Strong insertion of a contra-continuous function between two comparable real-valued functions

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Abstract. Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.

M.S.C. 2010: Primary 54C08, 54C10, 54C50; Secondary 26A15, 54C30.
Key words: Insertion, strong binary relation; $C$-open set; semi-preopen set, $\alpha$-open set; contra-continuous function; lower cut set.

1 Introduction

The concept of a $C$-open set in a topological space was introduced by E. Hatir, T. Noiri and S. Yksel in [12]. The authors define a set $S$ to be a $C$-open set if $S = U \cap A$, where $U$ is open and $A$ is semi-preclosed. A set $S$ is a $C$-closed set if its complement (denoted by $S^c$) is a $C$-open set or equivalently if $S = U \cup A$, where $U$ is closed and $A$ is semi-preopen. The authors show that a subset of a topological space is open if and only if it is an $\alpha$-open set and a $C$-open set or equivalently a subset of a topological space is closed if and only if it is an $\alpha$-closed set and a $C$-closed set. This enables them to provide the following decomposition of continuity: a function is continuous if and only if it is $\alpha$-continuous and $C$-continuous or equivalently a function is contra-continuous if and only if it is contra-$\alpha$-continuous and contra-$C$-continuous.

Recall that a subset $A$ of a topological space $(X, \tau)$ is called $\alpha$-open if $A$ is the difference of an open and a nowhere dense subset of $X$. A set $A$ is called $\alpha$-closed if its complement is $\alpha$-open or equivalently if $A$ is the union of a closed and a nowhere dense set. Sets which are dense in some regular closed subspace are called semi-preopen or $\beta$-open. A set is semi-preclosed or $\beta$-closed if its complement is semi-preopen or $\beta$-open.

In [7] it was shown that a set $A$ is $\beta$-open if and only if $A \subseteq Cl(Int(Cl(A)))$. A generalized class of closed sets was considered by Maki in [20]. He investigated the sets that can be represented as union of closed sets and called them $V$-sets. Complements of $V$-sets, i.e., sets that are intersection of open sets are called $\Lambda$-sets [20].
Recall that a real-valued function $f$ defined on a topological space $X$ is called $A$-continuous [25] if the preimage of every open subset of $\mathbb{R}$ belongs to $A$, where $A$ is a collection of subsets of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A$-continuity. However, for unknown concepts the reader may refer to [4, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [5] introduced a new class of mappings called contra-continuity. S. Jafari and T. Noiri in [13, 14] exhibited and studied among others a new weaker form of this class of mappings called contra-$\alpha$-continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 23].

Hence, a real-valued function $f$ defined on a topological space $X$ is called contra-continuous (resp. contra-$C$-continuous, contra-$\alpha$-continuous) if the preimage of every open subset of $\mathbb{R}$ is closed (resp. $C$-closed, $\alpha$-closed) in $X$ [5].

Results of Katětov [15, 16] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that $\Lambda$-sets or kernel of sets are open [20].

If $g$ and $f$ are real-valued functions defined on a space $X$, we write $g \preceq f$ in case $g(x) \leq f(x)$ for all $x$ in $X$.

The following definitions are modifications of conditions considered in [17].

A property $P$ defined relative to a real-valued function on a topological space is a cc-property provided that any constant function has property $P$ and provided that the sum of a function with property $P$ and any contra-continuous function also has property $P$. If $P_1$ and $P_2$ are cc-properties, the following terminology is used: (i) A space $X$ has the weak cc-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \preceq f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a contra-continuous function $h$ such that $g \preceq h \preceq f$. (ii) A space $X$ has the strong cc-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \preceq f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a contra-continuous function $h$ such that $g \preceq h \preceq f$ and if $g(x) < f(x)$ for any $x$ in $X$, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space whose $\Lambda$-sets or kernel of sets are open, is given a sufficient condition for the weak cc-insertion property. Also for a space with the weak cc-insertion property, we give necessary and sufficient conditions for the space to have the strong cc-insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the insertion of a contra-continuous function between two comparable contra-precontinuous real-valued functions has also recently considered by the author in [21].

2 The main result

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.
The abbreviations $cc$, $coc$ and $cCc$ are used for contra-continuous, contra-$\alpha$-continuous and contra-$C$-continuous, respectively.

**Definition 2.1.** Let $A$ be a subset of a topological space $(X, \tau)$. We define the subsets $A^\alpha$ and $A^V$ as follows:

$A^\alpha = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.

In [6, 19, 22], $A^\alpha$ is called the *kernel* of $A$.

The family of all $\alpha$--open, $\alpha$--closed, $C$--open and $C$--closed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $CO(X, \tau)$ and $CC(X, \tau)$, respectively.

We define the subsets $\alpha(A^\alpha), \alpha(A^V), C(A^\alpha)$ and $C(A^V)$ as follows:

$\alpha(A^\alpha) = \cap\{O : O \supseteq A, O \in \alpha O(X, \tau)\}$,

$\alpha(A^V) = \cup\{F : F \subseteq A, F \in \alpha C(X, \tau)\}$,

$C(A^\alpha) = \cap\{O : O \supseteq A, O \in CO(X, \tau)\}$ and

$C(A^V) = \cup\{F : F \subseteq A, F \in CC(X, \tau)\}$.

$\alpha(A^\alpha)$ (resp. $C(A^\alpha)$) is called the $\alpha$ -- kernel (resp. $C$ -- kernel) of $A$.

The following first two definitions are modifications of conditions considered in [15, 16].

**Definition 2.2.** If $\rho$ is a binary relation in a set $S$ then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $x \rho y$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any $u$ and $v$ in $S$.

**Definition 2.3.** A binary relation $\rho$ in the power set $P(X)$ of a topological space $X$ is called a *strong binary relation* in $P(X)$ in case $\rho$ satisfies each of the following conditions:

1) If $A_i \rho B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set $C$ in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A \bar{\rho} B$.

3) If $A \rho B$, then $A^\alpha \subseteq B$ and $A \subseteq B^V$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 2.4.** If $f$ is a real-valued function defined on a space $X$ and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number $\ell$, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of $f$ at the level $\ell$.

We now give the following main result:

**Theorem 2.1.** Let $g$ and $f$ be real-valued functions on the topological space $X$, in which kernel of sets are open, with $g \leq f$. If there exists a strong binary relation $\rho$ on the power set of $X$ and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of $f$ and $g$ at the level $t$ for each rational number $t$ such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$, then there exists a contra-continuous function $h$ defined on $X$ such that $g \leq h \leq f$.

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$ such that $g \leq f$. By
hypothesis there exists a strong binary relation \( \rho \) on the power set of \( X \) and there exist lower indefinite cut sets \( A(f,t) \) and \( A(g,t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \) then \( A(f,t_1) \rho A(g,t_2) \).

Define functions \( F \) and \( G \) mapping the rational numbers \( \mathbb{Q} \) into the power set of \( X \) by \( F(t) = A(f,t) \) and \( G(t) = A(g,t) \). If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then \( F(t_1) \rho F(t_2) \), \( G(t_1) \rho G(t_2) \), and \( F(t_1) \rho G(t_2) \). By Lemmas 1 and 2 of [16] it follows that there exists a function \( H \) mapping \( \mathbb{Q} \) into the power set of \( X \) such that if \( t_1 \) and \( t_2 \) are any rational numbers with \( t_1 < t_2 \), then \( F(t_1) \rho H(t_2), H(t_1) \rho H(t_2) \) and \( H(t_1) \rho G(t_2) \).

For any \( x \in X \), let \( h(x) = \inf \{ t \in \mathbb{Q} : x \in H(t) \} \).

We first verify that \( g \leq h \leq f \): If \( x \) is in \( H(t) \) then \( x \) is in \( G(t') \) for any \( t' > t \); since \( x \) is in \( G(t') = A(g,t') \) it follows that \( g(x) \leq t' \), it follows that \( g(x) \leq t \). Hence \( g \leq h \). If \( x \) is not in \( H(t) \), then \( x \) is not in \( F(t') \) for any \( t' < t \); since \( x \) is not in \( F(t') = A(f,t') \) it follows that \( f(x) > t' \), it follows that \( f(x) > t \). Hence \( h \leq f \).

Also, for any rational numbers \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \), we have \( h^{-1}(t_1,t_2) = H(t_2)^V \setminus H(t_1)^h \). Hence \( h^{-1}(t_1,t_2) \) is closed in \( X \), i.e., \( h \) is a contra-continuous function on \( X \).

The above proof used the technique of theorem 1 in [15].

If a space has the strong \( cc \)-insertion property for \( (P_1,P_2) \), then it has the weak \( cc \)-insertion property for \( (P_1,P_2) \). The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak \( cc \)-insertion property to satisfy the strong \( cc \)-insertion property.

**Theorem 2.2.** Let \( P_1 \) and \( P_2 \) be \( cc \)-property and \( X \) be a space that satisfies the weak \( cc \)-insertion property for \( (P_1,P_2) \). Also assume that \( g \) and \( f \) are functions on \( X \) such that \( g \leq f \), \( g \) has property \( P_1 \) and \( f \) has property \( P_2 \). The space \( X \) has the strong \( cc \)-insertion property for \( (P_1,P_2) \) if and only if there exist lower cut sets \( A(f-g,2^{-n}) \) and there exists a sequence \( \{F_n\} \) of subsets of \( X \) such that (i) for each \( n \), \( F_n \) and \( A(f-g,2^{-n}) \) are completely separated by contra-continuous functions, and (ii) \( x \in X : (f-g)(x) > 0 \} \cup \bigcup_{n=1}^{\infty} F_n \).

**Proof.** Suppose that there is a sequence \( \{A(f-g,2^{-n})\} \) of lower cut sets for \( f-g \) and suppose that there is a sequence \( \{F_n\} \) of subsets of \( X \) such that

\[
\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n
\]

and such that for each \( n \), there exists a contra-continuous function \( k_n \) on \( X \) into \([0,2^{-n}] \) with \( k_n = 2^{-n} \) on \( F_n \) and \( k_n = 0 \) on \( A(f-g,2^{-n}) \). The function \( k \) from \( X \) into \([0,1/4] \) which is defined by

\[
k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x)
\]

is a contra-continuous function by the Cauchy condition and the properties of contra-continuous functions, (1) \( k^{-1}(0) = \{x \in X : (f-g)(x) = 0\} \) and (2) if \( (f-g)(x) > 0 \) then \( k(x) < (f-g)(x) \). In order to verify (1), observe that if \( (f-g)(x) = 0 \), then \( x \in A(f-g,2^{-n}) \) for each \( n \) and hence \( k_n(x) = 0 \) for each \( n \). Thus \( k(x) = 0 \).
Conversely, if \((f - g)(x) > 0\), then there exists an \(n\) such that \(x \in F_n\) and hence \(k_n(x) = 2^{-n}\). Thus \(k(x) \neq 0\) and this verifies (1). Next, in order to establish (2), note that

\[ \{x \in X : (f - g)(x) = 0\} = \bigcap_{n=1}^{\infty} A(f - g, 2^{-n}) \]

and that \((A(f - g, 2^{-n}))\) is a decreasing sequence. Thus if \((f - g)(x) > 0\) then either \(x \notin A(f - g, 1/2)\) or there exists a smallest \(n\) such that \(x \notin A(f - g, 2^{-n})\) and \(x \in A(f - g, 2^{-j})\) for \(j = 1, \ldots, n - 1\).

In the former case,

\[ k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x) \leq 1/4 \sum_{n=1}^{\infty} 2^{-n} < 1/2 \leq (f - g)(x), \]

and in the latter,

\[ k(x) = 1/4 \sum_{j=n}^{\infty} k_j(x) \leq 1/4 \sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \leq (f - g)(x). \]

Thus \(0 \leq k \leq f - g\) and if \((f - g)(x) > 0\) then \((f - g)(x) > k(x) > 0\). Let \(g_1 = g + (1/4)k\) and \(f_1 = f - (1/4)k\). Then \(g \leq g_1 \leq f_1 \leq f\) and if \(g(x) < f(x)\) then \(g(x) < g_1(x) < f_1(x) < f(x)\).

Since \(P_1\) and \(P_2\) are \(cc\)-properties, then \(g_1\) has property \(P_1\) and \(f_1\) has property \(P_2\). Since by hypothesis \(X\) has the weak \(cc\)-insertion property for \((P_1, P_2)\), then there exists a contra-continuous function \(h\) such that \(g_1 \leq h \leq f_1\). Thus \(g \leq h \leq f\) and if \(g(x) < f(x)\) then \(g(x) < h(x) < f(x)\). Therefore \(X\) has the strong \(cc\)-insertion property for \((P_1, P_2)\). (The technique of this proof is by Lane \[17\].)

Conversely, assume that \(X\) satisfies the strong \(cc\)-insertion for \((P_1, P_2)\). Let \(g\) and \(f\) be functions on \(X\) satisfying \(P_1\) and \(P_2\) respectively such that \(g \leq f\). Thus there exists a contra-continuous function \(h\) such that \(g \leq h \leq f\) and such that if \(g(x) < f(x)\) for any \(x\) in \(X\), then \(g(x) < h(x) < f(x)\). We follow an idea contained in Powderly \[24\]. Now consider the functions 0 and \(f - h, 0\) satisfy property \(P_1\) and \(f - h\) satisfies property \(P_2\). Thus there exists a contra-continuous function \(h_1\) such that \(0 \leq h_1 \leq f - h\) and if \(0 < (f - h)(x)\) for any \(x\) in \(X\), then \(0 < h_1(x) < (f - h)(x)\).

We next show that

\[ \{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}. \]

If \(x\) is such that \((f - g)(x) > 0\), then \(g(x) < f(x)\). Therefore \(g(x) < h(x) < f(x)\). Thus \(f(x) - h(x) > 0\) or \((f - h)(x) > 0\). Hence \(h_1(x) > 0\). On the other hand, if \(h_1(x) > 0\), then since \((f - h) \geq h_1\) and \(f - g \geq f - h\), therefore \((f - g)(x) > 0\). For each \(n\), let \(A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}\),

\(F_n = \{x \in X : h_1(x) \geq 2^{-n+1}\}\) and

\(k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}\).

Since \(\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}\), it follows that

\[ \{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n. \]
We next show that $k_n$ is a contra-continuous function which completely separates $F_n$ and $\mathcal{A}(f - g, 2^{-n})$. From its definition and by the properties of contra-continuous functions, it is clear that $k_n$ is a contra-continuous function. Let $x \in F_n$. Then, from the definition of $k_n$, $k_n(x) = 2^{-n}$. If $x \in \mathcal{A}(f - g, 2^{-n})$, then since $h_1 \leq f - h \leq f - g, h_1(x) \leq 2^{-n}$. Thus $k_n(x) = 0$, according to the definition of $k_n$. Hence $k_n$ completely separates $F_n$ and $\mathcal{A}(f - g, 2^{-n})$. ■

**Theorem 2.3.** Let $P_1$ and $P_2$ be $cc$-properties and assume that the space $X$ satisfied the weak $cc$-insertion property for $(P_1, P_2)$. The space $X$ satisfies the strong $cc$-insertion property for $(P_1, P_2)$ if and only if $X$ satisfies the strong $cc$-insertion property for $(P_1, cc)$ and for $(cc, P_2)$.

**Proof.** Assume that $X$ satisfies the strong $cc$-insertion property for $(P_1, cc)$ and for $(cc, P_2)$. If $g$ and $f$ are functions on $X$ such that $g \leq f$, $g$ satisfies property $P_1$, and $f$ satisfies property $P_2$, then since $X$ satisfies the weak $cc$-insertion property for $(P_1, P_2)$ there is a contra-continuous function $h$ such that $g \leq h \leq f$. Also, by hypothesis there exist contra-continuous functions $h_1$ and $h_2$ such that $g \leq h_1 \leq k$ and if $g(x) < k(x)$ then $g(x) < h_1(x) < k(x)$ and such that $k \leq h_2 \leq f$ and if $k(x) < f(x)$ then $k(x) < h_2(x) < f(x)$. If a function $h$ is defined by $h(x) = (h_2(x) + h_1(x))/2$, then $h$ is a contra-continuous function, $g \leq h \leq f$, and if $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. Hence $X$ satisfies the strong $cc$-insertion property for $(P_1, P_2)$.

The converse is obvious since any contra-continuous function must satisfy both properties $P_1$ and $P_2$. (The technique of this proof is by Lane [18].) □

## 3 Applications

Before stating the consequences of Theorems 2.1, 2.2 and 2.3 we suppose that $X$ is a topological space whose kernel of sets are open.

**Corollary 3.1.** If for each pair of disjoint $\alpha$-open (resp. $C$-open) sets $G_1, G_2$ of $X$, there exist closed sets $F_1$ and $F_2$ of $X$ such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then $X$ has the weak $cc$-insertion property for $(cc, cc)$ (resp. $(cC, cCc)$).

**Proof.** Let $g$ and $f$ be real-valued functions defined on $X$ such that $f$ and $g$ are $cc$ (resp. $cCc$), and $g \leq f$. If a binary relation $\rho$ is defined by $A \rho B$ in case $\alpha(A^X) \subseteq \alpha(B^X)$ (resp. $C(A^X) \subseteq C(B^X)$), then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$\mathcal{A}(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq \mathcal{A}(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is an $\alpha$-open (resp. $C$-open) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha$-closed (resp. $C$-closed) set, it follows that $\alpha(\mathcal{A}(f, t_1)^X) \subseteq \alpha(\mathcal{A}(g, t_2)^Y)$ (resp. $\mathcal{C}(A(f, t_1)^X) \subseteq \mathcal{C}(A(g, t_2)^Y)$). Hence $t_1 < t_2$ implies that $\mathcal{A}(f, t_1) \rho \mathcal{A}(g, t_2)$. The proof follows from Theorem 2.1. ■

**Corollary 3.2.** If for each pair of disjoint $\alpha$-open (resp. $C$-open) sets $G_1, G_2$, there exist closed sets $F_1$ and $F_2$ such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then every contra-$\alpha$-continuous (resp. contra-$C$-continuous) function is contra-continuous.

**Proof.** Let $f$ be a real-valued contra-$\alpha$-continuous (resp. contra-$C$-continuous)
function defined on $X$. Set $g = f$, then by Corollary 3.1, there exists a contra-
continuous function $h$ such that $g = h = f$.  

**Corollary 3.3.** If for each pair of disjoint $\alpha$--open (resp. $C$--open) sets $G_1, G_2$ of $X$
, there exist closed sets $F_1$ and $F_2$ of $X$ such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$
and $F_1 \cap F_2 = \emptyset$ then $X$ has the strong $cc$--insertion property for ($cCc$, $cCc$) (resp. ($cCc$, $cCc$)).

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ and $g$
are $cCc$ (resp. $cCc$), and $g \leq f$. Set $h = (f + g)/2$, thus $g \leq h \leq f$
and if $g(x) < f(x)$ for any $x$ in $X$, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since $g$
and $f$ are contra-continuous functions hence $h$ is a contra-continuous function.

**Corollary 3.4.** If for each pair of disjoint subsets $G_1, G_2$ of $X$, such that $G_1$
is $\alpha$--open and $G_2$ is $C$--open, there exist closed subsets $F_1$ and $F_2$ of $X$ such that
$G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then $X$ have the weak $cc$--insertion property for
($cCc$, $cCc$) and ($cCc$, $cCc$).

**Proof.** Let $g$ and $f$ be real-valued functions defined on $X$, such that $g$ is $cCc$ (resp. $cCc$
and $f$ is $cCc$ (resp. $cCc$), with $g \leq f$.If a binary relation $\rho$ is defined by $A \rho B$
in case $C(A^\lambda) \subseteq \alpha(B^\lambda)$ (resp. $\alpha(A^\lambda) \subseteq C(B^\lambda)$), then by hypothesis $\rho$
is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $Q$
with $t_1 < t_2$, then
\[
A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);
\]
since $\{x \in X : f(x) \leq t_1\}$ is a $C$--open (resp. $\alpha$--open) set and since $\{x \in X : g(x) < t_2\}$
is an $\alpha$--closed (resp. $C$--closed) set, it follows that $C(A(f, t_1)^\lambda) \subseteq \alpha(A(g, t_2)^\lambda)$
(resp. $\alpha(A(f, t_1)^\lambda) \subseteq C(A(g, t_2)^\lambda)$). Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$.
The proof follows from Theorem 2.1.

Before stating consequences of Theorem 2.2, we state and prove the necessary
lemmas.

**Lemma 3.1.** The following conditions on the space $X$ are equivalent:

(i) For each pair of disjoint subsets $G_1, G_2$ of $X$, such that $G_1$ is $\alpha$--open and $G_2$
is $C$--open, there exist closed subsets $F_1, F_2$ of $X$ such that $G_1 \subseteq F_1, G_2 \subseteq F_2$
and $F_1 \cap F_2 = \emptyset$.

(ii) If $G$ is a $C$--open (resp. $\alpha$--open) subset of $X$ which is contained in an
$\alpha$--closed (resp. $C$--closed) subset $F$ of $X$, then there exists a closed subset $H$ of $X$
such that $G \subseteq H \subseteq H^\lambda \subseteq F$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $G \subseteq F$, where $G$ and $F$ are $C$--open (resp. $\alpha$--open)
and $\alpha$--closed (resp. $C$--closed) subsets of $X$, respectively. Hence, $F^c$ is an
$\alpha$--open (resp. $C$--open) and $G \cap F^c = \emptyset$.

By (i) there exists two disjoint closed subsets $F_1, F_2$ such that $G \subseteq F_1$ and $F^c \subseteq F_2$.
But
\[
F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,
\]
and
\[
F_1 \cap F_2 = \emptyset \Rightarrow F_1 \subseteq F_2^c
\]

hence
\[
G \subseteq F_1 \subseteq F_2^c \subseteq F
\]
and since $F_2^c$ is an open subset containing $F_1$, we conclude that $F_1^\Lambda \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq F_1^\Lambda \subseteq F.$$  

By setting $H = F_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that $G_1, G_2$ are two disjoint subsets of $X$, such that $G_1$ is $\alpha$–open and $G_2$ is $C$–open.

This implies that $G_2 \subseteq G_1^c$ and $G_1^c$ is an $\alpha$–closed subset of $X$. Hence by (ii) there exists a closed set $H$ such that $G_2 \subseteq H \subseteq H^\Lambda \subseteq G_1^c$.

But

$$H \subseteq H^\Lambda \Rightarrow H \cap (H^\Lambda)^c = \emptyset$$

and

$$H^\Lambda \subseteq G_1^c \Rightarrow G_1 \subseteq (H^\Lambda)^c.$$ 

Furthermore, $(H^\Lambda)^c$ is a closed subset of $X$. Hence $G_2 \subseteq H, G_1 \subseteq (H^\Lambda)^c$ and $H \cap (H^\Lambda)^c = \emptyset$. This means that condition (i) holds.

**Lemma 3.2.** Suppose that $X$ is a topological space. If each pair of disjoint subsets $G_1, G_2$ of $X$, where $G_1$ is $\alpha$–open and $G_2$ is $C$–open, can be separated by closed subsets of $X$ then there exists a contra-continuous function $h : X \to [0,1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.

**Proof.** Suppose $G_1$ and $G_2$ are two disjoint subsets of $X$, where $G_1$ is $\alpha$–open and $G_2$ is $C$–open. Since $G_1 \cap G_2 = \emptyset$, hence $G_2 \subseteq G_1^c$. In particular, since $G_1^c$ is an $\alpha$–closed subset of $X$ containing the $C$–open subset $G_2$ of $X$, by Lemma 3.1, there exists a closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq H_{1/2}^\Lambda \subseteq G_1^c.$$ 

Note that $H_{1/2}$ is also an $\alpha$–closed subset of $X$ and contains $G_2$, and $G_1^c$ is an $\alpha$–closed subset of $X$ and contains the $C$–open subset $H_{1/2}^\Lambda$ of $X$. Hence, by Lemma 3.1, there exists closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq H_{1/4}^\Lambda \subseteq H_{1/2} \subseteq H_{1/2}^\Lambda \subseteq H_{3/4} \subseteq H_{3/4}^\Lambda \subseteq G_1^c.$$ 

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of $2$, we obtain closed subsets $H_t$ with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function $h$ on $X$ by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_1$ and $h(x) = 1$ for $x \in G_1$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., $h$ maps $X$ into $[0,1]$. Also, we note that for any $t \in D, G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that $h$ is a contra-continuous function on $X$. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \bigcup\{H_t : t < \alpha\}$, hence, they are closed subsets of $X$. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = \bigcup\{(H_t)^c : t > \alpha\}$ hence, every of them is a closed subset. Consequently $h$ is a contra-continuous function.

**Lemma 3.3.** Suppose that $X$ is a topological space. If each pair of disjoint subsets $G_1, G_2$ of $X$, where $G_1$ is $\alpha$–open and $G_2$ is $C$–open, can separate by closed subsets
of $X$, and $G_1$ (resp. $G_2$) is a closed subsets of $X$, then there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and $h(G_2) = \{1\}$ (resp. $h(G_1) = \{1\}$).

**Proof.** Suppose that $G_1$ (resp. $G_2$) is a closed subset of $X$. By Lemma 3.2, there exists a contra-continuous function $h : X \to [0,1]$ such that, $h(G_1) = \{0\}$ (resp. $h(G_2) = \{0\}$) and $h(X \setminus G_1) = \{1\}$ (resp. $h(X \setminus G_2) = \{1\}$). Hence, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and since $G_2 \subseteq X \setminus G_1$ (resp. $G_1 \subseteq X \setminus G_2$), therefore $h(G_2) = \{1\}$ (resp. $h(G_1) = \{1\}$). 

**Lemma 3.4.** Suppose that $X$ is a topological space such that every two disjoint $C$–open and $\alpha$–open subsets of $X$ can be separated by closed subsets of $X$. The following conditions are equivalent:

(i) For every two disjoint subsets $G_1$ and $G_2$ of $X$, where $G_1$ is $\alpha$–open and $G_2$ is $C$–open, there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and $h^{-1}(1) = G_2$ (resp. $h^{-1}(1) = G_1$).

(ii) Every $\alpha$–open (resp. $C$–open) subset of $X$ is a closed subsets of $X$.

(iii) Every $\alpha$–closed (resp. $C$–closed) subset of $X$ is an open subsets of $X$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $G$ is an $\alpha$–open (resp. $C$–open) subset of $X$. Since $\varnothing$ is a $C$–open (resp. $\alpha$–open) subset of $X$, by (i) there exists a contra-continuous function $f : X \to [0,1]$ such that, $f^{-1}(0) = G$. Set $F_n = \{x \in X : f(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, $F_n$ is a closed subset of $X$ and $\bigcap_{n=1}^{\infty} F_n = \{x \in X : f(x) = 0\} = G$.

(ii) $\Rightarrow$ (i) Suppose that $G_1$ and $G_2$ are two disjoint subsets of $X$, where $G_1$ is $\alpha$–open and $G_2$ is $C$–open. By Lemma 3.3, there exists a contra-continuous function $f : X \to [0,1]$ such that, $f^{-1}(0) = G_1$ and $f(G_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two open subsets of $X$ and $(G \cup F) \cap G_2 = \varnothing$. By Lemma 3.3, there exists a contra-continuous function $g : X \to [\frac{1}{2},1]$ such that, $g^{-1}(1) = G_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define $h$ by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then $h$ is well-defined and a contra-continuous function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence $h$ defined on $X$ and maps to $[0,1]$. Also, we have $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$.

(iii) $\Leftrightarrow$ (iii) By De Morgan law and noting that the complement of every open subset of $X$ is a closed subset of $X$ and complement of every closed subset of $X$ is an open subset of $X$, the equivalence is hold. 

**Corollary 3.5.** If for every two disjoint subsets $G_1$ and $G_2$ of $X$, where $G_1$ is $\alpha$–open (resp. $C$–open) and $G_2$ is $C$–open (resp. $\alpha$–open), there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$ then $X$ has the strong cc–insertion property for $(cc, cCc)$ (resp. $(Cc, cCc)$).

**Proof.** Since for every two disjoint subsets $G_1$ and $G_2$ of $X$, where $G_1$ is $\alpha$–open (resp. $C$–open) and $G_2$ is $C$–open (resp. $\alpha$–open), there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, define $F_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $F_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then $F_1$ and $F_2$ are two disjoint closed subsets of $X$ that contain $G_1$ and $G_2$, respectively. Hence by Corollary 3.4, $X$ has the weak cc–insertion property for $(cc, cCc)$ and $(Cc, cCc)$. Now, assume that $g$ and $f$ are functions on $X$ such that $g \leq f$, $g$ is $cac$ (resp. $cCc$) and $f$ is $cc$. Since $f - g$ is $cac$ (resp. $cCc$), therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$
is an \( \alpha \)-open (resp. \( C \)-open) subset of \( X \). Now setting \( H_n = \{ x \in X : (f-g)(x) > 2^{-n} \} \) for every \( n \in \mathbb{N} \), then by Lemma 3.4, \( H_n \) is an open subset of \( X \) and we have \( \{ x \in X : (f-g)(x) > 0 \} = \bigcup_{n=1}^{\infty} H_n \) and for every \( n \in \mathbb{N} \), \( H_n \) and \( A(f-g, 2^{-n}) \) are disjoint subsets of \( X \). By Lemma 3.2, \( H_n \) and \( A(f-g, 2^{-n}) \) can be completely separated by contra-continuous functions. Hence by Theorem 2.2, \( X \) has the strong \( cc \)-insertion property for \( (coc, cc) \) (resp. \( (cCc, cc) \)).

By an analogous argument, we can prove that \( X \) has the strong \( cc \)-insertion property for \( (cc, cCc) \) (resp. \( (cc, coc) \)). Hence, by Theorem 2.3, \( X \) has the strong \( cc \)-insertion property for \( (coc, cCc) \) (resp. \( (cCc, cac) \)).

Acknowledgement
This research was partially supported by Centre of Excellence for Mathematics(University of Isfahan).

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