Duality between integrable Stäckel systems.

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For the Stäckel family of the integrable systems a non-canonical transformation of the time variable is considered. This transformation may be associated to the ambiguity of the Abel map on the corresponding hyperelliptic curve. For some Stäckel’s systems with two degrees of freedom the $2 \times 2$ Lax representations and the dynamical $r$-matrix algebras are constructed. As an examples the Henon-Heiles systems, integrable Holt potentials and the integrable deformations of the Kepler problem are discussed in detail.

1 Introduction

In her celebrated papers \cite{1} published in 1889 S.Kowalewski discovered a necessary condition for an $n$-dimensional system to be completely integrable. This criterion enabled her to classify all integrable solid body motion about a fixed point and introduce the separation variables for her celebrated top. Evidently, it was the first application of singularity analysis to a concrete physical problem. Recall, that the method of singularity analysis associates integrability with the Kowalewski-Painlevé property, i.e. a movable polelike singularity $(t - t_0)^{-m}$ in the solution of the equations of motion, here space and time must be thought of as complex \cite{2, 3}.

There exist cases, however, of integrable hamiltonian system with rational integrals of the motion, whose analytic structure permits solutions with algebraic singularities of the type $(t - t_0)^{1/k}$, $(k$ being a positive integer larger than one). This led to the introduction of the "weak" Painlevé property \cite{2}. A simple change of the independent variable, in and of itself, does not turn a "weak" Painlevé system into one that satisfies the usual Kowalewski-Painlevé criterion. In particular, the simple-minded idea to take $(t - t_0)^{1/k}$ as a new independent variable does not lead to a Painlevé expansion for all the solutions. According by \cite{2}, any transformation that modifies the nature of the singular expansions must also involve a change of the dependent variables in order to reestablish the Painlevé property. Such transformations, however, if they exist, are expected to be quite nontrivial and difficult to generalize to other examples.

Of course, transformations, which use change of the independent time variable $t$, are non-canonical transformations. The classical example of the such non-canonical transformations is the duality of the Kepler problem to the geodesic motion on the sphere \cite{4} or to the harmonic oscillator \cite{5}.

Another important to us example is the Kolosoff transformations for the Kowalewski top \cite{6}. In this case the Kowalewski separation variables coincide to the standard elliptic coordinates $\{q_1, q_2\}$ at the plane $(x, y)$ after the non-canonical change of the time

$$d\bar{t} = (q_1(t) + q_2(t)) \, dt.$$  

(1.1)

Recently, the non-canonical transformations relate the Kowalewski top with the geodesic motion on $SO(4)$ \cite{7} and with the Neumann system on the sphere $S^2$ \cite{8}.

In this paper we consider the fortiori integrable systems with the known separation variables. Since we shall not discuss singularity analysis \cite{2} and theory of algebraic completely integrable systems \cite{3} in detail. Recall, the variable separation method permits one to reduce an integration
problem with several degrees of freedom to a sequence of one-dimensional integration problems. The inverse problem of obtaining various classes of completely integrable systems starting from a set of separated one-dimensional problems was started in the lectures by Jacobi. In framework of this approach, Liouville and Stäckel introduced a family of the simple systems integrable in quadratures (a Liouville family is a particular case of a Stäckel family).

Here for the Stäckel system we introduce non-canonical transformations of the time variable associated to the ambiguity of the Abel map on the hyperelliptic curve. All these transformations depend on coordinates only and, therefore, they are closed to the Kolosoff change of the time \((1.1)\). For the some Stäckel systems we propose the Lax pairs and \(r\)-matrix algebras. As an examples the Henon-Heiles systems, integrable Holt potentials and the integrable deformations of the Kepler problem are discussed in detail.

2 Duality between the Stäckel systems

Before proceeding father it is useful to recall the classical work of Stäckel \([9]\). The system associated with the name of Stäckel \([9]\) is a holonomic system on the phase space \(\mathbb{R}^{2n}\) equipped with the canonical variables \(\{p_j, q_j\}_{j=1}^n\), with the standard symplectic structure \(\Omega_n\) and with the following Poisson brackets

\[
\Omega_n = \sum_{j=1}^n dp_j \wedge dq_j, \quad \{p_j, q_k\} = \delta_{jk}.
\]  

(2.1)

The nondegenerate \(n \times n\) Stäckel matrix \(S\), whose \(j\) column \(s_{kj}\) depends only on \(q_j\)

\[
\det S \neq 0, \quad \frac{\partial s_{kj}}{\partial q_m} = 0, \quad j \neq m
\]

defines \(n\) functionally independent integrals of motion

\[
I_k = \sum_{j=1}^n c_{jk}(p_j^2 + U_j), \quad c_{jk} = \frac{s_{kj}}{\det S}.
\]  

(2.2)

which are quadratic in momenta. Here \(C = [c_{ik}]\) denotes inverse matrix to \(S\) and \(s_{kj}\) be cofactor of the element \(s_{kj}\).

Each integral \(I_k\) \((2.2)\) may be associated to the time variable \(t_k\), such that for any function \(\xi(p, q)\) one gets

\[
\frac{d\xi(p, q)}{dt_k} = \{I_k, \xi(p, q)\}.
\]

By definition the first integral \(I_1 = H\) be the Hamilton function associated to the time \(t\). The common level surface of the integrals \((2.2)\)

\[
M_\alpha = \left\{z \in \mathbb{R}^{2n} : I_k(z) = \alpha_k, \quad k = 1, \ldots, n\right\}
\]

(2.3)

is diffeomorphic to the \(n\)-dimensional real torus and one immediately gets

\[
p_j^2 = \left(\frac{\partial S}{\partial q_j}\right)^2 = \sum_{k=1}^n \alpha_k s_{kj}(q_j) - U_j(q_j),
\]  

(2.4)

where \(S(q_1, \ldots, q_n)\) is a reduced action function \([10]\). If this real torus is a part of complex algebraic torus, then the corresponding mechanical system is called an algebraic completely integrable system \([3]\).

The corresponding Hamilton-Jacobi equation on \(M_\alpha\)

\[
\frac{\partial S}{\partial t} + H(t, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, q_1, \ldots, q_n) = 0, \quad \Rightarrow \quad c_{j1} \frac{\partial S}{\partial q_j} \frac{\partial S}{\partial q_j} = E,
\]  

(2.5)

admits the variable separation

\[
S(q_1, \ldots, q_n) = \sum_{j=1}^n S_j(q_j), \quad S_j(q_j) = \int \sqrt{F_j(q_j)} \; dq_j.
\]  

(2.6)
Here the functions $F_j(\lambda)$ depend on $n$ parameters $\{\alpha_k\}_{k=1}^n$

$$F_j(\lambda) = \sum_{k=1}^n \alpha_k s_{kj}(\lambda) - U_j(\lambda).$$

Coordinates $q_j(t, \alpha_1, \ldots, \alpha_n)$ are determined from the equation explicitly depending on time

$$\sum_{j=1}^n \int_{\gamma_0(p_0, q_0)}^{\gamma_j(p_j, q_j)} \frac{s_{1j}(\lambda) d\lambda}{\sqrt{\sum_{k=1}^n \alpha_k s_{1j}(\lambda) - U_j(\lambda)}} = \beta_1 = t, \quad (2.7)$$

and from other $n-1$ equations

$$\sum_{j=1}^n \int_{\gamma_0(p_0, q_0)}^{\gamma_j(p_j, q_j)} \frac{s_{kj}(\lambda) d\lambda}{\sqrt{\sum_{k=1}^n \alpha_k s_{kj}(\lambda) - U_j(\lambda)}} = \beta_k, \quad k = 2, \ldots, n. \quad (2.8)$$

The solutions of the problem is thus reduced to solving a sequence of one-dimensional problems, which is the essence of the method of separation of variables.

Now we turn to the non-canonical change of the time and prove the following

**Proposition 1** If the two Stäckel matrices $S$ and $\tilde{S}$ be distinguished the first row only $s_{kj} = \tilde{s}_{kj}$, $k \neq 1$, the corresponding Stäckel systems with the following Hamilton functions

$$\tilde{H} = v(q) H, \quad v(q) = \frac{\det S(q_1, \ldots, q_n)}{\det S(q_1, \ldots, q_n)}, \quad (2.9)$$

are related by non-canonical change of the time.

In fact, the corresponding Hamilton functions $H$ and $\tilde{H}$ obey to the equation (2.9), which follows from the definitions of the hamiltonians

$$H = \sum_{j=1}^n c_{1j} \left( p_j^2 + U_j(q_j) \right) \quad (2.10)$$

and entries of the inverse matrix

$$c_{1j} = \frac{S_{1j}}{\det S} = \frac{1}{\det S} \frac{\partial \det S}{\partial s_{1j}}.$$ 

Equation (2.9) defines an implicit change of the time $t \to \tilde{t}$ associated to the integrals $H = I_1$ and $\tilde{H} = I_{\tilde{1}}$, respectively. On the other hand the equation (2.7) may be considered as an explicit determination of this transformation $t \to \tilde{t}$. In contrast with the general coupling constant metamorphosis discussed in (2.3) equation (2.9) is independent on the any constant entering in the potential $U$.

Obviously, by using row by row transformations of the Stäckel matrices with the associated $t_k \to \tilde{t}_k$ transformations we can reduce the given Stäckel system to any other Stäckel system on $\mathbb{R}^{2n}$.

As an example, let us consider three matrices

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} q_1 & q_2 \\ 1 & -1 \end{pmatrix}, \quad \hat{S} = \begin{pmatrix} q_1^2 & q_2^2 \\ q_1 & q_2 \end{pmatrix}. \quad (2.11)$$

The corresponding hamiltonians $H$, $\tilde{H}$ and $\hat{H}$ defined by (2.10) are dual (2.9)

$$\tilde{H} = \frac{1}{2} \left( q_1 + q_2 \right)^{-1} H, \quad (2.12)$$

$$\hat{H} = \frac{1}{2} \left( q_1^2 + q_2^2 \right)^{-1} H = \frac{q_1 + q_2}{q_1^2 + q_2^2} \tilde{H},$$
for any potentials \( U \). For the Hamiltonians \( \tilde{H} \) and \( \hat{H} \) the change of the time (2.12) is closed to the Kolosoff transformation (1.4) and for any function \( \xi(q) \) depending on coordinates only one gets
\[
\frac{d\xi(q)}{dt} = \{H, \xi(q)\} = \frac{1}{2(q_1 + q_2)} \frac{d\xi(q)}{dt}
\]  
(2.13)
For instance, let us consider uniform cubic potential
\[
U(q_j) = 2\alpha^2 q_j^3 + \beta q_j^2 + \gamma q_j + \delta,
\]
which gives rise to the Hamiltonian \( H \)
\[
H = \frac{1}{4} (p_1^2 + p_2^2) + \alpha^2 (q_1^4 + q_2^4) + \beta (q_1^2 + q_2^2) + \gamma (q_1 + q_2) + \delta.
\]
Using canonical transformation
\[
q_1 = \frac{x + y}{2}, \quad p_1 = px + py,
\]
\[
q_2 = \frac{x - y}{2}, \quad p_2 = px - py,
\]
for the first system, the more complicated transformation
\[
q_1 = \frac{3}{4} \frac{x^{2/3} + py}{3\alpha}, \quad p_1 = px \frac{x^{1/3} - \frac{3}{2} y}{2},
\]
\[
q_2 = \frac{3}{4} \frac{x^{2/3} - py}{3\alpha}, \quad p_2 = px \frac{x^{1/3} + \frac{3}{2} y}{2},
\]
for the system associated to \( \tilde{S} \) and the following change of variables in the third case
\[
q_1 = \sqrt{x - \sqrt{y}}, \quad p_1 = px \sqrt{x - p_y \sqrt{y}},
\]
\[
q_2 = -i (\sqrt{x} + \sqrt{y}), \quad p_2 = i (px \sqrt{x} + p_y \sqrt{x}),
\]
one gets the Hamilton functions in the natural form
\[
H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{\alpha^2}{4} x (x^2 + 3y^2) + \frac{\beta}{4} (x^2 + y^2) + \frac{\gamma}{2} x + \delta,
\]
\[
\tilde{H} = \frac{1}{2} (p_x^2 + p_y^2) + \frac{9\alpha^2}{8} x^{-2/3} (\frac{3}{4} x^2 + y^2) + \delta x^{-2/3} + \frac{3\gamma}{4}, \quad \text{only by} \quad \beta = 0,
\]
\[
\hat{H} = \frac{1}{2} px p_y - \frac{\beta}{2} \frac{1}{\sqrt{xy}} + \frac{\gamma}{4} \left( \frac{1 + i}{\sqrt{x}} - \frac{1 - i}{\sqrt{y}} \right) + \delta, \quad \text{by} \quad \alpha = 0.
\]
under restriction \( \beta = 0 \) for the second case.

The system with the first Hamiltonian \( H \) is so-called first integrable case of the Henon-Heiles system [3]. The second Hamiltonian \( \tilde{H} \) is related to so-called Holt potential [2]. Note, the second integral of motion is the polynomial of the third order in momenta for the Holt system. The system with the third Hamiltonian \( \hat{H} \) may be considered as an integrable deformation of the Kepler problem.

Duality between the Henon-Heiles and the Holt systems with the Hamiltonians \( H \) and \( \tilde{H} \) may be considered as the known coupling constant metamorphosis with respect to the constant \( \gamma \) [2]. The second known duality between the harmonic oscillator and the Kepler problem with the Hamiltonians \( H \) by \( \alpha = 0, \gamma = 0 \) and \( \tilde{H} \) may be considered as the coupling constant metamorphosis with respect to another constant \( \delta \) [2, 3].

We can see that in practical circumstances the St"ackel approach is not very useful because it is usually unknown which canonical transformation have to be used in order to transform a Hamiltonian (2.10) to the natural form \( H = T + V \) [1]. This problem was partially solved for the uniform systems \( U_j = U, \ j = 1, \ldots, n \) with polynomial potentials by using the corresponding Lax pairs [11]. Note, that the movable branch points of the type \( (t - t_0)^{1/k} \) appear in the expansions of the physical variables \( (x, y) \) after canonical transformations.

Henceforth, we shall restrict our attention to the uniform St"ackel systems, where all the polynomial potentials \( U_j(q_j) = U(q_j) \) and associated hyperelliptic curves \( C_j \) (2.4) are equal.
3 Duality and Abel map.

Let us briefly recall some necessary facts about the Abel map and the inverse Jacobi problem.

The set of point $\mathcal{C}(z, \lambda)$ satisfying

$$
\mathcal{C} : \quad z^2 = F(\lambda) = \sum_{k=0}^{2g+1} e_k \lambda^k = \prod_{j=1}^{2g+1} (\lambda - \lambda_j),
$$

(3.1)
is a model of a plane hyperelliptic curve of genus $g$. Here $F(\lambda)$ is polynomial without multiple zeros. Let us denote by $\text{Div}(\mathcal{C})$ the Abelian divisor group and denote by $J(\mathcal{C})$ the Jacobian of the curve $\mathcal{C}$. The Abel map puts into correspondence the point $D \in \text{Div}(\mathcal{C})$ and the point $u \in J(\mathcal{C})$

$$
\mathcal{U} : \quad \text{Div}(\mathcal{C}) \to J(\mathcal{C}),
$$

(3.2)
The set of all effective divisors $D = \gamma_1 + \cdots + \gamma_n$ (the $\gamma_j$'s may be not mutually distinct) of deg $n$ of $\mathcal{C}$ is called the $n$th symmetric product of $\mathcal{C}$, and is denoted by $\mathcal{C}^{(n)} = S^n\mathcal{C}$. The $\mathcal{C}^{(n)}$ can be identified with the set of all unordered $n$-tuples $\{\gamma_1, \ldots, \gamma_n\}$, where $\gamma_j$ are arbitrary elements of $\mathcal{C}$.

Now consider restriction of the Abel map (3.2) to $\mathcal{C}^{(n)}$

$$
\mathcal{U} : \quad \mathcal{C}^{(n)} \to J(\mathcal{C}),
$$

(3.3)

where

$$
\mathcal{U}(\gamma_1, \gamma_2, \ldots, \gamma_g) = \mathcal{U}(\gamma_1) + \mathcal{U}(\gamma_2) + \cdots + \mathcal{U}(\gamma_g).
$$

According to the Abel-Jacobi theorem this map is surjective and generically injective if $n = g$ only

(2) [13]. If $n \neq g$ the Abel map is either lack of uniqueness or degenerate. The corresponding Stäckel system either has a dual system associated with the same curve or it is a superintegrable system [11].

Suppose that point $D = \gamma_1 + \cdots + \gamma_k, \quad k \leq g$ belongs to $\mathcal{C}^{(k)}$. The differential of the Abel-Jacobi map (3.3) at the point $D$ is a linear mapping from the tangent space $T_D(\mathcal{C}^{(n)})$ of $\mathcal{C}^{(n)}$ at the point $D$ into the tangent space $T_{U(D)}(J(\mathcal{C}))$ of $J(\mathcal{C})$ at the point $U(D)$

$$
U_D^* : \quad T_D(\mathcal{C}^{(n)}) \to T_{U(D)}(J(\mathcal{C})).
$$

Now suppose that $D$ is a generic divisor, and $x_j$ is a local coordinate on $\mathcal{C}$ near the point $\gamma_j$. Then $(x_1, \ldots, x_n)$ yields a local coordinate system near the point $D$ in $\mathcal{C}^{(n)}$. Let $dw_k$ ($k = 1, \ldots, g$) is a basis for a space $\mathcal{H}_1(\mathcal{C})$ of holomorphic differentials on $\mathcal{C}$, and near $\gamma_j$

$$
dw_k = \phi_{kj}(x_j)dx_j,
$$

(3.4)

where $\phi_{kj}(x_j)$ is holomorphic. It follows that the Abel-Jacobi map $\mathcal{U}$ can be expressed near $D$ as

$$
\mathcal{U}(z_1, \ldots, z_n) = \left( \sum_{j=1}^{n} \int_{\gamma_0}^{x_j} \phi_{1j}(x_j)dx_j, \ldots, \sum_{j=1}^{n} \int_{\gamma_0}^{x_j} \phi_{nj}(x_j)dx_j \right).
$$

Hence

$$
U_D^* = \left( \begin{array}{cccc}
\phi_{11}(\gamma_1) & \cdots & \phi_{1g}(\gamma_1) \\
\vdots & \ddots & \vdots \\
\phi_{k1}(\gamma_n) & \cdots & \phi_{kn}(\gamma_n)
\end{array} \right),
$$

(3.5)
is the so-called Brill-Noether matrix [14]. Henceforth, we shall restrict our attention to the special divisors $D_n$, such that coefficients in the expansion (3.4) are independent on the point $\gamma_j$

$$
dw_k = \phi_k(x_j)dx_j.
$$

In this case all rows of the symmetric Brill-Noether matrix depend on local coordinate $\{x_1, \ldots, x_n\}$ identically.

The Jacobi inversion problem (2.8)) is formulated as follows: for a given point

$$
u = (\beta_1, \beta_2, \ldots, \beta_n) \in J(\mathcal{C})$$

on page 5.
find $n$ points $\gamma_1, \gamma_2, \ldots, \gamma_n$ on the genus $g$ Riemann surface $C$ such that

$$
\sum_{k=1}^{g} \int_{\gamma_0}^{\gamma_k} dw_j = \beta_j, \quad j = 1, \ldots, n. \tag{3.6}
$$

Here we shall tacitly assume that the base point $\gamma_0 \in C$ has already been fixed \[12\].

If $n = g$ for almost all points $u \in J(C)$ the solution $D = \gamma_1 + \cdots + \gamma_n$ exist and is uniquely determined by system \(3.6\) (for the unordered set of points $\gamma_j$) \[12\]. However, if the degree $n < g$ of the symmetric product $C^{(n)}$ is less than genus $g$ of $C$, the Abel map is lack of uniqueness. In this case we can propose that two different points $u, \bar{u} \in J(C)$ have the one Abel preimage $\{\gamma_1, \ldots, \gamma_n\} \in C^{(n)}$.

The Abel preimage of the point $u \in J(C)$ is given by set \{(p_1, q_1), \ldots, (p_n, q_n)\} $\in C^{(n)}$, where \{q_1, \ldots, q_n\} are zeros of the Bolza equation \[15, 13\]. The Abel preimage of the point $u_\lambda$ on the genus $g$ curve $\{p_1, \ldots, p_n\}$ has a common Abel preimage $\{\gamma_1, \ldots, \gamma_n\} \in C^{(n)}$.

Here vector $u$ belongs to Jacobian $J(C)$ and $\varphi_{k,j}(u)$ is the Kleinian $\varphi$-function \[15, 13\].

Now we turn to the uniform St"ackel systems. We can regard each expression \(2.4\) as being defined on the genus $g$ Riemann surface

$$
C : \quad y_j^2 = F(\lambda), \quad F(\lambda) = \sum_{k=1}^{n} \alpha_k s_{kj}(\lambda) - U(\lambda), \tag{3.9}
$$

which depends on the values $\alpha_k$ of integrals of motion. For the St"ackel systems on $\mathbb{P}^{2n}$ the minimum admissible genus $g$ of the curve $C$ is equal to $g = [(n - 1)/2]$.

The $n$th symmetric product of $C$ defines the $n$-dimensional Lagrangian submanifold in the complete symplectic manifold $\mathbb{R}^{2n}$

$$
C^{(n)} : \quad C(p_1, q_1) \times C(p_2, q_2) \times \cdots \times C(p_n, q_n). \tag{3.10}
$$

Then, the integration problem \(2.7-2.8\) for equation of motion is reduced to inverse Jacobi problem \(3.3\) on Lagrangian submanifold \(3.10\). The corresponding holomorphic differentials $dw_k$ are equal to

$$
dw_k = s_{kj}(\lambda) d\lambda / z(\lambda). \tag{3.11}
$$

The set of these differentials either form a basis in the space of holomorphic differentials $H_1(C)$ \[12\] or may be complement to a basis. The corresponding $n \times n$ St"ackel matrix be the $n \times n$ block of the transpose Brill-Noether matrix $U^*_n$.

The different blocks are determined the dual St"ackel systems. In this case vectors differing the first entry only

$$
u = \{\beta_2, \ldots, \beta_n\} \in J(C), \quad \bar{u} = \{\bar{e}_2, \ldots, \beta_n\} \in J(C)
$$

have a common Abel preimage \{(p_1, q_1), \ldots, (p_n, q_n)\} $\in C^{(n)}$.

Let us consider the standard basis of holomorphic differentials in $H_1(C)$

$$
dw_j = \frac{\lambda^{j-1}}{z(\lambda)} d\lambda, \quad j = 1, \ldots, g. \tag{3.12}
$$

Recall, that derivative $U^*_n$ bears a great resemblance to the canonical map $C \rightarrow \mathbb{P}^{g-1}$ and, therefore, to the Veronese map $\mathbb{P}^{1} \rightarrow \mathbb{P}^{g-1}$ given by a basis for the polynomial ring of degree $g - 1$. With respect to the basis of $H_1(C)$ \(3.12\), the Veronese map of $C$ has an extremely simple expression

$$
(y, \lambda) \rightarrow \lambda \rightarrow [\lambda^{g-1}, \lambda^{g-2}, \ldots, \lambda, 1].
$$
By using the corresponding symmetric Brill-Noether matrix $U_D^*$ (3.3), we shall determine the Stäckel matrices as $(n \times n)$ blocks of the following $(g \times n)$ matrix
\[
\begin{pmatrix}
q_1^{g-1} & q_2^{g-1} & \cdots & q_n^{g-1} \\
q_1^{g-2} & q_2^{g-2} & \cdots & q_n^{g-2} \\
\vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1
\end{pmatrix}.
\]
(3.13)

Evidently, all the Stäckel matrices can not be obtained from the symmetric Brill-Noether matrices. For instance, the Stäckel matrices (2.11) do not belong to the set of symmetric matrices.

4 Lax representation.

Henceforth, we shall restrict our attention to the basis (3.12) and the symmetric matrix (3.13). For the corresponding Stäckel systems let us look for the Lax representation as
\[
L = \begin{pmatrix}
h(\lambda, p, q) & e(\lambda, q) \\
e(\lambda, p, q) & -h(\lambda, p, q)
\end{pmatrix}.
\]
(4.1)

Hereafter, by abuse of notation, we shall omit the some arguments at the entries of the Lax matrix.

Let us fix hyperelliptic genus $g$ curve $C$ and dimension of the phase space $n \leq g$. Then we extract the $(n \times n)$ Stäckel matrix $S$ from the matrix (3.13) and define the Hamilton function $H$ (2.10) with $U = 0$.

To construct the Lax matrix let us determine function $e(\lambda, u)$ (3.7) initially
\[
e(\lambda, q) = \prod_{j=1}^{n} (\lambda - q_j),
\]
(4.2)

with $n$ zeroes, which are solution of the inverse Jacobi problem.

In the second step let us introduce the second entry of the Lax matrix as
\[
h(\lambda) = -\frac{1}{2v(\lambda, q)} \frac{d e(\lambda)}{d t} + w(\lambda, p, q) e(\lambda).
\]
Here function $v(\lambda, q)$ is calculated by using the second Bolza equation (3.8)
\[
\left. h(\lambda) \right|_{\lambda=q_k} = p_k = \left( \frac{1}{2v} \frac{d e(\lambda)}{d t} \right)_{\lambda=q_k} = - \frac{\partial e(\lambda)}{\partial u_n} \bigg|_{\lambda=q_k}.
\]
(4.3)

Let the third entry of the Lax matrix takes the form
\[
f(\lambda) = \frac{1}{v} \frac{d h(\lambda)}{d t}.
\]
Here the single unknown function $w(\lambda, p, q)$ is determined such, that the spectral curve of the Lax matrix (4.1)
\[
C : \quad z^2 = F(\lambda) = -\det L_0(\lambda) = h^2(\lambda) + e(\lambda) f(\lambda)
\]
(4.4)

be the same as initial algebraic curve $C$ (2.4) by $U = 0$.

The constructed above matrix $L_0(\lambda)$ (4.1) reads as
\[
L_0(\lambda) = \begin{pmatrix}
-\frac{1}{2v} e_t(\lambda) + w(\lambda, p, q) e(\lambda) & e(\lambda) \\
\frac{1}{v} h_t(\lambda) & \frac{1}{2v} e_t(\lambda) - w(\lambda, p, q) e(\lambda)
\end{pmatrix},
\]
(4.5)

where
\[
e_t = \frac{d e(\lambda)}{d t} = \{ H, e(\lambda) \}, \quad h_t = \frac{d h(\lambda)}{d t} = \{ H, h(\lambda) \},
\]
obeys the Lax equation

\[
\frac{dL_0}{dt} = \{H, L_0\} = \left[A_0, L_0\right]
\]

with the second matrix

\[A_0 = v(\lambda, q) \begin{pmatrix} w(\lambda, p, q) & 1 \\ 0 & -w(\lambda, p, q) \end{pmatrix} .\]

By definition of the Lax matrix all the pairs of separation variables \(\gamma_j = (p_j, q_j)\) \([4.2][4.3]\) lie on the spectral curve \(C\) \([4.3]\) of the matrix \(L_0\) \([4.3]\)

\[\varepsilon^2(\gamma_j) = p_j^2 = h^2(\lambda)\big|_{\lambda=q_j} = F(\lambda = q_j) = F(\lambda)_{\gamma_j} .\]

For the systems with polynomial potential \(U \neq 0\) we propose to change the entry \(f(\lambda)\) in \([4.3]\) as

\[f(\lambda) = \frac{1}{v} \frac{d\lambda}{dt} + u(\lambda, q)e(\lambda) ,\]

where we add new function \(u(\lambda, q)\) depending on coordinates only. Of course, to construct the Lax matrix here

\[L(\lambda) = \begin{pmatrix} -\frac{1}{2v} e_t(\lambda) + w(\lambda, p, q) e(\lambda) & e(\lambda) \\ \frac{1}{v} h_t(\lambda) + u(\lambda, q) e(\lambda) & \frac{1}{2v} e_t(\lambda) + w(\lambda, p, q) e(\lambda) \end{pmatrix} . \quad (4.6)
\]

we have to use the complete Hamiltonian with \(U \neq 0\). The associated second Lax matrix reads as

\[A = A_0 + \begin{pmatrix} 0 & 0 \\ v(\lambda, q) & u(\lambda, q) \end{pmatrix} = v(\lambda, q) \begin{pmatrix} w(\lambda, p, q) & 1 \\ u(\lambda, q) & -w(\lambda, p, q) \end{pmatrix} . \quad (4.7)
\]

To consider the corresponding Lax equation

\[\frac{dL(\lambda)}{dt} = \left[A(\lambda), L(\lambda)\right] ,\]

we can assume that the common factor \(v(\lambda, q)\) in front of the matrix \(A\) may be associated to the change of time for the Stäckel systems.

In general the proof of existence functions \(v, w\) and \(u\) requires an application of the method of algebraic geometry \([13]\). By definition of the Lax matrices \(L(\lambda)\) \([4.6]\) and \(A(\lambda)\) \([4.7]\) this problem may be reduced to the solution of the single equation

\[\frac{df(\lambda)}{dt} - 2v(uh - wf) = 0 , \quad \iff \quad \frac{dF(\lambda, e, v, u)}{dt} = 0 , \quad (4.8)
\]

for the given function \(e(\lambda)\) \([1.2]\) and the given Hamiltonian \(H\) \([2.10]\).

If we consider the lower \((n \times n)\) block of the matrix \([3.13]\), the differentials \([3.11]\) span a whole space \(H_1(C)\) and the Abel map is the one-to-one correspondence. In this case from equations \([3.8]\) and \([4.3]\) follows that

\[u_t(\lambda, q) = 0 , \quad w(\lambda, p, q) = 0 .\]

If we put \(v = 1\), rename \(t = x\) and introduce "new" time variable \(\tau\), the equation \((4.8)\) is rewritten as

\[\frac{\partial u(x, \tau, \lambda)}{\partial \tau} = \left[ \frac{1}{4} \partial_x^2 + u(\lambda) \partial_x + \frac{1}{2} u_x(\lambda) \right] e(\lambda) = 0 , \quad x = t . \quad (4.9)
\]

This equation may be identified with equation on the finite-band stationary solutions \(\frac{\partial u(x, \tau, \lambda)}{\partial \tau} = 0\) of the nonlinear soliton equations. In this theory equation \((4.9)\) is called the generating equation. For different choices of the form of \(e(\lambda)\) and \(u(\lambda)\), this procedure leads to different hierarchies of integrable equations, as an example to the KdV, nonlinear Shrödinger and sine-Gordon hierarchies or to the Dym hierarchy (see references within \([1]\)).

Function \(u(\lambda, q)\) in \([4.3]\) is constructed by using function \(e(\lambda)\) \([3.7][4.2]\)

\[u(\lambda, q_1, \ldots, q_n) = \left[ \phi(\lambda) e^{-2}(\lambda) \right]_{MN} . \quad (4.10)\]
Here \( \phi(\lambda) \) is a parametric function on spectral parameter and \( [\xi]_N \) is the linear combinations of the following Taylor projections

\[
[\xi]_N = \left[ \sum_{k=-\infty}^{+\infty} z_k \lambda^k \right]_N = \sum_{k=0}^{N} \xi_k \lambda^k, \quad (4.11)
\]

or the Laurent projections [17, 11].

If the differentials \( [\xi]_N \) span the whole space \( \mathcal{H}_1(C) \) the corresponding Stäckel systems describe all the possible systems, which separable in the orthogonal curvilinear coordinate systems in \( \mathbb{R}^n \) [11]. Let us consider the Stäckel systems which are dual to these systems. To apply equation (2.13) to the function \( e(\lambda) \) [12] and by using definition (4.3) one gets

\[
p_k = \left. \tilde{h}(\lambda) \right|_{\lambda=q_k} = \left( -\frac{1}{2v} \left\{ H, e(\lambda) \right\} \right)_{\lambda=q_k} = \frac{\det S}{\det S} \left( -\frac{1}{2v} \left\{ H, e(\lambda) \right\} \right)_{\lambda=q_k}
\]

\[
= \frac{\det S}{\det S} \left( \frac{v}{v} h(\lambda) \right)_{\lambda=q_k} = p_k \frac{\det S}{\det S} \left( \frac{v}{v} \right)_{\lambda=q_k}, \quad (4.12)
\]

Recall that \( v = 1 \) for the integrable system with the Hamiltonian \( H \) associated to the lower \( (n \times n) \) block of the matrix (3.13).

Thus, according to (4.12), below we shall consider the Stäckel systems with following functions \( v(q) \) only

\[
v(q) = \frac{\det S(q_1, \ldots , q_m)}{\det S(q_1, \ldots , q_m)},
\]

The corresponding change of the time (2.13) depending on coordinates only is closed to the Kolosoff transformation [3, 9].

Let us briefly discuss canonical transformation which transforms a Hamiltonian (2.11) to the natural form \( H = T + V \). For integrable systems separable in the orthogonal curvilinear coordinate systems on \( \mathbb{R}^n \) the Abel map is one-to-one correspondence and \( \nu_t = \{ H, v \} = 0 \). In this case we can put \( v = 1 \) and introduce function \( B(\lambda) \)

\[
B^2(\lambda) = e(\lambda), \quad (4.13)
\]

which was proposed in the theory of the soliton equations [16]. It allows us to rewrite generating function of integrals of motion

\[
F(\lambda) = -B^3 B_{tt} + u(\lambda, q) B^4. \quad (4.14)
\]

as a Newton equation for the function \( B \)

\[
\dot{B}(\lambda, q) = -F(\lambda, \alpha_1, \ldots , \alpha_n) B^{-3}(\lambda, q) + u(\lambda, q) B(\lambda, q). \quad (4.15)
\]

To expand function \( B(\lambda) \) at the Laurent set

\[
B = \sum_{j=0}^{N} x_j \lambda^j
\]

it is easy to prove that coefficients \( x_j \) obey the Newton equation of motion (4.15) (see references within [14, 11]). Here we reinterpret the coefficients of the function \( F(\lambda) \) in (4.15) not as functions on the phase space, but rather as integration constants \( \alpha_j [2,3] \).

In general by \( \nu_t \neq 0 \) the generating function \( F(\lambda) = -\det L(\lambda) [16] \) is equal to

\[
F(\lambda) = \frac{1}{4\nu^2} \left( e_2^2 - 2 e e_{tt} \right) + \left( \frac{\nu_t}{2\nu^2} - w \right) \frac{e_t e}{v} + \left( w^2 + \frac{\nu}{v} \right) e^2.
\]

In this case the suitable canonical transformations, which transforms any Hamiltonian (2.10) to the natural form, are unknown.

Although we can not proof validity of the presented Lax representation in general, this construction works for the many well-known mechanical systems. In the next Section we consider some two-dimensional Stäckel systems in detail.
5 Examples.

Let us consider four orthogonal systems of coordinates on plane: elliptic, parabolic, polar and cartesian [3]. The Lax matrix $L_0(\lambda)$ (5.5) by $U = 0$ is transformed to the Lax matrix $L(\lambda)$ (5.6) by $U \neq 0$ by using the outer automorphism of the space of infinite-dimensional representations of underlying algebra $sl(2)$ [7, 11]. Since, we shall consider the Lax representations for the geodesic motion by $U = 0$ more extensively.

1. Parabolic and cartesian coordinate systems ($w(\lambda, p, q) = 0$).

Let us consider two hyperelliptic curves

$$C^{(1)}: \quad z^2 = \prod_{i=1}^{2g+1} (\lambda - \lambda_i),$$

$$C^{(2)}: \quad z^2 = \lambda^{-1} \prod_{i=1}^{2g+1} (\lambda - \lambda_i).$$

If we choose the standard basis in the space of holomorphic differentials one gets the following symmetric matrices (3.13) for two-dimensional systems

$$U_t^i(q_1, q_2) = \begin{pmatrix} q_1^{q-1} & q_2^{q-1} \\ \vdots & \vdots \\ q_1^2 & q_2^2 \\ q_1 & q_2 \\ -1 & -1 \end{pmatrix}, \quad U_2^i(q_1, q_2) = \begin{pmatrix} q_1^{g-2} & q_2^{g-2} \\ \vdots & \vdots \\ q_1 & q_2 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}.$$ (5.2)

Different $(2 \times 2)$ blocks of the matrices $U_t$ determine different Stäckel systems.

Let us consider two blocks for the each matrices, such that the corresponding change of the time will be same as the Kolosoff transformation (5.3). So, for the curve $C^{(1)}$ we shall consider the following matrices

$$S_1 = \begin{pmatrix} q_1 & q_2 \\ -1 & -1 \end{pmatrix}, \quad \bar{S}_1 = \begin{pmatrix} q_1^2 & q_2^2 \\ -1 & -1 \end{pmatrix}.$$ (5.3)

For the second curve $C^{(2)}$ the associated Stäckel matrices are equal to

$$S_2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \bar{S}_2 = \begin{pmatrix} q_1 & q_2 \\ -1 & -1 \end{pmatrix}.$$ (5.4)

Introduce the Hamilton functions (5.10) by $U = 0$

$$H_0^{(1)} = \frac{P_1^2 - P_2^2}{q_1 - q_2}, \quad \bar{H}_0^{(1)} = (q_1 + q_2)^{-1} H_0^{(1)},$$

$$H_0^{(2)} = \frac{q_1 P_1^2 - q_2 P_2^2}{q_1 - q_2}, \quad \bar{H}_0^{(2)} = (q_1 + q_2)^{-1} H_0^{(2)}.$$ (5.5)

The corresponding second integrals of motion of the dual systems are related

$$j_0^{(k)} = j_0^{(k)} - \frac{q_1 q_2}{q_1 + q_2} H_0^{(k)}, \quad k = 1, 2.$$

The functions $e(\lambda, u)$ (5.7)

$$e_1(\lambda) = (\lambda - q_1)(\lambda - q_2), \quad e_2(\lambda) = \frac{(\lambda - q_1)(\lambda - q_2)}{\lambda}.$$ (5.6)
These variables obviously related to the cartesian coordinate system. For the second curve the corresponding equation defines the standard parabolic coordinate system

\[ e_1(\lambda) = (\lambda - q_1)(\lambda - q_2) = B^2(\lambda), \quad B(\lambda) = \lambda - \frac{x}{2} - \frac{y}{4\lambda}, \]

immediately yields the following canonical transformation

\[ q_1 = \frac{x - \sqrt{2} y}{2}, \quad p_1 = p_x - \sqrt{2} y p_y, \]
\[ q_2 = \frac{x + \sqrt{2} y}{2}, \quad p_2 = p_x + \sqrt{2} y p_y. \]

These variables obviously related to the cartesian coordinate system. For the second curve \( C^{(2)} \) the corresponding equation

\[ e_2(\lambda) = \lambda^{-1} (\lambda - q_1)(\lambda - q_2) = \lambda - x - \frac{y^2}{4\lambda} \]

defines the standard parabolic coordinate system

\[ q_1 = \frac{x - \sqrt{x^2 + y^2}}{2}, \quad p_1 = p_x - \sqrt{x^2 + y^2 + x} \frac{y}{y} p_y, \]
\[ q_2 = \frac{x + \sqrt{x^2 + y^2}}{2}, \quad p_2 = p_x + \sqrt{x^2 + y^2 - x} \frac{y}{y} p_y. \]

By \( U = 0 \) the Hamilton functions are given by

\[ H^{(1)}_0 = 4 p_x p_y, \quad H^{(2)}_0 = p_x^2 + p_y^2. \]

According to (4.3) and (4.12) functions \( v(q_1, q_2) \) entering in the Lax representation are equal to

\[ v(\lambda, q_1, q_2) = 1 \quad \text{for matrices} \quad S_{1,2}, \]
\[ v(\lambda, q_1, q_2) = (q_1 + q_2)^{-1} = \frac{1}{x} \quad \text{for matrices} \quad \tilde{S}_{1,2}. \]

In physical variables the Lax matrices are given by

\[ L^{(1)}_0 = \begin{pmatrix} p_x + (2\lambda - x) p_y & \lambda^2 - \lambda x + \frac{x^2 - 2 y}{4} \frac{y}{y} \\ -4 p_y^2 & -p_x - (2\lambda - x) p_y \end{pmatrix}, \]

\[ L^{(2)}_0 = \begin{pmatrix} p_x + \frac{1}{2\lambda} y p_y & \lambda - x - \frac{1}{4\lambda} y^2 \\ \frac{1}{\lambda} p_y^2 & -p_x - \frac{1}{2\lambda} y p_y \end{pmatrix}. \]

For the dual Stäckel systems the Lax matrices \( \tilde{L}^{(1,2)}_0 \) have the form

\[ \tilde{L}^{(1)}_0 = L^{(1)}_0 + \begin{pmatrix} 0 & 0 \\ 4 p_x p_y & 0 \end{pmatrix} = L^{(1)}_0 + \begin{pmatrix} 0 & 0 \\ \tilde{H}^{(1)}_0 & 0 \end{pmatrix}, \]

\[ \tilde{L}^{(2)}_0 = L^{(2)}_0 + \begin{pmatrix} p_x^2 + p_y^2 & 0 \\ \frac{1}{x} p_y^2 & 0 \end{pmatrix} = L^{(1)}_0 + \begin{pmatrix} 0 & 0 \\ \tilde{H}^{(2)}_0 & 0 \end{pmatrix}. \]

By using property \( \{ h_t(\lambda), v(q) \} = 0 \) of the function \( v(q) \) we can easy proof equation (5.8) for the dual systems by using the same equation for the system with \( v_t = 0 \)

\[ \tilde{f}_t = \{ \tilde{H}, f + \tilde{H} \} = f_t = 0. \]
Another consequence of this property is that the function \( w(\lambda, p, q) \) in (4.15) is equal to zero.

Note, in the works [3] and [4], devoted to the Kowalewski top, the common Lax matrices are proposed for the both dual systems after the non-canonical change of variables. Here we obtain different Lax matrices for the systems connected by non-canonical change of the time.

The spectral curves of the matrices \( L_0 \) (5.8) coincides with the initial curves \( C^{(1,2)} \) (2.4) by \( U = 0 \)

\[
z^2 = H^{(1)}_0 \lambda + J^{(1)}_0, \quad z^2 = H^{(2)}_0 + \frac{J^{(2)}_0}{\lambda}.
\]  
(5.10)

Here \( J^{(1,2)}_0 \) be the second integrals of motion (2.2). For the dual systems with the Hamilton functions \( \bar{H}^{(1,2)}_0 \) the corresponding spectral curves are equal to

\[
z^2 = \bar{H}^{(1)}_0 \lambda^2 + \bar{J}^{(1)}_0, \quad z^2 = \bar{H}^{(2)}_0 \lambda + \frac{\bar{J}^{(2)}_0}{\lambda}.
\]  
(5.11)

If for the system with the Hamiltonian \( H^{(1)}_0 \) the Abel map is one-to-one correspondence on the curve (5.10), then for the same system on the curve

\[
z^2 = e_2 \lambda^2 + e_1 \lambda + e_0
\]

the associated Abel map is lack of uniqueness in general. So, on this curve we can introduce the second Stäckel system with the dual Hamiltonian \( \bar{H}^{(1)}_0 \).

Let us briefly consider systems with polynomial potentials \( U \neq 0 \). As an example, introduce different potentials for the curves \( C^{(1,2)} \) (5.1):

\[
U^{(1)}(q_j) = \alpha^2 q_j^5 + \beta q_j^3, \quad U^{(1)}(q_j) = \alpha^2 q_j^3 + \beta q_j.
\]  
(5.12)

To describe these potentials we have to put \( N = 6 \) and \( N = 4 \) in (4.11) and have to use the following parametric functions

\[
\phi^{(1)}(\lambda) = -\alpha^2 \lambda^5 \quad \text{and} \quad \phi^{(2)}(\lambda) = -\alpha^2 \lambda^3
\]

for the curves \( C^{(1)} \) and \( C^{(2)} \), respectively. For the both curves the common function \( u(\lambda, q_1, q_2) \) is given by

\[
u^{(1,2)} = -\alpha^2 (\lambda + 2 x).
\]  
(5.13)

Here we restrict ourselves the presentation of the function \( u \) only, the complete Lax matrices \( L(\lambda) \) may be constructed by the rule (4.6).

The spectral curves of the corresponding matrices (5.10) coincides with the initial curves (5.8). For instance, curves for the systems with dual Hamiltonians \( \bar{H}^{(1,2)} \) are

\[
C^{(1)}: \quad z^2 = \alpha^2 \lambda^5 + \beta \lambda^3 - \bar{H} \lambda - \bar{J},
\]

\[
C^{(2)}: \quad z^2 = \alpha^2 \lambda^3 - \bar{H} \lambda + \beta - \frac{\bar{J}}{\lambda}.
\]

The Poisson bracket relations for the Lax matrix (5.8) coincide s with the initial curves (3.9).

By \( \nu_t = 0 \) for the systems related to the matrices \( S^{[1,2]} \) the corresponding \( r \)-matrices \( r_{ij}(\lambda, \mu) \) in (5.14) consist of two terms

\[
r_{ij} = r_{ij}^{(0)} + r_{ij}^{(u)}.
\]  
(5.15)
The first matrix is a standard $r$-matrix on the loop algebra $L(sl(2))$

$$r_{12}^p(\lambda, \mu) = \frac{\Pi}{\lambda - \mu} = \frac{1}{\lambda - \mu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ \(5.16\)

The second matrix may be associated to outer automorphism of the space of infinite-dimensional representations of underlying algebra $sl(2)$ \[17, 11\]. The corresponding dynamical $r_{ij}^u$-matrices depend on the coordinates only

$$r_{12}^u = \frac{u(\lambda, q) - u(\mu, q)}{\lambda - \mu} \sigma_{-} \otimes \sigma_{-}, \quad \sigma_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.17}$$

By $v_t \neq 0$ for the dual Stäckel systems related to the matrices $\tilde{S}_{1,2}$ we have to add to the $r$-matrices 5.15 the third term

$$r_{12}^v = v(q_1, q_2) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.18}$$

where the second matrix $r_{21}^v$ is defined by (5.14).

The matrix $r_{ij}^v$ may be connected with the Drinfeld twist for the Toda lattice associated to the root system $D_n$. Let us consider the Drinfeld twist \[19\] of the quantum $R$-matrix

$$\tilde{R} = FRF_{21}^{-1}, \quad F_{21} = \Pi F \Pi. \tag{5.19}$$

Here matrix $R$ satisfies the Yang-Baxter equation and matrix $F$ has the special property \[19\]. To introduce the corresponding linear $r$-matrix \[20\], one gets

$$R = I + 2\eta r^p + O(\eta^2), \quad F = I + \eta r^v + O(\eta^2).$$

Then we consider limit of the twisted matrix $\tilde{R}$ by $\eta \to 0$

$$\tilde{R}_{12} = I + \eta \left( r_{12}^p + r_{12}^v - \Pi (r_{12}^p + r_{12}^v) \Pi \right) + O(\eta^2). \tag{5.20}$$

Formally, coefficients by $\eta$ may be called twisted linear $r$-matrix.

By using generators $h, e, f$ of the underlying Lie algebra $sl(2)$

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h, \tag{5.21}$$

let us introduce an appropriate element $F \in U(sl(2)) \otimes U(sl(2))$

$$F = \exp(\xi \cdot e \otimes f), \quad \xi \in \mathbb{C}$$

belonging to a tensor product of the corresponding universal enveloping algebras $U(sl(2)) \otimes U(sl(2)) \otimes U(sl(2)) \otimes U(sl(2))$. In the fundamental spin-1/2 representation $\rho_\frac{1}{2}$ we have

$$F(\xi) = (\rho_\frac{1}{2} \otimes \rho_\frac{1}{2}) F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \xi & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$\(5.14\).

To substitute in (5.14) the Yang solution of the Yang-Baxter equation $R = I + \frac{\eta}{\lambda} \Pi$ we get a twisted $R$-matrix. If the element $\xi(q)$ be a suitable function on coordinates, this dynamical twisted $R$-matrix may be used to description of the Toda lattice associated with the $D_n$ root system \[21\].

Let us consider twisted dynamical matrix (5.19) by $\xi = v(q)$. We can see that the linear $r$-matrix associated to the dual Stäckel system (5.14) (5.15)

$$r_{12} = r_{12}^p + r_{12}^v, \quad r_{21} = -\Pi (r_{12}^p + r_{12}^v) \Pi$$
is equal to the half of the twisted linear matrix \([5.20]\).

Recall, for the Stäckel matrices \(S_1\) \([5.3]\) and \(S_2\) \([1.4]\) the corresponding differentials \([3.11]\) span \(H_1\). Since, the associated Hamilton functions have a natural form in physical variables. For instance, Hamiltonians with potentials \(5.12\) are given by

\[
H^{(1)} = 2 p_x p_y + \frac{\alpha^2}{4} (y^2 + 5 x^2 y + \frac{5}{4} x^4) + \frac{\beta}{2} \left( \frac{3}{2} x^2 + y \right),
\]

\[
H^{(2)} = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\alpha^2}{2} x (2 x^2 + y^2) + \beta.
\]

To consider the dual Stäckel systems we have to use additional transformation

\[
x = \sqrt{2} \bar{x}, \quad p_x = \bar{p}_x \sqrt{2 \bar{x}},
\]

for the first curve and the following more complicated transformation

\[
x = \frac{3}{2} \bar{x}^{2/3}, \quad p_x = \bar{p}_x \bar{x}^{1/3}, \quad y = \sqrt{2} \frac{\bar{p}_y}{\alpha}, \quad p_y = -\sqrt{2} \frac{\bar{\alpha} \bar{y}}{\beta}.
\]

for the second curve. After this canonical change of variables the Hamiltonians \(\bar{H}^{(1,2)}\) \([5.3]\) obtain the natural form

\[
\bar{H}^{(1)} = 2 \bar{p}_x \bar{p}_y + \frac{\alpha^2}{8} (y^2 + 10 y \bar{x} + 5 \bar{x}^2) \sqrt{\frac{2}{\bar{x}}} + \frac{\beta}{4} (3 \bar{x} + y) \sqrt{\frac{2}{\bar{x}}},
\]

\[
\bar{H}^{(2)} = \frac{1}{2} (\bar{p}_x^2 + \bar{p}_y^2) + \frac{3 \alpha^2}{4} \bar{x}^{-2/3} \left( \frac{9}{2} \bar{x}^2 + \bar{y}^2 \right) + \beta \bar{x}^{-2/3}.
\]

The system with the Hamiltonian \(H^{(2)}\) is so-called second integrable case of the Henon-Heiles system \([3]\). The dual system with the Hamiltonian \(\bar{H}^{(2)}\) is so-called Holt-type system \([2]\). Note, the second integral of motion is a polynomial of the fourth order in momenta for the Holt system.

Additional canonical transformation \([5.24]\) allows us to get natural Hamiltonians for the restricted class of the potentials \(U\) \([5.12]\) only. Unlike canonical transformation \([5.23]\) may be used for any potentials \(U\). As an example, rational potential

\[
U(q) = \frac{\alpha}{q^2} + \frac{\beta}{q} + \gamma q + \delta q^2 + \rho q^4
\]

give rise the following Hamiltonian

\[
\bar{H} = 2 \bar{p}_x \bar{p}_y - \frac{4 \alpha}{(x-y)^2} - \frac{\beta}{x-y} \sqrt{\frac{2}{x}} + \gamma \sqrt{\frac{2}{y}} + 1 + \frac{1}{2} \delta + \frac{\rho}{2} (x+y).
\]

Also we can add potential terms \([5.24]\) to this Hamiltonian.

By \(v = 1\) and \(w = 0\) the Lax representation \([4.3]\) for a system with an arbitrary number \(n\) of degrees of freedom may be regarded as a generic point at the loop algebra \(L(\text{sl}(2))\) in fundamental representation after an appropriate completion \([11]\). As an example, for the generalized parabolic coordinate systems function \(e(\lambda)\) is given by

\[
e(\lambda) = \prod_{j=1}^{n} \frac{\lambda - q_j}{\lambda - \delta_k} = \lambda - x_n + \sum_{k=1}^{n-1} \frac{x_k^2}{4 (\lambda - \delta_k)}, \quad \delta_k \in \mathbb{R}.
\]

To construct the Lax representation for a potential motion we can use the outer automorphism of the space of infinite-dimensional representations of \(\text{sl}(2)\) proposed in \([17]\).

By \(v \neq 0\) for the dual Stäckel systems the Lax representations may be constructed without any problem as well. For instance, let us consider system with the three degrees of freedom. To construct the Lax matrix by \([5.3],[1.4]\) with the function \(u\) given by \([5.13]\) one gets

\[
e(\lambda) = \begin{cases} 
\lambda - x - \frac{y^2}{\lambda} - \frac{z^2}{4 (\lambda - k)}, & k \in \mathbb{R}, \\
\frac{1}{x} (p_x^2 + p_y^2 + \frac{\alpha^2 k z^2}{4}) + \frac{\alpha^2}{2} (2 x^2 + y^2 + z^2).
\end{cases}
\]

\[14\]
After an additional canonical transformation \([5.24]\) extended on the \(p_z, z\) variables the Hamilton function takes the form

\[
\tilde{H} = p_x^2 + p_y^2 + p_z^2 \left(1 + \frac{k}{3} \bar{x}^{-2/3}\right) + \frac{3\alpha}{8} \bar{x}^{-2/3} \left(\frac{9}{2} x^2 + \bar{y}^2 + \bar{z}^2\right).
\]

So, the main unsolved problem is to introduce additional canonical transformation, which transform the dual Hamilton function \(\tilde{H}\) into the natural form.

2. **Elliptic and polar coordinates** \((w(p, q) \neq 0)\).

Recall, that the polar coordinate system may be obtained from elliptic coordinate system and, therefore, we shall consider elliptic coordinate systems in detail.

For the elliptic coordinate systems algebraic curve is given by

\[
C^{(3)} \quad z^2 = \prod_{i=1}^{2g+1} \frac{(\lambda - \lambda_i)}{(\lambda - k)(\lambda + k)}, \quad k \in \mathbb{C}.
\]

Let us consider two Stäckel matrices associated to this curve

\[
S_3 = \begin{pmatrix}
q_1 & q_2 \\
q_2^2 - k^2 & q_2^2 - k^2
\end{pmatrix}, \quad \tilde{S}_3 = \begin{pmatrix}
4q_1^2 & 4q_2^2 \\
q_2^2 - k^2 & q_2^2 - k^2
\end{pmatrix}, \quad (5.25)
\]

The corresponding non-canonical change of the time \((2.9)\) is closed to the Kolosoff transformation \([1.1] [8]\).

For the polar coordinate system the Stäckel matrices are non-symmetric matrices

\[
S_4 = \begin{pmatrix} 1 & 0 \\ \frac{1}{q_1^2} & \frac{1}{4(q_2^2 - k)} \end{pmatrix}, \quad \tilde{S}_4 = \begin{pmatrix} q_1^2 & 0 \\ \frac{1}{q_1^2} & \frac{1}{4(q_2^2 - k)} \end{pmatrix}, \quad (5.26)
\]

The corresponding non-canonical change of the time \((2.9)\) is closed to the Kepler change of the time \([7]\).

By \(U = 0\) the initial hyperelliptic curves \([3.4]\) for the matrices \(S_3\) and \(\tilde{S}_3\) are given by

\[
z^2 = \frac{H_0 \lambda + J}{\lambda - k^2}, \quad z^2 = \frac{4H_0 \lambda^2 + J_0}{\lambda - k^2}, \quad (5.27)
\]

with the following Hamiltonians

\[
H_0^{(3)} = \frac{p_1^2 (q_1^2 - k^2) - p_2^2 (q_2^2 - k^2)}{q_1 - q_2}, \quad \tilde{H}_0^{(3)} = \frac{1}{4(q_1 + q_2)} H, \quad (5.28)
\]

The Hamiltonians related to the matrices \(S_4\) and \(\tilde{S}_4\) read as

\[
H_0^{(4)} = p_1^2 - 4 \frac{q_2^2 - k}{q_1^2} p_2^2, \quad \tilde{H}_0^{(4)} = q_1^{-2} H_0^{(4)}. \quad (5.29)
\]

Let us fix elliptic coordinates by using equation

\[
e(\lambda) = \frac{(\lambda - q_1)(\lambda - q_2)}{(\lambda - k)(\lambda + k)} = 1 - \frac{x^2}{4(\lambda - k)} - \frac{y^2}{4(\lambda + k)}
\]

such that

\[
q_1 = \frac{x^2 + y^2}{8} + \frac{1}{2} \sqrt{(x^2 + y^2)^2 + 16 k (x^2 - y^2) + 64 k^2}, \quad q_2 = \frac{x^2 + y^2}{8} - \frac{1}{2} \sqrt{(x^2 + y^2)^2 + 16 k (x^2 - y^2) + 64 k^2}.
\]
The corresponding equation for the polar coordinates
\[
e(\lambda) = \frac{q_1 (\lambda - q_2)}{\lambda (\lambda - 1)} = \frac{x^2}{4\lambda} + \frac{4y^2}{\lambda - 1}
\]
immediately yields
\[
q_1 = r = \sqrt{x^2 + y^2}, \quad q_2 = \cos^2(\phi) = \frac{x^2}{x^2 + y^2}.
\]
In physical variables the Hamiltonians (5.28-5.29) have a common form
\[
H = p_x^2 + p_y^2, \quad \tilde{H} = \frac{p_x^2 + p_y^2}{x^2 + y^2}.
\]
To construct the Lax representations we begin with the calculation of the functions \(v(\lambda, q)\) by the rule (4.3-4.12)
\[
v = 1 \quad \text{for matrices} \quad S_{3,4}
\]
\[
v = \frac{1}{4} (q_1 + q_2)^{-1} = \frac{1}{x^2 + y^2} \quad \text{for matrix} \quad \tilde{S}_{3,4}, \quad (5.30)
\]
\[
v = \frac{1}{q_1^2} = \frac{1}{x^2 + y^2} \quad \text{for matrix} \quad \tilde{S}_4.
\]
So, for the Stäckel systems associated with the matrices \(S_{3,4} (5.24)\) and \(S_4 (5.26)\) one gets
\[
L_0(\lambda) = \begin{pmatrix}
\frac{x p_x}{2(\lambda - k)} + \frac{y p_y}{2(\lambda + k)} & \epsilon \lambda - \frac{x^2}{4(\lambda - k)} - \frac{y^2}{4(\lambda + k)} \\
\frac{p_x^2}{\lambda - k} + \frac{p_y^2}{\lambda + k} & -\frac{x p_x}{2(\lambda - k)} - \frac{y p_y}{2(\lambda + k)}
\end{pmatrix}.
\] (5.31)
Here \(\epsilon = 1\) for the elliptic coordinate system and \(\epsilon = 0\) for the parabolic coordinate system. The spectral curve of the Lax matrix \(L_0(\lambda)\) coincides to the initial curve (5.27).

For the dual system, in contrast to the cartesian and parabolic coordinates, the Lax matrix has the more complicated form. Both these Lax matrices may be constructed by the rule (4.6) with the following common function \(w(p, q)\)
\[
w(p, q) = 2 \sqrt{H}.
\] (5.32)
The Lax matrix reads as
\[
\tilde{L}_0(\lambda) = L_0(\lambda) + \begin{pmatrix}
w e(\lambda) & 0 \\
-2w \left[ h(\lambda) - w e(\lambda) - \epsilon w \right] & -w e(\lambda)
\end{pmatrix}, \quad \epsilon = 0, 1. \quad (5.33)
\]
Here \(e(\lambda)\) and \(h(\lambda)\) are entries of the corresponding matrices \(L_0(\lambda)\) (5.31) by \(\epsilon = 0, 1\). As above, the spectral curve of the Lax matrix \(L_0(\lambda)\) by \(\epsilon = 1\) coincides with the initial curve (5.27).

For the cartesian and parabolic coordinate systems we can get equation
\[
\begin{align*}
\{ \{ H, v^{-1}(q) \} , e(\lambda, q) \} &= \{ \{ H, (q_1 + q_2) \} , e(\lambda, q) \} = 2, \\
\end{align*}
\]
on the Hamiltonian \(H\), function \(e(\lambda)\) and function \(v(q)\) defining change of the time. For the polar and elliptic coordinate systems the corresponding equation is
\[
\{ \{ H, v^{-1}(q) \} , e(\lambda, q) \} = 8 \left( e(\lambda) - \epsilon \right), \quad \epsilon = 0, 1.
\]
Hence, from the equation (4.8) follows that the function \(w(p, q)\) in (4.5, 5.32) does not equal to zero. If we consider more complicated change of the time for the cartesian and parabolic coordinate systems, one gets non-zero function \(w (4.4)\) as well.
The quadratic $R$-term is given by

$$
\{ L(\lambda), L(\mu) \} = \left[ r_{12}, L(\lambda) \right] + \left[ r_{21}, L(\mu) \right]
$$

$$
+ R \frac{1}{2} L(\lambda) \frac{2}{2} L(\mu) + \frac{1}{\lambda} L(\lambda) \frac{2}{2} L(\mu) R - \frac{1}{\lambda} L(\lambda) \frac{2}{2} L(\mu) R - \frac{2}{\mu} L(\mu) R \frac{1}{\lambda} L(\lambda).
$$

Here linear $r$-matrix reads as

$$
r_{12}(\lambda, \mu) = r_{12}(\lambda - \mu) + 4 \epsilon r_{12}^w, \quad r_{21}(\lambda, \mu) = -\Pi r_{21}(\lambda, \mu) \Pi,
$$

where $r^p(\lambda - \mu)$ be the standard linear $r$-matrix on the loop algebra $L(sl(2))$. The second dynamical term is given by

$$
r_{12}^w = v(q) \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
w & 1 & 0 & 0 \\
0 & -w & 0 & 0
\end{pmatrix}.
$$

The quadratic $R$-matrix is closed to the twisted linear $r$-matrix (5.20)

$$
R = -\frac{2}{w} \left( r_{12}^w + r_{21}^w \right) = -\frac{2}{w} \left( r_{12}^w - \Pi r_{12}^w \Pi \right).
$$

For the systems with $U \neq 0$ functions $u(\lambda, q)$ may be constructed as usual [17, 11]. Note, the both dual Hamiltonians obtain a natural form after the following additional canonical transformation of variables

$$
x = \sqrt{x} - \sqrt{y}, \quad p_x = \sqrt{x} \tilde{p}_x - \sqrt{y} \tilde{p}_y,
$$

$$
y = -i (\sqrt{x} + \sqrt{y}), \quad p_y = i (\sqrt{x} \tilde{p}_x + \sqrt{y} \tilde{p}_y).
$$

As an example, for elliptic coordinate system the uniform potential

$$
U^{(3)}(q_j) = \alpha q_j^2 + \beta q_j
$$

give rise to the following dual Hamiltonian

$$
\tilde{H} = 2 \tilde{p}_x \tilde{p}_y + \frac{\alpha}{4} (\tilde{x} + k) (\tilde{y} + k) - \frac{\beta}{8 \sqrt{\tilde{x} \tilde{y}}} \left(2 \tilde{x} \tilde{y} + k \tilde{x} + k \tilde{y} \right).
$$

For the polar coordinate system we present the non-uniform degenerate potentials

$$
U_1^{(4)}(q_1) = \beta, \quad U_2^{(4)}(q_2) = 0,
$$

associated to the dual Hamiltonians in the form

$$
\tilde{H} = \frac{\tilde{p}_x \tilde{p}_y}{2} - \beta \left( \frac{16}{\sqrt{\tilde{x} \tilde{y}}} + \frac{1}{\tilde{x}} + \frac{1}{\tilde{y}} - \frac{2}{\tilde{x} \tilde{y}} \right).
$$

Both these systems may be considered as an integrable deformation of the Kepler problem.

6 Conclusions

In this paper we have considered the non-canonical relations between the different Stäckel systems. The proposed change of the time is related to ambiguity of the Abel map. For the two degree of freedom systems that were studied in this paper, we found the Lax representations and the $r$-matrix algebras. The corresponding dynamical $r$-matrices have the intriguing connections to the Drinfeld twists.

Of course, considered above particular family of the time transformations (2.9) does not exhausted all the set of the non-canonical changes of the time, which preserve the integrability. As an example, the complete Kolosoff transformation $\{ t, p, q \} \rightarrow \{ \tilde{t}, \tilde{p}, q \}$ connects the Stäckel system with the other integrable non-Stäckel system. So, it would be interesting to investigate another integrable systems connected with the Stäckel systems by non-canonical transformations.

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