Metric-affine Myrzakulov gravity theories with Gauss-Bonnet and boundary term scalars

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Abstract

In this paper, we consider some metric-affine Myrzakulov gravity (MG) theories with Gauss-Bonnet scalars. Also we consider the MG theories with the boundary term scalars. Note that these MG theories with the Gauss-Bonnet and boundary term scalars were proposed in [arXiv:1205.5266]. Some examples of Metric-Affine Gravity (MAG) theories are reviewed in the context of the $F(R, T, Q, T, D)$ type models. Then the generalized MAG theory with the curvature, torsion and nonmetricity (the so-called MG-VIII) was studied. For the FRW spacetime case, in particular, the Lagrangian, Hamiltonian and gravitational equations are obtained. The particular case $F(R, T) = \alpha R + \beta T + \mu Q + \nu T$ is investigated in detail. In quantum case, the corresponding Wheeler-DeWitt equation is obtained. Finally, some gravity theories with the curvature, torsion and nonmetricity are presented.

1 Introduction

At present, General Relativity (GR) is considered the best accepted fundamental theory describing gravity. GR is described in terms of the Levi-Civita connection, which is the basis of Riemannian geometry with the Ricci curvature scalar $R$. But GR can be described in terms of different geometries from the Riemannian one, for example, $F(R)$ gravity. There are several other alternative gravity theories. For example, one of the alternative gravity theory is the so-called teleparallel gravity with the torsion scalar $T$ or its generalization $F(T)$ gravity. Another possible alternative gravity theory is the symmetric teleparallel gravity with the nonmetricity scalar $Q$ or its generalization $F(Q)$ gravity. In this paper, we will consider the more general gravity theory, the so-called MG-VIII with the action

$$S = \int \sqrt{-g} d^4x [F(R, T, Q, T) + L_m]. \quad (1.1)$$

This paper is organized as follows. In Sec. 2, we briefly review the geometry of the underlying spacetime. In Sec. 3, we present a main information on the MG-VIII gravity. FRW cosmology of the MG-VIII is studied in Sec. 4. The specific model $F(R, T) = \alpha R + \beta T + \mu Q + \nu T$ is analyzed in Sec. 5. The cosmological power-law solution is obtained in Sec. 6. In Sec. 7, the Wheeler-DeWitt equation is derived. The relation with the soliton theory is considered in Sec. 8. Some other known gravity theories related with the curvature, torsion and nonmetricity are presented in Sec. 9. Final conclusions and remarks are provided in Sec. 10.

2 Preliminaries

2.1 Geometric setup

Consider a general spacetime with the curvature, torsion and nonmetricity. The corresponding connection is given by

$$\Gamma^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\mu\nu} + K^\rho_{\mu\nu} + L^\rho_{\mu\nu}, \quad (2.1)$$
where $\tilde{\Gamma}^\mu_{\rho\nu}$ is the Levi–Civita connection, $K^\rho_{\mu\nu}$ is the contorsion tensor and $L^\rho_{\mu\nu}$ is the disformation tensor. These three tensors have the following forms

$$
\tilde{\Gamma}^\mu_{\rho\nu} = \frac{1}{2} g^{\mu\sigma} (\partial_{\rho} g_{\nu\sigma} + \partial_{\nu} g_{\rho\sigma} - \partial_{\sigma} g_{\rho\nu}) \, ,
$$

$$
K^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\lambda\nu\mu}) = -K^\rho_{\nu\mu} \, ,
$$

$$
L^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (-Q_{\mu\lambda\nu} - Q_{\nu\lambda\mu} + Q_{\lambda\mu\nu}) = L^\rho_{\nu\mu} \, .
$$

Here

$$
T^\mu_{\rho\nu} = 2\Gamma^\mu_{|\rho\nu|} \, , \quad Q_{\rho\mu\nu} = \nabla_{\rho} g_{\mu\nu} \, .
$$

(2.5)

are the torsion tensor and the nonmetricity tensor, respectively. In this generalized spacetime with the curvature, torsion and nonmetricity, let us introduce three scalars as

$$
R = g^{\mu\nu} R_{\mu\nu} \, ,
$$

$$
T = S^\rho_{\nu\mu} T^\rho_{\mu\nu} \, ,
$$

$$
Q = -g^{\mu\nu} (L^\rho_{\mu\nu} T^\rho_{\nu\alpha} - L^\rho_{\beta\alpha} T^\rho_{\mu\nu}) \, ,
$$

(2.6, 2.7, 2.8)

where $R$ is the curvature scalar, $T$ is the torsion scalar and $Q$ is the nonmetricity scalar. Here

$$
R_{\rho_{ijk}} = \partial_i \Gamma^i_{jk} - \partial_j \Gamma^i_{ik} + \Gamma^i_{ip} \Gamma^p_{jk} - \Gamma^i_{jp} \Gamma^p_{ik} \, ,
$$

$$
S^\rho_{\nu\mu} = K^\rho_{\nu\mu} - g^{\rho\sigma} T^\rho_{\sigma\nu} + g^{\rho\mu} T^\rho_{\nu\sigma} \, ,
$$

$$
K^\nu_{\rho\mu} = \frac{1}{2} (T^\rho_{\nu \mu} + T^\rho_{\mu \nu} - T^\rho_{\nu\nu}) \, .
$$

(2.9, 2.10, 2.11)

are the Ricci tensor, the potential and the contorsion tensor, respectively. The key moment of our construction is following: as in our previous paper [12], here we assume that these three scalars have the following forms

$$
R = u + R_s \, ,
$$

$$
T = v + T_s \, ,
$$

$$
Q = w + Q_s \, .
$$

(2.12, 2.13, 2.14)

where $u = u(\Gamma^\rho_{\mu\nu}; x_i; g_{ij}, \tilde{g}_{ij}, ...; f_j)$, $v = v(\Gamma^\rho_{\mu\nu}; x_i; g_{ij}, \tilde{g}_{ij}, ...; g_j)$ and $w = w(\Gamma^\rho_{\mu\nu}; x_i; g_{ij}, \tilde{g}_{ij}, ...; h_j)$ are some real functions. Here: i) $R_s = R^{(LC)}$ is the curvature scalar corresponding to the Levi-Civita connection with the vanishing torsion and nonmetricity ($T = Q = 0$); ii) $T_s = T^{(WC)}$ is the torsion scalar for the purely Weitzenböck connection with the vanishing curvature and nonmetricity ($R = Q = 0$); iii) $Q_s = Q^{(NM)}$ is the nonmetricity scalar with the vanishing torsion and curvature ($R = T = 0$).

Consider the Friedmann-Robertson-Walker (FRW) spacetime. The flat FRW spacetime is described by the metric

$$
ds^2 = -N^2(t) dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) ,
$$

(2.15)

where $a = a(t)$ is the scale factor, $N(t)$ is the lapse function. The orthonormal tetrad components $e_i(x^\mu)$ are related to the metric through

$$
g_{\mu\nu} = \eta_{ij} e^i_{\mu} e^j_{\nu} \, .
$$

(2.16)

where the Latin indices $i, j$ run over $0, ..., 3$ for the tangent space of the manifold, while the Greek letters $\mu, \nu$ are the coordinate indices on the manifold, also running over $0, ..., 3$. With the FRW metric ansatz the three variables $R_s, T_s, Q_s$ look like (we assume that $N = 1$)

$$
R_s = R^{LC} = 6(\dot{H} + 2H^2) \, ,
$$

$$
T_s = T^{WC} = -6H^2 \, ,
$$

$$
Q_s = Q^{NM} = 6H^2 \, .
$$

(2.17, 2.18, 2.19)
where \( H = (\ln a)_t \) is the Hubble parameter. Therefore, these three scalars \((R, T, Q)\) of the metric-affine spacetime (in the FRW case) take the following forms

\[
R = u + 6(\dot{H} + 2H^2), \tag{2.20}
\]
\[
T = v - 6H^2, \tag{2.21}
\]
\[
Q = w + 6H^2, \tag{2.22}
\]

where \( u, v, w \) are some real functions of \( \Gamma^\rho_{\mu\nu}; t, a, \dot{a}, R_s, T_s, Q_s, \ldots \) and so on.

The concepts of torsion and non-metricity allow to classify different geometries on which we will make observations throughout this work.

### 2.2 Variations

#### 2.2.1 Variations of the torsion variables

Let us now derive the variations for the torsion tensor \((S^\alpha_{\mu\nu})\) and torsion vector \((S_\alpha \equiv S^\alpha_{\mu\nu})\) since we will be using them in the various theories we are going to study. Firstly, note that since the torsion does not depend on the metric, the \( \delta g^{\mu\nu} \) variation is identically zero, namely

\[
\delta g^{\mu\nu} S^\alpha_{\mu\nu} = 0 \tag{2.23}
\]

as well as

\[
\frac{1}{2} \delta g^{\mu\nu} S_\mu = 0 \tag{2.24}
\]

Now to proceed with the \( \Gamma \)-variation we recall that we want to have a common factor \( \delta \Gamma^\lambda_{\mu\nu} \) appearing in the variation. Thus, we express the torsion tensor as

\[
S^\lambda_{\alpha\beta} = \frac{1}{2} (\Gamma^\lambda_{\alpha\beta} - \Gamma^\lambda_{\beta\alpha}) = \frac{1}{2} (\delta^\mu_{\alpha} \delta^\nu_{\beta} \Gamma^\lambda_{\mu\nu} - \delta^\mu_{\beta} \delta^\nu_{\alpha} \Gamma^\lambda_{\mu\nu}) = \frac{1}{2} (\delta^\mu_{\alpha} \delta^\nu_{\beta} - \delta^\mu_{\beta} \delta^\nu_{\alpha}) \Gamma^\lambda_{\mu\nu} \Rightarrow S^\lambda_{\alpha\beta} = \delta^\mu_{\alpha} \delta^\nu_{\beta} \Gamma^\lambda_{\mu\nu} \tag{2.25}
\]

such that

\[
\delta \Gamma S^\alpha_{\mu\nu} = \delta^\mu_{\alpha} \delta^\nu_{\beta} \Gamma^\lambda_{\mu\nu} \tag{2.26}
\]

So long as the torsion vector is concerned we contract the above in \( \beta, \lambda \) to obtain

\[
\delta \Gamma S_\alpha = \delta \Gamma S^\lambda_{\alpha\lambda} = \delta \Gamma^\mu_{\alpha} \delta \Gamma^\lambda_{\mu\nu} \tag{2.27}
\]

and for the torsion pseudo-vector (in 4-dim)

\[
\delta \Gamma \tilde{S}_\alpha = \epsilon^{\mu\nu\alpha\lambda} \delta \Gamma^\lambda_{\mu\nu} \tag{2.28}
\]

Having performed the variations of the torsion, we know proceed to derive the variations of the non-metricity tensor with respect to both the metric tensor and the connection.

#### 2.2.2 Variations of the nonmetricity variables

Let us firstly obtain the variation of the non-metricity tensor with respect to the connection. To do so we single out a common \( \Gamma^\lambda_{\mu\nu} \) factor in the expression of the non-metricity as we did with the torsion. We have

\[
Q_{\rho\alpha\beta} = - \nabla_\rho g_{\alpha\beta} = - \partial_\rho g_{\alpha\beta} + \Gamma^\lambda_{\alpha\rho} g_{\lambda\beta} + \Gamma^\lambda_{\beta\rho} g_{\lambda\alpha} = - \partial_\rho g_{\alpha\beta} + \delta^\mu_{\alpha} \delta^\nu_{\beta} \Gamma^\lambda_{\mu\nu} + \delta^\mu_{\beta} \delta^\nu_{\alpha} \Gamma^\lambda_{\mu\nu} \Rightarrow
\]

\[
Q_{\rho\alpha\beta} = - \partial_\rho g_{\alpha\beta} + \delta^\mu_{\alpha} \delta^\nu_{\beta} (\partial_\rho g_{\lambda\beta} + \partial_\rho g_{\lambda\alpha}) \Gamma^\lambda_{\mu\nu} \Rightarrow
\]

\[\text{This is so because in order to form the torsion vector } S^\rho \text{ we need only contract an upper with a lower index without the use of any metric. Notice also that if we were to form another vector by contracting the first two indices of the torsion with the metric tensor, the result would yield zero due to the fact that the torsion is antisymmetric in its first two indices while the metric tensor is symmetric. In words, } S^\rho \equiv g^{\alpha\beta} S_{\alpha\beta}^\rho = 0.\]
Let us now vary with respect to the metric tensor. Using the above definition of non-metricity along with the identity
\[ \delta g_{\alpha\beta} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\mu\nu} \]  
(2.31)
it follows that
\[ \delta g_{\alpha\beta} = -\partial_\rho g_{\alpha\beta} + \Gamma^\lambda_{\alpha\rho}g_{\lambda\beta} + \Gamma^\lambda_{\beta\rho}g_{\lambda\alpha} = 
\]
\[ = \partial_\rho(g_{\mu\alpha}g_{\nu\beta}\delta g^{\mu\nu}) - \Gamma^\lambda_{\alpha\rho}g_{\lambda\nu}g_{\nu\beta}\delta g^{\mu\nu} - \Gamma^\lambda_{\beta\rho}g_{\mu\nu}g_{\nu\alpha}\delta g^{\mu\nu} = 
\]
\[ = \partial_\rho(g_{\mu\alpha}g_{\nu\beta}\delta g^{\mu\nu} - (\delta g^{\mu\nu})g_{\lambda\mu}g_{\nu\alpha}\delta g^{\mu\nu}) 
\]
Thus, one has
\[ \delta g_{\alpha\beta} = \partial_\rho(g_{\mu\alpha}g_{\nu\beta}\delta g^{\mu\nu}) - (\delta g^{\mu\nu})g_{\lambda\mu}g_{\nu\alpha}\delta g^{\mu\nu} \]  
(2.32)
We continue by varying the Weyl vector
\[ Q_\nu = -g^{\alpha\beta}\nabla_\nu g_{\alpha\beta} = -g^{\alpha\beta}\partial_\nu g_{\alpha\beta} + 2\Gamma^\lambda_{\lambda\nu} \]  
(2.33)
Variation with respect to the connection yields
\[ \delta_\Gamma Q_\nu = 2\delta\Gamma^\lambda_{\lambda\nu} = \delta\Gamma^\lambda_{\lambda\mu}\delta^\lambda_{\mu
u} \]
\[ \Rightarrow \delta_\Gamma Q_\rho = 2\delta\Gamma^\lambda_{\lambda\mu}\delta^\lambda_{\mu
u} \]  
(2.34)
While variation with respect to the metric tensor gives
\[ \delta_\rho Q_\rho = -(\delta g^{\mu\nu})\partial_\rho g_{\mu\nu} - g^{\alpha\beta}\partial_\rho g_{\alpha\beta} \]  
(2.35)
Now, expanding the second term, we have
\[ g^{\alpha\beta}\partial_\rho g_{\alpha\beta} = -g^{\alpha\beta}\partial_\rho(g_{\mu\alpha}g_{\nu\beta}\delta g^{\mu\nu}) = 
\]
\[ = -g_{\mu\nu}\partial_\rho(g_{\mu\alpha}g_{\nu\beta}\delta g^{\mu\nu}) - 2(\delta g^{\mu\nu})\partial_\rho g_{\mu\nu} \]  
(2.36)
such that
\[ \delta_\rho Q_\rho = -(\delta g^{\mu\nu})\partial_\rho g_{\mu\nu} + g_{\mu\nu}\partial_\rho\delta g^{\mu\nu} + 2(\delta g^{\mu\nu})\partial_\rho g_{\mu\nu} = 
\]
\[ = g_{\mu\nu}\partial_\rho\delta g^{\mu\nu} + (\delta g^{\mu\nu})\partial_\rho g_{\mu\nu} = \partial_\rho(g_{\mu\nu}\delta g^{\mu\nu}) \]
Thus, the \( g \)-variation of the Weyl vector has the handy form
\[ \delta_\rho Q_\rho = \partial_\rho(g_{\mu\nu}\delta g^{\mu\nu}) \]  
(2.37)
Let us now proceed by varying the second non-metricity vector \( 2\mu
u \). Recall that the latter is given by
\[ \tilde{Q}_\beta = g^{\alpha\beta}Q_{\rho\alpha\beta} = -g^{\alpha\beta}\partial_\rho g_{\alpha\beta} + (\delta g^{\mu\nu})g_{\beta\lambda}\Gamma^\lambda_{\mu\nu} \]  
(2.38)
Variation with respect to the connection immediately gives
\[ \delta_\Gamma \tilde{Q}_\beta = (g^{\mu\nu}g_{\beta\lambda} + \delta^\mu_{\beta}\delta^\lambda_{\nu})\delta\Gamma^\lambda_{\mu\nu} \]
while variation with respect to the metric tensor reads
\[ \delta_\rho \tilde{Q}_\beta = -(\delta g^{\mu\nu})\partial_\rho g_{\beta\lambda} - g^{\alpha\beta}\partial_\rho\delta g_{\alpha\beta} + (\delta g^{\mu\nu})g_{\beta\lambda}\Gamma^\lambda_{\mu\nu} + g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\delta g_{\beta\lambda} = 
\]
\[ = \delta g^{\mu\nu} - \partial_\rho g_{\beta\lambda} + g_{\lambda\beta}\Gamma^\lambda_{\mu\nu} - g^{\mu\nu}\partial_\rho g_{\alpha\beta} + g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\delta g_{\beta\lambda} \]  
(2.40)
\footnote{This may also be obtained by contracting \ref{2.30} with \( g^{\alpha\beta} \). Of course, this can be done because the \( \Gamma \)-variation commutes with the metric tensor. However, this is not true for the \( g \)-variation.}
Now using
\[ \delta g_{\alpha \beta} = -g_{\alpha \mu} g_{\beta \nu} \delta g^{\mu \nu} \]  
(2.41)
it can easily be shown that
\[ g^{\rho \sigma} \partial_{\rho} \delta g_{\alpha \beta} = -g_{\beta \nu} g^{\rho \sigma} (\partial_{\rho} g_{\mu \alpha}) \delta g^{\mu \nu} - \partial_{\mu} (g_{\nu \beta} \delta g^{\mu \nu}) \]  
(2.42)
as well as
\[ g^{\rho \mu} \Gamma^\lambda_{\rho \mu} \delta g_{\beta \lambda} = -g^{\rho \sigma} \Gamma^\alpha_{\rho \sigma} g_{\mu \alpha} g_{\nu \beta} \delta g^{\mu \nu} \]  
(2.43)
and upon using these, the g-variation of \( \tilde{Q}_\beta \) reads
\[ \delta g Q_{\alpha \beta} = \delta g_{\mu \nu} \left[ g_{\nu \beta} g^{\rho \sigma} (\partial_{\rho} g_{\mu \alpha}) + \Gamma^\lambda_{\mu \nu} g_{\lambda \beta} - g^{\rho \sigma} \Gamma^\alpha_{\rho \sigma} g_{\mu \alpha} g_{\nu \beta} \right] + g_{\nu \beta} (\partial_{\mu} \delta g^{\mu \nu}) \]  
(2.44)
Notice that there is a quicker and more elegant way to derive the g-variation of non-metricity. This comes about by first recalling that the general covariant derivative \( \nabla_\alpha \) does not depend on the metric tensor. Then, using the definition of the variation, one has
\[ \delta g Q_{\mu \nu} = -\nabla_\alpha (g_{\mu \nu} + \delta g_{\mu \nu}) + \nabla_\alpha g_{\mu \nu} = -\nabla_\alpha \delta g_{\mu \nu} \]  
(2.45)
and also
\[ \delta g Q_{\alpha}^{\mu \nu} = \nabla_\alpha (g^{\mu \nu} + \delta g^{\mu \nu}) - \nabla_\alpha g_{\mu \nu} = +\nabla_\alpha \delta g^{\mu \nu} \]  
(2.46)
So, when coupled to a tensor filed (or a tensor density) \( T^\alpha_{\mu \nu} \) we have
\[ T^\alpha_{\mu \nu} \delta g Q_{\mu \nu}^\alpha = \nabla_\alpha (T^\alpha_{\mu \nu} \delta g^{\mu \nu}) - (\delta g^{\mu \nu}) \nabla_\alpha T^\alpha_{\mu \nu} \]  
(2.47)
where we have employed Leibniz's rule for the covariant derivatives. Next we derive the variations of the Riemann tensor.

### 2.2.3 Variations of the Riemann variables

For the sake of completeness we also give here the variations of the Riemann tensor (and its related contractions) with respect to the independent connection and the metric. First notice that the prototype of the Riemann tensor
\[ R^\mu_{\nu \alpha \beta} := 2 \partial_{[\alpha} \Gamma^\mu_{\nu \beta]} + 2 \Gamma^\mu_{\rho \alpha} \Gamma^\rho_{\nu \beta} \]  
(2.48)
does not depend on the metric and therefore
\[ \delta g R^\mu_{\nu \alpha \beta} = 0 \]  
(2.49)
When the first index is brought down however we have a metric tensor dependence since
\[ R_{\rho \nu \alpha \beta} = g_{\mu \rho} R^\mu_{\nu \alpha \beta} \]  
(2.50)
and thus
\[ \delta g R_{\rho \nu \alpha \beta} = (\delta g_{\mu \rho}) R^\mu_{\nu \alpha \beta} = -(\delta g^{\mu \lambda}) g_{\mu \rho} g_{\nu \lambda} R^\mu_{\nu \alpha \beta} = -(\delta g^{\mu \lambda}) g_{\rho \lambda} R^\mu_{\nu \alpha \beta} \]  
(2.51)
Now, to derive the variation with respect to the connection we start by (2.48) and compute
\[ \delta \Gamma R^\mu_{\nu \alpha \beta} = R^\mu_{\nu \alpha \beta} \Gamma + \delta \Gamma [\Gamma + \delta \Gamma] - R^\mu_{\nu \alpha \beta} [\Gamma] \]  
(2.52)
and expanding \( R^\mu_{\nu \alpha \beta} [\Gamma + \delta \Gamma] \) to linear order in \( \delta \Gamma \) we finally arrive at
\[ \delta \Gamma R^\mu_{\nu \alpha \beta} = \nabla_\alpha (\delta \Gamma_{\nu \beta}) - \nabla_\beta (\delta \Gamma_{\nu \alpha}) - 2 S^\lambda_{\alpha \beta} \delta \Gamma_{\nu \lambda} \]  
(2.53)
Having obtained all he necessary setup we are now in a position to study Metric-Affine Theories of Gravity. We do so in what follows.
2.2.4 Variations of the geometrical scalars

2.2.5 Variations of the energy-momentum variables

3 Harko paper appendices

3.1 Calculation of $Q = -Q_{\alpha\mu\nu} P^{\alpha\mu\nu}$

According to Eq. (??) and Eq. (??), we have

$$Q = -g^{\mu\nu} \left( L^\alpha_{\beta\mu} L^{\beta}_{\nu\alpha} - L^\alpha_{\beta\alpha} L^{\beta}_{\mu\nu} \right), \quad (3.1)$$

$$L^\alpha_{\beta\mu} = -\frac{1}{2} g^{\alpha\lambda} (Q_{\mu\beta\lambda} + Q_{\beta\lambda\mu} - Q_{\lambda\mu\beta}), \quad (3.2)$$

$$L^{\beta}_{\nu\alpha} = -\frac{1}{2} g^{\beta\rho} (Q_{\alpha\nu\rho} + Q_{\nu\rho\alpha} - Q_{\rho\alpha\nu}), \quad (3.3)$$

$$L^\alpha_{\beta\alpha} = -\frac{1}{2} g^{\alpha\lambda} (Q_{\alpha\beta\lambda} + Q_{\beta\lambda\alpha} - Q_{\lambda\alpha\beta})$$

$$L^{\beta}_{\mu\nu} = -\frac{1}{2} g^{\beta\rho} (Q_{\nu\rho\alpha} + Q_{\rho\alpha\nu} - Q_{\alpha\nu\rho}). \quad (3.5)$$

Thus, we obtain

$$-g^{\mu\nu} L^\alpha_{\beta\mu} L^{\beta}_{\nu\alpha} = -\frac{1}{4} g^{\mu\nu} g^{\alpha\lambda} g^{\beta\rho} (Q_{\mu\beta\lambda} + Q_{\beta\lambda\mu} - Q_{\lambda\mu\beta}) \times (Q_{\alpha\nu\rho} + Q_{\nu\rho\alpha} - Q_{\rho\alpha\nu}) = \frac{1}{4} (Q_{\nu\rho\alpha} + Q_{\nu\rho\alpha} - Q_{\rho\alpha\nu}),$$

$$-Q^{\rho\alpha\nu} Q_{\nu\rho\alpha} + Q^{\alpha\nu\rho} Q_{\alpha\nu\rho} + Q^{\alpha\rho\nu} Q_{\alpha\rho\nu} - Q^{\alpha\rho\nu} Q_{\alpha\rho\nu} = -\frac{1}{4} (2Q^{\rho\alpha\nu} Q_{\nu\rho\alpha} - Q^{\alpha\rho\nu} Q_{\alpha\rho\nu}), \quad (3.6)$$

$$g^{\mu\nu} L^\alpha_{\beta\alpha} L^{\beta}_{\mu\nu} = \frac{1}{4} g^{\mu\nu} g^{\beta\rho} (Q_{\nu\rho\alpha} + Q_{\nu\rho\alpha} - Q_{\nu\rho\alpha}),$$

$$= \frac{1}{4} Q^\rho \left( 2\tilde{Q}_\rho - Q_\rho \right), \quad (3.7)$$

$$Q = -\frac{1}{4} \left( -Q^{\rho\alpha\nu} Q_{\alpha\nu\rho} + 2Q^{\rho\alpha\nu} Q_{\alpha\rho\nu} - 2Q^{\rho\alpha\nu} \tilde{Q}_\rho + Q^\rho Q_\rho \right). \quad (3.8)$$

Then, according to Eq. (??), we have

$$P^{\alpha\mu\nu} = \frac{1}{4} \left[ -Q^{\alpha\mu\nu} + Q^{\alpha\nu\mu} + Q^{\nu\alpha\mu} + Q^\alpha g^{\mu\nu} - Q^\alpha g^{\mu\nu} \right], \quad (3.9)$$

$$\frac{1}{2} (g^{\alpha\mu} Q^\nu + g^{\alpha\nu} Q^\mu),$$

$$-Q_{\alpha\mu\nu} P^{\alpha\mu\nu} = -\frac{1}{4} \left[ -Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + Q_{\alpha\mu\nu} Q^{\mu\alpha\nu} + Q_{\alpha\mu\nu} Q^{\nu\alpha\mu} + \frac{1}{2} Q_{\alpha\mu\nu} (g^{\alpha\mu} Q^\nu + g^{\alpha\nu} Q^\mu) \right] = -\frac{1}{4} (-Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + 2Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} = Q). \quad (3.10)$$

To obtain the above result we have used the relations $Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} = Q_{\alpha\mu\nu} Q^{\alpha\mu\nu}$, which is valid since $Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} = Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} = Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} = Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} = Q_{\alpha\mu\nu} Q^{\alpha\mu\nu}$. Hence, we have proved that $Q = -Q_{\alpha\mu\nu} P^{\alpha\mu\nu}$, a relation which is very useful in later calculations.
3.2 Calculation of the variation of $\delta Q$

Before the presentation of the detailed variation of $\delta Q$, we write down all the nonmetricity tensors for later applications. They are obtained as

$$Q_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu},$$  \hbox{(3.11)}
$$Q^\alpha_{\mu\nu} = g^{\alpha\beta} Q_{\beta\mu\nu} = g^{\alpha\beta} \nabla_{\beta} g_{\mu\nu} = \nabla^\alpha g_{\mu\nu},$$  \hbox{(3.12)}
$$Q_{\alpha}^{\mu\nu} = g^{\mu\rho} Q_{\alpha\rho\nu} = g^{\mu\rho} \nabla_{\alpha} g_{\rho\nu} = -g_{\mu\nu} \nabla_{\alpha} g^\mu,\quad Q_{\alpha}^\mu = g_{\alpha\beta} \nabla^\beta g^\mu = g_{\mu\rho} \nabla^\alpha g^\rho,$$  \hbox{(3.13)}
$$Q_{\alpha}^{\mu\nu} = g^{\mu\rho} Q_{\alpha\rho\nu} = g^{\mu\rho} \nabla_{\alpha} g_{\rho\nu} = -g_{\mu\nu} \nabla_{\alpha} g^\rho,$$  \hbox{(3.14)}
$$Q^\mu = g^{\mu\rho} g^{\nu\sigma} \nabla_{\alpha} g_{\rho\sigma} = -g_{\mu\nu} g_{\rho\sigma} \nabla^\alpha g^{\rho\sigma} = -\nabla_{\alpha} g^{\mu\nu},$$  \hbox{(3.15)}
$$Q^{\alpha\mu\nu} = -\nabla^\alpha g_{\mu\nu}.$$  \hbox{(3.16)}

Let us find the variation of $Q$ by using Eq. (3.3),

$$\delta Q = -\frac{1}{4} \delta \left( -Q^{\alpha\rho\mu\nu} Q_{\alpha\rho\mu\nu} + 2Q^{\alpha\rho\nu} Q_{\rho\alpha\nu\mu} - 2Q^\rho Q_{\nu} + Q^\rho Q_{\nu} \right)$$
$$= -\frac{1}{4} \left[ -Q^{\alpha\rho\mu\nu} Q_{\alpha\rho\mu\nu} - Q^{\alpha\rho\nu} Q_{\alpha\rho\nu\mu} + 2Q^\rho Q_{\nu} + Q^\rho Q_{\nu} \right]$$
$$= -\frac{1}{4} \left[ -Q^{\alpha\rho\mu\nu} Q_{\alpha\rho\mu\nu} - Q^{\alpha\rho\nu} Q_{\alpha\rho\nu\mu} + 2Q^\rho Q_{\nu} + Q^\rho Q_{\nu} \right]$$

In order to simplify the above equation we can use several useful equations, which are given
\[ \delta g_{\mu \nu} = -g_{\mu \rho } \delta g^{\rho \beta} g_{\beta \nu }, \quad (3.20) \]

\[ -Q^{\alpha \rho } \nabla _{\alpha } \delta g_{\rho \nu } = -Q^{\alpha \mu } \nabla _{\alpha } \left( -g_{\nu \lambda } \delta g^{\lambda \rho } g_{\rho \mu } \right) = 2Q^{\alpha \nu } \delta Q_{\alpha \lambda } g^{\lambda \rho } + Q_{\alpha \lambda } \nabla ^{\alpha } g^{\lambda \rho } = 2Q^{\alpha \sigma } \nu \nabla _{\alpha } \nabla ^{\sigma } g^{\nu \rho } + Q_{\alpha \nu } \nabla ^{\alpha } g^{\nu \rho } , \]

\[ 2Q^{\alpha \rho } \nabla _{\rho } \delta g_{\alpha \nu } = -4Q_{\mu } \alpha \nu Q_{\rho \nu } \delta g^{\mu \nu } - 2Q_{\rho \nu } \nabla ^{\alpha } g^{\nu \rho } , \quad (3.21) \]

\[ -2Q^{\alpha } \nabla ^{\lambda } \delta g_{\beta \lambda } = 2Q^{\alpha } Q_{\nu \alpha } \delta g^{\mu \nu } + 2Q_{\nu } \nabla ^{\alpha } g^{\nu \rho } + 2Q_{\nu } g_{\alpha } \nabla ^{\alpha } g^{\nu \rho } , \]

\[ (3.22) \]

Thus, Eq. (3.19) takes the form

\[ \delta Q = -\frac{1}{4} \left[ Q_{\alpha \nu } \nabla ^{\alpha } g^{\nu \rho } + 2Q^{\alpha } \nu Q_{\alpha \mu } \delta g^{\mu \nu } + Q_{\alpha \nu } \nabla ^{\alpha } g^{\nu \rho } - 2Q_{\alpha \nu } \nabla ^{\alpha } g^{\nu \rho } - 4Q_{\mu }^{\sigma } \nu Q_{\rho \sigma } \nabla ^{\alpha } g^{\nu \rho } - 2Q_{\nu } g_{\alpha } \nabla ^{\alpha } g^{\nu \rho } + 2Q_{\nu } Q^{\alpha } \nabla ^{\alpha } g^{\nu \rho } + 2Q_{\nu } Q_{\nu } \nabla ^{\alpha } g^{\nu \rho } - Q_{\alpha \nu } g_{\alpha } \nabla ^{\alpha } g^{\nu \rho } \right] = 2P_{\alpha \nu } \nabla ^{\alpha } g^{\nu \rho } - \left( P_{\nu \alpha } Q_{\nu } \nabla ^{\alpha } g^{\nu \rho } - 2Q^{\alpha \beta } \mu P_{\alpha \beta \nu } \right) \delta g^{\mu \nu } , \quad (3.24) \]

where we have used the relations

\[ 2P_{\alpha \nu } = -\frac{1}{4} \left[ 2Q_{\alpha \nu } - 2Q_{\rho \nu } - 2Q_{\nu } g_{\alpha } \right] , \quad (3.25) \]

\[ 4 \left( P_{\nu \alpha } Q_{\nu } \nabla ^{\alpha } g^{\nu \rho } - 2Q^{\alpha \beta } \mu P_{\alpha \beta } \right) = 2Q^{\alpha } \nu Q_{\alpha \beta } \nu \]

\[ -4Q_{\mu }^{\alpha } \nabla ^{\alpha } g^{\nu \rho } + 2Q_{\alpha }^{\alpha } Q_{\alpha } Q_{\nu } + 2Q_{\nu } Q_{\nu } \]

\[ + 2Q_{\nu } Q_{\nu } Q_{\nu } , \quad (3.26) \]

### 3.3 Variation of the gravitational action with respect to the connection

The full action of the $f(Q, T)$ theory supplemented with the Lagrangian multipliers is

\[ S = \int d^{4}x \left[ \frac{\sqrt{-g}}{16\pi} f(Q, T) + \mathcal{L}_{M} \sqrt{-g} 
+ \lambda_{\alpha } \delta \gamma \alpha \beta \gamma + \xi_{\alpha } \beta \mu \nu \right] . \quad (3.27) \]

We can vary the action separately, thus obtaining

\[ \delta \left[ \frac{\sqrt{-g}}{16\pi} f(Q, T) + \mathcal{L}_{M} \sqrt{-g} \right] = \left( \frac{4\sqrt{-g}}{16\pi} f_{\alpha } P^{\alpha } + H_{\alpha } \nabla ^{\alpha } \right) \delta \Gamma^{\alpha } \mu \nu , \quad (3.28) \]

\[ \delta \left( \lambda_{\alpha } \mu \nu T^{\alpha } \mu \nu \right) = 2\lambda_{\alpha } \mu \nu \delta \Gamma^{\alpha } \mu \nu , \quad (3.29) \]

\[ \delta \left( \xi_{\alpha } \beta \mu \nu \right) = \xi_{\alpha } \beta \mu \nu \left[ \nabla _{\mu } \left( \delta \Gamma^{\alpha } \nu \beta \right) - \nabla _{\nu } \left( \delta \Gamma^{\alpha } \mu \beta \right) \right] \]

\[ = 2\xi_{\alpha } \beta \mu \nu \nabla _{\beta } \delta \Gamma^{\alpha } \mu \nu \approx 2 \left( \nabla _{\beta } \xi_{\alpha } \beta \mu \nu \right) \delta \Gamma^{\alpha } \mu \nu . \quad (3.30) \]
Thus,

\[ \delta S = \int d^4x \left( \frac{4\sqrt{-g}}{16\pi} f_Q P_{\alpha \beta} P_{\mu \nu} + H_{\alpha \mu \nu} + 2\lambda_{\alpha \mu \nu} \right) \delta \hat{\Gamma}_{\alpha \mu \nu}. \]  

(3.31)

To eliminate the Lagrange multipliers, we take two covariant derivatives \( \nabla_\mu \nabla_\nu \) or \( \nabla_\nu \nabla_\mu \) (considering vanishing curvature tensor) of the integrand, and thus we finally arrive to Eq. (3.31).

### 3.4 Metric divergence of (1,1)-form field equations

The metric divergence of the gravitational field equation Eq. (3.31) of the \( f(Q,T) \) theory is

\[
\mathcal{D}_\mu \left[ f_T (T^\mu_\nu + \Theta^\mu_\nu) - 8\pi T^\mu_\nu \right] = \frac{1}{2} \partial_\nu f \\
+ \mathcal{D}_\mu \left( f_Q Q_{\nu}^{\alpha \beta} P_{\mu \alpha \beta} \right) + \mathcal{D}_\mu \left[ \frac{2}{\sqrt{-g}} \nabla_\nu \left( f_Q \sqrt{-g} P_{\alpha \mu} \right) \right],
\]

(3.32)

where we have

\[
\mathcal{D}_\mu \left( f_Q Q_{\nu}^{\alpha \beta} P_{\mu \alpha \beta} \right) = \nabla_\mu \left( f_Q Q_{\nu}^{\alpha \beta} P_{\mu \alpha \beta} \right) \\
+ \frac{1}{2} Q_\rho \left( f_Q Q_{\nu}^{\alpha \beta} P_{\mu \alpha \beta} \right) + L^\rho_\mu \left( f_Q Q_{\rho}^{\alpha \beta} P_{\nu \alpha \beta} \right),
\]

(3.33)

\[
\mathcal{D}_\mu \left[ \frac{2}{\sqrt{-g}} \nabla_\nu \left( f_Q \sqrt{-g} P_{\alpha \mu} \right) \right] \\
= \frac{2}{\sqrt{-g}} \mathcal{D}_\mu \left[ \nabla_\nu \left( f_Q \sqrt{-g} P_{\alpha \mu} \right) \right] \\
= \frac{2}{\sqrt{-g}} \nabla_\nu \nabla_\alpha \left( f_Q \sqrt{-g} P_{\alpha \mu} \right) \\
+ \frac{1}{\sqrt{-g}} Q_\rho \nabla_\alpha \left( f_Q \sqrt{-g} P_{\alpha \rho} \right) \\
+ \frac{2}{\sqrt{-g}} L^\rho_\mu \nabla_\nu \left( f_Q \sqrt{-g} P_{\alpha \rho} \right),
\]

(3.34)

which gives

\[
\mathcal{D}_\mu \left[ f_T (T^\mu_\nu + \Theta^\mu_\nu) - 8\pi T^\mu_\nu \right] + \frac{8\pi}{\sqrt{-g}} \nabla_\alpha \nabla_\mu H_{\nu \alpha \mu} \\
= \frac{1}{2} \partial_\nu f + \nabla_\mu \left( f_Q Q_{\nu}^{\alpha \beta} P_{\mu \alpha \beta} \right) + \frac{1}{2} Q_\rho \left( f_Q Q_{\nu}^{\alpha \beta} P_{\mu \alpha \beta} \right) \\
+ L^\rho_\mu \left( f_Q Q_{\rho}^{\alpha \beta} P_{\nu \alpha \beta} \right) + \frac{2}{\sqrt{-g}} L^\rho_\mu \nabla_\nu \left( f_Q \sqrt{-g} P_{\alpha \rho} \right) \\
+ \frac{1}{\sqrt{-g}} Q_\rho \nabla_\alpha \left( f_Q \sqrt{-g} P_{\alpha \rho} \right) = \sum_{i=1}^{10} E_i.
\]

(3.35)
For the sake of clarity, in the above equation we have defined

\[ E_1 = \frac{1}{2} \partial_\nu f, \]
\[ E_2 = \left( \nabla_\mu f \right) Q_{\nu \alpha \beta} P^{\mu \alpha \beta}, \]
\[ E_3 = f Q \left( \nabla_\mu Q_{\nu \alpha \beta} \right) P^{\mu \alpha \beta}, \]
\[ E_4 = f Q Q_{\nu \alpha \beta} \left( \nabla_\mu P^{\mu \alpha \beta} \right), \]
\[ E_5 = \frac{1}{2} f Q Q_\mu Q_{\nu \alpha \beta} P^{\mu \alpha \beta}, \]
\[ E_6 = f Q L^\rho_\mu Q_{\rho \alpha \beta} P^{\mu \alpha \beta}, \]
\[ E_7 = 2 \left( \nabla_\alpha f Q \right) L^\rho_\mu P^{\rho \mu}, \]
\[ E_8 = f Q_\alpha L^\rho_\mu P^{\rho \mu}, \]
\[ E_9 = 2 f Q L^\rho_\mu \nabla_\alpha P^{\rho \mu}, \]
\[ E_{10} = \frac{1}{\sqrt{-g}} Q_\mu \nabla_\alpha \left( f Q \sqrt{-g} P^{\rho \mu} \right). \]

Then, we can find the following relations

\[ E_2 + E_7 = \nabla_\mu f Q \left( Q_{\nu \alpha \beta} + 2 L_\beta^\alpha \right) P^{\mu \alpha \beta} = 0, \]
\[ E_5 + E_8 = \frac{1}{2} f Q Q_\mu \left( Q_{\nu \alpha \beta} + 2 L_\beta^\alpha \right) P^{\mu \alpha \beta} = 0, \]
\[ E_4 + E_9 = f Q \left[ \left( Q_{\nu \alpha \beta} + 2 L_\beta^\alpha \right) \nabla_\mu P^{\mu \alpha \beta} + 2 L^\rho_\mu Q_{\rho \alpha \beta} P^{\mu \alpha \beta} \right] \]
\[ = 2 f Q L^\rho_\mu Q_{\mu \alpha \beta} P^{\rho \mu}, \]
\[ E_4 + E_6 + E_9 \]
\[ = f Q \left( \nabla_\alpha Q_{\nu \alpha \beta} + 2 L^\rho_\alpha Q_{\rho \beta \mu} + L^\rho_\mu Q_{\rho \alpha \beta} \right) P^{\mu \alpha \beta} \]
\[ = \frac{1}{2} f Q D_\nu \left( Q_{\mu \alpha \beta} P^{\mu \alpha \beta} \right) = \frac{1}{2} f Q \partial_\nu Q. \]

Finally, we obtain

\[ D_\mu \left[ f T^\mu_\nu + \Theta^\mu_\nu \right] - 8 \pi T^\mu_\nu = \frac{8 \pi}{\sqrt{-g}} \nabla_\alpha \nabla_\mu H^{\alpha \mu} \]
\[ = \frac{1}{2} \partial_\nu f - \frac{1}{2} f Q \partial_\nu Q + \frac{1}{\sqrt{-g}} Q_\mu \nabla_\alpha \left( f Q \sqrt{-g} P^{\rho \mu}_\nu \right) \]
\[ = \frac{1}{2} f T^{\nu}_\mu + \frac{1}{\sqrt{-g}} Q_\mu \nabla_\alpha \left( f Q \sqrt{-g} P^{\rho \mu}_\nu \right). \]

### 3.5 Calculation of \( Q = 6H^2/N^2 \)

Recalling Eq. (3.10), we have

\[ Q = -\frac{1}{4} \left( - Q_{\alpha \mu \nu} Q^{\alpha \mu \nu} + 2 Q_{\alpha \mu \nu} Q^{\mu \alpha \nu} \right. \]
\[ \left. + Q_\alpha Q^\alpha - 2 Q_\alpha Q^\alpha \right). \]
By using the relations already presented in Appendix 3.2, for the case of the Friedmann-
Robertson-Walker metric we obtain
\begin{align}
-Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} &= \nabla_{\alpha}g_{\mu\nu}\nabla^{\alpha}g^{\mu\nu} = \frac{4}{N^2}(T^2 + 3H^2), \quad (3.52) \\
Q_{\alpha\mu\nu}Q^{\mu\alpha\nu} &= -\nabla_{\alpha}g_{\mu\nu}\nabla^{\mu}g^{\alpha\nu} = -\frac{4}{N^2}T^2, \quad (3.53) \\
Q_{\alpha}Q^{\alpha} &= (g_{\mu\nu}\nabla_{\alpha}g^{\mu\nu})(g_{\sigma\nu}\nabla_{\alpha}g^{\sigma\nu}) = \frac{4}{N^2}(T + 3H)^2, \quad (3.54) \\
Q_{\alpha}Q^{\alpha} &= (g_{\mu\nu}\nabla_{\alpha}g^{\mu\nu})(\nabla_{\beta}g^{\alpha\beta}) = \frac{4}{N^2}(T^2 + 3HT), \quad (3.55)
\end{align}

Thus, we have
\begin{align}
Q &= -\frac{1}{4}\left[\frac{4}{N^2}(T^2 + 3H^2) - \frac{4}{N^2}2T^2 - \frac{4}{N^2}(T + 3H)^2 + \frac{4}{N^2}(2T^2 + 6HT)\right] = \frac{6H^2}{N^2}. \quad (3.56)
\end{align}

4 Brief review metric-affine gravity theories

4.1 Theories with $F = F(X_1)$

4.1.1 $F(R)$ gravity

The action of the Myrzakulov $F(R,T,Q,T)$ gravity or the MG-VIII reads as \[31\]
\begin{align}
S &= \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(R) + 2\kappa L_m], \quad (4.1)
\end{align}

where $R$ is the curvature scalar, $T$ is the torsion scalar, $Q$ is the nonmetricity scalar and $T$ is the trace of the energy-momentum tensor (the trace of the stress-energy tensor). The MG-VIII is for example the unification of $F(R), F(T), F(Q)$ or $F(R,T), F(T), F(Q)$ theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations
\begin{align}
\frac{1}{2}g_{\mu\nu}F + F_{R}R_{(\mu\nu)} &= \kappa T_{\mu\nu}, \quad (4.2) \\
P_{\lambda}^{\mu\nu}(F_R) &= \kappa \Delta_{\lambda}^{\mu\nu}, \quad (4.3)
\end{align}

where
\begin{align}
P_{\lambda}^{\mu\nu}(F_R) &= -\frac{\nabla_{\lambda}(\sqrt{-g}F_{RG}g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_{\lambda}(\sqrt{-g}F_{RG}g^{\mu\nu}\delta_{\lambda}^{\nu})}{\sqrt{-g}} + 2F_{R}(S_{\lambda}g^{\mu\nu} - S^{\mu}\delta_{\lambda}^{\nu} - S_{\lambda}^{\mu\nu}). \quad (4.4)
\end{align}

4.1.2 $F(T)$ gravity

The action of the Myrzakulov $F(R,T,Q,T)$ gravity or the MG-VIII reads as \[32\]
\begin{align}
S &= \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(T) + 2\kappa L_m], \quad (4.5)
\end{align}

where $R$ is the curvature scalar, $T$ is the torsion scalar, $Q$ is the nonmetricity scalar and $T$ is the trace of the energy-momentum tensor (the trace of the stress-energy tensor). The MG-VIII is for example the unification of $F(R), F(T), F(Q)$ or $F(R,T), F(T), F(Q)$ theories. The variations of
the field equations \( \text{(30)} \)

\[
\frac{1}{2}g_{\mu\nu}F + F_T \left( 2S_{\alpha\beta\gamma} S_{\mu}^{\alpha\beta} - S_{\alpha\beta\mu} S_{\nu}^{\alpha\beta} + 2S_{\alpha\beta\gamma} S_{\mu}^{\beta\alpha} - 4S_{\mu} S_{\nu} \right) = \kappa T_{\mu\nu},
\]

(4.6)

\[
2F_T \left( S_{\mu\nu}^{\alpha\beta\gamma} - 2S_{\alpha}^{\mu\nu} - 4S_{\mu}^{\alpha\beta\gamma} \right) = \kappa \Delta_{\mu\nu}.
\]

(4.7)

### 4.1.3 \( F(Q) \) gravity

The action of the Myrzakulov \( F(R, T, Q, T) \) gravity or the MG-VIII reads as \( \text{(33)} \)

\[
S = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(Q) + 2\kappa L_m],
\]

(4.8)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( Q \) is the nonmetricity tensor and \( T \) is the trace of the energy-momentum tensor (the trace of the stress-energy tensor). The MG-VIII is for example the unification of \( F(R), F(T), F(Q) \) or \( F(R, T), F(T), F(Q) \) theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations \( \text{(30)} \)

\[
-\frac{1}{2}g_{\mu\nu}F + F_T \left( 2S_{\alpha\beta\gamma} S_{\mu}^{\alpha\beta} - S_{\alpha\beta\mu} S_{\nu}^{\alpha\beta} + 2S_{\alpha\beta\gamma} S_{\mu}^{\beta\alpha} - 4S_{\mu} S_{\nu} \right) = \kappa T_{\mu\nu},
\]

(4.9)

\[
F_Q \left( 2Q_{\mu\nu}^{\alpha\beta\gamma} + (q^\mu - Q^\mu)^{\alpha\beta\gamma} + Q_{\mu\nu}^{\alpha\beta} + \frac{1}{2}Q_{\mu\nu}^{\alpha\beta\gamma} \right) = \kappa \Delta_{\mu\nu},
\]

(4.10)

where

\[
\Omega_{\alpha\mu\nu} = \frac{1}{4}Q^{\alpha\mu\nu} - \frac{1}{2}Q^{\mu\nu\alpha} - \frac{1}{4}g^{\mu\nu}Q^\alpha + \frac{1}{2}g^{\alpha\mu}Q^\nu,
\]

(4.11)

\[
4\Delta_{\mu\nu} = (Q_{\mu\alpha\beta} - 2Q_{\alpha\beta\mu}) Q_{\nu}^{\alpha\beta} + (Q_{\mu} + 2q_{\mu}) Q_{\nu} + (2Q_{\mu\nu\alpha} - Q_{\alpha\mu\nu}) Q^\alpha - 4\Omega_{\mu\nu} Q_{\alpha\beta\mu} - 4\Omega_{\alpha\beta\mu} Q_{\mu\nu}^{\alpha\beta}.
\]

(4.12)

\[
J_{\mu\nu} := \sqrt{-g} \left( \frac{1}{4}Q_{\mu\nu}^{\alpha\beta\gamma} - \frac{1}{2}Q_{\mu\nu}^{\alpha\beta\gamma} + \Omega_{\mu\nu}^{\alpha\beta\gamma} \right), \quad \zeta_{\mu\nu} := \sqrt{-g} \left( \frac{1}{2}g_{\mu\nu} - \frac{1}{4}Q_{\mu\nu} \right).
\]

(4.13)

### 4.2 Theories with \( F = F(X_1, X_2) \)

#### 4.2.1 \( F(R, T) \) gravity

The action of the Myrzakulov \( F(R, T, Q, T) \) gravity or the MG-VIII reads as \( \text{(34)} \)

\[
S = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(R, T) + 2\kappa L_m],
\]

(4.14)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( Q \) is the nonmetricity tensor and \( T \) is the trace of the energy-momentum tensor (the trace of the stress-energy tensor). The MG-VIII is for example the unification of \( F(R), F(T), F(Q) \) or \( F(R, T), F(T), F(Q) \) theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations \( \text{(30)} \)

\[
-\frac{1}{2}g_{\mu\nu}F + F_R R_{\mu\nu} + F_T (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu},
\]

(4.15)

\[
P_{\lambda\nu}^{\mu\nu}(F_R) - F_T \Theta_{\mu}^{\mu\nu} = \kappa \Delta_{\mu\nu},
\]

(4.16)

where

\[
\nabla_\lambda := \frac{1}{\sqrt{-g}} (2S_{\lambda} - \nabla_\lambda), \quad \Theta_{\mu}^{\mu\nu} := -\frac{\delta T}{\delta T_{\mu\nu}}, \quad \Theta_{\mu\nu} := g_{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}.
\]

(4.17)

\[
P_{\lambda}^{\mu\nu}(F_R) = -\frac{\nabla_\lambda (\sqrt{-g} F_R g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_\lambda (\sqrt{-g} F_R g^{\mu\nu} \delta_{\lambda}^{\nu})}{\sqrt{-g}} + 2F_R (S_{\lambda} g^{\mu\nu} - S^{\mu} \delta_{\lambda}^{\nu} - S^{\nu} \delta_{\lambda}^{\mu}).
\]

(4.18)
4.2.2 \( F(T, T) \) gravity

The action of the Myrzakulov \( F(R, T, Q, T) \) gravity or the MG-VIII reads as \[35\]

\[
S = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(T, T) + 2\kappa L_m],
\]

(4.19)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar and \( T \) is the trace of the energy-momentum tensor (the trace of the stress-energy tensor). The MG-VIII is for example the unification of \( F(R), F(T), F(Q) \) or \( F(R, T), F(T), F(Q) \) theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations \[30\]

\[
F_T \left( 2S_{\alpha\beta} S_\mu^{\alpha\beta} - S_\alpha S_\beta S_\mu^{\alpha\beta} + 2S_{\alpha\beta} S_\mu^{\alpha\beta} - 4S_\mu S_\nu \right) - \frac{1}{2} g_{\mu\nu} F + F_T (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu},
\]

(4.20)

\[
2F_T \left( S_{\mu\nu}^{\alpha\beta} - 2S_{\alpha\beta}^{\mu\nu} - 4S_{\mu\nu}^{\alpha\beta} \right) = F_T \Theta_{\lambda}^{\mu\nu} = \kappa \Delta_{\lambda}^{\mu\nu},
\]

(4.21)

where

\[
\nabla_\lambda := \frac{1}{\sqrt{-g}} (2S_\lambda - \nabla_\lambda), \quad \Theta_{\lambda}^{\mu\nu} := -\frac{\delta T}{\delta \Gamma_{\mu\nu}}, \quad \Theta_{\mu\nu} := g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}.
\]

4.2.3 \( F(Q, T) \) gravity

The action of the Myrzakulov \( F(R, T, Q, T) \) gravity or the MG-VIII reads as \[36\]

\[
S = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(Q, T) + 2\kappa L_m],
\]

(4.23)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar and \( T \) is the trace of the energy-momentum tensor (the trace of the stress-energy tensor). The MG-VIII is for example the unification of \( F(R), F(T), F(Q) \) or \( F(R, T), F(T), F(Q) \) theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations \[30\]

\[
-\frac{1}{2} g_{\mu\nu} F + F_Q L_{(\mu\nu)} + \nabla_\lambda (F_Q J^\lambda_{(\mu\nu)}) + g_{\mu\nu} \nabla_\lambda (F_Q \zeta^\lambda) + F_T (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu},
\]

(4.24)

\[
F_Q \left( 2Q_{\alpha\beta}^{\mu\nu} - Q_{\lambda}^{\mu\nu} - (g^\nu - Q^\nu) \right) + Q_\lambda g^{\mu\nu} + \frac{1}{2} Q^{\mu\nu} \Delta_\lambda = F_T \Theta_{\lambda}^{\mu\nu} = \kappa \Delta_{\lambda}^{\mu\nu},
\]

(4.25)

where

\[
\nabla_\lambda := \frac{1}{\sqrt{-g}} (2S_\lambda - \nabla_\lambda), \quad \Omega^{\alpha\beta}_{\mu\nu} = \frac{1}{4} Q^{\alpha\mu\nu} - \frac{1}{2} Q^{\mu\nu\alpha} - \frac{1}{4} g^{\mu\nu} Q^\alpha + \frac{1}{2} g^{\alpha\beta} Q_{\nu}^{\mu}, \quad \Theta_{\mu\nu} := -\frac{\delta T}{\delta \Gamma_{\mu\nu}}.
\]

(4.26)

\[
4L_{\mu\nu} = (Q_{\mu\beta\gamma} - 2Q_{\alpha\beta} Q_{\nu})^{\alpha\beta} + (Q_{\mu} + 2Q_{\nu}) Q_{\nu} + (2Q_{\mu\nu\alpha} - Q_{\alpha\beta\mu}) Q_{\nu} - 4\Omega^{\alpha\beta\nu} Q_{\alpha\mu\beta} - 4\Omega_{\alpha\beta\mu} Q^{\alpha\beta}.
\]

(4.27)

\[
\Theta_{\mu\nu} := g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}, \quad J^\lambda_{\mu\nu} := \sqrt{-g} \left( \frac{1}{4} Q^\lambda_{\mu\nu} - \frac{1}{2} Q_{\mu\nu}^\lambda + \Omega^\lambda_{\mu\nu} \right) - \frac{3}{4} J^\lambda, \quad \zeta^\lambda = \sqrt{-g} \left( \frac{1}{2} Q^\lambda - \frac{3}{4} \zeta \right).
\]

(4.28)

4.2.4 MG-I

The action of the Myrzakulov \( F(R, T) \) gravity or the MG-I has the following form \[12\]

\[
S = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(R, T) + 2\kappa L_m],
\]

(4.29)
where $R$ is the curvature scalar, $T$ is the torsion scalar and $L_m$ is the matter Lagrangian. This MG-I is some kind generalizations of the well-known $F(R)$ and $F(T)$ gravity theories. If exactly, the MG-I is the unification of the $F(R)$ and $F(T)$ theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations \[30\]

\[-\frac{1}{2}g_{\mu\nu}F + FR_{(\mu\nu)} + FT\left(2S_{\alpha\beta\gamma\delta}\alpha^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}_{\nu} + 2S_{\alpha\beta\delta\gamma} - 4S_{\mu\nu}\right) = \kappa T_{\mu\nu}, \tag{4.30}\]

\[P_{\mu\nu}(F_R) + 2FT\left(S_{\mu\nu}^{\lambda} - 2S_{\lambda}^{\ [\mu\nu]} - 4S[\alpha\beta^{\lambda}]\right) = \kappa \Delta_{\mu\nu}^{\lambda}, \tag{4.31}\]

where \[30\]

\[\hat{\nabla}_{\lambda} := \frac{1}{\sqrt{-g}}(2S_{\lambda} - \nabla_{\lambda}), \tag{4.32}\]

\[4L_{\mu\nu} = (Q\mu\nu) - 2Q_{\alpha\beta\mu}Q_{\nu}^{\alpha\beta} + (Q_{\mu\nu} - Q_{\alpha\beta\mu}Q^{\alpha\beta}) - 4\Omega^{\alpha\beta}_{\mu}Q_{\alpha\beta\mu} - 4\Omega_{\alpha\beta\mu}Q^{\alpha\beta}, \tag{4.33}\]

\[P_{\mu\nu}(F_R) = -\frac{\hat{\nabla}_{\lambda}(\sqrt{-g}FRg^{\mu\nu})}{\sqrt{-g}} + \frac{\hat{\nabla}_{\alpha}(\sqrt{-g}FRg^{\mu\nu}Q^{\lambda})}{\sqrt{-g}} + 2FR(S_{\lambda}g^{\mu\nu} - S^{[\mu\nu]} - S_{\lambda}^{[\mu\nu]}). \tag{4.34}\]

### 4.2.5 MG-II

The action of the Myrzakulov $F(R, Q)$ gravity or the MG-II reads as \[12\]

\[S = \frac{1}{2\kappa} \int \sqrt{-g}d^4x[F(R, Q) + 2\kappa L_m], \tag{4.35}\]

where $R$ is the curvature scalar and $Q$ is the nonmetricity scalar. The MG-II is the unification of the $F(R)$ and $F(Q)$ theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations \[30\]

\[-\frac{1}{2}g_{\mu\nu}F + FR_{(\mu\nu)} + FQ_{\mu}L(\mu\nu) + \hat{\nabla}_{\lambda}(FQ J^{\lambda}_{(\mu\nu)}) + g_{\mu\nu}\hat{\nabla}_{\lambda}(FQ\zeta^{\lambda}) = \kappa T_{\mu\nu}, \tag{4.36}\]

\[P_{\mu\nu}(F_R) + FQ\left(2Q_{\mu}^{\ [\nu]} - Q_{\mu}^{\ [\nu]} + (q_{\mu} - Q_{\nu})\delta^{\mu}_{\nu} + Q_{\mu}g^{\mu\nu} + \frac{1}{2}Q^{\mu\nu}\right) = \kappa \Delta_{\mu\nu}^{\lambda}, \tag{4.37}\]

where \[30\]

\[\hat{\nabla}_{\lambda} := -\frac{1}{\sqrt{-g}}\nabla_{\lambda}, \quad \Omega^{\alpha\mu\nu} = \frac{1}{4}Q^{\alpha\mu\nu} - \frac{1}{2}Q^{\mu\alpha\nu} - \frac{1}{4}g^{\mu\nu}Q^{\alpha} + \frac{1}{4}g^{\alpha\mu}Q^{\nu}, \tag{4.38}\]

\[4L_{\mu\nu} = (Q_{\mu\nu}) - 2Q_{\alpha\beta\mu}Q_{\nu}^{\alpha\beta} + (Q_{\mu\nu} + Q_{\mu\nu} - Q_{\alpha\beta\mu}Q^{\alpha\beta}) - 4\Omega_{\mu}^{\alpha\beta}Q_{\alpha\beta\mu} - 4\Omega_{\alpha\beta\mu}Q^{\alpha\beta}, \tag{4.39}\]

\[J^{\lambda}_{\mu\nu} := \sqrt{-g}\left(\frac{1}{4}Q^{\lambda}_{\mu\nu} - \frac{1}{2}Q^{\mu\nu}_{\lambda} + \Omega^{\lambda}_{\mu\nu}\right), \quad \zeta^{\lambda} = \sqrt{-g}\left(\frac{1}{2}g^{\lambda} - \frac{1}{4}Q^{\lambda}\right), \tag{4.40}\]

\[P_{\mu\nu}(F_R) = -\frac{\hat{\nabla}_{\lambda}(\sqrt{-g}FRg^{\mu\nu})}{\sqrt{-g}} + \frac{\hat{\nabla}_{\alpha}(\sqrt{-g}FRg^{\alpha\mu\nu})}{\sqrt{-g}}. \tag{4.41}\]
4.2.6 MG-III

The action of the Myrzakulov $F(T,Q)$ gravity or the MG-III reads as \[ F \]
\[ S = \frac{1}{2\kappa} \int \sqrt{-g} d^4x \left[ F(T,Q) + 2\kappa L_m \right], \]  
(4.42)
where $T$ is the torsion scalar and $Q$ is the nonmetricity scalar. The MG-III is the unification of the $F(T)$ and $F(Q)$ theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations \[ 30 \]
\[ \frac{1}{2} g_{\mu\nu} F + F_T \left( 2S_{\nu\alpha\beta} S_{\mu \alpha} - S_{\alpha\beta\mu} S_{\alpha\beta} \right) + 4S_{\mu\nu} S_{\alpha\beta} - 4S_{\alpha\beta} S_{\mu\nu} + F_Q L_{(\mu\nu)} \]
\[ + \nabla_{(\mu} F_{Q J^\nu)} + g_{\mu\nu} \nabla_{(\mu} F_{Q(\nu)} = \kappa T_{\mu\nu}, \]
(4.43)
where \[ 30 \]
\[ 2F_T \left( S_{\nu\alpha} - 2S_{\alpha} \left[ \nu \right] - 4S_{\left[ \nu \right]} \right) + F_Q \left( 2Q^{\left[ \nu \right]} - \gamma_Q + \left( \eta^\nu - Q^\nu \right) \delta_Q^\nu + Q^\nu g^\mu + \frac{1}{2} Q^\nu g^\mu \right) = \kappa \Delta_{\mu}^{\nu}. \]
(4.44)

4.3 Theories with $F = F(X_1, X_2, X_3)$

4.3.1 MG-IV

The action of the Myrzakulov $F(R,T,T)$ gravity or the MG-IV has the following form \[ 12 \]
\[ S = \frac{1}{2\kappa} \int \sqrt{-g} d^4x \left[ F(R,T,T) + 2\kappa L_m \right], \]  
(4.48)
where $R$ is the curvature scalar, $T$ is the torsion scalar and $T$ is the trace of the energy-momentum tensor. The MG-IV is the unification of the $F(R,T)$ and $F(T)$ theories.
\[ F_{R R_{(\mu\nu)}} - \frac{1}{2} g_{\mu\nu} F + F_T \left( 2S_{\nu\alpha\beta} S_{\mu \alpha} - S_{\alpha\beta\mu} S_{\alpha\beta} \right) + 4S_{\mu\nu} S_{\alpha\beta} - 4S_{\alpha\beta} S_{\mu\nu} \]
\[ + F_T (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu}. \]
(4.49)
\[ P_{\lambda}^{\mu\nu} (F_R) + 2F_T \left( S_{\mu\nu} - 2S_{\lambda} \left[ \nu \right] - 4S_{\left[ \nu \right]} \right) = \kappa T_{\lambda}^{\mu\nu}, \]
(4.50)
where
\[ \nabla_{\lambda} := \frac{1}{\sqrt{-g}} (2S_{\lambda} - \nabla_{\lambda}), \quad \Theta_{\mu\nu} := - \frac{\delta T_{\mu\nu}}{\delta \gamma_{\mu\nu}}, \]
(4.51)
\[ \Theta_{\mu\nu} := g_{\mu\nu} \delta T_{\alpha\beta}, \]
(4.52)
\[ P_{\lambda}^{\mu\nu} (F_R) = - \nabla_{\lambda} \left( \sqrt{-g} F_{R R_{\mu\nu}} \right) \]
\[ + \frac{\nabla_{\alpha} \left( \sqrt{-g} F_{R R_{\mu\nu}} \delta_{\gamma}^\nu \right)}{\sqrt{-g}} \]
\[ + 2F_T (S_{\lambda} g_{\mu\nu} - S_{\mu} \delta_{\lambda}^\nu - S_{\nu} \delta_{\lambda}^\mu). \]
(4.53)
4.3.2 MG-V

The action of the Myrzakulov $F(R, T, Q)$ gravity or the MG-V is given by

$$S = \frac{1}{2\kappa} \int \sqrt{-g} d^4 x [F(R, T, Q) + 2\kappa L_m],$$

(4.54)

where $R$ is the curvature scalar, $T$ is the torsion scalar and $Q$ is the nonmetricity scalar. The MG-V is the unification of $F(R), F(T), F(Q)$ theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations [30]

$$-\frac{1}{2} g_{\mu\nu} F + F_R R(\mu\nu) + F_T \left( 2S_{\nu\alpha\beta}\delta_\mu^\alpha - S_{\alpha\beta\mu}^\alpha + 2S_{\nu\alpha\beta}\delta_\mu^\beta - 4S_{\mu\nu}\right) + F_Q L(\mu\nu)$$

$$+ \bar{\nabla}_\lambda (F_Q J^\lambda_{(\mu\nu)}) + g_{\mu\nu} \bar{\nabla}_\lambda (F_Q \zeta^\lambda) = \kappa T_{\mu\nu},$$

(4.55)

$$P^\mu_{\lambda\nu}(F_R) + 2F_T \left( S_{\lambda\alpha\nu}^{\lambda\kappa} - 2S_{\lambda\nu}^{[\mu\nu]} + 4S_{\nu\mu}^{[\lambda]} \right)$$

$$+ F_Q \left( 2Q^{[\mu\nu]} \right) - \bar{Q}_\lambda^{\mu\nu} + (q^\mu - Q^\nu) \delta^\mu_{\nu} + \bar{Q}_{\lambda}^{\mu\nu} + \frac{1}{2} \bar{Q}^{\mu\nu} = \kappa \Delta_{\lambda}^\mu, \nu,$$

(4.56)

where

$$\bar{\nabla}_\lambda := -\frac{1}{\sqrt{-g}} (2S_{\lambda} - \nabla_\lambda), \quad \Omega^{\mu\nu} = \frac{1}{4} Q^{\mu\nu} - \frac{1}{2} g^{\mu\nu} Q + \frac{1}{4} g^{\mu\nu} Q.$$  

(4.57)

$$4L_{\mu\nu} = (Q_{\mu\alpha\beta} - 2Q_{\alpha\beta\mu}^{\alpha\beta}) Q_{\nu\alpha\beta} + (Q_\mu + 2q_\mu) Q_\nu + (2Q_{\mu\alpha\nu} - Q_{\alpha\mu\nu}) Q^{\alpha\beta} - 4Q_{\rho\alpha\beta} Q_{\alpha\beta} - 4Q_{\alpha\beta\mu} Q^{\alpha\beta}.$$  

(4.58)

$$J^\lambda_{\mu\nu} := -\bar{g} \left( \frac{1}{4} Q^\lambda_{\mu\nu} - \frac{1}{2} Q^\lambda_{\mu\nu} + \Omega^\lambda_{\mu\nu} \right), \quad \zeta^\lambda = \sqrt{-g} \left( \frac{1}{2} g^\lambda - \frac{1}{4} Q^\lambda \right),$$

(4.59)

$$P^\lambda_{\mu\nu}(F_R) = -\frac{\nabla_{\lambda} (\sqrt{-g} F_R g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_{\alpha} (\sqrt{-g} F_R g^{\mu\nu})}{\sqrt{-g}} + 2F_R (S_{\lambda} g^{\mu\nu} - S_{\mu}^{\lambda} - S_{\nu}^{\lambda}).$$

(4.60)

4.3.3 MG-VI

The action of the Myrzakulov $F(R, Q, T)$ gravity or the MG-VI reads as

$$S = \frac{1}{2\kappa} \int \sqrt{-g} d^4 x [F(R, Q, T) + 2\kappa L_m],$$

(4.61)

where $R$ is the curvature scalar, $Q$ is the nonmetricity scalar and $T$ is the trace of the energy-momentum tensor. The MG-VI is the unification of $F(R, T)$ and $F(Q)$ theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations [30]

$$F_R R(\mu\nu) - \frac{1}{2} g_{\mu\nu} F + F_Q L(\mu\nu) + \bar{\nabla}_\lambda (F_Q J^\lambda_{(\mu\nu)}) + g_{\mu\nu} \bar{\nabla}_\lambda (F_Q \zeta^\lambda) + F_T (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu},$$

(4.62)

$$P^\mu_{\lambda\nu}(F_R) + + F_Q \left( 2Q^{[\mu\nu]} \right) - \bar{Q}_\lambda^{\mu\nu} + (q^\mu - Q^\nu) \delta^\mu_{\nu} + \bar{Q}_{\lambda}^{\mu\nu} + \frac{1}{2} \bar{Q}^{\mu\nu} = F_T \Theta_{\lambda}^{\mu\nu} + \kappa \Delta_{\lambda}^{\mu\nu},$$

(4.63)

where

$$\bar{\nabla}_\lambda := -\frac{1}{\sqrt{-g}} \nabla_\lambda, \quad \Omega^{\mu\nu} = \frac{1}{4} Q^{\mu\nu} - \frac{1}{2} g^{\mu\nu} Q + \frac{1}{4} g^{\mu\nu} Q,$$

(4.64)

$$\Theta_{\mu\nu} := g^{\rho\beta} \delta T_{\alpha\beta}^{\rho\beta} / \delta g^{\mu\nu}, \quad J^\lambda_{\mu\nu} := -\frac{1}{4} Q^\lambda_{\mu\nu} - \frac{1}{2} Q^\lambda_{\mu\nu} + \Omega^\lambda_{\mu\nu}, \quad \zeta^\lambda = \sqrt{-g} \left( \frac{1}{2} g^\lambda - \frac{1}{4} Q^\lambda \right),$$

(4.65)

$$P^\lambda_{\mu\nu}(F_R) = -\frac{\nabla_{\lambda} (\sqrt{-g} F_R g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_{\alpha} (\sqrt{-g} F_R g^{\mu\nu})}{\sqrt{-g}} + 2F_R (S_{\lambda} g^{\mu\nu} - S_{\mu}^{\lambda} - S_{\nu}^{\lambda}).$$

(4.66)
4.3.4 MG-VII

The action of the Myrzakulov \( F(T, Q, T) \) gravity or the MG-VII reads as [12]

\[
S = \frac{1}{2\kappa} \int \sqrt{-g} \, dt^4 [F(T, Q, T) + 2\kappa L_m],
\]

(4.68)

and \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar and \( T \) is the trace of the energy-momentum tensor. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations [30]

\[
\begin{align*}
-\frac{1}{2}g_{\mu\nu}F + F_T \left( 2S_{\alpha\beta\mu}^\alpha \right. & - S_{\alpha\beta\mu}^\alpha \left. - 2S_{\nu\alpha\beta}^\beta S_{\nu}^\alpha \right) + (F_Q J^A)_{(\mu\nu)} \\
& + \tilde{\nabla}_\lambda (F_Q J^A)_{(\mu\nu)} + g_{\mu\nu} \tilde{\nabla}_\lambda (F_Q \zeta^\lambda) + F_T (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu},
\end{align*}
\]

(4.69)

where

\[
\begin{align*}
\tilde{\nabla}_\lambda := & \frac{1}{\sqrt{-g}}(2S_{\lambda} - \nabla_\lambda), \\
\Omega_{\mu\nu} := & \frac{1}{4}Q_{\mu\nu} - \frac{1}{2}Q_{\mu\alpha\nu}^\alpha - \frac{1}{2}g_{\mu\nu} Q^\alpha - \frac{1}{2}g^\mu_\alpha Q^\nu, \\
\Theta_{\mu\nu} := & -\frac{\delta T}{\delta \Sigma_{\mu\nu}}.
\end{align*}
\]

(4.71)

4L_{\mu\nu} = (Q_{\mu\alpha\beta} - 2Q_{\alpha\beta\mu})Q_{\nu}^\alpha + (Q_{\alpha} + 2\Theta_{\mu\nu})Q_{\nu} + (2Q_{\mu\alpha\nu} - Q_{\alpha\nu} Q_{\alpha}) - 4Q_{\lambda} = 4\kappa L_{\mu\nu}.

(4.72)

where

\[
\begin{align*}
\Theta_{\mu\nu} := & g_{\alpha\beta} \frac{\delta T_{\mu\nu}}{\delta g_{\alpha\beta}}, \\
J_{\mu\nu} := & \sqrt{-g} \left( \frac{1}{4}Q_{\mu\nu} - \frac{1}{2}Q_{\mu\nu}^\lambda + \Omega_{\mu\nu}^\lambda \right), \\
\zeta^\lambda := & \sqrt{-g} \left( \frac{1}{4}Q^\lambda - \frac{1}{4}Q \right).
\end{align*}
\]

(4.73)

\[
\Theta_{\mu\nu} := -\frac{\delta T}{\delta \Sigma_{\mu\nu}}.
\]

(4.74)

4.4 Theories with \( F = F(X_1, X_2, X_3, X_4) \)

4.4.1 MG-VIII

The action of the Myrzakulov \( F(R, T, Q, T) \) gravity or the MG-VIII reads as [12]

\[
S = \frac{1}{2\kappa} \int \sqrt{-g} \, dt^4 [F(R, T, Q, T) + 2\kappa L_m],
\]

(4.75)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar and \( T \) is the trace of the energy-momentum tensor (the trace of the stress-energy tensor). The MG-VIII is for example the unification of \( F(R) \), \( F(T) \), \( F(Q) \) or \( F(R, T) \), \( F(T) \), \( F(Q) \) theories. The variations of the action (9) with respect to the metric tensor and the affine connection give the following set of the field equations [30]

\[
\begin{align*}
-\frac{1}{2}g_{\mu\nu}F + F_R R_{(\mu\nu)} + F_T \left( 2S_{\alpha\beta\mu}^\alpha - S_{\alpha\beta\mu}^\alpha \right. & - 2S_{\nu\alpha\beta}^\beta S_{\nu}^\alpha \left. + 2S_{\nu\alpha\beta}^\beta S_{\nu}^\alpha \right) + F_Q L_{(\mu\nu)} \\
& + \tilde{\nabla}_\lambda (F_Q J^A)_{(\mu\nu)} + g_{\mu\nu} \tilde{\nabla}_\lambda (F_Q \zeta^\lambda) + F_T (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu},
\end{align*}
\]

(4.76)

\[
\begin{align*}
P_{\mu\nu} (F_R) + 2F_T \left( S_{\mu\nu}^\gamma - S_{\mu\nu} \right. & - 2S_{\mu\nu}^\gamma - 4S_{\mu\nu}^\gamma \\
& + F_Q \left( 2Q_{\mu\nu}^\gamma - Q_{\mu\nu}^\gamma + (Q_{\mu\nu}^\gamma - Q_{\mu\nu}) \right. \\
& \left. + Q_{\lambda} g_{\mu\nu} + \frac{1}{2}Q_{\mu\nu}^\gamma \right) \right) = F_T \Theta_{\mu\nu} + \kappa \Theta_{\mu\nu}.
\end{align*}
\]

(4.77)
Here \[30\]

\[
\nabla_\lambda := \frac{1}{\sqrt{-g}} (2S_\lambda - \nabla_\lambda), \quad \Omega^{\mu \nu} = \frac{1}{4} Q^{\mu \nu} - \frac{1}{2} Q^{\mu \sigma} \Omega_{\sigma \nu} - \frac{1}{4} g^{\mu \nu} Q^\alpha + \frac{1}{2} g^{\mu \nu} Q^\alpha, \quad \Theta_{\lambda}^{\mu \nu} := - \frac{\delta T}{\delta \Gamma_{\mu \nu}^\lambda} (4.78)
\]

\[
4L_{\mu \nu} = (Q_{\mu \alpha \beta} - 2Q_{\alpha \beta \mu})Q^\nu_{\alpha \beta} + (Q_\mu + 2q_\mu)Q_\nu + (2Q_{\mu \nu \alpha} - Q_{\alpha \mu \nu})Q^\alpha - 4\Omega_{\nu \alpha} Q_{\alpha \beta \mu} - 4\Omega_{\alpha \beta \nu} Q_{\alpha \mu} (4.79)
\]

\[
\Theta_{\mu \nu} := g^{\alpha \beta} \frac{\delta T_{\alpha \beta}}{\delta g^{\mu \nu}}, \quad J^\lambda_{\mu \nu} := \sqrt{-g} \left( \frac{1}{4} Q^\lambda_{\mu \nu} - \frac{1}{2} Q^{\lambda \sigma} \Omega_{\sigma \nu} + \Omega^\lambda_{\mu \nu} \right), \quad \zeta^\lambda = \sqrt{-g} \left( \frac{1}{4} Q^\lambda - \frac{1}{2} Q^\lambda + \Omega^\lambda_{\mu \nu} \right) (4.80)
\]

\[
P^\lambda_{\mu \nu}(F_R) = - \frac{\nabla_\lambda (\sqrt{-g} F_R g_{\mu \nu})}{\sqrt{-g}} + \nabla_\alpha (\sqrt{-g} F_R g_{\mu \nu} \delta^\alpha_{\nu}) + 2F_R (S_\lambda g^{\mu \nu} - S^\mu \delta^\lambda_{\nu} - S^\lambda_{\mu \nu}) (4.81)
\]

### 4.5 Theories with \( F = F(X_1, X_2, X_3, X_4, X_5) \)

Here we present one example of the MAG theories with the five arguments the so-called metric-affine \( F(R, T, Q, T, D) \) \[30\]. Its action is given by \[30\]

\[
S[g, \Gamma, \phi] = S_g + S_m = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(R, T, Q, T, D) + 2\kappa L_m],
\]

where \( R \) stands for the Ricci scalar (curvature scalar), \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar and \( T \) is trace of the energy-momentum tensor of matter Lagrangian \( L_m, D \) is the dilaton current scalar. The field equations of this theory have the forms \[30\]

\[
\frac{1}{2} g_{\mu \nu} F + F_{\mu \rho} R^{\rho \lambda} + F_T \left( 2S_{\rho \nu \sigma} S_{\lambda}^{\rho \nu \sigma} - S_{\rho \nu} S_{\lambda}^{\rho \nu} + 2S_{\rho \nu} S_{\lambda}^{\rho \nu} - 4S_{\rho \nu} S_{\rho \nu} \right) + F_{Q} L_{\mu \nu} + \nabla_\lambda (F_{Q} J^\lambda_{\mu \nu}) + g_{\mu \nu} \nabla_\lambda (F_{Q} \zeta^\lambda) + F_T (\Theta_{\mu \nu}) + F_{D} M_{\mu \nu} = \kappa T_{\mu \nu} (4.83)
\]

\[
P^\lambda_{\mu \nu}(F_R) + 2F_T \left( S_{\mu \nu} - 2S_{[\mu \nu]} - 4S_{[\rho \delta]_{\mu \nu}} \right) - M_{\mu \nu} \delta_{\rho \sigma} F_{D} + F_Q \left( 2Q^{[\rho \nu]}_{\alpha} - Q^\rho_{\alpha \nu} + (q^\sigma - Q^\nu) \delta^\rho_{\alpha} + Q_{\alpha} g^{\rho \nu} + \frac{1}{2} Q^\rho \delta^\rho_{\alpha} \right) = F_T \Theta^\mu_{\nu} + \kappa D_{\mu \nu} (4.84)
\]

Here

\[
t = T + \frac{1}{2\sqrt{-g}} \partial_{\nu} (\sqrt{-g} \Delta^\nu), \quad \Delta^\nu = \Delta_{\mu \nu}, \quad D = \frac{1}{\sqrt{-g}} \partial_{\nu} (\sqrt{-g} \Delta^\nu) (4.85)
\]

\[
\nabla_\lambda := \frac{1}{\sqrt{-g}} (2S_\lambda - \nabla_\lambda), \quad \Omega^{\mu \nu} = \frac{1}{4} Q^{\mu \nu} - \frac{1}{2} Q^{\mu \sigma} \Omega_{\sigma \nu} - \frac{1}{4} g^{\mu \nu} Q^\alpha + \frac{1}{2} g^{\mu \nu} Q^\alpha, \quad \Theta_{\lambda}^{\mu \nu} = - \frac{\delta T}{\delta \Gamma_{\mu \nu}^\lambda} (4.86)
\]

\[
4L_{\mu \nu} = (Q_{\mu \alpha \beta} - 2Q_{\alpha \beta \mu})Q^\nu_{\alpha \beta} + (Q_\mu + 2q_\mu)Q_\nu + (2Q_{\mu \nu \alpha} - Q_{\alpha \mu \nu})Q^\alpha - 4\Omega_{\nu \alpha} Q_{\alpha \beta \mu} - 4\Omega_{\alpha \beta \nu} Q_{\alpha \mu} (4.77)
\]

\[
\Theta_{\mu \nu} := g^{\alpha \beta} \frac{\delta T_{\alpha \beta}}{\delta g^{\mu \nu}}, \quad M_{\mu \nu} := \frac{\delta D}{\delta g^{\mu \nu}}, \quad J^\lambda_{\mu \nu} := \sqrt{-g} \left( \frac{1}{4} Q^\lambda_{\mu \nu} - \frac{1}{2} Q^{\lambda \sigma} \Omega_{\sigma \nu} + \Omega^\lambda_{\mu \nu} \right), \quad \zeta^\lambda = \sqrt{-g} \left( \frac{1}{4} Q^\lambda - \frac{1}{2} Q^\lambda + \Omega^\lambda_{\mu \nu} \right) (4.88)
\]

\[
\Theta^\mu_{\nu} := - \frac{\delta T}{\delta \Gamma_{\mu \nu}^\lambda}, \quad M^\mu_{\nu \alpha} := \frac{\delta \Delta_{\mu \nu}}{\delta \Gamma_{\mu \nu}^\lambda}, \quad \Theta_{\mu \nu} := - \frac{\delta T}{\delta \Gamma_{\mu \nu}^\lambda}, \quad M^\mu_{\nu \alpha} := \frac{\delta \Delta_{\mu \nu}}{\delta \Gamma_{\mu \nu}^\lambda} (4.89)
\]

\[
P^\lambda_{\mu \nu}(F_R) = - \frac{\nabla_\lambda (\sqrt{-g} F_R g_{\mu \nu})}{\sqrt{-g}} + \nabla_\alpha (\sqrt{-g} F_R g_{\mu \nu} \delta^\alpha_{\nu}) + 2F_R (S_\lambda g^{\mu \nu} - S^\mu \delta^\lambda_{\nu} - S^\lambda_{\mu \nu}) (4.90)
\]
4.6 Other MAG theories

4.6.1 Metric-affine $F(R, R_{\mu\nu} R^{\mu\nu})$ gravity

The action of the metric-affine $F(R, R_{\mu\nu} R^{\mu\nu})$ gravity has the form (see e.g. [15] and references therein)

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4 x J F(R, R_{\mu\nu} R^{\mu\nu}) + 2\kappa^2 L_m = S_g[g, \Gamma] + S_m[g, \Gamma, \psi].$$  (4.91)

Variations of the action with respect to the metric tensor (\(\delta g S = 0\)) and the connection (\(\delta \Gamma S = 0\)), respectively, give the following set of two field equations

\[
F_{R R(\mu\nu)} - 0.5 F g_{\mu\nu} + F_{R_{\mu\nu} R^{\mu\nu}} (R_{\mu\alpha} R_{\nu}^\alpha + R_{\alpha\nu} R_{\mu}^\alpha) = \kappa^2 T_{\mu\nu},
\]

\[
\nabla_\alpha (\sqrt{-g} B^{\mu\nu}) \delta^\alpha_\lambda - \nabla_\lambda (\sqrt{-g} B^{\mu\nu}) + 2 \sqrt{-g} [B^{\mu\nu} S_\lambda - B^{\nu\lambda} (S_\lambda^\nu + S_\nu^\lambda)] = \kappa^2 H^{\mu\nu}_\lambda,
\]

where

$$B^{\mu\nu} = F R g^{\mu\nu} + 2 F_{R_{\mu\nu} R^{\mu\nu}} R^{\mu\nu}.$$  (4.94)

4.6.2 Metric-affine $F(g_{\mu\nu}, R^\alpha_{\beta\gamma\rho})$ gravity

One of examples of generalized metric-affine gravity theories is the metric-affine $F(g_{\mu\nu}, R^\alpha_{\beta\gamma\rho})$ gravity. Its action reads as (see e.g. [15] and references therein)

$$S[g, \Gamma] = S_g + S_m = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} L_g (g_{\mu\nu}, R^\alpha_{\beta\gamma\rho}) + \int d^4 x \sqrt{-g} L_m (g_{\mu\nu}, R^\alpha_{\beta\gamma\rho}, \psi),$$  (4.95)

The two metric-affine gravity theories presented in the previous two subsections (2.1.1) and (2.2.2) are particular cases of the more general metric-affine gravity theory given by the action [15]. Varying the action (4.100) with respect to the metric tensor and to the affine connection, we come to the following field equations [15]

\[
-0.5 L_g g_{\mu\nu} + \frac{\partial L_g}{\partial g_{\mu\nu}} = \kappa^2 T_{\mu\nu},
\]

\[
\frac{2}{\sqrt{-g}} [2 S_\alpha - \nabla_\alpha (\sqrt{-g} \Sigma^\mu_\alpha^\nu) - \sqrt{-g} \Sigma^\nu_\alpha^\gamma \Sigma^\gamma_\delta S^\delta_\gamma] = \kappa^2 H^\mu_\alpha^\nu,
\]

where

$$\Sigma^\mu_\alpha^\nu = \frac{\partial L_g}{\partial R^\mu_{\alpha\nu}}.$$  (4.98)

4.6.3 Metric-affine $F(g_{\mu\nu}, R^\alpha_{\beta\gamma\rho}, S^\lambda_{\mu\nu}, Q_{\alpha\mu\nu})$ gravity

One of most general examples of metric-affine gravity theories is the metric-affine $F(g_{\mu\nu}, R^\alpha_{\beta\gamma\rho}, S^\lambda_{\mu\nu}, Q_{\alpha\mu\nu})$ gravity. Its action is given by (see e.g. [15] and references therein)

$$S[g, \Gamma] = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} L_g (g_{\mu\nu}, R^\alpha_{\beta\gamma\rho}, S^\lambda_{\mu\nu}, Q_{\alpha\mu\nu}) + \int d^4 x \sqrt{-g} L_m (g_{\mu\nu}, R^\alpha_{\beta\gamma\rho}, S^\lambda_{\mu\nu}, Q_{\alpha\mu\nu}),$$  (4.99)

where

$$S_g = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} L_g (g_{\mu\nu}, R^\alpha_{\beta\gamma\rho}, S^\lambda_{\mu\nu}, Q_{\alpha\mu\nu}), \quad S_m = \int d^4 x \sqrt{-g} L_m (g_{\mu\nu}, R^\alpha_{\beta\gamma\rho}, S^\lambda_{\mu\nu}, Q_{\alpha\mu\nu}).$$  (4.100)

The two metric-affine gravity theories presented in the previous two subsections are particular cases of the more general metric-affine gravity theory given by the action (3.18). Varying the action (3.18) with respect to the metric tensor and to the affine connection, we come to the following field equations [15]

\[
-0.5 L_g g_{\mu\nu} + \frac{\partial L_g}{\partial g_{\mu\nu}} + \frac{1}{\sqrt{-g}} (2 S_\alpha - \nabla_\alpha) \sqrt{-g} \frac{\partial L_g}{\partial Q_{\alpha\mu} g_{\lambda\mu}} = \kappa^2 T_{\mu\nu},
\]

\[
-2 \nabla_\alpha (\sqrt{-g} \Sigma^\mu_\alpha^\nu) + 4 \Sigma^\mu_\alpha^\nu S_\alpha - \Sigma^\mu_\beta^\lambda S^\gamma_\beta S^\gamma_\nu + 2 W^\mu_{\lambda\nu} + V^\mu_{\lambda\nu} = \kappa^2 H^\mu_\lambda^\nu,
\]

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where

\[ \Sigma^\mu_{\lambda \alpha \nu} = \frac{\partial L_g}{\partial R^\lambda_{\mu \alpha \nu}}, \quad V^\mu_{\lambda \nu} = \frac{\partial L_g}{\partial S^\lambda_{\mu \nu}}, \quad W^{\alpha \mu \nu} = \frac{\partial L_g}{\partial Q_{\alpha \mu \nu}}. \]  

We consider some generalized metric-affine spacetime with the curvature, torsion, and nonmetricity. In the previous sections, we have considered the MG-VIII theory. In this section, we want to collect some other generalized and/or modified gravity theories.

### 4.6.4 Einstein-Ricci gravity

The equations of motion of the Einstein-Ricci gravity are given by [25]-[26]

\[ R_{ij} - 0.5 R g_{ij} - \kappa T_{ij} + \phi_{ij} = 0, \quad (4.104) \]
\[ g_{ij} + 2 n R_{ij} + f_{ij} = 0. \quad (4.105) \]

### 4.6.5 Einstein-Calabi gravity

The equations of motion of the Einstein-Calabi gravity (ECG) read as [25]-[26]

\[ R_{ij} - 0.5 R g_{ij} - \kappa T_{ij} + \phi_{ij} = 0, \quad (4.106) \]
\[ g_{ij} - n \frac{\partial^2 R}{\partial z^i \partial z^j} + f_{ij} = 0. \quad (4.107) \]

### 4.6.6 Einstein-Cartan gravity

The action of the Einstein-Cartan gravity (ECG) reads as

\[ S = \int L \sqrt{-g} d^4 x = \frac{1}{16 \pi G} \int R(\Gamma, g) \sqrt{-g} d^4 x + S_m. \quad (4.108) \]

The equations of motion of the ECG are given by [25]-[26]

\[ G_{ij} + 4 B^\alpha_{[\beta} B^{\alpha \beta]} + 2 B_{\beta \alpha \mu} e^{\beta \alpha} - B_{\mu \beta \alpha} e^{\beta \alpha} - 0.5 g_{ij} (4 B_{\beta [\lambda}^{\alpha} B^{\alpha \beta]} + B_{\alpha \beta \gamma} e^{\alpha \beta \gamma}) = \kappa T_{ij} \quad (4.109) \]
\[ -T_{\mu \nu}^\lambda + \delta_{\mu}^\lambda T_{\nu} - \delta_{\nu}^\lambda T_{\mu} = \kappa S^\lambda_{\mu \nu} \quad (4.110) \]
\[ \frac{\partial L}{\partial \phi} + (\nabla_\lambda - 2 T_{\lambda}) \frac{\partial L}{\partial \nabla_\phi} = 0 \quad (4.111) \]

where \( T_{\mu} = T_{\mu \lambda}, \)

\[ G_{ij} = R_{ij} - 0.5 R g_{ij}, \quad T_{ij} = \frac{\delta \sqrt{-g} L_m}{\delta g^0}, \quad S_{\mu \nu} = \frac{\delta L_m}{\delta T_{\mu \nu}}, \quad B_{\mu \nu} = T_{\mu \nu}^\lambda + \delta_{\mu}^\lambda T_{\nu} - \delta_{\nu}^\lambda T_{\mu}. \]  

### 4.6.7 Einstein-Yamabe gravity

The equations of motion of the Einstein-Yamabe gravity are given by [25]-[26]

\[ R_{ij} - 0.5 R g_{ij} - \kappa T_{ij} + \phi_{ij} = 0, \quad (4.113) \]
\[ g_{ij} + 2 n R_{ij} + f_{ij} = 0. \quad (4.114) \]

### 5 MG-VIII: Myrzakulov \( F(R, T, Q, \mathcal{T}) \) gravity

Let us consider the general spacetime with the curvature, torsion, and nonmetricity. In this spacetime, the action of the Myrzakulov \( F(R, T, Q, \mathcal{T}) \) gravity (or shortly the MG-VIII gravity) is given by [12]

\[ S = S_g + S_m = \int \sqrt{-g} \, d^4 x [F(R, T, Q, \mathcal{T}) + L_m] = \int \sqrt{-g} \, d^4 x F(R, T, Q, \mathcal{T}) + S_m, \quad (5.1) \]
where $R$ stands for the Ricci scalar (curvature scalar), $T$ is the torsion scalar, $Q$ is the nonmetricity scalar and $\mathcal{T}$ is trace of the energy-momentum tensor of matter Lagrangian $L_m$. These fourth scalars are given by

\begin{align}
R &= g^{\mu\nu}R_{\mu\nu}, \\
T &= S_\rho^{\ \ \mu\nu}T^\rho_{\mu\nu}, \\
Q &= -g^{\mu\nu}(L_3^\alpha L_{\alpha\beta} - L_3^\beta L_{\mu\nu}^\beta), \\
\mathcal{T} &= g^{\mu\nu}T_{\mu\nu}.
\end{align}

Here

\begin{align}
R_{\mu\nu} &= R_{\mu\nu}^\alpha, \\
T_{\mu\nu} &= -\frac{1}{\sqrt{-g}} \frac{\delta(-\sqrt{-g}L_m)}{\delta g^{\mu\nu}} = L_m g_{\mu\nu} - \frac{\delta L_m}{\delta g^{\mu\nu}}
\end{align}

are $R_{\mu\nu}^\alpha$ is the Riemann curvature tensor and $T_{\mu\nu}$ is the energy-momentum tensor, respectively. Note that in the action (5.1), we have three independent variables: the metric, the affine connection and the matter fields contained in $S_m$. In this case, the energy-momentum tensor and the hypermomentum tensor are given by

\begin{align}
The variation of the action gives
\delta S = \int [F_R \delta R + F_T \delta T + F_Q \delta Q + F_T \delta \mathcal{T} - 0.5F_{\mu\nu} \delta g^{\mu\nu} + \frac{2k}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} \delta g^{\mu\nu}] \sqrt{-g} dx. \tag{5.9}
\end{align}

Let us find $\delta T_{\mu\nu}$. We have

\begin{align}
\delta T_{\mu\nu} &= L_m \delta g_{\mu\nu} + g_{\mu\nu} \frac{\delta L_m}{\delta g^{\alpha\beta}} \delta g^{\alpha\beta} - 2 \frac{\delta^2 L_m}{\delta g^{\mu\nu} \delta g^{\alpha\beta}} \delta g^{\alpha\beta} = \tag{5.10} \\
&= -L_m g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta} + 0.5 g_{\mu\nu} (g_{\alpha\beta} L_m - T_{\alpha\beta}) \delta g^{\alpha\beta} - 2 \frac{\delta^2 L_m}{\delta g^{\mu\nu} \delta g^{\alpha\beta}} \delta g^{\alpha\beta}. \tag{5.11}
\end{align}

This equation gives

\begin{align}
\frac{\delta T_{\mu\nu}}{\delta g^{\alpha\beta}} = -L_m g_{\mu\alpha} g_{\nu\beta} + 0.5 g_{\mu\nu} (g_{\alpha\beta} L_m - T_{\alpha\beta}) - 2 \frac{\delta^2 L_m}{\delta g^{\mu\nu} \delta g^{\alpha\beta}}. \tag{5.12}
\end{align}

Note that the variation of $\mathcal{T}$ with respect to the metric tensor $g_{\mu\nu}$ is given by

\begin{align}
\frac{\delta \mathcal{T}}{\delta g^{\mu\nu}} = \frac{\delta (g^{\alpha\beta} T_{\alpha\beta})}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Theta_{\mu\nu}. \tag{5.13}
\end{align}

Hence we obtain

\begin{align}
\Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = -L_m g_{\mu\nu} + 2 g_{\mu\nu} L_m - 2 T_{\mu\nu} - 2 g^{\alpha\beta} \frac{\delta^2 L_m}{\delta g^{\mu\nu} \delta g^{\alpha\beta}}. \tag{5.14}
\end{align}

We now ready to write the gravitational field equation. We have

\begin{align}
F_R R_{\mu\nu} + \nabla^\alpha \nabla_\alpha (F_R g_{\mu\nu}) - \nabla_\mu \nabla_\nu F_R - 0.5 F g_{\mu\nu} + ... = (\kappa - F_T) T_{\mu\nu} - F_T \Theta_{\mu\nu}, \tag{5.15}
\end{align}

where $\kappa = 0.5$. Let us consider the perfect fluid with

\begin{align}
T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu}, \tag{5.16}
\end{align}

where $\rho$ and $p$ are the energy density and matter pressure of the fluid, respectively. The $u = (0, 0, 0, 1)$ is the components of the four velocity vector ($u_\mu$) in the co-moving coordinate system.

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which satisfies the conditions \( u^\mu u_\mu = 1 \) and \( u^\mu \nabla_\nu u_\mu = 0 \). We choose the perfect fluid matter as \( L_m = p \) in the action \((5.1)\). Therefore we obtain

\[
\Theta_{\mu\nu} = -2T_{\mu\nu} + pg_{\mu\nu}.
\] (5.17)

Substituting the obtained expressions into the field equations we finally get

\[
F_R R_{\mu\nu} + g_{\mu\nu} \nabla^\alpha \nabla_\alpha F_R - \nabla_\mu \nabla_\nu F_R - 0.5 F g_{\mu\nu} + ... = (\kappa - F_T) T_{\mu\nu} - F_T pg_{\mu\nu}.
\] (5.18)

Now we want rewrite the action of the MG-VIII with the lagrangian multipliers as

\[
S = \int \sqrt{-g} d^4 x \left[ F - \lambda_1 (R - R_s - u) - \lambda_2 (T - T_s - v) - \lambda_3 (Q - Q_s - w) - \lambda_4 (T_s - y) + L_m \right].
\] (5.19)

The variations with respect to \( R, T, Q, T \) of the action give \( \lambda_1 = F_R, \lambda_2 = F_T, \lambda_3 = F_Q, \lambda_4 = F_T \) respectively. Thus the action of the MG-VIII takes the form

\[
S = \int \sqrt{-g} d^4 x \left[ F - F_R (R - R_s - u) - F_T (T - T_s - v) - F_Q (Q - Q_s - w) - F_T (T_s - y) + L_m \right].
\] (5.20)

Let us find the variation of the curvature scalar \( R \). We obtain

\[
\delta R = \delta (g^{\mu\nu} R_{\mu\nu}) = R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \left[ \nabla_\lambda \delta \Gamma^{\lambda}_{\mu\nu} - \nabla_{\lambda} \Gamma^{\lambda}_{\mu\nu} \right].
\] (5.21)

Note that the variation of the affine connection is given by

\[
\delta \Gamma^{\lambda}_{\mu\nu} = 0.5 g^{\lambda\alpha} \left( \nabla_\mu \delta g_{\alpha\nu} + \nabla_\nu \delta g_{\alpha\mu} - \nabla_\alpha \delta g_{\mu\nu} \right).
\] (5.22)

Therefore, for the variation of the curvature scalar \( R \) we obtain

\[
\delta R = R_{\mu\nu} \delta g^{\mu\nu} + g_{\mu\nu} \nabla_\lambda \delta g^{\mu\lambda} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu}.
\] (5.23)

Using the Palatini formalism (see, for example, Refs \([23], [24]\)) and varying the action with respect to the metric and the affine connection, we obtain the following system of the two field equations

\[
F_R R_{\mu\nu} + g_{\mu\nu} \nabla^\alpha \nabla_\alpha F_R - \nabla_\mu \nabla_\nu F_R - 0.5 F g_{\mu\nu} + ... = (k - F_T) T_{\mu\nu} - F_T pg_{\mu\nu},
\] (5.24)

\[
\nabla_\mu \left[ \sqrt{-g} \delta \lambda^\mu_F R_{\mu\nu} - 0.5 \delta \lambda^\mu_F g_{\mu\nu} - 0.5 \delta \lambda^\mu_F F_R g_{\mu\nu} \right] + ... = H_{\mu\nu}.
\] (5.25)

Let us also here present one important equation. The trace of the field equation \((3.15)\) of the MG-VIII becomes

\[
F_R R - 2F + 3 \Box F_R + ... = \frac{1}{2} T + F_T T - 4p F_T.
\] (5.26)

### 6 FRW cosmological equations

For a simplicity, we consider the flat FLRW metric in the following form

\[
ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,
\] (6.1)

where \( a(t) \) stands for the scale factor. If we write down Lagrangian of \( F(R, T, Q, T) \) for this metric and if we assumed that the Universe is filled with matter fields with effective pressure \( p \) and energy density \( \rho \), we obtain \( T_x = 3p - \rho \). Therefore the trace of the field equation \((3.24)-(3.25)\) of the MG-VIII becomes as in \((3.26)\). In the FRW spacetime, the action of the MG-VIII reads as

\[
S = \int L dt,
\] (6.2)
where the point like Lagrangian of the MG-VIII after an integration by part takes the form

\[
\mathcal{L} = L + L_m = a^3 \left( F - RF_R - TF_T - QF_Q - TF_T \right)
\]

\[-6a\dot{a}^2 \left( F_R + F_T - F_Q \right) - 6F_{Rt}a^2 \ddot{a} + a^3 [uF_R + vF_T + wF_Q + (T_s + y)F_T + L_m]. \tag{6.3}\]

Here we suppose that \( L_m = -\epsilon \rho (a), \quad (\epsilon = \pm 1) \) and

\[
L = a^3 \left( F - RF_R - TF_T - QF_Q - TF_T \right) - 6a\dot{a}^2 \left( F_R + F_T - F_Q \right) - 6F_{Rt}a^2 \ddot{a} = a^3(1) - 6a\dot{a}^2(2) - 6F_{Rt}a^2 \ddot{a},
\]

\[
\dot{L}_m = a^3[uF_R + vF_T + wF_Q + (T_s + y)F_T + L_m]. \tag{6.4}\]

where

\[
(1) = F - RF_R - TF_T - QF_Q - TF_T, \tag{6.6}
\]

\[
(2) = F_R + F_T - F_Q. \tag{6.7}
\]

Here

\[
R = 6\dot{a}^2 + \ddot{a} + u, \tag{6.8}
\]

\[
T = -6\dot{a}^2 + v, \tag{6.9}
\]

\[
Q = 6\dot{a}^2 + w, \tag{6.10}
\]

\[
\mathcal{T} = 3\rho - \rho + y, \tag{6.11}
\]

where \( u, v, w, y \) are some real functions of \( a, \dot{a}, \ldots \). The associated Euler-Lagrange equations are given by

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = 0, \tag{6.12}
\]

where \( q \equiv \{a, R, T, Q, \mathcal{T} \} \). Let find the following derivatives

\[
\frac{\partial \mathcal{L}}{\partial a} = 3a^2(1) - 6a\dot{a}^2(2) - 12F_{Rt}a\dot{a} + \frac{\partial \dot{L}_m}{\partial a}, \tag{6.13}
\]

\[
\frac{\partial \mathcal{L}}{\partial \dot{a}} = -12a\dot{a}(2) - 6F_{Rt}a^2 + \frac{\partial \mathcal{L}_m}{\partial \dot{a}}, \tag{6.14}
\]

\[
\frac{\partial \mathcal{L}}{\partial R} = a^3(1)R - 6a\dot{a}^2(2) + \frac{\partial \mathcal{L}_m}{\partial R}, \tag{6.15}
\]

\[
\frac{\partial \mathcal{L}}{\partial \dot{R}} = -6F_{Rt}a^2 \ddot{a} + \frac{\partial \mathcal{L}_m}{\partial \dot{R}}, \tag{6.16}
\]

\[
\frac{\partial \mathcal{L}}{\partial T} = a^3(1)T - 6a\dot{a}^2(2)T - 6F_{RT}a^2 \ddot{a} + \frac{\partial \mathcal{L}_m}{\partial T}, \tag{6.17}
\]

\[
\frac{\partial \mathcal{L}}{\partial \dot{T}} = -6F_{RT}a^2 \ddot{a} + \frac{\partial \mathcal{L}_m}{\partial \dot{T}}, \tag{6.18}
\]

\[
\frac{\partial \mathcal{L}}{\partial Q} = a^3(1)Q - 6a\dot{a}^2(2)Q\dot{a} + \frac{\partial \mathcal{L}_m}{\partial Q}, \tag{6.19}
\]

\[
\frac{\partial \mathcal{L}}{\partial \dot{Q}} = -6F_{RQ}a^2 \ddot{a} + \frac{\partial \mathcal{L}_m}{\partial \dot{Q}}, \tag{6.20}
\]

\[
\frac{\partial \mathcal{L}}{\partial \mathcal{T}} = a^3(1)\mathcal{T} - 6a\dot{a}^2(2)\mathcal{T} - 6F_{RT}a^2 \ddot{a} + \frac{\partial \mathcal{L}_m}{\partial \mathcal{T}}, \tag{6.21}
\]

\[
\frac{\partial \mathcal{L}}{\partial \dot{\mathcal{T}}} = -6F_{RT}a^2 \ddot{a} + \frac{\partial \mathcal{L}_m}{\partial \dot{\mathcal{T}}}. \tag{6.22}
\]
Now we assume that

\[ \frac{\partial L_m}{\partial R} = \frac{\partial L_m}{\partial T} = \frac{\partial L_m}{\partial Q} = \frac{\partial L_m}{\partial T} = 0. \]  

(6.23)

Thus finally we obtain

\[ \frac{\partial \mathcal{L}}{\partial \dot{a}} = 3a^2(1) - 6a^2(2) - 12F_R a \ddot{a} + \frac{\partial L_m}{\partial \dot{a}}, \]  

(6.24)

\[ \frac{\partial \mathcal{L}}{\partial a} = -12a \ddot{a}(2) - 6F_R a^2 + \frac{\partial L_m}{\partial a}, \]  

(6.25)

\[ \frac{\partial \mathcal{L}}{\partial R} = a^3(1)_R - 6a^2(2)_R - 6F_{RR} a^2 \ddot{a} + \frac{\partial L_m}{\partial R}. \]  

(6.26)

\[ \frac{\partial \mathcal{L}}{\partial T} = -6F_{RR} a^2 \ddot{a}, \]  

(6.27)

\[ \frac{\partial \mathcal{L}}{\partial Q} = a^3(1)_Q - 6a^2(2)_Q - 6F_{RQ} a^2 \ddot{a} + \frac{\partial L_m}{\partial Q}, \]  

(6.28)

\[ \frac{\partial \mathcal{L}}{\partial Q} = -6F_{RQ} a^2 \ddot{a}, \]  

(6.29)

\[ \frac{\partial \mathcal{L}}{\partial T} = a^3(1)_T - 6a^2(2)_T - 6F_{RT} a^2 \ddot{a} + \frac{\partial L_m}{\partial T}, \]  

(6.30)

\[ \frac{\partial \mathcal{L}}{\partial T} = -6F_{RT} a^2 \ddot{a}. \]  

(6.31)

As result, we obtain the following five equations

\[ 6 (\ddot{a}^2 + 2a \dddot{a}) (\dot{a}^2 + 12a \dddot{a}) + 6 F_{RT} a^2 + \frac{\partial L_m}{\partial a} - \frac{\partial^2 L_m}{\partial \dot{a} \partial \dot{a}} = 0, \]  

(6.34)

\[ a^3(1)_R + 6a^2[2F_{RR} - (2)_R] + 6F_{RR} a^2 \ddot{a} + \frac{\partial L_m}{\partial R} = 0, \]  

(6.35)

\[ a^3(1)_T + 6a^2[2F_{RT} - (2)_T] + 6F_{RT} a^2 \ddot{a} + \frac{\partial L_m}{\partial T} = 0, \]  

(6.36)

\[ a^3(1)_Q + 6a^2[2F_{RQ} - (2)_Q] + 6F_{RQ} a^2 \ddot{a} + \frac{\partial L_m}{\partial Q} = 0, \]  

(6.37)

\[ a^3(1)_T + 6a^2[2F_{RT} - (2)_T] + 6F_{RT} a^2 \ddot{a} + \frac{\partial L_m}{\partial T} = 0. \]  

(6.38)

One more equation we get from the following Hamiltonian constraint

\[ \mathcal{H} = \dot{a} \frac{\partial \mathcal{L}}{\partial \dot{a}} + R \frac{\partial \mathcal{L}}{\partial R} + T \frac{\partial \mathcal{L}}{\partial T} + Q \frac{\partial \mathcal{L}}{\partial Q} + \dot{Q} \frac{\partial \mathcal{L}}{\partial \dot{T}} + \dot{\dot{T}} \frac{\partial \mathcal{L}}{\partial \dot{Q}} - \mathcal{L} = 0. \]  

(6.39)

This constraint gives

\[ 6a \dddot{a}(2) + 6a^2 \dddot{a}[F_{RR} \dddot{R} + F_{RT} \dddot{T} + F_{RQ} \dddot{Q} + F_{RT} \dddot{T}] - \dot{a} \frac{\partial L_m}{\partial \dot{a}} + L_m = 0. \]  

(6.40)
Finally we have the following system of the 6 gravitational equations

\[ 6(\ddot{a}^2 + 2a\ddot{a})(2) + 3a^2(1) + 12a\dot{a}(2)t + 6F_{RR}a^2 + \frac{\partial L_m}{\partial a} - \frac{\partial^2 L_m}{\partial t \partial a} = 0, \]  
\[ (6.41) \]

\[ a^3(1)R + 6a\dot{a}^2[2F_{RR} - (2)R] + 6F_{RR}a^2\ddot{a} + \frac{\partial L_m}{\partial R} = 0, \]  
\[ (6.42) \]

\[ a^3(1)T + 6a\dot{a}^2[2F_{RT} - (2)T] + 6F_{RT}a^2\ddot{a} + \frac{\partial L_m}{\partial T} = 0, \]  
\[ (6.43) \]

\[ a^3(1)Q + 6a\dot{a}^2[2F_{RQ} - (2)Q] + 6F_{RQ}a^2\ddot{a} + \frac{\partial L_m}{\partial Q} = 0, \]  
\[ (6.44) \]

\[ a^3(1)\bar{\sigma} + 6a\dot{a}^2[2F_{R\bar{\sigma}} - (2)\bar{\sigma}] + 6F_{R\bar{\sigma}}a^2\ddot{a} + \frac{\partial L_m}{\partial \bar{\sigma}} = 0, \]  
\[ (6.45) \]

\[ 6a\dot{a}^2(2) + 6a^2\ddot{a}[F_{RR}\ddot{R} + F_{RT}\ddot{T} + F_{RQ}\ddot{Q} + F_{R\bar{\sigma}}\ddot{\bar{\sigma}}] - \dot{a} \frac{\partial L_m}{\partial \dot{a}} + L_m = 0. \]  
\[ (6.46) \]

7 FRW cosmology of \( F = \alpha R + \beta T + \mu Q + \nu \bar{T} \)

To understand the physical and mathematical nature of the Myrzakulov \( F(R,T,Q,\bar{T}) \) gravity (that is the MG-VIII), in this section, we consider the following particular model

\[ F(R,T,Q,\bar{T}) = \alpha R + \beta T + \mu Q + \nu \bar{T}, \]  
\[ (7.1) \]

where \( \alpha, \beta, \mu, \nu \) are some real constants. Then

\[ (1) = 0, \quad (2) = \alpha + \beta - \mu = \sigma. \]  
\[ (7.2) \]

In this particular case, the Lagrangian (4.3) takes the form

\[ \mathcal{L} = -6\sigma a\dot{a}^2 + a^3[\alpha u + \beta v + \mu w + \nu(y + T_s) + L_m] = -6\sigma a\dot{a}^2 + a^3B, \]  
\[ (7.3) \]

where

\[ B = \alpha u + \beta v + \mu w + \nu(y + T_s) + L_m. \]  
\[ (7.4) \]

Let us find the following derivatives:

\[ \frac{\partial \mathcal{L}}{\partial a} = -6\sigma a\dot{a}^2 + \left[a^3[\alpha u + \beta v + \lambda w + \gamma(y + T_s) + L_m]\right]_a = -6\sigma a\dot{a}^2 + [a^3B]_a, \]  
\[ (7.5) \]

\[ \frac{\partial \mathcal{L}}{\partial \ddot{a}} = -12\sigma a\dot{a} + \left[a^3[\alpha u + \beta v + \lambda w + \gamma(y + T_s) + L_m]_a\right] = -12\sigma a\dot{a} + a^3B_\dot{a}, \]  
\[ (7.6) \]

\[ \left(\frac{\partial \mathcal{L}}{\partial \dot{a}}\right)_{\ddot{a}} = -12\sigma(\ddot{a}^2 + a\ddot{a}) + [a^3B_{\ddot{a}}]. \]  
\[ (7.7) \]

Hence from the Euler-Lagrange equation

\[ \frac{\partial \mathcal{L}}{\partial a} - \left(\frac{\partial \mathcal{L}}{\partial \dot{a}}\right)_{\ddot{a}} = 0, \]  
\[ (7.8) \]

we obtain the following first field equation

\[ 6\sigma(\ddot{a}^2 + 2a\ddot{a}) + [a^3B]_a - [a^3B_{\ddot{a}}] = 0. \]  
\[ (7.9) \]

From the Hamiltonian constraint

\[ \mathcal{H} = \dot{a} \frac{\partial \mathcal{L}}{\partial \dot{a}} - \mathcal{L} = 0, \]  
\[ (7.10) \]

we get the second field equation

\[ -12\sigma a\dot{a}^2 + a^3\dot{a}B_\dot{a} + 6\sigma a\ddot{a}^2 - a^3B = 0 \]  
\[ (7.11) \]
or
\[6\sigma a \dot{\bar{a}}^2 - a^3 \ddot{a}_B + a^3 B = 0.\] (7.12)

Finally we get the following system of the two field equations
\[6\sigma a \dot{\bar{a}}^2 - a^3 \ddot{a}_B + a^3 B = 0,\] (7.13)
\[6\sigma(\dot{a}^2 + 2a\ddot{a}) + [a^3 B]_a - [a^3 B]_t = 0.\] (7.14)

We can rewrite these two equations in the following standard forms
\[3H^2 = \rho,\] (7.15)
\[2\dot{H} = -(\rho + p),\] (7.16)

where the matter density \(\rho\) and the pressure \(p\) have the following forms
\[\rho = \frac{1}{2\sigma}[\dot{a}B_a - B],\] (7.17)
\[p = \frac{1}{6\sigma a^2}[(a^3 B)_a - (a^3 B)_t].\] (7.18)

The EoS have the form
\[\omega = \frac{p}{\rho} = \frac{1}{3a^2} \frac{(a^3 B)_a - (a^3 B)_t}{\dot{a}B_a - B}.\] (7.19)

8 Cosmological solutions

As example of the cosmological solutions, let us consider the power-law solution
\[a = a_0 t^n,\] (8.1)

where \(a_0, n\) are some constants. Then
\[\rho = \frac{3n^2}{t^2},\quad p = \frac{n(2 - 3n)}{t^2},\quad H = \frac{n}{t},\quad \dot{H} = -\frac{n}{t^2}.\] (8.2)

On the other hand, from (7.17) we obtain
\[\rho = \frac{1}{2\sigma} \frac{t}{n-1}B_t - B\] (8.3)

where we used the following formulas
\[B_a = \frac{t^{2-n}}{na_0(n-1)}B_t,\quad B_a = \frac{t^{1-n}}{a_0 n}B_t.\] (8.4)

From (5.18) we get the following expression for the pressure
\[p = \frac{1}{6\sigma} \frac{3B - (3 + n)t}{n(n-1)}B_t - \frac{t^2}{n(n-1)}B_t].\] (8.5)

Now we assume that \(B\) has the form
\[B = \frac{\delta}{t^2},\] (8.6)

where \(\delta = \text{const}\). Eqs. (8.2) and (8.3) for the density of energy give
\[\delta = \frac{6\sigma n^2 (1-n)}{1+n}.\] (8.7)

At the same time, the expressions of the pressure (8.2) and (8.5) give
\[\delta = \frac{6\sigma (2-3n)n(n-1)}{3n-1}.\] (8.8)

The last two equations give that \(n = 0\) that is \(a = a_0 = \text{const}\). Thus the power-law solution is the trivial at least for our assumptions.
9 Wheeler-DeWitt equation

In the Hamiltonian formulation of ordinary classical mechanics the key concept is the Poisson bracket (PB). In this formalism, the canonical coordinate system consists of canonical position $q_i$ and momentum $p_i$ variables which satisfy the following fundamental canonical PB relations

$$\{q_i, p_j\} = \delta_{ij}. \quad (9.1)$$

Here the PB reads as

$$\{f, g\} = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right), \quad (9.2)$$

where $f, g$ are the phase space functions. Correspondingly, the Hamilton equations have the following forms

$$\dot{q}_i = \{q_i, H\}, \quad (9.3)$$
$$\dot{p}_i = \{p_i, H\}, \quad (9.4)$$

which can be interpreted as the flow or orbit in phase space generated by $H$. In quantum case the $q, p$ are promoted to quantum operators $\hat{q}, \hat{p}$ on a Hilbert space with the following canonical commutation

$$[\hat{q}, \hat{p}] = i\hbar. \quad (9.5)$$

These operators satisfy the following equations

$$\hat{q}\psi(q) = q\psi(q), \quad (9.6)$$
$$\hat{p}\psi(q) = -i\hbar \frac{d}{dq}\psi(q). \quad (9.7)$$

Finally we get the following Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi = \hat{H}\psi, \quad (9.8)$$

where $\hat{H}$ is the operator form of the Hamiltonian $H$ with the usual replacements

$$q \mapsto q, \quad p \mapsto -i\hbar \frac{d}{dq}. \quad (9.9)$$

The momenta conjugate to variable $a$ is given by

$$\pi_1 = \frac{\partial L}{\partial \dot{a}} = -12\sigma a\dot{a} + a^3 B\dot{a}. \quad (9.10)$$

Hence we get

$$\dot{a} = -\frac{\pi_1 - a^3 B\dot{a}}{12\sigma a}. \quad (9.11)$$

Therefore the Hamiltonian takes the form

$$\hat{H} = \hat{a} \frac{\partial L}{\partial \dot{a}} - L = 6\sigma a\dot{a}^2 - a^3 \dot{a} B\dot{a} + a^3 B \quad (9.12)$$

or

$$\hat{H} = \frac{1}{24\sigma a}[\pi_1 - a^3 B\dot{a}]^2 + \frac{a^3 B a(\pi_1 - a^3 B\dot{a})}{12\sigma a} + a^3 B = \frac{1}{24\sigma a} \left( \pi_1^2 - a^6 B\dot{a}^2 + 24\sigma a^4 B \right). \quad (9.13)$$
The classical dynamics is governed by the following Hamiltonian equations

\[
\dot{a} = \{a, \hat{H}\} = \frac{\partial \hat{H}}{\partial \pi_1}, \quad (9.14)
\]

\[
\dot{\pi}_1 = \{\pi_1, \hat{H}\} = -\frac{\partial \hat{H}}{\partial a}. \quad (9.15)
\]

Therefore, we have

\[
\dot{a} = \frac{\pi_1}{12\sigma a}, \quad (9.16)
\]

\[
\dot{\pi}_1 = \left(\frac{\pi_1^2 - a^6 B^2_a + 24\sigma a^4 B}{24\sigma a^2} + \frac{(a^6 B^2_a - 24\sigma a^4 B)}{24\sigma a}\right). \quad (9.17)
\]

According to the Dirac quantization approach, the quantum states of the universe should be annihilated by the operator version of the Hamiltonian, that is

\[
\hat{H}\Psi = \left[\frac{1}{24\sigma a} \left(\pi_1^2 - a^6 B^2_a + 24\sigma a^4 B\right)\right]\Psi = 0, \quad (9.18)
\]

where \(\Psi = \Psi(a)\) is the wave function of the universe. We now use the standard representation \(\pi_1 \rightarrow -i\partial_a\). Then we obtain the Wheeler - DeWitt equation ((WDWE) \[19\]-\[20\])

\[
\hat{H}\Psi = \left[\frac{1}{24\sigma a} \left(-\frac{\partial^2}{\partial a^2} - a^6 B^2_a + 24\sigma a^4 B\right)\right]\Psi = 0 \quad (9.19)
\]

or

\[
\left[\frac{1}{24\sigma a} \left(\frac{\partial^2}{\partial a^2} + a^6 B^2_a - 24\sigma a^4 B\right)\right]\Psi = 0. \quad (9.20)
\]

10 Relation with the soliton theory

Let us rewrite the WDWE as

\[
L\Psi = -\left[\frac{\partial^2}{\partial a^2} - U\right]\Psi = \lambda\Psi, \quad (10.1)
\]

where

\[
U = -a^6 B^2_a + 24\sigma a^4 B. \quad (10.2)
\]

Introduce the operator \(A\) as

\[
A = 4\partial^3_a - 3[U\partial_a + \partial_a U]. \quad (10.3)
\]

Then the Lax equation

\[
L\Lambda = [L, A] \quad (10.4)
\]

gives the famous Korteweg-de Vries equation

\[
U_a + 6UU_a + U_{aaa} = 0. \quad (10.5)
\]

11 Metric-affine MG theories

In this section, some metric-affine Myrzakulov gravity theories are presented \[12\]. Consider the metric-affine spacetime with the affine connection \(\tilde{\Gamma}^\lambda_{\mu\nu}\). Then the torsion and nonmetricity tensors are given by

\[
T^\lambda_{\mu\nu} = 2\tilde{\Gamma}^\lambda_{[\mu\nu]}, \quad (11.1)
\]

\[
Q^\lambda_{\mu\nu} = \tilde{\nabla}\lambda g_{\mu\nu}. \quad (11.2)
\]
The corresponding covariant derivative of an arbitrary vector $v^\lambda$ can be split into a Riemannian contribution and a distortion tensor

$$\tilde{\nabla}_\mu v^\lambda = \nabla_\mu v^\lambda + N^\lambda_{\rho\mu} v^\rho,$$

(11.3)

where

$$N^\lambda_{\rho\mu} = K^\lambda_{\rho\mu} + L^\lambda_{\rho\mu}.$$  

(11.4)

Here the contortion and disformation tensors read as

$$K^\lambda_{\rho\mu} = \frac{1}{2} (T^\lambda_{\rho\mu} - T^\rho_{\mu\lambda} - T^\mu_{\lambda\rho}),$$  

(11.5)

$$L^\lambda_{\rho\mu} = \frac{1}{2} (Q^\lambda_{\rho\mu} - Q^\rho_{\mu\lambda} - Q^\mu_{\lambda\rho}),$$  

(11.6)

respectively. The commutation of the covariant derivatives takes the form

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] v^\lambda = \tilde{R}^\lambda_{\rho\mu\nu} v^\rho + T^\rho_{\mu\nu} \tilde{\nabla}_\rho v^\lambda,$$

(11.7)

where

$$\tilde{R}^\lambda_{\rho\mu\nu} = \partial_\mu \tilde{\Gamma}^\lambda_{\rho\nu} - \partial_\nu \tilde{\Gamma}^\lambda_{\rho\mu} + \tilde{\Gamma}^\lambda_{\sigma\mu} \tilde{\Gamma}^\sigma_{\rho\nu} - \tilde{\Gamma}^\lambda_{\sigma\nu} \tilde{\Gamma}^\sigma_{\rho\mu}. $$

(11.8)

Note that the geometric structure of the metric-affine spacetime is determined by three tensors: the metric tensor ($g_{\mu\nu}$), the torsion tensor ($T^\lambda_{\mu\nu}$) and the nonmetricity tensor ($Q^\lambda_{\mu\nu}$). The torsion tensor is the antisymmetric part of the connection and the nonmetricity tensor measures the failure of the connection to be metric compatible. Note that these three tensors can be computed once an affine connection $\tilde{\Gamma}^\alpha_{\beta\lambda}$ is given. In this metric-affine spacetime, let us introduce five scalars - $R$, $T$, $Q$, $G$, $B$, where $R$ is the metric-affine curvature scalar, $T$ is the metric-affine torsion scalar, $Q$ is the metric-affine nonmetricity scalar, $G$ is the metric-affine Gauss-Bonnet scalar, $B$ is the boundary term scalar. Below $T$ is the trace of the energy-momentum tensor. In the previous sections, we have considered the Myrzakulov gravity-I (MG-I) which has the following action

$$ S = \sqrt{-g} \int d^4 x [F(R, T) + L_m],$$

(11.9)

where $R$ is the curvature scalar, $T$ is the torsion scalar and $L_m$ is the matter Lagrangian. This MG-I is some kind generalization (unification) of the well-known $F(R)$ and $F(T)$ gravity theories. We now going to present some other examples of metric-affine Myrzakulov gravity theories, also abbreviated below as MG-N, where N=I, II, III, IV, ... (see, also, Table 1, Table 2 and Table 3).

### 11.1 MG-I

The action of the Myrzakulov gravity-I (MG-I) has the following form

$$ S = \frac{1}{2\kappa^2} \sqrt{-g} d^4 x [F(R, T) + 2\kappa^2 L_m],$$

(11.10)

where $R$ is the curvature scalar, $T$ is the torsion scalar and $L_m$ is the matter Lagrangian. This MG-I is some kind generalizations of the well-known $F(R)$ and $F(T)$ gravity theories. If exactly, the MG-I is the unification of the $F(R)$ and $F(T)$ theories.

### 11.2 MG-II

The action of the Myrzakulov gravity-II (MG-II) reads as

$$ S = \frac{1}{2\kappa^2} \sqrt{-g} d^4 x [F(R, Q) + 2\kappa^2 L_m],$$

(11.11)

where $R$ is the curvature scalar and $Q$ is the nonmetricity scalar of the metric-affine spacetime.
11.3 MG-III

The action of the Myrzakulov gravity - III (MG-III) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q) + 2\kappa^2 L_m], \]  

(11.12)

where \( T \) is the torsion scalar and \( Q \) is the nonmetricity scalar of the metric-affine spacetime.

11.4 MG-IV

The action of the Myrzakulov gravity - IV (MG-IV) has the following form

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, T) + 2\kappa^2 L_m], \]  

(11.13)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar and \( T \) is the trace of the energy-momentum tensor.

11.5 MG-V

The action of the Myrzakulov gravity - V (MG-V) is given by

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q) + 2\kappa^2 L_m], \]  

(11.14)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar and \( Q \) is the nonmetricity scalar of the metric-affine spacetime.

11.6 MG-VI

The action of the Myrzakulov gravity - VI (MG-VI) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, Q, T) + 2\kappa^2 L_m], \]  

(11.15)

where \( R \) is the curvature scalar, \( Q \) is the nonmetricity scalar and \( T \) is the trace of the energy-momentum tensor of our generalized spacetime.

11.7 MG-VII

The action of the Myrzakulov gravity - VII (MG-VII) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, T) + 2\kappa^2 L_m], \]  

(11.16)

and \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.

11.8 MG-VIII

The action of the Myrzakulov gravity - VIII (MG-VIII) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q, T) + 2\kappa^2 L_m], \]  

(11.17)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar and \( T \) is the trace of the energy-momentum tensor (the trace of the stress-energy tensor) of the metric-affine spacetime.
12 Metric-affine MG theories with the Gauss-Bonnet scalars

The metric-affine MG theories with the Gauss-Bonnet scalars \((G)\) were proposed in [12]. For our convenience, let us present these models (see e.g. Table 2).

### 12.1 MG-IX

The action of the Myrzakulov gravity - IX (MG-IX) has the following form

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4 x [F(R, T, G) + 2\kappa^2 L_m], \tag{12.1}
\]

where \(R\) is the curvature scalar, \(T\) is the torsion scalar, \(G\) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

### 12.2 MG-X

The action of the Myrzakulov gravity - X (MG-X) reads as

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4 x [F(R, Q, G) + 2\kappa^2 L_m], \tag{12.2}
\]

where \(R\) is the curvature scalar, \(Q\) is the nonmetricity scalar, \(G\) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

### 12.3 MG-XI

The action of the Myrzakulov gravity - XI (MG-XI) reads as

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4 x [F(T, Q, G) + 2\kappa^2 L_m], \tag{12.3}
\]

where \(T\) is the metric-affine torsion scalar, \(Q\) is the metric-affine nonmetricity scalar and \(G\) is the metric-affine Gauss-Bonnet scalar of our metric-affine spacetime.

### 12.4 MG-XII

The action of the Myrzakulov gravity - XII (MG-XII) has the following form

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4 x [F(R, T, G, T) + 2\kappa^2 L_m], \tag{12.4}
\]

where \(R\) is the metric-affine curvature scalar, \(T\) is the metric-affine torsion scalar, \(G\) is the metric-affine Gauss-Bonnet scalar and \(T\) is the trace of the energy-momentum tensor.
12.5 MG-XIII
The action of the Myrzakulov gravity - XIII (MG-XIII) is given by
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(R, T, Q, G) + 2\kappa^2 L_m \right], \] (12.5)
where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

12.6 MG-XIV
The action of the Myrzakulov gravity - XIV (MG-XIV) reads as
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(R, Q, G, T) + 2\kappa^2 L_m \right], \] (12.6)
where \( R \) is the metric-affine curvature scalar, \( Q \) is the metric-affine nonmetricity scalar, \( G \) is the metric-affine Gauss-Bonnet scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.

12.7 MG-XV
The action of the Myrzakulov gravity - XV (MG-XV) reads as
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(T, Q, G, T) + 2\kappa^2 L_m \right], \] (12.7)
and \( T \) is the metric-affine torsion scalar, \( Q \) is the metric-affine nonmetricity scalar, \( G \) is the metric-affine Gauss-Bonnet scalar and \( T \) is the trace of the energy-momentum tensor of our metric-affine spacetime.

12.8 MG-XVI
The action of the Myrzakulov gravity - XVI (MG-XVI) reads as
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(R, T, Q, G, T) + 2\kappa^2 L_m \right], \] (12.8)
where \( R \) is the metric-affine curvature scalar, \( T \) is the metric-affine torsion scalar, \( Q \) is the metric-affine nonmetricity scalar, \( G \) is the metric-affine Gauss-Bonnet scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.

12.9 MG-XVII
The action of the Myrzakulov gravity - XVII (MG-XVII) reads as
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(Q, G) + 2\kappa^2 L_m \right], \] (12.9)
where \( Q \) is the metric-affine nonmetricity scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

12.10 MG-XVIII
The action of the Myrzakulov gravity - XVIII (MG-XVIII) reads as
\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x \left[ F(R, T, G) + 2\kappa^2 L_m \right], \] (12.10)
where \( R \) is the metric-affine curvature scalar, \( T \) is the metric-affine torsion scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.
12.11 MG-XIX

The action of the Myrzakulov gravity - XIX (MG-XIX) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4 x [F(T, G, \mathcal{T}) + 2\kappa^2 L_m], \]  

(12.11)

where \( T \) is the metric-affine torsion scalar, \( G \) is the metric-affine Gauss-Bonnet scalar and \( \mathcal{T} \) is the trace of the energy-momentum tensor of the metric-affine spacetime.
## 13 Metric-affine MG theories with boundary term scalars

In this section, we would like to present some metric-affine MG theories with the boundary term scalars \((B)\). Note that these MG theories with the boundary term scalars were proposed in [12] (see e.g. Table 3).

### 13.1 MG-XX

The action of the Myrzakulov gravity - XX (MG-XX) has the following form

\[
S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(R, T, B) + 2\kappa^2 L_{m} \right],
\]  

where \(R\) is the curvature scalar, \(T\) is the torsion scalar, \(B\) is the boundary term scalar and \(L_{m}\) is the matter Lagrangian. This MG-I is some kind generalizations of the well-known \(F(R)\) and \(F(T)\) gravity theories. If exactly, the MG-I is the unification of the \(F(R)\) and \(F(T)\) theories.

### 13.2 MG-XXI

The action of the Myrzakulov gravity - XXI (MG-XXI) reads as

\[
S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(R, Q, B) + 2\kappa^2 L_{m} \right],
\]  

where \(R\) is the curvature scalar, \(B\) is the boundary term scalar and \(Q\) is the nonmetricity scalar of the metric-affine spacetime.

### 13.3 MG-XXII

The action of the Myrzakulov gravity - XXII (MG-XXII) reads as

\[
S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(T, Q, B) + 2\kappa^2 L_{m} \right],
\]  

where \(T\) is the torsion scalar, \(B\) is the boundary term scalar and \(Q\) is the nonmetricity scalar of the metric-affine spacetime.

### 13.4 MG-XXIII

The action of the Myrzakulov gravity - XXIII (MG-XXIII) has the following form

\[
S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(R, T, B, T) + 2\kappa^2 L_{m} \right],
\]  

where \(R\) is the curvature scalar, \(T\) is the torsion scalar, \(B\) is the boundary term scalar and \(T\) is the trace of the energy-momentum tensor.

---

**Table 2: Metric-affine Myrzakulov gravity theories with Gauss-Bonnet scalars**

| N | Name                          | Action                                                                 |
|---|-------------------------------|------------------------------------------------------------------------|
| 9 | Myrzakulov Gravity - IX (MG-IX) | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(R, T, B) + 2\kappa^2 L_{m} \right] \) |
| 10 | Myrzakulov Gravity - X (MG-X)   | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(R, Q, B) + 2\kappa^2 L_{m} \right] \) |
| 11 | Myrzakulov Gravity - XI (MG-XI) | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(T, Q, B) + 2\kappa^2 L_{m} \right] \) |
| 12 | Myrzakulov Gravity - XII (MG-XII) | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(R, T, G, T) + 2\kappa^2 L_{m} \right] \) |
| 13 | Myrzakulov Gravity - XIII (MG-XIII) | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(R, T, Q, G) + 2\kappa^2 L_{m} \right] \) |
| 14 | Myrzakulov Gravity - XIV (MG-XIV) | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(T, Q, G, T) + 2\kappa^2 L_{m} \right] \) |
| 15 | Myrzakulov Gravity - XV (MG-XV)   | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(T, Q, G, T) + 2\kappa^2 L_{m} \right] \) |
| 16 | Myrzakulov Gravity - XVI (MG-XVI) | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(R, T, Q, G) + 2\kappa^2 L_{m} \right] \) |
| 17 | Myrzakulov Gravity - XVII (MG-XVII) | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(Q, G, T) + 2\kappa^2 L_{m} \right] \) |
| 18 | Myrzakulov Gravity - XVIII (MG-XVIII) | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(T, Q, G) + 2\kappa^2 L_{m} \right] \) |
| 19 | Myrzakulov Gravity - XIX (MG-XIX)  | \( S = \frac{1}{2\kappa^2} \int d^{4}x \sqrt{-g} \left[ F(T, G, T) + 2\kappa^2 L_{m} \right] \) |
13.5 MG-XXIV
The action of the Myrzakulov gravity - XXIV (MG-XXIV) is given by

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q, B) + 2\kappa^2 L_m], \tag{13.5} \]

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( B \) is the boundary term scalar and \( Q \) is the nonmetricity scalar of the metric-affine spacetime.

13.6 MG-XXV
The action of the Myrzakulov gravity - XXV (MG-XXV) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, Q, B, T) + 2\kappa^2 L_m], \tag{13.6} \]

where \( R \) is the curvature scalar, \( Q \) is the nonmetricity scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor of our generalized spacetime.

13.7 MG-XXVI
The action of the Myrzakulov gravity - XXVI (MG-XXVI) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, B, T) + 2\kappa^2 L_m], \tag{13.7} \]

and \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.

13.8 MG-XXVII
The action of the Myrzakulov gravity - XXVII (MG-XXVII) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, G, B) + 2\kappa^2 L_m], \tag{13.8} \]

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( G \) is the boundary term scalar, \( B \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

13.9 MG-XXVIII
The action of the Myrzakulov gravity - XXVIII (MG-XXVIII) has the following form

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, G, B) + 2\kappa^2 L_m], \tag{13.9} \]

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( B \) is the boundary term scalar, \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

13.10 MG-XXIX
The action of the Myrzakulov gravity - XXIX (MG-XXIX) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, Q, G, B) + 2\kappa^2 L_m], \tag{13.10} \]

where \( R \) is the curvature scalar, \( Q \) is the nonmetricity scalar, \( B \) is the boundary term scalar, \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.
13.11  MG-XXX

The action of the Myrzakulov gravity - XXX (MG-XXX) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, G, B) + 2\kappa^2 L_m], \]  

(13.11)

where \( T \) is the metric-affine torsion scalar, \( Q \) is the metric-affine nonmetricity scalar, \( B \) is the boundary term scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of our metric-affine spacetime.

13.12  MG-XXXI

The action of the Myrzakulov gravity - XXXI (MG-XXXI) has the following form

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, G, B) + 2\kappa^2 L_m], \]  

(13.12)

where \( R \) is the metric-affine curvature scalar, \( T \) is the metric-affine torsion scalar, \( G \) is the metric-affine Gauss-Bonnet scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor.

13.13  MG-XXXII

The action of the Myrzakulov gravity - XXXII (MG-XXXII) is given by

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, G, B) + 2\kappa^2 L_m], \]  

(13.13)

where \( R \) is the curvature scalar, \( T \) is the torsion scalar, \( Q \) is the nonmetricity scalar, \( B \) is the boundary term scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

13.14  MG-XXXIII

The action of the Myrzakulov gravity - XXXIII (MG-XXXIII) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, Q, G, B, T) + 2\kappa^2 L_m], \]  

(13.14)

where \( R \) is the metric-affine curvature scalar, \( Q \) is the metric-affine nonmetricity scalar, \( G \) is the metric-affine Gauss-Bonnet scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.

13.15  MG-XXXIV

The action of the Myrzakulov gravity - XXXIV (MG-XXXIV) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, Q, G, B, T) + 2\kappa^2 L_m], \]  

(13.15)

and \( T \) is the metric-affine torsion scalar, \( Q \) is the metric-affine nonmetricity scalar, \( G \) is the metric-affine Gauss-Bonnet scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor of our metric-affine spacetime.

13.16  MG-XXXV

The action of the Myrzakulov gravity - XXXV (MG-XXXV) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, Q, G, B) + 2\kappa^2 L_m], \]  

(13.16)

where \( R \) is the metric-affine curvature scalar, \( T \) is the metric-affine torsion scalar, \( Q \) is the metric-affine nonmetricity scalar, \( G \) is the metric-affine Gauss-Bonnet scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.
13.17 MG-XXXVI

The action of the Myrzakulov gravity - XXXVI (MG-XXXVI) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(Q, G, B) + 2\kappa^2 L_m], \]  

(13.17)

where \( Q \) is the metric-affine nonmetricity scalar, \( B \) is the boundary term scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

13.18 MG-XXXVII

The action of the Myrzakulov gravity - XXXVII (MG-XXXVII) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(R, T, G, B) + 2\kappa^2 L_m], \]  

(13.18)

where \( R \) is the metric-affine curvature scalar, \( T \) is the metric-affine torsion scalar, \( B \) is the boundary term scalar and \( G \) is the metric-affine Gauss-Bonnet scalar of the metric-affine spacetime.

13.19 MG-XXXVIII

The action of the Myrzakulov gravity - XXXVIII (MG-XXXVIII) reads as

\[ S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [F(T, G, B, T) + 2\kappa^2 L_m], \]  

(13.19)

where \( T \) is the metric-affine torsion scalar, \( G \) is the metric-affine Gauss-Bonnet scalar, \( B \) is the boundary term scalar and \( T \) is the trace of the energy-momentum tensor of the metric-affine spacetime.
Table 3: Metric-affine MG theories with boundary term scalars

| N | Name                                  | Action                                                                 |
|---|---------------------------------------|------------------------------------------------------------------------|
| 1 | Myrzakulov Gravity - XX (MG-XX)       | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, B) + 2k^2 L_m \right]$ |
| 2 | Myrzakulov Gravity - XXI (MG-XXI)     | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, Q, B) + 2k^2 L_m \right]$ |
| 3 | Myrzakulov Gravity - XXII (MG-XXII)   | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, Q, B) + 2k^2 L_m \right]$ |
| 4 | Myrzakulov Gravity - XXIII (MG-XXIII) | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, B, T) + 2k^2 L_m \right]$ |
| 5 | Myrzakulov Gravity - XXIV (MG-XXIV)   | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, Q, B) + 2k^2 L_m \right]$ |
| 6 | Myrzakulov Gravity - XXV (MG-XXV)     | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, Q, B) + 2k^2 L_m \right]$ |
| 7 | Myrzakulov Gravity - XXVI (MG-XXVI)   | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, Q, B, T) + 2k^2 L_m \right]$ |
| 8 | Myrzakulov Gravity - XXVII (MG-XXVII) | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, Q, B, T) + 2k^2 L_m \right]$ |
| 9 | Myrzakulov Gravity - XXVIII (MG-XXVIII)| $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, G, B) + 2k^2 L_m \right]$ |
| 10| Myrzakulov Gravity - XXIX (MG-XXIX)   | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, Q, G, B) + 2k^2 L_m \right]$ |
| 11| Myrzakulov Gravity - XXX (MG-XXX)     | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, Q, G, B, T) + 2k^2 L_m \right]$ |
| 12| Myrzakulov Gravity - XXXI (MG-XXXI)   | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, G, B, T) + 2k^2 L_m \right]$ |
| 13| Myrzakulov Gravity - XXXII (MG-XXXII) | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, Q, G, B) + 2k^2 L_m \right]$ |
| 14| Myrzakulov Gravity - XXXIII (MG-XXXIII)| $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, Q, G, B, T) + 2k^2 L_m \right]$ |
| 15| Myrzakulov Gravity - XXXIV (MG-XXXIV) | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, Q, G, B, T) + 2k^2 L_m \right]$ |
| 16| Myrzakulov Gravity - XXXV (MG-XXXV)   | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, Q, G, B, T) + 2k^2 L_m \right]$ |
| 17| Myrzakulov Gravity - XXXVI (MG-XXXVI) | $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, Q, G, B, T) + 2k^2 L_m \right]$ |
| 18| Myrzakulov Gravity - XXXVII (MG-XXXVII)| $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, B, T) + 2k^2 L_m \right]$ |
| 19| Myrzakulov Gravity - XXXVIII (MG-XXXVIII)| $S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R(F, T, G, B, T) + 2k^2 L_m \right]$ |

14 Cosmology in metric-affine MG theories

Consider the FRW universe. The flat FRW spacetime is described by the metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$

(14.1)

where $a = a(t)$ is the scale factor. Let $R, T, Q$ are the Ricci, torsion, nonmetricity scalars. For the FRW metric they have the forms: i) $R = R_0$, where $T = Q = 0$; ii) $T = T_0$, where $R = Q = 0$; iii) $Q = Q_0$, where $R = T = 0$. For the FRW metric, they have the forms:

$$R_0 = 6(\dot{H} + 2H^2),$$
$$T_0 = -6H^2,$$
$$Q_0 = 6H^2,$$

(14.2) (14.3) (14.4)

where $H = (\ln a)'$ is the Hubble parameter. In the metric-affine spacetime case, we assume that the Ricci, torsion and nonmetricity scalars take the forms

$$R = 6(\dot{H} + 2H^2) + u,$$
$$T = -6H^2 + v,$$
$$Q = 6H^2 + w.$$

(14.5) (14.6) (14.7)

Similarly, we can write the boundary term scalar ($B$) and the GB scalar ($G$) as [12]

$$G = G_0 + p,$$
$$B = B_0 + f,$$

(14.8) (14.9)

where $u, v, w, p, f$ are some real functions of $t, a, \dot{a}, \ddot{a}.$

15 Spherically symmetric and black hole solutions in metric-affine MG theories

Let us now present our idea to study, for example, the black hole solutions of metric-affine MG theories. For this aim, we consider the following static and spherically symmetric metric [12]

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)(d\theta^2 + \sin^2 \theta d\phi^2),$$

(15.1)
where $A(r), B(r)$ and $C(r)$ are real functions of the radial coordinate $r$. Consider two connections: the Levi-Civita connection and the Weitzenb"ock connection. First, let us consider the Levi-Civita connection case. In this case, the nonmetricity and torsion scalar $s$ are equal to zero that is $T_{0} = Q_{0} = 0$. Then the corresponding Ricci scalar has the form

$$R_{0} = \frac{A''}{AB} + \frac{2C''}{BC} + \frac{A'C'}{ABC} - \frac{A'^{2}}{2AB^{2}} - \frac{A'B'}{2AB^{2}} - \frac{B'C'}{B^{2}C} - \frac{2}{C}. \quad (15.2)$$

Here and below primes denote differentiation with respect to the radial coordinate $r$. Let us now consider the Weitzenb"ock connection case. In this case, the Ricci scalar and nonmetricity scalar are equal to zero that is $R_{0} = Q_{0} = 0$ and the torsion scalar is given by

$$T_{0} = \frac{C'(2AC + AC')}{2ABC^{2}} \quad (15.3)$$

Similarly, we can calculate the nonmetricity scalar $Q_{0}$. For the metric (11.1), it has the form

$$Q_{0} = -\frac{C'(2AC + AC')}{2ABC^{2}}, \quad (15.4)$$

where $R_{0} = T_{0} = 0$. The geometry of the MG theories is the metric-affine spacetime. For that reason, now let us consider the more general case, namely, the metric-affine spacetime. For this metric-affine spacetime, we have the metric-affine connection. In this metric-affine connection case, the Ricci scalar, the torsion scalar and the nonmetricity scalar take the forms

$$R = R_{0} + u, \quad (15.5)$$

$$T = T_{0} + v, \quad (15.6)$$

$$Q = Q_{0} + w. \quad (15.7)$$

Here the metric-affine contributions are given by the following functions [12]

$$u = u(A, B, C, A', B', C', A'', B'', C''), \quad (15.8)$$

$$v = v(A, B, C, A', B', C', A'', B'', C''), \quad (15.9)$$

$$w = w(A, B, C, A', B', C', A'', B'', C''). \quad (15.10)$$

They are some real functions of the metric tensor components $g_{ij}$ (11.1).

16 Metric-affine MG theories with boundary term scalars

Next, we very briefly mention the main moments of metric-affine MG theories with the boundary term scalars [12]. According our idea, we assume that the boundary term scalar has the form [12]

$$B = B_{0} + f. \quad (16.1)$$

Similarly, we can write the GB scalar for the metric-affine spacetime as

$$G = G_{0} + p. \quad (16.2)$$

In the last two equations, $p$ and $f$ are metric-affine contributions and some functions of $A, B, C$ and their derivatives.

17 Conclusion

As we mentioned in the introduction, GR has several generalizations like $F(R), F(T)$ and so on. Among these generalizations of GR, the metric-affine gravity theories have a nice feature by extending to admit not only curvature but both torsion and nonmetricity. This means the MAG is described by a pseudo - Riemannian geometry. The geometrical structure of the MAG can be studied once a metric tensor and a connection are given. In this way, we can calculate
the affine connection for the underlying theory. In this paper, we have considered the so-called generalized Myrzakulov gravity or MG-VIII which can be considered as the particular case of the MAG. To simplify the problem, we consider the FRW spacetime case in detail. For this case the point-like Lagrangian and Hamiltonian of the theory is derived. Using this Lagrangian and the Euler-Lagrangian equation, the gravitational equations of the MG-VIII is presented. For simplicity, the particular case of the MG-VIII when $F = \alpha R + \beta T + \mu Q + \nu T$ is investigated. For this particular case, the gravitational equations is considered in detail. For the quantum case, the corresponding Wheeler - DeWitt equation is presented. The relation with the soliton theory is shortly discussed. These results show that altogether one can say that some ingredients of the MG-VIII are present and work as expected, but some other aspects remain to be properly understood. These aspects of the MG-VIII certainly worth further investigation (see e.i. refs. [4]-[8]).

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