Exact PPS Sampling with Bounded Sample Size

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Abstract

Probability proportional to size (PPS) sampling schemes with a target sample size aim to produce a sample comprising a specified number $n$ of items while ensuring that each item in the population appears in the sample with a probability proportional to its specified "weight" (also called its "size"). These two objectives, however, cannot always be achieved simultaneously. Existing PPS schemes prioritize control of the sample size, violating the PPS property if necessary. We provide a new PPS scheme that allows a different trade-off: our method enforces the PPS property at all times while ensuring that the sample size never exceeds the target value $n$. The sample size is exactly equal to $n$ if possible, and otherwise has maximal expected value and minimal variance. Thus we bound the sample size, thereby avoiding storage overflows and helping to control the time required for analytics over the sample, while allowing the user complete control over the sample contents. The method is both simple to implement and efficient, being a one-pass streaming algorithm with an amortized processing time of $O(1)$ per item.

Keywords: probability proportional to size, weighted sampling, unequal probability sampling

1. Introduction

Datasets are becoming ever larger, outpacing the growth in computer storage (Zwolenski and Weatherill, 2014) as well as outpacing Moore’s Law with respect to processing time [HiPEAC]. Thus sampling is often a key step in data analysis. One sampling scheme that is popular but challenging to implement is weighted random sampling without replacement, also known as sampling with probability proportional to size (PPS). There are well over 50 papers on this topic; see Hanif and Brewer (1980) for a review of the older literature. In a fixed-size PPS sampling scheme, each item $x_i$ is accompanied by an observable positive-valued auxiliary variable $w_i$, called the "weight" (or sometimes the "size"), and the goal is to output a sample $S$ containing a specified number $n$ of items, i.e., a sample of size $|S| = n$, where each item in the population appears in the sample with probability proportional to its weight. An early motivation for PPS sampling was that use of fixed-size PPS samples when estimating a population total $\sum_{i=1}^{N} y_i$ via the Horvitz-Thompson formula leads to highly precise estimates when the weight of each item $i$ approximates $y_i$ (Särndal et al., 1992, Sec. 3.6). Recent work has indicated the usefulness of PPS samples for online management of supervised learning models (Hentschel et al., 2019), where precise control over sample content is of paramount importance.

Unfortunately, it is not possible to simultaneously enforce both the sample-size requirement and the PPS property for all datasets. As a simple example, consider a dataset with six items $a_1, \ldots, a_6$ of weight 1 and six items $b_1, \ldots, b_6$ of weight 4, and suppose that we desire a sample of exactly 10 items. Denote by $p$ (resp., $q$) the appearance probability of an item $a_i$ (resp., $b_i$). Since the sample size $|S|$ equals 10 with probability 1, we have $E[|S|] = 6p + 6q = 10; \text{ if } q = 4p, \text{ then we must have } p = 1/3 \text{ and } q = 4/3, \text{ which is impossible since } q \text{ is a probability. If we try and fix this situation by choosing } q \leq 1 \text{ and setting } p = q/4, \text{ so that the PPS property holds, then } E[|S|] \leq 7.5 < 10.$

Existing algorithms either assume the given weights are such that this trade-off does not occur, or they prioritize achieving the target sample size $n$ over enforcing the PPS property. Specifically, suppose that we have a universe of $N$ items with positive weights $w_1, \ldots, w_N$. Systematic sampling and conditional Poisson sampling approaches
assume the existence of $\pi_i = \rho w_i$ with $0 \leq \pi_i \leq 1$ and $\sum_{i=1}^N \pi_i = n$ (see, e.g., Madow 1949, Hajek 1964); under these assumptions both goals are simultaneously satisfiable, but the conditions do not allow for arbitrary values of the $w_i$. Almost all other approaches include all items if $N \leq n$; otherwise, they implicitly or explicitly solve the equation $n = \sum_{i=1}^N \min(1, \tau w_i)$ for $\tau$ and include each item with probability $\min(1, \tau w_i)$ (see, e.g., Chromy 1979, Chao 1982, Rosén 1997, Cohen et al., 2011, Titte, 2019). When $N \geq n$, this always creates a sample of size $n$, but can violate the PPS property to an arbitrary degree when item weights differ significantly from each other. For instance, in our previous example, $\tau = 2/3$ and thus items of weight 1 are $2/3$ as likely to be in the sample as items of weight 4, instead of $1/4$ as likely, as was desired. The typical result is that items with weights much higher than the average appear in the sample with probability 1, i.e., are over-represented, as exemplified by items $b_1$-$b_6$ in our example. (Somewhat confusingly, such methods are sometimes referred to in the literature as enforcing “strict PPS.”) Sunter (1977, 1986) also considers the case where $n$ is an upper bound on the sample size, but allows over-representation of heavy items.

We extend the set of PPS sampling schemes to allow for a different trade-off between controlling the sample size and controlling the sample contents. Our new method, Exact and Bounded PPS (EB-PPS), strictly enforces the PPS property at all times while ensuring that the sample size never exceeds the target value $n$. That is, $n$ is now an upper bound on the sample size rather than the exact sample size. EB-PPS ensures that each item $x_i$ has appearance probability $\rho w_i$ as desired, where $\rho = \min(1/\max_i w_i, n/\sum_i w_i)$. If $\rho = n/\sum_i w_i$ then both a sample size of $n$ and the PPS property are simultaneously feasible, and our scheme yields the same appearance probabilities as those given above. If $\rho = 1/\max_i w_i$, then we prove that EB-PPS produces a PPS sample whose expected size is maximal and whose sample size variability is minimal over all possible PPS samples. Note that the latter situation occurs when $w_{\max}/\bar{w} \geq N/n$, where $w_{\max}$ and $\bar{w}$ are the maximum and average, respectively, of the $w_i$’s. The sample size therefore falls below the target $n$ in the presence of items having large relative weights.

Thus our sampling scheme EB-PPS reverses the usual assumptions and treats proper data representation as more important than obtaining the maximal sample size. By bounding the sample size, EB-PPS, like prior schemes, helps control the time required for analytics over the sample and avoids storage overflows, especially when many samples are being maintained in parallel. However, the appearance probabilities are now “as advertised,” promoting user trust, and the sample is kept as large and stable as possible when a full sample of size $n$ cannot be achieved.

EB-PPS has several important benefits that make it useful in practice. First, the algorithm works in the context of data streams: it views all items exactly once, does not need to know the dataset size in advance, and can forget all non-sampled items. For a fixed, finite data set, it follows that EB-PPS can produce a sample via a single pass through the data. Second, the algorithm is efficient: we prove that the amortized processing time is $O(1)$ per item, which matches the optimal complexity of Chromy’s algorithm (1979) for static datasets and improves upon the previous state-of-the-art for streaming data of $O(\log \log n)$ per item (Cohen et al., 2011). Finally, EB-PPS has low memory overhead and is simple to implement. The only data structure used throughout is an array of size $n$.

The algorithm presented below works sequentially over data streams of unknown size, so consider a sequence of items $x_1, x_2, \ldots$ with corresponding weights $w_1, w_2, \ldots$ and for $t \geq 1$ let $U_t = \{x_1, \ldots, x_t\}$ be the set of items scanned so far. For any $t \geq 1$ we want to be able to produce an exact bounded PPS sample $S_t$ from $U_t$.

The first goal of EB-PPS sampling is to ensure that the appearance probability of each item $x_i$ is proportional to $w_i$ at all times or, equivalently,

$$\frac{\Pr(x_i \in S_t)}{\Pr(x_j \in S_t)} = \frac{w_i}{w_j}$$

(1)
for $t \geq 1$ and $i, j \leq t$. The other goal is to ensure that at each step $t$ the sample size $|S_t|$ never exceeds $n$; the sample size should equal $n$ if feasible or, if not, then the sample size should have both maximal expected value and minimal variance relative to all possible bounded PPS samples.

As we have seen, rigorous enforcement of the PPS property conflicts with the goal of controlling the sample size. We therefore “relax” the problem by effectively allowing the sample size to take on fractional values; we then use a randomized procedure to deliver an integer-sized sample to the user. In more detail, EB-PPS maintains a data structure $L$ called a “latent sample,” from which we can extract an actual sample $S$ on demand. Formally, given a set $U$ of items, a latent sample of $U$ having real-valued latent size $C \geq 0$ is a triple $L = (A, \pi, C)$, where $A \subseteq U$ is a set of $|C|$ full items and $\pi \subseteq U$ is a (possibly empty) set containing at most one partial item; $\pi$ is nonempty if and only if $C > |C|$. In the following, we denote by $S, S', S_1$ samples extracted from $L, L', L_i$, by $(A, \pi, C), (A', \pi', C')$, $(A_1, \pi_1, C_1)$ the components of $L, L', L_i$, and so on. We slightly abuse notation and write $x \in L$ for $L = (A, \pi, C)$ if $x \in A \cup \pi$. We similarly say that latent samples $L_1$ and $L_2$ are disjoint if the sets $A_1 \cup \pi_1$ and $A_2 \cup \pi_2$ are disjoint.

Latent samples are described in detail in the subsequent section. Here we cover the functionality of their three main methods:

1. **OUTPUT:** For a latent sample $L$ having latent size $C$, the function $\text{OUTPUT}(L)$ produces a realized sample $S$ of expected size $C$ and of actual size either $|C|$ or $\lceil C \rceil$. Thus if $C$ is an integer then $S$ is of exactly size $C$.

2. **DOWNSAMPLE:** Given a latent sample $L$ having latent size $C$ and a real number $0 < \theta \leq 1$, the function $\text{DOWNSAMPLE}(L, \theta)$ produces a new latent sample $L'$ having latent size $C' = \theta C$ and satisfying

\[
\Pr(x \in S') = \theta \Pr(x \in S)
\]

for any item $x$ in the population.

3. **UNION:** Given two disjoint latent samples $L_1$ and $L_2$, $\text{UNION}(L_1, L_2)$ produces a new latent sample $L$ such that

- $\Pr(x \in S) = \Pr(x \in S_1)$ for all $x \in L_1$,
- $\Pr(x \in S) = \Pr(x \in S_2)$ for all $x \in L_2$, and
- $L$ has latent size $C = C_1 + C_2$.

Latent samples are reminiscent of the PMR samples in [Chromy, 1979], but differ in that latent samples can decrease in size; this is why our scheme can support exact PPS sampling.

Algorithm 1 gives the EB-PPS sampling scheme. A sample can be materialized at any step $t$ by calling $\text{OUTPUT}(L)$ after execution of line 11.

### Algorithm 1. EB-PPS

1. $n$: Sample-size bound
2. $(x_1, w_1), (x_2, w_2), \ldots$: Stream of sample items and weights
3. Initialize: $L = L' = (\emptyset, \emptyset, 0), w_{\max} = -\infty, W = 0$
   
   for $t \leftarrow 1, 2, \ldots$
   
   /* compute new proportionality constant */

   4. $w_{\max}' \leftarrow \max(w_1, w_{\max})$

   5. $W' \leftarrow W + w_t$

   6. $\rho' \leftarrow \min(1/w_{\max}', n/W')$

   /* downsample old items and new item, union result */

   7. if $W > 0$ then $L' \leftarrow \text{DOWNSAMPLE}(L, \rho'/\rho)$;

   8. $T \leftarrow \{x_2\}, \emptyset, 1\}$

   9. $T' \leftarrow \text{DOWNSAMPLE}(T, \rho' \cdot w_T)$

   10. $L \leftarrow \text{UNION}(L', T')$

   11. $(W, w_{\max}, \rho) \leftarrow (W', w_{\max}', \rho')$

The following result establishes the PPS property of the algorithm, as well as the sample size bound.
Theorem 1. For all \( t \geq 1 \) and \( x_i \) with \( 1 \leq i \leq t \), we have \( \Pr(x_i \in S_t) = \rho_t w_i \), where \( \rho_t = \min \left( \frac{1}{\max_{1 \leq i \leq t} w_i}, \frac{n}{\sum_{i=1}^{t} w_i} \right) \). Moreover, \( |S_t| \leq n \) for \( t \geq 1 \).

Proof. The proof is by induction. For \( t = 1 \), the algorithm sets \( \rho' = 1/w_1 \) in Line 6. The unique sample \( S' \) extracted from the latent sample \( T \) defined in line 3 satisfies \( \Pr(x_1 \in S') = 1 \) and the downsampling operation in Line 9 then yields \( \Pr(x_1 \in S'') = (\rho_1 w_1)\Pr(x_1 \in S') = 1 \) by (2), where \( S'' \) is a sample extracted from \( T' \). Note that \( L' = (\emptyset, \emptyset, 0) \) because \( L' \) is initialized to this value and the downsampling operation in Line 7 is not executed. It follows from the properties of the UNION function that \( \Pr(x_1 \in S_1) = \Pr(x_1 \in S'') \) and item \( x_1 \) is included in \( S_1 \) with probability \( \rho_1 w_1 = 1 \).

For \( t > 1 \), we have that \( \rho' = \rho_t \) from Line 6. Thus for \( i < t \) we have, after executing line 7, that \( \Pr(x_i \in S') = (\rho_t/\rho_{t-1})\Pr(x_i \in S_t) = (\rho_t/\rho_{t-1})\rho_{t-1} w_i = \rho_t w_i \) by (2) and the induction hypothesis, where \( S' \) is a sample extracted from \( L' \). Similarly, for \( i = t \), an argument similar to that given for \( t = 1 \) shows that \( \Pr(x_t \in S'') = \rho_t w_t \), where \( S'' \) is a sample extracted from \( T' \). The desired result then follows from the properties of the UNION function. Note that the downsampling operation on Line 9 is allowed since \( \rho_t/\rho_{t-1} \leq 1 \) by positivity of the weights, and the downsampling operation on Line 9 is allowed since \( \rho_t w_t \leq w_t/\max_{1 \leq i \leq t} w_i \leq 1 \). This proves the first assertion of the theorem. To prove the second assertion, observe that, at any step \( t \), we have

\[
E[|S_t|] = \sum_{i=1}^{t} \Pr(x_i \in S_t) = \sum_{i=1}^{t} \rho_t w_i \leq n
\]

by the first assertion and the definition of \( \rho_t \). This implies that the corresponding latent sample \( L_t \) contains at most \( n \) items, and thus any sample extracted from \( L_t \) can contain at most \( n \) items. \( \square \)

The following two theorems show that when EB-PPS sampling produces a sample of size less than \( n \), the expected sample size is the maximum possible under the PPS constraint in (1) and the sample-size variance is the minimum possible given maximal expected size.

Theorem 2. Let \( H \) be any weighted sampling scheme that satisfies (1) and denote by \( S_t \) and \( S_t^H \) the samples produced at step \( t \) by EB-PPS and by \( H \). If \( E[|S_t|] < n \), then \( E[|S_t^H|] \leq E[|S_t|] \).

Proof. Since \( H \) satisfies (1), it follows that for each item \( x_j \) with \( j \leq t \), the inclusion probability \( \Pr(x_j \in S_t^H) \) must be of the form \( r_j w_j \) for some constant \( r_j \) independent of \( j \). Since \( r_j w_j \leq 1 \), it follows that \( r_j \leq 1/\max_{1 \leq i \leq t} w_i \). The quantity on the right is exactly the constant Algorithm (1) uses for appearance probabilities when giving a sample of size less than \( n \), and so the result follows. \( \square \)

Theorem 3. Let \( H \) be any weighted sampling algorithm that satisfies (1) and has maximal expected sample size \( C_t < n \), and denote by \( S_t \) and \( S_t^H \) the samples produced at step \( t \) by EB-PPS and by \( H \). Then \( \var[|S_t^H|] \geq \var[|S_t|] \) for any \( t \geq 1 \).

Proof. Considering all possible distributions over the sample size having a mean value equal to \( C_t \), it is straightforward to show that variance is minimized by concentrating all of the probability mass onto \( \lceil C_t \rceil \) and \( \lfloor C_t \rfloor \). This is the sample-size distribution attained by EB-PPS. \( \square \)

3. Operations on Latent Samples

In this section we discuss the three methods OUTPUT, DOWNSAMPLE, and UNION. We use the notation \( \text{frac}(C) = C - \lfloor C \rfloor \) throughout. Proofs for all of the results in this section can be found in Hentschel et al. (2019).

OUTPUT: To create the sample \( S \) from the latent sample \( L = (A, \pi, C) \) we first include all \( \lfloor C \rfloor \) items of \( A \) with certainty. If \( \pi = \emptyset \), then \( E[|S|] = |S| = \lfloor C \rfloor = C \). Otherwise \( \pi = \{x\} \) for some element \( x \), and we generate \( U \) from a Uniform(0, 1) distribution and add \( x \) to \( S \) if \( U \leq \text{frac}(C) \). In this case, \( E[|S|] = (1 - \text{frac}(C)) \lfloor C \rfloor + \text{frac}(C)(\lfloor C \rfloor + 1) = \lfloor C \rfloor + \text{frac}(C) = C \).

DOWNSAMPLE: Given \( \theta \in [0, 1] \), the goal of downsampling \( L = (A, \pi, C) \) by a factor of \( \theta \) is to obtain an new latent sample \( L' = (A', \pi', \theta C) \) such that, if we generate \( S \) and \( S' \) from \( L \) and \( L' \) via OUTPUT, the appearance
Algorithm 2. Downsample

1. \( L = (A, \pi, C) \): input latent sample with \( C > 0 \)
2. \( \theta \): scaling factor with \( \theta \in [0, 1] \)
3. if \( \theta = 1 \) then return \( L' = (A, \pi, C) \)
4. \( U \leftarrow \text{UNIFORM}(1) \), \( C' = \theta C \)
5. if \( |C'| = 0 \) then
   //no full items retained
   6. if \( U > \text{frac}(C)/C \) then
     7. \( (A', \pi') \leftarrow \text{SWAP1}(A, \pi) \)
     8. \( A' \leftarrow \emptyset \)
   9. else if \( 0 < |C'| = |C| \) then
     10. if \( U > (1 - \theta \text{frac}(C))/(1 - \text{frac}(C')) \) then
         11. \( (A', \pi') \leftarrow \text{SWAP1}(A, \pi) \)
     12. else
         //items deleted: \( 0 < |C'| < |C| \)
         13. if \( U \leq \theta \text{frac}(C) \) then
             14. \( A' \leftarrow \text{SAMPLE}(A, |C'|) \)
             15. \( (A', \pi') \leftarrow \text{SWAP1}(A', \pi) \)
         16. else
             17. \( A' \leftarrow \text{SAMPLE}(A, |C'| + 1) \)
             18. \( (A', \pi') \leftarrow \text{MOVE1}(A', \pi) \)
     19. if \( C' = |C'| \) then
         20. \( \pi' \leftarrow \emptyset \)
         //no fractional item
   21. return \( L' = (A', \pi', C') \)

Probabilities are scaled down according to (2). Theorem 4 (later in this section) asserts that Algorithm 2 satisfies this property.

In the pseudocode for Algorithm 2, the UNIFORM function generates a random value from a Uniform(0,1) distribution and SAMPLE(A, n) samples n items uniformly and without replacement from A. The subroutine SWAP1(A, π) moves a randomly selected item from A to π and moves the current item in π to A. Similarly, MOVE1(A, π) moves a randomly selected item from A to π, replacing the current item in π (if any).

To gain some intuition for why the algorithm works, consider a simple special case, where the goal is to form a latent sample \( L' = (A', \pi', \theta C) \) from a latent sample \( L = (A, \pi, C) \), where C is an integer, and assume that \( C' = \theta C \) is non-integer, so that \( L' \) contains a partial item. In this case, we simply select an item at random from A to be the partial item in \( L' \) and then select \( \lfloor C' \rfloor \) of the remaining \( C - 1 \) items at random to be the full items in \( L' \). By symmetry, each item \( i \in L \) is equally likely to be included in \( S' \), so that the inclusion probabilities for the items in L are all scaled down by the same fraction, as required for (3). This scenario corresponds to lines 17 and 18 in the algorithm, where we carry out the above selections by randomly sampling \( \lfloor C' \rfloor + 1 \) items from A to form \( A' \) and then choosing a random item in \( A' \) as the partial item by moving it to π.

In the case where \( L \) contains a partial item \( x^* \) that appears in \( S \) with probability \( \text{frac}(C) \), the algorithm handles \( x^* \) first, thus reducing the remaining problem to the prior case. In particular, \( x^* \) should appear in \( S' \) with probability \( p = \theta P[x^* \in S] = \theta \text{frac}(C) \). Thus, with probability \( p \), lines 14–15 retain \( x^* \) and convert it to a full item so that it appears in \( S' \). Otherwise, in lines 17–18 \( x^* \) is removed from the sample when it is overwritten by a random item from \( A' \). In both cases, a new partial item is again chosen from A in a random manner to uniformly scale down the inclusion probabilities. Depending on whether \( x^* \) is kept or not, the problem then reduces to choosing \( \lfloor C' \rfloor \) or \( \lfloor C' \rfloor + 1 \) items and the uniformity of the selection preserves property (2) for all items in A.

The if-statements in lines 5 and 9 cover corner cases of the algorithm in which (i) \( L' \) does not retain any full items from \( L \) and (ii) no items are deleted from the latent sample, e.g., when \( C = 4.7 \) and \( C' = 4.2 \). These cases are handled similarly to the previous case but special care is taken either because π cannot become a full item or cannot be deleted.

Across all cases, property (2) holds, and so we have the following theorem.

**Theorem 4.** For \( \theta \in [0, 1] \), let \( L' = (A', \pi', \theta C) \) be the latent sample produced from a latent sample \( L = (A, \pi, C) \) via Algorithm 2 and let \( S' \) and \( S \) be samples produced from \( L' \) and \( L \) via OUTPUT. Then \( \Pr(x \in S') = \theta \Pr(x \in S) \).
for all \( x \in L \).

**Algorithm 3. Union**

1. \( L_1 = (A_1, \pi_1, C_1) \): fractional sample of size \( C_1 \)
2. \( L_2 = (A_2, \pi_2, C_2) \): fractional sample of size \( C_2 \)
3. \( C \leftarrow C_1 + C_2 \)
4. \( U \leftarrow \text{UNIFORM}() \)
5. if \( \text{frac}(C_1) = \text{frac}(C_2) = 0 \) then
   6. \( A \leftarrow A_1 \cup A_2 \)
   7. \( \pi \leftarrow 0 \)
5. else if \( \text{frac}(C_1) + \text{frac}(C_2) < 1 \) then
   8. \( A \leftarrow A_1 \cup A_2 \)
   9. if \( U \leq (\text{frac}(C_1) / (\text{frac}(C_1) + \text{frac}(C_2))) \) then \( \pi \leftarrow \pi_1 \) else \( \pi \leftarrow \pi_2 \)
   10. else if \( \text{frac}(C_1) + \text{frac}(C_2) = 1 \) then
        \( \pi \leftarrow 0 \)
   11. else if \( \text{frac}(C_1) + \text{frac}(C_2) > 1 \) then \( \pi \leftarrow 1 \)
   12. if \( U \leq (1 - \text{frac}(C_1)) / [(1 - \text{frac}(C_1)) + (1 - \text{frac}(C_2))] \) then
        \( \pi \leftarrow \pi_1 \)
   13. else \( \pi \leftarrow \pi_2 \)
   14. if \( U \leq (1 - \text{frac}(C_1)) / [(1 - \text{frac}(C_1)) + (1 - \text{frac}(C_2))] \) then
        \( A \leftarrow A_1 \cup A_2 \cup \pi \)
   15. else \( A \leftarrow A_1 \cup A_2 \cup \pi \)
16. return \( L = (A, \pi, C) \)

**Theorem 5.** Let \( L_1 = (A_1, \pi_1, C_1) \) and \( L_2 = (A_2, \pi_2, C_2) \), be disjoint latent samples, and let \( L = (A, \pi, C) \) be the latent sample produced from \( L_1 \) and \( L_2 \) by Algorithm 3. Let \( S_1, S_2, \) and \( S \) be random samples generated from \( L_1 \), and \( L_2 \), and \( L \) via OUTPUT. Then

(i) \( C = C_1 + C_2 = E[S] \);

(ii) \( \Pr(x \in S) = \Pr(x \in S_1) \) for all \( x \in L_1 \) and

(iii) \( \Pr(x \in S) = \Pr(x \in S_2) \) for all \( x \in L_2 \).

4. **Algorithmic Runtime**

The runtime performance of EB-PPS can be analyzed in terms of the average or maximum cost to process a scanned item. We focus on the cost of maintaining the latent sample as items are scanned and do not explicitly include the \( \Theta(n) \) cost of materializing samples for the user. We also assume that the latent sample can fit in memory and, for \( i \geq 0 \), that the contents of \( L_i \) can be freely overwritten when computing \( L_{i+1} \). Under these assumptions we have the following result for the average per-item processing cost.

**Theorem 6.** For any sequence of \( t \) items, the runtime of EB-PPS sampling is \( O(t) \) so that the amortized execution cost is \( O(1) \) per item.

**Proof.** First observe that the execution of the \textsc{Union} operator in line 10 is a constant-time operation that involves (potentially) adding the single element in \( T' \) to the current latent sample \( L' \). Similarly, all of the other steps in EB-PPS are constant time operations, except for the \textsc{Downsample} operation.
For DOWNSAMPLE, note that if, as in line 9 of EB-PPS, a latent sample with $C = 1$ is downsampled to a new latent size $C' \leq 1$, then either the algorithm immediately returns the original latent sample in line 3 or executes a single swap in line 7 so that the call to DOWNSAMPLE has an $O(1)$ cost. In general, the only steps in DOWNSAMPLE that are not $O(1)$ operations are the executions of the SAMPLE operator in lines 14 and 17. To analyze these costs, denote by $d$ the number of elements of $A$ that are discarded during a call to SAMPLE($A, m$). We can use the shuffle algorithm of Fisher and Yates as implemented by Durstenfeld (1964) to move all of the items not retained in the sample to the rightmost positions of the array that stores the elements of $A$; these elements can then be overwritten in subsequent steps. The Fisher-Yates shuffle requires $d$ swaps, so that the overall cost of executing DOWNSAMPLE is $O(d)$.

Thus, in EB-PPS, the downsampling operation in line 9 of Algorithm 1 has cost $O(1)$ as discussed above, and the cost incurred in line 7 is $O(d)$, where $d_i$ is the number of elements discarded from the latent sample when processing item $x_i$. Thus the total cost of processing items $x_1, \ldots, x_t$ is $O(D_t)$, where $D_t = \sum_{i=1}^t d_i$. The number of elements discarded from the latent sample is bounded by the number of elements inserted into the latent sample. Since at most one element is inserted per item scanned, we have $O(D_t) = O(t)$, and the desired result follows.

The maximum cost to process an item is $\Theta(n)$, and indeed some items can incur such relatively high execution costs. However, since producing an output sample also incurs a cost of $\Theta(n)$, this execution cost does not seem prohibitive. Moreover, the $\Theta(n)$ cost is only incurred when the sample reaches its maximum size of $n$ items and then loses many items due to a new heavy item, switching the constant $p_s$ from $n/\sum_{i=1}^t w_i$ to $1/\max_{1 \leq i \leq t} w_i$. When this scenario occurs, the sample is no longer capable of producing a sample of size $n$ without violating (1), and algorithms other than EB-PPS would have $\Theta(n)$ over-represented items.

5. Conclusion

We have provided a new scheme for weighted random sampling that prioritizes the PPS property over maintaining a fixed sample size. The resulting algorithm thus expands the set of known unequal-probability sampling schemes. The scheme enforces an upper bound on the sample size and, in an appropriate sense, keeps the sample size as large and as stable as possible. In addition, the scheme is simple to implement, is a one pass algorithm capable of working over data streams, and has the best known amortized execution cost per round.

In future work, we plan to consider the case of extremely large samples that do not fit in memory. The hope is to combine the ideas in this paper with the notion of a “geometric file” as introduced by Pol et al. (2008).

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