mathcal{O}$ are called definable sets and compact definable sets are called constructible sets. The subcollection of constructible subsets in $\mathcal{O}_d$ is denoted $\mathcal{CS}(\mathbb{R}^d)$.

Intuitively, any shape that can be faithfully represented via a mesh on a computer is a constructible set. This is because every constructible set is triangulable. This property also implies that certain algebraic topological signatures, such as homology, are well-defined for any constructible set.

2.2. Persistent Homology. A useful algebraic summary of a constructible set $M \in \mathcal{CS}(\mathbb{R}^d)$ is homology in degree $n$ with coefficients in a field $\mathbb{k}$, written $H_n(M; \mathbb{k})$. Typically, we assume that $\mathbb{k}$ is of characteristic 0 so that homology and cohomology are isomorphic, i.e. $H_n(M; \mathbb{k}) \cong H^n(M; \mathbb{k})$. For constructible sets all usual theories of (co)homology—singular,
simplicial and sheaf—agree. The dimension of the \(n\)th homology group of \(M\) is also called the \(n\)th Betti number \(\beta_n(M)\), which can be used to easily distinguish shapes such as a torus \(\mathbb{T}\) and a sphere \(\mathbb{S}^2\), e.g. \(\beta_1(\mathbb{T}) = 2\) and \(\beta_1(\mathbb{S}^2) = 0\).

Persistent homology is a powerful refinement of homology as it encodes how homology of \(M\) changes under an \(\mathbb{R}\)-indexed filtration, i.e. a collection of subsets \(\{M_t\}_{t \in \mathbb{R}}\). This encoding is more subtle than simply a graph of the Betti numbers as the filtration parameter changes. Specifically, the persistent homology barcode (or diagram) in degree \(n\), written \(\text{PH}^n(M)\), is a multi-set of intervals in \(\mathbb{R}\) that represents lifetimes of homology classes in degree \(n\). The space of all barcodes is written \(\text{Dgm}\) and can be equipped with a Wasserstein \(p\)-norm for any \(p \in [1, \infty]\). Typically \(p = \infty\), and the distance between barcodes is called the bottleneck distance. The persistent homology transform studies the persistent homology of a constructible subset \(M \in \mathcal{CS}(\mathbb{R}^d)\) by considering the filtration \(M_{v,t} = \{x \in M \mid x \cdot v \leq t\}\) for all directions \(v \in \mathbb{S}^{d-1}\).

**Definition 2.2** (PHT: Map Version). Let \(M \in \mathcal{CS}(\mathbb{R}^d)\) be a constructible set. The **persistent homology transform** of \(M\) is defined as the continuous map

\[
PHT(M) : \mathbb{S}^{d-1} \to \text{Dgm}^d \quad v \mapsto (\text{PH}^0(M, v), \text{PH}^1(M, v), \ldots, \text{PH}^d(M, v))
\]

Although homology is usually a lossy summary of a shape, knowing persistent homology of a shape in every direction completely determines the shape.

**Theorem 2.3** (PHT: Injectivity, cf. [13, 25]). If \(PHT(M) = PHT(N)\), then \(M = N\).

### 2.3. Sheaf Theory

Persistent homology can be viewed as a parameterized homology theory, where the base space for the parameter is \(\mathbb{R}\). In similar spirit, the persistent homology transform can be viewed as a parameterized homology theory where the base space is \(\mathbb{S}^{d-1} \times \mathbb{R}\). Understanding how (co)homology of a space varies with reference to a map to a base space was Leray’s motivation behind the development of sheaf theory, but it quickly became adapted for more general purposes over the intervening 80 years.

**Definition 2.4** (Pre-Sheaves and Sheaves). Let \(\mathcal{X}\) be a topological space and let \(\text{Open}(\mathcal{X})\) be the poset of open sets in \(\mathcal{X}\). A **pre-sheaf** of vector spaces on \(\mathcal{X}\) is a functor \(\mathcal{F} : \text{Open}(\mathcal{X})^{\text{op}} \to \text{Vect}\). A pre-sheaf is a **sheaf** if for every open set \(U\) and open cover \(\mathcal{U} = \{U_i\}_{i \in \Lambda}\) we have that

\[
\mathcal{F}(U) \cong H^0\left[\bigoplus \mathcal{F}(U_i) \to \bigoplus \mathcal{F}(U_{ij}) \to \bigoplus \mathcal{F}(U_{ijk}) \to \cdots\right]
\]

where \(U_{ij} = U_i \cap U_j\) and so on. In other words, for a sheaf, the value of \(\mathcal{F}\) on a large open set \(U\) can be computed in terms of Čech cohomology of the nerve of a cover \(\mathcal{U}\) with coefficients in \(\mathcal{F}\). More generally, one can define sheaves valued in categories that are not necessarily abelian, such as \(\text{Set}\), but where the categorical notion of limit\(^1\) makes sense. One then modifies the sheaf axiom to instead require that

\[
\mathcal{F}(U) \cong \lim_{\leftarrow} \left[\prod_i \mathcal{F}(U_i) \Rightarrow \prod_{i,j} \mathcal{F}(U_{ij}) \Rightarrow \cdots\right].
\]

**Definition 2.5.** Let \(\text{Dat}\) be a complete “data” category, i.e. all limits in \(\text{Dat}\) exist. Denote the category of pre-sheaves and sheaves on \(\mathcal{X}\) valued in \(\text{Dat}\) by \(\text{PShv}(\mathcal{X}; \text{Dat})\) and

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\(^1\)Kernels, equalizers, and pullbacks are all examples of limits.
\[ \text{Shv}(\mathbb{X}; \text{Dat}), \text{respectively. Since every sheaf is a pre-sheaf there is a natural inclusion of categories} \]

\[ \text{ps} : \text{Shv}(\mathbb{X}; \text{Dat}) \hookrightarrow \text{PShv}(\mathbb{X}; \text{Dat}). \]

Under suitable hypotheses\(^2\) on \(\text{Dat}\), the \(\text{ps}\) has a right adjoint called \textbf{sheafification}

\[ \text{sh} : \text{PShv}(\mathbb{X}; \text{Dat}) \to \text{Shv}(\mathbb{X}; \text{Dat}). \]

Typically we set \(\text{Dat} = \text{Vect}\) and let \(\text{Shv}(\mathbb{X})\) denote the category of sheaves of vector spaces.

\section*{Definition 2.6 (Leray Sheaves).} Suppose \(f : \mathbb{Y} \to \mathbb{X}\) is a proper continuous map of spaces, then the \(i^{th}\) \textbf{Leray sheaf of} \(f\), written \(R^i f_! k\) is the sheaf associated to the pre-sheaf

\[ \text{sh} \left[ U \mapsto H^i(f^{-1}U; k) \right]. \]

We now have enough language to describe the PHT sheaf-theoretically.

\section*{Definition 2.7 (PHT: Sheaf Version, cf. [13]).} Let \(M \in \mathcal{CS}(\mathbb{R}^d)\) be a constructible set. Associated to \(M\) is the \textbf{auxiliary total space}

\[ Z_M := \{ (x, v, t) \in M \times S^{d-1} \times \mathbb{R} \mid x \cdot v \leq t \}. \]

The \(i^{th}\) \textbf{persistent homology transform sheaf} of \(M\), written \(\text{PHT}_i(M)\), is the \(i^{th}\) Leray sheaf of the map \(f_M : Z_M \to S^{d-1} \times \mathbb{R}\) that projects onto the last two factors.

Since \(M\) is compact and \(f_M\) is a projection, we see that \(f_M\) is proper. By the proper base change theorem [27], the \(i^{th}\) Leray sheaf at \(v, t \in S^{d-1} \times \mathbb{R}\) is valued at the \(i^{th}\) cohomology of the fiber i.e. \(H^i(f_M^{-1}(v, t); k_{Z_M})\).

\section*{2.4. Derived Sheaf Theory.} There is a third way of describing the persistent homology transform that requires the language of derived categories. We briefly recall a definition of the derived category of an abelian category.

\section*{Definition 2.8.} Let \(\mathcal{A}\) be an abelian category, in particular every morphism has a kernel and cokernel. Consider the category of bounded chain complexes of objects in \(\mathcal{A}\), written \(\mathcal{C}^b(\mathcal{A})\). Associated to this category is the \textbf{homotopy category} \(\mathcal{K}^b(\mathcal{A})\) of chain complexes, which has the same objects as \(\mathcal{C}^b(\mathcal{A})\), but where morphisms are homotopy classes of chain maps. Recall that a chain map \(\varphi : (A^* , d_A) \to (B^* , d_B)\) is a \textbf{quasi-isomorphism} if it induces isomorphisms on all cohomology groups

\[ H^i(\varphi) : H^i(A^*) \to H^i(B^*). \]

Let \(\mathcal{Q}\) denote the class of quasi-isomorphisms. The \textbf{bounded derived category} of \(\mathcal{A}\) is the localization of \(\mathcal{K}^b(\mathcal{A})\) at the collection of morphisms \(\mathcal{Q}\), i.e.

\[ \mathcal{D}^b(\mathcal{A}) := \mathcal{K}^b(\mathcal{A})[\mathcal{Q}^{-1}]. \]

\section*{Remark 2.9.} An alternative definition of the derived category makes use of the assumption that \(\mathcal{A}\) has enough injectives, i.e. every object in \(\mathcal{A}\) has an injective resolution or, said differently, every object in \(\mathcal{A}\) is quasi-isomorphic to a complex of injective objects. Under this assumption, the derived category of \(\mathcal{A}\) is equivalently defined as the homotopy category of injective objects in \(\mathcal{A}\), i.e.

\[ \mathcal{D}^b(\mathcal{A}) \simeq \mathcal{K}^b(\text{Inj} - \mathcal{A}). \]

\(^2\)See condition (17.4.1) in [30].
Remark 2.9 provides an easier to understand prescription for working with the derived category. One simply takes an object, e.g. a sheaf, replaces it with its injective resolution and works with the resolution instead.

**Definition 2.10.** Suppose $F : A \rightarrow B$ is an additive and left-exact functor, i.e. it commutes with direct sums and preserves kernels, then the total right derived functor of $F$, written $RF : D^b(A) \rightarrow D^b(B)$ is defined by

$$RF(A^\bullet) := F(I^\bullet)$$

for $I^\bullet$ an injective resolution of $A^\bullet$. In general, one can substitute $I^\bullet$ with any $F$-acyclic resolution of $A^\bullet$. Such resolutions are said to be adapted to $F$.

We can now define the persistent homology transform as a derived sheaf.

**Definition 2.11 (PHT: Derived Version, cf. [13]).** Let $M \in CS(\mathbb{R}^d)$ be a constructible set. Let $Z_M$ be the auxiliary space construction from Definition 2.7. Let $f_M : Shv(Z_M) \rightarrow Shv(S^{d-1} \times \mathbb{R})$ be the pushforward (or direct image) functor along the projection map $f_M : Z_M \rightarrow S^{d-1} \times \mathbb{R}$. The derived PHT sheaf is

$$PHT(M) := Rf_M^*k_{Z_M} \in D^b(Shv(S^{d-1} \times \mathbb{R})).$$

More explicitly we can describe this right-derived pushforward as follows: For a topological space $X$ we let $S^p(U; k)$ denote the group of singular $p$-cochains of $U \subset X$ with coefficients in $k$. Define $\mathcal{F}^p(X, k) = sh(U \mapsto S^p(U; k))$ where $sh$ stands for sheafification. The constant sheaf $k_{Z_M}$ admits a flabby resolution by singular cochains:

$$0 \rightarrow k_{Z_M} \rightarrow \mathcal{F}^0(Z_M; k) \rightarrow \mathcal{F}^1(Z_M; k) \rightarrow \mathcal{F}^2(Z_M; k) \rightarrow \cdots$$

Because flabby resolutions form an adapted class for the pushforward functor [5] we can describe $PHT(M)$ as the pushforward of the complex of sheaves of singular cochains:

$$Rf_M^*k_{Z_M} := f_M^*\mathcal{F}^0(Z_M; k) \rightarrow f_M^*\mathcal{F}^1(Z_M; k) \rightarrow \cdots$$

3. A Homotopy Sheaf on Shape Space

As mentioned in the introduction, we want to build a shape space using a sheaf-theoretic construction on the poset of constructible sets $CS(\mathbb{R}^d)$. Naively one would like to prove that the association

$$\mathcal{F} : CS(\mathbb{R}^d)^{op} \rightarrow D^b(Shv(S^{d-1} \times \mathbb{R})) \quad M \mapsto PHT(M)$$

is a sheaf, but there are two main obstacles.

The first obstacle is that a topology on $CS(\mathbb{R}^d)$ needs to be specified. Although sheaves on posets are well-defined via the Alexandrov topology—see [14] for a modern treatment—the poset under consideration is infinite and using the Alexandrov topology here would imply that a shape can be determined via a cover by its points; this is clearly impossible as there is not enough of an interface between points to determine homology. The second obstacle is fatal for a naive sheaf-theoretic approach: the pre-sheaf $U \mapsto H^i(U)$ is not a sheaf for $i \geq 1$. Indeed, the connecting homomorphism in the Mayer-Vietoris long-exact sequence quantifies precisely the failure of the sheaf axiom. Both of these obstacles are addressed via tools from “higher” sheaf theory: Grothendieck topologies and homotopy sheaves. We recall this machinery now.
3.1. Sites and Homotopy Sheaves. Grothendieck topologies provide a way of generalizing sheaves to contravariant functors on a general category $\mathcal{C}$. Covers of an open set are replaced with collections of morphisms that have certain “cover-like” properties.

**Definition 3.1** (Grothendieck Pre-topology, cf. [1]). Let $\mathcal{C}$ be a category with pullbacks. A *basis for a Grothendieck topology* (or a *pre-topology*) on $\mathcal{C}$ requires specifying for each object $U \in \mathcal{C}$ a collection of admissible covers of $U$. This collection of covers must be closed under the following operations:

1. (Isomorphism) If $f : U' \rightarrow U$ is an isomorphism then $\{ f : U' \rightarrow U \}$ is a cover.
2. (Composition) If $\{ f_i : U_i \rightarrow U \}$ is a cover of $U$ and if for each $i$ we have a cover $\{ g_{i,j} : U_{i,j} \rightarrow U_i \}$ then the composition $\{ f_i \circ g_{i,j} : U_{i,j} \rightarrow U \}$ is also a cover.
3. (Base Change) If $\{ f_i : U_i \rightarrow U \}$ is a cover and $V \rightarrow U$ is any morphism then the pullback $\{ \pi_i : V \times_U U_i \rightarrow V \}$ is a cover as well.

As the name suggests, the above data specifies a genuine Grothendieck topology on $\mathcal{C}$. A category equipped with a Grothendieck topology is known as a *site*.

**Remark 3.2** (Sheaves on Sites). The classical definition of a pre-sheaf and sheaf can now be generalized to a site. A functor $F : \mathcal{C}^{\text{op}} \rightarrow \text{Dat}$ is a *pre-sheaf*. If $\text{Dat}$ has all limits, we say a pre-sheaf is a *sheaf* if for every object $U \in \mathcal{C}$ and cover $U = \{ f_i : U_i \rightarrow U \}$

$$F(U) = \lim \left[ \prod_i F(U_i) \Rightarrow \prod_{i,j} F(U_{i,j}) \right].$$

Here equality means isomorphic up to a unique isomorphism and $U_{i,j}$ is the pullback of $f_j : U_j \rightarrow U$ along $f_i : U_i \rightarrow U$ for any pair of morphisms $f_i$ and $f_j$ that participate in the cover $U$.

Unfortunately the functor $F$ specified in Theorem 1.1 is valued in the derived category of sheaves on $S^{d-1} \times \mathbb{R}$. It is well-known among experts that the derived category $\mathcal{D}^b(\mathcal{A})$ of an abelian category $\mathcal{A}$ is not abelian. Candidate kernels and co-kernels do not have canonical inclusion and projection maps, but one can work with so-called distinguished triangles instead. More generally, we can describe a sheaf axiom whenever the notion of a homotopy limit makes sense in the target category $\text{Dat}$. We recall a special case of this construction for $\text{Dat} = \mathcal{D}^b(\mathcal{A})$.

**Definition 3.3** (Homotopy Limits). Given an inverse system of objects $(K_n, f_n)$ in $\mathcal{D}^b(\mathcal{A})$

$$\cdots f_{n+1} : K_{n+1} \rightarrow K_n f_n : K_n \rightarrow K_{n-1} \cdots$$

an object $K$ is a *homotopy limit* if there is a distinguished triangle in the derived category

$$K \rightarrow \prod_n K_n \xrightarrow{\text{shift}} \prod_n K_n \rightarrow K[1].$$

The shift map being given by $(k_n) \mapsto (k_n - f_{n-1}(k_{n-1}))$. We note that the homotopy limit is not necessarily unique and so we say that $S$ is a homotopy limit rather than it is *the* homotopy limit.

We can now define sheaves valued in the derived category.
Definition 3.4 (Homotopy Sheaf). A pre-sheaf \( F : \mathcal{C}^{op} \to \mathcal{D}^b(\mathcal{A}) \) is a homotopy sheaf (or satisfies \( Č \)ech descent [10]) if for every object \( U \in \mathcal{C} \) and cover \( U = \{U_i \to U\} \) the following map is a quasi-isomorphism:

\[
F(U) \cong \text{holim} \left[ \prod_i F(U_i) \Rightarrow \prod_{i,j} F(U_{ij}) \Rightarrow \cdots \right]
\]

3.2. Gluing Results for the PHT. We can now prove our main results.

Lemma 3.5. The poset \( \mathcal{CS}(\mathbb{R}^d) \) admits the structure of a site.

Proof. For every object \( M \in \mathcal{CS}(\mathbb{R}^d) \) we say that \( \{M_i \hookrightarrow M\} \) is a covering if it is a finite closed cover of \( M \) in the usual sense, i.e. \( \bigcup M_i = M \). Pullbacks exist by virtue of the fact that \( \alpha \)-minimal sets are closed under intersection. \( \square \)

Lemma 3.6. The following assignment is a pre-sheaf

\[
F : \mathcal{CS}(\mathbb{R}^d)^{op} \to \mathcal{D}^b(\mathbf{Shv}(\mathbb{S}^{d-1} \times \mathbb{R})) \quad M \mapsto \text{PHT}(M)
\]

where \( \text{PHT}(M) \) is the derived sheaf version of the PHT; see Definition 3.4.

Proof. We want to show that \( F \) is a contravariant functor. Let \( i : M_1 \hookrightarrow M_2 \) be an inclusion of constructible sets of \( \mathbb{R}^d \). Note that \( M_1 \) is a closed subspace of \( M_2 \). This induces an inclusion of the auxiliary total spaces \( i : Z_{M_1} \hookrightarrow Z_{M_2} \) of Definition 2.7. This in turn determines a morphism of pre-sheaves \( f_{M_2*} \mathcal{S}^j(Z_{M_2}; \mathbb{k}) \to f_{M_1*} \mathcal{S}^j(Z_{M_1}; \mathbb{k}) \) for all \( j \). To see this, take an open \( U \subset \mathbb{S}^{d-1} \times \mathbb{R} \) and observe that \( f_{M_1}^{-1}(U) \) is open in \( f_{M_2}^{-1}(U) \). More generally, for \( U \subset V \) in \( \mathbb{S}^{d-1} \times \mathbb{R} \) we have a commutative diagram of cochain groups:

\[
\begin{array}{ccc}
\mathcal{S}^j(f_{M_1}^{-1}(U); \mathbb{k}) & \leftarrow & \mathcal{S}^j(f_{M_2}^{-1}(U); \mathbb{k}) \\
\uparrow & & \uparrow \\
\mathcal{S}^j(f_{M_1}^{-1}(V); \mathbb{k}) & \leftarrow & \mathcal{S}^j(f_{M_2}^{-1}(V); \mathbb{k})
\end{array}
\]

Since sheafification is a functor, we get a morphism \( f_{M_2*} \mathcal{S}^j(Z_{M_2}; \mathbb{k}) \to f_{M_1*} \mathcal{S}^j(Z_{M_1}; \mathbb{k}) \) for all \( j \). These fit together into a morphism between complexes of sheaves:

\[
\begin{array}{ccc}
f_{M_2*} \mathcal{S}^0(Z_{M_2}; \mathbb{k}) & \longrightarrow & f_{M_2*} \mathcal{S}^1(Z_{M_2}; \mathbb{k}) & \longrightarrow & f_{M_2*} \mathcal{S}^2(Z_{M_2}; \mathbb{k}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
f_{M_1*} \mathcal{S}^0(Z_{M_1}; \mathbb{k}) & \longrightarrow & f_{M_1*} \mathcal{S}^1(Z_{M_1}; \mathbb{k}) & \longrightarrow & f_{M_1*} \mathcal{S}^2(Z_{M_1}; \mathbb{k}) & \longrightarrow & \cdots
\end{array}
\]

The canonical functor from \( \mathcal{C}^b(\mathbf{Shv}(\mathbb{S}^{d-1} \times \mathbb{R})) \to \mathcal{D}^b(\mathbf{Shv}(\mathbb{S}^{d-1} \times \mathbb{R})) \) then induces the desired restriction morphism between derived PHT sheaves:

\[
F(M_2) := Rf_{M_2*} \mathbb{k}_{Z_{M_2}} \to Rf_{M_1*} \mathbb{k}_{Z_{M_1}} =: F(M_1)
\]

\( \square \)

The following is the main result of the paper, which was stated as Theorem 1.1 in the introduction. We give a direct proof below, but Remark 3.9 gives a more intuitive and computationally flavored proof using spectral sequences.

Theorem 3.7. The pre-sheaf \( F \) of Lemma 3.6 is a homotopy sheaf; see Definition 3.4.
Proof. We have already specified a Grothendieck topology on $\mathcal{CS}(\mathbb{R}^d)$ in Lemma 2.9. Let $\mathcal{M} = \{M_i\}_{i \in \Lambda}$ be a finite closed cover of $M$. Since $\mathcal{F}$ is a pre-sheaf we have an inverse system of derived sheaves:

$$
\bigoplus_{J \subseteq \Lambda \text{ s.t. } |J| = 1} Rf_{M_J}^* k_{Z_{M_J}} \cong \bigoplus_{J \subseteq \Lambda \text{ s.t. } |J| = 2} Rf_{M_J}^* k_{Z_{M_J}} \cong \bigoplus_{K \subseteq \Lambda \text{ s.t. } |K| = 3} Rf_{M_K}^* k_{Z_{M_K}} \to \cdots
$$

We wish to show that $Rf_{M_i}^* k_{Z_M}$ is the homotopy limit of the above inverse system of derived sheaves, i.e. we want to show that

$$
Rf_{M_i}^* k_{Z_{M_i}} \simeq \text{holim} \left[ \bigoplus_{|J| = 1} Rf_{M_J}^* k_{Z_{M_J}} \to \bigoplus_{|J| = 2} Rf_{M_J}^* k_{Z_{M_J}} \to \cdots \right].
$$

By replacing each $Rf_{M_i}^* k_{Z_{M_i}}$ with its flabby resolution by singular cochains it suffices to prove that the following is a distinguished triangle:

$$
f_{M_i}^* \mathcal{F}'(Z_M; k) \to \prod_{n} \bigoplus_{|I| = n} f_{M_i}^* \mathcal{F}'(Z_{M_i}; k) \xrightarrow{\text{shift}} \prod_{n} \bigoplus_{|I| = n} f_{M_i}^* \mathcal{F}'(Z_{M_i}; k)
$$

To show this we consider the following maps of complexes of sheaves:

$$
f_{M_i}^* \mathcal{F}'(Z_M) \to \prod_{n} \bigoplus_{|I| = n} f_{M_i}^* \mathcal{F}'(Z_{M_i}) \xrightarrow{\text{shift}} \prod_{n} \bigoplus_{|I| = n} f_{M_i}^* \mathcal{F}'(Z_{M_i})
$$

where we drop the coefficient $k$ for convenience. For every $(v, t) \in S^{d-1} \times \mathbb{R}$, these morphisms induce a sequence on stalks, which give rise to a sequence of cochain complexes

$$
S^* (M_{v,t}) \to \prod_{n} \bigoplus_{|I| = n} S^* ((M_I)_{v,t}) \xrightarrow{\text{shift}} \prod_{n} \bigoplus_{|I| = n} S^* ((M_I)_{v,t})
$$

where $(M_I)_{v,t}$ is the intersection of $M_I$ with the half-space $\{x \mid x \cdot v \leq t\}$. The kernel of the shift map at each stalk is clearly the cochain complex of small co-chains $S^*_{M_{v,t}} ((M_I)_{v,t})$; these are cochains supported on singular simplices that are individually contained in some cover element $(M_i)_{v,t}$ of the fiber $M_{v,t}$. Consequently, on the level of stalks we have distinguished triangles

$$
S^*_{M_{v,t}} ((M_I)_{v,t}) \to \prod_{n} \bigoplus_{|I| = n} S^* ((M_I)_{v,t}) \xrightarrow{\text{shift}} \prod_{n} \bigoplus_{|I| = n} S^* ((M_I)_{v,t})
$$

We now show that we can replace $S^*_{M_{v,t}} ((M_I)_{v,t})$ with $S^* ((M_I)_{v,t})$ above because the inclusion $S^*_{M_{v,t}} ((M_I)_{v,t}) \hookrightarrow S^* ((M_I)_{v,t})$ is a quasi-isomorphism, and hence an isomorphism in the derived category.

To prove this, we appeal to simplicial cohomology. By the Triangulation Theorem (Theorem 2.9 in [42]) we can triangulate $M_{v,t}$ in a way that is subordinate to the closed cover $\{(M_I)_{v,t}\}$ for arbitrary (yet finite) intersections $M_I$. Simplicial cochains for this triangulation
form a sub-cochain complex of $S_{\mathcal{M}_{v,t}}^*(M_{v,t})$, but the triangulation can be used to compute cohomology of $M_{v,t}$. This completes the proof. 

Remark 3.9 (Proof via Spectral Sequences). By Theorem 4.4.1 of Godemont [24] there is a resolution of $\mathbb{k}_{Z_M}$ using the cover of $Z_M$ by $\{Z_{M_i}\}$. As such there is a weak equivalence (quasi-isomorphism)

\begin{equation}
\mathbb{k}_{Z_M} \rightarrow \left[ \bigoplus_{|I|=1} \mathbb{k}_{Z_{M_I}} \rightarrow \bigoplus_{|J|=2} \mathbb{k}_{Z_{M_J}} \rightarrow \cdots \right]
\end{equation}

Applying the right derived pushforward functor preserves this weak-equivalence. This already proves, in essence, Čech descent for the PHT. More specifically, the homotopy sheaf axiom is witnessed via a first quadrant spectral sequence.

In practice, the spectral sequence gives a method of computing the PHT of $M$ at a point $(v, t)$. Passing to stalks the first quadrant of the $E_1$ page reads

\begin{align*}
\bigoplus_{|I|=1} H^2(M_{I,v,t}; \mathbb{k}) &\longrightarrow \bigoplus_{|J|=2} H^2(M_{J,v,t}; \mathbb{k}) \\
\bigoplus_{|I|=1} H^1(M_{I,v,t}; \mathbb{k}) &\longrightarrow \bigoplus_{|J|=2} H^1(M_{J,v,t}; \mathbb{k}) \\
\bigoplus_{|I|=1} H^0(M_{I,v,t}; \mathbb{k}) &\longrightarrow \bigoplus_{|J|=2} H^0(M_{J,v,t}; \mathbb{k})
\end{align*}

Call this complex $C^n(\mathcal{M}_{v,t}; H^q(\mathbb{k}_M))$ where $H^q(\mathbb{k}_M) : I \rightarrow H^q(M_{I,v,t})$ is a system of coefficients, i.e. a cellular sheaf on the nerve of $\mathcal{M}$. Taking cohomology of this complex horizontally gives us the $E_2$ page of the spectral sequence. Theorem 5.2.4 of [24] guarantees that we converge to the cohomology of $M_{v,t}$.

We now illustrate the power of the spectral sequence approach in the following corollary, which was previously stated as Theorem 1.2 in the introduction.

Corollary 3.11. Suppose $M \in \text{CS}(\mathbb{R}^d)$ is a polyhedron and suppose $\mathcal{M} = \{M_i\}_{i \in I}$ is a cover of $M$ by PL subspaces, then $\text{PHT}^n(M)$ is the $n$-th cohomology of the complex,

\begin{equation}
0 \rightarrow \bigoplus_{|I|=1} \text{PHT}^0(M_I) \rightarrow \bigoplus_{|J|=2} \text{PHT}^0(M_J) \rightarrow \cdots
\end{equation}

where the $\cdots$ represents $\text{PHT}^0$ of higher intersection terms.

Proof. By examining the $E_1$ page of the spectral sequence in Remark 3.9 one can see that for a PL cover, the higher homologies, i.e. the higher PHTs, all vanish. Consequently the spectral sequence collapses after the $E_1$ page.

3.3. Example Calculation. So far we have showed that we can construct a homotopy sheaf on shape space where each shape is assigned its persistent homology transform. In this section, we leverage the spectral sequence argument of Remark 3.9 to illustrate an explicit calculation of the gluing process.

Let $M = S^1$ be the circle in $\mathbb{R}^2$. Define a covering $\mathcal{M} = \{A, B\}$ by two closed half-circles, as indicated in Figure 3. First, we compute the PHT of each of the cover elements and their intersection. Because our PHT sheaves are on $S^1 \times \mathbb{R}$ we can project this cylinder onto the plane $\mathbb{R}^2$ by following the instructions in the caption of Figure 4.

Now for every point $(v, t) \in S^1 \times \mathbb{R}$ we write out the spectral sequence in Remark 3.9. For example, let $(v, t) = (\uparrow, 0)$, then the $E_1$ page of the spectral sequence works out to be:
Figure 3. $M = S^1$ with cover elements $A$ and $B$.

Figure 4. To visualise the sheaf we have mapped the cylinder $S^1 \times \mathbb{R}$ to the plane $\mathbb{R}^2$ by squashing down a tapering cylinder onto a plane in the following way: fix direction $v \in S^1$ and then map $\{v\} \times \mathbb{R}$ onto $(0, \infty)$. So every direction $v$ has a ray attached to it. For example, consider direction $w = \uparrow = (0, 1)$ and $t = 0$ then in figure (A) we see $H^0(A_{w,t}; \mathbb{k}) = \mathbb{k}$ (see the red square) since $A_{w,t} = \{x \in A| x \cdot w \leq t\}$ has one connected component.

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 \\
\mathbb{k} \oplus \mathbb{k} & \longrightarrow & \mathbb{k} & \longrightarrow & 0
\end{array}
\]

This spectral sequence collapses after the $E_2$ page and converges to $H^*(M_{v,t}; \mathbb{k})$. And so for this example taking cohomology horizontally gives us that $H^0(M_{v,t}; \mathbb{k}) = \mathbb{k}$ and $H^1(M_{v,t}; \mathbb{k}) = 0$. Since the PHT is a sheaf we can can do this at all $(v,t)$ to find $\text{PHT}^*(M)$. Figure 5 shows the PHT of $M$.

3.4. **Relative PHT.** In the previous section we showed how to construct the PHT of a shape by gluing PHTs that from a cover. Intuitively this corresponds to “adding” several
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Figure 5. PHT\(0\)(M)

PHTs together in a precise way. A natural question to consider is if there is a process for “subtracting” one PHT from another. This is accomplished by using relative cohomology.

Definition 3.13 (Relative PHT). Let \(M \in \mathcal{CS}(\mathbb{R}^d)\) be a constructible set and suppose \(N \subset M\) is a closed constructible subset of \(M\) The relative PHT is defined to be the sheaf over \(\mathbb{S}^{d-1} \times \mathbb{R}\) defined by

\[
PHT^i(M, N) = \text{sh} \left[ U \to H^i(f_M^{-1}(U), f_N^{-1}(U); k_{Z_M}) \right]
\]

The stalk at \((v, t) \in \mathbb{S}^{d-1} \times \mathbb{R}\) is the relative cohomology of the pair \((M_{v,t}, N_{v,t})\)

To prove that this definition suitably “subtracts” one PHT from another, consider the long exact sequence of pairs:

\[
\cdots \to PHT^i(M, N)_{v,t} \to PHT^i(M)_{v,t} \to PHT^i(N)_{v,t} \to PHT^{i+1}(M, N)_{v,t} \to \cdots
\]

Exactness at stalks implies exactness of sheaves and so we have the following long exact sequence of PHT sheaves:

\[
\cdots \to PHT^i(M, N) \to PHT^i(M) \to PHT^i(N) \to PHT^{i+1}(M, N) \to \cdots
\]

3.5. Euler Calculus Interpretation. In this section we show how the “addition” and “subtraction” operations on PHT sheaves described above corresponds to actual addition and subtraction once cohomology is replaced with Euler characteristic. Although the results in this section appear to be much simpler—e.g. the homotopy sheaf axiom reduces to a generalized inclusion-exclusion principle—the passage from sheaves to functions is actually part of a rich mathematical theory known as Euler Calculus; see [15] for an accessible introduction.

We start by reviewing how to go from a (derived) sheaf to an integer-valued function.

Definition 3.15 (Sheaf-to-Function Correspondence). Let \(\mathcal{F}^\bullet\) be a complex of (cohomologically) constructible sheaves on a constructible set \(X \in \mathcal{CS}(\mathbb{R}^d)\). The local Euler-Poincaré index is the piecewise constant integer-valued function defined by

\[
h(x) := \chi(\mathcal{F}^\bullet)(x) = \sum_i (-1)^i \dim(H^i\mathcal{F}_x).
\]

In the simplest setting, for constructible sheaves that are concentrated in a single cohomological degree, the local Euler-Poincaré index is just the Hilbert function—it records the dimension of the stalks of a sheaf. For complexes of sheaves, the dimension function is replaced by the point-wise Euler characteristic of the complex. This index was used by
Kashiwara to prove that the Grothendieck group of constructible sheaves is isomorphic to the group of constructible functions on $X$ [30, Thm 9.7.1].

For a constructible set $M$, the derived PHT sheaf $PHT(M) = R(f_M)_* \mathcal{K}_{Z_M}$ is constructible and so by the above correspondence there is a constructible function associated to it [13].

**Definition 3.16** (The Euler Characteristic Transform). The **Euler Characteristic Transform** (ECT) of a constructible set $M$ is the constructible function on $\mathbb{S}^{d-1} \times \mathbb{R}$ given by

$$\ECT(M)(v, t) = \chi(f_M^{-1}(v, t)) \quad \forall (v, t) \in \mathbb{S}^{d-1} \times \mathbb{R}.$$ 

Equivalently, it is the Euler-Poincaré index of the derived PHT sheaf $PHT(M) = R(f_M)_* \mathcal{K}_{Z_M}$.

Alternatively, one can view the Euler Characteristic Transform of $M$ as the Radon transform of its indicator function on $\mathbb{R}^d$ [13, 25]. This perspective requires using the operations of Euler Calculus to their full effect [38].

**Definition 3.17** (Operations on Constructible Functions). If $X \in \mathcal{CS}(\mathbb{R}^d)$, then a **constructible function** is an integer valued function $f : X \to \mathbb{Z}$ with only finitely many non-empty level sets, each of which are triangulable. If $\varphi : X \to Y$ is a (tame) mapping between constructible sets, then we have a **pushforward** $\varphi_* : \text{CF}(X) \to \text{CF}(Y)$ and **pullback** $\varphi^* : \text{CF}(Y) \to \text{CF}(X)$ operations on constructible functions via

$$\varphi_* f(y) := \int_{\varphi^{-1}(y)} f \, d\chi \quad \text{and} \quad \varphi^* g(x) := g(\varphi(x)) \quad \text{where} \quad \int_X f \, d\chi := \sum_{n=-\infty}^{\infty} n \cdot \chi(f^{-1}(n))$$

is integration with respect to compactly-supported Euler characteristic.

**Definition 3.18** (Radon Transform). Let $S \subset X \times Y$ be a closed constructible subset of the product of two constructible sets. Let $\pi_X$ and $\pi_Y$ be the projections onto the indicated factors. The **Radon Transform** with respect to $S$ is a group homomorphism $\mathcal{R}_S : \text{CF}(X) \to \text{CF}(Y)$ defined by

$$\mathcal{R}_S(\phi) := (\pi_Y)_* \left[ (\pi_X)^* \phi \right] \mathbf{1}_S \quad \text{where} \quad \mathcal{R}_S(\phi)(y) = \int_{\pi_Y^{-1}(y)} (\phi \circ \pi_X) \mathbf{1}_S \, d\chi.$$ 

Notice that by taking $\phi = \mathbf{1}_M$ and $S = Z_M \subset M \times \mathbb{S}^{d-1} \times \mathbb{R}$ the Euler characteristic transform of $M$ coincides with the Radon transform of its indicator function.

The statement that the Radon transform is a group homomorphism is simply the statement that it is linear, i.e. $\mathcal{R}_S(\phi + \psi) = \mathcal{R}_S(\phi) + \mathcal{R}_S(\psi)$. When $M \in \mathcal{CS}(\mathbb{R}^d)$ is covered by $\mathcal{M} = \{M_i\}_{i \in \Lambda}$, we can express its indicator function using a linear combination of indicator functions defined using this cover. This allows us to build up the ECT of a shape using the ECTs of its cover and provides a simplified expression of the homotopy sheaf axiom for the PHT sheaves.

**Theorem 3.19** (Decategorification of the Homotopy Sheaf Axiom). For a finite constructible cover $\mathcal{M} = \{M_i\}_{i \in \Lambda}$ of $M \subset \mathbb{R}^d$, the homotopy sheaf axiom reduces via the local Euler-Poincaré index to the statement

$$\ECT(M) = \sum_{I \subseteq \Lambda} (-1)^{|I|+1} \ECT(M_I)$$

where each $M_I$ denotes the intersection $M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_k}$ for $I = (i_1, \ldots, i_k)$.
Proof. The generalized-inclusion exclusion principle allows us to write
\[ 1_M = \sum_{I \subseteq \Lambda} (-1)^{|I|+1} 1_{M_I}, \]
which is exactly akin to the local Euler-Poincaré index of Godemont’s resolution from Equation 3.10. Linearity of the Radon transform allows us to write
\[ \mathcal{R}_{Z_M} 1_M = \sum_{I \subseteq \Lambda} (-1)^{|I|+1} \mathcal{R}_{Z_M} 1_{M_I}, \]
which is the expression using ECTs written above. This is exactly the local Euler-Poincaré index of the pushforward of the resolution in Equation 3.10. Checking on stalks reveals that for any \((v, t) \in S^d - 1 \times \mathbb{R}\)
\[ \chi( f_M^{-1}(v, t) ) = \sum_{I \subseteq \Lambda} (-1)^{|I|+1} \chi( f_{M_I}^{-1}(v, t) ). \]

Similarly, we can also interpret the long exact sequence of pairs given in Equation 3.14 from the point of view of Euler characteristic.

**Definition 3.20.** (Relative ECT) The relative ECT of a pair \((M, N)\) where \(N\) is a closed constructible subset of constructible \(M\), is the function associated to the relative PHT sheaf under the function-to-sheaf correspondence. That is for all \((v, t) \in S^d - 1 \times \mathbb{R}\)
\[ \text{ECT}(M, N)(v, t) = \chi( PHT(M, N)(v, t) ) = \sum_i (-1)^i \dim H^i( f_M^{-1}(v, t), f_N^{-1}(v, t) ). \]

Subtraction of ECTs results in the relative Euler Characteristic Transform.

**Lemma 3.21.** For closed and constructible \(N \subset M\)
\[ \text{ECT}(M, N)(v, t) = \chi( PHT(M, N)(v, t) ) = \sum_i (-1)^i \dim H^i( f_M^{-1}(v, t), f_N^{-1}(v, t) ). \]

Proof. Recall the LES of a pair from Equation 3.14
\[ \cdots \rightarrow PHT^i(M, N)_{v,t} \rightarrow PHT^i(M)_{v,t} \rightarrow PHT^i(N)_{v,t} \rightarrow PHT^{i+1}(M, N)_{v,t} \rightarrow \cdots \]
The long exact sequence implies that
\[ \chi( PHT(M)(v, t) ) = \chi( PHT(M, N)(v, t) ) + \chi( PHT(N)(v, t) ), \]
which can be rewritten as
\[ \text{ECT}(M)(v, t) = \text{ECT}(M, N)(v, t) + \text{ECT}(N)(v, t). \]

## 4. Stability and Approximations of the PHT

Aside from the intrinsic theoretical interest in a gluing result for the PHT, a practical motivation is to parallelize PHT computations over a cover. This parallelization inevitably becomes more complex if our cover elements have higher homology when viewed in certain directions and at certain filtration values. See Figure 2 for an example.

In this section we prove that up to some tolerance, we can always approximate a submanifold \(M \in \mathcal{CS}(\mathbb{R}^d)\) via a polyhedron \(K\) so that the PHT’s of \(M\) and \(K\) are arbitrarily close.
We do this in three steps. First, we prove that the persistent homology transform is stable under small perturbations of the underlying shape. This stability property reaffirms our belief that the PHT is a good summary statistic for shapes. Second, we use the sampling procedure based on Niyogi-Smale-Weinberger \cite{35} to approximate a submanifold of \( \mathbb{R}^d \) by a polyhedron. Third, we conclude from the stability theorem that the PHT of the polyhedron is close to the PHT of the submanifold.

4.1. Distances on PHTs. To define distances between the persistent homology transform of shapes we make use of the interleaving distance of sheaves in \cite{14}.

**Definition 4.1** (Interleaving of pre-sheaves). Define the \( \epsilon \) thickening of a pre-sheaf \( F \) via the formula \( F^\epsilon(U) := F(U^\epsilon) \). Let \( F, G : \text{Open}(X)^\text{op} \to D \) be two pre-sheaves on a metric space \( X \). We define an \( \epsilon \)-interleaving of \( F \) and \( G \) to be a pair of natural transformations \( \varphi_{\epsilon} : F^\epsilon \to G \) and \( \psi_{\epsilon} : G^\epsilon \to F \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
F^{2\epsilon} & \longrightarrow & F^\epsilon \\
\downarrow & & \downarrow \\
G^{2\epsilon} & \longrightarrow & G^\epsilon
\end{array}
\]

We can define the interleaving distance on pre-sheaves by

\[
d_I(F, G) := \inf\{ \epsilon \geq 0 | \exists \text{ } \epsilon \text{ - interleaving} \}.
\]

If no such interleaving exists, we define \( d_I(F, G) = \infty \).

We use this distance to define a distance between the PHT sheaves. We note that that this definition does not use homology in any way.

4.2. The Stability Theorem. We prove that if two shapes are of the same homotopy type and are \( \epsilon \)-close then the PHT of the shapes are also \( \epsilon \)-close.

**Theorem 4.2.** (Stability of PHT) Let \( M, N \subseteq \mathbb{R}^d \) be constructible sets and let \( \varphi : M \to N \) and \( \psi : N \to M \) be a homotopy equivalence. Further, for some \( \epsilon > 0 \) assume that \( \|x - \varphi(x)\|_{\mathbb{R}^d}^2 \leq \epsilon \) for all \( x \in M \) and \( \|y - \psi(y)\|_{\mathbb{R}^d}^2 \leq \epsilon \) for all \( y \in N \). Then the PHT of \( M \) and \( N \) are \( \epsilon \)-interleaved as presheaves.

**Proof.** We show the following diagram is commutative.

\[
\begin{array}{ccc}
(Rf_*k_{Z_M})^{2\epsilon} & \longrightarrow & (Rf_*k_{Z_N})^{2\epsilon} \\
\downarrow & & \downarrow \\
(Rf_*k_{Z_M})^\epsilon & \longrightarrow & (Rf_*k_{Z_N})^\epsilon \\
\downarrow & & \downarrow \\
(Rf_*k_{Z_M}) & \longrightarrow & (Rf_*k_{Z_N})
\end{array}
\]

We prove this for the left triangle as the commutativity of the right triangle follows from a similar argument. Let \( U \subset S^{d-1} \times \mathbb{R} \) be a test open set and so \( (Rf_*k_{Z_M})^\epsilon(U) = Rf_*k_{Z_M}(U^\epsilon) = [S^*(f_M^{-1}(U^\epsilon); k)] \) where \( [S^*(f_M^{-1}(U^\epsilon); k)] \) represents the class of complexes quasi-isomorphic to the singular cochain complex on \( f_M^{-1}(U^\epsilon) \).
We show that we have the following diagram of topological spaces.

\[
\begin{array}{ccc}
 f^{-1}_M(U^{2\epsilon}) & \xrightarrow{h} & f^{-1}_M(U) \\
 \uparrow & & \uparrow \\
 f^{-1}_M(U^\epsilon) & \xrightarrow{g} & f^{-1}_N(U^\epsilon) \\
 \uparrow & & \uparrow \\
 f^{-1}_M(U) & \xrightarrow{} & \end{array}
\]

The diagram is not commutative, in fact the map \(h \circ g\) is homotopic to \(\text{Id}_{f^{-1}_M(U)}\). So if we apply the singular cochain sheaf to this diagram and then take quotients by quasi-isomorphisms, we will get the desired commutative triangle in figure \(\Box\).

Now we explicitly describe the maps \(g\) and \(h\). For \((x, v, t) \in f^{-1}_M(U)\) define \(g\) such that \(g(x, v, t) = (\varphi(x), v, t + \epsilon)\). It remains to verify that \((\varphi(x), v, t)\) is in \(f^{-1}_N(U^\epsilon)\). Note that

\[\varphi(x) \cdot v - x \cdot v = (\varphi(x) - x) \cdot v \leq \|\varphi(x) - x\|^2 \leq \epsilon.\]

Since \(x \cdot v \leq t\), we have that \(\varphi(x) \cdot v - t \leq \varphi(x) \cdot v - x \cdot v \leq \epsilon\) and so \(\varphi(x) \cdot v \leq t + \epsilon\).

Similarly for \((y, v, t) \in f^{-1}_N(U^\epsilon)\), let \(h(y, v, t) = (\psi(y), v, t + \epsilon)\). On composing we get that \(h \circ g(x, v, t) = (\psi \circ \varphi(x), v, t + 2\epsilon)\) and since \(\psi \circ \varphi\) is homotopic to the identity on \(f^{-1}_M(U)\), \(h \circ g\) is homotopic to \(\text{Id}_{f^{-1}_M(U)}\).

\(\Box\)

4.3. Background on Sampling. In [35] the authors considered the problem of how to determine homology of a submanifold of \(\mathbb{R}^d\) using a point sample. Their result reads as follows:

**Theorem 4.3** (Theorem 3.1 in [35]). Let \(M\) be a compact submanifold of \(\mathbb{R}^d\) with condition number \(\tau\). Let \(x = \{x_1, \ldots, x_n\}\) be a set of \(n\) points drawn independently and identically from a uniform probability measure on \(M\). Let \(0 < \epsilon < \tau/2\). Let \(U = \bigcup_{x \in x} B_\epsilon(x)\) be the union of the open balls of radius \(\epsilon\) around the sample points. Then for all

\[n > \beta_1 \left( \log \beta_2 + \log \frac{1}{\delta} \right)\]

the homology of \(U\) equals the homology of \(M\) with probability \(> 1 - \delta\). Here \(\beta_1\) and \(\beta_2\) are constants that depend on the condition number \(\tau\), \(\epsilon\) and the volume of \(M\). The bound on \(n\) ensures that with high probability the sample is \(\epsilon\)-dense in \(M\).

We let \(K\) be the alpha complex of the balls \(U := \{B_\epsilon(x)\}\) produced by Theorem 4.3. By the nerve lemma, the alpha complex is homotopy equivalent to union of the balls \(U\) and so with high probability the homology of \(K\) is equal to homology of \(M\).

4.4. The Main Approximation Result. Now we are ready to bound the PHT distance between a submanifold \(M \in \mathcal{CS}(\mathbb{R}^d)\) and the sampled alpha complex \(K\) described in Section 4.3 using the Niyogi-Smale-Weinberger process.

**Corollary 4.4.** (Approximation) Let \(M\) be a compact submanifold of \(\mathbb{R}^d\) with condition number \(\tau\). Let \(x = \{x_1, \ldots, x_n\}\) be a set of \(n\) points drawn independently and identically from a uniform probability measure on \(M\). Let \(0 < \epsilon < \frac{\tau}{2}\). Let \(U = \bigcup_{x \in x} B_\epsilon(x)\) be the union of the
open balls of radius $\epsilon$ around the sample points. Let $K$ be the alpha complex of $U$. Then for all

$$n > \beta_1 \left( \log \beta_2 + \log \frac{1}{\delta} \right)$$

we have that, $d_1(PHT(M), PHT(K)) \leq \epsilon^2$ with high confidence i.e. probability $\leq 1 - \delta$.

**Proof.** We show that the assumptions of Theorem 4.2 are satisfied and then apply Theorem 4.2 to conclude the result.

Choose $\epsilon$ according to $0 < \epsilon < \tau/2$. We need to find a homotopy equivalence $\varphi : K \to M$ and $\psi : M \to K$ such that $\|x - \varphi(x)\| \leq \epsilon$ for all $x \in K$ and $\|y - \psi(y)\| \leq \epsilon$ for all $y \in M$.

We pass to the union of balls $U$ to construct the desired homotopy equivalence.

*Homotopy equivalence of $M$ and $U$.* Since the sample is $\epsilon/2$-dense in $M$, there is an inclusion of $M$ into $U$. Let $i$ be this inclusion and let $f$ be the projection that sends $x \mapsto \arg\min_{p \in M} \|p - x\|$. This map is a deformation retraction and can be seen by taking the homotopy $H_U(x, t) = tx + (1 - t)f(x)$ for all $x \in U$ and $t \in [0, 1]$.

*Homotopy equivalence of $U$ and $K$.* We have the inclusion map $j : K \to U$. There is a deformation retraction map $g : U \to K$ which can be seen in [20]. (Figure 6 gives a visual description of the homotopy map).

![Figure 6. The deformation retraction map from the union of balls to its alpha complex.](image)

Let $\varphi := f \circ j$ and $\psi := g \circ i$. On composing, $\varphi \circ \psi = f \circ j \circ g \circ i \sim f \circ Id_U \circ i = f \circ i \sim Id_M$ and similarly $\psi \circ \varphi \sim Id_K$.

The radius of balls are less than $\epsilon$ and so for $x \in K$, $\|x - f \circ j(x)\| \leq \epsilon$ and so $\|x - \varphi(x)\|^2 < \epsilon^2$. Since the sample points are $\epsilon/2$-dense, for $y \in M$, $\|y - g \circ i(y)\| \leq \epsilon/2 < \epsilon$ and so $\|y - \psi(y)\|^2 \leq \epsilon^2$. Apply Theorem 4.2 to get an $\epsilon^2$-interleaving of the PHTs of $M$ and $K$. \qed

5. Discussion

In this paper we have to our knowledge introduced the most general construction of a shape space. This shape space does not rely on diffeomorphisms or correspondences between
shapes. We require very minimal measurability conditions, o-minimality. The two classic shape space constructions of Kendall (based on landmarks) and Grenander (based on diffeomorphisms) can be formalized as consisting of a base manifold where each shape is a point on the base manifold and the shape is represented as a fiber bundle. We can also place our shape space in this framework consisting of a base representation, in our case a base poset, and instead of representing the shape as a fiber bundle we represent the shape as a sheaf.

The use of sheaves in modeling shapes and other data science applications has not been well developed and we consider this paper a preliminary example of how sheaves can be used to construct richer data representations that currently exist. There is also the very natural idea of unifying Kendall’s, Grenander’s, and our shape space, for example if the shapes in our setting are all diffeomorphic then our construction should reduce to the fiber bundle framework of shape space. We leave this for future work.

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