CUPLENGTH ESTIMATES IN MORSE COHOMOLOGY

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Abstract. The main goal of this paper is to give a unified treatment to many known cuplength estimates. As the base case, we prove that for $C^0$-perturbations of a function which is Morse-Bott along a closed submanifold, the number of critical points is bounded below in terms of the cuplength of that critical submanifold. As we work with rather general assumptions the proof also applies in a variety of Floer settings. For example, this proves lower bounds for the number of fixed points of Hamiltonian diffeomorphisms, Hamiltonian chords for Lagrangian submanifolds, translated points of contactomorphisms, and solutions to a Dirac-type equation.

1. Introduction

In this paper, we prove a rather general cuplength estimate in Morse homology and explain how this generalizes in the context of Floer theory. The main goal is to give a unified treatment of many known cuplength-type results in Floer theory including one improvement and a new result.

Let $M$ be a manifold, not necessarily compact, and $F : M \to \mathbb{R}$ be a smooth function. We assume that $Z \subset M$ is a closed, connected submanifold satisfying $Z \subset \text{Crit} F$ and $F$ is Morse-Bott along $Z$, that is, for all $z \in Z$ we have

$$\ker \text{Hess}_z F = T_z Z.$$ (1.1)

For convenience of notation, we also assume that $F|_Z = 0$ and that the spectral gap $\mathcal{S}$ of $F$ with respect to $Z$ is positive. Here, the spectral gap is defined by

$$\mathcal{S} = \mathcal{S}(F, Z) := \inf \{|F(x)| : x \in \text{Crit} F \setminus Z\} > 0.$$ (1.2)

If $F$ does not have critical points outside of $Z$, we set $\mathcal{S} := \infty$. In particular, this assumption guarantees that there are no critical points with value 0 outside of $Z$ and the smallest critical value different from 0 measured in absolute value is positive.

Let $h : M \to \mathbb{R}$ be a smooth function such that $\sup (h - F) \geq 0$ and $\inf (h - F) \leq 0$. These conditions are normalization conditions and can be achieved for any smooth function $h$ by adding an appropriate constant. This does not change the critical points and the gradient flow of $h$.

Since $M$ is not assumed to be compact, we have to make an assumption which guarantees that solution spaces to (generalized) gradient flow equations are well-behaved. This is Assumption 2 below. With these assumptions on $F$ and $h$, we have the following

Theorem 1.1. If $h$ is sufficiently close to $F$ in the sense that

$$\|h - F\| := \sup (h - f) - \inf (h - F) < \mathcal{S}$$ (1.3)

and Assumption 2 is satisfied, then $h$ has at least cuplength$(Z) + 1$ critical points with critical values in the interval $[-\|h - F\|, \|h - F\|]$. 

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We recall that the cuplength of the critical submanifold $Z$ is defined as
\[
\text{cuplength}(Z) := \max\{k \in \mathbb{N} \mid \exists \alpha_1, \ldots, \alpha_k \in H^{\geq 1}(Z) \text{ such that } \alpha_1 \cup \ldots \cup \alpha_k \neq 0\}
\] (1.4)
where $H^{\geq 1}(Z)$ denotes the cohomology in degree at least 1.

For a more detailed discussion of our Assumption 2, more technical definitions are needed, see Section 3.1 for the precise statement and Section 3.7 for how it implies compactness of all relevant moduli spaces. Assumption 2 holds if $M$ is compact or, more generally, if the functions involved are coercive.

Remark 1.2. The assumption that $F$ is Morse-Bott along $Z$ can be relaxed to an action energy estimate given in Assumption 1, which is easier to prove in the infinite dimensional cases of our Floer theoretic applications. If $F$ is Morse-Bott along $Z$, this estimate holds as is established in Lemma 3.3.

In Section 4, we also prove another version of Theorem 1.1 where the bounds on $h$ in terms of $F$ are replaced by index constraints for critical points of $F$, see Theorem 4.1.

2. Floer theoretic applications

We now illustrate Theorem 1.1 on Floer theoretic examples. Even though Theorem 1.1 is stated for finite-dimensional Morse theory we explain in detail in Section 5 how the proof of Theorem 1.1 has to be adjusted in the context of Floer theory. Even though Theorem 1.1 is of independent interest, our main focus is giving a unified explanation and proof of several cuplength estimates in Floer theory. Theorem 1.1 is applied via the analogy of Floer homology as a half-infinite dimensional Morse theory on the loop space of a symplectic manifold. To ensure that Assumption 2 holds in these cases, we now let $(W, \omega)$ be a closed symplectic manifold.

2.1. Symplectic aspherical manifolds. First we assume that $(W, \omega)$ is symplectically aspherical, that is $\omega|_{\pi_2(M)} = 0$. Then the symplectic area functional $\mathcal{A} : \Lambda W \to \mathbb{R}$ is defined on the space $\Lambda W := C^\infty_{\text{contr}}(S^1, W)$ of contractible loops by
\[
\mathcal{A}(x) := -\int_{D^2} u^* \omega
\] (2.1)
where the capping $u$ of $x$ is a smooth map $u : D^2 \to W$ with $u|_{S^1} = x$. In this situation, we set $F := \mathcal{A}$ and $M := \Lambda W$ and observe that $\text{Crit} F$ is the set constant loops and can be identified with $W$. We set
\[
Z := \text{Crit} F = W.
\] (2.2)
It follows from the non-degeneracy of the symplectic form that $F$ is indeed Morse-Bott along $Z$ and since $F$ has no other critical points, we have $\mathcal{G} = \infty$.

The function $h$ is then defined using Hamiltonian perturbations of $F$. We choose a function $H : S^1 \times W \to \mathbb{R}$ and set
\[
h(x) := \mathcal{A}_H(x) = F(x) + \int_0^1 H(t, x(t)) dt : \Lambda W \to \mathbb{R}.
\] (2.3)
This is the usual action functional of classical mechanics. We have
\[
\|h - F\| = \int_0^1 \left[ \max_W H(t, \cdot) - \min_W H(t, \cdot) \right] dt =: \|H\|_H,
\] (2.4)
is well-defined. Critical points of $F$ and $W$ have the same symplectic area. Then

$$\hat{x}(t) = X_H(t, x(t)).$$

(2.5)

This is a special case of a Theorem by Floer \cite{Flo89} and independently by Hofer \cite{Hof88}. In fact, they treated the following more general situation using Lagrangian Floer homology.

Let $L \subset W$ be a closed Lagrangian submanifold such that $\omega|_{\pi_2(W,L)} = 0$. We consider the space $P_0(W, L) := \{x \in C^\infty([0,1], W) \mid x(0), x(1) \in L, [x] = 0 \in \pi_1(W, L)\}$ of paths in $W$ starting and ending on $L$ and which are contractible relative to $L$. Then the symplectic area functional $\mathcal{A} : P_0(W, L) \to \mathbb{R}$ is defined by

$$\mathcal{A}(x) := -\int_{D^+} u^* \omega$$

(2.6)

where the capping $u : D^+ := \{z \in \mathbb{C} \mid |z| \leq 1, \text{Im}(z) \geq 0\} \to W$ is a smooth map with $u|_{S^1 \cap D^+} = x$ and $u|_{\mathbb{R} \cap D^+} \subset L$. Again we set $F := \mathcal{A}$ and $M := P_0(W, L)$ and observe that $\text{Crit} F$ is the set of constant paths which is identified with $L$, i.e.,

$$Z := \text{Crit} F = L.$$

(2.7)

It follows from the fact that $L$ is Lagrangian that $F$ is indeed Morse-Bott along $Z$ and since $F$ has no other critical points we have $\mathcal{G} = \infty$. As above, Hamiltonian perturbations of $F$ give rise to functions $h$ satisfying Assumption $2$. That is, we choose a function $H : S^1 \times W \to \mathbb{R}$ and set

$$h(x) := F(x) + \int_0^1 H(t, x(t)) dt : P_0(W, L) \to \mathbb{R}.$$

(2.8)

Again the condition that $\|h - F\| < \mathcal{G}$ is empty and it follows from Theorem 1.1 that $\mathcal{A}_H := h$ has at least cuplength($L$) + 1 critical points with critical values in the interval $[-\|H\|_H, \|H\|_H]$, see Theorem 5.3 for the exact statement. Of course, critical points of $\mathcal{A}_H$ are solutions of the Hamiltonian equation with Lagrangian boundary condition given by $L$, i.e.,

$$\begin{cases}
\dot{x}(t) = X_H(t, x(t)) \\
x(0), x(1) \in L
\end{cases}$$

(2.9)

of $H$. This is the result previously proved independently by Floer and Hofer.

2.2. Rational symplectic manifolds. More generally, we may assume that $\omega|_{\pi_2(W)} = \lambda Z$ for some $\lambda > 0$. The symplectic aspherical case corresponds to $\lambda = \infty$. Now the symplectic area functional is defined on the cover $M := \Lambda W$ consisting of equivalence classes $\tilde{x} := [x, u]$ where $x \in \Lambda W$ and $u$ is a capping of $x$. Pairs $(x, u)$ and $(x, v)$ are declared equivalent if $u$ and $v$ have the same symplectic area. Then

$$F(\tilde{x}) := \mathcal{A}(\tilde{x}) := -\int_{D^2} u^* \omega$$

(2.10)

is well-defined. Critical points of $F$ are of the form $[x, u]$ where $x$ is a constant loop and $u$ is some capping which is topologically a sphere in $W$. Thus the only critical points with critical value zero are the constants with trivial capping meaning the map $u$ is constant, too. This set $Z \subset \text{Crit} F$ is again a Morse-Bott component and $\mathcal{G} = \lambda > 0$ since $(W, \omega)$ is rational.
Theorem 1.1 implies that if $H : S^1 \times W \to \mathbb{R}$ is a Hamiltonian function with
\[ ||H||_H < \lambda \] then the number of 1-periodic Hamiltonian orbits of $H$ with action value in $[-\lambda, \lambda]$ is at least cuplength$(W) + 1$.

This has been proved by Schwarz in [Sch98]. The analogous statement in the Lagrangian case, with the condition $\omega|_{\pi_2} = \lambda Z$ replaced by $\omega|_{\pi_2(W,L)} = \lambda Z$ for some positive $\lambda$, is due to Liu, cf. [Liu05]. In Section 5 we phrase and prove Theorems 5.1 and 5.3 in this more general setting. The aspherical case is then included by setting $\lambda = \infty$.

**Remark 2.1.** The same proof can also be applied in more general settings. We only need that some version of Floer homology is defined and the necessary assumptions 2 for compactness and 1 are satisfied. For example, this is the case for symplectic manifolds which are convex at infinity or of bounded geometry. The same method also applies in the setting of non-resonant magnetic flows on tori, where Frauenfelder, Merry and Paternain have developed a Floer homology, see [FMP13]. Our Morse theoretic proof also applies in this case and gives a lower bound on the number of periodic orbits. More concretely, the proof shows that if the magnetic field on $T^{2N}$ is non-resonant in period $\tau$, the number of $\tau$-periodic orbits is at least $2N + 1$.

### 2.3. Translated points.

Now we also apply Theorem 1.1 to translated points in contact geometry. Translated points were introduced by Sandon in [San11] and related to contact rigidity phenomena. In [San12] Sandon conjectures (and proves in certain cases) that the number of translated points of a contactomorphism of a contact manifold $\Sigma$ is at least equal to the minimal number of critical points of a function on $\Sigma$. We want to explain how Theorem 1.1 together with the SFT-type compactness result explained in [AFM13] implies under additional assumptions a lower bound on the number of translated points in terms of cuplength$(\Sigma)$.

Let $(\Sigma, \alpha)$ be a closed contact manifold and $\varphi : \Sigma \to \Sigma$ be a contactomorphism which is contact isotopic to the identity. Note that $\varphi$ does not preserve the contact form $\alpha$, but we have $\varphi^*\alpha = \rho\alpha$ for some function $\rho : \Sigma \to \mathbb{R}_{>0}$. We call a point $q \in \Sigma$ a translated point with time-shift $\eta \in \mathbb{R}$ if
\[
\begin{cases}
\varphi(q) = \theta^\eta(q) \\
\rho(q) = 1,
\end{cases}
\]
where $\theta^t$ is the Reeb flow. We point out that the time-shift is not unique if $q$ lies on a closed Reeb orbit. We define the symplectization of $\Sigma$ to be $S\Sigma := \Sigma \times \mathbb{R}_{>0}$ equipped with the symplectic form $\omega := d(\rho\alpha)$ where $r \in \mathbb{R}_{>0}$. The unperturbed Rabinowitz action functional $A : AS\Sigma \times \mathbb{R} \to \mathbb{R}$ is defined as
\[
A(x, \eta) := \int_{S^1} ry^*\alpha - \eta \int_0^1 (r - 1) dt
\] (2.12)
where $x = (y, r) : S^1 \to S\Sigma$. Then $(x, \eta) \in \text{Crit}A$ if and only if $x(S^1) \subset \Sigma \times \{1\}$ and $\dot{y} = \eta R(y)$. Here $R$ is the Reeb vector field of $\alpha|_{\Sigma}$, i.e., we can identify a critical point $(x, \eta)$ with an $\eta$-periodic Reeb orbit where negative $\eta$ means that the $-\eta$-periodic Reeb orbit is traversed in the opposite direction and $\eta = 0$ corresponds to a constant loop in $\Sigma$. Constant loops in $\Sigma$ are as usual identified with $\Sigma$ itself. Again the set $Z := \Sigma$ is a Morse-Bott component, see [AF10 Lemma 2.12], and the corresponding spectral gap of $F := A$ is $\mathcal{G} = \text{minimal period of a contractible Reeb orbit.}
Using the function \( \rho : \Sigma \to \mathbb{R}_{>0} \) given by \( \varphi^* \alpha = \rho \alpha \), we obtain a non-compactly supported Hamiltonian diffeomorphisms \( \phi : S\Sigma \to S\Sigma \) by setting \( \phi(q, r) := (\varphi(q), r_2\rho(q)) \). As explained in [AM10], the Rabinowitz action functional \( F = \mathcal{A} \) can be perturbed by a certain cut-off of the lift \( \hat{\phi} \) of \( \varphi \) in such a way that the critical points of the perturbed functional \( h := \mathcal{A}_\varphi \) correspond to translated points of \( \varphi \), see Lemma 3.5 in [AM10]. This perturbation is supported inside \( \Sigma \times [e^{-\kappa(\varphi)}, e^{\kappa(\varphi)}] \subset S\Sigma \), where the constant \( \kappa(\varphi) \) is specified in Section 5.3. This implies that

\[
\|h - F\| \leq e^{\kappa(\varphi)} \|H\|_H
\]

(2.13)

where \( H : S^1 \times \Sigma \to \mathbb{R} \) is any contact Hamiltonian such that the induced contact isotopy \( \psi_t \) satisfies \( \varphi = \psi_1 \). Thus our main theorem implies that if \( e^{\kappa(\varphi)} \|H\|_H < \mathcal{G} \) the functional \( \mathcal{A}_\varphi \) has at least \( \text{cuplength}(S\Sigma) + 1 = \text{cuplength}(\Sigma) + 1 \) many critical points with critical values in the interval \([-e^{\kappa(\varphi)} \|H\|_H, e^{\kappa(\varphi)} \|H\|_H]\), see Theorem 5.4.

This result complements and strengthens the main result in [AM10], which concerned with the more general notion of leafwise intersections. Translated points of \( \varphi \) correspond to leafwise intersections of its lift \( \phi \). More concretely, as we work with the symplectization instead of a symplectic filling of \( \Sigma \), our bounds depend only on the contact manifold, where is [AM10], the lower bound was given by a relative cuplength of \( \Sigma \) in the filling. We refer the reader to the end of section 5.3 for further details.

### 2.4. Solutions to perturbed Dirac-type equations

Now we also apply Theorem 1.1 in the hyperkähler setting, which can be viewed as a generalization of Hamiltonian mechanics with multi-dimensional time.

Let the “time”-manifold be \( X \) either \( T^3 \) or \( S^3 \) and consider a volume form \( \mu \) on \( X \) and a divergence-free global frame \( v_1, v_2, v_3 \). For the considered homology theory to be defined, we need special choices of the global frame, which will be specified in Section 5.4 see also [HNS09], where the Floer homology in this setting is constructed.

Let \( Y \) be a compact, flat hyperkähler manifold with almost complex structures \( I, J, K \). We define symplectic forms \( \omega_i \) by choosing a Riemannian metric and choosing \( \omega_1(\cdot, \cdot) = \langle \cdot, I \cdot \rangle \), \( \omega_2(\cdot, \cdot) = \langle \cdot, J \rangle \) and \( \omega_3(\cdot, \cdot) = \langle \cdot, K \rangle \).

In this setting, we consider the manifold \( M \) to be the space of smooth maps from \( f : X \to Y \). The role of the function \( F \) is taken by the functional

\[
F(f) := \mathcal{A}(f) = -\sum_{l=1}^{3} \int_{[0,1] \times X} F^* \omega_l \wedge i_{v_l} \mu, \quad (2.14)
\]

where \( F \) is a homotopy from \( f \) to a constant map, i.e., an analog of the capping of a loop in Hamiltonian dynamics.

Critical points of \( F \) are solutions to the Dirac-type equation

\[
\hat{\delta} f := IL_{v_1} f + JL_{v_2} f + KL_{v_3} f = 0. \quad (2.15)
\]

As in the case of classical Hamiltonian dynamics described above, we consider a ”time”-dependent Hamiltonian in \( Y \). That is, the Hamiltonian is a function \( H : X \times Y \to \mathbb{R} \) and use this to define the Hamiltonian perturbation of \( F \) as the functional

\[
h(f) := \mathcal{A}_H(f) = -\sum_{l=1}^{3} \int_{[0,1] \times X} F^* \omega_l \wedge i_{v_l} \mu - \int_X H(f) \mu. \quad (2.16)
\]
For this functional, critical points are solutions to the equation
\[ \dot{f} = \nabla H(f). \] (2.17)

Using Fourier analysis in the form of the Peter-Weyl theorem, it was proved in [GH12] that the number of solutions to equation (2.17) is bounded below by cuplength(Y) + 1 and this result is reproduced by our Morse theoretic proof, see Theorem 5.5.

**Remark 2.2.** The action functional \( A \) can be defined in a more general setting, see [GH13], and it is a generalization of the action functional of classical Hamiltonian dynamics. Also the cuplength estimate for the number of critical points holds in this more general situation. However, for our Morse theoretic proof to apply, we need to work in the restricted setting described above as the hyperkähler Floer homology is only defined in this special case.

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3. Preliminaries

In this section, we describe the setting used in the proofs of existence of critical points in the Morse case in more detail. Furthermore, we establish a number of estimates on the energy and the Morse function along trajectories. These estimates are used to show compactness of the moduli spaces considered in the proofs.

3.1. Setting and basic assumptions. Let \( M \) be a manifold, not necessarily compact, and \( F: M \to \mathbb{R} \) be a smooth function. We assume that \( Z \subset M \) is a closed, connected submanifold satisfying \( Z \subset \text{Crit} F \). For convenience of notation, we also assume that \( F|_Z = 0 \). We define the spectral gap of \( F \) with respect to \( Z \) by
\[ \mathcal{S} = \mathcal{S}(Z) := \inf\{|F(x)| : x \in \text{Crit} F \setminus F^{-1}(0)\} \] (3.1)
and assume this value to be positive. In particular, this implies that there are no critical points outside \( Z \) with value 0 and that the smallest critical value different from 0 measured in absolute value is positive. If \( F \) does not have critical points outside of \( Z \), we set \( \mathcal{S} := \infty \).

Instead of assuming that \( F \) is Morse-Bott along \( Z \), we can only assume an action energy estimate. For this purpose, we define the energy density
\[ e(x) := \langle \nabla F(x), \nabla F(x) \rangle \text{ for } x \in M. \] (3.2)

**Assumption 1** (Action-energy estimate). There exists \( C > 0 \) such that for \( x \) sufficiently close to \( Z \), we have
\[ |F(x)| \leq C \, e(x). \] (3.3)

For all proof purposes later in the paper, we will use this condition instead of the Morse-Bott property in order to keep the proofs applicable in the most general setting as this estimate is true for many Morse and Floer type cohomology theories. For example, if \( F \) is the symplectic area functional the action-energy inequality is a consequence of the isoperimetric inequality and Hölder’s inequality. In Lemma 3.3 we prove this assumption in the case that \( F \) is Morse-Bott along \( Z \).

**Convention 3.1.** For \( k \in \mathbb{N} \) and \( r > 0 \), we fix a smooth family of functions \( \beta_r \in C^\infty(\mathbb{R}, [0, 1]) \) satisfying
(1) for $r \geq 1$: $\beta_r(s) = 1$ for $s \in [0, (k+1)r]$, and $\beta_r(s) = 0$ for $s \leq -1$ and $s \geq (k+1)r + 1$ and such that $\beta_r(s)$ is increasing on $(-1, 0)$ with a slope at most 2 and decreasing on ($(k + 1)r, (k + 1)r + 1$) with a slope at least $-2$.

(2) for $r \leq 1$: $\beta_r(s) \leq r$ for all $s \in \mathbb{R}$ and supp$\beta_r \subset [-1, k + 2]$ and $\beta'(s) \in (-2, 2)$.

(3) $\lim_{r \to \infty} \beta_r(s) =: \beta^\infty_Z(s)$ and $\lim_{r \to -\infty} \beta_r(s + (k + 1)r) =: \beta^\infty_Z(s)$ exist, where the limit is taken with respect to the $C^\infty_{loc}$ topology.

Later in the proofs, we will specify the value of $k$, but for now, we prove all Lemmas in the general case, where $k$ is any natural number.

Let $g$ be a Riemannian metric on $M$ and consider a smooth function $h: M \to \mathbb{R}$ such that sup$(h - F) \geq 0$ and inf$(h - F) \leq 0$. These conditions on $h$ are a normalization condition and can be achieved for any smooth function $h$ by adding a constant, i.e., without changing the critical points or the gradient flow. Now we define the function

$$G_{r,s}(x) := \beta_r(s)h(x) + (1 - \beta_r(s))F(x)$$

and the moduli space

$$\mathcal{M} := \left\{(r, \gamma) \mid r \geq 0, \gamma: \mathbb{R} \to M \text{ satisfies } \gamma'(s) + \nabla^g G_{r,s}(\gamma(s)) = 0 \text{ and } E(\gamma) < \infty \right\}.$$  (3.5)

Here, the energy of a gradient flow line is given by

$$E(\gamma) := \int_{-\infty}^{\infty} |\gamma'(s)|^2 ds.$$  (3.6)

Furthermore, we define

$$\mathcal{M}_{[0,R]}(Z) := \{(r, \gamma) \in \mathcal{M} \mid r \in [0, R] \text{ and } \lim_{s \to \pm \infty} \gamma(s) \in Z\}.$$  (3.7)

In order to guarantee compactness of this moduli space in the $C^\infty$-topology, we need to make another crucial assumption. This will be satisfied not only in the case of classical Morse-Bott case for suitable Morse-Bott functions as described above, but also in the infinite-dimensional cases of various Floer theories. Namely, we need to ensure $C^\infty_{loc}$-compactness of the space of trajectories $\gamma$ with finite energy with all reparametrizations by shifts.

**Assumption 2.** For all sequences $\{(r_n, \gamma_n)\}_{n \in \mathbb{N}} \in \mathcal{M}$ and $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$, the sequence

$$\{\gamma_n(\cdot - s_n)\}_{n \in \mathbb{N}}$$

converges (up to taking subsequences) in $C^\infty_{loc}(\mathbb{R}, M)$.

**Remark 3.2.** Even though we do not use this in our article, we point out that given this assumption, the Morse cohomology of $F$ is well-defined, see [CF09]. Since the manifold $M$ is not necessarily compact, the above compactness assumption is not automatic. It is not hard to verify this assumption for Morse-Bott functions $F$ on compact manifolds using the flow equation in the definition of $\mathcal{M}$ and the Arzelà-Ascoli theorem. In the Floer cases, a similar argument using the Floer equation shows the assumption for the analog moduli spaces of Floer trajectories. Detailed arguments for the Floer cases will be given in Section 5.1.

**Lemma 3.3.** If the function $F$ is Morse-Bott along the critical submanifold $Z$, Assumption 1 holds, i.e., we have $|F(x)| \leq C e(x)$ for some constant $C$.

**Proof.** We identify a neighborhood $U$ of $Z$ as a neighborhood of the zero-section in the normal bundle of $Z$ in $M$ and write $x = (z, y)$ for $x \in U$ with $z \in Z$ and $y$ being the normal component.
Now we estimate $e(x)$ and $F(x)$ separately. For $e(x)$, we use the Morse-Bott condition to find

$$e(x) = \langle \nabla F(x), \nabla F(x) \rangle$$

$$= \frac{1}{2} D^2 F(y, y) + O(|y|^3)$$

$$= |\text{Hess } F(y)|^2 + O(|y|^3)$$

$$(3.9)$$

$$\geq C_1 |y|^2 + O(|y|^3)$$

$$\geq C_2 (1 - |y|) |y|^2$$

Similarly, we find for $F(x)$, using again the Morse-Bott condition and that $F(Z) = 0$,

$$F(x) = \frac{1}{2} D^2 F(y, y) + O(|y|^3)$$

$$\Rightarrow |F(x)| \leq C_3 |y|^2 + O(|y|^3)$$

$$(3.10)$$

For sufficiently small $y$, we conclude

$$F(x) \leq \frac{C_4 (1 + |y|)}{C_2 (1 - |y|)} e(x) \leq C \ e(x)$$

as desired.  

$\square$

3.2. Estimates for Morse-Bott functions. In this section, we use Assumptions 1 and 2 to prove a number of estimates along curves from $M$ and $M_{[0,R]}(Z)$. Several of the energy computations will also be used later for other estimates needed in the proof of the main theorem. The first is an estimate for the values of $G_{r,s}$ along the flow lines in terms of the oscillation norm of $h - F$. We recall that the oscillation norm of a function $f : M \to \mathbb{R}$ is

$$||f|| := \sup f - \inf f$$

which is not really a norm since $||f|| = 0$ is equivalent to $f$ being constant.

**Lemma 3.4.** Let $(r, \gamma) \in M_{[0,R]}(Z)$. Then for all $s \in \mathbb{R}$, we have

$$|G_{r,s}(\gamma(s))| \leq ||h - F||.$$  

**Proof.** As a first step, we compute for $s \in \mathbb{R}$ the energy of part of the gradient flow line.

$$0 \leq E_s(\gamma) := \int_s^\infty |\gamma'(t)|^2 dt$$

$$= - \int_s^\infty \langle \nabla^g G_{r,t}(\gamma(t)), \gamma'(t) \rangle dt$$

$$= - \int_s^\infty dG_{r,t}(\gamma(t))[\gamma'(t)]dt$$

$$= - \int_s^\infty \frac{d}{dt} G_{r,t}(\gamma(t))dt + \int_s^\infty \frac{\partial G_{r,t}}{\partial t}(\gamma(t))dt$$

$$= - G_{r,\infty}(\gamma(\infty)) + G_{r,s}(\gamma(s)) + \int_s^\infty \beta'_r(t)[h - F](\gamma(t))dt$$

$$= G_{r,s}(\gamma(s)) + \int_s^\infty \beta'_r(t)[h - F](\gamma(t))dt.$$

$$(3.14)$$
Similarly, we compute the energy for the front end of the trajectory:

\[
0 \leq E^s(\gamma) := \int_{-\infty}^{s} |\gamma'(t)|^2 dt
= \int_{-\infty}^{s} \langle \gamma'(t), \gamma'(t) \rangle dt
= -\int_{-\infty}^{s} \langle \nabla^q G_{r,t}(\gamma(s)), \gamma'(t) \rangle dt
= -\int_{-\infty}^{s} dG_{r,t}(\gamma(t)) [\gamma'(s)] dt
= -\int_{-\infty}^{s} \frac{dG_{r,t}(\gamma(t))}{dt} dt + \int_{-\infty}^{s} \frac{\partial G_{r,t}}{\partial t} (\gamma(t)) dt
= -G_{r,s}(\gamma(s)) + G_{r,\infty} (\gamma(\infty)) + \int_{-\infty}^{s} \beta'_r(t) [h - F] (\gamma(t)) dt
= -G_{r,s}(\gamma(s)) + \int_{-\infty}^{s} \beta'_r(t) [h - F] (\gamma(t)) dt.
\]

Now we can estimate the integral terms to conclude the desired bounds and use the definition of \( \beta \) to find

\[
G_{r,s}(\gamma) \geq -\int_{s}^{\infty} \beta'_r(t) [h - F] (\gamma(t)) dt
\geq -\int_{-1}^{0} \beta'_r(t) \sup_{x \in M} (h(x) - F(x)) dt - \int_{(k+1)r}^{(k+1)1} \beta'_r(t) \inf_{x \in M} (h(x) - F(x)) dt
\geq -\sup_{x \in M} (h(x) - F(x)) + \inf_{x \in M} (h(x) - F(x))
\geq -\|h - F\|.
\]

In this computation, the inequality in the second line holds due to our normalization condition on \( h \) which determines the signs of \( \sup(h - F) \) and \( \inf(h - F) \). This ensures that both integrands are non-negative and we make the total expression at most smaller. Using now (3.15) instead of (3.14), we can use an analogous argument to show

\[
G_{r,s}(\gamma(s)) \leq \|h - F\|.
\]

This proves the lemma. \( \square \)

The computations (3.14) and (3.15) in this proof also give rise to energy estimates for trajectories in \( \mathcal{M}_{[0,R]}(Z) \). Namely, consider (3.15) for large \( s \). Then we have \( G_{r,s} = F \) and \( \gamma(s) \to Z \) and thus

\[
E^s(\gamma) = \int_{-\infty}^{s} \beta'_r(t) [h - F] (\gamma(t)) dt
= \int_{-1}^{0} \beta'_r(t) [h - F] (\gamma(t)) dt + \int_{(k+1)r}^{(k+1)1} \beta'_r(t) [h - F] (\gamma(t)) dt
\]

(3.18)
as these are the only intervals with $\beta' \neq 0$. Then the signs of $\beta'$ in these intervals show that

$$0 \leq E(\gamma) \leq \sup_{x \in M} (h(x) - F(x)) - \inf_{x \in M} (h(x) - F(x)) = \|h - F\|. \tag{3.19}$$

This energy estimate will be used in the Floer cases to exclude bubbling in the proof of compactness of the moduli spaces analog to $\mathcal{M}_{[0, R]}(Z)$ in the Morse case.

To prove later that trajectories cannot leave a certain neighborhood of $Z$, we will need an estimate of the function $F$ along the trajectories in $\mathcal{M}_{[0, R]}(Z)$. Here, we use some of the above computations to show the following

**Lemma 3.5.** Along curves $\gamma$ for $(\gamma, r) \in \mathcal{M}_{[0, R]}(Z)$, we have the following estimate:

$$|F(\gamma(s))| \leq \|h - F\| \tag{3.20}$$

for all $s \in \mathbb{R}$.

**Proof.** For different values of $s$, the proof works differently and we consider several cases.

First note that for $s \leq -1$ and $s \geq (k+1)r+1$, we have $G_{r,s}(x) = F(x)$ and thus $|F(\gamma(s))| \leq \|h - F\|$ by Lemma 3.4. Thus we only need to prove the lemma for $s \in (-1, (k+1)r+1)$. We will prove the upper and lower bound of $F(\gamma(s))$ separately as we need to consider different intervals for $s$.

For the lower bound, recall from (3.15) that

$$G_{r,s}(\gamma(s)) \geq -\int_s^\infty \beta'(t)[h - F](\gamma(t))dt \tag{3.21}$$

for all $s \in \mathbb{R}$. For $0 \leq s \leq (k+1)r+1$, we can combine this with the estimate

$$\int_s^\infty \beta'(t)[h - F](\gamma(t))dt \leq -\inf(h - F), \tag{3.22}$$

which follows from the definition of $\beta_r$. With these two estimates, we compute for $s \geq 0$ that

$$\inf(h - F) \leq -\int_s^\infty \beta'(t)[h - F](\gamma(t))dt \leq G_{r,s}(\gamma(s)) = \beta_r(s)h(\gamma(s)) + (1 - \beta_r(s))F(\gamma(s)) = F(\gamma(s)) + \beta_r(s)(h(\gamma(s)) - F(\gamma(s))) \leq F(\gamma(s)) + \sup(h - F). \tag{3.23}$$

For the last inequality, use that $\beta_r(s) \leq 1$. Thus for those $s$, we have

$$F(\gamma(s)) \geq -\sup(h - F) + \inf(h - F) = -\|h - F\| \tag{3.24}$$

which is the desired lower bound on $F$. 
It remains to show the lower bound for $s \in (-1, 0)$. Again, we use a computation similar to (3.15), but need to estimate more carefully:

$$F(\gamma(s)) + \beta_r(s) \sup(h - F) \geq F(\gamma(s)) + \beta_r(s) (h(\gamma(s)) - F(\gamma(s)))$$

$$= G_{r,s}(\gamma(s))$$

$$\geq - \int_{\gamma}^{\infty} \beta'_{r}(t)[h - F](\gamma(t))dt$$

$$= - \int_{s}^{0} \beta'_{r}(t)[h - F](\gamma(t))dt - \int_{0}^{\infty} \beta'_{r}(t)[h - F](\gamma(t))dt$$

$$\geq - \sup(h - F) \int_{s}^{0} \beta'_{r}(t)dt + \int_{0}^{\infty} (-\beta'_{r}(t))[h - F](\gamma(t))dt$$

$$\geq - \sup(h - F)(1 - \beta_r(s)) + \inf(h - F) \int_{0}^{\infty} -\beta'_{r}(t)dt$$

$$= - \sup(h - F)(1 - \beta_r(s)) + \inf(h - F).$$

Here, we use in the second last step that $-\beta'_{r}(t)$ is non-negative on $(0, \infty)$ and the integral over this interval is 1. This shows that

$$F(\gamma(s)) \geq - \sup(h - F) + \inf(h - F) = -\|h - F\|$$

also for $s \in (-1, 0)$ and thus for all $s \in \mathbb{R}$.

The proof of the upper bound works similarly. Here, we need to treat the interval where $\beta'_{r}$ is negative individually and use the upper bound on $G$ given in (3.14)

$$G_{r,s}(\gamma(s)) \leq \int_{-\infty}^{s} \beta'_{r}(t)[h - F](\gamma(t))dt. \quad (3.26)$$

Analogously to the lower bound, we find the desired upper bound on $F$ for $s \leq (k + 1)r$ by this inequality together with

$$\int_{-\infty}^{s} \beta'(t)(h(\gamma(t)) - F(\gamma(t))) dt \leq \sup(h - F). \quad (3.27)$$

With these two inequalities, we compute

$$\sup(h - F) \geq G_{r,s}(\gamma(s))$$

$$= F(\gamma(s)) + \beta_r(s)(h(\gamma(s)) - F(\gamma(s)))$$

$$= F(\gamma(s)) - \beta_r(s)(F(\gamma(s)) - h(\gamma(s)))$$

$$\geq F(\gamma(s)) + \sup(F - h)$$

$$= F(\gamma(s)) - \inf(h - F).$$

This shows that

$$F(\gamma(s)) \leq \sup(h - F) - \inf(h - F) = \|h - F\|,$$ \hspace{0.5cm} (3.29)

i.e., we have proven the desired upper bound for $s \notin ((k + 1)r, (k + 1)r + 1)$. 

\hspace{1cm}
For the remaining values of \(s\), we again need to estimate more carefully in the computation \((3.14)\) to find

\[
F(\gamma(s)) + \beta_r(s) \inf(h - F) \leq F(\gamma(s)) + \beta_r(s) (h(\gamma(s)) - F(\gamma(s)))
\]

\[
= G_{r,s}(\gamma(s)) 
\]

\[
\leq \int_{-\infty}^{s} \beta_r'(t) [h - F](\gamma(t)) dt
\]

\[
= \int_{-1}^{0} \beta_r'(t) [h - F](\gamma(t)) dt + \int_{(k+1)r}^{s} \beta_r'(t) [h - F](\gamma(t)) dt
\]

\[
\leq \sup(h - F) \int_{-1}^{0} \beta_r'(t) dt + \inf(h - F) \int_{(k+1)r}^{s} \beta_r'(t) dt
\]

\[
= \sup(h - F) + \inf(h - F) (\beta_r'(s) - 1).
\]

Rearranging this inequality, we find the desired upper bound

\[
F(\gamma(s)) \leq \sup(h - F) - \inf(h - F) = \|h - F\| \tag{3.30}
\]

also for \(s \in ((k + 1)r, (k + 1)r + 1)\) and thus for all \(s \in \mathbb{R}\). \(\square\)

The last lemma of this section has a slightly different flavor and does not primarily concern estimates on functions along trajectories, but the convergence to the critical submanifold \(Z\).

**Lemma 3.6.** Assume that for a trajectory \(\gamma \in M\), there is a sequence \(s_n \to \infty\) such that \(\lim_{n \to \infty} \gamma(s_n)\) converges to a point \(z^+ \in Z\). Then the gradient flow line \(\gamma\) converges exponentially fast to \(z^+\), i.e., for sufficiently large \(s\), we find the estimate

\[
|z^+ - \gamma(s)| \leq A e^{-Bs} \tag{3.31}
\]

for some constants \(A \geq 0\) and \(B > 0\).

**Proof.** The first step to prove this lemma is an estimate of the distance to \(Z\) in terms of the value of \(F\) along the flow line \(\gamma\). For any fixed large \(s\), we find \(N \in \mathbb{N}\) with \(s_N > s\) such that for all \(n \geq N\), we have

\[
|z^+ - \gamma(s_n)| < \frac{1}{N}, \tag{3.32}
\]

since \(\gamma(s_n)\) converges to \(z^+ \in Z\). The following estimate is originally taken from [AF13] and is used here with only minor changes. We use some \(n \geq N\) and the action-energy estimate
from Assumption \ref{assumption_1} to compute

\[
|z^+ - \gamma(s)| \leq |z^+ - \gamma(s_n)| + |\gamma(s_n) - \gamma(s)|
\]
\[
< \frac{1}{N} + \int_s^{s_n} |\gamma'(t)| \, dt
\]
\[
= \frac{1}{N} \int_s^{s_n} |\nabla F(\gamma(t))| \, dt
\]
\[
= \frac{1}{N} \int_s^{s_n} \frac{|\nabla F(\gamma(t))|^2}{|\nabla F(\gamma(t))|} \, dt
\]
\[
= \frac{1}{N} \int_s^{s_n} \frac{|\nabla F(\gamma(t))|^2}{\sqrt{e(\gamma(t))}} \, dt
\]
\[
\leq \frac{1}{N} + C \int_s^{s_n} \frac{|\nabla F(\gamma(t))|^2}{\sqrt{F(\gamma(t))}} \, dt
\]
\[
= \frac{1}{N} - C \int_s^{s_n} \frac{\langle \nabla F(\gamma(t)), \gamma'(t) \rangle}{\sqrt{F(\gamma(t))}} \, dt
\]
\[
= \frac{1}{N} - C \int_s^{s_n} \frac{dF(\gamma(t))}{\sqrt{F(\gamma(t))}} \, dt
\]
\[
= \frac{1}{N} - 2C \int_s^{s_n} \frac{d}{dt} \sqrt{F(\gamma(t))} \, dt
\]
\[
= \frac{1}{N} - 2C \left( \sqrt{F(\gamma(s_n))} - \sqrt{F(\gamma(s))} \right)
\]  

As the left hand side is independent of \( N \), we can now take the limit \( N \to \infty \). Then \( F(\gamma(s_n)) \) converges to 0, since we assumed \( n > N \) and \( \gamma(s_n) \) converges to \( z^+ \in Z \). In the limit, we find the estimate

\[
|z^+ - \gamma(s)| \leq 2C \sqrt{F(\gamma(s))}.
\]  

Now it suffices to calculate

\[
\frac{d}{ds} F(\gamma(s)) = -\langle \nabla F(\gamma(s)), \nabla F(\gamma(s)) \rangle
\]
\[
= -e(\gamma(s))
\]
\[
\leq -\frac{1}{C} F(\gamma(s)).
\]  

This differential inequality shows that

\[
F(\gamma(s)) \leq C' e^{-\frac{s}{2C'}},
\]  

and therefore the above computation gives the estimate

\[
|z^+ - \gamma(s)| \leq 2C \sqrt{F(\gamma(s))} \leq 2C \sqrt{C' e^{-\frac{s}{2C'}}}.
\]

This shows exponential convergence of \( \gamma(s) \) to \( z^+ \in Z \) and completes the proof of the lemma. \( \square \)
3.3. Compactness of moduli spaces. In this subsection, we will apply Assumptions 1 and 2 and the bounds given by Lemma 3.4 to prove compactness of $\mathcal{M}_{[0,R]}(Z)$.

**Theorem 3.7.** If
\[
\|h - F\| < \mathcal{G}
\]  
then for all $R \geq 0$ the moduli space $\mathcal{M}_{[0,R]}(Z)$ is compact in $C^\infty(\mathbb{R}, M)$.

**Proof of Theorem 3.7.** Let $(r_n, \gamma_n)$ be a sequence in $\mathcal{M}_{[0,R]}(Z)$. As a first step, we show that (a subsequence of) the sequence $(\gamma_n)$ converges in $C^\infty(\mathbb{R}, M)$ and then show that the limit is again in $\mathcal{M}_{[0,R]}(Z)$.

Setting $s_n = 0$ in Assumption 2, we see that a subsequence of $\gamma_n$ converges in $C^\infty_{loc}(\mathbb{R}, M)$ to some limit $\gamma$, since $\mathcal{M}_{[0,R]}(Z) \subset \mathcal{M}$. The compactness of $[0, R]$ implies that, by possibly passing to another subsequence, $(r_n, \gamma_n)$ converges to $(r, \gamma)$ with $r \in [0, R]$.

It remains to show that $(r, \gamma) \in \mathcal{M}_{[0,R]}(Z)$, i.e.,
- $(r, \gamma) \in \mathcal{M}$ and
- $\lim_{s \to \pm \infty} \gamma(s) \in Z$.

For the first assertion, we need to show that $\gamma' = \nabla G_{r,s}(\gamma)$ and $E(\gamma) < \infty$. The gradient flow equation is a local condition and therefore follows from $C^\infty_{loc}$-convergence of the sequence $\gamma_n$. To show that the energy of the limit is finite, observe that the energy values $E(\gamma_n)$ are uniformly bounded by $\|h - F\|$. Then also the energy of the limit cannot exceed this bound and is therefore finite.

The second assertion requires more work. We consider only the limit as $s \to +\infty$ as the case $s \to -\infty$ is analogous. The first step is to find a candidate for the limit of $\gamma(s)$ as $s \to \infty$ and then show that this point is indeed the limit and lies in $Z$. For this, we choose a sequence $(s_n) \to \infty$ such that $s_n > (k + 1)R + 1$ for all $n \in \mathbb{N}$. We only work with the limit curve $\gamma$, we define the new sequence $\bar{\gamma}_n$ by
\[
\bar{\gamma}_n(s) = \gamma(s + s_n).
\]

For this sequence (and the constant sequence $r_n = r$), we apply again Assumption 2 to find a subsequence of $\bar{\gamma}_n$ which converges to some gradient flow line $\gamma^+$. Now we can apply an argument as in Lemma 2.1 in [CF09] to show that $\gamma^+$ is constant for positive $s$. For all $s$, the argument is similar, but requires a more careful choice of the sequence $(s_n)$. For the sake of completeness, we include the argument here.

Assume $\gamma^+$ to be non-constant and observe that if $s$ is positive, $\gamma^+$ is a gradient flow line of $F$ by our choice of $s_n$. Since $\gamma^+$ is a gradient flow line, there is some $s_0 > 0$, such that
\[
F(\gamma^+(0)) - F(\gamma^+(s_0)) = \epsilon
\]
for some $\epsilon > 0$. By $C^\infty_{loc}$-convergence of $\gamma_n$ to $x^+$, this implies for some $n_0 \in \mathbb{N}$ that
\[
F(\gamma_n(0)) - F(\gamma_n(s_0)) \geq \epsilon/2
\]
for all $n \geq n_0$. We now use the definition of $\gamma_n$ to find
\[
F(\gamma_n(s_n)) - F(\gamma(s_n + s_n)) \geq \epsilon/2.
\]
We now define a subsequence of $s_n$ starting at the above $n_0$, as
\[
n_k = \min\{n \mid n_k - n_{k-1} \geq s_0\}.
\]
If we choose \( k_0 > 2E(\gamma)/\epsilon \), we can now compute for the energy \( E(\gamma) \) of the gradient flow line \( \gamma \):

\[
E(\gamma) = \int_{-\infty}^{\infty} |\partial_s \gamma|^2 \, ds \\
\geq \sum_{k=0}^{k_0-1} \int_{s_{nk}}^{s_{nk}+s_0} |\partial_s \gamma|^2 \, ds \\
= -\sum_{k=0}^{k_0-1} \int_{s_{nk}}^{s_{nk}+s_0} \frac{\partial}{\partial s} F(\gamma(s)) \, ds \\
\geq \sum_{k=0}^{k_0-1} F(\gamma(s_{nk}) - F(\gamma(s_{nk} + s_0)) \\
\geq \frac{k_0 \epsilon}{2} > E(\gamma).
\]

We conclude that \( \gamma^+ \) is indeed constant and thus a critical point of \( F \). Now the assumption \( \gamma^+ \) implies that \( \gamma^+ = z^+ = \lim_{n \to \infty} \gamma(s_n) \in Z \). This shows that we can apply Lemma 3.6 and therefore \( \lim_{s \to \infty} \gamma(s) \in Z. \) This shows that \( \gamma \in \mathcal{M}_{[0,R]}(Z) \) and therefore \( \mathcal{M}_{[0,R]}(Z) \) is compact.

4. CUP-LENGTH ESTIMATES

In this section, we will prove Theorem 1.1 in the setting described in Section 3.1. Recall that we need to show that the number of critical points of the smooth function \( h \) is bounded below in terms of the cuplength of the critical submanifold \( Z \), which is defined as

\[
\text{cuplength}(Z) := \max\{k \in \mathbb{N} \mid \exists \alpha_1, \ldots, \alpha_k \in H^1(Z) \text{ such that } \alpha_1 \cup \ldots \cup \alpha_k \neq 0\}. \tag{4.1}
\]

Here, \( H^1(Z) \) denotes the cohomology in degree at least 1. Furthermore, in Section 4.2 we will prove Theorem 4.1, which is an analogous result using slightly different conditions on the functions \( F \) and replacing the energy estimates above by index constraints.

4.1. Norm constraints. In this subsection, we prove Theorem 1.1 as stated in the introduction. From now on, we fix the function \( F \) having a Morse-Bott critical submanifold \( Z \). For convenience, we assume again that \( F|_{Z}=0 \) and that \( F \) has positive spectral gap an therefore does not have other critical points with value 0. Moreover, let \( h \) be a smooth function such that

\[
\|h - F\| < \mathcal{G} \tag{4.2}
\]

such that \( \sup(h - F) \geq 0 \) and \( \inf(h - F) \leq 0 \) hold. Finally, in the choice of the function \( \beta_r \) in Convention 3.1 we choose the integer \( k := \text{cuplength}(Z) \). We assume that Assumption 1 and 2 hold. We recall that we need to show that the function \( h \) has at least cuplength\((Z) + 1 \) critical points with critical values in the interval \([-\|h - F\|, \|h - F\|]\).

PROOF. We follow the line of proof from [AM10]. We choose Morse functions \( f_1, \ldots, f_k : Z \to \mathbb{R} \) and extend them to a neighborhood \( U \) of \( Z \) in \( M \) defined as follows. Denote \( S = \|h - F\| \) and let \( U \) be the connected component of \( F^{-1}([-S, S]) \) containing \( Z. \) \( Z \) and \(-S \) are regular values of \( F \) as \( S \in (0, \mathcal{G}) \) and \( \mathcal{G} \) is a lower bound for the absolute value of a critical value of \( F \) outside \( Z \). This shows that \( U \) is a submanifold of \( M \) with smooth boundary and that
there are no other critical points of $F$ in $U \setminus Z$. Therefore, $U$ is homotopy equivalent to $Z$ and the cohomology of $U$ is isomorphic to the cohomology of $Z$.

We extend the functions $f_i$ from $Z$ to $U$, such that near $\partial U$, the functions are the negative distance functions from $\partial U$. As the boundary of $U$ is a smooth submanifold of $M$, the distance function from $\partial U$ is smooth in a small neighborhood of the boundary. Inside $U$, we do not specify the extensions of $f_i$, but require that the extended functions are Morse in $U$.

Finally we extend the functions $f_i$ to be functions on $M$. We keep the notation $f_1, \ldots, f_k$ for the extensions and consider from now on $f_i: M \to \mathbb{R}$. Now we choose Riemannian metrics $g_1, \ldots, g_k$ on $M$. Additionally, we consider a Morse function $f_*: Z \to \mathbb{R}$ and a Riemannian metric $g_*$ on $Z$.

For a fixed $R \geq 0$, the set
\[ \mathcal{M}_R(Z) = \{ \gamma \mid (R, \gamma) \in \mathcal{M}_{[0, R]}(Z) \} \]
(4.3)
is the zero-set of a Fredholm section in a Banach space bundle. The above proof of compactness of $\mathcal{M}_{[0, R]}(Z)$ applies unchanged to $\mathcal{M}_R(Z)$. Thus, we can choose a small abstract perturbation of the Fredholm section so that the zero-set $\widetilde{\mathcal{M}}_R(Z)$ of the perturbation is a compact, smooth manifold of finite dimension. Moreover, by the Morse-Bott assumption on $F$ the Fredholm section is already transverse to the zero-section for $R = 0$ and thus, by compactness, also for small $R \geq 0$. Therefore, we may assume that $\widetilde{\mathcal{M}}_R(Z) = \mathcal{M}_R(Z)$ for $R$ sufficiently close to $0$. Analogously, we can perturb the moduli space $\mathcal{M}_{[0, R]}(Z)$ to obtain a smooth compact manifold $\widetilde{\mathcal{M}}_{[0, R]}(Z)$. Again, we may assume that this perturbation is so that, with respect to the natural projection to $[0, R]$, the fibers over $0$ resp. $R$ of $\widetilde{\mathcal{M}}_{[0, R]}(Z)$ are $\mathcal{M}_0(Z)$ resp. $\mathcal{M}_R(Z)$. The space $\widetilde{\mathcal{M}}_R(Z)$ carries a natural evaluation map $\widetilde{ev}_R: \widetilde{\mathcal{M}}_R(Z) \to M^k$ defined by
\[ \widetilde{ev}_R(\gamma) = (\gamma(R), \gamma(2R), \ldots, \gamma(kR)) . \]
(4.4)
If $R = 0$, this is the diagonal embedding of $M$ into $M^k$. For critical points $x_i$ of $f_i$, and $x^\pm$ of $f_*$, consider the moduli space
\[ \widetilde{\mathcal{M}}(R, x_1, \ldots, x_k, x^\pm, x^\pm) := \left\{ (R, \gamma) \in \widetilde{\mathcal{M}}_R(Z) \mid \begin{array}{l}
\gamma(\infty) \in W^u(x^\pm), \gamma(\infty) \in W^s(x^\pm), \\
\widetilde{ev}_R(\gamma) \in W^s(x_1, f_1) \times \cdots \times W^s(x_k, f_k)
\end{array} \right\} \]
where $W^u$ resp. $W^s$ are unstable resp. stable manifolds. For generic choices of the Morse functions $f_i$, $f_*$ and the Riemannian metrics $g_i, g_*$, the moduli spaces $\widetilde{\mathcal{M}}(R, x_1, \ldots, x_k, x^\pm, x^\pm)$ are smooth manifolds. We now define the cohomology operation
\[ \theta_R: \text{CM}^*(f_1) \otimes \cdots \otimes \text{CM}^*(f_k) \otimes \text{CM}_*(f_*) \to \text{CM}_*(f_*) \]
\[ x_1 \otimes \cdots \otimes x_k \otimes x_* \mapsto \sum_{x^\pm \in \text{Crit}(f_*)} \#_2 \widetilde{\mathcal{M}}(R, x_1, \ldots, x_k, x^-, x^+, x^+, x^+) \cdot x^+, \]
(4.5)
For $R = 0$ and under the assumption that the functions $f_i$ are extended to $M$ as described above, the argument in [AM10], see also [Sch93], shows that this is a Morse-theoretic realization of the usual products in cohomology. Namely, under the identification of $H^*(f_i)$ with $H^*(U)$ and $HM(f_*)$ with the cohomology of $Z$, the operation $\theta_0$ agrees with the map
\[ \Theta: H^*(U) \otimes \cdots \otimes H^*(U) \otimes H_*(Z) \to H_*(Z) \]
\[ a_1 \otimes \cdots \otimes a_k \otimes b \mapsto (a_1 \cup \cdots \cup a_k) \cap b, \]
(4.6)
We denote by \( i: U \to M \) the inclusion map. A priori, as discussed in [AM10], this operation gives the cohomology operation
\[
a_1 \otimes \ldots \otimes a_k \otimes b \mapsto i^* (a_1 \cup \ldots \cup a_k) \cap b \tag{4.7}
\]
in the cohomology of \( M \), since the critical points \( x_i \) are critical points of functions \( f_i: M \to \mathbb{R} \) and therefore, the \( x_i \) represent cohomology classes in \( H^*(M) \). In our case, we choose the critical point \( x_i \) of \( f_i \) to be in \( U \). Then all stable manifolds of the critical points \( x_i \) are contained in \( U \), since the extensions of \( f_i \) is chosen such that the gradient points outwards at \( \partial U \) and we work with the negative gradient flow.

Using Lemma 3.5 we also see that the curves \( \gamma \) for \( (R, \gamma) \in \mathcal{M}_{[0, R]}(Z) \) are contained in \( U \). This shows that we indeed have \( \theta_0 = \Theta \) in \( H^*(U) \). But since \( U \) is homotopy equivalent to \( Z \), the cohomology map induced by the inclusion of \( Z \) into \( U \) is an isomorphism. Therefore, we may also consider \( \theta_0 \) to be an operation on the cohomology of \( Z \), i.e., we may write
\[
\theta_0 = \Theta: H^*(Z) \otimes \ldots \otimes H^*(Z) \otimes H_*(Z) \to H_*(Z) \tag{4.8}
\]
and consider all operations to be in the cohomology of \( Z \).

Now we study the map \( \theta_R \) for \( R \neq 0 \) and show that it is chain homotopy equivalent to \( \theta_0 \). To construct the chain homotopy operator, we recall that the perturbation \( \tilde{\mathcal{M}}_{[0, R]}(Z) \) has as fibers over 0 resp. \( R \) the spaces \( \mathcal{M}_0(Z) \) resp. \( \tilde{\mathcal{M}}_R(Z) \). Now also the perturbed moduli space \( \tilde{\mathcal{M}}_{[0, R]}(Z) \) carries an evaluation map extending \( \partial \nu_0 \) and \( \partial \nu_R \) and which is transverse to all products of stable manifolds. As in [AM10], this gives rise to a chain homotopy between \( \theta_0 \) and \( \theta_R \). Again Lemma 3.5 implies that all relevant curves remain inside \( U \). This shows that also for \( R \neq 0 \), the map \( \theta_R \) is a Morse-theoretic realization of the usual cup product in the cohomology ring of \( Z \).

Having now defined the cohomology operations, we use them to prove the existence of critical points by making particular choices of the functions \( f_i \) on \( Z \). We can choose the Morse functions \( f_1, \ldots, f_k: Z \to \mathbb{R} \) such that there exist critical point \( x_i \in \text{Crit} f_i \) with
\[
\left( W^s(x_1; f_1) \times W^s(x_2; f_2) \times \cdots \times W^s(x_k; f_k) \right) \cap \Delta_k \subset M^k \tag{4.9}
\]
where
\[
\Delta_k := \{(x, \ldots, x) \in M^k \mid x \in M\} \tag{4.10}
\]
is the diagonal. Furthermore, we choose critical points \( x_i \) of \( f_i \) such that the stable manifolds of the critical points have dimension lower than \( \dim M \). This holds for generic choices of Morse functions \( f_i \) and Riemannian metrics \( g_i \).

We now specify the value of \( k \) to be \( k = \text{cuplength}(Z) \) and choose the functions \( f_i \) such that
\begin{enumerate}
  \item for all \( x_i \in \text{Crit} f_i \cap U \) with non-zero Morse index, \( W^s(x_i; f_i) \cap \text{Crit}(h) = \emptyset \),
  \item for all \( n \in \mathbb{N} \), the evaluation map \( \partial \nu_n: \tilde{\mathcal{M}}(n, x_1, \ldots, x_k, x_+, x^-) \to M^k \) is transverse to the product of all stable manifolds of \( (f_i, g_i) \) and
  \item the evaluation maps at \( \pm \infty \) are transverse to all stable and unstable manifolds critical points of \( (f_*, g_*) \).
\end{enumerate}

Now we choose critical points \( x_i \) of \( f_i \) representing cohomology classes \( a_i \) with \( a_1 \cup \ldots \cup a_k \neq 0 \), which is possible by the definition of \( k = \text{cuplength}(Z) \). This choice of critical points implies that for every \( n \in \mathbb{N} \), the map \( \theta_n \) is non-zero as \( \theta_n \) is homotopy equivalent to \( \theta_0 \) which represents the above non-zero product.
By definition of $\theta_n$, this shows that there are critical point $x_n^+$ of $f_*$, possibly depending on $n$, such that
\[
\mathcal{M}(n, x_1, \ldots, x_k, x_n^+, x_n^-) \neq \emptyset.
\] (4.11)
Since the whole construction of the perturbed moduli spaces does not depend on the particular perturbation, we can conclude that also the unperturbed moduli space is non-empty. If this space were empty, then this would also be true for small perturbations. We now choose curves
\[
\gamma_n \in \mathcal{M}(n, x_1, \ldots, x_k, x_n^+, x_n^-)
\] (4.12)
and consider the sequences
\[
\gamma_{n,j}(s) := \gamma_n(s + nj)
\] (4.13)
for $j = 0, \ldots, k + 1$. By Assumption 2 these sequences converge in $C^\infty_{loc}(\mathbb{R}, M)$, up to taking a subsequence, to curves $\gamma^{(j)}$ which solve the following equations:
\[
\gamma^{(0)} \text{ solves } \frac{d}{ds} \gamma^{(0)}(s) + \nabla g \left( \beta_n^+(s) h(\gamma^{(0)}(s)) + (1 - \beta_n^+(s)) F(\gamma^{(0)}(s)) \right) = 0,
\] (4.14)
\[
\gamma^{(k+1)} \text{ solves } \frac{d}{ds} \gamma^{(k+1)}(s) + \nabla g \left( \beta_n^-(s) h(\gamma^{(k+1)}(s)) + (1 - \beta_n^-(s)) F(\gamma^{(0)}(s)) \right) = 0.
\] (4.15)
For $j = 1, \ldots, k$,
\[
\gamma^{(j)} \text{ solves } \frac{d}{ds} \gamma^{(j)}(s) + \nabla g h(\gamma^{(j)}(s)) = 0
\] (4.16)
and the endpoints at $\pm \infty$ of those $\gamma^{(j)}$ and also $\gamma^{(0)}(\infty)$ and $\gamma^{(k+1)}(-\infty)$ are critical points of $h$. Denote these critical points by $y_j^\pm := \gamma^{(j)}(\pm \infty)$. Furthermore, it follows from the definition of the sequences $\gamma_{n,j}$, that the values of $h$ at these critical points are ordered as follows:
\[
h(y_0^+) \geq h(y_1^-) \geq h(y_1^+) \geq h(y_2^-) \geq \ldots \geq h(y_k^-) \geq h(y_{k+1}^-).
\] (4.17)
To prove now that we have at least $k + 1$ different critical points of $h$, we will show that the inequalities between positive and negative ends of the trajectories $\gamma^{(j)}$ are strict, i.e., that the trajectories $\gamma^{(j)}$ are non-constant.
Assume this is not the case and recall that by the definition of the moduli space, we have
\[
ev_n(\gamma_n) \in W^s(x_1, f_1) \times \ldots \times W^s(x_k, f_k).
\] (4.18)
In particular, this implies $\gamma_n(nj) \in W^s(x_j, f_j)$ for all $n \in \mathbb{N}$. By definition of the evaluation map and the sequence $\gamma_{n,j}$, this shows that
\[
\gamma^{(j)}(0) = \lim_{n \to \infty} \gamma_{n,j}(n) \in W^s(x_j, f_j).
\] (4.19)
The closure of a stable manifold is a union of stable manifolds of index at least the index of the critical point $x_j$. As we assume that $\gamma^{(j)}$ is constant, thus a critical point of $h$, this contradicts condition (1) in the choice of the Morse function $f_j$. To see why this is all in $U$, recall from Lemma 3.3 that $|F(\gamma_n(s))|$ is bounded by $S$ for all $s \in \mathbb{R}$. Then also $F(\gamma^{(j)}(0)) \in I = [-S, S]$ by continuity of $F$ and therefore $\gamma^{(j)}(0) \in F^{-1}(I) = U$.
Thus none of the $\gamma^{(j)}$ is constant and the corresponding inequalities in (4.17) are strict.
This shows that indeed the point $\tilde{y}_j = y_j^-$ for $j = 1, \ldots, k$ and $\tilde{y}_{k+1} = y_k^+$ are different critical points of $h$.
To see why the critical values are in the claimed interval $[-S, S]$, it suffices by (4.17) to look at $h(y_0^+)$ and $h(y_{k+1}^-)$. Using the definition of these critical points and the sequence $(\gamma_n)$, the required bounds are easy to see using Lemma 3.4. This proves the Theorem. □
4.2. **Index constraints.** In this section, we will show an analog of Theorem 1.1 removing the bounds on \( h \) in terms of \( F \). Instead, we impose constraints on the indices of critical points of \( F \). These index constraints are sufficient to provide compactness of moduli spaces. Then the above proof applies and gives a lower bound on the number of critical points of \( h \).

Let \( F \) be a Morse-Bott function and assume that \( F \) has a critical submanifold \( Z := F^{-1}(0) \) and no critical points outside of \( Z \) with index in the interval \((\text{ind } Z, \text{ind } Z + \text{dim } Z)\). Now let \( h \) be any smooth function on \( M \). Then we define the moduli spaces \( \mathcal{M} \) and \( \mathcal{M}_{[0,R]}(Z) \) and the evaluation maps \( \text{ev}_r : \mathcal{M} \to M^k \) as above. In this setting, we find the following analog of Theorem 1.1:

**Theorem 4.1.** Let \( F \) be as above with a Morse-Bott critical submanifold \( Z \) and let \( U \) be an open neighborhood of \( Z \) such that \( U \) contains the image of all evaluation maps \( \text{ev}_r \) for all \( r \geq 0 \). Then the number of critical points of any smooth function \( h \) is bounded below by \( \text{cuplength}(U) + 1 \).

Note that in this case, the cohomology of \( U \) need not be equal to the cohomology of \( Z \). We cannot use the function \( F \) to define \( U \), since we do not impose any restrictions on critical values of \( F \), but only on the indices of critical points.

In this setting, we do not restrict the set of functions \( h \) and therefore we cannot use the bounds on \( \|h - F\| \) to prove compactness of moduli spaces. Thus as a first step, we need to show the compactness result from Theorem 3.7 for this setting using only the index condition.

**Proposition 4.2.** In the above setting, the moduli space \( \mathcal{M}_{[0,R]}(Z) \) is compact.

**Proof of Theorem 4.1**. The proof of Theorem 1.1 applies also in this case using Proposition 4.2 instead of Theorem 3.7. As above, we can choose the Morse functions \( f_i \) such that the cohomology operations are the product in \( U \) and therefore give the lower bound of \( \text{cuplength}(U) + 1 \). The last step of the proof of Theorem 1.1 showing that this is the cuplength of \( Z \), is not needed (and in general not true) in this setting.

Now it remains to prove Proposition 4.2.

**Proof.** For \( \mathcal{M}_{[0,R]}(Z) \) to be compact, we need to show that trajectories in this moduli space cannot break at other critical points of \( F \). It suffices to show that for no critical point \( p \) of \( F \), there can be a gradient flow line from \( p \) to \( Z \) or from \( Z \) to \( p \).

Assume that such a trajectory \( \gamma \) exists. Using an auxiliary Morse-function \( f \) on \( Z \), we may assume that \( \gamma(\infty) \) is the minimum of \( f \), which has index \( \text{ind } Z \). Then the space of trajectories has positive dimension. This dimension is given by the index difference \( \text{ind } p - \text{ind } Z \), i.e., we necessarily have \( \text{ind } p > \text{ind } Z \). If we do not consider the minimum of \( f \), then the dimension of the moduli space is at most larger, i.e., the lower bound on the index of \( p \) is larger. Therefore, the minimum gives the lowest possible bound on \( \text{ind } p \).

Similarly, if \( \gamma \) is a trajectory from \( Z \) to \( p \), we may assume that \( \gamma(-\infty) \) is the maximum of the Morse function \( f \), which has index \( \text{ind } Z + \text{dim } Z \). Again we study the space of trajectories from this maximum to \( p \). Its dimension is given by \( \text{ind } Z + \text{dim } Z - \text{ind } p \). For this to be positive, we find the inequality \( \text{ind } p < \text{ind } Z + \text{dim } Z \). As before, the maximum gives the highest upper bound on the index of \( p \).

Together, we have found that for a trajectory from \( Z \) to a critical point \( p \) to exist, the point \( p \) has to have index in the interval \((\text{ind } Z, \text{ind } Z + \text{dim } Z)\). Such critical points of \( F \) do not exist by our assumed index condition on \( F \).
5. Applications to Floer theory

5.1. Fixed points of Hamiltonian diffeomorphisms. In this section, we apply Theorem 1.1 and also Theorem 4.1 to the action functional of Hamiltonian dynamics. The resulting bound on the number of critical points then yields a bound on the number of fixed points of Hamiltonian diffeomorphisms in terms of the cohomology of the underlying symplectic manifold. This reproduces previous proofs of the Arnold conjecture, see, e.g., [Sch98, LO96].

The proofs in the Floer case only differ in some details from the Morse theoretic proof above and we will point out the necessary changes in the proofs. The main differences are that different critical points can represent the same periodic orbit and that compactness can not only fail due to breaking trajectories as in the Morse case but also due to bubbling. Both issues can be excluded using the bounds on the Hamiltonian given by the condition on $h$ in Theorem 1.1.

We first describe the setting and identify the functionals which will take the role of the functions $F$ and $h$ from the Morse case.

Let $(W, \omega)$ be a closed, rational symplectic manifold, i.e. $\omega|_{\pi_2} = \lambda \mathbb{Z}$ for some positive $\lambda$. If $\omega|_{\pi_2} = 0$, we set $\lambda = \infty$. For this setting, we can now apply the Morse theoretic argument above to the Hamiltonian Floer action functional.

To do this, let $M = \Lambda W$ be the covering space of the space of contractible loops in $W$ and

$$F = A(\bar{x}) = -\int_{D^2} u^* \omega,$$

where $\bar{x}$ is a loop $x$ together with an equivalence class $[u]$ of cappings. Two cappings are equivalent, if the two discs have the same symplectic area.

For this functional, the critical points are the constant loops with all different equivalence classes of cappings. We define again

$$Z = F^{-1}(0) = W$$

(5.2)

to be the set of constant loops with constant cappings and identify this with the symplectic manifold $W$. Furthermore, for this functional, we find $\mathcal{S} = \lambda$.

To define the function $h$, we define for a Hamiltonian $H : S^1 \times W \to \mathbb{R}$ the action

$$A_H(\bar{x}) = -\int_{D^2} u^* \omega + \int_{S^1} H(x) \, dt.$$

(5.3)

Critical points of this functional are one-periodic orbits of the Hamiltonian flow of $H$, which can be identified with fixed points of the time one map $\varphi_H$. For those fixed points, we find the following analog of Theorem 1.1 in the current setting:

**Theorem 5.1** (Existence of fixed points). If $H$ has Hofer norm $\|H\| < \mathcal{S}$, then $H$ has at least $\text{cuplength}(W) + 1$ one-periodic orbits.

If $\lambda = \infty$, this lower bound on the number of one-periodic orbits holds for all Hamiltonians, as $W$ is compact and therefore all Hamiltonians have finite Hofer norm, i.e. $\|H\| < \mathcal{S} = \lambda$ is automatically satisfied.

**Proof.** Using the Hamiltonian Floer equation instead of the gradient equation, we still define the moduli spaces $\mathcal{M}$ and $\mathcal{M}_{[0,R]}(Z)$ as in the Morse case. For $r = 0$, the function $G_{r,s}$ is simply the unperturbed action $\bar{F}$ and all Floer trajectories are constant. The estimates in Lemma 3.4 also work in this case and give bounds in terms of the Hofer norm of the Hamiltonian $H$. As remarked above, Assumption 11 for the Floer case is a consequence of...
the isoperimetric inequality. Assumption 2 also holds for all Floer theoretic situations as this provides compactness in $C^\infty_{loc}$ convergence of sequences of trajectories. The only possibility of this compactness to fail is bubbling. Now the definition of $\lambda$ shows that every bubble must have at least energy $\lambda$ and therefore, there is not enough energy for bubbling to occur by (3.19). This completes the proof of Assumption 2 in this setting.

For the evaluation maps and the Morse functions $f_i$, however, we need to change the definitions. Namely, for the definition of the evaluation maps, note that the elements in $\mathcal{M}_{[0,R]}(Z)$ can now be considered not only as paths in the loop space, but also as cylinders in the symplectic manifold $W$, i.e., as maps $\gamma: S^1 \times \mathbb{R} \to W$, where $S^1 = \mathbb{R}/\mathbb{Z}$.

Then we define the evaluation map by

$$ev_r(\gamma) = (\gamma(0,r), \gamma(0,2r), \ldots, \gamma(0,kr)), \quad (5.4)$$

as a map $ev_r: \mathcal{M}_{[0,R]}(Z) \to W^k$.

With this definition, it suffices now to define the Morse functions $f_i$ to be functions on $W$ and we can define the moduli spaces $\mathcal{M}(r, x_1, \ldots, x_k, x^-_s, x^+_s)$ as before. In particular, we do not need to consider a neighborhood $U$ of the critical submanifold $Z = W$ as all evaluation maps already take values only in $W^k$ and therefore define the cup product in the cohomology of $W$.

To show why the moduli space $\mathcal{M}_{[0,R]}(Z)$ is compact, we need to exclude breaking of Floer trajectories at critical points of $A$ outside $Z$ and bubbling.

Breaking is excluded by the same argument as in the Morse case. To exclude bubbling, we recall the energy bound from (3.19). This shows that the energy is bounded above by $\|h - F\| = \mathcal{G} - \epsilon$, where $\mathcal{G} - \|H\| > \epsilon$. If we now consider a sequence of trajectories in the proof, we see that also the broken trajectory in the limit has energy at most $\mathcal{G} - \epsilon < \lambda$.

Now the definition of $\lambda$ shows that every bubble must have at least energy $\lambda$ and therefore there is not enough energy for bubbling to occur. This completes the proof of compactness for $\mathcal{M}_{[0,R]}(Z)$ for the action functional of Hamiltonian dynamics.

From here on, the proof of Theorem 1.1 goes through for the Floer case. As critical points of the functions $f_i$ represent cohomology classes of $W$, the proof gives the desired bound of at least cuplength($W$) + 1 critical points of the actions functional with action value in the interval $(-\lambda, \lambda)$.

To see why these critical points actually are different periodic orbits of $H$, i.e., that no periodic orbit is found twice with different cappings, we use the energy bound discussed above. Again the bound on $\|H\|$ shows that a (broken) Floer trajectory cannot connect a periodic orbit to itself with a different capping.

This shows that we indeed have found cuplength($W$) + 1 different one-periodic orbits. □

Also Theorem 4.1 can be applied to this setting, reproducing the lower bound on the number of periodic orbits given in [LO96].

**Theorem 5.2.** Let $(W^{2n}, \omega)$ be a negative monotone symplectic manifold with minimal Chern number at least $n$, i.e., $c_1|_{\pi_2(W)} = \lambda \omega|_{\pi_2(W)}$ and the positive generator $N$ of $c_1|_{\pi_2(W)}$ satisfies $N \geq n$. Then the Hamiltonian diffeomorphism generated by any Hamiltonian $H$ on $M$ has at least cuplength($W$) + 1 fixed points.

**Proof.** The conditions of this theorem are set such that the proof of Theorem 4.1 applies with the same modifications as above in the proof using Theorem 1.1 to prove Theorem 5.1.

The critical submanifold $Z$ is again taken to be the constant loops with constant capping. The dimension of this critical manifold is the dimension of the symplectic manifold $W$. To
apply Theorem 4.1, we therefore need to show that there are no critical points of the action \( A \) with index in the interval \((\text{ind } Z, \text{ind } Z + \dim Z)\).

Critical points of \( A \) are exactly the constant loops in \( W \) with all different cappings. Let \((x, [v])\) be a capped periodic orbit with \([v]\) being an equivalence class of cappings. The same periodic orbit \( x \) with a different capping can be written as \((x, [v] \# A)\) for some \( A \in \pi_2(M) \). Then the relation between the indices is

\[
\mu(x, [v] \# A) = (x, [v]) + 2c_1(A). \tag{5.5}
\]

Thus the index changes by at least \(2N \geq 2n\) and there are indeed no one-periodic orbits with index in the interval \( (0, 2n) \).

As before, we also need to exclude bubbling to achieve compactness of \( M_{[0,R]}(Z) \). If there were bubbling, the sphere would be \( J \)-holomorphic and therefore had positive symplectic area. Then the assumption that \( W \) is negative monotone implies that the bubble had negative index and thus would reduce the index of the Floer trajectory by at least \(2N\), as this is the lowest index of a sphere in \( W \). Now our assumption guarantees that all considered Floer trajectories have index less than \(2n\) and therefore there cannot be bubbling on all relevant trajectories.

From here on, we can continue the proof exactly as in the proof of Theorem 5.1 and find the desired lower bound on the number of fixed points of Hamiltonian diffeomorphisms. Again we need to show that no two of the critical points of the functional represent the same periodic orbits. In the proof of Theorem 5.1 this is done by energy estimates, which in this setting are replaced by an index argument. Two critical points representing the same periodic orbit would have two different cappings and therefore index difference at least \(2N\). As above, the total index of the connecting trajectory is less than \(2N\) and therefore, no trajectory can connect a periodic orbit to itself with a different capping. \(\square\)

5.2. Hamiltonian chords of a Lagrangian. As for periodic orbits, we can also apply Theorem 1.1 to find Hamiltonian chords of a Lagrangian submanifold to produce estimates similar to those in [Che98, Flo89, Hof88, Liu05].

For this setting, let \((W, \omega)\) be a symplectic manifold and \(L \subset W\) a Lagrangian. Assume that \(\omega|_{\pi_2(W,L)}\) is rational with rationality constant \(\lambda \in (0, \infty)\).

Then we define \( M \) as the set \( P_0(W,L) \) of paths \( x: [0,1] \to W \) with \( x(0) \) and \( x(1) \) in \( L \), which are contractible to a path in \( L \) and consider the action functional

\[
F = A(\bar{x}) = -\int_{D^2} u^* \omega. \tag{5.6}
\]

Here, \( \bar{x} \) is a path \( x \) together with an equivalence class \([u]\) of discs bounded by the path \( x \) and \( L \). Analogously to above, two cappings are equivalent if the two discs have the same symplectic area.

For this functional, the critical points are the constant paths on \( L \) with all different equivalence classes of cappings. We define again

\[
Z = F^{-1}(0) = L \tag{5.7}
\]

to be the set of critical points with constant cappings and identify this with the Lagrangian \( L \). Furthermore, for this functional, we find \( \mathcal{G} = \lambda \).

The functional \( h \) is defined using a Hamiltonian \( H: [0,1] \times W \to \mathbb{R} \). Namely, we take

\[
A_H(\bar{x}) = -\int_{D^2} u^* \omega + \int_{[0,1]} H \, dt. \tag{5.8}
\]
Critical points of this functional are Hamiltonian chords of $L$ together with an equivalence class of cappings.

The above setting is again analogous to the situation of Theorem 1.1 and we find the following

**Theorem 5.3 (Existence of Hamiltonian chords).** If $\|H\| < \mathcal{S}$, where $\| \cdot \|$ is the Hofer norm of $H$, then the Lagrangian $L$ has at least cuplength($L$) + 1 Hamiltonian chords.

**Proof.** The proof follows again the proof of Theorem 1.1 with similar modifications as in the case of fixed points of Hamiltonian diffeomorphisms.

As the functional is analogous to the one for periodic orbits of $H$, we again find that $\|h - F\|$ is the Hofer norm of $H$. The evaluation map is defined by evaluating a path $x$ at time 0 and by definition of $P_0(W, L)$, we have $x(0) \in L$. Therefore, we can choose the $f_1$ to be Morse functions on $L$ and find the cohomology operations to be the cup product in $L$ without need of considering the neighborhood $U$ from the Morse case in Section 4. With these small modifications, the proof of Theorem 1.1 applies also in this case and gives the desired lower bound for the number of critical points of $A_H$ in terms of the cuplength of $L$.

We still need to show that these critical points correspond indeed to different Hamiltonian chords. Two different critical points of the action functional can give rise to the same Reeb chord with a different capping, i.e., a different homotopy from the chord to a path in the Lagrangian $L$.

Similar to the case of periodic orbits, the requirement on the symplectic form and energy bounds in terms of $\|H\|$ guarantee that the broken trajectory does not have sufficient energy to connect a Hamiltonian chord to the same chord with a different capping. Namely, these two critical points of the action functional have action values that differ by at least $\lambda = \mathcal{S}$. If two Hamiltonian chords are connected by a (broken) Floer trajectory $u$, the energy $E(u)$ is equal to the difference in action value and bounded above by $\|H\| < \mathcal{S}$. Therefore, we indeed find the desired number of different Hamiltonian chords.

\[\square\]

5.3. **Translated points.** Next we apply Theorem 1.1 to translated points, a notion introduced by Sandon [San11]. Let $(\Sigma, \alpha)$ be a closed contact manifold. Let $\varphi : \Sigma \to \Sigma$ be a contactomorphism which is contact isotopic to the identity. We recall that a point $q \in \Sigma$ is a translated point with time-shift $\eta \in \mathbb{R}$

\[
\begin{align*}
\varphi(q) &= \theta^\eta(q) \\
\rho(q) &= 1.
\end{align*}
\]

We point out that the time-shift is not unique if $q$ lies on a closed Reeb orbit. The unperturbed has as critical points precisely the contractible Reeb orbits, see section 2.3 and thus its spectral gap is

\[\mathcal{S} = \text{minimal period of a contractible Reeb orbits.}\] (5.9)

Using the notation $\varphi^* \alpha = \rho \alpha$, where $\rho : \Sigma \to \mathbb{R}_{>0}$, a certain cut-off of the Hamiltonian diffeomorphisms $\phi : S\Sigma \to S\Sigma$, $\phi(q, r) := (\varphi(q), \frac{r}{\rho(q)})$ can be used to perturb the Rabinowitz action functional $F = A$ to obtain $h = A_\varphi$, see [AFMR], where it is proved in Lemma 3.5 that the critical points of $h$ correspond to translated points of $\varphi$. The perturbation is supported
inside $\Sigma \times \left[e^{-\kappa(\varphi)}, e^{\kappa(\varphi)}\right] \subset S\Sigma$ where

$$\kappa(\varphi) := \max_{t \in [0,1]} \left| \max_{x \in \Sigma} \frac{\rho_s(x)}{\rho'_s(x)} ds \right|.$$  

(5.10)

This implies that

$$\|h - F\| \leq e^{\kappa(\varphi)}\|H\|_H$$  

(5.11)

where $H : S^1 \times \Sigma \to \mathbb{R}$ is any contact Hamiltonian such that the induced contact isotopy $\psi_t$ satisfies $\varphi = \psi_1$.

**Theorem 5.4.** If $e^{\kappa(\varphi)}\|H\|_H < \mathcal{G}$ then there are at least $\cuplength(\Sigma) + 1$ many distinct translated points with time-shifts in the interval $[-e^{\kappa(\varphi)}\|H\|_H, e^{\kappa(\varphi)}\|H\|_H]$.

**Proof.** Compared to the proof given in [AM10], which was the idea behind the proof of Theorem 1.1 above, the only change is that we now work in the symplectization instead of assuming that the contact manifold $(\Sigma, \alpha)$ has a exact symplectic filling. The only new point to the proof is that in Example 1.1 above, the only change is that we now work in the symplectization of $\Sigma$ since the symplectization $S\Sigma$ has a negative end. The assumption $e^{\kappa(\varphi)}\|H\|_H < \mathcal{G}$ together with Stokes’ theorem precisely implies that Theorem 5.4 in [AFM13] is applicable. We conclude that there exists $\epsilon > 0$ such that for each element in $\mathcal{M}$ the $\mathbb{R}_{>0}$ component of its image in the symplectization $S\Sigma = \Sigma \times \mathbb{R}_{>0}$ is bounded by $\epsilon$. Thus, all elements in $\mathcal{M}$ stay inside a compact subset of $S\Sigma$. Now as in [AM10] and the Hamiltonian Floer case above, we define the evaluation maps by evaluating the trajectories at time 0 and apply the proof of Theorem 1.1 to obtain the lower bound for the number of translated points.

Translated points are a special case of the notion of leafwise intersection points introduced by Moser in [Mos78]. In [AM10], a lower bound on the number of leafwise intersections in terms of a relative cup length for Liouville fillable contact manifolds has been proved. This result can be improved as follows. If we consider a Hamiltonian diffeomorphism $\phi$ with support inside $\Sigma \times \left[e^{-\kappa}, e^{\kappa}\right] \subset S\Sigma$ for some $\kappa > 0$, then if $e^{\kappa(\varphi)}\|H\|_H < \mathcal{G}$ there exist at least $\cuplength(\Sigma) + 1$ leafwise intersections. Now $H : S^1 \times S\Sigma \to \mathbb{R}$ is any Hamiltonian function generating $\phi$. If the contact manifold $\Sigma$ is Liouville fillable then $\phi$ extends by the identity to the filling. We point out that in this situation $\cuplength(\Sigma)$ is always at least as big the relative cup length used in [AM10] and thus our result here reproduces and in general improves the bound given there.

### 5.4. Solutions to perturbed Dirac-type equations

In this section, we apply Theorem 1.1 to the hyperkähler Floer homology developed by Hohloch, Noetzel and Salamon in [HNS09] and reprove the cuplength estimate by Ginzburg and the second author in [GH12].

Recall that we take the “time”-manifold $X$ to be either $T^3$ or $S^3$ equipped with a volume form $\mu$ and a special choice of a global frame. If $X = T^3$, we choose the global frame $v_1, v_2, v_3$ on $X$ by $\hat{e}_i$ on $T^3$ for angular coordinates $t_1, t_2, t_3$ and the volume form be $\mu = dt_1 \wedge dt_2 \wedge dt_3$. If $X = S^3$, we identify $S^3$ with the unit quaternions and define $v_1(x) = ix, v_2(x) = jx$ and $v_3(x) = kx$ and choose the volume form $\mu$ to be the (probability) Haar measure on $S^3$.

Let $Y$ be a compact, flat hyperkähler manifold with almost complex structures $I, J, K$. This implies that $Y$ is some compact quotient of a hyperkähler vector space, i.e., a torus or a quotient of a torus by a finite group. On $Y$, we define symplectic forms $\omega_i$ by choosing a Riemannian metric and choosing $\omega_1(\cdot, \cdot) = \langle \cdot, I\cdot \rangle$, $\omega_2(\cdot, \cdot) = \langle \cdot, J\cdot \rangle$ and $\omega_3(\cdot, \cdot) = \langle \cdot, K\cdot \rangle$. 

In this setting, we consider the manifold $M$ to be the space of smooth maps from $f: X \to Y$ and define the action functional
\[
A(f) = -\sum_{l=1}^{3} \int_{[0,1] \times X} \hat{f}^* \omega_l \wedge i_{v_l} \mu,
\]
where $\hat{f}: [0,1] \times X \to Y$ is a homotopy from $f$ to a constant map. As the covering space of $Y$ is contractible, this functional is independent of the choice of $\hat{f}$ and only depends on the map $f$. This action functional is the one defined in [GH12] and agrees with the formulas given in [HNS09]. The differential of the action functional $A$ at $f$ is
\[
(dA)_f(w) = \sum_{l=1}^{3} \int_{M} \omega_l(L_{v_l} f, w) \mu
\]
and the $L^2$-gradient of $A$ at $f$ is given by $\hat{\nabla} f := IL_{v_1} f + JL_{v_2} f + KL_{v_3} f$. Critical points of $A$ are solutions to the Dirac-type equation
\[
\hat{\nabla} f = 0.
\]
The functional $A$ takes the role of the function $F$ from the Morse case. All critical points of $F$ are constant, i.e., we again identify the critical submanifold $Z$ with the manifold $Y$ and find that the spectral gap is $\mathcal{S} = \infty$.

Just as in the case of classical Hamiltonian dynamics described above, we consider a Hamiltonian perturbation for a Hamiltonian $H: X \times Y \to \mathbb{R}$. The function $h$ is then given by the functional
\[
h(f) := A_H(f) = -\sum_{l=1}^{3} \int_{[0,1] \times X} F^* \omega_l \wedge i_{v_l} \mu - \int_{X} H(f) \mu.
\]
For this functional, critical points are solutions to the equation
\[
\hat{\nabla} f = \nabla H(f).
\]

As in the case of classical Hamiltonian dynamics discussed above, Theorem 1.1 implies the following

**Theorem 5.5.** For all Hamiltonians $H$, the number of critical points of $A_H$ is bounded below by cuplength($Y$) + 1.

**Proof.** In this setting and with these choices of global frames, our Assumptions 1 and 2 are established in [HNS09], where also the hyperkähler Floer homology is defined. To define the evaluation maps, we evaluate the trajectories at $x = 1$ if $X = S^3$ or, if $X = T^3$, at $x = (0,0,0)$. Then the evaluation maps take values in $Y$ and we can choose the Morse functions $f_i$ to be functions on $Y$. As in the case of Hamiltonian Floer homology discussed above, the proof of Theorem 1.1 gives the lower bound for the number of critical points of $A_H$ to be cuplength($Y$) + 1 for all Hamiltonians $H$. \(\square\)

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