CONCORDANCE OF DECOMPOSITIONS GIVEN BY DEFINING SEQUENCES

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Abstract. We study the concordance and cobordism of decompositions associated with defining sequences and we relate them to some invariants of toroidal decompositions and to the cobordism of homology manifolds. These decompositions are often wild Cantor sets and they arise as nested intersections of knotted solid tori. We show that there are at least uncountably many concordance classes of such decompositions in the 3-sphere.

1. Introduction

We study equivalence classes of decompositions of $S^3$ and also decompositions of other manifolds. These decompositions are given by toroidal defining sequences (we use the term toroidal for a subspace of an $n$-dimensional manifold being homeomorphic to the disjoint union of finitely many copies of $S^{n-2} \times D^2$) although more generally it would be possible to get similar results by considering handlebodies instead of solid tori in the defining sequences. The problem of classifying decompositions was studied by many authors. By [Sh68] so-called Antoine decompositions in $\mathbb{R}^3$ are equivalently embedded if and only if their toroidal defining sequences can be mapped into each other by homeomorphisms of the stages. More generally [ALM68] for a decomposition $G$ of $\mathbb{R}^3$ given by an arbitrary defining sequence made of handlebodies the homeomorphism type of the pair $(\mathbb{R}^3/G, cl\pi_G(H_G))$, where $\pi_G$ is the decomposition map and $H_G$ is the union of the non-degenerate elements, is determined by the homeomorphism types of the consecutive stages of the defining sequence of $G$. By [GRWZ11] two Bing-Whitehead decompositions of $S^3$ are equivalently embedded if and only if the stages of the toroidal defining sequences are homeomorphic to each other after some number of iterations (counting only the Bing stages). Decompositions given by defining sequences are upper semi-continuous and many shrinkability conditions are known about them. For example, Bing-Whitehead decompositions are shrinkable under some conditions [AS89, KP14] just like Antoine’s necklaces, which are wild Cantor sets. In [Ze05] the maximal genus of handlebodies being associated with a defining sequence is used to study Cantor sets.

In the present paper we define the concordance of decompositions (see Section 2.3) which come with toroidal defining sequences. As for knots, slice decompositions play an important role in the classification: a decomposition is slice if each component of a defining sequence is slice in a way that the $D^{n-1} \times D^2$ thickened slice disk stages are nested into each other. Being concordant means the analogous concordance of the solid tori in the defining sequence and this makes the well-known knot and link concordance invariants possible to apply in order to distinguish between the concordance classes of such decompositions. For example, we show that the concordance group of decompositions of $S^3$, where the defining
sequences have some intrinsic properties, has at least uncountably many elements, see Theorem 3.7. The uncountably many elements that we find are represented by Antoine’s necklaces.

Decompositions appear in studying manifolds, where cell-like resolutions of homology manifolds [Qu82, Qu83, Qu87, Th84, Th04] provide a tool of obtaining topological manifolds. Decompositions also appear in the proof of the Poincaré conjecture in dimension four, see [Fr82, FQ90, BKKPR21], where a cell-like decomposition of a 4-dimensional manifold yields a decomposition space which is a topological manifold. In higher dimensions the decomposition space given by a cell-like decomposition of a compact topological manifold is a homology manifold being also a topological manifold if it satisfies the disjoint disk property [Ed16]. A particular result [Ca78, Ca79, Ed80, Ed06] is that the double suspension of every integral homology 3-sphere is homeomorphic to $S^5$, that is for every homology 3-sphere $H$ there is a cell-like decomposition $G$ of $S^5$ such that the decomposition space is the homology manifold $\Sigma^2 H$ and since $\Sigma^2 H$ satisfies the disjoint disk property, the decomposition $G$ is shrinkable (and this implies that the decomposition space is $S^5$).

Beside concordance, we also define and study another equivalence relation, which is the cobordism of decompositions, see Definition 2.15 and Section 3.2. This yields a cobordism group, which has a natural homomorphism into the cobordism group of homology manifolds [Mi90, Jo99, JR00]. We study how homological manifolds are related to the cobordism group of cell-like decompositions via taking the decomposition space. It turns out that every such decomposition space is cobordant to a topological manifold in the cobordism group of homology manifolds and they generate a subgroup isomorphic to the cobordism group of topological manifolds, see Proposition 3.11. Often we state and prove our results only for unoriented cobordisms but all the arguments obviously work for the oriented cobordisms as well giving the corresponding results.

The paper is organized as follows. In Section 2 we give some basic lemmas and the definitions of the most important notions and in Section 3 we state and prove our main results.

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2. Preliminaries

2.1. Cell-like decompositions. Throughout the paper we suppose that if $X$ is a compact manifold with boundary and $Y$ is a compact manifold with corners, then an embedding $e: Y \to X$ is such that the corners of $Y$ are mapped into $\partial X$ and the pairs of boundary components near the corners of $Y$ are mapped into $\text{int} X$ and into $\partial X$, respectively. We also suppose that $e(\text{int} Y) \subset \text{int} X$. If $Y$ has no corners, then $\partial X \cap e(Y) = \emptyset$. We generalize the notions of defining sequence, cellular set and cell-like set in the obvious way for manifolds with boundary as follows. Recall that a decomposition of a topological space $X$ is a collection of pairwise disjoint subsets of $X$ whose union is equal to $X$.

**Definition 2.1** (Defining sequence for a subset). Let $X$ be an $n$-dimensional manifold with possibly non-empty boundary. A *defining sequence* for a subset $C \subset X$ is a sequence

$$c: \mathbb{N} \to \mathcal{P}(X)$$

$$C_0, C_1, C_2, \ldots, C_n, \ldots$$

of compact $n$-dimensional submanifolds-with-boundary possibly with corners in $X$ such that
(1) every $C_{n+1}$ has a neighbourhood $U$ such that $U \subseteq C_n$,
(2) in every component of $C_n$ there is a component of $C_{n+1}$,
(3) $\cap_{n=0}^{\infty} C_n = C$ and
(4) if $\partial X \neq \emptyset$, then there is an $\varepsilon > 0$ such that $\partial X \times [0, \varepsilon)$ is a collar neighbourhood of $\partial X$ and for every $C_n$ such that $C_n \cap \partial X \neq \emptyset$ we have $C_n \cap (\partial X \times [0, \varepsilon)) = (C_n \cap \partial X) \times [0, \varepsilon)$.

A decomposition of $X$ defined by the defining sequence $c$ is the triple $(X, D, C)$, where $C = \cap_{n=0}^{\infty} C_n$ and the elements of $D \subseteq \mathcal{P}(X)$ are
(1) the connected components of $C$ and
(2) the points in $X - C$.

We denote the decomposition map by $\pi$.

Observe that for a decomposition $(X, D, C)$ the set $C$ is non-empty and each of the non-degenerate elements is a subset of $C$. There could be singletons in $C$ as well. For example in the case of an Antoine’s necklace there are no non-degenerate elements, we choose $C$ to be the Cantor set Antoine’s necklace itself and so $C$ consists of singletons. Every decomposition defined by some defining sequence is upper semi-continuous. A decomposition $D$ of a manifold induces a decomposition on its boundary by intersecting the decomposition elements with the boundary. The decomposition of the boundary $\partial X$ induced by a defining sequence in $X$ is upper semi-continuous. This induced decomposition is given by an induced defining sequence $C_n \cap \partial X$ if $D_{n,k} \cap \partial X \neq \emptyset$ for every component $D_{n,k}$ of each $C_n$. If all $C_n$ in a defining sequence are connected, then $\cap_{n=0}^{\infty} C_n$ is connected.

**Definition 2.2** (Cell-like set). A compact subset $C$ of a metric space $X$ is cell-like if for every neighbourhood $U$ of $C$ there is a neighbourhood $V$ of $C$ in $U$ such that the inclusion map $V \to U$ is homotopic in $U$ to a constant map. A decomposition is called cell-like if each of its decomposition elements is cell-like.

Cell-like sets given by defining sequences are connected because if the connected components could be separated by open neighbourhoods, then a homotopy could not deform the set into one single point in the neighbourhoods.

A space $X$ is finite dimensional if for every open cover $\mathcal{U}$ of $X$ there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ such that no points of $X$ lies in more than $K_X$ of the elements of $\mathcal{V}$, where $K_X$ is a constant depending only on $X$.

**Lemma 2.3.** Let $D$ be a decomposition of a manifold $X$ possibly with non-empty boundary given by a defining sequence. Then the decomposition space $X/D$ is finite dimensional.

**Proof.** If $X$ has no boundary, then the statement follows from Theorem 2 and Proposition 3 in [Da86] Chapter 34. If $X$ has non-empty boundary, then the argument is also similar. $\square$

2.2. **Homology manifolds.** Recall that a metric space $Y$ is an absolute neighbourhood retract (or ANR for short) if for every metric space $Z$ and embedding $i: Y \to Z$ such that $i(Y)$ is closed there is a neighbourhood $U$ of $i(Y)$ in $Z$ which retracts onto $i(Y)$, that is $r_{i(Y)} |_{i(Y)} = i(Y)$ for some map $r: U \to i(Y)$. It is a fact that every manifold is an ANR. A space is called a Euclidean neighbourhood retract (or ENR for short) if it can be embedded into a Euclidean space as a closed subset so that it is a retract of some of its neighbourhoods. It is well-known that a space is an ENR if and only if it is a locally compact, finite dimensional, separable ANR.
Definition 2.4 (Homology manifold). Let \( n \geq 0 \) and let \( X \) and \( Y \) be finite dimensional ANR spaces, where \( Y \) is a closed subset of \( X \). Suppose that for every \( x \in X \) we have

1. \( H_k(X, X - \{x\}) = 0 \) for \( k \neq n \) and
2. \( H_n(X, X - \{x\}) \) is isomorphic to \( \mathbb{Z} \) if \( x \in X - Y \) and it is isomorphic to 0 if \( x \in Y \).

Then \( X \) is an \( n \)-dimensional homology manifold. The set of points \( x \in Y \) are the boundary points of \( X \) and the set \( Y \) is denoted by \( \partial X \). A homology manifold is called closed if it is compact and has no boundary.

Since locally compact and separable homology manifolds are ENR spaces, a locally compact and separable homology manifold is called an ENR homology manifold. In [Mi90] it is proved that for \( n \geq 1 \) and for every compact and locally compact \( n \)-dimensional homology manifold \( X \) the set of boundary points \( \partial X \) is an \( (n - 1) \)-dimensional homology manifold.

Sometimes a space \( X \) without the ANR property but having \( H_k(X, X - \{x\}) = 0 \) for \( k \neq n \) and \( H_n(X, X - \{x\}) = \mathbb{Z} \) in the sense of Čech homology is also called a homology manifold. These spaces arise as quotient spaces of acyclic decompositions of topological manifolds [DW83] while ANR homology manifolds are often homeomorphic to quotients of cell-like decompositions [Qu82, Qu83, Qu87].

In the case of cell-like decompositions the decomposition spaces are homology manifolds if they are finite dimensional essentially because of the Vietoris-Begle theorem [DV09, Theorem 0.4.1]. In more detail, we will use the following. Let \( X' \) be a compact \( n \)-dimensional manifold with possibly non-empty boundary, let \( Y = \partial X' \times [0, 1] \) and attach \( Y \) to \( X' \) as a collar to get a manifold \( X \).

Lemma 2.5. Let \( D' \) be a cell-like decomposition of \( X' \) given by a defining sequence such that \( X' \) contains a small open set (intersecting the possibly non-empty boundary) which consists of singletons. Suppose that the induced decomposition on \( \partial X' \) is cell-like and it is given by the induced defining sequence. Suppose that in \( Y \) a cell-like decomposition \( E \) is given, where \( E \) is the product of the decomposition induced by \( D' \) on \( \partial X' \) and the trivial decomposition of \( [0, 1] \). Denote by \( D \) the resulting decomposition on \( X \). Then \( X/D \) is an \( n \)-dimensional ENR homology manifold with possibly non-empty boundary. The boundary points of \( X/D \) are exactly the points of the ENR homology manifold \( \pi(\partial X) \).

Proof. We have to show that the quotient space

\[ X/D \]

is an \( n \)-dimensional homology manifold with boundary the homology manifold \( \pi(\partial X) \). Take the closed manifold

\[ X \cup_\varphi X, \]

where \( \varphi: \partial X \to \partial X \) is the identity map.

The decomposition space \( X'/D' \) (that is the part of the decomposition space \( X/D \) which is obtained from \( X' \)) is finite dimensional by Lemma 2.3. The doubling of the decomposition \( D \) on \( X \cup_\varphi X \) yields a finite dimensional quotient space, we get this by using estimations for the covering dimension, see [HW41] and [Da86, Corollary 2.4A]. So the decomposition space \( P \) obtained by factorizing \( X \cup_\varphi X \) by the double of \( D \) is a closed finite dimensional homology manifold by [DV09, Proposition 8.5.1]. Since a small neighbourhood of a singleton results an open set in \( P \) homeomorphic to \( \mathbb{R}^n \), it is \( n \)-dimensional. We obtain the space \( X/D \) by cutting \( P \) into two pieces along \( \pi(\partial X) \).
Because of a similar argument the space $\pi(\partial X)$ is a closed $(n-1)$-dimensional homology manifold. The set $\pi(\partial X)$ is closed in the decomposition space $X/\mathcal{D}$ since $\mathcal{D}$ is upper semi-continuous and $\partial X$ is closed. Also, the homology group $H_n(X/\mathcal{D}; X/\mathcal{D} - \{p\})$ is equal to 0 for every $p \in \pi(\partial X)$. So $\pi(\partial X)$ is the boundary of $X/\mathcal{D}$.

Moreover the space $X/\mathcal{D}$ is a locally compact separable metric space because $X$ is so. By [DV09, Corollary 7.4.8] the space $X/\mathcal{D}$ is an ANR so it follows that it is an ENR. The same holds for $\pi(\partial X)$. □

**Definition 2.6** (Cobordism of homology manifolds). The closed $n$-dimensional homology manifolds $X_1$ and $X_2$ are cobordant if there exists a compact $(n+1)$-dimensional homology manifold $W$ such that $\partial W$ is homeomorphic to the disjoint union of $X_1$ and $X_2$. The induced cobordism group (the group operation is the disjoint union) is denoted by $\mathfrak{R}_n^H$.

In a similar way the induced oriented cobordism group is denoted by $\Omega_n^H$.

Note that the connected sum of homology manifolds does not always exist. Analogously let $\mathfrak{R}_n^E$ and $\Omega_n^E$ denote the cobordism group and oriented cobordism group of ENR homology manifolds (the cobordisms are also ENR), respectively.

Almost all oriented cobordism groups $\Omega_n^H$ are computed [BFMW96, Jo99, JR00]:

$$\Omega_n^H = \begin{cases} 
\mathbb{Z} & \text{if } n = 0 \\
0 & \text{if } n = 1, 2 \\
\Omega_n^{TOP}[8\mathbb{Z} + 1] & \text{if } n \geq 6,
\end{cases}$$

where $\Omega_n^{TOP}$ denotes the cobordism group of topological manifolds and the group $\Omega_n^{TOP}[8\mathbb{Z} + 1]$ denotes the group of finite linear combinations $\sum_{i \in 8\mathbb{Z} + 1} \omega_i$ of cobordism classes of topological manifolds. By [Ma71, Corollary 4.2] the oriented cobordism group of manifolds $\Omega_n^H$ is always a subgroup of $\Omega_n^H$.

A resolution of a homology manifold $N$ is a topological manifold $M$ and a cell-like decomposition of $M$ such that the decomposition space is homeomorphic to the homology manifold $N$, the quotient map $\pi$ is proper and $\pi^{-1}(\partial N) = \partial M$. By [Qu82, Qu83, Qu87] homology manifolds are resolvable if a local obstruction is equal to 1, more precisely we have the following.

**Theorem 2.7** ([Qu82, Qu83, Qu87]). For every $n \geq 4$ and every non-empty connected $n$-dimensional ENR homology manifold $N$ there is an integer local obstruction $i(N) \in 8\mathbb{Z} + 1$ such that

1. if $U \subset N$ is open, then $i(U) = i(N)$,
2. if $\partial N \neq \emptyset$, then $i(\partial N) = i(N)$,
3. $i(N \times N_1) = i(N)i(N_1)$ for any other homology manifold $N_1$,
4. if $\dim N = 4$ and $\partial N$ is a manifold, then there is a resolution if and only if $i(N) = 1$ and
5. if $\dim N \geq 5$, then there is a resolution if and only if $i(N) = 1$.

By [Th84, Th04] a closed 3-dimensional ENR homology manifold $N$ is resolvable if its singular set has general position dimension less than or equal to one, that is any map of a disk into $N$ can be approximated by one whose image meets the singular set (i.e. the set of non-manifold points) of $N$ in a 0-dimensional set.

**Lemma 2.8.** Let $M_1$ and $M_2$ be two closed $n$-dimensional manifolds, where $n \geq 4$. If both of them are resolutions of the ENR homology manifold $N$, then $M_1$ and $M_2$ are cobordant as manifolds.
Proof. If there are two resolutions \( f_1: M_1 \rightarrow N \) and \( f_2: M_2 \rightarrow N \) of a closed \( n \)-dimensional homology manifold \( N \), then as in the proof of [Qu82, Theorem 2.6.1] take a resolution

\[
Y \rightarrow X_{f_1} \cup X_{f_2}
\]

of the double mapping cylinder \( X_{f_1} \cup X_{f_2} \) of the maps \( f_1 \) and \( f_2 \) by applying [Qu83, Theorem 1.1] and [Qu87]. This resolution exists because \( X_{f_1} \cup X_{f_2} \) is an \((n+1)\)-dimensional ENR homology manifold and \( i(X_{f_1} \cup X_{f_2}) = 1 \). Let

\[
X_{f_1} \cup X_{f_2} \rightarrow N \times [-1,1]
\]

be the natural map of the double mapping cylinder onto \( N \times [-1,1] \), where the target \( N \) of the two mapping cylinders is mapped onto \( N \times \{0\} \).

It follows that the composition

\[
Y \rightarrow X_{f_1} \cup X_{f_2} \rightarrow N \times [-1,1]
\]

is a resolution, moreover by [Qu83, Theorem 1.1] the cell-like map \( Y \rightarrow X_{f_1} \cup X_{f_2} \) can be chosen so that it is a homeomorphism over the boundary hence \( Y \) is a cobordism between \( M_1 \) and \( M_2 \).

2.3. Concordance and cobordism of decompositions. We will study decompositions given by defining sequences \( C_0, C_1, C_2, \ldots \) such that each \( C_n \) is a disjoint union of solid tori. We remark that more generally all the following notions work for decompositions whose stages are handlebodies instead of just tori. In a closed \( n \)-dimensional manifold \( M \) instead of decompositions \((M, D, A)\) we will consider decompositions with some thickened link which contains the set \( A \) so in the following a decomposition in \( M \) is a quadruple \((M, D, A, L)\), where \( L \subset M \) is the thickened link and \( A \subset L \). For example an Antoine’s necklace is situated inside an unknotted solid torus while it can be knotted in many different ways in the solid torus.

Definition 2.9 (Concordance of decompositions). Let \( M_1 \) and \( M_2 \) be closed \( n \)-dimensional manifolds. The decompositions \((M_1, D_1, A, L_1)\) and \((M_2, D_2, B, L_2)\) are cylindrically related if there exist toroidal defining sequences \( C_0, C_1, C_2, \ldots \) for \( A \) and \( D_0, D_1, D_2, \ldots \) for \( B \) and there exists a defining sequence \( E_0, E_1, E_2, \ldots \) for a decomposition \( E \) of a compact \((n+1)\)-dimensional manifold \( W \) such that

1. \( C_0 = L_1 \) and \( D_0 = L_2 \),
2. \( \partial W = M_1 \cup M_2 \),
3. each \( E_i \) is homeomorphic to \( C_i \times [0,1] \) and
4. each \( E_i \) bounds the components of \( C_i \subset M_1 \) and \( D_i \subset M_2 \) that is \( C_i \times \{0\} \) corresponds to \( C_i \) and \( C_i \times \{1\} \) corresponds to \( D_i \).

Two decompositions \((M_1, D_1, A, L_1)\) and \((M_2, D_2, B, L_2)\) are concordant if there exist closed \( n \)-dimensional manifolds \( M'_1, \ldots, M'_k \) and decompositions \((M'_1, D'_1, A'_1, L'_1)\) for every \( i = 1, \ldots, k \) such that

1. \((M_1, D_1, A, L_1)\) is cylindrically related to \((M'_1, D'_1, A'_1, L'_1)\), also for \( i = 1, \ldots, k - 1 \) every \((M'_i, D'_i, A'_i, L'_i)\) is cylindrically related to \((M'_{i+1}, D'_{i+1}, A'_{i+1}, L'_{i+1})\) and \((M'_k, D'_k, A'_k, L'_k)\) is cylindrically related to \((M_2, D_2, B, L_2)\) and
2. for each \( A'_i \subset M'_i \), where \( i = 1, \ldots, k \), the two toroidal defining sequences \( C'_i, 0; C'_i, 1, \ldots \) and \( C''_i, 0; C''_i, 1, \ldots \) in \( M'_i \) appearing in these successive cylindrically related decompositions are such that the 0-th stages \( C'_i, 0 \) and \( C''_i, 0 \) are equal as subsets of \( M'_i \).
Being concordant is an equivalence relation and the equivalence classes are called concordance classes.

Hence being concordant implies that the two decompositions are in the same equivalence class of the equivalence relation generated by being cylindrically related, that is the two decompositions can be connected by a finite number of cylindrically related decompositions. Being concordant also implies that the 0-th stages of two toroidal defining sequences for the two decompositions are connected by a single concordance in the usual sense. Clearly in the definition each $E_i$ intersects some fixed collar of $\partial W$ as the defining sequence in (4) of Definition 2.1. The concordance classes form a commutative semigroup under the operation “disjoint union”. Moreover this semigroup is a monoid because the neutral element is the “empty manifold”, that is the empty set $\emptyset$. To have a more meaningful neutral element we define the following.

**Definition 2.10 (Slice decomposition).** Let $M$ be a closed $n$-dimensional manifold and let $(M, D, A, L)$ be a decomposition of $M$ such that there exists a toroidal defining sequence $C_0, C_1, C_2, \ldots$ with $C_0 = L$ for $A$. Then $(M, D, A, L)$ is slice if it is concordant to a decomposition $(M', D', A', L')$ with defining sequence $C'_0, C'_1, C'_2, \ldots$ with $C'_0 = L'$ such that there exists a defining sequence $E_0, E_1, E_2, \ldots$ for a decomposition $\mathcal{E}$ of the $(n+1)$-dimensional manifold $M' \times [0, 1)$, where each $E_i$ consists of finitely many $D^{n-1} \times D^2$ bounding the torus components $S^{n-2} \times D^2$ of $C'_i \subset M' \times \{0\}$.

Analogously to Definitions 2.9 and 2.10 we define the oriented concordance of decompositions by requiring all the manifolds to be oriented in the usual consistent way, in this way we also get a corresponding monoid. Observe that the set of concordance classes of slice decompositions is a submonoid of the monoid of concordance classes of decompositions. To obtain a group we factor out the concordance classes by the classes represented by the slice decompositions and also by the classes of the form $[(M, D, A, L)] + [(-M, D, A, L)]$, where $-M$ denotes the opposite orientation. Observe that all these classes form a submonoid.

**Definition 2.11 (Decomposition concordance group).** Define the relation $\sim$ on the set of concordance classes of decompositions by the following rule: $a \sim b$ exactly if there exist slice decompositions $s_1$ and $s_2$ and decompositions $(M, D, A, L)$ and $(M', D', A', L')$ such that

$$a + [s_1] + [(M, D, A, L)] + [(-M, D, A, L)] = b + [s_2] + [(M', D', A', L')] + [(-M', D', A', L')]$$

The relation $\sim$ is a congruence and we obtain a commutative group by factoring out by this congruence. We call this group the oriented decomposition concordance group and denote it by $\Gamma_n$.

If we confine the closed $n$-dimensional manifolds to $S^n$ and the cobordisms to $S^n \times [0, 1]$, then we obtain something similar to the classical link concordance. For the convenience of the reader we repeat the definitions.

**Definition 2.12 (Concordance group of decompositions in $S^n$).** Let $(S^n, D_1, A, L_1)$ and $(S^n, D_2, B, L_2)$ be decompositions of $S^n$ in the complement of $\infty$. They are cylindrically related if there exist toroidal defining sequences $C_0, C_1, C_2, \ldots$ for $A$ and $D_0, D_1, D_2, \ldots$ for $B$ and there exists a defining sequence $E_0, E_1, E_2, \ldots$ for a decomposition $\mathcal{E}$ of the
compact \((n + 1)\)-dimensional manifold \(S^n \times [0, 1]\) in the complement of \(\{\infty\} \times [0, 1]\) such that

1. \(C_0 = L_1\) and \(D_0 = L_2\).
2. each \(E_i\) is homeomorphic to \(C_i \times [0, 1]\) and
3. each \(E_i\) bounds the components of \(C_i \subset S^n \times \{0\}\) and \(D_i \subset S^n \times \{1\}\).

Two decompositions are concordant if

1. they are in the same equivalence class of the equivalence relation generated by being cylindrically related so the two decompositions can be connected by a finite number of cylindrically related decompositions and
2. the 0-th stages of the defining sequences appearing in this sequence of cylindrically related decompositions are concordant as thickened links in the usual sense.

The obtained equivalence classes are called concordance classes. If two decompositions of \(S^n\) are given by defining sequences, then in the connected sum (at \(\infty\)) of the two \(n\)-spheres the “disjoint union” induces a commutative semigroup operation on the set of concordance classes. Then by factoring out by the submonoid of classes of slice decompositions and classes of the form \([S^n, D, A, L] + [(-S^n, D, A, L)]\) we get a group called the decomposition concordance group in \(S^n\). We denote this group by \(\Delta_n\).

For example, the Whitehead decomposition in \(S^3\) is slice \cite{fr} and the Bing decomposition in \(S^3\) is also slice because the Bing double of the unknot is slice. Observe that the Bing decomposition \((S^3, B, C)\) has only singletons, where \(C\) is a wild Cantor set. As another example, a defining sequence in \(S^3\) given by the replicating pattern of a solid torus and inside of it a link made of a sequence of ribbon knots linked with each other circularly can yield a slice decomposition.

Since being concordant implies that the two decompositions can be connected by a finite number of cylindrically related decompositions, all invariants of concordance classes defined through defining sequences are invariant under choosing another defining sequence for the same decomposition (while leaving the 0-th stage unchanged). For \(n = 3\) in the following we restrict ourselves only to such toroidal defining sequences \(C_0, C_1, C_2, \ldots\) of decompositions of the closed \(n\)-dimensional manifolds in Definitions 2.9-2.12 which satisfy the following conditions:

**Definition 2.13 (Admissible defining sequences and decompositions).** Suppose

1. for \(m \geq 1\) each \(C_m\) has at least four components in a component of \(C_{m-1}\) and each component \(T\) of \(C_m\) is linked to exactly two other components of \(C_m\) in the ambient space \(S^3\) with algebraic linking number non-zero and the splitting number of \(T\) and each of the other components is equal to 0,
2. for \(m \geq 1\) the components \(A_1, \ldots, A_k\) of \(C_m\) which are in a component \(D\) of \(C_{m-1}\) are linked in such a way that if a component \(A_i\) is null-homotopic in a solid torus \(T\) whose boundary is disjoint from all \(A_i\), then all \(A_i\) are in this solid torus \(T\),
3. \(C_m\) is not separated by and not contained in any 2-dimensional sphere \(S\) for which \(S \subset C_m\) for some \(m\),
4. every embedded circle in the boundary of a component of \(C_m\) which bounds no 2-dimensional disk in this boundary cannot be shrunk to a point in the complement of \(\cap_{m=0}^{\infty} C_m\).

We call such defining sequences and decompositions admissible.
Proposition 2.14. In the connected sum (at $\infty$) of two 3-spheres the “disjoint union” as in Definition 2.12 of two admissible toroidal decompositions is an admissible toroidal decomposition.

Proof. Checking the conditions (1)-(4) in Definition 2.13 is obvious, details are left to the reader. $\square$

Then we denote the arising concordance group in $S^3$ by $\Delta^a_3$. For example, Antoine’s necklaces (or Antoine’s decompositions) for $n = 3$ have defining sequences satisfying these conditions [Sh68]. We note that by [Sh68] their defining sequences also have the property of simple chain type, which means that the torus components are unknotted and they are linked like the Hopf link. We have the natural group homomorphisms

$$\Delta^a_n \to \Delta_n$$

and also for arbitrary $n$ the group homomorphism

$$\Delta_n \to \Gamma_n.$$

We will show that the number of elements of the group $\Delta^a_3$ is at least uncountable.

Now we define cobordism of decompositions, where we restrict ourselves to cell-like decompositions (not necessarily admissible) at the cobordisms and at the representatives as well.

Definition 2.15 (Cobordism of decompositions). Let $M_1$ and $M_2$ be closed $n$-dimensional manifolds and let $(M_1, D_1, A)$ and $(M_2, D_2, B)$ be cell-like decompositions such that there exist toroidal defining sequences $C_0, C_1, C_2, \ldots$ for $A$ and $D_0, D_1, D_2, \ldots$ for $B$. Then $(M_1, D_1, A)$ and $(M_2, D_2, B)$ are coupled if there exists a defining sequence $E_0, E_1, E_2, \ldots$ for a cell-like decomposition $E$ of a compact $(n+1)$-dimensional manifold $W$ such that

1. $\partial W = M_1 \sqcup M_2$,
2. each $E_i$ is homeomorphic to the disjoint union of finitely many manifolds $P_j^{n-1} \times D^2$, $j = 1, \ldots, m_i$, where all $P_j^{n-1}$ are compact $(n-1)$-dimensional manifolds and
3. each $E_i$ bounds the components of $C_i$ and $D_i$.

We attach a collar $\partial W \times [0,1]$ to $W$ along its boundary and extend the decomposition $E$ to the collar by taking the product of $D_1$ and $D_2$ with the trivial decomposition on $[0,1]$, respectively. We say that this extended manifold $W \cup (\partial W \times [0,1])$ and its decomposition is a coupling between $(M_1, D_1, A)$ and $(M_2, D_2, B)$. Finally, two decompositions are cobordant if they are in the same equivalence class of the equivalence relation generated by being coupled. The generated equivalence classes are called cobordism classes.

Clearly each $E_i$ intersects some fixed collar of $\partial W$ as the defining sequence in (4) of Definition 2.11. The cobordism classes form a commutative group under the operation “disjoint union”. Denote this group by $B_n$.

We will show that for a cobordism between arbitrary given cell-like decompositions $D_{1,2}$ as in Definition 2.15 if we take the decomposition space, then we get a group homomorphism into the cobordism group of homology manifolds.

3. Results

3.1. Computations in the concordance groups. We are going to define invariants of elements of the group $\Delta^a_3$. With the help of these invariants, we will show that the group $\Delta^a_3$ has at least uncountably many elements.
Definition 3.1. For a given defining sequence $C_0, C_1, C_2, \ldots, C_n, \ldots$ in $S^3$ let

$$n_{C_0, C_1, C_2, \ldots} = (n_0, n_1, n_2, \ldots)$$

be the sequence of the numbers of components of the manifolds $C_0, C_1, C_2, \ldots$.

If two decompositions of $S^3$ as in Definition 2.12 are cylindrically related, then they have defining sequences $C_0, C_1, C_2, \ldots$ and $D_0, D_1, D_2, \ldots$ such that

$$n_{C_0, C_1, C_2, \ldots} = n_{D_0, D_1, D_2, \ldots}.$$

By [Sh68, Theorem 3] for canonical defining sequences of an Antoine’s necklace (or an Antoine decomposition) the sequence $n_{C_0, C_1, C_2, \ldots}$ uniquely exists (note that $C_0$ is only an unknotted solid torus which is not appearing in [Sh68]).

Proposition 3.2. Let $(S^3, D, A, C_0)$ be an admissible decomposition and let $C_0, C_1, C_2, \ldots$ and $D_0, D_1, D_2, \ldots$ be admissible defining sequences for $(S^3, D, A, C_0)$, where we suppose that $C_0 = D_0$. Then we have

$$n_{C_0, C_1, C_2, \ldots} = n_{D_0, D_1, D_2, \ldots}.$$

Proof. Suppose that $C_0, C_1, C_2, \ldots$ and $D_0, D_1, D_2, \ldots$ are admissible defining sequences for a decomposition $(S^3, D, A, C_0)$ such that $C_0 = D_0$. Of course

$$\cap_{n=0}^{\infty} C_n = A = \cap_{n=0}^{\infty} D_n.$$

We use an algorithm applied in [Sh68, Proof of Theorem 2]. We restrict ourselves to one component of $C_0$ and to the components of the defining sequences in it, the following argument works the same way for the other components. We can suppose that $\partial C_1 \cap \partial D_1$ is a closed 1-dimensional submanifold of $S^3$. Suppose some component $P$ of $\partial C_1 \cap \partial D_1$ bounds a 2-dimensional disk $Q \subset \partial D_1$. Also suppose that $P$ is an innermost component of $\partial C_1 \cap \partial D_1$ in $\partial D_1$ so $\text{int} Q \cap \partial C_1 = \emptyset$. By (4) in Definition 2.13 if $P$ does not bound a disk $Q'$ in $\partial C_1$, then $P$ is not homotopic to constant in the complement of $A$ but then $P$ cannot bound the disk $Q \subset \partial D_1$. Hence $P$ bounds a disk $Q' \subset \partial C_1$ as well. Then the interior of the sphere $Q \cup Q'$ does not intersect $A$ because of (3) in Definition 2.13. So we can modify $C_1$ by pushing $Q'$ through the sphere $Q \cup Q'$ by a self-homeomorphism of the complement of $A$ and hence we obtain fewer circles in the new $\partial C_1 \cap \partial D_1$. After repeating these steps finitely many times we obtain a new $C_1$ such that $\partial C_1 \cap \partial D_1$ contains no circles which bound disks on $\partial C_1 \cup \partial D_1$. Similarly, by further adjusting $C_1$ in the complement of $A$ as written on [Sh68, page 1198] in order to eliminate the circles in $\partial C_1 \cap \partial D_1$ which bound annuli we finally obtain a $C_1$ such that

- the intersection $\partial C_1 \cap \partial D_1$ is empty,
- no component of $C_1$ is disjoint from all the components of $D_1$ and vice versa,
- each component of $C_1$ is inside a component of $D_1$ or it contains some components of $D_1$.

Then we can see that there is a bijection between the number of components of $C_1$ and $D_1$ because of the following.

If a component of $C_1$ is in $\text{int} D_1$ and it is homotopic to constant in $\text{int} D_1$, then all the other components of $C_1$ are in the same component of $\text{int} D_1$ by (2) in Definition 2.13. This would result that no part of $A$ is in other components of $D_1$, which would contradict to (1) in Definition 2.13 so no component of $C_1$ in $\text{int} D_1$ is homotopic to constant in $\text{int} D_1$. The same holds if we switch the roles of $C_1$ and $D_1$. This means that
• the winding number of a component $T$ of $C_1$ in the component of $D_1$ which contains $T$ is not equal to 0 and the same holds for $D_1$ and $C_1$ with opposite roles.

Furthermore suppose that $T$ is some component of $D_1$ and $T$ contains at least two components $T_1$ and $T_2$ of $C_1$. Then $T$ is linking with other component $T'$ of $D_1$ by (1) in Definition 2.13 with algebraic linking number non-zero. Let $T_3$ be a component of $C_1$ such that $T_3 \subset T'$ or $T' \subset T_3$. If $T_3 \subset T'$, then $T_1$ and $T_2$ are linking with $T_3$ with algebraic linking number non-zero. If $T' \subset T_3$, then $T$ is not in $T_3$ because for example $T_1$ cannot be in $T_3$. But then $T$ is linking with $T_3$ with algebraic linking number non-zero since the same holds for $T$ and $T'$. So again we obtained that $T_1$ and $T_2$ are linking with $T_3$ with linking number non-zero. Now, there is a $T''$ component of $D_1$ which is linking with $T'$ with linking number non-zero and which is disjoint from all the previously mentioned tori ($T', T'' \subset T_3$ is impossible because then both of $T', T''$ are linking with $T$ and also with each other and this contradicts to (1) in Definition 2.13). Let $T_4$ be a component of $C_1$ such that $T_4 \subset T''$ or $T'' \subset T_4$. There are a number of cases to check. If $T_3 \subset T'$ and $T_4 \subset T''$, then $T_3$ is linking with $T_4$. If $T_3 \subset T'$ but $T'' \subset T_4$, then since $T_4$ cannot contain $T$ or $T'$, we have again that $T_3$ is linking with $T_4$. Finally, if $T' \subset T_3$, then since $T''$ cannot be in $T_3$, we have that $T_4 \subset T''$ implies that $T_3$ and $T_4$ are linking and $T'' \subset T_4$ implies that since $T_4$ is disjoint from all the other tori, again $T_4$ is linking with $T_3$. So we obtain that $T_1$ and $T_2$ are linking with $T_3$ and $T_3$ is linking with $T_4$ resulting that $T_3$ is linking with three other components of $C_1$ which contradicts to (1) in Definition 2.13. Summarizing, we obtained the following.

• The intersection $\partial C_1 \cap \partial D_1$ is empty,
• no component of $C_1$ is disjoint from all the components of $D_1$ and vice versa,
• every component of $C_1$ contains one component of $D_1$ or is contained in one component of $D_1$,
• no component of $C_1$ contains more than one component of $D_1$ and vice versa.

All of these imply that the number of components of $C_1$ is equal to the number of components of $D_1$. We repeat the same line of arguments for the components of $C_2$ and $D_2$ lying in each component of $C_1$ or $D_1$ separately, where we perform the previous algorithm in the larger component which contains the smaller one, and so on, in this way we get the result. \qed

**Remark 3.3.** If in (1) in Definition 2.13 we require having splitting number greater than 0 instead of having algebraic linking number non-zero, then the previous arguments could be repeated to get a similar result if we could prove that having two solid tori with splitting number greater than 0 and embedding one circle into each of these tori with non-zero winding numbers results that the splitting number of these two knots is greater than 0. For similar results about knots and their unknotting numbers, see [ST88, HLP22].

It follows that if two admissible decompositions of $S^3$ are in the same equivalence class of the equivalence relation generated by being cylindrically related, then they determine the same sequence of numbers of components. So if we define the operation

$$(n_0, n_1, \ldots) + (m_0, m_1, \ldots) = (n_0 + m_0, n_1 + m_1, \ldots)$$

on the set of sequences, then the induced map

$$[[S^3, D, A, C_0]] \mapsto n_{C_0, C_1, C_2, \ldots},$$

where $C_0, C_1, C_2, \ldots$ is some admissible defining sequence, is a monoid homomorphism.
Definition 3.4. For an equivalence class \( x \) represented by the admissible decomposition \((S^n, D, A, C_0)\) and for its admissible defining sequence \( C_0, C_1, \ldots \) let
\[
L(x) = (l_1, l_2, \ldots)
\]
be the sequence of numbers mod 2 of the components of \( C_m \) which have non-zero algebraic linking number with some other component of \( C_m \).

Lemma 3.5. The map \( L \) is well-defined i.e. admissible decompositions being concordant through finitely many cylindrically related admissible decompositions have the same value of \( L \).

Proof. If decompositions with defining sequences \( C_0, C_1, \ldots \) and \( D_0, D_1, \ldots \) are cylindrically related, then for every \( m \geq 0 \) the pairs of components of \( C_m \) and the pairs of corresponding components of \( D_m \) have the same algebraic linking numbers. Suppose for a decomposition there are two admissible defining sequences \( C_0, C_1, \ldots \) and \( D_0, D_1, \ldots \) such that \( C_0 = D_0 \), we have to show that the linking numbers are equal to 0 simultaneously for both of them (for the components of \( C_0 \) and \( D_0 \) this is obviously true). Of course we know that the components are in bijection with each other by the proof of Proposition 3.2 and in every component of \( C_0 \) after some deformation we have that
- the intersection \( \partial C_1 \cap \partial D_1 \) is empty,
- no component of \( C_1 \) is disjoint from all the components of \( D_1 \) and vice versa,
- every component of \( C_1 \) contains one component of \( D_1 \) or is contained in one component of \( D_1 \),
- no component of \( C_1 \) contains more than one component of \( D_1 \) and vice versa.

If a component \( T \) of \( C_1 \) is linked with a component \( T' \) of \( C_1 \) with linking number 0, then any knot in \( T' \) is linked with \( T \) with linking number 0. Also, if a knot in \( T' \) is linked with \( T \) with linking number 0, then \( T \) and \( T' \) are linked with linking number 0. For every \( m \geq 1 \) after a finite number of iterations of the algorithm in the proof of Proposition 3.2 we get the result. \( \square \)

Of course the map \( L \) is a monoid homomorphism moreover for a class \( x \) represented by a slice decomposition we have \( L(x) = (0, 0, \ldots) \). Also, for a class \( x \) of the form \([(S^n, D, A)] + [(-S^n, D, A)]\) we have \( L(x) = (0, 0, \ldots) \) since all the linking components appear twice.

Definition 3.6. We call the function
\[
\nu: \Delta_3^a \to \mathbb{Z}_2^N
\]
obtained by \( \nu([x]) = L(x) \) the mod 2 component number sequence of the elements of \( \Delta_3^a \).

Theorem 3.7. There are at least uncountably many different elements in the concordance group \( \Delta_3^a \). These can be represented by Antoine decompositions.

Proof. For every element \((l_0, l_1, \ldots) \in \mathbb{Z}_2^N\), where \( l_0 = 0 \), we have an Antoine decomposition representing a class \( x \) such that \( \nu([x]) = (l_0, l_1, \ldots) \). Hence we get uncountably many different classes in the concordance group. \( \square \)

3.2. Computations in the cobordism group.

Proposition 3.8. Suppose that \( n \geq 0 \) and \( M \) is a closed manifold. A closed \( n \)-dimensional homology manifold \( N \) having a resolution \( M \to N \) is cobordant in \( \Omega_n^E \) to \( M \).
Proof. Take $M \times [0, 1]$ and consider the cell-like decomposition $\mathcal{D}$ of $M$ which results the homology manifold $N$. If $\mathcal{S}(X)$ denotes the collection of singletons in a space $X$, then $\mathcal{D} \times \mathcal{S}([0, 1/2])$ union $\mathcal{S}(M \times (1/2, 1])$ is a cell-like decomposition of $M \times [0, 1]$, denote it by $\mathcal{E}$. We have to show that the quotient space

$$M \times [0, 1]/\mathcal{E}$$

is an $(n + 1)$-dimensional homology manifold with boundary homology manifolds $N$ and $M$. Take the closed manifold

$$M \times [0, 1] \cup \varphi M \times [0, 1],$$

where $\varphi : \partial(M \times [0, 1]) \to \partial(M \times [0, 1])$ is the identity map. Since $M/\mathcal{D}$ is $n$-dimensional, the doubling of the decomposition $\mathcal{E}$ on $M \times [0, 1] \cup \varphi M \times [0, 1]$ yields a finite dimensional quotient space, we get this by using estimations for the covering dimension, see [HW41] and [Da86, Corollary 2.4A]. So the decomposition space $\mathcal{P}$ obtained by factorizing $M \times [0, 1] \cup \varphi M \times [0, 1]$ by the double of $\mathcal{E}$ is a closed finite dimensional homology manifold by [DV09, Proposition 8.5.1]. Since this space has an open set homeomorphic to $\mathbb{R}^{n+1}$, it is $(n + 1)$-dimensional. We obtain the space $M \times [0, 1]/\mathcal{E}$ by cutting $\mathcal{P}$ into two pieces along two subsets homeomorphic to $M$ and $N$. This means that $M$ and $N$ are cobordant in $\mathcal{N}_n^E$. □

So if every 3-dimensional homology manifold is resolvable, then $\mathcal{N}_3^E = 0$. Also note that the decomposition space $S^3/\mathcal{W}$ of the Whitehead decomposition $\mathcal{W}$ is a null-cobordant 3-dimensional homology manifold, because $[S^3] = 0$.

Proposition 3.9. For $n \geq 4$ the cobordism group $\mathcal{N}_n$ is a subgroup of $\mathcal{N}_n^E$.

Proof. Let $M_1$ and $M_2$ be closed manifolds. If the two cobordism classes $[M_1]$ and $[M_2]$ in $\mathcal{N}_n^E$ coincide, then since $M_i$ are manifolds, we have $i(M_i) = 1$ hence a cobordism in $\mathcal{N}_n^E$ between $M_1$ and $M_2$ also has index 1 so this cobordism is resolvable. By [Qu83, Theorem 1.1] and Lemma 2.8 there is a manifold cobordism between $M_1$ and $M_2$. □

In Definition 2.15 for $i = 1, 2$ the space $M_i/\mathcal{D}_i$ is an $n$-dimensional ENR homology manifold and $W/\mathcal{E}$ is an $(n + 1)$-dimensional ENR homology manifold if we add the appropriate collars by Lemma 2.5. If $(M, \mathcal{D})$ is such a cell-like decomposition, then we can assign the cobordism class of the decomposition space $M/\mathcal{D}$ to the cobordism class of $(M, \mathcal{D})$. This map

$$\beta_n : \mathcal{B}_n \to \mathcal{N}_n^E$$

$$[(M, \mathcal{D})] \mapsto [M/\mathcal{D}]$$

is a group homomorphism. The image of $\beta_n$ contains the classes represented by topological manifolds since trivial decompositions always exist and it contains also the classes represented by homology manifolds having appropriate resolutions. For $n = 1, 2$ all the homology manifolds are topological manifolds [Wi79] so the homomorphism $\beta_n$ is surjective. Take the natural forgetting homomorphism

$$F_n : \mathcal{B}_n \to \mathcal{N}_n$$

$$[(M, \mathcal{D})] \mapsto [M].$$

For every $n \geq 0$ the diagram
is commutative by Proposition 3.8, where \( \varphi_n \) is the natural map assigning the cobordism class \([M] \in \mathcal{N}_n^E\) to the cobordism class \([M] \in \mathcal{N}_n\).

**Proposition 3.10.** For every \( n \geq 0 \) the image of \( \beta_n \) is equal to the subgroup of \( \mathcal{N}_n^E \) generated by the cobordism classes of topological manifolds.

**Proof.** The statement follows from the fact that \( F_n \) is surjective. \( \square \)

**Proposition 3.11.** For \( n \geq 1 \), we have \( \beta_n(B_n) = \mathcal{N}_n \) in \( \mathcal{N}_n^E \).

**Proof.** By Proposition 3.9 and Proposition 3.10 we have \( \beta_n(B_n) = \mathcal{N}_n \) for \( n \geq 4 \). For \( n = 3 \), since \( \mathcal{N}_3 = 0 \), the statement also holds. For \( n = 2 \), The group \( \mathcal{N}_2 \) is isomorphic to \( \mathbb{Z}_2 \) so by Proposition 3.10 it is enough to show that \( \beta_2(B_2) = \mathbb{Z}_2 \). But \([\mathbb{R}P^2]\) is not null-cobordant in \( \mathcal{N}_2^E \) because \( \mathbb{R}P^2 \) has a non-zero characteristic number as a smooth or topological manifold and then by [BH91] it cannot be null-cobordant. For \( n = 1 \), of course \( \mathcal{N}_1^E = \mathcal{N}_1 = 0 \). \( \square \)

**Remark 3.12.** Instead of cell-like decompositions, which result homology manifolds, it would be possible to study decompositions which are just homologically acyclic and nearly 1-movable, see [DWS83]. These result homology manifolds as well. Without being nearly 1-movable, these can result non-ANR homology manifolds.

As we could see, the class \( \beta_n([\{M,D\}]) = [M/D] \in \mathcal{N}_n \) could not expose a lot of things about the decomposition \( D \). If we add more details to the homology manifolds and their cobordisms, then we could obtain a finer invariant of the cobordism group of decompositions. Recall that the singular set of a homology manifold is the set of non-manifold points, which is a closed set.

**Definition 3.13** (0- and 1-singular homology manifolds). A homology manifold is 0-*singular* if its singular set is a 0-dimensional set. A compact homology manifold with collared boundary is 1-*singular* if its singular set \( S \) consists of properly embedded arcs such that \( S \) is a direct product in the collar. The closed \( n \)-dimensional 0-singular homology manifolds \( X_1 \) and \( X_2 \) are *cobordant* if there exists a compact \((n+1)\)-dimensional 1-singular homology manifold \( W \) such that \( \partial W \) is homeomorphic to the disjoint union of \( X_1 \) and \( X_2 \) and \( \partial W \cap S \) coincides with the singular set of \( X_1 \sqcup X_2 \) under this homeomorphism. The set of (oriented) cobordism classes is denoted by \( \mathcal{N}_n^S \) and \( \Omega_n^S \).

The set of cobordism classes \( \mathcal{N}_n^S \) and \( \Omega_n^S \) are groups with the disjoint union as group operation. Denote by \( \mathcal{N}_n^0 \) the cobordism group of 0-singular manifolds where the cobordisms are arbitrary but the singular set of the cobordisms is not the entire manifold.

Note that the representatives of the classes in \( \beta_n(B_n) \) are 0-singular and the cobordisms between them have not only singular points because the boundary has not only singular points since the singular set is a compact 0-dimensional set. There are natural homomorphisms

\[
i'_n: B'_n \to \mathcal{N}_n^S, \quad i_n: B_n \to \mathcal{N}_n^0 \quad \text{and} \quad B'_n \to B_n,
\]

where \( B'_n \) is the version of \( B_n \) yielding 0-singular spaces and 1-singular cobordisms, there is the forgetful map

\[
\varphi_n: \mathcal{N}_n^S \to \mathcal{N}_n^0
\]
and then the diagram

\[
\begin{array}{ccc}
\mathcal{B}'_n & \overset{i'_n}{\longrightarrow} & \mathcal{N}^S_n \\
\downarrow & & \downarrow \varphi_n \\
\mathcal{B}_n & \overset{i_n}{\longrightarrow} & \mathcal{B}'_0 \\
&\overset{\psi_n}{\longrightarrow} & \mathcal{N}^E_n
\end{array}
\]

commutes. Observe that \(\psi_n\) is injective, \(\varphi_n\) is surjective and since \(\beta_n(B_n) = \psi_n \circ i_n(B_n) = \mathcal{N}_n\), the image \(i'_n(B'_n)\) is in \(\varphi_n^{-1} \circ \psi_n^{-1}(\mathcal{N}_n)\), which could be a larger group than \(\mathcal{N}_n\).

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