Constraints on brane-world inflation from the CMB power spectrum: revisited

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Abstract. We analyze the Randal Sundrum brane-world inflation scenario in the context of the latest CMB constraints from Planck. We summarize constraints on the most popular classes of models and explore some more realistic inflaton effective potentials. The constraint on standard inflationary parameters changes in the brane-world scenario. We confirm that in general the brane-world scenario increases the tensor-to-scalar ratio, thus making this paradigm less consistent with the Planck constraints. Indeed, when BICEP2/Keck constraints are included, all monomial potentials in the brane-world scenario become disfavored compared to the standard scenario. However, for natural inflation the brane-world scenario could fit the constraints better due to larger allowed values of $e$-foldings $N$ before the end of inflation in the brane-world.

Keywords: CMBR theory, cosmology with extra dimensions

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1 Introduction

Inflation is required to solve various problems in standard big-bang cosmology. It also provides the primordial density fluctuations required for the formation of large scale structure [1, 2]. A number of inflationary models have been suggested (cf. [1]) since the first proposed inflationary paradigm. In the last few years observations by the Wilkinson Microwave Anisotropy Probe (WMAP) [3] and the Planck mission [4] have constrained the inflationary paradigm. In particular, the Planck15 inflation analysis [4] has ruled out many of the frequently adopted models.

The three most important parameters when constraining inflation models are the spectral index ($n_s$), the tensor-to-scalar ratio ($r$) and the running of the spectral index ($\alpha = dn_s/d\ln k$). It is now clear, for example, that models like the quadratic or quartic effective inflaton potentials are ruled out based upon the parameters deduced from the CMB power spectrum. Constraints on brane-world inflation have also previously been studied by many authors e.g. [4–12] (see also reviews in refs. [13, 14]). Although there are a number of braneworld paradigms, here we consider inflation in the context of the Randal Sundrum brane-world scenario (RS II) [15, 16]. In this paper we reinvestigate a number of inflationary potentials in the context of the newest Planck15 [4] constraints. There have been similar recent studies based upon the Planck15 [12] or the Planck15 plus WMAP polarization [3, 17], and BAO [18–20] data. We also consider the more stringent constraints based upon the combined Planck15 [4] + BICEP2/Keck [21] data, and we consider several monomial potentials that were not explicitly analyzed in previous studies.

In the brane-world scenario, the universe is a sub-manifold embedded in a higher-dimensional spacetime. Physical matter fields are confined to this sub-manifold. However gravity can reside in the higher-dimensional spacetime. This scenario was first proposed [15] to solve the hierarchy problem in the Standard Model of particle physics. The huge difference
between the electroweak and gravity scales can be solved in this scenario because the existence of a large extra dimension brings the scale of gravity down to the weak scale. This helps to eliminate the Planck scale hierarchy with respect to the electroweak scale by generating another difference of scale between the weak scale and the size of the extra dimensions. In particular, it has been shown [15, 16] that the hierarchy problem is solved by introducing non-compact extra dimensions. This is an alternative to the standard Kaluza-Klein compactification scheme. Specifically, the universe is described as a three-brane embedded in a five-dimensional anti-deSitter space $AdS_5$. This scheme guarantees the usual four dimensional spacetime in our 3-brane.

The cosmological evolution of the brane-world can be reduced [14, 22–26] to a generalized Friedmann equation whereby the cosmological expansion for an observer in the three brane is described by:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3} \rho - \frac{K}{a^2} + \frac{\Lambda_4}{3} + \frac{\kappa_5^4}{36} \rho^2 + \frac{\mu}{a^4}. \quad (1.1)$$

Here, $a(t)$ is the usual scale factor at time $t$, while $\rho$ is the energy density of matter in the normal 3 space. $G_N$ is the four dimensional normal gravitational constant and is related to it’s five-dimensional counterpart $\kappa_5$ by,

$$G_N = \kappa_5^2 \lambda/48\pi, \quad (1.2)$$

where $\lambda$ is the intrinsic tension on the brane, $\kappa_5^2 = M_5^{-3}$, and $M_5$ is the five-dimensional Planck mass. The $\Lambda_4$ in the third term is the four dimensional cosmological constant and is related to it’s five dimensional counterpart by,

$$\Lambda_4 = \Lambda = \kappa_5^2 \lambda^2/12 + 3\Lambda_5/4. \quad (1.3)$$

Note, that for $\Lambda_4$ to be close to zero $\Lambda_5$ should be negative.

The standard Friedmannian cosmology does not contain the fourth and the fifth terms of eq. (1.1). The fifth term scales like radiation with a constant $\mu$. It is called the dark radiation. This term derives from the electric part of the five-dimensional Weyl tensor. The coefficient $\mu$ is a constant of integration obtained by integrating the five-dimensional Einstein equations. The magnitude and sign of $\mu$ depend upon the initial conditions. The effects of the dark radiation term have been previously well studied [27, 28].

In this paper, however, we are primarily interested in the fourth term of eq. (1.1). The fourth term arises from the imposition of a junction condition for the scale factor at the surface of the brane. The modified Friedmann equation at high energy where this term dominates, makes the early universe cosmology different from the standard scenario [29]. During the post-reheating, radiation-dominated epoch this term vanishes very quickly as $a^{-8}$. Nevertheless, it can play a significant role in the inflationary era when the universe is dominated by vacuum energy. If the standard inflationary potentials are inserted as part of the $\rho^2$ term in eq. (1.1), there can be a significant differences between the inflation in the projected brane-world inflation and that of standard inflationary cosmology.

# 2 Inflationary paradigm in the brane-world

To analyze of the inflationary paradigm in the brane-world it is useful to simplify the Friedmann equation. To achieve this we first assume that the effect of the dark radiation term can be neglected during the very early time of the inflation epoch. This is justified since the
magnitude of this term is limited to be small \([27, 28]\) from the combined BBN and CMB constraints. We also take the curvature term \(K\) to be zero during inflation along with the cosmological constant term. This leads to:

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3} \rho + \frac{\kappa_4^2}{36} \rho^2 .
\]  

(2.1)

Then writing the equations in terms of the reduced Planck mass \(M_P\) and writing \(\rho\) as the inflation generating potential we obtain:

\[
H^2 = \frac{V(\phi)}{3M_P^2} \left(1 + \frac{V(\phi)}{\rho_0}\right) .
\]  

(2.2)

Here, \(V(\phi)\) is the inflaton potential, \(\phi\) is the usual inflaton field and the variable \(\rho_0\) is defined as:

\[
\rho_0 = \frac{12M_5^6}{M_P^2} .
\]  

(2.3)

Thus, the usual slow roll paradigm \([1]\) is altered due to this change in the evolution equation. It is worth mentioning that this scenario approaches the standard cosmology when, \(V/\rho_0 \ll 1\). In the following sections, we would carry out the calculations in the limit, \(V/\rho_0 \gg 1\). Since the inflaton is confined to the brane, the scalar perturbation obeys a similar evolution to that of the standard cosmology after taking into account \([10, 14]\) the modification of the Hubble expansion. Nevertheless, the gravitational power spectrum and the tensor spectrum change due to the presence of the extra dimension. This can be understood intuitively from the fact that the graviton resides in the extra dimension. Thus, there is an effect of the extra dimension in the tensor perturbation. In particular, the scalar perturbation becomes:

\[
P_s = \frac{9}{4\pi^2} \frac{H^6}{V^{7/2}} .
\]  

(2.4)

The spectral index \(n_s = d \ln P_s / d \ln k\) and the running of the spectral index \(\alpha = d n_s / d \ln k\) also change due to the modified slow roll parameters derived from the modified Hubble expansion \([14]\). The modified slow roll parameters are:

\[
\epsilon = \frac{\ln(H^2) V'}{6H^2}, \quad \eta = \frac{V''}{3H^2} .
\]  

(2.5)

Here, the prime stands for the derivative with respect to \(\phi\) and from now on, we take, \(M_P = 1\).

In standard inflation analytic expressions for \(n_s, r, \) and \(\alpha\) can be derived via simple calculations \([1]\). Extending this to the brane-world is straightforward \([5, 10, 12, 14]\) allowing for the modification of the Hubble parameter eq. (2.2). From this, the spectral index \(n_s\) and running of the spectral index \(\alpha\) become \([12]\),

\[
n_s = 1 - 6\epsilon + 2\eta, \quad \alpha = \frac{V'}{3H}(6\epsilon - 2\eta')
\]  

(2.6)

The most interesting change occurs for the tensor power spectra. Because gravity resides in the higher dimensional manifold, an extra correction term is required. Hence, the tensor power spectrum becomes \([30]\):

\[
P_T = 8 \left(\frac{H}{2\pi}\right)^2 F(x_0)^2 .
\]  

(2.7)
where the extra factor \( F(x_0) \) is written:

\[
F(x) = \left( \sqrt{1 + x^2} - x^2 \ln \left( \frac{1}{x} + \sqrt{1 + \frac{1}{x^2}} \right) \right)^{-1/2},
\]

with, \( x_0 = 2(3H^2/\rho_0)^{1/2} \). For \( x_0 \ll 1 \) this reduces to the standard cosmology. For, \( x_0 \gg 1 \) the correction factor can be approximated by \( \sqrt{3x_0/2} \). The tensor-to-scalar ratio is then just, \( r \equiv P_T/P_s \).

Finally, the number of e-folds before the end of inflation that the observable scales left the horizon can be calculated in the usual way [1] leading to:

\[
N = \int_{\phi_e}^{\phi_0} d\phi \frac{3H^2}{V'} = \int_{\phi_e}^{\phi_0} d\phi \frac{V}{V'} \left( 1 + \frac{V}{\rho_0} \right).
\]

Here, \( \phi_0 \) and \( \phi_e \) represent the field values at the horizon exit of the CMB modes and the end of inflation respectively. There is an independent bound [31] on this number of e-folds in the brane world of \( N \leq 75 \). However, that limit was calculated for a closed universe with a positive cosmological constant which does not re-collapse, while the present work is for a flat universe. Nevertheless, as discussed in [32], the number of e-foldings before the end of inflation that observable scales left the horizon may be larger in the brane-world scenario than in the standard scenario. As such, for illustration we consider \( N = 50, 60 \) and \( 70 \) in the case of the brane-world scenario. We note, however, that a detailed calculation of the number of e-folds in the brane-world scenario is still desired and we hope to address in a future work.

3 Monomial \( V(\phi) \propto \phi^n \) inflation models

Analytic expressions for \( n_s, r, \) and \( \alpha \) based upon monomial inflation in the braneworld can be found via simple calculations [5, 10, 14] as described in the previous section. A comparison of the Planck constraints with predictions for the RSII cosmology for powers \( n = 2/3, 2 \) can be found in [10]. In [12] \( n = 2 \) and \( 4 \) are analyzed along with other potentials. In this study we consider a broader range of monomial potentials \( n = 2/3, 1, 4/3, 2, 3, 4 \). Unlike other recent work [10, 12] we also consider the more stringent constraints from the combined Planck15 + BICEP2/Keck analysis.

3.1 \( V(\phi) \propto \phi \)

We begin our discussion of inflation in brane-world cosmology with the Linear Axion Monodromy potential [33]. Wrapped branes in string compactifications introduce a monodromy that extends the field range of individual closed-string axions to beyond the Planck scale. This leads to a general mechanism for chaotic inflation based upon monodromy-extended closed-string axions. At leading order the effective potential can be approximated:

\[
V(\phi) = \lambda_1 \phi.
\]

We studied this model both in the standard and brane-world scenarios. For the standard inflation scenario one has:

\[
n_s = 1 - \frac{6}{1 + 4N}, \quad r = \frac{16}{1 + 4N}, \quad \alpha = \frac{-24}{(1 + 4N)^2},
\]

where \( N \) is the number of e-folds before the end of inflation as defined by eq. (2.9).
Then taking into consideration the power spectrum measured $P_s(k_0) = 2.196 \times 10^{-9}$ (for the chosen pivot scale at $k_0 = 0.002$ Mpc$^{-1}$), we obtain,

$$\lambda_1[\text{Standard}] = \frac{24\sqrt{2\pi^2} \times 2.196 \times 10^{-9}}{(1 + 4N)^{3/2}} \text{ (GeV}^3).$$  \hspace{1cm} (3.3)

In the brane-world scenario, we deduce:

$$n_s = 1 - \frac{6}{1 + 3N}, r = \frac{24}{1 + 3N}, \alpha = \frac{-18}{(1 + 3N)^2}. \hspace{1cm} (3.4)$$

In this case there is a dependence of $\lambda_1$ on $M_5$. The analytic dependence of $\lambda_1$ is calculated to be,

$$\lambda_1[\text{Brane-world}] = \frac{1.76 \times 10^{-3} \times M_5^3}{(3N + 1)} \text{ (GeV}^3).$$  \hspace{1cm} (3.5)

As can be seen in figure 1, the spectral index is reduced in the brane-world scenario. Although the spectral index $n_s$, decreases with respect to the standard cosmology, there will be always an enhancement of the tensor-to-scalar ratio in the brane-world. This can be seen in equation (3.4).

Figure 2 illustrates the dependence of the running spectral index on the number of $e$-folds of inflation before the end of inflation that the observable scales left the horizon. The magnitude of the running of the spectral index $\alpha$ increases in the brane world. Figure 3 shows the dependence of $\lambda_1$ on $M_5$ for various values of $N$. One can see that the running spectral index varies with the number of $e$-folds before the end of inflation (cf. eq. (3.4)). Also, values for $\lambda_1$ vary significantly within the plotted range for $M_5$. We note that a lower bound to $M_5$ can be fixed by requiring $\rho_0^{1/4} > T_{\text{BBN}}$ at the epoch of big bang nucleosynthesis [12, 27].
However a more stringent range for $M_5$ is determined [12] by fits to the CMB Planck15 contours (cf. figure 1). This constraint is described in detail in section 5.

The effect of the brane-world scenario can even be seen on the scalar power spectrum through the running of the spectral index. The magnitude of the running increases (becomes more negative) in the braneworld scenario. Although the effect is small, there is a discernible difference with respect to the case of standard inflation (cf. equation (3.4)).

### 3.2 $V(\phi) \propto \phi^2$

Next we investigate the popular quadratic potential:

$$V(\phi) = \frac{1}{2} m^2 \phi^2. \quad (3.6)$$

For the standard scenario one can write,

$$n_s = 1 - \frac{4}{1 + 2N}, \quad r = \frac{16}{1 + 2N}, \quad \alpha = \frac{-8}{(1 + 2N)^2}, \quad (3.7)$$

Then again taking the measured power spectrum in consideration as before, we obtain the mass,

$$m[\text{Standard}] = \frac{2.196 \times 10^{-9} \times 12\pi^2}{(2N + 1)^2} \text{ (GeV)}. \quad (3.8)$$

while in the brane-world scenario we obtain,

$$n_s = 1 - \frac{5}{1 + 2N}, \quad r = \frac{24}{1 + 2N}, \quad \alpha = \frac{-10}{(1 + 2N)^2}. \quad (3.9)$$

The mass in the case of brane-world inflation is:

$$m[\text{Brane-world}] = \frac{2.69527 \times 10^{-3} M_5}{(N + \frac{1}{2})^5} \text{ (GeV)}. \quad (3.10)$$

Figure 4 shows the tensor-to-scalar ratio $r$ versus the spectral index $n_s$ as a function of $N$ for the case of $V(\phi) = (1/2)m^2\phi^2$ in both the brane world and standard inflation. These
Figure 4. Tensor-to-scalar ratio $r$ vs. spectral index $n_s$ for the case of $V(\phi) = (1/2)m^2\phi^2$ in the brane-world (black line) compared to that of standard (indigo line) inflation. Contours are the 1 and 2\(\sigma\) confidence limits from various versions of the Planck analysis [4] as labeled.

Figure 5. The running of the spectral index as a function of $N$ for the brane world (red line) and standard inflation (blue line) in the case of $V(\phi) = (1/2)m^2\phi^2$.

Figure 6. The mass term $m$ with as a function of $M_5$ for $N = 50$ (Blue), $N = 60$ (Red) and $N = 70$ (Green) in the case of $V(\phi) = (1/2)m^2\phi^2$.

are compared to the 1 and $2\sigma$ confidence limit contours from the Planck analysis [4]. Here we see again that in brane-world inflation for this potential the $r$ always exceeds that of standard inflation. Hence, although the standard inflation paradigm is marginally allowed at the $2\sigma$ level in the Planck TT,TE,EE+low L analysis [4], the brane-world inflation paradigm with this potential is ruled out at more than $2\sigma$.

Figures 5 illustrates the dependence of the running spectral index on $N$ for the quadratic effective potential. Figure 6 shows the dependence of the mass $m$ on $M_5$ for various values of $N$. One can see that values for $m$ vary linearly within the allowed range for $M_5$. Also, the dependence of the running of spectral index on $N$ is significant.
Figure 7. Tensor-to-scalar ratio $r$ vs. spectral index $n_s$ for the case of $V(\phi) = \lambda_3 \phi^3$ in the brane-world (black line) compared to that of standard (indigo line) inflation. Contours are the 1 and 2$\sigma$ confidence limits from various versions of the Planck analysis [4] as labeled.

3.3 $V(\phi) \propto \phi^3$

The cubic potential has previously been shown [4] to be well outside the allowed parameter space allowed by the Planck15 analysis. Nevertheless, for completeness we summarize the form of the spectral index, the tensor-to-scalar ratio, and the running of the spectral index. In the standard cosmology one obtains,

$$n_s = 1 - \frac{10}{3 + 4N}, \quad r = \frac{48}{3 + 4N}, \quad \alpha = \frac{-40}{(3 + 4N)^2}. \quad (3.11)$$

Again with the same constraint from the measured power spectrum, we obtain,

$$\lambda_3[\text{Standard}] = \frac{8.4931 \times 10^{11}}{(3 + 4N)^{5/2}} \text{ (GeV).} \quad (3.12)$$

While the brane-world scenario we deduce,

$$n_s = 1 - \frac{14}{3 + 5N}, \quad r = \frac{72}{3 + 5N}, \quad \alpha = \frac{-70}{(3 + 5N)^2}. \quad (3.13)$$

In the same manner, we have calculated $\lambda_1$. In this case there is a dependence of $\lambda_3$ on $M_5$. The analytic dependence of $\lambda_3$ is calculated to be,

$$\lambda_3[\text{Brane-world}] = 7.73846 \times 10^{-8} \times \left[\frac{M_5^{0.2}}{(5N + 3)^{2.8}}\right]^{1.8} \text{ (GeV}^{25/9}). \quad (3.14)$$

Figure 7 shows the tensor-to-scalar ratio $r$ versus the spectral index $n_s$ as a function of $N$ for the case of cubic potential. In figure 8, the dependence of $\alpha$ on $N$ are demonstrated for the cubic potential, while figure 9 demonstrates the dependence of $\lambda_3$ on $N$ and $M_5$. 

– 8 –
3.4 $V(\phi) \propto \phi^4$

For the case of the quartic potential $V(\phi) = \lambda_4 \phi^4$, in the standard cosmology we get,

$$n_s = 1 - \frac{6}{3 + 2N}, \quad r = \frac{32}{3 + 2N}, \quad \alpha = \frac{-12}{(3 + 2N)^2}. \quad (3.15)$$

For the quartic potential, we obtain,

$$\lambda_4[\text{Standard}] = \frac{8.225 \times 10^{-10} \pi^2}{(1 + N)^3}. \quad (3.16)$$

In the brane-world scenario we deduce,

$$n_s = 1 - \frac{9}{2 + 3N}, \quad r = \frac{48}{2 + 3N}, \quad \alpha = \frac{-27}{(2 + 3N)^2}. \quad (3.17)$$

Here again we have calculated $\lambda_4$. In this case there is no dependence of $\lambda_4$ on $M_5$. This is expected because in case of the quartic potential the normalization is unit-less (thus independent of $M_5$). This maintains the renormalizability of the potential. Hence, the results are very similar to the standard scenario. The analytic form of $\lambda_4$ is calculated to be,

$$\lambda_4[\text{Brane-world}] = \frac{3\pi^2 \times 2.196 \times 10^{-9}}{8(2 + 3N)^3}. \quad (3.18)$$

As shown in figure 10 there is not much shift in $n_s$ and $r$ between the brane-world scenario and the standard cosmology for this potential. In figure 11, the variation of $\alpha$ is plotted as a function of $N$.

3.5 $V(\phi) \propto \phi^{2/3}$

The $V(\phi) = \lambda_2/3 \phi^{2/3}$ is probably the most realistic effective potential among the simple monomial potentials. This is a lowest order approximation to axion monodromy [34]. It is a string-theory motivated version of natural inflation [35–38]. The analytic continuation on a compact manifold of this class of model makes it a more realistic candidate for the brane-world scenario. Therefore, before analyzing the original natural inflation, we investigate this case. This effective potential avoids the super-Planck scale width of the cosine part of the natural inflation potential [35–38].
Figure 10. Tensor-to-scalar ratio $r$ vs. spectral index $n_s$ for the case of $V(\phi) = \lambda_4 \phi^4$ in the brane-world (black line) compared to that of standard (indigo line) inflation. Contours are the 1 and 2σ confidence limits from various versions of the Planck analysis [4] as labeled.

Figure 11. The running of the spectral index as a function of $N$ for both the brane-world (blue line) and standard (red line) inflation scenarios and $V(\phi) = \lambda_4 \phi^4$.

In standard inflation with this potential one obtains,

$$n_s = 1 - \frac{8}{1+6N}, \quad r = \frac{16}{1+6N}, \quad \alpha = \frac{-48}{(1+6N)^2}. \quad (3.19)$$

Again we calculated the constant $\lambda_4^2$ in standard cosmology to obtain:

$$\lambda_4^2[\text{Standard}] = \frac{2.196 \times 10^{-9} \times 12 \times 6^2 \pi^2}{(6N+1)^{\frac{1}{3}}} (\text{GeV}^{10})^2. \quad (3.20)$$

In the brane-world scenario, however, we deduce,

$$n_s = 1 - \frac{7}{1+4N}, \quad r = \frac{24}{1+4N}, \quad \alpha = \frac{-28}{(1+4N)^2}. \quad (3.21)$$
Figure 12. Tensor-to-scalar ratio $r$ vs. spectral index $n_s$ for the case of $V(\phi) = \lambda_2/3 \phi^{2/3}$ in the brane-world (black line) compared to that of standard (indigo line) inflation. Contours are the 1 and 2$\sigma$ confidence limits from various versions of the Planck analysis [4] as labeled.

Again we calculated the constant $\lambda_2/3$ in brane-world cosmology at the pivot scale:

$$\lambda_2/3[\text{Brane-world}] = \frac{7.853 \times 10^{-2} \times M_5^{10}}{(4N+1)^{2/3}} \text{(GeV)}^{4/3}. \quad (3.22)$$

Figure 12 summarizes the tensor-to-scalar ratio $r$ versus $n_s$ for standard and brane-world inflation with the axion monodromy potential. The Planck analysis [4] tends to disfavor values for the spectral index that are close to unity $n_s \to 1$ as is the case for standard inflation with this potential. However, in brane-world inflation the scalar index falls near the optimum value for $n_s$ from the Planck TT + low-P analysis, even though the value of the tensor-to-scalar ratio is increased. The standard inflation values, however, remain a better fit when the BICEP2/Keck analysis is also included.

Figure 13 illustrates how the running of the coupling constant $\alpha$ changes as a function of $N$ in the brane-world model compared to the standard inflation. Figure 14 shows how the constant term $\lambda_4/3$ changes with $M_5$.

3.6 $V(\phi) \propto \phi^{4/3}$

The $\phi^{4/3}$ potential is studied next. For the standard inflation scenario,

$$n_s = 1 - \frac{20}{3 + 12N}, \quad r = \frac{64}{3 + 12N}, \quad \alpha = -\frac{80}{3(1 + 4N)^2}. \quad (3.23)$$

Again we calculated the constant $\lambda_4/3$ in standard cosmology.

$$\lambda_4/3[\text{Standard}] = \frac{2.196 \times 10^{-9} \times 16 \times 3^{2/3} \times \pi^2}{(4N+1)^{2/3}} \text{(GeV)}^{8/3}. \quad (3.24)$$

However, for the brane-world scenario we obtain,

$$n_s = 1 - \frac{11}{2 + 5N}, \quad r = \frac{48}{2 + 5N}, \quad \alpha = -\frac{55}{(2 + 5N)^2}. \quad (3.25)$$

– 11 –
Figure 13. The running of the spectral index as a function of $N$ for both the brane-world (blue line) and standard (red line) inflation scenarios and $V(\phi) = \lambda_{2/3} \phi^{2/3}$.

Figure 14. The constant term $\lambda_{2/3}$ as a function of $M_5$ for $N = 50$ (Blue), $N = 60$ (Red) and $N = 70$ (Green).

Figure 15. Tensor-to-scalar ratio $r$ vs. spectral index $n_s$ for the case of $V(\phi) = \lambda_{4/3} \phi^{4/3}$ in the brane-world (black line) compared to that of standard (indigo line) inflation. Contours are the 1 and 2$\sigma$ confidence limits from various versions of the Planck analysis [4] as labeled.

Again we calculated the constant $\lambda_{4/3}$ in brane-world cosmology.

$$\lambda_{4/3}[\text{Brane-world}] = \frac{1.423 \times 10^{-3} \times M_5^3}{(5N + 2)} \text{[GeV}^{8/3}] .$$  \hspace{1cm} (3.26)

Figure 15 summarizes the tensor-to-scalar ratio versus the spectral index for standard and brane-world inflation with this potential. In this case, the brane-world scenario is just outside the Planck 2015 TT, TE, EE + low P 2$\sigma$ limit, while the standard scenario is still allowed at 2$\sigma$. However, if we include BICEP2/Keck data, the standard scenario is also ruled out at 2$\sigma$. Figure 16 illustrates how the running of the coupling constant $\alpha$ changes as a function of $N$ in RS model compared to the standard inflation. Figure 17 shows how the constant $\lambda_{4/3}$ changes with $M_5$.

4 Natural inflation model

In natural inflation [35–38] the role of the inflaton is played by a pseudo Nambu-Goldstone boson that appears when an approximate global symmetry is broken spontaneously. The
The flatness of the potential is protected by a shift symmetry under $\phi \to \phi + \text{constant}$, which remains after the global symmetry is spontaneously broken.

The explicit breaking of the shift symmetry leads to a potential of the form [35]:

$$V(\phi) = \Lambda^4 [1 + \cos(\phi/f)].$$

The two mass scales $\Lambda$ and $f$ characterize the height and width of the potential, respectively. The mass scale of $f$ is typically taken as $f \sim m_{pl} \sim 10^{19}$ GeV and $\Lambda \sim M_{\text{GUT}} \sim 10^{16}$ GeV, as expected in particle physics. Natural inflation is a very well studied model. We take into account the possibility that the number of $e$-folds before the end of inflation can be increased in the brane-world scenario. A larger value of $N$ is allowed in the brane-world scenario as discussed at the end of section 2. The parameter $f$ determines the curvature of the potential. For standard inflation the symmetry breaking scale is constrained to be $f > 0.9 m_{pl}$ from the Planck analysis [4]. For the brane-world, we utilize the prior range on $f$ that was adopted in [4] of $0.3 \leq \log_{10}(f/m_{pl}) \leq 2.5$. This is chosen to encompass a reasonable range around the Planck scale and include the Planck15 allowed range as seen below. For the standard inflationary cosmology the calculation of the spectral index, tensor-to-scalar ratio and the running of the spectral index is much easier because we know that inflation occurs while the inflaton slowly evolves towards the potential minimum at $\phi = \pi f$. Also, for standard cosmology the slow roll parameters, and thus $n_s$, $r$ and $\alpha$, are dependent upon the width of the potential $f$, but not on the height $\Lambda$.

For standard inflation, approximate expressions for $n_s$, $r$ and $\alpha$ can be written as:

$$n_s \approx 1 + \frac{1}{f^2} - \frac{2}{N}, \quad r \approx \frac{8}{f^2} \left( \frac{f^2 - N}{N} \right), \quad \alpha \approx \frac{N - f^2}{f^2 N^2}. \quad (4.2)$$

In the brane-world scenario the derivation of the inflation parameters is much more complex as we now outline. In brane-world inflation, as in the other cases studied here, there is a tendency for higher values of the tensor-to-scalar ratio than that from standard inflation. However, the brane-world calculation is a bit more cumbersome for two reasons. For one, in standard natural the parameters are only affected by $f$ (cf. eq. (4.2)). However, in the RSII brane-world, there is an effect from $\Lambda$ at the end of inflation. Here also the value of $\Lambda$ is taken to be $\Lambda \sim M_{\text{GUT}} \sim 10^{16}$ GeV. Here, we have taken $\rho_0^{1/4} \sim 10^{10}$ GeV. Ultimately, the end of the inflation occurs approximately when:

$$\phi_{\text{end}} \approx \tan^{-1} \left( \frac{1}{2} \left( -2 - y - \sqrt{y^2 + 8y} \right) \right), \quad (4.3)$$
where
\[ y \equiv \frac{\rho_0}{f^2 \Lambda^4}. \] (4.4)

Now let us say,
\[ (-2 - y - \sqrt{y^2 + 8y}) = -(a + b), \quad a = 2 + y, \quad b = \sqrt{y^2 + 8y}. \] (4.5)

Then,
\[ \phi_{\text{end}} \approx \tan^{-1} \frac{-(a + b)}{2}. \] (4.6)

Eqs. (4.3) and (4.4) show that a dependence on \( \Lambda \) and \( f \) cannot be avoided. Secondly, as inflation occurs, the inflaton slowly evolves towards the minimum and starts to oscillate. Moreover, the presence of the bulk dimension causes the end of inflation based upon the condition of slow roll violation to be dependent upon \( M_5 \) through the \( \rho^2 \) term.

The expressions for \( n_s \), \( r \) and \( \alpha \) become much more complicated in the brane-world scenario. We have derived new relations for the spectral index, tensor-to-scalar ratio, and running of the spectral index in terms of a new auxiliary variable \( z \):
\[ n_s = 1 - \frac{y}{2(1 + z)^2}, \quad r = 8y \frac{-z}{(1 + z)^2}, \quad \alpha = 56y \frac{-z}{(1 + z)^2}, \] (4.7)

where, \( z \) is determined from a solution to the exponential equation,
\[ z e^z = d, \] (4.8)

and \( d \) is defined as:
\[ d = \left[ \frac{1}{8} \exp \left( -\left( 1 + \frac{1}{2} \left[ (2N + 1)(a - 2) - b/2 \right] \right) \left( b^2 - (a + 2)b + 8 \right) \right) \right]^{1/2}, \] (4.9)

with \( a \), and \( b \) defined in eq. (4.5). Eqs. (4.4)–(4.9) together contain an implicit dependence of the spectral index, tensor-to-scalar ratio and the running of the spectral index on \( f \) and \( \Lambda \) via eq. (4.4). Our solution for the brane-world natural inflation is consistent with the solution deduced in [10]. In our analysis there is a formally different choice of auxiliary variables. We utilize the auxiliary variables \( a \) and \( b \) as defined above. We then numerically solve for \( N \). Ref. [10] utilizes an auxiliary variable \( A_{RS} \) that is related to our parameter \( y = 1/(3A_{RS}) \), and hence \( a \) and \( b \). In that sense the derivations in the two papers are equivalent. Values of \( N \) in both approaches are then related to the adopted auxiliary variables and the value of \( \cos (\phi/f) \) at the end of inflation. Both papers then use numerical calculations to obtain the final forms for the spectral index and other observables.

Hence, although appearing formally different, our calculations are equivalent to those of [10] as noted below. To clarify the similarity of our result to the result obtained in [10], we point out that to first order the results are same. The parameter \( z \) in our calculation can be related to the parameter defined as \( x \) in the section 5.2.1 in [10] as:
\[ z \approx \frac{3x^2 - \sqrt{3} \sqrt{3x^4 + 8x^3 + 14x^2 + 16x + 7} + 4x + 5}{2(x - 1)}. \] (4.10)

The agreement between our results and those of ref. [10] then serves as a validation of both derivations. Figure 18 shows the tensor-to-scalar ratio \( r \) versus spectral index \( n_s \) for the brane-world and standard inflation scenarios. In this case the solutions are bands as indicated by lines on the figure due to the range of values for the width parameter \( f \).
From a comparison of figure 18 with figure 4 one can see that in the limit of large $f$ the results coincide with those of the $\phi^2$ potential as noted in [10]. As one can see on figure 18, for the same values of $N$ the standard inflation provides a better fit to the Planck constraints. Also, our results for $N = 50$ and 60 are nearly identical to the corresponding graph in [10]. However, we also point out, that because the brane-world natural inflation allows for a higher value for $N$, the region of large $f$ and $N = 70$ is displaced toward higher $n_s$ and better agrees with the CMB constraints than that of the standard inflation.

The running of the spectral index $\alpha$ depends explicitly on $N$ and $y$, while $y$ is related to the model parameters $f$ and $\Lambda$. Thus, it is worthwhile to graphically illustrate the resultant dependence of $\alpha$ upon the various model parameters as shown in figures 19 and 20. The running of the spectral index for the standard natural inflation as a function of $N$ and $f$ can be seen as a surface in three dimensions in figure 19. For the brane-world scenario, however, the dependence becomes more complicated. The running of the spectral index is not only a function of $N$ and $f$ but also the value of $\Lambda$. In figure 20 we show the dependence of $\alpha$ on $N$ and $y$ for $N = 50$–70.

5 Constraints on $M_5$

There is a model-independent lower bound on the 5-dimensional Planck mass of $M_5 > 8.8$ TeV from Big Bang Nucleosynthesis (BBN) [12] and the requirement that $\rho_0^{1/4} \geq T_{\text{BBN}}$. Of particular relevance to the present work is that precision measurements by Planck [4] also allow for constraints on $M_5$ for each of the inflation models considered here. This follows [12] from the requirement that the spectral index fall within the $2\sigma$ limits from the Planck analysis. In figures 21a–e, we show the values the spectral index $n_s$ as a function of $M_5$ for some of the models considered in the present work and various values of $N$. The corresponding constraints on $M_5$ are summarized in table 1. It is worth mentioning that in this section there is no initial assumption of $V/\rho_0 \gg 0$ as in the previous sections. This is to show the
variation with respect to $\rho_0$. For most cases there is only an upper limit to $M_5$. However, for the special case of $\phi^2$ and natural inflation with $N = 50$ a lower limit can also be deduced. A similar kind of analysis is carried out in [12] for some of the inflationary potentials. Though the only common model with ours is the $\phi^2$ potential, it should be mentioned here that the variation in that case is carried out for a set of fixed value of $n_s$. Whereas in this work $n_s$ is varied freely and then for an allowed range of $n_s$, the range of $M_5$ is estimated and represented in a log-linear plot and $M_5$ is expressed in terms of Planck units. Also in our analysis all the bounds on $M_5$ are kept in mind.

For all the cases on figures 21a–e, the horizontal dashed brown line shows the Planck central value ($n_s = 0.968$) for the TT+lowP+lensing data [4]. The pink band corresponds to the 1-$\sigma$ region ($\Delta n_s \sim 0.006$) and the orange band corresponds to 2-$\sigma$ region. The black band corresponds to a hypothetical future 1-$\sigma$ sensitivity if $\Delta n_s$ could ever be reduced to $\Delta n_s \sim 0.002$; assuming the central value remains unchanged [40, 41].

6 Conclusion

We have considered constraints on the brane-world inflation paradigm from the results of the Planck15 analysis [4]. We also consider the more stringent constraints based upon the combined Planck15 [4] + BICEP2/Keck [21], and we analyze a few monomial potentials that were not explicitly considered in previous studies (e.g. [10, 12]). We have confirmed previous analytic derivations in brane-world inflation for the spectral index, the tensor-to-scalar ratio, and the running of the spectral index for a variety of monomial inflation effective potentials and deduced constraints on natural inflation consistent with the previous results of [10]. We also deduce new limits on the five-dimensional Planck mass $M_5$ from the requirement that the spectral index reside within the 2-$\sigma$ limits deduced by the Planck analysis.

We confirm that in general the brane-world scenario increases the tensor-to-scalar ratio thus making this paradigm less consistent with the Planck constraints. However, in the case of the lowest order $\phi^{2/3}$ axion monodromy (if one only compares to the Planck15 TT + low -P constraint) there might be a slight improvement over the standard model in the case of large $N$. However this improvement vanishes when comparing to the combined constraint from BICEP2/Keck.

Standard natural inflation fits the Planck constraints better for the same values of $N$. However, due to the fact that larger values of $N$ are allowed in the brane-world paradigm,
the brane-world natural inflation could provide a better fit to the CMB spectrum. These results are encouraging since the axion monodromy and natural inflation are most consistent with an interpretation of brane-world cosmology as an approximation to string theory.

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\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
$V(\phi)$ & N & $M_5(M_p = 1)$ \\
\hline
$\propto \phi$ & 50 & $M_5 < 0.0128$ \\
& 60 & $M_5 < 0.0098$ \\
& 70 & $M_5 < 0.0079$ \\
\hline
$\propto \phi^{2/3}$ & 50 & $M_5 < 0.0411$ \\
& 60 & $M_5 < 0.0315$ \\
& 70 & $M_5 < 0.0411$ \\
\hline
$\propto \phi^2$ & 50 & $0.0059 < M_5 < 0.0196$ \\
& 60 & $M_5 < 0.0196$ \\
& 70 & $M_5 < 0.0170$ \\
\hline
$\propto \phi^{4/3}$ & 50 & $M_5 < 0.0211$ \\
& 60 & $M_5 < 0.0177$ \\
& 70 & $M_5 < 0.0211$ \\
\hline
$\propto 1 + \cos(\phi/f)$ & 50 & $0.0314 < M_5 < 0.075$ \\
& 60 & $M_5 < 0.0679$ \\
& 70 & $M_5 < 0.0658$ \\
\hline
\end{tabular}
\caption{Constraints on $M_5$ for various inflation effective potentials.}
\end{table}

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