Flat polymerized membranes at three-loop order

S. Metayer\textsuperscript{1}\textsuperscript{*}, D. Mouhanna\textsuperscript{2}\textsuperscript{†}, S. Teber\textsuperscript{1}\textsuperscript{‡}

\textsuperscript{1} Sorbonne Universit\'e, CNRS, Laboratoire de Physique Th\'eorique et Hautes Energies, LPTHE, F-75005 Paris, France
\textsuperscript{2} Sorbonne Universit\'e, CNRS, Laboratoire de Physique Th\'eorique de la Mati\`ere Condens\'ee, LPTMC, F-75005 Paris, France
E-mail: \textsuperscript{*}smetayer@lpthe.jussieu.fr, \textsuperscript{†}mouhanna@lptmc.jussieu.fr, \textsuperscript{‡}teber@lpthe.jussieu.fr

Abstract. In this conference report, we present a recent field theoretic renormalization group analysis of flat polymerized membranes at three-loop order by the present authors [Phys. Rev. E 105, L012603 (2022)].

1. Context

In this conference report, we present a brief overview of the methods and the main results of Ref. [1]. The latter focuses on a field theoretic study of the flat phase of polymerized membranes, see, e.g., [2, 3, 4, 5, 6, 7] for early studies. Such a model has been the subject of extensive studies during the last decades especially that it seems to adequately describe the elastic degrees of freedom of graphene and graphene-like materials, see, e.g., the textbook [8]. A major challenge in this context is an accurate determination of the renormalization group (RG) functions of the model and in particular the field anomalous dimension $\eta$. This critical exponent, alone, determines all the power-law scaling behaviors of the theory in the infra-red (IR) regime leading to an anomalous rigidity characteristic of the flat phase, see, e.g., [9] for a review.

Early, one-loop, computations were carried out in the seminal work of Aronovitz and Lubensky [3] and led to $\eta = 0.96$. Due to major computational challenges, over 30 years passed before the achievement of the full two-loop computations in [10] that made use of massless multi-loop techniques (see, e.g., [11] for a review) and led to $\eta = 0.9139$. During this time, a lot of other approaches have been carried out, including non-perturbative techniques such as NPRG [12] and SCSA [13, 9] as well as numerical Monte-Carlo techniques [14, 15, 16, 17]. All these methods rely on their own sets of approximations and have led to scattered values for $\eta$ in the range $[0.72, 0.88]$. Very recently, the field theoretic approach could be extended to three-loop order in [1] thanks to a full state of the art automation of the computations using packages originally developed for high-energy physics calculations [18, 19].

In the following, we will first introduce the model of a flat polymerized membrane. Then, we will briefly review the field theoretic RG approach used in our calculations. Finally, we will present our results and conclude.

2. Model

We consider a $D$-dimensional homogeneous and isotropic membrane embedded in a $d$-dimensional space. Each mass point of the membrane is indexed by $\vec{x} \in \mathbb{R}^D$. In $\mathbb{R}^d$, the
reference state of the membrane is the unperturbed flat state where each of these mass points is indexed by $\vec{R}^{(0)}(\vec{x}) = (\vec{x}, 0_d)$ where $0_d$ is the null vector of co-dimension $d_c = d - D \geq 0$. Allowing for small displacements inside the membrane, the latter are parameterized by a phonon field $\vec{u}(\vec{x}) \in \mathbb{R}^D$ and a flexuron field $\vec{h}(\vec{x}) \in \mathbb{R}^{d_c}$ such that the perturbed mass points are located at: $\vec{R}(\vec{x}) = (\vec{x} + \vec{u}(\vec{x}), \vec{h}(\vec{x}))$. The induced metric is then defined as: $g_{ij} = \partial_i \vec{R}(\vec{x}) \cdot \partial_j \vec{R}(\vec{x})$ ($g_{ij}^{(0)} = \delta_{ij}$ in the unperturbed state). The strain tensor is defined as:

$$u_{ij} = \frac{1}{2} \left( \partial_i \vec{R}(\vec{x}) \cdot \partial_j \vec{R}(\vec{x}) - g_{ij}^{(0)} \right) = \frac{1}{2} \left( \partial_i u_j + \partial_j u_i + \partial_i \vec{h} \cdot \partial_j \vec{h} + \partial_i \vec{u} \cdot \partial_j \vec{u} \right).$$

The Euclidean low-energy action of the membrane reads $[2, 3, 4, 5, 6, 7]$:

$$S[\vec{u}, \vec{h}] = \int d^D x \left[ \frac{1}{2} (\Delta \vec{h})^2 + \frac{\lambda}{2} u_{aa}^2 + \mu u_{ab}^2 \right],$$

where $\lambda$ and $\mu$ are Lamé elastic moduli. Moreover, the bending rigidity has been set to $\kappa = 1$ without restricting the generality of the problem.

Upon neglecting non-linearities in the phonon field $\vec{u}$, the action (2) yields the so-called two-field model that involves cubic and quartic interactions in the fields $\vec{u}$ and $\vec{h}$ and coupling constants $\lambda$ and $\mu$. Because this model is quadratic in the phonon field, the latter can be integrated over exactly. This then yields another, equivalent, model, the so-called flexural effective model, that depends only on the flexural field $\vec{h}$ and on coupling constants $\lambda$ and $b$, see [1] for more details.

Both of these models are highly derivative field theories that, after Fourier transform, result in non-trivial momentum dependencies of the Feynman rules, and ultimately, to very lengthy expressions for the Feynman diagrams. They also display rather non-trivial tensorial structures that need to be projected out carefully especially for the effective model. The massless nature of both models translates the long-range nature of elastic interactions that is responsible for the renormalization of the two sets of couplings ($\lambda, \mu$) and ($b, \mu$). Fortunately, it allows to apply powerful multiloop techniques, see the review [11]. Moreover, this theory is highly constrained by Ward identities that relate all renormalization constants to two-point correlation functions only, which further reduces the complexity of the problem.

3. Perturbative renormalization group approach

In [1], we have analysed the RG flows of both the two-field and the flexural effective models up to three-loop order. Following the two-loop computations of [10], the three-loop case has been achieved in [1] thanks to a full automation of the calculations. The latter can be summarized in the following steps:

(i) Compute the bare flexuron self-energy $\Sigma$ and the bare polarization $\Pi$ from Feynman diagram expansions.\(^1\) These computations were carried out for a membrane with arbitrary codimension $d_c$ using dimensional regularization in $D = 4 - 2\varepsilon$.\(^2\) A total of 61 distinct diagrams had to be computed in the two-field model and 32 distinct diagrams in the effective flexural model, see Table 1. The computational time was approximately the same for the two sets of diagrams due to the intricate tensorial structure of the effective model. Diagrams were generated using QGRAF [18] and then imported to MATHEMATICA where the numerator algebra and tensor manipulations were performed with home made codes. The reduction to master integrals was automated with the program LITERED [19] and the analytic expression of some complicated masters taken from, e.g., the review [11].

\(^1\) The polarization $\Pi$ corresponds to the phonon polarization in the two-field model (with longitudinal $\Pi_\parallel$ and transverse $\Pi_\perp$ projections) and to an effective polarization in the effective model (with projections $\Pi_N$ and $\Pi_M$ on irreducible tensors $N$ and $M$ that are defined, e.g., in the review [9]).

\(^2\) Note that the original paper [1] uses the convention $D = 4 - \varepsilon$. 

2438 (2023) 012141 doi:10.1088/1742-6596/2438/1/012141
(ii) Compute the renormalization constants $Z$, $Z_\lambda$, $Z_\mu$ in the two-field model and $Z$, $Z_b$, $Z_\mu$ in the effective model. They are defined as: $h = Z^{1/2} h_r$ (together with $u = Z u_r$ in the two-field model), $\lambda = M^{2\varepsilon} Z_\lambda \lambda_r$, $b = M^{2\varepsilon} Z_b b_r$ and $\mu = M^{2\varepsilon} Z_\mu \mu_r$ where $M$ is the renormalization mass scale. The constants were found with the help of simple algebraic relations (with no need to compute any additional counter-term diagrams) that read

$$\text{finite} = (p^4 - \Sigma) Z, \quad \text{finite} = (p^2 Z_\mu \mu_r - \Pi) Z^2, \quad \text{finite} = (p^2(Z\lambda + 2 Z_\mu \mu_r) - \Pi) Z^2,$$

for the two-field model and

$$\text{finite} = (p^4 - \Sigma) Z, \quad \text{finite} = ((Z_\mu \mu_r)^{-1} - \Pi_M) Z^{-2}, \quad \text{finite} = ((Z_b b_r)^{-1} - \Pi_N) Z^{-2},$$

for the effective model, where finite means of order $\varepsilon^0$.

(iii) Compute the RG functions in both models, consisting in the beta functions $\beta_x = M \partial_M Z_x$ ($x = \{\lambda, \mu, b\}$) and the field anomalous dimension $\eta = M \partial_M \log Z$. Note that, at this point, these functions differ in the two models.

(iv) Solve perturbatively the system of beta functions: $\{ \beta_\lambda(\lambda^*, \mu^*) = 0, \beta_\mu(\lambda^*, \mu^*) = 0 \}$ in the two-field model and $\{ \beta_\mu(\mu^*, b^*) = 0, \beta_b(\mu^*, b^*) = 0 \}$ in the effective flexural model. This allows to access the different scale invariant fixed points: $(\lambda^*, \mu^*)$ in the two-field model and $(\mu^*, b^*)$ in the effective model. Once again, the expressions of the fixed points differ in the two models.

(v) Compute the anomalous dimension $\eta$ at the various fixed-points for each model. A strong check of our calculations is that, for the special case of the non-trivial and IR stable fixed point (see more below), both $\eta(\lambda^*, \mu^*)$ in the two-field model and $\eta(\mu^*, b^*)$ in the effective model are equal. This is in accordance with the fact that the two models are identical and that these quantities are scheme-independent and universal.

4. Results

We now proceed on summarizing the results of the above calculations. In both models, four perturbative fixed points are obtained. In the case of the two-field model, they correspond to:

(i) $P_1$: the unstable gaussian fixed-point ($\lambda_1^* = \mu_1^* = 0$) with $\eta_1 = 0$.

(ii) $P_2$: the unstable shearless fixed-point ($\lambda_2^* = 32\pi^2\varepsilon/d_c$, $\mu_2^* = 0$) with $\eta_2 = 0$.

(iii) $P_3$: this fixed point has non-trivial values of $\lambda_3^*$ and $\mu_3^*$ (see [1]) that lead to a strictly negative bulk-modulus at three-loop ($B_3^* = \lambda_3^* + 2\mu_3^*/D < 0$); it therefore lies outside of the (mechanical) stability region of the model ($B \geq 0$). For the sake of comparison with the effective model, we nevertheless provide the expression of the corresponding anomalous dimension in the case $d_c = 1$:

$$\eta_3 = 0.9524 \varepsilon - 0.0711 \varepsilon^2 - 0.0698 \varepsilon^3 + O(\varepsilon^4),$$

(iv) $P_4$: the IR-stable non-trivial fixed point (see [1] for the expressions of $\lambda_4^*$ and $\mu_4^*$) with $\eta_4$ provided below in the physical case $d_c = 1$, see (7).
Table 2: Benchmarking other approaches using $\eta$

| $\eta$               | $\varepsilon = 1$ | (Re)-expanded in $\varepsilon$ |
|----------------------|-------------------|---------------------------------|
| 3-loop [1]           | 0.8872            | $0.96 \varepsilon - 0.0461 \varepsilon^2 - 0.0267 \varepsilon^3$ |
| SCSA [13, 9]         | 0.8209            | $0.96 \varepsilon - 0.0476 \varepsilon^2 - 0.0280 \varepsilon^3$ |
| NPRG [12]            | 0.8491            | $0.96 \varepsilon - 0.0367 \varepsilon^2 - 0.0266 \varepsilon^3$ |

In the case of the effective flexural model, the fixed points correspond to:

(i) $P_1$: the unstable gaussian fixed-point ($\mu_1^* = b_1^* = 0$) with $\eta_1 = 0$,
(ii) $P'_2$: an unstable shearless fixed-point ($\mu_2^* = 0$) with a non-trivial expression for $b_2^*$ [1].
Contrary to $P_2$, this fixed point has a non-trivial anomalous dimension that reads (in the case $d_c = 1$):

$$
\eta_2' = 0.8000 \varepsilon - 0.0053 \varepsilon^2 + 0.0110 \varepsilon^3 + O(\varepsilon^4),
$$

(6)

(iii) $P_3$: the infinitely compressible fixed point with $b_3^* = 0$ and a non-trivial expression for $\mu_3^*$ [1].
Contrary to the case of the two-field model, it now has a vanishing bulk modulus ($B_3^* = 0$) and is therefore located on the (mechanical) stability line of the model. Interestingly, the corresponding $\eta_3$ in the effective model corresponds exactly to $\eta_3$ in the two-field model, see (5) in the case $d_c = 1$.

(iv) $P_4$: the IR-stable non-trivial fixed point (see [1] for the expressions of $\mu_4^*$ and $b_4^*$) with $\eta_4$ exactly equal to the one found in the two-field model and provided below in the physical case $d_c = 1$, see (7).

From the above results, we see that the globally IR attractive fixed point that controls the physics of the flat phase is $P_4$ at three-loops (as was already the case at one- and two-loops). Both the two-field model and the effective flexural model yield the same field anomalous dimension at this fixed point. Referring to it simply as $\eta$, its expression in the physical case $d_c = 1$ reads:

$$
\eta \equiv \eta_4 = \frac{24}{25} \varepsilon - \frac{144}{3125} \varepsilon^2 + \frac{4(1286928 \zeta_3 - 568241)}{146484375} \varepsilon^3 + O(\varepsilon^4),
$$

with $\zeta_3 \approx 1.202$ being the Apéry constant. Numerically evaluating the coefficients yields:

$$
\eta = 0.9600 \varepsilon - 0.0461 \varepsilon^2 - 0.0267 \varepsilon^3 + O(\varepsilon^4).
$$

(7)

Interestingly, the coefficients of (8) are small and even decreasing with increasing loop order. The asymptotic series seems to be in a convergent regime for which an extrapolation to the case of interest ($\varepsilon = 1$) does not require any resummation. Order by order in the perturbative expansion, we therefore have (for $d_c = 1$ and $D = 2$):

$$
\eta_{1\text{-loop}} = 0.96 [3], \quad \eta_{2\text{-loop}} \approx 0.9139 [10], \quad \eta_{3\text{-loop}} \approx 0.8872 [1].
$$

(9)

Clearly, with increasing loop order, the result gets closer to the range of values obtained by other methods discussed in the introduction: [0.72, 0.88].

Because our results for $\eta$ are exact order by order and universal, we may use them as a benchmark for other methods, such as NPRG and SCSA. This is provided by Table 2 that displays a striking proximity between all approaches. Note that more detailed comparisons involving all of the fixed points can be found in [1].

---

3 This is due to the structure of the series that consists of very large denominators, see [1] for details.
5. Discussion

In this short conference report, we have reviewed recent calculations of RG functions in two equivalent models of flat polymerized membranes at three-loop order [1]. A full state of the art automation of all tasks was necessary to achieve such a goal as the corresponding field theories are highly derivative with rather intricate tensor structures.

A central result was the computation of the field anomalous dimension $\eta$ at the globally attractive fixed point controlling the physics of the flat phase in $D = 4 - 2\varepsilon$ and for arbitrary $d_c$. A strong check of our results was that the expression for $\eta$ was found to be the same in both models (despite the fact that intermediate steps differ). The associated (asymptotic) $\varepsilon$-series displayed remarkably small and decreasing coefficients with increasing loop order, see (8). Without any resummation needed, we found $\eta = 0.8872$ for the physical case $D = 2$ and $d_c = 1$. This result lies in the range of values obtained by other methods [0.72, 0.88]. It also revealed the striking ability of both SCSA and NPRG to numerically mimic the true perturbative expansion (Table 2).

Note: while writing this conference proceedings, the four-loop computation, in the two-field model, was achieved by Pikelner [20] using similar techniques, thereby confirming our three-loop results and leading to the new improved value $\eta_{4\text{-loop}} = 0.8670$.

Acknowledgments

S. Metayer thanks the organizers of the conference “Advanced Computing and Analysis Techniques” (ACAT) 2021 for providing the opportunity to present this work through two poster sessions, which lead to fruitful discussions with participants.

References

[1] Metayer S, Mouhanna D and Teber S 2022 Phys. Rev. E 105 L012603
[2] Nelson D R and Peliti L 1987 J. Phys. France 48 1085–1092
[3] Aronovitz J A and Lubensky T C 1988 Phys. Rev. Lett. 60(25) 2634–2637
[4] Guitter E, David F, Leibler S and Peliti L 1988 Phys. Rev. Lett. 61(26) 2949–2952
[5] Aronovitz J, Golubovic L and Lubensky T C 1989 J. Phys. France 50 609–631
[6] Guitter E, David F, Leibler S and Peliti L 1989 J. Phys. France 50 1787–1819
[7] Guitter E and Kardar M 1990 Europhysics Letters (EPL) 13 441–446
[8] Katsnelson M I 2012 Graphene: Carbon in Two Dimensions (Cambridge, U.K.: Cambridge University Press)
[9] Le Doussal P and Radzihovsky L 2018 Annals of Physics 392 340–410
[10] Coquand O, Mouhanna D and Teber S 2020 Phys. Rev. E 101 062104
[11] Kotikov A V and Teber S 2019 Physics of Particles and Nuclei 50 1–41
[12] Kownacki J P and Mouhanna D 2009 Phys. Rev. E 79(4) 040101
[13] Le Doussal P and Radzihovsky L 1992 Phys. Rev. Lett. 69(8) 1209–1212
[14] Zhang Z, Davis H T and Kroll D M 1993 Phys. Rev. E 48(2) R651–R654
[15] Bowick M J, Catterall S M, Falciomi M, Thorleifsson G and Anagnostopoulos K N 1996 J. Phys. I France 6 1321–1345
[16] Töster A 2013 Phys. Rev. B 87(10) 104112
[17] Los J H, Katsnelson M I, Yazyev O V, Zakharchenko K V and Fasolino A 2009 Phys. Rev. B 80(12) 121405
[18] Nogueira P 1993 Journal of Computational Physics 105 279–289 ISSN 0021-9991
[19] Lee R N 2014 Journal of Physics: Conference Series 523 012059 ISSN 1742-6596
[20] Pikelner A 2022 EPL 138 17002