Global Convergence of Policy Gradient Methods to (Almost) Locally Optimal Policies

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January 3, 2019

Abstract

Policy gradient (PG) methods are a widely used reinforcement learning methodology in many applications such as videogames, autonomous driving, and robotics. In spite of its empirical success, a rigorous understanding of the global convergence of PG methods is lacking in the literature. In this work, we close the gap by viewing PG methods from a nonconvex optimization perspective. In particular, we propose a new variant of PG methods for infinite-horizon problems that uses a random rollout horizon for the Monte-Carlo estimation of the policy gradient. This method then yields an unbiased estimate of the policy gradient with bounded variance, which enables the tools from nonconvex optimization to be applied to establish global convergence. Employing this perspective, we first recover the convergence results with rates to the stationary-point policies in the literature. More interestingly, motivated by advances in nonconvex optimization, we modify the proposed PG method by introducing periodically enlarged stepsizes. The modified algorithm is shown to escape saddle points under mild assumptions on the reward and the policy parameterization. Under a further strict saddle points assumption, this result establishes convergence to essentially locally-optimal policies of the underlying problem, and thus bridges the gap in existing literature on the convergence of PG methods. Results from experiments on the inverted pendulum are then provided to corroborate our theory, namely, by slightly reshaping the reward function to satisfy our assumption, unfavorable saddle points can be avoided and better limit points can be attained. Intriguingly, this empirical finding justifies the benefit of reward-reshaping from a nonconvex optimization perspective.

1 Introduction

In reinforcement learning (RL) [1][2], an autonomous agent moves through a state space and seeks to learn a policy which maps states to a probability distribution over actions to maximize a long-term accumulation of rewards. When the agent selects a given action at a particular state, a reward is revealed and a random transition to a new state occurs according to a probability density that only depends on the current state and action, i.e., state transitions are Markovian. This evolution process is usually modeled as a Markov decision process (MDP). Under this setting, the agent must evaluate the merit of different
actions by interacting with the environment. Two dominant approaches to reinforcement learning have emerged: those based on optimizing the accumulated reward directly from the policy space, referred to as “direct policy search”, and those based on finding the value function by solving the Bellman fixed point equations [3]. The goal of this work is to rigorously understand the former approach of direct policy search, specifically policy gradient (PG) methods [4]. Policy search has gained traction recently, thanks to its ability to scale gracefully to large and even continuous spaces [5, 6] and to incorporate deep networks as function approximators [7, 8].

Despite the increasing prevalence of policy gradient methods, their global convergence in the infinite-horizon discounted setting, which is conventional in dynamic programming [2], is not yet well understood. This gap stems firstly from the fact that obtaining unbiased estimates of the policy gradient through sampling is often elusive. Specifically, following the Policy Gradient Theorem [4], obtaining an unbiased estimate of the policy gradient requires two significant conditions to hold: (i) the state-action pair is drawn from the discounted state-action occupancy measure of the Markov chain under the policy; (ii) the estimate of the action-value (or Q) function induced by the policy is unbiased. This gap also results from the fact that the value function to be maximized in RL is in general nonconvex with respect to the policy parameter [9,10,11,12,13]. In the same vein as our work, there is a surging interest in studying the global convergence of PG methods, see the recent work [9,10,14], and concurrent work [11,12,13]. In particular, orthogonal to our work, these work considered convergence to the global optimum in several special RL settings: [9,10,14] considered the linear quadratic setting, [11,13] considered the tabular setting, [12,15] focused on the setting with overparameterized neural networks for function approximation, and [13] also considered the setting when the optimality gap of using certain policy class can be quantified. In contrast, our focus is on the case where the nonconvexity might be general, so that solving the problem can be NP-hard.

When one restricts the focus to episodic reinforcement learning, Monte-Carlo rollout may be used to obtain unbiased estimates of the Q-function. In particular, the rollout simulates the MDP under certain policy up to a finite time horizon, and then collects the rewards and state-action histories along the trajectory. However, this finite-horizon rollout, though generally used in practice, is known to introduce bias in estimating an infinite-horizon discounted value function. Such a bias in estimating the policy gradient for infinite-horizon problems has been identified in the earlier work [16,17], both analytically and empirically. To address this bias issue, we employ in this work random geometric time rollout horizons, a technique first proposed in [18]. This rollout procedure allows us to obtain unbiased estimates of the Q function, using only rollouts of finite horizons. Moreover, the random rollout horizon also creates an unbiased sampling of the state-action pair from the discounted occupancy measure [4]. With these two challenges addressed, the policy gradient can be estimated unbiasedly. Consequently, the policy gradient methods can be more naturally connected to the classical stochastic programming algorithms [19], where the unbiasedness of the stochastic gradient is a critical assumption. We refer to our algorithm as random-horizon policy gradient (RPG), to emphasize that the finite horizon of the Monte-Carlo rollout is random.

Leveraging this connection, we are able to address a noticeably open issue in policy gradient methods: a technical understanding of the effect of the policy parameterization on
both the limiting and finite-iteration algorithm behaviors. In particular, it is well known in nonconvex optimization that with only first-order information and no additional hypothesis, convergence to a stationary point with zero gradient-norm is the best one may hope to achieve [20]. Indeed, this is the type of points that most current PG methods are guaranteed to converge to, as pointed out by [13]. However, in some asymptotic analyses for policy gradient methods with function approximation [21], or their variant, actor-critic algorithms [22, 23, 24, 25], it was claimed that the limit points of the algorithms starting from any initialization constitute the locally-optimal policies, i.e., the algorithms enjoy global convergence to the local-optima. However, by the theory of stochastic approximation [26], such a claim can only be made locally, i.e., the local-optimality can only be obtained if the algorithm starts around a local minima, under the assumption that a strict Lyapunov function exists. Therefore, global convergence of PG methods to the actual locally-optimal policies, though claimed in words in some literature, is still an open question. Another line of theoretical studies of policy gradient methods only focuses on showing the one-step policy improvement [27, 21, 28], by choosing appropriate stepsizes and/or batch data sizes. Such one-step result still does not imply any global convergence result. In summary, the misuse of the term locally-optimal policy and the lack of studying global convergence property of PG methods motivate us to further investigate this problem from a nonconvex optimization perspective. Thanks to the analytical tools from optimization, we are able to first recover the asymptotic convergence, and then provide the convergence rate, to stationary-point policies.

Encouraged by this connection between nonconvex optimization and policy search, we then tackle a related question: what implications do recent algorithms that can escape saddle points for nonconvex problems ([29, 30]) have on policy gradient methods in RL? To answer this question, we identify several structural properties of RL problems that can be exploited to mitigate the underlying nonconvexity, which rely on some key assumptions on the policy parameterization and reward. Specifically, the reward needs to be bounded and either strictly positive or negative, and the policy parameterization need to be regular, i.e., its Fisher information matrix is positive definite (a conventional assumption in RL [31]). Under these mild conditions, we can establish that policy gradient methods can escape saddle points and converge to approximate second-order stationary points with high probability, when a periodically enlarged stepsize strategy is employed. We refer to the resulting method as Modified RPG (MRPG). Nevertheless, the strict positivity/negativity of reward function may amplify the variance of the gradient estimate, compared to the setting that has reward values with both signs but of smaller magnitude. This increased variance can be alleviated by introducing a baseline in the gradient estimate, as advocated by existing work [32, 33, 22]. Therefore, we propose two further modified updates that include the baselines, both shown to converge to approximate second-order stationary points as well.

Main Contribution: The main contribution of the present work is three-fold: i) we propose a series of random-horizon PG methods that unbiasedly estimate the true policy gradient for infinite-horizon discounted MDPs, which facilitates the use of analytical tools from nonconvex optimization to establish their convergence to stationary-point policies; ii) by virtue of such a connection of PG methods and nonconvex optimization, we pro-
pose modified RPG methods with periodically enlarged stepizes, with guaranteed convergence to actual *locally-optimal policies* under mild conditions on the reward functions and parametrization of the policies; iii) we connect the condition on the reward function to the reward-reshaping technique advocated in empirical RL studies, justifying its benefit, both analytically and empirically, from a nonconvex optimization perspective. Additionally, we believe such a perspective opens the door to exploiting more advancements in nonconvex optimization to improve the convergence properties of policy gradient methods in RL.

The rest of the paper is organized as follows. In §2 we clarify the problem setting of reinforcement learning and the technicalities of Markov Decision Processes. In §3 we develop the policy gradient method using random geometric Monte-Carlo rollout horizons, i.e., the RPG method. Further, we establish both its limiting (Theorem 4.2) and finite-sample (Theorem 4.3 and Corollary 4.4) behaviors under standard conditions. We note that Corollary 4.4 provides one of the first constant learning rate results in reinforcement learning. In §5 we focus on problems with positive bounded rewards and policies whose parameterizations are *regular*, and propose a variant of policy gradient method that employs a periodically enlarged stepsize scheme. The salient feature of this modified algorithm is that it is able to escape saddle points, an undesirable subset of stationary points, and converge to approximate second-order stationary points (Theorem 5.6). Numerical experiments in §6 corroborate our main findings: for Algorithm 3, the use of random rollout horizons avoids stochastic gradient bias and hence exhibits reliable convergence that matches the theoretically established rates; moreover, for the modified RPG algorithm, use of periodically enlarged stepsizes makes it possible to escape from undesirable saddle points and yields better limiting solutions. All proofs, which constitute an integral part of the paper, are relegated to nine appendices at the end of the paper, so as not to disrupt the flow of the presentation of the main results.

**Notations:** We denote the probability distribution over the space $S$ by $P(S)$, and the set of integers $\{1, \cdots, N\}$ by $[N]$. We use $\mathbb{R}$ to denote the set of real numbers, and $E$ to denote the expectation operator. We let $\| \cdot \|$ denote the 2-norm of a vector in $\mathbb{R}^d$, or the spectral norm of a matrix in $\mathbb{R}^{d \times d}$. We use $|A|$ to denote the cardinality of a finite set $A$, or the area of a region $A$, i.e., $|A| = \int_A da$. For any matrix $A \in \mathbb{R}^{d \times d}$, we use $A > 0$ and $A \succeq 0$ to denote that $A$ is positive definite and positive semi-definite, respectively. We use $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ to denote, respectively, the smallest and largest eigenvalues of some square symmetric matrix $A$, respectively. We use $E_X$ or $E_{X \sim f(x)}$ to denote the expectation with respect to random variable $X$. Otherwise specified, we use $\mathbb{E}$ to denote the full expectation with respect to all random variables.

## 2 Problem Formulation

In reinforcement learning, an autonomous agent moves through a state space $S$ and takes actions that belong to some action space $A$. Here the spaces $S$ and $A$ are allowed to be either finite sets, or compact real vector spaces, i.e., $S \subseteq \mathbb{R}^q$ and $A \subseteq \mathbb{R}^p$. An action at the state causes a transition to the next state, where the transition mapping that depends on the current state and action; every such transition generates a reward revealed by the en-
environment. The goal is for the agent to accumulate as much reward as possible in the long term. This situation can be formalized as a Markov decision process (MDP) characterized by a tuple \((S, A, P, R, \gamma)\) with Markov kernel \(P(s'|s, a) : S \times A \to \mathcal{P}(S)\) that determines the transition probability from \((s, a)\) to state \(s'\). \(\gamma \in (0, 1)\) is the discount factor. \(R(\cdot, \cdot)\) is the reward that is a function of \(s\) and \(a\).

At each time \(t\), the agent executes an action \(a_t \in A\) given the current state \(s_t \in S\), following a possibly stochastic policy \(\pi : S \to \mathcal{P}(A)\), i.e., \(a_t \sim \pi(\cdot | s_t)\). Then, given the state-action pair \((s_t, a_t)\), the agent observes a reward \(r_t = R(s_t, a_t)\). Thus, under any policy \(\pi\) that maps states to actions, one can define the value function \(V_\pi : S \to \mathbb{R}\) as

\[
V_\pi(s) = \mathbb{E}_{a_t \sim \pi(\cdot | s_t), s_{t+1} \sim P(\cdot | s_t, a_t)} \left( \sum_{t=0}^{\infty} \gamma^t r_t \middle | s_0 = s \right),
\]

which quantifies the long term expected accumulation of rewards discounted by \(\gamma\). We can further define the value \(V_\pi : S \times A \to \mathbb{R}\) conditioned on a given initial action as the action-value, or Q-function as \(Q_\pi(s, a) = \mathbb{E} \left( \sum_{t=0}^{\infty} \gamma^t r_t \middle | s_0 = s, a_0 = a \right)\). We also define \(A_\pi(s, a) = Q_\pi(s, a) - V_\pi(s)\) for any \(s, a\) to be the advantage function. Given any initial state \(s_0\), the goal is to find the optimal policy \(\pi\) that maximizes the long-term return \(V_\pi(s_0)\), i.e., to solve the following optimization problem

\[
\max_{\pi \in \Pi} V_\pi(s_0), \tag{2.1}
\]

when the model, i.e., the transition probability \(P\) and the reward function \(R\), is unknown to the agent. In this work, we investigate policy search methods to solve (2.1). In general, we must search over an arbitrarily complicated function class \(\Pi\) which may include those which are unbounded and discontinuous. To mitigate this issue, we propose to parameterize policies \(\pi\) in \(\Pi\) by a vector \(\theta \in \mathbb{R}^d\), i.e., \(\pi = \pi(\cdot | s_t, a_t)\), which gives rise to RL algorithms called policy gradient methods [34, 23, 35]. With this parameterization, we may reduce a search over arbitrarily complicated function class \(\Pi\) in (2.1) to one over the Euclidean space \(\mathbb{R}^d\). Nonparametric parameterizations are also possible [34, 35], but here we fix the parameterization in order to simplify exposition. For notational convenience, we define \(J(\theta) := V_{\pi_\theta}(s_0)\), then the vector-valued optimization problem can be written as

\[
\max_{\theta \in \mathbb{R}^d} J(\theta). \tag{2.2}
\]

Generally, the value function is nonconvex with respect to the parameter \(\theta\), meaning that obtaining a globally optimal solution to (2.2) is NP-hard, unless in several special RL settings that have been identified very recently [9, 11]. In fact, the limit point of most gradient-based methods to nonconvex optimization is a stationary solution, which could either be a saddle point or a local optimum. Usually the local optima achieve reasonably good performance, in some cases comparable to the global optima, whereas the saddle points are undesirable and can stall training procedures. Therefore, it is beneficial to design methods that may escape saddle points – see recent efforts on escaping saddle points.\footnote{\(R(s_t, a_t)\) may be a random variable given \((s_t, a_t)\). Here without loss of generality, we assume that it is deterministic for simplicity.}
points with first-order methods, e.g., perturbed gradient descent \[36, 29, 30\], and second-order methods \[37, 38\].

Our goal in this work is to develop stochastic gradient methods to maximize \(J(\theta)\) and rigorously understand the interplay between its limiting properties and the necessity of augmenting the algorithmic update, reward function, and policy parameterization, all toward escaping undesirable limit points. This issue was first observed and addressed in \[39\] by adding random perturbations in the reinforcement learning update (which may amplify variance), based on the asymptotic convergence results in \[40\]. Here we provide a modern perspective and incorporate the latest developments in nonconvex optimization.

### 3 Policy Gradient Methods

In this section, we connect stochastic gradient ascent, as it is called in stochastic optimization, with the policy gradient method, a flavor of direct policy search, in reinforcement learning. We start with the following standard assumption on the regularity of the MDP problem and the smoothness of the parameterized policy \(\pi_\theta\).

**Assumption 3.1.** Suppose the reward function \(R\) and the parameterized policy \(\pi_\theta\) satisfy the following conditions:

(i) The absolute value of the reward \(R\) is uniformly bounded, say by \(U_R\), i.e., \(|R(s,a)| \in [0, U_R]\) for any \((s,a) \in S \times A\).

(ii) The policy \(\pi_\theta\) is differentiable with respect to \(\theta\), and \(\nabla \log \pi_\theta(a|s)\), known as the score function corresponding to the distribution \(\pi_\theta(·|s)\), exists. Moreover, it is \(L_\Theta\)-Lipschitz and has bounded norm for any \((s,a) \in S \times A\),

\[
\|\nabla \log \pi_{\theta_1}(a|s) - \nabla \log \pi_{\theta_2}(a|s)\| \leq L_\Theta \cdot \|\theta_1 - \theta_2\|, \text{ for any } \theta_1, \theta_2, \tag{3.1}
\]

\[
\|\nabla \log \pi_\theta(a|s)\| \leq B_\Theta, \text{ for some constant } B_\Theta \text{ for any } \theta. \tag{3.2}
\]

for some constant \(B_\Theta > 0\).

Note that the boundedness of the reward function in Assumption 3.1(i) is standard in the literature of policy gradient/actor-critic algorithms \[22, 23, 35, 41, 42\]. The uniform boundedness of \(R\) also implies that the absolute value of the Q-function is upper bounded by \(U_R/(1 - \gamma)\), since by definition

\[
|Q_{\pi_\theta}(s,a)| \leq \sum_{t=0}^{\infty} \gamma^t \cdot U_R = U_R/(1 - \gamma), \text{ for any } (s,a) \in S \times A.
\]

The same bound also applies to \(V_{\pi_\theta}(s)\) for any \(\pi_\theta\) and \(s \in S\), and thus to the objective \(J(\theta)\) which is defined as \(V_{\pi_\theta}(s_0)\), i.e.,

\[
|V_{\pi_\theta}(s)| \leq U_R/(1 - \gamma), \text{ for any } s \in S, \quad |J(\theta)| \leq U_R/(1 - \gamma).
\]

In addition, the conditions (3.1) and (3.2) have also been adopted in several recent work on the convergence analysis of policy gradient algorithms \[35, 21, 43, 44\]. Both of
the conditions can be readily satisfied by many common parametrized policies such as the Boltzmann policy \(^{[39]}\) and the Gaussian policy \(^{[45]}\). For example, for Gaussian policy\(^2\) in continuous spaces, \(\pi_\theta(\cdot|s) = \mathcal{N}(\phi(s)\top\theta, \sigma^2)\), where \(\mathcal{N}(\mu, \sigma^2)\) denotes the Gaussian distribution with mean \(\mu\) and variance \(\sigma^2\), and \(\phi(s)\) is the feature vector that incorporates some domain knowledge to approximate the mean action at state \(s\). Then the score function has the form \([a - \phi(s)\top\theta]|\phi(s)/\sigma^2\), which satisfies (3.1) and (3.2) if the following three conditions hold: the norm of the feature \(||\phi(s)||\) is bounded; the parameter \(\theta\) lies in some bounded set; and the actions \(a \in A\) is bounded.

Under Assumption 3.1, the gradient of \(J(\theta)\) with respect to the policy parameter \(\theta\), given by the Policy Gradient Theorem \(^{[4]}\), has the following form\(^3\):

\[
\nabla J(\theta) = \int_{s \in S, a \in A} \sum_{t=0}^{\infty} \gamma^t \cdot p(s_t = s|s_0, \pi_\theta) \cdot \nabla \pi_\theta(a|s) \cdot Q_{\pi_\theta}(s,a) dsda
\]  \( (3.3) \)

\[
= \frac{1}{1 - \gamma} \int_{s \in S, a \in A} (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \cdot p(s_t = s|s_0, \pi_\theta) \cdot \nabla \pi_\theta(a|s) \cdot Q_{\pi_\theta}(s,a) dsda
\]

\[
= \frac{1}{1 - \gamma} \int_{s \in S, a \in A} \rho_{\pi_\theta}(s) \cdot \nabla \rho_\theta(a|s) \cdot \log \rho_\theta(a|s) \cdot Q_{\pi_\theta}(s,a) dsda
\]

\[
= \frac{1}{1 - \gamma} \cdot \mathbb{E}_{(s,a) \sim \rho_\theta(\cdot, \cdot)} \left[ \nabla \log \rho_\theta(a|s) \cdot Q_{\pi_\theta}(s,a) \right]. \quad (3.4)
\]

Here, we denote by \(p(s_t = s|s_0, \pi_\theta)\) the probability that state \(s_t\) equals \(s\) given initial state \(s_0\) and policy parameter \(\theta\), and the distribution \(\rho_{\pi_\theta}(s) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t p(s_t = s|s_0, \pi_\theta)\) which has been shown to be a valid probability measure over the state \(S\) in \([4]\). We refer to \(\rho_{\pi_\theta}(s)\) as the discounted state-occupancy measure hereafter. For notational convenience, we let \(\rho_\theta(s,a) = \rho_{\pi_\theta}(s) \cdot \pi_\theta(a|s)\), which denotes the discounted state-action occupancy measure.

In addition, based on the fact that for any function \(b : S \rightarrow \mathbb{R}\) independent of action \(a\),

\[
\int_{a \in A} \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) \cdot b(s) da = \nabla \int_{a \in A} \pi_\theta(a|s) da \cdot b(s) = \nabla 1 \cdot b(s) = 0, \quad \text{for any } s \in S,
\]

the policy gradient in (3.4) can be written as

\[
\nabla J(\theta) = \frac{1}{1 - \gamma} \cdot \mathbb{E}_{(s,a) \sim \rho_\theta(\cdot, \cdot)} \left[ \nabla \log \pi_\theta(a|s) \cdot [Q_{\pi_\theta}(s,a) - b(s)] \right],
\]

where \(b(s)\) is usually referred to as a baseline function. One common choice of the baseline is the state-value function \(V_{\pi_\theta}(s)\), which gives the following advantage-based policy gradient.

\[
\nabla J(\theta) = \frac{1}{1 - \gamma} \cdot \mathbb{E}_{(s,a) \sim \rho_\theta(\cdot, \cdot)} \left[ \nabla \log \pi_\theta(a|s) \cdot A_{\pi_\theta}(s,a) \right]. \quad (3.5)
\]

\(^2\)Note that in practice, the action space \(A\) is bounded, thus a truncated Gaussian policy over \(A\) is often used; see \([43]\).

\(^3\)Note that here we use \(\int\) to represent both summation over finite sets and integral over continuous spaces.
**Algorithm 1 EstQ: Unbiasedly Estimating Q-function**

**Input:** $s, a$, and $\theta$. Initialize $\hat{Q} \leftarrow 0$, $s_0 \leftarrow s$, and $a_0 \leftarrow a$.

Draw $T$ from the geometric distribution $\text{Geom}(1 - \gamma^{1/2})$, i.e., $P(T = t) = (1 - \gamma^{1/2})\gamma^{t/2}$.

**for all** $t = 0, \cdots, T-1$ **do**

Collect and add the instantaneous reward $R(s_t, a_t)$ to $\hat{Q}$, $\hat{Q} \leftarrow \hat{Q} + \gamma^{t/2} \cdot R(s_t, a_t)$.

Simulate the next state $s_{t+1} \sim \mathbb{P}(\cdot | s_t, a_t)$ and action $a_{t+1} \sim \pi(\cdot | s_{t+1})$.

**end for**

Collect $R(s_T, a_T)$ by $\hat{Q} \leftarrow \hat{Q} + \gamma^{T/2} \cdot R(s_T, a_T)$.

**return** $\hat{Q}$.

In this work, we devise methods that can use iterative updates based on the classical policy gradient (3.4) or its variant that makes use of the advantage function (3.5) through the aforementioned identity regarding baselines. First note that under Assumption 3.1, we can establish the Lipschitz continuity of the policy gradient $\nabla J(\theta)$ as in the following lemma, whose proof is deferred to §A.1.

**Lemma 3.2** (Lipschitz-Continuity of Policy Gradient). Under Assumption 3.1, the policy gradient $\nabla J(\theta)$ is Lipschitz continuous with some constant $L > 0$, i.e., for any $\theta^1, \theta^2 \in \mathbb{R}^d$

\[
\|\nabla J(\theta^1) - \nabla J(\theta^2)\| \leq L \cdot \|\theta^1 - \theta^2\|,
\]

where the value of the Lipschitz constant $L$ is defined as

\[
L := \frac{U_R \cdot L_\Theta}{(1 - \gamma)^2} + \frac{(1 + \gamma) \cdot U_R \cdot B_\Theta^2}{(1 - \gamma)^3}.
\] (3.6)

Next, we discuss how (3.4) and (3.5) can be used to develop first-order stochastic approximation methods to address (2.2). Unbiased samples of the gradient $\nabla J(\theta)$ are required to perform the stochastic gradient ascent, which hopefully converges to a stationary solution of the nonconvex optimization problem. Moreover, through the addition of carefully designed perturbations, we aim to attain a local optimum, namely, asymptotically stable stationary point, as in [39, 36, 29].

**Sampling the Policy Gradient:** In order to obtain an unbiased sample of $\nabla J(\theta)$, it is necessary to: i) draw state-action pair $(s, a)$ from the distribution $\rho_\theta(\cdot, \cdot)$; and ii) obtain an unbiased estimate of the Q-function $Q_{\pi_\theta}(s,a)$, or the advantage function $A_{\pi_\theta}(s,a)$ evaluated at $(s,a)$.

Both of the requirements can be satisfied by using a random horizon $T$ that follows certain geometric distribution in the sampling process. In particular, to ensure the condition i) is satisfied, we use the last sample $(s_T, a_T)$ of a finite sample trajectory $(s_0, a_0, s_1, \cdots, s_T, a_T)$ to be the sample at which $Q_{\pi_\theta}(\cdot, \cdot)$ and $\nabla \log \pi_\theta(\cdot | \cdot)$ are evaluated, where the horizon $T \sim \text{Geom}(1 - \gamma)$. It can be shown that $(s_T, a_T) \sim \rho_\theta(\cdot, \cdot)$. Moreover, given $(s_T, a_T)$, we perform Monte-Carlo rollouts for another horizon $T' \sim \text{Geom}(1 - \gamma^{1/2})$ independent of $T$, and estimate the Q-function value $Q_{\pi_\theta}(s,a)$ as follows by collecting the $\gamma^{1/2}$-discounted rewards.
**Algorithm 2 EstV: Unbiasedly Estimating State-Value function**

**Input:** \( s \) and \( \theta \). Initialize \( \hat{V} \leftarrow 0 \), \( s_0 \leftarrow s \), and draw \( a_0 \sim \pi_\theta(\cdot|s_0) \).

Draw \( T \) from the geometric distribution \( \text{Geom}(1 - \gamma^{1/2}) \).

**for all** \( t = 0, \cdots, T - 1 \) **do**

- Collect the instantaneous reward \( R(s_t, a_t) \) and add to value \( \hat{V} \); \( \hat{V} \leftarrow \hat{V} + \gamma^{t/2} \cdot R(s_t, a_t) \).
- Simulate the next state \( s_{t+1} \sim \mathcal{P}(\cdot|s_t, a_t) \) and action \( a_{t+1} \sim \pi(\cdot|s_{t+1}) \).

**end for**

Collect \( R(s_T, a_T) \) by \( \hat{V} \leftarrow \hat{V} + \gamma^{T/2} \cdot R(s_T, a_T) \).

**return** \( \hat{V} \).

along the trajectory:

\[
\hat{Q}_\pi(s, a) = \sum_{t=0}^{T'} \gamma^{t/2} \cdot R(s_t, a_t) \mid s_0 = s, a_0 = a.
\] (3.7)

Then, it can be shown that \( \hat{Q}_\pi(s, a) \) unbiasedly estimates \( Q_\pi(s, a) \) for any \( (s, a) \) (see Theorem 3.4 whose proof is given in Appendix A.2). The subroutine of estimating the Q-function is summarized as **EstQ** in Algorithm 1.

**Remark 3.3.** Thanks to randomness of the horizon, we note that the aforementioned sampling process creates the first unbiased estimate of the Q-function in the discounted infinite-horizon setting, using the Monte-Carlo rollouts of finite horizons. While in practice, usually finite-horizon rollouts are used to approximate the infinite-horizon Q-function, e.g., in the REINFORCE algorithm, which causes bias in the Q-function estimate, and hence the policy gradient estimate. Our sampling technique addresses this challenge, and ends up with an unbiased estimate of the policy gradient as to be introduced next. We note that the proposed sampling technique for estimating the Q-function improves the one in [18] that uses \( \text{Geom}(1 - \gamma) \) (instead of \( \text{Geom}(1 - \gamma^{1/2}) \)) to generate the rollout horizon \( T' \). In particular, the proposed Q-function estimate is almost surely bounded thanks to the \( \gamma^{1/2} \)-discount factor in (3.7), which later leads to almost sure boundedness of the stochastic policy gradient, a necessary assumption required in the convergence analysis to approximate second-order stationary points in §5.

Motivated by the form of policy gradient in (3.4), we propose the following stochastic estimate \( \hat{\nabla} J(\theta) \)

\[
\hat{\nabla} J(\theta) = \frac{1}{1 - \gamma} \cdot \hat{Q}_\pi(s_T, a_T) \cdot \nabla \log[\pi_\theta(a_T|s_T)].
\] (3.8)

In addition, we can also estimate the policy gradient using advantage functions as in (3.5), where the advantage function is estimated by either the difference between the value function and the action-value function, or the temporal difference (TD) error. In particular, we propose the following two stochastic policy gradients

\[
\hat{\nabla} J(\theta) = \frac{1}{1 - \gamma} \cdot [\hat{Q}_\pi(s_T, a_T) - \hat{V}_\pi(s_T)] \cdot \nabla \log[\pi_\theta(a_T|s_T)],
\] (3.9)

\[
\check{\nabla} J(\theta) = \frac{1}{1 - \gamma} \cdot [R(s_T, a_T) + \gamma \hat{V}_\pi(s'_T) - \hat{V}_\pi(s_T)] \cdot \nabla \log[\pi_\theta(a_T|s_T)],
\] (3.10)
where \( \hat{V}_{\pi_0}(s) \) is an unbiased estimate of the value function \( V_{\pi_0}(s) \), and \( s'_T \) is the next state given state \( s_T \) and \( a_T \). The process of estimating \( \hat{V}_{\pi_0}(s) \) employs the same idea as the EstQ algorithm, where \( \hat{V}_{\pi_0}(s) \) is obtained by collecting the \( \gamma^{1/2} \)-discounted rewards along the trajectory starting from \( s_0 \) (instead of a state-action pair \((s,a)\)) following \( a_t \sim \pi_\theta(\cdot|s_t) \), and of length \( T' \sim \text{Geom}(1-\gamma^{1/2}) \), i.e., \( \hat{V}_{\pi_0}(s) = \sum_{t=0}^{T'} \gamma^{1/2} \cdot R(s_t, a_t) | s_0 = s \). We refer to this subroutine as EstV, which is summarized in Algorithm 2. The reason for these alternate updates is that the off-set term can be used to reduce the variance of estimating the policy gradient [32].

We then establish in the following theorem, which states that all the stochastic policy gradients \( \hat{V}(\theta), \tilde{V}(\theta) \), and \( \bar{V}(\theta) \) are unbiased estimates of \( \nabla J(\theta) \) [cf. (3.4)]. Additionally, we can also establish the boundedness of \( \|\nabla J(\theta)\|, \|\hat{V}(\theta)\|, \) and \( \|\bar{V}(\theta)\| \), as well as \( \|\nabla J(\theta)\| \) for any \( \theta \in \Theta \). The proof is deferred to Appendix A.2.

**Theorem 3.4** (Properties of Stochastic Policy Gradients). For any \( \theta \), \( \hat{V}(\theta), \tilde{V}(\theta), \) and \( \bar{V}(\theta) \) obtained from (3.8), (3.9), and (3.10), respectively, are all unbiased estimates of \( \nabla J(\theta) \) in (3.4), i.e., for any \( \theta \)

\[
\mathbb{E}[\hat{V}(\theta) | \theta] = \mathbb{E}[\tilde{V}(\theta) | \theta] = \mathbb{E}[\bar{V}(\theta) | \theta] = \nabla J(\theta).
\]

where the expectation is with respect to the random horizon \( T' \), the trajectory along \((s_0, a_0, s_1, \cdots, s_{T'}, a_{T'})\), and the random sample \((s_T, a_T)\). Moreover, the norm of the policy gradient \( \nabla J(\theta) \) is bounded, and its stochastic estimates \( \hat{V}(\theta), \tilde{V}(\theta), \bar{V}(\theta) \) are all almost surely (a.s.) bounded, i.e.,

\[
\|\nabla J(\theta)\| \leq \frac{B_\Theta \cdot U_R}{(1-\gamma)^2}, \quad \|\hat{V}(\theta)\| \leq \hat{\ell} \quad \text{a.s.}, \quad \|\tilde{V}(\theta)\| \leq \tilde{\ell} \quad \text{a.s.}, \quad \|\bar{V}(\theta)\| \leq \bar{\ell} \quad \text{a.s.},
\]

for some constants \( \ell, \hat{\ell}, \tilde{\ell}, \bar{\ell} > 0 \), whose values are given in (A.19), (A.20), and (A.21) in Appendix A.2.

Henceforth in this section and the next, we will mainly focus on the convergence analysis for the RPG algorithm with the stochastic gradient \( \hat{V}(\theta) \) as defined in (3.8). The RPG algorithms with \( \hat{V}(\theta) \) and \( \bar{V}(\theta) \) will be discussed later in §5, where reducing the variance of RPG is of greater interest.

To this end, let \( k \) be the iteration index and \( \theta_k \) be the associated estimate for the policy parameter. Under Theorem 3.4, the policy gradient update for step \( k+1 \) is

\[
\theta_{k+1} = \theta_k + \alpha_k \hat{\nabla} J(\theta_k) = \theta_k + \frac{\alpha_k}{1-\gamma} \cdot \hat{Q}_{\pi_{\theta_k}}(s_{T_{k+1}}, a_{T_{k+1}}) \cdot \nabla \log[\pi_{\theta_k}(a_{T_{k+1}} | s_{T_{k+1}})],
\]

where \( \{\alpha_k\} \) is the stepsize sequence that can be either diminishing or constant, and \( \{T_k\} \) are drawn i.i.d. from Geom\((1-\gamma)\). The details of the policy gradient method, which we refer to as the random-horizon policy gradient algorithm, are summarized in Algorithm 3. Note that the estimate of \( \hat{Q}_{\pi_{\theta_k}}(s_{T_{k+1}}, a_{T_{k+1}}) \), i.e., Algorithm 3, is conducted in the inner-loop of the stochastic policy gradient update.

**Remark 3.5.** We note that in order to estimate the Q-function, it is not very sample-efficient to use Monte-Carlo rollouts to sample states, actions, and rewards. In fact, there
Algorithm 3 RPG: Random-horizon Policy Gradient Algorithm

**Input:** $s_0$ and $\theta_0$, initialize $k \leftarrow 0$.

**Repeat:**
- Draw $T_{k+1}$ from the geometric distribution $\text{Geom}(1 - \gamma)$.
- Draw $a_0 \sim \pi_{\theta_k}(|s_0)$
- **for all** $t = 0, \cdots, T_{k+1} - 1$ **do**
  - Simulate the next state $s_{t+1} \sim P(|s_t, a_t)$ and action $a_{t+1} \sim \pi_{\theta_k}(\cdot|s_{t+1})$.
- **end for**
- Obtain an estimate of $Q_{\pi_{\theta_k}}(s_{T_{k+1}}, a_{T_{k+1}})$ by Algorithm 1, i.e.,
  \[ \hat{Q}_{\pi_{\theta_k}}(s_{T_{k+1}}, a_{T_{k+1}}) \leftarrow \text{Est}Q(s_{T_{k+1}}, a_{T_{k+1}}, \theta_k). \]
- Perform policy gradient update
  \[ \theta_{k+1} \leftarrow \theta_k + \frac{\alpha_k}{1 - \gamma} \cdot \hat{Q}_{\pi_{\theta_k}}(s_{T_{k+1}}, a_{T_{k+1}}) \cdot \nabla \log[\pi_{\theta_k}(a_{T_{k+1}} | s_{T_{k+1}})]. \]
- Update the iteration counter $k \leftarrow k + 1$.

Until Convergence

exist some methods that can estimate the Q-function in parallel with the policy gradient update, which is usually referred to as *actor-critic* method [34, 23]. This online policy evaluation update is generally performed via *bootstrapping* algorithms such as temporal difference learning [46], which will introduce biases into the Q-function estimate, and thus the policy gradient estimate. In addition, such policy evaluation updates in concurrence with the policy improvement will inevitably cause correlation between consecutive stochastic policy gradients. Analyzing the non-asymptotic convergence performance of such *biased* RPG with *correlated noise* is still open and challenging, which is left as a future research direction.

In the next sections, we shift focus to analyzing the theoretical properties of the aforementioned policy learning methods, establishing their asymptotic and finite-time performances, as well as stepsize strategies designed to mitigate the challenges of non-convexity when certain reward structure is present.

4 Convergence to Stationary Points

In this section, we provide convergence analyses for the policy gradient algorithms proposed in §3. We start with the following assumption for the diminishing stepsize $\alpha_k$, which is standard in stochastic approximation.
Assumption 4.1. The sequence of stepsize $\{\alpha_k\}_{k \geq 0}$ satisfies the Robbins-Monro condition

$$
\sum_{k=0}^{\infty} \alpha_k = \infty, \quad \sum_{k=0}^{\infty} \alpha_k^2 < \infty.
$$

We first establish the convergence of Algorithm 3 in the following theorem under the aforementioned technical conditions.

Theorem 4.2 (Asymptotic Convergence of Algorithm 3). Let $\{\theta_k\}_{k \geq 0}$ be the sequence of parameters of the policy $\pi_{\theta_k}$ given by Algorithm 3. Then under Assumptions 3.1 and 4.1, we have $\lim_{k \to \infty} \theta_k \in \Theta^*$, where $\Theta^*$ is the set of stationary points of $J(\theta)$.

Theorem 4.2, whose proof is in Appendix A.3, shows that the random-horizon policy gradient update converges to the (first-order) stationary points of $J(\theta)$ almost surely. The proof of the theorem is relegated to §A.3. We note that the asymptotic convergence result here is established from an optimization perspective using supermartingale convergence theorem [47], which differs from the existing techniques that show convergence of actor-critic algorithms from dynamical systems theory (or ODE method) [26]. Such optimization perspective can be leveraged thanks to the unbiasedness of the stochastic policy gradients obtained from Algorithm 3.

An additional virtue of this style of analysis is that we can also establish convergence rate of the policy gradient algorithm without the need for sophisticated concentration inequalities. In contrast, the finite-iteration analysis for actor-critic algorithms is known to be quite challenging [48, 49]. By convention, we choose the stepsize to be either $\alpha_k = k^{-a}$ for some parameter $a \in (0, 1)$ or constant $\alpha > 0$. Note that for the diminishing stepsize, here we allow a more general choice than that in Assumption 4.1. Since $J(\theta)$ is generally nonconvex, we consider the convergence rate in terms of a metric of nonstationarity, i.e., the norm of the gradient $\|\nabla J(\theta_k)\|$. We then provide the convergence rates of Algorithm 3 for the setting of using diminishing and constant stepsizes in the following theorem and corollary, respectively. The proofs of the results are given in §A.4.

Theorem 4.3 (Convergence Rate of Algorithm 3 with Diminishing Stepsize). Let $\{\theta_k\}_{k \geq 0}$ be the sequence of parameters of the policy $\pi_{\theta_k}$ given by Algorithm 3. Let the stepsize be $\alpha_k = k^{-a}$ where $a \in (0, 1)$. Let

$$
K_\epsilon = \min \left\{ k : \inf_{0 \leq m \leq k} \mathbb{E} \|\nabla J(\theta_m)\|^2 \leq \epsilon \right\}.
$$

Then, under Assumption 3.1, we have $K_\epsilon \leq O(\epsilon^{-1/p})$, where $p$ is defined as $p = \min\{1-a, a\}$. By optimizing the complexity bound over $a$, we obtain $K_\epsilon \leq O(\epsilon^{-2})$ with $a = 1/2$.

Corollary 4.4 (Convergence Rate of Algorithm 3 with Constant Stepsize). Let $\{\theta_k\}_{k \geq 0}$ be the sequence of parameters of the policy $\pi_{\theta_k}$ given by Algorithm 3. Let the stepsize be $\alpha_k = \alpha > 0$. Then, under Assumption 3.1, we have

$$
\frac{1}{k} \sum_{m=1}^{k} \mathbb{E} \|\nabla J(\theta_m)\|^2 \leq O(\alpha L \hat{\ell}^2),
$$

where recall that $L$ is the Lipchitz constant of the policy gradient as defined in (3.6) in Lemma 3.2.
Theorem 4.3 illustrates that when diminishing stepsize is adopted, which essentially establishes a $1/\sqrt{k}$ convergence rate for the convergence of the expected gradient norm square $||\nabla J(\theta_k)||^2$. Corollary 4.4 shows that the average of the gradient norm square will converge to a neighborhood around zero with the rate of $1/k$. The size of the neighborhood is controlled by the stepsize $\alpha$. Moreover, (A.37) also implies that a smaller stepsize may decrease the size of the neighborhood, at the expense of the convergence speed. We note that both results are standard and recover the convergence properties of stochastic gradient descent for nonconvex optimization problems [19, 50]. In the next section, we propose modified stepsize rules, which under an appropriate hypothesis on the policy parameterization and reward structure of the problem, yield stronger limiting policies.

5 Convergence to Second-Order Stationary Points

In this section, we provide convergence analyses for several modified policy gradient algorithms based on Algorithm 3, which may escape saddle points and thus converge to the approximate second-order stationary points of the problem. In short, we propose a custom periodically enlarged stepsize rule, which under an additional hypothesis on the incentive structure of the problem and some other standard conditions (see §5.1), allow us to attain improved limiting policy parameters (see §5.2).

We start with the definition of (approximate) second-order stationary points [51].

**Definition 5.1.** An $(\epsilon_g, \epsilon_h)$-approximate-second-order stationary point $\theta$ is defined as

$$||\nabla J(\theta)|| \leq \epsilon_g, \quad \lambda_{\text{max}}[\nabla^2 J(\theta)] \leq \epsilon_h.$$

If $\epsilon_g = \epsilon_h = 0$, the point $\theta$ is a second-order stationary point.

The intuition for this definition is that a local maximum is one in which the gradient is null and the Hessian is negative semidefinite. When we relax the first criterion, we obtain the first inequality, whereas when we relax the second one, we mean that the Hessian is near negative semidefinite.

With the further assumption that all saddle points are strict (i.e., for any saddle point $\theta$, $\lambda_{\text{max}}[\nabla^2 J(\theta)] > 0$) [29, 36], all second-order stationary points ($\epsilon_g = \epsilon_h = 0$) are local maxima. In this case, converging to (approximate) second-order stationary points is equivalent to converging to approximate local minima, which is usually more desirable than converging to (first-order) stationary points.

5.1 Algorithm

The modified RPG (MRPG) algorithms are built upon the RPG algorithm (Algorithm 3) discussed in Section 3. These modifications can yield escape from saddle points under certain conditions, and hence convergence to approximate local extrema.

Note that Definition 5.1 is based on the maximization problem we consider here, which is slightly different from the definition for minimization problems where $\lambda_{\text{max}}[\nabla^2 J(\theta)] \leq \epsilon_h$ is replaced by $\lambda_{\text{min}}[\nabla^2 J(\theta)] \geq -\epsilon_h$. 

13
Algorithm 4 EvalPG: Calculating the Three Types of Stochastic Policy Gradients

Input: $s, a, \theta$ and the gradient type $\diamondsuit$.

if gradient type $\diamondsuit = \hat{\cdot}$ then

Obtain an estimate $\hat{Q}_{\pi_\theta}(s, a) \leftarrow \text{EstQ}(s, a, \theta)$.
Calculate $\hat{\nabla}J(\theta)$, i.e., let

$$g_\theta \leftarrow \frac{1}{1 - \gamma} \cdot \hat{Q}_{\pi_\theta}(s, a) \cdot \nabla \log \pi_\theta(a|s).$$

else if gradient type $\diamondsuit = \check{\cdot}$ then

Obtain estimates $\hat{Q}_{\pi_\theta}(s, a) \leftarrow \text{EstQ}(s, a, \theta)$ and $\hat{V}_{\pi_\theta}(s) \leftarrow \text{EstV}(s, \theta)$.
Calculate $\check{\nabla}J(\theta)$, i.e., let

$$g_\theta \leftarrow \frac{1}{1 - \gamma} \cdot [\hat{Q}_{\pi_\theta}(s, a) - \hat{V}_{\pi_\theta}(s)] \cdot \nabla \log \pi_\theta(a|s).$$

else if gradient type $\diamondsuit = \tilde{\cdot}$ then

Simulate the next state: $s' \sim P(\cdot|s, a)$.
Obtain estimates $\hat{V}_{\pi_\theta}(s) \leftarrow \text{EstV}(s, \theta)$ and $\hat{V}_{\pi_\theta}(s') \leftarrow \text{EstV}(s', \theta)$.
Calculate $\tilde{\nabla}J(\theta)$, i.e., let

$$g_\theta \leftarrow \frac{1}{1 - \gamma} \cdot [R(s, a) + \gamma \cdot \hat{V}_{\pi_\theta}(s') - \hat{V}_{\pi_\theta}(s)] \cdot \nabla \log \pi_\theta(a|s).$$

end if

return Stochastic policy gradient $g_\theta$

In order to reduce the variance of the RPG update (3.11), we employ the stochastic gradients $\check{\nabla}J(\theta)$ and $\tilde{\nabla}J(\theta)$ as defined in (3.9) and (3.10), respectively. Note that the evaluations of both $\check{\nabla}J(\theta)$ and $\tilde{\nabla}J(\theta)$ need to estimate the state-value function $\hat{V}_{\pi_\theta}(s)$ for any given $\theta$ and $s$. Built upon the subroutines EstQ and EstV, we summarize the subroutine for calculating all three types of stochastic policy gradients as EvalPG in Algorithm 4.

In order to converge to the approximate second-order stationary points, we modify the RPG algorithm, i.e., Algorithm 3, by periodically enlarging the constant stepsizes of the update, once every $k$ steps. The larger stepsize can amplify the variance along the eigenvector corresponding to the largest eigenvalue of the Hessian, which provides a direction for the update to escape at the saddle points. This idea was first introduced in [30] for general stochastic gradient methods, and is outlined in Algorithm 5. Note that $\alpha$ and $\beta$ are the constant stepsizes with $\beta > \alpha > 0$, whose values will be given in §5.2 to obtain certain convergence rates. To design this behavior while avoiding unnecessarily large variance, we propose updates that make use of the advantage function, i.e., $\check{\nabla}J(\theta)$ and $\tilde{\nabla}J(\theta)$. The resulting algorithm, with periodically enlarged stepsizes, and stochastic policy gradients that use advantage functions, is summarized as Algorithm 5. Subsequently, we shift focus to characterizing its policy learning performance analysis.
Algorithm 5 MRPG: Modified Random-horizon Policy Gradient Algorithm

**Input:** \( s_0, \theta_0, \) and the gradient type \( \phi \), initialize \( k \leftarrow 0 \), return set \( \hat{\Theta}^* \leftarrow \emptyset \).

**Repeat:**

Draw \( T_{k+1} \) from the geometric distribution \( \text{Geom}(1 - \gamma) \), and draw \( a_0 \sim \pi_{\theta_k}(\cdot|s_0) \).

for all \( t = 0, \ldots, T_{k+1} - 1 \) do

Simulate the next state \( s_{t+1} \sim P(\cdot|s_t, a_t) \) and action \( a_{t+1} \sim \pi_{\theta_k}(\cdot|s_{t+1}) \).

end for

Calculate the stochastic gradient \( g_k \leftarrow \text{EvalPG}(s_{T_k + 1}, a_{T_k + 1}, \theta_k, \phi) \).

if \((k \mod k_{\text{thre}}) = 0\) then

\[ \hat{\Theta}^* \leftarrow \hat{\Theta}^* \cup \{\theta_k\} \]

\[ \theta_{k+1} \leftarrow \theta_k + \beta \cdot g_k \]

else

\[ \theta_{k+1} \leftarrow \theta_k + \alpha \cdot g_k \]

end if

Update the iteration counter \( k \leftarrow k + 1 \).

Until Convergence

**return** \( \theta \) uniformly at random from the set \( \hat{\Theta}^* \).

5.2 Convergence Analysis

In this subsection, we provide a finite-iteration convergence result for the modified RPG algorithm, i.e., Algorithm 5. To this end, we first introduce the following condition, built upon Assumption 3.1 which is required in the sequel.

**Assumption 5.2.** The MDP and the parameterized policy \( \pi_{\theta} \) satisfy the following conditions:

(i) The reward \( R(s,a) \) is either positive or negative for any \((s, a) \in S \times A\). Thus, \(|R(s, a)| \in [L_R, U_R] \) with some \( L_R > 0 \).

(ii) The score function \( \nabla \log \pi_{\theta} \) exists, and its norm is bounded by \( \| \nabla \log \pi_{\theta} \| \leq B_{\Theta} \) for any \( \theta \). Also, the Jacobian of \( \nabla \log \pi_{\theta} \) has bounded norm and is Lipschitz continuous, i.e., there exist constants \( \rho_{\Theta} > 0 \) and \( L_{\Theta} < \infty \) such that for any \((s, a) \in S \times A\)

\[
\| \nabla^2 \log \pi_{\theta_1}(a|s) - \nabla^2 \log \pi_{\theta_2}(a|s) \| \leq \rho_{\Theta} \cdot \| \theta_1 - \theta_2 \|, \text{ for any } \theta_1, \theta_2,
\]

\[
\| \nabla^2 \log \pi_{\theta}(a|s) \| \leq L_{\Theta}, \text{ for any } \theta.
\]

(iii) The integral of the Fisher information matrix induced by \( \pi_{\theta}(\cdot|s) \) is positive-definite
uniformly for any \( \theta \in \mathbb{R}^d \), i.e., there exists a constant \( L_I > 0 \) such that

\[
\int_{s \in S, a \in A} \rho_\theta(s, a) \cdot \nabla \log \pi_\theta(\cdot | s) \cdot [\nabla \log \pi_\theta(\cdot | s)]^\top dads \succeq L_I \cdot I, \quad \text{for all } \theta \in \mathbb{R}^d. \tag{5.1}
\]

We note that Assumption 5.2 is indeed standard, and can be readily satisfied in practice. First, the strict positivity (or negativity) of the reward function in Assumption 5.2(i) can be easily satisfied by adding (or subtracting) an offset to the original non-negative and upper-bounded reward. In fact, it can be justified in the following lemma that adding any offset does not change the optimal policy of the original MDP.

**Lemma 5.3.** Given any MDP \( M = (S, A, P, R, \gamma) \), let \( \tilde{M} \) be a modified MDP of \( M \), such that \( \tilde{M} = (S, A, \tilde{P}, \tilde{R}, \gamma) \) and \( \tilde{R}(s, a) = R(s, a) + C \), for any \( s \in S, a \in A, \) and \( C \in \mathbb{R} \). Then, the sets of optimal policies for the two MDPs, \( M \) and \( \tilde{M} \), are equal.

The proof of the lemma is deferred to Appendix §A.5. The positivity (or negativity) of the rewards ensures that the absolute value of the Q-function is also lower-bounded, by the value of \( L_R/(1 - \gamma) \), which will benefit the convergence of the MRPG algorithm as to be specified shortly. Interestingly, such a reshape of the reward function can be shown to yield better convergence results. To our knowledge, our work appears to be the first theoretical study on the effect of reward-reshaping on the convergence property of policy gradient methods, although reward-reshaping is known to be useful for improving learned policies in practice.

On the other hand, we note that such positivity of \(|Q_\pi_\theta|\) will cause a relatively large variance in the original RPG update \((3.11)\). This makes the RPG with baseline, i.e., the use of \( \tilde{\nabla}J(\theta) \) and \( \tilde{\nabla}J(\theta) \), beneficial for variance reduction.

The latter conditions (ii)-(iii) in Assumption 5.2 can also be satisfied easily by commonly used policies such as Gaussian policies and Gibbs policies. For example, for a Gaussian policy, \( \nabla^2 \log \pi_\theta(a | s) \) reduces to the matrix \( \phi(s)\phi(s)^\top/\sigma^2 \), which is a constant function of \( \theta \) and thus satisfies condition (i). Such a condition is used to show the Lipschitz continuity of the Hessian matrix of the objective function \( J(\theta) = V_{\pi_\theta}(s_0) \), which is standard in establishing the convergence to approximate second-order stationary points in the nonconvex optimization literature [36, 29, 30, 38]. Formally, the Lipschitz continuity of the Hessian is substantiated in the following lemma, whose proof is relegated to Appendix §A.6.

**Lemma 5.4.** The Hessian matrix of the objective function \( \mathcal{H}(\theta) \) is Lipschitz continuous, i.e., with some constant \( \rho > 0 \),

\[
\| \mathcal{H}(\theta^1) - \mathcal{H}(\theta^2) \| \leq \rho \cdot \| \theta^1 - \theta^2 \|, \quad \text{for any } \theta^1, \theta^2 \in \mathbb{R}^d.
\]

The value of the Lipschitz constant \( \rho \) is given in \((A.68)\) in §A.6.

The third condition (iii) in Assumption 5.2 holds for many regular policy parameterizations, and has been assumed in prior works on natural policy gradient [31] and actor-critic algorithms [23]. We note that Assumption 5.2 implies Assumption 3.1. More specifically, the condition on the reward function in Assumption 3.1 only requires boundedness,
Table 1: List of parameter values used in the convergence analysis.

| Param. | Value | Order | Constraint | Equation | Const. |
|--------|-------|-------|------------|----------|--------|
| \( \beta \) | \( c_1 \varepsilon^2/(2\ell^2L) \) | \( O(\varepsilon^2) \) | \( \leq \varepsilon^2/(2\ell^2L) \) | (A.90) | \( c_1 = 1 \) |
| \( \beta \) | \( " \) | \( " \) | \( \leq [J_\text{three}]^1/2 \) | (A.92) | " |
| \( \beta \) | \( " \) | \( O(\varepsilon) \) | \( \leq \eta \lambda^2/(24\ell^2\rho) \) | (A.112) | " |
| \( J_\text{three} \) | \( c_2 \eta \varepsilon^2/(2\ell^2L) \) | \( O(\varepsilon^4) \) | \( \leq \beta \varepsilon^2/2 \) | (A.91) | \( c_2 = c_1/2 \) |
| \( J_\text{three} \) | \( " \) | \( " \) | \( \leq \eta \beta \lambda^2/(48\ell \rho) \) | (A.113) | " |
| \( \alpha \) | \( c_1 \varepsilon^2/(2\ell^2L \sqrt{\kappa_\text{three}}) \) | \( O(\varepsilon^{3/2}) \) | \( \leq \beta / \sqrt{\kappa_\text{three}} \) | (A.89) | " |
| \( \alpha \) | \( " \) | \( " \) | \( \leq \eta \beta \lambda^2/(24\ell^2 \rho) \) | (A.114) | " |
| \( \kappa_\text{three} \) | \( c_4 \frac{\log[24 \eta \beta \alpha \sqrt{\rho \varepsilon}]}{\alpha (\rho \varepsilon)^{1/2}} \) | \( \Omega(\varepsilon^{5/2} \log(1/\varepsilon)) \) | \( \geq \varepsilon^{log[24 \eta \beta \alpha \sqrt{\rho \varepsilon}]/\alpha (\rho \varepsilon)^{1/2}} \) | (A.115) | \( c_4 = c \) |
| \( K \) | \( c_5 \frac{J_\text{three} - J(\theta_0)}{\delta J_\text{three}} \) | \( O(\varepsilon^{-9} \log(1/\varepsilon)) \) | \( \geq 2 \frac{\eta(1 - 1/\varepsilon)}{\lambda} \) | (A.96) | \( c_5 = 2 \) |

without further requirement on its positivity/negativity; the boundedness of the norm \( \|\nabla^2 \log \pi_\theta(a|s)\| \) in Assumption 3.2 implies the \( L_\theta \)-Lipschitz continuity of the score function \( \nabla \log \pi_\theta(a|s) \) in Assumption 3.1. And the condition on the positive-definiteness of the Fisher information matrix is additional. We will show shortly that these stricter assumptions enable stronger convergence guarantees.

Now we show that all the three stochastic policy gradients \( \hat{\nabla} J(\theta), \tilde{\nabla} J(\theta), \) and \( \bar{\nabla} J(\theta) \) satisfy the so-termed correlated negative curvature (CNC) condition [30], which is crucial in the ensuing analysis. The proof of Lemma 5.5 is deferred to Appendix \( \S A.7 \).

**Lemma 5.5.** Under Assumption 5.2, all the three stochastic policy gradients \( \hat{\nabla} J(\theta), \tilde{\nabla} J(\theta), \) and \( \bar{\nabla} J(\theta) \) satisfy the correlated negative curvature condition, i.e., letting \( v_\theta \) be the unit-norm eigenvector corresponding to the maximum eigenvalue of the Hessian matrix \( \mathcal{H}(\theta) \), there exist constants \( \eta, \bar{\eta}, \bar{\eta} > 0 \) such that for any \( \theta \in \mathbb{R}^d \)

\[
\mathbb{E}[|v_\theta^\top \hat{\nabla} J(\theta)|^2 | \theta] \geq \eta, \quad \mathbb{E}[|v_\theta^\top \tilde{\nabla} J(\theta)|^2 | \theta] \geq \bar{\eta}, \quad \mathbb{E}[|v_\theta^\top \bar{\nabla} J(\theta)|^2 | \theta] \geq \bar{\eta}.
\]

The CNC condition, established in Appendix A.7 basically illustrates that the perturbation caused by the stochastic gradient is guaranteed to have variance along the direction with positive curvature, i.e., the escaping direction of the objective [30]. Such an escaping direction is dictated by the eigenvectors associated with the maximum eigenvalue of the Hessian matrix \( \mathcal{H}(\theta) \). The CNC condition here can be satisfied thanks to Assumption 5.2 primarily due to the strict positivity of the absolute value of the reward, and the positive-definiteness of the Fisher information matrix. To be more specific, recall the formula of stochastic policy gradients in (3.8)-(3.10), such two conditions ensure: i) the square of the Q-value/advantage function estimates is strictly positive and uniformly lower-bounded; ii) thus the expectation of the outer-product of the stochastic policy gradients is strictly positive-definite, which gives the lower bound in Lemma 5.5. The argument will be detailed in the proof of the lemma in Appendix A.7.

Now we are ready to lay out the following convergence guarantees of the modified RPG algorithm, i.e., Algorithm 5. The values of parameters used in the analysis are specified in Table 1.
Theorem 5.6. Under Assumption 5.2, Algorithm 5 returns an \((\epsilon, \sqrt{\rho \epsilon})\)-approximate second-order stationary point policy with probability at least \((1 - \delta)\) after
\[
O\left(\frac{\rho^{3/2}L\epsilon^{-9}}{\delta \eta} \log \left(\frac{\ell_g L}{\epsilon \eta \rho}\right)\right).
\] (5.2)
steps, where \(\delta \in (0, 1)\), \(\ell_g^2 := 2\ell^2 + 2B_\Theta^2 U_R^2/(1 - \gamma)^4\), \(B_\Theta, U_R\) are as defined in Assumption 5.2, \(\rho\) is the Lipschitz constant of the Hessian in Lemma 5.4, \(\ell\) and \(\eta\) take the values of \(\hat{\ell}, \ell, \ell\) in Theorem 3.4 and \(\hat{\eta}, \eta, \tilde{\eta}\) in Lemma 5.5, when the stochastic policy gradients \(\hat{\nabla}J(\theta)\), \(\tilde{\nabla}J(\theta)\), and \(\nabla J(\theta)\) are used, respectively.

The proof of Theorem 5.6 originates but improves the proof techniques in [30]\(^5\) and is relegated to Appendix §A.8. Note that we follow the convention of using \((\epsilon, \sqrt{\rho \epsilon})\) as the convergence criterion for approximate second-order stationary points \([51, 52, 29]\), which reflects the natural relation between the gradient and the Hessian. Theorem 5.6 concludes that it is possible for the policy gradient algorithm to escape the saddle points efficiently and retrieve an approximate second-order stationary point in a polynomial number of steps\(^6\). Additionally, if all saddle points are strict (cf. definition in [36]), the modified RPG algorithm will converge to an actual local-optimal policy. In the next section, we experimentally investigate the validity of our algorithms proposed in this section and the previous section, and probe whether reward-shaping to mitigate challenges of non-convexity is borne out empirically.

6 Simulations

In this section, we present several experiments to corroborate the results of the previous two sections. Focused on the discounted infinite-horizon setting, we use the Pendulum environment in the OpenAI gym [53] as the test environment. In particular, the pendulum starts in a random position, and the goal is to swing it up so that it stays upright. The state is a vector of dimension three, i.e., \(s_t = (\cos(\theta_t), \sin(\theta_t), \dot{\theta}_t)^\top\), where \(\theta_t\) is the angle between the pendulum and the upright direction, and \(\dot{\theta}_t\) is the derivative of \(\theta_t\). The action \(a_t\) is a one-dimensional scalar representing the joint effort. In addition, the reward \(R(s_t, a_t)\) is defined as
\[
R(s_t, a_t) := -(\theta^2 + 0.1 \ast \dot{\theta}^2 + 0.001 \ast a_t^2) - 0.5,
\] (6.1)
which lies in \([-17.1736044, -0.5]\), since \(\theta\) is normalized between \([-\pi, \pi]\) and \(a_t\) lies in \([-20, 20]\). Different from the reward in the original Pendulum environment, we shift the

\(^5\)Our convergence result corresponds to Theorem 2 in [30]. However, we have identified and informed the authors, and have been acknowledged, that there is a flaw in their proof, which breaks the convergence rate claimed in the original version of the paper (personal communication). At the time the current manuscript is prepared, the authors of [30] have corrected the proof in the Arxiv version using a similar idea to what we proposed in the personal communication.

\(^6\)Note that the number of steps here in (5.2) corresponds to the notion of iteration complexity in the iteration of optimization, which is not the total sample complexity since each step of our algorithm requires two rollouts with random but finite horizon. Thus, the expected number of samples, i.e., state-action-reward tuples, equals \(1/(1 - \gamma) + 1/(1 - \gamma^{1/2})\) times the expression in (5.2).
reward by $-0.5$, so that the negativity of $R(s, a)$ in Assumption 5.2 is satisfied, i.e., $L_R = 0.5$. The transition probability follows the physical rules of Newton’s Second Law. We choose the discounted factor $\gamma$ to be 0.97. We use Gaussian policy $\pi_\theta$ truncated over the support $[-20, 20]$, which is parameterized as $\pi_\theta(\cdot | s) = \mathcal{N}(\mu_\theta(s), \sigma^2)$, where $\sigma = 1.0$ and $\mu_\theta(s) : \mathcal{S} \rightarrow \mathcal{A}$ is a neural network with two hidden layers. Each hidden layer contains 10 neurons and uses softmax as activation functions. The output layer of $\mu_\theta(s)$ uses tanh as the activation function. One can verify that such parameterization satisfies Assumption 5.2.

We first compare the performance of our algorithms with that of the popular REINFORCE algorithm [54]. To make the comparison fair, we choose the length of the rollout horizon of REINFORCE to be the expected value of the geometric distribution with success probability $1 - \gamma^{1/2}$, i.e., $T = \gamma^{1/2} / (1 - \gamma^{1/2}) = 66$. Recall that the length of the rollout horizon for Q-function estimate in our algorithm is drawn from $\text{Geom}(1 - \gamma^{1/2})$. After each rollout, i.e., one episode, the policy parameter $\theta_k$ is updated and then evaluated by calculating the value of $J(\theta)$ using the Monte-Carlo method.

First, we compare the performance of RPG (Algorithm 3) with that of the popular REINFORCE algorithm [54]. Recall that REINFORCE creates bias in the policy gradient estimate. To make a fair comparison, we set the rollout horizon of REINFORCE to be the expected value of the geometric distribution with success probability $1 - \gamma^{1/2}$, the same distribution that the rollout horizon for Q-function estimate in Algorithm 1 is drawn from, i.e., $T = \gamma^{1/2} / (1 - \gamma^{1/2}) = 66$. For RPG, we test both diminishing and constant stepsizes, where the former is set as $\alpha_k = 1 / \sqrt{k}$ and the latter is set as $\alpha_k = 0.05$ for all $k \geq 0$.

Fig. 1 (left) plots the discounted return obtained along the iterations of REINFORCE and our proposed RPG algorithms. The return is estimated by running the algorithms 30 times. The bar areas represent the standard deviation region calculated using the 30 simulations. It is shown that our proposed algorithms perform slightly better than REINFORCE in terms of discounted return, but with higher variance. This is expected since our policy gradient estimates are unbiased, compared to REINFORCE. Moreover, the higher
Figure 2: Left: The convergence of discounted return \( J(\theta) \) when REINFORCE, the proposed RPG (Algorithm 3), and MRPG\(_1\) (Algorithm 5) algorithms are used. The MRPG\(_1\) algorithm is also evaluated for the setting with mixed reward, i.e., the reward can be both negative and positive. Right: the convergence of discounted return \( J(\theta) \) when REINFORCE and the proposed MRPG (Algorithm 5) algorithms with three types of policy gradients are used.

We also evaluate the convergence of the expected gradient norm square studied in Theorem 4.3 and Corollary 4.4. Fig. 1(right) plots the empirical estimates of \( \mathbb{E}\|\nabla J(\theta_m)\|^2 \) after 30 runs of the algorithms. It is verified that using diminishing stepsize results in convergence of the gradient norm to zero a.s. (the curve keeps decreasing), while using constant stepsizes leads to an error that is lower-bounded above zero (the curves stay mostly unchanged after certain episodes). Moreover, it is shown that a smaller constant stepsize indeed creates a smaller size of the error neighborhood. Convergence rates under both diminishing and constant stepsize choices are sublinear, as identified in our theoretical results.

We further evaluate the performance of Algorithm 5 that uses intermittently larger stepsizes with stochastic policy gradient \( \hat{\nabla} J(\theta) \) as MRPG\(_1\), which theoretically we expect to yield favorable performance under appropriately designed incentive structure. Thus, in order to verify the significance of the CNC condition in escaping saddle points, we also test the MRPG\(_1\) algorithm in the environment that has mixed reward, i.e., the reward can be both positive and negative. We generate such an environment by adding a constant 10.0 onto the reward defined in (6.1). Each learning curve in Figure 2 is run for 30 times, and the bar area in the figure represents plus or minus one sample standard deviation of 30 trajectories.

First, it can be seen from Figure 2(left) that RPG achieves almost identical performance as REINFORCE, which shows that the unbiasedness of the RPG update seems to not hold great advantages over the biased PG obtained from REINFORCE, in finding the first-order stationary points. On the other hand, Figure 2(left) illustrates that MRPG\(_1\) achieves greater return than RPG, substantiating the necessity of finding approximate second-order stationary points than first-order ones. To the best of our knowledge, this appears to be the first empirical observation in RL that saddle-escaping techniques may benefit the policy learning. Interestingly, when the reward is “mixed”, the MRPG\(_1\) algo-
rithm suffers from lower discounted return and larger variance across 30 trajectories. This may be explained by the fact that different trajectories may converge to different saddle points or stationary points that may be of very different qualities. This observation also justifies the necessity of escaping undesirable saddle points for policy gradient updates.

We have also evaluated the performance of the other two MRPG algorithms that use the policy gradients $\hat{\nabla}J(\theta)$ and $\tilde{\nabla}J(\theta)$, which we refer to as $MRPG_2$ and $MRPG_3$, respectively, in Figure 2(right). Recall that the key differences of these alternative gradient updates is that they subtract a baseline or use Bellman’s evaluation equation, respectively, to replace the $Q$ function that multiplies the score function with the advantage function. As shown in Figure 2(right), the update with baselines does not always benefit the variance reduction, at least in this experiment. In particular, the policy gradient $\hat{\nabla}J(\theta)$ that uses $V(s)$ as the baseline indeed outperforms the $MRPG_1$ algorithm; however, the policy gradient $\tilde{\nabla}J(\theta)$ that uses TD error to estimate the advantage function performs even worse. Even so, all the MRPG algorithms beat the REINFORCE algorithm in terms of discounted return, and $MRPG_1$ and $MRPG_2$ also beat REINFORCE in terms of variance.

7 Conclusions

Despite its tremendous popularity, policy gradient methods in RL have rarely been investigated in terms of their global convergence, i.e., there seems to be a gap in the literature regarding the limiting properties of policy search and how this is a function of the initialization. Motivated by this gap, we have adopted the perspective and tools from nonconvex optimization to clarify and partially overcome some of the challenges of policy search for MDPs over continuous spaces. In particular, we have developed a series of random-horizon policy gradient algorithms, which generate unbiased estimates of the policy gradient for the infinite-horizon setting. Under standard assumptions for RL, we have first recovered the convergence to stationary-point policies for such first-order optimization algorithms. Moreover, by virtue of the recent results in nonconvex optimization, we have proposed the modified RPG algorithms by introducing periodically enlarged step-sizes, which are shown to be able to escape saddle points and converge to actual local optimal policies under mild conditions that are satisfied for most modern reinforcement learning applications. Specifically, we have given an optimization-based explanation of why reward-reshaping is beneficial: it improves the curvature profile of the problem in neighborhoods of saddle points. On the inverted pendulum balancing task, we have experimentally corroborated our theoretical findings. Many enhancements are possible for future research directions via the link between policy search and nonconvex optimization: rate improvements through acceleration, trust region methods, variance reduction, and Quasi-Newton methods.
A Detailed Proofs

We provide in this appendix the proofs of some of the results stated in the main body of the paper.

A.1 Proof of Lemma 3.2

Proof. The proof proceeds by expanding the expression of $\nabla J(\theta)$, and upper-bounding the norm $\|\nabla J(\theta^1) - \nabla J(\theta^2)\|$ for any $\theta^1, \theta^2 \in \mathbb{R}^d$ by multiples of $\|\theta^1 - \theta^2\|$. To this end, we first substitute the definition of $Q_{\pi_{\theta}}$ into the expression of policy gradient in (3.4), which gives

$$\nabla J(\theta) = \sum_{t=0}^{\infty} \sum_{\tau=0}^{\infty} \gamma^{t+\tau} \int R(s_{t+\tau}, a_{t+\tau}) \cdot \nabla \log \pi_{\theta}(a_t | s_t) \cdot p_{\theta, 0:t+\tau} \cdot ds_{1:t+\tau} da_{0:t+\tau}, \quad (A.1)$$

where for brevity we have introduced

$$p_{\theta, 0:t+\tau} = \left[ \prod_{u=0}^{t+\tau-1} p(s_{u+1} | s_u, a_u) \right] \cdot \left[ \prod_{u=0}^{t+\tau} \pi_{\theta}(a_u | s_u) \right] \quad (A.2)$$

to represent the probability density of the trajectory $(s_0, a_0, \cdots, s_{t+\tau}, a_{t+\tau})$. Note that (A.2) follows from the Markov property of the trajectory. Hence, for any $\theta^1, \theta^2 \in \mathbb{R}^d$, we can analyze the difference of gradients through (A.1) as:

$$\|\nabla J(\theta^1) - \nabla J(\theta^2)\| = \left\| \sum_{t=0}^{\infty} \sum_{\tau=0}^{\infty} \gamma^{t+\tau} \left( \int R(s_{t+\tau}, a_{t+\tau}) \cdot \left( \nabla \log \pi_{\theta^1}(a_t | s_t) - \nabla \log \pi_{\theta^2}(a_t | s_t) \right) \cdot p_{\theta^1, 0:t+\tau} 
\right. 
\right. 
\left. \left. + \int R(s_{t+\tau}, a_{t+\tau}) \cdot \nabla \log \pi_{\theta^2}(a_t | s_t) \cdot (p_{\theta^1, 0:t+\tau} - p_{\theta^2, 0:t+\tau}) \right) \cdot ds_{1:t+\tau} da_{0:t+\tau} \right\|$$

$$\leq \sum_{t=0}^{\infty} \sum_{\tau=0}^{\infty} \gamma^{t+\tau} \left\{ \int R(s_{t+\tau}, a_{t+\tau}) \cdot \left\| \nabla \log \pi_{\theta^1}(a_t | s_t) - \nabla \log \pi_{\theta^2}(a_t | s_t) \right\| \cdot \left| p_{\theta^1, 0:t+\tau} - p_{\theta^2, 0:t+\tau} \right| \cdot ds_{1:t+\tau} da_{0:t+\tau} \right\}$$

$$\left. + \int \left| R(s_{t+\tau}, a_{t+\tau}) \right| \cdot \left\| \nabla \log \pi_{\theta^2}(a_t | s_t) \right\| \cdot \left| p_{\theta^1, 0:t+\tau} - p_{\theta^2, 0:t+\tau} \right| \cdot ds_{1:t+\tau} da_{0:t+\tau} \right\}, \quad (A.3)$$

where the first equality comes from adding and subtracting the term $\sum_{t=0}^{\infty} \sum_{\tau=0}^{\infty} \gamma^{t+\tau} \cdot \int R(s_{t+\tau}, a_{t+\tau}) \cdot \nabla \log \pi_{\theta^2}(a_t | s_t) \cdot p_{\theta^1, 0:t+\tau} \cdot ds_{1:t+\tau} da_{0:t+\tau}$, and the inequality follows from Cauchy-Schwarz inequality. The first term $I_1$ inside the summand on the right-hand side of (A.3) depends on a difference of score functions, whereas the second term $I_2$ depends on a difference between distributions induced by different policy parameters. We establish that both terms depend only on the norm of the difference between policy parameters.

By Assumption 3.1, we have $|R(s, a)| \leq U_R$ for any $(s, a)$, and

$$\left| \nabla \log \pi_{\theta^1}(a_t | s_t) - \nabla \log \pi_{\theta^2}(a_t | s_t) \right| \leq L_{\Theta} \cdot \|\theta^1 - \theta^2\|.$$
Hence, we can bound the term $I_1$ in (A.3) as
\[ I_1 \leq U_R \cdot L_\Theta \cdot \|\theta^1 - \theta^2\|. \tag{A.4} \]
To bound the term $I_2$, let $U_{t+t} = \{u : u = 0, \cdots, t+\tau\}$; we first have
\[ p_{\theta^1,0:t+\tau} - p_{\theta^2,0:t+\tau} = \left[ \prod_{u=0}^{t+\tau-1} p(s_{u+1} | s_u, a_u) \right] \left[ \prod_{u \in U_{t+\tau}} \pi_{\theta^1}(a_u | s_u) - \prod_{u \in U_{t+\tau}} \pi_{\theta^2}(a_u | s_u) \right]. \tag{A.5} \]
By Taylor expansion of $\prod_{u \in U_{t+\tau}} \pi_{\theta}(a_u | s_u)$, we have
\[
\begin{align*}
\left| \prod_{u \in U_{t+\tau}} \pi_{\theta^1}(a_u | s_u) - \prod_{u \in U_{t+\tau}} \pi_{\theta^2}(a_u | s_u) \right| &= \left( \theta^1 - \theta^2 \right)^T \left[ \sum_{m \in U_{t+\tau}} \nabla \pi_{\theta^1}(a_m | s_m) - \sum_{m \in U_{t+\tau}} \nabla \pi_{\theta^2}(a_m | s_m) \right] \\
&\leq \|\theta^1 - \theta^2\| \cdot \sum_{m \in U_{t+\tau}} \|\nabla \log \pi_{\theta^1}(a_m | s_m)\| \cdot \prod_{u \in U_{t+\tau}} \nabla \pi_{\theta^2}(a_u | s_u) \\
&\leq \|\theta^1 - \theta^2\| \cdot (t + \tau + 1) \cdot B_\Theta \cdot \prod_{u \in U_{t+\tau}} \nabla \pi_{\theta^2}(a_u | s_u), \tag{A.6}
\end{align*}
\]
where $\tilde{\theta}$ is a vector lying between $\theta^1$ and $\theta^2$, i.e., there exists some $\lambda \in [0, 1]$ such that $\tilde{\theta} = \lambda \theta^1 + (1 - \lambda)\theta^2$. Therefore, we can upper bound $|p_{\theta^1,0:t+\tau} - p_{\theta^2,0:t+\tau}|$ by substituting (A.6) into (A.5), which further upper-bounds the term $I_2$ in (A.3) by
\[
I_2 \leq \|\theta^1 - \theta^2\| \cdot U_R \cdot B^2_\Theta \cdot \int \left[ \prod_{u=0}^{t+\tau-1} p(s_{u+1} | s_u, a_u) \right] \cdot (t + \tau + 1) \cdot \prod_{u \in U_{t+\tau}} \nabla \pi_{\theta^2}(a_u | s_u) ds_1:t+\tau \, da_0:t+\tau \\
= \|\theta^1 - \theta^2\| \cdot U_R \cdot B^2_\Theta \cdot (t + \tau + 1), \tag{A.7}
\]
where the last equality follows from the fact that $\prod_{u=0}^{t+\tau-1} p(s_{u+1} | s_u, a_u) \cdot \prod_{u \in U_{t+\tau}} \nabla \pi_{\theta^2}(a_u | s_u)$ is a valid probability density function.

Combining the bounds for $I_1$ in (A.4) and that for $I_2$ in (A.7), we have
\[
\|\nabla J(\theta^1) - \nabla J(\theta^2)\| \leq \sum_{t=0}^{\infty} \sum_{\tau=0}^{\infty} \gamma^{t+\tau} \cdot U_R \cdot \left[ L_\Theta + B^2_\Theta \cdot (t + \tau + 1) \right] \cdot \|\theta^1 - \theta^2\| \\
\leq \left[ \frac{1}{(1 - \gamma)^2} \cdot U_R \cdot L_\Theta + \frac{1 + \gamma}{(1 - \gamma)^3} \cdot U_R \cdot B^2_\Theta \right] \cdot \|\theta^1 - \theta^2\|,
\]
where the last inequality uses the expression for the limit of a geometric series, given that $\gamma \in (0, 1)$:
\[
\sum_{t=0}^{\infty} \sum_{\tau=0}^{\infty} \gamma^{t+\tau} = \frac{1}{(1 - \gamma)^2}, \quad \sum_{t=0}^{\infty} \sum_{\tau=0}^{\infty} \gamma^{t+\tau} \cdot (t + \tau + 1) = \frac{1 + \gamma}{(1 - \gamma)^3}.
\]
Hence, we define the Lipschitz constant $L$ as follows
\[
L := \frac{U_R \cdot L_\Theta}{(1 - \gamma)^2} + \frac{(1 + \gamma) \cdot U_R \cdot B^2_\Theta}{(1 - \gamma)^3},
\]
which completes the proof. \qed
A.2 Proof of Theorem 3.4

Proof. We first establish unbiasedness of the stochastic estimates of the policy gradient. We start by showing unbiasedness of the Q-estimate, i.e., for any \((s, a) \in \mathcal{S} \times \mathcal{A}\) and \(\theta \in \mathbb{R}^d\),

\[
\mathbb{E}[\hat{Q}_{\pi_\theta}(s, a) | \theta, s, a] = Q_{\pi_\theta}(s, a).
\]

In particular, from the definition of \(\hat{Q}_{\pi_\theta}(s, a)\), we have

\[
\mathbb{E}[\hat{Q}_{\pi_\theta}(s, a) | \theta, s, a] = \mathbb{E}\left[ \sum_{t=0}^{T'} \gamma^{t/2} \cdot R(s_t, a_t) \mid \theta, s_0 = s, a_0 = a \right]
\]

where we have replaced \(T'\) by \(\infty\) since we use the indicator function \(\mathbb{I}\) such that the summand for \(t \geq T'\) is null.

Now we show that the inner-expectation over \(T'\) and summation in (A.8) can be interchanged. In fact, by Assumption 3.1 regarding the boundedness of the reward, for any \(N > 0\), we have

\[
\mathbb{E}_T\left( \left\| \sum_{t=0}^{N} \mathbb{I}_{0 \leq t \leq T'} \gamma^{t/2} \cdot R_t \right\| \right) \leq U_R \cdot \mathbb{E}_T\left( \sum_{t=0}^{N} \mathbb{I}_{0 \leq t \leq T'} \gamma^{t/2} \right),
\]  

(A.9)

Note that on the right-hand side of (A.9), the random variable in the expectation is monotonically increasing and the limit as \(N \to \infty\) exists. Thus, by the Monotone Convergence Theorem [55], we can interchange the limit with the integral, i.e., the sum and inner-expectation in (A.8) as follows

\[
\mathbb{E}\left\{ \sum_{t=0}^{\infty} \mathbb{I}_{T' \geq t \geq 0} \cdot \gamma^{t/2} \cdot R(s_t, a_t) \mid \theta, s_0 = s, a_0 = a \right\}
\]

\[
= \sum_{t=0}^{\infty} \mathbb{E}\left[ \mathbb{E}_T (\mathbb{I}_{T' \geq t \geq 0} \cdot \gamma^{t/2} \cdot R(s_t, a_t)) \mid \theta, s_0 = s, a_0 = a \right]
\]

\[
= \sum_{t=0}^{\infty} \mathbb{E}\left[ \gamma^t \cdot R(s_t, a_t) \mid \theta, s_0 = s, a_0 = a \right],
\]  

(A.10)

where we have also used in the first equality the fact that \(T'\) is drawn independently of the system evolution \((s_1, T', a_1, T')\), and \(d\) in the second equality the fact that \(T' \sim \text{Geom}(1 - \gamma^{1/2})\) and thus \(\mathbb{E}_T(\mathbb{I}_{T' \geq t \geq 0}) = \mathbb{P}(T' \geq t \geq 0) = \gamma^{t/2}\) in the second equality. Furthermore, since \(\sum_{t=0}^{N} \gamma^t R(s_t, a_t)\) \((N \to \infty)\) exists, by the Dominated Convergence Theorem [56], the right-hand side of (A.10) can be written as

\[
\sum_{t=0}^{\infty} \mathbb{E}\left[ \gamma^t \cdot R(s_t, a_t) \mid \theta, s_0 = s, a_0 = a \right] = \mathbb{E}\left[ \sum_{t=0}^{\infty} \gamma^t \cdot R(s_t, a_t) \mid \theta, s_0 = s, a_0 = a \right] = Q_{\pi_\theta}(s, a),
\]

which completes the proof of the unbiasedness of \(\hat{Q}_{\pi_\theta}(s, a)\).
Similar logic allows us to establish that $\hat{V}_{\pi_\theta}(s)$ is an unbiased estimate of $V_{\pi_\theta}(s)$, i.e., for any $s \in \mathcal{S}$ and $\theta \in \mathbb{R}^d$,

$$
\mathbb{E}\left[ \sum_{t=0}^{\infty} \mathbb{1}_{T \geq t \geq 0} \cdot \gamma^{t/2} \cdot R(s_t, a_t) \mid \theta, s_0 = s \right] = \mathbb{E}[\hat{V}_{\pi_\theta}(s) \mid \theta, s] = V_{\pi_\theta}(s),
$$

where the expectation is taken along the trajectory as well as with respect to the random horizon $T' \sim \text{Geom}(1 - \gamma^{t/2})$. Therefore, if $s' \sim \mathbb{P}(\cdot \mid s, a)$ and $a' \sim \pi_\theta(\cdot \mid s')$, we have

$$
\mathbb{E}[\hat{Q}_{\pi_\theta}(s, a) - \hat{V}_{\pi_\theta}(s) \mid \theta, s, a] = \mathbb{E}[R(s, a) + \gamma \hat{V}_{\pi_\theta}(s') - \hat{V}_{\pi_\theta}(s) \mid \theta, s, a] = A_{\pi_\theta}(s, a). \quad (A.11)
$$

That is, $\hat{Q}_{\pi_\theta}(s, a) - \hat{V}_{\pi_\theta}(s)$ and $R(s, a) + \gamma \hat{V}_{\pi_\theta}(s') - \hat{V}_{\pi_\theta}(s)$ are both unbiased estimates of the advantage function $A_{\pi_\theta}(s, a)$.

Now we are ready to show unbiasedness of the stochastic gradients $\hat{V}J(\theta), \hat{\nabla}J(\theta)$, and $\hat{\nabla}J(\theta)$. First for $\hat{V}J(\theta)$, we have from (A.10) that

$$
\mathbb{E}[\hat{V}J(\theta) \mid \theta] = \mathbb{E}_{T, (s_T, a_T)}\left[ \inf_{s'=1} \mathbb{E}_{T', (s_{1:T'}, a_{1:T'})}[\hat{V}J(\theta) \mid \theta, s_T = s, a_T = a] \right],
$$

$$
= \mathbb{E}_{T, (s_T, a_T)}\left( \mathbb{E}_{T', (s_{1:T'}, a_{1:T'})} \left\{ \frac{1}{1 - \gamma} \cdot \hat{Q}_{\pi_\theta}(s_T, a_T) \cdot \nabla \log[\pi_\theta(a_T \mid s_T)] \mid \theta, s_T = s, a_T = a \right\} \right),
$$

$$
= \mathbb{E}_{T, (s_T, a_T)}\left( \frac{1}{1 - \gamma} \cdot Q_{\pi_\theta}(s_T, a_T) \cdot \nabla \log[\pi_\theta(a_T \mid s_T)] \right) \mid \theta. \quad (A.12)
$$

By using the identity function $1_{t=T}$, (A.12) can be further written as

$$
\mathbb{E}[\hat{V}J(\theta) \mid \theta] = \frac{1}{1 - \gamma} \cdot \mathbb{E}_{T, (s_T, a_T)} \left( \sum_{t=0}^{\infty} 1_{t=T} \cdot Q_{\pi_\theta}(s_t, a_t) \cdot \nabla \log[\pi_\theta(a_t \mid s_t)] \right) \mid \theta. \quad (A.13)
$$

Note that by Assumption 3.1 $||\hat{V}J(\theta)||$ is directly bounded by $(1 - \gamma)^{-2} \cdot U_R \cdot B_\Theta$, since there is only one nonzero term in the summation in (A.13). Thus, by the Dominated Convergence Theorem, we can interchange the summation and expectation in (A.13) and obtain

$$
\mathbb{E}[\hat{V}J(\theta) \mid \theta] = \sum_{t=0}^{\infty} P(t = T) \cdot \mathbb{E}\left( Q_{\pi_\theta}(s_t, a_t) \cdot \nabla \log[\pi_\theta(a_t \mid s_t)] \right) \mid \theta,
$$

$$
= \sum_{t=0}^{\infty} \gamma^t \cdot \mathbb{E}\left( Q_{\pi_\theta}(s_t, a_t) \cdot \nabla \log[\pi_\theta(a_t \mid s_t)] \right) \mid \theta \quad \text{(A.14)}
$$

$$
= \sum_{t=0}^{\infty} \gamma^t \cdot \int_{s \in \mathcal{S}, a \in \mathcal{A}} p(s_t = s, a_t = a \mid s_0, \pi_\theta) \cdot Q_{\pi_\theta}(s, a) \cdot \nabla \log[\pi_\theta(a \mid s)] ds da, \quad (A.15)
$$

where (A.14) is due to the fact that $T \sim \text{Geom}(1 - \gamma)$ and thus $P(t = T) = (1 - \gamma)^t$, and in (A.15) we define $p(s_t = s, a_t = a \mid s_0, \pi_\theta) = p(s_t = s \mid s_0, \pi_\theta) \cdot \pi_\theta(a_t = a \mid s_t)$, with $p(s_t = s \mid s_0, \pi_\theta)$ being the probability of state $s_t = s$ given initial state $s_0$ and policy $\pi_\theta$. By the Dominated Convergence Theorem, we can further re-write (A.15) by interchanging the summation and the integral, i.e.,

$$
\mathbb{E}[\hat{V}J(\theta) \mid \theta] = \int_{s \in \mathcal{S}, a \in \mathcal{A}} \sum_{t=0}^{\infty} \gamma^t \cdot p(s_t = s \mid s_0, \pi_\theta) \cdot Q_{\pi_\theta}(s, a) \cdot \nabla \pi_\theta(a \mid s) ds da. \quad (A.16)
$$
Note that the expression in (A.16) coincides with the policy gradient given in (3.3), which completes the proof of unbiasedness of $\hat{V}J(\theta)$.

For $\hat{V}J(\theta)$, we have the following identity similar to (A.12):

$$
\mathbb{E}[\hat{V}J(\theta) | \theta] = \mathbb{E}_{T,(s_T,a_T)}[\mathbb{E}_{T',(s_{1:T'},a_{1:T'})}[\hat{V}J(\theta) | \theta, s_T = s, a_T = a] | \theta]
$$

$$
= \mathbb{E}_{T,(s_T,a_T)}\left( \mathbb{E}_{T',(s_{1:T'},a_{1:T'})}\left\{ \frac{\dot{Q}_{\pi_0}(s,a) - \dot{V}_{\pi_0}(s)}{1 - \gamma} \cdot \nabla \log[\pi_\theta(a_T | s_T)] | \theta, s_T = s, a_T = a \right\} | \theta \right)
$$

$$
= \mathbb{E}_{T,(s_T,a_T)}\left\{ \frac{1}{1 - \gamma} \cdot A_{\pi_0}(s_T,a_T) \cdot \nabla \log[\pi_\theta(a_T | s_T)] \right\}. \quad (A.17)
$$

From the definition of $A_{\pi_0}(s_T,a_T) = Q_{\pi_0}(s_T,a_T) - V_{\pi_0}(s_T),$ (A.17) further implies

$$
\mathbb{E}[\hat{V}J(\theta) | \theta] = \int_{s \in S, a \in A} \sum_{t=0}^{\infty} \gamma^t \cdot p(s_t = s | s_0, \pi_\theta) \cdot [Q_{\pi_0}(s,a) - V_{\pi_0}(s)] \cdot \nabla \pi_\theta(a | s) ds da, \quad (A.18)
$$

which follows from similar arguments as in (A.13)-(A.16). Note that (A.18) also coincides with the policy gradient given in (3.3), since $\int_{a \in A} \nabla \pi_\theta(a | s) da = 0$. Similar arguments also hold for the stochastic policy gradient $\hat{V}J(\theta)$, since (A.17) can also be obtained from $\mathbb{E}[\hat{V}J(\theta) | \theta]$. This proves unbiasedness of $\hat{V}J(\theta)$ and $\hat{V}J(\theta)$.

Now we establish almost sure boundedness of the stochastic policy gradients $\hat{V}J(\theta), \hat{V}J(\theta),$ and $\hat{V}J(\theta)$. In particular, from the definition of $\hat{V}J(\theta)$ in (3.8)

$$
\|\hat{V}J(\theta)\| = \left\| \frac{1}{1 - \gamma} \cdot \dot{Q}_{\pi_0}(s_T,a_T) \cdot \nabla \log[\pi_\theta(a_T | s_T)] \right\| \leq \frac{B_\Theta}{1 - \gamma} \left\| \sum_{t=0}^{T} \gamma^{t/2} \cdot R(s_t,a_t) \right\|
$$

$$
\leq \frac{B_\Theta}{1 - \gamma} \sum_{t=0}^{T} \gamma^{t/2} \cdot U_R \leq \frac{B_\Theta}{1 - \gamma} \sum_{t=0}^{\infty} \gamma^{t/2} \cdot U_R = \frac{B_\Theta U_R}{(1 - \gamma)(1 - \gamma^{1/2})} =: \hat{\ell}, \quad (A.19)
$$

where we have used Assumption 3.1 namely that $|R(s,a)| \leq U_R$ and $\|\nabla \log \pi_\theta(a | s)\| \leq B_\Theta$ for any $s,a$ and $\theta$. Similarly, we arrive at the following bounds

$$
\|\hat{V}J(\theta)\| \leq \frac{2B_\Theta}{1 - \gamma} \sum_{t=0}^{\infty} \gamma^{t/2} \cdot U_R = \frac{2B_\Theta U_R}{(1 - \gamma)(1 - \gamma^{1/2})} =: \hat{\ell}, \quad (A.20)
$$

$$
\|\hat{V}J(\theta)\| \leq \frac{B_\Theta}{1 - \gamma} \left[ 1 + (\gamma + 1) \left( \sum_{t=0}^{T} \gamma^{t/2} \right) \right] \cdot U_R \leq \frac{(2 + \gamma - \gamma^{1/2})B_\Theta U_R}{(1 - \gamma)(1 - \gamma^{1/2})} =: \bar{\ell}, \quad (A.21)
$$

which completes the proof.

\[ \square \]

### A.3 Proof of Theorem 4.2

**Proof.** Recall that the policy gradient method follows (3.11). At each iteration $k$, we define the random horizon used in estimating $\dot{Q}_{\pi_{\theta_k}}(s_{T_k+1},a_{T_k+1})$ in the inner-loop of Algorithm 1
as $T_{k+1}'$. We then introduce a probability measure space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_k\}_{k \geq 0}$ denote a sequence of increasing sigma-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \mathcal{F}_\infty \subset \mathcal{F}$, where

$$
\mathcal{F}_k = \sigma\left(\{\theta_\tau \}_{\tau=0:k}, \{T_\tau\}_{\tau=0:k}, \{(s_\tau, a_\tau)\}_{\tau=T_0:T_k}, \{(s_\tau, a_\tau)\}_{\tau=0:T'_0}, \cdots , \{(s_\tau, a_\tau)\}_{\tau=0:T'_k}\right).
$$

We also define the following auxiliary random variable $W_k$, which is essential to the analysis of Algorithm 3

$$
W_k = J(\theta_k) - L\hat{\ell}^2 \sum_{j=k}^{\infty} \alpha_k^2, \quad (A.22)
$$

where we recall that $L$ is the Lipschitz constant of $\nabla J(\theta)$ as defined in (3.6), and $\hat{\ell}$ is the upper bound of $\|\hat{\nabla} J(\theta_k)\|$ in Theorem 3.4. Noting that $J(\theta)$ is bounded and $\{\alpha_k\}$ is square-summable, we conclude that $W_k$ is bounded for any $k \geq 0$. In fact, we can show that $\{W_k\}$ is a bounded submartingale, as stated in the following lemma.

**Lemma A.1.** The objective function sequence defined by Algorithm 3 satisfies the following stochastic ascent property:

$$
\mathbb{E}[J(\theta_{k+1}) | \mathcal{F}_k] \geq J(\theta_k) + \mathbb{E}[\{(\theta_{k+1} - \theta_k) | \mathcal{F}_k\}^\top \nabla J(\theta_k) - L\alpha_k^2 \hat{\ell}^2] \geq J(\theta_k) + \mathbb{E}[\{(\theta_{k+1} - \theta_k) | \mathcal{F}_k\}^\top \nabla J(\theta_k) - L\alpha_k^2 \hat{\ell}^2] \geq J(\theta_k) + \mathbb{E}[\{(\theta_{k+1} - \theta_k) | \mathcal{F}_k\}^\top \nabla J(\theta_k) - L\alpha_k^2 \hat{\ell}^2] \geq J(\theta_k) + \mathbb{E}[\{(\theta_{k+1} - \theta_k) | \mathcal{F}_k\}^\top \nabla J(\theta_k) - L\alpha_k^2 \hat{\ell}^2]
$$

Moreover, the sequence $\{W_k\}$ defined in (A.22) is a bounded submartingale.

$$
\mathbb{E}(W_{k+1} | \mathcal{F}_k) \geq W_k + \alpha_k \|\nabla J(\theta_k)\|^2. \quad (A.24)
$$

**Proof.** Note that $W_k$ is adapted to the sigma-algebra $\mathcal{F}_k$. Consider the first-order Taylor expansion of $J(\theta_{k+1})$ at $\theta_k$. Then there exists some $\tilde{\theta}_k = \lambda \theta_k + (1 - \lambda)\theta_{k+1}$ for some $\lambda \in [0,1]$ such that $W_{k+1}$ can be written as

$$
W_{k+1} = J(\theta_k) + (\theta_{k+1} - \theta_k)^\top \nabla J(\tilde{\theta}_k) - L\hat{\ell}^2 \sum_{j=k+1}^{\infty} \alpha_k^2
$$

$$
= J(\theta_k) + (\theta_{k+1} - \theta_k)^\top \nabla J(\theta_k) + (\theta_{k+1} - \theta_k)^\top [\nabla J(\theta_k) - \nabla J(\tilde{\theta}_k)] - L\hat{\ell}^2 \sum_{j=k+1}^{\infty} \alpha_k^2
$$

$$
\geq J(\theta_k) + (\theta_{k+1} - \theta_k)^\top \nabla J(\theta_k) - L\|\theta_{k+1} - \theta_k\|^2 - L\hat{\ell}^2 \sum_{j=k+1}^{\infty} \alpha_k^2
$$

where the second equality comes from adding and subtracting $(\theta_{k+1} - \theta_k)^\top \nabla J(\theta_k)$, and the inequality follows from applying Lipschitz continuity of the gradient (Lemma 3.2), i.e.

$$
(\theta_{k+1} - \theta_k)^\top [\nabla J(\tilde{\theta}_k) - \nabla J(\theta_k)] \geq -\|\theta_{k+1} - \theta_k\| \cdot \|\nabla J(\tilde{\theta}_k) - \nabla J(\theta_k)\| \geq -\|\theta_{k+1} - \theta_k\| \cdot L\|\tilde{\theta}_k - \theta_k\|
$$

$$
= -\|\theta_{k+1} - \theta_k\| \cdot L(1 - \lambda) \cdot \|\theta_{k+1} - \theta_k\| \geq -L \cdot \|\theta_{k+1} - \theta_k\|^2,
$$

27
with the constant $L$ being defined in (3.6). By taking conditional expectation over $\mathcal{F}_k$ on both sides, we further obtain

$$
\mathbb{E}[W_{k+1} | \mathcal{F}_k] \geq J(\theta_k) + \mathbb{E}[(\theta_{k+1} - \theta_k) | \mathcal{F}_k]^\top \nabla J(\theta_k) - L \mathbb{E}[(\theta_{k+1} - \theta_k)^2 | \mathcal{F}_k) - L\ell^2 \sum_{j=k+1}^{\infty} \alpha_j^2
$$

$$
= J(\theta_k) + \mathbb{E}[(\theta_{k+1} - \theta_k) | \mathcal{F}_k]^\top \nabla J(\theta_k) - La_k^2 \mathbb{E}[\nabla J(\theta_k)^2 | \mathcal{F}_k) - L\ell^2 \sum_{j=k+1}^{\infty} \alpha_j^2
$$

$$
\geq J(\theta_k) + \mathbb{E}[(\theta_{k+1} - \theta_k) | \mathcal{F}_k]^\top \nabla J(\theta_k) - La_k^2 \ell^2 - L\ell^2 \sum_{j=k+1}^{\infty} \alpha_j^2, \tag{A.25}
$$

where the first inequality comes from substituting $\theta_{k+1} - \theta_k = \alpha_k \nabla J(\theta_k)$ and the second one uses the fact that $\mathbb{E}[|\nabla J(\theta_k)^2|] \leq \ell^2$. By definition of $J(\theta)$, we have

$$
\mathbb{E}[J(\theta_{k+1}) | \mathcal{F}_k) \geq J(\theta_k) + \mathbb{E}[(\theta_{k+1} - \theta_k) | \mathcal{F}_k]^\top \nabla J(\theta_k) - La_k^2 \ell^2,
$$

which establishes the first argument of the lemma.

In addition, note that

$$
\mathbb{E}[(\theta_{k+1} - \theta_k) | \mathcal{F}_k) = \alpha_k \mathbb{E}(\nabla J(\theta_k) | \mathcal{F}_k) = \alpha_k \nabla J(\theta_k),
$$

which we may substitute into the right-hand side of (A.25), and upper-bound the negative constant terms by null to obtain

$$
\mathbb{E}(W_{k+1} | \mathcal{F}_k) \geq W_k + \alpha_k \|\nabla J(\theta_k)\|^2.
$$

This concludes the proof. \qed

Now we are in a position to show that $\|\nabla J(\theta_k)\|$ converges to zero as $k \to \infty$. In particular, by definition, we have the boundedness of $W_k$, i.e., $W_k \leq J^*$, where $J^*$ is the global maximum of $J(\theta)$. Thus, (A.24) can be written as

$$
\mathbb{E}(J^* - W_{k+1} | \mathcal{F}_k) \leq (J^* - W_k) - \alpha_k \|\nabla J(\theta_k)\|^2,
$$

where $\{J^* - W_k\}$ is a nonnegative sequence of random variables. By applying the super-martingale convergence theorem [47], we have

$$
\sum_{k=1}^{\infty} \alpha_k \|\nabla J(\theta_k)\|^2 < \infty, \text{ a.s.}, \tag{A.26}
$$

Note that by Assumption 4.1, the stepsize $\{\alpha_k\}$ is non-summable. Therefore, the only way that (A.26) may be valid is if the following holds:

$$
\liminf_{k \to \infty} \|\nabla J(\theta_k)\| = 0. \tag{A.27}
$$

From here, we proceed to show that $\limsup_{k \to \infty} \|\nabla J(\theta_k)\| = 0$ by contradiction. To this end, we construct a sequence of $\{\theta_k\}$ that has two sub-sequences lying in two disjoint sets.
We aim to establish a contradiction on the sum of the distances between the points in the two sets. Specifically, suppose that for some random realization $\omega \in \Omega$, we have
\[
\limsup_{k \to \infty} \|\nabla J(\theta_k)\| = \epsilon > 0.
\] (A.28)

Then it must hold that $\|\nabla J(\theta_k)\| \geq 2\epsilon/3$ for infinitely many $k$. Moreover, (A.27) implies that $\|\nabla J(\theta_k)\| \leq \epsilon/3$ for infinitely many $k$. We thus can define the following sets $N_1$ and $N_2$ as
\[
N_1 = \{\theta_k : \|\nabla J(\theta_k)\| \geq 2\epsilon/3\}, \quad N_2 = \{\theta_k : \|\nabla J(\theta_k)\| \leq \epsilon/3\}.
\]

Note that since $\|\nabla J(\theta)\|$ is continuous by Lemma 3.2, both sets are closed in the Euclidean space. We define the distance between the two sets as
\[
D(N_1, N_2) = \inf_{\theta^1 \in N_1} \inf_{\theta^2 \in N_2} \|\theta^1 - \theta^2\|.
\]

Then $D(N_1, N_2)$ must be a positive number since the sets $N_1$ and $N_2$ are disjoint and closed. Moreover, since both $N_1$ and $N_2$ are infinite sets, there exists an index set $I$ such that the subsequence $\{\theta_k\}_{k \in I}$ of $\{\theta_k\}_{k \geq 0}$ crosses the two sets infinitely often. In particular, there exist two sequences of indices $\{s_i\}_{i \geq 0}$ and $\{t_i\}_{i \geq 0}$ such that
\[
\{\theta_k\}_{k \in I} = \{\theta_{s_i}, \cdots, \theta_{t_i-1}\}_{i \geq 0},
\]
with $\{s_i\}_{i \geq 0} \subseteq N_1, \{t_i\}_{i \geq 0} \subseteq N_2$, and for any indices $k = s_i + 1, \cdots, t_i - 1 \in I$ (not including $s_i$) in between the indices $\{s_i\}$ and $\{t_i\}$, we have
\[
\frac{\epsilon}{3} \leq \|\nabla J(\theta_k)\| \leq \frac{2\epsilon}{3} \leq \|\nabla J(\theta_{s_i})\|.
\]

Setting aside this expression for now, let us analyze the norm-difference of iterates $\theta_k$ associated with indices in $I$. By the triangle inequality, we may write
\[
\sum_{k \in I} \|\theta_{k+1} - \theta_k\| = \sum_{i=0}^{\infty} \sum_{k=s_i}^{t_i-1} \|\theta_{k+1} - \theta_k\| \geq \sum_{i=0}^{\infty} \|\theta_{s_i} - \theta_{t_i-1}\| \geq \sum_{i=0}^{\infty} D(N_1, N_2) = \infty. \quad (A.29)
\]

Moreover, (A.26) implies that
\[
\infty > \sum_{k \in I} \alpha_k \|\nabla J(\theta_k)\|^2 \geq \sum_{k \in I} \alpha_k \frac{\epsilon^2}{9},
\]
using the definition of $\epsilon$ in (A.28). We may therefore conclude that $\sum_{k \in I} \alpha_k < \infty$. Also from Theorem 3.4 we have that the stochastic policy gradient has a finite first moment: $\mathbb{E}(\|\hat{\nabla} J(\theta_k)\|) < \infty$. Taken together, we therefore have
\[
\sum_{k \in I} \mathbb{E}(\|\theta_{k+1} - \theta_k\|) = \sum_{k \in I} \alpha_k \mathbb{E}(\|\hat{\nabla} J(\theta_k)\|) < \infty.
\]
The monotone convergence theorem then implies that \( \sum_{k \in I} \| \theta_{k+1} - \theta_k \| < \infty \) almost surely, which contradicts (A.29). Therefore, (A.29) must be false, which implies that the hypothesis that the limsup is bounded away from zero, as in (A.28), is invalid. As a consequence, its negation must be true: the set of sample paths for which this condition holds has measure zero. This allows us to conclude

\[
\limsup_{k \to \infty} \| \nabla J(\theta_k) \| = 0, \quad a.s.
\]

This statement together with (A.27) allows us to conclude that \( \lim_{k \to \infty} \| \nabla J(\theta_k) \| = 0 \) a.s., which completes the proof. \( \square \)

### A.4 Proofs of Theorem 4.3 and Corollary 4.4

**Proof.** By the stochastic ascent property, i.e., (A.23) in Lemma A.1, we can write

\[
\mathbb{E}[J(\theta_{k+1}) | \mathcal{F}_k] \geq J(\theta_k) + \mathbb{E}[(\theta_{k+1} - \theta_k) | \mathcal{F}_k]^T \nabla J(\theta_k) - L \alpha_k^2 \ell^2
\]

\[
= J(\theta_k) + \alpha_k \| \nabla J(\theta_k) \|^2 - L \alpha_k^2 \ell^2.
\]

(A.30)

Let \( U(\theta) = J^* - J(\theta) \), where \( J^* \) is the global optimum\(^7\) of \( J(\theta) \). Then, we immediately have

\[ 0 \leq U(\theta) \leq 2U_R/(1 - \gamma) \]

since \( |J(\theta)| \leq U_R/(1 - \gamma) \) for any \( \theta \). Moreover, we may write (A.30) as

\[
\mathbb{E}[U(\theta_{k+1}) | \mathcal{F}_k] \leq U(\theta_k) - \alpha_k \| \nabla J(\theta_k) \|^2 + L \alpha_k^2 \ell^2.
\]

(A.31)

Let \( N > 0 \) be an arbitrary positive integer. By re-ordering the terms in (A.31) and summing over \( k - N, \ldots, k \), we have

\[
\sum_{m = k-N}^{k} \mathbb{E}[\| \nabla J(\theta_m) \|^2] \leq \sum_{m = k-N}^{k} \frac{1}{\alpha_m} \cdot \mathbb{E}[U(\theta_m)] - \mathbb{E}[U(\theta_{m+1})] + \sum_{m = k-N}^{k} L \alpha_m \ell^2
\]

(A.32)

where the equality follows from adding and subtracting an additional term \( \frac{1}{\alpha_{k-N-1}} \cdot \mathbb{E}[U(\theta_{k-N})] \). Now, using the fact that the value sub-optimality is bounded by \( 0 \leq U(\theta) \leq 2U_R/(1 - \gamma) \), we can further bound the right-hand side of (A.32) as

\[
\sum_{m = k-N}^{k} \left( \frac{1}{\alpha_m} - \frac{1}{\alpha_{m-1}} \right) \cdot \mathbb{E}[U(\theta_m)] - \frac{1}{\alpha_k} \cdot \mathbb{E}[U(\theta_{k+1})] + \frac{1}{\alpha_{k-N-1}} \cdot \mathbb{E}[U(\theta_{k-N})] + \sum_{m = k-N}^{k} L \alpha_m \ell^2
\]

\[
\leq \sum_{m = k-N}^{k} \left( \frac{1}{\alpha_m} - \frac{1}{\alpha_{m-1}} \right) \cdot \frac{2U_R}{1 - \gamma} + \frac{1}{\alpha_{k-N-1}} \cdot \frac{2U_R}{1 - \gamma} + \sum_{m = k-N}^{k} L \alpha_m \ell^2
\]

\[
\leq \frac{1}{\alpha_k} \cdot \frac{2U_R}{1 - \gamma} + \sum_{m = k-N}^{k} L \alpha_m \ell^2,
\]

(A.33)

\(^7\)Such an optimum is assumed to always exist for the parameterization \( \pi_\theta \).
where we drop the nonpositive term $-\mathbb{E}[U(\theta_{k+1})/\alpha_k]$ and upper-bound $\mathbb{E}[U(\theta_m)]$ by $2U_R/(1-\gamma)$ for all $m = k - N, \ldots, k$. We use the fact that the stepsize is non-increasing $\alpha_m \leq \alpha_{m-1}$, such that $1/\alpha_m \geq 1/\alpha_{m-1}$. By substituting $\alpha_k = k^{-a}$ into (A.33) and then (A.32), we further have

$$\sum_{m=k-N}^{k} \mathbb{E}[\|\nabla J(\theta_m)\|^2] \leq O\left(k^a \cdot \frac{2U_R}{1-\gamma} + L\hat{\ell}^2 \cdot [k^1 - (k-N)^1 - a]\right),$$

where we use the fact that

$$\sum_{m=k-N}^{k} m^{-a} \leq k^{1-a} - (k-N)^{1-a}$$

for $a \in (0, 1)$. Setting $N = k - 1$ and dividing by $k$ on both sides of (A.34), we obtain

$$\frac{1}{k} \sum_{m=1}^{k} \mathbb{E}[\|\nabla J(\theta_m)\|^2] \leq O\left(k^{a-1} \cdot \frac{2U_R}{1-\gamma} + L\hat{\ell}^2 \cdot [k^{-a} - k^{-1}]\right) \leq O(k^{-p}),$$

where $p = \min\{1-a,a\}$. By definition of $K_\epsilon$, we have

$$\mathbb{E}[\|\nabla J(\theta_k)\|^2] > \epsilon, \text{ for any } k < K_\epsilon,$$

which together with (A.35) gives us

$$\epsilon \leq \frac{1}{K_\epsilon} \sum_{m=1}^{K_\epsilon} \mathbb{E}[\|\nabla J(\theta_m)\|^2] \leq O(K_\epsilon^{-p}).$$

This shows that $K_\epsilon \leq O(\epsilon^{-1/p})$. Note that $\max_{a\in(0,1)} p(a) = 1/2$ with $a = 1/2$, which concludes the proof of Theorem 4.3.

From (A.32) in the proof of Theorem 4.3, we obtain that for any $k > 0$ and $0 \leq N < k$,

$$\sum_{m=k-N}^{k} \mathbb{E}[\|\nabla J(\theta_m)\|^2] \leq \sum_{m=k-N}^{k} \frac{1}{\alpha} \cdot \left(\mathbb{E}[U(\theta_m)] - \mathbb{E}[U(\theta_{m+1})]\right) + \sum_{m=k-N}^{k} L\alpha\hat{\ell}^2$$

$$= \frac{1}{\alpha} \cdot \left(\mathbb{E}[U(\theta_k)] - \mathbb{E}[U(\theta_{k-N+1})]\right) + \sum_{m=k-N}^{k} L\alpha\hat{\ell}^2 \leq \frac{1}{\alpha} \cdot \frac{2U_R}{1-\gamma} + (N+1) \cdot L\alpha\hat{\ell}^2,$$

where the equality follows from telescope cancellation and the second inequality follows from the fact that $0 \leq U(\theta) \leq 2U_R/(1-\gamma)$. By choosing $N = k - 1$ and dividing both sides of (A.36) by $k$, we obtain

$$\frac{1}{k} \sum_{m=1}^{k} \mathbb{E}[\|\nabla J(\theta_m)\|^2] \leq \frac{1}{k\alpha} \cdot \frac{2U_R}{1-\gamma} + L\alpha\hat{\ell}^2 \leq O(\alpha L\hat{\ell}^2),$$

which completes the proof of Corollary 4.4. \qed
A.5 Proof of Lemma 5.3

Proof. A policy \( \pi \) is an optimal policy for the MDP if and only if the corresponding Q-function satisfies the Bellman equation \([57]\), namely, for any \((s, a) \in S \times A\)

\[
Q_\pi(s, a) = R(s, a) + \gamma \cdot \mathbb{E}_{s'} \left[ \max_{a' \in A} Q_\pi(s', a') \right].
\]

For any \( C \in \mathbb{R} \), by adding \( C/(1 - \gamma) \) to both sides, we obtain

\[
Q_\pi(s, a) + \frac{C}{1 - \gamma} = R(s, a) + \gamma \cdot \mathbb{E}_{s'} \left[ \max_{a' \in A} Q_\pi(s', a') + \frac{C}{1 - \gamma} \right]
= \tilde{R}(s, a) + \gamma \cdot \mathbb{E}_{s'} \left[ \max_{a' \in A} \tilde{Q}_\pi(s', a') \right],
\]

where \( \tilde{Q}_\pi(s', a') = Q_\pi(s', a') + C/(1 - \gamma) \) is the Q-function corresponding to \( \tilde{R} \) under policy \( \pi \). Since \( C \in \mathbb{R} \) can be any value, we conclude the proof for the opposite direction. \( \square \)

A.6 Proof of Lemma 5.4

Proof. First, from Theorem 3 in \([58]\), we know that the Hessian \( \mathcal{H}(\theta) \) of \( J(\theta) \) takes the form

\[
\mathcal{H}(\theta) = \mathcal{H}_1(\theta) + \mathcal{H}_2(\theta) + \mathcal{H}_{12}(\theta) + \mathcal{H}_{12}^\top(\theta),
\]

(A.38)

where the matrices \( \mathcal{H}_1, \mathcal{H}_2, \) and \( \mathcal{H}_{12} \) have the form

\[
\mathcal{H}_1(\theta) = \int_{s \in S, a \in A} \rho_\theta(s, a) \cdot Q_\pi_\theta(s, a) \cdot \nabla \log \pi_\theta(a | s) \cdot \nabla \log \pi_\theta(a | s) ^\top \, dads
\]

(A.39)

\[
\mathcal{H}_2(\theta) = \int_{s \in S, a \in A} \rho_\theta(s, a) \cdot Q_\pi_\theta(s, a) \cdot \nabla^2 \log \pi_\theta(a | s) \, dads
\]

(A.40)

\[
\mathcal{H}_{12}(\theta) = \int_{s \in S, a \in A} \rho_\theta(s, a) \cdot \nabla \log \pi_\theta(a | s) \cdot \nabla Q_\pi_\theta(s, a) ^\top \, dads,
\]

(A.41)

and \( \nabla Q_\pi_\theta(s, a) \) here is the gradient of \( Q_\pi_\theta(s, a) \) with respect to \( \theta \). Recall that \( \rho_\theta(s, a) = \rho_\pi_0(s) \cdot \pi_\theta(a | s) \) and \( \rho_\pi_0(s) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t p(s_t = s | s_0, \pi_\theta) \) is the discounted state-occupancy measure over \( S \).

\( \mathcal{H}_1 \) is the Fisher information of the policy scaled by its value in expectation with respect to the discounted state-occupancy measure over \( S \). \( \mathcal{H}_2 \) is the Hessian of the log-likelihood of the policy, i.e., the gradient of the score function, again scaled by its value in expectation with respect to the discounted state-occupancy measure over \( S \). \( \mathcal{H}_{12} \) contains a product between the score function and the derivative of the action-value function with respect to the policy scaled in expectation with respect to the discounted state-occupancy measure over \( S \).
For any $\theta$ and $(s,a)$, we define the function $f_\theta(s,a)$ as

$$
\begin{align*}
   f_\theta(s,a) := Q_{\pi_\theta}(s,a) & \cdot \nabla \log \pi_\theta(a|s) \cdot \nabla \log \pi_\theta(a|s)^T + Q_{\pi_\theta}(s,a) \cdot \nabla^2 \log \pi_\theta(a|s) \\
   & \quad + \nabla \log \pi_\theta(a|s) \cdot \nabla Q_{\pi_\theta}(s,a)^T + \nabla Q_{\pi_\theta}(s,a) \cdot \nabla \log \pi_\theta(a|s)^T. \\
\end{align*}
$$

\[ A.42 \]

For notational convenience, we separate the terms in $f_\theta$ into $f_\theta^1, f_\theta^2, \text{ and } f_\theta^3$ as defined above, which are the terms inside the integrand of $\mathcal{H}_1, \mathcal{H}_2, \text{ and } \mathcal{H}_{12} + \mathcal{H}_{12}^\top$.

Note that by definition,

$$
\mathcal{H}(\theta) = \int_{s \in \mathcal{S}, a \in \mathcal{A}} \rho_\theta(s,a) \cdot f_\theta(s,a) \text{d}a\text{d}s.
$$

Then, for any $\theta^1, \theta^2$, we obtain from \[ A.38 \]-\[ A.41 \] that

$$
\| \mathcal{H}(\theta^1) - \mathcal{H}(\theta^2) \| \leq \int \| \rho_{\theta^1}(s,a) \cdot f_{\theta^1}(s,a) - \rho_{\theta^2}(s,a) \cdot f_{\theta^2}(s,a) \| \text{d}a\text{d}s
$$

$$
\leq \int [\| \rho_{\theta^1}(s,a) - \rho_{\theta^2}(s,a) \| \cdot \| f_{\theta^1}(s,a) \| + \| \rho_{\theta^2}(s,a) \| \cdot \| f_{\theta^1}(s,a) - f_{\theta^2}(s,a) \| ] \text{d}a\text{d}s,
$$

\[ A.43 \]

where the second inequality follows from adding and subtracting $\rho_{\theta^2}(s,a) \cdot f_{\theta^1}(s,a)$, and applying the Cauchy-Schwarz inequality. Now we proceed our proof by first establishing the boundedness and Lipschitz continuity of $f_\theta(s,a)$. To this end, we need the following technical lemma.

\textbf{Lemma A.2.} For any $(s,a)$, $Q_{\pi_\theta}(s,a)$ and $\nabla Q_{\pi_\theta}(s,a)$ are both Lipschitz continuous, with constants $L_Q := U_R \cdot B_\Theta \cdot \gamma/(1-\gamma)^2$ and

$$
L_{QGrad} := U_R \cdot \left[ B_\Theta^2 \cdot \frac{\gamma(1+\gamma)}{(1-\gamma)^3} + L_\Theta \cdot \frac{\gamma}{(1-\gamma)^2} \right],
$$

respectively. Further, the norm of $\nabla Q_{\pi_\theta}(s,a)$ is also uniformly bounded by $L_Q$.

\textbf{Proof.} By the definition of $Q_{\pi_\theta}(s,a)$, we have

$$
Q_{\pi_\theta}(s,a) = \sum_{i=0}^{\infty} \int \gamma^i R(s_t, a_t) \cdot p_\theta(h_t|s_0=s,a_0=a)ds_1:t da_1:t,
$$

where $h_t = (s_0,a_0,s_1,a_1,\ldots,s_t,a_t)$ denotes the trajectory until time $t$, and $p_\theta(h_t|s,a)$ is defined as

$$
p_\theta(h_t|s_0,a_0) = \prod_{u=0}^{t-1} p(s_{u+1}|s_u,a_u) \cdot \prod_{u=1}^{t} \pi_\theta(a_u|s_u). \tag{A.44}
$$
Therefore, the gradient $\nabla Q_{\pi_\theta}(s,a)$ has the following form

$$
\nabla Q_{\pi_\theta}(s,a) = \nabla \sum_{t=0}^\infty \gamma^t R(s_t,a_t) \cdot p_\theta(h_t|s_0 = s,a_0 = a)ds_1:t \cdot da_1:t \\
= \sum_{t=1}^\infty \int \gamma^t R(s_t,a_t) \cdot p_\theta(h_t|s_0 = s,a_0 = a) \cdot \sum_{u=1}^t \nabla \log \pi_\theta(a_u|s_u)ds_1:t \cdot da_1:t, \tag{A.45}
$$

where (A.45) is due to the facts that: i) the first term in the summation $R(s_0,a_0)$ does not depend on $\theta$; ii) for any $t \geq 1$,

$$
\nabla p_\theta(h_t|s_0,a_0) = \left[ \prod_{u=0}^{t-1} p(s_{u+1}|s_u,a_u) \right] \cdot \nabla \left[ \prod_{u=1}^t \pi_\theta(a_u|s_u) \right] \\
= \left[ \prod_{u=0}^{t-1} p(s_{u+1}|s_u,a_u) \right] \cdot \sum_{\tau=1}^t \left[ \prod_{u \neq \tau, u=1}^t \pi_\theta(a_u|s_u) \right] \cdot \nabla \pi_\theta(a_\tau|s_\tau) \\
= p_\theta(h_t|s_0,a_0) \cdot \sum_{\tau=1}^t \nabla \log \pi_\theta(a_\tau|s_\tau). \tag{A.46}
$$

Hence, from (A.45) we immediately have that for any $(s,a)$ and $\theta$,

$$
\left\| \nabla Q_{\pi_\theta}(s,a) \right\| \leq \sum_{t=1}^\infty \int \gamma^t |R(s_t,a_t)| \cdot p_\theta(h_t|s_0 = s,a_0 = a) \cdot \left\| \sum_{u=1}^t \nabla \log \pi_\theta(a_u|s_u) \right\| ds_1:t \cdot da_1:t \\
\leq \sum_{t=1}^\infty \int \gamma^t \cdot U_R \cdot p_\theta(h_t|s_0 = s,a_0 = a) \cdot \sum_{u=1}^t \nabla \log \pi_\theta(a_u|s_u) \left\| ds_1:t \cdot da_1:t \right\| \tag{A.47}
$$

$$
\leq \sum_{t=1}^\infty \int \gamma^t \cdot U_R \cdot p_\theta(h_t|s_0 = s,a_0 = a) \cdot B_\Theta \cdot t \cdot ds_1:t \cdot da_1:t = U_R \cdot B_\Theta \sum_{t=1}^\infty \gamma^t \cdot t, \tag{A.48}
$$

where (A.47) and (A.48) are due to the boundedness of $|R(s,a)|$ and $\|\nabla \log \pi_\theta(a_u|s_u)\|$, respectively. Let $S = \sum_{t=1}^\infty \gamma^t \cdot t$; then

$$(1 - \gamma) \cdot S = \gamma + \sum_{t=2}^\infty \gamma^t = \frac{\gamma}{1 - \gamma} \implies S = \frac{\gamma}{(1 - \gamma)^2}. \tag{A.49}$$

Combining (A.48) and (A.49), we further establish that

$$
\left\| \nabla Q_{\pi_\theta}(s,a) \right\| \leq U_R \cdot B_\Theta \cdot \frac{\gamma}{(1 - \gamma)^2}, \tag{A.50}
$$

which proves that $\nabla Q_{\pi_\theta}(s,a)$ has norm uniformly bounded by $U_R \cdot B_\Theta \cdot \gamma/(1 - \gamma)^2$. Moreover, (A.50) also implies that $Q_{\pi_\theta}(s,a)$ is Lipschitz continuous with constant $U_R \cdot B_\Theta \cdot \gamma/(1 - \gamma)^2$. 

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Now we proceed to show the Lipschitz continuity of $\nabla Q_{\pi_\theta}(s, a)$. For any $\theta^1, \theta^2 \in \mathbb{R}^d$, we obtain from (A.45) that

$$
\left| \nabla Q_{\pi_{\theta^1}}(s, a) - \nabla Q_{\pi_{\theta^2}}(s, a) \right| \leq \sum_{t=1}^{\infty} \gamma^t R(s_t, a_t) \cdot \left| p_{\theta^1}(h_t | s_0 = s, a_0 = a) \cdot \sum_{u=1}^{t} \nabla \log \pi_{\theta^1}(a_u | s_u) - p_{\theta^2}(h_t | s_0 = s, a_0 = a) \cdot \sum_{u=1}^{t} \nabla \log \pi_{\theta^2}(a_u | s_u) \right| ds_{1:t} da_{1:t}
$$

$$
\leq \sum_{t=1}^{\infty} \gamma^t U_R \cdot \left[ \left| p_{\theta^1}(h_t | s_0 = s, a_0 = a) - p_{\theta^2}(h_t | s_0 = s, a_0 = a) \right| \left\| \sum_{u=1}^{t} \nabla \log \pi_{\theta^1}(a_u | s_u) - \nabla \log \pi_{\theta^2}(a_u | s_u) \right\| \right] ds_{1:t} da_{1:t}.
$$

(A.51)

Now we upper bound $I_1$ and $I_2$ separately as follows. By Taylor expansion of $\prod_{u=1}^{t} \pi_{\theta}(a_u | s_u)$, we have

$$
\left| \prod_{u=1}^{t} \pi_{\theta^1}(a_u | s_u) - \prod_{u=1}^{t} \pi_{\theta^2}(a_u | s_u) \right| = \left| (\theta^1 - \theta^2)^\top \left[ \sum_{m=1}^{t} \nabla \pi_\theta(a_m | s_m) \prod_{u=m+1}^{t} \pi_{\theta^1}(a_u | s_u) \right] \right|
$$

$$
\leq \| \theta^1 - \theta^2 \| \cdot \sum_{m=1}^{t} \left\| \nabla \log \pi_\theta(a_m | s_m) \right\| \prod_{u=m+1}^{t} \pi_{\theta^1}(a_u | s_u)
$$

$$
\leq \| \theta^1 - \theta^2 \| \cdot t \cdot B_\Theta \cdot \prod_{u=1}^{t} \pi_{\theta^1}(a_u | s_u),
$$

(A.52)

where $\tilde{\theta}$ is a vector lying between $\theta^1$ and $\theta^2$, i.e., there exists some $\lambda \in [0, 1]$ such that $\tilde{\theta} = \lambda \theta^1 + (1 - \lambda) \theta^2$. Therefore, (A.52), combined with (A.44), yields

$$
\left| p_{\theta^1}(h_t | s_0, a_0) - p_{\theta^2}(h_t | s_0, a_0) \right| \leq \left[ \prod_{u=0}^{t-1} p(s_{u+1} | s_u, a_u) \right] \cdot \| \theta^1 - \theta^2 \| \cdot t \cdot B_\Theta \cdot \prod_{u=1}^{t} \pi_{\theta^1}(a_u | s_u)
$$

$$
= \| \theta^1 - \theta^2 \| \cdot t \cdot B_\Theta \cdot p_{\tilde{\theta}}(h_t | s_0, a_0).
$$

(A.53)

Therefore, the term $I_1$ can be bounded as follows by substituting (A.53)

$$
I_1 \leq \| \theta^1 - \theta^2 \| \cdot t \cdot B_\Theta \cdot p_{\tilde{\theta}}(h_t | s_0, a_0)
$$

$$
\leq \| \theta^1 - \theta^2 \| \cdot t \cdot B_\Theta \cdot p_{\tilde{\theta}}(h_t | s_0, a_0) \cdot t \cdot B_\Theta.
$$

(A.54)
In addition, $I_2$ can be bounded using the $L_\Theta$-Lipschitz continuity of $\nabla \log \pi_\theta(a|s)$, i.e.,

\[
I_2 \leq p_{\theta^2}(h_t|s_0 = s, a_0 = a) \cdot \sum_{i=1}^t \left\| \nabla \log \pi_\theta^1(a_u|s_u) - \nabla \log \pi_\theta^2(a_u|s_u) \right\|
\]

\[
\leq p_{\theta^2}(h_t|s_0 = s, a_0 = a) \cdot t \cdot L_\Theta \cdot \left\| \theta^1 - \theta^2 \right\|.
\]

Substituting (A.54) and (A.55) into (A.51), we obtain that

\[
\begin{align*}
\left\| \nabla Q_{\pi_\theta}(s,a) - \nabla Q_{\pi_\theta}(s,a) \right\| & \leq \sum_{i=1}^\infty \int \gamma^t U_R \cdot \left[ \left\| \theta^1 - \theta^2 \right\| \cdot t^2 \cdot B_{\Theta}^2 \cdot p_\theta(h_t|s_0 = s, a_0 = a) \\
& \quad + p_{\theta^2}(h_t|s_0 = s, a_0 = a) \cdot t \cdot L_\Theta \cdot \left\| \theta^1 - \theta^2 \right\| \right] ds_1 t da_1 t
\end{align*}
\]

\[
= \sum_{i=1}^\infty \gamma^t U_R \cdot \left[ t^2 \cdot B_{\Theta}^2 + t \cdot L_\Theta \right] \cdot \left\| \theta^1 - \theta^2 \right\|
\]

\[
= U_R \cdot \left[ B_{\Theta}^2 \cdot \frac{\gamma(1 + \gamma)}{(1 - \gamma)^3} + L_\Theta \cdot \frac{\gamma}{(1 - \gamma)^2} \right] \cdot \left\| \theta^1 - \theta^2 \right\|,
\]

where the first equality follows from that $\int p_\theta(h_t|s_0 = s, a_0 = a) ds_1 t da_1 t = 1$ for any $\theta$, and the last equality is due to (A.49) plus the fact that

\[
\sum_{i=1}^\infty \gamma^t t^2 = \frac{1}{1 - \gamma} \sum_{i=0}^\infty (1 - \gamma)^t t^2 = \frac{1}{1 - \gamma} \cdot \mathbb{E} T^2 = \frac{1}{1 - \gamma} \cdot \frac{\gamma(1 + \gamma)}{(1 - \gamma)^2}.
\]

Note that $T$ is a random variable following geometric distribution with success probability $1 - \gamma$. Hence, (A.56) shows the uniform Lipschitz continuity of $\nabla Q_{\pi_\theta}(s,a)$ for any $(s,a)$, with the desired constant $L_{QGrad}$ claimed in the lemma. This completes the proof.

Using Lemma A.2, we can easily obtain the boundedness and Lipschitz continuity of $f_\theta(s,a)$ (cf. definition in (A.42)). In particular, to show that the norm of $f_\theta(s,a)$ is bounded, we have

\[
\left\| f_\theta(s,a) \right\| \leq \left| Q_{\pi_\theta}(s,a) \right| \cdot \left\| \nabla \log \pi_\theta(a|s) \cdot \nabla \log \pi_\theta(a|s)^T + \nabla^2 \log \pi_\theta(a|s) \right\|
\]

\[
\leq \frac{U_R}{1 - \gamma} \cdot \left[ \left\| \nabla \log \pi_\theta(a|s) \right\|^2 + \left\| \nabla^2 \log \pi_\theta(a|s) \right\| \right] + 2 \cdot \left\| \nabla \log \pi_\theta(a|s) \right\| \cdot \left\| \nabla Q_{\pi_\theta}(s,a) \right\|
\]

\[
\leq \frac{U_R}{1 - \gamma} \cdot \left( B_{\Theta}^2 + L_\Theta \right) + 2 \cdot B_{\Theta} \cdot L_Q \cdot \frac{U_R (B_{\Theta}^2 + L_\Theta)}{1 - \gamma} + \frac{2 U_R B_{\Theta}^2}{(1 - \gamma)^2} \cdot B_f, \tag{A.57}
\]

where the second inequality follows from the fact\footnote{Note that by definition, for any two vectors $a,b \in \mathbb{R}^d$, $\|ab^T\| = \sup_{\|v\|=1} \sqrt{\|a^Tb^T\|} = \sup_{\|v\|=1} \|a^Tb^T\| \cdot \|v\| \leq \|a\| \cdot \|b\|$. Specially, if $a = b$, $\|aa^T\| \leq \|a\|^2$.} that for any vector $a,b \in \mathbb{R}^d$, $\|ab^T\| \leq \|a\| \cdot \|b\|$, and $|Q_{\pi_\theta}| \leq U_R/(1 - \gamma)$. We use $B_f$ to denote the bound of the norm $\|f_\theta(s,a)\|$. 

\[36\]
To show the Lipschitz continuity of \( f_\theta(s,a) \), we need the following straightforward but useful lemma.

**Lemma A.3.** For any two functions \( f_1, f_2 : \mathbb{R}^d \rightarrow \mathbb{R}^{m \times n} \), if, for \( i = 1, 2 \), \( f_i \) has norm bounded by \( C_i \) and is \( L_i \)-Lipschitz continuous, then \( f_1 + f_2 \) is \( L_m \)-Lipschitz continuous, and \( f_1 \cdot f_2^T \) is \( \bar{L}_m \)-Lipschitz continuous, with \( L_m = \max\{C_1, C_2\} \) and \( \bar{L}_m = \max\{C_1 L_2, C_2 L_1\} \).

**Proof.** The proof is straightforward, and is thus omitted here. \( \square \)

By Lemma A.3, we immediately have that \( \nabla \log \pi_\theta(a|s) \cdot \nabla \log \pi_\theta(a|s)^T \) is \( B_\Theta L_\Theta \)-Lipschitz continuous. Also, note that the norm of \( \nabla \log \pi_\theta(a|s) \cdot \nabla \log \pi_\theta(a|s)^T \) is bounded by \( B_\Theta^2 \).

Thus, recalling the definition in (A.42), we further obtain from Lemmas A.2 and A.3 that for any \( \theta^1, \theta^2 \),

\[
\|f_{\theta^1}(s,a) - f_{\theta^2}(s,a)\| \leq \max \left\{ \frac{U_R}{1 - \gamma} \cdot B_\Theta L_\Theta, \frac{U_R B_\Theta \gamma}{(1 - \gamma)^2} \cdot B_\Theta^2 \right\} \cdot \|\theta^1 - \theta^2\|. \tag{A.58}
\]

Similarly, we establish the Lipschitz continuity of \( f_{\theta^1}(s,a) \) and \( f_{\theta^2}(s,a) \) as follows

\[
\|f_{\theta^1}(s,a) - f_{\theta^2}(s,a)\| \leq \max \left\{ \frac{U_R}{1 - \gamma} \cdot \rho_\Theta, \frac{U_R B_\Theta \gamma}{(1 - \gamma)^2} \cdot L_\Theta \right\} \cdot \|\theta^1 - \theta^2\|, \tag{A.59}
\]

\[
\|f_{\theta^1}(s,a) - f_{\theta^2}(s,a)\| \leq 2 \cdot \max \left\{ \rho_\Theta, L_{\text{QGrad}}, \frac{U_R B_\Theta \gamma}{(1 - \gamma)^2} \cdot L_\Theta \right\} \cdot \|\theta^1 - \theta^2\|, \tag{A.60}
\]

where (A.59) is due to \( |Q_{\pi_\theta}(s,a)| \) being \( U_R / 1 - \gamma \)-bounded and \( U_R \cdot B_\Theta \cdot \gamma / (1 - \gamma)^2 \)-Lipschitz, and \( \nabla^2 \log \pi_\theta(a|s) \) being \( L_\Theta \)-bounded and \( \rho_\Theta \)-Lipschitz; (A.60) is due to \( |\nabla Q_{\pi_\theta}(s,a)| \) being \( U_R \cdot B_\Theta \cdot \gamma / (1 - \gamma)^2 \)-bounded and \( L_{\text{QGrad}} \)-Lipschitz, and \( \nabla^2 \log \pi_\theta(a|s) \) being \( B_\Theta \)-bounded and \( L_\Theta \)-Lipschitz. Combining (A.58)-(A.60) and the definition in (A.42), we finally obtain the Lipschitz continuity of \( f_\theta(s,a) \) with constant \( L_f \), i.e.,

\[
\|f_{\theta^1}(s,a) - f_{\theta^2}(s,a)\| \leq \frac{U_R B_\Theta}{1 - \gamma} \cdot \max \left\{ \rho_\Theta, L_\Theta, \frac{B_\Theta^2 (1 + \gamma) + L_\Theta (1 - \gamma) \gamma}{(1 - \gamma)^2} \right\} \cdot \|\theta^1 - \theta^2\|. \tag{A.61}
\]

By substituting (A.57) and (A.61) into (A.43), we arrive at

\[
\|\mathcal{H}(\theta^1) - \mathcal{H}(\theta^2)\| \leq \int \left[ |\rho_{\theta^1}(s,a) - \rho_{\theta^2}(s,a)| \cdot \|f_{\theta^1}(s,a)\| + |\rho_{\theta^2}(s,a)| \cdot \|f_{\theta^1}(s,a) - f_{\theta^2}(s,a)\| \right] d\pi_\theta(s,a) d\rho_\theta(s,a) d\gamma \leq B_f \cdot \|\theta^1 - \theta^2\| \cdot \int |\rho_{\theta^2}(s,a)| d\pi_\theta(s,a) d\rho_\theta(s,a) d\gamma = B_f \cdot \|\theta^1 - \theta^2\|. \tag{A.62}
\]
Now it suffices to show the Lipschitz continuity of \( \int |\rho_{\theta^1}(s,a) - \rho_{\theta^2}(s,a)| \, ds \, da \). By definition, we have

\[
\rho_{\theta}(s,a) = (1 - \gamma) \cdot \sum_{t=0}^{\infty} \gamma^t p(s_t = s \mid s_0, \pi_{\theta}) \pi_{\theta}(a \mid s)
\]

\[
= (1 - \gamma) \cdot \sum_{t=0}^{\infty} \gamma^t p(s_t = s, a_t = a \mid s_0, \pi_{\theta}). \tag{A.63}
\]

Note that

\[
p(s_{t}, a_{t} \mid s_{0}, \pi_{\theta}) = \int \left[ \prod_{u=0}^{t-1} p(s_{u+1} \mid s_{u}, a_{u}) \cdot \prod_{u=0}^{t} \pi_{\theta}(a_{u} \mid s_{u}) \right] ds_{1:t-1} \, da_{0:t-1}, \tag{A.64}
\]

where we define \( p_{\theta}(h_{t} \mid s_{0}) \) similarly to \( p_{\theta}(h_{t} \mid s_{0}, a_{0}) \) in \((A.44)\). Hence, for any \( \theta^1, \theta^2 \in \mathbb{R}^{d} \), \((A.63)\) yields

\[
\int \left| \rho_{\theta^1}(s,a) - \rho_{\theta^2}(s,a) \right| \, ds \, da
\]

\[
= (1 - \gamma) \cdot \sum_{t=0}^{\infty} \gamma^t \int \left| p(s_{t} = s, a_{t} = a \mid s_{0}, \pi_{\theta^1}) - p(s_{t} = s, a_{t} = a \mid s_{0}, \pi_{\theta^2}) \right| \, ds \, da
\]

\[
\leq (1 - \gamma) \cdot \sum_{t=0}^{\infty} \gamma^t \int |p_{\theta^1}(h_{t} \mid s_{0}) - p_{\theta^2}(h_{t} \mid s_{0})| ds_{1:t-1} \, da_{0:t-1} \, ds_{t} \, da_{t}, \tag{A.65}
\]

where the first equality interchanges the sum and the integral due to the monotone convergence theorem; the inequality follows by substituting \((A.64)\) and applying the Cauchy-Schwarz inequality. Now it suffices to bound \( |p_{\theta^1}(h_{t} \mid s_{0}) - p_{\theta^2}(h_{t} \mid s_{0})| \). Then, we can apply the same argument from \((A.52)\) to \((A.53)\) that bounds \( |p_{\theta^1}(h_{t} \mid s_{0}, a_{0}) - p_{\theta^2}(h_{t} \mid s_{0}, a_{0})| \). Note that the only difference between the definitions of \( p_{\theta}(h_{t} \mid s_{0}) \) and \( p_{\theta}(h_{t} \mid s_{0}, a_{0}) \) is one additional multiplication of \( \pi_{\theta}(a_{0} \mid s_{0}) \). Thus, we will first have

\[
\left| \prod_{u=0}^{t} \pi_{\theta^1}(a_{u} \mid s_{u}) - \prod_{u=0}^{t} \pi_{\theta^2}(a_{u} \mid s_{u}) \right| \leq \left\| \theta^1 - \theta^2 \right\| \cdot (t + 1) \cdot B_{\Theta} \cdot \prod_{u=0}^{t} \pi_{\tilde{\theta}}(a_{u} \mid s_{u}),
\]

where \( \tilde{\theta} \) is some vector lying between \( \theta^1 \) and \( \theta^2 \). Then, the bound for \( |p_{\theta^1}(h_{t} \mid s_{0}) - p_{\theta^2}(h_{t} \mid s_{0})| \) has the form of

\[
|p_{\theta^1}(h_{t} \mid s_{0}) - p_{\theta^2}(h_{t} \mid s_{0})| = \left\| \theta^1 - \theta^2 \right\| \cdot (t + 1) \cdot B_{\Theta} \cdot p_{\tilde{\theta}}(h_{t} \mid s_{0}). \tag{A.66}
\]

Combining \((A.65)\) and \((A.66)\), we obtain

\[
\int \left| \rho_{\theta^1}(s,a) - \rho_{\theta^2}(s,a) \right| ds \, da \leq (1 - \gamma) \cdot \sum_{t=0}^{\infty} \gamma^t \int \left\| \theta^1 - \theta^2 \right\| \cdot (t + 1) \cdot B_{\Theta} \cdot p_{\tilde{\theta}}(h_{t} \mid s_{0}) ds_{1:t} \, da_{0:t}
\]

\[
= (1 - \gamma) \cdot \sum_{t=0}^{\infty} \gamma^t \left\| \theta^1 - \theta^2 \right\| \cdot (t + 1) \cdot B_{\Theta} \cdot \frac{1}{(1 - \gamma)^2}. \tag{A.67}
\]
By substituting (A.67) into (A.62), we finally arrive at the desired result, i.e.,

$$\|H(\theta^1) - H(\theta^2)\| \leq B_f \cdot \int |\rho_{\theta^1}(s,a) - \rho_{\theta^2}(s,a)| \, d\alpha ds + L_f \cdot \|\theta^1 - \theta^2\|$$

$$\leq B_f \cdot \|\theta^1 - \theta^2\| \cdot \frac{B_\Theta}{1 - \gamma} + L_f \cdot \|\theta^1 - \theta^2\| = \left(\frac{B_f B_\Theta}{1 - \gamma} + L_f\right) \cdot \|\theta^1 - \theta^2\|,$$

where $B_f$ and $L_f$ are as defined in (A.57) and (A.61). In sum, the Lipschitz constant $\rho$ in the lemma has the following form

$$\rho := \frac{U_R B_\Theta L_\Theta}{(1 - \gamma)^2} + \frac{U_R B_\Theta^3 (1 + \gamma)}{(1 - \gamma)^3} + \frac{U_R B_\Theta}{1 - \gamma} \cdot \max \left\{ L_\Theta, \frac{B_\Theta^2 \gamma}{1 - \gamma}, \frac{L_\Theta \gamma}{1 - \gamma}, \frac{B_\Theta^2 (1 + \gamma) + L_\Theta (1 - \gamma) \gamma}{(1 - \gamma)^2} \right\}. \tag{A.68}$$

This completes the proof. \hfill \Box

### A.7 Proof of Lemma 5.5

**Proof.** We start with the proof for $\mathbb{E}\left[|v_\theta^T \hat{V}(\theta)|^2 \middle| \theta \right]$. By definition, we have that for any $v \in \mathbb{R}^d$ and $|v| = 1$,

$$\mathbb{E}\left[|v^T \hat{V}(\theta)|^2 \middle| \theta \right] = \mathbb{E}\left[\left| \hat{Q}_{\pi_\theta}(s_T, a_T) \cdot v^T \nabla \log \pi_\theta(a_T | s_T) \right|^2 \middle| \theta \right]$$

$$= \mathbb{E}_{T'(s_T, a_T)} \left[ \mathbb{E}_{T'(s_T', a_T') \mid T'} \left[ \hat{Q}_{\pi_\theta}(s_T', a_T') \left| T' = \tau \right. \right] \right]$$

For notational simplicity, we write $\mathbb{E}_{T'(s_T', a_T') \mid T'} \left[ \hat{Q}_{\pi_\theta}(s_T', a_T') \right]$ as $\mathbb{E}_{T'(s_T', a_T') \mid T'} \left[ \hat{Q}_{\pi_\theta}(s_T', a_T') \right]$, which is the conditional expectation over the sequence $(s'_T, a'_T)$ and the random variable $T'$ given $\theta$ and $s_T, a_T$. Then note that $\mathbb{E}_{T'(s_T', a_T') \mid T'} \left[ \hat{Q}_{\pi_\theta}(s_T', a_T') \right]$ is uniformly lower-bounded for any $(s_T, a_T)$ and any $\theta$, since the reward $|R|$ is lower-bounded by $L_R > 0$. In particular, we have

$$\mathbb{E}_{T'(s_T', a_T') \mid T'} \left[ \hat{Q}_{\pi_\theta}(s_T', a_T') \right] \geq \mathbb{E}_{T'} \left( \frac{1 - \gamma^{(T' + 1)/2}}{1 - \gamma^{1/2}} \cdot L_R \right)^2 \geq L_R^2 \cdot \sum_{\tau=0}^{\infty} \gamma^{\tau/2} (1 - \gamma^{1/2}) = L_R^2 \geq 0,$$

where the first inequality holds because $R(s,a)$ is either all positive or negative for any $s,a$, and the second inequality follows from the fact that $[1 - \gamma^{(T' + 1)/2}] \cdot (1 - \gamma^{1/2})^{-1} \geq 1$ for all $T' \geq 0$. Substituting the preceding expression into the first product term on the right-hand side of (A.69) and pulling out the vector $v$ yields

$$\mathbb{E}\left[|v^T \hat{V}(\theta)|^2 \middle| \theta \right] \geq \mathbb{E}_{T'} \left( \frac{1 - \gamma^{(T' + 1)/2}}{1 - \gamma^{1/2}} \cdot L_R \right)^2 \cdot \mathbb{E}_{T'(s_T, a_T)} \left[ \nabla \log \pi_\theta(a_T | s_T)^T \cdot v \right]$$

$$\geq \mathbb{E}_{T'} \left( \frac{1 - \gamma^{(T' + 1)/2}}{1 - \gamma^{1/2}} \cdot L_R \right)^2 \cdot \mathbb{E}_{T'(s_T, a_T)} \left[ \nabla \log \pi_\theta(a_T | s_T)^T \cdot v \right] \geq L_R^2 \cdot L_I := \bar{\eta} > 0,$$  \tag{A.70}
where the second inequality follows from the fact the Fisher information matrix is assumed to be positive definite (cf. (5.1)) in Assumption 5.2. Note that (A.70) holds for any unit-norm vector \( v \), and does also for any eigenvector \( v_\theta \) (may be more than one) that corresponds to the maximum eigenvalue of \( \mathcal{H}(\theta) \). This verifies that \( \mathbb{E}\left[ |v_\theta^T \hat{\nabla} J(\theta)|^2 \mid \theta \right] \geq \hat{\gamma} \) for some \( \hat{\gamma} \) defined in (A.70).

To establish that the CNC condition holds for \( \hat{\nabla} J(\theta) \), the steps are similar to those previously followed for \( \hat{\nabla} J(\theta) \). Specifically, we start with the expected value of the square of the inner product of \( \hat{\nabla} J(\theta) \) with a unit vector \( v \). By definition of \( \hat{\nabla} J(\theta) \), we have

\[
\mathbb{E}\left[ |v^T \hat{\nabla} J(\theta)|^2 \mid \theta \right] = \mathbb{E}\left[ |\hat{Q}_{\pi_\theta}(s_T, a_T) - \hat{V}_{\pi_\theta}(s_T)|^2 \cdot |v^T \nabla \log \pi_\theta(a_T \mid s_T)|^2 \mid \theta \right]
\]

where for notational simplicity we also write \( \mathbb{E}_{T', (s_{1:T'}, a_{1:T'})} \left[ |\hat{Q}_{\pi_\theta}(s_T, a_T) - \hat{V}_{\pi_\theta}(s_T)|^2 \mid \theta, s_T, a_T \right] \) as \( \mathbb{E}_{T', (s_{1:T'}, a_{1:T'})} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) - \hat{V}_{\pi_\theta}(s_T) \right]^2 \). We claim that \( \mathbb{E}_{T', (s_{1:T'}, a_{1:T'})} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) - \hat{V}_{\pi_\theta}(s_T) \right]^2 \) can also be uniformly lower-bounded. Specifically, we have

\[
\mathbb{E}_{T', (s_{1:T'}, a_{1:T'})} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) - \hat{V}_{\pi_\theta}(s_T) \right]^2 \geq \left\{ \mathbb{E}_{T', (s_{1:T'}, a_{1:T'})} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) - \hat{V}_{\pi_\theta}(s_T) \right]^2 + \text{Var} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) - \hat{V}_{\pi_\theta}(s_T) \right] \right\} \]

\[
\mathbb{E}_{T', (s_{1:T'}, a_{1:T'})} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) - \hat{V}_{\pi_\theta}(s_T) \right]^2 = \mathbb{E}_{T', (s_{1:T'}, a_{1:T'})} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) - \hat{V}_{\pi_\theta}(s_T) \right]^2 + \text{Var} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) \right] + \text{Var} \left[ \hat{V}_{\pi_\theta}(s_T) \right].
\]

where the first equality is due to \( \mathbb{E}X^2 = (\mathbb{E}X)^2 + \text{Var}(X) \), and the second one follows from the fact that \( \hat{Q}_{\pi_\theta}(s_T, a_T) \) and \( \hat{V}_{\pi_\theta}(s_T) \) are independent and unbiased estimates of \( Q_{\pi_\theta}(s_T, a_T) \) and \( V_{\pi_\theta}(s_T) \), respectively. Note that the first term in (A.72) may be zero, for example, when \( \pi_\theta \) is a degenerated policy such that when \( \pi_\theta(a \mid s_T) = 1_{a=a_T} \). Hence, a uniform lower-bound on the two variance terms in (A.72) need to be established. By definition of \( \text{Var}[\hat{Q}_{\pi_\theta}(s_T, a_T)] \), we have

\[
\text{Var}[\hat{Q}_{\pi_\theta}(s_T, a_T)] = \mathbb{E}_{T', (s_{1:T'}, a_{1:T'})} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) - Q_{\pi_\theta}(s_T, a_T) \right]^2 = \mathbb{E}_{T} \left( \mathbb{E}_{(s_{1:T'}, a_{1:T'})} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) - Q_{\pi_\theta}(s_T, a_T) \right]^2 \mid T' = \tau \right). \]

Given \( (s_T, a_T), \theta, \) and \( T' = \tau \), the conditional expectation in (A.73) can be expanded as

\[
\mathbb{E}_{(s_{1:T'}, a_{1:T'})} \left[ \hat{Q}_{\pi_\theta}(s_T, a_T) - Q_{\pi_\theta}(s_T, a_T) \right]^2 \mid T' = \tau \]

\[
= \mathbb{E}_{(s_{1:T'}, a_{1:T'})} \left( \sum_{t=0}^{T'} \gamma^{t/2} \cdot R(s_t, a_t) - Q_{\pi_\theta}(s_T, a_T) \right)^2 \mid T' = \tau \}
\]

Now we first focus on the case when \( R(s, a) \) are strictly positive, i.e., \( R(s, a) \in [L_R, U_R] \). In this case, \( Q_{\pi_\theta}(s_T, a_T) \) is a scalar that lies in the bounded interval between \( [L_R/(1-\gamma), U_R/(1-\gamma)] \). Also, notice that \( \mathbb{E}_{(s_{1:T'}, a_{1:T'})} \left[ \sum_{t=0}^{T'} \gamma^{t/2} \cdot R(s_t, a_t) \right] \) is a strictly increasing function of \( T' \)
since $R(s,a) \geq L_R > 0$ for any $(s,a)$. Moreover, notice that given $(s_T,a_T)$, $\mathbb{E}_{T'}(s_{1:T'}, a_{1:T'})[\sum_{t=0}^{T'} \gamma^{t/2} \cdot R(s_t,a_t)]$ is an unbiased estimate of $Q_{\pi_0}(s_T,a_T)$, and $T'$ follows the geometric distribution over non-negative support. Thus, there must exist a finite $T_\varepsilon \geq 0$, such that

$$\mathbb{E}_{(s_{1:T}, a_{1:T})}\left[\sum_{t=0}^{T_\varepsilon} \gamma^{t/2} \cdot R(s_t,a_t) \right] < Q_{\pi_0}(s_T,a_T) \leq \mathbb{E}_{(s_{1:T+1}, a_{1:T+1})}\left[\sum_{t=0}^{T_\varepsilon+1} \gamma^{t/2} \cdot R(s_t,a_t) \right]. \quad (A.74)$$

As a result, we can substitute (A.74) into the right-hand side of (A.73), yielding

$$\text{Var}\left[\hat{Q}_{\pi_0}(s_T,a_T)\right] = \sum_{\tau=0}^{\infty} \gamma^{\tau/2} (1 - \gamma^{1/2}) \cdot \mathbb{E}_{(s_{1:T_a}, a_{1:T_a})}\left[\sum_{t=0}^{\tau} \gamma^{t/2} \cdot R(s_t,a_t) - Q_{\pi_0}(s_T,a_T)\right]^2 \geq \sum_{\tau=0}^{\infty} \gamma^{\tau/2} (1 - \gamma^{1/2}) \cdot \left\{ \mathbb{E}_{(s_{1:T_a}, a_{1:T_a})}\left[\sum_{t=0}^{\tau} \gamma^{t/2} \cdot R(s_t,a_t) - Q_{\pi_0}(s_T,a_T)\right]\right\}^2 \geq \sum_{\tau=0}^{T_\varepsilon} \gamma^{\tau/2} (1 - \gamma^{1/2}) \cdot \left[ L_R \cdot \sum_{t=\tau+1}^{T_\varepsilon} \gamma^{t/2} \right]^2 + \sum_{\tau=T_\varepsilon+2}^{\infty} \gamma^{\tau/2} (1 - \gamma^{1/2}) \cdot \left[ L_R \cdot \sum_{t=\tau+1}^{\infty} \gamma^{t/2} \right]^2, \quad (A.75)$$

where the first inequality (A.75) uses $\mathbb{E}X^2 \geq (\mathbb{E}X)^2$, and the second inequality (A.76) follows by removing the term with $\tau = T_\varepsilon$ and $\tau = T_\varepsilon + 1$ in the summation in (A.75) that sandwiched $Q_{\pi_0}(s_T,a_T)$, and noticing the fact that the term $\mathbb{E}_{(s_{1:T_a}, a_{1:T_a})}\left[\sum_{t=0}^{\tau} \gamma^{t/2} \cdot R(s_t,a_t)\right]$ is at least $L_R \cdot \sum_{t=\tau+1}^{T_\varepsilon} \gamma^{t/2}$ away from $Q_{\pi_0}(s_T,a_T)$ when $\tau \geq T_\varepsilon + 2$, and at least $L_R \cdot \sum_{t=\tau+1}^{T_\varepsilon} \gamma^{t/2}$ away from $Q_{\pi_0}(s_T,a_T)$ when $\tau \leq T_\varepsilon$. Furthermore, multiplying the first term in (A.76) by $\gamma^{3/2}$ yields

$$\text{Var}\left[\hat{Q}_{\pi_0}(s_T,a_T)\right] \geq \gamma^{3/2} \cdot \sum_{\tau=0}^{T_\varepsilon} \gamma^{\tau/2} (1 - \gamma^{1/2}) \cdot \left[ L_R \cdot \frac{\gamma^{(\tau+1)/2} - \gamma^{(T_\varepsilon+1)/2}}{1 - \gamma^{1/2}} \right]^2 + \sum_{\tau=T_\varepsilon+2}^{\infty} \gamma^{\tau/2} (1 - \gamma^{1/2}) \cdot \left[ L_R \cdot \frac{\gamma^{(\tau+1)/2} - \gamma^{(T_\varepsilon+2)/2}}{1 - \gamma^{1/2}} \right]^2, \quad (A.77)$$

where the first inequality follows from the fact that $\gamma^{3/2} < 1$, and the equality is obtained by changing the starting point of the summation of the second term to $T_\varepsilon + 1$, and then

---

9Note that we define $\sum_{t=\tau+1}^{T_\varepsilon} \gamma^{t/2} = 0$ if $\tau + 1 < T_\varepsilon$. 

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pulling out \( \gamma^{1/2} \) from the square bracket. This way, we can further bound (A.77) as

\[
\Var[\hat{Q}_{\pi_\theta}(s_T, a_T)] \geq \gamma^{3/2} \cdot L_R^2 \cdot \Var[T'] \left\{ \frac{\gamma^{(T'+1)/2} - \gamma^{(T'+1)/2}}{1 - \gamma^{1/2}} \right\}^2 \\
\geq \gamma^{3/2} \cdot L_R^2 \cdot \Var[T'] \left\{ \frac{\gamma^{(T'+1)/2} - \gamma^{(T'+1)/2}}{1 - \gamma^{1/2}} \right\} = \gamma^{3/2} \cdot L_R^2 \cdot \Var[T'] \left\{ \frac{\gamma^{(T'+1)/2}}{1 - \gamma^{1/2}} \right\},
\]

(A.78)

where the first inequality follows by expressing the right-hand side of (A.77) as an expectation over \( T' \), the second inequality follows from \( \Var(X) \geq \Var(X) \), and the last equality is due to the fact that \( T' \) is deterministic and thus does not affect the variance given \((s_T, a_T)\) and \( \theta \). Note that \( \Var[\gamma^{(T'+1)/2}] \) can be uniformly bounded as

\[
\Var[\gamma^{(T'+1)/2}] = \mathbb{E}[\gamma^{(T'+1)/2} - \{\mathbb{E}[\gamma^{(T'+1)/2}]\}^2 = \gamma^{(1 - \gamma^{1/2})/2} - \left[ \frac{\gamma^{1/2}(1 - \gamma^{1/2})}{1 - \gamma} \right]^2 \\
= \frac{\gamma^{3/2} \cdot (1 - \gamma^{1/2})^3}{(1 - \gamma^{3/2}) \cdot (1 - \gamma)^2} > 0.
\]

(A.79)

Combining (A.78) and (A.79), we obtain

\[
\Var[\hat{Q}_{\pi_\theta}(s_T, a_T)] \geq \frac{\gamma^{3/2} \cdot L_R^2}{(1 - \gamma^{1/2})^2 \cdot (1 - \gamma^{3/2}) \cdot (1 - \gamma)^2} \cdot \frac{\gamma^{3/2} \cdot (1 - \gamma^{1/2})}{(1 - \gamma^{3/2}) \cdot (1 - \gamma)^2} = \frac{L_R^2 \cdot \gamma^3 \cdot (1 - \gamma^{1/2})}{(1 - \gamma^{3/2}) \cdot (1 - \gamma)^2}.
\]

(A.80)

By the same arguments as above, we can also obtain that

\[
\Var[\hat{V}_{\pi_\theta}(s_T)] \geq \frac{L_R^2 \cdot \gamma^3 \cdot (1 - \gamma^{1/2})}{(1 - \gamma^{3/2}) \cdot (1 - \gamma)^2}.
\]

(A.81)

Substituting (A.80) and (A.84) into (A.72), we arrive at

\[
\mathbb{E}_{T'\sim(s_1:T,a_1:T)}[ \hat{Q}_{\pi_\theta}(s_T, a_T) - \hat{V}_{\pi_\theta}(s_T)]^2 \geq \frac{2L_R^2 \cdot \gamma^3 \cdot (1 - \gamma^{1/2})}{(1 - \gamma^{3/2}) \cdot (1 - \gamma)^2}.
\]

(A.82)

Finally by combining (A.82) and (A.71), we conclude that

\[
\mathbb{E}\left\{ [v^T \hat{V}(\theta)]^2 \left| \theta \right. \right\} \geq \frac{2L_R^2 \cdot \gamma^3 \cdot (1 - \gamma^{1/2})}{(1 - \gamma^{3/2}) \cdot (1 - \gamma)^2} \cdot L_I =: \bar{\eta} > 0.
\]

The proof for the case when \( R(s,a) \in [-U_R, -L_R] \) is as the one above, with only some minor modifications due to sign flipping. For example, \( \mathbb{E}_{(s_1:T',a_1:T')}[\sum_{t=0}^{T'} \gamma^{t/2} \cdot R(s_t, a_t)] \) now becomes a strictly decreasing function of \( T' \) since \( R(s,a) \leq -L_R < 0 \). The remaining arguments are similar, and are omitted here to avoid repetition.

The proof of \( \mathbb{E}\left\{ [v^T \hat{V}(\theta)]^2 \left| \theta \right. \right\} \geq \bar{\eta} \) for some \( \bar{\eta} > 0 \) is very similar to the proofs above. First, we have by definition that

\[
\mathbb{E}\left\{ [v^T \hat{V}(\theta)]^2 \left| \theta \right. \right\} = \mathbb{E}\left\{ [v^T \hat{V}(\theta)]^2 \left| \theta \right. \right\} \mathbb{E}\left\{ [v^T \hat{V}(\theta)]^2 \left| \theta \right. \right\} = \mathbb{E}_{T'\sim(s_1:T',a_1:T')}[\hat{V}(s_T, a_T) - \hat{V}(s_T)]^2
\]

(A.83)
where we use $T'$ and $T''$ to represent the random horizon used in calculating $\hat{\nabla} \pi_0(s_T')$ and $\hat{\nabla} \pi_0(s_T)$, respectively, and recall that $s_T'$ is sampled from $\mathbb{P}(\cdot|s_T,a_T)$. Note that given $(s_T,a_T)$, we have

$$
\mathbb{E}_{s_T',T',T'',(s_1:T',a_1:T'),(s_1:T'',a_1:T'')}[ R(s_T,a_T) + \gamma \cdot \hat{\nabla} \pi_0(s_T') - \hat{\nabla} \pi_0(s_T) ]
$$

where $(A.80)$, we can lower-bound $(A.85)$ and thus further bound $(A.83)$ by

$$
\mathbb{E}_{s_T',T',T'',(s_1:T',a_1:T'),(s_1:T'',a_1:T'')}[ R(s_T,a_T) + \gamma \cdot \hat{\nabla} \pi_0(s_T') - \hat{\nabla} \pi_0(s_T) ]
$$

where $(A.84)$ and $(A.85)$ are due to the independence and unbiasedness of the estimates $\hat{\nabla} \pi_0(s_T')$ and $\hat{\nabla} \pi_0(s_T)$, respectively. Then, since the variance of $\hat{\nabla} \pi_0(s_T)$ has been lower-bounded by $(A.80)$, we can lower-bound $(A.85)$ and thus further bound $(A.83)$ by

$$
\mathbb{E}_{s_T')[v^T \hat{\nabla} \pi_0(s_T)] = \frac{(1 + \gamma^2) \cdot L_R^2 \cdot \gamma^3 \cdot (1 - \gamma^{1/2})}{(1 - \gamma^{3/2}) \cdot (1 - \gamma)^2} \cdot L_I =: \tilde{\eta} > 0,
$$

which completes the proof. \qed

### A.8 Proof of Theorem 5.6

**Proof.** We first note that we have listed the parameters to be used in our analysis below in Table 1 in the main body of the paper, which will be referred to in this section.

Now recall that in Algorithm 4, we use $g_0$ to unify the notation of the three stochastic policy gradients $\hat{\nabla} J(\theta)$, $\hat{\nabla} \pi_0(s_T)$, and $\hat{\nabla} \pi_0(s_T')$ (see the definitions in (3.8)-(3.10)). From Theorem 3.4, we know that all the three stochastic policy gradients are unbiased estimates of $\nabla J(\theta)$. Moreover, we have shown that all the three stochastic policy gradients have their norms bounded by some constants $\ell, \ell, \ell$ and $\ell > 0$, respectively, which are defined in Theorem 3.4.

To unify the notation in the ensuing analysis, we use a common $\ell$ to denote the bound of $g_0$, which takes the value of either $\ell, \ell, \ell$, or $\ell$, depending on which policy gradient is used. Also, as illustrated in Lemma 5.5, all the three stochastic policy gradients satisfy the correlated negative curvature condition. We thus use a common $\eta$ to represent the value of $\eta, \eta, \eta$, and $\eta$ correspondingly. Therefore, we have

$$
\| v_0 \| \leq \ell, \quad \text{and} \quad \mathbb{E}[v_0^T g_0] \geq \eta, \quad \text{for any} \ \theta,
$$

where $v_0$ is the unit-norm eigenvector corresponding to the maximum eigenvalue of the Hessian at $\theta$. In addition, recall from Lemmas 3.2 and 5.4 that $J(\theta)$ is both $L$-gradient Lipschitz and $\rho$-Hessian Lipschitz, i.e., there exist constants $L$ and $\rho$ (see the definitions in the corresponding lemmas), such that for any $\theta^1, \theta^2 \in \mathbb{R}^d$,

$$
\| \nabla J(\theta^1) - \nabla J(\theta^2) \| \leq L \cdot \| \theta^1 - \theta^2 \|, \quad \| H(\theta^1) - H(\theta^2) \| \leq \rho \cdot \| \theta^1 - \theta^2 \|.
$$

Our analysis is separated into three steps that characterize the convergence properties of the iterates in three different regimes, depending on the magnitude of the gradient and
the curvature of the Hessian. This type of analysis for convergence to approximate second-order stationary points in nonconvex optimization originated from [36], where isotropic noise is added to the update to escape the saddle points. Here we do not assume that the stochastic policy gradient has isotropic noise, since: 1) in RL the noise results from the sampling along the trajectory of the MDP, which do not necessarily satisfy the isotropic property in general; 2) the noise of policy gradients is notoriously known to be large, thus adding artificial noise may further degrade the performance of the RPG algorithm. An effort to improve the limit points of first-order methods for nonconvex optimization, while avoiding adding artificial noise, has appeared recently in [30]. However, we have identified that the proof in [30] is flawed and cannot be applied directly for the convergence of the RPG algorithms here. Thus, part of our contribution here is to provide a precise fix in its own right, as well as map it to the analysis of policy gradient methods in RL.

Note that Algorithm 5 returns the iterates that have indices \( k \) such that \( k \mod k_{\text{thre}} = 0 \), i.e., the iterates belong to the set \( \hat{\Theta}^* \). For notational convenience, we index the iterates in \( \hat{\Theta}^* \) by \( m \), i.e., let \( \tilde{\theta}_m = \theta_{m,k_{\text{thre}}} \) for all \( m = 0, 1, \ldots, \lfloor K/k_{\text{thre}} \rfloor \). Now we consider the three regimes of the iterates \( \{\tilde{\theta}_m\}_{m \geq 0} \).

**Regime 1: Large gradient**

We first introduce the following standard lemma that quantifies the increase of function values, when stochastic gradient ascent of a smooth function is adopted.

**Lemma A.4.** Let \( \theta_{k+1} \) be obtained by one stochastic gradient ascent step at \( \theta_k \), i.e., \( \theta_{k+1} = \theta_k + \alpha g_k \), where \( g_k = g_{\theta_k} \) is an unbiased stochastic gradient at \( \theta_k \). Then, for any given \( \theta_k \), the function value \( J(\theta_{k+1}) \) increases in expectation as

\[
\mathbb{E}[J(\theta_{k+1})] - J(\theta_k) \geq \alpha \frac{\|\nabla J(\theta_k)\|^2 - L\alpha^2\ell^2}{2}.
\]

**Proof.** By the \( L \)-smoothness of \( J(\theta) \), we have

\[
\mathbb{E}[J(\theta_{k+1})] - J(\theta_k) \geq \alpha \nabla J(\theta_k)^T \mathbb{E}(g_k | \theta_k) - \frac{L\alpha^2}{2} \|g_k\|^2 = \alpha \|\nabla J(\theta_k)\|^2 - \frac{L\alpha^2}{2} \|g_k\|^2,
\]

which completes the proof by using the fact that \( \|g_k\|^2 \leq \ell^2 \) almost surely. \qed

Therefore, when the norm of the gradient is large at \( \tilde{\theta}_m \), a large increase of \( J(\tilde{\theta}) \) from \( \tilde{\theta}_m \) to \( \tilde{\theta}_{m+1} \) is guaranteed, as formally stated in the following lemma.

**Lemma A.5.** Suppose the gradient norm at any given \( \tilde{\theta}_m \) is large such that \( \|\nabla J(\tilde{\theta}_m)\| \geq \epsilon \), for some \( \epsilon > 0 \). Then, the expected value of \( J(\tilde{\theta}_{m+1}) \) increases as

\[
\mathbb{E}[J(\tilde{\theta}_{m+1})] - J(\tilde{\theta}_m) \geq J_{\text{thre}},
\]

where the expectation is taken over the sequence from \( \theta_{m,k_{\text{thre}}+1} \) to \( \theta_{(m+1)-k_{\text{thre}}} \).

---

\(^{10}\)Note that the expectation here is taken over the randomness of \( g_k \).
The choice of \( J \) where the expectation is taken over the sequence from \( \theta \) which yields a lower-bound on the right-hand side of (A.88) as
\[
\mathbb{E}[J(\tilde{\theta}_{m+1})] - J(\tilde{\theta}_m) = \sum_{p=0}^{k_{thre}-1} \mathbb{E}[J(\theta_{m:k_{thre}+p}+1)] - \mathbb{E}[J(\theta_{m:k_{thre}+p})]
\]
\[
= \sum_{p=0}^{k_{thre}-1} \left\{ \mathbb{E}[J(\theta_{m:k_{thre}+p}+1)] - \mathbb{E}[J(\theta_{m:k_{thre}+p})] \right\} \theta_{m:k_{thre}+p},
\]
where \( \mathbb{E}[J(\theta_{m:k_{thre}})|\theta_{m:k_{thre}}] = J(\theta_{m:k_{thre}}) = J(\tilde{\theta}_m) \) for given \( \tilde{\theta}_m \). By Lemma A.4, we further have
\[
\mathbb{E}[J(\tilde{\theta}_{m+1})] - J(\tilde{\theta}_m) \geq \beta \|\nabla J(\theta_{m:k_{thre}})\|^2 - \frac{L\beta^2 \ell^2}{2} + \sum_{p=1}^{k_{thre}-1} \alpha \mathbb{E}[\|\nabla J(\theta_{m:k_{thre}+p})\|^2] - \frac{k_{thre} \alpha^2 \ell^2}{2}
\]
\[
\geq \beta \|\nabla J(\theta_{m:k_{thre}})\|^2 - \frac{L\beta^2 \ell^2}{2} - \frac{k_{thre} \alpha^2 \ell^2}{2} \geq \beta \|\nabla J(\theta_{m:k_{thre}})\|^2 - L\beta^2 \ell^2,
\] (A.88)
where the last inequality follows from Table I that
\[
\beta^2 \geq k_{thre} \cdot \alpha^2. \tag{A.89}
\]
Moreover, by the choice of the large stepsizes \( \beta \), we have
\[
\|\nabla J(\theta_{m:k_{thre}})\|^2 = \|\nabla J(\tilde{\theta}_m)\|^2 \geq \epsilon^2 \geq 2\ell^2 L\beta, \tag{A.90}
\]
which yields a lower-bound on the right-hand side of (A.88) as
\[
\mathbb{E}[J(\tilde{\theta}_{m+1})] - J(\tilde{\theta}_m) \geq \beta \|\nabla J(\theta_{m:k_{thre}})\|^2 - L\beta^2 \ell^2 \geq \beta \|\nabla J(\theta_{m:k_{thre}})\|^2/2 \geq \beta \epsilon^2/2 \geq J_{thre}. \tag{A.91}
\]
The choice of \( J_{thre} = \beta \epsilon^2/2 \) completes the proof.

**Regime 2: Near saddle points**

When the iterate reaches the neighborhood of saddle points, our modified RPG will use a larger stepsize \( \beta \) to find the positive eigenvalue direction, and then uses small stepsizes \( \alpha \) to follow this positive curvature direction. We establish in the following lemma that such an updating strategy also leads to a sufficient increase of function value, provided that the maximum eigenvalue of the Hessian \( \mathcal{H}(\tilde{\theta}_m) \) is large enough. This enables the iterate to escape the saddle points efficiently.

**Lemma A.6.** Suppose that the Hessian at any given \( \tilde{\theta}_m \) has a large positive eigenvalue such that \( \lambda_{\max}[\mathcal{H}(\tilde{\theta}_m)] \geq \sqrt{\ell} \epsilon \). Then, after \( k_{thre} \) steps we have
\[
\mathbb{E}[J(\tilde{\theta}_{m+1})] - J(\tilde{\theta}_m) \geq J_{thre},
\]
where the expectation is taken over the sequence from \( \theta_{m:k_{thre}+1} \) to \( \theta_{(m+1):k_{thre}} \).
Lemma A.6 asserts that after $k_{\text{thre}}$ steps, the expected function value increases by at least $J_{\text{thre}}$. Together with Lemma A.5, it can be shown that the expected return $\mathbb{E}[J(\tilde{\theta}_{m+1})]$ is always increasing, as long as the iterate $\tilde{\theta}_m$ violates the approximate second-order stationary point condition, i.e., $\|\nabla J(\tilde{\theta}_m)\| \geq \varepsilon$ or $\lambda_{\text{max}}[\mathcal{H}(\tilde{\theta}_m)] \geq \sqrt{\rho \varepsilon}$. The proof of Lemma A.6 is deferred to §A.9 to maintain the flow here.

**Regime 3: Near second-order stationary points**

When the iterate converges to the neighborhood of the desired second-order stationary points, both the norm of the gradient and the largest eigenvalue are small. However, due to the variance of the stochastic policy gradient, the function value may still decrease. By Lemma A.4 and (A.91), we can immediately show that such a decrease is bounded, i.e.,

$$\mathbb{E}[J(\tilde{\theta}_{m+1})] - J(\tilde{\theta}_m) \geq -L\beta^2 \ell^2 \geq -\delta J_{\text{thre}}/2,$$  

(A.92)

which is due to the choice of $J_{\text{thre}} \geq 2L(\ell\beta)^2/\delta$ as in Table 1.

Now we combine the arguments above to obtain a probabilistic lower-bound on the returned approximate second-order stationary point. Let $\mathcal{E}_m$ be the event that

$$\mathcal{E}_m := \{\|\nabla J(\tilde{\theta}_m)\| \geq \varepsilon \text{ or } \lambda_{\text{max}}[\mathcal{H}(\tilde{\theta}_m)] \geq \sqrt{\rho \varepsilon}\}.$$  

By Lemmas A.5 and A.6, we have

$$\mathbb{E}[J(\tilde{\theta}_{m+1})] - J(\tilde{\theta}_m) \geq J_{\text{thre}},$$  

(A.93)

where the expectation is taken over the randomness of both $\tilde{\theta}_{m+1}$ and $\tilde{\theta}_m$ given the event $\mathcal{E}_m$. Namely, after $k_{\text{thre}}$ steps, as long as $\tilde{\theta}_m$ is not an $(\varepsilon, \sqrt{\rho \varepsilon})$-approximate second-order stationary point, a sufficient increase of $\mathbb{E}[J(\tilde{\theta}_{m+1})]$ is guaranteed. Otherwise, we can still control the possible decrease of the return using (A.92), which yields

$$\mathbb{E}[J(\tilde{\theta}_{m+1})] - J(\tilde{\theta}_m) |_{\mathcal{E}_m^c} \geq -\delta J_{\text{thre}}/2,$$  

(A.94)

where $\mathcal{E}_m^c$ is the complement event of $\mathcal{E}_m$.

Let $P_m$ denote the probability of the occurrence of the event $\mathcal{E}_m$. Thus, the total expectation $\mathbb{E}[J(\tilde{\theta}_{m+1}) - J(\tilde{\theta}_m)]$ can be obtained by combining (A.93) and (A.94) as follows

$$\mathbb{E}[J(\tilde{\theta}_{m+1}) - J(\tilde{\theta}_m)] \geq (1 - P_m) \cdot \left(-\frac{\delta J_{\text{thre}}}{2}\right) + P_m \cdot J_{\text{thre}}.$$  

(A.95)

Suppose the iterate of $\theta_k$ runs for $K$ steps starting from $\theta_0$; then there are $M = \lfloor K/k_{\text{thre}} \rfloor$ of $\tilde{\theta}_m$ for $k = 1, \ldots, K$. Summing up all the $M$ steps of $\{\tilde{\theta}_m\}_{m=1, \ldots, M}$, we obtain from (A.95) that

$$\frac{1}{M} \sum_{m=1}^{M} P_m \leq \frac{J^* - J(\theta_0)}{MJ_{\text{thre}}} + \frac{\delta}{2} \leq \delta,$$

where $J^*$ is the global maximum of $J(\theta)$, and the last inequality follows from the choice of $K$ in Table 1 that satisfies

$$K \geq 2[J^* - J(\theta_0)]k_{\text{thre}}/(\delta J_{\text{thre}}).$$  

(A.96)
Therefore, the probability of the event $\mathcal{E}_m^c$ occurs, i.e., the probability of retrieving an $(\epsilon, \sqrt{\rho} \epsilon)$ approximate second-order stationary point uniformly over the iterates in $\hat{\Theta}^*$, can be lower-bounded by

$$1 - \frac{1}{M} \sum_{m=1}^{M} P_m \geq 1 - \delta.$$ 

This completes the proof.

\[ \square \]

### A.9 Proof of Lemma A.6

**Proof.** The proof is based on the improve or localize framework proposed in [59]. The basic idea is as follows: starting from some iterate, if the following iterates of stochastic gradient update do not improve the objective value to a great degree, then the iterates must not move much from the starting iterate. Our goal here is to show that after $k_{\text{thre}}$ steps, the objective value will increase by at least $J_{\text{thre}}$. In particular, the proof proceeds by contradiction: suppose the objective value does not increase by $J_{\text{thre}}$ from $\tilde{\theta}_m$ to $\tilde{\theta}_{m+1}$, then the distance between the two iterates can be upper-bounded by a polynomial function of the number of iterates in between, i.e., $k_{\text{thre}}$. On the other hand, due to the CNC condition (cf. Lemma 5.5), the distance between $\tilde{\theta}_m$ and $\tilde{\theta}_{m+1}$ can be shown to be lower-bounded by an exponential function of $k_{\text{thre}}$. This way, by choosing large enough $k_{\text{thre}}$ following Table 1, the lower-bound exceeds the upper-bound, which causes a contradiction and justifies our argument.

First, for notational convenience, we suppose $m = k = 0$ without loss of generality, and denote

$$J_p = J(\theta_p), \quad \nabla J_p = \nabla J(\theta_p), \quad \mathcal{H}_p = \nabla^2 J(\theta_p),$$

for any $p = 0, \cdots, k_{\text{thre}} - 1$. Suppose that starting from $\tilde{\theta}_0$, after 1 step iteration with large stepsize $\beta$ and $k_{\text{thre}} - 1$ steps with small stepsize $\alpha$, the expected return value does not increase by more than $J_{\text{thre}}$, i.e.,

$$\mathbb{E}(J_{\text{thre}}) - J_0 \leq J_{\text{thre}}.$$  \hspace{1cm} (A.98)

Then, for any $0 \leq p \leq k_{\text{thre}}$, we can establish that the expectation of the distance from $\theta_p$ to $\theta_0$ is upper-bounded, as formally stated in the following lemma.

**Lemma A.7.** Given any $\theta_0$, suppose (A.98) holds, for any $0 \leq p \leq k_{\text{thre}}$. Then, the expected distance between $\theta_p$ and $\theta_0$ can be upper-bounded as

$$\mathbb{E}\|\theta_p - \theta_0\|^2 \leq \left[4\alpha^2 \ell_g^2 + 4\alpha J_{\text{thre}} + 2L\alpha(\ell B)^2 + 2L\ell^2 \alpha^3 k_{\text{thre}}\right] \cdot p + 2\beta^2 \ell^2,$$

where $\ell_g^2 := 2\ell^2 + 2B_\Theta^2 U_k \cdot (1 - \gamma)^{-4}$.  \hspace{1cm} (A.99)
**Proof.** We have obtained from Lemma [A.4] and (A.88) (with \(m = 0\)) that

\[
\mathbb{E}(J_{thre}) - J_0 \geq \beta \|\nabla J_0\|^2 - \frac{L\beta^2\ell^2}{2} + \frac{\sum_{q=1}^{k_{thre}-1} \cdot \alpha \mathbb{E}\|\nabla J_q\|^2 - \frac{k_{thre}L^2\alpha^2\ell^2}{2}}{2}.
\]

since \(0 \leq \alpha < \beta\) and \(0 \leq p \leq k_{thre}\), where we note that the total expectation is taken along the sequence from \(\theta_1\) to \(\theta_{k_{thre}}\), and we write \(\|\nabla J_0\|^2 = \mathbb{E}\|\nabla J_0\|^2\) since \(\theta_0\) is given and deterministic. Combined with (A.98), we have

\[
J_{thre} \geq \alpha \sum_{q=0}^{p-1} \mathbb{E}\|\nabla J_q\|^2 - \frac{k_{thre}L^2\alpha^2\ell^2}{2} - \frac{L\beta^2\ell^2}{2},
\]

which implies that

\[
\sum_{q=0}^{p-1} \mathbb{E}\|\nabla J_q\|^2 \leq \frac{J_{thre}}{\alpha} + \frac{k_{thre}L^2\alpha^2\ell^2}{2} + \frac{L\beta^2\ell^2}{2\alpha}. \tag{A.100}
\]

Now, let us consider the distance between \(\theta_p\) and \(\theta_0\) that can be decomposed as follows:

\[
\mathbb{E}\|\theta_p - \theta_0\|^2 = \mathbb{E} \left\| \sum_{q=0}^{p-1} \theta_{q+1} - \theta_q \right\|^2 \leq 2\alpha^2\mathbb{E} \left\| \sum_{q=1}^{p-1} g_q \right\|^2 + 2\beta^2\mathbb{E}\|g_0\|^2, \tag{A.101}
\]

where the first equality comes from the telescopic property of the summand and the later inequality comes from \(\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2\).

For the first term on the right-hand side of (A.101), we have

\[
2\alpha^2\mathbb{E} \left\| \sum_{q=1}^{p-1} g_q \right\|^2 = 2\alpha^2\mathbb{E} \left\| \sum_{q=1}^{p-1} g_q - \nabla J_q + \nabla J_q \right\|^2
\]

\[
\leq 4\alpha^2\mathbb{E} \left\| \sum_{q=1}^{p-1} g_q - \nabla J_q \right\|^2 + 4\alpha^2\mathbb{E} \left\| \sum_{q=1}^{p-1} \nabla J_q \right\|^2
\]

\[
= 4\alpha^2\mathbb{E} \sum_{q=1}^{p-1} \left\| g_q - \nabla J_q \right\|^2 + 4\alpha^2\mathbb{E} \left\| \sum_{q=1}^{p-1} \nabla J_q \right\|^2, \tag{A.102}
\]

where the first inequality follows from \(\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2\), and the last equality uses the fact that \(\mathbb{E}[(g_q - \nabla J_p)^\top (g_q - \nabla J_q)] = 0\) for any \(p \neq q\), since the stochastic error \(g_p - \nabla J_q\) across iterations are independent, and \(g_p\) is an unbiased estimate of \(\nabla J_p\). Moreover, due to the boundedness of \(\|\nabla J_q\|\) and \(\|g_q\|\) for any value of \(\theta_q\) (cf. Theorem 3.4), we have

\[
\mathbb{E}\|g_q - \nabla J_q\|^2 \leq 2\mathbb{E}\|g_q\|^2 + 2\mathbb{E}\|\nabla J_q\|^2 \leq 2\ell^2 + \frac{2B^2 U^2 R}{(1 - \gamma)^4} =: \epsilon^2.
\]

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Thus, by the Cauchy-Schwarz inequality and (A.100), we can further upper-bound the right-hand side of (A.102) as

\[ 2\alpha^2 \mathbb{E} \left\| \sum_{q=1}^{p-1} g_q \right\|^2 \leq 4\alpha^2 \sum_{q=1}^{p-1} \mathbb{E} \left\| g_q - \nabla J_q \right\|^2 + 4\alpha^2 \mathbb{E} \left\| \sum_{q=1}^{p-1} \nabla J_q \right\|^2 \]

\[ \leq 4\alpha^2 \cdot (p - 1) \cdot \ell_g^2 + 4\alpha^2 \cdot (p - 1) \cdot \sum_{q=1}^{p-1} \mathbb{E} \left\| \nabla J_q \right\|^2 \]

\[ \leq 4\alpha^2 \cdot (p - 1) \cdot \ell_g^2 + 4\alpha^2 \cdot (p - 1) \cdot \left( J_{\text{thre}}^2 \frac{\alpha}{\alpha} + \frac{k_{\text{thre}} L \alpha \ell^2}{2} + \frac{L^2 \ell^2}{2} \right), \tag{A.103} \]

where we recall that the expectation is taken over the random sequence \( \{\theta_1, \cdots, \theta_{p-1}\} \).

For the second term on the right-hand side of (A.101), observe that \( \mathbb{E} \| g_0 \|^2 \leq \ell^2 \). Therefore, combined with (A.103), we may upper estimate (A.101) as

\[ \mathbb{E}(\| \theta_p - \theta_0 \|^2) \leq 4\alpha^2 \cdot (p - 1) \cdot \ell_g^2 + 4\alpha^2 \cdot (p - 1) \cdot \left( J_{\text{thre}}^2 \frac{\alpha}{\alpha} + \frac{k_{\text{thre}} L \alpha \ell^2}{2} + \frac{L^2 \ell^2}{2} \right) + 2\beta^2 \ell^2 \]

\[ \leq \left[ 4\alpha^2 \cdot \ell_g^2 + 4\alpha^2 \cdot \left( J_{\text{thre}}^2 \frac{\alpha}{\alpha} + \frac{k_{\text{thre}} L \alpha \ell^2}{2} + \frac{L^2 \ell^2}{2} \right) \right] \cdot p + 2\beta^2 \ell^2, \]

which completes the proof. \( \square \)

By substituting \( q = k_{\text{thre}} \), Lemma 7 asserts that the expected distance from \( \theta_{k_{\text{thre}}} \) to \( \theta_0 \) is upper-bounded by a quadratic function of \( k_{\text{thre}} \). As illustrated at the beginning of the proof, we proceed by providing a lower-bound on this distance, and show that the lower-bound exceeds the upper-bound given in Lemma 7. As a result, the assumption that (A.98) holds is not true, which implies a sufficient increase of no less than \( J_{\text{thre}} \) from \( J_0 \) to \( \mathbb{E}(J_{k_{\text{thre}}}) \).

To create such a lower-bound, we first note that for any \( \theta \) close to \( \theta_0 \), the function value \( J(\theta) \) can be approximated by some quadratic function \( Q(\theta) \), i.e.,

\[ Q(\theta) = J_0 + (\theta - \theta_0)^T \nabla J_0 + \frac{1}{2} (\theta - \theta_0)^T \mathcal{H}_0 (\theta - \theta_0). \tag{A.104} \]

This way, one can then bound the difference between the gradients of \( J \) and \( Q \) in the following lemma.

**Lemma A.8** (60). For any twice-differentiable, \( \rho \)-Hessian Lipschitz function \( J : \mathbb{R}^d \to \mathbb{R} \), using the quadratic approximation in (A.104), the following bound holds

\[ \| \nabla J(\theta) - \nabla Q(\theta) \| \leq \frac{\rho}{2} \cdot \| \theta - \theta_0 \|^2. \]

For convenience, we let \( \nabla Q_p = \nabla Q(\theta_p) \) for any \( p = 0, \cdots, k_{\text{thre}} - 1 \). Then, we can express the difference between any \( \theta \) and \( \theta_0 \) in terms of the difference between the gradients \( \nabla Q_p \)
and \( \nabla J_p \), and thus relate it back to the difference between \( \theta \) and \( \theta_0 \) from Lemma A.8. In particular, for any \( p \geq 0 \), we can decompose \( \theta_{p+1} - \theta_0 \) as follows:

\[
\theta_{p+1} - \theta_0 = \theta_p - \theta_0 + \alpha g_p = \theta_p - \theta_0 + \alpha \nabla Q_p + \alpha (\nabla J_p - \nabla J_p - \nabla J_p)
= (I + \alpha H_0)(\theta_p - \theta_0) + \alpha (\nabla J_p - \nabla Q_p + g_p - \nabla J_p + \nabla J_0)
= (I + \alpha H_0)^p(\theta_1 - \theta_0) + \alpha \sum_{q=1}^p (I + \alpha H_0)^{p-q}(\nabla J_q - \nabla Q_q)
+ \sum_{q=1}^p (I + \alpha H_0)^{p-q}\nabla J_0
+ \sum_{q=1}^p (I + \alpha H_0)^{p-q}(g_q - \nabla J_q),
\]

(A.105)

where \( I \) is the identity matrix, \( u_p, \delta_p, d_p \), and \( \xi_p \) are defined as above, and recall that \( H_0 = \nabla^2 J(\theta_0) \) denotes the Hessian matrix evaluated at \( \theta_0 \) as defined in (A.97). The first equality uses the update from \( \theta_p \) to \( \theta_{p+1} \), and the second one adds and subtracts \( \nabla J_q \) and \( \nabla Q_q \). The third equality uses the definition of \( \nabla Q_p \) from (A.104), and the last one follows by iteratively unrolling the third equation \( p \) times. As a result, we can lower-bound the distance \( \mathbb{E}\|\theta_{p+1} - \theta_0\|^2 \) by

\[
\mathbb{E}\|\theta_{p+1} - \theta_0\|^2 \geq \mathbb{E}\|u_p\|^2 + 2\alpha \mathbb{E}(u_p^\top \delta_p) + 2\alpha \mathbb{E}(u_p^\top d_p) + 2\alpha \mathbb{E}(u_p^\top \xi_p),
\]

\[
\geq \mathbb{E}\|u_p\|^2 - 2\alpha \mathbb{E}(\|u_p\|^2\|\delta_p\|^2) + 2\alpha \mathbb{E}(u_p^\top d_p) + 2\alpha \mathbb{E}(u_p^\top \xi_p),
\]

(A.106)

where the first inequality uses the fact that \( \|a + b\|^2 \geq \|a\|^2 + 2a^\top b \), and the second one is due to the Cauchy-Schwarz inequality and the fact that \( d_p \) is deterministic given \( \theta_0 \). Now we bound the terms on the right-hand side of (A.106) in the following lemmas.

**Lemma A.9** (Lower-Bound on \( \mathbb{E}\|u_p\|^2 \)). Suppose the conditions in Lemma A.6 hold. Then after \( p \geq 1 \) iterates starting from \( \theta_0 \), it follows that

\[
\mathbb{E}\|u_p\|^2 \geq \eta \beta^2 \kappa^{2p},
\]

where \( \eta \) is the lower-bound of \( \mathbb{E}(v_0^\top g_0)^2 \) for any \( \theta \) as defined in (A.86), and we also define

\[
\kappa := 1 + \alpha \cdot \max\{\lambda_{\max}(H_0), 0\}.
\]

(A.107)

**Proof.** The proof follows the proof of Lemma 11 in [30]. Let \( v \) denote the unit eigenvector corresponding to \( \lambda_{\max}(H_0) \) for \( H_0 \); then by the Cauchy-Schwarz inequality, \( \mathbb{E}\|u_p\|^2 = \mathbb{E}(\|v^\top\|^2\|u_p\|^2) \geq \mathbb{E}(v^\top u_p)^2 \). By definition of \( \kappa \) in (A.107) and the fact that \( v \) is one of the eigenvector corresponding to \( \lambda_{\max}(H_0) \), we have

\[
v^\top (I + \alpha H_0) = v^\top [I + \alpha \lambda_{\max}(H_0)] = v^\top \kappa.
\]

Therefore, we have

\[
\mathbb{E}\|u_p\|^2 \geq \mathbb{E}(v^\top (\theta_1 - \theta_0))^2 \cdot \kappa^{2p} \geq \eta \beta^2 \cdot \kappa^{2p},
\]

which completes the proof. \( \square \)
Lemma A.10 (Upper Bound on $\mathbb{E}(|u_p||\delta_p|)$). Suppose the conditions in Lemma A.6 hold, then after $p = 1, \cdots, k_{\text{thre}} - 1$ iterates starting from $\theta_0$, it follows that

$$\mathbb{E}(|u_p||\delta_p|) \leq \left(4\ell \alpha^2 \cdot \ell_g^2 + 4\ell \alpha f_{\text{thre}} + 2L\ell^3 \alpha^3 k_{\text{thre}} + 2L\alpha \beta^2 \ell^3 \right) \cdot \rho \beta \cdot \frac{\kappa^2 p}{(\alpha \lambda)^2} + 2\rho \beta^3 \ell^3 \cdot \frac{\kappa^2 p}{\alpha \lambda}. $$

Proof. By definition of $u_p$ and $\delta_p$ in (A.105), we have

$$
\mathbb{E}(|u_p||\delta_p|) = \mathbb{E}\left[\left|(I + \alpha \mathcal{H}_0)^p(\theta_1 - \theta_0)\right| \cdot \left\|\sum_{q=1}^{p} (I + \alpha \mathcal{H}_0)^{p-q}(\nabla J_q - \nabla Q_q)\right\|ight]
\leq \kappa^p \beta \cdot \mathbb{E}\left(\|g_0\| \cdot \frac{\rho}{2} \cdot \sum_{q=1}^{p} \kappa^{p-q} \|\theta_q - \theta_0\|^2\right) \leq \frac{\kappa^p \beta \rho \ell}{2} \cdot \sum_{q=1}^{p} \kappa^{p-q} \cdot \mathbb{E}\|\theta_q - \theta_0\|^2, \quad (A.108)
$$

where the first inequality follows from the fact that $\|I + \alpha \mathcal{H}_0\| \leq \kappa$ and Lemma A.8 that $\|\nabla \theta(\theta) - \nabla Q(\theta)\| \leq \rho/2 \cdot \|\theta - \theta_0\|^2$, and the second inequality uses the almost sure boundedness that $\|g_0\| \leq \ell$.

Moreover, by Lemma A.7, we can substitute the upper-bound of $\mathbb{E}\|\theta_q - \theta_0\|^2$ in (A.99), and further bound the right-hand side of (A.108) as

$$
\mathbb{E}(|u_p||\delta_p|) \leq \frac{\kappa^p \beta \rho \ell}{2} \cdot \sum_{q=1}^{p} \kappa^{p-q} \cdot \left[4\alpha^2 \ell_g^2 + 4\alpha f_{\text{thre}} + 2L \alpha (\ell \beta)^2 + 2L^2 \alpha^3 k_{\text{thre}}\right] \cdot q + 2\beta^2 \ell^2,
\leq \left[4\alpha^2 \ell_g^2 + 4\alpha f_{\text{thre}} + 2L \alpha (\ell \beta)^2 + 2L^2 \alpha^3 k_{\text{thre}}\right] \cdot \rho \beta \ell \cdot \frac{\kappa^2 p}{(\alpha \lambda)^2} + 2\beta^3 \ell^3 \cdot \frac{\kappa^2 p}{\alpha \lambda}, \quad (A.109)
$$

where the second inequality uses the fact that

$$
\sum_{q=1}^{p} \kappa^{p-q} \leq \frac{2\kappa^p}{\alpha \lambda}, \quad \sum_{q=1}^{p} \kappa^{p-q} q \leq \frac{2\kappa^p}{(\alpha \lambda)^2},
$$

with $\lambda := \max\{0, \lambda_{\max}(\mathcal{H}_0), 0\}$. This gives the formula in the lemma and completes the proof.

Lemma A.11 (Lower-Bound on $\mathbb{E}(u_p^T d_p)$). Suppose the conditions in Lemma A.6 hold, then after $p = 1, \cdots, k_{\text{thre}} - 1$ iterates starting from $\theta_0$, it follows that

$$\mathbb{E}(u_p^T d_p) \geq 0.$$ 

Proof. By definition of $u_p$ in (A.105), it follows that

$$\mathbb{E}(u_p) = (I + \alpha \mathcal{H}_0)^p \mathbb{E}(\theta_1 - \theta_0) = \beta(I + \alpha \mathcal{H}_0)^p \nabla J_0.$$ 

By choosing $\alpha \leq 1/L$, we have $I + \alpha \mathcal{H}_0 \geq 0$, which further yields

$$\mathbb{E}(u_p^T d_p) = \beta(\nabla J_0)^T (I + \alpha \mathcal{H}_0)^p \sum_{q=1}^{p} (I + \alpha \mathcal{H}_0)^{p-q} \nabla J_q = \beta(\nabla J_0)^T \sum_{q=1}^{p} (I + \alpha \mathcal{H}_0)^{2p-q} \nabla J_q \geq 0,$$

which completes the proof.
Moreover, due to unbiasedness of \( g_q \), we have
\[
\mathbb{E}\{\xi_p | \theta_0, \cdots, \theta_p\} = 0.
\]
Thus,
\[
\mathbb{E}(u_p^T \xi_p) = \mathbb{E}_{\theta_0, \cdots, \theta_p}[\mathbb{E}(u_p^T \xi_p | \theta_0, \cdots, \theta_p)] = \mathbb{E}_{\theta_0, \cdots, \theta_p}[u_p^T \mathbb{E}(\xi_p | \theta_0, \cdots, \theta_p)] = 0, \tag{A.110}
\]
where the last equation is due to the fact that \( u_p \) is \( \sigma(\theta_0, \cdots, \theta_p) \)-measurable.

Now we are ready to present the lower-bound on the distance \( \mathbb{E}\|\theta_{p+1} - \theta_0\|^2 \) using (A.106). In particular, we combine the results of Lemma A.9, Lemma A.10, Lemma A.11, and (A.110), and arrive at the following lower-bound
\[
\mathbb{E}\|\theta_{p+1} - \theta_0\|^2 \geq \eta \beta^2 \kappa^2 p - 2 \alpha \left[ \left( 4 \ell \alpha^2 \cdot \ell^2 g + 4 \ell \alpha J_{\text{thre}} + 2 \ell \alpha^3 k_{\text{thre}} + 2 \ell \alpha^2 \beta \ell^3 \right) \cdot \beta \kappa^2 p \right.
\]
\[
= \left( \eta \beta - \frac{8 \ell \alpha \ell^2 g \rho}{\lambda^2} - \frac{8 \ell J_{\text{thre}} \rho}{\lambda^2} - \frac{4 \ell \alpha^3 k_{\text{thre}} \rho}{\lambda^2} - \frac{4 \ell \alpha^2 \beta \ell^2 \rho}{\lambda^2} - \frac{4 \ell \alpha \beta \ell^3 \rho}{\lambda^2} \right) \cdot \beta \kappa^2 p. \tag{A.111}
\]

To establish contradiction, we need to show that the lower-bound on the distance \( \mathbb{E}(\|\theta_{p+1} - \theta_0\|^2) \) in (A.111) is greater than the upper bound in Lemma A.7. In particular, we may choose parameters as in Table 1 such that the terms in the bracket on the right-hand side of (A.111) are greater than \( \eta \beta / 6 \). To this end, we let
\[
\frac{8 \ell \alpha \ell^2 g \rho}{\lambda^2} \leq \eta \beta / 6, \quad \frac{8 \ell J_{\text{thre}} \rho}{\lambda^2} \leq \eta \beta / 6, \quad \frac{4 \ell \alpha^3 k_{\text{thre}} \rho}{\lambda^2} \leq \eta \beta / 6, \quad \frac{4 \ell \alpha \beta \ell^3 \rho}{\lambda^2} \leq \eta \beta / 6,
\]
which require
\[
\beta \leq \eta \lambda / (24 \ell \rho), \quad \beta \leq \eta \lambda^2 / (24 \ell \rho), \quad J_{\text{thre}} \leq \eta \beta \lambda^2 / (48 \ell^2 g \rho), \quad \alpha \leq \eta \beta \lambda^2 / (48 \ell^2 g \rho), \quad \alpha \leq \frac{\eta \beta \lambda^2}{(24 \ell \rho^3 k_{\text{thre}})} \right)^{1/2}. \tag{A.113}
\]
Note that the choice of \( \alpha \) depends on \( k_{\text{thre}} \), which is determined as follows. Specifically, we need to choose a large enough \( k_{\text{thre}} \), such that the following contradiction holds
\[
\frac{\eta \beta^2}{6} \cdot \kappa^2 k_{\text{thre}} \geq \left( 4 \alpha^2 \ell^2 g + 4 \alpha J_{\text{thre}} + 2 \ell \alpha \beta \ell^2 + 2 \ell \alpha^3 k_{\text{thre}} \right) \cdot p + 2 \beta^2 \ell^2,
\]
where the right-hand side follows from (A.99) by setting \( p = k_{\text{thre}} \). To this end, we need \( k_{\text{thre}} \) to satisfy
\[
k_{\text{thre}} \geq \frac{c}{\alpha \lambda} \cdot \log \left( \frac{L \ell g}{\eta \beta \alpha \lambda} \right), \tag{A.114}
\]
52
where $c$ is a constant independent of parameters $L, \lambda, \eta,$ and $\rho$. By substituting the lower-bound of (A.114) into (A.113), we arrive at

$$a \leq c' \eta \beta \lambda^3 / (24L\ell^3 \rho),$$

(A.115)

where $c' > \max\{[c \log(L\ell/\eta \beta \alpha \lambda)]^{-1}, 1\}$ is some large constant. This is satisfied by the choice of stepsizes in Table 1 and thus completes the proof of the lemma.

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