On The Beta–Function in N=2 Supersymmetric Yang-Mills Theory

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Abstract

The constraints of $N = 2$ supersymmetry, in combination with several other quite general assumptions, have recently been used to show that $N = 2$ supersymmetric Yang-Mills theory has a low energy quantum parameter space symmetry characterised by the discrete group $\Gamma_U(2)$. We show that if one also assumes the commutativity of renormalization group flow with the action of this group on the complexified coupling constant $\tau$, then this is sufficient to determine the non-perturbative $\beta$-function, given knowledge of its weak coupling behaviour. The result coincides with the outcome of direct calculations from the Seiberg-Witten solution.

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I. INTRODUCTION

In recent years there has been remarkable progress in understanding the non-perturbative dynamics of supersymmetric gauge theories. The powerful holomorphy constraints imposed by supersymmetry (SUSY) have allowed results, calculable at weak coupling, to be analytically continued to the strong coupling regime along quantum-mechanically flat directions in the space of inequivalent vacua (see e.g. [1]). In particular, the constraints of $N = 2$ SUSY have allowed an exact solution for the holomorphic two-derivative contribution to the Wilsonian effective action to be determined by Seiberg and Witten [2] for $N = 2$ supersymmetric Yang-Mills (SYM) theory and SQCD. More recently, it has been argued [3,4] that a number of the conjectures made in [2], in particular those of physical importance such as electric-magnetic duality, are strictly unnecessary in order to obtain a unique solution. Indeed the existence of an underlying discrete parameter-space symmetry group of the full quantum theory, by which we mean transformations the couplings of the theory which leave the vacuum state and full mass spectrum invariant, has been determined uniquely from several rather general requirements including unbroken $N = 2$ supersymmetry [4].

Quite generally, the existence of such parameter space symmetries is of great utility, as they act on the same space as renormalization group (RG) transformations of the theory, and thus place restrictions on the structure of RG flow. In particular, when discrete symmetries associated with a subgroup of the modular group $SL(2,\mathbb{Z})$ hold at all scales in a quantum theory, then the $\beta$–function must satisfy certain modular transformation properties. When the symmetry group is large enough, for example $SL(2,\mathbb{Z})$ itself, the constraint on the $\beta$–function is generally strong enough to force it to vanish, rendering the theory scale invariant. However, in cases where the symmetry group is smaller, the RG flow may still be nontrivial albeit highly constrained. In such cases, study of the required modular transformation properties of $\beta$ often provides nonperturbative information about the RG flow. Such arguments have been used to highly constrain the structure of the RG $\beta$–functions in statistical systems [5], and also the non-linear sigma model [6].

In this letter, we re-analyse renormalization group flow in $N = 2$ SYM from this perspective of parameter space symmetries. A particular nonperturbative definition of the $\beta$–function, associated with the flow of the couplings along the moduli space, arises naturally in this context. This $\beta$–function has been obtained previously from the Seiberg–Witten solution by Minahan and Nemeschansky [7], and also Bonelli and Matone [8]. Here we show that this result follows from the following conditions, along with knowledge of its weak-coupling behaviour. These conditions, although not in their most basic form, are given by:

1. The maximal parameter-space symmetry group of the low energy effective theory, when acting on the complexified coupling $\tau$, is given by $\Gamma_U(2)$\(^1\), the index–3 subgroup of $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$.

\(^1\)Throughout we shall use the notation of Rankin [9] for subgroups of $SL(2,\mathbb{Z})$ and their generators. Note that $\Gamma_U(2)$ is also commonly denoted $\Gamma_0(2)$.
2. This equivalence group commutes, in an appropriate sense to be defined below, with the flow of $\tau$ induced by the renormalization group.

In order to justify condition 1, we also require the conditions shown in [4] to be sufficient to ensure the uniqueness of this symmetry group. We shall discuss these additional assumptions shortly, although we note that no assumption about the existence of a dual magnetic description at strong coupling will be required.

The second condition above is crucial, and is the statement of compatibility between the action of the symmetry group and the renormalization group flow. Once satisfied, it implies that if the parameter space symmetry holds at one scale, then it will be preserved by the RG flow, and continue to hold at lower scales.

In Section 2, after briefly reviewing some general features of the effective theory, we recall the conditions imposed in [4] which were used to ensure condition 1. Although it is not clear at this stage if all these assumptions are strictly necessary, certain holomorphy conditions on the coupling, following from unbroken $N = 2$ SUSY, and the allowed singularity structure structure over the moduli space of vacua, will be important in the following analysis. In Section 3 we discuss the precise characterization of the renormalization group flow to be studied, and give a definition of $\beta(\tau)$. We then illustrate how condition 2 allows determination of $\beta(\tau)$ where all unknown parameters may be fixed purely via knowledge of the behaviour in the semi-classical limit.

II. THE MAXIMAL EQUIVALENCE GROUP

Unbroken supersymmetry ensures that the potential for the scalar component $\phi$ of the $N = 2$ multiplet vanishes, even when quantum corrections are included, when $\phi$ takes values in the Cartan subalgebra of the gauge group, which we shall take to be $SU(2)$. Classically, for non-zero $a \equiv \langle \phi \rangle$ the Higgs mechanism spontaneously breaks the gauge symmetry to $U(1)$, with the arbitrariness in $\langle \phi \rangle$, or more precisely a gauge invariant parameter such as $\langle Tr \phi^2 \rangle$, leading to a moduli space of vacua $M$. It is a consequence of $N = 2$ supersymmetry that the structure of the low energy Wilsonian effective action for the light $U(1)$ multiplet may be represented in terms of an $N = 2$ superfield $\mathcal{A} = \phi + \theta \psi + \cdots$ as [10]

$$\Gamma_W[\mathcal{A}] = \frac{1}{4\pi} \text{Im} \int d^4xd^2\theta_1d^2\theta_2 \mathcal{F}(\mathcal{A}) + \cdots,$$

up to non-holomorphic higher derivative terms, where $\mathcal{F}$ is the holomorphic prepotential. In $N = 1$ superspace this action takes the form

$$\Gamma_W[A, W_\alpha] = \frac{1}{4\pi} \text{Im} \int d^4x \left[ \int d^4\theta K(A, \overline{A}) + \int d^2\theta \frac{1}{2} \tau(A) W^\alpha W_\alpha \right] + \cdots,$$  

where $A$ and $W_\alpha$ are $N = 1$ chiral and vector superfields, $K \equiv \mathcal{F}'(A)\overline{A}$ is the Kähler potential, and

$$\tau(A) \equiv \frac{\partial^2 \mathcal{F}}{\partial A^2}.$$

Since \( \tau(A) \) is the coefficient of the kinetic term, its imaginary part must be positive. Its real part also plays a role similar to the theta parameter in the microscopic theory, and thus it is natural to define the corresponding effective parameters in the manner

\[
\tau(u) \equiv \frac{2\pi}{\theta_{\text{eff}}(u)} + \frac{4\pi i}{g_{\text{eff}}^2(u)},
\]

where \( u \) is a gauge invariant parameter labelling the moduli space. Careful analysis in [11] has shown that one may match these effective parameters to the underlying microscopic \( SU(2) \) parameters at small scales via an appropriate identification of the \( \Lambda \) scale of the effective theory with the renormalization-scheme-dependent dynamically generated microscopic scale.

Although we shall need the relationship only in the weak–coupling region, it has been shown quite generally [12] that for the pure Yang-Mills case \( u \) may be identified as \( u = \langle \text{Tr} \phi^2 \rangle \). Thus \( \langle \text{Tr} \phi^2 \rangle \) is a good global coordinate on the moduli space, as was conjectured in [2]. The point is that \( a \), when defined as the expectation value of the scalar component of the \( N=2 \) superfield \( \mathcal{A} \), only corresponds to a useful parametrization in the semi-classical domain.

An important insight is that the moduli space coordinatized by \( u \), may also be parametrized in terms of \( \tau \) which plays the role of a convenient uniformizing parameter. This requires knowledge of the multivaluedness of the relation \( \tau = \tau(u) \). While, for unitarity \( \tau \in \mathbb{H} \), the upper-half complex plane, it is clear from the presence of an effective theta parameter in (4) that there exist equivalence relations between various values of \( \tau \). In other words the mapping between \( \tau \) and \( u \) is not one-to-one. In particular, one may identify \( \tau \mapsto U(\tau) = \tau + 1 \), corresponding to a rotation of the effective \( \theta \)-angle by \( 2\pi \).

Following [4], we define the maximal equivalence group \( G \) as the group of all transformations of \( \tau \) which leave the vacuum, \( u(\tau) \in \mathcal{M} \), and the full mass spectrum, invariant. Since \( g_{\text{eff}}^2 \) must be positive, this ensures that \( G \subset SL(2,\mathbb{R}) \). In order to constrain \( G \) further, we note that \( N=2 \) SUSY implies that the mass of charged particles is BPS saturated [14], i.e. \( M = \sqrt{2}|Z| \), where \( Z \) is the SUSY central charge. It has been argued in [3,15,16] that quantum mechanically this relation has the form

\[
M = \sqrt{2}|q_e a + q_m a_D|,
\]

where \( a_D \equiv \mathcal{F}'(a) \), and \( q_e \) and \( q_m \) are integer valued charges. The invariance of this spectrum, and the fact that \( \tau \) satisfies \( \tau = da_D/da \) as follows from (3), then ensures that \( G \) is a subgroup of \( PSL(2,\mathbb{Z}) \), and thus the physical moduli space may be represented as the fundamental \( \tau \)-domain \( D = \mathbb{H}/G \).

Having presented evidence above for the nontriviality of \( G \subset PSL(2,\mathbb{Z}) \), we now recall the analysis of [4] which argued that the following assumptions constitute a set of sufficient conditions to determine \( G \) uniquely:

1. \( \tau \) takes all values in \( \mathbb{H} \).

\[2\] The situation is more complex for certain models with higher matter field content [13].

\[3\] Note that a more general definition is necessary when one adds matter hypermultiplets [17,16].
2. There are a finite number of singular points in $\mathcal{M}$.

3. The BPS mass $M$ is single-valued on $\mathcal{M}$.

4. The mass of the lightest charged field is finite except in the perturbative region.

With these assumptions, it was shown that $G$ is given by $\Gamma_U(2) \subset \Gamma(1) = SL(2, \mathbb{Z})$, and thus these requirements also serve as sufficient conditions to ensure the validity of our initial assumption that the symmetry group of the quantum theory is $\Gamma_U(2)$. Conditions 2 and 4 deserve further comment. In particular, condition 2 ensures that functions, or more generally sections, defined over the moduli space, will be meromorphic. Since singular points in the moduli space will be mapped to the vertices of the fundamental domain of $G$, this ensures that there are no singularities in the interior of this fundamental domain. Meanwhile, condition 4 is the expected behaviour for a theory with only one asymptotically free regime, as we observe in the present case.

III. DETERMINATION OF THE BETA–FUNCTION

The above definition of the equivalence group, as acting at a fixed point of the moduli space, is important in determining the precise characterisation of RG flow we shall use. In particular, in order to consistently apply the second assumption of Section 1, we need to consider the infinitesimal flow at a fixed point of the moduli space. Since the effective theory has been constructed via recourse only to the general constraints imposed by supersymmetry, there is no explicit cutoff scale, and the only dimensionful parameters of the theory are $u$ (or $a$) and $\Lambda$. In order to allow compatibility with the action of the equivalence group it is then appropriate, following [7,18,8], to define the $\beta$–function for the evolution of $\tau$ as

$$\beta(\tau) \equiv \Lambda \frac{\partial \tau}{\partial \Lambda} \bigg|_u = -2 \frac{u}{u'|\Lambda},$$

(6)

where $u'$ denotes $\partial u/\partial \tau$. The latter relation, noted in [8], follows from the fact that, since $\tau$ is dimensionless, $\beta = \beta(u/\Lambda^2)$ and the above definition is physically equivalent to considering the flow induced by motion over the moduli space, i.e. changing $u$ with $\Lambda$ fixed. The latter description is perhaps more physically relevant, and closer in spirit to a more standard weak-coupling definition such as [4], $\beta^a \equiv a \partial_a \tau|_\Lambda$, where the renormalization scale is chosen equal to the vev. The choice of $u$ as the scale in the nonperturbative definition (6) is motivated by the fact that $u$ is a more appropriate coordinate for the moduli space at strong coupling. However, while these definitions may differ in the strong coupling region one expects that they should be equivalent, at least up to an overall constant, in the perturbative region $a \gg \Lambda$. This may be verified by considering the appropriate differentials of $a$, $u$, and $\tau$ [8], from which one obtains the following relation,

$$\beta = \beta^a \left( \Lambda \frac{\partial}{\partial \Lambda} \ln a \bigg|_u - 1 \right) \rightarrow -\beta^a + \cdots.$$

(7)

The final limit is taken in the perturbative region, where the relation simplifies as expected since in this case $a \sim \sqrt{2u} + \cdots$ which is independent of $\Lambda$. Thus perturbatively the
definitions agree up to a sign, while non-perturbatively there is a discrepancy due to the
fact that \( a \) is not a good global coordinate on the moduli space.

At weak coupling, we may match this \( \beta \)–function for the effective coupling to the running
of the microscopic coupling. The 1–loop contribution is then given by \( \beta_{\text{pert}}(\tau) = -2i/\pi \) [19]. Note that there are no higher loop contributions [20,21,10], but instantons [21] lead to
additional nonperturbative effects as first discussed for \( N=2 \) theories by Seiberg in [10]. It
is these contributions to \( \beta(\tau) \) that we shall determine below.

It is important to note that \( \beta(\tau) \) inherits certain analyticity properties from those of \( \tau(u) \).
The holomorphy of \( \tau \) follows from its definition as the derivative of \( \mathcal{F} \) and the existence of
unbroken supersymmetry. Furthermore, as was noted in the previous section, since possible
singular points in the moduli space are mapped to the vertices of the fundamental domain
\( D \), \( \beta(\tau) \) must be regular in the interior of this domain.

We now concentrate on the global definition of \( \beta \) given in Eq. (6) and make use of the
second assumption of Section 1 in order to obtain an explicit expression. Recall that the
existence of a nontrivial equivalence group \( \Gamma_U(2) \) implies that the physical moduli space \( \mathcal{M} \)
reduces to the corresponding fundamental domain \( D = \mathbb{H}/\Gamma_U(2) \). The action of a general
element of this equivalence group, \( \gamma \cdot \tau = (a\tau + b)/(c\tau + d) \), where \( ad - bc = 1 \), \( a, b, c, d \in \mathbb{Z} \),
may be conveniently represented in terms of the generators \( U \) and \( VU^2 \) [9],

\[
U : \tau \rightarrow \tau + 1 \quad VU^2 V : \tau \rightarrow \frac{\tau}{1 - 2\tau}.
\] (8)
The assumed validity of this symmetry at all scales leads to the condition that the RG flow
commute with this action of the equivalence group. From the restrictions on its allowed
analytic structure, this implies that \( \beta(\tau) \) for \( \tau \in \mathbb{H} \), satisfies [3]

\[
\beta(\gamma \cdot \tau) = \frac{d(\gamma \cdot \tau)}{d\tau} \beta(\tau) = (c\tau + d)^{-2} \beta(\tau),
\] (9)
where \( \gamma \in \Gamma_U(2) \). That is to say, \( \beta(\tau) \) transforms as a modular form of \( \Gamma_U(2) \) of weight \(-2\). Note that the fact that \( \beta \) is not invariant under such transformations is due to its definition
as a contravariant vector field on the space of couplings.

Positive weight modular forms may be obtained constructively in terms of generalised
Eisenstein series. However, for negative weight forms no such construction exists. Neverthe-
less, we may use a general theorem, valid for all even weight forms [4], which states that
any weight \(-2\) modular form of a discrete group \( \Gamma \subset \Gamma(1) \), may be represented in terms of
a univalent automorphic function \( f \) of \( \Gamma \), via

\[
\beta(\tau) = \frac{1}{f'} \frac{P(f)}{Q(f)},
\] (10)
where \( f' \) denotes \( \partial f / \partial \tau \), and \( P \) and \( Q \) are polynomials in \( f \). In the present case \( \Gamma = \Gamma_U(2) \),
and a convenient automorphic function is given by \( f = F(\tau) = -\theta_3^4 \theta_4^4 / \theta_2^8 \), where \( \theta_i = \theta_i(0|\tau) \),
for \( i = 2, 3, 4 \), are the theta constants.

\[\text{Note that there is a relative minus sign compared to [19,10] due to the definition of the } \beta \text{–function in (8).}\]
We could now proceed to place constraints on the polynomials $P$ and $Q$ from the known weak-coupling asymptotics. However, an alternative approach is to recall that the definition of the $\beta$–function (1), when expressed in terms of $\tilde{u} = u/\Lambda^2$, has precisely the form (10), with $\tilde{u}$ playing the role of $f$. Furthermore, since $\tilde{u}$ is clearly automorphic from the definition of the equivalence group, and univalent due to its parametrisation of the moduli space, we may equally well identify $f = \tilde{u}(\tau)$.

Obtaining an explicit expression for (10) then reduces to determining the relation $\tilde{u} = \tilde{u}(\tau)$ for the automorphic function $\tilde{u}$. Using only the required transformation properties under $\Gamma_U(2)$, and the known weak coupling perturbative asymptotics for $\tau(u \sim a^2/2)$ arising from the 1–loop $\beta$–function, this relationship was obtained by Nahm [17]. These are the assumptions of the present paper, and thus we may use this result which, in corrected form and with a convenient normalization of the scale $\Lambda$, reads [17]:

\[
\left( \frac{u}{\Lambda^2} \right)^2 = 1 - 4F. \tag{11}
\]

This simple relationship between $F$ and $\tilde{u}^2$ is essentially demanded by their required transformation properties under $\Gamma_U(2)$. To gain a little more insight into this expression, we may also recover the functional relation, $F(\tilde{u})$, as follows. Since $\tilde{u}$ and $F$ are both univalent and automorphic under $\Gamma_U(2)$, they may be functionally related by a polynomial of degree determined by their singularity structure [9]. Importantly, since the singularity of $\tilde{u}$ is at the weak–coupling vertex, $\tau = i\infty$, of $D$, we may extract the order of the pole from weak coupling asymptotics. Introducing an elliptic modulus $k^2(\tau) = \theta_4^2/\theta_4^3$ which has a zero at $\tau = i\infty$, and using the weak coupling relation for $\tau(u \sim a^2/2)$, obtained by integrating the 1–loop $\beta$–function, we find that $u \rightarrow 2k^{-2}$ at the weak coupling vertex. Similarly, using the explicit representation for $F$ in terms of complete elliptic integrals [22], we find $F \rightarrow -k^{-4}$. Thus $F$ has a pole of order 2 at the pole of $\tilde{u}$, and consequently we may write [11]

\[
F = c_1 \left( \frac{u}{\Lambda^2} \right)^2 + c_2 \left( \frac{u}{\Lambda^2} \right) + c_3. \tag{12}
\]

The univalence of $F$ and $\tilde{u}$ implies that $c_2 = 0$, while the perturbative asymptotics, arising from the 1-loop $\beta$–function, $\beta_{\text{pert}}(\tau) = -2i/\pi$ [15], implies $c_1 = -1/4$. From the earlier discussion, we now expect that instanton contributions to $\beta$ are determined by the value of $c_3$.

Since $F$ and $\tilde{u}$ are univalent, we may fix $c_3$ by a choice of the zero. In the analysis of [17] discussed above, it was pointed out that the zero of $\tilde{u}(\tau)$ should lie at the orbifold vertex of $\Gamma_U(2)$, $\tau = (i - 1)/2$. This fixes $c_3 = 1/4$ and (12) then reduces to (11).

More generally, the zero of $\tilde{u}(\tau)$ may not be fixed by the group structure and thus, from a calculational point of view, it is helpful to consider an expansion of this result near the weak coupling vertex, without first fixing $c_3$,

\[
\beta = \frac{4c_3 - 4F}{F'} \sim -\frac{2i}{\pi} \left( 1 + \frac{1}{32} (32c_3 - 3) k^4 + \cdots \right). \tag{13}
\]

The $O(k^4)$ correction may be associated with a 1-instanton contribution [19]. This may be seen by noting that $k^2 \sim 2\exp(i\pi\tau) + \cdots$ in this limit and therefore, setting the $\theta$-angle to zero for clarity, we have
\[ \beta \sim -\frac{2i}{\pi} \left( 1 + \frac{1}{8} (32c_3 - 3) \exp \left( -\frac{8\pi^2}{g^2(u)} \right) + \cdots \right), \]  

(14)

which exhibits the standard 1-instanton exponential factor.

As a consequence, we may also fix the constant \( c_3 \) from knowledge of the 1-instanton correction which is calculable at weak coupling. In effect, when restricted to the 1-instanton level, the constant \( c_3 \) has implicitly been fixed by the choice of perturbative renormalization scheme. In order to see this more clearly we note that in the weak coupling limit we can identify \( \Lambda \) with the perturbative renormalization group invariant scale, given by \( \Lambda^4 = u^2 \exp(2i\pi \tau(u)) \). Thus we have \( k^4 \sim 4\Lambda^4/u^2 + \cdots \) and, from the structure of (13), we observe that a change of renormalization scheme corresponding to a change in \( \Lambda \) may be compensated by a change in \( c_3 \) at the 1-instanton level. The 1-instanton term is therefore scheme dependent and a choice of scheme fixes the coefficient of the associated exponential factor, and thus the value of \( c_3 \), unambiguously.

Therefore, the final constant may be fixed via knowledge of the 1-instanton contribution at weak coupling, calculable by saddle point methods in a scheme such as Pauli-Villars. Such a scheme was shown in [11] to be equivalent to the implicit scheme used in [2]. However, a direct instanton calculation [11] gives the first power correction to the perturbative result at a fixed value of \( a \), rather than \( u \). Nevertheless, while \( \beta|_u \) and \( \beta|_a \) differ at this order, the relationship, \( u = a^2/2 + \Lambda^4/(4a^2) \), is again calculable at weak coupling to the required 1-instanton order using the fact that \( u = \langle \text{Tr} \phi^2 \rangle \). Thus, converting from \( u \) to \( k \), the 1-instanton induced power correction to \( \beta \) takes the form

\[ \beta_{\text{weak coupling}} = -\frac{2i}{\pi} \left( 1 + \frac{5}{32} k^4 + \cdots \right), \]

(15)

where the dots represent higher order instanton contributions. Comparing (13) with (15) leads to the identification \( c_3 = 1/4 \), consistent with our earlier conclusion.

The final result for the \( \beta \)-function is then given by

\[ \beta(\tau) = \frac{1 - 4F}{F'} = -\frac{i}{\pi} \left( \frac{1}{\theta_3^4} + \frac{1}{\theta_4^4} \right). \]

(16)

After accounting for the alternative \( \theta \)-function notation used, one may readily verify that this result coincides with that obtained by Minahan and Nemeschansky [7] from the elliptic curve of the Seiberg-Witten solution. This expression may also be shown to coincide with the result obtained by Bonelli and Matone [8] from the Picard-Fuchs equation for the vevs \( a \) and \( a_D \) of the chiral superfield. However, this requires use of an alternative choice of boundary conditions in this equation in order to be consistent with the choice \( k^2 = \theta_3^4/\theta_4^4 \). The singularity structure of this \( \beta \)-function has been discussed previously in [11] and [8]. We recall here that the fixed points, located at \( \tau = (i - 1)/2 \), and the equivalent points under \( \Gamma_U(2) \), correspond to \( u = 0 \), where the full gauge symmetry is classically restored. This result is not unexpected recalling that \( \beta = \beta(u/\Lambda^2) \). Finally we note that the \( \beta \)-function is

\[ \text{Our normalisation of } \Lambda \text{ differs from that of [11] by a factor of } \sqrt{2}, \text{ i.e. } \Lambda^2 = 2\Lambda^2_{\text{DR}}, \text{ which fixes the renormalization scheme.} \]
singular at the vertices of the fundamental domain which lie on the real axis, i.e. $\tau \in \mathbb{Z}$ in general. The gauge coupling diverges at these points, which are associated with a breakdown of the effective theory due to the presence of extra massless monopoles and dyons.

Finally, we note that one may expand (16) to higher order allowing comparison of the 2-instanton coefficient with semi-classical calculations. Expanding (16) up to $O(k^8)$, which includes all 2-instanton effects plus partial corrections due to three instantons, we obtain

$$\beta \sim -\frac{2i}{\pi} \left(1 + \frac{5}{32} k^4 + \frac{5}{32} k^6 + \frac{1229}{8192} k^8 + \cdots \right).$$

(17)

In order to compare this result with saddle-point calculations, which are obtained at fixed $a$, rather than $u$, it is necessary to carefully convert the 2-instanton results [23] for $\mathcal{F}$ (or $\tau$) to functions of $u$ or $k$ (see e.g. [24]). Once one does this, and accounts for the different normalisation of $\Lambda$ in [23], one readily verifies the explicit 2-instanton coefficient obtained by evaluation of the induced vertex.

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