On holomorphic two-spheres with constant curvature in the complex Grassmann manifold $G(2, n)$

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Abstract. In this paper, we determine the value distribution of the curvature of all linearly full unramified holomorphic two-spheres with constant curvature in $G(2, n)$ satisfying that the generated harmonic sequence degenerates at position 2, and characterize these immersions completely in terms of harmonic sequences in the complex projective spaces. We also construct some families of non-homogeneous holomorphic two-spheres in $G(2, n)$ with constant curvature using our characterization. These results verify Conjecture 1 completely and Conjecture 2 partly in [9] for the special case.

Keywords and Phrases. Harmonic sequence, Holomorphic two-sphere, Gaussian curvature, Complex Grassmann manifold.

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1 Introduction

The Physicists Din and Zakrzewski [10] first gave the classification of harmonic maps of two-sphere in the complex projective space $\mathbb{CP}^n$. Inspired by their work, the classification and construction of harmonic maps of two-sphere in the complex Grassmann manifold $G(k, n)$ were obtained in a sequence of papers (cf. [29], [1], [6], [32], [4], [33]). Holomorphic curves of two-sphere in $G(k, n)$ are harmonic, which plays an important role in their classifications.

It is well known that a holomorphic curve in $\mathbb{CP}^n$ is determined by its first fundamental form, up to unitary equivalence (cf. [8], [17]). Using the above fact, Bolton et al. classified all linearly full minimal immersions of two-sphere into $\mathbb{CP}^n$ with constant curvature, which belong to the Veronese sequence, up to unitary equivalence (cf. [3]). Chi and Zheng [7] studied the rigidity of pseudo-holomorphic curves of constant curvature in $G(k, n)$ by moving frames. They classified all holomorphic two-spheres in $G(2, 4)$ with constant curvature 2 into two classes, up to unitary equivalence, in which none of the curves are congruent. The result illustrates that the rigidity of holomorphic curves in general $G(k, n) \ (k \geq 2)$ fails if we assume only they have the same first fundamental form. So it would be interesting and important to consider the classification of holomorphic curves of two-sphere with constant curvature in $G(k, n)$.

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In general, the classification problem is very difficult. So far only partial results were known. For the simplest case of homogeneous holomorphic immersions of two-sphere, Xu and Peng ([28]) and the first author ([13]) completely classified for $G(2,n)$ and $G(3,n)$ respectively. For the cases of $k = 2$ and $n = 4, 5$, Li and Yu [27] classified all holomorphic two-spheres in $G(2,4)$ with constant curvature. Jiao and Peng [23] classified all linearly full nonsingular holomorphic two-spheres in $G(2,5)$ with constant curvatures $K = 4, 2, 4/3, 1$ and $4/5$ into some classes, up to unitary equivalence. In 2015, the second author, Jiao and Zhou [19] completely classified all linearly full totally unramified holomorphic curves of constant curvature from two-sphere into $G(2,5)$. They determined two classes of linearly full non-homogeneous holomorphic curves from two-sphere into $G(2,5)$ with the same constant curvature $4/3$.

Theoretical physicists are also interested in this classification problem. In 2013, Delisle, Hussin and Zakrzewski ([9]) recovered the classification results in $G(2,4)$ and $G(2,5)$ mentioned above by their approach from the view point of Grassmannian sigma models, and proposed two conjectures regarding the value distributions of curvature and existence of such holomorphic solutions with constant curvature in $G(k,n)$. Recently, Hussin et al. explored the constant curvature holomorphic solutions of the supersymmetric Grassmannian sigma model $G(k,n)$ using the gauge invariance of the model (cf. [20], [21]). Here we state the two conjectures (c.f. [9]) explicitly as follows:

**Conjecture 1** For holomorphic solutions with constant curvature $K = 4/r$ in $G(k,n)$, the values distribution of the positive integer $r$ is \{1, 2, \ldots, k(n-k)\}.

**Conjecture 2** For $k$ fixed, holomorphic solutions with constant curvature in $G(k,n)$ can be constructed for all integer values of $r$ such that $1 \leq r \leq k(n-k)$.

In this paper, we will consider the values distribution of $r$ for the case of $G(2,n)$ and the constructions of non-homogeneous holomorphic two-spheres with constant curvature. The main tools used here are the method of moving frames and harmonic sequences in $\mathbb{C}P^n$, and the fact that there do not exist globally defined non-zero holomorphic differential forms on $S^2$.

Our paper is organized as follows. Preliminaries are given in Section 2. In Section 3, in terms of harmonic sequence in $\mathbb{C}P^n$, we characterize all linearly full unramified holomorphic curves from two-sphere in $G(2,n)$ with constant curvature satisfying that the generated harmonic sequence degenerates at position 2, and determine the values of curvature (See Theorem 3.10). As an application, we completely classify such immersions with constant length of second fundamental form (See Theorem 3.12). In Section 4, we construct some families of non-homogeneous holomorphic two-spheres in $G(2,n)$ with constant curvature by using our main theorem (See Proposition 4.2-4.5, Theorem 4.6).

## 2 Preliminaries

### 2.1 Geometry of holomorphic curves in $G(2,n)$

In this subsection, we introduce the geometry of holomorphic curves in $G(2,n)$ by the method of moving frames. More details can be found in [7] and [14]. Let $\varphi$ be a linearly full holomorphic immersion from $S^2$ into $G(2,n)$. Then $\varphi$ generates the following harmonic
sequence (cf. [4],[6])

\[ 0 \xrightarrow{\partial'} \varphi_0 = \varphi \xrightarrow{\partial'} \varphi_1 \xrightarrow{\partial'} \varphi_2 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \varphi_l \xrightarrow{\partial'} 0, \]  

(2.1)

where \( \varphi_i : S^2 \to G(k_i, n) \) is a harmonic map with \( k_i \leq 2, \partial^i \varphi_i = \varphi_{i+1} \) for \( 0 \leq i < l \), \( \partial^i \varphi_l = 0 \) and \( \sum_{i=0}^l k_i = n \). If \( k_1 = \cdots = k_{r-1} = 2 \) and \( k_i = 1 \) for \( r \leq i \leq l \), we say that \( \varphi \) degenerates at position \( r \) and \( l = n - r - 1 \). For example, \( \varphi \) degenerates at position 2 means \( k_1 = 2, k_i = 1 \) for all \( i = 2, \cdots, n - 3 \).

Since any member of \( \varphi_0, \cdots, \varphi_l \) are orthogonal, we can choose a local unitary frame \( e = \{ e_1, \cdots, e_n \} \) along \( \varphi \) so that

\[ \varphi_i = \text{span} \{ e_{2i+1}, e_{2i+2} \}, \text{ for } 0 \leq i \leq r - 1, \]
\[ \varphi_i = \text{span} \{ e_{r+i+1} \}, \text{ for } r \leq i \leq n - r - 1, \]
\[ \partial'[e_{2r-1}] = 0, \partial'[e_{2r}] = \varphi_r. \]

Under such frame, the pull back of (right invariant) Maurer-Cartan forms which are denoted by \( \omega = (\omega_{AB}) \) are

\[
\begin{pmatrix}
\Omega_1 & A_1 \phi \\
-A_1^* \phi & \Omega_2 & A_2 \phi \\
-A_2^* \phi & \Omega_3 & \Omega_4 \\
\ddots & \ddots & \ddots \\
\Omega_r & A_r \phi \\
-A_r^* \phi & \Omega_{r+1} & \ddots \\
\ddots & \ddots & \ddots \\
\omega_{n-1,n-1} & a_{n-1,n} \phi & \omega_{n,n} \\
\end{pmatrix},
\]

where \( \phi \) is a local unitary coframe of \( (1, 0) \) type with respect to the induced metric \( \varphi^* ds^2 \) (here \( ds^2 \) is the standard Kähler metric on \( G(2, n) \)), \( \Omega_i, 1 \leq i \leq r \) are \( u(2) \)-valued 1-forms, \( A_i = \begin{pmatrix} a_{2i-1,2i+1} & a_{2i-1,2i+2} \\ a_{2i,2i+1} & a_{2i,2i+2} \end{pmatrix} \) with rank \( A_i = 2 \) for \( 1 \leq i \leq r - 1 \), \( A_r = \begin{pmatrix} 0 & 0 \\ a_{2r,2r+1} & 0 \end{pmatrix} \).

The structure equations of \( \varphi^* ds^2 \) can be written as

\[ d\phi = -\sqrt{-1}\rho \wedge \phi, \quad d(\sqrt{-1}\rho) = \frac{K}{2} \phi \wedge \bar{\phi}, \]

where \( \rho \) is the real-valued connection form and \( K \) is the curvature.

Notice that the unitary frame we choose is determined up to a transformation of the group \( \underbrace{U(2) \times \cdots \times U(2)}_{r-1} \times \underbrace{U(1) \times \cdots \times U(1)}_{n-2(r-1)} \), so \( |\det A_i| \) \( (i = 1, 2, \cdots, r - 1) \) are global invariants of analytic type on \( S^2 \) vanishing only at isolated points, and away from their zeros, they satisfy (cf. [7], [14])

\[ \Delta \log |\det A_i| = 2K + 2(L_{i-1} - 2L_i + L_{i+1}), \text{ for } 1 \leq i \leq r - 1, \]

(2.2)

where \( L_{-1} = 0, L_i = tr A_i A_i^*, 1 \leq i \leq r \) are also globally defined invariants on \( S^2 \), and \( \Delta \) is Laplace-Beltrami operator with respect to \( \varphi^* ds^2 \).
Let $S$ be the square of the length of the second fundamental form. Then the Gauss equation of $\varphi$ (cf. [13],[15]) is

$$K = 4 - 8|\det A_1|^2 - \frac{1}{2}|S|.$$  \hspace{1cm} (2.3)

### 2.2 Harmonic sequence of two-spheres in $\mathbb{C}P^m$

In this subsection, we introduce the harmonic sequence of two-spheres in the complex projective space $\mathbb{C}P^m$, which will be used to choose proper frames of holomorphic two-spheres in $G(2,n)$. Let $\psi : S^2 \to \mathbb{C}P^m$ be a linearly full harmonic map. Eells and Wood’s result (cf. [11]) shows that the following sequence in $\mathbb{C}P^m$ is uniquely determined by $\psi$

$$0 \xleftarrow{\partial^*} \psi_0 \xrightarrow{\partial^*} \cdots \xrightarrow{\partial^*} \psi = \psi_1 \xrightarrow{\partial^*} \cdots \xrightarrow{\partial^*} \psi_m \xrightarrow{\partial^*} 0,$$ 

for some $i = 0, 1, \cdots, m$.

Under a local coordinate $z$, we choose a holomorphic section $f_0^{(m)}$ of $\psi_0^{(m)}$ such that $\partial f_0^{(m)} = 0$. Let $f_i^{(m)}$ be a local section of $\psi_i^{(m)}$ such that

$$f_i^{(m)} = \frac{\partial}{\partial z} f_{i-1}^{(m)} - \frac{\partial}{\partial \bar{z}} f_{i-1}^{(m)},$$

for $i = 1, \cdots, m$. Then we have some formulas as follows (cf. [3]):

$$\frac{\partial}{\partial z} f_i^{(m)} = f_{i+1}^{(m)} + \frac{\partial}{\partial z} \log |f_i^{(m)}|^2 f_i^{(m)}, \quad i = 0, \cdots, m-1,$$ \hspace{1cm} (2.5)

$$\frac{\partial}{\partial \bar{z}} f_i^{(m)} = -l_i^{(m)} f_{i-1}^{(m)}, \quad i = 1, \cdots, m,$$ \hspace{1cm} (2.6)

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log |f_i^{(m)}|^2 = l_i^{(m)} - l_{i-1}^{(m)},$$ \hspace{1cm} (2.7)

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log |l_i^{(m)}| = l_{i+1}^{(m)} - 2l_i^{(m)} + l_{i-1}^{(m)}, \quad i = 0, \cdots, m-1,$$ \hspace{1cm} (2.8)

where $l_i^{(m)} = |f_i^{(m)}|^2/|f_i^{(m)}|^2$ for $i = 0, \cdots, n$, and $l_{-1}^{(m)} = l_0^{(m)} = 0$.

Set $e_i^{(m)} = l_i^{(m)}/|f_i^{(m)}|$. Then from (2.5)-(2.8) and by a straightforward computation, we have

$$d e_i^{(m)} = -\sqrt{l_{i-1}^{(m)} e_i^{(m)}} + \theta_{ii}^{(m)} e_i^{(m)} + \sqrt{l_i^{(m)}} d z e_{i+1}^{(m)},$$ \hspace{1cm} (2.9)

where $\theta_{ii}^{(m)} = \frac{\partial}{\partial z} \log |f_i^{(m)}| d z - \frac{\partial}{\partial \bar{z}} \log |f_i^{(m)}| d \bar{z}$ is an imaginary 1-form.

The $p$-th osculating curve $\sigma_p : S^2 \to \mathbb{C}P^{(p+1)-1}$ of $\psi_0^{(m)}$ is defined as follows: Locally

$$F_p^{(m)} = f_0^{(m)} \wedge \cdots \wedge f_p^{(m)}, \quad 0 \leq p \leq m.$$

Here $F_p^{(m)}$ may have isolated zeros. At the singular points, factor out the common divisors, we write $F_p^{(m)} = h_{p,m} \tilde{F}_p^{(m)}$, where $\tilde{F}_p^{(m)}$ is a $\mathbb{C}^{(m+1)-1}$-valued holomorphic map without zeros.
So we can extend the definition of $\sigma_p$ at these points by $F_p^{(m)}$. Here $\sigma_p$ is a holomorphic map. By (2.7), we have

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log |F_p^{(m)}|^2 = l_i^{(m)}, \ 0 \leq p \leq m - 1.$$ (2.10)

The degree $\delta_p^{(m)}$ of $\sigma_p$ is given by

$$\delta_p^{(m)} := \int_{S^2} l_i^{(m)} dz \wedge d\bar{z} \cdot \frac{i}{2\pi}. \quad (2.11)$$

Let $r_p^{(m)}$ be the degree of the singular divisor of $\sigma_p^* ds^2_F \Sigma = l_i^{(m)} dz d\bar{z}$, then we have the global Plücker formula

$$\delta_{p-1}^{(m)} - 2 \delta_p^{(m)} + \delta_{p+1}^{(m)} = -2 - r_p^{(m)}. \quad (2.12)$$

We recommend readers to Bolton et al. [3] for details.

Particularly, let $\psi_i^{(m)} = [f_i^{(m)}] = [f_{i,0}, \cdots, f_{i,p}, \cdots, f_{i,m}]$, where $f_{i,p}$ is explicitly given by

$$f_{i,p}(z) = \frac{i!}{(1 + z\bar{z})^{i+1}} \sqrt{m \choose p} z^{p-i} \sum_{k} (-1)^k \binom{m-p}{i-k} \binom{m-p}{k} (z\bar{z})^k.$$

Such a map $\psi_{i}^{(m)} : S^2 \to \mathbb{CP}^m$ is a conformal minimal immersion with constant curvature and constant Kähler angle, which are given by

$$K_{i}^{(m)} = \frac{4}{m + 2i(m-i)}, \quad \cos \alpha_{i}^{(m)} = \frac{m - 2i}{m + 2i(m-i)}.$$

This harmonic sequence is well known as Veronese sequence in [3], which will be denoted by $V_m^{(m)} : S^2 \to \mathbb{CP}^m$ correspondingly.

## 3 Holomorphic two-spheres of constant curvature in $G(2, n)$

Let $\varphi$ be a linearly full holomorphic immersion from $S^2$ into $G(2, n)$ with constant curvature $K$. The second author in her Ph.D. Thesis ([18]) showed that if $\varphi$ degenerates at position 1, then up to $U(n)$, $\varphi = V_0^{(n-1)} \oplus V_1^{(n-1)}$ with $K = \frac{2}{n-2}$ or $\varphi = V_0^{(n-2)} \oplus \mathbb{C}v$ with $K = \frac{4}{n-2}$, where $v$ is a non-zero constant vector. In this section, we study holomorphic two-spheres of constant curvature in $G(2, n)$ for the case that the generated harmonic sequence degenerates at position 2. The following lemma will be used in this section.

**Lemma 3.1 ([12])** Let $(M, ds^2_M)$ be a surface and $z$ a conformal coordinate on some open subset $U$ of $M$. Let $u$ be a smooth complex valued function and $\eta$ a purely imaginary 1-form on $U$. Assume

$$du \equiv u\eta \mod dz.$$ 

Then $u$ is a function of analytic type, which means $u$ is identically zero or vanishes only at isolated points. Around the points where $u$ does not vanish, then it satisfies

$$\Delta \log |u| = 2d\eta,$$

where $\phi = \lambda dz$ is a local unitary coframe on $U$ and $\Delta$ is Laplace-Beltrami operator with respect to $ds^2_M$.
We give a characterization of a special kind of analytic type functions on \((S^2, ds^2)\). Here we assume that the curvature of \(ds^2\) is a constant \(K\). By stereographic projection, there exists a complex coordinate \(z\) on \(\mathbb{C} = S^3 \setminus \{\infty\}\) such that

\[
ds^2 = \frac{4/K}{(1 + z\bar{z})^2} dzd\bar{z}.
\]

We call such coordinate a canonical coordinate of \((S^2, ds^2)\). The following lemma appears in ([24], Lemma 2.8), see also ([27], Lemma 2.3).

**Lemma 3.2 ([24],[27])** Let \(u\) be a non-constant function of analytic type on \((S^2, ds^2)\) with constant curvature \(K\). If \(u\) satisfies

\[
\frac{1}{4} \Delta \log |u|^2 = c,
\]

where \(u \neq 0, \infty\), and \(c\) is a constant. Then in any canonical coordinate chart, \(|u|^2 = |g(z)|^2(1+z\bar{z})^\frac{4}{K}\), where

\[
g(z) = \frac{c_0(z-z_1) \cdots (z-z_p)}{(z-z_{p+1}) \cdots (z-z_{p+q})}
\]

is a rational polynomial of \(z\), \(c_0\) is a constant. \(z_1, \cdots, z_p\) and \(z_{p+1}, \cdots, z_{p+q}\) are zeros and poles of \(u\) in canonical coordinate \(\mathbb{C}\) respectively, counted multiplicity.

If \(\varphi\) generates the harmonic sequence (2.1) with \(r = 2\), then \(\varphi_2, \cdots, \varphi_{n-3}\) belong to the following harmonic sequence in \(\mathbb{C}P^m\) \((m = n-1, n-2, n-3)\)

\[
0 \xrightarrow{\varphi} \psi^{(m)}_0 \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} \psi^{(m)}_{m-n-5} = \varphi_2 \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} \psi^{(m)}_{m} = \varphi_{n-3} \xrightarrow{\varphi} 0. \tag{3.1}
\]

Now we assume that \(\varphi\) is unramified (cf.[19], Definition 2.4). Then the well-defined invariant \(|\det A_1|^2 \varphi^2 \varphi^{\partial^2}\) has no zeros on \(S^2\) and satisfies

\[
\Delta \log |\det A_1| = 2L_2 + 2K - 4 \tag{3.2}
\]

for \(L_1 = 1\).

In the following we discuss the values of \(K\) and the corresponding map \(\varphi\) in three cases \(m = n-1, n-2, n-3\) respectively.

**Case I:** \(m = n-1\).

In this case, we choose a local unitary frame \(e\) as follows

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
\vdots \\
e_n
\end{pmatrix} = \begin{pmatrix}
e^{(n-1)}_1 \\
e^{(n-1)}_2 \\
e^{(n-1)}_3 \\
0 \\
0 \\
\end{pmatrix}, \tag{3.3}
\]

where

\[
\begin{pmatrix}
e^{(n-1)}_1 \\
e^{(n-1)}_2 \\
e^{(n-1)}_3 \\
0 \\
0 \\
\end{pmatrix} = \begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
0 \\
0 \\
\end{pmatrix},
\]

for \(L_1 = 1\).
where \( U_1 = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \subseteq U(3) \), such that in the Maurer-Cartan forms \((\omega_{AB})\),

\[
A_1 = \begin{pmatrix} a_{13} & 0 \\ a_{23} & a_{24} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ a_{45} & 0 \end{pmatrix}.
\]

Taking exterior differentiation of (3.3) and using (2.9), we get

\[
\begin{pmatrix} 0 & 0 & 0 \\ a_{24} \phi & 0 & 0 \\ \omega_{34} & 0 & 0 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{l_2^{(n-1)}}dz & 0 \end{pmatrix},
\]

\[
\begin{pmatrix} \omega_{44} & a_{45} \phi & 0 \\ -\alpha_{45} \phi & \omega_{55} & \omega_{56} \\ 0 & -\omega_{56} & \omega_{66} \end{pmatrix} = \begin{pmatrix} \theta_{33}^{(n-1)} & \sqrt{l_3^{(n-1)}}dz & 0 \\ -\sqrt{l_3^{(n-1)}}d\bar{z} & \theta_{44}^{(n-1)} & \sqrt{l_4^{(n-1)}}dz \\ 0 & -\sqrt{l_4^{(n-1)}}d\bar{z} & \theta_{55}^{(n-1)} \end{pmatrix},
\]

and

\[
\Theta_1 = dU_1 U_1^* + U_1 \Theta_2 U_1^*,
\]

where

\[
\Theta_1 = \begin{pmatrix} \omega_{11} & \omega_{12} & a_{13} \phi \\ -\omega_{12} & \omega_{22} & a_{23} \phi \\ -a_{13} \phi & -a_{23} \phi & \omega_{33} \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} \theta_{30}^{(n-1)} & \sqrt{l_0^{(n-1)}}dz & 0 \\ -\sqrt{l_0^{(n-1)}}d\bar{z} & \theta_{11}^{(n-1)} & \sqrt{l_1^{(n-1)}}dz \\ 0 & -\sqrt{l_1^{(n-1)}}d\bar{z} & \theta_{22}^{(n-1)} \end{pmatrix}.
\]

It follows from (3.4) and (3.5) that,

\[
u_{13} = 0, \quad a_{24} \phi = \sqrt{l_1^{(n-1)}}u_{23}dz, \quad \omega_{34} = u_{33}\sqrt{l_3^{(n-1)}}dz, \quad a_{45} \phi = \sqrt{l_3^{(n-1)}}dz.
\]

Then from (3.6) and \(U_1 U_1^* = I_3\), we have

\[
a_{13} \phi = du_{11} \cdot \overline{\omega}_{31} + du_{12} \cdot \overline{\omega}_{32} + (\theta_{00}^{(n-1)} - \theta_{11}^{(n-1)})u_{11} \overline{\omega}_{31} - \sqrt{l_0^{(n-1)}}u_{12} \overline{u}_{31}d\bar{z} + (\sqrt{l_0^{(n-1)}}u_{11} \overline{u}_{32} + \sqrt{l_1^{(n-1)}}u_{12} \overline{u}_{33})dz,
\]

and

\[
a_{23} \phi = du_{21} \cdot \overline{\omega}_{31} + du_{22} \cdot \overline{\omega}_{32} + du_{23} \cdot \overline{\omega}_{33} + (\theta_{00}^{(n-1)} - \theta_{22}^{(n-1)})u_{21} \overline{\omega}_{31} + (\sqrt{l_0^{(n-1)}}u_{22} \overline{u}_{32} + \sqrt{l_1^{(n-1)}}u_{23} \overline{u}_{33})d\bar{z} + (\sqrt{l_0^{(n-1)}}u_{21} \overline{u}_{32} + \sqrt{l_1^{(n-1)}}u_{22} \overline{u}_{33})dz.
\]

In the following we will prove that there doesn’t exist this case.
Lemma 3.3  In the case $m = n - 1, u_{32} \neq 0$ and $u_{11} \neq 0$.

Proof: In this case, since $a_{24} \neq 0$, then $u_{23} \neq 0$. If we assume that $u_{32} = 0$, then $u_{22} = 0$ by $U_1 \in U(3)$. It means $|u_{12}|^2 = 1$ and $u_{11} = 0$. From (3.7) we have $a_{13}\phi = -l_0^{(n-1)}u_{12}\bar{u}_{31}d\bar{z} + l_1^{(n-1)}u_{12}\bar{u}_{33}dz$, which implies $u_{31} = 0$. Using $U_1 \in U(3)$ again, we get $u_{23} = 0$, which leads to a contradiction. Similarly, if $u_{11} = 0$, then $u_{32} = 0$ by $U_1 \in U(3)$. It contradicts the above argument. Thus the lemma is proved. \[\square\]

Lemma 3.4  $\eta_1 = \left(\frac{u_{11}
u_{31}^a_{45}}{\nu_{32}a_{13}}\right) dz$ is a globally defined holomorphic differential form on $S^2$.

Proof: We choose another unitary frame $\bar{e}$ satisfying (3.3), then

$$\bar{e} = \text{diag} \left\{ e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, 1, \ldots, 1 \right\} e.$$

Since

$$\bar{a}_{13} = a_{13} \cdot e^{i(\theta_1 - \theta_3)}, \bar{a}_{45} = a_{45}, \bar{u}_{11} = u_{11} \cdot e^{i\theta_1}, \bar{u}_{31} = u_{31} \cdot e^{i\theta_3}, \bar{u}_{32} = u_{32} \cdot e^{i\theta_3},$$

then we have

$$\frac{\bar{u}_{11}\bar{u}_{31}\bar{a}_{45}}{\bar{u}_{32}\bar{a}_{13}} = \frac{u_{11}\nu_{31}^a_{45}}{\nu_{32}a_{13}},$$

which implies that $\eta_1$ is globally defined on $S^2$.

Substituting $u_{12} = -u_{11} \cdot \frac{\bar{u}_{31}}{\bar{u}_{32}}$ into (3.7), we obtain

$$\frac{\partial}{\partial z} \left( \frac{\bar{u}_{31}}{\bar{u}_{32}} \right) = \frac{\bar{u}_{31}}{\bar{u}_{32}} \cdot \frac{\partial \log l_0^{(n-1)}}{\partial z} + \left( \frac{\bar{u}_{31}}{\bar{u}_{32}} \right)^2 \sqrt{l_0^{(n-1)}}, \tag{3.9}$$

and

$$\frac{a_{13}\phi}{u_{11}u_{32}} = \left\{ -\frac{\partial}{\partial z} \left( \frac{\bar{u}_{31}}{\bar{u}_{32}} \right) - \frac{\bar{u}_{31}}{\bar{u}_{32}} \cdot \frac{\partial \log l_0^{(n-1)}}{\partial z} + \sqrt{l_0^{(n-1)}} - \frac{\bar{u}_{31} \bar{u}_{33}}{\bar{u}_{32} \bar{u}_{32}} \cdot \sqrt{l_1^{(n-1)}} \right\} dz,$$

which implies by $\phi = \sqrt{l_3^{(n-1)}} dz$,

$$\frac{a_{13}\sqrt{l_3^{(n-1)}}}{u_{11}u_{31}a_{45}} = -\frac{\partial}{\partial z} \left( \frac{\bar{u}_{31}}{\bar{u}_{32}} \right) - \frac{\bar{u}_{31}}{\bar{u}_{32}} \cdot \frac{\partial \log l_0^{(n-1)}}{\partial z} + \sqrt{l_0^{(n-1)}} - \frac{\bar{u}_{31} \bar{u}_{33}}{\bar{u}_{32} \bar{u}_{32}} \cdot \sqrt{l_1^{(n-1)}}. \tag{3.10}$$

Substituting $u_{22} = -u_{21} \cdot \frac{\bar{u}_{41}}{\bar{u}_{42}} - u_{23} \cdot \frac{\bar{u}_{33}}{\bar{u}_{32}}$ into (3.8) and using (3.9), we have

$$\frac{\partial}{\partial z} \left( \frac{\bar{u}_{33}}{\bar{u}_{32}} \right) = \frac{\bar{u}_{33}}{\bar{u}_{32}} \cdot \frac{\partial \log l_1^{(n-1)}}{\partial z} + \frac{\bar{u}_{33} \bar{u}_{31}}{\bar{u}_{32} \bar{u}_{32}} \cdot \sqrt{l_0^{(n-1)}} - \sqrt{l_1^{(n-1)}}. \tag{3.11}$$
By (3.9), (3.10) and (3.11), a straightforward computation shows

\[
\frac{\partial}{\partial z} \left( \frac{u_{11}a_{45}}{u_{32}a_{13} \sqrt{l_0^{(n-1)}l_3^{(n-1)}}} \right) = 0. \tag{3.12}
\]

which implies that \( \eta_1 \) is holomorphic.

\[\square\]

**Lemma 3.5** In the case \( m = n - 1 \), \( u_{31} = 0 \).

**Proof:** Since \( S^2 \) has no holomorphic differential forms of positive degree except zero, then from Lemma 3.4, \( \eta_1 = 0 \), which gives \( u_{31} = 0 \). \[\square\]

Since \( u_{31} = 0 \), then \( u_{21}u_{23} = 0 \) by \( U_1^* U_1 = I_3 \). Then by \( u_{23} \neq 0 \) we have \( u_{21} = 0 \), which implies \( u_{12} = 0 \). Now we can choose \( e \) such that

\[
U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_{22} & u_{23} \\ 0 & -\bar{u}_{23} & \bar{u}_{22} \end{pmatrix} \in U(3).
\]

From (3.4)-(3.6), we get

\[
a_{13}\phi = -u_{23} \sqrt{l_0^{(n-1)}} \, dz, \quad a_{24}\phi = u_{23} \sqrt{l_2^{(n-1)}} \, dz, \quad \omega_{12} = \bar{u}_{22} \sqrt{l_0^{(n-1)}} \, dz, \quad \omega_{34} = \bar{u}_{22} \sqrt{l_2^{(n-1)}} \, dz,
\]

and

\[
dU_1 = \Theta_1 \cdot U_1 - U_1 \cdot \Theta_2,
\]

where \( \Theta_1, \Theta_2 \) is given by (3.6). It follows from (3.8) that

\[
a_{23}\phi = -du_{22} \cdot u_{23} + du_{23} \cdot u_{22} - u_{22}u_{23}(\theta_{11}^{(n-1)} - \theta_{22}^{(n-1)}) + (u_{23})^2 \sqrt{l_1^{(n-1)}} \, d\bar{z} + (u_{22})^2 \sqrt{l_1^{(n-1)}} \, dz,
\]

which implies that \( u_{22} \neq 0 \), and

\[
\frac{\partial}{\partial \bar{z}} \left( \frac{u_{22}}{u_{23}} \right) = -\frac{\partial \log \sqrt{l_1^{(n-1)}}}{\partial \bar{z}} \cdot \frac{u_{22}}{u_{23}} + \sqrt{l_1^{(n-1)}}. \tag{3.14}
\]

**Proposition 3.6** If \( \varphi : S^2 \to G(2,n) \) is a linearly full unramified holomorphic curve of constant curvature and generates the harmonic sequence (2.1) with \( r = 2 \), then \( m \neq n - 1 \).

**Proof:** The equation (3.14) is a first order linear differential equation with respect to \( \bar{z} \). Solving the differential equation, we get

\[
\frac{u_{22}}{u_{23}} = \frac{\int l_1^{(n-1)} \, d\bar{z} + c(z)}{\sqrt{l_1^{(n-1)}}},
\]

where \( c(z) \) is a holomorphic function. From (2.10) we have

\[
\frac{\partial}{\partial z} \left( \frac{\partial \log |F_1^{(n-1)}|^2}{\partial z} \right) = l_1^{(n-1)}.
\]
Then the above solution becomes

\[ \frac{u_{22}}{u_{23}} = \frac{\partial |F_1^{(n-1)}|^2}{\partial z} + c(z)|F_1^{(n-1)}|^2 \cdot \frac{1}{|F_0^{(n-1)}||F_2^{(n-1)}|}. \]  (3.15)

Since \( \varphi \) is an unramified holomorphic curve of constant curvature \( K \). Then we can choose a complex coordinate \( z \) on \( \mathbb{C} = \mathbb{S}^2 \setminus \{ p \} \) so that the induced metric \( \varphi^* ds^2 = \phi \bar{\phi} \) is given by

\[ \phi \bar{\phi} = \frac{4/K}{(1 + z^n)^2} dz d\bar{z}. \]

And

\[ |\det A_1|^2 \phi \bar{\phi}^2 = |a_{13}|^2 |a_{24}|^2 \phi \bar{\phi}^2 = |u_{23}|^4 l_0^{(n-1)} l_1^{(n-1)} l_2^{(n-1)} d\bar{z} d\bar{z}. \]

has no zeros on \( \mathbb{S}^2 \). Since \( |u_{22}|^2 + |u_{23}|^2 = 1 \), then \( |u_{23}|^2 \) has no poles, then \( |u_{23}|^2 \), \( l_0^{(n-1)} dz d\bar{z} \), \( l_2^{(n-1)} dz d\bar{z} \) all have no zeros on \( \mathbb{S}^2 \).

From the second rows of both sides of (3.13), we have

\[ du_{23} \equiv u_{23} \left( \omega_{22} - \theta_{22}^{(n-1)} \right) \mod dz. \]  (3.16)

It follows from Lemma 3.1 and (3.16) that \( u_{23} \) is a function of analytic type. Moreover,

\[ \frac{1}{4} \Delta \log |u_{23}|^2 \phi \wedge \bar{\phi} = d \left( \omega_{22} - \theta_{22}^{(n-1)} \right) \]

\[ = \omega_{21} \wedge \omega_{12} + \omega_{23} \wedge \omega_{32} + \omega_{24} \wedge \omega_{42} - d\theta_{22}^{(n-1)} \]

\[ = \left( |a_{12}|^2 - |a_{23}|^2 - |a_{24}|^2 \right) \phi \wedge \bar{\phi} - \left( l_1^{(n-1)} - l_2^{(n-1)} \right) dz \wedge d\bar{z} \]

\[ = -\phi \wedge \bar{\phi} + \left( l_0^{(n-1)} - l_1^{(n-1)} + l_2^{(n-1)} \right) dz \wedge d\bar{z}, \]  (3.17)

here we use \( L_1 = |a_{13}|^2 + |a_{23}|^2 + |a_{24}|^2 = 1 \). Integrating both sides of (3.17), we have by (2.11) that

\[ 0 = \frac{1}{4} \int_{S^2} \Delta \log |u_{23}|^2 \phi \wedge \bar{\phi} \cdot \frac{i}{2} \]

\[ = -\text{Area} + \int_{S^2} \left( l_0^{(n-1)} - l_1^{(n-1)} + l_2^{(n-1)} \right) dz \wedge d\bar{z} \cdot \frac{i}{2} \]

\[ = -\frac{4\pi}{K} + \left( \delta_0^{(n-1)} - \delta_1^{(n-1)} + \delta_2^{(n-1)} \right) \pi, \]  (3.18)

where in the first equation we use the fact that the analytic function \( u_{23} \) has no zeros or poles. Area is the area of \( \varphi \). Recall \( \varphi \) is of constant curvature \( K \), and in the last equation we use the Gauss-Bonnet Theorem. Hence from (3.18) we have

\[ \frac{4}{K} = \delta_0^{(n-1)} - \delta_1^{(n-1)} + \delta_2^{(n-1)}. \]  (3.19)

From (2.10) we have

\[ \frac{1}{4} \Delta \log \frac{|F_0^{(n-1)}|^2 |F_2^{(n-1)}|^2}{|F_1^{(n-1)}|^2} \phi \wedge \bar{\phi} = \left( l_0^{(n-1)} - l_1^{(n-1)} + l_2^{(n-1)} \right) dz \wedge d\bar{z}. \]  (3.20)
Since $l_0^{(n-1)}dzd\bar{z}$ has no zeros on $S^2$, then $|F_0^{(n-1)}|^2$, $|F_1^{(n-1)}|^2$ have no zeros. Here $|F_2^{(n-1)}|^2 = |h_{2,n-1}|^2 |\tilde{F}_2^{(n-1)}|^2$, where $|\tilde{F}_2^{(n-1)}|^2$ has no zeros and $h_{2,n-1} = c_0(z - z_1) \cdots (z - z_r)$ is a holomorphic polynomial that represents the zeros of $F_2^{(n-1)}$. Substituting (3.20) into (3.17), we get

$$\frac{1}{4} \Delta \log \frac{|F_0^{(n-1)}|^2 |\tilde{F}_2^{(n-1)}|^2}{|u_{23}|^2 |F_1^{(n-1)}|^2} = 1.$$ (3.21)

From (3.21) and Lemma 3.2 we have

$$\frac{|F_0^{(n-1)}|^2 |\tilde{F}_2^{(n-1)}|^2}{|u_{23}|^2 |F_1^{(n-1)}|^2} = c_1 (1 + z\bar{z})^\frac{4}{K},$$

where $c_1$ is a positive constant, which implies

$$|u_{23}|^2 = \frac{|F_0^{(n-1)}|^2 |\tilde{F}_2^{(n-1)}|^2}{c_1 |F_1^{(n-1)}|^2 (1 + z\bar{z})^\frac{4}{K}}.$$ (3.22)

Substituting (3.15) and (3.22) into $\frac{|u_{23}|^2}{|u_{23}|^2} + 1 = \frac{1}{|u_{23}|^2}$, we get

$$\left| \frac{\partial |F_1^{(n-1)}|^2}{\partial z} + c(z) |F_1^{(n-1)}|^2 \right|^2 + 1 = \frac{c_1 |F_1^{(n-1)}|^2 (1 + z\bar{z})^\frac{4}{K}}{|F_0^{(n-1)}|^2 |F_2^{(n-1)}|^2}.$$ (3.23)

It follows from (3.19) that the term of the right side of (3.23) is bounded on $S^2$. Set

$$T = \left| \frac{\partial |F_1^{(n-1)}|^2}{\partial z} + c(z) |F_1^{(n-1)}|^2 \right|^2 \frac{|F_0^{(n-1)}|^2 |F_2^{(n-1)}|^2}{|F_0^{(n-1)}|^2 |F_2^{(n-1)}|^2}.$$ (3.24)

From (3.23) we find the holomorphic function $c(z)$ must be in this form

$$c(z) = \frac{c_0(z - z_1') \cdots (z - z_{p'})}{(z - z_{p+1}') \cdots (z - z_{p+q}')}, \quad (p \geq 0).$$

We claim $c(z) = c_0(z - z_1') \cdots (z - z_{p'})$. Otherwise let $z \to z_{p+j}'$, then $T \to \infty$. It contradicts (3.23). From (2.12) we have

$$\delta_0^{(n-1)} - 2\delta_1^{(n-1)} + \delta_2^{(n-1)} = -2 - r_1^{(n-1)},$$

which implies

$$2\delta_1^{(n-1)} + p = (\delta_0^{(n-1)} + \delta_2^{(n-1)} + r_1^{(n-1)}) = p + 2 \geq 2.$$ (3.25)

It means that $T \to \infty$ as $z \to \infty$. It contradicts (3.23). So there doesn’t exist the case of $m = n - 1$, i.e. $m \neq n - 1$.  

□

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Case II: \( m = n - 2 \).

In this case, similarly we choose a local unitary frame \( e \) as follows

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_n
\end{pmatrix} = \begin{pmatrix}
u_{11} & u_{12} & u_{13} & 0 & \cdots & 0 \\
u_{21} & u_{22} & u_{23} & 0 & \cdots & 0 \\
u_{31} & u_{32} & u_{33} & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix} \begin{pmatrix}
v_0 \\
e_0^{(n-2)} \\
e_1^{(n-2)} \\
e_2^{(n-2)} \\
\vdots \\
e_{n-2}^{(n-2)}
\end{pmatrix}, \quad (3.24)
\]

where \( v_0 = (0, \cdots, 0, 1) \), so that \( A_1, A_2 \) have the same form as in Case I, here the trivial bundle \( S^2 \times \mathbb{C}^n \) over \( S^2 \) has a corresponding decomposition \( S^2 \times \mathbb{C}^n = S^2 \times \mathbb{C}^{n-1} \oplus S^2 \times \mathbb{C} \).

In this case we get the similar equation with \((3.4)\) and \((3.5)\), which implies

\[ u_{13} = 0, \quad a_{45} \phi = \sqrt{i_0^{(n-2)}} dz. \]

In the equation \((3.6)\), the matrix \( \Theta_2 \) becomes

\[
\Theta_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & \theta_{00}^{(n-2)} & \sqrt{i_0^{(n-2)}} dz \\
0 & -\sqrt{i_0^{(n-2)}} d\bar{z} & \theta_{11}^{(n-2)}
\end{pmatrix} \quad (3.25)
\]

Then from \((3.6)\) we have

\[
a_{13} \phi = d u_{11} \cdot \bar{u}_{31} + d u_{12} \cdot \bar{u}_{32} + u_{12} \bar{u}_{32} \theta_{00}^{(n-2)} + u_{12} \bar{u}_{33} \sqrt{i_0^{(n-2)}} dz, \quad (3.26)
\]

and

\[
a_{23} \phi = d u_{21} \cdot \bar{u}_{31} + d u_{22} \cdot \bar{u}_{32} + d u_{23} \cdot \bar{u}_{33} + u_{22} \bar{u}_{32} \theta_{00}^{(n-2)} + u_{23} \bar{u}_{33} \theta_{11}^{(n-2)} \\
- u_{23} \bar{u}_{32} \sqrt{i_0^{(n-2)}} d\bar{z} + u_{22} \bar{u}_{33} \sqrt{i_0^{(n-2)}} dz. \quad (3.27)
\]

**Lemma 3.7** In the case \( m = n - 2, \ u_{31} \neq 0 \) and \( u_{12} \neq 0 \).

**Proof:** Assume that \( u_{31} = 0 \), then \( u_{21} \bar{u}_{23} = 0 \) by \( U_1^* U_1 = I_3 \). Then by \( u_{23} \neq 0 \) we have \( u_{21} = 0 \), which implies \( u_{12} = 0 \). From \((3.26)\) we find \( a_{13} = 0 \). It’s a contradiction. If \( u_{12} = 0 \), then \( u_{31} = 0 \) by \( U_1^* U_1 = I_3 \). It contradicts the above argument. So the lemma is proved. \( \square \)

Similar to Case I, we have

**Lemma 3.8** \( \eta_2 = \left( \frac{u_{12} \bar{u}_{23} \phi_{13}}{u_{13} \sqrt{i_0^{(n-2)}}} \right) \) \( dz \) is a globally defined holomorphic differential form on \( S^2 \).
Proof: We choose another unitary frame $\tilde{e}$ satisfying (3.24), then by similar argument as in Case I, we have

$$\frac{\tilde{a}_{12}\tilde{u}_{32}\tilde{a}_{45}}{\tilde{a}_{13}} = \frac{u_{12}\overline{u}_{32}a_{45}}{a_{13}},$$

which implies that $\eta_2$ is globally defined on $S^2$.

Substituting $u_{11} = -u_{12} \cdot \frac{u_{32}}{\overline{u}_{31}}$ into (3.26), we obtain

$$\frac{\partial}{\partial z} (\frac{u_{32}}{u_{31}}) = -\frac{\partial \log |f_0^{(n-2)}|}{\partial z} \cdot \frac{u_{32}}{u_{31}},$$

(3.28)

and

$$\frac{a_{13}\phi}{u_{12}u_{31}} = \left\{ -\frac{\partial}{\partial z} \left( \frac{u_{32}}{u_{31}} \right) + \frac{\partial \log |f_0^{(n-2)}|}{\partial z} \cdot \frac{u_{32}}{u_{31}} + \sqrt{l_0^{(n-2)}} \cdot \frac{u_{32}}{u_{31}} \right\} dz,$$

which implies by $\phi = \sqrt{\frac{l_2^{(n-2)}}{a_{45}}} dz$,

$$\frac{a_{13}\sqrt{l_2^{(n-2)}}}{u_{12}u_{31}a_{45}} = -\frac{\partial}{\partial z} \left( \frac{u_{32}}{u_{31}} \right) + \frac{\partial \log |f_0^{(n-2)}|}{\partial z} \cdot \frac{u_{32}}{u_{31}} + \sqrt{l_0^{(n-2)}} \cdot \frac{u_{32}}{u_{31}}.$$

(3.29)

Substituting $u_{21} = -u_{22} \cdot \frac{u_{32}}{u_{31}} - u_{23} \cdot \frac{u_{33}}{u_{31}}$ into (3.27) and using (3.28), we have

$$\frac{\partial}{\partial z} \left( \frac{u_{33}}{u_{31}} \right) = -\frac{\partial \log |f_1^{(n-2)}|}{\partial z} \cdot \frac{u_{33}}{u_{31}} + \sqrt{l_0^{(n-2)}} \cdot \frac{u_{32}}{u_{31}}.$$

(3.30)

By (3.28), (3.29) and (3.30), a straightforward computation shows

$$\frac{\partial}{\partial z} \left( \frac{a_{13}|f_0^{(n-2)}| \sqrt{l_2^{(n-2)}}}{u_{12}u_{31}a_{45}} \right) = 0.$$

(3.31)

Combining (3.28) and (3.31), we find

$$\frac{\partial}{\partial z} \left( \frac{u_{12}u_{31}a_{45}}{a_{13}\sqrt{l_2^{(n-2)}}} \right) = 0.$$

(3.32)

So $\eta_2$ is also a holomorphic differential form on $S^2$. $\square$

As a corollary we have $\eta_2 = 0$, which gives $u_{32} = 0$. Then $u_{23}u_{33} = 0$ by $U_1^*U_1 = I_3$. Thus by $u_{23} \neq 0$ we have $u_{22} = 0$, which implies $u_{11} = 0$. Now we can choose $e$ such that

$$U_1 = \begin{pmatrix} 0 & 1 & 0 \\ u_{21} & 0 & u_{23} \\ -\overline{u}_{23} & 0 & \overline{u}_{21} \end{pmatrix} \in U(3).$$

Similar to (3.4)-(3.6), we get

$$a_{13}\phi = u_{21}\sqrt{l_0^{(n-2)}} dz, \ a_{24}\phi = u_{23}\sqrt{l_1^{(n-2)}} dz, \ \omega_{12} = \overline{u}_{23}\sqrt{l_0^{(n-2)}} dz, \ \omega_{34} = \overline{u}_{21}\sqrt{l_1^{(n-2)}} dz,$$

and
and
\[ dU_1 = \Theta_1 \cdot U_1 - U_1 \cdot \Theta_2, \]  
(3.33)
where \( \Theta_1 \) is the same form as (3.6) and \( \Theta_2 \) is given by (3.25).

Assume \( \varphi \) is an unramified holomorphic curve of constant curvature \( K \). Then we can choose a complex coordinate \( z \) on \( \mathbb{C} = S^2 \setminus \{ pt \} \) so that the induced metric \( \varphi^* ds^2 = \phi \bar{\phi} \) is given by
\[ \phi \bar{\phi} = \frac{4/K}{(1 + z\bar{z})^2} dz d\bar{z}. \]
And
\[ |\det A|^2 \phi \bar{\phi}^2 = |a_{13}|^2 |a_{24}|^2 \phi \bar{\phi}^2 = |u_{21}|^2 |u_{23}|^2 l_{0}^{(n-2)} l_{1}^{(n-2)} dz^2 d\bar{z}^2 \]
has no zeros on \( S^2 \). Since \( |u_{21}|^2 + |u_{23}|^2 = 1 \), then both \( |u_{21}|^2 \) and \( |u_{23}|^2 \) have no poles, then \( |u_{21}|^2, |u_{23}|^2, l_{0}^{(n-2)} d\bar{z} dz \) all have no zeros on \( S^2 \). In the following we will prove that there doesn’t exist this case \( m = n - 2 \).

**Proposition 3.9** If \( \varphi : S^2 \to G(2, n), n \geq 5 \) is a linearly full unramified holomorphic curve of constant curvature and generates the harmonic sequence (2.1) with \( r = 2 \), then \( m \neq n - 2 \).

**Proof:** From the second rows of both sides of (3.33), we have
\[
\begin{align*}
du_{21} &\equiv u_{21} \omega_{22} \mod dz, \\
du_{23} &\equiv u_{23} \left( \omega_{22} - \delta_{11}^{(n-2)} \right) \mod dz. \\
\end{align*}
\]
(3.34)
It follows from Lemma 3.1 and (3.34) that \( u_{21}, u_{23} \) are functions of analytic type. Moreover,
\[
\frac{1}{4} \Delta \log |u_{21}|^2 \phi \wedge \bar{\phi} = du_{22} \\
= \omega_{21} \wedge \omega_{12} + \omega_{23} \wedge \omega_{32} + \omega_{24} \wedge \omega_{42} \\
= \left( |a_{12}|^2 - |a_{23}|^2 - |a_{24}|^2 \right) \phi \wedge \bar{\phi} \\
= \left( -1 + |a_{12}|^2 + |a_{13}|^2 \right) \phi \wedge \bar{\phi} \\
= \left( -1 + l_{0}^{(n-2)} \cdot \frac{(1 + z\bar{z})^2}{4/K} \right) \phi \wedge \bar{\phi},
\]
(3.35)
here we use \( |a_{13}|^2 + |a_{23}|^2 + |a_{24}|^2 = 1 \). Similarly, we have
\[
\frac{1}{4} \Delta \log |u_{23}|^2 \phi \wedge \bar{\phi} = \left( -1 + l_{1}^{(n-2)} \cdot \frac{(1 + z\bar{z})^2}{4/K} \right) \phi \wedge \bar{\phi}.
\]
(3.36)
Integrating both sides of (3.35), we get by (2.11)
\[
0 = \frac{1}{4} \int_{S^2} \Delta \log |u_{21}|^2 \phi \wedge \bar{\phi} \cdot \frac{i}{2} \\
= \int_{S^2} \left( -1 + l_{0}^{(n-2)} \cdot \frac{(1 + z\bar{z})^2}{4/K} \right) \phi \wedge \bar{\phi} \cdot \frac{i}{2} \\
= -\text{Area} + \int_{S^2} l_{0}^{(n-2)} dz \wedge d\bar{z} \cdot \frac{i}{2} \\
= -\frac{4\pi}{K} + \delta_{0}^{(n-2)} \pi,
\]
(3.37)
where in the first equation we use the fact that the analytic function \( u_{21} \) has no zeros or poles. Area is the area of \( \varphi \). Recall \( \varphi \) is of constant curvature \( K \), and in the last equation we use the Gauss-Bonnet Theorem. Hence from (3.37) we have

\[
\frac{4}{K} = \delta_0^{(n-2)}. \tag{3.38}
\]

Similarly, integrating both sides of (3.36), we obtain

\[
\frac{4}{K} = \delta_1^{(n-2)}. \tag{3.39}
\]

Since \( i_0^{(n-2)} \,dz\,d\bar{z} \) has no zeros on \( S^2 \), then from (2.12) we have

\[
-2\delta_0^{(n-2)} + \delta_1^{(n-2)} = -2. \tag{3.40}
\]

From (3.38)-(3.40), we get \( \delta_0^{(n-2)} = 2 \). Since \( \delta_0^{(n-2)} \geq n - 2 \), then \( n \leq 4 \). It contradicts that \( n \geq 5 \). Thus there doesn’t exist the case of \( m = n - 2 \) if \( n \geq 5 \).

\[\square\]

**Case III:** \( m = n - 3 \).

In this case, the trivial bundle \( S^2 \times \mathbb{C}^n \) over \( S^2 \) has a corresponding decomposition \( S^2 \times \mathbb{C}^n = S^2 \times \mathbb{C}^{n-2} \oplus S^2 \times \mathbb{C}^2 \). Let \( G \) be a smooth section of \( S^2 \times \mathbb{C}^2 \). By the harmonic sequence (2.1) with \( r = 2 \) and (3.1) there exists a local section \( V = G + x_0 e_0^{(n-3)} \) such that \( \varphi_1 = \text{span} \{ V, e_1^{(n-3)} \} \). Since \( \partial \varphi_1 = \varphi_2 = \text{span} \{ e_2^{(n-3)} \} \), then a straightforward computation shows that \( \frac{\partial}{\partial z} G = \frac{\partial V}{|V|^2} G \), which implies that \( \text{span} \{ G \} \) is an anti-holomorphic line bundle of \( S^2 \times \mathbb{C}^2 \). So it belongs to the harmonic sequence in \( \mathbb{C}P^1 \) as follows

\[
0 \xrightarrow{\partial} \psi_0^{(1)} \xrightarrow{\partial} \psi_1^{(1)} = \text{span} \{ G \} \xrightarrow{\partial} 0. \tag{3.41}
\]

Let \( f_0^{(1)} \) be a nowhere zero holomorphic section of \( \text{Im} \psi_0^{(1)} \). Without loss of generality we assume that \( \partial f_0^{(1)}/\partial \bar{z} = 0 \), then by (2.5) we obtain \( f_1^{(1)} \), which is a local section of \( \text{Im} \psi_1^{(1)} \). Then we can choose a local unitary frame \( e \) as follows

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
\vdots \\
e_n
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & u_{22} & u_{23} & 0 & \cdots & 0 \\
0 & -\bar{u}_{23} & \bar{u}_{22} & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
e_0^{(1)} \\
e_1^{(1)} \\
e_0^{(n-3)} \\
e_1^{(n-3)} \\
\vdots \\
e_{n-3}^{(n-3)}
\end{pmatrix}, \tag{3.42}
\]

where \( \begin{pmatrix} u_{22} & u_{23} \\ -\bar{u}_{23} & \bar{u}_{22} \end{pmatrix} \in U(2) \). Here \( A_1, A_2 \) have the same form as in Case I.

In this case, we get the similar equation with (3.4) and (3.5), which implies

\[
a_{24}\phi = u_{23} \sqrt{\int_0^{l_{0}^{(n-3)}} dz}, \quad a_{45}\phi = \sqrt{\int_1^{l_{1}^{(n-3)}} dz}.
\]
In the equation (3.6), the matrix $\Theta_2$ becomes

$$\Theta_2 = \begin{pmatrix}
\theta_{00}^{(1)} & \sqrt{\ell_0^{(1)}} dz & 0 \\
-\sqrt{\ell_0^{(1)}} d\bar{z} & \theta_{11}^{(1)} & 0 \\
0 & 0 & \theta_{00}^{(n-3)}
\end{pmatrix}.$$

Then from (3.6) we have

$$a_{13} \phi = -u_{23} \sqrt{\ell_0^{(1)}} dz, \quad (3.43)$$

and

$$a_{23} \phi = -d u_{22} \cdot u_{23} + d u_{23} \cdot u_{22} - u_{22} u_{23} \theta_{11}^{(1)} + u_{23} u_{22} \theta_{00}^{(n-3)}. \quad (3.44)$$

Since $\phi$ is a holomorphic curve of constant curvature $K$, then we can choose a complex coordinate $z$ on $S^2 \setminus \{ pt \}$ so that the induced metric $\varphi^* ds^2 = \phi \overline{\phi}$ is given by

$$\phi \overline{\phi} = \frac{4/K}{(1 + z\overline{z})^2} dz d\overline{z}.$$  

Assume $\varphi$ is unramified, then

$$|\det A|^2 |\varphi|^2 = |a_{13}|^2 |a_{24}|^2 |\varphi|^2 = |u_{23}|^4 \ell_0^{(1)} \ell_0^{(n-3)} dz d\overline{z}^2$$

has no zeros on $S^2$. By $|u_{22}|^2 + |u_{23}|^2 = 1$, we know $|u_{23}|^4$ has no poles, then we obtain that $|u_{23}|^2$, $\ell_0^{(1)} dz d\overline{z}$, $\ell_0^{(n-3)} dz d\overline{z}$ all have no zeros on $S^2$. Since $\ell_0^{(1)} dz d\overline{z}$ has no zeros, it follows from section 3 of [3] that

$$\delta_0^{(1)} = \frac{1}{2\pi i} \int_{S^2} \ell_0^{(1)} d\overline{z} \wedge dz = 1,$$

then from (2.7),

$$\frac{1}{2\pi i} \int_{S^2} \frac{\partial^2}{\partial z \partial \overline{z}} \log |f_0^{(1)}|^2 d\overline{z} \wedge dz = 1,$$

which implies that the degree of the holomorphic curve $\psi_0^{(1)}$ is 1. Without loss of generality we assume that

$$|f_0^{(1)}|^2 = \alpha + \beta z + \beta \overline{z} + \delta z \overline{z}$$ \hspace{1cm} (3.45)

for some complex numbers $\alpha, \beta, \delta$ with $\alpha, \delta$ real, and $\alpha \delta - \beta \overline{\beta} > 0$. A straightforward computation shows that $|f_0^{(1)}|^2 |f_1^{(1)}|^2 = |f_0^{(1)}|^4 \cdot \frac{\partial^2}{\partial z \partial \overline{z}} \log |f_0^{(1)}|^2 = \alpha \delta - \beta \overline{\beta}$ is a positive constant denoted by $c_2^{(1)}$, which implies $\theta_{11}^{(1)} = -\frac{\partial}{\partial z} \log |f_0^{(1)}| dz + \frac{\partial}{\partial \overline{z}} \log |f_0^{(1)}| d\overline{z}$. From (3.44), we obtain

$$\frac{\partial}{\partial \overline{z}} \left( \frac{u_{22}}{u_{23}} \right) = -\frac{\partial}{\partial z} \log |f_0^{(1)}| |f_0^{(n-3)}| \cdot \frac{u_{22}}{u_{23}}, \quad (3.46)$$

and

$$\frac{a_{23}}{(u_{23})^2} = -\frac{\partial}{\partial z} \left( \frac{u_{22}}{u_{23}} \right) + \frac{\partial}{\partial z} \log |f_0^{(1)}| |f_0^{(n-3)}| \cdot \frac{u_{22}}{u_{23}}. \quad (3.47)$$
It follows from (3.46) that \( \frac{\partial}{\partial z}(u_{22} \cdot |f_0(1)||f_0^{(n-3)}|) = 0 \), which implies that the function \( u_{22} \cdot |f_0(1)||f_0^{(n-3)}| \) is a holomorphic function denoted by \( h(z) \). Then we have

\[
\frac{u_{22}}{u_{23}} = \frac{h}{|f_0(1)||f_0^{(n-3)}|}, \quad |u_{23}|^2 = \frac{|f_0(1)|^2|f_0^{(n-3)}|^2}{|h|^2 + |f_0(1)|^2|f_0^{(n-3)}|^2}.
\]

(3.48)

Assume the degree of the holomorphic curve \( \psi_0^{(n-3)} \) is a positive integer \( d \) (\( d \geq n-3 \)). Since \( l_0^{(n-3)}dzd\bar{z} \) has no zeros, it follows from section 3 of [3] that

\[
\delta^{(n-3)}_0 = d, \quad \delta^{(n-3)}_1 = 2d - 2.
\]

In this case since \( L_2\phi \bar{\phi} = l_1^{(n-3)}dzd\bar{z} \), then (3.2) becomes

\[
\frac{1}{4} \Delta \log |\det A_1| \phi \wedge \bar{\phi} = l_1^{(n-3)} \cdot \frac{1}{2} dz \wedge d\bar{z} + (K - 2) \cdot \frac{1}{2} \phi \wedge \bar{\phi}.
\]

(3.49)

Integrating both sides of (3.49), we have

\[
0 = \frac{1}{2} \int_{S^2} \Delta \log |\det A_1| \phi \wedge \bar{\phi} \cdot \frac{i}{2}
= \int_{S^2} l_1^{(n-3)} dz \wedge d\bar{z} \cdot \frac{i}{2} + (K - 2) \phi \wedge \bar{\phi} \cdot \frac{i}{2}
= \delta^{(n-3)}_1 \pi + (K - 2) \text{Area}
= (2d - 2) \pi + \frac{4(K - 2) \pi}{K},
\]

(3.50)

where in the first equation we use the fact that the analytic function \( \det A_1 \) has no zeros or poles. Area is the area of \( \varphi \). Recall \( \varphi \) is of constant curvature \( K \), and in the last equation we use the Gauss-Bonnet Theorem. Hence by (3.50), we have

\[
K = \frac{4}{d + 1}.
\]

(3.51)

Then the induced metric \( \varphi^s ds^2 = \phi \bar{\phi} \) is given by

\[
\phi \bar{\phi} = \frac{d + 1}{(1 + z\bar{z})^2} dz d\bar{z}.
\]

So we have

\[
|\det A_1| = |u_{23}|^2 \sqrt{\frac{l_0^{(1)}}{l_0^{(n-3)}}} \cdot \frac{(1 + z\bar{z})^2}{d + 1}, \quad 2L_2 = \Delta \log |f_0^{(n-3)}||f_1^{(n-3)}|,
\]

\[
2K - 4 = \frac{4 - 4d}{d + 1} = \Delta \log (1 + z\bar{z})^{1-d}.
\]

Using (3.2) again, we obtain

\[
\Delta \log \frac{|u_{23}|^2(1 + z\bar{z})^{d+1}}{|f_0(1)|^2|f_0^{(n-3)}|^2} = 0.
\]
Since \( \frac{|u_{23}|^2(1+z\bar{z})^{d+1}}{|f_0^{(1)}|^2|f_0^{(n-3)}|^2} \) is a globally defined nowhere zero function on \( S^2 \), then it follows that

\[
\frac{|u_{23}|^2(1+z\bar{z})^{d+1}}{|f_0^{(1)}|^2|f_0^{(n-3)}|^2} = \frac{1}{c},
\]

where \( c \) is a positive constant. So we get

\[
|u_{23}|^2 = \frac{|f_0^{(1)}|^2|f_0^{(n-3)}|^2}{c(1+z\bar{z})^{d+1}}. \tag{3.52}
\]

Combining (3.48) and (3.52), we have

\[
|h|^2 + |f_0^{(1)}|^2|f_0^{(n-3)}|^2 = c(1+z\bar{z})^{d+1}. \tag{3.53}
\]

Then in this case the corresponding holomorphic map \( \varphi \) is given as follows, by (3.42) and (3.48),

\[
\varphi = \text{span} \left\{ f_0^{(1)}, h f_1^{(1)} + c_0 f_0^{(n-3)} \right\} : S^2 \rightarrow G(2, n). \tag{3.54}
\]

From (3.47), (3.48), (3.52) and (3.53), a straightforward calculation shows \( (|a_{13}|^2 + |a_{24}|^2 + |a_{23}|^2)\phi \bar{\phi} = \frac{d+1}{(1+z\bar{z})^d}dzd\bar{z} \), which verifies that the holomorphic curve defined by (3.54) is of constant curvature \( K = \frac{4}{4+1} \). Since \( f_0^{(1)} = (\ast \ast 0 \cdots 0)^T \), using Plücker embedding

\[
\Phi_n : G(2, n) \rightarrow \mathbb{CP}^{n(n-1)}_{2n-4},
\]

we find that \( \Phi_n \circ \varphi \) is a holomorphic curve of constant curvature in \( \mathbb{CP}^{2n-4} \), then \( d + 1 \leq 2(n-2) \). Thus, we have completed the discussions of the three cases.

In summary, we have proved our main theorem:

**Theorem 3.10** Let \( \varphi : S^2 \rightarrow G(2, n) \) \( (n \geq 5) \) be a linearly full holomorphic curve satisfying that the harmonic sequence generated by \( \varphi \) degenerates at position 2. Let \( \psi_0^{(1)} : S^2 \rightarrow \mathbb{CP}^1 \) and \( \psi_0^{(n-3)} : S^2 \rightarrow \mathbb{CP}^{n-3} \) be holomorphic curves of degree 1 and \( d \) \( (d \geq n-3) \) respectively. Let \( f_0^{(1)} \) and \( f_0^{(n-3)} \) be nowhere zero holomorphic sections of \( \psi_0^{(1)} \) and \( \psi_0^{(n-3)} \) respectively. Set \( f_1^{(1)} = \frac{\partial}{\partial z} f_0^{(1)} - \frac{\partial \log |f_0^{(1)}|^2}{\partial z} f_0^{(1)} \) and \( |f_0^{(1)}||f_1^{(1)}| = c_0 \). If the holomorphic curve \( \varphi \) is unramified with constant curvature \( K \), then there exists a holomorphic polynomial \( h \) satisfying

\[
|f_0^{(1)}|^2|f_0^{(n-3)}|^2 + |h|^2 = c(1+z\bar{z})^{d+1}, \tag{3.55}
\]

where \( c \) is a positive constant, such that

\[
\varphi = \text{span} \left\{ f_0^{(1)}, h f_1^{(1)} - c_0 f_0^{(n-3)} \right\}
\]

with \( K = \frac{4}{4+1} \). Furthermore, \( d + 1 \leq 2(n-2) \).

**Remark 3.11** (i) In this theorem, if the condition of \( \varphi \) being unramified is improved to being totally unramified, that is, every element of the harmonic sequence generated by \( \varphi \) is unramified (cf.[19], Definition 2.4), then \( d = n-3 \) and \( K = \frac{4}{n-2} \). This result was firstly
obtained by Jiao and Yu in [25].

(ii) Through finding the solutions of the polynomial equation (3.55), we can get many examples of holomorphic two-spheres of constant curvature in \( G(2, n) \). If \( h = 0 \), by using Calabi Rigidity Theorem, the holomorphic map \( \varphi = V_0^{(1)} \oplus V_0^{(n-3)} \) with \( K = \frac{4}{n-2} \), up to \( U(n) \), which is homogeneous. And for non-trivial solutions of \( h \), we can check that the corresponding holomorphic curve are not homogeneous. In fact, in [19] the second author and her co-authors have given the symmetric solutions of the polynomial equation (3.55) in the case \( n = 5 \), then determine all totally unramified holomorphic two-spheres of constant curvature in \( G(2, 5) \), which include two families of non-homogeneous holomorphic two-spheres.

(iii) For \( n = 4 \) in Case II, up to \( U(4) \), \( \varphi = \text{span}\{V_0^{(2)}, \cos \theta V_0^{(1)} + \sin \theta v_0\} \), \( \theta \in [0, \pi/2] \), which was classified by Chi and Zheng ([7], see also [16]).

(iv) From analysis of Case III, we know that if \( \varphi \) is unramified with constant curvature then the equation (3.55) holds. But the solutions of (3.55) do not guarantee that the corresponding holomorphic curve \( \varphi \) is unramified (cf.[19]). So by solving the polynomial equation (3.55), we can also obtain ramified holomorphic two-spheres of constant curvature in \( G(2, n) \).

It is well known that the rigidity of holomorphic curves in \( G(k, n) \) \( (k \geq 2) \) does not hold if we only assume they have the same first fundamental form. However, if we further assume that \( \varphi \) has constant length of second fundamental form, then we have

**Theorem 3.12** Let \( \varphi : S^2 \rightarrow G(2, n) \) \( (n \geq 5) \) be a linearly full unramified holomorphic curve of constant curvature satisfying that the harmonic sequence generated by \( \varphi \) degenerates at position 2. If the holomorphic curve \( \varphi \) has constant length of second fundamental form, then \( \varphi = V_0^{(1)} \oplus V_0^{(n-3)} \) with \( K = \frac{4}{n-2} \), up to \( U(n) \).

**Proof:** From Gauss equation (2.3), we know \( |\det A_1| \) is also constant. In Case III, using (3.2), \( L_2 = 2 - K = \frac{2(d-1)}{d+1} \), which implies \( l_1^{(n-3)}dzd\bar{\zeta} = \frac{2(d-1)}{(1+z\bar{\zeta})^2}dzd\bar{\zeta} \). Thus the 1-th osculating curve \( \sigma_1 \) of \( \psi_0^{(n-3)} \) has constant curvature. By using the rigidity theorems for the Veronese sequences ([3],[31]), we know that \( \psi_0^{(n-3)} \) is the Veronese map \( V_0^{(n-3)} \), up to \( U(n-3) \), and \( |f_0^{(n-3)}|^2 = (1+z\bar{\zeta})^{n-3} \). It follows from (3.55) that \( 1+z\bar{\zeta} \) is a factor of \( |h|^2 \), which yields \( h = 0 \) since \( h \) is holomorphic function and \( 1+z\bar{\zeta} \) is an irreducible polynomial. Thus from Remark (3.11) (ii), up to \( U(n) \), \( \varphi = V_0^{(1)} \oplus V_0^{(n-3)} \) with \( K = \frac{1}{n-2} \). \( \square \)

4 Construction of non-homogeneous holomorphic two-spheres of constant curvature in \( G(2, n) \)

In this section we will construct examples of non-homogeneous holomorphic two-spheres of constant curvature in \( G(2, n) \), by finding the solutions of the polynomial equation (3.55).

At first, we give a family of solutions of the polynomial equation (3.55) as follows.

**Proposition 4.1** Let \( \psi_0^{(1)} : S^2 \rightarrow \mathbb{C}P^1 \) and \( \psi_0^{(n-3)} : S^2 \rightarrow \mathbb{C}P^{n-3} \) be holomorphic curves of degree 1 and \( d \) \( (d \geq n-3) \) respectively. Let \( f_0^{(1)} \) and \( f_0^{(n-3)} \) be nowhere zero holomorphic
sections of $\psi_0^{(1)}$ and $\psi_0^{(n-3)}$ respectively, let $h$ be a holomorphic polynomial, satisfying

$$|f_0^{(1)}|^2 = 1 + tz + t\bar{z} + z\bar{z}, \quad |f_0^{(n-3)}|^2 = \sum_{i,j=0}^{d} c_{ij} z^i \bar{z}^j, \quad h = \sum_{i=0}^{d+1} \alpha_i z^i,$$

(4.1)

where $c_{ij} = c_{ji} = c_{d-i,d-j}$, $\alpha_i$, $t$ are real coefficients, and $c_{ij}$ ($i \geq j$), $\alpha_i$ are given by

$$c_{ij} = \frac{\sum_{p=i-j}^{d} (-1)^p t^p \cdot \sum_{k} \binom{i+j-2k}{i-k} \binom{d-i-j+2k}{p-i-j+2k} \sum_{p=0}^{d} (-1)^p \binom{d+1}{p} t^p}{\sum_{p=0}^{d} (-1)^p \binom{d+1}{p} t^p},$$

$$\alpha_i \alpha_j = \frac{(-1)^{d+1} \binom{d+1}{i} \binom{d+1}{j} t^{d+1} \sum_{p=0}^{d} (-1)^p \binom{d+1}{p} t^p}{\sum_{p=0}^{d} (-1)^p \binom{d+1}{p} t^p}.$$

Then (4.1) gives a family of solutions of the polynomial equation (3.55), taking $t$ as a parameter in $(-1, 1)$ such that $|f_0^{(n-3)}|^2 > 0$.

**Proof:** Assume $|f_0^{(1)}|^2$, $|f_0^{(n-3)}|^2$, $h$ are given by (4.1). In order to prove they are solutions of the polynomial equation (3.55) it’s enough to prove that for $i = 0, \ldots, d+1$,

$$c_{ii} + 2t c_{i,i-1} + c_{i-1,i-1} + \alpha_i^2 - c \binom{d+1}{i} = 0 \quad (4.2)$$

holds and for $i > j$,

$$c_{ij} + tc_{i-1,j} + tc_{i,j-1} + c_{i-1,j-1} + \alpha_i \alpha_j = 0 \quad (4.3)$$

holds, where $c_{-1,j} = c_{d+1,j} = 0$ and $c = 1 + \alpha_0^2$.

Substituting (4.1) into the left right of (4.2), we get that (4.2) holds if and only if the following equations

$$\sum_{p=0}^{d} (-1)^p t^p \cdot \sum_{k} \binom{2i-2k}{i-k} \binom{d-2i+2k}{k} \binom{d+1}{p-2i+2k}$$

$$+ 2t \sum_{p=1}^{d} (-1)^p t^p \cdot \sum_{k} \binom{2i-1-2k}{i-1-k} \binom{d-2i+1+2k}{k} \binom{d+1}{p-2i+1+2k}$$

$$+ \sum_{p=0}^{d} (-1)^p t^p \cdot \sum_{k} \binom{2i-2-2k}{i-1-k} \binom{d-2i+2+2k}{k} \binom{d+1}{p-2i+2+2k}$$

$$+ (-1)^{d+1} \binom{d+1}{i} \binom{d+1}{i} t^{d+1} - \binom{d+1}{i} \sum_{p=0}^{d+1} (-1)^p \binom{d+1}{p} \cdot t^p = 0 \quad (4.4)$$

hold. In the following we prove (4.4) holds by proving the coefficient of the term $t^p$ is zero for $p = 0, \ldots, d+1$.

For $p = 0$, the coefficient of the term $t^0$ is given by

$$\binom{d}{i} + \binom{d}{i-1} - \binom{d+1}{i}.$$
which is obviously zero.

For $p = 1$, the coefficient of the term $t^1$ is given by

$$-\binom{d}{i} \binom{d+1}{1} - \binom{d}{i-1} \binom{d+1}{1} + \binom{d+1}{i} \binom{d+1}{1},$$

which is also obviously zero.

For $p = 2s$ ($s \geq 1, p \leq d$), the coefficient of the term $t^{2s}$ is given by

$$\sum_{k=i-1}^{i} \binom{2i-2k}{i-k} \binom{d-2i+2k}{k} \binom{d+1}{2s-2i+2k}$$

$$-2 \sum_{k=i-1}^{i-1} \binom{2i-1-2k}{i-1-k} \binom{d-2i+1+2k}{k} \binom{d+1}{2s-2i+2k}$$

$$+ \sum_{k=i-s-1}^{i-1} \binom{2i-2-2k}{i-1-k} \binom{d-2i+2+2k}{k} \binom{d+1}{2s-2i+2+2k} - \binom{d+1}{i} \binom{d+1}{2s}$$

$$= \sum_{q=1}^{s} \left\{ \binom{2q}{q} \binom{d-2q}{i-q} \binom{d+1}{2s-2q} - 2 \binom{2q-1}{q-1} \binom{d-2q+1}{i-q} \binom{d+1}{2s-2q} \right\}$$

$$+ \binom{2q}{q} \binom{d-2q}{i-q-1} \binom{d+1}{2s-2q}$$

$$+ \binom{d}{i} \binom{d+1}{2s} + \binom{d}{i-1} \binom{d+1}{2s} - \binom{d+1}{i} \binom{d+1}{2s}$$

$$= \sum_{q=1}^{s} \binom{2q}{q} \binom{d+1}{2s-2q} \left\{ \binom{d-2q}{i-q} - \binom{d-2q+1}{i-q} + \binom{d-2q}{i-q-1} \right\}$$

$$+ \binom{d+1}{2s} \left\{ \binom{d}{i} + \binom{d}{i-1} - \binom{d+1}{i} \right\},$$

which is zero.
For \( p = 2s + 1 \) \((s \geq 1, p \leq d)\), the coefficient of the term \( t^{2s+1} \) is given by

\[
- \sum_{k=1-s}^{i} \binom{2i - 2k}{i - k} \binom{d - 2i + 2k}{k} \binom{d + 1}{2s + 1 - 2i + 2k}
+ 2 \sum_{k=1-s}^{i-1} \binom{2i - 1 - 2k}{i - 1 - k} \binom{d - 2i + 1 + 2k}{k} \binom{d + 1}{2s + 1 - 2i + 2k}
- \sum_{k=1-s-1}^{i-1} \binom{2i - 2 - 2k}{i - 1 - k} \binom{d - 2i + 2 + 2k}{k} \binom{d + 1}{2s - 2i + 3 + 2k} + \binom{d + 1}{i} \binom{d + 1}{2s + 1}
\]

\[
= \sum_{q=1}^{s} \left\{- \binom{2q}{q} \binom{d - 2q}{i - q} \binom{d + 1}{2s + 1 - 2q} + 2 \binom{2q - 1}{q - 1} \binom{d - 2q + 1}{i - q} \binom{d + 1}{2s + 1 - 2q}
- \binom{2q}{q} \binom{d - 2q}{i - q - 1} \binom{d + 1}{2s + 1 - 2q} \right\}
- \binom{d + 1}{2s + 1} \left\{ \binom{d}{i} + \binom{d}{i - 1} - \binom{d + 1}{i} \right\},
\]

which is zero.

For \( p = d + 1 \), the coefficient of the term \( t^{d+1} \) is given by

\[
2(-1)^{d} \sum_{k=0}^{i-1} \binom{2i - 1 - 2k}{i - 1 - k} \binom{d - 2i + 1 + 2k}{k} \binom{d + 1}{d + 1 - 2i + 2k}
+ (-1)^{d+1} \binom{d + 1}{i} \binom{d + 1}{d + 1 - 2i + 2k}
\]

\[
= (-1)^{d} \sum_{k=0}^{i-1} \binom{d + 1}{2i - k} \binom{2i - k}{i - k} \binom{i}{k} + (-1)^{d+1} \binom{d + 1}{i} \binom{d + 1}{i - 1} - (-1)^{d+1} \binom{d + 1}{i}
\]

\[
= (-1)^{d} \sum_{k=0}^{i-1} \binom{d + 1}{i} \binom{d + 1 - i}{i - k} \binom{i}{k} + (-1)^{d+1} \binom{d + 1}{i} \binom{d + 1}{i} - (-1)^{d+1} \binom{d + 1}{i}
\]

\[
= (-1)^{d} \binom{d + 1}{i} \left\{ \binom{d + 1}{i} - 1 \right\} + (-1)^{d+1} \binom{d + 1}{i} \binom{d + 1}{i} - (-1)^{d+1} \binom{d + 1}{i},
\]

which is zero.

From the above cases, we know that (4.4) holds. It verifies (4.2).

Similarly, substituting (4.1) into the left right of (4.3), we get that (4.3) holds if and
only if the following equations hold. In the following we prove (4.5) holds by proving the coefficient of the term \( t^p \) is zero for \( p = i - j, \ldots, d + 1 \).

For \( p = i - j \), the coefficient of the term \( t^{i-j} \) is given by

\[
(-1)^{i-j} \left\{ \binom{d - i + j}{j} - \binom{d - i + j + 1}{j} + \binom{d - i + j}{j - 1} \right\},
\]

which is obviously zero.

For \( p = i - j + 1 \), the coefficient of the term \( t^{i-j+1} \) is given by

\[
(-1)^{i-j+1} \left( \binom{d + 1}{1} \right) \left\{ \binom{d - i + j}{j} - \binom{d - i + j + 1}{j} + \binom{d - i + j}{j - 1} \right\}.
\]

which is also obviously zero.
For \( p = i - j + 2s \) \((s \geq 1, p \leq d)\), the coefficient of the term \( t^{i-j+2s} \) is given by

\[
(-1)^{i-j} \sum_{k=j-s}^{j} \binom{i + j - 2k}{j-k} \frac{(d-i-j+2k)}{k} \frac{d+1}{2s-2j+2k} = (-1)^{i-j} \sum_{k=j-s}^{j} \binom{i + j - 2k}{j-k} \frac{(d-i-j+1+2k)}{k} \frac{d+1}{2s-2j+2k} \\
- (-1)^{i-j} \sum_{k=j-s}^{j-1} \binom{i + j - 2k}{j-k} \frac{(d-i-j+1+2k)}{k} \frac{d+1}{2s-2j+2k} \\
- (-1)^{i-j} \sum_{k=j-s}^{j-1} \binom{i + j - 2k}{j-1-k} \frac{(d-i-j+1+2k)}{k} \frac{d+1}{2s-2j+2k} \\
+ (-1)^{i-j} \sum_{k=j-s-1}^{j-2} \binom{i + j - 2k}{j-1-k} \frac{(d-i-j+2+2k)}{k} \frac{d+1}{2s-2j+2+2k} \\
+ (-1)^{i-j} \binom{d+1}{2s} \left\{ \binom{d-i+j}{j} - \binom{d-i+j+1}{j} + \binom{d-i+j}{j-1} \right\} \\
= (-1)^{i-j} \sum_{k=j-s}^{j-1} \binom{i + j - 2k}{j-k} \frac{(d-i-j+2k)}{k} \frac{d+1}{2s-2j+2k} \\
- (-1)^{i-j} \sum_{k=j-s}^{j-1} \binom{i + j - 2k}{j-k} \frac{(d-i-j+1+2k)}{k} \frac{d+1}{2s-2j+2k} \\
+ (-1)^{i-j} \sum_{k=j-s-1}^{j-2} \binom{i + j - 2k}{j-1-k} \frac{(d-i-j+2+2k)}{k} \frac{d+1}{2s-2j+2+2k} \\
= (-1)^{i-j} \sum_{q=1}^{s} \binom{i + j + 2q}{q} \frac{d+1}{2s-2q} \left\{ \binom{d-i+j+2q}{j-q} - \binom{d-i+j+1-2q}{j-q} \right\} + \binom{d-i+j+2q}{j-q-1},
\]

which is zero.
For \( p = i - j + 2s + 1 \) \((s \geq 1, p \leq d)\), the coefficient of the term \( t^{i-j+2s+1} \) is given by
\[
(-1)^{i-j+1}\sum_{q=0}^{s} \binom{i-j+2q}{2s+1-2q}\binom{d}{j-q}
\]
\[
\binom{d-i+j-2q}{j-q} + \binom{d-i+j-2q}{j-q-1},
\]
which is zero.

For \( p = d + 1 \), the coefficient of the term \( t^{d+1} \) is given by
\[
(-1)^d\sum_{k=0}^{j} \binom{i+j-1-2k}{j-k}\binom{d-i-j+1+2k}{k}\binom{d+1}{d+1-i-j+2k}
\]
\[
(-1)^d\sum_{k=0}^{j-1} \binom{i+j-1-2k}{j-1-k}\binom{d-i-j+1+2k}{k}\binom{d+1}{d+1-i-j+2k}
\]
\[
+(-1)^{d+1}\binom{d+1}{i}\binom{d+1}{j},
\]
which is zero.

From the above cases, we know that (4.5) holds. It verifies (4.3).

Next, using this family of solutions given by Proposition 4.1, we give the explicit expressions of \( f_0^{(1)} \) and \( f_0^{(n-3)} \), then characterize the corresponding holomorphic curve \( \varphi \) explicitly by Theorem 3.10.

For \( d+1 = 3 \). Since \( d+1 \geq n-2 \), then \( n \leq 5 \), so \( n = 5 \).

Set
\[
V_0^{(2)} = \begin{bmatrix} 1 & \sqrt{2}z & z^2 \end{bmatrix}^T.
\]
Set \( f_0^{(2)} = A_0^{(2)}V_0^{(2)} \), where \( A_0^{(2)} \in M(3; \mathbb{C}) \), then from (4.1), we have
\[
(A_0^{(2)})^* A_0^{(2)} = \begin{pmatrix} 1 & \frac{c_{10}}{\sqrt{2}} & \frac{c_{20}}{\sqrt{2}} \\ \frac{c_{10}}{\sqrt{2}} & \frac{c_{20}}{2} & \frac{c_{20}}{2} \\ \frac{c_{20}}{2} & \frac{c_{20}}{2} & 1 \end{pmatrix},
\]
where
\[
c_{10} = \frac{-t + 3t^2}{1 - 3t + 3t^2}, \quad c_{20} = \frac{t^2}{1 - 3t + 3t^2}, \quad c_{11} = \frac{2 - 6t + 8t^2}{1 - 3t + 3t^2}.
\]

By calculating the eigenvalues and corresponding eigenvectors of matrix \((A_0^{(2)})^* A_0^{(2)}\), we get
\[
(A_0^{(2)})^* A_0^{(2)} W_0^{(2)} = W_0^{(2)} (D_0^{(2)})^2,
\]
where
\[
W_0^{(2)} = \begin{pmatrix}
\frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\
\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\
\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2}
\end{pmatrix} \in U(3),
\]
\[
D_0^{(2)} = \frac{1}{\sqrt{1-3t+3t^2}} \begin{pmatrix}
1-t & 0 & 0 \\
0 & \sqrt{(1-t)(1-2t)} & 0 \\
0 & 0 & \sqrt{1-4t+7t^2}
\end{pmatrix}.
\]

Set \( U_0^{(2)} = A_0^{(2)} W_0^{(2)} (D_0^{(2)})^{-1} \), then \( U_0^{(2)} \in U(3) \), and \( A_0^{(2)} = U_0^{(2)} D_0^{(2)} (W_0^{(2)})^T \), thus we have
\[
f_0^{(2)} = U_0^{(2)} D_0^{(2)} (W_0^{(2)})^T V_0^{(2)}. \tag{4.6}
\]

Similarly, set
\[
V_0^{(1)} = [1 \ z]^T,
\]
then we get
\[
f_0^{(1)} = U_0^{(1)} D_0^{(1)} (W_0^{(1)})^T V_0^{(1)}, \tag{4.7}
\]
where \( U_0^{(1)} \in U(2) \),
\[
D_0^{(1)} = \begin{pmatrix}
\sqrt{1-t} & 0 \\
0 & \sqrt{1+t}
\end{pmatrix},
W_0^{(1)} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{pmatrix} \in U(2).
\]

In this case, from (4.1) we have
\[
h = \pm \sqrt{-t^3} \frac{1}{1-3t+3t^2}(1+z)^3, \ c_0 = \sqrt{1-t^2}.
\]

From Theorem 3.10 we know
\[
\varphi = \text{span} \left\{ f_0^{(1)}, h \frac{\partial f_0^{(1)}}{\partial z} + c_0 f_0^{(2)} \right\}.
\]

Using (4.6) and (4.7), we obtain a family of linearly full holomorphic curves in \( G(2,5) \) with \( K = \frac{4}{3} \), up to \( U(5) \), as follows,
\[
\varphi = \begin{pmatrix}
\sqrt{1-t}(1-z) & -\sqrt{-2t^3(1-t)}(1+z)^3 \\
\sqrt{1+t}(1+z) & \sqrt{-2t^3(1+t)}(1+z)^3 \\
0 & (1-t)\sqrt{1-t^2}(1-z)^2 \\
0 & -\sqrt{2(1-t)(1-2t)(1-t^2)}(1-z^2) \\
0 & \sqrt{(1-4t+7t^2)(1-t^2)(1+z^2)}
\end{pmatrix}, \tag{4.8}
\]
where \(-1 < t < \frac{1}{2}\).

From the above discussions, we have the following propositions.

**Proposition 4.2** For \( d+1 = 3 \), the solutions (4.1) give a family of linearly full non-homogeneous holomorphic curves of constant curvature \( K = \frac{4}{3} \) in \( G(2,5) \), that is (4.8), which is also given in [19].
II For $d + 1 = 4$. Since $d + 1 \geq n - 2$, then $n \leq 6$, so $n = 5, 6$.

II(1) For $d + 1 = 4$, $n = 6$. Set

$$V_0^{(3)} = [1 \ \sqrt{3}z \ \sqrt{3}z^2 \ z^3]^T.$$ 

Set $f_0^{(3)} = A_0^{(3)}V_0^{(3)}$, where $A_0^{(3)} \in M(4; \mathbb{C})$, then from (4.1), we have

$$\begin{pmatrix} A_0^{(3)} \end{pmatrix}^* A_0^{(3)} = \begin{pmatrix} 1 & c_{10} & c_{11} & c_{12} \\ c_{10} \sqrt{3} & c_{20} & c_{21} & c_{22} \\ c_{11} \sqrt{3} & c_{21} \sqrt{3} & c_{30} & c_{31} \\ c_{12} \sqrt{3} & c_{22} \sqrt{3} & c_{31} \sqrt{3} & c_{32} \end{pmatrix},$$

(4.9)

where

$$c_{10} = \frac{-t + 4t^2 - 6t^3}{1 - 4t + 6t^2 - 4t^3}, \ c_{20} = \frac{t^2 - 4t^3}{1 - 4t + 6t^2 - 4t^3}, \ c_{30} = \frac{-t^3}{1 - 4t + 6t^2 - 4t^3},$$

$$c_{11} = \frac{3 - 12t + 20t^2 - 20t^3}{1 - 4t + 6t^2 - 4t^3}, \ c_{21} = \frac{-2t + 8t^2 - 15t^3}{1 - 4t + 6t^2 - 4t^3}.$$

By calculating the eigenvalues and corresponding eigenvectors of matrix $\begin{pmatrix} A_0^{(3)} \end{pmatrix}^* A_0^{(3)}$, we get

$$\begin{pmatrix} A_0^{(3)} \end{pmatrix}^* A_0^{(3)} W_0^{(3)} = W_0^{(3)} \begin{pmatrix} D_0^{(3)} \end{pmatrix}^2,$$

where

$$W_0^{(3)} = \begin{pmatrix} -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix} \in U(4),$$

$$D_0^{(3)} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

with

$$\lambda_1 = \sqrt{\frac{(1-t)^3}{(1-2t)(1-2t+2t^2)}}, \ \lambda_2 = \sqrt{\frac{(1-t)^2(3-5t)}{3(1-2t)(1-2t+2t^2)}},$$

$$\lambda_3 = \sqrt{\frac{(1-t)(3-10t+11t^2)}{3(1-2t)(1-2t+2t^2)}}, \ \lambda_4 = \sqrt{\frac{(1-3t)(1+2t+5t^2)}{(1-2t)(1-2t+2t^2)}}.$$ 

Set $U_0^{(3)} = A_0^{(3)} W_0^{(3)} \left( D_0^{(3)} \right)^{-1}$, then $U_0^{(3)} \in U(3)$, and $A_0^{(3)} = U_0^{(3)} D_0^{(3)} \left( W_0^{(3)} \right)^T$, thus we have

$$f_0^{(3)} = U_0^{(3)} D_0^{(3)} \left( W_0^{(3)} \right)^T V_0^{(3)}.$$  

(4.10)
In this case, from (4.1) we have

\[ h = \pm \sqrt{\frac{t^4}{1 - 4t + 6t^2 - 4t^3}}(1 + z)^4. \]

Using (4.10) and (4.7), from Theorem 3.10 we obtain a family of linearly full holomorphic curves in \( G(2, 6) \) with \( K = 1 \), up to \( U(6) \), as follows,

\[
\varphi = \begin{bmatrix}
\sqrt{1 - t}(1 - z) & -2t^2\sqrt{(1 - t)(1 + z)^4} \\
\sqrt{1 + t}(1 + z) & 2t^2\sqrt{(1 + t)(1 + z)^4} \\
0 & -(1 - t)^2\sqrt{1 + t(1 - z)^3} \\
0 & (1 - t)\sqrt{(1 - t^2)(3 - 5t)(1 - z)^2(1 + z)} \\
0 & -(1 - t)\sqrt{(1 + t)(3 - 10t + 11t^2)(1 - z)(1 + z)^2} \\
0 & \sqrt{(1 - t^2)(1 - 3t)(1 - 2t + 5t^2)(1 + z)^3}
\end{bmatrix}, \tag{4.11}
\]

where \(-1 < t < \frac{1}{3}\).

**II(2)** For \( d + 1 = 4, \ n = 5 \). Set \( f_0^{(2)} = A_0^{(3)}V_0^{(3)} \), where \( A_0^{(3)} \in M(3 \times 4; \mathbb{C}) \), then from (4.1), we know that \( (A_0^{(3)})^*A_0^{(3)} \) is the same with (4.9). But in this case, \( A_0^{(3)} \) is a \((3 \times 4)\)-matrix, then we conclude that the matrix \( D_0^{(3)} \) is singular. So we get \( t = \frac{1}{3} \) in this case. Substituting \( t = \frac{1}{3} \) into (4.11) we obtain a holomorphic curve in \( G(2, 5) \) with \( K = 1 \), up to \( U(5) \), as follows,

\[
\varphi = \begin{bmatrix}
(1 - z) & -(1 + z)^4 \\
\sqrt{2}(1 + z) & \sqrt{2}(1 + z)^4 \\
0 & -2\sqrt{2}(1 - z)^3 \\
0 & 4(1 - z)^2(1 + z) \\
0 & -4(1 - z)(1 + z)^2
\end{bmatrix}. \tag{4.12}
\]

Hence we have the following conclusion.

**Proposition 4.3** For \( d + 1 = 4 \), the solutions (4.1) give a family of linearly full non-homogeneous holomorphic curves of constant curvature \( K = 1 \) in \( G(2, 6) \), that is (4.11), and a linearly full non-homogeneous holomorphic curve of constant curvature \( K = 1 \) in \( G(2, 5) \), that is (4.12).

**III** For \( d + 1 = 5 \). Then \( 5 \leq n \leq 7 \).

**III(1)** For \( d + 1 = 5, \ n = 7 \). Set

\[ V_0^{(4)} = \begin{bmatrix} 1 & 2z & \sqrt{6}z^2 & 2z^3 & z^4 \end{bmatrix}^T. \]

Set \( f_0^{(4)} = A_0^{(4)}V_0^{(4)} \), where \( A_0^{(4)} \in M(5; \mathbb{C}) \), then from (4.1), we have

\[
(A_0^{(4)})^*A_0^{(4)} = \begin{pmatrix}
1 & \frac{c_{10}}{2} & \frac{c_{20}}{\sqrt{6}} & \frac{c_{30}}{2} & \frac{c_{40}}{2} \\
\frac{c_{10}}{2} & \frac{c_{11}}{4} & \frac{c_{21}}{2\sqrt{6}} & \frac{c_{31}}{4} & \frac{c_{41}}{4} & \frac{c_{51}}{4} & \frac{c_{61}}{4} & \frac{c_{71}}{4} & \frac{c_{81}}{4} & \frac{c_{91}}{4} & \frac{c_{10}}{2} \\
\frac{c_{20}}{2\sqrt{6}} & \frac{c_{21}}{2\sqrt{6}} & \frac{c_{22}}{6} & \frac{c_{23}}{2\sqrt{6}} & \frac{c_{24}}{6} & \frac{c_{25}}{6} & \frac{c_{26}}{6} & \frac{c_{27}}{6} & \frac{c_{28}}{6} & \frac{c_{29}}{6} & \frac{c_{30}}{2} \\
\frac{c_{30}}{2} & \frac{c_{31}}{4} & \frac{c_{32}}{2\sqrt{6}} & \frac{c_{33}}{4} & \frac{c_{34}}{4} & \frac{c_{35}}{4} & \frac{c_{36}}{4} & \frac{c_{37}}{4} & \frac{c_{38}}{4} & \frac{c_{39}}{4} & \frac{c_{40}}{2} \\
\frac{c_{40}}{2} & \frac{c_{41}}{2} & \frac{c_{42}}{2\sqrt{6}} & \frac{c_{43}}{4} & \frac{c_{44}}{4} & \frac{c_{45}}{4} & \frac{c_{46}}{4} & \frac{c_{47}}{4} & \frac{c_{48}}{4} & \frac{c_{49}}{4} & \frac{c_{50}}{2}
\end{pmatrix} \tag{4.13}
\]
where
\[ c_{10} = \frac{-t + 5t^2 - 10t^3 + 10t^4}{1 - 5t + 10t^2 - 10t^3 + 5t^4}, \quad c_{20} = \frac{t^2 - 5t^3 + 10t^4}{1 - 5t + 10t^2 - 10t^3 + 5t^4}, \]
\[ c_{30} = \frac{-t^2 + 5t^4}{1 - 5t + 10t^2 - 10t^3 + 5t^4}, \quad c_{40} = \frac{t^4}{1 - 5t + 10t^2 - 10t^3 + 5t^4}, \]
\[ c_{11} = \frac{4 - 20t + 42t^2 - 50t^3 + 40t^4}{1 - 5t + 10t^2 - 10t^3 + 5t^4}, \quad c_{21} = \frac{-3t + 15t^2 - 33t^3 + 45t^4}{1 - 5t + 10t^2 - 10t^3 + 5t^4}, \]
\[ c_{31} = \frac{2t^2 - 10t^3 + 24t^4}{1 - 5t + 10t^2 - 10t^3 + 5t^4}, \quad c_{22} = \frac{6 - 30t + 64t^2 - 80t^3 + 76t^4}{1 - 5t + 10t^2 - 10t^3 + 5t^4}. \]

By calculating the eigenvalues and corresponding eigenvectors of matrix \( (A_0^{(4)})^* A_0^{(4)} \), we get
\[
(A_0^{(4)})^* A_0^{(4)} W_0^{(4)} = W_0^{(4)} (D_0^{(4)})^2,
\]
where
\[
W_0^{(4)} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}
\end{pmatrix} \in U(5),
\]
\[
D_0^{(4)} = \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{pmatrix}
\]
with
\[
\lambda_1 = \sqrt{\frac{(1 - t)^4}{\sum_{p=0}^{4} (-1)^p p^{(5)} t^p}}, \quad \lambda_2 = \sqrt{\frac{(1 - t)^4(2 - 3t)}{2 \sum_{p=0}^{4} (-1)^p p^{(5)} t^p}}, \quad \lambda_3 = \sqrt{\frac{(1 - t)^4(3 - 9t + 8t^2)}{3 \sum_{p=0}^{4} (-1)^p p^{(5)} t^p}},
\]
\[
\lambda_4 = \sqrt{\frac{(1 - t)(2 - 9t + 16t^2 - 13t^3)}{2 \sum_{p=0}^{4} (-1)^p p^{(5)} t^p}}, \quad \lambda_5 = \sqrt{\frac{1 - 6t + 16t^2 - 26t^3 + 31t^4}{\sum_{p=0}^{4} (-1)^p p^{(5)} t^p}}.
\]

Set \( U_0^{(4)} = A_0^{(4)} W_0^{(4)} (D_0^{(4)})^{-1} \), then \( U_0^{(4)} \in U(5) \), and \( A_0^{(4)} = U_0^{(4)} D_0^{(4)} (W_0^{(4)})^T \), thus we have
\[
f_0^{(4)} = U_0^{(4)} D_0^{(4)} (W_0^{(4)})^T V_0^{(4)}.
\]

In this case, from (4.1) we have
\[
h = \pm \sqrt{-\frac{-t^5}{1 - 5t + 10t^2 - 10t^3 + 5t^4}(1 + z)^5}.
\]

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Using (4.14) and (4.7), from Theorem 3.10 we obtain a family of linearly full holomorphic curves in $G(2,7)$ with $K = \frac{4}{5}$, up to $U(7)$, as follows,

$$
\varphi = \begin{bmatrix}
\sqrt{1-t}(1-z) & -2\sqrt{-2t^5(1-t)(1+z)^5} \\
\sqrt{1+t}(1+z) & 2\sqrt{-2t^5(1+t)(1+z)^5} \\
0 & (1-t)^2\sqrt{1-t^2(1-z)^4} \\
0 & -(1-t)^2\sqrt{2(1+t)(2-3t)(1-z)^3(1+z)} \\
0 & (1-t)\sqrt{2(1-t^2)(3-9t+8t^2)(1-z)^2(1+z)^2} \\
0 & -(1-t)\sqrt{2(1+t)(2-9t+16t^2-13t^3)(1-z)(1+z)^3} \\
0 & \sqrt{(1-t^2)(1-6t+16t^2-26t^3+31t^4)(1+z)} \\
\end{bmatrix},
\tag{4.15}
$$

where $-1 < t < t_0$ with $t_0$ being the unique zero of $1 - 6t + 16t^2 - 26t^3 + 31t^4$ in the interval $(0, \frac{2}{3})$.

**III(2)** For $d + 1 = 5$, $n = 6$. Set $f_0^{(3)} = A_0^{(4)}V_0^{(4)}$, where $A_0^{(4)} \in M(4 \times 5; \mathbb{C})$, then from (4.1), we know that $(A_0^{(4)})^*A_0^{(4)}$ is the same with (4.13). But in this case, $A_0^{(4)}$ is a $(4 \times 5)$-matrix, then we concludes that the matrix $D_0^{(4)}$ is singular and it’s multiplicity of zero eigenvalues is 1. So we get $t = t_0$ in this case. Substituting $t = t_0$ into (4.15) we obtain a holomorphic curve in $G(2,6)$ with $K = \frac{4}{5}$, up to $U(6)$, as follows,

$$
\varphi = \begin{bmatrix}
\sqrt{1-t_0}(1-z) & -2\sqrt{-2t_0^5(1-t_0)(1+z)^5} \\
\sqrt{1+t_0}(1+z) & 2\sqrt{-2t_0^5(1+t_0)(1+z)^5} \\
0 & (1-t_0)^2\sqrt{1-t_0^2(1-z)^4} \\
0 & -(1-t_0)^2\sqrt{2(1+t_0)(2-3t_0)(1-z)^3(1+z)} \\
0 & (1-t_0)\sqrt{2(1-t_0^2)(3-9t_0+8t_0^2)(1-z)^2(1+z)^2} \\
0 & -(1-t_0)\sqrt{2(1+t_0)(2-9t_0+16t_0^2-13t_0^3)(1-z)(1+z)^3} \\
0 & \sqrt{(1-t_0^2)(1-6t_0+16t_0^2-26t_0^3+31t_0^4)(1+z)} \\
\end{bmatrix},
\tag{4.16}
$$

**III(3)** For $d + 1 = 5$, $n = 5$. Set $f_0^{(2)} = A_0^{(4)}V_0^{(4)}$, where $A_0^{(4)} \in M(3 \times 5; \mathbb{C})$, then from (4.1), we know that $(A_0^{(4)})^*A_0^{(4)}$ is also the same with (4.13). But in this case, $A_0^{(4)}$ is a $(3 \times 5)$-matrix, then we concludes that the matrix $D_0^{(4)}$ is singular and it’s multiplicity of zero eigenvalues is 2. But by analyzing eigenvalues of the matrix $D_0^{(4)}$ we find there doesn’t exist such $t$ that it’s multiplicity of zero eigenvalues is 2. Hence this case doesn’t happen.

In summary, we get the following proposition.

**Proposition 4.4** For $d + 1 = 5$, the solutions (4.1) give a family of linearly full non-homogeneous holomorphic curves of constant curvature $K = \frac{4}{5}$ in $G(2,7)$, that is (4.15), and a linearly full non-homogeneous holomorphic curve of constant curvature $K = \frac{4}{5}$ in $G(2,6)$, that is (4.16).

Generally, we have the following conclusion.

**Proposition 4.5** For any integer $d + 1 \geq 3$, the solutions (4.1) can give a family of non-homogeneous holomorphic curves (may be not linearly full) of constant curvature $K = \frac{4}{d+1}$ in $G(2,d+3)$.

**Proof:** Let $\psi_0^{(1)} : S^2 \to \mathbb{C}P^1$ and $\psi_0^{(n-3)} : S^2 \to \mathbb{C}P^{n-3}$ be holomorphic curves of degree 1 and $d$ ($2n - 5 \geq d \geq n - 3$) respectively. Let $f_0^{(1)}$ and $f_0^{(n-3)}$ be nowhere zero holomorphic...
sections of \( \psi_0^{(1)} \) and \( \psi_0^{(n-3)} \) respectively, let \( h \) be a holomorphic polynomial, satisfying (4.1). Then by Theorem 3.10 we know
\[
\varphi = \text{span} \left\{ f_0^{(1)}, h \frac{\partial f_0^{(1)}}{\partial z} + c_0 f_0^{(n-3)} \right\}
\]
is a holomorphic curve of constant curvature \( K = \frac{4}{d+1} \), \((2n-4 \geq d+1 \geq n-2)\) in \( G(2,n) \).
In the following we prove that the solutions (4.1) can give the holomorphic curves (may be not linearly full) of constant curvature \( K = \frac{4}{d+1} \) in \( G(2,d+3) \).

For \( d+1 = n-2 \). Set
\[
V_0^{(n-3)} = \left[ 1 \; \sqrt{\binom{n-3}{1} z} \; \sqrt{\binom{n-3}{2} z^2} \; \cdots \; \sqrt{\binom{n-3}{n-3} z^{n-3}} \right]^T.
\]
Set \( f_0^{(n-3)} = A_0^{(n-3)} V_0^{(n-3)} \), where \( A_0^{(n-3)} \in M(n-2; \mathbb{C}) \), then from (4.1), we have
\[
A_0^{(n-3)^*} A_0^{(n-3)} = \begin{pmatrix}
1 & c_{10} & c_{20} & \cdots & c_{n-3,0} \\
c_{10} & \sqrt{\binom{n-3}{1}} & \sqrt{\binom{n-3}{2}} & \cdots & \sqrt{\binom{n-3}{n-3}} \\
c_{20} & \sqrt{\binom{n-3}{1}} & \sqrt{\binom{n-3}{2}} & \cdots & \sqrt{\binom{n-3}{n-3}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{n-3,0} & \sqrt{\binom{n-3}{1}} & \sqrt{\binom{n-3}{2}} & \cdots & 1
\end{pmatrix},
\]
where
\[
c_{ij} = \sum_{p=0}^{n-3} \frac{(-1)^p t^p}{p!} \sum_{k=0}^{i+j-2k} \binom{i+j-2k}{j-k} \binom{n-3-i-j+2k}{k} \binom{n-2}{p-i-j+2k} \frac{1}{p!}.
\]
Since \( A_0^{(n-3)^*} A_0^{(n-3)} \) is a Hermitian matrix, then there exist the eigenvalues and corresponding eigenvectors. Set
\[
A_0^{(n-3)^*} A_0^{(n-3)} W_0^{(n-3)} = W_0^{(n-3)} \left( D_0^{(n-3)} \right)^2,
\]
where \( W_0^{(n-3)} \in U(n-2) \) and
\[
D_0^{(n-3)} = \begin{pmatrix}
\lambda_0(t) & \lambda_1(t) & \cdots & \lambda_{n-3}(t) \\
\lambda_1(t) & \lambda_2(t) & \cdots & \lambda_{n-3}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{n-3}(t) & \lambda_{n-3}(t) & \cdots & \lambda_{n-3}(t)
\end{pmatrix},
\]
where \( \lambda_j(t) \) is a non-negative rational function of parameter \( t \). Assume for any \( j \) the value of \( \lambda_j(t) \) is non-negative. Define a set as follows,
\[
\Delta = \{ t \in (-1,1) \mid \text{ } \lambda_0(t) \lambda_1(t) \cdots \lambda_{n-3}(t) > 0 \}.
\]
We claim the set \( \Delta \) is nonempty. In fact, since \( g(t) = \lambda_0(t) \lambda_1(t) \cdots \lambda_{n-3}(t) \) is a continuous function except finite points in \((-1,1)\), and \( g(0) = 1 > 0 \), then there exists a enough small
neighborhood \((-\epsilon, \epsilon)\) such that for any \(t \in (-\epsilon, \epsilon)\), \(g(t) > 0\). Then for \(t \in \Delta\), the matrix \(D_0^{(n-3)}\) is invertible. Set \(U_0^{(n-3)} = A_0^{(n-3)} W_0^{(n-3)} \left( D_0^{(n-3)} \right)^{-1}\), then \(U_0^{(n-3)} \in U(n - 2)\), and \(A_0^{(n-3)} = U_0^{(n-3)} D_0^{(n-3)} \left( W_0^{(n-3)} \right)^T\), thus we have

\[
U_0^{(n-3)} = U_0^{(n-3)} D_0^{(n-3)} \left( W_0^{(n-3)} \right)^T V_0^{(n-3)}. \tag{4.17}
\]

From (4.1) we have

\[
h = \pm \sqrt{\frac{(-t)^{d+1}}{\sum_{p=0}^{d} (-1)^p (q+1)^{d+1} t^p (1 + z)^{d+1}, c_0 = \sqrt{1 - t^2}}.
\]

Then using (4.17) and (4.7), we obtain a family of linearly full holomorphic curves in \(G(2, d + 3)\) with \(K = \frac{d}{d+1}\), up to unitary equivalence.

Assume there exists some \(t = t_0\) such that the matrix \(D_0^{(n-3)}\) is singular and it’s multiplicity of zero eigenvalues is \(q\). Then we can obtain a linearly full holomorphic curve in \(G(2, d + 3 - q)\) with \(K = \frac{d}{d+1}\), up to unitary equivalence, which is not linearly full in \(G(2, d + 3)\).

Finally, combining Theorem 3.10 and the above discussions, we get the following theorem.

**Theorem 4.6** Let \(\varphi : S^2 \to G(2, n) \ (n \geq 5)\) be a linearly full holomorphic curve, satisfying that the harmonic sequence generated by \(\varphi\) degenerates at position 2. If the holomorphic curve \(\varphi\) is unramified with constant curvature \(K = \frac{4}{r}\), then \(r\) is an integer and satisfies \(n - 2 \leq r \leq 2n - 4\).

Moreover, if the multiplicity of zero eigenvalues of the coefficients matrix of \(|f_0^{(n-3)}|^2\) in Theorem 3.10 is \(q\), then the holomorphic curve with \(r = n - 2 + q\) can be constructed explicitly.

**Remark 4.7** The above theorem verifies Conjecture 1 completely and Conjecture 2 partly in [9] for linearly full non-degenerated unramified holomorphic two-spheres in \(G(2, n)\), generating the harmonic sequence that degenerates at position 2. In fact, in order to prove Conjecture 2 completely in this special case, we need to construct solutions of the polynomial equation (3.55) such that \(q\) can take 0, 1, \cdots, \(n-2\) respectively. By Propositions 4.2-4.5, the solutions (4.1) can only give the cases of \(q = 0\) and \(q = 1\).

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