Abstract. Black hole systems are among the most promising sources for a gravitational wave detection project. Now, China is planning to construct a space-based laser interferometric detector as a follow-on mission of LISA in the near future. Aiming to provide some theoretical support to this detection project on the numerical relativity side, we focus on black hole systems simulation in this work. Considering the globular galaxy, multiple black hole systems also likely to exist in our universe and play a role as a source for the gravitational wave detector we are considering. We will give a progress report in this paper on our black hole system simulation. More specifically, we will present triple black hole simulation together with binary black hole simulation. On triple black hole simulations, one novel perturbational method is proposed.

1. Introduction
Pioneered by Weber, gravitational wave detection is a tough but also a very important task for natural science development. More and more gravitational wave detectors are constructed or planned to construct including resonant bars such as EXPLORER [1], ALLEGRO [2], NIROBE [3], and AURIGA [4]; and laser interferometers such as LIGO [5], VIRGO [6], GEO600 [7], TAMA [8]) and the planned LISA [9]. Excitingly, China is also planning to construct a space-based laser interferometer as a follow-on mission of LISA. At present this project is considering shorter arm length than LISA to fill the frequency gap between LISA and LIGO detectors [10, 11]. This project is temporally named Advanced LISA mission (ALIA). Considering this project together with the above ones, the requirement for the theoretical prediction of gravitational wave signal becomes more and more urgent. As to the most important gravitational source—binary black hole(BBH) coalescence, numerical relativity(NR) plays a key role for both aiding gravitational wave data analysis [12] and helping to construct a phenomenological wave form [13] for a template bank. Aiming to give a theoretical support for the ALIA project on the NR side, we have spent much energy on black hole simulations. Here, in this paper, we will give a progress report on our BBH simulations. In previous theoretical predictions on BBH systems, BBHs are treated as an isolated system although they always locate in some galaxy with a supermassive black hole in the center. This is based on the idea that the gravitational potential induced by
the supermassive black hole is ignorable compared with the strong gravitational field introduced by BBH itself. Although the curved background of the supermassive black hole is much weaker compared with the BBH’s gravitational field, the nonlinearity of general relativity (GR) may make some difference due to the configuration change [14]. Moreover, the special environment of a globular galaxy makes multiple black hole systems also possibly exist. So multiple black hole systems are also interesting theoretically and realistically for NR.

NR corresponds to solving Einstein’s equation without any approximation or symmetry assumption but with the aid of a supercomputer. Besides the massive computational cost involved in NR, how to make the numerical calculation stable is not a trivial problem for Einstein’s equation. After the breakthroughs in 2005 and 2006 [15, 16, 17], NR developed fantastically fast. Now almost every NR group in the world can evolve black hole systems without any stability problem. Moreover, long term gravitational waves [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28], gravitational radiation induced black hole recoil [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44], estimates of final black hole mass and spin of BBH systems [45, 46, 47, 48, 49], and other interesting topics [50, 51, 52, 53, 54, 55] have been extensively studied in the last years. So as people hoped a long time ago, NR has become an efficient tool for research in GR and other kinds of gravity theory. But in any specific application, NR may meet numerical accuracy and efficiency problems.

Based on the requirement of ALIA project, we will present binary black hole simulation and make the simulation result meaningful. As to the multiple black hole systems, we try to investigate the effect of these gravitational potentials on the evolution of BBH and especially the gravitational wave form which is interesting for detection. But the large difference of the distances among the triple black holes makes full GR simulation too computationally costly. In order to attack this problem we propose a perturbational method for triple black hole systems. Besides triple black hole systems, our perturbational method is also possibly useful to other systems, say black hole systems in the asymptotically de Sitter space which represents a possible universe background.

This paper is organized as following: In Section 2, we give a brief description of our NR code and the numerical techniques involved. Then we present our simulation results about BBH in Section 3. There both head-on collision and inspiraling cases are included. Our code is also calibrated there. In Section 4 we give an interesting result about three black hole simulation with full GR but a new perturbational method is used. Here we only present the main result about our three black hole simulation. We refer the reader to [56] for details. At last we summarize and discuss our code and simulation result. For completeness and future reference, two appendixes about gravitational wave form $\Psi_4$ calculation are included.

2. Numerical techniques

2.1. Code description

We updated our code for solving Einstein’s equation based on the BSSN formalism from [57]. Besides the ICN method we have constructed routines to implement 4th order Runge-Kutta method in our new code. On the mesh refinement, we have developed our own infrastructure other than GrACE [58, 59] to implement these numerical techniques. Our infrastructure archives coarse-grained and fine-grained parallelism in a loose sense. We use MPI [60] to realize parallelization among different nodes and OpenMP [61] among the processors within the same node. This kind of combination of MPI and OpenMP has been applied extensively in the numerical relativity, for example the Cactus infrastructure [62, 63]. Our implementation follows very closely the ideas of Cactus about this combination. As to the mesh refinement boundary issue, we follow [64, 65] closely to deal with the interface of mesh levels. In order to make the initial data consistent with the 3rd order accuracy implemented in the prolongation and restriction during the numerical evolution, we need three initial time levels to do interpolation.
For this end, we adopt the scheme described in [66] to prepare the initial data. With GrACE, the time step for every mesh level is set to be the Courant factor times the finest spatial resolution; while with our new infrastructure, we set the time step following the recipe of [64, 65], i.e., the Berger-Oliger algorithm is implemented for the time updating. Typically, the Courant factor is taken from \[\{0.25, 0.5\}\]. The results of fixed mesh refinement with GrACE and with our own infrastructure are consistent to each other. In addition, our new infrastructure provides us a “moving box” style mesh refinement. In our code, we assume [16]

\[ \frac{dx_{BH}^i}{dt} = -\beta^i(x_{BH}^i), \]  

(1)
to determine the position of black holes, and then move the corresponding boxes.

As to the spatial derivatives, we have developed 2nd, 4th, 6th and 8th order finite difference routines which include corresponding centered finite difference and lopsided finite difference. Together with different finite difference order, we also adopt Kreiss-Oliger type numerical dissipation as [64, 65]

\[ \partial_t u \to \partial_t u + (-1)^{n/2} \sigma \sum_{i=1}^{3} h^{n+1} \frac{D^{n/2+1} D^{n/2+1}}{2h^{n+2}} u, \]  

(2)

where \( n \) is the finite difference order, \( h \) is the spatial spacing, \( D^+_u = (u_{i+1} - u_i)/h \), \( D^-_u = (u_i - u_{i-1})/h \), and the summation is refer to three spatial dimension. \( \sigma \) is the parameter to control the strength of numerical dissipation. As an example, \( D^+_u D^-_u u \) can be explicitly written out as

\[ \frac{1}{h^6} (u_{i-3} - 6u_{i-2} + 15u_{i-1} - 20u_i + 15u_{i+1} - 6u_{i+2} + u_{i+3}). \]  

(3)

We choose

\[ \partial_t \alpha = -2\alpha K + \beta^i \alpha^i, \]  

(4)

\[ \partial_t \beta^i = \frac{3}{4} B^i + \beta^j \beta^i_{,j}, \]  

(5)

\[ \partial_t B^i = \partial_t \dot{\Gamma}^i - \eta B^i + \beta^j B^i_{,j} - \beta^j \dot{\Gamma}^i_{,j}. \]  

(6)

with fixed \( \eta = 2 \) as our gauge condition in all of this work.

We have also constructed a multi-puncture initial data solver with the spectral method based on the LORENE library[67, 68]. The spatial infinity is included in the computational domain through compactification. Our solver is quite efficient so that we can always solve the full constraint equations for black hole systems of interest, especially the triple black hole systems in this work.

2.2. Perturbational method

As to the evolution part, the stability problem forces us to use traditional outgoing Sommerfeld boundary condition. So the large differences among the distances between black holes sets a nontrivial obstacle for multi black hole system simulation. As to the BBH system located in the supermassive black hole’s potential, we take the advantage of the specific configuration and propose a perturbational method to deal with it. As to the effect of the background gravitational potential, we deal with it as a perturbation considering the gravitational field induced by the potential as weak compared with the BBH system. Specifically, we let our computational domain cover only the part near the binary. So the boundary of our computational domain is definitely non-flat. In order to take the effects of the third black hole into account, we must modify the
usual Sommerfeld boundary condition. Our recipe is first to take away of the potential part introduced by the third black hole, then implement the Sommerfeld boundary condition, and finally add back the background potential part. It seems that our operation is only a linear operation although, the modification is essentially due to the nonlinear property of Einstein’s equation. Of course when the third black hole becomes father and farther away, we expect the behavior of the perturbation term to approach linearity when the potential background becomes weak enough.

From the construction of our perturbational method, it can be seen that it is also possibly useful to other systems, for example, black hole systems in the asymptotically de Sitter space which represents a possible universe background. Taking the effect of cosmology on the BBH systems into consideration, it is interesting to investigate more on this problem. This will be treated in our future work.

3. Binary black hole simulations

3.1. Head-on collision simulation

We have compared the results of head-on collision with several short separation as in Fig.6 of [57], by using GrACE and our new infrastructure respectively to implement fixed mesh refinement. The result is almost the same. To test our moving box part code, we enlarge the separation from 2.303 to 8.

Firstly we put two black holes at \((0, 0, \pm 4)\). This setting is same with the one in [69]. The octant symmetry is adopted for saving computation time. The outer boundary is at 128. We use 7 levels to do the simulation, and domains for each levels are initially set as

\[
0 : (0 : 128, 0 : 128, 0 : 128) \\
1 : (0 : 64, 0 : 64, 0 : 64) \\
2 : (0 : 32, 0 : 32, 0 : 32) \\
3 : (0 : 16, 0 : 16, 0 : 16)
\]
where levels #5 and #6 are movable. The finest resolution is $\frac{13}{2}$. Here we set the Courant factor 0.5 and $\sigma$ the numerical dissipation parameter 0.1 for moving levels while 0.5 for static levels. Under this setting, the gravitational wave $\Psi_4$ is dominated by $\ell = 2, m = 0$ mode. The result is plotted in Fig.1.

Secondly, we simulate the same system but put the two black holes at $(0, \pm 4, 0)$ initially. The grids setting is similar to the above case while putting the moving box along y axis. This simulation is same to one simulation of [38]. The resulted waveform for gravitational radiation is presented in Fig.2.

Theoretically, the results of above two simulations has a specific relation as deduced in Appendix B. In other words, we should have relation

$$C_{\ell m i} = -2C_{20 y} = \frac{4}{\sqrt{6}} C_{2 - 2 y} = \frac{4}{\sqrt{6}} C_{22 y},$$

(7)

where $C_{\ell m i}$ denotes the corresponding $\ell, m$ components of $\Psi_4$ of simulation with black holes collision along $i$ axis. The consistency of our result with the theoretical prediction is shown in Fig.3.

Figure 3. Waveform of head-on collision along $z$ and $y$ axis respectively. They are consistently related by the theoretical prediction (7).

Figure 4. The evolution with Cook & Pfeiffer’s initial data. The orbit of the two black holes in $x$-$y$ plane behaves a well quasi circular shape.

3.2. Inspiraling binary black hole along quasi-circular orbit

Our first simulation of a quasi-circular orbit BBH is based on FMR. Due to the restriction of FMR, we choose the BBH system with some short separation. Specifically, we use the initial data with separation 10 which is available online released by Cook and Pfeiffer [70]. This initial data was also used in the UIUC group[71]. As to the excised region in the original initial data, we use global fitting method with spherical harmonic function to fill this region. Instead of caring about the accuracy, we are only aiming to test our code, therefore, the outer boundary is set at 64 and only 4 levels are used for the simulation. In the equatorial symmetry, the domains
for these levels are set as

\[ \begin{align*}
0 & : (-64 : 64, -64 : 64, 0 : 64) \\
1 & : (-32 : 32, -32 : 32, 0 : 32) \\
2 & : (-16 : 16, -16 : 16, 0 : 16) \\
3 & : (-8 : 8, -8 : 8, 0 : 8).
\end{align*} \]

The finest resolution is a coarse $\frac{1}{8}$, sufficient to test our code. The quasi-circular shape of black holes' orbit can be seen very clearly in fig.4. The resulted gravitational wave is dominated by $l = 2, m = \pm 2$ modes. Since these two modes are only up to a complex conjugate, we plot only $\ell = m = 2$ mode in Fig.5 which is consistent with [71].

We also tested our code with approximate initial data deduced in [57, 72, 51] based on puncture type initial data for BBH [73]. We choose the same parameters as in Fig.1 of [51] for the BBH system. Specifically, the punctures have mass parameters $m_1 = m_2 = 0.4891$ and are placed at $x = \pm 4.27629813$ with momenta $p_y = \mp 0.1091832072$. In this simulation, we implement moving box techniques with our new infrastructure. We use 7 levels here and their domains are initially set as

\[ \begin{align*}
0 & : (-64 : 64, -64 : 64, 0 : 64) \\
1 & : (-36.2763 : 36.2763, -32.2763 : 32.2763, 0 : 32) \\
2 & : (-20.2763 : 20.2763, -20.2763 : 20.2763, 0 : 16) \\
3 & : (-12.2763 : 12.2763, -12.2763 : 12.2763, 0 : 8) \\
4 & : (-8.2763 : -8.2763, -8.2763 : 8.2763, 0 : 5) \\
5 & : (-6.2763 : -2.2763, -2 : 2, 0 : 2) \\
& \quad (2.2763 : 6.2763, -2 : 2, 0 : 2) \\
6 & : (-5.2763 : -3.2763, -1 : 1, 0 : 1) \\
& \quad (3.2763 : 5.2763, -1 : 1, 0 : 1)
\end{align*} \]
where levels #5 and #6 have two boxes respectively and are movable. The finest resolution is $\frac{1}{32}$. Similar to above case, we plot $\ell = m = 2$ mode of gravitational wave in Fig.6. Our result is consistent with that presented in Fig.1 of [51].

4. Triple black hole simulations
The considered triple black hole system has an obvious hierarchical structure as shown in Fig.7. The two smaller black holes form a small BBH system (denoted as SB), while this binary system and the third, massive, black hole form a bigger binary system (denoted as BB). From the point view of eccentricity $e$, the head-on collision ($e = 1$) and the (quasi-)circular-orbiting ($e = 0$) stand for the two limits of the astrophysical configurations existing in the universe. Here we would like to investigate the possible combinations of these limiting cases in order to offer some clues for understanding this kind of astrophysical systems. The aforementioned hierarchical structure indicates four possible limiting cases. They are the head-on-freefall case (both SB and BB are in the process of a head-on collision), the head-on-orbiting case (SB is under a head-on collision while BB is in a (quasi-)circular orbit), the inspiraling-freefall case (SB is in a (quasi-)circular orbit while BB is under a head-on collision), and the inspiraling-orbiting (both SB and BB are in a (quasi-)circular orbit). We will present the results for these four scenarios in the following subsections.

4.1. Head-on-freefall case
The results of a small BBH’s head-on collisions under the background of a third (massive) BH are compared with our earlier results (see [57] for details). Consider the small BBH’s head-on collision starting from the ISCO. The mass parameters of the small binary system are chosen

![Figure 7. A schematic illustration of the BBH system in the gravitational potential of a third (massive) BH as considered in this work.](image)

![Figure 8. Comparison of the typical gravitational waveforms from the head-on collisions of a small BBH system for the head-on-freefall case. The $\ell = 2, m = 0$ mode of the gravitational wave detected at $r = 40$ is plotted. The red-dashed (black-solid) line represents the case with (without) the third BH. From this figure we can see that the gravitational potential of the third BH causes mainly shifts in time, decreases in amplitude, and prolongation of the wavelength in the gravitational waveform.](image)
to be $m_1 = m_2 = 0.5$, and their positions are $(0,0,±1.1515)$. The settings are the same as in [57]. The third BH with mass $M$ is placed at $(-R,0,0)$. In this and the next subsections we will vary $M$ while fixing the coordinate distance at $R = 1000$.

Fig.8 shows a typical example, comparing the gravitational waveform from the small BBH’s head-on collision under the influence of a third BH with the case of an isolated BBH head-on collision. From this figure we can see that the gravitational potential of the third BH causes a time delay, a decrease of the amplitude, and a prolongation of the wavelength in the gravitational waveform. It is easy to understand that the decrease in the amplitude mainly comes from the gravitational redshift caused by the potential of the third BH. However, the time delay comes from three factors: (a) the delayed merger process in which the gravitational wave is produced near the source; (b) the prolonged proper distance for propagation of the gravitational wave from the source to the detector; (c) the change in the coordinate time. It can be understood that the last one is simply caused by the change of gauge due to the third BH.

As mentioned earlier, the redshift effect due to the existence of the third BH will decrease the amplitude as well as prolong the wavelength of the gravitational waveform. The amplitude decreases as the gravitational potential of the third BH increases. And the wavelength increases as the gravitational potential of the third BH increases.

We investigate the energy radiation carried by the gravitational wave and show the result in Fig.9. The plot shows that a stronger gravitational potential results in more energy radiation. This can be understood as follows: when the small BBH system is located in the gravitational potential of a third BH, the energy radiated during the head-on collision is enhanced by the potential. In our perturbational treatment, the curved background is fixed and thus there is no back-reaction between the small BBH and the third BH. In a more realistic case, the back-reaction will speed up the head-on collision of the big binary system (which includes the small BBH and the third BH.)

The behavior of a three-body system has been analyzed in a Newtonian framework in [75]. It was found that the system could introduce higher-order modes of the gravitational wave. We found a similar phenomenon in this work. Besides distorting the lower-order modes, the existence of the third BH will also excite higher-order modes in the small BBH’s gravitational radiation, especially, the $\ell = 3$ modes. The left panel of Fig.10 shows the waveform of the $\ell = 3, m = 1$ mode of the gravitational radiation excited in the current case with respect to the different strength of the gravitational potential. The right panel of Fig.10 shows the amplitudes of the $\ell = 3, m = 1$ mode of the gravitational wave, which demonstrates nonlinear growth with
Figure 10. Left panel: Comparison of the gravitational waveforms of the $\ell = 3$, $m = 1$ mode for different potential strengths of the third BH. Right panel: Amplitude of $\Psi_{4,31}$ with respect to different potential strengths, showing a nonlinear growth pattern.

Figure 11. Comparison of the $\text{Re}(r\Psi_{4,20})$ in four different situations. The solid line indicates the waveform for the head-on collision of an isolated BBH; the (red) dashed line indicates the one for the head-on collision of a BBH freefalling towards a third large BH; the (blue) dotted line indicates the one for the head-on collision of a boosted isolated BBH; the (green) dot-dashed line indicates the one for the head-on collision of a BBH orbiting around a third large BH. The results indicate that the relativistic Doppler effect on the behavior of the gravitational waveform is much less important than the gravitational redshift effect in the head-on-orbiting case.

Figure 12. Comparison of the gravitational waveforms for different potential strengths of the third BH.

respect to the strength of the third BH’s gravitational potential. In principle it is possible for the nonlinear growth of the $\ell = 3$, $m = 1$ mode to be used as a signature to distinguish a three-BH system from a BBH system (to some extent). However, the phenomenon could also complicate the identification of the source of the gravitational wave. Without a further detailed
understanding on the waveform pattern, the gravitational wave emitted from a three-BH system as in this work, could be misidentified as one from an unequal-mass BBH system in which the higher-order modes of gravitational wave also exist.

4.2. Head-on-orbiting case
In this subsection, we study the head-on collision of a BBH while the BBH system moves along a quasi-circular orbit around a third large BH. Initially, the orbiting velocity of the center of mass (COM) of the BBH is set according to the Newtonian gravity, i.e., \( v^2 \approx \frac{M}{R} \), in order to get a circular orbit. For the third large BH, we fix its coordinate distance at \( R = 1000 \) as in the previous subsection, and change the gravitational potential \( \frac{M}{R} \) by varying the mass parameter \( M \). Given the linear momentum, mass parameter and the position of these BHs, the puncture initial data could be constructed with the solver described in Section 2.1.

There are at least two effects on the gravitational waveform of the BBH orbiting in the gravitational potential well of a third large BH: the relativistic Doppler effect and the gravitational redshift. In order to identify the importance of these two effects on the time delay in the waveform in the current case, we compare the waveforms obtained from the following four scenarios: (i) the head-on collision of an isolated BBH; (ii) the head-on collision of a BBH freefalling towards a third large BH; (iii) the head-on collision of a boosted isolated BBH; (iv) the head-on collision of a BBH orbiting around a third large BH. Here the mass parameters of the third BH in cases (ii) and (iv) are both \( M = 14.4 \), which is larger than all other mass parameters used in this subsection, and the boosting velocity in (iii) is set to be the same as the orbiting velocity in (iv) the magnitude of which is 0.12 according to \( v^2 \approx \frac{M}{R} \) and \( R = 1000 \). The result is shown in Fig.11. In the plot we can see that the \( \text{Re}[\Psi_{4,20}] \) waveforms of (i) and (iii) are close to each other, and the waveforms of (ii) and (iv) are close to each other. This indicates that the gravitational redshift of the third large BH should have the major effect on the time delay of the gravitational wave in the merging process of a BBH. Furthermore, the relativistic Doppler effect is shown to have little influence on the time delay of the waveform. The result allows us to neglect the relativistic Doppler effect from the orbiting velocity of the COM of a BBH around a third large BH, and to focus mainly on its gravitational redshift effect.

In this case, in order not to let the small BBH move too fast, to stay inside the computational domain and to maintain the necessary accuracy during the generation of gravitational wave, the gravitational potential strength of the third BH is set to be smaller than that in the previous subsection. The numerical results for this case indicate that all the phenomena including the time delay of the waveform, the decrease of the wave amplitude, the prolongation of the wavelength, the enhancement of the energy radiated, and the excitation of the higher-order modes are qualitatively similar to the ones in the previous case. Therefore we only show here the decrease in amplitude of the gravitational waveform and the coordinate time delay of the highest peak of the \( \Psi_{4,20} \) mode with respect to different potential strengths; see Fig.12. It can be seen that the decrease in amplitude of the waveform is less than that obtained in the previous subsection due to the smaller gravitational potential strength of the third BH. Just as in the previous subsection, the orientation of the small BBH barely affects the result because of the weakness of the tidal force from the third BH.

4.3. Inspiraling-freefall case
In this and the next subsections, we will consider the effect on an inspiraling BBH of a third large BH. The two BHs initially have the irreducible masses \( m_1 = m_2 \approx 0.5 \) at \( (0,0,\pm \frac{D}{2}) \). The third BH with a mass parameter of \( M = 5 \times 10^4 \) is located at \( (R,0,0) \).

In this subsection, we study the inspiral-to-merger evolution of a BBH while the COM of the BBH freely falls towards the third large BH. The left panel of Fig.13 shows the evolution of a typical inspiraling trajectory of the BBH, with the initial coordinate separation \( D = 4.5 \),
Figure 13. Left panel: Comparison of the typical inspiral trajectory of a BBH under the influence of a third BH with the one without the influence in the inspiraling-freefall case. The solid curve indicates the trajectory of an isolated BBH and shows a quasi-circular inspiral shape. The (red) dashed curve indicates the trajectory of the BBH in the gravitational potential well of the third BH with $M = 5 \times 10^4$ which is located at $(R = 5 \times 10^6, 0, 0)$. There is a change in the eccentricity of the (red) dashed curve compared to the solid line. Right panel: Comparison of the typical gravitational waveforms from the inspiral of the small BBH for the inspiraling-freefall case. The real component of $r\Psi_{4,20}$ is plotted. The red dashed (black solid) line indicates the case with (without) the third BH. The detector is located at $r = 40$. It can be seen that the gravitational potential of the third BH causes a tiny time delay in the spurious part and the effect of a time advance in the major part of the waveform.

Affected by the gravitational potential of a third large BH located at $(R = 5 \times 10^6, 0, 0)$. From the plot we can see that the gravitational potential of the third large BH leads to an increase in the eccentricity of the trajectory of the BBH, compared to that of an isolated BBH. The right panel of Fig.13 shows a typical example of the difference in the gravitational waveforms between those of the BBH inspiraling under the influence of a third large BH and the inspiraling of an isolated BBH. In this figure it can be seen that the gravitational potential of the third large BH causes a time advance, a prolongation of the wavelength, and a “surprising” variation in the amplitude of the gravitational waveform.

As explained in Section 4.1, the time delay for the existence of a third large BH in a head-on collision arises from three factors: the delayed merger process, the prolonged proper distance, and the change of the coordinate time. For an inspiraling process of a BBH, there exists another effect, i.e., an increase in the eccentricity, as shown in the left panel of Fig.13. It is obvious that the increase in the eccentricity is mainly caused by the tidal effect of the third large BH. As a BBH system emits more gravitational radiation in an elliptical orbit than in a circular orbit [83, 84], the increase of eccentricity is expected to expedite the merger process of the BBH. Therefore, the time shift affecting the inspiraling process is the result of competition between the increase in eccentricity and the others. Here we look at this in more detail.

From the right panel of Fig.13 we can see that the waveform is affected by both the time delay and time advance phenomena: the spurious radiation part of the waveform is delayed, while the burst part is advanced. The beginning of the waveform, near $t = 50$, is a result of the spurious radiation hidden in the initial data. Thus, this part is somehow independent of the merging process of a BBH. However, we can still see a time delay, because of the prolonged proper distance from the BBH to the detector, at $r = 40$, due to the existence of the third BH. In the main part of the merging process, after a coordinate time of roughly $t = 100$, we can clearly
see the time advance effect. This is due to the increase in eccentricity under the influence of the gravitational potential of the third large BH. Obviously the effect of the eccentricity increase overtake other time delay factors in the competition.

Besides the effect of time advance, there is another feature shown in the waveform. If we take the highest peak in the wave-train of $\Psi_4$ roughly as the time of of merging [18], this divides the waveform in the gravitational wave-train into two kinds of different behaviors. Before the merger, the wave amplitude of $\Psi_4$ becomes larger as the gravitational potential of the third large BH becomes larger. This is because the increase in eccentricity of the BBH’s trajectory is the dominant effect in speeding up the inspiral-to-merger process compared to the isolated case. This results in stronger gravitational radiation. After the merger, the wave amplitude of $\Psi_4$ turns out to become smaller as the gravitational potential of the third BH becomes larger. The decrease in the amplitude of the gravitational waveform mainly arises from the stronger redshift effect due to the existence of the third BH. These phenomena are illustrated in the right panels of Fig.13.

**Figure 14.** Left panel: Comparison of the typical inspiral trajectory of a BBH under the influence of a third large BH with one without that influence in the inspiraling-orbiting case. The (red) dashed curve indicates the trajectory of the BBH in the gravitational potential well of the third BH with $M = 5 \times 10^4$ which is located at $(R = 2.5 \times 10^7, 0, 0)$. The solid curve indicates the trajectory of an isolated BBH and shows a quasi-circular inspiral shape. Right panel: Comparison of the typical gravitational waveforms from the inspiraling of a BBH for the inspiraling-orbiting case. The real component of $r\Psi_{4,20}$ is plotted. The red-dashed (black-solid) line indicates the case with (without) the third large BH. It can be seen that, besides the decrease in the wave amplitude, the gravitational potential of the third large BH causes a tiny time delay in the spurious part and a time advance in the major part of the waveform.

Compared with the head-on collision of a BBH as discussed in Section 4.1, the higher-order mode effect due to the existence of a third large BH in the inspiraling process is more evident. The higher-order mode, $\text{Re}(r\Psi_{4,33})$, shows a nonlinear dependence on the strength of the gravitational potential. This result is consistent with those detailed in the previous two subsections: This nonlinear phenomenon could be a signature distinguishing the three-BH system from the BBH system, as well as complicating the identification of the source of a gravitational wave.

4.4. **Inspiraling-orbiting case**

In this subsection, we investigate the inspiral-to-merger process of a BBH orbiting around a third large BH. Initially, the COM of the BBH moves around the third large BH in a quasi-circular
orbit. Therefore, the velocity of the COM of the BBH can be approximated using Newtonian mechanics. Given the linear momentum and position parameters of the three BHs, we use the LORENE library to obtain the initial data by numerically solving the Hamiltonian constraint with the puncture scheme. Similar to the above subsection, we set the two BHs with irreducible mass $m_1 = m_2 \approx 0.5$ and locate them at $(0, 0, \pm \frac{R}{2})$ initially. We put the third BH at $(R, 0, 0)$. In order to get a circular orbit for the BBH around the third large BH, we use the quantity $\sqrt{M/R}$ to evaluate the velocity of the COM of the BBH with respect to the third large BH. This velocity is then added vectorially to that of each individual BH of the BBH. For the third large BH, we fix its mass parameter $M = 5 \times 10^4$ as in the previous subsection while varying its coordinate distance $R$.

The left panel of Fig.14 shows a typical trajectory of the BBH with the initial coordinate separation $D = 4.5$ for the current case, where the third large BH is located at $(R = 2.5 \times 10^7, 0, 0)$. It can be understood from the plot that the COM of the BBH moves roughly along a Newtonian circular orbit around the third large BH. The trajectory is in fact the combination of the orbit of the COM of the BBH and the BBH’s inspiraling around the COM. Compared with the trajectory in Fig.13, it can be seen that the trajectory in Fig.14 (for the current case) is quite deformed compared to the one for an isolated BBH. This is mainly due to the BBH’s orbital velocity around the third large BH.

The typical resulting gravitational waveform of a BBH in the current case is presented in the right panel of Fig.14. We found that the resultant effect of the third large BH on the waveform is stronger in the current case than in the inspiraling-freefall cases. We can see in the plot that the gravitational potential of the third large BH causes a time delay in the spurious radiation but a time advance in the major part of the waveform. This is similar to the previous inspiraling-freefall case. The time advance of $\text{Re}(r\Psi_{4,20})$ due the existence of the third large black hole is clear and different from its spurious radiation part. This phenomenon is similar to the one in the inspiraling-freefall case. The amplitude of the waveform decreases more in the wave-train as the gravitational potential of the third BH becomes larger.

Similar to the previous case, as the initial separation of BBH increases, the phase change of the gravitational wave, due to the effect of the third large BH, transits from time-delay to time-advance. As in the previous case, a possible explanation for the change in the time difference of the inspiral-to-merger process of a BBH under the influence of a third large BH compared with the one of an isolated BBH, arises from the competition between the eccentricity effect and the gravitational redshift effect. Compared to the previous (inspiraling-freefall) case in which the $\ell = m = 3$ mode of $\Psi_4$ is dominant among the higher modes, the $\ell = 3, m = 0$ modes of $\Psi_4$ is dominant in the current case. The orbiting of the BBH along the $x$-$z$ plane accounts for this effect. In addition, the amplitude of the $\ell = 3, m = 0$ mode of $\text{Re}(\Psi_4)$ increases nonlinearly.

5. Summary and discussion
Motivated by the fact that all most BBH systems are located in the gravitational potential of a supermassive black hole hosted in the center of galaxy, we investigated the effect of this potential on the dynamics of the BBH. Special attention is paid to the gravitational wave form released from the binary system. To deal with this problem, we proposed a new perturbation method to take the effect of the supper-massive black hole into consideration. In our perturbation method, we ignore the back reaction of the BBH system on the supermassive black hole and look the third black hole as a background to the BBH system.

The effect of the gravitational potential includes: 1). delaying or advancing the merging process of the BBH which depends on the relative motion of the binary respect to the supermassive black hole; 2). making the gravitational wave signal propagate slower; 3). red-shifting the resulted gravitational wave form; 4). producing significantly the $\ell = m = 3$ mode of the gravitational wave form. Except the amplitude of $\ell = m = 3$ mode, all other quantities
depend on the gravitational potential \( \frac{M}{R} \) linearly which implies the validity of our perturbation method. And in most realistic cases the gravitational potential is much weaker than the ones considered in this paper. So our perturbation method is expected to be valid for almost realistic problems. In some sense the effect 4) is an evidence of the conjecture proposed in paper [75] in the full GR region. And it can provide valuable information for gravitational wave astronomy to distinguish if the detected BBH system is isolated or located in a potential of a supermassive black hole.

The code used in this work is an extension of our previous work. The consistence of the results obtained in this work also validate our code for BBH simulations. With this code we are ready to research other problems involved in black hole systems in the near future.

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**Appendix A. Calculation of \( \Psi_4 \)**

We update our method of calculating \( \Psi_4 \) from [57] to a new one. Associated with the Cartesian coordinates used in our simulation, we have a spherical coordinate \((r, \theta, \phi)\). We denote their orthonormal bases as \((\hat{r}, \hat{\theta}, \hat{\phi})\). In addition we denote the time-like unit vector normal to the constant coordinate time slice \(\hat{N}\). Then the null tetrad used in the calculation of \( \Psi_4 \) read as

\[
\begin{align*}
    l & \equiv \frac{1}{\sqrt{2}}(\hat{N} + \hat{r}), \\
    n & \equiv \frac{1}{\sqrt{2}}(\hat{N} - \hat{r}), \\
    m & \equiv \frac{1}{\sqrt{2}}(\hat{\phi} - i\hat{\theta}).
\end{align*}
\]

With

\[
\begin{align*}
    E_{ij} & \equiv R_{ij} - K_{ik}K_{j}^{k} + KK_{ij}, \\
    B_{ij} & \equiv \hat{r}^{k}[D_{k}K_{ij} - D_{(i}K_{j)k}],
\end{align*}
\]

\( \Psi_4 \) can be calculated through

\[
\Psi_4 = (B_{ij} - E_{ij})\hat{m}^{i}\hat{m}^{j}.
\]

**Appendix B. Transformation of \( \Psi_4 \)**

Now we consider relation of two head-on collision simulations, one is collision along \( z \) axis, the other is along \( y \) axis. These two simulations correspond to the same physical process. We use the Cartesian coordinate corresponds to the \( z \) axis collision simulation to do the analysis. Denote the corresponding spherical coordinate of \( z \) axis simulation \((r, \theta, \phi)\), while \( y \) axis simulation \((r', \theta', \phi')\). Then we have

\[
\begin{align*}
    x & = \sin \theta \cos \phi = \cos \theta', \\
    y & = \sin \theta \sin \phi = \sin \theta' \cos \phi', \\
    z & = \cos \theta = \sin \theta' \sin \phi',
\end{align*}
\]
and
\[
\hat{\phi} = \frac{1}{\sqrt{x^2 + y^2}} \left( \begin{array}{c} -y \\ x \\ 0 \end{array} \right),
\]
\[
\hat{\theta} = \frac{1}{\sqrt{x^2z^2 + y^2z^2 + (x^2 + y^2)^2}} \left( \begin{array}{c} xz \\ yz \\ -(x^2 + y^2) \end{array} \right),
\]
\[
\hat{\theta}' = \frac{1}{\sqrt{y^2 + z^2}} \left( \begin{array}{c} 0 \\ -z \\ y \end{array} \right),
\]
\[
\hat{\phi}' = \frac{1}{\sqrt{x^2y^2 + x^2z^2 + (y^2 + z^2)^2}} \left( \begin{array}{c} -(y^2 + z^2) \\ xy \\ xz \end{array} \right).
\]

\((\hat{\theta}, \hat{\phi})\) and \((\hat{\theta}', \hat{\phi}')\) are only up to a rotation
\[
\hat{\theta}' = -\frac{1}{\sqrt{\sin^2 \theta \sin^2 \phi + \cos^2 \theta}} (\cos \theta \cos \phi \hat{\phi} + \sin \phi \hat{\theta}),
\]
\[
\hat{\phi}' = -\frac{1}{\sqrt{\sin^2 \theta \sin^2 \phi + \cos^2 \theta}} (-\sin \phi \hat{\phi} + \cos \theta \cos \phi \hat{\theta}).
\]

With notation \(m \equiv \frac{1}{\sqrt{2}}(\hat{\phi} - i\hat{\theta})\) and \(m' \equiv \frac{1}{\sqrt{2}}(\hat{\phi}' - i\hat{\theta}')\), we have \(m' = e^{i\lambda} m\) with
\[
\cos \lambda = -\frac{\cos \theta \cos \phi}{\sqrt{\sin^2 \phi + \cos^2 \theta \cos^2 \phi}}, \quad (B.4)
\]
\[
\sin \lambda = -\frac{\sin \phi}{\sqrt{\sin^2 \phi + \cos^2 \theta \cos^2 \phi}}. \quad (B.5)
\]

Denoting the result of \(z\) axis simulation \(\Psi_4\) while \(x\) axis simulation \(\Psi_4'\), we have \(\Psi_4' = e^{-2i\lambda} \Psi_4\) considering the spin weight \(-2\) property of \(\Psi_4\). From numerical result we know \(\Psi\) is dominated by \(-2Y_{20}\) mode. so we have
\[
\Psi_4' = e^{-2i\lambda} C(t)\sqrt{6} \Psi_{-2Y_{20}}(\theta, \phi)
\]
\[
= C(t)[\frac{1}{2}\sqrt{6} Y_{-2Y_{20}}(\theta', \phi') - \frac{\sqrt{6}}{4} Y_{-2Y_{22}}(\theta', \phi')]. \quad (B.6)
\]

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