The free energies of six-vertex models and the $n$-equivalence relation.

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Abstract

The free energies of six-vertex models on general domain $D$ with various boundary conditions are investigated with the use of the $n$-equivalence relation which classifies the thermodynamic limit properties. It is derived that the free energy of the six-vertex model on the rectangle is unique in the limit $(\text{height}, \text{width}) \to (\infty, \infty)$. It is derived that the free energies of the model on $D$ are classified through the densities of left/down arrows on the boundary. Specifically the free energy is identical to that obtained by Lieb and Sutherland with the cyclic boundary condition when the densities are both equal to $1/2$. This fact explains several results already obtained through the transfer matrix calculations. The relation to the domino tiling (or dimer, or matching) problems is also noted.

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§ 1 Introduction

The six-vertex model is a solvable lattice model, usually introduced on the rectangle and considered with the use of the Bethe ansatz method or the Yang-Baxter relation. The model was first solved by Lieb[1-3] and generally by Sutherland[4], both assuming the cyclic boundary condition in the horizontal and the vertical directions. In this paper \( f_{\text{LS}} \) denotes the free energy obtained by Sutherland.

We have, on the other hand, an example[5] where the free energy is exactly obtained with another specific boundary condition. In this case the free energy is expressed in terms of the elementary functions, and thus it is apparently different from \( f_{\text{LS}} \). In the six-vertex model, the boundary condition is relevant even in the thermodynamic limit. This fact seems to be unusual comparing to other lattice models such as the Ising models.

In this paper we consider the free energies of the six-vertex models introduced generally on domain \( D \) with continuous boundary and with various boundary conditions. The free energies are investigated and classified, with the use of the \( n \)-equivalence relation, and our study includes the cases where the transfer matrix method cannot be directly applied. The main result of this paper is proposition 3 which states that the density of down arrows and that of left arrows on the boundary determine the free energy of the system on \( D \). The free energy is identical to \( f_{\text{LS}} \) when the two densities are both equal to 1/2. This result also means that the free energy is still intensive even if the boundary effect remains relevant in the thermodynamic limit.

Section 2.1 is a short summary on the six-vertex model and the corresponding transfer matrix treatment. In section 2.2, we introduce the domain \( D \) and, in section 2.3, introduce an equivalence relation of boundary conditions called the \( n \)-equivalence[6]. Two boundary conditions yield the identical free energy if they are \( n \)-equivalent. This \( n \)-equivalence is a generalization of the concept of boundary condition, classifies the infinite limit properties, and also corresponds to the irreducibility of the transfer matrix. In section 2.4, it is derived, with the use of the \( n \)-equivalence, that the free energy of the six-vertex model on the rectangle \( R \) with \( w \) columns and \( h \) rows is unique in the thermodynamic limit \( (w, h) \to (\infty, \infty) \), specifically independent of the ratio \( w/h \), independent of the order of two limits \( w \to \infty \) and \( h \to \infty \). The six-vertex model on a cylinder and a rectangle are considered and finally we obtain proposition 5.

There exist several exact calculations which yield \( f_{\text{LS}} \) with various boundary conditions. Our results explain why these free energies are equal to \( f_{\text{LS}} \) and, in addition to it, can determine the exact free energies of six-vertex models which have not yet been solved. The results can also be written in terms of the
domino tiling language and also we can certify that proposition 5 is consistent with the results in this area. These facts concerning the relations with other known results are summerized in section 3.

§ 2 The six-vertex models on \( D \)

2.1 The six-vertex model

Let us consider the square lattice and assign an arrow on each bond. The arrows are arranged such that two arrows come in and the other two go out at each site (the Ice rule). Then there exist six types of possible local arrow arrangements as shown in Fig.1. In this paper we are going to use the term 'vertex' as a site and four bonds around it. Each vertex is assumed to have finite energy. The energy is assumed to be unchanged by reversing all arrows on the four bonds. Then we have three energy parameters and hence three types of Boltzmann weights \( a, b \) and \( c \) assigned to the vertices (see again Fig.1). We also introduce the Boltzmann constant \( k_B \), the temperature \( T, \beta = 1/k_B T \) and the total number of sites \( N \). The partition function is

\[
Z = \sum_{\text{config.}} \prod_{i=1}^{N} e^{-\beta \epsilon_i}, \tag{2.1}
\]

where \( \epsilon_i \) is the energy of the \( i \)-th vertex and \( \sum_{\text{config.}} \) is taken over all the possible arrow configurations. The free energy \( f \) is obtained through \( -\beta f = \lim_{N \to \infty} N^{-1} \log Z \).

One can assign a line on each arrow pointing down or left (Fig.1). Then each arrow configuration corresponds to a line configuration on the lattice. The Ice rule corresponds to the restriction that each line begins from a bond on the boundary, continue until it reaches another bond on the boundary, and that the lines do not intersect each other.

Let us consider a rectangle \( R \) with \( w \) columns and \( h \) rows. Assume that \( h \) and \( w \) are even. Let \( \eta \equiv \{x_1, \ldots, x_m\} \) be a line configuration with \( m \) lines on a row of vertical bonds in \( R \), and let \( \eta' \equiv \{x'_1, \ldots, x'_m\} \) be that on the row below. The symbol \( \{x_1, x_2, \ldots, x_m\} \) denotes that there is a line on each \( x_k \)-th bond \( (k = 1, \ldots, m) \) and otherwise there is not. The \( (\eta', \eta) \)-element of the transfer matrix \( V \) is introduced as

\[
V_{\eta' \eta} \equiv \langle x'_1, \ldots, x'_m | V | x_1, \ldots, x_m \rangle = \sum_{\text{config.}} \prod_{k=1}^{w} e^{-\beta \epsilon_k}, \tag{2.2}
\]
where $|x_1, \ldots, x_m\rangle$ is the state corresponding to $\eta \equiv \{x_1, \ldots, x_m\}$, $\epsilon_k$ ($k = 1, \ldots, w$) is the energy of the $k$-th vertex on the row between the two rows of vertical bonds and here the sum $\sum_{\text{config.}}$ is taken over all the possible line configurations with fixed $\eta$ and $\eta'$.

When we identify the bond at the right end with that at the left end on each row (we call this the cyclic boundary condition in the horizontal direction), the transfer matrix is identical for all rows. If the boundary condition is also cyclic in the vertical direction the partition function $Z$ is written as

$$Z = \text{tr} \ V^h = \sum_i \lambda_i^h \sim \lambda_1^h \ (h \to \infty), \quad (2.3)$$

where $\lambda_i$'s are the eigenvalues of $V$ and $\lambda_1 \geq |\lambda_i|$ for all $i$.

Following the notations in [7] let us introduce $\Delta = (a^2 + b^2 - c^2)/2ab$. The transfer matrix is block diagonalized according to the number of lines $m$. The maximum eigenvalue of the transfer matrix lies in the block element with $m = 0$ or $w$ ($\Delta > 1$), with $m = w/2$ ($\Delta < 1$). When $\Delta > 1$, we have two types of frozen phase where specific line configurations are dominant. In this case the free energy is a constant and all the arguments in this paper become trivial. We thus concentrate on the case with $\Delta < 1$.

### 2.2 Domain $D$

We introduce the domain $D$. Let us consider a continuous and closed line $\gamma(t) = (x(t), y(t))$ ($0 \leq t \leq 1$) which satisfies $\gamma(t_1) \neq \gamma(t_2)$ if $t_1 \neq t_2$ except $\gamma(0) = \gamma(1)$. Let us assume that the sites are on the points $(n_1a_1, n_2a_2)$, where $n_1$ and $n_2$ are integers, $a_1$ and $a_2$ are the lattice spacings.

The sites inside $\gamma$ belong to $D$. The sites on the line $\gamma$ can be suitably defined to belong or not to belong to $D$. The vertices belong to $D$ when the corresponding sites belong to $D$. Each row and column in $D$ is assumed to be simply connected.

The bonds/vertices in $D$ are called the boundary bonds/vertices when they have non-zero intersection with $\gamma$. The sites of the boundary vertices are called the boundary sites. It is assumed that the number of vertices on the boundary divided by the total number of sites vanish in the thermodynamic limit.

We will take the following limit: fix the line $\gamma$ and take the thermodynamic limit $a_1, a_2 \to 0$. This corresponds to taking the limit $w, h \to \infty$ where $w$ and $h$ are the number of columns and rows in $D$. 

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**Note:** The above text appears to be a continuation of a mathematical or scientific discussion, possibly related to a specific field such as physics or mathematics. The context is not immediately clear without additional background information.
2.3 The n-equivalence

Now we introduce the n-equivalence. Assume that each site and bond takes one of a finite number of states. Let us consider the set of sites which lie in $D$ and can be reached from the boundary sites by $n$ steps ($n$ bonds) at minimum; we call these sites the $n$-boundary sites. Consider the set of bonds between $(n-1)$- and $n$-boundary sites, and call them the $n$-boundary bonds. At last let us consider the $n$-boundary sites together with the $n$-boundary bonds and call them the $n$-boundary. Configurations on the $n$-boundary are called $n$-boundary configurations. Let $\{\Gamma_i\}$ be the set of all the possible configurations on the $n$-boundary with a boundary condition $\Gamma$ on the actually boundary of $D$. Two boundary conditions $\Gamma$ and $\Gamma'$ are called $n$-equivalent when $\{\Gamma_i\} = \{\Gamma'_i\}$ as a set of $n$-boundary configurations.

**Proposition 1** Fix a sequence of finite lattices $\{D_N\}$, where $N$ is the number of sites and $D_N$ approaches to the thermodynamic limit as $N \to \infty$. Suppose that the boundary conditions $\Gamma$ and $\Gamma'$ are $n$-equivalent, for each $D_N$, with $n = o(N/N')$ where $N'$ is the number of boundary sites. Then the two free energies with $\Gamma$ and $\Gamma'$ are identical in the thermodynamic limit.

Proof: Let us write the partition function of the system on $D_N$ as $Z = \sum_i B_i Z_i$. The factor $Z_i$ is the partition function from the variables inside the $n$-boundary with a fixed $n$-boundary configuration $\Gamma_i$. The factor $B_i$ is the contribution from the other variables with the boundary condition $\Gamma$ and the fixed $\Gamma_i$. The factor $B_i Z_i$ is thus the partition function with the $\Gamma$ and the $\Gamma_i$. From the assumption that $n = o(N/N')$, we obtain $\log Z_i = -\beta N f_i + o(N)$ and $\log B_i = o(N)$. The index $i$ runs from 1 to $i_{\text{max}}$, where $i_{\text{max}}$ is the number of permitted configurations on the $n$-boundary and satisfies $i_{\text{max}} \leq O(rN')$ where $r$ is a constant. Then we obtain

$$\frac{1}{N} \log Z = \frac{1}{N} \log \left( \sum_i B_i Z_i \right)$$

$$= \frac{1}{N} \log B_1 Z_1 \left[ 1 + \sum_{i=2}^{i_1} \frac{B_i Z_i}{B_1 Z_1} + \sum_{i=i_1+1}^{i_{\text{max}}} \frac{B_i Z_i}{B_1 Z_1} \right]$$

$$= \frac{1}{N} \log B_1 Z_1 \left[ 1 + \sum_{i=2}^{i_1} e^{o(N)} + \sum_{i=i_1+1}^{i_{\text{max}}} e^{-\beta N (f_i - f_1) + o(N)} \right]$$

$$\to -\beta f_1 \quad (N \to \infty), \quad (2.4)$$

where $f_i = f_1 (1 \leq i \leq i_1)$, $f_1 < f_i (i_1 + 1 \leq i \leq i_{\text{max}})$. \[\square\]
Note that what is important is the set of possible configurations, we do not need to think about the number of ways in which each configuration is realized.

This $n$-equivalence is a generalization of the concept of boundary condition, and can be considered in other lattice models such as the 19-vertex model, and also can be introduced in stochastic processes\(^6\). All the boundary conditions are 1-equivalent, for example, in the case of Ising models with finite interactions, because the Ising spin states are independent of their nearest-neighbors. It follows that the free energies of the Ising models are independent of their boundary conditions.

The equivalence can be introduced in a more generalized form. We can consider a boundary condition $\Gamma$ on a subset of the boundary, and introduce the corresponding $n$-boundary and the corresponding $n$-equivalences. In propositions 2 and 5 we concentrate on one and two of the four edges of the rectangle $R$ and consider the corresponding $n$-equivalences.

### 2.4 Results

**Lemma 1** Consider a rectangle $R$ and assume the cyclic boundary condition in the horizontal direction. Then all the line configurations with $m$ lines on the upper edge (the first row of vertical bonds) of $R$ are $2m$-equivalent to each other.

Proof: Let $\{x_1, x_2, \ldots, x_m\}$, $x_i < x_{i+1}$, be a line configuration on the first row of $R$. Beginning from a line arrangement $\{x_1, x_2, x_4, x_5\} = \{2, 3, 5, 6, 9\}$ as shown in Fig.2(a), for example, one can introduce the shift of lines $\{2, 3, 5, 6, 9\} \rightarrow \{1, 2, 3, 4, 5\}$. When we generally have $\{1, 2, \ldots, i, x_j, x_{j+1}, \ldots, x_{m-k}\}$, $i+1 < x_j$, after $k$-th step, one can introduce the line arrangement $\{2, 3, \ldots, i, i + 1, x_j, \ldots, x_{n-(k+1)}\}$ as the next one. We need $m - l$ steps for the shift $\{x_1, x_2, \ldots, x_m\} \rightarrow \{1, 2, \ldots, m\}$ if $x_l \leq m < x_{l+1}$, and this means $\{x_1, x_2, \ldots, x_m\}$ is at most $m$-equivalent to $\{1, 2, \ldots, m\}$. Then all the line configurations $\{x'_1, x'_2, \ldots, x'_m\}$ are again possible on the $2m$-boundary.

Let us consider line configurations, on the left and right edges, with $m_2$ lines in the first $h_p$ bonds on the boundary, and $m_2$ lines in the next $h_p$ bonds, and so on. In this case one can introduce the line density $\rho_2$ on the boundary as $\rho_2 = m_2/h_p$.

**Lemma 2** Suppose that the boundary configurations on the right and the left edges of $R$ are identical and fixed with the density $\rho_2 = m_2/h_p$, where $0 < \rho_2 < 1$. 

Then all the line configurations with \( m \) lines on the upper edge of \( R \) are \( \bar{m} \)-equivalent where \( \bar{m} = \alpha m + \alpha' \) (\( \alpha, \alpha' \) are constant).

Proof: Let us introduce the shift of lines \( \{1, 2, \ldots, i, x_j, x_{j+1}, \ldots, x_{m-k}\} \rightarrow \{1, 2, \ldots, i, i + 1, x_j, \ldots, x_{m-(k+1)}\}, \ i + 1 < x_j \) on each row without lines on the right and left boundary bonds, and \( \{x'_1, x'_2, \ldots, x'_m\} \rightarrow \{x''_m, x'_1, \ldots, x'_{m-1}\} \) on each row with lines on the boundary bonds, as shown in Fig.2(b). The line configuration will be arranged as \( \{x_1, x_2, \ldots, x_m\} \rightarrow \{1, 2, \ldots, m\} \) within \( m' \) steps where \( m' = \lfloor m/(h_p - m_2) \rfloor h_p + h_p \)

Next we consider an arbitrary line configuration \( \{x''_1, x''_2, \ldots, x''_m\}, x''_i < x''_{i+1} \) and the shift \( \{1, 2, \ldots, m\} \rightarrow \{x''_1, x''_2, \ldots, x''_m\} \). Let us introduce \( \{x'_1, x'_2, \ldots, x'_m\} \) on each row without lines on the right and left boundary bonds, \( \{1, 2, \ldots, i, x''_{i+1}, \ldots, x''_m\} \rightarrow \{x''_m, 1, 2, \ldots, i - 1, x''_{i'}, x''_{i+1}, \ldots, x''_{m'}\} \), as shown in Fig.2(c), on each row with lines on the boundary bonds. Finally the configuration \( \{x''_1, x''_2, \ldots, x''_m\} \) is possible on the \( \bar{m} \)-boundary where \( \bar{m} \leq m' + m'' \) and \( m'' = \lfloor m/m_2 \rfloor h_p + h_p \).

Boundary line configurations with which no configuration is admitted on the whole of the lattice should be excluded from our argument, because in this case the system cannot be a six-vertex model. We also assume the convergence of the free energy in the sequential limits \( h \to \infty \) and \( w \to \infty \), and another sequential limits \( w \to \infty \) and \( h \to \infty \).

**Proposition 2** Consider the six-vertex model on a rectangle \( R \). Assume the cyclic boundary condition in the horizontal direction, or otherwise assume that the boundary configurations on the right and the left edges of \( R \) are identical, periodic with fixed period \( h_p \), with line density equal to \( \rho_2 = m_2/h_p \) satisfying \( 0 < \rho_2 < 1 \). Assume that \( h \) is always a multiple of \( h_p \). Then,

i) the transfer matrix of each row, or the product of the transfer matrices of sequential \( h_p \) rows, respectively, is block-diagonalized according to the number of lines and each block element is irreducible,

ii) the free energy is unique in the limit \( (w, h) \to (\infty, \infty) \), specifically the limit is independent of the order of two limiting procedures \( w \to \infty \) and \( h \to \infty \), and also independent of the ratio \( w/h \) when one take \( h \to \infty \) with fixed \( w/h \).

Proof: First let us assume that the boundary condition is cyclic in the horizontal direction. The case with the fixed boundary condition can be treated similarly, and will be considered at the last of the proof. Let \( m \) be the number of lines on the upper edge (the first row of vertical bonds) of \( R \). Then there are \( m \) lines
on every row of $R$ because of the line conservation and the boundary condition. Thus the transfer matrix $V$ is block-diagonalized according to $m$. Let $V_m$ be the block element of $V$ with a fixed line number $m$. All the elements of $V_m$ are non-negative because they are sums of Boltzmann weights. From lemma 1, all the line configurations with $m$ lines are $2m$-equivalent. Hence there always exist allowed line arrangements on the lattice for any line configuration with $m$ lines on the upper boundary (the first row of vertical bonds) and for any line configuration with $m$ lines on the $n$-boundary with $n \geq 2m + 2m$, and thus we find that all the elements of $V_m^{4m}$ are positive. (All the elements of $V_m^{2m}$ are already positive, which is obvious from the proof of lemma 1.) Hence we find that $V_m$ is irreducible, because any matrix is irreducible if all of its elements are positive. It follows that $V_m$ is irreducible because $V_m^{4m}$ cannot be irreducible if $V_m$ is not, this proves i). Then the Frobenius theorem works and we know the followings. There exists a non-degenerate eigenvalue $\lambda_1(w) > 0$ such that $\lambda_1(w) \geq |\lambda_i(w)|$ where $\lambda_i(w)$ $(i \geq 2)$ are the other eigenvalues of $V_m$. We also know that there exists an eigenvector associated with $\lambda_1(w)$ with all the elements being positive, i.e. the projections satisfy $\langle x_1, \ldots, x_m | \text{max} \rangle > 0$ where $|\text{max}\rangle$ is the eigenstate associated with the maximum eigenvalue of $V_m$. These results are valid for every finite $m$.

The partition function $Z$ with finite $w$ and $h$ is written as

\begin{align*}
Z &= \langle x'_1, \ldots, x'_m | V^h | x_1, \ldots, x_m \rangle \\
&= c_1 \lambda_1(w)^h + \sum_{i \geq 2} c_i \lambda_i(w)^h \quad (2.5)
\end{align*}

where $\{x_1, \ldots, x_m\}$ and $\{x'_1, \ldots, x'_m\}$ are the line configurations on the upper and the lower edges of $R$, respectively, $c_1 = \langle x'_1, \ldots, x'_m | \text{max} \rangle \langle \text{max} | x_1, \ldots, x_m \rangle$ is positive and independent of $h$ and here we have assumed $|\text{max}\rangle$ is normalized. (The coefficients $c_i$ are independent of $h$ when $V$ is diagonalizable. Otherwise $V$ is expressed in Jordan form, $c_i$ $(i \geq 2)$ are asymptotically bounded by polynomials of $h$ with finite degrees, and the following argument remains still valid.) We will consider the limit of

\begin{align*}
- \beta f_{h,w} &= \frac{1}{hw} \log Z = \frac{1}{w} \log \lambda_1(w) + \frac{1}{h} z'(h,w) \\
z'(h,w) &= \frac{1}{w} \log c_1 [ 1 + \sum_{i \geq 2} \frac{c_i}{c_1} (\frac{\lambda_i}{\lambda_1})^h ] \quad (2.6)
\end{align*}

The factor $z'(h,w)$ is finite when $h$ and $w$ are finite, $z'(h,w)$ depends on $h$ only through the second term and so remains finite in the limit $h \to \infty$ with fixed $w$. When we take the limit $h \to \infty$ in (2.6), the term $(hw)^{-1} \log Z$
is convergent, because the term \( w^{-1} \log \lambda_1(w) \) is independent of \( h \) and the factor \( h^{-1} z'(h, w) \) converges to zero. Taking \( w \to \infty \) afterwards, one find that \( w^{-1} \log \lambda_1(w) \) is convergent because \( (hw)^{-1} \log Z \) is assumed to be convergent in this limit.

On the otherhand, when one first take the limit \( w \to \infty \) in (2.6), one find that the factor \( z'(h, w) \) is convergent, because \( (hw)^{-1} \log Z \) is assumed to be convergent and now it can be used that \( w^{-1} \log \lambda_1(w) \) is also convergent in this limit. Therefore \( z'(h, w) \) is bounded by a factor \( C \) which is independent of \( w \), because convergent series are always bounded. Again note that \( z'(h, w) \) is finite in the limit \( h \to \infty \), and therefore the factor \( C \) can be taken as a constant independent of \( w \) and \( h \).

Hence from (2.6) we have found that

\[
|- \beta f_{h,w} - \frac{1}{w} \log \lambda_1(w)| \leq \frac{C}{h} \tag{2.8}
\]

which means that the convergence of \( f_{h,w} \) in the limit \( h \to \infty \) is uniform throughout \( w \). This proves ii) when we consider the known fact that the uniformity of the convergence yields uniqueness of the limit of double series.

The last fact is well known but we are going to show a proof of it. We have shown that the convergence in the limit \( h \to \infty \) is uniform: there exists a number \( f_{\infty,w} \) which satisfies that, for every \( \epsilon > 0 \) there is an integer \( h_0(\epsilon) \) which is independent of \( w \), such that \( |f_{h,w} - f_{\infty,w}| < \epsilon \) for all \( h \geq h_0(\epsilon) \). The free energy is convergent in the next limit \( w \to \infty \): there exists a number \( f \) which satisfies that, for every \( \epsilon > 0 \) there is an integer \( w_0(\epsilon) \) such that \( |f_{\infty,w} - f| < \epsilon \) for all \( w \geq w_0(\epsilon) \). Then for all \( h, w \geq \max\{h_0(\epsilon), w_0(\epsilon)\} \) we have

\[
|f_{h,w} - f| \leq |f_{h,w} - f_{\infty,w}| + |f_{\infty,w} - f| < 2\epsilon, \tag{2.9}
\]

which means

\[
\lim_{(h,w) \to (\infty,\infty)} f_{h,w} = f \tag{2.10}
\]

as a double series. We assumed the convergence \( f_{h,w} \to f_{h,\infty} \) \((w \to \infty)\), thus taking \( w \to \infty \) in (2.9) one obtains \( |f_{h,\infty} - f| \leq 2\epsilon \) which means

\[
\lim_{h \to \infty} \lim_{w \to \infty} f_{h,w} = f. \tag{2.11}
\]

Taking \( h \to \infty \) also in (2.9) one obtains \( |f_{\infty,w} - f| \leq 2\epsilon \) which means

\[
\lim_{w \to \infty} \lim_{h \to \infty} f_{h,w} = f. \tag{2.12}
\]

At last let us consider the case where the boundary line configurations on the right and the left edges of \( R \) are identical, fixed and the lines are located periodically with the period \( h_p \) on the edges, with the line density being
\( \rho_2 = m_2/h_p \) satisfying \( 0 < \rho_2 < 1 \). Then one can introduce a transfer matrix \( V = V_1 V_2 \cdots V_{h_p} \) where \( V_1, \ldots, V_{h_p} \) are the transfer matrices of sequential \( h_p \) rows of \( R \), respectively. The partition function is expressed as a linear combination of \( \lambda_i(w)^{h/h_p} \), where \( \lambda_i(w) \) are now the eigenvalues of \( V = V_1 V_2 \cdots V_{h_p} \). Then lemma 2 works and we obtain the same result.

Note that the \( 2m \)-equivalence of line configurations corresponds to the irreducibility of the block element \( V_{m}^{4m} \), and hence to the irreducibility of \( V_m \). The matrix \( V \) is irreducible if the line configurations used as a bases for the matrix representation of \( V \) are \( n \)-equivalent to each other for some finite \( n \). When the boundary is cyclic in the horizontal direction, \( \lambda_1(w) \) is already known and we have \( \lim_{w \to \infty} w^{-1} \log \lambda_1(w) = -\beta f_{LS} \). Hence the free energy itself is actually obtained in the proof of proposition 2. Here we show what is obtained from our formula using only the \( n \)-equivalence and the uniform convergence of the free energy.

**Lemma 3** Consider the six-vertex model on a rectangle \( R \). Assume the cyclic boundary condition in the horizontal direction, and assume that the number of lines on the upper and the lower edges are \( m = m(w) \). Then all the boundary line configurations with the same \( m(w) \) yield the identical free energy in the limit \( (w, h) \to (\infty, \infty) \). Specifically we have \( f = f_{LS} \) when \( m = w/2 \).

Proof: The line configuration on the upper edge is \( 2m \)-equivalent to arbitrary line configurations in the block element of \( V_{m}^{4m} \) with \( m \) lines, and hence \( 2m \)-equivalent to the cyclic boundary with \( m \) lines. We take the limit \( h \to \infty \) with fixed \( w \), and obtain the free energy with \( (w, h) \to (w, \infty) \). The resulted functions are the same for all of these fixed and the cyclic boundary conditions with \( m \) lines on the upper and the lower edges. Next taking the limit \( w \to \infty \), the free energies remain identical. In particular, we obtain the maximum eigenvalue of \( V \) and the known free energy \( f_{LS} \) when \( m = w/2 \).

When the system is cyclic in two directions, we know that the maximum eigenvalue of the the row to row transfer matrix lies in the block-element with \( w/2 \) lines in each row. Because of the symmetry of the system, the maximum eigenvalue of the the column to column transfer matrix also lies in the block element with \( h/2 \) lines in each column, otherwise we have contradictions.

When the boundary is cyclic in the horizontal direction with fixed \( h/2 \) lines in each column, the 'alternate' line configuration with \( \rho_2 = 1/2 \) with the period \( h_p = 2 \) is possible on the right and left edges, hence lemma 2 works, and in lemma 3 \( f_{LS} \) appears with the restriction that the number of lines is \( h/2 \) in each column. (Note that proposition 1 is valid when at least one necessary configuration is possible on the lattice.)
Lemma 4 In lemma 3, \( f_{LS} \) appears with the restriction that the number of lines is \( h/2 \) in each column.

Let us consider a fixed boundary line configuration with a periodic pattern on the boundary of \( R \). We assume that \( w \) and \( h \) are multiples of \( w_p \) and \( h_p \), respectively, and the line density is \( \rho_1 = m_1/w_p \) on the upper and the lower edges, \( \rho_2 = m_2/h_p \) on the left and the right edges. The boundary line configuration on the upper edge is identical to that on the lower edge, and the configuration on the right edge is identical to that on the left edge. The limit will be taken with fixed \( w_p \) and \( h_p \). In the proof of the next proposition we do not need the explicit form of \( \lambda_1(w) \).

Lemma 5 With these conditions the line densities \( \rho_1 \) and \( \rho_2 \) determine the free energy of the six-vertex model on \( R \): \( f = f(\rho_1, \rho_2) \). Specifically \( f(1/2,1/2) = f_{LS} \).

Proof: First assume \( 0 < \rho_1 < 1 \) and \( 0 < \rho_2 < 1 \). Lemma 2 yields that all the line configurations on the upper edge with the density \( \rho_1 \) are \( n \)-equivalent with some \( n \) which depends only on \( w \), and all the line configurations on the lower edge with the density \( \rho_1 \) are also \( n \)-equivalent with the same \( n \). The boundary effect is relevant to the limit only through \( \rho_1 \) when we take \( h \to \infty \) with fixed \( w \). Next taking \( w \to \infty \) we obtain the thermodynamic limit. This argument is also valid for the right and left edges with the density \( \rho_2 \) taking the limit \( w \to \infty \) with fixed \( h \) at first, and next \( h \to \infty \). The limit is unique because of the proposition 2.

Consider the case with \( \rho_1 = \rho_2 = 1/2 \). The boundary configurations on the upper and the lower edges are \( n \)-equivalent to the cyclic boundary with \( w/2 \) lines with some \( n \) which depends only on \( w \). Taking \( h \to \infty \) with fixed \( w \), we obtain the limit identical to that obtained with the cyclic boundary condition in the vertical direction with \( w/2 \) lines together with fixed boundary configurations with \( \rho_2 = 1/2 \) on the right and left edges. Next taking \( w \to \infty \) we obtain the thermodynamic limit, and from lemma 4 and proposition 2 the limit is identical to \( f_{LS} \).

When \( \rho_2 = 0 \) the free energy is \( (1 - \rho_1)\epsilon_1 + \rho_1\epsilon_2 \), and when \( \rho_1 = 0 \) we have \( (1 - \rho_2)\epsilon_1 + \rho_2\epsilon_2 \). As for the cases with the densities equal to 1, one can use the fact that \( f(\rho_1, \rho_2) = f(1 - \rho_1, 1 - \rho_2) \) which comes from the symmetry of vertex energies.

Let us consider the sequence of vertical boundary bonds on the line \( \gamma \). Let us assume a fixed boundary line configuration with a periodic pattern on this sequence with the line density \( \rho_1 = n_1/w_p \), i.e. \( n_1 \) lines on every \( w_p \) vertical
bonds on $\gamma$. Also assume a fixed and periodic boundary line configuration on the sequence of horizontal bonds on $\gamma$ with the line density $\rho_2 = n_2/h_p$, i.e. $n_2$ lines on every $h_p$ horizontal bonds on $\gamma$. Assume that there exists a line on a horizontal bond on the left edge of $D$ if and only if there is a line on the horizontal bond of the same row on the right edge of $D$. Assume also that there exists a line on a vertical bond on the lower edge of $D$ if and only if there is a line on the vertical bond of the same column on the upper edge of $D$. The limit will be taken with fixed $w_p$ and $h_p$. With these conditions we can derive the following:

**Proposition 3** The line densities $\rho_1$ and $\rho_2$ determine the free energy of the six-vertex model on $D$: $f = f(\rho_1, \rho_2)$. Specifically we obtain $f(1/2, 1/2) = f_{LS}$.

Proof: Let $D_0$ be a rectangle of the width $\Delta x$ and the height $\Delta y$ sufficiently large satisfying $D \subset D_0$. The rectangle $D_0$ is divided into small rectangles $R_i$ of the width $\Delta x$ and the height $\Delta y_i = \Delta h$, where the lower edge of $R_i$ coincide with the upper edge of $R_{i+1}$. Let $D' = \cup_i R'_i$, where $R'_i$ is a rectangle of the height $\Delta y_i$ and of the width $\Delta x_i$, satisfying $R'_i \subset R_i$ and $R'_i \subset D$, and satisfying that the number of columns in each $R'_i$ is a multiple of $w_p$ taking its maximum with the restriction that $R'_i \subset D$. The sites in $R'_i$ and those on the edges of $R'_i$, if any, are assumed to belong to $R'_i$, with the exception that the sites on $R'_i \cap R'_{i+1}$ belong to $R'_i$. The sum of the line numbers on the upper and the right edges of each $R'_i$ is generally equal to the sum of the line numbers on the lower and the left edges of $R'_i$, because of the line conservation property. Assume that the boundary line configuration on each edge of $D'$ is identical to the corresponding boundary line configuration of $D$, i.e. there exists a line on a horizontal bond on a right edge of $D'$ if and only if there is a line on the same row at the right edge of $D$, and so on for other edges.

We have assumed in each $R'_i$ that the number of columns is a multiple of $w_p$. In addition to it, we assume in each $R'_i$ that the number of rows is also a multiple of $h_p$. The latter choice corresponds to introducing a sequence $a_2 \to 0$ where each $a_2$ satisfies $\Delta h = n'h_p a_2$ with $n' = 1, 2, \ldots$. It is sufficient to consider this specific sequence because the difference from the remaining cases of $a_2$ vanish as $N \to \infty$ together with the ratio $N'/N$ where $N'$ is the number of boundary sites.

Because of the identical line configuration on the right and left edges, the number of lines on the upper and the lower edges of $R'_i$ are the same. All the possible configurations on the upper edge are $n$-equivalent with some $n$ independent of $a_2$, all the configurations on the lower edge are also $n$-equivalent, and hence they yield the identical free energy in each $R'_i$ in the limit $a_2 \to 0$. Next taking $a_1 \to 0$, we find that each rectangle $R'_i$ yields $f(\rho_1, \rho_2)$ which is the
free energy obtained in lemma 5. Thus we obtain \( f_{D'} = f(\rho_1, \rho_2) \) where \( f_{D'} \) is the free energy on \( D' \).

The result is independent of the ratio \( \Delta h/\Delta x_i \) and the ratio can be taken sufficiently small. The ratio of the energy contribution, the contribution from \( D \setminus D' \) over that from \( D, \) goes to zero in the limit \( \Delta h \to 0, \) because the line \( \gamma, \) which determines the boundary of \( D, \) is continuous. Therefore we obtain \( |f_D - f_{D'}| < \epsilon \) for arbitrary positive \( \epsilon. \)

The result means that the free energy is still additive even in the situation where the boundary condition remains relevant in the thermodynamic limit.

These results are applications of [6] to the case of the six-vertex model, and give sufficient conditions to have \( f_{LS}. \) It should be noted that we didn’t need diagonalize sequential transfer matrices but it was sufficient for us to consider the configurations which are \( n\)-equivalent, for the purpose to classify the thermodynamic limit properties.

§ 3 Conclusion

Our propositions explain several results already obtained and also able to determine the exact free energies of six-vertex models which have not been solved.

One can introduce boundary conditions, such as the cyclic boundary condition, in which various boundary configurations are admitted. If our vertex energies and the temperature satisfy \( \Delta < 1, \) and if we can find configurations being \( n\)-equivalent to those with \( \rho_1 = \rho_2 = 1/2, \) then the free energy is \( f_{LS}. \) If we are in the parameter region with \( \Delta > 1, \) and if one of the boundary configurations with \( (\rho_1, \rho_2) = (0, 0) \) or \( (1, 1) \) is admitted, the system falls in trivial frozen phases.

Owczarek and Baxter [8] solved (by the Bethe ansatz method) the six-vertex model with the cyclic boundary and a 'free' boundary condition, respectively, in two directions. Batchelor et al. [9] solved (by the Yang-Baxter relation) the six-vertex model on the rectangle \( R \) with a specific boundary condition in which they have assumed that the arrow at one end of a row points right (left) if that on the other end of the same row points left (right), and also assumed the cyclic boundary condition in the vertical direction. One can find that the boundary configuration with the line density equal to 1/2 is realized with the restrictions in both [8] and [9], and hence it can be directly derived from our results that the free energies of these systems are \( f_{LS}. \) Furthermore one can easily construct a bunch of \( n\)-equivalent cases which have not yet been solved and are extremely difficult to solve directly by the Bethe ansatz method or the
Yang-Baxter relation, but we now know that the free energies of all of these cases should be $f_{LS}$.

We would like to note that equivalences of boundary conditions in the six-vertex model is also investigated in [10] yielding $f_{LS}$.

One of the most interesting problems related to our results is the domino tiling (see [11,12]). The problem is to find the number of possible ways to cover a region completely using dominos ($1 \times 2$ rectangles). It is of course equivalent to find the number of dimer coverings on a given lattice.

Each configuration of domino is expressed in terms of a height function $h(x)$. The correspondence is unique except an overall constant. More precisely we introduce $h(x)$ as follows: color the squares in a checker-board pattern, and $h(x)$ increases one in each unit moving anti-clockwise around black squares on the boundary of dominos, and decreases one around white squares, as shown in Fig.3(a). Not all the regions can be tiled using dominos. The necessary and sufficient condition for tilability is written in terms of the height function as follows: $|h(x) - h(y)| \leq d(x,y)$ for all $x$ and $y$ on the boundary of the region where $d(x,y)$ is the minimal number of steps moving from $x$ to $y$ with only black squares on its left.

Kasteleyn [13] and independently Temperley and Fisher [14] derived the number of tilings on the $m \times n$ rectangle and obtained that in the thermodynamic limit it behaves $\exp(mnG/\pi)$ where $G$ denotes the Catalan’s constant $G = 1/1^2 - 1/3^2 + 1/5^2 - 1/7^2 + \cdots$. The number of tilings on the Aztec diamond (the 'square' rotated $\pi/4$ tiled by horizontal and vertical dominos) is also obtained [15] exactly as $2^{n(n+1)/2}$ where $n$ is the half of the diameter of the region. This is completely different from that of the $m \times n$ rectangle. We recognize that the number of possible ways of tiling strongly depends on the shape of the boundary.

Cohn, Kenyon and Propp [16] showed a variational principle for the number of tilings: assume that the region is tilable and sufficiently 'fat', and assume that the slope of the height function $(s, t) = (\partial h/\partial x, \partial h/\partial y)$ is asymptotically constant on the boundary, then the asymptotic number of tilings per domino is a function of $(s, t)$.

It is known [5,17] that the number of possible domino tilings per domino is equal to the partition function of the six-vertex model with $a = b = 1$ and $c = \sqrt{2}$, i.e. $\Delta = 0$. The equivalence is obtained from the correspondence of configurations of dominos and vertices shown in Fig.3(b).

The boundary of the $m \times n$ rectangle has constant slope $(0, 0)$ and, as a six-vertex model, line densities are $\rho_1 = 1$ and $\rho_2 = 0$ on the upper edge. Mixing two boundaries shown in Fig.3(c) periodically, we find that the line densities vary as $\rho_1 = 1 - \epsilon$ and $\rho_2 = 0 + \epsilon$ while the slope remains $(0, 0)$. Taking $\epsilon = 1/2$
we have $\rho_1 = \rho_2 = 1/2$. This modification can also be done for other three edges and we find, from proposition 8 that the number of tilings is obtained from the partition function by Lieb and Sutherland with $a = b = 1$, $c = \sqrt{2}$ that means $\Delta = 0$. The free energy is (see for example [7])

$$f_{LS} = -k_B T \int_{-\infty}^{+\infty} \frac{\sinh^2 \frac{\pi}{2} x}{2x \sinh \pi x \cosh \frac{\pi}{2} x} dx.$$  \hspace{1cm} (3.1)

Counting the residues at $z = i, 3i, 5i, \ldots$ on the imaginary axis, we find the Catalan’s constant $G$ and obtain that $-\beta f = 2G/\pi$. The factor 2 corresponds to the fact that the number of vertices is equal to the number of dominos and twice the number of squares, and the result from our proposition 3 is consistent with that previously obtained by Kasteleyn and by Temperley-Fisher. The limit $m \to \infty$ and $n \to \infty$ in the domino case is unique, which is also consistent with our proposition 2.

The six-vertex model with the domain wall boundary condition[5] corresponds to the Aztec diamond. In this case the boundary line densities do not satisfy our condition, the free energy is not identical to $f_{LS}$.

Quite recently, S.Sheffield informed me his work[18] in which he derived a variational principle for the systems with gradient Gibbs potential. His argument is based on the DLR condition, equivalent to the variational principle[16] in the case of the domino tiling problems, and corresponds to proposition 3 in the case of the six-vertex model while the treatments of the thermodynamic limit is different.

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Figure Captions:

Fig.1: Six vertices, corresponding line configurations and their Boltzmann weights.

Fig.2: Shift of lines introduced in the proof of Lemma 1 and Lemma 2.

Fig.3: (a) The height function for domino configurations. (b) Correspondence between vertices and domino tilings. The horizontal and the vertical lines are the lattice for the six-vertex model, while the lines rotated $\pm \pi/4$ are the edges of dominos. (c) Two boundaries with constant tilt (0,0) but different line densities.
Fig. 1

Fig. 2

Fig. 3