Size-dependent vibration of multi-span micropipes conveying fluid based on modified couple stress theory

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Abstract. In this article, the size-dependent free vibration and stability of multi-span micropipes conveying fluid are studied. To take the size effect into account, the modified couple stress theory is used to derive the governing equation. And the dynamic stiffness method (DSM) is adopted to determine the natural frequencies. Subsequently, some examples are presented to investigate the effects of fluid velocity and material length scale parameter on stability of the multi-span micropipes conveying fluid. It’s found that the natural frequencies and the critical velocities obtained by the modified couple stress theory are higher than those predicted by the classical theory.

1. Introduction
Being the simplest form of fluid structure interaction problem, dynamics of pipes conveying fluid has attracted enormous attentions from researchers due to the rich and interesting dynamic behavior. Numerous works have been carried out on the vibration and stability of pipes conveying fluid [1, 2]. Although various aspects of this problem have been studied in the past half-century, most of the publications are limited to the model of single-span pipe conveying fluid. But in real working environment, the piping systems are always arbitrary multi-span pipes, and the works concerning the arbitrary multi-span pipe conveying fluid are quite limited [3,4]. As pointed out by Wu and Shih [3], the main reason is that the classical mode method is easily applicable only to the special cases where the functions for the mode shapes are obtainable, such as single-span or the periodical multi-span pipes. Finite element method (FEM) may be the most versatile method for the dynamics of multi-span pipes conveying fluid, but it is laborious when the number of element is very large [5].

With the quick growth in science and engineering, fluid-conveying pipes have been made in microscale and applied in micro-electromechanical systems (MEMS), fluid storage, fluid transport and drug deliver [6,7]. Therefore, understanding the dynamic behavior of micropipes conveying fluid is currently a subject of great interest. However, the dynamic behavior of microstructures has been proved to be size dependent and the classical continuum theories are unable to capture the size effect of microstructures. To overcome this problem, a size-dependent nonclassical continuum theory, modified couple stress theory [8], has been proposed to capture the size effect. And the modified couple stress theory has been widely utilized by many researchers to study the dynamic behavior of microstructures [9, 10].

In this work, the size-dependent free vibration and stability of a multi-span micropipe conveying fluid are investigated. The modified couple stress theory and the Hamilton’s principle are used to drive
the governing equation. The natural frequencies and the critical velocities are determined by using the dynamic stiffness method (DSM). The effects of fluid velocity and material length scale parameter on dynamic behavior of multi-span micropipes conveying fluid are discussed.

2. Modified couple stress theory and governing equation

![Figure 1. Geometric model of a micropipe conveying fluid.](image)

A micropipe with mean radius $R$, length $L$ and fluid velocity $U$ is depicted in Figure 1. Displacement components in $x$-, $y$-, and $z$-directions denote as $u$, $v$ and $w$, respectively. According to the modified couple stress theory, the strain energy density is a function of both strain tensor (conjugated with stress tensor) and curvature tensor (conjugated with couple stress tensor) [8]. For a linear elastic material occupying region $\Omega$, the strain energy $U_m$ can be expressed as:

$$
U_m = \frac{1}{2} \int_{\Omega} \left( \sigma_{ij} \epsilon_{ij} + m_{ij} \chi_{ij} \right) dv \quad (i, j = 1, 2, 3)
$$

In which, $\sigma_{ij}$ is the stress tensor, $\epsilon_{ij}$ is the strain tensor. The deviatoric part of couple stress tensor $m_{ij}$, and the symmetric curvature tensor $\chi_{ij}$ are given by [10]

$$
m_{ij} = 2l^2 G \chi_{ij}, \quad \chi_{ij} = \left( \theta_{i,j} + \theta_{j,i} \right) / 2
$$

Where $\delta_{ij}$ is the Kronecker’s delta function, $l$ is material length scale parameter and $G$ is shear modulus. $\theta_i$ is the rotation vector and it can be written as:

$$
\theta_i = \text{curl}(u_i) / 2
$$

Where $u_i$ is the displacement vector. According to Euler-Bernoulli beam theory, the displacement components are given by

$$
u = 0, \quad w = w(x, t)
$$

Where $\psi$ is the rotation of pipe cross-section In view of the small deformation assumption, the following approximation can be applied

$$\psi(x) \approx w(x)'$$

where the prime represents differentiation with respect to $x$.

Based on Euler-Bernoulli beam theory, the strains can be obtained

$$
\epsilon_{xx} = -z w', \quad \epsilon_{yy} = \epsilon_{zz} = \epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0
$$
Accordingly, the stresses can be written as
\[
\sigma_{xx} = -E\varepsilon_{xx}, \quad \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0
\] (7)

It should be mentioned that the Poisson effect is neglected to facilitate the formulation of a simple beam theory in above equations [10].

Combination Eqs. (3)-(5), yields
\[
\theta_y = -w', \quad \theta_z = \theta = 0
\] (8)

Substituting Eq. (8) to Eq. (2), yields
\[
\chi_{yy} = -w'/2, \quad \chi_{zz} = \chi_{zy} = \chi_{yx} = \chi_{yz} = 0
\] (9)

Substitution of Eq. (9) to Eq. (2) gives
\[
 m_{yy} = -Gl^2w', \quad m_{zz} = m_{zy} = m_{yx} = m_{yz} = m_{zx} = 0
\] (10)

Substituting Eqs. (6)-(7) and (9)-(10) to Eq. (1), one obtains
\[
U_m = \frac{1}{2} \int_L \left( EI + GAf^2 \right) (w^*)^2 \, dx
\] (11)

Where, \( I \) is the moment of inertia.
The kinetic energies of pipe and fluid can be expressed as [1]
\[
T_p = \frac{m}{2} \int_L (\dot{w})^2 \, dx, \quad T_f = \frac{M}{2} \int_L \left[ (\dot{w} + \dot{U}w')^2 + U^2 \right] \, dx
\] (12)

where the dot represents differentiation with respect to \( t \).

According to Paidoussis [1], Hamilton’s principle for both ends supported pipe conveying fluid can be written as
\[
\int_0^L \delta \left( T_p + T_f - U_m \right) \, dt
\] (13)

Substituting Eqs. (11) and (12) into Eq. (13), and applying the variational techniques to Eq. (13), the governing equation of the micropipes conveying fluid can be obtained
\[
\left( EI + GAf^2 \right) w^{**} + 2M_f U\dot{w} + \left( M_f + m \right) \ddot{w} = 0
\] (14)

Compared with governing equation of classical pipes conveying fluid [1], the flexural rigidity is replaced by the equivalent flexural rigidity \( EI + GAf^2 \).

3. Application of DSM to multi-span pipes conveying fluid

In this section, the DSM is utilized to determine the natural frequencies of multi-span pipes conveying fluid. The general solution for Eq. (14) can be written as
\[
w(x, t) = \hat{w}(x, \omega) e^{i\omega t}
\] (15)
Where $\omega$ is the circular frequency and $i = \sqrt{-1}$.
Substituting Eq. (15) into Eq. (14), yields

$$\left( EI + GAf^2 \right) \ddot{w} + M_f U^2 \dot{w} + i2\alpha M_f U\dot{w} - \alpha^2 \left( M_f + m \right) \ddot{w} = 0 \quad (16)$$

Then, the displacement in frequency domain can be set as

$$\hat{w} = ce^{j\pi x} \quad (17)$$

By substituting Eq. (16) to Eq. (17), one obtains

$$\left( EI + GAf^2 \right) k^4 - M_f U^2 k^2 - 2\alpha M_f U k - \alpha^2 \left( M_f + m \right) = 0 \quad (18)$$

The transverse displacement in frequency domain can be rewritten as

$$\hat{w} = \sum_{j=1}^{d} w_j e^{i\theta_j} \quad (19)$$

Accordingly, the rotation, moment and shear force in frequency domain can be obtained

$$\hat{\theta} = \sum_{j=1}^{d} ik_j w_j e^{i\theta_j}, \hat{\mathcal{M}} = \sum_{j=1}^{d} (EI + GAf^2) k^4 w_j e^{i\theta_j}, \hat{\mathcal{Q}} = \sum_{j=1}^{d} (EI + GAf^2) ik_j w_j e^{i\theta_j} \quad (20)$$

Figure 2. Displacements of the $m$th span element.

The $m$th span element of a multi-span micropipe is shown in Figure 2. The nodal displacements and forces in local coordinate have the following relationships

$$\hat{w}_{ml} = \hat{w}(0), \quad \hat{\theta}_{ml} = \hat{\theta}(0), \quad \hat{w}_{mr} = \hat{w}(l_m), \quad \hat{\theta}_{mr} = \hat{\theta}(l_m) \quad (21)$$

$$\hat{\mathcal{M}}_{ml} = -\hat{\mathcal{M}}(0), \quad \hat{\mathcal{Q}}_{ml} = -\hat{\mathcal{Q}}(0), \quad \hat{\mathcal{M}}_{mr} = \hat{\mathcal{M}}(l_m), \quad \hat{\mathcal{Q}}_{mr} = \hat{\mathcal{Q}}(l_m) \quad (22)$$

where, the subscripts $l$ and $r$ represent the left and right ends of the element, respectively. And the subscript $m$ indicates the $m$th span element. According to Eqs. (19) and (20), the nodal displacements and nodal forces can be expressed as

$$\mathbf{D}_m = \mathbf{Y}_m(\omega) \mathbf{w}_m, \quad \mathbf{F}_m = \mathbf{X}_m(\omega) \mathbf{w}_m \quad (23)$$

The detailed expressions can be written as
\[
\begin{bmatrix}
\dot{w}_m \\
\dot{u}_m \\
\dot{v}_m \\
\dot{\theta}_m
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\phi^{k_1} & \phi^{k_2} & \phi^{k_3} & \phi^{k_4} \\
\phi^{\beta_1} & \phi^{\beta_2} & \phi^{\beta_3} & \phi^{\beta_4}
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{bmatrix},
\begin{bmatrix}
\dot{Q}_m \\
\dot{M}_m
\end{bmatrix}
= \begin{bmatrix}
-\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 \\
-\beta_1 & -\beta_2 & -\beta_3 & -\beta_4
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
\] (24)

Where, \( \lambda_j = ik_j \), \( \beta_j = (EI + GAL^2)k_j^2 \) (j=1, 2, 3, 4). Based on Eq. (23), we can get

\[
F_n = K_n(\omega)D_n
\] (25)

Where \( K_n = X_n Y_n^{-1} \) is the dynamic stiffness matrix of the \( m \)th span element. The global dynamic stiffness matrix can be assembled by the local dynamic stiffness matrices. And the relationship between the nodal forces and nodal displacements for multi-span micropipes can be written as

\[
F_g = K_g(\omega)D_g
\] (26)

where the subscript \( g \) represents the global coordinate system. After applying the boundary conditions to Eq. (26), the dynamic stiffness matrix \( K_g \) can be used to calculate the natural frequencies. And it has been demonstrated that the DSM is valid and efficient for dynamic analysis of multi-span pipelines conveying fluid [4].

4. Numerical results and discussions

For analysis convenience, two non-dimensional quantities are introduced [1]

\[
u = \left( M_f/El \right)^{\frac{1}{2}} LU, \quad \sigma = \left[ \left( M_f + m \right)/El \right]^{\frac{1}{2}} L^2 \omega
\] (27)

Where \( u \) is dimensionless fluid velocity and \( \sigma \) is dimensionless natural frequency.

**Figure 3.** A three-span micropipe conveying fluid.

A typical three-span micropipe conveying fluid is plotted in Figure 3. The material properties of the pipe are \( E=70 \text{ GPA} \), \( v=0.23 \), \( \rho_p=2700 \text{ kg/m}^3 \), \( \rho_f=1000 \text{ kg/m}^3 \) and the length scale parameter \( l \) is assumed to be 15 \( \mu \text{m} \) [10]. The other geometrical parameters are \( L_3=2L_1=2L_2 \), \( Do/Di=0.9 \), and \( L/Do=40 \). \( Di \) and \( Do \) represent the inner and outer diameters. \( L \) is the length of the whole piping system.
To investigate the effects of material length scale parameter and fluid velocity on stability of micropipes conveying fluid, the fundamental natural frequency as a function of fluid velocity \( u \) with different \( D_0/l \) is depicted in Figure 4. The critical velocities for different \( D_0/l \) are listed in Table 1. It should be mentioned that the real component of natural frequency, \( \text{re}(\omega) \), is the oscillation frequency, while the imaginary component of natural frequency, \( \text{Im}(\omega) \), is related to exponential growth or decay. It can be seen that the real components decrease with increasing fluid velocity \( u \). When the real part of fundamental natural frequency falls to zero, the conservative piping system loses its stability due to the divergence. And the corresponding velocity is called the critical velocity \( u_c \). Obviously, it can be seen from Figure 4 that the natural frequencies predicted by the modified couple stress theory are generally higher than those determined by the classical theory. It’s also found that by increasing \( D_0/l \), the results predicted by the modified couple stress theory get closer to those obtained by the classical theory. For example, the dimensionless critical velocity for \( D_0/l = 1 \) is 22.54. When the parameter \( D_0/l = 10 \), the dimensionless critical velocity decreases to 10.71 which has very closed to 10.57 predicted by the classical theory. Therefore, the results in Figure 4 and Table 1 demonstrate that the effects of material length scale parameter on the natural frequencies and critical velocities of micropipes conveying fluid are significant, especially when the diameter of micropipe is comparable to the material length scale parameter. However, for a higher value of \((D_0/l > 10)\), natural frequencies and critical velocities obtained by the modified couple stress theory converge to the classical ones.

## 5. Conclusion

In this paper, based on the modified couple stress theory, the governing equation of micropipes conveying fluid is derived by the Hamilton’s principle. The natural frequencies are determined by the DSM. It is found that the effects of material length scale parameter on natural frequencies and critical velocities are significant. And it makes the micropipe conveying fluid more stable, especially when the diameter of the micropipe is comparable to the material length scale parameter.
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