Elementary derivation of Weingarten functions of classical groups

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Abstract
Integration of polynomials over the classical groups of unitary, orthogonal and symplectic matrices can be reduced to basic building blocks known as Weingarten functions. We present a new derivation of these functions, more elementary than previous ones.

1 Introduction

1.1 Background
Integration with respect to matrix ensembles is the core of random matrix theory [1] and an important problem in many areas of mathematical physics [2]. Classical Lie groups of unitary, orthogonal and symplectic matrices, endowed with the corresponding Haar measure, constitute important classes of matrix ensembles. Integrals of functions which are polynomials in the matrix elements can be reduced to the general form

\[ \int_G \prod_{k=1}^{n_1} U_{i_k j_k} \prod_{\ell=1}^{n_2} U_{m_\ell r_\ell}^* dU \equiv \left\langle \prod_{k=1}^{n} U_{i_k j_k} \prod_{\ell=1}^{n_2} U_{m_\ell r_\ell}^* \right\rangle_G, \]

where \( G \) denotes one of the groups and \( dU \) the corresponding Haar measure.

Initial investigations in the physics literature considered the unitary group \( U(N) \).[3] It was found that the basic integral can be written as a double sum over the symmetric group,

\[ \left\langle \prod_{k=1}^{n} U_{i_k j_k} U_{m_k r_k}^* \right\rangle_{U(N)} = \sum_{\tau,\sigma \in S_n} Wg^U(\tau^{-1}\sigma) \prod_{k=1}^{n} [i_k = m_{\sigma(k)}][j_k = r_{\tau(k)}], \]

where \([i = m]\) is the same as \( \delta_{i,m} \) and \( Wg^U \) is a function which depends only on the cycle structure of its argument. Independently of the physicists, Collins [10] rediscovered the problem and suggested \( Wg^U \) be called the Weingarten function of \( U(N) \).

The corresponding question for the orthogonal and symplectic groups has also been much studied.[11] [12] [13] [14]. They too can be written as double sums over permutations, and the corresponding Weingarten functions \( Wg^O \) and \( Wg^{Sp} \) have been found.[15] [16]. Later
developments include new approaches (e.g. from Jucys-Murphy elements \([17, 18]\), generalizations (e.g. to compact symmetric spaces \([19, 20]\), connections (e.g. to factorizations of permutations \([21]\)) and applications (e.g. to different polynomial integrals \([22]\), to quantum mechanics \([23, 24, 25]\)).

Previous works where Weingarten functions were obtained were based either on representation theory and Schur-Weyl duality, \([10, 15]\) the theory of Gelfand pairs, \([20]\) or Jucys-Murphy elements.\([18]\) In contrast, we here derive Weingarten functions for the classical compact groups by means of some elementary direct calculations (although we rely on some classical results that can, of course, be interpreted very naturally in the light of those theories).

Before presenting details, we briefly outline the idea. It consists of five steps: 1) write the integrand as the derivative of a power sum function; 2) change basis from power sums to Schur functions; 3) perform the group integral; 4) revert back to power sums; 5) take the derivative to arrive at the result.

### 1.2 Notation, terminology, known facts

We use traditional notation and some well known results. More detailed explanations of the concepts introduced below can be found, for example, in \([20]\).

\(U(N)\) is the group of complex \(N \times N\) matrices satisfying \(U^\dagger U = 1\), where \(U^\dagger\) is the transpose conjugate of \(U\). The group \(O(N)\) is the subgroup of \(U(N)\) of real matrices. Let

\[ J = \begin{pmatrix} 0^N & 1^N \\ -1^N & 0^N \end{pmatrix}, \]

where \(0^N\) and \(1^N\) are, respectively, the zero and the identity matrix in \(N\) dimensions. The group \(Sp(2N)\) is the subgroup of \(U(2N)\) of complex matrices satisfying \(U^D U = 1\), where \(U^D = JU^T J^T\).

\(\lambda \vdash n\) means \(\lambda\) is a partition of \(n\), the number of parts being \(\ell(\lambda)\). The partition with \(n\) unit parts is denoted \(1^n\). We denote \(2\lambda \equiv (2\lambda_1, 2\lambda_2, \ldots)\) and \(\lambda \cup \lambda \equiv (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots)\). \(S_n\) is the group of permutations of \(n\) elements. The cycle type of a permutation is the partition whose parts are the lengths of its cycles. The identity in \(S_n\) has cycle type \(1^n\). The quantity \(\chi_\lambda(\mu) \equiv \chi_\lambda(\pi)\) is the character of the irreducible representation of \(S_n\) labeled by \(\lambda\), calculated for a permutation \(\pi\) of cycle type \(\mu\).

Power sum symmetric functions of matrix argument are given by \(p_\lambda(X) = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(X)\), with \(p_{\nu}(X) = \text{Tr}(X^\nu)\). If \(\pi \in S_n\) has cycle type \(\lambda\), then \(p_\pi(X) = p_\lambda(X)\). Power sums and Schur functions are related by

\[ s_\lambda(X) = \sum_{\pi \in S_n} \chi_\lambda(\pi)p_\pi(X), \quad p_\lambda(X) = \sum_{\mu \vdash n} \chi_\lambda(\mu)s_\mu(X). \quad (3) \]

Schur functions are irreducible characters of the unitary group, and satisfy

\[ \int_{U(N)} s_\lambda(AU)s_\mu(BU^\dagger)\,dU = \frac{s_\lambda(AB)}{s_\lambda(1^N)}, \quad (4) \]

if \(\ell(\lambda) \leq N\) (the integral vanishes otherwise).

The group \(S_{2n}\) has a subgroup called the hyperoctahedral, \(H_n\), which can be seen as the stabilizer of the permutation \((12)(34) \cdots (2n-1 \ 2n)\). Its order is \(|H_n| = 2^n n!\). The coset \(S_{2n}/H_n = M_n\) can be represented by permutations called matchings: if \(\sigma \in M_n\) then \(\sigma(2i-1) < \sigma(2i)\) and \(\sigma(2i-1) < \sigma(2i+1)\).
Given a permutation $\tau \in S_{2n}$, let $G_\tau$ be the graph whose vertices are labeled from 1 to $2n$ and whose edges are of the forms $\{2i-1, 2i\}$ or $\{\tau(2i-1), \tau(2i)\}$, $1 \leq i \leq n$. Since each vertex belongs to one edge of each form, all connected components of $G_\tau$ are cycles of even length. The coset type of $\tau$ is the partition of $n$ whose parts are half the number of edges in the connected components of $G_\tau$. Two permutations $\tau, \sigma$ in $S_{2n}$ have the same coset type if and only if $\tau = h_1 \sigma h_2$ with $h_1, h_2, \in H_n$.

The average $\omega_\lambda(\tau) = \frac{1}{|H_n|} \sum_{\xi \in H_n} \chi_{2\lambda}(\tau \xi)$ is called a zonal spherical function. It is obviously invariant under (left and right) action of the hyperoctahedral and hence depends only on the coset type of its argument. In terms of these functions, another symmetric function called zonal polynomial $Z_\lambda$ (actually a particular case of Jack polynomials) can be defined as

$$Z_\lambda(X) = \sum_{\mu \vdash n} \omega_\lambda(\mu) p_\mu(X). \quad (5)$$

These functions appear when Schur functions are integrated over the orthogonal group: the average $\langle s_\mu(AU) \rangle_{O(N)}$ is zero unless $\mu$ has only even parts, in which case

$$\langle s_{2\lambda}(AU) \rangle_{O(N)} = \frac{Z_\lambda(A^T A)}{Z_\lambda(1^N)} \quad (\ell(\lambda) \leq N). \quad (6)$$

The average $\psi_\lambda(\tau) = \frac{1}{|H_n|} \sum_{\xi \in H_n} \chi_{\lambda \cup \lambda}(\tau \xi) s(\xi)$, where $s(\xi)$ is the sign of $\xi$, is called a twisted zonal spherical function. The corresponding twisted zonal polynomials $Z'_\lambda$ (another particular case of Jack polynomials) are defined as

$$Z'_\lambda(X) = \sum_{\mu \vdash n} \psi_\lambda(\mu) p_\mu(X). \quad (7)$$

These functions appear when Schur functions are integrated over the symplectic group: the average $\langle s_\mu(AU) \rangle_{Sp(2N)}$ is zero unless $\mu$ has repeated parts, in which case

$$\langle s_{\lambda \cup \lambda}(AU) \rangle_{Sp(2N)} = \frac{Z'_\lambda(A^P A)}{Z'_\lambda(1^N)} \quad (\ell(\lambda) \leq N). \quad (8)$$

The only new construction we need in this work is a map from $S_n$ to $S_{2n}$, taking a permutation $\pi$ into another permutation $\overline{\pi}$, which fixes odd numbers and permutes even numbers according to

$$\overline{\pi}(2j - 1) = 2j - 1, \quad \overline{\pi}(2j) = 2\pi^{-1}(j), \quad 1 \leq j \leq n. \quad (9)$$

It is easy to see that the coset type of $\overline{\pi}$ coincides with the cycle type of $\pi$.

## 2 Unitary Group

We start from the basic identity

$$\prod_{k=1}^n U_{i_kj_k} = \frac{1}{n!} \prod_{k=1}^n \frac{\partial}{\partial A_{i_kj_k}^\dagger} p_{1^n}(A^\dagger U), \quad (10)$$
which can be easily verified. It allows us to write
\[
\left\langle \prod_{k=1}^{n} U_{i_k j_k} U_{m_k r_k}^* \right\rangle_{\mathcal{U}(N)} = \frac{1}{n!^2} \prod_{k=1}^{n} \frac{\partial}{\partial A_{i_k j_k}^*} \frac{\partial}{\partial B_{m_k r_k}} \langle p_1^n (A^\dagger U) p_1^n (B U^\dagger) \rangle_{\mathcal{U}(N)}. \quad (11)
\]

Expressing power sums in terms of Schur functions, we arrive at
\[
\langle p_1^n (A^\dagger U) p_1^n (B U^\dagger) \rangle_{\mathcal{U}(N)} = \sum_{\lambda, \mu \vdash n} \chi_{\lambda}(1^n) \chi_{\mu}(1^n) \langle s_\lambda(A^\dagger U) s_\mu(B U^\dagger) \rangle_{\mathcal{U}(N)}. \quad (12)
\]

Using the orthogonality of Schur functions, we arrive at
\[
\sum_{\lambda \vdash n} [\chi_{\lambda}(1^n)]^2 \frac{s_\lambda(A^\dagger B)}{s_{\lambda}(1^n)}. \quad (13)
\]

Reverting back to power sums, this is
\[
\langle p_1^n (A^\dagger U) p_1^n (B U^\dagger) \rangle_{\mathcal{U}(N)} = \frac{1}{n!} \sum_{\lambda \vdash n} [\chi_{\lambda}(1^n)]^2 \sum_{\pi \in S_n} \chi_{\lambda}(\pi) p_\pi(A^\dagger B). \quad (14)
\]

In order to take the derivative, we expand \( p_\pi \) as a trace,
\[
\prod_{k=1}^{n} \frac{\partial}{\partial A_{i_k j_k}^*} \frac{\partial}{\partial B_{m_k r_k}} p_\pi(A^\dagger B) = \sum_{\sigma_1, ..., \sigma_n=1}^{N} \sum_{\ell \in S_n} \prod_{k=1}^{n} \frac{\partial}{\partial A_{i_k j_k}^*} \frac{\partial}{\partial B_{m_k r_k}} A_{a_k b_k}^\dagger B_{b_k a_{\pi(k)}} \chi_{\lambda}(1^n), \quad (15)
\]

which immediately leads to
\[
\sum_{\sigma_1, ..., \sigma_n=1}^{N} \sum_{\ell \in S_n} \prod_{k=1}^{n} [b_k = m_{\rho}(k)] [a_{\pi(k)} = r_{\rho(k)}] [b_k = i_{\theta(k)}] [a_k = j_{\theta(k)}]. \quad (16)
\]

Summing over the \( a \)'s and \( b \)'s produces
\[
\sum_{\theta, \rho \in S_n} \prod_{k=1}^{n} [i_k = m_{\rho^{-1}(k)}] [j_k = r_{\rho^{-1} \theta^{-1}(k)}]. \quad (17)
\]

Finally, we change variables as \( \rho = \sigma \theta \) and then \( \tau = \sigma \theta^{-1} \theta^{-1} \). This last change preserves cycle type, so that \( \chi_{\lambda}(\pi) = \chi_{\lambda}(\sigma^{-1} \tau) = \chi_{\lambda}(\tau^{-1} \sigma) \). The sum over \( \pi \) is then just \( n! \), and we arrive at
\[
\left\langle \prod_{k=1}^{n} U_{i_k j_k} U_{m_k r_k}^* \right\rangle = \sum_{\tau \in S_n} \text{Wg}^U(\tau^{-1} \sigma) \prod_{k=1}^{n} [i_k = m_{\sigma(k)}] [j_k = r_{\tau(k)}], \quad (18)
\]

where
\[
\text{Wg}^U(\tau^{-1} \sigma) = \frac{1}{n!^2} \sum_{\lambda \vdash n} [\chi_{\lambda}(1^n)]^2 \frac{s_{\lambda}(1^n)}{s_{\lambda}(1^n)} \chi_{\lambda}(\tau^{-1} \sigma), \quad (19)
\]

is the Weingarten function of \( \mathcal{U}(N) \).
3 Orthogonal Group

In the same spirit of the previous section, we start by writing

$$\prod_{k=1}^{2n} O_{i_k,j_k} = \frac{1}{(2n)!} \prod_{k=1}^{2n} \frac{\partial}{\partial A_{i_k,j_k}} p_{12n}(A^T O).$$  \hspace{1cm} (20)

Changing to Schur functions, we get

$$\left\langle \prod_{k=1}^{2n} O_{i_k,j_k} \right\rangle_{\mathcal{O}(N)} = \frac{1}{(2n)!} \prod_{k=1}^{2n} \frac{\partial}{\partial A_{i_k,j_k}} \sum_{\mu^i \geq 2n} \chi_{\mu}(1^2n) \left\langle s_\mu(A^T O) \right\rangle_{\mathcal{O}(N)}. \hspace{1cm} (21)$$

Performing the group integral, we arrive at

$$\frac{1}{(2n)!} \sum_{\ell \in \mathcal{L}^{-n}_{\lambda} \leq N} \frac{\chi_{2\lambda}(1^2n)}{Z_{\lambda}(1^N)} \prod_{k=1}^{2n} \frac{\partial}{\partial A_{i_k,j_k}} Z_{\lambda}(A^T A). \hspace{1cm} (22)$$

We must return to power sums in order to take the derivative. Taking \( \pi \in S_n \) to be a permutation with cycle type \( \mu \), we write

$$\left\langle \prod_{k=1}^{2n} O_{i_k,j_k} \right\rangle_{\mathcal{O}(N)} = \frac{1}{(2n)!} \sum_{\ell \in \mathcal{L}^{-n}_{\lambda} \leq N} \frac{\chi_{2\lambda}(1^2n)}{Z_{\lambda}(1^N)} \sum_{\mu^i \geq n} \omega_{\lambda}(\mu) \prod_{k=1}^{2n} \frac{\partial}{\partial A_{i_k,j_k}} p_\pi(A^T A). \hspace{1cm} (23)$$

The derivative needed is

$$\prod_{k=1}^{2n} \frac{\partial}{\partial A_{i_k,j_k}} p_\pi(A^T A) = \sum_{a_1,\ldots,a_n=1}^N \prod_{r=1}^{2n} \frac{\partial}{\partial A_{i_1(2r-1),j_1(2r-1)} \partial A_{i_1(2r),j_1(2r)}} A^T_{a_r b_r} A_{b_r a_{\pi(r)}} \hspace{1cm} (24)$$

or

$$\sum_{a_1,\ldots,a_n=1}^N \prod_{r=1}^{2n} \left[ b_r = i_\tau(2r-1) \right] \left[ b_r = i_\tau(2r) \right] \left[ a_r = j_\tau(2r-1) \right] \left[ a_{\pi(r)} = j_\tau(2r) \right]. \hspace{1cm} (25)$$

Summing over the \( a \)'s and \( b \)'s gives

$$\sum_{\tau \in S_{2n}} \prod_{r=1}^{2n} \left[ i_\tau(2r-1) = i_\tau(2r) \right] \left[ j_\tau(2r-1) = j_\tau(2\pi^{-1}(r)) \right]. \hspace{1cm} (26)$$

The product over \( i \)'s is already in the form of a matching. The product over \( j \)'s is also a matching, i.e. there exists a unique \( \sigma \in M_n \) such that

$$\prod_{r=1}^{2n} \left[ j_\tau(2r-1) = j_\tau(2\pi^{-1}(r)) \right] = \prod_{k=1}^{n} \left[ j_\sigma(2k-1) = j_\sigma(2k) \right]. \hspace{1cm} (27)$$

Fortunately, we do not need to understand \( \sigma \). All we need to realize is that the permutation \( \tau^{-1}\sigma \) has the same coset type as the permutation \( \tilde{\pi} \) we defined in (9). Therefore, the coset type of \( \tau^{-1}\sigma \) is equal to the cycle type of \( \pi \) and \( \omega_{\lambda}(\mu) = \omega_{\lambda}(\tau^{-1}\sigma) \). We can thus write

$$\left\langle \prod_{k=1}^{2n} O_{i_k,j_k} \right\rangle_{\mathcal{O}(N)} = \frac{1}{(2n)!} \sum_{\ell \in \mathcal{L}^{-n}_{\lambda} \leq N} \frac{\chi_{2\lambda}(1^2n)}{Z_{\lambda}(1^N)} \sum_{\tau \in S_{2n}} \sum_{\sigma \in M_n} \omega_{\lambda}(\tau^{-1}\sigma) \Delta_\tau(i) \Delta_\sigma(j). \hspace{1cm} (28)$$
where
\[ \Delta_{\tau}(i) = \prod_{r=1}^{n}[i_{\tau(2r-1)} = i_{\tau(2r)}]. \] (29)

The sum over \( \tau \) can be split into a sum over \( M_n \) and a sum over \( H_n \). Since the zonal spherical function \( \omega \) and the \( \Delta \) function are both invariant under action of the hyperoctahedral, the final result is
\[
\left\langle \prod_{k=1}^{2n} O_{ik,jk} \right\rangle_{O(N)} = \sum_{\tau,\sigma \in M_n} Wg^O(\tau^{-1}\sigma)\Delta_{\tau}(i)\Delta_{\sigma}(j),
\] (30)

where
\[
Wg^O(\tau^{-1}\sigma) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{\chi_{\lambda}(2\mu)}{Z_{\lambda}(1^n)} \omega_{\lambda}(\tau^{-1}\sigma)
\] (31)
is the Weingarten function of \( O(N) \).

4 Symplectic Group

Let \( \langle u, v \rangle = u^T J v \) (matrix \( J \) was defined in section 1.2) so that \( \langle u, X v \rangle = \langle X^P u, v \rangle \). Calculations analogous to those in the previous section lead to
\[
\left\langle \prod_{k=1}^{2n} S_{ik,jk} \right\rangle_{Sp(2N)} = \frac{1}{(2n)!} \sum_{\lambda \vdash n} \frac{\chi_{\lambda}(2\mu)}{Z_{\lambda}(1^n)} \sum_{\mu \vdash n} \psi_{\lambda}(\mu) \prod_{k=1}^{2n} \frac{\partial}{\partial A_{ik,jk}} p_\pi(A^P A).
\] (32)

Choosing \( \pi \in S_n \) to have cycle type \( \mu \), we need to evaluate
\[
\prod_{k=1}^{2n} \frac{\partial}{\partial A_{ik,jk}} p_\pi(A^P A) = \sum_{a_1,\ldots,a_n = 1}^{N} \sum_{b_1,\ldots,b_n = 1}^{N} \prod_{r=1}^{n} \frac{\partial}{\partial A_{i(2r-1)j(2r-1)}} \frac{\partial}{\partial A_{i(2r)j(2r)}} A_{a_r,b_r} A_{b_r,a_{\pi(r)}},
\] (33)
or
\[
\sum_{a_1,\ldots,a_n = 1}^{N} \sum_{b_1,\ldots,b_n = 1}^{N} \prod_{r=1}^{n} \langle b_r, i_{\tau(2r-1)} \rangle \langle b_r, i_{\tau(2r)} \rangle \langle a_r, j_{\tau(2r-1)} \rangle \langle a_{\pi(r)}, j_{\tau(2r)} \rangle.
\] (34)

Summing over the \( a \)'s and \( b \)'s gives
\[
\sum_{\tau \in S_{2n}} \prod_{r=1}^{n} \langle i_{\tau(2r-1)}, i_{\tau(2r)} \rangle \langle j_{\tau(2r-1)}, j_{\tau(2r)} \rangle.
\] (35)

Once again, there exists a unique \( \sigma \in M_n \) such that
\[
\prod_{r=1}^{n} \langle j_{\tau(2r-1)} : j_{\tau(2\pi^{-1}(r))} \rangle = \prod_{k=1}^{n} \langle j_{\sigma(2k-1)} : j_{\sigma(2k)} \rangle \equiv \Delta_{\sigma}'(j)
\] (36)

and the coset type of \( \tau^{-1}\sigma \) is equal to the cycle type of \( \pi \). Therefore,
\[
\left\langle \prod_{k=1}^{2n} S_{ik,jk} \right\rangle_{Sp(2N)} = \sum_{\tau,\sigma \in M_n} Wg^{Sp}(\tau^{-1}\sigma)\Delta_{\tau}'(i)\Delta_{\sigma}'(j),
\] (37)
where
\[
W_{g}^{Sp}(\tau^{-1}\sigma) = \frac{2^{n}n!}{(2n)!} \sum_{\lambda \vdash n, \ell(\lambda) \leq N} \chi_{\lambda \cup \lambda}(1^{2n}) \frac{Z'_{\lambda}(1^{N})}{Z_{\lambda}(1^{N})} \psi_{\lambda}(\tau^{-1}\sigma)
\]  
(38)
is the Weingarten function of \(Sp(2N)\).

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