CONNECTEDNESS EXTENSIONS FOR ABELIAN VARIETIES

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1. Introduction

Suppose $A$ is an abelian variety defined over a field $F$, $\ell$ is a prime number, and $\ell \neq \text{char}(F)$. Let $F^s$ denote a separable closure of $F$, let $T_\ell(A) = \lim \leftarrow A_\ell^r$ (the Tate module), let $V_\ell(A) = T_\ell(A) \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell$, and let $\rho_{A,\ell}$ denote the $\ell$-adic representation

$$
\rho_{A,\ell} : \text{Gal}(F^s/F) \to \text{Aut}(T_\ell(A)) \subseteq \text{Aut}(V_\ell(A)).
$$

If $L$ is an extension of $F$ in $F^s$, let $G_{L,A}$ denote the image of $\text{Gal}(F^s/L)$ under $\rho_{A,\ell}$. Let $G_{F,A}$ denote the algebraic envelope of the image of $\rho_{A,\ell}$, i.e., the Zariski closure of $G_{L,A}$ in $\text{Aut}(V_\ell(A)) \cong \text{GL}_{2d}(\mathbb{Q}_\ell)$, where $d = \dim(A)$. Let $F_{\Phi,\ell}(A)$ be the smallest extension $F'$ of $F$ such that $G_{F',A}$ is connected. We call this extension the $\ell$-connectedness extension, or connectedness extension.

The algebraic group $G_{F,A}$ and the field $F_{\Phi,\ell}(A)$ were introduced by Serre ([15], [16], [17]), who proved that if $F$ is a global field or a finitely generated extension of $\mathbb{Q}$, then $F_{\Phi,\ell}(A)$ is independent of $\ell$ (see also [6], [7], [8]). In such cases, we will denote the field $F_{\Phi,\ell}(A)$ by $F_{\Phi}(A)$. For every integer $n \geq 3$ we have

$$
F_{\Phi}(A) \subseteq F(A_n)
$$

(see [3], [5], Proposition 3.6 of [2], and [23]). Larsen and Pink [8] recently proved that for every integer $n \geq 3$,

$$
F_{\Phi}(A) = \bigcap_{\text{prime } p \geq n} F(A_p).
$$

In [23] we found conditions for the connectedness of $G_{\ell}(F, A)$, while in [24] we used connectedness extensions and Serre’s $\ell$-independence results to obtain $\ell$-independence results for the intersection of $G_{\ell}(F, A)(\mathbb{Q}_\ell)$ with the torsion subgroup of the center of $\text{End}(A) \otimes \mathbb{Q}$.

Let $F(\text{End}(A))$ denote the smallest extension of $F$ over which all the endomorphisms of $A$ are defined. Then (see Proposition 2.10 of [24]),

$$
F(\text{End}(A)) \subseteq F_{\Phi,\ell}(A).
$$

Therefore, $G_{\ell}(F, A)$ fails to be connected when the ground field is not a field of definition for the endomorphisms of $A$. For example, if $F$ is a subfield of $\mathbb{C}$, and $A$ is an elliptic curve over $F$ with complex multiplication by an imaginary quadratic field $K$ which is not contained in $F$, then $F \neq KF = F(\text{End}(A)) \subseteq F_{\Phi,\ell}(A)$. More generally, if $A$ is an abelian variety of CM-type, and $\tilde{K}$ is the reflex CM-field, then $F(\text{End}(A)) \supseteq \tilde{K}$; if $\tilde{K}$ is not contained in $F$ then $F \neq F(\text{End}(A)) \subseteq F_{\Phi,\ell}(A)$. It is

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therefore natural to enlarge the ground field \( F \) so that it is a field of definition for the endomorphisms of \( A \).

By enlarging the ground field, we may assume that \( F = F(\text{End}(A)) = F_{\kappa, \ell}(A) \). We then consider the \( F \)-forms \( B \) of \( A \) such that \( F = F(\text{End}(B)) \). For such \( B \), we describe the connectedness extensions \( F_{\kappa, \ell}(B)/F \) (see [23] especially Theorem 3.1 and Corollary 3.2). Properties of Mumford-Tate groups given in [23] allow us to obtain explicit information about the connectedness extensions \( F_{\kappa, \ell}(B)/F \) under additional conditions (see Theorems 3.4 and 3.5). Our conditions in Theorems 3.4 and 3.5 are based on Weil’s philosophy in [25] whereby exceptional Hodge classes arise from certain abelian varieties that have a CM-field embedded in their endomorphism algebras. In §4 we use the results of §3 to explicitly compute non-trivial connectedness extensions in special cases.

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2. Definitions, notation, and lemmas

Let \( \mathbb{Z} \), \( \mathbb{Q} \), and \( \mathbb{C} \) denote respectively the integers, rational numbers, and complex numbers. If \( r \) is an integer, then \( \mathbb{Q}(r) \) denotes the rational Hodge structure of weight \(-2r\) on \( \mathbb{Q} \) (see §1 of [4]). If \( a \) and \( b \) are integers, let \( (a, b) \) denote the greatest common divisor of \( a \) and \( b \). If \( F \) is a field, let \( F^s \) denote a separable closure of \( F \) and let \( \bar{F} \) denote an algebraic closure of \( F \). If \( A \) is an abelian variety over a field \( F \), write \( \text{End}_F(A) \) for the set of endomorphisms of \( A \) which are defined over \( F \), let \( \text{End}(A) = \text{End}_F(A) \), and let \( \text{End}^0(A) = \text{End}(A) \otimes \mathbb{Z} \mathbb{Q} \). Let \( Z_A \) denote the center of \( \text{End}(A) \). If \( G \) is an algebraic group, let \( G^0 \) denote the identity connected component.

**Lemma 2.1** (Lemma 2.7 of [23]). If \( A \) is an abelian variety over a field \( F \), \( L \) is a finite extension of \( F \) in \( F^s \), and \( \ell \) is a prime number, then

\[
\mathfrak{G}_\ell(L, A) \subseteq \mathfrak{G}_\ell(F, A) \text{ and } \mathfrak{G}_\ell(L, A)^0 = \mathfrak{G}_\ell(F, A)^0.
\]

In particular, if \( \mathfrak{G}_\ell(F, A) \) is connected, then \( \mathfrak{G}_\ell(F, A) = \mathfrak{G}_\ell(L, A) \).

**Lemma 2.2.** Suppose \( A \) and \( B \) are abelian varieties over a field \( F \), \( L \) is a finite extension of \( F \) in \( F^s \), \( \ell \) is a prime number, \( \ell \neq \text{char}(F) \), \( \mathfrak{G}_\ell(F, A) \) is connected, and \( A \) and \( B \) are isomorphic over \( L \). Then:

(i) \( \mathfrak{G}_\ell(F, B)^0 = \mathfrak{G}_\ell(F, A) \), and

(ii) \( \mathfrak{G}_\ell(L, B) \) is connected, i.e., \( F_{\kappa, \ell}(B) \subseteq L \).

**Proof.** Since \( A \) and \( B \) are isomorphic over \( L \), and \( \mathfrak{G}_\ell(F, A) \) is connected, we have

\[
\mathfrak{G}_\ell(L, B) = \mathfrak{G}_\ell(L, A) = \mathfrak{G}_\ell(F, A) = \mathfrak{G}_\ell(F, A)^0
\]

\[
= \mathfrak{G}_\ell(L, A)^0 = \mathfrak{G}_\ell(L, B)^0 = \mathfrak{G}_\ell(F, B)^0,
\]

using Lemma 2.1. The result follows. \( \square \)

**Proposition 2.3.** Suppose \( A \) and \( B \) are abelian varieties over a field \( F \), \( L \) is a field extension of \( F \) in \( F^s \), and \( f : A \to B \) is an isomorphism defined over \( L \). Suppose that for every \( \sigma \in \text{Gal}(F^s/F) \), the element \( f^{-1} \sigma(f) \) of \( \text{Aut}(A) \) commutes with every element of \( \text{End}_L(A) \). Then \( \text{End}_F(A) \cong \text{End}_F(B) \).
Proof. Define an isomorphism \( \varphi : \text{End}_L(A) \to \text{End}_L(B) \) by \( \varphi(\beta) = f \beta f^{-1} \). For every \( \beta \in \text{End}_L(A) \) and \( \sigma \in \text{Gal}(F^s/F) \), we have \( f^{-1} \sigma(f) \beta = \beta f^{-1} \sigma(f) \). Therefore, \( \sigma(f^{-1} \beta f^{-1}) = f \beta f^{-1} \). Thus, \( \beta \in \text{End}_F(A) \) if and only if \( f \beta f^{-1} \in \text{End}_F(B) \). In other words, the restriction of \( \varphi \) to \( \text{End}_F(A) \) induces an isomorphism onto \( \text{End}_F(B) \).

As a corollary we have the following result. See also Lemma 5.1 of [22].

**Corollary 2.4.** Suppose \( A \) is an abelian variety over a field \( F \). If an element of \( H^1(\text{Gal}(F^s/F), \text{Aut}(A)) \) is represented by a cocycle \( c \) with values in the center of \( \text{End}^0(A) \), and \( B \) is the twist of \( A \) by \( c \), then \( \text{End}_F(A) \cong \text{End}_F(B) \).

**Proof.** The cocycle \( c \) defines an isomorphism \( f : A \to B \) such that for every \( \sigma \in \text{Gal}(F^s/F) \), \( f^{-1} \sigma(f) = c(\sigma) \). We apply Proposition 2.3.

**Lemma 2.5.** Suppose \( A \) is an abelian variety over a field \( F \), \( c \) is a cocycle on \( \text{Gal}(F^s/F) \) with values in \( \text{Aut}(A) \), \( B \) is the twist of \( A \) by \( c \), and \( F = F(\text{End}(A)) = F(\text{End}(B)) \). Then \( c \) is a character with values in \( Z^\times_A \), where \( Z_A \) denotes the center of \( \text{End}(A) \).

**Proof.** Since \( \text{Gal}(F^s/F) \) acts trivially on \( \text{End}(A) \), the cocycle \( c \) is a homomorphism. Let \( f : A \to B \) be the isomorphism induced by \( c \). Then \( c(\sigma) = f^{-1} \sigma(f) \) for every \( \sigma \in \text{Gal}(F^s/F) \). Since \( F = F(\text{End}(A)) = F(\text{End}(B)) \), it easily follows that \( c(\sigma) \in Z_A \) and \( c(\sigma)^{-1} \in Z_A \).

**Remark 2.6.** If an abelian variety \( B \) over \( F \) is the twist of an abelian variety \( A \) by \( c \in H^1(\text{Gal}(F^s/F), \text{Aut}(A)) \) then one may easily check that the Galois module \( B(F^s) \) is the twist by \( c \) of the Galois module \( A(F^s) \), and therefore the Galois module \( V_\ell(B) \) is the twist by \( c \) of the Galois module \( V_\ell(A) \).

We define the Mumford-Tate group of a complex abelian variety \( A \) (see §2 of [13] or §6 of [26]). If \( A \) is a complex abelian variety, let \( V = H_1(A(\mathbb{C}), \mathbb{Q}) \) and consider the Hodge decomposition \( V \otimes \mathbb{C} = H_1(A(\mathbb{C}), \mathbb{C}) = H^{-1,0} \oplus H^{0,-1} \). Define a homomorphism \( \mu : \mathbb{G}_m \to GL(V) \) as follows. For \( z \in \mathbb{C} \), let \( \mu(z) \) be the automorphism of \( V \otimes \mathbb{C} \) which is multiplication by \( z \) on \( H^{-1,0} \) and is the identity on \( H^{0,-1} \).

**Definition 2.7.** The Mumford-Tate group \( MT_A \) of \( A \) is the smallest algebraic subgroup of \( GL(V) \), defined over \( \mathbb{Q} \), which after extension of scalars to \( \mathbb{C} \) contains the image of \( \mu \).

It follows from the definition that \( MT_A \) is connected.

Define a homomorphism \( \varphi : \mathbb{G}_m \times \mathbb{G}_m \to GL(V) \) as follows. For \( z, w \in \mathbb{C} \), let \( \varphi(z, w) \) be the automorphism of \( V \otimes \mathbb{C} \) which is multiplication by \( z \) on \( H^{-1,0} \) and is multiplication by \( w \) on \( H^{0,-1} \). Then \( MT_A \) can also be defined as the smallest algebraic subgroup of \( GL(V) \), defined over \( \mathbb{Q} \), which after extension of scalars to \( \mathbb{C} \) contains the image of \( \varphi \). The equivalence of the definitions follows easily from the fact that \( H^{-1,0} \) is the complex conjugate of \( H^{0,-1} \). (See §3 of [13], where \( MT_A \) is called the Hodge group. See also §6 of [26].)

If \( A \) is an abelian variety over a subfield \( F \) of \( \mathbb{C} \), we fix an embedding of \( \bar{F} \) in \( \mathbb{C} \). This gives an identification of \( V_\ell(A) \) with \( H_1(A, \mathbb{Q}) \otimes \mathbb{Q}_\ell \), and allows us to view \( MT_A \times \mathbb{Q}_\ell \) as a linear \( \mathbb{Q}_\ell \)-algebraic subgroup of \( GL(V_\ell(A)) \). Let

\[
MT_{A,\ell} = MT_A \times \mathbb{Q}_\ell.
\]
Proof in the literature, we have included one for the benefit of the reader.

\[ \tilde{\rho} \text{ induces an isomorphism from } \]

structure of weight \( q \) \( E \) \( MT \) algebraic subgroup of \( GL \) \( C \) subfield of \( E \) Lemma 2.10. We will denote this \( p \) polarized Hodge structure of weight \( E \) \( \nu \). If \( \nu \) denotes the cyclotomic character. If \( r \) \( \tilde{\rho} \text{ Definition 2.11.} \]

\[ \begin{align*}
\text{Let } \tilde{\rho} & : \text{Gal}(F^s/F) \to Z_{\ell}^\times \subset Q_{\ell}^\times \\
\text{denote the cyclotomic character. If } r & \text{ is an integer, then the Gal}(F^s/F)\text{-module } \]

\[ Q_{\ell}(r) \text{ is the Gal}(F^s/F)\text{-module } \]

\[ Q_{\ell}(r) = Q(r) \otimes Q_{\ell} \text{ we have } \]

\[ Q(r) \otimes Q_{\ell} \text{ (see } \S 1 \text{ of } [5]) \]. Suppose \( A \) is an abelian variety over \( F \). Let \( V_\ell = V_\ell(A) \) and let \( V_\ell^* \) be the dual of \( V_\ell \). If \( \nu \in G_m \), let \( \nu \) act on \( Q(1) \) as \( \nu^{-1} \), and we obtain a canonical action of \( GL(V) \times G_m \) on \( T \). (Note that \( V^* \cong V \otimes Q(1) \), since \( V \) is a polarized Hodge structure of weight \(-1\).)

**Definition 2.9.** The group \( \tilde{MT}_A \) is the subgroup of \( GL(V) \times G_m \) consisting of the elements which fix all rational tensors of bidegree \((0,0)\) belonging to any \( T \).

**Lemma 2.10** (Proposition 3.4 of [5]). The algebraic group \( \tilde{MT}_A \) is the smallest algebraic subgroup of \( GL(V) \times G_m \) defined over \( Q \) which, after extension of scalars to \( C \), contains the image of \( (\mu, id) : G_m \to GL(V) \times G_m \).

If \( F \) is a field and \( \ell \) is a prime number different from \( \text{char}(F) \), let

\[ \chi_{\ell} : \text{Gal}(F^s/F) \to Z_{\ell}^\times \subset Q_{\ell}^\times \]

denote the cyclotomic character. If \( r \) is an integer, then the \( \text{Gal}(F^s/F) \text{-module } Q_{\ell}(r) \) is the \( Q_{\ell}\text{-vector space } Q_{\ell} \) with Galois action defined by the character \( \chi_{\ell}^r \).

We have \( Q_{\ell}(r) = Q(r) \otimes Q_{\ell} \) (see \( \S 1 \) of [5]). Suppose \( A \) is an abelian variety over \( F \). Let \( V_\ell = V_\ell(A) \) and let \( V_\ell^* \) be the dual of \( V_\ell \). If \( \nu \in G_m \), let \( \nu \) act on \( Q_{\ell}(1) \) as \( \nu^{-1} \). We obtain a canonical action of \( GL(V_\ell) \times G_m \) on \( V_\ell^{\otimes p} \otimes (V_\ell^*)^{\otimes q} \otimes Q_{\ell}(r) \).

Define

\[ \tilde{\rho}_{A,\ell} : \text{Gal}(F^s/F) \to Aut(V_\ell) \times Q_{\ell}^\times = Aut(V_\ell) \times G_m(Q_{\ell}) \]

by \( \tilde{\rho}_{A,\ell}(\sigma) = (\rho_{A,\ell}(\sigma), \chi_{\ell}^{-1}(\sigma)) \).

**Definition 2.11.** Let \( \tilde{\Theta}_{\ell}(F,A) \) denote the smallest \( Q_{\ell}\text{-algebraic subgroup of } \)

\[ GL(V_\ell) \times G_m \]

whose group of \( Q_{\ell}\text{-points contains the image of } \tilde{\rho}_{A,\ell} \).

If \( A \) is a complex abelian variety, then a polarization on \( A \) (i.e., the imaginary part of a Riemann form) produces an element \( E \) of \( \text{Hom}(\wedge^2 V, Q(1)) \) which is a rational tensor of bidegree \((0,0)\). If \( A \) is an abelian variety over an arbitrary field \( F \), then a polarization on \( A \) defined over \( F \) defines a \( \text{Gal}(F^s/F) \text{-invariant element } E_\ell \text{ of } \text{Hom}(\wedge^2 V_\ell, Q_{\ell}(1)) \) (since the Weil pairing is \( \text{Gal}(F^s/F) \text{-equivariant}). If \( F \) is a subfield of \( C \), and we fix a polarization on \( A \) defined over \( F \), then the line generated by \( E_\ell \) in \( \text{Hom}(\wedge^2 V_\ell, Q_{\ell}(1)) \) is the extension of scalars to \( Q_{\ell} \) of the line generated by \( E \) in \( \text{Hom}(\wedge^2 V, Q(1)) \). (See p. 237 of [11], especially the last sentence.)

The following result implies that the projection map \( GL(V) \times G_m \to GL(V) \) induces an isomorphism from \( \tilde{MT}_A \) onto \( MT_A \). Since we were not able to find a proof in the literature, we have included one for the benefit of the reader.
**Proposition 2.12.** If $A$ is a complex abelian variety, then there exists a (unique) character $\gamma : MT_A \to G_m$ such that $\tilde{MT}_A$ is the graph of $\gamma$.

**Proof.** Let $p_1$ and $p_2$ denote the projection maps from $GL(V) \times G_m$ onto $GL(V)$ and $G_m$, respectively. By Lemma 2.10, $MT_A$ is the image of $\tilde{MT}_A$ under $p_1$. Fix a polarization on $A$. The polarization generates a line $D$ in the $Q$-vector space $\text{Hom}(\wedge^2 V, Q(1))$, on which $\tilde{MT}_A$ acts trivially. Let $D(-1) = D \otimes Q(-1)$, a line in $\text{Hom}(\wedge^2 V, Q)$. Since $\tilde{MT}_A$ acts trivially on $D$, $\tilde{MT}_A$ acts on $D(-1)$ via $p_2$. Let $B = \{\alpha \in GL(V) : \alpha D(-1) \subseteq D(-1)\}$ and let the character $\gamma : B \to \text{Aut}(D(-1)) = G_m$ be induced by the action of $GL(V)$ on $\text{Hom}(\wedge^2 V, Q)$. The action of $GL(V) \times G_m$ on $\text{Hom}(\wedge^2 V, Q)$ factors through $GL(V)$. Therefore $MT_A \subseteq B$, and we have a commutative diagram

$$
\begin{array}{ccc}
\tilde{MT}_A & \xrightarrow{p_1} & MT_A \\
p_2 \downarrow & & \downarrow \\
G_m
\end{array}
$$

which gives the desired result. \qed

**Proposition 2.13.** If $A$ is an abelian variety over a field $F$, $\ell$ is a prime number, and $\ell \neq \text{char}(F)$, then there exists a (unique) character $\gamma_\ell : \tilde{G}_\ell(F, A) \to G_m$ such that

(i) $\tilde{G}_\ell(F, A)$ is the graph of $\gamma_\ell$,

(ii) the restriction of $\gamma_\ell$ to $G_{F,A}$ is $\chi_\ell^{-1}$,

(iii) if $\tilde{F}$ is a subfield of $C$, then $\gamma_\ell = \gamma$ on $MT_{A,\ell} \cap \tilde{G}_\ell(F, A)$.

**Proof.** Let $\pi_1$ and $\pi_2$ denote the projection maps from $GL(V_\ell) \times G_m$ onto $GL(V_\ell)$ and $G_m$, respectively. By the definitions, $\tilde{G}_\ell(F, A)$ is the image of $\tilde{G}_\ell(F, A)$ under $\pi_1$. Fix a polarization on $A$ defined over $F$. The polarization generates a line $D_\ell$ in the $Q_\ell$-vector space $\text{Hom}(\wedge^2 V_\ell, Q_\ell(1))$. Let $D_\ell(-1) = D_\ell \otimes Q_\ell(-1)$, a line in $\text{Hom}(\wedge^2 V_\ell, Q_\ell)$. Since the Weil pairing is $\text{Gal}(F^s/F)$-equivariant, $\text{Gal}(F^s/F)$ acts trivially on $D_\ell$. Therefore $\tilde{G}_\ell(F, A)$ acts trivially on $D_\ell$, and acts via $\pi_2$ on $D_\ell(-1)$. Let $B_\ell = \{\alpha \in GL(V_\ell) : \alpha D_\ell(-1) \subseteq D_\ell(-1)\}$ and let the character $\gamma_\ell : B_\ell \to \text{Aut}(D_\ell(-1)) = G_m$ be induced by the action of $GL(V_\ell)$ on $\text{Hom}(\wedge^2 V_\ell, Q_\ell)$. The action of $GL(V_\ell) \times G_m$ on $\text{Hom}(\wedge^2 V_\ell, Q_\ell)$ factors through the action of $GL(V_\ell)$. Therefore $\tilde{G}_\ell(F, A) \subseteq B_\ell$, and we have a commutative diagram

$$
\begin{array}{ccc}
\tilde{G}_\ell(F, A) & \xrightarrow{\pi_1} & G_\ell(F, A) \\
\pi_2 \downarrow & & \downarrow \gamma_\ell \\
G_m
\end{array}
$$

which gives (i). Since the restriction of $\pi_2$ to $G_{F,A}$ is $\chi_\ell^{-1}$, we have (ii). Now suppose $\tilde{F}$ is a subfield of $C$. Using the fixed polarization, define $D$, $D(-1)$, $B$, and $\gamma$ as in the proof of Theorem 2.12. Then $B_\ell = B \times Q_\ell$, and therefore $MT_{A,\ell} \subseteq B_\ell$. Since $\gamma$
(respectively, \(\gamma\)) is induced by the action of \(GL(V)\) on \(\text{Hom}(\wedge^2 V, \mathbb{Q})\) (respectively, \(GL(V)\) on \(\text{Hom}(\wedge^2 V, \mathbb{Q}_\ell)\)), and \(V_\ell = V \otimes \mathbb{Q}_\ell\), we have (iii).

Write \(\tilde{MT}_{A,\ell}\) for the \(\mathbb{Q}_\ell\)-algebraic subgroup \(\tilde{MT}_{A} \times \mathbb{Q}_\ell\) of \(GL(V_\ell) \times \mathbb{G}_m\). Then \(\tilde{MT}_{A}(\mathbb{Q}_\ell) = \tilde{MT}_{A,\ell}(\mathbb{Q}_\ell)\). We state a reformulation of Theorem 2.8, which we will use in \(\S\) 3.

**Theorem 2.14.** If \(A\) is an abelian variety over a finitely generated extension \(F\) of \(\mathbb{Q}\), then \(\tilde{G}_{\ell}(F,A)^0 \subseteq \tilde{MT}_{A,\ell}\).

**Proof.** The result follows directly from Theorem 2.8 and Propositions 2.12 and 2.13.

3. Connectedness extensions

**Theorem 3.1.** Suppose \(A\) is an abelian variety over a field \(F\), \(\ell\) is a prime number not equal to \(\text{char}(F)\), \(c : \text{Gal}(F^s/F) \to \text{Aut}_F(A) \subseteq \text{Aut}(V_\ell(A))\) is a homomorphism, \(B\) is the twist of \(A\) by the cocycle determined by \(c\), and \(F = F(\text{End}(A)) = F_{\Phi,\ell}(A)\).

Then:

(i) \(c\) induces an isomorphism \(\text{Gal}(F_{\Phi,\ell}(B)/F) \cong \text{Im}(c)/\text{Im}(c) \cap \tilde{G}_{\ell}(F,A)(\mathbb{Q}_\ell))\),

(ii) \(G_{\ell}(F,B)\) is connected if and only if \(\text{Im}(c) \subseteq \tilde{G}_{\ell}(F,A)(\mathbb{Q}_\ell))\),

(iii) if \(M\) is the abelian extension of \(F\) in \(F^s\) cut out by \(c\), then \(c\) induces an isomorphism \(\text{Gal}(M/F_{\Phi,\ell}(B)) \cong \text{Im}(c) \cap \tilde{G}_{\ell}(F,A)(\mathbb{Q}_\ell))\).

**Proof.** By Lemma 2.3, \(F_{\Phi,\ell}(B) \subseteq M\). The character \(c\) induces isomorphisms \(\text{Gal}(M/F) \cong \text{Im}(c)\) and \(\text{Gal}(M/F_{\Phi,\ell}(B)) \cong \text{Im}(c) \cap \tilde{G}_{\ell}(F,B)^0(\mathbb{Q}_\ell))\).

By Lemma 2.2, we have \(\tilde{G}_{\ell}(F,B)^0 \cong \tilde{G}_{\ell}(F,A)\), and the result follows.

**Corollary 3.2.** Suppose \(A\) is an abelian variety over a field \(F\), \(\ell\) is a prime number not equal to \(\text{char}(F)\), \(B\) is the twist of \(A\) by a cocycle \(c : \text{Gal}(F^s/F) \to \text{Aut}(A) \subseteq \text{Aut}(V_\ell(A))\), and \(F = F(\text{End}(A)) = F_{\Phi,\ell}(A) = F(\text{End}(B))\).

Then:

(i) \(c\) is a character with values in \(Z_A^X\) (where \(Z_A\) denotes the center of \(\text{End}(A)\)),

(ii) \(c\) induces an isomorphism \(\text{Gal}(F_{\Phi,\ell}(B)/F) \cong \text{Im}(c)/\text{Im}(c) \cap \tilde{G}_{\ell}(F,A)(\mathbb{Q}_\ell))\),

(iii) \(G_{\ell}(F,B)\) is connected if and only if \(\text{Im}(c) \subseteq \tilde{G}_{\ell}(F,A)(\mathbb{Q}_\ell))\).
Proof. By Lemma 2.5 and the assumption that $\sigma$ is an embedding of $k$ into $C$, let

\[ n_\sigma = \dim_C \{ t \in \text{Lie}(A) \otimes_F C : \iota(\alpha)t = \sigma(\alpha)t \text{ for all } \alpha \in k \}. \]

Write $\bar{k}$ for the composition of $\sigma$ with the involution complex conjugation of $k$.

**Definition 3.3.** If $A$ is an abelian variety over an algebraically closed field $C$ of characteristic zero, $k$ is a CM-field, and $\iota : k \hookrightarrow \text{End}_F^0(A)$ is an embedding, we say $(A,k,\iota)$ is of Weil type if $n_\sigma = n_\bar{k}$ for all embeddings $\sigma$ of $k$ into $C$.

Although we do not use this fact, we remark that $(A,k,\iota)$ is of Weil type if and only if $\iota$ makes $\text{Lie}(A) \otimes_F C$ into a free $k \otimes \mathbb{Q}C$-module (see p. 525 of [13] for the case where $k$ is an imaginary quadratic field). Using the semisimplicity of the $F$-algebra $k \otimes \mathbb{Q}F$ and the $C$-algebra $k \otimes \mathbb{Q}C$, one may easily deduce that $\iota$ makes $\text{Lie}(A) \otimes_F C$ into a free $k \otimes \mathbb{Q}C$-module if and only if $\iota$ makes $\text{Lie}(A)$ into a free $k \otimes \mathbb{Q}F$-module.

Suppose $(A,k,\iota)$ is of Weil type, and we have an element of

\[ H^1(\text{Gal}(\bar{F}/F), \text{Aut}(A)) \]

which is represented by a cocycle $c$ with values in the center of $\text{End}_F^0(A)$. Let $B$ be the twist of $A$ by $\iota$, and let $\varphi$ be the isomorphism from $\text{End}_F(A)$ to $\text{End}_F(B)$ obtained in Corollary 2.3 and Proposition 2.3. Since $(A,\iota)$ and $(B,\varphi \circ \iota)$ are isomorphic over $C$, it follows that $(B,k,\varphi \circ \iota)$ is of Weil type.

Note that if $(A,k,\iota)$ is of Weil type, then $\dim(A)$ is divisible by $[k : \mathbb{Q}]$.

**Theorem 3.4.** Suppose $A$ is an abelian variety over a finitely generated extension $F$ of $\mathbb{Q}$, $\ell$ is a prime number, $k$ is a CM-field, and $\iota : k \hookrightarrow \text{End}_F^0(A)$ is an embedding into the center of $\text{End}_F^0(A)$ such that $(A,k,\iota)$ is of Weil type, $c : \text{Gal}(\bar{F}/F) \to k^\times$ is a character of finite order $n$, $r = 2\dim(A)/[k : \mathbb{Q}] \in \mathbb{Z}$, $M$ is the $\mathbb{Z}/n\mathbb{Z}$-extension of $F$ cut out by $c$, and $B$ is the twist of $A$ by $c$. Suppose $F = \text{End}(A)$, $\iota \circ c$ takes values in $\text{Aut}(A)$, $r$ is even, and $n$ does not divide $r$. Then

(i) $F = \text{End}(B)$,
(ii) either $F \neq F_\varphi(A)$ or $F \neq F_\varphi(B)$,
(iii) if $F_\varphi(A) = F$, then $F_\varphi(B) \subseteq M$ and $[M : F_\varphi(B)]$ divides $n, 2r$,
(iv) if $F_\varphi(A) = F$ and $(n, 2r) = 2$, then $[M : F_\varphi(B)] = 2$.

Proof. The Galois module $V_\ell(B)$ is the twist of $V_\ell(A)$ by $c$ (see Remark 2.6). By applying Corollary 2.4 to the cocycle induced by $c$, we deduce (i) and we obtain an isomorphism $\varphi$ from $\text{End}_F(A)$ onto $\text{End}_F(B)$ such that $(B,k,\varphi \circ \iota)$ is of Weil type.

Let $k_\ell = k \otimes \mathbb{Q}_\ell$. For $U = A$ or $B$, let

\[ W_U = \text{Hom}_{\mathbb{Q}}(\wedge^k H_1(U, \mathbb{Q}), \mathbb{Q}_\ell(\frac{r}{2})), \quad W_{U,\ell} = \text{Hom}_{\mathbb{Q}_\ell}(\wedge^{k_\ell} V_\ell(U), \mathbb{Q}_\ell(\frac{r}{2})), \]

where $\text{Hom}_E$ means homomorphisms of $E$-vector spaces, if $E$ is a field. Then $W_U$ is a one-dimensional $k$-vector space and $W_{U,\ell}$ is a free rank-one $k_\ell$-module. The
elements of $W_U$ are called Weil classes for $U$. Since $V_\ell(U) = H_1(U, \mathbb{Q}) \otimes \mathbb{Q} \mathbb{Q}_\ell$, we have $W_{U,\ell} = W_U \otimes \mathbb{Q} \mathbb{Q}_\ell$. Consider the action of the Galois group $\text{Gal}(\bar{F}/F)$. The Galois module $W_{B,\ell}$ is the twist of the Galois module $W_{A,\ell}$ by the character $c^{-r}$. Since $n$ does not divide $r$, this is a non-trivial twist, so the Galois modules $W_{B,\ell}$ and $W_{A,\ell}$ cannot be simultaneously trivial.

By pp. 52–54 of [1] (see also Lemma 2.8 of [14] and p. 423 of [25]), the elements of $W_U$ are Hodge classes (since we are dealing with abelian varieties of Weil type).

Since $\tilde{\mathcal{M}}T_{U,\ell}(\mathbb{Q}_\ell)$ acts trivially on $W_U$, $\tilde{\mathcal{M}}T_{U,\ell}(\mathbb{Q}_\ell)$ acts trivially on $W_{U,\ell} = W_U \otimes \mathbb{Q} \mathbb{Q}_\ell$. Suppose now that $\mathcal{G}_0(F, A)$ and $\mathcal{G}_0(F, B)$ are both connected. Then $\mathcal{G}_0(F, A)$ and $\mathcal{G}_0(F, B)$ are both connected (by Proposition 2.13). It follows from Theorem 2.14 that $\mathcal{G}_0(F, U) \subseteq \tilde{\mathcal{M}}T_{U,\ell}$. Therefore, $W_{B,\ell}$ and $W_{A,\ell}$ are both trivial as Gal($\bar{F}/F$)-modules. This is a contradiction. We therefore have (ii).

Suppose that $F_\Phi(A) = F$. Then $\mathcal{G}_0(F, A)$ is connected, so $\mathcal{G}_0(F, B)$ is disconnected. By Lemma 2.7, $\mathcal{G}_0(M, B)$ is connected. Therefore, $F_\Phi(B) \subseteq M$. By Corollary 2.2, $\text{Gal}(M/F_\Phi(B)) \cong \text{Im}(c) \cap \mathcal{G}_0(F, A)(\mathbb{Q}_\ell)$.

Let $\mu_\sigma(k)$ denote the group of $\sigma$-th roots of unity in $k^\times$. We have

$$\text{Im}(c) = \mu_\sigma(k) \cong \mathbb{Z}/n\mathbb{Z}.$$ 

Suppose $\alpha \in \text{Im}(c) \cap \mathcal{G}_0(F, A)(\mathbb{Q}_\ell)$. Then $\alpha^n = 1$. By Theorem 2.8 and the facts that $\mathcal{G}_0(F, A) = \mathcal{G}_0(F, A)^0$ and $\alpha \in \text{End}^0(A)$, we have $\alpha \in \text{MT}_A(\mathbb{Q})$. Applying the character $\gamma$ of Theorem 2.12 we have that $\gamma(\alpha)$ is an $n$-th root of unity in $\mathbb{Q}_\ell^\times$, and therefore $\gamma(\alpha)$ is 1 or $-1$. By the definition of $W_{A,\ell}$, $\alpha$ acts on $W_{A,\ell}$ as multiplication by $\alpha^{-r}\gamma(\alpha)^{-r/2}$. Since $\alpha \in \text{MT}_A(\mathbb{Q}_\ell)$, $\alpha$ acts trivially on $W_{A,\ell}$. Therefore $\alpha^{-r}\gamma(\alpha)^{-r/2} = 1$, so $\alpha^{2r} = 1$. Let $t = (n, 2r)$. Then

$$\text{Gal}(M/F_\Phi(B)) \cong \text{Im}(c) \cap \mathcal{G}_0(F, A)(\mathbb{Q}_\ell) \subseteq \mu_\sigma(k) \cap \mu_{2r}(k) = \mu_t(k).$$

Therefore, $[M : F_\Phi(B)]$ divides $t$. Since $\mathcal{G}_0(F, A)$ contains the homotheties $G_m$ (see 2.3 of [14]), we have $-1 \in \mathcal{G}_0(F, A)(\mathbb{Q}_\ell)$. So $-1 \in \text{Im}(c) \cap \mathcal{G}_0(F, A)(\mathbb{Q}_\ell)$ if and only if $-1 \in \text{Im}(c)$, i.e., if and only if $n$ is even. Thus if $t = 2$, then

$$\text{Gal}(M/F_\Phi(B)) \cong \{\pm 1\}.$$ 

\[\square\]

**Theorem 3.5.** Suppose $X$ and $Y$ are abelian varieties over a finitely generated extension $F$ of $\mathbb{Q}$, $\ell$ is a prime number, $\text{Hom}(X, Y) = 0$, $F = F(\text{End}(X)) = F(\text{End}(Y))$, $k$ is a CM-field, $[k : \mathbb{Q}] = 2\dim(Y)$, and $\dim(X) = \ell\dim(Y)$ for some odd positive integer $\ell$. Suppose $\iota_X$ and $\iota_Y$ are embeddings of $k$ into $\text{End}^0(X)$ and $\text{End}^0(Y)$, respectively, and $(X \times Y, k, \iota_X \times \iota_Y)$ is of Weil type. Suppose $c$ is the non-trivial character associated to a quadratic extension $M$ of $F$, let $Y^c$ denote the twist of $Y$ by $c$, let $A = X \times Y$, and let $B = X \times Y^c$. Then

(i) $F = F(\text{End}(B))$,

(ii) either $F(\text{End}(A)) \neq F_\Phi(A)$ or $F(\text{End}(B)) \neq F_\Phi(B)$,

(iii) if $F_\Phi(A) = F$, then $F_\Phi(B) = M$.

**Proof.** We have $F = F(\text{End}(A))$. Since $\text{Hom}(X, Y) = 0$, we have $\text{End}^0(A) = \text{End}^0(X) \oplus \text{End}^0(Y)$ and $\text{Aut}(A) = \text{Aut}(X) \times \text{Aut}(Y)$. Consider the cocycle $c$ that sends $\sigma \in \text{Gal}(\bar{F}/F)$ to $(1, c(\sigma)) \in \text{Aut}(X) \times \text{Aut}(Y) = \text{Aut}(A)$. All the values of $c$ are of the form $(1, \pm 1)$, and therefore belong to the center of $\text{End}^0(A)$. The abelian
For viewing the Tate modules as free

Let

\( \mathcal{O} \) (abelian variety of CM-type \((K, \mathbf{Z})\) Let \( \Psi \) be the subset of \( \mathbb{Q} \)\n
must be the quadratic extension \( M \) complex multiplication by an imaginary quadratic field \( K \)

have (i), and we obtain an isomorphism \( \varphi \) from \( \text{End}_F(A) \) onto \( \text{End}_F(B) \) such that

\((B, k, \varphi \circ (\tau_X \times \tau_Y)) \) is of Weil type. Let \( k_\ell = k \otimes \mathbb{Q}_\ell \). We have

\[ V_\ell(A) = V_\ell(X) \oplus V_\ell(Y), \quad V_\ell(B) = V_\ell(X) \oplus V_\ell(Y^c). \]

Viewing the Tate modules as free \( k_\ell \)-modules, we have

\[ \wedge^{t+1} V_\ell(A) = \wedge^{t}_k V_\ell(X) \otimes_{k_\ell} V_\ell(Y), \quad \wedge^{t+1} V_\ell(B) = \wedge^{t}_k V_\ell(X) \otimes_{k_\ell} V_\ell(Y^c). \]

For \( U = A \) or \( B \), let

\[ W_{U, \ell} = \text{Hom}_{\mathbb{Q}_\ell}(\wedge^{t+1} V_\ell(U), \mathbb{Q}_\ell((t+1)/2)). \]

The Galois module \( V_\ell(Y^c) \) is the twist of the Galois module \( V_\ell(Y) \) by the character \( c \) (see Remark 2.6), and the Galois module \( W_{B, \ell} \) is the twist of the Galois module \( W_{A, \ell} \) by \( c^{-1} = c \). Since \( c \) is non-trivial, the Galois modules \( W_{B, \ell} \) and \( W_{A, \ell} \) cannot be simultaneously trivial. As in the proof of Theorem 3.4, it follows that \( \Phi_\ell(F, A) \) and \( \Phi_\ell(F, B) \) cannot both be connected. If \( F_\mathfrak{p}(A) = F \), then \( \Phi_\ell(F, B) \) is connected, so \( \Phi_\ell(F, B) \) is disconnected. By Lemma 2.2, \( \Phi_\ell(M, B) \) is connected, and so \( F_\mathfrak{p}(B) \) must be the quadratic extension \( M \) of \( F \).

**Remark 3.6.** Suppose \( F \) is a subfield of \( \mathbb{C} \), \( Y \) is an elliptic curve over \( F \) with complex multiplication by an imaginary quadratic field \( K \), and \( X \) is an absolutely simple 3-dimensional abelian variety over \( F \) with \( K \) embedded in its endomorphism algebra. Then we can always ensure (by taking complex conjugates if necessary) that the two embeddings of \( K \) into \( \mathbb{C} \) occur with the same multiplicity in the action of \( K \) on the tangent space of the 4-dimensional abelian variety \( X = X \times Y \).

Note that the hypotheses of Theorem 3.4 (or of Theorem 3.3) cannot be simultaneously satisfied with \( \dim(A) < 4 \). In Example 4.2 we exhibit 4-dimensional abelian varieties satisfying the hypotheses of Theorem 3.3.

### 4. Examples

Using Theorems 3.4 and 3.3, we can construct examples of abelian varieties \( B \) such that \( \Phi_\ell(F, B) \) is disconnected, and compute the connectedness extensions.

**Example.** Let \( k = \mathbb{Q}(\sqrt{-3}) \) and let \( K \) be the CM-field which is the compositum of \( \mathbb{Q}(\sqrt{-3}) \) with the maximal totally real subfield \( L \) of \( \mathbb{Q}(\zeta_{17}) \). Then

\[ \text{Gal}(K/\mathbb{Q}) \cong \text{Gal}(k/\mathbb{Q}) \times \text{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}. \]

Let \( \Psi \) be the subset of \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \cong \text{Gal}(K/\mathbb{Q}) \) defined by

\[ \Psi = \{(0, 0), (0, 1), (0, 4), (0, 7), (1, 2), (1, 3), (1, 5), (1, 6)\}. \]

Let \( \mathcal{O}_K \) denote the ring of integers of \( K \). Let \((A, \iota_K)\) be an 8-dimensional CM abelian variety of CM-type \((K, \Psi)\) constructed from the lattice \( \mathcal{O}_K \) as in Theorem 3 on p. 46 of [20], and defined over a number field \( F \) (this can be done by Proposition 26 on p. 109 of [20]). Then \( A \) is absolutely simple, by the choice of \( \Psi \) and Proposition 26 on p. 69 of [20], and \( \text{End}(A) = \mathcal{O}_K \) (see Proposition 6 on p. 42 of [20]). Further, the reflex field of \((K, \Psi)\) is \( K \). Take the number field \( F \) to be sufficiently large so that \( F_\mathfrak{p}(A) = F \). Let \( \iota \) be the restriction of \( \iota_K \) to \( k \). By the definition of \( \Psi \), if \( \sigma \in \text{Gal}(k/\mathbb{Q}) \) then \( n_\sigma = 4 \). Therefore \((A, k, \iota)\) is of Weil type. Let \( c : \text{Gal} \left( \tilde{F}/F \right) \to k^\times \) be a non-trivial cubic character associated to a cubic extension
$M$ of $F$, and let $B$ denote the twist of $A$ by $c$. Applying Theorem 3.4(ii) with $n = 3$ and $r = 8$, then $F(\text{End}(B)) = F$ and $F_B(B) = M$.

4.2. Example. Let $J$ be the Jacobian of the genus 3 curve

$$y^7 = x(1 - x),$$

and let $E$ be the elliptic curve $X_0(49)$. A model for $E$ is given by the equation

$$y^2 + xy = x^3 - x^2 - 2x - 1.$$  

Let $d$ be a non-zero square-free integer. If $d \neq 1$ let $E^{(d)}$ be the twist of $E$ by the non-trivial character of $\mathbb{Q}(\sqrt{d})$, and if $d = 1$ let $E^{(d)} = E$. Let

$$A = J \times E, \quad A^{(d)} = J \times E^{(d)}.$$  

The abelian varieties $A^{(d)}$ are defined over $\mathbb{Q}$. Let $\zeta_7$ be a primitive seventh root of unity and let

$$K = \mathbb{Q}(\zeta_7), \quad L_d = K(\sqrt{d}), \quad k = \mathbb{Q}(\sqrt{-7}).$$

If $d = 1$ or $-7$ then $K = L_d$; otherwise, $[L_d : K] = 2$. The abelian variety $J$ is a simple abelian variety with complex multiplication by $K$, and the elliptic curves $E^{(d)}$ have complex multiplication by the subfield $k$ of $K$. We have $\text{Gal}(K/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_6\}$ where $\sigma_1(\zeta_7) = \zeta_7^4$. The CM-type of $J$ is $(K, \{\sigma_1, \sigma_2, \sigma_3\})$ (see p. 34 of [3] or §15.4.2 of [20]), and the reflex CM-type is $(K, \{\sigma_4, \sigma_5, \sigma_6\})$ (see §8.4.1 of [20]). We can identify $\text{End}(A^{(d)})$ with the direct sum of $\text{End}(J)$ and $\text{End}(E^{(d)})$. By Proposition 30 on p. 74 of [20], the smallest extension of $\mathbb{Q}$ over which all the elements of $\text{End}(J)$ are defined is the reflex CM-field of the CM-type of $J$, which is $K$. Similarly, $k$ is the smallest extension of $\mathbb{Q}$ over which all the elements of $\text{End}(E^{(d)})$ are defined. We therefore have

$$K = \mathbb{Q}(\text{End}(A^{(d)})).$$

Next, we will prove that $L_d = \mathbb{Q}_8(A^{(d)})$.

Write $\mathcal{O}_\Omega$ for the ring of integers of a number field $\Omega$. If $q$ is a prime number, let $\mathcal{O}_q = \mathcal{O}_\Omega \otimes \mathbb{Z}_q$.

Lemma 4.2.1. If $K'$ is a finite abelian extension of $K$ which is unramified away from the primes above 7, then $[K' : K]$ is a power of 7.

Proof. We have $-1 - \zeta_7 = (1 - \zeta_7^2)/(\zeta_7 - 1) \in \mathcal{O}_K^\times$. Let $\mathcal{P}$ be the prime ideal of $K$ above 7. The reduction map

$$\mathcal{O}_K^\times \to (\mathcal{O}_K/\mathcal{P})^\times \cong (\mathbb{Z}/7\mathbb{Z})^\times$$

is surjective, since $-1 - \zeta_7$ maps to $-2$, a generator of $(\mathbb{Z}/7\mathbb{Z})^\times$. Moreover, the class number of $K$ is one. Therefore by class field theory, there is no non-trivial abelian extension of $K$ of degree prime to 7 and unramified away from the primes above $\mathcal{P}$. \hfill \square

Lemma 4.2.2. If $p$ is a prime and $p \equiv 3 \pmod{7}$, then the only field $K'$ such that

(i) $K \subseteq K' \subseteq K(A_p)$, and

(ii) $K'/K$ is unramified away from the primes above 7,

is $K$ itself.
Proof. Since $K$ is a field of definition for the endomorphisms of the CM abelian varieties $J$ and $E$, the extension $K(J_n)/K$ is abelian for every integer $n$ (see Corollary 2 on p. 502 of [18]). Suppose $p$ and $K'$ satisfy the hypotheses of Lemma 4.2.2. Let $I_p \subseteq \text{Gal}(K(J_p)/K)$ be the inertia subgroup at $p$. We will first show

$$\#(I_p) = \frac{p^6 - 1}{p^2 + p + 1}. \tag{1}$$

The image of $\mathcal{O}_p^\times$ in $\text{Gal}(K(J_p)/K)$ under the Artin map of class field theory is $I_p$, and we have natural homomorphisms

$$\text{Gal}(K(J_p)/K) \hookrightarrow \text{Aut}_{\mathcal{O}_k}(J_p) \cong (\mathcal{O}_K/p\mathcal{O}_K)^\times \cong \mathcal{O}_p^\times/(1 + p\mathcal{O}_p).$$

We therefore obtain maps

$$\mathcal{O}_p^\times \to I_p \to \mathcal{O}_p^\times/(1 + p\mathcal{O}_p). \tag{2}$$

Since the first map of (2) is surjective, the order of $I_p$ is the order of the image of the composition. Since $p \equiv 3 \pmod{7}$, we know that $p$ is inert in $K/Q$, so $(\mathcal{O}_K/p\mathcal{O}_K)^\times$ is a cyclic group of order $p^6 - 1$. Since the greatest common divisor of $p^6 - 1$ and $p^2(p^2 + p + 1) = p^2 + p + 1$, equation (2) will be proved when we show that the composition of maps in (2) sends $u \in \mathcal{O}_p^\times$ to $u^{-p^2(p^2 + p + 1)} \pmod{1 + p\mathcal{O}_p}$. We can view elements of $\text{Gal}(K/Q)$ as automorphisms of $\mathcal{O}_p^\times$. Proposition 7.40 on p. 211 of [19] implies that the image of $u$ in $\text{Gal}(K/Q)$ is of the form $\alpha(u)/\eta(u)$ (mod $1 + p\mathcal{O}_p$) where $\eta(u) = \sigma_4(u)\sigma_5(u)\sigma_6(u)$ and $\alpha(u) = K^\times \lambda$ for the idele group of $K$, and for each archimedean prime $\lambda$ of $K$, define a Grössencharacter $\psi_\lambda : K^\times \lambda \to \mathbb{C}^\times$ by $\psi_\lambda(x) = (\alpha(x)/\eta(x))_.$. View $\mathcal{O}_p^\times$ as a subgroup of $K^\times \lambda$. Since $J$ has good reduction outside 7, we have $\psi_\lambda(\mathcal{O}_p^\times) = 1$, by Theorem 7.42 of [19]. For $u \in \mathcal{O}_p^\times$, we have $1 = \psi_\lambda(u) = \alpha(u)\lambda = \alpha(u)$. Therefore the image of $u$ in $\mathcal{O}_p^\times/(1 + p\mathcal{O}_p)$ is $1/\eta(u)$ (mod $1 + p\mathcal{O}_p$). Since $p$ is inert in $K/Q$ we have $\text{Gal}(K/Q) \cong \text{Gal}((\mathcal{O}_K/p)/\mathbb{Z}/p) = D_p$, where $D_p$ is the decomposition group at $p$. The latter group is a cyclic group of order 6 generated by the Frobenius element, and we compute that

$$\sigma_4(u) \equiv u^{p^2}, \quad \sigma_5(u) \equiv u^{p^3}, \quad \sigma_6(u) \equiv u^{p^3} \pmod{1 + p\mathcal{O}_p}$$

(since $p^4 \equiv 4 \pmod{7}$, $p^5 \equiv 5 \pmod{7}$, and $p^6 \equiv 6 \pmod{7}$). Therefore

$$1/\eta(u) \equiv u^{-p^2(p^2 + p + 1)} \pmod{1 + p\mathcal{O}_p},$$

as desired.

We have

$$\text{Gal}(K(E_p)/K) \hookrightarrow \text{Aut}_{\mathcal{O}_k}(E_p) \cong (\mathcal{O}_k/p\mathcal{O}_k)^\times.$$ 

The order of $(\mathcal{O}_k/p\mathcal{O}_k)^\times$ is $p^2 - 1$, which is not divisible by 7. Therefore $[K(E_p) : K(J_p)]$ is not divisible by 7. By Lemma 4.2.1, $[K' : K]$ is a power of 7. Therefore $K' \subseteq K(J_p)$. Since $K'/K$ is unramified at $p$, we have $I_p \subseteq \text{Gal}(K(J_p)/K')$. Suppose $K' \neq K$. Then $\#(I_p)$ divides $(p^6 - 1)/7$. By (1), $(p^6 - 1)/(p^2 + p + 1)$ divides $(p^6 - 1)/7$. Therefore 7 divides $p^2 + p + 1$, which contradicts the assumption that $p \equiv 3 \pmod{7}$. Therefore, $K' = K$. \hfill \Box

Suppose $p$ and $q$ are distinct odd primes, and $p \equiv 3 \pmod{7}$. Let $K' = K(A_p) \cap K(A_q)$. Since $A$ has good reduction outside 7, the extension $K'/K$ is unramified.
away from the primes above 7. By Lemma \[4.2.2\] we have \( K' = K \). As mentioned in the introduction, for every integer \( n \geq 3 \) we have
\[
K_\Phi(A) \subseteq K(A_n).
\]
We therefore obtain
\[
K = K_\Phi(A) = Q_\Phi(A).
\]
It follows from Theorem \[3.5\] that
\[
L_d = Q_\Phi(A^{(d)}).
\]
Note that Shioda (see Theorem 4.4 of \[21\]) proved the Hodge Conjecture for \( A \), and therefore also for \( A^{(d)} \). Thus, the Weil classes on \( A^{(d)} \) are algebraic. It follows easily that \( L_d \) is the smallest extension of \( Q \) over which all the algebraic cycle classes on all powers of \( A^{(d)} \) are defined.

**Remark 4.2.3.** If \( A \) is an abelian variety over a finitely generated extension \( F \) of \( Q \), and if the (as yet unproved) Tate Conjecture is true for all powers of \( A \) over \( F_\Phi(A) \), then the field \( F_\Phi(A) \) is the smallest extension of \( F \) over which all the algebraic cycle classes on all powers of \( A \) are defined.

**References**

[1] Borovoi, M.: The action of the Galois group on the rational cohomology classes of type \((p, p)\) of abelian varieties (Russian). Mat. Sbornik (N. S.) 94 (136), 649–652 (1974) = Math. USSR Sbornik 23, 613–616 (1974)

[2] Borovoi, M.: The Shimura-Deligne schemes \( MC(G, h) \) and the rational cohomology classes of type \((p, p)\) of abelian varieties (Russian). In: Problems of group theory and homological algebra (Russian) (No. 1, pp. 3–53) Yaroslavl’: Yaroslav. Gos. Univ. 1977

[3] Chi, W.: \( \ell \)-adic and \( \lambda \)-adic representations associated to abelian varieties defined over number fields. Amer. J. Math. 114, 315–353 (1992)

[4] Deligne, P. (notes by J. Milne): Hodge cycles on abelian varieties. In: P. Deligne, et al.: Hodge cycles, motives, and Shimura varieties (Lecture Notes in Mathematics, vol. 900, pp. 9–100) Berlin Heidelberg New York: Springer 1982

[5] Lang, S.: Complex Multiplication (Grundlehren Math. Wiss. Bd. 255) New York Berlin Heidelberg Tokyo: Springer 1983

[6] Larsen, M., Pink, R.: On \( \ell \)-independence of algebraic monodromy groups in compatible systems of representations. Invent. math. 107, 603–636 (1992)

[7] Larsen, M., Pink, R.: Abelian varieties, \( \ell \)-adic representations, and \( \ell \)-independence. Math. Ann. 302, 561–579 (1995)

[8] Larsen, M., Pink, R.: A connectedness criterion for \( \ell \)-adic representations. To appear in Israel J. Math.

[9] Milne, J. S.: Shimura varieties and motives. In: U. Jannsen et al.: Motives (Proc. Symp. Pure Math. vol. 55 , Part 2, pp. 447–523) Providence: Amer. Math. Soc. 1994

[10]Moonen, B., Zarhin, Yu. G.: Hodge classes and Tate classes on simple abelian fourfolds. Duke Math. J. 77, 553–581 (1995)

[11] Mumford, D.: Abelian varieties, Second Edition (Tata Lecture Notes) London: Oxford Univ. Press 1974

[12] Piatetski-Shapiro, I. I.: Interrelations between the Tate and Hodge conjectures for abelian varieties (Russian), Mat. Sbornik 85, 610–620 (1971) = Math. USSR Sbornik 14, 615–625 (1971)

[13] Ribet, K.: Hodge classes on certain types of abelian varieties. Amer. J. Math. 105, 523–538 (1983)

[14] Serre, J-P.: Repr´esentations \( \ell \)-adiques. In: S. Iyanaga: Algebraic Number Theory (Proceedings of the International Taniguchi Symposium, Kyoto, 1976) (pp. 177–193) Tokyo: Japan Society for the Promotion of Science 1977 = # 112 of Œuvres (Vol. III, pp. 384–400) Berlin Heidelberg New York Tokyo: Springer 1986

[15] Serre, J-P.: Letters to K. Ribet, Jan. 1, 1981 and Jan. 29, 1981
[16] Serre, J.-P.: Résumé des cours de 1984–1985, Résumé des cours de 1985–1986, Collège de France

[17] Serre, J.-P.: Propriétés conjecturales des groupes de Galois motiviques et des représentations $\ell$-adiques. In: U. Jannsen et al.: Motives (Proc. Symp. Pure Math. vol. 55, Part 2, pp. 377–400) Providence: Amer. Math. Soc. 1994

[18] Serre, J.-P., Tate, J.: Good reduction of abelian varieties. Ann. of Math. 88, 492–517 (1968)

[19] Shimura, G.: Introduction to the arithmetic theory of automorphic functions, Princeton: Princeton Univ. Press 1971

[20] Shimura, G., Taniyama, Y.: Complex multiplication of abelian varieties and its applications to number theory (no. 6) Publ. Math. Soc. Japan 1961

[21] Shioda, T.: Algebraic cycles on abelian varieties of Fermat type. Math. Ann. 258, 65–80 (1981)

[22] Silverberg, A., Zarhin, Yu. G.: Isogenies of abelian varieties. J. Pure and Applied Algebra 90, 23–37 (1993)

[23] Silverberg, A., Zarhin, Yu. G.: Connectedness results for $\ell$-adic representations associated to abelian varieties. Comp. math. 97, 273–284 (1995)

[24] Silverberg, A., Zarhin, Yu. G.: Images of $\ell$-adic representations and automorphisms of abelian varieties. Preprint.

[25] Weil, A.: Abelian varieties and the Hodge ring (1977c). Œuvres scientifiques (Vol. III, pp. 421–429) New York Heidelberg Berlin: Springer 1979

[26] Zarhin, Yu. G.: Weights of simple Lie algebras in the cohomology of algebraic varieties (Russian). Izv. Akad. Nauk SSSR Ser. Mat. 48, 264–304 (1984) = Math. USSR - Izv. 24, 245–282 (1985)

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