Generalization and refinement of Khintchin’s inequality.

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Abstract.

We derive the exponential as well as power decreasing tail estimations for normed sums of centered independent identical distributed (or not) random variables on the Khintchine’s form.

We consider arbitrary, in particular, non-Rademacher’s variables and not only Lebesgue - Riesz rearrangement invariant norms for the random variables.

We intend to calculate the exact value of correspondent limit.

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1 Statement of problem. Notations. Previous results.

Let \((\Omega = \{\omega\}, \mathcal{B}, \mathbb{P})\) be certain probability space with expectation \(\mathbb{E}\) and variance \(\text{Var}\). It will be presumed that it is sufficiently rich; so that there exists considered further random sequences.

The ordinary Lebesgue - Riesz \(L_p = L_p(\Omega)\) for the numerical valued random variable \(\eta\) is defined as usually
\[
||\eta||_p \overset{\text{def}}{=} \left[ \mathbb{E}|\eta|^p \right]^{1/p}, \quad p \geq 1.
\]

Let also \(\{\xi_i\}, i = 1, 2, \ldots\) be a sequence of centered (mean zero) independent identical distributed (i., i.d.) random variables (r.v.); \(\xi = \xi_1\). The law of distribution of the r.v. \(\xi = \xi_1\) will be denoted by \(L(\xi)\).

Denote by \(D(n)\), \(n = 1, 2, \ldots\) the \(n\) - tuple (vector) of the deterministic numerical sequences of the form
\[
D(n) \overset{\text{def}}{=} \{ (a_1, a_2, \ldots, a_n) \} = \{ \vec{a} \},
\]
for which
\[
\left[ \sum_{k=1}^{n} a_k^2 \right]^{1/2} = ||\vec{a}||_2 = ||a||_2 = 1,
\]
and put
\[
||a||_p = ||\vec{a}||_p \overset{\text{def}}{=} \left[ \sum_{k=1}^{n} |a_k|^p \right]^{1/p}, \quad p \geq 2;
\]
\[
D := \bigcup_{n=1}^{\infty} D(n);
\]
\[
S_n = S_n[\xi] = S_n[D(n)] = S_n[D(n)](\{\xi_k\}) \overset{\text{def}}{=} \frac{\sum_{k=1}^{n} a_k \xi_k}{\sqrt{\sum_{k=1}^{n} a_k^2}} = \sum_{k=1}^{n} a_k \xi_k,
\]
as long as we assume henceforth
\[
a = \vec{a} = (a_1, a_2, \ldots, a_n) \in D(n).
\]

Definition 1.1. Recall the definition of the following important Khintchin’s constants
\[
B[L(\xi)](p) \overset{\text{def}}{=} \sup_{n=1,2,\ldots} \sup_{\{a_k\} \in D(n)} \frac{||S_n[D(n)](\{\xi_k\})||_p}{\sqrt{\sum_{k=1}^{n} a_k^2}},
\]
\[
A[L(\xi)](p) \overset{\text{def}}{=} \inf_{n=1,2,\ldots} \inf_{\{a_k\} \in D(n)} \frac{||S_n[D(n)](\{\xi_k\})||_p}{\sqrt{\sum_{k=1}^{n} a_k^2}}.
\]
These constants was introduced by A.Ya.Khintchin in [22] for the Rademacher’s distribution of the (independent) r.v. - s \(\{\xi_k\}\) and for the Lebesgue - Riesz spaces \(L_p = L_p(\Omega, P)\):

\[
P(\xi_k = 1) = P(\xi_k = -1) = \frac{1}{2}.
\]

They was investigated in many works: [14], [15], [16], [19], [20], [21], [23], [25], [26], [27], [39], [41], [43], [44], [45], [46] etc. In particular, was obtained the exact values of these constants as well as its upper and lover bounds, as a rule, for the Rademacher’s variables.

It was considered also the case when the r.v. \(\xi\), as well as its copies, belongs to some separable Banach space.

The aim of this preprint is twofold: a generalization of the mentioned results into the case of an arbitrary distribution of the r.v. - s and into the another rearrangement invariant Banach spaces, instead the classical Lebesgue - Riesz ones, in particular, into the Grand Lebesgue Spaces (GLS), builded on the source probability space.

We clarify also the known results and obtain sometimes the exact values of correspondent constants, see for example theorem 3.1.

The case when the r.v. - s \(\{\xi_k\}\) forms a sequence of martingale differences was investigated in [29], [31], [34], [36], [39], [40]. Note that the correspondent Khintchine estimate is formulated in slightly different terms.

In detail, let \(F\) be some Banach rearrangement invariant functional space builded on the source probability space equipped with the norm \(||\eta||_F\) on the centered r.v. \(\eta\), and let \(\eta_j, j = 1, 2, \ldots\) be independent copies of \(\eta\).

**Definition 1.2.** Define the following important for us *generalized Khintchine’s constants*

\[
B[L(\eta)]\{F\} \overset{\text{def}}{=} \sup_{n=1,2,\ldots} \sup_{\{a_k\} \in D(n)} \frac{||S_n[D(n)]\{\eta_k\}||_F}{\sqrt{\sum_{k=1}^n a_k^2}},
\]

\[
A[L(\eta)]\{F\} \overset{\text{def}}{=} \inf_{n=1,2,\ldots} \inf_{\{a_k\} \in D(n)} \frac{||S_n[D(n)]\{\eta_k\}||_F}{\sqrt{\sum_{k=1}^n a_k^2}}.
\]

These constants may be named as *generalized* Khintchine’s constants defined for the r.v. - s \(\{\eta_j\}\) and for the space \(F\).

**Proposition 1.1. Preliminary bounds.** Note for the beginning that always \(B[L(\eta)]\{F\} \geq ||\eta||_F\). Further, denote \(\sigma^2 := \text{Var}(\eta)\) and introduce also the Gaussian distributed centered r.v. \(\zeta\) with parameters \((0, \sigma^2) : \text{Law}(\zeta) = N(0, \sigma^2)\), if of course \(\sigma^2 \in (0, \infty)\).

It follows from the classical CLT that \(B[L(\eta)]\{F\} \geq ||\zeta||_F\). Thus,
\[
B[L(\eta)]\{F\} \geq \max(||\zeta||F, ||\eta||F).
\] (6)

Quite analogously
\[
A[L(\eta)]\{F\} \leq \min(||\zeta||F, ||\eta||F).
\] (7)

Notice that the best values of the Khinchine’s constant for the classical Rademacher’s series (on the real line) of the r.v. \( \{ \xi_j \} \) was found by U. Haagerup in [16]; in this case both the estimations (6) and (7) are exact for the Lebesgue-Riesz spaces \( F = L_p \) for all the greater values \( p \).

2 Brief description of the theory of Grand Lebesgue Spaces (GLS), with addition.

We will deal with the so-called Grand Lebesgue Spaces (GLS). Recall briefly some used further facts from the theory of these spaces.

**Definition 2.1**, see [24], [30], chapter 1, sections 1.1 - 1.3; [1], [8], [9], [10], [11], [12], [13]. Let \( \phi = \phi(\lambda) \). \( \lambda \in (-\lambda_0, \lambda_0), \lambda_0 = \text{const} \in (0, \infty) \) be even twice continuous differentiable convex function, such that \( \lambda \to 0 \Rightarrow \phi(\lambda) \approx \lambda^2 \), strictly increasing on the interval \( [0, \lambda_0) \). We impose the following condition on these functions in the case when \( \lambda_0 = \infty \):

\[
\lim_{\lambda \to \infty} \frac{\phi(\lambda)}{\lambda} = \infty.
\]

The set of all such a functions will be denoted by \( \Phi \), \( \Phi = \{ \phi \} \).

By definition, the random variable \( \zeta \) belongs to the space \( B(\phi) \), for certain fixed function \( \phi \in \Phi \), if and only if there exists a non-negative finite constant \( \tau \) such that

\[
\forall \lambda : |\lambda| < \lambda_0 \Rightarrow E \exp(\lambda \zeta) \leq \exp(\phi(\lambda \tau)).
\] (8)

The minimal value of the constant \( \tau \) which satisfies the inequality (8) is said to be the \( B(\phi) \) norm of the r.v. \( \zeta \):

\[
||\zeta||B(\phi) \overset{def}{=} \max_{\phi^{-1}} \sup_{\lambda \in [0, \lambda_0)} \{ \ln E \exp(\pm \lambda \zeta) \} / |\lambda|,
\]
so that

\[
\forall \lambda : |\lambda| < \lambda_0 \Rightarrow E \exp(\lambda \zeta) \leq \exp(\phi(\lambda ||\zeta|| B(\phi))).
\] (9)

We suppose in fact that the r.v. \( \zeta \) satisfies the well-known Cramer’s condition:
\[ \exists c > 0 \Rightarrow P(|\zeta| > x) \leq \exp(-cx), \ x \geq 0. \]

then \( \lambda_0 > 0. \) In this case the generated function \( \phi(\cdot) \) may be introduced naturally. Namely, the so-called natural (generating) function for the r.v. \( \zeta \) satisfying the Cramer’s condition is defined as follows

\[ \phi_\zeta(\lambda) := \max_{\pm} \ln E \exp(\pm \lambda \zeta). \]

The function \( \lambda \to E \exp(\lambda \zeta), \ \lambda = \text{const} \) is said to be a moment generating function for the r.v. \( \zeta \), of course, if there exists in certain non-zero neighborhood of origin.

The natural function \( \phi_\zeta(\lambda) \) play a very important role, in particular, in the theory of Large Deviations (L.D.)

These \( B(\phi) \) spaces are complete Banach functional and rearrangement invariant, as well as considered further Grand Lebesgue Spaces. They were introduced at first in the article [24]. The detail investigation of these spaces may be found in the monographs [2] and [30], chapters 1,2.

It is known that \( 0 \neq \zeta \in B(\phi) \) if and only if \( E\zeta = 0 \) and

\[ \exists K = \text{const} \in (0, \infty) \Rightarrow \max \{P(\zeta \geq u), P(\zeta \leq -u)\} \leq \exp \{-\phi^*(u/K)\}, \ u > 0, \]

where \( \phi^*(u) \) denotes the famous Young - Fenchel, or Legendre transform of the function \( \phi : \)

\[ \phi^*(u) \overset{\text{def}}{=} \sup_{|\lambda| < \lambda_0} (\lambda u - \phi(\lambda)); \]

and herewith

\[ ||\zeta||_B(\phi) \leq C_1(\phi)K \leq C_2(\phi)||\zeta||_B(\phi). \]

More exactly, if \( 0 < ||\zeta||_B(\phi) = ||\zeta|| < \infty, \) then \( \forall u \geq 0 \Rightarrow \)

\[ \max \{P(\zeta \geq u), P(\zeta \leq -u)\} \leq \exp (-\phi^*(u/||\zeta||)). \]

Define the following Young - Orlicz \( N- \) function

\[ N[\phi](u) := \exp \phi^*(u) - 1. \quad (10) \]

It is proved in particular in [24], [30], chapter 1, that if \( \lambda_0 = \infty, \) then the space \( B(\phi) \) coincides up to norm equivalence with the closed subspace of exponential Orlicz space \( L(N[\phi]) \) builded on the source probability space and consisted only on the centered random variables.

Recall yet, see e.g. [1], [8], [9], [10], [11], [12], that the so-called Grand Lebesgue Space (GLS) \( G\psi \) equipped with the norm \( ||\zeta||_{G\psi} \) of the r.v. \( \zeta \) is defined as follows
$\|\zeta\|_{G^\psi} \overset{\text{def}}{=} \sup_{p \geq 2} \left[ \frac{\|\zeta\|_p}{\psi(p)} \right]$. 

Here $\psi = \psi(p)$, $p \geq 1$ is measurable bounded from below function, which is names as ordinary as generating function for this space.

If the r.v. $\zeta$ belongs to some $B(\phi)$ space, then it belongs also to certain $G^\psi$ space with

$$\psi = \psi_\phi(p) = \frac{\phi^{-1}(p)}{p}, \quad p \geq 2.$$ 

The inverse conclusion in not true. Namely, the mean zero r.v. $\zeta$ can has finite all the moments $|\zeta|_p < \infty$, $p \geq 2$, but may not satisfy the Cramer’s condition.

A very popular class of these spaces form the subgaussian random variables, i.e. for which $\phi(\lambda) = \phi_2(\lambda) \overset{\text{def}}{=} 0.5\lambda^2$ and $\lambda_0 = \infty$. For instance, every centered Gaussian distributed r.v. is also subgaussian.

The correspondent $\psi$ function has a form $\psi(p) = \sqrt{p}$.

More generally, suppose

$$\phi(\lambda) = \phi_m(\lambda) = |\lambda|^{m}/m, \quad |\lambda| \geq 1, \quad \lambda_0 = \infty, \quad m = \text{const} \geq 1.$$ 

The correspondent $\psi$ function has a form

$$\psi(p) = \psi_m(p) = p^{1/m}$$

and the correspondent tail estimate is follow:

$$\max \{ \mathbb{P}(\zeta \geq u), \mathbb{P}(\zeta \leq -u) \} \leq \exp \left\{ -(u/K)^m \right\}, \quad u > 0.$$

These space are used in particular for obtaining of the exponential estimates for sums of independent random variables, see e.g. [24], [30], sections 1.6, 2.1 - 2.5.

Indeed, introduce for any function $\phi(\cdot)$ from the set $\Phi$ a new function $\overline{\phi}(\cdot)$ which belongs also at the same set:

$$\overline{\phi}(\lambda) \overset{\text{def}}{=} \sup_{n=1,2,...} \left[ \phi\{\lambda/\sqrt{n}\} \right].$$

It is easily to see that

$$\sup \mathbb{E} \exp(\lambda S(n)/\sqrt{n}) \leq \exp[\overline{\phi}(\lambda)]$$

with correspondent uniform relative the variable $n$ exponential tail estimate.

For instance, if $\mathbb{E}\xi = 0$ and for some value $m = \text{const} > 0$

$$\max \{ \mathbb{P}(\xi \geq u), \mathbb{P}(\xi \leq -u) \} \leq \exp \left\{ -u^m \right\}, \quad u > 0,$$

then

$$\sup_n \max \left[ \mathbb{P}(S(n)/\sqrt{n} \geq u), \mathbb{P}(S(n)/\sqrt{n} \leq -u) \right] \leq$$
\[ \exp \left\{ -C(m)u^{\min(m,2)} \right\}, \ u > 0, \ C(m) \in (0, \infty), \]

and the last estimate is essentially non-improvable.

We will use the following fact, see [30], chapter 1, section 1.6, theorem 1.6.1. Let \( \{\xi_i, i = 1, 2, \ldots, n\} \) be independent r.v. belonging to the certain space \( B(\phi) \). Denote the set of all convex functions by \( \text{Conv} \).

Let us impose the following important condition on this function (CONDITION TRIANGLE) of order \( r, r = \text{const} \in [1, 2] \).

**Definition 1.4.**, see [30] chapter 1, section 1.6. We will write that the function \( \phi(\cdot) \) from the set \( \Phi \) belongs to the set \( \text{Conv}_r \), write \( \phi(\cdot) \in \text{Conv}_r \), iff the function \( \lambda \rightarrow \phi(|\lambda|^{1/r}) \) is convex.

For instance, the classical subgaussian function \( \phi_2(\lambda) = 0.5\lambda^2, \ \lambda \in \mathbb{R} \) belongs to the set \( \text{Conv}_2 \).

It is proved ibid that if the sequence of independent r.v. \( \eta_j \) belong to the space \( B(\phi) \) with \( \phi(\cdot) \in \text{Conv}_2 \), then

\[ ||n \sum_{j=1}^n \eta_j||^2 B\phi \leq \sum_{j=1}^n ||\eta_j||^2 B(\phi), \quad (11) \]

the Pythagoras inequality.

More generally, let \( \phi(\cdot) \) be arbitrary function from the set \( \Phi \). Define the following its transformation

\[ \hat{\phi}(\lambda) \overset{\text{def}}{=} \sup_n \sup_{a \in D(n)} \phi \left( \lambda \sum_{j=1}^n a_j \right). \quad (12) \]

This function \( \hat{\phi}(\cdot) \) obeys a following sense. Let \( \{\eta_j\} \) be independent r.v. - s from the set \( B\phi \) and have an unit norm in this space: \( ||\eta_j||B\phi = 1 \). Then for all the values \( n = 1, 2, \ldots \)

\[ ||n \sum_{j=1}^n a_j \eta_j||B\hat{\phi} \leq 1, \quad \{a_j\} \in D(n). \]

### 3 Main result. Exact Khinchine’s constant calculation.

**Theorem 3.1.** Suppose that the source random variable \( \xi \) belongs to some \( B(\phi), \ \exists \phi \in \Phi \) space and assume besides that the function \( \phi(\cdot) \) belongs to the class \( \text{Conv}_2 \). Then
\[ B[L(\xi)]\{B(\phi)\} = ||\xi||B(\phi). \tag{13} \]

**Proof** is simple. Let \( \xi \in B(\phi), \ \phi \in \text{Conv}_2 \). Let also \( \sum_{j=1}^{n} a_j^2 = 1 \). We have the following upper estimate for the arbitrary integer positive value \( n \)

\[ ||S_n||B(\phi) \leq ||\xi||B(\phi) \cdot \sqrt{\sum_{j=1}^{n} a_j^2} = ||\xi||B(\phi). \]

On the other hand, we have the following lower estimate, choosing the value \( n = 1 \)

\[ ||S_1||B(\phi) = ||\xi||B(\phi), \]

This completes the proof of (13).

**Example 3.1.** Let the r.v. \( \theta \) has a Rademacher’s distribution; then

\[ E e^{\lambda \theta} = \cosh \lambda, \ \lambda \in \mathbb{R}. \]

As long as \( \cosh \lambda \leq \exp(\lambda^2/2) \), we observe that the r.v. \( \theta \) is subgaussian and has an unit norm in the space \( B\phi_2 \). Since the function \( \phi_2 \) belongs to the set \( \text{Conv}_2 \), one can apply the proposition of theorem 3.1.

Further, the subgaussian \( B\phi_2 \) norm of the r.v. is equivalent to the Grand Lebesgue norm with the generating function \( \psi_2(p) = \sqrt{p}, \ p \geq 1 \); and we obtain the known result

\[ \sup_n ||S_n[\theta]||_p \leq C \sqrt{p}, \]

or equally

\[ \sup_n ||S_n[\theta]||B\phi_2 < \infty. \]

**Example 3.2.** Suppose that the source (centered) variable \( \nu \) belongs to the space \( B\phi_m, \ m \geq 1 \). Denote \( m' := \min(m, 2) \). We conclude

\[ \sup_n ||S_n[\nu]||B\phi_{m'} < \infty \]

or equally

\[ \sup_n ||S_n[\nu]||_p \leq C_1(m) p^{1/m'}, \ p \geq 1. \]

Let us consider a more general case of arbitrary moment generating function \( \phi(\cdot) \in \Phi \).

**Theorem 3.2.**
\[ B[L(\xi)]\{B(\hat{\phi})\} \leq ||\xi||B\phi. \] (14)

**Proof** follows immediately from the direct definition of \( \hat{\phi} \). Indeed, let \( \xi \in B(\phi), \phi \in \Phi \). Let also as above \( \sum_{j=1}^{n} a_j^2 = 1 \). We have the following upper estimate for the arbitrary integer positive value \( n \)

\[ ||S_n||B(\hat{\phi}) \leq ||\xi||B(\phi) \cdot \sqrt{n} \sum_{j=1}^{n} a_j^2 = ||\xi||B(\phi). \]

4 Case of non-identical distributed variables.

It is no hard to generalize obtained before results into the case of the non-identical distributed variables.

Let now the r.v. - s \( \{\xi_k\}, k = 1, 2, \ldots \) be a sequence of independent centered but not necessary identical distributed random variables. Suppose that the each r.v. \( \xi_k \) belongs to some space \( B\phi_k, \phi_k \in \Phi \):

\[ E\exp(\lambda \xi_k) \leq \exp(\phi_k(\lambda)), |\lambda| \leq \lambda_0, \exists \lambda_0 > 0. \]

Of course, one can choose all the functions \( \{\phi_k(\cdot)\}, k = 1, 2, \ldots \) as a natural ones for the r.v. - s \( \xi_k \).

As before, \( S_n := \sum_{k=1}^{n} a_k \xi_k \). Define the following function

\[ \kappa(\lambda) \text{ def } \sup_n \sup_{a \in D(n)} \sum_{k=1}^{n} \phi_k(a_k \lambda). \] (15)

**Theorem 4.1.** Assume that \( \kappa(\cdot) \in \Phi \). Then

\[ B[L\{\xi_k\}]G\kappa \leq 1. \] (16)

**Proof.** Indeed, we have for arbitrary value \( a \in D(n) \)

\[ E\exp(\lambda S_n) = \prod_{k=1}^{n} E\exp(\lambda a_k \xi_k) \leq \prod_{k=1}^{n} \exp(\phi_k(a_k \lambda)) \leq \exp(\kappa(\lambda)), \]

or equally

\[ ||S_n||B(\kappa) \leq 1. \]
uniformly in $a, n$.

**Remark 4.1.** The proposition of theorem 3.1 is a particular case of considered here, indeed, when the functions $\phi_k(\cdot)$ are equal: $\phi_k(\lambda) = \phi(\lambda)$.

Therefore, it is also essentially non-improvable.

5 Grand Lebesgue Space approach.

We suppose now only that the centered r.v. $\xi$ belongs to certain Grand Lebesgue Space (GLS) $G\psi$, $\psi \in \Psi$. As above, $\{\xi_j\}$ are independent copies $\xi$.

We will apply the famous Rosenthal’s [42] inequality for the variable $C(p)$, $p \geq 2$, where

$$C(p) \overset{def}{=} \sup_{\{\eta_j\}} \sup_n \frac{\|\sum_{j=1}^n \eta_j\|_p}{\max\left(\|\sum_{j=1}^n \eta_j\|_2, \left(\sum_{j=1}^n \|\eta_j\|_p\right)^{1/p}\right)} < \infty. \quad (17)$$

Here $\{\eta_j\}$, $j = 1, 2, \ldots$ are independent copies of the centered variable $\eta = \eta_1$ such that $\eta \in L(p, \Omega)$.

Indeed, there are huge numbers of works devoted to evaluate of these "constants", see e.g. [5], [17], [18], [28], [42] etc. In the article [28] was obtained the ultimate optimal order of the value of $C(p)$:

$$C(p) \leq C_R \frac{p}{e \ln p}, \quad p \geq 2, \quad (18)$$

where

$$C_R \approx 1.776379 < 1.77638. \quad (19)$$

Note that for the symmetrical distributed r.v. $\{\eta_j\}$ this constant is equal (approximately) to 1.53572.

**Remark 5.1.** Both the last estimates are attainable: in the first case when the r.v. $\eta$ has a form $\eta = \rho - 1$, the r.v. $\rho$ has a standard Poisson distribution with parameter 1, in the second case $\eta$ has a form $\eta = \rho_1 - \rho_2$, where $\rho_1, \rho_2$ are independent Poisson distributed with parameter 0.5; see [17], [18], [28].

Assume now that the source centered r.v. $\xi$ belongs to some GLS $G\psi$, $\psi \in \Psi$. Introduce the auxiliary such a function

$$\psi_R(p) := C_R \frac{p}{e \ln p} \psi(p), \quad p \geq 2. \quad (20)$$

**Theorem 5.1.**
Proof. Let \( \xi \in G\psi \); one can suppose without loss of generality \( ||\xi||G\psi = 1 \).

Therefore, for all the acceptable values \( p \Rightarrow ||\xi||p \leq \psi(p) \).

We intend to apply the estimate (17) with clarification (19), substituting the r.v. \( a_j \xi_j \) instead \( \eta_j \). We have denoting \( \sigma^2 = \text{Var}(\xi) \), taking into account the equality \( a \in D(n) : \)

\[
|| \sum_{j=1}^{n} a_j \xi_j \|_2^2 = \sigma^2; 
\]

\[
\sum_{j=1}^{n} |a_j|^p ||\xi_j||_p^p \leq \psi^p(p) \sum_{j=1}^{n} |a_j|^p = \psi^p(p) \|a\|_p^p \leq \psi^p(p) \|a\|_2^p, 
\]

as long as for \( p \geq 2 \Rightarrow \|a\|_p \leq \|a\|_2 = 1 \). Note yet \( \sigma \leq \psi(p) \), as long as \( p \geq 2 \).

We obtain by virtue of (17)

\[
|| \sum_{j=1}^{n} a_j \xi_j ||_p \leq C(p) \max \left[ \sigma \left( \sum_{j=1}^{n} a_j^2 \right)^{1/2}, \|a\|_p \psi(p) \right] \leq 
C_R \frac{p}{e \ln p} \psi(p) = \psi_R(p),
\]

Q.E.D.

6 Khintchine’s inequality in the space of continuous functions.

Let \( Z = \{z\} \) be arbitrary set; the semi-distance function \( \rho = \rho(z_1, z_2), \) \( z_{1,2} \in Z \) on this set will be introduced below. Recall that the semi-distance function is non-negative symmetrical function vanishing in the diagonal \( \rho(z, z) = 0 \), satisfying the triangle inequality but in general case the relation \( \rho(z_1, z_2) = 0 \) does not imply that \( z_2 = z_1 \).

The (Banach) space of all the continuous numerical valued functions will be denoted as ordinary \( C(Z) = C(Z, \rho) \); it is equipped with the uniform norm

\[
||f||C(Z) := \sup_{z \in Z} |f(z)|.
\]

Let \( \eta = \eta(z), \ z \in Z \) be separable centered: \( \mathbf{E} \eta(z) = 0 \) numerical valued random field (r.f.). Let also \( \eta_i = \eta_i(z), \ i = 1, 2, \ldots \) be independent copies of \( \eta(z) \).

Let us impose the following subgaussian condition on the r.f. \( \eta = \eta(z) : \)
\[
\sigma := \sup_{z \in \mathcal{Z}} ||\eta(z)||B(\phi_2) < \infty.
\]

Introduce therefore the following bounded semi-distance on the set \( \mathcal{Z} \):

\[
\rho(z_1, z_2) \overset{\text{def}}{=} ||\eta(z_1) - \eta(z_2)||B(\phi_2).
\]

Denote also by \( H(\epsilon) = H(\mathcal{Z}, \rho, \epsilon) \), \( 0 < \epsilon \leq C_5 \), the metric entropy of the whole set (space) \( \mathcal{Z} \) relative the metric \( \rho \), i.e. the (natural) logarithm of the minimal amount of the closed ball in the distance \( \rho \), which cover this set \( \mathcal{Z} \).

Put as above for arbitrary tuple \( a = \bar{a}(n) \in D(n) \)

\[
Y_{a(n)}(z) \overset{\text{def}}{=} \sum_{i=1}^{n} a_i \eta_i(z), \quad a(n) = \bar{a}(n) \in D(n),
\]

and define the r.v.

\[
\beta \overset{\text{def}}{=} \sup_{a(n) \in D(n)} ||\sup_{z \in \mathcal{Z}} Y_{a(n)}(z)||B(\phi_2).
\]

**Theorem 6.1.** Suppose that the following entropic integral convergent:

\[
\int_0^1 H^{1/2}(\mathcal{Z}, \rho, \epsilon) \, d\epsilon < \infty
\]

the famous Dudley condition, [6], [7]. Then all the random fields \( Y_{a(n)}(z) \) are \( \rho \) – continuous with probability one:

\[
P \left( Y_{a(n)}(\cdot) \in C(\mathcal{Z}, \rho) \right) = 1
\]

and the r.v. \( \beta \) is subgaussian: \( \beta \in B(\phi_2) \) or equally:

\[
\exists C_0 = C_0(H, \sigma) = \text{const} < \infty \Rightarrow ||\beta||_p \leq C_0 \sqrt{p} < \infty.
\]

**Proof.** Let us consider the subgaussian random field \( Y_{a(n)}(z) \). We have taking into account the equality \( \sum_i a^2(i) = 1 \) and properties of the subgaussian norm

\[
\sup_{z \in \mathcal{Z}} \sup_{n} \max_{a(n) \in D(n)} ||Y_{a(n)}(z)||B(\phi_2) \leq \sigma
\]

and

\[
\sup_{n} \max_{a(n) \in D(n)} ||Y_{a(n)}(z_1) - Y_{a(n)}(z_2)||B(\phi_2) \leq \rho(z_1, z_2).
\]

Both the propositions of theorem 6.1 follows immediately from Theorem 3.17.1 of monograph [30], chapter 3, section 17; see also [3], [4].
7 Concluding remarks.

It is interest in our opinion to generalize the obtained results on the sequence of the centered (stationary or not) random variables satisfying one or another mixing condition.

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