Eigenvalues of $K$-invariant Toeplitz Operators on Bounded Symmetric Domains

Harald Upmeier

Abstract. We determine the eigenvalues of certain “fundamental” $K$-invariant Toeplitz type operators on weighted Bergman spaces over bounded symmetric domains $D = G/K$, for the irreducible $K$-types indexed by all partitions of length $r = \text{rank}(D)$.

Mathematics Subject Classification. Primary 39B82; Secondary 32025.

Keywords. Toeplitz operators, Symmetric domains, Eigenvalues, Dimension formula.

1. Introduction

The well-known Toeplitz–Berezin calculus, acting on the Bergman space $H^2(D)$ of a bounded domain $D \subseteq \mathbb{C}^d$, is covariant under the biholomorphic automorphism group $G$ of $D$. Actually, Berezin [3] considered two kinds of symbolic calculus (contravariant and covariant symbols) which are related by the Berezin transform. For a bounded symmetric domain $D = G/K$ of rank $r$, where $G$ acts transitively on $D$ and $K$ is a maximal compact subgroup of $G$, one has a more general covariant Toeplitz–Berezin calculus acting on the weighted Bergman spaces $H^2_\nu(D)$ over $D$. Here $\nu$ is a scalar parameter for the (scalar) holomorphic discrete series of $G$ and its analytic continuation. Since $G$ acts irreducibly on $H^2_\nu(D)$, there are no non-trivial $G$-invariant operators in the $C^*$-algebra generated by Toeplitz operators. On the other hand, there exist interesting $K$-invariant Toeplitz type operators, which have been studied in relation to complex and harmonic analysis [2,6]. These operators are uniquely determined by a sequence of eigenvalues indexed over all partitions of length $r$. In this paper, we determine the eigenvalues of certain “fundamental” $K$-invariant Toeplitz type operators, both for the covariant and contravariant symbol. While the covariant symbol is treated as a direct generalization of [2], the contravariant symbol eigenvalue formula requires

The author was supported by an Infosys Visiting Chair Professorship at the Indian Institute of Science, Bangalore.
more effort. Here a crucial ingredient is the dimension formula for the irreducible $K$-types.

2. $K$-invariant Toeplitz Operators

In the following we use the Jordan theoretical description of bounded symmetric domains. For more details, see [1,5,10,11,16]. Each irreducible bounded symmetric domain $D$ of rank $r$ and dimension $d$ can be realized as the (spectral) open unit ball of a hermitian Jordan triple $Z \approx \mathbb{C}^d$. Let $G$ be the identity component of the biholomorphic automorphism group of $D$, and let $K$ be the stabilizer subgroup at $0 \in D$. Then $K$ is a compact linear group consisting of Jordan triple automorphisms of $Z$.

The Shilov boundary $S$ of $D$ consists of all tripotents in $Z$ of maximal rank $r$. Let $(z|w)$ denote the unique $K$-invariant inner product on $Z$ such that $(e|e) = r$ for any $e \in S$. For any maximal tripotent $e \in Z$ the so-called Peirce 2-space $Z_2(e)$ [10] is a hermitian Jordan triple of tube type. Put

$$d_e := \dim Z_2(e).$$

Let $e_1, \ldots, e_r$ be a frame of orthogonal minimal tripotents, and put $e = e_1 + \ldots + e_r$. The joint Peirce decomposition [10] gives rise to two numerical invariants $a, b$ such that

$$\frac{d_e}{r} = 1 + \frac{a}{2}(r - 1),$$

$$d = 1 + \frac{a}{2}(r - 1) + b = \frac{d_e}{r} + b.$$  

The tube type case is characterized by $d_e = d$ or, equivalently, $b = 0$. The triple $(r, a, b)$ characterizes $D$ up to isomorphism. As an important special case, the spin factor $Z$ of dimension $d \geq 3$ has the invariants $r = 2$, $a = d - 2$ and $b = 0$. Here the normalized inner product on $Z \approx \mathbb{C}^d$ is $(z|w) = 2z \cdot \overline{w}$, and the unit element $e = (1, 0, \ldots, 0)$.

Let $\mathcal{P}(Z)$ denote the algebra of all (holomorphic) polynomials on $Z$. The natural action

$$(k \cdot f)(z) := f(k^{-1}z)$$

of $K$ on functions $f$ on $Z$ (or $D$), induces a Peter–Weyl decomposition [12]

$$\mathcal{P}(Z) = \sum_m \mathcal{P}^m(Z)$$

ranging over all integer partitions $m = (m_1 \geq \ldots \geq m_r \geq 0)$ of length $r$. Here $\mathcal{P}^m(Z)$ denotes the irreducible $K$-module of all polynomials on $Z$ of type $m$ [14]. By irreducibility, any two $K$-invariant inner products are proportional on each submodule $\mathcal{P}^m(Z)$.

Let $H^2(Z)$ denote the Fock space of all entire functions on $Z$, with Fischer-Fock inner product

$$\langle \phi|\psi \rangle = \int_Z \frac{dz}{\pi^d} e^{-|z|^2} \overline{\phi(z)} \psi(z)$$
and reproducing kernel
\[ \mathcal{K}(z, w) = e^{-\langle z | w \rangle}. \]

Note that for function spaces we use inner products which are conjugate-linear in the first variable.

Consider the classical Pochhammer symbol
\[ (\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} \]
which equals \( \lambda(\lambda + 1) \cdots (\lambda + k - 1) \) if \( k \in \mathbb{N} \). For any scalar parameter \( \nu \) and \( r \)-tuple \( \mu = (\mu_1, \ldots, \mu_r) \) we define the multi-variable Pochhammer symbol
\[ (\nu)_{\mu} := \prod_{j=1}^{r} (\nu - \frac{d}{r} (j - 1))_{\mu_j}. \]  
(2.4)

The numerical invariant \( p := \frac{1}{r}(d_e + d) = 2 + a(r - 1) + b \) is called the genus of \( D \). For a scalar parameter \( \nu > p - 1 \) the weighted Bergman space \( H^2_{\nu}(D) \) consists of all holomorphic functions \( \phi \) on \( D \) which are square-integrable for the inner product
\[ (\phi | \psi)_{\nu} = c_\nu \int_D \frac{dz}{\pi d} \Delta(z, z)^{\nu-p} \phi(z) \overline{\psi(z)}. \]  
(2.5)

Here \( \Delta(z, w) \) denotes the Jordan triple determinant, which is a sesqui-polynomial uniquely determined by the property
\[ \Delta(z, z) = \prod_{i=1}^{r} (1 - t_i^2) \]
for all \( z = \sum_{i=1}^{r} t_i e_i \in Z \), where \( e_1, \ldots, e_r \) is any frame of minimal orthogonal tripotents and \( t_1, \ldots, t_r \) are the (non-negative) “singular values” of \( z \). For a Lie theoretic definition see [5, p. 262] (for tube type domains) and [4, (3.4)-(3.6)] (general case). The normalizing constant \( c_\nu \), giving rise to a probability measure, is
\[ c_\nu = \prod_{j=1}^{r} \Gamma\left(\nu - \frac{a}{2}(j - 1)\right). \]
The reproducing kernel is
\[ \mathcal{K}_{\nu}(z, w) = \Delta(z, w)^{-\nu}. \]  
(2.6)

For the continuous part of the so-called Wallach set, explicitly given by the condition
\[ \nu > \frac{a}{2}(r - 1) \]
[4,8,9], the kernel (2.6) is strictly positive on \( D \times D \), and the associated reproducing kernel Hilbert space, still denoted by \( H^2_{\nu}(D) \), contains \( \mathcal{P}(Z) \) as a dense subspace. As a special case, the parameter \( \nu = \frac{d}{r} = 1 + b + \frac{a}{2}(r - 1) \) corresponds to the Hardy space
\[ H^2_{d/r}(D) = H^2(S) \]
on the Shilov boundary $S$ of $D$.

Under the Fischer-Fock inner product (2.3) each finite-dimensional Hilbert space $\mathcal{P}^m(Z)$ has a reproducing kernel

$$E^m(z, w) = \sum_{\alpha} \psi^m_\alpha(z) \overline{\psi^m_\alpha(w)}$$

for any orthonormal basis $\psi^m_\alpha \in \mathcal{P}^m(Z)$. The Hilbert spaces $H^2(Z)$ and $H^2_\nu(D)$ (in the continuous Wallach set) are invariant under the action (2.1) of $K$. The Faraut–Korányi binomial formula [4]

$$K^\nu(z, w) = \Delta(z, w)^{-\nu} = \sum_m (\nu)_m E^m(z, w) \quad (2.7)$$

implies that the inner product (2.5) on $H^2_\nu(D)$ and the Fischer-Fock inner product (2.3) are related by

$$(p|q)_\nu = \frac{(p|q)}{(\nu)_m} \quad (2.8)$$

for each partition $m$ and all $p, q \in \mathcal{P}^m(Z)$, using the multi-variable Pochhammer symbol (2.4).

We now introduce Toeplitz operators. Let $F(z, w)$ be a sequi-holomorphic symbol function, written as a sum (or series)

$$F(z, w) = \sum_i \phi_i(z) \overline{\psi_i(w)}$$

for holomorphic functions $\phi_i, \psi_i$. We are mainly concerned with sesqui-polynomials, where $\phi_i, \psi_i \in \mathcal{P}(Z)$ and the sum is finite. We write $F_w(z) := F(z, w)$ for fixed $w$.

Let $M_\phi$ (resp., $M^\nu_\phi$) denote the multiplication operator by a polynomial $\phi$ acting on $\mathcal{P}(Z)$ or $H^2_\nu(D)$, respectively. The $\nu$-th Toeplitz operator $T^\nu_F$ on $H^2_\nu(D)$ with symbol function $F$ (where $F$ denotes the restriction of $F$ to the diagonal) has the form

$$T^\nu_F = \sum_i M^{\nu*}_{\psi_i} M^\nu_{\phi_i}. \quad (2.9)$$

We define similarly

$$T_F = \sum_i M^{*}_{\psi_i} M_{\phi_i}$$

acting on $\mathcal{P}(Z)$, or as a densely defined unbounded operator on the Fock space $H^2(Z)$. Here $M^*_\psi$ is the constant coefficient differential operator associated with a polynomial $\psi$ via the normalized inner product. Thus

$$M^*_\psi e^{(z|w)} = \overline{\psi(w)} e^{(z|w)}$$

for all $z, w \in Z$. For $p, q, \psi \in \mathcal{P}(Z)$ we have

$$(p|q) = M^*_\psi q(0)$$

and

$$(p|M^*_\psi q) = (M^*_\psi p|q) = (\psi \cdot p|q).$$

Note that $M^*_\psi$ depends in a conjugate-linear way on $\psi$. 

The Toeplitz calculus is sometimes called the “anti-Wick” calculus. On the other hand, the “Wick” functional calculus (normal ordering, where annihilation operators are moved to the right) yields the operator

\[ F_{\nu} := \sum_i M_{\phi_i}^\nu M_{\psi_i}^{\nu*} \]
on $H_\nu^2(D)$, and, similarly,

\[ F := \sum_i M_{\phi_i} M_{\psi_i}^* , \]
acting on $\mathcal{P}(Z)$ or as a densely defined unbounded operator on $H^2(Z)$.

If $F$ is $K$-invariant in the sense that

\[ F(kz, kw) = F(z, w) \]
for all $k \in K$, then the operators $T_F$, $T_F^\nu$ and $F_T$, $F_T^\nu$ commute with the $K$-action (2.1). Since the decomposition (2.2) is multiplicity free, it follows that $K$-invariant operators form a commutative algebra, and every such operator $T$ is a block-diagonal operator uniquely determined by its sequence of eigenvalues $\lambda_{\nu}^{(m)}$ defined by

\[ \lambda_{\nu}^{(m)} = \langle p| F p \rangle_{\nu} \]

for all $p \in \mathcal{P}^m(Z)$. For $T_F^\nu$, with $F$ $K$-invariant, we obtain

\[ T_F^\nu p =: T_{F^\nu}^\nu (m) p \]
for all $p \in \mathcal{P}^m(Z)$, with eigenvalues given by

\[ T_{F^\nu}^\nu (m) (p|p)_{\nu} = (p|T_{F^\nu}^\nu (m) p)_{\nu} = (p|T_{F^\nu}^\nu p)_{\nu} = (p|F p)_{\nu} \]

\[ = c_{\nu} \int_D \frac{dz}{\pi^d} \frac{p(z)}{\overline{p(z)}} \Delta(z, z)^{\nu-p} F(z, z) p(z). \]

The Jordan triple determinant $\Delta(z, w) = \Delta_w(z)$ is $K$-invariant, and for its powers we obtain

**Lemma 1.**

\[ T_{\Delta^\beta}^\nu (m) = \frac{c_{\nu}}{c_{\nu+\beta}} \frac{(\nu)m}{(\nu+\beta)m} \]

**Proof.** Let $p \in \mathcal{P}^m(Z)$. Then (2.8) implies

\[ T_{\Delta^\beta}^\nu (m) (p|p)_{\nu} = c_{\nu} \int_D \frac{dz}{\pi^d} \frac{p(z)}{\overline{p(z)}} \Delta(z, z)^{\nu-p} \Delta(z, z)^{\beta} p(z) \]

\[ = \frac{c_{\nu}}{c_{\nu+\beta}} c_{\nu+\beta} \int_D \frac{dz}{\pi^d} \frac{p(z)}{\overline{p(z)}} \Delta(z, z)^{\nu+\beta-p} p(z) \]

\[ = \frac{c_{\nu}}{c_{\nu+\beta}} (p|p)_{\nu+\beta} = \frac{c_{\nu}}{c_{\nu+\beta}} \frac{(\nu)m}{(\nu+\beta)m} (p|p)_{\nu} \]

\[ \square \]
Proposition 2. For a $K$-invariant sesqui-holomorphic function $F$, let $F^{\nu}$ denote the $\nu$-th Berezin transform (defined via the Berezin symbol of operators on $H^2_{\nu}(D)$). Then

$$ T_F^\nu(m) = F^{\nu}_{T_F}(m). $$

Proof. Using Einstein summation convention, the identity

$$ F(z, w) K^\nu(z, w) = \sum_m (\nu)_m F^{\nu}(m) E^m(z, w) \quad (2.9) $$

follows from the computation

$$ F(z, w) K^\nu(z, w) = \phi_i(z) \psi_i(w) (K^\nu_{z|w})_\nu = (\phi_i(z) K^\nu_{z|w})_\nu = (M^\nu_{\phi_i} M^\nu_{\psi_i} K^\nu_{z|w})_\nu = \sum_m (\nu)_m (K^\nu_{z|w})_m F^{\nu}(m) E^m(z, w) $$

Similarly, the identity

$$ F^{\nu}(z, w) K^\nu(z, w) = \sum_m (\nu)_m T_F^{\nu}(m) E^m(z, w) \quad (2.10) $$

follows from the computation

$$ F^{\nu}(z, w) K^\nu(z, w) = (K^\nu_{z|w})_\nu = \sum_m (\nu)_m (K^\nu_{z|w})_m F^{\nu}(m) E^m(z, w) $$

Comparing coefficients in (2.9) and (2.10), the assertion follows. □

In general, the Berezin transform of a $K$-invariant function is difficult to compute. For powers of $\Delta$ we obtain with Lemma 1

Corollary 3.

$$ (\Delta^\beta)^{\nu}(z, w) K^\nu(z, w) = \frac{c_{\nu}}{c_{\nu+\beta}} \sum_m (\nu)_m^2 \frac{(\nu)_m^{\nu+\beta}}{(\nu+\beta)_m} E^m(z, w) $$

3. The First Eigenvalue Formula

The Jordan triple determinant has a decomposition

$$ \Delta(z, w) = \sum_{\ell=0}^r (-1)^\ell \Delta^{(\ell)}(z, w) $$
into sesqui-polynomials $\Delta^{(\ell)}$ which are homogeneous of bi-degree $(\ell, \ell)$. For $\ell = 1$ we obtain the normalized $K$-invariant inner product

$$\Delta^{(1)}(z, w) = (z|w).$$

If $Z$ is of tube type with unit element $e$, then

$$\Delta^{(r)}(z, w) = N(z) \overline{N(w)},$$

where $N$ is the Jordan algebra determinant normalized by $N(e) = 1$. The first eigenvalue formula gives the eigenvalues of the $K$-invariant operators $\Delta^{(\ell)}_{T^\nu}$ for $0 \leq \ell \leq r$. This generalizes the approach in [2] for $\ell = 1$.

Consider the fundamental partitions

$$(\ell) := (1, \ldots, 1, 0, \ldots, 0)$$

with $\ell$ ones. Let $\psi^{(\ell)}_\alpha$ be an orthonormal basis of $\mathcal{P}^{(\ell)}(Z)$ and consider the Fischer-Fock reproducing kernel

$$E^{(\ell)}(z, w) = \sum_\alpha \psi^{(\ell)}_\alpha(z) \overline{\psi^{(\ell)}_\alpha(w)}$$

of $\mathcal{P}^{(\ell)}(Z)$. Then $\mathcal{P}^{(0)}(Z) = \mathbb{C}$ consists of constant functions and $\Delta^{(0)} = E^{(0)} = 1$. In the first interesting case $\ell = 1$ we obtain the dual space $\mathcal{P}^{(1)}(Z) = Z^*$ of all linear forms on $Z$, and

$$\Delta^{(1)}(z, w) = E^{(1)}(z, w) = (z|w) = \sum_i (z|u_i)(u_i|w)$$

for any orthonormal basis $u_i \in Z$.

**Lemma 4.**

$$\Delta^{(\ell)}(z, w) = E^{(\ell)}(z, w) \prod_{j=1}^{\ell} (1 + \frac{a}{2}(j - 1)).$$

**Proof.** The Faraut–Korányi formula (2.7) applied to the parameter $-1$ yields

$$\Delta^{(\ell)}(z, w) = (-1)^\ell (-1)^{(\ell)} E^{(\ell)}(z, w)$$

with (positive) constant given by

$$(-1)^\ell (-1)^{(\ell)} = (-1)^\ell \prod_{j=1}^{\ell} (-1 - \frac{a}{2}(j - 1))$$

$$= (-1)^\ell \prod_{j=1}^{\ell} (-1 - \frac{a}{2}(j - 1)) = \prod_{j=1}^{\ell} (1 + \frac{a}{2}(j - 1)).$$

Lemma 4 implies that

$$\Delta^{(\ell)}_{T^\nu} = \prod_{j=1}^{\ell} (1 + \frac{a}{2}(j - 1)) E^{(\ell)}_{T^\nu} = \prod_{j=1}^{\ell} (1 + \frac{a}{2}(j - 1)) \sum_{\alpha} M^{\nu}_{\psi^{(\ell)}_\alpha} M^{\nu*}_{\psi^{(\ell)}_\alpha}.$$
and
\[
\Delta_T^{(\ell)} = \prod_{j=1}^{\ell} (1 + \frac{a}{2} (j-1)) E_T^{(\ell)} = \prod_{j=1}^{\ell} (1 + \frac{a}{2} (j-1)) \sum_{\alpha} M_{\psi_\alpha} M_{\psi_\alpha}^*.
\]
In particular, for \(\ell = 1\),
\[
\Delta_T^{(1)} = E_T^{(1)} = \sum_{\nu} M_{(\nu|u)} M_{(\nu|u)}^*.
\]

For any partition \(m\) and \(1 \leq k \leq r\) we put
\[
m'_k := m_k - \frac{a}{2} (k-1). \tag{3.4}
\]
This notation (unrelated to the ‘dual’ partition sometimes also denoted by \(m'\)) will be used throughout the paper. For any \(\ell\)-element subset \(L \subset \{1, \ldots, r\}\), with characteristic function
\[
\chi_L^i := \begin{cases} 1 & i \in L \\ 0 & i \notin L \end{cases},
\]
we define
\[
\alpha_m^L := \prod_{k \in L \neq h} \left(1 + \frac{a/2}{m'_k - m'_h}\right) \tag{3.5}
\]
whenever \(m + \chi_L\) is also a partition, and put \(\alpha_m^L := 0\) otherwise. Similarly, we define
\[
\beta_m^L := \prod_{k \in L \neq h} \left(1 - \frac{a/2}{m'_k - m'_h}\right) \tag{3.6}
\]
whenever \(m - \chi_L\) is also a partition, and put \(\beta_m^L := 0\) otherwise.

**Proposition 5.** Assume that \(Z\) is of tube type, with unit element \(e\). Then the spherical polynomial \(\Phi^m\) of type \(m\) satisfies
\[
\Delta_e^{(\ell)} \Phi^m = \sum_{|L| = \ell} \Phi^{m + \chi_L} \alpha_m^L \tag{3.7}
\]
for \(0 \leq \ell \leq r\), with summation over all \(\ell\)-element subsets \(L \subset \{1, \ldots, r\}\) such that \(m + \chi_L\) is a partition.

**Proof.** Choose a frame \(e_1, \ldots, e_r\) with \(e = e_1 + \ldots + e_r\). For \(t = (t_1, \ldots, t_r) \in \mathbb{C}^r\) we put \(t \cdot e := t_1 e_1 + \ldots + t_r e_r\). Then
\[
\Delta_e (t \cdot e) = \Delta(t \cdot e, e) = \prod_{i=1}^{r} (1 - t_i).
\]
It follows that
\[
\Delta_e^{(\ell)} (t \cdot e) = \sigma_\ell (t) \tag{3.8}
\]
is the \( \ell \)-th elementary symmetric polynomial. Now consider the ‘Selberg-Jack’ symmetric functions \( s_m^{a/2} \), for any partition \( m \) and parameter \( k = a/2 \), as defined in [7] (cf. also [13]). Putting

\[
f_r^{a/2}(m) := \prod_{i<j}(m'_i - m'_j)_{a/2}
\]
as in [7, (1.2)], it is shown in [7, identity (S) on p. 69] that

\[
s_m^{a/2}(1^r) = \frac{f_r^{a/2}(m)}{f_r^{a/2}((0))}.
\]

Therefore

\[
\Phi_m(t \cdot e) = \frac{1}{s_m^{a/2}(1^r)} s_m^{a/2}(t) = \frac{f_r^{a/2}(0))}{f_r^{a/2}(m)} s_m^{a/2}(t), \tag{3.9}
\]

By [7, identity (U) on p. 44 and (5.2)] we have a Pieri formula

\[
\sigma_\ell(t) s_m^{a/2}(t) = \sum_{|L| = \ell} U^{a/2}(m + \chi_L/m) s_m^{a/2}(t)
\]

where, according to [7, (3.15)], the coefficients are given by

\[
U^{a/2}(m + \chi_L/m) = \frac{f_r^{a/2}(m)}{f_r^{a/2}(m + \chi_L/m)} \prod_{i<j} \left( 1 + \frac{a}{2} \frac{\chi_i^L - \chi_j^L}{m'_i - m'_j} \right).
\]

In view of (3.9) this implies

\[
\Delta^{(\ell)}(t \cdot e) \Phi_m(t \cdot e) = \sum_{|L| = \ell} \Phi_{m+\chi_L}(t \cdot e) \prod_{i<j} \left( 1 + \frac{a}{2} \frac{\chi_i^L - \chi_j^L}{m'_i - m'_j} \right).
\]

For any \( i \neq j \) we have \( \chi_i^L - \chi_j^L = 0 \) whenever both \( i, j \) belong to \( L \) or belong to its complement. On the other hand

\[
\chi_i^L - \chi_j^L = \begin{cases} 
1 & i \in L, \ j \notin L \ 
-1 & i \notin L, \ j \in L.
\end{cases}
\]

It follows that

\[
\prod_{i<j} \left( 1 + \frac{a}{2} \frac{\chi_i^L - \chi_j^L}{m'_i - m'_j} \right) = \prod_{L \ni i < j \notin L} \left( 1 + \frac{a/2}{m'_i - m'_j} \right) \prod_{L \ni i < j \in L} \left( 1 - \frac{a/2}{m'_i - m'_j} \right) = \prod_{k \in L \notin h} \left( 1 + \frac{a/2}{m'_k - m'_h} \right) = \alpha_m^L. \tag{3.10}
\]

Thus (3.7) holds for all \( t \cdot e \in Z \). Since the spherical polynomials are uniquely determined by their values on the “diagonal” \( t \cdot e \), the assertion follows. \( \square \)

For \( \ell = 1 \) we use singletons \( L = \{k\} \) and obtain

\[
(z \cdot e) \Phi^m(z) = \sum_{k=1}^r \Phi_{m+\chi^{(k)}}^m(z) \alpha_m^{\{k\}} = \sum_{k=1}^r \Phi_{m+\chi^{(k)}}^m(z) \prod_{h \neq k} \frac{m'_k - m'_h + a/2}{m'_k - m'_h}
\]
in accordance with [2, Lemma 4.2].
Example 6. For the spin factor of rank \( r = 2 \) and \( m \in \mathbb{N} \), \( m \geq 1 \), we obtain

\[
(z|e) \Phi^{(m,0)}(z) = \Phi^{(m+1,0)}(z) \frac{m + a}{m + \frac{a}{2}} + \Phi^{(m,1)}(z) \frac{m}{m + \frac{a}{2}}.
\]

Remark 7. Evaluating (3.7) at \( e \) yields the non-obvious identity

\[
\binom{r}{\ell} \prod_{1 \leq i < j \leq r} (m_i - m_j + (j-i)\frac{a}{2}) = \sum_{|L| = \ell} \prod_{1 \leq i < j \leq r} (m_i + \frac{a}{2} \chi_i^L - m_j - \frac{a}{2} \chi_j^L + (j-i)\frac{a}{2}).
\]

With (3.4) this yields

\[
\binom{r}{\ell} = \sum_{|L| = \ell} \prod_{1 \leq i < j \leq r} \frac{m_i' + \frac{a}{2} \chi_i^L - m_j' - \frac{a}{2} \chi_j^L}{m_i' - m_j'}
\]

which, by (3.10), implies (3.11).

For any parameter \( \nu \) define

\[
(\nu)_m^L := \prod_{k \in L} (\nu + m_k').
\]

(3.12)

Lemma 8.

\[
\frac{(\nu)_{m+\chi^L}}{(\nu)_m} = (\nu)_m^L.
\]

In particular, for a singleton \( L = \{k\} \)

\[
\frac{(\nu)_{m+\chi^{(k)}}}{(\nu)_m} = (\nu)_m^{\{k\}} = \nu + m_k'
\]

Proof.

\[
\frac{(\nu)_{m+\chi^L}}{(\nu)_m} = \prod_{k=1}^{r} \frac{(\nu - \frac{a}{2}(k-1))_{m+k+\chi_k^L}}{(\nu - \frac{a}{2}(k-1))_{m_k}} = \prod_{k \in L} \frac{(\nu - \frac{a}{2}(k-1))_{m_k+1}}{(\nu - \frac{a}{2}(k-1))_{m_k}} = \prod_{k \in L} (\nu - \frac{a}{2}(k-1) + m_k) = \prod_{k \in L} (\nu + m_k')
\]

The dimension \( d_m := \dim \mathcal{P}^m(Z) \) has been computed in [15, Lemma 2.6] (for tube domains) and [15, Lemma 2.7] (general case). It satisfies

\[
C \frac{(d_e/r)m}{(d/r)m} d_m = \prod_{i < j} (m_i' - m_j') (m_i' - m_j' + 1 - \frac{a}{2} a-1)
\]
\[
\begin{align*}
= \prod_{i<j} (m'_i - m'_j) \frac{\Gamma(m'_i - m'_j + \frac{a}{2})}{\Gamma(m'_i - m'_j + 1 - \frac{a}{2})} \\
= \prod_{i<j} \frac{m'_i - m'_j}{m'_i - m'_j - \frac{a}{2}} \frac{\Gamma(m'_i - m'_j + \frac{a}{2})}{\Gamma(m'_i - m'_j - \frac{a}{2})}.
\end{align*}
\]

(3.13)

Here the constant
\[
C = \prod_{i<j} (0'_i - 0'_j) (0'_i - 0'_j + 1 - \frac{a}{2})_{a-1}
= \prod_{i<j} \left( \frac{a}{2}(j - i) \right) (1 + \frac{a}{2}(j - i - 1))_{a-1}
\]

(3.14)
is determined by the condition \(d_{(0)} = 1\), corresponding to \(\mathcal{P}^{(0)}(Z) = C\).

**Lemma 9.** For the partitions \((\ell)\) the dimension is given by

\[
d_{(\ell)} = \binom{r}{\ell} \frac{(d/r)_{(\ell)}}{\prod_{j=1}^{\ell} (1 + \frac{a}{2}(j - 1))} = \binom{r}{\ell} \prod_{j=1}^{\ell} \frac{1 + b + \frac{a}{2}(r - j)}{1 + \frac{a}{2}(j - 1)}
\]

**Proof.** This follows, with (3.13) and (3.14), from the computation

\[
\prod_{i<j} \left( \frac{a}{2}(j - i) \right) \frac{(\ell)'_i - (\ell)'_j}{(\ell)'_i - (\ell)'_j + 1 - \frac{a}{2})_{a-1}
= \prod_{i<j} (\ell)_i - (\ell)_j + \frac{a}{2}(j - i) \frac{(\ell)'_i - (\ell)'_j + 1 + \frac{a}{2}(j - i - 1))_{a-1}}{(1 + \frac{a}{2}(j - i - 1))_{a-1}}
= \prod_{i \leq j < \ell} \frac{1 + \frac{a}{2}(j - i)}{\frac{a}{2}(j - i) \frac{(2 + \frac{a}{2}(j - i - 1))_{a-1}}{(1 + \frac{a}{2}(j - i - 1))_{a-1}}}
= \prod_{i \leq j < \ell} \frac{1 + \frac{a}{2}(j - i)}{\frac{a}{2}(j - i) \frac{\frac{a}{2}(j - i + 1)}{1 + \frac{a}{2}(j - i - 1)}}
= \prod_{i=1}^{\ell} \frac{1 + \frac{a}{2}(r - i)}{1 + \frac{a}{2}(\ell - i)} \frac{\frac{a}{2}(r + 1 - i)}{\frac{a}{2}(\ell + 1 - i)}
= \frac{(d_{(\ell)}/r)_{(\ell)}}{\prod_{k=1}^{\ell} (1 + \frac{a}{2}(k - 1))} \binom{r}{\ell}
\]

using

\[
\prod_{i=1}^{\ell} \frac{\frac{a}{2}(r + 1 - i)}{\frac{a}{2}(\ell + 1 - i)} = \prod_{i=1}^{\ell} \frac{r + 1 - i}{\ell + 1 - i} = \frac{r(r - 1) \cdots (r - \ell)}{\ell!} = \binom{r}{\ell}
\]
and 
\[ \prod_{i=1}^{\ell} (1 + \frac{a}{2}(r - i)) = \prod_{i=1}^{\ell} (1 + \frac{a}{2}(r - 1) - \frac{a}{2}(i - 1)) = (d_e/r)(\ell). \]

For \( \ell = 1 \) we obtain 
\[ d_{(1,0,\ldots,0)} = d_{(1)} = \binom{r}{1} (d/r)(1) = r \frac{d}{r} = d \]
for \( P^{(1)}(Z) = Z^* \). Here the above computation (say, for tube type domains) simplifies to 
\[ d_{(1,0,\ldots,0)} = \prod_{j=2}^{r} \frac{1 + \frac{a}{2}(j - 1)}{1 + \frac{a}{2}(j - 2)} = \prod_{j=2}^{r} \frac{1 + \frac{a}{2}(j - 1)}{1} \cdot \frac{\frac{a}{2}j}{1} = \frac{1 + \frac{a}{2}(r - 1)}{1} \cdot \frac{\frac{a}{2}r}{1} = (1 + \frac{a}{2}(r - 1)) \cdot r = d. \]

**Example 10.** For the spin factor \( Z \) and \( m \in \mathbb{N} \), \( P^{(m,0)}(Z) \) is the space of all \( m \)-homogeneous harmonic polynomials in \( d \) variables. Since \( a = d - 2 \) and \( b = 0 \) in this case, we obtain the well-known dimension formula 
\[ d_{(m,0)} = \frac{m + a}{2} \cdot \frac{\Gamma(m + a)}{\Gamma(m + 1)} \cdot \frac{\Gamma(1)}{\Gamma(a)} = \frac{2m + a}{a} \cdot \frac{(m + a - 1)!}{m!(a - 1)!} = \frac{(2m + d - 2)(m + d - 3)!}{m!(d - 2)!}. \]

**Proposition 11.** Suppose \( m \) and \( m + \chi^L \) are partitions. Then 
\[ \frac{d_m}{d_{m + \chi^L}} = \frac{(d_e/r)^L_m}{(d_e/r)^L_{m + \chi^L}} \cdot \frac{\beta^L_{m + \chi^L}}{\alpha^L_m}. \]

**Proof.** For any \( i \neq j \) we have \( (m + \chi^L)_i - (m + \chi^L)_j = m_i - m_j \) whenever both \( i, j \) belong to \( L \) or belong to its complement. On the other hand 
\[ (m + \chi^L)_i - (m + \chi^L)_j = \begin{cases} m_i + 1 - m_j & i \in L, \ j \notin L \\ m_i - m_j - 1 & i \notin L, \ j \in L \end{cases}. \]

By (3.13) we have 
\[ \frac{(d/r)^L_m}{(d_e/r)^L_m} \cdot \frac{d_m}{d_{m + \chi^L}} = \frac{(d/r)^L_m}{(d_e/r)^L_m} \cdot \frac{(d/r)^L_{m + \chi^L}}{d_{m + \chi^L}} = \frac{(d/r)^L_m}{(d_e/r)^L_m} \cdot \frac{(d/r)^L_{m + \chi^L}}{d_{m + \chi^L}} = \prod_{L \ni i < j \notin L} \frac{m_i - m_j}{m_i - m_i - 1} \cdot \frac{(m_i - m_j + 1 - \frac{a}{2})_{a-1}}{(m_i - m_j - \frac{a}{2})_{a-1}} \cdot \prod_{L \ni i < j \in L} \frac{m_i - m_j}{m_i - m_i - 1} \cdot \frac{(m_i - m_j + 1 - \frac{a}{2})_{a-1}}{(m_i - m_j - \frac{a}{2})_{a-1}}. \]
\[ E_w \Delta_e(\ell) = \sum_{|L|=\ell} \frac{(d/e)^L_m}{(d/r)^m_e} \mathcal{E}^{m+\chi^L} E_e^{m+\chi^L}. \quad (3.15) \]

**Proposition 12.** Let \( 0 < \ell \leq r \) and \( w \in Z \). Then

\[ \Delta_w^{(\ell)} E_w^m = \sum_{|L|=\ell} (d/e)^L_m \beta^{L\chi^L}_{m+\chi^L} E_w^{m+\chi^L}. \]

**Proof.** We first show that (3.15) holds for any maximal tripotent \( w = e \in Z \). Assume first that \( Z \) is of tube type. Since

\[ \Delta_e(\ell) = \left( \begin{array}{c} r \\ \ell \end{array} \right) \Phi(\ell) \]

by (3.8) and

\[ E_e^m = \frac{d_m}{(d/r)^m_e} \Phi^m \]

as a consequence of Schur orthogonality [5], it follows that

\[ \Delta_e(\ell) E_e^m = \frac{d_m}{(d/r)^m_e} \Delta_e(\ell) \Phi^m = \frac{d_m}{(d/r)^m_e} \sum_{|L|=\ell} \alpha^L_m \Phi^{m+\chi^L} \]

\[ = \frac{d_m}{(d/r)^m_e} \sum_{|L|=\ell} \frac{(d/r)^m_{m+\chi^L}}{(d/r)^m_e} \alpha^L_m E_e^{m+\chi^L} \]

\[ = \sum_{|L|=\ell} \frac{(d/r)^L_m}{(d/r)^m_e} \frac{(d/e)^L_m}{(d/r)^m_e} \beta^{L\chi^L}_{m+\chi^L} \alpha^L_m E_e^{m+\chi^L} \]

\[ = \sum_{|L|=\ell} \frac{(d/e)^L_m}{(d/r)^m_e} \beta^{L\chi^L}_{m+\chi^L} E_e^{m+\chi^L}. \]
If $Z$ is not of tube type, we have $\Delta^{(\ell)}(z) = \Lambda^{(\ell)}(Pz)$ and $E^m_z(z) = E^m_z(Pz)$, where $P : Z \to Z_2(e)$ is the Peirce 2-projection. Thus (3.15) for $w = e$ holds for $Z$. Since both sides of (3.15) are $K$-invariant and the orbit $S = K \cdot e$ is a set of uniqueness for (anti)-holomorphic functions, the assertion follows for all $w \in Z$. □

**Lemma 13.** For $0 \leq \ell \leq r$ we have

$$\Delta^{(\ell)} = \left(\begin{array}{c} r \\ \ell \end{array}\right) \frac{(d/r)(\ell)}{d(\ell)} E^{(\ell)}.$$  

**Proof.** This follows from $\Delta^{(\ell)} = \left(\begin{array}{c} r \\ \ell \end{array}\right)$ and $E^{(\ell)} = \frac{d(\ell)}{(d/r)(\ell)} E^{(\ell)}$. As a double check, the same result is obtained by combining Lemmas 9 and 4. □

For any polynomial $p \in \mathcal{P}(Z)$ we denote by $p_m \in \mathcal{P}^m(Z)$ its $m$-th component under the Peter–Weyl decomposition (2.2).

**Proposition 14.** Let $p \in \mathcal{P}^m(Z)$. Then for each $\ell$-element subset $L$ such that $m - \chi^L$ is a partition, we have (on the Fock space)

$$\sum_{\alpha} \psi^{(\ell)}_{\alpha}(M^*_{\psi,\ell})p_{m-\chi^L} = p \left(\sum_{\alpha} \frac{d(\ell)}{r(\ell)(d/r)(\ell)} \right) \left(\frac{d(\ell)}{(d/r)(\ell)} \right) \beta^{L}_m$$

and

$$\sum_{\alpha} \| (M^*_{\psi,\ell})p_{m-\chi^L} \| ^2 = \| p \| ^2 \left(\sum_{\alpha} \frac{d(\ell)}{r(\ell)(d/r)(\ell)} \right) \left(\frac{d(\ell)}{(d/r)(\ell)} \right) \beta^{L}_m$$

**Proof.** We follow the argument, for $\ell = 1$, contained in [2]. For fixed $w$ we have

$$E^{(\ell)}_w = \sum_{\alpha} \psi^{(\ell)}_{\alpha} \frac{p_{\gamma}}{\psi^{(\ell)}_{\alpha}}(w)$$

by (3.3). With Lemma 13 and Proposition 12 it follows that

$$\sum_{\alpha} \psi^{(\ell)}_{\alpha}(w)(M^*_{\psi,\ell})p_{m-\chi^L}(w) = \psi^{(\ell)}_{\alpha}(w)(E^m_w - \chi^L | M^*_{\psi,\ell})p$$

$$= \sum_{\alpha} \psi^{(\ell)}_{\alpha}(w)(\psi^{(\ell)}_{\alpha} E^m_w - \chi^L | p)$$

$$= \sum_{\alpha} \left(\frac{\psi^{(\ell)}_{\alpha}(w)}{E^m_w - \chi^L | p} \right)$$

$$= (E^{(\ell)}_w E^m_w - \chi^L | p) = \left((E^{(\ell)}_w E^m_w - \chi^L)_m | p\right)$$

$$= \left(\frac{d(\ell)}{(r(\ell)(d/r)(\ell))} \left(\frac{d(\ell)}{(d/r)(\ell)} \beta^{L}_m \right) \left(E^m_w | p\right)$$

$$= \frac{d(\ell)}{(r(\ell)(d/r)(\ell))} \left(\frac{d(\ell)}{(d/r)(\ell)} \beta^{L}_m \right) p(w).$$

For the second assertion,

$$\sum_{\alpha} \| (M^*_{\psi,\ell})p_{m-\chi^L} \| ^2 = \sum_{\alpha} \| (M^*_{\psi,\ell})p_{m-\chi^L} | (M^*_{\psi,\ell})p_{m-\chi^L} \|$$
\[= \sum_{\alpha} (M_{\psi_{\alpha}}^* p)(M_{\psi_{\alpha}}^* p)_{m-\chi_L}\]
\[= \sum_{\alpha} (p|\psi_{\alpha}^{(\ell)}(M_{\psi_{\alpha}}^* p)_{m-\chi_L})\]
\[= \frac{d(\ell)}{(\ell)! (d/r)(\ell)} (d_e/r)^L_{m-\chi_L} \beta_m^L \|p\|^2\]

\[\square\]

For the weighted Bergman spaces we obtain

**Lemma 15.** Let \( p \in \mathcal{P}^m(Z) \) and \( \phi \in \mathcal{P}^{(\ell)}(Z) \). Then

\[(M_{\phi}^{\nu*} p)_{m-\chi_L} = \left( \frac{(\nu)_{m-\chi_L}}{(\nu)_m} (M_{\phi}^* p)_{m-\chi_L} = \frac{1}{(\nu)_m^{L}} (M_{\phi}^* p)_{m-\chi_L}.\]

**Proof.** Let \( q \in \mathcal{P}^{m-\chi_L}(Z) \). Then, with (2.8),

\[\left( (M_{\phi}^{\nu*} p)_{m-\chi_L} | q \right)_{\nu} = (M_{\phi}^{\nu*} p | q)_{\nu} = (p|\phi q)_{\nu} = \frac{1}{(\nu)_m} (p|\phi q) = \frac{1}{(\nu)_m^{L}} (M_{\phi}^* p | q)\]

\[= \left( \frac{(\nu)_{m-\chi_L}}{(\nu)_m} (M_{\phi}^* p | q)_{\nu} = \frac{(\nu)_{m-\chi_L}}{(\nu)_m^{L}} ((M_{\phi}^{\nu*} p)_{m-\chi_L} | q)_{\nu}.\]

Since \( q \) is arbitrary, the assertion follows. \[\square\]

The **first eigenvalue formula** is the following:

**Theorem 16.** For \( 0 \leq \ell \leq r \) the \( K \)-invariant operators \( \Delta^{(\ell)}_T \) and \( \Delta^{(\ell)}_{T^*} \) have the eigenvalues

\[\Delta^{(\ell)}_T(m) = \sum_{|L|=\ell} (d_e/r)^L_{m-\chi_L} \beta_m^L\]

and

\[\Delta^{(\ell)}_{T^*}(m) = \sum_{|L|=\ell} (d_e/r)^L_{m-\chi_L} \beta_m^L,\]

with \( \beta_m^L \) defined in (3.6).

**Proof.** If \( p \in \mathcal{P}^m(Z) \) then

\[E^{(\ell)}_{T^*} p = \sum_{\alpha} \psi_{\alpha}^{(\ell)}(M_{\psi_{\alpha}}^* p) = \sum_{|L|=\ell} \sum_{\alpha} \psi_{\alpha}^{(\ell)}(M_{\psi_{\alpha}}^* p)_{m-\chi_L}.\]

In view of Lemma 13, the first assertion for the Fock space follows from Proposition 14. For the second assertion, we use Lemma 15 and obtain, for each subset \( L \) such that \( m-\chi_L \) is a partition

\[\sum_{\alpha} M_{\psi_{\alpha}}^*(M_{\psi_{\alpha}}^{\nu*} p)_{m-\chi_L} = \sum_{\alpha} \psi_{\alpha}^{(\ell)}(M_{\psi_{\alpha}}^{\nu*} p)_{m-\chi_L}\]

\[= \frac{1}{(\nu)_m^{L}} \sum_{\alpha} \psi_{\alpha}^{(\ell)}(M_{\psi_{\alpha}}^* p)_{m-\chi_L}\]

\[= \frac{1}{(\nu)_m^{L}} \frac{d(\ell)}{(\ell)! (d/r)(\ell)} (d_e/r)^L_{m-\chi_L} \beta_m^L p.\]
Now the assertion follows from
\[ \Delta^{(\ell)}_{T^{\nu}} = \left( \frac{d}{r} \right) \left( \frac{d}{r(\ell)} \right) E^{(\ell)}_{T^{\nu}} = \left( \frac{d}{r} \right) \left( \frac{d}{r(\ell)} \right) \sum_\alpha M^\nu_{\psi_\alpha}(M^\nu_{\psi_\alpha}). \]

\[ \square \]

**Example 17.** For \( \ell = 1 \) we obtain the formula
\[
\Delta^{(1)}_{T^{\nu}}(m) = \left( \sum_i M^\nu_{(z|u_i)}(M^\nu_{z|u_i}) \right)(m) = \sum_{k=1}^r \frac{m_k' + \frac{d}{r} - 1}{m_k' + \nu - 1} \beta^{\{k\}}_m
\]

previously obtained in [2, Proposition 4.4].

**Example 18.** For \( \ell = r, L = \{1, \ldots, r\} \) we have \( \beta^L_m = 1 \) (empty product). If \( Z \) is of tube type then \( d_c = d \). Using (3.1) Theorem 16 simplifies to
\[
(M_N M^*_N)(m) = \Delta^{(r)}_{T} (m) = \frac{(d/r)m}{(d/r)m-1},
\]
\[
(M^\nu_N M^*_N)(m) = \Delta^{(r)}_{T^{\nu}} (m) = \frac{(d/r)m}{(d/r)m-1} \frac{(\nu)m-1}{(\nu)m}.
\]

### 4. The Second Eigenvalue Formula

The second eigenvalue formula gives the eigenvalues of the \( K \)-invariant operators \( T^{\nu}_{\Delta(\ell)} \) and \( T_{\Delta(\ell)} \), for \( 0 \leq \ell \leq r \). In this case the previous arguments, based on reproducing kernel identities, do not apply immediately.

**Lemma 19.** Under the \( K \)-action (2.1) on polynomials, we have
\[
M^*_p(k^{-1} \cdot q) = k^{-1} \cdot (M^*_p q).
\]

**Proof.** It suffices to check for linear polynomials \( p(z) = (z|u) \), where \( u \in Z \). We have
\[
(M^*_p(k^{-1} \cdot q))(z) = (k^{-1} \cdot q)'(z)u = (q \circ k)'(z)u
\]
\[
= q'(kz)ku = (M^*_p(kz)(z)) = k^{-1} \cdot (M^*_p(kz|u)q)(z).
\]

Since \( (k \cdot p)(z) = p(k^{-1}z) = (k^{-1}z|u) = (z|ku) \), the assertion follows. \( \square \)

**Lemma 20.** Let \( \phi_\alpha \) and \( \psi_\beta \) be orthonormal bases of \( P^{(\ell)}(Z) \). Then for any sesqui-linear form \( \langle \phi|\psi \rangle \) on \( P^{(\ell)}(Z) \) we have
\[
\sum_\alpha \langle \phi_\alpha|\phi_\alpha \rangle = \sum_\beta \langle \psi_\beta|\psi_\beta \rangle
\]

**Proof.** Using Einstein summation convention to simplify notation, we have \( \phi_\alpha = A^\beta_\alpha \psi_\beta \) for a unitary ’matrix’ \( A \). Then
\[
\langle \phi_\alpha|\phi_\alpha \rangle = \langle \Lambda^\sigma_\alpha \psi_\sigma|\Lambda^\tau_\alpha \psi_\tau \rangle = \Lambda^\sigma_\alpha \langle \psi_\sigma|\psi_\tau \rangle \Lambda^\tau_\alpha
\]
\[
= \langle \psi_\sigma|\psi_\tau \rangle (\Lambda^\tau \Lambda^\sigma)^\tau = \langle \psi_\sigma|\psi_\tau \rangle \delta^\tau_\sigma = \langle \psi_\sigma|\psi_\sigma \rangle.
\]

\( \square \)
For any \( p, q \in \mathcal{P}(Z) \) the map \( (\phi, \psi) \mapsto \langle \phi | \psi \rangle := (p|M^*_\phi q)(M^*_\psi q | p) \) is sesqui-linear. Hence Lemma 20 implies for each \( k \in K \)

\[
\sum_{\alpha} (p|M^*_{k, \psi^*_\alpha} q)(M^*_{k, \psi^*_\alpha} q | p) = \sum_{\alpha} \langle k \cdot \psi^*_\alpha | k \cdot \psi^*_\alpha \rangle
= \sum_{\alpha} \langle \psi^*_\alpha | \psi^*_\alpha \rangle
= \sum_{\alpha} (p|M^*_\psi q)(M^*_\psi q | p), \tag{4.1}
\]

since \( k \cdot \psi^*_\alpha \) is also an orthonormal basis.

\textbf{Proposition 21.} \textit{Let} \( \mathbf{m} \) \textit{and} \( \mathbf{m} + \chi^L \) \textit{be partitions. Then we have (on the Fock space)}

\[
\sum_{\alpha} \| (\psi^*_\alpha p)_{m+\chi^L} \|^2 = \| p \|^2 \frac{d(\ell)}{(\ell)(d/r)(\ell)} (d/r)_m \alpha^L_m
\]

\textit{and}

\[
\sum_{\alpha} M^*_\psi^*_\alpha (\psi^*_\alpha p)_{m+\chi^L} = p \frac{d(\ell)}{(\ell)(d/r)(\ell)} (d/r)_m \alpha^L_m
\]

\textit{for all} \( p \in \mathcal{P}^m(Z) \).

\textbf{Proof.} \textit{Let} \( q \in \mathcal{P}^{m+\chi^L}(Z) \). Schur orthogonality applied to \( \mathcal{P}^{m+\chi^L}(Z) \) yields for each \( \alpha \)

\[
\frac{\| q \|^2}{d_{m+\chi^L}} \| (\psi^*_\alpha p)_{m+\chi^L} \|^2
= \int \limits_K dk \left( (\psi^*_\alpha p)_{m+\chi^L} | k^{-1} \cdot q \right) \left( k^{-1} \cdot q | (\psi^*_\alpha p)_{m+\chi^L} \right)
= \int \limits_K dk \left( \psi^*_\alpha p | k^{-1} \cdot q \right) \left( k^{-1} \cdot q | \psi^*_\alpha p \right)
= \int \limits_K dk \left( p | M^*_{\psi^*_\alpha} (k^{-1} \cdot q) \right) \left( M^*_{\psi^*_\alpha} (k^{-1} \cdot q) | p \right)
= \int \limits_K dk \left( p | k^{-1} \cdot (M^*_{k, \psi^*_\alpha} q) \right) \left( k^{-1} \cdot (M^*_{k, \psi^*_\alpha} q) | p \right)
= \int \limits_K dk \left( k \cdot p | M^*_{k, \psi^*_\alpha} q \right) \left( M^*_{k, \psi^*_\alpha} q | k \cdot p \right).
\]

With (4.1) and Schur orthogonality applied to \( \mathcal{P}^m(Z) \) we obtain

\[
\frac{\| q \|^2}{d_{m+\chi^L}} \sum_{\alpha} \| (\psi^*_\alpha p)_{m+\chi^L} \|^2
= \int \limits_K dk \sum_{\alpha} \left( k \cdot p | M^*_{k, \psi^*_\alpha} q \right) \left( M^*_{k, \psi^*_\alpha} q | k \cdot p \right)
\]
\[\begin{align*}
&= \int dk \sum_{\alpha} (k \cdot p | M_{\psi_\alpha}^* q) (M_{\psi_\alpha}^* q | k \cdot p) \\
&= \sum_{\alpha} \int_k dk (k \cdot p | (M_{\psi_\alpha}^* q)_m) ((M_{\psi_\alpha}^* q)_m | k \cdot p) \\
&= \frac{\|p\|^2}{d_m} \sum_{\alpha} \| (M_{\psi_\alpha}^* q)_m \|^2 \\
&= \frac{\|p\|^2}{d_m} \frac{d(\ell)}{(d/r)(\ell)} (d_e/r)_m^L \beta_m^L \|q\|^2,
\end{align*}\]

where in the last step we use Proposition 14. It follows that

\[\sum_{\alpha} \| (\psi_\alpha^{(\ell)} p)_m + \chi \|^2 = \frac{\|p\|^2}{d_m} \frac{d(\ell)}{(d/r)(\ell)} (d_e/r)_m^L \beta_m^L \|q\|^2,\]

This proves the first assertion. The second assertion follows, since \(\sum_{\alpha} M_{\psi_\alpha}^* (\psi_\alpha^{(\ell)} p)_m + \chi \) is a multiple of \(p\) and

\[\sum_{\alpha} \left( p | M_{\psi_\alpha}^* (\psi_\alpha^{(\ell)} p)_m + \chi \right) = \sum_{\alpha} (\psi_\alpha^{(\ell)} p | (\psi_\alpha^{(\ell)} p)_m + \chi) = \sum_{\alpha} (\psi_\alpha^{(\ell)} p)_{m+\chi} | (\psi_\alpha^{(\ell)} p)_{m+\chi} = \sum_{\alpha} \| (\psi_\alpha^{(\ell)} p)_{m+\chi} \|^2.\]

\[\square\]

The second eigenvalue formula is the following:

**Theorem 22.** For \(0 \leq \ell \leq r\) the K-invariant operators \(T_{\Delta^{(\ell)}}\) and \(T_{\nu^{(\ell)}}\) have the eigenvalues

\[T_{\Delta^{(\ell)}}(m) = \sum_{|L|=\ell} (d/r)_m^L \alpha_m^L\]

and

\[T_{\nu^{(\ell)}}(m) = \sum_{|L|=\ell} (d/r)_m^L (\nu)_m^L \alpha_m^L\]

with \(\alpha_m^L\) defined in (3.5).
Proof. Let \( p \in \mathcal{P}^m(Z) \). By Lemma 15 applied to \( m + \chi^L \) we have
\[
\sum_{\alpha} M^\nu_{\psi_{\alpha}}(M^\nu_{\psi_{\alpha}}p)_{m+\chi^L} = \sum_{\alpha} M^\nu_{\psi_{\alpha}}(\psi_{\alpha}^p)_{m+\chi^L} = \frac{1}{(\nu)^L_m} \sum_{\alpha} M^\nu_{\psi_{\alpha}}(\psi_{\alpha}^p)_{m+\chi^L} = p \frac{d(\ell)}{(r)_\ell (d/r)_{\ell}(\nu)^L_m} \alpha_m^L.
\]
Since
\[
T^\nu_{\Delta(\ell)}p = \frac{(r)_{\ell}(d/r)_{\ell}}{d(\ell)} T^\nu_{E(\ell)}p = \frac{(r)_{\ell}(d/r)_{\ell}}{d(\ell)} \sum_{|L|=\ell} \sum_{\alpha} M^\nu_{\psi_{\alpha}}(M^\nu_{\psi_{\alpha}}p)_{m+\chi^L}
\]
the assertion follows by summing over all \( \ell \)-element subsets \( L \) such that \( m + \chi^L \) is a partition. The proof for the Fock space is similar.

For the Hardy space \( H^2(S) \) over the Shilov boundary \( S \) of \( D \), corresponding to \( \nu = \frac{d}{r} \), the above formula combined with (3.11) simplifies to
\[
T^{d/r}_{\Delta(\ell)}(m) = \sum_{|L|=\ell} \alpha_m^L = \binom{r}{\ell}.
\]
This, however, is trivial since \( \Delta(\ell)(z,z) = \binom{r}{\ell} \) is constant on \( S \).

Example 23. For \( \ell = 1 \), with \( \Delta^{(1)}(z,z) = (z|z) \), we obtain as a special case
\[
T^\nu_{(z|z)}(m) = \sum_{k=1}^r \frac{(d/r)^k}{(\nu)^k_m} \alpha_m^k = \sum_{k=1}^r \frac{m_k^r + \frac{d}{r}}{m_k^r + \nu} \prod_{h \neq k} \frac{m_h^r - m_h^r + \frac{a}{2}}{m_h^r - m_h^r}
\]
for all partitions \( m \). This formula was conjectured in [6] and proved there, by a different argument, for all bounded symmetric domains of rank \( r = 2 \).

Besides the spin factors, which correspond to the rank 2 domains of tube type, there exist three types of Jordan triples of rank 2 which are not of tube type: (i) the space of all complex \((2 \times N)\)-matrices with \( N > 2 \), where \( d = 2N \) and \( a = 2 \), (ii) the space of all complex anti-symmetric \((5 \times 5)\)-matrices, where \( d = 10 \) and \( a = 4 \), and (iii) the exceptional domain of dimension \( d = 16 \), where \( a = 6 \).

Example 24. For \( \ell = r, L = \{1, \ldots, r\} \) we have \( \alpha_m^L = 1 \) (empty product). If \( Z \) is of tube type, then \( d = d_e \). Using (3.1) Theorem 22 simplifies to
\[
(M^*_N M_N)(m) = T^{(r)}_{\Delta(\nu)}(m) = \frac{(d/r)^{m+1}}{(d/r)_{m}},
\]
\[
(M^*_N M^*_N)(m) = T^\nu_{\Delta(\nu)}(m) = \frac{(d/r)^{m+1}}{(d/r)_{m}} \frac{(\nu)_m}{(\nu)_{m+1}}.
\]

Funding Open Access funding enabled and organized by Projekt DEAL.
Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Arazy, J.: A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains. Contemp. Math. 185, 7–65 (1995)
[2] Arazy, J., Zhang, G.: Homogeneous multiplication operators on bounded symmetric domains. J. Funct. Anal. 202, 44–66 (2003)
[3] Berezin, F.: Quantization in complex symmetric spaces. Math. USSR-Izv. 9, 341–379 (1975)
[4] Faraut, J., Korányi, A.: Function spaces and reproducing kernels on bounded symmetric domains. J. Funct. Anal. 88, 64–89 (1990)
[5] Faraut, J., Korányi, A.: Analysis on Symmetric Cones. Clarendon Press, Oxford (1994)
[6] Ghara, S., Kumar, S., Pramanick, P.: K-homogeneous tuple of operators on bounded symmetric domains. Israel J. Math. (to appear)
[7] Kadell, K.: The Selberg–Jack symmetric functions. Adv. Math. 130, 33–102 (1997)
[8] Lassalle, M.: Noyau de Szegö, K-types et algèbres de Jordan. C.R. Acad. Sci. Paris 303, 1–4 (1986)
[9] Lassalle, M.: Algèbres de Jordan et ensemble de Wallach. Invent. Math. 89, 375–393 (1987)
[10] Loos, O.: Jordan Pairs. Springer Lect. Notes in Math. 460 (1975)
[11] Loos, O.: Bounded Symmetric Domains and Jordan Pairs. Univ. of California, Irvine (1977)
[12] Schmid, W.: Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen. Invent. Math. 8, 61–80 (1969)
[13] Stanley, R.P.: Some combinatorial properties of the Jack symmetric functions. Adv. Math. 77, 76–115 (1989)
[14] Upmeier, H.: Jordan algebras and harmonic analysis on symmetric spaces. Am. J. Math. 108, 1–25 (1986)
[15] Upmeier, H.: Toeplitz operators on bounded symmetric domains. Trans. Am. Math. Soc. 280, 221–237 (1983)
[16] Upmeier, H.: Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics, CBMS Series in Math. 67, Am. Math. Soc. (1987)

Harald Upmeier (✉)
Department of Mathematics
University of Marburg
39052 Marburg
Germany
e-mail: upmeier@mathematik.uni-marburg.de

Accepted: April 27, 2021.