Remarks on the global regularity issue of the two-and-a-half-dimensional Hall-magnetohydrodynamics system

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Abstract. Whether or not the solution to the 2$\frac{1}{2}$-dimensional Hall-magnetohydrodynamics system starting from any smooth initial data preserves its regularity for all time remains a challenging open problem. Although the research direction on component reduction of regularity criteria for Navier–Stokes equations and magnetohydrodynamics system has caught much attention recently, the Hall term has presented many difficulties. In this manuscript, we discover a certain cancellation within the Hall term and obtain various new regularity criteria: first, in terms of a gradient of only the third component of the magnetic field; second, in terms of only the third component of the current density; third, in terms of only the third component of the velocity field; fourth, in terms of only the first and second components of the velocity field. As another consequence of the cancellation that we discovered, we are able to prove the global well-posedness of the 2$\frac{1}{2}$-dimensional Hall-magnetohydrodynamics system with hyper-diffusion only for the magnetic field in the horizontal direction; we also obtained an analogous result in the three-dimensional case via the discovery of additional cancellations. These results extend and improve various previous works.

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1. Introduction

1.1. Motivation from physics and mathematics

Initiated by Alfvén [2] in 1942, the study of magnetohydrodynamics (MHD) concerns the properties of electrically conducting fluids. For example, while fluid turbulence is often investigated through Navier–Stokes (NS) equations, MHD turbulence occurs in laboratory settings such as fusion confinement devices (e.g., reversed field pinch), as well as astrophysical systems (e.g., solar corona), and the conventional system of equations for such a study is that of the MHD. Lighthill in 1960 introduced the Hall-MHD system by adding a Hall term which arises upon writing the current density as the sum of the ohmic current and a Hall current that is perpendicular to the magnetic field (see [30, equation (94)]). Ever since, the Hall-MHD system has found tremendous attractions due to its applicability: study of the sun (e.g., [4]); magnetic reconnection (e.g., [17,25]); turbulence (e.g., [34]); star formation (e.g., [40]). In particular, the statistical study of magnetic reconnection events in [17] by Donato et al. focused on the two-and-a-half-dimensional (2$\frac{1}{2}$-D) Hall-MHD system, necessarily because the Hall term decouples from the rest of the system in the 2-D case (see equation (2)). It is well-known that despite many advances made in the mathematical analysis of the Hall-MHD system in the past decades, the global well-posedness of the 2$\frac{1}{2}$-D Hall-MHD system remains open, e.g., “Contrary to the usual MHD the global well-posedness in the 2$\frac{1}{2}$ -dimensional Hall-MHD is wide open” from [9, Abstract] by Chae and Lee. The purpose of this manuscript is to present some new cancellations within the Hall term (see (35), (38), (88), (91), and
(94)) and obtain new regularity criteria in terms of only a few components of the solution, as well as prove the global well-posedness of the $2\frac{1}{2}$-D Hall-MHD system with hyper-diffusion for the magnetic field only in the horizontal direction (see also Theorem 2.5 in the 3-D case). In Remark 2.1, we also list some interesting open problems for future works.

1.2. Previous works

Let us denote $\partial_i \triangleq \frac{\partial}{\partial t}, \partial_j \triangleq \frac{\partial}{\partial x_j}$ for $j \in \{1, 2, 3\}$, components of any vector by sub-index such as $x = (x_1, \ldots, x_D)$, and $A \lesssim B$ to imply the existence of a constant $C \geq 0$ of no dependence on any important parameter such that $A \leq CB$ due to the equation ($\cdot$). The spatial domain $\Omega$ of our interest will be $T^D$ or $\mathbb{R}^D$ (see [1, Remark 1] concerning the difficulty in the case of a general domain). For clarity let us first present the 3-D Hall-MHD system. With $u : \mathbb{R}_{\geq 0} \times \Omega \mapsto \mathbb{R}^3$, $b : \mathbb{R}_{\geq 0} \times \Omega \mapsto \mathbb{R}^3$, $\pi : \mathbb{R}_{\geq 0} \times \Omega \mapsto \mathbb{R}$, and $j \triangleq \nabla \times b$, representing, respectively, velocity field, magnetic field, pressure field, and current density, as well as $\nu \geq 0$, $\eta \geq 0$, and $\epsilon \geq 0$, respectively, the viscosity, magnetic diffusivity, and a Hall parameter, this system reads

$$\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla \pi &= \nu \Delta u + (b \cdot \nabla)b t > 0, \\
\partial_t b + (u \cdot \nabla)b &= \eta \Delta b + (b \cdot \nabla)u - \epsilon \nabla \times (j \times b) t > 0,
\end{align*}$$

starting from initial data $(u_0, b_0) \triangleq (u, b)|_{t=0}$. The case $\epsilon = 0$ reduces (1) to the MHD system. Furthermore, taking $b \equiv 0$ and assuming $\nu > 0$ transform (1) to the NS equations; additionally taking $\nu = 0$ leads us to the Euler equations. Throughout the rest of this manuscript, we will assume $\nu > 0$ and $\eta > 0$. It can be immediately seen from (1a)–(1b) that the 2-D case of $u(t, x) = (u_1, u_2)(t, x_1, x_2), \ b(t, x) = (b_1, b_2)(t, x_1, x_2)$ result in the decoupling of the Hall term from the rest; for this reason, physicists such as Donato et al. [17] have turned to the 2$\frac{1}{2}$-D case in which a pair of

$$
u > 0, \nu \cdot b = 0, t > 0,$$

solves (1) (see also [32, Section 2.3.1]).

We now review recent developments in the mathematical theory of the Hall-MHD system. Acheritogaray, Degond, Frouvelle, and Liu [1] proved the global existence of a weak solution to the 3-D Hall-MHD system in case $\Omega = T^3$ by using a vector calculus identity

$$(\Theta \times \Psi) \cdot \Theta = 0 \forall \Theta, \Psi \in \mathbb{R}^3$$

which led to the zero contribution from the Hall term on the energy identity. This pioneering work led to many others, some of which we list and refer readers to their references: [8] on various well-posedness results in $\mathbb{R}^3$; [10] on temporal decay in $\mathbb{R}^3$; [13,14] for partial regularity results; [11] for local well-posedness in case the magnetic diffusion $-\Delta b$ is generalized to $(-\Delta)^\alpha b$ for any $\alpha > \frac{1}{2}$; [12] for singularity formation in case of zero magnetic diffusion; [18] for local well-posedness in Besov spaces.

Considering that the global regularity problem of the 3-D and even 2$\frac{1}{2}$-D Hall-MHD systems remain completely open, one natural approach would be to pursue Prodi–Serrin-type regularity criteria ([19,35,37]). Such results have been extended to the MHD system (e.g., [23,48] in terms of only $u$), as well as the Hall-MHD system. For example, Chae and Lee obtained various blow-up criterion such as

$$\limsup_{t \nearrow T^*}(\|u(t)\|_{H^m}^2 + \|b(t)\|_{H^m}^2) = \infty \quad \text{if and only if} \quad \int_0^{T^*}(\|u\|_{BMO}^2 + \|\nabla b\|_{BMO}^2)dt = \infty$$

What is the global well-posedness problem of the 3-D Hall-MHD system?
for $m > \frac{5}{2}$ that is an integer for the 3-D Hall-MHD system (see [9, Theorem 2]) while

$$
\limsup_{t \to T^*} \left( \|u(t)\|_{H^m}^2 + \|b(t)\|_{H^m}^2 \right) = \infty \quad \text{if and only if} \quad \int_0^{T^*} \|j\|_{BMO}^2 dt = \infty \tag{5}
$$

for $m > 2$ that is an integer for the $2\frac{1}{2}$-D Hall-MHD system (see [9, Remark 3]). Furthermore, [9, Theorem 1] stated a Prodi–Serrin-type regularity criteria for the 3-D Hall-MHD system, specifically that $u, \nabla b \in L_T^r L^p_x$ for $\frac{2}{r} + \frac{2}{p} \leq 1, \ p \in (3, \infty]$ (also [47, Theorem 1.1]), and it can be immediately generalized to show that $2\frac{1}{2}$-D Hall-MHD system is globally well-posed as long as

$$
\nabla b \in L_T^r L^p_x \quad \text{for} \quad \frac{2}{p} + \frac{2}{r} \leq 1, \ p \in (2, \infty]. \tag{6}
$$

Remark 1.1. The intuition behind (4) concerning why a bound on $\nabla b$ is needed when its counterpart is $u$, as well as why we only need a bound on $\nabla b$ in (6) is because the Hall term $\nabla \times (j \times b)$ is precisely more singular than those in the MHD system by one derivative and the $2\frac{1}{2}$-D MHD system is known to be globally well-posed (the proof follows the 2-D case verbatim, e.g., [36]).

In the past few decades, a research direction on component reduction of such Prodi–Serrin-type regularity criteria has flourished. As far back as 1999, Chae and Choe [7] proved a regularity criterion in terms of only two components of the vorticity vector field $\omega \triangleq \nabla \times u$ for the 3-D NS equations; we also refer to [27] on a criterion in terms of $\partial_3 u$ and [5] on a criterion in terms of $u_3$. In particular, let us briefly elaborate on the method of Kukavica and Ziane [26] which inspired many more results on related equations. Denoting a horizontal gradient and a horizontal Laplacian, respectively, by $\nabla h \triangleq (\partial_1, \partial_2, 0)$ and $\Delta_h \triangleq \sum_{k=1}^{2} \partial_k^2$, Kukavica and Ziane observed that a solution to the 3-D NS equations satisfies

$$
\frac{1}{2} \partial_t \| \nabla h u \|_{L^2}^2 + \nu \| \nabla \nabla h u \|_{L^2}^2 \lesssim \int \| u_3 \| \| \nabla h u \| dx, \tag{7a}
$$

$$
\frac{1}{2} \partial_t \| \nabla u \|_{L^2}^2 + \nu \| \Delta u \|_{L^2}^2 \lesssim \int \| \nabla h u \| \| \nabla u \| dx. \tag{7b}
$$

It follows that one can assume a bound on $u_3$, apply it in (7a) to get a bound on $\nabla h u$, and then use the bound on $\nabla h u$ in (7b) to attain the desired bound on $\nabla u$, although this extra step has always prevented one from attaining the criterion at a scaling-invariant level. The interesting part is how one can separate $u_3$ in (7a); it turns out that using divergence-free property of $u$, one can compute

$$
\frac{1}{2} \partial_t \| \nabla h u \|_{L^2}^2 + \nu \| \nabla \nabla h u \|_{L^2}^2 = \int \langle u \cdot \nabla \rangle u \cdot \Delta_h u dx
$$

$$
= - \sum_{i, l, k = 1}^{2} \int \partial_k u_i \partial_l u_i \partial_k u_l dx + \sum_{l, k = 1}^{2} \int u_3 \partial_l (\partial_3 u_l \partial_k u_l) dx + \sum_{i = 1}^{3} \sum_{k = 1}^{2} \int u_3 \partial_l (\partial_3 u_i \partial_k u_3) dx. \tag{8}
$$

The second and third integrals already have $u_3$ separated. For the remaining integral of $\sum_{i, l, k = 1}^{2} \int \partial_k u_i \partial_l u_i \partial_k u_l dx$ which do not seem to have any $u_3$, strategic couplings and making use of the divergence-free property lead to the desired bound (7a), e.g., the sum of two terms $(i, l, k) = (1, 1, 2)$ and $(i, l, k) = (2, 1, 2)$ leads to

$$
\int \partial_2 u_1 \partial_1 u_1 \partial_2 u_1 + \partial_2 u_2 \partial_2 u_1 \partial_2 u_1 dx = \int (\partial_2 u_1)^2 (-\partial_3 u_3) dx \lesssim \int \| u_3 \| \| \nabla h u \| dx. \tag{9}
$$
For the MHD system, specifically (1) with $\epsilon = 0$, lengthy computations and strategic couplings of terms in $(u \cdot \nabla)u$ with $(u \cdot \nabla)b$, as well as $(b \cdot \nabla)b$ with $(b \cdot \nabla)u$, led to discovery of various hidden identities (e.g., [42, Proposition 3.1]). However, such cancellations seemed virtually impossible for the Hall-MHD system; a curl operator makes it one more derivative singular and there is no other nonlinear term of the same order to couple together.

Subsequently, Chemin and Zhang [15] employed anisotropic Littlewood–Paley theory to obtain a regularity criterion for the 3-D NS equations in terms of only $u^3$ in a scaling-invariant space, although in Sobolev space $H^{4+\frac{1}{p}}(\mathbb{R}^3)$ for $p \in (4, 6)$ rather than Lebesgue space $L^p(\mathbb{R}^3)$ (see also [16]). This approach of anisotropic Littlewood–Paley theory was also successfully extended to the MHD system in [44] by the discovery of some crucial cancellations, e.g., [44, equations (21)–(22)] (see also [21,31]). Again, to the best of the authors’ knowledge, an attempt to extend this approach to the Hall-MHD system has not been done due to the difficulty of the Hall term (see also [39] for recent development on another approach).

2. Statement of main results

We overcame the challenges described in Sect. 1.2 and obtained component reduction results of regularity criteria for the $2\frac{1}{2}$-D Hall-MHD system. For clarity, we work on $\mathbb{R}^2$ hereafter although all of our works may be instantly extended to the case $x \in \mathbb{T}^2$ by straight-forward modifications. One immediate convenience of working on the $2\frac{1}{2}$-D Hall-MHD system is that an $H^1(\mathbb{R}^2)$-bound on $(u, b)$ immediately implies higher regularity. This can be seen from the fact that an $H^1(\mathbb{R}^2)$-bound gives $\int_0^T \|\Delta b\|_{L^2}^2 dt < \infty$ from the magnetic diffusive term, that $BMO(\mathbb{R}^d)$ norm may be bounded by $H^{\frac{2}{3}}(\mathbb{R}^d)$ norm (e.g., [3, Theorem 1.48]), and (5); this also shows that $H^1(\mathbb{R}^2)$-bound does not suffice for a bootstrap to higher regularity in the 3-D case considering (4). For clarity, let us note that in the $2\frac{1}{2}$-D case

$$j = \nabla \times b = (\partial_2 b_3 - \partial_1 b_3, \partial_1 b_2 - \partial_2 b_1)^T.$$  \hspace{1cm} (10)

Our first result is a criterion in terms of $\nabla b_3$ or equivalently $j_1$ and $j_2$; because we get it at the level of $\frac{2}{p} + \frac{2}{r} \leq 1$, $p \in (2, \infty)$, these are direct improvements of (6). The heuristic that led us to pursue this result is as follows. On the one hand, the $2\frac{1}{2}$-D MHD system is globally well-posed in $H^m(\mathbb{R}^2)$ for any $m > 2$; on the other hand, if $b_3 \equiv 0$, then the Hall term within the $2\frac{1}{2}$-D Hall-MHD system decouples and the resulting system is globally well-posed. Such intuitions indicate that even if $b_3$ does not necessarily vanish, as long as it is sufficiently small or bounded in some norm, the global well-posedness of the $2\frac{1}{2}$-D Hall-MHD system may still hold; the following result answers this question positively.

**Theorem 2.1.** Suppose that $(u_0, b_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ where $m > 2$ is an integer and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. If $(u, b)$ is a corresponding local smooth solution to the $2\frac{1}{2}$-D Hall-MHD system (1) emanating from $(u_0, b_0)$ and

$$\int_0^T \|\nabla b_3\|_{L^p}^2 dt = \sum_{k=1}^2 \int_0^T \|j_k\|_{L^p}^2 dt < \infty \quad \text{where} \quad \frac{2}{p} + \frac{2}{r} \leq 1, \ p \in (2, \infty),$$  \hspace{1cm} (11)

then for all $t \in [0, T]$ \hspace{1cm} (12)

$$\|u(t)\|_{H^m} + \|b(t)\|_{H^m} < \infty.$$  \hspace{1cm}

The proof of Theorem 2.1 is not a two-step approach as in [26] (recall (7)); we directly work on the $H^1(\mathbb{R}^2)$-estimate and discover crucial cancellations (35) and (38). Thus, we are able to attain the criterion at the level of $\frac{2}{p} + \frac{2}{r} \leq 1$, $p \in (2, \infty)$, somewhat similarly to the works of [28, 41].
Next, having obtained a criterion in terms of $j_1$ and $j_2$ rather than $j$, a natural question is whether we can reduce to only one component of the current density which would be a further improvement of (5); that is the content of our next result.

**Theorem 2.2.** Suppose that $(u_0, b_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ where $m > 2$ is an integer and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. If $(u, b)$ is a corresponding local smooth solution to the $2\frac{1}{2}$-D Hall-MHD system (1) emanating from $(u_0, b_0)$ and

$$\int_0^T \|j_3\|_{L^p}^r dt < \infty \text{ where } \frac{2}{p} + \frac{2}{r} \leq 1, p \in (2, \infty],$$  \hspace{1cm} (13)

then for all $t \in [0, T]$ (12) holds.

(Cf. [42] in which a regularity criterion for 3-D MHD system in terms of $u_3$ and $j_3$ was obtained).

Next, having obtained a criterion in terms of $j_1$ and $j_2$ or just $j_3$ alone, a natural question is whether we can obtain a similar result in terms of components of velocity or vorticity $\omega = \nabla \times u$. Intuitively, this seems very difficult. As we mentioned, the $2\frac{1}{2}$-D MHD system is globally well-posed and the only difference with the $2\frac{1}{2}$-D Hall-MHD system is the Hall term $\nabla \times (j \times b)$. Upon any energy estimate such as the bounds on $L^2(\mathbb{R}^2), L^p(\mathbb{R}^2)$ for $p \in (2, \infty], \dot{H}^1(\mathbb{R}^2)$ or $\dot{H}^2(\mathbb{R}^2)$, the Hall term is always multiplied by a magnetic field again; therefore, any bound on $u$ does not seem to offer any help to handle the Hall term. The trick is that, as we will see in (45), we may rewrite the Hall term as $\nabla \times ((b \cdot \nabla)b)$ and realize that this term appears naturally in the equation of vorticity $\omega$ (see equation (43)). Hence, upon any estimate on the vorticity $\omega$, this “Hall term within the equation of vorticity” gets multiplied by vorticity $\omega$, giving us a chance to show that a sufficient bound on some components of vorticity may allow one to bound this nonlinear term. The following result answers this question positively.

**Theorem 2.3.** Suppose that $\nu = \eta = \epsilon = 1$. Suppose that $(u_0, b_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ where $m > 2$ is an integer and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$.

1. If $(u, b)$ is a corresponding local smooth solution to the $2\frac{1}{2}$-D Hall-MHD system (1) emanating from $(u_0, b_0)$ and

$$\int_0^T \|\Delta(\Delta u_3)\|_{L^p}^r dt < \infty \text{ where } \frac{2}{p} + \frac{2}{r} \leq 1, p \in (2, \infty],$$  \hspace{1cm} (14)

or

$$\int_0^T \|\Delta(\Delta u_3)\|_{BMO}^2 dt < \infty,$$  \hspace{1cm} (15)

then for all $t \in [0, T]$ (12) holds.

2. If $(u, b)$ is a corresponding local smooth solution to the $2\frac{1}{2}$-D Hall-MHD system (1) emanating from $(u_0, b_0)$ and

$$\int_0^T \|\Delta(\Delta u_k)\|_{L^{p_k}}^{r_k} dt < \infty \text{ where } \frac{2}{p_k} + \frac{2}{r_k} \leq 1, p_k \in (2, \infty),$$  \hspace{1cm} (16)

or

$$\int_0^T \|\Delta(\Delta u_k)\|_{BMO}^2 dt < \infty,$$  \hspace{1cm} (17)

for both $k \in \{1, 2\}$, then for all $t \in [0, T]$ (12) holds.
After this work was completed, we were informed of the results in [24] in which the authors also obtained several regularity criteria of the 3-D Hall-MHD system in terms of \( u \) in a high-order norm, e.g., \( \omega \in L^2_t \dot{H}^m_x \) for \( m > \frac{3}{2} \) in [24, Theorem 1.3]. In comparison, while Theorem 2.3 is concerned with the \( 2\frac{1}{2} \)-D case, its criteria are on only a few components of \( u \).

Our last result in the \( 2\frac{1}{2} \)-D case is about global well-posedness of a certain system of equations with magnetic hyper-diffusion. In [45, Theorem 2.3], the global well-posedness of the following system was shown:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla \pi &= \nu \Delta u + (b \cdot \nabla) b & t > 0, \\
\partial_t b + (u \cdot \nabla) b + \eta \Lambda^3 b &= (b \cdot \nabla) u - \epsilon \nabla \times (j \times b) & t > 0,
\end{align*}
\]

where \( \Lambda^r \triangleq (-\Delta)^{\frac{r}{2}} \) for \( r \in \mathbb{R} \) is a Fourier operator with a symbol of \( |\xi|^r \), i.e., \( \mathcal{F}(\Lambda^r f)(\xi) = |\xi|^r \mathcal{F}(f)(\xi) \) with \( \mathcal{F} \) representing the Fourier transform. The intuition behind why we need precisely one more derivative in the magnetic diffusion, namely \( \Lambda^3 b \) rather than \( -\Delta b \), is similar to the explanation in Remark 1.1. Inspired partially by [46] concerning the 3-D NS equations with various powers of fractional diffusion in different directions, we consider the following system:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla \pi &= \nu \Delta u + (b \cdot \nabla) b & t > 0, \\
\partial_t b + (u \cdot \nabla) b + \eta_1 \Lambda^3 b + \eta_2 \Lambda^2 b_v &= (b \cdot \nabla) u - \epsilon \nabla \times (j \times b) & t > 0,
\end{align*}
\]

where

\[
b_h \triangleq (b_1 \ b_2 \ 0)^T \quad \text{and} \quad b_v \triangleq (0 \ 0 \ b_3)^T.
\]

Clearly, the system (19) has weaker magnetic diffusion in the vertical direction than (18); in fact, it has the same magnetic diffusion strength as the classical Hall-MHD system. Nonetheless, we are able to prove its global well-posedness as a consequence of the proofs of Theorems 2.1–2.3.

**Theorem 2.4.** Suppose that \((u_0, b_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)\) where \( m > 2 \) is an integer and \( \nabla \cdot u_0 = \nabla \cdot b_0 = 0 \). Then, there exists a unique solution

\[
u \in L^\infty((0, \infty); H^m(\mathbb{R}^2)) \cap L^2((0, \infty); H^{m+1}(\mathbb{R}^2)),
\]

\[
b_h \in L^2((0, \infty); H^{m+\frac{1}{2}}(\mathbb{R}^2)),
\]

\[
b_v \in L^2((0, \infty); H^{m+1}(\mathbb{R}^2))
\]

such that

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla \pi + \nu \Lambda^\frac{3}{2} u &= (b \cdot \nabla) b, & t > 0, \\
\partial_t b + (u \cdot \nabla) b + \eta_1 \Lambda^\frac{3}{2} b_h + \eta_2 \Lambda^\frac{3}{2} b_v &= (b \cdot \nabla) u - \epsilon \nabla \times (j \times b), & t > 0,
\end{align*}
\]

As we emphasized, the crux of the proofs of Theorems 2.1–2.4 rely on the cancellations (35) and (38). After obtaining Theorems 2.1–2.4, the second author Yamazaki shared these results with Prof. Mimi Dai and her comments inspired us to further try to generalize to the 3-D case. Before hearing her comments, we thought that the cancellations that we discovered, specifically (35) and (38), are valid only in the \( 2\frac{1}{2} \)-D case due to the simplicity in low dimension. Hence, we were quite surprised when we realized that the cancellations that we discovered can indeed be extended to the 3-D case (see (88), (91), and (94)), shedding light on the structure of the Hall term in general. As we elaborated already, an \( H^1(\mathbb{R}^3) \)-bound on the 3-D Hall-MHD system does not suffice to lead to higher regularity; thus, at the time of writing this manuscript, it is not clear how to extend Theorems 2.1–2.3 to the 3-D case. On the other hand, via new cancellations (88), (91), and (94), we can extend Theorem 2.4 to the 3-D case as follows. Under same notations of \( b_h \) and \( b_v \) from (20), we consider

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla \pi + \nu \Lambda^\frac{3}{2} u &= (b \cdot \nabla) b, & t > 0, \\
\partial_t b + (u \cdot \nabla) b + \eta_1 \Lambda^\frac{3}{2} b_h + \eta_2 \Lambda^\frac{3}{2} b_v &= (b \cdot \nabla) u - \epsilon \nabla \times (j \times b), & t > 0.
\end{align*}
\]
Theorem 2.5. Suppose that \((u_0, b_0) \in H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3)\) where \(m > \frac{5}{2}\) is an integer and \(\nabla \cdot u_0 = \nabla \cdot b_0 = 0\). Then, there exists a unique solution
\[
u \in L^\infty((0, \infty); H^m(\mathbb{R}^3)) \cap L^2((0, \infty); H^{m+\frac{5}{4}}(\mathbb{R}^3)), b \in L^\infty((0, \infty); H^m(\mathbb{R}^3))
\]
such that
\[
b_h \in L^2((0, \infty); H^{m+\frac{7}{4}}(\mathbb{R}^3)), b_v \in L^2((0, \infty); H^{m+\frac{5}{4}}(\mathbb{R}^3))
\]
to (23), and \((u, b)|_{t=0} = (u_0, b_0)\).

We note that Theorem 2.5 improves [47, Corollary 1.4] which claimed the global well-posedness of the system (23) when \(\Lambda \frac{5}{2} b_v\) in (23b) is replaced by \(\Lambda \frac{7}{2} b_v\).

Remark 2.1. To the best of our knowledge, Theorems 2.1–2.3 present first component reduction results for regularity criteria of the Hall-MHD system. Moreover, Theorems 2.4–2.5 present first global well-posedness results for the Hall-MHD system with magnetic diffusion that is weaker than the critical threshold for the MHD system added by one more derivative, i.e., \(\eta_h \Lambda^3 b_h + \eta_v \Lambda^2 b_v\) in (18b) and \(\eta_h \Lambda^\frac{5}{2} b_h + \eta_v \Lambda^\frac{7}{2} b_v\) in (23b) are both weaker by one derivative in the vertical component than the hyper-diffusion that we previously thought we need to attain global well-posedness (e.g., [45, Theorem 2.3] and [47, Corollary 1.4]).

1. Concerning Theorems 2.1–2.3, a natural question may be whether or not one can obtain a regularity criterion in terms of partial derivatives of the pressure \(\pi\) (see, e.g., [5, 49] on a criterion in terms of \(\partial_3 \pi\) for the 3-D NS equations, and [6] for the 3-D MHD system). Another direction will be to extend such component reduction results to the 3-D Hall-MHD system (see, e.g., [43] concerning the regularity criteria in terms of \(u_3\) and \(u_4\) for the 4-D NS equations).

2. Concerning Theorems 2.4–2.5, reducing the strength of the hyper-diffusion of magnetic field furthermore will certainly be of great interest and significance.

3. Finally, it would be of interest to extend Theorems 2.1–2.4 to the Hall-MHD system with an ion-slip effect (see, e.g., [22]).

3. Proofs of Theorems 2.1–2.4

Before we start our proofs, for the convenience of readers we recall Gronwall’s inequality in both differential and integral form as follows:

Lemma 3.1. 1. (e.g., [20, Appendix B]) Let \(\eta\) be a non-negative, absolutely continuous function on \([0, T]\) which satisfies for almost every \(t \in [0, T]\)
\[
\eta'(t) \leq \phi(t) \eta(t) + \psi(t)
\]
where \(\phi(t)\) and \(\psi(t)\) are non-negative, integrable functions on \([0, T]\). Then,
\[
\eta(t) \leq e^{\int \phi(s)ds} \left[ \eta(0) + \int_0^t \psi(s)ds \right] \quad \forall \ t \in [0, T].
\]

2. (e.g., [38, p. 317]) Let \(\phi\) be a continuous function in \([0, T]\) which satisfies for every \(t \in [0, T]\)
\[
\phi(t) \leq \int_0^t h(s) \phi(s)ds + \alpha
\]
where \(\alpha \in \mathbb{R}\) and \(h(t)\) is a non-negative and integrable function on \([0, T]\). Then,
\[
\phi(t) \leq \alpha e^{\int_0^t h(s)ds}.
\]
We also list special cases of the Gagliardo–Nirenberg inequalities that will be used throughout our proofs: for any \( f \) that is sufficiently smooth,

\[
\|f\|_{L^4(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \tag{26a}
\]

\[
\|f\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}^{\frac{6}{7}} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{7}} \text{ if } p \in (2, \infty), \tag{26b}
\]

\[
\|f\|_{L^\infty(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\Delta f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \tag{26c}
\]

\[
\|f\|_{L^4(\mathbb{R}^3)} \lesssim \|f\|_{H^\frac{1}{4}(\mathbb{R}^3)} \|\Lambda f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}, \tag{26d}
\]

We refer to [33] for details concerning the Gagliardo–Nirenberg inequality and to [3] for more modern treatments via Fourier analysis. We will also rely on Hölder’s and Young’s inequalities, which can be found in various textbooks such as [20, Appendix B].

### 3.1. Proof of Theorem 2.1

We take \( L^2(\mathbb{R}^2) \)-inner products on (1a)–(1b) with \((u, b)\) and use the fact that

\[
\int_{\mathbb{R}^2} \nabla \times (j \times b) \cdot b dx = \int_{\mathbb{R}^2} (j \times b) \cdot j dx = 0 \tag{27}
\]

to deduce the energy identity for all \( t \in [0, T] \):

\[
\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2 \int_0^t \nu\|\nabla u\|_{L^2}^2 + \eta\|\nabla b\|_{L^2}^2 ds = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{28}
\]

Next, we take \( L^2(\mathbb{R}^2) \)-inner products on (1a)–(1b) with \((-\Delta u, -\Delta b)\) to deduce

\[
\frac{1}{2} \partial_t (\|\nabla u\|^2_{L^2} + \|\nabla b\|^2_{L^2}) + \nu\|\Delta u\|^2_{L^2} + \eta\|\Delta b\|^2_{L^2} = \sum_{i=1}^{5} I_i, \tag{29}
\]

where

\[
I_1 \triangleq \int_{\mathbb{R}^2} (u \cdot \nabla) u \cdot \Delta u dx, \tag{30a}
\]

\[
I_2 \triangleq \int_{\mathbb{R}^2} (u \cdot \nabla) b \cdot \Delta b dx, \tag{30b}
\]

\[
I_3 \triangleq - \int_{\mathbb{R}^2} (b \cdot \nabla) b \cdot \Delta u dx, \tag{30c}
\]

\[
I_4 \triangleq - \int_{\mathbb{R}^2} (b \cdot \nabla) u \cdot \Delta b dx, \tag{30d}
\]

\[
I_5 \triangleq \epsilon \int_{\mathbb{R}^2} \nabla \times (j \times b) \cdot \Delta b dx. \tag{30e}
\]

The estimates on \( I_1 - I_4 \) are immediate as follows: by integration by parts, Hölder’s and Young’s inequalities, as well as the Gagliardo–Nirenberg inequality (26a)

\[
I_1 + I_2 + I_3 + I_4 \lesssim \|\nabla u\|_{L^2}(\|\nabla u\|_{L^4}^2 + \|\nabla b\|_{L^4}^2) \lesssim \|\nabla u\|_{L^2}(\|\nabla u\|_{L^2} \|\Delta u\|_{L^2} + \|\nabla b\|_{L^2} \|\Delta b\|_{L^2}).
\]
\[ \frac{\nu}{2} \| \Delta u \|_{L^2}^2 + \frac{\eta}{4} \| \Delta b \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 (\| \nabla u \|_{L^2}^2 + \| \nabla b \|_{L^2}^2). \]  

(31)

The heart of the matter is, of course, the Hall term. For clarity, we first split

\[ I_5 = I_{5,1} + I_{5,2} \quad \text{where} \quad I_{5,1} \triangleq \epsilon \int_{\mathbb{R}^2} \nabla \times (j \times b) \cdot \partial^2_i b dx \quad \text{and} \quad I_{5,2} \triangleq \epsilon \int_{\mathbb{R}^2} \nabla \times (j \times b) \cdot \partial^2_j b dx. \]  

(32)

Now, we carefully compute

\[ I_{5,1} \overset{(32)}{=} \epsilon \int_{\mathbb{R}^2} (j \times b) \cdot \partial^2_i j dx = -\epsilon \int_{\mathbb{R}^2} (\partial_1 j \times b) \cdot \partial_1 j + (j \times \partial_1 b) \cdot \partial_1 j dx \]

\[ = -\epsilon \int_{\mathbb{R}^2} (j \times \partial_1 b) \cdot \partial_1 j dx = \sum_{i=1}^{6} I_{5,1,i} \]  

(33)

where

\[ I_{5,1,1} \triangleq -\epsilon \int_{\mathbb{R}^2} j_2 \partial_1 b_1 \partial_1 j_1 dx, \]

(34a)

\[ I_{5,1,2} \triangleq \epsilon \int_{\mathbb{R}^2} j_3 \partial_1 b_2 \partial_1 j_1 dx, \]

(34b)

\[ I_{5,1,3} \triangleq \epsilon \int_{\mathbb{R}^2} j_1 \partial_1 b_3 \partial_1 j_2 dx, \]

(34c)

\[ I_{5,1,4} \triangleq -\epsilon \int_{\mathbb{R}^2} j_3 \partial_1 b_1 \partial_1 j_2 dx, \]

(34d)

\[ I_{5,1,5} \triangleq -\epsilon \int_{\mathbb{R}^2} j_1 \partial_1 b_2 \partial_1 j_3 dx, \]

(34e)

\[ I_{5,1,6} \triangleq \epsilon \int_{\mathbb{R}^2} j_2 \partial_1 b_1 \partial_1 j_3 dx. \]

(34f)

We make the key observation that \( I_{5,1,1} \) and \( I_{5,1,3} \) together cancel out as follows:

\[ I_{5,1,1} + I_{5,1,3} \overset{(34)}{=} -\epsilon \int_{\mathbb{R}^2} j_2 \partial_1 b_3 \partial_1 j_1 - j_1 \partial_1 b_3 \partial_1 j_2 dx \]

\[ = -\epsilon \int_{\mathbb{R}^2} -\partial_1 b_3 \partial_1 j_1 - \partial_2 b_3 \partial_2 j_1 + \partial_3 b_3 \partial_3 j_1 \partial_1 b_3 dx \]

\[ = -\epsilon \int_{\mathbb{R}^2} -\partial_1 b_3 \frac{1}{2} \partial_2 (\partial_1 b_3)^2 + \partial_2 b_3 \frac{1}{2} \partial_1 (\partial_1 b_3)^2 dx \]

\[ = -\epsilon \int_{\mathbb{R}^2} \frac{1}{2} \partial_1 \partial_3 b_3 (\partial_1 b_3)^2 - \frac{1}{2} \partial_1 \partial_2 b_3 (\partial_1 b_3)^2 dx = 0. \]  

(35)

Analogously, we compute

\[ I_{5,2} \overset{(32)}{=} \epsilon \int_{\mathbb{R}^2} (j \times b) \cdot \partial^2_j j dx = -\epsilon \int_{\mathbb{R}^2} (\partial_2 j \times b) \cdot \partial_2 j + (j \times \partial_2 b) \cdot \partial_2 j dx \]
\[
\begin{align*}
\mathcal{I}_{5,2,1} & \triangleq -\epsilon \int_{\mathbb{R}^2} j_2 \partial_2 b_3 \partial_2 j_1 \, dx, \\
\mathcal{I}_{5,2,2} & \triangleq \epsilon \int_{\mathbb{R}^2} j_3 \partial_2 b_2 \partial_2 j_1 \, dx, \\
\mathcal{I}_{5,2,3} & \triangleq \epsilon \int_{\mathbb{R}^2} j_1 \partial_2 b_3 \partial_2 j_2 \, dx, \\
\mathcal{I}_{5,2,4} & \triangleq -\epsilon \int_{\mathbb{R}^2} j_3 \partial_2 b_1 \partial_2 j_2 \, dx, \\
\mathcal{I}_{5,2,5} & \triangleq -\epsilon \int_{\mathbb{R}^2} j_1 \partial_2 b_2 \partial_2 j_3 \, dx, \\
\mathcal{I}_{5,2,6} & \triangleq \epsilon \int_{\mathbb{R}^2} j_2 \partial_2 b_1 \partial_2 j_3 \, dx.
\end{align*}
\]

Again, we make the key observation that \(\mathcal{I}_{5,2,1}\) and \(\mathcal{I}_{5,2,3}\) together cancel out as follows:

\[
\mathcal{I}_{5,2,1} + \mathcal{I}_{5,2,3} \overset{(37)}{=} -\epsilon \int_{\mathbb{R}^2} j_2 \partial_2 b_3 \partial_2 j_1 - j_1 \partial_2 b_3 \partial_2 j_2 \, dx
\]

\[
= -\epsilon \int_{\mathbb{R}^2} -\partial_1 b_3 \partial_2 b_3 \partial_3 \partial_2 b_2 + \partial_2 b_3 \partial_3 \partial_2 b_2 \partial_1 b_3 \, dx
\]

\[
= -\epsilon \int_{\mathbb{R}^2} -\partial_1 b_3 \frac{1}{2} \partial_2 (\partial_2 b_2) + \partial_2 b_3 \frac{1}{2} \partial_1 (\partial_2 b_2) \, dx
\]

\[
= -\epsilon \int_{\mathbb{R}^2} \frac{1}{2} \partial_1 \partial_2 b_3 (\partial_2 b_2) - \frac{1}{2} \partial_1 \partial_2 b_3 (\partial_2 b_2) \, dx = 0. \tag{38}
\]

Thanks to (35) and (38), we only have to estimate \(\mathcal{I}_{5,1,i}\) and \(\mathcal{I}_{5,2,i}\) for \(i \in \{2, 4, 5, 6\}\) and it turns out that we can immediately obtain from (34) and (37)

\[
\begin{align*}
\mathcal{I}_{5,1,2} & = \epsilon \int_{\mathbb{R}^2} j_3 \partial_1 b_2 \partial_1 j_1 \, dx = -\epsilon \int_{\mathbb{R}^2} \partial_1 (j_3 \partial_1 b_2 \partial_1 j_1) \, dx \lesssim \int_{\mathbb{R}^2} |j_1| ||\nabla b|| \nabla^2 b| \, dx, \tag{39a}
\\
\mathcal{I}_{5,1,4} & = -\epsilon \int_{\mathbb{R}^2} j_3 \partial_1 b_1 \partial_1 j_2 \, dx = \epsilon \int_{\mathbb{R}^2} \partial_1 (j_3 \partial_1 b_1) j_2 \, dx \lesssim \int_{\mathbb{R}^2} |j_2| ||\nabla b|| \nabla^2 b| \, dx, \tag{39b}
\\
\mathcal{I}_{5,1,5} & = -\epsilon \int_{\mathbb{R}^2} j_1 \partial_1 b_2 \partial_1 j_3 \, dx \lesssim \int_{\mathbb{R}^2} |j_1| ||\nabla b|| \nabla^2 b| \, dx, \tag{39c}
\\
\mathcal{I}_{5,1,6} & = \epsilon \int_{\mathbb{R}^2} j_2 \partial_1 b_1 \partial_1 j_3 \, dx \lesssim \int_{\mathbb{R}^2} |j_2| ||\nabla b|| \nabla^2 b| \, dx. \tag{39d}
\end{align*}
\]
In fact, the cancellations (35) and (38) are not necessary for this proof of Theorem 2.1 because we can immediately accomplish this task as follows: from (34) and (37)

\[ I_{5,2,2} = \epsilon \int \nabla b \, dx = \epsilon \int \nabla b \, dx \lesssim \int |j_1| \, dx, \]

(39e)

\[ I_{5,2,4} = -\epsilon \int \nabla b \, dx = \epsilon \int \nabla b \, dx \lesssim \int |j_2| \, dx, \]

(39f)

\[ I_{5,2,5} = -\epsilon \int j_1 \, dx \lesssim \int |j_1| \, dx, \]

(39g)

\[ I_{5,2,6} = \epsilon \int j_2 \, dx \lesssim \int |j_2| \, dx. \]

(39h)

In fact, the cancellations (35) and (38) are not necessary for this proof of Theorem 2.1 because we can bound

\[ I_{5,1,1} + I_{5,2,1} \lesssim \int |j_2| \, dx \quad \text{and} \quad I_{5,1,3} + I_{5,2,3} \lesssim \int |j_1| \, dx. \]

Nonetheless, they simplified our computations and will become absolutely necessary in the proofs of Theorems 2.2, 2.3 (1), and 2.4 (see Remarks 3.1, 3.2, and 3.3).

Now we are ready to conclude the proof of Theorem 2.1; for \( p \in (2, \infty) \), interpreting \( \frac{2p}{p-2} = 2 \) when \( p = \infty \), we compute via Hölder’s and Young’s inequalities, as well as the Gagliardo–Nirenberg inequality (26b),

\[ I_5 \overset{(32)\,(33)\,(36)}{=} \sum_{i=1}^{6} I_{5,1,i} + I_{5,2,i} \overset{(39)}{\lesssim} \int \nabla b \, dx \lesssim \sum_{i=1}^{6} \int j_i \, dx \lesssim \sum_{i=1}^{6} \int j_i \, dx. \]

(39)

Applying (40) and (31) to (29) and Gronwall’s inequality and relying on that \( \nabla u, \nabla b \in L^p_x \) from (28) complete the proof of Theorem 2.1.

### 3.2. Proof of Theorem 2.2

The proof of Theorem 2.2 almost immediately follows from that of Theorem 2.1. We continue to rely on all the computations (28)–(38) so that the proof of Theorem 2.1 shows that we only have to bound \( I_{5,1,i} \) and \( I_{5,2,i} \) for \( i \in \{2, 4, 5, 6\} \) by constant multiples of \( \int |j_3| \nabla b \, dx \) this time, in contrast to (39). We immediately accomplish this task as follows: from (34) and (37)

\[ I_{5,1,2} = \epsilon \int j_3 \, dx \lesssim \int |j_3| \, dx, \]

(41a)

\[ I_{5,1,4} = -\epsilon \int j_3 \, dx \lesssim \int |j_3| \, dx, \]

(41b)

\[ I_{5,1,5} = -\epsilon \int j_1 \, dx \lesssim \int |j_1| \, dx, \]

(41c)

\[ I_{5,1,6} = \epsilon \int j_2 \, dx \lesssim \int |j_2| \, dx. \]

(41d)
\[ I_{5,2,2} = \epsilon \int_{\mathbb{R}^2} j_3 \partial_2 b_2 \partial_2 j_1 \, dx \lesssim \int_{\mathbb{R}^2} |j_3| |\nabla b| |\nabla^2 b| \, dx, \]  
(41e)  
\[ I_{5,2,4} = -\epsilon \int_{\mathbb{R}^2} j_3 \partial_2 b_1 \partial_2 j_2 \, dx \lesssim \int_{\mathbb{R}^2} |j_3| |\nabla b| |\nabla^2 b| \, dx, \]  
(41f)  
\[ I_{5,2,5} = -\epsilon \int_{\mathbb{R}^2} j_1 \partial_2 b_2 \partial_2 j_3 \, dx = \epsilon \int_{\mathbb{R}^2} \partial_2 (j_1 \partial_2 b_2) j_3 \, dx \lesssim \int_{\mathbb{R}^2} |j_3| |\nabla b| |\nabla^2 b| \, dx, \]  
(41g)  
\[ I_{5,2,6} = \epsilon \int_{\mathbb{R}^2} j_2 \partial_2 b_1 \partial_2 j_3 \, dx = -\epsilon \int_{\mathbb{R}^2} \partial_2 (j_2 \partial_2 b_1) j_3 \, dx \lesssim \int_{\mathbb{R}^2} |j_3| |\nabla b| |\nabla^2 b| \, dx. \]  
(41h)

**Remark 3.1.** Here, we stress that the cancellations (35) and (38) are crucial because \( j_3 \) is absent in any of \( I_{5,1,1} \) and \( I_{5,1,3} \) in (34) and \( I_{5,2,1} \) and \( I_{5,2,3} \) in (37).

Following the same argument in the proof of Theorem 2.1 immediately completes the proof of Theorem 2.2.

### 3.3. Proof of Theorem 2.3

First, we note that the energy identity (28) with \( \nu = \eta = 1 \) remains valid providing us the regularity of

\[ u, b \in L^\infty_T L^2_x \cap L^2_T \dot{H}^1_x. \]  
(42)

The proof will consist of a few steps; for clarity we divide it into three propositions. For the first proposition, let us write down the equation of the motion of vorticity \( \omega = \nabla \times u \):

\[ \partial_t \omega + (u \cdot \nabla) \omega = \Delta \omega + (\omega \cdot \nabla) u + \nabla \times ((b \cdot \nabla)b). \]  
(43)

**Proposition 3.2.** Under the hypothesis of Theorem 2.3, let \( \omega \) be a smooth solution to (43) over \([0,T]\).

Then

\[ \omega \in L^\infty_T L^2_x \cap L^2_T \dot{H}^1_x. \]  
(44)

**Proof of Proposition 3.2.** We recall the vector calculus identity of

\[ (\nabla \times \Theta) \times \Theta = -\nabla \left( \frac{|\Theta|^2}{2} \right) + (\Theta \cdot \nabla) \Theta \quad \forall \quad \Theta \in \mathbb{R}^3 \]

so that we may rewrite the Hall term as

\[ \nabla \times (j \times b) = \nabla \times \left[ -\nabla \left( \frac{|b|^2}{2} \right) + (b \cdot \nabla) b \right] = \nabla \times ((b \cdot \nabla)b). \]  
(45)

Then we define

\[ z^1 \triangleq b + \omega; \]  
(46)

we note that this trick was used by Chae and Wolf in [13, equation (2.7)]. Then we see from (1), (43), and (45) that the equation of motion of \( z^1 \) is given by

\[ \partial_t z^1 + (u \cdot \nabla) z^1 = (z^1 \cdot \nabla) u + \Delta z^1. \]  
(47)

We take \( L^2(\mathbb{R}^2) \)-inner products with \( z^1 \) in (47) and compute by relying on Hölder’s and Young’s inequalities, as well as the Gagliardo–Nirenberg inequality (26a),

\[ \frac{1}{2} \partial_t \|z^1\|^2_{L^2} + \|\nabla z^1\|^2_{L^2} = \int_{\mathbb{R}^2} (z^1 \cdot \nabla) u \cdot z^1 \, dx \]
We see that the governing equation of \( z^1 \) is
\[
\partial_t z^1 + (u \cdot \nabla) z^1 = -(\omega \cdot \nabla) z^1 + \Delta z^2 + (z^1 \cdot \nabla) \omega + (z^2 \cdot \nabla) u + 2 \begin{pmatrix} 0 \\ 0 \\ \partial_t z^1 \cdot \partial_2 u - \partial_2 z^1 \cdot \partial_1 u \end{pmatrix}.
\] (52)

**Proposition 3.3.** Under the hypothesis of Theorem 2.3, let \( z^2 \) be a smooth solution to (52) over \([0, T]\).

Then,
\[
z^2 \in L^\infty_T L^2_x \cap L^2_T \dot{H}^1_x.
\] (53)

**Proof of Proposition 3.3.** We take \( L^2(\mathbb{R}^2) \)-inner products on (52) with \( z^2 \) and estimate via H"older’s and Young’s inequalities, as well as the Gagliardo–Nirenberg inequality (26a)
\[
\frac{1}{2} \partial_t \| z^2 \|_{L^2}^2 + \| \nabla z^2 \|_{L^2}^2 \\
\leq \| \omega \|_{L^4} \| \nabla z^2 \|_{L^2} \| z^1 \|_{L^4} + \| z^1 \|_{L^4} \| \nabla z^2 \|_{L^2} \| \omega \|_{L^4} + \| z^2 \|_{L^4} \| \nabla u \|_{L^2} + \| \nabla z^1 \|_{L^2} \| \nabla u \|_{L^4} \| z^2 \|_{L^4}
\]
\[
\leq \| \omega \|_{L^2} \| \nabla z^2 \|_{L^2} \| z^1 \|_{L^2} \| z^1 \|_{L^2} \| \nabla \omega \|_{L^2} + \| z^2 \|_{L^2} \| \nabla u \|_{L^2} + \| \nabla z^1 \|_{L^2} \| \omega \|_{L^2} \| \nabla \omega \|_{L^2} \| z^2 \|_{L^2} \\
+ \| z^2 \|_{L^2} \| \nabla \omega \|_{L^2} \| z^1 \|_{L^2} \| z^1 \|_{L^2} \| \nabla \omega \|_{L^2} \| z^2 \|_{L^2} \\
\leq \frac{1}{2} \| \nabla z^2 \|_{L^2}^2 + C(\| \nabla \omega \|_{L^2}^2 + \| \nabla \omega \|_{L^2}^2 + 1)(\| \omega \|_{L^2}^2 + \| z^1 \|_{L^2}^2 + 1) \| z^2 \|_{L^2}^2 + 1).
\] (54)

After subtracting \( \frac{1}{2} \| \nabla z^2 \|_{L^2}^2 \) from both sides of (54), relying on (44) and (49) and applying Gronwall’s inequality allow us to deduce (53).

Now, first we prove Theorem 2.3 part (1) with a criterion in terms of \( \Delta u_3 \) in (14). Considering the definition of \( z^2 \) in (51) and (53), we realize that the \( H^1(\mathbb{R}^2) \)-bound of \( u, b \) is attained once we obtain an \( L^2(\mathbb{R}^2) \)-bound of \( \nabla \times \omega \) that has the governing equation of
\[
\partial_t \nabla \times \omega + \nabla \times ((u \cdot \nabla) \omega) = \Delta \nabla \times \omega + \nabla \times ((\omega \cdot \nabla) u) + \nabla \times \nabla \times ((b \cdot \nabla) b).
\] (55)

In fact, we consider only the third component of (55), namely
\[
\partial_t (\nabla \times \omega)_3 + (\nabla \times ((u \cdot \nabla) \omega)_3 = \Delta (\nabla \times \omega)_3 + (\nabla \times ((\omega \cdot \nabla) u)_3 + (\nabla \times \nabla \times ((b \cdot \nabla) b)_3.
\] (56)

**Proposition 3.4.** Under the hypothesis of Theorem 2.3 (1), let \( \nabla \times \omega \) be a smooth solution to (55) over \([0, T]\). Then,
\[
(\nabla \times \omega)_3 \in L^\infty_T L^2_x \cap L^2_T \dot{H}^1_x.
\] (57)
Proof of Proposition 3.4. We take \( L^2(\mathbb{R}^2) \)-inner products on (56) with \((\nabla \times \omega)_3\) to deduce
\[
\frac{1}{2} \partial_t \| (\nabla \times \omega)_3 \|_{L^2}^2 + \| (\nabla \times \omega)_3 \|_{L^2}^2 = \sum_{i=1}^3 \Pi_i
\]
(58)
where
\[
\Pi_1 \equiv - \int_{\mathbb{R}^2} (\nabla \times ((u \cdot \nabla))_3 (\nabla \times \omega)_3) dx,
\]
(59a)
\[
\Pi_2 \equiv \int_{\mathbb{R}^2} (\nabla \times ((\omega \cdot \nabla))_3 (\nabla \times \omega)_3) dx,
\]
(59b)
\[
\Pi_3 \equiv \int_{\mathbb{R}^2} (\nabla \times \nabla \times ((b \cdot \nabla))_3 (\nabla \times \omega)_3) dx.
\]
(59c)
For convenience in further computations, we note simple identities: any \( f, g \) that are \( \mathbb{R}^3 \)-valued and do not depend on \( x_3 \) satisfy
\[
\int_{\mathbb{R}^2} (\nabla \times f)_1 g_1 + (\nabla \times f)_2 g_2 dx = \int_{\mathbb{R}^2} f_3 (\nabla \times g)_3 dx;
\]
(60)
additionally, if \( g \) is divergence-free, then they satisfy
\[
\int_{\mathbb{R}^2} (\nabla \times f)_3 (\nabla \times g)_3 dx = - \int_{\mathbb{R}^2} (f_1 f_2 0)^T \cdot (\Delta g_1 \Delta g_2 0)^T dx.
\]
(61)
Applying (60)–(61) to (59) leads us to
\[
\Pi_1 = \int_{\mathbb{R}^2} ((u \cdot \nabla) \omega_1 (u \cdot \nabla) \omega_2 0)^T \cdot (\Delta \omega_1 \Delta \omega_2 0)^T dx,
\]
(62a)
\[
\Pi_2 = - \int_{\mathbb{R}^2} ((\omega \cdot \nabla) u_1 (\omega \cdot \nabla) u_2 0)^T \cdot (\Delta \omega_1 \Delta \omega_2 0)^T dx,
\]
(62b)
\[
\Pi_3 = - \int_{\mathbb{R}^2} (b \cdot \nabla) b_3 (\nabla \times \omega)_3 dx.
\]
(62c)
We estimate via Hölder’s and Young’s inequalities, the Gagliardo–Nirenberg inequalities (26a) and (26c), and the fact that \( \omega_1 = \partial_2 u_3, \omega_2 = - \partial_1 u_3, \)
\[
\Pi_1 \lesssim \sum_{k=1}^2 \| u \|_{L^\infty} \| \nabla \omega_k \|_{L^2} \| \Delta \omega_k \|_{L^2} \lesssim \| u \|_{L^2} \| \Delta u \|_{L^2} \| \nabla u_3 \|_{L^2} \| \Delta u_3 \|_{L^2} \| \Delta \nabla u_3 \|_{L^2}
\]
\[
\lesssim \frac{1}{4} \| \nabla (\nabla \times \omega)_3 \|_{L^2}^2 + C \| u \|_{L^2} \| \Delta u \|_{L^2} \| (\nabla \times \omega)_3 \|_{L^2}^2,
\]
(63)
and
\[
\Pi_2 \lesssim \sum_{k=1}^2 \| \omega \|_{L^4} \| \nabla u_k \|_{L^4} \| \Delta \omega_k \|_{L^2}
\]
\[
\lesssim \| \nabla u \|_{L^2} \| \Delta u \|_{L^2} \| \Delta u_3 \|_{L^2} \leq \frac{1}{4} \| \nabla (\nabla \times \omega)_3 \|_{L^2}^2 + C \| \nabla u \|_{L^2} \| \Delta u \|_{L^2}^2.
\]
(64)
For \( \Pi_3 \), we first consider \( p \in (2, \infty) \) and estimate via Hölder’s inequality and the Gagliardo–Nirenberg inequality (26b)
\[
\Pi_3 \leq \| b \|_{L^{\frac{2p}{2-p}}} \| \nabla b_3 \|_{L^2} \| \Delta (\nabla \times \omega)_3 \|_{L^p} \lesssim \| b \|_{L^2} \| \nabla b_3 \|_{L^2} \| \Delta (\nabla \times \omega)_3 \|_{L^p}.
\]
(65)
Applying (63), (64), and (65) to (58) gives us
\[ \partial_t \| (\nabla \times \omega)_3 \|^2_{L^2} + \| \nabla (\nabla \times \omega)_3 \|^2_{L^2} \leq \|(\nabla \times \omega)_3\|^2_{L^2} \| u \|_{L^2} \| \Delta u \|_{L^2} + \| \nabla u \|^2_{L^2} \| \Delta u \|^2_{L^2} + \| b \|^2_{L^2} \| \nabla b_3 \|^2_{L^2} \| \Delta (\nabla \times \omega)_3 \|_{L^p}. \] (66)

Integrating (66) over time \([0, t]\) and relying on Hölder’s inequality give us
\[ \left\| (\nabla \times \omega)_3 (t) \right\|^2_{L^2} + \int_0^t \left\| (\nabla \times \omega)_3 \right\|^2_{L^2} ds \leq \left\| (\nabla \times \omega)_3 (0) \right\|^2_{L^2} + \int_0^t \left\| (\nabla \times \omega)_3 \right\|^2_{L^2} ds + 1 + \left( \int_0^t \| \Delta (\Delta u_3) \|^2_{BMO} ds \right)^{\frac{2p}{2p-2}}. \] (67)

To obtain a criterion in terms of \(BMO(\mathbb{R}^2)\)-norm in (15), we denote by \(H^1(\mathbb{R}^2)\) the Hardy space and rely on the dual of \(H^1(\mathbb{R}^2)\) being \(BMO(\mathbb{R}^2)\), as well as the div-curl lemma (e.g., [29, Theorem 12.1]) to estimate
\[ \Pi_3 \lesssim \| (b \cdot \nabla) b \|_{H^1} \| \Delta (\nabla \times \omega)_3 \|_{BMO} \lesssim \| b \|_{L^2} \| \nabla b \|_{L^2} \| \Delta (\Delta u_3) \|_{BMO}. \] (68)

We apply (63), (64), and (68) to (58), integrate the resulting equation over time \([0, t]\) and rely on Hölder’s inequality to attain similarly to (67)
\[ \left\| (\nabla \times \omega)_3 (t) \right\|^2_{L^2} + \int_0^t \left\| (\nabla \times \omega)_3 (s) \right\|^2_{L^2} ds \leq \left\| (\nabla \times \omega)_3 (0) \right\|^2_{L^2} + \int_0^t \left\| (\nabla \times \omega)_3 \right\|^2_{L^2} ds + 1 + \left( \int_0^t \| \Delta (\Delta u_3) \|^2_{BMO} ds \right)^{\frac{1}{2}}. \] (69)

Due to the hypothesis of (14) and (15), we see that applying Gronwall’s inequality on (67) and (69) implies the desired result of (57) and hence the proof of Proposition 3.4 is complete.

We are ready to conclude the proof of the first part of Theorem 2.3. By Propositions 3.3–3.4, we realize that
\[ j_3 = \frac{51}{5} \left( z_3^2 - (\nabla \times \omega)_3 \right) \in L_T^\infty L_x^2 \cap L_T^2 H_x^1. \] (70)

We can apply the Gagliardo–Nirenberg inequality (26a) to deduce from (70)
\[ \int_0^T \| j_3 \|^2_{L^1} ds \lesssim \sup_{s \in [0, T]} \| j_3 (s) \|^2_{L^2} \int_0^T \| \nabla j_3 \|^2_{L^2} ds \lesssim 1. \] (71)

Therefore, \( j_3 \in L_T^4 L_x^4 \); this allows us to apply Theorem 2.2 and deduce (12) as desired.

**Remark 3.2.** As we emphasized in Remark 3.1, the cancellations (35) and (38) were crucial in the proof of Theorem 2.2. Because the proof of Theorem 2.3 (1) relied on Theorem 2.2, the proof of Theorem 2.3 (1) in turn also relied crucially on the cancellations (35) and (38).

Next, the proof of the second part of Theorem 2.3 follows same line of reasonings for the first part. In short, we continue to rely on Propositions 3.2–3.3, obtain analogous estimates to Proposition 3.4 in components \( k = 1, 2 \) instead of \( k = 3 \), and thereafter rely on Theorem 2.1 instead of Theorem 2.2. Due to similarity, we leave this in Appendix for completeness.
3.4. Proof of Theorem 2.4

Local existence of the unique solution to (19) starting from the given \((u_0, b_0)\) can be proven following previous works such as [8, Theorem 2.2]. Moreover, the proof of the blow-up criterion from (5) can be easily seen to go through for (19) because (19) has more diffusive strength than the classical Hall-MHD system. Therefore, it suffices to obtain \(H^1(\mathbb{R}^2)\)-bound again because it will imply \(\int_0^T \|\Delta b\|^2_{L^2} dt < \infty\) from the magnetic diffusion which can bound \(\int_0^T \|j\|^2_{BMO} dt < \infty\) (e.g., [3, Theorem 1.48]). Hence, we are able to follow the same line of reasoning in the proof of Theorem 2.1. In fact, as we will emphasize in Remark 3.3, the key to the proof of Theorem 2.4 is once again the cancellations in (35) and (38).

First, relying on (27) we obtain the following energy identity:

\[
\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2 \int_0^t \nu \|\nabla u\|_{L^2}^2 + \eta_h \|\Lambda^{\frac{3}{2}} b_h\|_{L^2}^2 + \eta_v \|\nabla b_v\|_{L^2}^2 ds = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{72}
\]

We take \(L^2(\mathbb{R}^2)\)-inner products on (19) with \((-\Delta u, -\Delta b)\) to deduce

\[
\frac{1}{2} \partial_t (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \nu \|\Delta u\|_{L^2}^2 + \eta_h \|\Lambda^{\frac{3}{2}} b_h\|_{L^2}^2 + \eta_v \|\Delta b_v\|_{L^2}^2 = \sum_{i=1}^5 I_i \tag{73}
\]

where \(I_i\) for \(i \in \{1, \ldots, 5\}\) are same as those in (30), which we recall here for convenience:

\[
I_1 \triangleq \int_{\mathbb{R}^2} (u \cdot \nabla) u \cdot \Delta u dx,
\]

\[
I_2 \triangleq \int_{\mathbb{R}^2} (u \cdot \nabla) b \cdot \Delta b dx,
\]

\[
I_3 \triangleq - \int_{\mathbb{R}^2} (b \cdot \nabla) b \cdot \Delta u dx,
\]

\[
I_4 \triangleq - \int_{\mathbb{R}^2} (b \cdot \nabla) u \cdot \Delta b dx,
\]

\[
I_5 \triangleq \epsilon \int_{\mathbb{R}^2} \nabla \times (j \times b) \cdot \Delta b dx.
\]

Similarly to (31) we can estimate by Hölder’s and Young’s inequalities, the Gagliardo–Nirenberg inequality (26a), as well as Sobolev embedding \(H^\frac{3}{2}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)\),

\[
I_1 + I_2 + I_3 + I_4 \lesssim \|\nabla u\|_{L^2} \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2} \|\nabla b_h\|_{L^2}^2 + \|\nabla u\|_{L^2} \|\nabla b_v\|_{L^2}^2
\]

\[
\lesssim \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} + \|\nabla u\|_{L^2} \|\Lambda^{\frac{3}{2}} b_h\|_{L^2}^2 + \|\nabla u\|_{L^2} \|\nabla b_v\|_{L^2} \|\Delta b_v\|_{L^2}
\]

\[
\lesssim \frac{\nu}{2} \|\Delta u\|_{L^2}^2 + \frac{\eta_v}{2} \|\Delta b_v\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}} b_h\|_{L^2}^2). \tag{74}
\]

For the Hall term \(I_5\) in (30), we split again

\[
I_5 \overset{(32)}{=} I_{5,1} + I_{5,2} \overset{(33)\text{(36)}}{=} \sum_{i=1}^6 I_{5,1,i} + I_{5,2,i} \tag{75}
\]
where $I_{5,1,1} + I_{5,1,3} = I_{5,2,1} + I_{5,2,3} = 0$ due to (35) and (38) so that we only need to estimate $I_{5,1,i}$ and $I_{5,2,i}$ for $i \in \{2, 4, 5, 6\}$ from (34) and (37). It turns out that we may bound them all as follows:

$$I_{5,1,2} = \epsilon \int_{\mathbb{R}^2} j_3 \partial_1 b_2 \partial_1 j_1 dx = \epsilon \int_{\mathbb{R}^2} (\partial_1 b_2 - \partial_2 b_1) \partial_1 b_2 \partial_1 \partial_2 b_3 dx$$

$$= - \epsilon \int_{\mathbb{R}^2} \partial_1 [(\partial_1 b_2 - \partial_2 b_1) \partial_1 b_2] \partial_2 b_3 dx \lesssim \int_{\mathbb{R}^2} |\nabla b_h| |\nabla^2 b_h| |\nabla b| dx,$$

(76a)

$$I_{5,1,4} = - \epsilon \int_{\mathbb{R}^2} j_3 \partial_1 b_1 \partial_1 j_3 dx = \epsilon \int_{\mathbb{R}^2} (\partial_1 b_2 - \partial_2 b_1) \partial_1 b_1 \partial_3 b_3 dx$$

$$= - \epsilon \int_{\mathbb{R}^2} \partial_1 [(\partial_1 b_2 - \partial_2 b_1) \partial_1 b_1] \partial_1 b_3 dx \lesssim \int_{\mathbb{R}^2} |\nabla b_h| |\nabla^2 b_h| |\nabla b| dx,$$

(76b)

$$I_{5,1,5} = - \epsilon \int_{\mathbb{R}^2} j_1 \partial_1 b_2 \partial_3 j_1 dx$$

$$= - \epsilon \int_{\mathbb{R}^2} \partial_3 b_3 \partial_1 b_2 \partial_1 (\partial_1 b_2 - \partial_2 b_1) dx \lesssim \int_{\mathbb{R}^2} |\nabla b| |\nabla b_h| |\nabla^2 b_h| dx,$$

(76c)

$$I_{5,1,6} = \epsilon \int_{\mathbb{R}^2} j_2 \partial_1 b_1 \partial_1 j_3 dx$$

$$= - \epsilon \int_{\mathbb{R}^2} \partial_1 b_3 \partial_1 b_1 \partial_1 (\partial_1 b_2 - \partial_2 b_1) dx \lesssim \int_{\mathbb{R}^2} |\nabla b| |\nabla b_h| |\nabla^2 b_h| dx,$$

(76d)

$$I_{5,2,2} = \epsilon \int_{\mathbb{R}^2} j_3 \partial_2 b_2 \partial_2 j_1 dx = \epsilon \int_{\mathbb{R}^2} (\partial_1 b_2 - \partial_2 b_1) \partial_2 b_2 \partial_2 \partial_3 b_3 dx$$

$$= - \epsilon \int_{\mathbb{R}^2} \partial_2 [(\partial_1 b_2 - \partial_2 b_1) \partial_2 b_2] \partial_3 b_3 dx \lesssim \int_{\mathbb{R}^2} |\nabla b_h| |\nabla^2 b_h| |\nabla b| dx,$$

(76e)

$$I_{5,2,4} = - \epsilon \int_{\mathbb{R}^2} j_3 \partial_2 b_1 \partial_2 j_3 dx = \epsilon \int_{\mathbb{R}^2} (\partial_1 b_2 - \partial_2 b_1) \partial_2 b_1 \partial_2 \partial_1 b_3 dx$$

$$= - \epsilon \int_{\mathbb{R}^2} \partial_2 [(\partial_1 b_2 - \partial_2 b_1) \partial_2 b_1] \partial_1 b_3 dx \lesssim \int_{\mathbb{R}^2} |\nabla b_h| |\nabla^2 b_h| |\nabla b| dx,$$

(76f)

$$I_{5,2,5} = - \epsilon \int_{\mathbb{R}^2} j_1 \partial_2 b_2 \partial_2 j_3 dx$$

$$= - \epsilon \int_{\mathbb{R}^2} \partial_2 b_3 \partial_2 b_2 \partial_2 (\partial_1 b_2 - \partial_2 b_1) dx \lesssim \int_{\mathbb{R}^2} |\nabla b| |\nabla b_h| |\nabla^2 b_h| dx,$$

(76g)

$$I_{5,2,6} = \epsilon \int_{\mathbb{R}^2} j_2 \partial_2 b_1 \partial_2 j_3 dx$$

$$= - \epsilon \int_{\mathbb{R}^2} \partial_1 b_3 \partial_2 b_1 \partial_2 (\partial_1 b_2 - \partial_2 b_1) dx \lesssim \int_{\mathbb{R}^2} |\nabla b| |\nabla b_h| |\nabla^2 b_h| dx.$$

(76h)
Remark 3.3. Here, we emphasize again the importance of the cancellations in (35) and (38), e.g., it is not clear how we can bound

\[ I_{5,1,1} = -\epsilon \int_{\mathbb{R}^2} j_2 \partial_1 b_3 \partial_1 j_1 \, dx = \epsilon \int_{\mathbb{R}^2} (\partial_1 b_3)^2 \partial_1 \partial_2 b_3 \, dx \]

by a constant multiple of \( \int |\nabla b||\nabla b_h||\nabla^2 b_h| \, dx \).

Due to (76), we are able to bound by Hölder’s inequality, Sobolev embedding \( H^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2) \), and Young’s inequality

\[
I_5 \lesssim \|\nabla b_h\|_{L^4} \|\nabla^2 b_h\|_{L^4} \|\nabla b\|_{L^2} \leq \frac{\eta_\nu}{2} \|\Lambda^\frac{3}{2} b_h\|_{L^2}^2 + C \|\Lambda^\frac{3}{2} b_h\|_{L^2}^2 \|\nabla b\|_{L^2}^2.
\] (77)

Applying (74) and (77) to (73) gives us

\[
\partial_t (\|u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \nu \|\Delta u\|_{L^2}^2 + \eta_\nu \|\Lambda^\frac{3}{2} b_h\|_{L^2}^2 + \eta_\nu \|\Delta b_h\|_{L^2}^2 \\
\lesssim (1 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) (\|\nabla u\|_{L^2}^2 + \|\Lambda^\frac{3}{2} b_h\|_{L^2}^2).
\] (78)

Applying Gronwall’s inequality on (78) and relying on (72) complete the proof of the \( H^1(\mathbb{R}^2) \)-bound and therefore that of Theorem 2.4.

### 3.5. Proof of Theorem 2.5

Local existence of the unique solution to (23) starting from the given \((u_0, b_0)\) can be proven following previous works such as [8, Theorem 2.2]. We obtain the following energy identity by relying on (27):

\[
\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2 \int_0^t \nu \|\Lambda^\frac{3}{2} u\|_{H^m}^2 + \eta_\nu \|\Lambda^\frac{3}{2} b_h\|_{H^m}^2 + \eta_\nu \|\Lambda^\frac{3}{2} b\|_{H^m}^2 \, ds = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.
\] (79)

Although an \( H^1(\mathbb{R}^3) \)-bound does not suffice for the solution to the 3-D Hall-MHD system to bootstrap to higher regularity, it does for (23) due to its hyper-diffusion; that is the content of the following proposition, the proof of which is left in Appendix for completeness.

**Proposition 3.5.** Under the hypothesis of Theorem 2.5, if \((u, b)\) is a solution to (23) starting from the given \((u_0, b_0)\) over \([0, T]\), then for all \(t \in (0, T]\), \((u, b)\) satisfies

\[
\|u(t)\|_{H^m}^2 + \|b(t)\|_{H^m}^2 + \int_0^T \nu \|\Lambda^\frac{3}{2} u\|_{H^m}^2 + \eta_\nu \|\Lambda^\frac{3}{2} b_h\|_{H^m}^2 + \eta_\nu \|\Lambda^\frac{3}{2} b\|_{H^m}^2 \, ds \\
\lesssim (1 + \|u_0\|_{H^m}^2 + \|b_0\|_{H^m}^2) e^\int_0^T (\|\Lambda^\frac{3}{2} u\|_{L^2}^2 + \|\Lambda^\frac{3}{2} b\|_{L^2}^2) \, ds.
\] (80)

Due to Proposition 3.5 and (79), the proof of Theorem 2.5 boils down to proving that \( \int_0^T \|\Lambda^\frac{3}{2} b\|_{L^2}^2 \, ds < \infty \). In this endeavor, we take \( L^2(\mathbb{R}^3) \)-inner products on (23) with \((-\Delta u, -\Delta b)\) to deduce

\[
\frac{1}{2} \partial_t (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \nu \|\Lambda^\frac{3}{2} u\|_{L^2}^2 + \eta_\nu \|\Lambda^{\frac{1}{2}} b_h\|_{L^2}^2 + \eta_\nu \|\Lambda^\frac{3}{2} b\|_{L^2}^2 = \sum_{i=1}^5 \text{IV}_i
\] (81)
where

\begin{align*}
IV_1 & \triangleq \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx, \\
IV_2 & \triangleq \int_{\mathbb{R}^3} (u \cdot \nabla) b \cdot \Delta b dx, \\
IV_3 & \triangleq - \int_{\mathbb{R}^3} (b \cdot \nabla) b \cdot \Delta u dx, \\
IV_4 & \triangleq - \int_{\mathbb{R}^3} (b \cdot \nabla) u \cdot \Delta b dx, \\
IV_5 & \triangleq \epsilon \int_{\mathbb{R}^3} \nabla \times (j \times b) \cdot \Delta b dx.
\end{align*}

Using Hölder’s inequality, the Sobolev embedding $H^{\frac{4}{3}}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$, the Gagliardo–Nirenberg inequality (26d), and Young’s inequality, we can compute similarly to (74)

\begin{align*}
IV_1 + IV_2 + IV_3 + IV_4 & \lesssim \|\nabla u\|_{L^2} \|\nabla u\|_{L^4}^2 + \|\nabla u\|_{L^2} \|\nabla b\|_{L^2} \|\nabla b\|_{L^4}^2 \\
& \quad + \|\nabla u\|_{L^2} \|\Lambda^{\frac{2}{3}} b\|_{L^2}^2 + \|\nabla u\|_{L^2} \|\Lambda^{\frac{2}{3}} b\|_{L^2} \|\Lambda^{\frac{2}{3}} b\|_{L^2} \\
& \leq \frac{\nu}{2} \|\Lambda^{\frac{2}{3}} u\|_{L^2}^2 + \frac{\eta}{2} \|\Lambda^{\frac{2}{3}} b\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) (1 + \|\Lambda^{\frac{2}{3}} u\|_{L^2}^2 + \|\Lambda^{\frac{2}{3}} b\|_{L^2}^2 + \|\Lambda^{\frac{2}{3}} b\|_{L^2}^2). & (82)
\end{align*}

Now we decompose the Hall term carefully as follows:

\begin{align*}
IV_5 = \sum_{k=1}^{3} \epsilon \int_{\mathbb{R}^3} \nabla \times (j \times b) \cdot \partial_k^2 b dx & (83)
\end{align*}

where for any $k \in \{1, 2, 3\}$,

\begin{align*}
\int_{\mathbb{R}^3} \nabla \times (j \times b) \cdot \partial_k^2 b dx & = - \int_{\mathbb{R}^3} \partial_k (j \times b) \cdot \partial_k j dx \\
& \quad \overset{(3)}{=} - \int_{\mathbb{R}^3} j \times \partial_k b \cdot \partial_k j dx = \sum_{i=1}^{6} V_{k,i} & (84)
\end{align*}

and

\begin{align*}
V_{k,1} & \triangleq - \int_{\mathbb{R}^3} j_2 \partial_b b_3 \partial_k j_1 dx, & (85a) \\
V_{k,2} & \triangleq \int_{\mathbb{R}^3} j_2 \partial_b b_3 \partial_k j_1 dx, & (85b) \\
V_{k,3} & \triangleq \int_{\mathbb{R}^3} j_1 \partial_b b_3 \partial_k j_2 dx, & (85c) \\
V_{k,4} & \triangleq - \int_{\mathbb{R}^3} j_3 \partial_b b_1 \partial_k j_2 dx, & (85d)
\end{align*}
\[ V_{k,5} = - \int_{\mathbb{R}^3} j_1 \partial_k b_2 \partial_k j_3 dx, \]  
\[ V_{k,6} = \int_{\mathbb{R}^3} j_2 \partial_k b_1 \partial_k j_3 dx. \]  
(85e)

(85f)

We take the lessons we learned from (35) and (38) and first couple strategically from (85)

\[ V_{k,1} + V_{k,3} = - \int_{\mathbb{R}^3} j_2 \partial_k b_3 \partial_k j_1 - j_1 \partial_k b_3 \partial_k j_2 dx \]
\[ = - \int_{\mathbb{R}^3} \partial_k b_3 (j_2 \partial_k j_1 - j_1 \partial_k j_2) dx = \sum_{l=1}^{8} VI_{k,l} \]  
(86)

where

\[ VI_{k,1} = \int_{\mathbb{R}^3} \partial_k b_3 \partial_1 b_3 \partial_2 b_3 dx, \]  
(87a)

\[ VI_{k,2} = - \int_{\mathbb{R}^3} \partial_k b_3 \partial_1 b_3 \partial_2 b_2 dx, \]  
(87b)

\[ VI_{k,3} = - \int_{\mathbb{R}^3} \partial_k b_3 \partial_2 b_3 \partial_1 b_3 dx, \]  
(87c)

\[ VI_{k,4} = \int_{\mathbb{R}^3} \partial_k b_3 \partial_1 b_3 \partial_2 b_2 dx, \]  
(87d)

\[ VI_{k,5} = - \int_{\mathbb{R}^3} \partial_k b_3 \partial_2 b_3 \partial_1 b_3 dx, \]  
(87e)

\[ VI_{k,6} = \int_{\mathbb{R}^3} \partial_k b_3 \partial_2 b_3 \partial_1 b_2 dx, \]  
(87f)

\[ VI_{k,7} = \int_{\mathbb{R}^3} \partial_k b_3 \partial_2 b_3 \partial_1 b_3 dx, \]  
(87g)

\[ VI_{k,8} = - \int_{\mathbb{R}^3} \partial_k b_3 \partial_3 b_3 \partial_1 b_3 dx. \]  
(87h)

Then, we observe that analogously to (35) and (38)

\[ VI_{k,1} + VI_{k,5} = \int_{\mathbb{R}^3} \partial_k b_3 \partial_1 b_3 \partial_2 b_3 - \partial_k b_3 \partial_2 b_3 \partial_1 b_3 dx \]
\[ = \int_{\mathbb{R}^3} \partial_1 b_3 \left( \frac{1}{2} \partial_2 (\partial_k b_3)^2 - \partial_2 b_3 \frac{1}{2} \partial_1 (\partial_k b_3)^2 \right) dx \]
\[ = \frac{1}{2} \int_{\mathbb{R}^3} \partial_1 \partial_2 b_3 (\partial_k b_3)^2 - \partial_1 \partial_2 b_3 (\partial_k b_3)^2 dx = 0. \]  
(88)
Remarkably, we are able to find two more cancellations upon coupling within (85) appropriately. Let us couple

\[ V_{k,2} + V_{k,5} = \int_{\mathbb{R}^3} j_3 \partial_k b_2 \partial_k j_1 - j_1 \partial_k b_2 \partial_k j_3 dx \]

\[ = \int_{\mathbb{R}^3} \partial_k b_2 (j_3 \partial_k j_1 - j_1 \partial_k j_3) dx = \sum_{l=1}^{8} V_{k,l} \]  

(89)

where

\[ V_{I,k,1} \triangleq \int_{\mathbb{R}^3} \partial_k b_2 \partial_1 b_2 \partial_k \partial_2 b_3 dx, \]  

(90a)

\[ V_{I,k,2} \triangleq -\int_{\mathbb{R}^3} \partial_k b_2 \partial_1 b_2 \partial_k \partial_3 b_2 dx, \]  

(90b)

\[ V_{I,k,3} \triangleq -\int_{\mathbb{R}^3} \partial_k b_2 \partial_2 b_1 \partial_3 b_2 dx, \]  

(90c)

\[ V_{I,k,4} \triangleq \int_{\mathbb{R}^3} \partial_k b_2 \partial_2 b_1 \partial_k \partial_3 b_2 dx, \]  

(90d)

\[ V_{I,k,5} \triangleq -\int_{\mathbb{R}^3} \partial_k b_2 \partial_3 b_2 \partial_1 b_2 dx, \]  

(90e)

\[ V_{I,k,6} \triangleq \int_{\mathbb{R}^3} \partial_k b_2 \partial_3 b_2 \partial_k \partial_2 b_1 dx, \]  

(90f)

\[ V_{I,k,7} \triangleq \int_{\mathbb{R}^3} \partial_k b_2 \partial_3 b_2 \partial_2 b_1 dx, \]  

(90g)

\[ V_{I,k,8} \triangleq -\int_{\mathbb{R}^3} \partial_k b_2 \partial_3 b_2 \partial_1 b_2 dx. \]  

(90h)

We observe that \( V_{I,k,2} + V_{I,k,7} \) vanishes as follows:

\[ V_{I,k,2} + V_{I,k,7} = \int_{\mathbb{R}^3} -\partial_k b_2 \partial_1 b_2 \partial_k \partial_3 b_2 + \partial_k b_2 \partial_3 b_2 \partial_k \partial_1 b_2 dx \]

\[ = \int_{\mathbb{R}^3} -\partial_1 b_2 \frac{1}{2} \partial_3 (\partial_k b_2)^2 + \partial_3 b_2 \frac{1}{2} \partial_1 (\partial_k b_2)^2 dx \]

\[ = \frac{1}{2} \int_{\mathbb{R}^3} \partial_1 \partial_3 b_2 (\partial_1 b_2)^2 - \partial_1 \partial_3 b_2 (\partial_3 b_2)^2 dx = 0. \]  

(91)

Third, we couple from (85)

\[ V_{k,4} + V_{k,6} = -\int_{\mathbb{R}^3} j_3 \partial_k b_1 \partial_k j_2 - j_2 \partial_k b_1 \partial_k j_3 dx \]

\[ = -\int_{\mathbb{R}^3} \partial_k b_1 (j_3 \partial_k j_2 - j_2 \partial_k j_3) dx = \sum_{l=1}^{8} \text{VIII}_{k,l} \]  

(92)
where

\[
\begin{align*}
\text{VIII}_{k,1} & \triangleq \int_{\mathbb{R}^3} \partial_k b_1 \partial_1 b_2 \partial_k \partial_1 b_3 dx, \\
\text{VIII}_{k,2} & \triangleq - \int_{\mathbb{R}^3} \partial_k b_1 \partial_1 b_2 \partial_k \partial_3 b_1 dx, \\
\text{VIII}_{k,3} & \triangleq - \int_{\mathbb{R}^3} \partial_k b_1 \partial_2 b_1 \partial_k \partial_1 b_3 dx, \\
\text{VIII}_{k,4} & \triangleq \int_{\mathbb{R}^3} \partial_k b_1 \partial_2 b_1 \partial_k \partial_3 b_1 dx, \\
\text{VIII}_{k,5} & \triangleq - \int_{\mathbb{R}^3} \partial_k b_1 \partial_1 b_3 \partial_k \partial_1 b_2 dx, \\
\text{VIII}_{k,6} & \triangleq \int_{\mathbb{R}^3} \partial_k b_1 \partial_1 b_3 \partial_k \partial_2 b_1 dx, \\
\text{VIII}_{k,7} & \triangleq \int_{\mathbb{R}^3} \partial_k b_1 \partial_3 b_1 \partial_k \partial_1 b_2 dx, \\
\text{VIII}_{k,8} & \triangleq - \int_{\mathbb{R}^3} \partial_k b_1 \partial_3 b_1 \partial_k \partial_2 b_1 dx.
\end{align*}
\] (93a) – (93h)

We observe that \text{VIII}_{k,4} + \text{VIII}_{k,8} vanishes as follows:

\[
\text{VIII}_{k,4} + \text{VIII}_{k,8} = \int_{\mathbb{R}^3} \partial_k b_1 \partial_2 b_1 \partial_k \partial_3 b_1 - \partial_k b_1 \partial_1 b_3 \partial_k \partial_2 b_1 dx \\
= \int_{\mathbb{R}^3} \partial_2 b_1 \frac{1}{2} \partial_3 (\partial_k b_1)^2 - \partial_3 b_1 \frac{1}{2} \partial_2 (\partial_k b_1)^2 dx \\
= \frac{1}{2} \int_{\mathbb{R}^3} -\partial_2 \partial_3 b_1 (\partial_k b_1)^2 + \partial_2 \partial_3 b_1 (\partial_k b_1)^2 dx = 0. \tag{94}
\]

Therefore, we have shown that

\[
\text{IV}_5 \overset{(83)}{=} \sum_{k=1}^{3} \epsilon \int_{\mathbb{R}^3} \nabla \times (j \times b) \cdot \partial_k^2 b dx \overset{(84)}{=} \epsilon \sum_{k=1}^{3} \sum_{l=1}^{6} V_{k,l} \\
\overset{(86)(89)(92)}{=} \epsilon \sum_{k=1}^{3} \sum_{l=1}^{8} \text{VI}_{k,l} + \text{VI}_k + \text{VIII}_{k,l} \\
\overset{(88)(91)(94)}{=} \epsilon \sum_{k=1}^{3} \left( \sum_{l \in \{2,3,4,6,7,8\}} \text{VI}_{k,l} + \sum_{l \in \{1,3,4,5,6,8\}} \text{VI}_k + \sum_{l \in \{1,2,3,5,6,7\}} \text{VIII}_{k,l} \right). \tag{95}
\]
We wish to bound all these terms by a constant multiples of \( \int |\nabla b||\nabla b_h||\nabla^2 b_h| dx \) similarly to (76). We start with \( \sum_{l \in \{2,3,4,6,7,8\}} \sum_{k=1}^{3} V_{k,l} \) which actually presents the most difficulty. First,

\[
\sum_{k=1}^{3} \left( V_{k,4} + V_{k,8} \right) \lesssim \sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_k b_3 \partial_3 b_1 \partial_3 b_2 - \partial_2 b_3 \partial_3 b_2 \partial_k \partial_3 b_1 \, dx \]

Second, we integrate by parts and rely on divergence-free property so that \( \partial_3 b_3 = -\partial_1 b_1 - \partial_2 b_2 \) to deduce

\[
\sum_{k=1}^{3} \left( V_{k,2} + V_{k,6} \right) \lesssim \sum_{k=1}^{3} \int_{\mathbb{R}^3} (\partial_k \partial_3 b_3 \partial_3 b_1 + \partial_2 b_3 \partial_3 b_2 \partial_3 b_1) \, dx \]

Finally, we integrate by parts and rely on \( \partial_3 b_3 = -\partial_1 b_1 - \partial_2 b_2 \) again to deduce

\[
\sum_{k=1}^{3} \left( V_{k,3} + V_{k,7} \right) \lesssim \sum_{k=1}^{3} \int_{\mathbb{R}^3} (\partial_2 b_3 \partial_3 b_1 \partial_3 b_2 + \partial_3 b_3 \partial_3 b_2 \partial_3 b_1) \, dx \]

The rest of the terms in the right hand side of (95) are easier to bound as follows:

\[
\epsilon \sum_{k=1}^{3} \left( \sum_{l \in \{1,3,4,5,6,8\}} V_{k,l} + \sum_{l \in \{1,2,3,5,6,7\}} VIII_{k,l} \right) \]

\[
\epsilon \sum_{k=1}^{3} \left( \sum_{l \in \{1,2,3,5,6,7,8\}} V_{k,l} \right) \lesssim \int_{\mathbb{R}^3} |\nabla b||\nabla b_h||\nabla^2 b_h| dx.
\]
+ \int_{\mathbb{R}^3} \partial_k b_1 (\partial_1 b_2 \partial_k \partial_1 b_3 - \partial_1 b_2 \partial_k \partial_3 b_1 - \partial_2 b_1 \partial_k \partial_1 b_3 \\
- \partial_1 b_3 \partial_k \partial_1 b_2 + \partial_1 b_3 \partial_k \partial_2 b_1 + \partial_3 b_1 \partial_k \partial_1 b_2) dx \lesssim \int_{\mathbb{R}^3} \| \nabla b \| \| \nabla b_k \| \| \nabla^2 b_k \| \, dx \tag{99}
\end{align}
where we integrated by parts on the four terms that have second partial derivatives on \( b_3 \), namely \( \partial_k \partial_2 b_3 \) and \( \partial_k \partial_1 b_3 \). We apply (96), (97), (98), and (99) to (95) to conclude by the Sobolev embedding of \( H^4(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3) \) and Young’s inequality that
\begin{align}
IV_5 \lesssim \int_{\mathbb{R}^3} \| \nabla b \| \| \nabla b_k \| \| \nabla^2 b_k \| \, dx \lesssim \| \nabla b \|_{L^2} \| \Lambda^{\frac{3}{2}} b_k \|_{L^2} \| \Lambda^{\frac{1}{2}} b_k \|_{L^2} \leq \frac{\eta}{2} \| \Lambda^{\frac{1}{2}} b_k \|_{L^2} + C \| \nabla b \|_{L^2} \| \Lambda^{\frac{5}{2}} b_k \|_{L^2}.
\end{align}
Applying (82) and (100) to (81) gives us
\begin{align}
\partial_t \left( \| \nabla u \|_{L^2}^2 + \| \nabla b \|_{L^2}^2 + \nu \| \Lambda^{\frac{3}{2}} u \|_{L^2}^2 + \eta \| \Lambda^{\frac{1}{2}} b \|_{L^2}^2 + \eta \| \Lambda^{\frac{1}{2}} b \|_{L^2}^2 \right) \\
\lesssim (1 + \| \nabla u \|_{L^2}^2 + \| \nabla b \|_{L^2}^2)(1 + \| \Lambda^{\frac{3}{2}} u \|_{L^2}^2 + \| \Lambda^{\frac{5}{2}} b \|_{L^2}^2).
\end{align}
Applying Gronwall’s inequality on (101) and relying on (79) allow us to deduce in particular that
\( \int_0^T \| A^2 b \|_{L^2}^2 \, dt < \infty \); applying this bound to (80) completes the \( H^m(\mathbb{R}^3) \)-bound of \((u, b)\) and hence the proof of Theorem 2.5.

4. Appendix: Proof of Theorem 2.3 (2)

As we mentioned, Propositions 3.2-3.3 remain valid for us. We obtain the following proposition instead of Proposition 3.4:

**Proposition 4.1.** Under the hypothesis of Theorem 2.3 (2), let \( \nabla \times \omega \) be a smooth solution to (55) over \([0, T]\). Then for both \( k \in \{1, 2\} \),
\[(\nabla \times \omega)_k \in L^\infty_T L^2_x \cap L^2_T \dot{H}^1_x, \tag{102}\]

**Proof of Proposition 4.1.** For clarity, let us prove Theorem 2.3 (2) when \( \Delta(\Delta u_1) \in L^p_1 L^{p_1}_x \) with \( \frac{2}{p_1} + \frac{2}{r_1} \leq 1, p_1 \in (2, \infty) \) and \( \Delta(\Delta u_2) \in L^p_1 BMO_x \) as the other cases can be proven similarly. We take \( L^2(\mathbb{R}^2) \)-inner products on (55) with \(( (\nabla \times \omega)_1, (\nabla \times \omega)_2 )^T \) to deduce
\[ \sum_{k=1}^2 \frac{1}{2} \partial_t \| (\nabla \times \omega)_k \|_{L^2}^2 + \| (\nabla (\nabla \times \omega)_k \|_{L^2}^2 = \sum_{k=1}^3 \sum_{i=1}^3 \Pi_{k, i} \tag{103}\]
where
\begin{align}
\Pi_{k, 1} & \triangleq - \int_{\mathbb{R}^2} (\nabla \times ((u \cdot \nabla) \omega))_k (\nabla \times \omega)_k \, dx, \tag{104a} \\
\Pi_{k, 2} & \triangleq \int_{\mathbb{R}^2} (\nabla \times ((\omega \cdot \nabla) u))_k (\nabla \times \omega)_k \, dx, \tag{104b} \\
\Pi_{k, 3} & \triangleq \int_{\mathbb{R}^2} (\nabla \times \nabla \times ((b \cdot \nabla) b))_k (\nabla \times \omega)_k \, dx. \tag{104c}
\end{align}
Using (60)–(61) we can rewrite using the fact that \( \nabla \times \omega = -\Delta u \)

\[
\sum_{k=1}^{2} \Pi_{k,1} = \int_{\mathbb{R}^2} (u \cdot \nabla) \omega_3 \Delta \omega_3 \, dx,
\]

\[
\sum_{k=1}^{2} \Pi_{k,2} = -\int_{\mathbb{R}^2} (\omega \cdot \nabla) u_3 \Delta \omega_3 \, dx,
\]

\[
\sum_{k=1}^{2} \Pi_{k,3} = -\int_{\mathbb{R}^2} ((b \cdot \nabla)b_1 (b \cdot \nabla)b_2 0)^T \cdot (\Delta(\nabla \times \omega)_1 \Delta(\nabla \times \omega)_2 0)^T \, dx.
\]

We can estimate \( \sum_{k=1}^{2} \Pi_{k,1} \) and \( \sum_{k=1}^{2} \Pi_{k,2} \) by Hölder’s and Young’s inequalities, and the Gagliardo–Nirenberg inequalities (26a) and (26c),

\[
\sum_{k=1}^{2} \Pi_{k,1} \lesssim \|u\|_{L^\infty} \|\nabla \omega_3\|_{L^2} \|\Delta(\partial_1 u_2 - \partial_2 u_1)\|_{L^2} \lesssim \sum_{k,l=1}^{2} \|u\|^\frac{1}{2} \|\Delta u\|^\frac{1}{2} \|\Delta u_k\|_{L^2} \|\Delta u_l\|_{L^2}
\]

\[
\lesssim \frac{1}{4} \sum_{k=1}^{2} \|\nabla(\nabla \times \omega)_k\|^2_{L^2} + C \|u\|_{L^2} \|\Delta u\|_{L^2} \sum_{k=1}^{2} \|\Delta \nabla u_k\|_{L^2},
\]

\[
\sum_{k=1}^{2} \Pi_{k,2} \lesssim \|\omega\|_{L^4} \|\nabla u_3\|_{L^4} \|\Delta \omega_3\|_{L^2} \lesssim \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \sum_{k=1}^{2} \|\Delta \nabla u_k\|_{L^2}
\]

\[
\lesssim \frac{1}{4} \sum_{k=1}^{2} \|\nabla(\nabla \times \omega)_k\|^2_{L^2} + C \|u\|^\frac{1}{2} \|\Delta u\|^\frac{1}{2} \|\Delta u_l\|^2_{L^2}.
\]

For \( p_1 \in (2, \infty) \) we estimate via Hölder’s inequality, the Gagliardo–Nirenberg inequality (26b), and the div-curl lemma,

\[
\sum_{k=1}^{2} \Pi_{k,3} \lesssim \|b\|_{L^{\frac{2p_1}{p_1-1}}} \|\nabla b_1\|_{L^2} \|\Delta(\nabla \times \omega)_1\|_{L^{p_1}} + \|b \cdot \nabla b_2\|_{BMO} \|\Delta(\nabla \times \omega)_2\|_{BMO}
\]

\[
\lesssim \|b\|_{L^{\frac{2p_1}{p_1-1}}} \|\nabla b_1\|_{L^2} \|\Delta(\nabla u_1)\|_{L^{p_1}} + \|b\|_{L^2} \|\nabla b\|_{L^2} \|\Delta(\nabla u_2)\|_{BMO}.
\]

Thus, we can now apply (106), (107), and (108) to (103), integrate over \([0, t]\), and use Hölder’s inequality to deduce

\[
\sum_{k=1}^{2} \|((\nabla \times \omega)_k\|_{L^2}^2 + \int_0^t \|((\nabla \times \omega)_k\|_{L^2}^2 \, ds \leq \sum_{k=1}^{2} \|((\nabla \times \omega)_k(0)\|_{L^2}^2
\]

\[
+ C \left( \int_0^t \sum_{k=1}^{2} \|((\nabla \times \omega)_k\|_{L^2}^2 \|\Delta u\|_{L^2} \, ds + \int_0^t \|\Delta(\nabla u_1)\|_{L^{p_1}}^\frac{2p_1}{p_1-2} \, ds \right) + \left( \int_0^t \|\Delta(\nabla u_2)\|_{BMO}^2 \, ds \right)^{\frac{1}{2}}
\]

from which Gronwall’s inequality completes the proof of Proposition 4.1.

\[ \square \]

By Propositions 3.3 and 4.1, we realize that for both \( k \in \{1, 2\} \)

\[
j_k \overset{(51)}{=} z_k^2 - (\nabla \times \omega)_k \in L^\infty_T L^2_x \cap L^2_T H^1_x.
\]
We can apply the Gagliardo–Nirenberg inequality (26a) to deduce from (110) that for both \( k \in \{1, 2\} \)
\[
\int_0^T \| j_k \|_{L^4}^4 ds \lesssim \sup_{s \in [0, t]} \| j_k(s) \|_{L^2}^2 \int_0^T \| \nabla j_k \|_{L^2}^2 ds \lesssim 1.
\]
(111)
Therefore, \( j_k \in L^4_T L^4_x \) for both \( k \in \{1, 2\} \), allowing us to apply Theorem 2.1 to deduce (12) as desired. The proof of Theorem 2.3 (2) is now complete.

5. Appendix: Proof of Proposition 3.5

We apply \( D^\alpha \) on (23) with \( \alpha \) being any multi-index such that \( |\alpha| \leq m \), take \( L^2(\mathbb{R}^3) \)-inner products with \((D^\alpha u, D^\alpha b)\), sum over all such \( \alpha \) and obtain an identity of
\[
\frac{1}{2} \partial_t (\| u \|^2_{H^m} + \| b \|^2_{H^m}) + \nu \| \Lambda^{\frac{3}{2}} u \|^2_{H^m} + \eta_h \| \Lambda^{\frac{3}{2}} b_h \|^2_{H^m} + \eta_c \| \Lambda^{\frac{3}{2}} b_c \|^2_{H^m} = \sum_{i=1}^{4} \text{III}_i
\]
(112)
where
\[
\text{III}_1 \triangleq - \sum_{|\alpha| \leq m} \int \mathcal{D}^\alpha [(u \cdot \nabla) u] \cdot D^\alpha u \, dx,
\]
(113a)
\[
\text{III}_2 \triangleq - \sum_{|\alpha| \leq m} \int \mathcal{D}^\alpha [(u \cdot \nabla) b] \cdot D^\alpha b \, dx,
\]
(113b)
\[
\text{III}_3 \triangleq \sum_{|\alpha| \leq m} \int \mathcal{D}^\alpha [(b \cdot \nabla) b] \cdot D^\alpha u + \mathcal{D}^\alpha [(b \cdot \nabla) u] \cdot D^\alpha b \, dx,
\]
(113c)
\[
\text{III}_4 \triangleq - \epsilon \sum_{|\alpha| \leq m} \int \mathcal{D}^\alpha [(j \times b)] \cdot D^\alpha j \, dx.
\]
(113d)
Making use of the well-known identities,
\[
\int_{\mathbb{R}^3} (u \cdot \nabla) D^\alpha u \cdot D^\alpha u \, dx = 0, \quad \int_{\mathbb{R}^3} (u \cdot \nabla) D^\alpha b \cdot D^\alpha b \, dx = 0,
\]
\[
\int_{\mathbb{R}^3} (b \cdot \nabla) D^\alpha b \cdot D^\alpha u + (b \cdot \nabla) D^\alpha u \cdot D^\alpha b \, dx = 0, \quad \int_{\mathbb{R}^3} D^\alpha j \times b \cdot D^\alpha j \, dx^{(3)} = 0,
\]
we can estimate by using Hölder’s inequality and the Sobolev embeddings of \( \dot{H}^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow L^{\frac{6}{5}}(\mathbb{R}^3) \) and \( \dot{H}^{\frac{5}{2}}(\mathbb{R}^3) \hookrightarrow L^{12}(\mathbb{R}^3) \)
\[
\text{III}_1 \lesssim \| u \|_{H^m} \| \Lambda^{\frac{3}{2}} u \|_{L^2} \| \Lambda^{\frac{3}{2}} u \|_{H^m},
\]
(114a)
\[
\text{III}_2 \lesssim (\| \Lambda^{\frac{3}{2}} u \|_{H^m} \| \Lambda^{\frac{3}{2}} b \|_{L^2} + \| \Lambda^{\frac{3}{2}} u \|_{L^2} \| \Lambda^{\frac{3}{2}} b \|_{H^m}) \| b \|_{H^m},
\]
(114b)
\[
\text{III}_3 \lesssim (\| u \|_{H^m} + \| b \|_{H^m})(\| \Lambda^{\frac{3}{2}} u \|_{H^m} + \| \Lambda^{\frac{3}{2}} b \|_{H^m})(\| \Lambda^{\frac{3}{2}} u \|_{L^2} + \| \Lambda^{\frac{3}{2}} b \|_{L^2}),
\]
(114c)
\[
\text{III}_4 \lesssim \| \Lambda^{\frac{3}{2}} b \|_{L^2} \| b \|_{H^m} \| \Lambda^{\frac{3}{2}} b \|_{H^m}.
\]
(114d)
Applying (114) to (112) and relying on Young’s inequality give us for any \( \delta > 0 \),
\[
\frac{1}{2} \partial_t (\| u \|^2_{H^m} + \| b \|^2_{H^m}) + \nu \| \Lambda^{\frac{3}{2}} u \|^2_{H^m} + \eta_h \| \Lambda^{\frac{3}{2}} b_h \|^2_{H^m} + \eta_c \| \Lambda^{\frac{3}{2}} b_c \|^2_{H^m}
\]
\[
\leq \frac{\nu}{2} \| \Lambda^{\frac{3}{2}} u \|^2_{H^m} + \delta \| \Lambda^{\frac{3}{2}} b \|^2_{H^m} + C(\| u \|^2_{H^m} + \| b \|^2_{H^m})(\| \Lambda^{\frac{3}{2}} u \|^2_{L^2} + \| \Lambda^{\frac{3}{2}} b \|^2_{L^2})
\]
(115)
where Plancherel theorem and Young’s inequality allow us to write
\[
\delta \| \Lambda^\frac{5}{4} b \|^2_{H^m} \leq 2\delta (\| \Lambda^\frac{5}{4} bh \|^2_{H^m} + \| \Lambda^\frac{5}{4} bv \|^2_{H^m}) \leq 2\delta (C(\| bh \|^2_{H^m} + \| \Lambda^\frac{5}{4} bh \|^2_{H^m} + \| \Lambda^\frac{5}{4} bv \|^2_{H^m}),
\]
as well as \( \| \Lambda^\frac{5}{4} b \|^2_{L^2} \lesssim \| b \|^2_{L^2} + \| \Lambda^\frac{9}{4} b \|^2_{L^2} \). Thus, taking \( \delta > 0 \) sufficiently small and applying Gronwall’s inequality on (115) lead us to (80) and complete the proof of Proposition 3.5.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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