$gl(2|2)$ Current Superalgebra and Non-unitary Conformal Field Theory

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Abstract

Motivated by application of current superalgebras in the study of disordered systems such as the random XY and Dirac models, we investigate $gl(2|2)$ current superalgebra at general level $k$. We construct its free field representation and corresponding Sugawara energy-momentum tensor in the non-standard basis. Three screen currents of the first kind are also presented.
1 Introduction

Virasoro algebra and current algebras are algebraic structures in conformal field theories (CFTs) [1, 2, 3] and string theory [4]. According to the Zamolodchikov c-theorem [5], a unitary CFT corresponds to a fixed point of the renormalization group flow. So unitary CFTs and current algebras also play an important role in the study of 2D critical behaviours of statistical mechanics models.

Recently current superalgebras and their corresponding non-unitary CFTs have attracted much attention. The reason is partly as follows. One the one hand, representations of superalgebras are often indecomposable, and such indecomposable representations lead to logarithmic singularities in correlation functions: a character of so-called logarithmic CFTs (e.g. see e.g. [6] and reference therein). On the other hand, current superalgebras with zero superdimension have found various physical applications ranging from condensed matter physics to high energy physics. Particularly interesting are the applications in topological field theory [7] and the supersymmetric method [8] to the study of disordered systems [9, 10, 11, 12, 13, 14]. There the vanishing of Virasoro central charges and the existence of negative dimensional primary fields are essential [11, 9, 15].

Free field realization [16] is a common approach used in CFTs, representation theory of current (super)algebras and applications [17, 18, 19, 20, 21, 22, 23, 24, 25]. From the viewpoint of application to disordered systems, most interesting are \( \mathfrak{osp}(N|N) \) and \( \mathfrak{gl}(N|N) \) current superalgebras [11, 14]. The representations of \( \mathfrak{gl}(1|1) \) and \( \mathfrak{osp}(2|2) \) current superalgebras at general level \( k \) have been studied in details in [26, 9, 12, 25].

In this paper we investigate the non-semisimple \( \mathfrak{gl}(2|2) \) current superalgebra at general level \( k \), relevant to the two-species random \( XY \) and Dirac models. We construct its free field representation in the non-standard basis. Let us point out that a free field realization for the semisimple \( \mathfrak{sl}(2|2) \) current superalgebra in the standard basis was obtained in [27]. We moreover construct the Sugawara energy-momentum tensor and obtain three screen currents of the first kind.

2 Notations

Unlike bosonic algebras, the simple root system of superalgebras is not unique. In the case of \( \mathfrak{gl}(N|N) \), the standard (distinguished) basis has a single fermionic simple root. From physical application point of view, the non-standard basis where all simple roots are fermionic is more useful. In this paper, we will adapt the non-standard basis. In this basis, \( \mathfrak{gl}(2|2) \) has simple raising generators \( E_{13}, E_{32} \) and \( E_{24} \), and simple lowering generators \( E_{31}, E_{23} \) and \( E_{42} \). The corresponding Cartan subalgebra is generated by

\[
H_1 = E_{11} + E_{33}, \quad H_2 = E_{22} + E_{33}, \quad H_3 = E_{22} + E_{44}, \\
H_4 = E_{11} + E_{22} - E_{33} - E_{44} + \alpha(E_{11} + E_{22} + E_{33} + E_{44}),
\]

where \( \alpha \) is an arbitrary parameter. That \( H_4 \) is not uniquely determined is a consequence of the fact that \( \mathfrak{gl}(2|2) \) is non-semisimple. The defining representation of \( \mathfrak{gl}(2|2) \) in the non-standard basis is given by

\[
E_{13} = e_{13}, \quad E_{32} = e_{32}, \quad E_{24} = e_{24}, \\
E_{31} = e_{31}, \quad E_{23} = e_{23}, \quad E_{42} = e_{42},
\]
where $e_{ij}$ are the matrix with entry 1 at the $i$-th row and $j$-th column, and zero elsewhere, The generators obey the (anti-) commutation relations:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - (-1)^{(i+j)(k+l)} \delta_{il} E_{jk}, \quad (2.3)$$

where $[1] = [2] = 0$, $[3] = [4] = 1$ and

$$E_{11} = \frac{1}{4}((3 - \alpha)H_1 - 2H_2 + (1 - \alpha)H_3 + H_4),$$
$$E_{22} = \frac{1}{4}(-(1 + \alpha)H_1 + 2H_2 + (1 - \alpha)H_3 + H_4),$$
$$E_{33} = \frac{1}{4}((1 + \alpha)H_1 + 2H_2 - (1 - \alpha)H_3 - H_4),$$
$$E_{44} = \frac{1}{4}((1 + \alpha)H_1 - 2H_2 + (3 + \alpha)H_3 - H_4). \quad (2.4)$$

The quadratic Casimir of $gl(2|2)$ is

$$C_1 = \sum_{A,B} (-1)^{g(B)} E_{AB} E_{BA}$$

$$= \frac{1}{2} H_1 (H_1 - 2) - \frac{1}{2} H_3 (H_3 + 2) - (H_1 - H_3) H_2 + \frac{1}{2} (H_1 + H_3) H_4$$
$$+ 2 E_{12} E_{21} - 2 E_{13} E_{31} - 2 E_{24} E_{42} - 2 E_{14} E_{41} + 2 E_{32} E_{23} - 2 E_{34} E_{43}$$
$$\frac{\alpha}{2} (H_1 + H_3)^2. \quad (2.5)$$

Because $gl(2|2)$ is non-semisimple, $\sum_A E_{AA}$ is a central element and thus there is another quadratic Casimir

$$C_2 = \sum_{A,B} E_{AA} E_{BB} = (E_{11} + E_{22} + E_{33} + E_{44})^2 = (H_1 + H_3)^2. \quad (2.6)$$

These two Casimir elements are useful in the following for the construction of Sugawara energy-momentum tensor.

The $gl(2|2)$ current superalgebra at general level $k$ can be written as

$$J_{ij}(z)J_{kl}(w) = k \frac{str(E_{ij}E_{kl})}{(z-w)^2} + \frac{1}{z-w} \left( \delta_{jk} J_{il}(w) - (-1)^{(i+j)(k+l)} \delta_{il} J_{jk}(w) \right), \quad (2.7)$$

where $J_{ij}(z)$ are currents corresponding to generators $E_{ij}$ and

$$J_{11}(z) = \frac{1}{4}((3 - \alpha)J_1(z) - 2J_2(z) + (1 - \alpha)J_3(z) + J_4(z)),$$
$$J_{22}(z) = \frac{1}{4}(-(1 + \alpha)J_1(z) + 2J_2(z) + (1 - \alpha)J_3(z) + J_4(z)),$$
$$J_{33}(z) = \frac{1}{4}((1 + \alpha)J_1(z) + 2J_2(z) - (1 - \alpha)J_3(z) - J_4(z)),$$
$$J_{44}(z) = \frac{1}{4}((1 + \alpha)J_1(z) - 2J_2(z) + (3 + \alpha)J_3(z) - J_4(z)).$$

Here $J_i(z)$ are currents associated with $H_i$. 

2
3 Free field realization

To obtain a free field realization, we first construct Fock space representation of $gl(2|2)$. The Fock space is constructed by the actions of the raising operators $E_{13}$, $E_{32}$, $E_{24}$, $E_{12}$, $E_{34}$ and $E_{14}$ on the lowest weight state. Define the lowest weight state $|\Lambda >$ of $gl(2|2)$ by

$$E_{13}|\Lambda > = E_{32}|\Lambda > = E_{24}|\Lambda > = E_{12}|\Lambda > = E_{34}|\Lambda > = E_{14}|\Lambda > = 0,$$
$$H_i|\Lambda > = \Lambda_i|\Lambda >.$$

Then in the non-standard basis, the action of the operator $e^A$ with

$$A = \theta_{i3}E_{13} + \theta_{i2}E_{32} + \theta_{i4}E_{24} + x_{12}E_{12} + x_{34}E_{34} + \theta_{i4}E_{14}$$

on the lowest weight state $|\Lambda >$ generates a coherent state of the algebra, where $x_{ij}$ are bosonic coordinates satisfying $x_{12}x_{34} = x_{34}x_{12}$ and $[\partial_{x_{ij}}, x_{kl}] = \delta_{ik}\delta_{jl}$ and $\theta_{ij}$ are fermionic coordinates obeying $\theta_{ij}\theta_{kl} = -\delta_{kl}\theta_{ij}$ and $\{\partial_{\theta_{ij}}, \theta_{kl}\} = \delta_{ik}\delta_{jl}$.

We write

$$ge^A|\Lambda > = d_g e^A|\Lambda >,$$

where $g$ is a generator of $gl(2|2)$ and $d_g$ is the corresponding differential operator. By using the defining relations of $gl(2|2)$ and the Baker-Campbell-Hausdorff formula, after a long algebraic manipulation we obtain

$$d_{E_{14}} = \partial_{\theta_{14}},$$
$$d_{E_{12}} = \partial_{x_{12}} - \frac{1}{2}\theta_{24}\partial_{\theta_{14}},$$
$$d_{E_{34}} = \partial_{x_{34}} + \frac{1}{2}\theta_{13}\partial_{\theta_{14}},$$
$$d_{E_{13}} = \partial_{\theta_{13}} + \frac{1}{2}\theta_{32}\partial_{x_{12}} - \frac{1}{2}x_{34}\partial_{\theta_{14}} - \frac{1}{12}\theta_{32}\theta_{24}\partial_{\theta_{14}},$$
$$d_{E_{32}} = \partial_{\theta_{32}} + \frac{1}{2}\theta_{13}\partial_{x_{12}} + \frac{1}{2}\theta_{24}\partial_{x_{34}} - \frac{1}{6}\theta_{13}\theta_{24}\partial_{\theta_{14}},$$
$$d_{E_{24}} = \partial_{\theta_{24}} + \frac{1}{2}\theta_{32}\partial_{x_{34}} + \frac{1}{2}x_{12}\partial_{\theta_{14}} + \frac{1}{12}\theta_{32}\theta_{13}\partial_{\theta_{14}},$$
$$d_{H_{1}} = \Lambda_1 - \theta_{32}\theta_{32} - x_{12}\partial_{x_{12}} - x_{34}\partial_{x_{34}} - \theta_{13}\partial_{\theta_{14}},$$
$$d_{H_{2}} = \Lambda_2 + \theta_{13}\partial_{\theta_{13}} - \theta_{24}\partial_{\theta_{24}} + x_{12}\partial_{x_{12}} - x_{34}\partial_{x_{34}},$$
$$d_{H_{3}} = \Lambda_3 + \theta_{32}\partial_{\theta_{32}} + x_{12}\partial_{x_{12}} + x_{34}\partial_{x_{34}} + \theta_{14}\partial_{\theta_{14}},$$
$$d_{H_{4}} = \Lambda_4 - 2(\theta_{13}\partial_{\theta_{13}} - \theta_{32}\partial_{\theta_{32}} + \theta_{24}\partial_{\theta_{24}} + \theta_{14}\partial_{\theta_{14}}),$$
$$d_{E_{31}} = \theta_{13}\Lambda_1 - x_{12}\partial_{\theta_{32}} + \theta_{14}\partial_{x_{34}} - \frac{1}{2}\theta_{13}(\theta_{32}\partial_{\theta_{32}} + x_{12}\partial_{x_{12}} + x_{34}\partial_{x_{34}} + \theta_{14}\partial_{\theta_{14}})$$
$$+ \frac{1}{12}\theta_{13}(\theta_{32}\theta_{24}\partial_{x_{34}} - x_{12}\theta_{24}\partial_{x_{34}}),$$
$$d_{E_{23}} = \theta_{32}\Lambda_2 + x_{12}\partial_{\theta_{13}} - x_{34}\partial_{\theta_{24}} + \frac{1}{2}\theta_{32}(\theta_{13}\partial_{\theta_{13}} - \theta_{24}\partial_{\theta_{24}} + x_{12}\partial_{x_{12}} - x_{34}\partial_{x_{34}})$$
$$+ \frac{1}{6}\theta_{32}(\theta_{13}x_{34}\partial_{\theta_{14}} + x_{12}\theta_{24}\partial_{\theta_{14}}),$$
$$d_{E_{42}} = \theta_{24}\Lambda_3 + x_{34}\partial_{\theta_{32}} + \theta_{14}\partial_{x_{12}} + \frac{1}{2}\theta_{24}(\theta_{32}\partial_{\theta_{32}} + x_{12}\partial_{x_{12}} + x_{34}\partial_{x_{34}} + \theta_{14}\partial_{\theta_{14}})$$
$$+ \frac{1}{12}\theta_{24}(\theta_{32}\theta_{13}\partial_{x_{34}} - x_{12}\theta_{13}\partial_{x_{34}}).$$
It is straightforward to prove that the above differential operators satisfy the algebraic relations of $gl(2|2)$ algebra. From the differential operators realization, we see that the parameter $\alpha$ involved term has no contribution to the result.

With the help of the differential operator representation we can find the Wakimoto realization of $gl(2|2)$ current superalgebra in terms of two bosonic $\beta-\gamma$ pairs, four fermionic $b-c$ type systems and four free scalar fields $\phi_i$. The free fields obey the following OPEs:

$$
\beta_{ij}(z)\gamma_{kl}(w) = -\gamma_{kl}(z)\beta_{ij}(w) = \frac{\delta_{ik}\delta_{jl}}{z-w},
$$

$$
\psi_{ij}(z)\psi^\dagger_{kl}(w) = \psi^\dagger_{kl}(z)\psi_{ij}(w) = \frac{\delta_{ik}\delta_{jl}}{z-w},
$$

$$
\phi_i(z)\phi_j(w) = -\text{str}(H_i H_j) \ln(z - w). \quad (3.11)
$$

The free field realization of the $gl(2|2)$ current superalgebra is obtained by the substitution:

$$
d_{E_{ij}} \rightarrow J_{ij}(z), \quad d_{H_i} \rightarrow J_i(z), \quad x_{ij} \rightarrow \gamma_{ij}(z), \quad \partial_{x_{ij}} \rightarrow \beta_{ij}(z),
$$
in the differential operator realization of \( gl(2|2) \) and a subsequent addition of anomalous terms linear in \( \partial \psi_{ij}(z) \) and \( \partial \gamma_{ij}(z) \) in currents associated with the lowering generators of \( gl(2|2) \). It also turns out that one needs to add the “anomalous” term \( \partial \phi_1(z) + \partial \phi_3(z) \) with an appropriate coefficient to the current \( J_4(z) \). The result is

\[
J_{14}(z) = \psi_{14}^\dagger(z),
\]

\[
J_{12}(z) = \beta_{12}(z) - \frac{1}{2} \psi_{24}(z) \psi_{14}^\dagger(z),
\]

\[
J_{34}(z) = \beta_{34}(z) + \frac{1}{2} \psi_{13}(z) \psi_{14}^\dagger(z),
\]

\[
J_{13}(z) = \psi_{13}^\dagger(z) + \frac{1}{2} \psi_{32}(z) \beta_{12}(z) - \frac{1}{2} \left( \gamma_{34}(z) + \frac{1}{6} \psi_{32}(z) \psi_{24}(z) \right) \psi_{14}^\dagger(z),
\]

\[
J_{32}(z) = \psi_{32}^\dagger(z) + \frac{1}{2} \psi_{13}(z) \beta_{12}(z) + \frac{1}{2} \psi_{24}(z) \left( \beta_{34}(z) + \frac{1}{3} \psi_{13}(z) \psi_{14}^\dagger(z) \right),
\]

\[
J_{24}(z) = \psi_{24}(z) + \frac{1}{2} \psi_{32}(z) \beta_{34}(z) + \frac{1}{2} \left( \gamma_{12}(z) - \frac{1}{6} \psi_{13}(z) \psi_{32}(z) \right) \psi_{14}^\dagger(z),
\]

\[
J_1(z) = i \sqrt{k} \partial \phi_1(z) - \psi_{32}(z) \psi_{14}^\dagger(z) - \beta_{12}(z) \gamma_{12}(z) - \beta_{34}(z) \gamma_{34}(z) - \psi_{14}(z) \psi_{14}^\dagger(z),
\]

\[
J_2(z) = i \sqrt{k} \partial \phi_2(z) + \psi_{13}(z) \psi_{14}^\dagger(z) - \psi_{24}(z) \psi_{24}^\dagger(z) + \beta_{12}(z) \gamma_{12}(z) - \beta_{34}(z) \gamma_{34}(z),
\]

\[
J_3(z) = i \sqrt{k} \partial \phi_3(z) + \psi_{32}(z) \psi_{32}^\dagger(z) + \beta_{12}(z) \gamma_{12}(z) + \beta_{34}(z) \gamma_{34}(z) + \psi_{14}(z) \gamma_{14}(z),
\]

\[
J_4(z) = i \sqrt{k} \partial \phi_4(z) - \frac{2i}{\sqrt{k}} (\partial \phi_1(z) + \partial \phi_3(z))
\]

\[
- 2 \left( \psi_{13}(z) \psi_{14}^\dagger(z) - \psi_{32}(z) \psi_{24}^\dagger(z) + \psi_{24}(z) \psi_{24}^\dagger(z) + \psi_{14}(z) \psi_{14}^\dagger(z) \right),
\]

\[
J_{31}(z) = i \sqrt{k} \partial \phi_1(z) \psi_{13}(z) - \gamma_{12}(z) \psi_{32}^\dagger(z) + \psi_{14}(z) \beta_{34}(z)
\]

\[
- \frac{1}{2} \psi_{13}(z) \left( \psi_{32}(z) \psi_{14}^\dagger(z) + \beta_{12}(z) \gamma_{12}(z) + \beta_{34}(z) \gamma_{34}(z) + \psi_{14}(z) \psi_{14}^\dagger(z) \right)
\]

\[
+ \frac{1}{12} \psi_{13}(z) \left( \psi_{32}(z) \psi_{24}(z) \beta_{34} - \gamma_{12}(z) \psi_{24}(z) \beta_{34} + k \partial \psi_{13}(z) \right)
\]

\[
J_{23}(z) = i \sqrt{k} \partial \phi_2(z) \psi_{32}(z) + \gamma_{12}(z) \psi_{13}^\dagger(z) - \gamma_{34}(z) \psi_{24}^\dagger(z)
\]

\[
+ \frac{1}{2} \psi_{32}(z) \left( \psi_{13}(z) \psi_{14}^\dagger(z) - \psi_{24}(z) \psi_{24}^\dagger(z) + \beta_{12}(z) \gamma_{12}(z) - \beta_{34}(z) \gamma_{34}(z) \right)
\]

\[
+ \frac{1}{6} \psi_{32}(z) \left( \psi_{13}(z) \gamma_{34}(z) \psi_{14}^\dagger(z) + \gamma_{12}(z) \psi_{24}(z) \psi_{14}^\dagger(z) \right) - k \partial \psi_{32}(z),
\]

\[
J_{42}(z) = i \sqrt{k} \partial \phi_3(z) \theta_{24} + \gamma_{34}(z) \psi_{32}^\dagger(z) + \psi_{14}(z) \beta_{12}(z)
\]

\[
+ \frac{1}{2} \psi_{24}(z) \left( \psi_{32}(z) \psi_{32}^\dagger(z) + \beta_{12}(z) \gamma_{12}(z) + \beta_{34}(z) \gamma_{34}(z) + \psi_{14}(z) \psi_{14}^\dagger(z) \right)
\]

\[
+ \frac{1}{12} \psi_{24}(z) \left( \psi_{13}(z) \psi_{32}(z) \beta_{12} - \psi_{13}(z) \gamma_{34}(z) \psi_{14}^\dagger(z) \right) + k \partial \psi_{24}(z),
\]

\[
J_{21}(z) = i \sqrt{k} \partial \phi_1(z) \gamma_{12}(z) - \frac{1}{2} \psi_{13}(z) \psi_{32}(z) \right) - i \sqrt{k} \partial \phi_2(z) \gamma_{12}(z) + \frac{1}{2} \psi_{13}(z) \psi_{32}(z))
\]

\[
- \gamma_{12}(z) \left( \psi_{13}(z) \psi_{14}^\dagger(z) + \psi_{32}(z) \psi_{32}^\dagger(z) + \beta_{12}(z) \gamma_{12}(z) \right)
\]

\[
- \left( \psi_{14}(z) - \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) - \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) - \frac{1}{6} \psi_{13}(z) \psi_{32}(z) \psi_{24}(z) \right) \psi_{24}(z)
\]

\[
+ \frac{1}{2} \psi_{32}(z) \left( \psi_{14}(z) + \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) + \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) \right) \beta_{34}(z)
\]

\[\text{(3.12)}\]
\[-\frac{1}{2} \gamma_{12}(z) \left( \psi_{14}(z) + \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) + \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) \right) \psi_{14}^{\dagger}(z) \]
\[+ \frac{1}{12} \psi_{13}(z) \psi_{23}(z) \left( \psi_{14}(z) - \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) \right) \psi_{14}^{\dagger}(z) + (k - 1/2) \partial \gamma_{12}(z) \]
\[+ \frac{1}{2} \left( k + 1/6 \right) \psi_{32}(z) \partial \psi_{13}(z) + \frac{1}{2} \left( k + 1/3 \right) \psi_{13}(z) \partial \psi_{32}(z), \]
\[J_{13}(z) = i \sqrt{\kappa} \partial \phi_{2}(z) (\gamma_{34}(z) - \frac{1}{2} \psi_{32}(z) \psi_{24}(z)) - i \sqrt{\kappa} \partial \phi_{3}(z) (\gamma_{34}(z) + \frac{1}{2} \psi_{32}(z) \psi_{24}(z)) \]
\[- \gamma_{34}(z) \left( \psi_{32}(z) \psi_{32}^{\dagger}(z) + \psi_{24}(z) \psi_{24}^{\dagger}(z) + \beta_{34}(z) \gamma_{34}(z) \right) \]
\[- \left( \psi_{14}(z) + \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) + \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) - \frac{1}{6} \psi_{13}(z) \psi_{32}(z) \psi_{24}(z) \right) \psi_{13}^{\dagger}(z) \]
\[- \frac{1}{2} \psi_{32}(z) \left( \psi_{14}(z) + \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) - \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) \right) \beta_{34}(z) \]
\[- \frac{1}{2} \gamma_{34}(z) \left( \psi_{14}(z) - \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) - \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) \right) \psi_{14}^{\dagger}(z) \]
\[- \frac{1}{12} \psi_{32}(z) \psi_{24}(z) \left( \psi_{14}(z) + \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) \right) \psi_{14}^{\dagger}(z) - (k + 1/2) \partial \gamma_{34}(z) \]
\[- \frac{1}{2} \left( k - 1/3 \right) \psi_{24}(z) \partial \psi_{32}(z) - \frac{1}{2} \left( k - 1/6 \right) \psi_{32}(z) \partial \psi_{24}(z), \]
\[J_{11}(z) = i \sqrt{\kappa} \partial \phi_{1}(z) \left( \psi_{14}(z) + \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) + \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) - \frac{1}{6} \psi_{13}(z) \psi_{32}(z) \psi_{24}(z) \right) \]
\[- i \sqrt{\kappa} \partial \phi_{2}(z) \left( \psi_{14}(z) + \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) - \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) + \frac{1}{3} \psi_{13}(z) \psi_{32}(z) \psi_{24}(z) \right) \]
\[+ i \sqrt{\kappa} \partial \phi_{3}(z) \left( \psi_{14}(z) - \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) - \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) - \frac{1}{6} \psi_{13}(z) \psi_{32}(z) \psi_{24}(z) \right) \]
\[+ \psi_{13}(z) \left( \psi_{14}(z) + \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) \right) \psi_{13}^{\dagger}(z) + \frac{1}{6} \gamma_{12}(z) \psi_{24}(z) \psi_{13}(z) \gamma_{34}(z) \psi_{14}^{\dagger}(z) \]
\[- \left( \gamma_{12}(z) \gamma_{34}(z) - \frac{1}{2} \gamma_{12}(z) \psi_{32}(z) \psi_{24}(z) + \frac{1}{2} \psi_{13}(z) \psi_{32}(z) \gamma_{34}(z) \right) \psi_{13}^{\dagger}(z) \]
\[- \gamma_{12}(z) \left( \psi_{14}(z) + \frac{1}{2} \gamma_{12}(z) \psi_{24}(z) + \frac{1}{2} \psi_{13}(z) \psi_{32}(z) \gamma_{34}(z) \right) \beta_{12} \]
\[+ \gamma_{34}(z) \left( \psi_{14}(z) - \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) + \frac{1}{12} \psi_{13}(z) \psi_{32}(z) \psi_{24}(z) \right) \beta_{34}(z) \]
\[- \psi_{24}(z) \left( \psi_{14}(z) - \frac{1}{2} \psi_{13}(z) \gamma_{34}(z) \right) \psi_{24}^{\dagger}(z) + \frac{1}{2} \left( k - 1 \right) \psi_{24}(z) \partial \gamma_{12}(z) \]
\[- \frac{k}{2} \gamma_{12}(z) \partial \psi_{24}(z) - \frac{1}{2} \left( k + 1 \right) \psi_{13}(z) \partial \gamma_{34}(z) + \frac{2}{3} \gamma_{34}(z) \partial \psi_{13}(z) \]
\[- \frac{1}{6} \left( k + 1/2 \right) \psi_{32}(z) \psi_{24}(z) \partial \psi_{13}(z) - \frac{k}{3} \psi_{13}(z) \psi_{24}(z) \partial \psi_{32}(z) \]
\[- \frac{1}{6} \left( k - 1/2 \right) \psi_{13}(z) \psi_{32}(z) \partial \psi_{24}(z) + k \partial \psi_{14}(z). \quad (3.13) \]

We remark that even though the parameter \( \alpha \) does not occur in the expression of the currents, it appears in the following OPE

\[J_{4}(z) J_{4}(w) = k \frac{8 \alpha}{(z - w)^2} + \ldots.\]

So different choice of \( \alpha \) will give different OPE of \( J_{4}(z) \) with itself. It is straightforward to
check that the above currents satisfy the OPEs of the \( gl(2|2) \) current superalgebra given in the last section.

## 4 Energy-momentum tensor

The CFT associated with the \( gl(2|2) \) current superalgebra is obtained by constructing the Sugawara energy-momentum tensor. The Sugawara tensor corresponding to the quadratic Casimir \( C_1 \) is given by

\[
T_1(z) = \frac{1}{2k} \sum_{i,j} (-1)^{g(i)} J_{ij}(z) J_{ji}(z) :
\]

\[
\begin{align*}
&= \beta_{12}(z) \partial \gamma_{12}(z) + \beta_{34}(z) \partial \gamma_{34}(z) - \psi_{13}^\dagger(z) \partial \psi_{13}(z) \\
&\quad - \psi_{32}^\dagger(z) \partial \psi_{32}(z) - \psi_{24}^\dagger(z) \partial \psi_{24}(z) - \psi_{14}^\dagger(z) \partial \psi_{14}(z) \\
&\quad - \frac{1}{4} \left( (\partial \phi_1(z))^2 - (\partial \phi_3(z))^2 + (\partial \phi_1(z) + \partial \phi_3(z)) \partial \phi_4(z) \right) \\
&\quad + \frac{1}{2} (\partial \phi_1(z) - \partial \phi_3(z)) \partial \phi_2(z) + \frac{i}{2\sqrt{k}} \left( \partial^2 \phi_1(z) + \partial^2 \phi_3(z) \right) \\
&\quad + \frac{4 + k\alpha}{4k} (\partial \phi_1(z) + \partial \phi_3(z))^2.
\end{align*}
\]

(4.14)

With respect to this tensor, all currents except \( J_4(z) \) are primary fields. However, its OPE with \( J_4(z) \) reads

\[
T_1(z) J_4(w) = \frac{1}{(z-w)^2} J_4(w) + \frac{1}{(z-w)} \partial J_4 \\
\quad - \frac{1}{(z-w)^2} \frac{4i}{\sqrt{k}} (\partial \phi_1(w) + \partial \phi_3(w)) \\
\quad - \frac{1}{(z-w)} \frac{4i}{\sqrt{k}} \left( \partial^2 \phi_1(w) + \partial^2 \phi_3(w) \right).
\]

(4.15)

This means that \( T_1(z) \) is not a current energy-momentum tensor of the theory.

Now the Sugawara tensor associated with the Casimir \( C_2 \) is

\[
T_2(z) = \frac{1}{2k} \sum_{i,j} J_{ii}(z) J_{jj}(z) := -\frac{1}{2} (\partial \phi_1(z) + \partial \phi_3(z))^2.
\]

(4.16)

It has a non-trivial OPE with current \( J_4(z) \):

\[
T_2(z) J_4(w) = \frac{4i\sqrt{k}}{(z-w)^2} (\partial \phi_1(w) + \partial \phi_3(w)) + \frac{4i\sqrt{k}}{(z-w)} \left( \partial^2 \phi_1(w) + \partial^2 \phi_3(w) \right).
\]

(4.17)

Comparing the two above OPEs, we see that if we define

\[
T(z) = T_1(z) + \frac{1}{k} T_2(z)
\]

(4.18)

then all currents become primary with respect to \( T(z) \) and moreover

\[
T(z) T(w) = -\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w},
\]

(4.19)
where the central charge $c = 0$. So $T(z)$ is the energy-momentum tensor of the $gl(2|2)$ current superalgebra. In terms of the free fields $T(z)$ reads

$$
T(z) = \beta_{12}(z)\partial\gamma_{12}(z) + \beta_{34}(z)\partial\gamma_{34}(z) - \psi_{13}^\dagger(z)\partial\psi_{13}(z) - \psi_{32}^\dagger(z)\partial\psi_{32}(z) - \psi_{24}^\dagger(z)\partial\psi_{24}(z) - \psi_{14}^\dagger(z)\partial\psi_{14}(z)
- \frac{1}{4} \left( (\partial\phi_1(z))^2 - (\partial\phi_3(z))^2 + (\partial\phi_1(z) + \partial\phi_3(z))\partial\phi_4(z) \right)
+ \frac{1}{2} (\partial\phi_1(z) - \partial\phi_3(z))\partial\phi_2(z) + \frac{i}{2\sqrt{k}} \left( \partial^2\phi_1(z) + \partial^2\phi_3(z) \right)
+ \frac{\alpha}{4} (\partial\phi_1(z) + \partial\phi_3(z))^2.
$$

(4.20)

5 Screening currents

An important object in the free field approach is screening current. Screening currents are primary fields with conformal dimension 1, and their integrations give the screening charges. Screening currents commute with current generators up to a total derivative. These properties ensure that screening charges may be inserted into correlators while the conformal or affine ward identities remain intact. For the present case, we find three screening currents,

$$
\begin{align*}
 s_{13}(z) &= \left( -\psi_{13}^\dagger(z) + \frac{1}{2} \beta_{12}(z)\psi_{32}(z) - \frac{1}{2} \left( \gamma_{34}(z) - \frac{1}{6} \psi_{32}(z)\psi_{24}(z) \right)\psi_{14}(z) \right) \tilde{s}_{13}(z), \\
 s_{32}(z) &= \left( -\psi_{32}^\dagger(z) + \frac{1}{2} \beta_{12}(z)\psi_{13}(z) + \frac{1}{2} \psi_{24}(z) \left( \beta_{34}(z) - \frac{1}{3} \psi_{13}(z)\psi_{14}(z) \right) \right) \tilde{s}_{32}(z), \\
 s_{24}(z) &= \left( -\psi_{24}^\dagger(z) + \frac{1}{2} \beta_{34}(z)\psi_{32}(z) + \frac{1}{2} \left( \gamma_{12}(z) + \frac{1}{6} \psi_{13}(z)\psi_{32}(z) \right)\psi_{14}(z) \right) \tilde{s}_{24}(z),
\end{align*}
$$

(5.21)

where

$$
\begin{align*}
 \tilde{s}_{13}(z) &= \exp\{-\frac{i}{\sqrt{k}} \phi_1(z)\}, & \tilde{s}_{32}(z) &= \exp\{\frac{i}{\sqrt{k}} \phi_2(z)\}, \\
 \tilde{s}_{24}(z) &= \exp\{-\frac{i}{\sqrt{k}} \phi_3(z)\}.
\end{align*}
$$

(5.22)

The OPEs with the currents are

$$
\begin{align*}
 J_{31}(z)s_{13}(w) &= \partial_w \left( \frac{k}{z-w} \tilde{s}_{13}(w) \right), \\
 J_{23}(z)s_{13}(w) &= \ldots, & J_{42}(z)s_{13}(w) &= \ldots, \\
 J_{21}(z)s_{13}(w) &= \partial_w \left( \frac{k}{z-w} \psi_{32}(w)\tilde{s}_{13}(w) \right), & J_{43}(z)s_{13}(w) &= \ldots, \\
 J_{41}(z)s_{13}(w) &= \partial_w \left( \frac{k}{z-w} \left( \gamma_{34}(w) - \frac{1}{2} \psi_{32}(w)\psi_{24}(w) \right)\tilde{s}_{13}(w) \right), \\
 J_{23}(z)s_{32}(w) &= -\partial_w \left( \frac{k}{z-w} \tilde{s}_{32}(w) \right), \\
 J_{31}(z)s_{32}(w) &= \ldots, & J_{42}(z)s_{32}(w) &= \ldots,
\end{align*}
$$

8
\[
\begin{align*}
J_{21}(z)s_{32}(w) &= \partial_w \left( \frac{k}{z-w} \psi_{13}(w) \tilde{s}_{32}(w) \right), \\
J_{43}(z)s_{32}(w) &= -\partial_w \left( \frac{k}{z-w} \psi_{24}(w) \tilde{s}_{32}(w) \right), \\
J_{41}(z)s_{32}(w) &= -\partial_w \left( \frac{k}{z-w} \psi_{13}(w) \psi_{24}(w) \tilde{s}_{32}(w) \right), \\
J_{42}(z)s_{24}(w) &= \partial_w \left( \frac{k}{z-w} \tilde{s}_{24}(w) \right), \\
J_{31}(z)s_{24}(w) &= \ldots, \quad J_{23}(z)s_{24}(w) = \ldots, \\
J_{43}(z)s_{24}(w) &= -\partial_w \left( \frac{k}{z-w} \psi_{32}(w) \tilde{s}_{24}(w) \right), \quad J_{12}(z)s_{24}(w) = \ldots, \\
J_{11}(z)s_{24}(w) &= -\partial_w \left( \frac{k}{z-w} (\gamma_{12}(w) + \frac{1}{2} \psi_{13}(w) \psi_{32}(w)) \tilde{s}_{24}(w) \right). 
\end{align*}
\]

(5.23)

All other OPEs are trivial.

6 Discussions

We have studied the non-semisimple current superalgebra \(gl(2|2)_k^{(1)}\) at the general level \(k\). We have constructed its Wakimoto free field representation and Sugawara energy-momentum tensor in the non-standard basis. We have found three screen currents of the first kind.

The motivation for this study is the application of the \(gl(2|2)\) current superalgebra to disordered systems. To fully take the advantage of the CFT method, we need to construct its primary fields. There are two types of representations for this current superalgebra: typical and atypical representations. Atypical representations have no counterpart in the bosonic algebra setting and our understanding to such representations is still very much incomplete. So it is a highly non-trivial task to construct primary fields corresponding to the atypical representations. On the other hand, typical representations are similar to those appearing in a bosonic algebra, and so primary fields associated with the typical representations can be constructed by the usual procedure. Results for the primary fields will be published in a separate paper.

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