The rigidity of embedded constant mean curvature surfaces

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Abstract

We study the rigidity of complete, embedded constant mean curvature surfaces in \( \mathbb{R}^3 \). Among other things, we prove that when such a surface has finite genus, then intrinsic isometries of the surface extend to isometries of \( \mathbb{R}^3 \) or its isometry group contains an index two subgroup of isometries that extend.

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1 Introduction.

In this paper we discuss some global results for certain complete embedded surfaces \( M \) in \( \mathbb{R}^3 \) which have constant mean curvature \( H \). If this mean curvature is zero, we call \( M \) a minimal surface and if it is nonzero, we call \( M \) a CMC surface. Our main theorems deal with the rigidity of complete, embedded constant mean curvature surfaces in \( \mathbb{R}^3 \) with finite genus.

Recall that an isometric immersion \( f : \Sigma \to \mathbb{R}^3 \) of a Riemannian surface \( \Sigma \) is congruent to another isometric immersion \( h : \Sigma \to \mathbb{R}^3 \), if there exists an isometry \( I : \mathbb{R}^3 \to \mathbb{R}^3 \) such that \( f = I \circ h \). We say that an isometric immersion \( f : \Sigma \to \mathbb{R}^3 \) with constant mean curvature \( H \) is rigid, if whenever \( h : \Sigma \to \mathbb{R}^3 \) is another isometric immersion with constant mean curvature \( H \) or \(-H\), then \( f \) is congruent to \( h \).

In general, if \( f : M \to \mathbb{R}^3 \) is an isometric immersion of a simply-connected surface with constant mean curvature \( H \) and \( f(M) \) is not contained in a round sphere or a plane, then there exists a smooth one-parameter deformation of the immersion \( f \) through non-congruent isometric immersions with mean curvature \( H \); this family contains all noncongruent isometric immersions of \( M \) into \( \mathbb{R}^3 \) with constant mean curvature \( H \) or \(-H\). Thus, the rigidity of simply-connected, constant mean curvature immersed surfaces fails in a rather natural way. On the other hand, the main theorems

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†We require that \( M \) is equipped with a Riemannian metric and that the inclusion map \( i : M \to \mathbb{R}^3 \) preserves this metric.
presented in this paper affirm that some complete, nonsimply-connected, embedded constant mean curvature surfaces in $\mathbb{R}^3$ are rigid. More precisely, we prove the following theorems.

**Theorem 1.1 (Finite Genus Rigidity Theorem)** Suppose $M \subset \mathbb{R}^3$ is a complete, embedded, constant mean curvature surface of finite genus.

1. If $M$ is a minimal surface which is not the helicoid, then $M$ is rigid.
2. If $M$ is a CMC surface with bounded Gaussian curvature, then $M$ is rigid.

**Theorem 1.2 (Finite Genus Isometry Extension Theorem)** Let $M \subset \mathbb{R}^3$ be a complete, embedded, constant mean curvature surface of finite genus and let $\sigma : M \rightarrow M$ be an isometry.

1. If $M$ has bounded Gaussian curvature or if $M$ is minimal, then $\sigma$ extends to an isometry of $\mathbb{R}^3$.
2. If $\sigma$ fails to extend to an isometry of $\mathbb{R}^3$, then the isometry group of $M$ contains a subgroup of index two, consisting of those isometries which do extend to $\mathbb{R}^3$. In particular, if $\sigma$ fails to extend, then $\sigma^2$ extends.

The first relevant result in the direction of revealing the rigidity of certain constant mean curvature surfaces is a theorem of Choi, Meeks, and White. In [3] they proved that any properly embedded minimal surface in $\mathbb{R}^3$ with more than one end admits a unique isometric minimal immersion into $\mathbb{R}^3$. One of the outstanding conjectures in this subject states that, except for the helicoid, the inclusion map of a complete, embedded constant mean curvature surface $M$ into $\mathbb{R}^3$ is the unique such isometric immersion with the same constant mean curvature up to congruence. Since extrinsic isometries of the helicoid extend to ambient isometries, the validity of this conjecture implies the closely related conjecture that the intrinsic isometry group of any complete, embedded constant mean curvature surface in $\mathbb{R}^3$ is equal to its ambient symmetry group. These two rigidity conjectures were made by Meeks; see Conjecture 15.12 in [9] and the related earlier Conjecture 22 in [7] for properly embedded minimal surfaces. The theorems presented in this paper demonstrate the validity of these rigidity conjectures under some additional hypotheses.

The proofs of Theorems 1.1 and 1.2 rely on the classification of isometric immersions of simply-connected constant mean curvature surfaces in $\mathbb{R}^3$. A key ingredient in the proof of these theorems is our Dynamics Theorem for CMC surfaces in $\mathbb{R}^3$ together with the Minimal Element Theorem which are proven in [14]. Among other things, in [14] we prove that, under certain hypotheses, a CMC surface in $\mathbb{R}^3$ contains an embedded Delaunay surface at infinity. The fact that embedded Delaunay surfaces are rigid is applied in the proofs of our main theorems.

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2 The Delaunay surfaces are CMC surfaces of revolution which were discovered and classified by Delaunay [4] in 1841. When these surfaces are embedded, they are called *unduloids* and when they are nonembedded, they are called *nodoids*.  

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Additionally, using techniques similar to those applied to prove Theorems 1.1 and 1.2, we also demonstrate the following related rigidity theorem. This theorem is well known in the special case that the surface has finite topology (see for example, [5]).

**Theorem 1.3** Suppose that $M \subset \mathbb{R}^3$ is a complete embedded CMC surface with bounded Gaussian curvature. If

$$\lim\inf \left( \frac{\text{Area}[M \cap B(R)]}{R^2} \right) = 0 \quad \text{or} \quad \lim\inf \left( \frac{\text{Genus}[M \cap B(R)]}{R^2} \right) = 0,$$

then $M$ is rigid.

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## 2 Background on the Dynamics Theorem.

Before stating results which we need from [14] we introduce some definitions; see also [11] for some related results for minimal surfaces.

**Definition 2.1** Suppose $W$ is a complete flat three-manifold with boundary $\partial W = \Sigma$ together with an isometric immersion $f: W \to \mathbb{R}^3$ such that $f$ restricted to the interior of $W$ is injective. We call the image surface $f(\Sigma)$ a strongly Alexandrov embedded CMC surface if $f(\Sigma)$ is a CMC surface and $W$ lies on the mean convex side of $\Sigma$.

We note that, by elementary separation properties, any properly embedded CMC surface in $\mathbb{R}^3$ is always strongly Alexandrov embedded. Furthermore, by item 1 of Theorem 2.3 below, any strongly Alexandrov embedded CMC surface in $\mathbb{R}^3$ with bounded Gaussian curvature is properly immersed in $\mathbb{R}^3$. We remind the reader that the Gauss equation implies that a surface $M$ in $\mathbb{R}^3$ with constant mean curvature has bounded Gaussian curvature if and only if its principal curvatures are bounded in absolute value; thus, $M$ having bounded Gaussian curvature is equivalent to $M$ having bounded second fundamental form.

**Definition 2.2** Suppose $M \subset \mathbb{R}^3$ is a connected, strongly Alexandrov embedded CMC surface with bounded Gaussian curvature.

1. $T(M)$ is the set of all connected, strongly Alexandrov embedded CMC surfaces $\Sigma \subset \mathbb{R}^3$, which are obtained in the following way. There exists a sequence of points $p_n \in M$, $\lim_{n \to \infty} |p_n| = \infty$, such that the translated surfaces $M - p_n$ converge $C^2$ on compact subsets of $\mathbb{R}^3$ to a strongly Alexandrov embedded CMC surface $\Sigma'$, and $\Sigma$ is a connected component of $\Sigma'$ passing through the origin. Actually we consider the immersed surfaces in $T(M)$ to be pointed in the sense that if such a surface is not embedded at the origin, then we consider the surface to represent two different surfaces in $T(M)$ depending on a choice of one of the two preimages of the origin.
2. $\Delta \subset T(M)$ is called $T$-invariant, if $\Sigma \in \Delta$ implies $T(\Sigma) \subset \Delta$.

3. A nonempty subset $\Delta \subset T(M)$ is called a minimal $T$-invariant set, if it is $T$-invariant and contains no smaller nonempty $T$-invariant subsets.

4. If $\Sigma \in T(M)$ and $\Sigma$ lies in a minimal $T$-invariant subset of $T(M)$, then $\Sigma$ is called a minimal element of $T(M)$.

The following theorem is a collection of results and special cases of statements taken from the CMC Dynamics Theorem, the Minimal Element Theorem and Theorem 4.1 in [14]; in the statement of this theorem, $B(R)$ denotes the open ball of radius $R$ centered at the origin in $\mathbb{R}^3$.

**Theorem 2.3** Let $M$ be a connected, noncompact, strongly Alexandrov embedded CMC surface with bounded Gaussian curvature. Then:

1. $M$ is properly immersed in $\mathbb{R}^3$.

2. $T(M)$ is nonempty and $T$-invariant.

3. Every nonempty $T$-invariant subset of $T(M)$ contains a nonempty minimal $T$-invariant subset. In particular, since $T(M)$ is itself a nonempty $T$-invariant set, $T(M)$ always contains minimal elements.

4. Suppose
   $$\liminf \left( \frac{\text{Area}[M \cap B(R)]}{R^2} \right) = 0 \quad \text{or} \quad \liminf \left( \frac{\text{Genus}[M \cap B(R)]}{R^2} \right) = 0.$$

Then $T(M)$ contains a minimal element $\Sigma$ which is an embedded Delaunay surface.

5. Suppose $M$ has a plane of Alexandrov symmetry and more than one end. Then $T(M)$ contains a minimal element $\Sigma$ which is an embedded Delaunay surface.

### 3 Background on Calabi’s and Lawson’s Rigidity Theorems.

In this section we review the classical rigidity theorems of Calabi and Lawson for simply-connected constant mean curvature surfaces in $\mathbb{R}^3$ (see also [1]). The rigidity theorem of Lawson is motivated by the earlier result of Calabi [2] who classified the set of isometric minimal immersions of a simply-connected Riemannian surface $\Sigma$ into $\mathbb{R}^3$; we now describe Calabi’s classification theorem.

Suppose $f: \Sigma \to \mathbb{R}^3$ is a isometric minimal immersion and $\Sigma$ is simply-connected. In this case the coordinate functions $f_1, f_2, f_3$ are harmonic functions which are the real parts of corresponding holomorphic functions $h_1, h_2, h_3$ defined on $\Sigma$. For any
\( \theta \in [0, 2\pi) \), the map \( f_\theta = \text{Re}(e^{i\theta}(h_1, h_2, h_3)): \Sigma \to \mathbb{R}^3 \) is an isometric minimal immersion of \( \Sigma \) into \( \mathbb{R}^3 \); the immersions \( f_\theta \) are called \textit{associate immersions} to \( f \). Many classical examples of minimal surfaces arise from this associate family construction. For example, simply-connected regions on a catenoid are the images of regions in the helicoid under the associate map for \( \theta = \frac{\pi}{2} \); in this case the corresponding coordinate functions on these domains are conjugate harmonic functions and consequently, the catenoid and the helicoid are called \textit{conjugate} minimal surfaces.

Calabi’s classification or rigidity theorem states that if \( \Sigma \) is not flat, then for any isometric minimal immersion \( F: \Sigma \to \mathbb{R}^3 \), there exists a unique \( \theta \in (0, \pi) \) such that \( F \) is congruent to \( f_\theta \), i.e. there exists an isometry \( I: \mathbb{R}^3 \to \mathbb{R}^3 \) such that as mappings, \( F = I \circ f_\theta \). This notion of rigidity does not mean that the image surface \( f(\Sigma) \) cannot be congruent to the image of an associate surface \( f_\theta(\Sigma) \), where \( \theta \in (0, \pi) \). For example, let \( f: \mathbb{C} \to \mathbb{R}^3 \) be a parametrization of the classical Enneper surface and let \( f_{\pi/2} \) be the conjugate mapping. Then the images of these immersions are congruent as subsets of \( \mathbb{R}^3 \) but these immersions are not congruent as mappings. In fact, the rotation \( R_{\pi/2} \) counter clockwise by \( \frac{\pi}{2} \) in the usual parameter coordinates \( \mathbb{C} \) for Enneper’s surface is an intrinsic isometry of this surface which does not extend to an isometry of \( \mathbb{R}^3 \) and \( f \circ R_{\pi/2} \) is congruent to the immersion \( f_{\pi/2} \).

Calabi condition for rigidity of \( \Sigma \) is equivalent to the property that the conjugate harmonic coordinate functions are well defined, which by Cauchy’s theorem is equivalent to the property that the flux vector\(^3\) of any simple closed curve on \( \Sigma \) is zero.

Lawson’s Rigidity Theorem that we referred to in the previous paragraph appears in Theorem 8 in [6] and holds in space forms other than \( \mathbb{R}^3 \) as well. We will not need his theorem in its full generality and we state below the special case which we will apply in the next section.

**Theorem 3.1 (Lawson’s Rigidity Theorem for \( \text{CMC} \) surfaces in \( \mathbb{R}^3 \))** If \( f: \Sigma \to \mathbb{R}^3 \) is an isometric \( \text{CMC} \) immersion with mean curvature \( H \) and \( \Sigma \) is simply-connected, then there exists a differentiable \( 2\pi \)-periodic family of isometric immersions

\[
f_\theta: \Sigma \to \mathbb{R}^3
\]

of constant mean curvature \( H \) called associate immersion to \( f \). Moreover, up to congruences the maps \( f_\theta \), for \( \theta \in [0, 2\pi) \), represent all isometric immersions of \( \Sigma \) into \( \mathbb{R}^3 \) with constant mean curvature \( H \) or \(-H \) and these immersions are non-congruent to each other if \( f(M) \) is not contained in a sphere.

Note that if \( A \) represents the second fundamental form of \( f \) and \( A_\theta \) the one of \( f_\theta \), these forms are related by the following equation, see [16]:

\[
A_\theta = \cos(\theta)(A - HI) + \sin(\theta)J(A - HI) + HI, \tag{1}
\]

where \( I \) is the identity matrix and \( J \) is the almost complex structure on \( M \).

\(^3\)The flux of an oriented unit speed curve \( \gamma \subset \Sigma \) is the integral of \( J(\gamma') \) along \( \gamma \), where \( J \) is the complex structure.
4 Proofs of the main theorems.

Our rigidity theorems are motivated by several classical results on the rigidity of certain complete embedded minimal surfaces. The first relevant result in this direction is a theorem of Choi, Meeks, and White [3] who proved that any properly embedded minimal surface in $\mathbb{R}^3$ with more than one end admits a unique isometric minimal immersion into $\mathbb{R}^3$; their result proved a special case of the conjecture of Meeks [7] that the inclusion map of a properly embedded, nonsimply-connected minimal surfaces in $\mathbb{R}^3$ is the unique minimal immersion of the surface into $\mathbb{R}^3$ up to congruence.

We now prove the following important special case of Theorem 1.1 regarding properly embedded minimal surfaces.

**Theorem 4.1** If $M$ is a connected, properly embedded minimal surface in $\mathbb{R}^3$ with finite genus which is not a helicoid, then $M$ is rigid.

**Proof.** If $M$ has more than one end, then the result of Choi, Meeks, and White implies that $M$ satisfies the conclusions of the theorem.

Suppose $M$ has finite genus, one end and $M$ is not a helicoid. The main theorem of Meeks and Rosenberg in [12] then implies that $M$ is a plane or is asymptotic to a helicoid $\Sigma$. In the latter case, $M$ can be thought of as being a helicoid with a finite positive number of handles attached. Since Theorem 4.1 holds for planes, assume now that $M$ is not a plane. Since $M$ is asymptotic to a helicoid $\Sigma$, any plane $P$ orthogonal to the axis of $\Sigma$ intersects $M$ in an analytic set with each component of $M \cap P$ having dimension one and such that outside of some ball in $\mathbb{R}^3$, $M \cap P$ consists of two proper arcs asymptotic to the line $\Sigma \cap P$. Since $M$ has finite positive genus, elementary Morse theory implies that for a certain choice of $P$, $M \cap P$ is a one-dimensional analytic set with a vertex contained in the intersection of two open analytic arcs in $M \cap P$. An elementary combinatorial argument implies that $P \cap M$ contains a simple closed oriented curve $\Gamma$ bounding a compact disk whose interior is disjoint from $M$. It follows that the integral of the conormal to $\Gamma$ has a nonzero dot product with the normal to $P$. The existence of $\Gamma$ implies that for at least one of the coordinate functions $x_i$ of $M$, the conjugate harmonic function of $x_i$ is not well-defined (for example, if $\Gamma$ lies in the $(x_1, x_2)$-plane, then the conjugate harmonic function of the $x_3$-coordinate function is not well-defined on $M$). From our discussion of the Calabi Rigidity Theorem in the previous section, it follows that the inclusion map of $M$ into $\mathbb{R}^3$ is the unique isometric minimal immersion of $M$ into $\mathbb{R}^3$ up to congruence. This completes the proof of the theorem. \(\square\)

We will now apply the above theorem and the results described in Sections 2 and 3 to prove Theorems 1.1, 1.2 and 1.3.

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4This asymptotic to a helicoid property of $M$ follows directly from the proof of the uniqueness of the plane and the helicoid as being the only properly embedded simply-connected minimal surfaces. The proof of this asymptotic argument is explained briefly in the last section of [12]. In the last section of their paper, Meeks and Rosenberg inadvertently describe a stronger analytic statement which is false. In any case, a simplification of the asymptotic to a helicoid property of $M$ is also proved in [3].
In what follows, let $M \subset \mathbb{R}^3$ be a complete, embedded, constant mean curvature surface of finite genus.

**Proof of Theorem 1.1.** We first prove item 1. Suppose $M$ is a minimal surface which is not a helicoid. If $M$ has positive injectivity radius, then $M$ is properly embedded by the Minimal Lamination Closure Theorem in [13] and the result is a consequence of Theorem 1.1. If $M$ fails to have positive injectivity radius, then the local picture theorem on the scale of the topology in [10] implies that there exists a sequence of compact domains $\Delta_n \subset M$ such that, after scaling and a rigid motion of $\mathbb{R}^3$, the new domains $\tilde{\Delta}_n$ converge smoothly with multiplicity one to a properly embedded genus zero minimal surface $M_{\infty}$ in $\mathbb{R}^3$ with injectivity radius one (and hence not simply-connected) or there exist closed geodesics $\gamma_n \subset \tilde{\Delta}_n$ with nontrivial flux (the integral of the conormal is nonzero). The latter happens when the $\tilde{\Delta}_n$ converge smoothly away from two vertical lines $L_1$ and $L_2$ to a foliation $\mathcal{F}$ by horizontal planes of $\mathbb{R}^3$. These lines are the singular sets of convergence to $\mathcal{F}$. Such a picture is called a parking garage structure on $\mathbb{R}^3$ and the geodesics $\gamma_n$ correspond to “connection” curves between the “columns” $L_1$ and $L_2$ (see [10]). On the one hand, if $M_{\infty}$ is a properly embedded genus zero minimal surface with injectivity radius one, then by Theorem 4.1, $M_{\infty}$ is rigid and a compactness argument implies that $M$ is also rigid (see the proof of Theorem 1.3 for this type of argument). On the other hand, minimal surfaces with nontrivial flux are rigid, by our discussion in Section 3. This completes the proof of item 1.

Since finite genus implies that $\lim \inf (\frac{\text{Genus}(M \cap B(R))}{R^2}) = 0$, item 2 is a consequence of Theorem 1.3, which is proved below. This completes the proof of Theorem 1.1. \qed

**Proof of Theorem 1.2.** Since rigidity implies that intrinsic isometries extend to extrinsic isometries, item 1 is a simple consequence of Theorem 1.1. We now prove item 2. Suppose $M$ is a CMC surface and does not have bounded Gaussian curvature. Let $i: M \rightarrow \mathbb{R}^3$ be the inclusion map. If $\sigma$ is an isometry of $M$, then $i \circ \sigma$ is congruent to a unique associate surface $i_{\theta(\sigma)}$, $\theta(\sigma) \in [0, 2\pi)$. By defining $F(\sigma) = \theta(\sigma)$, we obtain a homomorphism $F: \text{Isom}(M) \rightarrow S^1$. We are going to show that $F: \text{Isom}(M) \subset \{0, \pi\} = \mathbb{Z}_2 \subset S^1$ which will imply item 2 of the theorem.

Assume that $\sigma: M \rightarrow M$ is an isometry that does not extend to $\mathbb{R}^3$. By Theorem 3.1 and the previous discussion we know that $i \circ \sigma: M \rightarrow \mathbb{R}^3$ must be congruent to an associate immersion $i_{\theta}: M \rightarrow \mathbb{R}^3$, where $\theta \in (0, 2\pi)$.

The local picture theorem on the scale of curvature in [11] states that there exists a sequence of points $p_n \in M$ and positive numbers $\varepsilon_n, l_n$ such that:

1. $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, $\lim_{n \rightarrow \infty} l_n = \infty$ and $\lim_{n \rightarrow \infty} l_n \varepsilon_n = \infty$.

2. The component $M_n$ of $M \cap B(p_n, \varepsilon_n)$ that contains $p_n$ is compact with $\partial M_n \subset \partial B(p_n, \varepsilon_n)$.

3. The second fundamental forms of the surfaces $\tilde{M}_n = l_n M_n \subset l_n B(p_n, \varepsilon_n) \subset \mathbb{R}^3$ are uniformly bounded and are equal to one at the related points $\tilde{p}_n$. 

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4. The translated surfaces $\tilde{M} - p_n$ converge with multiplicity one to a connected, properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ with bounded curvature and genus zero.

Suppose first that $M_\infty$ is a helicoid. We will use the embeddedness property of $M$ to show that $\theta = \pi$. If $M_\infty$ is a helicoid, then the associate surfaces $(\tilde{M} - p_n)_{\theta}$ can be chosen to approximate a large region of the related associate surface $(M_\infty)_{\theta}$ to the helicoid $M_\infty$; this can be seen by letting $H$ go to zero in equation (1). If $\theta \neq \pi$, then $(M_\infty)_{\theta}$ intersects itself transversely which means that $(\tilde{M} - p_n)_{\theta}$ is not embedded. This contradiction concludes the argument.

It remains to show the case when $M_\infty$ is not the helicoid. If $M_\infty$ is not a helicoid, then Theorem 3.1 implies $(M_\infty)_{\theta}$ is not well-defined unless $\theta = \pi$.

**Proof of Theorem 1.3.** Suppose $M$ is a CMC surface with bounded Gaussian curvature such that either $\lim \inf(\frac{\text{Area}(M \cap B(R))}{R^2}) = 0$ or $\lim \inf(\frac{\text{Genus}(M \cap B(R))}{R^2}) = 0$. Without loss of generality we will assume $H_M = 1$. By item 3 of Theorem 2.3, $T(M)$ contains an embedded Delaunay surface $\Sigma$. More precisely, for $n \in \mathbb{N}$, there exist compact annular domains $\Delta_n \subset M$ and points $p_n \in \Delta_n$ such that the translated surfaces $\Delta_n - p_n$ converge $C^2$ to $\Sigma$ on compact subsets of $\mathbb{R}^3$. For concreteness, suppose $g_1$ denotes the inclusion map of $\Sigma$ into $\mathbb{R}^3$. First we show that the immersion $g_1$ is rigid.

Let $\pi: \tilde{\Sigma} \to \Sigma$ denote the universal covering of $\Sigma$. Consider $\tilde{\Sigma}$ with the induced Riemannian metric and let $f = g_1 \circ \pi: \tilde{\Sigma} \to \mathbb{R}^3$ be the related isometric immersion. Let $f_{\theta}: \tilde{\Sigma} \to \mathbb{R}^3$ be the associate immersion for angle $\theta \in [0, 2\pi)$, given in Theorem 3.1; note $f_0 = f$. Suppose $g_2$ is another isometric immersion of $\Sigma$ into $\mathbb{R}^3$ which is not congruent to $g_1$. This implies that $g_2 \circ \pi$ is congruent to $f_{\overline{\theta}}$ for a certain $\overline{\theta} \in (0, 2\pi)$.

Let $\tilde{\gamma} \subset \tilde{\Sigma}$ be a lift of the shortest geodesic circle $\gamma \subset \Sigma$. We will prove that for any $\theta \in (0, 2\pi)$ the endpoints of $f_{\theta}(\tilde{\gamma})$ are distinct, which means that the associate immersions to $g_1$ do not exist. We will accomplish this by describing the geometry of $f_{\theta}(\tilde{\gamma})$.

A computation shows that for the geodesic $\tilde{\gamma}$ and the immersion $f_{\theta}$, the curvature $k_{\theta}$ and torsion $\tau_{\theta}$ of $f_{\theta}(\tilde{\gamma})(t)$ are given by

\[
k_{\theta} = \langle A_{\theta}(\tilde{\gamma}'(t)), \tilde{\gamma}'(t) \rangle \quad \text{and} \quad \tau_{\theta} = -\langle A_{\theta}(\tilde{\gamma}'(t)), J(\tilde{\gamma}'(t)) \rangle.
\]

Furthermore, since $\tilde{\gamma}$ is a lift of the shortest geodesic circle in $\Sigma$, there exist $s \leq 0 < 2 \leq r$, $r + s = 2$ such that in the $\tilde{\gamma}'$, $J\tilde{\gamma}'$ basis, the second fundamental form along $\tilde{\gamma}$ is expressed as the matrix

\[
A = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}.
\]

Consequently, equations (1) and (2) imply that

\[
k_{\theta} = \cos(\theta)(r - 1) + 1 \quad \text{and} \quad \tau_{\theta} = \sin(\theta)(1 - r).
\]

5The curve $\tilde{\gamma}$ is a compact embedded arc in $\tilde{\Sigma}$ which is the image of a lift of map $\gamma: [0, 1] \to \Sigma$. 


In particular, \( k_\theta \) and \( \tau_\theta \) are constants. If \( \theta \neq \pi \), then \( f_\theta(\tilde{\gamma}) \) is contained in a helix, while if \( \theta = \pi \), it is contained in circle of radius \( R = |2 - r| = |s| < |r| \). Since the length of \( f_\pi(\tilde{\gamma}) \) is \( \frac{2\pi}{r} \), it follows that in either case the endpoints of \( f_\theta(\tilde{\gamma}) \) are distinct.

Since the compact annuli \( \Delta_n - p_n \) converge \( C^2 \) to the embedded Delaunay surface \( \Sigma \) as \( n \to \infty \), we conclude that the associate immersions for \( \theta \in (0, 2\pi) \) are not well-defined on \( \Delta_n - p_n \) for \( n \) large. By Theorem 3.1, Theorem 1.1 now follows. \( \square \)

**Remark 4.2** In the above proof, we showed that any embedded Delaunay surface is rigid. However, the same computations prove that if \( f \) represents the inclusion map of a nodoid into \( \mathbb{R}^3 \), then the associate immersions \( f_\theta, \theta \in (0, \pi) \) are never well-defined and the associate immersions \( f_\pi \) are well-defined for an infinite countable set of nodoids.

**Remark 4.3** The conclusions of the Finite Genus Rigidity Theorem (Theorem 1.1) should hold without the hypothesis that \( M \) have bounded Gaussian curvature. This improvement would be based on techniques we are developing in [15] to prove curvature estimates for certain embedded CMC surfaces in \( \mathbb{R}^3 \).

**Remark 4.4** The full generality of the CMC Dynamics Theorem and the Minimal Element Theorem can be used to prove rigidity of embedded CMC surfaces under hypotheses which imply the existence of an embedded Delaunay surface at “infinity.” For instance, item 4 of Theorem 2.3 implies that if \( M \) has bounded Gaussian curvature, a plane of Alexandrov symmetry and more than one end, then \( M \) is rigid.

**Remark 4.5** It also makes sense to talk about “local rigidity” of a CMC surface. The full generality of the CMC Dynamics Theorem and the Minimal Element Theorem can be used to prove local rigidity of certain embedded CMC surfaces (see [17]).

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