Integrating out strange quarks in ChPT: terms at order $p^6$

J. Gasser$^a$, Ch. Haefeli$^a$, M.A. Ivanov$^b$ and M. Schmid$^a$

$^a$Center for Research and Education in Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstr. 5, CH–3012 Bern, Switzerland

$^b$Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna (Moscow region), Russia

Abstract

Chiral perturbation theory in the two–flavour sector allows one to analyse Green functions in QCD in a limit where the strange quark mass is considered to be large in comparison to the external momenta and to the light quark masses $m_u$ and $m_d$. In this framework, the low–energy constants of SU(2)$_R \times$ SU(2)$_L$ depend on the value of the heavy quark masses. In a recent article, we have worked out, for the coupling constants $l_i$ which occur at order $p^4$ in the chiral expansion, the dependence on the strange quark mass at two–loop accuracy. Here, we provide analogous relations for some of the couplings $c_i$ which are relevant at order $p^6$. To keep the calculations somewhat reasonable in size, we consider only those $28$ couplings which enter the Green functions built from vector- and axial vector quark currents in the chiral limit $m_u = m_d = 0, m_s \neq 0$. This provides the matching for $27$ linear combinations of the $28$ couplings.

Key words: Chiral symmetries, Chiral perturbation theory, Chiral Lagrangians

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1

At low energies and small quark masses, the Green functions of quark currents can be analysed in the framework of chiral perturbation theory (\chi PT$^2$). The method allows one to work out the momentum and quark mass dependence of the quantities of interest in a systematic and coherent manner. It is customary to perform the quark mass expansion either around $m_u = m_d = 0$, with the strange quark mass held fixed at its physical value (\chi PT$_2$), or to consider an expansion in all three quark masses around $m_u = m_d = m_s = 0$ (\chi PT$_3$). The corresponding effective Lagrangians contain low–energy constants (LECS) that parametrise the degrees of freedom which are integrated out. The
two expansions are not independent: one can express the LECs in the two–
flavour case through the ones in $\chi$PT, referred to as matching in the following.
In Ref. [3], the pertinent relations for the couplings $l_i$ – which occur at order $p^4$ in $\chi$PT – were worked out at one–loop order. Recently, this matching has been performed at two–loop order [4].

In this article, we investigate the analogous relations for the LECs $c_i$ which enter the effective Lagrangian of $\chi$PT at order $p^6$. The structure of the expansion is the following,

$$c_i = \frac{d_{i2}F^4}{M_K^4} + \frac{d_{i1}F^2}{M_K^2} + d_{i0} + O(m_s), \ i = 1, \ldots, 56 . \quad (1)$$

Here, $F(M_K)$ denotes the pion decay constant (the kaon mass) in the chiral limit. The constants $d_{im}$ are the coefficients we are after: the $d_{i2}, d_{i1}, d_{i0}$ require a tree, one–loop and two–loop calculation, in order. Furthermore, the $d_{i0}$ are linear in the $p^6$ couplings $C_i$ from $\chi$PT. This shows that, in order to have the relation between the $c_i$ and $C_i$ at leading order correct, a two–loop evaluation of the local terms in the effective action of $\chi$PT at order $p^6$ is needed. For the corresponding relations between $l_i$ and $L_i$ at leading order, the expansion of the one–loop action of $\chi$PT at order $p^4$ suffices.

It turns out that the required calculations are very complex. We circumvent the problem at the cost of loosing some information: we confine ourselves to the investigation of those LECs that occur in the Green functions of vector– and axial vector currents in the chiral limit. This allows one to remove the external scalar and pseudoscalar sources in the effective Lagrangian of the three flavour framework nearly altogether: it suffices to set $s=\text{diag}(0,0,m_s)$, and $p=0$. This simplifies the calculations considerably. On the other hand, as we will see, we can provide the matching for only 27 linear combinations of the 28 LECs that occur in the Green functions mentioned.

We comment on related works available in the literature, aside from the ones already mentioned. i) The strange quark mass expansion of the $\chi$PT LEC $B (F^2 B)$ was provided at two–loop accuracy in Ref. [5] (6). ii) Matching of the order $p^6$ LECs in the parity–odd sector was performed recently in Ref. [7]. iii) Analogous work was done in the baryon sector in Ref. [8], and for electromagnetic interactions in Refs. [9][10][11][12]. iv) The authors of Refs. [13][14] investigate what happens if chiral symmetry breaking exhibits different patterns in $\chi$PT and $\chi$PT. The literature on the subject may be traced from Ref. [14]. In this scenario, a substantial strange quark mass dependence may show up, as a result of which $\chi$PT must be reordered and the effect of vacuum fluctuations of $\bar{s}s$ pairs summed up. Whether the relations provided below favour such a

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1 Throughout this article, we denote by chiral limit the case where $m_u = m_d = 0, m_s \neq 0$. 
situation is not investigated here – the present work just provides the algebraic
dependences of the $\chi$PT$_2$ LECs on the strange quark mass, at two–loop order.

The article is structured as follows. In section 2, we illustrate the matching
considered here with the vector–vector correlator, where a two–loop calcula-
tion in $\chi$PT$_3$ is available [15], and where the two–loop result of $\chi$PT$_2$ can be
read off easily, as a result of which the matching becomes almost trivial. In
section 3, we turn to the general case and outline the method used. In section 4 we
first present a linear combination between the local polynomials at
order $p^6$ which holds in the restricted framework. Afterwards, we display the
matching relations. The final section 5 contains concluding remarks.

2

In principle, a matching of the LECs can be achieved by comparing pertinent
matrix elements, calculated in both theories up to two–loop order, in the
chiral limit: One then simply needs to expand the amplitudes from $\chi$PT$_3$ in
small momenta up to the relevant order and establish the relations between
the SU(2)– and SU(3)– LECs by equating the results of the two calculations.
To illustrate the procedure, consider the matching in the case of the vector–vector
correlator $\Pi_{V\pi}$. This quantity was evaluated in the framework of $\chi$PT$_3$
to two loops in Refs. [16,15]. In the chiral limit, the corresponding expression
in the framework of $\chi$PT$_2$ may be easily obtained from the three flavour one
by dropping the kaon contributions and replacing the SU(3)– LECs with the
ones of SU(2), e.g. $L^r_9 \to -\frac{1}{2} l^r_6$, $C^r_{93} \to c^r_{56}$, etc. In momentum space,

\begin{align}
\Pi^{\text{SU}(2)}_{V\pi}(t) &= 8h^r_2 - 4B_{V\pi}(t) \\
&\quad + \frac{t}{F^2} \left[8B_{V\pi}(t) \left(l^r_6 + B_{V\pi}(t)\right) - 4c^r_{56}\right] + O(F^{-4}), \\
\Pi^{\text{SU}(3)}_{V\pi}(t) &= -2(2H^r_1 + L^r_{10}) - 4B_{V\pi}(t) - 2B_{V\pi}(t) \\
&\quad + \frac{M^2_K}{F^2_0} \left[\frac{4}{N} \ell_K \left(L^r_9 + L^r_{10}\right) - 32C^r_{62}\right] \\
&\quad + \frac{t}{F^2_0} \left[-8 \left(2B_{V\pi}(t) + B_{V\pi}(t)\right)L^r_9 \right.
\quad + \left.2 \left(2B_{V\pi}(t) + B_{V\pi}(t)\right)^2 - 4C^r_{93}\right] + O(F_0^{-4}),
\end{align}

where

\begin{align}
B_{V\pi}(t) &= \frac{1}{6} \nu_K - \frac{1}{120N} \frac{t}{M^2_K} + O(t^2), \\
\nu_K &= \frac{1}{2N} (\ell_K + 1), \quad \ell_K = \ln(\bar{M}_K^2/\mu^2), \quad N = 16\pi^2.
\end{align}
Here, $F_0$ denotes the pion decay constant $F$ at $m_s = 0$. The square of the kaon mass in the chiral limit reads

$$\bar{M}_K^2 = B_0m_s\left[1 + \frac{B_0m_s}{F_0^2}\left(\frac{4}{9N}\ln\frac{4B_0m_s}{3\mu^2} + 16(2L_6^r - L_4^r) + 8(2L_8^r - L_5^r)\right) + O(m_s^2)\right].$$

By requiring that \(\Pi^{SU(2)}_{V\pi}(t) = \Pi^{SU(3)}_{V\pi}(t)\), we find

\[
\begin{align*}
&h_2^r = -\frac{1}{2}H_1^r - \frac{1}{4}L_1^r - \frac{1}{24}\nu_K \\
&+ \frac{\bar{M}_K^2}{F_0^2}\left[\frac{1}{2N}\ell_K(L_9^r + L_{10}^r) - 4C_{62}^r\right] + O(m_s^2), \\
&c_{56}^r = -\frac{1}{240N}\frac{F^2}{\bar{M}_K^2} + \frac{1}{3}\nu_K L_9^r - \frac{1}{72}\nu_K^2 + C_{53}^r + O(m_s).
\end{align*}
\]

The first line of Eq. (6) was already derived in Ref. [3], whereas the terms proportional to $m_s$ in the second line have been calculated while working on [4]. The result Eq. (7) will be derived by a general method again below. Note that the relation (7) involves a term proportional to $1/m_s$, a situation similar to the case of $l_7$ [3]. There, the singular term stems from a tree-level contribution, whereas for $c_{56}^r$, it originates from the momentum expansion of the loop function $B_{VK}$ in Eq. (4). Matching relations for the $c_i$ may involve terms proportional to $1/m_s^2$ as well. However, these occur only in monomials related to the sources $s$ and $p$.

We now outline a systematic method which allows one to obtain matching relations without the need to evaluate a large number of matrix elements.

3

The idea is to restrict the physics of $\chi PT_3$ to the one of $\chi PT_2$. To this end, we impose the following restrictions, collectively referred to as \textit{two-flavour limit}:

\begin{itemize}
  \item[i)] the external sources of $\chi PT_3$ are restricted to the two-flavour subspace, with $m_s$ kept at its physical value;
  \item[ii)] the matching is performed in the chiral limit;
  \item[iii)] external momenta are restricted to values below the threshold of the massive fields, $|p^2| \ll \bar{M}_K^2$.
\end{itemize}

The matching relations can then be read off from equating the pertinent generating functionals in $\chi PT_3$ and $\chi PT_2$. [An analogous method was established in Ref. [17] in the context of the linear sigma model.]

It is straightforward to apply it at one-loop level to $\chi PT$ and to obtain the relations presented in Ref. [3] for the pertinent LECs. We have extended it
to the two–loop level and established the relations between the $\chi$PT$_2$ LECs – which appear in the effective lagrangians $L_2$ and $L_4$ – and the corresponding $\chi$PT$_3$ LECs [13,4]. This technique was also applied to determine the strange quark mass dependence of the electromagnetic two–flavour LECs [12].

In the present work we do not deal with the full $\chi$PT. Rather, we switch off the sources $s$ and $p$ (while retaining $m_s$). This yields the following simplifications:

i) the solution of the classical EOM for the eta–field is trivial, $\eta = 0$;

ii) there is no mixing between the $\eta$ and the $\pi^0$ fields.

Point i) greatly simplifies the transition from $\chi$PT$_3$ building blocks of the monomials to those of two flavours, as it suppresses any effects from the eta, whereas point ii) eliminates many possible graphs and hence considerably reduces the requested labour. Indeed, in this restricted framework, the one–particle reducible graphs (two one–loop diagrams linked by a single propagator) do not contribute to the matching: due to strangeness conservation, the linking propagator cannot be a kaon. Since we can concentrate on local terms only, we can drop the pions as candidates, too. The remaining one–particle reducible diagrams do not contribute to the matching at this order, as can be verified by working out the algebra of the vertices linking the single eta propagator with the one–loop part.

Aiming for the $L_6$-monomials in the generating functional requires rather many graphs with sunset–like topology. In the two–flavour limit, where one is interested in the local contributions only, one can simplify the loop calculations by using a short distance expansion for the massive propagators. This simplifies drastically the involved loop integrals; however, the contributions from individual graphs are not chirally invariant. Collecting terms stemming from different graphs to obtain a manifestly chirally invariant result is rather cumbersome. Since we are interested in the local terms only, we use a shortcut which is based on gauge invariance: one may choose a gauge such that at some fixed space–time point $x_0$, the totally symmetric combination of up to three derivatives acting on the chiral connection vanish,

\[ \Gamma_\mu(x_0) = 0, \partial_\{\mu \Gamma_\nu\}(x_0) = 0, \partial_\{\mu \partial_\nu \Gamma_\rho\}(x_0) = 0, \partial_\{\mu \partial_\nu \partial_\rho \Gamma_\sigma\}(x_0) = 0. \quad (8) \]

Up to four ordinary derivatives are then indistinguishable from the fully symmetric combinations of covariant derivatives:

\[ \partial_\mu f(x_0) = \nabla_\mu f(x_0), \partial_\mu \partial_\nu f(x_0) = \frac{1}{2} \{\partial_\mu, \partial_\nu\} f(x_0) = \frac{1}{2} \{\nabla_\mu, \nabla_\nu\} f(x_0), \text{etc.} \]

\[ \text{(9)} \]

\[ ^2 \text{We are grateful to H. Leutwyler for pointing out this possibility to us.} \]
This allows us to write even intermediate results in a manifestly chiral invariant manner.

To check our calculations, we matched the available SU(2)– and SU(3)–results for the vector–vector correlator [15] (already discussed above) and for the pion form factor, worked out in Refs. [19] and [20]. We found that the obtained relations for \( c_{56} \) and \( c_{51} - c_{53} \) agree with our findings. Furthermore, we verified the scale independence of the found relations.

As already stated in Ref. [21], the monomial \( P_{27} \) can be discarded from the \( p^6 \)–Lagrangian for \( \chi_{PT} \). Therefore, the matching relations will certainly be a combination of some \( c_i^r \) and \( c_{27}^r \). Due to the restricted framework, only relations for LECs not involving monomials dependent on the sources \( s \) or \( p \) are nontrivial. Moreover, in the restricted framework, there is an additional relation among the remaining SU(2)–monomials:

\[
\begin{align*}
\frac{4}{3}P_1 - \frac{1}{3}P_2 + P_3 - \frac{14}{3}P_{24} + \frac{4}{3}P_{25} + 2P_{26} - \frac{3}{3}P_{28} - \frac{1}{2}P_{29} \\
+ \frac{1}{2}P_{30} - P_{31} + 2P_{32} - \frac{1}{2}P_{33} + \frac{4}{3}P_{36} - \frac{4}{3}P_{37} - \frac{11}{6}P_{39} \\
+ \frac{5}{3}P_{40} + \frac{5}{3}P_{41} - \frac{4}{3}P_{42} - \frac{3}{2}P_{43} + \frac{1}{2}P_{44} - \frac{1}{2}P_{45} - P_{51} - P_{53} = 0 .
\end{align*}
\] (10)

Because the EOM is different in the full framework, this relation is no longer valid there. We used Eq. (10) to exclude the monomial \( P_1 \) from our consideration. As a result, we give the matching for the \( 27 \) combinations of \( c_i^r \), as shown in table 1. In the full framework, an additional matching relation (apart from the ones for the monomials involving the sources \( s \) and \( p \)) for \( c_1^r \) can be worked out, yielding the only missing piece in the matching for the \( 28 \) LECs worked out here.

To render the formulae more compact, we found it convenient to express the bare kaon mass squared \( B_0 m_s \) through its equivalent \( \bar{M}_{2K}^2 \) in the chiral limit, cf. (5). Then, the final result may be written in the form

\[
x_i = p_i^{(0)} + p_i^{(1)} \ell_{\bar{K}} + p_i^{(2)} \ell_{\bar{K}}^2 + O(m_s) ,
\] (11)

where \( x_i \) denotes one of the \( 27 \) linear combinations of the \( c_i^r \) displayed in table 1. The explicit expressions for the polynomials \( p_i^{(n)} \) in the \( \chi_{PT} \)–LECs are displayed in tables 2 and 3. We use the abbreviations

\[
Z_s = \frac{F^2}{16\pi^2 M_{\bar{K}}^2} , \quad \rho_1 = \sqrt{2} \text{Cl}_2(\arccos(1/3)) \sim 1.41602 ,
\]

\[
\text{Cl}_2(\theta) = -\frac{1}{2} \int_0^\theta d\phi \ln (4 \sin^2 \frac{\phi}{2}) .
\] (12)
In summary, we have worked out the strange quark mass dependence of two–flavour LECs at order $p^6$. The calculation is performed at two–loop order. To simplify the procedure, we have restricted the evaluation to LECs that occur in the axial and vector Green functions, in the chiral limit. This concerns 28 out of the 56 LECs at this order. The calculation of the pertinent relations for the remaining 28 LECs would require a very considerable amount of work.

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| $i$ | $x_i$ | $i$ | $x_i$ | $i$ | $x_i$ |
|-----|-------|-----|-------|-----|-------|
| 1   | $c_2^r + \frac{1}{4} c_1^r$ | 10  | $c_{32}^r - \frac{3}{2} c_1^r - c_{27}^r$ | 19  | $c_{43}^r + \frac{9}{8} c_1^r + \frac{1}{4} c_{27}^r$ |
| 2   | $c_3^r - \frac{3}{4} c_1^r$  | 11  | $c_{33}^r + \frac{3}{8} c_1^r + \frac{1}{4} c_{27}^r$ | 20  | $c_{44}^r - \frac{3}{8} c_1^r - \frac{1}{4} c_{27}^r$ |
| 3   | $c_{24}^r + \frac{5}{2} c_1^r$ | 12  | $c_{36}^r - c_1^r$  | 21  | $c_{45}^r + \frac{3}{8} c_1^r + \frac{1}{4} c_{27}^r$ |
| 4   | $c_{25}^r - c_1^r$  | 13  | $c_{37}^r + c_1^r$  | 22  | $c_{50}^r$ |
| 5   | $c_{26}^r - \frac{3}{2} c_1^r$ | 14  | $c_{38}^r$  | 23  | $c_{51}^r + \frac{3}{4} c_1^r + \frac{1}{2} c_{27}^r$ |
| 6   | $c_{28}^r + 2 c_1^r - c_{27}^r$ | 15  | $c_{39}^r + \frac{11}{8} c_1^r + \frac{1}{4} c_{27}^r$ | 24  | $c_{52}^r$ |
| 7   | $c_{29}^r + \frac{3}{8} c_1^r + \frac{1}{4} c_{27}^r$ | 16  | $c_{40}^r - \frac{5}{8} c_1^r - \frac{1}{4} c_{27}^r$ | 25  | $c_{53}^r + \frac{3}{4} c_1^r + \frac{1}{2} c_{27}^r$ |
| 8   | $c_{30}^r - \frac{3}{8} c_1^r - \frac{1}{4} c_{27}^r$ | 17  | $c_{41}^r - \frac{7}{4} c_1^r - \frac{1}{2} c_{27}^r$ | 26  | $c_{55}^r$ |
| 9   | $c_{31}^r + \frac{3}{4} c_1^r + \frac{1}{2} c_{27}^r$ | 18  | $c_{42}^r + c_1^r$  | 27  | $c_{56}^r$ |

Table 1: The quantities $x_i$ in Eq. (11)
\[
\begin{array}{|c|l|}
\hline
i & p_i^{(0)} \\
\hline
1 & -\frac{24271}{589824N^2} - \frac{1}{1920}Z_s - \frac{231}{262144N^2} \ln \frac{4}{3} + \frac{1}{3N}L_1^r + \frac{1}{12N}L_2^r + \frac{11}{96N}L_3^r - \frac{1}{24N}L_4^r \\
& + \frac{1}{4}C_1^r + \frac{1}{2}C_2^r + C_3^r - \frac{2285}{1572864N^2} \rho_1 \\
\hline
2 & \frac{30193}{589824N^2} + \frac{1}{480}Z_s + \frac{1099}{786432N^2} \ln \frac{4}{3} - \frac{1}{N}L_1^r - \frac{1}{4N}L_2^r - \frac{29}{96N}L_3^r + \frac{1}{8N}L_4^r \\
& - \frac{3}{2}C_1^r - \frac{3}{2}C_2^r + C_4^r + \frac{5651}{1572864N^2} \rho_1 \\
\hline
3 & -\frac{927}{32768N^2} - \frac{1}{576}Z_s - \frac{2893}{393216N^2} \ln \frac{4}{3} + \frac{1}{2N}L_1^r - \frac{1}{6N}L_2^r + \frac{11}{48N}L_3^r - \frac{1}{4N}L_4^r \\
& + \frac{5}{2}C_1^r + 5C_2^r + C_3^r + 2C_4^r + C_5^r + C_6^r + C_7^r - \frac{3223}{262144N^2} \rho_1 \\
\hline
4 & -\frac{18085}{147456N^2} - \frac{3}{64}Z_s + \frac{841}{19608N^2} \ln \frac{4}{3} + \frac{1}{2N}L_1^r - \frac{1}{4N}L_2^r + \frac{17}{48N}L_3^r - \frac{1}{4N}L_4^r - C_1^r \\
& - 2C_2^r + C_3^r + 2C_4^r + C_5^r - \frac{1951}{393216N^2} \rho_1 \\
\hline
5 & \frac{18091}{98304N^2} + \frac{11}{2880}Z_s + \frac{2747}{393216N^2} \ln \frac{4}{3} - \frac{2}{N}L_1^r - \frac{1}{3N}L_2^r - \frac{31}{48N}L_3^r + \frac{1}{4N}L_4^r \\
& - \frac{3}{2}C_1^r - 3C_2^r + C_3^r + C_4^r - C_7^r + \frac{8963}{786432N^2} \rho_1 \\
\hline
6 & \frac{6875}{13728N^2} + \frac{7}{380}Z_s - \frac{1223}{98304N^2} \ln \frac{4}{3} - \frac{10}{3N}L_1^r - \frac{1}{3N}L_2^r - \frac{3}{4N}L_3^r + \frac{2}{3N}L_4^r \\
& + \frac{1}{4N}L_1^r + 2C_1^r + 4C_2^r + 2C_4^r + 2C_7^r + 2C_9^r - C_5^r + C_6^r - \frac{101}{65536N^2} \rho_1 \\
\hline
7 & -\frac{22535}{393216N^2} + \frac{1}{1280}Z_s - \frac{2205}{32256N^2} \ln \frac{4}{3} + \frac{1}{6N}L_1^r - \frac{1}{8N}L_2^r - \frac{5}{192N}L_3^r + \frac{5}{48N}L_4^r \\
& - \frac{1}{8N}L_1^r + \frac{3}{8}C_1^r + \frac{1}{4}C_2^r + \frac{1}{4}C_5^r - \frac{1}{4}C_7^r + C_8^r + 2C_9^r - \frac{5781}{1048576N^2} \rho_1 \\
\hline
8 & \frac{260927}{353844N^2} + \frac{7}{3840}Z_s + \frac{4055}{1572864N^2} \ln \frac{4}{3} - \frac{2}{2N}L_1^r - \frac{1}{4N}L_2^r - \frac{29}{192N}L_3^r + \frac{1}{16N}L_4^r \\
& - \frac{1}{8N}L_1^r - \frac{3}{8}C_1^r + \frac{1}{8}C_2^r - \frac{1}{4}C_4^r + \frac{1}{4}C_5^r + C_6^r + \frac{35263}{3145728N^2} \rho_1 \\
\hline
9 & -\frac{73685}{589824N^2} + \frac{11}{1920}Z_s - \frac{2327}{768432N^2} \ln \frac{4}{3} + \frac{1}{N}L_1^r + \frac{5}{12N}L_2^r + \frac{37}{96N}L_3^r - \frac{1}{8N}L_4^r \\
& - \frac{1}{4N}L_1^r + \frac{3}{4}C_1^r + \frac{1}{2}C_2^r + \frac{1}{2}C_5^r - \frac{1}{2}C_7^r + C_8^r + \frac{3841}{1572864N^2} \rho_1 \\
\hline
10 & \frac{6245}{32768N^2} + \frac{1}{192}Z_s + \frac{2687}{393216N^2} \ln \frac{4}{3} - \frac{2}{N}L_1^r + \frac{1}{2N}L_2^r - \frac{25}{48N}L_3^r + \frac{1}{4N}L_4^r - \frac{1}{12N}L_9 \\
& - \frac{3}{2}C_1^r - 3C_2^r + C_3^r + C_4^r + C_5^r + C_6^r + C_7^r + \frac{3623}{768432N^2} \rho_1 \\
\hline
11 & -\frac{165839}{353844N^2} - \frac{7}{1280}Z_s - \frac{1511}{1572864N^2} \ln \frac{4}{3} + \frac{2}{2N}L_1^r + \frac{24}{21N}L_2^r + \frac{29}{192N}L_3^r - \frac{1}{16N}L_4^r \\
& + \frac{1}{6N}L_1^r + \frac{3}{8}C_1^r + \frac{1}{4}C_2^r + \frac{1}{4}C_5^r - \frac{1}{4}C_7^r + C_8^r + \frac{5455}{3145728N^2} \rho_1 \\
\hline
12 & \frac{3015}{4423868N^2} + \frac{1}{192}Z_s + \frac{587}{19608N^2} \ln \frac{4}{3} - \frac{8}{3N}L_1^r - \frac{1}{2N}L_2^r - \frac{19}{21N}L_3^r + \frac{1}{3N}L_4^r \\
& - \frac{1}{21N}L_1^r - C_1^r - 2C_2^r + C_6^r + \frac{1}{2}C_8^r + \frac{1043}{393216N^2} \rho_1 \\
\hline
13 & \frac{13001}{4423868N^2} - \frac{1}{480}Z_s - \frac{1009}{19608N^2} \ln \frac{4}{3} + \frac{4}{3N}L_1^r + \frac{1}{3N}L_2^r + \frac{11}{24N}L_3^r - \frac{1}{6N}L_4^r \\
& + C_1^r + 2C_5^r + C_6^r + \frac{2359}{393216N^2} \rho_1 \\
\hline
\end{array}
\]

Table 2: The polynomial \( p_i^{(0)} \) as defined in Eq. (11)

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Table 2: The polynomial $p_i^{(0)}$ as defined in Eq. (3) (cont.)
| $i$ | $p_i^{(1)}$ | $p_i^{(2)}$ |
|-----|-------------|-------------|
| 1   | $-\frac{913}{27648N^2} + \frac{1}{3N} L_1 + \frac{1}{12N} L_2 + \frac{3}{32N} L_3^r - \frac{25}{2904N^2}$ | |
| 2   | $\frac{1483}{27648N^2} - \frac{3}{4N} L_1 - \frac{1}{4N} L_2 - \frac{23}{96N} L_3^r - \frac{5}{256N^2}$ | |
| 3   | $-\frac{785}{15824N^2} + \frac{1}{2N} L_1 - \frac{1}{6N} L_2 + \frac{5}{48N} L_3^r - \frac{1}{8N^2}$ | |
| 4   | $-\frac{329}{3456N^2} + \frac{1}{N} L_1 - \frac{1}{4N} L_2 + \frac{11}{48N} L_3^r - \frac{73}{2304N^2}$ | |
| 5   | $\frac{2407}{13824N^2} - \frac{3}{2N} L_1 - \frac{3}{8N} L_2 - \frac{25}{48N} L_3^r + \frac{11}{288N^2}$ | |
| 6   | $\frac{79}{1152N^2} - \frac{2}{N} L_1 - \frac{1}{3N} L_2^r - \frac{5}{12N} L_3^r + \frac{1}{4N} L_9^r - \frac{17}{288N^2}$ | |
| 7   | $-\frac{113}{2048N^2} + \frac{1}{3N} L_1 - \frac{1}{8N} L_2 + \frac{5}{64N} L_3^r - \frac{1}{8N} L_{10}^r - \frac{11}{1536N^2}$ | |
| 8   | $\frac{4187}{55296N^2} - \frac{3}{8N} L_1 - \frac{1}{24N} L_2^r - \frac{23}{192N} L_3^r - \frac{1}{8N} L_{10}^r - \frac{61}{4096N^2}$ | |
| 9   | $-\frac{889}{3456N^2} + \frac{1}{3N} L_1^r + \frac{5}{12N} L_2^r + \frac{31}{96N} L_3^r - \frac{1}{4N} L_{10}^r - \frac{19}{768N^2}$ | |
| 10  | $\frac{149}{4096N^2} - \frac{3}{2N} L_1^r + \frac{1}{6N} L_2^r - \frac{19}{48N} L_3^r - \frac{1}{12N} L_{10}^r - \frac{35}{1152N^2}$ | |
| 11  | $-\frac{2395}{55296N^2} + \frac{3}{8N} L_1 - \frac{1}{24N} L_2^r + \frac{23}{192N} L_3^r + \frac{1}{6N} L_9^r - \frac{23}{1536N^2}$ | |
| 12  | $\frac{605}{6912N^2} - \frac{2}{N} L_1 - \frac{1}{2N} L_2 - \frac{5}{8N} L_3 - \frac{1}{24N} L_9^r + \frac{47}{1152N^2}$ | |
| 13  | $-\frac{229}{6912N^2} + \frac{1}{N} L_1 + \frac{1}{3N} L_2^r + \frac{3}{8N} L_3^r - \frac{13}{576N^2}$ | |
| 14  | $\frac{29}{864N^2} - \frac{4}{3N} L_1^r - \frac{1}{6N} L_2^r - \frac{1}{3N} L_3^r - \frac{1}{12N} L_{10}^r + \frac{5}{288N^2}$ | |
| 15  | $\frac{707}{6144N^2} - \frac{3}{3} L_1^r + \frac{1}{24N} L_2^r + \frac{23}{192N} L_3^r - \frac{1}{12N} L_{10}^r + \frac{1}{8N} L_9^r - \frac{73}{4096N^2}$ | |
| 16  | $\frac{419}{55296N^2} - \frac{5}{8N} L_1^r - \frac{1}{8N} L_2^r - \frac{41}{192N} L_3^r + \frac{1}{12N} L_{10}^r - \frac{7}{512N^2}$ | |
| 17  | $\frac{1997}{9216N^2} - \frac{7}{4N} L_1^r - \frac{5}{12N} L_2^r - \frac{59}{96N} L_3^r + \frac{1}{12N} L_{10}^r - \frac{109}{2304N^2}$ | |
| 18  | $-\frac{29}{2304N^2} - \frac{1}{N} L_1^r - \frac{1}{24N} L_3^r - \frac{6}{12N} L_9^r + \frac{5}{576N^2}$ | |
| 19  | $\frac{3745}{55296N^2} + \frac{1}{8N} L_1^r + \frac{1}{8N} L_2^r + \frac{7}{64N} L_3^r - \frac{1}{12N} L_{10}^r - \frac{55}{4096N^2}$ | |
| 20  | $\frac{1273}{18432N^2} - \frac{3}{3} L_1^r + \frac{1}{24N} L_2^r - \frac{23}{192N} L_3^r + \frac{1}{6N} L_{10}^r - \frac{21}{4096N^2}$ | |
| 21  | $-\frac{841}{18432N^2} + \frac{3}{8N} L_1^r + \frac{1}{24N} L_2^r + \frac{23}{192N} L_3^r - \frac{1}{12N} L_{10}^r + \frac{37}{4608N^2}$ | |
| 22  | $\frac{19}{1152N^2} - \frac{1}{12N} L_9^r - \frac{1}{576N^2}$ | |
| 23  | $-\frac{985}{9216N^2} + \frac{3}{4N} L_1^r + \frac{1}{12N} L_2^r + \frac{23}{96N} L_3^r - \frac{5}{256N^2}$ | |
| 24  | $\frac{17}{576N^2} + \frac{1}{12N} L_3^r - \frac{7}{24N} L_{10}^r - \frac{7}{1152N^2}$ | |
| 25  | $-\frac{3067}{27648N^2} + \frac{3}{4N} L_1^r + \frac{1}{12N} L_2^r + \frac{31}{96N} L_3^r + \frac{1}{24N} L_{10}^r - \frac{43}{2304N^2}$ | |
| 26  | $\frac{1}{36N^2} - \frac{2}{3N} L_9^r + \frac{1}{72N^2}$ | |
| 27  | $-\frac{1}{144N^2} + \frac{1}{6N} L_9^r - \frac{1}{288N^2}$ | |

Table 3: The polynomials $p_i^{(1)}$ and $p_i^{(2)}$ as defined in Eq. (11)
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