Kashiwara Algebras and Imaginary Verma Modules for $U_q(\hat{g})$

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Abstract. We consider imaginary Verma modules for quantum affine algebra $U_q(\hat{g})$, where $\hat{g}$ is of type 1 i.e. of non-twisted type, and construct Kashiwara type operators and the Kashiwara algebra $K_q$. We show that a certain quotient $N_q^-$ of $U_q(\hat{g})$ is a simple $K_q$-module.

1. Introduction

Let $\hat{g}$ be an affine Lie algebra and $\Delta$ denote the set of roots with respect to the Cartan subalgebra $\hat{h}$. Then we have a natural (standard) partition of $\Delta = \Delta_+ \cup \Delta_-$ into set of positive and negative roots. With respect to this standard partition we have a standard Borel subalgebra from which we may induce the standard Verma module. A partition $\Delta = S \cup -S$ of the root system $\Delta$ is said to be a closed partition if whenever $\alpha, \beta \in S$ and $\alpha + \beta \in \Delta$ we have $\alpha + \beta \in S$. It is well known that for any finite dimensional complex simple Lie algebra, all closed partitions of the root system are Weyl group conjugate to the standard partition. However, this is not the case for affine Lie algebras. The classification of closed subsets of the root system for affine Lie algebras was obtained by Jakobsen and Kac [JK85, JK89], and independently by Futorny [Fut90, Fut92]. In fact for affine Lie algebras there exists a finite number ($\geq 2$) of inequivalent Weyl orbits of closed partitions. Corresponding to each such non-standard partitions we have non-standard Borel subalgebras from which we can induce other non-standard Verma-type modules and these typically contain both finite and infinite dimensional weight spaces. The imaginary Verma module [Fut94] is a non-standard Verma-type module associated with the simplest non-standard partition of the root system $\Delta$ which is the focus of our study in this paper.

For generic $q$, the quantum affine algebra $U_q(\hat{g})$ is the $q$-deformations of the universal enveloping algebras of $\hat{g}$ ([Dri85], [Jim85]). It is known [Lus88] that integrable highest weight modules of $\hat{g}$ can be deformed to those over $U_q(\hat{g})$ in such a way that the dimensions of the weight spaces are invariant under the deformation.

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Following the framework of [Lus88] and [Kan95], quantum imaginary Verma modules for $U_q(\hat{\mathfrak{g}})$ were constructed in ([CFKM97], [FGM98]) and it was shown that these modules are deformations of those over the universal enveloping algebra $U(\hat{\mathfrak{g}})$ in such a way that the weight multiplicities, both finite and infinite-dimensional, are preserved.

Lusztig [Lus90] from a geometric viewpoint and Kashiwara [Kas91] from an algebraic viewpoint introduced the notion of canonical bases (equivalently, global crystal bases) for standard Verma modules $V_q(\lambda)$ and integrable highest weight modules $L_q(\lambda)$. The crystal base ([Kas90, Kas91]) can be thought of as the $q = 0$ limit of the global crystal base or canonical base. An important ingredient in the construction of crystal base by Kashiwara in [Kas91], is a subalgebra $B_q$ of the quantum group which acts on the negative part of the quantum group by left multiplication. This subalgebra $B_q$, which we call the Kashiwara algebra, played an important role in the definition of the Kashiwara operators which defines the crystal base.

In this paper we construct an analog of Kashiwara algebra $K_q$ for the imaginary Verma module $\tilde{M}_q(\lambda)$ for the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ by introducing certain Kashiwara-type operators. Then we prove that certain quotient $N^-_q$ of $U_q(\hat{\mathfrak{g}})$ is a simple $K_q$-module. This generalizes the corresponding result in [CFM10] for the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(2))$. However, it is worth pointing out that some of the arguments involving explicit calculations in [CFM10] do not extend to this general case.

The paper is organized as follows. In Sections 2 and 3 we recall necessary definitions and some results that we need. In Section 4 we recall some facts about the imaginary Verma modules for $U_q(\hat{\mathfrak{g}})$. In particular, for any dominant weight $\lambda$ with $\lambda(c) = 0$ we give a necessary and sufficient condition for the reduced imaginary Verma module $\tilde{M}_q(\lambda)$ to be simple. In Section 5 we define certain operators we call $\Omega$-operators acting on certain subalgebra $N^-_q$ of $\tilde{M}_q(\lambda)$ and prove generalized commutation relations among them. We define the Kashiwara algebra $K_q$ in terms of certain Drinfeld generators and the $\Omega$-operators in Section 6 and show that $N^-_q$ is a left $K_q$-module and define a symmetric invariant bilinear form on $N^-_q$. Finally, in Section 7 we prove that $N^-_q$ is simple as a $K_q$-module and that the form defined in Section 6 is nondegenerate.

2. The affine Lie algebra $\hat{\mathfrak{g}}$.

We begin by recalling some basic facts and constructions for the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ and its imaginary Verma modules. See [Kac90] for Kac-Moody algebra terminology and standard notations.

2.1. Let $I = \{0, \ldots, N\}$ and $A = (a_{ij})_{0 \leq i, j \leq N}$ be a generalized affine Cartan matrix of type 1 for an untwisted affine Kac-Moody algebra $\hat{\mathfrak{g}}$. Let $D = (d_0, \ldots, d_N)$ be a diagonal matrix with relatively prime integer entries such that the matrix $DA$ is symmetric. Then $\hat{\mathfrak{g}}$ has the loop space realization

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\mathfrak{g}$ is the finite dimensional simple Lie algebra over $\mathbb{C}$ with Cartan matrix $(a_{ij})_{1 \leq i, j \leq N}$, $c$ is central in $\hat{\mathfrak{g}}$; $d$ is the degree derivation, so that $[d, x \otimes t^n] = nx \otimes t^n$
for any \( x \in \mathfrak{g} \) and \( n \in \mathbb{Z} \), and \( [x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + \delta_{n+m,0} n(x|y)c \) for all \( x, y \in \mathfrak{g}, n, m \in \mathbb{Z} \).

An alternative Chevalley-Serre presentation of \( \hat{\mathfrak{g}} \) is given by defining it as the Lie algebra with generators \( e_i, f_i, h_i \) (\( i \in I \)) and \( d \) subject to the relations

\[
\begin{align*}
(h_i, h_j) &= 0, \quad [d, h_i] = 0, \\
([h_i, e_j] = a_{ij} e_j, \quad [d, e_j] &= \delta_{0,j} e_j, \\
([h_i, f_j] &= -a_{ij} f_j, \quad [d, f_j] &= -\delta_{0,j} f_j, \\
([e_i, f_j] &= \delta_{ij} h_i, \\
(\text{ade}_i)^{1-a_{ij}}(e_j) &= 0, \quad (\text{ad} f_j)^{1-a_{ij}}(f_j) = 0, \quad i \neq j.
\end{align*}
\]

We set \( \mathfrak{h} \) to be the span of \( \{h_0, \ldots, h_N, d\} \).

Let \( \Delta_0 \) be the set of roots of \( \mathfrak{g} \) with chosen set of positive/negative roots \( \Delta_{0,\pm} \). Let \( Q_0 \) be the free abelian group with basis \( \alpha_i, 1 \leq i \leq N \) which is the root lattice of \( \mathfrak{g} \). Let \( \tilde{Q}_0 = \sum \mathbb{Z} \tilde{\alpha}_i \) be the coroot lattice of \( \mathfrak{g} \). The co-weight lattice is defined to be \( \tilde{P}_0 = \text{Hom}(Q_0, \mathbb{Z}) \) with basis \( \omega_i \) defined by \( \langle \omega_i, \alpha_j \rangle = \delta_{i,j} \). The simple reflections \( s_i : \tilde{P}_0 \to \tilde{P}_0 \) are defined by \( s_i(x) = x - \langle \alpha_i, x \rangle \tilde{\alpha}_i \). The \( s_i \) also act on \( Q_0 \) by \( s_i(y) = y - \langle y, \tilde{\alpha}_i \rangle \alpha_i \). The Weyl group of \( \mathfrak{g} \) is defined as the subgroup \( W_0 \) of \( \text{Aut} \tilde{P}_0 \) generated by \( s_1, \ldots, s_N \). The affine Weyl group is defined as \( W = W_0 \ltimes \tilde{Q}_0 \).

Let \( \theta \) be the longest positive root and set \( s_0 = (s_0, -\theta) \). Then \( W \) is generated by \( s_0, \ldots, s_N \). Let \( \tilde{W} = W_0 \ltimes \tilde{P}_0 = W \ltimes T \) be the generalized affine Weyl group where \( T \) is the group of Dynkin diagram automorphisms.

Let \( \Delta \) be the root system of \( \hat{\mathfrak{g}} \) with positive/negative set of roots \( \Delta_{\pm} \) and simple roots \( \Pi = \{\alpha_0, \ldots, \alpha_N\} \). Define \( \delta = \alpha_0 + \theta \). Extend the root lattice \( \tilde{Q}_0 \) of \( \mathfrak{g} \) to the affine root lattice \( Q := Q_0 \oplus \mathbb{Z} \delta \), and extend the form \( (\cdot, \cdot) \) to \( Q \) by setting \( (q|\delta) = 0 \) for all \( q \in Q_0 \) and \( (\delta|\delta) = 0 \). The generalized affine Weyl group \( \tilde{W} \) acts on \( Q \) as an affine transformation group. In particular if \( z \in \tilde{P}_0 \) and \( 1 \leq i \leq N \), then \( z(\alpha_i) = \alpha_i - (z, \alpha_i) \delta \). Let \( \tilde{Q}_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \oplus \mathbb{Z}_{\geq 0} \delta \).

The root system \( \Delta \) of \( \hat{\mathfrak{g}} \) is given by

\[
\Delta = \{ \alpha + n \delta \mid \alpha \in \Delta_0, n \in \mathbb{Z} \} \cup \{ k \delta \mid k \in \mathbb{Z}, k \neq 0 \}.
\]

The roots of the form \( \alpha + n \delta \), \( \alpha \in \Delta, n \in \mathbb{Z} \) are called real roots, and those of the form \( k \delta, \delta \), \( \delta \in \mathbb{Z}, k \neq 0 \) are called imaginary roots. We let \( \Delta^r \) and \( \Delta^m \) denote the sets of real and imaginary roots, respectively. The set of positive real roots of \( \hat{\mathfrak{g}} \) is \( \Delta^r_+ = \Delta_{0,+} \cup \{ \alpha + n \delta \mid \alpha \in \Delta_0, n > 0 \} \) and the set of positive imaginary roots is \( \Delta^m_+ = \{ k \delta \mid k > 0 \} \). The set of positive roots of \( \hat{\mathfrak{g}} \) is \( \Delta_+ = \Delta^r_+ \cup \Delta^m_+ \). Similarly, on the negative side, we have \( \Delta_- = \Delta^r_- \cup \Delta^m_- \), where \( \Delta^r_- = \Delta_{0,-} \cup \{ \alpha + n \delta \mid \alpha \in \Delta_0, n < 0 \} \) and \( \Delta^m_- = \{ k \delta \mid k < 0 \} \). The weight lattice \( P \) of \( \hat{\mathfrak{g}} \) is \( P = \{ \lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}, i \in I, \lambda(d) \in \mathbb{Z} \} \). Let \( B \) denote the associated braid group with generators \( T_0, T_1, \ldots, T_N \).

2.2. Consider the partition \( \Delta = S \cup -S \) of the root system of \( \hat{\mathfrak{g}} \) where \( S = \{ \alpha + n \delta \mid \alpha \in \Delta_{0,+}, n \in \mathbb{Z} \} \cup \{ k \delta \mid k > 0 \} \). This is a non-standard partition of the root system \( \Delta \) in the sense that \( S \) is not Weyl equivalent to the set \( \Delta_+ \) of positive roots.

3. The quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \)
3.1. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is the $\mathbb{C}(q^{1/2})$-algebra with 1 generated by

$$E_i, F_i, K_\alpha, \gamma^{\pm 1/2}, D^{\pm 1} \quad 0 \leq i \leq N, \alpha \in Q,$$

defining relations:

$$DD^{-1} = D^{-1}D = K_i K_i^{-1} = K_i^{-1} K_i = \gamma^{1/2} \gamma^{-1/2} = 1,$$

$$[\gamma^{\pm 1/2}, U_q(\hat{\mathfrak{g}})] = [D, K_i^{\pm 1}] = [K_i, K_j] = 0,$$

$$(\gamma^{\pm 1/2})^2 = K_\delta^{\pm 1},$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$K_\alpha E_i K_\alpha^{-1} = q^{(\alpha(\alpha))} E_i, \quad K_\alpha F_i K_\alpha^{-1} = q^{-(\alpha(\alpha))} F_i,$$

$$DE_i D^{-1} = q^{\delta_{i0}} E_i, \quad DF_i D^{-1} = q^{-\delta_{i0}} F_i,$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s E_i^{(1-a_{i,j}-s)} E_j E_i^{(s)} = 0 = \sum_{s=0}^{1-a_{ij}} (-1)^s F_i^{(1-a_{i,j}-s)} F_j F_i^{(s)}, \quad i \neq j.$$

where

$$q_i := q^{d_i}, \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! := \prod_{k=1}^{n} [k]_i,$$

and $K_i = K_\alpha$, $E_i^{(s)} = E_i/[s]_i!$ and $F_i^{(s)} = F_i/[s]_i!$ (see [Bec94a] and [Lus88]).

The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is a Hopf algebra with a comultiplication given by

(3.1) $\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1},$

(3.2) $\Delta(D^{\pm 1}) = D^{\pm 1} \otimes D^{\pm 1}, \quad \Delta(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2}$

(3.3) $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$

(3.4) $\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$

and an antipode given by

$$s(E_i) = -E_i K_i^{-1}, \quad s(F_i) = -K_i F_i,$$

$$s(K_i) = K_i^{-1}, \quad s(D) = D^{-1}, \quad s(\gamma^{1/2}) = \gamma^{-1/2}.$$

Let $\Phi : U_q(\hat{\mathfrak{g}}) \to U_q(\hat{\mathfrak{g}})$ be the $\mathbb{C}$-algebra automorphism defined by

(3.5) $\Phi(E_i) = F_i, \quad \Phi(F_i) = E_i, \quad \Phi(K_\alpha) = K_\alpha,$

$$\Phi(D) = D, \quad \Phi(\gamma^{1/2}) = \gamma^{1/2}, \quad \Phi(q^{1/2}) = q^{1/2},$$

and let $\tilde{\Omega} : U_q(\hat{\mathfrak{g}}) \to U_q(\hat{\mathfrak{g}})$ be the $\mathbb{C}$-algebra anti-automorphism defined by

(3.6) $\tilde{\Omega}(E_i) = F_i, \quad \tilde{\Omega}(F_i) = E_i, \quad \tilde{\Omega}(K_\alpha) = K_\alpha^{-1},$

$$\tilde{\Omega}(D) = D^{-1}, \quad \tilde{\Omega}(\gamma^{1/2}) = \gamma^{-1/2}, \quad \tilde{\Omega}(q^{1/2}) = q^{-1/2},$$

(see [Bec94a, Section 1]).
There is an alternative realization for $U_q(\hat{g})$, due to Drinfeld [Dri85], which we shall also need. We will use the formulation due to J. Beck [Bec94a]. Let $U_q(\hat{g})$ be the associative algebra with 1 over $\mathbb{C}(q^{1/2})$-generated by

$$x_{ir}^{\pm 1}, \ h_{is}, \ K_i^{\pm 1}, \ \gamma^{\pm 1/2}, \ D^{\pm 1} \ 1 \leq i \leq N, \ r, \ s \in \mathbb{Z}, \ s \neq 0,$$

with defining relations:

\begin{equation}
DD^{-1} = D^{-1}D = K_i K_i^{-1} = K_i^{-1} K_i = \gamma^{1/2} \gamma^{-1/2} = 1,
\end{equation}

\begin{equation}
[\gamma^{\pm 1/2}, U_q(\hat{g})] = [D, K_i^{\pm 1}] = [K_i, K_i] = [K_i, h_{jk}] = 0,
\end{equation}

\begin{equation}
D h_{ir} D^{-1} = q^r h_{ir}, \ D x_{ir}^{\pm} D^{-1} = q^r x_{ir}^{\pm},
\end{equation}

\begin{equation}
K_i x_{jr}^{\pm} K_i^{-1} = q^{x_{ij}^{\alpha(\alpha)}} x_{jr}^{\pm},
\end{equation}

\begin{equation}
[h_{ik}, h_{jl}] = \delta_k, -\frac{1}{k} \left[ k \alpha_{ij} \right], \frac{\gamma^k - \gamma^{-k}}{q^j - q^{-1}},
\end{equation}

\begin{equation}
\frac{1}{k} \left[ k \alpha_{ij} \right] \gamma^{\pm |k|/2} x_{j,k+1},
\end{equation}

\begin{equation}
x_{i,k+1}^{\pm} \gamma^{-1} - q^{x_{ik}^{\alpha(\alpha)}} x_{i,k+1}^{\pm} = q^{x_{ik}^{\alpha(\alpha)}} x_{i,k+1}^{\pm} - x_{j,l+1}^{\pm} x_{j,k}^{\pm},
\end{equation}

\begin{equation}
\left[ x_{ik}^{\pm}, x_{jl}^{\pm} \right] = \delta_{ij} \left( \frac{1}{q_i - q_j} \left( \gamma^i \gamma^j \psi_{i,k+l} - \gamma^i \gamma^j \phi_{i,k+l} \right) \right),
\end{equation}

where

\begin{equation}
\sum_{k=0}^{\infty} \psi_{ik} z^k = K_i \exp \left( (q_i - q_i^{-1}) \sum_{l>0} h_{il} z^l \right), \text{ and}
\end{equation}

\begin{equation}
\sum_{k=0}^{\infty} \phi_{i,k} z^{-k} = K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{l>0} h_{il} z^{-l} \right).\end{equation}

For $i \neq j, \ n := 1 - \alpha_{ij}$

\begin{equation}
\text{Sym}_{k_1, k_2, \ldots, k_n} \sum_{r=0}^{n} (-1)^r \left[ \begin{array}{c}
\frac{n}{p} \\
\end{array} \right] x_{i,k_1}^{\pm} \cdots x_{i,k_r}^{\pm} x_{i,j}^{\pm} x_{i,k_{r+1}}^{\pm} \cdots x_{i,k_n}^{\pm} = 0.
\end{equation}

Note that Beck’s paper [Bec94a] on page 565 has a typo in it where he has $\phi_{i,k} z^k$ instead of $\phi_{i,k} z^{-k}$.

In the above last relation Sym means symmetrization with respect to the indices $k_1, \ldots, k_n$. Also in Drinfeld’s notation one has $e^{h_{il} z^2} = \gamma$ and $e^{h_{il} z^2} = q$.

The algebras given above and in §3.1 are isomorphic [Dri85]. If one uses the formal sums

\begin{equation}
\phi_i(u) = \sum_{p \in \mathbb{Z}} \phi_{ip} u^{-p}, \ \psi_i(u) = \sum_{p \in \mathbb{Z}} \psi_{ip} u^{-p}, \ x_{i}^{\pm}(u) = \sum_{p \in \mathbb{Z}} x_{ip}^{\pm} u^{-p}
\end{equation}
Drinfeld’s relations (3.11)–(3.14) can be written as

\begin{align}
(3.18) \quad [\phi_i(u), \phi_j(v)] &= 0 = [\psi_i(u), \psi_j(v)] \\
(3.19) \quad \phi_i(u)\psi_j(v)\phi_i(u)^{-1}\psi_j(v)^{-1} &= g_{ij}(uv^{-1}\gamma^{-1})/g_{ij}(uv^{-1}\gamma) \\
(3.20) \quad \phi_i(u)x_j^\pm(v)\phi_i(u)^{-1} &= g_{ij}(uv^{-1}\gamma^{1/2})^{\pm1}x_j^\pm(v) \\
(3.21) \quad \psi_i(u)x_j^\pm(v)\psi_i(u)^{-1} &= g_{ji}(vu^{-1}\gamma^{1/2})^{\pm1}x_j^\pm(v) \\
(3.22) \quad (u - q^{\pm(\alpha,\alpha)}v)x_j^\pm(u)x_j^\pm(v) &= (q^{\pm(\alpha,\alpha)}u - v)x_j^\pm(u)x_j^\pm(v) \\
(3.23) \quad [x_j^+(u), x_j^-(v)] &= \delta_{ij}(q_i - q_i^{-1})^{-1}(\delta(u/v\gamma)\psi_i(v\gamma^{1/2}) - \delta(u\gamma/v)\phi_i(u\gamma^{1/2}))
\end{align}

where \(g_{ij}(t) = g_{ij,q}(t)\) is the Taylor series at \(t = 0\) of the function \((q^{(\alpha,\alpha)}t - 1)/(t - q^{(\alpha,\alpha)})\) and \(\delta(z) = \sum_{k \in \mathbb{Z}} z^k\) is the formal Dirac delta function.

**3.3.** Let \(U_q^+ = U_q^+(\mathfrak{g})\) (resp. \(U_q^- = U_q^-(\mathfrak{g})\)) be the subalgebra of \(U_q(\mathfrak{g})\) generated by \(E_i\) (resp. \(F_i\)), \(i \in I\), and let \(U_q^0 = U_q^0(\mathfrak{g})\) denote the subalgebra generated by \(K_i^\pm\) (\(i \in I\)) and \(D^\pm\).

Beck in [Bec94a] and [Bec94b] has given a total ordering of the root system \(\Delta\) and a PBW like basis for \(U_q(\mathfrak{g})\). Below we follow the construction developed by Damiani [Dam98], Gavarini [Gav99] and [BK96] and let \(E_\beta\) denote the root vectors for each \(\beta \in \Delta_+\) counting with multiplicity for the imaginary roots. One defines \(F_\beta = E_{-\beta} := 0(E_\beta)\) for \(\beta \in \Delta_+\) (refer to (3.6)).

For any affine Lie algebra \(\mathfrak{g}\), there exists a map \(\pi : \mathbb{Z} \to I\) such that, if we define

\[\beta_k = \begin{cases} 
\pi(0)\pi(-1)\cdots\pi(k-1)(\alpha_k) & \text{for } k < 0, \\
\alpha_{\pi(0)} & \text{for } k = 0, \\
\alpha_{\pi(1)} & \text{for } k = 1, \\
\pi(1)\pi(2)\cdots\pi(k-1)(\alpha_k) & \text{for } k > 1,
\end{cases}\]

then the map \(\pi^{' : \mathbb{Z} \to \Delta_+^{\text{im}}}\) given by \(\pi'(k) = \beta_k\) is a bijection. Note that the map \(\pi^{'}, \text{ and hence the total ordering, is not unique. We fix } \pi \text{ so that } \{\beta_k | k \leq 0\} = \{\alpha + n\delta | \alpha \in \Delta_{0,+}, n \geq 0\} \text{ and } \{\beta_k | k \geq 1\} = \{-\alpha + n\delta | \alpha \in \Delta_{0,+}, n > 0\}.\) One also defines the set of imaginary roots with multiplicity as

\[\Delta_+(\text{im}) := \Delta_+^{\text{im}} \times I_0,\]

where \(I_0 = \{1, \ldots, N\}\).

It will be convenient for us to invert Beck’s original ordering of the positive roots (see [BK96, §1.4.1]). Let

\begin{equation}
(3.24) \beta_0 > \beta_{-1} > \beta_{-2} > \cdots > \delta > 2\delta > \cdots > \beta_2 > \beta_1,
\end{equation}

(see [Gav99, §2.1] for this ordering). We define \(-\alpha < -\beta\) if \(\beta > \alpha\) for all positive roots \(\alpha, \beta\), so we obtain a corresponding ordering on \(\Delta_-\).

The following elementary observation on the ordering will play a crucial role later. Write \(A < B\) for two sets \(A\) and \(B\) if \(x < y\) for all \(x \in A\) and \(y \in B\). Then Beck’s total ordering of the positive roots can be divided into three sets:

\[\{\alpha + n\delta | \alpha \in \Delta_{0,+}, n \geq 0\} > \{k\delta | k > 0\} > \{-\alpha + k\delta | \alpha \in \Delta_{0,+}, k > 0\}.\]

Similarly, for the negative roots, we have,

\[\{-\alpha - n\delta | \alpha \in \Delta_{0,+}, n \geq 0\} < \{-k\delta | k > 0\} < \{\alpha - k\delta | \alpha \in \Delta_{0,+}, k > 0\}.\]
The action of the braid group generators $T_i$ on the generators of the quantum group $U_q(\mathfrak{g})$ is given by the following.

$$
T_i(E_i) = -F_iK_i, \quad T_i(F_i) = -K_i^{-1}E_i,
$$

$$
T_i(E_j) = \sum_{r=0}^n (-1)^r \eta_i^{-r} a_{ij} E_i(\eta_i^{-r}) E_j E_i(r), \quad \text{if } i \neq j,
$$

$$
T_i(F_j) = \sum_{r=0}^n (-1)^r \eta_i^{-r} a_{ij} F_i(\eta_i^{-r}) F_j F_i(r), \quad \text{if } i \neq j,
$$

$$
T_i(K_j) = K_j K_i^{-a_{ij}}, \quad T_i(K_j^{-1}) = K_j^{-1} K_i^{a_{ij}},
$$

$$
T_i(D) = D K_i^{-\delta,0}, \quad T_i(D^{-1}) = D^{-1} K_i^{\delta,0}.
$$

For each $\beta_k \in \Delta^{\text{re}}_\mathfrak{g}$, define the root vector $E_{\beta_k}$ in $U_q(\mathfrak{g})$ by

$$
E_{\beta_k} = \begin{cases} 
T_{\pi(0)}^{-1} T_{\pi(1)}^{-1} \cdots T_{\pi(k)}^{-1}(E_{\pi(k)}) & \text{for all } k < 0, \\
E_{\pi(0)} & k = 0, \\
E_{\pi(1)} & k = 1, \\
T_{\pi(1)} T_{\pi(2)} \cdots T_{\pi(k-1)}(E_{\pi(k)}) & \text{for all } k > 1.
\end{cases}
$$

Orient the Dynkin diagram of $\mathfrak{g}$ by defining a map $o : V \to \{\pm 1\}$ so that for adjacent vertices $i$ and $j$ one has $o(i) = -o(j)$. Beck defines $\hat{T}_{\omega}$ as $o(i) T_{\omega_i}$, and obtains ([Bec94a, Section 4]) for $i \in I$ and $k \in \mathbb{Z}$,

$$
x_{ik}^- := \hat{T}_{\omega_i}^k(F_i), \quad x_{ik}^+ := \hat{T}_{\omega_i}^{-k}(E_i).
$$

The following result is due to Iwahori, Matsumoto and Tits (see [Bec94a], Section 2).

**Proposition 3.3.1.** Suppose $w \in \hat{W}$ and $w = \tau s_{i_1} \cdots s_{i_n}$ is a reduced decomposition in terms of simple reflections. Then $T_w = \tau T_{i_1} \cdots T_{i_n}$ does not depend on the reduced decomposition of $w$ chosen, but rather only on $w$.

Fix $i \in I_0$ and $k \geq 0$. The proposition above in the particular case of the reduced decomposition of $\omega_i = \tau s_{i_1} \cdots s_{i_n} \in \hat{P}_0 \subset \hat{W}$ where $\tau$ is a diagram automorphism and the $s_i$ are simple reflections, gives

$$
x_{ik}^+ = \hat{T}_{\omega_i}^{-k}(E_i) = o(i)^k (\tau T_{i_{1}} \cdots T_{i_{n}})^{-k}(E_i) = o(i)^k T_{j_{1}} \cdots T_{j_{m}} \tau^{-k}(E_i),
$$

for some $j_i \in I$.

Fixing still $i \in I_0$ and $k \geq 0$, choose now $w_{\alpha, +k\delta} \in \hat{W}$, and $j \in I$, such that $w_{\alpha, +k\delta}(\alpha_j) = \alpha_i + k\delta$. Writing $w_{\alpha, +k\delta} = s_{i_1} \cdots s_{i_m}$ as a reduced decomposition of simple reflections, Beck defines

$$
E_{\alpha, +k\delta} := T_{w_{\alpha, +k\delta}}(E_j) = T_{j_{1}} \cdots T_{j_{m}}(E_j),
$$

which according to Lusztig is independent of the choice of $w_{\alpha, +k\delta}$, its reduced decomposition and $j \in I$. In particular we can choose $j = \tau^{-k}(i)$ and $w = s_{j_1} \cdots s_{j_m}$, so that $s_{j_1} \cdots s_{j_m}(\alpha_j) = s_{j_1} \cdots s_{j_m}(\alpha_i - \tau^{-k}(i)) = \alpha_i + k\delta$. Then

$$
E_{\alpha, +k\delta} = T_{j_{1}} \cdots T_{j_{m}}(E_{\tau^{-k}(i)}) = o(i)^k x_{ik}^+.
$$
Now one defines

\[ F_{\alpha_i + k\delta} = \Omega(E_{\alpha_i + k\delta}) = o(i)^k \Omega(x_{ik}^+) = o(i)^k \Omega(\bar{T}^{-k}(E_i)) \]

\[ = o(i)^k \bar{T}^{-k}_\omega(\Omega(E_i)) = o(i)^k \bar{T}^{-k}_\omega(E_i) = o(i)^k x_{i-k}, \]

as \( T_\omega \Omega = \Omega T_\omega \) and \( T_\tau \Omega = \Omega T_\tau \).

If \( k < 0 \) and \( i \in I_0 \), then \(-\alpha_i - k\delta \in \Delta^*_r \), so that \(-\alpha_i - k\delta = \beta_l = s_{\pi(1) \cdots s_{\pi(l-1)}(\alpha_{\pi(i)})} \) for \( l > 1 \) and \(-\alpha_i - k\delta = \beta_l = \alpha_{\pi(1)} \) if \( l = 1 \). Then for \( l > 1 \),

\[ E_{-\alpha_i - k\delta} = E_{-\beta_l} = T_{\omega_i}^{-k} T_i^{-1}(E_i) = -T_{\omega_i}^{-k}(K_i^{-1}E_i) \]

\[ = -o(i)^k T_{\omega_i}^{-k}(K_i^{-1})x_{i-k}^- = -o(i)^k K_i^{-1}\gamma^{-k}x_{i-k}^- \]

as \( \omega_i(-\alpha_i) = -\alpha_i + \delta \) (see \( \S 2.1 \)) so that \( \omega_i^{-k}s_i(\alpha_i) = \omega_i^{-k}(-\alpha_i) = -\alpha_i - k\delta \) and \( T_{\omega_i}(K_i^{-1}) = K_{-\alpha_i - \delta} \). Now

\[ F_{-\alpha_i - k\delta} = \Omega(E_{-\alpha_i - k\delta}) = -o(i)^k \Omega(K_i^{-1}\gamma^{-k}x_{i-k}^-) \]

\[ = -o(i)^k K_i \gamma^k x_{i-k}^+. \]

Then as shown in \([\text{Bec94a}, \text{Theorem 4.7}]\), for \( k > 0 \)

\[ \psi_{ik} = (q_i - q_i^{-1}) \gamma^{k/2} E_i, \bar{T}_{\omega_i}(F_i) = (q_i - q_i^{-1}) \gamma^{k/2} [E_i, x_{ik}], \]

\[ \phi_{i-k} = (q_i - q_i^{-1}) \gamma^{-k/2} [F_i, \bar{T}_{\omega_i}E_i] = (q_i - q_i^{-1}) \gamma^{-k/2} [F_i, x_{i-k}^+], \]

\( \psi_{i,0} := K_i, \phi_{i,0} := K_i^{-1} \), and for any \( \tau \in T \),

\[ T_\tau(E_i) := E_{\tau(i)}, \quad T_\tau(F_i) := F_{\tau(i)}, \quad T_\tau(K_i) := K_{\tau(i)}. \]

One writes \( \tau \) for \( T_\tau \). Note also that \( \tau s_i \tau^{-1} = s_{\tau(i)} \) for all \( 0 \leq i \leq n \).

Each real root space is 1-dimensional, but each imaginary root space is \( N \)-dimensional. Hence, for each positive imaginary root \( k\delta \) \( (k > 0) \) one defines the \( N \) imaginary root vectors, \( E_{k\delta}^{(i)} \) \( (i \in I_0) \) by

\[ \exp \left( (q_i - q_i^{-1}) \sum_{k=1}^{\infty} E_{k\delta}^{(i)} z^k \right) = 1 + (q_i - q_i^{-1}) \sum_{k=1}^{\infty} K_i^{-1} [E_i, x_{ik}^+] z^k \]

\[ = 1 + \sum_{k=1}^{\infty} K_i^{-1} \psi_{ik} \left( \gamma^{-1/2} z \right)^k \]

\[ = \exp \left( (q_i - q_i^{-1}) \sum_{l>0} h_{il} \gamma^{-l/2} z^l \right). \]

So \( E_{k\delta}^{(i)} = h_{ik} \gamma^{-k/2} \) for all \( k > 0 \). For \( k < 0 \) we also define \( E_{k\delta}^{(i)} := \bar{\Omega}(E_{-k\delta}^{(i)}) = h_{ik} \gamma^{k/2} \). Our definition of \( E_{k\delta}^{(i)} \) is the same as \([\text{Dam98}], \text{Definition 7}\). Recall that the \( R^- \) “matrices” are defined having values in \( U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \) (see \([\text{Lus93}]\) for the
definition of $U_q(\widehat{\mathfrak{g}})\otimes U_q(\widehat{\mathfrak{g}})$ and [Bec94a, Section 5]) for $1 \leq i \leq N$ by

$$R_i = \sum_{n \geq 0} (-1)^n q_i^{-n(n-1)/2} (q_i - q_i^{-1})^n [n]! T_i(E_i^{(n)}) \otimes T_i(E_i^{(n)}),$$

$$= \sum_{n \geq 0} (q_i^{-1} - q_i)^n q_i^{-n(n-1)/2} E_i^n K^{-n} \otimes F_i^n K_i^n,$$

(3.33)  $$\tilde{R}_i = T_i^{-1} \otimes T_i^{-1} \circ R_i^{-1} = \sum_{n \geq 0} q_i^{n(n-1)/2} (q_i - q_i^{-1})^n [n]! F_i^{(n)} \otimes E_i^{(n)}.$$  

These operators have inverses

$$R_i^{-1} = \sum_{n \geq 0} (q_i - q_i^{-1})^n q_i^{-n(n+1)/2} E_i^n K^{-n} \otimes F_i^n K_i^n,$$

$$\tilde{R}_i^{-1} = \sum_{n \geq 0} q_i^{-n(n+1)/2} (q_i^{-1} - q_i)^n F_i^n \otimes E_i^n.$$  

Suppose $w \in \tilde{W}$ and $\tau s_{i_1} \cdots s_{i_k}$ is a reduced presentation for $w$ where $\tau$ is defined as in (3.30). Beck defines the following “$R$-matrices”:

$$R_w = \tau(S_{i_1} S_{i_2} \cdots S_{i_{r-1}}(R_{i_r}) \cdots S_{i_2}(R_{i_1}) R_{i_1}),$$

(3.35)  $$\tilde{R}_w = \tau(S_{i_r}^{-1} S_{i_{r-1}}^{-1} \cdots S_{i_2}^{-1}(R_{i_1})^{-1} \cdots S_{i_r}^{-1}(R_{i_r})^{-1}) \widetilde{R}_{i_r}.$$  

Using the root partition $S = \{\alpha + k\delta \mid \alpha \in \Delta_{0,+}, \ k \in \mathbb{Z}\} \cup \{l\delta \mid l \in \mathbb{Z}_{>0}\}$ from Section 2.3, we define:

$U_q^+(S)$ to be the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $x_{i,k}^+ (1 \leq i \leq N, \ k \in \mathbb{Z})$ and $h_{i,l}^+ (1 \leq i \leq N, \ l > 0);$  

$U_q^-(S)$ to be the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $x_{i,k}^- (1 \leq i \leq N, \ k \in \mathbb{Z})$ and $h_{i,l}^- (1 \leq i \leq N, \ l > 0),$ and  

$U_q^0(S)$ to be the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $K_i^{\pm 1} (1 \leq i \leq N), \gamma^{\pm 1/2},$ and $D^{\pm 1}$ (which all commute).

3.4. Let $\omega$ denote the standard $\mathbb{C}(q^{1/2})$-linear antiautomorphism of $U_q(\widehat{\mathfrak{g}}),$ and set $E_{-\alpha} = \omega(E_{\alpha})$ for all $\alpha \in \Delta_+.$ Then $U_q$ has a basis of elements of the form $E_\pm H E_\pm,$ where $E_\pm$ are ordered monomials in the $E_\alpha, \alpha \in \Delta_\pm,$ and $H$ is a monomial in $K_i^{\pm 1}, \gamma^{\pm 1/2},$ and $D^{\pm 1}$ (which all commute).

Furthermore, this basis is, in Beck’s terminology, convex, meaning that, if $\alpha, \beta \in \Delta_+$ and $\beta > \alpha,$ then

$$E_\beta E_\alpha - q^{(\alpha\mid\beta)} E_\alpha E_\beta = \sum_{\alpha < \gamma_1 < \cdots < \gamma_r < \beta} c_\gamma E_{\gamma_{1}}^{a_1} \cdots E_{\gamma_{r}}^{a_r},$$

for some integers $a_1, \ldots, a_r$ and scalars $c_\gamma \in \mathbb{C}[q, q^{-1}], \gamma = (\gamma_1, \ldots, \gamma_r)$ (see [BK96, Proposition 1.7c], [LS90]), and similarly for the negative roots. The above is called the Levendorski and Soibelman’s convexity formula.
A form such that:

\[
\gamma \left( K_i^n \right) \leq n, \quad \forall i \in I,
\]

which gives the bijection

\[
\gamma \left( U \right) \rightarrow \gamma (s) \left( n \right), \quad \forall s, n \in \mathbb{Z}.
\]

For 1 \leq i \leq N, r, s \in \mathbb{Z}, s \neq 0 where following [Lus88], for each i \in I, s \in \mathbb{Z} and n \in \mathbb{Z}_+, we define the Lusztig elements in \( U_q(\mathfrak{g}) \):

\[
\gamma^\pm \left( s \right) \left( n \right) = \gamma \left( K_i^n \right) \leq n, \quad \forall i \in I,
\]

and

\[
\gamma^\pm \left( s \right) \left( n \right) = \gamma \left( K_i^n \right) \leq n, \quad \forall i \in I.
\]

where \( q_0 = q^{a_0} \). This \( A \)-form can be shown to be the same as that in [FGM98] with the exception that we have added the generators \( \gamma^\pm \left( s \right) \left( n \right) \) and \( \gamma^\pm \left( s \right) \left( n \right) \).

Let \( U^+_{\mathfrak{h}} \) (resp. \( U^-_{\mathfrak{h}} \)) denote the subalgebra of \( U_{\mathfrak{h}} \) generated by the \( x_i^+ \), \( h_i \), where \( k \in \mathbb{Z}, l \in \mathbb{N} \setminus \{0\}, 1 \leq i \leq N \) (resp. \( x_i^- \), \( h_i \), where \( k \in \mathbb{Z}, l \in -\mathbb{N} \setminus \{0\}, 1 \leq i \leq N \)), \( i \in I \), and let \( U^\pm_{\mathfrak{h}} \) denote the subalgebra of \( U_{\mathfrak{h}} \) generated by the elements

\[
\gamma^\pm \left( s \right) \left( n \right) = \gamma \left( K_i^n \right) \leq n, \quad \forall i \in I.
\]

Note that if \( k+l > 0 \) (resp. \( k+l < 0 \)), then

\[
\gamma^\pm \left( s \right) \left( n \right) \in U^+_{\mathfrak{h}} \quad \text{(resp.} \quad \gamma^\pm \left( s \right) \left( n \right) \in U^-_{\mathfrak{h}} \text{)}.
\]

Let \( \text{Aut}(\Gamma) \) be the set of automorphisms of the affine Dynkin diagram \( \Gamma \). Recall \( I_0 = \{1, \ldots, N\} \), and let \( \pi : \mathbb{Z} \ni r \mapsto \pi_r \in I, N_1, \ldots, N_n \in \mathbb{N}, \tau_1, \ldots, \tau_n \in \text{Aut}(\Gamma) \) be such that:

i). \( N_i = \sum_{j=1}^i l(\omega_j) \forall i \in I_0 \) (where \( \langle \omega_i, \alpha_j \rangle = \delta_{ij} \) for all \( i, j \in I_0 \));

ii). \( s_{\pi_1} \cdots s_{\pi_N} \tau_i = \sum_{j=1}^i \omega_j \forall i \in I_0 \);

iii). \( \tau_{r+r_{\mathfrak{h}}} = \tau_{r} \tau_{r_{\mathfrak{h}}} \forall r \in \mathbb{Z} \);

These conditions imply that for all \( r, r' \in \mathbb{Z} \), \( s_{\pi_r} s_{\pi_{r+1}} \cdots s_{\pi_{r'-1}} s_{\pi_r} \) is a reduced expression, see [IM65] and [Kac90].

Then \( \pi \) induces a map

\[
\mathbb{Z} \ni r \mapsto w_r \in W
\]

defined by

\[
w_r = \begin{cases} 
  s_{\pi_0} \cdots s_{\pi_{r+1}} & \text{if } r < 0, \\
  1 & \text{if } r = 0, 1, \\
  s_{\pi_1} \cdots s_{\pi_{r-1}} & \text{if } r > 1,
\end{cases}
\]

which gives the bijection

\[
\mathbb{Z} \ni r \mapsto \beta_r = w_r(\alpha_{\pi_r}) \in \Phi^r_+.
\]

Of course we also have a bijection: \( \{\pm\} \times \mathbb{Z} \leftrightarrow \Phi^r_+ \).
For all $\alpha = \beta_r \in \Phi_+^c$ as in (3.25) the root vectors $E_\alpha$ can now be written as:

$$E_{\beta_r} = \begin{cases} T_{w_r}^{-1}(E_{\pi_r}) & \text{if } r < 0, \\ E_{\pi(0)} & \text{if } r = 0, \\ E_{\pi(1)} & \text{if } r = 1, \\ T_{w_r}(E_{\pi_r}) & \text{if } r > 1 \end{cases}$$

and we define

$$\tilde{F}_\alpha = \Omega(E_\alpha).$$

For $r \in \mathbb{Z}$, we define

$$\beta_r^\pm = \begin{cases} \pm \beta_r & \text{if } r \leq 0 \\ \mp \beta_r & \text{if } r > 0 \end{cases}$$

then of course

$$\{\beta_r^+ | r \in \mathbb{Z}\} = \{m\delta + \alpha | m \in \mathbb{Z}, \alpha \in Q_{0,+}\},$$
$$\{\beta_r^- | r \in \mathbb{Z}\} = \{m\delta - \alpha | m \in \mathbb{Z}, \alpha \in Q_{0,+}\}.$$

The root vectors do depend on $\pi$ (for example if $a_{ij} = a_{ji} = -1$ we have $T_i(E_j) \neq T_j(E_i)$). What is independent of $\pi$ are the root vectors relative to the roots $m\delta \pm \alpha_i$:

$$E_{m\delta + \alpha_i} = T_{w_i}^{-m}(E_i), \quad E_{m\delta - \alpha_i} = T_{w_i}^m T_i^{-1}(E_i).$$

Let $m \in \mathbb{Z}$, $\alpha \in Q_{0,+}$ be such that $m\delta \pm \alpha \in \Delta$; consider the following modified root vectors:

$$X_{m\delta + \alpha} = \begin{cases} E_{m\delta + \alpha} & \text{if } m \geq 0, \\ -F_{-m\delta - \alpha} K_{-m\delta - \alpha} & \text{if } m < 0, \end{cases}$$

$$X_{m\delta - \alpha} = \begin{cases} -K_{m\delta - \alpha}^{-1} E_{m\delta - \alpha} & \text{if } m > 0, \\ F_{-m\delta + \alpha} & \text{if } m \leq 0, \end{cases}$$

($\Omega(X_{m\delta \pm \alpha}) = X_{-m\delta \mp \alpha}$).

Equivalently

$$X_{\beta_r^\pm} = \begin{cases} E_{\beta_r} & \text{if } r \leq 0 \\ -F_{\beta_r} K_{\beta_r} & \text{if } r > 1 \end{cases} \quad X_{\beta_r^+} = \begin{cases} F_{\beta_r} & \text{if } r \leq 0 \\ -K_{\beta_r}^{-1} E_{\beta_r} & \text{if } r > 1 \end{cases}$$

**Theorem 3.4.1 ([CDFM13])**. Given $m : \mathbb{Z} \ni r \mapsto m_r \in \mathbb{N}$ such that $\# \{ r \in \mathbb{Z} | m_r \neq 0 \} < \infty$ define

$$X^{-}(m) = \prod_{r \in \mathbb{Z}} X_{m_r}^{-}, \quad X^{+}(m) = \prod_{r \in \mathbb{Z}} X_{m_r}^{+}$$

where one chooses a fixed ordering for the products.

Given $\mathcal{L} : \Delta_+(\im\mathfrak{g}) \to \mathbb{N}$ such that $\# \{ (r\delta, i) \in \Delta_+(\im\mathfrak{g}) | l_{(r\delta, i)} \neq 0 \} < \infty$ define

$$E^{im}(\mathcal{L}) = \prod_{(r\delta, i) \in \Delta_+(\im\mathfrak{g})} E^{l_{(r\delta, i)}}(\mathcal{L}), \quad F^{im}(\mathcal{L}) = \Omega(E^{im}(\mathcal{L})),$$

where $E^{(r\delta, i)} = E^{(r\delta, i)}$. Then the set

$$\{ X^{-}(m) F^{im}(\mathcal{L}) K_{\alpha} D^r \gamma^{n/2} E^{im}(\mathcal{L}) X^{+}(m') \}, \quad r, s \in \mathbb{Z}, \quad \alpha \in Q_0$$

is a basis of $U_q(\hat{\mathfrak{g}})$. 

4. Imaginary Verma Modules

The algebra $\hat{\mathfrak{g}}$ has a triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_+$, where $\hat{\mathfrak{g}}_+ = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ and $\mathfrak{h}$ is defined in §2.2. Let $U(\hat{\mathfrak{g}}_+)$ (resp. $U(\hat{\mathfrak{g}}_-)$) denote the universal enveloping algebra of $\hat{\mathfrak{g}}_+$ (resp. $\hat{\mathfrak{g}}_-$).

Let $\lambda \in P$, where $P$ is the weight lattice of $\hat{\mathfrak{g}}$. A weight (with respect to $\hat{\mathfrak{h}}$) $U(\hat{\mathfrak{g}})$-module $V$ is called an $S$-highest weight module with highest weight $\lambda$ if there is some nonzero vector $v \in V$ such that

(i). $u^+ \cdot v = 0$ for all $u^+ \in \hat{\mathfrak{g}}_+$;
(ii). $V = U(\hat{\mathfrak{g}}) \cdot v$.

Let $\lambda \in P$. We make $\mathbb{C}$ into a 1-dimensional $U(\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{h}})$-module by picking a generating vector $v$ and setting $(x + h) \cdot v = \lambda(h)v$, for all $x \in \hat{\mathfrak{g}}_+, h \in \hat{\mathfrak{h}}$. The induced module

$$M(\lambda) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{h}})} \mathbb{C}v = U(\hat{\mathfrak{g}}_-) \otimes \mathbb{C}v$$

is called the imaginary Verma module with $S$-highest weight $\lambda$. Imaginary Verma modules are in many ways similar to ordinary Verma modules except they contain both finite and infinite-dimensional weight spaces. They were studied in [Fut94], from which we summarize.

**Proposition 4.0.2 ([Fut94], Proposition 1, Theorem 1).** Let $\lambda \in P$, and let $M(\lambda)$ be the imaginary Verma module of $S$-highest weight $\lambda$. Then $M(\lambda)$ has the following properties.

(i). The module $M(\lambda)$ is a free $U(\hat{\mathfrak{g}}_-)$-module of rank 1 generated by the $S$-highest weight vector 1 of weight $\lambda$.
(ii). $M(\lambda)$ has a unique maximal submodule.
(iii). Let $V$ be a $U(\hat{\mathfrak{g}})$-module generated by some $S$-highest weight vector $v$ of weight $\lambda$. Then there exists a unique surjective homomorphism $\phi: M(\lambda) \twoheadrightarrow V$ such that $\phi(1 \otimes 1) = v$.
(iv). $\dim M(\lambda)_\lambda = 1$. For any $\mu = \lambda - k\delta$, $k$ a positive integer, $0 < \dim M(\lambda)_\mu < \infty$. If $\mu \neq \lambda - k\delta$ for any integer $k \geq 0$ and $\dim M(\lambda)_\mu \neq 0$, then $\dim M(\lambda)_\mu = \infty$.
(v). The module $M(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.

4.1. The Subalgebras $U_q(-S)$ and $U_q^+(S)$ of $U_q(\hat{\mathfrak{g}})$. Let $U_q(\pm S)$ be the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by $\{X_{r+ \delta}, X_{i \delta} \mid r \in \mathbb{Z}\} \cup \{E_{\pm k\delta}^{(i)} \mid 1 \leq i \leq N, k > 0\}$, and let $B_q^+$ denote the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by $U_q(S) \cup U_q^0(\hat{\mathfrak{g}})$ (the superscript $r$ is used to remind us that it is generated in part by root vectors). Let $U_q^+(S)$ be the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by $\{X_{r+ \delta}, X_{i \delta} \mid 1 \leq i \leq N, k \in \mathbb{Z}\} \cup \{h_i \mid 1 \leq i \leq N, l \in \mathbb{Z}\}$, and let $B_q^d$ denote the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by $U_q^+(S) \cup U_q^0(\hat{\mathfrak{g}})$. (The superscript $d$, is used to remind us that the respective subalgebras are generated in part by Drinfeld generators).

Let $\lambda \in P$. A $U_q(\hat{\mathfrak{g}})$ weight module $V_q^\lambda$ is called an $S$-highest weight module with highest weight $\lambda$ if there is a non-zero vector $v \in V_q^\lambda$ of weight $\lambda$ such that:

(i). $u^+ \cdot v = 0$ for all $u^+ \in U_q(S) \setminus \mathbb{C}(q^{1/2})^*$;
(ii). $V_q^\lambda = U_q(\hat{\mathfrak{g}}) \cdot v$.

Let $\mathbb{C}(q^{1/2}) \cdot v$ be a 1-dimensional vector space. Let $\lambda \in P$, and set $X_{r+ \delta} \cdot v = 0$, for all $r \in \mathbb{Z}$ and $E_{k\delta}^{(i)} \cdot v = 0$ for $k < 0$ and $1 \leq i \leq N$, $K_i^{\pm 1} \cdot v = q^{\pm \lambda(h_i)}v$. 


we obtain

\[ (4.1) \]

\[
\{ X^-(w) F^{\text{imm}}(\hat{v}) \}.
\]

In particular, \( M_q^r(\lambda) \) has a basis consisting of the ordered monomials.

We obtain the following refinement of [FGM05, Theorem 3.5]:

**Theorem 4.1.1 ([CDFM13]).** As a vector space, \( M_q^r(\lambda) \) has a basis consisting of the ordered monomials.

Recall the notation from §3.3. Let \( M_q^d(\lambda) = U_q / L_q^d \) where \( L_q^d \) is the left ideal generated by the Drinfeld generators \( x_{ik}^\pm, h_{ij}, i \in I_0, k \in \mathbb{Z}, l > 0, \lambda \in \mathbb{C} \) together with \( K_i^{\pm 1} - q^{\pm \lambda(h_i)} \), \( \gamma^{\pm 1/2} - q^{\pm \lambda(c)/2} \) and \( D^{\pm 1} - q^{\pm \lambda(d)} \). Let \( B_q^d \) be the subalgebra of \( U_q \) generated by \( U_q^{-}(S) \) and \( U_q^0(\hat{g}) \) and \( \mathbb{C}(q^{1/2})_\lambda \) be the one dimensional \( B_q^d \)-module where \( x_{ik}^\pm 1 = 0, h_{ij} 1 = 0, K_i^{\pm 1} 1 = q^{\pm \lambda(h_i)} 1, i \in I_0, k \in \mathbb{Z}, l > 0, \gamma^{\pm 1/2} 1 = q^{\pm \lambda(c)/2} 1 \) and \( D^{\pm 1} 1 = q^{\pm \lambda(d)} 1 \). Note that \( B_q^d \subseteq B_q^d \) as \( E_{\alpha_i+k\delta} = o(i)^k x_{ik}^\pm \) for \( k \geq 0, F_{-\alpha_i-k\delta} = -o(i)^k K_i \gamma^k x_{ik}^\pm \) for \( k < 0 \), and \( E_{k\delta} = \gamma^{-k/2} h_{ik} \) (see (3.26) and (3.29)).

By universal mapping properties of quotients and the tensor products one has

\[
M_q^d(\lambda) \cong U_q \otimes_{B_q^d} \mathbb{C}(q^{1/2})_\lambda.
\]

Since \( L_q^d \subseteq L_q^r \), there is a surjective \( U_q \)-module homomorphism \( \pi : M_q^d(\lambda) \to M_q^r(\lambda) \).

**Corollary 4.1.2 ([CDFM13]).** \( M_q^d(\lambda) \) is isomorphic to \( M_q^r(\lambda) \) as \( U_q \)-modules.

We have immediately from [FGM05], Corollary 6.5.

**Corollary 4.1.3.** \( M_q^d(\lambda) \) is irreducible if and only if \( \lambda(c) \neq 0 \).

### 4.2. Reduced imaginary Verma modules

Let \( \lambda \in P \). Suppose now that \( \lambda(c) = 0 \). Then \( \gamma^{\pm 1/2} \) acts on \( M_q^d(\lambda) \) by 1. Denote by \( J_q^d(\lambda) \) the left ideal of \( U_q = U_q(\hat{g}) \) generated by \( L_q^d \) and \( h_{ij} \) for all \( l \) and all \( i \in \hat{I} \). Set

\[
\tilde{M}_q(\lambda) = U_q / J_q^d(\lambda).
\]

Then \( \tilde{M}_q(\lambda) \) is a homomorphic image of \( M_q^d(\lambda) \) which we call the **reduced imaginary Verma module**. The module \( \tilde{M}_q(\lambda) \) has a \( P \)-gradation:

\[
\tilde{M}_q(\lambda) = \bigoplus_{\xi \in P} \tilde{M}_q(\lambda)_{\xi},
\]

where \( \tilde{M}_q(\lambda)_{\xi} \) is spanned by

\[
E_{-\beta_1} m_1 \delta \cdots E_{-\beta_l} m_l \delta E_{-\gamma_1 + k_1 \delta} \cdots E_{-\gamma_r + k_r \delta} \quad m_i \geq 0, k_i > 0, \beta_i, \gamma_i \in \Delta_+ \cdot 1
\]

for

\[
\xi = -\sum_{i=1}^{l} \beta_i - \sum_{j=1}^{r} \gamma_j + \left( -\sum_{i=1}^{l} m_i + \sum_{j=1}^{r} k_j \right) \delta.
\]

Applying [FGM05, Theorem 7.1] and Corollary 4.1.2 we obtain
Theorem 4.2.1. Let $\lambda \in P$ such that $\lambda(c) = 0$. Then module $\tilde{M}_q(\lambda)$ is simple if and only if $\lambda(h_i) \neq 0$ for all $i \in \hat{I}$.

5. $\Omega$-operators and their relations

Recall that the Schur polynomials $S_k(x)$, $k \in \mathbb{Z}$ are defined to be polynomials in $\mathbb{C}[x_1, x_2, \ldots]$ given by

$$\sum_{k \in \mathbb{Z}} S_k(x)z^k = \exp\left(\sum_{l=1}^{\infty} x_l z^l\right)$$

Consider now the subalgebra $\mathcal{N}_q^-$, generated by $\gamma^{\pm 1/2}$, and $x_{i,l}^-$, $l \in \mathbb{Z}$, $1 \leq i \leq N$. Note that the corresponding relations (3.13) hold in $\mathcal{N}_q^-$. 

Lemma 5.0.2. Fix $k \in \mathbb{Z}$ and $1 \leq i \leq N$. Then for any $P \in \mathcal{N}_q^-$, there exists unique

$$Q(i, k, p), R(i, k, r) \in \mathcal{N}_q^-, \quad p, r \in \mathbb{Z},$$

such that

$$[x_{i,k}^+, P] = K_i \sum_{q} \frac{S_{i,p}^+ Q(i, k, p)}{q_i - q_i^{-1}} + K_i^{-1} \sum_{q} \frac{S_{i,k}^- R(i, k, r)}{q_i - q_i^{-1}}. \tag{5.1}$$

where

$$S_{i,k}^+ := S_k((q_i - q_i^{-1})E_{1}^{(i)}, (q_i - q_i^{-1})E_{2}^{(i)}, \ldots),$$

$$S_{i,k}^- := S_k((q_i - q_i^{-1})E_{1}^{(i)} - (q_i - q_i^{-1})E_{-2}^{(i)}, \ldots).$$

Note that the $S_{i,k}$ have degree $k$ with respect to $D$.

Proof. For the existence we have the following: Now any element in $\mathcal{N}_q^-$ is a sum of elements of the form

$$P_{m_1, \ldots, m_k} = \gamma^{l/2} x_{j_1, m_1}^- \cdots x_{j_k, m_k}^-,$$

where $m_i \in \mathbb{Z}, k \geq 0$, $l \in \mathbb{Z}, 1 \leq j_i \leq N$ and such a product is a summand of

$$P = P(v_1, \ldots, v_k) := \gamma^{l/2} x_{j_1}^-(v_1) \cdots x_{j_k}^-(v_k)$$

Set $\tilde{P} = x_{j_1}^-(v_1) \cdots x_{j_k}^-(v_k)$ and $\tilde{P}_l = x_{j_1}^-(v_1) \cdots x_{j_{l-1}}^-(v_{l-1}) x_{j_{l+1}}^- \cdots x_{j_k}^-(v_k)$.

Then we have by (3.20) and (3.21),

$$x_{j_1}^-(v_1) \cdots x_{j_{l-1}}^-(v_{l-1}) \psi_i(v_i \gamma^{1/2}) = \prod_{m=1}^{l-1} g_{i,j_m}(v_m v_{l-1}^{-1})^{-1} \psi_i(v_i \gamma^{1/2}) x_{j_1}^-(v_1) \cdots x_{j_{l-1}}^-(v_{l-1})$$

$$x_{j_1}^-(v_1) \cdots x_{j_{l-1}}^-(v_{l-1}) \phi_i(u \gamma^{1/2}) = \prod_{m=1}^{l-1} g_{i,j_m}(u \gamma v_{j_m}^{-1}) \phi_i(u \gamma^{1/2}) x_{j_1}^-(v_1) \cdots x_{j_{l-1}}^-(v_{l-1}).$$
so that by (3.23)

\[
[x_i^+(u), x_j^-(v_1) \cdots x_k^-(v_k)] = \sum_{l=1}^{k} \delta_{i,j_l} x_j^-(v_1) \cdots \left( \frac{\delta(u/v_i \gamma) \psi_i(v_i \gamma^{1/2}) - \delta(u/v_i \gamma) \phi_i(u \gamma^{1/2})}{q_i - q_i^{-1}} \right) \cdots x_j^-(v_k)
\]

\[
= \sum_{l=1}^{k} \delta_{i,j_l} x_j^-(v_1) \cdots x_{j_l-1}^-(v_{l-1}) \psi_i(v_l \gamma^{1/2}) x_{j_l}^-(v_{l+1}) \cdots x_j^-(v_k) \delta(u/v_i \gamma) q_i - q_i^{-1}
\]

\[
= \sum_{l=1}^{k} \delta_{i,j_l} x_j^-(v_1) \cdots x_{j_l-1}^-(v_{l-1}) \phi_i(u \gamma^{1/2}) x_{j_l}^-(v_{l+1}) \cdots x_j^-(v_k) \frac{\delta(u/v_i \gamma)}{q_i - q_i^{-1}}
\]

\[
= \psi_i(u \gamma^{1-k/2}) \frac{q_i - q_i^{-1}}{q_i - q_i^{-1}} \sum_{l=1}^{k} \delta_{i,j_l} x_j^-(v_1) \cdots x_{j_l-1}^-(v_{l-1}) \psi_i(v_l \gamma^{1/2}) x_{j_l}^-(v_{l+1}) \cdots x_j^-(v_k) \delta(u/v_i \gamma)
\]

\[
= \phi_i(u \gamma^{1-k/2}) \frac{q_i - q_i^{-1}}{q_i - q_i^{-1}} \sum_{l=1}^{k} \delta_{i,j_l} x_j^-(v_1) \cdots x_{j_l-1}^-(v_{l-1}) \phi_i(v_l \gamma^{1/2}) x_{j_l}^-(v_{l+1}) \cdots x_j^-(v_k) \delta(u/v_i \gamma).
\]

Note that \(\psi_{i,k}(u \gamma^{-k/2})\) and \(\phi_{i,k}(u \gamma^{k/2})\) do not depend on \(P\). By (3.15) we can rewrite

\[
\psi_i(u \gamma^{-1/2}) = \sum_{k=0}^{\infty} \psi_{i,k} \gamma^{k/2} u^{-k} = K_i \exp \left( (q_i - q_i^{-1}) \sum_{l>0} \gamma^{l/2} u^{-l} \right) = K_i \left( \sum_{k=0}^{\infty} S_{+,i,k}^+ u^{-k} \right),
\]

so that \(\psi_{i,k} \gamma^{k/2} = K_i S_{+,i,k}^+\) and similarly \(\phi_{i,k} \gamma^{-k/2} = K_i^{-1} S_{-,i,k}^-\). Thus

\[
[x_{im}^+, x_{j_1,n_1}^- \cdots x_{j_k,n_k}^-] = K_i \sum_{i,k,p} S_{+,i,k}^+ Q(i,k,p) q_i - q_i^{-1} + K_i^{-1} \sum_{i,k,r} S_{-,i,k}^- R(i,k,r) q_i - q_i^{-1},
\]

where \(Q(i,k,p), R(i,k,r) \in N_q^-\). This proves existence.

Uniqueness is proven as follows: The components of \(\tilde{P}_1(u)\) have the form

\[
x_{j_1,n_1}^- \cdots x_{j_{l-1},n_{l-1}}^- x_{j_{l+1},n_{l+1}}^- \cdots x_{j_k,n_k}^{-}
\]

and after using the Levendorski and Soibelman’s convexity formula (3.36) (possibly after applying a Lusztig automorphism \(T_w\)) we can rewrite this component as a linear combination of elements of the form \(X^- (m)\). Let \(F^-\) be the span of the
X^-(m) over \( \mathbb{Q}(q^{1/2}) \). Then

\[
[x_{im}^+, x_{j_1,n_1}^- \cdots x_{j_k,n_k}^-] = K_i \sum S_{i,p}^+ \tilde{Q}(i,k,p) q_i - q_i^{-1} + K_i^{-1} \sum S_{i,r}^- \tilde{R}(i,k,r) q_i - q_i^{-1},
\]

where \( \tilde{Q}(i,k,p), \tilde{R}(i,k,r) \in \mathcal{F}^- \).

If also

\[
[x_{im}^+, x_{j_1,n_1}^- \cdots x_{j_k,n_k}^-] = K_i \sum S_{i,p}^+ Q^i(i,k,p) q_i - q_i^{-1} + K_i^{-1} \sum S_{i,r}^- R^i(i,k,r) q_i - q_i^{-1},
\]

for some \( Q^i(i,k,p), R^i(i,k,r) \in \mathcal{N}^q_- \), then from Theorem 4.1.1 we must have

\[
Q^i(i,k,p) = \tilde{Q}(i,k,p) = Q(i,k,p), \quad R^i(i,k,r) = \tilde{R}(i,k,r) = R(i,k,r)
\]
as the \( S_{i,p}^+, S_{i,r}^- \) have leading terms \( (E_{\pm}^{(i)})^k \) that are distinct PBW basis elements and \( \tilde{Q}(i,k,p) \) and \( \tilde{R}(i,k,r) \) are sums of PBW basis elements.

Lemma 5.0.2 motivates the definition of a family of operators as follows. Set

\[
G_{il} = G_{il}^{1/q} = G_{il}^{1/q}(v_{j_1}, \ldots, v_{j_l}, v_{l}) := \delta_{l,1} \prod_{j=1}^{l-1} g_{i,j,m,q}(v_{j_m}/v_{l}),
\]

\[
G_{il}^q = G_{il}(v_{j_1}, \ldots, v_{j_l}, v_{l}) := \delta_{l,1} \prod_{j=1}^{l-1} g_{i,j,m}(v_{l}/v_{j_m})
\]

where \( G_{il} := \delta_{i,j}. \) Now define a collection of operators \( \Omega_{\psi_i}(k), \Omega_{\phi_i}(k) : \mathcal{N}^q_- \rightarrow \mathcal{N}^q_- \), \( k \in \mathbb{Z} \), in terms of the generating functions

\[
\Omega_{\psi_i}(u) = \sum_{l \in \mathbb{Z}} \Omega_{\psi_i}(l) u^{-l}, \quad \Omega_{\phi_i}(u) = \sum_{l \in \mathbb{Z}} \Omega_{\phi_i}(l) u^{-l}
\]

by

\[
\Omega_{\psi_i}(u)(\bar{P}) := \sum_{l=1}^{k} G_{il} \bar{P} \delta(u/v_{l})
\]

(5.2)

\[
\Omega_{\phi_i}(u)(\bar{P}) := \sum_{l=1}^{k} G_{il}^q \bar{P} \delta(u/v_{l}).
\]

(5.3)

Then we can write the above computation in the proof of Lemma 4.0.2 as

\[
[x_i^+(u), \bar{P}] = (q_i - q_i^{-1})^{-1} \left( \psi_i(u\gamma^{-1/2}) \Omega_{\psi_i}(u)(\bar{P}) - \phi_i(u\gamma^{1/2}) \Omega_{\phi_i}(u)(\bar{P}) \right).
\]

(5.4)

Note that \( \Omega_{\psi_i}(u)(1) = \Omega_{\phi_i}(u)(1) = 0. \) More explicitly let us write

\[
\bar{P} = x_{j_1}(v_1) \cdots x_{j_k}(v_k) = \sum_{n_1 \in \mathbb{Z}} \cdots \sum_{n_k \in \mathbb{Z}} x_{j_1,n_1}^- \cdots x_{j_k,n_k}^- v_1^{-n_1} \cdots v_k^{-n_k}
\]

Then

\[
\psi_i(u\gamma^{-1/2}) \Omega_{\psi_i}(u)(\bar{P})
\]

\[
= \sum_{l \geq 0} \sum_{p \in \mathbb{Z}} \sum_{n_1 \in \mathbb{Z}} \gamma^{l/2} \psi_{il} \Omega_{\psi_i}(p)(x_{j_1,n_1}^- \cdots x_{j_k,n_k}^-) v_1^{-n_1} \cdots v_k^{-n_k} u^{-l-p}
\]

\[
= \sum_{n_1 \in \mathbb{Z}} \cdots \sum_{n_k \in \mathbb{Z}} \gamma^{l/2} \psi_{il} \Omega_{\psi_i}(m - l)(x_{j_1,n_1}^- \cdots x_{j_k,n_k}^-) v_1^{-n_1} \cdots v_k^{-n_k} u^{-m}
\]
while
\[ [x_i^+(u), \bar{P}] = \sum_{m \in \mathbb{Z}, n_1, n_2, \ldots, n_k \in \mathbb{Z}} [x_{i,m}^+, x_{j_1,n_1}^- \cdots x_{j_k,n_k}^-]u_i^{-n_1} \cdots v_k^{-n_k} u^{-m}. \]

Thus for a fixed \( m \) and \( k \)-tuple \(((j_1, n_1), \ldots, (j_k, n_k))\) the sum
\[
\sum_{l \geq 0} \frac{1}{l!} \psi_l \Omega_{\psi_i}(m - l)(x_{j_1,n_1}^- \cdots x_{j_k,n_k}^-)
\]
must be finite. Hence
\[
\Omega_{\psi_i}(m - l)(x_{j_1,n_1}^- \cdots x_{j_k,n_k}^-) = 0,
\]
for \( l \) sufficiently large.

**Proposition 5.0.3.** Consider \( x_i^-(v) = \sum_m x_{im}^- v^{-m} \) as a formal power series of left multiplication operators \( x_{im}^- : \mathcal{N}_q \to \mathcal{N}_q^- \). Then
\[
\Omega_{\psi_i}(u)x_i^-(v) = \delta_{i,m} \delta(u/v) + g_{i,m,q-1}(v^\gamma/u)x_i^-(v)\Omega_{\psi_m}(u),
\]
(5.6)
\[
\Omega_{\phi_m}(u)x_i^-(v) = \delta_{i,m} \delta(u/v) + g_{i,m}(u/v)x_i^-(v)\Omega_{\phi_m}(u)
\]
(5.7)
\[
(q^{(\alpha_j|\alpha_k)}u_1 - u_2)\Omega_{\psi_j}(u_1)\Omega_{\psi_k}(u_2) = (u_1 - q^{(\alpha_j|\alpha_k)}u_2)\Omega_{\psi_j}(u_2)\Omega_{\psi_k}(u_1)
\]
(5.8)
\[
(q^{(\alpha_j|\alpha_k)}u_1 - u_2)\Omega_{\phi_j}(u_1)\Omega_{\phi_k}(u_2) = (u_1 - q^{(\alpha_j|\alpha_k)}u_2)\Omega_{\phi_j}(u_2)\Omega_{\phi_k}(u_1)
\]
(5.9)
\[
q^{(\alpha_j|\alpha_k)}\gamma^2 u_1 - u_2)\Omega_{\phi_j}(u_1)\Omega_{\phi_k}(u_2) = (\gamma^2 u_1 - q^{(\alpha_j|\alpha_k)}u_2)\Omega_{\phi_j}(u_2)\Omega_{\phi_k}(u_1)
\]
(5.10)

**Proof.** Setting \( \bar{P} = x_{j_1}(v_1) \cdots x_{j_k}(v_k) \) we get
\[
\Omega_{\psi_m}(u)x_i^-(v)(\bar{P}) = \delta_{i,m} x_{j_1}(v_1) \cdots x_{j_k}(v_k)\delta(u/v) + x_i^-(v)\sum_{l=1}^k g_{i,m,q-1}(v/v_l)G_{ml}\bar{P}\delta(u/v_l)
\]
\[= \delta_{i,m} \bar{P}\delta(u/v) + x_i^-(v)g_{i,m,q-1}(v^\gamma/u)\Omega_{\psi_m}(u)\bar{P}.\]

Similarly
\[
\Omega_{\phi_m}(u)x_i^-(v)(\bar{P}) = \delta_{i,m} x_{j_1}(v_1) \cdots x_{j_k}(v_k)\delta(u/v)
\]
\[+ x_i^-(v)\sum_{l=1}^k g_{i,m}(v_l/v)G_{ml}\bar{P}\delta(u/v_l)
\]
\[= \delta_{i,m} \bar{P}\delta(v/u) + x_i^-(v)g_{i,m}(u/v^\gamma)\Omega_{\phi_m}(u)\bar{P}.\]

One can prove (5.8) and (5.9) directly from their definitions, (5.2) and (5.3), but there is another way (due to Kashiwara) to prove this identity and it goes as follows:
\[
\Omega_{\psi_j}(u_1)\Omega_{\psi_k}(u_2)x_i^-(v) = \delta_{j,i}\Omega_{\psi_j}(u_1)\delta(v^\gamma/u_2) + \Omega_{\psi_j}(u_1)x_i^-(v)g_{j,i,k,q-1}(v^\gamma/u_2)\Omega_{\psi_k}(u_2)
\]
\[= \delta_{j,i}\Omega_{\psi_j}(u_1)\delta(v^\gamma/u_2) + \delta_{j,i}g_{j,i,k,q-1}(v^\gamma/u_2)\Omega_{\psi_j}(u_1)x_i^-(v)\Omega_{\psi_k}(u_2)
\]
and on the other hand
\[
\Omega_{\psi_k}(u_2)\Omega_{\psi_j}(u_1)x_i^-(v) = \delta_{j,i}\Omega_{\psi_j}(u_2)\delta(v^\gamma/u_1) + \delta_{j,i}g_{j,i,q-1}(v^\gamma/u_1)\Omega_{\psi_j}(u_1)x_i^-(v)\Omega_{\psi_k}(u_2)
\]
\[+ g_{j,i,q-1}(v^\gamma/u_1)g_{j,i,q-1}(v^\gamma/u_2)x_i^-(v)\Omega_{\psi_k}(u_2)\Omega_{\psi_j}(u_1).\]
Hence and on the other hand

Thus setting $S = (u_1 - q^{-(\alpha_j|\alpha_k)}u_2)\Omega_{\phi_j}(u_1)\Omega_{\phi_k}(u_2) - (q^{-(\alpha_j|\alpha_k)}u_1 - u_2)\Omega_{\phi_k}(u_2)\Omega_{\phi_j}(u_1)$ we get

$$S x^-_i(v) = (u_1 - q^{-(\alpha_j|\alpha_k)}u_2)\delta_{i,k}\Omega_{\phi_j}(u_1)\delta(v/u_2) + (u_1 - q^{-(\alpha_j|\alpha_k)}u_2)\delta_{i,j}g_{k,i,q^{-1}}(v\gamma/u_2)\Omega_{\phi_k}(u_2)\delta(v\gamma/u_1) + (u_1 - q^{-(\alpha_j|\alpha_k)}u_2)g_{k,i,q^{-1}}(v\gamma/u_2)\delta_{j,i}\Omega_{\phi_k}(u_2)\delta(v\gamma/u_1) - (q^{-(\alpha_j|\alpha_k)}u_1 - u_2)\delta_{j,i}\Omega_{\phi_k}(u_2)\delta(v\gamma/u_1)

Thus setting $(j,i,q^{-1})\Omega_{\phi_k}(u_2)\delta(v\gamma/u_1) - (q^{-(\alpha_j|\alpha_k)}u_1 - u_2)\delta_{j,i,q^{-1}}(v\gamma/u_1)\Omega_{\phi_j}(u_1)\delta(v\gamma/u_2) - (q^{-(\alpha_j|\alpha_k)}u_1 - u_2)g_{j,i,q^{-1}}(v\gamma/u_1)g_{q^{-1}}(v\gamma/u_2)\delta_{j,i}\Omega_{\phi_k}(u_2)\Omega_{\phi_j}(u_1)

$$S x^-_i(v) = g_{k,i,q^{-1}}(v\gamma/u_2)g_{j,i,q^{-1}}(v\gamma/u_1)\delta(v\gamma/u_1)\Omega_{\phi_k}(u_2)\Omega_{\phi_j}(u_1)

Hence

$$S x^-_i(v_1)\ldots x^-_n(v_n) = \prod_{i=1}^n g_{k,i,q^{-1}}(v_i\gamma/u_1)g_{j,i,q^{-1}}(v_j\gamma/u_2)\Omega_{\phi_j}(u_1)\Omega_{\phi_k}(u_2),$$

which implies, after applying this to 1, that $S = 0.$

Next we have

$$\Omega_{\phi_j}(u_1)\Omega_{\phi_k}(u_2)x^-_i(v) = \delta_{i,k}\Omega_{\phi_j}(u_1)\delta(v/u_2) + \delta_{j,i}g_{k,i}(u_2\gamma/v)\Omega_{\phi_k}(u_2)\delta(v/u_1) + g_{k,i}(u_2\gamma/v)g_{j,i}(u_1\gamma/v)x^-_i(v)\Omega_{\phi_j}(u_1)\Omega_{\phi_k}(u_2)$$

and on the other hand

$$\Omega_{\phi_k}(u_2)\Omega_{\phi_j}(u_1)x^-_i(v) = \delta_{j,i}\Omega_{\phi_k}(u_2)\delta(v/u_1) + \delta_{k,i}g_{j,i}(u_1\gamma/v)\Omega_{\phi_j}(u_1)\delta(v/u_2) + g_{j,i}(u_1\gamma/v)g_{k,i}(u_2\gamma/v)x^-_i(v)\Omega_{\phi_k}(u_2)\Omega_{\phi_j}(u_1)$$
So if we set \( S = (u_1 - q^{-(\alpha_j | \alpha_k)} u_2) \Omega_{\phi_j}(u_1) \Omega_{\phi_k}(u_2) - (q^{-\alpha_j | \alpha_k} u_1 - u_2) \Omega_{\phi_k}(u_2) \Omega_{\phi_j}(u_1) \)
we get

\[
S x_i^- (v) = (u_1 - q^{-(\alpha_j | \alpha_k)} u_2) \delta_{k,i} \Omega_{\phi_j}(u_1) \delta(v/u_2 \gamma) \\
+ (u_1 - q^{-(\alpha_j | \alpha_k)} u_2) \delta_{j,i} g_{k,i} (u_2 \gamma / v) \Omega_{\phi_k}(u_2) \delta(v/u_1 \gamma) \\
+ (u_1 - q^{-(\alpha_j | \alpha_k)} u_2) g_{k,i} (u_2 \gamma / v) g_{j,i} (u_1 \gamma / v) x_i^- (v) \Omega_{\phi_j}(u_1) \Omega_{\phi_k}(u_2) \\
- (q^{-(\alpha_j | \alpha_k)} u_1 - u_2) \delta_{i,j} \Omega_{\phi_k}(u_2) \delta(v/u_1 \gamma) \\
- (q^{-(\alpha_j | \alpha_k)} u_1 - u_2) \delta_{i,k} g_{j,i} (u_1 \gamma / v) \Omega_{\phi_j}(u_1) \delta(v/u_2 \gamma) \\
- (q^{-(\alpha_j | \alpha_k)} u_1 - u_2) g_{j,i} (u_1 \gamma / v) g_{k,i} (u_2 \gamma / v) x_i^- (v) \Omega_{\phi_k}(u_2) \Omega_{\phi_j}(u_1) \\
= \left( (u_1 - q^{-(\alpha_j | \alpha_k)} u_2) - (q^{-(\alpha_j | \alpha_k)} u_1 - u_2) g_{j,i} (u_1 \gamma / v) \right) \delta_{k,i} \Omega_{\phi_j}(u_1) \delta(v/u_2 \gamma) \\
+ \left( (u_1 - q^{-(\alpha_j | \alpha_k)} u_2) g_{k,i} (u_2 \gamma / v) - (q^{-(\alpha_j | \alpha_k)} u_1 - u_2) \right) \delta_{j,i} \Omega_{\phi_k}(u_2) \delta(v/u_1 \gamma) \\
+ g_{k,i} (u_2 \gamma / v) g_{j,i} (u_1 \gamma / v) x_i^- (v) \\
\times \left( (u_1 - q^{-(\alpha_j | \alpha_k)} u_2) \Omega_{\phi_j}(u_1) \Omega_{\phi_k}(u_2) - (q^{-2} u_1 - u_2) \Omega_{\phi_k}(u_2) \Omega_{\phi_j}(u_1) \right) \\
= g_{k,i} (u_2 \gamma / v) g_{j,i} (u_1 \gamma / v) x_i^- (v) S.
\]

As in the calculation for (5.8) we get \( S = 0 \).

Moreover

\[
\Omega_{\phi_j}(u_1) \Omega_{\psi_k}(u_2) x_i^- (v) = \delta_{k,i} \Omega_{\phi_j}(u_1) \delta(v/u_2 \gamma) + \Omega_{\phi_j}(u_1) x_i^- (v) g_{k,i} (u_1 \gamma / v) g_{\psi_k}(u_2) \\
= \delta_{k,i} \Omega_{\phi_j}(u_1) \delta(v/u_2 \gamma) + \delta_{j,i} g_{k,i} (u_1 \gamma / v) \Omega_{\phi_k}(u_2) \delta(u_1 \gamma / v) \\
+ g_{k,i} (u_1 \gamma / v) g_{j,i} (u_1 \gamma / v) x_i^- (v) \Omega_{\phi_j}(u_1) \Omega_{\psi_k}(u_2)
\]

and

\[
\Omega_{\psi_k}(u_2) \Omega_{\phi_j}(u_1) x_i^- (v) = \delta_{j,i} \Omega_{\psi_k}(u_2) \delta(u_1 \gamma / v) + \Omega_{\psi_k}(u_2) x_i^- (v) g_{j,i} (u_1 \gamma / v) \Omega_{\phi_j}(u_1) \\
= \delta_{j,i} \Omega_{\psi_k}(u_2) \delta(u_1 \gamma / v) + \delta_{k,i} g_{j,i} (u_1 \gamma / v) \Omega_{\phi_j}(u_1) \delta(v/u_2 \gamma) \\
+ g_{k,i} (u_1 \gamma / v) g_{j,i} (u_1 \gamma / v) x_i^- (v) \Omega_{\psi_k}(u_2) \Omega_{\phi_j}(u_1)
\]
Set \( S = (q^{(\alpha)|\alpha}) \gamma^2 u_1 - u_2) \Omega_{\phi_j}(u_1) \Omega_{\psi_i}(u_2) - (\gamma^2 u_1 - q^{(\alpha)|\alpha}) u_2) \Omega_{\psi_k}(u_2) \Omega_{\phi_j}(u_1). \)

Then
\[
S x_i^-(v) = (q^{(\alpha)|\alpha}) \gamma^2 u_1 - u_2) \delta_{k,i} \Omega_{\phi_j}(u_1) \delta((v\gamma)/u_2)
+ (q^{(\alpha)|\alpha}) \gamma^2 u_1 - u_2) \delta_{j,i} g_{k,i,q^-1}(v\gamma/u_2) \Omega_{\phi_k}(u_2) \delta((u_1\gamma/v)
+ (q^{(\alpha)|\alpha}) \gamma^2 u_1 - u_2) g_{k,i,q^-1}(v\gamma/u_2) g_{j,i} (u_1\gamma/v) x_i^-(v) \Omega_{\phi_j}(u_1) \Omega_{\psi_k}(u_2)
- (\gamma^2 u_1 - q^{(\alpha)|\alpha}) u_2) \delta_{j,i} \Omega_{\psi_k}(u_2) \delta((u_1\gamma/v)
- (\gamma^2 u_1 - q^{(\alpha)|\alpha}) u_2) \delta_{j,i} g_{j,i} (u_1\gamma/v) \Omega_{\phi_j}(u_1) \delta((v\gamma)/u_2)
- (\gamma^2 u_1 - q^{(\alpha)|\alpha}) u_2) g_{k,i,q^-1}(v\gamma/u_2) g_{j,i} (u_1\gamma/v) x_i^-(v) \Omega_{\phi_j}(u_2) \Omega_{\psi_k}(u_1)

= \left((q^{(\alpha)|\alpha}) \gamma^2 u_1 - u_2) - (\gamma^2 u_1 - q^{(\alpha)|\alpha}) u_2) g_{j,i} (u_1\gamma/v) \right) \delta_{k,i} \Omega_{\phi_j}(u_1) \delta((v\gamma)/u_2)
+ \left((q^{(\alpha)|\alpha}) \gamma^2 u_1 - u_2) g_{k,i,q^-1}(v\gamma/u_2) - (\gamma^2 u_1 - q^{(\alpha)|\alpha}) u_2) \right) \delta_{j,i} \Omega_{\psi_k}(u_2) \delta((u_1\gamma/v)
+ g_{k,i,q^-1}(v\gamma/u_2) g_{j,i} (u_1\gamma/v) x_i^-(v)
\times \left((q^{(\alpha)|\alpha}) \gamma^2 u_1 - u_2) \Omega_{\phi_j}(u_1) \Omega_{\psi_k}(u_2) - (\gamma^2 u_1 - q^{(\alpha)|\alpha}) u_2) \Omega_{\psi_k}(u_2) \Omega_{\phi_j}(u_1)\right)

= g_{k,i,q^-1}(v\gamma/u_2) g_{j,i} (u_1\gamma/v) x_i^-(v) S.

As in the previous calculations we get that \( S = 0 \) and thus the last statement of the proposition hold.

The identities (5.6), (5.7) in Proposition 5.0.3 can be rewritten as

\[ (q^{(\alpha)|\alpha}) v\gamma - u) \Omega_{\psi_j}(u) x_i^-(v) = (q^{(\alpha)|\alpha}) v\gamma - u) \delta_{i,j} \delta(v\gamma/u) + (q^{(\alpha)|\alpha}) v\gamma - u) x_i^-(v) \Omega_{\psi_j}(u), \]
\[ (q^{(\alpha)|\alpha}) v - u\gamma) \Omega_{\phi_j}(u) x_i^-(v) = (q^{(\alpha)|\alpha}) v - u\gamma) \delta_{i,j} \delta(v/u\gamma) + (v - q^{(\alpha)|\alpha}) u\gamma) x_i^-(v) \Omega_{\phi_j}(u) \]

which may be written out in terms of components as

\[ q^{(\alpha)|\alpha}) \gamma \Omega_{\psi_j}(m) x_{i,n+1}^- - \Omega_{\psi_j}(m + 1) x_i^- \]
\[ = (q^{(\alpha)|\alpha}) \gamma - 1) \delta_{i,j} \delta_{m,n-1} + \gamma x_{i,n+1}^- \Omega_{\psi_j}(m) - q^{(\alpha)|\alpha}) x_{i,n}^- \Omega_{\psi_j}(m + 1), \]
\[ q^{(\alpha)|\alpha}) \Omega_{\phi_j}(m) x_{i,n+1}^- - \gamma x_{i,n}^- \Omega_{\psi_j}(m + 1) \]
\[ = (q^{(\alpha)|\alpha}) - \gamma) \delta_{i,j} \delta_{m,n-1} + x_{i,n+1}^- \Omega_{\psi_j}(m) - q^{(\alpha)|\alpha}) x_{i,n}^- \Omega_{\psi_j}(m + 1), \]

We can also write (5.6) in terms of components and as operators on \( N_q^- \)

\[ \Omega_{\psi_j}(k) x_{i,m}^- = \delta_{i,j} \delta_{k,-m} \gamma^k + \sum_{r \geq 0} g_{i,j,q^-1}(r) x_{i,m+r}^- \Omega_{\psi_j}(k - r) \gamma^r. \]
The sum on the right hand side turns into a finite sum when applied to an element in $\mathcal{N}_q^-$, due to (5.5).

We also have by (5.10)

\begin{equation}
\Omega_{\psi_i}(k)\Omega_{\phi_j}(m) = \sum_{r \geq 0} g_{i,j}(r)\gamma^{2r}\Omega_{\phi_j}(r + m)\Omega_{\psi_i}(k - r),
\end{equation}

as operators on $\mathcal{N}_q^-$. 

6. The Kashiwara algebra $K_q$  

The Kashiwara algebra $K_q$ is defined to be the $\mathbb{F}(q^{1/2})$-algebra with generators $\Omega_{\psi_i}(m), x_i^n(n), \gamma^{\pm 1/2}, m, n \in \mathbb{Z}, 1 \leq i, j \leq N$, where $\gamma^{\pm 1/2}$ are central and the defining relations are

\begin{align}
q^{(\alpha_i|\alpha_j)}\gamma\Omega_{\psi_i}(m)x_{i,n+1}^- - \Omega_{\psi_i}(m+1)x_{i,n}^- & = q^{(\alpha_i|\alpha_j)}\gamma - 1\delta_{i,j}\delta_{m,-n-1} + \gamma x_{i,n+1}^-\Omega_{\psi_j}(m) - q^{(\alpha_i|\alpha_j)}x_{i,n}^-\Omega_{\psi_j}(m+1) \\
q^{(\alpha_i|\alpha_j)}\Omega_{\psi_i}(k+1)\Omega_{\psi_j}(l) - \Omega_{\psi_i}(l)\Omega_{\psi_j}(k) & = \Omega_{\psi_i}(k)\Omega_{\psi_j}(l+1) - q^{(\alpha_i|\alpha_j)}\Omega_{\psi_i}(l+1)\Omega_{\psi_j}(k)
\end{align}

(which comes from (6.5), (6.8) written out in terms of components), and

\begin{equation}x_{i,k+1}^-x_{j,l}^- - q^{(\alpha_i|\alpha_j)}x_{j,l}^-x_{i,k+1}^- = q^{(\alpha_i|\alpha_j)}x_{i,k}^-x_{j,l+1}^- - x_{j,l+1}^-x_{i,k}^-
\end{equation}

together with

$\gamma^{1/2}\gamma^{-1/2} = 1 = \gamma^{-1/2}\gamma^{1/2}$.

**Lemma 6.0.4.** The $\mathbb{F}(q^{1/2})$-linear map $\tilde{\alpha} : K_q \to K_q$ given by

$\tilde{\alpha}(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2}, \quad \tilde{\alpha}(x_{i,m}^-) = \Omega_{\psi_i}(-m), \quad \tilde{\alpha}(\Omega_{\psi_i}(m)) = x_{i,-m}^-$

for all $m \in \mathbb{Z}$ is an involutive anti-automorphism.

**Proof.** We have

$\tilde{\alpha}\left(x_{i,k+1}^-x_{j,l}^- - q^{(\alpha_i|\alpha_j)}x_{j,l}^-x_{i,k+1}^-ight) = \Omega_{\psi_j}(-l)\Omega_{\psi_i}(-k - 1) - q^{(\alpha_i|\alpha_j)}\Omega_{\psi_i}(-k - 1)\Omega_{\psi_j}(-l) = q^{(\alpha_i|\alpha_j)}\Omega_{\psi_i}(-l - 1)\Omega_{\psi_j}(-k) - \Omega_{\psi_i}(-k)\Omega_{\psi_j}(-l - 1)$

$= \tilde{\alpha}\left(q^{(\alpha_i|\alpha_j)}x_{i,k}^-x_{j,l+1}^- - x_{j,l+1}^-x_{i,k}^-ight)$

and

$\tilde{\alpha}\left(q^{(\alpha_i|\alpha_j)}\gamma\Omega_{\psi_i}(m)x_{i,n+1}^- - \Omega_{\psi_i}(m+1)x_{i,n}^-ight) = q^{(\alpha_i|\alpha_j)}\gamma\Omega_{\psi_i}(-n - 1)x_{j,m,-n}^- - \Omega_{\psi_i}(-n)x_{j,m,-n-1}^- = (q^{(\alpha_i|\alpha_j)}\gamma - 1)\delta_{i,j}\delta_{m,-n+1} + \gamma x_{j,-m}^-\Omega_{\psi_i}(-n - 1) - q^{(\alpha_i|\alpha_j)}x_{j,-m-1}^-\Omega_{\psi_j}(-n)

= \tilde{\alpha}\left((q^{(\alpha_i|\alpha_j)}\gamma - 1)\delta_{i,j}\delta_{m,-n-1} + \gamma x_{i,n+1}^-\Omega_{\psi_i}(m) - q^{(\alpha_i|\alpha_j)}x_{i,n}^-\Omega_{\psi_j}(m+1)\right)$

$\square$
Lemma 6.0.5. $\mathcal{N}_q^-$ is a left $K_q$-module and

$$\mathcal{N}_q^- \cong K_q/\sum_{i=1}^{N} \sum_{k \in \mathbb{Z}} K_q\Omega_{\psi_i}(k).$$

Proof. Proposition 5.0.3 implies that $\mathcal{N}_q^-$ is a left $K_q$-module. We have an induced left $K_q$-module epimorphism from $K_q$ to $\mathcal{N}_q^-$ which sends 1 to 1. Since the $\Omega_{\psi_i}(k)$ annihilates 1 for all $k$ and $1 \leq i \leq N$, we get an induced left $K_q$-module epimorphism

$$K_q/\sum_{i=1}^{N} \sum_{k \in \mathbb{Z}} K_q\Omega_{\psi_i}(k) \twoheadrightarrow \mathcal{N}_q^-.$$

Let $C$ denote the subalgebra of $K_q$ generated by $x_{i,m}, \gamma^{\pm 1/2}$. Then we have a surjective homomorphism

$$C \twoheadrightarrow K_q/\sum_{i=1}^{N} \sum_{k \in \mathbb{Z}} K_q\Omega_{\psi_i}(k)$$

The composition $\eta \circ \mu$ is surjective and since $\mathcal{N}_q^-$ is defined by generators $x_{i,n}, 1 \leq i \leq N, \gamma^{\pm 1/2}$ and relations (3.13), we get an induced map $\nu : \mathcal{N}_q^- \to C$ splitting the surjective map $\eta \circ \mu$. Since the composition $\nu \circ \eta \circ \mu$ is the identity, we get that $\eta \circ \mu$ is an isomorphism and thus $q$ is an isomorphism. \qed

Proposition 6.0.6. There is a unique symmetric form $(\ , \ )$ defined on $\mathcal{N}_q^-$ satisfying

$$(x_{i,m}a, b) = (a, \Omega_{\psi_i}(-m)b), \quad (1, 1) = 1.$$

Proof. Using the anti-automorphism $\bar{\alpha}$ we can make $M = \text{Hom}(\mathcal{N}_q^-, \mathbb{F}(q^{1/2}))$ into a left $K_q$-module by defining

$$(x_{i,m}f)(a) = f(\Omega_{\psi_i}(-m)a), \quad (\Omega_{\psi_i}(m)f)(a) = f(x_{i,-m}a),$$

$$(\gamma^{\pm 1/2}f)(a) = f(\gamma^{\pm 1/2}a).$$

for $a \in \mathcal{N}_q^-$ and $f \in M$.

Consider the element $\beta_0 \in M$ satisfying $\beta_0(1) = 1$ and

$$\beta_0 \left( \sum_{i=1}^{N} \sum_{k \in \mathbb{Z}} x_{i,m}K_q \right) = 0.$$ 

Then $\Omega_{\psi_i}(m)\beta_0 = 0$ for any $m \in \mathbb{Z}, 1 \leq i \leq N$, we get an induced homomorphism of $K_q$-modules

$$\tilde{\beta} : \mathcal{N}_q^- \cong K_q/\sum_{i=1}^{N} \sum_{m \in \mathbb{Z}} K_q\Omega_{\psi_i}(m) \to M,$$

where $\tilde{\beta}(1) = \beta_0$. Define the bilinear form $(\ , \ ) : \mathcal{N}_q^- \times \mathcal{N}_q^- \to \mathbb{F}(q^{1/2})$ by

$$(a, b) = (\tilde{\beta}(a))(b).$$

This form satisfies $(1, 1) = 1$ and

$$(x_{i,m}a, b) = (a, \Omega_{\psi_i}(-m)b), \quad (\Omega_{\psi_i}(m)a, b) = (a, x_{i,-m}b),$$

$$(\gamma^{\pm 1/2}a, b) = (a, \gamma^{\pm 1/2}b).$$

Since $\mathcal{N}_q^-$ is generated by $x_{i,m}$ and $\gamma^{\pm 1/2}$ we get that the form is the unique form satisfying these three conditions. The form is symmetric since the form defined by $(a, b)' = (b, a)$ also satisfies the above conditions. \qed
7. Simplicity of $\mathcal{N}_q^-$ as a $\mathcal{K}_q$-module

We will show that $\mathcal{N}_q^-$ is simple as a module over $\mathcal{K}_q$.

**Lemma 7.0.7.** Let $P \in \mathcal{N}_q^-$. If $\Omega_{\psi_i}(s)P = 0$ for all $s \in \mathbb{Z}$ and all $1 \leq i \leq N$, then $P$ is a constant multiple of 1.

**Proof.** We may assume without loss of generality that $P$ is a homogeneous element, say $P \in (\mathcal{N}_q^-)_{\lambda-\xi}$. We assume that $\xi \neq 0$. Then $\xi = \sum_{i=1}^{N} n_i \alpha_i + m \delta$, $n_i \geq 0$, $\sum n_i^2 \neq 0$, $m \in \mathbb{Z}$. Set $|\xi| = n = \sum_i n_i$. We shall prove the lemma by induction on $|\xi|$.

Suppose $|\xi| = 1$. Then $P = x_{i,m}^-$ for some $i$ and

$$\Omega_{\psi_i}(s)(x_{i,m}^-) = \delta_{i,j} \delta_{s,-m} \gamma^s + \sum_{r' \geq 0} g_{i,j,q-r'}(r)x_{i,m+r'}^- \Omega_{\psi_j}(s-r') \gamma^{r'}1$$

$$= \delta_{i,j} \delta_{s,-m} \gamma^s$$

Hence $\Omega_{\psi_i}(-m)(P) \neq 0$ unless $P = 0$.

Suppose $|\xi| > 1$. We have by hypothesis $\Omega_{\psi_i}(l)(P) = 0$ for any $l \in \mathbb{Z}$ and all $1 \leq i \leq N$. Then we use (5.18) so that for all $k$, $m \in \mathbb{Z}$, and $1 \leq i,j \leq N$ we get

$$\Omega_{\psi_i}(k)\Omega_{\psi_j}(m)(P) = \sum_{r \geq 0} g_{i,j}(r) \gamma^{2r} \Omega_{\psi_j}(r + m) \Omega_{\psi_i}(k - r)(P) = 0.$$

Hence by the induction hypothesis $\Omega_{\psi_i}(m)(P) = 0$ as $\Omega_{\psi_i}(m)(P) \in (\mathcal{N}_q^-)_{\lambda-\xi+1}$. Then $[x_{i,m}^+, P] = 0$ by (5.4).

Consider the imaginary Verma module $M'_q(\lambda)$ with $\lambda(c) = 0$ and choose $\lambda$ such that $\lambda(h_i) \neq 0$ for some $h_i \in \mathfrak{h}$. Then $\hat{M}_q(\lambda)$ is the unique irreducible quotient of $M'_q(\lambda)$ and $v = Pu\lambda$ is a nonzero element of the module $\hat{M}_q(\lambda)$.

Thus

$$x_{i,s}^+ v = [x_{i,s}^+, P] u\lambda + Px_{i,s}^+ u\lambda = 0$$

for all $s \in \mathbb{Z}$ and all $1 \leq i \leq N$.

Consider $V = \mathcal{N}_q^- v \subset \hat{M}_q(\lambda)$. Then $V$ is a nonzero proper submodule of $\hat{M}_q(\lambda)$ which is a contradiction by Theorem 4.2.1. This completes the proof. □

Lemma 7.0.7 implies immediately the following result.

**Theorem 7.0.8.** The algebra $\mathcal{N}_q^-$ is simple as a $\mathcal{K}_q$-module.

**Corollary 7.0.9.** The form $(\ , \ )$ defined in Proposition 6.0.6 is non-degenerate.

**Proof.** By Proposition 6.0.6 the radical of the form $(\ , \ )$ is a $\mathcal{K}_q$-submodule of $\mathcal{N}_q^-$ and since $(1, 1) = 1$, the radical must be zero. □

We remark that in [Kas91], Kashiwara introduced the algebra $\mathcal{B}_q$ and showed that $U_q^-(\mathfrak{g})$ is a simple $\mathcal{B}_q$-module. This in turn played an important role in showing the existence of crystal base for $U_q^-(\mathfrak{g})$, hence the standard Verma module. We expect to show in a future publication that the Kashiwara algebra $\mathcal{K}_q$ will play a similar role in constructing a crystal-like base for the reduced imaginary Verma module.
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