NO LOCAL DOUBLE EXPONENTIAL GRADIENT GROWTH IN HYPERBOLIC FLOW FOR THE EULER EQUATION

VU HOANG AND MARIA RADOSZ

Abstract. We consider smooth, double-odd solutions of the two-dimensional Euler equation in \([-1, 1)^2\) with periodic boundary conditions. It is tempting to think that the symmetry in the flow induces possible double-exponential growth in time of the vorticity gradient at the origin, in particular when conditions are such that the flow is “hyperbolic”. This is because examples [12] of solutions with \(C^{1,\gamma}\)-regularity were already constructed with exponential gradient growth. We analyze the flow in a small box around the origin in a strongly hyperbolic regime and prove that the compression of the fluid induced by the hyperbolic flow alone is not sufficient to create double-exponential growth of the gradient.

1. Introduction

The question whether solutions of the two-dimensional Euler equation in vorticity form

\[
\omega_t + u \cdot \nabla \omega = 0
\]

(1)

can exhibit strong gradient growth in time is a topic of ongoing interest. The best known upper bound predicts double-exponential growth in time:

\[
\|\nabla \omega\|_\infty \leq C_1 \exp(C_2 \exp(C_3 t))
\]

with constants \(C_i\) depending on the initial data. A natural and important question is: Are there flows for which this upper bound is attained? The problem can be considered in bounded domains with no-flow boundary conditions or in domains without a natural boundary (e.g. on the torus). For domains with boundary, a recent breakthrough by A. Kiselev and V. Šverák [8] answers the question affirmatively. For smooth solutions on the torus, the best known result so far was given by S. Denisov. In [4], he shows that at least superlinear gradient growth is possible and in [5] he provides an example of double-exponential growth for an arbitrarily long, but finite time interval. In the recent paper [12], A. Zlatós constructs initial data leading to exponential gradient growth, his solution is however in \(C^{1,\gamma}\) for some \(\gamma \in (0, 1)\) and not in \(C^2\).

In [8] the construction is based on imposing certain symmetries on the solution leading to a hyperbolic flow scenario. The presence of a boundary and the hyperbolic flow work nicely together, allowing the construction of examples with double-exponential gradient growth. Considering double-odd solutions, i.e.

\[
\omega(-x_1, x_2) = -\omega(x_1, x_2), \ \omega(x_1, -x_2) = -\omega(x_1, x_2),
\]

(2)

is a possible, natural way to replace the physical wall from [8] by the \(x_1\)-axis in order to try to create strong gradient growth in the bulk. This construction was employed in [12]. In [5], a perturbation argument starting from a non-smooth double-odd stationary

\[\text{Date: June 10, 2014.}\]
solution (see [1]) was used. Creating infinite-time double-exponential growth away from the boundary, however, is met with considerable difficulties.

It is interesting to notice that the result [8] is in some sense analogous to the still open blowup problem for the more singular surface quasigeostrophic equation. In SQG blowup means that the solution becomes singular in finite time whereas for the 2d Euler equation “blowup” would mean maximal (double-exponential) gradient growth on an infinite time interval. There are important conditional regularity results for the SQG equation such as [2], [3], where one studies a certain blowup scenario, in order to finally exclude it. An analogous “conditional regularity result” for 2d Euler equation would be to show that in certain scenarios maximal gradient growth does not occur. Since the possible motions of fluids are various and in general very complicated, studying scenarios is an invaluable method to gain insight into regularity problems of fluid mechanics.

Our goal in this paper is to prove such a conditional regularity result in the sense that a hyperbolic flow cannot create maximal gradient growth near the origin by itself when we start with double-odd $C^2$ initial data, provided a certain “upstream” control is assumed on the flow. This is an important step into understanding the double-odd hyperbolic scenario since we rule out the most promising candidate for a mechanism creating maximal gradient growth, i.e. the local hyperbolic compression. Our result does not imply impossibility of double-exponential growth in general, but makes the construction of examples much harder.

In some sense, the scenario considered here is complementary to the one considered by D. Cordoba for the SQG equation in [2], where a closing hyperbolic saddle is considered. There the solution stays smooth except for the possible closing of the saddle. In our scenario for 2d Euler, the hyperbolic saddle is fixed due to the symmetry ($\omega = 0$ on the coordinate axes), and we are asking if blowup can happen in another way.

Finally, we would like to mention the recent preprint [7], where a different approach is proposed to study whether double-exponential gradient growth can occur at an interior point (see also T. Tao’s blog [9] for a related discussion).

1.1. **Main result.** We consider (1) on $\mathbb{T} = [-1,1]^2$ with periodic boundary conditions and double-odd $C^2$ initial data $\omega_0$. The double-odd symmetry is preserved by the evolution and (2) implies that the origin is a stagnant point of the flow field for all times. Moreover, the flow on each coordinate axis is always directed along that axis. When considering smooth solutions $\omega \in C^1([0,\infty), C^2(\mathbb{T}))$, (2) also implies
\[ \omega = 0 \]
on the coordinate axes.

We will studying the flow in boxes of the form
\[ D = (0, \delta_1) \times (0, \delta_2), \quad \hat{D} = (0, \delta_1 + \delta_3) \times (0, \delta_2), \]
where $\delta_j$ are positive, but small and
\[ 0 < \delta_1 < \delta_2 < \delta_1 + \delta_3. \]
In a hyperbolic flow, which we will explain in detail in section 1.2, fluid particles are supposed to constanty enter the box $D$ from the right and leave on the top. Therefore we call $\hat{D} \setminus D$ *feeding zone*. The following definition formalizes the control we assume on the solution in the feeding zone. The meaning of the parameter $\alpha$ will become clear later.
Definition 1.1. Let \( \alpha \in (0, \frac{1}{4}) \). The box \( \hat{D} \) is said to satisfy the conditions of controlled feeding, with feeding parameter \( R \geq 0 \) if
\[
|\partial x_2, \omega(x, t)| \leq Rx_2^{1-\alpha}, \quad |\partial x_1, \omega(x, t)| \leq R \quad (x \in \hat{D} \setminus D)
\]
for all times \( t \geq 0 \).

We can think of the first inequality in (5) as a Hölder-version of a bound on \( \partial x_2, \omega \), keeping in mind that \( \partial x_1, \omega(x_1, 0, t) = 0 \) for all times. The concept of controlled feeding conditions allows us to study the evolution of \( \omega \) in \( D \) independent of the remaining flow.

Our main result is the following theorem.

Theorem 1.2. Fix \( 0 < \alpha < \frac{1}{4}, 0 < \delta_3 \). Let \( \omega \) be a smooth, double-odd solution of the Euler equation, and suppose the flow is hyperbolic near the origin. Let \( R > 0 \) be given. There exist small \( \delta_1, \delta_2 > 0 \), such that if \( \hat{D} \) satisfies the controlled feeding conditions with parameter \( R \), then
\[
\|\nabla \omega\|_{\Delta, \infty} \leq C_1 \exp(C_2 t) \quad (t \in [0, \infty))
\]
for some \( C_1, C_2 > 0 \).

This means that in this situation for maximal gradient growth near the origin one cannot rely on the hyperbolic compression alone but rather has to create in some other way a scenario where the feeding conditions are violated, i.e. there has to be a compression in \( x_2 \)-direction in the feeding zone.

1.2. The hyperbolic scenario. In order to give a definition of hyperbolic flow suitable for our purposes, we introduce the following important quantity. Let \( \alpha \in (0, \frac{1}{4}) \) be fixed. For a smooth, periodic function \( \omega \) we set
\[
M(x, t) := \max_{0 \leq y_1, y_2 \leq \max(x_1, x_2)} \left\{ \left| y_1^\alpha \frac{\partial \omega}{\partial x_1}(y, t) \right|, \left| y_2^\alpha \frac{\partial \omega}{\partial x_2}(y, t) \right| \right\}.
\]
Note that \( M(x, t) \) also depends on \( \omega \) and \( \alpha \). The velocity field \( u(x, t) := \nabla^\perp (-\Delta)^{-1} \omega \) for double-odd \( \omega \) (\( \omega \) with mean zero over \( \mathbb{T} \)) can be written in the form
\[
u_1(x, t) = -x_1Q_1(x, t), \quad u_2(x, t) = x_2Q_2(x, t)
\]
where \( Q_1, Q_2 \) are scalar fields given by certain integral operators (see (20)) acting on \( \omega \). The following definition says we regard the flow as hyperbolic if both \( Q_1 \) and \( Q_2 \) essentially have a positive lower bound, up to a term controlled by the quantity \( M(x, t) \).

Definition 1.3. Let \( \omega \) be a smooth solution of the Euler equation, and let \( \alpha \in (0, \frac{1}{4}) \) be fixed. We say that the flow is hyperbolic near the origin if there are constants \( \rho, A, \beta_0 > 0 \) for which the following condition is satisfied
\[
Q_i(x, t) + A |x|^{1-\alpha} M(x, t) \geq \beta_0 > 0 \quad (0 \leq x_1, x_2 \leq \rho)
\]
where \( i = 1, 2 \), and for all \( t \in [0, \infty) \).

By choosing the initial data \( \omega_0 \) suitably, we can ensure hyperbolic flow. One possible choice is, for example, choosing \( \omega_0 \) to be nonnegative in \([0, 1]^2\) and such that \( \omega_0 = 1 \) on a set of sufficiently large measure, as it was done in [8], [12]. This creates a situation where (9) is satisfied. The proof will be given in section 4. Physically, we then have compression of the fluid in the \( x_1 \)-direction and expansion of the fluid in \( x_2 \)-direction.
2. Gradient growth in the hyperbolic scenario

Before describing our approach, let us explain first why at first sight the hyperbolic scenario seems to be a good candidate for double-exponential growth. Namely, for $Q_1, Q_2$ we have the upper bounds

$$Q_1(x, t), Q_2(x, t) \lesssim \|\omega\|_\infty |\log(x_1^2 + x_2^2)|.$$  

If it were possible to create a situation where a lower bound of roughly the same order holds, i.e. $Q_1 \geq C|\log(x_1^2 + x_2^2)|$ over an infinitely long time interval, then for the particle trajectories lying on the $x_1$-axis (i.e. $X_2 = 0$)

$$X_1(t) \leq \exp(-C_1 \exp(C_2 t))$$

would hold, as seen by solving the ODE $\dot{X}_1 = -X_1 Q_1$. If, moreover one could arrange for the initial data $\omega_0$ to have suitable nontrivial values on the $x_1$-axis, then this would create double exponential gradient growth. However, and the simultaneous requirements of smoothness and double-odd symmetry of $\omega$, necessarily imply $\omega = 0$ on the axes. Moreover, it is highly unclear how a such strong lower bound on $Q_1$ could be achieved. As we shall see later, a certain amount of smoothness of $\omega$ and the vanishing of $\omega$ on the axes lead to a better upper bound, without the logarithmic behavior which is crucial for the double-exponential growth.

Another way one might hope to get double exponential growth is to consider a “projectile”, i.e. to track the movement of a small domain close to the origin on which $\omega = 1$, as it was done in [8]. There the self-interaction of the projectile was able to create enough growth in the values of $Q_1$ to allow double-exponential growth. Namely, while the projectile approaches the origin, the values of $Q_1$ on it get larger, this fact being connected to a certain logarithmically divergent integral. Our Theorem 1.2 shows that in general this is not possible for double-odd solutions, unless there is some compression in $x_2$-direction in the feeding zone. Thus a scenario with maximal gradient growth must be much more complicated than just using the self-interaction of the projectile.

In fact, provided the feeding condition holds, the steady fluid compression guaranteed by (9) will turn out to stabilize the flow in the neighborhood of the origin. That is, the hyperbolicity condition (9) - essentially a lower bound on $Q_i$ - is converted in the proof of Theorem 1.2 into an upper bound for $Q_i$. This is what finally leads to a bound on the gradient growth in $D$.

2.1. Heuristic considerations. We now present an intuitive discussion of our result. Fluid particles carried by the hyperbolic flow will constantly enter the box $D$ from the right and leave on the top (see figure 1). All particles except for those moving on the axes spend a finite time in the box. As for the particles on the $x_1$-axis, these move towards the left, approaching the origin asymptotically as $t \to \infty$. Particle trajectories $t \mapsto X(t) = (X_1(t), X_2(t))$ for which $X_2(0)$ is small approximate the straight trajectories of the particles on the $x_1$-axis, before going steeply upward. The time a particle spends in $D$ goes to infinity as $X_2(0) \to 0$. We now consider the trajectory of a particle $X$. The particle may have started inside $D$ at time $t = 0$, or may have entered the box at some time $T_0 > 0$, in which case $X(T_0) \in \partial D$. Also, assume that the particle exits the box $D$ at some time $T_e$, i.e. $X_2(T_e) = \delta_2$. The evolution of the gradient of $\omega$ along the trajectory
is given by an ODE of the form
\[
\frac{d}{dt} \nabla \omega(X(t), t) = (-\nabla u)^T (X(t), t) \nabla \omega(X(t), t)
\]
where $\nabla u$ is the velocity gradient. The relation (11) is simply obtained by differentiating the Euler equation. The key is now to use the structure (8) of the velocity field. Combining with (11), we obtain
\[
\frac{d}{dt} \nabla \omega(X(t), t) = \begin{pmatrix}
Q_1 + x_1 \frac{\partial Q_1}{\partial x_1} & -x_2 \frac{\partial Q_2}{\partial x_1} \\
-x_1 \frac{\partial Q_1}{\partial x_2} & -Q_2 - x_2 \frac{\partial Q_2}{\partial x_2}
\end{pmatrix} \nabla \omega(X(t), t)
\]
We write the matrix in (12) as
\[
\begin{pmatrix}
a(t) & c(t) \\
b(t) & -a(t)
\end{pmatrix}
\]
evaluating all matrix entries along the given trajectory $X$ (note that the matrix has trace zero, since the velocity field $u$ is divergence free). Since in a sufficiently small box $x_1 \frac{\partial Q_1}{\partial x_1}, x_2 \frac{\partial Q_2}{\partial x_2}$ should be rather “small” (due to the prefactors $x_1, x_2$), $a$ should be positive and bounded away from zero along the hyperbolic trajectory. Roughly speaking, the the form of (12) implies that $\omega_{x_1}$ grows in time like $e^{\int_{T_0}^t a(X(s))ds}$ whereas $\omega_{x_2}$ should decay in time like $e^{-\int_{T_0}^t a(X(s))ds}$. This would be exactly true if (12) were a diagonal system.

To gain some insight, we consider the case of a particle moving close to the $x_1$-axis, i.e. with small $X_2(T_0) > 0$. We expect that $c = x_2 \frac{\partial Q_2}{\partial x_2}, b = -x_1 \frac{\partial Q_1}{\partial x_2}$ are “small”. This suggest to neglect $b, c$ and set $b, c = 0$ in (12), so that we have a diagonal system. Denoting $\xi(t) = \nabla \omega(X(t))$ the solution would be given by
\[
\xi_1(t) = e^{A(t)} \xi_1(T_0), \quad \xi_2(t) = e^{-A(t)} \xi_2(T_0).
\]
where $A(t) = \int_{T_0}^t a(X(s))ds$. (13) shows that, in general, the gradient in $x_1$-direction grows along the particle trajectory. However, there is an effect which allows us to cancel
the growing factor $e^A$. Assume for the sake of the discussion that the following stronger feeding conditions hold:

\begin{equation}
|\partial_{x_1} \omega(x,t)| \leq Rx_2, \ |\partial_{x_2} \omega(x,t)| \leq R
\end{equation}

These imply

\begin{equation}
|\xi_1(t)| \leq R e^A(t) X_2(T_0).
\end{equation}

Now we observe that

\begin{equation}
A(t) \approx \int_{T_0}^t Q_2(s) \, ds
\end{equation}

temporarily neglecting the term $x_2 \frac{\partial Q_2}{\partial x_2}$. Now from (8) we have the differential equation

\begin{equation}
\dot{X}_2 = X_2 Q_2, \text{ so that } X_2(T_0) = X_2(T_e) \exp \left( - \int_{T_0}^{T_e} Q_2(X(s)) ds \right)
\end{equation}

Combining (17), (15) and (16), we get

\begin{equation}
|\xi_1(t)| \leq \delta_2 R \exp \left( - \int_{t}^{T_e} Q_2(X(s)) ds \right) \leq \delta_2 R
\end{equation}

(we assume $Q_2 \geq 0$ for this heuristic discussion), suggesting that the gradient in $x_1$-direction does not grow at all in time. Our rigorous result does not give such a strong conclusion, but we will be able to prove that the gradient grows at most exponentially in time. In Remark 4.6 we explain why we actually do not use (14).

The heuristics appear deceivingly simple, but in order to make the argument rigorous, we have to overcome a number of formidable technical difficulties. To begin with, the coefficients of (12) depend on the solution $\omega$ through the integral operators $Q_1, Q_2$. The derivatives $\frac{\partial Q_1}{\partial x_1}, \frac{\partial Q_2}{\partial x_2}$ are given by singular integral operators. These can be controlled if one has control over the first derivatives $\frac{\partial \omega}{\partial x_1}, \frac{\partial \omega}{\partial x_2}$ of $\omega$ inside the box, and thus one has a certain control over the coefficients of the ODE system (12).

Of course, none of the coefficients may be neglected, and we have to produce sufficiently good estimates on the solutions of the full ODE system (12). A major obstacle in getting good estimates, however, is caused by the unstable nature of (12). This may be seen, e.g. by setting $c = 0$, but keeping $b$, so that we get a supposedly better approximation than the diagonal system. In this model, the solutions can be calculated explicitly, and we get

\begin{equation}
\xi_1(t) = e^A(t) \xi_1(T_0), \ \xi_2(t) = e^{-A(t)} \left[ \xi_2(T_0) + \xi_1(T_0) \int_0^t b(s) e^{2A(s)} \, ds \right].
\end{equation}

This shows that not only the derivative in $x_1$-direction but also the derivative in $x_2$-direction of $\omega$ may potentially grow in time (due to the contribution $e^{-A(t)} \int_0^t b(s) e^{2A(s)} \, ds$), making things worse, since a possible strong growth in $\frac{\partial \omega}{\partial x_2}$ is coupled back into the coefficients of the ODE (12) via our estimates on $\frac{\partial Q_1}{\partial x_1}, \frac{\partial Q_2}{\partial x_2}$. On the other hand, the factor $\xi_1(T_0)$ may help as before, via the feeding condition (14). We need therefore to proceed with extreme care, looking to cancel the growing factor $e^A$ with the decaying factor $e^{-A}$ whenever possible.
3. Notation

3.1. Euler velocity field. For \( x = (x_1, x_2) \) we write \( \tilde{x} = (-x_1, x_2) \) and \( \bar{x} = (x_1, -x_2) \). The velocity field for the Euler equation is

\[
\begin{align*}
  u(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(y - x)\perp}{|y - x|^2} \omega(y, t) \, dy, \\
  \omega \in C^2(\mathbb{T}) &\text{ is periodically extended to all of } \mathbb{R}^2. 
\end{align*}
\]

where \( \omega \) is the velocity field is \( \nabla^2 (\Delta)^{-1} \omega \), where \( -\Delta \) is the periodic Laplacian on the Torus \( \mathbb{T} \). A simple calculation using the double-odd symmetry of \( \omega \) leads to

\[
\begin{align*}
  u_1(x, t) &= -x_1 Q_1(x, t), \\
  u_2(x, t) &= x_2 Q_2(x, t)
\end{align*}
\]

where \( Q_1, Q_2 \) are the following integral operators

\[
\begin{align*}
  Q_1(x, t) &= c_0 \int_{[0,1]^2} \left[ G^1_1(x, y) + G^2_1(x, y) \right] \omega(y) \, dy + Q^*_1(x, t) \\
  Q_2(x, t) &= c_0 \int_{[0,1]^2} \left[ G^1_2(x, y) + G^2_2(x, y) \right] \omega(y) \, dy + Q^*_2(x, t)
\end{align*}
\]

with kernels

\[
\begin{align*}
  G^1_1(x, y) &= \frac{y_1(y_2 - x_2)}{|y - x|^2 |y - \tilde{x}|^2}, \\
  G^2_1(x, y) &= \frac{y_1(y_2 + x_2)}{|y + x|^2 |y - \tilde{x}|^2}, \\
  G^1_2(x, y) &= \frac{y_2(y_1 + x_1)}{|y + x|^2 |y - \bar{x}|^2}, \\
  G^2_2(x, y) &= \frac{y_2(y_1 - x_1)}{|y - x|^2 |y - \bar{x}|^2},
\end{align*}
\]

where \( c_0 \) denotes the right constant. The expression \( Q^*_1 \) is given by the following (limit in the mean) integral

\[
c_0 \int_{\mathbb{R}^2 \setminus [0,1]^2} \left[ G^1_1(x, y) + G^2_1(x, y) \right] \omega(y) \, dy,
\]

a similar formula holding for \( Q^*_2 \).

3.2. Convention for estimates. The notation \( f \lesssim g \) means

\[
f \leq C g,
\]

where \( C \) may depend on \( \alpha, \beta, \|\omega\|_{\infty} \) and on universal constants, e.g. geometrical characteristics of the domain \( \mathbb{T} \). \( C \) does not depend on \( \delta_1, \delta_2, \delta_3 \). When using this notation, we shall always imply that \( C < \infty \) for all \( \alpha \in (0, \frac{1}{4}) \).

4. Potential theory of \( Q_1, Q_2 \)

4.1. Sufficient conditions for hyperbolic flow. We will be working with boxes of the form

\[
\begin{align*}
  D &= (0, \delta_1) \times (0, \delta_2) \\
  \hat{D} &= (0, \delta_1 + \delta_3) \times (0, \delta_2)
\end{align*}
\]

with the following restriction:

\[
0 < \delta_1 < \delta_2 < \delta_1 + \delta_3.
\]
and $\delta_j$ so small that $\hat{D} \subset [0,1]^2$. We also write
\begin{equation}
(24) \quad d(x) = \delta_2 - x_2
\end{equation}
which is the distance of the point $x$ to the top of the box. We write $\delta = (\delta_1, \delta_2)$, $|\delta|^2 = \delta_1^2 + \delta_2^2$.

We define
\begin{equation}
(25) \quad M_D(t) := \max_{y \in D} \left\{ \left| y_1 \frac{\partial \omega}{\partial x_1}(y,t) \right|, \left| y_2 \frac{\partial \omega}{\partial x_2}(y,t) \right| \right\}
\end{equation}
and $M_{\tilde{D}}$ for the analogous quantity, but where the maximum over $D$ is replaced by a maximum over $\tilde{D}$. Note that $M_D$ and $M_{\tilde{D}}$ depend on $\omega$ and $\alpha$.

As mentioned before, the flow near the origin can be made hyperbolic, with compression in the $x_1$-direction and expansion in $x_2$-direction by choosing the initial data such that $\omega_0 \geq 0$ on $[0,1]^2$ such that
\begin{equation}
(26) \quad m := |\{ x : \omega_0(x) = \|\omega_0\|_\infty \}|
\end{equation}
is sufficiently large. This is a consequence of theorem [4.2]

Remark 4.1. The periodicity and double-oddness of $\omega(\cdot, t)$ imply also the reflection symmetries
\begin{equation}
\omega(1+x_1,x_2,t) = -\omega(1-x_1,x_2,t), \quad \omega(x_1,1+x_2) = -\omega(x_1,1-x_2).
\end{equation}
Consequently, the four corner points of $[-1,1] \times [-1,1]$ are also stagnant points of the flow, the flow being confined in $[0,1]^2$. Hence $\omega_0 \geq 0$ on $[0,1]^2$ implies $\omega(x,t) \geq 0$ on $[0,1]^2$ for all times, a fact we shall use below.

Theorem 4.2. Suppose $\omega_0(x) \geq 0$ on $[0,1]^2$. There exists a universal $0 < m_0 < 1, 0 < K$ such that if $m_0 < m < 1$, there are $\beta_0 > 0, A > 0$ such that the following estimate holds for all times
\begin{align}
Q_2(x,t) + AM(x,t)|x|^{1-\alpha} &\geq \beta_0 \\
Q_1(x,t) + AM(x,t)|x|^{1-\alpha} &\geq \beta_0
\end{align}
for $|x| \leq K(1-m)$, i.e. the flow is hyperbolic near the origin.

To prove this, we need the following lemma, which is an adaption of a result in [12].

Lemma 4.3. Let $\Omega(2x) := [2x_1,1] \times [2x_2,1]$. Suppose $\omega(x) \geq 0$ for $x \in [0,1]^2$. Then the estimate
\begin{equation}
(28) \quad Q_i(x) \geq c_0 \int_{\Omega(2x)} \frac{y_1y_2}{|y|^4} \omega(y) \, dy - M(x,t)|x|^{1-\alpha} - C_2\|\omega\|_\infty \quad (x \in D, \ i = 1,2)
\end{equation}
holds, with universal $C_2 > 0$.

Proof. We write $G_2 = G^1_2 + G^2_2$ and prove the result for $Q_2$. The proof for $Q_1$ is similar. We have
\begin{align*}
Q_2(x) \geq &c_0 \int_{\Omega(2x)} \frac{y_1y_2}{|y|^4} \omega(y) \, dy + \int_{\Omega(2x)} \left[ G^2_2(x,y) - c_0 \frac{y_1y_2}{|y|^4} \right] \omega(y) \, dy \\
&+ \int_{[0,1]^2 \setminus \Omega(2x)} G^2_2(x,y)\omega(y) \, dy - C_1\|\omega\|_\infty,
\end{align*}
throwing away the nonnegative contribution from $G_2^1$ and estimating $Q_2^1$ by $C_1\|\omega\|_\infty$. First, note that straightforward calculations and estimations give

$$G_2(x, y) - c_0 \frac{y_1 y_2}{|y|^4} \lesssim \left( |y - x| + |y|^2 \right) \left( |x| + |y| \right) \frac{|y|^2 |y - x|^2 |y - \bar{x}^2|}{|y|^2 |x| + |y|}.$$ \hspace{1cm} (29)

$y \in \Omega(2x)$ implies that $|y - x| \geq \frac{1}{2} |y|, |y - \bar{x}| \geq \frac{1}{2} |y|$ so that

$$\left| G_2(x, y) - c_0 \frac{y_1 y_2}{|y|^4} \right| \lesssim (|y|^{-4} + |y|^{-2})(|x|^2 + |x||y|)$$ \hspace{1cm} (30)

and hence the integral over $\Omega(2x)$ is bounded in absolute value by

$$|x|^2 \int_{|y| \geq 2|x|^2} (|y|^{-4} + |y|^{-2}) \, dy + |x| \int_{|y| \geq 2|x|^2} (|y|^{-3} + |y|^{-1}) \, dy$$

$$\leq C|x|^2(|x|^{-2} + \log |x|) + C|x|(|x|^{-1} + 1) \leq C.$$ \hspace{1cm} For the estimation of the integral with domain of integration $[0, 1]^2 \setminus \Omega(2x)$, we distinguish two cases. The more difficult case is given by the condition $x_2 \leq x_1$, and we split the domain of integration up into the three parts $[2x_1, 1] \times [0, 2x_2], [0, 2x_1] \times [2x_1, 1]$ and $[0, 2x_1] \times [0, 2x_1]$. For the integral over $[2x_1, 1] \times [0, 2x_2]$, estimate $\omega$ by its $L^\infty$-norm and in the remaining integral we substitute $y_j = x_j + z_j$.

$$\int_{x_1}^{x_1} \int_{x_1}^{x_2} \frac{z_1(x_2 + z_2)}{(z_1^2 + z_2^2)(z_1^2 + (2x_2 + z_2)^2)} \, dz \leq \frac{2z_1x_2}{(z_1^2 + z_2^2)(z_1^2 + x_2^2)} \, dz \, dz_1$$

$$\leq C \int_0^1 \frac{z_1x_2}{(z_1^2 + x_2^2)} \, dz_1 \leq C \arctan(1/x_2) \leq C.$$ \hspace{1cm} The same strategy for the integral over $[0, 2x_1] \times [2x_1, 1]$ leads to

$$\int_{-z_1}^{x_1} \int_{z_1}^{x_2} \frac{|z_1(x_2 + z_2)}{(z_1^2 + z_2^2)(z_1^2 + z_2^2)} \, dz \leq \frac{2z_1x_2}{(z_1^2 + z_2^2)(z_1^2 + (2x_2 + z_2)^2)} \, dz.$$ \hspace{1cm} Noting

$$\int_0^1 \frac{z_1x_2}{(z_1^2 + z_2^2)(z_1^2 + (2x_2 + z_2)^2)} \, dz_2 \, dz_1$$

$$\leq C \int_0^1 \frac{x_2}{z_1^2 + x_2^2} \, dz_1 \leq C \arctan(x_1/x_2) \leq C$$ \hspace{1cm} and

$$\int_0^1 \int_{z_1}^{x_2} \frac{z_1z_2}{(z_1^2 + z_2^2)(z_1^2 + (2x_2 + z_2)^2)} \, dz \leq \frac{x_1}{(z_1^2 + z_2^2)} \, dz_2 \, dz_1$$

we can estimate the integral in question by $C\|\omega\|_\infty$.\hspace{1cm}
It remains to estimate the integral over \([0, 2x_1] \times [0, 2x_1]\). First note that
\[
\int_{[0, 2x_1] \times [0, 2x_1]} G_2^2(x, y) \omega(y) \, dy \geq \int_{[0, x_1] \times [0, 2x_1]} G_2^2(x, y) \omega(y) \, dy.
\]
since \(\omega \geq 0\) and \(G_2^2(x, y) \geq 0\) if \(y_1 \leq x_1\). We will estimate the integral over \([0, x_1] \times [0, 2x_1]\) in absolute value, splitting it again into \([0, x_1] \times [0, x_1]\) and \([0, x_1] \times [x_1, 2x_1]\). First, writing \(M = M(x, t)\),
\[
\left| \int_{[0, x_1] \times [0, x_1]} G_2^2(x, y) \omega(y) \, dy \right| \leq \int_{[0, x_1] \times [0, x_1]} \frac{M y_2^{1-\alpha}}{|y - x||y - \bar{x}|} \, dy \leq \int_{[0, x_1] \times [0, x_1]} M |y - x|^{1-\alpha} \, dy \leq \int_{B(x, r)} M |y - x|^{1-\alpha} \, dy \leq M r^{1-\alpha}
\]
where \(B(x, r)\) is the smallest ball around \(x\) containing \([0, 2x_1] \times [0, 2x_1]\). Clearly \(r \lesssim x_1\), so the integral is \(\lesssim M x_1^{1-\alpha}\).

Next, for the remaining part over \([0, x_1] \times [x_1, 2x_1]\), we estimate \(\omega\) by \(|\omega|_\infty\). We need to bound
\[
\int_{[0, x_1] \times [x_1, 2x_1]} |G_2^2(x, y)| \, dy = \int_{-x_1}^{0} \int_{x_1 - x_2}^{2x_1 - x_2} \frac{|z_1| |z_2|}{(z_1^2 + z_2^2)(z_1^2 + (2x_2 + z_2)^2)} \, dz \, dx
\]
and
\[
\int_{x_1}^{0} \int_{x_1 - x_2}^{2x_1 - x_2} \frac{|z_1| |z_2|}{(z_1^2 + z_2^2)(z_1^2 + (2x_2 + z_2)^2)} \, dz \, dx.
\]
For the integral containing \(|z_1| |z_2|\) we distinguish two cases. In case \(x_2 \leq \frac{1}{2} x_1\), we use \(z_1^2 + (2x_2 + z_2)^2 \geq z_1^2 + z_2^2\), leading to a bound on the form \(\log(1 + \frac{x_1}{z_1 - x_2}) \leq C\). If \(x_2 \geq \frac{1}{2} x_1\), we use \(z_1^2 + (2x_2 + z_2)^2 \geq (x_2 + z_2)^2\) in the denominator and \(z_2 \leq (z_2 + x_2)\) in the nominator and get the bound \(C x_2^{-1} x_1 \leq C\). The integral with \(|z_1| |z_2|\) is estimated as before.

If \(x_1 \leq x_2\), we split \([0, 1]^2 \setminus \Omega(2x)\) into \([0, 1] \times [0, x_2]\), \([0, 2x_1] \times [2x_2, 1]\) and perform similar calculations. In this case, we do not need to use \(M(x, t)\). \qed

**Proof.** (of theorem 4.2) Following [8, 12] we observe that the integral \(\int_{\Omega(2x)} y_1 y_2 |y|^{-4} \omega(y, t) \, dy\) can be bounded away from zero by an expression of the form \(C_1 |\omega|_\infty \log(1 - m)|\), for \(|x| \leq K(1 - m)\). with universal \(C_1, K > 0\). Hence we obtain (27). \qed

### 4.2. Upper bounds.
The following lemma gives an upper bound on \(Q_1, Q_2\), in terms of \(M_B(t)\). Recall that \(d(x)\) is the distance to the top of the box, so the upper bound given blows up close to the top of the box. This is, however not a problem, since we mostly have to integrate \(Q_1, Q_2\) along particle trajectories (see the proof Theorem 6.3).

**Lemma 4.4.** For \(x \in D\),
\[
Q_i(x) \lesssim C |\omega|_\infty (1 + |\log d(x)|) + M_B(t) |\delta|^{1-\alpha} \quad (i = 1, 2)
\]
**Proof.** We bound \(Q_2\), the calculation for \(Q_1\) is analogous. First we note
\[
|G_k^2| \lesssim |x - y|^{-1} |y - \bar{x}|^{-1} \quad (k = 1, 2)
\]
for \( y, x \in [0, 1]^2 \). We write \( M = M_D(t) \), and split the integral in question into two parts:

\[
\int_{[0,1]^2} G_2^k(x, y) \omega(y) \, dy = \int_{\hat{D}} \ldots + \int_{[0,1]^2 \setminus \hat{D}} \ldots
\]

Since \( |\omega(y)| \lesssim My_2^{1-\alpha} \),

\[
\left| \int_{\hat{D}} G_2^k(x, y) \omega(y) \, dy \right| \lesssim M \int_{\hat{D}} y_2^{1-\alpha} |x - y|^{-1} |y - \bar{x}|^{-1} \, dy
\]

\[
\lesssim M \int_{\hat{D}} |x - y|^{-1} |y - \bar{x}|^{-\alpha} \, dy \lesssim M \int_{\hat{D}} |x - y|^{-1-\alpha} \, dy
\]

\[
\leq M \int_{B(x, r)} |x - y|^{-1-\alpha} \, dy \leq Mr^{-\alpha}
\]

where \( B(x, r) \) is the smallest ball centered at \( x \) containing \( \hat{D} \). Obviously \( r \lesssim |\delta| \), so the part over \( \hat{D} \) is dominated by \( M|\delta|^{1-\alpha} \).

For the part over \( [0, 1]^2 \setminus \hat{D} \), we have

\[
\left| \int_{[0,1]^2 \setminus \hat{D}} G_2^k(x, y) \omega(y) \, dy \right| \lesssim ||\omega||_\infty \int_{[0,1]^2 \setminus \hat{D}} |y - x|^{-2} dy
\]

\[
\lesssim ||\omega||_\infty \int_{B(x,10) \setminus B(x, d(x))} |y - x|^{-2} dy \lesssim ||\omega||_\infty |\log d(x)|
\]

where we have used \( |G_2^k| \lesssim |x - y|^{-1} |y - \bar{x}|^{-1} \) and \( |y - \bar{x}| \geq |y - x| \) for \( x, y \in [0, 1]^2 \). Note also that for \( x \in D, [0, 1]^2 \setminus \hat{D} \) is completely contained in \( B(x, 10) \setminus B(x, d(x)) \) because of (23).

\[\square\]

The following important lemma allows us to control the coefficients of the ODE system (12) in terms of the quantity \( M_D \). Recall that \( d(x) \) is the distance from \( x \in D \) to the top of the box.

**Lemma 4.5.** We have the following estimates for \( x \in D \):

\[
|c(x)| \leq C(\alpha)M_Dx_2^{1-\alpha} + C(\alpha, \gamma_1, \gamma_2)x_2^{1-\gamma_1-\gamma_2}x_1^{\gamma_2}d(x)^{-1+\gamma_1+\gamma_2},
\]

\[
|b(x)| \leq C(\alpha)M_Dx_1^{1-\alpha}(1 + |\log d(x)|) + C(\alpha, \gamma)x_1^{1-\gamma}d(x)^{-1+\gamma},
\]

\[
\left| x_i \frac{\partial Q_i(x)}{\partial x_i} \right| \leq C(\alpha)M_Dx_i^{1-\alpha}(1 + |\log d(x)|) + C(\alpha, \gamma)x_i^{1-\gamma}d(x)^{-1+\gamma}
\]

where \( \gamma, \gamma_1, \gamma_2 \in (0, 1), \gamma_1 + \gamma_2 = 1, i = 1, 2 \) and the constants do not depend on \( \delta_1, \delta_2, \delta_3 \).

**Proof.** This is a consequence of Proposition 8.9 (see appendix) and the definition of \( c, b, x_i\partial_{x_i} Q_i(x) \). Note that we have

\[
\left| \frac{\partial Q_i^r(x)}{\partial x_j} \right| \leq C||\omega||_\infty
\]

for \( x \in D \). \[\square\]
Remark 4.6. It is not possible to set $\alpha = 0$ in the estimates of Lemma 4.5, i.e. if we replace $M_{\tilde{D}}$ by $\|\nabla \omega\|_{D,\infty}$, then e.g. the first term on the right-hand side of the estimate for $c$ would contain a logarithmic expression

$$\|\nabla \omega\|_{D,\infty} \cdot x_{2} \log x_{2}.$$ 

This is the main reason why we do not adopt the stronger feeding condition (14), since we do not know how to deal with the logarithmic terms in our main argument.

5. Perturbation theory for an ordinary differential equation

In this section we derive estimates for an ODE system of the form

$$\dot{\xi}(t) = \begin{pmatrix} a(t) & c(t) \\ b(t) & -a(t) \end{pmatrix} \xi(t)$$

where $a, b, c$ are given smooth functions on a time interval $[T_{0}, T_{e}]$. For simplicity of notation, we set $T_{0} = 0$. This part is independent of the actual structure of $a, b, c$ from the ODE (12).

The idea will be to perturb from the system with $c \equiv 0$. We write

$$P(t) := \begin{pmatrix} a(t) & 0 \\ b(t) & -a(t) \end{pmatrix}, \quad S(t) := \begin{pmatrix} 0 & c(t) \\ 0 & 0 \end{pmatrix}$$

Definition 5.1. Let the integral operators $\hat{P}, \hat{S}$ be given by

$$\hat{P} \xi(t) = \int_{0}^{t} P(\tau) \xi(\tau) \, d\tau, \quad \hat{S} \xi(t) = \int_{0}^{t} S(\tau) \xi(\tau) \, d\tau$$

Recall that $A(t) = \int_{0}^{t} a(s)ds$. It is convenient to introduce the following operators:

$$F^{+} g(t) = g(t) + e^{A} \int_{0}^{t} a e^{-A} g(s) ds$$

$$F^{-} g(t) = g(t) - e^{-A} \int_{0}^{t} a e^{A} g(s) ds.$$ 

Proposition 5.2. (a) The operator $(I - \hat{P})$ is bounded and bijective as an operator from $C[0,T]$ into $C[0,T]$.

(b) Consider the Volterra integral equation

$$\phi = \hat{P} \phi + g$$

with given $g \in C([0,T], \mathbb{R}^2)$. The solution $\phi = (I - \hat{P})^{-1} g$ is given by

$$\phi_{1}(t) = F^{+} g_{1}$$

$$\phi_{2}(t) = F^{-} g_{2} + e^{-A} \int_{0}^{t} e^{A} b(F^{+} g_{1})(s) ds$$

Proof. The statement (a) is standard. Statement (b) is an easy calculation, noting that (39) is equivalent to the ODE system $\dot{\xi} = P \xi + \dot{g}$ for $g \in C^{1}$. \qed
The initial value problem for the system
\[ \dot{\xi} = (P + S)\xi, \quad \xi(0) \text{ given} \]
is equivalent to the Volterra integral equation
\[ \xi = (\hat{P} + \hat{S})\xi + \xi(0). \tag{41} \]
We can write \( \xi = (I - \hat{P})^{-1}w \) for some \( w \in C[0, T] \). This leads to
\[ w = \hat{S}(I - \hat{P})^{-1}w + \xi(0). \tag{42} \]
The following proposition gives a representation of the solution \( \xi \) in terms of \( w \):

**Proposition 5.3.** Let \( \xi \in C[0, T] \) solve the integral equation (41) with given \( \xi(0) \). Then
\[ \begin{align*}
\xi_1(t) &= \xi_1(0) + \xi_2(0) \int_0^t e^{-A}ds + \int_0^t e^{-A} c \int_0^s e^{A b(F^+ w_1)(\tau)} d\tau, \\
\xi_2(t) &= \xi_2(0) + e^{-A} \int_0^t e^{A b(F^+ w_1)} ds.
\end{align*} \tag{43} \]

**Proof.** First note that
\[ \hat{S}(I - \hat{P})^{-1}w = \hat{S}(\xi) = (\int_0^t c(s) \xi_2(s) ds, 0) \tag{44} \]
and hence by (42), \( w_2(t) = \xi_2(0) \) (the second line of (43)). Recalling \( \xi = (I - \hat{P})^{-1}w \) and using (40), we get the following relation:
\[ \begin{align*}
\xi_2(t) &= \xi_2(0)[1 - e^{-A} \int_0^t e^{A s} ds] + e^{-A} \left[ \int_0^t e^{A b w_1(s)} ds + \int_0^t e^{2A b} \int_0^s e^{-A w_1} d\tau \right] \\
&= e^{-A} \left[ \xi_2(0) + \int_0^t e^{A b w_1(s)} ds + \int_0^t e^{2A b} \int_0^s e^{-A w_1} d\tau \right] \\
&\quad \tag{45}
\end{align*} \]
(44) and (42) together give,
\[ w_1(t) = \xi_1(0) + \int_0^t c(s) \xi_2(s) ds. \]
By inserting (45), we get the relation
\[ w_1(t) = \xi_2(0) \xi_1(0) + \int_0^t e^{-A} c \\
+ \int_0^t e^{-A} c \int_0^s e^{A b w_1} + \int_0^t e^{-A} c \int_0^s e^{2A b} \int_0^\tau e^{-A w_1}, \]
which is the first line of (43). \qed

We will need the following Gronwall-type inequality by Wilett [11]:

**Lemma 5.4.** Let \( z, f_0, f_1, f_2, v_1, v_2, v_3 \) are nonnegative, integrable functions on \([0, T]\) and suppose \( z \) satisfies the following integral inequality:
\[ z(t) \leq f_0(t) + f_1(t) \int_0^t v_1 z + f_2(t) \int_0^t v_2 z. \tag{46} \]
Then \( z \leq Hf_0 \), where \( H \) is the following functional

\[
(Hf_0)(t) = f_0 + f_1 \exp \left( \int_0^t v_1 f_1 \right) \int_0^t v_1 f_0 \\
+ \left[ f_2(t) + f_1(t) \exp \left( \int_0^t v_1 f_1 \right) \int_0^t v_1 f_2 \right] \\
\times \exp \left( \int_0^t v_2 \left[ f_2(s) + f_1(s) \exp \left( \int_0^s v_1 f_1 \right) \int_0^s v_1 f_2 \right] \right) \\
\times \int_0^t v_2 \left[ f_0(s) + f_1(s) \exp \left( \int_0^s v_1 f_1 \right) \int_0^s v_1 f_0 \right]
\]

(47)

We write \( Hf_0 \) to emphasize the linear dependency on \( f_0 \).

**Proof.** We give the proof for reference. Recall first the following form basic of Gronwall’s integral inequality: suppose \( z, r, f_1, v_1 \) are nonnegative functions on \([0, T]\) satisfying the integral inequality

\[
z(t) \leq r(t) + f_1(t) \int_0^t v_1 z,
\]

(48) then

\[
z(t) \leq r(t) + f_1(t) \exp \left( \int_0^t v_1 f_1 \right) \int_0^t v_1 r \quad (t \in [0, T]).
\]

(49)

Set \( r = f_0 + f_2 \int_0^t v_2 z \) and apply (49). This leads to the following bound for \( z \):

\[
z(t) \leq f_0 + f_2 \int_0^t v_2 z + f_1(t) \exp \left( \int_0^t v_1 f_1 \right) \int_0^t v_1 \left[ f_0 + f_2 \int_0^s v_2 z \right].
\]

(50)

Note that

\[
\int_0^t v_1 f_2 \int_0^s v_2 z = -\int_0^t \left( \int_0^s v_1 f_2 \right) v_2 u + \left( \int_0^t v_1 f_2 \right) \int_0^t v_2 z
\leq \left( \int_0^t v_1 f_2 \right) \int_0^t v_2 z
\]

since \( v_1, f_1, z, v_2 \geq 0 \). Thus (50) implies

\[
z(t) \leq f_0 + f_1 \exp \left( \int_0^t v_1 f_1 \right) \int_0^t v_1 f_0 + \\
+ \left[ f_2(t) + f_1(t) \exp \left( \int_0^t v_1 f_1 \right) \left( \int_0^t v_1 f_2 \right) \right] \int_0^t v_2 z.
\]

Applying (49) again, this time with \( r = f_0 + f_1 \exp \left( \int_0^t v_1 f_1 \right) \int_0^t v_1 f_0 \), yields the result (47). □
Lemma 5.5. Let \( \xi \in C[0,T] \) solve the integral equation (41) with given \( \xi(0) \). Then the estimates

\[
|\xi_1(t)| \leq (Hf_0)(t) + e^A \int_0^t |a|e^{-A}Hf_0
\]

(51)

\[
|\xi_2(t)| \leq |e^{-A}\xi_2(0)| + e^{-A} \left[ \int_0^t e^A|b|Hf_0 + \int_0^t e^{2A}|b| \int_0^t |a|e^{-A}Hf_0 \right],
\]

hold, where \( H \) is the functional (47) and where

\[
f_1(t) = \int_0^t e^{-A}|c| \int_0^s e^{2A}|b| ds,
\]

\[
f_2(t) = \int_0^t e^{-A}|c|,
\]

\[
f_0(t) = |\xi_1(0)| + f_2(t)|\xi_2(0)|,
\]

\[
v_1(t) = |a(t)|e^{-A},
\]

\[
v_2(t) = |b(t)|e^A.
\]

Proof. Using obvious estimations, we get from (43) the following integral inequality for \( |w_1| \):

\[
|w_1(t)| \leq |\xi_1(0)| + |\xi_2(0)| \int_0^t e^{-A}|c| + \int_0^t e^{-A}|c| ds \int_0^t e^A|b||w_1| ds
\]

\[
+ \int_0^t e^{-A}|c| \int_0^s e^{2A}|b| d\tau ds \int_0^t |a|e^{-A}|w_1|
\]

\[
= f_0(t) + f_1(t) \int_0^t v_1|w_1| ds + f_2(t) \int_0^t v_2|w_1| ds,
\]

where the expressions \( f_0, f_1, f_2, v_1, v_2 \) are given as in the statement of the lemma. Now using lemma 5.4, we obtain \( |w_1(t)| \leq Hf_0 \) on \([0,T]\). The inequalities (51) follow from \( \xi = (I - \hat{P})^{-1}w \) and the formulas (40). \( \square \)

6. Main argument

6.1. The main technical result. In order to formulate our main technical result, we introduce a notion of harmless nonlinear bound.

Definition 6.1. A function \( \mathcal{N} = \mathcal{N}(R, \beta, \alpha, \delta, M) \) where all arguments are nonnegative numbers is a harmless nonlinear function if for fixed \( \alpha \in (0,1) \), \( \beta > 0 \), \( \delta_3 > 0 \) the following holds: for any given \( R > 0 \), there exists \( \delta_2(R) > 0 \) and a number \( \delta_1 = \delta_1(R, \delta_2) \) such that for all \( \delta_2 \leq \delta_2, \delta_1 \leq \delta_1(\delta_2) \) the inequality

\[
\mathcal{N}(R, \beta, \alpha, \delta, R) < R
\]

(52)

holds.

Recall the box \( \hat{D} \) is said to satisfy the conditions of controlled feeding if there is a \( R \geq 0 \) with

\[
|\partial_{x_1}\omega(x,t)| \leq Rx_1^{-\alpha}, \quad |\partial_{x_2}\omega(x,t)| \leq R \quad (x \in \hat{D} \setminus D)
\]

(53)

for all times \( t \geq 0 \). \( R \) is called feeding parameter. For convenience, we introduce the following definition.
**Definition 6.2.** Let $T > 0, \beta > 0$. We say that the flow is $\beta$-hyperbolic in the box $D$ on $[0,T]$ if
\begin{equation}
Q_i(x,t) \geq \beta \quad (x \in D, \ t \in [0,T], \ i = 1,2).
\end{equation}

**Theorem 6.3.** Let $0 < \alpha < 1/4$. There exists a harmless nonlinear function $N = N(R,\beta,\alpha,\delta,M)$ with the following properties. If $\omega$ is a solution of the Euler equation, $\hat{D}$ a box defined by (22) with parameters $\delta_1,\delta_2,\delta_3 > 0$ satisfying (23) and $T > 0$ is such that
(i) the flow is $\beta$-hyperbolic in the box $D$ on the time interval $[0,T]$,
(ii) the box $\hat{D}$ satisfies the conditions of controlled feeding with parameter $R > 0$
(iii) for the initial data,
\begin{equation}
M_D(0) < R,
\end{equation}
\begin{equation}
\left| \frac{\partial \omega_0}{\partial x_1}(x) \right| \leq Rx_2^{1-\alpha}, \quad \left| \frac{\partial \omega_0}{\partial x_2}(x) \right| \leq R \quad (x \in D),
\end{equation}
(iv) there exists a number $K$ such that
\begin{equation}
M_D(t) \leq K \quad (t \in [0,T]),
\end{equation}
then
\begin{equation}
M_D(t) \leq N(R,\alpha,\beta,\delta,K)
\end{equation}
holds.

6.2. **Estimates along particle trajectories.** We now begin the proof of our technical main result, theorem 6.3. Therefore, let $\omega$ be a given double-odd solution of the Euler equation that is in $C^1([0,\infty),C^2(\mathcal{T}))$. Moreover, let $\hat{D}$ be a box depending on the parameters $\delta_1,\delta_2,\delta_3 > 0$ satisfying the conditions (23).

Suppose also that for the remainder of this section, (i)-(iv) from theorem 6.3 are satisfied. For abbreviation, we write in the following
\begin{equation}
M := \max\{K,R\}.
\end{equation}

We observe the following important fact: since $\delta_1,\delta_2,\delta_1 \leq 1$,
\begin{equation}
M_D(t) \leq M
\end{equation}
holds.

We consider associated particle trajectories, which are the solutions of
\begin{equation}
\dot{X}_1 = -X_1Q_1, \quad \dot{X}_2 = X_2Q_2.
\end{equation}

More precisely, we define the particle trajectories as follows: for any $(x_0,t_0) \in \mathcal{D} \times [0,\infty)$ we take the maximal solution of $t \mapsto X(t)$ of (58) which passes through $(x_0,t_0)$, and lies $\mathcal{D}$. $X$ is defined on an interval $[T_0,T_e]$ such that
(i) $X(t) \in \mathcal{D}$ for all $T_0 \leq t \leq T_e$,
(ii) either $T_0 = 0$ or $T_0 > 0$, in which case necessarily $X(T_0) \in \partial D$,
(iii) $X(T_e) \in \partial D$.

Observe that $X$ is given by
\begin{equation}
X_1(t) = X_1(T_0) \exp \left( - \int_{T_0}^{t} Q_1(X(t),s) \, ds \right)
\end{equation}
\begin{equation}
X_2(t) = X_2(T_0) \exp \left( \int_{T_0}^{t} Q_2(X(t),s) \, ds \right).
\end{equation}
We call $T_0$ the entry time and $T_e$ the exit time of a particle trajectory. $T_0 = 0$ if the particle starts in $D$ for $t = 0$.

The next proposition gives an upper bound for the time a particle can spend in the upper half of the box $D$, provided the flow is $\beta$-hyperbolic.

**Proposition 6.4.** Suppose that the flow is $\beta$-hyperbolic in the box $D$ on the time interval $[0, T]$. Let $X$ be a particle trajectory whose entry time $T_0$ is $< T$. Then if

(i) $X_2(T_0) \neq 0$,
(ii) $T_0 < T_e$,

there is a either time $T_e > T_1 \geq T_0$ such that

$$X_2(t) \geq \frac{1}{2} \delta_2 \quad (t \in [T_1, T])$$

or

$$X_2(t) \leq \frac{1}{2} \delta_2 \quad (t \in [T_0, T]).$$

If $T_1$ exists, we have the estimate

$$T_e - T_1 \leq \beta^{-1} \log(2).$$

**Definition 6.5.** We call a function $g = g(\alpha, \beta, \delta, M)$ harmless generic factor if it has the following property: there exists a $p > 0$ such that for fixed $\alpha, \beta, M$

$$g(\alpha, \beta, \delta^2, \delta_2, M)$$

is bounded as $\delta_2 \to 0$.

For example, a function of the form

$$g = C(\alpha, \beta) \left[ \delta_2^{\gamma_3} M (1 + |\log \delta_2|) + \delta_2^{\gamma_1} \delta_2^{\gamma_2} + 1 \right]^{\gamma_4}$$

($\gamma_j > 0$) is a harmless generic factor, and $e^g$ is also a harmless generic factor if $g$ is one. When performing estimations, we shall often absorb harmless generic factors into one another, so the actual meaning of $g$ may change from line to line.

Our goal will be to obtain estimates for the quantities $f_0, f_1, f_2, v_1, v_2$ along a single particle trajectory, up to the given time $T$, so that we can apply our ODE estimates. The crucial point is that our bounds depend not directly on $\omega, T, T_e$ but only on $\beta, \alpha, X(T_0)$. For the estimations below we often refer to a fixed particle trajectory with entry time $T_0$, along which we evaluate integrals over time of the quantities $Q_1, Q_2, c$ etc. To make the notation more compact, we often skip $X$ in the arguments of the integrands, e.g. we write

$$\int_{T_0}^{t} |c| e^{-A} ds = \int_{T_0}^{t} |c| e^{-A(X(s), t)} ds = \int_{T_0}^{t} |c(X(s), s)| \exp \int_{T_0}^{s} a(X(\tau), \tau) d\tau \ ds.$$

**Lemma 6.6.** For any $T^* \leq T_e$,

$$X_2(T_0) \leq \delta_2 \exp \left( - \int_{T_0}^{T^*} Q_2(X) ds \right).$$

**Proof.** Since the particle trajectory lies in $D$ for $t \in [T_0, T_e]$,

$$\delta_2 \geq X_2(T^*) = X_2(T_0) \exp \left( \int_{T_0}^{T^*} Q_2(X) ds \right)$$

holds. \(\square\)
Let $\phi : [0, \infty) \to [0, \infty)$ be a function with the properties
\begin{equation}
\phi(s) \leq 1 - e^{-s}
\end{equation}
and $\phi$ monotone nondecreasing on $[0, \infty)$, $\phi$ linear on $[0, s^*]$ and $\phi$ constant on $[s^*, \infty)$ for some $s^*$. We fix such a function $\phi$ for the following.

**Proposition 6.7.** Along a particle trajectory in a $\beta$-hyperbolic flow in $D$, we have the following for $t \in [T_0, \min\{T_e, T\}]:$

(i)
\begin{align*}
X_1(t) &\leq \delta_1 \exp(-\beta(t - T_0)) \\
X_2(t) &\leq \delta_2 \exp(-\beta(\min\{T_e, T\} - t)) ,
\end{align*}

(ii)
\begin{equation}
d(X(t)) \geq \delta_2 \phi \left( \int_t^{T_e} Q_2 \, ds \right) \geq \delta_2 \phi(\beta(\min\{T, T_e\} - t)),
\end{equation}

(iii) For any $\gamma \in (0, 1)$, $t \in [T_1, \min\{T_e, T\}],$
\begin{align*}
\int_{T_1}^t d(X(s))^{-1+\gamma} \, ds &\leq C(\gamma, \beta)\delta_2^{-1+\gamma}, \\
\int_{T_1}^t |\log d(X(s))| \, ds &\leq C(\gamma, \beta)|\log \delta_2|
\end{align*}
with a $C(\gamma, \beta)$ independent of the trajectory.

**Proof.** For (i), recall that under the assumption of $\beta$-hyperbolic flow, $Q_2 \geq \beta$. From (59), we get
\begin{equation}
X_2(t) = X_2(T_0) \exp \left( \int_{T_0}^{\min\{T_e, T\}} Q_2 \, ds - \int_t^{\min\{T_e, T\}} Q_2 \, ds \right)
\end{equation}
\begin{equation}
= X_2(\min\{T_e, T\}) \exp \left( -\int_t^{\min\{T_e, T\}} Q_2 \, ds \right)
\end{equation}
\begin{equation}
\leq \delta_2 \exp(-\beta(\min\{T_e, T\} - t)),
\end{equation}
noting that $X_2(\min\{T_e, T\}) \leq \delta_2$. The bound for $X_1$ is analogous.

Now we show (ii). Recall that $d(X) = \delta_2 - X_2(t)$. Hence by (63)
\begin{equation}
\delta_2 - X_2(t) \geq \delta_2 \left( 1 - \exp \left( -\int_t^{\min\{T_e, T\}} Q_2 \, ds \right) \right) \geq \delta_2 \phi \left( \int_t^{\min\{T_e, T\}} Q_2 \, ds \right).
\end{equation}

(iii) We split the integrals by introducing the time $T^*$ defined as follows: $T^*$ is the maximum of all $T_1 \leq t$ such that
\begin{equation}
\phi(\exp(\beta(\min\{T_e, T\} - s))) = \phi(s^*).
\end{equation}
If there are no such $t$, we set $T^* = T_1$. Thus we split as follows:
\begin{equation}
\int_{T_1}^t = \int_{T_1}^{T^*} \ldots + \int_{T^*}^t \ldots
\end{equation}
if \( t \geq T^* \), otherwise we have only one integral from \( T_1 \) to \( t \). We calculate

\[
\int_{T_1}^{T^*} d(X(s))^{-1+\gamma} \, ds \leq \delta_2^{-1+\gamma} \int_{T_1}^{T^*} \phi(\beta(\min\{T_e, T\} - s))^{-1+\gamma} \, ds
\]

\[
\leq \delta_2^{-1+\gamma} (T_e - T_1) \phi(s^*)^{-1+\gamma} \, ds \leq C(\beta, \gamma) \delta_2^{-1+\gamma}
\]

\[
\int_{T^*}^{t} d(X(s))^{-1+\gamma} \, ds \leq \delta_2^{-1+\gamma} \int_{T^*}^{t} \phi(\beta(\min\{T_e, T\} - s))^{-1+\gamma} \, ds
\]

\[
\lesssim \delta_2^{-1+\gamma} \beta^{-1+\gamma} \int_{T^*}^{t} (\min\{T_e, T\} - s)^{-1+\gamma} \, ds
\]

\[
\lesssim \delta_2^{-1+\gamma} \beta^{-1+\gamma} \int_{T_1}^{\min\{T_e, T\}} (\min\{T_e, T\} - s)^{-1+\gamma} \, ds
\]

\[
\lesssim \delta_2^{-1+\gamma} \beta^{-1+\gamma} \int_{0}^{T_e - T_1} z^{-1+\gamma} \, dz \lesssim \delta_2^{-1+\gamma} C(\beta, \gamma).
\]

using (ii), Proposition 6.4 to estimate \( T_e - T_1 \) and the fact that \( \phi \) is linear on \([0, s^*]\). The second integral is treated analogously. \( \square \)

**Lemma 6.8.** Along a particle trajectory, we have, for \( T_0 \leq t \leq \min\{T, T_e\} \),

\[
e^{\pm A(t)} \leq g(\alpha, \beta, \delta, M) \exp \left( \pm \int_{T_0}^{t} Q_1(\alpha, \beta, \delta, M) ds \right),
\]

\[
e^{\pm \int_{T_0}^{t} Q_1 ds} \leq g(\alpha, \beta, \delta, M) e^{\pm \int_{T_0}^{t} Q_2 ds}
\]

where \( g(\alpha, \beta, \delta, M) \) are harmless factors depending only on the quantities indicated.

**Proof.** We prove the second inequality of the lemma, the other ones being analogous. Recall \( a(t) = Q_2(t) + X_2 \partial_{x_2} Q_2(t) \) and thus

\[
\pm A \leq \pm \int_{T_0}^{t} Q_2(s) ds + \int_{T_0}^{t} |X_2(s) \partial_{x_2} Q_2(s)| \, ds.
\]

We now use lemma 4.3

\[
\int_{T_0}^{t} |X_2(s) \partial_{x_2} Q_2(s)| \, ds \lesssim M \int_{T_0}^{\min\{T, T_e\}} X_2^{1-\alpha} (1 + |\log d(X)|) \, ds
\]

\[
\quad + C(\gamma_1, \gamma_2) \int_{T_0}^{\min\{T, T_e\}} X_2^{1-\gamma_1-\gamma_2} X_1^{\gamma_2} d(X)^{-1+\gamma_1} \, ds
\]

(note that the interval of integration has been enlarged). We split the interval of integration into \([T_0, T_1]\) and \([T_1, \min\{T, T_e\}]\) provided \( \min\{T, T_e\} \geq T_1 \). The case \( \min\{T, T_e\} < T_1 \) is analogous.
In the part over \([T_0, T_1]\), while \(d(X) \geq \frac{1}{2} \delta_2\), we cannot control the length of the time interval, so we estimate as follows:

\[
\int_{T_0}^{T_1} X_2^{1-\alpha} (1 + |\log d(X)|) \leq \delta_2^{1-\alpha} \int_{T_0}^{T_1} e^{-(1-\alpha)\beta(\min\{T, T_e\} - s)} (C + |\log \delta_2|) \, ds
\]

\[
\lesssim \delta_2^{1-\alpha} |\log \delta_2| \int_0^\infty e^{-(1-\alpha)\beta z} \, dz
\]

\[
\lesssim C(\alpha, \beta) \delta_2^{1-\alpha} |\log \delta_2|,
\]

using part (i) of Proposition 6.7 and \(d(X(s)) \geq \frac{1}{2} \delta_2\) for \(s \in [T_0, T_1]\), and \(\delta_2\) sufficiently small. In the part over \([T_1, \min\{T, T_e\}]\) the length of the time interval is bounded but \(d(X)\) is unbounded, so we proceed differently:

\[
\int_{T_1}^{\min\{T, T_e\}} X_2^{1-\alpha} (1 + |\log d(X)|) \lesssim \delta_2^{1-\alpha} \int_{T_1}^{\min\{T, T_e\}} |\log d(X)| \, ds
\]

\[
\lesssim \delta_2^{1-\alpha} |\log \delta_2|.
\]

using statement (iii) of Proposition 6.7 and \(X_2 \leq \delta_2\).

For the second integral involving \(X_2^{1-\gamma_1+\gamma_2} X_1^{\gamma_2} d(X)^{-1+\gamma_1}\), we note

\[
\int_{T_0}^{T_1} X_2^{1-\gamma_1-\gamma_2} X_1^{\gamma_2} d(X)^{-1+\gamma_1} \lesssim \delta_2^{\gamma_2} \delta_2^{1-\gamma_1-\gamma_2} \delta_2^{1+\gamma_1} \int_{T_0}^{T_1} e^{-(1-\gamma_1-\gamma_2)\beta(\min\{T, T_e\} - s)} \, ds
\]

\[
\lesssim \delta_2^{\gamma_2} \delta_2^{1-\gamma_2}
\]

\[
\int_{T_1}^{\min\{T, T_e\}} X_2^{1-\gamma_1-\gamma_2} X_1^{\gamma_2} d(X)^{-1+\gamma_1} \lesssim \delta_2^{\gamma_2} \delta_2^{1-\gamma_2}
\]

by Proposition 6.7 (i) and (iii) and moreover using \(X_1 \leq \delta_1, X_2 \leq \delta_2\). This yields finally

\[
\int_{T_0}^{t} |X_2(s) \partial_x Q_2(s)| \, ds \leq C(\alpha, \beta) [M \delta_2^{1-\alpha} |\log \delta_2| + (\delta_1/\delta_2)^{\gamma_2}]
\]

implying the result, since the factor in square brackets is a harmless generic factor. □

**Lemma 6.9.** The following estimates hold for \(T_0 \leq t \leq \min\{T, T_e\}\):

(64) \(f_2(t) \leq g(\alpha, \beta, \delta, M) X_2(T_0)^{1-\alpha} \left[M + \delta_1^{\alpha/2} \delta_2^{-1+\alpha/2}\right]\),

(65) \(f_0(t) \leq R g(\alpha, \beta, \delta, M) X_2(T_0)^{1-\alpha} \left[M + \delta_1^{\alpha/2} \delta_2^{-1+\alpha/2}\right]\).

**Proof.** We write \(g = g(\alpha, \beta, \delta, M)\) for any occurring harmless factor. Using Lemma 4.5

\[
f_2(t) = \int_{T_0}^{t} e^{-A|c|} \lesssim M \int_{T_0}^{\min\{T, T_e\}} e^{-A X_2^{1-\alpha}} \, ds
\]

\[
+ C(\alpha) \int_{T_0}^{\min\{T, T_e\}} e^{-A X_2^{1-\alpha} X_1^{\alpha/2} d(X)^{-1+\alpha/2}} \, ds
\]
(choose $\gamma_1 = \gamma_2 = \frac{1}{2}\alpha$). Observe first that we can use Lemma 6.8 to replace $e^{-A}$ by
\[ e^{-A}X_2(s)^{1-\alpha} \leq gX_2(T_0)^{1-\alpha} \exp \left( -\alpha \int_{T_0}^{t} Q_2 \, d\tau \right) \]
(66)
\[ \leq gX_2(T_0)^{1-\alpha} \exp ( -\alpha \beta (s - T_0) ) \]
(we use again $Q_2 \geq \beta$).

Use (66) to estimate the integral containing $e^{-A}X_2^{1-\alpha}$:
\[ \int_{T_0}^{\min \{ T, T_e \}} e^{-A}X_2^{1-\alpha} \, ds \leq gX_2(T_0)^{1-\alpha} \int_{T_0}^{\infty} e^{-\alpha \beta (s-T_0)} \, ds \]
\[ \leq gX_2(T_0)^{1-\alpha} C(\alpha, \beta). \]

For integral containing $e^{-A}X_2^{1-\alpha}X_1^{\alpha/2}d(X)^{1-\alpha/2}$, we use (66) again and estimate
\[ \int_{T_0}^{\min \{ T, T_e \}} e^{-A}X_2^{1-\alpha}X_1^{\alpha/2}d(X)^{1-\alpha/2} \, ds \leq \]
\[ gX_2(0)^{1-\alpha} \delta_1^{\alpha/2} \int_{T_0}^{\min \{ T, T_e \}} e^{-\alpha \beta (s-T_0)} d(X)^{-1+\alpha/2} \, ds. \]

As in the proof of Lemma 6.8, we split the interval of integration into $[T_0, T_1]$ and $[T_1, \min \{ T, T_e \}]$ in case $T_1 \leq \min \{ T, T_e \}$, obtaining
\[ \int_{T_0}^{T_1} e^{-\alpha \beta (s-T_0)} d(X)^{-1+\alpha/2} \, ds \lesssim \delta_2^{-1+\alpha/2} C(\alpha, \beta), \]
\[ \int_{T_1}^{\min \{ T, T_e \}} e^{-\alpha \beta (s-T_0)} d(X)^{-1+\alpha/2} \, ds \lesssim \int_{T_1}^{\min \{ T, T_e \}} d(X)^{-1+\alpha/2} \, ds \]
\[ \lesssim \delta_2^{-1+\alpha/2} \]

where we have used $d(X) \geq \frac{1}{2} \delta_2, e^{-\alpha \beta (s-T_0)} \leq 1$ and Proposition 6.7. In the case $T_1 \geq \min \{ T, T_e \}$, we are left with only integral and deal with it in the same way.

To estimate $f_0$, we use that the feeding condition holds and that assumption (iii) from Theorem 6.3 holds. This gives
\[ |\xi_1(T_0)| = |\partial_{x_1} \omega(X(T_0), T_0)| \leq RX_2(T_0)^{1-\alpha}, \]
\[ |\xi_2(T_0)| = |\partial_{x_2} \omega(X(T_0), T_0)| \leq R \]
for both of the cases $T_0 = 0$ (particle starts in $D$) and $T_0 > 0$ (particle starts in feeding zone). Now use the definition of $f_0$ and the estimate (64) for $f_2$. \qed

**Lemma 6.10.** For $T_0 \leq t \leq \min \{ T, T_e \}$,
\[ f_1(t) \leq g(\alpha, \beta, \delta, M) \delta_1^{1-\alpha} \delta_2^{1-2\alpha} \alpha^\alpha f_{1,0}^T Q_2 ds \]
with a universal factor $g$ depending on the quantities indicated.
Proof. We abbreviate again $g = g(\alpha, \beta, \delta, M)$. First we claim that
\begin{equation}
\int_{T_0}^{t} e^{2A} |b| \, ds \leq g C(\alpha, \beta) X_1(0)^{1-\alpha} \exp \left( (1 + \alpha) \int_{T_0}^{t} Q_1 \, ds \right) \times \left[ M(1 + |\log \delta_2|) + \delta_2^{2\alpha} \delta_2^{-1+2\alpha} \right].
\end{equation}
We treat the case $T_1 \leq t \leq \min\{T, T_e\}$. Using Lemma 4.5 (recall $M_{\omega, \delta} \leq M$), with $\gamma = 2\alpha$, and Lemma 6.8 we get
\begin{align*}
e^{2A} |b| &\leq e^{2A} X_1^{1-\alpha} \left[ M(1 + |\log d(X)|) + X_1^{2\alpha} d(X)^{-1+2\alpha} \right] \\
&\leq g e^{(1+\alpha) \int_{T_0}^{t} Q_1 \, ds} X_1(0)^{1-\alpha} \left[ M(1 + |\log d(X)|) + \delta_1^{2\alpha} d(X)^{-1+2\alpha} \right].
\end{align*}
We integrate this bound from $T_0$ to $t$ and split into two integrals from $T_0$ to $T_1$ and $T_1$ to $t$:
\begin{align*}
g X_1(0)^{1-\alpha} \left[ M(1 + |\log \delta_2|) + \delta_1^{2\alpha} \delta_2^{-1+2\alpha} \right] \int_{T_0}^{T_1} e^{(1+\alpha) \int_{t_0}^{t} Q_1 \, ds} \, ds.
\end{align*}
Observe for the integral:
\begin{align*}
\int_{T_0}^{T_1} e^{(1+\alpha) \int_{T_0}^{T_1} Q_1 \, ds} \, ds &= \int_{T_0}^{T_1} \frac{Q_1}{Q_1} e^{(1+\alpha) \int_{t_0}^{t} Q_1 \, ds} \, ds \\
&\leq \beta^{-1}(1 + \alpha)^{-1} e^{(1+\alpha) \int_{T_0}^{T_1} Q_1 \, ds} \bigg|_{s=T_1}^{s=T_1} \leq e^{(1+\alpha) \int_{T_0}^{T_1} Q_1 \, ds}.
\end{align*}
Hence
\begin{align*}
\int_{T_0}^{T_1} e^{2A} |b| \, ds \lesssim g X_1(0)^{1-\alpha} \left[ M(1 + |\log \delta_2|) + \delta_1^{2\alpha} \delta_2^{-1+2\alpha} \right] e^{(1+\alpha) \int_{T_0}^{T_1} Q_1 \, ds}.
\end{align*}
For the remaining part $\int_{T_1}^{t} e^{2A} |b| \, ds$, we use Proposition 6.7 again, and find the bound
\begin{align*}
g e^{(1+\alpha) \int_{T_0}^{T_1} Q_1 \, ds} X_1(0)^{1-\alpha} \int_{T_1}^{t} \left[ M(1 + |\log d(X)|) + \delta_1^{2\alpha} \delta_2^{-1+2\alpha} \right] \, ds \\
\lesssim g X_1(0)^{1-\alpha} \left[ M(1 + |\log \delta_2|) + \delta_1^{2\alpha} \delta_2^{-1+2\alpha} \right] e^{(1+\alpha) \int_{T_0}^{T_1} Q_1 \, ds}.
\end{align*}
The claim follows for the case $T_1 \leq t \leq \min\{T, T_e\}$. The calculation for $t \leq T_1$ is similar (and slightly simpler). Next, using again Lemma 4.5 with $\gamma = 2\alpha$,
\begin{align*}
e^{-A} |c| &\leq e^{-A} X_1^{2-\alpha} \left[ M + X_2^{\alpha/2} X_1^{\alpha/2} d(X)^{-1+\alpha/2} \right] \\
&\leq g e^{-\alpha \int_{T_0}^{T_1} Q_1 \, ds} X_2(0)^{1-\alpha} \left[ M + \delta_1^{\alpha/2} \delta_2^{-1+\alpha/2} \right] \\
&\times \int_{T_0}^{T_1} e^{-\alpha \int_{T_0}^{T_1} Q_1 \, ds} X_2(0)^{1-\alpha} \left[ M + \delta_1^{\alpha/2} \delta_2^{-1+\alpha/2} \right] \, ds.
\end{align*}
Continuing the estimation, we have
\[
\int_{T_0}^t e^{-\int_{T_0}^s f_{T_0}^s Q_{2dr} e^{(1+\alpha) f_{T_0}^s Q_1 dr}} [M + \delta_2^{\alpha/2} \delta_1^{\alpha/2} d(X)^{-1+\alpha/2}] \, ds
\]
\[
\leq g \int_{T_0}^t e^{-\int_{T_0}^s f_{T_0}^s Q_{2ds} e^{(1+\alpha) f_{T_0}^s Q_2 ds}} [M + \delta_2^{\alpha/2} \delta_1^{\alpha/2} d(X)^{-1+\alpha/2}] \, ds
\]
\[
\leq g \int_{T_0}^t e^{\int_{T_0}^s f_{T_0}^s Q_{2dr} [M + \delta_2^{\alpha/2} \delta_1^{\alpha/2} d(X)^{1+\alpha/2}]} \, ds
\]
\[
\leq g C(\alpha, \beta) e^{\int_{T_0}^t Q_{2ds}} [M + \delta_2^{\alpha/2} \delta_1^{\alpha/2} \delta_2^{-1+\alpha/2}]
\]
where we have used Lemma 6.8 in the second line to exchange \( e^{-\int_{T_0}^s Q_1 dr} \) for \( e^{-\int_{T_0}^s Q_2 dr} \) and used the familiar splitting at \( T_1 \) to estimate the integral. Thus, finally we get
\[
\int_{T_0}^t e^{-A} |c| \int_{T_0}^s e^{2A} |b| \lesssim g X_1(0)^{1-\alpha} \left[ M(1 + \log \delta_2) + \delta_1^{\alpha/2} \delta_2^{-1+\alpha/2} \right]
\]
\[
\times [ M + \delta_1^{\alpha/2} \delta_2^{-1+\alpha} ] e^{\int_{T_0}^s Q_{2ds}} X_2(0)^{1-\alpha}.
\]
It remains to apply key Lemma 6.6 to estimate the factor \( e^{\int_{T_0}^t Q_{2ds}} X_2(0)^{1-\alpha} \), which is less than
\[
\delta_2^{1-\alpha} e^{\int_{T_0}^t Q_{2ds} e^{(1+\alpha) f_{T_0}^s Q_{2ds} -(1+\alpha) f_{T_0}^s Q_{2ds}} \, ds} \lesssim \delta_2^{1-\alpha} e^{\int_{T_0}^t Q_{2ds}}.
\]
Now observe that the factor \( \delta_2^{1-\alpha} \) on the right-hand side of (69) can be combined with
\[
\left[ M(1 + \log \delta_2) + \delta_1^{\alpha/2} \delta_2^{-1+\alpha/2} \right],
\]
\[
\delta_2^{1-2\alpha} g.
\]
\[\square\]

**Lemma 6.11.**
\[
v_1 \lesssim g \left[ Q_2 + M X_2^{1-\alpha}(1 + |\log d(X)|) + X_2^{1-\alpha} d(X)^{1-\alpha} \right] e^{-f_{T_0}^s Q_{2ds}}
\]
\[
v_2(t) \leq g X_1(T_0)^{1-\alpha} \left[ M(1 + |\log d(X)|) + \delta_1^{\alpha/2} d(X)^{-1+\alpha/2} \right] e^{\alpha f_{T_0}^t Q_1 ds},
\]
\[
\int_{T_0}^{\min(T, T_e)} v_1 \leq g[M(1 + \log \delta_2) + 1],
\]
\[
\int_{T_0}^{\min(T, T_e)} v_2 e^{\alpha f_{T_0}^t Q_1 ds} \leq g e^{2\alpha f_{T_0}^t Q_{2ds}}
\]
\[
\int_{T_0}^{\min(T, T_e)} v_1 f_1 \lesssim g \delta_1^{1-\alpha} \delta_2^{1-\alpha/2},
\]
\[
\int_{T_0}^{\min(T, T_e)} v_2 f_1 \lesssim g \delta_1^{1-\alpha} \delta_2^{1-\alpha} X_1(T_0)^{1-\alpha} e^{2\alpha f_{T_0}^t Q_{2ds}}.
\]
\[
\int_{T_0}^{\min(T, T_e)} v_1 f_2 \leq g X_2(0)^{1-\alpha},
\]
\[g = g(\alpha, \beta, \delta, M) \text{ a harmless factor.}\]
Proof. The estimates for \( v_1 \) and \( v_2 \) follow from Lemma 4.5 and Lemma 6.8. By Proposition 6.7 and the usual splitting of the interval of integration,

\[
\int_{T_0}^{\min\{T,T_e\}} v_1 \leq g[M(1 + |\log \delta_2|) + 1].
\]

The integral \( \int_{T_0}^{\min\{T,T_e\}} v_1 e^{-A} f_0^t Q_1 ds \) is estimated similarly.

Using Lemma 6.10 and Lemma 4.5 we get

\[
v_1 f_1 = |a| e^{-A} f_1 \leq g\delta_1^{1-\alpha} \delta_2^{1-\alpha/2} e^{-A} f_0^t Q_2 ds e^{-A}
\]

\[
\times \left[ Q_2 + M X_1^{1-\alpha} (1 + |\log d(X)|) + X_1^{1-\alpha} d(X)^{1-\alpha} \right]
\]

\[
\leq g\delta_1^{1-\alpha} \delta_2^{1-\alpha/2} (-1+\alpha) f_0^t Q_2 ds
\]

\[
\times \left[ Q_2 + M \delta_2^{1-\alpha} (1 + |\log d(X)|) + \delta_2^{1-\alpha} d(X)^{-1+\alpha} \right].
\]

Note how the growing factor \( e^{\alpha} f_0^t Q_2 ds \) was cancelled by \( e^{-A} \). By integrating, we can bound \( \int_{T_0}^{\min\{T,T_e\}} v_1 f_1 \):

\[
\leq g\delta_1^{1-\alpha} \delta_2^{1-2\alpha} e^{-A} f_0^t Q_2 ds
\]

\[
\times \left[ Q_2 + M \delta_2^{1-2\alpha} (1 + |\log d(X)|) + \delta_2^{1-\alpha} d(X)^{-1+\alpha} \right]
\]

\[
\leq g\delta_1^{1-\alpha} \delta_2^{1-2\alpha} \left[ M \delta_2^{1-\alpha} |\log \delta_2| + 1 \right]
\]

yielding the desired estimate for the integral with integrand \( v_2 f_1 \).

Proceeding analogously, we bound \( v_2 f_1 = e^{A} |b| f_1 \) by

\[
g\delta_1^{1-\alpha} \delta_2^{1-\alpha/2} e^{(1+\alpha) f_0^t Q_2 ds} X_1^{1-\alpha} \left[ M (1 + |\log d(X)|) + X_1^{\alpha/2} d(X)^{1-\alpha/2} \right]
\]

\[
\leq g\delta_1^{1-\alpha} \delta_2^{1-\alpha/2} X_1(T_0)^{1-\alpha} e^{2\alpha f_0^t Q_2 ds} \left[ M (1 + |\log d(X)|) + \delta_1^{\alpha/2} d(X)^{1-\alpha/2} \right]
\]

where we have used \( e^{(1+\alpha) f_0^t Q_2 ds} X_1^{1-\alpha} \leq g X_1(T_0)^{1-\alpha} e^{(1+\alpha) f_0^t Q_2 ds} e^{-(1-\alpha) f_0^t Q_1 ds} \) and used Lemma 6.8 to replace \( Q_1 \) by \( Q_2 \). Integration now yields

\[
\int_{T_0}^{t} v_2 f_1 \leq g\delta_1^{1-\alpha} \delta_2^{1-\alpha/2} X_1(T_0)^{1-\alpha} e^{2\alpha f_0^t Q_2 ds} \left[ M |\log \delta_2| + \delta_1^{\alpha/2} \delta_2^{1-\alpha/2} \right].
\]

\[
\square
\]

Lemma 6.12. Along a particle trajectory, for \( T_0 \leq t \leq \min\{T,T_e\} \),

\[
(H f_0)(t) \leq g \|f_0\|_{\infty} \left[ 1 + e^{\alpha f_0^t Q_2 ds} \right]
\]

holds, where \( \|f_0\|_{\infty} = \|f_0\|_{\infty,[T_0,t]} \).
Proof. First we estimate the expression \( f_0 + f_1 \exp \left( \int_{T_0}^t v_1 f_1 \right) \int_{T_0}^t v_1 f_0 \). Using Lemmas 6.10, 6.11 we get the following bound

\[
\|f_0(t) + g \delta_1^{1-\alpha} \delta_2^{1-2\alpha} e^\alpha \int_{T_0}^t Q_2 ds \exp (g \delta_1^{1-\alpha} \delta_2^{1-2\alpha}) \|f_0\|_\infty \int_{T_0}^{T_e} v_1 ds \leq \|f_0\|_\infty (1 + g \delta_1^{1-\alpha} \delta_2^{1-2\alpha} e^\alpha \int_{T_0}^t Q_2 ds \mathbb{M}(1 + |\log \delta_2|)) \leq \|f_0\|_\infty (1 + g \delta_1^{1-\alpha} \delta_2^{1-3\alpha} e^\alpha \int_{T_0}^t Q_2 ds) \leq g e^\alpha \int_{T_0}^t Q_2 ds
\]

where we have absorbed the harmless factors \( \exp (g \delta_1^{1-\alpha} \delta_2^{1-2\alpha}), \delta_2^\alpha \mathbb{M}(1 + |\log \delta_2|) \) into \( g \).

Next we consider

\[
f_2 + f_1 \exp \left( \int_{T_0}^t v_1 f_1 \right) \int_{T_0}^t v_1 f_2,
\]

for which we get the following bound (Lemmas 6.9, 6.10)

\[
gX_2(T_0)^{1-\alpha} + g \delta_1^{1-\alpha} \delta_2^{1-2\alpha} e^\alpha \int_{T_0}^t Q_2 ds X_2(T_0)^{1-\alpha} \\
\leq gX_2(T_0)^{1-\alpha} + g \delta_1^{1-\alpha} \delta_2^{1-2\alpha} X_2(T_0)^{1-2\alpha} \\
\leq g(\delta_2^\alpha + \delta_1^{1-\alpha} \delta_2^{1-2\alpha}) X_2(T_0)^{1-2\alpha} \leq gX_2(T_0)^{1-2\alpha} \leq gX_2(T_0)^{1-2\alpha}
\]

using, in the first step, the key Lemma 6.6 to cancel of \( e^\alpha \int_{T_0}^t Q_2 ds \) using the factor \( X_2(T_0)^\alpha \).

So for \( v_2 \left[ f_2 + f_1 \exp \left( \int_{T_0}^t v_1 f_1 \right) \int_{T_0}^t v_1 f_2 \right] \) we obtain the upper bound

\[
v_2 gX_2(T_0)^{1-2\alpha} \leq g e^\alpha \int_{T_0}^t Q_1 X_2(T_0)^{1-2\alpha} \left[ \mathbb{M}(1 + |\log d(X)|) + \delta_1^{1/2} d(X)^{-1+\alpha/2} \right] \\
\leq g \delta_2^\alpha X_2(T_0)^{1-3\alpha} \left[ \mathbb{M}(1 + |\log d(X)|) + \delta_1^{1/2} d(X)^{-1+\alpha/2} \right]
\]

using the key lemma 6.6 again to cancel \( e^\alpha \int_{T_0}^t Q_1 \). Thus we see that

\[
\exp \left( \int_{T_0}^{\min(T,T_e)} v_2 \left[ f_2 + f_1 \exp \left( \int_{T_0}^s v_1 f_1 \right) \int_{T_0}^s v_1 f_2 \right] \right) \leq g
\]

Finally, by (71) and Lemma 6.11

\[
\int_{T_0}^t v_2 \left[ f_0 + f_1 \exp \left( \int_{T_0}^s v_1 f_1 \right) \int_{T_0}^s v_1 f_0 \right] \leq g \|f_0\|_\infty \int_{T_0}^t v_2 e^\alpha \int_{T_0}^t Q_2 ds \leq g \|f_0\|_\infty e^\alpha \int_{T_0}^t Q_2 ds.
\]
Thus, in total we get
\[
f_0 + f_1 \exp \left( \int_{T_0}^t v_1 f_1 \right) \int_{T_0}^t v_1 f_0 \leq g \| f_0 \|_\infty e^{\alpha \int_0^t Q_2 ds}
\]
\[
f_2 + f_1 \exp \left( \int_{T_0}^t v_1 f_1 \right) \int_{T_0}^t v_1 f_2 \leq gX_2(T_0)^{1-2\alpha}
\]
(72)
\[
\exp \left( \int_{T_0}^{\min\{T,T_e\}} v_2 \left[ f_2 + f_1 \exp \left( \int_{T_0}^s v_1 f_1 \right) \int_{T_0}^s v_1 f_2 \right] \right) \leq g
\]
\[
\int_{T_0}^t v_2 \left[ f_0 + f_1 \exp \left( \int_{T_0}^s v_1 f_1 \right) \int_{T_0}^s v_1 f_0 \right] \leq g \| f_0 \|_\infty e^{\alpha \int_0^t Q_2 ds}.
\]
Combining the inequalities in (72),
\[
\left( f_2 + f_1 \exp \left( \int_{T_0}^t v_1 f_1 \right) \int_{T_0}^t v_1 f_2 \right)
\times \exp \left( \int_{T_0}^{T_e} v_2 \left[ f_2 + f_1 \exp \left( \int_{T_0}^s v_1 f_1 \right) \int_{T_0}^s v_1 f_2 \right] \right)
\times \int_{T_0}^t v_2 \left[ f_0 + f_1 \exp \left( \int_{T_0}^s v_1 f_1 \right) \int_{T_0}^s v_1 f_0 \right]
\leq gX_2(T_0)^{1-4\alpha} \| f_0 \|_\infty \leq g \| f_0 \|_\infty
\]
using again the key lemma to get rid of the factor $e^{2\alpha \int_0^t Q_2}$, and in the very last step we used $\alpha \in (0, 1/4)$. In view of (47), (70) now follows. \(\square\)

We can now complete the proof of our main technical result, Theorem 6.3. At time $t = T$, any $x \in D$ is occupied by a particle, i.e. $x = X(T)$ for some particle trajectory. Let us write
\[
\partial_{x_j} \omega(X(t), t) = \xi_j(t)
\]
along that particle trajectory, and so by (51),
(73)
\[
|\xi_1(t)| \leq (H f_0)(t) + e^A \int_{T_0}^t |a| e^{-A}(H f_0)(s) ds
\]
First note that by Lemmas 4.5, 6.12, 6.9, 6.11
\[
e^A \int_{T_0}^t v_1(s)(H f_0)(s) ds \lesssim e^A g \| f_0 \|_\infty \int_{T_0}^t v_1 ds
\]
(74)
\[
\leq g \| f_0 \|_\infty (M |\log \delta_2| + 1)e^A \leq gRX_2(T_0)^{1-\alpha} e^A(M |\log \delta_2| + 1)
\]
Note that the growing exponential factor $e^{\alpha \int_0^t Q_2 ds}$ is cancelled by the decaying one $e^{-\int_0^t Q_2 ds}$ and that the integral can be estimated using the familiar splitting technique. Moreover again by Lemmas 6.12, 6.9
\[
(H f_0)(t) \leq gRX_2(T_0)^{1-\alpha} \left[ 1 + e^{\alpha \int_0^t Q_2 ds} \right]
\]
and so in view of (73),
\[
|\xi_1(t)|X_1(t)^\alpha \leq g R X_2(T_0)^{1-\alpha} \left[ 1 + e^{\alpha \int_0^1 Q_2 ds} \delta_1 e^{-\alpha \int_0^1 Q_1 ds} \right]
\]
\[
+ g R (M |\log \delta_2| + 1) e^A X_2(T_0)^{1-\alpha} \delta_1 e^{-\alpha \int_0^1 Q_1 ds} \delta_1^\alpha
\]
(75)
where we used the key Lemma 6.6 to cancel $e^A$
\[
e^A X_2(T_0)^{1-\alpha} e^{-\alpha \int_0^1 Q_1 ds} \leq g \delta_2^{1-\alpha}
\]
and combined the factor $\delta_2^{1-\alpha}$ with $(M |\log \delta_2| + 1)$ to get a harmless generic factor. In fact, this was the most critical estimate in the whole proof, since the dangerous factor $e^A$ was barely cancelled.

We now derive a similar estimate for $|\xi_2(t)|X_2(t)^\alpha$. From the second line of (51),
\[
|\xi_2(t)| \leq Re^{-A} + e^{-A} \|H f_0\| \int_t^\infty e^A |b| ds + e^{-A} \int_0^t e^{2A} |b| \int_s^t |a| e^{-A} H f_0.
\]
First we observe that by Lemmas 6.12, 6.9, 6.11
\[
\|H f_0\| e^{-A} \int_0^t e^A |b| ds \leq \|H f_0\| e^{-A} \int_0^t v_2 ds
\]
\[
\leq g \delta_1^{1-\alpha} R X_2(T_0)^{1-\alpha} \left[ 1 + e^{\alpha \int_0^1 Q_2 ds} \right] e^{-\alpha \int_0^1 Q_1 ds} \alpha \int_0^1 Q_2 ds
\]
\[
\leq g \delta_1^{1-\alpha} R X_2(T_0)^{1-\alpha}.
\]
Moreover, using (74) and Lemma 6.10
\[
e^{-A} \int_0^t e^{2A} |b| \int_s^t |a| e^{-A} H f_0 \leq g R X_2(T_0)^{1-\alpha} (M |\log \delta_2| + 1) e^{-A} \int_0^t e^{2A} |b|
\]
\[
= g R X_2(T_0)^{1-\alpha} (M |\log \delta_2| + 1) \delta_1^{1-\alpha} \delta_2^{1-2\alpha} e^{-A} e^{(1+\alpha) \int_0^1 Q_2 ds}
\]
\[
\leq g R \delta_1^{1-\alpha} X_2(T_0)^{1-\alpha} e^{\alpha \int_0^1 Q_2 ds}
\]
\[
\leq g R \delta_1^{1-\alpha} X_2(T_0)^{1-2\alpha}.
\]
Hence from (76),
\[
|\xi_2(t)| \leq g R
\]
and thus
\[
|\xi_2(t)|X_2^\alpha \leq g R \delta_2^\alpha.
\]
(78)
follows. Inequalities (78) and (75) imply
\[
M_D(T) \leq g(\alpha, \beta, \delta, M) R \delta_2^\alpha =: N(R, \alpha, \beta, \delta, M).
\]
It remains to show that $N$ is a harmless nonlinearity. Therefore, let $\alpha, \beta, R$ be given. Recall that $g$ has the property that $g(\alpha, \beta, \delta, \delta_2, R)$ is bounded as $\delta_2 \to 0$, with some $p > 0$. Hence
\[
g(\alpha, \beta, \delta, \delta_2, R) R \delta_2^\alpha < R
\]
for sufficiently small $\delta_2 > 0$. 

NO LOCAL DOUBLE EXPONENTIAL GROWTH IN HYPERBOLIC FLOW 27
6.3. Proof of the main result. We are now ready to prove Theorem 1.2. So let \( \alpha \in (0, \frac{1}{4}) \) and \( R \geq 0 \) be a given nonnegative number. Let \( \mathcal{N} \) be the harmless nonlinear function from theorem 6.3. Fix small positive \( \delta_1, \delta_2 \) such that the following set of inequalities hold true:

\[
\delta_1, \delta_2 \leq \rho, \quad \beta_0 - A|\delta|^{1-\alpha}R \geq \frac{1}{2}\beta_0.
\]

\[
M_D(0) < R, \quad \left| \frac{\partial \omega_0}{\partial x_1} \right| \leq Rx_2^{1-\alpha}, \quad \left| \frac{\partial \omega_0}{\partial x_2} \right| \leq R,
\]

\[
\mathcal{N}(R, \alpha, \frac{1}{2}\beta_0, \delta_1, \delta_2, R) < R,
\]

where \( A, \beta_0, \rho \) are the numbers from the definition of the hyperbolicity of the flow.

Note that the box can be chosen so small that that (80) holds. This a consequence of \( \frac{\partial \omega_0}{\partial x_2}(0, x_2) = 0 \) and the \( C^2 \)-smoothness of \( \omega_0 \).

We claim now that if the box \( \hat{D} \) satisfies the controlled feeding conditions with parameter \( R \), then we have the bound

\[
M_D(t) \leq R \quad (t \in [0, \infty))
\]

for all times. Assume (82) is not true for all times. Since \( M_D(0) < R \), and the solution \( \omega \) is sufficiently smooth in time by assumption, there exists a time \( T > 0 \) such that \( M_D(t) < R \) holds on \([0, T)\) and \( M_D(T) = R \). Moreover, by (79), the flow is \( \frac{1}{2}\beta_0 \)-hyperbolic in the box \( D \) on the time interval \([0, T]\). Observe that because of (23) and the feeding conditions, \( M(x, t) \leq M_D(t) \leq R \) for all \( x \in D \) and \( t \in [0, T] \).

(80) implies that (ii) in the formulation of Theorem 6.3 holds. Also, on \([0, T]\), we have \( M_D(t) \leq R \). Applying Theorem 6.3 (choose \( K = R \) and (81), we get

\[
M_D(T) \leq \mathcal{N}(R, \alpha, \frac{1}{2}\beta_0, \delta_1, \delta_2, R) < R.
\]

In combination with (81), this gives

\[
M_D(T) < R,
\]

a contradiction. This proves (82). Now we prove the exponential bound on the gradient growth. At an arbitrary time \( t \geq 0 \), each \( x \in D \) is occupied by a particle \( X(t) \) that has entered the box at some earlier time \( T_0 \). The same calculation leading to (75), using (73) and (74) yields

\[
\left| \frac{\partial \omega}{\partial x_1}(X(t), t) \right| \leq g(\alpha, \beta, \delta, R)R\delta_2^{1-\alpha} \left[ 1 + e^{\alpha \int_{T_0}^t Q_1 ds} \right].
\]

for all \( t \in [T_0, T] \) on account of (82).

We apply now Lemma 4.4

\[
\int_{T_0}^t Q_1 ds \leq \int_{T_0}^t (C\|\omega\|_{\infty} + R|\delta|^{1-\alpha})(t - T_0) + \|\omega\|_{\infty} \int_{T_0}^t |\log d(X)| ds
\]

The integral containing the logarithmic term can be estimated using the familiar splitting at \( T_1 \) and gives

\[
\int_{T_0}^t |\log d(X)| ds \leq C(t - T_0)|\log \delta_2| + C(\alpha, \beta).
\]
Thus, finally,
\[
\left| \frac{\partial \omega}{\partial x_1}(X(t), t) \right| \leq g(\alpha, \beta, \delta, R) R \delta_2^{1-\alpha} e^{a(C\|\omega\|_\infty + R|\delta|^{1-\alpha})t}.
\]

The derivative in \(x_2\)-direction is bounded by \(\|\|\|\). This concludes the proof of Theorem 1.2.

7. Acknowledgments

The authors cordially thank A. Kiselev for suggesting the problem and a great number of helpful discussions. VH would like to express his gratitude to the Deutsche Forschungsgemeinschaft (German Research Foundation), without whose financial support (FOR HO 5156/1-1) the present research could not have been undertaken.

8. Appendix

8.1. Appendix A.

**Proposition 8.1.** For all \(x, y \in [0, 1]^2\), \(x \neq y\) the following estimates hold.

\[
|G^k_i(x, y)| \lesssim |y - x|^{-1} x_i^{-1}
\]

(83)

\[
\left| \frac{\partial G^k_i}{\partial x_j}(x, y) \right| \lesssim |y - x|^{-3}
\]

(84)

\((i, k = 1, 2)\).

The proofs are straightforward calculations based on the identities in Appendix B, and the reflection identities:

\[
|y - \bar{x}| \geq |y - x|, |y - \bar{x}| \geq |y - x|, |y + x| \geq |y - \bar{x}|
\]

holding for \(x, y \in [0, 1]^2\). Also, use the obvious inequalities

\[
x_2 \leq |y - \bar{x}|, x_1 \leq |y - \bar{x}|.
\]

We observe some useful relations for the kernels \(G^k_i\) and their derivatives. Let \(G\) stand for any \(G^k_i\) and let

\[
\Omega_x = (-x_1, 1 - x_1) \times (-x_2, 1 - x_2).
\]

\(G\) has the form \(G(x, y) = \tilde{G}(y - x, x, y)\), where \(\tilde{G}(z, \eta, \mu)\) is smooth provided \(\eta, \mu \in (0, 1)^2, z \in \Omega_x \setminus \{0\}\). For example, if \(G = G^1_1\) then

\[
\tilde{G}(z, \eta, \mu) = \frac{\mu_1 z_1}{|z|^2|\mu - \eta|^2}.
\]

Note that for \(x \neq y, x, y \in (0, 1)^2\),

\[
(\partial_{z_j} G)(x, y) = (\partial_{\eta_j} \tilde{G})(y - x, x, y) - (\partial_{z_j} \tilde{G})(y - x, x, y)
\]

\[
(\partial_{y_j} G)(x, y) = (\partial_{\mu_j} \tilde{G})(y - x, x, y) + (\partial_{z_j} \tilde{G})(y - x, x, y)
\]

so that

\[
(\partial_{z_j} G)(x, y) = -(\partial_{y_j} G)(x, y) + (\partial_{\eta_j} \tilde{G})(y - x, x, y) + (\partial_{\mu_j} \tilde{G})(y - x, x, y).
\]
Moreover, we always have

\[ |\tilde{G}(z, x, y)|, \left| \frac{\partial \tilde{G}}{\partial \eta_j}(z, x, x + z) \right|, \left| \frac{\partial \tilde{G}}{\partial \mu_j}(z, x, x + z) \right| \leq C(\eta)|z|^{-1}. \tag{87} \]

where \(C(\eta)\) is uniformly bounded if \(\eta\) varies in a compact subset of \((0, 1)^2\).

**Proposition 8.2.**

\[ \frac{\partial G^k_i}{\partial x_j} = -\frac{\partial G^k_i}{\partial y_j} + x_i^{-2}\delta_{ij}O(\|y - x\|^{-1}) \tag{88} \]

**Proof.** This is a straightforward calculation using (86). \(\square\)

**Proposition 8.3** (Derivatives of \(Q_i\)).

\[ \frac{\partial Q_i}{\partial x_j} = c_0 P.V. \int_{[0,1]^2} \left[ \frac{\partial G^1_i}{\partial x_j} + \frac{\partial G^2_i}{\partial x_j} \right] \omega(y) \, dy \]

\[ -\omega(x) \lim_{\delta \to 0^+} \int_{\partial B(\delta, x)} G^i_i \cdot \nu_j \, d\sigma + \frac{\partial Q^3_i}{\partial x_j} \]

**Proof.** Write \((G^1_i + G^2_i)(x, y) := G(x, y)\). \(G\) has again the form \(G(x, y) = \tilde{G}(y - x, x, y)\), where \(\tilde{G}(z, \eta, \mu)\) is smooth provided \(\eta, \mu \in (0, 1)^2, z \in \Omega_x \setminus \{0\}\). Also (85), (87) hold for \(\tilde{G}\). Now

\[ \frac{\partial}{\partial x_j} \int_{\Omega_x} \tilde{G}(z, x, x + z) \omega(x + z) \, dz = \int_{\Omega_x} \tilde{G}(z, x, x + z) \frac{\partial \omega}{\partial z_j}(x + z) \, dz \]

\[ + \int_{\Omega_x} \partial_{z_j}(\tilde{G}(z, x, x + z)) \omega(x + z) \, dz \]

\[ -\int_{\partial \Omega_x} \tilde{G}(z, x, x + z) \omega(x + z) \nu_j \, d\sigma \tag{89} \]

where \(\nu_j\) denotes the \(j\)-th component of the unit outer normal. This is a standard differentiation result (note the bounds (87)).

Now consider the integral in the line (89), exclude the singularity and integrate by parts:

\[ \int_{\Omega_x} \tilde{G}(z, x, x + z) \frac{\partial \omega}{\partial z_j}(x + z) \, dz = -\int_{\Omega_x \setminus B(0, \delta)} \partial_{z_j}(\tilde{G}(z, x, x + z)) \omega(x + z) \, dz \]

\[ + \int_{\partial \Omega_x} \tilde{G}(z, x, x + z) \omega(x + z) \nu_j \, d\sigma \]

\[ -\int_{\partial B(0, \delta)} \tilde{G}(z, x, x + z) \omega(x + z) \nu_j \, d\sigma \tag{90} \]

Observe that by (85),

\[ -\partial_{z_j}(\tilde{G}(z, x, x + z)) + \partial_{x_j}(\tilde{G}(z, x, x + z)) = (\partial_{x_j} G)(z, x, x + z). \]
So combining (89) and (90), we finally get
\[
\frac{\partial}{\partial x_j} \int_{\Omega_x} \tilde{G}(z, x, x + z) \omega(x + z) \, dz = -\int_{\partial B(0, \delta)} \tilde{G}(z, x, x + z) \omega(x + z) \nu_j \, d\sigma \\
+ \int_{\Omega_x} (\partial_x G)(z, x, x + z) \omega(x + z) \, dz + \int_{B(0, \delta)} \partial_x (\tilde{G}(z, x, x + z)) \omega(x + z) \, dz.
\]
Replacing \(x + z\) by \(y\) and sending \(\delta \to 0\) yields the statement. \(\square\)

Recall that
\[d_1(x) = \min\{x_1, x_2\}\]
is the distance of the point \(x\) to the coordinate axes. Observe also that
\[\frac{1}{2} x_r \leq y_r \leq \frac{3}{2} x_r\]
for \(y \in B(\frac{1}{2} d_1(x), x), r = 1, 2\). For the entire appendix, we shall write that \(M = M_\delta\), i.e.
\[|\frac{\partial \omega}{\partial x_j}(x)| \leq M x_j^{-\alpha} \quad (x \in \hat{D}, j = 1, 2)\]
holds, implying also the inequalities
\[|\omega(x)| \lesssim M x_j^{-\alpha} \quad (x \in \hat{D}, j = 1, 2)\]
(by the fact that \(\omega\) vanishes identically on the coordinate axes). Figure 2 illustrates the domains we need in the proof of the following propositions.

**Figure 2.** Domains of integration in Proposition 8.4.

**Proposition 8.4.** Let \(I = B(\frac{1}{2} d_1(x), x) \cap \hat{D}\). Then
\[|P.V. \int_I \frac{\partial (G_i^1 + G_i^2)}{\partial x_j} \omega(y) \, dy| \lesssim M x_i^{-\alpha} (1 + \delta_j \log d(x)) \quad (i \neq j)\]
Proof. First let $i \neq j$ and $0 < \delta < \frac{1}{2}d_1(x)$. By Proposition 8.2 and integration by parts,

$$\left| \int_{I \setminus B(\delta, x)} \frac{\partial G^k_i}{\partial x_j} \omega(y) \, dy \right| = \left| \int_{I \setminus B(\delta, x)} \frac{\partial G^k_i}{\partial y_j} \omega(y) \, dy \right|$$

where $\nu = (\nu_1, \nu_2)$ is the unit outward pointing normal on $\partial I$. We first take care of the integral over $\partial I$. Observe that for $x \in D$, $\partial I$ is either a full circle is the union of a part of a circle and a flat part $\Sigma$. Hence

$$\left| \int_{\partial I} G^k_i \nu_j \, d\sigma \right| \leq \int_{\Sigma} |G^k_i| |\delta_{2j}| \, d\sigma + \int_{\partial B(0, \frac{1}{2}d_1(x))} |G^k_i(\varphi)| \, d\sigma$$

For all sufficiently small $\varepsilon > 0$,

$$\int_{\Sigma} |G^k_i| \, d\sigma \leq \int_{\Sigma \cap \{|y_1 - x_1| \leq \varepsilon\}} |G^k_i| \, d\sigma + \int_{\Sigma \cap \{|y_1 - x_1| \geq \varepsilon\}} |G^k_i| \, dy_1$$

$$\lesssim \int_{\Sigma \cap \{|y_1 - x_1| \leq \varepsilon\}} \frac{x_i^{-1}}{|y - x|} \, dy_1 + \int_{\Sigma \cap \{|y_1 - x_1| \geq \varepsilon\}} \frac{x_i^{-1}}{|y - x|} \, dy_1$$

$$\lesssim x_i^{-1} \int_{x_1 - \varepsilon}^{x_1 + \varepsilon} \frac{1}{|x_2 - \delta_2|} \, dy_1 + x_i^{-1} \int_{1>|x_1 - y_1| > \varepsilon} \frac{1}{|y_1 - x_1|} \, dy_1$$

$$\lesssim \frac{x_i^{-1}\varepsilon}{|x_2 - \delta_2|} + x_i^{-1} \int_{1}^{1} \frac{1}{y_1} \, dy_1$$

$$\lesssim \frac{x_i^{-1}\varepsilon}{|x_2 - \delta_2|} + x_i^{-1} |\log \varepsilon|.$$ 

Here we used proposition 8.1 again. Choosing $\varepsilon = |x_2 - \delta_2| = d(x)$ we get

$$\int_{\Sigma} |G^k_i| \, d\sigma \lesssim x_i^{-1}(1 + |\log d(x)|).$$

The other part is estimated by (using proposition 8.1 again)

$$\int_{\partial B(0, \frac{1}{2}d_1(x))} |G^k_i(\varphi)| \, d\sigma \lesssim x_i^{-1} \int_{0}^{2\pi} |y - x|^{-1} d_1(x) \, d\varphi \lesssim x_i^{-1}.$$ 

Therefore we get for the integral over $\partial I$, using (94) and (92):

$$\left| \int_{\partial I} G^k_i \omega(y) \nu_j \, d\sigma \right| \lesssim \int_{\partial I} |G^k_i \nu_j| |\omega(y)| \, d\sigma \lesssim M \int_{\partial I} |G^k_i \nu_j| y_1^{1-\alpha} \, d\sigma$$

$$\lesssim M x_i^{-\alpha} \int_{\partial I} |G^k_i \nu_j| \, d\sigma \lesssim M x_i^{1-\alpha} x_i^{-1}(1 + \delta_{j2} |\log d(x)|).$$
Similar estimates yield that the contribution from the integral over $\partial B(\delta, x)$ is $\lesssim M x_i^{-\alpha}$, with universal constants independent of $\delta$. For the remaining integral we have, using (92):

$$\left| \int_{I \setminus B(\delta, x)} G_i^k \frac{\partial \omega}{\partial y_j}(y) \ dy \right| \lesssim M x_i^{-\alpha} \int_{I \setminus B(\delta, x)} |G_i^k| \ dy \lesssim M x_i^{-\alpha} \int_{I \setminus B(\delta, x)} |y - x|^{-1} x_i^{-1} \ dy$$

$$\lesssim M x_i^{-\alpha} x_i^{-1} \int_\delta^{d_1(x)} \frac{1}{\rho} \rho \, d\rho \lesssim M x_i^{-\alpha} x_i^{-1} \int_\delta^{d_1(x)} \frac{1}{\rho} \rho \, d\rho$$

$$\lesssim M x_i^{-\alpha} x_i^{-1} d_1(x).$$

Since $d_1(x) \leq x_j$ we get:

$$\left| \int_{I \setminus B(\delta, x)} G_i^k \frac{\partial \omega}{\partial y_j}(y) \ dy \right| \lesssim M x_i^{-\alpha}.$$

This concludes the case $i \neq j$, since all the estimates are independent of $\delta$.

Now let $i = j$. We argue as before, but in (96) now an additional term of the form

$$x_i^{-2} \int_{I \setminus B(\delta, x)} O(|y - x|^{-1})|\omega(y)| \, dy$$

appears, for which we obtain the estimate $\lesssim M x_i^{-\alpha}$ by the same methods. 

\begin{lemma}
Let $\gamma \in (0, 1)$ and $x_1 \geq 0$. Then

$$y_1^\gamma \leq |y_1 - x_1|^\gamma + x_1^\gamma \quad (y_1 \geq 0).$$

\end{lemma}

\begin{proof}
If $y_1 \leq x_1$, the inequality is obvious. For $y_1 > x_1$ we have $y_1 \geq y_1 - x_1 > 0$ and hence $\gamma y_1^{\gamma - 1} \leq \gamma (y_1 - x_1)^{\gamma - 1}$ so that

$$y_1^\gamma - x_1^\gamma \leq \gamma \int_{x_1}^{y_1} s^{\gamma - 1} \, ds \leq \gamma \int_{x_1}^{y_1} (s - x_1)^{\gamma - 1} \, ds = (y_1 - x_1)^{\gamma}.$$

\end{proof}

\begin{proposition}
Let $II = \widehat{D} \setminus I$ with $I$ as in proposition 8.4 Then

$$\left(97\right) \quad \left| \int_{II} \frac{\partial G_i^k}{\partial x_j} \omega(y) \ dy \right| \lesssim M x_i^{-\alpha}$$

\end{proposition}

\begin{proof}
Using proposition 8.1, (94) and lemma 8.5 we have

$$\left| \int_{II} \frac{\partial G_i^k}{\partial x_j} \omega(y) \ dy \right| \lesssim \int_{II} \left| \frac{\partial G_i^k}{\partial x_i} \right| |\omega(y)| \, dy$$

$$\lesssim \int_{II} |y - x|^{-3} y_i^{-1-\alpha} \, dy \lesssim \int_{II} |y - x|^{-3} (|y_i - x_i|^{-1-\alpha} + x_i^{-1-\alpha})$$

$$\lesssim \int_{II} |y - x|^{-2-\alpha} \, dy + x_i^{-1-\alpha} \int_{II} |y - x|^{-3}$$

$$\lesssim \int_0^\infty \frac{1}{\rho^{2+\alpha}} \rho \, d\rho + x_i^{-1-\alpha} \int_0^\infty \frac{1}{\rho^{3}} \rho \, d\rho$$

$$\lesssim d_1(x)^{-\alpha} + x_i^{-1-\alpha} d_1(x)^{-1} \lesssim x_i^{-\alpha} + x_i^{-1-\alpha} x_i^{-1} \lesssim x_i^{-\alpha}.$$

\end{proof}
Proposition 8.7. For $i \neq j$, we have
\begin{equation}
\left| \frac{\partial (G_i^1 + G_j^2)}{\partial x_j} \right| \lesssim x_i^{-\gamma_1-\gamma_2} x_j^{\gamma_2} |y - x|^{-(3-\gamma_1)}.
\end{equation}
where $\gamma_1, \gamma_2 \in [0, 1], \gamma_1 + \gamma_2 \leq 1$.

Proof. The proof of the proposition is based on a cancellation property of the kernels $G_i^1$ and $G_j^2$ and requires quite tedious computations. First calculate the sum of $\partial x_i G_i^2$ and $\partial x_1 G_1^3$ and see that it can be grouped into three expressions:
\begin{align*}
&\frac{y_2(y_1 - x_1)^2}{|y - x|^4 |y - x|^2} - \frac{y_2(y_1 + x_1)^2}{|y + x|^4 |y - x|^2} = (A) \\
&\frac{y_2(y_1 - x_1)^2}{|y - x|^2 |y - x|^4} - \frac{y_2(y_1 + x_1)^2}{|y + x|^2 |y + x|^4} = (B) \\
&\frac{y_2}{|y - x|^2 |y + x|^2} - \frac{y_2}{|y - x|^2 |y - x|^2} = (C)
\end{align*}
For convenience, these can be further written as
\begin{align*}
&\frac{y_2(y_1 - x_1)^2}{|y - x|^4 |y - x|^2} \left[ |y - x|^{-4} - |y - \bar{x}|^{-4} \right] + \frac{y_2}{|y - \bar{x}|^4} \left( \frac{(y_1 - x_1)^2}{|y - x|^2} - \frac{(y_1 + x_1)^2}{|y + x|^2} \right) \\
&\quad = (1) + (2) \\
&\frac{y_2}{|y - \bar{x}|^4} \left[ \frac{(y_1 - x_1)^2}{|y - x|^2} - \frac{(y_1 + x_1)^2}{|y - \bar{x}|^2} \right] + \frac{y_2}{|y - x|^2} \left[ \frac{(y_1 + x_2)^2}{|y - x|^4} + \frac{(y_1 + x_1)^2}{|y + x|^4} \right] \\
&\quad = (3) + (4) \\
&\frac{y_2}{|y + x|^2} \left[ |y - \bar{x}|^{-2} - |y - x|^{-2} \right] + \frac{y_2}{|y - x|^2} \left[ |y + x|^{-2} - |y - \bar{x}|^{-2} \right] \\
&\quad = (5) + (6)
\end{align*}
Let us estimate expression (1). Using $|y - \bar{x}|^2 - |y - x|^2 = 2x_1 y_1$ and the relations $y_2 \leq |y - \bar{x}|$, $(y_1 - x_1)^2 \leq |y - x|^2$, $y_1 \leq (y_1 + x_1)$, we arrive at
\begin{equation}
|(1)| \lesssim \frac{x_1}{|y - \bar{x}| |y - x|} \left[ |y - x|^{-2} + |y - \bar{x}|^{-2} \right].
\end{equation}
Write $\gamma = \gamma_1 + \gamma_2$ and noting that $|y - \bar{x}| \geq x_2^\gamma |y - \bar{x}|^{1-\gamma}$, $|y - \bar{x}| \geq x_1^{1-\gamma_2} |y - \bar{x}|$ and the reflection relations $|y - \bar{x}|, |y - \bar{x}| \geq |y - x|$ for $y \in [0, 1]^2$, we arrive at
\begin{equation}
|(1)| \lesssim x_2^{-\gamma} x_1^{\gamma_2} |y - x|^{-(3-\gamma_1+\gamma_2)}.
\end{equation}
To estimate (2), we use the relation
\begin{equation}
|y + x|^2 (y_1 - x_1)^2 - |y - \bar{x}|^2 (y_1 + x_1)^2 = -4x_1 y_1 (y_2 + x_2)
\end{equation}
and similar estimations as above to arrive at
\begin{equation}
|(2)| \lesssim \frac{y_2 y_1 x_1 (y_2 + x_2)^2}{|y - \bar{x}|^4 |y - \bar{x}|^2 |y + x|^2} \lesssim \frac{x_1}{|y - \bar{x}| |y - \bar{x}|^3} \lesssim x_2^{-\gamma} x_1^{\gamma_1} |y - x|^{-3-\gamma_2+\gamma}.
\end{equation}
(3) is similar to the above. (4) is slightly different, since expressions containing $x_1^2$ may appear. In (4), we use
\[
|y + x|^4 - |y - \bar{x}|^4 \lesssim [y_1 - x_1|x_1 + x_2^2][y + x^2 + |y - \bar{x}|^2]
\]
to get
\[
|(4)| \leq \frac{|y_1 - x_1|x_1}{|y - \bar{x}||y - \bar{x}|^3|y + x|} + \frac{x_1^2}{|y - \bar{x}||y - \bar{x}|^3|y + x|} + \frac{|y_1 - x_1|x_1}{|y - \bar{x}||y - \bar{x}|^3|y + x|} + \frac{x_1^2}{|y - \bar{x}||y - \bar{x}|^3|y + x|}.
\]
The terms not containing $x_1^2$ are handled in a familiar manner. Concerning the others, we have for example
\[
x_1^2 |y - \bar{x}||y - \bar{x}|^3|y + x| \leq x_2^{-\gamma}x_1^{\gamma\gamma_2}(x_1 + y_1) \frac{(x_1 + y_1)}{|y - \bar{x}||y - \bar{x}|^3|y + x|} \leq x_2^{-\gamma}x_1^{\gamma\gamma_2}|y - x|^{-3 + \gamma - \gamma_2}.
\]
The treatment of (5), (6) parallels the above. □

**Proposition 8.8.** For $i \neq j$,
\[
\left| \int_{[0,1]^2 \setminus \hat{D}} \left[ \frac{\partial G_i^1}{\partial x_j} + \frac{\partial G_i^2}{\partial x_j} \right] \omega(y) \, dy \right| \leq C(\gamma_1, \gamma_2)x_i^{-(\gamma_1+\gamma_2)}x_j^{\gamma_2}d(x)^{-1+\gamma_1}
\]
where $\gamma_1 \in (0, 1), \gamma_2 \in [0, 1), \gamma_1 + \gamma_2 < 1$. Also,
\[
\left| \int_{[0,1]^2 \setminus \hat{D}} \left[ \frac{\partial G_i^i}{\partial x_1} + \frac{\partial G_i^2}{\partial x_j} \right] \omega(y) \, dy \right| \leq C(\gamma)x_i^{-\gamma}d(x)^{-1+\gamma_1}
\]

**Proof.** As a preparation, we note that for $x \in D, 0 < \gamma_1 < 1$,
\[
(99) \quad \int_{[0,1]^2 \setminus \hat{D}} |y - x|^{-3+\gamma_1} \lesssim d(x)^{-1+\gamma_1}.
\]
This follows from
\[
\int_{[0,1]^2 \setminus \hat{D}} |y - x|^{-3+\gamma_1} \leq \int_{[0,1]^2 \setminus B(x,d(x))} |y - x|^{-3+\gamma_1} \leq \int_{B(x,10) \setminus B(x,d(x))} |y - x|^{-3+\gamma_1},
\]
since $[0,1]^2 \setminus \hat{D}$ is contained in $[0,1]^2 \setminus B(x,d(x))$ because of $\delta_2 < \delta_3, \delta_1 < \delta_2$.

From Proposition 8.7, we get in case $i \neq j$
\[
\left| \int_{[0,1]^2 \setminus \hat{D}} \left[ \frac{\partial G_i^1}{\partial x_1} + \frac{\partial G_i^2}{\partial x_j} \right] \omega(y) \, dy \right| \leq x_i^{-(\gamma_1+\gamma_2)}x_j^{\gamma_2} \int_{[0,1]^2 \setminus \hat{D}} |y - x|^{-1+\gamma_1} \leq x_i^{-(\gamma_1+\gamma_2)}x_j^{\gamma_2}d(x)^{-1+\gamma_1},
\]
according to (99).

For the second inequality of the Proposition, we note that (see Proposition 8.1)
\[
\left| \frac{\partial G_i^k}{\partial x_i} \right| \lesssim x_i^{-\gamma}|y - x|^{-3+\gamma},
\]
Proposition 8.9. For $x \in \Omega$, 

\[
\left| P.V. \int_{[0,1]^2} \left[ \frac{\partial G_i}{\partial x_j} + \frac{\partial G_j}{\partial y_j} \right] \omega(y) \, dy \right| \leq M x_i^{-1} x_j^{\alpha} (1 + \delta_j \log d(x))
\]

\[
+ C(\gamma_1, \gamma_2) x_i^{-(\gamma_1 + \gamma_2)} x_j^{\gamma_2} d(x)^{-1 + \gamma_1} \quad (i \neq j)
\]

\[
\left| P.V. \int_{[0,1]^2} \left[ \frac{\partial G_i}{\partial x_i} + \frac{\partial G_j}{\partial x_j} \right] \omega(y) \, dy \right| \leq M x_i^{-\alpha} (1 + \delta_i \log d(x)) + C(\gamma_1) x_i^{-\gamma_1} d(x)^{-1 + \gamma_1}
\]

with $\gamma, \gamma_1 \in (0, 1)$, $\gamma_2 \in [0, 1)$, $\gamma_1 + \gamma_2 < 1$.

Proof. We split the integral into a principal value integral over $\hat{D}$ and a convergent integral over $[0,1]^2 \setminus \hat{D}$. The integral over $\hat{D}$ is further split into integrals over the domains $I = B(x, \frac{1}{2} d_1(x))$ and $II = \hat{D} \setminus I$, which are estimated by Propositions 8.4 and 8.6. The part over $[0,1]^2 \setminus \hat{D}$ is estimated by Proposition 8.8. \qed

8.2. Appendix B.

Proposition 8.10. The following relations hold:

\[
\frac{\partial G_1}{\partial x_1} = - \frac{2 y_1 (y_1 + x_1) (y_2 - x_2)}{|x - x|^2 |y - \bar{x}|^4} + \frac{2 y_1 (y_1 - x_1) (y_2 - x_2)}{|x - x|^2 |y - \bar{x}|^2}
\]

\[
\frac{\partial G_1}{\partial x_2} = \frac{2 y_1 (y_2 - x_2)^2}{|x - x|^2 |y - \bar{x}|^4} + \frac{2 y_1 (y_2 - x_2)^2}{|x - x|^2 |y - \bar{x}|^2} - \frac{y_1}{|x - x|^2 |y - \bar{x}|^2}
\]

\[
\frac{\partial G_1}{\partial y_1} = \frac{2 y_1 (y_1 + x_1) (y_2 - x_2)}{|x - x|^2 |y - \bar{x}|^4} - \frac{2 y_1 (y_1 - x_1) (y_2 - x_2)}{|x - x|^2 |y - \bar{x}|^2} + \frac{y_2 - x_2}{|x - x|^2 |y - \bar{x}|^2}
\]

\[
\frac{\partial G_2}{\partial y_2} = \frac{2 y_1 (y_2 - x_2)^2}{|x - x|^2 |y - \bar{x}|^4} - \frac{2 y_1 (y_2 - x_2)^2}{|x - x|^2 |y - \bar{x}|^2} + \frac{y_1}{|x - x|^2 |y - \bar{x}|^2}
\]

\[
\frac{\partial G_1}{\partial x_1} = - \frac{2 y_1 (y_1 + x_1) (y_2 + x_2)}{|x + x|^2 |y - \bar{x}|^2} + \frac{2 y_1 (y_1 - x_1) (y_2 + x_2)}{|x + x|^2 |y - \bar{x}|^4}
\]

\[
\frac{\partial G_1}{\partial x_2} = - \frac{2 y_1 (y_2 + x_2)^2}{|x + x|^2 |y - \bar{x}|^4} - \frac{2 y_1 (y_2 + x_2)^2}{|x + x|^2 |y - \bar{x}|^2} + \frac{y_1}{|x + x|^2 |y - \bar{x}|^2}
\]

\[
\frac{\partial G_1}{\partial y_1} = - \frac{2 y_1 (y_1 + x_1) (y_2 + x_2)}{|x + x|^2 |y - \bar{x}|^4} - \frac{2 y_1 (y_1 - x_1) (y_2 + x_2)}{|x + x|^2 |y - \bar{x}|^2} + \frac{y_2 + x_2}{|x + x|^2 |y - \bar{x}|^2}
\]

\[
\frac{\partial G_2}{\partial y_2} = - \frac{2 y_1 (y_2 + x_2)^2}{|x + x|^2 |y - \bar{x}|^4} - \frac{2 y_1 (y_2 + x_2)^2}{|x + x|^2 |y - \bar{x}|^2} + \frac{y_1}{|x + x|^2 |y - \bar{x}|^2}
\]
\[
\begin{align*}
\frac{\partial G_1}{\partial x_1} &= -\frac{2y_2(y_1 + x_1)^2}{|y + x|^4|y - \bar{x}|^2} - \frac{2y_2(y_1 + x_1)^2}{|y + x|^2|y - \bar{x}|^2} + \frac{y_2}{|y + x|^2|y - \bar{x}|^2}, \\
\frac{\partial G_2}{\partial y_2} &= -\frac{2y_2(y_1 + x_1)^2}{|y + x|^4|y - \bar{x}|^2} - \frac{2y_2(y_1 + x_1)^2}{|y + x|^2|y - \bar{x}|^2} + \frac{y_2}{|y + x|^2|y - \bar{x}|^2}.
\end{align*}
\]

References

1. H. Bahouri, J.-Y. Chemin, Equations de transport relatives a des champs de vecteurs non-Lipschitziens et mechanique de fluides. (French) Arch. Rational Mech. Anal. 127, 2, 159-181 (1994).
2. D. Cordoba: Nonexistence of Simple Hyperbolic Blow-Up for the Quasi-Geostrophic Equation. Ann. Math., Second Series, Vol. 148, No. 3 (1998), 1135-1152
3. D. Cordoba, C. Fefferman: Growth of Solutions for QG and 2D Euler Equations. Journal of the American Mathematical Society, 15, 3, p. 665-670 (2002).
4. S. Denisov, Infinite superlinear growth of the gradient for the two-dimensional Euler equation. Discrete Contin. Dyn. Syst. A 23 (2009), 755-764.
5. S. Denisov, Double-exponential growth of the vorticity gradient for the two-dimensional Euler equation. Proceedings of the AMS, to appear. Preprint arXiv:1201.1771v2.
6. S. Denisov, The sharp corner formation in 2D Euler dynamics of patches: infinite double exponential rate of merging, preprint arXiv:1201.2210v3.
7. N. H. Katz and A. Tapay, A model for studying double exponential growth in the two-dimensional Euler equations, preprint arXiv:1403.6867.
8. A. Kiselev and V. Sverák, Small scale creation for solutions of the incompressible two dimensional Euler equation, preprint arXiv:1310.4799v2.
9. http://terrytao.wordpress.com/2007/03/18/why-global-regularity-for-navier-stokes-is-hard/ post by Nets Katz on 20 March, 2007 at 12:21 am and the following thread.
10. W. Walter, Differential and Integral Inequalities, Springer 1970.
11. D. Wilett, A Linear Generalization of Gronwall’s inequality. Trans. Am. Math. Soc, 16, 774-778 (1965).
12. A. Zlatos, Exponential Growth of the vorticity gradient for the Euler equation on the torus, preprint arXiv:1310.6128v1.
