Nonlinear Field Space Cosmology

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We consider the FRW cosmological model in which the matter content of universe (playing a role of inflaton or quintessence) is given by a novel generalization of the massive scalar field. The latter is a scalar version of the recently introduced Nonlinear Field Space Theory (NFST), where physical phase space of a given field is assumed to be compactified at large energies. For our analysis we choose the simple case of a field with the spherical phase space and endow it with the generalized Hamiltonian analogous to the XXZ Heisenberg model, normally describing a system of spins in condensed matter physics. Subsequently, we study both the homogenous cosmological sector and linear perturbations of such a test field. In the homogenous sector we find that nonlinearity of the field phase space is becoming relevant for large volumes of universe and then it can lead to a recollapse, and possibly also at very high energies, leading to the phase of a bounce. Quantization of the field is performed in the limit where nontrivial nature of its phase space can be neglected, while there is a non-vanishing contribution from the Lorentz symmetry breaking term of the Hamiltonian. As a result, in the leading order of the XXZ anisotropy parameter, we find that the inflationary spectral index remains unmodified with respect to the standard case but the total amplitude of perturbations is subject to a correction. The Bunch-Davies vacuum state also becomes appropriately corrected. The proposed new approach is bringing cosmology and condensed matter physics closer together, which may turn out to be beneficial for both disciplines.

I. INTRODUCTION

Scalar fields play a significant role in our understanding of the cosmological dynamics and structure formation. Both the inflationary epoch and the current phase of dark energy domination can be modeled using the scalar matter (an inflaton or quintessence field). On the other hand, the increasingly more precise astronomical observations provide the opportunity to test various modifications of the ordinary scalar field theories. The modified models turn out to be useful because the standard approach is insufficient to explain all subtleties in the observational data. In particular, the explored research directions include String Theory-inspired cosmological models, such as the Dirac-Born-Infeld [1, 2] and multifield models [3]. Furthermore, recent results obtained within loop quantum cosmology [4] provide novel solutions to the problems of classical cosmology, which can also lead to rethinking of the established methods.

The Dirac-Born-Infeld and multifield models share certain features with the Nonlinear Field Space Theory (NFST) [5, 6], which has recently been proposed in a broad theoretical context. The essential idea of the NFST framework is rather simple. Namely, standard linear (i.e. affine) phase space of a given field theory is considered to be only a local approximation of some more general (nonlinear) field phase space, which also leads to a generalization of the field Hamiltonian. The freedom in the choice of nontrivial topology or geometry of phase spaces is in principle very large. For the mathematical and physical consistency it is enough to require that the nonlinear phase space remains a symplectic manifold, while the ordinary field theory is completely recovered in the appropriate limit. We also have to mention that field theories known as the non-linear sigma models, especially in the Tseytlin formulation [7], can be seen as a special type of NFST, although in the approach of [7] they are defined on string worldsheets rather than spacetime and their target spaces are not Riemannian.

One of the advantages of NFST is that it allows to naturally implement the “Principle of finiteness”, which was the original motivation behind the Born-Infeld theory [8]. In the Born-Infeld theory the time derivative of a field is constrained analogously to the velocity of a relativistic particle. Besides, from the modern, string-theoretic point of view, the value of a Born-Infeld scalar field is associated with the distance between branes in the higher dimensional space [1, 2]. The required conservation of the causal structure leads to a constraint on the velocity with which a brane can move, which translates into the corresponding constraint on the velocity (i.e. rate of change) of the Born-Infeld field. The restriction implies that the field Lagrangian is given by the Lorentz factor for the field velocity. On the other hand, in the NFST framework the allowed field values (as well as their conjugate momenta) are determined by the nontrivial structure of an assumed nonlinear phase space. On the latter we have to find the Hamiltonian that appropriately generalizes a given standard field theory.

The defining feature of NFST is the nonlinearity of field phase spaces. In this sense multi-field models, where the configuration subspace of field phase space usually has non-vanishing curvature, can be seen as a special subclass of NFST. However, a NFST model can also be constructed from a single-component scalar field, as we will discuss in a moment. In the case of multi-field models the nontrivial geometry is not introduced in the field’s momentum space, which ensures that the Lorentz symmetry of the field is preserved. In turn, in NFST the whole phase space is generally non-trivial and this may
not even allow a global decomposition of phase space into the configuration and momentum spaces. While symmetries of the background spacetime are not modified, the Lorentz symmetry of the field will usually be violated.

One still might ask whether there are any strong reasons to generalize the standard linear phase spaces of fields. A number of arguments and motivations, related especially to quantum gravity, can be found in the original reference \[6\]. Let us here mention two important premises. The first one is related to special properties of compact phase spaces. Namely, when the area of phase space is finite, it can accommodate only a finite number of degrees of freedom, each occupying a cell with the area of $2\pi\hbar$.\[4\] This implies the finite dimensionality of the Hilbert space in the corresponding quantum theory (see below), which is the quantum counterpart of the Principle of finiteness and may naturally solve the problem of UV divergences in quantum field theory. The second premise concerns the particular example of a compact phase space that is given by a sphere and therefore is equivalent to the phase space of a spin. The latter relation leads to an exact analogy between fields (with compact phase spaces) and spin systems, first observed in \[10\], where it has been called the Spin-Field Correspondence. In principle, this kind of duality should allow us to find the spin system counterparts for different types of field theories. Such an approach is in the spirit of the broadly considered analog condensed matter models of gravity \[11\]. However, it should not only allow to design a condensed matter system that, in the linear limit, emulates a given field theory, but also to hypothesize about the fundamental origin of physical fields.

In order to introduce NFST in the cosmological context we first have to discuss a scalar field theory defined on Minkowski spacetime. In such a case phase space of the field at any point of space, or for any Fourier mode, is a linear space $\mathbb{R}^2$. Then the simplest construction of a compact NFST model is to reinterpret the field phase space $\mathbb{R}^2$ as the small field approximation of a sphere $S^2$ that is equipped with the appropriate symplectic form and Hamiltonian. In \[7\] (see also \[8\]) it has been done at the level of Fourier modes and in \[10\] at the level of space points, which gives us two inequivalent models where the total field phase space is an infinite collection of spheres. In this paper we will restrict to the model constructed in the position representation, since it has a direct connection with spin systems in condensed matter physics.

There are three parametrizations of the spherical phase space that will be of interest to us here (see Fig.\[8\]). The first one is given by usual angular coordinates $\phi \in [-\pi, \pi]$ and $\theta \in [0, \pi]$. The second uses the physical phases space variables $\varphi$ and $\pi_{\varphi}$, with the origin at $(\varphi, \theta) = (0, \pi/2)$. More details on the description of phase space in this parametrization can be found in \[6, 10\]. Finally, the third possibility is the Cartesian parametrization, in which coordinates represent components of a spin vector $\mathbf{S} = (S_x, S_y, S_z)$ and in terms of the two other parameterizations they can be expressed as:

$$S_x := S \sin \theta \cos \phi = S \cos \left( \frac{\pi \varphi}{R_2} \right) \cos \left( \frac{\varphi}{R_1} \right),$$

$$S_y := S \sin \theta \sin \phi = S \cos \left( \frac{\pi \varphi}{R_2} \right) \sin \left( \frac{\varphi}{R_1} \right),$$

$$S_z := S \cos \theta = S \sin \left( \frac{\pi \varphi}{R_2} \right),$$

(1)

(2)

(3)

together with the obvious condition $S_x^2 + S_y^2 + S_z^2 = S^2$, where $S$ is the sphere’s radius. Moreover, the dimensionful constants $R_1$ and $R_2$ satisfy the relation $R_1 R_2 = S$, so that the standard symplectic form on a sphere in terms of the $\varphi$ and $\pi_{\varphi}$ variables is given by

$$\omega = \cos \left( \frac{\pi \varphi}{R_2} \right) d\pi_{\varphi} \wedge d\varphi.$$  

(4)

Since $\omega$ is equal to the area two-form, integrating it over the whole phase space one obtains the total area of the latter: $\text{Ar}(S^2) = \int S^2 \omega = 4\pi S$. Meanwhile, in the ordinary quantum theory every degree of freedom occupies the area $2\pi\hbar$ and hence we infer that $S$ is subject to the quantization condition $S = j\hbar$, with $2j \in \mathbb{N}$. This allows us to relate dimension of the spin Hilbert space $\text{dim}(H_j) = 2j + 1$ with the area of phase space in the semiclassical regime.

We now need to choose the Hamiltonian on the discussed phase space. According to what we mentioned above, a spherical phase space of the field, which is attached at each point of space, can also be seen as describing a fictitious spin. Therefore, we may try to apply here the known models from condensed matter physics. In this paper we will restrict our investigations to the XXZ generalization of the (continuous) Heisenberg model. As discussed in \[10\], the XXX Heisenberg model coupled to a constant magnetic field emulates the non-relativistic scalar NFST with the quadratic dispersion relation. On the other hand, it has been shown \[12\] that the generalization of the Heisenberg XXX model to the XXZ model allows us to recover the relativistic scalar NFST, in the limit of vanishing anisotropy parameter $\Delta$.

Let us first write the Hamiltonian for the discrete XXZ Heisenberg model of spins on a cubic lattice in three spatial dimensions:

$$H_{XXZ} = -J \sum_{i,j} \left( S^{(i)}_x S^{(j)}_x + S^{(i)}_y S^{(j)}_y + \Delta S^{(i)}_z S^{(j)}_z \right) - \mu \sum_i \mathbf{B} \cdot \mathbf{S}^{(i)},$$

(5)

where the first sum is performed over the nearest neighbors. $J$ and $\mu$ denote the coupling constants, $\mathbf{B}$ is an external magnetic field and $\Delta$ the dimensionless anisotropy

1 The area may change if the uncertainty relation is deformed.
parameter, defined so that for $\Delta = 1$ the XXZ Heisenberg model reduces to the XXX model. The interaction of spins $S^{(1)}$ with the magnetic field $B$ leads to the spin precession, which plays a crucial role in the duality between a spin system and NFST that has been mentioned above (see also the next Section). In the continuum limit the Hamiltonian $H$ becomes

$$H^{\text{cont}}_{\text{XXZ}} = -\tilde{J} \int d^3x \left[ (\nabla S_x)^2 + (\nabla S_y)^2 + \Delta (\nabla S_z)^2 \right]$$

$$- \tilde{\mu} \int d^3x B \cdot S,$$

with the new coupling constants $\tilde{J}$ and $\tilde{\mu}$. It should be stressed that in the continuous case the vector $S$ naturally gains the dimension of density (i.e. $1/|\text{length}|^3$ in the units of $\hbar = 1 = c$). However, for the later convenience the dimension can be absorbed into the definitions of $\tilde{J}$ and $\tilde{\mu}$.

There are two main objectives of this paper. Firstly, to study an application of the scalar NFST as an inflaton or quintessence field on the FRW background. Secondly, to examine a possibility of testing predictions of the considered framework with the use of present astronomical observations. In Sec. III we start by defining a homogenous cosmological model with a matter field corresponding to the continuous XXZ Heisenberg model. Its dynamics is analyzed in the Hamiltonian framework for an arbitrary value of $S$. Subsequently, in Sec. IIII we derive the leading order inhomogeneities, which correspond to the limit $S \to \infty$. The Hamiltonian obtained from the XXZ model includes the Lorentz symmetry breaking term, proportional to the anisotropy parameter $\Delta$. We study generation of quantum inflationary inhomogeneities taking this effect into account. Finally, in Sec. IV we summarize our results and discuss prospects of the further development of the considered framework.

II. HOMOGENEOUS COSMOLOGICAL MODEL

The purpose of this Section is to construct a homogeneous and isotropic cosmological model with the scalar matter field described by the XXZ Heisenberg model $[10]$. According to the results of $[10]$, one can expect that the Hamiltonian $H$ in the leading order is equivalent to the Hamiltonian of a massive scalar field on the FRW background, which is widely used to model the inflationary dynamics. Below we will relate the parameters appearing in the Heisenberg model with the scalar field’s mass and the cosmological scale factor, so that in the limit $S \to \infty$ we can recover the known expressions for an ordinary inflationary model. To this end let us begin with the case of a standard scalar field defined on the FRW background.

A. Symplectic form

Analogously to the case of Minkowski spacetime discussed in the Introduction, we first note that phase space of a scalar field on the FRW background is $\mathbb{R} \times \mathbb{R}$ equipped with the symplectic form $\omega_\varphi = V_0 d\pi_\varphi \wedge d\varphi$ (here $\varphi, \pi_\varphi$ are coordinates on $\mathbb{R}^2$). The factor $V_0$ denotes a fiducial volume over which the Hamiltonian density is integrated and we introduce it since the integration over the whole spatial slice $\mathbb{R}^3$ leads to an infinite result (i.e. IR divergence). Thereafter, $V_0$ will be absorbed into the definition of field variables and will not appear in the final results. Namely, the momentum $\pi_\varphi$ can be rescaled as follows $\pi_\varphi \to \pi_\varphi/V_0$, so that the symplectic form for the scalar field becomes $\omega_\varphi = d\pi_\varphi \wedge d\varphi$.

The other ingredient of an ordinary inflationary model is the dynamical FRW background. Its degrees of freedom are usually chosen to be the dimensionless scale factor $a$, which relates the comoving and physical distances, and the conjugate momentum. The scale factor can be normalized in such a way that $a = 1$ at some chosen moment of time. For convenience, in this paper we replace $a$ by the dimensionful volume variable $q \equiv V_0 a^3$, which measures the expanding (or contracting) volume of a region $V_0$. $q$ is complemented by the canonically conjugate momentum $p$.

Consequently, the total phase space of the model is four dimensional and its symplectic form is assumed to be composed of two independent contributions, namely

$$\omega_{\text{tot}} = \omega_{\text{FRW}} + \omega_\varphi,$$  \hspace{1cm} (7)
where \( \omega_\varphi = d\pi_\varphi \wedge dq \) and the gravitational component has the canonical form

\[
\omega_{\text{FRW}} = dp \wedge dq .
\]  (8)

Now we would like to generalize the standard symplectic form (7) to the case of a scalar field with the spherical phase space. The symplectic form for such a field defined on the Minkowski background is given by (7). Therefore, our straightforward approach for the FRW background is to consider the following two-form

\[
\omega = dp \wedge dq + \cos \left( \frac{\pi_\varphi}{R_2(q)} \right) d\pi_\varphi \wedge d\varphi ,
\]  (9)

where we allow the \( R_2 \) constant to become some function of \( q \). In the \( R_2 \to \infty \) limit the form (9) reduces to the symplectic form (7), as it should be for the model in the linear phase space limit. The basic requirement of the correspondence with the standard scalar field is, therefore, satisfied.

The second requirement is that the two-form (9) is symplectic, which is satisfied if and only if it is closed (i.e. \( d\omega = 0 \)). In the case of the canonical form (7) this condition is trivial. On the other hand, for the new symplectic form (9) it implies that

\[
R_2(q) = R_2 = \text{const} ,
\]  (10)

as in (7). The matter contribution to the total symplectic form is therefore expected to be independent of the variable \( q \). Actually, this is analogous to an ordinary scalar field theory, in which passing from the Minkowski to FRW background does not affect the form \( \omega_\varphi \).

Inverting the symplectic form (9) we can derive the corresponding Poisson bracket

\[
\{ \cdot , \cdot \} := (\omega^{-1})^{ij}(\partial_i)(\partial_j) = \left[ \frac{\partial}{\partial q} \frac{\partial q}{\partial p} - \frac{\partial p}{\partial q} \right] - \frac{1}{\cos \left( \frac{\pi_\varphi}{R_2} \right)} \left[ \frac{\partial}{\partial \varphi} \frac{\partial \varphi}{\partial \pi_\varphi} - \frac{\partial \pi_\varphi}{\partial \varphi} \right] ,
\]  (11)

which will be used in the further analysis of the model’s dynamics. Nevertheless, one might still wonder whether a different generalization of the standard symplectic form (7) to the case with the spherical phase space, leading to a different bracket than (12), could also give us valuable results. In the Appendix we briefly discuss one such possibility, which is obtained by relaxing the requirement that the two-form (7) is closed. The two-form is still constrained by the topology of phase space and has to be closely related to (12). In particular, a function other than cosine would either naturally correspond to a different topology or not lead to a sensibly defined phase space.

B. Matter Hamiltonian

Dynamics of an ordinary homogeneous massive scalar field is determined by the Hamiltonian

\[
H_\varphi = \int_{V_0} d^3x N a^3 \left( \frac{\pi_\varphi^2}{2a^6} + \frac{1}{2} m^2 \varphi^2 \right)
\]  

\[
= NV_0 a^3 \left( \frac{\pi_\varphi^2}{2a^6} + \frac{1}{2} m^2 \varphi^2 \right) ,
\]  (12)

where \( m \) is the field’s mass and \( N \) denotes the lapse function, which is associated with the time reparametrization symmetry. Due to the homogeneity of the integrand in (12), the integration reduces to \( \int_{V_0} d^3x = V_0 \). With the use of the \( q \) variable introduced in the previous Subsection and the rescaling \( \pi_\varphi \to \pi_\varphi/V_0 \) (necessary to obtain the well normalized symplectic form \( \omega_\varphi = d\pi_\varphi \wedge d\varphi \)), one can simplify the Hamiltonian (12) to

\[
H_\varphi = N q \left( \frac{\pi_\varphi^2}{2q^2} + \frac{1}{2} m^2 \varphi^2 \right) .
\]  (13)

This is the expression that should be recovered in the \( S \to \infty \) limit of a homogenous scalar NFST, which is described by the Hamiltonian analogous to the XXZ Heisenberg model (7).

The first term of (10) contains spatial derivatives and therefore will not contribute to the homogenous sector of the corresponding scalar field. What matters is the term describing interaction with a magnetic field \( B \). Similarly as in (7) (10), we choose the vector \( B \) to be oriented along the \( x \) axis (i.e. \( B := (B_x, 0, 0) \)), so that the precession of a spin \( S \) occurs around this direction (as depicted in Fig. 3). Then to obtain the scalar field Hamiltonian corresponding to \( -\tilde{\mu} \int d^3x B_x S_x \) we have to introduce the cosmological scaling and freedom of time reparametrization, which can be accomplished through multiplying the measure \( d^3x \) by \( Na^3 \). The resulting Hamiltonian is

\[
H_S = -N q \tilde{\mu} B_x S_x
\]  

\[
= N q \left( -\tilde{\mu} B_x S + \frac{\tilde{\mu} B_x S}{2R_1^2} \pi_\varphi^2 + \frac{\tilde{\mu} B_x S}{2R_1^2} \varphi^2 + O(4) \right) ,
\]  (14)

where we subsequently expanded the spin component \( S_x \) (given by the expression (11)) up to the quadratic order in the field variables \( \varphi, \pi_\varphi \).

While we have chosen \( R_2 = \text{const} \), \( R_1 \) can still depend on \( q \). Moreover, for \( q = V_0 \equiv q_0 \) (i.e. \( a = 1 \)), which corresponds to the Minkowski background, the condition \( R_1 R_2 = S \) has to be satisfied (10). Taking these issues into account, the comparison of (11) with (13) allows us to identify the following relations between the parameters.
of both Hamiltonians:

\[
\tilde{\mu} B_z = \left( \frac{q_0}{q} \right) \frac{m}{q},
\]

(15)

\[
R_1 = \frac{1}{q} \sqrt{\frac{S q_0}{m}},
\]

(16)

\[
R_2 = \sqrt{S q_0 m},
\]

(17)

and consequently \( R_1 R_2 = \frac{2 \pi}{q} S \). The value of \( q_0 \) could in principle be set to \( q_0 = 1 \) but we will leave it unspecified in order to keep track of the dimensions.

Finally, we note that the extra term \(-\tilde{\mu} B_z S = -Smq_0/q^2\) in the Hamiltonian \([14]\) can be eliminated by subtracting the constant \( S \) from \( S_z \), which leads to the properly normalized scalar field Hamiltonian of the form

\[
H_S = N m \left( \frac{q_0}{q} \right) (S - S_z)
\]

= \( N q \left( \frac{m^2 \varphi^2}{2q^2} + \frac{1}{2} m^2 \varphi^2 \right) + \mathcal{O}(4). \)  

(18)

The \( q \)-dependent energy shift in \([13]\) ensures that energy is positive definite and the Hamiltonian \( H_S \) vanishes at the classical minimum \((\varphi, \pi_\varphi) = (0, 0)\). This is analogous to the theory on the Minkowski background \([12]\).

Therefore, an ordinary massive scalar field is indeed recovered for small field values from the NFST Hamiltonian \([14]\) analogous to the XXZ Heisenberg model. Let us again stress that \([14]\) can be introduced as a generalization of the scalar field Hamiltonian \([13]\) due to the assumed spherical geometry of the field’s phase space at every point of spacetime, which is mathematically identical to the phase space of a spin. Furthermore, the form of the scalar field’s potential is determined by an interaction of such (probably) fictitious spins with a constant vector field that, in the context of condensed matter, plays a role of the external magnetic field.

On the other hand, if the negative term in the expansion of \([14]\) is not subtracted, the Hamiltonian \([18]\) becomes replaced by

\[
\tilde{H}_S = H_S + \delta H_S = -N m \left( \frac{q_0}{q} \right) S_x,
\]

(19)

which is negative in the regime \( S_x > 0 \), where the standard limit should be recovered. Furthermore, the \( \delta H_S \) term is always negative and can be perceived as a source of negative energy, with the energy density

\[
\rho_* = -Sm \frac{q_0}{q^2}.
\]

(20)

This contribution scales as \( 1/q^6 \) and therefore can play a dominant role in the early universe, while becoming irrelevant for the late time dynamics. In particular, it may lead to the phase of non-singular bounce, which replaces the big bang singularity (see Subsection \([11D]\)).

C. Equations of motion

According to the previous Subsections, the total Hamiltonian for the system under consideration is

\[
H_{tot} = H_{FRW} + H_S
\]

= \( N q \left( \frac{3}{4} \kappa p^2 + m \frac{q_0}{q} (S - S_x) \right), \)

(21)

where \( \kappa \equiv 8\pi G = 8\pi/m_{p1}^2 \), \( m_{p1} \) is the Planck mass and the matter field Hamiltonian is given by \([13]\). Using \([21]\) one can derive the Hamilton equations \( f = \{ f, H_{tot} \} \) for an arbitrary function \( f \) on phase space \((q, p, \varphi, \pi_\varphi)\). On the other hand, the Hamiltonian \([21]\) can also be seen as a constraint, imposed through the condition \( \partial_{\pi N} H_{tot} = 0 \) (which is equivalent to vanishing of the conjugate momentum of \( N \), i.e. \( p_N = 0 \)). The constraint can be written as:

\[
\frac{3}{4} \kappa p^2 = m \frac{q_0}{q} (S - S_x).
\]

(22)

In the usual way we also introduce the Hubble factor

\[
H \equiv \frac{1}{3q} \frac{\dot{q}}{q} - \frac{1}{3q} \{ q, H_{tot} \} = -\frac{1}{2} \kappa p,
\]

(23)

where we have chosen the gauge \( N = 1 \), which will be kept in the remaining part of this Section. In such a case the overdot “\( \dot{\} \)” denotes a differentiation with respect to the coordinate time \( t \).

Expressing \( p \) via \([23]\) and substituting it into the Hamiltonian constraint \([22]\) we obtain the Friedmann equation

\[
H^2 = \frac{1}{9} \left( \frac{\dot{q}}{q} \right)^2 = \kappa \frac{\rho}{3},
\]

(24)

with the matter energy density

\[
\rho = \frac{m}{q} \left( \frac{q_0}{q} \right) (S - S_x).
\]

(25)

The density \( \rho \) is positive definite and vanishes in the limit \( S_x \to S \).

Let us now derive the remaining equations of motion. We will do it for both the components of a spin vector \( S = (S_x, S_y, S_z) \) (as functions of \( \varphi \) and \( \pi_\varphi \)) and the field variables \( \varphi \) and \( \pi_\varphi \). The advantage of using the \( S_x, S_y, S_z \) variables is that, in contrast to \( \varphi \) and \( \pi_\varphi \), they are well defined on the whole \( S^2 \) phase space. As one can verify by a direct calculation, they naturally generate the so(3) algebra

\[
\{ S_x, S_y \} = S_z, \quad \{ S_z, S_x \} = S_y, \quad \{ S_y, S_z \} = S_x.
\]

(26)

Moreover, we find that their Poisson brackets with the gravitational variables \( q \) and \( p \) have the form

\[
\{ S_x, q \} = \{ S_y, q \} = \{ S_z, q \} = 0.
\]

(27)
and

\[
\{S_x, p\} = \frac{\partial S_x}{\partial q} = -\frac{S_q}{q} \arctan \frac{S_y}{S_x}, \quad (28)
\]

\[
\{S_y, p\} = \frac{\partial S_y}{\partial q} = \frac{S_x}{q} \arctan \frac{S_y}{S_x}, \quad (29)
\]

\[
\{S_z, p\} = \frac{\partial S_z}{\partial q} = 0. \quad (30)
\]

Using the above formulae we derive the evolution equations

\[
\dot{S}_x = \{S_x, H_{\text{total}}\} = \frac{3}{2} N \kappa p S_y \arctan \frac{S_y}{S_x}, \quad (31)
\]

\[
\dot{S}_y = \{S_y, H_{\text{total}}\} = N m S_z - \frac{3}{2} N \kappa p S_x \arctan \frac{S_y}{S_x} \quad (32)
\]

\[
\dot{S}_z = \{S_z, H_{\text{total}}\} = -N m S_y, \quad (33)
\]

and they naturally satisfy \(\partial_t S^2 = 2(S_x \dot{S}_x + S_y \dot{S}_y + S_z \dot{S}_z) = 0\). We also find that the equation of motion for \(p\) is

\[
\dot{p} = \frac{3}{4} N \kappa p^2 \left( \frac{q_1}{q} \right) \left( S - S_x - S_y \arctan \frac{S_y}{S_x} \right). \quad (34)
\]

On the other hand, for the \(\varphi\) and \(\pi_\varphi\) variables we calculate

\[
\dot{\varphi} = \frac{1}{\cos \left( \frac{\varphi}{R_2} \right)} \frac{\partial H_{\text{total}}}{\partial \pi_\varphi} = N \frac{R_2}{q} \tan \left( \frac{\varphi}{R_2} \right) \cos \left( \frac{\varphi}{R_1} \right), \quad (35)
\]

\[
\dot{\pi}_\varphi = -\frac{1}{\cos \left( \frac{\varphi}{R_2} \right)} \frac{\partial H_{\text{total}}}{\partial \varphi} = -N q R_1 m^2 \sin \left( \frac{\varphi}{R_1} \right). \quad (36)
\]

which are well posed on the hemisphere where \(\varphi \in (-\pi/2, \pi/2)\) and \(\varphi \in (-\pi/2, \pi/2)\). In the case of Minkowski spacetime \((q = \text{const})\), the exact solutions of the equations \((35)-(36)\) can actually be found (see \([4]\)).

In general, as one can easily verify, the standard equations for motion of a massive scalar field are recovered in the limit of \(R_1 \to \infty\), \(R_2 \to \infty\). Both of the limits are obtained when \(S \to \infty\). However, one has to keep in mind that \(R_1\) is actually a function of \(q\). From \([10]\) we infer that in the large volume limit \(q \to \infty\) we have \(R_1 \to 0\). Therefore, while taking \(S \to \infty\) always leads to the standard field dynamics, it is not obvious if this dynamics is recovered for some finite \(S\) in the large volume limit, \(q \to \infty\). This issue will be addressed in the next Subsection.

D. Basic features of the dynamics

In this Subsection we present a preliminary discussion of the features of dynamics described by evolution equations calculated above. The complete analysis is beyond the scope of this paper and will be a subject of the future investigations. Here we focus on the most basic properties of the considered model.

We start by deriving the \(O(1/S)\) corrections to the standard equations of motion expected for cosmology with a massive scalar field. To this end let us first remind that in the case of such an ordinary scalar field the expressions for its energy density and pressure respectively have the form

\[
\rho_\varphi := \frac{\pi^2}{2q^2} \left( \frac{\rho_\varphi}{2} - \frac{2m^2}{3} \varphi^2 \right) + O(1/S^2), \quad (37)
\]

\[
P_\varphi := \frac{\pi^2}{2q^2} - \frac{1}{2} m^2 \varphi^2. \quad (38)
\]

Applying the above definitions to the Friedmann equation \((24)\), where we expand the energy density \((23)\) as a series in \(1/S\), we obtain

\[
H^2 = \frac{\kappa}{3} \rho_\varphi - \frac{\kappa}{9} S q_0 m \left( \rho_\varphi - \frac{1}{2} P_\varphi \right) + O(1/S^2). \quad (39)
\]

As one can see, the leading order correction does not only depend on the field’s energy density but also on its pressure. These new contributions become more and more relevant with increasing \(q\). Therefore, we can expect that the spherical geometry of phase space modifies the late time dynamics. In particular, the correction in \((39)\) becomes negative if \(\rho_\varphi^2 - \frac{2}{3} P_\varphi > 0\), and this can lead to the effect of recollapse, as we will discuss below. For the special case of the barotropic equation of state \(P_\varphi = w \rho_\varphi\), the correction term remains negative if the condition \(|w| < \sqrt{2}\) is satisfied, which covers most of the types of matter considered in cosmology.

Furthermore, let us observe that expanding the equations of motion \((35)-(38)\) up to the first order in \(1/S\) we obtain

\[
\dot{\varphi} = \frac{\varphi}{q} + \frac{\pi_\varphi}{S m q_0} \left( \frac{\pi_\varphi^2}{3 \gamma^2} - \frac{m^2}{2} \varphi^2 \right) + O(1/S^2), \quad (40)
\]

\[
\dot{\pi}_\varphi = -q m^2 \varphi + \frac{q^2 m^3}{6 S q_0} \rho^3 + O(1/S^2). \quad (41)
\]

Combining the above equations we can derive the modified Klein-Gordon equation

\[
\dot{\varphi} + 3 H \dot{\varphi} + m^2 \varphi = -\frac{q^2 m}{S q_0} \left[ 3 H \dot{\varphi} \varphi^2 + 2 \varphi^2 - \frac{m^2}{3} \varphi^3 \right] + O(1/S^2), \quad (42)
\]

which is quite complicated and therefore we do not analyze it further here. In the leading order, the late time oscillation of the field at the bottom of the potential well is approximated by the solutions:

\[
\varphi \propto \frac{\cos(mt + \alpha)}{t} \quad \text{and} \quad q \propto t^2, \quad (43)
\]
where \( \alpha \) is a constant of integration. Hence one can show that at late times the average pressure is approximately zero \( \langle P_\varphi \rangle \approx 0 \), while energy density \( \rho_\varphi \sim 1/q \). In other words, in this regime the field effectively behaves like a dust matter.

Consequently, the modified Friedmann equation (39) at large \( q \) (i.e. late times) can be written as

\[
H^2 \approx \frac{\kappa}{3} \rho_\varphi \left( 1 - \frac{\rho_\varphi}{\rho_X} \right),
\]

where we neglected the pressure term, while

\[
\rho_X := \frac{3Sq_0}{q^2}.
\]

is the energy density scale. We note that \( \rho_X \) is inversely proportional to \( q^2 \). In other words, the equation (44) is obtained by taking the Friedmann equation (24) and expanding the energy density (25) (where \( S_\varphi \) is a trigonometric function of \( \varphi \) and \( \pi_\varphi \)) in terms of the standard expressions for the scalar field’s density and pressure (given by (37) and (38)). The simplification from (39) to (44) originates in the late time (oscillatory) evolution of the field, which allows us to average out the pressure contribution. Worth mentioning is also that the equation (44) has a similar structure to the effective Friedmann equation in loop quantum cosmology [1-5].

For the discussed late time dynamics, assuming that the terms \( O(1/S) \) are still negligible and do not affect significantly the approximate solutions, the Friedmann equation (44) with \( \rho_\varphi = c/q \) (where \( c \) is some constant) leads to

\[
H^2 = \frac{\kappa}{3} \frac{c}{q_0 m^3} - \frac{\kappa}{3} \frac{c^2}{3 S q_0 m} + O(1/S^2),
\]

where we fixed the current value of the scale factor as \( a = 1 \), which is equivalent to \( q = q_0 \). Surprisingly, while the first contribution to (46) describes the dust matter content (which possibly can play a role of dark matter), the second contribution is constant and can be interpreted as the negative cosmological constant term, namely

\[
\Lambda := -\frac{\kappa c^2}{3 S q_0 m}.
\]

Therefore, within our model both dark matter and negative cosmological constant possibly emerge as late time contributions of the scalar field. However, in case the inflationary period is driven by the field under consideration, its late time contribution to dark matter is naturally expected to be marginal.

Negative cosmological constant may eventually lead to a recollapse, occurring at the density scale \( \rho_\varphi = \rho_X \). This is under the assumption that the higher order terms will not spoil the discussed approximate dynamical behaviour. Writing the density in the form \( \rho_\varphi = \rho_0 \frac{a^3}{q} \) (such that \( c = \rho_0 q_0 \)), we find that the solution to the condition \( \rho_\varphi = \rho_X \) is

\[
q_{\text{collapse}} = \frac{3S m}{\rho_0}.
\]

Another interesting possibility concerns the slow-roll regime of the field dynamics, in which \( P_\varphi \approx -\rho_\varphi \) = const. Then the modified Friedmann equation (39) simplifies to

\[
H^2 = \frac{\kappa}{3} \rho_\varphi - \frac{\kappa}{18} \frac{q^2}{S q_0 m} \rho_\varphi^2 + O(1/S^2).
\]

In such a case, the first contribution is approximately constant, while the second one scales as \( q^2 \). The latter corresponds to the effective fluid characterized by the equation of state \( P = w_{\text{eff}} \rho \), with \( w_{\text{eff}} = -3 \), which describes the so-called phantom matter [14]. Consequently, the correction term is leading to a recollapse of universe, similarly as in the case of oscillatory regime.

It is worth stressing that at early times (small \( q \)) the NSFT corrections to the ordinary cosmological model are expected to be much smaller than for the late time dynamics. In particular, we expect that for sufficiently small \( q \) the standard scalar field dynamics is recovered, as one can infer from the expansions (40) and (41). The choice of initial conditions may, however, be affected by the nontrivial nature of the field phase space.

On the other hand, there is also a possibility that the negative energy density (20), which has been subtracted in the definition of the considered Hamiltonian [15], should actually be taken into account. In such a case it is necessary to balance (20) by an additional contribution to the total energy density (e.g. radiation or cosmological constant). As the result, \( \rho_\varphi \) will dominate the energy density at sufficiently small \( q \), leading to the phase of a cosmic bounce, while at the intermediate energy scales the standard scalar field approximation, with \( \rho \approx {\rho_\varphi} \), remains valid. At late times the \( O(1/S) \) corrections will start to prevail, triggering a recollapse. Briefly speaking, the above matter content may give us a non-singular oscillatory cosmological model. This interesting possibility will be investigated elsewhere.

III. PERTURBATIVE COSMOLOGICAL INHOMOGENEITIES

In the previous Section we introduced and discussed the homogeneous cosmological model employing a scalar field with the spherical phase space. The tentative analysis of its dynamics, which is determined by a Heisenberg model, was performed in the whole range of variability of the field values, including the region far beyond the domain where the linear phase space approximation is valid. Our next purpose is to study what are the consequences of applying such dynamics at the level of perturbative cosmological inhomogeneities. Similarly as in the homogeneous case, we will consider the scalar field theory
whose Hamiltonian is derived from the XXZ Heisenberg model. However, while previously we calculated the exact form of the evolution equations with an arbitrary $S$ and analyzing them we were using expansions up to the first order of $1/S$, here we will restrict to the effects that remain in $S \to \infty$ limit. This limit corresponds to the quadratic form of the Hamiltonian and the linear order of perturbations.

Let us again start with an inhomogeneous massive scalar field on the Minkowski spacetime background. Expressing the Hamiltonian $H$ of the continuous XXZ Heisenberg model in terms of the field variables $\varphi, \pi_\varphi$, via the relations \( \psi = \varphi \), and keeping only the leading terms of the \( 1/S \) expansion we obtain

$$H = \int d^3x \left[ \frac{\pi_\varphi^2}{2} + \frac{1}{2}(\nabla \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \Delta \frac{(\nabla \pi_\varphi)^2}{2m^2} \right],$$  \quad (50)

where $m$ is the field’s mass, being a function of the parameters of the Heisenberg model \( \Delta \), analogously to \( \Delta \). The scalar field Hamiltonian $H$ is a generalization of the one derived in \[] from the XXX Heisenberg model, which corresponds to $\Delta = 1$. The extra term \( \frac{\Delta}{2m^2}(\nabla \pi_\varphi)^2 \) for $\Delta = 1$ reflects the rotational symmetry of the XXX Heisenberg model. Breaking of this symmetry is controlled by the anisotropy parameter $\Delta$. While for $\Delta \to 0$ the standard relativistic scalar field theory is recovered, for $\Delta = 1$ we obtain the theory \( \Delta \) that is invariant under the Born reciprocity transformation: $\varphi \to \pi_\varphi/m$ and $\pi_\varphi \to -\varphi m$.

The detailed discussion of the case $\Delta = 1$ can be found in \[\] and the model with an arbitrary value of $\Delta$ will be studied in \[\]. In particular, the spherical phase space on which the Hamiltonian \( H \) is defined is equipped the Poisson bracket \( \{ f(x), g(y) \} = \int d^3z \frac{d^3x}{\cos(\pi_\varphi(z)/R_2)} \left( \frac{\delta f(x)}{\delta \varphi(z)} \frac{\delta g(y)}{\delta \pi_\varphi(z)} - \frac{\delta f(x)}{\delta \pi_\varphi(z)} \frac{\delta g(y)}{\delta \varphi(z)} \right), \quad (51) \)

which is the field theoretic generalization of the second term of the bracket \( \{ f(x), g(y) \} \).

In order to generalize the Hamiltonian \[\] to the FRW background we have to apply the following rescalings:

$$\pi_\varphi \to \frac{\pi_\varphi}{a^3}, \quad (52)$$
$$\nabla \to \frac{1}{a} \nabla, \quad (53)$$
$$d^3x \to Na^3d^3x, \quad (54)$$

where $a$ is the scale factor and $N$ the lapse function. As the result, the scalar field Hamiltonian \( H \) on the FRW background acquires the form

$$H_\varphi = \int d^3x H_\varphi = \int d^3x Na^3 \left[ \frac{\pi_\varphi^2}{2a^6} + \frac{1}{2a^2} (\nabla \varphi)^2 + \frac{2m^2}{a^2} \varphi^2 + \frac{\Delta}{2m^2a^2} (\nabla \pi_\varphi)^2 \right]. \quad (55)$$

In what follows we will choose the gauge $N = a$, so that we deal with the conformal time $\tau$, which is convenient in the studies of cosmological inhomogeneities. Accordingly, the prime “′” will denote a differentiation with respect to the conformal time.

Our analysis of the cosmological perturbations will also be simplified by the assumption that the inhomogeneous part of the field $\varphi$ can be treated as a test field, i.e. that it does not affect the background dynamics. Furthermore, we do not consider excitations of the gravitational degrees of freedom but focus only on the contributions from the matter field. While such an approach is quite restrictive, it will give us the first qualitative results concerning the statistical properties of quantum cosmological perturbations in a NFST model. The complete analysis, which would take into account the scalar gravitational degrees of freedom, is a next step for the future work.

Instead of using the variable $\varphi$ it is now convenient to introduce its cosmologically rescaled version $\psi := a \varphi$. Then the Hamiltonian density in \( H \) simplifies to the form in which, apart from the term proportional to $\Delta$, gravity manifests only as an effective modification of the field’s mass (see below). However, due to presence of the non-standard term $\frac{\Delta}{2m^2a^2} (\nabla \pi_\varphi)^2$, the relations between $\varphi, \pi_\varphi$ and the momentum $\pi_\psi$ canonically conjugate to $\psi$ can be expected to differ from the usual cosmological models. Therefore, in order to obtain the Hamiltonian $H(v, \pi_\psi)$ we will apply the following procedure:

1. We derive the Lagrangian $L(\varphi, \varphi')$ corresponding to the Hamiltonian $H(\varphi, \pi_\varphi)$.
2. We make the change of variables $\varphi = v/a$, which leads to the Lagrangian $L(v, v')$.
3. Finally, from $L(v, v')$ we calculate the Hamiltonian $H(v, \pi_\psi)$.

We begin by finding the equations of motion determined by the Hamiltonian \( \Delta \):

$$\varphi' = \frac{1}{a} \pi_\varphi - \frac{\Delta}{ma^3} \nabla^2 \varphi, \quad (56)$$
$$\pi_\varphi' = a^2 \nabla^2 \varphi - m^2 a^4 \varphi, \quad (57)$$

and we rewrite the first of them as

$$\pi_\varphi = \frac{a^2}{1 - \frac{\Delta}{ma^2} \nabla^2} \varphi' + O(\Delta^2). \quad (58)$$
The expansion around \( \Delta = 0 \) introduced here allows us to study deviations from the standard relativistic case, in agreement with what was discussed above. We will consider the terms up to the first order in \( \Delta \). In particular, using (58) we find that the Hamiltonian density

\[
\mathcal{H}_\varphi = \frac{\pi^2}{2a^2} + \frac{a^2}{2}(\nabla \varphi)^2 + \frac{1}{2}m^2a^4\varphi^2 + \frac{\Delta}{2m^2a^4}(\nabla \pi_\varphi)^2
\]

(59)
corresponds to the Lagrangian density

\[
\mathcal{L}_\varphi = \frac{a^2}{2}((\varphi')^2 - (\nabla \varphi)^2) - \frac{1}{2}m^2a^4\varphi^2 - \frac{\Delta}{2m^2a^2}(\nabla \varphi')^2 + \mathcal{O}(\Delta^2). 
\]

Therefore, we can calculate that the Hamiltonian density

\[
\mathcal{H}_v = \frac{1}{2}((v')^2 - (\nabla v)^2) - \left(a^2m^2 - \frac{a''}{a}\right)\frac{v^2}{2} - \frac{\Delta}{2ma^2}(\nabla v')^2 + \mathcal{O}(\Delta^2) 
\]

(61)
and hence we obtain the conjugate momentum

\[
\pi_v := \frac{\partial \mathcal{L}_v}{\partial v'} = v' + \frac{\Delta}{ma^2} \nabla^2 \left(v' - \frac{a'}{a}v\right) + \mathcal{O}(\Delta^2). 
\]

(62)
Therefore, we can calculate that the Hamiltonian density in terms of \( v \) and \( \pi_v \) has the form

\[
\mathcal{H}_v := v'\pi_v - \mathcal{L}_v = \frac{\pi_v^2}{2} + \frac{1}{2}(\nabla v)^2 + \frac{1}{2}m^2a^2v^2 + \frac{\Delta}{2ma^2}(\nabla \pi_v - \mathcal{H} \nabla v^2) + \mathcal{O}(\Delta^2), 
\]

(63)
where \( \mathcal{H} \equiv a'/a \) denotes the conformal Hubble factor and the quantity

\[
m^2_{\text{eff}} \equiv m^2a^2 - \frac{a''}{a}. 
\]

(64)
is the effective mass of the field.

The equations of motion resulting from (58) are

\[
v' = \pi_v + \frac{\Delta}{ma^2} (\mathcal{H} \nabla^2 v - \nabla^2 \pi_v) + \mathcal{O}(\Delta^2), 
\]

(65)
\[
\pi'_v = -m_{\text{eff}}^2v + \left(1 + \frac{\Delta}{ma^2} \mathcal{H}^2\right)\nabla^2 v - \frac{\Delta}{ma^2} \mathcal{H} \nabla^2 \pi_v + \mathcal{O}(\Delta^2). 
\]

(66)
Together they lead to the second order equation for \( v \):

\[
v'' - \nabla^2 v + m_{\text{eff}}^2 v \\
= \frac{\Delta}{ma^2} \left[(-2\mathcal{H} + m^2a^2) \nabla^2 v + 2\mathcal{H} \nabla^2 v' - \nabla^4 v \right], 
\]

(67)
To arrive at the starting point for quantization of our model we switch to the \( v \) and \( \pi_v \) variables, defined in the introductory part of this Section and then the bracket acquires the identical form

\[
\{ v(x), \pi_v(y) \} = \delta^{(3)}(x - y). \tag{72}
\]

After the canonical quantization \( v(x) \) and \( \pi_v(y) \) become quantum operators and (72) is replaced by the corresponding commutation relation

\[
[\hat{v}(x), \hat{\pi}_v(y)] = i\delta^{(3)}(x - y) \hat{1}. \tag{73}
\]

Let us remind here that we are using the units in which \( \hbar = 1 \).

The operators \( \hat{v}(x) \) and \( \hat{\pi}_v(y) \) can be now Fourier expanded:

\[
\hat{v}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{v}_k e^{ik\cdot x}, \tag{74}
\]

\[
\hat{\pi}_v(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{\pi}_{vk} e^{ik\cdot x}. \tag{75}
\]

In the Heisenberg picture we decompose their Fourier modes in the basis of creation and annihilation operators (it can be done because the Hamiltonian operator (79) introduced below is quadratic):

\[
\hat{v}_k(\tau) = f_k(\tau) \hat{a}_k + f^*_k(\tau) \hat{a}^\dagger_k, \tag{76}
\]

\[
\hat{\pi}_{vk}(\tau) = g_k(\tau) \hat{a}_k + g^*_k(\tau) \hat{a}^\dagger_k, \tag{77}
\]

where \( f_k(\tau), g_k(\tau) \) are complex functions, describing the time evolution of \( \hat{a}^\dagger_k \) and \( \hat{a}_k \). The creation and annihilation operators satisfy the standard commutation relation \( [\hat{a}_k, \hat{a}^\dagger_q] = \delta^{(3)}(k - q) \) (which would become algebraically deformed in the case of finite \( S \)).

Applying the decompositions (76, 77) to the Hamilton equation determined by (79) \( \dot{v}_k = -i[\hat{v}_k, \hat{H}_v] \), we derive the equation for mode functions

\[
f'_k = \left( 1 + \frac{\Delta k^2}{m^2a^2} \right) g_k - \frac{\Delta k^2}{m^2a^2} \mathcal{H} f_k + \mathcal{O}(\Delta^2). \tag{80}
\]

If one solves it for \( g_k \), it gives

\[
g_k = f'_k + \frac{\Delta k^2}{m^2a^2} (\mathcal{H} f_k - f'_k) + \mathcal{O}(\Delta^2). \tag{81}
\]

Finally, substituting (81) into (78) we obtain the modified Wronskian condition

\[
f_k(f'_k)^* - f'_k f^*_k = i \left( 1 + \frac{\Delta k^2}{m^2a^2} \right) + \mathcal{O}(\Delta^2), \tag{82}
\]

which has to be satisfied by functions \( f_k \).

### B. Vacuum normalization

The choice of the initial state of cosmological inhomogeneities is highly ambiguous. In practice, the only way to do it is to assume some particular form of the initial state and study its late time behavior. The vacuum state is often perceived as a distinguished choice, which is in agreement with the initial homogeneity of the early universe. The majority of results for the primordial perturbations have indeed been obtained in the case of the initial vacuum state. Therefore, finding the appropriate vacuum state should allow us to compare predictions of our model with results of the standard theory of a quantum scalar field on the cosmological background.

The initial vacuum state \( |0\rangle \) is defined as such a state that \( \hat{a}_k |0\rangle = 0 \) (for the time \( \tau \to -\infty \)). Furthermore, the vacuum state is assumed to be a ground energy state, in which the vacuum expectation value \( \langle 0 | \hat{H}_v | 0 \rangle \) achieves a minimum.

In order to find what the ground energy is, we express the quantum Hamiltonian (79) in terms of the creation and annihilation operators, using the decompositions (76, 77):
\[ \hat{H}_v = \frac{1}{2} \int d^3k \left( ((1 + A_k)(g_k^*))^2 + (\omega_k^2 + B_k)(f_k^*)^2 + 2C_k f_k g_k^* \right) \hat{a}_k \hat{a}_k^\dagger \\
+ ((1 + A_k)g_k^2 + (\omega_k^2 + B_k)f_k^2 + 2C_k f_k g_k) \hat{a}_k \hat{a}_k \\
+ ((1 + A_k)|g_k|^2 + (\omega_k^2 + B_k)|f_k|^2 + 2C_k f_k g_k) \hat{a}_k^\dagger \hat{a}_k \\
+ ((1 + A_k)|g_k|^2 + (\omega_k^2 + B_k)|f_k|^2 + 2C_k f_k g_k^* \hat{a}_k \hat{a}_k^\dagger \right), \]  
\tag{83} \]

where we denoted \( A_k \equiv \frac{\Delta \omega_k^2}{m^2 a^2} \), \( B_k \equiv \frac{\Delta \omega_k^2}{m^2 a^2} \mathcal{H} \) and \( C_k \equiv \frac{\Delta \omega_k^2}{m^2 a^2} \mathcal{H} \). Furthermore, as justified earlier the standard commutation relation \( [\hat{a}_k, \hat{a}_q^\dagger] = \delta^{(3)}(k - q) \) can be used. Consequently, the vacuum expectation value of (83) is calculated to be

\[ \langle 0 | \hat{H}_v | 0 \rangle = \frac{1}{2} \delta^{(3)}(0) \int d^3k \left( (1 + A_k)|g_k|^2 + (\omega_k^2 + B_k)|f_k|^2 + 2C_k f_k g_k^* \right) \]  
\tag{84} \]

and we identify

\[ E_k \equiv (1 + A_k)|g_k|^2 + (\omega_k^2 + B_k)|f_k|^2 + 2C_k f_k g_k^* \]  
\tag{85} \]

as the energy density of a given mode \( k \).

Let us now introduce the polar decomposition of each complex function \( f_k = r_k e^{i\alpha_k} \). From the Wronskian condition (82) we obtain the relation \( \alpha_k' = -A_k/(2r_k^2) \), which can be used to eliminate \( \alpha_k \) from (83), so that it becomes

\[ E_k = ((1 - A_k)r_k^2 + 4C_k r_k + (\omega_k^2 + B_k)r^2 \\
+ (1 + A_k)\frac{1}{4r^2} + iC_k + \mathcal{O}(\Delta^2). \]  
\tag{86} \]

Subsequently, to find a minimum of \( E_k \) we calculate its derivatives

\[ \frac{1}{\partial \alpha'} E_k = 2(1 - A_k)r' + 4C_k r, \]  
\tag{87} \]

\[ \frac{1}{\partial r} E_k = 4C_k r' + 2(\omega_k^2 + B_k)r - (1 + A_k)\frac{1}{2r^3} \]  
\tag{88} \]

and set both these expressions to zero. The solution of the resulting system of equations is given by

\[ r_k' = -2\mathcal{H} \frac{\Delta k^2}{m^2 a^2} r_k + \mathcal{O}(\Delta^2), \]  
\tag{89} \]

\[ r_k = \frac{1}{\sqrt{2\omega_k}} \left[ \frac{1}{4} \frac{\Delta k^2}{m^2 a^2} \left( 1 - \frac{\mathcal{H}^2}{\omega_k^2} \right) + \mathcal{O}(\Delta^2) \right] \]  
\tag{90} \]

(it should be stressed that \( \omega_k \) depends on time via \( m_{\text{eff}} \)) and consequently we also obtain

\[ \alpha_k = -\int d\tau \omega_k \left[ \frac{1}{2} \frac{\Delta k^2}{m^2 a^2} \left( 1 + \frac{\mathcal{H}^2}{\omega_k^2} \right) \right] + \mathcal{O}(\Delta^2). \]  
\tag{91} \]

In the UV limit \( k^2 \gg m^2 a^{-2} \) and also assuming the condition \( \frac{\Delta k^2}{m^2 a^2} \ll 1 \), we can use (90) and (91) to write the mode function in the form

\[ f_k = \frac{1}{\sqrt{2k}} \left[ 1 + \frac{\Delta k^2}{4 m^2 a^2} \right] \exp \left[ -ik\tau - i \frac{\Delta k^3}{m^2 a^2} \int \frac{d\tau}{a^2} \right] \\
+ \mathcal{O}(\Delta^2) \]  
\tag{92} \]

which describes the \( \Delta \)-modified version of the Bunch-Davies vacuum state.

C. Inflationary power spectrum

The Bunch-Davies vacuum normalization of the mode functions derived in the previous Subsection allows us to quantify the statistical properties of the vacuum field configuration. For the linear inhomogeneities, as the ones considered in this Section, the vacuum expectation values of the products of the physical field operators \( \hat{\phi} := \hat{\phi}/a \) carry the whole necessary information about correlations of quantum states. In particular, the two-point correlation function can be written in the form

\[ \langle 0 | \hat{\phi}(x, \tau) \hat{\phi}(y, \tau) | 0 \rangle = \int_0^\infty \frac{dk}{k} \sin kr P_\phi(k, \eta), \]  
\tag{93} \]

where the power spectrum is defined as

\[ P_\phi(k, \tau) := \frac{k^3}{2\pi^2} \left| \int \frac{d\tau}{a(\tau)} \right|^2 \]  
\tag{94} \]

and we denote \( r = |x - y| \).

As one can verify, the evolution equation for the mode functions \( f_k \) has the same form as the equation for the Fourier modes (60), namely

\[ f_k'' + 2\mathcal{H} \frac{\Delta k^2}{m^2 a^2} f_k' \\
+ \left( \omega_k^2 + \frac{\Delta k^2}{m^2 a^2} \right) f_k = 0. \]  
\tag{95} \]

We notice that in the expanding regime (when the conformal Hubble factor \( \mathcal{H} < 0 \)) the modes are additionally
dumped by the negative “friction term”, proportional to $\mathcal{H}$. This term can be eliminated from the equation by introducing the new variable

$$y_k := \exp \left( -\frac{1}{2} \frac{\Delta k^2}{m^2 a^2} \right) f_k,$$

in which (96) becomes

$$y_k'' + \left[ \omega_k^2 \left( 1 + \frac{\Delta k^2}{m^2 a^2} \right) + \frac{\Delta k^2}{m^2 a^2} \mathcal{H}^2 \right] y_k = 0.$$  (97)

As a specific example let us consider the de Sitter background, which is the leading order approximation (vanishing slow-roll parameters) of the inflationary period in cosmology. Such a period is expected to occur due to the ground, which is the leading order approximation (vanishing slow-roll parameters) of the inflationary period in cosmology. Consequently, the growing exponential factor leads to a slight shift of the amplitude of perturbations and agrees with the contribution expected from the $\Delta$-modified version of the Bunch-Davies vacuum. The magnitude of derived corrections is inversely proportional to $\eta$, and therefore $\eta$ has to be sufficiently large in order to avoid deviations from the known results. More precisely, the ratio $m/H$ has to satisfy the following consistency condition $m/H \gg \sqrt{\Delta}$.

On the basis of the above analysis one can conclude that the spectral index

$$n_s := \frac{d \ln \mathcal{P}_\phi(x = 1)}{d \ln k} = 0$$  (104)

has no leading order deviations in $\Delta$. In other words, the power spectrum remains scale-invariant, as expected for the de Sitter phase. The presence of $\Delta$ is manifest only in the amplitude of perturbations. However, the higher order corrections in $\Delta$ will have a non-vanishing contribution, similarly as the corrections linear in $\Delta$ but multiplied by the slow-roll parameters, which is not considered in the lowest order discussion presented here.

### IV. SUMMARY

This paper provides the first attempt to apply the recently introduced Nonlinear Field Space Theory (NFST) to the domain of cosmology. Let us stress that general relativity itself was not modified but we focused our attention on a scalar field describing the matter content of the standard cosmological model. The field was generalized to have the spherical phase space, on which we defined the appropriate symplectic form. In principle, other choices for a bilinear two-form on this phase space are possible as well. Some results for one of such forms are discussed in the Appendix below.

Using the analogy between a scalar NFST with the spherical phase space and a system of spins, we borrowed the Hamiltonian for the matter field from the XXZ Heisenberg model and adapted it to the FRW background. Then our considerations were restricted to a homogeneous cosmological model. As it was shown, the standard dynamics of a massive scalar field is recovered for sufficiently small volumes of universe. On the other hand, it was found that at late times the effects of NFST may become significant. Observational implications of this possibility and a detailed analysis of the discussed model deserve to be the subject of further investigations. The preliminary results suggest that the phase of a cosmic bounce, which replaces the big bang singularity, can also be obtained within our framework.
Subsequently, we studied a generation of primordial quantum inhomogeneities in the scalar field theory corresponding to the XXZ Heisenberg model. The leading order contributions of the anisotropy parameter \( \Delta \) were investigated. However, for the considered linear order of the perturbative analysis, the effects of nonlinearity of the field phase space were not taken into account. Such higher-order effects are unavoidably associated with the non-Gaussian features. Since non-linearity is the inherent feature of the NFST proposal, studies of the non-Gaussianity within this framework may provide a powerful tool to confront the predictions of NFST with the cosmological data.

It was shown that no corrections to the spectral index are expected at the linear order in \( \Delta \). However, it has to be stressed that we adopted certain simplifications in our calculations. In particular, a decomposition of the NFST scalar field into the background and perturbation contributions has to be investigated. Due to the field dependent function in the Poisson bracket, such a decomposition of the kinematics will require a subtle treatment. Only after it is done, the homogenous background can be considered a source of dynamics, on the top of which the inhomogeneous modes are introduced. In our simplified analysis we have not extracted the zero mode from the field dynamics, and assumed that \( \langle 0 \vert \tilde{\varphi}(x, \tau) \vert 0 \rangle = 0 \).

Worth stressing is that the results of this paper open a novel possibility of building relations between cosmology and condensed matter physics, thanks to the dual descriptions can be potentially introduced. In particular, the \( \Delta \rightarrow 0 \) case, which is the relativistic scalar NFST, is dual to the so-called XY model, which normally provides a description of the superconductive state of matter \([15]\). This relationship may turn out to be a source of new ideas for both cosmology and condensed matter physics. It concerns not only models with a scalar field, but also with other types of fields (such as spinor and gauge fields), for which the condensed matter dual descriptions can be potentially introduced.

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**APPENDIX**

Let us here consider the model in which we do not assume that the two-form \([9]\) on the total phase space has to be a closed form. Then to identify the standard Hamiltonian \([13]\) as the small field limit of \([14]\), under the assumption that \( R_1 R_2 \rightarrow S \) for \( q \rightarrow q_0 \), we need to impose the relations:

\[
\begin{align*}
\mu B_x &= \frac{m}{q}, \\
R_1 &= \sqrt{\frac{S}{qm}}, \\
R_2 &= \sqrt{S q m},
\end{align*}
\]

which give us simply \( R_1 R_2 = S = \text{const.} \) In this case \( q_0 \) does not appear in any formulae. Similarly as in \([13]\), taking into account the appropriate (constant) energy shift we obtain the proper Hamiltonian

\[
H_S = N m (S - S_x)
\]

\[
= N q \left( \frac{\pi_\varphi^2}{2 q^2} + \frac{1}{2} m^2 \varphi^2 \right) + O(4).
\]

Consequently, the expression for energy density \([25]\) becomes

\[
\rho = \frac{m}{q} (S - S_x)
\]

and the expansion of the (modified) Friedmann equation in \( 1/S \) can be written as

\[
H^2 = \frac{\kappa}{3} \rho_\varphi - \frac{\kappa}{9} \frac{q}{S m} \left( \rho_\varphi^2 - \frac{1}{2} \varphi^2 \right) + O(1/S^2).
\]

As one can observe, in the \( R_2 \rightarrow \infty \) limit \( \omega_{\varphi, q} \) reduces to the standard symplectic form \( \omega = d\pi_\varphi \wedge d\varphi \). However, \([111]\) is not a closed form, \( d\omega_{\varphi, q} \neq 0 \). This has a major impact on the corresponding algebra of phase space variables, whose bracket is determined by inverse of the total form \( \omega_{\text{FRW}} + \omega_{\varphi, q} \) and given by

\[
\{\cdot, \cdot\}_{\varphi, q} = \left[ \frac{\partial}{\partial q} \cdot \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \cdot \frac{\partial}{\partial q} \right]
+ \frac{1}{\cos \left( \frac{\pi_\varphi}{R_2(q)} \right)} \left[ \frac{\partial}{\partial \varphi} \cdot \frac{\partial}{\partial \pi_\varphi} - \frac{\partial}{\partial \pi_\varphi} \cdot \frac{\partial}{\partial \varphi} \right].
\]

The above algebra does not satisfy the Jacobi identity and therefore is not a Poisson algebra, as well as can not become the associative algebra of operators after quantization. Namely, for arbitrary functions on phase space \( f, g \) and \( h \), the so-called Jacobiator of the bracket \([112]\) is non-zero and has the form
where \([\cdot, \cdot]_\varphi = \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \pi_\varphi} \frac{\partial}{\partial \pi_\varphi}\). This is not a problem by itself, since such non-Poisson physical systems have already been considered from the theoretical perspective and can actually exist in nature, see e.g. \cite{16} and references therein. Moreover, they may lead to appearance of the fundamental length \cite{17} (see \cite{18} for a relation with string theory). These systems require a special treatment, especially regarding quantization, where one has to use the \(*\)-products or other refined constructions. However, in our investigations of the quantum regime in Sec. III we restricted to the limit \(S \rightarrow \infty\), where the form \cite{11} becomes a (closed) symplectic form. The remaining non-standard features of the theory are then a consequence of non-zero parameter \(\Delta\) in the scalar field Hamiltonian and therefore the results of Sec. III are also valid in the current case.

\[
\{f, g, h\} := \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{f, h\}\}
\]

\[
= \frac{\pi_\varphi}{2q\sqrt{Smq} \cos^2 \frac{\varphi}{\sqrt{Smq}}} \left( \frac{\partial f}{\partial p} [g, h]_\varphi + \frac{\partial h}{\partial p} [f, g]_\varphi + \frac{\partial g}{\partial p} [h, f]_\varphi \right)
\]

\[
= \left( \frac{\pi_\varphi^2}{2Smq^2} + \mathcal{O}(1/S^2) \right) \left( \frac{\partial f}{\partial p} [g, h]_\varphi + \frac{\partial h}{\partial p} [f, g]_\varphi + \frac{\partial g}{\partial p} [h, f]_\varphi \right),
\]

(113)

We note that a violation of the Jacobi identity in \cite{11} is associated only with the functions that depend on \(p\). Furthermore, the brackets between any pair of phase space variables \(S_x, S_y, S_z, q, p\) do not change except \(\{S_x, p\} = \frac{1}{2q} (-S_y \arctan \frac{Sy}{S_x} + \frac{S_x S_z}{\sqrt{S^2 - S_z^2}} \arcsin \frac{S_z}{S})\),

(114)

\(\{S_y, p\} = \frac{1}{2q} \left( S_x \arctan \frac{Sy}{S_x} + \frac{S_y S_z}{\sqrt{S^2 - S_z^2}} \arcsin \frac{S_z}{S} \right)\),

(115)

\(\{S_z, p\} = -\frac{1}{2q} \sqrt{S^2 - S_z^2} \arcsin \frac{S_z}{S}\).

Consequently, now the evolution equations for the \(S_x, S_y, S_z\) variables become

\[
\dot{S}_x = \{S_x, H_{\text{tot}}\} = -\frac{3}{4} N \kappa p \left(-S_y \arctan \frac{Sy}{S_x} + \frac{S_x S_z}{\sqrt{S^2 - S_z^2}} \arcsin \frac{S_z}{S}\right),
\]

(117)

\[
\dot{S}_y = \{S_y, H_{\text{tot}}\} = N m S_z - \frac{3}{4} N \kappa p \left(S_x \arctan \frac{Sy}{S_x} + \frac{S_y S_z}{\sqrt{S^2 - S_z^2}} \arcsin \frac{S_z}{S}\right),
\]

(118)

\[
\dot{S}_z = \{S_z, H_{\text{tot}}\} = -N m S_y + \frac{3}{4} N \kappa p \sqrt{S^2 - S_z^2} \arcsin \frac{S_z}{S},
\]

(119)

and the equation for \(p\) is

\[
\dot{p} = \frac{3}{4} N \kappa p^2 + N m \frac{1}{2q} \left(-S_y \arctan \frac{Sy}{S_x} + \frac{S_x S_z}{\sqrt{S^2 - S_z^2}} \arcsin \frac{S_z}{S}\right),
\]

(120)

up to the first order in \(1/S\), obtaining

\[
\dot{\varphi} = \frac{\pi_\varphi}{q} + \frac{\pi_\varphi}{Sm} \left( \frac{\pi_\varphi^2}{3q^2} - \frac{m^2}{2} \varphi^2 \right) + \mathcal{O}(1/S^2),
\]

(121)

\[
\ddot{\varphi} = -q m^2 \varphi + \frac{q^2 m^3}{6S} \varphi^3 + \mathcal{O}(1/S^2),
\]

(122)

On the other hand, the equations \cite{39, 40} in the current case have the identical form as before but with the implicit expressions for \(R_1\) and \(R_2\) given by \cite{100, 107}. The latter feature manifests itself if we expand the equations which differ with respect to \cite{10, 11} by the absent factor \(q/q_0\) in the first order terms. From the above equations we also derive the corresponding modified Klein-Gordon
equation
\[ \ddot{\varphi} + 3H \dot{\varphi} + m^2 \varphi = -\frac{q m}{S} \left[ \frac{H}{m^2} \varphi^3 + \frac{3H}{2} \dot{\varphi}^2 + 2 \dot{\varphi} \varphi^2 - \frac{2m^2}{3} \varphi^3 \right] + \mathcal{O}(1/S^2), \]
which has one additional term in comparison with \((123)\).

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