Non-Fermi-liquid-like behavior in various transport phenomena in high-$T_c$ cuprates is a key clue for solving the mystery of the ground state. For example, $R_H \propto T^{-1}$ is observed in hole-doped systems below $T_0 \sim 700K$, and $|R_H| \gg 1/ne$ ($n$ being the electron density) at lower temperatures [1]. Moreover, the magnetoresistance follows the relation $\Delta \rho / \rho_0 \propto R_H^2 \propto \omega^{-2}$, which is called the modified Kohler’s rule [2, 3]. A comprehensive understanding for them cannot be achieved by the relaxation time approximation (RTA) even if one assume a huge anisotropy of the relaxation time, $\tau_k$. Although $R_H \gg 1/ne$ is derived, then $\Delta \rho / \rho_0$ increases much faster than $T^{-4}$ so the modified Kohler’s rule is broken down [4, 5].

The modified Kohler’s rule is well reproduced if one take the vertex correction for the total current $J_k$, which is called the back-flow in Landau-Fermi liquid theory [6]. Due to the back-flow, $J_k$ becomes totally different from the quasiparticle velocity $v_{kq}$ in nearly AF Fermi liquid. Reflecting this fact, relations $R_H \propto \xi^2 \propto T^{-1}$ and $\Delta \rho / \rho_0 \propto \xi^4 \propto T^{-2}$ are derived, where $\xi$ is the antiferromagnetic (AF) correlation length. In the same way, anomalous behaviors of thermoelectric power and Nernst coefficient can be explained in the unified way [6, 7, 8].

AC transport phenomena give us further significant information on the electronic states. In HTSC’s, the extended-Drude (ED) relation given by the RTA,

$$\sigma_{xy}(\omega) \propto \{\sigma^0(\omega)\}^2 \propto (2\gamma_0(\omega) - iz^{-1}\omega)^{-2},$$

$$\gamma_0^{-1}(\omega) = \frac{2}{\omega} \int_{0}^{\gamma_0} d\epsilon [\gamma_k(\epsilon - \omega) + \gamma_k(\epsilon)]^{-1}_{FS}$$

is strongly violated even when $\omega \ll \gamma_0$ (in the far-infrared region, $\omega \sim 500cm^{-1}$). The imaginary part of the self-energy and $z(<1)$ is the mass-renormalization factor. The ED-relation seems to hold well in Cu and Au [9]. Note that we obtain a simple Drude form even when $\omega \gg \gamma_0$ (in the infrared region, $\omega \sim 1000cm^{-1}$), despite the fact that $\omega$-dependence of $\gamma_0(\omega)$ violates the Drude form of $\sigma_\omega(\omega)$. Previous theoretical works based on the RTA unable to give a comprehensive understanding for them [10, 11, 12]. Thus, anomalous AC transport phenomena in HTSC’s can be understood in terms of nearly AF Fermi liquid, if one take the CVC into account.

In this article, we develop the method of calculating $\sigma(\omega)$ and $\sigma_{xy}(\omega)$ using the FLEX approximation by taking the current vertex correction (CVC) to satisfy the conservation laws. We call it the FLEX+CVC approximation. We find that $\sigma_{xy}(\omega)$ deviates from the ED-form predominantly due to CVC when AF fluctuations are strong. We succeeded in reproducing the unexpected experimental behaviors of $R_H(\omega)$ as well as $\theta_H(\omega)$ at the same time. Thus, anomalous AC and DC transport phenomena in HTSC’s can be understood in terms of nearly AF Fermi liquid, if one take the CVC into account.

First, we study the self-energy $\Sigma_k(\omega_n)$ in the square-lattice Hubbard model using the FLEX approximation, which is one of the self-consistent spin-fluctuation theory [13]. This method can reproduce characteristic electronic properties in slightly under-doped systems above the pseudo-gap temperature [14, 15]. We put $(t, t’, t''; U) = (1, -0.1, 0.1; 5)$ and $n = 0.9$, where $t$, $t’$ and $t''$ are hopping integrals between the nearest neighbor, the second nearest and the third nearest, respectively. The dispersion of conduction electron is $\epsilon_k = -2t(\cos k_x + \cos k_y) - 4t’(\cos k_x \cos k_y - 2t”’(\cos 2k_x + \cos 2k_y))$. This set of parameters corresponds to slightly under-doped LSCO. We note that $\text{Im} \Sigma_k(-i\delta)$ on the FS takes the minimum value around $(\pi/2, \pi/2)$, which is called the ‘cold-spot’ in literature [16]. Its maximum (minimum) value on the FS is $0.46 (0.081)$ at $T = 0.02$: The ratio of anisotropy is $5.7$, which is too small to explain the enhancement of $R_H$: the CVC around the cold-spot magnifies $R_H$. Next, we study transport coefficients in the framework of the conservation approximation, by taking the CVC
into account [17]. According to the Kubo formula,
\[ \sigma_{\mu\nu}(\omega) = \frac{1}{i\omega} \left[ K_{\mu\nu}^{R}(\omega) - K_{\mu\nu}^{R}(0) \right], \] (3)
where \( K_{\mu\nu}^{R}(\omega) \) is the retarded correlation function, which is given by the analytic continuations of the following thermal Green functions with \( \omega \geq 0 \) [18,19] using the numerical Pade approximation:
\[ K_{xx}(i\omega_l) = -2e^2 T \sum_{n,k} v_{kz} n_{l}^{j} \Lambda_{kx}, \] (4)
\[ K_{xy}(i\omega_l) = i \cdot e^3 T \sum_{n,k} \sum_{\mu,\nu} \left[ G_{n}^{\mu} \frac{\partial}{\partial k_{\mu}} G_{n}^{\nu+1} \right] \times \left[ \Lambda_{kx}^{n_{l}^{j} \leftrightarrow n_{l}^{j}} \Lambda_{ky}^{n_{l}^{j}} \right] \cdot \epsilon_{\mu\nu z}, \] (5)
where \( [A \frac{\partial}{\partial k_{\mu}} B - B \frac{\partial}{\partial k_{\mu}} A] \) is a symmetric tensor with \( \epsilon_{xyz} = 1 \). \( \omega_l = 2\pi T l \) is a Matsubara frequency; here we promise that the \( \omega_l \) is an integer.

The velocity of a free electron, and \( \Lambda_{kx}^{n_{l}^{j}} \) is the dressed current. In the FLEX approximation, it is given by
\[ \Lambda_{kx}^{n_{l}^{j}} = v_k^{0} + T \sum_{n',p} \sum_{\mu,\nu} \left[ v_{k-p} \left( \omega_n - \omega_{n'} \right) \cdot \partial_{p} G_{n}^{\mu} \right] \cdot \partial_{k} \Lambda_{kx}^{n_{l}^{j}} \] (6)
where \( V_k(\omega) \) is given in eq.(9) of ref.[8].

This approximation is justified for DC-conductivities when the AF fluctuations for \( Q \approx (\pi, \pi) \) are dominant, as proved in ref.[8]. This will also be true for optical conductivities, due to the fact that the f-sum rules both for \( \sigma(\omega) \) and \( \sigma_{xy}(\omega) \) are well satisfied in the present study, as will be discussed below.

Pade approximation is less reliable when the function under consideration is strongly \( \omega \)-dependent, and when the temperature is high because the Matsubara frequency is sparse. To obtain reliable results, both \( \Sigma_k(\omega_n) \) and \( \Lambda_{kx}^{n_{l}^{j}} \) have to be converged so that the relative errors should be under 10\(^{-8}\). Moreover, we utilize the fact that the \( i\omega \)-linear term of \( K_{\mu\nu}(\omega) \) is equal to the DC value of \( \sigma_{\mu\nu} \), which can be obtained by the FLEX+CVC approximation with high accuracy, as performed in refs.[8,7,8].

As a result, we succeed in deriving the \( \sigma_{\mu\nu}(\omega) \) for any \( \omega \) with enough accuracy.

To confirm the accuracy of numerical results, we check the following f-sum rules [20]:
\[ \int_{0}^{\infty} d\omega \text{Re}\sigma(\omega) = \pi \epsilon^2 \sum_{k} \frac{\partial^2 v_k}{\partial k^2} n_k, \] (7)
\[ \int_{0}^{\infty} d\omega \text{Re}\sigma_{xy}(\omega) = 0. \] (8)
Because the FLEX+CVC approximation is a conserving approximation, f-sum rules should be satisfied in the present study if numerical error is absent. In Fig.1, \( \text{Int}\{\text{Re}\sigma(\omega)\} / \text{Int}\{\text{Re}\sigma_{xy}(\omega)\} \) represents the numerical result for the left-hand-side of eq. (7) (eq. (8)), and \( -E_{\text{opt}} \) is that for the right-hand-side of eq. (7). We see that the f-sum rules hold well, whose relative error is below 2.5%. This results assure the high reliability of the present numerical study.

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FIG. 1: f-sum rules both for \( \sigma(\omega) \) and \( \sigma_{xy}(\omega) \) are well satisfied if the CVC is correctly taken into account. This fact assures the reliability of the present numerical study.

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negative for $\omega > 0.1$, its absolute value is very small. It is naturally understood from the ED-form because $\gamma(\omega)$ increases with $\omega$. This large dip in $\text{Re}\sigma_{xy}(\omega)$ is naturally understood in terms of the $f$-sum rule, eq. (5), because $\text{Re}\sigma_{xy}(0) > 0$ takes an enhanced value due to the CVC. We also stress that $\text{Im}\sigma_{xy}(\omega)/\omega|_{\omega=0}$ is strongly enhanced due to the CVC; $\text{Im}\sigma_{xy}(\omega)$ at $T = 0.02$ takes the maximum value at $\omega_2 \sim 0.01$, which is about six times larger than $\omega_1$ for $\text{Im}\sigma(\omega)$.

The violation of the ED-form is apparent in $R_H(\omega) = \sigma_{xy}(\omega)/\sigma^2(\omega)$. We see that $R_H^0(\omega)$ by the RTA, $\sigma_{xy}^0(\omega)/\sigma^0(\omega)^2$, is almost $\omega$-independent and its imaginary part is tiny, which means that both $\sigma_{xy}^0(\omega)$ and $\sigma^0(\omega)$ follow the ED-forms given in eq. (4). On the other hand, $R_H(\omega)$ given by the FLEX+CVC approximation shows prominent frequency as well as temperature dependences. The obtained overall behavior is similar to the experimental observation in an optimally-doped YBCO at 95K [9]. $\text{Im}R_H(\omega)$ at $T = 0.02$ takes the maximum value at $\omega_3 \sim 0.01$, which is similar to $\omega_2$ for $\text{Im}\sigma_{xy}(\omega)$ and is six times larger than $\omega_1$ for $\text{Im}\sigma(\omega)$. The relation $\omega_1 \gg \omega_2 \sim \omega_3$ in the present study, which is consistent with experimental observation [9, 10], cannot be reproduced by the RTA: It can be explained only when the back-flow is taken into account [21].

IR optical Hall angle $\theta_H(\omega) = \sigma_{xy}(\omega)/\sigma(\omega)$ ($\omega \lesssim 1000\text{cm}^{-1} = 1440\text{K}$) has been intensively measured by Drew et al [11, 12]. They found that the following simple Drude form is well satisfied for the Hall angle:

$$\theta_H(\omega) = \frac{\Omega^*_H}{2\gamma^*_H - i\omega},$$

$$\gamma^*_H \propto T^{-d}; \quad d = 1.5 \sim 2$$

$$\Omega^*_H \propto T^0,$$

where $\gamma^*_H$ and $\Omega^*_H$ are $\omega$-independent for IR range ($\omega \lesssim 1000\text{cm}^{-1}$). This is highly nontrivial because $\gamma(\omega)$ in eq. (2) is approximately linear-in-$\omega$ for $\omega \gg T$ in HTSC’s [22]. $\Omega^*_H$ is almost independent of $\omega$ and $T$, which increases as the doping decreases. These unexpected results put very severe constraints on theories of HTSC.

From now on, we show that the Drude-type form of the Hall angle is reproduced quite well by the FLEX+CVC
Now, we introduce several results given by an approximate solution of the Bethe-Salpeter equation for $J_0$ at finite frequencies [21]. The general expressions for $\sigma(\omega)/\omega|_{\omega=0}$ and $\sigma_{xy}(\omega)/\omega|_{\omega=0}$ can be derived from the Kubo formula [21]. By analysing the CVC included in them, we obtain the relation $\sigma_{xy}(\omega) \approx a(\gamma_0)^{-1} + b \cdot iz^{-1}\omega(2\gamma_0)^{-2} + O(\omega^2)$, where $a \propto \xi^2$ and $b \propto \xi^3$ due to the CVC, whereas $a = b = 1$ in the RTA. As a result, $\sigma_{xy}(\omega)$ deviates form the ED-form due to the CVC. On the other hand, $\sigma(\omega) \approx (\gamma_0)^{-1} + iz^{-1}\omega(2\gamma_0)^{-2}$ even if the CVC is taken. Then, we obtain relations $\text{Im}(\sigma_{xy}(\omega))/\omega \propto z^{-1}\gamma_0 \propto T^{-2.6}$ and $\text{Im}(\sigma_{hh}(\omega))/\omega \propto z^{-1}\gamma_0 \propto T^{-0.4}$ and $z^{-1} \propto T^{-0.4}$ in the present FLEX approximation [21]. These relations are confirmed by the present numerical study. In addition, the cancellation of the $\omega$-dependences of $\gamma_0(\omega)$ and CVC’s result in the almost constant $\gamma_0$ in eq. (9).

In summary, we have calculated the optical conductivities for HTSC’s by the FLEX+CVC approximation, which had been serious problems associated intimately with the true electronic ground states in HTSC’s. Experimentally observed anomalous behaviors for $\sigma(\omega)$, $\sigma_{xy}(\omega)$, $\theta_H(\omega)$ and $R_H(\omega)$ are well reproduced for each wide range of frequencies and temperatures, without assuming any fitting parameters. They are consistent with the characteristic experimental results for HTSC’s reported by Drew et al. [4, 11, 12], which cannot be reproduced by previous theoretical works based on the RTA even if one assume extremely anisotropic $\tau_k$. The present study ensures that anomalous AC and DC transport phenomena in HTSC’s can be understood in terms of nearly AF Fermi liquid state.

There remain many important issues for future study. We will calculate the optical Hall conductivity for electron-doped HTSC’s [21]. We are also planning to study the optical conductivities in the pseudo-gap region based on the FLEX+T-matrix approximation, which ascribes the pseudo-gap phenomena in HTSC’s to the strong superconducting fluctuations [3, 14]. We expect that the non-Drude form of the far-IR ($\omega = 20 \sim 250$ cm$^{-1}$) Hall angle in YBa$_2$Cu$_3$O$_y$ below 150K might be explained by this study.

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