A Letter Highlighting Matrix Mapping in
Minimal 4D, $\mathcal{N} = 1$ On-Shell Supermultiplet Representations

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ABSTRACT

On the basis of comparing eigenvalues of an operator $\hat{\mathcal{C}}(\mathcal{R})$, that proved useful in distinguishing how off-shell 4D, $\mathcal{N} = 1$ supermultiplets become off-shell 4D, $\mathcal{N} = 2$ supermultiplets, the double tensor supermultiplet is shown to be radically different for other known multiplets. This suggests difficulties, if not impossibilities, to embed it into an off-shell structure.

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1 Introduction

It is the purpose of this letter to introduce into the literature approaches that allow on-shell supersymmetrical representations to be mapped (usually) into representations of Coxeter Groups [1, 2,3]. In a previous work [4] such a mapping operation was defined for off-shell minimal 4D, \( \mathcal{N} = 1 \) supersymmetrical multiplets. This allowed the super charges that act on supermultiplets to be mapped into Coxeter group elements. An enabling feature of this mapping operator is that the mapping of a supercharge is determined by the off-shell supermultiplet on which the supercharge acts.

The important lessons based on this work as well as that in [5,6]: starting with SUSY transformation laws generated by requiring invariances of on-shell actions (i.e. without auxiliary fields), typically leads to non-square \( L_i^{(R)} \) matrices. Non-square matrices can never describe the hidden Euclidean Clifford Algebras required for off-shell realizations of SUSY. A necessary but not sufficient condition for realizing higher extended SUSY is that some of the \( L_i^{(R)} \) matrices must be square. By judicious choices of constructing larger square matrices from smaller square matrices, higher extended off-shell SUSY’s can be realized.

We begin our presentation by reviewing results for the 4D, \( \mathcal{N} = 1 \) Chiral, Tensor, Vector, and Double Tensor supermultiplets found in the work of [7].

1.1 On-Shell 4D, \( \mathcal{N} = 1 \) Chiral Supermultiplet

We start with chiral supermultiplet (CM) where the supercovariant derivative (equivalent to the supercharge) acts according to

\[
D_a A = \psi_a , \\
D_a B = i (\gamma^5)_a^b \psi_b , \\
D_a \psi_b = i (\gamma^\mu)_{ab} \partial_\mu A - (\gamma^5\gamma^\mu)_{ab} \partial_\mu B .
\]

(1.1)

After a series of calculation one finds the following results

\[
\{ D_a, D_b \} A = i 2 (\gamma^\mu)_{ab} \partial_\mu A , \quad \{ D_a, D_b \} B = i 2 (\gamma^\mu)_{ab} \partial_\mu B , \\
\{ D_a, D_b \} \psi_c = i 2 (\gamma^\mu)_{ab} \partial_\mu \psi_c - i (\gamma^\mu)_{ab} (\gamma_\mu\gamma_\nu)_{cd} \partial_\nu \psi_d .
\]

(1.2)

1.2 On-Shell 4D, \( \mathcal{N} = 1 \) Tensor Supermultiplet

Next, in the tensor supermultiplet (TM) the supercovariant derivative acts according to

\[
D_a \varphi = \chi_a , \\
D_a B_{\mu\nu} = - \frac{1}{4} \left( \left[ \gamma_\mu, \gamma_\nu \right] \right)_a^b \chi_b , \\
D_a \chi_b = i (\gamma^\mu)_{ab} \partial_\mu \varphi - (\gamma^5\gamma^\mu)_{ab} \epsilon_\mu^{\rho\sigma\tau} \partial_\rho B_{\sigma\tau} .
\]

(1.3)

Using this realization of the D-operator yields results of the form

\[
\{ D_a, D_b \} \varphi = i 2 (\gamma^\mu)_{ab} \partial_\mu \varphi , \\
\{ D_a, D_b \} B_{\mu\nu} = i 2 (\gamma^\rho)_{ab} \partial_\mu B_{\mu\nu} + \partial_\mu q_{\nu \mu a b} - \partial_\nu q_{\mu \mu a b} , \\
\{ D_a, D_b \} \chi_c = i 2 (\gamma^\mu)_{ab} \partial_\mu \chi_c , \quad q_{\mu \mu a b} \equiv i 2 (\gamma^\nu)_{ab} [ B_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \varphi ] .
\]

(1.4)
1.3 On-Shell 4D, $\mathcal{N} = 1$ Vector Supermultiplet

In the vector supermultiplet ($VM$) the supercovariant derivative acts as

$$D_a A_\mu = (\gamma^\mu)_a^\ b \lambda_b \ ,$$
$$D_a \lambda_b = -i \frac{1}{4} ([\gamma^\mu, \gamma^\nu])_{ab} (\partial_\mu A_\nu - \partial_\nu A_\mu) \ ,$$

and once again we calculate the anti-commutator to find

$$\{ D_a , D_b \} A_\mu = i \ 2 (\gamma^\rho)_{ab} \partial_\rho A_\mu - \partial_\mu \tau_{ab} \ , \ \tau_{ab} \equiv i \ 2 (\gamma^\nu)_{ab} A_\nu \ ,$$
$$\{ D_a , D_b \} \lambda_c = i \ 2 (\gamma^\mu)_{ab} \partial_\mu \lambda_c - i \frac{1}{2} (\gamma^\mu)_{ab} (\gamma_\mu \gamma^\nu)_c^\ d \partial_\nu \lambda_d$$
$$+ i \frac{1}{16} (\gamma^\alpha , \gamma^\beta)_{ab} (\gamma_\gamma \gamma^\nu)_c^\ d \partial_\nu \lambda_d \ .$$

In both of the cases for the ($TM$) and ($VM$), when the Dirac equation is imposed on the spinor, the algebra takes the form of the usual supersymmetry definition.

1.4 On-Shell 4D, $\mathcal{N} = 1$ Double Tensor Supermultiplet

In the double tensor supermultiplet ($DTM$) the supercovariant derivative acts as

$$D_a X_{\mu \nu} = i \frac{1}{4} (\gamma^5 \gamma_\mu \gamma_\nu)_{ab} \lambda_b \ ,$$
$$D_a Y_{\mu \nu} = - \frac{1}{4} (\gamma_\mu \gamma_\nu)_{ab} \lambda_b \ ,$$
$$D_a \Lambda_c = i (\gamma^\mu)_{ab} \epsilon^{\rho \sigma \tau} \partial_\rho X_{\sigma \tau} - (\gamma^5 \gamma^\mu)_{ab} \epsilon^{\rho \sigma \tau} \partial_\rho Y_{\sigma \tau} \ ,$$

and yields a very different commutation algebra

$$\{ D_a , D_b \} X_{\mu \nu} = i \ 2 (\gamma^\rho)_{ab} \partial_\rho X_{\mu \nu} + \partial_\mu s_{\nu \ ab} - \partial_\nu s_{\mu \ ab}$$
$$- i [ \eta_{\alpha \mu} (\gamma_\nu)_{ab} - \eta_{\alpha \nu} (\gamma_\mu)_{ab} ] \epsilon^{\alpha \rho \sigma \tau} \partial_\rho Y_{\sigma \tau} \ ,$$
$$\{ D_a , D_b \} Y_{\mu \nu} = i \ 2 (\gamma^\rho)_{ab} \partial_\rho Y_{\mu \nu} + \partial_\mu t_{\nu \ ab} - \partial_\nu t_{\mu \ ab}$$
$$+ i [ \eta_{\alpha \mu} (\gamma_\nu)_{ab} - \eta_{\alpha \nu} (\gamma_\mu)_{ab} ] \epsilon^{\alpha \rho \sigma \tau} \partial_\rho X_{\sigma \tau} \ ,$$
$$s_{\mu \ ab} \equiv i \ 2 (\gamma^\nu)_{ab} X_{\mu \nu} \ , \ t_{\mu \ ab} \equiv i \ 2 (\gamma^\nu)_{ab} Y_{\mu \nu} \ ,$$
$$\{ D_a , D_b \} \Lambda_c = i \ 2 (\gamma^\mu)_{ab} \partial_\mu \Lambda_c + i (\gamma^\mu)_{ab} (\gamma^\nu)_c^\ d \partial_\nu \Lambda_d \ .$$

The closure of the algebra on the bosons $X_{\mu \nu}$ and $Y_{\mu \nu}$ include the terms

$$i [ \eta_{\alpha \mu} (\gamma_\nu)_{ab} - \eta_{\alpha \nu} (\gamma_\mu)_{ab} ] \epsilon^{\alpha \rho \sigma \tau} \partial_\rho Y_{\sigma \tau} \ ,$$
$$i [ \eta_{\alpha \mu} (\gamma_\nu)_{ab} - \eta_{\alpha \nu} (\gamma_\mu)_{ab} ] \epsilon^{\alpha \rho \sigma \tau} \partial_\rho X_{\sigma \tau} \ ,$$

respectively. These may be interpreted as equations of motion terms if there is an action that admits

$$\delta Z = -i \epsilon_\mu \epsilon^{\rho \sigma \tau} \left[ \left( \partial_\rho Y_{\sigma \tau} \right) \frac{\partial}{\partial X_{\mu \nu}} - \left( \partial_\rho X_{\sigma \tau} \right) \frac{\partial}{\partial Y_{\mu \nu}} \right] \ ,$$

as the generator of a symmetry.
2 From On-shell 4D, \( \mathcal{N} = 1 \) SUSY Multiplets to Non-Invertible Matrices

Following the procedure in \([6]\) a definition
\[
\hat{\gamma}_I^{(R)} = \frac{1}{2} (\sigma^1 + i\sigma^2) \otimes L_I^{(R)} + \frac{1}{2} (\sigma^1 - i\sigma^2) \otimes [L_I^{(R)}]^t ,
\]  
(2.1)
is utilized to form square matrices from any of the on-shell \( L_I^{(R)} \) quantities for all of the representations. Given the form of these on-shell \( L_I^{(R)} \) and \( R_I^{(R)} \) matrices, some of the quantities \( \hat{\gamma}_I^{(R)} \) are elements of Coxeter Groups. In any of the four on-shell representations, a quantity \( \hat{C}^{(R)} \) is defined as below from using each \( \hat{\gamma}_I^{(R)} \) representation,
\[
\hat{C}^{(R)} = \hat{\gamma}_1^{(R)} \ldots \hat{\gamma}_4^{(R)} .
\]  
(2.2)

In the works \([8,9]\), designed to quantify the relationships among adinkras related to one another by the raising or lowering of nodes, the eigenvalues of such operators are found to be the “height-yielding matrix numbers” (HYMN) that specify this data in matrices associated with each supermultiplet. However in the work of \([6]\) a different discovery was made.

The question in \([6]\) under investigation was whether matrices of the form of \( \hat{\gamma}_1^{(R)} \) and \( \hat{C}^{(R)} \) carried the data to distinguish when pairs of minimal off-shell 4D, \( \mathcal{N} = 1 \) supermultiplets could form off-shell 4D, \( \mathcal{N} = 2 \) supermultiplets? Here we ask the equivalent question about going from 4D, \( \mathcal{N} = 0 \) supermultiplets to 4D, \( \mathcal{N} = 1 \) supermultiplets.

Next we calculate the anti-commutators of the matrices define in Eq. (2.1). We find for \( (R) = (TM) \),
\[
\left\{ \hat{\gamma}_I^{(TM)} , \hat{\gamma}_J^{(TM)} \right\} = 2 \delta_{IJ} I_8 .
\]  
(2.3)

However for the on-shell \( (CM) \), \( (VM) \), and \( (DTM) \) representations one can prove it is impossible to satisfy an equation of this form. Nor can we find \([10]\)
\[
\left\{ \hat{\gamma}_I^{(R)} , \hat{\gamma}_J^{(R)} \right\} = 2 \delta_{IJ} I_d + \mathcal{N}_{IJ} \hat{\alpha}^{(R)} \kappa_{\hat{\alpha}}^{(R)} ,
\]  
(2.4)

with a set of calculable \( \mathcal{N}_{IJ} \hat{\alpha}^{(R)} \) coefficients and \( d \times d \) matrices \( \kappa_{\hat{\alpha}}^{(R)} \) \([5]\). The non-invertability intrudes.

3 Explicit Matrices Reported

We now turn to the explicit representation of the matrices \( \hat{\gamma}_1^{(R)} \) and \( \hat{C}^{(R)} \) for the well-known cases of the chiral, tensor, and vector supermultiplet, as the double tensor supermultiplet \([11,12,13,14]\).

3.1 On-shell Chiral Supermultiplet

\[
\hat{\gamma}_1^{(CM)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} , \quad \hat{\gamma}_2^{(CM)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,
\]
\[
\hat{\gamma}_{3}^{(CM)} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \hat{\gamma}_{4}^{(CM)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
\hat{C} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}.
\]

3.2 On-shell Tensor Supermultiplet

\[
\hat{\gamma}_{1}^{(TM)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \hat{\gamma}_{2}^{(TM)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
\hat{\gamma}_{3}^{(TM)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \hat{\gamma}_{4}^{(TM)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
\hat{C}^{(TM)} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
3.3 On-shell Vector Supermultiplet

\[ \hat{\gamma}_{1}^{(VM)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{\gamma}_{2}^{(VM)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ \hat{\gamma}_{3}^{(VM)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{\gamma}_{4}^{(VM)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ \hat{C}^{(VM)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 \end{bmatrix}. \]

3.4 On-shell Double Tensor Supermultiplet

\[ \hat{\gamma}_{1}^{(DTM)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \]
This final result is so distinctly different from the other cases, reinforcing the description of \((DTM)\) in [7] as "pathogenic." The eigenvalues of \(\hat{\gamma}(DTM)\) are \([-2, -2, -2, -1, 0, 0, 0, 0, 0, 0]\). In no other
case does a factor of 2 appear. In all the other cases, \( \tilde{C}(R) \) (for \((R) \in \{(CM), (TM), (VM)\}) \) is diagonal and with augmentation as described in [15] become invertible. This suggest there may be added difficulties in achieving a full off-shell version of the 4D, \( \mathcal{N} = 1 \) (DTM).

"An ocean traveller has even more vividly the impression that the ocean is made of waves than that it is made of water.'

- Arthur Eddington

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