INTERPOLATION IN MULTIVARIABLE DE
BRANGES-ROVNYAK SPACES

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ABSTRACT. We study a general metric constrained interpolation problem in a de Branges-Rovnyak space \( \mathcal{H}(K_S) \) associated with a contractive multiplier \( S \) between two Fock spaces along with its commutative counterpart, a de Branges-Rovnyak space associated with a Schur multiplier on the Drury-Arveson space of the unit ball of \( \mathbb{C}^n \).

1. INTRODUCTION

The functions analytic on the open unit disk \( \mathbb{D} \) and bounded by one in modulus (Schur-class functions) can be alternatively characterized as contractive multipliers of the Hardy space \( H^2 \): a function \( S \) belongs to the Schur class \( S \) if and only if the multiplication operator \( M_S : f \mapsto Sf \) is a contraction on \( H^2 \). The latter means that any \( S \in S \) gives rise to a positive kernel
\[
K_S(\lambda, \omega) = \frac{1 - S(\lambda)\overline{S(\eta)}}{1 - \lambda \overline{\eta}}
\]
and subsequently, to the reproducing kernel Hilbert space \( \mathcal{H}(S) := \mathcal{H}(K_S) \), the de Branges-Rovnyak space associated with \( S \). De Branges-Rovnyak spaces play a prominent role in operator model theory and Hilbert space approaches to \( H^\infty \)-interpolation. Interpolation theory in de Branges-Rovnyak spaces themselves has been initiated in [17].

To start let us recall the Nevanlinna-Pick problem in this setting: given \( S_0 \in S \), \( n \) distinct points \( \lambda_1, \ldots, \lambda_n \in \mathbb{D} \) and target values \( x_1, \ldots, x_n \in \mathbb{C} \) find all
\[
\tag{1.1}
f \in \mathcal{H}(S_0) \text{ such that } f(\lambda_k) = x_k \text{ for } k = 1, \ldots, n.
\]
The problem can be solved as follows. Let
\[
P = \left[ \frac{1 - S_0(\lambda_i)\overline{S_0(\lambda_j)}}{1 - \lambda_i \overline{\lambda_j}} \right]_{i,j=1}^n, \quad T = \begin{bmatrix} \overline{\lambda_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \overline{\lambda_n} & 0 \\ x_1 & \cdots & x_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},
\]
\[
E = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}, \quad N = \begin{bmatrix} S_0(\lambda_1) & \cdots & S_0(\lambda_n) \end{bmatrix}.
\]
Since the kernel \( K_{S_0} \) is positive on \( \mathbb{D} \times \mathbb{D} \), the matrix \( P \) is positive semidefinite. In fact, it is invertible unless \( S_0 \) is a Blaschke product of degree less than \( n \). In the
latter case, the problem (1.1) has a solution (which necessarily is unique) if and only if $x$ is in the range of $P$.

If $P$ is invertible, we can define the $2 \times 2$ matrix-function $\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}$ by the formula

$$
\Theta(\lambda) = I_2 + (\lambda - 1) \begin{bmatrix} E \\ N \end{bmatrix} (I_n - \lambda T)^{-1} (I_n - T^*)^{-1} \begin{bmatrix} E^* - N^* \end{bmatrix},
$$

which is $[0 \ 0 \ 1]$-inner in $D$. Furthermore, since $S_0 \in \mathcal{S}$, the function $E_0 = \theta_{22} S_0 - \theta_{12} \theta_{11} - \theta_{21} S_0$ belongs to the Schur class. Then all solutions $f$ to the problem (1.1) are parametrized by the formula

$$
f = f_0 + (\theta_{11} - \theta_{21} S_0) h, \quad h \in \mathcal{H}(E_0)
$$

where

$$
f_0 (\lambda) = (E - S_0 (\lambda) N) (I_n - \lambda T)^{-1} P^{-1} x
$$

and $h$ is a free parameter from the de Branges-Rovnyak space associated with $E_0 \in \mathcal{S}$. Moreover, the representation (1.4) turns out to be orthogonal in the metric of $\mathcal{H}(S_0)$ and in addition,

$$
\| f_0 \|_{\mathcal{H}(S_0)}^2 = x^* P^{-1} x \quad \text{and} \quad \|(\theta_{11} - \theta_{21} S_0) h\|_{\mathcal{H}(S_0)} = \| h \|_{\mathcal{H}(E_0)}
$$

for all $h \in \mathcal{H}(E_0)$. Therefore, for $f$ of the form (1.4), we have

$$
\| f \|_{\mathcal{H}(S_0)}^2 = x^* P^{-1} x + \| h \|_{\mathcal{H}(E_0)}^2
$$

which allows to solve the norm-constrained version of the problem (1.1) with no extra effort. We refer to [17] for the proof given there in the context of a more general Operator Argument interpolation Problem OAP with the interpolation condition given in terms of left-tangential evaluation calculus or equivalently, in terms of an observability operator associated with an output stable pair. In [18], similar results were shown to be true in the context of a vector-valued de Branges-Rovnyak space associated with an operator-valued Schur-class function; this case is briefly recalled in Section 2 below.

The main goal of the present paper is to extend some results from [18] to two multivariable settings: a free non-commutative setting, where we consider the de Branges-Rovnyak space associated with a contractive multiplier between two Fock spaces, discussed in Section 3, and the standard commutative multivariable setting dealing with the de Branges-Rovnyak space associated with a contractive multiplier between two Drury-Arveson spaces, discussed in Section 4. In each setting we consider the OAP with the interpolation condition given in terms of an appropriately defined left-tangential evaluation calculus, and get a parametrization of all solutions by formulas similar to (1.4). Along the way we make use of and clarify some results from [12] concerning the bi-contractivity of certain indefinite-metric Schur-class multipliers used to generate the linear-fractional maps used for the afore-mentioned parametrizations. This latter result appears to be new even for the definite case (thereby correcting a misconception that has appeared in the literature), as explained in Appendix B: the Schur-multiplier class is invariant under the conjugation map $S(z) \mapsto S^*(z) := S(\overline{z})^*$ (Fock-space noncommutative setting) and $S(\lambda) \mapsto S^*(\lambda) := S(\overline{\lambda})^*$ (Drury-Arveson-space commutative setting).
In the final section, Section 5, we consider the generalized de Branges-Rovnyak space $H(T)$ associated with an arbitrary contraction operator $T$ (rather than a contractive multiplication operator) between two Hilbert spaces along with an interpolation problem analogous to the OAP. At this level of generality, the general solution is still represented as the orthogonal sum of a particular minimal-norm solution and a general solution of the homogeneous problem. All solutions of the homogeneous problem, in turn, form a subspace of $H(T)$, which in the context of the problem (1.1) and in the multivariable problems considered in Sections 3 and 4 admit more detailed Beurling-type representations.

We conclude this section with some words on notations and terminology that are used at various places in the paper. All Hilbert spaces appearing in this paper are assumed to be separable. By $L(U, Y)$ we denote the space of bounded linear operators between Hilbert spaces $U$ and $Y$, abbreviated to $L(Y)$ in case $U = Y$.

If $X$ is a Hilbert space and $G$ is a selfadjoint operator on $X$, we use the notation $(X, G)$ to denote the indefinite inner product space, or Krein space $X_G$ with the indefinite inner product induced by $G$:

$$[x, y]_G := \langle Gx, y \rangle_X.$$  

In this paper we will be primarily interested in the case that the indefinite inner product is induced by a signature operator, which is an invertible operator $J \in L(X)$ with the property that $J = J^{-1} = J^*$. Given two signature operators $J_1 \in L(X_1)$ and $J_2 \in L(X_2)$, an operator $W \in L(X_1, X_2)$ is called a $(J_1, J_2)$-bi-contraction in case

$$(1.6) \quad W^*J_2W \preceq J_1 \quad \text{and} \quad WJ_1W^* \preceq J_2.$$  

In case only the first (second) inequality holds we say that $W$ is a $(J_1, J_2)$-contraction ($(J_1, J_2)$-s-contraction). Moreover, if the first (second) relation in (1.6) holds with equality then we say that $W$ is a $(J_1, J_2)$-isometry ($(J_1, J_2)$-coisometry) and if both relations hold with equality then $W$ is called $(J_1, J_2)$-unitary. Whenever $J_1 = J_2$ we will simply write $J_1$-bi-contraction, $J_1$-isometry, etc. More details on Krein spaces and a useful Krein space lemma will be given in Appendix A.

2. The classical vector-valued case

In this section we recall basic results concerning the OAP in vector-valued de Branges-Rovnyak spaces that we wish to extend to the multivariable setting. To fix notation, we denote by $H^2_Y$ the Hardy space of analytic $Y$-valued functions on $D$ with square-summable sequences of Taylor coefficients

$$H^2_Y = \left\{ f(\lambda) = \sum_{k=0}^{\infty} f_k \lambda^k : ||f||^2_{H^2_Y} := \sum_{k=0}^{\infty} ||f_k||^2_Y < \infty \right\}$$  

which turns out to be the reproducing kernel Hilbert space $H_Y(k)$ with reproducing Szegő kernel

$$k_{Sz}(\lambda, \eta) = (1 - \lambda\eta)^{-1},$$  

where we follow the convention that for any scalar positive kernel $k$ and Hilbert space $Y$ we set $H_Y(k) := H(kI_Y)$.

We next denote by $S(U, Y)$ the Schur class of analytic functions on $D$ whose values are contractive operators in $L(U, Y)$ and which are characterized, in particular,
as contractive multipliers from $H^2_D$ to $H^2_{E}$: a function $S : \mathbb{D} \to \mathcal{L}(U, \mathcal{Y})$ belongs to $\mathcal{S}(U, \mathcal{Y})$ if and only if the associated multiplication operator

$$M_S : f \to Sf$$

is a contraction from $H^2_D$ to $H^2_{E}$. The latter property translates to the de Branges-Rovnyak kernel

$$K_S(\lambda, \eta) = \frac{I_\mathcal{Y} - S(\lambda)S(\eta)^*}{1 - \lambda\bar{\eta}}$$

being a positive kernel on $\mathbb{D}$, i.e., $K_S$ has the property that

$$\sum_{i,j=1}^{N} (K_S(\lambda_i, \lambda_j)y_j, y_i) \geq 0$$

for all $\lambda_1, \ldots, \lambda_N \in \mathbb{D}, y_1, \ldots, y_N \in \mathcal{Y}$ and $N \geq 1$. This positive kernel in turn gives rise to a reproducing kernel Hilbert space $\mathcal{H}(S) := \mathcal{H}(K_S)$, the de Branges-Rovnyak space associated with $S$. Alternatively, $\mathcal{H}(S)$ can be defined as the range space $\text{Ran}(I - M_SM_S^*)$ with the lifted norm

$$\|(I - M_SM_S^*)f\|_{\mathcal{H}(S)} = \|(I - \pi)f\|_{H^2_{E}},$$

where $\pi$ is the orthogonal projection onto $\text{Ker}(I - M_SM_S^*)$.

Let us say that a pair $(E, T)$ with $E \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and $T \in \mathcal{L}(\mathcal{X})$ is output stable if the observability operator

$$O_{E,T} : x \mapsto E(I - \lambda T)^{-1}x = \sum_{k=0}^{\infty} \lambda^k ET^k x$$

maps $\mathcal{X}$ into $H^2_{E}$ and is bounded. Then a standard inner-product computation gives the formula for the adjoint operator

$$O_{E,T}^* f = \sum_{k=0}^{\infty} T^{*k} E^{*} f_k \text{ if } f(\lambda) = \sum_{k=0}^{\infty} f_k \lambda^k \in H^2_{\mathcal{Y}}$$

where the weak convergence of the operator series is guaranteed by the output-stability of the pair $(E, T)$. Thereby, any output stable pair $(E, T)$ gives rise to the left-tangential evaluation

$$f(\lambda) = \sum_{k=0}^{\infty} f_k \lambda^k \in H^2_{\mathcal{Y}},$$

which clearly makes sense for functions from the de Branges-Rovnyak space $\mathcal{H}(S) \subset H^2_{\mathcal{Y}}$ and extends to operator valued Schur-class functions $S \in \mathcal{S}(U, \mathcal{Y})$ via

$$(E^*S)^{\land L}(T^*) := O_{E,T}^* M_S|_{\mathcal{U}} = \sum_{k=0}^{\infty} T^{*k} E^{*} S_k \text{ if } S(\lambda) = \sum_{k=0}^{\infty} S_k \lambda^k.$$
If \( \dim \mathcal{X} = n \), \( \dim \mathcal{U} = \dim \mathcal{Y} = 1 \) and \( T \), \( E \) and \( x \) are chosen as in (1.2), then we have from (2.1)

\[
(E^* f)^\wedge L(T^*) = \sum_{k=0}^{\infty} \begin{bmatrix} \lambda_k^x \\ \vdots \\ \lambda_k^x \end{bmatrix} f_k = \begin{bmatrix} f(\lambda_1) \\ \vdots \\ f(\lambda_n) \end{bmatrix}
\]

from which we see that condition (2.2) collapses to the \( n \) conditions in (1.1).

The general case is handled as follows. We introduce \( N \in \mathcal{L}(\mathcal{X}, \mathcal{U}) \) and \( P \in \mathcal{L}(\mathcal{X}) \) by

\[
N^* := (E^* S_0)^\wedge L(T^*), \quad P := \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T}.
\]

It is not hard to show that since \( S_0 \) is a Schur-class function, the pair \((N, T)\) is output stable and the associated observability operator \( \mathcal{O}_{N,T} : \mathcal{X} \to \mathcal{H}_U^2 \) satisfies

\[
\mathcal{O}_{E,T}^* M_{S_0} = \mathcal{O}_{N,T}^*.
\]

Therefore, the operator \( P \) in (2.3) is bounded and positive semidefinite:

\[
P = \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T} = \mathcal{O}_{E,T}^* (I - M_{S_0} M_{S_0}^*) \mathcal{O}_{E,T} \succeq 0.
\]

It was shown in [18] that the problem (2.2) has a solution if and only if \( x \in \text{Ran} \ P^\frac{1}{2} \).

If \( P > 0 \) is strictly positive definite (i.e., positive semidefinite and boundedly invertible), one can construct an operator-valued function

\[
\Theta(\lambda) = \begin{bmatrix} \Theta_{11}(\lambda) & \Theta_{12}(\lambda) \\ \Theta_{21}(\lambda) & \Theta_{22}(\lambda) \end{bmatrix} : \mathcal{Y} \rightarrow \mathcal{Y}, \quad \mathcal{U} \rightarrow \mathcal{U} \quad (\lambda \in \mathbb{D})
\]

satisfying the identity

\[
\frac{J_{Y,U} - \Theta(\lambda) J_{Y,U} \Theta(\eta)^*}{1 - \lambda \eta} = \begin{bmatrix} E \\ N \end{bmatrix} (I - \lambda T)^{-1} P^{-1} (I - \eta T^*)^{-1} \begin{bmatrix} E^* \\ N^* \end{bmatrix}
\]

for all \( \lambda, \eta \in \mathbb{D} \), where \( J_{Y,U} \) denotes the signature operator

\[
J_{Y,U} := \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix}.
\]

The function \( \Theta \) is uniquely determined by (2.4) up to multiplication by a constant \( J_{Y,U} \)-unitary factor on the right. If \( 1 \notin \sigma(T^*) \), we can use formula (1.3) (with \( I_{\mathcal{X}} \) instead of \( I_{\mathcal{X}} \)). Otherwise, \( \Theta \) can be constructed via solving a certain Cholesky factorization problem as will be explained in the multivariable settings to come. It follows from (2.4) that

\[
J_{Y,U} - \Theta(\lambda) J_{Y,U} \Theta(\lambda)^* \succeq 0, \quad \lambda \in \mathbb{D}
\]

while a similar factorization as in (2.4) for \( \frac{J_{Y,U} - \Theta(\lambda) J_{Y,U} \Theta(\eta)^*}{1 - \lambda \eta} \) implies that

\[
J_{Y,U} - \Theta(\lambda)^* J_{Y,U} \Theta(\lambda) \succeq 0, \quad \lambda \in \mathbb{D}.
\]

Hence, we can conclude that \( \Theta \) is \((J_{Y,U}, J_{Y,U})\)-bi-contractive in \( \mathbb{D} \). Therefore, the formula

\[
\mathcal{E} \mapsto \mathcal{E}_\Theta[\mathcal{E}] := (\Theta_{11} \mathcal{E} + \Theta_{12}(\Theta_{21} \mathcal{E} + \Theta_{22}))^{-1}
\]

makes sense for all \( \mathcal{E} \in \mathcal{S}(\mathcal{U}, \mathcal{Y}) \) and defines a linear fractional map of \( \mathcal{S}(\mathcal{U}, \mathcal{Y}) \) into itself. Its range coincides with the solution set of a certain Schur class \( \mathcal{O}_{\mathcal{A}} \mathcal{P} \).

The following result has been established via various methods, cf., [34] [11] [35], including several where the de Branges-Rovnyak spaces played a prominent role. Let us mention that [11] as well as [20] consider the more general Bitangential
Operator-Argument interpolation Problem (with norm constraint), where a tangential operator-argument interpolation condition is considered both from the left and the right simultaneously.

**Theorem 2.1.** Given output stable pairs \((E,T)\) and \((N,T)\) with \(E \in \mathcal{L}(X,Y)\) and \(N \in \mathcal{L}(X,U)\), there exists a function

\[
S \in \mathcal{S}(U,Y) \quad \text{such that} \quad (E^*S)^{\perp L}(T^*) = N^*
\]

if and only if \(P := O_{E,T}^*O_{E,T} - O_{N,T}^*O_{N,T} \geq 0\). Furthermore, if \(P \succ 0\), then the solution set of the problem \((2.6)\) is parametrized by the formula

\[
S = \Sigma_0[E] := (\Theta_{11}E + \Theta_{12})(\Theta_{21}E + \Theta_{22})^{-1}
\]

with free parameter \(E \in \mathcal{S}(U,Y)\), where \(\Theta\) is a \((J_{Y,U},J_{Y,U})\)-contractive function subject to the identity \((2.4)\).

Returning to \(\text{OAP}_{\mathcal{H}(S_0)}\), in which \(S_0\) is now a given element of the Schur class \(\mathcal{S}(U,Y)\) and \(N\) is defined by \((2.3)\), it follows from the first formula in \((2.3)\) that \((2.7)\) holds with \(S = S_0\), so that \(S_0 = \Sigma_0[E_0]\) for some \(E_0\) in \(\mathcal{S}(U,Y)\).

We now present the solution to \(\text{OAP}_{\mathcal{H}(S_0)}\) for the case where \(P \succ 0\).

**Theorem 2.2.** The \(\text{OAP}_{\mathcal{H}(S_0)}\) has a solution if and only if \(x \in \text{Ran} P^\perp\). Assume that \(P \succ 0\). Let \(\Theta\) is a function satisfying \((2.4)\) and let \(E_0\) be a Schur-class function such that \(S_0 = \Sigma_0[E_0]\). Then

1. All solutions \(f\) of the \(\text{OAP}_{\mathcal{H}(S_0)}\) are parametrized by the formula

\[
(2.8) \quad f(\lambda) = f_0(\lambda) + (\Theta_{11}(\lambda) - S_0(\lambda)\Theta_{21}(\lambda))h(\lambda)
\]

where

\[
f_0(\lambda) = (E - S_0(\lambda)N)(I - \lambda T)^{-1}P^{-1}x
\]

and where \(h\) is a free parameter from the space \(\mathcal{H}(E_0)\).

2. The representation \((2.8)\) is orthogonal in the metric of \(\mathcal{H}(S_0)\); hence \(f_0\) is the minimal-norm solution to the \(\text{OAP}_{\mathcal{H}(S_0)}\).

3. The multiplication operator

\[
M_{\Theta_{11} - S_0\Theta_{21}} : \mathcal{H}(E_0) \rightarrow \mathcal{H}(S_0)
\]

is a partial isometry.

If in addition the operator \(T\) satisfies the condition

\[
(2.10) \quad \left( \bigcap_{k \geq 1} \text{Ran}(T^*)^k \right) \cap \text{Ker}(T^*) = \{0\},
\]

then

(a) The transformation \(\Sigma_0 : \mathcal{S}(U,Y) \rightarrow \mathcal{S}(U,Y)\) is injective and hence, \(S_0 = \Sigma_0[E_0]\) for a unique function \(E_0 \in \mathcal{S}(U,Y)\).

(b) The multiplication operator \((2.8)\) is an isometry and hence, for every \(f\) of the form \((2.8)\),

\[
\|f\|_{\mathcal{H}(S_0)}^2 = \|f_0\|_{\mathcal{H}(S_0)}^2 + \|f_{\perp}\|_{\mathcal{H}(S_0)}^2 = \|P^{-\frac{1}{2}}x\|^2_Y + \|h\|_{\mathcal{H}(E_0)}^2.
\]

The first part in Theorem 2.2 can be considered as a special case of Theorem 4.1 and Proposition 4.2 in [18], where the more general Abstract Interpolation Problem \(\text{AIP}_{\mathcal{H}(S_0)}\) was considered. The results in [18] cover the general case when \(P \preceq 0\).
is not necessarily strictly positive definite and rely on a Redheffer linear-fractional representation

\[ S_0 = \Sigma_{11} + \Sigma_{12} E_0 (I - \Sigma_{22} E_0)^{-1} \Sigma_{21} \]

of \( S_0 \) in terms of Schur-class functions \( E \) and \( \Sigma = [\Sigma_{11} \, \Sigma_{12}] \), which, however, can be written in terms of the chain-matrix linear-fractional formula (2.7) if \( P > 0 \). The “additional” part of the theorem follows from [15, Theorem 4.8]. Note that condition (2.10) holds automatically if \( \dim \mathcal{X} < \infty \) (in particular, for the Nevanlinna-Pick problem considered in Section 1). Also it holds if \( T^* \) is nilpotent or if \( \text{Ker}(T^*) = \{0\} \). Let us also mention that there is now a generalization of Theorem 2.2 (specialized to the scalar-valued setting) to the case where the de Branges-Rovnyak-type spaces are specified as sub-Bergman rather than sub-Hardy spaces (see [10]).

2.1. \( \text{OAP}_{H(S_0)} \) as a generalization of the Beurling-Lax theorem. As explained in Theorem 6.2 of [15] but in the context of a more general Abstract Interpolation Problem, one can view Theorem 2.2 as a generalization of the Beurling-Lax theorem (see e.g., [10]). Recall that any vector-valued Hardy space \( H_X^2 \) is equipped with the shift operator \( S_Y : f(\lambda) \mapsto \lambda f(\lambda) \). The Beurling-Lax theorem characterizes subspaces \( \mathcal{M} \subset H_X^2 \) which are invariant under \( S_Y \) as those which have a representation of the form \( B \cdot H_X^0 \), for some coefficient Hilbert space \( \mathcal{Y}_0 \) with \( \dim \mathcal{Y}_0 \leq \dim \mathcal{X} \) and \( B \) an inner function on the disk \( \mathbb{D} \) with values in \( \mathcal{L}(\mathcal{Y}_0, \mathcal{Y}) \) (i.e., such that the multiplication operator \( M_B : h(\lambda) \mapsto B(\lambda) h(\lambda) \) is an isometry from \( H_{\mathcal{Y}_0}^2 \) into \( H_{\mathcal{Y}}^2 \)).

To make the connection between Theorem 2.2 and the Beurling-Lax theorem, let us consider the special case of Theorem 2.2 where the coefficient space \( \mathcal{U} \) is taken to be the zero space \( \{0\} \) and hence the Schur class \( S(\{0\}, \mathcal{Y}) \) consists only of the zero function \( S_0(z) = 0 : \{0\} \rightarrow \mathcal{Y} \). Notice that \( \Sigma_{12} \) is also 0, the signature matrix (2.4) collapses to \( J = I_{\mathcal{Y}_0} \), the \( J \)-bi-inner function \( \Theta \) collapses to the \( J \)-inner function \( \Theta_{11} \) with values in \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \), the parameter function \( E_0 \) such that \( S_0 = \mathcal{X}_0[E_0] = \Theta_{11} E_0 \) is the zero function \( E_0(\lambda) : \{0\} \rightarrow \mathcal{Y} \), the de Branges-Rovnyak space \( \mathcal{H}(S_0) \) collapses to the Hardy space \( H_{\mathcal{Y}_0}^2 \). Let us take \( x = 0 \) so as to focus on the homogeneous problem. We are left with

\( \text{OAP}_{H_X^2} : \) Given an output stable pair \((E, T)\) with \( E \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) and \( T \in \mathcal{L}(\mathcal{X}) \), find all functions \( f \in H_X^2 \) so that

\[ (E^* f)^{\ast L}(T^*) := \mathcal{O}_{E,T}^* f = 0. \]

Let \( \mathcal{M}_{E,T} \subset H_X^2 \) denote the set of all solutions. From the intertwining property of \( \mathcal{O}_{E,T} \)

\[ S_Y^* \mathcal{O}_{E,T} = \mathcal{O}_{E,T} S_T, \]

we see that

\[ \mathcal{O}_{E,T}^* S_Y = T^* \mathcal{O}_{E,T}^* \]

from which it follows immediately that the subspace \( \mathcal{M}_{E,T} \) is \( S_Y \)-invariant. As a consequence of items (1) and (3) in Theorem 2.2 we see that

\[ \mathcal{M}_{E,T} = \{ \Theta_{11} h : h \in H_{\mathcal{U}}^2 \} \]

where \( \Theta_{11} \) is inner, i.e., \( M_{\Theta_{11}} : H_{\mathcal{U}}^2 \rightarrow H_X^2 \) is an isometry. Thus \( \Theta_{11} \) serves as a Beurling-Lax representer for the shift-invariant subspace \( \mathcal{M}_{E,T} \).

To get a proof of the Beurling-Lax theorem in general from this analysis, it remains only to argue that every shift-invariant subspace \( \mathcal{M} \) of \( H_X^2 \) has the form
$\mathcal{M} = \mathcal{M}_{E,T}$ for some output stable pair $(E,T)$. But this is well known from operator-model theory ideas as follows. Let us introduce the notation $\text{ev}_0$ for the operator from $H^2_\mathcal{Y}$ to $\mathcal{Y}$ given by evaluation at $0$: $\text{ev}_0: f \mapsto f(0)$. View the pair of operators

$$(\text{ev}_0, S^\mathcal{Y}_\mathcal{Y}) \in \mathcal{L}(H^2_\mathcal{Y}, \mathcal{Y}) \times \mathcal{L}(H^2_\mathcal{Y})$$

as an output pair and form the observability operator $O_{\text{ev}_0, S^\mathcal{Y}_\mathcal{Y}}: H^2_\mathcal{Y} \to H^2_\mathcal{Y}$. The computation

$$O_{\text{ev}_0, S^\mathcal{Y}_\mathcal{Y}}: f \mapsto \sum_{n=0}^\infty (\text{ev}_0 S^\mathcal{Y}_\mathcal{Y}^n f) \lambda^n = \sum_{n=0}^\infty f_n \lambda^n = f(\lambda)$$

shows that $O_{\text{ev}_0, S^\mathcal{Y}_\mathcal{Y}}$ is the identity operator on $H^2_\mathcal{Y}$. Now consider the restricted output pair $(E, T) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{X})$ given by

$$\mathcal{X} = \mathcal{M}^\perp, \quad E = \text{ev}_0|_\mathcal{X}, \quad T = S^\mathcal{Y}_\mathcal{Y}|_\mathcal{X}.$$  

A consequence of the preceding discussion is that $\mathcal{M}^\perp = \text{Ran} \mathcal{O}_{E,T}$. Hence $\mathcal{M} = (\mathcal{M}^\perp)^\perp$ is given by $\mathcal{M} = \text{Ker} \mathcal{O}_{E,T}$ as wanted.

In conclusion, we can view Theorem 2.2 as a generalization of the Beurling-Lax theorem to a more general non-homogeneous context.

3. THE FOCK SPACE SETTING

In this section we present the analogue of the single-variable results of Section 2 for multipliers between two Fock spaces. To define the Fock space, we let $\mathbb{F}^+_d$ denote the unital free semigroup (i.e., monoid) generated by the set of $d$ letters $\{1, \ldots, d\}$. Elements of $\mathbb{F}^+_d$ are words of the form $i_N \cdots i_1$ where $i_\ell \in \{1, \ldots, d\}$ for each $\ell \in \{1, \ldots, N\}$ with multiplication given by concatenation. The unit element of $\mathbb{F}^+_d$ is the empty word denoted by $\emptyset$. For $\alpha = i_N i_{N-1} \cdots i_1 \in \mathbb{F}^+_d$, we let $|\alpha|$ denote the number $N$ of letters in $\alpha$. Furthermore, we define the reversal of $\alpha$ to be the element $\alpha^\top \in \mathbb{F}^+_d$ given by

$$(3.1) \quad \alpha^\top = i_1 \cdots i_N \quad \text{if} \quad \alpha = i_N \cdots i_1.$$ 

We let $z = (z_1, \ldots, z_d)$ to be a collection of $d$ formal noncommuting variables and given a Hilbert space $\mathcal{Y}$, let $\mathcal{Y}(z)$ denote the set of noncommutative formal power series $\sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha$ where $f_\alpha \in \mathcal{Y}$ and where

$$(3.2) \quad z^\alpha = z_{i_N} \cdots z_{i_1} \quad \text{if} \quad \alpha = i_N \cdots i_1.$$ 

The Fock space $H^2_\mathcal{Y}(\mathbb{F}^+_d)$ is the Hilbert space given by

$$(3.3) \quad H^2_\mathcal{Y}(\mathbb{F}^+_d) = \left\{ f(z) = \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha \in \mathcal{Y}(z) : \|f\|_{H^2_\mathcal{Y}(\mathbb{F}^+_d)}^2 := \sum_{\alpha \in \mathbb{F}^+_d} \|f_\alpha\|_{\mathcal{Y}}^2 < \infty \right\}$$

with inner product

$$(f, g)_{H^2_\mathcal{Y}(\mathbb{F}^+_d)} := \sum_{\alpha \in \mathbb{F}^+_d} g_\alpha f_\alpha \quad \text{for} \quad f(z) = \sum_{\alpha \in \mathbb{F}^+_d} f_\alpha z^\alpha, \quad g(z) = \sum_{\alpha \in \mathbb{F}^+_d} g_\alpha z^\alpha \in H^2_\mathcal{Y}(\mathbb{F}^+_d),$$

and is equipped with the tuple $\mathbf{R}_z$ of right coordinate-variable multipliers

$$(3.4) \quad \mathbf{R}_z = (R_{z_1}, \ldots, R_{z_d}), \quad R_{sz_j} : f(z) \mapsto f(z)z_j.$$ 

The noncommutative Schur class $\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ is then defined as the set of contractive multipliers from $H^2_\mathcal{Y}(\mathbb{F}^+_d)$ to $H^2_\mathcal{Y}(\mathbb{F}^+_d)$, i.e., formal power series $S(z) = \sum_{\alpha \in \mathbb{F}^+_d} S_\alpha z^\alpha$ is contractive if

$$(3.5) \quad \sum_{\alpha \in \mathbb{F}^+_d} \|S_\alpha\|_{\mathcal{Y}}^2 < \infty.$$
we define the noncommutative observability operator

\[ O(z) = \sum_{\alpha \in \mathbb{F}_d^+} S_{\alpha} z^\alpha \]  

with coefficients \( S_{\alpha} \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \) such that the associated multiplication operator

\[ M_S : u(z) = \sum_{\alpha \in \mathbb{F}_d^+} u_{\alpha} z^\alpha \mapsto S(z)u(z) = \sum_{\alpha \in \mathbb{F}_d^+} \left( \sum_{\beta \gamma = \alpha} S_{\beta \gamma} u_{\gamma} \right) z^\alpha \]

is a contraction from \( H^2_\alpha(\mathbb{F}_d^+) \) to \( H^2_\alpha(\mathbb{F}_d^+) \). Similar to the single-variable case, a contraction operator \( \Psi : H^2_\alpha(\mathbb{F}_d^+) \to H^2_\alpha(\mathbb{F}_d^+) \) equals \( M_S \) for some \( S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y}) \) if and only if

\[ (R_z \otimes I_\mathcal{Y}) \Psi = \Psi(R_z \otimes I_\mathcal{U}) \quad \text{for} \quad j = 1, \ldots, d. \]

Given \( S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y}) \), the associated de Branges-Rovnyak space \( \mathcal{H}(S) \) is defined as the range space

\[ \mathcal{H}(S) = \text{Ran}(I - M_S M_S^*) \]

with the lifted norm defined by

\[ \|(I - M_S M_S^*)h\|_{\mathcal{H}(K_{d_2})}^2 = \langle (I - M_S M_S^*)h, h \rangle_{H^2_\alpha(\mathbb{F}_d^+)}. \]

This space also can be viewed as a reproducing kernel Hilbert space, but in the formal noncommutative sense of [24], which we recall in Subsection 3.2.

### 3.1. Interpolation problem

Extending the noncommutative functional calculus [3.2] to a \( d \)-tuple of operators \( T = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d \) by

\[ T^\alpha := T_{i_1} T_{i_{N-1}} \cdots T_{i_1} \quad \text{if} \quad \alpha = i_1 i_{N-1} \cdots i_1 \in \mathbb{F}_d^+, \]

and letting

\[ Z(z) = [z_1 \quad \cdots \quad z_d] \otimes I_\mathcal{X} \quad \text{and} \quad T = \begin{bmatrix} T_1 \\ \vdots \\ T_d \end{bmatrix}, \]

we define the noncommutative observability operator \( O_{E,T} \) of the output pair \( (E, T) \) with \( E \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) by the formula

\[ O_{E,T} : x \mapsto \sum_{\alpha \in \mathbb{F}_d^+} (ET^\alpha x) z^\alpha = E(I_X - Z(z)T)^{-1} x \]

and say that the pair \( (E, T) \) is output stable if the operator \( O_{E,T} : \mathcal{X} \to H^2_\alpha(\mathbb{F}_d^+) \) is bounded. In this case, we introduce the observability gramian

\[ G_{E,T} := O_{E,T}^* O_{E,T} = \sum_{\alpha \in \mathbb{F}_d^+} T^{*\alpha} E^* E T^\alpha \]

and note that the strong convergence of the series in (3.11) follows from the power-series expansion (3.10) for the observability operator together with the characterization (3.3) of the \( H^2_\alpha(\mathbb{F}_d^+) \)-norm.

For an output-stable pair \( (E, T) \) as above, we define a left-tangential functional calculus \( f \mapsto (E^* f)^{\wedge L}(T^*) \) on \( H^2_\alpha(\mathbb{F}_d^+) \) by

\[ (E^* f)^{\wedge L}(T^*) = \sum_{\alpha \in \mathbb{F}_d^+} T^{*\alpha} E^* f_\alpha \quad \text{if} \quad f = \sum_{\alpha \in \mathbb{F}_d^+} f_{\alpha} z^\alpha \in H^2_\alpha(\mathbb{F}_d^+). \]
The computation

\[ \sum_{\alpha \in \mathbb{F}_d^+} T^{*\alpha^T} E^* f_\alpha, x \bigg\} \mathcal{X} = \sum_{\alpha \in \mathbb{F}_d^+} \langle f_\alpha, E T^\alpha x \rangle \mathcal{Y} = \langle f, \mathcal{O}_{E,T} x \rangle_{H^2_d(\mathbb{F}_d^+)} \]

shows that the output-stability of the pair \((E,T)\) is exactly what is needed to verify that the infinite series in the definition (3.12) of \((E^* f)^{\wedge L}(T^*)\) converges in the weak topology on \(\mathcal{X}\). In fact the left-tangential evaluation with operator argument \(f \rightarrow (E^* f)^{\wedge L}(T^*)\) amounts to the adjoint of the observability operator:

\[ (E^* f)^{\wedge L}(T^*) = \mathcal{O}^*_{E,T} f^* \quad \text{for} \quad f \in H^2_d(\mathbb{F}_d^+). \]

The latter evaluation makes sense for all \(f \in \mathcal{H}(S) \subset H^2_d(\mathbb{F}_d^+)\) and on the other hand, extends to contractive multipliers \(S \in S_{nc,d}(\mathcal{U}, \mathcal{Y})\) by

\[ (E^* S)^{\wedge L}(T^*) = \mathcal{O}^*_{E,T} S \mid \mathcal{U} : \mathcal{U} \rightarrow \mathcal{X} \]

with \(\mathcal{U}\) in the restriction to be identified with the constant \(\mathcal{U}\)-valued functions in \(H^2_d(\mathbb{F}_d^+)\). The Operator Argument interpolation Problem in the de Branges-Rovnyak space \(\mathcal{H}(S)\) is formulated similarly to its single-variable prototype.

**OAP}_\mathcal{H}(S_0):** Given an output stable pair \((E,T) \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \times \mathcal{L} (\mathcal{X})^d\), a vector \(x \in \mathcal{X}\) and a Schur-class multiplier \(S_0 \in S_{nc,d}(\mathcal{U}, \mathcal{Y})\), find all \(f \in \mathcal{H}(S_0)\) such that

\[ (E^* f)^{\wedge L}(T^*) := \mathcal{O}^*_{E,T} f = x. \]

With a given output stable pair \((E,T)\) and a Schur-class multiplier

\[ S_0(z) = \sum_{\alpha \in \mathbb{F}_d^+} S_{0,\alpha} z^\alpha, \]

we associate the operator \(N \in \mathcal{L}(\mathcal{X}, \mathcal{U})\) defined by the formula

\[ N := (E^* S_0)^{\wedge L}(T^*) = \sum_{\alpha \in \mathbb{F}_d^+} S_{0,\alpha} E T^\alpha \]

or, equivalently, via its adjoint

\[ N^* = \mathcal{O}^*_{E,T} S_0 \mid \mathcal{U} : \mathcal{U} \rightarrow \mathcal{X}. \]

By [12, Proposition 4.1], the pair \((N,T)\) is output stable and the equality

\[ \mathcal{O}_{E,T} S_0 = \mathcal{O}^*_{N,T} \]

holds. Therefore,

\[ P := \mathcal{O}_{E,T} \mathcal{O}_{E,T} - \mathcal{O}_{N,T} \mathcal{O}_{N,T} = \mathcal{O}_{E,T}(I - M_{S_0} M_{S_0}^*) \mathcal{O}_{E,T} \succeq 0. \]

Using the formula (3.11) for \(\mathcal{O}_{E,T} \mathcal{O}_{E,T}\) and a similar formula for the gramian \(\mathcal{O}_{N,T} \mathcal{O}_{N,T}\) we get the infinite series representation

\[ P = \sum_{\alpha \in \mathbb{F}_d^+} T^{*\alpha^T} (E^* E - N^* N) T^\alpha, \]

from which it is easy to verify that \(P\) satisfies the Stein identity

\[ P - \sum_{j=1}^d T_j^* P T_j = E^* E - N^* N. \]
We next introduce the formal power series
\begin{equation}
F_{S_0}(z) = (E - S_0(z)N)(I - Z(z)T)^{-1}.
\end{equation}

\textbf{Lemma 3.1.} Let $P$ be defined as in (3.18) and $F_{S_0}$ as in (3.20). Then for any $x \in \mathcal{X}$, we have that $F_{S_0}x := F_{S_0}(.)x$ is in $H(S_0)$ and
\begin{equation}
\|F_{S_0}x\|_{H(S_0)} = \|P^\frac{1}{2}x\|_X.
\end{equation}

\textbf{Proof.} Indeed, by (3.20) and (3.17), the multiplication operator $M_{F_{S_0}}$ can be written in terms of observability operators as
\begin{equation}
M_{F_{S_0}} = \mathcal{O}_{E,T} - M_{S_0}O_{N,T} = (I_X - M_{S_0}M^*_0)\mathcal{O}_{E,T}
\end{equation}
which together with the range characterization (3.7) of $H(S_0)$ implies that $M_{F_{S_0}}$ maps $\mathcal{X}$ into $H(S_0)$. We now apply equality (3.8) to $h = \mathcal{O}_{E,T}x$ and make use of (3.18) to conclude
\begin{equation}
\|F_{S_0}x\|_{H(S_0)}^2 = \langle (I - M_{S_0}M^*_0)\mathcal{O}_{E,T}x, \mathcal{O}_{E,T}x \rangle_{H^2_d(\mathbb{F}_d^+)} = \langle P^\frac{1}{2}x, x \rangle_X
\end{equation}
which is equivalent to (3.21). \hfill $\Box$

\textbf{Remark 3.2.} For the operator $M_{F_{S_0}} : \mathcal{X} \to H(S_0) \subset H^2_d(\mathbb{F}_d^+)$ we will write $M^*_{F_{S_0}}$ for its adjoint in the metric of $H^2_d(\mathbb{F}_d^+)$ and we will denote by $M_{F_{S_0}}^\dagger$ the adjoint of $M_{F_{S_0}}$ in the metric of $\mathcal{H}(S_0)$. Note that these two adjoints are not the same unless $S_0$ is inner (i.e., the multiplication operator $M_{S_0} : H^2_d \to H^2_d$ is an isometry).

\textbf{Lemma 3.3.} Let $M_{F_{S_0}} : \mathcal{X} \to H(S_0)$ be defined as in (3.22). Then
\begin{equation}
M_{F_{S_0}}^\dagger = \mathcal{O}_{E,T}^*|_{H(S_0)} \quad \text{and} \quad M_{F_{S_0}}^\dagger M_{F_{S_0}} = P.
\end{equation}

\textbf{Proof.} The first formula is justified by the equalities
\begin{equation}
\langle M_{F_{S_0}}^\dagger f, x \rangle_X = \langle f, M_{F_{S_0}}x \rangle_{H(S_0)} = \langle f, (I - M_{S_0}M^*_0)\mathcal{O}_{E,T}x \rangle_{H(S_0)} = \langle f, \mathcal{O}_{E,T}x \rangle_{H^2_d(\mathbb{F}_d^+)} = \langle \mathcal{O}_{E,T}^*f, x \rangle_X
\end{equation}
holding for all $f \in H(S_0)$ and $x \in \mathcal{X}$. The second equality follows from (3.21). \hfill $\Box$

\textbf{Corollary 3.4.} The $\text{OAP}_H(S_0)$ has a solution if and only if $x \in \text{Ran} P^\frac{1}{2}$.

Indeed, by the first formula in (3.23), the interpolation condition (3.14) can be written as
\begin{equation}
M_{F_{S_0}}^\dagger f = x.
\end{equation}
Combining the latter condition with the second equality in (3.23) we see that the problem has a solution $f$ if and only if $x \in \text{Ran} M_{F_{S_0}}^\dagger = \text{Ran} P^\frac{1}{2}$.

3.2. Noncommutative formal reproducing kernel Hilbert spaces. To proceed we need formal positive kernels and associated reproducing kernel Hilbert spaces. Here we briefly recall the basic concepts and refer for more complete details to [24] and as well as [13 Section 2.1].

Given a coefficient Hilbert space $\mathcal{Y}$, we define the space $L(\mathcal{Y})((z,\bar{\zeta}))$ to consist of all formal power series (formal kernels)
\begin{equation}
K(z,\zeta) = \sum_{\alpha,\beta \in \mathbb{F}_d^+} K_{\alpha,\beta} z^\alpha \bar{\zeta}^\beta, \quad K_{\alpha,\beta} \in L(\mathcal{Y}),
\end{equation}
in two collections of formal noncommuting indeterminates \( z = (z_1, \ldots, z_d) \) and \( \zeta = (\zeta_1, \ldots, \zeta_d) \) such that each \( z_k \) commutes with each \( \zeta_j \). Here \( \beta^\top \) is defined as in (3.1). In the algebraic manipulations to follow we use the formal notations

\[
\zeta_j = \zeta_j, \quad (\zeta^\alpha)^\top = \zeta^\alpha^\top.
\]

The kernel \( K \) in (3.25) is called positive if for any finitely supported \( \mathcal{Y} \)-valued function \( \alpha \mapsto y_\alpha \) on \( \mathbb{F}_d^+ \),

\[
\sum_{\alpha, \beta \in \mathbb{F}_d^+} \langle K_{\alpha, \beta}, y_\alpha \rangle_{\mathcal{Y}} \geq 0.
\]

Such kernels are characterized by the existence of a Kolmogorov decomposition

\[
K(z, \zeta) = H(z) H(\zeta)^\top
\]

with \( H \) some operator-valued formal power series \( H \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \langle \langle z \rangle \rangle \), for some Hilbert space \( \mathcal{X} \), that is, \( H(z) \) is of the form

\[
H(z) = \sum_{\alpha \in \mathbb{F}_d^+} H_\alpha z^\alpha \quad \text{with } H_\alpha \in \mathcal{L}(\mathcal{X}, \mathcal{Y})\langle \langle z \rangle \rangle.
\]

We say that a Hilbert space \( \mathcal{H} \) consisting of power series \( f \) with coefficients in \( \mathcal{Y} \)

\[
f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \in \mathcal{Y}\langle \langle z \rangle \rangle,
\]

is a noncommutative formal reproducing kernel Hilbert space (NFRKHS) if, for each \( \beta \in \mathbb{F}_d^+ \), the linear operator \( \Phi_\beta: \mathcal{H} \to \mathcal{Y} \) defined by

\[
(3.26) \quad \Phi_\beta: f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \mapsto f_\beta
\]

is continuous.

To explain the reproducing-kernel property for this setting, we need to introduce more general formal inner-product pairings as follows. Let \( \zeta = (\zeta_1, \ldots, \zeta_d) \) be a second \( d \)-tuple of formal noncommuting indeterminates with properties as discussed in the previous paragraph. Let us introduce, for a general Hilbert space \( \mathcal{C} \), formal pairings

\[
\langle \cdot, \cdot \rangle_{\mathcal{C} \times \mathcal{C} \langle \langle \zeta \rangle \rangle} \to \mathcal{C} \langle \langle \zeta \rangle \rangle, \quad \langle \cdot, \cdot \rangle_{\mathcal{C} \langle \langle \zeta \rangle \rangle \times \mathcal{C} \langle \langle \zeta \rangle \rangle} \to \mathcal{C} \langle \langle \zeta \rangle \rangle
\]

by

\[
\langle c, \sum_{\beta \in \mathbb{F}_d^+} c_\beta \zeta^\beta^\top \rangle_{\mathcal{C} \times \mathcal{C} \langle \langle \zeta \rangle \rangle} = \sum_{\beta \in \mathbb{F}_d^+} \langle c, c_\beta \rangle_{\mathcal{C}} \zeta^\beta, \quad \langle \sum_{\alpha \in \mathbb{F}_d^+} c_\alpha \zeta^\alpha, c' \rangle_{\mathcal{C} \langle \langle \zeta \rangle \rangle \times \mathcal{C}} = \sum_{\alpha \in \mathbb{F}_d^+} \langle c_\alpha, c' \rangle_{\mathcal{C}} \zeta^\alpha.
\]

In particular, we have

\[
\langle c, \zeta_\beta \zeta^\beta \rangle_{\mathcal{C} \times \mathcal{C} \langle \langle \zeta \rangle \rangle} = \langle c, \zeta_\beta \rangle_{\mathcal{C}} \cdot \zeta^\beta.
\]

Now we assume that \( \mathcal{H} \) is a NFRKHS. With \( \Phi_\beta \) as defined in (3.26), we set

\[
H(z) = \sum_{\alpha \in \mathbb{F}_d^+} \Phi_\beta z^\alpha \quad \text{so that } \quad H(\zeta)^\top = \sum_{\beta \in \mathbb{F}_d^+} \Phi_\beta^\top \zeta^\beta^\top.
\]
Then for each $y \in \mathcal{Y}$ by definition $\Phi^*_\beta y \in \mathcal{H}$ and we can write

$$H(\zeta)^* y = \sum_{\beta \in \mathcal{F}^+_d} (\Phi^*_\beta y) \zeta^\top \in \mathcal{H}(\zeta).$$

For $f \in \mathcal{H}$ and $y \in \mathcal{Y}$ we can therefore compute the pairing

$$\langle f, H(\zeta)^* y \rangle_{\mathcal{H} \times \mathcal{H}(\zeta)} = \sum_{\beta \in \mathcal{F}^+_d} \langle f, \Phi^*_\beta y \rangle_{\mathcal{H}} \zeta^\beta = \sum_{\beta \in \mathcal{F}^+_d} \langle \Phi^*_\beta f, y \rangle_{\mathcal{Y}} \zeta^\beta = \left\langle \sum_{\beta \in \mathcal{F}^+_d} f \zeta^\beta, y \right\rangle_{\mathcal{Y}(\zeta) \times \mathcal{Y}}$$

where $\Phi^*_\beta y \in \mathcal{H}$ can be written more explicitly as the power series in $z$:

$$(\Phi^*_\beta y)(z) = \sum_{\alpha \in \mathcal{F}^+_d} (\Phi^*_\alpha \Phi^*_\beta y) z^\alpha.$$

If we now define $K(z, \zeta) \in \mathcal{L}(\mathcal{Y}(\zeta))$ by

$$K(z, \zeta) = \sum_{\alpha, \beta \in \mathcal{F}^+_d} \Phi^*_\alpha \Phi^*_\beta z^\alpha \zeta^\top = H(z) H(\zeta)^*,$$

then we observe that $K$ has the formal reproducing property for the NFRKHS $\mathcal{H}$:

(i) for each $y \in \mathcal{Y}$ we have $K(\cdot, \zeta) y \in \mathcal{H}(\zeta)$, and

(ii) for each $y \in \mathcal{Y}$ and $f \in \mathcal{H}$ we have $(f, K(\cdot, \zeta) y)_{\mathcal{H} \times \mathcal{H}(\zeta)} = \langle f(\zeta), y(Y(\zeta)) \rangle.$

In general, when $K$ and $\mathcal{H}$ are related in this way, we say that $K$ is the reproducing kernel for the NFRKHS $\mathcal{H}$ and we write $\mathcal{H} = \mathcal{H}(K)$.

**Example 3.5.** The Fock space $H^2_\mathcal{Y}(\mathcal{F}^+_d)$ introduced in [4,3] is the NFRKHS with reproducing kernel $k_{Sz}I_\mathcal{Y}$ where $k_{Sz}$ is the noncommutative Szegő kernel

$$(3.27) \quad k_{Sz}(z, \zeta) = \sum_{\alpha \in \mathcal{F}^+_d} z^\alpha \zeta^\top.$$

Indeed, for $f(z) = \sum_{\alpha \in \mathcal{F}^+_d} f_\alpha z^\alpha \in H^2_\mathcal{Y}(\mathcal{F}^+_d)$ and $y \in \mathcal{Y}$, we have

$$\langle f, k_{Sz}(\cdot, \zeta) y \rangle_{H^2_\mathcal{Y}(\mathcal{F}^+_d) \times H^2_\mathcal{Y}(\mathcal{F}^+_d)}(\zeta) = \sum_{\alpha \in \mathcal{F}^+_d} \langle f(z), y z^\alpha \rangle_{H^2_\mathcal{Y}(\mathcal{F}^+_d)} \zeta^\alpha = \sum_{\alpha \in \mathcal{F}^+_d} \langle f_\alpha, y \rangle \zeta^\alpha = \langle f(\zeta), y \rangle_{\mathcal{Y}(\zeta) \times \mathcal{Y}}.$$

Given two formal positive kernels $K$ and $K'$ with coefficients in $\mathcal{L}(\mathcal{Y})$ and $\mathcal{L}(\mathcal{U})$ respectively, together with the associated NFRKHSs $\mathcal{H}(K)$ and $\mathcal{H}(K')$, a formal power series $F \in \mathcal{L}(\mathcal{U}, \mathcal{Y})((z))$ is called a contractive multiplier from $\mathcal{H}(K')$ to $\mathcal{H}(K)$ if the left multiplication operator $M_F$ defined as in [4,3] is a contraction from $\mathcal{H}(K')$ to $\mathcal{H}(K)$. For details of the proof of the next result, we refer to Proposition 3.1.1 in [13].

**Proposition 3.6.** Let $K \in \mathcal{L}(\mathcal{Y})(\zeta, \zeta)$ and $K' \in \mathcal{L}(\mathcal{U})(\zeta, \zeta)$ be two positive formal kernels.
(1) A formal power series $F \in \mathcal{L}(\mathcal{U},\mathcal{Y})(z)$ is a \textit{contractive multiplier from} $\mathcal{H}(K')$ to $\mathcal{H}(K)$ if and only if
\[
K_F(z, \zeta) = K(z, \zeta) - F(z)K'(z, \zeta)F(\zeta)^* \in \mathcal{L}(\mathcal{Y})(z, \zeta)
\]
is a \textit{positive formal kernel}.

(2) $F$ is a \textit{coisometric multiplier from} $\mathcal{H}(K')$ to $\mathcal{H}(K)$ if and only if $K_F(z, \zeta) = 0$.

Specializing part (1) to the case where $K = k_{\mathcal{S}}I_{\mathcal{Y}}$ and $K' = k_{\mathcal{S}}I_{\mathcal{U}}$ gives the characterization of contractive multipliers from $\mathcal{H}_U(k_{\mathcal{S}}) = \mathcal{H}(k_{\mathcal{S}}I_{\mathcal{U}}) = H^2_2(\mathbb{F}_d^\to)$ to $\mathcal{H}_Y(k_{\mathcal{S}}) = \mathcal{H}(k_{\mathcal{S}}I_{\mathcal{Y}}) = H^2_2(\mathbb{F}_d^\leq)$ (i.e., the elements in $\mathcal{S}_{nc,\mathcal{d}}(\mathcal{U}, \mathcal{Y})$) in terms of the positive formal kernels: a formal power series $S$ belongs to the Schur class $\mathcal{S}_{nc,\mathcal{d}}(\mathcal{U}, \mathcal{Y})$ if and only if
\[
K_S(z, \zeta) := k_{\mathcal{S}}(z, \zeta)I_Y - S(z)(k_{\mathcal{S}}(z, \zeta)I_{\mathcal{U}})S(\zeta)^* \in \mathcal{L}(\mathcal{Y})(z, \zeta)
\]
is a \textit{positive formal kernel}.

One can show in much the same way as for the classical case that this FNRKHS $\mathcal{H}(K_S)$ with formal reproducing kernel (3.28) can also be viewed as the de Branges-Rovnyak space $\mathcal{H}(S) = \text{Ran}(I_{H^2_2(\mathbb{F}_d^\to)} - M_SM^*_S)\hat{\tau}$ with lifted norm given by (3.3).

### 3.3. Indefinite noncommutative Schur class

In order to describe the solutions to our interpolation problem $\text{OAP}_{\mathcal{H}(S_0)}$ we require Schur-class multipliers with respect to indefinite inner product spaces associated with two signature operators, which we define next. This subsection is partially a review of material from [12] and we refer there for further details and proofs. See the end of the introduction and Appendix A for more on indefinite inner product spaces, signature operators and the terminology used here. We are specifically interested in the case where the signature operators have the form
\[
J_{Y,\mathcal{U}} := \begin{bmatrix} I_Y & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix} \quad \text{or} \quad J_{Y,\mathcal{U}} := I_{H^2(\mathbb{F}_d^\to)} \otimes J_{Y,\mathcal{U}},
\]
for Hilbert spaces $\mathcal{Y}$ and $\mathcal{U}$.

**Definition 3.7.** Given coefficient Hilbert spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{F}$, the \textit{noncommutative indefinite Schur class} $\mathcal{S}_{nc,\mathcal{d}}(J_{Y,\mathcal{U}}, J_{\mathcal{F},\mathcal{U}})$ consists of formal power series
\[
\mathcal{A}(z) = \begin{bmatrix} \mathcal{A}_{11}(z) & \mathcal{A}_{12}(z) \\ \mathcal{A}_{21}(z) & \mathcal{A}_{22}(z) \end{bmatrix} \in \mathcal{L} \left( \begin{bmatrix} \mathcal{F} \\ \mathcal{U} \end{bmatrix}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \right) \left( (z) \right)
\]
such that the multiplication operator
\[
M_{\mathcal{A}} : (H^2_{\mathcal{F} \oplus \mathcal{U}}(\mathbb{F}_d^\to), J_{\mathcal{F},\mathcal{U}}) \to (H^2_{\mathcal{F} \oplus \mathcal{U}}(\mathbb{F}_d^\leq), J_{Y,\mathcal{U}})
\]
is a $(J_{\mathcal{F},\mathcal{U}}, J_{Y,\mathcal{U}})$-bi-contraction:
\[
M_{\mathcal{A}}^* J_{Y,\mathcal{U}} M_{\mathcal{A}} \leq J_{\mathcal{F},\mathcal{U}} \quad \text{and} \quad M_{\mathcal{A}} J_{Y,\mathcal{U}} M_{\mathcal{A}}^* \leq J_{Y,\mathcal{U}}.
\]

As the action of the operator $M_{\mathcal{A}}^* : H^2_{\mathcal{Y} \oplus \mathcal{U}}(\mathbb{F}_d^\to) \to H^2_{\mathcal{F} \oplus \mathcal{U}}(\mathbb{F}_d^\leq)$ on the kernel elements $k_{\mathcal{S}}(\cdot, \zeta)[u]$, for $y \in \mathcal{Y}$, $u \in \mathcal{U}$, is given by the formula
\[
M_{\mathcal{A}}^* k_{\mathcal{S}}(\cdot, \zeta)[u] = k_{\mathcal{S}}(\cdot, \zeta)\mathcal{A}(\zeta)^* [u],
\]
Proposition 3.8. (see the computation leading up to formula (3.3) in [13], it follows that

\[ k_{ Sz}(\cdot, \zeta) [ f' \ u'] M^*_{\mathfrak{A}} = k_{ Sz}(\cdot, \zeta) [ f' \ u'] \mathfrak{A}(\zeta)^* . \]

By linearity, the latter formula extends to linear combinations of kernel elements. Since the span of the kernel elements is dense in \( H^2_{\mathcal{Y} \oplus \mathcal{U}}(\mathbb{F}_d^J) \), we then see that the second condition in (3.30) (i.e., \( J_{ F, \mathcal{U}} - M_{\mathfrak{A}} J_{ F, \mathcal{U}} M^*_{\mathfrak{A}} \succeq 0 \)) is equivalent to the formal kernel

\[ K^{ J_{ F, \mathcal{U}} \cdot J_{ F, \mathcal{U}}}_M(z, \zeta) := k_{ Sz}(z, \zeta) J_{ F, \mathcal{U}} - \mathfrak{A}(z)(k_{ Sz}(z, \zeta) J_{ F, \mathcal{U}}) \mathfrak{A}(\zeta)^* \]

being positive. Therefore, the associated de Branges-Rovnyak space

\[ \mathcal{H}(K^{ J_{ F, \mathcal{U}} \cdot J_{ F, \mathcal{U}}}_M) \]

is a Hilbert space for any formal power series \( \mathfrak{A} \) satisfying just the second condition in (3.30).

To handle the first condition in (3.30), we use a duality trick as follows. To test for positivity of the operator \( J_{ F, \mathcal{U}} - M_{\mathfrak{A}} J_{ F, \mathcal{U}} M^*_{\mathfrak{A}} \), we let it act on the right on the space \( H^2_{\mathcal{Y} \oplus \mathcal{U}}(\mathbb{F}_d^J) \) consisting of formal power series with coefficients being vectors in the space \( \mathcal{F} \oplus \mathcal{U} \) of linear functionals acting on \( \mathcal{F} \oplus \mathcal{U} \) and resulting values in \( H^2_{\mathcal{Y} \oplus \mathcal{U}} \).

Again the operator \( M^*_{\mathfrak{A}} \) can be computed when acting on the right on a kernel element \( k_{ Sz}(\cdot, \zeta) [ f' \ u'] \), namely

\[ k_{ Sz}(\cdot, \zeta) [ f' \ u'] M^*_{\mathfrak{A}} = k_{ Sz}(\cdot, \zeta) [ f' \ u'] \mathfrak{A}(\zeta)^* . \]

Hence we can compute the operator explicitly on these dual kernel functions:

\[ k_{ Sz}(\cdot, \zeta) [ f' \ u'] (J_{ F, \mathcal{U}} - M^*_{\mathfrak{A}} J_{ F, \mathcal{U}} M_{\mathfrak{A}}) = k_{ Sz}(\cdot, \zeta) [ f' \ u'] J_{ F, \mathcal{U}} - k_{ Sz}(\cdot, \zeta) [ f' \ u'] \mathfrak{A}(\zeta)^* J_{ F, \mathcal{U}} \mathfrak{A}(\cdot) \]

By similar arguments as for the second condition in (3.30) it now follows that the operator-positivity condition \( J_{ F, \mathcal{U}} - M^*_{\mathfrak{A}} J_{ F, \mathcal{U}} M_{\mathfrak{A}} \succeq 0 \) is equivalent to the kernel

\[ K^{ J_{ F, \mathcal{U}} \cdot J_{ F, \mathcal{U}}}_M(\zeta, z) = k_{ Sz}(z, \zeta) J_{ F, \mathcal{U}} - \mathfrak{A}(\zeta)^*(k_{ Sz}(z, \zeta) J_{ F, \mathcal{U}}) \mathfrak{A}(z) \]

being positive. In conclusion, we have derived the following result.

**Proposition 3.8.** A given power series \( \mathfrak{A}(z) \in \mathcal{L}(\mathcal{F} \oplus \mathcal{U}, \mathcal{Y} \oplus \mathcal{U})\langle z, \bar{\zeta} \rangle \) is in the indefinite Schur class \( S^{ J_{ F, \mathcal{U}} \cdot J_{ F, \mathcal{U}}}_M \) if and only if both formal kernels \( K^{ J_{ F, \mathcal{U}} \cdot J_{ F, \mathcal{U}}}_M \) and \( K^{ J_{ F, \mathcal{U}} \cdot J_{ F, \mathcal{U}}}_M \) are positive.

\(^1\)This construct of letting an operator-valued function have both a right action on dual (row) vectors as well as a left action on (column) vectors comes up in the duality pairing between dual vector bundles used for the construction of an analogue of the Cauchy kernel for a vector bundle over an algebraic curve with a given determinantal representation (see [22] Proposition 2.2 and [23] formula (5.1)).
3.4. Linear-fractional maps associated with indefinite noncommutative Schur-class multipliers. We now describe how the indefinite noncommutative Schur-class multipliers described in the previous section can be used to define linear-fractional maps acting on a standard Schur class of multipliers. In the finite-dimensional, one-variable case, many of the arguments given here can also be found in [6], while for the infinite-dimensional case some pieces of these results can also be derived by applying the Potapov-Ginsburg transform to derive the corresponding results in the Redheffer (scattering) formalism (see [31, Theorem 1.3.4] and [12, Theorem 3.4]).

We start in a slightly more general setting. Given a block 2 \times 2 matrix formal power series

\[
A(z) = \begin{bmatrix}
A_{11}(z) & A_{12}(z) \\
A_{21}(z) & A_{22}(z)
\end{bmatrix} \in \mathcal{L} \left( \begin{bmatrix} F \\ U \end{bmatrix}, \begin{bmatrix} Y \\ U \end{bmatrix} \right)(\langle z \rangle)
\]

and a formal power series \( E(z) \in \mathcal{L}(U, F)(\langle z \rangle) \), let us say that \( E \) is in the domain of the linear fractional map \( \mathcal{T}_A \) if it is the case that \( A_{21}(z)E(z) + A_{22}(z) \) is invertible as a formal power series, i.e., its coefficient with index 0 is invertible in \( \mathcal{L}(U) \), in which case we define \( \mathcal{T}_A[E] \) to be the formal power series given by

\[
\mathcal{T}_A[E] = (A_{11}E + A_{12})(A_{21}E + A_{22})^{-1} \in \mathcal{L}(U, Y)(\langle z \rangle).
\]

For this discussion the following definition will be useful.

**Definition 3.9.** Let us say that the formal power series \( E \) is in the formal domain of \( \mathcal{T}_A \) if at least \( A_{21}E + A_{22} \) is invertible as a formal power series, thereby making \( \mathcal{T}_A[E] \) well defined as a formal power series. In case \( A \) is a multiplier, we say that \( E \) is in the multiplier domain of \( \mathcal{T}_A \) if \( E \) is a multiplier and \( A_{21}E + A_{22} \) is invertible as a multiplier on \( H_d^2(F^+) \).

Note that for the case where \( A \) is a multiplier, the multiplier domain of \( \mathcal{T}_A \) is contained in its formal domain, since any invertible multiplier is the multiplier arising from an invertible formal series.

In case \( A \) is an indefinite noncommutative Schur-class multiplier we get the following result.

**Theorem 3.10.** If \( A \in \mathcal{S}_{nc,d}(J_{y,\mathcal{U}}, J_{x,\mathcal{U}}) \) and \( E \in \mathcal{S}_{nc,d}(\mathcal{U}, F) \), then the linear fractional transform of \( E \)

\[
\mathcal{T}_A[E] = (A_{11}E + A_{12})(A_{21}E + A_{22})^{-1}
\]

is a well-defined element of \( \mathcal{S}_{nc,d}(\mathcal{U}, F) \). In particular, if \( A \in \mathcal{S}_{nc,d}(J_{y,\mathcal{U}}, J_{x,\mathcal{U}}) \) and \( E \in \mathcal{S}_{nc,d}(\mathcal{U}, F) \), then \( E \) is in the multiplier domain of \( \mathcal{T}_A \).

**Proof.** A consequence of the second relation in (3.30) is that

\[
M_{A_{21}}M_{A_{21}}^* - M_{A_{22}}^* M_{A_{22}} \preceq -I_{H_d^2(F^+)}
\]

which we can rearrange to

\[
I_{H_d^2(F^+)} \preceq M_{A_{21}}M_{A_{21}}^* + M_{A_{22}}^* M_{A_{22}} \preceq M_{A_{21}}M_{A_{21}}^* M_{A_{22}}^* M_{A_{22}}
\]

from which we conclude that \( M_{A_{22}} \) is surjective. From the first relation in (3.30) we get that

\[
M_{A_{12}}^* M_{A_{12}} - M_{A_{22}}^* M_{A_{22}} \preceq -I_{H_d^2(F^+)}
\]
which by parallel rearrangements as in the preceding argument leads to

$$I_{H^2_\mathbb{C}}(\mathcal{F}_d^+) \preceq I_{H^2_\mathbb{C}}(\mathcal{F}_d^+) + M_{\mathfrak{A}_{12}} M_{\mathfrak{A}_{12}} \preceq M_{\mathfrak{A}_{22}} M_{\mathfrak{A}_{22}}$$

leading to the conclusion that $M_{\mathfrak{A}_{22}}$ is also injective. We conclude that when both inequalities in (3.30) hold, it follows that $M_{\mathfrak{A}_{22}}$ is bijective with bounded inverse $M_{\mathfrak{A}_{22}}^{-1}$ given by $M_{\mathfrak{A}_{22}}^{-1} = M_{\mathfrak{A}_{22}}^{-1}$ (due to the characterization (3.6) of multiplier operators). Conjugating (3.35) by $M_{\mathfrak{A}_{22}}^{-1}$ then gives us

$$M_{\mathfrak{A}_{22}}^{-1} M_{\mathfrak{A}_{22}} (M_{\mathfrak{A}_{22}}^{-1})^* + M_{\mathfrak{A}_{22}}^{-1} (M_{\mathfrak{A}_{22}}^{-1})^* \preceq I_{H^2_\mathbb{C}}(\mathcal{F}_d^+)$$

from which we conclude that $\|M_{\mathfrak{A}_{22}}^{-1}\| < 1$ and hence also

$$\|M_{\mathfrak{A}_{22}}^{-1}\| < 1 \quad \text{for any} \quad \mathcal{E} \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{F}).$$

Then

$$I_{H^2_\mathbb{C}}(\mathcal{F}_d^+) + M_{\mathfrak{A}_{22}}^{-1} M_{\mathfrak{A}_{22}} \mathcal{E} = M_{\mathfrak{A}_{22}}^{-1} M_{\mathfrak{A}_{22}} + M_{\mathfrak{A}_{22}} \mathcal{E}$$

is invertible on $H^2_\mathbb{C}(\mathcal{F}_d^+)$, so that $M_{\mathfrak{A}_{22}}^{-1} M_{\mathfrak{A}_{22}} + M_{\mathfrak{A}_{22}} \mathcal{E}$ is also invertible on $H^2_\mathbb{C}(\mathcal{F}_d^+)$. Hence $\mathfrak{A}_{22} + \mathfrak{A}_{22} \mathcal{E}$ invertible as a multiplier on $H^2_\mathbb{C}(\mathcal{F}_d^+)$, making $\mathcal{E}$ in the multiplier domain of $\mathfrak{S}_\mathfrak{A}$. Since a formal power series is invertible as a formal power series whenever it is invertible as a multiplier, it follows that $\mathcal{E}$ is also in the formal domain.

This proves that any $\mathcal{E}$ in the Schur class $\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{F})$ is in the multiplier domain for the linear-fractional map $\mathfrak{S}_\mathfrak{A}$ and hence $\mathfrak{S}_\mathfrak{A}[\mathcal{E}]$ is a bounded multiplier from $H^2_\mathbb{C}(\mathcal{F}_d^+)$ to $H^2_\mathbb{C}(\mathcal{F}_d^+)$. To see that $\mathfrak{S}_\mathfrak{A}[\mathcal{E}]$ is in $\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$, we show that $\mathfrak{S}_\mathfrak{A}[\mathcal{E}]$ so defined has multiplier norm at most 1. To this end, multiply the first inequality of (3.30) with $[M^*_\mathcal{E} \ I]$ from the left and with $[M^*_\mathcal{E} \ I]^*$ from the right to see that

$$[M^*_\mathcal{E} \ I] M^*_\mathfrak{S}_{\mathfrak{A},\mathfrak{Y},\mathfrak{U}} M^*_\mathfrak{A} \begin{bmatrix} M^*_\mathcal{E} \\ I \end{bmatrix} \preceq [M^*_\mathcal{E} \ I] J^*_\mathfrak{Y,\mathfrak{U}} \begin{bmatrix} M^*_\mathcal{E} \\ I \end{bmatrix} = M^*_\mathcal{E} M^*_\mathcal{E} - I \preceq 0.$$

By writing out $M_{\mathfrak{A}}$ as a $2 \times 2$ block-multiplier, we see that

$$0 \preceq \begin{bmatrix} M^*_{\mathfrak{A}_{11},\mathcal{E} + \mathfrak{A}_{12}} & M^*_{\mathfrak{A}_{21},\mathcal{E} + \mathfrak{A}_{22}} \\ M^*_{\mathfrak{A}_{11},\mathcal{E} + \mathfrak{A}_{12}} & M^*_{\mathfrak{A}_{21},\mathcal{E} + \mathfrak{A}_{22}} \end{bmatrix} J^*_{\mathfrak{Y,\mathfrak{U}}} \begin{bmatrix} M^*_{\mathfrak{A}_{11},\mathcal{E} + \mathfrak{A}_{12}} \\ M^*_{\mathfrak{A}_{21},\mathcal{E} + \mathfrak{A}_{22}} \end{bmatrix}$$

$$= M^*_{\mathfrak{A}_{11},\mathcal{E} + \mathfrak{A}_{12}} M^*_{\mathfrak{A}_{11},\mathcal{E} + \mathfrak{A}_{12}} - M^*_{\mathfrak{A}_{21},\mathcal{E} + \mathfrak{A}_{22}} M^*_{\mathfrak{A}_{21},\mathcal{E} + \mathfrak{A}_{22}}$$

$$= M^*_{\mathfrak{A}_{11},\mathcal{E} + \mathfrak{A}_{12}} M^*_{\mathfrak{A}_{21},\mathcal{E} + \mathfrak{A}_{22}} (M^*_{\mathfrak{A}_{\mathfrak{A}}}[\mathcal{E}] M^*_{\mathfrak{A}_{\mathfrak{A}}}[\mathcal{E}] - I) M^*_{\mathfrak{A}_{21},\mathcal{E} + \mathfrak{A}_{22}}.$$

Since $M^*_{\mathfrak{A}_{21},\mathcal{E} + \mathfrak{A}_{22}}$ is invertible, it follows that $M^*_{\mathfrak{A}_{\mathfrak{A}}}[\mathcal{E}] M^*_{\mathfrak{A}_{\mathfrak{A}}}[\mathcal{E}] - I \preceq 0$, so that $\|M^*_{\mathfrak{A}_{\mathfrak{A}}}[\mathcal{E}]\| \leq 1$ and thus $\mathfrak{S}_\mathfrak{A}[\mathcal{E}]$ is in $\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$, as claimed. □

3.4.1. Injectivity (or lack thereof) of linear-fractional maps. An important feature used in the solution to the one-variable interpolation problem of Section 2 is that, under certain conditions, in the relation $S = \mathfrak{S}_\mathfrak{A}[\mathcal{E}]$ the Schur-class multiplier $\mathcal{E}$ is uniquely determined by $S$, that is, the linear fractional map $\mathfrak{S}_\mathfrak{A}$ is injective (see e.g., [18 Section 4.1]). We now investigate at least one scenario when this injectivity property holds for $\mathfrak{S}_\mathfrak{A}$ in the multivariable setting of the present section, starting with the general setting where $\mathfrak{A}$ is as in (3.33).

Theorem 3.11. Let $\mathfrak{A}$ be a formal power series as in (3.33) and assume that $\mathfrak{A}_{22}$ is invertible as a formal power series. For $\mathcal{E}$ in the formal domain of $\mathfrak{S}_\mathfrak{A}$, let $S$ be the formal power series given by $S = \mathfrak{S}_\mathfrak{A}[\mathcal{E}]$. Assume also that $\mathfrak{A}$ is invertible as a formal
power series, or equivalently (via Schur complement theory), that \( \mathcal{A}_{11} - \mathcal{A}_{12} \mathcal{A}^{-1}_{22} \mathcal{A}_{21} \)

is invertible as a formal power series. Then

\[
\mathcal{A}_{11} - \mathcal{S} \mathcal{A}_{21} = (\mathcal{A}_{11} - \mathcal{A}_{12} \mathcal{A}^{-1}_{22} \mathcal{A}_{21})(I + \mathcal{E} \mathcal{A}^{-1}_{22} \mathcal{A}_{21})^{-1}
\]

is invertible as a formal power series and we recover the formal power series \( \mathcal{E} \) from the formal power series \( \mathcal{S} \) according to the formula

\[
(3.36) \quad \mathcal{E} = (\mathcal{A}_{11} - \mathcal{S} \mathcal{A}_{21})^{-1}(\mathcal{S} \mathcal{A}_{22} - \mathcal{A}_{12}).
\]

**Proof.** Let us suppose that \( \mathcal{E} \) is in the formal domain of \( \mathcal{T}_3 \), set \( \mathcal{S} := \mathcal{T}_3[\mathcal{E}] \) and try to solve for \( \mathcal{E} \) in terms of \( \mathcal{S} \). From the definition of \( \mathcal{T}_3[\mathcal{E}] \) we find that

\[
\mathcal{S}(\mathcal{A}_{21} \mathcal{E} + \mathcal{A}_{22}) = \mathcal{A}_{11} \mathcal{E} + \mathcal{A}_{12}
\]

which we rearrange as

\[
(\mathcal{A}_{11} - \mathcal{S} \mathcal{A}_{21}) \mathcal{E} = \mathcal{S} \mathcal{A}_{22} - \mathcal{A}_{12}.
\]

If \( \mathcal{A}_{11} - \mathcal{S} \mathcal{A}_{21} \) is invertible, then we can solve for \( \mathcal{E} \):

\[
\mathcal{E} = (\mathcal{A}_{11} - \mathcal{S} \mathcal{A}_{21})^{-1}(\mathcal{S} \mathcal{A}_{22} - \mathcal{A}_{12})
\]

and we have a formula for the inverse linear-fractional map. Let us investigate when is \( \mathcal{A}_{11} - \mathcal{S} \mathcal{A}_{21} \) invertible under the assumption that \( \mathcal{S} = \mathcal{T}_3[\mathcal{E}] \) and that \( \mathcal{A}_{22} \) is invertible, as we have in the indefinite Schur-class multiplier case. Then we have

\[
\mathcal{A}_{11} - \mathcal{S} \mathcal{A}_{21} = \mathcal{A}_{11} - (\mathcal{A}_{11} \mathcal{E} + \mathcal{A}_{12})(\mathcal{A}_{21} \mathcal{E} + \mathcal{A}_{22})^{-1} \mathcal{A}_{21}
\]

\[
= \mathcal{A}_{11} - (\mathcal{A}_{11} \mathcal{E} + \mathcal{A}_{12})(I + \mathcal{A}_{22}^{-1} \mathcal{A}_{21} \mathcal{E})^{-1} \mathcal{A}_{22}^{-1} \mathcal{A}_{21}
\]

\[
= \mathcal{A}_{11} - (\mathcal{A}_{11} \mathcal{E} + \mathcal{A}_{12}) \mathcal{A}^{-1}_{22} \mathcal{A}_{21}(I + \mathcal{E} \mathcal{A}^{-1}_{22} \mathcal{A}_{21})^{-1}
\]

\[
= [\mathcal{A}_{11} + \mathcal{A}_{11} \mathcal{E} \mathcal{A}^{-1}_{22} \mathcal{A}_{21} - \mathcal{A}_{11} \mathcal{E} \mathcal{A}^{-1}_{22} \mathcal{A}_{21} - \mathcal{A}_{12} \mathcal{A}^{-1}_{22} \mathcal{A}_{21}](I + \mathcal{E} \mathcal{A}_{22} \mathcal{A}_{21})^{-1}
\]

\[
= (\mathcal{A}_{11} - \mathcal{A}_{12} \mathcal{A}^{-1}_{22} \mathcal{A}_{21})(I + \mathcal{E} \mathcal{A}^{-1}_{22} \mathcal{A}_{21})^{-1}.
\]

We have that \( I + \mathcal{E} \mathcal{A}^{-1}_{22} \mathcal{A}_{21} \) is indeed formal-power-series invertible, since \( \mathcal{E} \) is in the formal domain of \( \mathcal{T}_3 \). It thus follows that \( \mathcal{A}_{11} - \mathcal{S} \mathcal{A}_{21} \) is formal-power-series invertible if and only if the Schur complement of \( \mathcal{A} \) with respect to \( \mathcal{A}_{22} \), namely, \( \mathcal{A}_{11} - \mathcal{A}_{12} \mathcal{A}^{-1}_{22} \mathcal{A}_{21} \), is invertible. By standard Schur complement theory, the latter is equivalent to \( \mathcal{A} \) being formal-power-series invertible.

\[ \square \]

**Remark 3.12.** Let us now suppose that \( \mathcal{A} \) as in \( (3.33) \) is a multiplier and \( \mathcal{E} \) is in the multiplier domain of \( \mathcal{T}_3 \) (in the sense of Definition \( 3.9 \)). Assume that the multipliers \( \mathcal{A} \) and \( \mathcal{A}_{22} \) are at least formal-power-series invertible (and thus also the Schur complement \( \mathcal{A}_{11} - \mathcal{A}_{12} \mathcal{A}^{-1}_{22} \mathcal{A}_{21} \) is formal-power-series invertible). Let us assume that \( \mathcal{E} \) is in the multiplier domain of \( \mathcal{T}_3 \), so \( \mathcal{S} = \mathcal{T}_3[\mathcal{E}] \) is well-defined as a multiplier. As any multiplier can be viewed as arising from a formal power series, Theorem \( 3.11 \) applies to this situation and we conclude that we can recover \( \mathcal{E} \) from \( \mathcal{S} \) according to the formula \( (3.36) \). From inspection of the formula we only see that \( \mathcal{E} \) is a formal power series despite the fact that \( \mathcal{E} \) was originally chosen to be a multiplier.

Simple examples show that the invertibility assumption in Theorem \( 3.11 \) is too strong in general for getting injectivity results for the associated linear-fractional map, as illustrated in the next example.
Example 3.13. The degenerate case where one takes the input space $\mathcal{U}$ appearing in the definition (3.33) for the symbol $\mathfrak{A}$ for the linear-fractional map $\mathfrak{A}_N$ to be $\mathcal{U} = \{0\}$ is of interest as a special case. In this case $\mathfrak{A}$ collapses to $\mathfrak{A} = \mathfrak{A}_{11}$, $\mathfrak{A}_{22}$ is invertible vacuously, the Schur complement $\mathfrak{A}_{11} - \mathfrak{A}_{12}\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}$ collapses to $\mathfrak{A}$ itself, and the linear-fractional map $\mathfrak{A}_N$ collapses to the multiplication operator $M_{\mathfrak{A}_N} = M_N : f(z) \mapsto \mathfrak{A}(z)f(z)$. As an example let us consider the case where $\mathfrak{A}(z) = \mathfrak{A}_{11}(z) = [z_1 \cdots z_d]$ and $\mathfrak{A}_N = M_N$ is the multiplication operator

\begin{equation}
M_N : \begin{bmatrix} g_1(z) \\ \vdots \\ g_d(z) \end{bmatrix} \mapsto z_1g_1(z) + \cdots + z dg_d(z).
\end{equation}

A distinctive feature of the noncommutative Fock space setting is that this $M_N = M_{Z(\cdot)}$ with $Z(z) = [z_1 \cdots z_d]$ is injective, i.e., given a formal power series of the form $f(z) = z_1g_1(z) + \cdots + z dg_d(z)$, one can uniquely backsolve for the power series $g_1(z), \ldots, g_d(z)$ due to the fact that we have the orthogonal direct-sum decomposition

$$H^2(F^+_d) = \mathbb{C} \oplus \bigoplus_{j=1}^d z_j H^2(F^+_d).$$

Note that the range of $M_{Z(\cdot)}$ is the proper subspace $\mathcal{M}$ consisting of elements of $H^2(F^+_d)$ with $\emptyset$-coefficient $f_0$ equal to zero, and hence $M_{Z(\cdot)}$ is not surjective. In fact $Z(\cdot)$ amounts to the Beurling-Lax representer in the sense of Popescu [41] for the (proper) right-shift-invariant subspace $\mathcal{M}$. We shall discuss this example more generally in Theorem 3.19 below.

3.4.2. Construction of indefinite Schur-class multipliers. We next present the analogue of (2.4) in the setting of the present section. Closely related results appear in [12], specifically Theorems 3.4 and 3.7 there. Here we clarify the role of the second kernel (3.32) and use positivity of this kernel to avoid imposing as an assumption that the multiplier $M_{\mathfrak{A}_{22}}$ is invertible.

Theorem 3.14. Given output stable pairs $(E, T)$ and $(N, T)$ with $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $N \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ and $T = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d$ and given an operator $P > 0$ that satisfies the Stein identity (3.19), set $C := \begin{bmatrix} E_N \\ N \end{bmatrix} : \mathcal{X} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$ and define $T$ as in (3.33). Then we have:

1. There exists a Hilbert space $\mathcal{F}$ so that the $J$-Cholesky factorization problem

\begin{equation}
\begin{bmatrix} B \\ D \end{bmatrix} J_{\mathcal{F}, \mathcal{U}} \begin{bmatrix} B^* \\ D^* \end{bmatrix} = \begin{bmatrix} P^{-1} \otimes I_d \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} J_{\mathcal{Y}, \mathcal{U}} \end{bmatrix} - \begin{bmatrix} T \\ C \end{bmatrix} P^{-1} \begin{bmatrix} T^* \\ C^* \end{bmatrix}
\end{equation}

has an injective solution $\begin{bmatrix} B \\ D \end{bmatrix} : \mathcal{F} \oplus \mathcal{U} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \oplus \mathcal{U} \end{bmatrix}$.

2. The operator $[\mathcal{F}]$ is a $[P, \begin{bmatrix} I_d \otimes I_d \\ 0 \\ 0 \end{bmatrix}]$-isometry, and for any injective solution $[B]$ to the $J$-Cholesky factorization problem (3.38), the operator

\begin{equation}
U = \begin{bmatrix} T \\ C \end{bmatrix}
\end{equation}

is a $[P, \begin{bmatrix} I_d \otimes I_d \\ 0 \\ 0 \end{bmatrix}]$-unitary completion $[\mathcal{F}]$, and conversely, any such $U$ arises from an injective solution $[B]$ of (3.38). Explicitly $U$ satisfies
Proof. Our proof relies on a general Krein-space lemma which we prove in the appendix, namely Lemma A.2.

The requirement that \( V \in \mathbb{C}^{m \times m} \) in Lemma A.2 can be taken to be \( H \), hence Lemma A.2 indeed applies. The fact that the Krein space \((\mathcal{X}, J)\) in (3.39) satisfying relations (3.40) and (3.41) is just a rewriting of the Stein equation (3.19), which holds by assumption.

(3) For any injective solution \([B] \) to the \( J \)-Cholesky factorization problem (3.38), if we set \( \mathfrak{A} \) to equal to the transfer function of the colligation matrix \( \mathbb{U} \) in (3.39) having the form

\[
\mathfrak{A}(z) = D + C(I - Z(z)T)^{-1}Z(z)B,
\]

with \( Z(z) \) as in (3.39), then the formal kernels \( K_{\mathfrak{A}}^{J \mathcal{F}, \mathcal{U} \mathcal{Y}, \mathcal{U}}(z, \zeta) \) in (3.31) and \( \tilde{K}_{\mathfrak{A}}^{J \mathcal{F}, \mathcal{U} \mathcal{Y}, \mathcal{U}}(z, \zeta) \) in (3.32) associated with \( \mathfrak{A} \) have the explicit form

\[
K_{\mathfrak{A}}^{J \mathcal{F}, \mathcal{U} \mathcal{Y}, \mathcal{U}}(z, \zeta) = \begin{bmatrix} E \\ N \end{bmatrix} (I - Z(z)T)^{-1}P^{-1}(I - T^*Z(\zeta)^*)^{-1} \begin{bmatrix} E^* \\ N^* \end{bmatrix},
\]

(3.44) \[ \tilde{K}_{\mathfrak{A}}^{J \mathcal{F}, \mathcal{U} \mathcal{Y}, \mathcal{U}}(z, \zeta) = B^*(I - Z(\zeta)^*T^*)^{-1}(k_{\mathfrak{A}}(z, \zeta)(P \otimes I - Z(\zeta)^*PZ(z))(I - TZ(z))^{-1}B. \]

Both these formal kernels turn out to be positive kernels and hence \( \mathfrak{A} \) is in the indefinite Schur class \( \mathcal{S}_{nc, d}(J_{\mathcal{Y}, \mathcal{U}}, J_{\mathcal{F}, \mathcal{U}}). \)

Proof. Our proof relies on a general Krein-space lemma which we prove in the appendix, namely Lemma A.2.

The proof of points (1) and (2) is just an application of Lemma A.2 with \( (\mathcal{X}', J') := (\mathcal{X}, P), \quad (\mathcal{X}, J) := \left( \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \oplus \mathcal{U} \end{bmatrix}, \begin{bmatrix} P \otimes I_d & 0 \\ 0 & J_{\mathcal{Y}, \mathcal{U}} \end{bmatrix} \right), \]

\[ V := \begin{bmatrix} T \\ C \end{bmatrix} \text{ with } C := \begin{bmatrix} E \\ N \end{bmatrix} \text{ and } T \text{ as in (3.39).} \]

The requirement that \( V \) be a \((J', J)\)-isometry then takes the form

\[
\sum_{j=1}^{d} T_j^*P T_j + E^*E - N^*N = P
\]

which is just a rewriting of the Stein equation (3.19), which holds by assumption. Hence Lemma A.2 indeed applies. The fact that the Krein space \((\mathcal{X}', J')\) claimed to exist in Lemma A.2 can be taken to be \( (\mathcal{X}', J') = (\mathcal{F} \oplus \mathcal{U}, J_{\mathcal{F}, \mathcal{U}}) \)

follows by item (2) of Corollary A.3, where we take \( (\mathcal{X}, J_1) = (\mathcal{X}, P) \) and \( (\mathcal{X}, J_2) = \left( \begin{bmatrix} \mathcal{X}^{d-1} \\ \mathcal{Y} \oplus \mathcal{U} \end{bmatrix}, \begin{bmatrix} P \otimes I_{d-1} & 0 \\ 0 & J_{\mathcal{Y}, \mathcal{U}} \end{bmatrix} \right), \)

noting that the \((J, J')\)-unitary operator serves as a Krein-space isomorphism.

Hence by Lemma A.2 the \((J', J)\)-isometric operator \( \begin{bmatrix} T \\ C \end{bmatrix} \) can be completed to a \((J', J)\)-unitary operator for \( J' = \begin{bmatrix} J' & 0 \\ 0 & J' \end{bmatrix} \), which yields precisely an operator \( \mathbb{U} \) as in (3.39) satisfying relations (3.40) and (3.41). That this provides a solution to the \( J \)-Cholesky factorization problem (3.38) as claimed in item (1) and the relation between solutions to the \( J \)-Cholesky factorization problem (3.38) and \((J', J)\)-unitary
completions of $\mathcal{T}$ now follows directly by the second claim of Lemma A.2. So, we proved items (1) and (2) and it remains to prove item (3).

A straightforward computation based solely on the equality (3.41) shows that the kernel $K_{A,J,F,U,Y,U}$ in (3.31) associated with the characteristic formal power series of the colligation (3.39) given by (3.42) can be factored as in (3.43).

A similar computation based solely on the equality (3.40) shows that the kernel $\tilde{K}_{A,J,Y,U,F,U}$ in (4.8) associated with the same power series can be factored as in (3.44) or equivalently, as

$$\tilde{K}_{A,J,Y,U,F,U} (z, \zeta) = B^* (I - Z(\zeta)^* T^*)^{-1} (P^*_z \otimes I_d) \times (k_{Sz}(z, \zeta) I_d - Z(\zeta)^* (k_{Sz}(z, \zeta) I_d Z(z)))$$

$$\times (P^*_z \otimes I_d) (I - T(z))^{-1} B.$$

These computations are worked out in detail for the definite case (see Theorems 2.2 and 2.3 in [21]).

From the factorization (3.43) for the first kernel, we see immediately that the first kernel $K_{A,J,F,U,Y,U}$ is a positive kernel.

To show that the second kernel (3.44) is a positive kernel, it suffices to verify that the kernel

$$(3.45) \quad \mathfrak{R}(z, \zeta) = [\mathfrak{R}_{ij}(z, \zeta)]_{i,j=1}^d := k_{Sz}(z, \zeta) I_d - Z(\zeta)^* k_{Sz}(z, \zeta) Z(z)$$

is positive. Since

$$k_{Sz}(z, \zeta) - \sum_{j=1}^d \zeta_j k_{Sz}(z, \zeta) z_j = 1,$$

we have

$$(3.46) \quad \mathfrak{R}_{ij}(z, \zeta) = \begin{cases} 1 + \sum_{m \neq j} \zeta_m k_{Sz}(z, \zeta) z_m, & \text{for } i = j, \\ -\zeta_i k_{Sz}(z, \zeta) z_j, & \text{for } i \neq j. \end{cases}$$

Let

$$k_{Sz}(z, \zeta) = H(z) H(\zeta)^*, \quad H(z) = \mathrm{Row}_{\alpha \in \mathbb{F}_d^+} z^\alpha$$

be the Kolmogorov decomposition of $k_{Sz}$ and let $R_j := R_{x_j}$ be the coordinate-variable multipliers from (3.3). Since

$$(3.47) \quad (R_j H)(z) = \mathrm{Row}_{\alpha \in \mathbb{F}_d^+} z^\alpha z_j \quad \text{for } j = 1, \ldots, d,$$

we have

$$(3.47) \quad ((R_j H)(z))(R_i H(\zeta))^* = \overline{\zeta}_i k_{Sz}(z, \zeta) z_j \quad \text{for } i, j = 1, \ldots, d.$$ 

Let $e_j$ denote the $j$-th column of the identity matrix $I_d$, let

$$(3.48) \quad F_{ij}(z) = e_i (R_j H)(z) - e_j (R_i H)(z) \quad \text{for } 1 \leq i < j \leq d.$$ 

We claim that

$$(3.49) \quad \mathfrak{R}(z, \zeta) = I_d + \sum_{1 \leq i < j \leq d} F_{ij}(z) F_{ij}(\zeta)^*,$$
which provides a Kolmogorov decomposition of \( R(z, \zeta) \) and proves that \( R(z, \zeta) \) is a positive kernel. Note first that, by (3.48) and (3.47), we have

\[
F_{ij}(z)F_{ij}(\zeta)^* = (e_i(R_j H)(z) - e_j(R_i H)(z))((R_i H(\zeta))^* e_i^* - (R_i H(\zeta))^* e_j^*)
\]

(3.50)

In either case, according to (3.50), we end up with

\[
- e_i^* \zeta_j k_{Sz}(z, \zeta) z_j e_j^* - e_j^* \zeta_i k_{Sz}(z, \zeta) z_i e_i^*.
\]

First we consider the diagonal entries of \( R(z, \zeta) \). From the identity (3.50), we see that the \((\ell, \ell)\)-entry in the matrix \( F_{ij}(z)F_{ij}(\zeta)^* \) is positive only if either \( i = \ell \) or \( j = \ell \). Comparing the \((\ell, \ell)\)-entries in (3.49) gives

\[
1 + \sum_{1 \leq i < j \leq d} \left[ F_{ij}(z)F_{ij}(\zeta)^* \right]_{\ell \ell} = 1 + \sum_{i < \ell} \left[ F_{i\ell}(z)F_{i\ell}(\zeta)^* \right]_{\ell \ell} + \sum_{j > \ell} \left[ F_{\ell j}(z)F_{\ell j}(\zeta)^* \right]_{\ell \ell}
\]

\[
= 1 + \sum_{i < \ell} \left[ e_i^* \zeta_r k_{Sz}(z, \zeta) z_r e_r^* + e_r^* \zeta_i k_{Sz}(z, \zeta) z_i e_i^* \right]_{\ell \ell}
\]

\[
- e_i^* \zeta_j k_{Sz}(z, \zeta) z_j e_j^* - e_j^* \zeta_i k_{Sz}(z, \zeta) z_i e_i^*.
\]

Hence, we see that (3.49) holds on the diagonal.

For \( \ell \neq r \) the \((\ell, r)\)-entry in the right hand side of (3.49) works out as

\[
\sum_{1 \leq i < j \leq d} \left[ F_{ij}(z)F_{ij}(\zeta)^* \right]_{\ell r} = \begin{cases} \left[ F_{r\ell}(z)F_{r\ell}(\zeta)^* \right]_{\ell r} & \text{if } \ell \neq r, \\ \left[ F_{\ell r}(z)F_{\ell r}(\zeta)^* \right]_{\ell r} & \text{if } r \neq \ell. \end{cases}
\]

In either case, according to (3.50), we end up with \(- \zeta_r k_{Sz}(z, \zeta) z_r\), which corresponds to the \((\ell, r)\)-entry of \( R(z, \zeta) \) according to (3.46). Hence we have established (3.49) and obtained that \( R(z, \zeta) \) is a positive kernel, and thus that \( K_{\zeta}^{J_{Y, U}, J_{Y, U}}(z, \zeta) \) is a positive kernel.

Since we now see that both kernels (3.43) and (3.44) are positive, we conclude by Proposition 3.8 that \( R(z) \) belongs to the indefinite Schur class \( S_{nc, d}(J_{Y, U}, J_{Y, U}) \), and the proof of item (3) is complete. \( \square \)

**Remark 3.15.** If we let \( R_m(Z, \zeta) \) denote the \( m \times m \)-matrix valued kernel of the form (3.45) but in variables \( z_1, \ldots, z_m \), i.e.,

\[
R_m(z, \zeta) = k_{Sz, m}(z, \zeta) I_m - Z_m(\zeta)^* k_{Sz, m}(z, \zeta) Z_m(z),
\]

then it turns out that

\[
R_m(z, \zeta) - \begin{bmatrix} R_{m-1}(z, \zeta) & 0 \\ 0 & 1 \end{bmatrix} = G(z) G(\zeta)^*
\]

(3.51)

where

\[
G(z) = \begin{bmatrix} (R_m H)(z) & 0 & \ldots & 0 \\ 0 & (R_m H)(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & (R_m H)(z) \end{bmatrix}.
\]

Here, \( H \) denotes the power series from the Kolmogorov decomposition

\[
k_{Sz, m}(z, \zeta) = H(z) H(\zeta)^*, \quad H(z) = \text{Row}_{\alpha \in \mathbb{P}^+} z^\alpha
\]
and $R_j := R_{z_j}$ are the coordinate-variable multipliers from (3.37). Upon making use of (3.37), we can write the kernel $G(z)G(\zeta)^*$ in more detail as

$$G(z)G(\zeta)^* = \begin{bmatrix} \zeta_m k_{Sz,m}(z, \zeta) z_m I_{m-1} \\
\vdots \\
\zeta_m k_{Sz,m}(z, \zeta) [z_1 z_2 \ldots z_{m-1}] \\
\sum_{j=1}^{m-1} \zeta_j k_{Sz,m}(z, \zeta) z_j \end{bmatrix}$$

which along with formulas (3.46) implies (3.51). Note that by (3.51),

$$\Re_m(z, \zeta) \geq \begin{bmatrix} \Re_{m-1}(z, \zeta) & 0 \\
0 & 1 \end{bmatrix} \text{ for all } m \geq 2,$$

and since $\Re_1(z, \zeta) \equiv 1 \geq 0$, the positivity of the kernel $\Re_m$ for all $m \geq 1$ alternatively follows by an induction argument.

In [12], the series (3.42) and the associated linear fractional transformation (3.34) with free parameter $E \in S_{nc,d}(U, F)$ were constructed to describe all $S \in \mathcal{L}(U, \mathcal{Y})$ solving the OAP $\mathcal{S}_{nc,d}(U, \mathcal{Y})$ for interpolation problems (3.52) with interpolation condition

$$(E^* S)^{\wedge L}(T^*) = \mathcal{O}_{E,T}^* M_S|_{U} = N^*$$

in case $P = \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T} > 0$. Although the formal power series $\mathcal{A}$ is still crucial in context of the OAP $\mathcal{H}(S_0)$, the latter Schur-class interpolation theorem can be by-passed as we will see in the next subsection.

3.5. OAP $\mathcal{H}(S_0)$: parametrization of the solution set. Throughout this subsection we assume that the operator $P$ given in (3.18) is strictly positive definite. With all the above preliminaries out of the way, we can now describe the solution set for the interpolation problem OAP $\mathcal{H}(S_0)$. We start with a specific solution.

**Proposition 3.16.** The power series

$$(3.52) \quad f_0(z) = (E - S_0(z)N)(I_N - Z(z)T)^{-1}P^{-1}x$$

is a solution to the OAP $\mathcal{H}(S_0)$. Furthermore, we have $\|f_0\|_{\mathcal{H}(S_0)} = \|P^{-\frac{1}{2}}x\|_X$.

**Proof.** Let $F_{S_0}$ be given by (3.20). Since $f_0 = M_{F_{S_0}}^{-1}P^{-1}x$, the formula for $\|f_0\|_{\mathcal{H}(S_0)}$ follows from (3.21). Furthermore, by (3.23), we have

$$(E^* f_0)^{\wedge L}(T^*) = \mathcal{O}_{E,T}^* M_{F_{S_0}}^{-1}P^{-1}x$$

which completes the proof. \qed

We next observe from (3.21) that the operator $M_{F_{S_0}}P^{-\frac{1}{2}} : \mathcal{X} \to \mathcal{H}(S_0)$ is an isometry and that the space

$$(3.53) \quad \mathcal{N} = \{ F_{S_0}(z) x : x \in \mathcal{X} \} \text{ with norm } \| F_{S_0} x \|_{\mathcal{N}} = \| P_{\frac{1}{2}} x \|_X$$

is isometrically included in $\mathcal{H}(S_0)$. Furthermore, the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}(S_0)$ is the range of the operator $I - M_{F_{S_0}}P^{-1}M_{F_{S_0}}$ with lifted norm of $\mathcal{H}(S_0)$ and hence it is a NFRKHS with reproducing kernel

$$(3.54) \quad K_{\mathcal{N}^\perp}(z, \zeta) = K_{S_0}(z, \zeta) - F_{S_0}(z)P^{-1}F_{S_0}(\zeta)^*.$$
On the other hand, $\mathcal{N}^\perp$ can be characterized as the solution set of the homogeneous OAP $\mathcal{H}(S_0)$:

$$\mathcal{N}^\perp = \{ h \in \mathcal{H}(S_0) : (E^* h)^{\perp L}(T^*) = 0 \}.$$ 

Indeed, for each $h \in \mathcal{H}(S_0)$ we have $(E^* h)^{\perp L}(T^*) = M^{[e]}_{h'} h = 0$ (see (3.24)) if and only if

$$\langle h, M_{F_S} x \rangle_{\mathcal{H}(S_0)} = 0 \quad \text{for all} \quad x \in \mathcal{N}.$$ 

By the latter characterization and by Proposition 3.1 we conclude that $f \in \mathcal{Y}(\langle z \rangle)$ solves the problem (3.44) if and only if it is of the form $f = f_0 + h$ where $h$ is an element in $\mathcal{N}^\perp = \mathcal{H}(K_{\mathcal{N}^\perp})$.

To get a more specific description for $\mathcal{N}^\perp$ we take a closer look at the kernel (3.54). We first use the definitions (3.28) and (3.20) to write

$$K_{\mathcal{N}^\perp}(z, \zeta) = \left[ I \quad -S_0(z) \right] \left( k_{S\mathcal{A}}(z, \zeta) J_{\mathcal{U}, \mathcal{M}} \right) \left[ \begin{array}{c} I \\ -S_0(\zeta)^* \end{array} \right]$$

$$- \left[ I \quad -S_0(z) \right] \left[ \begin{array}{c} E_{\mathcal{F}} \\ \mathcal{N} \end{array} \right] \left( I - Z(z)T \right)^{-1} P^{-1} (I - T^* Z(\zeta)^* )^{-1} \left[ \begin{array}{c} E^* \\ N^* \end{array} \right] \left[ \begin{array}{c} I \\ -S_0(\zeta)^* \end{array} \right].$$

Let $\mathcal{A} = \left[ \begin{array}{c} \mathcal{A}_{11} \\ \mathcal{A}_{12} \\ \mathcal{A}_{22} \end{array} \right]$ be the power series constructed in Theorem 3.14. Then by identity (3.43) we can represent the kernel $K_{\mathcal{N}^\perp}$ as

$$(3.55) \quad K_{\mathcal{N}^\perp}(z, \zeta) = \left[ I \quad -S_0(z) \right] \mathcal{A}(z)(k_{S\mathcal{A}}(z, \zeta) J_{\mathcal{U}, \mathcal{M}}) \mathcal{A}(\zeta)^* \left[ \begin{array}{c} I \\ -S_0(\zeta)^* \end{array} \right].$$

Now set

$$(3.56) \quad u(z) = \mathcal{A}_{11}(z) - S_0(z) \mathcal{A}_{21}(z), \quad v(z) = S_0(z) \mathcal{A}_{22}(z) - \mathcal{A}_{12}(z),$$

i.e.,

$$[u(z) \quad -v(z)] := \left[ I \quad -S_0(z) \right] \mathcal{A}(z) \in \mathcal{L}(\mathcal{F} \oplus \mathcal{U}, \mathcal{Y})(\langle z \rangle).$$

Then, on account of (3.29), we can rewrite (3.55) as

$$(3.57) \quad K_{\mathcal{N}^\perp}(z, \zeta) = u(z)(k_{S\mathcal{A}}(z, \zeta) J_{\mathcal{F}}) u(\zeta)^* - v(z)(k_{S\mathcal{A}}(z, \zeta) J_{\mathcal{U}}) v(\zeta)^*.$$ 

Since the kernel $K_{\mathcal{N}^\perp}$ is positive, it follows by the non-commutative Leech theorem (see [19] Section 3.3) that there is a $\mathcal{E}_0 \in \mathcal{S}_{\mathcal{A},d}(\mathcal{U}, \mathcal{F})$ so that $v = u \mathcal{E}_0$, which being combined with (3.56) gives

$$S_0 \mathcal{A}_{22} - \mathcal{A}_{12} = (\mathcal{A}_{11} - S_0 \mathcal{A}_{21}) \mathcal{E}_0.$$ 

We thus recover $S_0$ as $S_0 = \mathcal{A}_{22}[\mathcal{E}_0]$.

On the other hand, plugging in $v = u \mathcal{E}_0$ into (3.57) gives

$$K_{\mathcal{N}^\perp}(z, \zeta) = u(z)(k_{S\mathcal{A}}(z, \zeta) J_{\mathcal{F}} - \mathcal{E}_0(z) k_{S\mathcal{A}}(z, \zeta) \mathcal{E}_0(\zeta)^*) u(\zeta)^* = u(z) K_{\mathcal{E}_0}(z, \zeta) u(\zeta)^*.$$ 

Combining the latter equality with (3.55) gives

$$K_{\mathcal{E}_0}(z, \zeta) = F_{S_0}(z) P^{-1} F_{S_0}(\zeta)^* + u(z) K_{\mathcal{E}_0}(z, \zeta) u(\zeta)^*.$$ 

The first and the second terms on the right are reproducing formal kernels of the subspace $\mathcal{N}$ in (3.53) and its orthogonal complement $\mathcal{N}^\perp$ in $\mathcal{H}(S_0)$, respectively.

By part (2) in Proposition 3.6, $M_u : \mathcal{H}(\mathcal{E}_0) \to \mathcal{N}^\perp$ is a coisometry and hence $M_u : \mathcal{H}(\mathcal{E}_0) \to \mathcal{H}(S_0)$ is a partial isometry.

The space $\mathcal{M} := \mathcal{H}(\mathcal{E}_0) \ominus \text{Ker } M_u$ is isometrically included in $\mathcal{H}(\mathcal{E}_0)$ and the restriction of $M_u$ to this space is an isometry from $\mathcal{M}$ into $\mathcal{H}(K_{\mathcal{E}_0})$.

Putting all these pieces together, we arrive at the following result.
Theorem 3.17. Let \((E, T) \in \mathcal{L}(U, \mathcal{Y}) \times \mathcal{L}(\mathcal{X})^d\) be an output stable pair, \(x\) a vector in \(\mathcal{X}\) and \(S_0\) a Schur-class multiplier in \(\mathcal{S}_{nc,d}(U, \mathcal{Y})\). Assume that \(P\) defined as in \ref{3.11} is strictly positive definite. Let \(\mathfrak{A}(z)\) be the formal power series constructed in Theorem \ref{3.14}. Then there exists a \(E_0 \in \mathcal{S}_{nc,d}(U, \mathcal{F})\) such that \(S_0 = \mathfrak{T}_\mathcal{A}[E_0]\). Furthermore, let
\[
u(z) = \mathfrak{A}_{11}(z) - S_0(z)\mathfrak{A}_{21}(z) \quad \text{and} \quad M = \mathcal{H}(E_0) \ominus \text{Ker} M_u.
\]
Then \(f \in \mathcal{Y}(\langle z \rangle)\) is a solution to the OAP\(\mathcal{H}(S_0)\) if and only if \(f\) is of the form
\[
f(z) = f_0(z) + u(z)\sigma(z), \quad \sigma \in \mathcal{M},
\]
where \(f_0\) is defined in \ref{3.60} and \(\sigma\) is a free parameter from \(\mathcal{M}\). Furthermore, for \(f\) as in \ref{3.58} we have
\[
\|f\|^2_{\mathcal{H}(S_0)} = \|f_0\|^2_{\mathcal{H}(S_0)} + \|u\sigma\|^2_{\mathcal{H}(S_0)} = \|P^{-\frac{1}{2}}x\|^2_\mathcal{F} + \|\sigma\|^2_{\mathcal{H}(E_0)}.
\]

3.6. Interpolation in \(H^2_2(\mathbb{F}^d_+)^\perp\). Following the single-variable prototype in Subsection \ref{3.11} to the case where the coefficient space \(U\) is taken to be the zero space \(\{0\}\) and hence the only member of the Schur class \(\mathcal{S}_{nc,d}(\{0\}, \mathcal{F})\) is the zero power series \(S_0 = 0 : \{0\} \rightarrow \mathcal{Y}\). Then \(\mathcal{H}(S_0) = H^2_2(\mathbb{F}^d_+)\), \(N : \mathcal{X} \rightarrow \{0\}\) (by \ref{3.11}), and \(F^{S_0}(z) = E(I - Z(z)T)^{-1}\) (see formula \ref{3.20}), so that \(M_{F^{S_0}} = \mathcal{O}_{E,T}\mathcal{O}_{E,T}\) strictly positive definite, i.e., the pair \((E, T)\) is exactly observable, we get the following result on \(H^2_2(\mathbb{F}^d_+)\)-interpolation as a special case of Theorem \ref{3.17}. We recall that a power series \(\Phi(z) \in \mathcal{S}_{nc,d}(\mathcal{F}, \mathcal{Y})\) is called (strictly) inner if the multiplication operator \(M_{\Phi}\) is an isometry from \(H^2_2(\mathbb{F}^d_+)\) onto \(H^2_2(\mathbb{F}^d_+)\).

Theorem 3.18. Given an output stable exactly observable pair \((E, T) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{X})^d\) (i.e., \(P := \mathcal{O}_{E,T}\mathcal{O}_{E,T}^* > 0\)), there exists an auxiliary Hilbert space \(\mathcal{F}\) and a strictly inner \(\Phi(z) \in \mathcal{S}_{nc,d}(\mathcal{F}, \mathcal{Y})\) such that
\[
K_{\Phi}(z, \zeta) := k_{\mathcal{S}_d}(z, \zeta)I_\mathcal{Y} - \Phi(z)(k_{\mathcal{S}_d}(z, \zeta)I_\mathcal{F})\Phi(\zeta)^* = E(I - Z(z)T)^{-1}P^{-1}(I - T^*Z(\zeta)^*E^*)^{-1}E^*.
\]
Furthermore, for a given vector \(x \in \mathcal{X}\), a power series \(f \in H^2_2(\mathbb{F}^d_+)\) satisfies the interpolation condition
\[
(E^*f)^{\perp_L}(T^*) := \mathcal{O}_{E,T}^*f = x
\]
if and only if it is of the form
\[
f(z) = E(I - Z(z)T)^{-1}P^{-1}x + \Phi(z)\sigma(z),
\]
where \(\sigma\) is a free parameter from \(H^2_2(\mathbb{F}^d_+)\). Finally, for \(f\) as in \ref{3.60} we have
\[
\|f\|^2_{H^2_2(\mathbb{F}^d_+)} = \|P^{-\frac{1}{2}}x\|^2_\mathcal{F} + \|h\|^2_{H^2_2(\mathbb{F}^d_+)},
\]
Proof. To specify Theorem \ref{3.17} to the current particular case, we first note that for \(U = \{0\}\), the power series \(\mathfrak{A}\) in \ref{3.42} collapses to
\[
\Phi(z) := \mathfrak{A}_{11}(z) = D_1 + E(I - Z(z)T)^{-1}Z(z)B_1,
\]
where \([B_1; D_1] : \mathcal{F} \rightarrow \mathcal{X}_d^\perp\) is an injective solution to the Cholesky factorization problem \ref{3.38}, which in its turn, now amounts to
\[
[B_1; D_1] [B_1^*; D_1^*] = \begin{bmatrix} P^{-1} & 0 \\ 0 & I_\mathcal{Y} \end{bmatrix} - \begin{bmatrix} T \\ E \end{bmatrix} P^{-1} [T^* & E^*].
\]
Note that this can be rewritten as
\[
\begin{bmatrix}
T & B_1 \\
E & D_1
\end{bmatrix}
\begin{bmatrix}
P^{-1} & 0 \\
0 & I_F
\end{bmatrix}
\begin{bmatrix}
T^* & E^* \\
B_1^* & D_1^*
\end{bmatrix}
= 
\begin{bmatrix}
P^{-1} \otimes I_d & 0 \\
0 & I_Y
\end{bmatrix}.
\]

The “whole” identity (3.43) now amounts to the equality of the 11-blocks, that is to the identity (3.59). We next observe from (3.11) and the Stein identity (3.19), which in the present setting reduces to
\[
(P - \sum_{j=1}^d T_j^* P T_j) = E^* E,
\]
that in the weak operator topology we have
\[
P = \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} = \lim_{n \to \infty} \sum_{\alpha \in \mathbb{F}_d^+; |\alpha| < n} T^\alpha \top E^* E T^\alpha
\]
\[
= \lim_{n \to \infty} \sum_{\alpha \in \mathbb{F}_d^+; |\alpha| < n} T^\alpha \top (P - \sum_{j=1}^d T_j^* P T_j) T^\alpha
\]
\[
= \lim_{n \to \infty} \left( \sum_{\alpha \in \mathbb{F}_d^+; |\alpha| < n} T^\alpha \top P T^\alpha - \sum_{\alpha \in \mathbb{F}_d^+; |\alpha| \leq n} T^\alpha \top P T^\alpha \right)
\]
\[
= P - \lim_{n \to \infty} \sum_{\alpha \in \mathbb{F}_d^+; |\alpha| = n} T^\alpha \top P T^\alpha.
\]

Hence, for every \( x \in \mathcal{X} \),
\[
\lim_{n \to \infty} \sum_{\alpha \in \mathbb{F}_d^+; |\alpha| = n} \| P^\alpha T^\alpha x \| = 0.
\]
Since \( P \succ 0 \), we also have
\[
\lim_{n \to \infty} \sum_{\alpha \in \mathbb{F}_d^+; |\alpha| = n} \| T^\alpha x \| = 0,
\]
meaning that the tuple \( T \) is strongly stable. Due to equalities (3.63) and (3.64) and since the tuple \( T \) is strongly stable, the power series \( \Phi \) in (3.62) is strictly inner by [13, Theorem 3.2.11]. Moreover, since \( S_0 = 0 \), we obtain that \( u \) in Theorem 3.17 is given by \( u = \Phi \).

Another character in Theorem 3.17 is the Schur-class multiplier \( E_0 \in S_{nc,d}(U,F) \) such that \( S_0 = \mathcal{T}_A(E_0) \). In the present setting, we have \( 0 = S_0 = \Phi E_0 \) with \( \Phi \) strictly inner, so that \( E_0 = 0 : \{0\} \to F \), and hence \( H(E_0) = H^2_F(\mathbb{F}_d^+) \). Since \( \Phi \) is strictly inner, \( \text{Ker} M_{\Phi} = \{0\} \), and the parametrization formula (3.58) takes the form (3.60). Finally, the norm identity (3.61) now follows immediately.

As an example of an exactly observable pair, let us consider the pair \( (e_0, R^*_z) \), where \( R^*_z = (R^*_z, \ldots, R^*_z) \) is the backward-shift tuple on \( H^2_F(\mathbb{F}_d^+) \) consisting of the adjoint operators of right coordinate multipliers (3.4):
\[
R^*_z : \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \mapsto \sum_{\alpha \in \mathbb{F}_d^+} f_{\alpha j} z^\alpha \quad \text{for} \quad j = 1, \ldots, d.
\]
and \( \text{ev}_\emptyset \) is the free-coefficient evaluation operator

\[
\text{ev}_\emptyset : \sum_{\alpha \in \mathbb{N}_+^d} f_\alpha z^\alpha \mapsto f_\emptyset.
\]

The computation

\[
\mathcal{O}_{\text{ev}_\emptyset, \mathbb{R}^d} f = \sum_{\alpha \in \mathbb{N}_+^d} (\text{ev}_\emptyset \mathbb{R} z^\alpha f) z^\alpha = \sum_{\alpha \in \mathbb{N}_+^d} f_\alpha z^\alpha = f
\]

shows that \( \mathcal{O}_{\text{ev}_\emptyset, \mathbb{R}^d} \) is the identity operator on \( H^2_\mathbb{R}(\mathbb{F}_d^+) \). If \( \mathcal{M} \) is a closed subspace of \( H^2_\mathbb{R}(\mathbb{F}_d^+) \) which is \( \mathbb{R}_z \)-invariant, i.e.,

\[
\mathbb{R}_z \mathcal{M} \subset \mathcal{M} \quad \text{for} \quad j = 1, \ldots, d,
\]

then the restricted output pair \( (E, T) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{X})^d \) given by

\[
\mathcal{X} = \mathcal{M}^\perp, \quad E = \text{ev}_\emptyset|_\mathcal{X}, \quad T = \mathbb{R}_z^*|_\mathcal{X}
\]

is exactly observable and \( \mathcal{M}^\perp = \text{Ran} \mathcal{O}_{E, T} \). Therefore, \( \mathcal{M} = \text{Ker} \mathcal{O}_{E, T}^* \) coincides with the solution set of the homogeneous Operator Argument interpolation Problem \( \text{OAP}_{H^2_\mathbb{R}(\mathbb{F}_d^+)} \) with the interpolation condition

\[
(E^* f)^L(T^*) = \mathcal{O}_{E, T}^* f = 0.
\]

Then part (2) in Theorem 3.18 implies the non-trivial “only if” part in the following Beurling-Lax type theorem; see [41].

**Theorem 3.19.** A closed subspace \( \mathcal{M} \) of \( H^2_\mathbb{R}(\mathbb{F}_d^+) \) is \( \mathbb{R}_z \)-invariant if and only if there exist a Hilbert space \( \mathcal{F} \) and a strictly inner multiplier \( \Phi \in \mathcal{S}_{\text{nc}, d}(\mathcal{F}, \mathcal{Y}) \) such that \( \mathcal{M} = \Phi \cdot H^2_\mathbb{R}(\mathbb{F}_d^+) \).

### 4. The Drury-Arveson space setting

If we replace the tuple of freely noncommutative indeterminates \( z = (z_1, \ldots, z_d) \) by a commutative \( d \)-tuple of complex numbers \( \lambda = (\lambda_1, \ldots, \lambda_d) \), i.e., so that \( \lambda_i \lambda_j = \lambda_j \lambda_i \) for all \( i, j = 1, \ldots, d \), then the Fock space \( \mathbb{F}_d^+ \) becomes the Drury-Arveson space. To be more precise, let us recall the following (commutative) multivariable notation: for multi-integers \( n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d \) and points \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \) we set

\[
|n| = n_1 + n_2 + \cdots + n_d, \quad n! = n_1! n_2! \cdots n_d!, \quad z^n = z_1^{n_1} z_2^{n_2} \cdots z_d^{n_d}.
\]

Given a word \( \alpha = i_N i_{N-1} \cdots i_1 \in \mathbb{F}_d^+ \) we let \( \alpha(\alpha) \in \mathbb{Z}_+^d \) be the abelianization of \( \alpha \), given by

\[
\alpha(\alpha) = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d \quad \text{where} \quad n_j = \# \{ k : i_k = j \} \quad \text{for} \quad j = 1, \ldots, d
\]

(where in general \( \# \mathcal{E} \) denotes the cardinality of the set \( \mathcal{E} \)). Then the noncommutative and commutative multivariable functional calculus are related by \( z^\alpha \rightarrow \lambda^{\alpha(\alpha)} \).

If \( T = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d \) is a tuple of commuting Hilbert-space operators, we have

\[
T^\alpha := T_{i_N} \cdots T_{i_1} = T^{\alpha(\alpha)}.
\]

Observe that for a given \( n \in \mathbb{Z}_+^d \),

\[
\# \{ \alpha \in \mathbb{F}_d^+ : \alpha(\alpha) = n \} = \frac{|n|!}{n!}
\]
Using the latter combinatorial fact, we may compute the commutative version $k_d$ of the noncommutative Szegő kernel (3.27):

\[(4.1) \quad k_d(z, \zeta) \mapsto k_d(\lambda, \eta) = \sum_{\alpha \in \mathbb{F}_d^+} \lambda_\alpha \overline{\eta}_\alpha = \sum_{n \in \mathbb{Z}_d^+} \frac{n!}{n!} \lambda^n \eta^n.\]

We will write $\langle \lambda, \eta \rangle = \sum_{j=1}^d \lambda_j \overline{\eta}_j$ for the standard inner product in $\mathbb{C}^d$ and we will denote by $\mathbb{B}^d = \{ \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d : \langle \lambda, \lambda \rangle < 1 \}$ the unit ball of the Euclidean space $\mathbb{C}^d$. The power series in (4.1) converges on $\mathbb{B}^d \times \mathbb{B}^d$ and can be written in the closed form as

\[k_d(\lambda, \eta) = \frac{1}{1 - \langle \lambda, \eta \rangle}, \quad \lambda, \eta \in \mathbb{B}^d.\]

The reproducing kernel Hilbert space (RKHS) $\mathcal{H}_Y(k_d)$ associated with the positive kernel $k_d(z, \zeta)I_Y$, called the Drury-Arveson space, can be explicitly characterized as

\[\mathcal{H}_Y(k_d) = \left\{ f(\lambda) = \sum_{n \in \mathbb{Z}_d^+} f_n \lambda^n : \| f \|^2 = \sum_{n \in \mathbb{Z}_d^+} \frac{n!}{n!} \| f_n \|^2 < \infty \right\}. \]

Note that if we let the variables commute for an element $f$ of the Fock space $H_2^Y(\mathbb{F}_d^+)$, we get a map

\[(4.2) \quad f(z) = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \in H_2^Y(\mathbb{F}_d^+) \mapsto \sum_{n \in \mathbb{Z}_d^+} \left( \sum_{\alpha : a(\alpha) = n} f_\alpha \right) \lambda^n.\]

This map is not injective. However to make a canonical choice of $f \in H_2^Y(\mathbb{F}_d^+)$ we can consider the following Calculus of Variations problem: for a fixed index $n \in \mathbb{Z}_d^+$, given a coefficient $f_n \in \mathcal{Y}$, among all collections of coefficient vectors $\{ f_\alpha : \alpha \in a^{-1}(n) \}$ such that

\[(4.3) \quad \sum_{\alpha \in a^{-1}(n)} f_\alpha = f_n, \]

find the choice of the collection which minimizes $\sum_{\alpha \in a^{-1}(n)} \| f_\alpha \|^2$. The solution is to choose $f_\alpha$ to be independent of $\alpha$ subject to the constraint (4.3), i.e.,

\[f_\alpha = \frac{n!}{n!} f_n \quad \text{for all} \quad \alpha \in a^{-1}(n).\]

If $\mathcal{Y}$ is separable, as we assume throughout the paper, the problem reduces to the scalar-valued case, and in that case the solution follows by the classical arithmetic-quadratic mean inequality

\[\left| \frac{a_1 + \cdots + a_n}{n} \right| \leq \sqrt{\frac{|a_1|^2 + \cdots + |a_n|^2}{n}}.\]

If we restrict to the permutation-invariant Fock space, i.e., the subspace $H_2^Y(\Pi \mathbb{F}_d^+)$ of $H_2^Y(\mathbb{F}_d^+) \text{consisting of power series } f = \sum_{\alpha \in \mathbb{F}_d^+} f_\alpha z^\alpha \text{so that}

\[a(\alpha) = a(\alpha') \Rightarrow f_\alpha = f_{\alpha'},\]
it follows that the map \(\lambda \mapsto H_\lambda\) is an isometric-isomorphism of \(H^2_d(\Pi F^+_d)\) onto \(H^2_d(k_d)\) with inverse given by
\[
\sum_{n \in \mathbb{Z}^d_+} f_n \lambda^n \mapsto \sum_{n \in \mathbb{Z}^d_+} \sum_{\alpha : a(\alpha) = n} \frac{n!}{n!} f_n \lambda^n.
\]

In short we see that there are two distinct approaches to study the Drury-Arveson space: directly, or via making use of known structure for the Fock space to project down via the abelianization map \(1.2\) and arrive at results for the Drury-Arveson space. For example the papers of Arias-Popescu [4] and Davidson-Pitts [28] arrive at the solution of the Nevanlinna-Pick interpolation problem for Drury-Arveson space: directly, or via making use of known structure for the Fock space to project down via the abelianization map (4.2) and arrive at results for the Drury-Arveson space. In any case the tuples \((\mathcal{E}, \mathcal{T})\) are commutative (or at least when \(\mathcal{E} \circ \mathcal{T} = \mathcal{T} \circ \mathcal{E}\) for all \(\alpha, \beta \in \mathbb{F}^+_d\) such that \(a(\alpha) = a(\beta)\)) the two notions of output stability coincide, but in general, \(\mathcal{a}\)-output stability is a stronger notion than just output stability. Indeed, as was shown in [13] Proposition 3.8, if a pair \((E, T)\) is output stable, then
\[
\mathcal{O}^a_{E, \mathcal{T}} \mathcal{O}^a_{E, \mathcal{O}} \subseteq \mathcal{O}^a_{E, \mathcal{T}} \mathcal{O}_{E, \mathcal{T}}
\]
and hence, \((E, T)\) is also \(\mathcal{a}\)-output stable. Throughout this section, however, the tuple \(T\) is assumed to be commutative and we will drop the superscript \(\mathcal{a}\) and simply write output stable instead of \(\mathcal{a}\)-output stable and \(\mathcal{O}_{E, \mathcal{T}}\) instead of \(\mathcal{O}^a_{E, \mathcal{T}}\).

Remark 4.1. Given an output pair \((E, \mathcal{T}) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{X})^d\) as in Section 3 so with the entries \(T_j\) of \(\mathcal{T}\) not necessarily commutating, one can associate two different observability operators, leading to two notions of output stability: We can define \(\mathcal{O}_{E, \mathcal{T}}\) as in Section 3 via (3.10) and again call \((E, \mathcal{T})\) output stable if \(\mathcal{O}_{E, \mathcal{T}}\) determines a bounded map from \(\mathcal{X}\) into \(H^2_d(\Pi F^+_d)\), or work with the commutative observability operator
\[
\mathcal{O}^a_{E, \mathcal{T}} : x \mapsto E(I_{\mathcal{X}} - Z(\lambda)T)^{-1} x = \sum_{n \in \mathbb{Z}^d_+} E \left( \sum_{\alpha : a(\alpha) = n} T^\alpha \right) \lambda^n,
\]
with \(T\) and \(Z(\lambda)\) as in (3.10), and say that \(\mathcal{O}_{E, \mathcal{T}}\) is \(\mathcal{a}\)-output stability when \(\mathcal{O}^a_{E, \mathcal{T}}\) determines a bounded map from \(\mathcal{X}\) into \(H^2_d(\mathcal{K}_d)\). In case the tuple \(\mathcal{T}\) is commutative (or at least when \(E \circ \mathcal{T} = \mathcal{T} \circ E\) for all \(\alpha, \beta \in \mathbb{F}^+_d\) such that \(a(\alpha) = a(\beta)\)) the two notions of output stability coincide, but in general, \(\mathcal{a}\)-output stability is a stronger notion than just output stability. Indeed, as was shown in [13] Proposition 3.8, if a pair \((E, T)\) is output stable, then
\[
\mathcal{O}^a_{E, \mathcal{T}} \mathcal{O}^a_{E, \mathcal{T}} \subseteq \mathcal{O}^a_{E, \mathcal{T}} \mathcal{O}_{E, \mathcal{T}}
\]
To formulate the Operator Argument interpolation Problem $\text{OAP}_{\mathcal{H}(S_0)}$ in $\mathcal{H}(S_0)$, we need a left-tangential calculus, which can be obtained via abelianization of \textbf{(3.12)}. Given an output stable pair $(E, T) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{Y})^d$, see Remark 4.1 since for $x \in \mathcal{X}$, the vector $O_{E,T}x$ is now a vector-valued function analytic around the origin, we can compute its Taylor expansion

$$(O_{E,T}x)(\lambda) = E(IX - Z(\lambda)T)^{-1}x = E \sum_{\alpha \in \mathbb{F}^d_+} T^\alpha \lambda^\alpha x$$

$$= E \sum_{n \in \mathbb{Z}^d_+} \left( \sum_{\alpha \in \mathbb{F}^d_+: n(\alpha) = n} T^\alpha \lambda^\alpha \right) x = \sum_{n \in \mathbb{Z}^d_+} \frac{|n|!}{n!} E T^n \lambda^n x.$$

We then define a \textit{left-tangential functional calculus} $f \mapsto (E^*f)^{\wedge L}(T^*)$ on $\mathcal{H}_Y(k_d)$ for the output stable pair $(E, T)$ by

$$\text{(4.4)} \quad (E^*f)^{\wedge L}(T^*) := \sum_{n \in \mathbb{Z}^d_+} T^*n E^* f_n \text{ if } f(\lambda) = \sum_{n \in \mathbb{Z}^d_+} f_n \lambda^n \in \mathcal{H}_Y(k_d).$$

The computation similar to that in \textbf{(3.13)}:

$$\left\langle \sum_{n \in \mathbb{Z}^d_+} T^*n E^* f_n, x \right\rangle_{\mathcal{X}} = \sum_{n \in \mathbb{Z}^d_+} \left \langle f_n, ET^n x \right \rangle_{\mathcal{Y}}$$

$$= \sum_{n \in \mathbb{Z}^d_+} \frac{n!}{n!} \left \langle f_n, \frac{|n|!}{n!} ET^n x \right \rangle_{\mathcal{Y}} = \langle f, O_{E,T}x \rangle_{\mathcal{H}_Y(k_d)},$$

shows that the left-tangential evaluation \textbf{(4.4)} amounts to the adjoint of the observability operator:

$$(E^*f)^{\wedge L}(T^*) = O_{E,T}^* f \text{ for } f \in \mathcal{H}_Y(k_d)$$

and applies to Schur-class multipliers as well as to functions from a given de Branges-Rovnyak space $\mathcal{H}(S_0)$. The Operator Argument interpolation Problem in the de Branges-Rovnyak space $\mathcal{H}(S_0)$ \textbf{(OAP$_{\mathcal{H}(S_0)}$)} is now formulated as follows.

\textbf{OAP$_{\mathcal{H}(S_0)}$}: Given an output stable pair $(E, T)$ with $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and a commutative d-tuple $T = (T_1, \ldots, T_d) \in \mathcal{L}(\mathcal{Y})^d$, a vector $x \in \mathcal{X}$ and a Schur-class multiplier $S_0 \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, find all $f \in \mathcal{H}(S_0)$ such that

$$\text{(4.5)} \quad (E^*f)^{\wedge L}(T^*) = O_{E,T}^* f = x.$$

Following the lines in Section 3.1 we define the operator $N \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ by the formula

$$\text{(4.6)} \quad N := \sum_{n \in \mathbb{Z}^d_+} S_{0,n}^* E T^n,$$

or equivalently, via its adjoint, by formula \textbf{(3.10)}. By \textbf{[9] Proposition 3.1}, the pair $(N, T)$ is output stable and equality

$$O_{E,T}^* M_{S_0} = O_{N,T}^* : \mathcal{H}_Y(k_d) \to \mathcal{X}$$
holds. Then the operator $P$ defined as in (3.18) is positive semidefinite. It is now defined via abelianized infinite series

$$P := O^*_E,T O_E,T - O^*_N,T O_N,T = \sum_{n \in \mathbb{Z}_+^d} \frac{|n|!}{n!} T^n (E^*E - N^*N) T^n$$

and still satisfies the Stein identity (3.19). We next use the formula (3.20) to define the holomorphic $\mathcal{L}(\mathcal{X,Y})$-valued function $F_{Sl}$ such that for every $x \in \mathcal{X}$, the $\mathcal{Y}$-valued function $F_{Sl}(x) = F_{Sl}(\cdot) x$ belongs to the de commutative Branges-Rovnyak space $\mathcal{H}(S_0)$ and satisfies equality (3.21). The abelianized version of Lemma 3.3 establishes equalities (3.22) where now $M_{\sigma}^{[\sigma]}$ denotes the adjoint of the operator $M_{\sigma} : \mathcal{X} \to \mathcal{H}(S_d)$ in the metric of $\mathcal{H}(S_0)$. Hence, as in the non-commutative case, the interpolation condition (4.14) can be written as

$$O^*_E,T f = M_{\sigma}^{[\sigma]} f = x$$

implying that the problem $\text{OAP}_{\mathcal{H}(S_0)}$ has a solution if and only if $x \in \text{Ran} M_{\sigma}^{[\sigma]} = \text{Ran} P^{\perp}$. If we assume that the operator $P$ in (4.7) is strictly positive definite, we can use Lemma A.2 and Corollary A.3 to construct a $(\begin{bmatrix} P & 0 \\ 0 & J_{\mathcal{X},\mathcal{U}} \end{bmatrix}, \begin{bmatrix} P \otimes I_d & 0 \\ 0 & J_{\mathcal{Y},\mathcal{U}} \end{bmatrix})$-unitary operator $U = \begin{bmatrix} F & E \end{bmatrix}$ (where we set $C = \begin{bmatrix} F \\ E \end{bmatrix}$) which we can use as the colligation matrix to define an abelianized transfer function

$$\mathfrak{A}(\lambda) = D + C(I - Z(\lambda)A)^{-1}Z(\lambda)B$$

Then it is a matter of checking that an abelianized version of the algebra behind the proof of item (3) in Theorem 3.14 shows that both kernels

$$K^{J_{\mathcal{X},\mathcal{U}},J_{\mathcal{Y},\mathcal{U}}}(\lambda, \eta) = k_d(\lambda, \eta)J_{\mathcal{Y},\mathcal{U}} - \mathfrak{A}(\lambda)(k_d(\lambda, \eta)J_{\mathcal{X},\mathcal{U}})\mathfrak{A}(\eta)^*$$

and

$$\widetilde{K}^{J_{\mathcal{X},\mathcal{U}},J_{\mathcal{Y},\mathcal{U}}}(\lambda, \eta) = k_d(\lambda, \eta)J_{\mathcal{X},\mathcal{U}} - \mathfrak{A}(\eta)^*(k_d(\lambda, \eta)J_{\mathcal{Y},\mathcal{U}})\mathfrak{A}(\lambda)$$

are positive.

Furthermore, the first kernel (4.8) satisfies the identity

$$J_{\mathcal{X},\mathcal{U}} - \mathfrak{A}(\lambda)J_{\mathcal{Y},\mathcal{U}}\mathfrak{A}(\eta)^* = \begin{bmatrix} E \\ N \end{bmatrix} (I - Z(\lambda)T)^{-1}P^{-1}(I - T^*Z(\eta)^*)^{-1} \begin{bmatrix} E^* \\ N^* \end{bmatrix},$$

the abelianization of (3.43). Positivity of the kernels $K^{J_{\mathcal{X},\mathcal{U}},J_{\mathcal{Y},\mathcal{U}}}$ and $\widetilde{K}^{J_{\mathcal{X},\mathcal{U}},J_{\mathcal{Y},\mathcal{U}}}$ (more precisely, of their compressions to $U$) guarantees that the operator $\mathfrak{A}_{22}(\lambda)$ is invertible at every point $\lambda \in \mathbb{B}^d$ and moreover, that

$$\|\mathfrak{A}_{22}(\lambda)^{-1}\mathfrak{A}_{21}(\lambda)\| < 1 \text{ for } \lambda \in \mathbb{B}^d.$$

However we can follow the same operator-theoretic argument as used for the non-commutative Fock-space setting (not involving evaluations at interior points in the ball) to show that the multiplier-norm analogue of (4.10) holds, namely,

$$\|M_{\mathfrak{A}_{22}}^{-1}M_{\mathfrak{A}_{21}}\| < 1,$$

and that the Drury-Arveson linear-fractional map

$$\mathcal{E} \mapsto \mathfrak{A}[\mathcal{E}] = (\mathfrak{A}_{11}\mathcal{E} + \mathfrak{A}_{12})(\mathfrak{A}_{21}\mathcal{E} + \mathfrak{A}_{22})^{-1}$$

maps the Schur class $S_d(\mathcal{U}, \mathcal{F})$ into the Schur class $S_d(\mathcal{U}, \mathcal{Y})$ via the Drury-Arveson space analogue of Theorem 3.14.
Remark 4.2. Let us note that Theorem 3.11 and Remark 3.12 formally apply to the present commutative situation without change. However Example 3.13 is distinctively different in the commutative situation in that the commutative analogue of (3.33), namely the operator
\[
M_{Z_{\text{com}}} : \begin{bmatrix} g_1(\lambda) \\ \vdots \\ g_d(\lambda) \end{bmatrix} \mapsto \lambda_1 g_1(\lambda) + \cdots + \lambda_d g_d(\lambda)
\]
is definitely not injective as an operator from \(\mathcal{H}_{\mathbb{C}^d}(k_d)\) to \(\mathcal{H}(k_d)\). For example, for \(d = 2\), the Schur multiplier \(\begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}\) is in the kernel of \(M_{Z_{\text{com}}}\). It is the case that \(\begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}\) is the Beurling-Lax representer (in the sense of McCullough-Trent [39]) for the shift-invariant subspace \(\{f \in \mathcal{H}(k_d) : f(0) = 0\}\), but for this setting one must take the Beurling-Lax representer \(Z_{\text{com}}\) to be such that
\[
M_{Z_{\text{com}}} : \mathcal{H}_{\mathbb{C}^2}(k_2) \rightarrow \mathcal{H}(k_2)
\]
to be only a partial isometry rather than an isometry (i.e., to be what we call McCCT-inner). This is a particular instance of the homogeneous version of Theorem 4.3 discussed below.

The parametrization of the solution set for the OAP \(\mathcal{OAP}_{\mathcal{H}(S_0)}\) can be obtained along the same lines as in Section 3.4. We first verify that the function \(f_0\) defined by the formula (3.52) belongs to \(\mathcal{H}(S_0)\) and satisfies (4.4) with norm equality \(\|f_0\|_{\mathcal{H}(S_0)} = \|P^{-\frac{1}{2}}x\|_X\). We next introduce the subspace \(\mathcal{N}\) of \(\mathcal{H}(S_0)\) defined as in (3.53) and its orthogonal complement \(\mathcal{N}^\perp\) which is on the one hand the solution set for the homogeneous problem (4.5) with \(x = 0\) and on another hand a reproducing kernel Hilbert space with reproducing kernel the abelianization of (3.3). The same manipulations with \(K_{\mathcal{N}^\perp}\) as in the non-commutative case, but based on the identity (4.9) rather than (3.43), lead us to the formula
\[
K_{\mathcal{N}^\perp}(\lambda, \eta) = \begin{bmatrix} I & -S_0(\lambda) \end{bmatrix} \begin{bmatrix} \mathfrak{A}(\lambda) J_{\mathcal{F}, \mathcal{F}} \mathfrak{A}(\eta)^* \\ 1 - \langle \lambda, \eta \rangle \end{bmatrix} \begin{bmatrix} I \\ -S_0(\eta)^* \end{bmatrix}.
\]

With holomorphic operator-valued functions \(u\) and \(v\) defined by formulas (3.56), we write the previous equality as
\[
K_{\mathcal{N}^\perp}(\lambda, \eta) = \frac{u(\lambda)u(\eta)^* - v(\lambda)v(\eta)^*}{1 - \langle \lambda, \eta \rangle}, \quad \lambda, \eta \in \mathbb{B}^d.
\]

The positivity of the latter kernel on \(\mathbb{B}^d \times \mathbb{B}^d\) implies, by the commutative multivariable Leech theorem (see [3, 38]) that there is a \(E_0 \in \mathcal{S}_d(\mathcal{U}, \mathcal{F})\) so that \(v = uE_0\), from which we recover \(S_0 = T_0[E_0]\) and get the eventual representation formula
\[
K_{S_0}(\lambda, \eta) = F^{S_0}(\lambda) P^{-1} F^{S_0}(\eta)^* + u(\lambda)K_{E_0}(\lambda, \eta)u(\eta)^*.
\]

As in the non-commutative case, \(M_u : \mathcal{H}(E_0) \rightarrow \mathcal{M}^\perp\) is a coisometry and hence \(M_u : \mathcal{H}(E_0) \rightarrow \mathcal{H}(S_0)\) is a partial isometry, while the restriction of \(M_u\) to the space \(\mathcal{M} = \mathcal{H}(E_0) \cap \text{Ker} M_u\) maps this space isometrically into \(\mathcal{H}(S_0)\). Next we state the analog of Theorem 3.17.

Theorem 4.3. Let \((E, T) \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \times \mathcal{L}(\mathcal{X})^d\) be an output stable pair with \(T\) a commutative tuple, \(x\) a vector in \(\mathcal{X}\) and \(S_0\) a Schur-class multiplier in \(\in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})\).
Assume that $P > 0$. Let $\mathcal{A}(\lambda)$ be a function on $\mathbb{B}^d$ subject to the identity \[1.1\] with $N$ as in \[1.6\], let $\mathcal{E}_0 \in S_d(\mathcal{U}, \mathcal{F})$ be such that $S_0 = T_\mathcal{A}(\mathcal{E}_0)$, and let

$$u(\lambda) = \mathcal{A}_{11}(\lambda) - S_0(\lambda)\mathcal{A}_{21}(\lambda)$$

and $\mathcal{M} = \mathcal{H}(\mathcal{E}_0) \ominus \text{Ker} \, M_u$.

Then $f \in \mathcal{Y}((z))$ is a solution to the OAP $\mathcal{H}(S_0)$ if and only if $f$ is of the form

$$f(\lambda) = f_0(\lambda) + u(\lambda)h(\lambda), \quad h \in \mathcal{M},$$

where $f_0$ is defined in \[3.52\] and $h$ is a free parameter from $\mathcal{M}$. Furthermore, for $f$ defined by \[4.11\], we have

$$\|f\|^2_{\mathcal{H}(S_0)} = \|f_0\|^2_{\mathcal{H}(S_0)} + \|uh\|^2_{\mathcal{H}(S_0)} + \|P^{-\frac{1}{2}}x\|^2_\mathcal{X} + \|h\|^2_{\mathcal{H}(\mathcal{E}_0)}.$$

Let us say that a Schur-class multiplier $S \in S_d(\mathcal{F}, \mathcal{Y})$ is McCT-inner (referring to the authors of the seminal paper \[39\]) if the operator $M_S : \mathcal{H}_\mathcal{F}(k_d) \to \mathcal{H}_\mathcal{Y}(k_d)$ is a partial isometry.

If we specify Theorem \[4.3\] to the case where $\mathcal{U} = \{0\}$ (as we did in Section 3.5 in the non-commutative setting), then $\mathcal{A}$ collapses to the analytic function $\Phi(\lambda)$ given by the formula \[3.62\]. The assumption $P := O_{E,T}^*O_{E,T} > 0$ still implies that $T$ is strongly stable, i.e., that

$$\lim_{N \to \infty} \sum_{n \in \mathbb{Z}_+^d : |n| = N} \frac{n!}{|n|!} \|T^n x\|_{\mathcal{H}_\mathcal{Y}(k_d)}^2 = 0 \quad \text{for all } x \in \mathcal{X},$$

which this time guarantees only $\Phi$ be McCT-inner; see \[16\] Section 5]. The rest is the same as in Theorem \[3.18\].

**Theorem 4.4.** Given an output stable, exactly observable pair $(E, T) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{Y})^d$ (i.e., $P := O_{E,T}^*O_{E,T} > 0$) with $T$ a commutative tuple, there exists an auxiliary Hilbert space $\mathcal{F}$ and a McCT-inner function $\Phi(\lambda) \in S_d(\mathcal{F}, \mathcal{Y})$ such that

$$I_{\mathcal{Y}} - \frac{\Phi(\lambda)\Phi(\eta)^*}{1 - (\lambda, \eta)} = E(I - Z(\lambda)T)^{-1}P^{-1}(I - T^*Z(\eta)^*)^{-1}E^*.$$

Furthermore, for a given vector $x \in \mathcal{X}$, a function $f \in \mathcal{H}_\mathcal{Y}(k_d)$ satisfies the interpolation condition

$$(E^*f)^\wedge \mathcal{L}(T^*) := O_{E,T}^* f = x$$

if and only if it is of the form

$$f(\lambda) = E(I - Z(\lambda)T)^{-1}P^{-1}x + \Phi(\lambda)h(\lambda), \quad h \in \mathcal{H}_\mathcal{F}(k_d).$$

The homogeneous version of the latter theorem contains the $\mathcal{H}_\mathcal{Y}(k_d)$-version of the Beurling-Lax theorem \[39\]. Still writing $R_x$ for the $d$-tuple of coordinate-variable multipliers $R_{\lambda_j}$ in $\mathcal{H}_\mathcal{Y}(k_d)$, we note that their adjoints are now given by formulas

$$R_{\lambda_j}^* : \sum_{n \in \mathbb{Z}_+^d} f_n \lambda^n \mapsto \sum_{n \in \mathbb{Z}_+^d} f_{n+\varepsilon_j} \lambda^n \quad \text{for } j = 1, \ldots, d$$

where $\varepsilon_j$ is the element in $\mathbb{Z}_+^d$ having the $j$-th partial index equal to one and all other partial indices equal to zero. If we let $ev_0 : f(\lambda) \to f(0)$ denote the zero-evaluation operator on $\mathcal{H}_\mathcal{Y}(k_d)$, then it is easily verified that $ev_0 R_{\lambda}^n : \sum_{n' \in \mathbb{Z}_+^d} f_{n'} \lambda^{n'} \to f_n$ for
all \( n \in \mathbb{Z}_d^+ \) and subsequently, that the observability operator \( O_{\text{ev}_0, R_{\lambda}} \) equals the identity operator on \( \mathcal{H}_Y(k_d) \). Indeed, for any \( f \in \mathcal{H}_Y(k_d) \), we have

\[
O_{\text{ev}_0, R_{\lambda}} f = \sum_{n \in \mathbb{Z}_d^+} (\text{ev}_0 R_{\lambda}^n f) \lambda^n = \sum_{n \in \mathbb{Z}_d^+} f_n \lambda^n = f.
\]

If \( \mathcal{M} \) is a closed subspace of \( \mathcal{H}_Y(k_d) \) which is \( R_{\lambda} \)-invariant, then the restricted output pair \( (E, T) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{X})^d \) given by

\[
\mathcal{X} = \mathcal{M}^+, \quad E = \text{ev}_0|_{\mathcal{X}}, \quad T = R_{\lambda}|_{\mathcal{X}}
\]
is exactly observable and \( \mathcal{M} = \text{Ran} O_{E, T} \). Therefore, \( \mathcal{M} = \text{Ker} O_{E, T}^* \) coincides with the solution set of the homogeneous Operator Argument interpolation Problem \( \text{OAP}_{\mathcal{H}_Y(k_d)} \) with interpolation conditions

\[
(E^* f)^{\perp_{L}}(T^*) := O_{E, T}^* f = 0.
\]

Then it follows by Theorem 4.4 that there exist a Hilbert space \( \mathcal{F} \) and a McCT-inner multiplier \( \Phi \in \mathcal{S}_d(\mathcal{F}, \mathcal{Y}) \) such that \( \mathcal{M} = \Phi \cdot \mathcal{H}_Y(k_d) \).

5. An operator theoretical view

In this section we present a general, purely operator theoretical perspective on the problems considered above, relying on what we will call a generalized de Branges-Rovnyak space. Our approach in this section is to view the problem in the context of a Douglas Factorization Problem with respect to the lifted norms from the generalized de Branges-Rovnyak spaces. For that purpose we also derive some results on the classical Douglas factorization problem.

5.1. Generalized de Branges-Rovnyak spaces. With a given contraction operator \( T \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \) we may associate the operator range of the positive semidefinite operator \( I_{\mathcal{Y}} - TT^* \geq 0 \):

\[
\mathcal{H}(T) := \text{Ran}(I - TT^*)^{\frac{1}{2}} \subset \mathcal{Y}
\]

with the lifted norm

\[
\|(I - TT^*)^{\frac{1}{2}} y\|_{\mathcal{H}(T)} = \|(I - \pi)y\|_{\mathcal{Y}}
\]

where \( \pi \) is the orthogonal projection onto \( \text{Ker}(I - TT^*)^{\frac{1}{2}} \). It follows from (5.1) that \( \|y\|_{\mathcal{H}(T)} \geq \|y\|_{\mathcal{Y}} \) for every \( y \in \mathcal{H}(T) \) and thus \( \mathcal{H}(T) \) is contractively included in \( \mathcal{Y} \). Upon letting \( y = (I - TT^*)^{\frac{1}{2}} y' \) in the last formula we get

\[
\|(I - TT^*)^{\frac{1}{2}} y'\|_{\mathcal{H}(T)} = \langle (I - TT^*) y', y' \rangle_{\mathcal{Y}}.
\]

The original characterization of \( \mathcal{H}(T) \) as the space of all vectors in \( y \in \mathcal{Y} \) such that

\[
\kappa(y) := \sup_{u \in \mathcal{U}} \left\{ \|y + Tu\|_{\mathcal{Y}}^2 - \|u\|_{\mathcal{U}}^2 \right\}
\]
is finite and the identity \( \|y\|_{\mathcal{H}(T)}^2 = \kappa(y) \) is due to de Branges and Rovnyak [27]. Let us note that de Branges and Rovnyak worked with the special case where \( \mathcal{U} \) and \( \mathcal{Y} \) are replaced by Hardy spaces \( H^2_\mathcal{U} \) and \( H^2_\mathcal{Y} \) and the operator \( T: H^2_\mathcal{U} \to H^2_\mathcal{Y} \) is the operator \( T = M_S: f(\lambda) \to S(\lambda)f(\lambda) \) of multiplication by a Schur-class function \( S \in \mathcal{S}(\mathcal{U}, \mathcal{Y}) \). The spaces \( \mathcal{H}(T) \) (operator- rather than function-theoretic) were proposed by Sarason [43] and will here be referred to as generalized de Branges-Rovnyak spaces.
5.2. The solutions to Douglas Factorization Problem. Consider $A \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$. The following well-known result due to Douglas [30] describes when there exists an operator $Y \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ that satisfies

\[(5.3) \quad AY = B \quad \text{and} \quad \|Y\| \leq 1.\]

**Lemma 5.1.** There exists a $Y \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ satisfying (5.3) if and only if $AA^* \succeq BB^*$. In this case, there exists a unique $Y$ satisfying (5.3) and the additional constraints $\text{Ran } Y \subset \text{Ran } A^*$ and $\text{Ker } Y = \text{Ker } B$.

The next proposition contains several characterizations of the operators $Y \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ that satisfy (5.3).

**Proposition 5.2.** Given $A \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ and $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, let us assume that

\[(5.4) \quad Q := AA^* - BB^* \succeq 0.\]

Then there exist unique contractions $X_1 \in \mathcal{L}(\mathcal{U}, \overline{\text{Ran } A})$ and $X_2 \in \mathcal{L}(\mathcal{Y}, \overline{\text{Ran } A})$ so that

\[(5.5) \quad (AA^*)_X X_1 = B, \quad (AA^*)_Y X_2 = A, \quad \text{Ker } X_1 = \text{Ker } B, \quad \text{Ker } X_2 = \text{Ker } A,\]

with $X_2$ being a coisometry. Given $Y \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, define

\[F_Y := A^* - Y B^* \in \mathcal{L}(\mathcal{X}, \mathcal{Y}).\]

Then the following statements are equivalent:

1. $Y$ satisfies conditions (5.3).
2. The operator
\[
\mathbb{P} = \begin{bmatrix} Q & F_Y^* \\ F_Y & I_Y - YY^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}
\]
is positive semidefinite, or, which is the same, $Y$ is a contraction and $F_Y$ maps $\mathcal{X}$ into $\mathcal{H}(\mathcal{Y})$ with the property
\[
\|F_Y x\|_{\mathcal{H}(\mathcal{Y})} \leq \|Q^{1/2} x\|_{\mathcal{X}} \quad \text{for every } x \in \mathcal{X}.
\]
3. $Y$ is a contraction and $F_Y$ maps $\mathcal{X}$ into $\mathcal{H}(\mathcal{Y})$ with the property
\[
\|F_Y x\|_{\mathcal{H}(\mathcal{Y})} = \|Q^{1/2} x\|_{\mathcal{X}} \quad \text{for every } x \in \mathcal{X}.
\]
4. $Y$ is of the form
\[
Y = X_2^* X_1 + (I - X_2^* X_2)^{1/2} K (I - X_1^* X_1)^{1/2}
\]
where $X_1$ and $X_2$ are defined as in (5.3) and where the parameter $K$ is an arbitrary contraction from $\text{Ran } (I - X_1^* X_1)$ into $\text{Ran } (I - X_2^* X_2)$.

Moreover, there is a unique operator $Y \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ satisfying (5.3) if and only if $X_1$ is isometric on $\mathcal{U}$ or $X_2$ is isometric on $\mathcal{Y}$. Moreover, for $Y$ as in (5.4) we have

\[(5.8) \quad \|Y h\|^2 = \|X_2^* X_1 h\|^2 + \|(I - X_2^* X_2)^{1/2} K (I - X_1^* X_1)^{1/2} h\|^2, \quad h \in \mathcal{U}
\]
so that $X_2^* X_1$ is the minimal norm solution to the problem (5.3).
Proof. The existence and uniqueness of the contractions $X_1$ and $X_2$ satisfying \((5.3)\) is a direct consequence of Lemma 5.1. That $X_2$ is a coisometry can be seen as a consequence of the identity
\[
(\AA^*)^\frac{1}{2}(I_{\ran A} - X_2X_2^*) (\AA^*)^\frac{1}{2} = 0
\]
which in turn is a consequence of the second equation in \((5.5)\).

The equivalence of \((1)\), \((2)\) and \((4)\) was established in [18, Lemma 2.2] via Schur-complement arguments. The central observation there was that $Y \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ satisfies \((5.3)\) if and only if it satisfies the condition
\[
\begin{bmatrix}
    I_{\mathcal{H}_1} & B^* & Y^* \\
    B & AA^* & A \\
    Y & A^* & I_Y
\end{bmatrix} \geq 0.
\]
\[(5.9)\]
The implication \((3) \Rightarrow (2)\) is trivial. We verify \((1)\) as follows. If $Y$ satisfies conditions \((5.3)\), then for every $x \in \mathcal{X}$ we have
\[
F_y x = (A^* - YB^*)x = (A^* - YY^*A^*)x = (I - YY^*)A^*x.
\]
We now apply the formula \((5.2)\) to $y' = A^*x$ and then make subsequent use of \((5.3)\) and \((5.4)\) to get
\[
\|F_y x\|_{\mathcal{H}(Y)} = \langle (I - YY^*)A^*x, A^*x \rangle_{\mathcal{Y}}
= \langle A(I - YY^*)A^*x, x \rangle_{\mathcal{X}}
= \langle (AA^* - BB^*)x, x \rangle_{\mathcal{X}} = \langle Qx, x \rangle_{\mathcal{X}} = \|Q^\frac{1}{2}x\|_{\mathcal{X}},
\]
which is equivalent to \((5.6)\).

Finally, to see \((5.8)\) simply note that $(I - X_2^*X_2)^\frac{1}{2}$ is the orthogonal projection onto $\mathcal{Y} \ominus \Ker A = \mathcal{Y} \ominus \Ker X_1$ since $X_2$ is a coisometry.

\[\square\]

Remark 5.3. The equivalence of conditions \((5.3)\) and \((5.4)\) means that the Douglas factorization problem can be reformulated as a matrix completion problem:

Given a partially defined matrix
\[
\begin{bmatrix}
    I_{\mathcal{H}_1} & B^* & ?^* \\
    B & AA^* & A \\
    ? & A^* & I_{\mathcal{H}_2}
\end{bmatrix},
\]

find an operator $Y$ such that plugging in $Y$ for $?$ leads to a positive semidefinite operator matrix. This at various times has been an active area of research in its own right (see e.g. [37]).

5.3. Douglas Factorization Problem in generalized de Branges-Rovnyak spaces. We next consider a problem similar to \((5.3)\) but now $Y$ is contractive with respect to the norms induced by two generalized de Branges-Rovnyak spaces: given operators $A \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ and contractions $T_1 \in \mathcal{L}(\mathcal{U}, \mathcal{U})$ and $T_2 \in \mathcal{L}(\mathcal{Y}, \mathcal{Y})$, find an operator $Y : \mathcal{H}(T_1) \rightarrow \mathcal{H}(T_2)$ such that
\[
AY = B|_{\mathcal{H}(T_1)} \quad \text{and} \quad \|Y\| \leq 1.
\]
\[(5.10)\]

If $A$ and $B$ are a priori considered as operators in $\mathcal{L}(\mathcal{H}(T_2), \mathcal{X})$ and $\mathcal{L}(\mathcal{H}(T_1), \mathcal{X})$ respectively (or, equivalently, $T_1 = 0$ and $T_2 = 0$), then this is the classical Douglas Factorization Problem from the previous subsection. The following lemma shows that the case where $A \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ also can be translated to the
Proposition 5.5. The problem

If this is the case, then the following are equivalent:

\[ F_c = (I - TT^*)C^* : \mathcal{X} \rightarrow \mathcal{H} \]

is contained in \( \mathcal{H}(T) \subset \mathcal{H} \), and therefore, \( F_c \) can also be viewed as an operator in \( \mathcal{L}(\mathcal{X}, \mathcal{H}(T)) \). Following Remark 5.2, we write \( F_c^* \) for the adjoint of \( F_c \) in \( \mathcal{L}(\mathcal{H}, \mathcal{X}) \) and \( F_c^{[*]} \) for the adjoint of \( F_c \) in \( \mathcal{L}(\mathcal{H}(T), \mathcal{X}) \).

Lemma 5.4. Given \( C \in \mathcal{L}(\mathcal{H}, \mathcal{X}) \) and a contraction \( T \in \mathcal{L}(\mathcal{H}, \mathcal{X}) \), define \( F_c \) as in (5.11) Then

\[ F_c^{[*]}g = Cg \text{ for all } g \in \mathcal{H}(T) \text{ and } F_c^{[*]}F_c = C(I - TT^*)C^*. \]

Proof. For any \( g \in \mathcal{H}(T) \) and \( x \in \mathcal{X} \), we have

\[ \langle x, F_c^{[*]}g \rangle_{\mathcal{X}} = \langle F_c x, g \rangle_{\mathcal{H}(T)} = \langle (I - TT^*)C^*x, g \rangle_{\mathcal{H}(T)} = \langle C^*x, g \rangle_{\mathcal{H}} = \langle x, Cg \rangle_{\mathcal{X}}, \]

and the first equality in (5.12) follows. Letting \( g := F_c \bar{x} = (I - TT^*)C^* \bar{x} \) in this equality we get

\[ F_c^{[*]}F_c \bar{x} = C(I - TT^*)C^* \bar{x} \text{ for all } \bar{x} \in \mathcal{X} \]

which justifies the second equality in (5.12).

Given operators \( A \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \), \( B \in \mathcal{L}(\mathcal{U}, \mathcal{X}) \) and contractions \( T_1 \in \mathcal{L}(\mathcal{U}, \mathcal{U}) \) and \( T_2 \in \mathcal{L}(\mathcal{Y}, \mathcal{Y}) \), define the operators

\[ F_A := (I - T_2 T_2^*)A^* \quad \text{and} \quad F_B := (I - T_2 T_2^*)B^* \]

which can also be viewed as operators in \( \mathcal{L}(\mathcal{X}, \mathcal{H}(T_2)) \) and \( \mathcal{L}(\mathcal{X}, \mathcal{H}(T_1)) \), respectively. Using Lemma 5.4 it follows that (5.10) can be rewritten as

\[ F_A^{[*]}F_A = F_B^{[*]} \quad \text{and} \quad \|Y\| \leq 1. \]

In this form, Lemmas 5.1 and 5.2 are applicable with \( A \) and \( B \) replaced by \( F_A^{[*]} \) and \( F_B^{[*]} \), respectively, leading to the following result.

Proposition 5.5. The problem (5.10) has a solution if and only if

\[ Q := A(I - T_2 T_2^*)A^* - B(I - T_1 T_1^*)B^* \succeq 0. \]

If this is the case, then the following are equivalent:

1. \( Y \in \mathcal{L}(\mathcal{H}(T_1), \mathcal{H}(T_2)) \) satisfies (5.10).
2. The operator

\[ P = \begin{bmatrix} Q & F_A^{[*]} - F_B^{[*]}Y^* \\ F_A - Y F_B & H(T_2) - YY^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{H}(T_2) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{H}(T_2) \end{bmatrix} \]

is positive semidefinite or, which is the same, \( Y \) is a contraction and \( F_Y := F_A - Y F_B \) maps \( \mathcal{X} \) into \( \mathcal{H}(Y) \) with the property

\[ \|F_Y x\|_{\mathcal{H}(Y)} \leq \|Q^\frac{1}{2} x\|_{\mathcal{X}} \text{ for every } x \in \mathcal{X}. \]

3. \( Y \) is a contraction and \( F_Y \) maps \( \mathcal{X} \) into \( \mathcal{H}(Y) \) with the property

\[ \|F_Y x\|_{\mathcal{H}(Y)} = \|Q^\frac{1}{2} x\|_{\mathcal{X}} \text{ for every } x \in \mathcal{X}. \]
(4) \( Y \) is of the form
\[
Y = X_2^*X_1 + (I - X_2^*X_2)^{\frac{1}{2}}K(I - X_1^*X_1)^{\frac{1}{2}}
\]
where \( X_1 \) the contraction in \( L(\mathcal{H}(T_1), (\mathcal{Ker}(I - T_2 T_2^*) A^*)^{-1}) \) and \( X_2 \) the contraction in \( L(\mathcal{H}(T_2), (\mathcal{Ker}(I - T_2 T_2^*) A^*)^{-1}) \) that are uniquely determined by
\[
(A(I - T_2 T_2^*) A^*)^{\frac{1}{2}}X_1g_1 = Bg_1, \text{ for } g_1 \in \mathcal{H}(T_1),
\]
\[
(A(I - T_2 T_2^*) A^*)^{\frac{1}{2}}X_1g_2 = Ag_2, \text{ for } g_2 \in \mathcal{H}(T_2),
\]
with \( X_2 \) being a coisometry, and where the parameter \( K \) is an arbitrary contraction from \( \text{Ran}(I - X_1^*X_1) \) into \( \text{Ran}(I - X_2^*X_2) \).

Moreover, there is a unique operator \( Y \in L(\mathcal{H}(T_1), \mathcal{H}(T_2)) \) satisfying (5.10) if and only if \( X_1 \) is isometric on \( \mathcal{H}(T_1) \) or \( X_2 \) is isometric on \( \mathcal{H}(T_2) \). Furthermore, for \( Y \) as in (5.14) we have
\[
\|Yh\|^2 = \|X_2^*X_1h\|^2 + \|(I - X_2^*X_2)^{\frac{1}{2}}K(I - X_1^*X_1)^{\frac{1}{2}}h\|^2, \quad h \in \mathcal{H}(T_1)
\]
so that \( X_2^*X_1 \) is the minimal norm solution to the problem (5.10).

Of particular interest for this paper is the special case of the de Branges-Rovnyak Douglas Factorization Problem (5.10) with \( \mathcal{H}_1 = \mathcal{H}_1 = \mathbb{C} \) and \( T_1 \) the zero operator. Then \( \mathcal{H}(T_1) = \mathbb{C} \) and we identify \( L(\mathcal{H}(T_1), \mathcal{H}(T_2)) \) with \( \mathcal{H}(T_2) \) and \( L(\mathcal{H}_1, \mathcal{X}) \) with \( \mathcal{X} \). Then we arrive at the following problem: for \( A \in L(\mathcal{H}_2, \mathcal{X}), x \in \mathcal{X} \) and a contraction \( T \in L(\mathcal{H}_2, \mathcal{H}_2) \), find a vector \( g \in \mathcal{H}_2 \) so that
\[
(5.15) \quad g \in \mathcal{H}(T_2), \quad Ag = x, \quad \|g\|_{\mathcal{H}(T_2)} \leq 1.
\]

Hence we consider the problem (5.10) with \( B = x \) and \( Y = g \). We split the result of Proposition 5.5 for the case at hand into two results.

**Lemma 5.6.** Let \( A \in L(\mathcal{H}_2, \mathcal{X}), x \in \mathcal{X} \) and \( T_2 \in L(\mathcal{H}_2, \mathcal{H}_2) \) with \( T_2 \) a contraction. Define \( F_A \) as in (5.13). Then there exists a vector \( g \in \mathcal{H}_2 \) satisfying (5.15) if and only if \( P := F_A^*[g] F_A = A(I - T_2 T_2^*) A^* \geq xx^* \). Furthermore, a vector \( g \in \mathcal{H}_2 \) satisfies (5.13) if and only if
\[
P = \begin{bmatrix}
1 & x^* & g^*[\cdot] \\
x & P & F_A^*[\cdot] \\
g & F_A & I_{\mathcal{H}(T_2)}
\end{bmatrix} \succeq 0
\]
or equivalently, if and only if
\[
\begin{bmatrix}
1 & x^* & g^*[\cdot] \\
x & P & F_A^*[\cdot] \\
g & F_A & I_{\mathcal{H}(T_2)}
\end{bmatrix} \succeq 0.
\]

In the second result we provide the parametrization of the solutions. For this purpose, note that in case \( F_A^*[g] F_A \geq xx^* \), then by Lemma 5.1 there exist unique \( \bar{x} \in (\mathcal{Ker} F_A)^\perp \) and \( \bar{F}_A \in L((\mathcal{Ker} F_A)^\perp, \mathcal{H}(T_2)) \) so that
\[
(5.16) \quad x = (F_A^*[F_A])^{\frac{1}{2}} \bar{x} \quad \text{and} \quad F_A = \bar{F}_A (F_A^*[F_A])^{\frac{1}{2}},
\]
with \( \bar{F}_A \) being an isometry. The space
\[
(5.17) \quad \mathcal{N} := \text{Ran}(I_{\mathcal{H}(T_2)} - \bar{F}_A F_A^*[\cdot])^{\frac{1}{2}} = \mathcal{H}(\bar{F}_A)
\]
is isometrically included in $\mathcal{H}(T_2)$, and its orthogonal complement $N^\perp$ in $\mathcal{H}(T_2)$ can be characterized as

$$N^\perp = \{ F_\alpha x : x \in \mathcal{X} \} \quad \text{with norm} \quad \| F_\alpha x \|_{\mathcal{H}(T)} = \| (F_\alpha^{(s)} F_\alpha^{(t)})^{1/2} x \|_x.$$  

**Theorem 5.7.** Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{X})$, $x \in \mathcal{X}$ and $T_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_2)$ with $T_2$ a contraction. Define $F_\alpha$ as in (5.13) and assume that $F_\alpha^{(s)} F_\alpha \geq xx^*$. Define $\tilde{x}$ and $\tilde{F}_\alpha$ by (5.16) and $N$ as in (5.17). Then all solutions $g$ to the problem (5.15) are given by the formula

$$g = F_\alpha \tilde{x} + h$$

where $h$ is a free parameter from the subspace $\mathcal{N} = \mathcal{H}(\tilde{F}_\alpha) \subset \mathcal{H}(T_2)$ subject to

$$\| h \|_{\mathcal{H}(T_2)} \leq \sqrt{1 - \| \tilde{x} \|_x^2}.$$  

Furthermore, the problem has a unique solution if and only if $\| \tilde{x} \|_x = 1$ or $\tilde{F}_\alpha F_\alpha^{(s)} = I_{\mathcal{H}(T)}$. Furthermore, we have

$$\| g \|_{\mathcal{H}(T_2)}^2 = \| F_\alpha \tilde{x} \|_{\mathcal{H}(T_2)}^2 + \| h \|_{\mathcal{H}(T_2)}^2 = \| \tilde{x} \|_{\tilde{x}}^2 + \| h \|_{\mathcal{H}(T_2)}^2.$$  

**Proof.** In the present framework, the operators $X_1$ and $X_2$ in Proposition 5.5 amount to $\tilde{x}$ and $F_\alpha^{(s)}$ respectively, and therefore, the parametrization formula (5.7) takes the form

$$g = F_\alpha \tilde{x} + (I_{\mathcal{H}(T_2)} - \tilde{F}_\alpha F_\alpha^{(s)})^{1/2} K \sqrt{1 - \tilde{x}^* \tilde{x}}$$

where now $K$ is an arbitrary vector in $\mathcal{N}$ with $\| K \|_N = \| K \|_{\mathcal{H}(T_2)} \leq 1$. Therefore, the second term on the right side of (5.21) represents a generic vector $h \in \mathcal{N}$ subject the inequality (5.20). The uniqueness statement follows immediately from (5.21). $\Box$

**Remark 5.8.** We remark that the second term $h$ on the right hand side of (5.19) represents in fact the general solution of the homogeneous interpolation problem (with interpolation condition $F_\alpha^{(s)} g = 0$). If $h$ runs through the whole space $\mathcal{H}(T_2)$, then formula (5.19) produces all $g \in \mathcal{H}(T_2)$ such that $F_\alpha^{(s)} g = x$. This unconstrained interpolation problem has a solution if and only if $x \in \text{Ran} F_\alpha^{(s)} F_\alpha^{(t)}$ and has a unique solution if and only if $\tilde{F}_\alpha F_\alpha^{(s)} = I_{\mathcal{H}(T_2)}$. If $\tilde{F}_\alpha F_\alpha^{(s)} \neq I_{\mathcal{H}(T_2)}$, then the unconstrained problem has infinitely many solutions, and $F_\alpha \tilde{x}$ has the minimal possible $\mathcal{H}(T_2)$-norm, which is $\| \tilde{x} \|_x$ by (5.18). Thus, if $\| \tilde{x} \|_x = 1$, then uniqueness occurs since the minimal norm solution already has unit norm.

**Appendix A. A Kreĭn space lemma**

In this section we prove a general Kreĭn space lemma. See the end of the introduction for the basic definitions of Kreĭn spaces and some classes of Kreĭn space operators. Before we state the result, some further preliminaries are required. Given two Kreĭn spaces $(\mathcal{X}', \mathcal{J}')$ and $(\mathcal{X}, \mathcal{J})$ and given an operator $T \in \mathcal{L}(\mathcal{X}', \mathcal{X})$, we define the $(\mathcal{J}', \mathcal{J})$-adjoint of $T$ to be the operator $T^{(s)} \in \mathcal{L}(\mathcal{X}, \mathcal{X'})$ so that

$$[Tx, y]_{\mathcal{J}} = [x, T^{(s)} y]_{\mathcal{J}'} \quad \text{for all} \ x \in \mathcal{X}', \ y \in \mathcal{X}.$$
The elementary computation
\[ [x, T^*[y]]_J = [T x, y]_J = (J T x, y)_X = (x, T^* J y)_X = (J^* x, J^{-1} T^* J y)_X, \]
shows that
\[ T^*[y] = J^{-1} T^* J. \]
In case \((X', J') = (X, J)\) we use the term \(J\)-adjoint rather than \((J, J)\)-adjoint for \(T^*[\cdot]\). If it happens that furthermore \(T = T^*[\cdot]\), we say that \(T\) is \(J\)-self-adjoint.

Note that elsewhere in this paper the notation \(T^*[\cdot]\) is also used for the adjoint with respect to a de Branges-Rovnyak space (cf., Remark 3.2). We use the same notation here for a Kreın space adjoint since this notation is standard and is not used outside the current appendix.

Using the Kreın space adjoint notation we note that \(T\) is a \((J, J')\)-isometry if \(T^*[\cdot]T = 1_X\). We then say that \(T\) is a \((J, J')\)-coisometry if \(TT^*[\cdot] = 1_{X'}\). The latter corresponds to the identity \(TJ^{-1} T^* = J^{-1}\), which corresponds to the second entry in (1.1) with the inequality replaced by an equality in case \(J\) and \(J'\) are signature operators. As before, \(T\) is \((J, J')\)-unitary if \(T\) is both \((J, J')\)-isometric and \((J, J')\)-coisometric, i.e., when \(T\) is invertible with \(T^{-1} = T^*[\cdot]\).

A well-known property for the Hilbert-space case is that a surjective isometry is unitary. This property extends to the Kreın space setting, as we record in the following remark.

**Remark A.1.** An onto \((J', J)\)-isometry is in fact \((J', J)\)-unitary. Indeed, suppose that \(V: X' \rightarrow X\) is an onto \((J', J)\)-isometry. To show that \(V\) is \((J', J)\)-coisometric we must show that \(V J'^{-1} V^* = J^{-1}\). Given \(x \in X\), we can find \(y \in X'\) so that \(x = JVy\). We then compute
\[ V J'^{-1} V^* x = V J'^{-1} V^* (J V y) = V J'^{-1} (V^* J V) y = V J^{-1} J' y = V y = J^{-1} x, \]
implying that \(V\) is also \((J, J')\)-coisometric as claimed.

Finally let us say that two invertible self-adjoint operators \(J\) on \(X\) and \(J'\) on \(X'\) are congruent if there exists a linear bijection \(R \in \mathcal{L}(X', X)\) so that \(R^* J R = J'\). By taking inverses in this equation we see that if \(R\) implements a congruence from \(J'\) to \(J\), then \(R^{-1}\) implements a congruence from \(J'^{-1}\) to \(J^{-1}\).

The following lemma shows how a Kreın-space isometry \(V\) has a row extension \([V \ W]\) to a Kreın space unitary operator. It is a variation on [9, Theorem 2.3 (3)] in a more general setting. Let us also mention that there is an extensive discussion in Section 2.3 of [31] of row extensions of a Kreın-space bicontraction operator \(T\) to a Kreın space bicontraction operator on a larger space \([T \ S]\). For simplicity, we give a self-contained direct proof of the special case of interest here.

**Lemma A.2.** Suppose that \((X'_1, J'_1)\) and \((X, J)\) are Kreın spaces. Let \(V \in \mathcal{L}(X'_1, X)\) be a \((J'_1, J)\)-isometry. Then there exist a Kreın spaces \((X'_2, J'_2)\) and an operator \(W \in \mathcal{L}(X'_2, X)\) so that \(J\) is congruent with \(J' := \begin{bmatrix} J'_1 & 0 \\ 0 & J'_2 \end{bmatrix}\) and \([V \ W]\) is \((J, J')\)-unitary.

Moreover, given a Kreın space \((X'_2, J'_2)\), an operator \(W \in \mathcal{L}(X'_2, X)\) completes \(V\) to a \((J, J')\)-unitary operator if and only if \(W\) is an injective solution to the following indefinite-metric Cholesky factorization problem:

\[ \text{(A.1)} \quad W J'_2^{-1} W^* = J^{-1} - V J'^{-1} V^*. \]
Proof. Since $V$ is a $(J'_1, J)$-isometry, we have $V^*JV = J'_1$. For $x, y \in X'_2$ we then have that

$$[Vx, Vy]_J = \langle J V x, V y \rangle_X = \langle V^* J V x, y \rangle_{X'_1} = \langle J'_1 x, y \rangle_{X'_1} = [x, y]_{J'_1}.$$ 

Hence the restriction of the indefinite inner product of $(X, J)$ to $M := \text{Ran} V$ defines a Krein space on $M$, that is, $M$ is a regular (or Krein) subspace of $X$. Another formulation of regularity of the subspace $M$ as a subspace of $K$ (see e.g. [31, Theorem 1.1.1]) is that $M$ be the range of a $J$-orthogonal projection operator, i.e., a bounded $J$-self-adjoint idempotent operator on $X$. Explicitly for the case here with $M = \text{Ran} V$, one can check that $M = \text{Ran} P$ where the $J$-orthogonal projection operator $P$ is given by

$$P = V J'^{-1} V^* J,$$

i.e., one can check that $P = J^{-1} P^* J = P^2$, $P^2 = P$ and $PV = V$ (implying when combined with the definition of $P$ that $\text{Ran} P = \text{Ran} V$).

Another characterization of regularity of $M$ (see e.g. Theorem 1.3 in [32]) is that $X$ is obtained as the direct sum of $M$ and the Krein space orthogonal complement $M^\perp := \{ x \in X : [x, y]_J = 0 \text{ for all } y \in M \}$ of $M$ in $X$. Explicitly the associated $J$-orthogonal projection operator $Q$ such that $\text{Ran} Q = M^\perp$ is given as the $J$-orthogonal projection operator $Q$ $J$-complementary to $P$, namely

$$Q = I_X - P = I_X - V J'^{-1} V^* J. \quad \text{(A.2)}$$

Indeed one can check directly that

$$Q = Q^2, \quad Q^2 = Q, \quad QV = 0, \quad P + Q = I_X.$$

We can get another formula for the $J$-orthogonal projection operator with range equal to $M^\perp$ as follows. Set $\tilde{X}'_2 = M^\perp$ and let $\tilde{W} : \tilde{X}'_2 \to X$ be the inclusion map of $\tilde{X}'_2$ into $X$. Hence by construction $\tilde{W}$ is injective. Define $\tilde{J}'_2$ by

$$\tilde{J}'_2 = \tilde{W}^* J \tilde{W}.$$ 

Since $\text{Ran} \tilde{W} = M^\perp$ is a regular subspace of $X$, it follows that $\tilde{J}'_2$ is an invertible self-adjoint operator on $\tilde{X}'_2$. By an argument parallel to that done above for the operator $V$ having $\text{Ran} V = M$, we see that $Q$ (the $J$-orthogonal projection operator with range equal to $M^\perp$) is given by

$$Q = \tilde{W} \tilde{J}'_2^{-1} \tilde{W}^* J. \quad \text{(A.3)}$$

Combining (A.2) with (A.3) gives us $\tilde{W} \tilde{J}'_2^{-1} \tilde{W}^* J = I_X - V J'^{-1} V^* J$, or in more Hermitian form,

$$\tilde{W} \tilde{J}'_2^{-1} \tilde{W}^* = J^{-1} - V J'^{-1} V^*,$$

i.e., $\tilde{W}$ arises as an injective solution of an indefinite-metric Cholesky factorization problem of the type (A.1).

Since $\text{Ran} V = M$ is $J$-orthogonal to $\text{Ran} \tilde{W} = M^\perp$, we certainly have

$$0 = \tilde{W}^* JV, \quad 0 = V^* J \tilde{W}.$$

Let us next compute

$$\begin{bmatrix} V^* \\ \tilde{W}^* \end{bmatrix} J \begin{bmatrix} V & \tilde{W} \end{bmatrix} = \begin{bmatrix} V^* JV & V^* J \tilde{W} \\ \tilde{W}^* JV & \tilde{W}^* J \tilde{W} \end{bmatrix} = \begin{bmatrix} J'_1 & 0 \\ 0 & \tilde{J}'_2 \end{bmatrix},$$

\]
i.e., \( \begin{bmatrix} V & W \end{bmatrix} \) is a \( \begin{bmatrix} J' & 0 \\ 0 & J_2 \end{bmatrix} \)-isometry. But we also know that \( \text{Ran} \begin{bmatrix} V & W \end{bmatrix} = \mathcal{M} + \mathcal{M}^\perp = \mathcal{X} \), so by Remark A.4 it follows that in fact \( \begin{bmatrix} V & \bar{W} \end{bmatrix} \) is a \( \begin{bmatrix} J' & 0 \\ 0 & J_2 \end{bmatrix} \)-unitary operator, and we conclude that \( \bar{W} \) implements the desired completion of \( V \) to a Kreïın space unitary operator.

Now suppose that \( W \) is any injective solution of the \( J \)-Cholesky factorization problem \( \text{A.1} \). As observed in the first part of the proof, the operator to be factored on the right-hand side of \( \text{A.1} \) multiplied by \( W \), namely, \((J^{-1} - VJ_2^{-1}V^*)J = I_X - VJ_2^{-1}V^*J\), is equal to the \( J \)-orthogonal projection operator with range equal to \((\text{Ran} V)^\perp J\), and hence has a fairly substantial kernel, namely

\[
\text{Ker}(I_X - VJ_2^{-1}V^*) = \text{Ran} V.
\]

Hence the Hermitian operator \((J^{-1} - VJ_2^{-1}V^*)J\) has closed range and its restriction to \((\text{Ker}(J^{-1} - VJ_2^{-1}V^*))^\perp\) is bounded below. Using spectral theory we can then find a factorization

\[
A.4 \quad J^{-1} - VJ_2^{-1}V^* = WJ_2^{-1}W^*
\]

with \( J_2 \) a bounded, invertible signature operator on \( \mathcal{X}_2 = (\text{Ker}(J^{-1} - VJ_2^{-1}V^*))^\perp \), i.e., the pair \((W, J_2)\) is a solution of the \( J \)-Cholesky factorization problem \( \text{A.1} \) with \( W \) injective. Note that any other choice of \( J_2 \) in such a factorization is then uniquely determined up to a congruence with any particular such \( J_2 \), with a corresponding adjustment of \( \bar{W} \) to get \( W \):

\[
J_2' = XJ_2X^*, \quad W = \bar{W}X^{-1}
\]

for some invertible \( X : \mathcal{X}_2' \to \mathcal{X}_2 \).

The equality \( \text{A.4} \) then can be interpreted as saying that \( \text{Ran} W = \mathcal{M}^\perp J \) (where \( \mathcal{M} = \text{Ran} V \subset \mathcal{X} \)). We may then proceed as in the first part of the proof to see that \( \begin{bmatrix} V & W \end{bmatrix} = \begin{bmatrix} J' & 0 \\ 0 & J_2 \end{bmatrix} \), \( J \)-unitary as expected. \( \square \)

The following corollary adds some additional information which will be needed to complete the proof of Theorem 3.14 in Section 3.3. We shall make use of the following notation: for an invertible self-adjoint operator \( J \) on some separable Hilbert space \( \mathcal{X} \),

\[
\text{Inertia}(J) = (\kappa_+(J), \kappa_-(J))
\]

means that \( \kappa_+(J) \) and \( \kappa_-(J) \) denote the respective dimensions (with \( \infty \) allowed) of the positive (respectively negative) spectral subspaces of \( J \). Note that two Kreïın spaces \((\mathcal{X}, J) \) and \((\mathcal{X}', J') \) are Kreïın-space isomorphic if and only if \( \text{Inertia}(J) = \text{Inertia}(J') \).

**Corollary A.3.** Suppose that \((\mathcal{X}_1, J_1), (\mathcal{X}_2, J_2), (\mathcal{X}'_1, J'_1), (\mathcal{X}'_2, J'_2)\) are Kreïın spaces such that

(i) \((\mathcal{X}_1 \oplus \mathcal{X}_2, \begin{bmatrix} J_1' & 0 \\ 0 & J_2 \end{bmatrix})\) is Kreïın-space isomorphic to \((\mathcal{X}'_1 \oplus \mathcal{X}'_2, \begin{bmatrix} J_1' & 0 \\ 0 & J_2' \end{bmatrix})\), and

(ii) \((\mathcal{X}_1, J_1)\) is Kreïın-space isomorphic to \((\mathcal{X}'_1, J'_1)\).

Then:

1. If \( \dim \mathcal{X}_j < \infty \) and \( \dim \mathcal{X}'_j < \infty \) for \( j = 1, 2 \), then \((\mathcal{X}_2, J_2)\) is Kreïın-space isomorphic to \((\mathcal{X}'_2, J'_2)\), i.e.,

\[
\kappa_+(J_2) = \kappa_+(J'_2) \quad \text{and} \quad \kappa_-(J_2) = \kappa_-(J'_2).
\]
(2) If \( \kappa_-(J_1) = \kappa_-(J_2) = 0 \), i.e., \((X_1, J_1)\) and \((X'_1, J'_1)\) are Hilbert spaces of the same dimension, then we can at least still conclude that
\[
\begin{align*}
\kappa_+(J_1) + \kappa_+(J_2) &= \kappa_+(J'_1) + \kappa_+(J'_2), \\
\kappa_-(J_2) &= \kappa_-(J'_2).
\end{align*}
\]

\textbf{Proof.} In case (1), all the indices \( \kappa_\pm(J_j) \) \((j = 1, 2)\) are finite. Note that hypothesis (i) is the statement that
\[
\begin{align*}
\kappa_+(J_1) + \kappa_+(J_2) &= \kappa_+(J'_1) + \kappa_+(J'_2), \\
\kappa_-(J_1) + \kappa_-(J_2) &= \kappa_-(J'_1) + \kappa_-(J'_2),
\end{align*}
\]
while the hypothesis in (ii) is that
\[
\kappa_+(J_1) = \kappa_+(J'_1), \quad \kappa_-(J_1) = \kappa_-(J'_1).
\]

Subtraction of the first of (A.6) from the first of (A.5) and of the second of (A.6) from the second of (A.5) and using that all these integers are finite leads us to the two identities
\[
\begin{align*}
\kappa_+(J_2) &= \kappa_+(J'_2), \\
\kappa_-(J_2) &= \kappa_-(J'_2)
\end{align*}
\]
which in turn is that statement that \((X_2, J_2)\) is Kreĭn-space isomorphic to \((X'_2, J'_2)\). Take now the extra hypothesis to be as in case (2). In this case the quantity \(\kappa_+(J_1) + \kappa_+(J_2)\) can be identified as the dimension of a maximal positive subspace in \((X_1 \oplus X_2, \left[ \begin{smallmatrix} J_1 & 0 \\ 0 & J_2 \end{smallmatrix} \right])\) while the quantity \(\kappa_+(J'_1) + \kappa_+(J'_2)\) can be identified as the dimension of a maximal positive subspace in \((X'_1 \oplus X'_2, \left[ \begin{smallmatrix} J'_1 & 0 \\ 0 & J'_2 \end{smallmatrix} \right])\). The assumption that \((X_1 \oplus X_2, \left[ \begin{smallmatrix} J_1 & 0 \\ 0 & J_2 \end{smallmatrix} \right])\) is Kreĭn-space isomorphic to \((X'_1 \oplus X'_2, \left[ \begin{smallmatrix} J'_1 & 0 \\ 0 & J'_2 \end{smallmatrix} \right])\) then implies that these two quantities must be the same (finite or infinite). Similarly, one can identify \(\kappa_-(J_2)\) as the dimension of a maximal negative subspace in \((X_1 \oplus X_2, \left[ \begin{smallmatrix} J_1 & 0 \\ 0 & J_2 \end{smallmatrix} \right])\) (since \(X_1\) is a Hilbert space by assumption) and similarly \(\kappa_-(J'_2)\) is the dimension of a maximal negative subspace in \((X'_1 \oplus X'_2, \left[ \begin{smallmatrix} J'_1 & 0 \\ 0 & J'_2 \end{smallmatrix} \right])\). Again, the assumed Kreĭn-space isomorphism between \((X_1 \oplus X_2, \left[ \begin{smallmatrix} J_1 & 0 \\ 0 & J_2 \end{smallmatrix} \right])\) and \((X'_1 \oplus X'_2, \left[ \begin{smallmatrix} J'_1 & 0 \\ 0 & J'_2 \end{smallmatrix} \right])\) implies that we must have the equality \(\kappa_-(J_2) = \kappa_-(J'_2)\) (finite or infinite). \hfill \(\square\)

\textbf{Appendix B. Schur multipliers and their adjoints}

As is well known, any Schur-class function \(S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})\) gives rise to a Schur-class function \(S^\ast(\lambda) := S(\lambda^\ast) \in \mathcal{S}(\mathcal{Y}, \mathcal{U})\). As a consequence of (3.15) and (3.19), it follows that a similar result holds in multivariable settings.

\textbf{Theorem B.1.} If \(S(z) = \sum_{\alpha \in \mathbb{F}_d^+} S_\alpha z^\alpha\) belongs to \(\mathcal{S}_{ac,d}(\mathcal{U}, \mathcal{Y})\), then \(S^\ast(z) := \sum_{\alpha \in \mathbb{F}_d^+} S_\alpha^\ast z^\alpha\) belongs to \(\mathcal{S}_{ac,d}(\mathcal{Y}, \mathcal{U})\).

\textbf{Proof.} Since \(S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})\), i.e., the formal kernel (3.28) is positive, it follows (see e.g., [15] Theorem 3.1) that there exists a Hilbert space \(\mathcal{X}\) and a unitary connection operator \(U\) of the form
\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \end{bmatrix} \\ \begin{bmatrix} A_d & B_d \\ C & D \end{bmatrix} \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}
\]

\begin{equation}
\text{(B.1)}
\end{equation}
so that \( S(z) \) can be realized as a formal power series in the form

\[
S(z) = D + \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} C\alpha B_j z^\alpha \cdot z_j = D + C(I - Z(z)A)^{-1}Z(z)B,
\]

Then

\[
S^*(z) = D^* + \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} B_j^* A^{\alpha^\top} C^* z_j z^\alpha
\]

\[
= D^* + \sum_{j=1}^{d} \sum_{\alpha \in \mathbb{F}_d^+} B_j^* z_j A^{\alpha^*} C^* z^\alpha
\]

\[
= D^* + B^* Z(z)^\top (I - A^* Z(z)^\top)^{-1} C^* = D^* + B^* (I - Z(z)^\top A^*)^{-1} Z(z)^\top C^*,
\]

and since \( U \) is unitary, we have

\[
\begin{align*}
K_{S^*}(z, \zeta) &:= k_{S^*}(z, \zeta)I_d - S^*(z)(k_{S^*}(z, \zeta)I_d)S^*(\zeta)^* \\
&= B^* (I - Z(z)^\top A^*)^{-1} (k_{S^*}(z, \zeta)I_d + Z(z)^\top k_{S^*}(z, \zeta)Z(\zeta)) (I - AZ(\zeta))^{-1} B.
\end{align*}
\]

As we have seen in the proof of Theorem 3.14, the kernel \( k_{S^*}(z, \zeta)I_d + Z(z)^\top k_{S^*}(z, \zeta)Z(\zeta) \) is positive; therefore the kernel \( K_{S^*}(z, \zeta) \) is also positive and hence \( S^* \in \mathcal{S}_{nc,d}(\mathcal{Y}, \mathcal{U}) \).

In [7, p. 110], it was erroneously suggested that for an \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \), the function \( S^* \) does not have to belong to \( \mathcal{S}_d(\mathcal{Y}, \mathcal{U}) \). It was stated that for the Schur multiplier \( S(\lambda) = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \), the associated function \( S^*(\lambda) = \begin{bmatrix} \lambda_1 \lambda_2 \end{bmatrix} \) is not a Schur multiplier, since the kernel

\[
K^{S^*}(\lambda, w) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \lambda_1 & \lambda_2 \\ 1 - \lambda_1 w_1 - \lambda_2 w_2 & \lambda_1 & \lambda_2 & w_1 \\ w_2 & w_1 & w_2 & 1 \end{bmatrix}
\]

is not positive on \( \mathbb{B}^d \), as the matrix

\[
\begin{bmatrix}
K^{S^*}(w^{(1)}, w^{(1)}) & K^{S^*}(w^{(1)}, w^{(2)}) \\
K^{S^*}(w^{(2)}, w^{(1)}) & K^{S^*}(w^{(2)}, w^{(2)})
\end{bmatrix}, \quad w^{(1)} = \left( \frac{1}{2}, 0 \right), \quad w^{(1)} = \left( 0, \frac{1}{2} \right)
\]

is not positive semidefinite. More careful later inspection revealed that the latter matrix equals

\[
\begin{bmatrix}
1 & 0 & 1 & \frac{1}{4} \\
0 & \frac{1}{3} & 0 & 1 \\
1 & 0 & \frac{1}{4} & 0 \\
-\frac{1}{4} & 1 & 0 & 1
\end{bmatrix}
\]

and is positive definite. Moreover, the kernel \([B.3] \) is actually positive on \( \mathbb{B}^2 \) by the following abelianized version of Theorem 3.1.

**Theorem B.2.** If \( S(\lambda) \) belongs to \( \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \), then \( S^*(\lambda) := S(\lambda)^* \) belongs to \( \mathcal{S}_d(\mathcal{Y}, \mathcal{U}) \).
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