ABOUT THE SIMULTANEOUS CO-EXISTENCE OF INSTANTANEOUS AND RETARDED INTERACTIONS IN CLASSICAL ELECTRODYNAMICS

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In this paper it is proved that, contrary to the results found by A.E. Chubykalo and S.J. Vlaev (Int. J. Mod. Phys. 14, 3789 (1999)), the retarded electric and magnetic fields for an uniformly accelerated charge exactly satisfy Maxwell equations (ME). Furthermore it is shown that ME are correctly written in the usual form with the partial derivatives and thus not, as proposed by Chubykalo and Vlaev, with the total derivatives.

1. Introduction

Recently Chubykalo and Vlaev1 argued that the electromagnetic fields (obtained from the Liénard-Wiechert potentials (LW)) of a charge moving with a constant acceleration along an axis do not satisfy Maxwell equations (ME) if one considers exclusively a retarded interaction. According to their interpretation the retarded interaction is mathematically described by the implicit functional dependence of the electric E and magnetic B fields on the observation time t. Further they stated that ME will be satisfied "if and only if" one takes into account an explicit functional dependence of E and B on the observation time t together with the mentioned implicit dependence. This explicit dependence of fields on t then mathematically describes an instantaneous interaction. The same assertions were also reported in series of papers, e.g., Refs. 2-4.

In the paper5 we have argued that the "electrodynamics dualism concept"4 with the simultaneous coexistence of instantaneous and short range interaction is not correct. In this paper we show that the opinion presented in1–4 about the incorrectness of the retarded solutions emerges, in fact, from an incorrect mathematical procedure in treating the partial derivatives of continuous functions of several variables. Their1–4 physical interpretation with
the "electrodynamics dualism concept" is then a consequence of such an incorrect mathematical treatment. Further it will be proved that, contrary to the assertions from,\textsuperscript{1–4} the ME
\begin{align*}
\nabla E &= \rho/\epsilon_0 \\
\nabla B &= 0 \\
\nabla \times B &= \mu_0 j + \mu_0 \varepsilon_0 \partial E/\partial t \\
\nabla \times E &= -\partial B/\partial t
\end{align*} (1)
are correctly written in the usual form, Eqs. (1), i.e., with the partial derivatives and not with the total derivatives, and that the retarded fields of a charge \( q \) moving with a constant acceleration along the \( X \) axis, Eqs. (22)-(27) from Ref. 1, exactly satisfy ME (1).

2. Derivation of the Fields \( E \) and \( B \) Taking into Account the Retarded Interaction Only

First it will be discussed the calculation of the LW fields \( E \) and \( B \) from a charge \( q \) for which the retarded position \( x_{0i}(t_0) \) as a function of the retarded time \( t_0 \) is assumed given. These fields are
\begin{align*}
E &= Kq \left\{ (R - VR/c)(1 - V^2/c^2)/s^3 \right\}_{t_0} + \\
&\quad + Kq \left\{ [R \times ((R - RV/c) \times \dot{V})]/c^2 s^3 \right\}_{t_0}, \\
B &= \left\{ [R \times E]/Rc \right\}_{t_0},
\end{align*} (2)
where \( K = 1/4\pi \varepsilon_0 \) and \( s = R - VR/c \). (The notation is similar to that one in Ref. 1; \( \{ \ldots \} \) means that all functions of \( x, y, z, t \) in the braces \( \{ \} \) are taken at the moment of time \( t_0 \), that is determined from the retardation condition (6), see below.) \( E \) and \( B \) are determined in the usual way
\begin{align*}
E &= -\nabla \varphi - \partial A/\partial t, \quad B = \nabla \times A
\end{align*} (4)
from the LW potentials
\begin{align*}
\varphi(r,t) &= \{ Kq/s \}_{t_0}, \quad A(r,t) = \{ KqV/s \}_{t_0}.
\end{align*} (5)
We note that the field and source point variables are connected by the retardation condition
\begin{equation}
R(x_i, x_{0i}) = \left[ \sum_i (x_i - x_{0i})^2 \right]^{1/2} = c(t - t_0). \quad (6)
\end{equation}
This equation is Eq. (8) from Ref. 1, see also Ref. 7 Sec. 20-1, or Ref. 6 Secs. 62 and 63. Since $x_0$ is assumed given as a function of $t_0$ we can write

$$R(x_i, x_0(t_0)) = f(x, y, z, t_0) = c(t - t_0),$$

(7)

which is a functional relation between $x_i, t$ and $t_0$, see also Eq. (20-4). The LW potentials are expressed in terms of $R$ (6), which means that $\varphi(r, t)$ and $A(r, t)$ are not only implicit functions of $x, y, z$ through $t_0$ but the explicit functions as well. Although Landau and Lifshitz p. 161 stated that the LW potentials (5) are only implicit functions of $x, y, z$ through $t_0$ they really calculated the fields $E$ and $B$ taking into account that $R$, (6) or (7), is also the explicit function of the field coordinates $x, y, z$ in the same way as in Ref. 7. This can be clearly seen from the calculation of $\nabla t_0$ when $\nabla R$ is written as

$$\nabla R = -c\nabla t_0 = \nabla_1 R + (\partial R/\partial t_0)\nabla t_0,$$

(8)

which is Eq. (20-9) from Ref. 7 and the same is written in the unnumbered equation preceding the numbered equation (63.7) in Ref. 6. Then the equations which constitute the transformation of the differential operators from the coordinates of the field point to those of the radiator are given as

$$\partial/\partial t = (R/s)\partial/\partial t_0, \quad \nabla = \nabla_1 + (\nabla t_0)\partial/\partial t_0,$$

(9)

Eqs. (20-8) and (20-11) in Ref. 7. The differential operator $\nabla_1$ refers to the differentiation with respect to, e.g., the first argument of the function $f$ in Eq. (7), that is, differentiation at constant retarded time $t_0$. Expressing this statement in another way we can say that $\nabla_1$ denotes the differentiation with respect to the arguments $x, y, z$ of the function $f$, i.e., $R$, (6) or (7), (and thus of the potentials $\varphi$ and $A$ and the fields $E$ and $B$ as well), on which $f$ (consequently $\varphi, A, E$ and $B$) explicitly depend (not only implicitly through $t_0$). This consideration reveals that the relations (11) from Ref. 1 have to be correctly written as

$$\partial \varphi/\partial x_i = (\partial \varphi/\partial x_i) + (\partial \varphi/\partial t_0)(\partial t_0/\partial x_i),$$

$$\partial A/\partial t = (\partial A/\partial t_0)(\partial t_0/\partial t),$$

$$\partial A_k/\partial x_i = (\partial A_k/\partial x_i) + (\partial A_k/\partial t_0)(\partial t_0/\partial x_i),$$

(10)

where $\partial/\partial x_i$ denotes, as above $\nabla_1$, the differentiation with respect to the argument $x_i$ (i.e., $x$ or $y$ or $z$) on which $\varphi$ or $A_k$ explicitly depend. The
partial derivative, e.g., $\partial/\partial x_i$ on the l.h.s. of (10) can be called - the complete partial derivative - which takes into account both the explicit and the implicit dependence of the functions $\varphi$ or $A_k$ on the argument $x_i$. The partial derivative $\partial/\partial x_i$ in the first and the third equation on the r.h.s. of (10) takes into account only the explicit dependence of the functions $\varphi$ or $A_k$ on the argument $x_i$. For the sake of brevity we shall, from now on, call such type of the partial derivative - the explicit partial derivative.

When such procedure is applied to the calculation of $E$ and $B$ by Eq. (4) from the LW potentials $\varphi$ and $A_k$ then one finds the exact expressions for the usual LW fields (2) and (3) without need for addition of any new terms. We note that in Ref. 1 Eqs. (11) the partial derivatives which refer to the explicit dependence of $\varphi$ and $A$ on $x_i$, i.e., $(\partial \varphi/\partial x_i)$ and $(\partial A_k/\partial x_i)$, were not taken into account. As a consequence the authors of 1 had to add an additional term in their Eq. (21) to find the result (3).

3. The Retarded Electromagnetic Fields of a Charge Moving with a Constant Acceleration Do Satisfy Maxwell Equations

It is argued in Ref.1 that the electric and magnetic fields for a charge $q$ moving with a constant acceleration along the $X$ axis, their Eqs. (22)-(27), do not satisfy ME (1). These fields are the retarded fields (2) and (3) but written in components, and taking into account that the velocity and the acceleration of that charge $q$ have only $x$-components, i.e., $V(V,0,0)$ and $a(a,0,0)$. We shall, for brevity, quote here only $E_x$ and $B_y$ components, Eqs. (22) and (26) from Ref. 1,

\[
E_x(x,y,z,t) = Kq \left\{ \frac{V R/c}{(R - V(x - x_0)/c)^3} \right\}_{t_0} + Kq \left\{ \frac{a ((x - x_0)^2 - R^2)}{c^2 (R - V(x - x_0)/c)^3} \right\}_{t_0},
\]

\[
B_y(x,y,z,t) = -Kq \left\{ \frac{V z (1 - V^2/c^2)}{c^2 (R - V(x - x_0)/c)^3} \right\}_{t_0} - Kq \left\{ \frac{a R z/c^3 (R - V(x - x_0)/c)^3} {t_0} \right\}.\]

To prove the above mentioned assertion the authors of 1 supposed, as in their Sec. 2., that $E$ and $B$ fields, their Eqs. (22)-(27), are functions of $x, y, z, t$ only through $t_0$, and consequently used the differentiation rules as in their Eqs. (11). But from our discussion in Sec. 2., see the relation for $R$ (7), and
from the explicit expressions for \( E \) and \( B \) fields, (11) and (12) or Eqs. (22)-(27) from Ref.1, one unambiguously concludes that \( E \) and \( B \) fields are also the explicit functions of the field point \( x, y, z \). Thence the relations (28) and (30)-(32) from Ref.1 have to be changed in accordance with our discussion given in Sec. 2. Thus, e.g., the second relation in Eq. (28) in Ref. (1) has to be changed and written as

\[
\partial E \{or B\}_k / \partial x_i = (\partial E \{or B\}_k / \partial x_i) + (\partial E \{or B\}_k / \partial t_0)(\partial t_0 / \partial x_i), \quad (13)
\]

where the partial derivative on the l.h.s. of (13) is (in our terminology) - the complete partial derivative - which takes into account both the explicit and the implicit dependence of the functions \( E \{or B\}_k \) on the argument \( x, y, z \), while the partial derivative in the first term on the r.h.s. of (13) is - the explicit partial derivative, which takes into account only the explicit dependence of the functions \( E \{or B\}_k \) on the argument \( x_i \). Of course the relations (30)-(32) from Ref.1 have to be changed in the same way as explained above.

Then one finds that the term \( \nabla_1 \times E \) (due to the explicit dependence of \( E \) on the arguments \( x, y, z \)) exactly cancels the terms on the r.h.s. of Eqs. (35) and (36) from Ref. 1. In such a way we prove, as expected, that the Faraday law is exactly satisfied. It can be easily shown that not only the Faraday law but all ME (1) are exactly satisfied by the LW fields (2) and (3), and thus also by the fields from uniformly accelerated charge, (11) and (12) or Eqs. (22)-(27) from Ref. 1, when all partial derivatives are correctly treated.

At this place it is worth to discuss the results from Sec. 4 in Ref. 1, where the similar mathematically incorrect step has been made. The authors of Ref. 1 argued that in the calculations of \( E \) and \( B \) from (4) and in ME (1) as well all derivatives have to be considered as total ones. To explain the need for such change they calculate \( \partial t_0 / \partial t \) and \( \partial t_0 / \partial x_i \), Eqs. (14) in Ref. 1 using two different expressions for \( R \), our Eq. (3), or Eqs. (37) and (38) from Ref. 1. Then to calculate the above mentioned expressions they claim that the commonly used partial derivatives \( \partial R / \partial t \) and \( \partial R / \partial x_i \) have to be replaced by the total derivative \( dR / dt \) and \( dR / dx_i \), see Eqs. (39) in Ref. 1. Thus they also objected the use of the partial derivatives by Landau and Lifshitz in the calculation of \( \partial R / \partial t \) and \( \partial R / \partial x_i \) considering that Landau and Lifshitz used the symbol \( \partial \): "in order to emphasize that \( R \) depends also on other independent variables \( x, y, z,... \)". The justification for such assertions they find in the following statement: "The point is that if a given function is expressed by two different types of functional dependences, then exclusively
total derivatives of these expressions with respect to a given variable can be equated (contrary to the partial ones).” However this statement is not true. Namely (using our terminology) it is true that one cannot equate - the explicit partial derivatives, but one can equate - the complete partial derivatives of the mentioned expressions. Moreover when one calculates, e.g., $\partial R/\partial x$, then it has to be written as

$$\partial R/\partial x = (\partial R/\partial x_1) + (\partial R/\partial t_0)(\partial t_0/\partial x),$$

(14)

since according to (1) and (2) $R$ depends explicitly not only on $x$ but also on $y$ and $z$ (thence $\partial R/\partial x_1$). Further, we explain what is the most important for the use of the operator $\partial/\partial x$ on the l.h.s. of (14) (also (8), (9), (10) and (13)) and not the total derivative operator $d/dx$. It is the fact that $t_0$ in $R$ depends not only on the $x$ variable but also on other independent variables $y$ and $z$. Of course $\partial R/\partial x$ on the l.h.s. of (14) is (in our terminology) - the complete partial derivative of $R$, while $\partial R/\partial x_1$ (the first term on the r.h.s. of (14)) is - the explicit partial derivative of $R$. To see that this explanation is correct and to understand the difference between the total derivative and the partial derivative of the function (composite) of several variables one can consult some mathematical book, e.g., Ref. 8 Eqs. (8) and (9) in Sec. 153. Also one can see the nice explanation given in Ref. 7 Sec. (20-1). This means that the objections from Ref. 1 to the Landau and Lifshitz derivations and results are unfounded. Hence in Eqs. (39) from Ref. 1 the complete partial derivatives $\partial R/\partial t$ and $\partial R/\partial x_i$ have to be written and not the total derivatives $dR/dt$ and $dR/dx_i$. Obviously the partial derivatives have to be retained in all other expressions and they must not be replaced by the total derivatives. The proposition from Ref. 1 that partial derivatives have to be replaced by the total derivatives has been also extensively used for numerous conclusions in Refs. 2-4, see particularly Eqs. (43)-(46) and the discussion in Ref. 2. We see that such changes of ME are unnecessary and, in fact, incorrect.

4. Conclusion

The consideration presented in this paper unambiguously reveals that Maxwell equations (1) are correctly written in the usual way with the partial derivatives and not with the total derivatives, as argued in Ref. 1. Furthermore, as proved in our Sec. 3., the retarded Liénard-Wiechert fields for an uniformly accelerated charge, (11) and (12) or Eqs. (22)-(27) from Ref. 1, exactly
satisfy the usual form of ME [1]. Hence it is not true that: "there is a simultaneous and independent coexistence of instantaneous and retarded interactions which cannot be reduced to each other." The retarded solution is a full-value solution, i.e., it is a complete solution of ME, which is sufficient to describe the whole electromagnetic phenomenon. In addition, it has to be mentioned (see also Ref. 5) that it is possible to write the complete solution of ME in the present-time formulation, i.e., as an action-at-a-distance formulation, and such a solution is completely equivalent to the retarded (and advanced) solution. This is explicitly shown in the general case in Ref. 9 by means of Lagrange series expansion. We shall present the potentials and fields in the present-time formulation in a closed form for the case of an uniformly accelerated charge, but it will be reported elsewhere.

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