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Optical soliton solutions are recovered for magneto–optic waveguides that maintains anti–cubic form of nonlinear refractive index. The analytical scheme is Jacobi’s elliptic function approach. Once the solutions to the governing model are obtained in terms of Jacobi’s elliptic functions, the limiting value to it’s modulus of ellipticity reveals the complete spectrum of soliton solutions.

1. Introduction

One of the most viable applications of optical soliton sciences is in magneto–optic waveguides where an external magnetic field controls the effect of “soliton clutter”. This technological marvel has been implemented across the industry to enable a streamlined flow of soliton pulses across inter–continental distances. Therefore, it is imperative to study the governing model equation in magneto–optic waveguides and secure a soliton solution for looking into additional features such as conservation laws, numerical analysis and so on. There are a variety of mathematical approaches to study soliton dynamics across the platform and these include both analytical and numerical approaches [1–19]. The conserved quantities thus emerged are applicable to the study of soliton perturbation theory in magneto–optic waveguides and control the intra–channel collision of optical solitons. The current paper studies soliton dynamics in magneto–optic waveguides that maintains anti–cubic (AC) form of refractive index. This law of refractive index was first proposed during 2003 and later it was studied all across [3]. The details of the derivation of soliton solutions are enumerated by the aid of Jacobi’s elliptic function expansion approach.
The coupled system of nonlinear Schrödinger’s equation (NLSE) in magneto-optic waveguides having AC law nonlinearity is given by

\[ \begin{align*}
\mathrm{i}q_r + a_1 q_{xx} + \left( \frac{b_{11}}{|q|^2} + b_{12} |q|^4 + c_{11} |q|^4 + c_{12} |q|^2 + c_{13} |q|^4 \right) q &= Q_t r + \mathrm{i} \left[ a_1 q_r + \lambda_1 \left( |q|^2 \right)_r + v_1 \left( |q|^4 \right)_x + \theta_1 |q|^2 q_x \right], \\
\mathrm{i}r_r + a_2 r_{xx} + \left( \frac{b_{21}}{|r|^2} + b_{22} |r|^4 + c_{21} |r|^4 + c_{22} |r|^2 + c_{23} |r|^4 \right) r &= Q_t q + \mathrm{i} \left[ a_2 r_r + \lambda_2 \left( |r|^2 \right)_r + v_2 \left( |r|^4 \right)_x + \theta_2 |r|^2 r_x \right],
\end{align*} \]

(1)

(2)

where \(a_1, b_{11}, b_{12}, b_{13}, c_{11}, c_{12}, c_{13}, Q, \alpha, \lambda, v_1 \) and \( \theta_1 \) for \( l = 1, 2 \) are constants, while \( i = \sqrt{-1} \). The dependent variables \( q(x,t) \) and \( r(x,t) \) stand for complex valued wave profiles where the independent variables are the spatial \( x \) and temporal \( t \). Here \( a_l \) stand for the coefficients of chromatic dispersion (CD), while \( b_{11}, b_{12} \) and \( b_{22} \) are the coefficients of self-phase modulation (SPM). \( c_{11}, c_{21} \) and \( c_{23} \) are the cross-phase modulation (XPM) terms. On the right hand side, \( Q \) and \( \theta_l \) are respectively the coefficients of magneto-optic parameters and inter-modal dispersions (IMD), while \( \lambda_l \) gives the self-steepening (SS) terms. Finally \( v_l \) and \( \theta_l \) stand for the coefficients of nonlinear dispersion.

The purpose of this work is to recover bright, dark, singular soliton solutions as well as Jacobi and Weierstrass elliptic function solutions for magneto–optic waveguides with AC form of nonlinear refractive index. It should be noted that these revealed results are new and are being reported for the first time in this paper. The results for magneto–optic waveguides having AC law of nonlinearity have not been addressed employing the generalized Jacobi elliptic function expansion scheme. Also, it needs to be mentioned that the solitons secured to magneto-optic waveguides with AC law nonlinearity will be extremely beneficial in the fiber-optic transmission technology.

2. Mathematical analysis

To solve Eqs. (1) and (2), we pick the hypotheses given by

\[ q(x,t) = \phi_1(\xi) \exp[i\psi(x,t)], \]

(3)

\[ r(x,t) = \phi_2(\xi) \exp[i\psi(x,t)], \]

(4)

and

\[ \xi = x - vt, \quad \psi(x,t) = -\kappa x + \omega t + \theta_0, \]

(5)

where \( v, \kappa, \omega \) and \( \theta_0 \) are the speed, frequency, wave number and phase constant, respectively. Substitute (3) and (4) along with (5) into (1) and (2). Then, real parts give

\[ a_1 \phi_1' + \frac{b_{11}}{\phi_1} + b_{13} \phi_1^2 + \frac{c_{11}}{\phi_1^3} \phi_1 + c_{12} \phi_1 \phi_2^2 + \left( b_{12} - \kappa (\lambda_1 + \theta_1) \right) \phi_1' + c_{13} \phi_1 \phi_2^2 - \left( \omega + a_1 \kappa + a_1 \kappa^2 \right) \phi_1 - Q_2 \phi_2 = 0, \]

(6)

\[ a_2 \phi_2' + \frac{b_{21}}{\phi_2} + b_{23} \phi_2^2 + \frac{c_{21}}{\phi_1^3} \phi_2 + c_{22} \phi_2 \phi_1^2 + \left( b_{22} - \kappa (\lambda_2 + \theta_2) \right) \phi_2' + c_{23} \phi_2 \phi_1^2 - \left( \omega + a_2 \kappa + a_2 \kappa^2 \right) \phi_2 - Q_2 \phi_1 = 0, \]

(7)

while imaginary parts imply

\[ (v + 2a_1 \kappa + a_1) \phi_1' + (3\lambda_1 + 2\nu_1 + \theta_1) \phi_2^2 \phi_1 = 0, \]

(8)

\[ (v + 2a_2 \kappa + a_2) \phi_2' + (3\lambda_2 + 2\nu_2 + \theta_2) \phi_1^2 \phi_2 = 0. \]

(9)

The linearly independent principle is applied on (8) and (9) to get:

\[ v = -(2a_1 \kappa + a_1), \]

(10)

\[ 3\lambda_1 + 2\nu_1 + \theta_1 = 0, \]

(11)

and

\[ v = -(2a_2 \kappa + a_2), \]

(12)

\[ 3\lambda_2 + 2\nu_2 + \theta_2 = 0. \]

(13)

From (10) and (12), we have the frequency of soliton as
\[ \kappa = \frac{(a_2 - a_1)}{2(a_1 - a_2)}, \]  

provided
\[ a_1 \neq a_2, a_1 \neq a_2. \]  

Set
\[ \phi \xi = \chi \phi \xi, \]  

where \( \chi \) is a nonzero constant, such that \( \chi \neq 1 \). Then, Eqs. (6) and (7) transform to
\[ a_1 \phi \xi \phi \xi + \left( b_{11} + \frac{c_{11}}{\chi^2} \right) - (\omega + a_1 \kappa + a_1 \kappa^2 + Q_1 \chi) \phi \xi + \left[ a_1 \kappa^2 + b_{12} - \kappa(\lambda_1 + \theta_1) \right] \phi \xi + \left( b_{13} + c_1 \chi \right) \phi \xi = 0. \]  

\[ a_2 \phi \xi \phi \xi + \left( \frac{b_{12}}{\chi^2} + c_{21} \chi \right) - \left[ (\omega + a_1 \kappa + a_1 \kappa^2) \chi + Q_2 \right] \phi \xi - \left( b_{22} - \kappa(\lambda_2 + \theta_2) \right) \chi \phi \xi + \left( b_{22} \chi + c_{22} \chi \right) \phi \xi = 0. \]  

Under the constraint conditions
\[ a_1 = a_2 \chi, \]  

\[ b_{11} + \frac{c_{11}}{\chi^2} = \frac{b_{12}}{\chi^2} + c_{21} \chi, \]  

\[ \omega + a_1 \kappa + a_1 \kappa^2 + Q_1 \chi = [\omega + a_1 \kappa + a_1 \kappa^2] \chi + Q_2, \]  

\[ c_{12} \chi^2 + b_{12} - \kappa(\lambda_1 + \theta_1) = c_{22} \chi + [b_{22} - \kappa(\lambda_2 + \theta_2)] \chi, \]  

\[ b_{11} + c_1 \chi^2 = b_{22} \chi + c_{22} \chi. \]  

Eqs. (17) and (18) have the same form. Next, from (19) and (21), the wave number \( \omega \) is recovered as
\[ \omega = \frac{\chi Q_1 - Q_2 - \kappa(\lambda_2 - \lambda_1)}{(\chi - 1)}. \]  

Balancing \( \phi \xi \phi \xi \) and \( \phi \xi \) in Eq. (17), yields \( N = \frac{1}{2} \). By the transformation
\[ \phi \xi = [U \xi]^\frac{1}{2}, \]  

Eq. (17) shapes up
\[ a_1 \left( 2UU' - U'^2 \right) + 4(\Delta_0 + \Delta_2 U^2 + \Delta_3 U^3 + \Delta_4 U^4) = 0, \]  

where
\[ \Delta_0 = b_{11} + \frac{c_{11}}{\chi^2}, \]  

\[ \Delta_2 = -\left( \omega + a_1 \kappa + a_1 \kappa^2 + Q_1 \chi \right), \]  

\[ \Delta_3 = \left[ c_{12} \chi^2 + b_{12} - \kappa(\lambda_1 + \theta_1) \right], \]  

\[ \Delta_4 = b_{11} + c_1 \chi^2. \]  

Eq. (26) will now be examined to secure soliton and other solutions to the model.

3. **Generalized Jacobi’s elliptic function expansion**

Assume that the formal solution of Eq. (26) is structured as
\[ U \xi = \sum_{j=0}^{N} \left[ A_j F^j(\xi) + B_j F^j(\xi) F^{j-1}(\xi) \right], \]  

where \( N \) is a positive integer, \( A_j, B_j (j = 0, 1, \ldots, N) \) are constants to be fixed and \( F(\xi) \) ensures
\[ F^2(\xi) = l_0 + l_1 F^2(\xi) + l_2 F^4(\xi). \]
The parameters $l_0$, $l_2$ and $l_4$ are constants to be detected later. It is well known that Eq. (29) has many generalized Jacobi elliptic function solutions. Now, balancing $U'^r$ with $U^l$ in Eq. (26) yields $N = 1$. Then, Eq. (26) can be rewritten as

$$U(ξ) = A_0 + A_1 F(ξ) + \frac{F'(ξ)}{F(ξ)}[B_0 + B_1 F(ξ)],$$

(30)

where $A_0, A_1, B_0$ and $B_1$ are constants to be established, while $F(ξ)$ satisfies Eq. (29). Insert (30) along with (29) into (26). Then, collecting all the coefficients of $[F(ξ)]^r[F'(ξ)]^s (r = 0, 1, \ldots, 12, s = 0, 1)$ and setting these coefficients to zero, and solving the resulting system yields

$$A_0 = -\frac{3A_1}{8A_2}, \quad A_1 = \frac{e}{2} \sqrt{-\frac{3a_1l_4}{A_2}}, \quad B_0 = 0, \quad B_1 = 0, \quad l_2 = \frac{9A_1^2 - 32A_2A_4}{8A_2A_4},$$

$$\Delta_0 = \frac{9A_1^2(15A_1^2 - 64A_2A_4)}{4096A_4^3} - \frac{3c_2^2l_4}{16A_4}, \quad l_0 = l_0, \quad l_1 = l_1,$$

(31)

provided $a_1l_4A_4 < 0$ and $e = \pm 1$. Now, we will consider the following cases:

**Case-1.** If we substitute $l_0 = \frac{wm^2((m^2 - 1))}{l_4}, w^2 = \frac{1}{2m^2 - 1}, l_4 < 0$ where $0 < m < 1$ in Eq. (31), then one gets

$$A_0 = -\frac{3A_1}{8A_2}, \quad A_1 = \frac{e}{2} \sqrt{-\frac{3a_1l_4}{A_2}}, \quad B_0 = 0, \quad B_1 = 0, \quad l_2 = \frac{9A_1^2 - 32A_2A_4}{8A_2A_4},$$

$$\Delta_0 = \frac{9A_1^2(15A_1^2 - 64A_2A_4)}{4096A_4^3} - \frac{3m^2(m^2 - 1)(9A_1^2 - 32A_2A_4)^2}{1024A_4^3(m^2 - 1)^2},$$

(32)

provided $A_4a_1 > 0$ and $e = \pm 1$. Putting (32) into Eq. (30) where $F(ξ)$ is given by:

$$F(ξ) = \frac{wm}{\sqrt{l_4}} \text{cn}(wξ),$$

(33)

Jacobi elliptic function (JEF) solutions to Eqs. (1) and (2) are revealed as

$$q(x, t) = \left\{-\frac{3A_1}{8A_2} + \frac{em}{8A_4} \sqrt{6(9A_1^2 - 32A_2A_4)} \left(\sqrt{\frac{9A_1^2 - 32A_2A_4}{8A_4a_1}(2m^2 - 1)}(x - vt)\right)\right\}^{\frac{1}{2}} e^{(-ax - awt + B_0)},$$

(34)

$$r(x, t) = \frac{i}{q(x, t)},$$

(35)

provided $(9A_1^2 - 32A_2A_4)(2m^2 - 1) > 0$, $eA_4 > 0$, $A_4a_1 > 0$ and $A_4A_2 < 0$. In particular, if $m \to 1$, then bright solitons emerge

$$q(x, t) = \left\{-\frac{3A_1}{8A_2} + \frac{e}{\sqrt{6A_4}} \sqrt{\frac{9A_1^2 - 32A_2A_4}{8A_4a_1}} \left(\sqrt{\frac{9A_1^2 - 32A_2A_4}{8A_4a_1}}(x - vt)\right)\right\}^{\frac{1}{2}} e^{(-ax - awt + B_0)},$$

(36)

$$r(x, t) = \frac{i}{q(x, t)},$$

(37)

**Case-2.** If we substitute $l_0 = \frac{wm^2}{l_4}, w^2 = \frac{1}{2m^2 - 1}, l_2 < 0, l_4 > 0$ where $0 < m < 1$ in Eq. (31), then one recovers

$$A_0 = -\frac{3A_1}{8A_2}, \quad A_1 = \frac{e}{2} \sqrt{-\frac{3a_1l_4}{A_2}}, \quad B_0 = 0, \quad B_1 = 0, \quad l_2 = \frac{9A_1^2 - 32A_2A_4}{8A_2A_4},$$

$$\Delta_0 = \frac{9A_1^2(15A_1^2 - 64A_2A_4)}{4096A_4^3} - \frac{3m^2(9A_1^2 - 32A_2A_4)^2}{1024A_4^3(m^2 - 1)^2},$$

(38)

provided $A_4a_1 < 0$ and $e = \pm 1$. Substituting (38) into Eq. (30) where $F(ξ)$ is given by:

$$F(ξ) = \left\{\begin{array}{ll}
\frac{wm}{\sqrt{l_4}} \text{sn}(wξ), \\
\frac{wm}{\sqrt{l_4}} \text{cd}(wξ).
\end{array}\right.$$  

(39)

JEF solutions to the model are
\[ q(x,t) = \left\{ \begin{array}{l} \frac{3\Delta_4}{8\Delta_4} + \frac{e_m}{8\Delta_4} \sqrt{\frac{6(9\Delta_4^2 - 32\Delta_2\Delta_4)}{m^2 + 1}} \sn \left( \sqrt{\frac{9\Delta_4^2 - 32\Delta_2\Delta_4}{8\Delta_4a_1(1 + m^2)}}(x - vt) \right) \right\}^{-\frac{1}{2}} e^{i(-\alpha x + \omega t + \theta)}, \\
\end{array} \] (40)

or

\[ q(x,t) = \left\{ \begin{array}{l} -\frac{3\Delta_4}{8\Delta_4} + \frac{e_m}{8\Delta_4} \sqrt{\frac{6(9\Delta_4^2 - 32\Delta_2\Delta_4)}{m^2 + 1}} \cd \left( \sqrt{\frac{9\Delta_4^2 - 32\Delta_2\Delta_4}{8\Delta_4a_1(1 + m^2)}}(x - vt) \right) \right\}^{\frac{1}{2}} e^{i(-\alpha x + \omega t + \theta)}, \\
\end{array} \] (42)

\[ r(x,t) = \lambda q(x,t), \] (41)

\[ q(x,t) = \left\{ \begin{array}{l} \frac{3\Delta_4}{8\Delta_4} + \frac{e_m}{8\Delta_4} \sqrt{\frac{3(9\Delta_4^2 - 32\Delta_2\Delta_4)}{8\Delta_4}} \tanh \left( \frac{1}{4} \sqrt{\frac{9\Delta_4^2 - 32\Delta_2\Delta_4}{\Delta_4a_1}}(x - vt) \right) \right\}^{\frac{1}{2}} e^{i(-\alpha x + \omega t + \theta)}, \\
\end{array} \] (44)

\[ r(x,t) = \lambda q(x,t), \] (43)

provided \((9\Delta_4^2 - 32\Delta_2\Delta_4) > 0, \Delta_4 > 0, \Delta_4a_1 > 0\) and \(\Delta_2\Delta_4 < 0\). In particular, if \(m \to 1\) in (40) and (41), then dark solitons fall out

\[ q(x,t) = \left\{ \begin{array}{l} \frac{3\Delta_4}{8\Delta_4} + \frac{e_m}{8\Delta_4} \sqrt{\frac{3(9\Delta_4^2 - 32\Delta_2\Delta_4)}{8\Delta_4}} \tanh \left( \frac{1}{4} \sqrt{\frac{9\Delta_4^2 - 32\Delta_2\Delta_4}{\Delta_4a_1}}(x - vt) \right) \right\}^{\frac{1}{2}} e^{i(-\alpha x + \omega t + \theta)}, \\
\end{array} \] (47)

\[ r(x,t) = \lambda q(x,t), \] (48)

\[ F(\xi) = \frac{w}{\sqrt{-l_4}} \sn(w\xi), \] (49)

\[ F(\xi) = \frac{w}{\sqrt{l_4}} \cs(w\xi), \] (50)

JEF solutions to the model are procured as

\[ q(x,t) = \left\{ \begin{array}{l} \frac{3\Delta_4}{8\Delta_4} + \frac{e_m}{8\Delta_4} \sqrt{\frac{3(9\Delta_4^2 - 32\Delta_2\Delta_4)}{8\Delta_4}} \sn \left( \sqrt{\frac{9\Delta_4^2 - 32\Delta_2\Delta_4}{\Delta_4a_1(2 - m^2)}}(x - vt) \right) \right\}^{\frac{1}{2}} e^{i(-\alpha x + \omega t + \theta)}, \\
\end{array} \] (51)

\[ r(x,t) = \lambda q(x,t), \] (52)

provided \((9\Delta_4^2 - 32\Delta_2\Delta_4) > 0, \Delta_4 > 0, \Delta_4a_1 < 0\) and \(\Delta_2\Delta_4 < 0\). In particular, if \(m \to 1\), then singular solitons are derived as follows:
\[
q(x,t) = \left\{ \begin{array}{l}
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{-6(9\Delta_3^2 - 32\Delta_2\Delta_4)} \csc \left( \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{8\Delta_4 a_1}} (x - vt) \right) \\
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{-3(9\Delta_3^2 - 32\Delta_2\Delta_4)} \cot \left( \frac{1}{4} \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{\Delta_4 a_1}} (x - vt) \right)
\end{array} \right\}^{\frac{1}{2}} e^{(-x+\omega t+\phi_0)},
\]
(53)
\[
r(x,t) = \chi q(x,t).
\]
(54)

While, If \( m \to 0 \), then periodic wave solutions are listed as
\[
q(x,t) = \left\{ \begin{array}{l}
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{-6(9\Delta_3^2 - 32\Delta_2\Delta_4)} \csc \left( \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{8\Delta_4 a_1}} (x - vt) \right) \\
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{-3(9\Delta_3^2 - 32\Delta_2\Delta_4)} \cot \left( \frac{1}{4} \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{\Delta_4 a_1}} (x - vt) \right)
\end{array} \right\}^{\frac{1}{2}} e^{(-x+\omega t+\phi_0)},
\]
(55)
\[
r(x,t) = \chi q(x,t).
\]
(56)

**Case-5.** If we substitute \( l_0 = \frac{w^5}{2m^2 - 1} \), \( w^2 = \frac{b}{2m - 1} \), \( l_4 > 0 \) where \( 0 < m < 1 \) in Eq. (31), then one gets the same results (32). Inserting (32) into Eq. (30), where \( F(\xi) \) is given by:
\[
F(\xi) = \frac{w}{\sqrt{l_4}} \text{ds}(w\xi),
\]
(57)
JEF solutions to the model are secured as
\[
q(x,t) = \left\{ \begin{array}{l}
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{-6(9\Delta_3^2 - 32\Delta_2\Delta_4)} \csc \left( \frac{1}{4} \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{\Delta_4 a_1}} (x - vt) \right) \\
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{-3(9\Delta_3^2 - 32\Delta_2\Delta_4)} \cot \left( \frac{1}{4} \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{\Delta_4 a_1}} (x - vt) \right)
\end{array} \right\}^{\frac{1}{2}} e^{(-x+\omega t+\phi_0)},
\]
(58)
\[
r(x,t) = \chi q(x,t).
\]
(59)
provided \((2m^2 - 1)(9\Delta_3^2 - 32\Delta_2\Delta_4) \neq 0\), \( \epsilon \Delta_4 > 0 \), \( \Delta_4 a_1 < 0 \) and \( \Delta_3 \Delta_4 < 0 \). In particular, if \( m \to 1 \), the same singular soliton solutions (53) and (54) are obtained. While, if \( m \to 0 \), then periodic waves are:
\[
q(x,t) = \left\{ \begin{array}{l}
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{-6(9\Delta_3^2 - 32\Delta_2\Delta_4)} \csc \left( \frac{1}{4} \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{\Delta_4 a_1}} (x - vt) \right) \\
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{-3(9\Delta_3^2 - 32\Delta_2\Delta_4)} \cot \left( \frac{1}{4} \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{\Delta_4 a_1}} (x - vt) \right)
\end{array} \right\}^{\frac{1}{2}} e^{(-x+\omega t+\phi_0)},
\]
(60)
\[
r(x,t) = \chi q(x,t).
\]
(61)

**Case-6.** If we substitute \( l_0 = \frac{w^5}{2m^2 - 1} \), \( w^2 = \frac{b}{2m - 1} \), \( l_4 > 0 \) where \( 0 < m < 1 \) in Eq. (31), then one gets the same results (32). Putting (32) into Eq. (30), where \( F(\xi) \) is given by:
\[
F(\xi) = \frac{w\sqrt{1 - m^2}}{\sqrt{l_4}} \text{nc}(w\xi),
\]
(62)
JEF solutions to the model are derived as
\[
q(x,t) = \left\{ \begin{array}{l}
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{6(1 - m^2)(9\Delta_3^2 - 32\Delta_2\Delta_4)} \csc \left( \frac{1}{4} \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{\Delta_4 a_1}} (x - vt) \right) \\
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{-3(9\Delta_3^2 - 32\Delta_2\Delta_4)} \cot \left( \frac{1}{4} \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{\Delta_4 a_1}} (x - vt) \right)
\end{array} \right\}^{\frac{1}{2}} e^{(-x+\omega t+\phi_0)},
\]
(63)
\[
r(x,t) = \chi q(x,t).
\]
(64)
provided \((2m^2 - 1)(9\Delta_3^2 - 32\Delta_2\Delta_4) \neq 0\), \( \epsilon \Delta_4 > 0 \), \( \Delta_4 a_1 < 0 \) and \( \Delta_3 \Delta_4 < 0 \). In particular, if \( m \to 0 \), then periodic wave solutions are of the form
\[
q(x,t) = \left\{ \begin{array}{l}
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{6(1 - m^2)(9\Delta_3^2 - 32\Delta_2\Delta_4)} \csc \left( \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{\Delta_4 a_1}} (x - vt) \right) \\
\frac{3\Delta_4}{8\Delta_4} + \frac{\epsilon}{8\Delta_4} \sqrt{-3(9\Delta_3^2 - 32\Delta_2\Delta_4)} \cot \left( \sqrt{\frac{9\Delta_3^2 - 32\Delta_2\Delta_4}{\Delta_4 a_1}} (x - vt) \right)
\end{array} \right\}^{\frac{1}{2}} e^{(-x+\omega t+\phi_0)},
\]
(65)
\[
r(x,t) = \chi q(x,t).
\]
(66)

**Case-7.** If we substitute \( l_0 = \frac{w^5}{2m^2 - 1} \), \( w^2 = \frac{b}{2m - 1} \), \( l_4 < 0 \) where \( 0 < m < 1 \) in Eq. (31), then one gets the same results (46). Inserting (46) into Eq. (31), where \( F(\xi) \) is given by:
\[ F(\xi) = \frac{w\sqrt{1 - m^2}}{\sqrt{t_i}} \text{nd}(w\xi), \]  
\[ (67) \]

JEF solutions to the adopted model are:

\[ q(x, t) = \begin{cases} 
\frac{3\Delta_1}{8\Delta_i} + \frac{\varepsilon}{8\Delta_i} \sqrt{6(1 - m^2)(9\Delta_i^2 - 32\Delta_i\Delta_4)} \text{nd} \left( \sqrt{\frac{9\Delta_i^2 - 32\Delta_i\Delta_4}{8\Delta_i a_i(2 - m^2)}}(x - vt) \right) \right)^{\frac{1}{2}} e^{(-x+\omega t+\theta)}, 
\end{cases} \]  
\[ (68) \]

\[ r(x, t) = \chi q(x, t), \]  
\[ (69) \]

provided \((9\Delta_4^2 - 32\Delta_i\Delta_4)(0, \varepsilon\Delta_4 > 0, \Delta_4 a_i > 0 \text{ and } \Delta_3 \Delta_4 < 0).\)

Case-8. If we substitute \(l_i = \frac{w(x - m^2)}{l_i}, w^2 = \frac{l_i}{x - m^2}, l_i > 0 \) where \(0 < m < 1 \) in Eq. (31), then one gets the same results (46). Plugging (46) into Eq. (30), where \(F(\xi)\) is given by:

\[ F(\xi) = \frac{w\sqrt{1 - m^2}}{\sqrt{l_i}} \text{sc}(w\xi), \]  
\[ (70) \]

JEF solutions to the model are listed as below:

\[ q(x, t) = \begin{cases} 
\frac{3\Delta_1}{8\Delta_i} + \frac{\varepsilon}{8\Delta_i} \sqrt{3(9\Delta_i^2 - 32\Delta_i\Delta_4)} \tan \left( \frac{1}{4} \sqrt{\frac{9\Delta_i^2 - 32\Delta_i\Delta_4}{\Delta_i a_i}}(x - vt) \right) \right)^{\frac{1}{2}} e^{(-x+\omega t+\theta)}, 
\end{cases} \]  
\[ (71) \]

\[ r(x, t) = \chi q(x, t), \]  
\[ (72) \]

provided \((9\Delta_4^2 - 32\Delta_i\Delta_4)(0, \varepsilon\Delta_4 > 0, \Delta_4 a_i < 0 \text{ and } \Delta_3 \Delta_4 < 0).\) In particular, if \(m \to 0\), then periodic wave solutions are revealed as

\[ q(x, t) = \begin{cases} 
\frac{3\Delta_1}{8\Delta_i} + \frac{\varepsilon}{8\Delta_i} \sqrt{6(1 - m^2)(9\Delta_i^2 - 32\Delta_i\Delta_4)} \text{sc} \left( \sqrt{\frac{9\Delta_i^2 - 32\Delta_i\Delta_4}{8\Delta_i a_i(2 - m^2)}}(x - vt) \right) \right)^{\frac{1}{2}} e^{(-x+\omega t+\theta)}, 
\end{cases} \]  
\[ (73) \]

\[ r(x, t) = \chi q(x, t), \]  
\[ (74) \]

Case-9. If we substitute \(l_i = \frac{x}{m}, w^2 = -\frac{l_i}{x - m^2}, l_i < 0, l_i > 0 \) where \(0 < m < 1 \) in Eq. (31), then one gets the same results (38). Substituting (38) into Eq. (30), where \(F(\xi)\) is given by:

\[ F(\xi) = \begin{cases} 
\frac{w}{\sqrt{l_i}} \text{dc}(w\xi), 
\frac{w}{\sqrt{l_i}} \text{ns}(w\xi), 
\end{cases} \]  
\[ (75) \]

JEF solutions to (1) and (2) are:

\[ q(x, t) = \begin{cases} 
\frac{3\Delta_1}{8\Delta_i} + \frac{\varepsilon}{8\Delta_i} \sqrt{6(9\Delta_i^2 - 32\Delta_i\Delta_4)} \text{dc} \left( \sqrt{-\frac{9\Delta_i^2 - 32\Delta_i\Delta_4}{8\Delta_i a_i(1 + m^2)}}(x - vt) \right) \right)^{\frac{1}{2}} e^{(-x+\omega t+\theta)}, 
\end{cases} \]  
\[ (76) \]

\[ r(x, t) = \chi q(x, t), \]  
\[ (77) \]

or

\[ q(x, t) = \begin{cases} 
\frac{3\Delta_1}{8\Delta_i} + \frac{\varepsilon}{8\Delta_i} \sqrt{6(9\Delta_i^2 - 32\Delta_i\Delta_4)(1 + m^2)} \text{ns} \left( \sqrt{-\frac{9\Delta_i^2 - 32\Delta_i\Delta_4}{8\Delta_i a_i(1 + m^2)}}(x - vt) \right) \right)^{\frac{1}{2}} e^{(-x+\omega t+\theta)}, 
\end{cases} \]  
\[ (78) \]

\[ r(x, t) = \chi q(x, t), \]  
\[ (79) \]

provided \((9\Delta_4^2 - 32\Delta_i\Delta_4)(0, \varepsilon\Delta_4 > 0, \Delta_4 a_i < 0 \text{ and } \Delta_3 \Delta_4 < 0).\) In particular, if \(m \to 1\) in (78) and (79), singular soliton solutions fall out
\[ q(x,t) = \left\{ \frac{3\Delta_i}{8\Delta_i} + \frac{\epsilon}{8\Delta_i} \sqrt{3(9\Delta_i^2 - 32\Delta_2\Delta_1)} \coth \left( \frac{1}{4} \sqrt{\frac{9\Delta_i^2 - 32\Delta_2\Delta_1}{\Delta_i a_1}} (x - vt) \right) \right\}^{\frac{1}{2}} e^{(-x+vt+s\theta_j)}, \tag{80} \]

\[ r(x,t) = \chi q(x,t). \tag{81} \]

While, if \( m \to 0 \) in (76)-(79), then one acquires the same periodic wave solutions (65), (66), (60) and (61), respectively.

**Case-10.** If we substitute \( l_0 = \frac{w^2(1-m^2)}{4l_4} \), \( w^2 = \frac{2l_4}{1+m^2} l_4 < 0 \) where \( 0 < m < 1 \) in Eq. (31), then one gets the same results (32). Inserting (32) into Eq. (30), where \( F(\xi) \) is given by:

\[ F(\xi) = \frac{w\sqrt{1-m^2}}{\sqrt{-l_4}} \text{sd}(w\xi). \tag{82} \]

JEF solutions to Eqs. (1) and (2) are retrieved as

\[ q(x,t) = \left\{ -\frac{3\Delta_i}{8\Delta_i} + \frac{\epsilon m}{8\Delta_i} \sqrt{\frac{6(1-m^2)(9\Delta_i^2 - 32\Delta_2\Delta_1)}{(2m^2-1)}} \text{sd} \left( \sqrt{\frac{9\Delta_i^2 - 32\Delta_2\Delta_1}{8\Delta_1 a_1(2m^2-1)}} (x - vt) \right) \right\}^{\frac{1}{2}} e^{(-x+vt+s\theta_j)}, \tag{83} \]

\[ r(x,t) = \chi q(x,t). \tag{84} \]

provided \((2m^2 - 1)(9\Delta_i^2 - 32\Delta_2\Delta_1) > 0\), \( \epsilon \Delta_4 > 0\), \( \Delta_1 a_1 > 0 \) and \( \Delta_2 \Delta_4 < 0 \).

**Case-11.** If we substitute \( l_0 = \frac{w^2(1-m^2)}{16l_4} \), \( w^2 = \frac{2l_4}{1+m^2} l_4 < 0 \) where \( 0 < m < 1 \) in Eq. (31), then one obtains

\[ A_0 = \frac{3\Delta_i}{8\Delta_i}, \quad A_1 = \frac{\epsilon}{2} \sqrt{\frac{3\Delta_1 l_4}{\Delta_4}}, \quad B_0 = 0, \quad B_1 = 0, \quad l_2 = \frac{9\Delta_i^2 - 32\Delta_2\Delta_1}{8\Delta_1 a_1}, \quad l_4 = l_4, \quad \Delta_0 = \frac{9\Delta_i^2(15\Delta_i^2 - 64\Delta_2\Delta_1)}{4096\Delta_4^2} - \frac{3(m-1)^2(m+1)^2(9\Delta_i^2 - 32\Delta_2\Delta_1)^2}{4096\Delta_4^2(m^2+1)^2}. \tag{85} \]

provided \( a_1 \Delta_4 > 0 \) and \( \epsilon = \pm 1 \). Putting (85) into Eq. (30), where \( F(\xi) \) is given by:

\[ F(\xi) = \frac{w}{2\sqrt{-l_4}} \left[ \text{mcn}(w\xi) \pm \text{dn}(w\xi) \right], \tag{86} \]

JEF solutions to the model are recovered as

\[ q(x,t) = \left\{ -\frac{3\Delta_i}{8\Delta_i} + \frac{\epsilon m}{8\Delta_i} \sqrt{\frac{3(9\Delta_i^2 - 32\Delta_2\Delta_1)}{(1+m^2)}} \left[ \text{mcn} \left( \frac{1}{2} \sqrt{\frac{9\Delta_i^2 - 32\Delta_2\Delta_1}{\Delta_i a_1(1+m^2)}} (x - vt) \right) \right] \right\}^{\frac{1}{2}} e^{(-x+vt+s\theta_j)}, \tag{87} \]

\[ r(x,t) = \chi q(x,t). \tag{88} \]

provided \((9\Delta_i^2 - 32\Delta_2\Delta_1) > 0\), \( \epsilon \Delta_4 > 0\), \( \Delta_1 a_1 > 0 \) and \( \Delta_2 \Delta_4 < 0 \). In particular, if \( m \to 1 \), then the same bright soliton solutions (36) and (37) emerge.

**Case-12.** If we substitute \( l_0 = \frac{w^2(1-m^2)^2}{16l_4} \), \( w^2 = \frac{2l_4}{1+m^2} l_4 > 0 \) where \( 0 < m < 1 \) in Eq. (31), then one gets the same results (85). Substituting (85) into Eq. (30), where \( F(\xi) \) is given by:

\[ F(\xi) = \begin{cases} \frac{w\sqrt{1-m^2}}{2\sqrt{l_4}} \left[ \text{nc}(w\xi) \pm \text{sc}(w\xi) \right], \\ \frac{w\sqrt{1-m^2}}{2\sqrt{l_4}} \left[ \text{cn}(w\xi) \right] \end{cases}, \tag{89} \]

the recovered JEF solutions are:
\[
q(x, t) = \left\{ \frac{-3\Delta_{1t} + \varepsilon}{8\Delta_{4t}} \sqrt{3(1 - m^2)(9\Delta_{1t}^2 - 32\Delta_{1t}^4)} \left( x - vt \right) + \text{sc} \left\{ \frac{1}{2} \sqrt{\frac{9\Delta_{1t}^2 - 32\Delta_{1t}^4}{\Delta_{4t}(1 + m^2)}} \left( x - vt \right) \right\} \right\}^\frac{1}{2} \times e^{i(\omega_t - \omega_0) t},
\]
(90)

or
\[
q(x, t) = \left\{ \frac{-3\Delta_{1t} + \varepsilon}{8\Delta_{4t}} \sqrt{3(1 - m^2)(9\Delta_{1t}^2 - 32\Delta_{1t}^4)} \left( x - vt \right) + \text{cn} \left\{ \frac{1}{2} \sqrt{\frac{9\Delta_{1t}^2 - 32\Delta_{1t}^4}{\Delta_{4t}(1 + m^2)}} \left( x - vt \right) \right\} \right\}^\frac{1}{2} \times e^{i(\omega_t - \omega_0) t},
\]
(92)

\[
r(x, t) = \chi q(x, t),
\]
(91)

provided \((9\Delta_{1t}^2 - 32\Delta_{2t}^4)\langle 0, \varepsilon \Delta_{4t} > 0, \Delta_{4t} \Delta_{4t} < 0 \) and \(\Delta_{2t} \Delta_{4t} < 0\). In particular, if \(m \rightarrow 0\), then from (90)-(93) we have, respectively periodic wave solutions to Eqs. (1) and (2) in the form
\[
q(x, t) = \left\{ \frac{3\Delta_{1t}}{8\Delta_{4t}} + \frac{\varepsilon}{8\Delta_{4t}} \sqrt{-3(9\Delta_{1t}^2 - 32\Delta_{1t}^4)} \left( x - vt \right) + \tan \left\{ \frac{1}{2} \sqrt{\frac{9\Delta_{1t}^2 - 32\Delta_{1t}^4}{\Delta_{4t}(1 + m^2)}} \left( x - vt \right) \right\} \right\}^\frac{1}{2} \times e^{i(\omega_t - \omega_0) t},
\]
(94)

or
\[
r(x, t) = \chi q(x, t),
\]
(95)

or
\[
q(x, t) = \left\{ \frac{-3\Delta_{1t} + \varepsilon}{8\Delta_{4t}} \sqrt{-3(9\Delta_{1t}^2 - 32\Delta_{1t}^4)} \left( x - vt \right) + \tan \left\{ \frac{1}{2} \sqrt{\frac{9\Delta_{1t}^2 - 32\Delta_{1t}^4}{\Delta_{4t}(1 + m^2)}} \left( x - vt \right) \right\} \right\}^\frac{1}{2} e^{i(\omega_t - \omega_0) t},
\]
(96)

\[
r(x, t) = \chi q(x, t).
\]
(97)

**Case-13.** If we substitute \(l_0 = \frac{m^2}{8\Delta_{4t}}, \ w^2 = \frac{20}{m^2}, l_4 > 0\) where \(0 < m < 1\) in Eq. (31) and solve it employing the Maple, then one attains
\[
A_0 = \frac{3\Delta_{1t}}{8\Delta_{4t}}, \quad A_1 = \frac{\varepsilon}{2} \sqrt{\frac{3a_1l_4}{\Delta_{4t}}}, \quad B_0 = 0, \quad B_1 = 0, \quad l_2 = \frac{9\Delta_{1t}^2 - 32\Delta_{1t}^4}{8\Delta_{4t}}, \quad l_4 = l_4,
\]
(98)

\[
\Delta_0 = \frac{9\Delta_{1t}^2(15\Delta_{1t}^2 - 64\Delta_{4t}^2)}{4096\Delta_{4t}^2} - \frac{3m^4(9\Delta_{1t}^2 - 32\Delta_{1t}^4)^2}{4096\Delta_{4t}^2(m^2 - 2)},
\]
provided \(a_1 \Delta_{4t} < 0\) and \(\varepsilon = \pm 1\). Substituting (98) into Eq. (30), where \(F(\xi)\) is given by:
\[
F(\xi) = \frac{w}{2\sqrt{m}} \left[ \sqrt{1 + m^2} \text{cn}(w_\xi) \pm \text{dc}(w_\xi) \right],
\]
(99)

the obtained JEF solutions are:
provided \((9\Delta_2^2 - 32\Delta_2\Delta_4)/0, \varepsilon\Delta_4 > 0, \Delta_4a_1 < 0\) and \(\Delta_3\Delta_4 < 0\). In particular, if \(m \to 0\), then the same periodic wave solutions (65) and (66) are derived.

**Case-14.** If we substitute \(a_0 = \frac{\sqrt{1 - m^2}}{1 - m^2}, a_1 = \frac{\sqrt{1 - m^2}}{1 - m^2}, a_2 = \frac{\sqrt{1 - m^2}}{1 - m^2}, a_3 = \frac{\sqrt{1 - m^2}}{1 - m^2}\) in Eq. (31), then one gets the same results (85). Inserting (85) into Eq. (30), where \(F(\xi)\) is given by:

\[
F(\xi) = \frac{w\sqrt{1 - m^2}}{2(1 - m^2)} [\text{mdw}(\xi) \pm \text{ndw}(\xi)],
\]

JEF solutions for the considered model are:

\[
q(x,t) = \left\{ \frac{3\Delta_1}{8\Delta_4} + \varepsilon \frac{3\sqrt{(1 - m^2)(9\Delta_2^2 - 32\Delta_2\Delta_4)}}{m^2 + 1} \left[ \text{mds} \left( \frac{1}{2} \sqrt{\frac{9\Delta_2^2 - 32\Delta_2\Delta_4}{\Delta_4a_1(m^2 + 1)}} (x - vt) \right) + \text{nd} \left( \frac{1}{2} \sqrt{\frac{9\Delta_2^2 - 32\Delta_2\Delta_4}{\Delta_4a_1(m^2 + 1)}} (x - vt) \right) \right] \right\}^\frac{1}{2} e^{i\omega(x+vt+\theta_0)},
\]

\[
r(x,t) = \varepsilon(q(x,t)).
\]

**Case-15.** Plugging (31) into Eq. (30), where \(F(\xi)\) is given by:

\[
F(\xi) = \frac{3\varphi(\xi, g_2, g_3)}{\sqrt{6\varphi(\xi, g_2, g_3) + l_0}}, l_0 > 0.
\]

Here \(\varphi(\xi, g_2, g_3)\) is called a Weierstrass elliptic function which satisfies the Eq. \(\varphi'^2 = 4\varphi^3 - g_2\varphi - g_3\), such that \(g_2\) and \(g_3\) are called invariants of the Weierstrass elliptic function, in which \(\varphi = \frac{dy}{dx}\). Now, one recovers the Weierstrass elliptic function solutions to Eqs. (1) and (2) in the form

\[
q(x,t) = \left\{ \frac{3\Delta_1}{8\Delta_4} + \frac{12\varepsilon a_1 \sqrt{-3a_1\Delta_4} \varphi((x - vt), g_2, g_3)}{48a_1\Delta_4 \varphi((x - vt), g_2, g_3) + (9\Delta_2^2 - 32\Delta_2\Delta_4)} \right\}^\frac{1}{2} e^{i\omega(x+vt+\theta_0)},
\]

\[
r(x,t) = \varepsilon(q(x,t)).
\]

provided \(\varepsilon a_1 < 0, a_1 \Delta_4 < 0, \Delta_3\Delta_4 < 0\) and \((9\Delta_2^2 - 32\Delta_2\Delta_4)/0\). In these results

\[
g_2 = \frac{(9\Delta_2^2 - 32\Delta_2\Delta_4)^2 + 768\Delta_4^2 \Delta_1 l_0 d_4}{768\Delta_4^2 \Delta_1^2},
\]

and

\[
g_3 = \frac{(9\Delta_2^2 - 32\Delta_2\Delta_4) \left[2304\Delta_4^2 \Delta_1 l_0 d_4 - (9\Delta_2^2 - 32\Delta_2\Delta_4)^2 \right]}{110592a_1^2 \Delta_4^2}.
\]

**Case-16.** Inserting (31) into Eq. (30), where \(F(\xi)\) is given by:

\[
F(\xi) = \frac{\sqrt{l_0(6\varphi(\xi, g_2, g_3) + l_0)}}{3\varphi(\xi, g_2, g_3)}, l_0 > 0.
\]

one secures the Weierstrass elliptic function solutions to the model as

\[
q(x,t) = \left\{ \frac{3\Delta_1}{8\Delta_4} + \sqrt{\frac{3a_1 l_0 d_4}{\Delta_4} \left[ \frac{48a_1 \Delta_4 \varphi((x - vt), g_2, g_3) + (9\Delta_2^2 - 32\Delta_2\Delta_4)}{48a_1 \Delta_4 \varphi((x - vt), g_2, g_3)} \right]} \right\}^\frac{1}{2} e^{i\omega(x+vt+\theta_0)},
\]

\[
r(x,t) = \varepsilon(q(x,t)).
\]
provided $l_4 < 0$, $a_1 \Delta_4 > 0$, $\Delta_3 \Delta_4 < 0$ and $(9 \Delta_3^2 - 32 \Delta_2 \Delta_4) > 0$. Here $g_2$ and $g_3$ are given by (108) and (109), respectively.

**Case-17.** Putting (31) into Eq. (30), where $F(\xi)$ is given by:

$$F(\xi) = \left[ \frac{3 \wp(\xi, g_2, g_3) - l_1}{3a} \right]^\frac{1}{2},$$

one reveals the Weierstrass elliptic function solutions for the model as below:

$$q(x, t) = \left[ \frac{3 \Delta_3}{8 \Delta_4} + \frac{\epsilon}{16 \Delta_4} \sqrt{8 \left[ -24 a_1 \Delta_4 \wp((x - vt), g_2, g_3) + (9 \Delta_3^2 - 32 \Delta_2 \Delta_4) \right]} \right]^\frac{1}{2} e^{(i \delta x + \omega t + \phi_0)},$$

$$r(x, t) = \chi q(x, t),$$

provided $\Delta_3 \Delta_4 < 0$, $\epsilon \Delta_4 > 0$, $a_1 \Delta_4 < 0$ and $(9 \Delta_3^2 - 32 \Delta_2 \Delta_4) > 0$. Here

$$g_2 = \frac{(9 \Delta_3^2 - 32 \Delta_2 \Delta_4)^2 - 192 a_1^2 \Delta_2^2 l_4}{48 a_1^2 \Delta_4^2},$$

and

$$g_3 = \frac{(9 \Delta_3^2 - 32 \Delta_2 \Delta_4) \left[ 288 a_1^2 \Delta_2^2 l_4 (48 a_1^2 \Delta_4^2 - (9 \Delta_3^2 - 32 \Delta_2 \Delta_4)^2) \right]}{1728 a_1^2 \Delta_4^2}.$$  

**Case-18.** Substituting (31) into Eq. (30), where $F(\xi)$ is given by:

$$F(\xi) = \left[ \frac{3l_0}{3 \wp(\xi, g_2, g_3) - l_2} \right]^\frac{1}{2},$$

one has the Weierstrass elliptic function solutions to Eqs. (1) and (2) as

$$q(x, t) = \left[ \frac{3 \Delta_3}{8 \Delta_4} + 12 a_1 \sqrt{\frac{l_0 l_4}{8 \left[ -24 a_1 \Delta_4 \wp((x - vt), g_2, g_3) + (9 \Delta_3^2 - 32 \Delta_2 \Delta_4) \right]}} \right]^\frac{1}{2} e^{(i \delta x + \omega t + \phi_0)},$$

$$r(x, t) = \chi q(x, t),$$

provided $l_0 l_4 > 0$, $\Delta_3 \Delta_4 < 0$, $\epsilon a_1 > 0$, $a_1 \Delta_4 < 0$ and $(9 \Delta_3^2 - 32 \Delta_2 \Delta_4) > 0$. Here $g_2$ and $g_3$ is given by (116) and (117), respectively.

4. Numerical simulations

This section presents the graphs of some solutions for the coupled system (1) and (2). Let us now examine Figures (Fig. 1, Fig. 2,
Fig. 2. The numerical simulations of dark soliton solutions (44) and (45) with the parameter values \( \kappa = -1, \omega = 0.5, \nu = 1, Q_1 = 1, Q_2 = 0.5, a_1 = -2, a_2 = -1, \chi = 2, a_1 = -5, a_2 = -3, b_{13} = 1.5, c_{13} = 0.25, c_{12} = 0.5, b_{12} = 0.25, \lambda_1 = 0.5, \theta_1 = -5.45, \varepsilon = 1 \).

Fig. 3. The numerical simulations of Jacobi elliptic solutions (51) and (52) with the parameter values \( \kappa = -1, \omega = -3.5, \nu = 1, Q_1 = -1, Q_2 = 0, a_1 = -2, a_2 = -1, \chi = 2, a_1 = -5, a_2 = -3, b_{13} = 5.5, c_{13} = 0.25, c_{12} = 0.5, b_{12} = 0.25, \lambda_1 = 0.5, \theta_1 = -5.45, \varepsilon = 1, m = 0.1 \).

Fig. 4. The numerical simulations of Weierstrass elliptic function solutions (106) and (107) with the parameter values \( \kappa = 1, \omega = 2.5, \nu = 1, Q_1 = 1, Q_2 = 0, a_1 = 2, a_2 = 1, \chi = 2, a_1 = -5, a_2 = -3, b_{13} = -1.5, c_{13} = -0.25, c_{12} = 0.5, b_{12} = 0.25, \lambda_1 = 0.5, \theta_1 = 0.45, b_0 = 1, l_4 = 0.5, \varepsilon = -1 \).

Fig. 3, Fig. 4) as it illustrates some of our solutions obtained in this paper. To this aim, we select some special values of the parameters as follows:

From the above figures, one can see that the secured solutions possess bright soliton solutions, dark soliton solutions, Jacobi elliptic solutions and Weierstrass elliptic function solutions to the governing equation (1) and (2). Also, these figures express the behavior of these solutions which give some perspective readers how the behavior solutions are produced.
5. Conclusions

A wide spectrum of soliton solutions were recovered for magneto–optic waveguides with AC form of nonlinear refractive index. The results of the paper have varied applications in the field of optoelectronics. The bright soliton solutions will be a big asset in controlling the soliton clutter as mentioned in the introduction section. This means that the solitons can be converted to a state of separation from a state of attraction which would mean clearing the clutter. Therefore, this would bring a factor of “ease” to the Internet bottleneck that is a growing problem to the modern day telecommunications industry where the Internet is a daily essential for survival. During the current pandemic period of COVID-19, where all business activities are conducted online, it is imperative to have a smooth and uninterrupted flow of pulses for undisturbed Internet communications. Likewise, dark solitons are also going to benefit soliton transmission when a background wave is present. However, singular soliton solutions are just a fancy form of solitons and are only listed to display a complete spectrum of soliton solutions yielded from the model. These solutions are not applicable in fiber optic communications. These solitons are the basic elements for further study in this avenue. The soliton solutions will enable to reveal conserved quantities once the conserved densities are recovered by the aid of multiplier approach. Subsequently, the soliton perturbation theory will yield the adiabatic parameter evolution of the parameters with quasi–monochromaticity. Later, additional studies will also reveal the quasi–stationary soliton solutions in such waveguides. These are just a handful few avenues to explore in the long run.

Declaration of Competing Interest

The authors report no declarations of interest.

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The authors also declare that there is no conflict of interest.

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