Lazy open quantum walks

Garreth Kemp\textsuperscript{a}, Ilya Sinayskiy\textsuperscript{b,c}, Francesco Petruccione\textsuperscript{b,c,d}

\textsuperscript{a}Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park, 2006
\textsuperscript{b}Quantum Research Group, School of Chemistry and Physics, University of KwaZulu-Natal, Durban, KwaZulu-Natal, 4001, South Africa
\textsuperscript{c}National Institute for Theoretical Physics (NITheP), KwaZulu-Natal, 4001, South Africa
\textsuperscript{d}School of Electrical Engineering, KAIST, Daejeon, 34141, Republic of Korea

E-mail: gkemp@uj.ac.za, sinayskiy@ukzn.ac.za, petruccione@ukzn.ac.za

Abstract.

Open quantum walks (OQWs) describe a quantum walker on an underlying graph whose dynamics is purely driven by dissipation and decoherence. Mathematically, they are formulated as completely positive trace preserving (CPTP) maps on the space of density matrices for the quantum walker on the graph. A microscopic derivation of the open quantum walks has been achieved in which it has been shown that all OQWs must include the possibility of remaining on the same site on the graph when the map is applied. In this work we extend the CPTP map to describe a lazy open quantum walk on a \(d\)-dimensional lattice. We study the asymptotic behaviour of this model and generalize an already existing central limit theorem for OQWs on the lattice to now include the lazy case.

1. Introduction:

Random walks have been applied to a vast number of areas in science including physics, computer science, financial economics and biology \cite{1, 2, 3, 4, 5, 6}. Elevating the random walk onto the quantum level was first performed in the context of closed systems undergoing unitary evolution. Models for unitary quantum walks in discrete and continuous time have been proposed in \cite{7} and \cite{8}, respectively. Comprehensive overviews for some of these early quantum walk models can be found in \cite{9} and \cite{10}. These models are comprised of a walker on an underlying graph. The walker possesses internal degrees of freedom (for example spin or polarization) which play a non-trivial role in determining the probability distribution on the graph. The unitary operator driving the evolution performs a transformation of the walker’s internal degrees of freedom and then, depending on this resulting state, shifts the walker from one position on the graph to another. This unitary operator is applied at each time step and a coherent superposition between all the possible positions emerges.
One of the reasons why quantum walks are interesting is because they display very different behaviour compared to their classical counterparts. In particular, it is interesting to study and compare the asymptotic, or long time, behaviour of the walks. The unitary quantum walks propagate outwards from the initial position quadratically faster than the classical random walk. A central limit theorem was derived in [11] in which the limit distribution was found not to be Gaussian, as it is for the classical case. Instead the distribution density was a function of the form

\[ f(x) = \frac{\sqrt{1 - |a|^2}}{\pi (1 - x^2) \sqrt{|a|^2 - x^2}} (1 - \lambda x), \]  

where \( \lambda \) and \( a \) are constants. A significant feature omitted in these models is an additional non-zero probability for the walker to remain on the same site on the graph. Unitary quantum walks having a probability of staying on the same site were introduced in [12], with the title “Grover Search with Lackadaisical Quantum Walks”. The use of the term ‘lackadaisical’ was used to avoid confusion with [13] in which the author formulates his own lazy random quantum walk to study the transition between discrete and continuous time unitary quantum walks. The long time behaviour for the lazy unitary quantum walker in one dimension was studied in [14]. The variance, although displaying smaller values than the usual unitary walk, had the same functional dependence on the number of steps \( T \), i.e., that of \( \sigma^2 \sim T^2 \).

Unitary evolution is indicative of a closed quantum system. In this work, we will be concerned with a discrete time open quantum system random walk model. One in which the walk is driven by a dissipative environment. Open system quantum walk models were first introduced in [15], [16], and [17]. These open quantum walks (OQWs) describe a system comprising of the walker possessing internal degrees of freedom and the underlying graph. The evolution of the walker is driven by a dissipative environment, where the interaction with this environment takes place between any two connected nodes. These non-unitary dynamics are described mathematically by completely positive trace preserving (CPTP) maps. These maps transform the internal degrees of freedom while shifting the walker from one position on the graph to another, thus again building up a statistical mixture of terms for each possible position contributing to the system’s density matrix. The probability distribution of the walker’s position for large times is Gaussian, reminiscent of the classical random walk behaviour. In [18] the OQW was derived from a microscopic model in which the system and the environment, concretely chosen to be a bath of harmonic oscillators, together constituted a closed system. After a quantum master equation was derived for the system’s reduced density matrix, the discrete time OQW was then obtained through a discretisation procedure.

With an appropriate choice of map, the OQW reproduces the classical Markov chain. A ‘physical realisation’ procedure establishes a relation between the OQW and the unitary quantum walk [15]. An OQW formulation of dissipative quantum computing (DQC) was presented in [19], in which the OQW based algorithms converged faster to the desired steady state, and had a higher probability of detection, than
the canonical DQC models. Furthermore, the OQW allows for a quantum trajectory [20] description which, in turn, allows for a quantitative analysis of the long-time, or asymptotic, behaviour of the OQW. Using quantum trajectories the work of [21] formulated a central limit theorem (CLT) for the discrete time homogeneous OQW where the underlying graph is a lattice $\mathbb{Z}^d$. They further managed to derive an explicit formula for the variance of the corresponding Gaussian. Using the CLT [21], the work of [22] introduced a Fourier space dual process for the OQWs and from this, they were able to find formal expressions for the probability distribution and, for a range of OQWs, the mean and variance for the corresponding distributions. Continuous time OQWs were first formulated in [23], and the CLT was proved in [24]. The authors of [25] managed to generalize the CLT to some particular non-homogenous cases of the OQW on the lattice. Next, [26] studied the asymptotic probability distributions for OQWs on $\mathbb{Z}$ where the operators in the CPTP map are simultaneously diagonalizable. The asymptotic distributions were found to consist of, at most, two soliton-like solutions along with a certain number of Gaussians. Furthermore, they uncovered connections between the spectrum of the operators and properties of the asymptotic distributions. As will be elucidated below, the OQW quantum trajectories may be seen as classical Markov chains. Indeed many notions present in classical Markov chain theory, such as irreducibility, period and communicating classes, have been successfully introduced to OQWs through the quantum trajectory route [27, 28], and the notion of hitting time for the OQW was defined in [29]. Applying the generic results of [27] to homogeneous OQWs on $\mathbb{Z}^d$, [28] proved the CLT as well as formulated the large deviation principle for quantum trajectories for OQWs.

As with the unitary quantum walk case previously, a significant feature omitted thus far in the OQW models is the possibility of the walker to remain on the same site after the CPTP map is applied. The work of [18] explicitly shows that all microscopically derived OQWs must necessarily have a self-jumping term. In this work, we extend the CPTP map to include an additional operator to encode for the possibility of a lazy open quantum walker. This then raises an interesting question about the long-time behaviour of the new OQW. We extend the central limit theorem of [21] to our lazy discrete OQW model. Lastly, in the scaling limit, OQWs gave rise to a new class of Brownian motion, namely, Open Quantum Brownian Motion [30, 31]. These models do not exhibit Gaussian behaviour and no CLT is yet known. The detailed account of current status of the field of OQWs can be found in [32].

This paper is structured as follows. In section 2, we describe the discrete time homogenous lazy open quantum walk on the $d$-dimensional lattice, $\mathbb{R}^d$. We also introduce the Markov chain, through the quantum trajectory description, that will allow for the formulation of the new extended CLT. In section 3, we discuss the CLT for our lazy OQW, revise some important aspects of the microscopic derivation, and then connect the homogeneous OQW on the lattice to the quantities in the microscopic derivation. Lastly in this section, we study some examples in which we conduct non-trivial checks of the variance formula obtained from the CLT. Lastly, in section 4
2. Lazy open quantum walk formulation

2.1. The basic formulation

We first introduce the OQW on a general graph. The graph consists of a set of nodes $\mathcal{V}$ and we define the set of all oriented edges on the graph $\{(i, j) | i, j \in \mathcal{V}\}$. These oriented edges denote the possible transitions between the nodes in $\mathcal{V}$. Let the total number of nodes be $P$, where $P$ can either be finite or countably infinite. The Hilbert space consisting of states describing the position of the walker on the graph is $\mathcal{K} = \mathbb{C}^P$, for finite $P$, and $\mathcal{K} = l^2(\mathbb{C})$ for $P$ being infinite. Here, $l^2(\mathbb{C})$ is the space of square integrable functions. We will denote the orthonormal basis for $\mathcal{K}$ by $|i\rangle$, where $i \in \mathcal{V}$. The walker on this graph possesses internal degrees of freedom described by an $n$-dimensional Hilbert space $\mathcal{H}$. These internal degrees of freedom could represent spin, or polarisation or energy levels. The state of the walker’s internal degree of freedom is given by the operator $\tau \in B(\mathcal{H})$. To specify the state of the quantum walker, we need to specify its internal state and its position on the graph. The total state of the system is thus given by a density matrix $\rho$ on the tensor product space of $\mathcal{H} \otimes \mathcal{K}$. Thus, we have $\rho \in B(\mathcal{H} \otimes \mathcal{K})$.

We want the walk on the graph to be driven by dissipation. Between each two connected nodes on the graph, we envisage an external environment, for example a heat bath, interacting with the system. For each node, $j$, we define a completely positive trace preserving map $\mathcal{M}_j$ acting on the space of operators $B(\mathcal{H})$. $\mathcal{M}_j$ consists of bounded operators $B^i_j \in B(\mathcal{H})$ that transform the walker’s internal degree of freedom as the jump from site $j$ to site $i$ is made. In the Kraus representation,

$$\mathcal{M}_j(\tau) = \sum_{i \in \mathcal{V}} B^i_j \tau B^{i\dagger}_j, \quad \sum_{i \in \mathcal{V}} B^{i\dagger}_j B^i_j = I. \quad (2)$$

The condition on the $B^i_j$ operators in (2) ensures that the trace of $\tau$ is preserved under $\mathcal{M}_j$ and expresses the fact that probability must be conserved. Figure 1 shows an illustration of an OQW on a graph. Three nodes on the graph are labelled $i, j, k$ and the operators $B^i_j$, for example, describe the transformation of the walker’s internal degree of freedom as a transition from site $j$ to site $i$ is made.

So far, we have only described the dynamics on the space $\mathcal{H}$. To formulate the jumping process, and thus describe the dynamics on the full tensor product space, we extend the map $\mathcal{M}$ on $B(\mathcal{H})$ to a map on $B(\mathcal{H} \otimes \mathcal{K})$. We define

$$M^i_j = B^i_j \otimes |i\rangle\langle j|, \quad \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} M^{i\dagger}_j M^i_j = I. \quad (3)$$
Figure 1. The above figure is an illustration of the lazy OQW. Three sample nodes, labeled $i$, $j$, and $k$, are shown for the underlying graph. The transition from node $j$ to node $i$, for example, represented by the directed arrow between those two nodes is described by the $B_{ij}$ operator. $B_{ij}$ transforms of the walker’s internal degree of freedom as a transition from site $j$ to site $i$ is made. Since this is a lazy OQW, the walker also has the possibility of remaining on the same node. The operator $B_{jj}$ encodes for this possibility, transforming the internal degree of freedom when the walker remains on site $j$.

where the identity operator here is defined on $\mathcal{H} \otimes \mathcal{K}$. We can now define a CPTP map on the density matrix $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$

$$\mathcal{M}(\rho) = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} M_{ij} \rho M_{ij}^\dagger. \quad (4)$$

The CPTP map constitutes a discrete time open quantum walk on a graph. Starting from an arbitrary initial state at time $t = 0$ say, $\rho^{(0)} = \sum_{i,j} \tau_{ij}^{(0)} \otimes |i\rangle\langle j|$, one can show that the form of the state becomes diagonal in the position space $\mathcal{K}$ after a single application of $\mathcal{M}$:

$$\mathcal{M}(\rho^{(0)}) = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} M_{ij} \left( \sum_{k \in \mathcal{V}} \sum_{l \in \mathcal{V}} \tau_{kl}^{(0)} \otimes |k\rangle\langle l| \right) M_{ij}^\dagger \quad (5)$$

$$= \sum_{i \in \mathcal{V}} \left( \sum_{j \in \mathcal{V}} B_{ij} \tau_{ij}^{(0)} B_{ij}^\dagger \right) \otimes |i\rangle\langle i|. \quad (6)$$

The density matrix at time $t = 1$ then has the form

$$\rho^{(1)} = \sum_{i \in \mathcal{V}} \tau_{i}^{(1)} \otimes |i\rangle\langle i|, \quad \tau_{i}^{(1)} = \sum_{j \in \mathcal{V}} B_{ij} \tau_{ij}^{(0)} B_{ij}^\dagger. \quad (7)$$

This indicates that there is no mixing taking place between the positions on the graph in our OQW model. For this reason we restrict our attention to density matrices of
the form $\rho = \sum_i \tau_i \otimes |i\rangle \langle i|$. The density matrix at time $t = n$ may be obtained through iteration:

$$
\rho^{(n)} = \sum_{i \in V} \tau_i^{(n)} \otimes |i\rangle \langle i|,
\tau_i^{(n)} = \sum_{j \in V} B_j^{n-1} \tau_j B_j^{\dagger}.
$$

The probability that a position measurement, at time $t = n$, will yield a result of $X_n = i$ is $p(X_n = i) = \text{Tr}(\tau_i^{(n)})$, with the sum over all the positions $i$ in $p(X_n = i)$ equal to 1.

For a more comprehensive introduction to OQWs on graphs, see [15], [16], and [17].

For the rest of this work, we will consider a homogeneous discrete time open quantum walk whose underlying graph is a $d$-dimensional lattice. We employ the use of the canonical basis $\{e_1, e_2, ..., e_d\}$ on $\mathbb{Z}^d$, with $e_{d+j} = -e_j$ for all $j = 1, ..., d$. Thus, for each site on the lattice there are $2d$ adjacent sites for the walker to jump to - one for each direction corresponding to the $e_i$'s. We also will include $e_0 = 0$ in our formulation to encode the idea that the walker can remain on the same site. Then in total there are $2d + 1$ possible jumps. For a homogeneous walk, all of the $B$ operators along the positive $i$th direction in $\mathbb{Z}^d$ are identical and are denoted by $A_i$, while all $B$'s along the negative $i$th direction will be denoted by $A_{i+d}$. The precise relation between the $B$ operators and the $A$ operators is, for the $i$th direction $B_k^{k+1} = A_i$, and $B_k^{k-1} = A_{i+d}$ for all $k \in \mathbb{Z}$, where $k$ labels the nodes in the $e_i$ direction. The position space of the walker is the Hilbert space $\mathcal{K} = \mathbb{C}^{2d}$, the basis for which is denoted by $|i\rangle_i \in \mathbb{Z}^d$.

In previous formulations of open quantum walks on the lattice, a family of bounded operators $\{A_1, ..., A_{2d}\} \in \mathcal{B}(\mathcal{H})$ performed transformations on the state $\tau$ as the walker necessarily jumped to an adjacent site. We extend the family of bounded operators to include an extra operator $A_0$ representing the effect of remaining on the same site. Thus we have $\{A_0, A_1, ..., A_{2d}\}$ acting on $\mathcal{H}$ and satisfying

$$
\sum_{j=0}^{2d} A_j^\dagger A_j = I.
$$

The completely positive map on $\mathcal{B}(\mathcal{H})$ in the Kraus representation is now

$$
\mathcal{L}(\tau) = \sum_{j=0}^{2d} A_j \tau A_j^\dagger.
$$

We extend the map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ with

$$
M_i^j = A_j \otimes |i + e_j\rangle \langle i|.
$$

The operator acting on $\mathcal{K}$ in (11) describes the transition from lattice site $i$ either to the adjacent site in the $j$th direction for $j = 1, 2, ..., 2d$, or to the same site $i$ again for $j = 0$. We still have

$$
\sum_{i \in \mathbb{Z}^d} \sum_{j=0}^{2d} (M_i^j)^\dagger (M_i^j) = I.
$$
where $I$ here is on $H \otimes K$. The CPTP map, defining the discrete time homogeneous open quantum walk is

$$\mathcal{M}(\rho) = \sum_{i \in \mathbb{Z}^d} \sum_{j=0}^{2d} M^j_i \rho (M^j_i)^\dagger. \quad (13)$$

We will still be interested in density matrices of the form $\rho = \sum_{i \in \mathbb{Z}^d} \tau_i \otimes |i\rangle\langle i|$. For $\rho$ to be normalized, we must have $\sum_{i \in \mathbb{Z}^d} \text{Tr}(\tau_i) = 1$. If, at time $n$, the state of the system is $\rho(n) = \sum_{i \in \mathbb{Z}^d} \tau_i(n) \otimes |i\rangle\langle i|$, then, after applying $\mathcal{M}$, the state at time $n+1$ is

$$\rho^{(n+1)} = \mathcal{M}(\rho^{(n)}) = \sum_{i \in \mathbb{Z}^d} \tau_i^{(n+1)} \otimes |i\rangle\langle i|, \quad \tau_i^{(n+1)} = \sum_{j=0}^{2d} A_j \tau_{i-e_j} A_j^\dagger. \quad (15)$$

A very important ingredient in our formulation of the central limit theorem is the steady state $\rho_\infty \in H$, defined to be invariant under the CPTP map in (10) $\rho_\infty = \mathcal{L}(\rho_\infty)$. In our formulation of the lazy OQW $\rho_\infty$, due the inclusion of the $A_0$ term in $\mathcal{L}$, will indeed be different from the previous case where staying on the same site was not possible. The lazy open quantum walk with $A_0$ shows similar behaviour to the previous open quantum walk. For a particular example, indicative of the general behaviour, consider a lazy walk on the line ($d=1$) with a two-dimensional $H$ space. There will be three matrices $A_1, A_0$ and $A_2$ for moving forward, staying on the same site, and moving backwards, respectively. Figure 3 shows an illustration of this particular OQW. In this example, we take

$$A_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i\pi/3} & 1 \end{pmatrix}, \quad A_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & e^{2i\pi/3} \\ 1 & -1 \end{pmatrix}, \quad A_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & e^{-2i\pi/3} \\ 1 & e^{-i\pi/3} \end{pmatrix}. \quad (16)$$

Indeed, these operators satisfy $A_1^\dagger A_1 + A_0^\dagger A_0 + A_2^\dagger A_2 = I$. From Figure 2, with the initial state chosen to be $I/2 \otimes |0\rangle\langle 0|$, one can see that the probability distribution of the lazy OQW approaches a Gaussian distribution centred at the origin.

### 2.2. Quantum Trajectories

In this section we introduce the quantum trajectory for the OQW. Quantum trajectories are a convenient way to simulate the OQW. Furthermore, the CLT formulations of OQWs thus far have all been within the quantum trajectory framework. Beginning from the general formulation of the OQW, the idea for the quantum trajectory is the following. Consider some state at time $t = n$, $\rho^{[n]} = \tau_n \otimes |i_n\rangle\langle i_n|$, where the position of the walker is $X_n = i_n$. We apply the map $\mathcal{M}$ and then perform a measurement on the position space $K$. The state then, at the time $n+1$, jumps to

$$\rho^{[n+1]} = \frac{1}{P(X_{n+1} = i_{n+1}|X_n = i_n)} B^i_{i_n+1} \tau_n (B^i)^{i_{n+1}} \otimes |i_{n+1}\rangle\langle i_{n+1}|. \quad (17)$$
Lazy open quantum walks

Figure 2. In these four figures, we ran the OQW for the operators defined in equation 16. We ran the OQW for \( n = 10, 30, 60 \) and then \( n = 100 \) successive steps. The horizontal axis label \( x \) labels position on the \( x \)-axis, while \( P_x \) on the vertical axis denotes the probability. Top left: for \( n = 10 \). Top right: for \( n = 30 \). Bottom left: for \( n = 60 \). Bottom right: for \( n = 100 \). The key point to note here is that lazy OQW still approaches a Gaussian distribution, as it did in the non-lazy case.

Figure 3. This figure depicts the discrete homogeneous lazy OQW on the line. The operators \( A_1, A_2 \) shift the walker forwards and backwards respectively, while \( A_0 \) corresponds to remaining on the same site.

where

\[
\mathbb{P}(X_{n+1} = i_{n+1} | X_n = i_n) = \mathbb{P}(i_{n+1} | i_n) = \text{Tr} \left( B_{i_n}^{i_{n+1}} \tau_n (B_i^\dagger)^{i_{n+1}} \right)
\]

(18)

is the conditional probability of the position \( X \) at time \( n + 1 \) being equal to \( i_{n+1} \) given that \( X \) at time \( n \) was \( i_n \). Repetition of this process leads to a classical Markov chain valued in the set of states of the form \( \tau \otimes |i\rangle\langle i| \). One may denote this Markov chain as \((\tau_n, X_n)_{n \geq 0}\). Averaging over this quantum trajectory procedure simulates an OQW
Lazy open quantum walks

Master equation driven by $\mathcal{M}$ as can be seen from
\begin{align}
\mathcal{E} (\rho_{n+1}^{[n+1]}) = \sum_{i_{n+1}} \mathbb{P} (i_{n+1} | i_n) \frac{1}{\mathbb{P} (i_{n+1} | i_n)} B_{i_{n+1}}^{\tau_n} (B_{i_{n+1}}^{\dagger})_{i_n} \otimes |i_{n+1}\rangle \langle i_{n+1}| \\
= \sum_{i_{n+1}} B_{i_{n+1}}^{\tau_n} (B_{i_{n+1}}^{\dagger})_{i_n} \otimes |i_{n+1}\rangle \langle i_{n+1}| \\
= \mathcal{M} (\rho_{[n]}). \tag{19}
\end{align}

Extending the map to include $A_0$ also admits a quantum trajectory description with a Markov chain $(\tau_n, X_n)$. For our homogeneous OQW on the lattice, the quantum trajectory description is the following. Let the state of the system at time $t = n$ be $(\tau_n, X_n = i)$. Apply the open quantum walk map $\mathcal{M}$ to the state performing a measurement of the position directly after. The state then at time $t = n + 1$ jumps to
\begin{align}
\left( \frac{1}{\mathbb{P} (j, n)} A_j \tau_n A_j^{\dagger}, X_{n+1} = i + e_j \right), \tag{22}
\end{align}

with the probability $\mathbb{P} (j, n) = \text{Tr} \left( A_j \tau_n A_j^{\dagger} \right)$. Note that even if the walker does remain on the same site, the probability for which would be $\mathbb{P} (0, n) = \text{Tr} \left( A_0 \tau_n A_0^{\dagger} \right)$, its state in $\mathcal{H}$ still undergoes a transformation by $\frac{1}{\mathbb{P} (0, n)} A_0 \tau_n A_0^{\dagger}$. We assume the sequence $\frac{1}{n} \sum_{t=1}^{n} \tau_t$ converges almost surely to a unique steady state $\rho_{\infty}$ \cite{20} \cite{21}.

The central limit theorem formulated in the following section will be formulated in terms of the random variables $(\tau_n, \Delta X_n)$, where $\Delta X_n = X_n - X_{n-1} \in \{e_0, e_1, \ldots, e_{2d}\}$ and $n \neq 0$. This sequence $(\tau_n, \Delta X_n)_{n \geq 0}$ also forms a Markov chain. The transition operator from state $(\tau, e_i)$ to $(\tau', e_j)$ is given by
\begin{align}
P \left[ (\tau, e_i), (\tau', e_j) \right] = \begin{cases} 
\text{Tr} \left( A_j \tau A_j^{\dagger} \right), & \text{if } \tau' = \frac{A_j \tau A_j^{\dagger}}{\text{Tr} (A_j \tau A_j^{\dagger})}, \\
0, & \text{otherwise.}
\end{cases} \tag{23}
\end{align}

The random variables that tend to feature in central limit theorems are independent and identically distributed (iid). Once the OQW is in the steady state, it is exactly the set of $\Delta X_n$ that will play the role of the iid random variables.

3. The central limit theorem for the lazy walker

3.1. The central limit theorem

A central limit theorem was proved for the open quantum walk in \cite{21}, which we generalize to the lazy open quantum walk. Define the iid random variables $Y_k = \{e_0, e_1, \ldots, e_{2d}\}$. For these random variables to be iid, we require the system to be in the steady state, $\rho_{\infty}$. We define the mean $m \in \mathbb{R}^d$ to be
\begin{align}
m = \mathbb{E} (Y_k) = \sum_{i=0}^{2d} \mathbb{P} (i) e_i, \quad \mathbb{P} (i) = \text{Tr} \left( A_i \rho_{\infty} A_i^{\dagger} \right). \tag{24}
\end{align}
Since $e_0 = 0$, $m$ reduces to the same expression as in [21]. The position of the walker on the lattice at time $n$ is

$$X_n = X_0 + \sum_{i=1}^{n} Y_i,$$  \hspace{1cm} (25)

where the initial position $X_0$ can be chosen for convenience to be zero. A central limit theorem can now be proven for the quantity

$$\frac{1}{\sqrt{n}} (X_n - \mathbb{E}(X_n)) = \frac{1}{\sqrt{n}} (X_n - nm) = \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} Y_i - m \right).$$  \hspace{1cm} (26)

Essentially, for any vector $l \in \mathbb{R}^d$, the quantity $(X_n - nm) \cdot l$ may be decomposed into a martingale and a quantity which does not contribute to the law of large numbers and the central limit theorem at large times. The martingale is

$$M_n = \sum_{j=2}^{n} [f(\rho_j, \Delta X_j) - P f(\rho_{j-1}, \Delta X_{j-1})],$$  \hspace{1cm} (27)

where the function $f$ is $f(\rho, x) = \text{Tr}(\rho L l) + x \cdot l$ and the operator $L_l = L \cdot l$ is a solution to the equation

$$(L_l - L^\dagger (L_l)) = \sum_{i=1}^{d} \tilde{A}_i (e_i \cdot l) - (m \cdot l) I, \quad \tilde{A}_i = A_i^\dagger A_i - A_{i+d}^\dagger A_{i+d}.$$  \hspace{1cm} (28)

The quantity $M_n$ satisfies the defining condition for a martingale. Martingales feature prominently in probability theory [33, 34]. One of the fundamental notions of probability theory is that of a $\sigma$-space spanned by events to which probabilities are assigned. In the current quantum trajectory setting, the events are $(\rho_n, X_n)$. We define the filtration $(\mathcal{F}_n)_{n \geq 2}$, where $\mathcal{F}_n$ is the $\sigma$-space spanned by events $(\rho_j, X_j)$ for $j \leq n$. The defining condition for martingale $M_n$ with respect to $(\mathcal{F}_n)_{n \geq 2}$ is

$$\mathbb{E}[\Delta M_n | \mathcal{F}_{n-1}] = 0.$$  \hspace{1cm} (29)

Note that the dual map $L^\dagger$ in (28), defined as $L^\dagger(\tau) = \sum_{i=0}^{2d} A_i^\dagger \tau A_i$, is different for the lazy open quantum walk because of the inclusion of the $A_0$ operator. We further note here that equation (28) forms a degenerate system of equations. One way of seeing this is by vectorising (28) with the help of the reshaping operation. The reshaping operation stacks the rows of a matrix on top of each other in a row vector. So for an $m \times n$ matrix $A$, for example, we have

$$\text{vec}(A) = (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, a_{m2}, \ldots, a_{mn})^T.$$  \hspace{1cm} (30)

The left-hand-side of (28) then becomes

$$\left( I - \sum_{i=1}^{2d} A_i^\dagger \otimes A_i^T \right) \text{vec}(L_l).$$  \hspace{1cm} (31)

One can show, using (9), that $\sum_{i=1}^{2d} A_i^\dagger \otimes A_i^T$ has an eigenvector of vec $(I)$ with eigenvalue of 1. This means the determinant of the matrix in (31) vanishes, and the system of equations is degenerate. However, [21] proves that there exists a solution to (28) for
any \( l \in \mathbb{R}^d \), and that the difference between any two solutions is proportional to the identity. Furthermore, we note here that after taking the adjoint of (28), \( L_l = L_l^\dagger \) is also a solution. Since \( L_l + \alpha I \) also solves (28), we conclude that half of the \( D(D-1) \) off-diagonal entries in \( L_l \) are not independent, where \( D \) is the dimension of the coin space, \( \mathcal{H} \). Writing the vector \( l \) in terms of the canonical basis \( l = \sum_{i=1}^d l_i e_i \), we have
\[ L_l = \sum_{i=1}^d L_i l_i. \]
It follows that there is an \( L_i \) for each direction on the \( d \)-dimensional lattice satisfying
\[ L_i - L_i^\dagger (L_i) = \tilde{A}_i - m_i I. \]  

The martingale in (27) satisfies the necessary conditions for a central limit theorem for martingales to be applicable [21], [33]. Thus \( M_n / \sqrt{n} \) converges in distribution to a Gaussian \( \mathcal{N}(0, \sigma_l^2) \). It is remarkable that one may obtain an analytic expression for the covariance matrix of this distribution
\[ C_{ij} = \delta_{ij} \left( \text{Tr} \left( A_i \rho_\infty A_j^\dagger \right) + \text{Tr} \left( A_{i+d} \rho_\infty A_{i+d}^\dagger \right) \right) - m_i m_j 
+ \left( \text{Tr} \left( A_i \rho_\infty A_i^\dagger L_j \right) + \text{Tr} \left( A_j \rho_\infty A_j^\dagger L_i \right) \right) 
- \text{Tr} \left( A_{i+d} \rho_\infty A_{i+d}^\dagger L_j \right) - \text{Tr} \left( A_{j+d} \rho_\infty A_{j+d}^\dagger L_i \right) 
- (m_i \text{Tr} (\rho_\infty L_j) + m_j \text{Tr} (\rho_\infty L_i)), \]  
for each of the possible directions on the lattice \( i, j = \{1, 2, \ldots, d\} \). We find that this is the same expression as for the non-lazy case. It may, however, be extended to the lazy open quantum walk case with the following considerations. Extending the open quantum walk map with the set of operators \( \{A_1, \ldots, A_{2d}\} \) to include \( A_0 \) firstly means that the normalisation condition is now given according to (9). Secondly, the extended equation \( \rho_\infty = \mathcal{L}(\rho_\infty) \) will need to be solved for a new steady state \( \rho_\infty \). A new \( \rho_\infty \) implies new values for the mean values \( m_i \). The new \( m_i \) values may be calculated with the expression (24). Lastly, new \( L_i \) matrices will have to be solved for since, in equation (32), the dual map is also extended by the \( A_0 \) term, and the new \( m_i \) values feature on the right-hand-side. In what follows, we subject the variance formula to a variety of checks for the lazy open quantum walk.

3.2. The microscopic derivation

Any CPTP map, such as the open quantum walk map described in Section 2.1, may be thought of as a quantum channel. Given a quantum channel, the Stinespring dilation theorem [35] guarantees the existence of a physical system implementing the given map. Thus, one may ask what is a physical system giving rise to the OQW? The first few steps in this direction were undertaken in [36] and [37] culminating in [18]. The Hamiltonian for the total system may be written as the sum of the system, bath and system-bath interactions Hamiltonians,
\[ H = H_S + H_B + H_{SB}. \]
The system Hamiltonian describes the local free evolution of the walker’s internal degree of freedom as well as the position on the underlying graph. Thus,

\[ H_S = \sum_i \Omega_i \otimes |i\rangle\langle i| \, . \tag{35} \]

Concretely, the bath is thought of as a bath of harmonic oscillators with \( H_B \) expressed in terms of bosonic creation and annihilation operators

\[ H_B = \sum_{i \neq j} \sum_n \omega_{i,j,n} a_{i,j,n}^\dagger a_{i,j,n}. \tag{36} \]

The system-bath interaction describes the bath driven transitions from site to site on the graph and hence may be written as

\[ H_{SB} = \sum_{i \neq j} \sum_n Q_{i,j} \otimes X_{i,j} \otimes B_{i,j}. \tag{37} \]

The \( Q_{i,j} \in \mathcal{B}(\mathcal{H}) \) operators are responsible for transforming the internal degree of freedom when a transition involving sites \( i \) and \( j \) occurs. The \( X_{i,j} \in \mathcal{B}(\mathcal{K}) \) is responsible for implementing the steps between the sites. A simple Hermitian choice for \( X_{i,j} \) is

\[ X_{i,j} = |i\rangle\langle j| + |j\rangle\langle i|. \]

Lastly, the \( B_{i,j} = \sum_n (g_{i,j,n} a_{i,j,n} + g_{i,j,n}^* a_{i,j,n}^\dagger) \) describes the coupling of the walker with the local environment.

The microscopic derivation of the open quantum walk model, performed in \cite{18} for a graph with a general topology, employed the theory outlined in \cite{38}. Using the Born-Markov approximation the reduced density matrix of the system \( \rho_s (t) \), in the interaction picture, satisfies the equation

\[
\frac{d}{dt} \rho_s (t) = -\int_0^\infty d\tau \text{Tr}_B \left[ H_{SB} (t) \left[ H_{SB} (t - \tau) \otimes \rho_s (t) \otimes \rho_B \right] \right] \tag{38}
\]

where \( \text{Tr}_B \) stands for tracing out the bath degrees of freedom, and \( \rho_B \) denotes the density matrix of the bath. Assuming that the environment is in a thermal equilibrium state, \( \rho_B = \exp(-\beta H_B) / \text{Tr}[\exp(-\beta H_B)] \). We assume that each of the \( \Omega_i \)'s have a unique set of eigenvalues. Their spectral decomposition may be written in terms of their eigenvalues \( \lambda^{(i)} \) and orthogonal projectors \( \Pi_i (\lambda^{(i)}) \). The \( Q_{i,j} \) operators are then expressed in the basis associated with \( \Omega_i \) and \( \Omega_j \),

\[
Q_{i,j} (\omega) = \sum_{\lambda^{(i)} - \lambda^{(j)} = \omega < 0} \Pi_i (\lambda^{(i)}) Q_{i,j} \Pi_j (\lambda^{(j)}), \tag{39}
\]

\[
Q_{i,j}^\dagger (\omega') = Q_{i,j} (-\omega'). \tag{40}
\]

After transforming \( H_{SB} \) to the interacting picture, and using the rotating wave approximation for the transition frequencies \( \omega \) and \( \omega' \), the following form for the master equation for \( \rho_s (t) \) emerges

\[
\frac{d}{dt} \rho_s (t) = \sum_{i,j} \sum_\omega \left\{ \gamma_{i,j} (-\omega) \mathcal{D} [Q_{i,j} (\omega) \otimes |j\rangle\langle i|] \rho_s (t) \\
+ \gamma_{i,j} (\omega) \mathcal{D} [Q_{i,j}^\dagger (\omega) \otimes |i\rangle\langle j|] \rho_s (t) \right\} \\
+ \sum_{i,j} \sum_\omega \left\{ \gamma_{i,j} (-\omega') \mathcal{D} [Q_{i,j} (\omega') \otimes |i\rangle\langle j|] \rho_s (t) \\
+ \gamma_{i,j} (\omega') \mathcal{D} [Q_{i,j}^\dagger (\omega') \otimes |j\rangle\langle i|] \rho_s (t) \right\} \tag{41}
\]
Lazy open quantum walks

\[ + \gamma_{i,j}(\omega') \mathcal{D} \left[ Q_{i,j}^\dagger(\omega') \otimes |j\rangle\langle i| \right] \rho_s(t) \}

where \( \mathcal{D}(X) \rho \) denotes standard dissipative superoperator in Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form \([38, 39, 40]\)

\[
\mathcal{D}(X) \rho = X \rho X^\dagger - \frac{1}{2} X^\dagger X \rho - \frac{1}{2} \rho X^\dagger X,
\]

where the \( X \)'s here form a basis for the corresponding \( N \)-dimensional Liouville space \([38]\). The function \( \gamma(\omega) \) is the real part of the Fourier transformation of the bath correlation functions \( \langle B_{i,j}^\dagger(s) B_{i,j}(0) \rangle \). See \([18]\) for the full expression. After writing \( \rho_s(t) = \sum_i \rho_i(t) \otimes |i\rangle\langle i| \), one may derive a system of master equations

\[
\frac{d}{dt} \rho_i(t) = \sum_{j,\omega} \gamma_{j,i}(\omega) Q_{j,i}(\omega) \rho_j Q_{j,i}^\dagger(\omega) - \frac{\gamma_{i,j}(\omega)}{2} \{ Q_{i,j}^\dagger(\omega) Q_{i,j}(\omega) , \rho_i \}
+ \sum_{j,\omega} \gamma_{i,j}(\omega) Q_{i,j}^\dagger(\omega) \rho_j Q_{i,j}(\omega) - \frac{\gamma_{j,i}(\omega)}{2} \{ Q_{j,i}^\dagger(\omega) Q_{j,i}(\omega) , \rho_i \}
+ \sum_{j,\omega} \gamma_{i,j}(\omega') Q_{i,j}(\omega') \rho_j Q_{i,j}^\dagger(\omega') - \frac{\gamma_{j,i}(\omega')}{2} \{ Q_{j,i}(\omega') Q_{j,i}^\dagger(\omega') , \rho_i \}.
\]

This defines the continuous time OQW. To obtain the discrete time OQW of section 2.1, a time step \( \Delta \) is introduced and the time derivative in the differential equation is discretised in terms of \( \Delta \). The connection between the discretised version of (43) and the discrete time OQW is established by the following identifications

\[
B_{i}^{(1)}(\omega) = \sqrt{\Delta \gamma_{i,j}(\omega)} Q_{i,j}(\omega), \quad B_{j}^{(2)}(\omega) = \sqrt{\Delta \gamma_{j,i}(\omega)} Q_{j,i}^\dagger(\omega)
\]

\[
B_{j}^{(1)}(\omega') = \sqrt{\Delta \gamma_{j,i}(\omega')} Q_{j,i}(\omega'), \quad B_{j}^{(2)}(\omega') = \sqrt{\Delta \gamma_{j,i}(\omega')} Q_{j,i}^\dagger(\omega')
\]

\[
B_{i}^{(\omega)} = I_N - \frac{\Delta}{2} \sum_{j,\omega} \left( \gamma_{i,j}(\omega) Q_{i,j}^\dagger(\omega) Q_{i,j}(\omega) + \gamma_{j,i}(\omega) Q_{j,i}(\omega) Q_{j,i}^\dagger(\omega) \right)
- \frac{\Delta}{2} \sum_{j,\omega'} \left( \gamma_{i,j}(\omega') Q_{i,j}(\omega') Q_{i,j}^\dagger(\omega') + \gamma_{i,j}(\omega') Q_{i,j}(\omega') Q_{i,j}^\dagger(\omega') \right).
\]

One may now show that the OQW with the transition operators in equation (44) satisfies the normalization condition of (2) up to \( O(\Delta^2) \), and the iteration formula for \( \rho_i^{[n+1]} \), at time \( n + 1 \), is of the same form as (8). As one can see from the presence of the \( B_{i}^{(\omega)} \) transition operator in (44), all microscopically derived OQWs are lazy. The expressions for the transition operators \( B_{j}^{(\omega)} \) in (44), and as described in \([18]\), establish connections between the dynamical properties of the OQW and the thermodynamic properties of the environment.

In the remainder of this section, we specialise the microscopic derivation to a homogeneous discrete time OQW on the lattice \( \mathbb{R}^d \), which is necessary for our central limit theorem to be applicable. We assume that the local unitary Hamiltonians on each site are identical and are denoted by \( H_0 \). Thus, for all \( i \), \( \Omega_i = H_0 \). Next, recall for
the homogeneous OQW map of the lattice, all the operators transforming the walker’s internal degrees of freedom are identical for a given direction on the lattice. Thus, these operators were expressed in terms of the \( A_i \)'s and the \( A_{i+d} \)'s. In the microscopic derivation we now similarly have \( Q_i \) and \( Q_{i+d} \). Defining the relation

\[
A_i = \sqrt{\Delta} Q_i, \quad A_{i+d} = \sqrt{\Delta} Q_{i+d},
\]

we obtain the discrete time homogeneous OQW on the lattice \( \mathbb{Z}^d \) from a microscopic derivation. In (45), we have absorbed the \( \gamma \) functions into the definition of the \( Q \)'s. Up to \( O(\Delta^2) \), we have

\[
\sum_{i=0}^{2d} A_i^\dagger A_i = I. \quad (46)
\]

The CPTP map on \( \mathcal{H} \), after \( n \) iterations, is

\[
\tau_i^{(n+1)} = \sum_{j=0}^{2d} A_j \tau_i^{(n)} A_j^\dagger. \quad (47)
\]

By substitution of (45) into the steady state condition \( \rho_\infty = \mathcal{L}(\rho_\infty) \), it is straightforward to see

\[
0 = -i [H_0, \rho_\infty] + \sum_{j=1}^{2d} \left( Q_j \rho_\infty Q_j^\dagger \rho_\infty - \frac{1}{2} Q_j^\dagger Q_j \rho_\infty - \frac{1}{2} \rho_\infty Q_j^\dagger Q_j \right) \quad (48)
\]

up to \( O(\Delta^2) \). Note that the right-hand-side of equation (48) is precisely of GKSL form. The Lindbladian super-operator describes the time evolution of an open quantum system, with state \( \rho \), and is defined by

\[
\dot{\rho} = -i [H, \rho] + \sum_{j=1}^{N^2-1} \mathcal{D}(X_j) \rho. \quad (49)
\]

Since the steady state is time-independent (and thus its time derivative vanishes), we obtain the quantum master equation for \( \rho_\infty \) in GKSL form

\[
\dot{\rho}_\infty = \mathcal{L}_\text{in}(\rho_\infty). \quad (50)
\]

We note that equation (50) is independent of the time step \( \Delta \). The mean, once written in terms of the \( Q \) operators is

\[
m = \Delta \sum_{j=0}^{2d} \text{Tr} \left( Q_j \rho_\infty Q_j^\dagger \right) e_j = \Delta \sum_{j=1}^{d} \text{Tr} \left( \tilde{Q}_j \rho_\infty \right) e_j, \quad (51)
\]

where we have defined \( \tilde{Q}_j = Q_j^\dagger Q_j - Q_{j+d}^\dagger Q_{j+d} \).

Next, we study equation (28). The left-hand-side becomes

\[
L_l - \mathcal{L}^\dagger(L_l) = -\Delta \left[ i [H_0, L_l] + \sum_{j=1}^{2d} \left( Q_j L_l Q_j - \frac{1}{2} L_l Q_j^\dagger Q_j - \frac{1}{2} Q_j^\dagger Q_j L_l \right) \right], \quad (52)
\]
where the terms in braces define the adjoint of the Lindbladian, \( L_{in}^\dagger \) [38]. The right-hand-side of (28) becomes
\[
\sum_{i=0}^{2d} A_i^\dagger A_i (e_i \cdot l) - (m \cdot l) I = \Delta \sum_{j=1}^d \left[ \tilde{K}_j - \text{Tr} \left( \tilde{K}_j \rho_\infty \right) I \right] (e_j \cdot l).
\]
(53)
Thus, equation (28) describes the time evolution of the \( L_l \) operator in the Heisenberg picture
\[
\dot{L}_l = L_{in}^\dagger (L_l) = \sum_{j=1}^d \left[ \tilde{K}_j - \text{Tr} \left( \tilde{K}_j \rho_\infty \right) I \right] (e_j \cdot l).
\]
(54)
This equation is also independent of the time step size \( \Delta \).

3.3. Example 1
We turn now to some examples derived from the microscopic model. The first example, considered in [18], is the open quantum walk on the circle. The appropriate operators are
\[
B = \sqrt{\Delta \gamma} \left( \langle n \rangle + 1 \right) \sigma_-, \quad C = \sqrt{\Delta \gamma} \langle n \rangle \sigma_+,
\]
(55)
\[
A = I - \frac{\Delta}{2} \left[ \gamma \left( \langle n \rangle + 1 \right) \sigma_+ \sigma_+ + \gamma \langle n \rangle \sigma_- \sigma_+ \right] - i \lambda \Delta \vec{n}_\lambda \vec{\sigma},
\]
(56)
where \( \vec{n}_\lambda \vec{n}_\lambda = n_x^2 + n_y^2 + n_z^2 = 1 \). Solving for the mean \( m \) from
\[
m = \text{Tr} \left( B \rho_\infty B^\dagger \right) - \text{Tr} \left( C \rho_\infty C^\dagger \right)
\]
we obtain
\[
m = \Delta \frac{4 \left( 1 - n_z^2 \right) \gamma \lambda^2}{\gamma^2 (1 + 2 \langle n \rangle)^2 + 8 \left( 1 + n_z^2 \right) \lambda^2}.
\]
Using formula (33) for the variance we find
\[
\sigma^2 = \frac{4 \Delta t_- \gamma \lambda^2}{s_2 (s_2 \gamma^2 + 8 t_+ \lambda^2)^3} \left[ s_2^6 \gamma^4 + 8 s_2^2 t_+ \gamma^2 \lambda^2 \left( 5 n_z^2 + 8 s_1 \langle n \rangle - 1 \right) \right.
\]
\[
+ 64 \lambda^4 \left( s_2^2 + 4 n_z^2 (s_2 + 2 \langle n \rangle) + n_z^4 (4 s_1 \langle n \rangle - 1) \right),
\]
where \( t_\pm = 1 \pm n_z^2 \), and \( s_j = 1 + j \langle n \rangle \). An important check of formula (33) is that our expressions for the mean and variance agree with those in [18]. Both \( m \) and \( \sigma^2 \) indeed do reduce to the corresponding expressions in [18] when \( n_y = 1 \).

3.4. 2D examples of lazy OQWs

3.4.1. Generic notations In this subsection two examples of lazy OQWs in 2D will be presented. In both examples, the transition operators will follow from the outlined microscopic model for lazy OQWs in 2D. To make notations more clear, the following conventions will be used:
coordinates on the 2D lattice \( r = (i, j) \);

- possible movement from the \( r \) along the \( x \)-axis is denoted \( r_x = (i + 1, j) \) and along the \( y \)-axis \( r_y = (i, j + 1) \), respectively;

- set of possible movements form the position \( r \) is denoted as \( r' = \{r_x, r_y\} \); for example \( \sum_r f(r) \equiv \sum_{i,j} f(i,j) \) or \( \sum_{r,r'} f(r|r') \equiv \sum_r f(r|x) + f(r|y) \equiv \sum_{i,j} f(i, j + 1, j) + f(i, j | i, j + 1) \).

3.4.2. Example 2

Let us consider 2D array of two level atoms (trapped ultra cold atoms on an optical lattice) described by the following Hamiltonian:

\[
H_S = \sum_r \frac{\omega_0}{2} \sigma_z \otimes |r\rangle \langle r| + \lambda (\vec{n}_\lambda \vec{\sigma}) \otimes |r\rangle \langle r|,
\]  

(57)

where \( \sigma_z \) is Pauli \( z \) matrix and describes internal degree of freedom of the walker and \( |r\rangle \equiv |i,j\rangle \) describes position of the on 2D lattice. To end up with OQW on 2D one needs to assume an environment assisted transport between every connected node of the walk. Taking this into consideration the Hamiltonian of the bath reads,

\[
H_B = \sum_{r,r',n} \omega_{r,r',n} a_{r,r',n}^\dagger a_{r,r',n},
\]  

(58)

where operators \( a_{r,r',n}^\dagger \) and \( a_{r,r',n} \) denotes bosonic creation and annihilation operators of \( n \)-th mode of the thermal bath located between nodes \( r \) and \( r' \), the frequency of this mode is denoted by \( \omega_{r,r',n} \).

In this example, it is assumed that the transition of the walker along the \( x \)-axis is assisted via a dissipative coupling, while transition via \( y \)-axis is driven by the decoherent coupling. Under these assumptions the system-bath Hamiltonian \( H_{SB} \) reads,

\[
H_{SB} = \sum_{r,n} g_{r,r_x,n} a_{r,r_x,n}^\dagger \sigma_- \otimes |r_x\rangle \langle r| + \text{h.c.}
\]  

(59)

\[
+ \sum_{r,n} g_{r,r_y,n} a_{r,r_y,n}^\dagger \sigma_z \otimes |r_y\rangle \langle r| + \text{h.c.,}
\]
where coefficients $g_{r,r,n}$ denote the coupling strength between $n$-th mode of the bosonic bath located between nodes $r$ and $r_i$ with OQW walker. Following a generic microscopic derivation for OQWs [18] it is straightforward to end up with the following transition operators,

$$B_x = \sqrt{\Delta \gamma \langle (n) + 1 \rangle} \sigma_z, \quad B_y = \sqrt{\Delta \gamma \langle (n) \rangle} \sigma_z,$$

$$C_x = \sqrt{\Delta \gamma \langle (n) \rangle} \sigma_x, \quad C_y = \sqrt{\Delta \gamma \langle (n) \rangle} \sigma_y,$$

$$A = I - \frac{\Delta}{2} \left[ \gamma (\langle n \rangle + 1) \sigma_+ \sigma_- + \gamma \langle n \rangle \sigma_- \sigma_+ + \gamma_y I + \gamma_y I^* \right] - i \lambda \Delta \vec{n} \vec{\sigma}. \quad (62)$$

The mean in the $x$ and $y$ directions are

$$m_x = \frac{4 \gamma \Delta \lambda^2 t_- T}{8 \lambda^2 t_- T + \gamma s_2 (16 \lambda^2 n_s^2 + T^2)}$$

$$m_y = \Delta r_-,$$

where $r_\pm = \gamma_y \pm \gamma_y$, $T = \gamma s_2 + 4 r_+$, $n_s^2 = n_s^2 \pm 1$. The covariance matrix entries are

$$C_{xx} = \frac{-4 \gamma \Delta \lambda^2 \eta_z}{(\gamma s_2 T^2 + 8 \lambda^2 (\gamma s_2 \eta_z^2 - 4 r_+ \eta_z^2))} \left[ 64 \lambda^4 \left( -8 \gamma r_+ \eta_z \left( \langle (n) \rangle s_1 - s_2^2 + 1 \right) n_s^2 + s_2^2 \right) + \gamma^2 s_2 \left( s_2^2 (5 n_s^4 - 2 n_s^2 + 1) - 2 \langle (n) \rangle s_1 + 3 \right) n_s^2 \eta_z \right] + 16 r_+ s_2 (\eta_z^2)^2$$

$$+ 8 \gamma \lambda^2 (4 r_+ + \gamma s_2) \left( \gamma s_2 \left( \langle (n) \rangle (2 \langle n \rangle - s_1 + 2) + 5 \right) n_s^2 + \gamma (8 \langle n \rangle s_1 - 1) \right) \right. \nonumber$$

$$- 4 r_+ \left. \left( \langle (n) \rangle s_1 + 1 \right) \eta_z \right] + \gamma^2 s_2^3 (4 r_+ + \gamma s_2)^2 \quad (65)$$

$$C_{yy} = \Delta r_+ \quad (66)$$

$$C_{xy} = C_{yx} = -\frac{16 \gamma^2 \Delta \lambda^2 \eta_z r_- s_2 \left( -16 \lambda^2 n_s^2 + 16 r_+^2 + 8 \gamma r_+ + \gamma^2 s_2^2 \right)}{(8 \lambda^2 (2 n_s^2 s_2 - n_s^2 T^2 + T) + \gamma s_2 T^2)^2} \quad (67)$$

Note that as $\gamma_y^+$ tends toward $\gamma_y^-$, the off diagonal entries $C_{xy}$ tend to zero.

![Figure 5](image-url). The Gaussian distribution plotted from the theoretically predicted values for the two-dimensional OQW in example 2. For $n_y = 1, \lambda = 0.3, \gamma = 0.1, \gamma_y^- = 0.5, \gamma_y^+ = 0.5, \langle n \rangle = 1$ and $\Delta = 0.05$, we found

$$\vec{m} = (0.000895522, 0), \quad C = \begin{pmatrix} 0.00260837 & -0.000106928 \\ -0.000106928 & 0.05 \end{pmatrix}$$
is defining a lazy OQW on the line with a two-dimensional coin space. The steady state to the following form of the transition operators,

\[ A = I - \frac{\Delta}{2} \left[ (\gamma_x + \gamma_y) (\langle n \rangle + 1) \sigma_+ \sigma_- + (\gamma_x + \gamma_y) \langle n \rangle \sigma_- \sigma_+ \right] - i \lambda \Delta \vec{n} \lambda \vec{\sigma}, \]

where \( n_y = 1 \) and \( n_x = n_z = 0 \). With these definitions we obtain the OQW for this particular model. The results are

\[ m_x = \frac{4\gamma_x \Delta \lambda^2}{8\lambda^2 + (2\langle n \rangle + 1)^2(\gamma_x + \gamma_y)^2}, \quad m_y = \frac{4\gamma_y \Delta \lambda^2}{8\lambda^2 + (2\langle n \rangle + 1)^2(\gamma_x + \gamma_y)^2} \]

The \( L_x \) and \( L_y \) matrices are relatively simple to report

\[ C_{xx} = \frac{\Delta}{(8\lambda^2 + r^2 s_2^2)\gamma_x^2} \left[ 2\langle n \rangle r s_2 \gamma_x^2 (4\lambda^2 + (\langle n \rangle + 1) r^2 s_2^2) \right. \\
+ 8\lambda^2 r^2 \gamma_x^2 (-4\lambda^2 \langle n \rangle (\langle n \rangle + 2) + 3 - \langle n \rangle r^2 s_2^2) \\
+ s_1 \gamma_x (8\lambda^2 + r^2 s_2^2) (4\lambda^2 + (\langle n \rangle - r^2 s_2^2) + \langle n \rangle \gamma_x (4\lambda^2 + r^2 s_1 s_2) (8\lambda^2 + r^2 s_2^2) \right]. \]

In (73), \( r = \gamma_x + \gamma_y \). \( C_{yy} \) is the same expression but with \( \gamma_x \) and \( \gamma_y \) interchanged. The off-diagonal elements are

\[ C_{xy} = -\frac{2\Delta \gamma_x \gamma_y (8\lambda^4 (8\langle n \rangle + 2) + 2) r s_2 + \langle n \rangle r^5 s_1 s_2 + 6\lambda^4 \langle n \rangle r^3 s_1 s_2^3)}{(8\lambda^2 + r^2 s_2^2)^3}. \]

We note that \( C_{yx} = C_{xy} \) and that the off-diagonal elements are symmetric under interchanging \( \gamma_x \) and \( \gamma_y \).

### 3.5. A numerical example

In this section, we study a numerical example. We consider the matrices in equation (16)

\[ A_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 \\ 1 & e^{i\pi/3} \end{pmatrix}, \quad A_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & e^{2i\pi/3} \\ 1 & -1 \end{pmatrix}, \quad A_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & e^{-2i\pi/3} \\ 1 & e^{-i\pi/3} \end{pmatrix} \]

defining a lazy OQW on the line with a two-dimensional coin space. The steady state is

\[ \rho_\infty = \begin{pmatrix} 0.5 & 0.375 - 0.217i \\ 0.375 + 0.217i & 0.5 \end{pmatrix}, \]
leading to an $m$ value of zero. Equation (28), in vectorised form is (see equation (31) for the left-hand-side)

$$
\begin{pmatrix}
0.5 & -0.5 & -0.5 & -0.5 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-0.5 & -0.25 - 0.433i & -0.25 + 0.433i & 0.5
\end{pmatrix}
\begin{pmatrix}
L_{11} \\
L_{12} \\
L_{21} \\
L_{22}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0.25 + 0.433i \\
0.25 - 0.433i \\
0
\end{pmatrix}
$$

Solving this equation for the $L$ entries, we obtain

$$
L = \begin{pmatrix}
0.25 & 0.25 + 0.433i \\
0.25 - 0.433i & -0.25
\end{pmatrix}
$$

Applying formula (33) to calculate the variance, we obtain $C = 1.04167$. To check this value, we simulated the above OQW for 10, 100, 1000, 10000 and 50000 steps. For each of these steps, we calculated the variance from the probability distribution and then converted it to the variance $C_{\text{sim}}$ associated with the central limit theorem. Our results are summarized in Table 1.

4. Conclusion

Following the microscopic derivation of the OQW, we found that we needed to incorporate the possibility that the walker could remain on the same lattice site after each application of the CPTP map. This then raised the question concerning the asymptotic, or long time, behaviour of the new lazy OQW. We found that a central limit theorem may still be derived for this case. Furthermore, it is still possible to obtain an analytic formula for the variance of the associated Gaussian distribution. The overall expression for the variance is the same as in [21]. However, the quantities populating this expression do change in a non-trivial way. Extending the map $\mathcal{L}$ on $\mathcal{H}$ to include the
Table 1. Table showing variance results from the OQW as defined in equation (16). The \( n \) denotes the number of steps and the \( C_{\text{sim}} \) denotes the variance associated with the central limit theorem. The theoretical value for the variance was \( C = 1.04167 \). The table shows that the variance obtained from the simulation converges to the theoretical value as the number of steps \( n \) increases.

| \( n \) | \( C_{\text{sim}} \) |
|-------|----------------|
| 10    | 1.08333        |
| 100   | 1.04583        |
| 1000  | 1.04208        |
| 10000 | 1.04171        |
| 50000 | 1.04167        |

possibility of self-jumping leads to a different steady state \( \rho_\infty \), as well as quantities, such as the mean \( m \), that depend on it. The \( L_i \) operators will also change in the new model due to the mean and the dual map \( L^\dagger \) being present in its system of equations. We checked the analytic formula (33) for a number of examples. Three analytic examples were presented where the OQW was derived from a microscopic model. A numerical example was then considered in which evidence was presented for the convergence of the variance, calculated from the simulated trajectories, to the variance calculated using the formula.

Further insight was obtained into equation for the \( L_i \) operators in equation (28). We found that the system is degenerate and that, up to a multiple of the identity matrix, the \( L \) operators are Hermitian. For \( L \) being a \( D \times D \) matrix, this means that \( \frac{1}{2}D(D-1) \) of the off-diagonal entries are not independent. We derived the discrete time homogeneous lazy OQW on the lattice \( \mathbb{R}^d \) from the microscopic model. In terms of the operators from the microscopic model, we managed to write the time evolution for the steady state \( \rho_\infty \) in GKSL form. Also in terms of the microscopic model operators, we wrote the time evolution for \( L_i \) in the adjoint GKSL form.

We have formulated the CLT for a homogeneous OQW on a lattice that has a probability of remaining on the same site on the underlying lattice. One of the main assumptions in our work is that the OQW steady state is unique. The problem of formulating a central limit theorem for the case of a non-unique steady state is an interesting future avenue of research to pursue. One can indeed construct examples of OQWs that converge to multiple steady states. Indeed, an interesting pursuit would be to generalize the work of [27, 28] to the lazy OQW case. It is conceivable that a central limit theorem could potentially exist for each steady state, and that the corresponding analytic formulas would, in some way, depend on the initial state of the walk.

In summary, we have defined a lazy open quantum walk in which the walker has the possibility of remaining on the same site. We then derived and presented some evidence for a central limit theorem for our lazy open quantum walks on a homogeneous lattice. This work adds an important piece to the overall OQW framework.
Lazy open quantum walks

This work was supported by the South African Research Chair Initiative of the Department of Science and Technology and the National Research Foundation. IS acknowledge support in part by the National Research Foundation of South Africa (Grant No. 119345).

5. References

[1] Barber M, Nakanishi N, Ninham B and Ninham B 1970 Random and Restricted Walks: Theory and Applications Mathematics and its applications : a series of monographs and texts (Gordon and Breach)
[2] Knight F B 1962 Transactions of the American Mathematical Society 103 218–228
[3] H Weiss G and J Rubin R 2007 Adv. Chem. Phys. 52 363 – 505
[4] Papadimitriou C H 1994 Computational Complexity (Reading, MA: Addison-Wesley)
[5] Cootner P The random character of stock market prices
[6] Berg H Random Walks in Biology Princeton paperbacks ISBN 9780691000640
[7] Aharonov Y, Davidovich L and Zagury N 1993 Phys. Rev. A 48(2) 1687–1690
[8] Farhi E and Gutmann S 1998 Phys. Rev. A 58(2) 915–928
[9] Kempe J 2003 Contemporary Physics 44 307–327
[10] Venegas-Andraca S E 2012 Quant. Inf. Proc. 11 1015–1106
[11] Konno N J. Math. Soc. Japan 1179–1195
[12] Wong T G 2015 Journal of Physics A Mathematical General 48 435304 (Preprint 1502.04567)
[13] Childs A M 2010 Communications in Mathematical Physics 294 581–603
[14] Dan L, Gettrick M M, Wei-Wei Z and Ke-Jia Z Chinese Physics B 24 050305
[15] Attal S, Petruccione F, Sabot C and Sinayskiy I J. Stat. Phys. 147 832 – 852
[16] Attal S, Petruccione F and Sinayskiy I 2012 Physics Letters A 376 1545 – 1548 ISSN 0375-9601
[17] Petruccione F and Sinayskiy I Phys. Scr. T. 151 014077
[18] Sinayskiy I and Petruccione F 2015 Phys. Rev. A 92(3) 032105
[19] Sinayskiy I and Petruccione F 2012 Quantum Information Processing 11 1301–1309
[20] Kummerer B and Maassen H Journal of Physics A: Mathematical and General 37 11889
[21] Attal S, Guillo?it-Plantard N and Sabot C Ann. Henri Poincaré 16(15)
[22] Konno N and Yoo H J 2013 Journal of Statistical Physics 150 299–319 (Preprint 1209.1419)
[23] Pellegrini C 2014 Journal of Statistical Physics 154 838–865
[24] Bringuier H 2017 Annales Henri Poincaré 18 3167–3192
[25] Sadowski P and Pawela L 2016 Quantum Information Processing 15 2725–2743
[26] Sinayskiy I and Petruccione F 2012 Physica Scripta 2012 014077
[27] Carbone R and Pautrat Y 2016 Annales Henri Poincaré 17 99–135
[28] Carbone R and Pautrat Y 2015 Journal of Statistical Physics 160 1125–1153
[29] Lardizabal C F and Souza R R 2016 Journal of Statistical Physics 164 1122–1156
[30] Bauer M, Bernard D and Tilloy A 2013 Phys. Rev. A 88(6) 062340
[31] Bauer M, Bernard D and Tilloy A Journal of Statistical Mechanics: Theory and Experiment 2014 P09001
[32] Sinayskiy I and Petruccione F 2019 The European Physical Journal Special Topics 227 1869–1883 ISSN 1951-6401
[33] Hall P and Heyde C Martingale limit theory and its application Probability and mathematical statistics ISBN 9780123193506
[34] Williams D Probability with Martingales Cambridge mathematical textbooks ISBN 9780521406055
[35] Stinespring W F 1955 Proceedings of the American Mathematical Society 6 211–216
[36] Sinayskiy I and Petruccione F 2014 International Journal of Quantum Information 12 1461010
[37] Sinayskiy I and Petruccione F 2013 Open Syst Inf Dyn 20 1340007
[38] Breuer H and Petruccione F The Theory of Open Quantum Systems ISBN 9780199213900
[39] Gorini V, Kossakowski A and Sudarshan E C G 1976 Journal of Mathematical Physics 17 821–825
[40] Lindblad G 1976 Communications in Mathematical Physics 48 119–130