HEIGHTS AND GEOMETRIC INVARIANT THEORY

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Abstract. Let $K$ be a number field, $\mathcal{O}_K$ be its ring of integers. We introduce the notion of compactified representation of $GL_N(\mathcal{O}_K)$ and, we see how to associate to a hermitian vector bundle $\mathcal{E}$ over Spec$(\mathcal{O}_K)$ and a compactified representation $\mathcal{T}$, a hermitian tensor bundle $\mathcal{E}_T$. We can prove then that there exists a lower bound for the heights of points $x \in P(\mathcal{E}_T)$ with $SL_N(K)$–semistable generic fibre in terms of the degree of $\mathcal{E}$ and some universal constants depending only on the compactified representation. We give then three applications: a universal lower bound for general flag varieties, an application to the adjoint representation of $SL_N(K)$ and a construction of a height on the moduli space of semistable vector bundles over algebraic curves.

§1 Introduction

In the papers [Bo1], [Bo2], [Bu], [So], [Zh1], [Zh2], the different authors (J.B. Bost, J.F. Burnol, C. Soulé, S. Zhang) have shown the interesting relations between height theory and geometric invariant theory. In particular J.B. Bost [Bo1] and S. Zhang [Zh2] shown that, if $X \subset P^n(\mathbb{Q})$ is a closed variety which has $SL(N+1)$–semistable Chow point then the height of $X$ cannot be too small.

Here we continue their program; we consider a (quite) arbitrary linear action of $GL_N(K)$ on a projective space $\mathbb{P}^M$ and we study the height of $SL_N(K)$–semistable or unstable points under this action.

Let $K$ be a number field and $\mathcal{O}_K$ be its ring of integers.

Let $T : GL_N(\mathcal{O}_K) \to GL(W)$ be a linear representation and $\mathcal{E} \to \text{Spec}(\mathcal{O}_K)$ be an hermitian vector bundle of rank $N$. As in classical algebraic geometry (the function field case), from $\mathcal{E}$ and $T$ we would like to construct an associated hermitian tensor bundle $\mathcal{E}_T$. The problem is: what is the metric on $\mathcal{E}_T$?

We solve this problem by giving a definition which fits very well in the Arakelov geometry: the notion of compactified representation $\overline{T}$ of $GL_N(\mathcal{O}_K)$ (cfr. chapter 3). So, given a compactified representation $\overline{T}$ and an hermitian vector bundle $\overline{E}$ we can uniquely construct an associated hermitian tensor bundle $\overline{E}_T$.

Let $a \in \mathbb{Z}$, we will say that a representation $T : GL_N(K) \to GL(W)$ is homogeneous of degree $a$ if, for all $t \in K^*$ and all $w \in W$, $T(t \cdot Id_N)(w) = t^a \cdot w$.

We can now state the main Theorem of this paper:
Theorem 1. Let $\overline{T} = (T : GL_N(O_K) \to GL(W); \lambda_\sigma)$ be a homogeneous compactified representation of degree $a$; then there exists a explicitly computable constant $C = C(\overline{T})$ such that, for all hermitian vector bundle $\overline{E} \to \text{Spec}(O_K)$ and all point $p \in \mathbb{P}(\overline{E})(O_K)$ which has restriction to the generic fibre $p_K$ which is $SL_N(K)$-semistable, we have

$$h(p) \geq \frac{a}{[K : \mathbb{Q}]} \cdot \frac{\deg(\overline{E})}{N} + C.$$ 

$h(p)$ is the height of $p$ in $\mathbb{P}(\overline{E})$.

Conversely: let $x \in (W_K)$ be an $SL_N(K)$-unstable point, then there exist a sequence of vector bundles of rank $N$, $\overline{E}_n \to \text{Spec}(O_K)$, such that, if $x_{\overline{E}_n} \in \mathbb{P}(\overline{E}_n)$ is the point corresponding to $x$, then

$$\lim_{n \to \infty} \left( h(x_{\overline{E}_n}) - \frac{a}{[K : \mathbb{Q}]} \cdot \frac{\deg(\overline{E}_n)}{N} \right) = -\infty.$$ 

We can prove also an analogue statement for arbitrary compactified representations (Theorem 2).

We propose then three applications of Theorem 1:

- First we can give a lower bound for the height of the flag varieties in term of the degree of the hermitian vector bundle:

Proposition 5.1. Let $N$ be a positive integer and $\underline{n} = (n_1; \ldots; n_k)$ be a partition of $N$; There exist two universal constants $A(\underline{n})$ and $B(\underline{n})$ such that, if $\overline{E} \to \text{Spec}(O_K)$ is an hermitian vector bundle of rank $N$ and $\mathbb{P}(\underline{n})(\overline{E})$ is the flag variety associated to $\overline{E}$ and the partition $\underline{n}$, then

$$h(\mathbb{P}(\underline{n})(\overline{E})) \geq A(\underline{n}) \frac{\deg(\overline{E})}{N} + B(\underline{n}).$$

(See chapter 5, paragraph 1 for more details).

- As second application we can consider heights on the projective space associated to the Lie algebra $\mathfrak{sl}(N), \mathbb{P}(\mathfrak{sl}(N))$. Using a caracterisation (due to Mumford) of semistable points on $\mathbb{P}(\mathfrak{sl}(N))$ and Theorem 1, we can give a universal lower bound for the height of those points $x \in \mathbb{P}(\mathfrak{sl}(N))$ such that $ad(x)$ is not nilpotent. (See chapter 5 paragraph 2).

- The third application is a bit different from the others. Using the work of J.F. Burnol [Bu] and S. Zhang [Za1], we can construct a (non canonical) height on the moduli space of semistable vector bundles over an algebraic curve. This height seems quite computable and it is very natural in terms of the geometric invariant theoretic construction of such a moduli space. This third application may also be seen as an appendix to another paper on the argument [Ga]. In that paper we constructed a "canonical" height under some hypothesis on the curve (good reduction everywhere) and the degree and the rank of the bundles. That height has a very interesting interpretation in terms of the Arakelov geometry of the bundles. The height constructed here has (a priori) no Arakelovian interpretation, but we do not make any hypothesis on the curve, the degree and the rank. It might be interesting to study the relations between the two heights.
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§2 Quick review of heights theory of projective varieties and Geometric Invariant Theory

- Heights theory

Let $K$ be a number field, $\mathcal{O}_K$ be its ring of integers and $S_\infty$ the set of infinite places of $K$.

An hermitian vector bundle of rank $N$, $\mathcal{E}$ over $\text{Spec}(\mathcal{O}_K)$ is a couple $\mathcal{E} = (E; \langle \cdot; \cdot \rangle_\sigma)_{\sigma \in S_\infty}$ where:
- $E$ is a projective $\mathcal{O}_K$–module of rank $N$ (or, equivalently, a vector bundle over the arithmetic curve $\text{Spec}(\mathcal{O}_K)$).
- For each $\sigma \in S_\infty$, the $\mathbb{C}$–vector space $E \otimes_\sigma \mathbb{C} = E_\sigma$ is endowed with a hermitian metric $\langle \cdot; \cdot \rangle_\sigma$ (if $\sigma$ is the complex conjugate of $\tau$ then the metric $\langle \cdot; \cdot \rangle_\sigma$ is the complex conjugate of $\langle \cdot; \cdot \rangle_\tau$).

We can associate to $\mathcal{E}$ an arithmetic variety $X = \mathbb{P}(\mathcal{E})$. More precisely, in the language of Gillet and Soulé arithmetic intersection theory, $X$ is an ”Arakelov Variety”; indeed, for each $\sigma \in S_\infty$, $X_\sigma = X \times_\sigma \text{Spec}(\mathbb{C})$ is endowed with an hermitian metric: the Fubini–Study metric. Moreover $X$ is equipped with a canonical universal hermitian quotient line bundle $\mathcal{T} = \mathcal{O}_X(1)$.

In this situation (actually in a more general one) Gillet and Soulé defined an arithmetic intersection theory with values in $\mathbb{R}$ (cfr. [GS]); actually we have:
- for each integer $i \in [O; N]$, an arithmetic Chow group $\widehat{CH}^i(X)$;
- an intersection product: $\langle \cdot; \cdot \rangle : \widehat{CH}^i(X) \otimes \widehat{CH}^j(X) \to \widehat{CH}^{i+j}(X)$;
- an arithmetic degree map: $\widehat{\deg} : \widehat{CH}^N(X) \to \mathbb{R}$;
- for each cycle of dimension $p$, $Z \in Z_p(X)$, an arithmetic fundamental class $[Z] \in \widehat{CH}^{N-p}(X)$ (this class depends on the chosen metric on $E$).

By all this machinery we can define a height function on cycles of arbitrary dimension on $X$: let $Z \in Z_p(X)$, we define the height of $Z$ by the formula

$$h(Z) = \frac{1}{[K : \mathbb{Q}]} \widehat{\deg}(\mathcal{T}^d; [Z]).$$

For an extended account on height theory see [BoGS], [F].

Remark. Suppose $Z_K \in Z_{p-1}(X_K)$ is a cycle of dimension $p - 1$ on the generic fibre $X_K$ of $X$; then we define its degree by the formula $\widehat{\deg}(Z_K) = \widehat{\deg}(L_{K}^{p-1}; [Z_K])$ (where $[Z_K]$ is the geometric fundamental class). comparison between this formula and formula (1), shows that we can see the height as the arithmetic analogue of the geometric degree. And, if the last one measure the geometric complexity, the former (by analogy) measure the arithmetic complexity of a cycle in $X$.

Example: Let $p \in X(\mathcal{O}_K)$; then $p$ correspond to projective module $M$ of rank one over $\mathcal{O}_K$ with a surjection

$$E \to M \to 0.$$
if we put over $M$ the quotient metric induced by $\overline{E}$, we have the formula

$$h(p) = \frac{1}{[K : \mathbb{Q}]} \hat{\deg}(M)$$

where $\hat{\deg}(M)$ is the Arakelov degree defined by the formula

$$\hat{\deg}(M) = \log(M/m_O K) - \sum_\sigma \log \|m\|_\sigma$$

where $m \in M \setminus \{0\}$ and $\|m\|_\sigma = \langle m; m \rangle^{1/2}_\sigma$ (cfr. [Sz]).

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**Geometric Invariant Theory**

Let $G$ be the reductive group $GL_N(K)$ or $SL_N(K)$.

Let

$$T : G \to GL(W)$$

be an algebraic linear representation of $G$, where $W$ is a finite dimensional $K$–vector space.

**Definition.** A point $v \in W$ is said **semistable** with respect to $T$ if it verify one of these two equivalent conditions:

a) the Zariski closure of the orbit of $v$, $\overline{T(G)(v)}$ does not contains the zero element;

b) there exists a $G$–invariant homogeneous polynomial of positive degree $f$ such that $f(v) \neq 0$ ($f \in Sym^*(W^*)^G$).

If $v$ is not semistable, it is said unstable.

By linearity of the action $T$, the group $G$ acts on the projective space $\mathbb{P}(W)$. A point $p \in \mathbb{P}(W)$ is said **semistable** if there exists a vector $v \in W$ over $p$ which is semistable (and then all the other vectors $v'$ over $p$ are semistable too). Otherwise $p$ is said unstable.

The following facts hold:

– The set of semistable points of $W$ (resp. $\mathbb{P}(W)$) is a Zariski open set $W^{ss}$ (resp. $\mathbb{P}(W)^{ss}$). This open set is invariant under base change: if $K'$ is an extension of $K$ and $T' : G \otimes_K K' \to GL(W \otimes_K K')$ is the extended representation, then

$$W^{ss} \otimes_K K' = (W \otimes_K K')^{ss}$$

and $\mathbb{P}(W)^{ss} \otimes_K K' = (\mathbb{P}(W \otimes_K K')^{ss}$.

– The $K$–algebra $Sym^*(W^*)^G$ of the $G$–invariant polynomials over $W$, is a finitely generated $K$–algebra (and, as before, commute to base change).

**Remark.** The theory can be extended to arbitrary reductive algebraic group; but we will not use that generality here.

For more details on Geometric Invariant Theory see [GIT] or [Se].
§3 Representations of $GL(N)$ and hermitian vector bundles

Let $K$ be a field and $X$ be a variety over $K$; let $E \to X$ be a vector bundle of rank $N$ and $T : GL_N(K) \to GL(W)$ be an algebraic linear representation defined over $K$. It is well known that we can construct a new vector bundle $E_T$ over $X$ by using $E$ and $T$. It is the so-called “tensor bundle associated to $E$ and $T$”.

Roughly speaking $E_T$ is constructed in the following way: $E$ can be described by an open covering $U = \{U_i\}_{i \in I}$ and algebraic functions $g_{ij} : U_i \cap U_j \to GL_N(K)$ (called transition functions) which satisfy the well known cocycle conditions; the $E_T$ is described by the same open covering $U$ and as transition functions the functions $h_{ij} = T(g_{ij})$.

Example If $V$ is the standard representation of $GL_N(K)$ and $T = Sym^n(V)$ (resp. $T = \wedge^i(V)$) then $E_T = Sym^n(E)$ (resp. $E_T = \wedge^i(E)$).

We would like to have an analogue construction over arithmetic curves.

Given a linear algebraic representation $T : GL_N(O_K) \to GL(W)$ and a vector bundle $E \to \text{Spec}(O_K)$ it is not difficult to construct a new vector bundle $E_T \to \text{Spec}(O_K)$ which is the analogue of the tensor bundle described before; actually the same construction works.

Example Let $V$ be an hermitian space and $W = Sym^n(V)$. Over $W$ there are at least two natural hermitian metrics:

- There is a canonical surjective map $V^{\otimes n} \to Sym^n(V)$, which induces on $W$ the quotient metric of the tensor product metric.

- We can see $W$ as $H^0(\mathbb{P}(V); \mathcal{O}_\mathbb{P}(n))$, and over this last one there is the $L^2$ metric induced by the Fubini–Study, metric on $\mathbb{P}(V)$ (cfr. also [BoGS] page 984).

Never the less we have the following Proposition which can be useful to solve the problem when the representation is irreducible:

**Proposition 3.1.** Let $T : GL_N(\mathbb{C}) \to GL(W)$ be an irreducible representation, let $U(N) \subset GL_N(\mathbb{C})$ be the unitary group; then, up to a constant, there exist a unique $U(N)$–invariant hermitian metric over $W$.

**Proof.** The restriction $T|_{U(N)} : U(N) \to GL(W)$is still irreducible. Let $h$ and $g$ be two $U(N)$–invariant metrics on $W$. By Riesz representation Theorem, there exist a linear map $A : W \to W$ such that, for all $x$ and $y$ in $W$, $h(x; A(y)) = g(x; y)$. It is not difficult to see that $A$ is a $U(N)$–equivariant map. By Schur Lemma there is a constant $\lambda$ (which must be real positive) such that $A = \lambda \cdot \text{Id}_W$.

This Proposition can be easily generalized to representations on which every irreducible representation appears with multiplicity zero or one (but the constant will be a $t$–tuple). But the following example shows that we cannot hope better than this.

Example Let $\mathbb{C}$ be the trivial representation of $GL_N(\mathbb{C})$; then every hermitian metric on $\mathbb{C} \oplus \mathbb{C}$ is $U(N)$–invariant.

So we propose the following definition.
We recall that if $V$ is a finite dimensional $\mathbb{C}$–vector space, the set $\mathcal{H}(V)$ of the hermitian metrics on $V$ can be seen as a homogeneous space in the following way: $GL(V)$ acts on $\mathcal{H}(V)$: if $h \in \mathcal{H}(V)$ and $g \in GL(V)$, then $\alpha(g) h = \overline{g}^{-1} h g$. The action $\alpha$ is transitive and if $h \in \mathcal{H}(V)$, the isotropy group of $h$ is the unitary group $U(h)$. So

$$\mathcal{H}(V) = GL(V) / U(h).$$

Let $T : GL_N(\mathbb{C}) \to GL(W)$ be a linear representation and let $h \in \mathcal{H}(W)$ be a $U(N)$–invariant hermitian metric. Then, by using $T$ and $h$, we can define a map from $\mathcal{H}(\mathbb{C}^N)$ to $\mathcal{H}(W)$: since $h$ is $U(N)$–invariant, $T(U(N)) \subset U(h)$, so $T$ defines a map from $GL_N(\mathbb{C}) / U(N)$ to $GL(W) / U(h)$, so a map

$$\lambda_{h; T} : \mathcal{H}(\mathbb{C}^N) \to \mathcal{H}(W).$$

Let $V$ be a hermitian vector space of dimension $N$; by fixing anortonormal basis, it defines a point $a_V \in \mathcal{H}(\mathbb{C}^N)$ and a compact subgroup $U(V) \subset GL_N(\mathbb{C})$ (the point $a_V$ and the unitary group $U(V)$ do not depend on the chosen basis); so, by the map $\lambda_{h; T}$ it defines an hermitian metric $\lambda_{h; T}(a_V)$ on $W$. The hermitian metric $\lambda_{h; T}(a_V)$ is $U(V)$–invariant.

**Definition.** Let $K$ be a number field, $\mathcal{O}_K$ its ring of integers and $S_\infty$ the set of embedding of $K$ in $\mathbb{C}$. A compactified representation $\overline{T}$ of $GL_N(\mathcal{O}_K)$ is a couple $\overline{T} = (T; h_\sigma)_{\sigma \in S_\infty}$, where

- $T : GL_N(\mathcal{O}_K) \to GL(W)$ is a linear representation and $W$ is a free $\mathcal{O}_K$–module.
- For each $\sigma \in S_\infty$, $h_\sigma$ is a $U(N)$–invariant metric.

Let $E$ be an hermitian vector bundle over $Spec(\mathcal{O}_K)$ and $\overline{T} = (T : GL_N(\mathcal{O}_K) \to GL(W); h_\sigma)_{\sigma \in S_\infty}$ be a compactified representation. From these data we can construct a new hermitian vector bundle $\overline{E}_T$ over $Spec(\mathcal{O}_K)$ as follows:

- If $U = \{U_i\}_{i \in I}$ is covering of $Spec(\mathcal{O}_K)$ by affine open sets ($U_i = Spec(A_i)$; $U_i \cap U_j = Spec(A_{i j})$) where $E$ trivializes, and $g_{i j} : U_i \cap U_j \to GL_N(A_{i j})$ are the transition functions defining $E$; then $E_T$ is defined by the same open covering and as transition functions $h_{i j} = T(g_{i j}) : U_i \cap U_j \to GL(W|_{U_i \cap U_j})$.
- For each $\sigma \in S_\infty$ the hermitian metric on $\overline{E}_T(\sigma) \simeq W_\sigma$ is $\lambda_{h_\sigma; T}(E_\sigma)$.

**Definition.** The hermitian vector bundle $\overline{E}_T$ will be called the hermitian tensor bundle associated to $\overline{E}$ and $\overline{T}$.

**Examples**

- If $V$ is the standard representation of $GL_N(\mathcal{O}_K)$ and $h_\sigma$ ”is the identity”, then $\overline{E}_T \simeq \overline{E}$.
- If $T : GL_N(\mathcal{O}_K) \to GL(W)$ is an irreducible representation, then we can prove that there exists a positive integer $d$, an integer $k$ and a $GL_N(\mathcal{O}_K)$–equivariant surjective map

$$\varphi : (V \otimes^d) \otimes (det(V) \otimes^k) \to W \to 0$$

(cfr. the Weyl construction, [FH] chapter 6). So, if we fix $\varphi$ we can define $h_\sigma$ as the quotient metric induced by $\varphi$; then for every hermitian vector bundle of rank $N$, $\overline{E}$, the metric on $\overline{E}_T$ is the quotient metric given by the the induced map

$$\varphi_T : (\overline{E} \otimes^d) \otimes (det(\overline{E}) \otimes^k) \to \overline{E}_T \to 0.$$

By Proposition 3.1 every other ”compactification” of $T$ is obtained by multiplying by a positive real number such a compactification.
Corollary 3.2. Let $T$ be a irreducible representation of $GL_N(\mathcal{O}_K)$ and $\mathbf{E}$ be a hermitian vector bundle of rank $N$; then there exists a unique metric on $E_T$ which is $U(E_\sigma)$–invariant for every $\sigma \in S_\infty$ and the isomorphism
\[
\det(E_T) \simeq (\det(\mathbf{E}))^{a \cdot \text{rk}(W) / N}
\]
is an isometry ($a$ is the integer such that, if $t \in \mathbb{G}_m(K)$ and $v \in W_K$, then $T(t \cdot \text{Id}_N)(v) = t^a \cdot v$; it depends only on $T$, cfr. below).

Remark. Let $T : GL_N(\mathbb{C}) \to GL(W)$ be a linear representation, and $h_1$ and $h_2$ be two $U(N)$–invariant metrics; then there exists two positive constants $C_1 = C_1(h_1; h_2)$ and $C_2 = C_2(h_1; h_2)$ such that for every $N$–dimensional hermitian vector space $V$ and element $x \in W$
\[
C_1 \|x\|_{\lambda_{h_1}, T(a_V)} \leq \|x\|_{\lambda_{h_2}, T(a_V)} \leq C_2 \|x\|_{\lambda_{h_1}, T(a_V)}
\]
where $\|x\|_{\lambda_{h_i}, T(a_V)} = (\lambda_{h_i, T(a_V)}(x; x))^{1/2}$.

§4 Heights of semistable points.

In this chapter, we will prove, roughly speaking, that the height of a semistable point in the projective space $\mathbb{P}(E_T)$ cannot be too small.

Definition. Let $K$ be a field (of characteristic zero) and $T : GL_N(K) \to GL(W)$ be a linear representation; $T$ is said to be homogeneous of degree $a$, where $a$ is an integer, if, for all $t \in K^*$ and $v \in W$, we have
\[
T(t \cdot \text{Id}_N)(v) = t^a \cdot v.
\]

The class of homogeneous representations is quite big, indeed:
- every irreducible representation is homogeneous;
- if $T$ is homogeneous (of degree $a$) then $\text{Sym}^n(T), \wedge^i(T), T^*$ are homogeneous (of degree $na, ia, -a$ respectively);
- every subrepresentation of a homogeneous representation is homogeneous (so, using duality, every quotient representation of a homogeneous representation is homogeneous);
- if $T$ and $T'$ are homogeneous representations, then $T \otimes T'$ is homogeneous;

The direct sum of homogeneous representation of different degree is not homogeneous (and this is, more or less, the only method to construct non homogeneous representations).

Let $K$ be our number field and $\mathcal{O}_K$ be its ring of integers.

We can now state the principal Theorem of this paper:

Theorem 1. Let $\mathbf{T} = (T : GL_N(\mathcal{O}_K) \to GL(W); h_\sigma)_{\sigma \in S_\infty}$ be a compactified homogeneous representation of degree $a$; then there exist constant $C = C(\mathbf{T})$ such that, for every hermitian vector bundle $\mathbf{E} \to \text{Spec}(\mathcal{O}_K)$ of rank $N$ and for every point $p \in \mathbb{P}(\mathbf{E}_T)(\mathcal{O}_K)$ such that, the restriction $p_K$ of $p$ to the generic fibre is $\text{SL}_N(K)$–semistable, we have
\[
h(p) \geq \frac{a}{\text{rk}(\mathbf{E})} \cdot \frac{\text{deg}(\mathbf{E})}{N} + C.
\]
Conversely: let \( x \in \mathbb{P}(W_K) \). Let

\[
V = \left\{ E' \text{ herm. vect. bun. of rk } N \text{ over } \text{Spec}(\mathcal{O}_K) \right\},
\]

then, for every \( E' \in V \), \( x \) defines a point \( x_{E'} \) in \( \mathbb{P}(E'_T) \). If \( x \) is an \( SL_N(K) \)-unstable point then

\[
\inf_V \left\{ h(x_{E'}) - \frac{a}{[K : \mathbb{Q}]} \cdot \widehat{\deg}(E') \right\} = -\infty.
\]

Remark. The proof of this Theorem is very similar to the proof of Proposition 2.1 in [Bo1] and Proposition 4.2 in [Zh2].

Proof. Let \( S = \text{Sym}^*(W_K) \) be the symmetric algebra of \( W \); \( SL_N(K) \) acts linearly over \( S \); let \( S^{SL_N} \) be the subalgebra of the \( SL_N(K) \)-invariant elements of \( S \). By the first fundamental Theorem in Geometric Invariant Theory, \( S^{SL_N} \) is a \( K \)-algebra of finite type. Let \( \{P_1, \ldots, P_M\} \) be a basis of \( S^{SL_N} \) over \( K \). By clearing the denominators, we can suppose that \( P_i \in S(W) \) (the \( \mathcal{O}_K \)-symmetric algebra of \( W \)); moreover we can suppose that the \( P_i \)'s are homogeneous.

Let \( \sigma \in S_\infty \) and \( h_\sigma \) the hermitian metric on \( W_\sigma^* \) (the dual of \( W_\sigma \)) given by the dual compactification; we define then

\[
||P_i||_\sigma = \sup_{v \in W_\sigma^*} \frac{||P_i(v)||}{||v||_{h_\sigma}^{\deg(P_i)}}
\]

and

\[
B_\sigma = \max_{1 \leq i \leq M} \left\{ \left\{ ||P_i||_\sigma^{1/\deg(P_i)} \right\} \right\}.
\]

We remark that, for all \( v \in W_\sigma^* \), \( P_i(v) \in \mathbb{C} \), then we can speak about its norm.

Let \( \overline{E} \to \text{Spec}(\mathcal{O}_K) \) be an hermitian vector bundle of rank \( N \).

Let \( p \in \mathbb{P}(E_T) \) a point with restriction to the generic fibre \( p_K \) which is \( SL_N(K) \)-semistable. By functoriality \( p \) defines a line bundle \( M \) over \( \text{Spec}(\mathcal{O}_K) \) with a surjection

\[
E_T \to M \to 0;
\]

as before we put on \( M \) the quotient metric.

By dualizing (3) and tensorizing by \( M \), we get an isometric embedding

\[
0 \to \overline{\mathcal{O}} \xrightarrow{\varphi} (E_T)^* \otimes M
\]

where \( \overline{\mathcal{O}} \) is the trivial line bundle with trivial metrics (\( \mathcal{O}_K; ||1||_\sigma = 1 \)).

Since \( p_K \) is \( SL_N(K) \)-semistable, there exists a \( SL_N(K) \)-invariant polynomial of positive degree \( P \) such that \( P(\varphi_K) \neq 0 \). We can suppose that \( P \) is one of the \( P_i \)'s.

We remark that, for every \( \mathcal{O}_K \)-algebra \( A \), by tensor product, the representation \( T \) induces a representation

\[
T : GL_n(A) \to GL(W \otimes A)
\]
Since $P$ is homogeneous (of degree, say $D$) and $SL_N$–invariant; and since the only characters of $GL_N$ are the tensor powers of the determinant, for every $O_K$–algebra $A$, every $v \in W \otimes_{O_K} A$ and $g \in GL_N(A)$ we have

$$P(T_A(g)(v)) = (\det(g))^\frac{AD}{N} P(v).$$

Suppose that $\overline{E}$ and $M$ are defined by the open affine covering $U = \{U_i\}_{i \in I}$ (where we can suppose that $U_i = \text{Spec}(A_i)$ and $U_i \cap U_j = \text{Spec}(A_{ij})$) and transition functions $g_{ij}$ and $f_{ij}$ respectively. Then $\varphi$ is given by sections

$$\varphi_i \in \Gamma(U_i; (E_T)^* \otimes M) \simeq W \otimes_{O_K} A_i$$

with the relations

$$\varphi_i = T^*(g_{ij}) \cdot f_{ij}(\varphi_j)$$

over $U_i \cap U_j$ ($T^*$ is the dual representation of $T$).

Then we see that $P(\varphi)$ defines a non zero element in $(\bigwedge^N(E^*))^\frac{AD}{N} \otimes M^D$.

Since the norm $\|\varphi\|_\sigma = 1$ (because of the isometry (4)) we have that

$$\|P(\varphi)\|_\sigma \leq B^D.$$

So

$$Dh(p) = \frac{1}{[K:Q]} \cdot Da \cdot \frac{\overline{\deg(E)}}{N} = \frac{\overline{\deg(\bigwedge^N(E^*))}^\frac{AD}{N} \otimes M^D}{[K:Q]} \geq - \sum_{\sigma} D \log B_\sigma.$$

So we define $C(T)$ to be $-\sum_\sigma \log B_\sigma$.

Now we prove the converse.

Let $A = \{\text{hermitian metrics } k = (k_\sigma)_{\sigma \in S_\infty} \text{ on } O_K^{\oplus N} \text{ such that } (O_K^{\oplus N}; k) \simeq \overline{O}_K\}.$

For each element $k \in A$ we will denote $E_k$ the corresponding hermitian vector bundle over $\text{Spec}(O_K)$.

the set $A$ can be seen as a "homogeneous space"

$$\prod_{\sigma \in S_\infty} SL_N(\mathbb{C}) / \prod_{\sigma \in S_\infty} SU(N)$$

where $\prod_{\sigma \in S_\infty} SU(N)$ acts on $\prod_{\sigma \in S_\infty} SL_N(\mathbb{C})$ on the left. This can be seen in the following way: let $e_1; \ldots; e_N$ be the standard basis of $(K)^N$: then we send the element $(g_\sigma)_{\sigma \in S_\infty} \in \prod_{\sigma \in S_\infty} SL_N(\mathbb{C})$ to the metric $k = (k_\sigma)_{\sigma \in S_\infty}$ having $\{g_\sigma(e_i)\}$ as hortonormal basis.

Let $k \in A$ and $E_k$ the corresponding vector bundle; the associated tensor bundle $(E_k)_T$ (without metrics) is just $W$.

The action of $SL_N(O_K)$ on $\mathbb{P}(W) = \mathbb{P}((E_k)_T)$ can be described as a morphism

$$SL_N(O_K) \times \mathbb{P}(W) \overset{T}{\longrightarrow} \mathbb{P}(W)$$

which satisfies some cocycle conditions (cfr. [GIT]); let $L = O_\mathbb{P}(1)$ be the universal quotient bundle on $\mathbb{P}(W)$ and let $p_2 : SL_N(O_K) \times \mathbb{P}(W) \rightarrow \mathbb{P}(W)$ be the second projection. Since the action of $SL_N(O_K)$ is linear, we have an isomorphism

$$\phi : \pi^*(L) \overset{\sim}{\longrightarrow} T^*(L)$$

(5)
The metric $k$ induces a metric $\| \cdot \|_k$ on $L$; let $\mathcal{L}_k$ be the corresponding hermitian line bundle on $\mathbb{P}(W)$. The isomorphism (5) is not an isometry in general, then
\[ p^*_2(\mathcal{L}_k) \simeq T^* (\mathcal{L}_k) \otimes \mathcal{O}(\mu) \]
where $\mathcal{O}(\mu)$ is the trivial bundle with norm $\|1\|_\sigma(g; x) = \exp(-\mu_\sigma(g; x))$ where $\mu_\sigma : SL_N(C) \times \mathbb{P}(W_\sigma) \to \mathbb{R}$ is a function.

Now, let $x \in \mathbb{P}(W_K)$; let $h(x)$ be its height when we see it as a point of $\mathbb{P}((\mathcal{O}^\oplus N))$; let $g = (g_\sigma)_{\sigma \in S^\infty}$ be an element in $\mathcal{A}$ and $E_g$ the corresponding hermitian vector bundle; let $x_{E_g}$ be the point $x$ when we see it as a point of $\mathbb{P}((E_g)_T)$; we have then
\[ h(x_{E_g}) = h(x) + \frac{1}{[K : \mathbb{Q}]} \sum_{\sigma \in S^\infty} \mu_\sigma (g_\sigma^{-1}; x_\sigma). \]

Then we conclude by using Theorem 2.2 in [Zh2].

**Remarks.**

1) As we can see from the proof, the constant $C(T)$ is effective if we know a basis of the $SL_N(K)$–invariant polynomials of $T^*$. There are effective methods to construct such a basis.

2) Theorem 2.2 in [Zh2], roughly speaking, says that, if we fix $x$, the functions $\mu_\sigma(g; x)$ are bounded below if and only if $x$ is semistable.

We will now shortly analyze the case of an arbitrary representation of $GL_N(\mathcal{O}_K)$.

Let firstly prove an easy generalization of Proposition 3.1

**Lemma 4.2.** Let $T_i : GL_N(C) \to GL(W_i)$, $i = 1, 2$ be two linear representations with no isotypic common factors; let
\[ W = W_1 \oplus W_2 \]
be the direct sum representation and $h$ an $U(N)$–invariant metric on $W$. Then the decomposition (6) is an hortogonal decomposition with respect to $h$.

**Proof.** Let $g$ be an $U(N)$–invariant metric on $W$ for which the (6) is an hortogonal decomposition. Let $A : W \to W$ be a linear map such that, for all $x, y \in W$, $h(x; y) = g(x; A(y))$. As in proposition 3.1 $A$ is an $U(N)$–equivariant map. The maps
\[ \varphi_{ij} : W_i \xrightarrow{A} W \xrightarrow{pr_j} W_j \]
(pr$_j$ are the projections) are $U(N)$–equivariant maps. But, since the $W_i$’s have no isotypic common factors, $\varphi_{ij} = 0$ if $i \neq j$. So $A(W_i) \subset W_i$ and the Lemma is proved.

Let $\mathcal{T} = (T : GL_N(\mathcal{O}_K) \to GL(W); h_\sigma)_{\sigma \in S^\infty}$ be a compactified representation; then, by the Lemma, we can write $\mathcal{T}$ in a unique way as a direct sum of compactified representation $\mathcal{T} = \oplus_i T_i$, where $T_i : GL(\mathcal{O}_K) \to GL(W_i)$ are homogeneous of degree $a_i$ ($a_i \neq a_j$ if $i \neq j$). Let $E \to \text{Spec}(\mathcal{O}_K)$ be an hermitian vector bundle of rank $N$; again by the Lemma we can decompose the hermitian vector bundle $E_T$ as direct sum $E_T = \oplus_i E_{T_i}$.

We can now state the analogue of Theorem 1 for arbitrary representations.
**Theorem 2.** Let $\mathcal{T} = (T : GL_N(\mathcal{O}_K) \to GL(W); h_\sigma)_{\sigma \in S_\infty}$ be a compactified representation; let $\mathcal{T} = \bigoplus_i T_i$ be the decomposition of $\mathcal{T}$ as a direct sum of homogeneous representations, $\deg T_i = a_i$. Let $E \to \text{Spec}(\mathcal{O}_K)$ be an hermitian vector bundle of rank $N$ and let $E_T = \bigoplus E_T i$ be the hermitian tensor bundle associated to $E$ and $T$. Let $A_T(E) = \min_i \left\{ \frac{a_i}{[K:Q]} \cdot \frac{\deg(E)}{N} \right\}$. There exists a constant $C = C(T)$ depending only on $T$, such that the following holds: let $x \in \mathbb{P}(E_T)(\mathcal{O}_K)$ be a point which verifies one of these two properties:

a) $x \in \mathbb{P}(E_{T_i}) \subset \mathbb{P}(E_T)$ and $x_K$ is semistable;

b) $x \notin \mathbb{P}(E_{T_i})$ for all $i$, but there exists $i$ such that, if $p_i : \mathbb{P}(E_T) \to (E_{T_i})$ is the linear projection induced by the exact sequence

$$0 \to \bigoplus_{j \neq i} E_{T_j} \longrightarrow E_T \longrightarrow E_{T_i} \to 0;$$

$p_i(x)_K$ is semistable. Then

$$h(x) \geq A_T(E) + C.$$

**Remark.** If $\overline{\deg}(E) \geq 0$ then $A_T(E) = \min_i \{a_i\} \cdot \frac{1}{[K:Q]} \cdot \frac{\deg(E)}{N}$.

**Proof.** If $x$ verifies a) then we just apply Theorem 1 to $x$ in $\mathbb{P}(E_{T_i})$; if $x$ verifies b) we apply Theorem 1 to $p_i(x)$ in $\mathbb{P}(E_{T_i})$ and the comparison of the heights of projections as in [BoGS] 3.3.2.

**Remarks.** a) If $x \in \mathbb{P}(E_{T_i})$ is unstable, we can state an analogue of the converse of Theorem 1.

b) If $T_i$, $i = 1; 2$ are two representations and $x = x_1 + x_2 \in T = T_1 \oplus T_2$; the if one of the $x_i$’s is semistable then $x$ is semistable, but in general the converse is not true.

### §5 Applications

We will see three applications of Theorem 1; a lower bound for the height of flag varieties; heights of semistable points under the adjoint representation and a construction of a height on the moduli space of semistable vector bundles of fixed rank and degree over algebraic curves.

**a) Heights of flag varieties**

Let $N$ be a positive integer and let $\overline{n} = (n_1; \ldots; n_m)$ be a partition of $N$ ($n_i \in \mathbb{N}_{>0}$ and $\sum n_i = N$). Let $E \to \text{Spec}(\mathcal{O}_K)$ be an hermitian vector bundle of rank $N$, and let $\mathbb{F}(n)(E)$ be the flag variety associated to $E$ and $n$ (cfr. [Gr]). The variety $\mathbb{F}(\overline{n})(E)$ is the variety classifying flags of vector spaces of type $\overline{n}$. Namely, let $k$ be a algebraic closed field (with a morphism $\mathcal{O}_K \to k$) then a closed point $x \in \mathbb{F}(\overline{n})(E)(k)$ is a flag of $k$–vector spaces

$$\{0\} \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m = E \otimes k$$

such that $\dim_k (F_i / F_{i-1}) = n_i.
Example - If $\underline{n} = (N-1; 1)$ then $F(\underline{n})(\mathcal{E}) = \mathbb{P}(\mathcal{E})$
- If $\underline{n} = (N-p;p)$ ($p > 1$) then $F(\underline{n})(\mathcal{E}) = Gr(p; \mathcal{E})$, the Grassmannian of the vector bundles of rank $p$ quotient of $\mathcal{E}$.

It is well known (cfr. [Gr]) that there exist a representation $T : GL(E) \to GL(W)$ and a canonical embedding

$$i_{\underline{n}} : F(\underline{n})(\mathcal{E}) \to F(\mathcal{E}_T).$$

$T$ is of the form $\bigotimes_{i=1}^{m} \Lambda^{\alpha_i} E$ (for the exact values of the $\alpha_i$ see [Gr]) then it may be “quite naturally” compactified: we put on $E_T$ the tensor product of the exterior product metrics. So we can speak about the height of $F(\underline{n})(\mathcal{E})$.

**Proposition 5.1.** There exist two universal constants $A = A(\underline{n})$ and $B = B(\underline{n})$ such that

$$h(F(\underline{n})(\mathcal{E})) \geq A(\underline{n})\deg(\mathcal{E}) + B(\underline{n})$$

**Proof.** The group scheme $SL(E)$ naturally acts on $F(\underline{n})(\mathcal{E})$ and the embedding $i_{\underline{n}}$ is $SL(E)$-equivariant.

Let $d$ be the Krull dimension of $F(\underline{n})(\mathcal{E})$ and $\delta$ the degree of $F(\underline{n})(\mathcal{E})_K$ in $F(\mathcal{E}_T)_K$.

The theory of Chow forms (cfr. [Bo1] and [BoGS]) allows us to construct a point $\Phi_\underline{n} \in \mathbb{P}(Sym^d(E_T)^{\otimes \delta})$, the Chow point of $F(\underline{n})(\mathcal{E})$. $\Phi_\underline{n}$ is uniquely determined by $F(\underline{n})(\mathcal{E})$, and conversely there exists a closed scheme $\mathcal{C}h \subset \mathbb{P}(Sym^d(E_T)^{\otimes \delta})$ such that every effective cycle of dimension $d$ and degree $\delta$ in $F(\mathcal{E}_T)$ is determined by a point in $\mathcal{C}h$.

If we put on $Sym^d(E_T)^{\otimes \delta}$ the tensor product of the quotient metrics induced by the surjections $(E_T)^d \to Sym^d(E_T)$ we can prove that there exists a universal constant $C(\underline{n})$ such that

$$h(F(\underline{n})(\mathcal{E})) \geq h(\Phi_\underline{n}) + C(\underline{n})$$

cfr. [Bo] Prop. 1.3 and [BoGS] Theorem 4.3.8.

The group scheme $SL(E)$ acts on $F(\mathcal{E}_T)$ fixing $F(\underline{n})(\mathcal{E})$, so, by functoriality of the Chow point, the point $\Phi_\underline{n}$ is a fixed point of $\mathbb{P}(Sym^d(E_T)^{\otimes \delta})$ under the $SL(E)$ action. In particular $(\Phi_\underline{n})_K$ is $SL_N$-semistable.

So applying Theorem 1 and (7) we can find two constants $A(\underline{n})$ and $B(\underline{n})$ such that

$$h(F(\underline{n})(\mathcal{E})) \geq A(\underline{n})\deg(\mathcal{E}) + B(\underline{n}).$$

**Remarks.** a) The constant $A(\underline{n})$ is “completely geometric”, it depends on $\underline{n}, d$ and $\delta$. It do not depends on the (natural) compactification of the representation used in the proof; indeed $A(\underline{n}) = \prod_{k=1}^{h} (N-\sum_{i \leq j} n_{ij})^{\delta}$. We can also explcitely compute the dimension $d$ et the degree $\delta$ (cfr. [Fu] example 14.6.15): for instance $d = N^2 - \sum_{i=1}^{h} n_i (\sum_{j \leq i} n_{ij})$; there exist also an explicit formula for $\delta$ but it is much more complicated (if $\underline{n} = (N-p;p)$ then $\delta = \frac{12! \cdot \ldots \cdot (p-1)! (d-1)!}{(N-p)! (N-p+1)! \cdot \ldots \cdot (N-1)!}$).

b) The heights of Grassmannians $Gr(p; \mathbb{Z}^N)$ have been explicitly computed by V. Maillot [M]. Our result is less precise but it is true for for every number field and every hermitian vector bundle (in particular with arbitrary metric).
c) It is possible to prove Proposition 5.1 directly by using Theorem II in [Bo1] and the results in [Ke].

d) Using the method in the proof (or more precisely, the method used in [Bo1] Theorem II) we can find similar lower bounds for heights of cycles in projective tensor bundles (associated to compactified homogeneous representations) having semistable Chow points.

### b) Semistable points under the adjoint representation

Let \( \mathfrak{sl}(N) \) be the Lie algebra of \( SL(N) \). Let

\[
Ad : SL(N) \rightarrow GL(\mathfrak{sl}(N))
\]

be the adjoint representation; let \( \mathfrak{Ad} \) be a compactification of \( Ad \). Let \( E \) be a hermitian vector bundle of rank \( N \); then we will denote \( \mathfrak{sl}(E) \) the tensor bundle \( E_{\mathfrak{Ad}} \).

Then as a direct consequence of Theorem 1 and the characterization of semistable points of \( \mathbb{P}(\mathfrak{sl}(N)) \) under the adjoint representation (cfr. [Mu] Proposition 1.15) we find

**Proposition 5.2.** There exists a constant \( C \in \mathbb{R} \) depending only on the chosen compactification such that, if \( E \) is an hermitian vector bundle of rank \( N \) and \( x \in \mathbb{P}(\mathfrak{sl}(E)) \) is a point such that \( ad(x_K) \) is not nilpotent, then

\[
h(x) \geq C.
\]

Conversely, if \( x_K \in \mathbb{P}(\mathfrak{sl}(N)) \) is such that \( ad(x_K) \) is nilpotent, we can find a sequence of hermitian vector bundles \( E_n \) of rank \( N \); such that, if \( x_{E_n} \) is the point in \( \mathbb{P}(\mathfrak{sl}(E_n)) \) defined by \( x_K \), we have

\[
\lim_{n \to \infty} h(x_{E_n}) = -\infty.
\]

**Proof.** It suffices to remark that the adjoint representation is homogeneous of degree zero and \( x \in \mathfrak{sl}(N) \) is unstable if and only if \( ad(x) \) is nilpotent.

### c) Heights on moduli space of semistable vector bundles over algebraic curves

This third application is a little bit different from the others. Indeed, using Geometric Invariant Theory, we construct a height on the moduli space of semistable vector bundles of fixed rank and degree over an algebraic curve and we use Theorem 1 to give a lower bound for this height.

The height we construct is not very canonical indeed, but it has some advantages with respect to other constructions (cfr. [Ga] for another construction which is more canonical but needs some hypothesis on the curve, the rank and the degree); let's quote some of them:

- we will not make any hypothesis on the curve and on the rank and the degree;
- the height is strictly related with the construction of the moduli space;
- the height seems quite computable.

Let \( X \) be a projective smooth curve of genus \( g \geq 1 \) over a number field \( K \).
Remark. Sometime we will silently make finite extensions of the base field $K$; anyway the results we are giving are invariant under base change.

Let $E$ be a vector bundle of rank $r$ and degree $d$ over $X$; let $\mu(E) = \frac{d}{r}$ the slope of $E$.

The vector bundle $E$ is said to be semistable, if for every subbundle $F \subset E$ we have

$$\mu(F) \leq \mu(E).$$

If we fix $r \geq 1$ and $d \in \mathbb{Z}$, there exists a coarse moduli space $U_{X}(r; d)$ of semistable vector bundles of rank $r$ and degree $d$ over $X$. It is a projective variety of dimension $r^2(g - 1) + 1$.

Remark. If $L$ is a line bundle of degree $n$ over $X$, the map

$$U_{X}(r; d) \longrightarrow U_{X}(r; d + rn)$$

$$[E] \longrightarrow [E \otimes L]$$

is an isomorphism; so if we want to study $U_{X}(r; d)$ we can suppose $d$ very big.

It is well known (cfr. [Ne] chapter 5) that, if $r$ is fixed and $d$ is sufficiently big (how big, can be explicitly computed) we can find a quasi projective smooth variety $R$, a $K$–vector space $W$, a vector bundle $U$ of rank $r$ over $X \times R$ such that the following properties are verified

- There exist a surjective map o vector bundles over $X \times R$

$$W \otimes \mathcal{O}_{X \times R} \longrightarrow U \longrightarrow 0$$

- for all $q \in R(K)$, the restriction $U_q = U|_{X \times \{q\}}$ is a vector bundle of rank $r$ and degree $d$ over $X$ and the induced map

$$W \longrightarrow H^0(X; U_q)$$

is an isomorphism;

- If $E$ is a semistable vector bundle of rank $r$ and degree $d$ over $X$, there exists a $q \in R$ such that $E \simeq U_q$;

- The group $SL(W)$ acts on $R$ and $U_{q_1} \simeq U_{q_2}$ if and only if $q_1$ and $q_2$ re in the same orbit.

Remark. There are also others properties verified, but since we will not use explicitly here it is unusefull to recall them (cfr. [Ne] page 138).

Let $R^{ss}$ be the subset of $q \in R$ such that $U_q$ is a semistable bundle over $X$.

Let $Z_N$ be the variety $Gr(r; W) \times \cdots \times Gr(r; W)$ ($N$ times), where $Gr(r; W)$ is the grassmannian of quotient vector spaces of $W$ of dimension $r$. By the tensor product of the Plücker embedding we can find a $SL(W)$–equivariant embedding $Z_N \hookrightarrow \mathbb{P}(\bigotimes_{i=1}^{N} \wedge^r W)$; so we can speak about the semistable points under the diagonal action of $SL(W)$ on $Z_N$.

Let $Z_N^{ss}$ be the open set of $SL(W)$–semistable points of $Z_N$ under this action.

If $x \in X$, we can define the map

$$\tau_x : R \longrightarrow Gr(r; W)$$

$$q \mapsto U_q|_x$$
where $U_q|_x$ is the fibre at $x$ of the bundle $U_q$. The map $\tau_x$ is a $SL(W)$–equivariant map.

There is an integer $N = N(r; d; g)$ and $N$ points $x_1; \ldots; x_N$ on $X$ such that the following properties are verified (cfr. [Ne] page 141):

- The map $\tau_N = \tau_{x_1} \times \cdots \times \tau_{x_N} : R \to Z_N$
  $$q \to (U_q|_{x_1}; \ldots; U_q|_{x_N})$$

is a $SL(W)$–equivariant map;

- $R^{ss} = \tau_N^{-1}(Z_N^{ss})$.

Remark. Again, there are other properties but we will not quote them here (cfr. [Ne] page 142).

Now we start the arithmetic construction:

Let $O_K$ be the ring of integers of $K$ and $W$ be a locally free model of $W$ over $O_K$. We choose some metrics on $W$ and a compactification $\overline{T}$ of the $GL(W)$–representation $\bigotimes_{i=1}^N \Lambda^r W$.

Let $Gr(r; W)$ be the grassmannian of locally free quotients of rank $r$ of $W$ and $Z_N = (Gr(r; W))^N$. The arithmetic scheme $Z_N$ is a smooth projective model of the variety $Z_N$ over $\text{Spec}(O_K)$. The group scheme $SL(W)$ acts on $Z_N$ and as before we have an $SL(W)$–equivariant embedding

$$i_N : Z_N \to \mathbb{P}(\overline{W}_T)$$

which extends the $i_N$ defined over $K$.

Let $T = \mathbb{P}(1)$ be the universal hermitian line bundle over $\mathbb{P}(\overline{W}_T)$. Let $Z_N^{ss}$ be the open set of semistable points of $Z_N$ and let $Y$ be the categorical quotient of $Z_N^{ss}$ (cfr. [Se]). The scheme $Y$ is a projective scheme over $\text{Spec}(O_K)$ with a ample line bundle $L$ such that, if $\pi : Z_N^{ss} \to Y$ is the projection, $\pi^*(L) = L^d|_{Z_N^{ss}}$ for some $d > 0$.

we have then the following diagram

$$\begin{array}{ccc}
R^{ss} & \xrightarrow{\tau_N} & Z_N^{ss} \\
p & \downarrow & \pi \\
\mathcal{U}_X(r; d) & \xrightarrow{\alpha} & Y \\
\end{array}$$

Since $L$ is an hermitian line bundle, we can construct a metric on $L$ by the formula

$$\|m\|(x) = \sup_{y \in \pi^{-1}(x)} (\|\pi^*(m)\|(y))$$

(cfr. [Zh1]).

The line bundle $M = \alpha^*(\mathcal{L}_K)$ is an ample line bundle on $\mathcal{U}_X(r; d)$ and we have the formula $p^*(M) = \tau_N^d(L^d)$ (cfr.J [DN]). So for $[E] \in \mathcal{U}_X(r; d)$ we can define

$$h([E]) = \frac{1}{d} h_M([E]) = \frac{1}{d} h_L(\alpha([E])) = \inf_{y \in \pi^{-1}(\alpha([E]))} \{h_T(y)\}.$$ 

We remark that, $\alpha([E])$ is a point in the generic fibre of the projective scheme $Y$, so we can speak about its height as a precise real number (and not just as a number up to bounded function).

By using Theorem 1 we have then
Proposition 5.3. There exists a universal constant $C = C(g; r; d)$ such that, for every $[E] \in \mathfrak{U}(r; d)$ we have

$$h([E]) \geq \frac{rN}{[K : \mathbb{Q}]} \cdot \frac{\deg(W)}{rk(W)} + C.$$

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