On instability of solitons in the 2d cubic Zakharov–Kuznetsov equation

Luiz Gustavo Farah · Justin Holmer · Svetlana Roudenko

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Abstract
We consider the critical generalized Zakharov–Kuznetsov (ZK) equation, \( u_t + \partial_{x_1}(\Delta u + u^3) = 0 \), \((x_1, x_2) \in \mathbb{R}^2\). In Farah et al. (Instability of solitons in the 2d cubic Zakharov–Kuznetsov equation, arXiv:1711.05907, 2017), we proved that solitons are unstable for this equation following the strategy by Martel and Merle (GAFA Geom Funct Anal 11:74–123, 2001) in their study of the critical generalized Kortweg–de Vries equation. The main ingredient used in Farah et al. (Instability of solitons in the 2d cubic Zakharov–Kuznetsov equation, arXiv:1711.05907, 2017) was the new pointwise decay estimates in two dimensions together with monotonicity properties of solutions. In this paper, we show that using only monotonicity properties and not relying on pointwise estimates, thus, greatly simplifying the approach, we can prove an instability of solitons, though a slightly weaker version.

Keywords Zakharov–Kuznetsov equation · Instability of solitons · Monotonicity · Virial · \( L^2 \)-critical

Mathematics Subject Classification Primary: 35Q53 · 37K40 · 37K45 · 37K05
1 Introduction

In this paper we consider the Cauchy problem of the 2d cubic ZK equation (also referred to as the modified ZK, \(m\)ZK, or the generalized ZK, \(g\)ZK) with initial data \(u_0\):

\[
\begin{cases}
  u_t + \partial_{x_1} \left( \Delta_{(x_1,x_2)} u + u^3 \right) = 0, & (x_1, x_2) \in \mathbb{R}^2, \ t > 0, \\
  u(0, x_1, x_2) = u_0(x_1, x_2) \in H^1(\mathbb{R}^2). 
\end{cases}
\]  

(1.1)

For physically relevant applications of this model see Zakharov–Kuznetsov [17], Monro–Parkes [13,14] and Melkonian–Maslowe [11]. Rigorous derivation can be found in Lannes–Linares–Saut [7] and Han-Kwan [4].

The Eq. (1.1) is scale invariant and the rescaled solution is given by

\[ u_\lambda(t, x_1, x_2) = \lambda u(\lambda^3 t, \lambda x_1, \lambda x_2), \]

which makes the \(L^2\)-norm invariant, i.e.,

\[ \|u_\lambda(0, \cdot, \cdot)\|_{L^2} = \|u_0\|_{L^2}. \]

Moreover, the solutions \(u(t, x_1, x_2)\) to (1.1) conserve the mass and energy

\[
M[u(t)] = \int_{\mathbb{R}^2} [u(t, x_1, x_2)]^2 \, dx_1 dx_2 = M[u(0)] \tag{1.2}
\]

and

\[
E[u(t)] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t, x_1, x_2)|^2 \, dx_1 dx_2 - \frac{1}{4} \int_{\mathbb{R}^2} [u(t, x_1, x_2)]^4 \, dx_1 dx_2 = E[u(0)]. \tag{1.3}
\]

It is known that the generalized Zakharov–Kuznetsov equation has a family of special solutions. Indeed, if \(Q\) is the unique radial and positive solution in \(H^1(\mathbb{R}^2)\) of the elliptic equation

\[
-\Delta Q + Q - Q^5 = 0,
\]

then the function

\[ u(t, x_1, x_2) = Q_c(x_1 - ct, x_2) \]

is a solution of Eq. (1.1), which travels only in \(x_1\) direction. This solution sometimes is referred to as a traveling wave, or a solitary wave, or simply, a soliton.

In [3] we proved the following instability of the soliton \(u(t, x_1, x_2) = Q(x_1 - t, x_2)\).
**Theorem 1.1** [3] There exists $\alpha_0 > 0$ such that for any $\delta > 0$ there exists $u_0 \in H^1(\mathbb{R}^2)$ satisfying

$$\|u_0 - Q\|_{H^1} \leq \delta$$

and for the corresponding solution $u(t)$ with initial data $u_0 \in H^1(\mathbb{R}^2)$, there exists a positive time $t_0 = t_0(u_0) < \infty$ such that

$$\inf_{\vec{z} \in \mathbb{R}^2} \|u(t_0, \cdot) - Q(\cdot - \vec{z})\|_{H^1} \geq \alpha_0.$$

As a consequence of the above theorem, $Q$ is not stable in the following sense.

**Definition 1.2** We say that $Q$ is stable if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$\|u_0 - Q\|_{H^1} \leq \delta,$$

then the corresponding solution $u(t)$ of Eq. (1.1) with initial data $u_0$ is defined for all $t \geq 0$ and

$$\sup_{t \geq 0} \inf_{\vec{z} \in \mathbb{R}^2} \|u(t, \cdot) - Q(\cdot - \vec{z})\|_{H^1} < \varepsilon.$$

On the other hand, for certain other dispersive equations it is possible to prove that solitons are stable in the energy space using the following (stronger) definition.

**Definition 1.3** We say that $Q$ is (uniformly) stable if there exist $C_0, \varepsilon_0 > 0$ such that

$$\|u_0 - Q\|_{H^1} = \varepsilon \leq \varepsilon_0,$$

then the corresponding solution $u(t)$ of Eq. (1.1) with initial data $u_0$ is defined for all $t \geq 0$ and

$$\sup_{t \geq 0} \inf_{\vec{z} \in \mathbb{R}^2} \|u(t, \cdot) - Q(\cdot - \vec{z})\|_{H^1} < C_0 \varepsilon.$$

This stronger type of stability has been shown in other equations, for instance, see Bennett–Brown–Stansfield–Stroughair–Bona [1], Weinstein [16] and Kenig–Martel [5] where they proved stability of solitons following Definition 1.3 for the Benjamin–Ono equation. In this note, we want to show that this stronger stability does not hold for the Eq. (1.1).

**Theorem 1.4** $Q$ is not stable for the critical generalized Zakharov–Kuznetsov equation (1.1) in the sense of Definition 1.3.

Theorem 1.1 is obviously stronger than Theorem 1.4. However, to obtain Theorem 1.1, we designed in [3] a new virial-type quantity, revisited monotonicity properties and, most importantly, developed new pointwise decay estimates. The main goal of
this paper is to show that a slightly weaker instability property, Theorem 1.4, can be proved in a much simpler way, without relying on pointwise estimates, instead only using monotonicity properties of the solutions.

The rest of the paper is organized as follows. In Sect. 2, we collect several preliminary results already proved in [3]. Section 3 is devoted to the proof of Theorem 1.4.

2 Canonical decomposition and preliminary results

We start with the following canonical decomposition of $u$ around $Q$:

$$v(t, y_1, y_2) = \lambda(t) u(t, \lambda(t)y_1 + x_1(t), \lambda(t)y_2 + x_2(t)).$$

We study the difference $\varepsilon = v - Q$, more precisely,

$$\varepsilon(t, \vec{y}) = v(t, \vec{y}) - Q(\vec{y}), \quad \vec{y} = (y_1, y_2).$$

If we rescale time $t \mapsto s$ by $\frac{ds}{dt} = \frac{1}{\lambda^3}$, the equation for $\varepsilon$ is given by (see [3, Lemma 4.1])

$$\varepsilon_s = (L\varepsilon)_{y_1} + \frac{\lambda_s}{\lambda} \Lambda Q + \left( \frac{(x_1)_s}{\lambda} - 1 \right) Q_{y_1} + \frac{(x_2)_s}{\lambda} Q_{y_2}$$

$$+ \frac{\lambda_s}{\lambda} \Lambda \varepsilon + \left( \frac{(x_1)_s}{\lambda} - 1 \right) \varepsilon_{y_1} + \frac{(x_2)_s}{\lambda} \varepsilon_{y_2}$$

$$- 3(Q\varepsilon^2)_{y_1} - (\varepsilon^3)_{y_1}, \quad (2.4)$$

where $\Lambda f = f + \vec{y} \cdot \nabla f$ and $L$ is the linearized operator around $Q$:

$$L := -\Delta + 1 - 3 Q^2. \quad (2.5)$$

Recall that $Q \in C^\infty(\mathbb{R}^2)$, $\partial_r Q(r) < 0$ for any $r = |\vec{x}| > 0$, and for any multi-index $\alpha \in \mathbb{N}^n$

$$|\partial^\alpha Q(\vec{x})| \leq c(\alpha) e^{-|\vec{x}|} \quad \text{for any } \vec{x} = (x_1, x_2) \in \mathbb{R}^2. \quad (2.6)$$

The operator $L$ is well understood (see Kwong [6] for all dimensions, Weinstein [15] for dimension 1 and 3, also Maris [8] and Chang et al. [2]). The next theorem summarizes some important properties of the linearized operator.

Theorem 2.1 (Properties of $L$) The following holds for an operator $L$ defined in (2.5)

- $L$ is a self-adjoint operator and $\sigma_{\text{ess}}(L) = [\lambda_{\text{ess}}, +\infty)$ for some $\lambda_{\text{ess}} > 0$
- $\ker L = \text{span}\{Qx_1, Qx_2\}$
L has a unique single negative eigenvalue $-\lambda_0$ (with $\lambda_0 > 0$) associated to a positive radially symmetric eigenfunction $\chi_0$. Moreover, there exists $\delta > 0$ such that

$$|\chi_0(x)| \lesssim e^{-\delta|x|} \text{ for all } x \in \mathbb{R}^2.$$ 

Coming back to the solution $\varepsilon(s)$ of Eq. (2.4), using mass and energy conservation (1.2), (1.3), we deduce that this solution also enjoys the following properties (see [3, Lemma 4.2]).

**Lemma 2.2** For any $s \geq 0$ we have mass and energy conservations for $\varepsilon$

$$M[\varepsilon(s)] = 2 \int_{\mathbb{R}^2} Q(\vec{y}) \varepsilon(0, \vec{y}) \, d\vec{y} + \int_{\mathbb{R}^2} \varepsilon^2(0, \vec{y}) \, d\vec{y} = M_0, \quad \text{and}$$

$$E[Q + \varepsilon(s)] = \lambda^2(s) E[u_0]. \quad (2.7)$$

Next, we recall the modulation theory and parameter estimates. To this end, we first introduce the following set: for $\alpha > 0$, the neighborhood (or “tube”) of radius $\alpha$ around $Q$ (modulo translations) is defined by

$$U_\alpha = \left\{ u \in H^1(\mathbb{R}^2) : \inf_{\vec{y} \in \mathbb{R}^2} \| u(\cdot) - Q(\cdot + \vec{y}) \|_{H^1} \leq \alpha \right\}.$$ 

Assume that $u(t) \in U_\alpha$ for all $t \geq 0$ with $\alpha > 0$ sufficiently small. Using the Implicit Function Theorem, we can define functions $\lambda(t)$ and $x(t)$ such that (see [3, Section 5])

$$\varepsilon(t) = \varepsilon_{\lambda(t),x(t)} = \lambda(t) u(t, \lambda(t)y_1 + x(t), \lambda(t)y_2) - Q(y_1, y_2) \quad (2.8)$$

satisfies

$$\varepsilon_{\lambda(t),x(t)} \perp \chi_0 \quad \text{and} \quad \varepsilon_{\lambda(t),x(t)} \perp Q y_j, \quad j = 1, 2.$$ 

Moreover, there exists a constant $C_1 > 0$, such that if $u \in U_\alpha$, with $0 < \alpha < \overline{\alpha}$, then

$$\| \varepsilon(t) \|_{H^1} \leq C_1 \alpha \quad \text{and} \quad |\lambda(t) - 1| \leq C_1 \alpha. \quad (2.9)$$

Rescaling time $t \mapsto s$ by $\frac{ds}{dt} = \frac{1}{\lambda^3}$, we deduce that $\lambda$ and $x$ are $C^1$ functions of $s$, satisfying the following equations (see [3, Lemma 5.4])

$$-\frac{\lambda_s}{\lambda} \int (\vec{y} \cdot \nabla Q y_1) \varepsilon + \left( \frac{X_s}{\lambda} - 1 \right) \left( \int |Q y_1|^2 - \int Q y_1 y_1 \varepsilon \right)$$

$$= 6 \int \underbrace{Q Q^2 y_1 \varepsilon - 3 \int Q y_1 y_1 \varepsilon^2 Q - \int Q y_1 y_1 \varepsilon^3}_{\text{ }}.$$
and

\[
\frac{\lambda_s}{\lambda} \left( \frac{2}{\lambda_0} \int Q - \int (\bar{y} \cdot \nabla \chi_0)\varepsilon \right) - \left( \frac{x_s}{\lambda} - 1 \right) \int (\chi_0)_{y_1} \varepsilon \\
= \int L((\chi_0)_{y_1})\varepsilon - 3 \int (\chi_0)_{y_1} Q \varepsilon^2 - \int (\chi_0)_{y_1} \varepsilon^3.
\]

Furthermore, there exists a universal constant \( C_2 > 0 \) such that if \( \| \varepsilon(s) \|_2 \leq \alpha \), for all \( s \geq 0 \), where \( \alpha < \alpha \) (reducing \( \alpha \) if necessary), then

\[
\left| \frac{\lambda_s}{\lambda} \right| + \left| \frac{x_s}{\lambda} - 1 \right| \leq C_2 \| \varepsilon(s) \|_2.
\]

Next, we recall the Lyapunov functional used in [3]. Let \( \varphi \in C_0^\infty(\mathbb{R}) \) be a function with

\[
\varphi(y_1) = \begin{cases} 
1, & \text{if } y_1 \leq 1 \\
0, & \text{if } y_1 \geq 2.
\end{cases}
\]

For \( A \geq 1 \) we define

\[
\varphi_A(y_1) = \varphi \left( \frac{y_1}{A} \right).
\]

We next consider the function

\[
F(y_1, y_2) = \int_{-\infty}^{y_1} \Lambda Q(z, y_2) dz.
\]

We define the virial-type functional by

\[
J_A(s) = \int_{\mathbb{R}^2} \varepsilon(s, y_1, y_2) F(y_1, y_2) \varphi_A(y_1) dy_1 dy_2.
\]

From the properties of \( Q \), see (2.6), we deduce that (see [3, Section 6])

\[
|J_A(s)| \leq c(1 + A^{1/2}) \| \varepsilon(s) \|_2.
\]

The derivative of \( J_A(s) \) is given in the following lemma (see [3, Lemma 6.1]).

**Lemma 2.3** Suppose that \( \varepsilon(s) \in H^1(\mathbb{R}^2) \) for all \( s \geq 0 \). Then the function \( s \mapsto J_A(s) \) is \( C^1 \) and

\[
\frac{d}{ds} J_A = -\frac{\lambda_s}{\lambda} (J_A - \kappa) + 2 \left( 1 - \frac{1}{2} \left( \frac{x_s}{\lambda} - 1 \right) \right) \int \varepsilon Q + R(\varepsilon, A).
\]

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where
\[
\kappa = \frac{1}{2} \int y_2^2 \left( \int Q_{y_2}(y_1, y_2) dy_1 \right)^2 dy_2 < \infty \quad \text{(by decay properties of } Q\text{, see (2.6))},
\]
and, there exists a universal constant } C_3 > 0 \text{ such that, for } A \geq 1, \text{ we have}
\[
|R(\varepsilon, A)| \leq C_3 \left( \|\varepsilon\|_2^2 + 2\|\varepsilon\|_2^2\|\varepsilon\|_{H^1} + A^{-1/2}\|\varepsilon\|_2 
+ \left| \frac{x_s}{\lambda} - 1 \right| \left( A^{-1} + \|\varepsilon\|_2 \right)
+ \left| \frac{x_s}{\lambda} \right| \left( A^{-1} + \|\varepsilon\|_2 + A^{1/2}\|\varepsilon\|_{L^2(y_1 \geq A)} + \left| \int_{\mathbb{R}^2} y_2 F_{y_2} \varepsilon \varphi_A \right| \right) \right).
\]

The last tool in our proof is the monotonicity properties of the solutions. Following the works of Martel and Merle [9] and [12], we consider
\[
\psi(x_1) = \frac{2}{\pi} \arctan \left( e^{\frac{x_1}{M}} \right),
\]
for } M \geq 4. Now, let } (x_1(t), x_2(t)) \in C^1(\mathbb{R}, \mathbb{R}^2) \text{ and for } x_0, t_0 > 0 \text{ and } t \in [0, t_0] \text{ define}
\[
I_{x_0, t_0}(t) = \int u^2(t, x_1, x_2) \psi(x_1 - x_1(t_0)) + \frac{1}{2}(t_0 - t) - x_0) dx_1 dx_2,
\]
where } u \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \text{ is a solution of the gZK equation (1.1), satisfying}
\[
\|u(t, x_1 + x_1(t), x_2 + x_2(t)) - Q(x_1, x_2)\|_{H^1} \leq \alpha, \text{ for some } \alpha > 0.
\]

The key estimate is given in the following lemma (see [3, Lemma 7.2]).

**Lemma 2.4** Let } M \geq 4 \text{ fixed and assume that } x_1(t) \text{ is an increasing function satisfying}
\[
x_1(t_0) - x_1(t) \geq \frac{3}{4}(t_0 - t) \text{ for every } t_0, t \geq 0 \text{ with } t \in [0, t_0]. \text{ Also assume that}
\]
\[
x_1(t) \geq \frac{1}{2}t \text{ and } x_2(t) = 0 \text{ for all } t \geq 0. \text{ Moreover, let } u \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \text{ be a solution of the gZK Eq. (1.1) satisfying (2.15) with } \alpha < \overline{\alpha} \text{ (reducing } \overline{\alpha} \text{ if necessary)}
\]
and with the initial data } u_0 \text{ verifying } \int |u_0(x_1, x_2)|^2 dx_2 \leq c e^{-\delta|x_1|} \text{ for some } c > 0 \text{ and } \delta > 0. \text{ Fix } M \geq \max\{4, \frac{2}{3}\}, \text{ then there exists } C_4 = C(M, \delta) > 0 \text{ such that for all}
\[
t \geq 0 \text{ and } x_0 > 0
\]
\[
\int_{\mathbb{R}} \int_{x_1 > x_0} u^2(t, x_1 + x_1(t), x_2) dx_1 dx_2 \leq C_4 e^{-\frac{x_0}{M}}.
\]

The above result implies the } L^2 \text{ exponential decay on the right for } \varepsilon(s) \text{ (see [3, Corollary 13.1]).}
Corollary 2.5  Let $M \geq 4$. Under the assumptions of Lemma 2.4, if $\alpha > 0$ is sufficiently small, then there exists $C_5 = C(M, \delta) > 0$ such that for every $s \geq 0$ and $y_0 > 0$

$$\int_{\mathbb{R}} \int_{y_1 > y_0} \varepsilon^2(s, y_1, y_2) dy_1 dy_2 \leq C_5 e^{-\frac{y_0}{2M}}.$$

3 Instability of $Q$

We now have all tools to prove our main result of this note.

Proof of Theorem 1.4  Let $n \in \mathbb{N}$ and define

$$u^n_0 = Q + \varepsilon^n_0,$$

where

$$\varepsilon^n_0 = \frac{1}{n} (Q + a \chi_0), \quad (3.16)$$

and $a \in \mathbb{R}$ is such that $\varepsilon^n_0 \perp \chi_0$, that is,

$$a = -\int \chi_0 Q \| \chi_0 \|^2_2.$$

From Theorem 2.1, we have that for every $n \in \mathbb{N}$

$$\varepsilon^n_0 \perp \{Qy_1, Qy_2, \chi_0\}.$$

Moreover, for some $C_6 > 0$ we have

$$\|u^n_0 - Q\|_{H^1} \leq \frac{C_6}{n}.$$

Denote by $u^n(t)$ the corresponding solution of (1.1) with initial data $u^n_0$. Assuming by contradiction that $Q$ is stable in the sense of Definition 1.3, there exists $C_0 > 0$ such that

$$\sup_{t \geq 0} \inf_{\tilde{z} \in \mathbb{R}^2} \|u(t, \cdot) - Q(\cdot - \tilde{z})\|_{H^1} < \frac{C_0 C_6}{n}$$

for $n$ sufficiently large.

Moreover, increasing $n$ if necessary, there exist functions $\lambda^n(t)$ and $x^n(t)$ (with $\lambda^n(0) = 1$ and $x^n(0) = 0$) such that $\varepsilon^n(t)$, defined in (2.8), satisfies

$$\varepsilon^n(t) \perp \{Qy_1, Qy_j, \chi_0\}.$$
Rescaling the time $t \mapsto s$ by $\frac{ds}{dt} = \frac{1}{\lambda^3}$, from (2.9) we have

$$\|\varepsilon^n(s)\|_{H^1} \leq \frac{C_0 C_1 C_6}{n} \quad \text{and} \quad |\lambda^n(s) - 1| \leq \frac{C_0 C_1 C_6}{n}. \quad (3.17)$$

Again increasing $n$ if necessary, we deduce

$$\frac{1}{2} \leq \lambda^n(s) \leq \frac{3}{2}, \quad \text{for all} \quad s \geq 0. \quad (3.18)$$

Furthermore, in view of (2.10), if $n$ is large enough, we have

$$\left|\frac{\lambda^n_s}{\lambda^n} + \frac{x^n_s}{\lambda^n} - 1\right| \leq C_2\|\varepsilon^n(s)\|_2 \leq \frac{C_0 C_1 C_2 C_6}{n}. \quad (3.19)$$

Since $x_t = x_s/\lambda^3$, we can choose $n$ large enough, such that

$$\frac{3}{4} \leq x^n_s \leq \frac{5}{4}. \quad (3.20)$$

The last inequality implies that $x^n(t)$ is increasing and by the Mean Value Theorem

$$x^n(t_0) - x^n(t) \geq \frac{3}{4}(t_0 - t)$$

for every $t_0, t \geq 0$ with $t \in [0, t_0]$. Also, recalling $x^n(0) = 0$, another application of the Mean Value Theorem yields

$$x^n(t) \geq \frac{1}{2}t$$

for all $t \geq 0$. Finally, by assumption (3.16) and properties of $Q$, we have

$$|u^n_0(\vec{x})| \leq ce^{-\delta|\vec{x}|},$$

for some $c > 0$ and $\delta > 0$.

Therefore, $x^n(t)$ and $u^n_0$ satisfy all the assumptions in Lemma 2.4 and hence, for $M \geq 4$, from Corollary 2.5, we obtain

$$\|\varepsilon^n(s)\|_{L^2(y_1 \geq y_0)}^2 \leq C_5 e^{-\frac{30}{M}}. \quad (3.20)$$

Next, we define a virial-type quantity that will provide the contradiction. Recalling the definition of $J_A$ in (2.12), we consider

$$K^n_A(s) = \lambda^n(s)(J^n_A(s) - \kappa),$$
where
\[
J^n_A(s) = \int_{\mathbb{R}^2} \varepsilon^n(s, y_1, y_2) F(y_1, y_2) \varphi_A(y_1) \, dy_1 \, dy_2.
\]

From (2.13) and (3.17), it is clear that \( K^n_A(s) \) is uniformly bounded in \( n \in \mathbb{N} \) and \( s \geq 0 \). The contradiction will be achieved if we show that \( \frac{d}{ds} K^n_A \) has strictly positive lower bound for large \( A > 0 \) (independent of \( n \)). Indeed, from Lemma 2.3, we have
\[
\frac{d}{ds} K^n_A(s) = \lambda^n \left( 2 \left( 1 - \frac{1}{2} \left( \frac{x^n_s}{\lambda^n} - 1 \right) \right) \int \varepsilon^n Q + R(\varepsilon^n, A) \right).
\]

The definition of \( M_0 \) (2.7) yields
\[
\frac{d}{ds} K^n_A(s) = \lambda^n \left( 2 \left( 1 - \frac{1}{2} \left( \frac{x^n_s}{\lambda^n} - 1 \right) \right) \right) M_0 + \tilde{R}(\varepsilon^n, A),
\]
where \( \tilde{R}(\varepsilon^n, A) = R(\varepsilon^n, A) - \left( 1 - \frac{1}{2} \left( \frac{x^n_s}{\lambda^n} - 1 \right) \right) \| \varepsilon^n \|_{L^2} \).

Let
\[
b = \int (Q + a \chi_0) Q = \| Q \|_2^2 - \frac{(\int Q \chi_0)^2}{\| \chi_0 \|_2^2} > 0 \quad \text{(since } Q \notin \text{ span } \{ \chi_0 \}).
\]

Using again the definition of \( M_0 \), we have
\[
M_0 = 2 \int \varepsilon_0^n Q + \int (\varepsilon_0^n)^2 \geq 2 \int \varepsilon_0^n Q = \frac{2b}{n}.
\]

Therefore, from (3.18) and the fact that \( \left| \frac{x^n_s}{\lambda^n} - 1 \right| \leq 1 \) for \( n \) large (see inequality (3.19)), we deduce
\[
2\lambda^n \left( 1 - \frac{1}{2} \left( \frac{x^n_s}{\lambda^n} - 1 \right) \right) M_0 \geq \frac{b}{n}.
\]

It remains to control the error term \( \tilde{R}(\varepsilon^n, A) \). Indeed, using the inequalities (2.14) and (3.19), there exists a universal constant \( C_7 > 0 \), such that for \( A \geq 1 \) we have
\[
\lambda \tilde{R}(\varepsilon^n, A) \leq C_7 \| \varepsilon(s) \|_2 \left( \| \varepsilon(s) \|_2 + A^{-1/2} + A^{1/2} \| \varepsilon(s) \|_{L^2(y_1 \geq A)} + \left| \int_{\mathbb{R}^2} y_2^2 F_{y_2} \varepsilon \varphi_A \right| \right).
\]

Moreover, by the decay properties of \( Q \) (2.6), the definition of \( F \) (2.11) and \( \| \varphi_A \|_\infty = 1 \), we deduce
\[
\left| \int_{\mathbb{R}^2} y_2 F_{y_2} \varepsilon^n \varphi_A \right| \leq \int_{\mathbb{R}} \int_{y_1 < 0} |y_2 F_{y_2}(y_1, y_2) \varepsilon^n(s)| \, dy_1 dy_2 + \int_{\mathbb{R}} \int_0^{2A} |y_2 F_{y_2}(y_1, y_2) \varepsilon^n(s)| \, dy_1 dy_2 \\
\leq \|\varepsilon^n(s)\|_2 \|y_2 F_{y_2}\|_{L^2(y_1 < 0)} + A^{1/2} \int_{\mathbb{R}} \sup_{y_1} |y_2 F_{y_2}(y_1, y_2)| \left( \int_0^{2A} |\varepsilon^n(s)|^2 \, dy_1 \right)^{1/2} dy_2 \\
\leq c \|\varepsilon^n(s)\|_2 (1 + A^{1/2}).
\]

Therefore, using (3.19) and (3.20), there exists a universal constant \( C_8 > 0 \) such that

\[
\lambda \tilde{R}(\varepsilon^n, A) \leq \frac{C_8}{n} \left( \frac{1}{n} + A^{-1/2} + A^{1/2} e^{-\frac{A}{2M}} + \frac{A^{1/2}}{n} \right).
\]

Now, taking \( A \geq 1 \) sufficiently large such that

\[
C_8 \left( A^{-1/2} + A^{1/2} e^{-\frac{A}{2M}} \right) < \frac{b}{4}
\]

and, for this fixed \( A \geq 1, n \) sufficiently large such that

\[
C_8 \left( \frac{1}{n} + \frac{A^{1/2}}{n} \right) < \frac{b}{4},
\]

we conclude

\[
\frac{d}{ds} K^n_A(s) \geq \frac{b}{2n} > 0, \quad \text{for all } s \geq 1.
\]

Integrating both sides of the last inequality in the interval \([0, s]\) and the taking \( s \to +\infty \), we get

\[
\lim_{s \to \infty} K^n_A(s) = \infty,
\]

which is a contradiction to the fact that \( K^n_A(s) \) is uniformly bounded in \( n \in \mathbb{N} \) and \( s \geq 0 \). Hence, our original assumption that \( Q \) is stable in the sense of Definition 1.3 is not valid and we conclude the proof of the theorem. \( \square \)

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Compliance with ethical standards

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.
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