A NOTE ON THE NAVARRO CONJECTURE FOR
ALTERNATING GROUPS WITH ABELIAN DEFECT

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Abstract. G. Navarro proposed (in [8]) a refinement of the unsolved McKay conjecture involving certain Galois automorphisms. The author verified this new conjecture for the alternating groups $A(\Pi)$ when $p = 2$ (see [7]). For odd primes $p$ the conjecture is more difficult to study due to the complexities in the $p$-local character theory. We consider the principal blocks of $A(\Pi)$ with an abelian defect group when $p$ is odd: in this case the Navarro conjecture holds for $p$-singular characters.

1. McKay and Navarro conjectures

Let $G$ be a finite group, $|G| = n$, $p$ be a prime dividing $n$, $D$ a Sylow $p$-group of $G$, and $N_G(D)$ the normalizer of $D$ in $G$. Let $\text{Irr}(G)$ denote the irreducible characters of $G$, and $\text{Irr}_{p'}(G)$ the subset of characters whose degree is relatively prime to $p$. The following is a well-known conjecture.

**Conjecture 1.1.** (McKay, [1])

$$|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(N_G(D))|.$$

Recently G. Navarro strengthened the McKay conjecture in the following way. All irreducible complex characters of $G$ are afforded by a representation with values in the $n$th cyclotomic field $\mathbb{Q}_n/\mathbb{Q}$ (Lemma 2.15, [4]). Then the Galois group $\mathcal{G} = \text{Gal}(\mathbb{Q}_n/\mathbb{Q})$ permutes the elements of $\text{Irr}(G).$ We denote the action of $\sigma$ on $\chi \in \text{Irr}(G)$ by $\chi^\sigma.$ Then $\chi \in \text{Irr}(G)$ is $\sigma$-fixed if its values are fixed by $\sigma,$ that is, $\chi^\sigma = \chi.$ Let $e$ be a nonnegative integer and consider $\sigma_e \in \mathcal{G}$ where $\sigma_e(\xi) = \xi^{p^e}$ for all $p'$-roots of unity $\xi.$ Define $\mathcal{N}$ to be the subset of $\mathcal{G}$ consisting of all such $\sigma_e.$ Let $\text{Irr}_{p'}^\sigma(G)$ and $\text{Irr}_{p'}^\sigma(N_G(D))$ be the subsets of $\text{Irr}_{p'}(G)$ and $\text{Irr}_{p'}(N_G(D))$ respectively fixed by $\sigma \in \mathcal{N}.$

2000 Mathematics Subject Classification. Primary 20C30.
Conjecture 1.2. (Navarro, [8]) Let \( \sigma \in \mathcal{N} \). Then
\[
|\text{Irr}_{p'}^\sigma(G)| = |\text{Irr}_{p'}^\sigma(N_G(D))|.
\]

The Navarro conjecture follows from the existence of a bijection \( \phi \) from \( \text{Irr}_{p'}(G) \) to \( \text{Irr}_{p'}(N_G(D)) \) that commutes with \( \mathcal{N} \). That is, \( \phi(\chi^\sigma) = \phi(\chi)^\sigma \) for all \( \sigma \in \mathcal{N} \) and \( \chi \in \text{Irr}_{p'}(G) \). The author verified in [6] that the Navarro conjecture holds for the alternating groups \( A(\Pi) \) when \( p = 2 \). The verification when \( p \) is odd is more complicated since little is known about values of \( \text{Irr}_{p'}(N_{A(\Pi)}(D)) \). However in the special case that \( A(\Pi) \) has an abelian defect group (equivalently \( |\Pi| = n_0 + wp \) with \( w < p \)) this paper verifies that the Navarro conjecture holds for the \( p \)-singular characters of the principal block. The proof relies on results of P. Fong and M. Harris (see §4, [3]) on the irrationalities of the \( p \)-singular characters of \( N_{A(\Pi)}(D) \).

2. A local-global bijection

2.1. \( p' \)-splitting characters of \( G \).

Let \( n \in \mathbb{N} \). A partition \( \lambda \) of \( n \) is a non-increasing integer sequence \( (a_1, \ldots, a_m) \) satisfying \( a_i \geq \cdots \geq a_m \) and \( \sum_i a_i = n \). Then the Young diagram of \( \lambda \) is \( n \) nodes placed in rows such that the \( i \)th row of \( \lambda \) consists of \( a_i \) nodes. The \((i, j)\)-node of \( \lambda \) lies in the \( i \)th row and \( j \)th column of the Young diagram. The \((i, j)\)-hook \( h_{ij}^\lambda \) of \( [\lambda] \) and consists of the \((i, j)\)-node (or corner of \( h_{ij}^\lambda \)), all nodes in the same row and to the right of the corner, and all nodes in the same column and below the corner. The column-lengths of \( [\lambda] \) form the conjugate partition \( \lambda^* \) of \( n \). Partitions where \( \lambda = \lambda^* \) are self-conjugate. Let \( \lambda = \lambda^* \) and \( \delta(\lambda) = \{\delta_{jj}\} \) be the set of diagonal hooks of \( \lambda \) i.e. \( \delta_{jj} = h_{jj} \), which are necessarily odd. When there is no ambiguity we write \( h_{ij}^\lambda = h_{ij} \).

Every \( \lambda \) is expressed uniquely in terms of its \( p \)-core \( \lambda^0 \) and \( p \)-quotient \( (\lambda_0, \lambda_2, \cdots, \lambda_{p-1}) \). The \( p \)-core \( \lambda^0 \) is the unique partition that results when all possible hooks of length \( p \) are removed from \( \lambda \). The \( p \)-quotient \( \langle \lambda \rangle \) is a \( p \)-tuple of (sub-)partitions which encode the \( p \)-hooks of \( \lambda \).

Henceforth, let \( \Pi \) be a set of size \( n \) and \( G = S(\Pi) \) and \( G^+ = A(\Pi) \) be respectively the symmetric and alternating groups on \( \Pi \). The elements of \( \text{Irr}(G) \) are labeled by partitions \( \{\lambda \vdash n\} \). Then \( \text{Irr}(G^+) \) is obtained from \( \text{Irr}(G) \) by restriction. If \( \alpha \) is an irreducible character for some finite group \( J \), and \( K \) is a subgroup of \( J \), the notation \( \alpha|_K \) indicates restriction of the subgroup \( K \).

Theorem 2.1. The irreducible characters of \( G^+ \) arise from those of \( G \) in two ways. If \( \lambda \neq \lambda^* \) then \( \chi_{\lambda}|_{G^+} = \chi_{\lambda^*}|_{G^+} \) is in \( \text{Irr}(G^+) \). If \( \lambda = \lambda^* \) then \( \chi_{\lambda}|_{G^+} \) splits into two conjugate characters \( \chi_{\lambda}^+ \) and \( \chi_{\lambda}^- \) in \( \text{Irr}(G^+) \).
The conjugacy classes $\kappa$ of $S(\Pi)$ are labeled by cycle-types of permutations of $n$. If $\lambda = \lambda^*$ we let $\kappa_{\delta(\lambda)}$ be the conjugacy class determined by the cycle-type of $(\delta_{11}, \cdots, \delta_{dd})$. Then $\kappa_{\delta(\lambda)}$ splits into $\kappa_{\delta(\lambda),+}$ and $\kappa_{\delta(\lambda),-}$ when viewed as a class of $G^+$. Let $\text{Irr}^*(G)$ be the set of splitting characters, i.e. those that split into two conjugate characters when restricted to $G^+$. The following is a classical result of Frobenius (see e.g. Theorem (4A), [3]).

**Theorem 2.2.** Suppose $\chi_\lambda$ is an irreducible character of $G$ which splits on $G^+$. Let $g \in G^+$. Then $(\chi_{\lambda,+} - \chi_{\lambda,-})(g) \neq 0$ if and only if $g$ is in $\kappa_{\delta(\lambda)}$. Moreover, $\chi_{\lambda,\pm}$ and $\kappa_{\delta(\lambda),\pm}$ may be labeled so that

$$
\chi^\pm_\lambda(g) = \frac{1}{2}[\epsilon_\lambda + \sqrt{\epsilon_\lambda \prod_j \delta_{jj}}] \quad \text{if} \ g \in \kappa_{\delta(\lambda),\pm}
$$

$$
\chi^\pm_\lambda(g) = \frac{1}{2}[\epsilon_\lambda - \sqrt{\epsilon_\lambda \prod_j \delta_{jj}}] \quad \text{if} \ g \in \kappa_{\delta(\lambda),\mp}
$$

where $\epsilon_\lambda = (-1)^{\frac{a_n}{2}}$.

By extension, $\text{Irr}^*(G^+)$ is the set of (pairs) of characters that arise from restricting elements of $\text{Irr}^*(G)$. Suppose $n = wp$, where $w < p$. By a condition of Macdonald (see [3]), the elements of $\text{Irr}_{p'}(G)$ are labeled by partitions for whom $\sum |\lambda_\gamma| = \omega$. Then the $p'$-splitting characters are labeled by self-conjugate partitions that satisfy the Macdonald condition.

2.2. $p'$-splitting characters of $H$.

Let $B$ be a $p$-block of $G$ the defect group $D$ and $b$ the $p$-block of $N_G(D)$ which is the Brauer correspondent of $B$. Let $\nu$ be the exponential valuation of $\mathbb{Z}$ associated with $p$ normalized so $\nu(p) = 1$. The height of the $\chi$ in $B$ is the nonnegative integer $h(\chi)$ such that $\nu(\chi(1)) = \nu(|G|) - \nu(|D|) + h(\chi)$. The height of $\xi$ in $b$ is the nonnegative integer $h(\xi)$ such that $\nu(\xi) = \nu(|N_G(D)|) - \nu(|D|) + h(\xi)$. Let $M(B)$ and $M(b)$ be the characters of $B$ and $b$ of height zero. By the Nakayama conjecture a $p$-block $B$ of $G$ is parametrized by a $p$-core $\lambda^0$ so $\chi_\mu \in B$ if and only if $\lambda^0 = \mu^0$. In particular, $n = n_0 + wp$ where $n_0 = |\lambda_0|$. We suppose that $B$ has abelian defect group $D$ or equivalently $w < p$. Thus $\Pi = \Pi_0 \cup \Pi_1$ is the disjoint union of sets $\Pi_0$ and $\Pi_1$ of cardinality $n_0$ and $wp$. We may suppose $\Pi_1 = \Gamma \times \Omega$ where $\Gamma = \{1, 2, \cdots, p\}$ and $\Omega$ is a set of $w$ elements. Let $X = S(\Gamma)$ and $Y = N_X(P)$ where $P$ is a fixed Sylow $p$-subgroup of $X$. Note that the when $B$ is a Sylow subgroup the $p'$-irreducible characters agree with the height zero characters.

We take $D$ as the Sylow $p$-subgroup $P^\Omega$ of $S(\Pi_1)$ and set $H = N_G(D)$ so that $H = H_0 \times H_1$ with $H_0 = S(\Pi_0)$ and $H_1 = Y \wr S(\Omega)$. The Brauer
Let \( p \) be a partition of \( n \) with \( p \)-core \( \lambda^0 \) and \( p \)-quotient \( \langle \lambda \rangle = (\lambda_0, \ldots, \lambda_{p-1}) \) normalized as follows: if \( \mu = \lambda^* \) then \( \lambda_i = (\mu_{p-i-1})^* \).

Let \( p^* = \frac{p - 1}{2} \). Then \( \lambda = \lambda^* \) implies \( \lambda_{p^*} = \lambda_{p^*}^* \). Let \( Y^\gamma = \{ \xi_{\gamma} : 0 \leq \gamma \leq p - 1 \} \). The characters in \( H^\gamma \) have the form \( \chi_{\tau} \times \psi_{\lambda} \) where \( \tau \) is a \( p \)-core partition and \( \chi_{\tau} \in \text{Irr}(H_0) \) and \( \psi_{\lambda} \in \text{Irr}(H_1) \) and \( \Lambda \) is a mapping

\[
Y^\gamma \longrightarrow \{ \text{Partitions} \}, \, \xi_{\gamma} \mapsto \mu_{\gamma},
\]

such that \( \sum_{\gamma} |\mu_{\gamma}| = w \). We also represent \( \Lambda \) by the \( p \)-tuple \( (\mu_1, \ldots, \mu_p) \).

Then \( M(B) \) and \( M(b) \) are in bijection via \( f : \chi_{\lambda} \mapsto \chi_{\lambda^0} \times \psi_{\langle \lambda \rangle} \) (see [2] for details). Hence \( \text{Irr}_{p'}(G) \) and \( \text{Irr}_{p'}(H) \) are in bijection via \( f = \cup_B f_B \).

There is an induced bijection \( f^+ \) between \( \text{Irr}_{p'}(G^+) \) and \( \text{Irr}_{p'}(N_{G^+}(D)) \).

Let \( \text{sgn}_H = \text{sgn}_G |_{H} \) and \( \text{sgn}_Y = \text{sgn}_X |_{Y} \). If \( (f, \sigma) \) is an element of \( H = Y \wr S(\Omega) \) with \( f \in S(\Omega) \) and \( \gamma \in S(\Omega) \) and \( \sigma \in S(\Omega) \), then

\[
\text{sgn}_H(f, \sigma) = \text{sgn}_{S(\Omega)}(\sigma) \prod_{i \in \Omega} \text{sgn}_Y(f(i)).
\]

Let \( H^+ = N_{G^+}(D) \). Then \( \Lambda \) is a splitting mapping of \( H \) if \( \psi_{\lambda} \) splitting character of \( H \) i.e. \( (\psi_{\lambda})|_{H^+} = \psi_{\lambda^*, +} - \psi_{\lambda^*, -} \) where \( \psi_{\lambda^*, \pm} \in (H^+)^\gamma \). Let \( * \) be the duality \( \Lambda \mapsto \Lambda^* \) where \( \Lambda^* : \xi_{\gamma} \mapsto (\lambda_{p-\gamma})^* \). The following is Proposition (4D) in [3].

**Proposition 2.3.** Let \( \psi_{\lambda} \in \text{Irr}(H) \). Then \( \text{sgn}_H \psi_{\lambda} = \psi_{\lambda^*}^* \). In particular, \( \psi_{\lambda} \) is a splitting character if and only if \( \Lambda = \Lambda^* \).

Proposition 2.3 implies that map \( f^+ \) induced by \( f \) remains a bijection on splitting characters (and \( p' \)-splitting characters). That is, \( \text{Irr}_{p'}(G^+) \) is in bijection with \( \text{Irr}_{p'}(H^+) \). In particular, if \( \lambda \neq \lambda^* \) then \( \chi_{\lambda}|_{G^+} = \chi_{\lambda^*}|_{G^+} \) is mapped to \( \psi_{\lambda^*}|_{H^+} = \psi_{\lambda^*}^*|_{H^+} \) and if \( \lambda = \lambda^* \) then \( \chi_{\lambda}^\pm \) maps to \( \psi_{\lambda}^\pm \).

### 3. Values of \( p \)-singular characters

We say \( \lambda \) is \( p \)-singular if \( \lambda_{p^*} \neq 0 \) and \( \lambda_i = 0 \) for all \( i \in \{0, \ldots, p-1\} - p^* \). Then \( \chi_{\lambda} \in \text{Irr}_{p'}(G) \) is \( p \)-singular if \( \lambda \) is. The notation \( \text{Irr}_{p', \text{sing}}(G) \) denotes the \( p \)-singular \( p' \)-characters and \( \text{Irr}_{p', \text{sing}}(G^+) \) is the restrictions to \( G^+ \). Then \( \text{Irr}_{p', \text{sing}}(H) \) and \( \text{Irr}_{p', \text{sing}}(H^+) \) are defined analogously. It is immediate from the definition of \( f^+ \) that \( \text{Irr}_{p', \text{sing}}(G^+) \) and \( \text{Irr}_{p', \text{sing}}(H^+) \) are in bijection. We show that \( f^+ \) commutes with the action of \( \sigma \in \mathcal{N} \) on \( p \)-singular \( p' \)-characters by describing explicitly the relevant irrational character values.
In [6], the author describes how to obtain the set of diagonal hooks \( \delta(\lambda) \) of a symmetric partition \( \lambda = \lambda^* \) given just the \( p \)-core \( \lambda^0 \) and the \( p \)-quotient \( \langle \lambda \rangle \). The following special case (Theorem 4.3 in [6]) is relevant to the goals of this paper.

**Theorem 3.1.** Suppose \( \lambda^0 \) is empty and \((\emptyset, \ldots, \lambda_p^*, \ldots, \emptyset)\) such that \( \lambda_p^* = (\lambda_{p'}^*)^* \) and \( \delta(\lambda_p^*) = (\delta_{12}^*, \ldots, \delta_{dd}^*) \). Then \( \delta(\lambda) = (\delta_{12}^1 p, \ldots, \delta_{dd}^1 p) \).

A conjugacy class \( C \) of \( H \) is a splitting class if \( C \subseteq H^+ \) and \( C = C_\pm \) is the union of two conjugacy classes of \( H^+ \). There is a bijection between splitting mappings \( \Lambda \) and splitting classes \( C_\Lambda \) of \( H \) (see pg.3491, [3]). The following is Proposition (4F) in [3].

**Theorem 3.2.** Let \(|\Pi| = wp\). Suppose \( \Lambda \) is a splitting mapping of \( N_{S(\Pi)}(D) \) that equals its \( p \)-singular part i.e. \( \Lambda = (\emptyset, \ldots, \lambda_p^*, \ldots, \emptyset) \). Let \((f, \sigma) \in N_{A(\Pi)}(D)^+ \). Then \((\psi_{\Lambda,+} - \psi_{\Lambda,-})(f, \sigma) \neq 0 \) if and only if \((f, \sigma) \in C_\Lambda \). Moreover, \( \psi_{\Lambda,\pm} \) and \( C_{\Lambda,\pm} \) may be labeled so that

\[
(\psi_{\Lambda,+} - \psi_{\Lambda,-})(f, \sigma) = \pm (\sqrt{\epsilon_{p^e}})^d \sqrt{f_{\lambda^p}} \eta_{\lambda^p}
\]

for \((f, \sigma) \in C_{\Lambda,\pm} \), where, \( \epsilon_{\lambda^p} = (-1)^{d-1} \), \( d \) is the number of diagonal nodes in \( \lambda^p \) and \( \delta(\lambda_p^*) = (\eta_1, \ldots, \eta_{dd}) \).

Suppose \( \sigma \in Gal(\mathbb{Q}^{G^+}/\mathbb{Q}) \) is such that \( \sigma(\xi) = \xi^{p^e} \) for some \( e \in \mathbb{Z}^+ \) and \( \xi \) is a \( p' \)-root of unity. We define \( Irr_{p'}(B_1) \) and \( Irr_{p'}(b_1) \) to be the \( p' \)-characters of the principal block \( B_1 \) of \( A(\Pi) \) and its Brauer correspondent \( b_1 \) and \( Irr_{p',sing}(B_1) \) and \( Irr_{p',sing}(b_1) \) are defined by extension.

**Theorem 3.3.** Let \( A(\Pi) \) be the alternating group on \( \Pi \) and \( p \) is an odd prime such that \( A(\Pi) \) has an abelian defect group. Let \( \sigma \in N. \) Let \( B_1 \) be the principal block of \( A(\Pi) \), \( \chi \in Irr_{p'}(B_1) \) and \( b_1 \) its Brauer correspondent. Then the restriction of \( f^\sigma \) is a bijection between \( Irr_{p',sing}(B_1) \) and \( Irr_{p',sing}(b_1) \) that commutes with \( \sigma \). That is, \( f^\sigma(\chi) = f^\sigma(\chi^\sigma) \).

**Proof.** Since \( A(\Pi) \) has abelian defect, and we are considering only the principal block, we can assume \(|\Pi| = wp\). By the discussion above, we consider two cases.

1. Suppose \( \lambda \neq \lambda^* \). Then the restrictions \( \chi_\lambda|_{G^+} = \chi_{\lambda^*}|_{G^+} \) are in bijection with \( \psi_{\lambda^*}|_{H^+} = \psi_{\lambda^*}|_{H^+} \). Since the values of \( \chi_\lambda \) are all rational, \( \chi_{\lambda^*} \) is \( \sigma \)-fixed. Since \( N_{G^+}(X) = Y \wr S(\Omega) \) where \(|\Omega| = p \), \( \psi_{\lambda^*}|_{H^+} \) is also \( \sigma \)-fixed.

2. Suppose \( \lambda = \lambda^* \). Upon restriction, the pair \( \chi^\pm \) is in bijection with the pair \( \psi_{\Lambda}^\pm \) via \( \tilde{f} \). It remains to show that the values of \( \chi_{\lambda}^\pm \) and \( \psi_{\Lambda}^\pm \) on the splitting classes \( \kappa_{d(\lambda)}^\pm \) and \( C_{\Lambda}^\pm \) are both exchanged.
or fixed by $\sigma$. By Theorem 3.1, Theorem 2.2, and Theorem 3.2, 
\[ \sqrt{\eta_j p^d} = \sqrt{\delta_j}. \]
Since $p$ is odd, $(wp - d) \equiv (p - 1 + w - d) \pmod{2}$, so $\epsilon_{\lambda_p} \cdot \epsilon_p = \epsilon_{\lambda}$. This completes the proof.

\[ \square \]

Acknowledgements. The author is indebted to Paul Fong for his guidance and suggestions. This research was partially supported from a grant by PSC-CUNY.

REFERENCES

[1] J. Alperin. (1976) The main problem of block theory in Proc. of the Conference of Finite Groups, University of Utah, Park City, Utah:341–356.
[2] P. Fong, The Isaacs-Navarro conjecture for symmetric groups. Journal of Algebra, 250, No.1(2003)154–161.
[3] P. Fong, and M. Harris, On perfect isometries and isotypies in alternating groups. Transactions of the American Mathematical Society 349, No.9:3469–3516.
[4] I.M. Isaacs, (1994) Character Theory of Finite Groups Dover
[5] I.G. MacDonald, On the degrees of the irreducible representations of the symmetric groups, Bull. London Math. Soc., 3 (1971), 189-192
[6] R. Nath, On the diagonal hook lengths of symmetric partitions arXiv:0903.2494v1
[7] R. Nath, The Navarro conjecture for alternating groups, $p = 2$ J. Algebra and its Applications, Volume 6 (2009) 837-844
[8] G. Navarro, The McKay conjecture and galois automorphisms Annals of Mathematics 160:1129–1140.

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