Prior Distributions for the Bradley-Terry Model of Paired Comparisons

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Abstract: The Bradley-Terry model assigns probabilities for the outcome of paired comparison experiments based on strength parameters associated with the objects being compared. We consider different proposed choices of prior parameter distributions for Bayesian inference of the strength parameters based on the paired comparison results. We evaluate them according to four desiderata motivated by the use of inferred Bradley-Terry parameters to rate teams on the basis of outcomes of a set of games: invariance under interchange of teams, invariance under interchange of winning and losing, normalizability and invariance under elimination of teams. We consider various proposals which fail to satisfy one or more of these desiderata, and illustrate two proposals which satisfy them. Both are one-parameter independent distributions for the logarithms of the team strengths: 1) Gaussian and 2) Type III generalized logistic.

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1. Background

1.1. Paired Comparison Experiments and the Bradley-Terry Model

A paired comparison experiment is a set of binary comparisons between pairs out of a set of $t$ objects. The Bradley-Terry model (Zermelo, 1929; Bradley and Terry, 1952) assigns to each object $i$ ($i = 1, \ldots, t$) a strength parameter $\pi_i$, and defines

$$\theta_{ij} = \frac{\pi_i}{\pi_i + \pi_j}$$

as the probability that object $i$ will be preferred in any given comparison with object $j$. Note that $\theta_{ji} = 1 - \theta_{ij}$. If $n_{ij}$ the number of comparisons between $i$ and $j$, the probability of any particular set of outcomes $D$ which includes object
\( i \) being chosen over \( j \) \( w_{ij} \) times is

\[
p(D|\{\theta_{ij}\}) = p(D|\{\pi_i\}) = \prod_{i=1}^{t} \prod_{j=i+1}^{t} \theta_{ij}^{w_{ij}} (1 - \theta_{ij})^{n_{ij} - w_{ij}} \quad (1.2)
\]

The model has been used in a number of contexts, ranging from taste tests between different foods to games between chess players. In the context of the present paper, we are interested in sporting competitions, so we will henceforth refer to the objects as “teams” and the comparisons as “games”. \( w_{ij} \) is thus the number of games won by team \( i \) against team \( j \), and \( n_{ij} = w_{ij} + w_{ji} \) is the number of games between them.

### 1.2. Bayesian Approach

A typical problem is to make inferences about the strengths \( \{\pi_i\} \), or equivalently the log-strengths \( \{\lambda_i\} \), given the results \( D \). Under a Bayesian approach, in terms of the vector \( \pi \) of team strengths \( \{\pi_i| i = 1, \ldots, t\} \), the posterior probability distribution will be, up to a \( \pi \)-independent normalization constant,

\[
f_{\pi|D}(\pi|D, I) \propto p(D|\{\pi_i\}) f_{\pi}(\pi|I) \quad (1.3)
\]

We are concerned with choices of the prior distribution \( f_{\pi}(\pi|I) \), with a given choice represented symbolically by \( I \).

It is useful to define \( \lambda_i = \ln \pi_i \) and note that

\[
\ln \frac{\theta_{ij}}{1 - \theta_{ij}} = \lambda_i - \lambda_j =: \gamma_{ij} \quad (1.4)
\]

and

\[
\theta_{ij} = (1 + e^{-\gamma_{ij}})^{-1}, \quad (1 - \theta_{ij}) = (1 + e^{\gamma_{ij}})^{-1} \quad (1.5)
\]

Since the parameters are continuous, the probability density functions transform as follows:

\[
f_{\lambda_i}(\lambda_i) = e^{\lambda_i} f_{\pi_i}(e^{\lambda_i}) \quad (1.6)
\]

\[
f_{\pi_i}(\pi_i) = \frac{1}{\pi_i} f_{\lambda_i}(\ln \pi_i) \quad (1.7)
\]

and likewise

\[
f_{\gamma_{ij}}(\gamma_{ij}) = (1 + e^{-\gamma_{ij}})^{-1}(1 + e^{\gamma_{ij}})^{-1} f_{\theta_{ij}}([1 + e^{-\gamma_{ij}}]^{-1}) \quad (1.8)
\]

\[
f_{\theta_{ij}}(\theta_{ij}) = \theta_{ij}^{-1}(1 - \theta_{ij})^{-1} f_{\Gamma_{ij}}(- \ln[\theta_{ij}^{-1} - 1]) \quad (1.9)
\]

Note that the \( t \) strengths \( \{\pi_i\} \) are only relevant in their use to determine the probabilities \( \{\theta_{ij}\} \) (of which \( t - 1 \) are independent), so we consider two probability distributions \( f_{\pi}(\pi|I_1) \) and \( f_{\pi}(\pi|I_2) \) equivalent if they produce the same marginalized distribution for the \( \{\theta_{ij}\} \).
Definition 1.1. Let $\gamma$ represent a $t - 1$-dimensional vector of linearly independent combinations of the log-odds-ratios $\{\gamma_{ij}\}$ from which all $(t - 1)$ can be constructed according to

$$\gamma_{ij} = \sum_{\alpha=1}^{t-1} C_{ij,\alpha} \gamma_{\alpha}$$  (1.10)

Possible choices are

$$\gamma_{12}, \gamma_{23}, \ldots, \gamma_{(t-1),t}$$  (1.11)

or

$$\gamma_{1t}, \gamma_{2t}, \ldots, \gamma_{(t-1)t}$$  (1.12)

or

$$\frac{1}{\sqrt{2}} \gamma_{12}, \frac{1}{\sqrt{6}} (\gamma_{13} + \gamma_{23}), \ldots, \frac{1}{\sqrt{t(t-1)}} [\gamma_{12} + \gamma_{23} + \ldots - (t-1)\gamma_{(t-1),t}]$$  (1.13)

The advantage of working with the $\{\gamma_{ij}\}$ is that we need not specify which basis we are using for $\gamma$ because the Jacobian determinants for transformations between different bases are constant.

Definition 1.2. Two probability distributions are equivalent, $f_{\Pi}(\pi | I_1) \cong f_{\Pi}(\pi | I_2)$ (or $f_{\Lambda}(\lambda | I_1) \cong f_{\Lambda}(\lambda | I_2)$) if and only if $f_{\Gamma}(\gamma | I_1) = f_{\Gamma}(\gamma | I_2)$.

Lemma 1.1. A sufficient condition for $f_{\Pi}(\pi | I_1) \cong f_{\Pi}(\pi | I_2)$ is that there exists a scalar function $C(\pi)$ such that the transformation

$$\pi' = \pi C(\pi)$$  (1.14)

converts the probability density $f_{\Pi}(\pi | I_1)$ into $f_{\Pi}(\pi | I_2)$, i.e.,

$$f_{\Pi}(\pi' | I_1) = \frac{f_{\Pi}(\pi | I_1)}{\det \left\{ \frac{\partial \pi_i}{\partial \pi_j} \right\}} = f_{\Pi}(\pi' | I_2)$$  (1.15)

Proof. The transformation leaves $\theta_{ij}$ unchanged

$$\theta'_{ij} = \frac{\pi'_{i} \pi'_{j}}{\pi'_{i} + \pi'_{j}} = \frac{\pi_{i} C(\pi)}{\pi_{i} C(\pi) + \pi_{j} C(\pi)} = \frac{\pi_{i}}{\pi_{i} + \pi_{j}} = \theta_{ij}$$  (1.16)

and therefore $\gamma'_{ij} = \gamma_{ij}$ and the transformation $\gamma \rightarrow \gamma'$ leaves $f_{\Gamma}(\gamma | I)$ unchanged, and

$$f_{\Gamma}(\gamma | I_2) = f_{\Gamma}(\gamma | I_2) = f_{\Gamma}(\gamma' | I_1) = f_{\Gamma}(\gamma | I_1)$$  (1.17)
1.3. Motivation and Desiderata

The primary interest motivating this work is design of rating systems to evaluate teams based on the outcome of games between them. To that end, prior information which distinguishes between the teams is inappropriate, as it would be considered “unfair” to build such information into the rating system. We are interested in rating systems which obey as many as possibly of the following desiderata.

**Desideratum 1.1.** Invariance under interchange of teams. A transformation \( \pi \rightarrow \pi' \) which, for some \( i \) and \( j \), obeys \( \pi'_i = \pi_j, \pi'_j = \pi_i, \pi'_k = \pi_k \) for all other \( k \), should transform \( f_\Pi(\pi|I_1) \) into an equivalent distribution \( f_\Pi(\pi|I_2) \cong f_\Pi(\pi|I_1) \).

**Desideratum 1.2.** Invariance under interchange of winning and losing. The transformation \( \forall i : \pi_i \rightarrow \pi'_i = \frac{1}{\pi_i} \), which corresponds to \( \Lambda' = -\Lambda \), \( \forall i, j : \theta'_{ij} = 1 - \theta_{ij}, \) and \( \gamma' = -\gamma \), should transform \( f_\Pi(\pi|I_1) \) into an equivalent distribution \( f_\Pi(\pi|I_2) \cong f_\Pi(\pi|I_1) \). A distribution obeying this desideratum will satisfy \( f_\Gamma(\gamma|I) = f_\Gamma(-\gamma|I_1) \).

**Desideratum 1.3.** Normalizability. \( f_\Gamma(\gamma|I) \) should be a proper prior, which can be normalized to \( \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^{t-1} \Gamma f_\Gamma(\gamma|I) = 1 \).

**Desideratum 1.4.** Invariant under elimination of teams. This desideratum assumes that a given principle can be used to generate prior distributions for any number of teams. Let \( t > 2 \), and define \( \pi \) to be the vector of \( t \) strengths, and \( \pi' \) to be the \((t - 1)\)-element vector with \( \pi'_i = \pi_i \) for \( i = 0, \ldots, t - 1 \). Suppose the principle generates priors \( f_\Pi(\pi'|I_{t-1}) \) when there are \( t - 1 \) teams and \( f_\Pi(\pi|I_t) = f_{\Pi|I_t}(\pi', \pi_t|I_t) \) when there are \( t \). The prior associated with \( I_{t-1} \) should be equivalent to that associated with \( I_t \), marginalized over \( \pi_t \), i.e.

\[
f_\Pi(\pi'|I_{t-1}) \cong \int_0^\infty d\pi_t \; f_{\Pi|I_t}(\pi', \pi_t|I_t) = \int_0^\infty d\pi_t \; f_\Pi(\pi|I_t) \tag{1.18}
\]

1.4. Comparison via Prior Predictive Distribution

A convenient way to quantify the effects of a prior, and thus to compare different priors, is to construct the prior predictive distribution

\[
p(D|\pi, I) = \int_0^\infty \cdots \int_0^\infty d^t \pi p(D|\pi, n) f_\Pi(\pi|I) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^{t-1} \Gamma p(D|\gamma, n) f_\Gamma(\gamma|I)
\]

where the second equality holds because the \( t - 1 \) log-rating differences in \( \gamma \) determine the sampling distribution \( p(D|\pi, n) \).

**Lemma 1.2.** For any \( n \), \( p(D|n, I) = p(D|n, I') \) is a necessary condition for \( f_\Pi(\pi|I) \cong f_\Pi(\pi|I') \).

**Proof.** Assume \( f_\Pi(\pi|I) \cong f_\Pi(\pi|I') \). Then, by definition \( f_\Gamma(\gamma|I) = f_\Gamma(\gamma|I') \). By (1.19), \( p(D|n, I) \) can be constructed from \( f_\Gamma(\gamma|I) \) and \( p(D|\gamma, n) \), and therefore \( f_\Gamma(\gamma|I) = f_\Gamma(\gamma|I') \) implies \( p(D|n, I) = p(D|n, I') \).

\( \square \)
Proof. Since leaves the range of the integration variables unchanged. Because the change of variables \( \gamma \rightarrow \gamma' \) as in the statement of desideratum 1.1 \((\pi'_i = \pi_j, \pi'_j = \pi_i, \pi'_k = \pi_k \text{ for all } k \neq i, j)\), we can see \( p(D'|\gamma', I) = p(D'|\pi', n, I) = p(D|\pi', n', I) \) where as usual \( \gamma'_ij = \ln(\pi'_i/\pi'_j) \). If desideratum 1.1 holds, we have \( f_{\gamma'}(\gamma'|I) = f_{\gamma}(\gamma|I) \) and so

\[
p(D'|n, I) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^{q-1}_{\gamma'} p(D'|\gamma', n) f_{\gamma'}(\gamma'|I) \]
\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^{q-1}_{\gamma} p(D|\gamma, n) f_{\gamma}(\gamma|I) \]
\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^{q-1}_{\gamma} p(D|\gamma, n) f_{\gamma}(\gamma|I) = p(D|n, I) \tag{1.20}
\]

because the change of variables \( \gamma \rightarrow \gamma' \) has unit Jacobian determinant and leaves the range of the integration variables unchanged. \( \square \)

Lemma 1.4. For any \( n \), desideratum 1.2 implies \( p(D|n, I) = p(D'|n, I) \) where \( w'_ij = n_{ij} - w_{ij} \).

Proof. Since \( w'_ij = n_{ij} - w_{ij} = w_{ji}, n'_{ij} = w'_ij + w'_ji = w_{ji} + w_{ij} = n_{ij} \). The rest of the proof proceeds as with lemma 1.3, but with the appropriate definitions of \( D \rightarrow D' \) and \( \pi \rightarrow \pi' \). \( \square \)

Lemma 1.5. For any \( n \), desideratum 1.3 implies \( p(D|n, I) > 0 \) if \( D \) is a set of results consistent with \( n \).

Proof. Since \( p(D|\theta, I) > 0 \) for all \( \theta \) with \( 0 < \theta_{ij} < 1 \), and \( p(D|\gamma, I) = p(D|\theta, I) \), we have \( p(D|\gamma, I) > 0 \) for all \( \gamma \) with \( -\infty < \gamma_{ij} < \infty \). Since \( f_{\gamma}(\gamma|I) \geq 1 \) and \( \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^{q-1}_{\gamma} f_{\gamma}(\gamma|I) = 1 \), we must have

\[
p(D|n, I) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^{q-1}_{\gamma} p(D|\gamma, n) f_{\gamma}(\gamma|I) \geq 0 \tag{1.21}
\]

\( \square \)

Lemma 1.6. Given \( t \) teams and a \( n \) with \( n_{it} = 0 \), desideratum 1.4 implies \( p(D|n, I_t) = p(D|n, I_{t-1}) \) if \( D \) is a set of results consistent with \( n \).

Proof. If \( n_{it} = 0 \), \( \pi_i \) is irrelevant to the sampling distribution, and \( p(D|\pi, n) = p(D|\pi', n) \) where \( \pi' \) is the \((t-1)\)-element vector with \( \pi'_i = \pi_i \) for \( i = 0, \ldots, t-1 \).
as in the statement of desideratum 1.4. Thus
\[
p(D|n, I_t) = \int_0^\infty \cdots \int_0^\infty \, d^p p(D|\pi, n) f_{\Pi}(\pi|I_t)
= \int_0^\infty \cdots \int_0^\infty \, d^{-1} \pi' p(D|\pi', n) \int_0^\infty \, d\pi f_{\Pi}(\pi|I_t) (1.22)
= \int_0^\infty \cdots \int_0^\infty \, d^{-1} \pi' p(D|\pi', n) f_{\Pi}(\pi'|I_t)
\]

Desideratum 1.4 says that \( f_{\Pi'}(\pi'|I_t) \equiv f_{\Pi'}(\pi'|I_{t-1}) \), and lemma 1.2 states that this implies \( p(D|n, I_t) = p(D|n, I_{t-1}) \).

\[\Box\]

2. Choice of Prior Distribution

2.1. General Considerations for Special Cases

2.1.1. Two Teams

When \( t = 2 \), there is only one independent probability \( \theta_{12} = \frac{\pi_1}{\pi_1 + \pi_2} \), so any distribution \( f_{\Pi}(\pi_1, \pi_2) \) reduces to a function \( f_{\theta_{12}}(\theta_{12}) \) via marginalization

\[
f_{\Gamma_{12}}(\gamma_{12}) = \int_{-\infty}^\infty 
\]

where the transformation (1.6) means

\[
f_{\Lambda}(\lambda_1, \lambda_2) = e^{\lambda_1 + \lambda_2} f_{\Pi}(e^{\lambda_1}, e^{\lambda_2})
\]

and (1.9) means

\[
f_{\Theta_{12}}(\theta_{12}) = \theta_{12}^{-1} (1 - \theta_{12})^{-1} f_{\Gamma_{12}}(-\ln[\theta_{12}^{-1} - 1])
\]

For the case of two teams, desiderata 1.1 and 1.2 are equivalent, as both transformations reduce to \( \theta_{12} \to 1 - \theta_{12} \), or equivalently \( \gamma_{12} \to -\gamma_{12} \). They will be satisfied if and only if \( f_{\Gamma_{12}}(\gamma_{12}) \) is an even function, or equivalently if \( f_{\Theta_{12}}(\theta_{12}) = f_{\Theta_{12}}(1 - \theta_{12}) \). Desideratum 1.3 will be satisfied if and only if

\[
\int_0^1 \, d\theta_{12} f_{\Theta_{12}}(\theta_{12}) < \infty
\]

or equivalently

\[
\int_{-\infty}^\infty \, d\gamma_{12} f_{\Gamma_{12}}(\gamma_{12}) < \infty
\]

Suppose \( f_{\Theta_{12}}(\theta_{12}) \) belongs to the family of beta distributions (which is the conjugate prior family for the likelihood (1.2)),

\[
f_{\Theta_{12}}(\theta_{12}) \propto \theta_{12}^{\alpha-1} (1 - \theta_{12})^{\beta-1}
\]
then \( f_{\Gamma_{12}}(\gamma_{12}) \) is a generalized logistic distribution of Type IV (Prentice, 1976; Nassar and Elmasry, 2012)

\[
 f_{\Gamma_{12}}(\gamma_{12}) \propto (1 + e^{-\gamma_{12}})^{-\alpha}(1 + e^{\gamma_{12}})^{-\beta} \tag{2.7}
\]

Included in this family are

1. The Haldane prior (Haldane, 1932; Jeffreys, 1961) \( \alpha = \beta = 0 \), which is uniform \( \gamma_{12} \). This improper prior corresponds to “total ignorance”.
2. The Jeffreys prior (Jeffreys, 1961, 1946) \( \alpha = \beta = \frac{1}{2} \).
3. The Bayes-Laplace prior \( \alpha = \beta = 1 \), which is uniform in \( \theta_{12} \). This is also the maximum entropy prior, if we assume a measure uniform in \( \theta_{12} \).

For the beta family, desiderata 1.1 and 1.2 will be satisfied if \( \alpha = \beta \). Desideratum 1.3 will be satisfied if \( \alpha, \beta > 0 \).

With \( t = 2 \), the prior predictive probability for a set of results which include \( w_{12} \) wins for team 1 and \( w_{21} = n_{12} - w_{12} \) wins for team 2 will be

\[
 p(D|n_{12}) = \int_0^1 d\theta_{12} \theta_{12}^{w_{12}}(1 - \theta_{12})^{w_{21}} f_{\Theta_{12}}(\theta_{12}) \tag{2.8}
\]

If \( f_{\Theta_{12}}(\theta_{12}) \) is in the Beta family (2.6), it will be

\[
 p(D|n_{12}, I_{\alpha,\beta}) = \frac{\Gamma(\alpha + \beta) \Gamma(\alpha + w_{12}) \Gamma(\beta + w_{21})}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\alpha + \beta + n_{12})} \tag{2.9}
\]

In particular, for the Bayes-Laplace prior,

\[
 p(D|n_{12}, I_{1,1}) = \frac{\Gamma(1 + w_{12}) \Gamma(1 + w_{21})}{\Gamma(2 + n_{12})} = \left( \frac{n_{12} + 1}{w_{12}} \right)^{-1} \tag{2.10}
\]

For the Jeffreys prior,

\[
 p(D|n_{12}, I_{1/2,1/2}) = \frac{\Gamma(\frac{1}{2} + w_{12}) \Gamma(\frac{1}{2} + w_{21})}{\pi \Gamma(1 + n_{12})} = \frac{(2w_{12} - 1)!!(2w_{21} - 1)!!}{2^{n_{12}} n_{12}!} \tag{2.11}
\]

Viewing the Haldane prior as a limiting case, \( p(D|n_{12}, I_{1/2,1/2}) = 0 \) unless \( w_{12} = n_{12} \) or \( w_{21} = n_{12} \). The prior predictive probabilities for \( w_{12} = n_{12} \) and \( w_{21} = n_{12} \) depend on the order in which the limits \( \alpha \to 0 \) and \( \beta \to 0 \) are taken.

2.1.2. Three Teams

In the case \( t = 3 \), the \( \{\theta_{ij}\} \) are related by \( \theta_{ji} = 1 - \theta_{ij} \) as well as

\[
 \theta_{13}^{-1} - 1 = (\theta_{12}^{-1} - 1)(\theta_{23}^{-1} - 1) \tag{2.12}
\]

Although each \( \theta_{ij} \) is confined to the finite range \([0, 1]\), the surface defined by (2.12) is curved, which makes it difficult to display the two-dimensional probability distribution \( f_{\Theta}(\Theta) \) while preserving its intuitive interpretation. On the other hand, in terms of the \( \{\gamma_{ij}\} \) the constraints are \( \gamma_{ji} = -\gamma_{ij} \) and

\[
 \gamma_{13} = \gamma_{12} + \gamma_{23} \tag{2.13}
\]
An especially convenient set of coördinates for displaying probability distributions \( f_\Gamma(\gamma) \) is

\[
x = \frac{1}{\sqrt{3}} (\gamma_{12} + \gamma_{13}), \quad y = \gamma_{23},
\]

which can be inverted to give

\[
\gamma_{12} = \frac{\sqrt{3}}{2} x - \frac{1}{2} y, \quad \gamma_{23} = y, \quad \gamma_{13} = \frac{\sqrt{3}}{2} x + \frac{1}{2} y
\]

These two-dimensional coördinates on the space of log-odds-ratios \( \gamma \) are also two of the three coördinates on the space of strengths \( \lambda \), according to

\[
x = \frac{1}{\sqrt{3}} (2\lambda_1 - \lambda_2 - \lambda_3), \quad y = \lambda_2 - \lambda_3, \quad z = \frac{\sqrt{2}}{3} (\lambda_1 + \lambda_2 + \lambda_3)
\]

In these coördinates, to go from a distribution \( f_\Lambda(\lambda) \) to \( f_\Gamma(\gamma) \) one simply converts \( f_\Lambda(\lambda) \) into \( f_{XYZ}(x, y, z) \) (which involves a constant Jacobian determinant) and then marginalizes over \( z \). An example of such a plot is shown in Fig. 1.

### 2.2. Evaluation of Prior Distributions

We now consider several families of prior distributions which have been proposed, and evaluate them according to the desiderata in Sec. 1.3.

#### 2.2.1. Haldane Prior

Perhaps the simplest prior that can be chosen is the improper prior

\[
f_\Lambda(\lambda | I_0) = \text{constant},
\]

uniform in all of the log-strengths, which is the generalization of the Haldane prior considered in Sec. 2.1.1. Then the marginalized prior probability distribution for the log-odds-ratios is \( f_\Gamma(\gamma | I_0) = \text{constant} \). This prior obviously satisfies desiderata 1.1 and 1.2, as well as desideratum 1.4. Of course, since the prior \( f_\Gamma(\gamma | I_0) \) is improper, it violates desideratum 1.3. The mode of the posterior \( f_{\Lambda|D}(\lambda | D, I_0) \) will be the maximum likelihood solution, and the posterior will be normalizable under the conditions given by Ford (1957) for the existence of the maximum likelihood solution.

Note that while the improper prior \( f_\Pi(\pi | I_0') = \text{constant} \) produces a different prior on the individual strengths, since

\[
f_\Lambda(\lambda | I_0') \propto \exp \left( \sum_{i=1}^{t} \lambda_i \right)
\]

the two are equivalent, \( f_\Lambda(\lambda | I_0') \cong f_\Lambda(\lambda | I_0) \), because \( f_\Lambda(\lambda | I_0') \) depends only on the sum of the log-strengths.
Fig 1. Contour plot of the prior probability distribution $f_{\Gamma}(\gamma|I_{S3})$ arising from the Maximum entropy prescription of Sec. 2.2.2 when $t = 3$. The coordinates for the plot are the $x$ and $y$ defined in (2.14), which determine all of the $\{\gamma_{ij}\}$ and thus the predicted probabilities. The orthogonal direction $z = \sqrt{\frac{2}{3}} (\lambda_1 + \lambda_2 + \lambda_3)$ defined in (2.16) is irrelevant to the predictions of the model.

2.2.2. Maximum Entropy

The next prior of interest is one which maximizes the Shannon entropy. For the case $t = 2$, we defined the entropy as

$$S_2 = - \int_0^1 d\theta_{12} f_{\Theta_{12}}(\theta_{12}) \ln f_{\Theta_{12}}(\theta_{12})$$

which made the maximum entropy prior $f_{\Theta_{12}}(\theta_{12}|I_{S2}) = \text{const}$. In order for the entropy of a continuous distribution $f(x)$ to be invariant, it needs to be defined with a measure $\mu(x)$ which transforms as a density under reparametrization. Thus we can write

$$S_2 = - \int_0^1 d\theta_{12} f_{\Theta_{12}}(\theta_{12}) \ln f_{\Theta_{12}}(\theta_{12}) \frac{f_{\Theta_{12}}(\theta_{12})}{\mu_{\Theta_{12}}(\theta_{12})} = - \int_0^1 d\gamma_{12} f_{\Gamma_{12}}(\gamma_{12}) \ln f_{\Gamma_{12}}(\gamma_{12}) \frac{f_{\Gamma_{12}}(\gamma_{12})}{\mu_{\Gamma_{12}}(\gamma_{12})}$$

(2.20)
where we have assumed that \( \mu_{\Theta_{12}}(\theta_{12}) = \text{constant} \) and thus, using (1.8) to transform the measure,

\[
\mu_{\Gamma_{12}}(\gamma_{12}) \propto (1 + e^{-\gamma_{12}})^{-1}(1 + e^{\gamma_{12}})^{-1} \tag{2.21}
\]

If we maximize the entropy of a continuous distribution with normalization as the only constraint, the probability density is proportional to the measure, so

\[
f_{\Gamma_{12}}(\gamma_{12}|I_{S^2}) \propto (1 + e^{-\gamma_{12}})^{-1}(1 + e^{\gamma_{12}})^{-1} \tag{2.22}
\]

which is indeed of the form (2.7) with \( \alpha = \beta = 1 \).

For general \( t \), we could define by analogy a measure uniform in the \( \{\theta_{ij}\} \),

\[
\mu_{\Theta}(\theta) = \text{constant}, \quad \text{and then minimize the entropy}
\]

\[
S_t = - \int_0^1 \cdots \int_0^1 d^{(t-1)/2} \theta f_{\Theta}(\theta) \ln f_{\Theta}(\theta) \tag{2.23}
\]

subject to the constraints that the probability density vanishes unless the arguments satisfy

\[
\theta^{-1}_{ij} = (\theta^{-1}_{ik} - 1)(\theta^{-1}_{kj} - 1) \quad i = 1, \ldots, t; \quad j = i + 1, \ldots, t; \quad k = i + 1, \ldots, j - 1 \tag{2.24}
\]

It is equivalent, and more straightforward, to confine the distribution to the constraint surface by writing it in terms of the \( t - 1 \) unique \( \gamma_\alpha \) parameters:

\[
S'_t = - \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty d^{t-1} \gamma f_{\Gamma}(\gamma) \ln \frac{f_{\Gamma}(\gamma)}{\mu_{\Gamma}(\gamma)} \tag{2.25}
\]

where the measure is

\[
\mu_{\Gamma}(\gamma) \propto \prod_{i=1}^t \prod_{j=i+1}^t (1 + e^{-\gamma_{ij}})^{-1}(1 + e^{\gamma_{ij}})^{-1} \tag{2.26}
\]

As before, the maximum entropy distribution is \( f_{\Gamma}(\gamma|I_{S^t}) \propto \mu_{\Gamma}(\gamma) \), or

\[
f_{\Gamma}(\gamma|I_{S^t}) \propto \prod_{i=1}^t \prod_{j=i+1}^t \left[ 1 + \exp \left( - \sum_{\alpha=1}^{t-1} C_{ij,\alpha} \gamma_\alpha \right) \right]^{-1} \left[ 1 + \exp \left( \sum_{\beta=1}^{t-1} C_{ij,\beta} \gamma_\beta \right) \right]^{-1} \tag{2.27}
\]

We can see from the form of (2.26) that desiderata 1.1 and 1.2 are satisfied. It is also easy to see that \( f_{\Gamma}(\gamma|I_{S^t}) \) is exponentially suppressed as any linear combination of the \( \{\gamma_\alpha\} \) goes to infinity, and therefore desideratum 1.3. For example, as \( \gamma_\alpha \to \infty \),

\[
f_{\Gamma}(\gamma|I_{S^t}) \to \prod_{i=1}^t \prod_{j=i+1}^t e^{-|C_{ij,\alpha}|\gamma_\alpha} = \exp \left( -\gamma_\alpha \sum_{i=1}^t \sum_{j=i+1}^t |C_{ij,\alpha}| \right) \tag{2.28}
\]
However, we can see that desideratum 1.4 is not satisfied by considering the case \( t = 3 \) and showing that the marginal distribution

\[
\tilde{f}_{r_{23}}(\gamma_{23}|I_{S3}) \neq \tilde{f}_{r_{23}}(\gamma_{23}|I_{S2})
\]

(2.29)

Explicitly, using the coordinates (2.14), in which \( y = \gamma_{23} \),

\[
f_{r}(x, y|I_{S3}) \propto (1 + e^{\sqrt{3}x}e^{-y/2})^{-1}(1 + e^{-\sqrt{3}x}e^{y/2})^{-1}(1 + e^{-y})^{-1}(1 + e^{y})^{-1} \\
\times (1 + e^{-\sqrt{3}x}e^{y/2})^{-1}(1 + e^{\sqrt{3}x}e^{-y/2})^{-1},
\]

(2.30)

which is plotted in Fig. 1. The marginalization integral can be done by partial fractions to give

\[
f_{r_{23}}(\gamma_{23}|I_{S3}) = \int_{-\infty}^{\infty} f_{r}(x, \gamma_{23}|I_{S3}) \, dx \propto e^{\gamma_{23}[2(1 - e^{\gamma_{23}}) + \gamma_{23}(1 + e^{\gamma_{23}})]} \\
(\gamma_{23} - 1)^{1}(1 + e^{-\gamma_{23}})(1 + e^{\gamma_{23}})
\]

(2.31)

which is manifestly different from

\[
f_{r_{23}}(\gamma_{23}|I_{S2}) \propto \frac{1}{(1 + e^{-\gamma_{23}})(1 + e^{\gamma_{23}})}
\]

(2.32)

by more than just a normalization constant.

2.2.3. Jeffreys Prior

The Jeffreys prior construction (Jeffreys, 1946) can be carried out using the likelihood (1.2), to produce a prior

\[
f_{\Lambda}(\lambda|I_{J}) \propto \sqrt{\mathcal{I}(\lambda)}
\]

(2.33)

where \( \mathcal{I}(\lambda) \) is the Fisher information associated with the likelihood. Since the likelihood is written in terms of the \( \{\theta_{ij}\} \), or equivalently in terms of the \( \{\gamma_{ij}\} \), it is simpler to generate

\[
f_{r}(\gamma|I_{J}) \propto \sqrt{\mathcal{I}(\gamma)}
\]

(2.34)

directly. If we write the \( t - 1 \) independent elements of \( \gamma \) as

\[
\gamma_{ij} = \sum_{\alpha=1}^{t-1} C_{ij,\alpha} \gamma_{\alpha}
\]

(1.10)

we can write

\[
\mathcal{I}(\gamma) = -E \left[ \det \left\{ \frac{\partial^2 \ell(\gamma; D)}{\partial \gamma_{\alpha} \partial \gamma_{\beta}} \right\} \right]
\]

(2.35)

where

\[
\ell(\gamma; D) = \ln p_{D|\gamma}(D|\gamma) = \sum_{i=1}^{t} \sum_{j=i+1}^{t} (w_{ij} \gamma_{ij} - n_{ij} \ln[1 + e^{\gamma_{ij}}])
\]

(2.36)
The linear form of (1.10) allows us to determine the Fisher information matrix for the \( \{ \gamma_\alpha \} \) from that for the \( \{ \gamma_{ij} \} \) as

\[
\frac{\partial^2 \ell(\gamma; D)}{\partial \gamma_\alpha \partial \gamma_\beta} = \sum_{i=1}^{t} \sum_{j=i+1}^{t} C_{ij,\alpha} C_{ij,\beta} \frac{\partial^2 \ell(\gamma; D)}{\partial \gamma_{ij} \partial \gamma_{i'j'}}
\]

(2.37)

Since the log-likelihood is relatively simple written in terms of the \( \{ \gamma_{ij} \} \), we can write

\[
\frac{\partial \ell(\gamma; D)}{\partial \gamma_{ij}} = w_{ij} - n_{ij} e^{\gamma_{ij}} (1 + e^{\gamma_{ij}})^{-1} = w_{ij} - n_{ij} (1 + e^{-\gamma_{ij}})^{-1}
\]

(2.38)

and

\[
\frac{\partial^2 \ell(\gamma; D)}{\partial \gamma_{ij} \partial \gamma_{i'j'}} = -\delta_{ii'}\delta_{jj'} n_{ij} e^{-\gamma_{ij}} (1 + e^{-\gamma_{ij}})^{-2} = -\delta_{ii'}\delta_{jj'} n_{ij} (1 + e^{-\gamma_{ij}})^{-1} (1 + e^{\gamma_{ij}})^{-1}
\]

(2.39)

so

\[
-\frac{\partial^2 \ell(\gamma; D)}{\partial \gamma_\alpha \partial \gamma_\beta} = \sum_{i=1}^{t} \sum_{j=i+1}^{t} C_{ij,\alpha} C_{ij,\beta} n_{ij} (1 + e^{-\gamma_{ij}})^{-1} (1 + e^{\gamma_{ij}})^{-1}
\]

(2.40)

We can see by inspection of (2.40) that the Jeffreys prior always satisfies desideratum 1.2. We can verify that in the case \( t = 2 \), for which there is only one independent \( \gamma_\alpha \), the Jeffreys prior becomes

\[
f_{\Gamma}(\gamma_{12}|I_{22}) \propto (1 + e^{-\gamma_{12}})^{-1/2} (1 + e^{\gamma_{12}})^{-1/2}
\]

(2.41)

which is of the form (2.7) with \( \alpha = \beta = \frac{1}{2} \) as before.

If we write

\[
-\frac{\partial^2 \ell(\gamma; D)}{\partial \gamma_\alpha \partial \gamma_\beta} = \sum_{i=1}^{t} \sum_{j=i+1}^{t} C_{ij,\alpha} C_{ij,\beta} n_{ij} (e^{\gamma_{ij}/2} + e^{-\gamma_{ij}/2})^{-2}
\]

(2.42)

Note that if we make the specific choice \( \gamma_\alpha = \gamma_{\alpha,\alpha+1} \), we have

\[
\gamma_{ij} = \sum_{\alpha=i}^{j-1} \gamma_\alpha
\]

(2.43)

which makes

\[
C_{ij,\alpha} = \begin{cases} 1 & i \leq \alpha \leq j - 1 \\ 0 & \text{otherwise} \end{cases}
\]

(2.44)

and then the Fisher information matrix (2.40) is

\[
-\frac{\partial^2 \ell(\gamma; D)}{\partial \gamma_\alpha \partial \gamma_\beta} = \sum_{\alpha=i}^{i+j} \sum_{\beta=i}^{i+j} n_{ij} (e^{\gamma_{ij}/2} + e^{-\gamma_{ij}/2})^{-2}
\]

(2.45)
For $t > 2$, the Fisher information matrix (2.40) depends on the number of games $n_{ij}$ to be played between each pair of teams. However, in order to satisfy desideratum 1.1, we need to have the same $n_{ij}$ for each pair of teams, in which case this $n_{ij}$ becomes a constant which can be absorbed into the normalization, and the prescription becomes unique.

By explicitly examining $t = 3$, we will show that the Jeffreys prior with all $\{n_{ij}\}$ equal fails to satisfy desideratum 1.4. In the case $t = 3$, the vector of independent log-odds-ratios $\gamma$ is two-dimensional, and the elements of the Fisher information matrix are

\[
\begin{align*}
-\frac{\partial^2 \ell(\gamma; D)}{\partial \gamma_1 \partial \gamma_1} &= M_{12} + M_{13} \quad (2.46a) \\
-\frac{\partial^2 \ell(\gamma; D)}{\partial \gamma_1 \partial \gamma_2} &= M_{13} \quad (2.46b) \\
-\frac{\partial^2 \ell(\gamma; D)}{\partial \gamma_2 \partial \gamma_2} &= M_{13} + M_{23} \quad (2.46c)
\end{align*}
\]

where

\[
M_{ij} := n_{ij}(e^{\gamma_{ij}/2} + e^{-\gamma_{ij}/2})^{-2} =: n_{ij}m_{ij} \quad (2.47)
\]

The Fisher information is the determinant of this matrix

\[
I(\gamma) = (M_{12} + M_{13})(M_{13} + M_{23}) - M_{13}^2 = M_{12}M_{13} + M_{12}M_{23} + M_{13}M_{23} \quad (2.48)
\]

so the Jeffreys prior is

\[
f_r(\gamma|I_{33}) \propto \sqrt{M_{12}M_{13} + M_{12}M_{23} + M_{13}M_{23}} \propto \sqrt{m_{12}m_{13} + m_{12}m_{23} + m_{13}m_{23}} \quad (2.49)
\]

We can show that the Jeffreys prior fails to satisfy desideratum 1.4 by using the posterior predictive distribution and Lemma 1.6. Suppose $n_{12} = 2$ and $n_{13} = 0$. Then (2.11) implies

\[
p(D|n, I_{12}) = \begin{cases} 
0.125 & w_{12} = 1 \\
0.375 & w_{12} = 0 \text{ or } 2
\end{cases} \quad (2.50)
\]

We can evaluate $p(D|n, I_{13})$ numerically, and find

\[
p(D|n, I_{13}) \approx \begin{cases} 
0.108 & w_{12} = 1 \\
0.392 & w_{12} = 0 \text{ or } 2
\end{cases} \quad (2.51)
\]

showing explicitly that $p(D|n, I_{13}) \neq p(D|n, I_{12})$ and therefore desideratum 1.4 is violated.

\footnote{It can be seen to satisfy 1.3 by noting that as a linear combination of the $\{\gamma_n\}$ goes to infinity, at most one of the $\{m_{ij}\}$ can remain finite; the other two will be exponentially suppressed, and thus each term in the square root in (2.49) will go to zero exponentially.}
2.2.4. Dirichlet Distribution

Chen and Smith (1984) discuss Bayesian estimators for the Bradley-Terry model starting with a Dirichlet distribution

$$f_{\Pi}(\pi|\Delta) = \frac{\Gamma\left(\sum_{i=1}^{t} \alpha_i\right)}{\prod_{i=1}^{t} \Gamma(\alpha_i)} \left(\prod_{i=1}^{t} \pi_i^{\alpha_i-1}\right) \delta \left(1 - \sum_{i=1}^{t} \pi_i\right)$$  (2.52)

where $\delta(x)$ is the Dirac delta function. In particular, they note that the marginal distribution for any of the $\{\theta_{ij}\}$ is a beta distribution with parameters $\alpha = \alpha_i$ and $\beta = \alpha_j$:

$$f_{\theta_{ij}}(\theta_{ij}|\Delta) = \frac{\Gamma(\alpha_i + \alpha_j)}{\Gamma(\alpha_i)\Gamma(\alpha_j)} \theta_{ij}^{\alpha_i-1}(1 - \theta_{ij})^{\alpha_j-1}$$  (2.53)

The Dirichlet prior satisfies desideratum 1.3 as long as all of the $\{\alpha_i\}$ are positive; it also satisfies desideratum 1.1 if all of the parameters $\{\alpha_i\}$ are equal to the same value $\alpha$. Although the delta function enforcing the constraint $\sum_{i=1}^{t} \pi_i = 1$ means that the different $\{\pi_i\}$ are not independently distributed under $\Delta$, we can see that desideratum 1.4 is satisfied by defining a change of variables

$$\pi'_i = \frac{\pi_i}{1 - \pi_t} \quad i = 1, \ldots, t - 1$$  (2.54)

under which the probability density (2.52) becomes

$$f_{\Pi}(\pi'|\Delta) = \frac{\Gamma\left(\sum_{i=1}^{t-1} \alpha_i\right)}{\prod_{i=1}^{t-1} \Gamma(\alpha_i)} \left(\prod_{i=1}^{t-1} \pi_i^{\alpha_i-1}\right) \delta \left(1 - \sum_{i=1}^{t-1} \pi_i\right) (1 - \pi_t) \sum_{i=1}^{t-1} \pi_i^{\alpha_i-1}$$  (2.55)

which, when we marginalize over $\pi_t$, gives

$$f_{\Pi}(\pi'|\Delta) = \frac{\Gamma\left(\sum_{i=1}^{t-1} \alpha_i\right)}{\prod_{i=1}^{t-1} \Gamma(\alpha_i)} \left(\prod_{i=1}^{t-1} \pi_i^{\alpha_i-1}\right) \delta \left(1 - \sum_{i=1}^{t-1} \pi_i\right)$$  (2.56)

which is a Dirichlet distribution with the same parameters $\{\alpha_1, \ldots, \alpha_{t-1}\}$.

However, desideratum 1.2 is not satisfied, which we can see explicitly by considering $t = 3$ and assuming $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, so that

$$f_{\Pi}(\pi_1, \pi_2, \pi_3|\Delta) \propto (\pi_1 \pi_2 \pi_3)^{\alpha-1} \delta(\pi_1 + \pi_2 + \pi_3 - 1)$$  (2.57)

or equivalently

$$f_{A}(\lambda_1, \lambda_2, \lambda_3|\Delta) \propto e^{\alpha(\lambda_1 + \lambda_2 + \lambda_3)} \delta \left(e^{\lambda_1} + e^{\lambda_2} + e^{\lambda_3} - 1\right)$$  (2.58)

If we write

$$\pi_1 + \pi_2 + \pi_3 = (\pi_1 \pi_2 \pi_3)^{1/3} \left(\left(\frac{\pi_1}{\pi_2 \pi_3}\right)^{1/3} + \left(\frac{\pi_2}{\pi_1 \pi_3}\right)^{1/3} + \left(\frac{\pi_3}{\pi_1 \pi_2}\right)^{1/3}\right)$$

$$= e^{\lambda_1} + e^{\lambda_2} + e^{\lambda_3} = e^{\lambda_1 + \lambda_2 + \lambda_3} \left(e^{-\gamma_1 + \gamma_2} + e^{-\gamma_2 + \gamma_3} + e^{-\gamma_1 + \gamma_3}\right)$$  (2.59)
we can see
\[ \delta \left( e^{\lambda_1} + e^{\lambda_2} + e^{\lambda_3} - 1 \right) \propto \delta \left( \lambda_1 + \lambda_2 + \lambda_3 + 3 \ln \left[ e^{\gamma_{12} + \gamma_{13}} + e^{-\gamma_{12} + \gamma_{23}} + e^{-\gamma_{13} - \gamma_{23}} \right] \right) \]

and so marginalizing over the combination \( \lambda_1 + \lambda_2 + \lambda_3 \) leaves a prior distribution
\[ f_{\Gamma}(\gamma|I_{D3}) \propto \left( e^{\gamma_{12} + \gamma_{13}} + e^{-\gamma_{12} + \gamma_{23}} + e^{-\gamma_{13} - \gamma_{23}} \right)^{-3\alpha} \quad (2.61) \]

We see that, for non-zero \( \alpha \), \( f_{\Gamma}(\gamma|I_{D3}) \neq f_{\Gamma}(-\gamma|I_{D3}) \). This is illustrated explicitly for \( \alpha = \frac{1}{2} \) in Fig. 2. Another way of expressing this asymmetry if we’d started with an “anti-Dirichlet” prior, i.e., requiring \( \{1/\pi_i\} \) to be Dirichlet distributed, we’d have got the distribution
\[ f_{\Gamma}(\gamma|I_{D3}') \propto \left( e^{-\gamma_{12} + \gamma_{13}} + e^{\gamma_{12} - \gamma_{23}} + e^{\gamma_{13} + \gamma_{23}} \right)^{-3\alpha} \quad (2.62) \]
2.2.5. Conjugate Prior Families

Davidson and Solomon (1973) construct a conjugate prior family of the form

\[ f_\Pi(\pi|I_C) \propto \left( \prod_{i=1}^{t} \pi_i^{v_i^0} \right) \left( \prod_{i=1}^{t} \prod_{j=i+1}^{t} (\pi_i + \pi_j)^{-n_{ij}^0} \right) \delta \left( 1 - \sum_{i=1}^{t} \pi_i \right) \]  

(2.63)

where \( n_{ij}^0 = 0 \). In order to satisfy desideratum 1.1 (interchange of teams), we require that \( v_i^0 = v^0 \) and \( n_{ij}^0 = n^0 \) for all \( i \) and \( j \neq i \), so the prior becomes

\[ f_\Pi(\pi|I_C) \propto \left( \prod_{i=1}^{t} \pi_i \right)^{v^0} \left( \prod_{i=1}^{t} \prod_{j=i+1}^{t} (\pi_i + \pi_j)^{-n^0} \right) \delta \left( 1 - \sum_{i=1}^{t} \pi_i \right) \].  

(2.64)

Note that Davidson and Solomon (1973) motivate \( v_i^0 \) and \( n_{ij}^0 \) as coming from a matrix \( w_{ij}^0 \) (with \( w_{ii}^0 = 0 \)) via \( v_i^0 = \sum_{j=1}^{t} w_{ij}^0 \) and \( n_{ij}^0 = w_{ij}^0 + w_{ji}^0 \), which means that in particular \( \sum_{i=1}^{t} \sum_{j=1}^{t} n_{ij}^0 = 2 \sum_{i=1}^{t} v_{ij}^0 \), which in the case of single \( n^0 \) and \( v^0 \) parameters would require \( v^0 = (t-1)n^0/2 \). We will not impose that condition at this stage, however.

To convert \( f_\Pi(\pi|I_C) \) into \( f_\Lambda(\lambda|I_C) \), we note that

\[ \sum_{i=1}^{t} \pi_i = \left( \prod_{k=1}^{t} \pi_k \right)^{1/t} \sum_{i=1}^{t} \left( \prod_{j=1}^{t} \pi_j \right)^{1/t} = \exp \left( \frac{1}{t} \sum_{k=1}^{t} \lambda_k \right) \sum_{i=1}^{t} \exp \left( \frac{1}{t} \sum_{j=1}^{t} \gamma_{ij} \right) \]  

(2.65)

and since, for \( t > 0 \),

\[ \delta \left( 1 - e^{u/t} \right) = \frac{\delta(u)}{e^{u/t}} = t \delta(u) \],

(2.66)

\[ \delta \left( 1 - \sum_{i=1}^{t} \pi_i \right) = t \delta \left( \sum_{k=1}^{t} \lambda_k + t \ln \sum_{i=1}^{t} \exp \left( \frac{1}{t} \sum_{j=1}^{t} \gamma_{ij} \right) \right) \].  

(2.67)

Similarly, we can write

\[ \prod_{i=1}^{t} \prod_{j=i+1}^{t} (\pi_i + \pi_j) = \frac{1}{\prod_{k=1}^{t} \prod_{i=1}^{t} 2 \pi_k} \left( \prod_{k=1}^{t} \prod_{i=1}^{t} (\pi_i + \pi_j) \right) \]

\[ = \frac{1}{2^t \prod_{k=1}^{t} \pi_k} \left( \prod_{k=1}^{t} \pi_k \right)^{t} \left[ \prod_{i=1}^{t} \prod_{j=1}^{t} \left( 1 + \frac{\pi_i}{\pi_j} \right) \right] \]  

(2.68)

\[ = \frac{1}{2^t} \exp \left( \left( \frac{t}{2} - 1 \right) \sum_{k=1}^{t} \lambda_k \right) \left( \prod_{i=1}^{t} \prod_{j=1}^{t} \left[ 1 + e^{\gamma_{ij}} \right] \right)^{1/2} \]
which makes the prior (recalling (1.6))

\[
f_\Lambda(\lambda|I_C) \propto \exp \left[ v^0 + 1 - n^0 \left( \frac{t}{2} - 1 \right) \sum_{k=1}^{t} \lambda_k \right] \left( \prod_{i=1}^{t} \prod_{j=1}^{t} [1 + e^{\gamma_{ij}}] \right)^{-n^0/2} \times \delta \left( \sum_{k=1}^{t} \lambda_k + t \ln \sum_{i=1}^{t} \exp \left( \frac{1}{t} \sum_{j=1}^{t} \gamma_{ij} \right) \right)
\]

(2.69)

When we marginalize over \( \sum_{k=1}^{t} \lambda_k \), the Dirac delta function sets \( \exp \left( \sum_{k=1}^{t} \lambda_k \right) \) to \( \left( \sum_{i=1}^{t} \exp \left( \frac{1}{t} \sum_{j=1}^{t} \gamma_{ij} \right) \right)^{-t} \) and the prior becomes

\[
f_\Gamma(\gamma|I_C) \propto \left( \sum_{i=1}^{t} \exp \left( \frac{1}{t} \sum_{j=1}^{t} \gamma_{ij} \right) \right)^{-t \left( v^0 + 1 - n^0 \left[ \frac{t}{2} - 1 \right] \right)} \left( \prod_{i=1}^{t} \prod_{j=1}^{t} [1 + e^{\gamma_{ij}}] \right)^{-n^0/2}
\]

(2.70)

The quantity in large square brackets is in general not symmetric under the transformation \( \gamma \to -\gamma \), while the remainder of the expression is. This means, to satisfy desideratum 1.2 (win-loss inversion), we should have \( v^0 = (t-2)n^0/2 - 1 \). Note that this is not the same as the condition \( v^0 = (t-1)n^0/2 \) implied by Davidson and Solomon (1973)’s conditions on \( v^0 \) and \( n^0 \). The precise form of their restriction, however, comes from the fact they wrote down a “natural” conjugate prior family for \( f_\Pi(\pi|I_C) \); if they had started with \( f_\Lambda(\lambda|I_C) \), they would have ended up with, in the present notation, \( v^0 + 1 = (t-1)n^0/2 \), which would also not satisfy desideratum 1.2. However, if we hadn’t imposed the constraint \( \sum_{i=1}^{t} \pi_i = 1 \) (which is clearly not invariant under \( \pi_i \to 1/\pi_i \)) in the first place, the marginalization over \( \sum_{k=1}^{t} \lambda_k \), would have rendered \( v^0 \) irrelevant and left us with

\[
f_\Gamma(\gamma|I_C) \propto \left( \prod_{i=1}^{t} \prod_{j=1}^{t} [1 + e^{\gamma_{ij}}] \right)^{-n^0/2}
\]

(2.71)

in any event. We therefore take (2.71) as the form of the conjugate prior \( f_\Gamma(\gamma|I_C) \), and note that if \( n^0 = 2 \), this reduces to the maximum entropy distribution (2.27) [see also (2.26)] considered in 2.2.2.

We can show that (2.71) violates desideratum 1.4 for any \( n^0 \in (0, \infty) \) by considering the prior predictive distribution and invoking Lemma 1.6. First, note that for \( t = 2 \), (2.71) becomes

\[
f_\Gamma(\gamma|I_C_2) \propto \left( [1 + e^{\gamma_{12}}] [1 + e^{\gamma_{21}}] \right)^{-n^0/2}
\]

(2.72)

which is just the beta/generalized logistic prior (2.7) with \( \alpha = \beta = n^0/2 \). Therefore, if we define \( D \) to be a set of results for which \( w_{12} = 1 = w_{21} \), and let
Prior predictive probability for $w_{12} = w_{21}$

**Fig 3.** Prior predictive probability $p(D|n, IC)$ for the conjugate prior family of Davidson and Solomon (1973), for a sequence of results in which $w_{12} = w_{21} = 1$ and $n_{12} = 2$. In order to satisfy desiderata 1.1 and 1.2, we require $n_{ij} = n^0$ and $v^0 = (t - 2)n^0/2 - 1$, which produces the prior (2.71). For any finite positive $n^0$, the prior predictive probability from the three-team prior is larger than that of the two-team prior, indicating that desideratum 1.4 is not satisfied. (For $n^0 = 0$ both versions reduce to the Haldane prior and $p(D|n, IC) = 0$. As $n^0 \to \infty$, they become delta functions at $\gamma = 0$ and $p(D|n, IC) \to 0.25$.)

$n_{12} = 2$ and $n_{i3} = 0$, (2.9) implies

$$p(D|n, I_{C2}) = \frac{\Gamma(n^0)\Gamma(n^0/2 + 1)\Gamma(n^0/2 + 1)}{\Gamma(n^0/2)\Gamma(n^0/2 + 1)\Gamma(n^0 + 2)} = \frac{1}{4(1 + \frac{1}{n_0})}$$

We evaluate $p(D|n, I_{C3})$ numerically for a range of $n^0$ values, plotted in Fig. 3, and find that for any $0 < n^0 < \infty$, $p(D|n, I_{C3}) > p(D|n, I_{C2})$.

### 2.2.6. Multivariate Gaussian Distribution

Leonard (1977) proposed a multivariate Gaussian prior on the $\{\lambda_i\}$ of the form

$$f_\lambda(\lambda|I_G) \propto \exp \left( -\frac{1}{2} \sum_{i=1}^{t} \sum_{j=1}^{t} (\lambda_i - \mu_i)(\sigma^{-2}_{ij})(\lambda_j - \mu_j) \right)$$

where $\{\sigma^{-2}_{ij}\}$ are the elements of the inverse of a positive semi-definite covariance matrix with elements $\{\sigma^2_{ij}\}$.

Desideratum 1.3 will be satisfied if the covariance matrix $\{\sigma^2\}_{ij}$ is positive definite, so that the prior is normalizable.

Desideratum 1.1 requires that all of the $\{\mu_i\}$ have the same value $\mu$, all of the variances $\{\sigma^2\}_{ii}$ have the same value $\sigma^2$, and all of the cross-covariances
\{[\sigma^2]_{ij}|i \neq j\} have the same value \(\rho \sigma^2\). In order for the matrix \{[\sigma^2]_{ij}\} to be positive definite we must have \(\sigma^2 > 0\) and \(-\frac{1}{t^2} < \rho < 1\). These conditions guarantee that desideratum 1.2 is satisfied.

Since the distribution \(f_\Gamma(\gamma|I_G)\) is unchanged by the transformation \(\lambda_i \rightarrow \lambda_i - \mu\), we can assume without loss of generality that \(\mu = 0\). We are thus left with \(\rho\) and \(\sigma^2\) as the adjustable parameters of the distribution. However, if we make the transformation \(\lambda_i \rightarrow \lambda_i + a \sum_{j=1}^{t} \lambda_j\), which leaves \(f_\Gamma(\gamma|I_G)\) unchanged, \(f_\Lambda(\lambda|I_G)\) becomes a multivariate Gaussian with covariance matrix

\[
\sigma^2 \left[ \rho + \delta_{ij}(1 - \rho) + a(2 + at)(t + 1 - \rho) \right]
\]

(2.75)

if \(a = -\frac{1}{t} \left( 1 + \sqrt{\frac{1-\rho}{1+\rho-1}} \right)\) or \(a = -\frac{1}{t} \left( 1 - \sqrt{\frac{1-\rho}{1+\rho-1}} \right)\) the covariance matrix becomes diagonal, with a variance equal to \((1 - \rho)\sigma^2\). Either value for \(a\) is guaranteed to be real by the conditions on \(\rho\) which ensure a positive definite correlation matrix. Thus \(f_\Gamma(\gamma|I_G)\) is equivalent to a product of independent Gaussian distributions for each \(\lambda_i\). For simplicity, we refer to the variance of each of these distributions as \(\sigma^2\) rather than \((1 - \rho)\sigma^2\).

Since the prior \(f_\Gamma(\gamma|I_G)\) is equivalent to independent distributions on the \{\lambda_i\}, it is invariant under elimination of teams, and satisfies desideratum 1.4.

2.2.7. Separable Priors

Thus far, the only prior considered to satisfy all four of our desiderata is the Gaussian prior which is equivalent to

\[
f_\Lambda(\lambda|I_G) = \frac{1}{(2\pi\sigma^2)^{t/2}} \exp \left( -\sum_{i=1}^{t} \frac{\lambda_i^2}{2\sigma^2} \right)
\]

(2.76)

This is an example of a separable prior of the form

\[
f_\Lambda(\lambda|I) = \prod_{i=1}^{t} f_{\Lambda_i}(\lambda_i|I)
\]

(2.77)

A prior of this form, which assigns prior pdfs to the strengths (or log-strengths) of the teams is guaranteed by its construction to satisfy desideratum 1.4 (invariance under team-elimination). It will also satisfy desideratum 1.1 (interchange) if the distributions for the different \{\lambda_i\} are identical \((f_{\Lambda_i}(\lambda_i|I) = f_{\Lambda_i}(\lambda_i|I))\), desideratum 1.2 (win-loss interchange) if the distribution \(f_\Lambda(\lambda|I)\) is even \((f_\Lambda(-\lambda|I) = f_\Lambda(\lambda|I))\) and desideratum 1.3 (normalizable) if \(f_\Lambda(-\lambda|I)\) is a proper prior.

2.2.8. Beta-Separable Priors

One family of separable priors can be constructed by defining

\[
\zeta_i = \frac{\pi_i}{1 + \pi_i} = (1 + e^{-\lambda_i})^{-1}
\]

(2.78)
and assuming that $\zeta_i$ obeys a Beta distribution. Explicitly,

$$f_{Z_i}(\zeta_i|I_B) \propto \zeta_i^{\alpha_i-1}(1-\zeta_i)^{\beta_i-1}$$

(2.79)

Since $\zeta_i = \text{logistic}(\lambda_i)$, just as $\theta_{ij} = \text{logistic}(\gamma_{ij})$, the prior on $\lambda_i$ is a generalized logistic distribution of Type IV (Prentice, 1976; Nassar and Elmasry, 2012):

$$f_{\Lambda_i}(\lambda_i|I_B) \propto (1 + e^{-\lambda_i})^{-\alpha_i}(1 + e^{\lambda_i})^{-\beta_i}$$

(2.80)

To enforce desideratum 1.2, we require $\alpha_i = \beta_i$, and for desideratum 1.1, we require that all $\alpha_i$ and $\beta_i$ parameters are the same, $\alpha_i = \beta_i = \eta$, which makes the prior a generalized logistic distribution of Type III.

$$f_{\Lambda_i}(\lambda_i|I_B) \propto (1 + e^{-\lambda_i})^{-\eta}(1 + e^{\lambda_i})^{-\eta}$$

(2.81)

An appealing feature of Beta-separable prior is that its functional form is similar to the likelihood

$$p(D|\lambda) = \prod_{i=1}^t \prod_{j=i+1}^t (1 + e^{-\gamma_{ij}})^{-w_{ij}}(1 + e^{\gamma_{ij}})^{n_{ij} - w_{ij}}$$

(2.82)

In particular, the posterior is proportional to the likelihood function which would arise by adding to the actual game results a set of “fictitious games” corresponding to $\eta$ wins and $\eta$ losses for each actual team against a fictitious team assumed to have a strength of 1. So any method for obtaining maximum likelihood estimates such as that of Ford (1957) could be adapted to obtaining maximum a priori estimates with this prior. This method, with $\eta = \frac{1}{2}$, has been used by Buttler (1993) to ensure regularity of the estimates of Bradley-Terry strengths. Another “obvious” choice is $\eta = 1$, which is equivalent to a uniform prior on $\zeta_i$. (The Haldane prior is $\eta = 0$.)

### 3. Conclusions

We have considered various families of prior distributions for team strengths in the Bradley-Terry model. Motivated by the application of a Bayesian Bradley-Terry model to rate teams based only on their game results, we have evaluated these priors according to the desiderata of invariance under interchange of teams (1.1), interchange of winning and losing (1.2) and elimination of irrelevant teams from the model (1.4), as well as normalizability (1.3). A Haldane-like prior of complete ignorance is not normalizable (violation of 1.3), although it satisfies the other desiderata. A prior based on maximum entropy arguments, as well as one from the conjugate family of Davidson and Solomon (1973) which is required to obey the other desiderata, will depend on the number of teams for which it was constructed (violation of 1.4). The same is true for the Jeffreys prior. A Dirichlet prior on the team strengths (Chen and Smith, 1984) can be made to satisfy the other desiderata, but will not be invariant under interchange of the definitions of winning and losing (violation of 1.2). Distributions can
be constructed which satisfy the desiderata by imposing independent priors on the strengths of all of the teams. In particular, a multivariate Gaussian in the log-strengths (Leonard, 1977) which satisfies the desiderata is equivalent to identical independent Gaussian priors on each of the log-strengths. Another simple family of separable prior distributions imposes independent generalized logistic distributions on the log-strengths. In each of these last two cases, a single parameter remains. Phelan and Whelan (2017) consider the relationship between the two, and propose a hierarchical method to estimate these parameters rather than assuming values for them.

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