1 On the completeness of quantum mechanics†

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Abstract. Quantum cryptography, quantum computer project, space-time quantization program and recent computer experiments reported by Accardi and his collaborators show the importance and actuality of the discussion of the completeness of quantum mechanics (QM) started by Einstein more than 70 years ago. Many years ago we pointed out that the violation of Bell’s inequalities is neither a proof of completeness of QM nor an indication of the violation of Einsteinian causality. We also indicated how and in what sense a completeness of QM might be tested with the help of statistical nonparametric purity tests. In this paper we review and refine our arguments. We also point out that the statistical predictions of QM for two-particle correlation experiments do not give any deterministic prediction for a single pair. After beam is separated we obtain two beams moving in opposite directions. If the coincidence is reported it is only after the beams had interacted with corresponding measuring devices and two particles had been detected. This fact has implications for quantum cryptography. Namely a series of the measurements performed on the beam by Bob and converted into a string of bits (secret key) will in general differ, due to lack of strict anti-correlations, from a secret key found by Alice using the same procedure.

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1.1 1. Introduction

Let us imagine that we are sitting on a shore of an island on a lake watching a sunset. We see the birds flying, the leaves and branches are moving with a wind, a passage of a boat produces all interesting patterns on the surface of the water and finally we hear regular waves hitting the shore. Finally a big round circle of the sun is hiding under the horizon leaving a place for beautifully illuminated clouds and later for the planets and stars. All these physical phenomena are perceived by us in three dimensions and they are changing in time usually in the irreversible way.

To do the physics we have to construct mathematical models leading to the predictions concerning our observations and measurements this is why we
created concepts of material points, waves and fields. For Newton light was a stream of small particles for Maxwell light was an electromagnetic wave moving in a continuous invisible medium called ether, similarly to waves on a water. With abandon of the concept of ether in special theory of relativity the image of the propagation of light became less intuitive. A discovery of the fact that the exchanges of the energy and of the linear momentum between light and matter are quantized gave a temptation to represent again the light as a stream of indivisible photons moving rectilinearly and being deflected only on the material obstacles or absorbed and emitted by the atoms. This picture together with an assumption that each indivisible photon may pass only by one slit or another and that the interaction with a slit through which it is passing does not depend on a fact that the other slit is open or closed is clearly inconsistent with the observed interference pattern. Anyway photons are not localizable objects but the same argument could be repeated for a double slit electron experiment. Therefore we discover that the light and the matter may present wave and corpuscular behavior in the mutually exclusive (complementary) experimental arrangements.

Moreover there is the wholeness in the experiment: a source is prepared and calibrated, it interacts with the experimental arrangement and the modified source and/or the final numerical results the measurements are found. The only picture given is a black box picture. As an input we have an initial "beam" entering a box as an output we have a modified "beam" ("beams") or a set of counts of various detectors. Quantum mechanics (QM) does not give any intuitive spatio-temporal picture of what is physically happening in the box. The QM gives only the predictions about the final "beams" and about statistical distribution of the counts of the detectors. Let cite Bohr[1]: "Strictly speaking, the mathematical formalism of quantum mechanics and electrodynamics merely offers rules of calculation for the deduction of expectations pertaining to observations obtained under well-defined experimental conditions specified by classical physical concepts". This statement is valid not only for the description of standard atomic phenomena but also for the S-matrix description of all scattering processes of elementary particles and for the stochastic models describing the time evolution of trapped molecules, atoms or ions. The quantum mechanics and new stochastic approaches have no deterministic prediction for a single measurement or for a single time-series of events observed for a trapped ultracold atom. The predictions being of statistical or of stochastic character apply to the statistical distribution of the results obtained in long runs or in several repetitions of the experiment. We will give a careful epistemological discussion of the experiments with trapped atoms, quantum dots and qubits in the subsequent paper. In this paper we will limit our discussion to standard experiments and to standard QM.

For example in a two slit interference experiment with low intensity source of monochromatic light we can "measure" its intensity by the counts registered by a photon detector, we can control the intensity of the source by opening and closing regularly a collimator in order to send regular pulses of light. In this case we estimate an average intensity of the "beam" (number of clicks of the
detector interpreted as a number of photons absorbed). If our screen behind the slits is in the form of a panel of photon detectors, after waiting long enough, we find spatial statistical distribution of the counts registered by detectors consistent with the classical interference pattern. However we are not allowed to imagine a light as a beam of somewhat localized and separated photons moving rectilinearly hitting one after another a double slit screen, passing by only one of the slits and after continuing their way to the detectors.

Similarly if we return to a passage of the boat on a lake we can detect and even measure the energy and the momentum transferred by the regular waves hitting a buoy close to the shore. We could even tell that we observe a beam of "wavelons" hitting the shore. There would be no comparable transfer of the energy and momentum on any buoy on a deep water away from a shore. Therefore we could not make an image of the boat producing a beam of wavelons which after rectilinear propagation hit the shore. Of course we can see changes on the surface of the considerable portion of a lake but in quantum physics we do not see the "lake".

This example shows a danger of image making. Wrong images lead to contradictions and to wrong deductions. A classical mechanics also concentrates on the description of the observations without creating too many images. The Sun and the Earth are represented mathematically by material points characterized at each moment of time by their masses, positions and velocities. If in some inertial frame the initial positions and velocities are known Newton’s equations allow us to determine a subsequent motion of these points which agrees remarkably well with a real motion of Earth around the Sun. There is no speculation by what mechanism a change in the position of one body causes an instantaneous change in the acceleration of its far away partner but it does not harm a success of the model. Of course a quest for more detailed understanding of the mutual interactions between far apart bodies led to the progress in physics namely to the development of classical electrodynamics and to the creation of general theory of relativity.

In spite of the fact that the QM gives only statistical predictions on the outcomes of the various experiments a claim is made that QM gives a complete description of the physical phenomena and even the most complete description of the individual system. Einstein has never accepted this claim and in his famous paper written with Rosen and Podolsky[2] about EPR paradox he started a fruitful discussion on the completeness of QM and on general epistemological foundations of physics. This discussion continues till now.

Many physicists adopt the statistical interpretation of QM[3] in which a wave function describes only an ensemble of identically prepared physical systems and the wave function reduction is a passage from the description of the whole ensemble of these systems to the description of a sub-ensemble satisfying some additional conditions. The statistical interpretation is free of paradoxes because a single measurement does not produce the instantaneous reduction of the wave function. The statistical interpretation leaves a place for the introduction of the supplementary parameters (called often hidden variables) which would determine the behavior of each particular physical system during the ex-
Several theories with supplementary parameters (TSP) have been discussed [4].

The QM gives predictions for spin polarization correlation experiments (SPCE) dealing with pairs of electrons or photons produced in a singlet state. In order to explain these long range correlations Bell[5] analyzed a large family of TSP so-called local or realistic hidden variable theories (LRHV) and showed that their predictions must violate, for some configurations of the experimental set-up, the quantum mechanical predictions. Bell’s argument was put into experimentally verifiable form by Clauser, Horne, Shimony and Holt[6]. Several experiments in particular those by Aspect et al. [7] confirmed the predictions of quantum mechanics. Many physicists concluded that if a TSP wants to explain the experimental data it must allow for the faster than light communication between particles and violate Einstein’s separability. Even without deep reasoning one can see that this conclusion must be flawed. Let us imagine a huge volcanic eruption taking place somewhere in the middle of the Pacific Ocean, the tsunami waves hitting the shores of Japan and America will be correlated in a natural way. Long range correlations come from the memory of the past events and time evolution and they have nothing to with extra luminal communications between far away objects.

It was shown by many authors that the assumptions made in LRHV are more restrictive that they seem to be and the Bell’s inequalities may be violated not only by quantum experiments but also by macroscopic ones. Let us mention few of them. Accardi gave an extensive discussion of non-Kolmogorovian character of the quantum probabilities[8] and noticed that the most important assumption needed to prove the Bell inequalities is not a locality assumption but the use of the same probability space. Pitovsky constructed local hidden variable model [9] which could reproduce the quantum mechanical polarization predictions. Aerts[10] showed that non-Kolmogorovian character of the quantum probabilities is due to the indeterminacy on the measurements in contrast to the indeterminacy on the classical states De Baere[11] strongly claimed that the violation of Bell inequalities is due to non-reproducibility of a set of hidden variables in the subsequent experiments.

In 1976 we noticed that if one associates to each EPR pair a couple of bi-valued spin functions $S_1(a)$ and $S_2(b)$ where $a$ and $b$ are the unit direction vectors of polarizers it is not clear how we can use the integration over the finite dimensional space of hidden variables to describe all these random experiments. Besides we noticed that we can not prove rigorously the Bell inequalities for the empirical spin expectation functions because in the runs of the different experiments the sets of couples of spin functions may be different. Nevertheless it seemed plausible to us that after averaging the approximate Bell’s inequalities would still be valid. We communicated our comments to Bell during our short stay in Geneva in September 1976 but we did not publish them.

In 1982 Bell brought to our attention the Pitovsky’s paper. The model was using axiom of choice and was quite difficult to understand but it was able to reproduce QM predictions for the SPCE. We noticed that the model can be simplified and that by using the particle beams described by Pitovsky’s
particular spin functions one could reproduce quantum mechanical predictions and avoid Bell’s inequalities [12, 13]. We noticed also[14] that in all proofs of Bell’s theorem [15] one is using (directly or indirectly) the assumption that in the moment of production both members of each pair of quanta have unknown but well defined and strictly correlated spin projection values in all directions, distributed according to some joint and unknown probability distribution, and if we try to measure a spin projection in a particular direction a measuring device can register a correct value or fail to register it with a small probability. Only in this case one can obtain predictions for all different random experiments (A,B) where A denotes polarizer used for a particle 1 and B a polarizer used for a particle 2 by conditionalization from a single sample (probability) space (at the time being we did not know the Accardi paper[8]).

However the photons and the electrons are not small spinning balls and it is well known that the QM is a contextual theory. Namely a value of a physical observable, here a spin projection, associated with a pure quantum ensemble and in this way with an individual physical system, is not an attribute of the system revealed by a measuring apparatus; it turns out to be a characteristic of this ensemble created by its interaction with the measuring device. It is therefore meaningless to consider joint spin projection distributions in all directions and the quantum mechanical predictions can not be hoped to be reproduced from the TSP models of this type. In the modified Pitovsky model[13] a quantum has a spin up in A direction if it is a member of an ensemble of particles transmitted by a polarizer A. The spin functions, in the model, describing interactions of the quanta with the polarizers, have well defined values on all unit vectors on a sphere but a passage through a given polarizer depends on the probability distribution of the unit vectors representing this polarizer, distributed statistically around a macroscopic orientation vector A. For this reason there is no deterministic prediction on the behavior of two members of each EPR pair and strict anti-correlations may not be anticipated before being observed. Similar conclusions were formulated by Schroeck[16] who analyzed the EPR experiment using the measurement scheme of stochastic quantum mechanics[17].

In[14] we recalled the paradox of Bertrand[18] who clearly demonstrated the importance of a direct link of the probabilistic model with a random experiment which it wants to describe. We underlined that the different experiments are described by the probability density distributions defined on their own probability spaces and they can be described by conditionalization from a single probability space only if all of them can be performed simultaneously on each member of a statistical population.

In view of these observations we had no doubt that the violation of Bell inequalities did not mean that if a TSP wanted to explain the EPR or other quantum mechanical experiments it had to violate Einstein’s causality. Because we were quite satisfied with the statistical description of the phenomena and we were not interested in inventing a new TSP we stopped working on the subject.

With the advent of quantum cryptography introduced by Bennet and Brassard [19, 20] streams of photons were proposed to be used to transmit a secret key and the fact that any measurement affects the quantum state could be used
to detect eavesdropping. We find questionable the use of single photons but certainly a scheme is realizable by using the pulses of the light polarized in a particular fashion. There is a bigger problem with a model by Ekert [21] in which strictly correlated EPR pairs are used in order to transmit the same secret key to Alice and Bob and the Bell's theorem is used as a test for eavesdropping. According to us there is no strict anti-correlation on the individual level so the argument does not hold.

In 2001 we received a preprint of Accardi and Regoli[22] in which they presented the results of the computer experiment violating Bell's inequalities and gave many other arguments and references showing that there was no contradiction between quantum theory and locality. According to Accardi a violation of the Bell's theorem is due to the concept of chameleon effect [23] (the dynamical evolution of the system depends on the observable one is going to measure) which is closely related to the fact that the QM is a contextual theory. Accardi's biological comparison is nice. In fact a quantum particle has no attributes by itself. A quantum particle shows different behavior in the interactions with different experimental arrangements similarly to a chameleon whose color depends whether he is sitting on a leaf or on a trunk of a tree.

The tests of Bell's theorem led to many beautiful experiments[24]. However it seems that the epistemological implications of the demonstrated violation of Bell's inequalities are still not fully understood by the majority of the physicists[25].

Moreover we realized that our contributions, still valid, to the subject seem to be unknown, forgotten or not understood. In particular we introduced direct tests of the completeness of QM[12, 26] by means of the purity tests which have been completely neglected.

This is why we decided in this paper to refine our old reasoning [12, 13, 14, 26] and to add some new much simpler arguments.

1.2 2.Completeness of a statistical theory.

A statistical description is not a description of the objects but it is a description of the regularities observed in large populations or in a series of repeated random experiments performed with the objects. Let see it on examples.

We consider a series of coin flipping experiments. Instead of coins having head and tails we have coins with one side "blue" (B) and one side "red" (R). If we want to provide a complete description of a coin using the concepts of classical physics and mechanics we may say that a coin is a round disk of a given diameter etc.. We can find also its mass, volume, moment of inertia etc. All these attributes (values of classical observables) describe "completely" a coin from a classical point of view. We have also at our disposal various coin-flipping devices: D1, D2,.. All of them look from outside in the same way: you have a place to put a coin, one of the faces up, and a button to push on. A coin is projected and you see it flying, rotating and finally it falls on the observation plate.
EXPERIMENT 1 (E1). We start with a device D1 and we use only one coin. At first we do not pay attention what is a color of a face of the coin which we put up. For example we record a series of outcomes: BBRBRRRB... At the first sight it is a time series of events without any regularity. We decide now to be more systematic and to put always a face B up into the device. To our big satisfaction we obtain a simple time series: :RRRRRR... If instead we put a face R up into device we obtain: BBBB.... From an empirical point of view our description of the phenomenon is complete. A device D1 is a classical deterministic device such that if we insert into it a particular coin it changes a face B up of the coin into a face R up of the coin and vice versa.

However we do not see only the final result we see also the coin flying, revolving and landing. If you are a physicist you would like to understand why so complicated phenomenon gives a simple deterministic result. Let us imagine that we are allowed to see the interior of the device. If we see that D1 give always to the coin the same initial linear velocity and the same initial angular velocity then knowing the laws of classical mechanics and taking into consideration air resistance but neglecting the influence of the air turbulences, caused by the revolving coin, we can, with a help of a computer, reproduce a flight of the coin and deduce that the coin placed with one face up will land always on the observation plate with another face up. It would provide a complete description of the phenomenon. Even if we were unable to make calculations we could anticipate a result and we could say that we understood "completely" a phenomenon. Of course we took the Newton’s equations for granted but in some point looking for the explanation we have to stop asking a question: "Why?".

EXPERIMENT 2 (E2). We take the same coin and a device D2. On a basis of the previous experiment we start by placing the coin always with face B up and we perform several series of trials. To our surprise we get a time series of results BRBRBRB... or RBRBR... We obtain similar results if we place the face R up. A complete empirical explanation of the phenomenon is that D2 produces completely deterministic alternating series of outcomes. The only uncertainty is a first result. It shows that a device has some memory. For example a flipping mechanism of the D2 can be identical to the flipping mechanism of the D1 with one difference that the inserted coins are rotated around a horizontal axis before being flipped with a rotating mechanism keeping a memory of events: each 180° rotation is followed by 360° and vice versa. To understand "completely" the phenomenon we look in the interior of the device and we repeat the analysis we did for D1.

EXPERIMENT 3 (E3) We replace the device D2 by a device D3. We repeat several times the experiment with the face B up and after with the face R up. We obtain various time series which seem to be completely random. We call a colleague statistician for help. He checks that the observed time series is random. He observes that relative empirical frequencies of observing the face B in long runs are close to 0.5. It concludes that each experiment is a Bernoulli trial with a probability $p=0.5$. Using this assumption he can make predictions concerning the number of faces B observed in N-repetitions of the experiment and compare them with the data. A statistical description of the
observed time-series of results is complete and it may be resumed in the following rigorous way: Anytime we place the coin into the device D3 there are two outcomes possible each obtained with a probability 0.5. A probability 0.5 it is not the information about the coin. It is not the information about the device D3. It is only the information about the statistical distribution of outcomes of random experiments: inserting the coin into the device, pushing the button on and observing the result. This is why a statement: the coin, if flipped, has a probability 0.5 to land with the face B up is incorrect. All devices considered above are flipping devices but the statistical distribution of the results they produce are different. We could correct this statement by adding: if flipped with the device D3, but one has to remember what does it mean. Once again to understand completely the phenomenon we could look in the interior of the device D3. For example we might find that D3 is identical to the device D1 but before flipping there was some mechanism rotating the coin in a random (pseudo random) way. It would allow us to understand more "completely" the phenomenon but it would not give us any additional information about the statistical distribution of results.

There could be however an advantage of this "complete" description of the phenomena. Let us imagine that to each device considered above we add a ventilator blowing on the coin when it is flying. It would certainly modify statistical distribution of results in the experiments E1 and E2. From the empirical point of view the device D1 with a ventilator it is a new device D'1 so we have a new random experiment and a new statistical distribution to be found. However on this level we are unable to predict how this new description originates from a previous one. On the contrary a knowledge of the "complete" description of the phenomenon describing a flight of the coin produced by D1 could be used to predict the modifications induced by the wind produced by the ventilator. If we had a classical theory describing time evolution of the air turbulences and its interactions with the coin (which we don’t have) we could in principle determine the possible trajectories of the coin and deduce the changes in the statistical distribution of experimental results.

In all these experiments we saw the coin flying and we could look inside the experimental devices. If we did not have this knowledge but only the knowledge of final results the only unambiguous description would be a statistical one. Probably we could invent infinite number of "microscopic" hidden variable models agreeing with observation but we would not gain any better understanding of the results. This resembles to the situation in quantum mechanics. We have a stable source producing some beam. We place some detector in front of the beam which clicks regularly what makes us believe that we have a beam of some invisible "particles" having some constant intensity. We take the detector out and we pass our beam by the experimental arrangement (a device) and we observe a time-series of the possible final outcomes. QM gives us the algorithms to calculate the time independent probability distributions of the outcomes giving no information how the time-series is building up. Einstein understood very well the statistical description of the experiments given by QM but he believed that this statistical description should be "completed
by some "microscopic" description explaining how the observed time-series of the results is building up. It seems to us that if such description existed, it would be extremely complicated and not unique so perhaps not very useful.

Even if one does not think that such "microscopic" description is needed a hypothesis, that such description is possible, suggests that there is some information in the time-series of the results not accounted for by the statistical description given by QM. If it was true a careful analysis of time-series of results could reveal some structure not explained by QM what would imply that statistical description provides the incomplete statistical description of the data.

Therefore a question whether a particular statistical description of the phenomenon is complete or not, it is an experimental question which can be asked and answered independently of the existence of a "microscopic" description of the phenomenon. The answer can be obtained with the help of the purity tests which were proposed many years ago[12, 26] and which we will discuss in some detail in the moment. Let us continue a discussion of simple macroscopic experiments which hopefully will help to understand our point of view.

As we saw in the experiment E1 any time-series obtained could be interpreted as a series of the results of the consecutive repetitions of the identical Bernoulli trials each characterized completely by a probability p = 0.5. Let us consider now another random experiment.

EXPERIMENT 4 (E4). There is a box containing the coins but we do not see what is in the box. There are 51 blue coins and 51 red coins in the box. With closed eyes we mix well the coins, we draw one coin from the box, we place it on a table and finally we open the eyes and we record the color of the coin. We decide to repeat this experiment n-times up to n=100 and after 100 repetitions we return all coins drawn into the box without looking at them. If we use the various runs(samples) with n bigger for example than 95 to estimate the frequency of observing a blue face we will find that it is close to 0.5 so we are ready to conclude that a probability of drawing a blue face in each draw is equal to 0.5 and mathematically speaking the experiment E4 is the same as the experiment E3. Our friend statistician tells us not to jump into conclusion too fast because if initially in the box we have 2N coins (N red and N blue) on the average we find 50% of blue coins in a sample but a time-series is different and in principle we can discover it by more detailed statistical analysis of this series. In the case of the Bernoulli trials at each repetition the probability of obtaining B is the same. On the contrary the probability of drawing a blue coin in the k-th draw depends on a number of blue coins drawn already. Namely if among the k draws there were m blue coins in this case the probability of obtaining a blue coin in the next draw is

\[ p(k+1) = \frac{(N-m)}{(2N-k)} \]

Consequently we will have a hypergeometric distribution instead of binomial etc.

Thus in the E4 we have a succession of the different dependent random experiments when in the E3 we have a succession of the identical independent random experiments. The averages of two time-series are consistent but they have completely different structure. In this case a statement that a probability of drawing a blue face in each draw equal to 0.5 is not only incomplete but it is
also incorrect. If we modify the experiment E4 namely by returning the coin to the box after each draw our new experiment is, for samples of a size smaller than 102, completely equivalent to the experiment E3. On the "microscopic" level there is however one fundamental difference: in the E4 the coins in the box are always either blue or red when the coin in the experiment E3 is neither blue nor red but unfortunately this difference is not seen from the existing data. To see more easily how such "microscopic" differences could be detected by performing additional experiments we will discuss another macroscopic experiments with the coins.

EXPERIMENT 5 (E5) There is a box, which contains now 50 blue coins and 50 red coins having all other physical properties identical. A button is pushed and a mechanical arm picks at random one of the coins in a box and inserts it into the device D3. The result B or R is recorded and handed to the experimenter and a coin is returned to the box.

EXPERIMENT 6 (E6) The only difference with E5 is that instead of containing 50 red and 50 blue coins a box contains 100 two-sided coins identical to the coin used in the experiments E1-E3. All other physical properties of two-sided coins are the same as the physical properties of the coins in the E5.

These two experiments produce random time-series of results which do not allow to find any significant difference between them. Two physicists agree with this statement but they cannot agree how to interpret the results. One of them, a pupil of Einstein, says: we have a statistical mixture (mixed statistical ensemble) in the box of the same number of blue and red coins and because we draw the coins from the box with replacement thus on average we observe 50% of blue coins in each run of the experiment. A second, pupil of Bohr, says: it is nonsense we have simply a pure statistical ensembles of quantum coins each in the same pure quantum state, such that each of them has simply a probability 0.5 to become blue or red after interacting with the measuring device.

They meet a statistician who confirms that the experiments give the indistinguishable results and tells them that without performing supplementary experiments one cannot decide whose model is a correct one. He tells them that in a mixed statistical ensemble every of it sub-ensembles can in principle have different observable statistical properties. On the contrary if one has a pure statistical ensemble all of its sub-ensembles have the same properties as the initial ensemble.

Our physicists agree with the statistician and they make a hole in the boxes containing coins and they decide to remove the same number of coins in the E5 and in the E6 before proceeding with several repetitions of their experiment.

If they removed by chance the equal number of blue and red coins in the E5 no difference could be noticed but if they by chance changed a proportion of blue coins in their box they could see the difference in long runs of the experiment. If less coins were left in the box the differences could be bigger. For example with 4 blue and 6 red coins in the box the probability of B became 0.4 instead of 0.5. With one blue coin in a box they would get p=1.

Following the same protocol for the experiment E6 they did not register any significant difference in the results. They concluded that there is a "microscopic"
difference between the E5 and the E6 and that Einstein’s model apply to the E5 and Bohr’s model apply to E6.

Therefore a claim that the QM gives a complete description of the individual quantum system may not be disproved by any philosophical argument nor by a mathematical theorem it may only be disproved by the experimental data. The only situation when the statistical predictions on the results of the experiments performed on the ensemble of identically prepared individual systems can be said to describe completely the interaction of the individual system with the experimental device occurs when the observed time-series is random and the statistical ensemble is pure.

The assumption of the completeness of the statistical description provided by QM is not only unnecessary but it is counter-productive. The experimentalists are interested only in testing the statistical distributions of the experimental results in long runs without even trying to analyze the observed time-series. They eliminate the ”bad” experimental runs, sometimes without finding any logical reason for doing it, simply because in the theory there is no place for them. We have enormous amount of data accumulated. If we performed the tests of the randomness and the purity tests[26] on these data perhaps we would discover new statistical regularities in the time-series we had never thought of. Let us talk now about the origin of the purity tests.

1.3 3.Purity tests

The QM did recognize that the purity of the quantum ensemble is important if a claim was made that QM provides the complete description of the individual system. However to define the purity the QM concentrated on the preparation stage of the experiment. Namely a system was said to be in a pure quantum state if it passed by a maximal filter or if a complete set of commuting observables was measured on the system.

A purity of the statistical ensemble describing the interactions of such prepared state with some other experimental device was not considered. Moreover it was not clear how we could know that a filter is maximal and how do we construct it. Besides in the axiomatic quantum mechanics, initiated by a paper by Birkhoff and von Neumann[27], it was claimed that to each vector in the Hilbert space corresponds a realizable physical state of a physical system and that the Hilbert space description is general enough to describe all imaginable future phenomena. The last claim was nicely refuted by Mielnik[21] who showed that one can imagine infinitely many non-Hilbertian ”quantum” worlds. Inspired by first two Mielnik’s papers we decided to analyze various general experimental set-ups which could be used to investigate the phenomena characterizing the ensembles of particle-beams. We considered the sources of some hypothetical particle beams, detectors( counters), filters, transmitters and instruments This analysis[29] led us to the various conclusions. Let us list those which are pertinent to the topic of this paper:

1) Properties of the beams depend on the properties of the devices and vice-versa and are defined only in terms of the observed interactions between
them. For example a beam $b$ is characterized by the statistical distribution of outcomes obtained by passing by all the devices $d_i$. A device $d$ is defined by the statistical distributions of the results it produces for all available beams $b_i$. All observables are contextual and physical phenomena observed depend on the richness of the beams and of the devices.

2) In different runs of the experiments we observe the beams $b_i$ each characterized by its empirical probability distribution. Only if an ensemble $\beta$ of all these beams is a pure ensemble of pure beams we can associate the estimated probability distributions of the results with the beams $b_i \in \beta$ and with the individual particles members of these beams.

3) A general operational definition of a pure ensemble $\beta$:

A pure ensemble $\beta$ of pure beams $b$ is characterized by such probability distribution $s(r)$ which remains approximately unchanged:

(i) for the new ensembles $\beta_i$ obtained from the ensemble $\beta$ by the application of the $i$-th intensity reduction procedure on each beam $b_i \in \beta$

(ii) for all rich sub-ensembles of $\beta$ chosen in a random way

In 1974 we noticed[30, 33, 36] that if the initial two-particle states in strong-interaction physics were mixed with respect to some additional parameter, for example the impact parameter, and if we wrote the unitary S-matrix as $S=I \oplus S_1$ instead of $S=iT+I$ then the probability would be conserved but the optical theorem could not be proven. Because all extrapolations to the forward direction were unreliable[35, 36] therefore the only way to check our hypothesis was to find this particular impurity of initial states in high energy scattering. For this purpose we proposed various purity tests[31, 32, 34]. The experiments to test our hypothesis were never done. In 1984 we noticed [26] that the similar purity tests could be used to test the completeness of the quantum mechanics.

The idea is extremely simple and it was explained in the previous section: one has to study in detail time-series of the experimental results and look for some fine structure.

Besides one has to compare different runs of the same experiment looking for statistically significant discrepancies. Namely if $b_i$ is a beam of $m_i$ particles produced by a stable source $O$ in the time interval $[t_i, t_i + \Delta t]$ we obtain a sample $S_i$ by measuring an observable $\gamma X$ on the beam $b_i$. We may also obtain the families of the beams $b_i(j)$, where $j$ denotes $j$-th beam intensity reduction procedure applied to the beam $b_i$. Measuring $\gamma X$ on the beams $b_i(j)$ we obtain new samples $S_i(j)$.

To test the purity of the beams produced by $O$ we test a hypothesis $H_0$: All the samples $S_i$ and $S_i(j)$ are drawn from the same unknown statistical population of the random variable $X$ associated to the observable $\gamma X$.

An extensive use of the non-parametric statistical tests is needed[34]. We are in 2002 nobody did the purity tests. Unfortunately all the experiments confirming the validity of the quantum mechanical statistical predictions and the violation of Bell’s theorem eliminates LHRV but are unable to prove the completeness of QM. Let us see this in more detail.
1.4 Bell’s Theorem

To each random experiment we associate a random variable \( X \), a probability space \( S \) and a probability density function \( f_X(x) \) for all \( x \in S \).

If \( X \) is a discrete random variable \( \sum_x f_X(x) = 1 \) and \( P(X=x) = f_X(x) \) if \( X \) is a continuous random variable \( \int_S f_X(x)dx = 1 \) and

\[
P(a \leq X \leq b) = \int_a^b f_X(x)dx \tag{1}
\]

where \( P(a \leq X \leq b) \) is a probability of finding a value of \( X \) included between \( a \) and \( b \). Note that \( P(X=x) = 0 \) for all \( x \in S \).

If in a random experiment we can measure simultaneously values of \( k \) random variables \( X_1, \ldots, X_k \) we describe the experiment by a \( k \)-dimensional random variable \( X = (X_1, \ldots, X_k) \), a common probability space \( S \) and some joint probability density function \( f_{X_1 X_2 \ldots X_k}(x_1, \ldots, x_k) \). From the joint probability density function one can obtain various conditional probabilities and by integration over \( k-1 \) variables one obtains \( k \) marginal probability density functions \( f_{X_i}(x_i) \) describing \( k \) different random experiments each performed to measure only one random variable \( X_i \) and neglecting all the others. In this case we say that \( f_{X_i}(x_i) \) were obtained by conditionalization from a unique probability space \( S \). In general if the random variables \( X_i \) are dependent (correlated)

\[
f_{X_1 X_2 \ldots X_k}(x_1, \ldots, x_k) \neq f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_k}(x_k) \tag{2}
\]

Each spin polarization correlation experiment \((A, B)\) is defined by two macroscopic orientation vectors \( A \) and \( B \) being some average orientation vectors of the realistic polarizer. A polarizer \( A \) is defined by a probability distribution \( d\rho_A(a) \), where \( a \) are the microscopic direction vectors, \( a \in O_A = \{ a \in S^{(2)}; |1 - a \cdot \mathbf{A}| \leq \varepsilon_A \} \).

Similarly a polarizer \( B \) is defined by \( d\rho(b) \). The probability \( p(A, B) \) that a particle 1 passes through the polarizer \( A \) and a particle 2, correlated with the particle 1 passes through a polarizer \( B \) is given by

\[
p(A, B) = \int_{O_A} \int_{O_B} p_{12}(a, b) d\rho_A(a) d\rho(b) \tag{3}
\]

where \( p_{12}(a, b) \) is a probability density function given by QM : \( p_{12}(a, b) = \frac{1}{2} \sin^2(\theta_{ab}/2) \). In the reference [13] we used a slightly different but consistent notation. It is impossible to perform different experiments \((A, B)\) simultaneously on the same couple of the particles therefore it does not seem possible to use a unique probability space \( S \) and to obtain, by conditionalization, the probabilities \( p(A, B) \) for all such experiments.

Let us for example analyze a model used by Clauser and Horne [37] to prove their inequalities:

\[
p(A, B) = \int_{O_A} p_1(\lambda, A) \int_{O_B} p_2(\lambda, B) d\rho(\lambda) \tag{4}
\]

where \( p_1(\lambda, A) \) and \( p_2(\lambda, B) \) are the probabilities of detecting component 1 and component 2 respectively, given the state \( \lambda \) of the composite system.

We see from (4) that a state \( \lambda \) is determined by all the values of strictly correlated spin projections of two components for all possible orientations of the polarizers \( A \) and \( B \). The polarizers are not perfect therefore the detection probabilities have been introduced. Therefore it is assumed in the model that
even before the detection each component has well defined spin projection in all directions. Therefore a model is using a single probability space Λ and obtains the predictions on the probabilities p(A,B) measured in different experiments by conditionalization. As we told the same assumption was used in all other proofs of Bell’s theorem. Explicit description of states λ by the values of spin projections is also clearly seen in Wigner’s proof[38].

As we told the experiments (A,B) are mutually exclusive there is no justification for using such models.

If we try to prove the Bell’s inequalities by comparing only the experimental runs of different experiments we can not do it without some additional and questionable assumptions.

Let us simplify the argument we gave in[14]. We want to estimate a value of the spin expectation function E_{AB} for an experiment (A,B). We have to perform several runs of the length N and find the value of the empirical spin expectation function r_{N}(A,B) for each run and after to find E_{AB} by averaging over various runs. Let us associate with each member of a pair a spin function s_1(x) or s_2(x), taking the values 1 or -1, on the unit sphere S^{(2)} (representing the orientation vectors of various polarizers). We assume also that s_1(x) = - s_2(x) = s(x) for all vectors x ∈ S^{(2)}. We saw in (3) that the macroscopic directions A and B were not sharp therefore in each particular run we might have different direction vectors (a,b) representing them. If for the simplicity we neglect this possibility, we get:

\[ r_{N}(A,B) = - \frac{1}{N} \sum_{i} s_{i}(A)s_{i}(B) \]  (5)

where N functions s_i are drawn from some uncountable set of spin functions F_0.

If we consider a particular run of the same length from the experiment (A,C) we get

\[ r_{N}(A,C) = - \frac{1}{N} \sum_{j} s_{j}(A)s_{j}(C) \]  (6)

where N functions s_j are drawn from the same uncountable set of spin functions F_0.

A probability that we have the same sets of spin functions in both experimental runs is equal to zero. Therefore in general we have completely distinct sets of functions in (5) and (6). and we are unable to prove the Bell’s theorem make by using r_{N}(A,B) - r_{N}(A,C). If we used the same sets of spin functions in the runs from the different experiments then we could replace (6) by (7)

\[ r_{N}(A,C) = - \frac{1}{N} \sum_{i} s_{i}(A)s_{i}(C) \]  (7)

and we could easily reproduce the Bell’s proof finding the similar inequalities. However the formula is counter-factual and does not represent the experimental data.

Let us notice the act of passage of the i-th particle through a given polarizer A depends in a complicated way on its interaction with this polarizer. Therefore we
should not consider a spin function as describing a state of a particle independent of its interaction with $A$. The spin functions $s_i$ in the (5) and (6) resume the interactions of the subsequent particles with the polarizers in a particular experiment. Therefore if we want to be rigorous we should replace (5) by (8).

$$r_N(A,B) = -\frac{1}{N} \sum_{i} s_i A(a_i)s_i B(b_i)$$

where $a_i \in O_A$ and $b_i \in O_B$. If we use the formula (8) there is no possibility of proving Bell’s theorem. This formula is consistent with the probabilistic model (3) and with the contextual character of observables.

A particular trivial, but artificial, example when a common probability space $S$ could have been used is a case when we have 4 independent random variables $X_1, X'_1, X_2, X'_2$ each described by its probability density function. If all these variables have only the values $\pm 1$ a proof of Bell’s inequalities is extremely easy. The “spin” expectation function $E_{X_1,X_2}$ is a product of expectation values of $X_1$ and $X_2$: $E_{X_1,X_2} = \langle X_1 \rangle \langle X_2 \rangle$ and we immediately get

$$|\langle X_1 \rangle \langle X_2 \rangle - \langle X_1 \rangle \langle X'_2 \rangle| + |\langle X'_1 \rangle \langle X_2 \rangle + \langle X'_1 \rangle \langle X'_2 \rangle| \leq |\langle X'_1 \rangle - \langle X'_2 \rangle||\langle X'_1 \rangle + \langle X'_2 \rangle| \leq 2 \langle X_1 \rangle \langle X_2 \rangle$$

Of course if we assume the independence there are no correlations. The statistical independence is equivalent to separability of the statistical operator used recently by Krüger in his proofs of Bell’s inequalities in [39] and in unpublished paper presented at TH 2002.

We find all these arguments very convincing but it is well known that a single picture is better than thousand words. This picture was given by the computer pseudo-random experiments of Accardi et al.[22] which violate Bell’s inequalities giving an example of the family of random experiments which cannot be described using a unique probability space $S$.

Therefore the violation of Bell’s inequalities found in SPCE proves that the probabilistic model used by LRHV is inappropriate but it tells nothing about completeness of QM or about the impossibility of causal “microscopic” explanation of quantum experiments.

Let us examine the "microscopic" description of the classical experiment E3 which we discussed in one of the preceding sections. We see that this description does not depend on observed results B or R, it depends on other physical properties of the coin, on the mechanical properties of the device and even on the properties of the ambient air. All these “hidden” parameters explain in very complicated way the observed events. Similarly a pilot wave models of de Broglie[40], Bohm[41] and Vigier[42] reproduce in a complicated way some of the quantum mechanical predictions. We like a remark by Tartalia [43] that objects in the quantum world are like programmed machines capable of different behaviors according to the physical conditions locally triggering them.

Another important implication of (3) and (8) is that the observable value of the spin projection characterizes only the whole beam of the particles which
passed through a given polarizer A. Nearly 100% of the particles of this beam pass through a subsequent polarizer A, but we have no prediction concerning any individual particle from the beam. Therefore in SPCE $p(A,A') \neq 1$ and we have no strict spin anti-correlations between the members of each pair[13].

1.5 5.Bertrand’s paradox.

Many probabilists in 19th century believed that for each random experiment there exists a probability distribution which may be determined only by combinatorial or geometric considerations. In 1889 Bertrand showed[18] that the various equally good mathematical arguments, in case of the continuous random variables, may lead to completely different predictions on the probabilities. He considered two concentric circles on a plane with radii $R$ and $R/2$, respectively. He showed that there are different possible answers to a question: “What is the probability $P$ that a chord of the bigger circle chosen at random cuts the smaller one at least in one point?”. The various answers are[14]: if we divide the ensemble of all chords into sub-ensembles of parallel chords, we find $P=1/2$. If we consider the sub-ensembles having the same beginning, we find $P=1/3$. Finally if we look for the midpoints of the chords lying in the small circle, we find $P=1/4$. A solution of the paradox is simple: the different values of $P$ correspond to the different random experiments which may performed in order to find the experimental answer to the Bertrand’s question. Thus the probabilities have only a precise meaning if the random experiments used for their estimation are specified.

Let us suppose now that we have a straight stick of the length $2R$. We draw two concentric circles on the horizontal platform and we construct three machines $M_1$, $M_2$ and $M_3$ working according to appropriate different pseudo-random protocols corresponding to different reasonings presented above.

In the first experiment we insert a stick into the machine $M_1$ which picks up a "point" $Q$ on the large circle, then follows the diameter of the circle arriving to the point $Q_1$ located in the randomly chosen distance $r$ from the point $Q$ ($0 \leq r \leq 2R$). Next $M_1$ places the stick perpendicularly to the diameter joining $Q$ and $Q_1$ with the midpoint of a stick coinciding with the point $Q_1$. If the stick touches in at least one point the smaller circle we may say that the value of the random variable $X$ is equal to 1 otherwise it’s value is -1. After many repetitions of this experiment we find the probability $p(X=1)=1/2$.

Similarly in the second experiment we can obtain the probability $p(Y=1)=1/3$ and in the third $p(Z=1)=1/4$. It is feasible but unreasonable and artificial to introduce a unique probability space $S$ and the joint probability distribution of $X,Y$ and $Z$ in order to deduce, by conditionalization, the probability distributions of our three random experiments.

1.6 6.Conclusions

The experimental tests of Bell’s theorem can neither confirm the completeness of QM nor to prove that the only TSP models able to give a "microscopic"
description of the SPCE have to violate Einsteinian causality.

A question whether a statistical description provided by the quantum theory gives a complete description of the experimental data is fully justified. This question about completeness can not be answered by proving a mathematical theorem or by constructing ad hoc TSP model reproducing the quantum predictions.

This question of the completeness of quantum theory can be only answered by a detailed analysis of the time-series of the experimental results which can be done with the help of the purity tests which were proposed many years ago and never done.

If the deviations from the randomness were detected and some new regularities found, the standard statistical description given by the quantum theory should be completed by a description using probably the ideas of the stochastic processes. In some sense this change of the description has already been made in the stochastic approaches used to explain various phenomena involving trapped atoms, ions and molecules. In these approaches the wave function obeys a Schrödinger equation with an effective Hamiltonian separated by quantum jumps occurring at random times. The purity tests could be also used to check these new stochastic models which assume without checking the ergodicity of the observed time-series. The question of the completeness formulated in this way is independent of the existence or non-existence of a detailed "microscopic" description of the phenomena presenting this particular stochastic behavior.

From Bertrand's paradox we learned that we should not talk about the probabilities without referring to the random experiments used to determine them. Therefore the quantum theory providing the predictions for various probabilities should not lose its contact with the experiments it wants to describe.

If one forgets that the quantum theory does not give any "microscopic" images but it provides only the mathematical algorithms able to describe the statistical regularities observed in the data one is tempted to create incorrect mental "microscopic" images which lead to false paradoxes and to speculations which seem to be a pure science fiction.

The quantum observables are contextual what means that their values are not the attributes of the individual members of the quantum ensemble but they give only the information about the possible interactions of the whole ensemble with the measuring devices. If the ensemble is pure one can speak about the probabilistic information pertinent to the interaction of each individual system with the measuring device. To be able to do this one must check the purity of the ensemble using the purity tests.

There is no strict anti-correlations of two time-series of the results of measurements performed on two members of EPR pair in the SPCE thus these two time series may not be used in quantum cryptography to assure that Bob and Alice use the same secret key.

The purity tests are important and relatively simple, the data are available. We hope that this paper will convince some experimentalists to do them.
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