Remarks about weighted energy integrals over Minkowski spectral functions from Euclidean lattice data

Thomas DeGrand\textsuperscript{1}

\textsuperscript{1} Department of Physics, University of Colorado, Boulder, CO 80309 USA

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Abstract

I make some simple observations about the calculation of weighted averages over energy of Minkowski space spectral densities from weighted averages over time of Euclidean space correlation functions, measured in lattice simulations. The correlator of two vector currents is used as an example, where it appears that a determination of a weighted average of the spectral function near the rho pole at the five per cent level is possible from lattice simulations.

*Electronic address: thomas.degrand@colorado.edu
Finding connections between theoretical calculations done in Euclidean space and results of experiments done in Minkowski space is a longstanding problem in many areas of physics and involves many approaches. Lattice studies of QCD and other related systems are no exception. This short note describes a simple technique for extracting weighted averages over energy of Minkowski space spectral densities from Euclidean space lattice correlation functions. Examples are motivated by calculations of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment (with Ref. [1] as my primary reference) though there are obvious applications to many similar processes [2].

When the Euclidean correlation \( \Pi(Q) \) is related to the Minkowski space spectral function \( \rho(\omega) \) measured at energy \( \omega \) by a once-subtracted dispersion relation, the connection between a Euclidean space correlation function defined at Euclidean time \( t \), \( G_E(t) \), and the spectral function is

\[
G_E(t) = \frac{1}{2\pi} \int_0^\infty d\omega [\omega^2 \rho(\omega)] \exp(-\omega t). 
\]

(1)

Inverting Eq. (1) to predict \( \rho(\omega) \) from \( G_E(t) \) is a difficult problem. However, it seems easy to compare a weighted average of \( G_E(t) \) to a weighted average of \( \rho(\omega) \),

\[
\hat{\rho}(Q_0) \equiv \int_0^\infty R_E(Q_0, t)G(t)dt = \int_0^\infty d\omega \rho(\omega)T(Q_0, \omega)
\]

(2)

with the connection

\[
T(\omega) = \frac{\omega^2}{2\pi} \int_0^\infty e^{-\omega t} R_E(Q_0, t)dt.
\]

(3)

\( Q_0 \) is shorthand for possible tunable parameter(s) in the weighting function. The most prominent present-day example of such a connection is the calculation of the hadronic vacuum polarization contribution for the anomalous magnetic moment of the muon \( a_{\mu}^{HV P} \). The Euclidean weighting function \( R_E(Q_0, t) \) for \( a_{\mu} \) is specified by a QED calculation. But of course, one could imagine doing the weighting with any function \( R_E(Q_0, t) \).

Each choice of \( R_E(Q_0, t) \) amounts to its own (indirect) comparison of theory \( (G_E(t)) \) with experiment \( (\rho(\omega), \rho(Q_0)) \). Families of related \( R_E(Q_0, t) \)’s can be combined into more extensive views of the spectral density. I don’t want to speculate on whether this could do a better job of probing the spectral function than the standard technique of fitting \( G_E(t) \) to a functional form with a set of parameters (masses and coupling constants) and then continuing the fit function from Minkowski to Euclidean space. I just want to raise the possibility that analysis methods for \( a_{\mu}^{HV P} \) might have wider applications.

Some choices of \( R_E \) are going to be more interesting than others, and a desirable goal would be to find an \( R_E \) whose \( T(\omega) \) is peaked around some energy range. To jump to the conclusion, the dominant feature of an \( R_E(Q_0, t) \) which does that is a restriction to a range of \( t \) values \( t_{min} < t < t_{max} \); the overall shape of \( R_E(Q_0, t) \) does not seem to be important for the examples I display. And given what is published about the precision of \( a_{\mu}^{HV P} \) lattice results, it seems possible to make a lattice determination of a weighted average of \( \rho(\omega) \) with enough accuracy to be phenomenologically interesting. (I have in mind the few percent tension in the \( \pi\pi \) channel in the 0.6-0.9 GeV range described in Ref. [1], between the KLOE experiment [3] and other groups.)

I think I am saying obvious things, but I haven’t found a discussion of this approach in the literature.

The idea described here is just a trivial variation on the “coordinate space representation” for \( a_{\mu}^{HV P} \): there is an implicit assumption that \( R_E(t) \) is a smooth function of \( t \), and replacing
an integral over continuous $t$ by a sum over a set of discrete lattice points is no different than replacing any continuous integral by a grid sum. There is also a large literature proposing solutions to the “inverse problem:” given a $G_E(t_i)$ defined at a set of discrete $t_i$ values, various approaches have different criteria for defining and constructing a weighted $\hat{\rho}$. Often, no smoothness assumptions go into the choice and in fact the $R_E(t_i)$’s found in the literature are far from smooth. Recent references (a very incomplete set for this vast field) are Ref. [2], which uses the Backus - Gilbert method [4, 5], related work by Ref. [6], and Chebychev techniques by Refs. [7–9].

I will continue the note focussing on $a_{HVP}^\mu$. There is a small literature associated with modifications to its $R_E$. Probably the most prominent one is the “intermediate window method” of Ref. [10]. It is a time - sliced version of the $a_{HVP}^\mu$ weighting:

$$R_E(Q_0, t) = R_{E}^\mu(Q_0, t)[\Theta(t, t_{min}, \Delta) - \Theta(t, t_{max}, \Delta)]$$

where $\Theta(t, t_0, \Delta)$ is a smoothed step function. Another approach to weighting, called “finite energy sum rules,” starts by writing a dispersion relation for a reweighted $\Pi_E(Q)$. For a discussion, see Refs. [11, 12].

To set conventions, we are interested in the correlator of two vector currents

$$\Pi(q)_{\mu \nu} = \int d^4x e^{iqx} \langle 0 | J_\mu(x) J_\nu(0) | 0 \rangle.$$  \hspace{1cm} (5)

We remove the indices with a transverse projection,

$$\Pi_{\mu \nu} = [g_\mu g_\nu = g_\mu q^2] \Pi(q^2)$$

and then the spectral function $\rho(\omega)$ is proportional to the discontinuity of $\Pi$ across the real energy axis (setting $q_\mu = (\omega, \vec{0})$). It is also proportional to the R-ratio, $R(\omega) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$. The standard lattice vector - vector correlator contracts $\rho_{\mu \nu}$ against polarization vectors $\epsilon_i^\mu \epsilon_j^\nu$ where typically $\epsilon_i^\mu = (0, \vec{\epsilon})$ is a unit vector. This means that in Eq. 5 $\rho(\omega) = R(\omega)/(6\pi)$ and

$$G_E(t) = \sum_i \int d^3x \langle J_i(\vec{x}, t) J_i(0, 0) \rangle$$

where $J_i(x, t) = e_q \bar{\psi}(x, t) \gamma_i \psi(x, t)$ for a quark of charge $e_q$ (in units of the electric charge).

The two relevant pictures are shown in Fig. 1: the familiar plot of the R-ratio in panel (a) and the expected $G_E(t)$ in panel (b), using Eq. 1 to do the inversion. “Experiment” in these pictures are the phenomenological model for $\rho(\omega)$ from Ref. [13] (in black) and a compilation of $R(\omega)$ from a table in the Review of Particle Properties [14] (in red). Of course, the question to try to answer is: Given a calculation $G_E(t)$, what can one say about $\rho(\omega)$?

This question is partially answered by the one theoretical line in panel (b) of Fig. 1. The straight line is the contribution of a stable rho meson at 770 MeV with a decay constant $f_V = 0.25$:

$$G_E^V(t) = \frac{\langle q \rangle m_V^2 f_V^2}{2m_V} \exp(-m_V t).$$  \hspace{1cm} (8)

The quantity $\langle q \rangle$ is the expectation value of the quarks’ charges in the meson: $[2/3 - (-1/3)]/\sqrt{2} = 1/\sqrt{2}$ for the rho, 1/18 for the omega, 1/9 for the phi, and so on. $G_E(t)$ is
FIG. 1: (a) $R(\omega)$ the R-ratio from the phenomenological model for $\rho(\omega)$ from Ref. [13] (in black) and from the table in the Review of Particle Properties [14] (in red). (b) $G_E(t)$ from the phenomenological model for $\rho(\omega)$ from Ref. [13] (in black) and from the table in the Review of Particle Properties [14] (in red). The inversion uses Eq. [1]. The dotted line is the contribution from a stable rho meson with $f_V = 0.25$.

flatter than $G_V^E(t)$ at very large $t$ due to the contribution of two-pion states with an invariant mass smaller than the rho mass, and it is steeper than $G_V^E(t)$ at small $t$ due to the phi meson and to the flat high energy part of $R(\omega)$. Nowhere does $G_V^E(t)$ saturate $G_E(t)$.

I can rephrase the question to try to answer as: Given a lattice calculation of $G_E(t)$, what can one say about $\rho(\omega)$? Then there are more constraints. The large $\omega$ region, where $\omega > 1/a$ and $a$ is the lattice spacing, is contaminated by lattice artifacts, and is inaccessible to a lattice calculation. Unfortunately, so is the small $\omega$ or large $t$ region. There are two reasons for this. First, the lattice signal becomes noisy. This is a usual issue in lattice simulations [15–17]. The data in Ref. [18] provide an example – see their Fig. 2. The collaboration has data at lattice spacings between 0.15 and 0.06 fm. Their data is only usable out to distances $t \sim 2.5$ fm. This precludes, at least for the present, studies of $\rho(\omega)$ near threshold. This situation is well known and documented in the $a_{HV}^{HF}$ literature [1].

A second reason that the small $\omega$ region is difficult is that it is dominated by two pion states. Lattice simulations are done in a finite box (say, of size $L$) and particle momenta are quantized, $\vec{p}_n = 2\pi \vec{n}/L$ for integer valued $\vec{n}$, so that there is no two pion continuum in a lattice simulation, just a set of exponentially falling contributions $\propto \exp(-2\sqrt{\vec{p}_n^2 + m^2/4})$. Presumably, these contributions interpolate into the continuum result when the volume is taken sufficiently large. Finite volume $\rho(\omega)$'s are sums of delta functions, but the smearing washes out this behavior.

Parenthetically, the vector correlator presents a somewhat special case compared to most lattice studies, where the lightest state in (continuum) $\rho(\omega)$ is an isolated pole. Then, simply going to large $t$ gives a $G_E(t)$ which is dominated by properties of the pole. Standard lattice techniques (fits to exponentials) are more efficient at producing high quality results than the proposal of weighting $G_E(t)$ given here.

So we are pushed back to the region of $\omega$ near the rho mass. The physical rho meson is broad. Is it possible to say anything about $\rho(\omega)$ for $\omega$ near $m_\rho$? This seems to be a serious
FIG. 2: Fractional contributions to $G_E(t)$ where $\rho(\omega)$ is taken from the phenomenological model of Ref. [13]. The curves label (a) $2m_\pi < \omega < 4m_\pi$; (b) $4m_\pi < \omega < 6m_\pi$; (c) $6m_\pi < \omega < 8m_\pi$; (d) $8m_\pi < \omega < 10m_\pi$.

issue for $a_{\mu}^{HVP}$ determinations.

Dividing up the contributions to $G_E(t)$ from different energy intervals shows the way to go. See Fig. 2, which shows the fractional contributions to $G_E(t)$ from different $\omega$ regions. Here $\rho(\omega)$ is taken from the phenomenological model of Ref. [13]. What is noticeable is that there is a fairly wide region at intermediate $t$ where the region around the rho mass contributes heavily. Of course, this can be seen by eye in Fig. 1 (This is basically just the phenomenon of vector meson dominance.)

As an application of this remark, suppose that the experimental $\rho(\omega)$ is not precisely known over some energy range, that two experiments differ by a fraction $\delta \rho(\omega)/\rho(\omega)$. Assuming that the difference is confined to some small region of $\omega$, there will be a change in $G_E(t)$ (constructing it from Eq. 1 with each experimental $\rho(\omega)$) of $\delta G_E(t)/G_E(t) \sim f \delta \rho/\rho$ where $f$ is the fractional contribution of the $\omega$ region of $\rho(\omega)$ to $G_E(t)$.

A simple example comes from modifying the model for $\rho(\omega)$ from Ref. [13] over a range $\omega_{\text{min}} < \omega < \omega_{\text{max}}$, by multiplication by a weighting factor

$$w(\omega) = 1 + a \sin \pi \left( \frac{\omega - \omega_{\text{min}}}{\omega_{\text{max}} - \omega_{\text{min}}} \right).$$

(9)

The fractional change in $G_E(t)$ is shown in Fig. 3 for the choice $\omega_{\text{min}} = 0.6 \text{ GeV}$, $\omega_{\text{max}} = 0.9$
FIG. 3: The fractional change in $G_E(t)$ from a five per cent variation in the phenomenological model for $\rho(\omega)$ of Ref. [13] over the range 0.6-0.9 GeV, as described in the text.

GeV, $a = 0.05$.

Notice the qualifier “assuming that the difference is confined to some small region of $\omega$.” $G_E(t)$ at any $t$ value is built of contributions from all $\omega$, and a measurement of $G_E(t)$ at any $t$ or for any range of $t$ values does not make an absolute prediction about $\rho(\omega)$ at any particular $\omega$ value. However, lattice results could still be useful to distinguish between the different experimental $\rho(\omega)$’s.

Fig. 3 shows that a five per cent variation in $\rho(\omega)$ translates into a 2.7 per cent variation in $G_E(t)$ over a fairly wide range of $t$. This is really the end of the story I can tell – I cannot write about uncertainties in either $G_E(t)$ or $\rho(\omega)$. Lattice data for $G_E(t)$ is typically highly correlated and it is almost impossible to estimate correlation uncertainties in a lattice data set without access to it. Similarly, the experimental data sets which give $\rho(\omega)$ are highly correlated. But, 2.7 per cent seems to be an easy target, given that contemporary lattice measurements of $a_{\mu}^{\nu\nu P}$ are well under a per cent. It seems likely that lattice calculations could determine $\rho(\omega)$ over the range 0.6-0.9 GeV at the five per cent level.

To do this in practice, we need a weighting function. There seem to be many possible choices. But Figs. 2 and 3 indicate that all that is significant for $R_E(t)$ is that it has a cutoff at small and large $t$, that it is nonzero for $t_{min} < t < t_{max}$. Two choices of $R_E(t)$ illustrate that claim.

First consider a family of power laws, $R_E(t) = (t/t_0)^n/n!$ for a range $t_{min} < t < t_{max}$. $t_0$ and the $n!$ factor are just rescalings, useful for plots across $n$ or for comparing weighted
lattice data at different lattice spacings. (This is a hard cutoff; Fig. 3 indicates that a soft cutoff would perform similarly.) Fig. 4 shows the contribution of \(4m_\pi < \omega < 6m_\pi\) to the integral of Eq. 2 for a power law \(R_E(t) = (t/t_0)^n/n!\) with \(t_0 = 0.15\) fm, for a range \(t_{\text{min}} < t < t_{\text{max}}\) plotted versus \(t_{\text{min}}\) for \(t_{\text{max}} = 1.2, 1.44, 1.8, 2.4\) and 5 fm. (a) \(n = 0\); (b) \(n = 2\); (c) \(n = 4\); (d) \(n = 6\).

Fig. 5 shows the fractional change in the integral \(\hat{\rho}\) from the model weighting factor of Eq. 9. This is the analog of Fig. 3 and the result is the same – the sensitivity to variation in \(\rho(\omega)\) depends most on the range of \(t\) spanned by \(R_E(t)\).

Fig. 6 shows similar results for the smearing kernel used for \(a_{\mu}^{HVP}\) in its intermediate window guise. The figures are nearly identical to the ones for power law weighting. The conclusion seems to be that a lattice calculation of \(G_E(t)\) with an accuracy of 2-3 per cent
FIG. 5: Fractional change in the integral $\hat{\rho}$ of Eq. 2 under a five per cent variation in $\rho(\omega)$ for $4m_\pi < \omega < 6m_\pi$ parameterized as in Eq. 9 for a power law $R_E(t) = (t/t_0)^n/n!$ with $t_0 = 0.15$ fm, for a range $t_{\text{min}} < t < t_{\text{max}}$ plotted versus $t_{\text{min}}$ for $t_{\text{max}} = 1.2, 1.44, 1.8, 2.4$ and 5 fm. (a) $n = 0$; (b) $n = 2$; (c) $n = 4$; (d) $n = 6$.

(in the continuum limit, of course) over the range of 1-2 fm can distinguish a five per cent variation in $\rho(\omega)$ in the rho region.

At this point I should stop and hope for an analysis by one of the lattice groups using its own data sets. The idea I have presented is trivial, but it also seems simple to implement. I think that $\rho(\omega)$ (and related quantities) are interesting in and of themselves, and that trying to extract features of $\rho(\omega)$ which have nothing to do with $a_{\mu}^{\text{HVP}}$ from $G_E(t)$ could be a useful project. And, of course, identical weighting techniques can connect other inclusive processes with Euclidean correlators.

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FIG. 6: The analog of Figs. 4 and 5 for the smearing kernel for $a_{\mu}^{HVP}$. Panel (a) is the contribution of $4m_{\pi} < \omega < 6m_{\pi}$ to the integral of Eq. 2. Panel (b) is the fractional change in the integral $\hat{\rho}$ of Eq. 2 under a five per cent variation in $\rho(\omega)$ for $4m_{\pi} < \omega < 6m_{\pi}$ parameterized as in Eq. 9. Again, the curves are $t_{\text{min}} < t < t_{\text{max}}$ plotted versus $t_{\text{min}}$ for $t_{\text{max}} = 1.2, 1.44, 1.8, 2.4$ and 5 fm.

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