Mathematical Modeling of Traffic

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Abstract: This mathematical exploration is around the topic of traffic. Traffic is a constant problem for many small and large cities with a significant population density and costs people efficiency. Traffic congestion adversely impacts metropolitan cities’ living standards, reflecting on travelers’ time cost, the number of resulting accidents, and trivially fuel consumption. Therefore good transportation is called for contributions everywhere today in the modern world. In this essay, mathematical theory is used to integrate the most influential cause - the culprit of traffic congestions.

1. Introduction
Traffic conditions are diverse from one city to another. Many factors are actively affecting traffic, and organizing an efficient traffic system is undoubtedly one of the most complicated topics, also the most challenging job for the city manager. Mathematics in topics about traffic may give clues to improve the efficiency of transportation networks. In this essay, a few conventional mathematics models of traffic will be explored, in which a few basic rules could be summed up to make things easier.

Governments often derive traffic legislation from mathematical analytics to ensure more coordinated and safer use of public networks.

Mathematical modeling of traffic helps explain a specific type of traffic system and make predictions about its behavior given different conditions. According to the online lab manual “Traffic Flow Theory”, density is the number of vehicles presented on a given length of roadway. Flow rate is the number of vehicles passing a fixed point per unit of time; the flow rate is the product of density and speed given by \( q = kv \); the flow is one indicator of the traffic system [1].

2. Background
Greenshield’s Fundamental Model Diagrams is a basic model under ideal traffic conditions. It is frequently used for its simplicity and historical reasons.

The model is summarized in the diagrams in Figure 1. According to Greenshield’s diagrams, it is assumed that the traffic speed has a linear relationship with the density of the traffic flow [2]. Vehicles achieve a maximum free speed when density is close to zero, and when the density is at maximum, vehicles get zero speed.
This relation can be expressed by the equation, with $v_f$ representing vehicles’ free speed when density is zero, i.e. there are no or few vehicles on the road. $k_j$ is the jam density when vehicles are stuck and their speed is zero:

$$v = v_f - \frac{k}{k_j} v_f$$  \hspace{1cm} (1)

The flow rate is given by the following equation:

$$q = k \times v$$  \hspace{1cm} (2)

From (1) and (2) the relation between flow rate and density is known to be:

$$q = v_f k - \frac{k^2}{k_j} v_f$$  \hspace{1cm} (3)

So the maximum flow rate occurred at half of the free speed and half of maximum density, just as illustrated in Figure 1.

3. Model 1: Straight road with distancing (Adding Restriction to Greenshield’s Model)

The first model will be derived from Greenshield’s Fundamental Model, using the same assumption. A restriction rule is added to Greenshield’s model. The distancing of vehicles is forced by traffic regulations with the “2 second rule” [3]. The two-second rule is a rule of thumb by which a driver may maintain a safe following distance at any speed. The rule is that a driver should ideally stay at least two seconds behind any vehicle directly in front of the driver’s vehicle.

For simplicity, the road network is assumed to be only a straightforward road.

The following variables will be used for building model 1.

- $v$: speed of vehicles, in km/hour.
- $s_1$: length of vehicles, in kilometers.
- $s_2$: run distancing, in kilometers.
- $s$: $s_1 + s_2$
- $k$: density which is the number of vehicles per kilometer.
- $t_d$: fixed parameter, time for run distancing, in hours.
- $q$: flow rate, which is the product of speed and density, in veh.km/hour (vehicle kilometer per hour).

Flow rate is also the index of traffic efficiency.

Most regions legislated on vehicle distancing by 2-second rule. Further more, considering an extra reaction time of human as 1.6 seconds, take $t_d=3.6$seconds=$0.001$hour for distancing time of running.
vehicles, then add this restriction to Greenshield’s model, replacing the postulation of linear relationship between speed and density. Therefore:

\[ s_2 = t_d \times v \]  

(4)

where \( s_2 \) is the run distancing. Now, the density will be:

\[ k = \frac{1}{s} = \frac{1}{s_1 + s_2} \]  

(5)

substitute (4) into (5):

\[ k = \frac{1}{s_1 + vt_d} \]  

(6)

substitute (6) into (2):

\[ q = \frac{v}{s_1 + vt_d} \]  

(7)

When vehicles are running on the highway, the speed can be very high. While vehicles are running on urban roads, the speed is often limited low. There is a speed limit set in every country. The speed limits vary in countries, but most countries have a speed limit between 110 kph (kilometre per hour) and 130 kph [4]. Therefore, taking the average value of 120 kph as the speed limit, the speed of vehicles in this model ranges from 0 kph to 120 kph. The average length of a vehicle is \( s_1 = 6 \) m= 0.006 kilometer. Calculation can be made with the above equations, and the data table and graphs are presented below.

From the table and the graphs, it can be seen that when speed increases, the density decreases rapidly at low speed and decreases slowly at high speed. This explains that speed is often low within crowded cities with a high density of vehicles.

Table 1.Speed-Density-Flow Rate Data of Model 1

| Speed \( v \), km/h | \( s_1 \), m | \( s_2 \), m | \( s_1 + s_2 \), m | Density \( k \), veh. | Flow rate \( q \), veh.km/h |
|---------------------|-------------|-------------|----------------|----------------|------------------|
| 0.0                 | 6           | 0           | 6              | 166.7          | 0.0              |
| 10.0                | 6           | 10          | 16             | 62.5           | 625.0            |
| 20.0                | 6           | 20          | 26             | 38.5           | 769.2            |
| 30.0                | 6           | 30          | 36             | 27.8           | 833.3            |
| 40.0                | 6           | 40          | 46             | 21.7           | 869.6            |
| 50.0                | 6           | 50          | 56             | 17.9           | 892.9            |
| 60.0                | 6           | 60          | 66             | 15.2           | 909.1            |
| 70.0                | 6           | 70          | 76             | 13.2           | 921.1            |
| 80.0                | 6           | 80          | 86             | 11.6           | 930.2            |
| 90.0                | 6           | 90          | 96             | 10.4           | 937.5            |
| 100.0               | 6           | 100         | 106            | 9.4            | 943.4            |
| 110.0               | 6           | 110         | 116            | 8.6            | 948.3            |
| 120.0               | 6           | 120         | 126            | 7.9            | 952.4            |
4. Discussion of Model 1

Figure 2 shows that when speed increase, the flow rate increases rapidly at low speed, and increases slowly at high speed.

To deduce the rate of how flow rate varies with speed, the first derivative will be evaluated by substituting $s_1 = 0.006$, $t_d = 0.001$ into (7):

$$q = \frac{v}{s_1 + vt_d} = \frac{v}{0.006 + 0.001v} = 1000 \left(\frac{6000}{6 + v}\right)$$

(8)
From figure 4, it can be found that as the speed increases, the flow rate keeps increasing, but the rate of increase becomes slower and slower. Given that this is a reciprocal about speed $v$, the graph approaches a horizontal asymptote where flow rate $q = 1000$.

By comparing figure 1 and the above graphs of model 1, it appears that the outcomes of model 1 are different from Greenshield’s Fundamental Diagram. In model 1, the maximum flow rate does not happen at half of the maximum speed or half of the maximum density as illustrated in Greenshield’s models, but at maximum speed and minimum density.

According to figure 4, the key is that density is approaching zero with the increase of speed, meaning that the road will not become crowded as speed increases. Therefore, the higher the speed, the more cars pass by per unit of time. The speed limits the maximum density, or according to the regulators, the maximum speed is limited because safe distancing between vehicles must be kept.

The flow rate is inversely proportional to the density. This is quite the same as the general understanding that traffic flow worsens when traffic is denser. This model is more practical and is a closer approach to the actual traffic. However, this could be doubted because there should be a larger flow rate when there are more cars on the road. This negatively correlated relationship is valid only when the density is high enough that the distance between cars is not enough for braking. At that time, the higher the density, the lower the flow rate because all drivers slow down their speed to avoid collisions.

5. Model 2: Distancing and Junction (Adding Junction to Model 1)

Model 1, in general, reflects the traffic in real life to a certain extent, but there is still a step lacking in the actual situation. A straight road segment can only be found as a traffic unit of the whole traffic network. The entire traffic network in the city is constituted of vast numbers of road segments and crossroads.

In model 2, a crossroad will be added to the straight road in model 1. Model 2 focus on a typical traffic unit, including a straight road and a junction. At a junction, vehicles are controlled by traffic lights, vehicles are stopped by red signals, and are permitted to go when the green signal turns on. The added junction affects traffic mainly by the waiting time of vehicles.

Some parameters newly introduced in this model are the total length of the road segment $L$, average speed $v_{m_2}$, and waiting time $t_w$ for a vehicle before it can go through the crossroad, therefore:

$$v_{m_2} = \frac{L}{\frac{v}{v} + t_w}$$

(9)
where $v$ is the average speed in model 1’s case, and $v_{m2}$ refers to the average speed after waiting time is added.

Substitute (9) into (7), for model 2, the flow rate is:

$$q_{m2} = \frac{v_{m2}}{s_1 + v_{m2}t_d} \quad (10)$$

Substitute (9) into (10):

$$q_{m2} = \frac{v}{s_1 \left( \frac{vt_w}{L} + 1 \right) + vt_d} \quad (11)$$

From the observations made about crossroads, the waiting time varies from 30 seconds to 90 seconds depending on the width of the road. The typical configure of the waiting time of crossroad is set to be $t_w=54$ seconds $=0.015$ hour from one of the observations, the length of the road is $L=1$km. Then the flow rate of model 2 is integrated, while the other variables are consistent with the value in model 1.

6. Discussion of Model 2

From table II, a speed-flow rate graph of model 2 can be drawn. For a clearer view, the speed-flow rate graph of models 1 and 2 are drawn in the same coordinate system.

Table 2. Speed-Flow Rate Data Table at $L = 1$km for Model 2

| Speed $v$, km/h | $s_1$, m | $s_2$, m | $s_1 + s_2$, m | Flow rate $q$, veh.km/h |
|-----------------|----------|----------|-----------------|-------------------|
|                 |          |          |                 | **Model 1**       | **Model 2**       |
|                 |          |          |                 | **L = 1**         | **L = 1**         |
| 0.0             | 6        | 0        | 6               | 0.0               | 0.0               |
| 10.0            | 6        | 10       | 16              | 625.0             | 591.7             |
| 20.0            | 6        | 20       | 26              | 769.2             | 719.4             |
| 30.0            | 6        | 30       | 36              | 833.3             | 775.2             |
| 40.0            | 6        | 40       | 46              | 869.6             | 806.5             |
| 50.0            | 6        | 50       | 56              | 892.9             | 826.4             |
| 60.0            | 6        | 60       | 66              | 909.1             | 840.3             |
| 70.0            | 6        | 70       | 76              | 921.1             | 850.5             |
| 80.0            | 6        | 80       | 86              | 930.2             | 858.4             |
| 90.0            | 6        | 90       | 96              | 937.5             | 864.6             |
| 100.0           | 6        | 100      | 106             | 943.4             | 869.6             |
| 110.0           | 6        | 110      | 116             | 948.3             | 873.7             |
| 120.0           | 6        | 120      | 126             | 952.4             | 877.2             |
The determinant equation of flow rate of model 2, (11), shows that the average speed is affected by adding waiting time, and consequently, the flow is affected too. Table II and figure 6 show that the horizontal asymptote of the graph is lower, i.e., the speed and the flow decrease in model 2, compared to model 1. Therefore, it is clear that traffic signals are a vital factor for flow rate.

By comparing (7) and (11), the determinant equations of flow rate from models 1 and 2, it can be depicted that the deviation is determined by the ratio of waiting time to the length of the segment.

\[ q = \frac{v}{s_1 + vt_d} \quad (7) \]

\[ q_{m_2} = \frac{v}{s_1 \left( \frac{vt_w}{L} + 1 \right) + vt_d} \quad (11) \]

Besides the velocity, more factors such as “the length of the road segment” and “waiting time of the crossway” are introduced into this model. This model represents the very long segment of highway between cities without any crossroad interfering and the short segments in the urban traffic network with crossroads and junctions. Therefore, it is a practical model approaching real traffic, but model 2 still does not fit because it cannot show the different effects on flow rate by changing the size of vehicles, road width, or adding zebra crossing.

7. The Responsiveness of Flow Rate to Change in Variables

\[ q_{m_2} = \frac{v}{s_1 \left( \frac{vt_w}{L} + 1 \right) + vt_d} \quad (11) \]

From (11), it is evident that the flow rate depends on five parameters, among which \( t_s \) and \( s_1 \) are set to a fixed value. Other variables, \( L, t_w, v \), vary depending on the situation. The responsiveness of flow rate to change in variables can be integrated by deriving the first derivative.

Equation (11) can be rewritten as:

\[ q_{m_2} = \frac{vL}{s_1 vt_w + s_1 L + Lvt_d} \quad (12) \]

Therefore:

\[ q'_{m_2}(v) = \frac{s_1 L^2}{(s_1 vt_w + s_1 L + Lvt_d)^2} \quad (13) \]
\[ q'_{m_2}(L) = \frac{s_1 t_w v^2}{(s_1 v t_w + s_1 L + L v t_d)^2} \]  
(14)

\[ q'_{m_2}(t_w) = -\frac{s_1 L v^2}{(s_1 v t_w + s_1 L + L v t_d)^2} \]  
(15)

With \( t_d = 0.001 \) hour (3.6 seconds), \( s_1 = 0.006 \) km, and \( t_w = 0.015 \) hour (54 seconds), the resulting flow rate can be calculated.

When \( v = 60 \) km/h, \( L = 0.5 \) km, \( q'_{m_2}(v) = 1 \), \( q'_{m_2}(L) = 219.7 \), \( q'_{m_2}(t_w) = -7324.2 \). This is the situation where urban traffic network presents. There are short segments with crossroads and junctions.

When \( v = 120 \) km/h, \( L = 5 \) km, \( q'_{m_2}(v) = 0.4 \), \( q'_{m_2}(L) = 3.2 \), \( q'_{m_2}(t_w) = -1052.1 \). This is the condition of highway, a long road without interfering traffic.

By comparing the absolute value of the first derivatives of different variables, it can be depicted that the waiting time affects the flow rate the most.

8. Conclusion and Evaluation

From the graphs and the discussion of the above models, the following conclusions have been verified:

Firstly, the flow rate increases rapidly with speed when at low speed. This scenario can be found in urban traffic, where the speed is often limited by density. The flow rate increases very slowly with speed when at high speed. This scenario can be found in the highway traffic, where the flow rate maintains a high level.

Secondly, the waiting time of crossroads impacts a lot on the flow rate within a shorter road segment, which can be found in urban traffic.

The conclusions explained that many cities with dense traffic are pushing the regulating methods such as encourage mass transportation. This will keep the density of vehicles at a low level and thus improve the flow rate. Other solutions are enlarging the road network and building overpasses for the key crossroad with high densities, cutting down the waiting time, and improving the traffic significantly. To minimize the waiting time and improve traffic, the government should coordinate the traffic lights with high technology.

These models also have aspects that are not close to daily life. For example, the negative proportional relationship between traffic density and traffic flow only occurs when the distance between the car and the front and rear cars is so small that the driver intentionally slows down to avoid a collision. In other cases, when there are few cars on the road, all cars in the road section can move forward at full speed. At this time, the increase in density will only increase the traffic volume. Furthermore, these models do not consider vehicles of different sizes. Involving public buses and motorcycles makes a difference because the drivers of buses may be more cautious when driving, and motorcycles are faster. All of these create different conditions of traffic.

References

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