SPECIFICATION PROPERTY FOR STEP SKEW PRODUCTS

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Abstract. Step skew products with interval fibres and a subshift as a base are considered. It is proved that if the fibre maps are continuous, piecewise monotone, expanding and surjective and the subshift has the specification property and a periodic orbit such that the composition of the fibre maps along this orbit is mixing, then the corresponding step skew product has the specification property.

1. Introduction

An interesting topic combining nonautonomous dynamical systems, skew products and random dynamical systems is the following. Let $T_1$ and $T_2$ be two continuous selfmaps of the interval $I = [0, 1]$. If $x_0 \in I$, we decide, by tossing a coin, whether we take $x_1 = T_1(x_0)$ or $x_1 = T_2(x_0)$. After $(n-1)$ steps arriving at $x_{n-1}$ we again decide, tossing a coin, whether we take $x_n = T_1(x_{n-1})$ or $x_n = T_2(x_{n-1})$. We thus obtain a sequence $(x_n)_{n=0}^{\infty}$. It can be viewed as the trajectory of the point $x_0$ in the nonautonomous system given by the sequence of maps determined by the coin tossing, each of the maps being either $T_1$ or $T_2$. Since any choice of $\omega = \omega_0\omega_1\omega_2\cdots \in \Sigma_2^+ = \{1, 2\}^{\mathbb{Z}_+}$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, yields a nonautonomous system given by the sequence of maps $T_{\omega_0}, T_{\omega_1}, T_{\omega_2}, \ldots$, all such nonautonomous systems are in a sense present in the skew product $(\omega, x) \mapsto (S(\omega), T_{\omega_0}(x))$ where $S$ is the shift transformation $\Sigma_2^+ \to \Sigma_2^+$, $(S\omega)_n = \omega_{n+1}$.

As a straightforward generalization, one can consider $\Sigma_n^+ = \{1, 2, \ldots, n\}^{\mathbb{Z}_+}$ and $n$ continuous maps $T_1, T_2, \ldots, T_n$. Also, instead of the full shift $\Sigma_n^+$ one can consider a subshift $B \subseteq \Sigma_n^+$. The present paper deals with the step skew product $F: B \times I \to B \times I$ defined by

$$F(\omega, x) = (S(\omega), T_{\omega_0}(x))$$

where $S$ is the shift transformation $\Sigma_n^+ \to \Sigma_n^+$ and the continuous fibre map $T_\omega$ depends only on the first (i.e. beginning) coordinate $\omega_0$ of $\omega$. Clearly, $F$ is continuous.

The dynamics of step skew products with interval fibres has been studied by many authors, usually under additional assumptions on fibre maps, see e.g. the recent papers [8, 7] and references therein.

In the present paper we study the specification property of step skew products [1]. The specification property was introduced by Bowen [4], a chapter on it can be found in [6]. It is a very strong property; systems with this property have a dense subset of periodic points and are topologically mixing. For continuous maps on the interval, specification is equivalent with topological mixing [3, 5]. The specification property for step skew products with circle rotations in the fibres was studied in [9].

Our main result is Theorem 4.1 and Corollary 4.2. However, also Theorem 3.1 used in the proof of Theorem 4.1 and Example 4.2 are of some interest.

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2. Preliminaries

Let \( \mathbb{N} \) be the set of positive integers. Let \((Y,R)\) be a topological dynamical system, which means that \( R \) is a continuous transformation on the compact metric space \( Y \) with metric \( d \). The dynamical system \((Y,R)\) has the specification property, if for every \( \varepsilon > 0 \) there is an integer \( M = M(\varepsilon) \) with the following property: For any integer \( k \geq 2 \), and for any \((y_1,n_1),(y_2,n_2),\ldots,(y_k,n_k) \in Y \times \mathbb{N} \) there exists a point \( u \in Y \) with \( R^n(u) = u \) such that

\[
d(R^i(y_j), R^{n_j+i}(u)) \leq \varepsilon \quad \text{for} \quad 0 \leq i < n_j \quad \text{and} \quad 1 \leq j \leq k
\]

where \( r_0 = 0 \) and \( r_j = n_1 + n_2 + \ldots + n_j + jM \) for \( 1 \leq j \leq k \). We call \( M \) the gap length for the given \( \varepsilon \). We also say that the map \( R \) itself has the specification property.

Recall that a dynamical system \((X,T)\) or the map \( T \) itself is called (topologically) mixing if for every pair of nonempty open sets \( U \) and \( V \) there is a positive integer \( N \) such that \( T^n(U) \cap V \neq \emptyset \) for all \( n \geq N \). It is called locally eventually onto (or topologically exact) if for every nonempty open set \( U \) there is a positive integer \( n \) such that \( T^n(U) = X \).

By an interval we mean a nondegenerate interval. The length of an interval \( J \) is denoted by \( |J| \). A continuous map \( T: [0,1] \rightarrow [0,1] \) is called piecewise monotone, if there is a finite partition \( \mathcal{P} \) of \([0,1]\) into intervals, such that \( T|_P \) is monotone for all \( P \in \mathcal{P} \). The endpoints of the intervals in \( \mathcal{P} \) are called critical points. We say that \( T \) is expanding, if there is \( \alpha > 0 \) such that \( |T(x) - T(y)| \geq \alpha|x-y| \) holds for all \( x,y \) which are in the same element of \( \mathcal{P} \). In such a case we call \( \alpha \) the expansion rate. Recall also that a continuous piecewise monotone map \( T: [0,1] \rightarrow [0,1] \) is mixing if and only if it is locally eventually onto, see e.g. [2] p. 158.

**Lemma 2.1.** Let \( T \) be a mixing, piecewise monotone, continuous transformation on \([0,1]\). For every \( \gamma > 0 \) there is an integer \( m \) such that \( T^m(U) = [0,1] \) holds for all intervals \( U \) with \( |U| \geq \gamma \).

**Proof.** Let \( \{U_1,\ldots,U_k\} \) be a partition of \([0,1]\) into intervals whose lengths are smaller than \( \gamma/2 \). Since \( T \) is locally eventually onto, for \( i = 1,\ldots,k \) there is a positive integer \( m_i \) with \( f^{m_i}(U_i) = [0,1] \). Put \( m := \max\{m_1,\ldots,m_k\} \) and realize that if \( |U| \geq \gamma \) then the interval \( U \) contains at least one of the intervals \( U_i \). \( \square \)

A sequence \((f_i)_{i=0}^{\infty}\) of maps from \([0,1]\) to \([0,1]\) is called a nonautonomous system. It is called finite, if only finitely many different maps occur in this sequence. For \( i \geq 1 \) and \( j \geq 0 \) we set \( f_j^i = f_{j+i-1} \circ \cdots \circ f_{j+1} \circ f_j \) and \( f_j^0 \) denotes the identity.

Let \( \Sigma^*_n = \{1,2,\ldots,n\}^\mathbb{Z}_+ \) be the full \( n \)-shift with product topology and shift transformation \( S \). Let \( g \) be a metric on \( \Sigma^*_n \) which generates the product topology, such that

\[
g(\omega,\eta) < 1 \quad \text{implies that} \quad \omega \text{ and } \eta \text{ have the same first symbol.} \tag{2}
\]

We define then a metric on \( X = \Sigma^*_n \times [0,1] \) by \( d((\omega,x),(\eta,y)) = \max(g(\omega,\eta),|x-y|) \).

Any string \( u = w_1w_2\ldots w_k \) of elements from \( \{1,2,\ldots,n\} \) is called a block or a word of length \( k \). Concatenation of blocks is indicated by juxtaposition; if \( u = u_1 \cdots u_m \) and

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1. Thus, \( M = M(\varepsilon) \) is such that for every finite family of orbit segments, if all the gap lengths are prescribed to be equal to \( M \), an \( \varepsilon \)-tracing periodic point \( u \) does exist. This is equivalent with the definition, also often used, in which \( M = M(\varepsilon) \) is such that for every finite family of orbit segments, if all the gap lengths are prescribed and greater than or equal to \( M \), an \( \varepsilon \)-tracing periodic point \( u \) still does exist. One implication is trivial. To prove the other one, let a system have the specification property according to the former definition, with \( M = M(\varepsilon) \). To show that it has the specification property according to the latter definition, let \( \varepsilon > 0 \) be given. We claim that the same \( M = M(\varepsilon) \) works. Indeed, let us have a finite family of orbit segments with lengths \( n_i \) and consider any prescribed gap lengths \( M + L_1 \) with \( L_1 \geq 0 \). We replace the orbit segments of points \( y_i \) with lengths \( n_i \) by the orbit segments of the same points \( y_i \) with lengths \( n_i + L_1 \). Then, since \( M \) was taken from the former definition, an \( \varepsilon \)-tracing periodic point \( u \) exists for this new system of (longer) orbit segments and with all gap lengths equal to \( M \). This point \( \varepsilon \)-traces also the original (shorter) orbit segments, with the gap lengths \( M + L_1 \), as required.
Let \( v = v_1 \ldots v_n \) then \( uv = u_1 \ldots u_m v_1 \ldots v_n \). We will also use the notation \( w^n = w \ldots w \) (\( n \) times) and \( w^\infty = www \ldots \). Clearly, a point \( \alpha \in \Sigma^+_n \) is periodic for the shift transformation if and only if \( \alpha = (\alpha_0 \alpha_1 \ldots \alpha_{p-1})^\infty \) for some block \( \alpha_0 \alpha_1 \ldots \alpha_{p-1} \).

If \( B \subseteq \Sigma^+_n \) is nonempty, closed and invariant, i.e. \( S(B) \subseteq B \), then \( B \) with \( S \) restricted to \( B \) is again a dynamical system, called a subshift. If no confusion can arise, the phase space \( B \) itself is called a subshift. If \( w_0 w_1 w_2 \ldots \) is an element of the subshift \( B \), then every word \( w_n w_{n+1} \ldots w_{n+k} \) (\( n, k \geq 0 \)) is called a \( B \)-word.

### 3. Finite nonautonomous systems and non-shrinking of intervals

The following theorem will be used in the proof of our main result.

**Theorem 3.1.** Let finitely many expanding, piecewise monotone, continuous maps \( T_1, T_2, \ldots, T_n \) from \([0,1]\) to \([0,1]\) be given. Then for every \( \varepsilon > 0 \) there is \( \gamma > 0 \) such that for any finite nonautonomous system \( \{ f_i \}_{i=0}^{\infty} \) containing only maps from the given finite family, and for any interval \( U \) with \( |U| \geq \varepsilon \) we have \( \inf_{i \geq 0} |f_i^0(U)| \geq \gamma \).

**Proof.** Fix \( \varepsilon > 0 \). We are going to choose a suitable \( \gamma > 0 \). For \( 1 \leq j \leq n \) let \( \alpha_j > 1 \) be the expansion rate of the expanding map \( T_j \). Set \( \alpha = \min(\alpha_1, \alpha_2, \ldots, \alpha_n) \). We have \( \alpha > 1 \). We fix a positive integer \( m \) such that \( \alpha^m > 2 \).

Set \( \mathcal{F} = \{ T_1, T_2, \ldots, T_n \}^m \). For any \( \Delta = (H_1, H_2, \ldots, H_m) \in \mathcal{F} \) we find a number \( \beta_\Delta > 0 \) such that any interval \( U \subseteq [0,1] \) of length \( \leq \beta_\Delta \), which has a critical point of \( H_1 \) as endpoint, satisfies the condition that

the interval \( G_{j-1}(U) \) has no critical point of \( H_j \) in its interior for \( 1 \leq j \leq m \),

where \( G_0 \) is the identity and \( G_i = H_1 \circ \cdots \circ H_2 \circ H_1 \) for \( 1 \leq i \leq m \). Now we set \( \beta = \min_{\Delta \in \mathcal{F}} \beta_\Delta \). Since \( \mathcal{F} \) is finite we have \( \beta > 0 \). Finally we choose \( \gamma = \min(\varepsilon, \beta) \).

Now let \( \{ f_i \}_{i=0}^{\infty} \) be a finite nonautonomous system consisting of the maps \( T_1, T_2, \ldots, T_n \) and let \( U \) be an interval with \( |U| \geq \varepsilon \). We have to show that \( |f_i^0(U)| \geq \gamma \) for all \( i \geq 0 \).

Let \( j_1 \geq 0 \) be minimal such that the interval \( f_i^{j_1}(U) \) contains a critical point of \( f_{j_1} \) in its interior. Since \( f_i^0(U) \) does not contain a critical point of \( f_i \) in its interior for \( 0 \leq i < j_1 \) and all maps are expanding, we get

\[
|f_i^0(U)| \geq \varepsilon \geq \gamma \quad \text{for} \quad 0 \leq i < j_1.
\]

The interval \( f_i^{j_1}(U) \) has length \( \geq \varepsilon \) and contains a critical point of \( f_{j_1} \) in its interior. Because of \( \gamma \leq \frac{\varepsilon}{2^r} \) we find an interval \( V_1 \subseteq f_i^{j_1}(U) \), which has a critical point of \( f_{j_1} \) as endpoint and satisfies \( |V_1| = \gamma \). Let \( j_2 \geq 0 \) be minimal such that the interval \( f_i^{j_2}(V_1) \) contains a critical point of \( f_{s_2} \) in its interior, where \( s_2 = j_1 + j_2 \). Since \( f_i^{j_1}(V_1) \) does not contain a critical point of \( f_{j_1 + i} \) in its interior for \( 0 \leq i < j_2 \) and all maps are expanding with expansion rate at least \( \alpha \), we get

\[
|f_i^{j_1+i}(U)| \geq |f_i^{j_1}(V_1)| \geq \alpha^i \gamma \geq \gamma \quad \text{for} \quad 0 \leq i \leq j_2.
\]

By the choice of \( \beta \) and because of \( \gamma \leq \beta \) we get \( j_2 \geq m \). This implies \( |f_i^{j_2}(V_1)| \geq \alpha^m \gamma > 2 \gamma \).

Since the interval \( f_i^{j_2}(V_1) \) contains a critical point of \( f_{s_2} \) in its interior, we find an interval \( V_2 \subseteq f_i^{j_2}(V_1) \), which has a critical point of \( f_{s_2} \) as endpoint and satisfies \( |V_2| = \gamma \). Now we can continue as above. Let \( j_3 \geq 0 \) be minimal such that the interval \( f_i^{j_3}(V_2) \) contains a critical point of \( f_{s_3} \) in its interior, where \( s_3 = s_2 + j_3 \). Since \( f_i^{j_2}(V_2) \) does not contain a critical point of \( f_{s_2 + i} \) in its interior for \( 0 \leq i < j_3 \), we get as above

\[
|f_i^{s_2+i}(U)| \geq |f_i^{s_2}(V_2)| \geq \alpha^i \gamma \geq \gamma \quad \text{for} \quad 0 \leq i \leq j_3.
\]

Again we get \( j_3 \geq m \) and \( |f_i^{j_3}(V_2)| \geq \alpha^m \gamma > 2 \gamma \). And we find an interval \( V_3 \subseteq f_i^{j_3}(V_2) \), which has a critical point of \( f_{s_3} \) as endpoint and satisfies \( |V_3| = \gamma \).
Repeating this procedure we finally get \( |f_0^n(U)| \geq \gamma \) for all \( i \geq 0 \). Therefore, the theorem

is proved. \( \square \)

In Theorem 3.1 we assume that the piecewise monotone maps \( T_1, \ldots, T_n \) are expanding, i.e., they have expansion rates \( \alpha_j > 1, j = 1, \ldots, n \). We are interested in whether it works under the weaker assumption that \( |T_j(x) - T_j(y)| \geq |x - y| \) holds for every \( j = 1, \ldots, n \) and for all \( x, y \) which are in the same interval of monotonicity of \( T_j \). The next example shows that this is not the case, even if the maps are surjective and piecewise linear.

Example 3.2 (Theorem 3.1 does not work if \( \alpha_j \geq 1 \) instead of \( \alpha_j > 1, j = 1, \ldots, n \)). We are going to find a finite nonautonomous system \((f_i)_{i=0}^\infty\) on \([0,1]\) made of three piecewise linear, surjective, continuous maps \( \varphi, f \) and \( g \) with slopes (in absolute value) \( \geq 1 \) such that for some nondegenerate interval \( J \), \( \lim_{n \to \infty} |f_0^n(J)| = 0 \).

We will use the following notation for “connect the dots” maps. Let \( 0 = a_0 < a_1 < \cdots < a_n = 1 \) and \( b_i \in I = [0,1], i = 0, 1, \ldots, n \).

Then by \( \langle (a_0, b_0), \ldots, (a_n, b_n) \rangle \) we will denote the map \( I \to I \) which sends \( a_i \) to \( b_i \), \( i = 0, 1, \ldots, n \) and is linear on each of the intervals \([a_i, a_{i+1}], i = 0, 1, \ldots, n - 1\).

To construct the system, fix a small positive irrational number \( \xi \); in fact any irrational \( \xi \) with \( 0 < \xi < 1/4 \) is good for our purposes. Consider the following piecewise linear maps:

\[
\varphi = \langle (0, \xi), (1 - \xi, 1), (1, 0) \rangle,
\]
\[
f = \langle (0, 1), (1 - 2\xi, 0), (1, 2\xi) \rangle.
\]
\[
g = \langle (0, 1), (1/2, 0), (1, 1/2) \rangle.
\]

All the slopes of \( \varphi, f \) and \( g \) are (in absolute value) \( \geq 1 \). Put \( J = [0, \xi] \). The system \((f_i)_{i=0}^\infty\) will have the form

\[
(f_i)_{i=0}^\infty = \underbrace{\varphi, \varphi, \ldots, \varphi}_k, \underbrace{\psi_1, \varphi, \ldots, \varphi}_k, \underbrace{\psi_2, \varphi, \ldots, \varphi}_k, \ldots
\]

where each \( \psi_i \) is either \( f \) or \( g \). We choose

\[ k_1 = \min \{ k : \varphi^k(J) \text{ contains } 1/2 \text{ in its closed middle third or intersects } [1 - \xi, 1] \} . \]

Note that the interval \( \varphi^{k_1}(J) \) is obtained from \( J \) by translation by \( k_1\xi \) and so has length \( \xi \). If \( \varphi^{k_1}(J) \) contains \( 1/2 \) in its closed middle third, we choose \( \psi_1 = g \) and then \( \psi_1(\varphi^{k_1}(J)) \) is an interval whose left endpoint is 0 and has length \( \leq (2/3)\xi = (2/3)|J| \). Otherwise the interval \( \varphi^{k_1}(J) \) intersects \([1 - \xi, 1]\) and thus it is a subset of \([1 - 2\xi, 1]\). We choose \( \psi_1 = f \). Then \( \psi_1(\varphi^{k_1}(J)) \) is a subinterval of \([0, 2\xi]\) and has length \( \xi \); it is obtained from \( \varphi^{k_1}(J) \) by translation by \( 2\xi - 1 \).

In either case we have that \( J_1 = \psi_1(\varphi^{k_1}(J)) \) is a subinterval of \([0, 2\xi]\) and we choose

\[ k_2 = \min \{ k : \varphi^k(J_1) \text{ contains } 1/2 \text{ in its closed middle third or intersects } [1 - \xi, 1] \} . \]

Again, if \( \varphi^{k_2}(J_1) \) contains \( 1/2 \) in its closed middle third, we choose \( \psi_2 = g \) and then

\[ \left| \psi_2(\varphi^{k_2}(J_1)) \right| \leq \frac{2}{3} \left| \varphi^{k_2}(J_1) \right| = \frac{2}{3} |J_1|. \]

Otherwise the interval \( \varphi^{k_2}(J_1) \) intersects \([1 - \xi, 1]\) and we choose \( \psi_2 = f \). Then the interval \( \psi_2(\varphi^{k_2}(J_1)) \subseteq [0, 2\xi] \), obtained from \( \varphi^{k_2}(J_1) \) by translation by \( 2\xi - 1 \), has the same length as \( J_1 \). We continue in the same way with the interval \( J_2 = \psi_2(\varphi^{k_2}(J_1)) \), etc. By induction we get the system [3].

Note that when we iterate the interval \( J = [0, \xi] \) under this system, its length does not change except of the moments when we apply \( \psi_i = g \), when the length decreases at least by factor \( 2/3 \). We claim that \( \psi_i = g \) for infinitely many \( i \)'s, which immediately implies that \( \lim_{n \to \infty} |f_0^n(J)| = 0 \).
Suppose, on the contrary, that \( \psi_1 = g \) only for finitely many \( i \)'s. Then for large enough \( m \) the interval \( J_m \) (which is a subset of \( [0, 2\xi] \) and has length at most \( \xi \)) has the following property: We have \( J_m = f_0^n(J) \) for some \( n = n(m) \) and the middle thirds of the (nondegenerate) intervals
\[
J_m = f_0^n(J), f_0^{n+1}(J), f_0^{n+2}(J), \ldots
\]
do not contain \( 1/2 \). Therefore each of the intervals \( f_0^{n+i}(J), i = 1, 2, \ldots, \), is obtained from the previous interval in \( (1) \) by translation, either by \( \xi \) or by \( 2\xi - 1 \). By identifying 0 and 1 in \( [0,1] \), we get a circle. Then our iterative process, assigning the sequence \( (1) \) to \( J_m \), corresponds to the iterating \( J_m \) by the irrational circle rotation by the angle \( 2\pi\xi \). Since \( \xi \in (0,1/4) \), during the first of these two applications of the rotation by \( 2\pi\xi \) our interval does not hit \( 1/2 \), therefore, using the fact that the middle thirds of the intervals in \( (1) \) do not contain \( 1/2 \), we conclude that the middle third of \( J_m \) never hits \( 1/2 \) under the circle rotation by the angle \( 2\pi\xi \). This contradicts the irrationality of \( \xi \). Irrational rotations are minimal.

4. MAIN RESULT

Our main result is the following sufficient condition for the step skew product \( (1) \) to have the specification property.

**Theorem 4.1.** Let \( T_1, T_2, \ldots, T_n \) be piecewise monotone, continuous maps on \([0,1]\), which are expanding and surjective. Suppose that \( B \subseteq \Sigma^n_\alpha \) is a subshift which has the specification property and contains a periodic point \( \alpha = (\alpha_0\alpha_1\ldots\alpha_{p-1})^\infty \) such that \( T_{\alpha_{p-1}} \circ \cdots \circ T_{\alpha_1} \circ T_{\alpha_0} \) is mixing. Let \( F : X \to X \) be the step skew product with \( X = B \times [0,1] \) and \( F(\omega,x) = (S(\omega),T_\omega(x)) \), where \( S: B \to B \) is the shift transformation and \( T_\omega = T_q \), if \( q \) is the first symbol of \( \omega \). Then \( (X,F) \) has the specification property.

**Proof.** Fix \( \varepsilon \in (0,1) \). We have to show that there is an integer \( M \) which has the following property: For any integer \( k \geq 2 \), and for any
\[
(\omega_1, x_1, n_1), (\omega_2, x_2, n_2), \ldots, (\omega_k, x_k, n_k) \in B \times [0,1] \times \mathbb{N}
\]
there exists a point \( (\eta, z) \in B \times [0,1] \) with
\[
F^r(\eta, z) = (\eta, z)
\]
for \( 0 \leq i < n_j \) and \( 1 \leq j \leq k \),
\[
d(F^i(\omega_j, x_j), F^{r_j-i}(\eta, z)) \leq \varepsilon
\]
for \( 0 \leq i < n_j \) and \( 1 \leq j \leq k \).

In order to show this, we first choose \( M \), which depends only on the fixed \( \varepsilon > 0 \). For the maps \( T_1, T_2, \ldots, T_n \) and the fixed \( \varepsilon \) let \( \gamma > 0 \) be as in Theorem 3.1. For this \( \gamma \) and for the transformation \( T_{\alpha_{p-1}} \circ \cdots \circ T_{\alpha_1} \circ T_{\alpha_0} \) choose \( m \) as in Lemma 2.1. For the given \( \varepsilon \) let \( K \) be the gap length of the dynamical system \((B,S)\) which has the specification property. Then set \( M = mp + 2K \).

Since the dynamical system \((B,S)\) has specification property (with \( K \) being the gap length for the given \( \varepsilon \)’s), for \( (\omega_1, n_1), (\alpha, mp), (\omega_2, n_2), (\alpha, mp), \ldots, (\omega_k, n_k), (\alpha, mp) \in B \times \mathbb{N} \)
there exists a point \( \eta \in B \) such that
\[
S^r(\eta) = \eta \quad \text{and} \quad \rho(S^i(\omega_j), S^{r_j-i}(\eta)) \leq \varepsilon \quad \text{for} \quad 0 \leq i < n_j \quad \text{and} \quad 1 \leq j \leq k
\]
\[
\rho(S^i(\omega_j), S^{r_j-i+n_j+K+i}(\eta)) \leq \varepsilon \quad \text{for} \quad 0 \leq i < mp \quad \text{and} \quad 1 \leq j \leq k.
\]Therefore \( \eta \in B \) is already found with the desired properties.
Now let \((f_i)_{i=0}^\infty\) be the nonautonomous system determined by this point \(\eta \in \Sigma_n^+\), this means \(f_i = T_i\), if \(q\) is the symbol in the \(i\)-th place in \(\eta\) (we start counting with 0). Remember that for \(i \geq 1\) and \(n \geq 0\) we set \(f_n^i = f_{n+i-1} \circ \cdots \circ f_{n+1} \circ f_n\) and \(f_n^0\) denotes the identity. Now set

\[
\tilde{J}_j = \{ y \in [0,1] : |f_{r_{j-1}}^i(x_j) - f_{r_{j-1}}^i(y)| \leq \varepsilon \ \text{for} \ 0 \leq i < n_j\} \quad \text{for} \ 1 \leq j \leq k.
\]

Let \(J_j\) be the connected component of \(\tilde{J}_j\) which contains the point \(x_j\).

Since the maps \(f_i^n\) are continuous, for every \(1 \leq j \leq k\) we get that the set \(J_j\) is a closed interval which is a neighbourhood of \(x_j\) (in the topology of \([0,1]\)). Furthermore, for \(1 \leq j \leq k\) there is \(i < n_j\) with \(|f_{r_{j-1}}^i(J_j)| \geq \varepsilon\) (otherwise, by continuity, an interval properly containing \(J_j\) would be a component of \(\tilde{J}_j\), a contradiction). In the following we set \(s_j = n_j + M\) for \(1 \leq j \leq k\). We have then \(r_0 = 0\) and \(r_j = r_{j-1} + s_j\) for \(1 \leq j \leq k\).

We start with \(J_1\). Because of \(|f_0^{n_1+K}(J_1)| \geq \gamma\) for some \(n_1\) we have \(|f_{n_1+K}^1(J_1)| \geq \gamma\) by Theorem 3.1. By \(\text{(6)}\) and \(\text{(2)}\) we have \(f_{n_1+K}^m = U^m\) with \(U = T_{\alpha_{p-1}} \circ \cdots \circ T_{\alpha_1} \circ T_{\alpha_0}\). By Lemma 2.1 we get then \(f_{n_1+K}^m \circ f_0^{n_1+K}(J_1) = [0,1]\), which means \(f_{n_1+K}^m(J_1) = [0,1]\). Since the fibre maps are surjective and we have \(n_1 + K + mp < n_1 + M = s_1\), this implies \(f_0^{s_1}(J_1) = [0,1]\). We find an interval \(K_1 \subseteq J_1\) with \(f_0^{r_1}(K_1) = J_2\), where we have used that \(s_1 = r_1\).

Now we consider \(J_2\). In the same way as above we get \(f_0^{s_2}(J_2) = [0,1]\). Because of \(f_0^{s_2} \circ f_0^{s_1} = f_0^{s_3}\) this implies \(f_0^{r_2}(K_2) = J_3\). Next we consider \(J_3\). As above we get \(f_0^{s_3}(J_3) = [0,1]\). Because of \(f_0^{s_3} \circ f_0^{s_2} = f_0^{r_3}\) this implies \(f_0^{r_3}(K_3) = [0,1]\). We find an interval \(K_3 \subseteq K_2 \subseteq K_1 \subseteq J_1\) with \(f_0^{r_3}(K_3) = J_4\).

Finally we end with intervals \(K_{k-1} \subseteq K_{k-2} \subseteq \cdots \subseteq K_1 \subseteq J_1\) satisfying \(f_0^{r_k}(K_{k-1}) = [0,1]\) and \(f_0^{r_{j-1}}(K_{j-1}) = J_j\) for \(2 \leq j \leq k\). Because of \(f_0^{s_k}(K_{k-1}) = [0,1]\) we find a point \(z \in K_{k-1}\) with \(f_0^{r_k}(z) = z\). Furthermore, we have \(z \in J_1\) and for \(2 \leq j \leq k\) we have \(f_0^{r_{j-1}}(z) \in J_j\), since \(z \in K_{j-1}\) and \(f_0^{r_{j-1}}(K_{j-1}) = J_j\). By the definition of the sets \(J_j\) this implies

\[
|f_0^{r_{j-1}}(x_j) - f_0^{r_{j-1}+i}(z)| \leq \varepsilon \quad \text{for} \ 0 \leq i < n_j \quad \text{and} \ 1 \leq j \leq k.
\]

Together with \(\text{(5)}\) this gives using the definition of the nonautonomous system \((f_i)_{i=0}^\infty\) that \(F^{r_k}(\eta, z) = (\eta, z)\) and that

\[
d(F^i(\omega_j, x_j), F^{r_{j-1}+i}(\eta, z)) \leq \varepsilon \quad \text{for} \ 0 \leq i < n_j \quad \text{and} \ 1 \leq j \leq k.
\]

The theorem is proved. \(\square\)

**Corollary 4.2.** Let \(T_1, T_2, \ldots, T_n\) be piecewise monotone, continuous maps on \([0,1]\), which are expanding and mixing. Suppose that \(B \subseteq \Sigma_n^+\) is a subshift which has a fixed point. Let \(F : X \rightarrow X\) be the corresponding step skew product as in Theorem 4.1. Then \((X,F)\) has the specification property if and only if the subshift \((B,S)\) has the specification property.

**Proof.** One implication is trivial, because the specification property is preserved by passing to a factor. Now assume that the subshift \((B,S)\) has the specification property. Since we assume that the (piecewise monotone) fibre maps are mixing (hence, they are also surjective) and that \(S\) has a fixed point in \(B\), all the assumptions of Theorem 4.1 are fulfilled. Thus \((X,F)\) has the specification property. \(\square\)

**Remark 4.3.** Recall that, by \([1]\), a subshift \(B\) has the specification property if and only if it has a uniform transition length, meaning that there exists a positive integer \(M\) such that for any \(B\)-words \(u\) and \(v\) there exists a \(B\)-word \(w\) of length \(M\) such that \(uwv\) is a \(B\)-word.
For subshifts of finite type, the specification property is equivalent to mixing, see [4] or [6, Proposition 21.2 and Proposition 21.3]. For the specification property of sofic systems see [10].

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