Algebraic Aspects in Tropical Mathematics

Tal Perri
Mathematics Department, Bar-Ilan University
Under the supervision of Professor Louis Rowen

June 25, 2013
Abstract

Much like in the theory of algebraic geometry, we develop a correspondence between certain types of algebraic and geometric objects. The basic algebraic environment we work in is the a semifield of fractions $\mathbb{H}(x_1, \ldots, x_n)$ of the polynomial semidomain $\mathbb{H}[x_1, \ldots, x_n]$, where $\mathbb{H}$ is taken to be a bipotent semifield, while for the geometric environment we have the space $\mathbb{H}^n$ (where addition and scalar multiplication are defined coordinate-wise). We show that taking $\mathbb{H}$ to be bipotent makes both $\mathbb{H}(x_1, \ldots, x_n)$ and $\mathbb{H}^n$ idempotent which turn out to satisfy many desired properties that we utilize for our construction.

The fundamental algebraic and geometric objects having interrelations are called kernels, encapsulating congruences over semifields (analogous to ideals in algebraic geometry) and skeletons which serve as the analogue for zero-sets of algebraic geometry. As an analogue for the celebrated Nullstellensatz theorem which provides a correspondence between radical ideals and zero sets, we develop a correspondence between skeletons and a family of kernels called polars originally developed in the theory of lattice-ordered groups. For a special kind of skeletons, called principal skeletons, we have simplified the correspondence by restricting our algebraic environment to a very special semifield which is also a kernel of $\mathbb{H}(x_1, \ldots, x_n)$.

After establishing the linkage between kernels and skeletons we proceed to construct a second linkage, this time between a family of skeletons and what we call ‘corner-loci’. Essentially a corner locus is what is called a tropical variety in the theory of tropical geometry, which is a set of corner roots of some set of tropical polynomials. The relation between a skeleton and a corner-locus is that they define the exact same subset of $\mathbb{H}^n$, though in different ways: while a corner locus is defined by corner roots of tropical polynomials, the skeleton is defined by an equality, namely, equating fractions from $\mathbb{H}(x_1, \ldots, x_n)$ to 1. All the connections presented above form a path connecting a tropical variety to a kernel.

In this paper we also develop a correspondence between supertropical varieties (generalizing tropical varieties) introduced by Izhakian, Knebusch and Rowen to the lattice of principal kernels. Restricting this correspondence to a sublattice of kernels, whom we call regular kernels, yields the correspondence described above. The research we conduct shows that the theory of supertropical geometry is in
fact a natural generalization of tropical geometry.
We conclude by developing some algebraic structure notions such as composition series and convexity degree, along with some notions holding a geometric interpretation, like reducibility and hyperdimension.
## Contents

1. Overview ................................................................. 7
2. Background ................................................................. 9
   2.1 Basic notions in lattice theory .................................. 9
   2.2 Semifields .......................................................... 10
   2.3 Semifield with a generator and generation of kernels ........ 21
   2.4 Simple semifields .................................................. 27
   2.5 Irreducible kernels, maximal kernels and the Stone topology 28
   2.6 Distributive semifields .......................................... 32
   2.7 Idempotent semifields: Part 1 .................................... 33
   2.8 Affine extensions of idempotent archimedean semifields .... 39
3. Lattice-ordered groups, idempotent semifields and the semifield of fractions $\mathbb{H}(x_1, \ldots, x_n)$ .................................................. 41
   3.1 Lattice-ordered groups ........................................... 42
   3.2 Idempotent semifields versus lattice-ordered groups ........ 45
   3.3 Idempotent semifields: Part 2 .................................... 46
   3.4 Archimedean idempotent semifields ............................... 52
4. Skeletons and kernels of skeletons .................................... 56
   4.1 Skeletons .......................................................... 56
   4.2 Kernels of skeletons .............................................. 58
5. The structure of the semifield of fractions ......................... 61
   5.1 Bounded rational functions in the semifield of fractions .... 61
   5.2 The structure of $\langle \mathbb{H} \rangle$ ................................. 65
6. The polar-skeleton correspondence .................................... 69
   6.1 Polars .............................................................. 69
   6.2 The polar-skeleton correspondence ................................ 75
   6.3 Appendix: The Boolean algebra of polars and the Stone representation .............................................. 81
7. The principal bounded kernel - principal skeleton correspondence 84
| Section | Title                                                                 | Page |
|---------|----------------------------------------------------------------------|------|
| 8       | The coordinate semifield of a skeleton                               | 93   |
| 9       | Basic notions: Essentiality, reducibility, regularity and corner-integrality | 95   |
| 9.1     | Essentiality of elements in the semifield of fractions              | 95   |
| 9.2     | Reducibility of principal kernels and skeletons                     | 98   |
| 9.3     | Decompositions                                                      | 106  |
| 9.4     | Regularity                                                          | 111  |
| 9.5     | Corner-Integrality                                                  | 115  |
| 9.6     | Appendix: Limits of skeletons                                        | 117  |
| 10      | The corner loci - principal skeletons correspondence                | 120  |
| 10.1    | Corner loci                                                         | 120  |
| 10.2    | Corner loci and principal skeletons                                  | 124  |
| 10.3    | Example: The tropical line                                          | 135  |
| 11      | Corner-integrality revisited                                         | 137  |
| 12      | Composition series of kernels of an idempotent semifield             | 146  |
| 13      | The Hyperspace-Region decomposition and the Hyperdimension           | 150  |
| 13.1    | Hyperspace-kernels and region-kernels                               | 150  |
| 13.2    | Geometric interpretation of HS-kernels and region kernels and the use of logarithmic scale | 156  |
| 13.3    | A preliminary discussion                                            | 159  |
| 13.4    | The HO-decomposition                                                | 160  |
| 13.5    | The lattice generated by regular corner-integral principal kernels  | 169  |
| 13.6    | Convexity degree and hyperdimension                                  | 170  |

Bibliography | 186 |
1 Overview

In the following overview, we will establish a linkage between our construction and the widely known theory of algebraic geometry, in order to give the reader some additional insights.

As noted in the abstract, we develop a geometric structure corresponding to a semifield of fractions $\mathbb{H}(x_1, \ldots, x_n)$ over a bipotent semifield $\mathbb{H}$. We concentrate on the case where $\mathbb{H}$ is a divisible archimedean semifield, motivated by our interest to link the theory to tropical geometry. We also consider a divisible bipotent archimedean semifield $\mathcal{R}$, which is also complete (and thus isomorphic to the positive reals with max-plus operations).

The role of ideals is played by an algebraic structures called kernels. While ideals encapsulate the structure of the preimages of zero of rings homomorphisms, kernels encapsulate the preimages of $\{1\}$ with respect to semifield homomorphisms with the substantial difference that considering homomorphisms one gets a sublattice of kernels with respect to intersection and multiplication, $(\text{Con}(\mathcal{R}(x_1, \ldots, x_n)), \cdot, \cap)$.

For each kernel $K$ of $\text{Con}(\mathcal{R}(x_1, \ldots, x_n))$, we define an analogue for zero sets corresponding to ideals, namely, a skeleton $\text{Skel}(K) \subseteq \mathcal{R}^n$, which is defined to be the set of all points in $\mathcal{R}^n$ over which all the elements of the kernel $K$ are evaluated to be 1.

While in the classical theory all ideals are finitely generated, in our case we explicitly consider the finitely generated kernels in $\text{Con}(\mathcal{R}(x_1, \ldots, x_n))$. This family of finitely generated kernels in $\text{Con}(\mathcal{R}(x_1, \ldots, x_n))$ forms a sublattice of $(\text{Con}(\mathcal{R}(x_1, \ldots, x_n)), \cdot, \cap)$, which we denote as $\text{PCon}(\mathcal{R}(x_1, \ldots, x_n))$. At this point, the theory is considerably simplified, since a well-known result in the theory of semifields states that every finitely generated kernel is principal, i.e., generated as a kernel by a single element.

The generator of a principal kernel is not unique, though there is a designated set of generators which provides us with many tools for implementing the theory.

First we establish a Zariski-like correspondence between a special kind of ker-
nels called polars and general skeletons. Then we find some special semifield of \( \mathcal{R}(x_1, \ldots, x_n) \) over which the above correspondence gives rise to a simplified correspondence between principal (finitely generated) kernels of and principal (finitely generated) skeletons which hold most of our interest. We show that a skeleton is uniquely defined by any of its corresponding kernel generators, define a maximal kernel, and show that the maximal principal kernels correspond to points in \( \mathcal{R}^n \). Along the way, we provide some theory concerning general (not necessarily principal) kernels and skeleton.

The geometric interpretation established for the theory of semifields provides us with a topology, called the Stone Topology, defined on the family of the so-called irreducible kernels (analogous to prime ideals) and the family of maximal kernels. This topology is in essence the semifields version of the famous Zariski topology.

After establishing this geometric framework, we introduce a map linking supertropical varieties (cf. [4]), which we call ‘Corner loci’, to a subfamily of principal skeletons. We use the latter to construct a correspondence between principal supertropical varieties (the analogue for hypersurfaces of algebraic geometry) and a lattice of kernels generated by special principal kernels which we call corner-integral. In addition, we characterize a family of skeletons that coincide with the family of ‘regular’ tropical varieties via which we obtain a correspondence with a sublattice of principal kernels called regular kernels.

Finally, we establish a notion of reducibility and some other notions and properties of kernels and their corresponding skeletons, such as convex-dependence, dimensionality etc.

We begin our thesis by introducing the relevant results in the theory of semifields.
2 Background

2.1 Basic notions in lattice theory

Definition 2.1.1. A poset \((X, \leq)\) is \emph{directed} if for any pair of elements \(a, b \in X\) there exists \(c \in X\) such that \(a \leq c\) and \(b \leq c\), i.e., \(c\) is an upper bound for \(a\) and \(b\).

A poset \((X, \leq)\) is a \emph{lattice} or \emph{lattice-ordered set} if for \(a, b \in X\), the set \(\{a, b\}\) has a join \(a \lor b\) (also known as the least upper bound, or the supremum) and a meet \(a \land b\) (also known as the greatest lower bound, or the infimum).

Equivalently, a lattice can be defined as a directed poset \((X, \lor, \land)\), consisting of a set \(X\) and two associative and commutative binary operations \(\lor\) and \(\land\) defined on \(X\), such that for all elements \(a, b \in X\).

\[ a \lor (a \land b) = a \land (a \lor b) = a. \]

The first definition is derived from the second by defining a partial order on \(X\) by

\[ a \leq b \iff a = a \land b \quad \text{or equivalently,} \quad a \leq b \iff b = a \lor b. \]

Definition 2.1.2. A lattice \((X, \lor, \land)\) is said to be \emph{distributive} if the following condition holds for any \(a, b, c \in X\):

\[ a \land (b \lor c) = (a \land b) \lor (a \land c). \]

A fundamental result in lattice theory states that this condition is equivalent to the condition

\[ a \lor (b \land c) = (a \lor b) \land (a \lor c). \]

Definition 2.1.3. A \emph{conditionally complete} lattice is a poset \(P\) in which every nonempty subset that has an upper bound in \(P\) has a least upper bound (i.e., a supremum) in \(P\) and every nonempty subset that has an lower bound in \(P\) has
an infimum in $P$. A lattice $P$ is said to be *complete* if all its subsets have both a join and a meet.

**Example 2.1.4.** The poset of the real numbers $\mathbb{R}$ is conditionally complete, and become complete when adjoining $-\infty, \infty$. Likewise the positive reals $\mathbb{R}^+$ is conditionally complete, and become complete when adjoining $0, \infty$. Note that for example the set $\{1/n : n \in \mathbb{N}\} \subset \mathbb{R}^+$ is not bounded below in $\mathbb{R}^+$ since $0 \not\in \mathbb{R}^+$.

**Definition 2.1.5.** The subset $A$ of the poset $P$ is said to be *completely closed* in $P$ if $A$ contains the least upper bound or greatest lower bound of any of its nonempty subsets, if either exists in $P$.

**Definition 2.1.6.** Given two lattices $(X, \lor_X, \land_X)$ and $(Y, \lor_Y, \land_Y)$, a *homomorphism of lattices* or *lattice homomorphism* is a function $f : X \rightarrow Y$ such that

$$f(a \lor_X b) = f(a) \lor_Y f(b), \text{ and } f(a \land_X b) = f(a) \land_Y f(b).$$

### 2.2 Semifields

**Basic setting and assumptions**

**Definition 2.2.1.** A *semiring* $S$ is a set $S$ equipped with two binary operations $+$ and $\cdot$, such that $(S, +)$ is a commutative monoid with identity element 0, $(S, \cdot)$ is a monoid with identity element 1, multiplication left and right distributes over addition and multiplication by 0 annihilates $S$. $S$ is called commutative when $(S, \cdot)$ is commutative.

$S$ is called a *domain* when $S$ is multiplicatively cancellative.

**Definition 2.2.2.** A *semifield* is a semiring $(\mathbb{H}, +, \cdot)$ in which all nonzero elements have a multiplicative inverse. A semifield is said to be *proper* if it is not a field.
Note 2.2.3. Though in general, a semifield is not assumed to be commutative, in the scope of our study we consider commutative semifields. Thus, we always assume a semifield to be commutative. Nevertheless, we introduce some definitions and results in the wider context in which a semifield is not necessarily commutative. For example, the notion of a semifield-kernel is defined as a normal subgroup implying the definition refers to the wider context.

From now on, unless stated otherwise we assume a semifield to be commutative and proper.

Lemma 2.2.4. If $\mathbb{H}$ is a proper semifield then $a + b \neq 0$ for all $a, b \in \mathbb{H} \setminus \{0\}$.

Proof. Let $\mathbb{H}$ be a semifield and let $a \in \mathbb{H}$ be any element of $\mathbb{H}$. If there exists $b \in \mathbb{H}$ such that $a + b = 0$, then $1 + a^{-1}b = 0$. Thus $-1 = a^{-1}b \in \mathbb{H}$ and thus for any $c \in \mathbb{H}$, we have $c + (-1)c = 0$ and $-c \in \mathbb{H}$, yielding that $\mathbb{H}$ is a field. \qed

Note 2.2.5. Throughout this dissertation, we assume a semifield to be a proper semifield. In view of Lemma 2.2.4, this implies that every element of a semifield is not invertible with respect to addition. This makes the zero element somewhat redundant. Thus we generally assume a proper semifield to have no zero element. Whenever we choose to adjoin such an element we will indicate it.

Example 2.2.6. The following are some well-known examples for semifields:

1. The positive real numbers with the usual addition and multiplication form a (commutative) semifield.

2. The rational functions of the form $f/g$, where $f$ and $g$ are polynomials in one variable with positive coefficients, comprise a (commutative) semifield.

3. The max-plus algebra, or the tropical semiring, $(\mathbb{R}, \text{max}, +)$ is a semifield. Here the sum of two elements is defined to be their maximum, and the product to be their ordinary sum.

4. If $(A, \leq)$ is a lattice ordered group then $(A, +)$ is an additively idempotent semifield. The semifield sum is defined to be the sup of two elements. Conversely, any additively idempotent semifield $(A, +)$ defines a lattice-ordered group $(A, \leq)$, where $a \leq b$ if and only if $a + b = b$.  

11
Remark 2.2.7. As stated in Example 2.2.6(4) (additively) idempotent semifields correspond to lattice ordered groups. In the stated correspondence, the addition operation $+$ of the semifield coincides with the so-called 'join' operation $\lor$ defined on the underlying lattice structure of the lattice ordered group. Due to this and in order to emphasize the underlying lattice structure of the semifield, when considering idempotent semifields we denote addition by $\dot{+}$.

Definition 2.2.8. An idempotent semiring is a semiring $S$ such that

$$\forall a \in S : a + a = a.$$  

An idempotent semifield is an idempotent semiring which is a semifield. Note that considering a semifield the condition of idempotency is equivalent to demanding that $1 + 1 = 1$, as $a = 1a = (1 + 1)a = a + a$.

Definition 2.2.9. We define a bipotent semiring $S$ to be a semiring with $1$ (multiplicative identity element) admitting bipotent addition, i.e., $\alpha + \beta \in \{\alpha, \beta\}$ for any $\alpha, \beta \in S$. When $S$ is a semifield, i.e., every nonzero element of $S$ is invertible with respect to multiplication, we say that $S$ is a bipotent semifield.

Remark 2.2.10. A bipotent semifield (semiring) is a special case of an idempotent semifield (semiring).

Motivated by tropical geometry, we have a special interest in the following particular semiring.

Definition 2.2.11. Let $(H, \cdot, 1)$ be a lattice ordered monoid. Define addition on $H$ to be the operation of supremum, denoted by $\dot{+}$, i.e., for any $a, b \in H$

$$a \dot{+} b = \text{sup}(a, b).$$  \hfill (2.1)

Adjoin a zero element $0$ to $H$ such that $\forall a \in H : a \dot{+} 0 = 0 \dot{+} a = a$. Then $\mathbb{H} = (H, \cdot, \dot{+}, 1, 0)$ is a semiring. If $\mathbb{H}$ is totally ordered then supremum is maximum and the semiring is bipotent. Taking $\mathbb{H}$ to be a multiplicative group, $\mathbb{H}$ becomes an idempotent semifield. In such a case, for $a, b \in \mathbb{H}$, $(a \dot{+} b)^{-1} = \text{inf}(a^{-1}, b^{-1})$. Thus $\dot{+}$ induces a infimum operation, to be denoted by $\land$, such that

$$a \land b = (a^{-1} \dot{+} b^{-1})^{-1} = \text{inf}(a, b).$$

In particular, for any $a \in \mathbb{H}$, $(a \dot{+} a^{-1})^{-1} = a \land a^{-1}$.

If $\mathbb{H}$ is totally ordered infimum is minimum.
Remark 2.2.12. The semifield described in Definition 2.2.11 is idempotent (bipotent) and thus can be viewed as a commutative lattice ordered-group (totally ordered) \((\mathbb{H}, +, \cdot)\) where for \(a, b \in \mathbb{H}\), \(a \leq b \iff a + b = b\).

The following lemma and the subsequent remarks establish the connection between a semifield addition and a natural order it induces.

**Lemma 2.2.13.** For every proper semifield \(\mathbb{H}\), the following quasi-identity holds:

\[
a + b + c = a \Rightarrow a + b = a
\]

(2.2)

for \(a, b, c \in \mathbb{H}\).

**Proof.** Let \(a + b + c = a\). Multiplying both sides of the equation by \(a^{-1}ba^{-1}\) and then adding \(ca^{-1}\) to both parts yield that \((ba^{-1} + ca^{-1})(1 + ba^{-1}) = ba^{-1} + ca^{-1}\). Hence, as multiplicative cancellation holds, we get \(1 + ba^{-1} = 1\) and therefore \(a + b = a\). \(\square\)

**Remark 2.2.14.** Every (commutative with respect to addition) semifield \(\mathbb{H}\) is endowed with a partial order defined by

\[
a \leq b \iff a = b \text{ or } a + c = b \text{ for some } c \in \mathbb{H}
\]

(2.3)

and is ordered with respect to a natural order, i.e., \(a \leq b\) implies that \(a + c \leq b + c\), \(ac \leq bc\) and \(ca \leq cb\) for all \(a, b, c \in \mathbb{H}\). In the special case of idempotent semifields, the relation (2.3) can be rephrased as

\[
a \leq b \iff a + b = b.
\]

(2.4)

**Note 2.2.15.** In the literature (for example in [3]), semifields are sometimes not assumed to be commutative with respect to addition and thus do not always have a natural order. Since the semifields we consider are additively commutative (abelian), a semifield in our scope is always partially ordered with respect to the natural order.

**Note 2.2.16.** For two elements \(a, b\) of a semifield \(\mathbb{H}\), we say that \(a\) and \(b\) are comparable (or \(a\) is comparable to \(b\)) if \(a \leq b\) or \(b \leq a\).

A delicate point arises when considering functions over some semifield. For example, consider the semifield of fractions in one variable \(\mathbb{H}(x)\) with \(\mathbb{H}\) a semifield. Although \(x + 1 \geq x\) and \(x \neq x + 1\), one has \(x + 1 \neq x\) (for example take \(x = 1\)).
Definition 2.2.17. A semiring $S$ is said to satisfy the Frobenius property if for every $m \in \mathbb{N}$ and for every $a, b \in S$:

$$(a + b)^m = a^m + b^m.$$ 

Lemma 2.2.18. Every bipotent semifield satisfies the Frobenius property.

Proof. Since a bipotent semifield is totally ordered, for any pair of elements $a, b$ of the semifield one has that $a \leq b$ or $b \leq a$. Assume that $a \leq b$. Then $a^{j_1}b^{k_1} \leq a^{j_2}b^{k_2}$ for $j_1 + k_1 = j_2 + k_2 = m$ and $j_1 \geq j_2$ which in turn implies that $(a + b)^m = b^m$ and so, as $a^m \leq b^m$ we can write $(a + b)^m = a^m + b^m$. Analogously $b \leq a$ implies that $(a + b)^m = b^m = a^m + b^m$. 

Remark 2.2.19. The converse implication does not hold, namely, a semifield satisfying the Frobenius property is not necessarily bipotent. See Example 2.2.37.

Definition 2.2.20. Let $S_1, S_2$ be semirings. A map $\phi : S_1 \to S_2$ is a semiring homomorphism if for any $a, b \in S_1$ the following conditions hold:

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b) \text{ and } \phi(a + b) = \phi(a) + \phi(b).$$ 

Remark 2.2.21. A semiring homomorphism is order preserving, in the sense that for a semiring homomorphism $\phi : S_1 \to S_2$, if $a, b \in S_1$ such that $a \leq b$, then $\phi(a) \leq \phi(b)$.

Indeed, $a \geq b$ yields that there exists $c \in S_1$ such that $a = b + c$ thus $\phi(a) \geq \phi(b)$ since $\phi(a) = \phi(b + c) = \phi(b) + \phi(c)$.

Proposition 2.2.22. Let $S_1, \ldots, S_t, t \in \mathbb{N}$, be proper semifields. Then their direct product $S = S_1 \times \cdots \times S_t$, defined as the set

$$\{(s_1, \ldots, s_t) : s_i \in S_i, \ 1 \leq i \leq t\}$$

with component-wise addition and multiplication, is also a proper semifield.

Definition 2.2.23. A semiring $H$ is divisible (also called radicalizable) if for any $n \in \mathbb{N}$ and $\alpha \in H$, there exists some $\beta \in H$ such that $\beta^n = \alpha$.

Remark 2.2.24. A homomorphic image of a divisible semifield is divisible.

Proof. Let $H$ be a semifield and let $\phi : H \to \text{Im}(\phi)$ be a semifield homomorphism. Since a homomorphic image of a semifield is a semifield, we only need to show
that $\text{Im}(\phi)$ is divisible. Let $a = \phi(\alpha) \in \text{Im}(\phi)$. Since $H$ is divisible for any $n \in \mathbb{N}$ there exists some $\beta \in S$ such that $\beta^n = \alpha$. Taking $b = \phi(\beta) \in \text{Im}(\phi), b^n = \phi(\beta)^n = \phi(\beta^n) = \phi(\alpha) = a$, thus $\text{Im}(\phi)$ is divisible.

**Definition 2.2.25.** A po-group (partially ordered group) $(G, \cdot)$ is called archimedean if $a^\mathbb{Z} \leq b$ implies that $a = 1$. A semifield $(H, +, \cdot)$ is said to be archimedean if $H \setminus \{0\}$ archimedean as a po-group.

**Note 2.2.26.** The archimedean property is widely used in the context of totally ordered groups, where every pair of elements are comparable. Those who are used to working in the total order setting, may find the implications of this property in the wider context of partial order groups somewhat confusing.

**Definition 2.2.27.** Throughout this chapter we denote by $R$ the bipotent semifield defined in Definition 2.2.11 with the supplementary properties of being divisible, archimedean (as a po-group) and complete in the sense that the underlying lattice (with $\vee (+)$ and $\wedge$ as its operations) is conditionally complete.

**Definition 2.2.28.** A semimodule $M$ over a semifield $H$ is a semigroup $(M, +)$ endowed with scalar multiplication such that for every $\alpha \in H$ and $a \in M$, $\alpha \cdot a \in M$. A semialgebra $A$ over a semifield $H$ is a semimodule $(A, +)$ endowed with multiplication, such that $(A, \cdot)$ is a semigroup and distributivity of multiplication over addition holds.

**Remark 2.2.29.** If $M$ is a semimodule over an idempotent semifield $H$, then $M$ lacks inverses with respect to addition. Indeed, the idempotency of $H$ implies that $M$ is idempotent with respect to addition. Thus if for some $u \in M$ there exists $v \in M$ such that $u + v = 0$ we have that $u = u + 0 = u + (u + v) = (u + u) + v = u(1 + 1) + v = u + v = 0$, proving our claim.

In view of Remark 2.2.29, any semialgebra (in particular a semifield) over a bipotent semifield is inverse free with respect to addition.

**Definition 2.2.30.** A semiring $D$ is said to be an extension of a semifield $H$ if $D \supseteq H$ and $H$ is a subsemiring of $D$.

**Definition 2.2.31.** Let $H$ be a semifield, and let $D$ be a semiring extending $H$. We say that $D$ is generated by a subset $A \subset D$ over $H$ if every element $a \in D$ is of the form $\sum_{i=1}^{n} a_i \prod_{j=1}^{m} a_{i,j}^{k_{i,j}}$ with $a_{i,j} \in A$ and $k_{i,j} \in \mathbb{N}$. $D$ is said to be affine over $H$, or an affine extension of $H$, if $A$ is finite.
Note 2.2.32. Later in this section we introduce the notion of a ‘semifield with a generator’. In this definition, we define generators of a semifield to be elements generating it as a kernel. In order to avoid ambiguity, when we refer to generation as in Definition 2.2.31, we will indicate it explicitly. In any other case we consider generation as a kernel.

Definition 2.2.33. For any set $X$ and any semiring $S$, $\text{Fun}(X,S)$ denotes the set of functions $f : X \to S$. $\text{Fun}(X,S)$ also is a semiring, whose operations are given pointwise:

$$(fg)(a) = f(a)g(a), \quad (f + g)(a) = f(a) + g(a)$$

for all $a \in X$. The unit element of $\text{Fun}(X,S)$ is the constant function always taking on the value $1_S$.

Remark 2.2.34. If $S$ is a semifield without a zero element then $\text{Fun}(X,S)$ is a semifield (without a zero element). Indeed, for any $f \in \text{Fun}(X,S)$ we have that $f^{-1}(x) = f(x)^{-1}$ is the inverse function of $f$.

Recall that we always assume a semifield does not contain a zero element, unless stated otherwise.

Remark 2.2.35. For a semiring $S$, there are two distinct semiring structures arising on $S[x_1, \ldots, x_n]$. The first is obtained by considering $S[x_1, \ldots, x_n]$ as a subset of $\text{Fun}(\mathbb{D}^n, \mathbb{D})$ with $\mathbb{D}$ taken to be any extension of $S$, i.e., the elements of $S[x_1, \ldots, x_n]$ are considered as functions defined over $\mathbb{D}^n$. The second way is to consider the variables $x_1, \ldots, x_n$ as symbols rather than functions and taking the formal addition and multiplication operations on $S[x_1, \ldots, x_n]$.

Note 2.2.36. We always consider the polynomial semifield $\mathbb{H}[x_1, \ldots, x_n]$ (and its semifield of fractions) mapped to the semiring of functions.

Example 2.2.37. For a nontrivial bipotent semifield $\mathbb{H}$, the semiring $\mathbb{H}[x_1, \ldots, x_n]$ (considered as a subsemiring of $\text{Fun}(\mathbb{H}^n, \mathbb{H})$) is idempotent but not bipotent (for example the constant function $\alpha$ for any $\alpha \in \mathbb{H}$ and the function $x$ are incomparable). Since $\mathbb{H}$ is bipotent, for any pair of polynomials $f, g \in \mathbb{H}[x_1, \ldots, x_n]$ we have that $(f(a) + g(a))^m = f(a)^m + g(a)^m$ at any given point $a \in \mathbb{H}^n$. Thus $(f + g)^m = f^m + g^m$ globally over $\mathbb{H}^n$, i.e., as elements of $\mathbb{H}[x_1, \ldots, x_n]$. So $\mathbb{H}[x_1, \ldots, x_n]$ satisfies the Frobenius property though it is not bipotent. Note that the arguments introduced above apply more generally to the semiring of functions yielding that the Frobenius property holds there too.
The structure of a semifield

Like a normal subgroup in group theory and an ideal in commutative ring theory, the kernel encapsulates relations on semifields. In the following few paragraphs, we introduce the notion of a kernel along with some of its properties.

**Definition 2.2.38.** A subset $K$ of a semifield $\mathbb{H}$ is a *semifield-kernel* of $\mathbb{H}$ if $K$ is a normal subgroup in $\mathbb{H}$ with the convexity property that for every $x, y \in \mathbb{H}$ such that $x + y = 1$,

$$ a, b \in K \implies xa + by \in K. $$

(2.5)

The set of all the kernels of a semifield $\mathbb{H}$ is denoted by $\text{Con}(\mathbb{H})$.

**Note 2.2.39.** We note that the name ‘kernel’ and the notation $\text{Con}$ are customary in previous study of semifields. From now on we refer to a semifield-kernel simply as a ‘kernel’. In places where confusion arises with the notion of a kernel of a homomorphism, we provide clarification.

**Note 2.2.40.**

- Some may find the name ‘kernel’ not necessarily the best choice for a name for the above structure. We presume the motivation for the name is that a ‘kernel’ is a kernel of a semifield homomorphism. It obviously might cause a little confusion. When such confusion may occur we explicitly indicate to which notion of kernel we refer.

- Though the so-called ‘convexity’ condition (2.5) may give a somewhat misleading impression, kernels are nothing but a special kind of groups inside the semifield $\mathbb{H}$.

**Remark 2.2.41.** [9, Proposition 1.1] An equivalent definition of a kernel of a semifield $\mathbb{H}$ is the class $[1]_\rho$ of an arbitrary congruence $\rho$ on $\mathbb{H}$.

**Remark 2.2.42.** If $\mathbb{S}$ is an idempotent semifield since $1 + 1 = 1$, we get that for any kernel $K$ of $\mathbb{S}$, $a, b \in K \implies a + b = 1a + 1b \in K$, yielding that $K$ is itself a semifield. A particular case of interest is the semifield of fractions $\mathbb{H}(x_1, ..., x_n)$ of $\mathbb{H}[x_1, ..., x_n]$. If $\mathbb{H}$ is idempotent, then $\mathbb{H}(x_1, ..., x_n)$ is idempotent which yields that the kernels of $\mathbb{H}(x_1, ..., x_n)$ are subsemifields of $\mathbb{H}(x_1, ..., x_n)$.
Note 2.2.43. Throughout this dissertation, we work with an underlying semifield \( \mathbb{H} \) of \( \mathbb{H}(x_1, ..., x_n) \) which is both idempotent and archimedean. In particular, under the assumption of idempotency, all kernels are also semifields.

**Theorem 2.2.44.** [3, Theorem 3.6] The set \( \text{Con}(\mathbb{H}) \) of all kernels of a semifield \( \mathbb{H} \) forms a full modular lattice with respect to the operations of multiplication and the intersection of kernels, canonically isomorphic to the lattice of all possible congruencies on \( \mathbb{H} \).

**Remark 2.2.45.** [3] Let \( K_1 \) and \( K_2 \) be kernels of the semifield \( \mathbb{H} \). Then \( K_1 \cap K_2 \) and \( K_1 \cdot K_2 \) are kernels of \( \mathbb{H} \). Moreover \( K_1 \cdot K_2 \) is the smallest kernel in \( \mathbb{H} \) containing \( K_1 \cup K_2 \).

**Note 2.2.46.** Note that multiplication of kernels is formulated by

\[
K_1 \cdot K_2 = \{ab : a \in K_1, b \in K_2\}
\]
as customary in the theory of groups.

**Lemma 2.2.47.** [9, Lemma 4.1] The following equalities hold for arbitrary kernels \( A, B \) and \( K \) of a semifield \( \mathbb{H} \), among which at least one is a semifield:

\[
AK \cap BK = (A \cap B)K; \quad (2.6)
\]

\[
(A \cap K)(B \cap K) = AB \cap K. \quad (2.7)
\]

**Corollary 2.2.48.** Since every kernel of an idempotent semifield is also a semifield, we have by Lemma 2.2.47 that its lattice of kernels is distributive. Thus every idempotent semifield is distributive.

The following are the three fundamental isomorphism theorems.

**Theorem 2.2.49.** [3, Theorems 3.4 and 3.5] Let \( \mathbb{H}_1, \mathbb{H}_2 \) be semifields and let \( R \subset \mathbb{H}_1 \) be a subsemifield of \( \mathbb{H}_1 \). Let \( \phi : \mathbb{H}_1 \to \mathbb{H}_2 \) be a semiring homomorphism and let \( K \) be the homomorphism kernel of \( \phi \). Then the following hold:

1. \( \phi(R) \subset \mathbb{H}_2 \) is a subsemifield of \( \mathbb{H}_2 \). The homomorphism kernel of the restriction \( \phi : R \to \phi(R) \) is \( R \cap K \).

2. \( \phi^{-1}(\phi(R)) = KR \) which is a subsemifield of \( \mathbb{H}_1 \).

3. For any kernel \( L \) of \( \mathbb{H}_1 \), \( \phi(L) \) is a kernel of \( \phi(S_1) \).
4. For a kernel $K$ of $\phi(H_1)$, $\phi^{-1}(K)$ is a kernel of $H_1$. In particular, for any kernel $L$ of $H_1$ we have that $\phi^{-1}(\phi(L)) = KL$ is a kernel of $H_1$.

**Corollary 2.2.50.** As a special case of (4), taking $K = \{1\}$, we have that the homomorphism kernel $\phi^{-1}(1)$ of a semifield homomorphism $\phi : H_1 \to H_2$ is a kernel.

**Theorem 2.2.51.** [IT] Let $H$ be a semifield and $K$ a kernel of $H$.

1. If $U$ is a subsemifield of $H$, then $U \cap K$ is a kernel of $U$ and $K$ a kernel of the subsemifield $U \cdot K = \{u \cdot k : u \in U, k \in K\}$ of $H$ and one has the isomorphism

$$U/(U \cap K) \cong U \cdot K/K.$$

2. If $L$ is a kernel of $H$, then $L \cap K$ is a kernel of $L$ and $K$ a kernel of $L \cdot K$.

Now, one has in general only the group isomorphism

$$L/(L \cap K) \cong L \cdot K/K$$

which is a semifield isomorphism exactly in the case when $L$ is also a subsemifield of $H$.

**Theorem 2.2.52.** [IT] Let $H$ be a semifield and let $K$ and $L$ be kernels of $H$ satisfying $K \subseteq L$. Then $L/K$ is a kernel of $H/K$ and one has the semifield isomorphism

$$H/L \cong (H/K)/(L/K).$$

The following result concerns the induced order of the quotient semifield. It holds for any idempotent semifield $S$, and in particular for $H(x_1, ..., x_n)$ with $H$ an idempotent semifield.

**Theorem 2.2.53.** Let $H$ be a semifield and let $L \in \text{Con}(H)$ be a kernel of $H$. Every kernel of $H/L$ has the form $K/L$ for some kernel $K \in \text{Con}(H)$ uniquely determined such that $K \supseteq L$, and there is a 1:1 correspondence

$$\{\text{Kernels of } H/L\} \to \{\text{Kernels of } H \text{ containing } L\}$$

given by $K/L \mapsto K$.  

19
Proof. From the theory of groups we have that there is such a bijection for normal subgroups. To apply the theorem for kernels, we only need to show that a homomorphic image and preimage of a kernel are kernels, which in turn is true by Theorem 2.2.49.

Remark 2.2.54. Let $K$ be a kernel of an idempotent semifield $S$. Then the induced order on the quotient semifield $S/K$ is such that $aK \leq bK \iff (1 + ab^{-1}) \in K$.

Proof. Let $a, b$ be elements of $H$. The induced order on $H/K$ is given by: $aK \leq bK$ if and only if there exists some $c \in K$ such that $a \leq cb$. Now, the following identity holds in $H$: $a = (a \land b)(ab^{-1} + 1)$ (distributing the right hand side and the left hand side one at a time give opposite weak inequalities). Consequently, if $(ab^{-1} + 1) \in K$ then $a = (a \land b)(ab^{-1} + 1)$, implying that $aK = (a \land b)K \leq bK$. Conversely, let $c \in K$ such that $a \leq cb$. Then we have $(a \land b) \leq a \leq cb \Rightarrow (a \land b)b^{-1} \leq ab^{-1} \leq c \Rightarrow ab^{-1} + 1 \leq ab^{-1} \leq c \Rightarrow 1 + (ab^{-1} + 1) \leq (ab^{-1} + 1) \leq 1 + c \Rightarrow 1 \leq (ab^{-1} + 1) \leq 1 + c$. Since $1, 1 + c \in K$ we have that $(ab^{-1} + 1) \in K$.

We conclude this part with the definition of a large kernel.

Definition 2.2.55. Let $S$ be a semifield. A kernel $K$ of a semifield $S$ is said to be large in $S$ if $L \cap K \neq \{1\}$ for each kernel $L \neq \{1\}$ of $S$. 
2.3 Semifield with a generator and generation of kernels

**Definition 2.3.1.** Let \( A \) be a subset of a semifield \( H \). Denote by \( \langle A \rangle \) the smallest kernel in \( H \) containing \( A \). It is equal to the intersection of all kernels in \( H \) containing \( A \). If \( H = \langle A \rangle \), then \( A \) is called a set of generators of the semifield \( H \) (as a kernel).

A kernel \( K \) is said to be finitely generated if \( K = \langle A \rangle \) where \( A \) is a finite set of elements of \( H \). By Remark 2.2.45 if \( K \) is generated by \( \{a_1, ..., a_t\} \subset H \) then \( K = \langle a_1 \rangle \cdots \langle a_t \rangle \) (the smallest kernel containing \( \{a_1, ..., a_t\} \)). In such a case, we write \( K = \langle a_1, ..., a_t \rangle \) to indicate that \( K \) is generated by \( \{a_1, ..., a_t\} \). If \( K = \langle a \rangle \) for some \( a \in H \), then \( K \) is called a principal kernel. A semifield is said to be finitely generated if it is finitely generated as a kernel. If \( H = \langle a \rangle \) for some \( a \in H \), then \( a \) is called a generator of \( H \) and \( H \) is said to be a semifield with a generator. In other words, a semifield with a generator is a semifield which is principal as a kernel of itself.

**Lemma 2.3.2.** [9, Property 2.3] Let \( K \) be a kernel of a semifield \( H \). Then for \( a,b \in H \),
\[
a + a^{-1} \in K \quad \text{or} \quad a + a^{-1} + b \in K \quad \Rightarrow \quad a \in K.
\] (2.8)

**Proof.** Let \( a + a^{-1} \in K \). Then in the factor semifield \( H/K \), we get \( w + w^{-1} = 1 \) for each element \( w = aK \) which yields that \( w \leq 1, w^{-1} \leq 1 \), and thus \( w = 1 \), i.e., \( aK = K \) and so \( a \in K \). For the second condition just apply Lemma 2.2.13. \( \square \)

**Proposition 2.3.3.** [9] Let \( H \) be a semifield. Then for any \( a \in H \) such that the kernel generated by \( a \) is a semifield, the following equality holds:
\[
\langle a \rangle = \langle a + a^{-1} \rangle.
\]

In words, the kernel generated by \( a \) coincides with the kernel generated by \( a + a^{-1} \).

**Proof.** A direct consequence of Lemma 2.3.2 which implies that \( a \in \langle a + a^{-1} \rangle \), and thus \( \langle a \rangle \subseteq \langle a + a^{-1} \rangle \). The converse inclusion follows from the fact that \( \langle a \rangle \) is a semifield. \( \square \)
Remark 2.3.4. [9, Property 2.4] Every kernel $K$ of a semifield $\mathbb{H}$ is convex with respect to the natural order on $\mathbb{H}$: for $a, c \in K$ and $b \in \mathbb{H}$

\begin{equation}
    a \leq b \leq c \Rightarrow b \in K.
\end{equation}

Proof. $a \leq b \leq c$ implies that $a + u = b$ and $b + v = c$ for $u, v \in \mathbb{H}$, i.e., $a + u + v = c$. The equality $K + uK + vK = K$ holds in the factor semifield $\mathbb{H}/K = \{yK : y \in \mathbb{H}\}$. Then $bK = (a + u)K = aK + uK = K$ by Lemma 2.2.13. Thus $b \in K$. □

Proposition 2.3.5. Let $N$ be a convex normal subgroup of a semifield $\mathbb{H}$. If each $a \in N$ is comparable with 1, i.e., if $\leq$ is a total order on $N$, then $N$ is a kernel of $\mathbb{H}$.

Proof. In order to prove $N$ is a kernel, we need to show that for $a \in N$ and $s, t \in \mathbb{H}$ such that $s + t = 1$, $s + ta \in N$. By assumption, we have that $1 \leq a$ or $a \leq 1$. In the former case, we have $t \leq ta$ and $s \leq sa$, thus $1 = s + t \leq s + ta \leq sa + ta = (s + t)a = a$. Since $N$ is convex and $a, 1 \in N$ we get that $s + ta \in N$. The latter case yields that $a \leq s + ta \leq 1$ which implies the same. □

The following remark yields an important property of a kernel, which we call ‘power-radicality’, to be introduced shortly.

Remark 2.3.6. The multiplicative group of every proper semifield $\mathbb{H}$ is a torsion-free group, i.e., all of its elements that are not equal to 1 have infinite order.

Proof. If $a^n = 1$ for $a \in \mathbb{H}$ and $n \in \mathbb{N}$, then

\begin{align*}
    a(a^{n-1} + a^{n-2} + \cdots + a + 1) &= a^n + (a^{n-1} + \cdots + a) = 1 + (a^{n-1} + \cdots + a) \\
    &= a^{n-1} + \cdots + a + 1
\end{align*}

which yields that $a = 1$. □

The following remark is a straightforward consequence of Remark 2.3.6.

Remark 2.3.7. Let $K$ be a kernel of a proper semifield $\mathbb{H}$. For every $a \in \mathbb{H}$, if $a^n \in K$ for some $n \in \mathbb{N}$ then $a \in K$. We refer to this property of kernels by saying that a kernel is power-radical.
Proof. Indeed, if there exists an element \( a \in \mathbb{H} \) not admitting the stated property, then its image \( \phi(a) \) in \( \mathbb{H}/K \) under the quotient homomorphism is torsion, which by Remark 2.3.6 is not possible since \( \mathbb{H}/K \) is a semifield.

The following subsequent statements establish a connection between the (normal) group generated by a set of elements and the kernel generated by the set. Recall that, kernels are a specific kind of group, the (normal) subgroup generated by a set of elements need not be a kernel.

**Proposition 2.3.8.** [3, Proposition (3.13)] Let \( \mathbb{H} \) be a semifield and let \( N \) be a normal subgroup of \((\mathbb{H}, \cdot)\). Then the smallest kernel containing \( N \) is given by

\[
K(N) = \left\{ \sum_{i=1}^{n} s_i h_i : n \in \mathbb{N}, \ h_i \in N, \ s_i \in \mathbb{H} \text{ such that } \sum_{i=1}^{n} s_i = 1 \right\}. \tag{2.10}
\]

**Remark 2.3.9.** Let \( S \subset \mathbb{H} \). The kernel generated by \( S \) is \( \langle S \rangle = \mathcal{K}(\mathcal{G}(S)) \) where \( \mathcal{G}(S) \) is the (multiplicative) group generated by \( S \).

**Proof.** As a kernel is defined to be a multiplicative (normal) group, the assertion is immediate.

**Remark 2.3.10.** Let \( S_1, ..., S_r \subset \mathbb{H} \) and let \( G_1, ..., G_r \) be the groups generated by \( S_1, ..., S_r \) respectively. Then \( \langle \bigcup_{i=1}^{r} S_i \rangle = \mathcal{K}(\prod_{i=1}^{r} G_i) = \prod_{i=1}^{r} \mathcal{K}(G_i) = \prod_{i=1}^{r} \langle S_i \rangle \)

**Proof.** By definition, \( \prod_{i=1}^{r} \mathcal{K}(G_i) \) is a kernel, and thus a group, which contains all the groups \( G_i \) for \( i = 1, ..., r \), thus also contains the group \( \prod_{i=1}^{r} G_i \). Since \( \mathcal{K}(\prod_{i=1}^{r} G_i) \) is the smallest kernel containing the group \( \prod_{i=1}^{r} G_i \), we get that \( \mathcal{K}(\prod_{i=1}^{r} G_i) \subseteq \prod_{i=1}^{r} \mathcal{K}(G_i) \). Now, since \( G_i \subseteq \mathcal{K}(\prod_{i=1}^{r} G_i) \) for every \( i = 1, ..., r \), we have that \( \bigcup_{i=1}^{r} G_i \subseteq \mathcal{K}(\prod_{i=1}^{r} G_i) \). As \( \prod_{i=1}^{r} \mathcal{K}(G_i) \) is the smallest kernel containing \( \bigcup_{i=1}^{r} G_i \) (see Remark 2.2.45), we get the converse inclusion and thus equality. All other equalities are group theoretic basic equalities.

**Proposition 2.3.11.** [9, Proposition (3.1)] Let \( K = \langle a \rangle \), a principal kernel in a semifield \( \mathbb{H} \) with \( a \in \mathbb{H} \) such that \( a \geq 1 \). Then

\[
K = \{ x \in \mathbb{H} : \exists n \in \mathbb{N} \text{ such that } a^{-n} \leq x \leq a^n \}. \tag{2.11}
\]
Corollary 2.3.12. Every nontrivial semifield \( H \) with a generator has a generator \( a \geq 1 \) such that \( a \neq 1 \).

Proof. Let \( u \in H \setminus \{1\} \) be a generator of \( H \). By Lemma 2.3.2, the element \( u + u^{-1} \) is also a generator of \( H \) which yields that the element \( a = (u + u^{-1})^2 = u^2 + u^{-2} + 1 \geq 1 \) is a generator of \( H \) too by Proposition 2.3.11.

Remark 2.3.13. Every semifield \( H \) such that \( H \neq \{1\} \) has an element \( a \in H \) such that \( a > 1 \).

Proof. Indeed, \( H \neq \{1\} \), so there exists \( a \in H \setminus \{1\} \). Now, if \( a \) is not comparable with 1 then take \( 1 + a \). Note that \( 1 + a \neq 1 \) since otherwise, it would imply that \( a \leq 1 \), contradicting our assumption that \( a \) and 1 are not comparable. Thus \( 1 + a > 1 \). On the other hand, if \( a \) is comparable with 1 and if \( a < 1 \), take \( 1 < a^{-1} \in H \).

Corollary 2.3.14. If \( H \) is a semifield, then for any element \( a \in H \) we have that

\[
\langle a \rangle = \{ x \in H : \exists n \in \mathbb{N} \text{ such that } (a + a^{-1})^{-n} \leq x \leq (a + a^{-1})^n \}. \tag{2.12}
\]

Proof. This is a direct consequence of Proposition 2.3.11 and Proposition 2.3.3.

Note 2.3.15. Note that for an idempotent semifield the equality introduced in Corollary 2.3.14 can be restated as

\[
\langle a \rangle = \{ x \in H : \exists n \in \mathbb{N} \text{ such that } (a \land a^{-1})^n \leq x \leq (a \lor a^{-1})^n \} \tag{2.13}
\]

using the underlying lattice operation \( \land \).

By Corollary 2.3.14 we have

Remark 2.3.16. For any element \( a \in H \) we have that

\[
\langle a \rangle = \{ x \in H : \exists n \in \mathbb{N} \text{ such that } (x + x^{-1}) \leq (a + a^{-1})^n \}. \tag{2.14}
\]

Proof. Just take inverses in equation \( (a + a^{-1})^{-n} \leq x \leq (a + a^{-1})^n \) and sum up both sides of resulting weak inequalities.

Note 2.3.17. Equality (2.14) of Remark 2.3.16 can be written as

\[
\langle a \rangle = \{ x \in H : \exists n \in \mathbb{N} \text{ such that } |x| \leq |a|^n \}. \tag{2.15}
\]
Remark 2.3.18. Property (3.2)] The homomorphic image of a generator is a generator of the image. In particular, a homomorphic image of a semifield with a generator is a semifield with a generator.

Proof. If $\phi : H \rightarrow U$ is a homomorphism of a semifield $H$ onto a semifield $U$ and $H = \langle a \rangle$, then $U = \langle \phi(a) \rangle$, because the preimage of a kernel at a homomorphism of semifields is always a kernel.

Remark 2.3.19. Let $\langle a \rangle$ be a principal kernel of a semifield $H$, which is also a semifield and let $\phi : H \rightarrow U$ be a semifield epimorphism. Then

$$\phi(\langle a \rangle) = \langle \phi(a) \rangle = \{ b \in U : \exists n \in \mathbb{N} \text{ such that } |b| \leq |\phi(a)|^n \},$$

i.e., the image of $\langle a \rangle$ is the kernel generated by $\phi(a)$ in $U$.

Proof. As $\langle a \rangle$ is also a semifield, it is a semifield with a generator $a$. Thus by Remark 2.3.18 its homomorphic image is also a semifield with a generator $\phi(a)$ and by Theorem 2.2.19 it is a kernel. Thus, its homomorphic image is a principal kernel $\langle \phi(a) \rangle$ which is also a semifield.

We can apply Remark 2.3.19 and get

Corollary 2.3.20. Let $\phi : R(x_1, \ldots, x_n) \rightarrow U$ be a semifield epimorphism. Then for every principal kernel $\langle f \rangle$ of $R(x_1, \ldots, x_n)$, one has that

$$\phi(\langle f \rangle) = \langle \phi(f) \rangle_U = \{ g \in \phi(R(x_1, \ldots, x_n)) : \exists n \in \mathbb{N} \text{ such that } |g| \leq |\phi(f)|^n \},$$

(2.16)

Note 2.3.21. Note that if $\phi$ is not onto $U$, then the kernel generated by $\phi(f)$ in $U$, $\langle \phi(f) \rangle_U$, may contain elements that are not in the image of $\phi$. In general one has that $\langle \phi(f) \rangle_{\text{Im}(\phi)} \subseteq \langle \phi(f) \rangle_U$.

Theorem 2.3.22. If a semifield $H$ has a finite number of generators, then $H$ is a semifield with a generator.

Proof. Let $H = \langle u_1 \rangle \cdot \cdots \cdot \langle u_n \rangle$ with the finite set of generators $\{u_1, \ldots, u_n\}$. By Remark 2.3.2 $u_1, ..., u_n$ are contained in the kernel $K = \langle u \rangle \subseteq H$ where $u = u_1 + u_1^{-1} + \cdots + u_n + u_n^{-1}$, thus $H = \langle u \rangle$ as desired.
Remark 2.3.23. Let $\mathbb{H}$ be an idempotent semifield. Let $K = \mathbb{H}(a_1, ..., a_n)$ be an affine semifield extension of the semifield $\mathcal{R}$. Then $K \cong \mathbb{H}(x_1, ..., x_n)/K$ for some $K \in \text{Con}(\mathbb{H}(x_1, ..., x_n))$.

Proof. Let $\phi : \mathbb{H}(x_1, ..., x_n) \to K$ be the substitution map sending $x_i \mapsto a_i$. Then $\phi$ is an epimorphism. Taking $K = \ker \phi$ we have by the first isomorphism theorem that $K \cong \mathbb{H}(x_1, ..., x_n)/K$. \qed

Corollary 2.3.24. Every affine semifield over an idempotent semifield $\mathbb{H}$ is a semifield with a generator.

Proof. By Remark 2.3.23 an affine semifield is a homomorphic image of the semifield of fractions, which is a semifield with a generator, thus by Remark 2.3.18 is also a semifield with a generator. \qed

Remark 2.3.25. Let $\mathbb{H}$ be an archimedean semifield. Then

$$\langle \alpha \rangle = \langle \beta \rangle \in \text{PCon}(\mathbb{H}(x_1, ..., x_n))$$

for any $\alpha, \beta \in \mathbb{H} \setminus \{1\}$.

Proof. Indeed, since $\mathbb{H}$ is archimedean, Corollary 2.3.14 implies that $\alpha \in \langle \beta \rangle$ and $\beta \in \langle \alpha \rangle$ so $\langle \alpha \rangle = \langle \beta \rangle$. \qed

Notation 2.3.26. As it does not depend on the choice of constant generator $\alpha \in \mathbb{H} \setminus \{1\}$, we denote the kernel generated by $\alpha$ by $\langle \mathbb{H} \rangle$.

Note that if $\mathbb{H}$ is an idempotent semifield then the semifield $\mathbb{H}(x_1, ..., x_n)$ is also idempotent, so $\langle \mathbb{H} \rangle \in \text{PCon}(\mathbb{H}(x_1, ..., x_n))$ is a subsemifield of $\mathbb{H}(x_1, ..., x_n)$. Also note that the elements of $\langle \mathbb{H} \rangle$ are rational functions which are not necessarily constant. We discuss the structure of $\langle \mathbb{H} \rangle$ thoroughly in the subsequent sections.

Note 2.3.27. Henceforth we always assume affine extensions, in particular the semifield of fractions, to be defined over an idempotent semifield, which make the extensions idempotent.
2.4 Simple semifields

**Definition 2.4.1.** A kernel $K$ of a semifield $\mathbb{H}$ which contain no kernels but the trivial ones, $\{1\}$ and $K$ itself, is called *simple*. A semifield is *simple* if it is simple as a kernel of itself.

*Remark 2.4.2.* Every totally (linearly) ordered archimedean semifield (i.e., bipo-tent semifield) $\mathbb{H}$ has no kernels but the trivial ones, i.e., is simple.

*Proof.* We may assume $\mathbb{H} \neq \{1\}$. Let $a \in \mathbb{H}$ such that $a > 1$ (there exists such $a$ by Remark 2.3.13). Now, since $\mathbb{H}$ is a linearly (totally) ordered semifield, for every $b \in \mathbb{H}$ there exists $m \in \mathbb{N}$ such that $a^{-m} \leq b \leq a^m$. Then by Proposition 2.3.11 we have that $b \in \langle a \rangle$. Thus $\langle a \rangle = \mathbb{H}$ and our claim is proved. \hfill $\square$

*Remark 2.4.3.* Any simple semifield is a semifield with a generator.

*Proof.* Indeed, if $\mathbb{H}$ is trivial then the assertion is obvious. Assume $\mathbb{H} \neq \{1\}$, then there exist some $\alpha \in \mathbb{H} \setminus \{1\}$ and so $\langle 1 \rangle \subset \langle \alpha \rangle \subseteq \mathbb{H}$. Since $\mathbb{H}$ is simple we have that $\langle \alpha \rangle = \mathbb{H}$, so $\mathbb{H}$ is a semifield with a generator. \hfill $\square$

**Corollary 2.4.4.** The semifield $\mathcal{R}$ is simple and thus by Remark 2.4.3 a semifield with a generator.
2.5 Irreducible kernels, maximal kernels and the Stone topology

**Definition 2.5.1.** A proper (non-trivial) kernel $K$ of a semifield $H$ is said to be **irreducible** if for any pair of kernels $A, B$ of $H$

$$A \cap B \subseteq K \Rightarrow A \subseteq K \text{ or } B \subseteq K. \quad (2.17)$$

A kernel $K$ is called **weakly irreducible** if for any pair of kernels $A, B$ of $H$

$$A \cap B = K \Rightarrow A = K \text{ or } B = K. \quad (2.18)$$

$K$ is called **maximal** if for any kernel $A$ of $H$

$$K \subseteq A \Rightarrow K = A \text{ or } A = K. \quad (2.19)$$

**Definition 2.5.2.** A semifield $H$ is said to be **reduced** if for any pair of kernels $A$ and $B$ of $H$, $A \cap B = \{1\}$ implies that $A = \{1\}$ or $B = \{1\}$.

**Remark 2.5.3.** If $P$ be an irreducible kernel of $H$, then the quotient semifield $H/P$ is reduced.

**Proof.** Let $\phi : H \rightarrow H/P$ be the quotient map and let $A \neq \{1\}, B \neq \{1\}$ kernels of $H/P$ such that $A \cap B = \{1\}$. Then the kernels $A' = \phi^{-1}(A)$ and $B' = \phi^{-1}(B)$ admit $A' \cap B' \supseteq \phi^{-1}(A \cap B) = \phi^{-1}(\{1\}) = P$ which yields that either $A' \subseteq P$ or $B' \subseteq P$ so $A = \phi(A') \subseteq \phi(P) = \{1\}$ or $B = \phi(B') \subseteq \phi(P) = \{1\}$, contradicting our assumption that $A \neq \{1\}, B \neq \{1\}$. Thus $A = \{1\}$ or $B = \{1\}$ and $H/P$ is reduced. \[\square\]

**Theorem 2.5.4.** [9, Theorem (4.1)] Let $K$ be a proper kernel of a semifield $H$. Then there exists at least one irreducible kernel $P$ of $H$ such that $K \subseteq P$.

An immediate consequence of Theorem 2.5.4 is

**Remark 2.5.5.** Every maximal kernel is irreducible.

In the following we prove some assertions concerning maximal kernels.
Remark 2.5.6. Let $\mathbb{H}$ be a semifield. Let $K$ be a kernel of $\mathbb{H}$ and $S \subseteq \mathbb{H}$ a subset. By Remark 2.2.49, the smallest kernel of $\mathbb{H}$ containing both $K$ and $S$ is $\langle M \cup S \rangle = M \cdot \langle S \rangle$.

Remark 2.5.7. Let $M$ be a kernel of a semifield $\mathbb{H}$. $M$ is maximal if and only if for any $a \in \mathbb{H} \setminus M$, $\langle M \cup \{a\} \rangle = M \cdot \langle a \rangle = \mathbb{H}$.

Proof. Let $M$ be maximal kernel. Since $a \notin M$ and since $M$ is a kernel, we have that $M \subseteq \langle M \cup \{a\} \rangle$ and thus $\langle M \cup \{a\} \rangle = \mathbb{H}$. Conversely, assume $M$ is not maximal, then there exists a kernel $N \neq \mathbb{H}$ such that $M \subseteq N$, thus there exists $a \in N$ such that $a \notin M$ and so we get $M \subseteq \langle M \cup \{a\} \rangle = \mathbb{H} \subseteq N$, contradicting $N$ being a maximal kernel.

Corollary 2.5.8. For any semifield $\mathbb{H}$ and a kernel $K$ of $\mathbb{H}$, $K$ is a maximal kernel if and only if $\mathbb{H}/K$ is simple.

Proof. If $\mathbb{H}/K$ is not simple, then there exists a kernel $\{1\} \subset B \subset \mathbb{H}/K$.
If $\phi : \mathbb{H} \rightarrow \mathbb{H}/K$ is the quotient homomorphism, then by Theorem 2.2.49, $\phi^{-1}(B)$ is a kernel of $\mathbb{H}$ and $K = \phi^{-1}(\{1\}) \subseteq \phi^{-1}(B) \subseteq \phi^{-1}(\mathbb{H}/K) = \mathbb{H}$, so $\phi^{-1}(B)$ is proper and contains $K$. Assume $K$ is not maximal, then there is some kernel $M$ of $\mathbb{H}$ containing (not equal to) $K$. Now, by Theorem 2.2.49(4), $K \subseteq \phi^{-1}(\phi(M)) = KM = M$ (since $K \subseteq M$) which in turn yields that $\phi(M) \subset \mathbb{H}/K$ is a proper kernel of $\mathbb{H}/K$ (for otherwise $\phi^{-1}(\phi(M)) = \mathbb{H}$) and $\phi(M) \neq \{1\}$ (for otherwise by the above $M = K$) thus $\mathbb{H}/K$ is not simple.

Definition 2.5.9. The set $Spec(\mathbb{H})$ of all irreducible kernels of a semifield $\mathbb{H}$ is called the irreducible spectrum of $\mathbb{H}$. The subset of $Spec(\mathbb{H})$ consisting of all maximal kernels $Max(\mathbb{H})$ is called the maximal spectrum of $\mathbb{H}$.

We now introduce the Stone topology defined on $Spec(\mathbb{H})$:

Remark 2.5.10. The sets $D(A) = \{P \in Spec(\mathbb{H}) : A \not\subseteq P\}$ with $A$ a kernel of $\mathbb{H}$ are open in the Stone topology. Denote $D(\langle u \rangle)$ for $u \in \mathbb{H}$ by $D(u)$. Then $D(1) = \emptyset$, $D(\mathbb{H}) = Spec(\mathbb{H})$, and $D(\bigsqcup A_i) = \bigcup D(A_i)$ for any family $\{A_i\}$ of kernels of $\mathbb{H}$. Moreover, $D(A \cap B) = D(A) \cap D(B)$ for any kernels $A$, $B$ of $\mathbb{H}$. Thus the collection $\{D(u) : u \in \mathbb{H}\}$ is a basis of a topology on $Spec(\mathbb{H})$ called the Stone topology.

Remark 2.5.11. $Spec(\mathbb{H})$ is a topological space with respect to the Stone topology and $Max(\mathbb{H})$ is a subspace of $Spec(\mathbb{H})$ (by definition, with respect to the induced topology).
Corollary 2.5.12. If \( A \) is a kernel of a semifield \( \mathbb{H} \), then \( D(A) = \text{Spec}(\mathbb{H}) \) implies \( A = \mathbb{H} \).

Theorem 2.5.13. [9, Theorem (4.2)] The following conditions are equivalent for any semifield \( \mathbb{H} \):

1. \( \mathbb{H} \) is a semifield with a generator.

2. \( \text{Spec}(\mathbb{H}) \) is compact.

3. \( \text{Max}(\mathbb{H}) \) is compact, and every proper kernel of \( \mathbb{H} \) is contained in some maximal kernel.

Proposition 2.5.14. [9, Proposition (4.1)] Any irreducible kernel in a semifield \( \mathbb{H} \) contains a minimal irreducible kernel.

Proposition 2.5.15. Let \( \mathbb{H}(x_1, \ldots, x_n) \) be the semifield of fractions where \( \mathbb{H} \) is a bipotent semifield. Then for any \( \gamma_1, \ldots, \gamma_n \in \mathbb{H} \) the kernel \( \langle \frac{x_1}{\gamma_1}, \ldots, \frac{x_n}{\gamma_n} \rangle \) is a maximal kernel of \( \mathbb{H}(x_1, \ldots, x_n) \).

Proof. Let \( \mathbb{H}(x) \) be the semifield of fractions of \( \mathbb{H}[x] \) where \( \mathbb{H} \) is a bipotent semifield. We will now show that \( \langle x \rangle \) is a maximal kernel of \( \mathbb{H}(x) \). Consider the substitution homomorphism \( \psi: \mathbb{H}(x) \to \mathbb{H} \) defined by mapping \( x \mapsto 1 \). Write \( a = \frac{\sum_{i=1}^m \alpha_i x^i}{\sum_{j=1}^n \beta_j x^j} \) with \( \alpha_i, \beta_j \in \mathbb{H} \) for a general element of \( \mathbb{H}(x) \). Then \( \psi(a) = \frac{\sum_{i=1}^m \alpha_i}{\sum_{j=1}^n \beta_j} \) and thus \( \psi(a) = 1 \) if and only if \( \sum_{i=1}^m \alpha_i = \sum_{j=1}^n \beta_j \). We will show that if \( \psi(a) = 1 \) then \( a \in \langle x \rangle \). Indeed, assume \( \psi(a) = 1 \), denote \( \alpha = \sum_{i=1}^m \alpha_i, \beta = \sum_{j=1}^n \beta_j \), then by the above \( \alpha = \beta \). We have \( \frac{\sum_{i=1}^m \alpha_i x^i}{\sum_{j=1}^n \beta_j x^j} = (\frac{\alpha}{\beta}) \frac{\sum_{i=1}^m \alpha_i x^i}{\sum_{j=1}^n \beta_j x^j} \). Now, since \( x^i, x^j \in \langle x \rangle \), since \( \langle x \rangle \) is multiplicative group and since \( \sum_{i=1}^m \frac{\alpha_i}{\alpha} = \sum_{j=1}^n \frac{\beta_j}{\beta} = 1 \), we have that \( \sum_{i=1}^m \frac{\alpha_i}{\alpha} x^i, (\sum_{j=1}^n \frac{\beta_j}{\beta} x^j)^{-1} \in \langle x \rangle \). By assumption \( \frac{\alpha}{\beta} = 1 \), and thus \( a \in \langle x \rangle \) as desired.

As \( \mathbb{H} \) is a bipotent semifield and thus completely ordered and hence simple, we can use the assertions above and Corollary 2.5.8 to deduce that \( \langle x \rangle \) is a maximal kernel of \( \mathbb{H}(x) \).

Applying a change of variable \( y = \frac{x}{\gamma} \) for each \( \gamma \in \mathbb{H} \), the above proof implies that the kernel \( \langle \frac{x}{\gamma} \rangle \) is the kernel of the substitution map \( \psi_\gamma: \mathbb{H}(x) \to \mathbb{H} \) defined by \( x \mapsto \gamma \). Since \( \mathbb{H} \neq \{1\} \) is simple it is generated by any \( \gamma \in \mathbb{H} \) so \( \psi_\gamma \) is onto and consequently, by the same argument used above, \( \langle \frac{x}{\gamma} \rangle \) is maximal for any \( \gamma \in \mathbb{H} \).
Taking a general element $\sum_{i \in I} \alpha_i x_I(i)$ of $\mathbb{H}(x_1, \ldots, x_n)$ where $I, J$ are both finite sets of multi-indices and $x = (x_1, \ldots, x_n)$ and performing the exact same procedure described above, substituting $\vec{1}$ for $x$, and then $(\frac{x_1}{\gamma_1}, \ldots, \frac{x_n}{\gamma_n})$ for $x$. We get that for any choice of $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \mathbb{H}^n$ the kernel $\langle \frac{x_1}{\gamma_1}, \ldots, \frac{x_n}{\gamma_n} \rangle$ is a maximal kernel of $\mathbb{H}(x_1, \ldots, x_n)$ and it corresponds to the substitution map $\mathbb{H}(x_1, \ldots, x_n) \to \mathbb{H}$, defined by

$$(x_1, \ldots, x_n) \mapsto (\gamma_1, \ldots, \gamma_n).$$

The case where $\mathbb{H} = \{1\}$ is trivial since any homomorphism (in particular substitution) has all the domain $\mathbb{H}(x_1, \ldots, x_n)$ as its kernel.

\[ \Box \]

**Example 2.5.16.** Let $\mathbb{H}$ be an idempotent semifield. For $a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{H}^n$ let

$$\phi_a : \mathbb{H}(x_1, \ldots, x_n) \to \mathbb{H}$$

be the substitution homomorphism defined by $f \mapsto f(a)$. Then we have that $Ker(\phi_a) = L_a = \langle \alpha_1 x_1, \ldots, \alpha_n x_n \rangle$. Taking the constant fractions in $\mathbb{H}(x_1, \ldots, x_n)$, we have that $\phi_a$ is onto. By Theorem 2.2.49 we have that $\mathbb{H}(x_1, \ldots, x_n) = \phi_a^{-1}(\mathbb{H}) = \mathbb{H} \cdot L_a = \mathbb{H} \cdot \langle \alpha_1 x_1, \ldots, \alpha_n x_n \rangle$.

Intersecting both sides of the last equality with $\langle \mathbb{H} \rangle$, we get

$$\langle \mathbb{H} \rangle = (\mathbb{H} \cdot L_a) \cap \langle \mathbb{H} \rangle = \mathbb{H} \cdot \langle \alpha_1 x_1, \ldots, \alpha_n x_n \rangle \cap \langle \mathbb{H} \rangle = \mathbb{H} \cdot (\langle \alpha_1 x_1, \ldots, \alpha_n x_n \rangle \cap \langle \mathbb{H} \rangle)$$

$$= \mathbb{H} \cdot \langle |\alpha_1 x_1| \wedge |\alpha|, \ldots, |\alpha_n x_n| \wedge |\alpha| \rangle$$

for any $\alpha \in \mathcal{B} \setminus \{1\}$. 

31
2.6 Distributive semifields

As will be shown in the section concerning idempotent semifields and lattice-ordered groups, the lattice of kernels every of an idempotent semifield is distributive. Idempotent semifields play an important role in our theory. Here we introduce some of the properties of semifields which have a distributive lattice of kernels.

**Definition 2.6.1.** A semifield $\mathbb{H}$ is called *distributive* if the lattice $\text{Con}(\mathbb{H})$ is distributive.

**Proposition 2.6.2.** [9, Proposition (4.2)] If $\mathbb{H}$ is a distributive semifield, then the following statements hold:

1. All weakly irreducible kernels of the semifield $\mathbb{H}$ are irreducible.
2. All proper kernels of $\mathbb{H}$ are intersections of its irreducible kernels.
3. $D(A) \subseteq D(B) \iff A \subseteq B$ for any pair of kernels $A, B$ of $\mathbb{H}$.
4. $D(A) = D(B) \iff A = B$ for any pair of kernels $A, B$ of $\mathbb{H}$.

As every reducible kernel is weakly irreducible, the first assertion of Proposition 2.6.2 states that

**Note 2.6.3.** In a distributive semifield $\mathbb{H}$, a kernel $K$ of $\mathbb{H}$ is irreducible if and only if

$$A \cap B = K \Rightarrow A = K \text{ or } B = K$$

(2.20)

for any pair of kernels $A, B$ of $\mathbb{H}$.

The following is a nice example of utilizing of irreducible kernels.

**Remark 2.6.4.** Let $\mathbb{H}$ be a distributive semifield. Then, for any $a, b \in \mathbb{H}$ such that $\langle a \rangle \cap \langle b \rangle = \{1\}$,

$$\langle a \rangle \langle b \rangle = \langle ab \rangle.$$  

(2.21)

**Proof.** As $ab \in \langle a \rangle \langle b \rangle$, obviously $\langle ab \rangle \subseteq \langle a \rangle \langle b \rangle$. Since $\langle a \rangle \cap \langle b \rangle = \{1\}$, we have that $a \in P$ or $b \in P$ for any $P \in \text{Spec}(\mathbb{H})$ (by definition of irreducibility of a kernel and...
the fact that \( \{1\} \subseteq P \). If \( ab \in P \), then if \( a \in P \) we get \( b = a^{-1}(ab) \in P \) and if \( b \in P \) we get \( a = (ab)b^{-1} \in P \). In any case both \( a \) and \( b \) are in \( P \) and thus we have shown that \( \langle ab \rangle \subseteq P \) implies \( \langle a \rangle \langle b \rangle \subseteq P \). Now, as \( H \) is distributive, we have by Proposition \[2.6.2\] (1) that \( \langle a \rangle \langle b \rangle = \langle ab \rangle \) as they are both intersections of the irreducible kernels containing them.

Kernels having trivial intersection \(( \{1\} \)\) play a very important role in our theory.

**Remark 2.6.5.** [9] Let \( H \) be a distributive semifield. Then any proper kernel of \( H \) is the intersection of all irreducible kernels containing it. In particular, the intersection of all the irreducible kernels of \( H \) equals \( \{1\} \).

### 2.7 Idempotent semifields : Part 1

In the following subsection, we concentrate our attention on the theory of idempotent semifields. We will continue the study of idempotent in subsequent sections after introducing the theory of lattice-ordered groups.

It turns out that this special kind of semifield has some very nice additional properties to those of general semifields that make it quite easy to work with.

As we have already noted, kernels of an idempotent semifield are themselves semifields. We now introduce a few more interesting properties of such semifields.

**Remark 2.7.1.** Finitely generated kernels of an idempotent semifield are principal.

**Proof.** This follows directly from Theorem \[2.3.22\] as a kernel of an idempotent semifield is itself a semifield.

**Remark 2.7.2.** Since every finitely generated kernel of an idempotent semifield is itself a semifield with a generator, by Corollary \[2.3.12\], we have that for any such kernel we can choose a generator \( a \) such that \( |a| = a \geq 1 \). This issue is
important, as for any such an element \( 1 + a = \sup(1, a) = \max(1, a) = a \) and \( 1 \land a = \inf(1, a) = \min(1, a) = 1 \), which imply that \( 1 + (a^{-1}) = \sup(1, a^{-1}) = \max(1, a^{-1}) = 1 \) and \( 1 \land (a^{-1}) = \inf(1, a^{-1}) = \min(1, a^{-1}) = a^{-1} \). By convention, we always take a generator of a kernel of an idempotent semifield to be such a ‘positive’ generator unless stated otherwise. The importance of such generator is that it is comparable to 1.

**Remark 2.7.3.** For any kernel \( K \) of an idempotent semifield the following holds:

\[
|g||h| \in K \iff |g| + |h| \in K. \tag{2.22}
\]

**Proof.** Indeed, Since \(|g|, |h| \geq 1\), we have that \(|g| \leq |g||h|\) and \(|h| \leq |g||h|\) thus

\[
|g| + |h| = \sup(|g|, |h|) \leq |g||h|.
\]

On the other hand, we have that

\[
(|g| + |h|)^2 = |g|^2 + |g||h| + |h|^2 \geq |g||h|.
\]

So, by Remark 2.3.16 we have that \( \langle |g||h| \rangle = \langle |g| + |h| \rangle \) and equality (2.22) follows. \( \Box \)

**Remark 2.7.4.** For any kernel \( K \) of an idempotent semifield the following holds:

\[
g, h \in K \iff |g| + |h| \in K;
\]

\[
g, h \in K \iff |g||h| \in K.
\]

**Proof.** By Remark 2.7.3 we only have to prove the first equality. If \( g, h \in K \) then \(|g|, |h| \in K \) thus \(|g| + |h| \in K \). On the other hand, since \(|g|, |h| \leq |g| + |h|\), by Remark 2.3.16 we have that \(|g| + |h| \in K \) implies \(|g|, |h| \in K \), which in turn yields that \( g, h \in K \) proving our claim. \( \Box \)

**Proposition 2.7.5.** [8, Theorem 2.2.4(d)] Let \( S \) be an idempotent semifield. For \( X, Y \subset S \), denote by \( \langle X \rangle \) and \( \langle Y \rangle \) the kernels generated by \( X \) and \( Y \), respectively. Then for any \( X, Y \subset S \), \( K \in \text{Con}(S) \) and \( a, b \in S \) the following statements hold:

1. \( \langle X \rangle \cdot \langle Y \rangle = \langle X \cup Y \rangle = \langle \{|x| + |y| : x \in X, y \in Y \} \rangle \).
2. \( \langle X \rangle \cap \langle Y \rangle = \langle \{|x| \land |y| : x \in X, y \in Y \} \rangle \).
3. \( \langle K, a \rangle \cap \langle K, b \rangle = K \cdot \langle |a| \land |b| \rangle \), where \( \langle K, a \rangle \) denotes the kernel generated by the set \( K \cup \{a\} \).

**Corollary 2.7.6.** Let \( S \) be an idempotent semifield. Then the intersection and product of two principal kernels are principal kernels. Namely, for every \( f, g \in S \)

\[
\langle f \rangle \cap \langle g \rangle = \langle (f + f^{-1}) \land (g + g^{-1}) \rangle ; \langle f \rangle \cdot \langle g \rangle = \langle (f + f^{-1})(g + g^{-1}) \rangle. \quad (2.23)
\]

**Proof.** Taking \( X = \{f\} \) and \( Y = \{g\} \) in Proposition 2.7.5 yield the equalities, where for the second equality, applying Remark 2.7.3 yields that

\[
\langle f \rangle \cdot \langle g \rangle = \langle (f + f^{-1}) \land (g + g^{-1}) \rangle,
\]

and from there we apply the proposition.

A direct consequence of Corollary 2.7.6 is

**Corollary 2.7.7.** The set of principal kernels of an idempotent semifield forms a sublattice of the lattice of kernels (i.e., a lattice with respect to intersection and multiplication)

**Definition 2.7.8.** Denote the collection of principal kernels of an idempotent semifield \( S \) by \( \text{PCon}(S) \). In particular we denote the collection of principal kernels of \( \mathbb{H}(x_1, ..., x_n) \) by \( \text{PCon}(\mathbb{H}(x_1, ..., x_n)) \) (where \( \mathbb{H} \) is idempotent).

**Remark 2.7.9.** Let \( S \) be an idempotent semifield. Then the following equalities hold for \( \langle f \rangle, \langle g \rangle, \langle h \rangle \in \text{PCon}(S) \):

\[
\begin{align*}
(((f) \cdot (g)) \cap (h)) \cdot (h) &= (\langle f \rangle \cap \langle g \rangle) \cdot (h); \\
\langle (|f| \land |h|) \land (|g| \land |h|) \rangle &= \langle (|f| \land |g|) \land |h| \rangle; \\
((f) \cap (h)) \cdot ((g) \cap (h)) &= (\langle f \rangle \cdot (g)) \cap (h); \\
\langle (|f| \land |h|) \land (|g| \land |h|) \rangle &= \langle (|f| \land |g|) \land |h| \rangle.
\end{align*}
\]

**Proof.** Follows directly from Lemma 2.2.47 and Corollary 2.7.6 above. The proof can also be found in [8]

**Remark 2.7.10.** As principal kernels form a sublattice of \( \text{PCon}(S) \) for any idempotent semifield \( S \) and since for a semifield homomorphism \( \phi \) whose kernel is a principal kernel, both homomorphic images and preimages (which are
then a product of principal kernels by Theorem 2.2.49 with respect to $\phi$ are principal kernels, we have that all three isomorphism theorems apply to the restriction of $\text{Con}(S)$ to $\text{PCon}(S)$.

**Corollary 2.7.11.** In the setting of Theorem 2.2.53, let $L = \langle a \rangle \in \text{PCon}(H)$ and let $\phi_L : H \to H/L$ be the quotient epimorphism. Then by Remark 2.3.19 the image of a principal kernel of $H$ under $\phi_L$ is a principal kernel of $H/L$ and the preimage of a principal kernel $\langle b \rangle/L \in \text{PCon}(H/L)$ is $\langle b \rangle \cdot L = \langle b \rangle \cdot \langle a \rangle = \langle |b| + |a| \rangle$ which is a principal kernel. Thus we have that the correspondence of Theorem 2.2.53 applies to the principal kernels in $\text{PCon}(H)$ and the principal kernels in $\text{PCon}(H/L)$ containing $L$, namely, there is a $1:1$ correspondence

$$\{\text{Principal kernels of } H/L\} \to \{\text{Principal kernels of } H \text{ containing } L\}$$

given by $(b)/\langle a \rangle \mapsto \langle b \rangle$.

**Proposition 2.7.12.** Let $H$ be a bipotent archimedean semifield. Let $H(x_1, ..., x_n)$ be the semifield of fractions of $H[x_1, ..., x_n]$. Then $H(x_1, ..., x_n)$ is finitely generated by $\{x_1, ..., x_n\}$ as a semifield over $H$. $H(x_1, ..., x_n)$ is also finitely generated by $\{e, x_1, ..., x_n\}$ as a kernel for any $e \in H$ (by Remark 2.4.2 in case $H$ is trivial we can omit $e$, otherwise we can choose $e > 1$ such that $H \subseteq \langle e \rangle$). In fact, $H(x_1, ..., x_n) = \langle e \rangle \cdot \prod_{i=1}^{n} \langle x_i \rangle$ and by Theorem 2.3.22 we have that $H(x_1, ..., x_n)$ is a semifield with a generator $\sum_{i=1}^{n} |x_i| + |e|$.

**Proof.** First note that as $\langle e \rangle \cdot \prod_{i=1}^{n} \langle x_i \rangle$ is closed under multiplication and addition (since it is a semifield), by Remark 2.3.16 it is enough to prove that for any monomial $f$ in $H[x_1, ..., x_n]$ there exists some $k \in \mathbb{N}$ such that

$$|f| \leq \left( \sum_{i=1}^{n} |x_i| + |e| \right)^{k} = \sum_{i=1}^{n} |x_i|^{k} + |e|^{k}.$$ 

Let $f(x_1, ..., x_n) = \alpha x_1^{p_1} \cdots x_n^{p_n}$ where $\alpha \in H$. Since $H$ is a semifield with a generator, and $e$ is a generator of $H$, we have that $\alpha \in \langle e \rangle$. Thus there exists some $s$ such that $|\alpha| \leq |e|^{s}$. Now, $|x_1^{p_1} \cdots x_n^{p_n}| \leq |x_1 + \cdots + x_n|^{\sum_{i=1}^{n} |p_i|^{e}} = |x_1|^{\sum_{i=1}^{n} p_i} + \cdots + |x_n|^{\sum_{i=1}^{n} p_i}$. For $k = \max(s, \sum_{i=1}^{n} |p_i|^{e})$, we get $|f| \leq \sum_{i=1}^{n} |x_i|^{k} + |e|^{k}$. \hfill $\square$
The exact same construction yields that the kernel as a kernel over itself.\[\text{Example 2.7.13.}\]

Let $\mathbb{H}(x)$ be the semifield of fractions in one variable of $\mathbb{H}[x]$, where $\mathbb{H}$ is a bipotent semifield. Consider the kernel, $\langle x \rangle$, generated by $x$ in $\mathbb{H}(x)$. Since $\{x^k : k \in \mathbb{Z}\}$ is the (normal) group generated by $x$, by Proposition 2.3.8, $\langle x \rangle$ is

$$\left\{ \sum_{i=1}^{m} f_i(x)x^{I(i)} : m \in \mathbb{N}, f_i \in \mathbb{H}(x) \text{ such that } \sum_{i=1}^{m} f_i = 1 \right\} \quad (2.24)$$

where $I \subset \mathbb{Z}$ containing $n$ elements.

First, note that since $\mathbb{H}$ is idempotent ($1 + 1 = 1$), $\langle x \rangle$ is closed with respect to addition and thus a semifield. Now, $\sum_{i=1}^{n} f_i(x) = 1$ yields by the natural order that $f_i \leq 1$ for every $i = 1, ..., n$. Write $f_i(x) = \frac{h_i(x)}{g_i(x)}$ and let $g(x) = \prod_{i=1}^{n} g_i(x)$ and $s_i(x) = \prod_{j \neq i} g_j(x)$, then $\sum_{i=1}^{n} f_i(x) = 1$ so $g(x) = \sum_{i=1}^{n} h_i(x)s_i(x)$ and $f_i(x) = \frac{h_i(x)s_i(x)}{g(x)} = \frac{h_i(x)s_i(x)}{\sum_{j=1}^{n} h_i(x)s_i(x)}$. Thus we can assume $f_i(x)$ is of the form $\frac{r_i(x)}{\sum_{i=1}^{n} r_i(x)}$ with $r_i(x) \in \mathbb{H}[x]$.

For simplicity of notation, denote $x_0 = e$, where as above $e$ is the generator of $\mathbb{H}$ as a kernel over itself.

The exact same construction yields that the kernel $\langle x_k \rangle$ in $\mathbb{H}(x_1, ..., x_n)$ with $k = 0, 1, ..., n$ is of the form

$$\left\{ \frac{\sum_{i=1}^{m} x_k^{I(i)} \cdot f_i(x_1, ..., x_n)}{\sum_{i=1}^{m} f_i(x_1, ..., x_n)} : m \in \mathbb{N}, f_i \in \mathbb{H}[x_1, ..., x_n] \right\} \quad (2.25)$$

where $I \subset \mathbb{Z}$ containing $n$ elements. Notice that in the special case where $e = 1$ the kernel in (2.25) degenerates to $\{1\}$. Further, (2.25) applies to any element $f(x_1, ..., x_n)$ of $\mathbb{H}(x_1, ..., x_n)$ taken in place of $x_k$.

Now, for each $x_i$, $i = 0, 1, ..., n$ since $\langle x_i \rangle$ are principal kernels and since none of the elements in $\{x_0 = e, x_1, ..., x_n\}$ are comparable to each other, Proposition 2.3.11 yields that $x_i \notin \langle x_j \rangle$ for any $j \neq i$. Thus, none of these kernels is contained in another.

Example 2.7.14.

Consider the semifield of fractions $\mathcal{R}(x, y)$ and the substitution homomorphism $\phi : \mathcal{R}(x, y) \to \mathcal{R}(y)$ defined by $x \mapsto 1$. Then $\phi(\mathcal{R}(x, y)) = \mathcal{R}(y)$.

1. Since $\phi(y) = y$, we have that

$$\phi(\langle y \rangle_{\mathcal{R}(x,y)}) = \langle \phi(y) \rangle_{\mathcal{R}(y)} = \langle y \rangle_{\mathcal{R}(y)} = \{ f \in \mathcal{R}(y) : \exists n \in \mathbb{N} \text{ such that } |f| \leq |y|^n \}$$

where $\langle y \rangle_{\mathcal{R}(x,y)} = \{ f \in \mathcal{R}(x, y) : \exists n \in \mathbb{N} \text{ such that } |f| \leq |y|^n \}$. 37
2. Since \( \phi(|x| \land |y|) = |\phi(x)| \land |\phi(y)| = 1 \land |y| = 1 \), we have that
\[
\phi(\langle |x| \land |y| \rangle, R) = \langle \phi(|x| \land |y|) \rangle, R(y) = \langle 1 \rangle, R(y);
\]
This is expected as \( |x| \land |y| \in \langle x \rangle = \text{Ker} \phi \).

3. Since \( \phi(|x| + |y|) = |\phi(x)| + |\phi(y)| = 1 + |y| = |y| \), we have that
\[
\phi(\langle |x| + |y| \rangle, R) = \langle \phi(|x| + |y|) \rangle, R(y) = \langle |y| \rangle, R(y) = \langle y \rangle, R(y).
\]
Thus we have that \( \phi(\langle y \rangle, R) = \phi(\langle |x| + |y| \rangle, R) \).

By Theorem 2.2.49 \( \phi^{-1}(\langle y \rangle, R) = \langle x \rangle, R(x, y) \cdot \langle y \rangle, R(x, y) = \langle |x| + |y| \rangle, R(x, y) \), thus \( \langle |x| + |y| \rangle \) is a preimage of \( \phi \) while \( \langle y \rangle \) is not.

**Proposition 2.7.15.** Since \( \text{PCon} (\mathcal{A}(x_1, \ldots, x_n)) \) forms a sublattice of the lattice of kernels in \( \mathcal{A}(x_1, \ldots, x_n) \) with respect to multiplications and intersections, and since the kernels \( 1 = \langle 1 \rangle \) and \( \mathcal{A}(x_1, \ldots, x_n) \) are principal (since \( \mathcal{A}(x_1, \ldots, x_n) \) is a semifield with a generator), the Stone topology induces a topology on \( \text{PCon} (\mathcal{A}(x_1, \ldots, x_n)) \).

The collection \( \text{PSpec} (\mathcal{A}(x_1, \ldots, x_n)) \) of all irreducible principal kernels of the semifield \( \mathcal{A}(x_1, \ldots, x_n) \) is called the irreducible principal spectrum of \( \mathcal{A}(x_1, \ldots, x_n) \). It is a topological space with respect to the principal Stone topology. Its subspace \( \text{PMax} (\mathcal{A}(x_1, \ldots, x_n)), \) consisting of all maximal principal kernels, is called the maximal principal spectrum.

As noted above, \( 1, \mathcal{A}(x_1, \ldots, x_n) \in \text{PSpec} (\mathcal{A}(x_1, \ldots, x_n)) \). Thus we have that \( D(1) = \emptyset \), \( D(\mathcal{A}(x_1, \ldots, x_n)) = \text{PSpec} (\mathcal{A}(x_1, \ldots, x_n)) \) are in the topology. The sets
\[
D(\langle f \rangle) = \{ P \in \text{Spec} (\text{PCon} (\mathcal{A}(x_1, \ldots, x_n))) : \langle f \rangle \not\subseteq P \}
\]
are open in the induced topology.

**Definition 2.7.16.** We call the induced topology on \( \text{PCon} (\mathcal{A}(x_1, \ldots, x_n)) \) introduced in Proposition 2.7.15 the principal Stone topology.

**Remark 2.7.17.** The Stone topology on \( \text{Con} (\mathcal{A}(x_1, \ldots, x_n)) \) gives rise to the induced topology on the kernels of the semifield with a generator \( \langle \mathcal{A} \rangle \), the kernels of which, \( \text{Con} (\langle \mathcal{A} \rangle) \), form a lattice of kernels which embeds as a sublattice of \( \text{Con} (\mathcal{A}(x_1, \ldots, x_n)) \).
Remark 2.7.18. In the next section, we show that

\[ \text{PCon}(\mathcal{R}) = \{ (f) \cap \mathcal{R} : f \in \text{PCon}(\mathcal{R}(x_1, \ldots, x_n)) \} \supset \text{PCon}(\mathcal{R}(x_1, \ldots, x_n)) \]

which forms a sublattice of \( \text{PCon}(\mathcal{R}(x_1, \ldots, x_n)) \). Thus the principal Stone topology induces a topology on \( \text{PCon}(\mathcal{R}) \).

2.8 Affine extensions of idempotent archimedean semifields

In the following, we characterize the affine idempotent semifield extensions of an idempotent archimedean semifield \( \mathbb{H} \) which are divisible over \( \mathbb{H} \) (divisible extensions are the analogue for algebraic extensions in ring theory).

Definition 2.8.1. For an idempotent archimedean semifield \( \mathbb{H} \) satisfying the Frobenius property, we denote by \( \overline{\mathbb{H}} \) the divisible closure of \( \mathbb{H} \), i.e., the smallest divisible semifield containing \( \mathbb{H} \).

Note 2.8.2. In Section 3, concerning lattice ordered groups, we revisit the notion of divisible closure and show that the semifield \( \overline{\mathbb{H}} \) exists and is also idempotent and archimedean.

Lemma 2.8.3. If \( \mathbb{H} \) is an idempotent semifield satisfying the Frobenius property, then \( \overline{\mathbb{H}} \) also satisfies the Frobenius property.

Proof. For \( a, b \in \overline{\mathbb{H}} \) there exist some (minimal) \( k, m \in \mathbb{N} \) such that \( a^k = \alpha, b^m = \beta \) are elements of \( \mathbb{H} \). First note that for every \( \alpha, \beta \in \mathbb{H} \) and \( t \in \mathbb{N} \), \( \alpha^s \beta^{t-s} \leq \alpha^t + \beta^t \) for any \( 0 \leq s \leq t \) since \( (\alpha + \beta)^t = \alpha^t + \beta^t \). Now \( (a^s b^{t-s})^{km} = a^{sm} \beta^{km(t-s)} \leq (\alpha^m + \beta^k)^t \) by the former observation. Since both \( \alpha^m \in \mathbb{H} \) and \( \beta^k \in \mathbb{H} \) we have that the Frobenius property holds, so that \( \alpha^m + \beta^k \leq \alpha^m \beta^{km(t-s)} \leq \alpha^{km} + \beta^{km} \), as desired. \( \square \)
Remark 2.8.4. Let $\mathbb{H}$ be an idempotent semifield satisfying the Frobenius property. Let $K = \mathbb{H}(a_1, \ldots, a_n)$ be an affine semifield extension of $\mathbb{H}$ with $a_1, \ldots, a_n \in \mathbb{H}$. The substitution map $\phi : \mathbb{H}(x_1, \ldots, x_n) \to K$ sending $x_i \mapsto a_i$ is an epimorphism. For each $a_i$ let $\alpha_i$ be such that $\alpha_i = a_i^{k(i)}$ with $k(i) \in \mathbb{N}$ minimal such that $a_i^{k(i)} \in \mathbb{H}$. Consider the kernel

$$K = \langle \alpha_1^{-1}x_1^{k(1)}, \ldots, \alpha_n^{-1}x_n^{k(n)} \rangle \in \text{PCon}(\mathbb{H}(x_1, \ldots, x_n)).$$

$K$ is a principal kernel. We claim that $\text{Ker}\phi = K$.

First, as all generators of $K$ are mapped to 1 we have that $K \subseteq \text{Ker}\phi$. Let $\mathbb{H}$ be the divisible closure of $\mathbb{H}$. If $\mathbb{H} = \mathbb{H}$ (i.e., $\mathbb{H}$ is divisible) then for every $1 \leq i \leq n$ we have that $\alpha_i = a_i$. Thus by Proposition 2.5.15 we have that $K$ is a maximal kernel and $K = \text{ker}\phi$. Let us denote this last kernel of $\mathbb{H}(x_1, \ldots, x_n)$ by $\overline{K}$. Now, for each $1 \leq i \leq n$ we have that $\alpha_i^{-1}x_i^{k(i)} = a_i^{-k(i)}x_i^{k(i)} = (a_i^{-1}x_i)^{k(i)}$ thus $\alpha_i^{-1}x_i^{k(i)}$ is a generator of the kernel $\langle a_i^{-1}x_i \rangle$. Consider the restriction

$$\tilde{\phi} = \phi|_{\mathbb{H}(x_1, \ldots, x_n)} : \mathbb{H}(x_1, \ldots, x_n) \to \mathbb{H}(a_1, \ldots, a_n)$$

of $\phi : \mathbb{H}(x_1, \ldots, x_n) \to \mathbb{H}(a_1, \ldots, a_n)$ to $\mathbb{H}(x_1, \ldots, x_n)$. Then $\tilde{\phi}$ is epimorphism and by Theorem 2.2.51 (1) taking $\alpha_i^{-1}x_i^{k(i)}$ as a generator of $\overline{K}$ we have that $\text{Ker}\tilde{\phi} = \overline{K} \cap \mathbb{H}(x_1, \ldots, x_n) = K$. By the first isomorphism theorem we have that

$$\mathbb{H}(x_1, \ldots, x_n)/K \cong \mathbb{H}(a_1, \ldots, a_n),$$

proving our claim.
3 Lattice-ordered groups, idempotent semifields and the semifield of fractions $\mathbb{H}(x_1, \ldots, x_n)$.

An affine semifield over an idempotent semifield $\mathbb{H}$, in particular, the semifield of fractions $\mathbb{H}(x_1, \ldots, x_n)$, is an idempotent semifield. It turns out that $\mathbb{H}(x_1, \ldots, x_n)$ and generally every idempotent semifield can be considered as a special kind of group, called a lattice-ordered group, or $\ell$-group, in which the notion of kernels coincides with the notion of normal and convex $\ell$-subgroups. Although we consider $\mathbb{H}(x_1, \ldots, x_n)$, all results given below hold for any idempotent semifield, in particular for the kernels of $\mathbb{H}(x_1, \ldots, x_n)$ each of which is an idempotent semifield in its own right. A particularly important such kernel is $\langle H \rangle$ which denotes the kernel of $\mathbb{H}(x_1, \ldots, x_n)$ generated by any $\alpha \in \mathbb{H} \setminus \{1\}$.

Due to the central role idempotent semifields play in our theory, we hereby introduce some basic notions and some important results in the theory of lattice-ordered groups. All relevant definitions can be found in [8]. We note that in [8] the group operation is taken to be addition while in our context we take it to be multiplication, as we use $\oplus$ for sup, i.e., $\lor$. The order preserving map $x \mapsto a^x$ with $a$ a symbol is used to translate the statements presented there to our language where the group is taken to be multiplicative. In particular $(\mathbb{R}, +)$ is translated to $(\mathbb{R}^+, \cdot)$.
3.1 Lattice-ordered groups

**Definition 3.1.1.** A partially-ordered group (p.o group) is a group $G$ endowed with a partial order such that the group operation preserves the order on $G$, that is,

\[ a \leq b \Rightarrow \forall g \in G \ (ag \leq bg \text{ and } ga \leq gb). \]

\[ a \leq b \Rightarrow \forall g \in G \ (g(a \wedge b) = ga \wedge gb \text{ and } (a \wedge b)g = ag \wedge bg). \]

If the partial order on $G$ is directed, then $G$ is a directed group. If the partial order on $G$ is a lattice order, then $G$ is a lattice-ordered group or $\ell$-group. If the order on $G$ is a linear order, then $G$ is called a totally-ordered group or o-group.

**Definition 3.1.2.** An $\ell$-subgroup of a po-group is a subgroup which is also a sublattice.

**Remark 3.1.3.** Since $x \vee y = (xy^{-1} \vee 1)y$ and $x \wedge y = (x^{-1} \vee y^{-1})^{-1}$, $L$ is an $\ell$-subgroup of a po-group $G$ exactly when $z \in L$ implies that $z \vee 1 \in L$ (whenever $z \vee 1$ exists in $G$).

**Definition 3.1.4.** If $G$ and $H$ are $\ell$-groups, then a group homomorphism $\phi : G \to H$ which is also a lattice homomorphism (preserves $\wedge$ and $\vee$ (\$\vee\$ in our notation)) is called an $\ell$-homomorphism.

**Definition 3.1.5.** A subset $S$ of a poset $P$ is said to be convex if $a \leq p \leq b$ with $a, b \in S$ implies that $p \in S$.

**Remark 3.1.6.** [8, Theorem (2.2.3)] All fundamental group isomorphism theorems hold for normal convex $\ell$-subgroups of an $\ell$-group.

**Theorem 3.1.7.** [6, Theorem (2.2.3)] Let $N$ be a normal convex $\ell$-subgroup of an $\ell$-group $G$. The mapping $A \mapsto A/N$ is a lattice isomorphism between the lattice of convex $\ell$-subgroups of $G$ that contain $N$ and the lattice of convex $\ell$-subgroups of $G/N$.

**Remark 3.1.8.** [8, Proposition (4.3)] Let $G$ be an $\ell$-group. Let $K$ be a normal convex $\ell$-subgroup of $G$ and $L$ a normal convex $\ell$-subgroup of $K$. Then $L$ is a normal convex $\ell$-subgroup of $G$ if and only if $L$ is a normal subgroup of $G$. 

42
Proposition 3.1.9. [8, Theorem (2.2.5)] Let \( G \) be an \( \ell \)-group. The lattice of convex \( \ell \)-subgroups of \( G \) is a complete distributive sublattice of the lattice of subgroups of \( G \), and, in fact, it satisfies the infinite distributive law
\[
A \cap \left( \bigvee_i B_i \right) = \bigvee_i (A \cap B_i).
\]

Definition 3.1.10. An \( \ell \)-group \( G \) is a subdirect product of the family \( \{ G_i : i \in I \} \) of \( \ell \)-groups if there is a monomorphism \( \phi : G \to \prod G_i \) such that each composite \( \pi_i \circ \phi \) is an epimorphism, and \( G \) is subdirectly irreducible if, in any such representation of \( G \) there is an index \( i \) such that \( \pi_i \circ \phi \) is an isomorphism. We indicate that \( G \) is a subdirect product of the family \( \{ G_i : i \in I \} \) by writing \( G \xrightarrow{s.d} \prod G_i \).

Note 3.1.11. We use the same notation presented in Definition 3.1.10 in the context of idempotent semifields.

Remark 3.1.12. A function \( \phi : G \to \prod G_i \) on an \( \ell \)-group \( G \) is uniquely determined by the family \( \{ \pi_i \circ \phi : i \in I \} \), and is an \( \ell \)-homomorphism if and only if each \( \pi_i \circ \phi \) is an \( \ell \)-homomorphism. In this case \( \text{Ker}(\phi) = \bigcap_i \text{Ker}(\pi_i \circ \phi) \) and \( G/\text{Ker}(\pi_i \circ \phi) \cong \text{Im}(\pi_i \circ \phi) \). Consequently, each family \( \{ N_i : i \in I \} \) of normal convex \( \ell \)-subgroups of the \( \ell \)-group \( G \) determines a homomorphism \( G \to \prod G/N_i \) with kernel \( N = \bigcap_i N_i \), and \( G/N \) is a subdirect product of the family \( \{ N_i : i \in I \} \) and all subdirect product representations of \( G/N \) essentially arise in this way. Clearly, a nonzero \( \ell \)-group \( G \) is subdirectly irreducible if and only if it has a smallest nontrivial (i.e., \( \not= \{1\} \)) normal convex \( \ell \)-subgroup.

Theorem 3.1.13. [8] Each \( \ell \)-group is a subdirect product of a family of subdirectly irreducible \( \ell \)-groups.

Proof. Let \( G \) be an \( \ell \)-group. For \( a \in G \setminus \{1\} \) let \( N_a \) be a normal convex \( \ell \)-group of \( G \) which is maximal with respect to excluding \( a \). The existence of \( N_a \) is given by Zorn’s Lemma. Since each normal convex \( \ell \)-subgroup of \( G \) that properly contains \( N_a \) must contain \( a \), \( G/N_a \) is subdirectly irreducible (it has a smallest nontrivial (contains the coset \( aN_a \)) normal convex \( \ell \)-subgroup. But \( \bigcap_{a \not= 1} N_a = \{1\} \), so \( G \) is isomorphic to a subdirect product of the \( G/N_a \). \( \square \)
Shortly we will introduce a much stronger result for commutative ℓ-groups.

**Definition 3.1.14.** [8, Section (2.4)] A convex ℓ-subgroup $P$ of the ℓ-group $G$ is called a prime subgroup if whenever $a, b \in G$ with $a \land b \in P$, then $a \in P$ or $b \in P$.

**Note 3.1.15.** In the context of idempotent semifields, we use the notion ‘irreducible’ for ‘prime’.

**Proposition 3.1.16.** [8, Theorem (2.4.1)] The following statements are equivalent for the convex ℓ-subgroup $P$ of $G$:

1. $P$ is a prime subgroup.
2. If $a, b \in G$ with $a \land b = 1$ then $a \in P$ or $b \in P$.
3. The lattice of (left) cosets $G/P$ is totally ordered.
4. The lattice of convex ℓ-subgroups of $G$ that contain $P$ is totally ordered.

**Remark 3.1.17.** [8] Each subgroup $L$ that contains a prime subgroup is an ℓ-subgroup, and hence is itself prime if it is convex. For if $a \in L$, then, since $(a \lor 1) \land (a^{-1} \lor 1) = 1$, $(a \lor 1) = a(a^{-1} \lor 1) \in L$ and so by Remark 3.1.3 $L$ is an ℓ-subgroup. It can be easily seen that the intersection of any chain of prime subgroups is prime. In particular, if $P$ is a prime subgroup and $\{P_i : i \in I\}$ of prime subgroups, then $\bigcap P$ is a minimal prime contained in $P$.

**Proposition 3.1.18.** Each convex ℓ-subgroup is the intersection of prime subgroups.

**Proof.** See [8] Theorem (2.4.2). □

**Definition 3.1.19.** Let $G$ be an ℓ-group. For $a, b \in G$, $a$ and $b$ are said to be disjoint or orthogonal if $|a| \land |b| = 1$.

**Remark 3.1.20.** Let $G$ be a commutative ℓ-group. For any $a \in G$, the set

$$\perp a = \{ x \in G : |a| \land |x| = 1 \}$$

is a convex ℓ-subgroup of $G$. 44
The following theorems can be found in [2] (Theorems XIII.21 and XIII.22).

**Theorem 3.1.21.** A commutative $\ell$-group is either linearly ordered or subdirectly reducible.

**Theorem 3.1.22.** (Clifford) Any commutative $\ell$-group is a subdirect product of subdirectly irreducible linearly ordered $\ell$-groups.

### 3.2 Idempotent semifields versus lattice-ordered groups

Endowed with the natural order given in Remark 2.2.14, an idempotent semifield of fractions $\mathbb{H}(x_1, \ldots, x_n)$ can be considered as a $\ell$-group. The following results establish the connection between $\ell$-groups and additively commutative and idempotent semifields in general, and in particular $\mathbb{H}(x_1, \ldots, x_n)$.

There exists a correspondence between $\ell$-groups and additively commutative and idempotent semifields.

The following results can be found in section 4 of [10].

**Proposition 3.2.1.** [10] 1. Let $(A, \leq, \cdot, +)$ be an $\ell$-group and define

$$a + b = a \lor b = \sup\{a, b\} \text{ for all } a, b \in A.$$  

Then $(A, +, \cdot)$ is a semifield such that $(A, +, \cdot)$ is commutative and idempotent.

2. Conversely, let $(A, +, \cdot)$ be commutative and idempotent semifield and define $a \leq b$ whenever $a + b = b$ for $a, b \in A$. Then $(A, \leq, \cdot)$ is an $\ell$-group satisfying $a \lor b = a + b$ and $a \land b = (a^{-1} + b^{-1})^{-1}$.

3. This correspondence is bijective. Moreover, the following statements are equivalent in this context:

- $\phi$ is a semifield homomorphism of $(A, +, \cdot)$ into $(B, +, \cdot)$. 

• φ is a \( \lor \)-preserving group homomorphism of \((A, \leq, \cdot)\) into \((B, \leq, \cdot)\).

• φ is a \( \land \)-preserving group homomorphism of \((A, \leq, \cdot)\) into \((B, \leq, \cdot)\).

**Proposition 3.2.2.** Each kernel \( K \) of an additively commutative and idempotent semifield \((A, +, \cdot)\) is a normal and convex subgroup of \((A, \leq, \cdot)\) such that \( a \lor b \in K \) holds for all \( a, b \in K \), and conversely. In fact, \( K \) is a subsemifield of \((A, +, \cdot)\) and a sublattice of \((A, \lor, \land)\).

The following is a direct consequence of Remark 3.1.8:

**Corollary 3.2.3.** Let \( \mathbb{H} \) be an idempotent commutative semifield. If \( K \) is a kernel of \( \mathbb{H} \) and \( L \) is a kernel of \( K \) (viewing \( K \) as an idempotent semifield) then \( L \) is a kernel of \( \mathbb{H} \).

In view of the above, all statements presented in 3.2.1 and 3.2.2 referring to \( \ell \)-groups are true for idempotent semifields, changing notation from \( \lor \) to \( + \).

### 3.3 Idempotent semifields: Part 2

The structure of lattice-ordered groups is thoroughly studied in \([6]\) and \([1]\), and many results given there are applicable for a semifield of fractions \( \mathbb{H}(x_1, \ldots, x_n) \) with \( \mathbb{H} \) a bipotent semifield and generally for affine semifield extensions of \( \mathbb{H} \) (which are just quotients of \( \mathbb{H}(x_1, \ldots, x_n) \) by Remark 2.3.23).

The results we introduce in the remainder of this section are derived from known results in the theory of lattice-ordered groups.

The notions of an \( \ell \)-group and normal convex \( \ell \)-subgroups correspond to an idempotent semifield and their kernels. In our case the idempotent semifield is also commutative, thus normality of an \( \ell \)-subgroup is insignificant.
Definition 3.3.1. Let $S$ be an idempotent semifield (equivalently, an $\ell$-group). Define the positive cone of $S$ to be

$$S^+ = \{ s \in S : s \geq 1 \}$$

and the negative cone of $S$ to be

$$S^- = \{ s^{-1} : s \in S^+ \} = \{ s \in S : s \leq 1 \} .$$

In the special case of the idempotent semifield $\mathbb{H}(x_1, ..., x_n)$, we call $\mathbb{H}(x_1, ..., x_n)^+$ and $\mathbb{H}(x_1, ..., x_n)^-$ the positive cone of fractions and the negative cone of fractions and denote them by $P^+$ and $P^-$, respectively.

As the following Theorem demonstrates, the partial orders of the group $(\mathbb{H}(x_1, ..., x_n), \cdot)$ that make it into a po-group are in one-to-one correspondence with the positive cones of $\mathbb{H}(x_1, ..., x_n)$.

Theorem 3.3.2. [8, Theorem (2.1.1)] The following statements hold for the positive and negative cones of an idempotent semifield $S$ (equivalently an $\ell$-group):

1. $(S^+, \cdot)$ is a (normal) subsemigroup of $S^+$,

2. $S^+ \cap S^- = \{1\}$,

3. For every $f, g \in S$, $f \leq g \iff gf^{-1} \in S^+$.

Conversely, if $P$ is a (normal) subsemigroup of $S$ that satisfies (2), then the relation defined in (3) is a partial order of $S$ which makes $S$ into a partially ordered group with positive cone $P$.

Note 3.3.3. Theorem 3.3.2 gives another perspective for viewing quotient semifields.

Theorem 3.3.4. [8, Theorem (2.1.2)] Let $G$ be a po-group.

- $G$ is totally ordered if and only if $G = G^+ \cup G^-$.

- $G$ is directed if and only if $G^+$ generates $G$. Moreover, if $G^+$ generates $G$, then $G = G^+ \cdot G^- = \{ ab^{-1} : a, b \in G^+ \}$.

- $G$ is an $\ell$-group if and only if $G^+$ is a lattice and generates $G$ as a group.
Corollary 3.3.5. \( \mathcal{P}^+ \) is a lattice and generates \( \mathbb{H}(x_1, \ldots, x_n) \).

Proof. Note that \( \mathcal{P}^+ \) being a lattice means that for any \( f, g \in \mathcal{P}^+ \) both \( f \land g \) and \( f + g \) (i.e., \( f \lor g \)) are in \( \mathcal{P}^+ \). The assertion follows directly from Theorem 3.3.4. \( \square \)

Definition 3.3.6. A lattice \( L \) is infinitely distributive if whenever \( \{x_i\} \) is a subset of \( L \) for which \( \lor x_i \) exists, then, for each \( y \in L \), \( \lor(y \land x_i) \) exists and

\[
y \land \lor x_i = \lor(y \land x_i).
\]

The dual also holds.

The following is a consequence of \( \mathbb{H}(x_1, \ldots, x_n) \) being an \( \ell \)-group.

Proposition 3.3.7. [8, Theorem (2.1.3)]

1. \( \mathbb{H}(x_1, \ldots, x_n) \) is infinitely distributive and hence is distributive.

2. \( \mathbb{H}(x_1, \ldots, x_n) \) satisfies the property \( f^k \geq 1 \Rightarrow f \geq 1 \) for any \( f \in \mathbb{H}(x_1, \ldots, x_n) \) and any integer \( k \geq 1 \), hence it has no nonzero elements of finite order.

Remark 3.3.8. [8] Chapter 2] For any \( f, g \in \mathbb{H}(x_1, \ldots, x_n) \) the following equalities hold:

1. \( (f + g)^{-1} = f^{-1} \land g^{-1}, \)
2. \( (f \land g)^{-1} = f^{-1} + g^{-1}, \)
3. \( f(f + g)^{-1}g = f \land g, \)
4. \( f(f \land g)^{-1}g = f + g, \)
5. \( (f(f \land g)^{-1}) \land (g(f \land g)^{-1}) = 1, \)
6. \( |fg^{-1}| = (f + g)(f \land g)^{-1}. \)

Lemma 3.3.9. If \( \mathbb{H} \) is bipotent then \( \lor \) is distributive over \( \land \) in \( \mathbb{H}(x_1, \ldots, x_n) \), i.e., for any \( f, g, h \in \mathbb{H}(x_1, \ldots, x_n) \) the following equality holds:

\[
f + (g \land h) = (f + g) \land (f + h). \tag{3.1}
\]
Proof. We can rewrite (3.1) as \( \max(f, \min(g, h)) = \min(\max(f, g), \max(f, h)) \).

Let \( x \in \mathbb{H}^n \). If \( f(x) \geq g(x), h(x), h(x) \geq f(x) \geq g(x) \) or \( g(x) \geq f(x) \geq h(x) \), then both sides of the equality are equal to \( f(x) \). If \( g(x) \geq h(x) \geq f(x) \) then both sides of the equality are equal to \( h(x) \). If \( h(x) \geq g(x) \geq f(x) \) then both sides of the equality are equal to \( g(x) \).

As the above holds for every \( x \in \mathbb{H}^n \), (3.1) is an identity.

Remark 3.3.10. Let \( g, h \in \mathbb{H}(x_1, \ldots, x_n) \). For every \( f \in \mathbb{H}(x_1, \ldots, x_n) \) such that \( f \geq |g| \land |h| \) (thus \( f \geq 1 \) and \( f = |f| \)), if \( f \leq |g| \) and \( f \leq |h| \) then \( f = |g| \land |h| \).

Indeed, \( f = f + (|g| \land |h|) = (f + |g|) \land (f + |h|) = |g| \land |h| \). The first equality from the right follows the assumption that \( f \geq |g| \land |h| \), the second follows Lemma 3.3.9 and the last follows the assumptions \( |f| \leq |g| \) and \( |f| \leq |h| \).

Definition 3.3.11. For any \( f \in \mathbb{H}(x_1, \ldots, x_n) \) we define

\[
|f|_{\geq}(x) = f(x) + 1 = \begin{cases} 1 & f(x) \leq 1; \\ f(x) & f(x) > 1, \end{cases}
\]

and

\[
|f|_{\leq}(x) = (f(x) \land 1)^{-1} = \begin{cases} f(x)^{-1} & f(x) \leq 1; \\ 1 & f(x) > 1. \end{cases}
\]

By definition \( |f|_{\geq}, |f|_{\leq} \in \mathbb{H}(x_1, \ldots, x_n) \).

Remark 3.3.12. By Remark 3.3.8 we have that

\[
|f|_{\leq} = (f \land 1)^{-1} = f^{-1} + 1.
\]

Remark 3.3.13. [6, Chapter 2] For any \( f, g \in \mathbb{H}(x_1, \ldots, x_n) \) the following statements hold:

1. \( |fg|_{\geq} \leq |f|_{\geq}|g|_{\geq} \).
2. \( |fg|_{\geq} = f (f \land g^{-1})^{-1} \).
3. \( |f^n|_{\geq} = |f|_{\geq}^n \) and \( |f^n|_{\leq} = |f|_{\leq}^n \) for any \( n \in \mathbb{N} \).
Remark 3.3.14. For any $f, g \in \mathbb{H}(x_1, ..., x_n)$ the following statements hold:

1. $|f|_{\geq} \cdot |f|_{\leq}^{-1} = f$.
2. $|f|_{\geq} \cdot |f|_{\leq} = |f|_{\geq} + |f|_{\leq} = |f|$.
3. $|f|_{\geq} \wedge |f|_{\leq} = 1$.
4. $||f|| = |f|, ||f| \wedge |g|| = |f| \wedge |g|, ||f| + |g|| = |f| + |g|$.
5. $|f|_{\geq} = |f|_{\geq}, ||f|_{\leq} = |f|_{\leq}$.
6. $\langle f \rangle = \langle |f| \rangle$.
7. $|f|^k = |f^k|$ for any $k \in \mathbb{N}$.

Proof. 1. Since $f \cdot |f|_{\leq}^{-1} = f(f \wedge 1)^{-1} = f(f^{-1} + 1) = 1 + f = |f|_{\geq}$ we have that $|f|_{\geq} \cdot |f|_{\leq} = f$.
3. $|f|_{\geq} \wedge |f|_{\leq}^{-1} = (|f|_{\geq}|f|_{\leq}|f|_{\leq}^{-1}) \wedge |f|_{\leq}^{-1} = (|f|_{\geq}|f|_{\leq} \wedge 1)|f|_{\leq}^{-1} = (f \wedge 1)|f|_{\leq}^{-1} = |f|_{\geq}|f|_{\leq}^{-1} = 1$. The sixth statement follows from Proposition 2.3.3. The last statement is due to the fact that $|f(x)|^k = (f(x) + (f(x))^{-1})^k = f(x)^k + (f(x))^{-k}$ for any $x \in \mathbb{H}^n$. The rest of the statements are obvious and can be found in chapter 2 of [6].

Remark 3.3.15. Suppose $f, g, h \in \mathbb{H}(x_1, ..., x_n)$

1. The following are equivalent:
   - $f \wedge g = 1$.
   - $fg = f + g$.
   - $f = |fg^{-1}|_{\geq}$ and $g = |fg^{-1}|_{\leq}$.
2. (The Riesz decomposition property) If $f, g, h \in \mathcal{P}^+$ and $f \leq gh$, then $f = g'h'$ where $1 \leq g' \leq g$ and $1 \leq h' \leq h$.
3. If $f, g, h \in \mathcal{P}^+$, then $f \wedge (gh) \leq (f \wedge g)(f \wedge h)$.
4. If $f \wedge g = 1$ and $f \wedge h = 1$, then $f \wedge (gh) = 1$.  

50
Proof. We will only give the proof for the Riesz decomposition property. For all other properties see [8] (2.1.4). Let \( g' = f \land g \) and \( h' = (g')^{-1}f \). Then \( 1 \leq g' \leq g \), \( h' = (f^{-1} + g^{-1})f = 1 + g^{-1}f \leq h \), \( h' \geq 1 \), and \( f = g'h' \).

Remark 3.3.16. For any \( f, g \in \mathbb{H}(x_1, ..., x_n) \) the following relations hold:

1. \(|f + g| \leq |f| + |g|\), \(|f| \cdot |g|\).

2. \(|fg^{-1}| = (f + g)(f \land g)^{-1}|\).

3. \(|fg| \leq |f| \cdot |g|\).

Proof. See [6], Chapter 2. We note that the third statement is true due to the commutativity of \( \mathbb{H}(x_1, ..., x_n) \).

Structure of affine extensions of a bipotent semifield

Due to the correspondence between idempotent semifields and \( \ell \)-groups, we can apply Theorems 3.1.21 and 3.1.22 to \( \mathbb{H}(x_1, ..., x_n) \) and deduce

Corollary 3.3.17. \( \mathbb{H}(x_1, ..., x_n) \) is a subdirect product of simple semifields.

Proof. First note that any kernel of an idempotent semifield is itself an idempotent semifield. Now, linearly (totally) ordered \( \ell \)-groups have no proper convex \( \ell \)-subgroups and thus correspond to semifield having no proper kernels, i.e., simple semifields (see Remark 2.4.2).

Remark 3.3.18. [5] Lemmas (3.1.8),(3.1.16)] [8] Sec. 2.1] For a torsion free partially ordered abelian group \( G \) there exists a group \( \overline{G} \) which is the smallest divisible group containing \( G \) extending its order. If \( G \) is lattice ordered, directed or totally ordered then so is \( \overline{G} \), lattice ordered, directed or totally ordered, respectively. \( \overline{G} \) is called the divisible hull of \( G \).

Definition 3.3.19. Viewing an idempotent semifield \( \mathbb{H} \) as an \( \ell \)-group, we define the divisible closure \( \overline{\mathbb{H}} \) of \( \mathbb{H} \) to be its divisible hull. By the above, \( \overline{\mathbb{H}} \) is also an idempotent semifield (as it is an \( \ell \)-group).
The following two results will be considered and proved in our subsequent discussions:

**Proposition 3.3.20.** Let $\mathbb{H}$ be a bipotent semifield. As we have previously shown, any bipotent semifield is totally (linearly) ordered and thus a simple semifield. For each point $a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{H}^n$, let

$$L_a = \langle (\beta_1)^{-1}x_1^{k_1}, \ldots, (\beta_n)^{-1}x_n^{k_n} \rangle$$

where $k_i \in \mathbb{N}$ is minimal such that $\beta_i = (\alpha_i)^{k_i} \in \mathbb{H}$. Then $L = \bigcap_{a \in \mathbb{H}^n} L_a = \{1\}$ and

$$\mathbb{H}(x_1, \ldots, x_n) = \mathbb{H}(x_1, \ldots, x_n)/L \rightarrow \prod_{a \in \mathbb{H}^n} Q_a$$

where $Q_a = \mathbb{H}(x_1, \ldots, x_n)/L_a$. Each $Q_a$ is an affine bipotent (thus totally ordered) semifield extension of $\mathbb{H}$ namely $\mathbb{H}(\alpha_1, \ldots, \alpha_n)$ which is a simple. In the case where $\mathbb{H}$ is divisible, we have each $Q_a \cong \mathbb{H}$.

**Proof.** Let $f \in L = \bigcap_{a \in \mathbb{H}^n} L_a$. Then for each $a \in \mathbb{H}^n$ since $f \in L_a$ we have that $f(a) = 1$. Thus, since $\mathbb{H}$ is divisible and $f$ coincides with 1 over $\mathbb{H}^n$, we have that $f = 1$. By Remark 2.8.4, we have that $Q_a \cong \mathbb{H}(\alpha_1, \ldots, \alpha_n)$ is thus a bipotent semifield (extending $\mathbb{H}$).

**Remark 3.3.21.** The analogous construction holds for the subsemifield $\langle \mathbb{H} \rangle$ of $\mathbb{H}(x_1, \ldots, x_n)$ which is the principal kernel $\langle |\gamma| \rangle$ with $\gamma$ any element of $\mathbb{H} \setminus \{1\}$, taking

$$L_a \cap \langle \mathbb{H} \rangle = \langle |(\beta_1)^{-1}x_1^{k_1}| \land |\gamma|, \ldots, |(\beta_n)^{-1}x_n^{k_n}| \land |\gamma| \rangle$$

with $\gamma \in \mathbb{H} \setminus \{1\}$.

### 3.4 Archimedean idempotent semifields

All the statements introduced in this section were stated originally for additive $\ell$-groups and have been translated by us to the language of commutative idempotents semifields, where the group operation is multiplication instead of addition.
Recall that an idempotent semifield $H$ is said to be archimedean if it is archimedean as a po-group, i.e., for $a, b \in H$ if $a^Z \leq b$ then $a = 1$.

**Proposition 3.4.1.** If $H$ is archimedean, then $H(x_1, ..., x_n)$ is archimedean. Moreover, every $K \in \text{Con}(H(x_1, ..., x_n))$ is archimedean as a subsemifield of $H(x_1, ..., x_n)$.

**Proof.** Let $f, g \in H(x_1, ..., x_n)$ such that $f^Z \leq g$. If $a \in H^n$ then by our assumption $f(a)^k \leq g(a)$ for all $k \in \mathbb{Z}$. Since $H$ is archimedean and $f(a), g(a) \in H$, we have that $f(a) = 1$. Since this holds for any $a \in H^n$ we have $f(H^n) = 1$ implying $f = 1$. The arguments above hold for any kernel of $H(x_1, ..., x_n)$.

**Remark 3.4.2.** The arguments given for Proposition 3.4.1 yield that the semifield $\text{Fun}(H^n, H)$ is archimedean.

**Definition 3.4.3.** A idempotent semifield $H$ is said to be complete if its underlying lattice is conditionally complete, cf. Definition 2.1.3

**Remark 3.4.4.** If the idempotent semifield $H$ is complete, then $\text{Fun}(H^n, H)$ is complete.

**Proof.** Let $X \subset \text{Fun}(H^n, H)$ be bounded from below, say by $h \in \text{Fun}(H^n, H)$. Then for any $a \in X$ the set $\{f(a) : f \in X\}$ is bounded from below by $h(a)$ thus has an infimum $\bigwedge_{f \in X} f(a)$. It is readily seen that the function $g \in \text{Fun}(H^n, H)$ defined by $g(a) = \bigwedge_{f \in X} f(a)$ is an infimum for $X$, i.e., $g = \bigwedge_{f \in X} f$. Analogously, if $X$ is bounded from above then $(\bigvee_{f \in X} f)(a) = \bigvee_{f \in X} f(a)$ is the supremum of $X$.

**Definition 3.4.5.** A completion of the idempotent semifield $H$ is a pair $(H, \theta)$ where $H$ is a complete idempotent semifield and $\theta : H \rightarrow H$ is a monomorphism whose image is dense in $H$.

The following theorem states that each archimedean idempotent semifield has a unique completion.

**Theorem 3.4.6.** [8, Theorem 2.3.4] An idempotent semifield has a completion if and only if it is archimedean. If $(A, \theta_A)$ and $(B, \theta_B)$ are two completions of the idempotent semifield $H$, then there is a unique isomorphism $\rho : A \rightarrow B$ such that $\rho \circ \theta_A = \theta_B$. 

53
Definition 3.4.7. A subset $A$ of the poset $P$ is called co-final in $P$ if for every $x \in P$ there exists some $a \in A$ such that $x \leq a$. $A$ is said to be co-initial in $P$ if for every $x \in P$ there exists some $a \in A$ such that $a \leq x$.

Definition 3.4.8. The subset $A$ of the idempotent semifield $\mathbb{H}$ is called left dense in $\mathbb{H}$ if $A^+ \setminus \{1\}$ (where $A^+ = \{a \in A : a \geq 1\}$) is co-initial in $\mathbb{H}^+ \setminus \{1\}$, and $A$ is called right dense in $\mathbb{H}$ if $A^+$ is co-final in $\mathbb{H}^+$.

Theorem 3.4.9. [8, Theorem (2.3.6)] Let $\mathbb{H}$ be an archimedean subsemifield of the complete idempotent semifield $H$. Then the following statements are equivalent:

1. $H$ is the completion of $\mathbb{H}$.

2. $\mathbb{H}$ is left dense in $H$, and if $A$ is an idempotent subsemifield of $H$ that is complete and contains $\mathbb{H}$, then $A = H$.

Corollary 3.4.10. [8] Suppose that $\mathbb{H}$ is a left dense archimedean idempotent subsemifield of the complete idempotent semifield $H$. Then if $A$ is a kernel of $\mathbb{H}$, then the completion of $A$ is the kernel of $H$ generated by $A$.

We now state the well-known result by Hölder for ℓ-groups:

Theorem 3.4.11. [8, Theorem (2.3.10)(Hölder)] The following statements are equivalent for an ℓ-group $G$.

1. The only convex ℓ-subgroups of $G$ are 1 and $G$.

2. $G$ is totally ordered and archimedean.

3. $G$ can be embedded in $(\mathbb{R}, +)$.

Proposition 3.4.12. [8] A divisible totally ordered archimedean group which is complete is isomorphic to $(\mathbb{R}, +)$.

Translating the above to the language of idempotent semifields and using the isomorphism $(\mathbb{R}, +) \rightarrow (\mathbb{R}^+, \cdot)$ defined by $x \mapsto e^x$ (where $\mathbb{R}^+$ is the set of positive real numbers) yield

Theorem 3.4.13. The following statements are equivalent for an idempotent semifield $\mathbb{H}$.

1. $\mathbb{H}$ is simple.
2. $\mathbb{H}$ is totally ordered and archimedean.

3. $\mathbb{H}$ can be embedded in $(\mathbb{R}^+, +, \cdot)$.

**Corollary 3.4.14.** A divisible totally ordered archimedean idempotent semifield which is complete is isomorphic to $(\mathbb{R}^+, +, \cdot)$.

In view of the above, we may regard the designated semifield $\mathcal{R}$ as being $(\mathbb{R}^+, +, \cdot)$. 
4 Skeletons and kernels of skeletons

In this section we introduce the notions of ‘skeletons’ and ‘kernels of skeletons’. We define a pair of operators \( \text{Skel} \) and \( \text{Ker} \) where \( \text{Skel} \) maps a kernel to its skeleton and \( \text{Ker} \) maps a skeleton to its corresponding kernel. Although we define these operators with respect to the semifield of fractions \( \mathbb{H}(x_1, \ldots, x_n) \), since we only use the fact that \( \mathbb{H}(x_1, \ldots, x_n) \) is a semifield to prove our assertions, all the properties proved in subsections 4.1 and 4.2 hold replacing \( \mathbb{H}(x_1, \ldots, x_n) \) by any subsemifield of the semifield of functions \( \text{Fun}(\mathbb{H}^n, \mathbb{H}) \).

4.1 Skeletons

We now define a geometric object in the semifield \( \mathbb{H}^n \) to which we aim to associate to a kernel of the semifield of fractions \( \mathbb{H}(x_1, \ldots, x_n) \). Namely, we define an operator

\[ \text{Skel} : \mathbb{P}(\mathbb{H}(x_1, \ldots, x_n)) \to \mathbb{H}^n \]

which associates a subset of \( \mathbb{H}^n \) to every kernel of \( \mathbb{H}(x_1, \ldots, x_n) \).

**Note** 4.1.1. In the field of computer vision the notion of skeleton (or topological skeleton) of a shape is a thin version of that shape that is equidistant to its boundaries.

Resembling the tropical variety in its shape, we have decided to call the next geometric object, which will be shown to generalize the notion of tropical variety, a ‘skeleton’.

**Definition 4.1.2.** Let \( S \) be a subset of \( \mathbb{H}(x_1, \ldots, x_n) \). Define the subset \( \text{Skel}(S) \) of \( \mathbb{H}^n \) to be

\[ \text{Skel}(S) = \{(a_1, \ldots, a_n) \in \mathbb{H}^n : f(a_1, \ldots, a_n) = 1, \ \forall f \in S\}. \]  

(4.1)

**Definition 4.1.3.** A subset \( S \) in \( \mathbb{H}^n \) is said to be a skeleton if there exists a subset \( S \subset \mathbb{H}(x_1, \ldots, x_n) \) such that \( S = \text{Skel}(S) \).
Definition 4.1.4. A skeleton $S$ in $\mathbb{H}^n$ is said to be a principal skeleton, if there exists $f \in \mathbb{H}(x_1, ..., x_n)$ such that $S = \text{Skel}\{f\}$.

Proposition 4.1.5. For $S_i \subset \mathbb{H}(x_1, ..., x_n)$ the following statements hold:

1. $S_1 \subseteq S_2 \Rightarrow \text{Skel}(S_2) \subseteq \text{Skel}(S_1)$.

2. $\text{Skel}(S_1) = \text{Skel}(\langle S_1 \rangle)$.

3. $\bigcap_{i \in I} \text{Skel}(S_i) = \text{Skel}(\bigcup_{i \in I} S_i)$ for any index set $I$ and in particular, $\text{Skel}(S) = \bigcap_{f \in S} \text{Skel}(f)$.

Proof. The first statement is set theoretically obvious and since $S_1 \subseteq \langle S_1 \rangle$, it implies that $\text{Skel}(\langle S_1 \rangle) \subseteq \text{Skel}(S_1)$ in the second assertion. To prove the opposite inclusion, we note that $(fg)(x) = f(x)g(x) = 1 \cdot 1 = 1$, $f^{-1}(x) = f(x)^{-1} = 1^{-1} = 1$ and $(f + g)(x) = f(x) + g(x) = 1 + 1 = 1$, for any $f, g$ such that $f(x) = g(x) = 1$ thus proving the skeleton of the semifield generated by $S_1$ is in $\text{Skel}(S_1)$. Now, we have to show that convexity is preserved. Since for $k_1, ..., k_t \in \mathbb{H}(x_1, ..., x_n)$ s.t. $\sum_{i=1}^t k_i = 1$ and for any $s_1, ..., s_i$ in the semifield generated by $S_1$ we have $(\sum_{i=1}^t k_i s_i)(x) = \sum_{i=1}^t k_i s_i(x) = \sum_{i=1}^t (k_i(x) \cdot 1) = \sum_{i=1}^t k_i(x) = 1(x) = 1$, we have that convexity holds. For the last assertion, by (1) we have that $\text{Skel}\bigcup_{i \in I} S_i \subseteq \text{Skel}(S_i)$ for each $i \in I$, thus $\text{Skel}\bigcup_{i \in I} S_i \subseteq \bigcap_{i \in I} \text{Skel}(S_i)$. The converse direction is obvious.

Remark 4.1.6. For $\mathbb{H}$ a bipotent semifield, the following statements hold for $f, g \in \mathbb{H}(x_1, ..., x_n)$:

1. $\text{Skel}(fg), \text{Skel}(f + g), \text{Skel}(f \land g) \supseteq \text{Skel}(f) \cap \text{Skel}(g)$.

2. $\text{Skel}(fg) = \text{Skel}(f + g) = \text{Skel}(f) \cap \text{Skel}(g)$ for all $f, g \geq 1$.

3. $\text{Skel}(f \land g) = \text{Skel}(f) \cup \text{Skel}(g)$ for all $f, g \leq 1$.

4. $\text{Skel}(f) = \text{Skel}(f^{-1}) = \text{Skel}(f + f^{-1}) = \text{Skel}(f \land f^{-1})$.

Proof. (1) If $x \in \text{Skel}(f) \cap \text{Skel}(g)$ then $f(x) = g(x) = 1$, so $f(x) + g(x) = f(x) \land g(x) = f(x)g(x) = 1$.

(2) We have that $(f + g)(x) = f(x) + g(x) = \sup(f(x), g(x)) = 1$. Since $f, g \geq 1$ we have that $\sup(f(x), g(x)) = 1$ if and only if $f(x) = 1$ and $g(x) = 1$. Analogously the same holds for $f(x)g(x) = 1$ when both $f, g \geq 1$.

(3) We have that $(f \land g)(x) = f(x) \land g(x) = \inf(f(x), g(x)) = 1$. Since $f, g \geq 1$
we have that inf\((f(x), g(x))\) = 1 if and only if \(f(x) = 1\) or \(g(x) = 1\).

(4) follows Proposition 4.1.5(2) since \(\langle f \rangle = \langle f^{-1} \rangle = \langle f + f^{-1} \rangle = \langle f \land f^{-1} \rangle\) (recalling that \(f \land f^{-1} = (f + f^{-1})^{-1}\)).

**Definition 4.1.7.** Denote the collection of skeletons in \(\mathbb{H}^n\) by \(Skl(\mathbb{H}^n)\) and the collection of principal skeletons in \(\mathbb{H}^n\) by \(PSkl(\mathbb{H}^n)\).

### 4.2 Kernels of skeletons

In the following discussion, we construct an operator \(Ker : \mathbb{H}^n \to \mathcal{P}(\mathbb{H}(x_1, ..., x_n))\) which associates a kernel of the semifield of fractions \(\mathbb{H}(x_1, ..., x_n)\) to any skeleton in \(Skl(\mathbb{H}^n)\). Then we will proceed to study the relation between this operator \(Ker\) and the operator \(Skel\) defined in the previous subsection.

Throughout this section, we take \(\mathbb{H}\) to be a bipotent (totally ordered) semifield.

**Definition 4.2.1.** Given a subset \(Z\) of \(\mathbb{H}^n\) define following subset of \(\mathbb{H}(x_1, ..., x_n)\):

\[
Ker(Z) = \{ f \in \mathbb{H}(x_1, ..., x_n) : f(a_1, ..., a_n) = 1, \forall (a_1, ..., a_n) \in Z \} \tag{4.2}
\]

**Remark 4.2.2.** For \(Z, Z_i \subset \mathbb{H}^n\) the following statements hold:

1. \(Ker(Z)\) is a kernel of \(\mathbb{H}(x_1, ..., x_n)\).
2. If \(Z_1 \subseteq Z_2\), then \(Ker(Z_2) \subseteq Ker(Z_1)\).
3. \(Ker(\bigcup_{i \in I} Z_i) = \bigcap_{i \in I} Ker(Z_i)\).
4. \(K \subseteq Ker(Skel(K))\) for any kernel \(K\) of \(\mathbb{H}(x_1, ..., x_n)\).
5. \(Z \subseteq Skel(Ker(Z))\).

**Proof.** The first assertion follows from the proof of (2) of Proposition 4.1.5. The second assertion is a trivial set theoretic fact, which in turn implies that \(Ker(\bigcup_{i \in I} Z_i) \subseteq \bigcap_{i \in I} Ker(Z_i)\). The second inclusion of (3) is trivial. Assertions (4) and (5) are straightforward consequences of the definitions of \(Skel\) and \(Ker\).
Proposition 4.2.3. 1. \(\text{Skel}(\text{Ker}(Z)) = Z\) if \(Z\) is a skeleton.

2. \(\text{Ker}(\text{Skel}(K)) = K\), if \(K = \text{Ker}(Z)\) for some \(Z \subseteq \mathbb{H}^n\).

Proof. In view of Proposition 4.1.5 (5), we need to show that \(\text{Skel}(\text{Ker}(Z)) \subseteq Z\).

Indeed, writing \(Z = \text{Skel}(S)\) we have by Proposition 4.1.5 that \(S \subset \text{Ker}(\text{Skel}(S))\) and thus

\[Z = \text{Skel}(S) \supseteq \text{Skel}(\text{Ker}(\text{Skel}(S))) = \text{Skel}(\text{Ker}(Z)).\]

For the second assertion, by Remark 4.2.2 (5), \(Z \subseteq \text{Skel}(\text{Ker}(Z))\), so

\[\text{Ker}(\text{Skel}(K)) = \text{Ker}(\text{Skel}(\text{Ker}(Z))) \subseteq \text{Ker}(Z) = K.\]

The reverse inclusion follows Remark 4.2.2 (4).

Definition 4.2.4. A \(K\)-kernel of \(\mathbb{H}(x_1,\ldots,x_n)\) is a kernel of the form \(\text{Ker}(Z)\), where \(Z\) is a suitable skeleton.

By the above assertions we have

Proposition 4.2.5. There is a \(1 : 1\) order reversing correspondence

\[
\{\text{skeletons of } \mathbb{H}^n\} \rightarrow \{K \text{-kernels of } \mathbb{H}(x_1,\ldots,x_n)\},
\]

(4.3)

given by \(Z \mapsto \text{Ker}(Z)\); the reverse map is given by \(K \mapsto \text{Skel}(K)\).

Proposition 4.2.6. Let \(E_1\) and \(E_2\) be kernels in \(\mathbb{H}(x_1,\ldots,x_n)\), and let \(S_1 = \text{Skel}(E_1)\) and \(S_2 = \text{Skel}(E_2)\) be their corresponding skeletons. Then the following statements hold:

\[\text{Skel}(E_1 \cdot E_2) = S_1 \cap S_2;\]  

(4.4)

\[S_1 \cup S_2 = \text{Skel}(E_1 \cap E_2).\]  

(4.5)

Proof. For the first assertion, denote \(K = E_1 \cdot E_2\). Since \(E_1\) and \(E_2\) are kernels, by Remark 4.2.2 (5), \(K\) also is a kernel and \(E_1, E_2 \subseteq K\). Thus, by Proposition 4.1.5 (1), \(\text{Skel}(K) \subseteq \text{Skel}(E_1)\) and \(\text{Skel}(K) \subseteq \text{Skel}(E_2)\), so \(\text{Skel}(K) \subseteq S_1 \cap S_2\).

Conversely, if \(x \in S_1 \cap S_2\) then \(\forall f \in E_1\) and \(\forall g \in E_2\), \(f(x) = g(x) = 1\) thus \((fg)(x) = f(x)g(x) = 1 \cdot 1 = 1\). Consequently,

\[S_1 \cap S_2 = \{x \in \mathbb{H}^n : f(x) = 1 \forall f \in E_1\} \cap \{x \in \mathbb{H}^n : g(x) = 1 \forall g \in E_2\}\]
\[ \{ x \in \mathbb{H}^n : f(x) = 1, g(x) = 1 \forall f \in E_1 \forall g \in E_2 \} \]
\[ \subseteq \{ x \in \mathbb{H}^n : (fg)(x) = f(x)g(x) = 1 \forall f \in E_1 \forall g \in E_2 \} \]
\[ = \{ x \in \mathbb{H}^n : h(x) = 1 \forall h \in K \} = \text{Skel}(K), \]

as desired.

For the second assertion, denote \( K = E_1 \cap E_2 \). Since \( K \subseteq E_1, E_2 \) we have by Proposition 4.1.5 that \( S_1 = \text{Skel}(E_1) \subseteq \text{Skel}(K) \) and \( S_2 = \text{Skel}(E_2) \subseteq \text{Skel}(K) \) thus \( S_1 \cup S_2 \subseteq \text{Skel}(K) \). Conversely, since \( E_1 \) and \( E_2 \) are kernels \( \langle E_1 \rangle = E_1 \) and \( \langle E_2 \rangle = E_2 \), thus Proposition 2.7.5 yields that \( K = \langle \{ |f| \land |g| : f \in E_1, g \in E_2 \} \rangle \) and so, by Proposition 4.1.5(2), \( \text{Skel}(K) = \text{Skel}(\{ |f| \land |g| : f \in E_1, g \in E_2 \}) \).

In view of the above we have that

\[ x \in \text{Skel}(K) \iff |f(x)| \land |g(x)| = 1 \forall f \in E_1, g \in E_2. \]

Now, let \( x \in \text{Skel}(K) \) and assume to the contrary that \( x \not\in \text{Skel}(E_1) \) and \( x \not\in \text{Skel}(E_2) \). Then there are \( f' \in E_1 \) and \( g' \in E_2 \) such that \( |f'(x)| > 1 \) and \( |g'(x)| > 1 \). Since \( \mathbb{H} \) is bipotent and \( |f'(x)|, |g'(x)| \in \mathbb{H} \) we have that \( (|f'| \land |g'|)(x) = |f'(x)| \land |g'(x)| = \min(|f'(x)|, |g'(x)|) > 1 \) which yields that \( x \not\in \text{Skel}(K) \). A contradiction.

As a special case of Proposition 4.2.6 we have

**Corollary 4.2.7.** For \( f, g \in \mathbb{H}(x_1, ..., x_n) \)

\[ \text{Skel}(\langle f \rangle \cdot \langle g \rangle) = \text{Skel}(f) \cap \text{Skel}(g); \quad (4.6) \]
\[ \text{Skel}(\langle f \rangle \cap \langle g \rangle) = \text{Skel}(f) \cup \text{Skel}(g). \quad (4.7) \]

Note that by convention \( \text{Skel}(f) = \text{Skel}(\langle f \rangle) \).
The structure of the semifield of fractions

5.1 Bounded rational functions in the semifield of fractions

Throughout this section we assume $\mathbb{H}$ to be a bipotent and divisible semifield. Any supplemental assumptions regarding $\mathbb{H}$ will be explicitly stated.

We begin by introducing an example which motivates our subsequent discussion in this section.

Example 5.1.1. Consider the principal kernel $\langle x \rangle$. Its corresponding skeleton is defined by the equation $x = 1$. Let $\alpha \in \mathbb{R}$ be such that $\alpha \neq 1$. The principal kernel $\langle x \rangle \cap \langle \alpha \rangle = \langle |x|\wedge|\alpha| \rangle$ where $|x|\wedge|\alpha| = \frac{|x||\alpha|}{|x|+|\alpha|} = \min(|x|,|\alpha|)$. Since $\min(|x|,|\alpha|) = 1 \iff |x| = 1$ we get that $\langle |x|\wedge|\alpha| \rangle$ defines exactly the same skeleton as $\langle x \rangle$. As $x \notin \langle x \rangle \cap \langle \alpha \rangle$ ($x$ cannot be bounded by a bounded function) we have that $\langle x \rangle \supset \langle x \rangle \cap \langle \alpha \rangle$.

The cause of this phenomenon is the kernels of the form $\langle \alpha \rangle$ with $\alpha \in \mathbb{R} \setminus \{1\}$. These kernels are not kernels of $\mathbb{R}$-homomorphisms, since for any such homomorphism $\phi$, one must have $\alpha = \alpha\phi(1) = \phi(\alpha)$ but as $\alpha \in \text{Ker}\phi$ we have that $\phi(\alpha) = 1$ too. Thus kernels containing them also fail to be kernels of $\mathbb{R}$-homomorphisms.

Definition 5.1.2. $f \in \mathbb{R}(x_1,\ldots,x_n)$ is said to be bounded from below if there exists some $\alpha > 1$ in $\mathbb{R}$ such that $|f| \geq \alpha$.

Remark 5.1.3. Let $\langle f \rangle$ be a principal kernel of $\mathbb{H}(x_1,\ldots,x_n)$. Then $f$ is bounded from below if and only if $\langle f \rangle \supseteq \langle \alpha \rangle$ for some $\alpha > 1$ in $\mathbb{H}$.

Proof. If $\langle f \rangle \supseteq \langle \alpha \rangle$ then in particular $\alpha \in \langle f \rangle$ thus there exists some $k \in \mathbb{N}$ such that $|\alpha| \leq |f|^k$ thus $\beta \leq |f|$ for $\beta \in \mathbb{H}$ such that $\beta^k = |\alpha|$ (such $\beta$ exists as $\mathbb{H}$ is divisibly closed). Thus $f$ is bounded from below. Conversely, if $f$ is bounded from below, then there exists some $\alpha > 1$ in $\mathbb{H}$ such that $|f| \geq \alpha = |\alpha|$ which yields that $\alpha \in \langle f \rangle$ and thus $\langle \alpha \rangle \subseteq \langle f \rangle$. 


Remark 5.1.4. Let \( \langle f \rangle \) be a principal kernel such that \( f \) is bounded from below. Then any generator \( g \in \langle f \rangle \) is bounded from below.

Proof. Since \( g \) is a generator, we in particular have that \( f \in \langle g \rangle \). Then, by Remark 2.3.16, there exists some \( k \in \mathbb{N} \) such that \( |f| \leq |g|^k \). Since \( f \) is bounded from below there exists some \( \alpha > 1 \) in \( \mathbb{H} \) such that \( |f| \geq \alpha \), thus \( |g|^k \geq |f| \geq \alpha \), which yields that \( g \) is bounded from below by \( \beta \in \mathbb{H} \) such that \( \beta^k = \alpha \) (and so \( \beta > 1 \)).

Definition 5.1.5. A principal kernel \( \langle f \rangle \) of \( \mathbb{H}(x_1, \ldots, x_n) \) is said to be bounded from below, if it is generated by a function bounded from below.

Let \( \mathbb{H} \) be a divisible bipotent semifield. We will begin by characterizing the principal kernels \( K = \langle f \rangle \subseteq \mathbb{H}(x_1, \ldots, x_n) \) for which \( \text{Skel}(f) = \emptyset \).

Remark 5.1.6. Let \( \langle f \rangle \) be a principal kernel of \( \mathbb{H}(x_1, \ldots, x_n) \). Then \( \text{Skel}(f) = \emptyset \) if and only if \( |f(x)| > 1 \) for every \( x \in \mathbb{H}^n \).

Proof. \( \text{Skel}(f) = \text{Skel}(|f|) = \{ x \in \mathbb{H}^n : h(x) = 1 \ \forall h \in \langle |f| \rangle \} \subseteq \{ x \in \mathbb{H}^n : |f(x)| = 1 \} = \emptyset \). Conversely, as the skeleton is determined by any generator of \( \langle f \rangle \), in particular \( |f| \), \( \text{Skel}(f) = \emptyset \) implies that there is no solutions to the equation \( |f(x)| = 1 \). Since \( |f(x)| \geq 1 \) for every \( x \in \mathbb{H}^n \), this implies that \( |f(x)| > 1 \) for every \( x \in \mathbb{H}^n \).

Proposition 5.1.7. Let \( \langle f \rangle \) be a principal kernel in \( \mathbb{H}(x_1, \ldots, x_n) \). Then the following statements are equivalent:

1. \( \text{Skel}(\langle f \rangle) = \emptyset \).

2. There exists a generator \( f' \) of \( \langle f \rangle \) such that \( f' = f' + \gamma \) with \( \gamma > 1 \).

3. There exists a generator of \( \langle f \rangle \) which is bounded from below.

Proof. (2) \( \iff \) (3) since \( f' = f' + \gamma \iff |f'| = f' \geq \gamma > 1 \).

We now prove (1) \( \iff \) (2). By Remark 5.1.6, \( \text{Skel}(\langle f \rangle) = \emptyset \) if and only if \( |f(x)| > 1 \) for every \( x \in \mathbb{H}^n \). We claim that \( |f(x)| > 1 \) for every \( x \in \mathbb{H}^n \) if and only if \( |f| = |f| + \gamma \) with \( \gamma > 1 \) or in other words, that \( |f| \geq \gamma \). If \( |f| = |f| + \gamma \), then \( |f(x)| > \gamma > 1 \) for every \( x \in \mathbb{H}^n \) and so \( \text{Skel}(f) = \emptyset \). Conversely, let \( |f| = \frac{h}{g} = \sum_{j=1}^{k} h_j g_j \) with \( h_i \) and \( g_j \) monomials in \( \mathbb{H}[x_1, \ldots, x_n] \). \( |f| : \mathbb{H}^n \to \mathbb{H} \) defines
a partition of $\mathbb{H}^n$ to a finite number of regions. Over each of these regions, $|f|$ has the form of a Laurent monomial $l = \frac{h_{i_0}}{g_{j_0}}$ where $h = h_{i_0}$ and $g = g_{j_0}$. Each region $R$ is either closed and bounded or unbounded. In the former case $|f|$, being continuous, attains a minimum value over $R$; thus the image of $|f|$ is bounded from below by some $\gamma > 1$. In the latter case, if $l$ is constant then it trivially attains a minimal value and is bounded from below by some $\gamma > 1$. The only possibility we are left with is that $l$ is not constant. Assume the image of $l$ is not bounded from below by any $\gamma > 1$. Then there exists a one dimensional curve over which $l$ is monotonically decreasing in a constant rate and thus must obtain the value $1$ contradicting the assumption that $|f(x)| > 1$ for all $x \in \mathbb{H}^n$. (It is convenient to consider $l$ in logarithmic scale so that $l$ takes the form of a linear form and the curve is a one dimensional affine space (a straight line) over which $l$ has a constant slope.)

**Corollary 5.1.8.** Let $\langle f \rangle$ be a principal kernel in $\mathbb{H}(x_1, \ldots, x_n)$. Then $\text{Skel}(\langle f \rangle) = \emptyset$ if and only if $\langle f \rangle$ is bounded from below.

**Proof.** If $\text{Skel}(\langle f \rangle) = \emptyset$ then by Proposition 5.1.7 $\langle f \rangle$ is generated by a bounded from below element, thus is a bounded from below kernel. Moreover, by Remark 5.1.4 we have that any generator of $\langle f \rangle$ is bounded from below. Conversely, if $\langle f \rangle$ is a bounded from below kernel, then $f'$ is bounded from below for any generator $f'$ of $\langle f \rangle$, in particular for $f' = f$. Thus there exists some $\alpha > 1$ in $\mathbb{H}$ such that $|f| \geq \alpha$, thus, by Remark 5.1.6, we have that $\text{Skel}(\langle f \rangle) = \emptyset$.

**Definition 5.1.9.** $f \in \mathbb{H}(x_1, \ldots, x_n)$ is said to be bounded from above (or simply bounded) if there exists some $\alpha \geq 1$ in $\mathbb{H}$ such that $|f| \leq \alpha$.

**Remark 5.1.10.** Let $\langle f \rangle$ be a principal kernel such that $f$ is bounded from above. Then any $g \in \langle f \rangle$ is bounded from above. In particular, any generator of $\langle f \rangle$ is bounded from above.

**Proof.** By Remark 2.3.16 for any $g \in \langle f \rangle$ there exists some $k \in \mathbb{N}$ such that $|g| \leq |f|^k$. Since $f$ is bounded there exists some $\alpha \geq 1$ in $\mathbb{H}$ such that $|f| \leq \alpha$, thus $|g| \leq |f|^k \leq \alpha^k$, which yields that $g$ is bounded.

**Definition 5.1.11.** A principal kernel $\langle f \rangle$ of $\mathbb{H}(x_1, \ldots, x_n)$ is said to be bounded from above if it is generated by a function bounded from above.

**Remark 5.1.12.** Let $\langle f \rangle$ be a principal kernel of $\mathbb{H}(x_1, \ldots, x_n)$. Then $f$ is bounded from above if and only if $\langle f \rangle \subseteq \langle \alpha \rangle$ for some $\alpha \in \mathbb{H}$.
Proof. If \( \langle f \rangle \subseteq \langle \alpha \rangle \) then in particular \( f \in \langle \alpha \rangle \). Thus there exists some \( k \in \mathbb{N} \) such that \( |f| \leq |\alpha|^k \), thus \( f \) is bounded from above. Conversely, if \( f \) is bounded from above, then there exists some \( \alpha \in \mathbb{H} ( \alpha \geq 1 \) such that \( |f| \leq \alpha = |\alpha| \) which yields that \( f \in \langle \alpha \rangle \) and thus \( \langle f \rangle \subseteq \langle \alpha \rangle \).

Remark 5.1.13. Let \( \langle f \rangle \) be a principal kernel of \( \mathbb{H}(x_1, ..., x_n) \). Then \( \langle f \rangle \) is bounded from below if and only if \( \mathbb{H} \subseteq \langle f \rangle \), if and only if there exists some \( \alpha > 1 \) in \( \mathbb{H} \) such that \( \alpha \in \langle f \rangle \).

Proof. If \( \langle f \rangle \) is bounded from below, then there exists some \( \alpha > 1 \) in \( \mathbb{H} \) such that \( |f| \geq \alpha \) thus by Remark 2.3.16 we have that \( \alpha \in \langle f \rangle \). Since any \( \alpha \neq 1 \) is a generator of \( \mathbb{H} \), we get that \( \mathbb{H} = \langle \alpha \rangle \subseteq \langle f \rangle \). Conversely, if \( \mathbb{H} \subseteq \langle f \rangle \) then in particular \( \alpha \in \langle f \rangle \) for any \( \alpha > 1 \) in \( \mathbb{H} \). If \( f \) is not bounded from below, then for any \( \beta > 1 \) there exists some \( a_\beta \in \mathbb{H} \) such that \( |f(a_\beta)| < \beta \). Now, as \( \alpha \in \langle f \rangle \), there exists some \( k \in \mathbb{N} \) such that \( \alpha \leq |f|^k \). Thus \( \beta \leq |f| \) for \( \beta \in \mathbb{H} \) such that \( \beta^k = \alpha \) (such \( \beta \) exists as \( \mathbb{H} \) is divisibly closed, and \( \beta > 1 \) since \( \alpha > 1 \)), a contradiction, since \( |f(a_\beta)| < \beta \).

Remark 5.1.14. Let \( \phi : \mathbb{H}(x_1, ..., x_n) \to \mathbb{H}(x_1, ..., x_n) \) be an \( \mathbb{H} \)-homomorphism. If \( f \in \mathbb{H}(x_1, ..., x_n) \) is bounded from below then \( \phi(f) \) is bounded from below, and if \( f \) is bounded from above then \( \phi(f) \) is bounded from above.

Proof. Let \( f \in \mathbb{H}(x_1, ..., x_n) \) be bounded from below. Then there exists some \( \alpha \geq 1 \) such that \( |f| + \alpha = |f| \). As \( |\phi(f)| = |\phi(|f| + \alpha)| = |\phi(|f|) + \phi(\alpha)| = |\phi(f)| + \alpha \), we have that \( \phi(f) \) is bounded from below.

Proposition 5.1.15. Let \( \phi : \mathbb{H}(x_1, ..., x_n) \to \mathbb{H}(x_1, ..., x_n) \) be an \( \mathbb{H} \)-homomorphism. Then \( \phi \) sends bounded from above principal kernels to bounded from above principal kernels and bounded from below principal kernels to bounded from below principal kernels.

Proof. The image of a kernel under a homomorphism is a kernel. Now, since \( \phi(\langle f \rangle) = \langle \phi(f) \rangle \) and since a principal kernel is bounded from above or bounded from below if and only if its generator is, the result follows from Remark 5.1.14.
Remark 5.1.16.

\[ \langle \mathbb{H} \rangle = \{ f \in \mathbb{H}(x_1, ..., x_n) : f \text{ is bounded from above } \} . \]

Proof. First note that for \( \alpha \in \mathbb{H} \) such that \( \alpha \neq 1 \), \( \alpha \) generates \( \mathbb{H} \) as a semifield and thus \( \langle \alpha \rangle = \langle \mathbb{H} \rangle \). The assertion follows from Remark 5.1.12 since \( f \in \langle \alpha \rangle = \langle \mathbb{H} \rangle \) if and only if \( \langle f \rangle \subseteq \langle \alpha \rangle \).

As we shall now show, there is a close connection between bounded from above and bounded from below kernels:

Let \( \langle f \rangle \in \text{PCon}(\mathbb{H}(x_1, ..., x_n)) \) be a principal kernel. By Theorem 2.2.51(2) we have that

\[ \langle f \rangle/(\langle f \rangle \cap \langle \mathbb{H} \rangle) \cong \langle f \rangle \cdot \langle \mathbb{H} \rangle/\langle \mathbb{H} \rangle. \]

In view of the preceding discussion in this section, \( \langle f \rangle \cdot \langle \mathbb{H} \rangle \) is the smallest bounded from below kernel containing \( \langle f \rangle \) while \( \langle f \rangle \cap \langle \mathbb{H} \rangle \) is the largest bounded from above kernel contained in \( \langle f \rangle \). One can consider the quotient \( \langle f \rangle/(\langle f \rangle \cap \langle \mathbb{H} \rangle) \) as a measure for “how far a kernel is from being bounded”.

The kernel \( \langle \mathcal{R} \rangle \) plays an important role in our theory. It actually contains all the kernels needed to form a correspondence between principal kernels and principal skeletons. In essence, every principal kernel \( \langle f \rangle \) in \( \text{PCon}(\mathcal{R}(x_1, ..., x_n)) \) has an bounded from above “copy” in the family of principal kernels contained in \( \langle \mathcal{R} \rangle \), which possesses the exact same skeleton. Our so called “Zariski correspondence” will take place between the principal kernels contained in \( \langle \mathcal{R} \rangle \) and the skeletons in \( \mathcal{R}^n \).

5.2 The structure of \( \langle \mathbb{H} \rangle \)

Let \( \mathbb{H} \) be a bipotent divisible archimedean semifield. In the following discussion, we study the structure of the subsemifield and kernel \( \langle \mathbb{H} \rangle \) of \( \mathbb{H}(x_1, ..., x_n) \),
which by Remark 2.3.25 is generated, as a kernel, by any \( \alpha \in \mathbb{H} \setminus \{1\} \). As we have shown in Remark 5.1.16, \( \langle \mathbb{H} \rangle \) is comprised of bounded from above elements of \( \mathbb{H}(x_1, \ldots, x_n) \). Note that \( \mathbb{H} \) is always assumed to be a bipotent divisible semi-field while any additional assumptions regarding \( \mathbb{H} \) will be stated when necessary.

**Proposition 5.2.1.** For any principal kernel \( (f) \in \text{PCon}(\langle \mathbb{H} \rangle) \) bounded from above, there exists an unbounded from above kernel \( (f') \in \text{PCon}(\mathbb{H}(x_1, \ldots, x_n)) \) such that

\[
\langle f \rangle = \langle f' \rangle \cap \langle \mathbb{H} \rangle.
\]

In particular, \( (f') \supset (f) \) and \( \text{Skel}(f') = \text{Skel}(f) \).

**Proof.** Let \( f(x_1, \ldots, x_n) \in \langle \mathbb{H} \rangle \) be bounded from above. Then there exists some \( \beta_1 \in \mathbb{H} \) such that \( |f(x_1, \ldots, x_n)| = |\alpha_1| \in \mathbb{H} \) for every \( |x_i| \geq \beta_1 \) for otherwise \( f \) will not be bounded from above since for every \( \gamma \in \mathbb{H} \) there exists some \( a = (a_1, \ldots, a_n) \in \mathbb{H}^n \) with \( |\alpha_1| > \beta(\gamma) \) such that \( |f(a, x_2, \ldots, x_n)| > \gamma \). Similarly for each \( 2 \leq i \leq n \) there exists some \( \beta_i \in \mathbb{H} \) such that \( |f(x_1, \ldots, x_n)| = |\alpha_i| \in \mathbb{H} \) for every \( |x_i| \geq \beta_i \). As \( |f| \) is continuous we have that \( \alpha_i = \alpha \) are all the same. Now define the following function

\[
f'(x_1, \ldots, x_n) = |\beta^{-1}|x_1| \wedge \ldots \wedge \beta^{-1}|x_n| + 1| + |f(x_1, \ldots, x_n)|
\]

where \( \beta = \sum_{i=1}^{n} |\beta_i| \). Write \( g(x_1, \ldots, x_n) = \beta^{-1}|x_1| + \ldots + \beta^{-1}|x_n| + 1 \). Let

\[
S = \{x = (x_1, \ldots, x_n) \in \mathbb{H}^n : |x_i| > |\beta_i| \forall i \}.
\]

Then for every \( a \in S \), \( f'(a) = g(a) + |\alpha| \). Moreover, for every \( b = (b_1, \ldots, b_n) \notin S \) there exists some \( j \) such that \( |b_j| \leq |\beta_j| \) thus we have that \( |\beta_j^{-1}|b_j| + \ldots + \beta^{-1}|b_n|| \leq 1 \) and so \( g(b) = 1 \). By construction \( \text{Skel}(f) \subseteq \text{Skel}(g) \), so \( \text{Skel}(f') = \text{Skel}([g] + |f|) = \text{Skel}(g) \cap \text{Skel}(f) = \text{Skel}(f) \). Finally as \( |g| \) is not bounded and \( |f'| = |g| + |f| \geq |g| \), we have that \( f' \) is not bounded. Now, as \( |f'| = |g| + |f| \) we have that \( |f| \leq |f'| \), so \( f \in \langle f' \rangle \). On the other hand, since \( f' \) is not bounded from above, Remark 5.1.10 implies that \( f' \notin \langle f \rangle \). Finally, \( g(a) \geq 1 \) for any \( a \in S \). Thus, \( f'(a) \wedge |\alpha| = (g(a) + |\alpha|) \wedge |\alpha| = |\alpha| \), while for \( a \notin S \) \( f'(a) \wedge |\alpha| = (g(a) + |f(a)|) \wedge |\alpha| = (1 + |f(a)|) \wedge |\alpha| = |f(a)| \wedge |\alpha| = |f(a)| \), since \( |f| \leq |\alpha| \). So we get that \( |f'| \wedge |\alpha| = |f| \) which means that \( \langle f \rangle = \langle f' \rangle \cap \langle \mathbb{H} \rangle \) (Note that \( f' = |f'| \) by definition, since \( f' \geq 1 \)). 

\[\square\]
Corollary 5.2.2. For every \( f \in \mathbb{H}(x_1, \ldots, x_n) \) there exists some \( f' \in \mathbb{H}(x_1, \ldots, x_n) \) such that \( f' \) is not bounded from above, \( \langle f \rangle \subseteq \langle f' \rangle \) and \( \text{Skel}(f') = \text{Skel}(f) \).

Proof. By Proposition 5.2.1 for every \( f \) bounded from above there exists such \( f' \). On the other hand, if \( f \) is not bounded from above, take \( f' = f \). \hfill \square

The following definition generalizes the notion of principal kernels bounded from above and from below introduced earlier.

Definition 5.2.3. Let \( \mathbb{H} \) be a bipotent semifield. A kernel \( K \in \text{Con}(\mathbb{H}(x_1, \ldots, x_n)) \) is said to be \textit{bounded} if \( K = K \cap \langle \mathbb{H} \rangle \). \( K \) is said to be \textit{bounded from below} if \( \langle \mathbb{H} \rangle \subseteq K \) or equivalently \( K = K \cdot \langle \mathbb{H} \rangle \).

By definition 5.2.3 every kernel \( K \subseteq \langle \mathbb{H} \rangle \) is a bounded kernel and vice versa.

Remark 5.2.4. Bounded kernels form a sublattice of \( (\text{PCon}(\mathbb{H}(x_1, \ldots, x_n)), \cdot, \cap) \) (with respect to multiplication and intersection) and principal bounded kernels form a sublattice of \( (\text{PCon}(\mathbb{H}(x_1, \ldots, x_n)), \cdot, \cap) \).

Proof. For any \( A, B \in \text{Con}(\mathbb{H}(x_1, \ldots, x_n)) \)

\[
(A \cap \langle \mathbb{H} \rangle) \cap (B \cap \langle \mathbb{H} \rangle) = (A \cap B) \cap \langle \mathbb{H} \rangle.
\]

By Lemma 2.2.47

\[
(A \cap \langle \mathbb{H} \rangle)(B \cap \langle \mathbb{H} \rangle) = A \cdot B \cap \langle \mathbb{H} \rangle.
\]

Thus \( (B, \cdot, \cap) \) is a sublattice of \( (\text{Con}(\mathbb{H}(x_1, \ldots, x_n)), \cdot, \cap) \).

Let \( \langle f \rangle \in \text{PCon}(\mathbb{H}(x_1, \ldots, x_n)) \). Then \( \langle f \rangle \cap \langle \mathbb{H} \rangle = \langle |f| + |\alpha| \rangle \) for any \( \alpha \in \mathbb{H} \setminus \{1\} \), thus \( \langle f \rangle \cap \langle \mathbb{H} \rangle \) is a bounded principal kernel. \hfill \square

Remark 5.2.5. For any \( f \in \mathbb{H}(x_1, \ldots, x_n) \) and \( \alpha \in \mathbb{H} \setminus \{1\} \)

\[
|f| \wedge |\alpha| = 1 \iff f = 1.
\]

Proof. \( (\Leftarrow) \) is obvious.

\( (\Rightarrow) \) is true since for any \( a \in \mathbb{H}^n \) \( |f(a)| \wedge |\alpha| = 1 \) if and only if \( |f(a)| = 1 \). \hfill \square

Reminder 5.2.6. Let \( \mathbb{S} \) be a semifield. A kernel \( K \) of a semifield \( \mathbb{S} \) is said to be \textit{large} in \( \mathbb{S} \) if \( L \cap K \neq \{1\} \) for each kernel \( L \neq \{1\} \) of \( \mathbb{S} \).
Corollary 5.2.7. Let $\langle f \rangle \in \text{PCon}(\mathbb{H}(x_1, \ldots, x_n))$. Then

$$\langle f \rangle \cap \langle \mathbb{H} \rangle = \{1\} \iff \langle f \rangle = \{1\}.$$ 

Consequently,

$$K \cap \langle \mathbb{H} \rangle = \{1\} \iff K = \{1\}$$

for any $K \in \text{Con}(\mathbb{H}(x_1, \ldots, x_n))$. Thus $\langle \mathbb{H} \rangle$ is large kernel in $\mathbb{H}(x_1, \ldots, x_n)$.

Proof. The first statement follows from the equality $\langle f \rangle \cap \langle \mathbb{H} \rangle = \langle |f| \wedge |\alpha| \rangle$ and Remark 5.2.5. For the second statement, if $K \neq \{1\}$ then taking $f \in K \setminus \{1\}$ we have

$$K \cap \langle \mathbb{H} \rangle \supseteq \langle f \rangle \cap \langle \mathbb{H} \rangle \neq \{1\}.$$ 

Remark 5.2.8. The following hold for the kernels of the semifield $\langle \mathbb{H} \rangle$:

$$\text{Con}(\langle \mathbb{H} \rangle) = \{K \cap \langle \mathbb{H} \rangle : K \in \text{Con}(\mathbb{H}(x_1, \ldots, x_n))\},$$

and

$$\text{PCon}(\langle \mathbb{H} \rangle) = \{\langle f \rangle \cap \langle \mathbb{H} \rangle : \langle f \rangle \in \text{PCon}(\mathbb{H}(x_1, \ldots, x_n))\}.$$ 

Moreover, every kernel of $\langle \mathbb{H} \rangle$ is also a kernel of $\mathbb{H}(x_1, \ldots, x_n)$.

Proof. Since $\langle \mathbb{H} \rangle$ is a subsemifield of $\mathbb{H}(x_1, \ldots, x_n)$ by Theorem 2.2.51(1) for any kernel $K$ of $\mathbb{H}(x_1, \ldots, x_n)$, $K \cap \langle \mathbb{H} \rangle$ is a kernel of $\langle \mathbb{H} \rangle$, i.e., $K \cap \langle \mathbb{H} \rangle \in \text{Con}(\langle \mathbb{H} \rangle)$. Conversely, as $\mathbb{H}(x_1, \ldots, x_n)$ is idempotent and commutative, by Remark 3.2.3 we have that any kernel $L$ of $\langle \mathbb{H} \rangle$ is a kernel of $\mathbb{H}(x_1, \ldots, x_n)$. Moreover, since $L \subseteq \langle \mathbb{H} \rangle$ we have that $L = L \cap \langle \mathbb{H} \rangle$. The second equality holds since for any principal kernels $\langle f \rangle \in \text{PCon}(\mathbb{H}(x_1, \ldots, x_n))$, $\langle f \rangle \cap \langle \mathbb{H} \rangle = \langle |f| \wedge |\alpha| \rangle \in \text{PCon}(\langle \mathbb{H} \rangle).$ 

\[\Box\]
6 The polar-skeleton correspondence

In subsection 4.2 we introduced the notion of a $K$-kernels of $\mathbb{H}(x_1,\ldots,x_n)$ whom have been shown to correspond to skeletons in $\mathbb{H}^n$ (Proposition 4.2.5). Our next aim is to characterize these special kind of kernels. In analogy to algebraic geometry, we are looking for our ‘radical ideals’ which are shown to correspond to algebraic sets (zero sets) by the Nullstellensatz theorem. Unfortunately in our context ‘radicality’ does help much since by Remark 2.3.7 every kernel is power-radical. It actually turns out that in our theory, the role of radical ideals is played by polars, a notion originating from the theory of lattice ordered groups. One can think of the theory introduced in this section as the analogue to the celebrated Nullstellensatz theorem.

We begin our discussion here considering general idempotent semifields. We will later restrict ourselves to the designated semifield of fractions $\mathbb{H}(x_1,\ldots,x_n)$ with $\mathbb{H}$ a bipotent semifield.

In the following section, $\mathbb{S}$ is always assumed to be an idempotent semifield.

6.1 Polars

In this subsection we introduce the notion of a polar, borrowed from the theory of lattice ordered groups ([8, section (2.2)]). We will see that polars are a special kind of kernels and use them to construct the so called $K$-kernels introduced in the previous section.

Translating Proposition 3.1.9 to the language of idempotent semifields we have:

**Proposition 6.1.1.** The lattice of kernels $\text{Con}(\mathbb{S})$ of an idempotent semifield $\mathbb{S}$ is complete distributive and it satisfies the infinite distributive law:

$$A \cap \left( \prod_i B_i \right) = \prod_i (A \cap B_i)$$

(6.1)

for any $A, B_i \in \text{Con}(\mathbb{S})$.

**Proof.** Viewing $\mathbb{S}$ as an $\ell$-group, since $\mathbb{S}$ is commutative every convex $\ell$-subgroup is normal and thus a kernel. Thus Proposition 3.1.9 applies to $\text{Con}(\mathbb{S})$. \qed
Definition 6.1.2. Let $A$ be a subset of $\mathcal{S}$. The polar of $A$ is the set

$$A^\perp = \{ x \in \mathcal{S} : |x| \wedge |a| = 1 ; \forall a \in A \}. \quad (6.2)$$

For $a \in \mathcal{S}$ we shortly write $a^\perp$ for $\{a\}^\perp$. The set of all polars in $\mathcal{S}$ is denoted by $\mathcal{B}(\mathcal{S})$.

Remark 6.1.3. The following statements are direct consequences of Definition 6.1.2. For any $K, L \subseteq \mathcal{S}$

1. $K \subseteq L \Rightarrow K^\perp \supseteq L^\perp$.
2. $K \subseteq K^\perp^\perp$.
3. $K^\perp = K^\perp^\perp^\perp$.

The following definition generalizes Definition 3.3.1:

Definition 6.1.4. Let $K$ be a kernel of an idempotent semifield $\mathcal{S}$. The positive cone of $K$ is the set

$$K^+ = \{ k \in K : k \geq 1 \}.$$ 

In particular the positive cone of $\mathcal{S}$ is $\mathcal{S}^+ = \{ h \in \mathcal{S} : h \geq 1 \}$.

Remark 6.1.5. For a family of groups $\{G_i : i \in I\}$, with an arbitrary index set $I$

$$\prod_{i \in I} G_i = \{ g_{i_1} \cdots g_{i_k} : g_{i_j} \in G_{i_j}, k \in \mathbb{N} \}.$$ 

Let $G$ be a group. A subgroup $H \leq G$ is said to be generated by a family of subgroups $\{G_i : i \in I\}$ of $G$, with an arbitrary index set $I$, if

$$H = \prod_{i \in I} G_i.$$ 

In particular for every $h \in H$ there are some $i_1, ..., i_k$ such that $h = g_{i_1} \cdots g_{i_k}$ where $g_{i_1} \in G_{i_1}, ..., g_{i_k} \in G_{i_k}$.

Proposition 6.1.6. [8, Theorem (2.2.4)(c)] The subgroup $K$ of an idempotent semifield $\mathcal{H}$ generated by a family of kernels $\{K_i : i \in I\}$ (where $I$ is an arbitrary index set) is a kernel, and its positive cone $K^+ = \{ k \in K : k \geq 1 \}$ is the subsemigroup of $\mathcal{H}^+$ generated by the corresponding family of positive cones.
Corollary 6.1.7. For any kernel $K$ of an idempotent semifield $\mathbb{H}$

$$K = \prod_{f \in K} \langle f \rangle = \bigcup_{f \in K} \langle f \rangle.$$ 

Proof. For $a \in \prod_{f \in K} \langle f \rangle$, there are some $f_1, \ldots, f_k \in K$ such that $a = g_1 \cdot \ldots \cdot g_k$ where $g_1 \in \langle f_1 \rangle, \ldots, g_k \in \langle f_k \rangle$. Since $K$ is a kernel, for any $f \in K$, $\langle f \rangle \subseteq K$ so $a = g_1 \cdot \ldots \cdot g_k \in K$. So $K \supseteq \prod_{f \in K} \langle f \rangle$. The opposite inclusion is obvious as each $f \in K$ is by definition in $\prod_{f \in K} \langle f \rangle$. Since the kernel generated by a kernel $K$ (as a set) is the kernel $K$ itself, the last equality holds.

Remark 6.1.8. Let $\mathbb{H}$ be an idempotent semifield and let $g \in \mathbb{H}$. Then $K \cap \langle g \rangle = \{1\}$ if and only if $\langle f \rangle \cap \langle g \rangle = \{1\}$ for every $f \in K$.

Proof. If $K = \bigcup_{f \in K} \{f\}$ then $K = \bigoplus_{f \in K} \langle f \rangle$. Since the lattice of kernels of $\mathbb{S}$ admit the infinite distributive law [6.1] we have that

$$K \cap \langle g \rangle = \left( \bigoplus_{f \in K} \langle f \rangle \right) \cap \langle g \rangle = \prod_{f \in K} (\langle f \rangle \cap \langle g \rangle)$$

from which the statement follows at once.

Remark 6.1.9. Let $K$ be a kernel of $\mathbb{S}$. The polar of $K$ is the set

$$K^\perp = \{ x \in \mathbb{S} : \langle x \rangle \cap K = \{1\} \}. \quad (6.3)$$

Proof. For any $a, b \in \mathbb{S}$ since $\langle a \rangle \cap \langle b \rangle = \langle |a| \land |b| \rangle$, we have that $|a| \land |b| = 1 \iff \langle a \rangle \cap \langle b \rangle = \{1\}$. Moreover, for any kernel $K$ and any $x \in \mathbb{S}$, $K \cap \langle x \rangle = \{1\}$ if and only if $\langle a \rangle \cap \langle x \rangle = \{1\}$ for all $a \in K$.

Theorem 6.1.10. 1. For any subset $A$ of $\mathbb{S}$, $A^\perp$ is a kernel of $\mathbb{S}$.

2. If $L \in \text{Con}(\mathbb{S})$, then for any $K \in \text{Con}(\mathbb{S})$

$$L \cap K = \{1\} \iff K \subseteq L^\perp.$$ 

3. $(\mathbb{S}, \cdot, \lor, \perp, \{1\}, \mathbb{S})$ is a complete Boolean algebra.
Proof. 1. By Theorem (2.2.4)(e) in [8], $A^\perp$ is a convex $\ell$-subgroup. Since $S$ is also commutative, we have that $A^\perp$ is normal and thus a kernel of $S$.

2. See Theorem (2.2.4)(d) in [8].

3. See [8], Theorem (2.2.5). If $\{A_i\}$ is a collection of subsets of $S$ over some index set, then

$$\left(\bigcup_i A_i\right)^\perp = \bigcap_i A_i^\perp.$$ 

Closure under negation is a consequence of (2). \qed

Remark 6.1.11. [8] Let $A$ be a subset of the $S$. Then

$$A^\perp = G(A)^\perp = SF(A)^\perp = \langle A \rangle^\perp$$

where $G(A)$ and $SF(A)$ are the subgroup and subsemifield generated by $A$ in $S$.

Remark 6.1.12. For any polar $P$, $P^\perp\perp = P$.

Indeed, $P$ is a polar then $P = V^\perp$ for some $V \subseteq S$. So $P^\perp\perp = (V^\perp)^{\perp\perp} = V^\perp = P$.

Proposition 6.1.13. For any subset $S \subseteq S$, $S^\perp\perp$ is the minimal polar containing $S$.

Proof. By the definition of a polar, $S^\perp\perp$ is a polar of $S^\perp$ and $S \subseteq S^\perp\perp$. Let $P$ be a polar such that $S \subseteq P$. Then since polar is inclusion reversing and $S \in P$ we have that $S^\perp \supseteq P^\perp$ and so $S^\perp\perp = (S^\perp)^\perp \subseteq (P^\perp)^\perp$. As $P$ is a polar we have by Remark 6.1.12 that $(P^\perp)^\perp = P$ and thus $S^\perp\perp \subseteq P$. \qed

Definition 6.1.14. Let $S \subseteq S$. We say that a polar $P$ is generated by $S$ if $P = S^\perp\perp$. If $S = \{a\}$ then we also write $a^\perp\perp$ for the polar generated by $\{a\}$.

Remark 6.1.15. The following statements hold:

1. A polar $B$ is generated by itself.

2. For any subset $A \subseteq S$

$$\langle A \rangle^\perp\perp = A^\perp\perp = G(A)^\perp\perp = SF(A)^\perp\perp$$

where $G(A)$ and $SF(A)$ are the subgroup and subsemifield generated by $A$ in $S$. 

72
Proof. The first assertion follows directly from Remark 6.1.12 while the second one is a direct consequence of Remark 6.1.11.

Definition 6.1.16. A polar $P$ of $S$ is said to be principal if there exists some $a \in S$ such that $P = a^\perp\perp$, i.e., $P$ is the polar generated by $a$. We denote the collection of principal polars of a semifield $S$ by $\mathcal{PB}(S)$.

Proposition 6.1.17. \cite[Section (2.2) q-13]{source}
The function $\phi : |a| \mapsto |a|^{\perp\perp}$ is a lattice homomorphism from $S^+$ to $\mathcal{B}(S)$. Namely, for any $a, b \in S$

\[ (|a| \wedge |b|)^{\perp\perp} = \phi(|a| \wedge |b|) = \phi(|a|) \cap \phi(|b|) = |a|^{\perp\perp} \cap |b|^{\perp\perp} \]

and

\[ (|a| + |b|)^{\perp\perp} = \phi(|a| + |b|) = \phi(|a|) \cdot \phi(|b|) = |a|^{\perp\perp} \cdot |b|^{\perp\perp}. \]

Proof. Indeed, by Theorem 6.1.10 and Remark 6.1.11 the following holds:

\[ (|a| \wedge |b|)^{\perp\perp} = (\langle |a| \wedge |b| \rangle)^{\perp\perp} = (\langle a \rangle \cap \langle b \rangle)^{\perp\perp} = (\langle a \rangle \perp \langle b \rangle)^{\perp} = \langle a \rangle^{\perp\perp} \cap \langle b \rangle^{\perp\perp} = |a|^{\perp\perp} \cap |b|^{\perp\perp} \]

The second assertion is proved analogously.

Remark 6.1.18. $(\mathcal{PB}(S), \cdot, \cap)$ is a sublattice of $(\mathcal{B}(S), \cdot, \cap)$.

Proof. Since for any $a, b \in S$ \((|a| \wedge |b|)^{\perp\perp} = (\langle |a| \wedge |b| \rangle)^{\perp\perp} = (\langle a \rangle \cap \langle b \rangle)^{\perp\perp}, (|a| + |b|)^{\perp\perp} = (\langle |a| + |b| \rangle)^{\perp\perp} = (\langle a \rangle \cdot \langle b \rangle)^{\perp\perp}\) and since $|a|^{\perp\perp} = a^{\perp\perp}$ we have that

\[ (\langle a \rangle \cap \langle b \rangle)^{\perp\perp} = a^{\perp\perp} \cap b^{\perp\perp} \]

and

\[ (\langle a \rangle \cdot \langle b \rangle)^{\perp\perp} = a^{\perp\perp} \cdot b^{\perp\perp}. \]

Corollary 6.1.19. $\mathcal{PB}(\mathbb{H}(x_1, ..., x_n)), \cdot, \cap)$ is a sublattice of $(\mathcal{B}(\mathbb{H}(x_1, ..., x_n)), \cdot, \cap)$ having $\mathbb{H}(x_1, ..., x_n)$ and $\{1\}$ as its maximal and minimal elements respectively.

Proof. $\mathbb{H}(x_1, ..., x_n) = \alpha^{\perp\perp}$ for any $\alpha \in \mathbb{H} \setminus \{1\}$ since $|f| \wedge |\alpha| = 1$ if and only if $f = 1$ and $1^{\perp} = \mathbb{H}(x_1, ..., x_n)$. $\{1\} = 1^{\perp\perp} = 1^{\perp\perp} = \mathbb{H}(x_1, ..., x_n) \subseteq \alpha^{\perp\perp} \subseteq \{1\}$. Thus $\{1\}, \mathbb{H}(x_1, ..., x_n) \in \mathcal{PB}(\mathbb{H}(x_1, ..., x_n))$. The rest of the assertion is a special case of Remark 6.1.18.
Proposition 6.1.20. [A section (2.2)q-13] The following statements hold for any idempotent semifield $S$:

1. If $\{B_i : i \in I\} \subseteq \mathcal{B}(S)$ then $\prod_i B_i = (\bigcup_i B_i)_{-1}^½$.

2. If $B \in \mathcal{B}(S)$ then $B = \prod_{b \in B} b_{-1}^½$.

3. For any $a, b \in S$ $(|a| \cdot |b|)_{-1}^½ = (|a| + |b|)_{-1}^½$.

Recall that a kernel $K$ of an semifield $S$ is a large kernel if $K \cap L \neq \{1\}$ for each kernel $L$ of $S$.

**Remark 6.1.21.** Let $K$ be a kernel of an idempotent semifield $S$. Then $K$ is large as a subkernel of $K_{-1}^½$.

**Proof.** First $K \subset K_{-1}^½$ and $K_{-1}^½ \in \text{Con}(S)$ thus $K \in \text{Con}(K_{-1}^½)$. If $L \in \text{Con}(K_{-1}^½)$ then in particular $L \in \text{Con}(S)$ (since $\text{Con}(K_{-1}^½) \subseteq \text{Con}(S)$). If $K \cap L = \{1\}$ then $L \subseteq K^½$ but also $L \subseteq K_{-1}^½$, thus $L \cap L = \{1\}$ which yields that $L = \{1\}$. So, $K \cap L = \{1\} \Rightarrow L = \{1\}$ for any $L \in \text{Con}(K_{-1}^½)$ and $K$ is a large kernel in $K_{-1}^½$. \qed

**Theorem 6.1.22.** [A Theorem (2.3.7)] Consider the following conditions on the kernel $K$ of the idempotent semifield $S$:

1. $K$ is a polar.

2. $K$ is completely closed in $S$.

Then (1) implies (2), and, if $S$ is complete, (2) implies (1).
6.2 The polar-skeleton correspondence

Recall $\mathcal{R}$ is our designated semifield defined to be bipotent, divisible, archimedean and complete. Also recall that the semifield of fractions $\mathcal{R}(x_1, ..., x_n)$ is considered as mapped to the semiring of functions $Fun(\mathcal{R}^n, \mathcal{R})$ as defined in 2.2.33.

Recall that since $\mathcal{R}$ is idempotent and archimedean semifield $Fun(\mathcal{R}^n, \mathcal{R})$ is an idempotent and archimedean semifield and so is $\mathcal{R}(x_1, ..., x_n)$ (cf. Proposition 3.4.1). Moreover, $Fun(\mathcal{R}^n, \mathcal{R})$ is also complete since $\mathcal{R}$ is complete (cf. Remark 3.4.4).

By Theorem 3.4.6, $\mathcal{R}(x_1, ..., x_n)$ has a unique completion to a complete archimedean idempotent semifield $\overline{\mathcal{R}}(x_1, ..., x_n)$ in $Fun(\mathcal{R}^n, \mathcal{R})$. By Theorem 3.4.9 $\mathcal{R}(x_1, ..., x_n)$ is dense in $\overline{\mathcal{R}}(x_1, ..., x_n)$.

In this section we concentrate our attention to $\mathcal{R}(x_1, ..., x_n)$. Doing so, we consider the natural extensions to $\mathcal{R}(x_1, ..., x_n)$ of the operators $Skel$ and $Ker$ defined in Subsection 4.2 with respect to $\mathcal{R}(x_1, ..., x_n)$. We write $Ker_{\mathcal{R}}(x_1, ..., x_n)$ to denote the restriction of $Ker$ to $\mathcal{R}(x_1, ..., x_n)$.

**Proposition 6.2.1.** Let $K$ be a kernel of $\mathcal{R}(x_1, ..., x_n)$. If $\langle g \rangle \cap K = \{1\}$ for some $g \in \mathcal{R}(x_1, ..., x_n)$ then

$$Skel(g) \cup Skel(K) = \mathcal{R}^n.$$ 

**Proof.** If $K \cap \langle g \rangle = \{1\}$ then $\langle f \rangle \cap \langle g \rangle = \{1\}$ for every $f \in K$. Thus as $Skel(K) = \bigcap_{f \in K} Skel(f)$ we have that $Skel(K) \cup Skel(g) = (\bigcap_{f \in K} Skel(f)) \cup Skel(g)$$

$= \bigcap_{f \in K} (Skel(f) \cup Skel(g)) = \bigcap_{f \in K} \mathcal{R}^n = \mathcal{R}^n$. Note that infinite distributive laws hold for sets with respect to intersections and unions.

Note that the same arguments hold taking $\overline{\mathcal{R}}(x_1, ..., x_n)$ instead of $\mathcal{R}(x_1, ..., x_n)$.

**Remark 6.2.2.** As any kernel $K \in Con(\mathcal{R}(x_1, ..., x_n))$ is a bipotent archimedean semifield in its own right, it has a completion too, which we denote by $\bar{K}$ and $K \subset \bar{\mathcal{R}}(x_1, ..., x_n)$. By Corollary 3.4.10 $\bar{K}$ is a kernel of $\bar{\mathcal{R}}(x_1, ..., x_n)$.
The question arises which kernels of $R(x_1, \ldots, x_n)$ are completions of kernels of $\mathcal{R}(x_1, \ldots, x_n)$. The answer to this question is all kernels of $\mathcal{R}(x_1, \ldots, x_n)$ that are completely closed. Indeed, $\mathcal{R}(x_1, \ldots, x_n)$ is a subsemifield of $\mathcal{R}(x_1, \ldots, x_n)$, so by Theorem 2.2.51(1) we have that

$$\text{Con}(\mathcal{R}(x_1, \ldots, x_n)) = \{ K \cap \mathcal{R}(x_1, \ldots, x_n) : K \in \text{Con}(\mathcal{R}(x_1, \ldots, x_n)) \}.$$ 

Since $\mathcal{R}(x_1, \ldots, x_n)$ is dense in $\mathcal{R}(x_1, \ldots, x_n)$ for every kernel $K$ of $\mathcal{R}(x_1, \ldots, x_n)$, the kernel $L = K \cap \mathcal{R}(x_1, \ldots, x_n)$ of $\mathcal{R}(x_1, \ldots, x_n)$ is dense in $K$. Thus $\bar{L} \supseteq K$ where $\bar{L}$ is the completion of $L$ and so, since $L \subseteq K$, we have that $\bar{L} = K$, i.e., $\bar{L} = K$ if and only if $K$ is completely closed.

**Corollary 6.2.3.** By Remark 6.2.2, each completely closed kernel $K$ of $\mathbb{H}(x_1, \ldots, x_n)$ defines a unique kernel of $\mathbb{H}(x_1, \ldots, x_n)$ given by $L = K \cap \mathbb{H}(x_1, \ldots, x_n)$ for which $\bar{L} = K$.

**Example 6.2.4.** Consider the kernel $K = \langle |x| \wedge |\alpha| \rangle \in \text{PCon}(\mathcal{R}(x))$ and the sub-
set $X = \{ |x| \wedge \alpha^n : n \in \mathbb{N} \}$. Then since $X \subseteq K$ one has that $|x| = \bigvee_{f \in X} f \in \overline{K}$, thus $\langle x \rangle \subseteq \overline{K}$ which yields that $\langle |x| \wedge |\alpha| \rangle = \langle |x| \rangle$.

**Remark 6.2.5.** For every $K \in \mathcal{R}(x_1, \ldots, x_n)$

$$\text{Skel}(\bar{K}) = \text{Skel}(K)$$

where $\bar{K}$ is the completion of $K$ in $\mathcal{R}(x_1, \ldots, x_n)$.

**Proof.** First note that since $K \subseteq \bar{K}$ in $\mathcal{R}(x_1, \ldots, x_n)$ we have that $\text{Skel}(\bar{K}) \subseteq \text{Skel}(K)$. Now, let $Z = \text{Skel}(K)$ and let $S$ be any nonempty subset of $K$. If $\bigvee_{s \in S} s \in \mathcal{R}(x_1, \ldots, x_n)$ then for any $a \in Z$ $\langle \bigvee_{s \in S} s(a) = \bigvee_{s \in S} 1 = 1 \rangle$, yielding that $\text{Skel}(\bigvee_{s \in S} s) \supseteq \bigvee_{s \in S} 1 = 1$, yielding that $\text{Skel}(\bigvee_{s \in S} s) \supseteq Z$ (cf. Remark 3.4.4). Similarly, if $\bigwedge_{s \in S} s \in \mathcal{R}(x_1, \ldots, x_n)$ then for any $a \in Z$ we have that $\langle \bigwedge_{s \in S} s(a) = \bigwedge_{s \in S} 1 = 1 \rangle$, yielding that $\text{Skel}(\bigwedge_{s \in S} s) \supseteq Z$. Thus as none of its supplementary elements reduces the size of $Z$, we have that $\text{Skel}(\bar{K}) = \text{Skel}(K)$.

**Note 6.2.6.** Let $S$ be an idempotent semifield and let $H$ be a subset of $S$. For a subset $A$ of $S$ let

$$A^H = \{ h \in H : |h| \wedge |a| = 1 \ \forall a \in A \},$$

76
then $A^H = A^S \cap H$.

We only write $A^H$ when $H$ is a proper subset of the semifield in which $A$ is considered. For example if $K \in \text{Con}(\mathcal{R}(x_1,\ldots,x_n))$ then $K^H = K^H(x_1,\ldots,x_n)$.

**Definition 6.2.7.** Let $K$ be a kernel of $\mathcal{R}(x_1,\ldots,x_n)$. Then $K$ is said to be a $K$-kernel if

$$K = \text{Ker}(\text{Skel}(K)).$$

In other words, $K$ is a preimage of its skeleton with respect to the map

$$\text{Ker}: \mathbb{P}(\mathbb{H}^n) \to \text{Con}(\overline{\mathcal{R}(x_1,\ldots,x_n)})$$

where $\mathbb{P}(X)$ is the powerset of the set $X$.

**Proposition 6.2.8.** A polar in $\mathcal{R}(\mathcal{R}(x_1,\ldots,x_n))$ is a $K$-kernel.

**Proof.** Let $K = V^H$ for some $V \subset \mathcal{R}(x_1,\ldots,x_n)$ and let $g \in \mathcal{R}(x_1,\ldots,x_n)$ such that $\text{Skel}(g) \supseteq \text{Skel}(K)$. Assume $g \not\in K$, then there exists some $v \in V$ such that $|g| \wedge |v| \neq 1$ so $\text{Skel}(g) \cup \text{Skel}(v) = \text{Skel}((g \cap \{v\}) = \text{Skel}(|g| \wedge |v|) \neq \mathbb{H}^n$. Since $K = V^H$, by definition $K \cap \{v\} = \{1\}$, so, Proposition 6.2.1 implies that $\text{Skel}(K) \cup \text{Skel}(v) = \mathbb{H}^n$. But $\text{Skel}(g) \supseteq \text{Skel}(K)$ implies that $\text{Skel}(g) \cup \text{Skel}(v) \supseteq \text{Skel}(K) \cup \text{Skel}(v) = \mathbb{H}^n$. A contradiction. Thus $g \in K$. \hfill \Box

**Proposition 6.2.9.** Every $K$-kernel of $\mathcal{R}(x_1,\ldots,x_n)$ is completely closed.

**Proof.** By Proposition 4.2.3 if $Z = \text{Skel}(K)$ then $\text{Skel}(\text{Ker}(Z)) = Z$. Let $Z = \text{Skel}(K) \subseteq \mathcal{R}^n$ where $K = \text{Ker}(Z) \in \text{Con}(\mathcal{R}(x_1,\ldots,x_n))$. Let $S \subseteq K$ be any nonempty subset of $K$ then for any $s \in S$ we have that $\text{Skel}(s) \supseteq Z$. Now, if $\bigvee_{s \in S} s \in \mathcal{R}(x_1,\ldots,x_n)$ and $\bigwedge_{s \in S} s \in \mathcal{R}(x_1,\ldots,x_n)$, then by Remark 3.4.3 for any $a \in Z$

$$\bigvee_{s \in S} s(a) = \bigvee_{s \in S} s = \bigvee_{s \in S} 1 = 1 \quad \text{and} \quad \bigwedge_{s \in S} s(a) = \bigwedge_{s \in S} s = \bigwedge_{s \in S} 1 = 1.$$

Thus

$$\text{Skel}(\bigvee_{s \in S} s), \text{Skel}(\bigwedge_{s \in S} s) \supseteq Z.$$

So $\bigvee_{s \in S} s, \bigwedge_{s \in S} s \in K$ and $K$ is completely closed in $\mathcal{R}(x_1,\ldots,x_n)$. \hfill \Box

**Remark 6.2.10.** Proposition 6.2.9 is not true when considering $K$-kernels of $\mathcal{R}(x_1,\ldots,x_n)$ instead of $\mathcal{R}(x_1,\ldots,x_n)$. Let $\alpha \in \mathcal{R}$ such that $\alpha > 1$. Consider
the subset

\[ X = \{|x^n| \land \alpha : n \in \mathbb{N}\}, \]

then \( X \subseteq \langle x \rangle \). \( \forall f \in X \ f = \alpha \) (where \( \alpha \) is the constant function) and \( \text{Skel}(f) = \{1\} \subseteq \mathcal{R} \) for every \( f \in X \). Thus \( \alpha = (\alpha)(1) = (\bigvee_{f \in X} f)(1) \neq \bigvee_{f \in X} f(1) = 1 \) and \( \alpha \) is not in the preimage of \( \text{Skel}(x) \) (see figure 6.1).

So we deduce that \( K \)-kernels of \( \mathcal{R}(x_1, \ldots, x_n) \) are not necessarily completely closed in \( \mathcal{R}(x_1, \ldots, x_n) \).

Since every polar is completely closed (cf. Theorem 6.1.22), by the example given above we have that \( \alpha \in x^{\perp\perp} \) which yields that \( x^{\perp} = x^{\perp\perp\perp} = (x^{\perp\perp})^{\perp} = \{1\} \).

**Corollary 6.2.11.** Every \( K \)-kernel of \( \mathcal{R}(x_1, \ldots, x_n) \) is a polar.

**Proof.** By Proposition 6.2.9, every \( K \)-kernel of \( \mathcal{R}(x_1, \ldots, x_n) \) is completely closed and thus by Theorem 6.1.22 is a polar since \( \mathcal{R}(x_1, \ldots, x_n) \) is complete. \( \square \)

Summarizing the above assertions we have

**Corollary 6.2.12.** Let \( K \) be a kernel of \( \mathcal{R}(x_1, \ldots, x_n) \). Then the following statements are equivalent:

1. \( K \) is a \( K \)-kernel.
2. \( K \) is a polar.
3. \( K \) is completely closed.
Proposition 6.2.13. For any \( f \in \mathcal{H}(x_1, \ldots, x_n) \),

\[
\text{Skel}(f) = \text{Skel}(f^\perp) \text{ and } f^\perp = \text{Ker}(\text{Skel}(f)).
\]

Proof. Since \( f^\perp \supseteq \langle f \rangle \) we have that \( \text{Skel}(f^\perp) \subseteq \text{Skel}(f) \). Let \( K \) be the \( K \)-kernel such that \( \text{Skel}(K) = \text{Skel}(f) \). Then \( f \in K \) and \( K \) is a polar by Corollary 6.2.11. By Proposition 6.1.13 \( f^\perp \) is the minimal polar containing \( f \) thus \( K \supseteq f^\perp \) and so \( \text{Skel}(f) = \text{Skel}(K) \subseteq \text{Skel}(f^\perp) \subseteq \text{Skel}(f) \). Finally, as \( f^\perp \) is a polar it is a \( K \)-kernel and so \( f^\perp = \text{Ker}(\text{Skel}(f^\perp)) = \text{Ker}(\text{Skel}(f)) \). \( \square \)

Theorem 6.2.14. There is a 1 : 1 correspondence

\[
\mathcal{B}(\mathcal{H}(x_1, \ldots, x_n)) \leftrightarrow \text{Skl}(\mathcal{H}^n)
\]

(6.4)

between the skeletons in \( \mathcal{H}^n \) and the polars of \( \mathcal{H}(x_1, \ldots, x_n) \) given by \( B \mapsto \text{Skel}(B) \) and \( Z \mapsto \text{Ker}(Z) \).

This correspondence restricts to a correspondence

\[
\mathcal{PB}(\mathcal{H}(x_1, \ldots, x_n)) \leftrightarrow \text{PSkl}(\mathcal{H}^n)
\]

(6.5)

between the principal skeletons in \( \mathcal{H}^n \) and the principal polars of \( \mathcal{H}(x_1, \ldots, x_n) \).

Proof. By Corollary 6.2.11 and Proposition 6.2.8 \( B \) is a polar if and only if \( B \) is a \( K \)-kernel thus \( \text{Ker}(\text{Skel}(B)) = B \). If \( B = f^\perp \) for some \( f \in \mathcal{H}(x_1, \ldots, x_n) \), so by Proposition 6.2.13 \( \text{Skel}(B) = \text{Skel}(f) \) and \( \text{Ker}(\text{Skel}(f^\perp)) = \text{Ker}(\text{Skel}(f)) = f^\perp \). \( \square \)

Corollary 6.2.15. Let

\[
B = \{ B \cap \mathcal{H}(x_1, \ldots, x_n) : B \in \mathcal{B}(\mathcal{H}(x_1, \ldots, x_n)) \}
\]

and

\[
PB = \{ B \cap \mathcal{H}(x_1, \ldots, x_n) : B \in \mathcal{PB}(\mathcal{H}(x_1, \ldots, x_n)) \}.
\]

Since \( (\mathcal{B}(\mathcal{H}(x_1, \ldots, x_n)), \cdot, \cap) \) is a lattice and \( \mathcal{PB}(\mathcal{H}(x_1, \ldots, x_n)), \cdot, \cap) \) is a sublattice of \( (\mathcal{B}(\mathcal{H}(x_1, \ldots, x_n)), \cdot, \cap) \), \( B \) is a lattice and \( PB \) is a sublattice of \( B \).
By Corollary 6.2.3, the correspondence in Theorem 6.7 yields a correspondence

\[ \text{B} \leftrightarrow \text{Skel}(R^n) \]  

(6.6)

given by \( K \mapsto \text{Skel}(K) \) and \( Z \mapsto \text{Ker}(Z) \cap R(x_1, ..., x_n) = \text{Ker}(x_1, ..., x_n)(Z) \).

By Corollary 6.2.12 and Remark 6.2.2 for a kernel \( K \in \text{Con}(R(x_1, ..., x_n)) \) we have that \( \text{Ker}(x_1, ..., x_n)(\text{Skel}(K)) = K \cap R(x_1, ..., x_n) \).

By Theorem 6.2.14, (6.6) restricts to a correspondence

\[ \text{PB} \leftrightarrow \text{PSkl}(R^n). \]  

(6.7)

**Example 6.2.16.** For the principal kernel \( \langle x \rangle \in \text{PCon}(R(x)) \) we have that \( \text{Ker}(x_1, ..., x_n)(\text{Skel}(\langle x \rangle)) = \langle x \rangle \cap R(x) = \langle x \rangle \). Analogously, for the principal kernel \( \langle x \rangle \cap \langle S \rangle = \langle |x| \wedge |\alpha| \rangle \in \text{PCon}(R(x)) \), in view of Example 6.2.4 we have that \( \text{Ker}(x_1, ..., x_n)(\text{Skel}(\langle |x| \wedge |\alpha| \rangle)) = \langle |x| \wedge |\alpha| \rangle \cap R(x) = \langle x \rangle \cap R(x) = \langle x \rangle \).

Thus the \( K \)-kernel corresponding to \( \langle x \rangle \) is the same as the \( K \)-kernel corresponding to the (bounded from above) kernel \( \langle |x| \wedge |\alpha| \rangle \).

The following example illustrates the necessity of working in \( R(x_1, ..., x_n) \) instead of \( R(x_1, ..., x_n) \).

**Example 6.2.17.** Consider the subset \( X = \{ \alpha |x|^{-n} \wedge 1 : n \in \mathbb{N} \} \) in \( R(x) \) where \( \alpha \in R, \alpha > 1 \). Then

\[
\sup X = f(x) = \begin{cases} 
\alpha & x = 1; \\
1 & x \neq 1;
\end{cases}
\]

Evidently \( f \in R(x) \setminus R(x) \).

While \( x_{\perp R(x)} = \{ 1 \} \) we have that \( f \in x_{\perp R(x)} \) thus we get that \( x_{\perp R(x) \perp R(x)} = R(x) \) whereas \( x_{\perp R(x) \perp R(x)} \subset R(x) \). For example, since \( |\beta \cdot 1| \wedge |f(1)| = |\beta| \wedge |\alpha| > 1 \) we get that \( |\beta x| \wedge |f| \neq 1 \) for any \( \beta \neq 1 \), so \( |\beta x| \not\in x_{\perp R(x) \perp R(x)} \).

I thank Prof. Kalle Karu for pointing out the example regarding the polar of \( \langle x \rangle \).
6.3 Appendix: The Boolean algebra of polars and the Stone representation

As stated in Theorem 6.1.10 for any idempotent semifield $S$, the collection of polars of $S$, $\mathcal{P}(S)$ forms a complete Boolean algebra with respect to intersection multiplication (corresponding to union) and $\perp$ (corresponding to negation). There is a representation for Boolean algebras called the Stone representation. We will now provide a brief overview of this representation applied to the complete Boolean algebra of polars of $S$. The complete construction can be found in [8], (Section (3.2)).

Definition 6.3.1. Let $X$ be a topological space. The collection of all clopen (closed which are also open) subsets of $X$ is a Boolean subalgebra of the Boolean algebra of all subsets of $X$. This collection is called the dual algebra of $X$.

Definition 6.3.2. A compact Hausdorff space whose clopen sets form a base is called a Boolean space.

Remark 6.3.3. A closed subset $Y$ of a Boolean space $X$ is also a Boolean space (with respect to the induced subspace topology) and each clopen set in $Y$ is the intersection with $Y$ of a clopen set in $X$.

Definition 6.3.4. A Stone space is a totally disconnected Boolean space which means that every open set is the union of clopen sets.

Definition 6.3.5. A Boolean homomorphism between two Boolean algebras is a lattice homomorphism that preserves complements.

Definition 6.3.6. The set $2 = \{0, 1\}$ is the totally ordered Boolean algebra which will also be considered as a topological space, giving it the discrete topology. Given the Boolean algebra $B$, the set $\text{Bool}[B, 2]$ of all 2-valued homomorphisms on $B$ is a subspace of the product space $2^B$. $\text{Bool}[B, 2]$ is called the dual space of $B$.

Remark 6.3.7. • For a Boolean algebra $B$, the dual space $\text{Bool}[B, 2]$ is a Boolean space. If $B$ is complete then $\text{Bool}[B, 2]$ is a Stone space.
• Each Boolean algebra $B$ is isomorphic to its second dual, i.e., the dual algebra $B$ of the Boolean space $\text{Bool}[B, 2]$. The isomorphism $\alpha : B \to B$ is given by

$$\alpha(b) = \{ f \in \text{Bool}[B, 2] : f(b) = 1 \}$$

• An element of $\text{Bool}[B, 2]$ is completely determined by its kernel which is a maximal ideal of $B$. Thus $\text{Bool}[B, 2]$ may be replaced by the set $\text{Spec}(B)$ which consists of all of the maximal ideals of $B$. The basic clopen sets of $\text{Spec}(B)$ are of the form $\{ V(b) : M \in \text{Spec}(B) : b \notin M \}$ where $b \in B$, $V$ is an isomorphism between $B$ and the algebra of clopen sets in $\text{Spec}(B)$. This topology on $\text{Spec}(B)$ is called the Zariski topology of $\text{Spec}(B)$.

• Each Boolean space is isomorphic to its second dual. Namely, let $A$ be the dual algebra of a Boolean space $X$, and let $Y = \text{Bool}[A, 2]$ be the dual space of $A$. Then the function $\beta : X \to Y$ given by

$$\beta(x)(P) = 1 \text{ if } x \in P \text{ and } \beta(x)(P) = 0 \text{ if } x \notin P$$

is a homeomorphism.

In view of the above, $\text{Spec}(\mathcal{B}(S))$ is a Stone space.

**Definition 6.3.8.** Let $X$ be a topological space. Define

$$E(X) = \{ f : X \to \bar{\mathbb{R}} : f \text{ is continuous} \}$$

where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. Also define

$$D(X) = \{ f \in E(X) : f^{-1}(\mathbb{R}) \text{ is dense in } X \}$$

the set of those continuous functions which are real-valued on a dense (and open) subset of $X$.

**Remark 6.3.9.** $E(X)$ is a poset with respect to the coordinate-wise partial order: $f \leq g$ if $f(x) \leq g(x)$ for each $x \in X$, and, in fact, $E(X)$ is a sublattice of the product $(\bar{\mathbb{R}})^X$. Note that $f \leq g$ provided that $f(x) \leq g(x)$ for each $x$ in some dense subset of $X$. The subset $D(X)$ is a sublattice of $E(X)$ since the intersection of two dense open subsets is dense.
Definition 6.3.10. A monomorphism $\phi : H \to U$ of idempotent semifields is a cl-essential monomorphism if $\phi(H) \cap K \neq \{1\}$ for every kernel $K \in \text{Con}(U)$.

Theorem 6.3.11. [8, Theorem (2.3.23)] Let $S$ be an archimedean idempotent semifield, and let $X = \text{Spec}(\mathcal{B}(S))$ be the Stone space of the Boolean algebra $\mathcal{B}(S)$ of polars of $S$. Then there is a (complete) cl-essential monomorphism from $S$ into $D(X)$. 

83
7 The principal bounded kernel - principal skeleton correspondence

Recall that the semifield \( R \) is defined to be a bipotent divisible archimedean complete semifield. By Corollary 3.4.14 one can regard \( R \) as being \((\mathbb{R}^+, +, \cdot)\).

Throughout this section, \( H \) is assumed to be a bipotent divisible semifield. Any supplementary assumptions on \( H \) will be explicitly stated.

Although we have already established a correspondence between skeletons and polars which restricts to a correspondence between principal skeletons and principal polars, we proceed to find a correspondence between principal skeletons to principal kernels of a very special kernel of \( R(x_1, \ldots, x_n) \), namely, the kernel \( \langle R \rangle \) presented above which will now serve us as a semifield of its own right. It turns out that \( \langle R \rangle \) possesses just enough distinct bounded copies of the principal kernels of \( R(x_1, \ldots, x_n) \) to represent the principal skeleton without any ambiguity.

**Remark 7.0.12.** The restriction of the image of the operator \( \text{Ker} : \mathbb{R}^n \to \mathbb{P}(\mathbb{R}(x_1, \ldots, x_n)) \) to \( \mathbb{P}(\langle R \rangle) \) is

\[
\text{Ker}(\langle R \rangle)(Z) = \{ f \in \langle R \rangle : f(a_1, \ldots, a_n) = 1, \forall (a_1, \ldots, a_n) \in Z \} = \text{Ker}(Z) \cap \langle R \rangle. 
\]

(7.1)

Additionally, let \( \text{Skel}|\langle R \rangle : \mathbb{P}(\langle R \rangle) \to \mathbb{R}^m \) be the restriction of \( \text{Skel} : \mathbb{P}(\mathbb{R}(x_1, \ldots, x_n)) \to \mathbb{R}^m \) to \( \mathbb{P}(\langle R \rangle) \). Then all the statements introduced in the section ‘Skeletons and kernels of skeletons’ apply to \( \text{Ker}|\langle R \rangle \) and \( \text{Skel}|\langle R \rangle \).

**Note 7.0.13.** As \( \text{Ker}|\langle R \rangle \) and \( \text{Skel}|\langle R \rangle \) are our actual object of interest, we also denote them by \( \text{Ker} \) and \( \text{Skel} \). The distinction will be carried out through the context of discussion. If an ambiguity arises we will explicitly say to which \( \text{Ker} \) or \( \text{Skel} \) we refer.

We now show that restricting \( \text{Skel} \) and \( \text{Ker} \) does not affect the collection of resulting skeletons and that each \( \mathcal{K} \)-kernel of \( \mathbb{R}(x_1, \ldots, x_n) \) has a \( \mathcal{K} \)-kernel (with respect to the restriction) in \( \langle R \rangle \).
Proposition 7.0.14. If $K \in \text{Con}(\mathcal{R}(x_1, \ldots, x_n))$ is a $\mathcal{K}$-kernel, then there exists a unique kernel $K' \in \text{Con}(\langle \mathcal{R} \rangle)$ such that $\text{Skel}(K) = \text{Skel}(K')$.

Proof. Let $\alpha \in \mathcal{R} \setminus \{1\}$. By assumption $K$ is a $\mathcal{K}$-kernel, thus $K = \text{Ker}(\text{Skel}(K))$. Define $K' = K \cap \langle \mathcal{R} \rangle$. By Proposition 2.7.5(2) we have that $K \cap \langle \mathcal{R} \rangle = \langle X \rangle \cap \langle \alpha \rangle = \langle \{|f| \land |\alpha| : f \in X\} \rangle$ where $X$ is any set generating $K$ as a kernel, in particular, one can take $X = K$. Now, for any $f \in \mathcal{R}(x_1, \ldots, x_n)$, in particular in $K$ we have that $f(x) = 1$ for some $x \in \mathbb{R}^n$ if and only if $f(x) \land |\alpha| = 1$ (since $|\alpha| > 1$) so $\text{Skel}(K') = \text{Skel}(K)$. Thus $K' = K \cap \langle \mathcal{R} \rangle = \text{Ker}(\text{Skel}(K')) \cap \langle \mathcal{R} \rangle = \text{Ker}_{\langle \mathcal{R} \rangle}(\text{Skel}(K'))$, and so $K'$ is a $\mathcal{K}$-kernel in $\text{Con}(\langle \mathcal{R} \rangle)$. \qed

Note that these restricted kernels are exactly the kernels of the (domain) restriction to $\langle \mathcal{R} \rangle$ of the restriction homomorphism, $\mathcal{R}(x_1, \ldots, x_n) \rightarrow \mathcal{R}(x_1, \ldots, x_n)/\langle f \rangle$ defined by $g \mapsto g|_{\text{Skel}(f)}$.

By Proposition 4.2.5 and the above discussion we have

Proposition 7.0.15. There is a $1 : 1$ order reversing correspondence

$$\{\text{skeletons of } \mathcal{R}^n\} \leftrightarrow \{\mathcal{K} - \text{kernels of } \langle \mathcal{R} \rangle\}, \quad (7.2)$$

given by $Z \mapsto \text{Ker}(Z) \cap \langle \mathcal{R} \rangle$; the reverse map is given by $K \mapsto \text{Skel}(K)$ with $K \in \text{Con}(\langle \mathcal{R} \rangle)$.

Let $\mathbb{S}$ be an idempotent semifield. Recall that the positive cone of $\mathbb{S}$ is

$$\mathbb{S}^+ = \{a \in \mathbb{S} : a \geq 1\} = \{|a| : a \in \mathbb{S}\}.$$

By Theorems 3.3.2 and 3.3.4 (applied to idempotent semifields), $\mathbb{S}^+$ is a subsemigroup of $\mathbb{S}$ which is a lattice (i.e., closed with respect to $+$ (i.e., $\lor$) and $\land$.

Let $f \in \mathbb{H}(x_1, \ldots, x_n)$ and fix $\alpha \neq 1$ in $\mathbb{H}$. Then

$$\langle |f| \land |\alpha| \rangle = \langle f \rangle \cap \langle \alpha \rangle \subseteq \langle \alpha \rangle = \langle \mathbb{H} \rangle.$$

Note that for any $f \in \mathbb{H}(x_1, \ldots, x_n)$, $|f|$ is a generator of $\langle f \rangle$. 85
We define the map \( \omega : \text{H}(x_1, ..., x_n)^+ \rightarrow \langle \text{H} \rangle^+ \) by 
\[ \omega(|f|) = |f| \land |\alpha|. \]

Then \( \omega \) is a lattice homomorphism. Indeed,
\[ \omega(|f| \lor |g|) = (|f| \lor |g|) \land |\alpha| = (|f| \land |\alpha|) \lor (|g| \land |\alpha|) = \omega(|f|) \lor \omega(|g|) \]
and
\[ \omega(|f| \land |g|) = (|f| \land |g|) \land |\alpha| = (|f| \land |\alpha|) \land (|g| \land |\alpha|) = \omega(|f|) \land \omega(|g|). \]

\( \omega \) induces a map
\[ \Omega : \text{PCon}(\text{H}(x_1, ..., x_n)) \rightarrow \text{PCon}(\text{H}(x_1, ..., x_n)) \cap \langle \text{H} \rangle \]
such that \( \Omega(\langle f \rangle) = \langle \omega(|f|) \rangle = \langle |f| \land |\alpha| \rangle = \langle f \rangle \cap \langle \text{H} \rangle. \)

Let us study the map \( \Omega \) and its image.

Remark 7.0.16. 1. By the above discussion we get that \( \Omega \) respects both intersection and product of kernels, and thus it maps the lattice of principal kernels of \( \text{H}(x_1, ..., x_n) \), \( \text{PCon}(\text{H}(x_1, ..., x_n)) \), onto the lattice of kernels
\[ \{ \langle f \rangle \cap \langle \text{H} \rangle : f \in \text{PCon}(\text{H}(x_1, ..., x_n)) \} = \text{PCon}(\langle \text{H} \rangle) \]
by Remark 5.2.8.

2. If \( \langle f \rangle \) is a bounded from below kernel, then \( \langle f \rangle \cap \langle \text{H} \rangle = \langle \text{H} \rangle \) by Remark 5.1.13. In fact, by the above discussion of bounded from below functions, we see that every principal kernel whose skeleton is the empty set is mapped to \( \langle \text{H} \rangle \).

3. As \( \text{Skel}(\langle \alpha \rangle) = \emptyset \) and since \( \text{Skel}(\langle f \rangle \cap \langle g \rangle) = \text{Skel}(\langle f \rangle) \cup \text{Skel}(\langle g \rangle) \), for any principal kernel \( \langle f \rangle \) we have that
\[ \text{Skel}(\Omega(\langle f \rangle)) = \text{Skel}(\langle f \rangle \cap \langle H \rangle) = \text{Skel}(\langle f \rangle) \cup \emptyset = \text{Skel}(\langle f \rangle). \]

Thus \( \Omega \) does not affect the skeleton of a kernel, the skeleton is fixed.
4. As any $\alpha \neq 1$ generates $\langle \mathbb{H} \rangle$, any proper subkernel $K$ of $\langle \alpha \rangle$ must admit $K \cap \mathbb{H} = \{1\}$, for otherwise, if there exists $\alpha \in K \cap \mathbb{H}$ such that $\alpha \neq 1$, we get $\langle \mathbb{H} \rangle \supseteq K \supseteq \langle \alpha \rangle = \langle \mathbb{H} \rangle$.

5. $\langle \mathbb{H} \rangle$ is an idempotent semifield of $\mathbb{H}(x_1, ..., x_n)$ as the latter is idempotent (since $\mathbb{H}$ is idempotent). Now, by Remark 5.2.8 we get that any kernel $K$ of $\mathbb{H}(x_1, ..., x_n)$ such that $K \subseteq \langle \mathbb{H} \rangle$ is a kernel of $\langle \mathbb{H} \rangle$. In particular, $\langle \mathbb{H} \rangle$ is a semifield with a generator (generated as a kernel over itself by a single element) with any $\alpha \neq 1$ as a generator.

6. By Proposition 5.2.1 we have that

$$\Omega(PCon(\mathbb{H}(x_1, ..., x_n)) \setminus PCon(\langle \mathbb{H} \rangle)) = PCon(\langle \mathbb{H} \rangle).$$

Summarizing the results introduced in Remark 7.0.16 for the designated semifield $R$ we have that

$$\Omega : PCon(R(x_1, ..., x_n)) \to PCon(\langle R \rangle)$$

is a lattice homomorphism of $(PCon(R(x_1, ..., x_n)), \cdot, \cap)$ onto $(PCon(\langle R \rangle), \cdot, \cap)$, such that $Skel(\langle f \rangle) = Skel(\Omega(\langle f \rangle))$.

Let $f \in PCon(\langle R \rangle)$ and let $A = \{g \in R : \Omega(\langle g \rangle) = f\}$.

Define $K = \langle A \rangle \in Con(R(x_1, ..., x_n))$. Then by Remark 7.0.16 if $g \in A$ then $Skel(g) = Skel(f)$.

As we shall shortly show, there exists a correspondence

$$\langle f \rangle \in PCon(\langle R \rangle) \leftrightarrow Skel(f)$$

between the principal skeletons in $R^n$ and the kernels in $PCon(\langle R \rangle)$.

If $Skel(g) = Skel(f)$ then since $Skel(g) = Skel(\Omega(g))$ we have that $Skel(\Omega(g)) = Skel(f) \in PCon(\langle R \rangle)$. Thus in view of the above $\Omega(\langle g \rangle) = \langle f \rangle$. Consequently we have that $Skel(K) = Skel(f)$ and $K$ is the maximal kernel of $R(x_1, ..., x_n)$ having this property.

The choice of $\langle R \rangle$ for our algebraic infrastructure is a natural choice, as we have $R$ as our basic semifield, with respect to which homomorphisms are taken. Taking the semifield in $R(x_1, ..., x_n)$ generated by $R$ as a kernel, we get the
only semifield homomorphism \( \langle R \rangle \to H \) (where \( H \) is a semifield containing \( R \)) sending \( \alpha \) to 1 is the trivial one \( \langle R \rangle \to \{1\} \). All proper subkernels \( K \) of \( \langle R \rangle \) admit \( K \cap R = \{1\} \) which is a necessary condition for a kernel of an \( R \)-homomorphism (which is otherwise not well defined, as any \( \alpha \neq 1 \) is required to be mapped to itself). One can view the elements of \( \langle R \rangle \setminus \{1\} \) as playing the role of invertible elements in rings, in the sense that any ideal containing an invertible element is the ring itself.

**Note 7.0.17.** Since the principal kernels in \( \text{PCon}(\langle R \rangle) \) are in particular principal kernels in \( \text{PCon}(\langle R(\mathbf{x}_1, \ldots, \mathbf{x}_n) \rangle) \), in the remainder of this paper we generally study \( \text{PCon}(\langle R(\mathbf{x}_1, \ldots, \mathbf{x}_n) \rangle) \) where the results are true in particular for \( \text{PCon}(\langle R \rangle) \). In our subsequent discussions we develop the notion of reducibility, regularity and corner-integrality for a principal kernel in \( \text{PCon}(\langle R(\mathbf{x}_1, \ldots, \mathbf{x}_n) \rangle) \). We develop the theory of reducibility for general sublattices of \( \text{PCon}(\langle R(\mathbf{x}_1, \ldots, \mathbf{x}_n) \rangle) \), thus in particular for sublattices of \( \text{PCon}(\langle R \rangle) \). In the sections to follow, we introduce the notions of regularity and corner-integrality of kernels which are oriented to skeletons, in the sense that we consider points for which a generator \( f \) (equivalently \( |f| \)) of the kernel attains the value 1. Since passing to \( |f| \wedge |\alpha| \) for some \( \alpha \neq 1 \) in \( \text{PCon}(\langle R \rangle) \), in the sense that \( |f| \) is regular (corner-integral) if and only if \( |f| \wedge |\alpha| \) is regular (corner-integral). In those places where it is necessary, we explicitly restrict ourselves to \( \text{PCon}(\langle R \rangle) \).

**Proposition 7.0.18.** If \( \langle f \rangle \) is a principal kernel generated by \( f \in \langle R(\mathbf{x}_1, \ldots, \mathbf{x}_n) \rangle \), then \( \text{Skel}(f) = \text{Skel}(\langle f \rangle) \). Moreover, \( \text{Skel}(f) = \text{Skel}(f') \) for any generator \( f' \) of \( \langle f \rangle \).

**Proof.** Although the first statement was proved in Proposition 4.1.5(2), we present here a somewhat more elegant proof. By Corollary 2.3.14 we have that \( g \in \langle R(x_1, \ldots, x_n) \rangle \) is in \( \langle f \rangle \) if and only if there exists some \( n \in \mathbb{N} \) such that \( (f + f^{-1})^{-n} \leq g \leq (f + f^{-1})^n \), or by different notation, \( |f|^{-n} \leq g \leq |f|^n \). Now, since over \( \text{Skel}(f) \), \( f = f^{-1} = 1 \), we have \( 1 = 1^{-n} \leq g \leq 1^n = 1 \) over \( \text{Skel}(f) \) and thus \( g = 1 \) for every \( g \in \langle f \rangle \).

For the second assertion, let \( f' \) be a generator of \( \langle f \rangle \) then by Corollary 2.3.14 for some \( k \in \mathbb{N} \), \( |f'|^{-k} \leq f \leq |f'|^n \), which yields, using the first statement that for any \( x \in \mathbb{R}^n \), \( f'(x) = 1 \) if and only if \( f(x) = 1 \), so \( \text{Skel}(f') = \text{Skel}(f) \). \( \square \)
Proposition 7.0.19. Let \( h \in \mathcal{R}(x_1, ..., x_n) \). If \( h \) is a generator of \( \langle f \rangle \), then \( \text{Skel}(h) = \text{Skel}(f) \).

Proof. First, note that by Proposition 4.1.5(1), we have that \( \text{Skel}(f) \subseteq \text{Skel}(h) \) since \( \langle h \rangle \subseteq \langle f \rangle \). The ‘only if’ part of the assertion follows from the fact that if \( h \) is a generator then \( \langle h \rangle = \langle f \rangle \) and thus their skeletons coincide.

Note 7.0.20. In the following proposition we use the property of \( \mathcal{R} \) being complete, in the sense that the underlying lattice of \( \mathcal{R} \) is conditionally complete (see Definition 2.2.27).

Proposition 7.0.21. Let \( \langle f \rangle \subseteq \langle \mathcal{R} \rangle \). If \( h \in \langle f \rangle \) is such that \( \text{Skel}(h) = \text{Skel}(f) \), then \( h \) is a generator of \( \langle f \rangle \).

Proof. The assertion is obvious in the case where \( \langle f \rangle = \langle 1 \rangle = \{1\} \). So, as \( \text{Skel}(h) = \text{Skel}(f) \) we can assume \( f \) and \( h \) to be not equal to 1. If \( \langle f \rangle = \langle a \rangle \) for some \( a \neq 1 \), then \( \text{Skel}(h) = \text{Skel}(f) = \emptyset \) implies by Remark 7.0.16(3) that \( \langle h \rangle = \langle \mathcal{R} \rangle = \langle f \rangle \).

Before we continue, note that \( \mathcal{R} \) is a totally ordered semifield, thus for any \( a \in \mathcal{R}^n \), either \( f(a) \leq f(a) \) or \( f(a) \leq h(a) \), for any pair of functions \( f, h \) in \( \mathcal{R}(x_1, ..., x_n) \).

Let \( h \in \langle f \rangle \) such that \( h \) is not a generator of \( \langle f \rangle \) while \( \text{Skel}(h) = \text{Skel}(f) \). By Corollary 2.3.14 we have that for each \( k \in \mathbb{N} \) there exists some \( x_k \in \mathcal{R}^n \) for which \( |f(x_k)| > |h(x_k)|^k \) (*). Note that \( f \) and \( h \) are rational polynomials thus continuous and so are \( |f|^s \) and \( |h|^s \) (by definition of \( |\cdot| \)) for any \( s \in \mathbb{N} \).

For any \( k \in \mathbb{N} \), define the set \( U_k = \{ x : |f(x)| > |h(x)|^k \} \). As \( \mathcal{R} \) is assumed to be (ordered) divisibly closed semifield, it is dense, so, for any \( x \in U_k \) there exists a neighborhood \( B_x \subset U_k \) containing \( x \) such that for all \( x' \in B_x \), \( |f(x')| > |h(x')|^k \). Now, since \( h \) and \( f \) are bounded from above rational functions both not equal to 1, \( U_k \) are bounded regions inside \( \mathcal{R}^n \). Taking the closure of \( U_k \) we may assume it is closed. Since \( \text{Skel}(h) = \text{Skel}(f) \), \( |f(x)| > |h(x)|^k \) implies that \( |h(x)|, |f(x)| > 1 \), so, by the definition of \( U_k \) we get the sequence of strict inclusions \( U_1 \supset U_2 \supset \cdots \supset U_k \supset \cdots \) where by our assumption (*), \( U_i \neq \emptyset \). Thus, since \( \mathcal{R} \) is complete, there exists an element \( y \in \mathcal{R} \) such that \( y \in B = \bigcap_{k \in \mathbb{N}} B_k \).

Now, for \( x \notin \text{Skel}(h) \), \( |h(x)| > 1 \) thus there exists some \( r = r(x) \in \mathbb{N} \) such that \( |h(x)|^r > |f(x)| \) thus \( x \notin B \) thus \( y \notin \mathcal{R}^n \setminus \text{Skel}(h) \). On the other hand, if \( y \in \text{Skel}(h) \) then \( y \in \text{Skel}(f) \) so \( 1 = |f(y)| \leq |h(y)| = 1 \). Thus \( \bigcap_{k \in \mathbb{N}} B_k = \emptyset \). A contradiction. **
Note 7.0.22. In the last proof we could analogously argue that since $y \not\in Skel(h)$, $f(y) > h(y)^k$ for every natural number $k$ where $h(y) > 1$, which yields that $f(y) \not\in R = \langle h(y) \rangle$.

Proposition 7.0.23. Let $\langle f \rangle \subseteq R(x_1, \ldots, x_n)$. If $h \in \langle f \rangle$ is such that $Skel(h) = Skel(f)$ and $|h|$ has an essential expansion as $|h| = \sum_{i=1}^{k} s_i|f|^{d(i)}$ with $d(i) \in \mathbb{Z}$ and $s_1, \ldots, s_k \in \mathcal{R}(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{k} s_i = 1$, then $h$ is a generator of $\langle f \rangle$.

Proof. As $|f|$ is a generator of $\langle f \rangle$ and $|h|$ a generator of $\langle h \rangle$, we may consider $|f|$ and $|h|$ instead of $f$ and $h$ and moreover, we may assume $|f|$ and $|h|$ to be in essential form. By Proposition 2.3.8, there exist some $s_1, \ldots, s_k \in \mathcal{R}(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{k} s_i = 1$ and $|h| = \sum_{i=1}^{k} s_i|f|^{d(i)}$ with $d(i) \in \mathbb{Z}_{\geq 0}$ (since $|h| \geq 1$, $d(i) \geq 0$).

As $|h|$ is in essential form, $k$ is minimal. Now, since $Skel(|f|) = Skel(|h|)$ we have that $1 = |h(x)| = \sum_{i=1}^{k} s_i(x)|f(x)|^{d(i)} = \sum_{i=1}^{k} s_i(x) \cdot 1 = \sum_{i=1}^{k} s_i(x)$ for every $x \in Skel(h)$. Thus for every $x \in Skel(h)$ there exists $1 \leq j \leq k$ such that $h(x) = s_j(x) = 1$, so $h(x) = 1$ if and only if $h(x) = s_j(x) = 1$ for some $1 \leq j \leq k$. Let $i_0$ be such that $d(i_0) = 0$. Since $|h|$ is in essential form, and $s_{i_0} \leq \sum_{i=1}^{k} s_i = 1$, $s_{i_0}$ must be dominate at some point $x_0 \in Skel(h)$, in the sense that $s_{i_0} > s_i$ for every $i \neq i_0$. As $|h|$ is continuous there exists a neighborhood $\epsilon$ of $x_0$ such that $s_{i_0}(y) \geq \sum_{i \neq i_0} s_i(y)|f(y)|^{d(i)}$ for every $y \in \epsilon$. But $\sum_{i=1}^{k} s_i = 1$ thus $s_{i_0}(y) = 1$ for every $y \in \epsilon$. Without loss of generality, take $i_0 = 1$. If $\epsilon \not\subseteq Skel(f)$ then, as $\mathcal{R}$ is divisibly closed, there exists a point $y_1 \in \epsilon \setminus Skel(f)$ such that $|h(y_1)| = s_1(y_1) + \sum_{j=2}^{k} s_j(y_1)|f(y_1)|^{d(j)} = s_1(y_1) = 0$, which contradicts the assumption that $Skel(h) = Skel(f)$. So $\epsilon \subseteq Skel(f)$.

Thus over $R = Cl(Skel(f)^c)$ we have that $|h(x)| = \sum_{i=2}^{k} s_i(x)|f(x)|^{d(i)}$ where $\sum_{i=1}^{k} s_i(x) = 1$ for every $x \in R$ (since $\epsilon \cap R = \emptyset$ there always exists some $2 \leq j \leq k$ for which $s_j(x) = 1$). Now, take $x \in R$. Then there exists some $2 \leq j_0 \leq k$ such that $s_{j_0}(x) = 1$, thus we have that $|h(x)| = \sum_{i=2}^{k} s_i(x)|f(x)|^{d(i)} \geq s_{j_0}(x)|f(x)|^{d(j_0)} = |f(x)|^{d(j_0)}$. Consequently, since $|f(x)| \geq 1$ we have that $|h(x)| \geq |f(x)|^{d}$ with $d = \min\{d(j) : 2 \leq j \leq k\}$, $d > 0$. As also $1 = |h(x)| \geq |f(x)|^d = 1^d = 1$ over $Skel(f)$, we get that $|h(x)| \geq |f(x)|^d \geq |f(x)|$ over $R \cup Skel(f) = \mathcal{R}^n$, i.e., $|h| \geq |f|$. Thus, by Remark 2.3.16, $|f| \subseteq \langle |h| \rangle$, so $\langle f \rangle = \langle |f| \rangle \subseteq \langle |h| \rangle = \langle h \rangle$. Finally, as $h \in \langle f \rangle$, we have that $\langle h \rangle = \langle f \rangle$, i.e., $h$ is a generator of $\langle f \rangle$ as desired. \qed

The proof of Proposition 7.0.23 does not apply to any element of $\langle f \rangle$, just to the element which can be written essentially in the form $a = \sum_{i=1}^{k} s_i|f|^{d(i)}$ with $\sum_{i=1}^{k} s_i = 1$. For example, the element $|x| \wedge \alpha \in \langle x \rangle$ with $\alpha > 1$ can be written...
as \((\frac{|x|}{\alpha + |x|}) \cdot 1 + (\frac{\alpha}{\alpha + |x|}) \cdot |x|\) with \(a_1(x) = \frac{|x|}{\alpha + |x|}\) and \(a_2(x) = \frac{\alpha}{\alpha + |x|}\), but this is not in an essential form since the first term never dominates.

**Remark 7.0.24.** By Proposition [4.1.5], we have that a skeleton \(S\) is a principal skeleton, i.e., \(S = \text{Skel}(f)\) for some \(f \in \langle \mathcal{R} \rangle\), if and only if \(S = \text{Skel}(\langle f \rangle)\).

**Proposition 7.0.25.** Let \(\langle f \rangle\) be a principal kernel in \(\text{PCon}(\langle \mathcal{R} \rangle)\). Then \(\langle f \rangle\) is a \(K\)-kernel.

**Proof.** We need to show that \((\text{Ker}(\text{Skel}(f)) = ) \text{Ker}(\text{Skel}(\langle f \rangle)) \subseteq \langle f \rangle\).

Let \(h \in \langle \mathcal{R} \rangle\) such that \(h \in \text{Ker}(\text{Skel}(f))\). Then \(h(x) = 1\) for every \(x \in \text{Skel}(f)\) and so \(\text{Skel}(f) \subseteq \text{Skel}(h)\). If \(|h| \leq |f|^k\) for some \(k \in \mathbb{N}\) then \(h \in \langle f \rangle\). Thus in particular we may assume that \(h \neq 1\). Now, by Corollary [4.2.7] we have that \(\text{Skel}(\langle f \rangle \cap \langle h \rangle) = \text{Skel}(f) \cup \text{Skel}(h) = \text{Skel}(h)\). Since \(h \neq 1\), \(\text{Skel}(h) \neq \mathcal{R}^n\) and thus \(\langle f \rangle \cap \langle h \rangle \neq \{1\}\). Again by Corollary [4.2.7] we have that \(\text{Skel}(\langle f \rangle \cdot \langle h \rangle) = \text{Skel}(f) \cap \text{Skel}(h) = \text{Skel}(f)\). Thus \(\langle f, h \rangle = \langle f \rangle \cdot \langle h \rangle \neq \langle \mathcal{R} \rangle\) for otherwise \(\text{Skel}(f) = \emptyset\). Consequently the kernel \(K = \langle g \rangle = \langle f \rangle \cap \langle h \rangle\), where \(g = |f| \wedge |h|\), admits \(\{1\} \neq K \subseteq \langle f \rangle\). So, we have that \(g \in \langle f \rangle\) and \(\text{Skel}(g) = \text{Skel}(h)\). Thus By Proposition [7.0.21] \(g\) is a generator of \(\langle h \rangle\), so, we have that \(\langle h \rangle = K \subseteq \langle f \rangle\) as desired. \(\square\)

**Corollary 7.0.26.** There is a \(1 : 1\) order reversing correspondence

\[
\{\text{principal skeletons of } \mathcal{R}^n\} \leftrightarrow \{\text{principal kernels of } \langle \mathcal{R} \rangle\}, \quad (7.3)
\]

given by \(Z \mapsto \text{Ker}_{\langle \mathcal{R} \rangle}(Z)\); the reverse map is given by \(K \mapsto \text{Skel}(K)\).

**Proof.** Every principal kernel gives rise to a principal skeleton by the definition of \(\text{Skel}\). The reverse direction follows Proposition [7.0.21] as every principal kernel which produces a principal skeleton using \(\text{Skel}\) is in fact a \(K\)-kernel. \(\square\)

In Proposition [2.5.13], we have shown using a substitution homomorphism \(\psi\) that any point \(a = (\alpha_1, ..., \alpha_n) \in \mathcal{R}^n\) corresponds to the maximal kernel

\[
\langle \frac{x_1}{\alpha_1}, ..., \frac{x_n}{\alpha_n} \rangle = \langle |\frac{x_1}{\alpha_1}| + ..., |\frac{x_n}{\alpha_n}| \rangle = \langle \frac{x_1}{\alpha_1} \cdot ..., \frac{x_n}{\alpha_n} \rangle = \langle \frac{x_1}{\alpha_1} \rangle \cdot ..., \langle \frac{x_n}{\alpha_n} \rangle .
\]

Let \(\psi : \mathbb{H}(x) \rightarrow \mathbb{H}\) be defined by sending \(x \mapsto 1\). Consider the restriction homomorphism \(\psi|_{\langle \mathcal{R} \rangle} : \langle \mathcal{R} \rangle \rightarrow \psi(\langle \mathcal{R} \rangle) = \mathcal{R}\). Then by Theorem [2.2.49] we have that \(\text{Ker}\psi|_{\langle \mathcal{R} \rangle} = \text{Ker}\psi \cap \langle \mathcal{R} \rangle = \langle x \rangle \cap \langle \mathcal{R} \rangle\). Thus, the result applies to \(\langle \mathcal{R} \rangle\) where the maximal kernel is \(\langle x \rangle \cap \langle \mathcal{R} \rangle\).
We will now show that any maximal kernel of \( \langle R \rangle \) is of that form.

**Proposition 7.0.27.** If \( K \) is a maximal kernel in \( \text{Con}(\langle H \rangle) \), then \( K = \Omega(\langle \frac{x_1}{\alpha_1}, \ldots, \frac{x_n}{\alpha_n} \rangle) \) for some \( \alpha_1, \ldots, \alpha_n \in R \).

**Proof.** Denote \( L_a = (|\frac{x_1}{\alpha_1}| + \ldots + |\frac{x_n}{\alpha_n}|) \land |\alpha| \) with \( \alpha \neq 1 \), for \( a = (\alpha_1, \ldots, \alpha_n) \).

By Remark 7.0.16 we may assume \( \text{Skel}(K) \neq \emptyset \), since the only kernel corresponding to the empty set is \( \langle R \rangle \) itself. If \( a \in \text{Skel}(K) \), then as \( \text{Skel}(L_a) = \{a\} \subseteq \text{Skel}(K) \), we have that \( \langle L_a \rangle \supseteq K \). Thus, the maximality of \( K \) implies that \( K = \langle L_a \rangle \). \( \Box \)
8 The coordinate semifield of a skeleton

In this section we define the coordinate semifield corresponding to a skeleton. Being the most relevant to the development achieved in the reminder of this work, we first perform the construction for principal skeletons, using the principal kernels of \( \langle R \rangle \). Later on we introduce a more general construction of the coordinate semifield of a (principal) skeleton using its corresponding (principal) polar.

**Definition 8.0.28.** Let \( V = \text{Skel}(\langle f \rangle) \) be a (principal) skeleton in \( \mathbb{R}^n \). Define

\[
R[V] = \{ f|_V(x_1, \ldots, x_n) : f \in \langle R \rangle \}.
\] (8.1)

We call \( R[V] \) the **coordinate semifield** of \( V \).

**Proposition 8.0.29.** For any (principal) skeleton \( V = \text{Skel}(\langle f \rangle) \subseteq \mathbb{R}^n \), define \( \phi_V : \langle R \rangle \to R[V] \) to be the restriction map \( f \mapsto f|_V \). Then \( \phi_V \) is a homomorphism and

\[
\langle R \rangle / \langle f \rangle \cong R[V].
\] (8.2)

**Proof.** For any \( g, h \in \langle R(x_1, \ldots, x_n) \), since \( \phi_V \) is a restriction map, we have that \( \phi_V(g + h)|_V = g|_V + h|_V = \phi_V(g) + \phi_V(h) \) and \( \phi_V(g \cdot h)|_V = g|_V \cdot h|_V = \phi_V(g) \cdot \phi_V(h) \) so \( \phi_V \) is a semiring homomorphism. It is trivially onto, by the definition of \( R[V] \). Now, By Proposition 7.0.25 we have that \( \text{Ker}(\phi_V) = \{ g \in \langle R \rangle : g|_V = 1 \} = \{ g \in \langle R \rangle : g \in \langle f \rangle \} = \langle f \rangle \). Thus by the isomorphism theorem 2.2.51 we have that \( \langle R \rangle / \langle f \rangle \cong \text{Im}(\phi_V) = R[V] \), as desired. \( \square \)

**Proposition 8.0.30.** Let \( K_1, K_2 \) be kernels of the semifield \( \langle R \rangle \) such that \( \langle R \rangle = K_1 \cdot K_2 \). Then

\[
\langle R \rangle / (K_1 \cap K_2) \cong \langle R \rangle / K_1 \times \langle R \rangle / K_2
\]
as groups.

**Proof.** \( \langle R \rangle / (K_1 \cap K_2) = (K_1 / (K_1 \cap K_2)) \cdot (K_2 / (K_1 \cap K_2)) = \bar{K}_1 \cdot \bar{K}_2 \) where \( \bar{K}_i \) is the homomorphic image of \( K_i \) under the quotient map \( \langle R \rangle \to \langle R \rangle / (K_1 \cap K_2) \).
Since $\overline{K}_1 \cap \overline{K}_2 = \{1\}$ we have that as groups $\overline{K}_1 \cdot \overline{K}_2 \cong \overline{K}_1 \times \overline{K}_2$. Now, by the second isomorphism theorem for kernels $\overline{K}_1 = (K_1/(K_1 \cap K_2)) \cong (K_1 \cdot K_2)/K_2 = \langle \mathcal{R} \rangle / K_2$ and similarly $\overline{K}_2 = \langle \mathcal{R} \rangle / K_1$. Thus we have that $G/(K_1 \cap K_2) \cong G/K_1 \times G/K_2$ as groups. 

Corollary 8.0.31. If $V_1, V_2$ are principal skeletons in $\mathcal{R}^n$ such that $V_1 \cap V_2 = \emptyset$, then

$$\mathcal{R}[V_1 \cup V_2] \cong \mathcal{R}[V_1] \times \mathcal{R}[V_2]$$

as groups.

Proof. Let $\langle f_1 \rangle$ and $\langle f_2 \rangle$ be the kernels in $\text{PCon}(\langle \mathcal{R} \rangle)$ such that $V_1 = \text{Skel}(\langle f_1 \rangle)$ and $V_2 = \text{Skel}(\langle f_2 \rangle)$. Then $\text{Skel}(\langle f_1 \rangle \cdot \langle f_2 \rangle) = \text{Skel}(\langle f_1 \rangle) \cap \text{Skel}(\langle f_2 \rangle) = V_1 \cap V_2 = \emptyset$ thus $\langle f_1 \rangle \cdot \langle f_2 \rangle = \langle \mathcal{R} \rangle$. Since $V_1 \cup V_2 = \text{Skel}(\langle f_1 \rangle \cap \langle f_2 \rangle)$ we have by Proposition 8.0.30 that $\mathcal{R}[V_1 \cup V_2] \cong \langle \mathcal{R} \rangle / (\langle f_1 \rangle \cap \langle f_2 \rangle) \cong \langle \mathcal{R} \rangle / \langle f_1 \rangle \times \langle \mathcal{R} \rangle / \langle f_2 \rangle \cong \mathcal{R}[V_1] \times \mathcal{R}[V_2]$. 

The following result is analogue to Proposition 8.0.29 using $\mathcal{R}(x_1, \ldots, x_n)$ and $\mathcal{B}$ defined in Corollary 6.2.15 instead of $\langle \mathcal{R} \rangle$ and $\text{Con}(\langle \mathcal{R} \rangle)$, respectively.

Proposition 8.0.32. For any skeleton $V = \text{Skel}(S) \subseteq \mathcal{R}^n$ with $S \subseteq \mathcal{R}(x_1, \ldots, x_n)$ define

$$\phi_V : \mathcal{R}(x_1, \ldots, x_n) \rightarrow \mathcal{R}[V]$$

to be the restriction map $f \mapsto f|_V$. Then $\phi_V$ is a homomorphism and

$$\mathcal{R}(x_1, \ldots, x_n)/K_S \cong \mathcal{R}[V]. \quad (8.3)$$

where $K_S = S^\perp \cap \mathcal{R}(x_1, \ldots, x_n)$ with $S^\perp$ is taken in the completion $\mathcal{R}(x_1, \ldots, x_n)$ of $\mathcal{R}(x_1, \ldots, x_n)$ in $\text{Fun}(\mathcal{R}^n, \mathcal{R})$.

Proof. Note that by Corollary 6.2.15 we have that $\text{Ker}(\phi_V) = \{ g \in \mathcal{R}(x_1, \ldots, x_n) : g|_V = 1 \} = \{ g \in \mathcal{R}(x_1, \ldots, x_n) : g \in S^\perp \} = K_S$. The rest of the proof is as in Proposition 8.0.29.
9 Basic notions: Essentiality, reducibility, regularity and corner-integrality

In the previous sections we have proved some correspondences between skeletons and kernels. In particular we have shown a correspondence between principal skeletons and principal kernels. We now turn to find the connection between tropical varieties (which we call corner loci) and skeletons.

Before diving into the core of the theory we develop some tools and notion to facilitate our construction.

Throughout this section, \( \mathbb{H} \) is assumed to be a bipotent divisible semifield. Any supplementary assumptions on \( \mathbb{H} \) will be explicitly stated. We also recall that our designated semifield \( \mathbb{R} \) is defined to be bipotent, divisible, archimedean and complete the prototype being \( (\mathbb{R}^+, +, \cdot) \) by Corollary 3.4.14.

9.1 Essentiality of elements in the semifield of fractions

In the theory of tropical geometry, there exists a notion of essentiality of monomials in a given polynomial.

**Definition 9.1.1.** A monomial \( m_1(x_1, \ldots, x_n) \in \mathbb{H}[x_1, \ldots, x_n] \) is said to be inessential in a polynomial \( p(x_1, \ldots, x_n) = \sum_{i=1}^{k} m_i(x_1, \ldots, x_n) \in \mathbb{H}[x_1, \ldots, x_n] \) over a domain \( D \subset \mathbb{H}^n \), if at any \( x \in D \) there exists some \( j \neq 1 \) such that \( m_j(x) \geq m_1(x) \), i.e., \( m_1 \) never solely dominates \( p \) over \( D \).

**Note 9.1.2.** In the following section, we introduce a notion of essentiality for elements of \( \mathbb{H}(x_1, \ldots, x_n) \). This notion differs from the tropical one, in fact it generalizes it in some sense, as we will show shortly. Generally, the context
will imply the relevant notion among the two. In case ambiguity arises, we will explicitly indicate the one we refer to.

Let \( f \in \mathbb{H}(x_1, \ldots, x_n) \), consider the skeleton defined by \( f \), \( \text{Skel}(f) \). We will now characterize for which \( g \in \mathbb{H}(x_1, \ldots, x_n) \), \( \text{Skel}(f + g) = \text{Skel}(f) \). For this purpose, we introduce the following definition.

**Definition 9.1.3.** Let \( g \in \mathbb{H}(x_1, \ldots, x_n) \). Define the following sets

\[
\text{Skel}_-(g) = \{ x \in \mathbb{H}^n : g(x) < 1 \}, \quad \text{Skel}_+(g) = \{ x \in \mathbb{H}^n : g(x) > 1 \}. \tag{9.1}
\]

Notice that \( \mathbb{H}^n = \text{Skel}_-(g) \cup \text{Skel}_+(g) \cup \text{Skel}(g) \). We call \( \text{Skel}_+(g) \) and \( \text{Skel}_-(g) \) the *positive* and *negative* regions of \( g \), respectively.

**Remark 9.1.4.** If \( f, g \in \mathbb{H}(x_1, \ldots, x_n) \), then \( \text{Skel}(f + g) = \text{Skel}(f) \) if and only if the following holds:

\[
\text{Skel}(f) \setminus \text{Skel}(g) \subset \text{Skel}_-(g) \quad \text{and} \quad \text{Skel}(g) \setminus \text{Skel}(f) \subset \text{Skel}_+(f). \tag{9.2}
\]

Note that when \( \text{Skel}(f) \subseteq \text{Skel}(g) \), i.e., \( \langle g \rangle \subseteq \langle f \rangle \), condition 9.2 takes the form \( \text{Skel}(g) \setminus \text{Skel}(f) \subset \text{Skel}_+(f) \). When \( \text{Skel}(g) \subseteq \text{Skel}(f) \), i.e., \( \langle f \rangle \subseteq \langle g \rangle \) condition 9.2 takes the form \( \text{Skel}(f) \setminus \text{Skel}(g) \subset \text{Skel}_-(g) \) and when \( \text{Skel}(f) \cap \text{Skel}(g) = \emptyset \) then \( \text{Skel}(g) \subset \text{Skel}_+(f) \) and \( \text{Skel}(f) \subset \text{Skel}_-(g) \).

**Proof.** This statement is a direct consequence of the definitions. \( \square \)

**Definition 9.1.5.** Let \( f, g \in \mathbb{H}(x_1, \ldots, x_n) \). \( g \) is said to be *inessential* for \( f \) if

\[
\text{Skel}(f + g) = \text{Skel}(f);
\]

otherwise \( g \) is *essential* for \( f \). Let \( f = \sum_{i=1}^k f_i \in \mathbb{H}(x_1, \ldots, x_n) \) and let \( j \in \{1, \ldots, n\} \). Then \( f_j \) is said to be inessential in \( f \) if \( f_j \) is inessential for \( \sum_{i \neq j} f_i \). Otherwise \( f_j \) is essential in \( f \).

**Note 9.1.6.** Note that inessentiality defined in Definition 9.1.5 differs from the notion of inessentiality in tropical geometry. In tropical geometry, a monomial of a polynomial is considered inessential if it is not dominant anywhere, in the sense that it does not attain *solely* the maximal value of the polynomial.
**Definition 9.1.7.** Let \( f \in \mathbb{H}(x_1, ..., x_n) \). Then we say \( f \) has the essentiality property if for any additive decomposition of \( f \), \( f = \sum_{i=1}^{k} f_i \) with \( f_i \in \mathbb{H}(x_1, ..., x_n) \), the following condition holds:

For any \( 1 \leq j \leq k \), \( \text{Skel}(f) \neq \text{Skel}(h_j) \) where \( h_j = \sum_{i=1; i \neq j}^{k} f_i \).

In words each \( f_i \) is essential in \( f \).

**Definition 9.1.8.** Let \( f \in \mathbb{H}(x_1, ..., x_n) \). Write \( f = \frac{h}{g} = \sum_{i=1}^{k} h_i \sum_{j=1}^{m} g_j \) where \( g_j \) and \( h_i \) are monomials in \( \mathbb{H}[x_1, ..., x_n] \). Then \( f \) is said to be of reduced form or essential form if for any \( I \subseteq \{1, ..., k\} \) and \( J \subseteq \{1, ..., m\} \) where \( I \) and/or \( J \) is a proper subset,

\[
\text{Skel}(f) \neq \text{Skel}(\tilde{f}) \quad \text{where} \quad \tilde{f} = \sum_{i \in I} h_i \sum_{j \in J} g_j.
\]

**Remark 9.1.9.** Note that in order for a monomial \( h_i \) to be essential, there is no need for the occurrence of some \( g_j \) such that \( h_i(x) = g_j(x) \) for some \( x \in \mathbb{H}^n \) and viceversa. The reason for this is that \( h_i \) can affect the skeleton by preventing another monomial \( h' \) of the numerator to dominate \( h \) in a point \( x \in \mathbb{H}^n \) where \( h'(x) = g(x) \).

The inessential monomials \( h_i \) and \( g_j \) in \( f = \frac{h}{g} \) are characterized as follows:

A monomial \( h' \) of \( h \) is inessential in \( f \) if one of the following two conditions holds for all \( x \in \mathbb{H}^n \):

1. \( h'(x) < h(x) \).
2. \( h'(x) = h(x) \) and \( h'(x) \neq g(x) \).

Analogously, a monomial \( g' \) of \( g \) is inessential in \( f \) if one of the following two conditions holds for all \( x \in \mathbb{H}^n \):

1. \( g'(x) < g(x) \).
2. \( g'(x) = g(x) \) and \( g'(x) \neq h(x) \).

It can easily be seen that \( h' \) and \( g' \) admitting the above criterion do not affect the skeleton of \( f \). Moreover, if a monomial \( g' \) is inessential in \( g \) then taking \( \tilde{f} = \frac{h}{g} \)
where \( \tilde{g} \) is defined to be \( g \) with \( g' \) omitted, we have that any monomial \( g'' \neq g' \) of \( g \) and any monomial \( h' \) of \( h \) are essential in \( \tilde{f} \) if and only if they are essential in \( f \).

In view of the above, we can define \( f_e \in \mathbb{H}(x_1, ..., x_n) \) to be the rational function obtained from \( f = \frac{g}{h} \) by omitting all inessential monomials in \( f \). By the above, \( f_e \) is well-defined regardless of the order with respect to which the monomials are omitted.

In view of the above discussion the following observations hold:

**Remark 9.1.10.** \( f \in \mathbb{H}(x_1, ..., x_n) \) is of essential form if and only if both \( f \) and \( f^{-1} \) admit the essentiality property.

**Remark 9.1.11.** Let \( f = \frac{h}{g} \in \mathbb{H}(x_1, ..., x_n) \) where \( h, g \in \mathbb{H}[x_1, ..., x_n] \) are polynomials. If \( h \) or \( g \) are inessential (in the tropical sense) then \( f \) is not of essential form.

**Proof.** Indeed, if \( h \) is inessential one of the composing monomials of \( h \), say \( h' \), does not affect the values \( h \) obtains and thus does not affect the skeleton of \( f \) and the summand \( \frac{h'}{g} \) can be omitted from \( f \) without changing \( \text{Skel}(f) \). If \( g \) is inessential, then since \( \text{Skel}(f) = \text{Skel}(f^{-1}) \) we can consider \( \frac{h}{g'} \) what brings us back to the previous case considered. Namely, if \( g' \) is an inessential monomial of \( g \), \( \frac{h}{g'} \) can be omitted from \( \frac{h}{g} \). Now, taking the inverse of the resulting fraction brings us back to \( f \) with the monomial \( g' \) omitted. \( \square \)

### 9.2 Reducibility of principal kernels and skeletons

In this section we consider the notion of reducibility with respect to a sublattice of kernels of the semifield \( \mathbb{H}(x_1, ..., x_n) \). The sublattice of kernels that is of interest to us are actually contained inside the lattice \( \text{PCon}(\langle R \rangle) \). Note that \( \text{PCon}(\langle R \rangle) \) is both a sublattice of \( \text{Con}(\langle R \rangle) \) and of \( \text{Con}(R(x_1, ..., x_n)) \).
Definition 9.2.1. Let \( S \) be a semifield. A subset \( \Theta \) of \( \text{Con}(S) \) is said to be a \textit{sublattice of kernels} if for every pair of kernels \( K_1, K_2 \in \Theta \),

\[
K_1 \cap K_2 \in \Theta \quad \text{and} \quad K_1 \cdot K_2 \in \Theta.
\]

Example 9.2.2. \( \text{Con}(\mathcal{R}(x_1,\ldots,x_n)) \), \( \text{PCon}(\mathcal{R}(x_1,\ldots,x_n)) \) are sublattices of kernels of \( \mathcal{R}(x_1,\ldots,x_n) \). \( \text{Con}(\langle \mathcal{R} \rangle) \), \( \text{PCon}(\langle \mathcal{R} \rangle) \) are sublattices of kernels of \( \mathcal{R}(x_1,\ldots,x_n) \) and of \( \langle \mathcal{R} \rangle \).

Definition 9.2.3. Let \( \Theta \) be a sublattice of kernels of a semifield \( S \). A proper (non-trivial) kernel \( K \in \Theta \) is called \( \Theta \)-irreducible if for any pair of kernels \( A,B \in \Theta \)

\[
A \cap B \subseteq K \Rightarrow A \subseteq K \quad \text{or} \quad B \subseteq K. \tag{9.3}
\]

A kernel \( K \) is called \textit{weakly} \( \Theta \)-irreducible if for any pair of kernels \( A,B \) of \( S \)

\[
A \cap B = K \Rightarrow A = K \quad \text{or} \quad B = K. \tag{9.4}
\]

\( K \) is called \( \Theta \)-maximal if for any kernel \( A \in \Theta \)

\[
K \subseteq A \Rightarrow K = A \quad \text{or} \quad A = K. \tag{9.5}
\]

Note that if a kernel \( K \in \Theta \) is \( \Theta \)-irreducible then \( K \) is weakly \( \Theta \)-irreducible.

Definition 9.2.4. Let \( S \) be a semifield and let \( \Theta \) be a sublattice of kernels. Then \( S \) is said to be \( \Theta \)-irreducible if for any pair of kernels \( K_1 \in \Theta \) and \( K_2 \in \Theta \) such that \( K_1 \cap K_2 = \{1\} \), either \( K_1 = \{1\} \) or \( K_2 = \{1\} \).

Remark 9.2.5. If \( K \) is an \( \Theta \)-irreducible kernel of \( S \), then the quotient semifield \( U = S/K \) is \( \Theta \)-irreducible.

\textit{Proof.} Let \( K_1 \) and \( K_2 \) be two kernels of \( U \). Then \( \phi^{-1}(K_1) = K \cdot K_1 \) and \( \phi^{-1}(K_2) = K \cdot K_2 \) are in \( \text{Con}(S) \), where \( \phi : S \to U \) is the quotient map. Assume \( K_1 \neq \{1\} \) and \( K_2 \neq \{1\} \) are distinct kernels such that \( K_1 \cap K_2 = \{1\} \). Then \( L_1 = K \cdot K_1 \) and \( L_2 = K \cdot K_2 \) are two distinct kernels in \( \Theta \) properly containing \( K \) and

\[
L_1 \cap L_2 = \phi^{-1}(K_1) \cap \phi^{-1}(K_2) \subseteq \phi^{-1}(K_1 \cap K_2) = \phi^{-1}(1) = K;
\]

contradicting the irreducibility of \( K \). \( \square \)
From 9.2.6. In particular, taking $S$ to be an idempotent semifield one can take $\theta$ to be the sublattice $\text{PCon}(S)$, i.e., restrict the notion of reducibility to the principal kernels of $S$.

Having definitions for reducibility and irreducibility of (principal) kernels, we now turn to define the analogous geometric notion for principal skeletons (for the case $S = \mathbb{H}(x_1, ..., x_n)$).

**Definition 9.2.7.** Let $\Theta$ be a sublattice of kernels in $\mathbb{H}(x_1, ..., x_n)$. A skeleton $S$ is said to be a $\Theta$-skeleton if there exists some kernel $K \in \Theta$ such that $S = \text{Skel}(K)$.

**Definition 9.2.8.** Let $\Theta$ be a sublattice of kernels in $\mathbb{H}(x_1, ..., x_n)$. A $\Theta$-skeleton $S$ is said to be $\Theta$-reducible if there exist some $\Theta$-skeletons $S_1$ and $S_2$ such that $S = S_1 \cup S_2$ and $S \neq S_1$ and $S \neq S_2$; otherwise $S$ is $\Theta$-irreducible.

**Remark 9.2.9.** By Definition 9.2.8, $\text{Skel}(f)$ is $\Theta$-irreducible if for any pair of $\Theta$-skeletons $\text{Skel}(g)$ and $\text{Skel}(h)$ such that $\text{Skel}(f) = \text{Skel}(g) \cup \text{Skel}(h)$, either $\text{Skel}(f) = \text{Skel}(g)$ or $\text{Skel}(f) = \text{Skel}(h)$. Translating this last statement to the kernels $\langle f \rangle$, $\langle g \rangle$ and $\langle h \rangle$ in $\text{PCon}(\mathbb{H})$, and using the principal kernels - principal skeletons correspondence, we get the condition stated in Definition 9.2.3 of $\Theta$-irreducible kernels.

By the last remark we have

**Corollary 9.2.10.** For $\Theta \subseteq \text{PCon}(\mathbb{H})$ a sublattice of kernels of $\mathbb{H}$. $\langle f \rangle$ is $\Theta$-irreducible if and only if $\text{Skel}(f)$ is $\Theta$-irreducible.

We now turn to study more closely the notion of reducibility of a kernel, and introduce adequate geometric interpretations for reducibility of its skeleton. We begin our discussion by introducing the notion of a reducible element of $\mathbb{R}(x_1, ..., x_n)$ corresponding to reducibility of the principal kernel it defines.
In the theory of commutative rings a generator of a principal ideal is unique up to multiplication by an invertible element, i.e., the association class of a generator of an ideal is unique. Recall that for two elements \( a \) and \( b \) of a commutative ring \( R \), \( a \) and \( b \) are associates if and only if \( a | b \) and \( b | a \). In our setting things are slightly more complicated. We consider \( \sim_K \) equivalence classes of generators of kernels. The equivalence \( \sim_K \) will be shown to be induced by a certain order relation, \( \succeq \), defined on the elements of the semifield and plays analogous role to that of \( | \).

**Definition 9.2.11.** Let \( S \) be a semifield and let \( \Theta \) be a sublattice of kernels in \( S \). An element \( a \in S \) is said to be a \( \Theta \)-element if \( \langle a \rangle \in \Theta \).

**Remark 9.2.12.** Following Definition **9.2.11** any generator \( b \in S \) of \( \langle a \rangle \) is a \( \Theta \)-element.

**Remark 9.2.13.** Let \( \Theta \subseteq \text{PCon}(\mathbb{H}(x_1, \ldots, x_n)) \) be a sublattice of kernels of \( \mathbb{H}(x_1, \ldots, x_n) \). If \( f, g \in \mathbb{H}(x_1, \ldots, x_n) \) are \( \Theta \)-elements then so are

\[
|f| + |g|, \quad |f||g| \quad \text{and} \quad |f| \wedge |g|.
\]

**Proof.** Indeed, \( \langle |f| + |g| \rangle = \langle |f||g| \rangle = \langle |f| \rangle \cdot \langle |g| \rangle = \langle f \rangle \cdot \langle g \rangle \in \Theta \) and \( \langle |f| \wedge |g| \rangle = \langle |f| \rangle \cap \langle |g| \rangle = \langle f \rangle \cap \langle g \rangle \in \Theta \).

We proceed in developing a relation on elements of \( \mathbb{H}(x_1, \ldots, x_n) \), using \( \text{PCon}(\mathbb{H}(x_1, \ldots, x_n)) \), which naturally induces a relation on \( \Theta \)-element for any sublattice of kernels \( \Theta \subseteq \text{PCon}(\mathbb{H}(x_1, \ldots, x_n)) \).

**Notation 9.2.14.** Throughout the rest of this subsection we continue taking \( \Theta \subseteq \text{Con}(\mathbb{H}(x_1, \ldots, x_n)) \) to be a sublattice of kernels.

**Definition 9.2.15.** Let \( S \) be a semifield and let \( a, b \in S \). Define the following relation on \( S \)

\[
a \sim_K b \iff \langle a \rangle = \langle b \rangle.
\]

This is clearly an equivalence relation, the classes of which are

\[
[a] = \{a' : a' \text{ is a generator of } \langle a \rangle \}.
\]

Define the partial relation \( \succeq \) on \( S \) as follows:

\[
a \succeq b \iff \exists a' \in [a] \exists b' \in [b] \text{ such that } |a'| \geq |b'|.
\]
Definition 9.2.16. Let $S$ be a semifield and let $a, b \in S$. We say that $a$ and $b$ are $k$-comparable if $a \succeq b$ or $b \succeq a$, i.e., if there exist some $a' \sim_K a$ and $b' \sim_K b$ such that $|a'|$ and $|b'|$ are comparable $|a'| \leq |b'|$ or $|b'| \leq |a'|$.

We introduce explicitly the translation of Definition 9.2.15 for the case where the semifield $S$ is $\mathbb{H}(x_1, ..., x_n)$ with $\mathbb{H}$ a bipotent divisible semifield.

Remark 9.2.17. For every $h, g \in \mathbb{H}(x_1, ..., x_n)$ such that $g, h \geq 1$, $g \geq h \iff h = g \wedge w$ for some $1 \leq w \in \mathbb{H}(x_1, ..., x_n)$.

Proof. If $g \geq h$ then taking $w = h \geq 1$ we have $h = g \wedge h$. Conversely, if $h = g \wedge w$ then $g \geq g \wedge w = h$. Moreover $w$ must admit $w \geq 1$ for otherwise if $w(a) < 1$ for some $a \in \mathbb{H}$ then $h(a) = g(a) \wedge w(a) \leq w(a) < 1$ contradicting the assumption that $h \geq 1$.

Definition 9.2.18. Let $f, g \in \mathbb{H}(x_1, ..., x_n)$. Then

$$f \sim_K g \iff \langle f \rangle = \langle g \rangle,$$

(9.8)

the classes with respect to $\sim_K$ are $[g] = \{g' : \text{ } g' \text{ is a generator of } \langle g \rangle \}$.

Since $h \sim_K |h| \geq 1$ for any $h \in \mathbb{H}(x_1, ..., x_n)$, by Remark 9.2.17 the relation $\succeq$ on $\mathbb{H}(x_1, ..., x_n)$ can be stated as follows:

$$f \succeq g \iff \exists w \in \mathbb{H}(x_1, ..., x_n) \exists f' \in [f] \text{ such that } w, f' \geq 1, \text{ } g \sim_K f' \wedge w.$$  (9.9)

Remark 9.2.19. In view of Corollary 2.3.14 the partial relation (9.9) of Definition 9.2.18 can be rephrased as

$$f \succeq g \iff \exists w \in \mathbb{H}(x_1, ..., x_n) \exists k \in \mathbb{N} \text{ such that } w \geq 1, g \sim_K |f|^k \wedge w.$$  (9.10)

Indeed, if $f' \in [f]$ then there exists $k \in \mathbb{N}$ such that $|f'| \leq |f|^k$, and thus $|f'| = |f|^k \wedge v$ for some $v \in \mathbb{H}(x_1, ..., x_n)$ where $v \geq 1$. Assume $f \succeq g$, then, by (9.9), $g \sim_K |f'| \wedge w$ for some $w \geq 1$. So, $g \sim_K |f'| \wedge w = (|f|^k \wedge v) \wedge w = |f|^k \wedge (v \wedge w)$, since $v, w \geq 1$ we have that $v \wedge w \geq 1$ which means that (9.10) holds for $g$. The converse direction is obvious, since $|f|^k \in [f]$ and $|f|^k \geq 1$ for any $k \in \mathbb{N}$.
Proposition 9.2.20. For any \( f, g \in H(x_1, ..., x_n) \),

\[ g \succeq f \iff \langle g \rangle \supseteq \langle f \rangle. \]  
(9.11)

**Proof.** Let \( f, g \in H(x_1, ..., x_n) \) such that \( g \succeq f \). Then, by (9.10) we have that \( g \succeq f \) if and only if \( f \sim_K |g|^k \wedge w \) for some \( w \geq 1 \) and \( k \in \mathbb{N} \), if and only if there exists some \( f' \sim f \) such that \( f' = |g|^k \wedge w \). Note that \( |g|^k \geq 1 \) and \( w \geq 1 \) so \( f' = |g|^k \wedge w \geq 1 \) and thus \( |f'| = f' \). Finally, \( |f'| = |g|^k \wedge w \iff |f'| \in \langle |g| \rangle \iff f' \in \langle g \rangle \iff f \in \langle g \rangle \). \( \square \)

**Corollary 9.2.21.** For any \( f, g \in H(x_1, ..., x_n) \)

\[ |g| \geq |f| \iff \langle g \rangle \subseteq \langle f \rangle. \]  
(9.12)

**Proof.** Follows from Remark 9.2.20 along with the property that \( |f| \in [f] \) or equivalently \( \langle f \rangle = \langle |f| \rangle \). \( \square \)

**Remark 9.2.22.** For \( f, w \in H(x_1, ..., x_n) \), the following conditions are equivalent

1. \( f \succeq w \) and \( w \not\succeq f \).
2. \( \langle w \rangle \subset \langle f \rangle \) (with strict inclusion).

**Proof.** By condition (9.10) and Proposition 9.2.20 \( \square \)

**Corollary 9.2.23.** For any \( f, g \in H(x_1, ..., x_n) \)

\[ f \sim_K g \iff f \succeq g \text{ and } g \succeq f. \]  
(9.13)

**Proof.** \( f \sim_K g \iff \langle f \rangle = \langle g \rangle \), which, by Proposition 9.2.20 is equivalent to \( f \succeq g \) and \( g \succeq f \). \( \square \)

As stated above, the set of principal kernels of \( \mathcal{R}(x_1, ..., x_n) \), \( \text{PCon}(H(x_1, ..., x_n)) \), forms a lattice with respect to multiplication and intersection. Irreducibility of a kernel is translated there as follows:

**Definition 9.2.24.** A principal kernel \( \langle f \rangle \) of the semifield \( H(x_1, ..., x_n) \) is \( \Theta \)-irreducible if for \( \Theta \) kernels \( \langle g \rangle, \langle h \rangle \supseteq \langle f \rangle \)

\[ \langle f \rangle = \langle g \rangle \cap \langle h \rangle \Rightarrow \langle f \rangle = \langle g \rangle \text{ or } \langle f \rangle = \langle h \rangle; \]  
(9.14)
otherwise, it is \(\Theta\)-reducible. When \(\Theta = \text{PCon}(\mathbb{H}(x_1, \ldots, x_n))\) we simply say reducible (irreducible).

**Remark 9.2.25.** Let \(f, g, h \in \mathbb{H}(x_1, \ldots, x_n)\) such that \(g \succeq f\) and \(h \succeq f\). Then
\[
f \succeq |g| \land |h| \Rightarrow f \sim |g| \land |h|.
\]

**Proof.** A consequence of Remark 3.3.10. \(\Box\)

**Definition 9.2.26.** Let \(f \in \mathbb{H}(x_1, \ldots, x_n)\). \(f\) is said to be irreducible if
\[
f \succeq |g| \land |h| \Rightarrow f \succeq |g| \text{ or } f \succeq |h|. \tag{9.15}
\]
Otherwise \(f\) is reducible. A \(\Theta\)-element \(f\) is said to be \(\Theta\)-irreducible if both \(g\) and \(h\) are restricted to being \(\Theta\) elements of \(\mathbb{H}(x_1, \ldots, x_n)\). Otherwise \(f\) is \(\Theta\)-reducible.

**Remark 9.2.27.** Since \(\mathbb{H}(x_1, \ldots, x_n)\) is an idempotent semifield it is distributive, thus Proposition 2.6.2(1) holds for \(\mathbb{H}(x_1, \ldots, x_n)\) implying that the following condition is equivalent to condition 9.15:
\[
f \sim_K g \land h \Rightarrow f \sim_K g \text{ or } f \sim_K h. \tag{9.16}
\]

**Remark 9.2.28.** By transitivity of \(\succeq\), if \(f_1 \succeq g_1, f_1 \sim_K f_2\) and \(g_1 \sim_K g_2\), then \(f_2 \succeq g_2\).

**Remark 9.2.29.** For \(\Theta\)-elements \(f\) and \(f'\), such that \(f' \sim_K f\), \(f'\) is \(\Theta\)-irreducible if and only if \(f\) is \(\Theta\)-irreducible.

**Proof.** \(f' \in [f]\) (or equivalently \(f' \sim_K f\)) if and only if \(f \succeq f'\) and \(f' \succeq f\). Assume \(f\) is \(\Theta\)-irreducible. If \(g\) and \(h\) are \(\Theta\)-elements such that \(g \land h \succeq f'\), then \(g \succeq f'\) and \(h \succeq f'\). Since \(f' \succeq f\) we get that \(g \succeq f\) and \(h \succeq f\). By \(\Theta\)-irreducibility of \(f\) we have that \(f \succeq g\) or \(f \succeq h\). Now, as \(f' \succeq f\) we get that \(f' \succeq g\) or \(f' \succeq h\) as desired. The arguments of the proof in the opposite direction of the assertion are symmetric. \(\Box\)

**Note 9.2.30.** Remark 9.2.29 actually follows irreducibility being defined by \(\succeq\) (while \(\succeq\) respects \(\sim_K\)). Nevertheless, we prove it explicitly.

The following Proposition establishes the connection between irreducible principal kernels and irreducible elements of \(\mathbb{H}(x_1, \ldots, x_n)\).
Proposition 9.2.31. For any principal kernel \( \langle f \rangle \), with \( f \in \mathbb{H}(x_1, \ldots, x_n) \), \( \langle f \rangle \) is irreducible if and only if \( f \) is irreducible.

Proof. If \( f \sim_K |g| \wedge |h| \) then \( \langle f \rangle = \langle |g| \wedge |h| \rangle = \langle g \rangle \cap \langle h \rangle \). Since \( \langle f \rangle \) is irreducible we have that either \( \langle g \rangle \subseteq \langle f \rangle \) or \( \langle h \rangle \subseteq \langle f \rangle \), which is equivalent, by Proposition 9.2.20, to \( f \supseteq |g| \) or \( f \supseteq |h| \). Now, as \( |g| \geq |g| \wedge |h| \) and \( |h| \geq |g| \wedge |h| \) we get that \( f \sim_K |g| \) or \( f \sim_K |h| \) thus \( f \) is irreducible. Conversely, assume \( f \) is irreducible.

Let \( \langle f \rangle = \langle g \rangle \cap \langle h \rangle \). Then \( \langle f \rangle = \langle |g| \wedge |h| \rangle \), and thus \( f \sim_K |g| \wedge |h| \). As \( f \) is irreducible, we have that \( f \sim_K |g| \) or \( f \sim_K |h| \) and so \( \langle f \rangle = \langle |g| \rangle = \langle g \rangle \) or \( \langle f \rangle = \langle |h| \rangle = \langle h \rangle \) as desired. \( \square \)

The proof of the following Corollary is completely analogous.

Corollary 9.2.32. For any principal kernel \( \langle f \rangle \in \Theta \), with \( f \in \mathbb{H}(x_1, \ldots, x_n) \) (a \( \Theta \)-element), \( \langle f \rangle \) is \( \Theta \)-irreducible if and only if \( f \) is \( \Theta \)-irreducible.

Example 9.2.33. Let \( a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{H}^n \) and let \( m_1, m_2 \in \text{PCon}(\mathbb{H}(x_1, \ldots, x_n) \) be \( m_1(x_1, \ldots, x_n) = |\alpha_1^{-1}x_1| + \cdots + |\alpha_n^{-1}x_n| \) and \( m_2(x_1, \ldots, x_n) = |\alpha_1^{-1}x_1| \cdots |\alpha_n^{-1}x_n| \) both admit \( m_1(a) = 1 \) and \( m_2(a) = 1 \). Since \( |\alpha_i^{-1}x_i| \geq 1 \) for each \( 1 \leq i \leq n \) and due to the property that for every \( w_1, w_2 \geq 1 \), \( w_1w_2 \leq (w_1 + w_2)^2 \) and \( w_1 + w_2 \leq w_1w_2 \), we get that \( m_1 \sim_K m_2 \).

Remark 9.2.34. \( f \in \mathbb{H}(x_1, \ldots, x_n) \) is reducible if and only if there exist \( g, h \in \mathbb{H}(x_1, \ldots, x_n) \), \( g, h \leq 1 \), such that \( f \sim_K g + h \) where \( f \not\sim_K g \) and \( f \not\sim_K h \). Equivalently, \( f \) is irreducible if for any such \( g \) and \( h \), \( f \sim_K g \) or \( f \sim_K h \)

Proof. The definition of \( \wedge \) implies that \( g^{-1} \wedge h^{-1} = (g + h)^{-1} \sim_K g + h \). Now, since \( s^{-1} \sim_K s \) for every \( s \in \mathbb{H} \), by definition of irreducibility we have that \( f \sim_K g^{-1} \) or \( f \sim_K h^{-1} \) if and only if \( f \sim_K g \) or \( f \sim_K h \) which yields the stated conclusion. \( \square \)

Lemma 9.2.35. Let \( f \in \mathbb{H}(x_1, \ldots, x_n) \) be a rational function. We can write \( f = \sum_{i=1}^k f_i \) where each \( f_i \), with \( i = 1, \ldots, k \), is of the form \( g_i h_i \) where \( g_i, h_i \in \mathbb{H}[x_1, \ldots, x_n] \) and \( g_i \) is a monomial. Then \( \text{Skel}(f) = \text{Skel}(\bigwedge_{i=1}^k |f_i|) \) if and only if for every \( 1 \leq i \leq k \) the following condition holds:

\[ f_i(x) = 1 \Rightarrow f_j(x) \leq 1, \forall j \neq i. \] (9.17)
Proof. Denote \( \tilde{f} = \wedge_{i=1}^{k} |f_i| \). If \( x \in \mathbb{H}^n \) such that \( f(x) = 1 \), then there exists some \( i \in \{1, ..., n\} \) such that \( f_i(x) = 1 \) and \( f_j(x) \leq 1 \) for every \( j \neq i \). Thus \( |f_i|(x) = |f_i(x)| = 1 \) and \( |f_j|(x) = |f_j(x)| \geq 1 \), yielding that

\[
\tilde{f}(x) = \inf \{|f_1(x)|, ..., |f_k(x)|\} = \min \{|f_1(x)|, ..., |f_k(x)|\} = |f_i(x)| = 1.
\]

Conversely, if \( \tilde{f}(x) = 1 \) then there exists some \( i \in \{1, ..., n\} \) such that \( |f_i(x)| = 1 \) and \( |f_j(x)| \geq 1 \) for every \( j \neq i \). Now, \( |f_i(x)| = 1 \) if and only if \( f_i(x) = 1 \) and as condition (9.17) holds, we get that \( f_j(x) \leq 1 \) for all \( j \neq i \). Thus

\[
f(x) = \sup \{f_1(x), ..., f_k(x)\} = \max \{f_1(x), ..., f_k(x)\} = f_i(x) = 1.
\]

\( \square \)

9.3 Decompositions

Lemma [9.2.35] provides us with some insight about reducible kernels. Let \( f \) be a rational function in \( \mathbb{H}(x_1, ..., x_n) \). We can write \( f = \sum_{i=1}^{k} f_i \), where each \( f_i \) is of the form \( \frac{g_i}{h_i} \) with \( g_i, h_i \in \mathbb{H}[x_1, ..., x_n] \) and \( g_i \) is a monomial. If each time the value 1 is attained by one of the terms \( f_i \) in this expansion and all other terms attain values smaller or equal to 1, then \( \tilde{f} = \wedge_{i=1}^{k} |f_i| \) defines the same skeleton as \( f \). Moreover, if \( f \in \langle \mathbb{H} \rangle \) then \( \tilde{f} \wedge |\alpha| \in \langle \mathbb{H} \rangle \), for \( \alpha \in \mathbb{H} \setminus \{1\} \) is also a generator of \( \langle f \rangle \). The reason we take \( \tilde{f} \wedge |\alpha| \) is that we have no guarantee that each of the \( f_i \)'s in the above expansion is bounded.

We can generalize this idea as follows:

Let \( f \in \mathbb{H}(x_1, ..., x_n) \) be essential. Then \( f \) is reducible if and only if there exists an additive expansion of \( f \) of the form \( f = \sum_{i=1}^{k} f_i \), where each \( f_i = \frac{g_i}{h_i} \), \( g_i, h \in \mathcal{R}[x_1, ..., x_n] \), (\( h \) being the common denominator derived from the \( h_i \)'s above), such that for every \( 1 \leq i \leq k \) the following condition holds:

\[
f_i(x) = 1 \Rightarrow f_j(x) \leq 1 \ \forall j \neq i.
\]
Definition 9.3.1. Let \( f \in \mathbb{H}(x_1, ..., x_n) \). A decomposition of \( f \) is an equality of the form
\[
|f| = |u| \land |v|
\] (9.18)
with \( u, v \in \mathbb{H}(x_1, ..., x_n) \).

The decomposition (9.18) is said to be trivial if \( f \sim_K u \) or \( f \sim_K v \) (equivalently \(|f| \sim_K |u| \) or \(|f| \sim_K |v|\)). Otherwise, if \( f \not\sim_K u, v \) (equivalently \(|f| \not\sim_K |u|, |v|\)), (9.18) is said to be non-trivial.

A decomposition (9.18) is said to be a \( \Theta \)-decomposition if both \( u \) and \( v \) (equivalently \(|u| \) and \(|v|\)) are \( \Theta \)-elements (and thus, so is \( f \)).

Lemma 9.3.2. If \( f \in \mathbb{H}(x_1, ..., x_n) \) is a \( \Theta \)-element, then \( \langle f \rangle \) is \( \Theta \)-reducible if and only if there exists some generator \( f' \) of \( \langle f \rangle \) such that \( f' \) has a nontrivial \( \Theta \)-decomposition.

Proof. If \( \langle f \rangle \) is \( \Theta \)-reducible then there exist some kernels \( \langle u \rangle \) and \( \langle v \rangle \) in \( \Theta \) such that \( \langle f \rangle = \langle u \rangle \cap \langle v \rangle \) where \( \langle f \rangle \neq \langle u \rangle \) and \( \langle f \rangle \neq \langle v \rangle \). Since \( \langle u \rangle \cap \langle v \rangle = \langle |u| \land |v| \rangle \) we have that \( f' = |u| \land |v| \) is a generator of \( \langle f \rangle \) which by the above is a nontrivial \( \Theta \)-decomposition of \( f' \). Conversely, assume \( f' = |u| \land |v| \) is a nontrivial \( \Theta \)-decomposition for some \( f' \sim_K f \). Then \( \langle f \rangle = \langle f' \rangle = \langle |u| \land |v| \rangle = \langle u \rangle \cap \langle v \rangle \). Since \( f' = |u| \land |v| \) is nontrivial, we have that \( u \not\sim_K f' \) and \( v \not\sim_K f' \) thus \( \langle |u| \rangle = \langle u \rangle \neq \langle f' \rangle = \langle f \rangle \) and similarly \( \langle v \rangle \neq \langle f \rangle \). Thus, by definition, \( \langle f \rangle \) is \( \Theta \)-reducible.

We can equivalently rephrase Lemma 9.3.2 as follows:

Remark 9.3.3. \( f \) is \( \Theta \)-reducible if and only if there exists some \( f' \sim_K f \) such that \( f' \) has a \( \Theta \)-decomposition.

There is an immediate question arising from Definition 9.3.1 and Lemma 9.3.2:

If \( f \in \mathbb{H}(x_1, ..., x_n) \) has a non-trivial \( \Theta \)-decomposition and \( g \sim_K f \), does \( g \) have a non-trivial \( \Theta \)-decomposition too, and if so, what is the relation between this pair of decompositions?

In the following few paragraphs we will give an answer to both of these questions in the case \( \theta = \text{PCon}(\langle \mathcal{A} \rangle) \).
Remark 9.3.4. By definition $a \land b = \inf(a, b)$ and $a \lor b = \sup(a, b)$ for any $a, b \in \mathbb{H}(x_1, \ldots, x_n)$. The following properties are immediate:

1. $(a \land b)^k = a^k \land b^k$ for any $k \in \mathbb{Z}_{\geq 0}$.

2. For any $s_1, \ldots, s_k, a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{H}(x_1, \ldots, x_n)$, The following holds:

$$\sum_{i=1}^{k} s_i(a_i \land b_i) = (\sum_{i=1}^{k} s_i a_i) \land (\sum_{i=1}^{k} s_i b_i).$$

3. For any $s_1, \ldots, s_k \in \mathbb{H}(x_1, \ldots, x_n)$ and $d(i) \in \mathbb{Z}_{\geq 0}$,

$$\sum_{i=1}^{k} s_i(a \land b)^{d(i)} = \left(\sum_{i=1}^{k} s_i a^{d(i)}\right) \land \left(\sum_{i=1}^{k} s_i b^{d(i)}\right).$$

Note that (3) is a consequence of (1) and (2). Indeed, by (1) we have that

$$\sum_{i=1}^{k} s_i(a \land b)^{d(i)} = \sum_{i=1}^{k} s_i((a^{d(i)}) \land (b^{d(i)}))$$

and by (2),

$$\sum_{i=1}^{k} s_i((a^{d(i)}) \land (b^{d(i)})) = \left(\sum_{i=1}^{k} s_i a^{d(i)}\right) \land \left(\sum_{i=1}^{k} s_i b^{d(i)}\right).$$

Remark 9.3.5. Let $h_1, \ldots, h_k \in \mathbb{H}(x_1, \ldots, x_n)$ be such that $h_i \geq 1$. Then $\sum_{i=1}^{k} s_i h_i \geq 1$ for every $s_1, \ldots, s_k \in \mathbb{H}(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{k} s_i = 1$. Indeed, the statement holds since we have that $\sum_{i=1}^{k} s_i h_i \geq (\bigwedge_{i=1}^{k} h_i)(\sum_{i=1}^{k} s_i) \geq \sum_{i=1}^{k} s_i = 1$.

Theorem 9.3.6. Let $\theta = \text{PCon}(\mathcal{R})$. If $\langle f \rangle$ is a (principal) $\Theta$-reducible kernel, then there exists a pair of $\Theta$-elements $g, h \in \mathbb{H}(x_1, \ldots, x_n)$ such that $|f| = |g| \land |h|$ and $|f| \not\sim_{K} |g|, |h|$.

Proof. If $\langle f \rangle$ is a principal reducible kernel, then there exists some $f' \sim_{K} f$ such that $f' = |u| \land |v| = \min(|u|, |v|)$ for some $\Theta$-elements $u, v \in \langle \mathcal{R} \rangle$ where $f' \not\sim_{K} |u|, |v|$. Since $f'$ is a generator of $\langle f \rangle$, we have that $|f| \in \langle f' \rangle$, so there exists some $s_1, \ldots, s_k \in \mathbb{H}(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{k} s_i = 1$ and $|f| = \sum_{i=1}^{k} s_i (f')^{d(i)}$...
with \( d(i) \in \mathbb{Z}_{\geq 0} \) \( (d(i) \geq 0 \text{ since } |f| \geq 1) \).

Thus

\[
f = \sum_{i=1}^{k} s_i(|u| \wedge |v|)^{d(i)} = \sum_{i=1}^{k} s_i(\min(|u|, |v|))^{d(i)} = \min \left( \sum_{i=1}^{k} s_i u^{d(i)}, \sum_{i=1}^{k} s_i v^{d(i)} \right)
\]

\[= |g| \wedge |h|
\]

where \( g = |g| = \sum_{i=1}^{k} s_i |u|^{d(i)}, h = |h| = \sum_{i=1}^{k} s_i |v|^{d(i)} \). Now, by the above we have that \( \langle |f| \rangle \subseteq \langle |g| \rangle \subseteq \langle |u| \rangle \) and \( \langle |f| \rangle \subseteq \langle |h| \rangle \subseteq \langle |v| \rangle \), thus \( \text{Skel}(f) \supseteq \text{Skel}(g) \supseteq \text{Skel}(u) \) and \( \text{Skel}(f) \supseteq \text{Skel}(h) \supseteq \text{Skel}(v) \). We claim that \( |g| \) and \( |h| \) generate \( \langle |u| \rangle \) and \( \langle |v| \rangle \), respectively. Since \( f' \sim_K |f| \) we have that \( \text{Skel}(f') = \text{Skel}(|f|) \), thus for any \( x \in \mathbb{H}^n \), \( f'(x) = 1 \leftrightarrow |f|(x) = 1 \). Let \( s_j(f'^{d(j)}) \) be a dominant term of \( |f| \) at \( x \), i.e.,

\[
|f| = \sum_{i=1}^{k} s_i(x)(f'(x))^{d(i)} = s_j(x)(f'(x))^{d(j)}.
\]

Then we have that \( f'(x) = 1 \leftrightarrow s_j(x)(f'(x))^{d(j)} = 1 \), thus \( f'(x) = 1 \leftrightarrow s_j(x) = 1 \) (since by Remark 2.3.6 \( (f'(x))^{d(j)} = 1 \) if and only if \( f'(x) = 1 \)). Now, consider \( x \in \text{Skel}(g) \). Then we have that \( g(x) = 1 \), i.e., \( \sum_{i=1}^{k} s_i |u|^{d(i)} = 1 \). Let \( s_i |u|^{d(i)} \) be a dominant term of \( g \) at \( x \). If \( s_i(x) = 1 \) then \( |u|^{d(i)} = 1 \) and thus \( u = 1 \) and \( x \in \text{Skel}(u) \) otherwise \( s_i(x) < 1 \) (since \( \sum_{i=1}^{k} s_i = 1 \)) and so, by the above \( s_i(f'^{d(i)}) \) is not a dominant term of \( |f| \) at \( x \). Since \( s_i(x) < 1 \) we have that

\[
|u(x)|^{d(j)} = s_j(x)|u(x)|^{d(j)} < s_t(x)|u(x)|^{d(j)} = g(x) = 1,
\]

for any dominant term of \( |f| \) at \( x \). Thus

\[
s_j(x)(f'(x))^{d(j)} = s_j(x)(|u|(x) \wedge |v|(x))^{d(j)} \leq s_j(x)|u(x)|^{d(j)} < 1. \tag{9.19}
\]

On the other hand, as \( \text{Skel}(f) \supseteq \text{Skel}(g) \), we have that \( f'(x) = 1 \) and thus \( s_j(x)(f'(x))^{d(j)} = 1 \), contradicting (9.19). So, we have that \( \text{Skel}(g) \subseteq \text{Skel}(u) \), so, by the above \( \text{Skel}(g) = \text{Skel}(u) \) which in turn yields that \( g \) is a generator of \( \langle |u| \rangle = \langle u \rangle \). The proof for \( h \) and \( |v| \) is analogous. Consequently, we have that \( g \sim_K |g| \sim_K |u| \) and \( h \sim_K |h| \sim_K |v| \), so, as \( |f| \sim_K f' \not\sim_K |u|, |v| \) we have that \( |f| \not\sim_K |g|, |h| \). \( \square \)
Corollary 9.3.7. If \( f \in \langle \mathcal{R} \rangle \) and if \(|f| = \bigwedge_{i=1}^{s} |f_i| \) for some \( f_i \in \langle \mathcal{R} \rangle \), then for any \( g \) such that \( g \sim_K f \), \(|g| = \bigwedge_{i=1}^{s} |g_i| \) with \( g_i \sim_K f_i \) for \( i = 1, \ldots, s \).

**Proof.** Follows successive application of Theorem 9.3.6. \( \square \)

**Remark 9.3.8.** Let \( \mathbb{H} \) be a bipotent divisible semifield. Let \( f \in \mathbb{H}(x_1, \ldots, x_n) \), and let \( g \in \langle f \rangle \) such that \( g \sim_K f \). As \( g \in \langle f \rangle \), \( g \) can be written as \( g = \sum_{i=1}^{k} s_i f^{d(i)} \) for some \( s_1, \ldots, s_k \in \mathbb{H}(x_1, \ldots, x_n) \) such that \( \sum_{i=1}^{k} s_i = 1 \) with \( d(i) \in \mathbb{Z} \). We call \( \sum_{i=1}^{k} s_i f^{d(i)} \) a **convex expansion** of \( g \) with respect to \( f \). Then for any \( x \in \mathbb{H}^n \), \( g(x) = 1 \) if and only if \( s_j(x) = 1 \) for any leading term \( s_j f^{d(j)} \) of \( g \) at \( x \).

Corollary 9.3.9. If \( \langle f \rangle \) is a kernel in \( \Theta \), then \( \langle f \rangle \) has a nontrivial \( \Theta \)-decomposition \( \langle f \rangle = \langle g \rangle \cap \langle h \rangle \) if and only if \(|f| \) has a non-trivial decomposition \(|f| = |g'| \land |h'|\) with \(|g'| \sim_K g \) and \(|h'| \sim_K h \).

**Proof.** If \(|f| = |g'| \land |h'| \) then, as \(|g'| \sim_K g \) and \(|h'| \sim_K h \) we have that \( \langle f \rangle = \langle |f| \rangle = \langle |g'| \land |h'| \rangle = \langle |g'| \rangle \cap \langle |h'| \rangle = \langle g \rangle \cap \langle h \rangle \). The converse follows the proof of Theorem 9.3.6. \( \square \)

Corollary 9.3.9 ensures us that there is a \( \Theta \)-decomposition of \(|f|\) for every generator \( f \) of a \( \Theta \)-reducible kernel in \( \Theta \).

**Remark 9.3.10.** By Corollary 2.7.6 we have that

\[ \langle f \rangle \cap \langle g \rangle = \langle (f + f^{-1}) \land (g + g^{-1}) \rangle = \langle |f| \land |g| \rangle. \]

But, in fact, as for any \( g' \sim_K g \) and \( h' \sim_K h \), \( \langle f \rangle \cap \langle g \rangle = \langle f' \rangle \cap \langle g' \rangle \), we could have taken \(|g'| \land |f'|\) instead of \(|g| \land |f|\) on the righthand side of the equality, e.g., \( \langle |f|^k \land |g|^m \rangle \) for any \( m, k \in \mathbb{Z} \setminus \{0\} \).
Since $|g| \land |h| = \min(|g|, |h|)$ we can utilize Remark 9.3.10 to get the following observation:

**Proposition 9.3.11.** A $\Theta$-element $f \sim_K |g| \land |h| \in \mathbb{H}(x_1, ..., x_n)$ is $\Theta$-reducible if the $\Theta$-elements $g$ and $h$ are not $K$-comparable.

**Proof.** If $g$ and $h$ are $k$-comparable, then there exist some $g' \sim_K g$ and $h' \sim_K h$ such that $|g'| \geq |h'|$ or $|h'| \geq |g'|$. Without loss of generality, assume $|g'| \geq |h'|$. Then $\langle |g| \land |h| \rangle = \langle |g| \rangle \cap \langle |h| \rangle = \langle |g'| \rangle \cap \langle |h'| \rangle = \langle \min(|g'|, |h'|) \rangle = \langle |g'| \rangle = \langle g \rangle$. Thus $\langle f \rangle = \langle g \rangle$ so $f \sim_K g$ yielding that $f$ is $\Theta$-irreducible. Conversely, if $g$ and $h$ are not $k$-comparable then $g \not\preceq h$ and $h \not\preceq g$, so by Proposition 9.2.20, $\langle h \rangle \not\subseteq \langle g \rangle$ and $\langle g \rangle \not\subseteq \langle h \rangle$ respectively. Then $\langle f \rangle = \langle g \rangle \cap \langle h \rangle \neq \langle g \rangle, \langle h \rangle$, which yields that $f \not\sim_K g$ and $f \not\sim_K h$, and so $f$ is $\Theta$-reducible.

### 9.4 Regularity

Recall that our designated semifield $\mathcal{R}$ is defined to be bipotent, divisible, archimedean and complete, the prototype being $(\mathbb{R}^+, +, \cdot)$ by Corollary 3.4.14. Thus $\mathcal{R}$ (respectively $\mathcal{R}^n$) can be thought of as a topological metric subspace of $\mathbb{R}$ (respectively $\mathbb{R}^n$) with the induced topology derived from the usual (Euclidian) topology of $\mathbb{R}$ (respectively $\mathbb{R}^n$).

There are two general types of nontrivial principal skeletons in $\mathcal{R}^n$: Skeletons not containing a region of dimension $n$ and skeletons that do contain a region of dimension $n$. These two types of principal skeletons emerge from two distinct types of kernels, characterized by their generators. The first type of skeletons correspond to principal kernels generated by an element of $\mathcal{R}(x_1, ..., x_n)$ which we call *regular* while the second type of skeletons correspond to principal kernels generated by an irregular element of $\mathcal{R}(x_1, ..., x_n)$.

Principal kernels encapsulate a relation of the form $f = 1$ for some $f \in \mathcal{R}(x_1, ..., x_n)$, which is induced on the quotient semifield. The relation $f = \frac{\sum g_i}{\sum h_i} = 1$ is local by
nature in the following sense: Let \( x \in \mathbb{R}^n \) be any point. Then there is at least one monomial \( f_{i0} \) of the numerator and at least one monomial \( g_{j0} \) of the denominator which are dominant at \( x \). If more than one monomial is dominant in each case, say \( \{f_{ik}\}_{k=1}^{s} \) and \( \{g_{jm}\}_{m=1}^{t} \), then we have some set of additional relations of the form \( f_{i0} = f_{ik} \) and \( g_{j0} = g_{jm} \). Now, the non-dominant monomials of both numerator and denominator define some order relations on the variables, which in turn define a region of \( \mathbb{R}^n \) over which the relations \( \{f_{ik} = g_{jm} : 0 \leq k \leq s, 0 \leq m \leq t\} \) hold. Every such relation translates by multiplying by inverses of variables to a relation of the form \( 1 = \phi(x_1, \ldots, x_n) \) with \( \phi \in \mathbb{R}[x_1, \ldots, x_n] \setminus \mathbb{R} \) a Laurent monomial, and thus reduces the dimension. Note that in the special case in which \( f_{i0} \) and \( g_{j0} \) singly dominate and are the same monomial, no relation is imposed on the region described above, so we are left only with order relations defining the region. In essence, leaving only dominant monomials at a neighborhood of a point, such local relations fall into two distinct cases:

- An order relation of the form \( 1 + g = 1 \) with \( g \in \mathbb{R}(x_1, \ldots, x_n) \), which in fact describes a relation of the form \( s + t = s \) with \( s, t \in \mathbb{R}[x_1, \ldots, x_n] \). The resulting quotient semifield \( \mathbb{R}(x_1, \ldots, x_n)/\langle 1 + g \rangle \) does not reduce the dimension of \( \mathbb{R}(x_1, \ldots, x_n) \), but only imposes new order relations on the variables.

- A ‘regular’ relation, in the sense that it is not an order relation. Such a relation reduces the dimensionality of the image of \( \mathbb{R}(x_1, \ldots, x_n) \) in the quotient semifield.

Both of these categories are illustrated below and will be studied in Section 13 concerning dimensionality of kernels and skeletons. We define a regular element of \( \mathbb{R}(x_1, \ldots, x_n) \) to be such that does not translate (locally) to order relations but only to regular relations (locally). This observation will allow us to characterize those relations which correspond to corner loci (tropical varieties in tropical geometry). The kernels corresponding to these relations will be shown to form a sublattice of the lattice of principal kernels (which is itself a sublattice of the lattice of kernels).

In the following, we will characterize the generators of principal kernels of \( \mathbb{R}(x_1, \ldots, x_n) \) which correspond to corner loci, which we call corner integral rational functions. In this subset of elements of \( \mathbb{R}(x_1, \ldots, x_n) \), the regular elements correspond to the traditional tropical varieties considered in tropical geometry,
which are precisely the supertropical varieties defined by tangible polynomials (see subsection 10 and [4]), while the irregular correspond to supertropical varieties defined by supertropical nontangible polynomials. Evidently ‘tangible’ polynomials form a multiplicative subset in the domain of supertropical polynomials.

**Example 9.4.1.** Consider the quotient map \( \phi : R(x) \to R(x)/⟨x+1⟩ \). This map imposes the relation \( x + 1 = 1 \) on \( R(x) \), which is just the order relation \( x \leq 1 \). Under the map \( \phi \), \( x \) is sent to \( x = x(x+1) \), where now, in \( \text{Im} (\phi) = R(\bar{x}) \), \( x \) and \( \bar{1} \) are comparable as opposed to the situation in \( R \), where \( x \) and \( 1 \) are not comparable, i.e., do not admit any order relation. If instead of considering \( x+1 \) we consider \( |x| + 1 = x + x^{-1} + 1 \), then as \( |x| \geq 1 \) the relation \( |x| + 1 = 1 \) is in fact \( |x| = 1 \) which yields the substitution map sending \( x \) to \( \bar{x} = x(x+1) \). As seen in Proposition 2.3.11 and Corollary 2.3.14, order relations affect the structures of kernels in a semifield. For instance, consider a principal kernel in a semifield \( R \) generated by an element \( a \in R \). Then \( b \notin ⟨a⟩ \) for any element \( b \in R \) such that \( b \) is not comparable to \( a \).

**Definition 9.4.2.** Let \( S \) be a bipotent semifield. Let \( f = \frac{h}{g} = \sum_{i=1}^{k} \frac{h_{i}}{\sum_{j=1}^{m} g_{j}} \in S(x_{1}, ..., x_{n}) \) such that \( h_{i} \) and \( g_{j} \) are monomials for all \( 1 \leq i \leq k \) and \( 1 \leq j \leq m \). Then \( f \) is said to be regular if for each \( x \in S^{n} \), there exist \( h_{i} \) and \( g_{j} \) such that \( h(x) = h_{i}(x) \), \( g(x) = g_{j}(x) \), and \( h_{i} \neq g_{j} \). Otherwise \( f \) is called irregular.

**Notation 9.4.3.** Let \( \text{Reg}(R(x_{1}, ..., x_{n})) \) denote the set of regular elements in \( R(x_{1}, ..., x_{n}) \).

**Remark 9.4.4.** If \( f, g \in \text{Reg}(R(x_{1}, ..., x_{n})) \) such that \( f \neq 1 \) and \( g \neq 1 \), then the following elements are also in \( \text{Reg}(R(x_{1}, ..., x_{n})) \):

\[ f^{-1}, \; f^{k} \text{ with } k \in \mathbb{Z}, \; f + g, \; |f| = f + f^{-1}, \; f \wedge g. \]
Proof. $f^{-1}$ is regular since the definition of regularity is invariant taking inverses. $f^k$ is regular follows easily from the regularity of $f$. We will now prove that $f + g$ is regular. Let $f, g \in \text{Reg}(\mathcal{R}(x_1, \ldots, x_n))$. Write $f = \frac{h}{w} = \frac{\sum_{j=1}^{k} h_j}{\sum_{j=1}^{w_j}}$ and $g = \frac{t}{s} = \frac{\sum_{j=1}^{l} t_j}{\sum_{j=1}^{s_j}}$. Then

$$f + g = \frac{u}{v} \frac{hs + tw}{ws} = \frac{\sum_{i=1}^{k} \sum_{j=1}^{r} h_i s_j + \sum_{i=1}^{l} \sum_{j=1}^{m} t_i w_j}{\sum_{i=1}^{m} \sum_{j=1}^{r} w_i s_j}.$$

Let $x \in \mathcal{R}^n$. If $u(x) = h_i(x) s_j(x)$ for some $1 \leq i \leq k$ and $1 \leq j \leq r$. Then in particular $s_j$ dominates $s$ at $x$ and since $f \neq 1$ there exists some $w_i \neq h_i$ such that $w_i(x) = w(x)$ thus $w_i(x) s_j(x) = ws(x) = v(x)$. Since we have multiplicative cancellation (invertibility) in $\mathcal{R}(x_1, \ldots, x_n)$, $h_i s_j \neq w_i s_j$. If $u(x) = t_i(x) w_j(x)$ for some $1 \leq j \leq r$ then in particular $w_j$ dominates $w$ at $x$ and since $f \neq 1$ there exists some $s_i \neq t_i$ such that $s_i(x) = s(x)$ thus $s_i(x) w_j(x) = ws(x) = v(x)$. Again, as noted above we have that $t_i w_j \neq s_i w_j$ as desired. Finally, $f + f^{-1}$ and $f \& g = (f^{-1} + g^{-1})^{-1}$ which yields that they are regular. \qed

**Definition 9.4.5.** Let $S$ be a bipotent semifield. A principal kernel $K = \langle f \rangle$ of $S(x_1, \ldots, x_n)$ is said to be regular if $f \in S(x_1, \ldots, x_n)$ is regular. The Skeleton $\text{Skel}(f)$ corresponding to $K$ is said to be regular also. We denote the family of regular skeletons by $\text{RegSkl}(S^n)$ and the family regular principal skeletons by $\text{RegPSkl}(S^n)$.

We need to show that regularity of a principal kernel is well-defined, i.e., that it is independent of the choice of the generator of the kernel.

**Remark 9.4.6.** Let $f' \in \langle f \rangle$ be a generator of $\langle f \rangle$. Then as we have previously shown $\text{Skel}(f') = \text{Skel}(f)$. Let $f' = \frac{h}{g} = \frac{\sum_{j=1}^{k} h_j}{\sum_{j=1}^{g_j}}$. Assume that $f'$ is irregular. Then there exists some $a \in \mathcal{R}^n$ such that $h(a) = h_{i_0}(a) = g_{j_0}(a) = g(a)$ and $h_{i_0} = g_{j_0}$ where for every $i \neq i_0$ and $j \neq j_0$ $h_i(a) < h_{i_0}(a)$ and $g_j(a) < g_{j_0}(a)$, respectively. Now, since $f'$ is continuous and by the definition of $\mathcal{R}$ (being isomorphic to $\mathbb{R}^+$) there is a neighborhood $\varepsilon(a)$ of $a$, $\{a\} \subset \varepsilon(a) \subseteq \mathcal{R}^n$ for which the above holds implying that $\text{Skel}(f') \neq \text{Skel}(f)$ since by the definition of regularity no such neighborhood exists.

**Example 9.4.7.** If $f \in \mathcal{R}(x_1, \ldots, x_n)$ such that $f \neq f + 1$ (i.e., 1 is essential in $f + 1$), then $f + 1 = \frac{f + 1}{1}$ is not regular since 1 being essential in the numerator coincides with the denominator over some nonempty region.
Note 9.4.8. Note that there may exist a mutual monomial of \( h \) and \( g \), but in such a case, it can not dominate both \( h \) and \( g \) at the same point.

**Corollary 9.4.9.** The set of regular principal kernels forms a sublattice of \( \mathcal{PCon}(\mathcal{R}(x_1, \ldots, x_n)) \).

*Proof.* This follows directly from Corollary 2.7.6 and Remark 9.4.4.

**Definition 9.4.10.** We denote by \( \text{Reg} \mathcal{PCon}(\mathcal{R}(x_1, \ldots, x_n)) \) the sublattice of regular principal kernels in \( \mathcal{PCon}(\mathcal{R}(x_1, \ldots, x_n)) \).

### 9.5 Corner-Integrality

**Definition 9.5.1.** Let \( S \) be a bipotent semifield and let \( f \) be an element of \( S(x_1, \ldots, x_n) \). Write \( f = \frac{h}{g} = \sum_{i=1}^{k} h_i \sum_{j=1}^{m} g_j \) where \( h_i \) and \( g_j \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, m \) are the component monomials in \( S[x_1, \ldots, x_n] \) of the numerator and denominator of \( f \), respectively. We say \( f \) is *corner-integral* if the following pair of conditions holds for every \( x \in S^n \):

\[
\exists i \neq j \in \{1, \ldots, k\} \text{ such that } h_i(x) = h_j(x) \Rightarrow \quad (9.20)
\]

\[
\exists t \in \{1, \ldots, m\} \text{ s.t. } h_i(x) \leq g_t(x) \text{ or } \exists s \in \{1, \ldots, k\} \setminus \{i, j\} \text{ s.t. } h_s(x) > h_i(x).
\]

\[
\exists i \neq j \in \{1, \ldots, m\} \text{ such that } g_i(x) = g_j(x) \Rightarrow \quad (9.21)
\]

\[
\exists t \in \{1, \ldots, k\} \text{ s.t. } g_i(x) \leq h_t(x) \text{ or } \exists s \in \{1, \ldots, m\} \setminus \{i, j\} \text{ s.t. } g_s(x) > g_i(x).
\]

In other words, \( f \in S(x_1, \ldots, x_n) \) is corner integral if for any \( x \in S^n \), if \( x \) is a corner root of \( h \) then \( g(x) \geq h(x) \) (i.e., \( g \) surpasses \( h \) at \( x \)) and if \( x \) is a corner root of \( g \) then \( h(x) \geq g(x) \) (i.e., \( h \) surpasses \( g \) at \( x \)).

**Definition 9.5.2.** Let \( S \) be a bipotent semifield. A principal kernel \( K = \langle f \rangle \) of \( S(x_1, \ldots, x_n) \) is said to be *corner-integral* if \( f \in S(x_1, \ldots, x_n) \) is corner-integral. In other words, a principal kernel is corner-integral if it has a corner-integral generator. The skeleton \( \text{Skel}(K) \) corresponding to \( K \) is said to be a corner-integral skeleton.
Remark 9.5.3. When \( f \in S(x_1, ..., x_n) \) is said to be corner integral we always take \( f \) to be a reduced fraction. For example, \( x \in \mathbb{R}(x) \) is trivially corner-integral, though \( \frac{(x+\alpha)x^2}{x+\alpha} \) for \( \alpha > 1 \) is not - since substituting \( \alpha \) for \( x \) we get \( \frac{x^2+\alpha^2}{\alpha+\alpha} \). Thus \( \alpha \) is a corner root of the numerator which is not surpassed by the denominator since \( \alpha^2 > \alpha \).

Notation 9.5.4. We denote by \( CI(S(x_1, ..., x_n)) \) the set of corner-integral elements of \( S(x_1, ..., x_n) \).

Remark 9.5.5. Let \( f \in \mathbb{R}(x_1, ..., x_n) \) be corner-integral. Then \( f^{-1} \), \( f^k \), for any \( k \in \mathbb{N} \), and \( \sum_{i=1}^{m} f^{d(i)} \) with \( d(i) \in \mathbb{Z} \), also are corner integrals.

Proof. First, \( f^{-1} \) is corner integral as the definition of corner integrality is invariant w.r.t taking inverses. Write \( f = \frac{h}{g} = \sum_{j=1}^{k} \frac{h_i}{g_j} \) where \( h_i, g_j \in \mathbb{R}[x_1, ..., x_n] \) are monomials composing \( f \)'s numerator and denominator. Then \( f^k = \frac{h^k}{g^k} \). Since the corner roots of \( h \) and \( g \) coincide with the corner roots of \( h^k \) and \( g^k \), respectively, corner integrality is preserved. For the last assertion, we consider two different cases: (1) If \( x \in \mathbb{R}^n \) then \( f'(x) = \sum_{i=1}^{m} f(x)^{d(i)} = \left(\frac{h(x)}{g(x)}\right)^{d(j)} \) for some \( j \in \{1, ..., m\} \). If \( x \) is such that \( f' = \sum_{i \in I} f(x)^{d(i)} \) for some subset \( I \subseteq \{1, ..., m\} \) where \( |I| \geq 2 \) where for any \( s, t \in I \), \( f(x)^{d(s)} = f(x)^{d(t)} \) and \( d(s) \neq d(t) \). W.l.o.g., assume \( d(t) > d(s) \), thus \( f(x)^{d(t)-d(s)} = 1 \) which by Remark 2.3.6 yields that \( f(x) = 1 \), so \( h(x) = g(x) \). Now, write \( d = \max_{i \in I} \{d(i)\} \), then \( f' = \frac{\sum_{i \in I} h(x)^{d(i)} g(x)^{d-d(i)}}{g(x)^{d}} \) and since \( h(x) = g(x) \), \( f' = \left(\frac{h(x)}{g(x)}\right)^{d} \) and integrality obviously holds. (2) If \( x \) is not as in (1) then \( f'(x) = \sum_{i=1}^{m} f(x)^{d(i)} = f(x)^{d(j)} \) for exactly one \( j \in \{1, ..., m\} \). If \( d(j) = 0 \) corner- integrality is trivial, thus we can assume \( d(j) \geq 1 \) for otherwise just take \( f^{-1} \), and by the above \( f' \) is corner integral at \( x \) as \( f^{d(j)} \) is corner-integral (at any point). Since one of the above options is true for any \( x \in \mathbb{R}^n \) we get that \( f' \) is corner-integral. \( \square \)

Remark 9.5.6. It can be shown that if \( f, g \in \mathbb{R}(x_1, ..., x_n) \) are corner-integral then \( |f|+|g| \) may not be corner-integral. Thus the collection of corner-integral principal kernels is not a lattice. In our study we thus take the lattice generated by principal corner-integral kernels which contains elements which are not corner-integral. These elements will be shown to correspond to finitely generated corner loci (to be introduced shortly) which are not principal.
9.6 Appendix : Limits of skeletons

The following discussion gives some of the flavor of the structure and behavior of irregular principal kernels of an archimedean and bipotent semifield \( \mathbb{H} \). We show that every principal kernel is a certain kind of limit of irregular principal kernels.

**Remark 9.6.1.** Consider the polynomial \( f(x) = x \). As \( |x| = x + x^{-1} \) generates the kernel \( \langle x \rangle \), the skeleton corresponding to \( f \) is

\[
\text{Skel}(f) = \text{Skel}(|f|) = \{ x \in \mathbb{H}^n : |f(x)| = 1 \},
\]

i.e., the vertical line for which \( x = 1 \). We will now consider a pair of skeletons related to \( \text{Skel}(f) \). Let \( \alpha, \beta \in \mathbb{H} \) such that \( \alpha, \beta \geq 1 \). Define the following rational functions:

\[
f_\alpha(x) = \frac{1}{\alpha} f(x) + 1 = \begin{cases} 1 & f(x) \leq \alpha; \\
\frac{f(x)}{\alpha} & f(x) \geq \alpha; \end{cases}
\]

\[
\beta f(x) = \frac{1}{\beta} f^{-1}(x) + 1 = \begin{cases} \frac{f(x)^{-1}}{\beta} & f(x) \leq \frac{1}{\beta}; \\
1 & f(x) \geq \frac{1}{\beta}. \end{cases}
\]

Define the rational function \( f_{\alpha,\beta}(x) = f_\alpha(x) + \beta f(x) \), i.e.,

\[
f_{\alpha,\beta}(x) = \begin{cases} \frac{f(x)^{-1}}{\beta} & f(x) \leq \frac{1}{\beta}; \\
1 & \frac{1}{\beta} \leq f(x) \leq \alpha; \\
\frac{f(x)}{\alpha} & f(x) \geq \alpha. \end{cases}
\]

Then \( \text{Skel}(f_{\alpha,\beta}) \) is the stripe \( \{ x : \frac{1}{\beta} \leq f(x) \leq \alpha \} \) containing \( \text{Skel}(f) \). Taking \( \alpha = \beta \), we get that

\[
f_{\alpha,\alpha}(x) = f_\alpha(x) + \alpha f(x) = \left( \frac{1}{\alpha} f(x) + 1 \right) + \left( \frac{1}{\alpha} f^{-1}(x) + 1 \right)
\]

\[
= \frac{1}{\alpha} (f(x) + f^{-1}(x)) + 1 = \frac{1}{\alpha} |f(x)| + 1.
\]

Note that \( f_\alpha, f_\beta f \) are irregular functions, and so is \( f_{\alpha,\beta} \) if either \( \alpha \neq 1 \) or \( \beta \neq 1 \). Moreover, when \( \alpha = \beta = 1 \), we have that \( f_{\alpha,\beta} = |f| \).
Assume \( \mathbb{H} \) is divisible then \( \mathbb{H} \) is dense. In such a case for \( \alpha_i, \beta_i \in \mathbb{H} \) such that \( \alpha_i, \beta_i \geq 1, \ i = 1, 2, \alpha_1 > \alpha_2, \beta_1 > \beta_2 \) we have that \( \text{Skel}(f_{\alpha_1, \beta_1}) \supset \text{Skel}(f_{\alpha_2, \beta_2}) \) (proper containment).

**Lemma 9.6.2.** If \( f \in \mathbb{H}(x_1, ..., x_n) \), then

\[
|f| = \lim_{\alpha \to +1, \beta \to +1} f_{\alpha, \beta}.
\]  

(9.22)

(Note that \( f_{\alpha, \beta} \) converge uniformly to \( |f| \) and \( \alpha \to +1 \) means \( \alpha \geq 1 \), i.e., one sided (‘positive’) limit.) Consequently,

\[
\langle f \rangle = \bigcup_{\substack{\alpha > 1 \\ \beta > 1}} \langle f_{\alpha, \beta} \rangle,
\]  

(9.23)

and

\[
\text{Skel}(f) = \bigcap_{\substack{\alpha > 1 \\ \beta > 1}} \text{Skel}(f_{\alpha, \beta}).
\]  

(9.24)

Moreover, if \( \mathbb{H} \) is divisible, we have that for \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{H} \) such that \( \alpha_i, \beta_i \geq 1, \ i = 1, 2, \alpha_1 > \alpha_2, \beta_1 > \beta_2 \) we have that

\[
\text{Skel}(f_{\alpha_1, \beta_1}) \supset \text{Skel}(f_{\alpha_2, \beta_2}).
\]  

(9.25)

and thus

\[
\text{Skel}(f) \subset \text{Int}(\text{Skel}(f_{\alpha, \beta})) \ \forall \alpha, \beta > 1.
\]  

(9.26)

Here \( \text{Int}(A) \) is the interior of \( A \) for a set \( A \).

**Proof.** The assertions follow directly from the construction of Remark 9.6.1 and Corollary 2.7.6. We note that, for \( \alpha, \beta \in \mathbb{H} \) such that \( \alpha > \beta \), since \( \alpha^{-1} \beta < 1 \) we have that \( (\alpha^{-1} \beta) + 1 = 1 \) and so \( \beta f + 1 = (\alpha^{-1} \beta)\alpha f + 1 \in \langle f \rangle \), thus \( \langle f_{\beta, \beta} \rangle \subseteq \langle f_{\alpha, \alpha} \rangle \). \( \square \)
We conclude with the following consequences for $\mathcal{R}(x_1, \ldots, x_n)$:

**Corollary 9.6.3.** Every principal kernel in $\mathcal{R}(x_1, \ldots, x_n)$ is a limit of irregular principal kernels. Every skeleton in $\mathcal{R}^n$ is a limit (with respect to inclusion) of irregular skeletons.

**Corollary 9.6.4.** Let $\langle f \rangle$ be a principal regular kernel in $\mathcal{R}(x_1, \ldots, x_n)$. Then there exists an irregular principal kernel $\langle g \rangle$ such that $\langle g \rangle \subset \langle f \rangle$ and $\text{Int}(\text{Skel}(\langle g \rangle)) \supset \text{Skel}(\langle f \rangle)$.

*Proof.* The existence of $\langle g \rangle$ follows from the above discussion. \qed
10 The corner loci - principal skeletons correspondence

In this section we will establish a connection between a geometric object, which is a subset of $\mathbb{R}^n$, called ‘corner-locus’ and a certain kind of principal skeletons. In fact, corner locus and what will be shown to be its corresponding principal kernel are two distinct ways to define the exact same subset of $\mathbb{R}^n$.

10.1 Corner loci

**Definition 10.1.1.** Let the supertropical semiring of polynomials $\mathcal{F}(\mathbb{R}[x_1, \ldots, x_n])$ be the supertropical polynomial semiring (defined in [4], Definition (4.1))

$$(R[x_1, \ldots, x_n], \mathcal{G}[x_1, \ldots, x_n], \nu)$$

where $R = \mathbb{R} \cup \nu(\mathbb{R})$, $\mathcal{G} = \nu(\mathbb{R})$ is called the ghost ideal with $\nu : R \to \mathcal{G}$ an idempotent endomorphism of semirings such that $\nu|_{\mathcal{G}} = id_{\mathcal{G}}$. $\nu$ is called the ghost map. The elements of $\mathcal{G}$ are called ghosts while the elements of $\mathbb{R}$ are called tangibles. For any monomial $f \in \mathcal{F}(\mathbb{R}[x_1, \ldots, x_n])$, $f^\nu = \nu(f) = f + f \in \mathcal{G}[x_1, \ldots, x_n]$ is called a ghost monomial. A monomial $f \in \mathcal{F}(\mathbb{R}[x_1, \ldots, x_n])$ is called tangible if $f \in \mathbb{R}[x_1, \ldots, x_n]$ (equivalently, $f \notin \mathcal{G}[x_1, \ldots, x_n]$). For monomials $f \in \mathcal{F}(\mathbb{R}[x_1, \ldots, x_n])$ and $g \in \mathcal{G}[x_1, \ldots, x_n]$ one has that

$$f^\nu = f^\nu + f \quad \text{and} \quad f \cdot g \in \mathcal{G}[x_1, \ldots, x_n].$$

A polynomial $f \in \mathcal{F}(\mathbb{R}[x_1, \ldots, x_n])$ is called tangible if none of its component monomials is ghost; otherwise we say $f$ is non-tangible.

**Note 10.1.2.** In our study we consider the evaluations of a supertropical polynomial $f \in \mathcal{F}(\mathbb{R}[x_1, \ldots, x_n])$ on the tangibles, i.e., over $\mathbb{R}^n$, i.e., we consider
\( \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]) \) as mapped to the semiring of functions \( \text{Fun}(\mathcal{R}^n, R) \) where \( R = \mathcal{R} \cup \nu(\mathcal{R}) \) as in Definition 10.1.1.

**Remark 10.1.3.** Let \( f \in \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]) \) be a supertropical polynomial. Then \( f \) can be written uniquely in the form

\[
f = \sum_{i=1}^{t} h_i + \sum_{j=1}^{s} g_j^\nu
\]  

(10.1)

where \( h_i, g_j \in \mathcal{R}[x_1, \ldots, x_n] \) are distinct monomials for \( 1 \leq i \leq t \) and \( 1 \leq j \leq s \). The polynomial \( h = \sum_{i=1}^{t} h_i \in \mathcal{R}[x_1, \ldots, x_n] \) is called the tangible part of \( f \) and the polynomial \( g^\nu = g + g \in \mathcal{G}[x_1, \ldots, x_n] \) for \( g = \sum_{j=1}^{s} g_j \) is called the ghost part of \( f \). Thus \( f \) can be expressed uniquely in the form:

\[
f = h + g^\nu = h + (g + g) = \sum_{i=1}^{t} h_i + \sum_{j=1}^{s} (g_j + g_j).
\]

In view of this we can consider an element of \( \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]) \) as a polynomial each of whose component monomials occurs either once, if it is a component monomial of the tangible part, or twice, if it is a component monomial of the ghost part.

**Note 10.1.4.** In what follows we consider a supertropical polynomial \( f \) as a sum of tangible monomials in \( \mathcal{R}[x_1, \ldots, x_n] \) where a component monomial occurs once if it belongs to the tangible part of \( f \) and twice if it belongs to the ghost part of \( f \).

**Example 10.1.5.** The supertropical polynomial \( f(x, y) = y^\nu + x + 1^\nu \) is considered as \( x + y + y + 1 + 1 \) where \( h(x, y) = x \) is its tangible part and \( g(x, y) = y + y + 1 + 1 \) is its ghost part.

We begin by introducing the well known notion of tropical geometry called `corner root`.

**Definition 10.1.6.** Let \( f \in \mathcal{R}[x_1, \ldots, x_n] \) be a polynomial. Then \( f = \sum_{i=1}^{k} f_i \) where each \( f_i \) is a monomial. A point \( a \in \mathcal{R}^n \) is said to be a corner-root of \( f \) if there exist two distinct monomials \( f_i \) and \( f_s \) of \( f \) such that \( f(a) = f_s(a) = f_t(a) \).
In [4, Section (5.2)] Izhakian and Rowen have generalized the notion of (tangible) corner-root to $F(\mathbb{R}[x_1, \ldots, x_n])$ as follows:

**Definition 10.1.7.** Let $f \in F(\mathbb{R}[x_1, \ldots, x_n])$ be a supertropical polynomial. Write $f = \sum_{i=1}^{k} f_i$ where each $f_i$ is a monomial. A point $a \in \mathbb{R}^n$ is said to be a corner-root of $f$ if $f(a) \in \mathcal{G}$, i.e., if $f$ obtains a ghost value at $a$. This happens in one of the following cases:

1. There exist two distinct monomials $f_t$ and $f_s$ of $f$ such that $f(a) = f_s(a) = f_t(a)$.
2. There exists a ghost monomial $f_t$ of $f$ such that $f(a) = f_t(a)$.

**Definition 10.1.8.** A set $A \subseteq \mathbb{R}^n$ is said to be a generalized corner-locus if $A$ is a set of the form

$$A = \{ x \in \mathbb{R}^n : \forall f \in S, \ x \text{ is a corner root of } f \} \quad (10.2)$$

for some $S \subset F(\mathbb{R}[x_1, \ldots, x_n])$.

We write $Cor(S)$ for the corner locus defined by $S$. In the case where $S = \{f_1, \ldots, f_r\}$ is finite, we write $A = Cor(f_1, \ldots, f_r)$ to indicate that $A$ is a corner locus defined by the mutual corner roots of $f_1, \ldots, f_r \in F(\mathbb{R}[x_1, \ldots, x_n])$, and say that $A$ is a finitely generated corner locus.

A corner locus $A \subseteq \mathbb{R}^n$ is called principal if there exists a supertropical polynomial $f \in F(\mathbb{R}[x_1, \ldots, x_n])$ such that $A = Cor(f)$. A corner locus $A \subseteq \mathbb{R}^n$ is called regular if $A = Cor(S)$ for $S \subset R[x_1, \ldots, x_n]$ (i.e., $S$ contains only tangible polynomials). A corner locus not indicated to be regular is assumed to be generalized.

In view of Definition 10.1.8 we define an operator $Cor: \mathbb{P}(F(\mathbb{R}[x_1, \ldots, x_n])) \to \mathbb{R}^n$ (where $\mathbb{P}(F(\mathbb{R}[x_1, \ldots, x_n]))$ is the power set of $F(\mathbb{R}[x_1, \ldots, x_n])$)

$$Cor: S \subset F(\mathbb{R}[x_1, \ldots, x_n]) \mapsto Cor(S). \quad (10.3)$$

We now proceed to study the behavior of the $Cor$ operator.

As a trivial consequence of Definition 10.1.8 we have the following

**Remark 10.1.9.** If $a, b \in \mathbb{R}^n$ are corner roots of $f, g \in F(\mathbb{R}[x_1, \ldots, x_n])$ respectively, then both $a$ and $b$ are corner roots of $f \cdot g$. 
Lemma 10.1.11. For the case where

$$S_A \subseteq S_B \Rightarrow \Cor(S_B) \subseteq \Cor(S_A).$$

(10.4)

Let \( \{S_i\}_{i \in I} \) be a family of subsets of \( \mathcal{F}(\mathbb{R}[x_1, \ldots, x_n]) \) for some index set \( I \). Then \( \bigcap_{i \in I} \Cor(S_i) \) is a corner locus and

$$\bigcap_{i \in I} \Cor(S_i) = \Cor\left( \bigcup_{i \in I} S_i \right) ; \bigcup_{i \in I} \Cor(S_i) \subseteq \Cor\left( \bigcap_{i \in I} S_i \right).$$

(10.5)

In particular, \( \Cor(S) = \bigcap_{f \in S} \Cor(f) \).

Proof. First, equality (10.4) is a set-theoretical direct consequence of the definition of corner loci. In turn this implies that \( \Cor(\bigcup_{i \in I} S_i) \subseteq \Cor(S_i) \) for each \( i \in I \) and thus \( \Cor(\bigcup_{i \in I} S_i) \subseteq \bigcap_{i \in I} \Cor(S_i) \). Conversely, if \( x \in \mathbb{R}^n \) is in \( \bigcap_{i \in I} \Cor(S_i) \) then \( x \in \Cor(S_i) \) for every \( i \in I \), which means that \( x \) is a common corner root of \( \{f : f \in S_i\} \). Thus \( x \) is a common corner root of \( \{f : f \in \bigcup_{i \in I} S_i\} \) which yields that \( x \in \Cor(\bigcup_{i \in I} S_i) \). For the second equation (inclusion) in (10.5), for each \( j \in I \), \( \bigcap_{i \in I} S_i \subseteq S_j \). Thus, by (10.4), \( \Cor(S_j) \subseteq \Cor(\bigcap_{i \in I} S_i) \), and so, \( \bigcup_{i \in I} \Cor(S_i) \subseteq \Cor(\bigcap_{i \in I} S_i) \).

Lemma 10.1.11. For the case where \( A \) and \( B \) are finitely generated. If

\( A = \Cor(f_1, \ldots, f_s) \) and \( B = \Cor(g_1, \ldots, g_t) \), then \( A \cap B = \Cor(f_1, \ldots, f_s, g_1, \ldots, g_t) \), and \( A \cup B = \Cor(\{f_i \cdot g_j\}_{i=1,j=1}^{s,t}) \).

Proof. This is a consequence of Definition 10.1.8 and Remarks 10.1.9 and 10.1.10. As the first equality is obvious, we only prove \( A \cup B = \Cor(\{f_i \cdot g_j\}_{i=1,j=1}^{s,t}) \).

Indeed, for each \( 1 \leq i \leq s \) and each \( 1 \leq j \leq t \), \( A \subseteq \Cor(f_i) \) and \( B \subseteq \Cor(g_j) \). Thus \( A \cup B \subseteq \Cor(f_i) \cup \Cor(g_j) = \Cor(\{f_i \cdot g_j\}) \) and so

\[ A \cup B \subseteq \bigcap_{i=1}^{s} \bigcup_{j=1}^{t} \Cor(\{f_i \cdot g_j\}) = \Cor(\{f_i \cdot g_j\}_{i=1,j=1}^{s,t}). \]

On the other hand, if \( a \not\in A \cup B \) then there exist some \( i_0 \) and \( j_0 \) such that \( a \not\in \Cor(f_{i_0}) \) and \( a \not\in \Cor(g_{j_0}) \). Thus \( a \not\in \Cor(f_{i_0}) \cup \Cor(g_{j_0}) = \Cor(f_{i_0} \cdot g_{j_0}) \).

So \( a \not\in \bigcap_{i,j=1}^{s,t} \Cor(\{f_i \cdot g_j\}) = \Cor(\{f_i \cdot g_j\}_{i=1,j=1}^{s,t}) \) proving the opposite inclusion.
Remark 10.1.12. In view of Lemma 10.1.11 it is obvious that if $A = \text{Cor}(f_1, \ldots, f_s)$ and $B = \text{Cor}(g_1, \ldots, g_t)$ are regular then $A \cap B$ and $A \cup B$ are regular (i.e., defined by tangible polynomials).

Definition 10.1.13. Denote the collection of corner loci in $\mathbb{R}^n$ by $CL(\mathbb{R}^n)$ and by $RCL(\mathbb{R}^n)$ the family of regular corner loci. Denote the collection of finitely generated corner loci by $FCL(\mathbb{R}^n) \subset CL(\mathbb{R}^n)$ and the collection of principal corner loci by $PCL(\mathbb{R}^n) \subset FCL(\mathbb{R}^n)$. Analogously we denote the collection of finitely generated regular corner loci by $FRCL(\mathbb{R}^n) \subset RCL(\mathbb{R}^n)$ and the collection of principal regular corner loci by $PRCL(\mathbb{R}^n) \subset FRCL(\mathbb{R}^n)$.

Remark 10.1.14. By Remark 10.1.10 and Lemma 10.1.11 $CL(\mathbb{R}^n)$ is closed under intersections, while $FCL(\mathbb{R}^n)$ is also closed under finite unions. Taking $f = 1 + 1 = 1^\nu$ and $g = \alpha$ with $\alpha \neq 1$, we get that $\mathbb{R}^n = \text{Cor}(f)$ and $\emptyset = \text{Cor}(g)$ are in $FCL(\mathbb{R}^n)$. By Remark 10.1.12 $RFCL(\mathbb{R}^n)$ is a sublattice of $FCL(\mathbb{R}^n)$.

10.2 Corner loci and principal skeletons

Having defined a corner locus, we proceed to construct the connection between corner loci and principal skeletons. We begin by constructing it for the special case of principal corner locus.

Proposition 10.2.1. Any principal corner locus of $\mathbb{R}^n$ (a set of corner roots of a supertropical polynomial) is a principal skeleton. In fact, there is a map $f \mapsto \hat{f}$ sending a supertropical polynomial $f$ to a rational function $\hat{f}$ such that $x \in \mathbb{R}^n$ is a corner root of $f$ if and only if $\hat{f}(x) = 1$.

Proof. Let $f = \sum_{i=1}^{k} f_i \in \mathcal{F}(\mathbb{R}[x_1, \ldots, x_n])$ be a polynomial represented as the sum of its component monomials $f_i$, $i = 1, \ldots, k$, where some of them may not be distinct (in fact, appear exactly twice, see Note 10.1.4). Define the following element of $\mathbb{R}(x_1, \ldots, x_n)$:

$$\hat{f} = \sum_{i=1}^{k} \frac{f_i}{\sum_{j \neq i} f_j}.$$  \hfill (10.6)
Then $x \in \mathcal{R}^n$ is a corner root of $f$ if and only if $\hat{f}(x) = 1$. Moreover, if $x$ is a corner root of multiplicity $m$ then $\hat{f}(x) = [m+1] 1$ (the notation $[m+1]1$ indicates that 1 occurs $m+1$ times).

If $x$ is a corner root of $f$, then there exists a subset of $2 \leq r \leq m$ monomials, say $f_1, ..., f_r$ such that $f_i(x) = ... = f_r(x) = f(x)$ and $f_i(x) > f_j(x)$ for every $i = 1, ..., r$ and $j = r + 1, ..., m$. Notice that, as $r > 1$, each denominator in $\hat{f}$ must contain as a summand some $f_i$ with $1 \leq i \leq r$. Thus, every denominator equals $f(x)$. Now, the number of numerators in $\hat{f}$ obtaining the value $f(x)$ at $x$ is exactly $r$, one for each monomial $f_i$ such that $1 \leq i \leq r$, so there are exactly $r$ summands of $\hat{f}$ obtaining the value $f(x)/f(x) = 1$ at $x$. As every other summand is of the form $\frac{f_i(x)}{\sum_{j \neq i} f_j(x)}$ with $s \in \{r+1, ..., m\}$, since $f_s(x) < f(x)$, all these summands are inessential at $x$, yielding the first direction of our claim.

Conversely, let $\hat{f}(x) = [m+1] 1$ for some $x \in \mathcal{R}$, then there are $m+1$ summands $g_1 = \frac{f_1(x)}{\sum_{j \neq i} f_j(x)}$, say $g_1, ..., g_{m+1}$ such that $g_1(x) = ... = g_{m+1}(x) = 1$. Then, for each $i = 1, ..., m+1$, we have $f_i(x) = \sum_{j \neq i} f_j(x)$ so there exists at least one essential $f_j$ with $j \neq i$ such that $f_j(x) = f_i(x)$. Notice that this last observation yields that $f_i(x)$ is essential in $\sum_{i=1}^m f_i(x)$ and that also $g_j(x) = 1$. Consequently, there are exactly $m+1$ essential monomials at $x$, $f_1, ..., f_{m+1}$ obtaining the same value at $x$, which in turn yields that $x$ is a corner root of multiplicity $m = (m+1) - 1$, as desired.

Finally, by the above, $\text{Skel}(\hat{f}) = \text{Cor}(f)$, i.e., the skeleton defined by $\hat{f}$ is exactly the corner locus of $f$.

\textbf{Corollary 10.2.2.} $f \in \mathcal{F}(\mathcal{R}[x_1, ..., x_n])$ is non-tangible iff $\hat{f} \in \mathcal{R}(x_1, ..., x_n)$ is irregular.

\textbf{Proof.} If we take $f \in \mathcal{F}(\mathcal{R}[x_1, ..., x_n])$ to be non-tangible, say $f_k$ and $f_s$ are the same monomial $g$ for some $1 \leq s, k \leq m$, then the terms $\frac{f_k}{\sum_{j \neq k} f_j}$ and $\frac{f_s}{\sum_{j \neq s} f_j}$ of $\hat{f}$ would be irregular since $g$ occurs as the monomial in the numerator and as one of the summands of the denominator. The essentiality of $g$ in $f$ implies that at least one of those terms is essential in $\hat{f}$, making it an irregular element of $\mathcal{R}(x_1, ..., x_n)$. Reversing the last arguments yields that if $\hat{f}$ is irregular then $f$ is non-tangible, i.e., has a component monomial which is ghost. 

\hfill $\Box$
Proposition 10.2.3. Write \( \hat{f} = \sum_{i=1}^{k} h_i \) where each \( h_i = \frac{f_i}{\sum_{j \neq i} f_j} \) for \( i = 1, \ldots, k \).
Then, for every \( 1 \leq i \leq k \),
\[
h_i(x) = 1 \Rightarrow h_j(x) \leq 1, \; \forall j \neq i.
\] (10.7)

Proof. If \( x \in \mathbb{R}^n \) be such that \( h_i(x) = 1 \), then there exists \( k \neq i \) such that \( f_k(x) = f_i(x) \) and \( f_k(x) \geq f_t(x) \) for every \( t \in \{1, \ldots, n\} \setminus \{i, k\} \). Thus \( h_k(x) = h_i(x) = 1 \) and \( h_t(x) = \frac{f_t}{\sum_{j \neq t} f_j} \leq \frac{f_k}{f_k} \leq 1 \). So \( h_j(x) \leq 1 \) for every \( j \in \{1, \ldots, n\} \).

Remark 10.2.4. Let \( \Phi^* : PCL(\mathbb{R}^n) \to PSkl(\mathbb{R}^n) \) denote the map
\[
\Phi^* : Cor(f) \mapsto Skel(\hat{f})
\] (10.8)
induced by the map \( f \mapsto \hat{f} \) given in Proposition 10.2.1. For \( S \subset F(\mathbb{R}[x_1, \ldots, x_n]) \), let \( \hat{S} = \{ \hat{f} = \Phi^*(f) : f \in S \} \subseteq \mathbb{R}(x_1, \ldots, x_n) \). Then by Proposition 4.1.5 and Remark 10.1.10 we have that
\[
Cor(S) = \bigcap_{f \in S} Cor(f) = \bigcap_{g \in \hat{S}} Skel(g) = Skel(\hat{S}).
\]
Thus, \( \Phi^* \) extends to a map
\[
\Phi : CL(\mathbb{R}^n) \to Skl(\mathbb{R}^n)
\]
where \( \Phi : CL(f) \mapsto Skl(\hat{f}) \). In particular, taking only finite generated corner loci, and recalling that finite intersections and unions of principal skeletons are principal skeletons, \( \Phi \) restricts to the map \( \Phi|_{FCL(\mathbb{R}^n)} : FCL(\mathbb{R}^n) \to PSkl(\mathbb{R}^n) \).
As our interest is in the latter map we will denote \( \Phi|_{FCL(\mathbb{R}^n)} \) by \( \Phi \).
Note that since the map \( f \mapsto \hat{f} \) sends tangible elements of \( F(\mathbb{R}[x_1, \ldots, x_n]) \) (i.e., elements of \( \mathbb{R}[x_1, \ldots, x_n] \)) to regular elements of \( \mathbb{R}(x_1, \ldots, x_n) \) we also have that \( \Phi|_{FRCL(\mathbb{R}^n)} : FRCL(\mathbb{R}^n) \to RegPSkl(\mathbb{R}^n) \).

Lemma 10.2.5. Let \( f = \sum_{i=1}^{k} f_i \in F(\mathbb{R}[x_1, \ldots, x_n]) \) be a polynomial represented as the sum of its component monomials \( f_i, \; i = 1, \ldots, k \).
For any \( i = 1, \ldots, k \), denote the \( i \)'th summand of \( \hat{f} \) defined in Proposition 10.2.1 by \( A_i = \frac{f_i}{\sum_{j \neq i} f_j} \in \mathbb{R}(x_1, \ldots, x_n) \).
Then for \( 1 \leq i, s \leq k \) such that \( i \neq s \)
\[
A_i = A_s \iff A_i = A_s = 1 \; \text{or} \; A_i = A_s \text{ are inessential in } \hat{f}.
\] (10.9)
Proof. Assume \( A_i = A_s \), i.e., \( \sum_{j \neq i} f_j = \sum_{j \neq s} f_j \). Since we have multiplicative cancellation we can multiply both sides by the product of the denominators without changing the equality. Thus we can rephrase this equation as

\[
f_i(f_i + A) = f_s(f_s + A); \quad A = \sum_{j \neq i, s} f_j
\] (10.10)

Now, we can partition \( \mathbb{R}^n \) into the following distinct regions:

1. \( f_i \) is the only essential monomial, in which case (10.10) has the form \( f_i^2 = f_s \cdot f_i \) and thus \( f_i = f_s \) which cannot hold (since \( f_i \) is the only essential monomial).

2. \( f_s \) is the only essential monomial, in which case (10.10) has the form \( f_i \cdot f_s = f_s^2 \) and thus \( f_i = f_s \) which cannot hold.

3. \( f_i = f_s \) are both essential.

4. Both \( f_i \) and \( f_s \) are non-essential, in which case (10.10) has the form \( f_i \cdot A = f_s \cdot A \) and thus \( f_i = f_s \).

Thus, the only possible solutions are \( f_i = f_s \) both essential and \( f_i = f_j < A \). When \( f_i = f_s \) are essential we have that \( A_i = \frac{f_i}{f_s} = \frac{f_s}{f_s} = A_s = 1 \). When \( f_i = f_j < A \) we have that \( A_j = \frac{f_j}{A} = \frac{f_j}{A} = A_s < 1 \). Since there is always some \( 1 \leq j \leq k \) such that \( A_j \geq 1 \) (see the proof of Proposition 10.2.1) we have that both \( A_i \) and \( A_s \) are inessential in \( \hat{f} \). The converse direction of (10.10) is obvious. \( \Box \)

Corollary 10.2.6. By Proposition 10.2.3 we can strengthen (10.10) as follows:

\[
A_i = A_s \text{ iff } A_i = A_s = 1 \text{ and are essential in } \hat{f} \text{ or } A_i = A_s \text{ are inessential in } \hat{f}.
\] (10.11)

We now establish the connection between tropical essentiality as described in Definition 9.1.1 and the essentiality introduced in Definition 9.1.5. We show that the map \( \Phi \) of Remark 10.2.4 sending a finitely generated corner locus to its corresponding principal skeleton, sends tropical inessential terms of a defining polynomial \( f \) of a principal locus to inessential terms in the defining element \( \hat{f} \) of the skeleton, and can be generally be omitted from the definition of \( \hat{f} \) without changing the skeleton defined by \( \hat{f} \).
Proposition 10.2.7. Let \( f = \sum_{i=1}^{k} f_i \in \mathcal{F}(R[x_1, \ldots, x_n]) \) be a polynomial represented as the sum of its component monomials \( f_i, \ i = 1, \ldots, k \). Let \( \hat{f} = \sum_{i=1}^{k} \frac{f_i}{\sum_{j \neq i, j} f_j} \). If \( f_m \) is inessential in \( f \), then \( \frac{f_m}{\sum_{j \neq m, j} f_j} \) is inessential in \( \hat{f} \).

Proof. Denote \( f' = \sum_{i=1, i \neq m}^{k} \frac{f_i}{\sum_{j \neq i, j} f_j} \), for each \( 1 \leq i \leq k \) denote \( g_i = \frac{f_i}{\sum_{j \neq i, j} f_j} \) and for \( i \neq m \) define \( g'_i = \frac{f_i}{\sum_{j \neq i, m} f_j} \) (\( g_i \) after omitting \( f_m \) from the denominator). If \( f_m \) is inessential in \( f \), then for every \( x \in R^n \) there exists a monomial \( f_t \), \( t = t(x) \neq m \) of \( f \) such that \( f_m(x) \leq f_t(x) \). Thus \( g'_i(x) = g_i(x) \) for every \( x \in R^n \) and we can write \( f' = \sum_{i=1, i \neq m}^{k} g'_i \). Consequently \( g_m = \frac{f_m}{\sum_{j \neq m, j} f_j} \leq \frac{f_m}{f_t} \leq 1 \). So \( g_m \) does not surpass any ‘roots’ of \( f' \). Now we have to show that \( g_m \) does not contribute ‘roots’ either. Indeed, if \( g_m(x) = 1 \) then there exists some \( f_s \) with \( s \neq m \) such that \( f_s(x) = f_m(x) \) and \( f_s(x) \geq f_j(x) \) for all \( j \in \{1, \ldots, k\} \setminus \{m, s\} \). Since \( f_m \) is inessential there must exists some \( l \notin \{s, m\} \) such that \( f_l(x) \geq f_s(x), f_m(x) \) (and so, \( f_l(x) = f_s(x) \)), for otherwise \( f_s \) and \( f_m \) are the only monomials defining a corner root of \( f \) at \( x \) which yields that \( f_m \) is essential in \( f \), contradicting our assumption. Thus \( g'_i(x) = \frac{f_i(x)}{\sum_{j \neq i, m} f_j(x)} = \frac{f_i(x)}{f_s(x)} = 1 \). Note that we could also take \( g'_i \) instead of \( g_i \). We have proved that \( g_m \) and \( f_m \) could be omitted from \( \hat{f} \) without affecting its skeleton and thus inessential by definition. \( \square \)

Lemma 10.2.8. Any one of the summands \( \frac{f_i}{\sum_{j \neq i, j} f_j} \) of the fractional function

\[
\hat{f} = \sum_{i=1}^{k} \frac{f_i}{\sum_{j \neq i, j} f_j} \in \mathcal{F}(x_1, \ldots, x_n)
\]

of Proposition 10.2.7 can be omitted, without affecting the skeleton of \( \hat{f} \).

Proof. For each \( 1 \leq i \leq k \) denote \( g_i = \frac{f_i}{\sum_{j \neq i, j} f_j} \). Without loss of generality, consider the term \( g_1 = \frac{f_1}{\sum_{j \neq 1, j} f_j} \). Then if \( g_1(x) = 1 \) for \( x \in R^n \), then there exists \( s \neq 1 \) such that \( f_s(x) = f_1(x) \) and \( f_s(x) \) is dominant in the denominator. Thus \( f_i(x) \leq f_1(x) = f_s(x) \) for all \( i \notin \{1, s\} \). This last observation yields that also \( g_s(x) = 1 \). So, we have shown that \( Skel(g_1) \subseteq Skel(\sum_{i=2}^{k} g_i) \). Now, it remains to show that \( g_1 \) does not surpass any point of the skeleton defined by \( \sum_{i=2}^{k} g_i \). Assume \( g_j(x) = 1 \) for some \( j \neq 1 \). Then as above there exists \( s \neq j \) such that \( f_s(x) = f_j(x) \) and \( f_s(x) \) is dominant in the denominator and \( f_i(x) \leq f_j(x) = f_s(x) \) for all \( i \neq j, s \). If \( s \neq 1 \) then we get that \( g_1(x) \leq 1 \) and
thus \( g_j \) is not surpassed by \( g_1 \) at \( x \). Thus assume \( s = 1 \). In such a case we get \( g_1(x) = \frac{f_1(x)}{f_1(x)} = 1 \), and once again \( g_j \) is not surpassed by \( g_1 \) at \( x \). As the occurrence of \( g_1 \) does not add or delete any point of the skeleton of \( \sum_{i=2}^{k} g_i \), it can be omitted. Since \( \text{Skel}(\sum_{i=2}^{k} g_i) = \text{Skel}(\tilde{f}) \) we have that \( \langle \sum_{i=2}^{k} g_i \rangle = \langle f \rangle \), thus \( \sum_{i=2}^{k} g_i \) is a generator of \( \langle \tilde{f} \rangle \).

\[ \] **Remark 10.2.9.** Let \( f = \sum_{i=1}^{k} f_i \in \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]) \) be a supertropical polynomial represented as the sum of its component monomials \( f_i \), \( i = 1, \ldots, k \). In Proposition 10.2.1 we considered the fractional function \( \tilde{f} = \sum_{i=1}^{k} \frac{f_i}{\sum_{j \neq i} f_j} \in \mathcal{R}(x_1, \ldots, x_n) \) and showed that \( \text{Skel}(\tilde{f}) = \text{Cor}(f) \). Now, consider the element \( \tilde{f} = \bigwedge_{i=1}^{k} \left| \frac{f_i}{\sum_{j \neq i} f_j} \right| = \min_{i=1}^{k} \left\{ \frac{f_i}{\sum_{j \neq i} f_j} \right\} \) of \( \mathcal{R}(x_1, \ldots, x_n) \). By definition \( \tilde{f} \geq 1 \). Moreover, by Proposition 10.2.3 \( \tilde{f} \) fulfills the condition (9.17) of Lemma 9.2.35 so \( \tilde{f}(x) = 1 \Leftrightarrow \tilde{f}(x) = 1 \). Thus \( \text{Skel}(\tilde{f}) = \text{Skel}(\tilde{f}) \).

In view of the above, we can say the following: the skeleton of \( \tilde{f} \) is \( \text{Cor}(f) \). By the correspondence between principal kernels and skeletons we have that \( \langle \tilde{f} \rangle = \langle \tilde{f} \rangle \), or equivalently, \( \tilde{f} \sim_K \tilde{f} \).

As a consequence of Remark 10.2.9 we get the following corollary:

**Corollary 10.2.10.** A corner locus of \( f = \sum_{i=1}^{k} f_i \in \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]) \) such that \( \tilde{f} \) contains more than 2 (distinct) essential terms \( g_i = \frac{f_i}{\sum_{j \neq i} f_j} \), is a reducible skeleton (with respect to principal skeletons), i.e., the kernel \( \langle \tilde{f} \rangle \) is reducible (with respect to the sublattice of principal kernels \( \text{PCon}(\mathcal{R}(x_1, \ldots, x_n)) \)).

**Proof.** Let \( \tilde{f} \) be in reduced form. \( \tilde{f} = \bigwedge_{i=1}^{k} \left| \frac{f_i}{\sum_{j \neq i} f_j} \right| \) implies that \( \langle \tilde{f} \rangle = \bigcap_{i=1}^{k} \left\{ \frac{f_i}{\sum_{j \neq i} f_j} \right\} \) and so \( \text{Skel}(\tilde{f}) = \text{Skel}(\tilde{f}) = \bigcup_{i=1}^{k} \text{Skel}(\frac{f_i}{\sum_{j \neq i} f_j}) \). For each \( 1 \leq i \leq k \), denote \( g_i = \frac{f_i}{\sum_{j \neq i} f_j} \). Assume that \( \tilde{f} \) is irreducible, then there exists \( 1 \leq t \leq k \) such that \( \text{Skel}(\tilde{f}) = \text{Skel}(g_t) \). Then \( \text{Skel}(g_t) + \sum_{i=1; i \neq t}^{k} g_i = \text{Skel}(\tilde{f}) = \text{Skel}(g_t) \), and thus \( \sum_{i=1; i \neq t}^{k} g_i \) is inessential, contradicting our assumption that \( \tilde{f} \) contains more than two essential terms \( g_i \).

**Remark 10.2.11.** We now describe the kernels corresponding to regular corner loci composed of at most two monomials. Assume \( f \in \mathcal{R}[x_1, \ldots, x_n] \) is of the form \( f = f_1 + f_2 \) where \( f_1 \) and \( f_2 \) are two distinct monomials. Then by the construction introduced in Proposition 10.2.1, \( f \) translates to \( \tilde{f} = \frac{f_1}{f_2} + \frac{f_2}{f_1} = (\frac{f_1}{f_2}) + (\frac{f_2}{f_1})^{-1} = |\frac{f_1}{f_2}| \). Note that \(|\frac{f_1}{f_2}| \not\sim_K |g| \land |h| = |\frac{|g|+|h|}{|g|}| |h| \) for any \(|g|, |h| \in \mathcal{R}(x_1, \ldots, x_n) \) such that \(|g| \land |h| \not\sim_K \{ |g|, |h| \} \) (i.e., \( \text{Skel}(g) \not\subseteq \text{Skel}(h) \) and \( \text{Skel}(h) \not\subseteq \text{Skel}(g) \)), since the monomials \( f_1 \) and \( f_2 \) dominate the numerator and denominator, respectively, over
all of $R^n$ while $|g| \land |h|$ has at least two monomials dominating its numerator. Thus $\hat{f}$ is irreducible and so corresponds to an irreducible kernel. In the case where $f = f_1$ is a single monomial, $f$ translates to $\hat{f} = \frac{f_1}{g_1} + \frac{0}{f_1} = |0|$, and thus $\{\hat{f} = 1\} = \emptyset$.

We now give an example of applying our theory to the well-known tropical line $y + x + 1$.

**Example 10.2.12.** Consider the polynomial $f = y + x + 1$ in $R(x, y)$. Then

$$\hat{f} = \frac{y}{x+1} + \frac{x}{y+1} + \frac{1}{x+y}. $$

This form is not reduced, since each of the component fractions can be omitted from the sum without affecting the skeleton of $\hat{f}$. Geometrically these three terms correspond to the three angles formed by the tropical line, of which one can be evidently omitted. $\hat{f}$ is corner integral and regular and thus $\tilde{f} = \left|\frac{y}{x+1}\right| \land \left|\frac{x}{y+1}\right| \land \left|\frac{1}{x+y}\right|$ is also a generator of $\langle \hat{f} \rangle$, i.e. $\hat{f} \sim \tilde{f}$. In reduced form, we can take $\hat{f} = \frac{y}{x+1} + \frac{x}{y+1}$ or $\hat{f} = \frac{1}{x+y} + \frac{x}{y+1}$. Say we take the former. Then $\tilde{f} = \left|\frac{y}{x+1}\right| \land \left|\frac{x}{y+1}\right|$. It can be easily seen that no more than one of the three terms introduced above can be omitted from $\hat{f}$ without changing the skeleton.

**Definition 10.2.13.** Let $f \in R(x_1, ..., x_n)$. Write $f = \sum_{i=1}^{k} f_i$, with each $f_i \in R(x_1, ..., x_n)$ having monomial numerator for $i = 1, ..., k$. We say that $f$ admits the bound property if for any $x \in R^n$ the following condition holds:

$$\forall 1 \leq i, j \leq k \text{ such that } i \neq j : f_i(x) = f_j(x) \iff f_i(x) = f_j(x) = 1$$

or $\exists s \in \{1, ..., k\} \setminus \{i, j\}$ such that $f_i(x), f_j(x) < f_s(x)$.

The latter option means that $f_i$ and $f_j$ are not essential in $f$ at $x$.

**Remark 10.2.14.** Let $h \in R(x_1, ..., x_n)$. Then, if $h$ admits the bound property then $h$ admits condition (11.1) and $h^{-1}$ admits condition (11.2).

**Proof.** Clearly, the second assertion follows the first, by taking inverse elements. Write $h = \frac{f}{g} = \sum_{j=1}^{k} \frac{f_j}{g_j}$, where $f_i, g_j \in R[x_1, ..., x_n]$ are monomials composing $h$’s numerator and denominator. Then $h = \sum_{i=1}^{k} \left(\frac{f_i}{g}\right)$. Assume there exists some
$x \in R^n$ such that $f_i(x) = f_j(x)$, then we have that $\frac{f_i(x)}{g(x)} = \frac{f_j(x)}{g(x)}$. So by the boundary property this yields that either $\frac{f_i(x)}{g(x)} = 1 = \frac{f_j(x)}{g(x)}$ or $\frac{f_i(x)}{g(x)} = \frac{f_j(x)}{g(x)}$ are inessential in $h$. The former yields that $f_i(x) = g(x) = f_j(x)$ so there is some $t \in \{1, ..., m\}$ such that $f_i(x) \leq g_t(x)$. The latter implies that $f_i(x)$ is inessential, as desired. 

\textbf{Proposition 10.2.15.} Let $f \in F_R[x_1, ..., x_n]$ be a supertropical polynomial. Then $\hat{f}$ defined in Proposition 10.2.1 is corner integral. Moreover, if $f$ is tangible (no monomial occurs twice) then $\hat{f}$ is regular.

\textit{Proof.} As a consequence of Lemma 10.2.5 we get that $\hat{f}$ admits the bound property. Thus by Remark 10.2.14 we have that condition (11.1) holds. For condition (11.2), writing $\hat{f} = h/g$ we have that every corner root of $g$ is surpassed by some summand of $h$. First note that the corner roots of $g$ are exactly those of the polynomial $w = \sum_{i=1}^{k} f_i(\sum_{l \neq i} f_l(x))$ and $g = \prod_{i=1}^{k} f_i(\sum_{l \neq i} f_l(x))$. We want to show that every corner root of $g$ is surpassed by some summand of $h$. First note that the corner roots of $g$ are exactly those of the polynomial $w = \sum_{i=1}^{k} f_i$. Indeed, if $x \in R^n$ is a corner root of $w$ then is a corner root of precisely $k - 2$ factors ($\sum_{l \neq i} f_l$) with $i$ an index of any chosen pair of essential monomials. Conversely, if $x \in R^n$ is a corner root of $g$ then it is a corner root of $k - 2$ of its factors not involving the essential monomials, say $f_i$ and $f_j$, which yields essentiality of $f_i$ and $f_j$ in $w$, proving our claim. In view of the last assertion, we can restrict our attention to the corner roots of $w$. Now, assume $f_i(x) = f_j(x)$ for some $i, j \in \{1, ..., k\}$. If there exists $s \in \{1, ..., k\} \setminus \{i, j\}$ such that $f_s(x) > f_i(x)$, take the one attaining the maximal value at $x$, the $f_s(\sum_{l \neq i} f_l(x)) \geq g(x)$. If no such $s$ exists, then $f_i(\sum_{l \neq i} f_l(x)) = f_j(\sum_{l \neq i} f_l(x))$ as desired.

By the construction of $\hat{f}$ and Corollary 10.2.2 irregularity of $\hat{f}$ implies a multiple occurrence of the same monomial in $f$, so $f$ is non-tangible thus the second assertion follows. \qed
Proposition 10.2.16. For \( h = \frac{\sum_{i=1}^{k} f_i}{\sum_{j=1}^{m} g_j} \in \mathcal{R}(x_1, \ldots, x_n) \) where \( f_i, g_j \in \mathcal{R}[x_1, \ldots, x_n] \), define the polynomial

\[
\bar{h} = \sum_{i=1}^{k} f_i + \sum_{j=1}^{m} g_j \in \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]).
\] (10.12)

Let \( Z = \text{Skel}(h) \subseteq \mathbb{R}^n \) be a principal corner integral skeleton. Then \( Z \) corresponds to a corner locus of the supertropical polynomial \( h \in \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]) \), in the sense that the generalized corner locus \( \text{Cor}(h) \) coincides with \( Z \) in \( \mathbb{R}^n \). If \( h \) is also regular then \( h \) is tangible so \( \text{Cor}(h) \in \text{PRCL}(\mathbb{R}^n) \) is a regular corner locus.

Proof. The corner locus of (10.12) is

\[
U = \bigcup_{i=1,j=1}^{k,m} \{ f_i = g_j, f_i, g_j \geq f_s, g_t \forall s \neq i, t \neq j \}
\]

\[
\bigcup_{i=1,j=1}^{k} \{ f_i = f_j, f_i, f_j \geq g_t \forall t \in \{1, \ldots, m\} \}
\]

\[
\bigcup_{i=1,j=1}^{m} \{ g_i = g_j, g_i, g_j \geq f_s \forall s \in \{1, \ldots, k\} \}.
\]

As \( h \) is corner integral we have that

\[
U = \bigcup_{i=1,j=1}^{k,m} \{ f_i = g_j, f_i, g_j \geq f_s, g_t \forall s \neq i, t \neq j \}
\]

as all corner roots of the numerator and denominator of \( h \) are surpassed. Now, the skeleton defined by \( h \) is

\[
\{(\alpha_1, \ldots, \alpha_n) : h(\alpha_1, \ldots, \alpha_n) = 1\} = \{(\alpha_1, \ldots, \alpha_n) : \frac{\sum_{i=1}^{k} f_i(\alpha_1, \ldots, \alpha_n)}{\sum_{j=1}^{m} g_j(\alpha_1, \ldots, \alpha_n)} = 1\}
\]

\[
= \{(\alpha_1, \ldots, \alpha_n) : \sum_{i=1}^{k} f_i(\alpha_1, \ldots, \alpha_n) = \sum_{j=1}^{m} g_j(\alpha_1, \ldots, \alpha_n)\}
\]

\[
= \bigcup_{i=1,j=1}^{k,m} \{ f_i = g_j, f_i, g_j \geq f_s, g_t \forall s \neq i, t \neq j \} = U.
\]

If \( h \) is taken to be also regular, then all multiple occurrences of monomials in
\[ h = \sum_{i=1}^{k} f_i + \sum_{j=1}^{m} g_j \] are inessential and can be omitted, and thus \( h \) is tangible. \( \square \)

**Definition 10.2.17.** Let \( f, g \in \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]) \) be a pair of supertropical polynomials. Define the following relation

\[ f \sim_L g \iff \text{Cor}(f) = \text{Cor}(g). \tag{10.13} \]

The relation \( \sim_L \) is obviously reflexive, symmetric and transitive, thus is an equivalence relation on \( \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]) \).

**Proposition 10.2.18.** There is a \( 1:1 \) correspondence between principal corner-integral skeletons and principal corner-loci which restricts to a correspondence between principal regular corner-integral skeletons and principal regular corner-loci. This correspondence induces a correspondence between principal (regular) corner-integral kernels and principal (regular) corner-loci.

*Proof.* Let \( f \in \mathcal{R}(x_1, \ldots, x_n) \) be corner integral, and let \( f' \sim_K f \). Then

\[ \text{Skel}(f') = \text{Skel}(f) = \text{Cor}(f) = \text{Skel}(\hat{f}). \]

So, \( f' \sim_K \hat{f} \).

Conversely, let \( g \in \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]) \) and let \( g' \sim_L g \). Then

\[ \text{Cor}(g') = \text{Cor}(g) = \text{Skel}(\hat{g}) = \text{Cor}(\hat{g}), \]

where the last equality holds since we have shown \( \hat{g} \) to be corner integral. So \( g' \sim_L \hat{g} \).

The restriction to regular skeletons, their corresponding kernels and corner loci follows the propositions introduced above. \( \square \)

**Definition 10.2.19.** Let \( \Omega \) be the lattice generated by principal corner integral kernels with respect to (finite) multiplications and intersections.
Remark 10.2.20. Now, using the procedure introduced in Remark 10.2.4 and the correspondence introduced in Corollary 7.0.26, we get that $\Omega$ corresponds to the lattice of finitely generated generalized corner loci. Intersecting $\Omega$ with the lattice of regular kernels yields a lattice $\Theta \subset \text{PCon}(\mathcal{R}(x_1, \ldots, x_n))$ generated by regular corner-integral principal kernels which corresponds to the lattice of regular finitely generated corner loci $\text{FRCL}(\mathcal{R}^n)$.

Note 10.2.21. We are mainly interested in the sublattice $\Theta \subset \text{PCon}(\langle \mathcal{R} \rangle)$ of corner-integral principal kernels in $\text{Con}(\langle \mathcal{R} \rangle)$, for which the above holds too. $\Theta$ also is a sublattice of the lattice of corner-integral principal kernels in $\text{PCon}(\mathcal{R}(x_1, \ldots, x_n))$.

Remark 10.2.22. $\Theta$ defined above is a sublattice of kernels of $\text{PCon}(\mathcal{R}(x_1, \ldots, x_n))$ (or of $\text{PCon}(\langle \mathcal{R} \rangle)$). Thus all results of the section concerning reducibility is applicable to it.

In the subsequent section concerning kernel dimension, we will show that the lattice generated by corner integral regular kernels is in fact the lattice of regular kernels! Thus the lattice of corner loci corresponds to the lattice of regular kernels.
10.3 Example: The tropical line

Note 10.3.1. In the following example we consider the rational function \( \hat{f} = \left| \frac{x}{y+1} + \frac{y}{x+1} + \frac{1}{x+y} \right| \wedge |\alpha| \in \langle R \rangle \) for any \( \alpha \in R \setminus \{1\} \). As taking \( \wedge |\alpha| \) does not affect the computations below, we prefer omitting it, and work with \( \frac{x}{y+1} + \frac{y}{x+1} + \frac{1}{x+y} \) instead of its ‘copy’ in \( \langle R \rangle \).

Example 10.3.2. Let \( f = x+y+1 \) be the tropical line. Its corresponding skeleton is defined by the rational function \( \hat{f} = \frac{x}{y+1} + \frac{y}{x+1} + \frac{1}{x+y} \), and so, its corresponding kernel in \( R(x, y) \) is \( \langle \hat{f} \rangle \). As shown above \( \hat{f} \sim_{K} \left| \frac{x}{y+1} \right| \wedge \left| \frac{y}{x+1} \right| \wedge \left| \frac{1}{x+y} \right| \). Moreover, any of the three terms above can be omitted. Thus we have that

\[
\langle \hat{f} \rangle = \left( \frac{x}{y+1} \right) \cap \left( \frac{y}{x+1} \right) \cap (x+y),
\]

where each of the kernels comprising the intersection is contained in both of the remaining kernels (in the last kernel we chose to take \( x+y \) as a generator instead of its inverse). Now, taking logarithms, it can be seen that \( \text{Skel}(\langle x+y \rangle) \) is exactly the union of the bounding rays of the third quadrant. As

\[
x + y \sim_{K} |x+y| = |x+y| + (|x| \wedge |y|) = (|x+y| + |x|) \wedge (|x+y| + |y|),
\]

we have that \( \langle x+y \rangle = \langle |x+y| + |x| \rangle \cap \langle |x+y| + |y| \rangle \). Notice that \( \frac{1}{x+y} \) does not admit the corner integrality condition, as the corner roots of \( x \) and \( y \) are not surpassed by 1, thus does not correspond to a tropical hypersurface. We proceed by considering \( \frac{x}{y+1} + \frac{y}{x+1} \) omitting the last term of \( \hat{f} \). Computing its representation as a single fraction, we get \( \frac{x^2 + x^2 + y^2 + y}{(x+1)(y+1)} = \frac{(x+y)(x+y+1)}{(x+1)(y+1)} \). The corner roots of the numerator are \( \{x = 1, y = 1\} \) and \( \{x = y\} \), and the corner roots of the denominator are \( \{x = 1\} \) and \( \{y = 1\} \). The corner root \( \{x = y\} \) is surpassed by the denominator as \( (x+1)(y+1) = (x+y)+(xy+1) \geq (x+y) \) thus this is a regular skeleton (or equivalently a regular kernel) as expected. We urge the reader not to try to put negative values into the last equation, since they, of course, do not exist in a semifield. Finally, we discuss the above wedge decomposition of \( |x+y| \). It is a quite natural one. The geometric locus of the equation \( |x| \wedge |y| = \min(|x|, |y|) = 1 \) in a logarithmic scale is exactly the axes, as a union of the \( x \)-axis corresponding to \( |x| = 1 \) and the \( y \)-axis corresponding to \( |y| = 1 \). Intersecting it with the geometric...
locus of $|x + y|$ leaves the latter untouched as the former locus contains it. In fact, using such methods of intersections we can define any segment and ray in $\mathbb{R}^2$ using principal kernels, so only points in the plane are irreducible skeletons. This of course is not a problem, since we are still free to consider lattices inside the lattice of principal kernels, generated by designated subsets, which will be considered prime or irreducible. In fact, principal kernels leave us with maximal ‘flexibility’ in our hands.
11 Corner-integrality revisited

In this section we study the notion of corner-integrality introduced in subsection 9.5. We specify a procedure for finding a corner-integral kernel containing a given kernel or equivalently a supertropical hypersurface containing a given skeleton. We show that under this procedure corner-integral kernels are left unchanged.

Let $f \in \mathcal{R}(x_1, \ldots, x_n)$ be a rational function. We start by expressing corner-integrality of $f$ and thus of $\langle f \rangle$, the kernel generated by $f$, by a condition involving kernels. Write $f = \frac{h}{g}$, where $h = \sum_{i=1}^{k} h_i$ and $g = \sum_{j=1}^{m} g_j$ where $h_i$ and $g_j$ for $i = 1, \ldots, k$ and $j = 1, \ldots, m$ are the component monomials in $\mathcal{R}[x_1, \ldots, x_n]$ of the numerator and denominator of $f$, respectively. For simplicity, we assume that $f$ is in essential form, so each monomial affects $\text{Skel}(f)$, the skeleton defined by $f$. As noted in Subsection 9.5, $f$ is corner-integral if the following two conditions hold for every $x \in \mathcal{R}^n$:

$$\exists i \neq j \in \{1, \ldots, k\} \text{ such that } h_i(x) = h_j(x) \Rightarrow \quad (11.1)$$

$$\exists t \in \{1, \ldots, m\} \text{ s.t } h_i(x) \leq g_t(x) \text{ or } \exists s \in \{1, \ldots, k\} \setminus \{i, j\} \text{ s.t } h_s(x) > h_i(x).$$

$$\exists i \neq j \in \{1, \ldots, m\} \text{ such that } g_i(x) = g_j(x) \Rightarrow \quad (11.2)$$

$$\exists t \in \{1, \ldots, k\} \text{ s.t } g_i(x) \leq h_t(x) \text{ or } \exists s \in \{1, \ldots, m\} \setminus \{i, j\} \text{ s.t } g_s(x) > g_i(x).$$

Equivalently, $f$ is corner integral if condition (11.1) holds for both $f$ and $f^{-1}$ (exchanging the $g_i$’s and the $h_j$’s in the condition).

**Proposition 11.0.3.** Let $f \in \mathcal{R}(x_1, \ldots, x_n)$. Then $f$ is corner integral if and only if $|f|$ is corner-integral.

**Proof.** Write $f = \frac{h}{g}$ where $f, g \in \mathcal{R}[x_1, \ldots, x_n]$. Then $|f| = \frac{h}{g} + \frac{g}{h} = \frac{h^2 + g^2}{gh}$. Since $|f| \geq 1$ we only need to check that the corner roots of the numerator of $|f|$ are surpassed by its denominator. If $f$ is corner integral then in the numerator all corner roots of $g$ are surpassed by $h$ and vice versa so we are left with the scenario

137
of a corner root $a$ such that $h^2(a) = g^2(a)$ which yields that $h(a) = g(a)$ (note that it is possible that $a$ will also be a corner root of $g^2$ or $h^2$). In such a case $h^2(a) = g^2(a) = gh(a)$ thus the value of the numerator of $|f|$ at $a$ is surpassed by the value of the denominator, proving that $|f|$ is corner-integral. Conversely, assume $|f|$ is corner integral then $h^2 + g^2$ is surpassed by $gh$, in particular any corner root of $h^2$ is surpassed by $gh$ which yields that any corner root of $h$ is surpassed by $g$, and any corner root of $g^2$ is surpassed by $gh$ which yields that any corner root of $g$ is surpassed by $h$, thus $f$ is corner-integral.

**Corollary 11.0.5.** An intersection of corner-integral kernels is a corner-integral kernel.

**Proof.** Let $K_1$ and $K_2$ be corner-integral (principal) kernels and let $u_1$ and $u_2$ be a pair of corner-integral generators for $K_1$ and $K_2$, respectively. By Proposition [11.0.3] we may assume $u_1, u_2 \geq 1$. Take $f = u_1 \wedge u_2$, then by Proposition [11.0.4] $f = u \wedge v = |u| \wedge |v|$ is corner-integral thus by definition $\langle f \rangle$ is corner-integral.

**Corollary 11.0.6.** Let $f \in \mathcal{R}(x_1, \ldots, x_n)$. Then $f$ is corner-integral if and only if $|f| \wedge |\alpha|$ is corner integral for any $\alpha \neq 1$ in $\mathcal{R}$.

**Proof.** By Proposition [11.0.3] we may assume $f \geq 1$, i.e. $f = |f|$. Since $|\alpha|$ is trivially corner-integral by Proposition [11.0.4] the corner-integrality of $f$ implies the corner-integrality of $f \wedge |\alpha|$. Conversely, if $f$ is not corner-integral then
\[ f = \frac{h_1 + h_2}{g} \] such that \( h_1(x_0) = h_2(x_0) > g(x_0) \) for some \( x_0 \in \mathbb{R}^n \). Then \( f \wedge |\alpha| = \frac{|\alpha|(h_1 + h_2)}{|\alpha|g + (h_1 + h_2)} \). Since \(|\alpha| > 1\) we have that \(|\alpha|(h_1 + h_2) > (h_1 + h_2)(x_0)) \) and by assumption \(|\alpha|(h_1(x_0) + h_2(x_0)) > |\alpha|g(x_0)\) so \((f \wedge |\alpha|)\) is not corner-integral.

Recall that a \( k\)-kernel is a kernel in \( \text{Con}(\mathcal{R}(x_1, \ldots, x_n)) \) which is a preimage of a skeleton \( Z = \text{Skel}(S) \subseteq \mathcal{R} \) for some subset \( S \subseteq \mathcal{R}(x_1, \ldots, x_n) \) under the map \( \text{Ker} : \mathbb{P}(\mathcal{R}^n) \to \mathcal{R}(x_1, \ldots, x_n) \). We have shown that \( \text{Skel}(S) = \text{Skel}(\langle S \rangle) \) where \( \langle S \rangle \) is the kernel generated by the elements of \( S \). We have found these \( k\)-kernels for the restriction \( \text{Ker}|_{\langle \mathcal{R} \rangle} : \mathbb{P}(\mathcal{R}^n) \to \langle \mathcal{R} \rangle \). We have shown that the preimage in \( \langle \mathcal{R} \rangle \) of a principal skeleton to be \( \text{Ker}(\text{Skel}(f)) = \langle f \rangle \cap \langle \mathcal{R} \rangle \) so that for \( \alpha \neq 1 \) \( \text{Ker}(\text{Skel}(\langle |f| \wedge |\alpha| \rangle)) = \langle |f| \wedge |\alpha| \rangle = \langle f \rangle \cap \langle \mathcal{R} \rangle \). Since corner integrality of \( f \) implies corner integrality of \( |f| \wedge |\alpha| \) we refer to \( \langle f \rangle \) rather then to \( \langle f \rangle \cap \langle \mathcal{R} \rangle \) in our computations.

We will now translate condition (11.1) introduced above to the language of kernels.

Let \( C_i(h) \) be the set of corner roots of \( h \) attained by the monomial \( h_i \) (and some other monomials) and denote by \( h = \sum_{j \neq i} h_j \). Then condition (11.1) is equivalent to saying that if \( x \in \mathbb{R}^n \) is a corner root of \( h \) then \( h(x) \leq g(x) \) (i.e \( g \) surpasses \( h \) at \( x \)). So, for every \( 1 \leq i \leq k \) we can formulate the condition by

\[
\begin{align*}
\{ x \in \mathbb{R}^n : x \in C_i(h) \} &\subseteq \{ x \in \mathbb{R}^n : h(x) \leq g(x) \} \\
= \{ x \in \mathbb{R}^n : h_i(x) = h_i(x) \} &\subseteq \{ x \in \mathbb{R}^n : h(x) \leq g(x) \} \\
= \left\{ x \in \mathbb{R}^n : \frac{h_i}{h_i}(x) = 1 \right\} &\subseteq \left\{ x \in \mathbb{R}^n : \frac{h(x)}{g(x)} + 1 = 1 \right\} \\
= \text{Skel}\left( \frac{h_i}{h_i} \right) &\subseteq \text{Skel}\left( \frac{h_i}{g(x)} + 1 \right) = \text{Skel}(f + 1).
\end{align*}
\]

(11.3)

This means that \( f \) is corner-integral if all the corner roots of its numerator \( h \) are contained in the skeleton defined by \( f + 1 \), i.e. in the region of \( \mathcal{R} \) over which \( f \leq 1 \). Since \( \text{Skel}(K_1) \subseteq \text{Skel}(K_2) \Leftrightarrow K_2 \subset K_1 \) for any pair of \( K\)-kernels \( K_1, K_2 \) we have

\[
\text{Skel}\left( \frac{h_i}{h_i} \right) \subseteq \text{Skel}(f + 1) \Leftrightarrow \langle f + 1 \rangle \subseteq \langle \frac{h_i}{h_i} \rangle
\]

(11.4)
Now, intersecting both sides of the expression obtained in (11.3) by \(\text{Skel}(f^{-1} + 1)\) we get

\[
\text{Skel}\left(\frac{h_i}{h_i}\right) \subseteq \text{Skel}(f + 1)
\]

\[
\Leftrightarrow \text{Skel}\left(\frac{h_i}{h_i}\right) \cap \text{Skel}(f^{-1} + 1) \subseteq \text{Skel}(f + 1) \cap \text{Skel}(f^{-1} + 1)
\]

\[
\Leftrightarrow \text{Skel}\left(\frac{h_i}{h_i}\right) \cap \text{Skel}(f^{-1} + 1) \subseteq \text{Skel}(f).
\]

Note that the second transition is an equivalence rather then implication since \(\text{Skel}(f^{-1} + 1) \cup \text{Skel}(f + 1) = \mathbb{R}^n\).

Again, translating the resulting skeletons expression to kernels yields

\[
\text{Skel}\left(\frac{h_i}{h_i}\right) \cap \text{Skel}(f^{-1} + 1) \subseteq \text{Skel}(f) \Leftrightarrow \langle f \rangle \subseteq \left\langle \frac{h_i}{h_i} \right\rangle \cdot (f^{-1} + 1).
\]

Moreover, since the above inclusion holds for every \(i \in \{1, ..., k\}\) we have that

\[
\left(\bigcup_{i=1}^{k} \text{Skel}\left(\frac{h_i}{h_i}\right)\right) \cap \text{Skel}(f^{-1} + 1) \subseteq \text{Skel}(f)
\]

and so

\[
\langle f \rangle \subseteq (f^{-1} + 1) \cdot \left(\bigcap_{i=1}^{k} \frac{h_i}{h_i}\right)
\]

Interchanging the roles of \(h\) and \(g\) we get that the second condition 11.2 translates to

\[
\text{Skel}\left(\frac{g_j}{g_j}\right) \subseteq \text{Skel}(f^{-1} + 1)
\]

which is equivalent to

\[
\text{Skel}\left(\frac{g_j}{g_j}\right) \cap \text{Skel}(f + 1) \subseteq \text{Skel}(f).
\] (11.5)

and so since the inclusion in (11.5) holds for every \(j \in \{1, ..., m\}\) we have that

\[
\left(\bigcup_{j=1}^{m} \text{Skel}\left(\frac{g_j}{g_j}\right)\right) \cap \text{Skel}(f + 1) \subseteq \text{Skel}(f)
\]

140
Using the notation introduced above we have the following necessary and sufficient conditions hold:

**Proposition 11.0.7.** \( f = \frac{h}{g} \in \mathcal{R}(x_1, ..., x_n) \) where \( g, h \in \mathcal{R}[x_1, ..., x_n] \) is corner-integral if and only if the following conditions hold

\[
\left( \bigcup_{i=1}^{k} \text{Skel} \left( \frac{h_i}{h_1} \right) \right) \cap \text{Skel}(f^{-1} + 1) \subseteq \text{Skel}(f) \tag{11.6}
\]

\[
\left( \bigcup_{j=1}^{m} \text{Skel} \left( \frac{g_j}{g_1} \right) \right) \cap \text{Skel}(f + 1) \subseteq \text{Skel}(f). \tag{11.7}
\]

**Remark 11.0.8.** Recall that for \( f = \sum_{i=1}^{k} f_i \in \mathcal{F}(\mathcal{R}[x_1, ..., x_n]) \) where \( \mathcal{F}(\mathcal{R}[x_1, ..., x_n]) \) is the supertropical semiring of polynomials, the map \( f \mapsto \hat{f} = \sum_{i=1}^{k} \frac{f_i}{\sum_{j \neq i} h_j} \) is sending a supertropical polynomial \( f \) to a rational function \( \hat{f} \) such that \( x \in \mathcal{R}^n \) is a corner root of \( f \) if and only if \( \hat{f}(x) = 1 \), i.e \( \text{Skel}(\hat{f}) = \text{Cor}(f) \). Recall also that \( \text{Skel}(\hat{f}) = \text{Skel}(\tilde{f}) \) where \( \tilde{f} = \bigwedge_{i=1}^{k} \left| \frac{f_i}{\sum_{j \neq i} h_j} \right| \). Notice that

\[
\bigcup_{i=1}^{k} \text{Skel} \left( \frac{h_i}{h_1} \right) = \text{Skel} \left( \bigwedge_{i=1}^{k} \left| \frac{f_i}{\sum_{j \neq i} h_j} \right| \right) = \text{Skel}(\tilde{h})
\]

and similarly

\[
\bigcup_{j=1}^{m} \text{Skel} \left( \frac{g_j}{g_1} \right) = \text{Skel}(\tilde{g}).
\]

Thus we can rewrite (11.6) and (11.7) in the form

\[
\text{Skel} \left( \tilde{h} \right) \cap \text{Skel}(f^{-1} + 1) \subseteq \text{Skel}(f) \quad \text{and} \quad \text{Skel}(\tilde{g}) \cap \text{Skel}(f + 1) \subseteq \text{Skel}(f).
\]

or in the form

\[
\text{Skel} \left( \tilde{h} \right) \cap \text{Skel}(f^{-1} + 1) \subseteq \text{Skel}(f) \quad \text{and} \quad \text{Skel}(\tilde{g}) \cap \text{Skel}(f + 1) \subseteq \text{Skel}(f).
\]

Finally, we have proved that \( \hat{f} \) is corner-integral for any supertropical polynomial \( f \in \mathcal{F}(\mathcal{R}[x_1, ..., x_n]) \).
In view of Proposition 11.0.7, given \( f = \frac{h}{g} \in \mathcal{R}(x_1, \ldots, x_n) \), in order to obtain a corner-integral fraction whose skeleton contains \( \text{Skel}(f) \) one must adjoin both

\[
\left( \bigcup_{i=1}^{k} \text{Skel} \left( \frac{h_i}{h_i} \right) \right) \cap \text{Skel}(f^{-1} + 1)
\]

and

\[
\left( \bigcup_{j=1}^{m} \text{Skel} \left( \frac{g_j}{g_j} \right) \right) \cap \text{Skel}(f + 1)
\]

to the skeleton of \( f \).

Define the map \( \Phi_{CI} : \mathcal{R}(x_1, \ldots, x_n) \rightarrow \mathcal{R}(x_1, \ldots, x_n) \) by taking \( \Phi_{CI}(f) \), where \( f = \frac{h}{g} \), to be the fraction whose skeleton is formed by adjoining all the necessary points required for \( f \) to admit corner integrality to \( \text{Skel}(f) \). Namely

\[
\Phi_{CI}(f) = |f| \land \left( |f^{-1} + 1| + \tilde{h} \right) \land (|f + 1| + \tilde{g}). \tag{11.8}
\]

Then since \(|f^{-1} + 1|, |f + 1| \leq |f^{-1} + 1| + |f + 1| = |f|\) we have that

\[
\Phi_{CI}(f) = \left( |f^{-1} + 1| + (|f| \land \tilde{h}) \right) \land (|f + 1| + (|f| \land \tilde{g})).
\]

By this definition we have that

\[
\langle \Phi_{CI}(f) \rangle = \left( \langle f^{-1} + 1 \rangle \cdot \left( \langle f \rangle \cap \bigcap_{i=1}^{k} \left( \frac{h_i}{h_i} \right) \right) \right) \land \left( \left( \langle f + 1 \rangle \cdot \left( \langle f \rangle \cap \bigcap_{j=1}^{m} \left( \frac{g_j}{g_j} \right) \right) \right) \right).
\]

**Proposition 11.0.9.** Let \( f = \frac{h}{g} \in \mathcal{R}(x_1, \ldots, x_n) \) be a rational function, where \( h = \sum_{i=1}^{k} h_i \) and \( g = \sum_{j=1}^{m} g_j \) where \( h_i \) and \( g_j \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, m \) are the component monomials in \( \mathcal{R}[x_1, \ldots, x_n] \) of the numerator and denominator of \( f \), respectively. Then

\[
\text{Skel}(h + g) = \text{Skel}(f)
\]

\[
\cup \left( \left( \bigcup_{i=1}^{k} \text{Skel} \left( \frac{h_i}{h_i} \right) \right) \cap \text{Skel}(f^{-1} + 1) \right)
\]

\[
\cup \left( \left( \bigcup_{j=1}^{m} \text{Skel} \left( \frac{g_j}{g_j} \right) \right) \cap \text{Skel}(f + 1) \right). \tag{11.9}
\]
Thus $\text{Skel}(\hat{h} + g) = \text{Skel}(\Phi_{CI}(f))$.

Proof. Let $a \in R^n$ be a corner roots of $h + g$ then $a$ admits one of the following disjoint characterizations:

1. $h(a) = g(a)$ which is equivalent to saying that $a \in \text{Skel}(f)$.

2. $a$ is a corner root of $h$ and $g(a) \leq h(a)$ (i.e. $f^{-1}(a) + 1 = 1$) which is equivalent to saying that $a \in \text{Skel}(\hat{h}) \cap \text{Skel}(f^{-1} + 1) = \text{Skel}(\hat{h} + |f^{-1} + 1|)$.

3. $a$ is a corner root of $g$ and $h(a) \leq g(a)$ (i.e. $f(a) + 1 = 1$) which is equivalent to saying that $a \in \text{Skel}(\hat{g}) \cap \text{Skel}(f + 1) = \text{Skel}(\hat{g} + |f + 1|)$.

Consequently

$$\text{Skel}(\hat{h} + g) = \text{Skel}(f) \cup \left(\text{Skel}(\hat{h}) \cap \text{Skel}(f^{-1} + 1)\right) \cup \left(\text{Skel}(\hat{h}) \cap \text{Skel}(f + 1)\right).$$

Thus by Remark 11.0.8 the equality in (11.9) holds.

Corollary 11.0.10. Let $f \in R(x_1, \ldots, x_n)$. Then $\langle \Phi_{CI}(f) \rangle$ is corner-integral, $\text{Ske}(\Phi_{CI}(f)) \supseteq \text{Skel}(f)$ and $\text{Ske}(\Phi_{CI}(f)) = \text{Skel}(f)$ if and only if $f$ is corner-integral.

Proof. The first claim follows Proposition 11.0.9 from which we have that $\Phi_{CI}(f) \sim_K \hat{f}$ where $\hat{f}$ is corner-integral by Proposition 10.2.15. The second claim is straightforward from the definition of $\Phi_{CI}(f)$ since

$$\text{Ske}\left(\left|f\right| \land \left(|f^{-1} + 1| + \hat{h}\right) \land \left(|f + 1| + \hat{g}\right)\right) = \text{Skel}(f)$$

$$\cup \text{Ske}\left(\left|f^{-1} + 1\right| + \hat{h}\right) \cup \text{Ske}\left(\left|f + 1\right| + \hat{g}\right).$$

The last statement follows Proposition 11.0.7.

We can rephrase Corollary 11.0.10 as follows:

Corollary 11.0.11. Let $f \in R(x_1, \ldots, x_n)$. Then $\langle \Phi_{CI}(f) \rangle$ is a corner-integral kernel contained in $\langle f \rangle$. Moreover, $f$ is corner-integral if and only if $\langle \Phi_{CI}(f) \rangle = \langle f \rangle$. Equivalently $\text{Ske}(\Phi_{CI}(f))$ is a principal corner-integral skeleton containing $\text{Ske}(f)$ which yields that $\text{Ske}(\Phi_{CI}(f))$ is supertropical hypersurface containing $\text{Ske}(f)$.
By Remark 9.5.5 if \( f \) is corner-integral then so is \( f' = \sum_{i=1}^{m} f^{d(i)} \) with \( d(i) \in \mathbb{Z} \) and thus \( \langle \Phi_{CI}(f) \rangle = \langle f \rangle = \langle f' \rangle = \langle \Phi_{CI}(f') \rangle \). In particular this applies to \( f' = f^{-1} + f = |f| \). It turns out that this equality \( \langle \Phi_{CI}(f) \rangle = \langle \Phi_{CI}(|f|) \rangle \) holds for any \( f \) as we prove in the following remark.

**Remark 11.0.12.** For any \( f = \frac{h}{g} \in \mathcal{R}(x_1, \ldots, x_n) \)

\[
\langle \Phi_{CI}(\sum_{i=1}^{k} f^{d(i)}) \rangle = \langle \Phi_{CI}(f) \rangle \tag{11.10}
\]

where \( d(i) \in \mathbb{Z} \) is monotonically increasing for \( i = 1, \ldots, k, \; d(1) < 0, \; d(k) > 0 \).

\[
\langle \Phi_{CI}(f^k) \rangle = \langle \Phi_{CI}(f) \rangle \tag{11.11}
\]

for any \( k \in \mathbb{Z} \setminus \{0\} \).

**Proof.** Due to the Frobenious property \( \sum_{i=1}^{k} f^{d(i)} = \sum_{i=1}^{k} \frac{h^{s+t} + g^{s+t}}{h^s g^t} = \frac{h^{s+t} + g^{s+t}}{h^s g^t} \)

where \( t = |d(k)| \) and \( s = |d(1)| \). So

\[
\Phi_{CI}(\sum_{i=1}^{k} f^{d(i)}) = \Phi_{CI}(\frac{h^{s+t} + g^{s+t}}{h^s g^t}) = h^{s+t} + g^{s+t} + h^s g^t = h^{s+t} + g^{s+t} = \hat{h} + g^{s+t}.
\]

Since \( s + t \neq 0 \) we have that \( \hat{h} + g^{s+t} \) is a generator of \( \langle \hat{h} + g \rangle \) thus \( \langle \hat{h} + g^{s+t} \rangle = \langle \hat{h} + g \rangle = \langle \Phi_{CI}(f) \rangle \). The latter equality holds since \( \Phi_{CI}(\hat{h} + g^{k}) = \hat{h} + g^{k} = \hat{h} + g^{s+t} - k \). Since \( k \neq 0 \), \( \hat{h} + g^{k} \) is a generator of \( \langle \hat{h} + g \rangle \) thus \( \langle \hat{h} + g^{k} \rangle = \langle \hat{h} + g \rangle = \langle \Phi_{CI}(f) \rangle \). \( \square \)

**Corollary 11.0.13.** Let \( f \in \mathcal{R}(x_1, \ldots, x_n) \) be such that \( f = u_1 \land \cdots \land u_k \) where \( u_1, \ldots, u_k \in \mathcal{R}(x_1, \ldots, x_n) \) are each corner-integral. The \( f \) is corner-integral and

\[
\Phi_{CI}(f) \sim_{K} \Phi_{CI}(u_1) \land \cdots \land \Phi_{CI}(u_k) \tag{11.12}
\]

**Proof.** \( f \) is corner-integral by induction on Proposition 11.0.4. First we assume \( f \geq 1 \) (thus so are the \( u_i \)'s). Since \( u_i \) is corner-integral \( \Phi_{CI}(u_i) \sim_{K} u_i \) for \( i = 1, \ldots, k \), thus \( \Phi_{CI}(u_1) \land \cdots \land \Phi_{CI}(u_k) \sim_{K} u_1 \land \cdots \land u_k = f \) (note that \( a \sim_{K} b \iff \langle a \rangle = \langle b \rangle \) thus \( \langle a \rangle \cap \langle b \rangle = \langle |a| \cap |b| \rangle \) for any \( a' \sim_{K} a, b' \sim_{K} b \)). Since \( f \) is corner-integral \( f \sim \Phi_{CI}(f) \) so \( (11.12) \) holds. For any \( g \in \mathcal{R}(x_1, \ldots, x_n) \) taking \( || \) does not change corner-integrality and \( \Phi_{CI}(|g|) \sim_{K} \Phi_{CI}(g) \). Thus for any \( f \) we can consider \( |f| \) instead and apply the first case. \( \square \)
Example 11.0.14. Let \( f = \sum_{i=1}^{k} f_i \in \mathcal{F}(\mathcal{R}[x_1, \ldots, x_n]) \) be a supertropical polynomial represented as the sum of its component monomials \( f_i \), \( i = 1, \ldots, k \). Then \( \tilde{f} = \bigwedge_{i=1}^{k} \left| \frac{f_i}{\sum_{j \neq i} f_j} \right| \) is corner-integral.

First note that \( \tilde{f} \geq 1 \). Now, for any given \( i \in \{1, \ldots, k\} \) denote \( D_i = \frac{f_i}{\sum_{j \neq i} f_j} \), then by Remark 11.0.12 we have that \( \Phi_{CI}(D_i) \sim_K \Phi_{CI}(D_i) = \hat{f} \). Then \( \Phi_{CI}(\tilde{f}) \sim_K \bigwedge_{i=1}^{k} \Phi_{CI}(D_i) = \bigwedge_{i=1}^{k} \hat{f} = \hat{f} \sim_K \tilde{f} \) so \( \Phi_{CI}(\tilde{f}) \sim_K \tilde{f} \) and \( \tilde{f} \) is corner-integral.

Remark 11.0.15. An \( \mathcal{R} \)-homomorphic image of corner-integral element of \( \mathcal{R}(x_1, \ldots, x_n) \) is corner-integral. Thus the \( \mathcal{R} \)-homomorphic image of a corner-integral kernel is corner-integral.

Proof. Let \( f = \frac{h}{g} \) and let \( a \in \mathcal{R}^n \) a corner root of \( h \) so that \( h_1(a) = h_2(a) \) for some component monomials \( h_1, h_2 \) of \( h \). Then for a semifield epimorphism \( \phi \), \( h_1(\phi(a)) + h_2(\phi(a)) = \phi(h_1(a)) + \phi(h_2(a)) = \phi(h_1(a) + h_2(a)) = \phi(h(a)) = h(\phi(a)) \) thus \( \phi(a) \) is a corner root of \( \phi(h) \) and since \( \phi \) is onto every corner root of \( \phi(h) \) is of the form \( \phi(a) \) for a corner root \( a \) of \( h \). Since \( f \) is corner-integral \( h(a) \leq g(a) \), and as \( \phi \) is order preserving \( h(\phi(a)) = \phi(h(a)) \leq \phi(g(a)) = g(\phi(a)) \) i.e., \( \phi(g) \) surpasses \( \phi(h) \). The symmetric argument switching \( h \) and \( g \) along with the assertions above yield the corner-integrality of \( \phi(f) \). \( \square \)
12 Composition series of kernels of an idempotent semifield

In this section, we restrict our discussion to idempotent semifields. We write an analogue to the theory of composition series of modules, just for kernels of an idempotent semifield. The kernels of an idempotent semifield are also subsemifields, thereby allowing us to utilize the isomorphism theorems to prove our assertions.

Remark 12.0.16. Let \( S \) be an idempotent semifield and let \( K_1, K_2 \) be kernels of \( S \) such that \( K_1 \subseteq K_2 \). Then \( K_1 \) is a kernel of \( K_2 \). In such a case we say that \( K_1 \) is a subkernel of \( K_2 \) and write \( K_1 \leq K_2 \). If \( K_1 \) is strictly contained in \( K_2 \), we write \( K_1 < K_2 \).

Proof. \( K_2 \) is a subsemifield of \( S \), so, by Theorem 2.2.51 (1), \( K_1 = K_2 \cap K_1 \) is a kernel of \( K_2 \).

Remark 12.0.17. As it was previously shown, in Corollary 2.7.7, the family of principal kernels of an idempotent semifield is a sublattice of kernels. Moreover, by remark 2.3.19, homomorphic images of principal kernel are principal kernels. Thus the Isomorphism Theorems 2.2.51 and 2.2.52 hold for principal kernels of an idempotent semifield. Consequently, all subsequent assertions hold for principal kernels too (considering idempotent semifields).

Definition 12.0.18. Let \( L \) be a kernel of an idempotent semifield \( S \). A descending chain

\[
L = K_0 \supset K_1 \supset \cdots \supset K_t
\]

(12.1)

of subkernels \( K_i \) of \( L \) for \( 1 \leq i \leq t \) is said to have length \( t \). The factors of the chain are the kernels \( K_{i-1}/K_i \), for \( 1 \leq i \leq t \). The chain in (12.1) is said to be a composition series for \( L \) if \( K_t = \langle 1 \rangle \) and each factor is a simple kernel.

We say that two chains of kernels are equivalent if they have isomorphic factors.

Remark 12.0.19. Let \( L \) be a kernel of an idempotent semifield \( S \). If \( L = K_0 \supset K_1 \supset \cdots \supset K_t \) is a chain \( C \) of kernels and \( K \leq K_t \), then the chain \( C' \) given by \( L/K = K_0/K \supset K_1/K \supset \cdots \supset K_t/K \) is equivalent to \( C \).
Proof. By Theorem 2.2.52, we have that \( K_{i-1}/K_i \cong (K_{i-1}/K)/(K_i/K) \), and thus the factors of \( C \) and \( C' \) are isomorphic. \( \square \)

Remark 12.0.20. If \( K_{i-1} \supset K_i \) are kernels such that the semifield \( K_{i-1}/K_i \) is not simple, then there exists some kernel \( N \) between them, i.e., \( K_{i-1} \supset N \supset K_i \). The process of inserting such an extra subkernel \( N \) into the chain is called refining the chain. Consequently, any chain that is not a composition series can be refined.

Remark 12.0.21. For any simple subkernel \( S \) and any \( K < L \), by Theorem 2.2.51 (2), we have that

\[(K \cdot S)/K \cong S/(K \cap S)\]

which is either isomorphic to \( S \) or \( \langle 1 \rangle = \{1\} \). It follows that \( L \) is a finite product of simple subkernels \( \{S_i : 1 \leq i \leq t\} \). Then letting \( L_k = \prod_{i=1}^{t-k} S_i \) we get a composition series

\[L = L_0 \supset L_1 \supset \cdots \supset L_{t-1} \supset \langle 1 \rangle\]

(discarding duplications).

Remark 12.0.22. Let \( L \) be a kernel of an idempotent semifield \( S \). Define a composition chain \( C(L, K) \) from \( L \) to a subkernel \( K \) to be a chain

\[L = L_0 \supset L_1 \supset \cdots \supset L_t = K\]

such that each factor is simple. By Remark 12.0.19, \( C(L, K) \) is equivalent to the composition series

\[L/K \supset L_1/K \supset \cdots \supset L_t/K = 1\]

of \( L/K \). It follows that if \( P \) is a subkernel of a kernel \( N \) for which \( N/P \cong L/K \), then there is a composition chain \( C(N, P) \) equivalent to \( C(L, K) \).

The following is the well-known Jordan–Hölder theorem for kernels:

**Theorem 12.0.23.** Let \( L \) be a kernel of an idempotent semifield \( S \). Suppose \( L \) has a composition series

\[L = L_0 \supset L_1 \supset \cdots \supset L_t = \langle 1 \rangle\]

which we denote by \( \mathcal{C} \). Then
1. Any arbitrary finite chain of subkernels

\[ L = K_0 \supset K_1 \supset \cdots \supset K_s \]

(denoted as \( \mathcal{D} \)), can be refined to a composition series equivalent to \( \mathcal{C} \). In particular \( s \leq t \).

2. Any two composition series of \( L \) are equivalent.

3. \( \ell(L) = \ell(K) + \ell(L/K) \) for every subkernel \( K \) of \( L \). In particular, every subkernel and every homomorphic image of a kernel with composition series has a composition series.

Proof. We prove the theorem by induction on \( t \). If \( t = 1 \) then \( L \) is simple and the theorem is trivial, so we assume the whole theorem is true for kernels having a composition series of length \( \leq t - 1 \).

(1) Let \( C_1 = C_1(L_1) \) denote the composition series \( L_1 \supset \cdots \supset L_t = \langle 1 \rangle \) of \( L_1 \); \( C_1 \) has length \( t - 1 \). If \( K_1 \subseteq L_1 \), then by induction on \( t \), the chain \( L_1 \supset K_1 \supset K_2 \supset \cdots \supset K_s \) can be refined to a composition series of \( L_1 \) equivalent to \( C_1 \), yielding (1) at once (by tagging on \( L \supset L_1 \)). Thus, we may assume \( K_1 \not\subseteq L_1 \), so \( L_1 \cap K_1 \subset K_1 \).

Also, by Remark 2.5.7, \( L_1 \cdot K_1 = L \) since \( L_1 \) is maximal in \( L \). Note that we have two ways of descending from \( L \) to \( L_1 \cap K_1 \); either via \( L_1 \) or via \( K_1 \). But these two routes are equivalent, in the sense that

\[ L/K_1 = (M_1 \cdot K_1)/K_1 \cong L_1/(L_1 \cap K_1), \quad (12.2) \]

\[ K_1/(L_1 \cap K_1) \cong (K_1 \cdot L_1)/L_1 = L/L_1. \quad (12.3) \]

By induction on \( t \), the chain \( L_1 \supset L_1 \cap K_1 \supset \langle 1 \rangle \) refines the composition series \( \mathcal{E}_1(L_1) \) equivalent to \( C_1 \) (of length \( t - 1 \)). This is comprised of \( \mathcal{E}'_1(L_1, L_1 \cap K_1) \), a composition series from \( L_1 \) to \( L_1 \cap K_1 \) of some length \( t_1 \) followed by a composition series \( \mathcal{E}'_2(L \cap K_1) \) of some length \( t_2 \), where \( t_1 + t_2 = t - 1 \). Since \( t_1 \geq 1 \), we see \( t_2 \leq t - 2 \). Furthermore, since \( L_1 \) is maximal in \( L \), by Corollary 2.5.8 we have that \( L/L_1 \) is simple, thus (12.3) shows that \( K_1/(L_1 \cap K_1) \) is simple. So the chain \( K_1 \supset L_1 \cap K_1 \) followed by \( \mathcal{E}'_2 \) is a composition series \( \mathcal{F}_1 \) of \( K_1 \) having length \( t_2 + 1 \leq t - 1 \). By induction, the chain \( K_1 \supset K_2 \supset \cdots \supset K_s \) refines to a composition series \( \mathcal{D}_1(K_1) \) equivalent to \( \mathcal{F}_1 \). The isomorphism (12.2) enables us to transfer \( \mathcal{E}'_1(L_1, L_1 \cap K_1) \) to an equivalent composition series \( \mathcal{E}''_1(L, K_1) \) from \( L \).
to $K_1$, also of length $t_1$. Tracking this onto $\mathcal{D}_1(K_1)$ yields the desired composition series refining $\mathcal{D}$ which is equivalent to $\mathcal{C}$. In conclusion, we have passed from our original composition series $\mathcal{C}_1$ through the following equivalent composition series:

1. $L \supset L_1$ followed by $\mathcal{E}'_1(L_1, L_1 \cap K_1)$ and $\mathcal{E}'_2$;
2. $\mathcal{E}''_1(L, K_1)$ followed by $K_1 \supset L_1 \cap K_1$ and $\mathcal{E}'_2$;
3. $\mathcal{E}''_1(L, K_1)$ followed by $\mathcal{D}_1$, which defines $\mathcal{D}$, as desired.

(2) is immediate from (1).

(3) Refine the chain $L > K > \langle 1 \rangle$ to a composition series, and apply Remark 12.0.22.

\[ \square \]

**Definition 12.0.24.** Given a kernel $L$ of an idempotent semifield $\mathbb{S}$, we define its composition length $\ell(L)$ to be the length of a composition series for $L$, if such exists.

**Remark 12.0.25.** Let $\mathbb{S}$ be an idempotent semifield. All the results stated in this section hold taking any sublattice of kernels $\Theta$ of $\text{Con}(\mathbb{S})$ in the sense of Definition 9.2.1 instead of $\text{Con}(\mathbb{S})$. Note that, given $\Theta$, maximal kernels are taken to be maximal $\Theta$-kernels, i.e., maximal elements of $\Theta$. For example, one can consider the sublattice of principal kernels of $\mathbb{S}$, $\text{PCon}(\mathbb{S})$. 

149
13 The Hyperspace-Region decomposition and the Hyperdimension

13.1 Hyperspace-kernels and region-kernels.

Remark 13.1.1. Though we consider the semifield of fractions $\mathcal{R}(x_1, \ldots, x_n)$, most of the results introduced in this section are applicable to any finitely generated semifield $\mathcal{R}(a_1, \ldots, a_n)$ over $\mathcal{R}$, where $\{a_1, \ldots, a_n\}$ are generators of $\mathcal{R}(a_1, \ldots, a_n)$ as a semifield. We explicitly indicate whenever a condition needs to be imposed on $\{a_1, \ldots, a_n\}$ to hold for the semifield $\mathcal{R}(a_1, \ldots, a_n)$. In particular, $\langle \mathcal{R} \rangle \subset \mathcal{R}(x_1, \ldots, x_n)$ is just another case of a finitely generated semifield over $\mathcal{R}$, taking $a_i = x_i \wedge |\alpha|$ for $1 \leq i \leq n$ and $\alpha \in \mathcal{R} \setminus \{1\}$. In this case, the generators $a_i$ are bounded from above (or simply $|a_i|$ are bounded), and we specifically designate the results that are true only for unbounded generators.

Definition 13.1.2. An element $f \in \mathcal{R}(x_1, \ldots, x_n)$ is said to be a hyperplane-fraction, or HP-fraction, if $f \sim_{K} \frac{h}{g}$ with $h, g \in \mathcal{R}[x_1, \ldots, x_n]$ distinct monomials and $\langle \frac{h}{g} \rangle \cap \mathcal{R} \subseteq \{1\}$.

Remark 13.1.3. The condition $\langle \frac{h}{g} \rangle \cap \mathcal{R} \subseteq \{1\}$ ensures us that $\text{Skel}(f) \neq \emptyset$. Moreover, this condition can be rephrased as $\langle \mathcal{R} \rangle \not\subseteq \langle \frac{h}{g} \rangle$. This is consistent with our interest in proper subkernels in $\langle \mathcal{R} \rangle$. We could equivalently take $f$ to be an element of $\langle \mathcal{R} \rangle$.

Remark 13.1.4. One can choose to view an HP-fraction simply as a nonconstant Laurent monomial in $\mathcal{R}(x_1, \ldots, x_n)$.

Remark 13.1.5. HP-fractions in $\mathcal{R}(a_1, \ldots, a_n)$ where $a_i$ are unbounded fractions, are not bounded; i.e., for any HP-fraction $f$ there exists no $\alpha \in \mathcal{R}$ such that $|f| \leq |\alpha|$.

Analogously we can prove the following assertion:

Remark 13.1.6. HP-fractions in $\mathcal{R}(x_1, \ldots, x_n)$ are not bounded; i.e., for any HP-fraction $f$ there exists no $\alpha \in \mathcal{R}$ such that $|f| \leq |\alpha|$.
Proof. Let $f' = \frac{h}{g}$ where $h, g \in R[x_1, ..., x_n]$ are distinct monomials such that $\langle \frac{h}{g} \rangle \cap R \subseteq \{1\}$. Then $f'$ contains at least one free (unbounded) indeterminate. Thus $f' \neq 1$ and so $|f'| \leq 1$. So we may assume $\alpha \neq 1$. Now, since $R$ is divisibly closed there exists some $a_k \in R^n$ such that $f'(a_k) = |\alpha|^k$ for any natural $k$. As $\alpha \neq 1$ is a generator of $R$ as a kernel, for any $|\beta|$ with $\beta \in R$ there exists some $k$ for which $|f'(a_k)| = f'(a_k) = |\alpha|^k > |\beta|$. Thus $f'$ is not bounded. By Remark 5.1.4 for $f \sim_K f'$ we have that $f$ is bounded if and only if $f'$ is bounded, concluding our claim. 

Corollary 13.1.7. Since the condition for a function $f$ to be bounded depends solely on $|f|$, if $f$ is not bounded then both $|f|$ and $f^{-1}$ are not bounded since $||f|| = |f^{-1}| = |f|$ (here $||f||$ denotes $| \cdot |$ applied to the function $|f|$).

Definition 13.1.8. An element $f \in R(x_1, ..., x_n)$ is said to be a hyperspace-fraction, or HS-fraction, if $f \sim_K \sum_{i=1}^t |f_i|$ where for each $1 \leq i \leq t$, $f_i = \frac{h_i}{g_i}$ with $h_i, g_i \in R[x_1, ..., x_n]$ distinct monomials such that $\langle \frac{h_i}{g_i} \rangle \cap R \subseteq \{1\}$.

Remark 13.1.9. HS-fractions in $R(a_1, ..., a_n)$ with $a_i \in R(x_1, ..., x_n)$ unbounded, are not bounded.

We can analogously prove

Remark 13.1.10. HS-fractions in $R(x_1, ..., x_n)$ are not bounded.

Proof. The claim follows from the inequality $|f_j| \leq \sum_{i=1}^t |f_i| = f'$ and the invariance of boundness under $\sim_K$. 

Definition 13.1.11. A principal kernel of $R(x_1, ..., x_n)$ is said to be a hyperplane-fraction kernel (or shortly, HP-kernel) if it is generated by a hyperplane fraction.

Remark 13.1.12. An HP-Kernel is regular.

Indeed, as regularity is preserved under $\sim_K$ we may assume $f = \frac{h}{g}$ with $h$ and $g$ distinct monomials as both numerator and denominator of a hyperplane fraction are monomials, the condition for regularity holds trivially.

Definition 13.1.13. A principal kernel of $R(x_1, ..., x_n)$ is said to be a hyperspace-fraction kernel (or shortly, HS-kernel) if it is generated by a hyperspace fraction.
Remark 13.1.14. A principal kernel is an HS-kernel if and only if it is a product of HP-kernels.

Proof. Every hyperplane fraction is a hyperspace fraction comprised of a single summand, thus every HP-kernel is an HS-kernel. Conversely, if ⟨f⟩ is an HS-kernel, then ⟨f⟩ = ⟨\sum_{i=1}^{t} |f_i|⟩ = \prod_{i=1}^{t} \langle f_i \rangle where h_i, g_i ∈ ℝ[x_1, ..., x_n] are distinct monomials for each 1 ≤ i ≤ t. Thus, by definition ⟨f_i⟩ is an HP-kernel for each i ∈ {1, ..., t} proving our claim. □

Corollary 13.1.15. An HS-Kernel is regular.

Proof. Since an HP-kernel is regular and since, by Remark 13.1.14, an HS-kernel is a product of HP-kernels, the assertion follows by Remark 9.4.4 and Corollary 9.4.9. □

Definition 13.1.16. A skeleton in ℝ^n is said to be a hyperplane-fraction skeleton (shortly HP-skeleton) if it is defined by a hyperplane fraction. A skeleton in ℝ^n is said to be a hyperspace-fraction skeleton (shortly HS-skeleton) if it is defined by a hyperspace fraction.

Corollary 13.1.17. A skeleton is an HS-skeleton if and only if it is an intersection of HP-skeletons.

Proof. As Skel(⟨f⟩ · ⟨g⟩) = Skel(⟨f⟩) ∩ Skel(⟨g⟩), the assertion follows directly from Remark 13.1.14. □

Proposition 13.1.18. Let ⟨f⟩ be an HP-kernel. Then w ∈ ⟨f⟩ is an HP-fraction if and only if w^s = f^k for some s, k ∈ ℤ \ {0}.

Proof. If w^s = f^k then w ∈ ⟨f⟩ and w^s = f^k ∼_K (\frac{h}{g})^k = \frac{h^k}{g^k} where h, g ∈ ℝ[x_1, ..., x_n] are distinct monomials such that \{\frac{h}{g}\} ∩ ℝ ⊆ \{1\}. Since h, g ∈ ℝ[x_1, ..., x_n] are distinct monomials such that \{\frac{h}{g}\} ∩ ℝ ⊆ \{1\}, we have that h^k, g^k are distinct monomials and since ℝ is divisibly closed we also have that \{\frac{h^k}{g^k}\} = \frac{h^k}{g^k} ∩ ℝ ⊆ \{1\} (for otherwise the condition will not hold for \frac{h}{g} too). Thus w^s and so also w is an HP-fractions. Conversely, let w ∈ ⟨f⟩ be an HP-fraction, then w ∼_K w' = \frac{r}{s} with r, s ∈ ℝ[x_1, ..., x_n] are distinct monomials such that \{\frac{r}{s}\} ∩ ℝ ⊆ \{1\}. As w ∼_K w' we can prove the claim for w = w'. Similarly, we can assume f = \frac{h}{g} where h, g ∈ ℝ[x_1, ..., x_n] are distinct monomials such that \{\frac{h}{g}\} ∩ ℝ ⊆ \{1\}. By assumption ⟨w⟩ ⊆ ⟨f⟩, thus Skel(w) ⊇ Skel(f).
Assume \( w^s \neq f^k \) for any \( s, k \in \mathbb{Z} \setminus \{0\} \). We will show that there exists some \( a \in \mathcal{R}^n \) such that \( a \in \text{Skel}(f) \setminus \text{Skel}(g) \) for some \( g \in \langle f \rangle \).

Let \( (p_1, \ldots, p_n), (q_1, \ldots, q_n) \in \mathbb{Z}^n \) be the vectors of powers of \( x_1, \ldots, x_n \) in the Laurent monomials \( f \) and \( w \). Since \( w \) and \( f \) are nonconstant, \( (p_1, \ldots, p_n) \neq (0), (q_1, \ldots, q_n) \neq (0) \). By Remark 2.3.7 we may assume that \( \gcd(p_1, \ldots, p_n) = \gcd(q_1, \ldots, q_n) = 1 \) (since \( \mathcal{R} \) is divisible, the constant terms of \( f \) and \( w \) can be adequately adjusted). Since \( w \in \langle f \rangle \) we have that \( |w| \leq |f|^m \) for some \( m \in \mathbb{N} \), so if \( x_i \) occurs in \( w \) it must also occur in \( f \). Finally, since \( w^s \neq f^k \) for any \( k \in \mathbb{Z} \setminus \{0\} \) we can also assume that \( (p_1, \ldots, p_n) \neq (q_1, \ldots, q_n) \), for otherwise \( w = \alpha f \) for some \( \alpha \neq 1 \) and thus

\[
\text{Skel}(f) \cap \text{Skel}(w) = \text{Skel}(f) \cap \text{Skel}(\alpha f) = \emptyset,
\]

contradicting our assumption that \( \text{Skel}(w) \supseteq \text{Skel}(f) \) and \( \text{Skel}(w) \neq \emptyset \). Let \( x_j^l \) occur in \( w \) for some \( l \in \mathbb{Z} \setminus \{0\} \) such that \( x_j \) is not identically 1 over \( \text{Skel}(w) \) (and thus also on \( \text{Skel}(f) \)). Thus there exists some \( k \in \mathbb{Z} \setminus \{0\} \) such that \( x_j^{l+k} \) occurs in \( f \). Define the Laurent monomial \( g = w^{-1}f^k \in \langle f \rangle \), then \( x_j \) does not occur in \( g \). Without loss of generality, assume \( j = 1 \) where the exponent of \( x_1 \) in \( f \) is \( p_1 \). If \( a = (\alpha_1, \ldots, \alpha_n) \in \text{Skel}(f) \), then \( g(a) = w(a)^{-1}f(a) = 1 \). By our assumption that \( w^s \neq f^k \) there exists \( x_i \) occurring in \( f \) with some power \( p_i \in \mathbb{Z} \setminus \{0\} \) and not in \( w \) (for otherwise they both contain only \( x_j \)). Take \( b = (1, \alpha_2, \ldots, \beta, \ldots, \alpha_n) \) with \( \beta \in \mathcal{R} \) occurring in the \( t \)'th place, such that \( \beta^{p_1} = \frac{\alpha_1^{p_1}}{\alpha_j^{p_1}} \) (there exists such \( \beta \) since \( \mathcal{R} \) is divisible). Then as \( x_j \) is not identically 1 over \( \text{Skel}(f) \), we can choose \( a \in \text{Skel}(w) \) such that \( f(b) = 1 \) and \( g(b) \neq 1 \).

\[ \blacksquare \]

**Corollary 13.1.19.** Let \( \langle f \rangle \) be an HP-kernel. Then \( w \in \langle f \rangle \) is an HP-fraction if and only if \( w \) is a generator of \( \langle f \rangle \).

**Proof.** The claim follows from Remark 13.1.18 and the property that \( \langle g^k \rangle = \langle g \rangle \) for any principal kernel of a semifield. \[ \blacksquare \]

**Definition 13.1.20.** We define an order-fraction \( o \) in the semifield \( \mathcal{R}(x_1, \ldots, x_n) \) to be an element of the form \( o = 1 + f \) for some HP-fraction \( f \neq 1 \). We say that \( o \) is the order fraction defined by \( f \).

**Definition 13.1.21.** We define an order-Kernel of the semifield \( \mathcal{R}(x_1, \ldots, x_n) \) to be a principal kernel of the form \( \langle o \rangle \) for some order-fraction \( o = 1 + f \). We say that \( \langle o \rangle \) is the order kernel defined by \( f \).
Remark 13.1.22. Let $O = \{1 + f\}$ be an order kernel of $\mathcal{R}(x_1, \ldots, x_n)$, where $f$ is its defining HP-fraction. Then there exists an order kernel $O^c$ of $\mathcal{R}(x_1, \ldots, x_n)$ such that

$$O \cap O^c = \{1\} \text{ and } O \cdot O^c = \langle f \rangle.$$  

Proof. Take $O^c = 1 + f^{-1}$ then since $f$ is an HP-fraction so is $f^{-1}$ and thus $O^c$ is an order kernel. Now, $O \cap O^c = \langle |1 + f| + |1 + f^{-1}| \rangle = \langle |1 + f|, |1 + f^{-1}| \rangle = \langle 1 \rangle$ and $O \cdot O^c = \langle |1 + f| + |1 + f^{-1}| \rangle = \langle 1 + f + f^{-1} \rangle = \langle |f| \rangle = \langle f \rangle$ (noting that $(1 + f), (1 + f^{-1}) \geq 1$ implies $|1 + f| = 1 + f$ and $|1 + f^{-1}| = 1 + f^{-1}$). □

Definition 13.1.23. In the notation of Remark 13.1.22, $O^c$ is said to be the complementary order kernel of $O$ and $1 + f^{-1}$ the complementary order fraction of $o = 1 + f$, denoted $o^c$.

Remark 13.1.24. By definition $(O^c)^c = O$.

Definition 13.1.25. An element $r \in \mathcal{R}(x_1, \ldots, x_n)$ is said to be a region-fraction, if $r \sim_K \sum_{i=1}^{t} |o_i|$ where $o_i$ is an order-fraction for every $1 \leq i \leq t$ and $o_j \neq o_i^c$ for any $1 \leq i, j \leq t$.

Remark 13.1.26. We will now clarify the reason behind the last condition posed on the order-fractions comprising the region fraction. For $i = 1, \ldots, t$ let $o_i = 1 + g_i$ with $g_i$ the HP-fraction defining the order fraction $o_i$.

Then since $1 + f_i \geq 1$ for every $i$, we have that

$$\sum_{i=1}^{t} |o_i| = \sum_{i=1}^{t} |1 + f_i| = \sum_{i=1}^{t} (1 + f_i) = 1 + \sum_{i=1}^{t} f_i.$$  

Thus a region-fraction $r$ can be defined as $r \sim_K 1 + \sum_{i=1}^{t} f_i$, so the last condition of the definition can be stated as $f_j \neq f_i^{-1}$ for any $1 \leq i, j \leq t$. One can see that if there exist $k$ and $m$ for which $f_m \neq f_k^{-1}$, we get that

$$\sum_{i=1}^{t} |o_i| = |f_k| + (1 + \sum_{i \neq k, m} f_i) = |f_k| + \left(1 + \sum_{i \neq k, m} f_i\right),$$

thus $\text{Skel}(r) = \text{Skel}(\sum_{i=1}^{t} |o_i|) = \text{Skel}(f_k) \cap \text{Skel}(1 + \sum_{i \neq k, m} f_i) \subseteq \text{Skel}(f_k)$.

We aim for a region fraction to define a skeleton containing some neighborhood in $\mathcal{R}^n$, thus the latter condition is required by the above discussion.

Definition 13.1.27. A principal kernel $R \in \text{PCon}(\mathcal{R}(x_1, \ldots, x_n))$ is said to be a region kernel if it is generated by a region fraction.
Remark 13.1.28. \( R \) is a region kernel if and only if it is of the form
\[
R = \prod_{i=1}^{v} O_i
\]
for some order kernels \( O_1, ..., O_v \in \text{PCon}(R(x_1, ..., x_n)) \).

Proof. Let \( r \) be a generating region fraction of \( R \). So,
\[
R = \langle r \rangle = \langle \sum_{i=1}^{t} |o_i| \rangle = \prod_{j=1}^{v} \langle o_i \rangle.
\]
Since \( o_i \) is an order fraction for each \( 1 \leq i \leq v \), we have by definition that the \( O_i = \langle o_i \rangle \) are order kernels. Conversely, if \( R = \prod_{i=1}^{v} O_i \) then taking \( o_i \) to be an order fraction generating \( O_i \), we get that \( r = \sum_{i=1}^{t} |o_i| \) is a region fraction generating \( R \), since \( \langle \sum_{i=1}^{t} |o_i| \rangle = \prod_{i=1}^{v} O_i = R \).

Lemma 13.1.29. Any HP-kernel is corner-integral, and any order kernel is corner-integral.

Proof. As an HP-fraction has a single monomial in its numerator and denominator, there are no corner roots to surpass, and thus it is trivially corner integral. For an order kernel, as it is generated by an element of the form \( 1 + \frac{h}{g} = \frac{g+h}{g} \) for monomials \( g \) and \( h \), it has a single corner root at the numerator (\( x \) such that \( g(x) = h(x) \)) which is surpassed by the denominator \( g(x) \). Finally as corner integrality does not depend on the choice of the generator, our claim is proved.

Remark 13.1.30. If \( \langle f \rangle \neq \langle 1 \rangle \) is an HP-kernel and \( \langle g \rangle \) is an order kernel, then
\[
\langle f \rangle \cdot \langle g \rangle = \langle |f| + |g| \rangle
\]
is regular.

Proof. As regularity does not depend on the choice of a generator of the kernel and since \( \langle |f| + |g| \rangle = \langle f \rangle \cdot \langle g \rangle \), we may consider \( |f| + |g| \). Write \( f = \frac{u}{v} \) and \( g = 1 + \frac{a}{b} \) with \( u, v, a \) and \( b \) monomials in \( R[x_1, ..., x_n] \). Then \( |f| + |g| = \frac{|u|}{|v|} + |1 + \frac{a}{b}| \). Since \( 1 + \frac{a}{b} \geq 1 \), we have that \( |f| + |g| = \frac{|u|}{|v|} + 1 + \frac{a}{b} \) and since \( \frac{|u|}{|v|} \geq 1 \) we get that
\[
|f| + |g| = \left( \frac{|u|}{|v|} + 1 \right) + \frac{a}{b} = \frac{|u|}{v} + \frac{a}{b}.
\]
Now, \[ \frac{u}{v} + \frac{v}{u} = \frac{u^2 + v^2}{uv} = \frac{(u^2 + v^2)b + auv}{buv}. \] Since \( (u^2 + v^2)b \geq uv \), we have that \( (u^2 + v^2)b \geq uvb \). Thus either \( u(x)^2b(x) \geq u(x)v(x)b(x) \) or \( v(x)^2b(x) \geq u(x)v(x)b(x) \) for any \( x \in \mathbb{R}^n \). By assumption, \( f \neq 1 \) and thus \( u \neq v \), so \( u^2b \neq uvb \) and \( v^2b \neq uvb \). Thus for any \( x \in \mathbb{R}^n \) there is always a monomial in the numerator distinct from the one in the denominator. Note that \( auv \) dominates the numerator if \( a(x)u(x)v(x) > (u(x)^2 + v(x)^2)b(x) \geq u(x)v(x)b(x) \), and thus essentiality of \( auv \) implies \( auv \neq buv \), i.e., \( a \neq b \) concluding our proof.

**Remark 13.1.31.** Induction yields that

\[ \langle f \rangle \cdot \langle g_1 \rangle \cdots \langle g_k \rangle \]

is regular, for any HP-kernel \( \langle f \rangle \neq 1 \) and order kernels \( \langle g_1 \rangle, \ldots, \langle g_k \rangle \).

### 13.2 Geometric interpretation of HS-kernels and region kernels and the use of logarithmic scale

**Definition 13.2.1.** Let \( \gamma \in \mathbb{H} \), and let \( k \in \mathbb{N} \). A \( k \)-th root of \( \gamma \), if exists, is an element \( \beta \in \mathbb{H} \) such that \( \beta^k = \alpha \).

**Remark 13.2.2.** For any \( k \in \mathbb{N} \) and \( \gamma \in \mathbb{H} \) the \( k \)-th root of \( \gamma \) is unique.

**Proof.** Let \( \alpha, \beta \in \mathbb{H} \) such that \( \alpha^k = \beta^k \) for some \( k \in \mathbb{N} \). Then multiplying both sides of the equality by \( \beta^{-k} \), we get that \( (\beta^{-1}\alpha)^k = 1 \). By Remark 2.3.6, we have \( \beta^{-1}\alpha = 1 \) and so \( \alpha = \beta \). \( \square \)

Let \((\mathbb{H}, \cdot, +)\) be a divisible semifield. By Remark 13.2.2, we can uniquely define any rational power of the elements of \( \mathbb{H} \). In such a way, \( \mathbb{H} \) becomes a vector space over \( \mathbb{Q} \), rewriting the multiplicative operation \( \cdot \) on \( \mathbb{H} \) as addition and defining

\[ (m/n) \cdot \alpha = \alpha^{\frac{m}{n}}. \quad (13.1) \]

In this way we can apply linear algebra techniques to \((\mathbb{H}, \cdot)\). When considering \( \mathbb{H} \) in such a way we will denote the original addition of \( \mathbb{H} \) (in the idempotent case) by \( + \) or \( \lor \) in order to avoid ambiguity. We call the representation given by (13.1) the logarithmic representation of \((\mathbb{H}, \cdot)\). \( \mathbb{H} \) viewed as just described,
any HP-fraction may be considered as a linear functional over \( \mathbb{Q} \) and thus an HP-kernel may be considered as a kernel generated by a linear functional (notice that for an HP-fraction \( f \), the equation \( f = 1 \) over \( (\mathbb{H}, \cdot)^n \) translates to \( F = 0 \) over \( (\mathbb{H}, +)^n \) where \( F \) is the linear form obtained from \( f \) by applying (13.1), i.e., \( F \) is the logarithmic form of \( f \).)

**Remark 13.2.3.** In the special case of a semifield \( \mathcal{R} \), by Theorem 3.4.13, the above interpretation allows us to consider \( (\mathcal{R}, \cdot) \) as being \( (\mathbb{R}, +) \) and \( \mathcal{R}^n \) as being \( \mathbb{R}^n \) with coordinate-wise addition and scalar multiplication over \( \mathbb{Q} \).

As a consequence of the above discussion, viewing an HP-fraction as a linear functional defining an \( n \)-dimensional affine subspace of \( \mathcal{R}^{n+1} \), we have the following statement:

**Remark 13.2.4.** If \( f \) is an HP-fraction in \( \mathcal{R}(x_1, ..., x_n) \), then \( f \) is completely determined by the set \( \{p_0, ..., p_n\} \) for any \( p_i = (\alpha_{i,1}, ..., \alpha_{i,n}, f(\alpha_{i,1}, ..., \alpha_{i,n})) \in \mathcal{R}^{n+1} \) where
\[
\{a_i = (\alpha_{i,1}, ..., \alpha_{i,n})\} \subset \mathcal{R}^n \text{ such that } p_0, ..., p_n \text{ are in general position (are not contained in an } (n-1) \text{-dimensional affine subspace of } \mathcal{R}^{n+1}).
\]

**Proof.** Writing \( f(x_1, ..., x_n) = \alpha \prod_{i=1}^{n} x_i^{k_i} \) with \( k_i \in \mathbb{Z} \), then \( \alpha = f(a_0) \prod_{i=1}^{n} a_{0,i}^{-k_i} \). After \( \alpha \) is determined, since \( p_0, ..., p_n \) are in general position the set
\[
\{a_1, ..., a_n, b = (f(a_1), ..., f(a_n))\} \subset \mathcal{R}^n
\]
define a linearly independent set of \( n \) linear equations in the variables \( k_i \), and thus determine them uniquely. \( \square \)

Consider an HS-kernel of \( \mathcal{R}(x_1, ..., x_n) \) defined by the HS-fraction \( f = \sum_{i=1}^{t} |f_i| \) where \( f_1, ..., f_t \) are HP-fractions. Then \( f = 1 \) if and only if \( f_i = 1 \) for each \( i = 1, ..., t \). Thus \( f = 1 \) gives rise to a homogenous system of rational linear equations of the form \( F_i = 0 \) where \( F_i \) is the logarithmic form of \( f_i \). This way \( \text{Skel}(f) \subset \mathcal{R}^n \cong (\mathbb{R}^+)^n \) is identified with an affine subspace of \( \mathbb{R}^n \) which is just the intersection of the \( t \) affine hyperplanes defined by \( F_i = 0, 1 \leq i \leq t \). Analogously, for an order kernel defined by \( o = 1 \wedge g \) for some HP-fraction \( g \), \( o = 1 \) if and only if \( g \leq 1 \) which gives rise to the rational half space of \( \mathbb{R} \) defined by the weak inequality \( g \leq 0 \). Thus, the region kernel defined by \( r = \sum_{i=1}^{t} |o_i| = 1 \sum_{i=1}^{t} g_i \) where \( o_1 = 1 \wedge g_1, ..., o_t = 1 \wedge g_t \) are order fractions, yields a nondegenerate
polyhedron formed as an intersection of the affine half spaces each of which is defined by $G_i \leq 0$, where $G_i$ is the logarithmic form of the HP-fraction $g_i$ defining $o_i$ (i.e., $o_i = 1 + g_i$).
13.3 A preliminary discussion

We introduce the following example to motivate our subsequent discussion. Consider a point \( a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \). Then \( a = \text{Skel}(f_a) \) where

\[
f_a(x_1, \ldots, x_n) = \left| \frac{x_1}{\alpha_1} \right| + \cdots + \left| \frac{x_n}{\alpha_n} \right| \in \mathcal{R}(x_1, \ldots, x_n).
\]

We would like \( \langle f_a \rangle \) to encapsulate the dimension reduction from \( \mathbb{R}^n \) to \( \{a\} \).

Consider the following chain of principal HS-kernels

\[
\langle f_a \rangle = \left\langle \left| \frac{x_1}{\alpha_1} \right| + \cdots + \left| \frac{x_n}{\alpha_n} \right| \right\rangle \supset \left\langle \left| \frac{x_1}{\alpha_1} \right| + \cdots + \left| \frac{x_{n-1}}{\alpha_{n-1}} \right| \right\rangle \supset \cdots \supset \left\langle \left| \frac{x_1}{\alpha_1} \right| \right\rangle \supset \langle 1 \rangle = \{1\}.
\]

(13.2)

For each \( 1 \leq k \leq n \) denote \( f_k = \left| \frac{x_k}{\alpha_k} \right| + \cdots + \left| \frac{x_n}{\alpha_n} \right| \) and \( f_0 = 1 \). Each of the HS-kernels \( \langle f_{k-1} \rangle \) in the chain is also a semifield which is a subsemifield of the preceding kernel \( \langle f_{k-1} \rangle \). The factors defined by the quotients \( \langle f_k \rangle / \langle f_{k-1} \rangle \) are the quotient semifields

\[
\left\langle \left| \frac{x_1}{\alpha_1} \right| + \cdots + \left| \frac{x_k}{\alpha_k} \right| \right\rangle / \left\langle \left| \frac{x_1}{\alpha_1} \right| + \cdots + \left| \frac{x_{k-1}}{\alpha_{k-1}} \right| \right\rangle = \prod_{j=1}^{k} \left\langle \frac{x_j}{\alpha_j} \right\rangle / \prod_{j=1}^{k-1} \left\langle \frac{x_j}{\alpha_j} \right\rangle \cong \left\langle \frac{x_k}{\alpha_k} \right\rangle / \left( \prod_{j=1}^{k-1} \left\langle \frac{x_j}{\alpha_j} \right\rangle \cap \left\langle \frac{x_k}{\alpha_k} \right\rangle \right).
\]

Note that on the right hand side of the equality we have a homomorphic nontrivial image of an HP-kernel, namely, an HP-kernel of the quotient semifield

\[
\mathcal{R}(x_1, \ldots, x_n) / \left( \prod_{j=1}^{k-1} \left\langle \frac{x_j}{\alpha_j} \right\rangle \cap \left\langle \frac{x_k}{\alpha_k} \right\rangle \right).
\]

There are some questions arising from the above construction:

Can this chain of HS-kernels be refined to a longer descending chain of principal kernels descending from \( \langle f_a \rangle \)? Are the lengths of descending chains of principal kernels beginning at \( \langle f_a \rangle \) bounded, and if so, can any chain can be refined to a chain of maximal length? The following example provides a positive answer to the first question:

Consider the kernels \( \langle |x| + |y| \rangle \) and \( \langle |x| \rangle = \langle x \rangle \). Both are semifields over the trivial semifield and \( \langle |x| \rangle \) is a subkernel of \( \langle |x| + |y| \rangle \). Consider the substitution map \( \phi \) sending
The kernel $\langle |y| \rangle$ is not simple in the lattice of principal kernels of the semifield $\langle |y| \rangle \mathfrak{A}(y)$ as we have the chain $\langle |y| \rangle \supset \langle |1 + y| \rangle \supset \langle 1 \rangle$, which is the image of the refinement

$$\langle |x| + |y| \rangle \supset \langle |x + y| + |x| \rangle \supset \langle |y| \rangle$$

(since $\phi(|x + y| + |x|) = |\phi(x) + \phi(y)| + |\phi(x)| = |1 + y| + |1| = |1 + y|$). One can notice that $\langle 1 + y \rangle$ is an order kernel which induces the order $y \leq 1$ on the semifield $\langle |y| \rangle$.

In view of the above example we would like to restate the questions posed above as follows: Can this chain of HS-kernels be refined to a longer descending chain of HS-kernels descending from $\langle f_a \rangle$? Are the lengths of descending chains of HS-kernels beginning at $\langle f_a \rangle$ bounded, and if so, can any chain of HS-kernels be refined to such a chain of maximal length?

In the next section we provide answers to these three questions by which the chain introduced above is of maximal unique length common to all chains of HS-kernels descending from $\langle f_a \rangle$.

### 13.4 The HO-decomposition

In the following we describe an explicit decomposition of a principal kernel $\langle f \rangle$ as an intersection of kernels of two types: The first, to be named an HO-kernel, is a product of some HS-kernel and a region kernel. The second is a product of a region kernel and a bounded from below kernel.

While the first type defines the skeleton of $\langle f \rangle$, the second type has no effect on it as it corresponds to the empty set. This latter type is the source of ambiguity in relating a skeleton to a kernel, preventing the kernel corresponding to a skeleton from being principal. When intersecting with $\langle \mathcal{R} \rangle$, the kernels of the second type in the decomposition are chopped off, in the sense that they all become equal to $\langle \mathcal{R} \rangle$. This restriction to $\langle \mathcal{R} \rangle$ thus removes the ambiguity making each HO-kernel (intersected with $\langle \mathcal{R} \rangle$) in a $1 : 1$ correspondence with its skeleton (which is, in fact, a segment in the skeleton defined by $\langle f \rangle$). Subsequently, the ‘HO-part’ is unique and independent of the choice of the kernel generating the skeleton.

Geometrically, the decomposition to be described below is just a fragmentation of
a principal skeleton defined by \( \langle f \rangle \) to the “linear” fragments comprising it. Each fragment is attained by bounding an affine subspace of \( \mathbb{R}^n \) defined by an appropriate HS-fraction (which in turn generates an HS-kernel) using a region fraction (generating a region kernel). Note that the HS-fraction may be 1 and so may the region fraction. By the discussion above, although the HS-fraction and region fraction defining each segment may vary moving from one generator of the principal kernel to the other, the HS-kernels and region kernels they define stay intact as they correspond to the fragments of the skeleton of \( \langle f \rangle \). Thus we may form the next construction, though explicit, using any generator without affecting the resulting HO-kernels.

We now move forward to introduce the construction which will be illustrated by two subsequent examples.

Construction 13.4.1. Consider an element \( f \in \mathbb{R}(x_1, \ldots, x_n) \), the kernel it generates, \( \langle f \rangle \), and its corresponding skeleton \( \text{Skel}(f) \). Taking \( |f| \) we may assume \( f \geq 1 \). Write \( f = \frac{h}{g} = \sum_{i=1}^{k} \frac{h_i}{g_i} \) where \( h_i \) and \( g_j \) are monomials in \( \mathbb{R}[x_1, \ldots, x_n] \). Assume \( \text{Skel}(f) \neq \emptyset \).
Let \( a \) be a point of \( \text{Skel}(f) \). Since \( f(a) = 1 \), there exists a subset \( H_a \subseteq H = \{ h_i : 1 \leq i \leq k \} \) and a subset
\[
G_a \subseteq G = \{ g_j : 1 \leq j \leq m \}
\]
for which \( g'(a) = h'(a) \) for any \( h' \in H_a \) and \( g' \in G_a \). Denote by \( H_a^C \) and \( G_a^C \) the complementary subsets of monomials of \( H \) and \( G \) respectively, i.e., \( H_a^C = H \setminus H_a \) and \( G_a^C = G \setminus G_a \). Then, for any \( h' \in H_a \) and \( h'' \in H_a^C \) we have
\[
h'(a) + h''(a) = h'(a),
\]
i.e., \( h'(a) \geq h''(a) \). We can write these last relations equivalently as \( 1 + \frac{h''(a)}{h'(a)} = 1 \) for all \( h' \in H_a \) and all \( h'' \in H_a^C \). Similarly, for any \( g' \in G_a \) and \( g'' \in G_a^C \) we have that \( g'(a) + g''(a) = g'(a) \). We can write these last relations equivalently as \( 1 + \frac{g''(a)}{g'(a)} = 1 \) for all \( g' \in G_a \) and all \( g'' \in G_a^C \).

Thus for any such \( a \) we obtain the following relations:
\[
\frac{h'}{g} = 1, \quad \forall h' \in H_a, g' \in G_a, \quad (13.3)
\]
\[
1 + \frac{h''}{h'} = 1 ; \quad 1 + \frac{g''}{g'} = 1, \quad \forall h' \in H_a, h'' \in H_a^C, g' \in G_a, g'' \in G_a^C. \quad (13.4)
\]
Varying \( a \in \text{Skel}(f) \), evidently there are only finitely many possibilities for relations in (13.3) and (13.4) as there are only finitely many monomials \( h_i \) and \( g_j \) comprising \( f \).

Any set of relations comprised of the relations in (13.3) and (13.4), which will hereby be denoted by \( \theta_1 \) and \( \theta_2 \), corresponds to a kernel generated by the corresponding elements

\[
\frac{h'}{g'}, \left(1 + \frac{h''}{h'}\right), \text{ and } \left(1 + \frac{g''}{g'}\right),
\]

where \( \{\frac{h'}{g'} = 1\} \in \theta_1 \) and \( \{g + \frac{g''}{g} = 1\}, \{1 + \frac{h''}{h} = 1\} \in \theta_2 \). The finite collection of pairs \((\theta_1(i), \theta_2(i))\) for \( i = 1, \ldots, s \), is formed by considering all points \( a \) in \( \text{Skel}(f) \).

Moreover, every point admitting the relations in \((\theta_1(i), \theta_2(i))\) for any \( i \in \{1, \ldots, s\} \) is in \( \text{Skel}(f) \). Thus this collection supplies a complete covering of \( \text{Skel}(f) \). Denote by \( K_i \), for \( i = 1, \ldots, s \), the kernel generated by the elements of \( \theta_1(i) \) and \( \theta_2(i) \).

Then since \( \text{Skel}(\langle f \rangle \cap \langle R \rangle) = \text{Skel}(f) = \bigcup_{i=1}^{s} \text{Skel}(K_i) = \bigcup_{i=1}^{s} \text{Skel}(K_i \cap \langle R \rangle) \)

where \( \langle f \rangle \cap \langle R \rangle \) and \( \bigcup_{i=1}^{s} K_i \cap \langle R \rangle \) are in \( \text{PCon}(\langle R \rangle) \) we have that \( \langle f \rangle \cap \langle R \rangle = \bigcap_{i=1}^{s} K_i \cap \langle R \rangle \).

In essence \( \bigcap_{i=1}^{s} K_i \) provides a local description of \( f \) in a neighborhood of its skeleton.

We now proceed to supply an insight of the structure of the kernel \( \langle f \rangle \) to put the construction above into a broader context.

In Construction 13.4.1 we used the skeleton of \( \langle f \rangle \) to construct \( \bigcap_{i=1}^{s} K_i \). Considering all points \( a \) in \( \mathcal{R}^n \) might add some regions, complementary to the regions defined by (13.4) in \( \theta_2(i) \) for \( i = 1, \ldots, s \), over which \( \frac{h'}{g'} \neq 1, \quad \forall h' \in H_a, \)

\( \forall g' \in G_a \), i.e., regions over which the dominating monomials never meet. Continuing the construction above using \( a \in \mathcal{R}^n \setminus \text{Skel}f \) similarly produces a finite collection of, say \( t \in \mathbb{Z}_{\geq 0} \), kernels generated by elements from (13.4) and their complementary order fractions and by elements of the form (13.3) (where now \( \frac{h'}{g'} \neq 1 \) over the considered region). A principal kernel \( N_j = \langle q_j \rangle \ 1 \leq j \leq t \), of this complementary set of kernels has the property that \( \text{Skel}(N_j) = \emptyset \), thus by Corollary 5.1.8 \( N_j \) is bounded from below. As there are finitely many such kernels there exists \( \gamma \in \mathcal{R}, \gamma > 1 \) and small enough, such that \( |q_j| \wedge \gamma = \gamma \) for \( j = 1, \ldots, t \). Thus \( \bigcap_{j=1}^{t} N_j \) is bounded from below and thus by Remark 5.1.3 we have that \( \bigcap_{j=1}^{t} N_j \supseteq \langle R \rangle \).
As now the extension coincides with \( f \) over all of \( \mathcal{R}^n \) and as \( \mathcal{R} \) is divisible, we have

\[
\langle f \rangle = \bigcap_{i=1}^{s} K_i \cap \bigcap_{j=1}^{t} N_j. \tag{13.5}
\]

So, \( \langle f \rangle \cap \langle \mathcal{R} \rangle = \bigcap_{i=1}^{s} K_i \cap \bigcap_{j=1}^{t} N_j \cap \langle \mathcal{R} \rangle = \bigcap_{i=1}^{s} K_i \cap \langle \mathcal{R} \rangle. \)

In view of the last discussion, we see that intersecting a principal kernel \( \langle f \rangle \) with \( \langle \mathcal{R} \rangle \) ‘chops off’ all of its comprising bounded from below kernels (the \( N_j \)'s above). This way it eliminates ambiguity in the kernel corresponding to \( \text{Skel}(f) \).

Finally we note that if \( \text{Skel}(f) = \emptyset \) then \( \langle f \rangle = \bigcap_{i=1}^{s} N_j \) for appropriate kernels \( N_j \) and \( \langle f \rangle \cap \langle \mathcal{R} \rangle = \langle \mathcal{R} \rangle \).

A few notes concerning the construction:

**Remark 13.4.2.**

1. If \( K_1 \) and \( K_2 \) are such that \( K_1 \cdot K_2 \cap \mathcal{R} = \{1\} \) (i.e., \( \text{Skel}(K_1) \cap \text{Skel}(K_2) \neq \emptyset \)), then the sets of HP-fractions \( \theta_1 \) of \( K_1 \) and of \( K_2 \) are not equal (though one may contain the other), for otherwise they would be combined via the construction to form a single kernel.

2. Let \( \langle f \rangle \cap \langle \mathcal{R} \rangle = \bigcap_{i=1}^{s} (K_i \cap \langle \mathcal{R} \rangle) = \bigcap_{i=1}^{s} |k_i| \wedge |\alpha| \) with \( \alpha \in \mathcal{R} \setminus \{1\} \). By Corollary 9.3.9 for any generator \( f' \) of \( \langle f \rangle \cap \langle \mathcal{R} \rangle \) we have that \( |f'| = \bigwedge_{i=1}^{s} |k'_i| \) with \( k'_i \sim_K |k_i| \wedge |\alpha| \) for every \( i = 1, \ldots, s \). In particular, \( \text{Skel}(k'_i) = \text{Skel}(|k_i| \wedge |\alpha|) = \text{Skel}(k_i) \). Thus the above construction of the \( K_i \) kernels is independent of the choice of the generator as it is totally defined by the fragments \( \text{Skel}(k_i) \) of the skeleton \( \text{Skel}(f) \).

We now provide two examples for the construction introduced above. We make use of the notation above for the different types of kernels involved in the construction.

**Example 13.4.3.** Let \( f = |x| \wedge \alpha \in \mathcal{R}(x, y) \) for some \( \alpha > 1 \) in \( \mathcal{R} \). Then \( f = \frac{\alpha|x|}{\alpha + |x|} \).

The order relation \( \alpha \leq |x| \) translates to the relation \( \alpha + |x| = |x| \) or equivalently to \( \alpha|x|^{-1} + 1 = 1 \). Over the region defined by the last relation we have \( f = \frac{\alpha|x|}{\alpha} = |x| \).

Similarly, the complementary order relation \( \alpha \geq |x| \) translates to \( \alpha^{-1}|x| + 1 = 1 \) (via \( |x| + \alpha = \alpha \)) over which region \( f = \frac{\alpha|x|}{x} = \alpha \). So

\[
\langle f \rangle = K_1 \cap K_2 = (R_{1,1} \cdot L_{1,1}) \cap (R_{2,1} \cdot N_{2,1})
\]
where \( R_{1,1} = \langle \alpha |x|^{-1} + 1 \rangle \), \( L_{1,1} = \langle |x| \rangle \), \( R_{2,1} = \langle \alpha^{-1} |x| + 1 \rangle \) and \( N_{2,1} = \langle \alpha \rangle \). Geometrically \( R_{1,1} \) is a strip containing the axis \( x = 1 \) and \( R_{2,1} \) is its complementary region. The restriction of \( f \) to \( R_{1,1} \) gives it the form \(|x|\) while restricting to \( R_{2,1} \), \( f \) equals \( \alpha \). Furthermore, we see that over \( R_{2,1} \), \( f \) is bounded from below by \( \alpha \). Omitting \( N_{2,1} \) we still have \( Skel(f) = Skel(R_{1,1} \cdot L_{1,1}) \) though \( R_{1,1} \cdot L_{1,1} \supset \langle f \rangle \) and equality do not hold. Intersecting \( \langle f \rangle \) with \( \langle R \rangle \) leaves \( \langle f \rangle \) intact while \( (R_{1,1} \cdot L_{1,1}) \cap \langle R \rangle = (R_{1,1} \cdot L_{1,1} \cap R_{2,1} \cdot N_{2,1}) \cap \langle R \rangle \). Thus \( \langle f \rangle = (f) \cap \langle R \rangle = (R_{1,1} \cdot L_{1,1}) \cap \langle R \rangle \).

**Example 13.4.4.** Let \( f = |x + 1| \land \alpha \in \mathcal{A}(x, y) \) for some \( \alpha > 1 \) in \( \mathcal{A} \). First note that since \( x + 1 \geq 1 \) we have that \( |x + 1| = x + 1 \), allowing us to rewrite \( f \) as \((x + 1) \land \alpha\). Then \( f = \frac{\alpha(x + 1)}{\alpha + x} = \frac{\alpha x + \alpha}{\alpha + x} \). The order relation \( \alpha \leq x \) translates to the relation \( \alpha + x = x \) or equivalently to \( \alpha x^{-1} + 1 = 1 \). Over the region defined by the last relation, we have \( f = \frac{\alpha x + \alpha}{\alpha + x} = \alpha + \frac{\alpha}{x} = \alpha \). Similarly, the complementary order relation \( \alpha \geq x \) translates to \( \alpha^{-1} x + 1 = 1 \) over which \( f = \frac{\alpha x + \alpha}{\alpha + x} = \frac{\alpha x + \alpha}{\alpha} = x + 1 \). So

\[
\langle f \rangle = K_1 \cap K_2 = (R_{1,1} \cdot R_{1,2}) \cap (R_{2,1} \cdot N_{2,1}) = R_{1,2} \cap R_{2,1} \cdot N_{2,1}
\]

where \( R_{1,1} = \langle \alpha^{-1} x + 1 \rangle \), \( R_{1,2} = \langle x + 1 \rangle \), \( R_{2,1} = \langle \alpha x^{-1} + 1 \rangle \) and \( N_{2,1} = \langle \alpha \rangle \). Since \( N_{2,1} \subset R_{2,1} \cdot N_{2,1} \) we have that \( Skel(R_{2,1} \cdot N_{2,1}) \subset Skel(N_{2,1}) = \emptyset \). So

\[
Skel(f) = Skel(R_{1,2}) \cup Skel(R_{2,1} \cdot N_{2,1}) = Skel(x + 1) \cup \emptyset = Skel(x + 1).
\]

As can be easily seen from examples [13.4.3] and [13.4.4] by substituting any HP-fraction for \( x \) and any order fraction for \((x + 1)\), the intersection of a kernel \( K = L \cdot R = \prod L_i \cdot \prod O_j \) (\( L_i \) and \( O_j \) are HP-kernels and order kernels respectively) with \( \langle R \rangle \) yields

\[
K' = K \cap \langle R \rangle = \prod (L_i \cap \langle R \rangle) \cdot \prod (O_j \cap \langle R \rangle)
= \prod ((L_i \cdot R_i) \cap (N_i \cdot R_i^c)) \cdot \prod ((O_j \cdot R_j) \cap (M_j \cdot R_j^c)) = \left( \prod L_i \cdot \prod O_j \right) \cap N
= (L \cdot R') \cap N
\]

where \( R', L_i \) and \( R_j \) are region kernels. \( R_i^c \) and \( R_j^c \) are \( R_i \)'s and \( R_j \)'s complementary region kernels respectively. \( N_i, M_j \) and \( N \) are bounded from below kernels. Note that the \( O_j \)'s involve the \( R_i \)'s, the \( R_j \)'s and the \( O_j \)'s, while \( N \) is derived from the bounded from below kernels, namely the \( N_i \)'s and \( M_j \)'s. Also note that intersecting with \( \langle R \rangle \) keeps the HS-kernel unchanged in the new decomposition.
As the bounded from below kernels, the $N_j$s in (13.3), do not affect $\text{Skel}(f)$ we leave them aside for the time being and proceed to study the structure of the kernels $K_i$ and their corresponding skeletons.

Let $K$ be one of the above kernels $K_i$. First note that every element of the set generating $K$ which is specified above, is either an HP-fraction (a nonconstant Laurent monomial) of the form $\frac{h'}{g'}$, or an order element of the form $1 + \frac{h''}{g''}$ (or $1 + \frac{g''}{h''}$). Let $L_i$ with $i = 1, \ldots, u$ and $O_j$ with $j = 1, \ldots, v$, be the kernels generated by each of the HP-kernels and the order kernels respectively. Then we can write

$$K = L \cdot R = \prod_{i=1}^{u} L_i \cdot \prod_{j=1}^{v} O_j \quad \text{(13.6)}$$

where $L = \prod_{i=1}^{u} L_i$ is an HS-kernel and $R = \prod_{j=1}^{v} O_j$ is a region kernel. Note that by the assumption of the construction above $\text{Skel}(K) \neq \emptyset$ (as there is at least one point of the skeleton used for constructing it). Moreover, there are no distinct HS-kernels $M_1$ and $M_2$ such that $L = M_1 \cap M_2$ for otherwise the construction would have produced two distinct kernels, one with $M_1$ as its HS-kernel and the other with $M_2$ as its HS-kernel instead of producing $K$ in the first place.

**Definition 13.4.5.** An element $f \in \mathcal{R}(x_1, \ldots, x_n)$ is said to be an HO-fraction if

$$f = l(f) + o(f)$$

with $l(a) = \sum_{i=1}^{k} |l_i|$ an HS-fraction and $o(a) = \sum_{j=1}^{k} |o_j|$ a region fraction, where the $l_i$'s are HP-fractions and the $o_j$'s are order-fractions. We say that $l(f)$ and $o(f)$ are an HS-fraction and a region-fraction corresponding to $f$.

**Definition 13.4.6.** A principal kernel $K \in \text{PCon}(\mathcal{R}(x_1, \ldots, x_n))$ is said to be an HO-kernel if it is generated by an HO-fraction.

**Remark 13.4.7.** A kernel $K \in \text{PCon}(\mathcal{R}(x_1, \ldots, x_n))$ is an HO-kernel if and only if $K = L \cdot R$ where $R$ is a region kernel and $L$ is an HS-kernel.

**Proof.** If $K$ is an HO-kernel, then $K = \langle f \rangle$ where $f = l(f) + o(f)$ is an HO-fraction. Thus $K = \langle l(f) + o(f) \rangle = \langle l(f) \rangle \cdot \langle o(f) \rangle = L \cdot R$ where $L = \langle l(f) \rangle$ is an HS-kernel and $R = \langle o(f) \rangle$ is a region kernel. Conversely, taking the HO-fraction $f = l + r$ where $l$ is an HS-fraction generating $L$ and $r$ is a region fraction generating $R$, we get that $\langle f \rangle = \langle l + r \rangle = \langle l \rangle \cdot \langle r \rangle = L \cdot R = K$. \)

165
Remark 13.4.8. Let \( K = L \cdot R \) be an HO-kernel with \( R \) a region kernel and \( L \) an HS-kernel. By Remarks [13.1.14] and [13.1.28] we have that \( L = \prod_{i=1}^{u} L_i \) for some HP-kernels \( L_1, \ldots, L_u \) and \( R = \prod_{j=1}^{v} O_j \) for some order kernels \( O_1, \ldots, O_v \). Thus \( K \) is of the form

\[
K = L \cdot R = \prod_{i=1}^{u} L_i \cdot \prod_{j=1}^{v} O_j
\]  

(13.7)

where \( u \in \mathbb{Z}_{\geq 0} \), \( v \in \mathbb{N} \), and \( O_1, \ldots, O_v \) are order kernels. Note that every region kernel and every HS-kernel are by definition an HO-kernel, taking \( u = 0 \) for a region-kernel and \( v = 1 \) with \( O_1 = \langle 1 + a \rangle \) where \( L = \langle a \rangle \) is the HS-kernel.

Remark 13.4.9. Let \( K_1 \) and \( K_2 \) be region-kernels (respectively HS-kernels) such that \( K_1 \cdot K_2 \cap \mathcal{R} = \{1\} \). Then \( K_1 \cdot K_2 \) is a region-kernel (respectively HS-kernel). Consequently, if \( K_1 \) and \( K_2 \) are HO-kernels such that \( K_1 \cdot K_2 \cap \mathcal{R} = \{1\} \), then \( K_1 \cdot K_2 \) is an HO-kernel. Indeed, the assertions follow from the decomposition \( K_s = L_s \cdot O_s = \prod_{i=1}^{u_s} L_{s,i} \cdot \prod_{j=1}^{v_s} O_{s,j} \) for \( s = 1, 2 \) so that

\[
K_1 \cdot K_2 = (L_1 L_2) \cdot (O_1 O_2) = \left( \prod_{i=1}^{u_1} L_{1,i} \prod_{i=1}^{u_2} L_{2,i} \right) \cdot \left( \prod_{j=1}^{v_1} O_{1,j} \prod_{j=1}^{v_2} O_{2,j} \right) = L \cdot O,
\]

with the appropriate \( u_s, v_s \) taken for \( s = 1, 2 \).

By the above discussion we have

**Theorem 13.4.10.** Every principal kernel \( \langle f \rangle \) of \( \mathcal{R}(x_1, \ldots, x_n) \) can be written as an intersection of finitely many principal kernels

\[
\{K_i : i = 1, \ldots, s\} \text{ and } \{N_j : j = 1, \ldots, m\},
\]

where each \( K_i \) is a product of an HS-kernel and a region kernel

\[
K_i = L_i \cdot R_i = \prod_{j=1}^{t} L_{j,i} \prod_{l=1}^{t} O_{l,i}
\]

(13.8)

while each \( N_j \) is a product of bounded from below kernels and (complementary) region kernels. For \( \langle f \rangle \in \text{PCon}(\mathcal{R}) \), the \( N_j \) can be replaced by \( \langle \mathcal{R} \rangle \) without affecting the resulting kernel.

- If \( \langle f \rangle \) is an HS-kernel, then the decomposition degenerates to \( \langle f \rangle = K_1 \) with \( K_1 = L_1 = \langle f \rangle \).
- If \( \langle f \rangle \) is a region kernel, then \( \langle f \rangle = K_1 \) with \( K_1 = R_1 = \langle f \rangle \).
- \( \langle f \rangle \) is an irregular kernel if and only if there exists some \( i_0 \in \{1, \ldots, s\} \) such that \( K_{i_0} = R_{i_0} = \prod_{l=1}^{t} O_{l,i_0} \).
• \( f \) is a regular kernel if and only if \( K_i \) is comprised of at least one HP-kernel, for every \( i = 1, \ldots, s \).

**Proof.** The last three assertions are direct consequences of the construction above, namely, if \( \langle f \rangle \) is either an HS-kernel or a region kernel, \( \langle f \rangle \) is already in the form of its decomposition. The fourth is equivalent to the third.

If \( \langle f \rangle \) is an HS-kernel then by Remark 13.1.14, \( \langle f \rangle = \prod_{j=1}^{t} L_j \) where \( L_j \) is an HP-kernel for each \( j \). Write \( f = \sum_{h, g} \) with \( h, g \in \mathcal{R}[x_1, \ldots, x_n] \) monomials. If \( \langle f \rangle \) is irregular, then by definition of irregularity we have some \( i_0 \) and \( j_0 \) for which \( h_{i_0} = g_{j_0} \) where \( g_{j_0} \neq h_i, g_j \) for every \( i \neq i_0 \) and \( j \neq j_0 \), at some neighborhood of a point \( a \in \mathcal{R}^n \). The kernel corresponding to (the closure) of this region has its relation (13.3) degenerating to \( 1 = 1 \) as \( \frac{h_{i_0}}{g_{j_0}} = 1 \) over the region, thus is given only by its order relations of (13.4).

**Definition 13.4.11.** We call the decomposition given in Theorem 13.4.10 of a principal kernel \( \langle f \rangle \in \text{PCon}(\mathcal{R}(x_1, \ldots, x_n)) \) the **HO-decomposition** of \( \langle f \rangle \). In the special case where \( \langle f \rangle \in \text{PCon}(\langle \mathcal{R} \rangle) \), all bounded from below terms of the intersection are equal to \( \langle \mathcal{R} \rangle \).

**Definition 13.4.12.** For a subset \( S \subseteq \mathcal{R}(x_1, \ldots, x_n) \), denote by \( \text{HO}(S) \) the family of HO-fractions in \( S \), by \( \text{HS}(S) \) the family of HS-fractions in \( S \), and by \( \text{HP}(S) \) the family of HP-fractions in \( S \).

**Remark 13.4.13.** Since every HP-fraction is an HS-fraction and every HS-fraction is an HO-fraction, we have that

\[
\text{HP}(S) \subset \text{HS}(S) \subset \text{HO}(S)
\]

for any \( S \subseteq \mathcal{R}(x_1, \ldots, x_n) \).

**Example 13.4.14.** Consider the kernel \( \langle f \rangle \) where \( f = \frac{x}{y+1} \in \mathcal{R}(x_1, \ldots, x_n) \). The points on the skeleton of \( f \) define three distinct HS-kernels: \( \langle \frac{x}{y} \rangle \) (corresponding to the equality \( x = y \)) over the region \( \{y \geq 1\} \) which is defined by the region kernel \( \langle 1+y^{-1} \rangle \), \( \langle x \rangle = \langle \frac{x}{1} \rangle \) (corresponding to \( x = 1 \)) over the region \( \{y \leq 1\} \) which is defined by the region kernel \( \langle 1+y \rangle \), and \( \langle |x|+|y| \rangle \) (corresponding to the point defined by \( x = 1 \) and \( x = y \)). Thus by Construction 13.4.1

\[
\langle f \rangle = \left( \langle \frac{x}{y} \rangle \cdot \langle 1+y^{-1} \rangle \right) \cap \langle x \rangle \cdot \langle 1+y \rangle \cap \langle |x|+|y| \rangle \cdot \langle 1 \rangle
\]
Figure 13.2: $\text{Skel}(\langle x \rangle \cap (1 + y^{-1})) = \text{Skel}(\langle x \rangle \cap (1 + y)) \cup \text{Skel}(\langle |x| + |y| \rangle)$

and

$\text{Skel}(f) = \left( \text{Skel} \left( \frac{x}{y} \right) \cap \text{Skel}(1 + y^{-1}) \right) \cup (\text{Skel}(x) \cap \text{Skel}(1 + y)) \cup (\text{Skel}(|x| + |y|) \cap \mathbb{R}^2)$. 

Note that the third component of the decomposition (i.e., the HS-kernel $\langle |x| + |y| \rangle$) can be omitted without effecting $\text{Skel}(f)$.

The decomposition is shown (in logarithmic scale) in Figure 13.2 where the first two components are the rays beginning at the origin and the third component is the origin itself.
13.5 The lattice generated by regular corner-integral principal kernels

Recall Remark 13.1.31 which states that the principal kernel

\[ \langle f \rangle \cdot \langle g_1 \rangle \cdot \cdots \cdot \langle g_k \rangle \]

is regular, for any HP-kernel \( \langle f \rangle \neq 1 \) and order kernels \( \langle g_1 \rangle, \ldots, \langle g_k \rangle \).

**Corollary 13.5.1.** Let \( K \in \text{PCon}(\langle \mathcal{R} \rangle) \) and let

\[ K = (L \cdot R) \cap \langle \mathcal{R} \rangle = \left( \prod_{i=1}^{u} L_i \cdot \prod_{j=1}^{v} O_j \right) \cap \langle \mathcal{R} \rangle \]

be the decomposition of \( K \), as given in (13.6), where \( L_1, \ldots, L_u \) are some HP-kernels and \( O_1, \ldots, O_v \) are some order kernels. If \( u \neq 0 \), i.e., \( L \neq \langle 1 \rangle \) then \( K \) is regular.

**Proof.** Indeed, \( K = \prod_{i=2}^{u} L_i \cdot (L_1 \cdot \prod_{j=1}^{v} O_j) \cap \langle \mathcal{R} \rangle. \) By Remark 13.1.31 we have that \( (L_1 \cdot \prod_{j=1}^{v} O_j) \) is regular as \( L_1 \) is regular as an HP-kernel. Thus since a product of regular kernels is regular and since intersection with \( \langle \mathcal{R} \rangle \) does not affect regularity, we have that \( K \) is a regular kernel. \( \square \)

**Theorem 13.5.2.** The lattice generated by principal corner integral kernels in \( \text{PCon}(\langle \mathcal{R} \rangle) \) is the lattice of principal kernels \( \text{PCon}(\langle \mathcal{R} \rangle) \) and the lattice generated by regular principal corner integral kernels is the lattice of regular principal kernels.

**Proof.** Let \( \langle f \rangle \) be a principal kernel and let

\[ \langle f \rangle = \left( \bigcap_{i=1}^{s} K_i \right) \cap \langle \mathcal{R} \rangle, \quad K_i = \prod_{j=1}^{t} L_{j,i} \prod_{l=1}^{l} O_{l,i} \]

be its HO-decomposition. By Lemma 13.1.29 each HP-kernel \( L_{j,i} \) and each order kernel \( O_{l,i} \) are corner integral. Thus \( \langle f \rangle \) as a finite product of principal corner integral kernels is in the lattice generated by principal corner-integral kernels. As any principal corner integral kernel is in particular principal, the lattice of principal kernels contains the lattice generated by principal corner-integral kernels. As for the second assertion, if \( \langle f \rangle \) is regular, then by Theorem 13.4.10 for every \( 1 \leq i \leq s \),
we have that $L_{1,i} \neq 1$. Thus by Corollary 13.5.1 we have that each $K_i$ is a product of principal regular corner-integral kernels. Thus $\langle f \rangle$ is in the lattice generated by principal regular corner-integral kernels. As any regular corner integral kernel is in particular regular, the lattice of principal regular kernels contains the lattice generated by principal regular corner-integral kernels. Thus the second assertion holds.

Corollary 13.5.3. By Theorem 13.5.2 and the correspondence between principal (regular) corner-loci and principal (regular) corner-integral kernels of $\langle R \rangle$, defined by composing the correspondence between the kernels with their principal skeletons $K \mapsto Skel(K)$ introduced in Corollary 7.0.26 with the correspondence between these principal skeletons and their corresponding corner loci introduced in Proposition 10.2.18, we have that the lattice of (regular) finitely generated corner loci corresponds to the lattices of principal (regular) kernels of $\langle R \rangle$.

Corollary 13.5.4. By Corollary 13.5.3, we have that supertropical varieties correspond to principal skeletons and kernels while tropical varieties correspond to regular principal skeletons and kernels.

13.6 Convexity degree and hyperdimension

In this section, $K$ is a semifield which is an affine extension of the bipotent semifield $R$, i.e., $K$ is of the form $R(x_1, ..., x_n)/L$ for some kernel $L \in \text{Con}(R(x_1, ..., x_n))$ (see Remark 2.3.23). In particular $K$ is idempotent.

Proposition 13.6.1. Let $S$ be an idempotent semifield. Let $M$ be a kernel of $S$ such that $M = K_1 \cap K_2 \cap \cdots \cap K_t$ for distinct kernels $K_i$ of $S$. Then $S/M$ is subdirectly reducible and

$$
S/M \rightarrow_{s.d.} \prod_{i=1}^{t} S/K_i.
$$

Proof. Consider the quotient semifield $\bar{S} = S/M$. Let $\phi : S \to \bar{S}$ be the quotient map. Denote by $\bar{K}_i = \phi(K_i) = K_i/M$ the images of $K_1, ..., K_n$ under $\phi$ which are kernels of $\bar{S}$. Since $M = K_1 \cap K_2 \cap \cdots \cap K_t$, we have that $\bar{K}_1 \cap \bar{K}_2 \cap \cdots \cap \bar{K}_t = \{1\}$ in $\bar{S}$ (note that by Corollary 2.3.23 there is a lattice isomorphism between the kernels of $\bar{S}$ and the kernels of $S$ that contain $M$).
Thus, by Remark 3.1.12 we have $\bar{S}$ is subdirectly reducible and as $\bar{S}/\bar{K}_i = (S/M)/(K_i/M) \cong S/K_i$ (by the second isomorphism theorem 2.2.52) we have that

$$\bar{S} \xrightarrow{s.d} \prod_{i=1}^{t} \bar{S}/\bar{K}_i \cong \prod_{i=1}^{t} S/K_i.$$ 

\[\square\]

**Corollary 13.6.2.** In particular, Proposition 13.6.1 applies to the idempotent semifield $H(x_1,\ldots,x_n)$ where $H$ is a bipotent semifield (or any of its kernels considered as a semifield) and to any principal kernel $M = \langle f \rangle \in \text{PCon}(H(x_1,\ldots,x_n))$ such that $M = K_1 \cap K_2 \cap \cdots \cap K_t$ for some $K_i \in \text{PCon}(H(x_1,\ldots,x_n))$.

Let $\langle f \rangle \in \langle \mathcal{R} \rangle$ be a principal kernel and let $\langle f \rangle = \bigcap_{i=1}^{s} K_i$, where

$$K_i = (L_i \cdot R_i) \cap \langle \mathcal{R} \rangle = (L_i \cap \langle \mathcal{R} \rangle) \cdot (R_i \cap \langle \mathcal{R} \rangle) = L'_i \cdot R'_i$$

is its (full) HO-decomposition; i.e., for each $1 \leq i \leq s$ $R_i \in \text{PCon}(\mathcal{R}(x_1,\ldots,x_n))$ is a region kernel and $L_i \in \text{PCon}(\mathcal{R}(x_1,\ldots,x_n))$ is either an HS-kernel or bounded from below (in which case $L'_i = \langle \mathcal{R} \rangle$). Then by Corollary 13.6.2 we have the subdirect decomposition

$$\langle \mathcal{R} \rangle/\langle f \rangle \xrightarrow{s.d} \prod_{i=1}^{t} (\langle \mathcal{R} \rangle/K_i) = \prod_{i=1}^{t} (\langle \mathcal{R} \rangle/L'_i \cdot R'_i)$$

where $t \leq s$ is the number of kernels $K_i$ for which $L'_i \neq \langle \mathcal{R} \rangle$ (for otherwise $\langle \mathcal{R} \rangle/K_i = \{1\}$ and can be omitted from the subdirect product).

The last discussion motivates us to study the semifields $\langle \mathcal{R} \rangle/K_i$ as building blocks for the algebraic structure of the quotient semifield $\langle \mathcal{R} \rangle/\langle f \rangle$ which in turn is the coordinate semifield corresponding to the skeleton $\text{Skel}(f)$.

**Example 13.6.3.** Consider the principal kernel $\langle x \rangle \in \text{PCon}(\mathcal{R}(x,y))$. For $\alpha \in \mathcal{R}$ such that $\alpha > 1$, we have the following infinite strictly descending chain of principal kernels

$$\langle x \rangle \supset \langle |x| + |y + 1| \rangle \supset \langle |x| + |\alpha^{-1}y + 1| \rangle \supset \langle |x| + |\alpha^{-2}y + 1| \rangle \supset \cdots$$

$$\supset \langle |x| + |\alpha^{-k}y + 1| \rangle \supset \cdots$$

and the strictly ascending chain of skeletons corresponding to it (see figure 2.3)

$$\text{Skel}(x) \subset \text{Skel}(|x| + |y + 1|) \subset \cdots \subset \text{Skel}(|x| + |\alpha^{-k}y + 1|) \subset \cdots =$$

$$\text{Skel}(x) \subset \text{Skel}(x) \cap \text{Skel}(y + 1) \subset \cdots \subset \text{Skel}(x) \cap \text{Skel}(\alpha^{-k}y + 1) \subset \cdots.$$
Example 13.6.4. Again, consider the principal kernel $\langle x \rangle \in \text{PCon}(\mathcal{R}(x, y))$. Then $\langle x \rangle = \langle |x| + (|y| + 1) \rangle = \langle (|x| + |y| + 1) \rangle = \langle (|x| + |y| + 1) \rangle = \langle (|x| + |y| + 1) \rangle$. So, we have that a nontrivial decomposition of $\text{Skel}(x)$ as $\text{Skel}(|x| + |y| + 1) \cap \text{Skel}(|x| + |y| + 1)$ (note that $\text{Skel}(|x| + |y| + 1) = \text{Skel}(x) \cap \text{Skel}(y + 1)$ and $\text{Skel}(|x| + |y| + 1) = \text{Skel}(x) \cap \text{Skel}(\frac{1}{y} + 1)$). In a similar way, using complementary order kernels, one can show that every principal kernel can be non-trivially decomposed to a pair of principal kernels.

Examples [13.6.3] and [13.6.4] demonstrate that the lattice of principal kernels $\text{PCon}(\mathcal{R}(x_1, ..., x_n))$ (resp. $\text{PCon}(\langle \mathcal{R} \rangle)$) is too rich to define reducibility or dimensionality. Moreover, these examples suggest that this richness is caused by order kernels. This motivates us to consider $\Theta$-reducibility for some sublattice of kernels $\Theta \subset \text{PCon}(\mathcal{R}(x_1, ..., x_n))$ (resp. $\Theta \subset \text{PCon}(\langle \mathcal{R} \rangle)$). There are various families of kernels that one can utilize to define the notions of reducibility, dimensionality, etc. We consider the sublattice generated by HP-kernels. Our choice is made due to its connection to the (local) dimension of the linear spaces (in logarithmic scale) defined by the skeleton corresponding to a kernel. Namely, HP-kernels, and more generally HS-kernels, define affine subspaces in $\mathcal{R}^m$ (see [13.2]).
Definition 13.6.5. Let $\mathbb{K}$ be a semifield as defined in the beginning of this section. Define $S(\mathbb{K})$ to be the set elements of $\mathbb{K}$ generated by the collection

$$\{|f| : f \in \mathbb{K} ; f \text{ is an HP-fraction} \}$$

with respect to the operations $\land$ and $\lor$ (equivalently $\lor$). Since $\mathbb{K}$ is a semifield it is closed with respect to $\land$ and $\lor$ and $|\cdot|$, thus $S(\mathbb{K}) \subset \mathbb{K}$. By definition, all HS-fractions in $\mathbb{K}$ are elements of $S(\mathbb{K})$. Define

$$\Gamma(\mathbb{K}) = S(\mathbb{K})/\sim_\mathbb{K}.$$  

Namely, for $f, g \in \Gamma(\mathbb{K})$, $f \sim_\mathbb{K} g$ if and only if $\langle f \rangle = \langle g \rangle$ in $\text{PCon}(\mathbb{K})$. Note that $1 \in \Gamma(\mathbb{K})$.

Let $\Omega(\mathbb{K}) \subset \text{PCon}(\mathbb{K})$ be the lattice of kernels generated by the collection of all HP-kernels of $\mathbb{K}$, i.e., every element $\langle f \rangle \in \Omega(\mathbb{K})$ is obtained by a finite intersections and products of HP-kernels. By definition, every HS-kernel in $\text{PCon}(\mathbb{K})$ is an element of $\Omega(\mathbb{K})$ as it is a product of finite set of HP-kernels. Note that $1 \in \Omega(\mathbb{K})$.

Remark 13.6.6. Let $\Gamma = \Gamma(\mathbb{K})$ and $\Omega = \Omega(\mathbb{K})$. There is a correspondence $\rho$ between $(\Gamma, \land, \lor)$ and $(\Omega, \cap, \cdot)$ defined by

$$\rho(f) = \langle f \rangle.$$ (13.9)

This is a lattice homomorphism in the sense that for any $f, g \in \Gamma$

$$\rho(f \land g) = \rho(f) \cap \rho(g), \text{ and } \rho(f \lor g) = \rho(f) \cdot \rho(g).$$
Note that \( f = |f| \) and \( g = |g| \) by the definition of \( \Gamma \). If \( f, g \in \Gamma \) such that \( \rho(f) = \rho(g) \), then by definition \( \langle f \rangle = \langle g \rangle \) and thus \( f \sim_K g \). So, \( \rho \) is injective. Finally, \( \rho \) is onto by the definition of HP-kernels.

As \( \Omega(K) \) is a sublattice of kernels of \( \text{PCon}(\mathbb{K}) \), we have the notion of \( \Omega(K) \)-irreducible kernel as developed in Subsection 9.2 concerning ‘reducibility of principal kernels and skeletons’ (taking \( \Theta = \Omega(K) \)). Namely \( \langle f \rangle \in \Omega(K) \) is reducible if there exist some \( \langle g \rangle, \langle h \rangle \in \Omega(K) \) such that \( \langle f \rangle \supseteq \langle g \rangle \cap \langle h \rangle \) while \( \langle f \rangle \not\supseteq \langle g \rangle \) and \( \langle f \rangle \not\supseteq \langle h \rangle \).

**Lemma 13.6.7.** By the construction of \( \Omega(K) \), the condition stated above is equivalent to the condition \( \langle f \rangle = \langle g \rangle \cap \langle h \rangle \) while \( \langle f \rangle \not= \langle g \rangle \) and \( \langle f \rangle \not= \langle h \rangle \).

**Proof.** Assume \( \langle f \rangle \) admits the stated condition. If \( \langle f \rangle \supseteq \langle g \rangle \cap \langle h \rangle \), then \( \langle f \rangle = \langle f \rangle \cdot \langle f \rangle = \langle g \rangle \cdot \langle f \rangle \cap \langle h \rangle \cdot \langle f \rangle \). Thus \( \langle f \rangle = \langle g \rangle \cdot \langle f \rangle \) or \( \langle f \rangle = \langle h \rangle \cdot \langle f \rangle \) and so \( \langle f \rangle \supseteq \langle g \rangle \) or \( \langle f \rangle \supseteq \langle h \rangle \). The converse is obvious. \( \square \)

**Note 13.6.8.** For the rest of this section, we refer to \( \Omega(K) \)-irreducibility as irreducibility.

**Definition 13.6.9.** Define the irreducible hyperspace spectrum of \( K \), \( H\text{Spec}(K) \), to be the family of irreducible kernels in \( \Omega(K) \).

**Remark 13.6.10.** \( H\text{Spec}(K) \) is the family of HS-kernels in \( \Omega(K) \) which is exactly the family of HS-kernels of \( K \).

**Definition 13.6.11.** A chain \( P_0 \subset P_1 \subset \cdots \subset P_t \) in \( H\text{Spec}(K) \) means an ascending chain of HS-kernels of \( K \), and is said to have length \( t \). An HS-kernel \( P \) has height \( t \) (denoted \( hgt(P) = t \)) if there is a chain of length \( t \) in \( H\text{Spec}(K) \) terminating at \( P \), but no chain of length \( t + 1 \) terminates at \( P \).

**Remark 13.6.12.** Let \( L \) be a kernel in \( \text{PCon}(\mathbb{K}) \). Consider the quotient homomorphism \( \phi_L : \mathbb{K} \to \mathbb{K}/L \). As the image of a principal kernel is the principal kernel generated by the image of any of its generators, we have that for an HP-kernel \( \langle f \rangle \) we have \( \phi_L(\langle f \rangle) = \langle \phi_L(f) \rangle \). Choosing \( f \) to be in canonical HP-fraction, since \( \phi_L \) is an \( R \)-homomorphism, we have that \( \langle \phi_L(f) \rangle \) is a nontrivial HP-kernel in \( \mathbb{K}/L \) if and only if \( \phi_L(f) \not\in \mathcal{R} \). Thus the set of HP-kernels of \( \mathbb{K} \) mapped to HP-kernels of \( \mathbb{K}/L \) is

\[
\left\{ \langle g \rangle : \langle g \rangle \cdot \langle \mathcal{R} \rangle \supseteq \phi_L^{-1}(\langle \mathcal{R} \rangle) = L \cdot \langle \mathcal{R} \rangle \right\}.
\]  

(13.10)

Since \( \phi_L \) is an \( \mathcal{R} \)-homomorphism and onto, we have that one of the preimages of a canonical HP-fraction \( g \in \mathbb{K}/L \) (represented by \( g \)) is \( g \in \mathbb{K} \) itself. As \( \phi_L \) is an \( \mathcal{R} \)-homomorphism it respects \( \lor, \land \) and \( | \cdot | \) thus \( \phi_L(\Gamma(\mathbb{K})) = \Gamma(\mathbb{K}/L) \).
In fact by Corollary 2.7.11 we have a correspondence identifying \( \text{HSpec}(\mathbb{K}/L) \) with the subset of \( \text{HSpec}(\mathbb{K}) \) which consists of all HS-kernels \( P \) of \( \mathbb{K} \) such that \( P \cdot \langle P \rangle \supseteq L \cdot \langle P \rangle \).

Moreover, by the same considerations and since \( \wedge \) is preserved under a homomorphism we have that the above correspondence extends to a correspondence identifying \( \Omega(\mathbb{K}/L) \) with the subset \( (13.10) \) of \( \Omega(\mathbb{K}) \). Under this correspondence, the maximal (HS) kernels of \( \mathbb{K}/L \) correspond to maximal (HS) kernels of \( \mathbb{K} \) and reducible kernels of \( \mathbb{K}/L \) correspond to reducible kernels of \( \mathbb{K} \). Indeed, the latter assertion is obvious since \( \wedge \) is preserved under a homomorphism. For the former assertion, by the second isomorphism theorem \( (\mathbb{K}/L)/(P/L) \cong \mathbb{K}/P \) so simplicity of the quotients is preserved. Thus so is maximality of \( P \) and \( P/L \).

**Definition 13.6.13.** The Hyperdimension of \( \mathbb{K} \), written \( \text{Hdim}\mathbb{K} \) (if it exists), is the maximal height of the HS-kernels in \( \mathbb{K} \).

**Definition 13.6.14.** Let \( A \subset \text{HS}(\mathbb{K}) \) be any set of HS-fractions and let \( f \in \text{HS}(\mathbb{K}) \). Then \( f \) is said to be \( \mathcal{R} \)-convexly dependent on \( A \) if

\[
f \in \langle \{ g : g \in A \} \rangle \cdot \langle \mathcal{R} \rangle,\]

(13.11)

otherwise \( f \) is said to be \( \mathcal{R} \)-convexly-independent of \( A \). A subset \( A \subset \text{HS}(\mathbb{K}) \) is said to be \( \mathcal{R} \)-convexly independent if for every \( a \in A \), \( a \) is \( \mathcal{R} \)-convexly independent of \( A \setminus \{ a \} \) over \( \mathcal{R} \). Note that by assuming \( g \in \mathbb{K} \setminus \{ 1 \} \) for some \( g \in A \), the condition in \( (13.11) \) simplifies to \( f \in \langle \{ g : g \in A \} \rangle \). Indeed, under this last assumption we have that \( \langle \mathcal{R} \rangle \subseteq \langle \{ g : g \in A \} \rangle \) and so \( \langle \{ g : g \in A \} \rangle \cdot \langle \mathcal{R} \rangle = \langle \{ g : g \in A \} \rangle \).

**Note 13.6.15.** If \( \{ a_1, ..., a_n \} \) is \( \mathcal{R} \)-convexly dependent (independent), then we also say that \( a_1, ..., a_n \) are \( \mathcal{R} \)-convexly dependent (independent).

**Remark 13.6.16.** By the definition, we have that an HS-fraction \( f \) is \( \mathcal{R} \)-convexly dependent on \( \{ g_1, ..., g_t \} \subset \text{HS}(\mathbb{K}) \) if and only if

\[
\langle |f| \rangle = \langle f \rangle \subseteq \langle g_1, ..., g_t \rangle \cdot \langle \mathcal{R} \rangle = \left( \sum_{i=1}^{t} |g_i| \right) \cdot \langle \mathcal{R} \rangle = \left( \sum_{i=1}^{t} |g_i| + |\alpha| \right)
\]

where \( \alpha \) is any element of \( \mathbb{K} \setminus \{ 1 \} \).

**Example 13.6.17.** For any \( \alpha \in \mathcal{R} \) and any \( f \in \mathcal{R}(x_1, ..., x_n) \),

\[
|\alpha f| \leq |f|^2 + |\alpha|^2 = (|f| + |\alpha|)^2.
\]

Thus \( \alpha f \in \langle (|f| + |\alpha|)^2 \rangle = \langle |f| + |\alpha| \rangle = \langle f \rangle \cdot \langle \mathcal{R} \rangle \). In particular, if \( f \) is an HS-fraction then \( \alpha f \) is \( \mathcal{R} \)-convexly dependent on \( f \).
As a consequence of Proposition \[\text{13.1.18}\] we have that for two HP-fractions \(f, g\), if \(g \in \langle f \rangle\) then \(\langle g \rangle = \langle f \rangle\). In other words, either \(\langle g \rangle = \langle f \rangle\) or \(\langle g \rangle \not\subseteq \langle f \rangle\) and \(\langle f \rangle \not\subseteq \langle g \rangle\). This motivates us to restrict our attention to the convex dependence relation on the set of HP-fractions. This will be justified later by showing that for each \(\mathcal{R}\)-convexly independent subset of HS-fractions of size \(t\) in \(\mathbb{K}\), there exists an \(\mathcal{R}\)-convexly independent subset of HP-fractions of size \(\geq t\) in \(\mathbb{K}\).

**Proposition 13.6.18.** Let \(A \subseteq HP(\mathbb{K})\) and let \(f \in HP(\mathbb{K})\). Then

1. If \(f \in A\) then \(f\) is \(\mathbb{H}\)-convexly-dependent on \(A\).

2. If \(f\) is \(\mathcal{R}\)-convexly-dependent on \(A\) and \(A_1\) is a set such that \(a\) is \(\mathcal{R}\)-convexly-dependent on \(A_1\) for each \(a \in A\), then \(f\) is \(\mathcal{R}\)-convexly dependent on \(S_1\).

3. If \(f\) is \(\mathcal{R}\)-convexly-dependent on \(A\), then \(f\) is \(\mathcal{R}\)-convexly-dependent on \(A_0\) for some finite subset \(A_0\) of \(A\).

**Proof.** (1) Since \(f \in A\) we have that \(f \in \langle A \rangle \subseteq \langle A \rangle \cdot \langle \mathcal{R} \rangle\).

2. If \(a\) is convexly-dependent on \(S_1\) for each \(a \in A\), then \(A \subseteq \langle A_1 \rangle \cdot \langle \mathcal{R} \rangle\). Thus \(\langle A \rangle \subseteq \langle A_1 \rangle \cdot \langle \mathcal{R} \rangle\). If \(f\) is \(\mathcal{R}\)-convexly dependent on \(A\) then \(f \in \langle A \rangle \cdot \langle \mathcal{R} \rangle \subseteq \langle A_1 \rangle \cdot \langle \mathcal{R} \rangle\), so, \(f\) is \(\mathcal{R}\)-convexly dependent on \(A_1\).

3. \(a \in \langle A \rangle \cdot \langle \mathcal{R} \rangle\), so by Proposition \[\text{2.3.3}\] there exist some \(s_1, \ldots, s_k \in \mathbb{K}\) and \(g_1, \ldots, g_k \in G(A \cup \mathcal{R}) \subseteq \langle A \rangle \cdot \langle \mathcal{R} \rangle\), where \(G(A \cup \mathcal{R})\) is the group generated by \(A \cup \mathcal{R}\), such that \(\sum_{i=1}^{k} s_i = 1\) and \(a = \sum_{i=1}^{k} s_i g_i^d(i)\) with \(d(i) \in \mathbb{Z}\). Thus \(a \in \langle g_1, \ldots, g_k \rangle\) and \(A_0 = \{g_1, \ldots, g_k\}\). \(\square\)

**Remark 13.6.19.** Note that \(\langle g_1, \ldots, g_k \rangle \cdot \langle \mathcal{R} \rangle\) is the smallest kernel containing the semifield \(SF(g_1, \ldots, g_k)\) generated (as a semifield) by \(g_1, \ldots, g_k\) over \(\mathcal{R}\).

Indeed, \(\mathcal{R} \subseteq SF(g_1, \ldots, g_k)\) and \(g_1, \ldots, g_k \in SF(g_1, \ldots, g_k)\). Thus any kernel containing \(SF(g_1, \ldots, g_k)\) must contain \(\langle g_1, \ldots, g_k \rangle \cdot \langle \mathcal{R} \rangle\), being the smallest kernel containing \(\mathcal{R}\) and \(\{g_1, \ldots, g_k\}\).

Since Remark \[\text{13.6.19}\] is similar to algebraic dependence, we are led to try to show that convex dependence is an abstract dependence.

**Remark 13.6.20.** For \(f \in HP(\mathbb{K})\) the following hold:

1. \(\langle \mathcal{R} \rangle \not\subseteq \langle f \rangle\).

2. If \(\mathbb{K}\) is not bounded then \(\langle f \rangle \not\subseteq \langle \mathcal{R} \rangle\).

Indeed, by definition an HP-fraction is not bounded from below. Thus \(\langle f \rangle \cap \mathcal{R} = \{1\}\) or equivalently \(\langle \mathcal{R} \rangle \not\subseteq \langle f \rangle\). For the second assertion, by Remark \[\text{13.1.13}\] an HP-kernel is not bounded when \(\mathbb{K}\) is not bounded, so we have that \(\langle f \rangle \not\subseteq \langle \mathcal{R} \rangle\).
A direct consequence of Remark 13.6.20 is

**Remark 13.6.21.** If $\mathbb{K}$ is not bounded then any proper HS-kernel (i.e., not $\langle 1 \rangle$) is $\mathcal{R}$-convexly independent.

**Proof.** By Remark 13.6.20 the assertion is true for HP-kernels, and thus for HS-kernels, since every HS-kernel contains some HP-kernel. □

**Proposition 13.6.22** (Exchange axiom). Let $S = \{b_1, ..., b_t\} \subset \text{HP}(\mathbb{K})$ and let $f$ and $b$ be elements of $\text{HP}(\mathbb{K})$. Then if $f$ is convexly-dependent on $S \cup \{b\}$ and $f$ is $\mathcal{R}$-convexly independent of $S$, then $b$ is $\mathcal{R}$-convexly-dependent on $S \cup \{f\}$.

**Proof.** We may assume that $\alpha \in S$ for some $\alpha \in \mathcal{R}$. Since $f$ is $\mathcal{R}$-convexly independent of $S$, by definition $f \notin \langle S \rangle$ this implies that $\langle S \rangle \subset \langle S \rangle \cdot \langle f \rangle$ (for otherwise $\langle f \rangle \subseteq \langle S \rangle$ yielding that $f$ is $\mathcal{R}$-convexly dependent on $S$). Since $f$ is $\mathcal{R}$-convexly-dependent on $S \cup \{b\}$, we have that $f \in \langle S \cup \{b\} \rangle = \langle S \rangle \cdot \langle b \rangle$. In particular, we get that $b \notin \langle S \rangle \cdot \langle f \rangle$ for otherwise $f$ would be dependent on $S$. Consider the quotient map $\phi : \mathbb{K} \to \mathbb{K}/\langle S \rangle$. Since $\phi$ is a semifield epimorphism and $f, g \notin \langle S \rangle \cdot \langle f \rangle = \phi^{-1}(\langle f \rangle)$, we have that $\phi(f)$ and $\phi(b)$ are not in $\mathcal{R}$ thus are HP-fractions in the semifield $\text{Im}(\phi) = \mathbb{K}/\langle S \rangle$. By the above, $\phi(f) \neq 1$ and $\phi(f) \in \phi(\langle b \rangle) = \langle \phi(b) \rangle$. Thus, by Corollary 13.1.19 we have that $\langle \phi(f) \rangle = \langle \phi(b) \rangle$. So $\langle S \rangle \cdot \langle f \rangle = \phi^{-1}(\langle \phi(f) \rangle) = \phi^{-1}(\langle \phi(b) \rangle) = \langle S \rangle \cdot \langle b \rangle$, consequently $b \in \langle S \rangle \cdot \langle b \rangle = \langle S \rangle \cdot \langle f \rangle = \langle S \cup \{f\} \rangle$, i.e., $b$ is $\mathcal{R}$-convexly-dependent on $S \cup \{f\}$. □

**Definition 13.6.23.** Let $A \subseteq \text{HP}(\mathbb{K})$. The convex-span of $A$ over $\mathcal{R}$ is the set

$$\text{ConSpan}_\mathcal{R}(A) = \{a \in \text{HP}(\mathbb{K}) : a \text{ is } \mathcal{R}\text{-convexly dependent on } A\}. \quad (13.12)$$

Let $\mathbb{K} \subseteq \mathbb{K}$ be a subsemifield such that $\mathcal{R} \subseteq \mathbb{K}$. Then a set $A \subseteq \text{HP}(\mathbb{K})$ is said to convexly span $\mathbb{K}$ over $\mathcal{R}$ if

$$\text{HP}(\mathbb{K}) = \text{ConSpan}_\mathcal{R}(A).$$

In view of Propositions 13.6.18 and 13.6.22 convex-dependence on $\text{HP}(\mathbb{K})$ is a (strong) dependence relation. Then by [7, Chapter 6], we have that:

**Corollary 13.6.24.** Let $V \subseteq \text{HP}(\mathbb{K})$. Then $V$ contains a basis $B_V \subset V$, which is a maximal convexly independent subset of unique cardinality such that

$$\text{ConSpan}(B_V) = \text{ConSpan}(V).$$
\textbf{Definition 13.6.25.} Let $V \subset \text{HP}(\mathbb{K})$ be a set of HP-fractions. We define the \textit{convexity degree} of $V$, \textit{condeg}(V), to be $|B|$ where $B$ is a basis for $V$.

\textbf{Remark 13.6.26.} If $S \subset \text{HP}(\mathbb{K})$, then for any $f, g \in \mathbb{K}$ such that $f, g \in \text{ConSpan}(S)$

\[ |f| + |g| \in \text{ConSpan}(S) \quad \text{and} \quad |f| \wedge |g| \in \text{ConSpan}(S). \]

\textbf{Proof.} First we prove that $|f| + |g| \in \text{ConSpan}(S)$. Since $\langle f \rangle \subseteq \langle S \rangle$ and $\langle g \rangle \subseteq \langle S \rangle$, we have $\langle f, g \rangle = \langle |f| + |g| \rangle = \langle f \rangle \cdot \langle g \rangle \subseteq \langle S \rangle \cdot \langle S \rangle$. For $|f| \wedge |g| \in \text{ConSpan}(S)$, $\langle |f| \wedge |g| \rangle = \langle f \rangle \cap \langle g \rangle \subseteq \langle S \rangle \subseteq \langle S \rangle \cdot \langle S \rangle$. \hfill \Box

\textbf{Remark 13.6.27.} Let $S = \{f_1, ..., f_m\}$ a finite set of HP-fractions. Then

\[ \text{ConSpan}(S) = \langle f_1, ..., f_m \rangle \cdot \langle \mathcal{R} \rangle. \]

\textbf{Proof.} A straightforward consequence of \textbf{Definition 13.6.23}. \hfill \Box

\textbf{Remark 13.6.28.} If $K$ is an HS-kernel, then $K$ is generated by an HS-fraction $f \in \mathbb{K}$ of the form $f = \sum_{i=1}^{t} |f_i|$ where $f_1, ..., f_t$ are HP-fractions. So,

\[ \text{ConSpan}(K) = \langle \mathcal{R} \rangle \cdot K = \langle \mathcal{R} \rangle \cdot \langle f \rangle = \langle \mathcal{R} \rangle \cdot \langle \sum_{i=1}^{t} |f_i| \rangle = \langle \mathcal{R} \rangle \cdot \prod_{i=1}^{t} \langle f_i \rangle \]

and so, $\{f_1, ..., f_t\}$ convexly spans $\langle \mathcal{R} \rangle \cdot K$.

\textbf{Remark 13.6.29.} Let $f$ be an HS-fraction. Then $f \sim_K \sum_{i=1}^{t} |f_i|$ where $f_i$ are HP-fractions. Since $\langle f \rangle = \prod_{i=1}^{t} \langle f_i \rangle = \langle \{f_1, ..., f_t\} \rangle$, we have that $f$ is $\mathcal{R}$-convexly dependent on $\{f_1, ..., f_t\}$.

\textbf{Lemma 13.6.30.} If $\{b_1, ..., b_m\}$ is a set of HS-fractions, such that $b_i \sim_K \sum_{j=1}^{t_i} |f_{i,j}|$ where $f_{i,j}$ are HP-fractions, then $b_i$ is $\mathcal{R}$-convexly dependent on $\{b_2, ..., b_m\}$ if and only if all its summands $f_{1,r}$ for $1 \leq r \leq t_1$ are $\mathcal{R}$-convexly dependent on $\{b_2, ..., b_m\}$.

\textbf{Proof.} If $b_1$ is $\mathcal{R}$-convexly dependent on $\{b_2, ..., b_m\}$, then

\[ \prod_{j=1}^{t_1} \langle f_{i,j} \rangle = \left\langle \sum_{j=1}^{t_1} |f_{i,j}| \right\rangle = \langle b_1 \rangle \subseteq \langle \{b_1, ..., b_m\} \rangle. \]

Since $\langle f_{1,r} \rangle \subseteq \prod_{j=1}^{t_1} \langle f_{1,j} \rangle$ for every $1 \leq r \leq t_1$, we have that $f_{1,r}$ is $\mathcal{R}$-convexly dependent on $\{b_1, ..., b_m\}$ and by Remark \textbf{13.6.29} $f_{1,r}$ is $\mathcal{R}$-convexly dependent on
Lemma 13.6.31. Let $V = \{f_1, \ldots, f_m\}$ be a $\mathcal{R}$-convexly independent set of HS-fractions, such that $f_i \sim_k \sum_{j=1}^{t_i} |f_{i,j}|$ where $f_{i,j}$ are HP-fractions. Then there exist a $\mathcal{R}$-convexly independent subset $S_0 \subseteq S = \{f_{i,j} : 2 \leq i \leq m; 1 \leq j \leq t_i\}$ such that $|S_0| \geq |V|$ and $\text{ConSpan}(S_0) = \text{ConSpan}(V)$.

Proof. By Remark 13.6.29, $f_i$ is dependent on $\{f_{i,j} : 1 \leq j \leq t_i\} \subset S$ for each $1 \leq i \leq m$, thus $\text{ConSpan}(S) = \text{ConSpan}(V)$. By Corollary 13.6.24, $S$ contains a maximal $\mathcal{R}$-convexly independent subset $S_0$ such that $\text{ConSpan}(S_0) = \text{ConSpan}(S)$. By Lemma 13.6.30 for each $i = 1, \ldots, m$, there exists some HP-fraction $g_i = f_{i,j}$, such that $g_i$ is $\mathcal{R}$-convexly independent of $V \setminus \{f_i\}$, for otherwise $f_i$ would be $\mathcal{R}$-convexly dependent on $V \setminus \{f_i\}$, thus $\{g_1, \ldots, g_m\} \subset S$ is $\mathcal{R}$-convexly independent, so $|S_0| \geq m = |V|$.

Now that our restriction to HP-fractions is justified, we move forward with our construction.

Remark 13.6.32. Let $K \in \mathbb{K}$ be an HS-kernel. Consider the following set

$$\text{HP}(K) = \{f \in K : f \text{ is an HP-fraction} \}.$$ 

$K$ is an HS-fraction, so by definition there are some HP-fractions $f_1, \ldots, f_t$ such that $L = \sum_{i=1}^{t} |f_i|$. By Remark 13.6.28, $\text{ConSpan}(K)$ is convexly-spanned by $f_1, \ldots, f_t$. Now, Since $\text{ConSpan}(K) = \text{ConSpan}(f_1, \ldots, f_t)$ and $\{f_1, \ldots, f_t\} \subset \text{HP}(K) \subset \text{HP}(K)$, by 13.6.24, $\{f_1, \ldots, f_t\}$ contains a basis $B = \{b_1, \ldots, b_s\} \subset \{f_1, \ldots, f_t\}$ of $\mathcal{R}$-convexly independent elements such that $\text{ConSpan}(B) = \text{ConSpan}(f_1, \ldots, f_t) = \text{ConSpan}(K)$. Note that $s \in \mathbb{N}$ is finite and is uniquely determined by $K$.

Definition 13.6.33. Let $K \in \mathbb{K}$ be an HS-kernel. In the notation of Remark 13.6.32, we define the convexity degree of $\text{ConSpan}(K)$, $\text{condeg}(K)$ to be $s$, the number of elements in a basis $B$.

Remark 13.6.34. By Example 13.6.40, we have that $\text{condeg}(\mathcal{R}(x_1, \ldots, x_n)) = n$.

Notation 13.6.35. For any semifield homomorphism $\phi : \mathbb{H} \to \mathbb{S}$, we denote the image of $h \in \mathbb{H}$ under $\phi$ by $\bar{h} = \phi(h)$.
Proposition 13.6.36. Let $O$ be an order-kernel of $\mathcal{R}(x_1, ..., x_n)$. Then if a set of HP-fractions $\{h_1, ..., h_t\}$ is $\mathcal{R}$-convexly dependent in $\mathcal{R}(x_1, ..., x_n)$, then $\{h_1, ..., h_t\}$ is $\mathcal{R}$-convexly dependent in the quotient semifield $\mathcal{R}(x_1, ..., x_n)/O$.

Proof. Denote by $\phi_O : \mathcal{R}(x_1, ..., x_n) \to \mathcal{R}(x_1, ..., x_n)/O$ the quotient $\mathcal{R}$-homomorphism. First note that since $\phi_O$ is an $\mathcal{R}$ homomorphism, we have that $\phi_O(\langle \mathcal{R} \rangle) = \langle \phi_O(\mathcal{R}) \rangle = (\mathcal{R}, \mathcal{R}(x_1, ..., x_n)/O)$. Now, if $h_1, ..., h_t$ are $\mathcal{R}$-convexly dependent then there exist some $j$, say without loss of generality $j = 1$, such that $h_1 \in \langle h_2, ..., h_t \rangle \cdot (\mathcal{R})$. By assumption,

$$\tilde{h}_1 = \phi_O(h_1) \in \phi_O((h_2, ..., h_t, \alpha)) = \langle \phi_O(h_2), ..., \phi_O(h_t), \phi_O(\alpha) \rangle = \langle \tilde{h}_1, ..., \tilde{h}_t, \alpha \rangle = \langle \tilde{h}_1, ..., \tilde{h}_t \rangle \cdot (\mathcal{R})$$

(the equalities hold by Remark 2.3.19 and $\phi_O$ being an $\mathcal{R}$-homomorphism). Thus $\tilde{h}_1$ is $\mathcal{R}$-convexly dependent on $\{\tilde{h}_2, ..., \tilde{h}_t\}$. □

Conversely, we have:

Lemma 13.6.37. Let $O$ be an order-kernel of $\mathcal{R}(x_1, ..., x_n)$. Let $\{h_1, ..., h_t\}$ be a set of HP-fractions. If $\tilde{h}_1, ..., \tilde{h}_t$ are $\mathcal{R}$-convexly dependent in the quotient semifield $\mathcal{R}(x_1, ..., x_n)/O$ and $\sum_{i=1}^t |\tilde{h}_1| \cap \mathcal{R} = \{1\}$, then $h_1, ..., h_t$ are $\mathcal{R}$-convexly dependent in $\mathcal{R}(x_1, ..., x_n)$.

Proof. Note that $\sum_{i=1}^t |\tilde{h}_1| \cap \mathcal{R} = \{1\}$ if and only if $\bigcap_{i=1}^t \text{Skel}(h_1) \cap \text{Skel}(O) \neq \emptyset$. Translating the variables by a point $a \in \bigcap_{i=1}^t \text{Skel}(h_1) \cap \text{Skel}(O)$, we may assume that the constant coefficient of each HP-fraction $h_i$ is 1. Assume $\tilde{h}_1, ..., \tilde{h}_t$ are $\mathcal{R}$-convexly dependent. W.l.o.g. we may assume that $h_1$ is $\mathcal{R}$-convexly dependent on $\{\tilde{h}_2, ..., \tilde{h}_{t+1}\}$. Taking $h_{t+1} \in \mathcal{R}$, we may write $\tilde{h}_1 \in \langle \tilde{h}_2, ..., \tilde{h}_t, \tilde{h}_{t+1} \rangle$. Considering the pre-images of the quotient map, we have that

$$\langle h_1 \rangle \cdot O \subseteq \langle h_2, ..., h_t, h_{t+1} \rangle \cdot O.$$ 

Thus $|h_2| + \cdots + |h_{t+1}| + |1 + g| \geq |h_1| + |1 + g|$ with $1 + g$ a generator of $O$. So, by Remark 2.3.16 there exists some $k \in \mathbb{N}$ such that

$$|h_1| + |1 + g| \leq (|h_2| + \cdots + |h_{t+1}| + |1 + g|)^k = |h_2|^k + \cdots + |h_{t+1}|^k + |1 + g|^k. \ (13.13)$$

As $1 + g \geq 1$ we have that $|1 + g| = 1 + g$, and the right hand side of equation (13.13) equals

$$|h_2|^k + \cdots + |h_{t+1}|^k + (1 + g)^k = |h_2|^k + \cdots + |h_{t+1}|^k + 1 + g^k = |h_2|^k + \cdots + |h_{t+1}|^k + g^k.$$ 

180
The last equality is due to the fact that $\sum |h_i|^k \geq 1$ so that 1 is absorbed. The same arguments applied to the left hand side of equation (13.13) yields that

$$|h_1| + g \leq |h_2|^k + \cdots + |h_{t+1}|^k + g^k.$$  \hspace{1cm} (13.14)

Assume on the contrary that $h_1$ is $R$-convexly independent of $\{h_2, \ldots, h_t\}$. Then

$$\langle h_1 \rangle \not\subseteq \langle h_2, \ldots, h_{t+1} \rangle = \left( \sum_{i=2}^{t+1} |h_i| \right).$$

Thus for any $m \in \mathbb{N}$ there exists some $x_m \in R^m$ such that

$$|h_1(x_m)| > \left| \sum_{i=2}^{t+1} |h_i(x_m)| \right|^m = \sum_{i=2}^{t+1} |h_i(x_m)|^m.$$  

Thus by equation (13.14) and the last observation we get that

$$\sum_{i=2}^{t} |h_i(x_m)|^m + g(x_m) < |h_1(x_m)| + g(x_m) \leq \sum_{i=2}^{t} |h_i(x_m)|^k + g(x_m)^k,$$

i.e., there exists some fixed $k \in \mathbb{N}$ such that for any $m \in \mathbb{N}$,

$$\sum_{i=2}^{t} |h_i(x_m)|^m < \sum_{i=2}^{t} |h_i(x_m)|^k + g(x_m)^k.$$  \hspace{1cm} (13.15)

For $m > k$, since $|\gamma|^k \leq |\gamma|^m$ for any $\gamma \in R$, we get that $\sum_{i=2}^{t} |h_i(x_m)|^m \geq \sum_{i=2}^{t} |h_i(x_m)|^k$. Write $g^k(x) = g(1)g'(x)$. Since $g^k$ is an HP-kernel, $g(1)$ is the constant coefficient of $g$ and $g'$ is a Laurent monomial with coefficient 1. Now, by the way $x_m$ were chosen we have that $\sum_{i=2}^{t} |h_i(x_m)|^m > 1$ and $\sum_{i=2}^{t} |h_i(x_m)|^m < g(x_m)^k$, and thus for sufficiently large $m_0$ we have that $g(1) < g'(x_{m_0})$. Since $g'$ is a Laurent monomial with coefficient 1, we have that $g'(x_{m_0}^{-1}) = g'(x_{m_0})^{-1}$ so $g^k(x_{m_0}^{-1}) = g(1)g'(x_{m_0}^{-1}) = g(1)g'(x_{m_0})^{-1} < 1$. Thus by (13.15) we must have $\sum_{i=2}^{t} |h_i(x_{m_0}^{-1})|^m < \sum_{i=2}^{t} |h_i(x_{m_0}^{-1})|^k$. A contradiction. \hfill $\blacksquare$

**Proposition 13.6.38.** Let $R$ be a region-kernel of $\mathcal{R}(x_1, \ldots, x_n)$. Let $\{h_1, \ldots, h_t\}$ be a set of HP-fractions such that $(R \cdot \langle h_1, \ldots, h_t \rangle) \cap \mathcal{R} = \{1\}$. Then $h_1 \cdot R, \ldots, h_t \cdot R$ are $\mathcal{R}$-convexly dependent in the quotient semifield $\mathcal{R}(x_1, \ldots, x_n) / R$ if and only if $h_1, \ldots, h_t$ are $\mathcal{R}$-convexly dependent in $\mathcal{R}(x_1, \ldots, x_n)$.

**Proof.** The ‘if’ part of the assertion follows Proposition 13.6.36. As $R = \prod_{i=1}^{m} O_i$ for some order-kernels $\{O_i\}_{i=1}^{m}$, the ‘only if’ part follows from Lemma 13.6.37 by applying it repeatedly to each of the $O_i$’s comprising $R$. \hfill $\blacksquare$
**Proposition 13.6.39.** Let $L \in HS(\mathbb{K})$ and let $R \in \text{PCon}(\mathbb{K})$ be a region kernel. Let $\phi_R : \mathbb{K} \to \mathbb{K}/R$ be the quotient map. Then

$$\text{condeg}(L) = \text{condeg}(L \cdot R).$$

**Proof.** Since $L$ is a subsemifield of $\mathbb{K}$, by Theorem 2.2.49 we have that $\phi_R^{-1}(\phi_R(L)) = R \cdot L$ and $\text{condeg}(L) \geq \text{condeg}(\phi_R(L))$ by Proposition 13.6.38 while by Lemma 13.6.37 we have that $\text{condeg}(\phi_R(L)) \geq \text{condeg}(\phi_R^{-1}(\phi_R(L)))$. Thus $\text{condeg}(L) \geq \text{condeg}(R \cdot L)$. $L \subseteq R \cdot L$ implies that $\text{condeg}(L) \leq \text{condeg}(R \cdot L)$. Thus equality holds. \qed

**Example 13.6.40.** As we have previously shown, the maximal kernels in $\text{PCon}(\mathcal{R}(x_1, ..., x_n))$ are HS-fractions of the form $L_{(\alpha_1, ..., \alpha_n)} = \langle \alpha_1 x_1, ..., \alpha_n x_n \rangle$ for any $\alpha_1, ..., \alpha_n \in \mathcal{R}$. We have previously shown that $\mathcal{R}(x_1, ..., x_n) = \mathcal{R} \cdot L_{(\alpha_1, ..., \alpha_n)} = \langle \mathcal{R} \rangle \cdot L_{(\alpha_1, ..., \alpha_n)}$. Thus $\mathcal{R}(x_1, ..., x_n) = \text{ConSpan}(\{\alpha_1 x_1, ..., \alpha_n x_n\})$, i.e., $\{\alpha_1 x_1, ..., \alpha_n x_n\}$ convexly spans $\mathcal{R}(x_1, ..., x_n)$ over $\mathcal{R}$. Now, since there is no order relation between $\alpha_i x_i$ and the elements of $\{\alpha_j x_j : j \neq i\} \cup \{\alpha : \alpha \in \mathcal{R}\}$ we have that $\alpha_k x_k \notin \bigcup_{j \neq k} \alpha_j x_j \cdot \langle \mathcal{R} \rangle$. Thus $\{\alpha_1 x_1, ..., \alpha_n x_n\}$ is $\mathcal{R}$-convexly independent constituting a basis for $\mathcal{R}(x_1, ..., x_n)$ for any chosen $\alpha_1, ..., \alpha_n \in \mathcal{R}$.

**Remark 13.6.41.** Let $R$ be a region kernel of $\mathbb{K}$ and let

$$A = \{\langle g \rangle : \langle g \rangle \cdot \langle \mathcal{R} \rangle \supseteq R \cdot \langle \mathcal{R} \rangle\}.$$  

In view Remark 13.6.42 and of Proposition 13.6.39 we have that $\text{condeg}(\mathbb{K}/R) = \text{condeg}(A)$. As $L_a \supseteq A$ for any $a \in \text{Skel}(R) \neq \emptyset$, we have that $\text{condeg}(A) = \text{condeg}(\mathbb{K})$.

**Proposition 13.6.42.** If $R$ be a region kernel and $L \in HS(\mathbb{K})$ of $\mathbb{K}$, then

$$\text{condeg}(\mathbb{K}/LR) = \text{condeg}(\mathbb{K}) - \text{condeg}(L).$$

In particular,

$$\text{condeg}(\mathcal{R}(x_1, ..., x_n)/LR) = n - \text{condeg}(L).$$

**Proof.** By the third isomorphism theorem, we have that $\mathbb{K}/LR \cong (\mathbb{K}/R)/(L \cdot R/R)$. We can always choose a basis for $\text{HP}(\mathbb{K}/R)$ containing a basis for $\text{HP}(L \cdot R/R)$. Taking $L$ instead of $\mathbb{K}$ in Remark 13.6.41 we have that $\text{condeg}(L \cdot R/R) = \text{condeg}(\phi_R(L))$. Note that $L \cdot R \cap \mathcal{R} = \{1\}$, thus By Proposition 13.6.38 $\text{condeg}(\phi_R(L)) = \text{condeg}(L)$.  

182
So $\text{condeg}(\mathbb{K}/LR) = \text{condeg}(A) - \text{condeg}(L) = \text{condeg}(\mathbb{K}) - \text{condeg}(L)$.

Taking $\mathbb{K} = \mathcal{R}(x_1, ..., x_n)$ in the above setting, we get that

\[
\text{condeg}(\mathcal{R}(x_1, ..., x_n)/LR) = \text{condeg}(A) - \text{condeg}(L) = n - \text{condeg}(L).
\]

\[\square\]

**Proposition 13.6.43.** Let $L$ be an HS-kernel in $\mathbb{K}$. Let \{h_1, ..., h_t\} be a set of HP-fractions in $\text{HSpec}(\mathbb{K})$ such that $\text{ConSpan}(h_1, ..., h_t) = \text{ConSpan}(L)$ and let $L_i = \langle h_i \rangle$. Consider the following descending chain of HS-kernels where $\text{Skel}(K) \neq \emptyset$

\[
L = \prod_{i=1}^{u} L_i \supseteq \prod_{i=1}^{u-1} L_i \supseteq \cdots \supseteq \prod_{i=1}^{1} L_i \supseteq \prod_{i=1}^{1} L_i \supseteq \prod_{i=1}^{1} L_i \supseteq \langle 1 \rangle.
\]

(13.16)

Then the chain (13.16) is a strictly descending chain of HS-kernels if and only if $h_1, ..., h_u$ are $\mathcal{R}$-convexly independent.

**Proof.** Since $\prod_{i=1}^{t} L_i \subseteq K$, $\text{Skel}(\prod_{i=1}^{t} L_i) \supseteq \text{Skel}(L) \neq \emptyset$ for every $0 \leq t \leq u$, which in turn implies that $(\prod_{i=1}^{t} L_i) \cap \mathcal{R} = \{1\}$ for every $0 \leq t \leq u$ (otherwise $\text{Skel}(\prod_{i=1}^{t} L_i) = \emptyset$). If \{h_1, ..., h_u\} is $\mathcal{R}$-convexly independent then $h_u$ is $\mathcal{R}$-convexly independent of the set \{h_1, ..., h_{u-1}\} thus $L_u = \langle h_u \rangle \not\subseteq \prod_{i=1}^{u-1} L_i \cdot \langle \mathcal{R} \rangle$. Thus the inclusions of the chain (13.16) are strict, i.e., it is strictly descending. On the other hand if $h_u$ is $\mathcal{R}$-convexly dependent on \{h_1, ..., h_{u-1}\} then $L_u = \langle h_u \rangle \subseteq \prod_{i=1}^{u-1} L_i \cdot \langle \mathcal{R} \rangle$. Assume $L_u = \langle h_u \rangle \not\subseteq \prod_{i=1}^{u-1} L_i$, then $\langle \mathcal{R} \rangle \subseteq \prod_{i=1}^{u} L_i$ implying that $\prod_{i=1}^{u} L_i$ is not an HS-kernel. Thus $L_u = \langle h_u \rangle \subseteq \prod_{i=1}^{u-1} L_i$ and the chain is not strictly descending. $\square$

**Proposition 13.6.44.** If $L \in \text{HSpec}(\mathbb{K})$, then $\text{hgt}(L) = \text{condeg}(L)$. Moreover, every factor of a descending chain of maximal length is an HP-kernel.

**Proof.** By Proposition 13.6.43 we have that the maximal length of a chain of HS-kernels descending from an HS-kernel $L$ equals the number of elements in a basis of $\text{ConSpan}(L)$; thus we have that the chain is of unique length $\text{condeg}(K)$, i.e., $\text{hgt}(L) = \text{condeg}(L)$. Moreover, by Theorem 2.2.51(2) we have that

\[
\prod_{i=1}^{j} L_i / \prod_{i=1}^{j-1} L_i \cong L_j / \left( L_j \cap \prod_{i=1}^{j-1} L_i \right).
\]

Since $L_j \cdot (L_j \cap \prod_{i=1}^{j-1} L_i) = L_j \cap \prod_{i=1}^{j-1} L_i \subset \prod_{i=1}^{j} L_i$ and $(\prod_{i=1}^{j} L_i) \cap \mathcal{R} = \{1\}$, we have that $(L_j \cdot (L_j \cap (\prod_{i=1}^{j-1} L_i))) \cap \mathcal{R} = \{1\}$. So the homomorphic image of the HP-kernel $L_j$ under the quotient map $\mathcal{R}(x_1, ..., x_n) \to \mathcal{R}(x_1, ..., x_n)/(L_j \cap (\prod_{i=1}^{j-1} L_i))$ is an HP-kernel. Thus every factor of the chain is an HP-kernel. $\square$

183
Theorem 13.6.45. If $\mathbb{K}$ is an affine semifield, then

$$Hdim(\mathbb{K}) = condeg(\mathbb{K}).$$

Proof. A consequence of Definition 13.6.13 and Proposition 13.6.44. □

Definition 13.6.46. A set of region kernels $\{R_1, ..., R_t\} \subset PCon(\mathcal{R}(x_1, ..., x_n))$ is said to be a cover of regions if $R_1 \cap R_2 \cap \cdots \cap R_t = \{1\}$. In other words, $\{R_1, ..., R_t\}$ is a cover of regions if $\text{Skel}(R_1) \cup \text{Skel}(R_2) \cup \cdots \cup \text{Skel}(R_t) = \mathcal{R}^n$.

Remark 13.6.47. If $\{R_1, ..., R_t\} \subset PCon(\mathcal{R}(x_1, ..., x_n))$ is a cover of regions, then we can express $\mathcal{R}(x_1, ..., x_n)$ as a subdirect product

$$\mathcal{R}(x_1, ..., x_n) = \mathcal{R}(x_1, ..., x_n)/(R_1 \cap R_2 \cap \cdots \cap R_t) \rightarrow \prod_{i=1}^t \mathcal{R}(x_1, ..., x_n)/R_i.$$

Let $K \in \text{Con}(\mathcal{R}(x_1, ..., x_n))$. Then $R_1 \cap R_2 \cap \cdots \cap R_t \cap K = \bigcap_{i=1}^t (R_i \cap K) = \{1\}$. Since $K$ itself is an idempotent semifield, we have that

$$K = K/\bigcap_{i=1}^t (R_i \cap K) \cong \prod_{i=1}^t K/(R_i \cap K) \cong \prod_{i=1}^t R_i K/R_i.$$

As we have seen for every principal regular kernel $\langle f \rangle \in (\mathcal{R}(x_1, ..., x_n))$, there exists an explicitly formulated cover of regions $\{R_{1,1}, ..., R_{1,s}, R_{2,1}, ..., R_{2,t}\}$ such that

$$\langle f \rangle = \bigcap_{i=1}^s K_i \cap \bigcap_{j=1}^t N_j$$

where $K_i = L_i \cdot R_{1,i}$ for $i = 1, ..., s$ and appropriate HS-kernels $L_i$ and $N_j = B_j \cdot R_{2,j}$ for $j = 1, ..., t$ and appropriate bounded from below kernels $B_j$. If $\langle f \rangle \in PCon(\mathcal{R})$, as we have shown, then $B_j = \langle \mathcal{R} \rangle$ for every $j = 1, ..., t$. Note that over the different regions in $\mathcal{R}^n$, corresponding to the region kernels $R_{i,j}$, $f$ is locally represented by distinct elements of $HSpec(\mathcal{R}(x_1, ..., x_n))$ (HS-fractions). In fact the regions themselves are defined such that the local HS-representation of $f$ will stay invariant over each. Thus the $R_{i,j}$’s, defining the partition of the space, are uniquely determined as the minimal set of regions over each of which $\langle f \rangle$ comes from a unique HS-kernel.

For each $j = 1, ..., t$, we have that

$$condeg(N_j) = Hdim(N_j) = 0$$
and
\[
\text{condeg}(\mathcal{R}(x_1, \ldots, x_n)/N_j) = H \text{dim}(\mathcal{R}(x_1, \ldots, x_n)/N_j) = n.
\]
For each \(i = 1, \ldots, s\), we have that
\[
\text{condeg}(K_i) = \text{condeg}(L_i) = H \text{dim}(L_i) \geq 1
\]
and
\[
\text{condeg}(\mathcal{R}(x_1, \ldots, x_n)/K_i) = H \text{dim}(\mathcal{R}(x_1, \ldots, x_n)/K_i) = n - H \text{dim}(L_i) < n.
\]

**Definition 13.6.48.** Using the notations used in the discussion above, let \(\langle f \rangle\) be a regular principal kernel in \(\text{PCon}(\mathcal{R})\). Define the **Hyper-dimension** of \(\langle f \rangle\) to be
\[
H \text{dim}(\langle f \rangle) = (H \text{dim}(L_1), \ldots, H \text{dim}(L_s))
\]
and
\[
H \text{dim}(\langle f \rangle) = (H \text{dim}(\mathcal{R}(x_1, \ldots, x_n)/L_1), \ldots, H \text{dim}(\mathcal{R}(x_1, \ldots, x_n)/L_s)).
\]

**Remark 13.6.49.** In view of the discussion in Subsection 13.2, each term \(\mathcal{R}(x_1, \ldots, x_n)/L_i\) in Definition 13.6.48 corresponds to the linear subspace of \(\mathbb{R}^n\) (in logarithmic scale) defined by the linear constraints endowed on the quotient \(\mathcal{R}(x_1, \ldots, x_n)/L_i\) by the HS-kernel \(L_i\). One can think of these terms as an algebraic description of the affine subspaces which locally comprise the skeleton \(\text{Skel}(f)\).
Bibliography

[1] M. Anderson and T. Feil, *Lattice Ordered Groups: An Introduction*, D. Reidel Publishing Company, Dordrecht, Holland, 1988.

[2] G. Birkhoff, *Lattice Theory, XXV, 3rd edition*, American Mathematical Society, 1967.

[3] H. Hutchins and H. Weinert, *Homomorphisms and kernels of semifields*, Periodica Mathematica Hungarica 21 (2) (1990), 113–152.

[4] Z. Izhakian and L. Rowen, *Supertropical algebra*, Adv. in Math 225 (2010), 2222–2286.

[5] D. Marker, *Model Theory: An Introduction*, Springer, 2002.

[6] N. Medvedev and V. Kopytov, *The Theory of Lattice Ordered Groups*, Kluwer, Dordrecht, The Netherlands, 1994.

[7] L. Rowen, *Graduate Algebra : Commutative View*, American Mathematical Society, 2006.

[8] S. Steinberg, *Lattice-ordered Rings and Modules*, Springer Science, 2010.

[9] E. Vechtomov and A. Cheraneva, *Semifields and their Properties*, (2009).

[10] H. Weinert and R. Wiegandt, *On the structure of semifields and lattice-ordered groups*, Periodica Mathematica Hungarica 32(1-2) (1996), 129–147.

[11] ———, *A new Kurosh-Amitsur radical theory for proper semifields*, Mathematica Pannonica 14(1) (2003), 3–28.