Compactness and Bubbles Analysis for 1/2-harmonic Maps

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Abstract

In this paper we study compactness and quantization properties of sequences of 1/2-harmonic maps $u_k: \mathbb{R} \rightarrow S^{m-1}$ such that $\|u_k\|_{H^{1/2}(\mathbb{R}, S^{m-1})} \leq C$. More precisely we show that there exist a weak 1/2-harmonic map $u_\infty: \mathbb{R} \rightarrow S^{m-1}$, a possible empty set $\{a_1, \ldots, a_\ell\}$ in $\mathbb{R}$ such that up to subsequences

\[
\left(\|(-\Delta)^{1/4}u_k\|^2 \rightarrow \|(-\Delta)^{1/4}u_\infty\|^2\right)dx + \sum_{i=1}^\ell \lambda_i \delta_{a_i}, \quad \text{in Radon measure,}
\]

as $k \rightarrow +\infty$, with $\lambda_i \geq 0$.

The convergence of $u_k$ to $u_\infty$ is strong in $W^{1/2,p}_{loc}(\mathbb{R} \setminus \{a_1, \ldots, a_\ell\})$, for every $p \geq 1$. We quantify the loss of energy in the weak convergence and we show that in the case of non-constant 1/2-harmonic maps with values in $S^2$ one has $\lambda_i = 2\pi n_i$, with $n_i$ a positive integer.

Key words. Fractional harmonic maps, nonlinear elliptic PDE’s, regularity of solutions, commutator estimates.

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1 Introduction

In the paper [9] Rivière and the author started the investigation of the following 1-dimensional quadratic Lagrangian

\[ L(u) = \int_{\mathbb{R}} |(-\Delta)^{1/4} u(x)|^2 dx, \]

where \( u : \mathbb{R} \to \mathcal{N} \), \( \mathcal{N} \) is a smooth \( k \)-dimensional submanifold of \( \mathbb{R}^m \) which is at least \( C^2 \), compact and without boundary. We observe that (1) is a simple model of Lagrangian which is invariant under the trace of conformal maps that keep invariant the half space \( \mathbb{R}^2_+ \): the Möbius group.

Precisely let \( \phi : \mathbb{R}^2_+ \to \mathbb{R}^2_+ \) in \( W^{1,2}(\mathbb{R}^2_+, \mathbb{R}^m) \) be a conformal map of degree 1, i.e. it satisfies

\[
\begin{cases}
    |\frac{\partial \phi}{\partial x}| = |\frac{\partial \phi}{\partial y}|
    \\
    \langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \rangle = 0
    \\
    \det \nabla \phi \geq 0 \text{ and } \nabla \phi \neq 0.
\end{cases}
\]

Here \( \langle \cdot, \cdot \rangle \) denotes the standard Euclidean inner product in \( \mathbb{R}^m \).

We denote by \( \tilde{\phi} \) the restriction of \( \phi \) to \( \mathbb{R} \). Then we have \( L(u \circ \tilde{\phi}) = L(u) \).

Moreover \( L(u) \) in (1) coincides with the semi-norm \( \| u \|^2 H_{1/2}(\mathbb{R}) \) and the following identity holds

\[
\int_{\mathbb{R}} |(-\Delta)^{1/4} u(x)|^2 dx = \inf \left\{ \int_{\mathbb{R}^2_+} |\nabla \tilde{u}|^2 dx : \tilde{u} \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^m), \text{ trace } \tilde{u} = u \right\}.
\]

The Lagrangian \( L \) extends to map \( u \) in the following function space

\[ \dot{H}^{1/2}(\mathbb{R}, \mathcal{N}) = \{ u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) : u(x) \in \mathcal{N}, \text{ a.e.} \} \text{.} \]

The operator \( (-\Delta)^{1/4} \) on \( \mathbb{R} \) is defined by means of the the Fourier transform as follows

\[ (-\Delta)^{1/4} u = |\xi|^{1/2} \hat{u}, \]

(given a function \( f \), \( \hat{f} \) denotes the Fourier transform of \( f \).

We denote by \( \pi_\mathcal{N} \) the orthogonal projection from \( \mathbb{R}^m \) onto \( \mathcal{N} \) which happens to be a \( C^\ell \) map in a sufficiently small neighborhood of \( \mathcal{N} \) if \( \mathcal{N} \) is assumed to be \( C^{\ell+1} \). We now introduce the notion of 1/2-harmonic map into a manifold.

**Definition 1.1** A map \( u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N}) \) is called a weak 1/2-harmonic map into \( \mathcal{N} \) if for any \( \phi \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m) \) there holds

\[
\frac{d}{dt} L(\pi_\mathcal{N}(u + t\phi))_{|t=0} = 0 \text{.}
\]
In short we say that a weak $1/2$--harmonic map is a critical point of $L$ in $\dot{H}^{1/2}(\mathbb{R},\mathcal{N})$ for perturbations in the target.

We next give some geometric motivations related to the study of the problem (1). First of all variational problems of the form (1) appear as a simplified model of renormalization area in hyperbolic spaces, see for instance [2]. There are also some geometric connections which are being investigated in the paper [11] between $1/2$-harmonic maps and the so-called free boundary sub-manifolds and optimization problems of eigenvalues. With this regards we refer the reader also to the papers [14, 15]. Finally $1/2$-harmonic maps into the circle $S^1$ might appear for instance in the asymptotic of equations in phase-field theory for fractional reaction-diffusion such as

$$\varepsilon^2 (-\Delta)^{1/2} u + u(1 - |u|^2) = 0,$$

where $u$ is a complex valued ”wave function”.

In this paper we consider the case $\mathcal{N} = S^{m-1}$. It can be shown (see, [9]) that every weak $1/2$-harmonic map satisfies the following Euler-Lagrange equation

$$(-\Delta)^{1/2} u \wedge u = 0 \text{ in } \mathcal{D}'(\mathbb{R}).$$  \hspace{1cm} (4)

One of the main achievements of the paper [9] is the rewriting of the equation (1) in a more “tractable” way in order to be able to investigate regularity and compactness property of weak $1/2$-harmonic maps. Precisely in [9] the following two results have been proved:

**Proposition 1.1** A map $u$ in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ is a weak $1/2$-harmonic map if and only if it satisfies the following equation

$$(-\Delta)^{1/4} (u \wedge (-\Delta)^{1/4} u) = T(u \wedge, u),$$   \hspace{1cm} (5)

where, in general for arbitrary positive integers $n, m, \ell$, for every $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}\ell_m(\mathbb{R})) \ell \geq 1^{(1)}$ and $u \in \dot{H}^{1/2}(\mathbb{R}^n, \mathbb{R}^m)$, $T$ is the operator defined by

$$T(Q, u) := (-\Delta)^{1/4} [Q (-\Delta)^{1/4} u] - Q(-\Delta)^{1/2} u + (-\Delta)^{1/4} Q (-\Delta)^{1/4} u.$$  \hspace{1cm} (6)

\[ \square \]

The equation (5) has been completed by the following ”structure equation” which is a consequence of the fact that $u \in S^{m-1}$ almost everywhere:

**Proposition 1.2** All maps in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ satisfy the following identity

$$(-\Delta)^{1/4} (u \cdot (-\Delta)^{1/4} u) = S(u, u) - R((-\Delta)^{1/4} u \cdot R(-\Delta)^{1/4} u).$$   \hspace{1cm} (7)

\[ ^{(1)} \mathcal{M}\ell_m(\mathbb{R}) \text{ denotes, as usual, the space of } \ell \times m \text{ real matrices.} \]
where, in general for arbitrary positive integers \(n, m, \ell\), for every \(Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{\ell \times m}(\mathbb{R}))\), \(\ell \geq 1\) and \(u \in \dot{H}^{1/2}(\mathbb{R}^n, \mathbb{R}^m)\), \(S\) is the operator given by

\[
S(Q, u) := (-\Delta)^{1/4}[Q(-\Delta)^{1/4}u] - \mathcal{R}(Q \nabla u) + \mathcal{R}((-\Delta)^{1/4}Q \mathcal{R}(-\Delta)^{1/4}u)
\]  

and \(\mathcal{R}\) is the Fourier multiplier of symbol \(m(\xi) = i\frac{\xi}{|\xi|}\).

We call the operators \(T\), \(S\) three-terms commutators and in [9] the following estimates have been established: for every \(u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)\) and \(Q \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{\ell \times m}(\mathbb{R}))\) we have

\[
\|T(Q, u)\|_{\dot{H}^{-1/2}((\mathbb{R}))} \leq C \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|u\|_{\dot{H}^{1/2}(\mathbb{R})},
\]

\[
\|S(Q, u)\|_{\dot{H}^{-1/2}((\mathbb{R}))} \leq C \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|u\|_{\dot{H}^{1/2}(\mathbb{R})},
\]

and

\[
\|\mathcal{R}((-\Delta)^{1/4}u \cdot \mathcal{R}(-\Delta)^{1/4}u)\|_{\dot{H}^{-1/2}((\mathbb{R}))} \leq C \|u\|^2_{\dot{H}^{1/2}(\mathbb{R})}.
\]

What has been discovered is a sort of "gain of regularity" in the r.h.s of the equations (5) and (7) in the sense that, under the assumptions \(u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)\) and \(Q \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{\ell \times m}(\mathbb{R}))\) each term individually in \(T\) and \(S\) - like for instance \((-\Delta)^{1/4}[Q(-\Delta)^{1/4}u]\) or \(Q(-\Delta)^{1/2}u\) - is not in \(H^{-1/2}\) but the special linear combination of them constituting \(T\) and \(S\) is in \(H^{-1/2}\). The same phenomenon appears in in dimension 2 in the context of harmonic maps, for the Jacobians \(J(a, b) := \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \frac{\partial b}{\partial x}\) (with \(a, b \in \dot{H}^1(\mathbb{R}^2)\)) which satisfy as a direct consequence of Wente’s theorem (see [5, 25])

\[
\|J(a, b)\|_{\dot{H}^{-1/2}(\mathbb{R}^2)} \leq C \|a\|_{\dot{H}^1(\mathbb{R}^2)} \|b\|_{\dot{H}^1(\mathbb{R}^2)}
\]

whereas, individually, the terms \(\frac{\partial a}{\partial x} \frac{\partial b}{\partial y}\) and \(\frac{\partial a}{\partial y} \frac{\partial b}{\partial x}\) are not in \(H^{-1}(\mathbb{R}^2)\).

The estimates (9) and (10) imply in particular that if \(u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1})\) is a 1/2-harmonic map then

\[
\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \leq C \|(-\Delta)^{1/2}u\|_{L^2(\mathbb{R})}^2
\]

where the constant \(C\) is independent on \(u\).

From the inequality (13) it follows that if \(C\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} < 1\) then the solution is constant. This the so-called bootstrap test and it is the key observation to prove Morrey-type estimates and to deduce Hölder regularity of 1/2-harmonic maps, see [9].

We mention here that since the paper [9] several extensions have been considered. The regularity of solutions to nonlocal linear Schrödinger systems with applications to 1/2-harmonic maps with values into general manifolds have been studied by Rivière and the author in [10]. n/2-harmonic maps in odd dimension \(n\) has been considered in [23] and [6] respectively in the case of values into the \(m-1\) dimensional sphere and into general manifolds and the case of \(\alpha\)-harmonic maps in \(W^{\alpha,p}(\mathbb{R}^n, S^{m-1})\), with \(\alpha p = n\), has been recently studied by Schikorra and the author in [12]. Finally Schikorra [24] has also studied the partial regularity of weak solutions to nonlocal linear systems with an
antisymmetric potential in the supercritical case (namely where \( \alpha p < n \)) under a crucial monotonicity assumption on the solutions which allows to reduce to the critical case.

In this paper we address to the issue of understanding the behaviour of sequences \( u_k \) of weak \( 1/2 \)-harmonic maps. We observe that as in the case of harmonic maps the bootstrap test [13] implies that if the energy is small then the system behaves locally like a linear system of the form \((-\Delta)^{1/2} u = 0\) (namely the r.h.s is “dominated” by the l.h.s of the equation). As a consequence we obtain that any sequence \( u_k \) of weak \( 1/2 \)-harmonic maps with uniformly bounded energy weakly converges to a weak \( 1/2 \)-harmonic map \( u_\infty \) and strongly converges to \( u_\infty \) away from a finite (possibly empty) set \( \{a_1, \ldots, a_\ell\} \subset \mathbb{R} \).

Namely we have (up to a subsequence)

\[
(\|(-\Delta)^{1/4} u_k\|^2 \rightarrow \|(-\Delta)^{1/4} u_\infty\|^2)dx + \sum_{i=1}^{\ell} \lambda_i \delta_{a_i}, \quad \text{in Radon measure,}
\]

as \( k \rightarrow +\infty \), with \( \lambda_i \geq 0 \). It remains the question to understand how the convergence at the concentration points \( a_i \) fails to be strong. A careful analysis shows that the loss of energy during the weak convergence is not only concentrated at the points \( a_i \) but it is also quantized: this amount of energy is given by the sum of energies of non-constant \( 1/2 \)-harmonic maps (the so-called bubbles). More precisely we get the following result

**Theorem 1.1** Let \( u_k \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1}) \) be a sequence of \( 1/2 \)-harmonic maps such that \( \|u_k\|_{\dot{H}^{1/2}} \leq C \). Then it holds:

1. There exist \( u_\infty \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1}) \) and a possibly empty set \( \{a_1, \ldots, a_\ell\}, \ell \geq 1 \), such that up to subsequence
   \[
u_n \rightarrow u_\infty \quad \text{in} \quad W^{1/2,p}_{\text{loc}}(\mathbb{R} \setminus \{a_1, \ldots, a_\ell\}), \quad p \geq 2 \quad \text{as} \quad k \rightarrow +\infty \tag{14}
\]
   and
   \[
   (-\Delta)^{1/2} u_\infty \wedge u_\infty = 0, \quad \text{in} \quad \mathcal{D}'(\mathbb{R}) \tag{15}
   \]

2. There is a family \( \tilde{u}_{i,j}^{\infty} \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1}) \) of \( 1/2 \)-harmonic maps \( (i \in \{1, \ldots, \ell\}, j \in \{1, \ldots, N_i\}) \), such that up to subsequence
   \[
   \|(-\Delta)^{1/4}(u_k - u_\infty - \sum_{i,j} \tilde{u}_{i,j}^{\infty})\|^2_{L^2_{\text{loc}}(\mathbb{R})} \rightarrow 0, \quad \text{as} \quad k \rightarrow +\infty. \tag{16}
   \]

Theorem 1.1 says that for every \( i \), \( \lambda_i = \sum_{j=1}^{N_i} L(\tilde{u}_{i,j}^{\infty}) \). Therefore there is no dissipation of energy in the region between \( u_\infty \) and the bubbles and between the bubbles themselves (the so-called neck-regions).

We would like now to mention a result obtained in the paper [11] on the characterization of \( 1/2 \) harmonic maps \( u: S^1 \rightarrow S^2 \) which permits us to deduce that in the case of \( 1/2 \) harmonic maps with values in \( S^2 \) one has \( \lambda_i = 2\pi n_i \), with \( n_i \) a positive integer and also to provide a simple example showing that the quantization may actually occur, namely the set \( \{a_1, \ldots, a_\ell\} \) may be nonempty.
Theorem 1.2 \[\text{[11]}\] i) \(u: S^1 \rightarrow S^1\) is a weak \(1/2\)-harmonic map if and if its harmonic extension \(\tilde{u}: D^2 \rightarrow \mathbb{R}^2\) is holomorphic or anti-holomorphic.

ii) \(u: S^1 \rightarrow S^2\) is a weak \(1/2\)-harmonic map if and if the composition of weak \(1/2\)-harmonic map \(u: S^1 \rightarrow S^1\) and an isometry \(T: S^2 \rightarrow S^2\).

\[\square\]

We remark that because of the invariance of the Lagrangian \([11]\) with respect to the trace of conformal transformations we can study without restrictions the problem in \(S^1\) instead of \(\mathbb{R}^2\).

From Theorem 1.2 it follows that \(1/2\)-harmonic maps \(u: S^1 \rightarrow S^1\) with \(\deg(u) = 1\) coincide with the trace of Möbius transformations of the disk \(D^2 \subseteq \mathbb{R}^2\). Moreover every non-constant weak \(1/2\)-harmonic map \(u: S^1 \rightarrow S^2\) satisfies

\[
\int_{S^1} |(-\Delta)^{1/4} u|^2 dx = 2\pi k < +\infty.
\]

where \(k\) is a positive integer which coincides with \(|\deg(u)|\).

Let us consider now the following sequence of \(1/2\)-harmonic maps

\[
u_n: S^1 \rightarrow S^1, \quad \nu_n(z) = \frac{z - a_n}{1 - \overline{a}_n z},
\]

with \(|a_n| = 1\) and \(a_n \rightarrow 1\) as \(n \rightarrow +\infty\). In this case we have \(u_n \rightarrow -1\) in \(C^\infty_{\text{loc}}(\mathbb{R} \setminus \{1\})\), thus the set of concentration points is nonempty. Theorem \([11]\) yields the existence of one Bubble \(\tilde{u}_\infty\) such that

\[
\|(-\Delta)^{1/4}(u_k - \tilde{u}_\infty)\|_{L^2_{\text{loc}}} \to 0, \quad \text{as} \quad n \to +\infty.
\]

We explain now the method we have used to prove the main Theorem \([11]\).

In order to get the quantization of the energy we exploit a “functional analysis” method introduced by Lin and Riviè re in \([20]\) in the context of harmonic maps in non-conformal dimensions. Such a method consists in the use of the interpolation Lorentz spaces in the special case where the r.h.s of the equation can be written as linear combination of Jacobians.

This techniques has been recently applied in \([18, 19]\) and in \([3]\) for the quantization analysis respectively of linear Schrödinger systems with antisymmetric potential in 2-dimension, of bi-harmonic maps in 4-dimensions, and of Willmore surfaces. We refer the reader to the papers \([21, 20]\) for an overview of the bubbling and quantization issues in the literature.

We describe briefly the key steps to get the quantization analysis.

1. First of all we will make use of a general result proved in \([3]\) which permits to split the domain (in our case \(\mathbb{R}\)) into the converging region (which is the complement of small neighborhoods of the \(a^j\)), bubbles domains and neck-regions (which are unions of degenerate annuli).
2. We prove that the $L^2$ norm of $(-\Delta)^{1/4}u_k$ in the neck regions is arbitrary small, (see Theorem 3.1). Thanks to the duality of the Lorentz spaces $L^{2,1} - L^{2,\infty}$, this is reduced in estimating the $L^{2,\infty}, L^{2,1}$ norms of $u_k$ in these regions. Precisely we first show that the $L^{2,\infty}$ norm of the $u_k$ is arbitrary small in degenerate annuli (see Lemma 3.2) and as far as the $L^{2,1}$ norm is concerned, we use the following improved estimate on the operators $T$ and $S$ which is proved in the Appendix.

**Theorem 1.3** Let $u, Q \in H^{1/2}(\mathbb{R}^n)$. Then $T(Q,u), S(Q,u) \in H^1(\mathbb{R}^n)$ and

\[ \|T(Q,u)\|_{H^1(\mathbb{R}^n)} \leq C\|Q\|_{H^{1/2}(\mathbb{R}^n)}\|u\|_{H^{1/2}(\mathbb{R}^n)}. \]  
\[ \|S(Q,u)\|_{H^1(\mathbb{R}^n)} \leq C\|Q\|_{H^{1/2}(\mathbb{R}^n)}\|u\|_{H^{1/2}(\mathbb{R}^n)}. \]  

In a forthcoming paper [8] we are going to investigate bubbles and quantization issues in the case of nonlocal Schrödinger linear systems with applications to $1/2$-harmonic maps with values into manifolds. The difficulty there is to succeed in getting a uniform $L^{2,1}$ estimate as well on degenerate annuli as in the local case (see [18]). It would be also very interesting to understand the geometric properties of the bubbles in the case of more general manifolds.

This paper is organized as follows.

In Section 1 we address to the compactness issue which is the first part of Theorem 1.1. In Section 2 we prove $L^2$ estimates on degenerate annual domains. In Section 3 we prove the second part of Theorem 1.1. In the Appendix we prove Theorem 1.3.

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*(2)* The $L^{2,\infty}(\mathbb{R})$ is the space of measurable functions $f$ such that

\[ \sup_{\lambda>0} \lambda |\{x \in \mathbb{R} : |f(x)| \geq \lambda\}|^{1/2} < +\infty. \]

$L^{2,1}(\mathbb{R})$ is the Lorentz space of measurable functions satisfying

\[ \int_0^{+\infty} |\{x \in \mathbb{R} : |f(x)| \geq \lambda\}|^{1/2} d\lambda < +\infty. \]

*(3)* $H^1(\mathbb{R}^n)$ denotes the Hardy space which is the space of $L^1$ functions $f$ on $\mathbb{R}^n$ satisfying

\[ \int_{\mathbb{R}^n} \sup_{t \in \mathbb{R}} |\phi_t * f(x)| \, dx < +\infty, \]

where $\phi_t(x) := t^{-n} \phi(t^{-1}x)$ and where $\phi$ is some function in the Schwartz space $S(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \phi(x) \, dx = 1$.

For more properties on the Hardy space $H^1$ we refer to [16] and [17].
2 Compactness

In this Section we prove the first part of Theorem [1]. The result is based on the following \( \varepsilon \)-regularity property whose proof can be found in [9, 10] and in [7].

Lemma 2.1 (\( \varepsilon \)-regularity ) Let \( u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1}) \) be a 1/2-harmonic map. Then there exists \( \varepsilon_0 > 0 \) such that if for some \( \gamma > 0 \)

\[
\sum_{j \geq 0} 2^{-j\gamma} \|(-\Delta)^{1/4} u\|_{L^2(B(x,2^j r))} \leq \varepsilon_0,
\]

then there is \( p > 2 \) such that for every \( x \in \mathbb{R}, y \in B(x,r/2) \) we have

\[
\left( \frac{r^{p/2-1}}{\int_{B(y,r/2)} |(-\Delta)^{1/4} u(x)|^p dx \right)^{1/p} \leq C \|u\|_{H^{1/2}(\mathbb{R})}.
\]  

(19)

By bootstrapping into the equations (5) and (17) and by localizing Theorems A.1 and A.2 (see [7]) one can show the following:

Corollary 2.1 Let \( u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1}) \) be a 1/2-harmonic map. Then \( (-\Delta)^{1/4} u \) is in \( L^p_{\text{loc}}(\mathbb{R}) \cap L^\infty_{\text{loc}}(\mathbb{R}) \) for every \( p \geq 2 \) with

\[
r^{1/2-1/p} \|(-\Delta)^{1/4} u\|_{L^p(B(x,r))} \leq C \|u\|_{H^{1/2}(\mathbb{R})},
\]

(20)

\[
r^{1/2} \|(-\Delta)^{1/4} u\|_{L^\infty(B(x,r))} \leq C \|u\|_{H^{1/2}(\mathbb{R})},
\]

(21)

for all \( x \in \mathbb{R} \) and \( r > 0 \).

We will use also the localized version of the following result whose proof can be found in [1] page 78:

Lemma 2.2 Let \( 0 < \alpha < 1 \) and \( f \in L^p(\mathbb{R}) \), \( 1 < p < \infty \). Then there is a constant \( C > 0 \) independent on \( f \), such that

\[
\|\Delta^{-\frac{\alpha}{p}} f\|_{L^r(\mathbb{R}^n)} \leq C \|\Delta^{-\frac{\alpha}{p}} f\|^\theta_{L^s(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}^{1-\theta},
\]

for \( 0 < \theta < 1, 1 \leq s \leq \infty, \frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{p} \).

Next we show that if \( \|u_k\|_{H^{1/2}(B(x,\rho))} \leq \varepsilon_0 \) where \( \varepsilon_0 > 0 \) is the small constant appearing in the \( \varepsilon \)-regularity Lemma 2.1 then \( u_k \in \dot{W}^{1/2,1}_{\text{loc}}(\mathbb{R}) \).

Proposition 2.1 Let \( \rho > 0 \) be such that \( \|u_k\|_{H^{1/2}(B(x,\rho))} \leq \varepsilon_0 \) with \( \varepsilon_0 > 0 \) given in Lemma 2.1. Then for all \( q > 2 \) there exists \( C > 0 \) (independent on \( x, k \)) such that

\[
\|(-\Delta)^{\frac{1}{2}+\frac{1}{q}} u_k\|_{L^2(B(x,\rho/2))} \leq C \|u_k\|_{H^{1/2}(\mathbb{R})}.
\]

(22)
Proof of Proposition 2.1. We set \( v_k = (-\Delta)^{1/4}u_k \). From Lemma 2.1 it follows that there exists \( q > 2 \) (independent on \( k \)) such that for every \( y \in B(x, \rho/2) \)

\[
\|v_k\|_{L^q(B(y,\rho/4))} \leq C\|u_k\|_{\dot{H}^{1/2}(\mathbb{R})} \tag{23}
\]
and bootstrapping into the equations (5) and (7) one gets

\[
\|(-\Delta)^{1/4}v_k\|_{L^{2q/(q+2)}(B(y,\rho/2))} \leq C\|u_k\|_{\dot{H}^{1/2}(\mathbb{R})}. \tag{24}
\]

Now we set \( f_k := (-\Delta)^{1/4}v_k \). By applying Lemma 2.2 in \( B(y,\rho/4) \) with \( f := v_k \) and \( p = q, r = 2, s = \frac{2q}{q+2}, \alpha = \frac{1}{2} \) and \( \theta = \frac{q^2}{q+2} \) we obtain

\[
\|(-\Delta)^{-\frac{q^2-2}{4q}}f_k\|_{L^2(B(y,\rho/4))} \leq C\|f_k\|_{L^{2q/(q+2)}(B(y,\rho/4))}\|v_k\|_{L^q(B(y,\rho/4))} \tag{25}
\]

In particular we get that \( v_k \in W^{1,2}_q(B(y,\rho/4)) \) and hence \( u_k \in W^{1,2}_q(B(y,\rho/4)) \) with

\[
\|u_k\|_{W^{1,2}_q(B(y,\rho/4))} = \|(-\Delta)^{1/2}u_k\|_{L^2(B(y,\rho/4))} \leq C. \tag{26}
\]

Actually one can show that the estimate (26) holds for every \( q > 2 \). This concludes the proof. \( \square \)

We show now a singular point removability type result for 1/2-harmonic maps.

Proposition 2.2 [Singular point removability] Let \( u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1}) \) be a 1/2-harmonic map in \( D'(\mathbb{R}\setminus\{a_1,\ldots,a_\ell\}) \). Then

\[
u \land (-\Delta)^{1/2}u = 0 \quad \text{in} \; D'(\mathbb{R}).
\]

Proof of Proposition 2.2. The fact that \( u \land (-\Delta)^{1/2}u = 0 \) in \( D'(\mathbb{R}\setminus\{a_1,\ldots,a_\ell\}) \)

implies that

\[
(-\Delta)^{1/4}(u \land (-\Delta)^{1/4}u) = T(u \land, u) \quad \text{in} \; D'(\mathbb{R}\setminus\{a_1,\ldots,a_\ell\}),
\]

where \( T(u\land, u) \in \dot{H}^{-1/2}(\mathbb{R}) \) and

\[
\|T(u\land, u)\|_{\dot{H}^{-1/2}(\mathbb{R})} \leq C\|u\|_{\dot{H}^{1/2}(\mathbb{R})}^2. \tag{27}
\]

The distribution \( \phi := (-\Delta)^{1/4}(u \land (-\Delta)^{1/4}u) - T(u\land, u) \) is of order \( p = 1 \) and supported in \( \{a_1,\ldots,a_\ell\} \). Therefore by Schwartz Theorem [1] one has

\[
\phi = \sum_{|\alpha| \leq 1} c_\alpha \partial^\alpha \delta_{a_i}.
\]
Since $\phi \in \dot{H}^{-\frac{1}{2}}(\mathbb{R})$, then the above implies that $c_\alpha = [4]$ is and thus
$$( -\Delta )^{1/4} ( u \wedge ( -\Delta )^{1/4} u ) = T ( u \wedge u ) \text{ in } \mathcal{D}' ( \mathbb{R} ).$$

We conclude the proof of Proposition 2.2. □

The proof of the first part of Theorem 1.1 concerning the compactness of uniformly bounded 1/2-harmonic maps is contained in the following Lemma.

**Lemma 2.3** Let $u_k \in \dot{H}^{1/2} ( \mathbb{R}, S^{m-1} )$ be a sequence of 1/2-harmonic maps such that $\| u_k \|_{\dot{H}^{1/2}} \leq C$. Then there exist a sequence $u_{k'}$ of $u_k$, a function $u_\infty \in \dot{H}^{1/2} ( \mathbb{R}, S^{m-1} )$ and $\{ a_1, \ldots, a_\ell \}$, $\ell \geq 1$, such that

$$u_{k'} \rightarrow u_\infty \text{ as } k' \rightarrow +\infty \text{ in } \dot{H}^{1/2}_{loc} ( \mathbb{R} \setminus \{ a_1, \ldots, a_\ell \} ) \quad \text{for } p \geq 2,$$

and

$$( -\Delta )^{1/2} u_\infty \wedge u_\infty = 0, \text{ in } \mathcal{D}' ( \mathbb{R} ).$$

**Proof of Lemma 2.3.**

1. First of all there exists a subsequence $u_{k'}$ of $u_k$, a function $u_\infty \in \dot{H}^{1/2} ( \mathbb{R}, S^{m-1} )$ such that $u_{k'} \rightarrow u_\infty$ as $k' \rightarrow +\infty$.

2. If $\| u_k \|_{\dot{H}^{1/2} ( B ( x, \rho ) )} \leq \varepsilon_0$ for all $k \geq 1$ then from Lemma 2.1 and the Rellich-Kondrachov Theorem (if $\Omega \subset \mathbb{R}$ is a bounded subset then the embedding $W^{1/2,\ell} ( \Omega ) \hookrightarrow W^{1/2,t} ( \Omega )$ is compact for all $t < \frac{2q}{q-2}$) it follows that

$$u_{k'} \rightarrow u_\infty \text{ as } k' \rightarrow +\infty \text{ in } \dot{H}^{1/2} ( B ( x, \rho / 4 ), S^{m-1} ).$$

for all $x \in \mathbb{R}$. In particular we have

$$( -\Delta )^{1/2} u_{k'} \rightarrow ( -\Delta )^{1/2} u_\infty \text{ as } k' \rightarrow +\infty \text{ in } \dot{H}^{-1/2} ( B ( x, \rho / 4 ), S^{m-1} ).$$

Hence

$$( -\Delta )^{1/2} u_{k'} \wedge u_{k'} \rightarrow ( -\Delta )^{1/2} u_\infty \wedge u_\infty \text{ as } k' \rightarrow +\infty \text{ in } \mathcal{D}' ( B ( x, \rho / 4 ) ),$$

and

$$( -\Delta )^{1/2} u_\infty \wedge u_\infty = 0 \text{ in } \mathcal{D}' ( B ( x, \rho / 4 ) ).$$

3. **Claim 1:** There are only finitely many points $\{ a_1, \ldots, a_\ell \}$ such that

$$( -\Delta )^{1/2} u_\infty \wedge u_\infty = 0, \text{ in } \mathcal{D}' ( \mathbb{R} \setminus \{ a_1, \ldots, a_\ell \} )$$

(4) Suppose by contradiction that a distribution $\phi \in H^{-\frac{1}{2}}(\mathbb{R})$ satisfies $\phi = \sum_{|\alpha|\leq 1} c_\alpha \partial^\alpha \delta_\alpha$. We can write $\phi = ( -\Delta )^{1/4} f$ for some $f \in L^2(\mathbb{R})$. Then $\mathcal{F}[\phi](\xi) = |\xi|^{1/2} \mathcal{F}[f](\xi) = \sum_{|\alpha|\leq 1} c_\alpha (i)^{|\alpha|} \xi^\alpha$ and this is not possible since $\mathcal{F}[f] \in L^2(\mathbb{R})$. Therefore $c_\alpha = 0$. 

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Proof of the claim 1. We associate to every $x$ the number $\rho^n_x > 0$ such that $\|u_k\|_{H^{1/2}(B(x, \rho^n_x))} = \varepsilon_0$ where $\varepsilon_0$ is as in Lemma 2.1.

For every $M > 0$ and $n \geq 1$ we set

$$I^M_k := \{x : \rho^n_x < \frac{1}{M}\}$$

and

$$F^M_k := \{B(x, \frac{1}{M}) \cap \mathbb{R}^n : x \in I^M_k\}.$$

By Vitali-Besicovitch Covering Theorem (see for instance [13]), we can find an at most countable family of points $(x^j_{k,M})_{j \in I^M_k} \subset I^M_k$ and $I^M_k \subseteq \bigcup_{j \in I^M_k} B(x^j_{k,M}, \frac{1}{M})$.

Moreover every $x \in I^M_k$ is contained in at most $K$ balls, $K$ being a number depending only on the dimension of the space.

Now we observe that

$$C \geq \|u_k\|_{H^{1/2}(B)}^2 \geq \sum_{j \in I^M_k} \int \int_{B(x^j_{k,M}, \frac{1}{M}) \times B(x^j_{k,M}, \frac{1}{M})} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^2} \, dx \, dy \geq \sum_{j \in I^M_k} \varepsilon_0^2 = |J^M_k|\varepsilon_0^2.$$

Thus $|J^M_k| < +\infty$ for every $k$ and $M$ and this implies that for $k$ and $M$ large enough $|J^M_k| = C$, with $C$ independent on $k$ and $M$. In particular there exists $k_0 > 0$ such that

$$I^M_k \subseteq \bigcup_{j=1}^{k_0} B(x^n_{j,M}, \frac{1}{M}).$$

By definition we have $I^{M+1}_k \subseteq I^M_k$ for all $n$ and $M$. By using a diagonal procedure we can subtract a subsequence $k' \to +\infty$ such that $x^{k',M}_j \to x^\infty_j$ for all $M > 0$ and $j$ and

$$I^\infty_k \subseteq \bigcup_{j=1}^{k_0} B(x^\infty_{j,M}, \frac{1}{M}).$$

Now we let $M \to +\infty$ and get

$$I^0_\infty \subseteq J^\infty_0 := \{x^\infty_j\}_{j=1,...,n_0}.$$

Claim 2. If $x \notin J^\infty_0$ then there exits $\tilde{r} > 0$ such that

$$u_\infty \wedge (-\Delta)^{\frac{1}{2}} u_\infty = 0 \quad \text{in } D'(B(x, \tilde{r})).$$
Proof of the claim 2. We assume that $x_{j_0}^{\infty,0} \neq \infty$ for all $j = 1, \ldots, n_0$. Let $\gamma = \text{dist}(x, J_{\infty,0})$ and $K > 0$ be such that $2K^{-1} < \gamma$. Let $M > 0$ be such that for all $M \geq \hat{M}$ and for all $j = 1, \ldots, n_0$ we have

$$|x_j^{\infty,0} - x_j^{\infty,M}| < \frac{1}{4K}. \quad (31)$$

Let $k > 0$ be such that for all $k' \geq k$ and for all $j = 1, \ldots, n_0$ we have

$$|x_j^{\infty,M} - x_j^{k',M}| < \frac{1}{4K}. \quad (32)$$

By combining (31) and (32) we get

$$|x - x_j^{k',M}| \geq |x - x_j^{\infty,0}| - |x_j^{\infty,0} - x_j^{\infty,M}| - |x_j^{\infty,M} - x_j^{k',M}| \geq \frac{2}{K} - \frac{1}{4K} - \frac{1}{4K} = \frac{3}{2K} > \frac{1}{M}.$$ 

Therefore $x \notin \bigcup_{j=1}^{n_0} B(x_j^{k',M}, \frac{1}{M})$, and $x \notin I_j^{k'}$ for all $k \geq \hat{k}$. In particular $\rho_{k',x} \geq M^{-1}$ and (up to subsequence) $\rho_{k',x} \rightarrow \rho_{\infty,x} > 0$. Now let $0 < \tilde{r} < \rho_{\infty,x}$. Then

$$B(x, \tilde{r}) \subseteq B(x, \rho_{k',x}), \quad \text{for } k \text{ large}.$$ 

Since we have $\|u_k\|_{H^{1/2}(B(x, \rho_{k',x}))} = \varepsilon_0$, then by applying Step 2 we get

$$u_\infty \land (-\Delta)^{1/2} u_\infty = 0 \quad \text{in } D'(B(x, \tilde{r})).$$

This concludes the proof of the claim 2 by setting $a_i := x_i^{\infty}$ and $\ell = n_0$.

Now we apply Proposition 2.2 and we get that

$$u_\infty \land (-\Delta)^{1/2} u_\infty = 0 \quad \text{in } D'(\mathbb{R}).$$

We can conclude the proof of the Lemma 2.3 and the first part of Theorem 1.1. \qed

3 $L^2, \infty$ and $L^2$ estimates in degenerate annuli

In this Section we will prove some energy estimates of 1/2-harmonic maps in degenerate annuli. Such estimates are crucial in the next Section in order to get the quantization analysis in the neck regions.

The main result of this Section is
Theorem 3.1 There exists $\tilde{\delta} > 0$ such that for any $1/2$-harmonic maps $u \in \dot{H}^\frac{1}{2}(\mathbb{R}, S^{m-1})$, for any $\delta < \tilde{\delta}$ and $\lambda, \Lambda > 0$ with $\lambda < (2\Lambda)^{-1}$ satisfying
\[
\sup_{\rho \in [\lambda, (2\Lambda)^{-1}]} \left( \int_{B(0, 2\rho) \setminus B(0, \rho)} |(-\Delta)^{1/4} u|^2\, dx \right)^{1/2} \leq \delta,
\] (33)
we have
\[
\int_{B(0, \Lambda^{-1}) \setminus B(0, \lambda)} |(-\Delta)^{1/4} u|^2\, dx \leq C \sup_{\rho \in [\lambda, (2\Lambda)^{-1}]} \left( \int_{B(0, 2\rho) \setminus B(0, \rho)} |(-\Delta)^{1/4} u|^2\, dx \right)^{1/2}. \tag{34}
\]

The proof of Theorem 3.1 consists in three steps:
1) first we show that we can control in degenerate annuli the $L^q$ norm of $(-\Delta)^{1/4} u$ for some $q > 2$ by
\[
\sup_{\rho \in [\lambda, (2\Lambda)^{-1}]} \left( \int_{B(0, 2\rho) \setminus B(0, \rho)} |(-\Delta)^{1/4} u|^2\, dx \right)^{1/2} \leq C \sup_{\rho \in [\lambda, (2\Lambda)^{-1}]} \left( \int_{B(0, 2\rho) \setminus B(0, \rho)} |(-\Delta)^{1/4} u|^2\, dx \right)^{1/2}. \tag{35}
\]
2) then we estimate the $L^{2,\infty}$ norm of $(-\Delta)^{1/4} u$ in degenerate annuli in terms of (35),
3) finally we use the global $L^{2,1} - L^{2,\infty}$ estimates obtained in the appendix (see Theorem 1.3) and the duality $L^{2,1} - L^{2,\infty}$ in order to conclude.

Lemma 3.1 ($L^q$-estimates) There exists $\tilde{\delta} > 0$ such that for any $1/2$-harmonic maps $u \in \dot{H}^\frac{1}{2}(\mathbb{R})$, for any $\delta < \tilde{\delta}$, $\lambda, \Lambda > 0$ with $2\lambda < (4\Lambda)^{-1}$ such that
\[
\sup_{\rho \in [\lambda, (2\Lambda)^{-1}]} \left( \int_{B(0, 2\rho) \setminus B(0, \rho)} |(-\Delta)^{1/4} u|^2\, dx \right)^{1/2} \leq \delta. \tag{36}
\]
then there exists $q > 2$ (independent on $\lambda, \Lambda, u$) such that
\[
\sup_{\rho \in [2\lambda, (4\Lambda)^{-1}]} \left( \rho^{q/2-1} \int_{B(0, 2\rho) \setminus B(0, \rho)} |(-\Delta)^{1/4} u|^q\, dx \right)^{1/q} \leq C \sup_{\rho \in [\lambda, (2\Lambda)^{-1}]} \left( \int_{B(0, 2\rho) \setminus B(0, \rho)} |(-\Delta)^{1/4} u|^2\, dx \right)^{1/2}. \tag{37}
\]

Proof of Lemma 3.1. We choose $\delta = \frac{\varepsilon_0}{2}$ where $\varepsilon_0 > 0$ is the constant appearing in the $\varepsilon$-regularity Lemma 2.1.

Step 1. There exist $p > 2$ (independent on $\lambda, \Lambda, u$) such that
\[
\sup_{\rho \in [2\lambda, (4\Lambda)^{-1}]} \left( \rho^{p/2-1} \int_{B(0, 2\rho) \setminus B(0, \rho)} |(-\Delta)^{1/4} u|^p\, dx \right)^{1/p} \leq C \|u\|_{\dot{H}^\frac{1}{2}(\mathbb{R})}. \tag{37}
\]
Proof of Step 1.
Let \( r > 0 \) be such that
\[
\left( \int_{B(0,2r) \setminus B(0,r)} \left| (-\Delta)^{1/4} u \right|^p \, dx \right)^{1/p} < \delta.
\]

Claim: There exists \( p > 2 \) (independent on \( \delta \) and \( r \)) such that
\[
\left( r^{p/2-1} \int_{B(0,\frac{3}{2}r) \setminus B(0,\frac{5}{4}r)} \left| (-\Delta)^{1/4} u \right|^p \, dx \right)^{1/p} \leq C \| u \|_{H^{1/2}(\mathbb{R})}.
\] (38)

Let \( y \in B(0, \frac{3}{2}r) \setminus B(0, \frac{5}{4}r) \), (we clearly have \( \text{dist}(y, \partial(B(0, 2r) \setminus B(0, r))) \geq 1/4 \)).

Let \( j_0 \geq 3 \) such that \( 2^{-j_0/2} \left( \int_{\mathbb{R}^n} \left| (-\Delta)^{1/4} u \right|^2 \, dx \right)^{1/2} \leq \delta \), and \( B(y, 2^{-j_0}r) \subset (B(0, 2r) \setminus B(0, r)) \) for all \( y \in B(0, \frac{3}{2}r) \setminus B(0, \frac{5}{4}r) \).

Estimate of \( \sum_{h \geq 0} 2^{-h/2} \| (-\Delta)^{1/4} u \|_{L^2(B(0,2^h(2^{-3j_0}r)))} \cdot \)
\[
\sum_{h \geq 0} 2^{-h/2} \| (-\Delta)^{1/4} u \|_{L^2(B(0,2^h(2^{-3j_0}r)))}
= \sum_{h=0}^{j_0} 2^{-h/2} \| (-\Delta)^{1/4} u \|_{L^2(B(0,2^h(2^{-3j_0}r)))} + \sum_{h=j_0+1}^{\infty} 2^{-h/2} \| (-\Delta)^{1/4} u \|_{L^2(B(0,2^h(2^{-3j_0}r)))}
\leq \delta \sum_{h=0}^{\infty} 2^{-h/2} + 2^{-(j_0+1)/2} \left( \int_{\mathbb{R}^n} \left| (-\Delta)^{1/4} u \right|^2 \, dx \right)^{1/2}
\leq \delta + \delta = 2\delta = \varepsilon_0.
\]

Now we apply Lemma 2.1: there exists \( p > 2 \) and \( C_{j_0} > 0 \) such that
\[
\left( r^{p/2-1} \int_{B(y,2^{-j_0+1}r)} \left| (-\Delta)^{1/4} u \right|^p \, dx \right)^{1/p} \leq C_{j_0} \| u \|_{H^{1/2}(\mathbb{R})}.
\] (39)

By covering the annulus \( B(0, \frac{3}{2}r) \setminus B(0, \frac{5}{4}r) \) by a finite number of balls \( B(y, 2^{-j_0+1}r) \) we finally get
\[
\left( r^{p/2-1} \int_{B(0,\frac{3}{2}r) \setminus B(0,\frac{5}{4}r)} \left| (-\Delta)^{1/4} u \right|^p \, dx \right)^{1/p} \leq \tilde{C}_{j_0} \| u \|_{H^{1/2}(\mathbb{R})}^2,
\]
and the proof of Claim 1 is concluded.

Hence
\[
\sup_{\rho \in [2\lambda, (4\lambda)^{-1}]} \left( r^{p/2-1} \int_{B(0,2\rho) \setminus B(0,\rho)} \left| (-\Delta)^{1/4} u \right|^p \, dx \right)^{1/p} \leq C \| u \|_{H^{1/2}(\mathbb{R})}.
\] (40)
We thus conclude the proof of Step 1.

**Step 2.** There exists \( q > 2 \) (independent on \( \lambda, \Lambda, u \) and dependent on \( p \)) such that

\[
\sup_{\rho \in [2\lambda,(4\lambda)^{-1}]} \left( \frac{\rho^{q/2-1}}{\rho^q} \int_{B(0,2\rho)\setminus B(0,\rho)} |(-\Delta)^{1/4}u|^q \, dx \right)^{1/q} \\
\leq C \sup_{\rho \in [\lambda,(2\lambda)^{-1}]} \left( \int_{B(0,2\rho)\setminus B(0,\rho)} |(-\Delta)^{1/4}u|^2 \, dx \right)^{1/2}.
\]

**Proof of Step 2.** Let us take \( q^{-1} = \theta p^{-1} + (1 - \theta)2^{-1} \). Then by Hölder Inequality and by using (39) we get

\[
\left( \frac{\rho^{q/2-1}}{\rho^q} \int_{B(0,2\rho)\setminus B(0,\rho)} |(-\Delta)^{1/4}u|^p \, dx \right)^{1/q} \\
\leq \left( \frac{\rho^{q/2-1}}{\rho^q} \int_{B(0,2\rho)\setminus B(0,\rho)} |(-\Delta)^{1/4}u|^p \, dx \right)^{1/p} \left( \int_{B(0,2\rho)\setminus B(0,\rho)} |(-\Delta)^{1/4}u|^2 \, dx \right)^{1/2} \\
\leq C \|u\|_{\dot{H}^{\frac{1}{4}}(\mathbb{R})} \sup_{\rho \in [\lambda,(2\lambda)^{-1}]} \left( \int_{B(0,2\rho)\setminus B(0,\rho)} |(-\Delta)^{1/4}u|^2 \, dx \right)^{1/2} \\
\leq C \sup_{\rho \in [\lambda,(2\lambda)^{-1}]} \left( \int_{B(0,2\rho)\setminus B(0,\rho)} |(-\Delta)^{1/4}u|^2 \, dx \right)^{1/2}.
\]

This concludes the proof of Step 2 and of Lemma 3.1.

**Lemma 3.2 (\( L^{2,\infty} \) estimates)** There exists \( \tilde{\delta} > 0 \) such that for any \( 1/2 \)-harmonic maps \( u \in \dot{H}^{\frac{1}{4}}(\mathbb{R}) \), for any \( \delta < \tilde{\delta} \) and \( \lambda, \Lambda > 0 \) with \( \lambda < (2\Lambda)^{-1} \) satisfying

\[
\sup_{\rho \in [\lambda,(2\lambda)^{-1}]} \left( \int_{B(0,2\rho)\setminus B(0,\rho)} |(-\Delta)^{1/4}u|^2 \, dx \right)^{1/2} \leq \delta
\]

then

\[
\|(-\Delta)^{1/4}u\|_{L^{2,\infty}(B(0,(2\lambda)^{-1})\setminus B(0,\lambda))} \leq C \sup_{\rho \in [\lambda,(2\lambda)^{-1}]} \left( \int_{B(0,2\rho)\setminus B(0,\rho)} |(-\Delta)^{1/4}u|^2 \, dx \right)^{1/2}.
\]

where \( C \) is independent on \( \rho, u, \lambda, \Lambda \).

**Proof of Lemma 3.2.** We set \( f = (-\Delta)^{1/4}u \) in \( B(0,(4\Lambda)^{-1}) \setminus B(0,2\lambda) \) and \( f = 0 \) otherwise.

Let \( \delta < \tilde{\delta}/4 \) where \( \tilde{\delta} \) is the constant appearing in Theorem 3.1. From Lemma 3.1 it follows that for all \( \lambda, \Lambda > 0 \) with \( 2\lambda < (4\Lambda)^{-1} \) if

\[
\sup_{\rho \in [\lambda,(2\lambda)^{-1}]} \left( \int_{B(0,2\rho)\setminus B(0,\rho)} |(-\Delta)^{1/4}u|^2 \, dx \right)^{1/2} \leq \delta
\]
then there exists \( q > 2 \), such that

\[
\sup_{\rho \in [2\lambda, (4\Lambda)^{-1}]} \left( \rho^{q/2-1} \int_{B(0,2\rho) \setminus B(0,\rho)} \left| (-\Delta)^{1/4} u \right|^q \right)^{1/q} 
\leq C \sup_{\rho \in [\lambda, (2\Lambda)^{-1}]} \left( \int_{B(0,2\rho) \setminus B(0,\rho)} \left| (-\Delta)^{1/4} u \right|^2 \right)^{1/2}.
\]

We set

\[
\gamma = C \sup_{\rho \in [2\lambda, (4\Lambda)^{-1}]} \left( \int_{B(0,2\rho) \setminus B(0,\rho)} \left| (-\Delta)^{1/4} u \right|^2 \right)^{1/2}
\]

We observe that for all \( \rho \in [2\lambda, (4\Lambda)^{-1}] \) one has:

\[
\gamma^q \geq \rho^{q/2-1} \int_{B(0,2\rho) \setminus B(0,\rho)} |f|^q \, dx 
\geq \rho^{q/2-1} \alpha^q \left| \{ x \in B(0,2\rho) \setminus B(0,\rho) : |f| > \alpha \} \right|.
\]

Let \( k \in \mathbb{Z} \), then the following estimate holds

\[
\alpha^2 \sum_{j \geq k} \left| \{ x \in B(0,2^{j+1}\alpha^{-2}) \setminus B(0,2^j\alpha^{-2}) : |f| > \alpha \} \right|
\leq \alpha^2 \sum_{j \geq k} \left( \frac{2^{j+1}\alpha^{-2}}{\alpha^q} \right)^{1-q/2} \gamma^q = \gamma^q \sum_{j \geq k} 2^{(j+1)(1-q/2)}
\leq \gamma^q 2^{1-q/2} 2^k (1 - 2^{1-q/2})^{-1}
= \gamma^q 2^k (1-q/2) (2^{q/2-1} - 1)^{-1}.
\]

Therefore

\[
\alpha^2 \left| \{ x \in \mathbb{R} : |f| > \alpha \} \right| \leq \gamma^q 2^{1-q/2} 2^k (1-q/2) + \alpha^2 |B(0,2^k\alpha^{-2})|
\leq \gamma^q 2^k (1-q/2) (2^{q/2-1} - 1)^{-1} + \alpha^2 2^k \alpha^{-2}.
\]

Now we choose \( k \) in such a way that \( 2^k = \gamma^q/2 \). It follows that

\[
\alpha^2 \left| \{ x \in \mathbb{R} : |f| > \alpha \} \right| \leq \frac{\gamma^2}{2} (2^{q/2-1} - 1)^{-1} + \gamma^2 = \frac{2^{q/2} - 1}{2^{q/2} - 2} \gamma^2.
\]
Hence

\[ \|(-\Delta)^{1/4} u\|_{L^2,\infty(B(0, (4\Lambda)^{-1}) \setminus B(0, 2\lambda))} \]

\[ = \sup_{\alpha > 0} (\alpha^2 \{ x \in B(0, (4\Lambda)^{-1}) \setminus B(0, 2\lambda) : |(-\Delta)^{1/4} u(x) > \alpha \})^{1/2} \quad (43) \]

\[ \leq \left( \frac{2q/2 - 1}{2q/2 - 2} \right)^{1/2} \gamma. \]

By combining (43) and the fact that the \( L^\infty \) norms of \((-\Delta)^{1/4} u \) in the annuli \( B(0, (4\Lambda)^{-1}) \setminus B(0, 2\lambda) \) and \( B(0, 2\lambda) \setminus B(0, \lambda) \) are controlled by the respective \( L^2 \) norms we get the estimate (42) and we conclude the proof of the Lemma 3.2.

Now we can prove Theorem 3.1.

**Proof of Theorem 3.1.** From Theorem 1.3 it follows that any \( 1/2 \)-harmonic map \( u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1}) \) satisfies \( \|(-\Delta)^{1/4} u\|_{L^{2,1}(\mathbb{R})} \leq C \) where \( C \) depends on \( \|u\|_{\dot{H}^{1/2}(\mathbb{R}, S^{m-1})} \).

Now it is enough to use the duality \( L^{2,1} - L^\infty \) and Lemma 3.2 to get

\[ \int_{B(0, (\Lambda)^{-1}) \setminus B(0, \lambda)} \|(-\Delta)^{1/4} u\|^2 dx \leq \|u\|_{\dot{H}^{1/2}(\mathbb{R})} \|(-\Delta)^{1/4} u\|_{L^{2,\infty}(B(0, (2\Lambda)^{-1}) \setminus B(0, \lambda))} \]

\[ \leq C \sup_{\rho \in [(\Lambda, (2\Lambda)^{-1})]} \left( \int_{B(0, 2\rho) \setminus B(0, \rho)} \|(-\Delta)^{1/4} u\|^2 dx \right)^{1/2}. \]

We can conclude the proof of Theorem 3.1.

Now we can prove Theorem 3.1. Form Theorem 1.3 it follows that any \( 1/2 \)-harmonic map \( u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1}) \) satisfies \( \|(-\Delta)^{1/4} u\|_{L^{2,1}(\mathbb{R})} \leq C \) where \( C \) depends on \( \|u\|_{\dot{H}^{1/2}(\mathbb{R}, S^{m-1})} \).

Now it is enough to use the duality \( L^{2,1} - L^\infty \) and Lemma 3.2 to get

\[ \int_{B(0, (\Lambda)^{-1}) \setminus B(0, \lambda)} \|(-\Delta)^{1/4} u\|^2 dx \leq \|u\|_{\dot{H}^{1/2}(\mathbb{R})} \|(-\Delta)^{1/4} u\|_{L^{2,\infty}(B(0, (2\Lambda)^{-1}) \setminus B(0, \lambda))} \]

\[ \leq C \sup_{\rho \in [(\Lambda, (2\Lambda)^{-1})]} \left( \int_{B(0, 2\rho) \setminus B(0, \rho)} \|(-\Delta)^{1/4} u\|^2 dx \right)^{1/2}. \]

We can conclude the proof of Theorem 1.1.

**4 Bubbles and neck-regions**

In the proof of the first part of Theorem 1.1 (see Lemma 2.3) we have shown (up to a subsequence) that

\[ |(-\Delta)^{1/4} u_k|^2 \rightharpoonup |(-\Delta)^{1/4} u_\infty|^2 dx + \sum_{i=1}^\ell \lambda_i \delta_{a_i}, \text{ in Radon measure}. \]

The aim of this Section is to show that for every \( i \in \{1, \ldots, \ell \} \) there exist bubbles \( (\tilde{u}_\infty^{i,j}) \), \( j \in \{1, \ldots, N_i\} \) such that \( \lambda_i = \sum_{j=1}^{N_i} \int_{\mathbb{R}} |(-\Delta)^{1/4} \tilde{u}_\infty^{i,j}|^2 dx \).

We first give the following definitions.

**Definition 4.1 (Bubble)** A Bubble is a non-constant \( 1/2 \)-harmonic map \( u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1}) \).
Definition 4.2 (Neck region) A neck region for a function \( f \in L^2(I, R) \) is the union of finite degenerate annuli of the type \( A_k(x) = B(x, R_k) \setminus B(x, r_k) \) with \( r_k \to 0 \) and \( \frac{R_k}{r_k} \to +\infty \) as \( k \to +\infty \) satisfying the following property: for all \( \delta > 0 \) there exists \( \Lambda > 0 \) such that
\[
\left( \sup_{\rho \in [2r_k, 2\Lambda]^{-1}R_k} \int_{B(x, 2\rho) \setminus B(x, \rho)} |f|^2 \, dx \right)^{1/2} \leq \delta.
\]

Proof of the second part of Theorem 1.1.
We have to show that there is a family \( \tilde{u}_{i,j}^\infty \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1}) \), of non-constant \( 1/2 \)-harmonic maps \( (i \in \{1, \ldots, \ell \}, j \in \{1, \ldots, N_i \}) \), such that up to subsequence
\[
\|(-\Delta)^{1/4}(u_k - u_\infty - \sum_{i,j} \tilde{u}_{i,j}^\infty)\|_{L^2_{\text{loc}}(\mathbb{R})} \to 0, \quad \text{as } n \to \infty. \tag{44}
\]

We first observe that the bootstrap test \([13]\) implies that each bubble has a bounded from below energy \( c_0 > 0 \). Therefore for every \( i, N_i < +\infty \).

For simplicity we assume that \( \ell = 1 \) and that there are at most two bubbles.

Now let us take \( \delta < \tilde{\delta} \) such that \( C\delta < \varepsilon_0 \) (the constants \( C \) and \( \tilde{\delta} \) are the one appearing in the statement of Theorem 3.1).

We also set \( \gamma = \min\left(\frac{\delta}{2}, \frac{\varepsilon_0}{2}\right) \).

Step 1. For every \( n \geq 1 \) we set
\[
\rho_k^1 = \inf\{\rho > 0 : \exists x \in B(a_1, 1) : \int_{B(x, \rho)} |(-\Delta)^{1/4}u_k|^2 \, dx = \gamma\}
\]

There are two cases:

Case 1.: \( \lim \inf_{k \to +\infty} \rho_k^1 > 0 \)

In this case there is not concentration of the energy, namely \( \lambda_1 = 0 \).

Case 2.: \( \lim_{k \to +\infty} \rho_k^1 = 0 \).

For every \( k \geq 1 \), let \( x_1^k \in B(a_1, 1) \) be the point such that
\[
\int_{B(x_1^k, \rho_k^1)} |(-\Delta)^{1/4}u_k|^2 \, dx = \gamma. \quad \text{We have (up to subsequence) } x_1^k \to a_1 \text{ as } k \to +\infty.
\]

Now we choose a subsequence of \( u_k \) (that we still denote by \( u_k \)) and a fixed radius \( \alpha > 0 \) such that
\[
\limsup_{n \to \infty} \left[ \sup_{0 < r < \alpha} \left\{ \int_{B(a_1, \alpha) \setminus B(a_1, r)} |(-\Delta)^{1/4}u_k(y)|^2 \, dy = \gamma \right\} \right] = 0.
\]

Now we borrow the idea in \([3]\) to split the annulus \( B(x_1^k, \alpha) \setminus B(x_1^k, \rho_k^1) \) in domains of unbounded conformal class where the energy is small and domains of bounded conformal class where the energy is bounded from below.

Precisely by applying Lemma 3.2 in \([3]\), we can find a sequence of family of radii
\[
R_0^j = \alpha > R_1^j > \ldots > R_{N_i}^j = \rho_k^1
\]
with \( \{1, \ldots, N_1\} = I_0 \cup I_1 \). For every \( i_\ell \in I_0 \) one has
\[
\lim_{k \to +\infty} \log \left( \frac{R_k^{i_\ell}}{R_k^{i_\ell+1}} \right) < +\infty \quad \text{and} \quad \int_{B(x_{1,k}, R_k^{i_\ell+1}) \setminus B(x_{1,k}, R_k^{i_\ell})} |(\Delta)^{1/4} \tilde{u}_k(y)|^2 \, dy \geq \gamma, 
\]
and for every \( i_\ell \in I_1 \), one has
\[
\lim_{k \to +\infty} \log \left( \frac{R_k^{i_\ell}}{R_k^{i_\ell+1}} \right) = +\infty \quad \text{and} \quad \forall \rho \in (R_k^{i_\ell}, R_k^{i_\ell+1}/2), \quad \int_{B(x_{1,k}, 2\rho) \setminus B(x_{1,k}, \rho)} |(\Delta)^{1/4} \tilde{u}_k(y)|^2 \, dy \leq 2\gamma.
\]
We consider the smallest annulus \( A_k^{i_\ell} := B(x_{1,k}, R_k^{i_\ell}) \setminus B(x_{1,k}, R_k^{i_\ell+1}) \) of the first type \( i_\ell \in I_0 \). For such an \( i_\ell \) we define
\[
r_k^{i_\ell} = \inf \{ r < R_k^{i_\ell+1} : \exists x \in A_k^{i_\ell} : \int_{B(x,r)} |(\Delta)^{1/4} u_k|^2 \, dx = \gamma \}.
\]
We consider the following two cases.

**Case 1.** There exists a subsequence of \( r_k^{i_\ell} \) such that
\[
\lim_{k \to +\infty} \frac{r_k^{i_\ell}}{R_k^{i_\ell}} > 0.
\]
In this case there is not concentration of the energy in \( A_k^{i_\ell} \) and we pass to the next \( A_k^{i_\ell} \) (if there is any).

**Case 2.** We have
\[
\lim_{k \to +\infty} \frac{r_k^{i_\ell}}{R_k^{i_\ell}} = 0.
\]
In this case we have once again concentration. Let \( x_{2,k} \in A_k^{i_\ell} \) such that
\[
\int_{B(x_{2,k}, r_k^{i_\ell})} |(\Delta)^{1/4} u_k|^2 \, dx = \gamma.
\]
and we set \( \rho_k^{i_\ell} = r_k^{i_\ell} \).

We separate two sub-cases:

**Case of two “separated” bubbles** \( \lim \inf_{k \to +\infty} \frac{\rho_k^{i_\ell}}{\rho_k^{i_\ell}} > 0 \). In this case the following two conditions hold
\[
\begin{cases}
\lim_{n \to \infty} \frac{|x_{1,k} - x_{2,k}|}{\rho_k^{i_\ell}} = +\infty \\
\lim_{n \to \infty} \frac{|x_{1,k} - x_{2,k}|}{\rho_k^{i_\ell}} = +\infty.
\end{cases}
\]
In this case the bubbles \( \tilde{u}_{2,\infty} \) and \( \tilde{u}_{1,\infty} \) are “independent”.
Let us consider the two “separated” balls $B(x_{1,k}, \rho_{1,k}^k)$ and $B(x_{2,k}, \rho_{2,k}^k)$, with

\[
\lim_{k \to \infty} \frac{|x_{1,k} - x_{2,k}|}{\rho_{1,k}^k + \rho_{2,k}^k} = +\infty.
\]

For every $\alpha$ we set

\[
N_{1,k}^\alpha = B(a_1, \alpha) \setminus \left( B(x_{1,k}, \alpha^{-1}\rho_{1,k}^k) \cup B(x_{2,k}, \alpha^{-1}\rho_{2,k}^k) \right).
\]

The above construction gives the existence of $\alpha$ small enough independent of $k$ such that

\[
\begin{cases}
\text{for } j = 1, 2 \text{ and for all } \rho \text{ such that } B(x_{j,k}, 2\rho) \setminus B(x_{j,k}, 2\rho) \subseteq N_{1,k}^\alpha(\alpha) \\
\int_{B(x_{j,k}, 2\rho) \setminus B(x_{j,k}, \rho)} |(\Delta)^{1/4} u_k|^2 dx \leq 2\gamma \\
\text{and} \\
\int_{B(x_{j,k}, \rho)} |(\Delta)^{1/4} u_k|^2 dx = \gamma.
\end{cases}
\]

Claim: the region $N_{1,k}^\alpha(\alpha)$ is a neck-region.

Proof of the Claim: it is a consequence of the following general property.

Lemma 4.1 Let $A_k = B(x_k, R_k) \setminus B(x_k, r_k)$ an annulus satisfying $r_k \to 0$, $\frac{R_k}{r_k} \to +\infty$ and $x_k \to x_{\infty}$ as $k \to +\infty$, and

\[
\sup_{r_k \leq \rho \leq \frac{R_k}{r_k}} \int_{B(x_k, 2\rho) \setminus B(x_k, \rho)} |(\Delta)^{1/4} u_k|^2 dx \leq 2\gamma.
\]

Then for all $\eta > 0$ there exists $\Lambda > 0$ such that

\[
\sup_{r \in \Lambda \leq r \leq \frac{1}{\Lambda} \cdot R_k} \int_{B(x_n, 2r) \setminus B(x_n, r)} |(\Delta)^{1/4} u_k|^2 dx \leq \eta.
\]

Proof of Lemma 4.1. Suppose by contradiction that there exists $\eta > 0$ and two sequence $\Lambda_k \to +\infty$ as $k \to +\infty$ and $\Lambda_k r_k \leq \tilde{r}_k \leq (\Lambda_k)^{-1} R_k$ such that

\[
\int_{B(x_k, 2\tilde{r}_k) \setminus B(x_k, \tilde{r}_k)} |(\Delta)^{1/4} u_k|^2 dx > \eta
\]

We define $\tilde{u}_k(y) = u(\tilde{r}_ky + x_{1,k})$. From condition (47) and Theorem 3.1 it follows that

\[
\int_{B(0, \frac{\tilde{r}_k}{r_k}) \setminus B(0, \frac{\tilde{r}_k}{R_k})} |(\Delta)^{1/4} \tilde{u}_k|^2 dx \leq 2\gamma.
\]
Lemma 2.1 and Lemma 2.3 imply that \( \tilde{u}_k \rightarrow \tilde{u}_\infty \) in \( W^{1/2,p}_{loc}(\mathbb{R}\setminus \{0\}) \) for all \( p \geq 1 \), where \( \tilde{u}_\infty \) is a nontrivial 1/2-harmonic maps \( \int_{B(0,2)\setminus B(0,1)} |(-\Delta)^{1/4} \tilde{u}_\infty|^2 dx > \eta \). On the other hand the condition (17) gives

\[
\int_{\mathbb{R}} |(-\Delta)^{1/4} \tilde{u}_\infty|^2 dx \leq C\delta < \varepsilon_0.
\]

The bootstrap test yields that \( \tilde{u}_\infty \) is trivial which is a contradiction.

We conclude the proof of Lemma 4.1 and of the claim. \( \square \)

By applying Theorem 1.1 we get that for all \( \eta > 0 \) small enough

\[
\int_{\mathbb{R}} |(-\Delta)^{1/4} u_k|^2 dx \leq \eta.
\]

**Case of bubble over bubble.** \( \liminf_{k \to +\infty} \frac{\rho_2^4}{\rho_1^4} = 0 \). We define \( \tilde{u}_{2,k}(y) = u(\rho_2^2 y + x_{2,k}) \) for all \( p \geq 1 \) and \( \int_{B(0,2)\setminus B(0,1)} |(-\Delta)^{1/4} \tilde{u}_{2,\infty}|^2 dx \geq \delta \). Therefore \( \tilde{u}_{2,\infty} \) is a new bubble (case Bubble over Bubble: the bubble \( \tilde{u}_{2,\infty} \) contains \( \tilde{u}_{1,\infty} \) and for \( k \) large enough \( x_{1,k} \in B(x_{1,2}, \rho_k^2) \)). For every \( \alpha \) we set

\[
N_{k}^{1,2}(\alpha) = (B(x_1,k, \alpha \rho_{-1}^2) \setminus B(x_1,k, \alpha^{-1} \rho_k^4))
\]

and

\[
N_k(\alpha) = N_k^{1}(\alpha) \cup N_k^{1,2}(\alpha).
\]

By arguing as above one can show that \( N_k(\alpha) \) is a neck region.

Since we have assumed that there are at most two bubbles, the procedure stops here. Otherwise one has to continue the procedure until annuli of the type \( I_0 \) have been explored.

Therefore for every \( \eta > 0 \) we get

1. case of independent bubbles:

\[
\lim_{k \to +\infty} \int_{\mathbb{R}} |(-\Delta)^{1/4} u_k|^2 dx = \lim_{k \to +\infty} \int_{N_k^{1}(\alpha)} |(-\Delta)^{1/4} u_k|^2 dx
\]

\[
+ \sum_{j=1}^{2} \lim_{k \to +\infty} \int_{B(x_j,k, \alpha^{-1} \rho_k^4)} |(-\Delta)^{1/4} u_k|^2 dx
\]

\[
+ \lim_{k \to +\infty} \int_{\mathbb{R} \setminus B(a_1, \alpha)} |(-\Delta)^{1/4} u_k|^2 dx
\]

\[
\leq \eta + \sum_{j=1}^{2} \int_{B(0, \alpha^{-1})} |\Delta^{1/4} \tilde{u}_j|^2 dx
\]

\[
+ \int_{\mathbb{R} \setminus B(a_1, \alpha)} |(-\Delta)^{1/4} u_\infty|^2 dx.
\]
2. case of bubble over bubble

\[ \lim_{k \to +\infty} \int_{\mathbb{R}} |(-\Delta)^{1/4} u_k|^2 dx \leq \lim_{k \to +\infty} \int_{N_k} |(-\Delta)^{1/4} u_k|^2 dx \]

\[ + \lim_{k \to +\infty} \left[ \int_{B(x_k, \alpha^{-1} \rho_k^2) \setminus B(x_k, \alpha \rho_k^2)} |(-\Delta)^{1/4} u_k|^2 dx + \int_{B(x_k, \alpha^{-1} \rho_k^2) \setminus B(x_k, \alpha \rho_k^2)} |(-\Delta)^{1/4} u_k|^2 dx \right] \]

\[ + \lim_{k \to +\infty} \int_{\mathbb{R} \setminus B(0, \alpha \rho_k^2)} |(-\Delta)^{1/4} \tilde{u}_k|^2 dx + \int_{\mathbb{R} \setminus B(0, \alpha \rho_k^2)} |(-\Delta)^{1/4} \tilde{u}_k|^2 dx \]

By taking in (51) and (52) the lim for \( \alpha, \eta \to 0 \) we get the desired quantization estimate (44). This concludes the proof of the second part of Theorem 1.1 \( \blacksquare \).

A Commutator estimates: Proof of Theorem 1.3

In this Section we prove Theorem 1.3. To this end we shall make use of the Littlewood-Paley dyadic decomposition of unity that we recall here. Such a decomposition can be obtained as follows. Let \( \phi(\xi) \) be a radial Schwartz function supported in \( \{\xi \in \mathbb{R}^n : |\xi| \leq 2\} \), which is equal to 1 in \( \{\xi \in \mathbb{R}^n : |\xi| \leq 1\} \). Let \( \psi(\xi) \) be the function given by

\[ \psi(\xi) := \phi(\xi) - \phi(2\xi) . \]

\( \psi \) is then a "bump function" supported in the annulus \( \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\} \).

Let \( \psi_0 = \phi \), \( \psi_j(\xi) = \psi(2^{-j} \xi) \) for \( j \neq 0 \). The functions \( \psi_j \), for \( j \in \mathbb{Z} \), are supported in \( \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\} \) and they realize a dyadic decomposition of the unity:

\[ \sum_{j \in \mathbb{Z}} \psi_j(x) = 1 . \]

We further denote

\[ \phi_j(\xi) := \sum_{k=-\infty}^{j} \psi_k(\xi) . \]

The function \( \phi_j \) is supported on \( \{\xi, |\xi| \leq 2^{j+1}\} \).

For every \( j \in \mathbb{Z} \) and \( f \in \mathcal{S}'(\mathbb{R}) \) we define the Littlewood-Paley projection operators \( P_j \) and \( P_{\leq j} \) by

\[ \widehat{P_j f} = \psi_j \hat{f} \quad \widehat{P_{\leq j} f} = \phi_j \hat{f} . \]
Informally $P_j$ is a frequency projection to the annulus $\{2^{j-1} \leq |\xi| \leq 2^j\}$, while $P_{\leq j}$ is a frequency projection to the ball $\{|\xi| \leq 2^j\}$. We will set $f_j = P_j f$ and $f^j = P_{\leq j} f$.

We observe that $f^j = \sum_{k=-\infty}^{+\infty} f_k$ and $f = \sum_{k=\infty}^{-\infty} f_k$ (where the convergence is in $S'(\mathbb{R})$).

Given $f, g \in S'(\mathbb{R})$ we can split the product in the following way

$$fg = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g),$$

where

$$\Pi_1(f, g) = \sum_{-\infty}^{+\infty} f_j \sum_{k \leq j-4} g_k = \sum_{-\infty}^{+\infty} f_j g^{j-4};$$

$$\Pi_2(f, g) = \sum_{-\infty}^{+\infty} f_j \sum_{k \geq j+4} g_k = \sum_{-\infty}^{+\infty} g_j f^{j-4};$$

$$\Pi_3(f, g) = \sum_{-\infty}^{+\infty} f_j \sum_{|k-j| < 4} g_k.$$

We observe that for every $j$ we have

$$\text{supp} \mathcal{F}[f^j g] \subset \{2^{j-2} \leq |\xi| \leq 2^{j+2}\};$$

$$\text{supp} \mathcal{F} \left[ \sum_{k=j-3}^{j+3} f_j g_k \right] \subset \{|\xi| \leq 2^{j+5}\}.$$

The three pieces of the decomposition (53) are examples of paraproducts. Informally the first paraproduct $\Pi_1$ is an operator which allows high frequencies of $f$ ($\sim 2^j$) multiplied by low frequencies of $g$ ($\ll 2^j$) to produce high frequencies in the output. The second paraproduct $\Pi_2$ multiplies low frequencies of $f$ with high frequencies of $g$ to produce high frequencies in the output. The third paraproduct $\Pi_3$ multiply high frequencies of $f$ with high frequencies of $g$ to produce comparable or lower frequencies in the output. For a presentation of these paraproducts we refer to the reader for instance to the book [17].

The following two Lemmae will be often used in the sequel.

**Lemma A.1** For every $f \in S'$ we have

$$\sup_{j \in \mathbb{Z}} |f^j| \leq M(f).$$

**Lemma A.2** Let $\psi$ be a Schwartz radial function such that $\text{supp}(\psi) \subset B(0, 4)$. Then for every $s \geq \left[ \frac{n}{2} \right] + 1$ we have

$$\|(-\Delta)^s \mathcal{F}^{-1} \psi\|_{L^1} \leq C_{\psi, n}(1 + s^{n+1})4^{2s},$$

where $C_{\psi, n}$ is a positive constant depending on the $C^2$ norm of $\psi$ and the dimension.
Lemma A.3 Let \( f \in B^0_{\infty, \infty}(\mathbb{R}^n) \). Then for all \( s \geq \left[ \frac{n}{2} \right] + 1 \) and for all \( j \in \mathbb{Z} \) we have
\[
2^{-2^j \| (-\Delta)^s f_j \|_{L^\infty}} \leq C_{\psi, n}(1 + s^{n+1})4^{2^j} \| f \|_{B^0_{\infty, \infty}(\mathbb{R}^n)}.
\]

For the proof of Lemma A.3 we refer to [9] and of Lemmata A.2 and A.3 we refer to [6].

Given \( u, Q \) we introduce the following pseudodifferential operators
\[
T(Q, u) := (-\Delta)^{1/4}(Q(-\Delta)^{1/4}u) - Q(-\Delta)^{1/2}u + (-\Delta)^{1/4}u(-\Delta)^{1/4}Q \quad (54)
\]
and
\[
S(Q, u) := (-\Delta)^{1/4}[Q(-\Delta)^{1/4}u] - \mathcal{R}(Q\nabla u) + \mathcal{R}((-\Delta)^{1/4}Q\mathcal{R}(-\Delta)^{1/4}u) \quad (55)
\]
and \( \mathcal{R} \) is the Fourier multiplier of symbol \( m(\xi) = i\frac{\xi}{|\xi|} \). We prove in this Section some estimates on the operators (54) and (55).

**Proof of Theorem 1.3**

We make the proof for \( n = 1 \). The case \( n > 1 \) is analogous (for the details we refer to [7]).

- Estimate of \( \| \Pi_1((-\Delta)^{1/4}(Q(-\Delta)^{1/4}u)) \|_{H^1} \).

\[
\| \Pi_1((-\Delta)^{1/4}(Q(-\Delta)^{1/4}u)) \|_{H^1} = \int_{\mathbb{R}^n} \left( \sum_{j=0}^{\infty} 2^j Q_j^2((-\Delta)^{1/4}u_j^{-4})^2 \right)^{1/2} dx \quad (56)
\]
\[
\leq \int_{\mathbb{R}^n} \sup_j \| (-\Delta)^{1/4}u_j^{-4}|(\sum_j 2^j Q_j^2)^{1/2} dx \quad (57)
\]
\[
\leq \left( \int_{\mathbb{R}^n} (M((-\Delta)^{1/4}u_j)^2 dx)^{1/2} \right)^{1/2} \left( \int_{\mathbb{R}^n} \sum_j 2^j Q_j^2 dx \right)^{1/2}
\]
\[
\leq C \| Q \|_{H^{1/2}(\mathbb{R})} \| u \|_{H^{1/2}(\mathbb{R})}.
\]

- Estimate of \( \| \Pi_1((-\Delta)^{1/4}Q(-\Delta)^{1/4}u) \|_{H^1} \).

\[
\| \Pi_1((-\Delta)^{1/4}Q(-\Delta)^{1/4}u) \|_{H^1} = \int_{\mathbb{R}^n} \left( \sum_{j=0}^{\infty} ((-\Delta)^{1/4}Q J_j)^2((-\Delta)^{1/4}u_j^{-4})^2 \right)^{1/2} dx \quad (58)
\]
\[
\leq \int_{\mathbb{R}^n} \sup_j \| (-\Delta)^{1/4}u_j^{-4}|(\sum_j ((-\Delta)^{1/4}Q J_j)^2)^{1/2} dx \quad (59)
\]
\[
\leq \left( \int_{\mathbb{R}^n} (M((-\Delta)^{1/4}u_j)^2 dx)^{1/2} \right)^{1/2} \left( \int_{\mathbb{R}^n} \sum_j ((-\Delta)^{1/4}Q J_j)^2 dx \right)^{1/2}
\]
\[
\leq C \| Q \|_{H^{1/2}(\mathbb{R})} \| u \|_{H^{1/2}(\mathbb{R})}.
\]

(5) The homogeneous Besov space \( \dot{B}^0_{\infty, \infty}(\mathbb{R}^n) \) is the space of tempered distribution \( u \) for which \( \| u \|_{\dot{B}^0_{\infty, \infty}(\mathbb{R}^n)} := \sup_{j \in \mathbb{R}} \| \mathcal{F}^{-1}[\psi_j \mathcal{F}[u]] \|_{L^\infty(\mathbb{R}^n)} \) is finite, (see for the precise definition of the Besov spaces [22])

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• Estimate of $\|\Pi_2((-\Delta)^{1/4}Q(-\Delta)^{1/4}u)\|_{H^1}$. It is as in \eqref{58}.

• Estimate of $\Pi_3((-\Delta)^{1/4}(Q(-\Delta)^{1/4}u))$.

We show that it is in $B_{1,1}^0$. We observe that if $h \in B_{\infty,\infty}^0$ then $(-\Delta)^{1/4}h \in B_{\infty,\infty}^{-1/2}$ and thus

\[
\sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} (-\Delta)^{1/4}(Q_j(-\Delta)^{1/4}u_k)h
= \sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \frac{1}{\ell_j} \int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} (-\Delta)^{1/4}(Q_j(-\Delta)^{1/4}u_k)[(-\Delta)^{1/4}h^j - 6 + \sum_{t=j-5}^j (-\Delta)^{1/4}h_t]dx
\]

We have

\[
\sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} (-\Delta)^{1/4}(Q_j(-\Delta)^{1/4}u_k)h^j - 6dx
= \sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} (Q_j(-\Delta)^{1/4}u_k)(-\Delta)^{1/4}h^j - 6dx
\]

\[
\leq C \sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \|h\|_{B_{\infty,\infty}^0} \int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} 2^{j/2}Q_j(-\Delta)^{1/4}u_kdx
\]

\[
\leq C \left( \int_{\mathbb{R}^n} \sum_{j} 2^{j/2}Q_j^2dx \right)^{1/2} \left( \int_{\mathbb{R}^n} \sum_{j} ((-\Delta)^{1/4}u_j)^2 dx \right)^{1/2}
\]

By analogous computations we get

\[
\sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} (-\Delta)^{1/4}(Q_j(-\Delta)^{1/4}u_k)[\sum_{t=j-5}^j (-\Delta)^{1/4}h_t]dx
\]

\[
\leq C \|Q\|_{H^{1/2}(\mathbb{R}^n)} \|u\|_{H^{1/2}(\mathbb{R}^n)}
\]

• Estimate of $\Pi_3((-\Delta)^{1/4}Q(-\Delta)^{1/4}u) - Q((-\Delta)^{1/2}u)$.

\[
\|\Pi_3((-\Delta)^{1/4}Q(-\Delta)^{1/4}u) - Q((-\Delta)^{1/2}u)\|_{B_{1,1}^0}
\]

\[
= \sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} \left[(-\Delta)^{1/4}(Q_j(-\Delta)^{1/4}u_k) - Q_j(-\Delta)^{1/2}u_k \right]h^j - 6 + \sum_{t=j-5}^j h_t]dx
\]

(60)
We only estimate the terms with $h^{j-6}$, being the estimates with $h_t$ similar.

\[
\sup_{\|h\|_{\mathcal{B}_{0,\infty}} \leq 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j| \leq 3} \left[ (-\Delta)^{1/4}(Q_j(-\Delta)^{1/4}u_k) - Q_j(-\Delta)^{1/2}u_k \right] [h^{j-6}] dx \tag{61}
\]

\[
= \sup_{\|h\|_{\mathcal{B}_{0,\infty}} \leq 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j| \leq 3} \mathcal{F}[h^{j-6}] \mathcal{F}[(\mathcal{F}[Q_j(-\Delta)^{1/4}u_k - Q_j(-\Delta)^{1/2}u_k] dx
\]

\[
= \sup_{\|h\|_{\mathcal{B}_{0,\infty}} \leq 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j| \leq 3} \mathcal{F}[h^{j-6}] \left[ \int_{\mathbb{R}^n} \mathcal{F}[Q_j](y) \mathcal{F}[(\mathcal{F}[(-\Delta)^{1/4}u_k](x-y)(|y|^{1/2} - |x-y|^{1/2})dy \right] dx.
\]

Now we observe that in (61) we have $|x| \leq 2^{j-3}$ and $2^{j-2} \leq |y| \leq 2^{j+2}$. Thus $|x| \leq \frac{1}{2}$.

Hence

\[
|y|^{1/2} - |x - y|^{1/2} = |y|^{1/2}[1 - |1 - \frac{x}{y}|^{1/2}] = |y|^{1/2} \frac{x}{y} [1 + |1 - \frac{x}{y}|^{1/2}]^{-1} = |y|^{1/2} \sum_{k=0}^{\infty} \frac{c_k}{k!} \left( \frac{x}{y} \right)^{k+1}.
\]

We may suppose that $\sum_{k=0}^{\infty} \frac{c_k}{k!} \left( \frac{x}{y} \right)^{k+1}$ is convergent if $\left| \frac{x}{y} \right| \leq \frac{1}{2}$, otherwise one may consider a different Littlewood-Paley decomposition by replacing the exponent $j - 4$ with $j - s$, $s > 0$ large enough. We introduce the following notation: for every $k \geq 0$ we set

\[
S_k g = \mathcal{F}^{-1}[\xi^{-(k+1)}|\xi|^{1/2} \mathcal{F} g].
\]

We note that if $h \in B^s_{\infty,\infty}$ then $S_k h \in B^{s+1/2+k}_{\infty,\infty}$ and if $h \in H^s$ then $S_k h \in H^{s+1/2+k}$.

Moreover if $Q \in H^{1/2}$ then $\nabla_k^{k+1}(Q) \in H^{-k-1/2}$.

We continue the estimate (61).
\[
(61) \quad \sup_{\|h\|_{B^\infty_{2,\infty}} \leq 1} \int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} \mathcal{F}[h^{j-6}] \\
\left[ \int_{\mathbb{R}^n} (\mathcal{F}[Q]_j(y) \mathcal{F}[(\nabla)^{1/4} u_k] (x-y)(|y|^{1/2} - |x-y|^{1/2}) dy \right] dx
\]

\[
= \sup_{\|h\|_{B^\infty_{2,\infty}} \leq 1} \sum_{\ell=0}^{\infty} \frac{C_{\ell}}{\ell!} \int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} (-i)^{\ell+1} \mathcal{F}[\nabla^{\ell+1} h^{j-6}] \mathcal{F}[S_\ell Q_j(\nabla)^{1/4} u_k)](x) dx
\]

\[
\leq \sup_{\|h\|_{B^\infty_{2,\infty}} \leq 1} \sum_{\ell=0}^{\infty} \frac{C_{\ell}}{\ell!} \int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} |\nabla^{\ell+1} h^{j-6} [S_\ell Q_j(\nabla)^{1/4} u_k)](x) dx
\]

by Lemma [A.3]

\[
\leq C \sup_{\|h\|_{B^\infty_{2,\infty}} \leq 1} \sum_{\ell=0}^{\infty} \frac{C_{\ell}}{\ell!} 2^{-6\ell} 4^{\ell+1} \|h\|_{B^\infty_{2,\infty}} \\
\int_{\mathbb{R}^n} \sum_{j} \sum_{|k-j| \leq 3} 2^{(\ell+1)j} |S_\ell Q_j(\nabla)^{1/4} u_k)](x) dx
\]

\[
\leq C \sum_{\ell=0}^{\infty} \frac{C_{\ell}}{\ell!} 2^{-6\ell} 4^{\ell+1} \left( \int_{\mathbb{R}^n} \sum_{j} 2^{2(\ell+1)j} |S_\ell Q_j|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^n} \sum_{j} |(\nabla)^{1/4} u_j|^2 dx \right)^{1/2}
\]

by Plancherel Theorem

\[
\leq C \sum_{\ell=0}^{\infty} \frac{C_{\ell}}{\ell!} 2^{-6\ell} 4^{\ell+1} \left( \int_{\mathbb{R}^n} \sum_{j} 2^{2(\ell+1)j} |\mathcal{F}[S_\ell Q_j]|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^n} \sum_{j} |(\nabla)^{1/4} u_j|^2 dx \right)^{1/2}
\]

\[
\leq C \sum_{\ell=0}^{\infty} \frac{C_{\ell}}{\ell!} 2^{-6\ell} 4^{\ell+1} 2^{1(j+1/2)} |\mathcal{F}[Q_j]|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^n} \sum_{j} |(\nabla)^{1/4} u_j|^2 dx \right)^{1/2}
\]

\[
\leq C \sum_{\ell=0}^{\infty} \frac{C_{\ell}}{\ell!} 2^{-6\ell} 4^{\ell+1} \left( \int_{\mathbb{R}^n} \sum_{j} 2^{2Q_j^2 dx} \right)^{1/2} \left( \int_{\mathbb{R}^n} \sum_{j} |(\nabla)^{1/4} u_j|^2 dx \right)^{1/2}
\]

\[
\leq C \sum_{\ell=0}^{\infty} \frac{C_{\ell}}{\ell!} 2^{-3\ell} \|Q\|_{H^{1/2}(\mathbb{R})} \|h\|_{H^{1/2}(\mathbb{R})}
\]

- Estimate of $\Pi_2((-\nabla)^{1/4} (Q(-\nabla)^{1/4} u) - Q(-\nabla)^{1/2} u))$. 

\hspace{1cm} 27
\[ \| \Pi_2((-\Delta)^{1/4}Q(-\Delta)^{1/4}u) - Q(-\Delta)^{1/4}u \|_{H^{1/4}} \]  

\[ = \sup_{\|h\|_{H^{\infty}} \leq 1} \left\{ \int_{\mathbb{R}^n} \sum_{j} \sum_{|t-j| \leq 3} [(-\Delta)^{1/4}(Q^{j-4}(-\Delta)^{1/4}u_j) - (-\Delta)^{1/2}(Q^{j-4}u_j)h_t dx \right. \]

\[ = \sup_{\|h\|_{H^{\infty}} \leq 1} \left\{ \int_{\mathbb{R}^n} \sum_{j} \sum_{|t-j| \leq 3} \mathcal{F}[Q^{j-4}]\mathcal{F}((-\Delta)^{1/4}u_j)(-\Delta)^{1/4}h_t - (-\Delta)^{1/2}u_jh_t dx \right\} \]

\[ = \sup_{\|h\|_{H^{\infty}} \leq 1} \left\{ \int_{\mathbb{R}^n} \sum_{j} \sum_{|t-j| \leq 3} (-i)^{\ell+1} \mathcal{F}[\nabla^{\ell+1}Q^{j-4}]\mathcal{F}[S_\ell(-\Delta)^{1/4}u_j]h_t dx \right\} \]

\[ \leq \sup_{\|h\|_{H^{\infty}} \leq 1} \left\{ \int_{\mathbb{R}^n} \sum_{j} \sum_{|t-j| \leq 3} (-i)^{\ell+1} \mathcal{F}[\nabla^{\ell+1}Q^{j-4}]\mathcal{F}[S_\ell(-\Delta)^{1/4}u_j] dx \right\} \]

\[ \leq \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \sum_{j} 2^{-(k+1/2)j} \nabla^{\ell+1}Q^{j-4} \mathcal{F}[S_\ell(-\Delta)^{1/4}u_j] dx \right\} \]

\[ \leq C \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \left\{ \int_{\mathbb{R}^n} \sum_{j} 2^{-(k+1/2)j} \nabla^{\ell+1}Q^{j-4} \mathcal{F}[S_\ell(-\Delta)^{1/4}u_j] dx \right\} \]

by Plancherel Theorem

\[ = C \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \left\{ \int_{\mathbb{R}^n} \sum_{j} 2^{-(k+1/2)j} |\xi|^{2\ell} |\mathcal{F}[\nabla Q^{j-4}]|^2 dx \right\} \]

\[ \leq C \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \left\{ \int_{\mathbb{R}^n} \sum_{j} 2^{-(k+1/2)j} |\xi|^{2\ell} |\mathcal{F}[\nabla Q^{j-4}]|^2 dx \right\} \]

\[ \leq C \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} 2^{-3\ell} \left\{ \int_{\mathbb{R}^n} \sum_{j} 2^{-j} |\mathcal{F}[\nabla Q^{j-4}]|^2 dx \right\} \]

\[ \leq C \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} 2^{-3\ell} \|Q\|_{H^{1/4}(\mathbb{R}^n)} \|u\|_{H^{1/4}(\mathbb{R}^n)} \]

The proof of the following Theorems and its localized version can be found in [7].

**Theorem A.1** Let \( u, Q \in \dot{W}^{1/2,q}(\mathbb{R}^n) \), with \( q > 2 \). Then \( T(Q, u), S(Q, u) \in L^{q/2}(\mathbb{R}^n) \) and

\[ \| T(Q, u) \|_{L^{q/2}} \leq C \|(-\Delta)^{1/4}Q\|_{L^{q}} \|(-\Delta)^{1/4}u\|_{L^{q}} ; \]  

(64)
\[ \|S(Q,u)\|_{L^{q/2}} \leq C\|(-\Delta)^{1/4}Q\|_{L^{q/2}}\|(-\Delta)^{1/4}u\|_{L^{q/2}}. \quad (65) \]

**Theorem A.2** Let \( Q \in \dot{H}^{1/2}(\mathbb{R}^n) \), \( u \in \dot{W}^{1/2,q}(\mathbb{R}^n) \) with \( q > 2 \). Then \( T(Q,u), S(Q,u) \in L^{\frac{2q}{q+2}}(\mathbb{R}^n) \) and
\[
\|T(Q,u)\|_{L^{\frac{2q}{q+2}}} \leq C\|(-\Delta)^{1/4}Q\|_{L^{q/2}}\|(-\Delta)^{1/4}u\|_{L^{q/2}}; \quad (66)
\]
\[
\|S(Q,u)\|_{L^{\frac{2q}{q+2}}} \leq C\|(-\Delta)^{1/4}Q\|_{L^{q/2}}\|(-\Delta)^{1/4}u\|_{L^{q/2}}. \quad (67)
\]

**Remark A.1** Actually Theorems A.1 and A.2 hold for the 2-terms commutators
\[
\tilde{T}(Q,u) = T(Q,u) - (-\Delta)^{1/4}Q(-\Delta)^{1/4}u = (-\Delta)^{1/4}(Q(-\Delta)^{1/4}u) - Q\Delta^{1/2}u
\]
and
\[
\tilde{S}(Q,u) = S(Q,u) - \mathcal{R}((-\Delta)^{1/4}Q\mathcal{R}(-\Delta)^{1/4}u) = (-\Delta)^{1/4}[Q(-\Delta)^{1/4}u] - \mathcal{R}(Q\nabla u). \quad \square
\]

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**References**

[1] D.R. Adams & L. I. Hedberg, *Function Spaces and Potential Theory*, 1996, Springer, Berlin.

[2] S. Alexakis & Rafe Mazzeo, *The Willmore functional on complete minimal surfaces in H3: boundary regularity and bubbling*, arXiv:1204.4955v2.

[3] Y. Bernard & T. Rivière, *Energy Quantization for Willmore Surfaces and Applications*, arXiv:1106.3780 (2011).

[4] J.M. Bony, *Cours d’analyse*, Éditions de l’École polytechnique, 2011.

[5] R. Coifman, R. Rochberg, G. Weiss, *Factorization theorems for hardy spaces in several variables*. Ann. of Math. 103 (1976), 611-635.

[6] F. Da Lio, *Fractional Harmonic Maps into Manifolds in odd dimension n > 1*, arXiv:1012.2741v1, Calculus of Variations and PDEs, 2012, DOI: 10.1007/s00526-012-0556-6.

[7] F. Da Lio, *Habilitation Thesis*, in preparation.

[8] F. Da Lio, *In preparation*.

[9] F. Da Lio & T. Riviere, *3-Commutators Estimates and the Regularity of 1/2-Harmonic Maps into Spheres*. APDE 4-1 (2011), 149–190. DOI 10.2140/apde.2011.4.149.
[10] F. Da Lio & T. Riviere, *Sub-criticality of non-local Schrödinger systems with antisymmetric potentials and applications to 1/2-harmonic maps*, Advances in Mathematics 227, (2011), 1300-1348.

[11] F. Da Lio & T. Riviere, *Fractional harmonic maps and free boundaries problems*, in preparation.

[12] F. Da Lio & A. Schikorra, *\((n,p)\)-harmonic maps: regularity for the sphere case*, arXiv:1202.1151v1, Advances in Calculus of Variations, to appear.

[13] L.C Evans & R.F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[14] A. Fraser, & R. Schoen, *The first Steklov eigenvalue, conformal geometry, and minimal surfaces*, Advances in Mathematics, 226, no. 5, 2011, 4011-4030.

[15] A. Fraser, & R. Schoen, *Eigenvalue bounds and minimal surfaces in the ball*, arXiv:1209.3789.

[16] L. Grafakos, *Classical Fourier Analysis*. Graduate Texts in Mathematics 249, Springer (2009).

[17] L. Grafakos, *Modern Fourier Analysis*. Graduate Texts in Mathematics 250, Springer (2009).

[18] P. Laurain & T. Rivièrè, *Angular Energy Quantization for Linear Elliptic Systems with Antisymmetric Potentials and Applications*, arXiv:1109.3599 (2011).

[19] P. Laurain & T. Rivièrè, *Energy Quantization for Biharmonic Maps*, arXiv:1112.5393 (2011).

[20] F.H. Lin & T. Rivièrè *Quantization property for moving Line vortices*, Comm. Pure App. Math., 54, (2001), 826-850

[21] T. Rivièrè, *Bubbling, quantization and regularity issues in geometric non-linear analysis*, ICM Beijing (2002)

[22] T. Runst & W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*. Walter de Gruyter, Berlin , 1996

[23] A. Schikorra, *Regularity of n/2 harmonic maps into spheres*, Journal of Differential Equations 252 (2012) 1862–1911.

[24] A. Schikorra, *epsilon-regularity for systems involving non-local, antisymmetric operators*, preprint.
[25] H.C. Wente, *An existence theorem for surfaces of constant mean curvature.* J. Math. Anal. Appl. 26 1969 318–344.