G-NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY AND NON-LIPSCHITZ COEFFICIENTS

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Abstract. This paper has two parts. In part I, existence and uniqueness theorem is established for solutions of neutral stochastic differential equations with variable delays driven by \( G \)-Brownian motion (VNSDDEGs in short) under global Carathéodory conditions. In part II, a simplified VNSDDEGs for the original one is proposed. And the convergence both in \( L^p \)-sense and capacity between the solutions of the simplified and original VNSDDEGs are established in view of the approximation theorems. Two examples are conducted to justify the theoretical results of the approximation theorems.

1. Introduction. The theory of nonlinear expectation paved the path to numerous applications in uncertainty problems, risk measures, the super hedging in finance and so on. Especially, the fundamental time-consistent theory of \( G \)-expectation and \( G \)-conditional expectation with \( G \) is the infinitesimal generator of a non-linear heat equation has been established by Peng [21]. In [21, 22], Peng provided the so-called \( G \)-normal distribution and set up the Itô stochastic calculus of it’s associated \( G \)-Brownian motion. Since then, a growing body of literature that deals with the stochastic analysis with respect to \( G \)-Brownian motion has been carried out (see, e.g., [6, 11, 14, 25, 29, 32, 38]). Gao in [8] proved the existence and uniqueness theorem of the solution to stochastic differential equations driven by \( G \)-Brownian motion (SDEGs in short) with Lipschitzian coefficients, which has been generalized by Bai and Lin [4] under integral Lipschitz conditions. Ren and Hu [24] studied the existence results of SDEGs under global Carathéodory conditions. For further work on SDEGs, the reader is refereed to [12, 15, 39, 40] and references therein.

Moreover, stochastic differential equations driven by Brownian motion (shortly, SDEs) are widely applied in engineering, system, ecological sciences and other areas. Central to the entire discipline of SDEs is the concept of stochastic functional differential equations (SFDEs) and neutral SFDEs (NSFDEs) describing systems with past history (delay) [19]. For the existence and uniqueness of solutions to SFDEs and NSFDEs, we refer to [31, 26, 5, 28, 27] and references therein. Following them,
Ren et al. [23] established the existence, uniqueness and exponential estimate of the solution to SFDEs with $G$-Brownian motion (SFDEGs in short) under the linear growth and Lipschitz conditions by successive approximation. Faizullah et al. in [7] provided the existence and mean-square stability of solutions to non-linear neutral SFDEs with $G$-Brownian motion (NSFDEGs) under the weakened linear growth and non-Lipschitz conditions. Very recently, Li and Yang [13] derived a numerical solutions for non-linear neutral stochastic delay differential equations driven by $G$-Brownian (NSDDEGs) using the stochastic theta and backward Euler-Maruyama (BEM) schemes and they proved that the BEM numerical solution is asymptotically mean-square stable under suitable conditions.

Recently, the approximation theorems as an averaging principle are fast becoming a key instrument in simplifying nonlinear dynamical systems to obtain approximate solutions for differential equations arising from mechanics, mathematics, physics, control and other areas. We refer the reader to [9, 20, 30, 35, 34, 37] and the references therein for a wealth of reference materials on the subject. Very recently, Abouagwa and Ji in [1] established two approximation theorems for solutions of non-Lipschitz stochastic fractional differential equations of Itô-Doob type. In particular, they proved the convergence in mean-square and probability between the solutions of the simplified and the original equations. Under polynomial growth conditions, He et al. [10] proposed the approximation theorems to solutions of NSDDEGs in view of the convergence in $L^p$-sense and capacity between the simplified and the original equations.

Based on the above discussion and to our best knowledge, no previous study has investigated the existence, uniqueness and approximation theorems to neutral stochastic differential equations driven by $G$-Brownian motion and variable delay (VNSDDEGs in short). Motivated by [3, 16, 17, 18], this paper aims to establish the existence and uniqueness of solutions in the sense of $L^p$-norm to VNSDDEGs under some Carathéodory conditions by means of Carathéodory approximation. Moreover, the approximation properties of the solutions to VNSDDEGs will be studied from the approximation theorems point of view. In particular, we will prove that the solution of the simplified VNSDDEGs converges to that of the original one both in $L^p$-sense and capacity. It is worth mentioning that some results in He et al. [10] are generalized and improved.

The present article organized as follows. In section 2, we present some preliminaries and several lemmas. Section 3 is devoted to stating the assumptions and proving the existence and uniqueness theorem of solutions to VNSDDEGs. We prove the approximation theorems of solutions to VNSDDEGs in section 4. In the last section, two examples are given to illustrate the effectiveness of the theoretical results obtained by the approximation theorems.

2. Preliminaries.

2.1. Itô calculus of $G$-Brownian motion. We recall some basic definitions and notations from [21, 22, 6, 11, 32], which will be needed in the sequel. Assume $\mathcal{H}$ be a linear space of real valued functions defined on a nonempty basic space $\Omega$ such that $c \in \mathcal{H}$ is constant and for all $\psi \in \mathcal{C}_{t,\text{lip}}(\mathbb{R}^n)$, $x_1, x_2, ..., x_n \in \mathcal{H}$, $\psi(x_1, x_2, ..., x_n) \in \mathcal{H}$, where $\mathcal{C}_{t,\text{lip}}(\mathbb{R}^n)$ is the space of linear functions $\psi$ defined as\
$$\mathcal{C}_{t,\text{lip}}(\mathbb{R}^n) = \left\{ \psi : \mathbb{R}^n \rightarrow \mathbb{R} | \exists C \in \mathbb{R}^+, m \in N s.t. |\psi(x) - \psi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \right\},$$
for \( x, y \in \mathbb{R}^n \). We consider that \( \mathcal{H} \) is the space of random variables.

**Definition 2.1.** An expectation \( \hat{E} : \mathcal{H} \rightarrow \mathbb{R} \) is said to be sub-linear expectation if the following properties are satisfied:

(i) Monotonicity: \( \hat{E}[x] \geq \hat{E}[y] \), for \( x \geq y \);
(ii) Preserving of constant: \( \hat{E}[c] = c \);
(iii) Subadditivity: \( \hat{E}[x + y] \geq \hat{E}[x] + \hat{E}[y] \) or \( \hat{E}[x] - \hat{E}[y] \geq \hat{E}[x - y] \);
(iv) Positive homogeneity: \( \hat{E}[\lambda x] = \lambda \hat{E}[x] \),

for all \( x, y \in \mathcal{H} \), \( c \in \mathbb{R} \) and \( \lambda \geq 0 \).

The triple \((\Omega, \mathcal{H}, \hat{E})\) is said to be sub-linear expectation space. Moreover, it is relevantly called nonlinear expectation space if the first two properties, (i) and (ii) are only satisfied.

**Definition 2.2.** A random variable \( x \in \mathcal{H} \) in a sub-linear expectation space \((\Omega, \mathcal{H}, \hat{E})\), with \( \sigma^2 = \hat{E}[x^2] \) and \( \bar{\sigma}^2 = -\hat{E}[-x^2] \), is called \( G \)-normal distributed or \( \mathcal{N}(0; [\sigma^2, \bar{\sigma}^2]) \)-distributed; if for all \( a, b \geq 0 \) and \( y \in \mathcal{H} \),

\[
ax + by \sim \sqrt{a^2 + b^2} x,
\]

where \( y \) is independent and identically distributed with \( x \) \((y \sim x)\).

Let \((\Omega, \rho)\) be a metric space under the norm

\[
\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i} \left( \max_{t \in [0,i]} |\omega^1_t - \omega^2_t| \wedge 1 \right), \quad \omega^1, \omega^2 \in \Omega
\]

where \( \Omega = C_0(\mathbb{R}^+) \) is the space of all \( \mathbb{R} \)-valued continuous functions \( \{\omega_i\}_{i \in \mathbb{R}^+} \) with \( \omega_0 = 0 \). For each \( \omega \in \Omega \), we have the canonical process \( B_t(\omega) = \omega_t \), for all \( t \in [0, \infty) \), which generates a filtration defined by \( \mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\} \), \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \). Then, for each fixed \( T \in [0, \infty) \), we have

\[
L_{ip}(\Omega_T) = \{ \psi(B_{t_1}, B_{t_2}, ..., B_{t_n}) : t_1, ..., t_n \in [0, T], \psi \in \mathcal{C}_{l,ip}(\mathbb{R}^n), n \in \mathbb{N} \},
\]

where \( L_{ip}(\Omega_T) \subseteq L_{ip}(\Omega_T), \ t \leq T \) and \( L_{ip}(\Omega_T) = \bigcup_{m=1}^{\infty} L_{ip}(\Omega_m) \).

**Definition 2.3.** The canonical process \( \{B_t, t \geq 0\} \) defined on a sub-linear expectation space \((\Omega, L_{ip}(\Omega), \hat{E})\) and satisfying

(i) \( B_0 = 0 \);
(ii) The increment \( B_{t+s} - B_t \) is independent of \( B_{t_1}, B_{t_2}, ..., B_{t_n} \), for every \( n \in \mathbb{Z}^+ \), and \( 0 \leq t_1 \leq ... \leq t_n \leq t \);
(iii) The increment \( B_{t+s} - B_t \) is \( \mathcal{N}(0; [s\sigma^2, s\bar{\sigma}^2]) \)-distributed,

for all \( 0 \leq s < t \), is called \( G \)-Brownian motion provided that the expectation \( \hat{E} : L_{ip}(\Omega) \rightarrow \mathbb{R} \) is a \( G \)-expectation.

Denote by \( L^p_{G}(\Omega), \ p \geq 1 \) to the completion of \( L_{ip}(\Omega) \) under the norm \( ||x||_p = \langle \hat{E}[|x|^p] \rangle^{\frac{1}{p}} \) and \( L^p_{G}(\Omega_T) \subseteq L^p_{G}(\Omega_T) \subseteq L^p_{G}(\Omega) \) for all \( 0 \leq t \leq T < \infty \). A partition \( \pi_T, T \in \mathbb{R}^+ \) of \([0, T]\) is a finite-ordered subset \( \pi = \{t_1, ..., t_N\} \) such that \( 0 = t_0 < t_1 < ... < t_N = T \). For fixed \( p \geq 1 \), define

\[
M_{G}^{p,0}(0, T) = \bigg\{ \eta(\omega) = \eta(t, \omega) = \sum_{i=1}^{N} \zeta_{i-1}(\omega) I_{[t_{i-1}, t_i]}(t); \zeta_{i-1} \in L^p_{G}(\mathcal{F}_{t_{i-1}}), t_{i-1} < t_i, i = 1, 2, ..., N, t_0 = 0, t_N = T, N \geq 1 \bigg\}.
\]
where \( L^p_G(\mathcal{F}_t) = \{ \zeta \in L^1_G(\mathcal{F}_t); \mathbb{E} |\zeta|^p < \infty \} \). For every \( p \geq 1 \), \( M^p_G(0, T) \) refers to the completion of \( M^p_G(0, T) \) under the norm
\[
||\eta||_{M^p_G(0,T)} := \frac{1}{T} \left( \int_0^T \mathbb{E}[|\eta_t|^p] dt \right)^{\frac{1}{p}}.
\]

**Definition 2.4.** For \( \eta_t(\omega) = \sum_{i=0}^{N-1} \zeta_i(\omega) I_{[t_i,t_{i+1})}(t) \in M^p_G(0, T) \), the Itô integral with respect to \( G \)-Brownian motion is defined as
\[
I(\eta) = \int_0^T \eta_t dB_t = \sum_{i=0}^{N-1} \zeta_i(B_{t_{i+1}} - B_{t_i}).
\]

Note that the mapping \( I : M^{2,0}_G(0, T) \to L^2_G(\mathcal{F}_T) \) is linear and continuous. Therefore, it can be extended continuously to \( I : M^p_G(0, T) \to L^p_G(\mathcal{F}_T) \).

**Definition 2.5.** For \( t > 0 \), a sequence of partitions \( \pi^N = \{ t_0^N, t_1^N, ..., t_N^N \} \), \( N = 1, 2, ... \) of \([0,T]\) with the mesh \( \mu(\pi^N) = \max_{0 \leq i \leq N} |t_i^N - t_{i-1}^N| \to 0 \) as \( N \to \infty \), define the quadratic variation process of the \( G \)-Brownian motion by
\[
\langle B \rangle_t := \lim_{\mu(\pi^N) \to 0} \sum_{i=0}^{N-1} (B_{t_{i+1}}^{B^N} - B_{t_i}^{B^N})^2 = \langle B \rangle_t^2 - 2 \int_0^t B_s dB_s.
\]

For any \( \eta_t \in M^{1,0}_G([0,T]), \) define the map \( Q_{0,T}(\eta) : M^{1,0}_G([0,T]) \to L^1_G(\mathcal{F}_T) \) by
\[
Q_{0,T}(\eta) = \int_0^T \eta_t dB_t := \sum_{i=0}^{N-1} \zeta_i (\langle B \rangle_{t_{i+1}} - \langle B \rangle_{t_i}),
\]
then \( Q_{0,T}(\eta) \) is linear and continuous. Hence, it can be extended continuously to \( M^{1,0}_G([0,T]) \).

**Definition 2.6.** Assume \( \mathcal{B}(\Omega) \) be the Borel \( \sigma \)-algebra of \( \Omega \) and \( \mathcal{P} \) be a weakly compact family of probability measures \( P \) defined on \( (\Omega, \mathcal{B}(\Omega)) \); then, the capacity \( \check{c}(\cdot) \) associated to \( \mathcal{P} \) is defined as
\[
\check{c}(A) = \sup_{P \in \mathcal{P}} P(A), \ A \in \mathcal{B}(\Omega).
\]

A set \( A \in \mathcal{B}(\Omega) \) is said to be polar if \( \check{c}(A) = 0 \), and a property holds quasi-surely (q.s.) if it holds outside a polar set.

**2.2. Several lemmas.**

**Lemma 2.7.** [18] Let \( p \geq 2, \alpha > 0 \) and \( a,b \in \mathbb{R} \), then
\[
|a + b|^p \leq \left( 1 + \alpha \frac{|b|}{|a|} \right)^{p-1} |a|^p + \frac{|b|^p}{\alpha}.
\]

**Lemma 2.8.** [14, 24] Let \( p \geq 1 \) and \( \phi \in M^p_G([0,T]; \mathbb{R}^d) \). Then, for all \( T \geq 0 \),
\[
\mathbb{E} \left[ \int_0^T |\phi_t|^p dt \right] \leq T^{p-1} \int_0^T \mathbb{E}[|\phi_t|^p] dt,
\]
\[
\mathbb{E} \left[ \left| \int_0^T \phi_t dB_t \right|^p \right] \leq T^{p-1} \int_0^T \mathbb{E}[|\phi_t|^p] dt.
\]
and for every \( p \geq 2 \),
\[
\hat{\mathbb{E}} \left| \int_0^T \phi_t dB_t \right|^p \leq C_p T^\frac{p}{2} \int_0^T \hat{\mathbb{E}} |\phi_t|^p dt,
\]
where \( 0 < C_p < \infty \) is positive constant and independent of \( T \).

The following lemma is taken from [23].

**Lemma 2.9. (Doob’s G-martingale inequality)** For \( p > 1 \), any \( \mathbb{R}^d \)-value G-martingale \( x \in \mathcal{M}_G^p([0, T]; \mathbb{R}^d) \) and \( a \leq t \leq b \) such that \([a, b]\) is a bounded interval of \( \mathbb{R}^+ \), we have
\[
\hat{\mathbb{E}} \left( \sup_{a \leq t \leq b} |x(t)|^p \right) \leq \hat{C}_p \mathbb{E}|x(b)|^p,
\]
where \( \hat{C}_p = \left( \frac{p}{p-1} \right)^p > 0 \) is constant.

The following lemma gives Markov inequality in the G-framework.

**Lemma 2.10.** [4] Let \( x \in \mathcal{M}_G^p([0, T]; \mathbb{R}^d) \), then for some \( p > 0 \), each \( \alpha > 0 \), \( \hat{\mathbb{E}}|x(t)|^p < \infty \), we have
\[
\hat{\mathbb{E}} \left( \{ |x(t)| > \alpha \} \right) \leq \frac{\hat{\mathbb{E}}|x(t)|^p}{\alpha^p}.
\]

3. NSDEGs with variable delay.

3.1. **Notations and assumptions.** Let \( \tau > 0 \) and \( C([\tau, 0]; \mathbb{R}^d) \) be the family of continuous \( \mathbb{R}^d \)-valued functions \( \varphi \) defined on \([\tau, 0]\) with norm
\[
||\varphi||_\infty = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|.
\]

Consider the following \( d \)-dimensional VNSDDEGs:
\[
d[X(t) - G(X(t - \delta(t)))] = b(t, X(t), X(t - \delta(t)))dt + g(t, X(t), X(t - \delta(t)))d(B, B)_t + h(t, X(t), X(t - \delta(t)))dB_t, \quad t \in [0, T]
\]
with initial condition \( \xi(0) \in \mathbb{R}^d \) and initial data \( X(0) = \xi = \{ \xi(\theta) : -\tau \leq \theta \leq 0 \} \) is an \( \mathcal{F}_0 \)-measurable \( C([\tau, 0]; \mathbb{R}^d) \)-valued random variable such that \( \mathbb{E}||\xi||^p < \infty \). \( B_t = (B_1^T, B_2^T, \ldots, B_m^T) \) is an \( m \)-dimensional G-Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, P)\), with a filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) satisfying the usual conditions, \( \{ (B, B)_t, t \geq 0 \} \) is the quadratic variation process of G-Brownian motion and \( \delta : [0, T] \rightarrow [0, \tau] \). The functions \( G : \mathbb{R}^d \rightarrow \mathbb{R}^d, b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) and \( g, h : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m} \) are Borel measurable functions as well as \( G, b, g, h \in \mathcal{M}_G^p([\tau, T]; \mathbb{R}^d) \).

**Definition 3.1.** An \( \mathbb{R}^d \)-valued stochastic process \( \{ X(t) \}_{0 \leq t \leq T} \) is called a unique solution to Eq. (1) if it has the following properties:

(i) \( \{ X(t) \} \) is \( t \)-continuous and \( \mathcal{F}_t \) adapted;

(ii) \( \int_0^T |X(t)|^p < \infty \), q.s. for any \( p \geq 2 \);
(iii) $X(0) = \xi$ and, for all $t \in [0, T]$,
\[
X(t) = \xi(0) - G(\xi(-\delta(t))) + G(X(t - \delta(t))) + \int_{0}^{t} b(s, X(s), X(s - \delta(s)))ds
\]
\[
+ \int_{0}^{t} g(s, X(s), X(s - \delta(s)))d\langle B, B \rangle_{s}
\]
\[
+ \int_{0}^{t} h(s, X(s), X(s - \delta(s)))dB_{s} \text{, q.s.}
\]
(2)

(iv) For any other solution $\bar{X}(t)$, we have $P\{X(t) = \bar{X}(t), \forall -\tau \leq t \leq T\} = 1$.

For studying the existence and uniqueness theorem of solutions to Eq. (1), we consider the following hypotheses:

(H1). There exists a function $F(t, u) : [0, +\infty) \times \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$ such that
(a) $F(t, u)$ is locally integrable in $t \geq 0$ for each fixed $u \in \mathbb{R}^{+}$ and is continuous, nondecreasing and concave in $u$ for each fixed $t \geq 0$. Moreover, $F(t, 0) = 0$;
(b) For $p \geq 2$, all $x_{1}, x_{2}, y_{1}, y_{2} \in \mathbb{R}^{d}$ and $t \in [0, T]$, we have
\[
\mathbb{E}|b(t, x_{1}, y_{1}) - b(t, x_{2}, y_{2})|^{p} + \mathbb{E}|g(t, x_{1}, y_{1}) - g(t, x_{2}, y_{2})|^{p}_{HS}
\]
\[
+ \mathbb{E}|h(t, x_{1}, y_{1}) - h(t, x_{2}, y_{2})|^{p}_{HS}
\]
\[
\leq F \left(t, \mathbb{E}|x_{1} - x_{2}|^{p} + \mathbb{E}|y_{1} - y_{2}|^{p}\right);
\]
(c) If a non-negative continuous function $Z(t)$, $0 \leq t \leq T$, satisfies
\[
Z(t) \leq D \int_{0}^{t} F(s, Z(s))ds, \quad 0 \leq t \leq T,
\]
where $D > 0$ is positive constant, then $Z(t) \equiv 0$ for all $t \in [0, T]$.

(H2). There exists a function $H(t, u) : [0, +\infty) \times \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$ such that
(a) $H(t, u)$ is locally integrable in $t \geq 0$ for each fixed $u \in \mathbb{R}^{+}$ and is continuous monotone, nondecreasing and concave in $u$ for each fixed $t \geq 0$;
(b) For $p \geq 2$, all $x, y \in \mathbb{R}^{d}$ and $t \in [0, T]$, we have
\[
\mathbb{E}|b(t, x, y)|^{p} + \mathbb{E}|g(t, x, y)|^{p}_{HS} + \mathbb{E}|h(t, x, y)|^{p}_{HS} \leq H \left(t, \mathbb{E}|x|^{p} + \mathbb{E}|y|^{p}\right);
\]
(c) The deterministic differential equation
\[
\frac{du}{dt} = KH(t, u),
\]
has a global solution $u_{t}$ for any initial value $u_{0}$ and positive constant $K$.

(H3). For all $x, y \in \mathbb{R}^{d}$, there exists a constant $k_{0} \in (0, 1)$ such that
\[
|G(x) - G(y)| \leq k_{0}|x - y|, \quad \text{and} \quad G(0) = 0.
\]

3.2. Existence and uniqueness theorem. We establish the existence and uniqueness of solutions to Eq. (1) under Carathéodory type conditions.

Define the Carathéodory approximation as follows. For $T > 0$, any integer $n \geq 2/\tau$, define $X_{n}(t) : [-\tau, T] \times \Omega \rightarrow \mathbb{R}^{d}$ by
\[
X_{n}(t) = \xi(\theta), \quad -\tau \leq \theta \leq 0,
\]
and
\[ X_n(t) = \xi(0) - G(\xi(-\delta(t))) + G(X_n(t - \delta(t))) \]
\[ + \int_0^t 1_{D_n}(s) \left( s, X_n \left( s - \frac{1}{n} \right), X_n(s - \delta(s)) \right) ds \]
\[ + \int_0^t 1_{D_n}(s) g \left( s, X_n \left( s - \frac{1}{n} \right), X_n(s - \delta(s)) \right) dB_s \]
\[ + \int_0^t 1_{D_n}(s) h \left( s, X_n \left( s - \frac{1}{n} \right), X_n(s - \delta(s)) \right) dB_s \]
\[ + \int_0^t 1_{D_n}(s) \left( s, X_n \left( s - \frac{1}{n} \right), X_n \left( s - \delta(s) - \frac{1}{n} \right) \right) ds \]
\[ + \int_0^t 1_{D_n}(s) g \left( s, X_n \left( s - \frac{1}{n} \right), X_n \left( s - \delta(s) - \frac{1}{n} \right) \right) d(B, B)_s \]
\[ + \int_0^t 1_{D_n}(s) h \left( s, X_n \left( s - \frac{1}{n} \right), X_n \left( s - \delta(s) - \frac{1}{n} \right) \right) dB_s, t \in [0, T]. \]

where \( D_n = \{ t : \delta(t) < \frac{1}{n}, 0 \leq t \leq T \} \) and \( D_n^c = [0, T] - D_n \). The functions \( 1_{D_n} \) and \( 1_{D_n^c} \) denote the indicator functions of \( D_n \) and \( D_n^c \), respectively.

**Theorem 3.2.** Let \( p \geq 2 \) and assume that hypotheses (H1)-(H3) are satisfied, then Eq. (1) has a unique solution \( X(t) \) on \([0, T]\) in the sense of \( L^p \)-norm.

In order to prove this theorem, let us present two useful lemmas.

**Lemma 3.3.** Under hypotheses (H2) and (H3), the sequence \( \{X_n(t), n \geq 2/\tau\} \) is bounded.

**Proof.** For any \( \alpha > 0 \), it immediately follows from Lemma 2.7 that
\[ |X_n(t)|^p = |G(X_n(t - \delta(t))) + X_n(t) - G(X_n(t - \delta(t)))|^p \]
\[ \leq \left( 1 + \alpha \frac{T}{p^\tau} \right)^{p-1} \left[ |X_n(t) - G(X_n(t - \delta(t)))|^p + \frac{|G(X_n(t - \delta(t)))|^p}{\alpha} \right]. \]

Letting \( \alpha = \left( \frac{k_0}{1-k_0} \right)^{p-1} \) and by using (H3)
\[ |X_n(t)|^p \leq k_0 |X_n(t - \delta(t))|^p + \frac{1}{(1-k_0)^{p-1}} |X_n(t) - G(X_n(t - \delta(t)))|^p. \]

Taking G-expectation, we have
\[ \mathbb{E} \left( \sup_{0 \leq s \leq t} |X_n(s)|^p \right) \leq k_0 \mathbb{E} \left( \sup_{0 \leq s \leq t} |X_n(s - \delta(s))|^p \right) \]
\[ + \frac{\mathbb{E} \left( \sup_{0 \leq s \leq t} |X_n(s) - G(X_n(s - \delta(s)))|^p \right)}{(1-k_0)^{p-1}}. \]

on the other hand, we have
\[ \mathbb{E} \left( \sup_{0 \leq s \leq t} |X_n(s - \delta(s))|^p \right) \leq \mathbb{E} \left( \sup_{-\tau \leq s \leq t} |X_n(s)|^p \right) \]
\[ \leq \mathbb{E} |\xi|^p + \mathbb{E} \left( \sup_{0 \leq s \leq t} |X_n(s)|^p \right) \]
\[ (5)\]
Combining Eqs. (4) and (5), we obtain
\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} |X_n(s)|^p \right) \leq \frac{k_0}{1 - k_0} \mathbb{E} |\xi|^p \\
+ \frac{1}{(1 - k_0)^p} \mathbb{E} \left( \sup_{0 \leq s \leq t} |X_n(s) - G(X_n(s - \delta(s)))|^p \right) \tag{6}
\]

By Eq. (3) and using the inequality
\[
|x_1 + x_2 + ... + x_m|^q \leq m^{q-1}(|x_1|^q + |x_2|^q + ... + |x_m|^q),
\tag{7}
\]
we have
\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} |X_n(s) - G(X_n(s - \delta(s)))|^p \right) \\
\leq 7^{p-1} \mathbb{E} \left( \sup_{0 \leq s \leq t} |\xi(0) - G(\xi(-\delta(t)))|^p \right) \\
+ 7^{p-1} \mathbb{E} \left( \sup_{0 \leq s \leq t} \left| \int_0^s 1_{D_n}(u) b \left( u, X_n \left( u - \frac{1}{n} \right), X_n(u - \delta(u)) \right) du \right|^p \right) \\
+ 7^{p-1} \mathbb{E} \left( \sup_{0 \leq s \leq t} \left| \int_0^s 1_{D_n}(u) \right| d(B,B)u \right|^p \right) \\
+ 7^{p-1} \mathbb{E} \left( \sup_{0 \leq s \leq t} \left| \int_0^s 1_{D_n}(u) \right| dB_u \right|^p \right) \\
+ 7^{p-1} \mathbb{E} \left( \sup_{0 \leq s \leq t} \left| \int_0^s 1_{D_n}(u) \right| dB_u \right|^p \right) \\
:= 7^{p-1} \sum_{i=1}^7 I_i \tag{8}
\]

Now, useful estimates will be introduced for the above terms. Taking \( \alpha = k_0^{p-1} \), we have by hypothesis (H3) and Lemma 2.7 that
\[
I_1 \leq \mathbb{E} \left( \sup_{0 \leq s \leq t} \left( 1 + \alpha^{\frac{1}{p - 1}} \right)^{p-1} \left( |\xi(0)|^p + \frac{|G(\xi(-\delta(t)))|^p}{\alpha} \right) \right) \\
\leq (1 + k_0)^{p-1} \mathbb{E} \left[ ||\xi||^p + \frac{k_0^p ||\xi||^p}{k_0^{p-1}} \right] \leq (1 + k_0)p \mathbb{E} ||\xi||^p \tag{9}
\]
By using Hölder’s inequality, Lemmas 2.8 and 2.9, inequality (7) and hypothesis (H2)b, one gets

\[ I_2 + I_3 + I_4 \]
\[ \leq T^{p-1} \int_0^t 1_{D_n}(s) \mathbb{E} \left| b \left( s, X_n \left( s - \frac{1}{n} \right), X_n(s - \delta(s)) \right) \right|^p ds \]
\[ + T^{p-1} \hat{C}_p \int_0^t 1_{D_n}(s) \mathbb{E} \left| g \left( s, X_n \left( s - \frac{1}{n} \right), X_n(s - \delta(s)) \right) \right|^p ds \]
\[ + T^{p-1} \hat{C}_p \int_0^t 1_{D_n}(s) \mathbb{E} \left| h \left( s, X_n \left( s - \frac{1}{n} \right), X_n(s - \delta(s)) \right) \right|^p ds \]
\[ \leq T^{\frac{p}{2} - 1} \left[ T^{\frac{p}{2}} (1 + \hat{C}_p) + \hat{C}_p C_p \right] \]
\[ \times \int_0^t 1_{D_n}(s) H \left( s, \mathbb{E} \left| X_n \left( s - \frac{1}{n} \right) \right|^p + \mathbb{E} \left| X_n(s - \delta(s)) \right|^p \right) ds. \quad (10) \]

Similarly, for \( I_5, I_6 \) and \( I_7 \), we get

\[ I_5 + I_6 + I_7 \]
\[ \leq T^{\frac{p}{2} - 1} \left[ T^{\frac{p}{2}} (1 + \hat{C}_p) + \hat{C}_p C_p \right] \]
\[ \times \int_0^t 1_{D_n}(s) H \left( s, \mathbb{E} \left| X_n \left( s - \frac{1}{n} \right) \right|^p + \mathbb{E} \left| X_n(s - \delta(s)) \right|^p \right) ds. \quad (11) \]

By using Eqs. (8)-(11), we obtain

\[ \mathbb{E} \left( \sup_{0 \leq s \leq t} |X_n(s) - G(X_n(s - \delta(s)))|^p \right) \]
\[ \leq \theta^{p-1} (1 + k_0)^p \mathbb{E}||\xi||^p + \theta^{p-1} T^\frac{p}{2} - 1 \left[ T^\frac{p}{2} (1 + \hat{C}_p) + \hat{C}_p C_p \right] \]
\[ \times \int_0^t 1_{D_n}(s) H \left( s, \mathbb{E} \left| X_n \left( s - \frac{1}{n} \right) \right|^p + \mathbb{E} \left| X_n(s - \delta(s)) \right|^p \right) ds \]
\[ + \theta^{p-1} T^\frac{p}{2} - 1 \left[ T^\frac{p}{2} (1 + \hat{C}_p) + \hat{C}_p C_p \right] \]
\[ \times \int_0^t 1_{D_n}(s) H \left( s, \mathbb{E} \left| X_n \left( s - \frac{1}{n} \right) \right|^p + \mathbb{E} \left| X_n(s - \delta(s)) \right|^p \right) ds \]
\[ \leq \theta^{p-1} (1 + k_0)^p \mathbb{E}||\xi||^p + \theta^{p-1} T^\frac{p}{2} - 1 \left[ T^\frac{p}{2} (1 + \hat{C}_p) + \hat{C}_p C_p \right] \]
\[ \times \int_0^t H \left( s, 2\mathbb{E}||\xi||^p + 2\mathbb{E} \left( \sup_{0 \leq u \leq s} |X_n(u)|^p \right) \right) ds \quad (12) \]

Substituting from Eq. (12) into Eq. (6), we have

\[ \mathbb{E} \left( \sup_{0 \leq s \leq t} |X_n(s)|^p \right) \]
\[ \leq \frac{k_0(1 - k_0)^{p-1} + \theta^{p-1}(1 + k_0)^p \mathbb{E}||\xi||^p}{(1 - k_0)^p} \]
\[ + \frac{\theta^{p-1} T^\frac{p}{2} - 1 \left[ T^\frac{p}{2} (1 + \hat{C}_p) + \hat{C}_p C_p \right]}{(1 - k_0)^p} \]
\[ \times \int_0^t H \left( s, 2\mathbb{E}||\xi||^p + 2\mathbb{E} \left( \sup_{0 \leq u \leq s} |X_n(u)|^p \right) \right) ds \]
Moreover,
\[
\hat{E} \left( 2||\xi||^p + 2 \sup_{0 \leq s \leq t} |X_n(s)|^p \right) \\
\leq \frac{2(1 - k_0)^p + 2[k_0(1 - k_0)^{p-1} + 7^{p-1}(1 + k_0)^p]}{(1 - k_0)^p} \hat{E}||\xi||^p \\
+ \frac{7^{p-1} T^{\frac{p}{p-1}} \left[ 2T \hat{\xi} (1 + \hat{C}_p) + 2\hat{C}_p C_p \right]}{(1 - k_0)^p} \\
\times \int_0^t H \left( s, \hat{E} \left( 2||\xi||^p + 2 \sup_{0 \leq u \leq s} |X_n(u)|^p \right) \right) \mathrm{d}s.
\]
Then, hypothesis (H2)c implies that there is a solution \( u_t \) satisfying
\[
u_t \leq \frac{2(1 - k_0)^p + 2[k_0(1 - k_0)^{p-1} + 7^{p-1}(1 + k_0)^p]}{(1 - k_0)^p} \hat{E}||\xi||^p \\
+ \frac{7^{p-1} T^{\frac{p}{p-1}} \left[ 2T \hat{\xi} (1 + \hat{C}_p) + 2\hat{C}_p C_p \right]}{(1 - k_0)^p} \int_0^t H \left( s, u_s \right) \mathrm{d}s.
\]
Since \( \hat{E}||\xi||^p < \infty \), we deduce that
\[
\hat{E} \left( \sup_{0 \leq s \leq t} |X_n(s)|^p \right) \leq u_t \leq u_T < \infty,
\]
which, shows the boundedness of the sequence \( \{X_n(t), n \geq 2/\tau\} \). Hence, proved. \( \square \)

**Lemma 3.4.** Assume that hypotheses (H2) and (H3) are satisfied. Then there exists a positive constant \( C_1 \) such that
\[
\hat{E}|X_n(t) - X_n(s)|^p \leq C_1(t - s),
\]
for all \( 0 \leq s < t \leq T \) and \( n \geq 2/\tau \).

**Proof.** Let
\[
X_n(t) - X_n(s) = G(X_n(t - \delta(t))) - G(X_n(s - \delta(s))) + J_n(t, s).
\]
Then lemma 2.7 reads
\[
|X_n(t) - X_n(s)|^p \\
\leq (1 + \alpha^{\frac{1}{p-1}})^{p-1} \left[ |J_n(t, s)|^p + \frac{|G(X_n(t - \delta(t))) - G(X_n(s - \delta(s)))|^p}{\alpha} \right].
\]
Letting \( \alpha = \left( \frac{k_0}{1 - k_0} \right)^{p-1} \), we have by hypothesis (H3)
\[
\hat{E}|X_n(t) - X_n(s)|^p \leq k_0 \hat{E}|X_n(t) - X_n(s)|^p + \frac{1}{(1 - k_0)^{p-1}} \hat{E}|J_n(t, s)|^p \\
\leq \frac{1}{(1 - k_0)^p} \hat{E}|J_n(t, s)|^p, \quad (14)
\]
where
\[
\hat{E}|\mathcal{J}_n(t, s)|^p \\
\leq 6^{p-1} \hat{E} \left| \int_s^t 1_{D_n}(u)b \left( u, X_n \left( u - \frac{1}{n} \right), X_n(u - \delta(u)) \right) du \right|^p \\
+ 6^{p-1} \hat{E} \left| \int_s^t 1_{D_n}(u)g \left( u, X_n \left( u - \frac{1}{n} \right), X_n(u - \delta(u)) \right) d(B, B)_u \right|^p \\
+ 6^{p-1} \hat{E} \left| \int_s^t 1_{D_n}(u)h \left( u, X_n \left( u - \frac{1}{n} \right), X_n(u - \delta(u)) \right) dB_u \right|^p \\
+ 6^{p-1} \hat{E} \left| \int_s^t 1_{D_n}(u)g \left( u, X_n \left( u - \frac{1}{n} \right), X_n(u - \delta(u) - \frac{1}{n}) \right) d(B, B)_u \right|^p \\
+ 6^{p-1} \hat{E} \left| \int_s^t 1_{D_n}(u)h \left( u, X_n \left( u - \frac{1}{n} \right), X_n(u - \delta(u) - \frac{1}{n}) \right) dB_u \right|^p \\
:= 6^{p-1} \sum_{i=1}^{6} J_i
\]

By using Hölder’s inequality, Lemmas 2.8 and hypothesis (H2), we conclude
\[
J_1 + J_2 + J_3 \leq (T - s)^{\frac{p}{2} - 1} \left( 2(T - s)^{\frac{p}{2}} + C_p \right) \int_s^t 1_{D_n}(u) \\
× H \left( u, \hat{E} \left| X_n \left( u - \frac{1}{n} \right) \right|^p + \hat{E} |X_n(u - \delta(s))|^p \right) du. \tag{16}
\]

Similarly with Eq. (16), we get
\[
J_4 + J_5 + J_6 \leq (T - s)^{\frac{p}{2} - 1} \left( 2(T - s)^{\frac{p}{2}} + C_p \right) \\
× \int_s^t 1_{D_n}(u)H \left( u, \hat{E} \left| X_n \left( u - \frac{1}{n} \right) \right|^p \right) \\
+ \hat{E} \left| X_n \left( u - \delta(u) - \frac{1}{n} \right) \right|^p du. \tag{17}
\]

Then from Eqs. (15)-(17) and Lemma 3.3, we obtain
\[
\hat{E}|\mathcal{J}_n(t, s)|^p \leq 6^{p-1}(T - s)^{\frac{p}{2} - 1} \left( 2(T - s)^{\frac{p}{2}} + C_p \right) \\
× \int_s^t H \left( u, 2\hat{E} \left( \sup_{0 \leq v \leq u} |X_n(v)|^p \right) \right) du \\
\leq 6^{p-1}(T - s)^{\frac{p}{2} - 1} \left( 2(T - s)^{\frac{p}{2}} + C_p \right) \\
× \left( \sup_{0 \leq t \leq T} H(t, 2C) \right) (t - s), \tag{18}
\]

where the constant $C$ comes from the boundedness of the sequence $\{X_n(t), n \geq 2/\tau\}$ by Lemma 3.3.
Combining Eqs. (14) and (18), the required result will be obtained with a positive constant $C_1 = 6^{n-1}(T-s)^\frac{n}{2} (2(T-s)^\frac{n}{2} + C_n) \left( \sup_{0 \leq s \leq T} H(t,2C) \right)$ and the proof of Lemma 3.4 is completed.

**Proof of Theorem 3.2.** We claim that $\{X_n(t), n \geq 2/\tau \}$ is a cauchy sequence in $M^p_G([0,T];\mathbb{R}^d)$. For $m > n \geq 2/\tau$, by similar analysis of Eq. (14), Lemma 2.7 and taking $G$-expectation to $|X_m(t) - X_n(t)|^p$, one may get

$$
\mathbb{E} \left( \sup_{0 \leq s \leq t} |X_m(s) - X_n(s)|^p \right) \leq \frac{1}{(1-k_0)^p} \mathbb{E} \left( \sup_{0 \leq s \leq t} |\Lambda_n(s)|^p \right),
$$

where

$$
\Lambda_n(t) = \int_0^t \left\{ 1_{D_n^m}(s)b \left( s, X_m \left( s - \frac{1}{m} \right), X_m(s - \delta(s)) \right) \\
-1_{D_n^m}(s)b \left( s, X_n \left( s - \frac{1}{n} \right), X_n(s - \delta(s)) \right) \right\} ds \\
+ \int_0^t \left\{ 1_{D_n^m}(s)g \left( s, X_m \left( s - \frac{1}{m} \right), X_m(s - \delta(s)) \right) \\
-1_{D_n^m}(s)g \left( s, X_n \left( s - \frac{1}{n} \right), X_n(s - \delta(s)) \right) \right\} d(B,B)_s \\
+ \int_0^t \left\{ 1_{D_n^m}(s)h \left( s, X_m \left( s - \frac{1}{m} \right), X_m(s - \delta(s)) \right) \\
-1_{D_n^m}(s)h \left( s, X_n \left( s - \frac{1}{n} \right), X_n(s - \delta(s)) \right) \right\} dB_s
$$

$$
:= \sum_{i=1}^6 \Lambda_i
$$

Now, the above terms will be estimated. By using the technique of plus and minus, we get

$$
\Lambda_1 = \int_0^t 1_{D_n^m}(s)b \left( s, X_m \left( s - \frac{1}{m} \right), X_m(s - \delta(s)) \right) ds \\
- \int_0^t 1_{D_n^m}(s)b \left( s, X_n \left( s - \frac{1}{n} \right), X_n(s - \delta(s)) \right) ds
\[
\begin{align*}
&+ \int_0^t 1_{D_m^c}(s)b\left(s, X_m \left(s - \frac{1}{m}\right), X_m(s - \delta(s))\right) ds \\
&- \int_0^t 1_{D_m^c}(s)b\left(s, X_m \left(s - \frac{1}{m}\right), X_m(s - \delta(s))\right) ds \\
&+ \int_0^t 1_{D_m^c}(s)b\left(s, X_n \left(s - \frac{1}{m}\right), X_n(s - \delta(s))\right) ds \\
&- \int_0^t 1_{D_m^c}(s)b\left(s, X_n \left(s - \frac{1}{m}\right), X_n(s - \delta(s))\right) ds \\
&= \int_0^t 1_{D_m^c}(s)\left\{b\left(s, X_m \left(s - \frac{1}{m}\right), X_m(s - \delta(s))\right) \\
&- b\left(s, X_n \left(s - \frac{1}{m}\right), X_n(s - \delta(s))\right)\right\} ds \\
&+ \int_0^t 1_{D_m^c}(s)\left\{b\left(s, X_n \left(s - \frac{1}{m}\right), X_n(s - \delta(s))\right) \\
&- b\left(s, X_n \left(s - \frac{1}{m}\right), X_n(s - \delta(s))\right)\right\} ds \\
&+ \int_0^t 1_{D_m^c - D_n^c}(s)b\left(s, X_m \left(s - \frac{1}{m}\right), X_m(s - \delta(s))\right) ds \\
&\end{align*}
\]

Then taking $G$-expectation, using inequality (7), Hölder’s inequality, hypotheses $(H1)b$ and $(H2)b$ and Lemma 3.3, we obtain

\[
\hat{E}\left( \sup_{0 \leq s \leq t} |A_1|^p \right)
\leq (3T)^{p-1} \int_0^t 1_{D_m^c}(s)F\left(s, \hat{E}\left| X_m \left(s - \frac{1}{m}\right) - X_n \left(s - \frac{1}{m}\right)\right|^p \right) ds \\
+ (3T)^{p-1} \int_0^t 1_{D_m^c}(s)F\left(s, \hat{E}\left| X_n \left(s - \frac{1}{m}\right) - X_n \left(s - \frac{1}{m}\right)\right|^p \right) ds \\
+ (3T)^{p-1} \int_0^t 1_{D_m^c - D_n^c}(s)H\left(s, \hat{E}\left| X_m \left(s - \frac{1}{m}\right) - X_n \left(s - \frac{1}{m}\right)\right|^p \right) ds \\
\leq (3T)^{p-1} \int_0^t 1_{D_m^c}(s)F\left(s, 2\hat{E}\left( \sup_{0 \leq u \leq s} |X_m(u) - X_n(u)|^p \right) \right) ds \\
+ (3T)^{p-1} \int_0^t 1_{D_m^c}(s)F\left(s, \hat{E}\left| X_n \left(s - \frac{1}{m}\right) - X_n \left(s - \frac{1}{m}\right)\right|^p \right) ds \\
+ 3^{p-1}T^{p-1} \left( \sup_{0 \leq t \leq T} H(t, 2C) \right) \mu(D_m^c - D_n^c). \tag{21}
\]

Similarly, by using inequality (7), Hölder’s inequality, Lemmas 2.8 and 2.9 and hypotheses $(H1)b$ and $(H2)b$ for $\Lambda_2$ and $\Lambda_3$ we get

\[
\hat{E}\left( \sup_{0 \leq s \leq t} |A_2|^p \right) + \hat{E}\left( \sup_{0 \leq s \leq t} |A_3|^p \right)
\]
\[
\leq 3^{p-1} [T^{p-1} \hat{C}_p + T^{\frac{p}{2} - 1} \hat{C}_p C_p] \int_0^T 1_{D_n}(s)
\times F \left( s, \hat{E} \left| X_n \left( s - \frac{1}{m} \right) - X_n \left( s - \frac{1}{n} \right) \right|^p \right) ds
+ 3^{p-1} [T^{p-1} \hat{C}_p + T^{\frac{p}{2} - 1} \hat{C}_p C_p] \int_0^t 1_{D_n}(s)
\times F \left( s, 2\hat{E} \left( \sup_{0 \leq u \leq s} |X_m(u) - X_n(u)|^p \right) \right) ds
+ 3^{p-1} [T^p \hat{C}_p + T^q \hat{C}_p C_p] \left( \sup_{0 \leq t \leq T} H(t, 2C) \right) \mu(D_n - D_m). \tag{22}
\]

By similar analysis of Eqs. (21) and (22), we obtain
\[
\hat{E} \left( \sup_{0 \leq s \leq t} |A_1|^p \right) + \hat{E} \left( \sup_{0 \leq s \leq t} |A_5|^p \right) + \hat{E} \left( \sup_{0 \leq s \leq t} |A_6|^p \right)
\leq 3^{p-1} [T^{p-1} (1 + \hat{C}_p) + T^{\frac{p}{2} - 1} \hat{C}_p C_p] \int_0^t 1_{D_n}(s)
\times F \left( s, 2\hat{E} \left( \sup_{0 \leq u \leq s} |X_m(u) - X_n(u)|^p \right) \right) ds
+ 3^{p-1} [T^{p-1} (1 + \hat{C}_p) + T^{\frac{p}{2} - 1} \hat{C}_p C_p] \int_0^T 1_{D_n}(s)
\times F \left( s, \hat{E} \left| X_n \left( s - \frac{1}{m} \right) - X_n \left( s - \frac{1}{n} \right) \right|^p \right) ds
+ \hat{E} \left| X_n \left( s - \delta(s) - \frac{1}{m} \right) - X_n \left( s - \delta(s) - \frac{1}{n} \right) \right|^p ds
+ 3^{p-1} [T^p (1 + \hat{C}_p) + T^q \hat{C}_p C_p] \left( \sup_{0 \leq t \leq T} H(t, 2C) \right) \mu(D_n - D_m). \tag{23}
\]

Combining Eqs. (19)-(23), we get
\[
\hat{E} \left( \sup_{0 \leq s \leq t} |X_m(s) - X_n(s)|^p \right)
\leq C_2 \int_0^t F \left( s, 2\hat{E} \left( \sup_{0 \leq u \leq s} |X_m(u) - X_n(u)|^p \right) \right) ds
+ C_2 \int_0^T 1_{D_n}(s) F \left( s, \hat{E} \left| X_n \left( s - \frac{1}{m} \right) - X_n \left( s - \frac{1}{n} \right) \right|^p \right) ds
+ C_2 \int_0^T 1_{D_n}(s) F \left( s, \hat{E} \left| X_n \left( s - \frac{1}{m} \right) - X_n \left( s - \frac{1}{n} \right) \right|^p \right) ds
+ \hat{E} \left| X_n \left( s - \delta(s) - \frac{1}{m} \right) - X_n \left( s - \delta(s) - \frac{1}{n} \right) \right|^p ds
+ C_3 \left( \sup_{0 \leq t \leq T} H(t, 2C) \right) \mu(D_n - D_m), \tag{24}
\]

where \(C_2 = \frac{(18)^{p-1} [T^{p-1} (1 + \hat{C}_p) + T^{\frac{p}{2} - 1} \hat{C}_p C_p]}{(1-k_0)^p}\) and \(C_3 = 2C_2 T\) are two positive constants. On the other hand, using Lemmas 3.3 and 3.4, we can estimate
\[
\int_0^T 1_{D_n}(s) F \left( s, \hat{E} \left| X_n \left( s - \frac{1}{m} \right) - X_n \left( s - \frac{1}{n} \right) \right|^p \right) ds
\]
Let by a similar way, we can estimate
\[ E \rightarrow \infty \]
Then, by the fact that \( \mu \) (3), we conclude that \( X \in G \) in the framework (see [4], Lemma 2.11) yield
\[ \leq F(s, 2^p \hat{E} |\xi|^p + 2p C) \left( \frac{1}{n} + \frac{1}{m} \right) + TF \left( s, 2\hat{C} 1 \left( \frac{1}{n} \right) \right). \]  
(25)

By a similar way, we can estimate
\[ \int_0^T 1_{D_m}(s)F \left( s, \hat{E} \left| X_n \left( s - \frac{1}{m} \right) - X_m \left( s - \frac{1}{n} \right) \right|^p \right) ds \]
\[ + \int_0^T 1_{C_1}(s) \left| X_n \left( s - \frac{1}{m} \right) - X_m \left( s - \frac{1}{n} \right) \right|^p ds \]
\[ \leq F \left( s, 2^{p+1} \hat{E} |\xi|^p + 2p C \right) \left( \frac{1}{n} + \frac{1}{m} \right) + TF \left( s, 2\hat{C} 1 \left( \frac{1}{n} \right) \right). \]
(26)

Let
\[ Z(t) = \limsup_{m,n \to \infty} 2\hat{E} \left( \sup_{0 \leq s \leq t} |X_m(s) - X_n(s)|^p \right). \]
(27)
Then, by the fact that \( \mu(D_n - D_m) \to 0 \) as \( n,m \to \infty \) and \( F(s,0) = 0 \) in hand and using Eqs. (25) and (26), Eqs. (24) and (27) together with the Fatous lemma in the G-framework (see [4], Lemma 2.11) yield
\[ Z(t) \leq 2C_2 \int_0^t F(s, Z(s)) ds. \]
(28)
Lastly, through Eq. (28) and Hypothesis (H1)c, it is immediate to obtain
\[ Z(t) = \limsup_{m,n \to \infty} 2\hat{E} \left( \sup_{0 \leq s \leq t} |X_m(s) - X_n(s)|^p \right) = 0, \]
which, gives
\[ \limsup_{m,n \to \infty} \hat{E} \left( \sup_{0 \leq s \leq t} |X_m(s) - X_n(s)|^p \right) = 0, \]
indicating that \( \{X_n(t), n \geq 2^m \} \) is a Cauchy sequence in \( M_G^p([0,T];\mathbb{R}^d) \) and converges to a limit say, \( X(t) \). According to the Borel Cantelli lemma, \( X_n(t) \to X(t) \), as \( n \to \infty \) uniformly for each \( t \in [0,T] \). Hence, taking limits on both sides of Eq. (3), we conclude that \( X(t) \) is a solution to Eq. (1) with the property
\[ \hat{E} \left( \sup_{0 \leq s \leq t} |X(s)|^p \right) = 0, \quad 0 \leq t \leq T. \]
(29)

Now, the existence has been proved. The uniqueness of solutions could be established by the same technique as the existence. And the proof of Theorem 3.2 has been completed. \( \square \)
Remark 1. When we take $\delta(t) \equiv \tau$, a positive constant, Eq. (1) will reduce to the neutral stochastic differential equations driven by $G$-Brownian motion and finite delay (NSDDEGs in short), which has been investigated by He et al. \[10\] under polynomial growth conditions by successive approximation. We point out that our global Carathéodory conditions imply the polynomial growth conditions used in \[10\]. However, this implication can not be reversed in general. Moreover, our results are obtained by Carathéodory approximation which is more complicated. Hence, our results generalize and improve that of \[10\].

Remark 2. Let

$$F(t, x) = \nu(t) \bar{F}(x), \quad 0 \leq t \leq T,$$

where $\nu(t) \geq 0$ is locally integrable and $\bar{F}(x) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a concave nondecreasing function such that $\bar{F}(0) = 0$, $\bar{F}(x) > 0$ for all $x > 0$ and $\int_{0+}^{1} \frac{1}{\bar{F}(x)} \, dx = \infty$. Hence, the condition (H1)c holds by employing the comparison theorem of differential equations.

For some concrete examples of the function $\bar{F}(\cdot)$, assume $\varsigma > 0$ and $\mu \in (0, 1)$ be sufficiently small, and define

$$\bar{F}_1(x) = \varsigma x, \quad x \geq 0,$$

$$\bar{F}_2(x) = \begin{cases} x \log(x^{-1}), & 0 \leq x \leq \mu, \\ \mu \log(\mu^{-1}) + \bar{F}_2^\prime(\mu^{-})(x - \mu), & x > \mu, \end{cases}$$

where $\bar{F}_2^\prime$ denotes the derivative of $\bar{F}_2$. They are all concave and non-decreasing functions satisfying $\int_{0+}^{1} \frac{1}{\bar{F}_i(x)} \, dx = +\infty (i = 1, 2)$.

Remark 3. It should be pointed out that by using the properties of the concave function like in Eq. (40), condition (H2)b will generate the linear growth condition.

Remark 4. It is well known that under Remarks 2 and 3 and the above examples, our Carathéodory conditions (H1) and (H2) could be relaxed to give the conditions (Lipschitz and linear growth conditions) used in \[16, 18\] and the conditions (non-Lipschitz and linear growth conditions) used in \[3, 17\] as special cases. Therefore, some existing results in \[3, 16, 17, 18\] are generalized and improved.

Remark 5. We would like to point out that the proofs of convergence in \[3, 1, 2\] were in second moment, but our proofs in the present article are in the $p$-th moment, which attain a much bigger degree of freedom for the parameter $q$ in study of $L^q(0 < q \leq p)$ convergence and possesses a good robustness. Moreover, the variable delay lag $\delta(\cdot)$ in the present model make the proofs more complicated than that of \[3, 1, 2\].

4. Approximation theorems. In this section, we shall present two approximation theorems for VNSDDEGs to study the convergence in $L^p$-sense and capacity between the standard form of Eq. (1) and the simplified one.

For fixed $T > 0$, consider the following standard VNSDDEGs:

$$X_\epsilon(t) = \xi(0) - G(\xi(-\delta(t))) + G(X_\epsilon(t - \delta(t)))$$

$$+ \epsilon \int_0^t b(s, X_\epsilon(s), X_\epsilon(s - \delta(s))) \, ds$$

$$+ \sqrt{\epsilon} \int_0^t g(s, X_\epsilon(s), X_\epsilon(s - \delta(s))) \, dB_s$$

$$+ \sqrt{\epsilon} \int_0^t h(s, X_\epsilon(s), X_\epsilon(s - \delta(s))) \, dB_s$$

(30)
with initial value $X_\epsilon(0) = \xi = \{\xi(\theta) : \tau \leq \theta \leq 0\} \in C([-\tau, 0]; \mathbb{R}^d)$ and the coefficients have the same definitions and conditions as in Eq. (1). The parameter $\epsilon \in (0, \epsilon_0]$ is a positive with $\epsilon_0$ a fixed number. According to Theorem 3.2, Eq. (30) also has a unique solution $X_\epsilon(t)$ in the sense of $L^p$-norm for every fixed $\epsilon \in (0, \epsilon_0]$.

Consider the following simplified VNSDDEGs which corresponds to the original standard form (30):

$$Y_\epsilon(t) = \xi(0) - G(\xi(-\delta(t))) + G(Y_\epsilon(t - \delta(t))) + \epsilon \int_0^t \tilde{h}(Y_\epsilon(s), Y_\epsilon(s - \delta(s)))ds$$

$$+ \sqrt{\epsilon} \int_0^t \tilde{g}(Y_\epsilon(s), Y_\epsilon(s - \delta(s)))d(B, B)_s$$

$$+ \sqrt{\epsilon} \int_0^t \tilde{h}(Y_\epsilon(s), Y_\epsilon(s - \delta(s)))dB_s$$  \hspace{1cm} (31)

Obviously, Eq. (31) also has a unique solution $Y_\epsilon(t)$ in the sense of $L^p$-norm under similar hypotheses as Eq. (30).

We also impose the following hypothesis:

**(H4).** For any $x, y \in \mathbb{R}^d$ and $p \geq 2$, there exists a function $\varphi_1(T_1) > 0$ satisfying $\sup_{T_1 \geq 0} \varphi_1(T_1) < \infty$ such that

$$\frac{1}{T_1} \int_0^{T_1} |\tilde{b}(t, x, y) - \tilde{b}(x, y)|^p dt + \frac{1}{T_1} \int_0^{T_1} ||g(t, x, y) - \tilde{g}(x, y)||^p_{HS}dt$$

$$+ \frac{1}{T_1} \int_0^{T_1} ||h(t, x, y) - \tilde{h}(x, y)||^p_{HS}dt \leq \varphi_1(T_1)\kappa(|x|^p + |y|^p).$$

where $\kappa(.) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous non-decreasing and concave function.

Now, we are concerned with the relationship between the solution processes $X_\epsilon(t)$ and $Y_\epsilon(t)$. The convergence both in $L^p$-sense and capacity between the standard form (30) and the simplified one (31) will be considered in view of the following two approximation theorems, respectively:

**Theorem 4.1.** Assume that hypotheses (H1)-(H4) hold. Then, for a given arbitrarily small number $\delta_1 > 0$ and a small $L > 0$, $\beta \in (0, 1)$, there exists a number $\epsilon_1 \in (0, \epsilon_0]$ such that for all $\epsilon \in (0, \epsilon_1]$,

$$\mathbb{E} \left( \sup_{t \in [0, L\epsilon^{-\beta}]} |X_\epsilon(t) - Y_\epsilon(t)|^p \right) \leq \delta_1.$$  \hspace{1cm} (32)

**Proof.** Considering the difference between Eqs. (30) and (31), we have

$$X_\epsilon(t) - Y_\epsilon(t) = G(X_\epsilon(t - \delta(t))) - G(Y_\epsilon(t - \delta(t))) + \Gamma(t),$$  \hspace{1cm} (33)

with

$$\Gamma(t) = \epsilon \int_0^t [b(s, X_\epsilon(s), X_\epsilon(s - \delta(s))) - \tilde{b}(Y_\epsilon(s), Y_\epsilon(s - \delta(s))))]ds$$

$$+ \sqrt{\epsilon} \int_0^t [g(s, X_\epsilon(s), X_\epsilon(s - \delta(s))) - \tilde{g}(Y_\epsilon(s), Y_\epsilon(s - \delta(s))))]d(B, B)_s$$

$$+ \sqrt{\epsilon} \int_0^t [h(s, X_\epsilon(s), X_\epsilon(s - \delta(s))) - \tilde{h}(Y_\epsilon(s), Y_\epsilon(s - \delta(s))))]dB_s.$$  \hspace{1cm} (33)

For $t \in [0, u] \subseteq [0, T]$, by similar analysis of Eq. (14) and using hypothesis (H3), we have for Eq. (32)
\[ \hat{E} \left( \sup_{0 \leq t \leq u} |X_\varepsilon(t) - Y_\varepsilon(t)|^p \right) \leq k_0 \hat{E} \left( \sup_{0 \leq t \leq u} |X_\varepsilon(t) - Y_\varepsilon(t)|^p \right) \]

\[ + \frac{1}{(1 - k_0)^{p-1}} \hat{E} \left( \sup_{0 \leq t \leq u} |\Gamma(t)|^p \right) \]

\[ \leq \frac{1}{(1 - k_0)^p} \hat{E} \left( \sup_{0 \leq t \leq u} |\Gamma(t)|^p \right) , \]

which, by Eq. (33), yields

\[ \hat{E} \left( \sup_{0 \leq t \leq u} |X_\varepsilon(t) - Y_\varepsilon(t)|^p \right) \leq \frac{3^{p-1} \varepsilon^p}{(1 - k_0)^p} \hat{E} \left( \sup_{0 \leq t \leq u} \left| \int_0^t [b(s, X_\varepsilon(s), X_\varepsilon(s - \delta(s))] - \hat{b}(Y_\varepsilon(s), Y_\varepsilon(s - \delta(s))) \right|^p ds \right) \]

\[ + \frac{3^{p-1} \varepsilon^p}{(1 - k_0)^p} \hat{E} \left( \sup_{0 \leq t \leq u} \left| \int_0^t [g(s, X_\varepsilon(s), X_\varepsilon(s - \delta(s))] - \hat{g}(Y_\varepsilon(s), Y_\varepsilon(s - \delta(s))) \right|^p ds \right) \]

\[ + \frac{3^{p-1} \varepsilon^p}{(1 - k_0)^p} \hat{E} \left( \sup_{0 \leq t \leq u} \left| \int_0^t [h(s, X_\varepsilon(s), X_\varepsilon(s - \delta(s))] - \hat{h}(Y_\varepsilon(s), Y_\varepsilon(s - \delta(s))) \right|^p ds \right) \]

\[ \leq \Pi_1 + \Pi_2 + \Pi_3. \] (34)

For \( \Pi_1 \), we have by inequality (7)

\[ \Pi_1 \leq \frac{(6u)^{p-1} \varepsilon^p C_p}{(1 - k_0)^p} \int_0^u \hat{E} |b(s, X_\varepsilon(s), X_\varepsilon(s - \delta(s))) - \hat{b}(Y_\varepsilon(s), Y_\varepsilon(s - \delta(s)))|^p ds \]

\[ + \frac{(6u)^{p-1} \varepsilon^p C_p}{(1 - k_0)^p} \int_0^u \hat{E} |g(s, X_\varepsilon(s), X_\varepsilon(s - \delta(s))) - \hat{g}(Y_\varepsilon(s), Y_\varepsilon(s - \delta(s)))|^p ds \]

\[ := \Pi_{11} + \Pi_{12}. \] (35)

Similarly for \( \Pi_2 \) and \( \Pi_3 \), we obtain

\[ \Pi_2 \leq \frac{(6u)^{p-1} \varepsilon^p C_p}{(1 - k_0)^p} \]

\[ \times \int_0^u \hat{E} |g(s, X_\varepsilon(s), X_\varepsilon(s - \delta(s))) - \hat{g}(Y_\varepsilon(s), Y_\varepsilon(s - \delta(s)))|^p_{H_S} ds \]

\[ + \frac{(6u)^{p-1} \varepsilon^p C_p}{(1 - k_0)^p} \]

\[ \times \int_0^u \hat{E} |g(s, X_\varepsilon(s), X_\varepsilon(s - \delta(s))) - \hat{g}(Y_\varepsilon(s), Y_\varepsilon(s - \delta(s)))|^p_{H_S} ds \]

\[ := \Pi_{21} + \Pi_{22}. \] (36)

and

\[ \Pi_3 \leq \frac{6^{p-1} u^{p-1} \varepsilon^p C_p}{(1 - k_0)^p} \]
\[
\times \int_0^u \hat{E} \left[ |h(s, X_s(s), X_s(s - \delta(s))) - h(s, Y_s(s), Y_s(s - \delta(s)))| |_{H^p}^p \right] ds \\
+ \frac{6^{p-1}u^{1-p}C_p C_p}{(1 - k_0)^p} \\
\times \int_0^u \hat{E} \left[ |h(s, X_s(s), X_s(s - \delta(s))) - h(Y_s(s), Y_s(s - \delta(s)))| |_{H^p}^p \right] ds
\]
\[:= \Pi_{31} + \Pi_{32}. \quad (37)\]

Then, we have by hypothesis (H1) b

\[
\Pi_{11} + \Pi_{21} + \Pi_{31} \\
\leq C_4 \epsilon \int_0^u F \left( s, \hat{E}|X_s(s) - Y_s(s)|^p + \hat{E}|X_s(s - \delta(s)) - Y_s(s - \delta(s))|^p \right) ds \\
\leq C_4 \epsilon \int_0^u F \left( s, 2\hat{E} \left( \sup_{0 \leq v \leq s} |X_s(v) - Y_s(v)| \right) \right) ds, \quad (38)
\]

where \( C_4 = \frac{6^{p-1}(1+\epsilon u)^{1-p}C_p C_p}{(1 - k_0)^p} \) is positive constant.

By hypothesis (H3) and the solution property given by Eq. (29), we obtain

\[
\Pi_{12} + \Pi_{22} + \Pi_{32} \leq C_5 \kappa (\hat{E}|Y_s(s)|^p + \hat{E}|Y_s(s - \delta(s))|^p) \epsilon u \\
\leq C_5 \kappa (2\hat{E}|\xi|^p + 2C) \epsilon u, \quad (39)
\]

where \( C_5 = C_4 \epsilon_2(u) \) is positive constant.

Given that \( F(t, v) \) is concave in \( v \) for any fixed \( t \geq 0 \) with \( F(t, 0) = 0 \), there exist \( a(t) \geq 0, b(t) \geq 0 \) such that

\[
F(t, v) \leq a(t) + b(t)v, \quad v \geq 0, \quad \int_0^T a(t) dt < \infty, \quad \int_0^T b(t) dt < \infty, \quad (40)
\]

Combining Eqs. (34)-(39), using Eq. (40) and Gronwall’s inequality, we get

\[
\hat{E} \left( \sup_{0 \leq t \leq u} |X_s(t) - Y_s(t)|^p \right) \leq C_4 \epsilon \int_0^u F \left( s, 2\hat{E} \left( \sup_{0 \leq v \leq s} |X_s(v) - Y_s(v)| \right) \right) ds \\
+ C_5 \kappa (2\hat{E}|\xi|^p + 2C) \epsilon u \\
\leq C_4 \epsilon \int_0^u [a(s) + 2b(s)\hat{E} \left( \sup_{0 \leq v \leq s} |X_s(v) - Y_s(v)| \right) ] ds \\
+ C_5 \kappa (2\hat{E}|\xi|^p + 2C) \epsilon u \\
\leq [C_4 \left( \sup_{0 \leq t \leq T} a(t) \right) + C_5 \kappa (2\hat{E}|\xi|^p + 2C)] \epsilon u \\
+ C_6 \epsilon \int_0^u \hat{E} \left( \sup_{0 \leq v \leq s} |X_s(v) - Y_s(v)|^p \right) ds \\
\leq [C_4 \left( \sup_{0 \leq t \leq T} a(t) \right) + C_5 \kappa (2\hat{E}|\xi|^p + 2C)] \epsilon u C_6 \epsilon u (41)
\]

where \( C_6 = 2C_4 \left( \sup_{0 \leq t \leq T} b(t) \right) \) is positive constant.

Choose \( \beta \in (0, 1) \) and \( L > 0 \) such that for all \( t \in [0, L e^{-\beta}] \subseteq [0, T] \), we have

\[
\hat{E} \left( \sup_{t \in [0,L e^{-\beta}]} |X_s(t) - Y_s(t)|^p \right) \leq \hat{K} L e^{1-\beta} \quad (42)
\]
Theorem 4.2. Assume that hypotheses (H1)-(H4) hold. Then, for a given arbitrarily small number \( \delta_2 > 0 \), we have
\[
\lim_{\epsilon \to 0} \hat{c} \left( \sup_{t \in [0, Lc^{-\beta}]} |X_\epsilon(t) - Y_\epsilon(t)| > \delta_2 \right) = 0,
\]
where \( L \) and \( \beta \) are from Theorem 4.1.

Proof. By Theorem 4.1 and Lemma 2.10, we have for any given number \( \delta_2 > 0 \)
\[
\hat{c} \left( \sup_{t \in [0, Lc^{-\beta}]} |X_\epsilon(t) - Y_\epsilon(t)| > \delta_2 \right) \leq \frac{1}{\delta_2^p} \hat{E} \left( \sup_{t \in [0, Lc^{-\beta}]} |X_\epsilon(t) - Y_\epsilon(t)|^p \right)
\leq \frac{\hat{K}Lc^{1-\beta}}{\delta_2^p},
\]
which, tends to 0 by taking \( \epsilon \to 0 \). \( \square \)

Remark 6. Theorem 4.1 shows that the solution of the simplified VNSDDEGs converges to that of the original standard VNSDDEGs in the \( L^p \)-sense. While, Theorem 4.2 means the convergence of these two solutions in capacity.

Remark 7. When we take \( \delta(t) \equiv \tau \), a positive constant, Eq. (1) reduces to NSDDEGs. The approximation theorems for this equation has been set up in [10] under polynomial growth conditions. But in our case, it has been established under global Carathéodory conditions. Therefore, the results of [10] are generalized and improved.

5. Examples. The following two examples give the approximation theorems for neutral stochastic differential delay equation driven by G-Brownian motion (NSDDEGs).

Example 5.1. Consider the following one-dimensional NSDDEGs.
\[
d \left[ X_\epsilon(t) - \frac{1}{4} \sin(X_\epsilon(t-1)) \right] = \epsilon \left[ -\frac{1}{2} X_\epsilon(t) + \lambda_1 t \sin(X_\epsilon(t-1)) \right] dt
+ \sqrt{\epsilon t} \cos(X_\epsilon(t-1)) d(B, B)_t
+ \sqrt{\epsilon t^2} \sin^2(X_\epsilon(t-1)) dB_t,
\quad t \in [0, T],
\]
with initial data \( X_\epsilon(t) = t + 1, t \in [-1, 0] \), where \( \lambda_1 \in \mathbb{R} \) and \((B_t)_{t \geq 0}\) is a one-dimensional G-Brownian motion with \( B_1 \sim N(\{0\} \times [\sigma^2, \sigma^2]) \). Here
\[
b(t, x, y) = -\frac{1}{2} x + \lambda_1 t \sin(y), \quad g(t, x, y) = t \cos(y), \quad h(t, x, y) = t^2 \sin^2(y).
\]
Then,
\[ \bar{b}(x,y) = \frac{1}{T} \int_0^T b(t,x,y) dt = \frac{1}{T} \int_0^T \left[ -\frac{1}{2} x + \lambda_1 t \sin(y) \right] dt = \frac{1}{2} \left[ \lambda_1 T \sin(y) - x \right], \]
and
\[ \bar{g}(x,y) = \frac{T}{2} \cos(y), \quad \bar{h}(x,y) = \frac{T^2}{3} \sin^2(y). \]

Then, the corresponding simplified equation is given by
\[ d\left[ Y_\epsilon(t) - \frac{1}{4} \sin(Y_\epsilon(t - 1)) \right] = \frac{\epsilon}{2} \left[ -Y_\epsilon(t) + \lambda_1 t \sin(Y_\epsilon(t - 1)) \right] dt \]
\[ + \sqrt{\frac{\epsilon}{2}} \cos(Y_\epsilon(t - 1)) d\langle B, B \rangle_t \]
\[ + \sqrt{\frac{\epsilon t^2}{3}} \sin^2(Y_\epsilon(t - 1)) dB_t \]
when \( t \in [0,1] \), we have
\[ d\left[ Y_\epsilon(t) - \frac{1}{4} \sin(t) \right] = \frac{\epsilon}{2} \left[ -Y_\epsilon(t) + \lambda_1 t \sin(t) \right] dt \]
\[ + \sqrt{\frac{\epsilon}{2}} \cos(t) d\langle B, B \rangle_t \]
\[ + \sqrt{\frac{\epsilon t^2}{3}} \sin^2(t) dB_t. \]

Let \( \tilde{Y}_\epsilon(t) = Y_\epsilon(t) - \frac{1}{4} \sin(t) \), then
\[ d\tilde{Y}_\epsilon(t) = \frac{\epsilon}{2} \left[ -\tilde{Y}_\epsilon(t) + \left( \lambda_1 t \sin(t) - \frac{1}{4} \right) \sin(t) \right] dt \]
\[ + \sqrt{\epsilon} \frac{t}{2} \cos(t) d\langle B, B \rangle_t \]
\[ + \sqrt{\epsilon t^2} \frac{2}{3} \sin^2(t) dB_t. \]

Obviously, the last equation is a linear in the narrow sense, and it’s solution is
\[ \tilde{Y}_\epsilon(t) = X_\epsilon(0)e^{-\frac{\epsilon}{2}t} e^{-\frac{\epsilon}{2}(t-s)} ds \]
\[ + \sqrt{\epsilon} \frac{t}{2} \int_0^t \cos(s) e^{-\frac{\epsilon}{2}(t-s)} d\langle B, B \rangle_s \]
\[ + \sqrt{\epsilon} \frac{t^2}{3} \int_0^t (s \sin(s))^2 e^{-\frac{\epsilon}{2}(t-s)} dB_s. \]

Then, when \( t \in [0,1] \), the solution of NSDDEGs is
\[ Y_\epsilon(t) = \frac{1}{4} \sin(t) + X_\epsilon(0)e^{-\frac{\epsilon}{2}t} e^{-\frac{\epsilon}{2}(t-s)} ds \]
\[ + \sqrt{\epsilon} \frac{t}{2} \int_0^t \cos(s) e^{-\frac{\epsilon}{2}(t-s)} d\langle B, B \rangle_s \]
\[ + \sqrt{\epsilon} \frac{t^2}{3} \int_0^t (s \sin(s))^2 e^{-\frac{\epsilon}{2}(t-s)} dB_s. \]
Repeating this procedure over the intervals $[1, 2]$, $[2, 3]$, etc, we obtain the solution on the entire interval $[0, T]$.

Noting that
\[
\frac{1}{T} \int_0^T |b(t, x, y) - \bar{b}(x, y)|^p dt = \frac{1}{T} \int_0^T |\frac{1}{2} \lambda_1 t \sin(y)|^p dt \leq \frac{1}{p+1} \left( \frac{\lambda_1}{2} \right)^p T^p < \infty.
\]

It is easy to see that the conditions $(H1)$-$(H4)$ are satisfied, so Theorems 4.1 and 4.2 hold.

**Example 5.2.** Consider the following one-dimensional NSDDEGs.
\[
d \left[ X_s(t) - \frac{1}{2} \sin(X_s(t) - 1) \right] = \epsilon \left[ -\frac{1}{2} X_s(t) + \frac{1}{4} X_s(t-1) \cos(t) \right] dt + \sqrt{\epsilon} \lambda_1 d(B, B)_t + \sqrt{\epsilon} dB_t,
\]
for $t \in [0, T]$, with initial data $X_s(t) = t + 1$, $t \in [-1, 0]$, where $\lambda_1 \in \mathbb{R}$ and $(B_t)_{t \geq 0}$ is a one-dimensional G-Brownian motion with $B_0 \sim N\{0\} \times [\sigma^2, \sigma^2]$. Here
\[
b(t, x, y) = -\frac{1}{2} x + \frac{1}{4} y \cos(t), \quad g(t, x, y) = \lambda_1, \quad h(t, x, y) = 1.
\]

Let
\[
\tilde{b}(x, y) = \frac{1}{T} \int_0^T b(t, x, y) dt = -\frac{1}{2} x + \frac{1}{4T} y \sin(T), \quad \tilde{g}(x, y) = \lambda_1, \quad \tilde{h}(x, y) = 1.
\]

Hence, we have the corresponding simplified NSDDEGs as follows:
\[
d \left[ Y_s(t) - \frac{1}{2} \sin(Y_s(t-1)) \right] = \epsilon \left[ -\frac{1}{2} Y_s(t) + \frac{1}{4} Y_s(t-1) \sin(t) \right] dt + \sqrt{\epsilon} \lambda_1 d(B, B)_t + \sqrt{\epsilon} dB_t
\]
when $t \in [0, 1]$, we have
\[
d \left[ Y_s(t) - \frac{1}{2} \sin(t) \right] = \epsilon \left[ -\frac{1}{2} Y_s(t) + \frac{1}{4} \sin(t) \right] dt + \sqrt{\epsilon} \lambda_1 d(B, B)_t + \sqrt{\epsilon} dB_t.
\]

Let $\check{Y}_s(t) = Y_s(t) - \frac{1}{2} \sin(t)$, the linear NSDDEGs becomes a linear SDEGs
\[
d \check{Y}_s(t) = -\frac{\epsilon}{2} \check{Y}_s(t) dt + \sqrt{\epsilon} \lambda_1 d(B, B)_t + \sqrt{\epsilon} dB_t.
\]

Obviously, $\check{Y}^\epsilon(t)$ is the well-known Ornstein-Uhlenbeck process, and it can be described as
\[
\check{Y}_s(t) = X_s(0)e^{-\frac{1}{2}t} + \sqrt{\epsilon} \left[ \lambda_1 \int_0^t e^{-\frac{1}{2}(t-s)} d(B, B)_s + \int_0^t e^{-\frac{1}{2}(t-s)} dB_s \right].
\]

when $t \in [0, 1]$, the solution of NSDDEGs is
\[
Y_s(t) = \frac{1}{2} \sin(t) + X_s(0)e^{-\frac{1}{2}t} + \sqrt{\epsilon} \left[ \lambda_1 \int_0^t e^{-\frac{1}{2}(t-s)} d(B, B)_s + \int_0^t e^{-\frac{1}{2}(t-s)} dB_s \right].
\]

Repeating this procedure over the intervals $[1, 2]$, $[2, 3]$, etc, we obtain the solution on the entire interval $[0, T]$.

Noting that
\[
\frac{1}{T} \int_0^T |b(t, x, y) - \bar{b}(x, y)|^p dt = \frac{1}{T} \int_0^T |\cos(t) - \frac{1}{t} \sin(t)|^p dt |\frac{1}{4} y|^p,
\]
and
\[
\frac{1}{T} \int_0^T |\cos(t) - \frac{1}{t} \sin(t)|^p dt \leq \frac{1}{T} \int_0^T \left( 1 + \frac{1}{t} \right)^p dt \leq 2^{p-1} \left[ 1 + \frac{T^{-p}}{1-p} \right] < \infty.
\]

It is easy to see that the conditions (H1)-(H4) are satisfied, so Theorems 4.1 and 4.2 hold.

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