RELATIVE LOG CONVERGENT COHOMOLOGY AND
RELATIVE RIGID COHOMOLOGY III

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ABSTRACT. In this paper, we prove the generic overconvergence of relative rigid cohomology with coefficient, by using the semistable reduction conjecture for overconvergent $F$-isocrystals (which is recently shown by Kedlaya).

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INTRODUCTION

Let $k$ be a perfect field of characteristic $p > 0$, let $V$ be a complete discrete valuation ring of mixed characteristic with residue field $k$ endowed with a lift of Frobenius $\sigma : V \to V$ and let $X$ be a scheme separated of finite type over $k$. Then, as we wrote in the introduction in the previous paper [Sh4], it is expected that the correct $p$-adic analogue on $X$ of the notion of the local systems (or smooth $\mathbb{Q}_l$-sheaves) should be the notion of overconvergent $F$-isocrystals. Based on this expectation, Berthelot conjectured in [Be2] that, for a proper smooth morphism $f : X \to Y$ of schemes of finite type over $k$ and an overconvergent $F$-isocrystal $\mathcal{E}$ on $X$, the higher direct image of $\mathcal{E}$ by $f$ (= relative rigid cohomology) should have a canonical structure of an overconvergent $F$-isocrystal. A version of this conjecture was proved in [Sh4].

The purpose of this paper is to consider the analogue of Berthelot’s conjecture in the case where the given morphism $f : X \to Y$ is no longer proper nor smooth. In this case, we cannot expect that the relative rigid cohomology of $\mathcal{E}$ has a structure of an overconvergent $F$-isocrystal on $Y$, because the higher direct image of an local system on $X$ should be only a constructible sheaf on $Y$ in this case. However, since a constructible sheaf is a local system on a dense open subset, it is natural to conjecture

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that the relative rigid cohomology of $\mathcal{E}$ has a structure of an overconvergent $F$-isocrystal on a dense open subset of $Y$. The purpose of this paper is to prove that a version of this conjecture is true, if we admit the validity of the semi-stable reduction conjecture for overconvergent $F$-isocrystals, which was first conjectured in [Sh2, 3.1.8] as a higher-dimensional analogue of quasi-unipotent conjecture (= a $p$-adic local monodromy theorem of André([A]), Mebkhout([M]) and Kedlaya([Ke1])). We would like to note that this result is already proved in the previous paper [Sh4] in the case where $f$ is proper, without any conjectural hypothesis. So the main point in this paper is to treat the case where $f$ is not proper.

Now we explain our conjecture and the main result in this paper in detail. Let us put $S := \text{Spec} k, S := \text{Spf} V$ and assume that all the pairs (resp. all the triples) are separated of finite type over $(S, S)$ (resp. $(S, S, S)$). (As for the terminology concerning pairs and triples, see [Ch-T], [Sh3], [Sh4].) Then the precise form of our conjecture is as follows:

**Conjecture 0.1.** Assume given a morphism of pairs $f : (X, \overline{X}) \longrightarrow (Y, \overline{Y})$ such that $\overline{X} \longrightarrow \overline{Y}$ is proper. Then there exist an open dense subset $U \subseteq Y$ and a subcategory $\mathcal{C}$ of the category of $(U, \overline{Y})$-triples over $(S, S, S)$ such that, for any overconvergent $F$-isocrystal $\mathcal{E}$ on $(X, \overline{X})/S_K$ and for any $q \geq 0$, there exists uniquely an overconvergent isocrystal $\mathcal{F}$ satisfying the following condition: For any $(Z, \overline{Z}, Z) \in \mathcal{C}$ such that $Z$ is formally smooth over $S$ on a neighborhood of $Z$, the restriction of $\mathcal{F}$ to $I^1((Z, \overline{Z})/\mathcal{S}_K, Z)$ is given functorially by $(R^q f_{(X \times_Y Z, \overline{X} \times_{\overline{Y}} \overline{Z})/Z, \text{rig} \mathcal{E}}, \epsilon)$, where $\epsilon$ is given by

$$p_2^* R^q f_{(X \times_Y Z, \overline{X} \times_{\overline{Y}} \overline{Z})/Z, \text{rig} \mathcal{E}} \cong R^q f_{(X \times_Y Z, \overline{X} \times_{\overline{Y}} \overline{Z})/Z, \text{rig} \mathcal{E}} \cong p_1^* R^q f_{(X \times_Y Z, \overline{X} \times_{\overline{Y}} \overline{Z})/Z, \text{rig} \mathcal{E}}.$$

(Here $I^1((Z, \overline{Z})/\mathcal{S}_K, Z)$ denotes the category of overconvergent isocrystals on $(Z, \overline{Z})/\mathcal{S}_K$ over $Z$, $R^q f_{(X \times_Y Z, \overline{X} \times_{\overline{Y}} \overline{Z})/Z, \text{rig} \mathcal{E}}$, $R^q f_{(X \times_Y Z, \overline{X} \times_{\overline{Y}} \overline{Z})/Z, \text{rig} \mathcal{E}}$ are relative rigid cohomologies and $p_i$ is the morphism $[Z_{\times_S Z} \longrightarrow \overline{Z}][Z]$ induced by the $i$-th projection.) Also, $\mathcal{F}$ admits a Frobenius structure which is induced by the Frobenius structure of $\mathcal{E}$.

**Remark 0.2.** (1) One can also consider a stronger version, which requires the category $\mathcal{C}$ to be the category of all the $(Y, \overline{Y})$-triples over $(S, S, S)$.

(2) If $f$ is strict (that is, $f^{-1}(Y) = X$ holds), then Conjecture 0.1 is shown in [Sh4].

Then the main result in this paper is as follows (for more precise form, see Corollary 5.3(1)):

**Theorem 0.3.** Conjecture 0.1 is true.

In the proof of Theorem 0.3, we use the semi-stable reduction conjecture of overconvergent $F$-isocrystals, which is recently proved by Kedlaya ([Ke4], [Ke5], [Ke6], [Ke7]) for $\mathcal{E}$. In fact, this is the only place where we use the Frobenius structure on $\mathcal{E}$. So we can also form a variant of Theorem 0.3 without using Frobenius structure, by introducing the notion of ‘potentially semi-stable overconvergent isocrystals’. (See Theorem 5.1.) We remark also that we can state the theorem in such a way that...
claims ‘a kind of constructibility’ of relative rigid cohomology. (See Corollary 5.2, Corollary 5.3(2).)

Now we give an outline of the proof of Theorem 0.3 and explain the content of this paper. Roughly speaking, the semi-stable reduction conjecture for overconvergent $F$-isocrystals claims that any overconvergent $F$-isocrystal should be extendable to the boundary logarithmically after taking pull-back by a certain proper surjective generically étale morphism. So, as a first step, we consider the coherence and the overconvergence of relative rigid cohomology in the case where the morphism $f : \overline{X} \to \overline{Y}$ admits a nice log structure, that is, the case where $f$ comes from a nice morphism of log schemes $(\overline{X}, M_{\overline{X}}) \to (\overline{Y}, M_{\overline{Y}})$. In Section 1, we give a review of the residues of log-$\nabla$-modules and isocrystals which are developed in [Ke4] and prove some preliminary facts which we need in Section 2, where we prove the coherence and the overconvergence of the relative rigid cohomology of $(X, \overline{X}) \to (Y, \overline{Y})$ in the above-mentioned case. Let $D$ be the closure of $f^{-1}(Y) \setminus X$ in $\overline{X}$. (In ‘nice case’ we are treating, $\overline{X}$ is smooth over $k$ and $D$ is a normal crossing divisor in $\overline{X}$.) In this case, we prove the overconvergence of the relative rigid cohomology of $(X, \overline{X}) \to (Y, \overline{Y})$ by relating it to the relative log analytic cohomology of $(\overline{X}, M_{\overline{X}}) \to (\overline{Y}, M_{\overline{Y}})$ via an intermediate cohomology, the relative log rigid cohomology of $((\overline{X} - D, M_{\overline{X}}|_{\overline{X} - D}), (\overline{X}, M_{\overline{X}})) \to (\overline{Y}, M_{\overline{Y}})$: We compare the relative rigid cohomology of $(X, \overline{X}) \to (Y, \overline{Y})$ with the relative log rigid cohomology of $((\overline{X} - D, M_{\overline{X}}|_{\overline{X} - D}), (\overline{X}, M_{\overline{X}})) \to (\overline{Y}, M_{\overline{Y}})$ by the method which is a generalization to the relative case of what we have developed in [Sh2, 2.4], and we relate the relative log rigid cohomology of $((\overline{X} - D, M_{\overline{X}}|_{\overline{X} - D}), (\overline{X}, M_{\overline{X}})) \to (\overline{Y}, M_{\overline{Y}})$ to the relative rigid cohomology of $(X, \overline{X}) \to (Y, \overline{Y})$ by the method we have developed in [Sh3, §5]. In Section 3, we prove the invariance of relative log analytic cohomology under log blow-ups. We need this property in the proof of the main theorem, mainly by technical reason. In Section 4, we prove a result of altering a given proper morphism $\overline{X} \to \overline{Y}$ of schemes over $k$ endowed with an open subscheme $X \subset \overline{X}$ to a certain simplicial morphism of schemes whose components admit nice log structures, by using results of de Jong [dJ1], [dJ2]. The results in this section is a slight generalization of what we have shown in [Sh4, §6]. This result allows us to reduce the proof of the main theorem to the case we considered in Section 2. In Section 5, we prove the main result by combining the results in the previous sections, by the method analogous to [Sh4, §7].

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Convention

(1) Throughout this paper, $k$ is a field of characteristic $p > 0$, $V$ is a complete discrete valuation ring of mixed characteristic with residue field $k$, and $K$ is the fraction field of $V$. Note that we will assume that $k$ is perfect in some results. In particular, as we will write in the text, we will assume that $k$ is perfect in Sections 4, 5. We put $S := \text{Spec } k$ and $\mathcal{S} := \text{Spf } V$. Let us fix a power $p^a$ of $p$ and assume
that we have an endomorphism $\sigma : V \to V$ which is a lift of $p^n$-th power Frobenius endomorphism on $k$. (However, we need the existence of $\sigma$ only when we speak of Frobenius structures on isocrystals.) All the log formal schemes are fine log (not necessarily $p$-adic) formal schemes which are separated and topologically of finite type over $S$. We call such a log formal scheme a fine log formal $S$-scheme. If it is $p$-adic, we call it a $p$-adic fine log formal $S$-scheme. If it is a log scheme, we call it a fine log $S$-scheme, and if it is defined over $S$, we call it a fine log formal $S$-scheme.

If the log structure is fs, we replace ‘fine’ by ‘fs’ and if the log structure is trivial, we omit the word ‘fine log’.

(2) For a formal $S$-scheme $T$, we denote the rigid analytic space associated to $T$ by $T_K$.

(3) In this paper, we freely use the terminologies concerning log structures defined in [Ka1], [Sh1] and [Sh2]. For a fine log (formal) scheme $(X, M_X)$, $(X, M_X)_{\text{triv}}$ denotes the maximal open sub (formal) scheme of $X$ on which the log structure $M_X$ is trivial. A morphism $f : (X, M_X) \to (Y, M_Y)$ is said to be strict if $f^* M_Y = M_X$ holds.

(4) In this paper, we freely use the notations and terminologies concerning (log) pairs, triples, isocrystals on relative log convergent site, overconvergent isocrystals, relative log analytic cohomologies, relative rigid cohomologies and relative log rigid cohomologies which are developed in the previous papers [Sh3], [Sh4]. All the pairs in this paper are separated of finite type over $(S, S)$ and all the triples in this paper are separated of finite type over $(S, S, S)$. Frobenius structures on isocrystals are always the ones with respect to $\sigma$ defined in (1).

(5) Fiber products of log formal schemes are taken in the category of fine log formal schemes.

1. Residues of log-$\nabla$-modules and isocrystals

In this section, we give a review of definitions and results by Kedlaya [Ke4] on log-$\nabla$-modules and isocrystals and prove some preliminary results on the residue of log-$\nabla$-modules and isocrystals. Finally, we recall the semi-stable reduction conjecture for overconvergent $F$-isocrystals.

First we recall the definition of log-$\nabla$-modules and the residues of them ([Ke4, 2.3]). Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of rigid analytic spaces over $K$ and let $x_1, \ldots, x_r \in \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$. Then we define the relative differential module $\Omega^1_{\mathfrak{X}/\mathfrak{Y}}$ by $\Omega^1_{\mathfrak{X}/\mathfrak{Y}}/\mathfrak{X}/K/f^* \Omega^1_{\mathfrak{Y}/K}$ (where $\Omega^1_{\mathfrak{X}/\mathfrak{Y}}, \Omega^1_{\mathfrak{Y}/K}$ denotes the sheaf of continuous differential forms of $\mathfrak{X}, \mathfrak{Y}$, respectively) and we define the relative log differential module $\omega^1_{\mathfrak{X}/\mathfrak{Y}}$ (which is denoted by $\Omega^1_{\mathfrak{X}/\mathfrak{Y}}$ in [Ke4]) by

$$
\omega^1_{\mathfrak{X}/\mathfrak{Y}} := (\Omega^1_{\mathfrak{X}/\mathfrak{Y}} \oplus \bigoplus_{i=1}^r \mathcal{O}_\mathfrak{X} \text{dlog} x_i)/N,
$$

where $N$ is the sub $\mathcal{O}_\mathfrak{X}$-module locally generated by $(dx_i, 0) - (0, x_i \text{dlog} x_i)$. A log-$\nabla$-module $(E, \nabla)$ on $\mathfrak{X}$ with respect to $x_1, \ldots, x_r$ relative to $\mathfrak{Y}$ is a locally free $\mathcal{O}_\mathfrak{X}$-module $E$ endowed with an integrable log connection $\nabla : E \to E \otimes_{\mathcal{O}_\mathfrak{X}} \omega^1_{\mathfrak{X}/\mathfrak{Y}}$. 

When \( \mathfrak{Y} = S_K \) holds, we omit the word ‘relative to \( \mathfrak{Y} \)’. If we put \( \mathfrak{D}_i := \{ x_i = 0 \} \subseteq \mathfrak{X} \) and \( M_i := \text{Im}(\Omega^1_{\mathfrak{X}/\mathfrak{Y}} \oplus \bigoplus_{j \neq i} \mathcal{O}_\mathfrak{X} \text{dlog} x_j \rightarrow \omega^1_{\mathfrak{X}/\mathfrak{Y}}) \) \( (1 \leq i \leq r) \), the composite

\[
E \xrightarrow{\nabla} E \otimes_{\mathcal{O}_\mathfrak{X}} \omega^1_{\mathfrak{X}/\mathfrak{Y}} \rightarrow E \otimes_{\mathcal{O}_\mathfrak{X}} (\omega^1_{\mathfrak{X}/\mathfrak{Y}}/M_i) \cong E \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_{\mathfrak{D}_i} \text{dlog} x_i = E \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_{\mathfrak{D}_i}
\]

naturally induces an element of \( \text{End}_{\mathcal{O}_{\mathfrak{D}_i}}(E \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_{\mathfrak{D}_i}) \). We call this element the residue of \((E, \nabla)\) along \( \mathfrak{D}_i \). \((E, \nabla)\) is said to have nilpotent residues if the residue of \((E, \nabla)\) along \( \mathfrak{D}_i \) is nilpotent for all \( 1 \leq i \leq r \). We denote the category of log-\(\nabla\)-modules on \( \mathfrak{X} \) with respect to \( x_1, \ldots, x_r \) relative to \( \mathfrak{Y} \) having nilpotent residues by \( \text{LNM}_{\mathfrak{X}/\mathfrak{Y}} \).

When \( \mathfrak{Y} = S_K \) holds, we write \( \text{LNM}_\mathfrak{X} \) instead of \( \text{LNM}_{\mathfrak{X}/\mathfrak{Y}} \).

**Remark 1.1.** The log differential module \( \omega^1_{\mathfrak{X}/\mathfrak{Y}} \) and the residues of log-\(\nabla\)-modules are unchanged if we replace \( x_i \)'s by \( u_i x_i \)'s for some \( u_i \in \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X}^* ) \) \( (1 \leq i \leq r) \).

Following [Ke4, 3.1], we call a subinterval \( I \) of \([0, \infty)\) aligned if any endpoint at which it is closed is in \( p^Q \cup \{ 0 \} \) (it is equal to \( \Gamma^* \) in [Ke4] in our case) and for an aligned interval \( I \), we define the polyannulus \( A^m_K(I) \) by

\[
A^m_K(I) := \{ (t_1, \ldots, t_n) \mid t_i \in I \, (1 \leq i \leq n) \}.
\]

Let \( \mathfrak{X} \) be a smooth rigid space endowed with sections \( x_1, \ldots, x_m \in \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \) whose zero loci are smooth and meet transversally. For an aligned interval \( I \) with \( 0 \notin I \) and \( \eta \in [0, \infty) \), a log-\(\nabla\)-module \((E, \nabla)\) on \( \mathfrak{X} \times A^m_K(I) \) with respect to \( x_1, \ldots, x_m \) is called \( \eta \)-convergent with respect to \( t_1, \ldots, t_n \) if, for any \( e \in \Gamma(\mathfrak{X} \times A^m_K(I), E) \), the multisequence

\[
\begin{vmatrix}
1 & \frac{\partial^j}{\partial t_1^{i_1}} & \cdots & \frac{\partial^n}{\partial t_n^{i_n}} (e)
\end{vmatrix}
\]

converges to zero. (cf. [Ke4, 2.4.2]. We slightly changed the terminology in order to make it shorter.) For \( a \in p^Q < 0 \cup \{ 0 \} \), a log-\(\nabla\)-module \((E, \nabla)\) on \( \mathfrak{X} \times A^m_K(a, 1) \) or \( \mathfrak{X} \times A^m_K(a, 1) \) with respect to \( x_1, \ldots, x_m, t_1, \ldots, t_n \) is called \( \eta \)-convergent if, for any \( \eta \in (0, 1) \), there exists some \( b \in (a, 1) \cap p^Q \) such that, for any \( c \in [b, 1) \cap p^Q \), \((E, \nabla)\) is \( \eta \)-convergent with respect to \( t_1, \ldots, t_n \) on \( \mathfrak{X} \times A^m_K(b, c) \) ([Ke4, 3.6.6]).

Let \( \mathfrak{X}, x_1, \ldots, x_m \) be as above and let \( I \) be an aligned interval. Then \((E, \nabla) \in \text{LNM}_{\mathfrak{X} \times A^m_K(I)}\) (here \( \mathfrak{X} \times A^m_K(I) \) is endowed with sections \( x_1, \ldots, x_m, t_1, \ldots, t_n \) ) is unipotent relative to \( \mathfrak{X} \) if there exists a filtration of \((E, \nabla)\) by sub log-\(\nabla\)-modules such that each graded quotient is again a log-\(\nabla\)-module and has the form \( \pi^*(F, \nabla_F) \) (\( \pi \) denotes the projection \( \mathfrak{X} \times A^m_K(I) \rightarrow \mathfrak{X} \)) for some log-\(\nabla\)-module \((F, \nabla_F)\) on \( \mathfrak{X} \) with respect to \( x_1, \ldots, x_m \) ([Ke4, 3.2.5]).

Then, by [Ke4, 3.6.2, 3.6.9], we have the following:

**Proposition 1.2.** Let \( a \in p^Q < 0 \), let \( \mathfrak{X} \) be a smooth rigid space endowed with sections \( x_1, \ldots, x_m \in \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \) whose zero loci are smooth and meet transversally and let \((E, \nabla) \) be an object in \( \text{LNM}_{\mathfrak{X} \times A^m_K(a, 1)} \) which is convergent. Then \((E, \nabla) \) is unipotent relative to \( \mathfrak{X} \) if and only if it extends to an object \((\overline{E}, \overline{\nabla}) \) in \( \text{LNM}_{\mathfrak{X} \times A^m_K(0, 1)} \). Moreover, the extension is unique and \((\overline{E}, \overline{\nabla}) \) is also unipotent relative to \( \mathfrak{X} \).

Next, let \( X \) be a smooth scheme over \( k \), let \( D := \bigcup_{i=1}^r D_i \) be a simple normal crossing divisor in \( X \) (each \( D_i \) is assumed to be smooth) and let \( M_X \) be the log
structure on $X$ associated to $D$. Let $\mathcal{E}$ be a locally free isocrystal on log convergent site $(X/S)^{\log}_{\text{conv}} = ((X, M_X)/S)_{\text{conv}}$. Zariski locally on $X$, we can form the Cartesian diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\subseteq} & \mathcal{D} \\
\cap & \Downarrow & \cap \\
X & \xrightarrow{\subseteq} & \mathcal{X},
\end{array}
$$

(1.1)

where all arrows are closed immersions, $\mathcal{X}$ is a $p$-adic formal $S$ scheme formally smooth over $S$ with $\mathcal{X} \times_S S = X$ and $\mathcal{D} = \bigcup_{i=1}^r \mathcal{D}_i$ is a relative simple normal crossing divisor on $\mathcal{X}$ (each $\mathcal{D}_i$ is assumed to be formally smooth over $S$) such that $\mathcal{D}_i \times_S S = D_i$ holds and that each $\mathcal{D}_i$ is defined by a single element $t_i \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Then, $\mathcal{E}$ induces a log-$\nabla$-module on $\mathcal{X}_K = X[\chi]$ with respect to $t_1, \ldots, t_r$. $\mathcal{E}$ is said to have nilpotent residues if there exists an open covering $X = \bigcup_j X_j$ such that each $X_j$ admits a diagram like (1.1) (with $X$ replaced by $X_j$) and that the induced log-$\nabla$-module on $\mathcal{X}_K$ has nilpotent residues ([Ke4, 6.4.4]).

Keep the notation in the diagram (1.1) and let $\mathcal{E}$ be a locally free isocrystal on $(X/S)^{\log}_{\text{conv}}$ having nilpotent residues such that the induced log-$\nabla$-module $(E, \nabla)$ on $\mathcal{X}_K = X[\chi]$ with respect to $t_1, \ldots, t_r$ has nilpotent residues. For a subset $I$ of $\{1, \ldots, r\}$, we define $D_I := \bigcap_{i \in I} D_i$, $\mathcal{D}_I := \bigcap_{i \in I} \mathcal{D}_i$. Then, we can restrict $(E, \nabla)$ to

$$
|D_I[\chi] \cong |D_I[D_i \times A^r_K[0, 1]) (= D_I[K \times A^r_K[0, 1]) |
$$

(The coordinate of $A^r_K[0, 1)$ is given by $t_i (i \in I)$.) Then we have the following:

**Proposition 1.3.** $(E, \nabla)|_{D_I[D_i \times A^r_K[0, 1)}$ is unipotent relative to $|D_I[D_I]$.\\

**Proof.** We may work Zariski locally on $\mathcal{X}$. So we may assume that $\mathcal{X} := \text{Spf} A$ is affine and $\mathcal{X}$ admits a formally etale morphism $f : \mathcal{X} \longrightarrow \text{Spf} V\{t_1, \ldots, t_n\}$ for some $n \geq r$ such that the image of $t_i \in V\{t_1, \ldots, t_n\}$ in $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is the element $t_i$ defined above for $1 \leq i \leq r$. Let $M_{\mathcal{X}}$ be the log structure on $\mathcal{X}$ associated to $\mathcal{D}$ and let $(\mathcal{X}(1), M_{\mathcal{X}(1)}) := (\mathcal{X}, M_{\mathcal{X}}) \times_S (\mathcal{X}, M_{\mathcal{X}})$. Then we have the morphism

$$
f(1) : \mathcal{X}(1) \longrightarrow \text{Spf} V\{t^{(1)}_1, \ldots, t^{(1)}_n, t^{(2)}_1, \ldots, t^{(2)}_n\}
$$

naturally induced by $f$. Let us define $\mathcal{X}(1)'$ by

$$
\mathcal{X}(1)' := \mathcal{X}(1) \times_{\text{Spf} V\{t^{(1)}_1, \ldots, t^{(1)}_n, t^{(2)}_1, \ldots, t^{(2)}_n\}} \text{Spf} V\{t^{(1)}_1, \ldots, t^{(1)}_n, u_1^{\pm 1}, \ldots, u_n^{\pm 1}\},
$$

where $u_i := t^{(2)}_i / t^{(1)}_i (1 \leq i \leq r)$, $u_i := t^{(2)}_i + 1 (r + 1 \leq i \leq n)$ and let $M_{\mathcal{X}(1)'}$ be the log structure on $\mathcal{X}(1)'$ associated to the pre-log structure $\mathbb{N}^r \longrightarrow \mathcal{O}_{\mathcal{X}(1)'} (m_1, \ldots, m_r) \rightarrow t^{(1)}_1 t^{(1)}_2 \cdots t^{(1)}_r$. Then the closed immersion $(X, M_X) \hookrightarrow (\mathcal{X}(1), M_{\mathcal{X}(1)})$ (induced by $X \hookrightarrow \mathcal{X}$ and the diagonal map $\mathcal{X} \hookrightarrow \mathcal{X}(1)$) naturally factors as

$$(X, M_X) \hookrightarrow (\mathcal{X}(1)', M_{\mathcal{X}(1)'}) \longrightarrow (\mathcal{X}(1), M_{\mathcal{X}(1)})$$

where the first map is an exact closed immersion and the second map is formally log etale. So we have $|X[\chi(1)] = |X[\chi(1)'].$
Let \( p_i : X^{\log}_{\mathcal{X}(1)} = X_{\mathcal{X}(1)^{\prime}} \to X \) be the projection. Then \( \mathcal{E} \) induces the isomorphism \( \epsilon : p_2^* E \xrightarrow{\sim} p_1^* E \). If we denote the completion of \( \mathcal{X}(1)^{\prime} \) along \( X \) by \( \hat{\mathcal{X}}(1) \), the morphism \( \hat{\mathcal{X}}(1) \to \mathcal{X} \) induced by the first projection \( \mathcal{X}(1) \to \mathcal{X} \) has the form \( \text{Spf} A[[u_1, \cdots, u_n - 1]] \to \text{Spf} A \). So, the morphism \( p_1 \) can be identified with the projection
\[
\to X_{\mathcal{X} \times A_K^1} \to X_{\mathcal{X}}
\]
where the coordinate of \( A_K^1(0, 1) \) is given by \( u_i - 1 (1 \leq i \leq n) \). By this identification, the isomorphism \( \epsilon \) can be written as
\[
1 \otimes e \mapsto \sum_{i_1, \cdots, i_n = 0}^{\infty} \left( \prod_{j=1}^{r} \prod_{l=0}^{i_j - 1} \left( t_j \frac{\partial}{\partial t_j} - l \right) \prod_{j=r+1}^{n} \prod_{j=1}^{\frac{(u_j - 1)^{i_j}}{i_j!}} \right) \otimes \left( \prod_{j=1}^{n} \right)
\]
by [Ke4, 6.4.1], [Ka1, 6.7.1]. (There is a slight mistake in [Ke4, 6.4.1].) So the multisequence
\[
\left\| \frac{1}{i_1! \cdots i_n!} \left( \prod_{j=1}^{r} \prod_{l=0}^{i_j - 1} \left( t_j \frac{\partial}{\partial t_j} - l \right) \prod_{j=r+1}^{n} \prod_{j=1}^{\frac{(u_j - 1)^{i_j}}{i_j!}} \right) (e) \right\|_{i_1 + \cdots + i_n}
\]
converges to zero for any \( e \in \Gamma([X, E]) \) and \( \xi \in [0, 1] \).

Now we consider the restriction of \( (E, \nabla) \) to \( \mathcal{D}_1[X] = \mathcal{D}_1[\mathcal{D}_1 \times A_K^1(0, 1)] \). For any \( \eta \in [0, 1] \), take any \( c \in (\sqrt{\eta}, 1) \cap p\mathbb{Q} \). Then, on \( \mathcal{D}_1[\mathcal{D}_1 \times A_K^1(\sqrt{\eta}, c)] \), we have the following inequality for any \( i_j \in \mathbb{N} (j \in I) \) and \( e \in \Gamma([X, E]) \):
\[
\left\| \frac{1}{\prod_{j \in I} t_j^{i_j}} \prod_{j \in I} \prod_{l=0}^{i_j - 1} \left( t_j \frac{\partial}{\partial t_j} - l \right) (e) \right\|_{\eta \sum_{j \in I} i_j} \geq \left\| \prod_{j \in I} \prod_{l=0}^{i_j - 1} \right\|_{\eta \sum_{j \in I} i_j}
\]
Since the multisequence (with respect to \( i_j (j \in I) \)) on the left hand side of the above inequality converges to zero, the multisequence on the right hand side of the above inequality also converges to zero. Since the trivial log-\( \nabla \)-module \( (\mathcal{O}, d) \) on \( \mathcal{D}_1[\mathcal{D}_1 \times A_K^1(\sqrt{\eta}, c)] \) is \( \eta \)-convergent with respect to \( t_j (j \in I) \) by [Ke4, 3.6.1], we can show (by using Leibniz rule) that the multisequence on the right hand side of the above inequality converges to zero for any \( e \in \Gamma([\mathcal{D}_1[\mathcal{D}_1 \times A_K^1(\sqrt{\eta}, c)], E]. \) So the log-\( \nabla \)-module \( (E, \nabla)|_{\mathcal{D}_1[\mathcal{D}_1 \times A_K^1(0, 1)]} \) is convergent. Since it has nilpotent residues, it is unipotent relative to \( \mathcal{D}_1[\mathcal{D}_1] \) by Proposition 1.2. \( \square \)

We define the notion of ‘having nilpotent residues’ for isocrystals in a slightly generalized situation.

**Definition 1.4.** Let \( X \) be a smooth scheme over \( k \) and let \( D \) be a normal crossing divisor on \( X \). (\( D \) is not necessarily a simple normal crossing divisor, that is, some irreducible component of \( D \) may have self-intersection.) A locally free isocrystal \( E \) on \( (X/S)^{\log}_{\text{conv}} \) is said to have nilpotent residues if there exists an etale covering
\[ \prod_j X_j \rightarrow X \text{ such that each } X_j \text{ admits a diagram like (1.1) (with } X \text{ replaced by } X_j \text{) and that the induced log-\nabla\text{-module on } X_{\kappa} \text{ has nilpotent residues.} \]

We prove the compatibility of the above definition with [Ke4, 6.4.4] and the fact that the above definition is independent of the choice of an etale covering \[ \prod_j X_j \rightarrow X \] and diagrams like (1.1). First we recall the notion of admissible closed immersion ([Na-Sh, 2.1.7]). (We also introduce the notion of strongly admissible closed immersion.)

**Definition 1.5.** Let \( X \) be a smooth scheme over \( k \), let \( D = \bigcup_{i=1}^{r} D_i \) be a simple normal crossing divisor (where \( D_i \) are smooth divisors) and denote the log structure on \( X \) associated to \( D \) by \( M_X \). Then, an exact closed immersion \( (X, M_X) \hookrightarrow (X', M_{X'}) \) into a \( p \)-adic fine log formal \( S \)-scheme \( (X', M_{X'}) \) is called an admissible closed immersion if \( X \) is formally log smooth over \( S \), the log structure \( M_{X'} \) is induced by a relative simple normal crossing divisor \( \mathcal{D} = \bigcup_{i=1}^{r} D_i \) (where \( D_i \) are formally smooth over \( S \)) such that \( D_i = X \times_X D_i \) holds. Moreover, it is called a strongly admissible closed immersion if each \( D_i \) is defined by a single element \( t_i \in \Gamma(X, O_X) \).

Let the notation be as above and assume that \( (X, M_X) \hookrightarrow (X', M_{X'}) \) is a strongly admissible closed immersion. Then an isocrystal \( E \) on \((X'/S)_{\text{log}}\) induces a log-\nabla\text{-module on } \left| X \right|_X \text{ with respect to } t_1, \cdots, t_r.

**Lemma 1.6.** Let \( X \) be a smooth scheme over \( k \), let \( D = \bigcup_{i=1}^{r} D_i \) be a simple normal crossing divisor (where \( D_i \) are smooth divisors) and denote the log structure on \( X \) associated to \( D \) by \( M_X \). Let us assume given the following diagram

\[
\begin{array}{ccc}
(X', M_{X'}) & \xrightarrow{i'} & (X', M_{X'}) \\
\downarrow f & & \downarrow g \\
(X, M_X) & \xrightarrow{i} & (X', M_{X'}),
\end{array}
\]

where \( f \) is a strict etale morphism, \( i, i' \) are strictly admissible closed immersions and \( M_X \) (resp. \( M_{X'} \)) is induced by a relative simple normal crossing divisor \( \mathcal{D} = \bigcup_{i=1}^{r} D_i \) (resp. \( \mathcal{D}' := \bigcup_{i=1}^{r} D'_i \)) with \( g'^*D_i = D'_i \) and let \( t_i \in \Gamma(X, O_X) \) be an element defining \( D_i \). Let \( (E, \nabla) \) be a log-\nabla\text{-module on } \left| X \right|_X \text{ with respect to } t_1, \cdots, t_r, \text{ and denote the pull-back of } (E, \nabla) \text{ by the morphism } \left| X' \right|_X \rightarrow \left| X \right|_X \text{ induced by } g \text{ by } (E', \nabla'). \text{ Then we have the following:}

(1) If \( (E, \nabla) \) has nilpotent residues, \( (E', \nabla') \) also has nilpotent residues.
(2) If the homomorphisms \( \Gamma(\left| D_i \right|_{D_i}, O_{\left| D_i \right|_{D_i}}) \rightarrow \Gamma(\left| D'_i \right|_{D'_i}, O_{\left| D'_i \right|_{D'_i}}) \) \( (1 \leq i \leq r) \) induced by \( g \) are all injective, the converse is also true.

**Proof.** The residue of \( (E, \nabla) \) along \( \left| D_i \right|_{D_i} \) (= zero locus of \( t_i \)) is sent to the residue of \( (E', \nabla') \) along \( \left| D'_i \right|_{D'_i} \) by the homomorphism

\[
\text{End}_{O_{\left| D_i \right|_{D_i}}}(E_{\left| D_i \right|_{D_i}}) \rightarrow \text{End}_{O_{\left| D'_i \right|_{D'_i}}}(E_{\left| D'_i \right|_{D'_i}})
\]

induced by \( g \), and this map is injective if the map

\[
\Gamma(\left| D_i \right|_{D_i}, O_{\left| D_i \right|_{D_i}}) \rightarrow \Gamma(\left| D'_i \right|_{D'_i}, O_{\left| D'_i \right|_{D'_i}})
\]
is injective. The assertion of the lemma is easily deduced by these facts. □

**Lemma 1.7.** The condition in Lemma 1.6(2) is satisfied in the following cases.

1. The case where \( f \) is the identity map and \( g \) is strict formally smooth.
2. The case \( g \) is formally etale, the diagram (1.2) is Cartesian and the images of the morphisms \( D'_i := X' \times_X D_i \to D_i \) induced by \( f \) are dense for all \( 1 \leq i \leq r \).

**Proof.** First we prove (1). We may work Zariski locally. Let \( \hat{D}_i, \hat{D}'_i \) be the completion of \( D_i, D'_i \) along \( D_i \) and assume that \( \hat{D}_i = \text{Spf } A \) is affine. Then, by [Sh3, 2.31], we have the isomorphism \( \hat{D}'_i \cong \text{Spf } A[[x_1, \ldots, x_n]] \) for some \( n \) Zariski locally on \( D_i \). So the map

\[
\Gamma(D_i[D_i, O_{D_i[D_i}] = \Gamma(\hat{D}_i, K, O_{\hat{D}_i, K}) \to \Gamma(\hat{D}'_i, K, O_{\hat{D}'_i, K}) = \Gamma(D'_i[D'_i, O_{\hat{D}'}_i])
\]

is injective.

Next we prove (2). We may assume that \( D_i \) is affine. Let \( \hat{D}_i, \hat{D}'_i \) be the completion of \( D_i, D'_i \) along \( D_i, D'_i \) respectively. Zariski locally on \( D_i \), we can take a Cartesian diagram

\[
\begin{array}{ccc}
D_i & \xrightarrow{m} & D_i \\
\downarrow & & \downarrow \\
\text{Spec } k[x_1, \ldots, x_m] & \xrightarrow{m} & \text{Spf } V \{x_1, \ldots, x_n\}
\end{array}
\]

for some \( m \leq n \), where the right vertical arrow is formally etale and the lower horizontal arrow is defined by \( x_{m+1} = \cdots = x_n = 0 \) and the closed immersion \( \text{Spec } k \hookrightarrow \text{Spf } V \). Let us define \( D_{i,0} \subset D_i \) as the zero locus \( x_{m+1} = \cdots = x_n = 0 \). Then \( D_{i,0} \) is formally smooth over \( S \) and \( D_i = D_{i,0} \times_S S \) holds. Moreover, we have the isomorphism \( \hat{D}_i \cong \text{Spf } A[[x_{m+1}, \ldots, x_n]] \), where \( D_{i,0} := \text{Spf } A \). So we have a retraction \( \hat{D}_i \to D_{i,0} \) defined by the natural inclusion \( A \hookrightarrow A[[x_{m+1}, \ldots, x_n]] \). Let \( D'_{i,0} \to D_{i,0} \) be the unique formally etale morphism lifting \( D'_i \to D_i \). Then both \( \hat{D}'_i \to \hat{D}_i \) and \( D'_{i,0} \times_{D_{i,0}} \hat{D}_i \to \hat{D}_i \) are formally etale morphisms lifting \( D'_i \to D_i \). So they are isomorphic. So we have \( \Gamma(D'_i, O_{\hat{D}'_i}) = \Gamma(D'_{i,0}, O_{\hat{D}'_{i,0}}) \hat{\otimes}_A A[[x_{m+1}, \ldots, x_n]] \).

Hence, to prove that the map

\[
\Gamma(D_i[D_i, O_{D_i[D_i}] = \Gamma(\hat{D}_i, K, O_{\hat{D}_i, K}) \to \Gamma(D'_{i,0}, K, O_{\hat{D}'_{i,0}, K}) = \Gamma(D'_{i,0}, K, O_{\hat{D}'_{i,0}, K}) \to \Gamma(D'_i[D'_i, O_{\hat{D}'_i}] = \Gamma(D'_i[D'_i, O_{\hat{D}'_i}])
\]

is injective, it suffices to prove that the map \( \Gamma(D_{i,0}, K, O_{D_{i,0}, K}) \to \Gamma(D'_{i,0}, K, O_{D'_{i,0}, K}) \) is injective. Using the formal etaleness of \( D'_{i,0} \to D_{i,0} \), the denseness of the image of \( D'_{i,0} \) in \( D_i \) and the \( p \)-torsion freeness of \( \Gamma(D_{i,0}, O_{D_{i,0}}) \) and \( \Gamma(D'_{i,0}, O_{D'_{i,0}}) \), we see easily the injectivity of the map \( \Gamma(D_{i,0}, K, O_{D_{i,0}, K}) \to \Gamma(D'_{i,0}, K, O_{D'_{i,0}, K}) \). So we are done. □
Lemma 1.8. Let us assume given a diagram (1.2) and assume that \( g \) is formally smooth and that the images of the morphisms \( D'_i := X' \times_X D_i \longrightarrow D_i \) induced by \( f \) are dense for all \( 1 \leq i \leq r \). Then \((E, \nabla)\) has nilpotent residues if and only if \((E', \nabla')\) has nilpotent residues. (Here \((E, \nabla), (E', \nabla')\) are as in Lemma 1.6.)

Proof. We may replace \((\mathcal{X}', M_{\mathcal{X}'})\) by a Zariski open covering of it. So we may assume by [Sh2, claim in p.81] that there exists a strictly formally etale morphism \((\mathcal{X}''', M_{\mathcal{X}'''}) \longrightarrow (\mathcal{X}, M_{\mathcal{X}})\) such that \((\mathcal{X}', M_{\mathcal{X}'}) = (X, M_X) \times_{(\mathcal{X}, M_{\mathcal{X}})} (\mathcal{X}''', M_{\mathcal{X}'''})\) holds. Let us put \((\mathcal{X}''', M_{\mathcal{X}'''}) := (\mathcal{X}', M_{\mathcal{X}'}) \times_{(\mathcal{X}, M_{\mathcal{X}})} (\mathcal{X}''', M_{\mathcal{X}'''})\), and let us denote the restriction of \((E, \nabla)\) to \([X'[\mathcal{X}'], X'']\), \((E'', \nabla'')\), \((E''', \nabla''')\) respectively. Then \((E, \nabla)\) has nilpotent residues if and only if \((E'', \nabla'')\) by Lemma 1.6 and Lemma 1.7(2), and \((E'', \nabla'')\) has nilpotent residues if and only if \((E''', \nabla''')\) by Lemma 1.6 and Lemma 1.7(1), and \((E''', \nabla''')\) has nilpotent residues if and only if \((E', \nabla')\) again by Lemma 1.6 and Lemma 1.7(1).

Proposition 1.9. Let \(X\) be a smooth scheme over \(k\), let \(D\) be a normal crossing divisor on \(X\) and denote the log structure associated to \(D\) by \(M_X\). Let us assume given a diagram

\[
\begin{array}{ccc}
(X', M_{X'}) & \longrightarrow & (\mathcal{X}', M_{\mathcal{X}'}) \\
\downarrow & & \downarrow \\
(X, M_X)
\end{array}
\]

where \(f\) is a strict etale morphism and \(i\) is a strongly admissible closed immersion. Let \(\mathcal{D}' = \bigcup \mathcal{D}'_i\) be relative the simple normal crossing divisor on \(\mathcal{X}'\) associated to \(M_{\mathcal{X}'}\) such that each \(\mathcal{D}'_i\) is a formally smooth divisor defined by \(t_i \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})\) \((1 \leq i \leq r)\) and put \(D'_i := X' \times_X \mathcal{D}_i\). Let \(E\) be a locally free isocrystal on \((X/\mathcal{S})^\text{log}_{\text{conv}}\) and denote the log-\(\nabla\)-module on \([X'[\mathcal{X}], X'']\) with respect to \(t_1, \ldots, t_r\) induced by \(E\) by \((E, \nabla)\). Then, if \(E\) has nilpotent residues in the sense of Definition 1.4, \((E, \nabla)\) has nilpotent residues and it is unipotent on

\[
(1.3) \quad [D'_I[X] \cong D'_I[D] \times A_K^{[I]}[0, 1]
\]

for any \(I \subseteq \{1, \ldots, r\}\) (where \(D'_I := \bigcap_{i \in I} D'_i, \mathcal{D}'_I := \bigcap_{i \in I} \mathcal{D}'_i\)).

Proof. Let us take a strict etale covering \(\coprod_j(X_j, M_{X_j}) \longrightarrow (X, M_X)\) and a strongly admissible closed immersions \((X_j, M_{X_j}) \hookrightarrow (\mathcal{X}_j, M_{\mathcal{X}_j})\) such that the log-\(\nabla\)-module \((E_j, \nabla_j)\) induced by \(E\) has nilpotent residues. (There exists such morphisms by the definition of ‘having nilpotent residues’ for isocrystals.) Let us put \((X'_j, M_{X'_j}) := (X', M_{X'}) \times_{(X, M_X)} (X_j, M_{X_j})\). By replacing \((X'_j, M_{X'_j})\) by Zariski open covering of it, we may assume that there exist strict formally etale morphisms \((\mathcal{X}', M_{\mathcal{X}'}) \longrightarrow (\mathcal{X}_j, M_{\mathcal{X}_j})\) such that \((X'_j, M_{X'_j}) = (X', M_{X'}) \times_{(\mathcal{X}', M_{\mathcal{X}'})} (\mathcal{X}_j, M_{\mathcal{X}_j})\) holds. Moreover, we may assume that the closed immersions \((X'_j, M_{X'_j}) \hookrightarrow (\mathcal{X}'_j, M_{\mathcal{X}'_j}), (X'_j, M_{X'_j}) \hookrightarrow (\mathcal{Y}_j, M_{\mathcal{Y}_j})\) are strongly admissible closed immersion and that the relative simple normal crossing divisor \(\mathcal{C}'_j, \mathcal{C}_j\) on \(\mathcal{Y}', \mathcal{Y}_j\) corresponding to the log structure \(M_{\mathcal{Y}'}, M_{\mathcal{Y}_j}\) has the same number of irreducible smooth components. Let us put \(C'_i := \bigcup_{i=1}^s C'_i, C_j := \)
\[ \bigcup_{i=1}^{n} C_j_i. \] Then let us define \((Y'_j, M_{Y'_j})\) to be ‘the logarithmic product’ of \((Y', M_Y)\) and \((Y_j, M_{Y_j})\) over \(S\): That is, we define \(Y'_j\) by
\[
Y'_j := \text{blow-up of } Y' \times_S Y_j \text{ along } \bigcup_{i=1}^{r} \left( \text{pr}_1^{-1}(C'_i) \cap \text{pr}_2^{-1}(C_j,i) \right)
\]
and define the log structure \(M_{Y'_j}\) to be the log structure associated to the inverse image of \(C'\) (which is equal to the inverse image of \(C_j\)). Then we have the commutative diagram
\[
\begin{array}{ccc}
(X', M_{X'}) & \leftarrow & \coprod_j (X'_j, M_{X'_j}) \\
\cap & \cap & \cap \\
(X', M_{X'}) & \leftarrow & (Y'_j, M_{Y'_j}) \rightarrow (X_j, M_{X_j})
\end{array}
\]
where the vertical arrows are (the disjoint union of) strongly admissible closed immersions, the upper horizontal arrows are strict etale surjective morphisms and the lower horizontal arrow is strict formally smooth morphisms.

Let \((E'_j, \nabla_j)\) be the restriction of \((E, \nabla)\) to \(]X'_j[\text{Y}'_j, which is the same as the restriction of \((E_j, \nabla_j)\) to \(]X'_j[\text{Y}'_j. Then, by Lemma 1.8 and the nilpotence of the residues of \((E_j, \nabla_j), (E'_j, \nabla'_j)\) has nilpotent residues, and by using Lemma 1.8 again, we see that \((E, \nabla)\) has nilpotent residues. So we have proved the former assertion of the proposition.

Next we prove the latter assertion. We may work Zariski locally on \(X'\) by [Ke4, 3.3.5]. So we may assume the existence of a Cartesian diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{c} & X' \\
\downarrow & & \downarrow \\
\text{Spec } k[t_1, \ldots, t_m] & \xrightarrow{c} & \text{Spf } V\{t_1, \ldots, t_n\}
\end{array}
\]
for some \(r \leq m \leq n\), where the right vertical arrow is formally etale, the lower horizontal arrow is defined by \(t_{m+1} = \cdots = t_n = 0\) and the closed immersion \(\text{Spec } k \hookrightarrow \text{Spf } V\) and each \(D'_i(1 \leq i \leq r)\) is the zero locus of \(t_i\). Let \(X'_0\) be the zero locus \(t_{m+1} = \cdots = t_n = 0\) in \(X'\), denote the completion of \(X'\), \(X'_0\) along \(D'_i\) by \(\hat{X}'_i, \hat{X}'_0\), respectively and denote the pull-back of \(D'_i\) to \(\hat{X}'_i, \hat{X}'_0\) by \(\hat{D}'_i, \hat{D}'_{0,i}\), respectively.

For disjoint subsets \(L, L'\) of \(\{1, \ldots, n\}\), let us define \(A_k^L := \text{Spec } k[t_i (i \in L)], \hat{A}_V^{L,L'} := \text{Spf } V\{t_i (i \in L)\}[t_i (i \in L')]\). Then, if we put \(I^c := \{1, \ldots, m\} - I, J := \{m + 1, \ldots, n\}\), we have the following Cartesian diagrams
\[
\begin{array}{ccc}
D'_I & \xrightarrow{c} & \hat{X}'_0 \\
\downarrow & & \downarrow \\
A_k^c & \xrightarrow{c} & \hat{A}_V^{c,I} \\
\end{array}
\quad
\begin{array}{ccc}
D'_I & \xrightarrow{c} & \hat{D}'_{0,I} \\
\downarrow & & \downarrow \\
A_k^c & \xrightarrow{c} & \hat{A}_V^{c,I,J}
\end{array}
\quad
\begin{array}{ccc}
D'_I & \xrightarrow{c} & \hat{D}'_I \\
\downarrow & & \downarrow \\
A_k^c & \xrightarrow{c} & \hat{A}_V^{c,0} \\
\end{array}
\quad
\begin{array}{ccc}
D'_I & \xrightarrow{c} & \hat{A}_V^{c,J}
\end{array}
\]
where the vertical arrows are formally etale and the horizontal arrows are natural closed immersions. Let us define the morphisms

\[
\hat{A}_V^{I, J} \leftrightarrow s_J \hat{A}_V^{I, J, J}
\]

as the morphism induced by the natural inclusion of rings of the form \(V\{t_i (i \in L')\}[t_i (i \in L')] \hookrightarrow V\{t_i (i \in L')\}[t_i (i \in L'')]\), and let us put \(s := s_J \circ s_J = s_J \circ s_J\). Then, both \(D'_0, i \times \hat{A}_V^{I, J} \rightarrow \hat{A}_V^{I, J, J}, \hat{A}_V^{I, J} \rightarrow \hat{A}_V^{I, J, J}\)

are formally etale morphism lifting the morphism \(D'_0 \rightarrow \hat{A}_k\). So there exists an isomorphism \(\hat{D}'_0, i \times \hat{A}_V^{I, J, J} \cong \hat{A}_V^{I, J, J}\), and it induces the isomorphisms \(\hat{D}'_0, i \times \hat{A}_V^{I, J, J} \cong \hat{A}_V^{I, J, J} \cong \hat{D}'_0\). Thus we have the isomorphisms \(\varphi_0 : \hat{X}_0' \leftarrow \hat{D}'_0, i \times \hat{A}_V^{I, J, J} \rightarrow \hat{D}'_0, i \times \hat{A}_V^{I, J, J}\).

Let us define the morphisms \(\pi, \pi'\) by

\[
\pi : \hat{D}'_0 \rightarrow \hat{D}'_0, i \times \hat{A}_V^{I, J, J} \xrightarrow{id \times s_J} \hat{D}'_0, i,
\]

\[
\pi' : \hat{X}_0' \leftarrow \hat{D}'_0, i \times \hat{A}_V^{I, J, J} \xrightarrow{id \times s_J} \hat{D}'_0, i \times \hat{A}_V^{I, J, J} \rightarrow \hat{X}_0'.
\]

Then we have the commutative diagram

\[
\begin{array}{ccc}
\hat{X}_0' & \xrightarrow{\varphi} & \hat{D}'_0, i \times \hat{A}_V^{I, J, J} \\
\downarrow \pi' & & \downarrow \pi \times \text{id} \\
\hat{X}_0' & \rightarrow & \hat{D}'_0, i \times \hat{A}_V^{I, J, J} \\
\end{array}
\]

and it induces the commutative diagram of rigid spaces

\[
\begin{array}{ccc}
|D'_0, i'\{\mathcal{X}'\}_{0'} & \rightarrow & |D'_0, i \times \mathcal{A}_k^{I, J}\{0, 1\} \\
\downarrow \pi' \times \text{id} & & \downarrow \pi \times \text{id} \\
|D'_0, i'\{\mathcal{X}'\}_{0'} & \rightarrow & |D'_0, i \times \mathcal{A}_k^{I, J}\{0, 1\}.
\end{array}
\]

Let \((E_0, \nabla_0)\) be the log-\(\nabla\)-module on \(|X'\{\mathcal{X}'\}_{0'}\) with respect to \(t_1, \ldots, t_r\) induced by \(\mathcal{E}\). Then it is unipotent on \(|D'_0, i \times \mathcal{A}_k^{I, J}\{0, 1\}\) relative to \(|D'_0, i \times \mathcal{A}_k^{I, J}\{0, 1\}\) by Proposition 1.3.

By pulling-back by \(\pi \times \text{id}\), we see that \((E, \nabla)\) is unipotent on \(|D'_0, i \times \mathcal{A}_k^{I, J}\{0, 1\}\) relative to \(|D'_0, i \times \mathcal{A}_k^{I, J}\{0, 1\}\). So we have proved the latter assertion of the proposition.

**Corollary 1.10.** Let \(X\) be a smooth scheme over \(k\), let \(D\) be a normal crossing divisor on \(X\), let \(M_X\) be the log structure on \(X\) associated to \(D\) and let \(\mathcal{E}\) be a locally free isocrystal on \((X/S)\)\(_{\text{conv}}^\log = ((X, M_X)/S)_{\text{conv}}\). Then:
(1) When \(D\) is a simple normal crossing divisor, \(\mathcal{E}\) has nilpotent residues in the sense in Definition 1.4 if and only if it has nilpotent residues in the sense in [Ke4, 6.4.4].

(2) The definition of ‘having nilpotent residues’ is independent of the choice of an etale covering \(\prod X_j \to X\) and a diagram like (1.1) chosen in Definition 1.4.

Next we prove the functoriality of the notion of ‘having nilpotent residues’:

**Proposition 1.11.** Let \(X, X'\) be smooth schemes over \(k\), let \(D\) (resp. \(D')\) be a normal crossing divisor on \(X\) (resp. \(X')\), let \(M_X\) (resp. \(M_{X'}\)) be the log structure on \(X\) (resp. \(X')\) associated to \(D\) (resp. \(D')\) and let us assume given a morphism \(f : (X', M_{X'}) \to (X, M_X)\). Then, for a locally free isocrystal \(\mathcal{E}\) on \((X/S)^{\log}_{\text{conv}}\) having nilpotent residues, the pull-back \(f^* \mathcal{E}\) of \(\mathcal{E}\) to \((X'/S)^{\log}_{\text{conv}}\) also has nilpotent residues.

**Proof.** We may work etale locally on \(X'\). So we may assume that \(D, D'\) are simple normal crossing divisors and that there exist strongly admissible closed immersions \((X, M_X) \to (\mathcal{X}, M_{\mathcal{X}}), (X', M_{X'}) \to (\mathcal{X}', M_{\mathcal{X}'})\) satisfying \((X, M_X) = (\mathcal{X}, M_{\mathcal{X}}) \times_S S, (X', M_{X'}) = (\mathcal{X}', M_{\mathcal{X}'}) \times_S S\). Then, locally on \(X'\), we have a morphism \(g : (\mathcal{X}', M_{\mathcal{X}'}) \to (\mathcal{X}, M_{\mathcal{X}})\) lifting \(f\). Let \((E, \nabla)\) be the log-\(\nabla\)-module on \(X'\) induced by \(\mathcal{E}\) and let \((E', \nabla')\) be the log-\(\nabla\)-module on \(X'\) induced by \(f^* \mathcal{E}\). Then \((E', \nabla')\) is nothing but the pull-back of \((E, \nabla)\) by \(g\). We may assume that \(E\) is free on \(X'\).

Let \(\mathcal{D} = \bigcup_{i=1}^r D_i\) (resp. \(\mathcal{D}' := \bigcup_{i=1}^r D'_i\)) be the relative simple normal crossing divisor on \(\mathcal{X}\) (resp. \(\mathcal{X}'\)) corresponding to \(M_X\) (resp. \(M_{X'}\)) and let \(t_i \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})\) \((1 \leq i \leq r)\) be an element defining \(D_i\). For fixed \(i \leq i \leq r\), we may replace \(\mathcal{D}', \mathcal{X}\) by \(\mathcal{D}' - \bigcup_{j \neq i} D'_j, \mathcal{X} - \bigcup_{j \neq i} D'_j\) to prove the nilpotence of the residue of \((E', \nabla')\) along \(D'_i\). (See the proof of Lemma 1.7.) So we may assume that \(\mathcal{D}'\) is a smooth divisor. Let \(s \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})\) be an element defining \(\mathcal{D}'\). Since \(g\) is a morphism of log schemes, \(g^* t_i\) has the form \(u_i s^{n_i}\) for some \(u_i \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})\) and \(n_i \in \mathbb{N}\).

Let us denote the residue of \((E, \nabla)\) along \(D_i\) by \(\text{res}\) and let us put \(I := \{i \mid n_i > 0\}\). For \(i \in I\), denote the map \(\text{End}(E|_{\mathcal{D}_i}) \to \text{End}(E'|_{\mathcal{D}'_i})\) (induced by \(f\) and \(g\)) by \(f_i^*\). If we put \(D_i := \cap_{i \in I} D_i, D'_i := \cap_{i \in I} D'_i\), the map \(f_i^*\) for \(i \in I\) factors as

\[
\text{End}(E|_{\mathcal{D}_i}) \to \text{End}(E|_{\mathcal{D}'_i}) \to \text{End}(E'|_{\mathcal{D}'_i}),
\]

where the first map is the restriction and the second map is the one induced by \(f\) and \(g\) which is independent of \(i \in I\). Since \((E, \nabla)\) is integrable, we have \(\text{res}_i \circ \text{res}_j = \text{res}_j \circ \text{res}_i\) in \(\text{End}(E|_{\mathcal{D}_i})\). So we have \(f_i^* (\text{res}_i) \circ f_j^* (\text{res}_j) = f_j^* (\text{res}_j) \circ f_i^* (\text{res}_i)\).

Simple computation implies that the residue \(\text{res}'\) of \((E', \nabla')\) along \(\mathcal{D}'_i\) is given by \(e \mapsto \sum_{i \in I} n_i f_i^* (\text{res}_i)(e)\). Then, if we take a positive integer \(N\) satisfying \(\text{res}_i^N = 0\) for \(1 \leq i \leq r\), we have

\[
(\text{res}')^N (\prod_{i \in I} n_i f_i^* (\text{res}_i)) = 0.
\]

So we are done. \(\square\)

Next we prove a relation between Frobenius structure and the nilpotence of residues. (This is already remarked, for example, in [Ke4, 7.1.4].)
Proposition 1.12. Let \( X \) be a smooth scheme over \( k \), let \( D \) be a normal crossing divisor on \( X \), let \( M_X \) be the log structure on \( X \) associated to \( D \) and let \((\mathcal{E},\alpha)\) be an \( F \)-isocrystal on \((X/S)_{\text{conv}}=((X,M_X)/S)_{\text{conv}}\). Then \( \mathcal{E} \) is locally free and it has nilpotent residues.

Proof. By [Sh2, 2.4.3], \( \mathcal{E} \) is locally free. To show that \( \mathcal{E} \) has nilpotent residues, we may assume that \( D \) is a simple normal crossing divisor, and we may assume the existence of a strongly admissible closed immersion \((X,M_X) \hookrightarrow (X',M_{X'})\) with \((X,M_X) = (X',M_{X'}) \times_S S\) (by [Sh2, 2.4.3]).

Let \( \mathcal{D} = \bigcup_{i=1}^r D_i \) and \( D_i \) is defined by \( t_i \) and \( D_i = \bigcup_{i=1}^r D_i \) endowed with a lift \( F : (X',M_{X'}) \to (X,M_X) \) of Frobenius on \((X,M_X)\) compatible with \( \sigma \) satisfying \( F^*t_i = t_{i}^{p} (1 \leq i \leq r) \). We may assume that the log-\( \nabla \)-module \((E,\nabla)\) on \([X]_s\) induced by \( \mathcal{E} \) is free. Moreover, by the same argument as the proof of Proposition 1.11, we may assume that \( \mathcal{D} \) is a smooth divisor (hence \( r = 1 \)). Let us denote the residue of \((E,\nabla)\) along \([D]_P\) by \( \mathcal{E} \). Then, on

\[
[D_X =] D_P \times A_K^1[0,1],
\]

we have, by [Ba-Ch, 1.5.3], a polynomial \( P(x) \in K[x] \) such that \( P(\mathcal{E}) = 0 \) holds. Take \( P(x) \) to be minimal and monic. By considering the image of \( \mathcal{E} \) in \( \text{End}(F^*E|_{[D]_P}) \), we see that \( P^a(p^a\mathcal{E}) = 0 \) holds, where \( \mathcal{E}' \) is the residue of \( F^*(E,\nabla) \) along \([D]_P\). Since \( F^*(E,\nabla) \) is isomorphic to \((E,\nabla)\) via the Frobenius structure \( \alpha \), we have \( P^a(p^a\mathcal{E}) = 0 \). So \( P^a(p^a) \) is equal to \( P(x) \) up to a multiplication by a non-zero constant, and so we have \( P(x) = x^m \) for some \( m \). Hence \( \mathcal{E} \) is nilpotent.

Finally in this section, we recall the statement of the semi-stable reduction conjecture for overconvergent \( F \)-isocrystals, which is shown recently by Kedlaya ([Ke4], [Ke5], [Ke6], [Ke7]).

Theorem 1.13 (Semistable reduction conjecture, Kedlaya). Let \((X,\mathcal{X})\) be a pair over \( k \) and let \( \mathcal{E} \) be an overconvergent \( F \)-isocrystal on \((X,\mathcal{X})/S_K\). Then there exists a strict morphism of pairs \( \varphi : (X',\mathcal{X}') \to (X,\mathcal{X}) \) such that \( \varphi : \mathcal{X}' \to \mathcal{X} \) is proper surjective and generically etale, \( D' := \mathcal{X}' - X' \) is a simple normal crossing divisor on \( \mathcal{X}' \), and if we denote the log structure on \( \mathcal{X}' \) associated to \( D' \) by \( M_{\mathcal{X}'} \), there exists an \( F \)-isocrystal \( \mathcal{F} \) on \((\mathcal{X}'/\mathcal{S})_{\text{conv}}=((\mathcal{X}',M_{\mathcal{X}'})/\mathcal{S})_{\text{conv}}\) satisfying \( \varphi^*\mathcal{E} = j^!\mathcal{F} \), where \( j^! \) is the functor of restriction to overconvergent isocrystals on \((X',\mathcal{X}')/S_K\).

As pointed out in [Ke4, 7.1.4], the conjecture remains true if we require \( \mathcal{F} \) to be a locally free isocrystal (without Frobenius structure) having nilpotent residues: Indeed, by Proposition 1.12, any \( F \)-isocrystal on \((\mathcal{X}'/\mathcal{S})_{\text{conv}}\) is locally free and has nilpotent residues, and if we have a locally free isocrystal \( \mathcal{F} \) on \((\mathcal{X}'/\mathcal{S})_{\text{conv}}\) having nilpotent residues satisfying \( \varphi^*\mathcal{E} = j^!\mathcal{F} \), we can extend the Frobenius structure on \( \varphi^*\mathcal{E} \) to that on \( \mathcal{F} \) by using [Ke4, 6.4.5].

To state the main result in this paper without using Frobenius structure, we introduce a terminology of ‘potential semi-stability’ of overconvergent isocrystals:

Definition 1.14. Let \((X,\mathcal{X})\) be a pair over \( k \) and let \( \mathcal{E} \) be an overconvergent isocrystal on \((X,\mathcal{X})/S_K\). \( \mathcal{E} \) is called potentially semi-stable if there exists a strict morphism

\[
\varphi : (X',\mathcal{X}') \to (X,\mathcal{X}) \]
of pairs $\varphi : (X', \overline{X'}) \to (X, \overline{X})$ such that $\varphi : \overline{X} \to \overline{X}$ is proper surjective and generically étale, $D' := \overline{X} - X'$ is a simple normal crossing divisor on $\overline{X}'$, and if we denote the log structure on $\overline{X}'$ associated to $D'$ by $M_{\overline{X}'},$ there exists a locally free isocrystal $\mathcal{F}$ on $(\overline{X}/S)^{\log}_{\text{conv}} = ((\overline{X}, M_{\overline{X}})/S)_{\text{conv}}$ having nilpotent residues satisfying $\varphi^* \mathcal{E} = j^! \mathcal{F},$ where $j^!$ is the functor of restriction to overconvergent isocrystals on $(X', \overline{X})/S_K$.

2. Overconvergence in log smooth case

In this section, we prove the overconvergence of relative rigid cohomology for (not necessarily strict) morphisms of certain pairs which admit ‘nice’ log structures.

Let us assume given a diagram

$$
(X, M_{\overline{X}}) \xrightarrow{f} (Y, M_{\overline{Y}}) \xrightarrow{\iota} (Y, M_{\overline{Y}}),
$$

where $f$ is proper log smooth, $\iota$ is an exact closed immersion, $X, Y$ are smooth over $S$, $M_{\overline{X}}$ (resp. $M_{\overline{Y}}$) is the log structure induced by some normal crossing divisor (resp. simple normal crossing divisor) and $(Y, M_{\overline{Y}})$ is of Zariski type and formally log smooth over $S$. We denote the normal crossing divisor corresponding to $M_{\overline{X}}$ by $D$. We assume moreover that $D$ has the decomposition $D = D^h \cup D^v$ into sub normal crossing divisors $D^h, D^v$ satisfying the following condition $(\ast)$:

$$(\ast): \quad \text{If we denote the log structure on } \overline{X} \text{ associated to } D^v \text{ by } M_{\overline{X}}^v \text{ and define } D_{[i]} (i \geq 0) \text{ by }$$

$$D[0] := \overline{X}, \quad D[1] := \text{the normalization of } D^h,$$

$$D[i] := i \text{-fold fiber product of } D[1] \text{ over } \overline{X},$$

$$(D[i], M_{\overline{X}}^v|_{D[i]}) \text{ is log smooth over } (\overline{Y}, M_{\overline{Y}}) \text{ for any } i \geq 0.$$

Then we have the following comparison theorem between the log analytic cohomology of $(X, M_{\overline{X}})/(\mathcal{O}_{\overline{Y}})$ and the log rigid cohomology of $((\overline{X} - D^h, M_{\overline{X}|_{\overline{X} - D^h}}), (X, M_{\overline{X}}))/(\mathcal{O}_{\overline{Y}})$.

**Theorem 2.1.** In the situation above, let $\mathcal{E}$ be a locally free isocrystal on $(\overline{X}/S)^{\log}_{\text{conv}} = ((\overline{X}, M_{\overline{X}})/S)_{\text{conv}}$ having nilpotent residues and denote the restriction of $\mathcal{E}$ to $\mathcal{I}^{\text{conv}}_{\mathcal{O}_{\overline{Y}}}$ by the same symbol. Assume moreover the following condition $(\ast):$

$$(\ast): \quad R^q f_{\overline{X}/\overline{Y}, \text{an}*} \mathcal{E} \text{ is a coherent } \mathcal{O}_{\overline{Y}}\text{-module for any } q \in \mathbb{N}.$$

Then, if we denote the restriction functor

$$\mathcal{I}^{\text{conv}}_{\mathcal{O}_{\overline{Y}}}(\overline{X}/\mathcal{O}_{\overline{Y}})^{\log} \to \mathcal{I}^!((\overline{X} - D^h, \overline{X})/\mathcal{O}_{\overline{Y}})^{\log}$$

by $j^!$, we have the isomorphism

$$R^q f_{\overline{X}/\overline{Y}, \text{an}*} \mathcal{E} \xrightarrow{\sim} R^q f_{(\overline{X} - D^h, \overline{X})/\mathcal{O}_{\overline{Y}}\text{-rig}*} j^! \mathcal{E} \quad (q \geq 0).$$
Proof. First we prove the existence of a ‘nice’ chart of the morphism $f$. (We take a chart Zariski locally on $\tilde{Y}$ and etale locally on $\tilde{X}$.) Let us fix a chart $\tilde{\beta} : Q \to M_Y$ of $M_Y$ and restrict it to the chart $\beta : Q \to M_{\tilde{Y}}$ of $M_{\tilde{Y}}$. (Note that we can take this chart Zariski locally on $Y$.) Let us consider etale locally and assume that $D^h := \bigcup_{i=1}^r D_i$ is a simple normal crossing divisor and that $D_i$ is defined by $t_i \in \Gamma(X, \mathcal{O}_X)$. Put $D_o := \bigcap_{i=1}^r D_i$, $M_{D_o} := M_{X}|_{D_o}$ and $\mathcal{I} := \text{Ker}(O_X \to O_{D_o})$. Then, since $(D_o, M_{D_o})$ is strict etale over $(D_{[r]}, M_{D_{[r]}})$, it is log smooth over $(\tilde{Y}, M_{\tilde{Y}}).$ So we have the following locally split exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \omega^1_{(X,M_{\tilde{X}})/\tilde{Y},M_{\tilde{Y}}} \big|_{D_o} \longrightarrow \omega^1_{D_o/\tilde{Y}} \longrightarrow 0$$

by [Na-Sh, 2.1.3]. Since $\mathcal{I}/\mathcal{I}^2$ is freely generated by $t_1, \cdots, t_r$ locally, we see that there exists a basis of $\omega^1_{(X,M_{\tilde{X}})/\tilde{Y},M_{\tilde{Y}}}$ locally, consisting of a lift of a basis of $\omega^1_{D_o/\tilde{Y}}$ of the form $\{d\log m_i\}_{i=1}^d$ and $\{dt_i\}_{i=1}^r$. Using this and arguing as in [Ka1, 3.13] (see also [Na-Sh, 2.1.4, 2.1.6]), one can see that there exists a chart $(P \xrightarrow{\alpha} M_X, \beta)$ of the morphism $(\tilde{X}, M_{\tilde{X}}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}})$ extending the given chart $Q \xrightarrow{\beta} M_{\tilde{Y}}$ such that $\gamma$ is injective, $|\text{Coker}(\gamma)|$ is prime to $p$ and that the morphism $(\tilde{X}, M_{\tilde{X}}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}})$ factors as

$$(\tilde{X}, M_{\tilde{X}}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}}) \times_{ \text{Spec} k[Q], M_{\tilde{Q}}} (\text{Spec} k[P \oplus N^r], M_P |_{\text{Spec} k[P \oplus N^r]}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}}),$$

where the first map, which is induced by $\alpha$ and $N \ni e_i \mapsto t_i$, is strict etale and the second map is the one induced by $\beta$ and the composite $Q \xrightarrow{\gamma} \tilde{P} \longrightarrow P \oplus N^r$ (which we denote by $\tilde{\gamma}$). (For a fine monoid $N$ and a ring $R$ or an adic topological ring $R$, $M_N$ denotes the log structure on $\text{Spec} R[N]$ or $\text{Spf} R[N]$ associated to the pre-log structure $N \longrightarrow R[N]$ or $N \longrightarrow R\{N\}$.) By adding the log structure $M_X^h$ associated to $D^h$, the above factorization induces the following factorization of the morphism $f$

$$(2.2) \quad (\tilde{X}, M_{\tilde{X}}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}}) \times_{ \text{Spec} k[Q], M_{\tilde{Q}}} (\text{Spec} k[P \oplus N^r], M_P |_{\text{Spec} k[P \oplus N^r]}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}}),$$

where the first map is strict etale and the second map is induced by $\beta$ and $\tilde{\gamma}$. So we obtain a chart $(P \oplus N^r \to M_{\tilde{X}}, Q \xrightarrow{\beta} M_{\tilde{Y}}, Q \xrightarrow{\tilde{\gamma}} P \oplus N^r)$ of $f$.

Using this chart, we can form the Cartesian diagram

$$(2.3) \quad (\tilde{X}, M_{\tilde{X}}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}}) \times_{ \text{Spec} k[Q], M_{\tilde{Q}}} (\text{Spec} k[P \oplus N^r], M_P |_{\text{Spec} k[P \oplus N^r]}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}})$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$(X, M_X) \longrightarrow (Y, M_Y) \times_{ \text{Spec} V(Q), M_{\tilde{Q}}} (\text{Spf} V(P \oplus N^r), M_P |_{\text{Spec} V(P \oplus N^r)}) \longrightarrow (Y, M_Y),$$

where the top horizontal line is as in (2.2), the vertical arrows are exact closed immersions, the lower right arrow is defined by $\beta$ and $\tilde{\gamma}$, and the lower left arrow is a strict formally etale morphism which makes the left square Cartesian. (The existence of such a morphism follows from [Sh2, claim in p.81]). Since $\gamma$ is injective and $|\text{Coker}(\gamma)|$ is prime to $p$, the lower right arrow in (2.3) is formally log smooth.
So the morphism $(\mathcal{X}, M_\mathcal{X}) \to (\mathcal{Y}, M_\mathcal{Y})$ is a formally log smooth lift of $(\overline{X}, M_{\overline{X}}) \to (\overline{Y}, M_{\overline{Y}})$. So we have shown the existence of a formally log smooth lift of $f$ etale locally on $\overline{X}$.

So we can take a strict etale surjective map $(\overline{X}^{(0)}, M_{\overline{X}^{(0)}}) \to (\overline{X}, M_{\overline{X}})$ and an exact closed immersion $(\overline{X}^{(0)}, M_{\overline{X}^{(0)}}) \hookrightarrow (\mathcal{X}^{(0)}, M_{\mathcal{X}^{(0)}})$ over $(\mathcal{Y}, M_\mathcal{Y})$ such that $(\mathcal{X}^{(0)}, M_{\mathcal{X}^{(0)}})$ is of Zariski type, formally log smooth over $(\mathcal{Y}, M_{\mathcal{Y}})$ and that $(\overline{X}^{(0)}, M_{\overline{X}^{(0)}}) = (\overline{Y}, M_{\overline{Y}}) \times_{(\mathcal{Y}, M_{\mathcal{Y}})} (\mathcal{X}^{(0)}, M_{\mathcal{X}^{(0)}})$ holds. For $m \in \mathbb{N}$, let $(\overline{X}^{(m)}, M_{\overline{X}^{(m)}})$ (resp. $(\mathcal{X}^{(m)}, M_{\mathcal{X}^{(m)}})$) be the $(m+1)$-fold fiber product of $(\overline{X}^{(0)}, M_{\overline{X}^{(0)}})$ (resp. $(\mathcal{X}^{(0)}, M_{\mathcal{X}^{(0)}})$) over $(\overline{X}, M_{\overline{X}})$ (resp. $(\mathcal{Y}, M_{\mathcal{Y}})$). Then we have the embedding system

$$\tag{2.4} (\overline{X}, M_{\overline{X}}) \hookrightarrow (\overline{X}^{(*)}, M_{\overline{X}^{(*)}}) \hookrightarrow (\mathcal{X}^{(*)}, M_{\mathcal{X}^{(*)}}).$$

Let us take a diagram

$$\begin{array}{ccc}
(\overline{Y}, M_{\overline{Y}}) & \xrightarrow{\iota'} & (\mathcal{Y}, M_{\mathcal{Y}}) \\
\downarrow & \swarrow \varphi & \\
(\overline{Y}, M_{\overline{Y}}) & \xrightarrow{\iota} & (\mathcal{Y}, M_{\mathcal{Y}})
\end{array}$$

such that $\iota'$ is a homeomorphic exact closed immersion and that the induced morphism of rigid analytic spaces $\mathcal{Y}^{\dagger}_K = \overline{Y}_{\mathcal{Y}}[y] \to \overline{Y}[y]$ is an admissible open immersion. Let us denote the pull-back of the diagram (2.4) by the morphism $\varphi$ by

$$\tag{2.6} (\overline{X}, M_{\overline{X}}) \hookrightarrow (\overline{X}^{(*)}, M_{\overline{X}^{(*)}}) \hookrightarrow (\mathcal{X}^{(*)}, M_{\mathcal{X}^{(*)}})$$

and put $D^{\text{th}} := D^{\text{th}} \times_{\overline{X}} \overline{X}^{\dagger}$. Let us denote the morphism $(\mathcal{X}^{(*)}, M_{\mathcal{X}^{(*)}}) \to (\mathcal{Y}, M_{\mathcal{Y}})$ by $g$, the morphism $\overline{X}^{(*)} \log_{\mathcal{X}^{(*)}} \to \overline{X}^{(*)}$ by $\text{sp}^{(*)}$ and the morphism $\overline{Y}[y] \to \mathcal{Y}$ by $\text{sp}$. We prove the following claim:

**claim.** The morphism

$$R_{\text{sp}} R f_{\overline{X} / \mathcal{Y}}^{(*)} \text{DR}(\overline{X}^{(*)} \log_{\mathcal{X}^{(*)}} / \mathcal{Y}^{\dagger}_K, \mathcal{E})$$

$$\to R_{\text{sp}} R f_{\mathcal{X} / \mathcal{Y}}^{(*)} \text{DR}(\mathcal{X}^{(*)} \log_{\mathcal{X}^{(*)}} / \mathcal{Y}^{\dagger}_K, j^! \mathcal{E})$$

$$= R_{\text{sp}} R f_{(\overline{X} - D^{\text{th}}, \overline{X}) / \mathcal{Y}^{\dagger}_K, \text{log-log} \text{rig}, j^! \mathcal{E}$$

is a quasi-isomorphism.

To prove the claim, it suffices to prove that the map

$$R_{\text{sp}}^m \text{DR}(\overline{X}^{(m)} \log_{\mathcal{X}^{(m)}} / \mathcal{Y}^{\dagger}_K, \mathcal{E}) \to R_{\text{sp}}^m \text{DR}(\mathcal{X}^{(m)} \log_{\mathcal{X}^{(m)}} / \mathcal{Y}^{\dagger}_K, j^! \mathcal{E})$$

is a quasi-isomorphism. In the following (until the end of the proof of the claim), we omit to write the superscript $(m)$. (So we prove that

$$\tag{2.7} R_{\text{sp}} \text{DR}(\overline{X}^{(*)} \log_{\mathcal{X}^{(*)}} / \mathcal{Y}^{\dagger}_K, \mathcal{E}) \to R_{\text{sp}} \text{DR}(\mathcal{X}^{(*)} \log_{\mathcal{X}^{(*)}} / \mathcal{Y}^{\dagger}_K, j^! \mathcal{E})$$

is a quasi-isomorphism.)
To prove that (2.7) is a quasi-isomorphism, we may work Zariski locally on \( X' \), and by [Sh3, 4.5] and [Sh4, claim in 4.7], the both hand sides of (2.7) are unchanged if we replace the closed immersion \((\overline{X}', M_{\overline{X}'}) \hookrightarrow (X', M_X)\) by another closed immersion into a \( p \)-adic fine log formal \( S \)-scheme which is formally log smooth over \((Y', M_{Y'})\). So we may assume the existence of the Cartesian diagram

\[
\begin{array}{ccc}
(X', M_{X'}) & \longrightarrow & (X', M_{X'}) \\
\downarrow & & \downarrow \\
(\overline{X}'(0), M_{\overline{X}'(0)}) & \longrightarrow & (\overline{X}'(0), M_{\overline{X}'(0)})
\end{array}
\]

such that the right vertical arrow is written as a composite morphism

\[
(X', M_{X'}) \longrightarrow (X'(0), M_{X'(0)}) \longrightarrow (X'(0), M_{X'(0)}),
\]

where the first morphism is strict formally etale and the second morphism is the canonical morphism. Since we may work Zariski locally on \( X' \), we may shrink \( X'(0) \) in order that the exact closed immersion \((\overline{X}'(0), M_{\overline{X}'(0)}) \hookrightarrow (X'(0), M_{X'(0)})\) is a strongly admissible closed immersion. Let \( D^{(0)} = \bigcup_{i=1}^r D_i^{(0)} \) be the relative simple normal crossing divisor corresponding to \( M_{X'(0)} \) and for \( I \subseteq \{1, \ldots, r\} \), let us put \( D_I^{(0)} := \cap_{i \in I} D_i^{(0)} \). Let us denote the pull-back of \( D_I^{(0)} \) to \( X'(0), X', X' \) by \( D_I^{(0)}, D_I, D_I', \) respectively.

Note that \( R^qsp_*(DR([X']_{\log}/Y'_K, E)) \) is the sheaf associated to the presheaf

\[
X' \supseteq \mathcal{U} \mapsto H^q(\mathcal{U}, DR([X']_{\log}/Y'_K, E))
\]

and that \( R^qsp_*(DR^d([X']_{\log}/Y'_K, j^! E)) \) is the sheaf associated to the presheaf

\[
X' \supseteq \mathcal{U} \mapsto H^q(\mathcal{U}, DR^d([X']_{\log}/Y'_K, j^! E)).
\]

Since we may replace \( X' \) by \( \mathcal{U} \), we see that it suffices to prove that the map

\[
H^q(X'_K, DR([X']_{\log}/Y'_K, E)) \longrightarrow H^q(X'_K, DR^d([X']_{\log}/Y'_K, j^! E))
\]

is an isomorphism. If we put the admissible open immersion \( X'_K \rightarrow (D^{\text{hs}})_X, \lambda \hookrightarrow X'_K \) by \( j_\lambda \), we have

\[
H^q(X'_K, DR^d([X']_{\log}/Y'_K, j^! E)) = H^q(X'_K, \lim_{\lambda \rightarrow 1} j_{\lambda *} j_\lambda^* DR([X']_{\log}/Y'_K, E)) = \lim_{\lambda \rightarrow 1} H^q(X'_K, j_{\lambda *} j_\lambda^* DR([X']_{\log}/Y'_K, E)).
\]

So it suffices to prove that the map

\[
H^q(X'_K, DR([X']_{\log}/Y'_K, E)) \longrightarrow H^q(X'_K, j_{\lambda *} j_\lambda^* DR([X']_{\log}/Y'_K, E))
\]

is an isomorphism for \( \lambda \in (0, 1) \). In the following, we denote \( DR([X']_{\log}/Y'_K, E) \) simply by DR.

Let \((E, \nabla)\) be the log-\(\nabla\)-module on \([X']_{\log}/[X']_{\log} = X'_K\) induced by \( E \), and let \((\overline{E}, \overline{\nabla})\) be the log-\(\nabla\)-module on \(X'_K\) relative to \(Y'_K\) induced by \((E, \nabla)\). Then DR is nothing but the de Rham complex associated to \((\overline{E}, \overline{\nabla})\). On the other hand, let
(E^{(0)}, \nabla^{(0)}) be the log-\nabla-module on \([\nabla^{(0)}_{\mathcal{X}^{(0)}}]) induced by \mathcal{E}. Then, by Proposition 1.9, \((E^{(0)}, \nabla^{(0)})\) is unipotent on

\[ |D^{(0)}_{I, \mathcal{X}^{(0)}}| \cong |D^{(0)}_{I, \mathcal{P}^{(0)}}| \times A^{[1]}_K[0, 1]. \]

Then \((E, \nabla)\) is unipotent on

\[ |D_{I, \mathcal{X}^{(0)}}| \cong |D_{I, \mathcal{P}^{(0)}}| \times A^{[1]}_K[0, 1]. \tag{2.8} \]

Next we consider a certain admissible covering of \(\mathcal{X}_K^{(0)}\) introduced by Baldassarri and Chiarellotto ([Ba-Ch2, 4.2]). (See also [Sh2, 2.4].) Let us fix \(\eta \in (\lambda, 1) \cap p^Q\) and for \(I \subseteq \{1, \ldots, r\}\), put

\[
\begin{align*}
P_{I, \eta} := \{x \in \mathcal{X}_K' | |t_i(x)| < 1 (i \in I), |t_i(x)| \geq \eta (i \notin I)\}, \\
V_{I, \eta} := \{x \in \mathcal{X}_K' | |t_i(x)| = 0 (i \in I), |t_i(x)| \geq \eta (i \notin I)\}.
\end{align*}
\]

Then we have the admissible open covering \(\mathcal{X}_K' = \bigcup_{I, \eta} P_{I, \eta}\). For \(m \in \mathbb{N}\), let \(\mathcal{I}_m\) be the set \(\{(I_0, \ldots, I_m) | I_j \subseteq \{1, \ldots, r\}\}\). For \(\mathcal{I} := (I_0, \ldots, I_m) \in \mathcal{I}_m\), let us put

\[ \mathcal{I} := \bigcap_{j=0}^m I_j \text{ and } P_{\mathcal{I}, \eta} := \bigcap_{j=0}^m P_{I_j, \eta}. \]

Then, since we have the spectral sequence

\[ E_1^{s,t} = \bigoplus_{I \in \mathcal{I}_s} H^t(P_{I, \eta}, \mathcal{P}^{\mathcal{I}}, \mathcal{P}^{\mathcal{I}}) \Rightarrow H^{s+t}(\mathcal{X}_K, \mathcal{P}^{\mathcal{I}}), \]

it suffices to prove the map

\[ H^q(P_{I, \eta}, \mathcal{P}^{\mathcal{I}}) \longrightarrow H^q(P_{I, \eta}, j_{\lambda, \mathcal{P}^{\mathcal{I}}}^* \mathcal{P}^{\mathcal{P}^{\mathcal{I}}}) \tag{2.9} \]

is an isomorphism. Note that we have

\[ P_{I, \eta} := \{x \in \mathcal{X}_K' | |t_i(x)| < 1 (i \in \bigcup_{j=0}^m I_j), |t_i(x)| \geq \eta (i \notin \mathcal{I})\}. \]

Let us define \(V_{I, \eta}\) by

\[ V_{I, \eta} := \{x \in \mathcal{X}_K' | |t_i(x)| = 0 (i \notin \mathcal{I}), |t_i(x)| \geq \eta (i \notin \mathcal{I}), \]

\[ |t_i(x)| < 1 (i \in \bigcup_{j=0}^m I_j - \mathcal{I})\}. \]

Then \(P_{I, \eta}\) is a quasi-Stein admissible open set of \([D_{I, \mathcal{X}^{(0)}}]|_\mathcal{P}^{\mathcal{I}}\), \(V_{I, \eta}\) is a quasi-Stein admissible open set of \([D_{I, \mathcal{P}^{(0)}}]|_\mathcal{P}^{\mathcal{I}}\), and the isomorphism (2.8) (\(I\) replaced by \(\mathcal{I}\)) induces the isomorphism

\[ P_{\mathcal{I}, \eta} \cong V_{\mathcal{I}, \eta} \times A^{[1]}_K[0, 1]. \]

Let us take an admissible open covering \(V_{I, \eta} = \bigcup_{j=1}^\infty V_j\) by increasing affinoid admissible open sets. Then we have

\[ H^q(P_{I, \eta}, \mathcal{P}^{\mathcal{I}}) = H^q(R \lim_j \mathcal{P}^{\mathcal{I}}(V_j \times A^{[1]}_K[0, 1], \mathcal{P}^{\mathcal{I}})), \]

\[ H^q(P_{I, \eta}, j_{\lambda, \mathcal{P}^{\mathcal{I}}}^* \mathcal{P}^{\mathcal{P}^{\mathcal{I}}}) = H^q(R \lim_j \mathcal{P}^{\mathcal{P}^{\mathcal{I}}}(V_j \times A^{[1]}_K[0, 1], j_{\lambda, \mathcal{P}^{\mathcal{P}^{\mathcal{I}}}})) \]

and we are reduced to showing that the map

\[ H^q(V_j \times A^{[1]}_K[0, 1], \mathcal{P}^{\mathcal{P}^{\mathcal{I}}}) \longrightarrow H^q(V_j \times A^{[1]}_K[0, 1], j_{\lambda, \mathcal{P}^{\mathcal{P}^{\mathcal{I}}}}) \tag{2.10} \]
is an isomorphism for each $q \geq 0$. Then, since $V_j \times A_W^1(0,1)$ and the map $j_{\lambda}$ are quasi-Stein, the map (2.10) is identical with the map

$$
H^q(\Gamma(V_j \times A_W^1(0,1), DR)) \rightarrow H^q(\Gamma(V_j \times A_W^1(\lambda,1), DR)).
$$

So it suffices to prove that the map (2.11) is an isomorphism. Let $(\mathcal{E}, \nabla)$ be the log-$\nabla$-module on $V_j \times A_W^1(0,1)$ relative to $V_j$ induced by $(\mathcal{E}, \nabla)$. Then, by using the spectral sequence of Katz-Oda type for $V_j \times A_W^1(0,1) \rightarrow V_j \rightarrow Y_K$, we may replace DR by the log de Rham complex associated to $(\mathcal{E}, \nabla)$. Since the restriction of $(E, \nabla)$ to $V_j \times A_W^1(0,1)$ is unipotent relative to $V_j$, $(\mathcal{E}, \nabla)$ is a successive extension of trivial log-$\nabla$-module $(\mathcal{O}, d)$ on $V_j \times A_W^1(0,1)$ relative to $V_j$. So, to prove that the map (2.11) is isomorphic, we may replace DR by the log de Rham complex associated to $(\mathcal{O}, d)$. In this case, the map is shown to be isomorphic in [Ba-Ch2, §6]. So we have proved the claim.

Now we prove the theorem. We prove that the map

$$
R^q f_{X/Y, \text{an}*} \mathcal{E} \rightarrow R^q f_{(X - D^{h}, X)/Y, \text{log-rig}, j^!} \mathcal{E}
$$

is an isomorphism, by induction on $q$, using the claim. Assume that (2.12) is an isomorphism up to $q - 1$. Let us take any diagram as in (2.5). Then, by restricting the isomorphism to $[Y']_{dY'} = Y'_K$, we have the isomorphism

$$
R^q f_{X'/Y', \text{an}*} \mathcal{E} \rightarrow R^q f_{(X - D^{h}, X)/Y', \text{log-rig}, j^!} \mathcal{E}
$$

for $t < q$. In particular, both hand sides are coherent. So we have

$$
R^s \text{sp}_* R^q f_{X'/Y', \text{an}*} \mathcal{E} = R^s \text{sp}_* R^q f_{(X - D^{h}, X)/Y', \text{log-rig}, j^!} \mathcal{E} = 0
$$

for $s > 0, t < q$. By this and the claim, we can deduce that the map

$$
\text{sp}_* R^q f_{X'/Y', \text{an}*} \mathcal{E} \rightarrow \text{sp}_* R^q f_{(X - D^{h}, X)/Y', \text{log-rig}, j^!} \mathcal{E}
$$

is an isomorphism. Since this isomorphism holds for any $Y'$ as in the diagram (2.5), we can conclude that the map (2.12) is an isomorphism for $q$. So we are done. □

Next we compare the relative log analytic cohomology and the relative rigid cohomology, using Theorem 2.1. Assume that we are in the situation of Theorem 2.1, and assume given open immersions $X \hookrightarrow X, j_Y : Y \hookrightarrow Y$ such that $X \subseteq (\mathcal{X}, M_X)_{\text{triv}}, Y \subseteq (\mathcal{Y}, M_Y)_{\text{triv}}$ and $f^{-1}(Y) = X$ holds. Then we have $X - D^{h} \subseteq (\mathcal{X}, M_X)_{\text{triv}}$. So, by [Sh3, §5], the open immersion $X - D^{h} \hookrightarrow \mathcal{X}$ (which we denote by $j_X$) induces the functor

$$
j_X^! : I_{\text{conv}}(\mathcal{Y}) \rightarrow I^!(X - D^{h}, \mathcal{X}/\mathcal{Y}).
$$

On the other hand, we have the exact functor

$$
j_Y^! : \text{Mod}(\mathcal{O}_{\mathcal{Y}}) \rightarrow \text{Mod}(j_Y^! \mathcal{O}_{\mathcal{Y}})
$$

which sends coherent modules to coherent modules. Then we have the following:
**Theorem 2.2.** In the above situation, we have the isomorphism
\[ j_Y^! R^q f_{X/Y, an}^* E \cong R^q f_{(X-D^n, X)/Y, rig, j_X^!] E } \]
for any \( q \geq 0 \). In particular, \( R^q f_{(X-D^n, X)/Y, rig, j_X^!] E } \) is a coherent \( j_Y^!] O_Y \)-module.
(Note that we assume the condition (*) in Theorem 2.1.)

**Proof.** The proof is similar to [Sh3, 5.13]. Let us take the embedding system (2.4) as in the proof of Theorem 2.1. Then we have the diagram of rigid analytic spaces
\[
\begin{array}{ccc}
|X_\log|_{X_\log}^\times & \xrightarrow{h_\log} & |Y_\log^\times | \\
\downarrow & & \downarrow \\
|X_\log|_{X_\log}^\times & \xrightarrow{h} & |Y_\log^\times |
\end{array}
\]
Also, we put \( X_\log^\times := X \times_{\mathcal{X}} \mathcal{X}_\log^\times \), \( D_\log^\times := \mathcal{D} \times_{\mathcal{X}} \mathcal{X}_\log^\times \) and denote the open immersion \( X_\log^\times \hookrightarrow \mathcal{X}_\log^\times \), \( \mathcal{X}_\log^\times - D_\log^\times \hookrightarrow \mathcal{X}_\log^\times \), \( X_\log^\times - D_\log^\times \hookrightarrow \mathcal{X}_\log^\times \) by \( j_{X,1}^\times, j_{X,2}^\times, j_X^\times \), respectively.

Then we have admissible open immersions
\[
\begin{align*}
& j_{X,1}^\log : [X_\log|_{X_\log}^\times ] \hookrightarrow [X_\log|_{X_\log}^\times ] \\
& j_{X,2}^\log : [X_\log|_{X_\log}^\times ] - D_\log^\times [X_\log|_{X_\log}^\times ] \hookrightarrow [X_\log|_{X_\log}^\times ] \\
& j_X^\log : [X_\log|_{X_\log}^\times ] - D_\log^\times [X_\log|_{X_\log}^\times ] \hookrightarrow [X_\log|_{X_\log}^\times ]
\end{align*}
\]
and it induces the exact functors \( j_{X,1}^{\log, \dagger}, j_{X,2}^{\log, \dagger}, j_X^{\log, \dagger} \) in the standard way (see [Sh3, §5]) satisfying \( j_X^{\log, \dagger} = j_{X,1}^{\log, \dagger} \circ j_{X,2}^{\log, \dagger} \). On the other hand, we have also an admissible open immersion \( |X_\log|_{X_\log}^\times - D_\log^\times [X_\log|_{X_\log}^\times ] \hookrightarrow [X_\log|_{X_\log}^\times ] \) induced by \( j_X \) and it induces the functor \( j_X^\dagger \) also in the standard way.

Since \( (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \) is of Zariski type and the morphism \( (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \hookrightarrow (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \) is an exact closed immersion, \( (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \) is of Zariski type. Hence so is \( (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \). Also, since \( (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \) and \( (\mathcal{Y}, M_\mathcal{Y}) \) are of Zariski type, so is \( (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \). Hence, Zariski locally on \( \mathcal{X}_\log^\times \), the closed immersion \( (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \hookrightarrow (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \) admits a factorization
\[
(\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \hookrightarrow (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times}) \hookrightarrow (\mathcal{X}_\log^\times, M_{\mathcal{X}_\log^\times})
\]
such that the first map is an exact closed immersion and the second map is a formally etale morphism. So, by the argument of [Sh3, 5.9], we see the equalities of functors \( \varphi_\mathcal{X} \circ j_X^{\log, \dagger} = j_X^\dagger \circ \varphi_\mathcal{X} \) (which we denote by \( j_X^\dagger \)), \( Rj_X^\dagger = j_X^\dagger \) for coherent \( \mathcal{O}|_{\mathcal{X}_\log^\times}|_{X_\log|_{X_\log}^\times} \)-modules. Then we have the following diagram:
\[
\begin{align*}
& j_Y^! Rf_{X/Y, an}^* E = j_Y^! Rh_{\log}^* DR(\mathcal{X}_\log^\times)^\log|_{X_\log|_{X_\log}^\times}/\mathcal{Y}_K, \mathcal{E}) \\
& \quad = Rj_Y^! Rh_{\log}^* DR(\mathcal{X}_\log^\times)^\log|_{X_\log|_{X_\log}^\times}/\mathcal{Y}_K, \mathcal{E}) \\
& \quad = Rj_Y^! Rh_{\log}^* DR(\mathcal{X}_\log^\times)^\log|_{X_\log|_{X_\log}^\times}/\mathcal{Y}_K, \mathcal{E}) \quad \text{(Theorem 2.1)}
\end{align*}
\]
overconvergent isocrystal (normal crossing divisor in $D$ decomposition $\cdot$ divisor $q \geq 0$ where $22$ ATSUSHI SHIHO

triple which admit ‘nice’ log structures:

Theorem 2.3. Let us assume given a proper log smooth morphism $f : (\overline{X}, M_{\overline{X}}) \rightarrow (\overline{Y}, M_{\overline{Y}})$ such that $M_{\overline{X}}$ (resp. $M_{\overline{Y}}$) is the log structure induced by a normal crossing divisor $D$ (resp. simple normal crossing divisor $E$). We assume that $D$ has the decomposition $D = D^h \cup D^\nu$ into sub normal crossing divisors $D^h, D^\nu$ satisfying the condition $(\ast)$ in the beginning of this section. Assume moreover that we are given open immersions $X \hookrightarrow \overline{X}, Y \hookrightarrow \overline{Y}$ satisfying $X \subseteq (\overline{X}, M_{\overline{X}})_{\text{triv}}, Y \subseteq (\overline{Y}, M_{\overline{Y}})_{\text{triv}}$ and $f^{-1}(Y) = X$. Denote the open immersion $X - D^h \hookrightarrow \overline{X}$ by $j_X$. Then, for any $q \geq 0$ and for any locally free isocrystal $E$ on $(\overline{X}/S)_{\text{conv}} = ([\overline{X}, M_{\overline{X}}]/S)_{\text{conv}}$ which has nilpotent residues and satisfies the condition $(\ast)'$ below, there exists uniquely an overconvergent isocrystal $F$ on $(Y, \overline{Y})/S_K$ satisfying the following: For any $(Y, \overline{Y})$-triple $(Z, \overline{Z}, Z)$ over $(S, S, S)$ such that $\overline{Z}$ is smooth over $k$, $E \times_{\overline{Y}} \overline{Z}$ is a simple normal crossing divisor in $\overline{Z}$ and that $\overline{Z}$ is formally smooth over $S$, the restriction of $F$ to $I^\dagger((Z, \overline{Z})/S_K, Z)$ is given functorially by $(R^q f_{((X-D^h)\times_Y Z, \overline{X} \times_{\overline{Y}} \overline{Z})/Z, \text{rigs}_J \overline{X} E, \epsilon)$, where $\epsilon$ is given by

$$p_2^* R^q f_{((X-D^h)\times_Y Z, \overline{X} \times_{\overline{Y}} \overline{Z})/Z, \text{rigs}_J \overline{X} E} \cong R^q f_{((X-D^h)\times_Y Z, \overline{X} \times_{\overline{Y}} \overline{Z})/Z \times_{S} \overline{Z}, \text{rigs}_J \overline{X} E} \cong p_1^* R^q f_{((X-D^h)\times_Y Z, \overline{X} \times_{\overline{Y}} \overline{Z})/Z, \text{rigs}_J \overline{X} E}.$$ (Here $p_i$ is the morphism $[Z \times_{S} \overline{Z} \longrightarrow \overline{Z}$ induced by the $i$-th projection.)

The condition $(\ast)'$ for $E$ in the statement of the theorem is given as follows:
(\star)': For any diagram

\[
\begin{array}{ccc}
(Y, M_Y) & \leftarrow & (X, M_X) \\
\downarrow f & & \downarrow \psi_X \\
(Y_1, M_{Y_1}) & \leftarrow & (X_1, M_{X_1})
\end{array}
\]

such that \(\varphi_Y\) is strict, the square is Cartesian, \(i_1\) is an exact closed immersion and \((Y_1, M_{Y_1})\) is formally log smooth over \(S\), the log analytic cohomology \(R^q f_{\overline{X}_1 \rightarrow Y_1, \an \ast} \varphi_X^* \mathcal{E}\) is a coherent \(\mathcal{O}_{Y_1} - \text{module}\).

**Proof.** Let us take a diagram

\[
\begin{array}{ccc}
(Y, M_Y) & \leftarrow & (Y^{(0)}, M_Y^{(0)}) \\
\downarrow \varphi_Y & & \downarrow \psi_Y \\
(Y_1, M_{Y_1}) & \leftarrow & (Y^{(0)}, M_{Y^{(0)}})
\end{array}
\]

where \(\varphi_Y^{(0)}\) is a strict Zariski covering and \(\psi_Y^{(0)}\) is an exact closed immersion into a \(p\)-adic fine log formal \(B\)-scheme \((Y^{(0)}, M_{Y^{(0)}})\) such that \(Y^{(0)}\) is formally smooth over \(S\) and \(M_{Y^{(0)}}\) is the log structure defined by a relative simple normal crossing divisor. For \(n \in \mathbb{N}\), let \((Y^{(n)}, M_{Y^{(n)}})\) be the \((n + 1)\)-fold fiber product of \((Y^{(0)}, M_{Y^{(0)}})\) over \((Y, M_Y)\) and let \(Y^{(n)}\) be the \((n + 1)\)-fold fiber product of \(Y^{(0)}\) over \(S\). Then, for each \(n\), there exists a log structure \(M_{Y^{(n)}}\) (resp. \(M_{Y^{(n)} \times_S Y^{(n)}}\)) on \(Y^{(n)}\) (resp. \(Y^{(n)} \times_S Y^{(n)}\)) such that the canonical closed immersion \(Y^{(n)} \hookrightarrow Y^{(n)}\) (resp. \(Y^{(n)} \hookrightarrow Y^{(n)} \times_S Y^{(n)}\)) comes from an exact closed immersion \((Y^{(n)}, M_{Y^{(n)}}) \rightarrow (Y^{(n)}, M_{Y^{(n)}})\) (resp. \((Y^{(n)}, M_{Y^{(n)}}) \rightarrow (Y^{(n)} \times_S Y^{(n)}, M_{Y^{(n)} \times_S Y^{(n)}})\)) and \((Y^{(n)}, M_{Y^{(n)}})\) (resp. \((Y^{(n)} \times_S Y^{(n)}, M_{Y^{(n)} \times_S Y^{(n)}})\)) is formally log smooth over \(S\). Let us denote the pull-back of \((X, M_X), X, D^h\) by the morphism \((Y^{(n)}, M_{Y^{(n)}}) \rightarrow (Y, M_Y)\) by \((X^{(n)}, M_{X^{(n)}}), X^{(n)}, D^{h(n)}\), respectively. Then, by Theorem 2.2 and the condition \((\star)\), \(R^q f_{\overline{X}^{(n)} \rightarrow D^{h(n)}, Y^{(n)}} \mathcal{E}\) (resp. \(R^q f_{\overline{X}^{(n)} \rightarrow D^{h(n)}, Y^{(n)} \times_S Y^{(n)}}, \mathcal{E}\)) is a coherent \(j_Y^! \mathcal{O}_{\overline{Y}^{(n)}}, Y^{(n)}\)-module (resp. a coherent \(j_Y^! \mathcal{O}_{\overline{Y}^{(n)}}, Y^{(n)} \times_S Y^{(n)}\)-module). Then, by the base change theorem of Tsuzuki ([T2, 2.3.1]), the arrows in the diagram

\[
\begin{array}{ccc}
p_2^* R^q f_{\overline{X}^{(n)} \rightarrow D^{h(n)}, Y^{(n)}, \an \ast} j_X^! \mathcal{E} & \rightarrow & R^q f_{\overline{X}^{(n)} \rightarrow D^{h(n)}, Y^{(n)} \times_S Y^{(n)}}, \an \ast} j_X^! \mathcal{E} \\
\downarrow & & \downarrow \psi_Y \\
p_1^* R^q f_{\overline{X}^{(n)} \rightarrow D^{h(n)}, Y^{(n)} \times_S Y^{(n)}}, \an \ast} j_X^! \mathcal{E}
\end{array}
\]

(where \(p_i\) denotes the \(i\)-th projection \(\overline{Y}^{(n)} \rightarrow Y^{(n)} \times_S Y^{(n)}\)) are isomorphisms.

If we denote the composite of the above arrows by \(\epsilon^{(n)}\), one can see (by using [T2, 2.3.1] again) that \(\mathcal{F}^{(n)} := (R^q f_{\overline{X}^{(n)} \rightarrow D^{h(n)}, Y^{(n)}}, \an \ast} \mathcal{E}, \epsilon^{(n)})\) defines an overconvergent isocrystal on \((Y^{(n)}, Y^{(n)})/S_K\) and that \(\mathcal{F}^{(n)}\) is compatible with respect to \(n\). So \(\{\mathcal{F}^{(n)}\}_{n=0,1,2}\) descents to an overconvergent isocrystal \(\mathcal{F}\) on \((Y, Y)/S_K\).

Since the triple \((Y^{(n)}, Y^{(n)}, Y^{(n)})\) satisfies the condition required for \((Z, Z, Z)\) in the theorem, we see that the image of \(\mathcal{F}\) in \(I^\dagger((Y^{(n)}, Y^{(n)})/S_K)\) should be functorially
isomorphic to $\mathcal{F}^{(n)}$ in the previous paragraph. So we see that the condition in the statement of the theorem characterizes $\mathcal{F}$ uniquely.

Finally we prove that the overconvergent isocrystal $\mathcal{F}$ satisfies the required condition. For a triple $(\mathcal{Z}, \mathcal{Z}, \mathcal{Z})$ as in the statement of the theorem, let us put $M_{\mathcal{Z}} := M_{\mathcal{Y}}|_{\mathcal{Z}}$ (this is nothing but the log structure associated to the simple normal crossing divisor $E \times_{\mathcal{Y}} \mathcal{Z}$). Then, Zariski locally on $\mathcal{Z}$, there exists a fine log structure $M_{\mathcal{Z}}$ on $\mathcal{Z}$ such that the closed immersion $\mathcal{Z} \hookrightarrow \mathcal{Z}$ comes from an exact closed immersion $(\mathcal{Z}, M_{\mathcal{Z}}) \hookrightarrow (\mathcal{Z}, M_{\mathcal{Z}})$ and that $(\mathcal{Z}, M_{\mathcal{Z}})$ is formally log smooth over $S$. Then, by Theorem 2.2, $R^q f_{((X - D'h) \times Y, Z, \mathcal{Z})/Z, \text{rig}}^* j^!_{X, Y} E$ is a coherent $j^! O_{\mathcal{Z}}$-module. Then, by [T2, 2.3.1] again, we see that the restriction of $\mathcal{F}$ to $I^!(((\mathcal{Z}, \mathcal{Z})/S_K, \mathcal{Z})$ is given by $(R^q f_{((X - D'h) \times Y, Z, \mathcal{Z})/Z, \text{rig}}^* j^!_{X, \mathcal{Z}}, \nu)$ as in the statement of the theorem. Finally, the functoriality of the expression above is proved as the proof of [Sh3, 4.8]. So we are done. □

As for Frobenius structure, we have the following:

**Theorem 2.4.** With the situation in Theorem 2.3, assume moreover that $k$ is perfect and that $j^!_{X, \mathcal{Z}}$ has a Frobenius structure. Then the overconvergent isocrystal $\mathcal{F}$ on $(Y, \mathcal{Y})/S_K$ has a canonical Frobenius structure.

**Proof.** The proof is similar to [Sh3, 5.16] (see also [T2, 3.3.3, 4.1.4]). First, by the argument developed in [Sh3, 5.16], we may reduce to the case that $Y = \mathcal{Y}$ is equal to $S$. (In the proof of [Sh3, 5.16], we once used the analytically flat base change theorem of log analytic cohomology. Here we use Tsuzuki’s base change theorem ([T2, 2.3.1]) instead of it.) So, we are reduced to the claim that the endomorphism on $H^q_{\text{rig}}(X - D'h/K, j^!_{X, \mathcal{Z}} E)$ induced by the Frobenius structure on $j^!_{X, \mathcal{Z}} E$ is an isomorphism. We can prove this by imitating the proof of [Ke3, 9.1.2]: Let us consider the following two claims:

(a) The Frobenius endomorphism on $H^q_{\text{rig}}(X/K, G)$ is an isomorphism for any $X$ smooth over $k$ of dimension at most $n$ and any overconvergent $F$-isocrystal $G$ on $X$.

(b) The Frobenius endomorphism on $H^q_{\text{rig}}(X/K, G)$ is an isomorphism for any closed immersion $Z \hookrightarrow X$ of a geometrically reduced scheme $Z$ over $k$ of dimension at most $n$ into a scheme $X$ smooth over $k$ and any overconvergent $F$-isocrystal $G$ on $X$.

$((a)_0$ is obvious and $(b)_{-1}$ is vacuous.) Then, by a similar argument to the proof of [Ke3, 9.1.2], we can prove the implications $(b)_{n-1} \implies (a)_n$ and $(a)_{n-1} + (b)_{n-1} \implies (a)_n$: The main point is that all the arguments in the proof of [Ke3, 9.1.2] are compatible with Frobenius. (The most important point is the compatibility of Gysin isomorphism with Frobenius, which is proved in [T1, 4.1.1].) So the Frobenius endomorphism on the rigid cohomology of smooth variety with coefficient is always an isomorphism. So we are done. □
Remark 2.5. The conditions $(\ast)$ and $(\ast)'$ are satisfied if we assume $f$ is integral, by [Sh3, 4.7]. We will provide a slightly different situation where the conditions $(\ast)$ and $(\ast)'$ are satisfied, in the next section.

3. LOG BLOW-UP INVARIANCE OF RELATIVE LOG ANALYTIC COHOMOLOGY

In this section, we prove an invariance property of relative log analytic cohomology under log blow-ups in certain cases.

First we recall the notion of log blow-up. (See [I, 6.1], [Sa, 2.1], [Ni, 4].) Let $(X, M_X)$ be an fs log scheme. A sheaf of ideals $I \subseteq M_X$ is called a coherent ideal if, etale locally on $X$, there exists a chart $P \to M_X$ of $M_X$ by a torsion-free fs monoid $P$ and an ideal $I$ of $P$ such that $IM_X = I$ holds. Then the log blow-up $(X_I, M_X)$ of $(X, M_X)$ along $I$ is locally defined as follows: Let $I^\sat := \{a \in P_{\gp} \mid \exists n \geq 1, a^n \in I^n\}$ (the saturation of the ideal $I$). Then $X_I$ is defined to be $X \times_{\Spec \Z[P]} \Proj \Z[\oplus_{n=0}^\infty (I^\sat)^n]$. It has an open covering by the schemes of the form $X \times_{\Spec \Z[P]} \Spec \Z[P_a^\sat] (a \in P)$, where $P_a := \bigcup a^n I^n$ and $P_a^\sat := \{x \in P_a^\gp \mid \exists n \geq 1, x^n \in P_a\}$ (the saturation of $P_a$). We define the log structure $M_{X_I}$ as the fs log structure whose restriction to $X \times_{\Spec \Z[P]} \Spec \Z[P_a^\sat]$ is given as the pull-back of the canonical log structure $M_{P_a^\sat}$ on $\Spec \Z[P_a^\sat]$. Log blow-ups are log etale. We can define the notion of low blow-up also for formal fs log schemes.

Let $R$ be a ring (resp. adic topological ring), let $P$ be a torsion-free fs monoid and let $I$ be an ideal of $P$. Let us denote the canonical log structure on $\Spec R\{P\}$ (resp. $\Spf R\{P\}$) by $M_P$. Then we denote the log blow-up of $(\Spec R\{P\}, M_P)$ (resp. $(\Spf R\{P\}, M_P)$) along the coherent ideal $IM_P$ by $(\Bl_P(P), M_{P,I})$.

Next let us recall the notion of log regular log scheme ([Ni, 2.2], [Ka2, 2]) and several properties of it. For a log scheme $(X, M_X)$ and $x \in X$, let us denote the ideal of $O_{X,x}$ generated by the image of $M_{X,x} - O_{X,x}^* \times \bar{\tau}$ by $I(M_X, \bar{\tau})$. An fs log scheme $(X, M_X)$ is called log regular if, for any $x \in X$, $O_{X,x}/I(M_X, \bar{\tau})$ is regular and the equality $\dim(O_{X,x}) = \dim(O_{X,x}/I(M_X, \bar{\tau})) + \text{rk}(M_{X,x}^\gp/O_{X,x}^*)$ holds. If $(X, M_X)$ is log regular, $X$ is normal and $M_X = j_* O_{(X,M_X)\triv} \cap O_X$ holds, where $j$ denotes the open immersion $(X, M_X)_{\triv} \hookrightarrow X$. An fs log scheme which is log smooth over a log regular scheme is again log regular. In particular, any fs log scheme which is log smooth over $k$ is log regular. In this case, the converse is also true if $k$ is perfect. For a log regular log scheme $(X, M_X)$, $X$ is regular if and only if, for any $x \in X$, $M_{X,x}/O_{X,x}^*$ is isomorphic to $\N^r(x)$ for some $r(x)$ (depending on $x$). In this case, the log structure $M_X$ is the one associated to some normal crossing divisor $D$ on $X$.

It is known that, for a log regular scheme $(X, M_X)$, there exists a log blow-up $(\tilde{X}, M_{\tilde{X}}) \to (X, M_X)$ such that $\tilde{X}$ is regular ([Ni, 5.7,5.10]).

Now we prove a log blow-up invariance property for log analytic cohomology:

Theorem 3.1. Let us assume given the diagram

$$
(\overline{X}, M_{\overline{X}}) \xrightarrow{\varphi} (\overline{X}, M_{\overline{X}}) \xrightarrow{\iota} (\overline{Y}, M_{\overline{Y}}) \xrightarrow{i} (\overline{Y}, M_{\overline{Y}}),
$$

where $(\overline{X}, M_{\overline{X}}), (\overline{Y}, M_{\overline{Y}})$ are fs log schemes, $(\overline{Y}, M_{\overline{Y}})$ is an fs log formal $S$-scheme of Zariski type formally log smooth over $S$, $f$ is log smooth, $i$ is an exact closed
immersion and \( \varphi \) is a log blow-up of \( (\overline{X}, M_{\overline{X}}) \) by some coherent ideal of \( M_X \). Then, for a locally free isocrystal \( \mathcal{E} \) on \( (\overline{X}/Y)_{\text{log}}^{\text{conv}} \), we have the quasi-isomorphism

\[
Rf_{\overline{X}/Y, \text{an}*} \mathcal{E} \Rightarrow Rf_{\overline{X}/Y, \text{an}} \varphi^* \mathcal{E}.
\]

**Proof.** We may work etale locally on \( \overline{X} \) and Zariski locally on \( Y \). So we may assume that there exists a chart \((Q \to M_Y, R \to M_X, Q \overset{\alpha}{\to} R)\) of \( \iota \circ f \) such that \( \alpha \) is injective, \( |\text{Coker}(\alpha_{\text{gp}}^\text{tor})| \) is prime to \( p \) and that \( f \) factors as

\[
(\overline{X}, M_{\overline{X}}) \to (\overline{Y}, M_{\overline{Y}}) \times_{(\text{Spec}\ k[Q], M_Q)} (\text{Spec}\ k[R], M_R) \to (\overline{Y}, M_{\overline{Y}})
\]

with the first map strict etale. Then we can form the Cartesian diagram

\[
\begin{array}{ccc}
(\overline{X}, M_{\overline{X}}) & \to & (\overline{Y}, M_{\overline{Y}}) \\
\downarrow & & \downarrow \\
(\mathcal{X}, M_X) & \to & (\mathcal{Y}, M_Y)
\end{array}
\]

with the lower left horizontal arrow strict formally etale, by shrinking \( \overline{X} \). Then \((\mathcal{X}, M_X) \to (\mathcal{Y}, M_Y)\) is a log smooth lift of the morphism \( f \). We may also assume that the log structure of \( \mathcal{X} \) is fs.

Next note that, etale locally on \( \overline{X} \), there exists a chart \( \psi : P \to M_{\mathcal{X}} \) of \( M_{\mathcal{X}} \) by a torsion-free fs monoid \( P \), an ideal \( I \subseteq P \) and a Cartesian diagram

\[
\begin{array}{ccc}
(\text{Bl}_I(P)_k, M_{P,I}) & \leftarrow & (\overline{X}', M_{\overline{X}}) \\
\downarrow & & \varphi \\
(\text{Spec}\ k[P], M_P) & \leftarrow & (\overline{X}, M_{\overline{X}})
\end{array}
\]  

(3.1)

where the lower horizontal arrow is the strict morphism induced by the chart. Note that, for \( x \in \overline{X} \), we can lift the homomorphism \( \psi_{\text{gp}} : P_{\text{gp}} \to M_{X,x}^\text{gp} \) to a homomorphism \( \psi' : P_{\text{gp}} \to M_{X,x}^\text{gp} \) such that the composite \( P_{\text{gp}} \overset{\psi'}{\to} M_{X,x}^\text{gp} \to M_{X,x}^\text{gp}/\mathcal{O}_{X,x}^\times(= M_{X,x}^\text{gp}/\mathcal{O}_{X,x}^\times) \) is surjective. So, if we put \( P' := \psi'^{-1}(M_X) \), \( \psi' : P' \to M_{X,x} \) extends to a chart of \( M_X \) on an etale neighborhood of \( x \) ([Ka1, 2.10]). Moreover, by the exactness of \((X, M_X) \leftarrow (\mathcal{X}, M_X)\), we have \( P \subseteq P' \) and \( P' \to M_X \to M_{\mathcal{X}} \) is again a chart of \( M_{\mathcal{X}} \). Also, the Cartesian diagram (3.1) induces the similar Cartesian diagram with \( P, I \) replaced by \( P', I P' \). As a conclusion, we may assume that (by putting \( P := P', I := IP' \)) the chart \( \psi : P \to M_{\overline{X}} \) of \( M_{\overline{X}} \) extends to a chart \( \psi' : P \to M_X \) of \( M_X \). Now we put

\[
(\mathcal{X}', M_{\mathcal{X}'}) := (\mathcal{X}, M_X) \times_{(\text{Spec}\ V\{P\}, M_P)} (\text{Bl}_I(P)_V, M_{P,I}) \overset{\varphi'}{\to} (\mathcal{X}, M_X).
\]

Then \( \varphi' \) is a formally log etale lift of \( \varphi \). Let us denote the map of rigid analytic spaces \( [\overline{X}[x] \to [\overline{X}[x], [\overline{X}[x] \to [Y]_Y] \] by \( \varphi'_K, h \), respectively. Then we have the map

\[
Rf_{\overline{X}/Y, \text{an}*} \mathcal{E} = R\text{h}_*\text{DR}([\overline{X}[x]/\mathcal{Y}_K, \mathcal{E}]) \to R\text{h}_*R\varphi'_K, h^* \mathcal{O}_{\mathcal{X}} \text{DR}([\overline{X}[x]/\mathcal{Y}_K, \mathcal{E}) = Rf_{\overline{X}/Y, \text{an}*} \mathcal{E}.
\]

We prove that this is a quasi-isomorphism. To prove this, it suffices to prove the equality \( R\varphi'_K, h^* \mathcal{O}_{\overline{X}}[x] = \mathcal{O}_{\overline{X}}[x] \), and it is reduced to the equality \( R\varphi'_* \mathcal{O}_X = \mathcal{O}_X \).
Let us denote the uniformizer of $V$ by $\pi$ and let $\varphi' : X'_n \to X_n$ be $\varphi' \otimes_V V/\pi^nV$. Then it suffices to prove the equality
\[(3.2) \quad R(\varphi'_n)_*\varphi'^*\mathcal{O}_{X_n} = \mathcal{O}_{X_n}.\]

Note that we have $X'_n = X'_n \times_{\text{Spec} \mathbb{Z}[P]} \text{Bl}_1(P)_{\mathbb{Z}}$, and note also that we have the vanishing $\text{Tor}_i^Z[\mathbb{Z}[P], \mathcal{O}_{X_n}] = 0$ for any $i \geq 1$, $n \in \mathbb{N}$ and injective homomorphism $P \to P'$ of integral monoids: Indeed, if $n = 1$, this follows from [Ka2, 6.1(ii)], since $(X_1, M_{X_1}|_{X_1})$, being log smooth over $k$, is log regular with a chart $P \to M_X \to M_{X_1}$. For general $n$, we can see by induction, using the exact sequence
\[0 \to \mathcal{O}_{X_{n-1}} \overset{\pi}{\to} \mathcal{O}_{X_n} \to \mathcal{O}_{X_1} \to 0.\]

(Note that, since $(X_n, M_{X_n})$ is log smooth over $\text{Spf} V/\pi^nV$, it is flat over $V/\pi^nV$.) By this vanishing of Tor, (3.2) is reduced to the equality $R(\varphi'_n)^*\mathcal{O}_{\text{Spec} \mathbb{Z}[P]} = \mathcal{O}_{\text{Spec} \mathbb{Z}[P]}$, where $\varphi'$ is the map $\text{Bl}_1(P)_{\mathbb{Z}} \to \text{Spec} \mathbb{Z}[P]$. This is true by [KKMS, Ch I, §3, Cor. 1]. (There they treat the case of $\text{Spec} k[P]$, but the arguments work for $\text{Spec} \mathbb{Z}[P]$. See also [Ka2, 11.3].) So the proof is finished. 

Next we deduce a consequence of Theorem 3.1, which is useful in later sections. To describe this, first we define the notion of log normal crossing divisor as follows:

**Definition 3.2.** Let $(X, M_X^\nu)$ be a log regular log scheme. Then a Cartier divisor $D \subseteq X$ is called a log normal crossing divisor if, for any $x \in D$, $D \times_X \text{Spec} \mathcal{O}_{X, x} \subseteq \text{Spec} \mathcal{O}_{X, x}$ is defined by the equation of the form $t_1^{e_1} \cdots t_r^{e_r} = 0 (t_i \in \mathcal{O}_{X, x})$ such that the loci $\{t_i = 0\}$ $(1 \leq i \leq r)$ is a regular divisor in $\text{Spec} \mathcal{O}_{X, x}/(M_X^\nu, x)$ and that the locus $\{t_1 \cdots t_r = 0\}$ is a simple normal crossing divisor in $\text{Spec} \mathcal{O}_{X, x}/(M_X^\nu, x)$.

**Remark 3.3.** The conditions required for $t_i$’s in the above definition is equivalent to the following condition: For any subset $I$ of $\{1, \cdots, r\}$, the locus $\{t_i = 0 (i \in I)\}$ is a regular closed subscheme of $\text{Spec} \mathcal{O}_{X, x}/(M_X^\nu, x)$ of codimension $|I|$.

**Proposition 3.4.** In the situation in Definition 3.2, let us define new log structures $M_X^\nu, M_X$ on $X$ by $M_X^\nu := j_*\mathcal{O}_{X, D}^\nu \cap \mathcal{O}_X$ and $M_X := M_X^\nu \oplus \mathcal{O}_X^\nu$. Then:

1. Etale locally, We have $M_X^{\nu} = \mathcal{O}_X^{\nu} t_1^{e_1} \cdots t_r^{e_r}$ and it is associated to the monoid homomorphism $\varphi : \mathbb{N}^r \to \mathcal{O}_X$; $e_i \mapsto t_i$. (Here $e_i (1 \leq i \leq r)$ is a canonical basis of $\mathbb{N}^r$.)

2. The log scheme $(X, M_X)$ is log regular. Also, if we define $D_{[i]} (i \geq 0)$ by

$$D_{[0]} := X, \quad D_{[1]} := \text{the normalization of } D,$$

$$D_{[i]} := \text{i-fold fiber product of } D_{[1]} \text{ over } X,$$

the log scheme $(D_{[i]}, M_{D_{[i]}}^\nu) := (D_{[i]}, M_X^\nu|_{D_{[i]}})$ is also log regular.

**Proof.** We may work etale locally. For $I \subseteq \{1, \cdots, r\}$, let us denote the closed subscheme of $X$ defined by the equation $t_i = 0 (i \in I)$ by $D_I$. First we prove that $(D_I, M_X^\nu|_{D_I})$ is log regular by induction on $|I|$. Write $I = \{i\} \bigsqcup I'$ with $i \notin I'$. Then we have

$$\dim(\mathcal{O}_{D_I, x}) = \dim(\mathcal{O}_{D_I, x}/(M_X^\nu|_{D_I, x})) + \text{rk } (M_X^\nu|_{D_I, x}/\mathcal{O}_{D_I, x}^\nu)$$
by induction hypothesis. Then, since $t_i$ is non-zero and non-invertible in $\mathcal{O}_{D_i, \pi}/I(M_X^i|_{D_i}, \pi)$, so is $t_i$ in $\mathcal{O}_{D_i, \pi}$. Since $\mathcal{O}_{D_i, \pi}$ is an integral domain (by induction hypothesis), we have

$$\dim(\mathcal{O}_{D_i, \pi}) = \dim(\mathcal{O}_{D_i, \pi}) - 1,$$

$$\dim(\mathcal{O}_{D_i, \pi}/I(M_X^i|_{D_i}, \pi)) = \dim(\mathcal{O}_{D_i, \pi}/I(M_X^i|_{D_i}, \pi)) - 1.$$

So we see the equality

$$\dim(\mathcal{O}_{D_i, \pi}) = \dim(\mathcal{O}_{D_i, \pi}/I(M_X^i|_{D_i}, \pi)) + \text{rk} (M_X^{i, \text{gp}}/\mathcal{O}_{D_i, \pi}),$$

and $\mathcal{O}_{D_i, \pi}/I(M_X^i|_{D_i}, \pi)$ is regular by assumption. So $(D_i, M_X|_{D_i})$ is log regular. In particular, $D_{(i)}$ is normal. Then we see that any local section $f$ in $M_X^i$ can be written uniquely as $f = u t_1^{n_1} \cdots t_r^{n_r}$ for some $u \in \mathcal{O}_X^\times$, $n_i \in \mathbb{N}$. So we have the assertion (1). Also, we see that $D_{(i)}$ is locally written as the disjoint union of $D_i$’s with $|I| = i + 1$. So $(D_{[i]}, M_{D_{[i]}})$ is log regular.

Finally we prove that $(X, M_X)$ is log regular. If we take a chart $\psi : P \to \mathcal{O}_X$ of $M_X^i$ with $P$ an fs monoid, we see by (1) that the homomorphism $P \oplus \mathbb{N}^r \to \mathcal{O}_X$; $(p, n) \mapsto \psi(p) \varphi(n)$ is a chart of $M_X$. So $M_X$ is an fs log structure. Then, since we have

$$\dim(\mathcal{O}_{X, \pi}/I(M_X, \pi)) = \dim(\mathcal{O}_{X, \pi}/I(M_X^i, \pi) + (t_1, \cdots, t_r)) = \dim(\mathcal{O}_{X, \pi}/I(M_X^i, \pi)) - r,$$

$$\text{rk} (M_X^{i, \text{gp}}/\mathcal{O}_{X, \pi}) = \text{rk} (M_X^{i, \text{gp}}/\mathcal{O}_{X, \pi}) + r,$$

we have the equality

$$\dim(\mathcal{O}_{X, \pi}) = \dim(\mathcal{O}_{X, \pi}/I(M_X, \pi)) + \text{rk} (M_X^{i, \text{gp}}/\mathcal{O}_{X, \pi}).$$

Since $\mathcal{O}_{X, \pi}/I(M_X, \pi) = \mathcal{O}_{X, \pi}/(I(M_X^i, \pi) + (t_1, \cdots, t_r))$ is regular by assumption, we see that $(X, M_X)$ is log regular. \hfill \Box

**Example 3.5.** Let $X$ be a smooth scheme over $k$ and let $D^h, D^v$ are normal crossing divisors on $X$ such that $D := D^h \cup D^v$ is again a normal crossing divisor. Then, if we denote the log structure on $X$ associated to $D^v$ by $M_X^h$, $D^h$ is a log normal crossing divisor on $(X, M_X^h)$. Moreover, $M_X^h$ (resp. $M_X^v$) above is nothing but the log structure associated to $D^h$ (resp. $D^v$).

Now let us assume given a proper log smooth integral morphism of fs log schemes $f' : (\overline{X}', M_{\overline{X}}') \to (\overline{Y}, M_\overline{Y})$ over $S$ such that $\overline{Y}$ is smooth over $S$ and that $M_\overline{Y}$ is the log structure associated to a simple normal crossing divisor on $\overline{Y}$. (Then $(\overline{X}', M_{\overline{X}}')$, being log smooth over $S$, is log regular.) Let $D' \subseteq \overline{X}'$ be a log normal crossing divisor, and let us define $M^h_{\overline{X}}$, $M_{\overline{X}}'$, $D'_i (i \geq 0)$ as above. Assume moreover the following conditions:

1. $(\overline{X}', M_{\overline{X}}')$ is log smooth over $(\overline{Y}, M_\overline{Y})$.
2. For any $i \geq 0$, $(D'_i, M^h_{\overline{X}}, D'_i)$ is log smooth over $(\overline{Y}, M_\overline{Y})$.

Take a log blow-up $\varphi^v : (\overline{X}, M_{\overline{X}}') \to (\overline{X}', M_{\overline{X}}')$ of $(\overline{X}', M_{\overline{X}}')$ such that $\overline{X}$ is regular, and put

$$(\overline{X}, M_{\overline{X}}) := (\overline{X}, M^h_{\overline{X}}) \times (\overline{X}', M_{\overline{X}}') \xrightarrow{\varphi} (\overline{X}', M_{\overline{X}}'),$$

$$(\overline{X}, M_{\overline{X}}) := (\overline{X}, M^h_{\overline{X}}) \times (\overline{X}', M_{\overline{X}}') \xrightarrow{\varphi} (\overline{X}', M_{\overline{X}}'),$$
where \( \varphi = \varphi^v \times \text{id} \). (It is also a log blow-up.) Let us put \( D^h := \varphi^* D' \) and let \( D^v \) be a normal crossing divisor corresponding to \( M^v_X \). Then \( D := D^h \cup D^v \) is a normal crossing divisor corresponding to \( M^v_X \) and they satisfies the condition (*) in the beginning of Section 2. (So the morphism \( f := \varphi \circ f' : (\overline{X}, M^v_X) \to (\overline{Y}, M^v_Y) \) is in the situation we treated in Section 2.) With this notation, we have the following:

**Proposition 3.6.** With the above notation, let \( \mathcal{E}_0 \) be a locally free isocrystal on 
\[ (\overline{X}/S)^{\log}_{\text{conv}} = ((\overline{X}, M^v_X)/S)_{\text{conv}} \]
and put \( \mathcal{E} := \varphi^* \mathcal{E}_0 \). Then:

1. For any exact closed immersion \((\overline{Y}, M^v_Y) \hookrightarrow (\overline{Y}, M^v_Y)\) such that \((\overline{Y}, M^v_Y)\) is formally log smooth over \( S \), \( \mathcal{E} \) satisfies the condition (*) in Theorem 2.1.
2. \( \mathcal{E} \) satisfies the condition \((*)'\) in Theorem 2.3.

**Proof.** We only prove (1). (The proof of (2) is similar.) Since \((\overline{X}, M^v_X) \to (\overline{Y}, M^v_Y)\) is proper log smooth integral and \((\overline{Y}, M^v_Y)\) is log smooth over \( k \), the relative log analytic cohomology \( R^q f_{\overline{X}/\overline{Y}, \text{an}}^* \mathcal{E}_0 \) is coherent by [Sh3, 4.7]. Moreover, by Theorem 3.1, we have the isomorphism \( R^q f_{\overline{X}/\overline{Y}, \text{an}}^* \mathcal{E}_0 = R^q f_{\overline{X}/\overline{Y}, \text{an}}^* \mathcal{E} \). So \( R^q f_{\overline{X}/\overline{Y}, \text{an}}^* \mathcal{E} \) is coherent. \( \square \)

**Remark 3.7.** As a consequence of Proposition 3.6, we have the following: In the situation of Proposition 3.6, assume moreover that \( \mathcal{E} \) has nilpotent residues. Then we have Theorems 2.1, 2.2 and 2.3 for \( \mathcal{E} \).

**Remark 3.8.** The reason we need Proposition 3.6 is that it is not always true that the morphism \( f \) is integral.

Finally, we prove a proposition concerning log normal crossing divisors which we use in later sections:

**Proposition 3.9.** Let \( f : (X, M^v_X) \to (Y, M^v_Y) \) be a log smooth integral universally saturated morphism of fs log schemes such that \((Y, M^v_Y)\) is log smooth over \( S \) and let \( D \subseteq X \) be a log normal crossing divisor. Define \( M^v_X, M^v_X, D_{[i]} \) as in Proposition 3.4 and assume that \((X, M^v_X), (D_{[i]}, M^v_X|_{D_{[i]}}) \) are log smooth over \((Y, M^v_Y)\). Let us assume given the Cartesian diagram

\[
\begin{array}{ccc}
(X', M^v_{X'}) & \longrightarrow & (Y', M^v_{Y'}) \\
\downarrow & & \downarrow \\
(X, M^v_X) & \longrightarrow & (Y, M^v_Y)
\end{array}
\]

such that \((Y', M^v_{Y'})\) is again an fs log scheme which is log smooth over \( S \). Put \( D' := D \times_X X' \). Then \( D' \) is a log normal crossing divisor in \( X' \) and if we define \( M^v_{X'}, M^v_{X'}, D'_{[i]} \) using \( D' \), \((X', M^v_{X'})\) and \((D'_{[i]}, M^v_{X'}|_{D'_{[i]}})\) are log smooth over \((Y', M^v_{Y'})\).

**Proof.** Let \( x' \in X' \), let \( x \) be its image in \( X \) and let us take \( t_1, \cdots, t_r \in \mathcal{O}_{X, \overline{x}} - \mathcal{O}_{X, \overline{x}} \) such that \( D \times_X \text{Spec} \mathcal{O}_{X, \overline{x}} \) is defined by \( t_1 \cdots t_r = 0 \) and that they satisfy the condition required in Remark 3.3. Then \( D' \times_{X'} \text{Spec} \mathcal{O}_{X', \overline{x'}} \) is also defined by \( t_1 \cdots t_r = 0 \) in \( \text{Spec} \mathcal{O}_{X', \overline{x'}} \). To prove the proposition, it suffices to prove that the elements \( t_i (1 \leq i \leq r) \) satisfy the condition required in Remark 3.3 (with \( D, X, x \)
replaced by $D', X', x'$: Indeed, this assertion implies that $D'$ is a log normal crossing divisor in $X'$ and that we have the isomorphisms $(X', M_{X'}) = (X, M_X) \times_{(Y', M_Y)} (Y', M_Y)$, $(D'_{[i]}, M_{X'}|_{D'_{[i]}}) = (D_{[i]}, M_X|_{D_{[i]}}) \times_{(Y', M_Y)} (Y', M_Y)$, and these immediately implies the assertions of the proposition. So we prove the above claim.

For $I \subseteq \{1, \cdots, r\}$, let us put $D_I := \{t_i = 0 (i \in I)\}$ and put $D'_I := D_I \times_Y Y'$. Then, since $(D'_I, M_{X'}|_{D'_I})$ is log smooth over $(Y', M_Y)$, it is log regular. In particular, $\mathcal{O}_{D'_I, X}/I(M_{X'}|_{D'_I}, \mathcal{F})$ is regular. So it suffices to prove the equality

$$\dim \mathcal{O}_{D'_I, X}/I(M_{X'}|_{D'_I}, \mathcal{F}) = \dim \mathcal{O}_{D'_I, X}/I(M_{X'}|_{D'_I}, \mathcal{F}) - 1$$

for any $J = \{j\} \cup I$ with $j \notin I$. Since $(D'_I, M_{X'}|_{D'_I}), (D'_J, M_{X'}|_{D'_J})$ are log regular, we have the equalities

$$\dim \mathcal{O}_{D'_I, X} = \dim \mathcal{O}_{D'_I, X}/I(M_{X'}|_{D'_I}, \mathcal{F}) + \text{rk} M_{X', \mathcal{F}}^{\text{gp}}/\mathcal{O}_{X', \mathcal{F}}^{\text{gp}},$$

$$\dim \mathcal{O}_{D'_J, X} = \dim \mathcal{O}_{D'_J, X}/I(M_{X'}|_{D'_J}, \mathcal{F}) + \text{rk} M_{X', \mathcal{F}}^{\text{gp}}/\mathcal{O}_{X', \mathcal{F}}^{\text{gp}}.$$

Now let us note the equality

$$\dim \mathcal{O}_{D'_I, X} = \dim \mathcal{O}_{D'_J, X} + 1.$$ 

This follows from the equalities $\text{rel.dim}(D'_I/Y') = \text{rel.dim}(D_I/Y), \text{rel.dim}(D'_J/Y') = \text{rel.dim}(D_J/Y)$, which are true because $D_I \longrightarrow Y, D_J \longrightarrow Y$ are smooth (in classical sense) on dense open subset. Combining (3.4), (3.5) and (3.6), we obtain (3.3). So we are done. □

4. Alteration and hypercovering

In this section, we prove the existence of certain diagrams involving hypercovering, which is a slight generalization of that treated in [Sh4, §6]. In this section, all the schemes are assumed to be defined over $S$ and from now on, we always assume that the field $k$ is perfect.

First let us prove the following proposition, which is a slight generalization of [Sh3, 6.4] and a consequence of the papers [dJ1], [dJ2]:

**Proposition 4.1.** Let $f : X \longrightarrow Y$ be a proper morphism of integral schemes whose generic fiber is geometrically irreducible and let $D \subseteq X$ be a Cartier divisor. Then there exists a diagram

$$\begin{array}{ccc}
X & \xleftarrow{\psi} & X' \\
\downarrow f & & \downarrow g \\
Y & \xleftarrow{\psi} & Y'
\end{array}$$

such that horizontal arrows are alterations with $X', Y'$ regular and a normal crossing divisor $D'$ (resp. $E'$) on $X'$ (resp. $Y'$) with a decomposition $D' = D'^h \cup D'^v$ into sub normal crossing divisors $D'^h, D'^v$, satisfying the following conditions:

1. $D'^v = g^{-1}(E)_{\text{red}}$.
2. $\psi^{-1}(X \setminus D) \cap (X' \setminus D'^v) = (X' \setminus D'^h) \cap (X' \setminus D'^v)$.
If we denote the log structure on $X'$ (resp. $Y'$) associated to $D'$ (resp. $E'$) by $M_{X'}$ (resp. $M_{Y'}$), the morphism $(X', M_{X'}) \to (Y', M_{Y'})$ induced by $g$ is proper log smooth integral and universally saturated.

If we denote the log structure on $D$ by $M_D$, the morphism $(D, M_D, v_D) \to (X, M_X, g_X)$ associated to $g$ is proper log smooth integral and universally saturated.

Now let us define $D'$ as $(\psi^{-1}(D))_{\text{red}} \cup D'^\nu$ and let $D'^h$ be the closure of $D' - D'^\nu$. Then we see that all the required conditions are satisfied.

Proof. By using [dJ2, Thm 5.9] and [dJ2, Prop 5.11] and arguing as [Sh4, 6.4], we see that there exist a diagram (4.1) and a normal crossing divisor $D''$ (resp. $E''$) on $X'$ (resp. $Y'$) satisfying the following conditions:

1. $\psi^{-1}(D)_{\text{red}} \subseteq D''$.
2. $D'^\nu := g^{-1}(E)_{\text{red}}$ is a sub normal crossing divisor of $D''$.
3. For any sub normal crossing divisor $D''' \subseteq D''$ containing $D'^\nu$, the condition (3) in the statement of the proposition is true if we replace $D'$ by $D'''$.
4. For any sub normal crossing divisor $D''' \subseteq D''$ containing $D'^\nu$, the condition (4) in the statement of the proposition is true if we replace $D'^h$ by $D'^{h'} = \text{closure of } D''' - D'^\nu$.

Now let us define $D'$ as $(\psi^{-1}(D))_{\text{red}} \cup D'^\nu$ and let $D'^h$ be the closure of $D' - D'^\nu$. Then we see that all the required conditions are satisfied. \qed

Next we prove the following lemma, which is a slight generalization of [Sh4, 6.9]:

**Lemma 4.2.** Let $f : \overline{X} \to \overline{Y}$ be a proper morphism of schemes, let $U \subseteq Y$ be a dense open subscheme and let $X \subseteq \overline{X}$ be an open subscheme. Then we have the diagram consisting of strict morphism of pairs

$$(U_{\overline{X}}, \overline{X}) \leftarrow (U_{\overline{Y}}, \overline{Y})$$

$$(U_{\overline{Y}}, \overline{Y})$$

$$(U_{\overline{X}}, \overline{X})$$

$$(f \downarrow)$$

$$(f' \downarrow)$$

with $(U_{\overline{X}}, \overline{X}) = \coprod_{j=1}^b (U_{\overline{X}_j}, \overline{X}_j)$, $(U_{\overline{Y}}, \overline{Y}) = \coprod_{j=1}^b (U_{\overline{Y}_j}, \overline{Y}_j)$ (decomposition into connected components with $f'(\overline{X}_j) \subseteq \overline{Y}_j$) satisfying the following conditions:

1. $U_{\overline{Y}}$ is contained in $U$ and dense in $\overline{Y}$.
2. $\overline{X}_j$'s, $\overline{Y}_j$'s are integral, and if we put $X' := X \times_{\overline{X}} \overline{X}_j$, $X'_j - X' \subseteq X'_j$ is a Cartier divisor (possibly empty).
3. The map $\overline{g}_X : (X' \cap U_{\overline{X}} \times_{\overline{Y}} \overline{X}) \to ((X \cap U_{\overline{X}}) \times_{U_{\overline{Y}}} U_{\overline{Y}} \times_{\overline{Y}} \overline{Y})$ induced by $(g_X, f')$ is a proper covering and the map $g_Y$ is a good strongly proper covering.
(4) For each $i, j$, the generic fiber of $f'_|x_{ji} : x'_{ji} \rightarrow y'_j$ is non-empty and geometrically irreducible. (Attention: It is possible that $r_j = 0$ holds for some $j$.)

Proof. Let $x_\text{red} = \bigcup_l x_l$ be the decomposition of $x_\text{red}$ into irreducible components and put $x_l := x \cap x_l$. For $l$ with $x_l \neq \emptyset$, let us take an alteration $x''_l \leftarrow x_l$ so that $x''_l$ is smooth over $S$ and that the complement of $x''_l := x_l \times x x''_l$ is a simple normal crossing divisor in $x''_l$. Let us put $x'' := \bigsqcup_{l, x_l \neq \emptyset} x''_l, x'' := x \times x x''$. Then the natural morphism of pairs $(x'', x') \rightarrow (x, x)$ is a proper covering. Then, by using [Sh4, 6.9] for the morphism $x'' \rightarrow y$, we can take a diagram consisting of strict morphisms of pairs

\[
\begin{array}{ccc}
(U, x) & \leftarrow & (U', x') \\
\downarrow & & \downarrow \\
(U, y) & \leftarrow & (U', y')
\end{array}
\]

such that the right square satisfies the conclusion of [Sh4, 6.9]. Then one can see that the diagram (4.2) induced by (4.3) satisfies the required conditions. □

Next we prove the following proposition, which is a generalization of [Sh4, 6.10]:

Proposition 4.3. Let $f : x \rightarrow y$ be a proper morphism of schemes, let $U \subseteq y$ be a dense open subscheme and let $x \subseteq x$ be an open subscheme. Then we have the diagram consisting of strict morphism of pairs

\[
\begin{array}{ccc}
(U, x) & \leftrightarrow & (U', x') \\
(f_X) & \leftarrow & (f_Y)
\end{array}
\]

satisfying the following conditions:

1. $U$ is contained in $U$ and dense in $y$.
2. $y$ is regular.
3. The map $g_X : (X \cap U, x) \rightarrow ((X \cap x) \times y U, x \times y y)$ induced by $(g_X, f)$ is a proper covering (where we put $X := x \times x x$) and the map $g_Y$ is a good strongly proper covering.
4. There exist fs log structures $M^v_x, M^v_y$ on $x, y$ respectively such that $M^v_x$ is associated to a simple normal crossing divisor on $y$, $U_x \subseteq (x, M^v_x)_{\text{triv}}, U_y \subseteq (y, M^v_y)_{\text{triv}}$ holds and that there exists a proper log smooth integral universally saturated morphism $(x, M^v_x) \rightarrow (y, M^v_y)$ whose underlying morphism of schemes is the same as $f$. 
(5) There exists a log normal crossing divisor $D \subseteq \hat{X}$ such that, if we define $M_{\hat{X}} \mid D[i] (i \geq 0)$ as in Proposition 3.4 using $D$, the induced morphisms

$$\begin{align*}
\hat{X}, M_{\hat{X}} \rightarrow (\hat{Y}, M_{\hat{Y}}), \\
(D[i], M_{\hat{X}} \mid D[i]) \rightarrow (\hat{Y}, M_{\hat{Y}}) (i \geq 0)
\end{align*}$$

are proper log smooth integral universally saturated, and we have $\hat{X} \cap (\hat{X}, M_{\hat{Y}})_{\text{triv}} = (\hat{X} - D) \cap (\hat{X}, M_{\hat{Y}})_{\text{triv}}$. (In particular, we have $\hat{X} \cap U_{\hat{X}} = (\hat{X} - D) \cap U_{\hat{X}}$.)

Proof. The proof is similar to [Sh4, 6.10], although the proof here is slightly more complicated. First let us take the diagram consisting of strict morphism of pairs

$$\begin{array}{ccc}
(V, X) & \leftarrow & (U, X) \\
| f & | & | f' |
\end{array}$$

$$\begin{array}{ccc}
(V', Y) & \leftarrow & (U', Y) \\
| f' & | & | f'' |
\end{array}$$

satisfying the conclusion of Lemma 4.2, and let $Y' = \coprod_{j=1}^{s} Y_j, X' = \coprod_{j=1}^{s} (\coprod_{i=1}^{r_j} X_{ji})$ be the decomposition of $Y', X'$ into connected components with $f'(X_{ji}) \subseteq Y_j$. Also let us put $X' := X \times_{\hat{X}} \hat{X}'$.

Let us fix $j$. We prove the following claim:

claim For any $1 \leq s \leq r_j$, there exists a diagram consisting of strict morphism of pairs

$$\begin{array}{ccc}
\coprod_{i=1}^{r_j} (X_{ji}, O_{X_{ji}}) & \leftarrow & \coprod_{i=1}^{r_j} (X'_{ji}, O'_{X_{ji}}) \\
| f' & | & | f'' |
\end{array}$$

(4.5)

$$\begin{array}{ccc}
(Y_j', O'_{Y_j}) & \leftarrow & (Y_j', O'_{Y_j}) \\
| f' & | & | f'' |
\end{array}$$

(we put $X''_{ji} := X' \times_{\hat{X}} X_{ji}$) satisfying the following conditions:

(1) $O'_{Y_j}$ is dense in $Y_j'$.

(2) $Y_j'$ is regular and connected (so it is integral).

(3) The upper horizontal arrow is a proper covering and the lower horizontal arrow is a good strongly proper covering.

(4) There exist $s$ log structures $M_{Y_j'} \mid \coprod_{i=1}^{r_j} X_{ji}, M_{Y_j'}$ on $\coprod_{i=1}^{r_j} X_{ji}$, $Y_j'$ respectively and a proper log smooth integral universally saturated morphism

$$g : \coprod_{i=1}^{r_j} (X_{ji}, O_{X_{ji}}) \rightarrow (Y_j', M_{Y_j'})$$

whose underlying morphism of schemes is the same as $f'' \mid \coprod_{i=1}^{r_j} X_{ji}$ such that $M_{Y_j'}$ is associated to a simple normal crossing divisor on $Y_j'$ and that $g^{-1}((Y_j', M_{Y_j'})_{\text{triv}})$ is contained in $(\coprod_{i=1}^{r_j} X_{ji}, M_{X_{ji}} \mid D[i])_{\text{triv}}$. 

(5) There exists a log normal crossing divisor \( D \subseteq \coprod_{i=1}^s \mathcal{X}''_{ji} \) such that, if we define \( M_{\coprod_{i=1}^s \mathcal{X}''_{ji}, D_{[i]} (i \geq 0)} \) as in Proposition 3.4 using \( D \), the induced morphisms

\[
\left( \coprod_{i=1}^s \mathcal{X}''_{ji}, M_{\coprod_{i=1}^s \mathcal{X}''_{ji}} \right) \longrightarrow (Y''_j, M_{Y''_j}), \quad (D_{[i]}, M_{\coprod_{i=1}^s \mathcal{X}''_{ji}}|_{D_{[i]}}) \longrightarrow (Y''_j, M_{Y''_j}) (i \geq 0)
\]

are proper log smooth integral universally saturated, and we have \( (\coprod_{i=1}^s \mathcal{X}''_{ji}, M_{\coprod_{i=1}^s \mathcal{X}''_{ji}})^{\text{triv}} = (\coprod_{i=1}^s \mathcal{X}''_{ji} - D) \cap (\coprod_{i=1}^s \mathcal{X}''_{ji}, M_{\coprod_{i=1}^s \mathcal{X}''_{ji}})^{\text{triv}}. \)

(6) For each \( i \geq s + 1, \) \( \mathcal{X}''_{ji} \) is integral, \( \mathcal{X}''_{ji} - \mathcal{X}''_{ji} \) is a Cartier divisor and the generic fiber of \( f''|_{\mathcal{X}''_{ji}} : \mathcal{X}''_{ji} \longrightarrow Y''_j \) is non-empty, geometrically irreducible.

We prove the claim by induction on \( s \): In the case \( s = 1 \), we apply Proposition 4.1 to the morphism \( \mathcal{X}''_{j1} \longrightarrow Y''_j \) to obtain the diagram

\[
\begin{array}{ccc}
\mathcal{X}''_{j1} & \longrightarrow & \mathcal{X}''_{j1} \\
\downarrow & & \downarrow \\
Y''_j & \longrightarrow & Y''_j
\end{array}
\]

satisfying the conclusion of Proposition 4.1. Then there exist log structures \( M_{\mathcal{X}''_{j1}, Y''_j} \) on \( \mathcal{X}''_{j1}, Y''_j \) respectively and a log normal crossing divisor \( D \) in \( \mathcal{X}''_{j1} \) satisfying (2), (4) and (5) (for \( s = 1 \)). Then, by the argument of the proof of the claim in [Sh4, 6.10], we can take an open subscheme \( O_{Y_j}^s \subseteq Y''_j \) and the diagram (4.5) satisfying the other assertions. So the claim is true for \( s = 1 \).

If the claim is true for \( s - 1 \), we have the diagram consisting of strict morphisms of pairs

\[
\coprod_{i=1}^s (\mathcal{X}''_{ji}, O_{\mathcal{X}''_{ji}}) \longrightarrow \coprod_{i=1}^s (\mathcal{X}''_{ji}, O_{\mathcal{X}''_{ji}}),
\]

\[
\begin{array}{ccc}
\coprod_{i=1}^s (\mathcal{X}''_{ji}, O_{\mathcal{X}''_{ji}}) & \longrightarrow & \coprod_{i=1}^s (\mathcal{X}''_{ji}, O_{\mathcal{X}''_{ji}}) \\
\downarrow & & \downarrow \\
(\mathcal{Y}_j, O_{\mathcal{Y}_j}) & \longrightarrow & (\mathcal{Y}_j, O_{\mathcal{Y}_j}),
\end{array}
\]

fs log structures \( M_{\coprod_{i=1}^s \mathcal{X}''_{ji}, \mathcal{Y}_j} \) on \( \coprod_{i=1}^s \mathcal{X}''_{ji}, \mathcal{Y}_j \) respectively, a proper log smooth integral universally saturated morphism \( g : (\coprod_{i=1}^s \mathcal{X}''_{ji}, M_{\coprod_{i=1}^s \mathcal{X}''_{ji}}) \longrightarrow (\mathcal{Y}_j, M_{\mathcal{Y}_j}) \) and a log normal crossing divisor \( D_1 \subseteq \coprod_{i=1}^s \mathcal{X}''_{ji} \) satisfying the properties stated in the claim (for \( s - 1 \)). Let us apply Proposition 4.1 to the morphism \( \mathcal{X}''_{js} \longrightarrow \mathcal{Y}_j'' \) to obtain the diagram

\[
\begin{array}{ccc}
\mathcal{X}''_{js} & \longrightarrow & \mathcal{X}''_{js} \\
\downarrow & & \downarrow \\
\mathcal{Y}_j' & \varphi & \mathcal{Y}_j'',
\end{array}
\]
log structures $M'^{\Psi}_{X_j}, M'^{\eta}_{Y_j}$ on $X'^{\eta}_{j}, Y'^{\eta}_{j}$ respectively and a log normal crossing divisor $D_2 \subseteq Y'^{\eta}_{j}$ which satisfy the analogue of the conditions (4), (5) for ‘s-component’. Let us define the reduced closed subscheme $Z \subseteq Y'^{\eta}_{j}$ as the scheme whose underlying space is the union of the simple normal crossing divisor corresponding to $M'^{\Psi}_{Y_j}$ and the inverse image of the simple normal crossing divisor corresponding to $M'^{\eta}_{Y_j}$, and let us take an alteration $\psi : Y'^{\eta}_{j} \to Y'^{\eta}_{j}$ such that $Y'^{\eta}_{j}$ is smooth over $S$ and that $\psi^{-1}(Z)_{\text{red}}$ is a simple normal crossing divisor on $Y'^{\eta}_{j}$. Then, if we denote the log structure associated to $\psi^{-1}(Z)_{\text{red}}$ by $M'^{\eta}_{Y_j}$, the morphisms $\varphi \circ \psi, \psi$ induce the morphisms of log schemes $\varphi \circ \psi : (Y'^{\eta}_{j}, M'^{\eta}_{Y_j}) \to (Y'^{\eta}_{j}, M'^{\eta}_{Y_j}), \psi : (Y'^{\eta}_{j}, M'^{\eta}_{Y_j}) \to (Y'^{\eta}_{j}, M'^{\eta}_{Y_j})$. Now let us put

$$
\left( \bigsqcup_{j=1}^{s-1} X'^{\eta}_{j}, M^\Psi_{X_j} \bigsqcup_{j=1}^{s-1} X'^{\eta}_{j} \right) := \left( \bigsqcup_{j=1}^{s-1} Y'^{\eta}_{j}, M^\Psi_{Y_j} \bigsqcup_{j=1}^{s-1} X'^{\eta}_{j} \right) \times (Y'^{\eta}_{j}, M'^{\eta}_{Y_j}) (Y'^{\eta}_{j}, M'^{\eta}_{Y_j}),
$$

$$
(X'^{\eta}_{j}, M^{\Psi}_{X_j}) := (X'^{\eta}_{j}, M^\Psi_{X_j}) \times (Y'^{\eta}_{j}, M'^{\eta}_{Y_j}) (Y'^{\eta}_{j}, M'^{\eta}_{Y_j}),
$$

$$
\left( \bigsqcup_{j=1}^{s} X'^{\eta}_{j}, M^\Psi_{X_j} \bigsqcup_{j=1}^{s} X'^{\eta}_{j} \right) := \left( \bigsqcup_{j=1}^{s-1} X'^{\eta}_{j}, M^\Psi_{X_j} \bigsqcup_{j=1}^{s-1} X'^{\eta}_{j} \right) \bigsqcup (X'^{\eta}_{j}, M^{\Psi}_{X_j})
$$

and let $D \subseteq \bigsqcup_{j=1}^{s} X'^{\eta}_{j}$ be the disjoint union of the inverse images of $D_1, D_2$ to $\bigsqcup_{j=1}^{s} X'^{\eta}_{j}$. Then these data satisfy the analogue of (2), (4) and (5). (To see that $D$ is a log normal crossing divisor, we use Proposition 3.9.) Then, by the argument in the proof of the claim in [Sh4, 6.10], we see that we can form the diagram like (4.5) for $s$. So the proof of the claim is finished.

By the claim for $s = r_j$ (for each $j$), we see that there exists a diagram consisting of strict morphism of pairs

$$
\begin{array}{ccc}
(X', O_{X'}) & \leftarrow & (\hat{X}, O_{\hat{X}}) \\
\downarrow & & \downarrow \hat{f} \\
(Y', O_{Y'}) & \leftarrow & (\hat{Y}, O_{\hat{Y}})
\end{array}
$$

(we put $\hat{X} := X' \times_{Y'} \hat{Y}$) satisfying the following conditions:

1. $\overline{Y_Y} \subseteq Y'$ is dense open.
2. $\overline{Y_Y}$ is regular.
3. The upper horizontal arrow is a proper covering and the lower horizontal arrow is a good strongly proper covering.
4. There exist fs log structures $M^\Psi_X, M^\Psi_Y$ on $\hat{X}, \hat{Y}$ respectively and a proper log smooth integral universally saturated morphism $(\hat{X}, M^\Psi_X) \to (\hat{Y}, M^\Psi_Y)$ whose underlying morphism of schemes is $\hat{f}$ such that $M^\Psi_Y$ is associated to a
simple normal crossing divisor on \( \tilde{Y} \) and that \( \tilde{f}^{-1}((\tilde{Y}, M_{\tilde{Y}})_{\text{triv}}) \) is contained in \((\tilde{X}, M_{\tilde{X}})_{\text{triv}}\).

(5) There exists a log normal crossing divisor \( D \subseteq \tilde{X} \) such that, if we define \( M_{\tilde{X}} D^{[i]} \) \((i \geq 0)\) as in Proposition 3.4 using \( D \), the induced morphisms

\[
(\tilde{X}, M_{\tilde{X}}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}}), \quad (D^{[i]}, M_{\tilde{X}}|_{D^{[i]}}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}}) \quad (i \geq 0)
\]

are proper log smooth integral universally saturated, and we have the equality \( \tilde{X} \cap (\tilde{X}, M_{\tilde{X}})_{\text{triv}} = (\tilde{X} - D) \cap (\tilde{X}, M_{\tilde{X}})_{\text{triv}} \).

Then, by the argument in the proof of [Sh4, 6.10], we can take an open subset \( U_{\tilde{Y}} \subseteq U \) and the diagram (4.4) in order that all the required conditions are satisfied. So we are done. \( \square \)

Using Proposition 4.3, we can prove the following theorem:

**Theorem 4.4.** Let \( f : X \rightarrow Y \) be a proper morphism of schemes, let \( X \subseteq X \) be an open subscheme and let \( q \in \mathbb{N} \). Then we have the diagram consisting of strict morphism of pairs

\[
(U_{\tilde{X}}, \tilde{X}) \leftarrow (U_{\tilde{X}}, \tilde{X}) \leftarrow (U_{\tilde{X}}, \tilde{X}) \leftarrow (U_{\tilde{X}}, \tilde{X})
\]

(4.7)

\[
(U_{\tilde{Y}}, \tilde{Y}) \leftrightarrow (U_{\tilde{Y}}, \tilde{Y}) \leftrightarrow (U_{\tilde{Y}}, \tilde{Y}) \leftrightarrow (U_{\tilde{Y}}, \tilde{Y})
\]

(we put \( \tilde{X} := X \times_X \tilde{X}, \tilde{X}^{(*)} := X \times_X \tilde{X}^{(*)} \)) satisfying the following conditions:

1. \( U_{\tilde{Y}} \) is dense in \( \tilde{Y} \).
2. \( \tilde{Y} \) is regular.
3. The left square is Cartesian, \( g_{\tilde{Y}} \) is a good strongly proper covering and the morphism

\[
(\tilde{X} \cap U_{\tilde{X}}, \tilde{X}) \leftarrow (\tilde{X}^{(*)} \cap U_{\tilde{X}}, \tilde{X}^{(*)})
\]

induced by \( h \) is a \( q \)-truncated proper hypercovering by a \( q \)-truncated split simplicial pair.

4. There exist \( \text{fs log structures } M_{\tilde{X}}^{(n)}, M_{\tilde{Y}}^{(n)} \) on \( \tilde{X}^{(n)}, \tilde{Y} \) respectively \((n \leq q)\) such that \( M_{\tilde{Y}}^{(n)} \) is associated to a simple normal crossing divisor on \( \tilde{Y} \), \( U_{\tilde{X}}^{(n)} \subseteq (\tilde{X}^{(n)}, M_{\tilde{X}}^{(n)})_{\text{triv}}, U_{\tilde{Y}}^{(n)} \subseteq (\tilde{Y}, M_{\tilde{Y}}^{(n)})_{\text{triv}} \) holds and that for each \( n \leq q \), there exists a proper log smooth integral universally saturated morphism \((\tilde{X}^{(n)}, M_{\tilde{X}}^{(n)}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}}^{(n)})\) whose underlying morphism of schemes is the same as \( \tilde{f}^{(n)} \).
(5) For each $n \leq q$, there exists a log normal crossing divisor $D^{(n)} \subseteq \widetilde{X}^{(n)}$ such that, if we define $M_{\widetilde{X}^{(n)}}, D^{(n)}_i (i \geq 0)$ as in Proposition 3.4 using $D^{(n)}$, the induced morphisms

$$(\widetilde{X}^{(n)}, M_{\widetilde{X}^{(n)}}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}}), \quad (D^{(n)}_i, M^{\text{triv}}_{\tilde{X}^{(n)}}|_{D^{(n)}_i}) \longrightarrow (\tilde{Y}, M_{\tilde{Y}}) (i \geq 0)$$

are proper log smooth integral universally saturated, and we have $\widetilde{X}^{(n)} \cap (\mathcal{X}^{(n)}_Y, M^{\text{triv}}_{\mathcal{X}^{(n)}_Y}) = (\tilde{X}^{(n)} - D^{(n)}) \cap (\mathcal{X}^{(n)}_Y, M^{\text{triv}}_{\mathcal{X}^{(n)}_Y})$. (In particular, we have $\widetilde{X}^{(n)} \cap U^{(n)}_\mathcal{X} = (\tilde{X}^{(n)} - D^{(n)}) \cap U^{(n)}_\mathcal{X}$.)

Proof. We can prove the theorem in the same way as [Sh4, 6.11], by using Proposition 4.3 instead of [Sh4, 6.10]. The detail is left to the reader. \hfill \square

5. Generic overconvergence

In this section, we prove the main theorem of this paper, that is, the generic overconvergence of relative rigid cohomology for any morphism when the coefficient is a potentially semistable overconvergent isocrystal. In particular, we obtain the generic overconvergence of relative rigid cohomology when the coefficient is an overconvergent $F$-isocrystal, by using Theorem 1.13.

Note that we keep the assumption that $k$ is perfect also in this section. The precise statement of the main theorem is as follows:

**Theorem 5.1.** Let us assume given a morphism of pairs $f : (X, \mathcal{X}) \longrightarrow (Y, \mathcal{Y})$ such that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is proper. Then, there exists a dense open set $U_{\mathcal{Y}}$ of $Y$, a proper surjective map $\tilde{Y} \longrightarrow Y$ and a simple normal crossing divisor $E$ on $\tilde{Y}$ with $E \cap U_{\mathcal{Y}} = \emptyset$ (where we put $U_{\mathcal{Y}} := U_{\mathcal{Y}} \times_{\mathcal{Y}} \tilde{Y}$) such that, for any potentially semistable overconvergent isocrystal $\mathcal{E}$ on $(X, \mathcal{X})/\mathcal{S}_K$ and $q \in \mathbb{N}$, there exists the unique overconvergent isocrystal $\mathcal{F}$ on $(U_{\mathcal{Y}}, \mathcal{Y})/\mathcal{S}_K$ satisfying the following condition: For any $(U_{\mathcal{Y}}, \tilde{Y})$-triple $(Z, \mathcal{Z}, \mathcal{Z})$ over $(S, S, S)$ such that $\mathcal{Z}$ is smooth over $S$, $(E \times_{\mathcal{Y}} \mathcal{Z})_{\text{red}}$ is contained in a simple normal crossing divisor on $\mathcal{Z}$ and that $\mathcal{Z}$ is formally smooth over $S$, the restriction of $\mathcal{F}$ to $\mathcal{I}((Z, \mathcal{Z})/\mathcal{S}_K, \mathcal{Z})$ is given functorially by $(R^q f_{(X \times_{\mathcal{Y}} Z, X \times_{\mathcal{Y}} \mathcal{Z})/Z, \text{rig}, \mathcal{E}), \epsilon)$, where $\epsilon$ is an isomorphism

$$p_i^* R^q f_{(X \times_{\mathcal{Y}} Z, X \times_{\mathcal{Y}} \mathcal{Z})/Z, \text{rig}, \mathcal{E}} \sim R^q f_{(X \times_{X \times Z} X \times_{\mathcal{Y}} \mathcal{Z})/Z \times S Z, \text{rig}, \mathcal{E}} \sim p_i^* R^q f_{(X \times_{\mathcal{Y}} Z, X \times_{\mathcal{Y}} \mathcal{Z})/Z, \text{rig}, \mathcal{E}}$$

($p_i$ denotes the $i$-th projection $\mathcal{Z} \times_{S \times Z} \mathcal{Z} \longrightarrow \mathcal{Z}$).

Proof. First, by the argument in the beginning of the proof of [Sh4, 7.4], there exists a positive integer $q_0$ such that, for any overconvergent isocrystal $\mathcal{E}$ on $(X, \mathcal{X})/\mathcal{S}_K$ and any $(Y, \mathcal{Y})$-triple $(Z, \mathcal{Z}, \mathcal{Z})$ over $(S, S, S)$, we have

$$R^q f_{(X \times_{\mathcal{Y}} Z, X \times_{\mathcal{Y}} \mathcal{Z})/Z, \text{rig}, \mathcal{E}} = 0$$

for any $q > q_0$. 
Let us take a proper hypercovering \((X^{(\bullet)}, \underline{X}^{(\bullet)}) \rightarrow (X, \underline{X})\) such that each \(\underline{X}^{(n)}\) is smooth over \(k\) and, each \(\underline{X}^{(n)} - X^{(n)}\) is a simple normal crossing divisor on \(\underline{X}^{(n)}\) and that the restriction of \(\mathcal{E}\) to \(\mathcal{F}((X^{(n)}, \underline{X}^{(n)})/\mathcal{S}_K)\) extends to a locally free isocrystal \(\mathcal{E}\) on \(\underline{(X^{(n)}/\mathcal{S})}_{\text{conv}} = ((\underline{X}^{(n)}, M_{\underline{X}^{(n)})}/\mathcal{S})_{\text{conv}}\) having nilpotent residues, where \(M_{\underline{X}^{(n)}}\) denotes the log structure on \(\underline{X}^{(n)}\) associated to \(\underline{X}^{(n)} - X^{(n)}\). (This is possible by \([\text{dJ1]}\) and the potential semi-stability of \(\mathcal{E}\).

Next let us put \(q_1 := q_0(q_0 + 1)/2, q_2 := q_1(q_1 + 1)/2\). By applying Theorem 4.4 to the proper map \(\prod_{n=0}^{q_2} \underline{X}^{(n)} \rightarrow \overline{Y}\) and the open subscheme \(\prod_{n=0}^{q_1} X^{(n)} \subset \prod_{n=0}^{q_1} \underline{X}^{(n)}\) for \(q = q_2\), we have the following: For each \(0 \leq n \leq q_1\), we have the following diagram consisting of strict morphisms of pairs

$$
\begin{array}{c}
(U_{\underline{X}^{(n)}}, \underline{X}^{(n)}) & \leftarrow & (U_{\underline{X}^{(n)}}, \underline{X}^{(n)}) & \leftarrow & (U_{\underline{X}^{(n)}}\langle\bullet\rangle, \underline{X}^{(n)<(\bullet)>)}) \\
\downarrow & & \downarrow & & \downarrow \\
(U_{\overline{Y}}, \overline{Y}) & \leftarrow & (U_{\overline{Y}}, \overline{Y}) & \leftarrow & (U_{\overline{Y}}\langle\bullet\rangle, \overline{Y})
\end{array}
$$

(5.1)

which satisfies the conclusion of Theorem 4.4 for \(q = q_2\) and that the lower horizontal line in (5.1) is independent of \(n\). In particular, there exist fs log structures \(M_{\overline{Y}}^{(n)}\langle\bullet\rangle\rangle, M_{\overline{Y}}\rangle\rangle\) on \(\underline{X}^{(n)<(\bullet)>)}, \overline{Y}\) respectively such that \(M_{\overline{Y}}\rangle\rangle\) is associated to a simple normal crossing divisor, \(U_{\underline{X}^{(n)<(\bullet)>)}} \subseteq \langle\underline{X}^{(n)<(\bullet)>)}, M_{\underline{X}^{(n)<(\bullet)>)}}\rangle\rangle\) and \(U_{\overline{Y}} \subseteq \langle\overline{Y}, M_{\overline{Y}}\rangle\rangle\rangle\) hold, and that there exists a proper log smooth integral universally saturated morphism \((\underline{X}^{(n)<(m)\rangle\rangle}, M_{\underline{X}^{(n)<(m)\rangle\rangle}}) \rightarrow (\overline{Y}, M_{\overline{Y}})\rangle\rangle\) whose underlying morphism of schemes is the same as \(f^{(n)<(m)\rangle\rangle}\) (which we also denote by \(f^{(n)<(m)\rangle\rangle}\)). Moreover, there exists a log normal crossing divisor \(D^{(n)<(m)\rangle\rangle}\) in \(\underline{X}^{(n)<(m)\rangle\rangle}\) such that, if we define \(M_{\underline{X}^{(n)<(m)\rangle\rangle}}, D^{(n)<(m)\rangle\rangle}\) as in Proposition 3.4, the morphisms

\[
(\underline{X}^{(n)<(m)\rangle\rangle}, M_{\underline{X}^{(n)<(m)\rangle\rangle}}) \rightarrow (\overline{Y}, M_{\overline{Y}}),
\]

\[
(D^{(n)<(m)\rangle\rangle}, M_{\underline{X}^{(n)<(m)\rangle\rangle}}\langle\bullet\rangle\rangle |_{D^{(n)<(m)\rangle\rangle}}) \rightarrow (\overline{Y}, M_{\overline{Y}})
\]

induced by \(f^{(n)<(m)\rangle\rangle}\) are proper log smooth integral universally saturated, and that we have the equality \(\underline{X}^{(n)<(m)\rangle\rangle} \cap \langle\underline{X}, M_{\underline{X}^{(n)<(m)\rangle\rangle}}\rangle\rangle = \langle\underline{X}^{(n)<(m)\rangle\rangle} - D^{(n)<(m)\rangle\rangle}\rangle \cap \langle\underline{X}, M_{\underline{X}^{(n)<(m)\rangle\rangle}}\rangle\rangle\), where we denoted the inverse image of \(X^{(n)}\) in \(\underline{X}^{(n)<(m)\rangle\rangle}\) by \(\underline{X}^{(n)<(m)\rangle\rangle}\).

Note that the properties in Theorem 4.4 for the diagram (5.1) remains true if we take a morphism of log schemes \(\varphi : (\overline{Y}, M_{\overline{Y}}) \rightarrow (\overline{Y}, M_{\overline{Y}})\rangle\rangle\) such that \(\overline{Y}\rangle\rangle\) is smooth over \(k\) and that \(M_{\overline{Y}}\rangle\rangle\) is a simple normal crossing divisor in \(\overline{Y}\rangle\rangle\) which does not meet \(U_{\overline{Y}} := \varphi^{-1}(U_{\overline{Y}})\rangle\rangle\), and replace the right square of the diagram (5.1) by the pull-back
of it by \( \varphi : (U_{\overline{Y}}, \overline{Y}) \longrightarrow (U_{\overline{Y}}, \overline{Y}) \). (We use Proposition 3.9.) We can take \( \varphi \) to be the alteration in order that \( \overline{Y}' \) is quasi-projective smooth over \( S \), \( \overline{Y}' - U_{\overline{Y}} \) is a simple normal crossing divisor and \( M_{\overline{Y}'} \) is the log structure associated to \( \overline{Y}' - U_{\overline{Y}} \).

So, in the diagram (5.1), we may assume that \( \overline{Y} \) is quasi-projective smooth over \( S \), \( E := \overline{Y} - U_{\overline{Y}} \) is a simple normal crossing divisor and the log structure \( M_{\overline{Y}} \) is the one associated to \( \overline{Y} - U_{\overline{Y}} \).

Next we replace \( U_{\overline{Y}} \) by \( U_{\overline{Y}} \cap Y \): Then all the conditions in Theorem 4.4 except the denseness of \( U_{\overline{Y}} \) in \( \overline{Y} \) remains true. (Note that \( U_{\overline{Y}} \) is still dense in \( Y \). So, if \( Y \) is dense in \( \overline{Y} \), \( U_{\overline{Y}} \) remains to be dense in \( \overline{Y} \).

Let us take an \((U_{\overline{Y}}, Y)\)-triple \((Z, \mathcal{Z}, \mathcal{E})\) over \((S, S, \mathcal{S})\) as in the statement of the theorem. Let us denote the open immersion \( Z \hookrightarrow \mathcal{Z} \) by \( j_{Z} \). We prove the following claim:

**Claim.** \( R^q f_{(X \times Y Z, X \times Y Z) / Z, \text{rig}^* \mathcal{E}} \) is a coherent \( j_{Z}^! \mathcal{O}_{\mathcal{Z}} \)-module.

Since we have \( R^q f_{(X \times Y Z, X \times Y Z) / Z, \text{rig}^* \mathcal{E}} = 0 \) for \( q > q_0 \), we may assume \( 0 \leq q \leq q_0 \). Let us put \( E_{Z} := (\text{inverse image of } E)_{\text{red}} \subseteq Z \). By assumption on the triple \((Z, \mathcal{Z}, \mathcal{E})\), it is a simple normal crossing divisor in \( Z \). Denote the log structure associated to \( E_{Z} \) by \( M_{\mathcal{Z}} \). Since the claim is Zariski local on \( Z \), we may assume that there exists a fine log structure \( M_{\mathcal{Z}} \) on \( Z \) such that the closed immersion \( \mathcal{Z} \hookrightarrow Z \) comes from an exact closed immersion \((\mathcal{Z}, M_{\mathcal{Z}}) \hookrightarrow (Z, M_{Z})\).

Let us put \( \widetilde{X} := X \times_{\overline{Y}} \overline{Y}, U_{\overline{X}} := X \times_{\overline{Y}} U_{\overline{Y}} \). Also, we put \( \overline{X}^{(n)(\bullet)} := \text{cosk}_{Y_{\mathcal{Z}}} \overline{X} \) and denote the inverse image of \( U_{\overline{X}^{(n)}} \) in \( \overline{X}^{(n)(\bullet)} \) by \( U_{\overline{X}^{(n)(\bullet)}} \). We denote the inverse image of \( X \) in \( \overline{X}, \overline{X}^{(n)}, \overline{X}^{(n)(\bullet)}, \overline{X}^{(n)(\bullet)} \) by \( \overline{X}, \overline{X}^{(n)}, \overline{X}^{(n)(\bullet)}, \overline{X}^{(n)(\bullet)} \) respectively. With this notation, we have \((X \times_{\overline{Y}} Z, \overline{X} \times_{\overline{Y}} Z) = ((\overline{X} \cap U_{\overline{X}}) \times_{\overline{Y}} Z, \overline{X} \times_{\overline{Y}} Z)\).

Since \((X^{(\bullet)}, \overline{X}^{(\bullet)}) \longrightarrow (X, \overline{X})\) is a proper hypercovering, so is \((\overline{X}^{(\bullet)} \cap U_{\overline{X}^{(\bullet)}}, \overline{X}^{(\bullet)}) \longrightarrow (\overline{X} \cap U_{\overline{X}}, \overline{X})\).

Hence

\[
((\overline{X}^{(\bullet)} \cap U_{\overline{X}^{(\bullet)}}) \times_{\overline{Y}} Z, \overline{X}^{(\bullet)} \times_{\overline{Y}} Z) \longrightarrow ((\overline{X} \cap U_{\overline{X}}) \times_{\overline{Y}} Z, \overline{X} \times_{\overline{Y}} Z) = (X \times_{\overline{Y}} Z, \overline{X} \times_{\overline{Y}} Z)
\]

is a proper hypercovering. So, by [T3], we have the spectral sequence

\[
E_{1}^{s,t} := R^s f_{((\overline{X}^{(\bullet)} \cap U_{\overline{X}^{(\bullet)}}) \times_{\overline{Y}} Z, \overline{X}^{(\bullet)} \times_{\overline{Y}} Z) / Z, \text{rig}^* \mathcal{E}} \Longrightarrow R^{s+t} f_{(X \times_{\overline{Y}} Z, \overline{X} \times_{\overline{Y}} Z) / Z, \text{rig}^* \mathcal{E}}.
\]

So, to prove the claim, it suffices to prove that \( R^t f_{((\overline{X}^{(\bullet)} \cap U_{\overline{X}^{(\bullet)}}) \times_{\overline{Y}} Z, \overline{X}^{(\bullet)} \times_{\overline{Y}} Z) / Z, \text{rig}^* \mathcal{E}} \)

is a coherent \( j_{Z}^! \mathcal{O}_{\mathcal{Z}} \)-module for \( 0 \leq s, t \leq q_1 \). Next, since \((U_{\overline{X}^{(\bullet)}}, \overline{X}^{(\bullet)}(\bullet) \longrightarrow \)
\((U_X^{(s)}, X^{(s)})\) is a proper hypercovering for for \(0 \leq s \leq q_1\), so is \((\tilde{X}^{(s)}(\bullet) \cap U_X^{(s)}(\bullet), X^{(s)}(\bullet)) \to (\tilde{X}^{(s)}(\cdot) \cap U_X^{(s)}(\cdot), X^{(s)}).\) Hence

\[
((\tilde{X}^{(s)}(\bullet) \cap U_X^{(s)}(\bullet)) \times_{\tilde{X}} Z, X^{(s)}(\bullet) \times_{\tilde{X}} Z) \to (\tilde{X}^{(s)}(\cdot) \cap U_X^{(s)} \times_{\tilde{X}} Z, X^{(s)} \times_{\tilde{X}} Z)
\]
is a proper hypercovering. So we have the spectral sequence

\[
E_{1}^{u,v} := R^u f \left[ \left( (\tilde{X}^{(s)}(\bullet) \cap U_X^{(s)}(\bullet)) \times_{\tilde{X}} Z, X^{(s)}(\bullet) \times_{\tilde{X}} Z \right) \right] / Z, \text{rig}^* \mathcal{E} \to \cdots
\]

So, to prove the claim, it suffices to prove that \(R^u f \left[ \right] \) is a coherent \(j_Z^{\dagger} \mathcal{O}_{\overline{Z}}\text{-module}\) for \(0 \leq s \leq q_1, 0 \leq u, v \leq q_2\). For such \(s, u\), we have \(\tilde{X}^{(s)(u)} \cap U_{X^{(s)(u)}}^{(s)} = \tilde{X}^{(s)(u)} \cap U_{X^{(s)(u)}}^{(s)} = X^{(s)(u)}.\) So it suffices to prove that \(R^u f \left[ \right] \) is a coherent \(j_Z^{\dagger} \mathcal{O}_{\overline{Z}}\text{-module}\) for \(0 \leq s \leq q_1, 0 \leq u, v \leq q_2\).

Now let us take a log blow-up \(\psi' : (\overline{X}, M^v) \to (\overline{X}^{(s)(u)}, M^v_{\overline{X}^{(s)(u)}}) \times_{\overline{Y}, M^v_{\overline{Y}}} (Z, M_Z)\) such that \(\overline{X}\) is regular. Then \(M^v_{\overline{X}}\) corresponds to a normal crossing divisor on \(\overline{X}\), which we denote by \(D^v\). Let us define \((\overline{X}, M^v)\) in order that the following diagram is Cartesian:

\[
\begin{array}{ccc}
(\overline{X}, M^v) & \xrightarrow{\psi} & (\overline{X}^{(s)(u)}, M^v_{\overline{X}^{(s)(u)}}) \times_{\overline{Y}, M^v_{\overline{Y}}} (Z, M_Z) \\
\downarrow & & \downarrow \\
(\overline{X}, M^v_{\overline{X}}) & \xrightarrow{\psi'} & (\overline{X}^{(s)(u)}, M^v_{\overline{X}^{(s)(u)}}) \times_{\overline{Y}, M^v_{\overline{Y}}} (Z, M_Z).
\end{array}
\]

Let us denote the pull-back of \(D^{(s)(u)} \times_{\overline{X}} Z\) in \(\overline{X}\) by \(D^h\). Since \(D^{(s)(u)}\) is a log normal crossing divisor in \(\overline{X}^{(s)(u)}\), \(D^{(s)(u)} \times_{\overline{X}} Z\) is a log normal crossing divisor in \(\overline{X}^{(s)(u)} \times_{\overline{X}} Z\) by Proposition 3.9. Then, by definition of \(\psi'\) above, we see that \(D^h \cup D^v\) is the normal crossing divisor corresponding to \(M_X, D^h, D^v\) are sub normal crossing divisors of it. Moreover, \((\overline{X}, M^v_{\overline{X}})\) is log smooth over \((Z, M_Z)\), and if we define \(D_{[i]}\) by

\[
D_{[0]} := \overline{X}, \quad D_{[1]} := \text{the normalization of } D^h, \quad D_{[i]} := i\text{-fold fiber product of } D_{[1]} \text{ over } \overline{X},
\]

\((D_{[i]}, M^v_{\overline{X}(D_{[i]})})\) is log smooth over \((Z, M_Z)\) for any \(i \in \mathbb{N}\). That is, the morphism \((\overline{X}, M^v) \to (Z, M_Z)\) and \(D^h, D^v\) are as in the situation in the beginning of Section 2.
Next, note that we have \( \widetilde{X}^{(s)(u)} \cap \tilde{X}^{(s)(u)} = (\tilde{X}^{(s)(u)} - D^{(s)(u)}) \cap (\widetilde{X}^{(s)(u)}, M_{\tilde{X}^{(s)(u)}})_{\text{triv}} \). This implies that the inverse image of \( X^{(s)} \) by the morphism \( \tilde{X}^{(s)(u)} \times \tilde{Z} \longrightarrow X^{(s)} \) contains \( ((\tilde{X}^{(s)(u)}, M_{\tilde{X}^{(s)(u)}}) \times (\tilde{Z}, M_{\tilde{Z}}))_{\text{triv}} \). So the diagram
\[
\tilde{X} \xrightarrow{\psi} \tilde{X}^{(s)(u)} \times \tilde{Z} \longrightarrow \tilde{X}^{(s)}
\]
comes from the diagram of log schemes
\[
(\tilde{X}, M_{\tilde{Z}}) \xrightarrow{\psi} (\tilde{X}^{(s)(u)}, M_{\tilde{X}^{(s)(u)}}) \times (\tilde{Z}, M_{\tilde{Z}}) \longrightarrow (\tilde{X}^{(s)}, M_{\tilde{X}^{(s)}}).
\]

Then the locally free isocrystal \( \mathcal{E} \) on \( (X^{(s)}/S)_{\text{conv}}^{\log} \) having nilpotent residues induces, by pull-back, a locally free isocrystal on \( ((\tilde{X}^{(s)(u)} \times \tilde{Z})/S)_{\text{conv}}^{\log} \), and the induced locally free isocrystal on \( (\tilde{X}/S)_{\text{conv}}^{\log} \) (which we denote also by \( \mathcal{E} \), by abuse of notation) has nilpotent residues by Proposition 1.11. In particular, the diagram
\[
(\tilde{X}, M_{\tilde{Z}}) \xrightarrow{\psi} (\tilde{X}^{(s)(u)}, M_{\tilde{X}^{(s)(u)}}) \times (\tilde{Z}, M_{\tilde{Z}}) \longrightarrow (\tilde{Z}, M_{\tilde{Z}}) \leftarrow (Z, M_{Z})
\]
and \( \mathcal{E} \) are in the situation of Proposition 3.6 and Remark 3.7.

Finally, note that we have
\[
\widetilde{X}^{(s)(u)} \cap U_{\tilde{X}^{(s)(u)}} = (\tilde{X}^{(s)(u)} - D^{(s)(u)}) \cap U_{\tilde{X}^{(s)(u)}} \subseteq (\tilde{X}^{(s)(u)}, M_{\tilde{X}^{(s)(u)}})_{\text{triv}}
\]
and \( Z \subseteq (\tilde{Z}, M_{\tilde{Z}})_{\text{triv}} \) (which follows from the equality \( U_{\tilde{Z}} \cap E = \emptyset \).) So, on \( \tilde{X}^{(s)(u)} \cap U_{\tilde{X}^{(s)(u)}} \times \tilde{Z}, \) the log structure of \( (\tilde{X}^{(s)(u)}, M_{\tilde{X}^{(s)(u)}}) \times (\tilde{Z}, M_{\tilde{Z}}) \) is trivial. Hence the morphism \( \psi \) is identity on it. That is, we have the isomorphism
\[
\tilde{X} \times \tilde{Z} - \mathbb{D}^{h} = (\tilde{X} - \mathbb{D}^{h}) \times \tilde{Z} \cong (\tilde{X}^{(s)(u)} - D^{(s)(u)}) \times \tilde{Z} = (\tilde{X}^{(s)(u)} \cap U_{\tilde{X}^{(s)(u)}}) \times \tilde{Z}.
\]

Let us denote the open immersion \( \tilde{X} \times \tilde{Z} - \mathbb{D}^{h} \hookrightarrow \tilde{X} \) by \( j_{X} \). Then, by combining the above results, we obtain the equalities
\[
R^{u} f_{((\tilde{X}^{(s)(u)} \cap U_{\tilde{X}^{(s)(u)}}) \times \tilde{Z}, \tilde{X})/\tilde{Z}, \text{rig}_{*}} \mathcal{E} = R^{u} f_{(\tilde{X} \times \tilde{Z} - \mathbb{D}^{h}, \tilde{X})/\tilde{Z}, \text{rig}_{*}} \mathcal{E} = R^{u} f_{(\tilde{X} \times \tilde{Z} - \mathbb{D}^{h}, \tilde{X})/\tilde{Z}, \text{rig}_{*}} j_{X}^{\dagger} \mathcal{E} = j_{Z}^{\dagger} R^{u} f_{\tilde{X} / Z, \text{an}_{*}} \mathcal{E} \quad \text{ (Theorem 2.2, Proposition 3.6, Remark 3.7)}
\]
and it is a coherent \( j_{Z}^{\dagger} \mathcal{O}_{\tilde{Z}} \text{-module}. \) So we have proved the claim.
By using the claim, we see also that $R^i f_{(X \times Y Z, X \times \overline{Y})/Z \times \overline{Z}} \mathcal{E}$ is a coherent $j^{*} O_{(Z\times \overline{Z}, \mathcal{E})}$-module. So, by [T2, 2.3.1], we see that the morphisms

$$p^i_{\ast} R^i f_{(X \times Y Z, X \times \overline{Y})/Z \times \overline{Z}} \mathcal{E} \longrightarrow R^i f_{(X \times Y Z, X \times \overline{Y})/Z \times \overline{Z}} \mathcal{E} \ (i = 1, 2)$$

in the definition of $\epsilon$ in the statement of the theorem are isomorphisms, and the pair $(R^i f_{(X \times Y Z, X \times \overline{Y})/Z \times \overline{Z}}, \mathcal{E}, \epsilon)$ defines an object in $I^i((Z, \overline{Z})/S_K, Z)$. We denote it by $\mathcal{F}_{Z}$. Now let us take a hypercovering $(U_{\overline{Y}}, \overline{Y}) \leftarrow (U_{\overline{Y}}, \overline{Y}^{(0)})$ such that each $\overline{Y}^{(n)}$ is quasi-projective smooth over $S$ and that $\overline{Y}^{(1)} - U_{\overline{Y}}^{(0)}$ is a simple normal crossing divisor in $\overline{Y}^{(1)}$. For each $n \leq 2$, let us take a closed immersion $\overline{Y}^{(n)} \hookrightarrow Y^{(n)}$ such that $Y^{(n)}$ is formally smooth over $S$. (It is possible because $\overline{Y}^{(n)}$ is quasi-projective over $S$.) Then, following [T4, 7.3.1], we define the simplicial formal scheme $\Gamma^{\ast} Y^{(m)} \leq m$ by

$$[n] \mapsto \prod_{\gamma: [m] \rightarrow [n]} Y^{(m)},$$

where $[n]$ denotes the set $\{0, 1, \cdots, n\}$, $\gamma$ runs through the non-decreasing maps $[m] \rightarrow [n]$ and the transition maps are defined in natural way. Using this, we define the 2-truncated simplicial formal scheme $Y^{(*)}$ by

$$Y^{(*)} := \text{sk}_2(\prod_{0 \leq m \leq 2} \Gamma^{\ast} Y^{(m)} \leq m)$$

with the transition maps induced by those of $\Gamma^{\ast} Y^{(m)} \leq m$'s. Then, for $0 \leq n \leq 2$, the triple $Y^{(n)} := (U_{\overline{Y}}^{(n)}, \overline{Y}^{(n)}, Y^{(n)})$ is a $(U_{\overline{Y}}, \overline{Y})$-triple satisfying the condition in the statement of the theorem required for $(Z, \overline{Z}, Z)$. So we have overconvergent isocrystals $\mathcal{F}_{Y^{(n)}}$ on $(U_{\overline{Y}}^{(n)}, \overline{Y}^{(n)})/S_K$ for $n = 0, 1, 2$. Moreover, by [T2, 2.3.1], it is compatible with respect to $n$. So, by proper descent for overconvergent isocrystals ([Sh4, 7.3] and [SD, 3.3.4.2]), the compatible family $\{\mathcal{F}_{Y^{(n)}}\}_{n=0,1,2}$ descents to an overconvergent isocrystal on $(U_{\overline{Y}}, \overline{Y})/S_K$, which we denote by $\mathcal{F}$. The uniqueness of the overconvergent isocrystal $\mathcal{F}$ follows from the uniqueness of the compatible family $\{\mathcal{F}_{Y^{(n)}}\}_{n=0,1,2}$ and proper descent for overconvergent isocrystals. We check that, for any $(Z, \overline{Z}, Z)$ as in the statement of the theorem, the restriction of $\mathcal{F}$ to $I^i((Z, \overline{Z})/S_K, Z)$ is given by $\mathcal{F}_{Z} := (R^i f_{(X \times Y Z, X \times \overline{Y})/Z \times \overline{Z}} \mathcal{E}, \epsilon)$ as in the statement of the theorem. (The proof is similar to [Sh3, 4.8].) Let us take $(Z, \overline{Z}, Z)$.
as above and let us consider the following diagram

\[(5.2)\]

\[
I^\dagger((U_{\overline{Y}}, S_K)/S_K) \longrightarrow I^\dagger((U_{\overline{Y}}, \overline{Y})/S_K) \longrightarrow I^\dagger((U_{\overline{Y}}, \overline{Y})/S_K, Y^{(0)})
\]

\[
I^\dagger((Z, \overline{Z})/S_K, Z) \longrightarrow I^\dagger((Z, \overline{Z})/S_K, Z \times_S Y^{(0)}),
\]

where all the functors are restrictions. Then, by definition, \(F\) is sent to \(F_{Y^{(0)}}\) by the composite of horizontal arrows and it is sent to \(F_{Z \times_S Y^{(0)}}\) by the right vertical arrow (by functoriality of relative rigid cohomology). On the other hand, the over-convergent isocrystal \(F_Z\) is sent to \(F_{Z \times_S Y^{(0)}}\) by the lower horizontal arrow (again by functoriality of relative rigid cohomology). So, by the commutativity of the diagram \((5.2)\), we see that the restriction of \(F\) to \(I^\dagger((Z, \overline{Z})/S_K, Z)\) is given by \(F_Z\).

Finally we prove the functoriality of the above expression. Let \(\varphi: (Z', \overline{Z}', Z') \longrightarrow (Z, \overline{Z}, Z')\) be a morphism of \((U_{\overline{Y}}, \overline{Y})\)-triples over \((S, S, S)\) satisfying the condition in the statement of the theorem required for \((Z, \overline{Z}, Z)\). Then we have two morphisms of the form

\[\varphi^* F_Z \longrightarrow F_{Z'}:\]

One is the morphism induced by the functoriality of the relative rigid cohomology and the other is the morphism induced by the isocrystal structure of \(F\). We should prove that they are equal. Let us consider the following commutative diagram

\[
I^\dagger((Z, \overline{Z})/S_K, Z) \xrightarrow{\varphi^*} I^\dagger((Z, \overline{Z})/S_K, Z \times_S Y^{(0)})
\]

\[
I^\dagger((Z', \overline{Z}')/S_K, Z') \xrightarrow{\varphi'^*} I^\dagger((Z', \overline{Z}')/S_K, Z' \times_S Y^{(0)}),
\]

where \(\varphi^*, \varphi'^*\) are the restriction functors induced by \(\varphi\) and \(\pi, \pi'\) are the restriction functors induced by \(Z \times_S Y^{(0)} \longrightarrow Z, Z' \times_S Y^{(0)} \longrightarrow Z'\), respectively. Then, by looking at the argument in the previous paragraph, we see that either of the morphisms \((5.3)\) fits into the following diagram

\[
\pi'^* \varphi^* F_Z \longrightarrow \varphi'^* F_{Z \times_S Y^{(0)}}
\]

\[
\pi'^* F_{Z'} \longrightarrow F_{Z' \times_S Y^{(0)}},
\]

where the left vertical arrow is the pull-back of the morphism \((5.3)\) by \(\pi'^*\) and the other arrows are induced by the functoriality of relative rigid cohomology. Since \(\pi'^*\) is an equivalence of categories, we see that the two morphisms \((5.3)\) are equal. So we have proved the functoriality and the proof of the theorem is now finished. □

By using Theorem 5.1 repeatedly, we obtain the following corollary, which says a kind of constructibility of relative rigid cohomology:
Corollary 5.2. Let us assume given a morphism of pairs \( f : (X, \overline{X}) \rightarrow (Y, \overline{Y}) \) such that \( f : \overline{X} \rightarrow \overline{Y} \) is proper. Then, there exists a stratification \( Y = \coprod_{i=0}^{d} Y_i \) of \( Y \) by locally closed subschemes \( Y_i \) with \( Y_i = \coprod_{j \geq i} Y_j \) (where \( Y_i \) denotes the closure of \( Y_i \) in \( Y \)) and proper surjective maps \( \overline{Y}_i \rightarrow \overline{Y}_i (0 \leq i \leq d) \) (where \( \overline{Y}_i \) denotes the closure of \( Y_i \) in \( \overline{Y} \)) and a simple normal crossing divisor \( E_i \) on \( \overline{Y}_i \) with \( E_i \cap \overline{Y}_i = \emptyset \) (where we put \( \overline{Y}_i := Y_i \times_{\overline{X}_i} \overline{Y}_i \)) such that, for any potentially semistable overconvergent isocrystal \( E \) on \( (X, \overline{X})/S_K \) and \( q \in \mathbb{N} \), there exist uniquely the overconvergent isocrystals \( F_i \) on \( (Y_i, \overline{Y}_i)/S_K \) (0 \leq i \leq d) satisfying the following condition: For any \( (\overline{Y}_i, Y_i) \)-triple \( (Z, Z, \overline{Z}) \) over \( (S, S, S) \) such that \( \overline{Z} \) is smooth over \( S \), \( (E_i \times_{\overline{X}_i} \overline{Z})_{\text{red}} \) is contained in a simple normal crossing divisor on \( Z \) and that \( Z \) is formally smooth over \( S \), the restriction of \( F_i \) to \( I^\dagger((Z, Z)/S_K, Z) \) is given functorially by \( (R^q f_{(X \times_Y Z \times_{\overline{Y}_i} Z, Z, \overline{Z})/Z, \text{rig}^\ast} E, \epsilon) \), where \( \epsilon \) is an isomorphism

\[
p_j^q R^q f_{(X \times_Y Z \times_{\overline{Y}_i} Z, Z, \overline{Z})/Z, \text{rig}^\ast} E \sim R^q f_{(X \times_Y Z \times_{\overline{Y}_i} Z, Z, \overline{Z})/Z, \text{rig}^\ast} E \sim p_j^q R^q f_{(X \times_Y Z \times_{\overline{Y}_i} Z, Z, \overline{Z})/Z, \text{rig}^\ast} E
\]

\((p_j \text{ denotes the } j\text{-th projection } |Z|_{Z, \overline{Z}} \longrightarrow |Z|_Z).\)

Proof. We may replace \( \overline{Y} \) by the closure of \( Y \) in \( \overline{Y} \). As we remarked in the proof in Theorem 5.1, we can take the open set \( U_{\overline{Y}} \) in Theorem 5.1 to be dense in \( \overline{Y} \) if \( Y \) is dense in \( \overline{Y} \). So we can take a dense open set \( Y_0 \subseteq Y \) such that the conclusion of the corollary holds for \( i = 0 \). Then put \( Y_i := Y - Y_0, \overline{Y}_i \) its closure in \( \overline{Y} \) and apply Theorem 5.1 to the pull-back of \( f \) by \( (Y_i, \overline{Y}_i) \rightarrow (Y, \overline{Y}) \): Then we have an open dense subscheme \( Y_i \) in \( Y_i \) such that the conclusion of the corollary is true for \( i = 1 \). Repeating this process, we can prove the corollary. \( \Box \)

Combining with Theorem 1.13, we obtain the following corollary.

Corollary 5.3. Let us assume given a morphism of pairs \( f : (X, \overline{X}) \rightarrow (Y, \overline{Y}) \) such that \( f : \overline{X} \rightarrow \overline{Y} \) is proper. Then,

1. There exists a dense open set \( U_{\overline{Y}} \) of \( Y \), a proper surjective map \( \overline{Y} \rightarrow \overline{Y} \) and a simple normal crossing divisor \( E \) on \( \overline{Y} \) with \( E \cap U_{\overline{Y}} = \emptyset \) (where we put \( U_{\overline{Y}} := U_{\overline{Y}} \times_{\overline{X}} \overline{Y} \)) such that, for any overconvergent \( F \)-isocrystal \( E \) on \( (X, \overline{X})/S_K \) and \( q \in \mathbb{N} \), there exists the unique overconvergent isocrystal \( F \) on \( (U_{\overline{Y}}, \overline{Y})/S_K \) satisfying the condition of Theorem 5.1.

2. Moreover, there exists a stratification \( Y = \coprod_{i=0}^{d} Y_i \) of \( Y \) by locally closed subschemes \( Y_i \) with \( Y_i = \coprod_{j \geq i} Y_j \) (where \( Y_i \) denotes the closure of \( Y_i \) in \( Y \)) and proper surjective maps \( \overline{Y}_i \rightarrow \overline{Y}_i (0 \leq i \leq d) \) (where \( \overline{Y}_i \) denotes the closure of \( Y_i \) in \( \overline{Y} \)) and a simple normal crossing divisor \( E_i \) on \( \overline{Y}_i \) with \( E_i \cap \overline{Y}_i = \emptyset \) (where we put \( \overline{Y}_i := Y_i \times_{\overline{X}_i} \overline{Y}_i \)) such that, for any overconvergent \( F \)-isocrystal \( E \) on \( (X, \overline{X})/S_K \) and \( q \in \mathbb{N} \), there exist uniquely the overconvergent isocrystals \( F_i \) on \( (Y_i, \overline{Y}_i)/S_K \) (0 \leq i \leq d) satisfying the conclusion of Corollary 5.2.
Furthermore, the overconvergent isocrystals $\mathcal{F}, \mathcal{F}_i$ have Frobenius structures which are induced by the Frobenius structure of $\mathcal{E}$.

Proof. The assertions except the last one follow immediately from Theorem 5.1, Corollary 5.2 and Theorem 1.13. As for the last assertion, we can argue as in the proof of [Sh3, 5.16] (see also Theorem 2.4) to reduce to the case $Y = \overline{Y}$ is equal to $S$, and in this case, the assertion follows from the bijectivity of the Frobenius endomorphism of the (absolute) rigid cohomology: In smooth case, it is shown in Theorem 2.4 and we can reduce the general case to the smooth case by using proper descent of rigid cohomology. So we are done.

\[ \Box \]

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