ASYMPTOTIC DYNAMICS OF HERMITIAN RICCATI DIFFERENCE EQUATIONS

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(dedicated to Prof. Sze-Bi Hsu in appreciation of his inspiring ideas)

Abstract. In this paper, we consider the hermitian Riccati difference equations. Analogous to a Riccati differential equation, there is a connection between a Riccati difference equation and its associated linear difference equation. Based on the linear difference equation, we can obtain an explicit representation for the solution of the Riccati difference equation and define the extended solution. Further, we can characterize the asymptotic state of the extended solution and the rate of convergence. Constant equilibrium solutions, periodic solutions and closed limit cycles are exhibited in the investigation of asymptotic behavior of the hermitian Riccati difference equations.

1. Introduction. Riccati difference equations often arise in many applications such as optimal control problems, computation of polynomial factorization..., etc (see [5] and [10]). Denote

\[ M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \]

with constant submatrices \( M_{11} \in \mathbb{C}^{m \times m} \), \( M_{12} \in \mathbb{C}^{m \times n} \), \( M_{21} \in \mathbb{C}^{n \times m} \), and \( M_{22} \in \mathbb{C}^{n \times n} \). Consider the Riccati difference equation

\[
\begin{align*}
W(k+1) &= -M_{21} - M_{22}W(k)[I - M_{12}W(k)]^{-1}M_{11}, \\
W(0) &= W_0.
\end{align*}
\]  

(1.1)

If \( M_{11} \) is invertible, we associate with \( M \) the matrix

\[
T(=T(M)) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},
\]

\[
:= \begin{bmatrix} I_m & 0 \\ -M_{21} & I_n \end{bmatrix} \begin{bmatrix} M_{11}^{-1} & 0 \\ 0 & -M_{22} \end{bmatrix} \begin{bmatrix} I_m & -M_{12} \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} M_{11}^{-1} & -M_{11}^{-1}M_{12} \\ -M_{21}M_{11}^{-1} & -M_{22} + M_{21}M_{11}^{-1}M_{12} \end{bmatrix}.
\]

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Similar to a Riccati differential equation, there is a connection between a Riccati difference equation and a linear difference equation. Introduce the linear difference equation
\[ Z(k + 1) = T Z(k), \quad Z(0) = \begin{bmatrix} I_m \\ W_0 \end{bmatrix}, \quad k \geq 0, \quad (1.2) \]
where \( Z(k) = \begin{bmatrix} X(k) \\ Y(k) \end{bmatrix} \) with \( X(k) \in \mathbb{C}^{m \times m} \) and \( Y(k) \in \mathbb{C}^{n \times m} \). Based on the following lemma, we can obtain an explicit solution formula of (1.1a) and (1.1b) in terms of the solution of (1.2). The linear difference equation (1.2) and the Riccati difference equation (1.1a) and (1.1b) are equivalent in the following sense:

**Lemma 1.1.** ([6]) (i) If \( \{ Z(k) \}_{0 \leq k \leq \nu} \) satisfies (1.2) and \( \det X(k) \neq 0 \) for \( 0 \leq k \leq \nu \), then \( \{ W(k) \}_{0 \leq k \leq \nu} \) with \( W(k) := Y(k)X(k)^{-1} \) satisfies (1.1a), in particular \( W(0) = W_0 \).

(ii) If \( \{ W(k) \}_{0 \leq k \leq \nu} \) satisfies (1.1a) and (1.1b), then the sequence \( \begin{bmatrix} X(k) \\ Y(k) \end{bmatrix} \) defined by
\[ X(k + 1) = M_{11}^{-1}X(k) - M_{11}^{-1}M_{12}W(k)X(k), \quad X(0) = I_m, \]
and
\[ Y(k) = W(k)X(k), \quad 0 \leq k \leq \nu, \]
\[ \text{satisfies the linear difference equation (1.2).} \]

In particular, we are interested in the Riccati difference equation with
\[ M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -AH \end{bmatrix}, \]
where \( A \) and \( S, Q \) with \( S = S^H \) and \( Q = Q^H \) are \( n \times n \) matrices. Then the corresponding Riccati difference equation (1.1a) is called the hermitian Riccati difference equation and (1.1a) and (1.1b) can be written as
\[ W(k + 1) = Q + AHW(k)[I + SW(k)]^{-1}A, \quad (1.3a) \]
\[ W(0) = W_0. \quad (1.3b) \]
Let \( A \) be assumed to be invertible. Then the associated operator \( T \) in (1.2) is
\[ T = \begin{bmatrix} A^{-1} & A^{-1}S \\ QA^{-1} & AH + QA^{-1}S \end{bmatrix}, \quad (1.4) \]
which is a symplectic matrix.

In terms of the solution \( Z(k) = [X(k)^T, Y(k)^T]^T \) of the linear difference equation (1.2), \( W(k) \) can be represented by
\[ W(k) = Y(k)X(k)^{-1} \quad \text{if } \det(X(k)) \neq 0. \]
\( W(k) \) blows up if \( \det(X(k)) = 0 \). Notice that if \( T \) is invertible, the linear equation (1.2) can generate the sequence \( \{ \text{Im} Z(k) \}_{k \geq 0} \) of points on the Grassmann manifold \( G^n(\mathbb{C}^{2n}) \) where \( \text{Im} Z \) is the column space spanned by the matrix \( Z \) and
\[ G^n(\mathbb{C}^{2n}) = \left\{ \text{Im} \left( \begin{bmatrix} A \\ B \end{bmatrix} \right) \mid A, B \in \mathbb{C}^{n \times n} \text{ and } \text{rank} \left( \begin{bmatrix} A \\ B \end{bmatrix} \right) = n \right\}. \]
Analogous to the continuous-time case (see [1]), the sequence \( \{ \text{Im} Z(k) \}_{k \geq 0} \) can be interpreted as the solution of the extended Riccati equation on \( G^n(\mathbb{C}^{2n}) \) which is defined for any initial value on \( G^n(\mathbb{C}^{2n}) \). In the discrete-time case, the extended Riccati equation and the linear difference equation (1.2) are equivalent. Therefore, we can define the sequence
\[
\{ W(k) \triangleq Y(k)X(k)^{-1} \}_{k \in K},
\]
where \( K \triangleq \{ k \in \{ 0 \} \cup \mathbb{N} \mid \det(X(k)) \neq 0 \} \), (1.5)
as the extended solution of (1.3a) and (1.3b) in this paper. Furthermore, the description of asymptotic behavior of the extended solution (1.5) draws our attention.

There have been a great amount of works on the study of Riccati equations. Some sufficient conditions of the existence of solutions for the corresponding hermitian algebraic Riccati difference equation
\[
W = Q + A^HW[I + SW]^{-1}A
\]
are given in [10], [12] and [14]. The existence and convergence properties of the solutions for symmetric Riccati difference equations are studied in [2], [3] and [4]. There are investigations of the convergence and dynamics for the hermitian Riccati difference equations in [6] and [7].

The operator \( T \) in (1.4) is a symplectic matrix. Inspired by the structure of the symplectic Jordan canonical form for a symplectic matrix, which has been deeply studied in [13], we focus on four elementary cases of \( T \) and give the characterization of asymptotic behavior of the extended solution as follows.

1. If \( T \) has only eigenvalue \( \lambda \) and \( \lambda^{-1} \) with \( |\lambda| > 1 \), the extended solution \( \{ W(k) \}_{k \in K} \) tends to an equilibrium solution at an exponential rate.
2. If \( T \) has one unimodular eigenvalue which has one Jordan block with size \( 2n \), the extended solution \( \{ W(k) \}_{k \in K} \) tends to an equilibrium solution at a polynomial rate.
3. If \( T \) has one unimodular eigenvalue but it has two Jordan blocks of size \( 2n_1 + 1 \) and \( 2n_2 + 1 \) \( (n_1 + n_2 + 1 = n) \), respectively, the extended solution \( \{ W(k) \}_{k \in K} \) tends to an equilibrium solution at a polynomial rate.
4. If \( T \) has two distinct unimodular eigenvalues \( \gamma \) and \( \delta \) with Jordan blocks of size \( 2n_1 + 1 \) and \( 2n_2 + 1 \) \( (n_1 + n_2 + 1 = n) \), respectively, either the extended solution \( \{ W(k) \}_{k \in K} \) asymptotically approaches a periodic solution or its trajectory is dense in a closed orbit which forms a limit cycle of \( \{ W(k) \}_{k \in K} \).

The paper is organized as follows. In section 2, the main theorem is presented. There are properties of symplectic matrices and preliminaries for the proof of the main theorem in section 3. We give a proof of the main theorem in section 4.

2. Main Theorem. Let \( S \) be the matrix such that
\[
J = S^{-1}TS,
\]
where \( J \) is the Jordan canonical form of \( T \) in (1.4). Define the \( l \times l \) nilpotent matrix
\[
N_l \triangleq \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 1 & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix},
\]
and let \( N_l(\lambda) \triangleq \lambda I_l + N_l \) be the Jordan block of the eigenvalue \( \lambda \) with size \( l \). In this paper, we assume that \( J \) is one of the following
elementary forms:

Case 1:
\[
J = \begin{bmatrix} N_n(\lambda) & 0 \\ 0 & N_n(1/\lambda) \end{bmatrix}
\]
with \(|\lambda| > 1; (2.2)

Case 2:
\[
J = N_{2n}(\lambda) \quad \text{with } |\lambda| = 1; (2.3)
\]

Case 3:
\[
J = \begin{bmatrix} N_{2n_1+1}(\gamma) & 0 \\ 0 & N_{2n_2+1}(\gamma) \end{bmatrix}
\]
with \(|\gamma| = 1\) and \(n_1 + n_2 + 1 = n; (2.4)

Case 4:
\[
J = \begin{bmatrix} N_{2n_1+1}(\gamma) & 0 \\ 0 & N_{2n_2+1}(\delta) \end{bmatrix}
\]
with \(\gamma \neq \delta, |\gamma| = |\delta| = 1\) and \(n_1 + n_2 + 1 = n; (2.5)

\textbf{Theorem 2.1} (Main Theorem). Let \(W(k) = Y(k)X(k)^{-1}\) be the extended solution of the hermitian Riccati difference equation (1.3a) and (1.3b), where \((Z(k) = \begin{bmatrix} X(k)^T & Y(k)^T \end{bmatrix}^T)\) is the solution of the associated linear difference equation (1.2). \(J\) is the matrix defined in (2.1). Then the following assertions hold.

(I) Define
\[
\begin{bmatrix} W_1^T, W_2^T \end{bmatrix} \triangleq S^{-1} [I_n, W_0^T]^T, (2.6)
\]
and suppose that the matrix \(S\) in (2.1) is partitioned as
\[
S = \begin{bmatrix} U_1 & V_1 \\ U_2 & V_2 \end{bmatrix} (2.7)
\]
with \(U_1, U_2, V_1, V_2 \in \mathbb{C}^{n \times n}\).

Case 1: \(J\) is of the form in (2.2). Assume \(W_1\) and \(U_1\) are both nonsingular. Then
\[
W(k) = U_2U_1^{-1} + O(k^{2(n-1)} / |\lambda|^{2k}) \quad \text{as } k \to \infty.
\]

Case 2: \(J\) is of the form in (2.3). Assume \(W_2\) and \(U_1\) are both nonsingular. Then
\[
W(k) = U_2U_1^{-1} + O(k^{-1}) \quad \text{as } k \to \infty.
\]

(II) Set \(\Theta \triangleq \Theta_1\Theta_2\) with
\[
\Theta_1 = \begin{bmatrix} I_{n_1+1} \oplus (-i\beta P_{n_1}) & 0 \\ 0 & I_{n_2+1} \oplus (i\beta P_{n_2}) \end{bmatrix},
\]
\[
\Theta_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} / 2 & 0 & 0 & \sqrt{2} i\beta \\ 0 & 0 & 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & \sqrt{2} / 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} i\beta \end{bmatrix},
\]
where the matrices \(P_{n_1}\) and \(P_{n_2}\) are defined in (3.1) and \(\beta \in \{1, -1\}\). Define
\[
\begin{bmatrix} W_1^T, W_2^T \end{bmatrix} \triangleq (S\Theta)^{-1} [I_n, W_0^T]^T, (2.8)
\]
and suppose that the matrix $S\Theta$ is partitioned as

$$S\Theta = \begin{bmatrix} U_1 & u_1 \\ U_2 & u_2 \end{bmatrix} \begin{bmatrix} V_1 & v_1 \\ V_2 & v_2 \end{bmatrix} \in \mathbb{C}^{2n \times 2n}$$  \hspace{1cm} (2.9)

where $U_1, U_2, V_1, V_2 \in \mathbb{C}^{n \times (n_1 + n_2)}$ and $u_1, u_2, v_1, v_2 \in \mathbb{C}^n$.

Case 3: $J$ is of the form in (2.4). Assume $W_2$ is nonsingular. Then there exist constants $f_u, f_v, g_u, g_v \in \mathbb{C}$ with $\bar{U}_1 \triangleq [U_1 f_u + f_v] \in \mathbb{C}^{n \times n}$ and $\zeta_i \triangleq g_u u_i + g_v v_i \in \mathbb{C}^n$, $i = 1, 2$, such that

$$W(k) = \left( \bar{U}_2 + \frac{(\zeta_2 - \bar{U}_2 \bar{U}_1^{-1} \zeta_1) e_n^H}{1 + e_n^H \bar{U}_1^{-1} \zeta_1} \right) \bar{U}_1^{-1} + O(k^{-1})$$

as $k \to \infty$ whenever $\bar{U}_1$ is nonsingular and $1 + e_n^H \bar{U}_1^{-1} \zeta_1 \neq 0$.

Case 4: $J$ is of the form in (2.5). Assume $W_2$ is nonsingular and $\gamma = e^{i\theta_1}, \delta = e^{i\theta_2}$ with $0 \leq \theta_1, \theta_2 < 2\pi$. Let $\theta \triangleq \theta_1 - \theta_2$. Then there exist constants $f_u, f_v, g_u, g_v \in \mathbb{C}$ with $\bar{U}_1 \triangleq [U_1 f_u + f_v] \in \mathbb{C}^{n \times n}$ and $\zeta_i \triangleq g_u u_i + g_v v_i \in \mathbb{C}^n$, $i = 1, 2$, such that the extended solution $W(k)$ behaves as the following descriptions whenever $\bar{U}_1$ is nonsingular.

- When $\frac{\theta}{\pi} = \frac{q}{p}, p, q \in \mathbb{Z}$, $W(k)$ tends to the periodic orbit

$$\bar{U}_2 \bar{U}_1^{-1} + \frac{1}{e^{-2\pi i k} + e_n^H \bar{U}_1^{-1} \zeta_1} (\zeta_2 - \bar{U}_2 \bar{U}_1^{-1} \zeta_1) e_n^H \bar{U}_1^{-1}, \hspace{1cm} k = 0, 1, 2, \ldots (p - 1);$$

- When $\frac{\theta}{\pi} \in \mathbb{Q}^c$, the asymptotic trajectory of $W(k)$ is dense in the closed orbit

$$\bar{U}_2 \bar{U}_1^{-1} + \frac{1}{e^{-2\pi i \nu} + e_n^H \bar{U}_1^{-1} \zeta_1} (\zeta_2 - \bar{U}_2 \bar{U}_1^{-1} \zeta_1) e_n^H \bar{U}_1^{-1}, \hspace{1cm} \nu \in [0, 1),$$

and hence it forms the limit cycle of $W(k)$.

3. Preliminaries. The operator $T$ in (1.4) is a symplectic matrix. To analyze the extended solution $\{W(k)\}_{k \in K}$ of the hermitian Riccati difference equation (1.3a) and (1.3b), the spectrum and the symplectic Jordan canonical form of a symplectic matrix is stated in the Lemma 3.1 and Theorem 3.1.

**Lemma 3.1.** ([1]) Let $T \in \mathbb{C}^{2n \times 2n}$ be symplectic. Then $T$ is nonsingular and its spectrum is symmetric relative to the unit circle, i.e.

$$\lambda_0 \in \sigma(T) \hspace{1cm} \text{implies} \hspace{1cm} \lambda_0^{-1} \in \sigma(T).$$

Moreover, the sizes and the numbers of the Jordan blocks with eigenvalue $\lambda_0$ and the sizes and the numbers of the Jordan blocks with eigenvalue $\lambda_0^{-1}$ are the same.
Theorem 3.1. ([13]) (Symplectic Jordan canonical form). Given a complex symplectic matrix \( S \). Then there exists a symplectic matrix \( U \) such that

\[
U^{-1}SU = \begin{bmatrix}
R_r & R_e & 0 & D_e & D_e \\
R_e & R_d & 0 & R_e^{-H} & D_e \\
0 & 0 & R_e^{-H} & R_e^{-H} & S_d
\end{bmatrix},
\]

where the matrix blocks have the following structures.

1. The blocks with index \( r \) are associated with the pairwise distinct eigenvalues \( \sigma_1, \cdots, \sigma_\mu, \bar{\sigma}_1^{-1}, \cdots, \bar{\sigma}_\mu^{-1} \), such that \( |\sigma_k| \neq 1 \). The blocks have the structures

\[
R_r = \text{diag}(R_{r1}^1, \cdots, R_{r\mu}^\mu),
R_e^k = \text{diag}(N_{d_k,1}(\sigma_k), \cdots, N_{d_k,q_k}(\sigma_k)), \quad k = 1, \cdots, \mu.
\]

2. The blocks with index \( e \) are associated with unimodular eigenvalues \( \theta_1, \cdots, \theta_\mu_e \) and they are obtained from even-size Jordan blocks of \( \theta_k \). The structures of the blocks are

\[
R_e = \text{diag}(R_{e1}^1, \cdots, R_{e\mu_e}^\mu),
D_e = \text{diag}(D_{e1}^1, \cdots, D_{e\mu_e}^\mu),
R_e^k = \frac{1}{2} \text{diag}(\beta_{k,1}^e e_{k,1}^H N_{d_k,1}(\theta_k)^{-H}, \cdots, \beta_{k,q_k}^e e_{k,q_k}^H N_{d_k,q_k}(\theta_k)^{-H}),
\]

where \( k = 1, \cdots, \mu_e \), \( j = 1, \cdots, q_k \) and \( \beta_{k,j}^e \in \{-1, 1\} \).

3. The blocks with index \( c \) are associated with unimodular eigenvalues \( \theta_1, \cdots, \theta_\mu_c \) and they come from combining two odd-size Jordan blocks of \( \theta_k \). The structures of the blocks are

\[
R_c = \text{diag}(R_{c1}^1, \cdots, R_{c\mu_c}^\mu),
D_c = \text{diag}(D_{c1}^1, \cdots, D_{c\mu_c}^\mu),
R_c^k = \text{diag}(B_{k,1}, \cdots, B_{k, r_k}),
D_c^k = \text{diag}(D_{k,1}, \cdots, D_{k, r_k}),
\]

where for \( k = 1, \cdots, \mu_c \) and \( j = 1, \cdots, r_k \), we have

\[
B_{k,j} = \begin{bmatrix}
N_{mk,j}(\theta_k) & 0 & -\frac{\sqrt{2}}{\theta_k e_{mk,j}} \\
0 & N_{nk,j}(\theta_k) & -\frac{\sqrt{2}}{\theta_k e_{nk,j}} \\
0 & 0 & \theta_k
\end{bmatrix},
D_{k,j} = i\sqrt{2} e_{k,j}^H \begin{bmatrix}
f(\theta_k) e_{mk,j}^H N_{mk,j}(\theta_k)^{-H} & 0 & -\frac{\sqrt{2}}{\theta_k e_{mk,j}} \\
0 & f(\theta_k) e_{nk,j}^H N_{nk,j}(\theta_k)^{-H} & -\frac{\sqrt{2}}{\theta_k e_{nk,j}} \\
-\frac{\sqrt{2}}{\theta_k e_{mk,j}} N_{nk,j}(\theta_k)^{-H} & \frac{\sqrt{2}}{\theta_k e_{nk,j}} N_{mk,j}(\theta_k)^{-H} & 0
\end{bmatrix},
\]

with \( \beta_{k,j}^c \in \{-1, 1\} \) and the function \( f(\sigma) \) defined by

\[
f(\sigma) \triangleq \begin{cases}
\frac{\sigma}{\sigma - 1} & \text{for } \sigma \neq 1, \\
\frac{1}{2} & \text{for } \sigma = 1.
\end{cases}
\]

4. The blocks with index \( d \) are associated with two disjoint sets of unimodular eigenvalues \( \{\gamma_1, \cdots, \gamma_{\mu_d}\} \) and \( \{\delta_1, \cdots, \delta_{\mu_d}\} \) and they come from combining odd-size
Jordan blocks of $\gamma_k$ and $\delta_k$. The corresponding block structures are

$$R_d = \text{diag}(R_{d_1}^1, \cdots, R_{d_{\mu_d}}^d), \quad D_d = \text{diag}(D_{d_1}^1, \cdots, D_{d_{\mu_d}}^d),$$
$$S_d = \text{diag}(S_{\mu_1}^d, \cdots, S_{\mu_d}^d), \quad G_d = \text{diag}(G_{\mu_1}^d, \cdots, G_{\mu_d}^d),$$

where for $k = 1, \cdots, \mu_d$, we have

$$R_k^d = \begin{bmatrix}
N_{s_k}(\gamma_k) & 0 & -\frac{\sqrt{2}}{2} \gamma_k e_{s_k} \\
0 & N_{t_k}(\delta_k) & -\frac{\sqrt{2}}{2} \delta_k e_{t_k} \\
0 & 0 & \frac{1}{2}(\gamma_k + \delta_k)
\end{bmatrix},$$

$$D_k^d = i \beta_k^d \begin{bmatrix}
f(\gamma_k)e_{s_k} e_{s_k}^H N_{s_k}(\gamma_k)^{-H} & 0 & -\frac{\sqrt{2}}{2} \gamma_k e_{s_k} \\
0 & -f(\delta_k) e_{t_k} e_{t_k}^H N_{t_k}(\delta_k)^{-H} & -\frac{\sqrt{2}}{2} \delta_k e_{t_k} \\
-\frac{\sqrt{2}}{2} e_{s_k}^H N_{s_k}(\gamma_k)^{-H} & \frac{\sqrt{2}}{2} e_{t_k}^H N_{t_k}(\delta_k)^{-H} & \frac{1}{2}(\gamma_k + \delta_k)
\end{bmatrix},$$

$$S_k^d = \begin{bmatrix}
\gamma_k N_{s_k}(\gamma_k)^{-H} & 0 & 0 \\
0 & \delta_k N_{t_k}(\delta_k)^{-H} & 0 \\
\frac{\sqrt{2}}{2} e_{s_k}^H N_{s_k}(\gamma_k)^{-H} & \frac{\sqrt{2}}{2} e_{t_k}^H N_{t_k}(\delta_k)^{-H} & \frac{1}{2}(\gamma_k + \delta_k)
\end{bmatrix},$$

$$G_k^d = i \beta_k^d \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{2}(\gamma_k - \delta_k)
\end{bmatrix},$$

with $\beta_k^d \in \{-1, 1\}$.

We denote some notations. For $1 \leq i, j \leq l$,

$$(P_l)_{ij} = \begin{cases}
(-1)^i & \text{for } j = l + 1 - i, \\
0 & \text{otherwise},
\end{cases}$$

$$\Phi_l(t) = e^{N_l t} = \begin{bmatrix}
1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{l-1}}{(l-1)!} \\
1 & t & \cdots & \cdots & \frac{t^2}{2!} \\
& \cdots & \cdots & \frac{t^2}{2!} & t \\
1 & t & \cdots & \frac{t^2}{2!} & 1
\end{bmatrix},$$

$$\phi_l(t) = \begin{bmatrix}
\frac{t}{1!} \\
\frac{t^2}{2!} \\
\cdots \\
\frac{t^l}{l!}
\end{bmatrix}, \quad \psi_l(t) = \begin{bmatrix}
\frac{t}{1!} \\
\frac{t^2}{2!} \\
\cdots \\
\frac{t^l}{l!}
\end{bmatrix}, \quad (3.1)$$

$$\Gamma_{l-1}^{l-1} = \begin{bmatrix}
\frac{t^{l-1}}{(l-1)!} & \frac{t^{l-1+1}}{(l-1)!+1} & \cdots & \frac{t^2}{2!} \\
\frac{t^{l-1}}{(l-1)!+1} & \frac{t^{l-1}}{(l-1)!+1} & \cdots & \frac{t^2}{2!} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{t^{l-1-l-2}}{(l-1-l)!} & \cdots & \cdots & \frac{t^1}{1!}
\end{bmatrix},$$

$$\tilde{\Phi}_l(t) = P_l^{-1} \Phi_l(t) P_l, \quad \tilde{\Gamma}_l^{l-1}(t) = \Gamma_l^{l-1}(t) P_l, \quad \tilde{\psi}_l^T(t) = \psi_l^T(t) P_l.$$

The following lemma states the asymptotic estimate for the powers of Jordan block with the eigenvalue $\lambda$ and size $l$, and properties related the matrix $P_l$ and the nilpotent matrix $N_l$. 


Lemma 3.2. (i) For $\lambda \neq 0$ and $k \in \mathbb{N}$ sufficiently large, we have
\[ N_t(\lambda)^k = (\lambda I_t + N_t)^k \]
\[ = \lambda^k C_j^{(1 + O(k^{-1}))} \]
\[ = \lambda^k e^{(N_t - \Phi)(1 + O(k^{-1}))} \]
\[ = \lambda^k e^{(N_t - \Phi)}(k/\lambda); \]
(ii) $P_t^{-1} = P_t^T$, $P_t^{-1} N_t P_t = -N_t^T$; (iii) $\Phi_t(t) = P_t^{-1} \Phi_t(t) P_t = e^{-N_t^T \Phi} \Phi_t(t)(= (\Phi_t(t))^{-1})$.

Proof. (i) By definition, we have
\[ N_t(\lambda)^k = (\lambda I_t + N_t)^k \]
\[ = \lambda^k I_t + \sum_{j=1}^{k} C_j^{(1 + O(k^{-1}))} \]
\[ = \lambda^k I_t + \sum_{j=1}^{k} \frac{k!(k-j)!}{j!} \lambda^{k-j} N_t^j \]
\[ = \lambda^k I_t + \sum_{j=1}^{k} \frac{k!(k-j)!}{j!} \lambda^{k-j} N_t^j \]
\[ = \lambda^k I_t + \sum_{j=1}^{k} \frac{k!(k-j)!}{j!} \lambda^{k-j} N_t^j \]
\[ = \lambda^k e^{(N_t - \Phi)}(k/\lambda), \]
for $k \in \mathbb{N}$ sufficiently large. The assertions (ii) and (iii) can be obtained by direct computations. 

For given integers $k, l, l_1$ and $l_2$ satisfying $0 \leq k, 0 \leq l, k \neq l$ and $0 < l_1 < l_2 < 2l_1$, we denote
\[ \Xi_{k,l}(t) = \begin{cases} \text{diag}(t^k, t^{k+1}, \ldots, t^l) & \text{if } k < l, \\ \text{diag}(t^l, t^{l-1}, \ldots, t^l) & \text{if } k > l, \end{cases} \]
\[ F_{t_1}^{l_2} = \begin{bmatrix} 1/t_1! & 1/(t_1+1)! & \cdots & 1/t_2! \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(l_1-1)!} & \frac{1}{l_1!} & \cdots & \frac{1}{(l_2-1)!} \\ \frac{1}{(2l_1-t_2)!} & \frac{1}{(2l_1-t_2)!} & \cdots & \frac{1}{t_2!} \end{bmatrix}. \tag{3.2} \]
The matrix $F_{t_1}^{l_2}$ is invertible (see Lemma 3.4 in [11]). Then the matrix $\Gamma_{t_1}^{l_2}(t)$ defined in (3.1) and its inverse can be written in terms of $\Xi_{k,l}(t)$ and $F_{t_1}^{l_2}$ in (3.2) as
\[ \Gamma_{t_1}^{l_2}(t) = t^{2l-t_2 l_2} \Xi_{l_2-\ell_1}(t) F_{t_1}^{l_2} \Xi_{l_2-\ell_1}(t), \tag{3.3} \]
\[ (\Gamma_{t_1}^{l_2}(t))^{-1} = t^{-2l_2+\ell_2} (\Xi_{l_2-\ell_1}(t))^{-1} (F_{t_1}^{l_2})^{-1} (\Xi_{l_2-\ell_1}(t))^{-1}. \tag{3.4} \]
The matrices $(\Xi_{0, l-1}(t))^{-1}$ and $(\Xi_{l-1, 0}(t))^{-1}$ shall be used to help eliminate the $t$ powers of $\Phi_t(t)$ and $\phi_t(t)$ in the proofs of the following lemmas.
Lemma 3.3. Let \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \). Then
\[
(i) \quad (\Gamma_n^{2n-1}(t))^{-1}\Phi_n(t) = O(|t|^{-1}), \quad \Phi_n(t)(\Gamma_n^{2n-1}(t))^{-1} = O(|t|^{-1})
\]
and \( \Phi_n(t)(\Gamma_n^{2n-1}(t))^{-1}\Phi_n(t) = O(|t|^{-1}) \);
\[
(ii) \quad \Phi_n(t)[\Phi_n(t)C + \Gamma_n^{2n-1}(t)]^{-1} = O(|t|^{-1})
\]
as \( |t| \to \infty \), where \( C \in \mathbb{C}^{n \times n} \) is a constant matrix.

Proof. The proof of this lemma is similar to the proof of Lemma 3.5 in [11]. \( \square \)

For simplicity of expression for \( T^k \) in case 4 of the proof, we introduce the following notations
\[
\Phi_{m,n}(k) \triangleq \gamma^k \Phi_m(k/\gamma) \oplus \delta^k \Phi_n(k/\delta),
\]
\[
\hat{\Phi}_{m,n}(k) \triangleq \gamma^k \Phi_m^T(k/\gamma) \oplus \delta^k \Phi_n^T(k/\delta)
\]
\[
= \gamma^k P_{m}^{-1} \Phi_m(k/\gamma) P_m \oplus \delta^k P_n^{-1} \Phi_n(k/\delta) P_n,
\]
\[
\phi_{m,n}^1(k) \triangleq \frac{\sqrt{2}}{2} \begin{bmatrix} \gamma^k \phi_m(k/\gamma) \\ \delta^k \phi_n(k/\delta) \end{bmatrix},
\]
\[
\phi_{m,n}^2(k) \triangleq \frac{\sqrt{2}}{2} i\beta \begin{bmatrix} \gamma^k \phi_m(k/\gamma) \\ -\delta^k \phi_n(k/\delta) \end{bmatrix},
\]
\[
\psi_{m,n}^1(k) \triangleq \frac{\sqrt{2}}{2} i\beta \begin{bmatrix} \gamma^k \psi_m(k/\gamma) \\ -\delta^k \psi_n(k/\delta) \end{bmatrix} = \left[ \begin{array} {c} \gamma^k \psi_m(k/\gamma) \\ -\delta^k \psi_n(k/\delta) \end{array} \right],
\]
\[
\psi_{m,n}^2(k) \triangleq -\frac{\sqrt{2}}{2} i\beta \begin{bmatrix} \gamma^k \psi_m(k/\gamma) \\ \delta^k \psi_n(k/\delta) \end{bmatrix} = \left[ \begin{array} {c} \gamma^k \psi_m(k/\gamma) \\ \delta^k \psi_n(k/\delta) \end{array} \right],
\]
\[
\hat{\psi}_{m,n}^1(k) \triangleq \frac{\sqrt{2}}{2} i\beta \begin{bmatrix} \gamma^k \hat{\psi}_m(k/\gamma) \\ -\delta^k \hat{\psi}_n(k/\delta) \end{bmatrix} = \left[ \begin{array} {c} \gamma^k \hat{\psi}_m(k/\gamma) \\ -\delta^k \hat{\psi}_n(k/\delta) \end{array} \right],
\]
\[
\hat{\psi}_{m,n}^2(k) \triangleq \frac{\sqrt{2}}{2} i\beta \begin{bmatrix} \gamma^k \hat{\psi}_m(k/\gamma) \\ \delta^k \hat{\psi}_n(k/\delta) \end{bmatrix} = \left[ \begin{array} {c} \gamma^k \hat{\psi}_m(k/\gamma) \\ \delta^k \hat{\psi}_n(k/\delta) \end{array} \right],
\]
(3.5)

The variable \( k \) of the notations introduced in (3.5) will be omitted wherever it is not necessary to specify it.

Lemma 3.4. Let \( n_1, n_2 \) and \( k \) be positive integers. Then
\[
(\Upsilon + \Gamma_n^{2n_1,2n_2})^{-1} = O(k^{-2}), \quad (\Upsilon + \Gamma_n^{2n_1,2n_2})^{-1}\phi_{n_1,n_2} = O(k^{-2}),
\]
\[
(\Upsilon + \Gamma_n^{2n_1,2n_2})^{-1}\tilde{\psi}_{n_1,n_2} = O(k^{-2}), \quad \tilde{\psi}_{n_1,n_2}^T (\Upsilon + \Gamma_n^{2n_1,2n_2})^{-1} = O(k^{-2}),
\]
\[
\tilde{\phi}_{n_1,n_2}^T (\Upsilon + \Gamma_n^{2n_1,2n_2})^{-1} = O(k^{-2}), \quad \tilde{\phi}_{n_1,n_2}^T (\Upsilon + \Gamma_n^{2n_1,2n_2})^{-1}\phi_{n_1,n_2} = O(k^{-2}),
\]
as \( k \to \infty \), where \( \Gamma_n^{2n_1,2n_2}, \Phi_{n_1,n_2}, \hat{\Phi}_{n_1,n_2}, \phi_{n_1,n_2} \) and \( \hat{\psi}_{n_1,n_2} \), \( j = 1, 2 \), are defined in (3.5) and \( \Upsilon = \Phi_{n_1,n_2}W + \delta^j \phi_{n_1,n_2}^T \omega^T \) with \( W \in \mathbb{C}^{(n_1+n_2) \times (n_1+n_2)} \) and \( \omega \in \mathbb{C}^{n_1+n_2} \). Moreover, we also have
\[
\hat{\psi}_{n_1,n_2}^T (\Upsilon + \Gamma_n^{2n_1,2n_2})^{-1}\phi_{n_1,n_2} = \hat{\psi}_{n_1,n_2}^T (\Gamma_n^{2n_1,2n_2})^{-1}\phi_{n_1,n_2} = O(k^{-1})
\]
as \( k \to \infty \), where \( j, l \in \{1, 2\} \).

**Proof.** Using the notations in (3.5) and the expression (3.4), we can obtain
\[
(\hat{\Gamma}_{n+1}^{2n+2})^{-1} = \mathcal{O}(k^{-2}),
\]
\[
(\hat{\Gamma}_{n+1}^{2n+2})^{-1} \Phi_{n_1, n_2} = \mathcal{O}(k^{-2}),
\]
\[
(\hat{\Gamma}_{n+1}^{2n+2})^{-1} \phi_{n_1, n_2}^j = \mathcal{O}(k^{-1}), \quad j = 1, 2
\]
due to for \( i = 1, 2 \),
\[
(\Xi_{n-i, 0})^{-1}(t) \Phi_{n_1}(t) = \mathcal{O}(1),
\]
\[
(\Xi_{n-i, 0})^{-1}(t) \phi_{n_1}(t) = \mathcal{O}(|t|)
\]
as \( |t| \to \infty \). These imply
\[
(\hat{\Gamma}_{n+1}^{2n+2})^{-1} \Upsilon = \mathcal{O}(k^{-1})
\]
and
\[
(\Upsilon + \hat{\Gamma}_{n+1}^{2n+2})^{-1} = [(\hat{\Gamma}_{n+1}^{2n+2})^{-1}(\hat{\Gamma}_{n+1}^{2n+2})^{-1} \Upsilon + I]^{-1}
\]
\[
= [I + (\hat{\Gamma}_{n+1}^{2n+2})^{-1} \Upsilon]^{-1} (\hat{\Gamma}_{n+1}^{2n+2})^{-1}
\]
\[
= [I + \sum_{l=1}^{\infty} (-1)^l ((\hat{\Gamma}_{n+1}^{2n+2})^{-1} \Upsilon)^l (\hat{\Gamma}_{n+1}^{2n+2})^{-1}]
\]
\[
= \mathcal{O}(k^{-2}).
\]

We can also obtain
\[
\hat{\psi}_{n_1, n_2}^T (\hat{\Gamma}_{n+1}^{2n+2})^{-1} = \mathcal{O}(k^{-1}),
\]
\[
\hat{\psi}_{n_1, n_2}^T (\hat{\Gamma}_{n+1}^{2n+2})^{-1} \Phi_{n_1, n_2} = \mathcal{O}(k^{-1}),
\]
\[
\hat{\psi}_{n_1, n_2}^T (\hat{\Gamma}_{n+1}^{2n+2})^{-1} \phi_{n_1, n_2}^j = \mathcal{O}(1),
\]
\[
\hat{\Phi}_{n_1, n_2} (\hat{\Gamma}_{n+1}^{2n+2})^{-1} = \mathcal{O}(k^{-2}),
\]
\[
\hat{\Phi}_{n_1, n_2} (\hat{\Gamma}_{n+1}^{2n+2})^{-1} \Phi_{n_1, n_2} = \mathcal{O}(k^{-2}),
\]
\[
\hat{\Phi}_{n_1, n_2} (\hat{\Gamma}_{n+1}^{2n+2})^{-1} \phi_{n_1, n_2}^j = \mathcal{O}(k^{-1}).
\]

The rest of the lemma can be proved by using (3.6) and (3.7). \( \square \)

**Lemma 3.5.** Given \( n \in \mathbb{N} \). Let \( \kappa_n = \hat{\psi}_{n}^T(t) (\hat{\Gamma}_{n+1}^{2n+2})^{-1} \phi_n(t) \), where \( \hat{\psi}_{n}^T(t), \phi_n(t) \) and \( \hat{\Gamma}_{n+1}^{2n+2}(t) \) are defined in (3.1). Then
\[
\kappa_n = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\]

**Proof.** The proof of this lemma is the same with the proof of Lemma 3.7 in [11]. \( \square \)

4. **Proof of the main theorem.** In terms of the solution of the associated linear difference equation (1.2)
\[
\begin{bmatrix} X(k+1) \\ Y(k+1) \end{bmatrix} = T \begin{bmatrix} X(k) \\ Y(k) \end{bmatrix} = T^{k+1} \begin{bmatrix} X(0) \\ Y(0) \end{bmatrix} = T^{k+1} \begin{bmatrix} I_n \\ W_0 \end{bmatrix},
\]
we can get an explicit solution representation of the hermitian Riccati difference equation (1.3a) and (1.3b)
\[
W(k+1) = Y(k+1) X(k+1)^{-1}.
\]
Proof of case 1. $T$ has only eigenvalues $\lambda$ and $\bar{\lambda}^{-1}$ with $|\lambda| \neq 1$ and each of the two eigenvalues has only one Jordan block. Here we assume $|\lambda| > 1$. From (2.1), we have

$$S^{-1}TS = \left[ \begin{array}{cc} N_n(\lambda) & 0 \\ 0 & N_n(1/\bar{\lambda}) \end{array} \right].$$

Based on the assumptions (2.6) and (2.7), the solution of the associated linear difference equation (1.2) can be written as

$$\begin{bmatrix} X(k) \\ Y(k) \end{bmatrix} = T^k \begin{bmatrix} I_n \\ W_0 \end{bmatrix}$$

$$= S \begin{bmatrix} N_n(\lambda)^k & 0 \\ 0 & N_n(1/\bar{\lambda})^k \end{bmatrix} S^{-1} \begin{bmatrix} I_n \\ W_0 \end{bmatrix}$$

$$= \begin{bmatrix} U_1 & V_1 \\ U_2 & V_2 \end{bmatrix} \begin{bmatrix} N_n(\lambda)^k & 0 \\ 0 & N_n(1/\bar{\lambda})^k \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$

$$= \begin{bmatrix} U_1 & V_1 \\ U_2 & V_2 \end{bmatrix} \begin{bmatrix} N_n(\lambda)^k W_1 \\ N_n(1/\bar{\lambda})^k W_2 \end{bmatrix}.$$ 

It yields that

$$X(k) = U_1 N_n(\lambda)^k W_1 + V_1 N_n(1/\bar{\lambda})^k W_2,$$
$$Y(k) = U_2 N_n(\lambda)^k W_1 + V_2 N_n(1/\bar{\lambda})^k W_2.$$

By Lemma 3.2(i), we have

$$N_n(\lambda)^k \sim \lambda^k e^{(N_n \cdot \frac{k}{\bar{\lambda}})} = \mathcal{O}(|\lambda|^k \cdot \frac{k^{n-1}}{|\lambda|^{n-r}})$$

for $k$ sufficiently large. Furthermore, due to

$$e^{(N_n \cdot \frac{k}{\bar{\lambda}})^{-1}} = e^{(-N_n \cdot \frac{k}{\bar{\lambda}})} = \mathcal{O}\left(\frac{k^{n-1}}{|\lambda|^{n-1}}\right)$$

and Lemma 3.2(i), we can also obtain

$$[N_n(\lambda)^k]^{-1} = \mathcal{O}(|\lambda|^{-k} \cdot \frac{k^{n-1}}{|\lambda|^{n-r}}).$$

Then it follows that for $k$ sufficiently large

$$\| V_1 N_n(1/\bar{\lambda})^k W_2 W_1^{-1} [N_n(\lambda)^k]^{-1} \| = \mathcal{O}(\frac{k^{2(n-1)}}{|\lambda|^{2k}})$$

under the assumption that $W_1$ is invertible. In addition, by applying Sherman-Morrison-Woodbury formula, we can get

$$\{U_1 + V_1 N_n(1/\bar{\lambda})^k W_2 W_1^{-1} [N_n(\lambda)^k]^{-1}\}^{-1} = U_1^{-1} + \mathcal{O}(\frac{k^{2(n-1)}}{|\lambda|^{2k}})$$

if $U_1$ is invertible and $k$ is sufficiently large. Therefore, for $|\lambda| > 1,$

$$W(k) = Y(k)X(k)^{-1}$$

$$= \{U_2 + V_2 N_n(1/\bar{\lambda})^k W_2 W_1^{-1} [N_n(\lambda)^k]^{-1}\} \{U_1 + V_1 N_n(1/\bar{\lambda})^k W_2 W_1^{-1} [N_n(\lambda)^k]^{-1}\}^{-1}$$

$$= \{U_2 + \mathcal{O}(\frac{k^{2(n-1)}}{|\lambda|^{2k}})\} \{U_1^{-1} + \mathcal{O}(\frac{k^{2(n-1)}}{|\lambda|^{2k}})\}$$

$$= U_2 U_1^{-1} + \mathcal{O}(\frac{k^{2(n-1)}}{|\lambda|^{2k}}).$$
as $k$ is large. Hence, the extended solution $W(k)$ approaches a constant equilibrium $U_2U_1^{-1}$ at an exponential rate as $k \to \infty$.

**Proof of case 2.** $T$ has one unimodular eigenvalue $\lambda$ ($|\lambda| = 1$) which has one Jordan block of size $2n$. From (2.1), we have

$$S^{-1}TS = N_{2n}.$$

Under the assumptions (2.6) and (2.7) and Lemma 3.2(i), the solution of the associated linear difference equation (1.2) is

$$X(k) = T^k \begin{bmatrix} I_n \\ W_0 \end{bmatrix} = SN_{2n} \lambda^k \begin{bmatrix} I_n \\ W_0 \end{bmatrix} \approx \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \cdot \lambda^k \begin{bmatrix} \Phi_n(k/\lambda) \\ \Phi_n(k/\lambda) \end{bmatrix} \cdot \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \lambda^k \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} \Phi_n(k/\lambda)W_1W_2^{-1} + \Gamma_n^{-1}(k/\lambda) \\ \Phi_n(k/\lambda) \end{bmatrix} W_2$$

if $k$ is sufficiently large and $W_2$ is invertible. Define

$$\Omega(t) = \Phi_n(t)W_1W_2^{-1} + \Gamma_n^{-1}(t).$$

Notice that $\Gamma_n^{-1}(t)$ is invertible for $t \neq 0$ and $\| \Gamma_n^{-1}(t)^{-1} \Phi_n(t) \| = O(|t|^{-1})$ by Lemma 3.3. Hence, for $k$ sufficiently large, $\Omega(k/\lambda)$ is invertible and

$$X(k) \approx \lambda^k [U_1 \Omega(k/\lambda)W_2 + V_1 \Phi_n(k/\lambda)W_2]$$

$$= \lambda^k [U_1 + V_1 \Phi_n(k/\lambda)\Omega(k/\lambda)^{-1}] \Omega(k/\lambda)W_2,$$

$$Y(k) \approx \lambda^k [U_2 \Omega(k/\lambda)W_2 + V_2 \Phi_n(k/\lambda)W_2]$$

$$= \lambda^k [U_2 + V_2 \Phi_n(k/\lambda)\Omega(k/\lambda)^{-1}] \Omega(k/\lambda)W_2.$$

Lemma 3.3 also implies the asymptotic estimate $\Phi_n(t)\Omega(t)^{-1} = O(|t|^{-1})$. Then we can get

$$W(k) = Y(k)X(k)^{-1} \approx [U_2 + V_2 \Phi_n(k/\lambda)\Omega(k/\lambda)^{-1}] [U_1 + V_1 \Phi_n(k/\lambda)\Omega(k/\lambda)^{-1}]^{-1}$$

$$= U_2U_1^{-1} + O(k^{-1})$$

as $k$ is large. Hence, the extended solution $W(k)$ approaches a constant equilibrium $U_2U_1^{-1}$ at a polynomial rate as $k \to \infty$.

**Proof of case 3.** $T$ has only one unimodular eigenvalue $\lambda$ ($|\lambda| = 1$) which has two Jordan blocks of odd sizes. This is a special one of case 4 with $\lambda = \gamma = \delta$.

**Proof of case 4.** $T$ has two distinct unimodular eigenvalues $\gamma$ and $\delta$, ($|\gamma| = |\delta| = 1$), with Jordan blocks of sizes $2n_1+1$ and $2n_2+1$, respectively. Here $n_1 + n_2 + 1 = n$. From (2.1), we have

$$S^{-1}TS = \begin{bmatrix} N_{2n_1+1}(\gamma) & 0 \\ 0 & N_{2n_2+1}(\delta) \end{bmatrix}.$$
The solution of the associated linear difference equation (1.2) is represented as, by Lemma 3.1(i),

\[
\begin{bmatrix}
X(k) \\
Y(k)
\end{bmatrix} = T^k \begin{bmatrix}
I_n \\
W_0
\end{bmatrix}
\]

\[
= S \begin{bmatrix}
[N_{2n+1}(\gamma)]^k \\
0
\end{bmatrix} S^{-1} \begin{bmatrix}
I_n \\
W_0
\end{bmatrix}
\]

\[
\simeq S \begin{bmatrix}
\gamma^k \Phi_{2n+1}(k/\gamma) \\
0
\end{bmatrix} S^{-1} \begin{bmatrix}
I_n \\
W_0
\end{bmatrix}
\]

for \( k \) sufficiently large. Let \( \Theta \triangleq \Theta_1 \Theta_2 \) with

\[
\Theta_1 = \begin{bmatrix}
I_{n+1} \oplus (-i \beta P_{n+1}) & 0 \\
0 & I_{n+1} \oplus (i \beta P_{n+1})
\end{bmatrix},
\]

\[
\Theta_2 = \begin{bmatrix}
I_{n_1} & 0 & 0 & 0 \\
0 & 0 & -\sqrt{2} & 0 \\
0 & 0 & 0 & I_{n_2} \\
0 & 0 & -\sqrt{2} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{2} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{2} & 0
\end{bmatrix}.
\]

With the aid of the matrices \( \Theta_1 \) and \( \Theta_2 \), we can obtain

\[
\Theta^{-1} \begin{bmatrix}
\gamma^k \Phi_{2n+1}(k/\gamma) \\
0
\end{bmatrix} \Theta
\]

\[
= \begin{bmatrix}
\Phi_{n_1,n_2} & 0 & 0 & 0 \\
\phi_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\phi_2 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\triangleq \begin{bmatrix}
B & D \\
G & E
\end{bmatrix}
\]

in terms of the notations defined in (3.5). Then by the assumptions (2.8) and (2.9), we have, for \( k \) sufficiently large,

\[
\begin{bmatrix}
X(k) \\
Y(k)
\end{bmatrix} \simeq S \Theta \cdot \left( \Theta^{-1} \begin{bmatrix}
\gamma^k \Phi_{2n+1}(k/\gamma) \\
0
\end{bmatrix} \Theta \right) \cdot (S\Theta)^{-1} \begin{bmatrix}
I_n \\
W_0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
U_1 & u_1 & V_1 & v_1 & B & D \\
U_2 & u_2 & V_2 & v_2 & G & E
\end{bmatrix} \begin{bmatrix}
W_1 \\
W_2
\end{bmatrix}.
\]

Since \( W_2 \) is invertible, we can get from above

\[
\begin{bmatrix}
X(k) \\
Y(k)
\end{bmatrix} W_2^{-1} \simeq \begin{bmatrix}
U_1 & u_1 & V_1 & v_1 & B W_1 W_2^{-1} + D \\
U_2 & u_2 & V_2 & v_2 & G W_1 W_2^{-1} + E
\end{bmatrix}.
\]

Set

\[
W \triangleq W_1 W_2^{-1} = \begin{bmatrix}
W_{1,1} \\
W_{1,2} \\
W_{2,1} \\
W_{2,2}
\end{bmatrix}.
\]
where $W_{1,1} \in \mathbb{C}^{(n_1+n_2) \times (n_1+n_2)}$, $w_{1,2} \in \mathbb{C}^{(n_1+n_2) \times 1}$, $w_{2,1} \in \mathbb{C}^{1 \times (n_1+n_2)}$ and $w_{2,2} \in \mathbb{C}$. By direct computation, we can obtain

$$BW_{1,2}^{-1} + D = BW + D = \begin{bmatrix} \Phi_{n_1,n_2} & \phi_{n_1,n_2}^1 \\ 0 & \omega_{11} \end{bmatrix} \begin{bmatrix} W_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{bmatrix} + D$$

$$= \begin{bmatrix} \Upsilon_{n_1,n_2} & \Phi_{n_1,n_2}w_{1,2} + \phi_{n_1,n_2}w_{2,2} \\ \omega_{11}w_{2,1} & \omega_{11}w_{2,2} \end{bmatrix} + D \quad (4.2)$$

where

$$\Upsilon_{n_1,n_2} = \begin{bmatrix} \Upsilon_{n_1,n_2} + \frac{\omega_{11}w_{2,1}}{\omega_{11}w_{2,1}^2 + \omega_{12}} & p_{n_1,n_2} \\ \phi_{n_1,n_2}^1 & \phi_{n_1,n_2}^2 \end{bmatrix}.$$

Similarly, we have

$$GW_{1,2}^{-1} + E = GW + E = \begin{bmatrix} 0 & 0 \\ 0 & \omega_{21} \end{bmatrix} \begin{bmatrix} W_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{bmatrix} + E \quad (4.3)$$

$\Phi_{n_1,n_2}$

Let

$$\Omega(k) = \begin{bmatrix} (\Upsilon_{n_1,n_2} + \frac{\omega_{11}w_{2,1}}{\omega_{11}w_{2,1}^2 + \omega_{12}})^{-1}, & -(\Upsilon_{n_1,n_2} + \frac{\omega_{11}w_{2,1}}{\omega_{11}w_{2,1}^2 + \omega_{12}})^{-1}p_{n_1,n_2} \\ 0 & 1 \end{bmatrix}.$$

by the asymptotic estimates in Lemma 3.4. Hence $\Omega(k)$ is well defined for $k$ large. Then we have

$$\begin{bmatrix} \Lambda_1(k) \\ \Lambda_2(k) \end{bmatrix} = \begin{bmatrix} X(k) \\ Y(k) \end{bmatrix} W_{1,2}^{-1} \Omega(k)$$

where

$$\begin{bmatrix} U_1 & V_1 \\ U_2 & V_2 \end{bmatrix} \begin{bmatrix} \Phi_{n_1,n_2} & \phi_{n_1,n_2}^1 \\ \omega_{11}w_{2,1} & \omega_{11}w_{2,2} \end{bmatrix} = \begin{bmatrix} \Upsilon_{n_1,n_2} + \frac{\omega_{11}w_{2,1}}{\omega_{11}w_{2,1}^2 + \omega_{12}} & p_{n_1,n_2} \\ \phi_{n_1,n_2}^1 & \phi_{n_1,n_2}^2 \end{bmatrix}.$$
From the asymptotic estimates in Lemma 3.4, we have
\[ \Phi_n(T_n, \omega_1, \omega_2) = O(k^{-1}), \]
\[ \Phi_n(T_{n+1}, \omega_1, \omega_2) = O(k^{-2}), \]
and
\[ (\hat{T}_n^2 T_n + \hat{T}^2_n)^{-1} = O(k^{-1}), \]
\[ (\hat{T}_n^2 T_{n+1} + \hat{T}^2_n)^{-1} = O(k^{-1}), \]
\[ (\hat{T}_n^2 T_n + \hat{T}^2_n + \omega_1 w_1^{-1} w_2) (\hat{T}_n + \hat{T}^2_n)^{-1} = O(k^{-1}), \]
for \( i = 1, 2 \).

Moreover, due to \( (\hat{T}_n^2 T_{n+1} + \hat{T}^2_n)^{-1} = O(k^{-1}) \), we have by Sherman-Morrison-Woodbury formula
\[ (T_n + \hat{T}_n^2 T_{n+1} + \hat{T}^2_n)^{-1} = I_k + (T_n + \hat{T}_n^2 T_{n+1} + \hat{T}^2_n)^{-1} - (T_n + \hat{T}_n^2 T_{n+1} + \hat{T}^2_n)^{-1} \]
for \( k \) sufficiently large. Then by the above identity, Lemma 3.4 and (3.7), it can be obtained that the asymptotic leading terms of \( -(\hat{T}_n^2 T_n + \hat{T}^2_n)^{-1} = O(1) \) and \( (\hat{T}_n^2 T_{n+1} + \hat{T}^2_n)^{-1} = O(k^{-1}) \).

To this end, we can obtain
\[ \Lambda_i(k) = [U_i] [u_i v_i] \]
\[ = \begin{bmatrix} I_{n+1} & 0 \\ -\psi_{n+1}^T (\hat{T}_n^2 T_{n+1} + \hat{T}^2_n)^{-1} (\phi_{n+2}^T w_2 + \phi_{n+2}) + \omega_1 w_2 + \omega_1^2 \\ 0 \end{bmatrix} \]
\[ + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \]
\[ = O(k^{-1}) \]
for \( i = 1, 2 \) by Lemma 3.4. If we focus on the \( O(1) \) term of \( \Lambda_i(k) \), using the notations (3.5) and Lemma 3.5, a direct calculation yields
\[ \begin{bmatrix} -\psi_{n+1}^T (\hat{T}_n^2 T_{n+1} + \hat{T}^2_n)^{-1} (\phi_{n+2}^T w_2 + \phi_{n+2}) + \omega_1 w_2 + \omega_1^2 \\ 0 \end{bmatrix} \]
\[ = \begin{bmatrix} \frac{1}{\delta} \begin{bmatrix} f_\omega g_\omega \\ f_\omega g_\omega \end{bmatrix} \begin{bmatrix} \gamma^k \\ \delta^k \end{bmatrix} \end{bmatrix} \]
for \( i = 1, 2 \).
Here, \( f_u, f_v, g_u \) and \( g_v \) are constants independent of \( k \). Due to \( |\gamma| = 1 \) and \( |\delta| = 1 \), we represent \( \gamma \) and \( \delta = e^{i\theta} \) with \( 0 \leq \theta_i < 2\pi \), \( i = 1, 2 \). Therefore, we can get for \( i = 1, 2 \)
\[
\mathbf{A}_i(k) = \left[ U_i \left[ \begin{array}{c} u_i \ v_i \end{array} \right] \left[ \begin{array}{c} f_u \\ f_v \end{array} \right] \left[ \begin{array}{c} e^{\theta_1 k} \\ e^{\theta_2 k} \end{array} \right] \right] + \mathbf{M}_i(k)
\]
\[
= \left[ U_i e^{i\theta_1 k} (f_u u_i + f_v v_i) \right] + \left[ 0 e^{i\theta_2 k} (g_u u_i + g_v v_i) \right] + \mathbf{M}_i(k)
\]
\[
= \left[ U_i (f_u u_i + f_v v_i) \right] + e^{i\theta_1 k} \zeta_i e_n^H + \mathbf{M}_i(k) \cdot (I_{n_1+n_2} \oplus e^{i\theta_1 k})
\]
\[
= \left( \tilde{U}_i + e^{i\theta_1 k} \zeta_i e_n^H + \mathbf{M}_i(k) \right) \cdot (I_{n_1+n_2} \oplus e^{i\theta_1 k}),
\]
where
\[
\zeta_i \triangleq g_u u_i + g_v v_i,
\]
\[
\mathbf{M}_i(k) \triangleq \mathbf{M}_1(k) \cdot (I_{n_1+n_2} \oplus e^{-i\theta_1 k}),
\]
\[
\tilde{U}_i \triangleq \left[ U_i \left[ \begin{array}{c} f_u u_i + f_v v_i \end{array} \right] \right],
\]
\[
\theta \triangleq \theta_2 - \theta_1.
\]
Set
\[
\hat{\mathbf{A}}_i(k) \triangleq \tilde{U}_i + e^{i\theta_1 k} \zeta_i e_n^H + \mathbf{M}_i(k)
\]
for \( i = 1, 2 \). If \( \tilde{U}_1 \) is invertible and \( 1 + e^{i\theta_0 k} e_n^H \tilde{U}_1^{-1} \zeta_1 \neq 0 \), then Sherman-Morrison-Woodbury formula implies that \( \hat{\mathbf{A}}_i(k) \) is invertible when \( k \) is large and
\[
\hat{\mathbf{A}}_i^{-1}(k) = \left( \tilde{U}_1 + \mathbf{M}_i(k) \right)^{-1} - \frac{\left( \tilde{U}_1 + \mathbf{M}_i(k) \right)^{-1} e^{i\theta_1 k} \zeta_i e_n^H (\tilde{U}_1 + \mathbf{M}_i(k))^{-1}}{1 + e^{i\theta_0 k} e_n^H (\tilde{U}_1 + \mathbf{M}_i(k))^{-1} \zeta_1} + O(k^{-1}).
\]
Then due to \( \hat{\mathbf{M}}_2(k) = O(k^{-1}) \), we can obtain
\[
W(k) = Y(k)X(k)^{-1} = \hat{\mathbf{A}}_2(k) \hat{\mathbf{A}}_1^{-1}(k)
\]
\[
= \left( \tilde{U}_2 + e^{i\theta_0 k} \zeta_2 e_n^H + \mathbf{M}_2(k) \right) \cdot \tilde{U}_1^{-1} - e^{i\theta_1 k} \tilde{U}_2 \tilde{U}_1^{-1} \zeta_1 e_n^H \tilde{U}_1^{-1} + O(k^{-1})
\]
\[
= \tilde{U}_2 \tilde{U}_1^{-1} - e^{i\theta_1 k} \tilde{U}_2 \tilde{U}_1^{-1} \zeta_1 e_n^H \tilde{U}_1^{-1}
\]
\[
+ e^{i\theta_0 k} e_n^H \tilde{U}_1^{-1} - \frac{e^{i\theta_0 k} e_n^H \tilde{U}_1^{-1} \zeta_1} {1 + e^{i\theta_0 k} e_n^H \tilde{U}_1^{-1} \zeta_1} + O(k^{-1})
\]
\[
= \tilde{U}_2 \tilde{U}_1^{-1} - e^{i\theta_1 k} \tilde{U}_2 \tilde{U}_1^{-1} \zeta_1 e_n^H \tilde{U}_1^{-1} + e^{i\theta_0 k} \frac{\zeta_2 e_n^H \tilde{U}_1^{-1}} {1 + e^{i\theta_0 k} e_n^H \tilde{U}_1^{-1} \zeta_1} + O(k^{-1})
\]
\[
= \tilde{U}_2 \tilde{U}_1^{-1} + \frac{e^{i\theta_0 k}} {1 + e^{i\theta_0 k} e_n^H \tilde{U}_1^{-1} \zeta_1} (\zeta_2 - \tilde{U}_2 \tilde{U}_1^{-1} \zeta_1) e_n^H \tilde{U}_1^{-1} + O(k^{-1})
\]
as \( k \) is large. Therefore, the asymptotic behavior of \( W(k) \) in this case is related to orbits of a rotation map (see Theorem 1.2.1 in [8]). We can conclude that
(i) when \( \frac{\theta_0}{\theta_1} = \frac{p}{q} \), \( p, q \in \mathbb{Z} \), \( W(k) \) tends to the periodic orbit
\[
\tilde{U}_2 \tilde{U}_1^{-1} + \frac{1}{e^{-2\pi i k} + e_n^H \tilde{U}_1^{-1} \zeta_1} (\zeta_2 - \tilde{U}_2 \tilde{U}_1^{-1} \zeta_1) e_n^H \tilde{U}_1^{-1}, \quad k = 0, 1, 2, \ldots (p - 1);
(ii) when $\frac{\theta}{2\pi} \in \mathbb{Q}$, the asymptotic trajectory of $W(k)$ is dense in the closed orbit
$$3\overline{U}_2 \overline{U}_1^{-1} + \frac{1}{e^{-2\pi i \nu} + e^{H \overline{U}_1^{-1} \zeta_1}}(\zeta_2 - 3\overline{U}_2 \overline{U}_1^{-1} \zeta_1)e^{H \overline{U}_1^{-1}}, \quad \nu \in [0, 1),$$
and hence it forms the limit cycle of $W(k)$.

5. **Appendix. Sherman-Morrison-Woodbury formula [9]:**

Given an invertible $n \times n$ matrix $A$, an $n \times k$ matrix $U$, and a $k \times n$ matrix $V$, let $B$ be an $n \times n$ matrix such that $B = A + UV$. Then, assuming $(I_k + VA^{-1}U)$ is invertible, we have
$$B^{-1} = A^{-1} - A^{-1}U(I_k + VA^{-1}U)^{-1}VA^{-1}.$$

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