Transport Coefficients of the QGP

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Outline

1 Transport Coefficients
   - Heavy Ion Collisions
   - Strongly Coupled QGP
   - Conductivity from Lattice QCD

2 On the Lattice
   - Details on the Action
   - Conserved Current
   - MEM

3 Results
   - Spectral functions
   - Stability Tests
   - Conductivity
Introduction

- Quark gluon plasma: a phase of matter when $T$ is raised up to 150,000 times the one at the core of the sun.
- When the temperature reaches $T > T_c$ quarks and gluons becomes the degrees of freedom $\rightarrow$ QGP
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Why do we study the QGP?

- Dynamic properties of QGP are relevant to constrain early universe cosmology models
- Understand the output of heavy ion collisions experiments at RHIC and CERN
Heavy Ion Collisions

- Effective theories to study the evolution of the QGP:
  → Input parameters: transport coefficients.
- Experimental evidence for a strongly coupled QGP:
  → perturbation theory fails (see results for $\eta$).
- First principles calculation is needed:
  → Lattice QCD.
Electrical Conductivity

Electromagnetic current (only up/down contribution)

\[ j_{\text{em}}^\mu = \frac{2}{3} \bar{u}\gamma^\mu u - \frac{1}{3} \bar{d}\gamma^\mu d \]
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- Electromagnetic current (only up/down contribution)

\[ j_{em}^\mu = \frac{2}{3} \bar{u} \gamma^\mu u - \frac{1}{3} \bar{d} \gamma^\mu d \]

- Euclidean Correlator

\[ G_{\mu \nu}(\tau) = \int d^3 x \left\langle j_{em}^\mu (\tau, x) j_{em}^\nu (0, 0)^\dagger \right\rangle \]
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- Euclidean Correlator \( \Rightarrow \) spectral function
  \[ G_{\mu\nu}(\tau) = \int d^3 x \langle j_{em}^\mu(\tau, x) j_{em}^\nu(0, 0) \rangle = \int_0^\infty \frac{d\omega}{2\pi} K(\omega, \tau) \rho^{\mu\nu}(\omega, p), \]
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- Kubo’s Formula for Conductivity \( \sigma \)

\[ \sigma = \lim_{\omega \to 0} \frac{1}{6} \frac{\rho^{ii}(\omega)}{\omega} \Rightarrow \text{Important for evolution of EM fields in the QGP} \]
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- Non-zero \( \sigma \) forces magnetic fields to freeze in the plasma.

[K. Tuchin, 2013]
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Clover Action - $N_f = 2 + 1$

\[
\hat{M}[U] = \hat{m}_0 + \gamma_t \hat{W}_t + \frac{1}{\gamma_f} \sum_s \gamma_s \hat{W}_s - \frac{c_t}{2} \sum_s \sigma_{ts} \hat{F}_{ts} - \frac{c_s}{2\gamma_g} \sum_{s<s'} \sigma_{ss'} \hat{F}_{ss'}
\]

[2009, Lin, Edwards, Joo]

- Bare gauge/fermion anisotropy $\gamma_g, \gamma_f$
  - Tuned to give a fixed value for $\xi = a_s/a_t = 3.5$
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  → Tree-level conditions: $c_t = 0.9027$, $c_s = 1.5893$
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- Stout-smeared gauge links:
  $\rightarrow \rho = 0.15$, $n_\rho = 2$
## Configurations

| $N_s$ | $N_τ$ | $T$ [MeV] | $T/T_c$ | $N_{\text{CFG}}$ | $N_{\text{SRC}}$ |
|-------|-------|-----------|---------|-------------------|------------------|
| 32    | 16    | 350       | 1.89    | 1059              | 4                |
| 24    | 20    | 280       | 1.52    | 1001              | 4                |
| 32    | 24    | 235       | 1.26    | 500               | 4                |
| 32    | 28    | 201       | 1.08    | 502               | 4                |
| 32    | 32    | 176       | 0.95    | 501               | 4                |
| 24    | 36    | 156       | 0.84    | 501               | 4                |
| 24    | 40    | 140       | 0.76    | 523               | 4                |
| 32    | 48    | 117       | 0.63    | 601               | 1                |

- Two spatial lattice extension available $N_s = 24, 32$
- $a_s = 0.1227(8)$ fm and $a_t = 0.03506(23)$ fm
- $m_s$ physical and $m_{u,d}$ with $M_\pi/M_\rho = 0.446(3)$
- $T = 120 \sim 350$ MeV, with $T_c = 186(2)$ MeV
Conserved Current on the Lattice

- Conserved vector current \( \kappa_4 = \frac{1}{2}, \kappa_i = \frac{1}{2\gamma_f} \)

\[
V^C_{\mu}(n) = \kappa_{\mu} \left[ \bar{\psi}(n + \hat{\mu})(1 + \gamma_{\mu}) U^{\dagger}_{\mu}(n) \psi(n) 
- \bar{\psi}(n)(1 - \gamma_{\mu}) U_{\mu}(n) \psi(n + \hat{\mu}) \right]
\]

- Ward identity protects the current from renormalization

\[ Z_{VC} \equiv 1 \]

- Improvement pattern given by

\[
V^{CI}_{\mu} - V^C_{\mu} \equiv \frac{1}{4} \sum_{\rho} (\delta_{\rho,0} + \nu \delta_{\rho,i}) a_\rho \partial^- \bar{\psi}(x) \sigma_{\mu\rho} \psi(x) \xrightarrow{p \to 0} 0
\]
Conserved Current - Spatial Component

\[ G_{ii}(\tau) = \sum_{\vec{y}} \langle V_i^C(\vec{x}, x_0) V_i^C(\vec{y}, \tau + x_0)^\dagger \rangle \]
Conserved Current - Volume Effects

\[ G_{ii}(\tau) = \sum_{\vec{y}} \langle V_i^C(\vec{x}, x_0) V_i^C(\vec{y}, \tau + x_0) \rangle \]
An ill posed problem

\[ G_{ii}(\tau) = \int_{0}^{\infty} d\omega \rho(\omega) \cosh(\frac{\beta}{2} - \tau) \frac{\cosh(\omega(\frac{\beta}{2} - \tau))}{\sinh(\beta\omega/2)} \]

\[ \sim O(10) \quad \sim O(1000) \]
An ill posed problem

\[ G_{ii}(\tau_j) = \Delta \omega \sum_{i=0}^{N_\omega} \rho_i K_{ij} \]

\[ \sim O(10) \quad \sim O(1000) \]

[\text{Karsch et al. 2002}] [\text{Gupta, 2004}]
An ill posed problem

\[ G_{ii}(\tau_j) = \Delta \omega \sum_{i=0}^{N_\omega} \rho_i K_{ij} \]

\[ \sim O(10) \quad \sim O(1000) \]

- Standard \( \chi^2 \)-fit fails: non unique solution.
- Need to use Bayesian probability theory.

[Karsch et al. 2002] [Gupta, 2004]
Bayesian Probability Theory

Conditional Probability

\[ P[\rho|DH] = \frac{P[D|\rho H]P[\rho|H]}{P[D|H]} \]
Bayesian Probability Theory

Conditional Probability

\[ P[\rho|DH] = \frac{P[D|\rho H]P[\rho|H]}{P[D|H]} \]

- \( P[D|\rho H] \) - likelihood function \( \exp(-L) \)
- \( P[D|H] \) - normalization

\[ L = \frac{1}{2} \sum_{i,j} (G(\tau_i) - F_i) C_{ij}^{-1} (G(\tau_j) - F_j) \]

\[ F_j = \Delta \omega \sum_{i}^{N_{\omega}} \rho_i K_{ij} \]
Bayesian Probability Theory

Conditional Probability

\[ P[\rho|DH] = \frac{P[D|\rho H]P[\rho|H]}{P[D|H]} \]

- \( P[D|\rho H] \) - likelihood function \( \exp(-L) \)
- \( P[D|H] \) - normalization
- \( P[\rho|H] \) - prior probability: Entropy \( \exp(-\alpha S) \)

\[ S = \int_0^{\infty} \frac{d\omega}{2\pi} \left[ \rho(\omega) - m(\omega) - \rho(\omega) \ln \frac{\rho(\omega)}{m(\omega)} \right] \]

Default Model: \( m(\omega) = m_0\omega(b + \omega) \) finite intercept of \( \rho(\omega)/\omega \)
Bayesian Probability Theory

Conditional Probability

\[ P[\rho|DH] \propto \exp(-L + \alpha S) \]

- \( P[D|\rho H] \) - likelihood function \( \exp(-L) \)
- \( P[D|H] \) - normalization
- \( P[\rho|H] \) - prior probability: Entropy \( \exp(-\alpha S) \)

Solution given by \( \delta P[\rho|DH] = 0 \):
  - Modification of Bryan’s algorithm [Aarts et al. - 2007]
    \( \Rightarrow \) Fixes kernel instabilities at low \( \omega \)
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Spectral Functions from MEM

First Peak: $\rho$-particle

Structures at $3 - 6$ GeV: lattice artefacts

Inset: intercept shows a $T$-dependent conductivity

$$\sigma = \lim_{\omega \to 0} \frac{1}{6} \frac{\rho^{ii}(\omega)}{\omega}$$
Default Model Dependence

- We check the dependence of the result on $b$:

$$m(\omega) = m_0 \omega (b + a_t \omega)$$
Stability Tests - $\Delta \tau$

- Comparing results with same $\tau$-slices but different $T$
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Stability Tests - $\Delta \tau$

- Comparing results with same $\tau$-slices but different $T$
Stability Tests - Anisotropy

- Comparing results when using only a subset of $\tau$-slices

anisotropy pays off at higher temperatures
Conductivity - Final Result

\[ \sigma/T \]

- \( N_f = 0 \) (Aarts et al. - 2007)
- \( N_f = 0 \) (Ding et al. - 2011)
- \( N_f = 2 \) (Brandt et al. - 2012)
- DS (Si-xue Qin - 2013)
- \( 24^3 \) \( N_f = 2+1 \)
- \( 32^3 \) (this work)
Conclusions

- Electrical Conductivity plays an important role in the evolution of EM fields in Heavy Ion Collisions;
- Inside QGP phase results comparable with previous ones;
- First analysis with conserved current of conductivity with different temperatures;
- New observation: increase of $\sigma/T$ already in the confined phase.

Next:
- Strange quark contribution;
- Magnetic Field influence.
Thanks
Strongly Coupled QGP

- Elliptic flow
  \[ v_2 = \left\langle \frac{p_X^2 - p_Y^2}{p_X^2 + p_Y^2} \right\rangle \]

- \( v_2 \) found very large: a direct measure of collectivity.

- Dissipative hydrodynamic: \( v_2(P_T) \leftrightarrow \eta \) shear viscosity

- \( \eta/s \) found smaller than other system:
  → strongly interacting.
  → perturbation theory fails.

- First principle calculation of transport coefficients is needed:
  → Lattice QCD.
Conserved Current - Temporal Component

\[ G_{00}(\tau) = \sum_{\vec{y}} \langle V_0^C(\vec{x}, x_0) V_0^C(\vec{y}, \tau + x_0) \rangle \]

\[ \begin{array}{cccccc}
0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \\
G_{00}/T^3 & 1.89 & 1.52 & 1.26 & 1.08 & 0.95 & 0.84 & 0.76 \\
T/T_c & 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \\
\end{array} \]
Spectral Functions from MEM

- Plot of $\rho(\omega)/\omega$ with different models: Bryan, Historic, Default.
- Distribution of $P(\alpha|D_m)$
- Graph of $\log(C(t))$ with data points and Default Model curve.
Outline

- Retarded Correlator
- Hydrodynamics Evolution
Linear Response

- Classical external source coupled to $O$

$$H(t) = H_0 - H_{ext}(t) \quad \text{with} \quad H_{ext} = \int d\mathbf{x} \, f(\mathbf{x}, t) \, O(\mathbf{x}, t)$$

- The evolution for $O$ is

$$i \frac{\partial}{\partial t} O(t) = -[H(t), O(t)]$$

- To linear order in $f$

$$\delta \langle O(\mathbf{x}, t) \rangle = \int_{-\infty}^{t} dt' d\mathbf{x}' G(\mathbf{x} - \mathbf{x}', t - t') f(\mathbf{x}', t') + O(f^2)$$

where $G(\mathbf{x}, t) = i \langle [O(\mathbf{x}, t), O(0, 0)] \rangle$
Linear Response

- Classical external source coupled to $O$

$$H(t) = H_0 - H_{ext}(t) \quad \text{with} \quad H_{ext} = \int d\bm{x} \, f(\bm{x}, t) \, O(\bm{x}, t)$$

- The evolution for $O$ is

$$i \frac{\partial}{\partial t} O(t) = -[H(t), O(t)]$$

- To linear order in $f$

$$+ \int dt \, d\bm{x} \, e^{i\omega t - \bm{x} \cdot \bm{k}}$$

$$\delta \langle \tilde{O}(\bm{k}, \omega) \rangle = \tilde{G}_R(\omega, \bm{k}) \tilde{f}(\omega, \bm{k})$$

where $\tilde{G}_R(\omega, \bm{k}) = i \int_0^\infty dt \, e^{i\omega t - \bm{x} \cdot \bm{k}} \langle [O(t, \bm{x}), O(0, 0)] \rangle$
Particle Number Diffusion

- Perturbation of Particle number

\[ H_\mu = H_0 - \int d\mathbf{x} \mu(\mathbf{x}, t) n(\mathbf{x}, t) \quad \text{with} \quad \mu(\mathbf{x}, t) = \mu(\mathbf{x}) e^{\epsilon t} \theta(-t) \]
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- Hydrodynamics: conservation law + constitutive equation

\[ \partial_t n + \nabla \cdot j = 0 \quad j = -D\nabla n \quad \text{(Fick’s Law)} \]
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- Hydrodynamics: conservation law + constitutive equation

\[ \partial_t n + \nabla \cdot \mathbf{j} = 0 \quad \mathbf{j} = -D \nabla n \quad \text{(Fick’s Law)} \]

- Diffusion equation

\[ \partial_t n(\mathbf{x}) = D \nabla^2 n(\mathbf{x}) \quad \rightarrow \quad \tilde{n}(\omega, k) = \frac{n(0, k) D k^2}{-i\omega + D k^2} \]

- Static susceptibility

\[ \chi_s^N = n(0, \mathbf{x}) = \int_0^\infty \! \! dt \, e^{-\epsilon t} \int \! \! d\mathbf{x}' \, G^{nn}(\mathbf{x} - \mathbf{x}', t) \mu(\mathbf{x}') \]
Kubo’s Formula

- Substituting $f \leftarrow \mu$ and $O \leftarrow n$ in the linear response for $\delta \langle \tilde{O}(k, \omega) \rangle$

$$\tilde{G}^{nn}_{R}(\omega, k) = \frac{(Dk^2)^2 + i\omega Dk^2}{\omega^2 + (Dk^2)^2}$$

- Spectral function

$$\rho^{nn}(\omega, \mathbf{x}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \, e^{i\omega t} \langle [n(\mathbf{x}, t), n(0, 0)] \rangle = \frac{1}{\pi} \text{Im} \, G^{nn}_{R}(\omega, \mathbf{x})$$

- Diffusion coefficient extracted by

$$D\chi_{s}^{N} = \pi \lim_{\omega \to 0} \lim_{k \to 0} \frac{\rho_{L}(\omega, k)}{\omega}$$
Technical Issues

- SVD Decomposition of the Kernel [Bryan, 1989]

\[ K^T = U W V^T \text{ with } W = \text{diag}(w_1, \ldots, w_{N\omega}) \]

but \( w_{N_s+i} \ll 1 \Rightarrow \bar{\rho}^* = \sum_{i=0}^{N_s} b_i \bar{u}_i \text{ with } N_s < N_T \ll N_\omega \)
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- Bryan approach: integrate over all \( \alpha \)

\[ \rho_{\text{out}} = \int d\alpha \rho_\alpha(\omega) P[\alpha|DHm] \]

\[ \Rightarrow \text{Fix Kernel instabilities at low } \omega \text{ [Aarts, Allton, Hands, Foley, 2007]} \]

\[ K \sim \frac{1}{\omega} \Rightarrow \overline{K}(\omega, \tau) = \frac{\omega}{2T}K(\omega, \tau), \quad \overline{\rho}(\omega) = \frac{2T}{\omega}\rho(\omega) \]
SVD Decomposition of the Kernel [Bryan, 1989]

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Estimates of $\eta$

- **Experiments** [Teaney, 2009]

\[
\left( \frac{\eta}{s} \right)_{\text{pheno}} \lesssim 0.40
\]

- **Perturbative** [Arnold, Moore, Yaffe, 2000]

\[
\left( \frac{\eta}{s} \right)_{\text{leading log}} = \frac{c}{g^4 \log(1/g)} \approx \alpha_s = 0.15 \approx 2.0
\]

- **SYM at infinite coupling** [Policastro, Son, Starinets, 2001]

\[
\left( \frac{\eta}{s} \right)_{\mathcal{N}=4, \lambda=\infty} = \frac{1}{4\pi}
\]