THE DEGENERATE PRINCIPAL SERIES REPRESENTATIONS OF EXCEPTIONAL GROUPS OF TYPE $E_6$ OVER $p$-ADIC FIELDS

BY

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ABSTRACT

In this paper, we study the reducibility of degenerate principal series of the simple, simply-connected exceptional group of type $E_6$. Furthermore, we calculate the maximal semi-simple subrepresentation and quotient of these representations.

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1. Introduction

One of the main problems in the representation theory of $p$-adic groups is the question of reducibility and structure of parabolic induction. More precisely, let $G$ be a $p$-adic group and let $P$ be a parabolic subgroup of $G$. Let $M$ denote the Levi subgroup of $P$ and let $\sigma$ be a smooth irreducible representation of $M$. One can ask:

- Is the parabolic induction $\pi = \text{Ind}_P^G \sigma$ reducible?
- If the answer to the previous question is positive, what is Jordan–Hölder series of $\pi$?

In this paper we completely answer the first question for degenerate principal series of the exceptional group of type $E_6$. We further calculate the maximal semi-simple subrepresentation and quotient of such representations which partially answers the second question.

More precisely, let $G$ be a simple, split, simply-connected $p$-adic group of type $E_6$ and let $P$ be a maximal parabolic subgroup of $G$. For a one-dimensional representation $\Omega$ of $P$, we consider the normalized parabolic induction $\pi = \text{Ind}_P^G \Omega$. We determine for which $\Omega$, $\pi$ is reducible and its maximal semi-simple subrepresentation and maximal semi-simple quotient. Our main result, Theorem 4.3, is summarized by the following corollary.

**Corollary 1.1:** With the exception of one case, for any maximal Levi subgroup $M$ of $G$ (of type $E_6$) and any one-dimensional representation $\Omega$ of $M$, if the degenerate principal series representation $\pi = \text{Ind}_P^G \Omega$ is reducible, then $\pi$ admits a unique irreducible subrepresentation and a unique irreducible quotient.

In the one exception, up to contragredience, $\Omega = (\chi \circ \omega_4) |\omega_4|^{1/2}$, where $\omega_4$ is the 4th fundamental weight and $\chi$ is a cubic character. In this case, $\pi$ admits a maximal semi-simple quotient of length 3 and a unique irreducible subrepresentation.

Similar studies were performed for both classical and smaller exceptional groups. For example, see:

- [23, 4] for general linear groups. (It should be noted that the scope of these works goes beyond degenerate principal series.)
- [11] for symplectic groups.
- [2, 13] for orthogonal groups.
- [15] for type $G_2$.
- [7] for type $F_4$. 

The reason that such a study was not performed for groups of type $E_n$ before, is that Weyl groups of these types are extremely big and have complicated structure. For that reason, in Section 3 we describe an algorithm, to calculate the reducibility of degenerate principal series. This algorithm was implemented by us using Sagemath \cite{21}.

The study of degenerate principal series for groups of type $E_7$ and $E_8$, using the algorithm described in this paper, is a work in progress.

This paper is organized as follows:

- Section 2 introduces the notations used in this paper.
- Section 3 introduces the algorithm used by us to calculate the reducibility of $\pi = \text{Ind}_P^G \Omega$ and the maximal semi-simple subrepresentation and quotient of $\pi$.
- Section 4 begins with a short introduction of the group of $G$ and contains our main result, Theorem 4.3, the reducibility of degenerate principal series of $E_6$, their subrepresentations and their quotients.
- Appendix A contains a few technical results which are useful in the implementation of the algorithm in Section 3.
- Appendix B contains an example of the application of the irreducibility test described in Section 3.

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2. Preliminaries and notations

Let $k$ be a locally-compact non-Archimedean field and $G$ the group of $k$-rational points of a split, simply-connected, reductive, algebraic group over $k$. Fix a Borel subgroup $B$ of $G$ with a maximal split torus $T$ and assume that $n = \text{rank}(G)$. Let $\Phi$ denote the set of roots of $G$ with respect to $T$ and let $\Delta$ denote the set of simple roots with respect to $B$. Also, let $\Phi^+$ denote the set of positive roots of $G$ with respect to $B$. For $\alpha \in \Delta$, let $\alpha^\vee$ denote the associated
co-character and let $\omega_\alpha$ denote the associated fundamental weight. Namely,

$$\langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha, \beta} \quad \forall \alpha, \beta \in \Delta,$$

where $\delta_{\alpha, \beta}$ denotes the usual Kronecker delta function on $\Delta$. We also denote by $A_G$ the center of $G$.

Let $W = W_G$ denote the Weyl group of $G$ with respect to $T$; it is generated by simple reflections $s_\alpha$ with respect to the simple roots $\alpha \in \Delta$.

For $\Theta \subset \Delta$, let $P_\Theta$ denote the associated standard parabolic subgroup of $G$ and denote its Levi subgroup by $M_\Theta$. In this case, let $\Delta_M = \Theta$. Let

$$\Phi_M = \Phi \cap \text{Span}_\mathbb{Z}(\Delta_M)$$

be the set of roots of $M$ with respect to $T$,

$$\Phi_M^+ = \Phi^+ \cap \Phi_M$$

be the set of positive roots of $M$ and let

$$\rho_M = \rho_M^G = \frac{1}{2} \sum_{\alpha \in \Phi_M^+} \alpha$$

be the half-sum of positive roots in $M$. Also, let $W_M$ denote the Weyl group of $M$ relative to $T$; it is the subgroup of $W_G$ generated by $\{ s_\alpha \mid \alpha \in \Delta_M \}$.

For any $\alpha \in \Delta$ we denote $L_\alpha = M_{\{\alpha\}}$ and $M_\alpha = M_{\Delta \setminus \{\alpha\}}$.

For a Levi subgroup $M$ of $G$, we denote its complex manifold of characters by $X(M)$. As $X(M)$ is a commutative group, it is convenient to adopt additive notations for its elements. In particular, for $\lambda \in X(M)$, $-\lambda$ denotes its inverse. The trivial element in $X(M)$ will be denoted by either $1$ or $0$. We further denote by $X^*(M)$ the ring of rational characters of $M$ and let $a_{\mathbb{M}, \mathbb{C}}^* = X^*(M) \otimes \mathbb{Z} \mathbb{C}$.

By abuse of notations, given $\lambda \in X(M)$, $\alpha \in \Phi$ and $s \in \mathbb{C}$ such that $\lambda \circ \alpha^\vee(t) = |t|^s$ for all $t \in \mathbb{G}_m(k)$, we write $\langle \lambda, \alpha^\vee \rangle = s$.

Within $X(M)$, we distinguish the real and imaginary characters on $M$,

$$\mathcal{Re}(X(M)) = \{ \lambda \in X(M) \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{R} \forall \alpha \in \Delta \},$$

$$\mathcal{Im}(X(M)) = i \mathcal{Re}(X(M)).$$

Then $a_{\mathbb{M}, \mathbb{C}}^* = \mathcal{Re}(X(M)) \oplus \mathcal{Im}(X(M))$. For $\mu \in X(M)$, we write $\mathcal{Re}(\mu)$ for the projection of $\mu$ to $\mathcal{Re}(X(M))$ and $\mathcal{Im}(\mu)$ for the projection of $\mu$ to $\mathcal{Im}(X(M))$.

Let $\mu \in X(M)$, and assume that there exists $k \in \mathbb{N}$ such that $\mu^k = 1$. We say that $\mu$ is of finite order and denote the minimal such $k$ by $\text{ord}(\mu)$, the order of $\mu$. In particular, $\text{ord}(1) = 1$ and, for a quadratic character $\mu$, $\text{ord}(\mu) = 2$. 
One can choose a direct sum complement $X(M) = a^*_{M,C} \oplus X_{M,0}$, where the characters in $X_{M,0}$ are of finite order. In particular, we may write,

$$X(\mathbb{G}_m(k)) = \mathbb{C} \oplus X_{\mathbb{G}_m(k),0}.$$  

We note that restriction to $T$ gives rise to the following inclusions:

$$\iota_M : X(M) \hookrightarrow X(T), \quad X^*(M) \hookrightarrow X^*(T),$$

$$a^*_{M,C} \hookrightarrow a^*_{T,C}, \quad X_{M,0} \hookrightarrow X_{T,0}.$$  

The image of these embeddings can be identified by restriction to the derived group $M^{\text{der}}$. Namely, for $\chi \in X(T), \chi \in X(M)$ if and only if $\langle \chi, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta_M$. Similarly for $X^*(M), a^*_{M,C}$ and $X_{M,0}$.

Recall that $a^*_{M,C} \cong \mathbb{C}^{n-|\Delta_M|}$, We use this identification and write elements of $a^*_{M,C}$ as vectors in $\mathbb{C}^{n-|\Delta_M|}$ whose basis elements are the fundamental weights $\omega_\alpha$ for $\alpha \in \Delta \setminus \Delta_M$. In fact, by abuse of notations, we may write any element $\Omega \in X(M)$ in the form

$$\Omega = \Omega_C + \sum_{\alpha \in \Delta \setminus \Delta_M} \Omega_\alpha \circ \omega_\alpha,$$

where $\Omega_\alpha \in X(\mathbb{G}_m(k))$ for all $\alpha \in \Delta \setminus \Delta_M$ and $\Omega_C \in X(A_G)$.

Let $\lambda \in X(T)$. We say that $\lambda$ is dominant if

$$\Re(\langle \lambda, \alpha^\vee \rangle) \geq 0 \quad \forall \alpha \in \Delta.$$

In other words, $\Re(\lambda)$ is in the (closed) positive Weyl chamber. We say that $\lambda$ is anti-dominant if $-\lambda$ is dominant.

Let $\text{Rep}(G)$ denote the category of complex admissible representations of $G$. Let $M$ be the Levi subgroup of a parabolic subgroup $P$ of $G$. We recall two functors associating representations of $M$ and $G$:

- For a representation $\Omega$ of $M$, we denote the normalized parabolic induction of $\Omega$ to $G$ by $i^G_M \Omega$. In particular, $i^G_M : \text{Rep}(M) \to \text{Rep}(G)$.

- For a representation $\pi$ of $G$, we denote the normalized Jacquet functor of $\pi$ along $P$ by $r^G_M \pi$. In particular, $r^G_M : \text{Rep}(G) \to \text{Rep}(M)$.

These two functors are adjoint to each other, namely, they satisfy the Frobenius reciprocity

$$\text{Hom}_G(\pi, i^G_M \sigma) \cong \text{Hom}_M(r^G_M \pi, \sigma).$$

Note that for a 1-dimensional representation $\Omega$ of $M$,

$$r^G_M \Omega = \iota_M(\Omega) - \rho^M_B \cap M.$$
In most of this work, we consider representations of a group $H$ as elements in the Grothendieck ring $\mathcal{R}(H)$ of $H$. In particular, we write $\pi = \pi_1 + \pi_2$ if, for any irreducible representation $\rho$ of $H$,

$$\text{mult}(\rho, \pi) = \text{mult}(\rho, \pi_1) + \text{mult}(\rho, \pi_2).$$

Here, $\text{mult}(\rho, \pi)$ denotes the multiplicity of $\rho$ in the Jordan–Hölder series of $\pi$. Furthermore, we write $\pi \leq \pi'$ if, for any irreducible representation $\rho$ of $H$,

$$\text{mult}(\rho, \pi) \leq \text{mult}(\rho, \pi').$$

**Remark 2.1:** Both $\mathbf{X}(M) \subset \text{Rep}(M)$ and $\mathcal{R}(M)$ are commutative groups of interest here. Furthermore, there is a natural map

$$\text{Rep}(M) \longrightarrow \mathcal{R}(M).$$

Therefore, elements of $\mathbf{X}(M)$ can be identified with elements in $\mathcal{R}(M)$. However, this is not a group homomorphism. Namely, the sum of two characters in $\mathbf{X}(M)$ is a character in $\mathbf{X}(M)$, while a sum of two characters in $\mathcal{R}(M)$ is a two-dimensional representation of $M$.

Hence, in order to avoid confusion, given $\Omega \in \text{Rep}(M)$ we will write $[\Omega]$ for the image of $\Omega$ in $\mathcal{R}(M)$. In that spirit, we have:

- $\Omega_1 + \Omega_2 \in \mathbf{X}(M)$ is the product of $\Omega_1$ and $\Omega_2$ in $\mathbf{X}(M)$, namely,

$$ (\Omega_1 + \Omega_2)(m) = \Omega_1(m)\Omega_2(m). $$

- $n \cdot \Omega \in \mathbf{X}(M)$ is the character given by

$$ (n \cdot \Omega)(m) = \Omega(m^n). $$

- $-\Omega$ is the inverse of $\Omega \in \mathbf{X}(M)$.

However, on the other hand:

- $[\Omega_1] + [\Omega_2]$ is the class, in $\mathcal{R}(M)$, of the direct sum $\Omega_1 \oplus \Omega_2$.
- $n \times [\Omega] \in \mathcal{R}(M)$ is the class in $\mathcal{R}(M)$ of $\bigoplus_{i=1}^{n} \Omega$.
- $-[\Omega]$, with $\Omega \in \text{Rep}(M)$, is a virtual representation.

We recall the following fundamental result (see [5, lem. 2.12], [6, Theorem 6.3.6]):

**Lemma 2.2 (Geometric Lemma):** For Levi subgroups $L$ and $M$ of $G$, let

$$W^{M,L} = \{ w \in W \mid w(\Phi^+_M) \subset \Phi^+, w^{-1}(\Phi^+_L) \subset \Phi^+ \}$$

be the set of shortest representatives in $W$ of $W_L \backslash W / W_M$. 

For a smooth representation $\Omega$ of $M$, the Jacquet functor $r^G_L i^G_M \Omega$, as an element of $\mathcal{R}(L)$, is given by
\begin{equation}
[r^G_L i^G_M \Omega] = \sum_{w \in W^{M,L}} [i^L_L \circ w \circ r^M_M \Omega],
\end{equation}
where, for $w \in W^{M,L}$,
\begin{align*}
M' &= M \cap w^{-1}(L), \\
L' &= w(M) \cap L.
\end{align*}

In particular, for an admissible representation $\Omega$ of $M$, the Jacquet functor $r^M_T \Omega$, as an element of $\mathcal{R}(T)$, is a finite sum of one-dimensional representations of $T$. Each such representation is called an exponent of $\Omega$. If $\lambda = r^M_T \Omega$ is one-dimensional, then $\lambda$ is said to be the leading exponent of $i^G_M \Omega$.

Remark 2.3: It should be noted that the leading exponent of $\pi = i^G_M \Omega$ depends on the particular realization of $\pi$ and not the equivalence class of $\pi$.

For example, given $\lambda \in X(T)$ such that $i^G_T \lambda$ is irreducible, it follows that
\begin{equation}
i^G_T \lambda \cong i^G_T w \cdot \lambda
\end{equation}
for all $w \in W$. In particular, $\lambda$ is the leading exponent of $i^G_T \lambda$ but not of $w \cdot \lambda$ when $\lambda \neq w \cdot \lambda$.

Let $G$ be a simple group, so $A_G$ is finite. Let $M = M_{\Delta \setminus \{\alpha\}}$ denote the Levi subgroup of a standard maximal parabolic subgroup $P$ of $G$. We can identify $X(M)$ with $X(A_M)$ and $X(G_m(k))$. In particular, by Equation (2.3), any element in $X(M)$ is of the form
\begin{equation}
\Omega_{M,\chi,s} = (s + \chi) \circ \omega_\alpha,
\end{equation}
where $s \in \mathbb{C}$ and $\chi \in X(G_m(k),0)$ as in Equation (2.1). We denote
\begin{equation}
\chi_{M,s} = r^M_T \Omega_{M,\chi,s}
\end{equation}
and
\begin{equation}
I_P(\chi,s) = i^G_M \Omega_{M,\chi,s}.
\end{equation}
It follows from Equation (2.4) that
\begin{equation}
I_P(\chi,s) \hookrightarrow i^G_T \chi_{M,s}.
\end{equation}

In this paper we study the degenerate principal series representations of a simply-connected, simple, split group $G$ of type $E_6$. Namely, we study the
reducibility of $I_P(\chi, s)$, when $P$ is a maximal parabolic subgroup of $G$. Since the algorithm described in Section 3 is applicable to other groups, we postpone the discussion on the structure of this group to Section 4.

3. Method outline

In this section we explain the algorithm used to compute the reducibility of degenerate principal series and the mathematical background behind it.

Throughout this section, we fix $G$. For a maximal parabolic subgroup $P$ of $G$ with Levi subgroup $M = M_\alpha$, $s \in \mathbb{C}$ and $\chi \in \mathbf{X}_{M,0} \cong \mathbf{X}_{G_m(k),0}$, let $\pi = I_P(\chi, s)$. For any such triple $(P, s, \chi)$ we wish to determine the reducibility of $I_P(\chi, s)$ and the structure of its maximal semi-simple subrepresentation and quotient.

The algorithm we implement for doing this has four parts:

1. For regular $I_P(\chi, s)$ determine whether $I_P(\chi, s)$ is reducible or not. This part is completely determined by the results of [6, 5].
2. Apply various reducibility criteria for non-regular $I_P(\chi, s)$.
3. Apply an irreducibility test for non-regular $I_P(\chi, s)$ by applying a chain of branching rules (which will be introduced in Subsection 3.3).
4. For a reducible representation, determine its maximal semi-simple subrepresentation and quotient.

For the convenience of the reader, at the end of each subsection, we outline the way in which we implemented the relevant calculations.

Also, we wish to point out that, as it turns out, the results of the following calculations depend on $\text{ord}(\chi)$ but not the particular choice of $\chi$.

3.1. Part I—Regular Case. We say that $\lambda \in \mathbf{X}(T)$ is regular if

$$\text{Stab}_W(\lambda) = \{e\}.$$ 

By abuse of notations, we say that $I_P(\chi, s)$ and $\Omega_{M,\chi,s}$ are regular if $\chi_{M,s} = r_\mathcal{C}^G \Omega_{M,\chi,s}$ is.

Remark 3.1: We note the following facts regarding stabilizers of elements in $\mathbf{X}(M)$:

- The stabilizer of a character $\lambda \in \mathfrak{a}^*_{T,\mathbb{R}}$ is generated by the reflections $s_\beta$ for $\beta \in \Phi$ orthogonal to $\lambda$. 

• If $I_P(\chi, s)$ is non-regular, then the imaginary part must satisfy $\text{Im}(s) \in 2\pi \cdot \mathbb{Q}$. Namely, $\text{Im}(s)$ (as an element of $X(\mathbb{G}_m(k))$) has finite order. In particular, absorbing the imaginary part into $\chi$, we may assume that $s \in \mathbb{R}$.
• Since $\text{Stab}_W(\chi_{M,s}) = \text{Stab}_W(\text{Re}(\chi_{M,s})) \cap \text{Stab}_W(\text{Im}(\chi_{M,s}))$, it follows that if $I_P(\chi, s)$ is non-regular, then $\text{Stab}_W(\text{Re}(\chi_{M,s}))$ is non-trivial.

If $w \in \text{Stab}_W(\chi_{M,s})$, then
$$\chi \circ (w \cdot \omega_{\alpha} - \omega_{\alpha}) \equiv 1.$$ Writing
$$\omega_{\alpha} - w \cdot \omega_{\alpha} = \sum_{\beta \in \Delta} n_{\beta} \omega_{\beta},$$
where $M = M_{\alpha}$, one gets
$$\chi \circ \left( \sum_{\beta \in \Delta} n_{\beta} \omega_{\beta} \right) = \sum_{\beta \in \Delta} \chi n_{\beta} \circ \omega_{\beta} = 0.$$ In particular, $\chi n_{\beta} = 1$ for all $\beta \in \Delta$.

• Given $M$ and $s \in \mathbb{R}$, $\text{Stab}_W(\chi_{M,s})$ depends only on the order of $\chi$.
• Note that there are only finitely many triples $(P, \chi, s)$ such that $I_P(\chi, s)$ is non-regular.

For $\pi = I_P(\chi, s)$ and a Levi subgroup $L$ of $G$ we let its **Bernstein–Zelevinsky set** be

$$BZ_L(\pi) = \{ i_{L'}^L \circ w \circ \tau | w \in W^{M,L}, M' = M \cap w^{-1}(L), L' = w(M) \cap L, \tau \leq r_{M'}^M(\Omega_{M,\chi,s}) \text{ is irreducible} \}.$$ (3.1)

We quote a corollary of [11, Theorem 3.1.2] for this case.

**Corollary 3.2:** The following are equivalent for a regular $\pi = I_P(\chi, s)$:

• $\pi$ is irreducible.
• $\sigma$ is irreducible for any $\beta \in \Delta$ and $\sigma \in BZ_{L_{\beta}}(\pi)$.

For a Levi subgroup $L_{\alpha}$ of semi-simple rank 1 and an element $w \in W^{M,L_{\beta}}$, the Levi factor $L'_{\beta}$ of $L_{\beta}$ is either $T$ or $L_{\beta}$. The Levi factor $M'$ of $M$ is either $T$ or $w^{-1}(L_{\beta})$. More precisely:

• If $w^{-1}(\beta) \in \Phi_M$ then $L'_{\beta} = L_{\beta}$ and $M' = w^{-1}(L_{\beta})$.
• If $w^{-1}(\beta) \notin \Phi_M$ then $L'_{\beta} = T$ and $M' = T$.

We recall that, for a regular $\mu \in X(T)$, $i_T^{L_{\alpha}} \mu$ is reducible if and only if $\langle \mu, \alpha^\vee \rangle = \pm 1$ since $L_{\alpha}$ has semi-simple rank 1.
Calculating non-regular points. In order to find all non-regular $I_P(\chi, s)$ we proceed with the following steps:

- We start by making a list $X$ of all values of $s$ such that $\text{Stab}_W(\text{Re}(\chi_{M,s}))$ is non-trivial. We do this by listing all $s \in \mathbb{R}$ such that

\[ \text{Re}(\langle \chi_{M,s}, \beta^\vee \rangle) = 0 \]

for some $\beta \in \Phi^+$. 

- We then make a list $Y$ of all values $m \in \mathbb{N}$ such that $\text{Stab}_W(\text{Im}(\chi_{M,s}))$ is non-trivial. For any $w \in W^M, T$, we write

\[ \omega_\alpha - w \cdot \omega_\alpha = \sum_{\beta \in \Delta_G} n_\beta \omega_\beta \]

and let $m_0 = \gcd(n_\beta)_{\beta \in \Delta_G}$. We list all the positive divisors $m$ of $m_0$.

- For all candidates $(s, m) \in X \times Y$, we check if $\chi_{M,s}$ is non-regular, where $\chi$ is of order $m$. 
  - Let $w \in W$ such that $w \cdot \text{Re}(\chi_{M,s})$ is anti-dominant and denote $\lambda_{a.d.} = w \cdot \chi_{M,s}$.
  - The stabilizer of $\text{Re}(\lambda_{a.d.})$ is generated by the simple reflections in

\[ \{ s_\beta \mid \beta \in \Delta, \langle \text{Re}(\lambda_{a.d.}), \beta^\vee \rangle = 0 \} \]

- The imaginary part of $\lambda_{a.d.}$ is given by $\text{Im}(\lambda_{a.d.}) = \chi \circ (w \cdot \omega_\alpha)$.
  - If there exists a non-trivial $u \in \text{Stab}_W(\text{Re}(\chi_{M,s}))$ such that

\[ \chi \circ (w \cdot \omega_\alpha) = \chi \circ (uw \cdot \omega_\alpha), \]

then $I_P(\chi, s)$ is non-regular.

Calculating the regular points of reducibility. First, given $\alpha \in \Delta$ and $\chi$ of order $m$, we calculate all points $s \in \mathbb{R}$ where $BZ_{L_\beta}(\pi)$ is reducible, where

\[ \pi = i^G_M \Omega_{M,\chi,s}. \]

By the above discussion, it is given by

\[ \{ s \in \mathbb{R} \mid \exists \beta \in \Phi_G^+ \setminus \Phi_M^+ : \langle \chi_{M,s}, \beta^\vee \rangle = \pm 1 \}. \]

The list of reducible regular degenerate principal series is given by the intersection between this set and the set of $s \in \mathbb{R}$ such that $I_P(\chi, s)$ is regular.

We now wish to determine whether a non-regular $I_P(\chi, s)$ is reducible.
3.2. Part II—Reducibility Tests. We consider the following reducibility criterion ([20, Lemma 3.1]).

**Lemma 3.3 (RC):** Let $\pi = i_M^\mathbf{G}\Omega$. Assume there exist smooth representations $\Pi$ and $\sigma$ of $\mathbf{G}$ of finite length and a Levi subgroup $L$ of $\mathbf{G}$ such that:

1. $\pi \leq \Pi, \sigma \leq \Pi$.
2. $r^G_L\pi + r^G_L\sigma \not\leq r^G_L\Pi$.
3. $r^G_L\pi \not\leq r^G_L\sigma$.

Then $\pi$ is reducible and admits a common irreducible subquotient with $\sigma$.

We also have:

**Lemma 3.4:** For $\lambda_{a.d.} \in \mathbf{W} \cdot \chi_{M,s}$, such that $\mathfrak{Re}(\lambda_{a.d.})$ is anti-dominant, then
\[
\text{mult}(\lambda_{a.d.}, r^T_T\pi) = \# \text{Stab}_W(\lambda_{a.d.}).
\]

**Proof.** We recall an algorithm (see [14, Section 5.3.2]) to find all the reduced Weyl words $w \in \mathbf{W}$ such that, for a given $\lambda \in \mathfrak{a}^*_{T,\mathbb{R}}$, $w \cdot \lambda$ is in the closed fundamental Weyl chamber. Write $\lambda = \sum_{\beta \in \Delta} a_{\beta}\omega_{\beta}$. Choose $\beta \in \Delta$ such that $a_{\beta} < 0$ and apply the simple reflection $s_{\beta}$. Repeating this process will produce a reduced Weyl word $w = s_{\beta_k} \cdots s_{\beta_1}$ such that $w \cdot \lambda$ is dominant.

A similar approach can be used to find all Weyl elements $w \in \mathbf{W}$ so that $w \cdot \lambda$ is in the negative Weyl chamber (namely, anti-dominant).

We say that a Weyl element $w \in \mathbf{W}$ starts with $s_{\beta}$ if $\text{len}(ws_{\beta}) < \text{len}(w)$.

Write $\mathfrak{Re}(\chi_{M,s}) = \sum_{\beta \in \Delta} a_{\beta}\omega_{\beta}$. As $M = M_\alpha$, we have $a_{\beta} = -1$ for all $\beta \neq \alpha$. Assume that $a_\alpha > 0$. By the above-mentioned algorithm, any Weyl word $w \in \mathbf{W}$ such that $w \cdot \mathfrak{Re}(\chi_{M,s})$ is anti-dominant starts with $s_\alpha$, as $a_\alpha > 0$, and does not start with $s_{\beta}$ for $\beta \neq \alpha$ since $a_{\beta} < 0$. On the other hand,
\[
W^{M,T} = \{ w \in \mathbf{W} \mid l(ws_{\beta}) > l(w) \ \forall \beta \in \Delta \setminus \{\alpha\}\}.
\]

It follows that
\[
\{ w \in \mathbf{W} \mid w \cdot \mathfrak{Re}(\chi_{M,s}) \text{ anti-dominant} \} \subset W^{M,T}.
\]

If $a_\alpha = 0$, then $\chi_{M,s}$ is anti-dominant and
\[
\{ w \in \mathbf{W} \mid w \cdot \mathfrak{Re}(\chi_{M,s}) \text{ anti-dominant} \} \subset \{ e, s_\alpha \} \subset W^{M,T}.
\]

If, on the other hand, $a_\alpha < 0$, then $\chi_{M,s}$ is regular and anti-dominant. Hence,
\[
\{ w \in \mathbf{W} \mid w \cdot \mathfrak{Re}(\chi_{M,s}) \text{ anti-dominant} \} = \{ e \} \subset W^{M,T}.
\]
Remark 3.5: Given two degenerate principal series $i_M^G \Omega$ and $i'_M^G \Omega'$ such that $W \cdot r_T^M \Omega \cap W \cdot r_T^{M'} \Omega' \neq \emptyset$, it follows that

$$W \cdot r_T^M \Omega = W \cdot r_T^{M'} \Omega'.$$

Furthermore, Lemma 3.4 implies that for any anti-dominant $\lambda_{a.d.} \in r_G^Ti_M^G \Omega$ we have

$$\text{mult}(\lambda_{a.d.}, r_G^Ti_M^G \Omega) = \text{mult}(\lambda_{a.d.}, r_G^Ti'_M^G \Omega') = \# \text{Stab}_W(\lambda_{a.d.}).$$

Furthermore, assumption (2) in Lemma 3.3 holds automatically for $\pi = i_M^G \Omega$, $\sigma = i'_M^G \Omega'$ and $\Pi = \text{Ind}^G_{B} \chi_M s$.

Namely, $\pi$ and $\sigma$ share a common subquotient.

In particular, if the inducing data is unramified, $\pi$ and $\sigma$ both contain the unique irreducible spherical subquotient in $\Pi$.

For a given non-regular representation $I_P(\chi, s)$, let $v_{P, \chi, s}$ denote the set of all $w \cdot \chi_M s$, for $w \in W$ such that $\Re(w \cdot \chi_M s)$ is anti-dominant. We call the elements of $v_{P, \chi, s}$ the anti-dominant exponents of $I_P(\chi, s)$.

**IMPLEMENTATION.**

Implementing the reducibility criterion for non-regular points. We implement the reducibility test in two steps:

1. Consider two triples $(P, \chi, s)$ and $(P', \chi', s')$ such that $I_P(\chi, s)$ and $I_{P'}(\chi', s')$ share an anti-dominant exponent $\lambda_{a.d.}$. In order to show that $\pi$ is reducible, it will be enough to show that

$$\pi = I_P(\chi, s), \quad \sigma = I_{P'}(\chi', s') \quad \text{and} \quad \Pi = \text{Ind}^G_{B} \chi_M s$$

satisfy the conditions in (RC). Conditions (1) and (2) automatically follow from Equation (2.7) and Lemma 3.4. Condition (3) can be verified using the following methods:

- If $\# W(G, P) > \# W(G, P')$, (3) holds.
- If $\text{mult}(\chi_M s, r_G^P \pi) > \text{mult}(\chi_M s, r_G^P \sigma)$, (3) holds.
- If $\sum_{W_M, T} w \circ \chi_M s \not\subseteq \sum_{W_{M'}, T} w \circ \chi_{M'} s'$, (3) holds.

We note here that these tests could be performed to check the reducibility of $I_P(\chi, s)$ and $I_{P'}(\chi', s')$ simultaneously. Also, we have performed these tests in this order, as each test requires more computations than the previous one.
(2) For triples \((P, \chi, s)\) such that reducibility could not be verified using the previous comparisons and irreducibility could not be verified either (see Subsection 3.3), we performed the following reducibility test.

Given \(\lambda \in W \cdot \chi_{M,s}\) and

\[\Theta = \{ \beta \in \Delta \mid \langle \lambda, \beta^\vee \rangle = -1\},\]

\(\lambda = r_T^{M_\Theta} \Omega',\) with \(\Omega' \in X(M_\Theta)\). If the conditions of (RC) hold with respect to

\[\pi = I_P(\chi, s), \quad \sigma = i_{M_\Theta}^G \Omega' \quad \text{and} \quad \Pi = \text{Ind}_B^G \chi_{M,s},\]

then it follows that \(\pi\) is reducible.

We note here that usually, in this case, there is a maximal Levi subgroup \(M'\) of \(G\) such that \(i_{M_\Theta}^{M'} \Omega'\) is an irreducible degenerate principal series of \(M'\).

Remark 3.6: For computational reasons, we applied step (2) only after the irreducibility test introduced in Subsection 3.3.

3.3. PART III—IRREDUCIBILITY TEST. After applying the reducibility test (1) explained above, one is left with a list of triples \((P, \chi, s)\) such that \(I_P(\chi, s)\) is expected to be irreducible. We now explain the method we used to test irreducibility of such representations.

The main idea we use to show that \(\pi = I_P(\chi, s)\) is irreducible goes as follows:

- Let \(v \leq r_T^G \pi\) be an exponent such that \(\Re(v)\) is anti-dominant and let \(\pi' \leq \pi\) be an irreducible representation such that \(v \leq r_T^G \pi'\).
- The Jacquet functor is exact and hence \(r_T^G \pi' \leq r_T^G \pi\).
- If one can show that \(r_T^G \pi' = r_T^G \pi\) (for example, by showing that \(\dim_C r_T^G \pi' \geq \#W(G, P) = \dim_C r_T^G \pi\)), then \(\pi' = \pi\) is irreducible.

Indeed, if \(r_T^G \pi = r_T^G \pi'\) and \(\pi = \pi' + \pi''\), then \(\pi''\) is both cuspidal and has cuspidal support along \(T\) and hence \(\pi'' = 0\). It follows that \(\pi = \pi'\) is irreducible.

In order to prove that \(r_T^G \pi' = r_T^G \pi\) we use branching rules.

Branching Rules. Let \(L\) be a Levi subgroup of \(G\) and \(\lambda \leq r_T^G \pi'\) such that there exists a unique irreducible representation \(\sigma\) of \(L\) such that \(\lambda \leq r_L^L \sigma\). Then the following hold:
• The multiplicity \( \text{mult}(\lambda, r_T^L\sigma) \) divides the multiplicity \( \text{mult}(\lambda, r_T^G\pi') \).

• If \( n = \frac{\text{mult}(\lambda, r_T^G\pi')}{\text{mult}(\lambda, r_T^L\sigma)} \) then \( n \times [r_T^L\sigma] \leq [r_T^G\pi'] \).

Proof. Indeed, write
\[
[r_T^G\pi'] = \sum_{i=1}^{k} n_i \times [\sigma_i],
\]
where \( \sigma_i \) are disjoint irreducible representations of \( L \). Since \( r_T^G = r_T^L r_T^G \) it follows that
\[
[\lambda] \leq [r_T^G\pi'] = \sum_{i=1}^{k} n_i \times [r_T^L\sigma_i].
\]
Without any loss of generality, \( \lambda \leq r_T^L\sigma_1 \). Since \( \sigma \) is the unique irreducible representation of \( L \) such that \( \lambda \leq r_T^L\sigma \) then:

• \( \sigma_1 = \sigma \).
• \( \lambda \not\leq r_T^L\sigma_i \) for \( i > 1 \).
• \( \text{mult}(\lambda, r_T^G\pi') = n_1 \cdot \text{mult}(\lambda, r_T^L\sigma) \).
• \( n_1 \times [r_T^L\sigma] \leq r_T^G\pi' \). \[\square\]

A list of branching rules, which were used by us in Section 4, can be found in Appendix A. These branching rules are associated with Levi subgroups \( M \) such that \( M^{\text{der}} \) is of type \( A_1, A_2, A_3 \) or \( D_4 \).

Implementation.

Implementing the irreducibility criterion for non-regular points. We start by choosing an anti-dominant exponent of \( I_P(\chi, s) \). Namely, we choose an anti-dominant \( \lambda_{a.d.} \) in \( r_T^G(I_P(\chi, s)) \). Also, let \( \pi' \leq \pi \) be an irreducible subquotient such that \( \lambda_{a.d.} \leq r_T^G\pi' \).

We then create a list of pairs \([\lambda, n_\lambda]\), where \( \lambda \) is an exponent of \( I_P(\chi, s) \) and \( n_\lambda \) is a lower bound on the multiplicity of \( \lambda \) in \( \pi' \). For any \( \lambda \) in the list we check which of the branching rules (see Appendix A) can be applied to \( \lambda \) and update the bounds \( n_\lambda \) accordingly. The process is terminated when either
\[
\sum_{\lambda} n_\lambda \times [\lambda] = r_T^G\pi,
\]
in which case \( \pi = \pi' \) is irreducible, or no more branching rules can be further applied, in which case the algorithm is inconclusive.
Remark 3.7: In Appendix we give an example of a proof of irreducibility using this algorithm.

Remark 3.8: It follows from Equation (A.1) that, if $\chi = 1$, then $\pi'$ is the unique subquotient of $\pi$ with that property and that

$$|\text{Stab}_W(\lambda_{a.d.})| \times \lambda_{a.d.} \leq r_T^G \pi'.$$

In that case, $\pi'$ is the unique spherical subquotient of $i_T^G \lambda_{a.d.}$.

3.4. Part IV—Irreducible Quotients and Subrepresentations. We now describe the tools used to calculate the maximal semi-simple subrepresentation of $\pi = i_M^G \Omega$ and its maximal semi-simple quotient.

First, note that, by contragredience, it is enough to find its maximal semi-simple subrepresentation. Furthermore, note the following bound on its length:

**Lemma 3.9:** For $\Omega = \Omega_{M,X,s}$ and $\pi = i_M^G(\Omega)$, the number of irreducible subrepresentations of $\pi$ is bounded by

$$\text{mult}(\chi_{M,s}, r_T^G \pi').$$

**Proof.** We prove that for any irreducible $\pi'$ such that

$$\pi' \hookrightarrow \pi = i_M^G \Omega$$

then $\chi_{M,s} \leq r_T^G \pi'$. Note that

$$\pi' \hookrightarrow i_M^G \Omega \hookrightarrow i_T^G \chi_{M,s}.$$

Hence, $\pi'$ is a subrepresentation of $i_T^G \chi_{M,s}$.

On the other hand, by Frobenius reciprocity,

$$\text{Hom}_G(\pi', i_T^G \chi_{M,s}) \cong \text{Hom}_T(r_T^G \pi', \chi_{M,s}).$$

Since the left-hand side is non-trivial, so is the right-hand side. It follows that

$$\chi_{M,s} \leq r_T^G \pi'.$$

The claim then follows. 


Corollary 3.10: If \( \text{mult}(\chi_{M,s}, r^G_T \pi) = 1 \), then \( \pi = i^G_M(\Omega) \) admits a unique irreducible subrepresentation, which appears in \( \pi \) with multiplicity one.

In particular, this holds when \( \Omega = \Omega_{M,\chi,s} \) is regular.

We now describe the various methods used by us for calculating the maximal semi-simple subrepresentation of \( \pi \).

Case I: By Corollary 3.10 If \( \text{mult}(\chi_{M,s}, r^G_T \pi) = 1 \), then \( \pi = i^G_M(\Omega) \) admits a unique irreducible subrepresentation, \( \pi_0 \), which appears in \( \pi \) with multiplicity one.

Remark 3.11: We note here that in all of the relevant cases, if \( s > 0 \), then \( \text{mult}(\chi_{M,s}, r^G_T \pi) = 1 \) so \( I_P(\chi, s) \) admits a unique irreducible subrepresentation. This, in fact, follows directly from [3, Theorem 6.3].

It is thus enough to consider \( s \leq 0 \).

Case II: Assume that \( m = \text{mult}(\chi_{M,s}, r^G_T \pi) \) and that, using a sequence of branching rules, one can show that there exists an irreducible representation \( \pi_0 \leq \pi \) of \( G \) such that

\[
m \times [\chi_{M,s}] \leq [r^G_T \pi_0].
\]

Then, \( \pi = i^G_M(\Omega) \) admits a unique irreducible subrepresentation. This subrepresentation, \( \pi_0 \), appears with multiplicity 1 in \( \pi \) as

\[
\text{mult}(\chi_{M,s}, r^G_T \pi) = \text{mult}(\chi_{M,s}, r^G_T \pi_0).
\]

We now describe one more possible case, which occurred in the course of this work. Due to the scarcity of its use, we did not implement it as part of our algorithm. The triples \( (M, s, \chi) \) in which these cases were relevant will be further discussed in Section 4.

Case III: Let:

- \( M' \) be a Levi subgroup of \( G \).
- \( M'' = M \cap M' \).
- \( \Omega'' = r^M_{M''} \Omega \)
- Assume that \( i^M_{M''} \Omega'' = \bigoplus \sigma_i \) is semi-simple.
By induction in stages, it follows that
\[ i^G_M \Omega \hookrightarrow i^G_M (i^M_M' \Omega'') \]
\[ \cong i^G_M' \Omega'' \]
\[ \cong i^G_M' (i^M_M'' \Omega'') = \bigoplus i^G_{M'} \sigma_i. \]

Furthermore, assume that each of the $i^G_{M'} \sigma_i$ admits a unique irreducible subrepresentation $\pi_i$. It follows that the maximal semi-simple subrepresentation of $\pi$ is a subrepresentation of $\bigoplus \pi_i$. It remains to determine which of the $\pi_i$-s is a subrepresentation of $\pi$ and the equivalencies between them. Both of these can often be done by a comparison of Jacquet modules and multiplicities of certain exponents in them.

We finish this section by noting a useful fact, which other sources sometimes refer to as a central character argument. This is, in a sense, a counterpart of Lemma 3.9.

**Lemma 3.12:** Let $\pi'$ be a smooth irreducible representation of $G$. If $\lambda \leq r^G_T \pi'$ then
\[ \pi' \hookrightarrow i^G_T \lambda. \]

**Proof.** Let $\rho = r^G_T \pi'$ and for any $\omega : T \to \mathbb{C}^\times$ let
\[ V_{\omega,n} = \{ v \in V \mid (\rho(t) - \omega(t))^n v = 0 \ \forall t \in T \}, \]
where $V$ denotes the underlying (finite-dimensional) vector space of $\rho$. Furthermore, let
\[ V_{\omega,\infty} = \bigcup_{n \geq 1} V_{\omega,n}, \quad V_\omega = V_{\omega,1}. \]
From [6, Proposition 2.1.9] (note that $Z_T = T$) it follows that
\[ V = \bigoplus_{\omega : T \to \mathbb{C}^\times} V_{\omega,\infty}, \tag{3.2} \]
where $V_{\omega,\infty} \neq \{0\}$ for only finitely many $\omega$. Namely,
\[ \rho = \bigoplus_{i \in I} V_{\omega_i,\infty}, \]
for a finite set $I$.

By Frobenius reciprocity,
\[ \text{Hom}_G(\pi', i^G_T \lambda) \cong \text{Hom}_T(r^G_T \pi', \lambda) = \text{Hom}_T(\rho, \lambda). \]
Since \( \lambda \leq \rho \), it follows that \( V_\lambda \neq \{0\} \). By Equation (3.2),
\[
\text{Hom}_T(r^G_T \pi, \lambda) \cong \bigoplus_{i \in I} \text{Hom}_T(\rho_i, \infty, \lambda)
\cong \text{Hom}_T(\rho_{\lambda, \infty}, \lambda) \neq \{0\}.
\]

We conclude that \( \text{Hom}_G(\pi, i^G_T \lambda) \neq \{0\} \) and hence \( \pi \hookrightarrow i^G_T \lambda \).

**Implementation.**

*Implementing Case I.* This only requires calculating \( \text{mult}(\chi_{M,s}, r^G_T \pi) \) and showing that it is 1. If that is not the case, we move to the next cases.

*Implementing Case II.* This calculation follows the ideas of Subsection 3.3. Here we do not aim to show that \( \sum_\lambda n_\lambda \times [\lambda] = [r^G_T \pi] \); rather we aim to show that \( n_{\chi_{M,s}} = \text{mult}(\chi_{M,s}, r^G_T \pi) \).

As in the irreducibility criterion, we start with an anti-dominant exponent \( \lambda_{a.d.} \leq r^G_T \pi \) and a subquotient \( \pi_0 \) of \( \pi \) such that \( \lambda_{a.d.} \leq r^G_T \pi_0 \). We then apply branching rules on \( \lambda_{a.d.} \) and attempt to prove that \( n_{\chi_{M,s}} = \text{mult}(\chi_{M,s}) \), in which case \( \pi_0 \) is the unique irreducible subrepresentation of \( \pi \).

This computation would terminate either if it found that \( n_{\chi_{M,s}} = \text{mult}(\chi_{M,s}, r^G_T \pi) \) or if no branching rule can be further applied, in which case this test is inconclusive.

4. Degenerate principal series representations of \( E_6 \)

Let \( G \) be the simple, split and simply-connected \( F \)-group of type \( E_6 \).

In this section, we describe the structure of the degenerate principal series of \( G \) using the algorithm described in Section 3. Namely, we determine the following:

- All regular reducible degenerate principal series \( I_P(\chi, s) \) of \( G \).
- All non-regular degenerate principal series \( I_P(\chi, s) \) of \( G \).
- For each non-regular \( I_P(\chi, s) \), we determine whether it is reducible or irreducible.
- For reducible \( I_P(\chi, s) \) we determine its maximal semi-simple subrepresentation and quotient. In fact, we show that, with only one exception, all \( I_P(\chi, s) \) admit a unique irreducible quotient and a unique irreducible subrepresentation.
4.1. The Exceptional Group of Type $E_6$. We start by describing the structure of $G$. We consider $G$ as a simply-connected Chevalley group of type $E_6$ (see [18, p. 21]). Namely, we fix a Borel subgroup $B$ whose Levi subgroup is a torus $T$. This gives rise to a set of 72 roots $\Phi$, containing a set of simple roots

$$\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}.$$ 

The Dynkin diagram of type $E_6$ is

$$\begin{array}{cccccc}
\alpha_1 & - & \alpha_2 & - & \alpha_3 & - \\
& & & & & \\
\alpha_4 & - & \alpha_5 & - & \alpha_6 & \\
\end{array}$$

The group $G$ is generated by the symbols

$$\{x_\alpha(r) \mid \alpha \in \Phi, r \in F\}$$

subject to the Chevalley–Steinberg relations (see [18, p. 66]).

We record the isomorphism classes of standard Levi subgroups $M_\Theta$ of $G$, with $\Theta$ inducing a connected sub-Dynkin diagram, in the following lemma.

**Lemma 4.1:** Let $\Theta \subset \Delta$ induce a connected sub-diagram. Then, $\Theta$ is either $A_n$ (with $1 \leq n \leq 5$), $D_4$ or $D_5$. Furthermore, $M_\Theta$ is classified as follows:

- If $\Theta$ is of type $A_n$, with $n \leq 4$, then $M_\Theta \cong GL_{n+1} \times GL_5^{-n}$.
- If $\Theta$ is of type $D_5$, then $M_\Theta \cong GSpin_{10}$.
- If $\Theta$ is of type $D_4$, then $M_\Theta \cong GL_1 \times GSpin_8$.
- If $\Theta$ is of type $A_5$, then

$$M_\Theta = M_2 \cong GL_6(k)^0 = \{g \in GL_6(k) \mid \det g \in (k^\times)^2\}.$$ 

**Proof.** If $\Theta$ is of type $D_5$, then this is the Levi subgroup of the parabolic subgroup of $G$ whose nilradical is Abelian, considered in [22]. If $\Theta$ is of type $D_4$ or $A_n$ with $n \leq 4$, this follows from [11] together with the fact that $\Theta$ is contained in a sub-Dynkin diagram of type $D_5$. If $\Theta$ is of type $A_5$, this follows from a direct calculation. □

Let $P_i$ denote the maximal standard parabolic subgroup associated to $

$$\Theta_i = \Delta \setminus \{\alpha_i\}.$$ 

Let $M_i = M_{\alpha_i} = M_{\Theta_i}$ denote the Levi subgroup of $P_i$. 

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Lemma 4.2: The maximal Levi subgroups of $G$ are given as follows:

$M_1 \cong M_6 \cong G \text{Spin}_{10}$.

$M_2 \cong \{g \in GL_6(k) \mid \det g \in (k^\times)^2\}$.

$M_3 \cong M_5 \cong \{(g_1, g_2) \in GL_2 \times GL_5 \mid \det g_1 = \det g_2\}$.

$M_4 \cong \{(g_1, g_2, g_3) \in GL_3(k) \times GL_2(k) \times GL_3(k) \mid \det g_1 = \det g_2 = \det g_3\}$.

Let $W$ denote the Weyl group of $G$ associated with $T$. We have that $|W| = 51,840$. For convenience, we write $s_i = s_{\alpha_i}$ for the simple reflections generating $W$ and we write $w_{i_1 i_2 \cdots i_k}$ for a Weyl word $s_{i_1} \cdots s_{i_k}$.

The Weyl group $W_{M_i}$ of the maximal Levi subgroup $M_i$ has the following cardinality:

$$|W_{M_i}| = \begin{cases} 
1,920, & \text{if } i = 1, 6, \\
720, & \text{if } i = 2, \\
240, & \text{if } i = 3, 5, \\
72, & \text{if } i = 4.
\end{cases}$$

Hence, the set $W^{M_i, T}$ of minimal representatives of $W/W_{M_i}$ has the following cardinality:

$$|W^{M_i, T}| = |W/W_{M_i}| = \frac{|W|}{|W_{M_i}|} = \begin{cases} 
27, & \text{if } i = 1, 6, \\
72, & \text{if } i = 2, \\
216, & \text{if } i = 3, 5, \\
720, & \text{if } i = 4.
\end{cases}$$

We note here that for a 1-dimensional representation $\Omega$ of $M_i$, $r_T^G i_{M_i}^G \Omega$ has dimension $|W^{M_i, T}|$.

We also note here that any $\lambda \in \mathbf{X}(T)$ can be written as the following combination:

$$\sum_{i=1}^6 \Omega_i \circ \omega_{\alpha_i}.$$ 

As a shorthand, we will write

$$(4.1) \quad \begin{pmatrix} \Omega_1 & \Omega_2 & \Omega_4 & \Omega_5 \\ \Omega_3 & \Omega_6 \end{pmatrix} = \sum_{i=1}^6 \Omega_i \circ \omega_{\alpha_i}.$$
Also, let 
\[
\begin{bmatrix}
\Omega_2 \\
\Omega_1 & \Omega_3 & \Omega_4 & \Omega_5 & \Omega_6
\end{bmatrix}
\]
denote the class in \( \mathcal{A}(T) \) of \((\Omega_1, \Omega_3, \Omega_4, \Omega_5, \Omega_6)\).

### 4.2. The Degenerate Principal Series of \(E_6\)

Before stating the results, we make a few general comments.

- By contragredience, it is enough to state the results for \( s \leq 0 \).
- For short, we write \([i, s, k]\) for any \( I_P(\chi, s) \) with \( P = P_i \) and \( \chi \) of order \( k \). In particular, the calculations below will demonstrate that only the order of \( \chi \), and not the particular \( \chi \) itself, will matter to the question of the reducibility/irreducibility of \( \pi \) and to the length of its maximal semi-simple subrepresentation.
- By Remark 3.11, \( I_P(\chi, s) \) admits a unique irreducible quotient when \( s < 0 \).
- By the action of the outer automorphism group, \( \mathbb{Z}/2\mathbb{Z} = \{1, \vartheta\} \) of \( \text{Dyn}(E_6) \), it is enough to consider the parabolic subgroups \( P_1, P_2, P_3 \) and \( P_4 \). The results for \( P_5 \) and \( P_6 \) are similar to those of \( P_1 \) and \( P_3 \), respectively.

**Theorem 4.3:** For any \( 1 \leq i \leq 4 \), all reducible regular \( I_P(\chi, s) \) and all non-regular \( I_P(\chi, s) \) are given in the following tables:

- For \( P = P_1 \) and \( P = P_6 \):

| ord(\chi) | \(-6\) | \(-5\) | \(-4\) | \(-3\) | \(-2\) | \(-1\) | \(0\) |
|------------|------|------|------|------|------|------|------|
| reg. red.  | non-reg. irr. | non-reg. irr. | non-reg. red. | non-reg. irr. | non-reg. irr. | non-reg. irr. |

- For \( P = P_2 \):

| ord(\chi) | \(-\frac{11}{2}\) | \(-\frac{9}{2}\) | \(-\frac{7}{2}\) | \(-\frac{5}{2}\) | \(-\frac{3}{2}\) | \(-\frac{1}{2}\) | \(0\) |
|------------|------|------|------|------|------|------|------|
| reg. red.  | non-reg. irr. | non-reg. irr. | non-reg. red. | non-reg. irr. | non-reg. irr. | non-reg. irr. |

| ord(\chi) | \(-\frac{11}{2}\) | \(-\frac{9}{2}\) | \(-\frac{7}{2}\) | \(-\frac{5}{2}\) | \(-\frac{3}{2}\) | \(-\frac{1}{2}\) | \(0\) |
|------------|------|------|------|------|------|------|------|
| reg. irr.  | reg. irr. | reg. irr. | reg. irr. | reg. irr. | reg. irr. | non-reg. irr. |
For $P = P_3$ or $P = P_5$:

| $\text{ord}(\chi)$ | $s$  | $-\frac{9}{2}$ | $-\frac{7}{2}$ | $-\frac{5}{2}$ | $-\frac{3}{2}$ | $-1$ | $-\frac{1}{2}$ | $0$ |
|---------------------|------|----------------|----------------|----------------|----------------|-----|--------------|-----|
|                     | 1    | reg. red.      | non-reg. red.  | non-reg. red.  | non-reg. irr.  | non-reg. irr. | non-reg. irr. |     |
|                     | 2    | reg. irr.      | reg. irr.      | reg. irr.      | non-reg. irr.  | non-reg. irr. | non-reg. irr. |     |

For $P = P_4$:

| $\text{ord}(\chi)$ | $s$  | $-\frac{7}{2}$ | $-\frac{5}{2}$ | $-\frac{3}{2}$ | $-1$ | $-\frac{1}{2}$ | $0$ |
|---------------------|------|----------------|----------------|----------------|-----|--------------|-----|
|                     | 1    | reg. red.      | non-reg. red.  | non-reg. red.  | non-reg. irr.  | non-reg. irr. | non-reg. irr. |
|                     | 2    | reg. irr.      | reg. irr.      | reg. irr.      | non-reg. irr.  | reg. irr.     | non-reg. irr. |
|                     | 3    | reg. irr.      | reg. irr.      | reg. irr.      | non-reg. irr.  | non-reg. irr. | reg. irr.     |

All of the above representations admit a unique irreducible subrepresentation and a unique irreducible quotient, with the exception of $[4, -\frac{1}{2}, 3]$, in which case $I_P(\chi, s)$ admits a unique irreducible quotient and a maximal semi-simple subrepresentation of the form $\sigma_1 \oplus \sigma_2 \oplus \sigma_3$ (where the $\sigma_i$ are inequivalent).

Furthermore, for all cases, any irreducible subrepresentation or quotient of $I_P(\chi, s)$ appears in $I_P(\chi, s)$ with multiplicity 1.

**Remark 4.4:** We note here that our results for the parabolic subgroup $P_1$ agree with the results of [22] (see page 297). In particular, the unique irreducible subrepresentation, $\Pi_{\text{min}}$, of $I_{P_1}(1, -3)$ is the minimal representation of $G$. By examining the anti-dominant exponent of $\Pi_{\text{min}}$, one sees that this is also the unique irreducible subrepresentation of $I_{P_2}(1, -\frac{7}{2})$, $I_{P_0}(1, -3)$ and $I_{P_4}(1, -\frac{5}{2})$. This fact gives rise to a Siegel–Weil identity between the relevant Eisenstein series at these points (see [10, p. 68]).
Proof. We deal with the question of reducibility separately from the calculation of maximal semi-simple subrepresentations.

Reducibility and Irreducibility

For most \( I_P(\chi, s) \), the algorithm provided in Section 3 suffices to yield the results stated in the theorem. In particular, reducibility of regular \( I_P(\chi, s) \) and irreducibility of non-regular \( I_P(\chi, s) \) is completely determined by the tools described there. The branching rules used in this proof are listed in Appendix A. See Appendix B for an example of a proof of the irreducibility of \( I_P(\chi, s) \), in the case \([1, -2, 1]\), using branching rules.

We note that in the case \([3, -\frac{3}{2}, 1]\), the branching rules supplied in Appendix did not cover all of \( r_I^G \pi \). One can still prove irreducibility as will be explained later.

For most non-regular reducible \( I_P(\chi, s) \), reducibility can be determined by (RC) (Lemma 3.3) by setting \( \Pi = i^G T \chi_M, s \) and \( \sigma \) being another degenerate principal series. In such a case, we quote one triple \([i, s, k]\) relevant for the proof in the following table. Usually, there is more than one such triple, in which case we recorded only one.

A few cases required the application of (RC) with respect to an induction from a non-maximal parabolic \( M_\Theta \) such that \( |\Theta| = 4 \). In such a case, we write \([i_1, i_2], [s_1, s_2], [k_1, k_2]\) to represent the induction from \( M = M_\Theta \) with \( \Theta = \Delta \setminus \{\alpha_{i_1}, \alpha_{i_2}\} \) with initial exponent

\[
- \left( \sum_{j \neq i_1, i_2} \omega_{\alpha_j} \right) + (s_1 + k_1 \cdot \chi) \circ \omega_{\alpha_{i_1}} + (s_2 + k_2 \cdot \chi) \circ \omega_{\alpha_{i_2}},
\]

where \( \chi \) is of order \( k \). In this case, again, there is more than one possible choice of data \([i_1, i_2], [s_1, s_2], [k_1, k_2]\) that will yield a proof of reducibility; we record only one such choice.

The reducibility in the remaining case, \([4, -\frac{1}{2}, 3]\), follows from the fact that, in that case, \( I_P(\chi, s) \) admits a maximal semi-simple subrepresentation of length 3 as will be shown later.

- For \( P = P_1 \):

| ord(\chi) | \(-6\) | \(-5\) | \(-4\) | \(-3\) | \(-2\) | \(-1\) | \(0\) |
|---------|-----|-----|-----|-----|-----|-----|-----|
|         |     |     |     |     |     |     |     |
• For $P = P_2$:

$$\begin{array}{cccccc}
\text{ord}(\chi) & s & -\frac{11}{2} & -\frac{9}{2} & -\frac{7}{2} & -\frac{5}{2} & -\frac{1}{2} \\
1 & & [1, -3, 1] & [1, 0, 1] & [1, 6, [1, 6, [0, 0]]] & \\
\end{array}$$

• For $P = P_3$:

$$\begin{array}{cccccc}
\text{ord}(\chi) & s & -\frac{9}{2} & -\frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & -1 & -\frac{1}{2} \\
1 & & [1, -4, 1] & [6, -1, 1] & [2, -\frac{1}{2}, 1] & \\
\end{array}$$

• For $P = P_4$:

$$\begin{array}{cccccc}
\text{ord}(\chi) & s & -\frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & -1 & -\frac{1}{2} & -\frac{1}{6} \\
1 & & [1, -3, 1] & [2, -\frac{1}{2}, 1] & [3, 0, 1] & [1, 5, [1, 1, 1, 0]] & \\
2 & & [3, 0, 2] & [3, 6, [1, 6, 1, 1]] & [0, 0] & \\
3 & & & & * & \\
\end{array}$$

It remains to explain how to prove the irreducibility of $\pi = I_P(\chi, s)$ in the case $[3, -\frac{1}{2}, 1]$. The anti-dominant exponent in this case is given by

$$\lambda_{a.d.} = \begin{pmatrix}
-1 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.$$ 

As mentioned above, applying the branching rules from Appendix A on exponents, starting with $\lambda_{a.d.}$, does not result in the full Jacquet module $r_T^G \pi$. Still, irreducibility follows from that calculation, as will be explained now. Let

$$\lambda_0 = \begin{pmatrix}
-1 & 0 \\
-1 & 3 & -1 & 1 & -1
\end{pmatrix}, \quad \lambda_1 = \begin{pmatrix}
-1 & -1 & -1 & 4 & -1
\end{pmatrix}$$

denote the leading exponents of $\pi$ and its image $\hat{\pi} = I_{P_5}(1, \frac{1}{2})$ under Iwahori–Matsumoto involution, in the sense of [12, Remark 2.2.5]. Since

$$\operatorname{mult}(\lambda_0, \pi) = \operatorname{mult}(\lambda_1, \pi) = 1,$$
it follows that $\pi$ admits a unique irreducible subrepresentation and a unique irreducible quotient. Let $\pi_0 \leq \pi$ be the irreducible subquotient of $\pi$ such that $\lambda_{a.d.} \leq r^G_{\pi_0}$. Applying the branching rules in Appendix A on $\lambda_{a.d.}$ yields that $\lambda_0, \lambda_1 \leq r^G_{\pi_0}$. Hence, $\pi_0$ is both the unique irreducible subrepresentation and the unique irreducible quotient of $\pi$. It follows that $\pi = \pi_0$ is irreducible.

**Subrepresentations**

As explained in Remark 3.11 for all $I_P(\chi, s)$ with $s > 0$ (regardless of regularity and reducibility), the fact that they admit a unique irreducible subrepresentation follows from Corollary 3.10 By contragredient, $I_P(\chi, s)$, with $s < 0$, admits a unique irreducible quotient. Using the same reasoning, we see that this quotient appears in $I_P(\chi, s)$ with multiplicity 1.

We now treat the maximal semi-simple subrepresentation of reducible non-regular $I_P(\chi, s)$, with $s \leq 0$. First of all, note that, by Corollary 3.10 when $I_P(\chi, s)$ is regular, it admits a unique irreducible subrepresentation.

When $I_P(\chi, s)$ is non-regular, the fact that $I_P(\chi, s)$ admits a unique irreducible subrepresentation follows from the considerations explained in Subsection 3.4 In the following tables, we list all cases which follow from Case I and Case II or Case III. For points where we used Case III ([4, $-\frac{1}{2}$, 1] and [4, $-\frac{1}{2}$, 3]), the argument is detailed at the end of this proof.

- For $P = P_1$:

| ord($\chi$) | $s$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1           |     |     |     |     |     |     |     | Case I |

- For $P = P_2$:

| ord($\chi$) | $s$ | $-\frac{1}{2}$ | $-\frac{9}{2}$ | $-\frac{7}{2}$ | $-\frac{5}{2}$ | $-\frac{3}{2}$ | $-\frac{1}{2}$ | $0$ |
|-------------|-----|----------------|----------------|----------------|----------------|----------------|----------------|-----|
| 1           |     |               |               |               |               |               |               | Case I |

- For $P = P_3$:

| ord($\chi$) | $s$ | $-\frac{9}{2}$ | $-\frac{7}{2}$ | $-\frac{5}{2}$ | $-\frac{3}{2}$ | $-1$ | $-\frac{1}{2}$ | $0$ |
|-------------|-----|----------------|----------------|----------------|----------------|-----|----------------|-----|
| 1           |     |               |               |               |               | Case II |               | Case II |
For $P = P_4$:

\[
\begin{array}{ccccccc}
\text{ord}(\chi) & s & -\frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & -1 & -\frac{1}{2} & -\frac{1}{6} \ 1 & \text{Case II} & \text{Case II} & \text{Case II} & \text{Case III} \ 2 & \text{Case II} & \text{Case II} & \text{Case II} & \text{Case III} \ 3 & \text{Case III} & \text{Case III} & \text{Case III} & \text{Case III} \\
\end{array}
\]

Also, we note that when $I_P(\chi, s)$ admits a unique irreducible subrepresentation, $\tau$, then it is the unique subquotient of $I_P(\chi, s)$ such that $r^G_T \tau$ contains an anti-dominant exponent. It follows that $\tau$ appears in $I_P(\chi, s)$ with multiplicity 1.

It remains to deal with the cases $[4, -\frac{1}{2}, 1]$ and $[4, -\frac{1}{2}, 3]$. Both calculations are of the form suggested in Subsection 3.4 as Case III. However, for $[4, -\frac{1}{2}, 1]$ we show that $\pi = I_P(\chi, -\frac{1}{2})$ admits a unique irreducible subrepresentation while for $[4, -\frac{1}{2}, 3]$ we show that the maximal semi-simple subrepresentation is of length 3.

- Consider $\pi = I_P(\chi, -\frac{1}{2})$, where $P = P_4$ and $\chi = 1$. Let $M' = M_6$ and $M'' = M \cap M' = M_{1,2,3,5}$. Furthermore, let $\Omega'' = r^{M'}_{M''} \Omega$, where $\Omega = \Omega_{M,\chi,s}$, and note that

\[
\lambda_0 = r^M_T \Omega = r^{M''}_{M'} \Omega'' = \begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}.
\]

As explained in Subsection 3.4

\[i^G_M \Omega \hookrightarrow i^G_{M'} (i^{M''}_{M'} \Omega'').\]

It can be shown, using the algorithm in Subsection 3.3 that $i^{M''}_{M'} \Omega''$ is a unitary irreducible representation of $M'$. We wish to point out that this fact is in agreement with the results of [2, Theorem 5.3].

We then note that $\lambda_{a.d.} \leq r^{M'}_T (i^{M''}_{M'} \Omega'')$ and hence, by Lemma 3.12

\[i^{M''}_{M'} \Omega'' \hookrightarrow i^{M'}_T \lambda_{a.d.}.\]

It follows that

\[i^G_M \Omega \hookrightarrow i^G_{M'} (i^{M'}_T \lambda_{a.d.}) \cong i^G_T \lambda_{a.d.}.\]

Since $i^G_T \lambda_{a.d.}$ admits a unique irreducible subrepresentation, then so does $i^G_M \Omega$.

- Consider $\pi = I_P(\chi, -\frac{1}{2})$, where $P = P_4$ and $\chi$ is of order 3. We show that the maximal semi-simple subrepresentation of $\pi$ has length 3. In particular, this proves that $\pi$ is reducible.
The leading exponent in this case is
\[
\lambda_0 = \begin{pmatrix}
-1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
2 & -1 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix} + \chi \circ \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix},
\]
while
\[
\lambda_{a.d.} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} + \chi \circ \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
is an anti-dominant exponent in \( r_T^G \pi \).

We note that \( \lambda_{a.d.} = w_{42354} \cdot \lambda_0 \) and that the standard normalized intertwining operator \( N(w_{42354}, \lambda_0) \) is an isomorphism, as a composition of 5 isomorphisms associated with simple reflections (see [16] Subsection 2.5] for a survey on these intertwining operators). It follows that
\[
I_P(\chi, \frac{1}{2}) \hookrightarrow i_G T \lambda_{a.d.}.
\]

One checks that
\[
\text{Stab}_W(\lambda_{a.d.}) = \langle w_{3165} \rangle \cong \mathbb{Z}/3\mathbb{Z}.
\]

Following the analysis in [9] [19] we have
\[
i_M T \lambda_{a.d.} = \bigoplus_{\xi \in \mathbb{Z}/3\mathbb{Z}} \sigma_\xi.
\]

For any \( \xi \in \mathbb{Z}/3\mathbb{Z} \), \( i_M^G \sigma_\xi \) is a standard module in the subrepresentation setting of the Langlands classification theorem (see [3]). Hence, \( i_M^G \sigma_\xi \) admits a unique irreducible subrepresentation \( \pi_\xi \).

So \( \bigoplus_{\xi \in \mathbb{Z}/3\mathbb{Z}} \pi_\xi \) is the maximal semi-simple subrepresentation of \( i_T^G \lambda_{a.d.} \). We prove that it is a subrepresentation of \( \pi \).

Note that
\[
\text{mult}(\lambda_{a.d.}, r_T^G i_T^G \lambda_{a.d.}) = 3
\]
and
\[
\text{mult}(\lambda_{a.d.}, r_T^G \pi_\xi) = 1
\]
for any \( \xi \in \mathbb{Z}/3\mathbb{Z} \). Namely, these are the only subquotients of \( i_T^G \lambda_{a.d.} \) containing the exponent \( \lambda_{a.d.} \). Since \( \text{mult}(\lambda_{a.d.}, r_T^G \pi) = 3 \), \( \bigoplus_{\xi \in \mathbb{Z}/3\mathbb{Z}} \pi_\xi \) is the maximal semi-simple subrepresentation of \( \pi \).
Appendix A. Branching rules coming from small Levi subgroups

In this section, we describe the branching rules (see Subsection 3.3) used in Section 4. A branching rule stemming from a Levi subgroup $M$, whose derived subgroups $M^{\text{der}}$ is of type $X_n$, will be referred to as an $X_n$-branching rule. We will record only the relevant results on the representation theory of $M$ and the branching rules following from it. For further discussion on the representations of $GL_n$ for small $n$, see [17].

We note that since the representations $I_P(\chi, s)$ have cuspidal support on $T$, then the same holds for all the constituents of $r^G_M I_P(\chi, s)$ for any standard Levi subgroup $M$ of $G$. Namely, $r^G_M I_P(\chi, s)$ does not contain supercuspidals.

A.1. “Golden Rule”—Induction from the Trivial Character of the Torus. Let $\pi$ be an irreducible representation of $G$ with $\lambda \leq r^G_T \pi$ and let

$$\Theta_\lambda = \{ \alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle = 0 \} \subset \Delta.$$

We note that $\lambda \not\perp \langle \alpha \mid \alpha \in \Theta_\lambda \rangle$ and hence

$$i_{M_{\Theta_\lambda}}^M \lambda = (i_{M_{\Theta_\lambda}}^M 1) \otimes \lambda,$$

where, by abuse of notations, we consider $\lambda$ as a character of $M_{\Theta_\lambda}$. We also note that $i_{M_{\Theta_\lambda}}^M 1$ is irreducible. On the other hand,

$$[r^G_{M_{\Theta_\lambda}} (i_{M_{\Theta_\lambda}}^M \lambda)] = |W_{M_{\Theta_\lambda}}| \times [\lambda],$$

and hence $i_{M_{\Theta_\lambda}}^M \lambda$ is the unique irreducible representation of $M_{\Theta_\lambda}$ such that $\lambda \leq r^G_{M_{\Theta_\lambda}} \pi$. We conclude that

(A.1) $\lambda \leq r^G_T \pi \implies |W_{M_{\Theta_\lambda}}| \times [\lambda] \leq |r^G_T \pi|.$

A.2. $A_1$-Branching Rules. Let $M = M_{\{\alpha\}}$ be a Levi subgroup of $G$ whose derived subgroup $M^{\text{der}}$ is isomorphic to $SL_2$. It follows that $i^M_T \lambda$ is irreducible if $\langle \lambda, \alpha^\vee \rangle \neq \pm 1$. We then have the following $A_1$-branching rule:

(A.2) $\lambda \leq r^G_T \pi, \langle \lambda, \alpha^\vee \rangle \neq \pm 1 \implies |[\lambda] + [s_{\alpha} \cdot \lambda] \leq |r^G_T \pi|.$
A.3. $A_2$-Branching Rules. Let $M = M_{\{\alpha, \beta\}}$ be a Levi subgroup of $G$ whose derived subgroup $M^{\text{der}}$ is isomorphic to $SL_3$. Assume that $\langle \lambda, \alpha^\vee \rangle = \pm 1$ and $\langle \lambda, \beta^\vee \rangle = 0$. In such a case, $i^M_T \lambda$ has length two. We write $i^M_T \lambda = \sigma_1 + \sigma_2$. Up to conjugating induces, we have
\[
[r^G_T \sigma_1] = [\lambda] + [s_\beta \cdot \lambda] + [s_\alpha s_\beta \cdot \lambda] = 2 \times [\lambda] + [s_\alpha \cdot \lambda],
\]
\[
[r^G_T \sigma_2] = [s_\alpha \cdot \lambda] + [s_\beta s_\alpha \cdot \lambda] + [s_\alpha s_\beta s_\alpha \cdot \lambda] = 2 \times [s_\beta s_\alpha \lambda] + [s_\alpha \cdot \lambda].
\]
We then have the following $A_2$-branching rule:
\[
(A.3) \quad \lambda \leq r^G_T \pi, \quad \langle \lambda, \alpha^\vee \rangle = \pm 1, \quad \langle \lambda, \beta^\vee \rangle = 0 \implies 2 \times [\lambda] + [s_\alpha \cdot \lambda] \leq [r^G_T \pi].
\]

A.4. $A_3$-Branching Rules. Let $M = M_{\{\alpha, \beta, \gamma\}}$ be a Levi subgroup of $G$ whose derived subgroup $M^{\text{der}}$ is isomorphic to $SL_4$. Assume that $\beta$ is a neighbor of both $\alpha$ and $\gamma$ in $\text{Dyn}(G)$, the Dynkin diagram of $G$. Further assume that $\langle \lambda, \alpha^\vee \rangle = 1$, $\langle \lambda, \beta^\vee \rangle = 0$ and $\langle \lambda, \gamma^\vee \rangle = -1$. There exists a unique irreducible representation $\sigma$ of $M$ such that $\lambda \leq r^G_T \sigma$. Furthermore,
\[
[r^G_T \sigma] = 2 \times [\lambda] + [s_\alpha \cdot \lambda] + [s_\gamma \cdot \lambda] + 2 \times [s_\alpha s_\gamma \cdot \lambda].
\]
We conclude the following $A_3$-branching rule:
\[
(A.4) \quad \lambda \leq r^G_T \pi, \quad \langle \lambda, \alpha^\vee \rangle = 1, \quad \langle \lambda, \beta^\vee \rangle = 0, \quad \langle \lambda, \gamma^\vee \rangle = -1
\]
\[
\implies 2 \times [\lambda] + [s_\alpha \cdot \lambda] + [s_\gamma \cdot \lambda] + 2 \times [s_\alpha s_\gamma \cdot \lambda] \leq [r^G_T \pi].
\]

A.5. $D_4$-Branching Rules. Let $M = M_{\{\alpha, \beta, \gamma, \delta\}}$ be a Levi subgroup of $G$ whose derived subgroup $M^{\text{der}}$ is isomorphic to $Spin_8$ or $SO_8$. The representation theory of these groups was studied in [2] and [8]. Assume that $\beta$ is a neighbor of both $\alpha$, $\gamma$ and $\delta$ in $\text{Dyn}(G)$ and let
\[
w_0 = s_\beta s_\alpha s_\gamma s_\delta s_\beta.
\]
Further assume that $\langle \lambda, \beta^\vee \rangle = -1$ and $\langle \lambda, \alpha^\vee \rangle = \langle \lambda, \gamma^\vee \rangle = \langle \lambda, \delta^\vee \rangle = 0$. There exists a unique irreducible representation $\sigma$ of $M$ such that $\lambda \leq r^G_T \sigma$. Furthermore,
\[
[r^G_T \sigma] = \sum_{w \in W^M_{\{s_\beta\}, T} \setminus \{1, s_\beta\}} [w \cdot (w_0 \cdot \lambda)].
\]
We conclude the following $D_4$-branching rule:

\[
\lambda \leq r_T^G \pi, \quad \langle \lambda, \beta^\vee \rangle = -1, \quad \langle \lambda, \alpha^\vee \rangle = \langle \lambda, \gamma^\vee \rangle = \langle \lambda, \delta^\vee \rangle = 0
\]

\[\Rightarrow \quad \sum_{w \in W_{M_{\{\beta, \gamma\}}} \setminus \{1, s_\beta\}} [w \cdot (w_0 \cdot \lambda)] \leq [r_T^G \sigma].\]

**Appendix B. Example of Application of Branching Rules to Prove the Irreducibility of $I_P(\chi, s)$**

In this section, we consider a detailed example of the application of branching rules in order to prove the irreducibility of $I_{P_1}(1, -2)$.

We start by writing the initial exponent of $I_{P_1}(1, -2)$:

\[
\lambda_0 = \begin{bmatrix} -1 \\ 3 & -1 & -1 & -1 & -1 \end{bmatrix}.
\]

The anti-dominant exponent in the $W$-orbit of $\lambda_0$ is

\[
\lambda_{a.d.} = \begin{bmatrix} 0 \\ -1 & -1 & 0 & -1 \end{bmatrix}.
\]

Let $\pi$ be an irreducible representation of $G$ such that $\pi \leq I_{P_1}(1, -2)$ and $\lambda_{a.d.} \leq r_T^G \pi$. Note that $\dim_{\mathbb{C}} r_T^G I_{P_1}(1, -2) = 27$ and hence $\dim_{\mathbb{C}} r_T^G \pi \leq 27$. In particular, if we show that $\dim_{\mathbb{C}} r_T^G \pi = 27$, then it would follow that $\pi = I_{P_1}(1, -2)$ is irreducible.

We apply branching rules as follows:

- Since $|\text{Stab}_W(\lambda_{a.d.})| = 4$, it follows from Equation (A.1) that
  
  \[4 \times \lambda_{a.d.} \leq r_T^G \pi.\]

- Applying the $A_2$-branching rules on $\lambda_{a.d.}$ with respect to the Levi subgroups $M_{\{\alpha_2, \alpha_4\}}$ (or $M_{\{\alpha_4, \alpha_5\}}$) and $M_{\{\alpha_5, \alpha_6\}}$ yields
  
  \[2 \times \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} + 2 \times \begin{bmatrix} 0 \\ -1 & -1 \end{bmatrix} \leq r_T^G \pi.\]

- Applying, on $[ -1 -2 \ -1 \ -1 ]$, the $A_1$-branching rule associated to $M_{\{\alpha_3\}}$ yields
  
  \[2 \times \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \leq r_T^G \pi.\]
• Applying the $A_1$-branching rule associated to $M_{\{\alpha_1\}}$ on this exponent yields

$$2 \times \begin{bmatrix} -1 \\ 3 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \leq r^G_T \pi.$$  

• Applying on $[-1 \ -1 \ 0 \ -1 \ -1]$ the $A_2$-branching rule associated to $M_{\{\alpha_2, \alpha_4\}}$ yields

$$\begin{bmatrix} -1 \\ -1 \\ -2 \\ 1 \\ -2 \\ 1 \end{bmatrix} \leq r^G_T \pi.$$  

• We now apply a sequence of $A_1$-branching rules associated with entries which are not 0 or $\pm 1$ on $[-1 \ -2 \ -1 \ -2 \ 1]$ and subsequent exponents. This yields

$$\begin{bmatrix} -1 \\ -3 \\ 2 \\ -1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ -2 \\ -1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \\ 2 \\ -3 \\ 2 \\ -1 \end{bmatrix}$$

$$+ \begin{bmatrix} -1 \\ 3 \\ -1 \\ -1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ -1 \\ -3 \\ 2 \\ -1 \end{bmatrix}$$

$$+ \begin{bmatrix} -4 \\ 3 \\ -4 \\ 3 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \\ 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$$+ \begin{bmatrix} -4 \\ 3 \\ -4 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$+ \begin{bmatrix} 4 \\ 3 \\ -5 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$+ \begin{bmatrix} -1 \\ -1 \\ 5 \\ -6 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ -1 \\ 6 \\ -7 \end{bmatrix}$$

$$+ \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 7 \end{bmatrix} \leq r^G_T \pi.$$
We conclude that
\[
4 \times \lambda_{a.d.} + 2 \times \left[ \begin{array}{cccc}
-1 & -2 & 1 & -1 \\
-3 & 2 & -1 & -1
\end{array} \right] + 2 \times \left[ \begin{array}{cccc}
-1 & -1 & -1 & -1 \\
3 & -1 & -1 & -1
\end{array} \right] \\
+ 2 \times \left[ \begin{array}{cccc}
1 & -1 & 1 & -1 \\
2 & -1 & 2 & -1
\end{array} \right] + 2 \times \left[ \begin{array}{cccc}
1 & -1 & 1 & -1 \\
2 & -1 & 2 & -1
\end{array} \right] \\
+ 2 \times \left[ \begin{array}{cccc}
-2 & 1 & 2 & -1 \\
-3 & 2 & -1 & 2
\end{array} \right] + 2 \times \left[ \begin{array}{cccc}
-2 & 1 & 2 & -1 \\
-3 & 2 & -1 & 2
\end{array} \right] \\
+ 2 \times \left[ \begin{array}{cccc}
3 & -1 & -1 & -2 \\
-1 & -3 & 2 & -1
\end{array} \right] + 2 \times \left[ \begin{array}{cccc}
3 & -1 & -1 & -2 \\
-1 & -3 & 2 & -1
\end{array} \right] \\
+ 2 \times \left[ \begin{array}{cccc}
-4 & 3 & -1 & -1 \\
3 & -4 & 3 & -1
\end{array} \right] + 2 \times \left[ \begin{array}{cccc}
-4 & 3 & -1 & -1 \\
3 & -4 & 3 & -1
\end{array} \right] \\
+ 2 \times \left[ \begin{array}{cccc}
-1 & 4 & -1 & -1 \\
-1 & 4 & -1 & -1
\end{array} \right] + 2 \times \left[ \begin{array}{cccc}
-1 & 4 & -1 & -1 \\
-1 & 4 & -1 & -1
\end{array} \right] \\
+ 2 \times \left[ \begin{array}{cccc}
4 & -5 & -1 & -1 \\
4 & -5 & -1 & -1
\end{array} \right] + 2 \times \left[ \begin{array}{cccc}
4 & -5 & -1 & -1 \\
4 & -5 & -1 & -1
\end{array} \right] \\
+ 2 \times \left[ \begin{array}{cccc}
-1 & 5 & -6 & -1 \\
-1 & 5 & -6 & -1
\end{array} \right] + 2 \times \left[ \begin{array}{cccc}
-1 & 5 & -6 & -1 \\
-1 & 5 & -6 & -1
\end{array} \right] \\
+ 2 \times \left[ \begin{array}{cccc}
-1 & -1 & -1 & -7 \\
-1 & -1 & -1 & -7
\end{array} \right] \leq r_G^G \pi.
\]

Since we have proven that \(27 \leq \dim \mathbb{C} r_G^G \pi\), it follows that \(\pi = I_{P_1}(1, -2)\) is irreducible.

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