GENERATORS VERSUS PROJECTIVE GENERATORS IN ABELIAN CATEGORIES

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Abstract. Let \( \mathcal{A} \) be an essentially small abelian category. We prove that if \( \mathcal{A} \) admits a generator \( M \) with \( \text{End}_\mathcal{A}(M) \) right artinian, then \( \mathcal{A} \) admits a projective generator. If \( \mathcal{A} \) is further assumed to be Grothendieck, then this implies that \( \mathcal{A} \) is equivalent to a module category. When \( \mathcal{A} \) is Hom-finite over a field \( k \), the existence of a generator is the same as the existence of a projective generator, and in case there is such a generator, \( \mathcal{A} \) has to be equivalent to the category of finite dimensional right modules over a finite dimensional \( k \)-algebra. We also show that when \( \mathcal{A} \) is a length category, then there is a one-to-one correspondence between exact abelian extension closed subcategories of \( \mathcal{A} \) and collections of Hom-orthogonal Schur objects in \( \mathcal{A} \).

1. Introduction

Let \( \mathcal{A} \) be an abelian category. A natural and fundamental problem is to determine whether \( \mathcal{A} \) is equivalent to a category of modules over a ring. It is well known that this is true if and only if \( \mathcal{A} \) is co-complete and admits a compact projective generator, that is, an object \( P \in \mathcal{A} \) which generates \( \mathcal{A} \) (see below for the definition) and such that \( \text{Hom}_{\mathcal{A}}(P, -) \) is exact and commutes with arbitrary direct sums.

In this paper, we consider the notion of a generator of \( \mathcal{A} \). An object \( M \) of \( \mathcal{A} \) is a generator of \( \mathcal{A} \) if for any object \( X \) of \( \mathcal{A} \), we have an epimorphism \( \bigoplus_{i \in I} M \rightarrow X \) where \( I \) is some index set. A (minimal) generator needs not, a priori, be projective, since \( \mathcal{A} \) does not necessarily have enough projective objects. In the first section of this paper, we will see that when \( \mathcal{A} \) has a generator \( M \) with \( \text{End}_\mathcal{A}(M) \) right artinian, then \( \mathcal{A} \) also has a projective generator. In case \( \mathcal{A} \) is further assumed to be Grothendieck, then it has to be equivalent to a module category over a right artinian ring. In case \( \mathcal{A} \) is a length category or is Hom-finite over a field, then \( \mathcal{A} \) is equivalent to the module category of finitely generated modules over a right artinian ring.

When \( \mathcal{A} \) is a length category, it need not have a generator. However, \( \mathcal{A} \) has simple objects and these objects can be used to build all objects of \( \mathcal{A} \) by successive extensions. Moreover, this set of objects need not be finite. In the second section, we consider Hom-orthogonal sets of Schur objects (or bricks) in \( \mathcal{A} \) and prove that these are in bijection with the exact abelian extension-closed subcategories of \( \mathcal{A} \). Finally, in the third section, we apply our results in the hereditary case, where the exact abelian extension closed subcategories are the same as the thick subcategories.

The paper is self-contained and all proofs are elementary. The author has been informed by Henning Krause that some results of Section 1 can be derived by the

\footnote{The author is thankful to Henning Krause for pointing out Remark 2.13(1).}
well known Gabriel-Popescu theorem. An outline of Krause’s argument will be given in Remark 2.13(1).

2. Generators, Projective Generators and Length Categories

Throughout, the symbol \( A \) always stands for an abelian category which is essentially small. We start by recalling some finiteness conditions on the objects of \( A \) and the notion of generator.

An object \( M \in A \) is \textit{artinian} if any descending chain of subobjects of \( M \) becomes stationary. The category \( A \) is \textit{artinian} if all objects of \( A \) are artinian. Similarly, an object \( M \in A \) is \textit{noetherian} if any ascending chain of subobjects of \( M \) becomes stationary. The category \( A \) is \textit{noetherian} if all objects of \( A \) are noetherian. A non-zero object \( X \in A \) is called \textit{simple} or \textit{minimal} if it has no proper non-zero subobject. If \( A \) is both artinian and noetherian, then it is called a \textit{length category}.

A \textit{non-zero object} \( X \in A \) is \textit{of finite length} if it is both artinian and noetherian. Thus, for a finite length object \( X \), there is a finite chain

\[ 0 = X_n \subset X_{n-1} \subset \cdots \subset X_1 \subset X_0 = X \]

of subobjects of \( X \) such that the quotients \( X_{i-1}/X_i \) are simple for all \( 1 \leq i \leq n \). Such a chain is called a \textit{composition series} of \( X \) and the length \( n \) of this series is uniquely determined by \( X \) and called the \textit{length} of \( X \). This is known as (the categorical version of) the Jordan-Hölder theorem.

A \textit{generator} of \( A \) is an object \( M \) of \( A \) such that for any \( X \in A \), there is an epimorphism \( \bigoplus_{i \in I} \rightarrow X \) for some index set \( I \). We will see that \( A \) having a generator \( M \) with \( \text{End}_A(M) \) right artinian imposes many restrictions on \( A \). We start with the following lemma.

**Lemma 2.1.** If \( A \) admits a generator and is artinian, then \( A \) has finitely many non-isomorphic simple objects.

**Proof.** Assume that \( A \) is artinian and admits a generator \( M \). Assume to the contrary that \( A \) has infinitely many simple objects, up to isomorphism. Let \( M = M_1 \oplus \cdots \oplus M_n \) be a decomposition of \( M \) into indecomposable direct summands (which is guaranteed by \( A \) being artinian). Let \( S \) be a simple object in \( A \). Since \( S \) is simple, there exists \( 1 \leq i \leq n \) such that there is an epimorphism \( M_i \rightarrow S \). Therefore, we may assume that there is \( 1 \leq i \leq n \) such that \( M_i \) has infinitely many non-isomorphic simple quotients. Let \( \{S_j\}_{j \geq 1} \) be such an infinite collection of non-isomorphic simple quotients of \( M_i \). For each \( j \), let \( K_j \) denote the kernel of a projection \( M_i \rightarrow S_j \). Consider the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K_1 \overset{f_1}{\longrightarrow} M_i \overset{p_1}{\longrightarrow} S_1 \longrightarrow 0 \\
0 & \longrightarrow & K_j \overset{f_j}{\longrightarrow} M_i \overset{p_j}{\longrightarrow} S_j \longrightarrow 0 
\end{array}
\]

where \( j > 1 \). Assume \( p_1 f_1 = 0 \). Then there is a morphism \( g : K_1 \rightarrow K_1 \) with \( f_j g = f_1 \). Passing to the cokernels in the above diagram yields a non-zero morphism from \( S_1 \) to \( S_j \), a contradiction. Thus, there is an epimorphism \( K_1 \rightarrow S_j \) for all \( j > 1 \). Set \( M_{i,1} := K_1 \). Repeating this process, for any \( j \geq 1 \), there is a proper subobject
$M_{i,j+1}$ of $M_{i,j}$ with $\text{Hom}_A(M_{i,j+1}, S_p) \neq 0$ for all $p > j + 1$. Therefore, we get a descending chain $\cdots \subset M_{i,2} \subset M_{i,1} \subset M$ of proper inclusions, a contradiction. □

**Remarks 2.2.** (1) Observe that the fact that $A$ is artinian is crucial. For instance, the category of finitely generated modules over $k[x]$ where $k$ is a field has a generator but is not artinian. It has infinitely many non-isomorphic simple objects indexed by the irreducible polynomials.

(2) The fact that $A$ has a generator is also crucial. Let $Q$ be a quiver with infinitely many vertices and no arrow and let $A$ be the category of finite dimensional representations of $Q$ over a field $k$. Then $A$ is artinian but has no generator. It has infinitely many non-isomorphic simple objects.

**Proposition 2.3.** If $M$ has finite length then $M$ decomposes into a finite direct sum of indecomposable objects with local endomorphism rings. Moreover, $\text{End}_A(M)$ is semiperfect.

**Proof.** It is clear that if $M$ is of finite length, then $M$ decomposes into a finite direct sum $M = M_1 \oplus \cdots \oplus M_r$ of indecomposable objects. Any $M_i$ is again of finite length. By Fitting’s lemma, any endomorphism in $\text{End}_A(M_i)$ is an isomorphism or is nilpotent. Therefore, we get the first part of the statement. For the second part, we refer the reader to [5, Prop. 1.2] or [4, Cor. 4.4]. □

Recall that a full subcategory of $A$ is exact abelian if it is closed under taking kernels and cokernels in the ambient category $A$. We denote by $\text{fl}(A)$ the full subcategory of $A$ of those objects of finite length. This category is exact abelian and extension-closed.

**Lemma 2.4.** Assume that $A$ admits a generator $M$ such that $\text{End}_A(M)$ is right artinian. Then $M$ is of finite length.

**Proof.** Observe first that for any non-zero morphism $f : X \to Y$ in $A$, since $M$ is a generator, there exists a morphism $g : M \to X$ such that $fg \neq 0$. Therefore, we see that $\text{Hom}_A(M, -)$ is faithful. Assume that $M$ is not artinian. Let $\cdots \subset M_2 \subset M_1 \subset M_0$ be an infinite strictly descending chain of subobjects of $M$. Using the fact that $\text{Hom}_A(M, -)$ is left exact and faithful, we get an infinite strictly descending chain $\cdots \subset \text{Hom}_A(M, M_2) \subset \text{Hom}_A(M, M_1) \subset \text{Hom}_A(M, M_0)$ of right $\text{End}_A(M)$-submodules of $\text{End}_A(M)$. This contradicts the fact that $\text{End}_A(M)$ is right artinian. The proof of the fact that $M$ is noetherian is similar since by the Hopkins-Levitzki theorem, the ring $\text{End}_A(M)$ is also right noetherian. □

**Remark 2.5.** Note that if $M$ is of finite length, then $\text{End}(M)$ need not be right artinian (although, as we have shown, it has to be semiperfect). For instance, let $B$ be the category of right modules over a right artinian ring $R$ that is not left artinian. Note that $R_R$ has finite length in $B$. Consider the category $A = B^{\text{op}}$. Now, $R_R$ also has finite length in $A$ and its endomorphism ring is isomorphic to $R^{\text{op}}$, which is left artinian but not right artinian.

**Theorem 2.6.** Assume that $A$ admits a generator $M$ such that $\text{End}_A(M)$ is right artinian. Then both $\text{fl}(A)$ and $A$ have a projective generator, which is a direct summand of $M$. 

Proof. By Lemma 2.3, we know that $M$ is of finite length. By Lemma 2.1, we know that $\mathrm{fl}(\mathcal{A})$ has finitely many simple objects. Start with any simple object, say $S = E_0$. If $\operatorname{Ext}^1_{\mathcal{A}}(E_0, -)$ vanishes on all simple objects, then $E_0$ is projective in $\mathrm{fl}(\mathcal{A})$. So assume otherwise. Let $S_1$ be a simple object with $\operatorname{Ext}^1_{\mathcal{A}}(E_0, S_1) \neq 0$. There is a non-split extension

$$0 \to S_1 \to E_1 \xrightarrow{g_1} E_0 \to 0$$

where $E_1$ is indecomposable. Clearly, $E_1$ has a unique simple quotient $S$. In general, assume that $E_i$ for $i \geq 1$ has been constructed, is indecomposable and has a unique simple quotient $S$. If $\operatorname{Ext}^1_{\mathcal{A}}(E_i, -)$ vanishes on all simple objects of $\mathcal{A}$, then $E_i$ is projective in $\mathrm{fl}(\mathcal{A})$. If not, let $S_{i+1}$ be a simple object with $\operatorname{Ext}^1_{\mathcal{A}}(E_i, S_{i+1}) \neq 0$. Consider the non-split short exact sequence

$$0 \to S_{i+1} \to E_{i+1} \xrightarrow{g_{i+1}} E_i \to 0$$

Let $g : E_{i+1} \to S'$ be an epimorphism with $S'$ simple. Since the sequence is non-split, $g$ factors through $g_i$ and, by induction, $S \cong S'$ is the unique simple quotient of $E_i$. Hence $E_{i+1}$ is indecomposable and has a unique simple quotient $S$. Assume that no $E_i$ is projective in $\mathrm{fl}(\mathcal{A})$. Since $\mathcal{A}$ has a generator $M$ and all $E_i$ have a unique simple quotient $S$, there is an epimorphism from $M$ to $E_i$ for all $i \geq 0$. This is a contradiction since the $E_i$ have unbounded lengths and $M$ has finite length. Therefore, for each simple $S$, there is a projective object $P_S$ in $\mathrm{fl}(\mathcal{A})$ with an epimorphism $P_S \to S$. If $S_1, \ldots, S_n$ is a complete list of the non-isomorphic simple objects of $\mathrm{fl}(\mathcal{A})$, then $P := \bigoplus_{1 \leq i \leq n} P_{S_i}$ is a projective generator of $\mathrm{fl}(\mathcal{A})$.

Now, there is $m \geq 1$ with an epimorphism $M^m \to P$ which gives, by the projective property of $P$, that $P$ is a direct summand of $M^m$. This gives $M^m \cong P \oplus P'$. Now, it follows from Proposition 2.1 that finite length objects decompose into finite direct sums of objects having local endomorphism rings. Therefore, we may use the Krull-Remak-Schmidt theorem for the above decomposition. Since the $P_{S_i}$ are all non-isomorphic, we get that $P$ is a direct summand of $M$. Since there is $r \geq 1$ with an epimorphism $P^r \to M$, we see that $P$ is a generator of $\mathcal{A}$. It remains to prove that $P$ is projective in $\mathcal{A}$. Equivalently, we need to prove that for $S$ a simple object, any epimorphism $f : X \to P_S$ splits. Since $P$ is a generator, we have an epimorphism $h : \bigoplus_{i \in I} P \to X$. To prove that $f$ splits, we need to prove that $fh : \bigoplus_{i \in I} P \to P_S$ splits. Let $u : P_S \to S$ be an epimorphism and, for $i \in I$, let $(fh)_i : P \to P_S$ be the restriction of $fh$ to the corresponding summand. Observe that if $u(fh)_i = 0$ for all $i \in I$, then $u fh = 0$, which is impossible. Therefore, there is some $i_0 \in I$ with $u(fh)_{i_0} \neq 0$, which gives that $(fh)_{i_0}$ is an epimorphism in $\mathrm{fl}(\mathcal{A})$ and hence, splits. This proves that $fh$ splits.

Remarks 2.7. (1) Let $k$ be a field and consider $\mathcal{A}$ the category of finitely presented $k$-representations of the quiver $Q$ having two vertices and infinitely many arrows from one vertex to the other. Clearly, $\mathcal{A}$ is abelian with a generator $M$ but $\operatorname{End}(M)$ is not right artinian. Observe that $\mathrm{fl}(\mathcal{A})$ has no projective generator.

(2) If $\mathcal{A}$ is a Hom-finite $k$-category where $k$ is a field, then $\operatorname{End}(M)$ is always right artinian.

Recall that $\mathcal{A}$ is Grothendieck if (it is abelian and) it admits a generator, has arbitrary coproducts and filtered colimits of exact sequences are exact. The well
known Gabriel-Popescu theorem \cite{2} implies that any such category is a full subcategory of a module category.

**Proposition 2.8.** Assume that $\mathcal{A}$ is a Grothendieck category having a generator $M$ with $\text{End}_{\mathcal{A}}(M)$ right artinian. Then $\mathcal{A}$ is a module category over a right artinian ring.

**Proof.** We know that $M$ is of finite length by Lemma 2.4. It follows from Theorem 2.6 that $M$ has a projective direct summand $P$ which is also a generator. In order to prove the statement, it suffices to prove that $P$ is compact. Observe that for a short exact sequence

$$0 \to X \to Y \to Z \to 0,$$

if $X, Z$ are compact, then so is $Y$. Therefore, since $P$ is of finite length, it suffices to prove that any simple object $S$ is compact. Let $S$ be simple. Let $f : S \to \oplus_{i \in I} Z_i$ be a non-zero morphism and let $Z := \oplus_{i \in I} Z_i$. For each $i \in I$, let $q_i : Z_i \to Z$ be the canonical injection. For each finite subset $J$ of $I$, let $Z_J := \sum_{j \in J} q_j(Z_j)$. Observe that the $Z_J$ for $J$ finite form a directed system with inclusions. Moreover, we have $Z = \sum_{J \subseteq I \text{ finite}} Z_J$. Since $\mathcal{A}$ is Grothendieck, we have

$$\text{Im} f \cap \sum_{J \subseteq I \text{ finite}} Z_J = \sum_{J \subseteq I \text{ finite}} \text{Im} f \cap Z_J.$$

Since the latter is simple, at least one summand $\text{Im} f \cap Z_{J'}$ is non-zero and simple and has to be equal to $\text{Im} f$. Therefore, we have $\text{Im} f \subseteq Z_{J'}$ which proves that $f$ factors through $\oplus_{j \in J'} Z_j$.

Restricting to length categories, we get the following.

**Proposition 2.9.** Assume that $\mathcal{A}$ is a length category having a generator $M$ with $\text{End}_{\mathcal{A}}(M)$ right artinian. Then $\mathcal{A}$ is equivalent to the module category of the finitely generated right modules over a right artinian ring.

**Lemma 2.10.** Let $\mathcal{A}$ be a Hom-finite abelian $k$-category. If $\mathcal{A}$ has a generator, then $\mathcal{A}$ is a length category.

**Proof.** Let $M$ be a generator. Since $\mathcal{A}$ is Hom-finite, $\text{End}_{\mathcal{A}}(M)$ is a finite dimensional $k$-algebra and hence is (right) artinian. Since $M$ is a generator and the category is Hom-finite, any object is a quotient of a finite direct sum of copies of $M$. Thus, all objects are of finite length since $M$ is of finite length by Lemma 2.4.

**Theorem 2.11.** Let $\mathcal{A}$ be a Hom-finite abelian $k$-category. The following are equivalent.

1. $\mathcal{A}$ has a generator.
2. $\mathcal{A}$ has a projective generator.
3. $\mathcal{A}$ is equivalent to the category of finite dimensional modules over a finite dimensional $k$-algebra.

**Proof.** It is clear that (2) implies (1). By Theorem 2.6 (1) implies (2). Clearly, (3) implies (2). The fact that (2) implies (3) follows from Proposition 2.9 and Lemma 2.10 by observing that for $P$ a projective generator, $\text{End}_{\mathcal{A}}(P)$ is a finite dimensional $k$-algebra and that the finitely generated modules over $\text{End}_{\mathcal{A}}(P)$ are the finite dimensional ones.
Corollary 2.12. Let $\mathcal{A}$ be an exact abelian extension-closed subcategory of a Hom-finite abelian $k$-category. Assume that $\mathcal{A}$ has finitely many indecomposable objects, up to isomorphism. Then $\mathcal{A}$ is equivalent to a module category over a finite dimensional $k$-algebra.

Proof. It is clear that $\mathcal{A}$ has a generator $M$ by taking the direct sum of all non-isomorphic indecomposable objects. The result now follows from Theorem 2.11. □

Remarks 2.13. (1) Assume that $\mathcal{A}$ is a length category or is Hom-finite over a field. Assume that $\mathcal{A}$ has a generator $M$ with $R := \text{End}_A(M)$ right artinian. The ind completion $\text{ind}\mathcal{A}$ of $\mathcal{A}$ is a Grothendieck category. By the Gabriel-Popescu theorem, $\text{ind}\mathcal{A}$ is equivalent to a Serre quotient of the category $\text{Mod}R$ of right generated right $R$-modules. Thus, $\mathcal{A}$ is equivalent to a Serre quotient of the category $\text{mod}R$ of finitely generated right $R$-modules. Any Serre subcategory of $\text{mod}R$ is uniquely determined by a set of simple modules of $\text{mod}R$. Let $e$ be the idempotent corresponding to these simple modules. Then $\mathcal{A}$ is equivalent to $\text{mod}(1-e)R(1-e)$ and thus has a projective generator.

(2) Start instead with a Grothendieck category having a generator $M$ with $R := \text{End}_A(M)$ right artinian. Again, by the Gabriel-Popescu theorem, we have that $\mathcal{A}$ is equivalent to a Serre quotient of the category $\text{Mod}R$. However, it is not clear that such a quotient has to be again a module category. In general, a Grothendieck category need not be equivalent to a module category. For instance, take $B = k$ where $k$ is a field and let $S$ be the subcategory of $\text{Mod}k$ of all finite dimensional $k$-vector spaces. Then $S$ is a Serre subcategory of $\text{Mod}k$ and $\text{Mod}k/S$ is not a module category as it has no indecomposable object.

3. Exact abelian extension-closed subcategories

An object $X$ in $\mathcal{A}$ is called Schur if $\text{End}_A(X)$ is a division ring. Clearly, any Schur object is indecomposable and any simple object is Schur, by Schur’s lemma. Let $\mathcal{A}$ be a length category. In this section, we describe all exact abelian extension-closed subcategories of $\mathcal{A}$ in terms of their simple objects.

Two objects $X, Y \in \mathcal{A}$ are Hom-orthogonal provided

$$\text{Hom}_\mathcal{A}(X, Y) = 0 = \text{Hom}_\mathcal{A}(Y, X).$$

Given a set of objects $\mathcal{O}$ in $\mathcal{A}$, we let $C(\mathcal{O})$ denote the smallest exact abelian extension-closed subcategory of $\mathcal{A}$ containing the objects from $\mathcal{O}$. Let $\mathcal{S}$ be the set such that an element $S \in \mathcal{S}$ is a collection of non-isomorphic Schur objects that are pairwise Hom-orthogonal. If $T$ is an exact abelian extension-closed subcategory of $\mathcal{A}$, we let $S(T)$ denote a complete set of representatives of the simple objects in $T$. Clearly, $S(T) \in \mathcal{S}$. For $S_1, S_2 \in \mathcal{S}$, we set $S_1 = S_2$ if the elements can be pairwise identified by isomorphisms.

Proposition 3.1. Assume that $\mathcal{A}$ is a length category. Then $\mathcal{S}$ in $\mathcal{S}$ forms the non-isomorphic simple objects of $C(\mathcal{S})$.

Proof. Let $S \in \mathcal{S}$. We define a full subcategory $\mathcal{B}$ of $C(\mathcal{S})$ as follows. We declare that $S \subseteq B$ and $0 \in B$. If $X$ in $C(\mathcal{S})$ is the middle term of a short exact sequence

$$0 \rightarrow X' \rightarrow X \rightarrow S \rightarrow 0$$

then $X \in \mathcal{S}$.
with $X' \in \mathcal{B}$ and $S \in \mathcal{S}$, then we declare that $X \in \mathcal{B}$. We prove that $\mathcal{B} = \mathcal{C}(\mathcal{S})$, from which the result will follow. It is sufficient to prove that $\mathcal{B}$ is closed under kernels, cokernels, and extensions. Let $f : X \to Y$ be a non-zero morphism with $X, Y \in \mathcal{B}$. We prove by induction on $\ell(X) + \ell(Y)$ that the kernel $K$ of $f$ and the cokernel $C$ of $f$ lie in $\mathcal{B}$ (length is taken in $\mathcal{A}$). Consider the short exact sequences

$$0 \to X' \xrightarrow{u_X} X \xrightarrow{\nu_X} S_1 \to 0$$

and

$$0 \to Y' \xrightarrow{u_Y} Y \xrightarrow{\nu_Y} S_2 \to 0$$

where $S_1, S_2 \in \mathcal{S}$. Note that $\ell(X') \leq \ell(X) - 1$ and $\ell(Y') \leq \ell(Y) - 1$. Assume first that $\nu_Y f u_X \neq 0$. Consider the commutative diagram

$$\begin{array}{ccc}
0 & \xrightarrow{} & X' \xrightarrow{u_X} X \xrightarrow{\nu_X} S_1 & \xrightarrow{} & 0 \\
\phantom{0} & \downarrow{f u_X} & \phantom{0} & \downarrow{f} & \phantom{0} \\
0 & \xrightarrow{} & Y' \xrightarrow{u_Y} Y & \xrightarrow{} & 0 \\
\end{array}$$

Set $K'$ the kernel of $f u_X$ and $C'$ its cokernel. Since $f u_X$ is a non-zero morphism and $\ell(X') + \ell(Y) < \ell(X) + \ell(Y)$, by induction, $K', C'$ lie in $\mathcal{B}$. Assume as a first case that the induced morphism $h : S_1 \to C'$ is non-zero. Now, $C$ is the cokernel of $h$ where $\ell(S_1) + \ell(C') < \ell(X) + \ell(Y)$. Therefore, $C$ has to be in $\mathcal{B}$ by induction. Let $Z$ denote the kernel of $h$. Again, we know that $Z$ lies in $\mathcal{B}$. If $Z$ is not in $\mathcal{S}$ and is non-zero, then by definition of $\mathcal{B}$, there is a proper subobject $Z'$ of $Z$ which is in $\mathcal{B}$. Since the length of $Z$ is finite, we see that $Z$ has to have a proper subobject in $\mathcal{S}$, and hence that $S_1$ has to have a proper subobject in $\mathcal{S}$, which contradicts that $\mathcal{S}$ is Hom-orthogonal. Therefore, $Z = 0$ and $h$ is a monomorphism. Thus, $K \cong K' \in \mathcal{B}$. Assume now that $h = 0$. Then $K$ is an extension of $S_1$ by $K' \in \mathcal{B}$ so $K \in \mathcal{B}$ by definition. Similarly, we get $C \in \mathcal{B}$. So assume that $\nu_Y f u_X = 0$. We get a commutative diagram

$$\begin{array}{ccc}
0 & \xrightarrow{} & X' \xrightarrow{u_X} X \xrightarrow{\nu_X} S_1 & \xrightarrow{} & 0 \\
\phantom{0} & \downarrow{f'} & \phantom{0} & \downarrow{f} & \phantom{0} \\
0 & \xrightarrow{} & Y' \xrightarrow{u_Y} Y \xrightarrow{\nu_Y} S_2 & \xrightarrow{} & 0 \\
\end{array}$$

If $f''$ is non-zero, then it needs to be an isomorphism. Therefore, we have $K \cong K'$, $C \cong C'$ and, by induction, $K, C \in \mathcal{B}$. Otherwise, $f'' = 0$. Either $K \cong K' \in \mathcal{B}$ or else, $K$ is an extension of $S_1$ by $K' \in \mathcal{B}$ so $K \in \mathcal{B}$. Similarly, $C \in \mathcal{B}$. It remains to prove that $\mathcal{B}$ is closed under extensions. Consider a short exact sequence

$$0 \to U \to V \to W \to 0$$

where $U, W \in \mathcal{B}$. We prove by induction on the length of $V$ that $V \in \mathcal{B}$. If $W$ is in $\mathcal{S}$, then we are done. Otherwise, $W$ has a proper subobject $W'$ in $\mathcal{B}$ with corresponding quotient an object $S \in \mathcal{S}$. Consider the pullback $E$ of the inclusion $W' \to W$ and the morphism $V \to W$. We have a short exact sequence

$$0 \to U \to E \to W' \to 0.$$ 

Since $\ell(E) = \ell(U) + \ell(W') < \ell(U) + \ell(W) = \ell(V)$, by induction, we have that $E \in \mathcal{B}$. Now, the short exact sequence

$$0 \to E \to V \to S \to 0.$$
yields $V \in \mathcal{B}$.

The following result follows from the last proposition.

**Theorem 3.2.** Assume that $\mathcal{A}$ is an abelian length category. Then there is a one-to-one correspondence between $\mathcal{S}$ and the exact abelian extension-closed subcategories of $\mathcal{A}$. If $\mathcal{S} \in \mathcal{S}$, then $\mathcal{C}(\mathcal{S})$ is the corresponding exact abelian extension-closed subcategory. If $\mathcal{T}$ is exact abelian extension-closed, then $\mathcal{S}(\mathcal{T})$ is the corresponding element in $\mathcal{S}$.

A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is **thick** if it is closed under direct summands, under extensions, under kernels of epimorphisms and cokernels of monomorphisms. Clearly, if $\mathcal{B}$ is exact abelian extension-closed, then $\mathcal{B}$ is thick. The converse is not true. However, if $\mathcal{A}$ is hereditary, thick is equivalent to being exact abelian and extension-closed; see [3], for instance. Hence, we get the following.

**Theorem 3.3.** Assume that $\mathcal{A}$ is a hereditary abelian length category. Then there is a one-to-one correspondence between $\mathcal{S}$ and the thick subcategories of $\mathcal{A}$. If $\mathcal{S} \in \mathcal{S}$, then $\mathcal{C}(\mathcal{S})$ is the corresponding thick subcategory. If $\mathcal{T}$ is thick, then $\mathcal{S}(\mathcal{T})$ is the corresponding element in $\mathcal{S}$.

**Remark 3.4.** Note that the assumption of $\mathcal{A}$ being a length category is essential. If $\mathcal{A}$ is not artinian or not noetherian, then $\text{fl}(\mathcal{A})$ is exact abelian extension-closed and has the same simple objects at the ones of $\mathcal{A}$ but $\text{fl}(\mathcal{A}) \neq \mathcal{A}$. Therefore, the exact abelian extension-closed subcategories are determined by their simple objects if and only if $\mathcal{A}$ is a length category.

4. **Hereditary categories with generators**

Let $\mathcal{A}$ be a Hom-finite hereditary abelian $k$-category where $k$ is an algebraically closed field. If $\mathcal{A}$ has a generator, then by Theorem 2.11 we know that $\mathcal{A}$ is equivalent to the module category of a finite dimensional algebra. Since $\mathcal{A}$ is hereditary and $k = \bar{k}$, this yields $\mathcal{A} \cong \text{rep}(Q)$ for some finite acyclic quiver $Q$. On the other hand, if $Q$ is a finite acyclic quiver, then the category $\mathcal{A} := \text{rep}(Q)$ of finite dimensional representations of $Q$ is an hereditary abelian $k$-category and is a length category. Hence, all the results obtained so far apply.

Assume now that $Q$ is a finite acyclic quiver having $n$ vertices $\{1, 2, \ldots, n\}$. To each $M \in \text{rep}(Q)$, we can associate its dimension vector $d_M \in (\mathbb{Z}_{\geq 0})^n$ such that, for $1 \leq i \leq n$, the $i$-th entry of $d_M$ is the dimension over $k$ of $M(i)$. The dimension vector of a Schur object in $\text{rep}(Q)$ is called a Schur root. Schur roots are extensively studied in geometric representation theory. Let $d = (d_1, \ldots, d_n) \in (\mathbb{Z}_{\geq 0})^n$. Consider $\text{rep}(Q, d)$ the space of all representations $M$ with $M(i) = k^{d_i}$. We can consider the full subcategory $\mathcal{A}(d)$ of $\text{rep}(Q)$ with

$$\mathcal{A}(d) = \{ X \in \text{rep}(Q) \mid \text{Hom}(X, N) = 0 = \text{Ext}^1(X, N) \text{ for some } N \in \text{rep}(Q,d) \}.$$  

It is proven in [6] that this subcategory is thick and that it has a projective generator if and only if $d = dv$ for some $V$ with $\text{Ext}^1(V, V) = 0$. This subcategory has the feature that if $X$ is a simple object of it with $\text{Ext}^1(X, X) \neq 0$, then there are infinitely many non-isomorphic simple objects with dimension vector $d_X$ in $\mathcal{A}(d)$.

**Theorem 4.1.** The following are equivalent.

1. The category $\mathcal{A}(d)$ has a generator,
(2) The category $\mathcal{A}(d)$ has a projective generator,
(3) There is a finite acyclic quiver $Q'$ with $\mathcal{A}(d) \cong \text{rep}(Q')$,
(4) We have that $d$ is the dimension vector of some $V$ with $\text{Ext}^1(V, V) = 0$.

Proof. If $\mathcal{A}(d)$ is equivalent to a category of finite dimensional modules over a finite dimensional $k$-algebra $A$, then $A$ has to be hereditary. Therefore, $A \cong kQ'$ for some finite acyclic quiver $Q'$, meaning that $\mathcal{A}(d) \cong \text{rep}(Q')$. Thus, the equivalence of the first three statements follow from Theorem 2.11. The equivalence of (2) and (4) follows from [6]. □

Example 4.2. Let $Q$ be the Kronecker quiver, that is, the quiver with two vertices and two arrows pointing in the same direction. Let $d = (1, 1)$. The category $\mathcal{A}(d)$ is the full subcategory of regular representations of $Q$. The simple objects of $\mathcal{A}(d)$ are indexed by $\mathbb{P}^1(k)$ and all have dimension vector $(1, 1)$. It is not hard to check that if $V \in \text{rep}(Q, d)$, then $\text{Ext}^1(V, V) \neq 0$. It follows from the last theorem that $\mathcal{A}(d)$ has no generator and no projective generator. Any subset of $\mathbb{P}^1(k)$ will give rise to a thick subcategory of $\text{rep}(Q)$ contained in $\mathcal{A}(d)$. In this special example, the simple objects of $\mathcal{A}(d)$ are all the Schur objects of $\mathcal{A}(d)$. Therefore, the thick subcategories of $\mathcal{A}(d)$ are indexed by the subsets of $\mathbb{P}^1(k)$. Since any simple object in $\mathcal{A}(d)$ has a self-extension, the only thick subcategory of $\mathcal{A}(d)$ that has a generator is the trivial one coming from $\emptyset \subseteq \mathbb{P}^1(k)$.

Example 4.3. Let $Q = (Q_0, Q_1)$ be the infinite quiver as follows. Its underlying graph is a binary tree where all vertices but one, say $a$, have weight 3. We choose the orientation of $Q$ so that $Q$ has a unique source vertex $a$ and all vertices but $a$ have one incoming arrow and two outgoing arrows. We consider the category $\text{rep}^+(Q)$ of finitely presented representations of $Q$. This is a Hom-finite hereditary abelian $k$-category; see [1]. Consider the projective representation $P_0$ at $a$. We have $P_0(x) = k$ for all $x \in Q_0$ and $P_0(\alpha) = 1$ for all $\alpha \in Q_1$. Then $P$ is neither noetherian nor artinian. By Lemma 2.10, $\text{rep}^+(Q)$ has no generator and no projective generator. Note that $\text{rep}^+(Q)$ has enough projective objects, though.

Example 4.4. Let $Q$ be any infinite quiver and let $\mathcal{A} := \text{Rep}(Q)$ the category of all representations of $Q$. This is a Grothendieck abelian $k$-category (but not Hom-finite). It clearly has a projective generator $P$, however, $P$ is not compact. In fact, any projective generator is not compact. Thus, $\text{Rep}(Q)$ is not equivalent to a module category.

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