EXEL’S CROSSED PRODUCT AND
CROSSED PRODUCTS BY COMPLETELY POSITIVE MAPS

BARTOSZ KOSMA KWAŚNIEWSKI

ABSTRACT. We introduce crossed products of a $C^*$-algebra $A$ by a completely positive map $\varphi : A \to A$ relative to an ideal in $A$. When $\varphi$ is multiplicative they generalize various crossed products by endomorphisms. When $A$ is commutative they include $C^*$-algebras associated to Markov operators by Ionescu, Muhly, Vega, and to topological relations by Brenken, but in general they are not modeled by topological quivers of Muhly and Solel.

We show that Exel’s crossed product $A \rtimes_{\alpha, L} \mathbb{N}$, generalized to the case where $A$ is not necessarily unital, is the crossed product of $A$ by the transfer operator $L$ relative to the ideal generated by $\alpha(A)$. We give natural conditions under which $\alpha(A)$ is uniquely determined by $L$, and hence $A \rtimes_{\alpha, L} \mathbb{N}$ depends only on $L$. Moreover, the $C^*$-algebra $\mathcal{O}(A, \alpha, L)$ associated to $(A, \alpha, L)$ by Exel and Royer always coincides with our unrelative crossed product by $L$.

As another non-trivial application of our construction we extend a result of Brownlowe, Raeburn and Vittadello, by showing that the $C^*$-algebra of an arbitrary infinite graph $E$ can be realized as a crossed product of the diagonal algebra $\mathcal{D}_E$ by a ‘Perron-Frobenious’ operator $L$. The important difference to the previous result is that in general there is no endomorphism $\alpha$ of $\mathcal{D}_E$ making $(\mathcal{D}_E, \alpha, L)$ an Exel system.

CONTENTS

1. Introduction 1
2. Preliminaries on positive maps and Exel’s crossed products 6
3. Crossed products by completely positive maps 9
4. A new look at Exel systems and their crossed products 18
5. Graph $C^*$-algebras as crossed products by completely positive maps 25
References 33

1. Introduction

In the present state of the art the theory of crossed products of $C^*$-algebras by endomorphisms breaks down into two areas that involve two different constructions. The first approach originated in late 1970’s in the work of Cuntz [10] and was developed by many authors [41], [46], [1], [38], [2], [28], [26]. Another approach was initiated by Exel [12] in the beginning of the present century and immediately received a lot of attention; in particular, Exel’s construction was extended in [9], [33], [14], [6]. By now, both of the approaches have proved to be useful in an innumerable variety of problems and their importance is well-acknowledged. They (or their semigroup versions) serve as tools to construct and analyse the most intensively studied $C^*$-algebras in recent years. These include: Cuntz algebras [10], Cuntz-Kriger algebras [12],

2010 Mathematics Subject Classification. 46L05,46L55.

Key words and phrases. Completely positive map, transfer operator, crossed product, Exel’s crossed product, graph $C^*$-algebra, Cuntz-Pimsner algebra.

This research was supported by NCN grant number DEC-2011/01/D/ST1/04112 and by a Marie Curie Intra European Fellowship within the 7th European Community Framework Programme; project ‘OperaDynaDual’ (2014-2016).
Exel-Laca algebras [14], graph algebras [9], [17], [25], higher-rank graph algebras [6], $C^*$-algebras arising from semigroups [1], number fields [31], [3], or algebraic dynamical systems [7]. Among the applications one could mention their significant role in classification of $C^*$-algebras [44], [27], study of phase transitions [30], or short exact sequences and tensor products [32].

In view of what has been said, it is somewhat surprising that the intersection of the two aforementioned approaches is relatively small: the two constructions coincide for injective corner endomorphisms [12] and more generally for systems called complete in [2], [23], and reversible in [26], [27]. Moreover, general relationship between these two lines of research is still shrouded in mystery, and definitely calls for clarification. One of the overall aims of the present paper is to cover this demand. We do it by showing that the two areas are different special cases of one natural construction; they can be unified in the realm of crossed products by completely positive maps. Since completely positive maps are ubiquitous in the $C^*$-theory and in quantum physics, the crossed products we introduce have an ample potential for further study and applications.

In particular, we hope that the present article will not only clear the decks but also give an impulse for such a development (see, for instance, our remarks concerning crossed products of commutative algebras; also study of ergodic properties of non-commutative Perron-Frobenius operators we introduce is of interest).

Let us explain our strategy in more detail. We define relative crossed products $C^*(A, g; J)$ of a $C^*$-algebra $A$ by a completely positive mapping $g : A \rightarrow A$ relative to an ideal $J$ in $A$. The unrelative crossed product is $C^*(A, g) := C^*(A, g, N^g_0)$ where $N^g_0$ is the largest ideal contained in $\ker g$. When $\alpha := g$ is multiplicative, hence an endomorphism of $A$, the crossed products $C^*(A, \alpha; J)$ cover the line of research we attributed to Cuntz. More specifically (see Subsection 3.4 below), $C^*$-algebras $C^*(A, \alpha; J)$ coincide with crossed products by endomorphisms studied in [28], for unital $A$, and in [26], [27], for extendible $\alpha$. In particular, if $\alpha$ is extendible $C^*(A, \alpha; A)$ is Stacey’s crossed product [40], and $C^*(A, \alpha; \{0\})$ is the partial isometric crossed product introduced, in a semigroup context, by Lindiarni and Raeburn [34]. Accordingly, $C^*(A, \alpha) = C^*(A, \alpha; \ker \alpha^+) +$ is a good candidate for the (unrelative) crossed product by an arbitrary endomorphism, cf. [28], [26], [27]. In contrast to this multiplicative case, we claim that Exel’s crossed product $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ is a crossed product by the transfer operator $\mathcal{L}$ (which as a rule is not multiplicative). In order to make this statement precise we need to thoroughly re-examine - take ‘a new look at’ Exel’s construction.

We recall that Exel introduced in [12] the crossed product $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ of a unital $C^*$-algebra $A$ by an endomorphism $\alpha : A \rightarrow A$ which also depends on the choice of a transfer operator, i.e. a positive linear map $\mathcal{L} : A \rightarrow A$ such that $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$, for all $a, b \in A$. This construction was generalized to non-unital case in [9], [33] were authors assumed that both $\alpha$ and $\mathcal{L}$ extend to strictly continuous maps on the multiplier algebra $M(A)$. We show however that extendability of $\mathcal{L}$ is automatic and since extendability of $\alpha$ does not play any role in the definition, in the present paper, we consider crossed products $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ for Exel systems $(A, \alpha, \mathcal{L})$ where $A$, $\alpha$ and $\mathcal{L}$ are arbitrary. Obviously, in a typical situation there are infinitely many different transfer operators for a fixed $\alpha$. On the other hand, under natural assumptions, such as faithfulness of $\mathcal{L}$, which usually appear in applications [12], [13], [8], [9], the endomorphism $\alpha$ is uniquely determined by a fixed transfer operator $\mathcal{L}$. Moreover, any transfer operator $\mathcal{L}$ is necessarily a completely positive map and therefore it is suitable to form a crossed product on its own. This provokes the question:

To what extent $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ depends on $\alpha$?

Before giving an answer, we need to stress that the pioneering Exel’s definition of $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$, [12, Definition 3.7], was to some degree experimental. In general it requires a modification. Namely, Brownlowe and Raeburn in [8] recognized $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ as a relative Cuntz-Pimsner algebra $O(K_\alpha, M_\mathcal{L})$ where $M_\mathcal{L}$ is a $C^*$-correspondence associated to $(A, \alpha, \mathcal{L})$ and $K_\alpha = \overline{A_\alpha(A)A \cap \mathcal{J}(M_\mathcal{L})}$ is the intersection of the ideal generated by $\alpha(A)$ and the ideal of elements that the left action $\phi$
of $A$ on $M_L$ sends to ‘compacts’. Then it follows from general results on relative Cuntz-Pimsner algebras \cite{35} that $A$ embeds into $A \rtimes_{\alpha, L} \mathbb{N}$ if and only if $K_0$ is contained in Katsura’s ideal $\ker \phi \perp J(M_L)$. But this is not always the case. In particular, the theory of Cuntz-Pimsner algebras indicates that Exel’s construction should be improved by replacing in \cite{12} Definition 3.7 the ideal $A\alpha(A)A$ with $\ker \phi \perp J(M_L)$. This is done by Exel and Royer in \cite{14}, cf. also \cite{6} Proposition 4.5, where they associate to $(A, \alpha, \mathcal{L})$ a $C^*$-algebra $\mathcal{O}(A, \alpha, \mathcal{L})$ which is isomorphic to Katsura’s Cuntz-Pimsner algebra $\mathcal{O}_{M_L}$ (as a matter of fact, authors of \cite{14} deal with more general Exel systems where $\alpha$ and $\mathcal{L}$ are only ‘partially defined’).

Turning back to our question, the results of the present paper give the following answer, which consists of three parts:

1. the modified Exel’s crossed product $\mathcal{O}(A, \alpha, \mathcal{L})$ always coincides with our unrelative crossed product $C^*(A, \mathcal{L})$ of $A$ by $\mathcal{L}$,
2. original Exel’s crossed product $A \rtimes_{\alpha, L} \mathbb{N}$, for regular systems, does not depend on $\alpha$, if we assume certain conditions assuring that $A$ embeds into $A \rtimes_{\alpha, L} \mathbb{N}$,
3. the three algebras $A \rtimes_{\alpha, L} \mathbb{N}$, $\mathcal{O}(A, \alpha, \mathcal{L})$, $C^*(A, \mathcal{L})$ coincide for most of systems appearing in applications (we give a number of conditions implying that below).

In connection with point (3) it is interesting to note that, in general, there seems to be no clear relation between the ideals

$$A\alpha(A)A, \quad (\ker \phi)^\perp, \quad J(M_L).$$

However, for many natural systems $(A, \alpha, \mathcal{L})$, for instance for all such systems arising from graphs (cf. Lemma \ref{5.9} below), we always have $A\alpha(A)A = (\ker \phi)^\perp \cap J(M_L)$ and consequently $A \rtimes_{\alpha, L} \mathbb{N} = \mathcal{O}(A, \alpha, \mathcal{L}) = C^*(A, \mathcal{L})$. This shows that (by incorporating the ideal $A\alpha(A)A$ into his original construction) Exel exhibited an incredibly good intuition; especially that, in contrast to $A\alpha(A)A$, determining $(\ker \phi)^\perp \cap J(M_L)$ is very hard in practice. In particular, it is an important task to identify Exel systems $(A, \alpha, \mathcal{L})$ for which $A \rtimes_{\alpha, L} \mathbb{N} = \mathcal{O}(A, \alpha, \mathcal{L}) = C^*(A, \mathcal{L})$. We find a large class of such objects in the present article.

We test the results of our findings on graph $C^*$-algebras. We recall that the main motivation for introduction of $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ in \cite{12} was to realize Cuntz-Krieger algebras as crossed products arising naturally from the associated one-sided Markov shifts. This result was adapted in \cite{9} to graph $C^*$-algebras $C^*(\mathcal{E})$ where $\mathcal{E}$ is a locally finite graph with no sinks or sources (by \cite{6} Proposition 4.6, it can be generalized to graphs admitting sinks). For such graph $\mathcal{E}$ the space of infinite paths $\mathcal{E}^\infty$ is a locally compact Hausdorff space and the one-sided shift $\sigma : \mathcal{E}^\infty \to \mathcal{E}^\infty$ is a surjective proper local homeomorphism. In particular, the formulas

$$\alpha(a)(\mu) = a(\sigma(\mu)), \quad \mathcal{L}(a)(\mu) = \frac{1}{|\sigma^{-1}(\mu)|} \sum_{\eta \in \sigma^{-1}(\mu)} a(\eta),$$

$a \in C_0(\mathcal{E}^\infty)$, $\mu \in \mathcal{E}^\infty$, yield well-defined mappings on $C_0(\mathcal{E}^\infty)$. Actually, $(C_0(\mathcal{E}^\infty), \alpha, \mathcal{L})$ is an Exel system and $C_0(\mathcal{E}^\infty)$ is naturally isomorphic to the diagonal $C^*$-subalgebra $\mathcal{D}_{\mathcal{E}}$ of $C^*(\mathcal{E})$. By \cite{9} Theorem 5.1, the isomorphism $C_0(\mathcal{E}^\infty) \cong \mathcal{D}_{\mathcal{E}}$ extends to the isomorphism $C_0(\mathcal{E}^\infty) \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \cong C^*(\mathcal{E})$. In order to generalize that result to arbitrary graphs one is forced to pass to boundary path space $\partial \mathcal{E}$ of $\mathcal{E}$, cf. \cite{15}. Then $C_0(\partial \mathcal{E}) \cong \mathcal{D}_{\mathcal{E}}$, but the analogues of maps given by \cite{11} are in general not well defined onto the whole of $C_0(\partial \mathcal{E})$. One possible solution, see \cite{6}, is to consider ‘partial’ Exel systems defined in \cite{14}. In the present paper, we circumvent this problem by studying a more general class of ‘Perron-Frobenious operators’:

$$\mathcal{L}_\lambda(a)(\mu) = \sum_{e \in \mathcal{E}, e \mu \in \partial \mathcal{E}} \lambda_e a(e \mu), \quad a \in C_0(\partial \mathcal{E}),$$

where $\lambda = \{\lambda_e\}_{e \in \mathcal{E}}$ is a family of strictly positive numbers indexed by the edges of $\mathcal{E}$. We find necessary and sufficient conditions on $\lambda$ assuring that $\mathcal{L}_\lambda : C_0(\partial \mathcal{E}) \to C_0(\partial \mathcal{E})$ is well-defined.
For any such $\lambda$ we get an isomorphism

$$C^*(E) \cong C^*(C_0(\partial E), \mathcal{L}_\lambda).$$

Moreover, the map induced on $\mathcal{D}_E \cong C_0(E^\infty)$ by $\mathcal{L}_\lambda$ extends in a natural way to a completely positive map on $C^*(E)$ which deserves a name of non-commutative Perron-Frobenius operator. This indicates, at least in the present context, a somewhat superior role of a Perron-Frobenius operator $\mathcal{L}_\lambda$ over the standard non-commutative Markov shift, cf., for instance, [19], which in general is not even well-defined.

Finally, we mention our findings concerning an arbitrary (necessarily complete) positive map $\varrho$ on a commutative $C^*$-algebra $A = C_0(D)$, where $D$ is a locally compact Hausdorff space. Any such map defines a relation on $D$:

$$(y, x) \in R \overset{\text{def}}{\iff} (\forall a \in A, a \geq 0) a(y) \neq 0 \implies a(x) \neq 0.)$$

If the set $R \subseteq D \times D$ is closed, then $\varrho$ give rise to a topological relation $\mu$ in the sense of [4] and a topological quiver $Q$ in the sense of [37]. Then we prove that for the corresponding $C^*$-algebras associated to $\mu$ and $Q$, in [4] and [37] respectively, we have $C^*(A, \varrho; A) \cong \mathcal{C}(\mu)$ and $C^*(A, \varrho) \cong C^*(Q)$. In particular, if $\varrho$ is a Markov operator in the sense of [18], the $C^*$-algebra $C^*(\varrho)$ considered in [18] coincides with $C^*(A, \varrho)$. However, as we explain in detail and show by concrete examples, when $R$ is not closed in $D \times D$, then $C^*(A, \varrho)$ cannot be modeled in any obvious way by the $C^*$-algebras studied in [37]. In particular, performing an analysis similar to that in [18] for general positive maps on commutative $C^*$-algebras would call for a generalization of the theory of topological quivers [37].

The content of the paper is organized as follows.

In Section 2 we gather certain facts on positive maps. We also present a definition of Exel’s crossed product $A \times_{\alpha, \mathcal{L}} \mathbb{N}$ for arbitrary Exel systems $(A, \alpha, \mathcal{L})$, and recall a definition of Exel-Royer’s crossed product $\mathcal{O}(A, \alpha, \mathcal{L})$ for such systems.

In Section 3 we introduce relative crossed products $C^*(A, \varrho; J)$ for a completely positive map $\varrho : A \to A$. We present three pictures of $C^*(A, \varrho; J)$: as a quotient of a certain Toeplitz algebra (Definition 3.5); as a relative Cuntz-Pimsner algebra associated with a GNS correspondence $X_\varrho$ of $(A, \varrho)$ (Theorem 3.13); and as a universal $C^*$-algebra generated by suitably defined covariant representations of $(A, \varrho)$ (Proposition 3.17). This allows us to determine the conditions under which $A$ embeds into $C^*(A, \varrho; J)$, and leads us to a version of gauge-invariant uniqueness theorem (Theorem 3.18). We finish this section by revealing relationships of our construction with various crossed products by endomorphisms (subsection 3.4), and with $C^*$-algebras associated to topological relations, topological quivers, and Markov operators (subsection 3.5).

In Section 4 we show that the Toeplitz algebra $\mathcal{T}(A, \alpha, \mathcal{L})$ of $(A, \alpha, \mathcal{L})$ coincides with the Toeplitz algebra $\mathcal{T}(A, \mathcal{L})$ of $(A, \mathcal{L})$ (Proposition 4.3), which leads us to identities $\mathcal{O}(A, \alpha, \mathcal{L}) = C^*(A, \mathcal{L})$ and $A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; A\alpha(A)A)$ (Theorem 4.7). In Subsection 4.2 we study Exel systems $(A, \alpha, \mathcal{L})$ with the additional property that $E := \alpha \circ \mathcal{L}$ is a conditional expectation onto $\alpha(A)$. For historical reasons, to be explained below, we call such systems regular. We give a number of characterizations and an intrinsic description of regular Exel systems, which, in particular, generalize the corresponding results from [23]. This leads us to convenient conditions implying that $A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L})$ (cf. Proposition 4.16). Such conditions are satisfied for instance by extendible regular Exel systems $(A, \alpha, \mathcal{L})$ with $\mathcal{L}$ faithful (Corollary 4.17) or $\alpha(A)$ hereditary (Theorem 4.21).

In the closing Section 5 we analyze the $C^*$-algebra $C^*(E) = C^*(\{p_v : v \in E^0\} \cup \{s_e : e \in E^1\})$ associated to an arbitrary infinite graph $E = (E^0, E^1, r, s)$. We briefly present Brownlowe’s [6] realization of $C^*(E)$ as Exel-Royer’s crossed product for a partially defined Exel system $(C_0(\partial E), \alpha, \mathcal{L})$. Then we characterize the cases when $\alpha$ can be treated as an honest map on $C_0(\partial E)$ (Proposition 5.3), and find conditions on the numbers $\lambda = \{\lambda_e\}_{e \in E^1}$ assuring that (2) defines a self-map on $C_0(\partial E)$. For any such choice of $\lambda$ we prove, using an algebraic picture of
the system \((C_0(\partial E), \mathcal{L}_{\lambda})\), that \(C^*(E) \cong C^*(\mathcal{D}_E, \mathcal{L})\), where \(\mathcal{L}(a) := \sum_{e \in E^1} \lambda_e s_e^* a_s e, a \in \mathcal{D}_E\) (see Theorem 5.6). If \(E\) is locally finite and without sources then the (non-commutative) Markov shift \(\alpha(a) := \sum_{e \in E^1} s_e a_e s_e^*\) is the unique endomorphism of \(\mathcal{D}_E\) such that \((\mathcal{D}_E, \alpha, \mathcal{L})\) is an Exel system and \(C^*(\mathcal{D}_E, \mathcal{L}) = \mathcal{D}_E \times_{\alpha, \mathcal{L}} \mathbb{N}\) (Theorem 5.6(ii)). In general there is no endomorphism making \((\mathcal{D}_E, \alpha, \mathcal{L})\) an Exel system (Theorem 5.6(ii)). One of possible interpretations of these results is that in order to associate a non-commutative shift to an arbitrary infinite graph one is forced to fix a certain measure system and encode the shift in its ‘transfer operator’, as the ‘composition endomorphism’ does not exist.

1.1. Conventions and notation. All ideals in \(C^*\)-algebras (unless stated otherwise) are assumed to be closed and two-sided. If \(I\) is an ideal in a \(C^*\)-algebra \(A\) we denote by \(I^\perp = \{a \in A : aI = 0\}\) the annihilator of \(I\). We denote by \(1\) the unit in the multiplier algebra \(M(A)\) of \(A\). Any approximate unit in \(A\) is assumed to compose of contractive positive elements. All homomorphisms between \(C^*\)-algebras are assumed to be \(\ast\)-preserving. For actions \(\gamma : A \times B \rightarrow C\) such as multiplications, inner products, etc., we use the notation:

\[
\gamma(A, B) = \{\gamma(a, b) : a \in A, b \in B\}, \quad \gamma(A, B) = \text{span}\{\gamma(a, b) : a \in A, b \in B\}.
\]

By Cohen-Hewitt Factorization Theorem we have \(\gamma(A, B) = \gamma(A, B)\) whenever \(\gamma\) can be interpreted as a continuous representation of a \(C^*\)-algebra \(A\) on a Banach space \(B\). We emphasize that we will use this fact without further warning. In particular, a \(C^*\)-subalgebra \(A\) of a \(C^*\)-algebra \(B\) is non-degenerate if \(AB = B\).

We assume the reader is familiar with the theory of Hilbert modules and Cuntz-Pimsner algebras (for an introduction we recommend, for instance, [29] and [51] Subsection 4.6). The following brief summary serves merely as a way of fixing notation and nomenclature.

Let \(A\) be a \(C^*\)-algebra. A (right) \(C^*\)-correspondence over \(A\) is a right Hilbert \(A\)-module \(X\) together with a left action of \(A\) on \(X\) given by a homomorphism \(\phi\) of \(A\) into the \(C^*\)-algebra \(\mathcal{L}(X)\) of all adjointable operators on \(X\); we write \(a \cdot x = \phi(a)x\). A representation \((\pi, \pi_X)\) of a \(C^*\)-correspondence \(X\) consists of a representation \(\pi : A \rightarrow B(H)\) in a Hilbert space \(H\) and a linear map \(\pi_X : A \rightarrow B(H)\) such that

\[
\pi_X(ax \cdot b) = \pi(a) \pi_X(x) \pi(b), \quad \pi_X(x)^* \pi_X(y) = \pi((x, y)_A), \quad a, b \in A, x, y \in X.
\]

Then \(\pi_X\) is automatically bounded. If \(\pi\) is faithful, then \(\pi_X\) is automatically isometric, and \((\pi, \pi_X)\) is called faithful. The \(C^*\)-subalgebra \(\mathcal{K}(X) \subseteq \mathcal{L}(X)\) of generalized compact operators is the closed linear span of the operators \(\Theta_{x,y}\) where \(\Theta_{x,y}(z) = x(y, z)_A\) for \(x, y, z \in X\). Any representation \((\pi, \pi_X)\) of \(X\) induces a homomorphism \((\pi, \pi_X)^{(1)} : \mathcal{K}(X) \rightarrow B(H)\) which satisfies

\[
(\pi, \pi_X)^{(1)}(\Theta_{x,y}) = \pi_X(x) \pi_X(y)^*, \quad (\pi, \pi_X)^{(1)}(T) \pi_X(x) = \pi_X(Tx)
\]

for \(x, y \in X\) and \(T \in \mathcal{K}(X)\), see [5] Proposition 4.6.3.

The Toeplitz algebra of \(X\) is the \(C^*\)-algebra \(T(X)\) generated by \(i_A(A) \cup i_X(X)\) where \((i_A, i_X)\) is a universal representation of \(X\) (universality means here that we have a natural one-to-one correspondence between representations of \(T(X)\) and of \(X\)). For any ideal \(J\) in \(J(X) := \phi^{-1}(\mathcal{K}(X))\) the relative Cuntz-Pimsner algebra \(\mathcal{O}(J, X)\), [35] Definition 2.18, is the quotient of \(T(X)\) by the ideal generated by \(\{i_A(a) - (i_A, i_X)^{(1)}(\phi(a)) : a \in J\}\). Katsura’s \(C^*\)-algebra associated to \(X\) is \(\mathcal{O}_X := \mathcal{O}((\ker \phi) \perp \cap J(X), X)\), [20] Definition 2.6.

If the \(C^*\)-correspondence \(X\) is essential, that is if \(AX = X\), we may restrict our attention only to representations \((\pi, \pi_X)\) where \(\pi\) is non-degenerate. This is due to the following statement, which was proved in [9] for a certain concrete \(C^*\)-correspondence \(X\). However, the proof uses only the fact that \(X\) is essential.

**Lemma 1.1** ([9], Lemma 3.4). For any representation \((\pi, \pi_X)\) of an essential \(C^*\)-correspondence \(X\) on the Hilbert space \(H\), the essential subspace \(K = \pi(A)H\) is reducing for \((\pi, \pi_X)\) and we have \(\pi|_{K^\perp} = 0\) and \(\pi_X|_{K^\perp} = 0\).
Since all the $C^*$-correspondences considered in the text will be essential, all the representations $(\pi, \pi_\chi)$ of $C^*$-correspondences will be assumed to be non-degenerate in the sense that $\pi$ is non-degenerate. It will force our universal morphisms to be also non-degenerate. We recall that a homomorphism $h : A \to B$ between two $C^*$-algebras is non-degenerate if $h(A)$ is non-degenerate in $B$, that is if $h(A)B = B$.

2. Preliminaries on positive maps and Exel’s crossed products

In this section, we present certain facts concerning completely positive maps. Most of them are known, but usually they are stated in the literature in the unital case. Moreover, since they hold for (not necessarily completely) positive maps, we present them in this generality. In the second part of this section, we introduce a definition of Exel crossed product for arbitrary Exel systems, and also recall the definition of crossed products associated to such systems in \cite{14}.

2.1. Positive maps. Throughout this subsection we fix a positive map $\varrho : A \to B$ between two $C^*$-algebras $A$ and $B$. This means that $\varrho : A \to B$ is linear and $\varrho(aa^*) \geq 0$ for every $a \in A$. Such $\varrho$ is automatically $*$-preserving: $\varrho(a^*) = \varrho(a)^*$, $a \in A$; and bounded, see \cite{29} Lemma 5.1.

We have the following formula for norm of $\varrho$, which is well known for completely positive maps, cf. \cite{29} Lemma 5.3(i)], and less known for positive maps.$^1$

**Lemma 2.1.** For any approximate unit $\{\mu_\lambda\}_{\lambda \in \Lambda}$ in $A$ we have $\|\varrho\| = \lim_{\lambda \in \Lambda} \|\varrho(\mu_\lambda)\|$.

**Proof.** Recall that the double dual $A^{**}$ of $A$ can be identified with enveloping the von Neumann algebra of, cf. \cite{5} Theorem 1.4.1. Similarly for $B$. Then the double dual $\varrho^{**} : A^{**} \to B^{**}$ of $\varrho : A \to B$ is a $\sigma$-weakly continuous extension of $\varrho$. As positive elements in $A$ are $\sigma$-weakly dense in the set of positive elements in $A^{**}$, $\varrho^{**}$ is positive. Hence Russo-Dye theorem implies, see \cite{29} Corollary 2.9], that $\|\varrho^{**}\| = \|\varrho^{**}(1)\|$. Moreover, since $\{\mu_\lambda\}_{\lambda \in \Lambda}$ converges $\sigma$-weakly to 1, $\{\varrho^{**}(\mu_\lambda)\}_{\lambda \in \Lambda}$ converges $\sigma$-weakly to $\varrho^{**}(1)$. Therefore

$$\|\varrho^{**}\| = \|\varrho^{**}(1)\| \leq \liminf_{\lambda \in \Lambda} \|\varrho^{**}(\mu_\lambda)\| = \liminf_{\lambda \in \Lambda} \|\varrho(\mu_\lambda)\|.$$  

As clearly we have $\limsup_{\lambda \in \Lambda} \|\varrho(\mu_\lambda)\| \leq \|\varrho\| \leq \|\varrho^{**}\|$, we get the desired equality. \hfill $\Box$

**Corollary 2.2.** We have $\|\varrho((ab)^*ab)\| \leq \|a^*a\| \cdot \|\varrho(b^*b)\|$ for all $a, b \in A$.

**Proof.** If $b \in A$ is fixed, then $\varrho_b(a) := \varrho(b^*ab), a \in A$, defines a positive map. Applying Lemma 2.1 to $\varrho_b$, we get $\|\varrho_b\| = \varrho(b^*b)$. This implies the assertion. \hfill $\Box$

**Corollary 2.3.** The set

$$(3) \quad N_\varrho := \{a \in A : \varrho((ab)^*ab) = 0 \text{ for all } b \in A\}$$

is the largest ideal in $A$ contained in the kernel of the mapping $\varrho : A \to B$.

**Proof.** Obviously, $N_\varrho$ is a closed right ideal in $A$. By Corollary 2.2 it is also a left ideal. In particular, if $a$ is a positive element in $N_\varrho$, then $a^{1/4} \in N_\varrho$ and therefore $\varrho(a) = \varrho((a^{1/4}a^{1/4})^*a^{1/4}a^{1/4}) = 0$. This implies that $N_\varrho \subseteq \ker \varrho$. Clearly, if $I$ is an ideal in $A$ contained in $\ker \varrho$, then $I \subseteq N_\varrho$. \hfill $\Box$

Corollary 2.3 inspire us to coin the following definition.

**Definition 2.4.** We call the ideal $N_\varrho$ the GNS-kernel of $\varrho : A \to B$.

The ideal introduced above is closely related to the notion of almost faithfulness introduced, in the context of Exel systems, in \cite{5}. Namely, following \cite{5}, we say that $\varrho$ is almost faithful on an ideal $I$ in $A$ if $a \in I$ and $\varrho((ab)^*ab) = 0$ for all $b \in A \implies a = 0$. 

$^1$the author thanks Paul Skoufranis for providing the following short proof
The above implication is equivalent to the equality $I \cap N_\varphi = \{0\}$. In other words,

$$\varphi$$ is almost faithful on $I \iff I \subseteq N_\varphi$. 

In particular, the annihilator $N_\varphi^\perp$ of the GNS-kernel of $\varphi$ is the largest ideal in $A$ on which $\varphi$ is almost faithful. We recall that $\varphi$ is faithful on a $C^*$-subalgebra $C \subseteq A$ if for any $a \in C$, $\varphi(a^*a) = 0$ implies $a = 0$. The following lemma sheds considerable light on the relationship between the two aforementioned notions.

**Lemma 2.5.** Let $C \subseteq A$ be a $C^*$-subalgebra and consider the following conditions:

1. $\varphi$ is faithful on the ideal $\overline{ACA}$,
2. $\varphi$ is faithful on the hereditary $C^*$-subalgebra $CAC$,
3. $\varphi$ is almost faithful on the ideal $\overline{ACA}$.

Then i) $\Rightarrow$ ii) $\Rightarrow$ iii) and if $A$ is commutative then the above conditions are equivalent.

**Proof.** The inclusion $CAC \subseteq \overline{ACA}$ yields the implication i) $\Rightarrow$ ii).

ii) $\Rightarrow$ iii). Let $a \in N_\varphi$. Consider an element $a_1c_2 \in ACA$ where $a_1, a_2 \in A$ and $c \in C$. Then $\varphi((a_1c_2)b)a_1c_2b) = 0$ for all $b \in A$. Taking $b = a^*_c$ and using faithfulness of $\varphi$ on $CAC$ we infer that $a_1c_2a^*_c = 0$. This implies that $a(a_1c_2) = 0$. Accordingly, $ACA \subseteq N_\varphi^\perp$ and since $N_\varphi^\perp$ is an ideal we get $\overline{ACA} \subseteq N_\varphi^\perp$.

Assume now that $\overline{ACA} \subseteq N_\varphi^\perp$ and $A$ is commutative. Consider an element $ac \in ACA = AC$, $a \in A$, $c \in C$, such that $\varphi((ac)^*ac) = 0$. By Corollary 2.2, for all $b \in A$ we have

$$\varphi((acb)^*acb) = \varphi((bac)^*bac) \leq ||b^*b\varphi((ac)^*ac) = 0.$$ 

Thus (by almost faithfulness) $ac = 0$. Hence $\varphi$ is faithful on $\overline{ACA} = CA$. \qed

There is a natural $C^*$-subalgebra of $A$, on which $\varphi$ is multiplicative, cf. [39, p. 38], [5, p. 12].

**Definition 2.6.** Let $\varphi : A \rightarrow B$ be a positive map. We call the set

$$(4) \quad MD(\varphi) := \{a \in A : \varphi(b)\varphi(a) = \varphi(ba) \text{ and } \varphi(a)\varphi(b) = \varphi(ab) \text{ for every } b \in A\}$$

the multiplicative domain of $\varphi$.

It is immediate that $MD(\varphi)$ is a $C^*$-subalgebra of $A$, and hence $\varphi : MD(\varphi) \rightarrow B$ is a homomorphism of $C^*$-algebras. In the literature, see e.g. [39, p. 38], [29, p. 12], multiplicative domains are considered for contractive completely positive maps, which is due to the fact we express in Proposition 2.7 below. We recall that $\varphi$ is completely positive if for every integer $n > 0$ the amplified map $\varphi^{(n)} : M_n(A) \rightarrow M_n(B)$ obtained by applying $\varphi$ to each matrix element: $\varphi^{(n)}((a_{ij})) = (\varphi(a_{ij}))$, is positive, see [39, p. 5], [29, p. 39], or [17, IV, Definition 3.3]. It is not hard to show, cf. [40, Remark 5.1], see [17, IV, Corollary 3.4], that a linear map $\varphi : A \rightarrow B$ is completely positive if and only if

$$\sum_{i,j=1}^n b_i^*\varphi(a_i^*a_j)b_j \geq 0,$$

for all $a_1, ..., a_n \in A$ and $b_1, ..., b_n \in B$.

The following fact is a generalization of [39, Theorem 3.18] to not necessarily unital completely positive maps on not necessarily unital $C^*$-algebras.

**Proposition 2.7.** Let $\varphi : A \rightarrow B$ be a contractive completely positive map between $C^*$-algebras. Then

$$(5) \quad MD(\varphi) = \{a \in A : \varphi(a)^*\varphi(a) = \varphi(a^*a) \text{ and } \varphi(a)\varphi(a)^* = \varphi(aa^*)\}.$$ 

In particular, $MD(\varphi)$ is the largest $C^*$-subalgebra of $A$ on which $\varphi$ restricts to a homomorphism.
Proof. To show the equality (5) note that the argument of the proof of \[39\] Theorem 3.18 applies, only modulo the fact that the Schwarz inequality
\[
g(2)(a^*)g(2)(a) \leq g(2)(a^*a), \quad a \in M_2(A),
\]
used there holds for arbitrary contractive completely positive maps, see \[29\] Lemma 5.3 (ii)]. Plainly, (5) implies that for any \(C^\ast\)-subalgebra \(C\) of \(A\) such that \(g : C \to B\) is a homomorphism we have \(C \subseteq MD(g)\). \(\square\)

We recall that any positive map \(g : A \to B\) is automatically completely positive whenever \(A\) or \(B\) is commutative \[17\] Corollary 3.5, Proposition 3.9]. Of course any homomorphism is a completely positive contraction. Also it is well known, cf., for instance, \[47\], III, Theorem 3.4, IV, Corollary 3.4], that if \(B\) is a \(C^\ast\)-subalgebra of \(A\) then for a linear idempotent \(E : A \to B\) we have
\[
E \text{ is contractive} \iff E \text{ is positive and } B \subseteq MD(E)
\]
\[
\iff E \text{ is completely positive and } B \subseteq MD(E)
\]
An idempotent \(E\) satisfying the above equivalent conditions is called a conditional expectation.

Definition 2.8. Let \(g : A \to B\) be a positive map. We say that \(g\) is strict if \(\{g(\mu_\lambda)\}_{\lambda \in \Lambda}\) is strictly convergent in \(M(A)\) for some approximate unit \(\{\mu_\lambda\}_{\lambda \in \Lambda}\) in \(A\). We say \(g\) is extendible if it extends to a strictly continuous mapping \(\overline{g} : M(A) \to M(B)\).

Remark 2.9. The positive elements in \(A\) are strictly dense in the set of positive elements in \(M(A)\). Thus if \(g\) is an extendible (completely) positive map, then \(\overline{g} : M(A) \to M(B)\) is also (completely) positive. Clearly, every extendible map is strict, and it is well known that for homomorphisms these notions are actually equivalent.

2.2. Exel’s and Exel-Royer’s crossed products. Initially, Exel defined his crossed product for unital \(C^\ast\)-algebras \[12\], and then it was generalized in \[9\], \[33\] to Exel systems that compose of extendible maps. Nevertheless, the definition of the crossed product makes sense for an arbitrary Exel system and can be expressed as follows.

Definition 2.10. Let \(\alpha : A \to A\) be an endomorphism of a \(C^\ast\)-algebra \(A\) and let \(\mathcal{L} : A \to A\) be a positive linear map such that
\[
(6) \quad \mathcal{L}(aa(b)) = \mathcal{L}(a)b, \quad \text{for all } a, b \in A.
\]
Then \(\mathcal{L}\) is called a transfer operator for \(\alpha\) and the triple \((A, \alpha, \mathcal{L})\) is an Exel system.

Definition 2.11. A representation of an Exel system \((A, \alpha, \mathcal{L})\) is a pair \((\pi, S)\) consisting of a non-degenerate representation \(\pi : A \to B(H)\) and an operator \(S \in B(H)\) such that
\[
(7) \quad S\pi(a) = \pi(\alpha(a))S \quad \text{and} \quad S^*\pi(a)S = \pi(\mathcal{L}(a)) \quad \text{for all } a \in A.
\]
A redundancy of a representation \((\pi, S)\) of \((A, \alpha, \mathcal{L})\) is a pair \((\pi(a), k)\) where \(a \in A\) and \(k \in \overline{\pi(A)SS^*\pi(A)}\) are such that
\[
\pi(a)\pi(b)S = k\pi(b)S, \quad \text{for all } b \in A.
\]
The Toeplitz algebra \(T(A, \alpha, \mathcal{L})\) of \((A, \alpha, \mathcal{L})\) is the \(C^\ast\)-algebra generated by \(i_A(A) \cup i_A(A)t\) for a universal representation \((i_A, t)\) of \((A, \alpha, \mathcal{L})\). Exel’s crossed product \(A \times_{\alpha, \mathcal{L}} \mathbb{N}\) of \((A, \alpha, \mathcal{L})\) is the quotient \(C^\ast\)-algebra of \(T(A, \alpha, \mathcal{L})\) by the ideal generated by the set
\[
\{i_A(a) - k : a \in A, k = 1\} \quad \text{and} \quad (i_A(a), k) \text{ is a redundancy of } (i_A, t).
\]
Existence of the universal representation \((i_A, t)\) and the \(C^\ast\)-algebra \(T(A, \alpha, \mathcal{L})\), considered above, can be obtained by realizing \(T(A, \alpha, \mathcal{L})\) as a Toeplitz algebra of a \(C^\ast\)-correspondence \(M_\mathcal{L}\), see \[12\], \[8\], \[9\], \[14\]. Let us recall the construction of \(M_\mathcal{L}\).
Let \((A, \alpha, \mathcal{L})\) be an Exel system. One makes \(A\) into a semi-inner product (right) \(A\)-module \(A\) by putting \(m \cdot a := ma(a), \langle m, n \rangle := \mathcal{L}(m^*n), m, n \in A, a \in A\), and defines \(M_\mathcal{L}\) to be the associated Hilbert \(A\)-module:

\[
M_\mathcal{L} := \frac{A\mathcal{L}}{N}, \quad N := \{m \in A\mathcal{L} : \langle m, m \rangle = 0\}.
\]

Denoting by \(q : A\mathcal{L} \to M_\mathcal{L}\) the quotient map one gets, cf. [12, 8], that

\[
a \cdot q(m) := q(am), \quad m \in A, \quad a \in A,
\]

yields a well defined left action of \(A\) on \(M_\mathcal{L}\) making \(M_\mathcal{L}\) into a \(C^\ast\)-correspondence, such that \(A \cdot M_\mathcal{L} = M_\mathcal{L}\).

The following fact was proved in [9, Lemmas 3.2, 3.3] for extendible Exel systems. However, the proofs exploit only extendability of \(L\) which as we will show (see Proposition 1.2) is automatic. Moreover, the following proposition will follow from our more general results, cf. Corollary 4.3 below.

**Proposition 2.12.** We have a one-to-one correspondence between representations \((\pi, S)\) of \((A, \alpha, \mathcal{L})\) and representations \((\pi, \pi M_\mathcal{L})\) of the \(C^\ast\)-correspondence \(M_\mathcal{L}\). In particular, \(T(A, \alpha, \mathcal{L}) \cong T(M_\mathcal{L})\).

Previous versions of the above result were a point of departure in [14]. More specifically, the authors of [14] considered ‘partial Exel systems’ \((A, \alpha, \mathcal{L})\) where \(\mathcal{L}\) is not everywhere defined and \(\alpha\) may attain values outside of \(A\). For such triples they defined a crossed product \(O(A, \alpha, \mathcal{L})\), in essence, simply to be \(O M_\mathcal{L}\), cf. [6, Proposition 4.5], where \(M_\mathcal{L}\) is a generalization of the \(C^\ast\)-correspondence defined above to the ‘partial case’. In the present paper we will only make use of [14, Definition 1.6] applied to ‘global’ Exel systems. Thus we adopt the following

**Definition 2.13.** The Exel-Royer’s crossed product \(O(A, \alpha, \mathcal{L})\) associated to an Exel system \((A, \alpha, \mathcal{L})\) is the quotient of \(T(A, \alpha, \mathcal{L})\) by the ideal generated by the set

\[
\{i_A(a) - k : a \in (\ker \phi)^\perp \cap J(M_\mathcal{L}) \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}
\]

where \(\phi\) is the homomorphism implementing left action of \(A\) on \(M_\mathcal{L}\).

### 3. Crossed products by completely positive maps

Throughout this section, we fix a completely positive map \(\varrho : A \to A\), and refer to the pair \((A, \varrho)\) as to a \(C^\ast\)-dynamical system. We introduce relative crossed products \(C^\ast(A, \varrho; J)\) as quotients of a certain Toeplitz algebra. Then we realize them as relative Cuntz Pimsner algebras and as universal \(C^\ast\)-algebras generated by appropriately defined covariant representations of \((A, \varrho)\). At the end of this section we discuss two important special cases when: 1) \(\varrho\) is multiplicative; 2) \(A\) is commutative.

#### 3.1. Crossed products

Following the original idea of Exel [12], we first define a Toeplitz algebra, and then construct crossed products by ‘eliminating redundancies’ in the latter.

**Definition 3.1.** A representation of \((A, \varrho)\) is a pair \((\pi, S)\) consisting of a non-degenerate representation \(\pi : A \to B(H)\) and an operator \(S \in B(H)\) such that

\[
(8) \quad S^\ast \pi(a)S = \pi(\varrho(a)) \quad \text{for all } a \in A.
\]

If \(\pi\) is faithful we call \((\pi, S)\) faithful. We denote by \(C^\ast(\pi, S)\) the \(C^\ast\)-algebra generated by \(\pi(A) \cup \pi(A)S\). We define the Toeplitz algebra of \((A, \varrho)\) to be the \(C^\ast\)-algebra \(T(A, \varrho) := C^\ast(i_A(A), t)\) where \((i_A, t)\) is the universal representation of \((A, \varrho)\). This means that for any representation \((\pi, S)\) of \((A, \varrho)\) the mappings

\[
(9) \quad i_A(a) \mapsto \pi(a), \quad i_A(a)t \mapsto \pi(a)S, \quad a \in A,
\]

define the epimorphism from \(T(A, \varrho)\) onto \(C^\ast(\pi, S)\).
Remark 3.2. Existence of the universal representation \((i_A, t)\) can be shown using Zorn’s lemma and the fact that direct sum of any set of representations of \((A, \varrho)\) is again a representation of \((A, \varrho)\) (the latter is clear because, by non-degeneracy of \(\pi\) and Lemma 2.1 [S], implies that \(\|S\|^2 = \|\pi \circ \varrho\| \leq \|\varrho\|\)). The uniqueness of \(\mathcal{T}(A, \varrho)\), up to a natural isomorphism, can be readily deduced from universality of \((i_A, t)\). Nevertheless, both existence of \((i_A, t)\) and uniqueness of \(\mathcal{T}(A, \varrho)\) will follow from Proposition 3.10 below, and the corresponding facts known for \(C^*\)-correspondences.

To study the structure of \(C^*(\pi, S) = C^*(\pi(A) \cup \pi(A)S)\) one needs to understand the relationship between the following ‘monomials’:

\[
\pi(a)S\pi(b)\pi(c), \quad \pi(a)\pi^*(b)\pi^*(c), \quad \pi(a)\pi^*(b)\pi(c), \quad \pi(a)\pi(b)\pi^*(c).
\]

As we will see in the course of our analysis, the first two behave like ‘simple tensors’, and by [S] the third one is in \(\pi(A)\). Establishing the relationship with the fourth ‘monomial’ requires determining additional data which is encoded in an ideal that we are about to introduce. This ideal is closely related to the notion of redundancy.

Definition 3.3. By a redundancy of a representation \((\pi, S)\) of \((A, \varrho)\) we mean a pair \((\pi(a), k)\) such that \(a \in A, k \in \pi(A)S\pi(A)\pi^*(A)\) and

\[
\pi(a)\pi(b)S = k\pi(b)S \quad \text{for all} \quad b \in A.
\]

Let \(J_{(\pi, S)}\) be the set of elements \(a \in A\) such that \((\pi(a), k)\) is a redundancy of \((\pi, S)\) with \(\pi(a) = k\). Clearly, it is an ideal in \(A\) given by

\[
J_{(\pi, S)} = \{a \in A : \pi(a) \in \pi(A)S\pi(A)\pi^*(A)\}.
\]

We call it the ideal of covariance for \((\pi, S)\).

Remark 3.4. Using [S], we see that \(\pi(A)S\pi(A)\pi^*(A)\) is a \(C^*\)-algebra that acts on the space \(\pi(A)S\). Moreover, this action is faithful. Thus, if \((\pi(a), k)\) is a redundancy of \((\pi, S)\), then \(k\) is uniquely determined by \(a\), and \(\pi(a) = k\) if and only if \(a \in J_{(\pi, S)}\).

Let us consider the GNS-kernel \(N_\varrho\) of \(\varrho\), see [M], and a representation \((\pi, S)\) of \((A, \varrho)\). If \(a \in N_\varrho\) then the pair \((\pi(a), 0)\) is necessarily a redundancy because \(\|\pi(a)\pi(b)S\|^2 = \|\varrho((ab)^*ab)\| = 0\), for all \(b \in A\). In particular, \(a \in J_{(\pi, S)} \cap N_\varrho\) implies \(\pi(a) = 0\). Accordingly, if \((\pi, S)\) is faithful then \(J_{(\pi, S)} \subseteq N_\varrho^\perp\). This explains the special role of the ideal \(N_\varrho^\perp\) in the following definition.

Definition 3.5. We define the crossed product \(C^*(A, \varrho)\) of \(A\) by \(\varrho\) to be the quotient of the Toeplitz \(C^*\)-algebra \(\mathcal{T}(A, \varrho)\) by the ideal generated by the set

\[
\{i_A(a) - k : a \in N_\varrho^\perp \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}.
\]

More generally, for any ideal \(J\) in \(A\) we define the relative crossed product \(C^*(A, \varrho; J)\) relative to \(J\) to be the quotient of the Toeplitz \(C^*\)-algebra \(\mathcal{T}(A, \varrho)\) by the ideal generated by the set

\[
\{i_A(a) - k : a \in J \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}.
\]

We denote by \((j_A, s)\) the representation of \((A, \varrho)\) that generates \(C^*(A, \varrho; J)\).

3.2. Crossed products as relative Cuntz-Pimsner algebras. A \(C^*\)-correspondence associated to a completely positive map was already considered by Paschke [10] section 5] and sometimes is called a GNS or a KSGNS-correspondence (for Kasparov, Stinespring, Gelfand, Naimark, Segal), cf. [29], [18]. Namely, we let \(X_\varrho\) to be a Hausdorff completion of the algebraic tensor product \(A \otimes A\) with respect to the seminorm associated to the \(A\)-valued sesquilinear form given by

\[
\langle a \otimes b, c \otimes d \rangle_\varrho := b^*\varrho(a^*c)d, \quad a, b, c, d \in A.
\]
In the sequel we use the symbol $a \otimes b$ to denote the image of the simple tensor $a \otimes b$ in $X_\varrho$. The space $X_\varrho$ becomes a $C^*$-correspondence over $A$ with the left and right actions determined by: $a \cdot (b \otimes c) = (ab) \otimes c$ and $(b \otimes c) \cdot a = b \otimes (ca)$ where $a, b, c \in A$.

**Definition 3.6.** We call $X_\varrho$ defined above the $C^*$-correspondence of $(A, \varrho)$.

**Remark 3.7.** Clearly, the $C^*$-correspondence $X_\varrho$ is essential. The GNS-kernel (3) of $\varrho$ coincides with the kernel of the left action of $A$ on $X_\varrho$. Hence the left action of $A$ on $X_\varrho$ is faithful if and only if $\varrho$ is almost faithful on $A$.

If $A$ is not unital we give a meaning to the symbol $a \otimes 1, a \in A$, using the following lemma.

**Lemma 3.8.** Let $\{\mu_\lambda\}_{\lambda \in A}$ be an approximate unit in $A$. Then, for any $a \in A$, the limit
\[
(11) \quad a \otimes 1 := \lim_{\lambda \in A} a \otimes \mu_\lambda
\]
equalspace exists and defines a bounded linear map $A \ni a \mapsto a \otimes 1 \in X_\varrho$ of norm $\|\varrho\|^\frac{1}{2}$.

**Proof.** Since $\|a \otimes (\mu_\lambda - \mu_{\lambda'})\|^2 = \| (\mu_\lambda - \mu_{\lambda'}) \varrho(a^*a)(\mu_\lambda - \mu_{\lambda'}) \|$ tends to zero, as $\lambda$ and $\lambda'$ tend to ‘infinity’, the net $\{a \otimes \mu_\lambda\}_{\lambda \in A}$ is Cauchy and hence convergent. Since
\[
\sup_{a \in A, \|a\|=1} \|a \otimes 1\|^2 = \sup_{a \in A, \|a\|=1} \|\varrho(a^*a)\| = \|\varrho\|,
\]
equalspace we see that $A \ni a \mapsto a \otimes 1 \in X_\varrho$ is a bounded operator of norm $\|\varrho\|^\frac{1}{2}$. $\square$

**Remark 3.9.** Let $\{\mu_\lambda\}_{\lambda \in A}$ be an approximate unit in $A$. The limit
\[
(12) \quad 1 \otimes a := \lim_{\lambda \in A} \mu_\lambda \otimes a, \quad a \in A,
\]
equalspace in general may not exist, but if $\varrho$ is strict then it does exist and the map $A \ni a \mapsto 1 \otimes a \in X_\varrho$ is linear bounded, again of norm $\|\varrho\|^\frac{1}{2}$, see [29] p. 50.

The mapping in the latter remark, which exists when $\varrho$ is strict, plays a key role in the construction of KSGNS-dilation of $\varrho$, cf. [29] Theorem 5.6]. We adjust this construction to get a description of representations of the $C^*$-correspondence $X_\varrho$ for arbitrary $\varrho$.

**Proposition 3.10.** We have a one-to-one correspondence between representations $(\pi, S)$ of $(A, \varrho)$ and representations $(\pi, \pi_X)$ of the $C^*$-correspondence $X := X_\varrho$ of $(A, \varrho)$ where
\[
(13) \quad \pi_X(a \otimes b) = \pi(a)S\pi(b), \quad a \otimes b \in X,
\]
equalspace and for any approximate unit $\{\mu_\lambda\}_{\lambda \in A}$ in $A$:
\[
(15) \quad S = s\lim_{\lambda \in A} \pi_X(\mu_\lambda \otimes 1)
\]
equalspace where the limit is taken in the strong operator topology. If $\varrho$ is strict, the limit in (15) is strictly convergent in $M(C^*(\pi, S))$, and the multiplier $S \in M(C^*(\pi, S))$ is determined by the formula $S\pi(a) = \pi_X(1 \otimes a), a \in A$, cf. Remark 3.9.

**Proof.** Let $(\pi, S)$ be a representation of $(A, \varrho)$ on $H$. The following computation
\[
\left( \sum_i \pi(a_i)S\pi(b_i) \right) \ast \sum_j \pi(c_j)S\pi(d_j) = \sum_{i,j} \pi(b_i^*c_j^*)S^*\pi(a_i^*c_j)S\pi(d_j) = \sum_{i,j} \pi(b_i^*\varrho(a_i^*c_j)d_j) = \pi(\sum_i a_i \otimes b_i, \sum_j c_j \otimes d_j)_{\varrho}
\]
equalspace
implies that the mapping (13) extends to a linear contractive map \( \pi_X : X \to B(H) \). It is evident that \((\pi, \pi_X)\) is a representation of \(X\). We have \( S^*\pi_X(a \otimes b) = S^*\pi(a)S\pi(b) = \pi(\varrho(a)b)\), for any \(a, b \in A\). Since the range of \(S\) is contained in \(\pi(A)SH = \pi(A)\pi(X)H \subseteq \pi_X(X)H\) we get \(S^*|_{(\pi_X(X)H)} \equiv 0\). Thus (14) holds. Moreover, using the operator ‘\(*1\)’ introduced in Lemma 3.8 we get

\[
\pi(A)S = \pi_X(A \otimes 1) \subseteq \pi_X(X) = (A)\pi(A).
\]

Hence \(C^*(\pi, S) = C^*(\pi(A) \cup \pi(A)S) = C^*(\pi(A) \cup \pi(A)\pi(A)) = C^*(\pi(A) \cup \pi_X(X))\).

Suppose now that \((\pi, \pi_X)\) is a representation of \(X\). We need to show that there exists an operator \(S \in B(H)\) such that \((\pi, S)\) is a representation of \((A, \varrho)\) satisfying (13). Let \(\{\mu_\lambda\}_{\lambda \in \Lambda}\) be an approximate unit in \(A\) and consider the net of bounded operators \(S_\lambda := \pi_X(\mu_\lambda \otimes 1)\), \(\lambda \in \Lambda\). Note that \(S_\lambda \pi(a) = \pi(\mu_\lambda \otimes a)\), \(a \in A\). To see that (15) determines a bounded operator it suffices to show that the net \(\{S_\lambda\}_{\lambda \in \Lambda}\) is strongly Cauchy. To this end, let \(a \in A\), \(h, h \in H\) and \(\lambda \leq \lambda', \lambda \in \Lambda\) in the directed set \(\Lambda\). Then

\[
\| (S_\lambda - S_{\lambda'})\pi(a)h\|^2 = \|\pi_X((\mu_\lambda - \mu_{\lambda'}) \otimes a)h\|^2
\]

\[
= \langle h, \pi((\mu_\lambda - \mu_{\lambda'}) \otimes a) \rangle = \langle h, \pi(\mu_\lambda \otimes a) \rangle
\]

\[
\leq \langle h, \pi(\mu_\lambda \otimes a) \rangle = \langle \pi(a)h, \pi(\mu_\lambda \otimes a) \rangle.
\]

Since the net \(\{\pi(\mu_\lambda)\}_{\lambda \in \Lambda}\) is strongly convergent the last expression tends to zero. Hence \(S := \varrho\lim_{\lambda \in \Lambda} S_\lambda\) defines a bounded operator. Let \(a, b \in A\). As \(\| (a - a\mu_\lambda) \otimes b\|^2 = \|b^* \varrho(\mu_\lambda a^*) \otimes (a - a\mu_\lambda)\|\) tends to zero, we get that \(a \otimes b = \varrho\lim_{\lambda \in \Lambda} a\mu_\lambda \otimes b\) in \(X\). Thus

\[
\pi_X(a \otimes b) = \varrho\lim_{\lambda \in \Lambda} \pi_X(a\mu_\lambda \otimes b) = \pi(a) \lim_{\lambda \in \Lambda} S_\lambda \pi(b) = \pi(a) S \pi(b),
\]

that is (16) holds. Moreover, for any \(a, b \in A\) and \(h, f \in H\) we have

\[
\langle \pi_X(a \otimes b)h, Sf \rangle = \varrho\lim_{\lambda \in \Lambda} \langle \pi_X(a \otimes b)h, \pi_X(\mu_\lambda \otimes 1)f \rangle
\]

\[
= \varrho\lim_{\lambda \in \Lambda} \langle \pi(\mu_\lambda a)h, f \rangle = \langle \pi(a)h, f \rangle.
\]

Hence \(S^*\pi_X(a \otimes b) = \pi(\varrho(a)b)\) and therefore \(S^*\pi_X(a \otimes b) = S^*\pi_X(a \otimes b) = \pi(\varrho(a)) \pi(b)\). Since \(\pi\) is non-degenerate this implies (15).

Suppose now \(\varrho\) is strict. Then, in view of Remark 3.9 for any \(a \in A\) the following limit exists

\[
\lim_{\lambda \in \Lambda} S_\lambda(\pi(a)) = \lim_{\lambda \in \Lambda} \pi_X(\mu_\lambda \otimes a) = \pi_X(1 \otimes a).
\]

Similarly, we get \(\lim_{\lambda \in \Lambda} \pi(a)S_\lambda = \pi_X(a \otimes 1)\). Thus, since \(\pi(A)\) is non-degenerate in \(C^*(\pi, S) = C^*(\pi(A) \cup \pi(A)\pi(A))\), the limit in (15) is strictly convergent. \(\square\)

**Remark 3.11.** Let \(j_\lambda, s\) be the representation of \((A, \varrho)\) that generates the crossed product \(C^*(A, \varrho; J)\), where \(J\) is an ideal in \(A\). By the above proposition \(s\) is an element of the enveloping von Neumann algebra \(C^*(A, \varrho; J)^{**}\). If \(\varrho\) is strict then actually \(s \in M(C^*(A, \varrho; J))\).

The following lemma is inspired by [3, Lemma 3.5]. It implies that when considering the relative crossed products it suffices to restrict attention to ideals \(J\) contained in the ideal \(J(X_\varrho) = \varphi^{-1}(\mathcal{K}(X_\varrho))\). It will also lead us to the main result of this subsection.

**Lemma 3.12.** Let \((\pi, S)\) be a faithful representation of \((A, \varrho)\) and let \((\pi, \pi_X)\) be the representation of \(X = X_\varrho\) with \(\pi_X\) given by (13). A pair \((\pi(a), k)\) is a redundancy of \((\pi, S)\) if and only if \(a \in J(X)\) and \(k = (\pi, \pi_X)(1)(\varphi(a))\).
Proof. Note that \((\pi, \pi_X)^{(1)} : \mathcal{K}(X) \to \overline{\pi_X(X)\pi_X(X)^*} = \overline{\pi(A)\pi(A)S^*\pi(A)}\). Thus if we suppose that \(a \in J(X)\) and \(k = (\pi, \pi_X)^{(1)}(\phi(a))\) then \(k \in \pi(A)\pi(A)S^*\pi(A)\). Moreover, for any \(b, c \in A\) we have
\[
\pi(a)\pi(b)\pi(c) = \pi(\pi_X(b \otimes c)) = \pi_X(\phi(a) b \otimes c) = (\pi, \pi_X)^{(1)}(\phi(a))\pi_X(b \otimes c)
\]
\[
= (\pi, \pi_X)^{(1)}(\phi(a))\pi(b)\pi(c).
\]
Hence by non-degeneracy of \(\pi\), we see that \((\pi(a), k)\) is a redundancy.

Now let \((\pi(a), k)\) be any redundancy. Then \(k = (\pi, \pi_X)^{(1)}(t)\) for a certain \(t \in \mathcal{K}(X)\), and for any \(b, c \in A\) we have
\[
\pi_X(\phi(a) b \otimes c) = \pi(a)\pi_X(b \otimes c) = \pi(a)\pi(b)\pi(c) = k(b)\pi(c)
\]
\[
= (\pi, \pi_X)^{(1)}(t)\pi_X(b \otimes c) = \pi_X(t(b \otimes c)).
\]
Since faithfulness of \(\pi\) implies injectivity of \(\pi_X\), we get \(\phi(a) b \otimes c = tb \otimes c\), for all \(b, c \in A\). Consequently, \(\phi(a) = t\) as desired. \(\Box\)

**Theorem 3.13.** Let \(X = X_\varphi\) be the \(C^*\)-correspondence of \((A, g)\). For any ideal \(J\) in \(A\) we have
\[
C^*(A, g; J) = C^*(A, g; J \cap J(X)) \cong \mathcal{O}(J \cap J(X), X).
\]
The universal homomorphism \(j_A : A \to C^*(A, g; J)\) is injective if and only if \(g\) is almost faithful on \(J \cap J(X)\), that is if and only if \(J \cap J(X) \subseteq N_\varphi^+\). In particular,
\[
C^*(A, g) \cong \mathcal{O}_X
\]
and \(j_A : A \to C^*(A, g)\) is injective.

**Proof.** By Proposition 3.10 Toeplitz algebras \(T(A, g)\) and \(T(X)\) are naturally isomorphic and identifying them explicitly we may assume that the universal representation \((i_A, i_X)\) of \(X\) in \(T(X)\) satisfies \(i_X(a \otimes b) = i_A(a)i_X(b)\) for \(a, b \in A\). Then \(T(A, g) = T(X)\) and by Lemma 3.12
\[
\{i_A(a) - k : a \in J \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}
\]
\[
= \{i_A(a) - k : a \in J \cap J(X) \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}
\]
\[
= \{i_A(a) - (i_A, i_X)^{(1)}(\phi(a)) : a \in J \cap J(X)\}.
\]
Hence the three algebras \(C^*(A, g; J)\), \(C^*(A, g; J \cap J(X))\) and \(\mathcal{O}(J \cap J(X), X)\) arise as quotients of the same algebra by the same ideal. Thus they are isomorphic. For the remaining part of the assertion it suffices to recall, see [35 Proposition 2.21] or [8 Lemma 2.2], that the universal homomorphism of \(A\) into \(\mathcal{O}(J \cap J(X), X)\) is injective if and only if \(J \cap J(X) \subseteq (\ker \phi)^+ = N_\varphi^+\). \(\Box\)

**Remark 3.14.** In [15] Subsection 3.3 a crossed product by a completely positive map \(g\) was defined as Pimsner’s (augmented) \(C^*\)-algebra associated to the \(C^*\)-correspondence \(X_\varphi\). Thus this crossed product coincides with our relative crossed product \(C^*(A, g, A) = C^*(A, g, J(X))\).

**Remark 3.15.** Using the notion of multiplicative domain, an interaction \((\mathcal{V}, \mathcal{H})\) on a \(C^*\)-algebra \(A\) introduced in [13] Definition 3.1] can be defined as a pair of positive maps \(\mathcal{V}, \mathcal{H} : A \to A\) such that
\[
\mathcal{V}\mathcal{H}\mathcal{V} = \mathcal{V}, \quad \mathcal{H}\mathcal{V}\mathcal{H} = \mathcal{H}, \quad \mathcal{V}(A) \subseteq MD(\mathcal{H}), \quad \mathcal{H}(A) \subseteq MD(\mathcal{V}).
\]
Then \(\mathcal{V}\) and \(\mathcal{H}\) are automatically contractive completely positive maps, see [13] Corollary 3.3. In [25] the author considered an interaction \((\mathcal{V}, \mathcal{H})\) on a unital \(C^*\)-algebra \(A\) such that the ranges \(\mathcal{V}(A)\), \(\mathcal{H}(A)\) are corners in \(A\). By [25] Corollary 2.16, the crossed product \(C^*(A, \mathcal{V}, \mathcal{H})\) defined in [25] coincides with the covariance algebra associated to \((\mathcal{V}, \mathcal{H})\) in [13]. By [25] Proposition 2.14, \(C^*(A, \mathcal{V}, \mathcal{H})\) can be realized as a crossed product by a Hilbert bimodule \(X_\mathcal{V}\). Thus combining Theorem 3.13 and [20] Proposition 3.7, we get
\[
C^*(A, \mathcal{V}, \mathcal{H}) \cong C^*(A, \mathcal{V}) \cong C^*(A, \mathcal{H}),
\]
where $C^*(A, \mathcal{V})$ and $C^*(A, \mathcal{H})$ are crossed products in the sense of Definition 3.5.

### 3.3. Universal description and gauge-invariant uniqueness theorem

We describe the $C^*$-algebra $C^*(A, g, J)$ as a universal object in the following way.

**Definition 3.16.** Let $J$ be an ideal in $A$. We say that a representation $(\pi, S)$ of $(A, g)$ is $J$-covariant if $J \cap J(X_g) \subseteq J(\pi, S)$ or just covariant if $N_0^+ \cap J(X_g) \subseteq J(\pi, S)$.

**Proposition 3.17.** Let $J$ be an ideal in $A$. The relative crossed product $C^*(A, g; J)$ is universal with respect to $J$-covariant representations of $(A, g)$, that is $(j_A, s)$ is $J$-covariant and for every $J$-covariant representation $(\pi, S)$ of $(A, g)$ the mapping

$$j_A(a) \mapsto \pi(a), \quad j_A(a)s \mapsto \pi(a)S, \quad a \in A,$$

extends to a (necessarily unique) epimorphism $\pi \times_J S : C^*(A, g; J) \to C^*(\pi, S)$.

**Proof.** That $(j_A, s)$ is $J$-covariant follows from the equality (16) and the definition of $C^*(A, g; J)$. Let $(\pi, S)$ be a $J$-covariant representation of $(A, g)$, and let $\alpha = \beta \times_{[0]} S : T(A, g) \to C^*(\pi, S)$ be the epimorphism determined by (9). Clearly, $\pi \times_{[0]} S$ maps redundancies of $(i_A, i_X)$ onto redundancies of $(\pi, S)$. Thus in view of (16), using $J \cap J(X_g) \subseteq J(\pi, S)$, we conclude that $\pi \times_{[0]} S$ factors through to the desired epimorphism $\pi \times_J S : C^*(A, g; J) \to C^*(\pi, S)$. \hfill \( \square \)

Using Katsura's gauge-invariant uniqueness theorem for Cuntz-Pimsner algebras [21, Theorem 6.4] we get the following version of this standard tool for $C^*(A, g)$. One could formulate a more general result for relative crossed products $C^*(A, g; J)$, cf. for instance [24, Theorem 9.1], but we will not need it here.

**Proposition 3.18.** Let $(\pi, S)$ be a covariant representation of $(A, g)$. The epimorphism given by (18) is an isomorphism:

$$C^*(\pi, S) \cong C^*(A, g)$$

if and only if $(\pi, S)$ is faithful and there exists a strongly continuous action $\beta : T \to \text{Aut}(C^*(\pi, S))$ such that $\beta_z(\pi(a)) = \pi(a)$ and $\beta_z(\pi(a)S) = z\pi(a)S$ for all $a \in A$ and $z \in T$.

**Proof.** If $(\pi, S)$ is a faithful covariant representation of $(A, g)$ then by Lemma 3.12 the corresponding representation $(\pi, \pi_X)$ of $X_g$ is covariant in the sense of [21, Definition 3.4]. Since $C^*(A, g) \cong \mathcal{O}_X$, by Theorem 3.13 it suffices to apply [21, Theorem 6.4]. \hfill \( \square \)

### 3.4. The case when $g$ is multiplicative

In this section we assume that $g$ is multiplicative and denote it by $\alpha$. In other words, we assume that $\alpha : A \to A$ is an endomorphism. We show that our relative crossed products $C^*(A, \alpha; J)$ coincide with various crossed products by endomorphisms appearing in the literature. The latter are typically studied in the case $\alpha$ is extendible.

**Lemma 3.19.** If $\alpha : A \to A$ is an endomorphism, then the $C^*$-correspondence $X_\alpha$ associated to $\alpha$ is isomorphic to the $C^*$-correspondence defined by the formulas

$$E_\alpha := \alpha(A)A, \quad (x, y)_A := x^*y, \quad a \cdot x \cdot b := \alpha(a)xb, \quad x, y \in \alpha(A)A, a, b \in A.$$

In particular, we have $J(X_\alpha) = J(E_\alpha) = A$.

**Proof.** We leave it to the reader, as an easy exercise, to check that $X_\alpha \ni a \otimes b \to \alpha(a)b \in E_\alpha$ determines the desired isomorphism of $C^*$-correspondences. For every $a \in A$ and $x \in E$ we can write $a = a_1a_2$, where $a_1, a_2 \in A$, and then $a \cdot x = \Theta_{\alpha(a_1), \alpha(a_2)}x$. Thus $J(E_\alpha) = A$. \hfill \( \square \)

**Proposition 3.20.** Let $\alpha : A \to A$ be an endomorphism and let $(\pi, S)$ be a representation of $(A, \alpha)$. Then we automatically have that $S$ is a partial isometry,

$$\pi(a)S = S\pi(\alpha(a)), \quad \text{for all } a \in A,$$

where $C^*(A, \mathcal{V})$ and $C^*(A, \mathcal{H})$ are crossed products in the sense of Definition 3.5.
and

\[(20) \quad J_{(\pi,S)} = \{ a \in A : SS^*\pi(a) = \pi(a) \}. \]

Thus if \( \alpha \) is extendible, then

i) relative crossed products introduced (with an increasing degree of generality) in [28 Definition 1.12], [28 Definition 4.8] and [27 Definition 2.9], coincide with crossed products \( C^*(A,\alpha; J) \) introduced in this paper;

ii) Stacey’s crossed product \( A \rtimes_1 \mathbb{N} \), [16 Definition 3.1], is naturally isomorphic to \( C^*(A,\alpha; A) \);

iii) the partial-isometric crossed product \( A \rtimes \mathbb{N} \) defined in [34, p. 73] is naturally isomorphic to the Toeplitz algebra \( T(A,\alpha) = C^*(A,\alpha; \{0\}) \).

**Proof.** Let \( \{\mu_\lambda\}_{\lambda \in \Lambda} \) be an approximate unit in \( A \). Multiplicativity of \( \alpha \) implies that the increasing net \( \{\pi(\alpha(\mu_\lambda))\}_{\lambda \in \Lambda} \) converges strongly to a projection in \( B(H) \). By \( \mathbf{5} \), and non-degeneracy of \( \pi \), we have \( \lim \pi(\alpha(\mu_\lambda)) = \lim S\pi(\alpha(\mu_\lambda))S = SS^* \). Hence \( S \) is a partial isometry. Now the argument in the proof of [28 Lemma 1.2] shows the commutation relation \( \mathbf{19} \). By \( \mathbf{19} \) and its adjoint version \( \pi(a) = \pi(\alpha(a))S^* \), \( a \in A \) we have \( \pi(A)S\pi(A)S^*\pi(A) = SS^*\alpha(\alpha(A)A\alpha(A))S^* \). Since \( S \) is partial isometry, this implies that \( J_{(\pi,S)} = \{ a \in A : SS^*\pi(a) = \pi(a) \} \). Conversely, for any \( a \in A \) such that \( SS^*\pi(a) = \pi(a) \), again by \( \mathbf{19} \), we have

\[ \pi(a) = SS^*\pi(a) = S\pi(\alpha(a))S^* \in \pi(\alpha(A)A\alpha(A))S^* = \pi(A)S\pi(A)S^*\pi(A). \]

This proves \( \mathbf{20} \). In particular, since \( J(X_{\alpha}) = A \), see Lemma 3.19, for any ideal \( J \) in \( A \) we have

\[(21) \quad (\pi,S) \text{ is } J\text{-covariant} \iff J \subseteq \{ a \in A : SS^*\pi(a) = \pi(a) \}. \]

Now assume that \( \alpha \) is extendible.

i). In view of \( \mathbf{21} \), \( (\pi,S) \) is a \( J \)-covariant representation if and only if \( (\pi,S^*) \) is a \( J \)-covariant representation of \( (A,\alpha) \) in the sense of [26 Definition 4.5], see also [27 Subsection 2.3]. Thus \( C^*(A,\alpha; J) \) coincides with the relative crossed product introduced, as the corresponding universal \( C^* \)-algebra, in [26 Definition 4.8] (for \( J \subseteq (\ker \alpha)^\perp = N_\alpha \)), and in [27 Definition 2.9] (for arbitrary \( J \)). In the case \( J \subseteq (\ker \alpha)^\perp \) there always exists a faithful representation \( (\pi,S) \) of \( (A,\alpha) \) such that \( J = \{ a \in A : SS^*\pi(a) = \pi(a) \} \), see [26 Proposition 4.12] or [28 Proposition 1.8]. Thus the crossed products introduced in [28 Definition 1.12], for \( A \) unital and \( J \subseteq (\ker \alpha)^\perp \), coincide with our \( C^* \)-algebras \( C^*(A,\alpha; J) \).

ii). That \( A \rtimes_1 \mathbb{N} \) is naturally isomorphic to \( C^*(A,\alpha; A) \) follows from item i) and [27 Proposition 2.14].

iii). If \( (\pi,S) \) is a representation of \( (A,\alpha) \), then using \( \mathbf{19} \), and its adjoint version, we get

\[ SS^*\pi(a) = S\pi(\alpha(a))S^* = \pi(a)SS^*, \quad \text{for all } a \in A. \]

Thus \( (\pi,S^*) \) is a covariant partial-isometric representation of \( (A,\alpha) \) in the sense of [34 Definition 4.1] (the second relation in [34 Formula (4.1)] is superfluous). Accordingly, the crossed product \( A \rtimes \mathbb{N} \) defined in [34, p. 73] (in the semigroup setting) is naturally isomorphic to \( T(A,\alpha) \).

\[ \square \]

**Remark 3.21.** It follows from \( \mathbf{21} \) and Remark 3.11 that if \( \alpha : A \to A \) is an endomorphism, then \( C^*(A,\alpha) \) can be viewed as a universal \( C^* \)-algebra generated by a copy of \( A \) and the space \( uA \) where \( u \in C^*(A,\alpha)^* \) is a partial isometry subject to relations

\[ \alpha(a) = uau^* \quad \text{and} \quad a \in (\ker \alpha)^\perp \implies u^*ua = a, \]

for all \( a \in A \). Moreover, if \( \alpha \) is extendible then \( u \in M(C^*(A,\alpha)) \).
3.5. The case when $A$ is commutative. In this subsection we assume that $A = C_0(D)$ is the algebra of continuous, vanishing at infinity functions on a locally compact Hausdorff space $D$. We let $\mathrm{Mes}(D)$ be the space of Radon positive measures on $D$ and treat $\mathrm{Mes}(D)$ as the subset of the dual space $A^*$ equipped with the $w^*$-topology. Let us start with a few simple observations.

Lemma 3.22. We have a one-to-one correspondence given by the relation
\[
\varrho(a)(x) = \int_D a(y) d\mu_x(y), \quad x \in D, a \in A,
\]
between positive maps $\varrho$ on $A$ and continuous, uniformly bounded maps
\[
(22) \quad D \ni x \mapsto \mu_x \in \mathrm{Mes}(D)
\]
that vanish at infinity in the $w^*$-sense, that is for every $a \in A$ and every $\varepsilon > 0$ the set $\{x : |\mu_x(a)| \geq \varepsilon\}$ is compact in $D$. Under this correspondence $\|\varrho\| = \max_{x \in X} \|\mu_x\|$. 

Proof. The assertion readily follows from Riesz theorem, cf., for instance, [4, Section 1]. In particular, using Lemma 2.1 we get $\|\varrho\| = \max_{x \in X} \|\mu_x\|$. □

We denote by $\mathrm{Closed}(D)$ the set of all closed subsets of $D$. A mapping $\Phi : D \to \mathrm{Closed}(D)$ is lower-semicontinuous if for every open $V \subseteq D$ the set $\{x \in D : V \cap \Phi(x) \neq \emptyset\}$ is open.

Lemma 3.23. Any continuous mapping $\Phi$ induces a lower-semicontinuous mapping
\[
(23) \quad D \ni x \mapsto \text{supp}\, \mu_x \in \mathrm{Closed}(D).
\]
In particular, the set $\text{Dom}(\Phi) := \{x \in D : \Phi(x) \neq \emptyset\}$ is open in $D$.

Proof. Assume that $V \cap \Phi(x_0) \neq \emptyset$ where $x_0 \in D$ and $V \subseteq D$ is open. Then there is a positive function $a \in C_c(D)$ that vanishes outside $V$ and such that $\mu_x(a) > 0$. Continuity of $\mu$ implies that $U := \{x \in D : \mu_x(a) > 0\}$ is open. Since $x_0 \in U \subseteq \{x \in D : V \cap \Phi(x) \neq \emptyset\}$, this proves the assertion. □

Let us fix a positive map $\varrho$ and the corresponding maps $\mu$ and $\Phi$ given by (22) and (23). We associate to $\Phi$ the following relation on $D$:
\[
(24) \quad R_\Phi := \bigcup_{x \in \text{Dom}(\Phi)} \Phi(x) \times \{x\}.
\]
If $R_\Phi$ is closed in $D \times D$, then $\mu$ is called a topological relation in [4]. In this case, as we show below, $\mu$ can also be viewed as a topological quiver. In general the relationship with topological quivers is subtle. We recall the relevant definition, see [36, Example 5.4], cf. [37, Definition 3.1] (comparing to the original, we exchange the roles of the maps $r, s$, as it fits better to conventions we adopt in Section 5).

Definition 3.24. A topological graph is a quadruple $E = (E^0, E^1, r, s)$ consisting of locally compact Hausdorff spaces $E^0$, $E^1$ and continuous maps $r, s : E^1 \to E^0$, where $s$ is additionally assumed to be open. An $s$-system on the graph $E$ is a family of Radon measures $\lambda = \{\lambda_v\}_{v \in E^0}$ on $E^1$ such that
\begin{align*}
\text{(Q1)} \quad & \text{supp}\, \lambda_v = s^{-1}(v) \text{ for all } v \in E^0, \\
\text{(Q2)} \quad & v \mapsto \int_{E^1} a(e) d\lambda_v(e) \text{ is an element of } C_c(E^0) \text{ for all } a \in C_c(E^1).
\end{align*}

The quintuple $(E^0, E^1, r, s, \lambda)$ where $E := (E^0, E^1, r, s)$ is a topological graph and $\lambda$ is an $s$-system on $E$ is called a topological quiver.

The following lemma and proposition should be compared with [18, Proposition 2.2] stated (essentially without a proof) in the context of Markov operators, see discussion below.
Lemma 3.25. Consider the quintuple
\[ Q = (D, \Phi, r, s, \lambda) \]
where \( R_\Phi \) is given by \( 24 \), \( s(y, x) := y, r(y, x) := x \) and \( \lambda_x \) is a measure supported on \( \Phi(x) \times \{ x \} \) given by \( \lambda_x(U \times \{ x \}) = \mu_x(U) \). Then \( Q \) satisfies all axioms of topological quiver, except that openness of the source map and axiom (Q1) hold for the restriction \( s|_{R_\Phi} \) to \( R_\Phi \), rather than for \( s : R_\Phi \to D \) itself.

**Proof.** Openness of \( s : R_\Phi \to D \) is equivalent to lower-semicontinuity of \( \Phi \) and thus follows from Lemma 3.28. To show the axiom (Q2) define a map \( \Psi : C_c(D) \otimes C_c(D) \to C_0(D) \) by the formula
\[
\Psi(\sum_i a_i \otimes b_i)(x) := \int \sum_i a_i \otimes b_i \, d\lambda_x = \sum_i \int a_i \mu_x b_i(x) = \left( \sum_i \varphi(a_i)b_i \right)(x).
\]

It is well defined because \( \sum_i g(a_i)b_i \in C_0(D) \) and it is linear because it is given by the integral. It is bounded with \( \| \Psi \| \leq \max_{x \in X} \| \lambda_x \| = \max_{x \in X} \| \mu_x \| = \| \varphi \|. \) Thus, since \( C_c(D) \otimes C_c(D) \) is uniformly dense in \( C_0(D \times D) \) we deduce that the formula \( \Psi(a)(x) = \int ad\lambda_x \) defines a bounded linear map \( \Psi : C_0(D \times D) \to C_0(D) \). Concluding, for any \( a \in C_c(R_\Phi) \) we see that \( x \to \int_{E_1} a(y, x) d\lambda_x(y) \) defines a continuous function on \( D \) which vanishes outside the compact set \( r(\text{supp}(a)) \). This proves (Q2). The rest is clear by construction.

□

Remark 3.26. By the above lemma the quintuple \( (D, R_\Phi, r, s, \lambda) \) is a topological quiver whenever \( R_\Phi \) is locally compact. However, if \( R_\Phi \) is not closed in \( D \times D \) then the mapping \( 25 \) below, with \( R_\Phi \) in place of \( R_\Phi \), is not well defined.

**Proposition 3.27.** The quintuple \( Q = (D, R_\Phi, r, s, \lambda) \) from Lemma 3.25 gives rise to a C*-correspondence \( X_Q \) which is the Hausdorff completion of the semi-inner C*-correspondence defined on \( C_c(R_\Phi) \) via
\[
a \otimes f \cdot b(y, x) = ((a \circ s)f(b \circ r))(y, x) = a(y)f(y, x)b(x),
\]
and
\[
(f, g)_Q(x) = \int_{R_\Phi} fg \, d\lambda_x = \int_{\Phi(x)} \int g(y, x) f(x, y) d\mu_x(y),
\]
\( f, g \in C_c(R_\Phi), a, b \in C_0(D) = C_0(D) \). Moreover, putting \( W(a \otimes b)(y, x) := a(y)b(x) \), for \( (y, x) \in R_\Phi \) the mapping determined by
\[
(25) \quad C_c(D) \otimes C_c(D) \ni a \otimes b \mapsto W(a \otimes b) \in C_c(R_\Phi),
\]

factors through and extends to an isomorphism of C*-correspondences \( X_\varphi \cong X_Q \).

**Proof.** It is a routine exercise to check that \( C_c(R_\Phi) \) is semi-inner product (right) \( A \)-module, equipped with a left \( A \)-module action by adjointable operators, \([20], \text{p. 3}\). Thus we get the C*-correspondence \( X_\varphi \). Furthermore, one readily checks that \( 25 \) determines a well-defined map \( W : C_c(D) \otimes C_c(D) \to C_c(R_\Phi) \) satisfying
\[
aW(f \otimes g)b = W(a \cdot gb), \quad (W(f \otimes g), W(f \otimes g))_Q = (f \otimes g, f \otimes g)_\varphi,
\]
for all \( a, b \in C_0(D) \), \( f, g \in C_c(D) \). Since the image of \( C_c(D) \otimes C_c(D) \) is dense in \( X_\varphi \) it follows that \( W \) factors trough and extends to an isometric homomorphism of C*-correspondences \( W : X_\varphi \to X_Q \). To see it is surjective, note that by Stone-Weierstrass theorem for any \( f \in C_c(R_\Phi) \) we can find a sequence \( \{ f_n \} \subseteq C_c(D) \otimes C_c(D) \) such that \( W(f_n) \) converges uniformly to \( f \), and thus all the more in the semi-norm induced by \( \langle \cdot, \cdot \rangle_Q \). Hence \( W \) is the desired isomorphism. □
Corollary 3.28. Let \( g : C_0(D) \to C_0(D) \) be a positive map and assume that the set \((24)\) is closed in \( D \times D \), so that \( \mu \) is a topological relation and the \( Q \) defined in Lemma \( 3.25 \) is a topological quiver. Then

\[
C^*(A; g; A) \cong C(\mu) \quad \text{and} \quad C^*(A, g) \cong C^*(Q)
\]

where \( C(\mu) \) is the \( * \)-algebra associated to \( \mu \) in \([3] \) Section 2, and \( C^*(Q) \) is the \( * \)-algebra associated to \( Q \) in \([37] \) Definition 3.17.

Proof. The \( * \)-correspondence \( X_Q \) coincides with the one associated to \( \mu \) in \([3] \) and to \( Q \) in \([37] \). The corresponding algebras were defined as \( C(\mu) = O(J(X_Q), X_Q) \) and \( C^*(Q) = O_X \). Thus we get the assertion by Proposition \( 3.27 \) and Theorem \( 3.13 \). \( \square \)

Suppose now that \( D \) is compact and \( g \) is unital (equivalently, every measure \( \mu_x, x \in D \), is a probability distribution). Then \( g \) is called a Markov operator in \([18] \) Definition 1], and the closure \( \overline{R}_g \) of the set \((24)\) coincides with the one associated to \( \mu \) defined in \([18] \) Definition 4. It seems that the authors of \([18] \) tacitly assumed that the corresponding set \( \overline{R}_g \) is closed, since they model their algebras via \( * \)-algebras associated to topological quivers. In this case we get

Corollary 3.29. Let \( g : C(D) \to C(D) \) be a Markov operator such that the set \((24)\) is closed. Then

\[
C^*(A; g) \cong C^*(g)
\]

where \( C^*(g) \) is the \( * \)-algebra associated to \( g \) in \([18] \) Definition 7].

Proof. Apply Corollary \( 3.28 \) and equality \( C^*(g) = C^*(Q) \). \( \square \)

We finish with concrete examples showing that in general neither of the quintuples \((D, R_\Phi, r, s, \lambda)\) nor \((D, \overline{R}_\Phi, r, s, \lambda)\), considered above, satisfies all of the axioms of topological quiver.

Example 3.30. Consider the following Markov operators \( g_1, g_2 \) on \( C([0, 1]) \):

\[
g_1(a)(x) = \begin{cases} \int_0^1 a(t)f_x(t) \, dt & \text{if } x \neq 0 \\ a(1), & \text{if } x = 0 \end{cases}, \quad f_x(t) = \frac{t^{1/x}}{\int_0^1 s^{1/x} \, ds};
\]

\[
g_2(a)(x) = f(x)a(0) + f(1-x)a(1), \quad f(x) = \chi_{[0,1/3]} + (2-3x)\chi_{[1/3,2/3]}.
\]

Then the corresponding relations on \([0, 1] \) are

\[
R_1 = (0, 1] \times [0, 1] \cup \{(0, 1)\}, \quad R_2 = [0, 2/3] \times \{0\} \cup (1/3, 1] \times \{1\}.
\]

Plainly, \( R_1 \) is not locally compact in \((0, 1)\), while the source map on \( \overline{R}_2 \) is not open.

4. A new look at Exel systems and their crossed products

Throughout this section, \((A, \alpha, \mathcal{L})\) denotes an Exel system. We show that Exel-Royer’s crossed product \( O(A, \alpha, \mathcal{L}) \) is the unrelative crossed product \( C^*(A, \mathcal{L}) \) and Exel’s crossed product \( A \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \) is the relative crossed product \( C^*(A, \mathcal{L}, \tilde{A}\alpha(A)A) \). We study in detail the structure of Exel systems with the property that \( \alpha \circ \mathcal{L} \) is a conditional expectation, and discuss cases when for such systems we have \( A \rtimes_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}) \).

4.1. Exel’s crossed products as relative crossed products by transfer operators. Let us start with the following simple but fundamental observation.

Lemma 4.1. Any transfer operator is a completely positive map.

Proof. Using \([6] \) and its symmetrized version: \( \mathcal{L}(\alpha(b)a) = b\mathcal{L}(a), a, b \in A \), for any \( a_i, b_i \in A, i = 1, \ldots, n \), we get

\[
\sum_{i,j=1}^n b_i^*\mathcal{L}(a_i^*a_j)b_j = \mathcal{L}\left((\sum_{i=1}^n a_i\alpha(b_i))^* (\sum_{j=1}^n a_j\alpha(b_j))\right) \geq 0.
\]
Hence $L$ is a completely positive map.

Authors of [9] and [33] considered Exel systems $(A, \alpha, L)$ under the additional assumption that both $\alpha$ and $L$ are extendible. It turns out that extendibility of $L$ is automatic.

**Proposition 4.2.** A transfer operator $L$ for $\alpha$ is extendible. Its strictly continuous extension $\overline{L} : M(A) \rightarrow M(A)$ is determined by the formula

$$\overline{L}(m\alpha(a)) = L(m\alpha(a)), \quad a \in A, \ m \in M(A).$$

In particular, $\overline{L}(1)$ is a positive central element in $M(A)$, and if $\alpha$ is extendible then the triple $(M(A), \pi, L)$ is an Exel system.

**Proof.** Fix $m \in M(A)$. We claim that (27) defines a multiplier, that is an adjointable mapping $\overline{L}(m) : A \rightarrow A$ where we view $A$ as the standard Hilbert $A$-module. Indeed, for any $a, b \in A$ we have

$$\overline{L}(m)^*b = L(m\alpha(a))^*b = L(\alpha(a)^*m)b = L(\alpha(a^*)\alpha(b)) = a^*(\overline{L}(m^*)b).$$

Hence $\overline{L}(m) \in M(A)$ and $\overline{L}(m)^* = L(m^*)$. Accordingly, (27) defines a $*$-preserving mapping $\overline{L} : M(A) \rightarrow M(A)$. It follows directly from (27) that $\overline{L}$ is a strictly continuous extension of $L : A \rightarrow A$. Moreover, for $a \in A$, we have $\overline{L}(1)a = L(1\alpha(a)) = L(\alpha(a))1 = a\overline{L}(1)$. Thus $\overline{L}(1)$ belongs to the commutant of $A$ in $M(A)$. This commutant coincides with the center of $M(A)$. If additionally $\alpha$ is extendible then $\overline{L}$ is a transfer operator for $\pi$ because (9) is preserved when passing to strict limits.

Another somehow unexpected fact is that the first relation in (7) is superfluous.

**Proposition 4.3.** Suppose $(A, \alpha, L)$ is an Exel system. For any representation $(\pi, S)$ of $(A, L)$ we automatically have

$$S\pi(a) = \pi(\alpha(a))S, \quad a \in A.$$

Thus the classes of representations of $(A, L)$ and $(A, \alpha, L)$ coincide and

$$T(A, L) = T(A, \alpha, L).$$

Moreover, the notions of redundancy for $(\pi, S)$ as a representation of $(A, L)$ and as a representation of $(A, \alpha, L)$ coincide.

**Proof.** Let $(\pi, S)$ be a representation of $(A, L)$, $\pi : M(A) \rightarrow B(H)$ the extension of $\pi$, and $\overline{L}$ the strictly continuous extension of $L$, which exists by Proposition 122. It readily follows that $(\pi, S)$ is a representation of $(M(A), \overline{L})$. In particular, $S^*S = \pi(\overline{L}(1))$. Using this and (27), one sees that each of the expressions

$$S^*\pi(\alpha(a^*))S\pi(a), \ S^*\pi(\alpha(a^*))\pi(\alpha(a))S, \ \pi(a^*)S^*S\pi(a), \ \pi(a^*)S^*\pi(\alpha(a))S$$

is equal to $\pi(\overline{L}(\alpha(a^*)))$, for any $a \in A$. Hence we get

$$\|S\pi(a) - \pi(\alpha(a))S\|^2 = \|(S^*\pi(\alpha(a^*)) - \pi(a^*)S^*) (S\pi(a) - \pi(\alpha(a))S)\| = 0.$$

This finishes the proof of the first part of the assertion. For the second part it suffices to show that $\pi(A)S\pi(A)S^*\pi(A) = \pi(A)S^*S\pi(A)$. In view of what we have just shown we have

$$\pi(A)S\pi(A)S^*\pi(A) = \pi(A)\pi(\alpha(a))SS^*\pi(A) \subseteq \pi(A)SS^*\pi(A).$$

Moreover, the last inclusion is the equality because $\pi(a)S = \lim_{\lambda \in A} \pi(a\alpha(\mu_\lambda))S$ for any approximate unit $\{\mu_\lambda\}_{\lambda \in A}$ in $A$. Indeed,

$$\|\pi(a)S - \pi(a\alpha(\mu_\lambda))S\|^2 = \|(L(a^*a) - \overline{L}(a^*a)\mu_\lambda - \mu_\lambda L(a^*a) + \mu_\lambda L(a^*a)\mu_\lambda)\|$$

clearly tends to 0. □

The above coincidence can be explained on the level of $C^*$-correspondences.
Lemma 4.4. The $C^*$-correspondence $M_\mathcal{L}$ associated to $(A, \alpha, \mathcal{L})$ and the $C^*$-correspondence $X_\mathcal{L}$ associated to $(A, \mathcal{L})$ are isomorphic, via the mapping determined by $a \otimes b \mapsto q(a\alpha(b))$, $a, b \in A$.

Proof. That $a \otimes b \mapsto q(a\alpha(b))$ yields a well defined isometry follows from the equality in \cite{20}. Clearly, it is a $C^*$-correspondence map. It is onto because $q(a\alpha(\mu))$ converges in $M_\mathcal{L}$ to $q(a)$ for any $a \in A$ and any approximate unit $\{\mu\}$ in $A$.

Corollary 4.5. For every Exel system $(A, \alpha, \mathcal{L})$ we have
\[ T(M_\mathcal{L}) \cong T(X_\mathcal{L}) \cong T(A, \mathcal{L}) = T(A, \alpha, \mathcal{L}). \]

Proof. $T(M_\mathcal{L}) \cong T(X_\mathcal{L})$ by Lemma 4.4. $T(X_\mathcal{L}) \cong T(A, \mathcal{L})$ by Proposition 3.10 and $T(A, \mathcal{L}) = T(A, \alpha, \mathcal{L})$ by Proposition 4.3.

Remark 4.6. The isomorphism $T(M_\mathcal{L}) \cong T(A, \alpha, \mathcal{L})$ was proved in \cite{8} Corollary 3.3], cf. \cite{22} Theorem 3.7], for $A$ unital, and in \cite{9} Proposition 3.1], for extendible Exel systems.

Now we are in a position to show the main result of this subsection.

Theorem 4.7. For any Exel system $(A, \alpha, \mathcal{L})$ we have
\[ \mathcal{O}(A, \alpha, \mathcal{L}) = C^*(A, \mathcal{L}), \quad A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; J), \]
where $J := A\alpha(A)A$. In particular, we have
\[ A \times_{\alpha, \mathcal{L}} \mathbb{N} \cong \mathcal{O}(X_{\mathcal{L}}, J \cap J(X_{\mathcal{L}})) \cong \mathcal{O}(M_\mathcal{L}, J \cap J(M_\mathcal{L})), \]
and the universal homomorphism $j_\mathcal{L} : A \to A \times_{\alpha, \mathcal{L}} \mathbb{N}$ is injective if and only if $\mathcal{L}$ is almost faithful on $\overline{A\alpha(A)A} \cap J(X_{\mathcal{L}})$.

Proof. Equality $A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; J)$ follows from Proposition 4.3. To get $\mathcal{O}(A, \alpha, \mathcal{L}) = C^*(A, \mathcal{L})$ combine Proposition 4.3, Lemma 4.4 and the first part of Theorem 3.13. The second part of the assertion follows now from Lemma 4.4 and Theorem 3.13.

Remark 4.8. The second part of the assertion in the above theorem generalizes \cite{8} Proposition 3.10 and Theorem 4.2] proved in unital case, and \cite{9} Theorems 4.1, 4.3] where authors assumed extendibility of the Exel system.

4.2. Regular transfer operators. Most of natural Exel systems appearing in applications, see \cite{12}, [15], [8], [9], [23], have the property that $\alpha \circ \mathcal{L}$ is a conditional expectation onto $\alpha(A)$. In \cite{12} Exel called transfer operators with that property non-degenerate. However, we have reasons to change this name. Firstly, the term ‘non-degenerate’ when referred to a positive operator is sometimes used to mean a faithful map \cite[page 60]{13} or a strict map \cite[subsection 3.3]{15}, which may be misleading. Secondly, there are historical reasons. Namely, transfer operators on unital commutative $C^*$-algebras, under the name averaging operators, were studied at least from 1950’s see \cite{12}, cf. \cite{23}. The averaging operators are called regular exactly when the corresponding composition $\alpha \circ \mathcal{L}$ is a conditional expectation, cf. \cite[Proposition 2.1.i)]{23}. Therefore we adopt the following definition.

Definition 4.9. Let $(A, \alpha, \mathcal{L})$ be an Exel system. We say that both the transfer operator $\mathcal{L}$ and the Exel system $(A, \alpha, \mathcal{L})$ are regular, if $E := \alpha \circ \mathcal{L}$ is a conditional expectation onto $\alpha(A)$.

Let us start with a simple fact.

Lemma 4.10. Let $(A, \alpha, \mathcal{L})$ be an Exel system. The range $\mathcal{L}(A)$ of the transfer operator $\mathcal{L}$ is a self-adjoint two-sided (not necessarily closed) ideal in $A$ such that $\ker \alpha \subseteq \mathcal{L}(A)^\perp$.

Proof. Since $\mathcal{L}$ is linear and $*$-preserving, $\mathcal{L}(A)$ is a self-adjoint linear space. The space $\mathcal{L}(A)$ is a two-sided ideal in $A$ by \cite{6}. For $a \in \ker \alpha$ we have $a\mathcal{L}(A) = \mathcal{L}(\alpha(a)A) = \mathcal{L}(0) = 0$. Hence
\[ \ker \alpha \subseteq \mathcal{L}(A)^\perp. \]

Suppose that the central positive element $\overline{\mathcal{L}}(1) \in M(A)$, described in Proposition 4.12, is a projection. Then by the above lemma the multiplier $\overline{\mathcal{L}}(1)$ projects $A$ onto an ideal contained in $(\ker\alpha)^\perp$. It turns out that $\mathcal{L}$ is regular exactly when $\overline{\mathcal{L}}(1)$ is a (possibly) maximal projection, i.e. when $\overline{\mathcal{L}}(1)$ projects $A$ onto $(\ker\alpha)^\perp$.

**Proposition 4.11.** Let $(A, \alpha, \mathcal{L})$ be an Exel system and let $\{\mu_\lambda\}$ be an approximate unit in $A$. The following conditions are equivalent:

i) $\mathcal{L}$ is regular, that is $E = \alpha \circ \mathcal{L} : A \to \alpha(A)$ is a conditional expectation,

ii) $\{\alpha(\mathcal{L}(\mu_\lambda))\}$ is an approximate unit in $\alpha(A)$,

iii) $\alpha \circ \mathcal{L} \circ \alpha = \alpha$,

iv) $\{\mathcal{L}(\mu_\lambda)\}$ converges strictly to a projection $\overline{\mathcal{L}}(1) \in M(A)$ onto $(\ker\alpha)^\perp$,

v) $(\alpha, \mathcal{L})$ is an interaction in the sense of [13, Definition 3.1], see Remark 3.15.

In particular, if the above equivalent conditions hold, then $\ker\alpha$ is a complemented ideal in $A$, $\overline{\mathcal{L}}(1)A = \mathcal{L}(A) = (\ker\alpha)^\perp$, $(1 - \overline{\mathcal{L}}(1))A = \ker\alpha$, and

\[(28) \quad \mathcal{L}(a) = \alpha^{-1}(E(a)), \quad \alpha(a) = \mathcal{L}^{-1}(\overline{\mathcal{L}}(1)a), \quad a \in A,
\]

where $\alpha^{-1}$ is the inverse to the isomorphism $\alpha : (\ker\alpha)^\perp \to \alpha(A)$, and $\mathcal{L}^{-1}$ is the inverse to the isomorphism $\mathcal{L} : \alpha(A) \to \mathcal{L}(A)$.

**Proof.** i) $\Rightarrow$ ii). For any $a \in \alpha(A)$ we have $\lim_{\lambda \in A} \alpha(\mathcal{L}(\mu_\lambda))a = \lim_{\lambda \in A} E(\mu_\lambda a) = E(a) = a$.

ii) $\Rightarrow$ iii). For any $a \in A$ we have

\[\alpha(a) = \lim_{\lambda \in A} \alpha(\mathcal{L}(\mu_\lambda))\alpha(a) = \lim_{\lambda \in A} \alpha(\mathcal{L}(\mu_\lambda)a) = \lim_{\lambda \in A} \alpha(\mathcal{L}(\mu_\lambda\alpha(a))) = \alpha(\mathcal{L}(\alpha(a))).\]

iii) $\Rightarrow$ iv). By Proposition 4.12 $\{\mathcal{L}(\mu_\lambda)\}$ converges strictly to a central element $\overline{\mathcal{L}}(1)$ in $M(A)$. In particular, using (27), for $a \in A$, we get

\[\overline{\mathcal{L}}(1)^2a = \overline{\mathcal{L}}(1)\mathcal{L}(\alpha(a)) = \mathcal{L}(\alpha(\mathcal{L}(\alpha(a)))) = \mathcal{L}(\alpha(a)) = \overline{\mathcal{L}}(1)a.
\]

Hence $\overline{\mathcal{L}}(1)$ is a projection. On one hand $(1 - \overline{\mathcal{L}}(1))A \subseteq \ker\alpha$ because

\[\alpha((1 - \overline{\mathcal{L}}(1))a) = \alpha(a) - \alpha(\mathcal{L}(\alpha(a))) = \alpha(a) - \alpha(a) = 0,
\]

for any $a \in A$. On the other hand, $\ker\alpha \subseteq (1 - \overline{\mathcal{L}}(1))A$ because if $a \in \ker\alpha$ then

\[(1 - \overline{\mathcal{L}}(1))a = a - \mathcal{L}(\alpha(a)) = a.
\]

Accordingly, $\ker\alpha = (1 - \overline{\mathcal{L}}(1))A$ and $(\ker\alpha)^\perp = \overline{\mathcal{L}}(1)A$.

iv) $\Rightarrow$ v). If $\overline{\mathcal{L}}(1)$ is a projection onto $(\ker\alpha)^\perp$ then in view of (27) for $a \in A$ we have

\[\alpha(a) = \alpha(\overline{\mathcal{L}}(1))\alpha(a) = \alpha(\mathcal{L}(\alpha(a))),\]

that is $\alpha = \alpha \circ \mathcal{L} \circ \alpha$. By Lemma 4.10 $\ker\alpha \subseteq \mathcal{L}(A)^\perp$. This implies that $\mathcal{L}(A) \subseteq (\mathcal{L}(A)^\perp)^\perp \subseteq (\ker\alpha)^\perp = \overline{\mathcal{L}}(1)A$. Consequently

\[\mathcal{L}(a) = \overline{\mathcal{L}}(1)\mathcal{L}(a) = \mathcal{L}(\alpha(\mathcal{L}(a))),\]

that is $\mathcal{L} \circ \alpha \circ \mathcal{L}$. We note that as $\overline{\mathcal{L}}(1)A = \mathcal{L}(\alpha(A)) \subseteq \mathcal{L}(A)$ we actually get $\mathcal{L}(A) = \overline{\mathcal{L}}(1)A = (\ker\alpha)^\perp$. Clearly, $\mathcal{L}(A) \subseteq MD(\alpha) = A$. We have $\alpha(A) \subseteq MD(\mathcal{L})$ because

\[\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b) = \overline{\mathcal{L}}(1)a\mathcal{L}(b) = \mathcal{L}(\alpha(a))\mathcal{L}(b),\]

and similarly $\mathcal{L}(b\alpha(a)) = \mathcal{L}(b)\mathcal{L}(\alpha(a))$, $a, b \in A$.

v) $\Rightarrow$ i). $E = \alpha \circ \mathcal{L}$ is a conditional expectation by [13, Corollary 3.3]. \hfill $\Box$

**Remark 4.12.** If $A$ is unital the equivalence of conditions i), ii), iii) above (with units in place of approximate units) was proved by Exel [12, Proposition 2.3], and in [23, Proposition 1.5] it was noticed that they imply that $\mathcal{L}(1)$ is a central projection with $\mathcal{L}(1)A = \mathcal{L}(A) = (\ker\alpha)^\perp$.

The following classification of regular transfer operators generalizes [23, Theorem 1.6] to non-unital case.
Proposition 4.13. Fix an endomorphism $\alpha : A \to A$. If $\alpha$ admits a regular transfer operator then its kernel is a complemented ideal in $A$ and the formulas
\[
\mathcal{L} = \alpha^{-1} \circ E, \quad E = \alpha \circ \mathcal{L}
\]
where $\alpha^{-1}$ is the inverse to $\alpha : (\ker \alpha)^\perp \to \alpha(A)$, establish a one-to-one correspondence between conditional expectations $E$ from $A$ onto $\alpha(A)$ and regular transfer operators $\mathcal{L}$ for $\alpha$.

In particular, if the range of $\alpha$ is a hereditary subalgebra of $A$, then $\alpha$ admits at most one regular transfer operator.

Proof. The first part follows immediately from Proposition 4.11. For the second part notice that every conditional expectation $E : A \to B \subseteq A$ is determined by its restriction to the hereditary $C^*$-subalgebra $BAB$ of $A$ generated by $B$.

Now, we reverse the situation and parametrize all regular Exel systems for a fixed transfer operator. To this end, we recall, cf. subsection 1.3, that a completely positive contraction $\varrho : A \to B$ is called a retraction if there exists a homomorphism $\theta : B \to A$ such that $\varrho \circ \theta = id_B$; then $\theta$ is called a section of $\varrho$.

Proposition 4.14. A completely positive mapping $\mathcal{L} : A \to A$ is a regular transfer operator for a certain endomorphism if and only if $\mathcal{L}(A)$ is a complemented ideal in $A$ and $\mathcal{L} : A \to \mathcal{L}(A)$ is a retraction.

If the above conditions hold, we have bijective correspondences between the following objects:

i) endomorphisms $\alpha : A \to A$ making $(A, \alpha, \mathcal{L})$ into a regular Exel system,
ii) sections $\theta : \mathcal{L}(A) \to A$ of $\mathcal{L} : A \to \mathcal{L}(A)$,
iii) $C^*$-subalgebras $B \subseteq MD(\mathcal{L})$ such that $\mathcal{L} : B \to \mathcal{L}(A)$ is a bijection.

These correspondences are given by the relations
\[
\alpha(a) = \theta(\overline{\alpha}(1)a), \quad a \in A, \quad B = \alpha(A) = \theta(\mathcal{L}(A)),
\]
where $\overline{\alpha}(1)$ is the projection onto $\mathcal{L}(A)$ and $\theta : \mathcal{L}(A) \to B$ is the inverse to $\mathcal{L} : B \to \mathcal{L}(A)$.

Proof. By virtue of Proposition 4.11 if $\mathcal{L}$ is a regular transfer operator for $\alpha$ then $\mathcal{L}(A)$ is a complemented ideal in $A$ and $\alpha$ is given by the first formula in (29). In particular, this formula yields a bijection between objects in items i) and ii). If $\theta$ is a section of $\mathcal{L} : A \to \mathcal{L}(A)$ and $\alpha$ is the corresponding endomorphism then $B := \alpha(A) = \theta(\mathcal{L}(A))$ is a $C^*$-subalgebra of $MD(\mathcal{L})$ by Proposition 4.11(k). By the same proposition $\mathcal{L} : B \to \mathcal{L}(A)$ is a bijection. Conversely, if $B$ is a $C^*$-algebra such that $B \subseteq MD(\mathcal{L})$ and $\mathcal{L} : B \to \mathcal{L}(A)$ is a bijection, then the inverse to $\mathcal{L} : B \to \mathcal{L}(A)$ is a section of $\mathcal{L} : A \to \mathcal{L}(A)$. This shows the correspondence between objects in ii) and iii).

Example 4.15 (cf. Example 4.7 in [8]). Let $\mathcal{L} : C([0, 2]) \to C([0, 2])$ be given by $\mathcal{L}(a)(x) = a(x/2)$. Regular Exel systems $(C([0, 2]), \alpha, \mathcal{L})$ are parametrized by continuous extensions of the mapping $[0, 1] \ni x \to 2x \in [0, 2]$; for any such system we have
\[
\alpha(a)(x) = \begin{cases} 
\alpha(2x), & \text{if } x \in [0, 1] \\
\alpha(\gamma(x)), & \text{if } x \in [1, 2], \quad a \in C([0, 2]),
\end{cases}
\]
where $\gamma : [1, 2] \to [0, 2]$ is continuous and $\gamma(1) = 2$. In other words, algebras $B$ in Proposition 4.14 correspond to continuous mappings on $[0, 2]$ whose restriction to $[0, 1]$ is $2x$.

4.3. Non-degenerate Exel’s crossed products for regular Exel systems. By Theorem 4.1, the crossed product $A \times_{\alpha, \mathcal{L}} N$ is non-degenerate, in the sense that the natural homomorphism $A \to A \times_{\alpha, \mathcal{L}} N$ is injective, if and only if $\overline{\alpha(A)A} \cap J(X_{\mathcal{L}}) \subseteq N_{\mathcal{L}}$. In this subsection, we consider regular Exel systems satisfying stronger, but easier to check in practice, condition: $\overline{\alpha(A)A} \subseteq N_{\mathcal{L}}^\perp$. The latter inclusion holds, for instance, for systems with faithful transfer.
operators or with corner endomorphisms. These are the cases when Exel's crossed product boasts its greatest successes, see [12], [15], [8], [9], and we show that for such systems $A \times_{\alpha, \mathcal{L}} \mathbb{N}$ is nothing but $C^*(A, \mathcal{L})$.

**Proposition 4.16.** Suppose that $(A, \alpha, \mathcal{L})$ is a regular Exel system such that $\mathcal{L}$ is faithful on $A\alpha(A)\overline{A}$. Then $\alpha$ is uniquely determined by $\mathcal{L}$ and

$$A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; A(ker \mathcal{L}|_{MD(\mathcal{L})})^\perp)$$

where $(ker \mathcal{L}|_{MD(\mathcal{L})})^\perp$ is the annihilator of the kernel of the homomorphism $\mathcal{L} : MD(\mathcal{L}) \to \mathcal{L}(A)$.

If $A(ker \mathcal{L}|_{MD(\mathcal{L})})^\perp A = N^\perp_L$, which holds for instance when $\alpha$ is non-degenerate, then

$$A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}).$$

**Proof.** On one hand, $\mathcal{L}$ is injective homomorphism on the $C^*$-algebra $A\alpha(A)\overline{A} \cap MD(\mathcal{L})$, which is an ideal in $MD(\mathcal{L})$. On the other hand, by Proposition [1.14] $\alpha(A) \subseteq A\alpha(A)\overline{A} \cap MD(\mathcal{L})$ and $\mathcal{L} : \alpha(A) \to \mathcal{L}(A)$ is an isomorphism. This implies that $\alpha(A) = A\alpha(A)\overline{A} \cap MD(\mathcal{L}) = (ker \mathcal{L}|_{MD(\mathcal{L})})^\perp$. Thus, by Proposition [1.14] $\alpha$ is uniquely determined by $\mathcal{L}$. By Theorem 4.7 we get $A \times_{\alpha, \mathcal{L}} N = C^*(A, \mathcal{L}; A(ker \mathcal{L}|_{MD(\mathcal{L})})^\perp A)$, and if $A(ker \mathcal{L}|_{MD(\mathcal{L})})^\perp A = N^\perp_L$, then actually $A \times_{\alpha, \mathcal{L}} N = C^*(A, \mathcal{L}; N^\perp_L) = C^*(A, \mathcal{L}; N^\perp_L \cap J(X_L)) = C^*(A, \mathcal{L})$. If $\alpha$ is non-degenerate then $A = A\alpha(A)\overline{A} = N^\perp_L$. \hfill $\square$

**Corollary 4.17.** If $(A, \alpha, \mathcal{L})$ is a regular Exel system such that $\mathcal{L}$ is faithful and $\alpha$ is extendible, then

$$A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}).$$

**Proof.** By Proposition [4.16] it suffices to show that $\alpha$ is non-degenerate. For any $a \in A$ we have

$$\mathcal{L}((a - a\overline{a}(1))^*(a - a\overline{a}(1))) = \mathcal{L}(a^*a) - \mathcal{L}(\overline{a}(1)a^*a) - \mathcal{L}(a^*a\overline{a}(1)) + \mathcal{L}(\overline{a}(1)a^*a\overline{a}(1)) = 0,$$

which by faithfulness of $\mathcal{L}$ implies that $a = \overline{a}(1)a$. Hence $A = \overline{a}(1)A = \alpha(A)A$. \hfill $\square$

Now we turn to Exel systems $(A, \alpha, \mathcal{L})$ where $\alpha$ and $\mathcal{L}$ have somehow equal rights. Algebras arising from such systems were studied for instance in [10], [12], [2], [23], [25], [26], [27].

**Definition 4.18.** We say that a regular Exel system $(A, \alpha, \mathcal{L})$ is a **corner system** if $\alpha(A)$ is a hereditary subalgebra of $A$.

The above terminology is justified by Lemma [4.19] and Remark [4.22] below. We note that corner systems $(A, \alpha, \mathcal{L})$ satisfy condition $A\alpha(A)\overline{A} \subseteq N^\perp_L$. Indeed, since $\alpha(A)$ is hereditary and $\mathcal{L}$ is faithful on $\alpha(A)$, Lemma [2.5] implies that $\mathcal{L}$ is almost faithful on $A\alpha(A)\overline{A}$.

**Lemma 4.19.** Let $(A, \alpha, \mathcal{L})$ be an Exel system. The following statements are equivalent:

i) $(A, \alpha, \mathcal{L})$ is a corner system.

ii) $\alpha$ is extendible and

$$\alpha(\mathcal{L}(a)) = \overline{a}(1)a\overline{a}(1) \quad \text{for all } a \in A.$$  \hfill (30)

iii) $\alpha$ has a complemented kernel and a corner range; $\mathcal{L}$ is a unique regular transfer operator for $\alpha$, it is given by the formula

$$\mathcal{L}(a) = \alpha^{-1}(pap), \quad a \in A,$$

where $p \in M(A)$ is a projection such that $\alpha(A) = pAp$, and $\alpha^{-1}$ is the inverse to the isomorphism $\alpha : (ker \alpha)^\perp \to pAp$.  \hfill (31)
iv) \( \mathcal{L}(A) \) is a complemented ideal in \( A \) and the annihilator \( (\ker \mathcal{L}|_{MD(\mathcal{L})})^\perp \) of the kernel of \( \mathcal{L} : MD(\mathcal{L}) \to \mathcal{L}(A) \) is a hereditary subalgebra of \( A \) mapped by \( \mathcal{L} \) onto \( \mathcal{L}(A) \); \( \alpha \) is given by the formulas:

\[
\alpha|_{\mathcal{L}(A)}^\perp \equiv 0 \quad \text{and} \quad \alpha(a) = \mathcal{L}^{-1}(a), \quad \text{for } a \in \mathcal{L}(A),
\]

where \( \mathcal{L}^{-1} \) is the inverse to the isomorphism \( \mathcal{L} : (\ker \mathcal{L}|_{MD(\mathcal{L})})^\perp \to \mathcal{L}(A) \).

Proof. i)⇒ii). Let \( \{\mu_\lambda\}_{\lambda \in \Lambda} \) be an approximate unit in \( A \). Since \( \mathcal{L} \) is isometric on \( \alpha(A) = \alpha(A)A\alpha(A) \), for any \( a \in A \), we have

\[
\| (\alpha(\mu_\lambda) - \alpha(\mu_{\lambda'}))a \|^2 = \| (\alpha(\mu_\lambda) - \alpha(\mu_{\lambda'}))aa^*(\alpha(\mu_\lambda) - \alpha(\mu_{\lambda'})) \| \\
= \| \mathcal{L}( (\alpha(\mu_\lambda) - \alpha(\mu_{\lambda'}))aa^* (\alpha(\mu_\lambda) - \alpha(\mu_{\lambda'}))) \| \\
\leq 2 \| \mathcal{L}(aa^*)(\alpha_\lambda - \alpha_{\lambda'}) \|,
\]

which is arbitrarily small for sufficiently large \( \lambda \) and \( \lambda' \). Accordingly, \( \{\alpha(\mu_\lambda)\}_{\lambda \in \Lambda} \) is strictly Cauchy and thereby strictly convergent. Hence \( \alpha \) is extendible and we have \( \alpha(A) = \overline{\alpha(1)A\alpha(1)} \). Since \( E(a) = \overline{\alpha(1)\alpha(1)} \) is the unique conditional expectation onto \( \alpha(A) \) we conclude, using Proposition 4.13, that (30) holds.

ii)⇒iii). Note that (30) implies that \( \alpha(A) = \overline{\alpha(1)A\alpha(1)} \) is a corner in \( A \). In particular, \((A, \alpha, \mathcal{L})\) is regular because \( E(a) = (\alpha \circ \mathcal{L})(a) = \overline{\alpha(1)\alpha(1)} \) is a conditional expectation onto \( \alpha(A) \). Thus by Proposition 4.13, ker \( \alpha \) is complemented and \( \mathcal{L}(a) = \alpha^{-1}(\overline{\alpha(1)\alpha(1)}))a \in A \).

iii)⇒iv). Decomposing \( A \) into parts \( pAp \), \( (1-p)Ap \), \( pA(1-p) \), \( (1-p)A(1-p) \), \( \mathcal{L} \) assumes the form

\[
\mathcal{L} \left( \begin{array}{cc}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array} \right) = \alpha^{-1}(a_{11}).
\]

Using, for instance, Proposition 2.7 one readily sees that \( pAp \oplus (1-p)A(1-p) \subseteq MD(\mathcal{L}) \). Moreover, if \( a = a_{12} + a_{21} \), where \( a_{12} \in (1-p)Ap \) and \( a_{21} \in pA(1-p) \), then using (3) we get

\[
\alpha^{-1}(a_{12}^*a_{12}) = 0 \quad \text{and} \quad \alpha^{-1}(a_{21}^*a_{21}) = 0.
\]

Since \( a_{12}^*a_{12} \) and \( a_{21}^*a_{21} \) belong to \( pAp \), it follows that \( a = a_{12} + a_{21} = 0 \). Hence \( MD(\mathcal{L}) = pAp \oplus (1-p)A(1-p) \). Consequently, \( (\ker \mathcal{L}|_{MD(\mathcal{L})})^\perp = pAp \). Now the formula (32) is immediate.

iv)⇒i). It follows readily from Proposition 4.13. \( \square \)

Remark 4.20. Transfer operators satisfying (30) are called complete transfer operators in [2, 20, 21, 22]. The pair \((A, \alpha)\) where \( \alpha \) is an endomorphisms satisfying condition iii) above, is called a reversible C*-dynamical system in [26, 27].

In the case \( A \) is unital, Exel systems satisfying (30) were considered in [2] and [22]. In particular, it was shown in [2] Theorem 4.16 that \( A \times_{\alpha, \mathcal{L}} \mathbb{N} \cong C^*(A, \alpha) \), and in [22] Theorem 3.14 that \( A \times_{\alpha, \mathcal{L}} \mathbb{N} \cong \mathcal{O}_{M_\mathcal{L}} \). By Theorem 4.17 we know that \( \mathcal{O}_{M_\mathcal{L}} \cong \mathcal{O}(A, \alpha, \mathcal{L}) = C^*(A, \mathcal{L}) \). Hence combining these results we get \( A \times_{\alpha, \mathcal{L}} \mathbb{N} \cong C^*(A, \mathcal{L}) \cong C^*(A, \alpha) \). We now generalize this fact to non-unital case.

Theorem 4.21. Suppose \((A, \alpha, \mathcal{L})\) is a corner Exel system. Then \( \alpha \) and \( \mathcal{L} \) determine each other uniquely and

\[
A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}) \cong C^*(A, \alpha).
\]

In particular, \( A \times_{\alpha, \mathcal{L}} \mathbb{N} \) can be viewed as a universal C*-algebra generated by \( j_A(A) \cup j_A(A)s \) subject to relations

\[
(33) \quad s j_A(a)s^* = j_A(\alpha(a)), \quad s^* j_A(a)s = j_A(\mathcal{L}(a)), \quad a \in A,
\]

where \( j_A : A \to A \times_{\alpha, \mathcal{L}} \mathbb{N} \) is a non-degenerate homomorphism and \( s \in M(A \times_{\alpha, \mathcal{L}} \mathbb{N}) \).
Proof. That $\alpha$ and $\mathcal{L}$ determine each other uniquely follows from Lemma 4.19. By Theorem 4.7 to prove the equality $A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L})$ it suffices to show that $A\alpha(A)\bar{A} = N^+_{\mathcal{L}} \cap J(X_{\mathcal{L}})$. To this end, we use the isomorphism $X_{\mathcal{L}} \cong M_{\mathcal{L}}$ from Lemma 4.4. For $x \in A$ we have $q(x) = q(x\mathcal{L}(1)) \in M_{\mathcal{L}}$. For any $x, y, z \in A$ we get

$$\Theta_{q(x), q(y)} q(z) = q(x)\mathcal{L}(y^*z) = q(x\alpha(\mathcal{L}(y^*z))) = q(x\mathcal{L}(1)y^*z\mathcal{L}(1)) = (x\mathcal{L}(1)y)q(z).$$

Thus $\phi(x\mathcal{L}(1)y) = \Theta_{q(x), q(y)}$. It follows that $\phi$ sends $A\alpha(A)\bar{A} = A\alpha(\mathcal{L}(1)) \subseteq N^+_{\mathcal{L}} = (\ker \phi)^\perp$ isometrically onto $\mathcal{K}(X_{\mathcal{L}}) \cong \mathcal{K}(M_{\mathcal{L}})$. Hence $A\alpha(A)\bar{A} = J(X_{\mathcal{L}}) \cap N^+_{\mathcal{L}}$ and we have $A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L})$.

In order to show that the first relation in (33) holds in $A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L})$ (the second holds trivially) it suffices to check that $(i_A(\alpha(a)), si_A(a)^*)$, for $a \in A$, is a redundancy of the Toeplitz representation $(i_A, \iota)$ (note that $\alpha(A) \in N^+_{\mathcal{L}}$). Invoking Proposition 4.3 we have $si_A(a) = i_A(\alpha(a))s$. Thus, using (30), for any $b, c \in A$ we get

$$(si_A(a)^*)i_A(b)s_iA(c) = si_A(a)A(c)b = i_A(\alpha(a)\alpha(c)b) = i_A(\alpha(a)\alpha(c)\alpha(c)b) = i_A(\alpha(a)b\alpha(c))s.$$ 

Since $i_A$ is non-degenerate this shows that $(i_A(\alpha(a)), si_A(a)^*)$ is a redundancy and thus (33) holds. Moreover, since the ideal $J(X_{\mathcal{L}}) \cap N^+_{\mathcal{L}} = A\alpha(\mathcal{L})\bar{A}$ is generated by $\alpha(A)$ we see that the kernel of the quotient map $\mathcal{T}(A, \mathcal{L}) \to C^*(A, \mathcal{L})$ is the ideal generated by differences $i_A(\alpha(a)) - si_A(a)^*$, $a \in A$. Hence $(C^*(A, \mathcal{L}), J, \iota, s)$ is universal with respect to relations (33), cf. Proposition 4.17. By [26] Proposition 4.6], cf. Proposition 3.20 $C^*(A, \alpha)$ is universal with respect to the same relations and thus $C^*(A, \alpha) \cong C^*(A, \mathcal{L})$. \hfill $\square$

Remark 4.22. If $A$ is unital then an Exel system $(A, \alpha, \mathcal{L})$ is a corner system if and only if $(\alpha, \mathcal{L})$ is a corner interaction studied in [25], see Proposition 4.11b). In particular, the isomorphism $C^*(A, \mathcal{L}) \cong C^*(A, \alpha)$ is an instance of the isomorphism (17). An examination of the argument leading to (17) shows that it holds also in the non-unital case if one defines a corner interaction as an interaction $(\mathcal{V}, \mathcal{H})$ over $A$ where both $\mathcal{V}$ and $\mathcal{H}$ are extendible and have corner ranges. Thus corner interactions give a symmetrized framework for corner Exel systems, and one could think of them as partial automorphism of $A$ whose domain and range are corners in $A$.

5. Graph $C^*$-algebras as crossed products by completely positive maps

In this section, we test Exel’s construction and the results of the present paper against the original idea standing behind [12] that Cuntz-Krieger algebras (or more generally graph $C^*$-algebras) should be viewed as crossed products associated to topological Markov shifts. We start by presenting Brownlowe’s [6] realization of graph $C^*$-algebras $C^*(E)$ as Exel-Royer’s crossed product for partially defined Exel system $(D_E, \alpha, \mathcal{L})$. Discussion of a possibility of extension of these maps to the whole of diagonal algebra $D_E$ leads us to a complete description of Perron-Frobenious operators on $D_E$ associated to quivers on $E$. We prove that the crossed product of $D_E$ by any such operator is isomorphic to $C^*(E)$.

5.1. Graph $C^*$-algebras as Exel-Royer’s crossed products. For graphs and their $C^*$-algebras we use the notation and conventions of [43, 9, 17]. Throughout this section, we fix an arbitrary countable directed graph $E = (E^0, E^1, r, s)$. Hence $E^0$ and $E^1$ are countable sets and the range and source maps $r, s : E^1 \to E^0$ arbitrary. We denote by $E^n$, $n > 0$, the set of finite paths $\mu = \mu_1...\mu_n$ satisfying $s(\mu_i) = r(\mu_{i+1})$, for all $i = 1, ..., n$. Then $|\mu| = n$ stands for the length of $\mu$ and $E^* = \bigcup_{n=0}^{\infty} E^n$ is the set of all finite paths (vertices are treated as paths of length zero). We put $E^\infty$ to be the set of infinite paths. The maps $r, s$ extend naturally to $E^*$ and $r$ extends also to $E^\infty$.

The graph $C^*$-algebra $C^*(E)$ is generated by a universal Cuntz-Krieger $E$-family consisting of partial isometries $\{s_e : e \in E^1\}$ and mutually orthogonal projections $\{p_v : v \in E^1\}$ such
that $s_e^* s_e = p_{s(e)}$, $s_e s_e^* \leq p_{r(e)}$ and $p_v = \sum_{r(e) = v} s_e s_e^*$ whenever the sum is finite (i.e. $v$ is a finite receiver). It follows that $C^*(E) = \bigcup_{n \geq 0} (s^{n-1}(s) : \mu, \nu \in E^*)$ where $s_\mu := s_{\mu_1} s_{\mu_2} \cdots s_{\mu_n}$ for $\mu = \mu_1 \cdots \mu_n \in E^n$, $n > 0$, and $s_\mu = p_\mu$ for $\mu \in E^0$. We denote by $D_E := \bigcup_{n \geq 0} (s^{n-1}(s) : \mu \in E^*)$ the diagonal $C^*$-subalgebra of $C^*(E)$.

It is attributed to folklore, see [16] or [48] for an extended discussion, that Gelfand spectrum of $D_E$ can be identified with the boundary space of $E$. To be more specific, we define $E_{\text{inf}} := \{ \mu \in E^* : |r^{-1}(s(\mu))| = \infty \}$ and $E^*_s := \{ \mu \in E^* : r^{-1}(s(\mu)) = \emptyset \}$, so $E_{\text{inf}}^*$ is the set of paths that start in infinite receivers, and $E^*_s$ is the set of paths that start in sources. For any $\eta \in E^* \setminus E^0$ let $\eta E^{\infty \infty} := \{ \mu = \mu_1 \cdots \in E^* \cup E^\infty : \mu_1 \cdots \mu_|\eta| = \eta \}$ and for $v \in E^0$ put $v E^{\infty \infty} := \{ \mu \in E^* \cup E^\infty : r(\mu) = v \}$. The boundary space of $E$, cf. [48, Section 2] or [6, Subsection 4.1], is the set

$$\partial E := E^\infty \cup E_{\text{inf}}^* \cup E^*_s$$

equipped with the topology generated by the ‘cylinders’ $D_\eta := \partial E \cap \eta E^{\infty \infty}$, $\eta \in E^*$, and their complements. In fact, the sets $D_\eta \setminus \bigcap_{\mu \in F} D_\mu$, where $\eta \in E^*$ and $F \subseteq \eta E^{\infty \infty} \cap E^*$ is finite, form a basis of compact and open sets for Hausdorff topology on $\partial E$, [48, Section 2] or [6, Section 2].

**Proposition 5.1** ([48, Theorem 3.7]). The formula

$$s_\mu s_\mu^* \mapsto \chi_{D_\mu}, \quad \mu \in E^*,$$

determines an isomorphism $D_E \cong C_0(\partial E)$.

**Proof.** It suffices to apply a dual description of the assertion in [48, Theorem 3.7]. Alternatively, one can note that the above mapping extends uniquely to a $*$-homomorphism

$$\text{span}\{s_\mu s_\mu^* : \mu \in E^*\} \ni \sum_{\mu \in F} c_\mu s_\mu s_\mu^* \mapsto \sum_{\mu \in F} c_\mu \chi_{D_\mu} \in C_c(\partial E),$$

whose range separates the points in $\partial E$. A more non-trivial task is to show that $\| \sum_{\mu \in F} c_\mu s_\mu s_\mu^* \| = \max_{v \in \partial E} |\sum_{\mu \in F} c_\mu \chi_{D_\mu}(v)|$. This can be achieved by representing the considered elements in an appropriate form, cf. the formula before [48, (3.3)]. Then Stone-Weierstrass theorem does the job. \hfill $\square$

The one-sided topological Markov shift associated to $E$ is the map $\sigma : \partial E \setminus E^0 \to \partial E$ defined, for $\mu = \mu_1 \mu_2 \cdots \in \partial E \setminus E^0$, by the formulas

$$\sigma(\mu) := \mu_2 \mu_3 \cdots \text{ if } \mu \notin E^1, \quad \text{and } \quad \sigma(\mu) := s(\mu_1) \text{ if } \mu = \mu_1 \in E^1.$$

By [6, Proposition 2.1] the shift $\sigma$ is a local homeomorphism. Furthermore, results of [6] imply the following (we adopt the convention that a sum over empty set is zero):

**Proposition 5.2.** The formulas

$$\alpha(a)(\mu) = a(\sigma(\mu)), \quad \mathcal{L}(a)(\mu) = \sum_{\nu \in \sigma^{-1}(\mu)} a(\nu)$$

define respectively a homomorphism $\alpha : C_0(\partial E) \to M(C_0(\partial E \setminus E^0))$ and a linear map $\mathcal{L} : C_c(\partial E \setminus E^0) \to C_c(\partial E)$. Moreover, the triple $(C_0(\partial E), \alpha, \mathcal{L})$ forms a $C^*$-dynamical system in the sense of [8, Definition 1.2], and we have an isomorphism

$$\mathcal{O}(C_0(\partial E), \alpha, \mathcal{L}) \cong C^*(E).$$

**Proof.** The first part of the assertion follows from [6, Proposition 2.1] and the discussion at the beginning of [6, Section 4]. The second part follows from [6, Proposition 4.4]. \hfill $\square$
The above mappings (35) have the following important algebraic description. The isomorphism \( \mathcal{D}_E \cong C_0(\partial E) \) from Proposition 5.1 gives rise to isomorphisms of *-algebras \( M(\text{span}\{s_\mu s_\mu^* : \mu \in E^* \setminus \{0\}\}) \cong M(C_0(\partial E \setminus E_0)) \) and span\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0\} \cong C_0(\partial E). Using these isomorphisms the mappings in (35) are intertwined respectively with a homomorphism \( \Phi : \mathcal{D}_E \to M(\text{span}\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0\}) \) and a linear map \( \Phi_s : \text{span}\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0\} \to \text{span}\{s_\mu s_\mu^* : \mu \in E^*\} \) which are given by the formulas

\[
\Phi(a) = \sum_{e \in E^1} s_e a s_e^*, \quad \Phi_s(a) = \sum_{e \in E^1} s_e^* a s_e
\]

When \( E \) has no infinite emitters (see Proposition 5.3 below) the formula \( \Phi(a) = \sum_{e \in E^1} s_e a s_e^* \) defines a completely positive map on \( C^*(E) \). In the literature, this mapping, usually considered when \( E \) is locally finite (i.e. \( r \) and \( s \) are finite-to-one), is called a non-commutative Markov shift and its ergodic properties are well studied, cf., for instance, [19].

5.2. Non-commutative Perron-Frobenius operators arising from quivers. The mappings \( \alpha \) and \( \mathcal{L} \) considered in Proposition 5.2 are viewed as partial mappings on \( C_0(\partial E) \). Now we discuss the problem of when formulas (35), or their analogues, define honest mappings on \( C_0(\partial E) \).

**Proposition 5.3.** The following conditions are equivalent:

i) the first of formula in (35) defines an endomorphism \( \alpha : C_0(\partial E) \to C_0(\partial E) \),

ii) \( \sigma : \partial E \setminus E^0 \to \partial E \) is a proper map (preimage of a compact set is compact),

iii) \( \sigma \) is finite-to-one mapping,

iv) there are no infinite emitters in \( E \),

v) the sum \( \sum_{\mu \in E^1} s_\mu a s_\mu^* \) converges in norm for every \( a \in C^*(E) \),

vi) the range of homomorphism \( \Phi \) given by (36) is contained in \( \text{span}\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0\} \subseteq \mathcal{D}_E \), and hence \( \Phi : \mathcal{D}_E \to \mathcal{D}_E \) is an endomorphism.

In particular, if the above equivalent conditions hold, then the first formula in (35) defines a completely positive map \( \Phi : C^*(E) \to C^*(E) \) which restricts to an endomorphism \( \Phi : \mathcal{D}_E \to \mathcal{D}_E \).

**Proof.** i)\(\Leftrightarrow\)ii). It is a well known general fact that a continuous mapping \( \tau : X \to Y \) between locally compact Hausdorff spaces \( X, Y \), gives rise to the composition operator from \( C_0(Y) \) to \( C_0(X) \) (rather then to \( C_0(X) = M(C_0(X)) \)) if and only if \( \tau \) is proper.

ii)\(\Rightarrow\)iii). If \( \sigma \) is a proper local homeomorphism then \( \sigma^{-1}(\mu), \mu \in \partial E \), is compact and cannot have a cluster point. Hence \( \sigma \) is finite-to-one.

iii)\(\Rightarrow\)iv). It follows readily from the definition of \( \sigma \).

iv)\(\Rightarrow\)v). Consider a net, indexed by finite sets \( F \subseteq E^1 \) ordered by inclusion, consisting of mappings \( \alpha_F : C^*(E) \to C^*(E) \) given by \( \alpha_F(a) := \sum_{e \in F} s_\mu s_\mu^* \). Since the projections \( s_\mu s_\mu^* \), \( e \in F \), are mutually orthogonal we get \( \|\alpha_F(a)\| = \max_{e \in F} \|s_\mu s_\mu^*\| \leq \|a\| \), and thus \( \alpha_F \) is a contraction. Let \( a \in C^*(E) \). For any \( \varepsilon > 0 \) there is a finite linear combination \( b = \sum_{\mu, \nu \in K} c_{\mu,\nu} s_\mu s_\nu^* \) such that \( \|a - b\| \leq \varepsilon \) (\( K \subseteq E^* \) is finite set). Since \( E \) has no infinite emitters the set

\[ F = \{ e \in E^1 : s(e) = r(\mu) \text{ for a certain } \mu \in K \} = \bigcup_{v \in \tau(K)} s^{-1}(v) \]

is finite. Clearly, for any finite set \( F' \subseteq E^* \) containing \( F \) we have \( \alpha_{F'}(b) = \alpha_F(b) \). Thus

\[ \|\alpha_{F'}(a) - \alpha_F(a)\| \leq \|\alpha_{F'}(a) - \alpha_{F'}(b)\| + \|\alpha_F(b) - \alpha_F(a)\| \leq 2\varepsilon. \]

Hence the net \( \{\alpha_F(a)\}_{F} \) is Cauchy and the sum \( \sum_{e \in E^1} s_\mu a s_\mu^* \) converges in norm.

v)\(\Rightarrow\)vi). It is straightforward.

vi)\(\Rightarrow\)i). Not that the isomorphism \( \mathcal{D}_E \cong C_0(\partial E) \) given by (34) intertwines \( \Phi \) and \( \alpha \).

\[\square\]
One can check that the second formula in (35) defines a mapping \( L : C_c(\partial E) \to C_c(\partial E) \) if and only if \( E \) has no infinite receivers. But even if the graph \( E \) is locally finite, this mapping might be unbounded. On the other hand, if \( E \) is locally finite, we can adjust the formula for \( L \) by adding averaging as in (1), and then \( L \) has norm one, so in particular it extends to a self-map of \( C_0(\partial E) \). This motivates us to consider slightly more general averagings, which will allow us to get a bounded positive operator on \( C_0(\partial E) \) for arbitrary graphs. Accordingly, we wish to consider strictly positive numbers \( \lambda = \{ \lambda_e \}_{e \in E^1} \) such that the formula
\[
L_\lambda(a)(\mu) = \sum_{e \in E^1, e\mu \in \partial E} \lambda_e a(e\mu)
\]
defines a mapping on \( C_0(\partial E) \). We note that fixing the family \( \{ \lambda_e \}_{e \in E^1} \) is equivalent to fixing a system of measures \( \{ \lambda_e \}_{e \in E^0} \) on \( E^1 \) making the graph \( E \) into a (topological) quiver. Indeed, the relation \( \lambda_e = \lambda_s(e) \{ e \} \) establishes a one-to-one correspondence between families \( \{ \lambda_e \}_{e \in E^1} \) of strictly positive numbers and \( s \)-systems of measures \( \{ \lambda_e \}_{e \in E^0} \) on \( E \), cf. Definition 3.24. In particular, if \( E \) has no infinite emitters one can put \( \lambda_e := |s^{-1}(s(e))|^{-1} \), \( e \in E^1 \), which corresponds to the situation where all the measures \( \{ \lambda_e \}_{e \in E^0} \) are uniform probability distributions. In this case one recovers from (37) the second formula in (1).

**Proposition 5.4.** Let \( \lambda = \{ \lambda_e \}_{e \in E^1} \) be a family strictly positive numbers. The following conditions are equivalent:

i) the formula (37) defines a bounded operator \( L_\lambda : C_0(\partial E) \to C_0(\partial E) \),

ii) the following conditions are satisfied:
\[
\left\{ \sum_{e \in \tilde{s}^{-1}(v)} \lambda_e \right\}_{v \in s(E^1)} \in \ell_\infty\left( s(E^1) \right),
\]
\[
\forall v \in s(E^1) \left\{ \sum_{e \in r^{-1}(v) \cap \tilde{s}^{-1}(w)} \lambda_e \right\}_{w \in \tilde{s}(r^{-1}(v))} \in c_0\left( s(r^{-1}(v)) \right),
\]

iii) the sum
\[
u_\lambda := \sum_{e \in E^1} \sqrt{\lambda_e} s_e
\]
converges strictly in \( M(C^*(E)) \),

iv) the sum \( \sum_{e,f \in E^1} \sqrt{\lambda_e \lambda_f} s_es_f^* \) converges in norm for every \( a \in C^*(E) \) and
\[
\Phi_{s,\lambda}(a) := \sum_{e,f \in E^1} \sqrt{\lambda_e \lambda_f} s_es_f^*, \quad a \in C^*(E),
\]
defines a completely positive map \( \Phi_{s,\lambda} : C^*(E) \to C^*(E) \).

If the above equivalent conditions hold, then we have \( \Phi_{s,\lambda}(a) = \nu_\lambda \eta_a \lambda_a \), \( a \in C^*(E) \), and the isomorphism \( D_E \cong C_0(\partial E) \) from Proposition 5.1 intertwines \( \Phi_{s,\lambda}|_{D_E} \) and \( L_\lambda \).

**Proof.** i)⇒ii). One readily sees that
\[
L_\lambda(\chi_{D_\eta}) = \lambda_{\eta_1} \chi_{D_{s(\eta)}}, \quad \text{for any } \eta = \eta_1 ... \in \partial E \setminus E^0.
\]
Hence for any \( v \in s(E^1) \) and any finite set \( F \subseteq s^{-1}(v) \) we get
\[
\sum_{e \in F} \lambda_e = \| \left( \sum_{e \in F} \lambda_e \right) \chi_{D_v} \| = \| L_\lambda \left( \sum_{e \in F} \chi_{D_e} \right) \| \leq \| L_\lambda \|,
\]
which implies condition (38). Now let \( v \in r(E^1) \) and note that, for any \( \mu \in \partial E \),
\[
\mathcal{L}_\lambda(\chi_{D_v})(\mu) = \sum_{e \in r^{-1}(v), e \in s^{-1}(r(\mu))} \lambda_e \chi_{D_v}(e\mu) = \sum_{e \in r^{-1}(v)} \lambda_e \chi_{D_{\mu}}(\mu) \cdot \sum_{e \in r^{-1}(v)} \lambda_e \chi_{D_{\mu}}(\mu).
\]
Since the sets \( D_{\mu} \) are disjoint and open, \( \mathcal{L}_\lambda(\chi_{D_{\mu}}) \in C_0(\partial E) \) implies condition (39). For future reference, note that in view of the above calculation we have (we treat empty sums as zero)
\[
(39) \quad \mathcal{L}_\lambda(\chi_{D_v}) = \sum_{e \in r^{-1}(v)} \lambda_e \chi_{D_{\mu}}, \quad v \in E^0.
\]
\( \text{ii}\Rightarrow \text{iii} \). Let \( v \in s(E^1) \). For any finite set \( F \subseteq s^{-1}(v) \) we have \( \| \sum_{e \in F} \chi_{s_e} \| = \| \sum_{e \in F} \lambda_e p_e \| = \sum_{e \in F} \lambda_e \). Since \( \sum_{e \in s^{-1}(v)} \lambda_e < \infty \), by (38), it follows that the sum \( u_v := \sum_{e \in s^{-1}(v)} \chi_{s_e} \) converges in norm. Thus for any finite set \( F \subseteq s(E^1) \) we have
\[
(40) \quad u_F := \sum_{v \in F} u_v = \sum_{v \in F} \sum_{e \in s^{-1}(v)} \chi_{s_e} = \sum_{e \in s^{-1}(F)} \sqrt{\lambda_e} s_e \in C^*(E).
\]
By (38), \( M := \sup_{v \in s(E^1)} \sum_{e \in s^{-1}(v)} \lambda_e \) is finite. The set of elements \( u_F \) is bounded:
\[
(41) \quad \| u_F \|^2 = \| u_F^* u_F \| = \| \sum_{w \in F} \sum_{e \in s^{-1}(v)} \lambda_e p_e \| = \max_{w \in F} \sum_{e \in s^{-1}(v)} \lambda_e \leq M.
\]
Condition (39) implies that for any \( v \in E^0 \) the sum \( \sum_{e \in r^{-1}(v)} \sqrt{\lambda_e} s_e \) converges in norm. Indeed, for any finite set \( F \subseteq r^{-1}(v) \) we have
\[
\| \sum_{e \in F} \sqrt{\lambda_e} s_e \|^2 = \| \sum_{e \in F} \lambda_e p_e \| = \max_{w \in F} \sum_{e \in s^{-1}(v)} \lambda_e \sqrt{\lambda_e} s_e a \in C^*(E).
\]
which by (39) can be made arbitrarily small by choosing \( F \) lying outside a sufficiently large finite subset of \( r^{-1}(v) \).

Now fix a (nonzero) finite linear combination \( a = \sum_{\mu, v \in K} \lambda_{\mu, v} s_{\mu}^* s_{\mu} \), where \( K \subseteq E^* \) is finite. Since we know, by (45), that \( \| \sum_{e \in F} \sqrt{\lambda_e} s_e \| \leq \sqrt{M} \) for every finite \( F \subseteq E^1 \), to prove the strict convergence of the sum in (40) it suffices to check the convergence in norm of the two series \( \sum_{e \in E^1} \sqrt{\lambda_e} s_e a \) and \( \sum_{e \in E^1} \sqrt{\lambda_e} s_e a \).

Firstly, note that for \( v \in E^0 \) we have \( u_v a = 0 \) unless \( v \in r(K) \). Hence for any finite set \( F \subseteq s(E^1) \) containing \( r(K) \) we get \( u_F a = u_{r(K)} a \). Recall, see (44), that \( u_{r(K)} = \sum_{e \in s^{-1}(r(K))} \sqrt{\lambda_e} s_e \) converges in norm. Therefore, for any \( \varepsilon > 0 \) there is a finite set \( F_0 \subseteq s^{-1}(r(K)) \cap E^1 \) such that for any finite \( F \subseteq E^1 \) disjoint with \( F_0 \) we have
\[
\| \sum_{e \in F} \sqrt{\lambda_e} s_e a \| = \| \sum_{e \in F \cap s^{-1}(r(K))} \sqrt{\lambda_e} s_e a \| \leq \varepsilon.
\]
This means that the sum \( \sum_{e \in E^1} \sqrt{\lambda_e} s_e a \) converges in \( C^*(E) \).

Secondly, note that for \( e \in E^1 \) we have \( s_{\mu}^* a = 0 \) unless \( \mu \in K \) for some \( \mu \in E^* \), or \( r(\varepsilon) \in K \cap E^0 \). Recall that the sum \( \sum_{e \in r^{-1}(v)} \sqrt{\lambda_e} s_e^* \) is norm convergent for all \( v \in E^0 \). Thus for a fixed \( \varepsilon > 0 \) we can find a finite set \( F_1 \subseteq E^1 \) such that for any \( F \) disjoint with \( F_1 \) we have
\[
\| \sum_{e \in r^{-1}(v) \cap F} \sqrt{\lambda_e} s_e^* \| \leq \frac{\varepsilon}{|K \cap E^0| \cdot \| a \|} \quad \text{for all } v \in K \cap E^0.
\]
Then for any finite $F \subseteq E^1$ lying outside the finite set $F_0 := \{ e \in E^1 : e\mu \in K, \mu \in E^* \} \cup F_1$ we get

$$\left\| \sum_{e \in F} \sqrt{\lambda_e} s_e^* a \right\| = \left\| \sqrt{\sum_{v \in K \cap E^0} \sum_{e \in \epsilon^{-1}(v) \cap F} \lambda_e s_e^* a} \right\| \leq \sum_{v \in K \cap E^0} \left\| \sqrt{\lambda_e s_e^* a} \right\| \leq \varepsilon.$$

Thus $\sum_{e \in E^1} \sqrt{\lambda_e} s_e^* a$ converges in $C^*(E)$. This shows that $u_\lambda = \sum_{e \in E^1} \sqrt{\lambda_e} s_e$ converges in strict topology in $M(C^*(E))$.

iii) $\Rightarrow$ iv). Plainly, as the sum $u_\lambda = \sum_{e \in E^1} \sqrt{\lambda_e} s_e$ is strictly convergent the sum $u_\lambda a u_\lambda^* = \sum_{e,f \in E^1} \sqrt{\lambda_e} \lambda_f s_es_f^*$ converges in norm for every $a \in C^*(E)$.

iv) $\Rightarrow$ i). Using relations (42), (43) one readily verifies that the isomorphism given by (34) intertwines the restriction $\Phi_{*,\lambda}|_{D_E}$ of $\Phi_{*,\lambda}$ to $D_E$ with a mapping $L_\lambda : C_0(\partial E) \to C_0(\partial E)$ given by (37).

**Remark 5.5.** For $u_\lambda$ given by (40) we have $u_\lambda^* u_\lambda = \sum_{e \in E^0} (\sum_{e \in s^{-1}(v)} \lambda_e) p_v$. Hence $u_\lambda$ is a partial isometry if and only if the measures $\{\lambda_v\}_{v \in E^0}$ arising from $\lambda = \{\lambda_e\}_{e \in E^1}$ are normalized, that is if and only if

$$\sum_{e \in s^{-1}(v)} \lambda_e = 1, \quad \text{for all } v \in E^0. \quad (46)$$

Clearly, (46) implies (38) and if no vertex in $E$ receives edges from infinitely many vertices then (39) is trivial. So in this case $u_\lambda$ can be chosen to be a partial isometry. Nevertheless, in general there might be no systems satisfying (46) for which the sum (40) is strictly convergent (e.g. consider the infinite countable graph with a vertex receiving one edge from each of the remaining ones). If $E$ is locally finite, one can let $\lambda_e, v \in E^0$, to be uniform probability distributions by putting $\lambda_e := |s^{-1}(s(e))|^{-1}, e \in E^1$. In the latter case and under the assumption that $E$ has no sinks or sources it was noted implicitly in [9] Theorem 5.1 and explicitly in [17] Section 5) that the formula (40) defines an isometry in $M(C^*(E))$. A detailed discussion of history and analysis of operators (40), (41) associated to systems of uniform probability measures for arbitrary finite graphs can be found in [25].

Let us note that $\Phi_{*,\lambda}$, given by (41), restricted to $D_E$ assumes the form

$$\Phi_{*,\lambda}(a) = \sum_{e \in E^1} \lambda_e s_e a s_e^*, \quad a \in D_E. \quad (47)$$

In particular, in view of the last part of Proposition 5.4 it is natural to call $\Phi_{*,\lambda} : C^*(E) \to C^*(E)$ the non-commutative Perron-Frobenius operator associated to the quiver $(E^1, E^0, r, s, \lambda)$.

### 5.3. Graph $C^*$-algebras as crossed products $C^*(D_E, \mathcal{L})$

Now we are ready to state and prove the main result of this section. In previous subsections we have shown that for positive numbers $\lambda = \{\lambda_e\}_{e \in E^1}$ satisfying (38), (39) we have two mappings $L_\lambda : C_0(\partial E) \to C_0(\partial E)$ and $\Phi_{*,\lambda} : D_E \to D_E$, given respectively by (37) and (47). These mappings are intertwined by the isomorphism $C_0(\partial E) \cong D_E$ determined by (34). Thus one could express the following statement equally well in terms of $(C_0(\partial E), L_\lambda)$ or $(D_E, \Phi_{*,\lambda})$. We choose the second system, as it is more convenient for our proofs. In order to shorten the notation we denote $\Phi_{*,\lambda}$ simply by $\mathcal{L}$. 
Theorem 5.6. Suppose \( E = (E^0, E^1, s, r) \) is an arbitrary directed graph and choose the numbers \( \lambda_e > 0, \ e \in E^1 \), such that the conditions \([38], [39]\) hold. Then the sum

\[
\mathcal{L}(a) := \sum_{e \in E^1} \lambda_e s_e^* a s_e, \quad a \in \mathcal{D}_E,
\]

is convergent in norm and defines a (completely) positive map \( \mathcal{L} : \mathcal{D}_E \to \mathcal{D}_E \) such that

\[
C^*(E) \cong C^*(\mathcal{D}_E, \mathcal{L}),
\]

with the isomorphism determined by \( a \mapsto j_{\mathcal{D}_E}(a), \ a u_\lambda \mapsto j_{\mathcal{D}_E}(a)s, \ a \in \mathcal{D}_E, \) where \( u_\lambda \) is given by the strictly convergent sum \([40]\). Further under these assumptions:

i) If \( E \) has no infinite emitters, then the following sum is convergent in norm:

\[
\alpha(a) := \sum_{e \in E^1} s_e a s_e^*, \quad a \in \mathcal{D}_E.
\]

It defines an endomorphism such that \((\mathcal{D}_E, \alpha, \mathcal{L})\) is an Exel system and

\[
C^*(\mathcal{D}_E, \mathcal{L}) = \mathcal{D}_E \rtimes_{\alpha, \mathcal{L}} \mathbb{N}.
\]

Moreover, \((\mathcal{D}_E, \alpha, \mathcal{L})\) is a regular Exel system if and only if \([46]\) holds.

ii) If \( E \) has no infinite receivers then \( \mathcal{L} \) is a transfer operator for a certain endomorphism \( \alpha \) if and only if \( E \) is locally finite. In this event \( \alpha \) given by \([49]\) is a unique endomorphism such that \((\mathcal{D}_E, \alpha, \mathcal{L})\) is an Exel system and \( \mathcal{L} \) is faithful on \( \alpha(\mathcal{D}_E)\mathcal{D}_E \).

iii) If \( E \) is locally finite and without sources then \( \mathcal{L} \) is faithful and \( \alpha \) given by \([49]\) is a unique endomorphism such that \((\mathcal{D}_E, \alpha, \mathcal{L})\) is an Exel system.

Remark 5.7. We comment on the corresponding items in the above theorem:

i) Recall that \([46]\) holds if and only if the operator \( u_\lambda \) is a partial isometry. In particular, the general question for which graphs \( E \) the numbers \( \lambda_e > 0, \ e \in E^1 \), can be chosen so that the Exel system \((\mathcal{D}_E, \alpha, \mathcal{L})\) is regular, seems to be a complex problem.

ii. One could conjecture that in general \( \mathcal{L} \) is transfer operator for a certain endomorphism \( \alpha \) if and only if \( E \) has no infinite emitters, and then this endomorphism is the (non-commutative) Markov shift given by \([49]\).

iii) If \( E \) is locally finite and without sources then \( \partial E = E^\infty \) and we can put \( \lambda_e := |s^{-1}(s(e))|^{-1}, \ e \in E^1 \). In this case, identifying \( \mathcal{D}_E \) with \( C_0(E^\infty) \), the mappings \([49]\) and \([48]\) coincide with those given by \([11]\). In particular, Theorem 5.6 yields an isomorphism

\[
C^*(E) \cong C_0(E^\infty) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}
\]

proved by Brownlowe in \([6, Proposition 4.6]\), and when \( E \) has no sinks by Brownlowe, Raeburn and Vitadello in \([9, Theorem 5.1]\).

The proof of Theorem 5.6 will rely on the following two lemmas. We fix the notation from the assertion of Theorem 5.6 and note that the map \( \mathcal{L} : \mathcal{D}_E \to \mathcal{D}_E \) is well defined by Proposition 5.4. We denote by \( E^0_s := \{ v \in E^0 : r^{-1}(v) = \emptyset \} \) and \( E_{\text{in} f} := \{ v \in E^0 : |r^{-1}(v)| = \infty \} \) the set of sources and the set of infinite receivers, respectively.

Lemma 5.8. Let \( X_E \) be the \( C^* \)-correspondence of \((\mathcal{D}_E, \mathcal{L})\). We have \( N^+_E = \overline{\text{span}}\{ s_\mu s_\mu^* : \mu \in E^* \setminus E^0_s \} \) and \( J(X_E) = \overline{\text{span}}\{ s_\mu s_\mu^* : \mu \in E^* \setminus E_{\text{in} f} \} \). Hence

\[
N^+_E \cap J(X_E) = \overline{\text{span}}\{ s_\mu s_\mu^* : \mu \in E^* \setminus E^0_s \}.
\]

Proof. Note that \( \mathcal{D}_E \) is a direct sum of two complemented ideals \( \overline{\text{span}}\{ p_v : v \in E^0_s \} \) and \( \overline{\text{span}}\{ s_\mu s_\mu^* : \mu \in E^* \setminus E^0_s \} \). One readily sees that \( \mathcal{L} \) vanishes on the first one and is faithful on the second one. Hence \( N_E = \overline{\text{span}}\{ p_v : v \in E^0_s \} \) and \( N^+_E = \overline{\text{span}}\{ s_\mu s_\mu^* : \mu \in E^* \setminus E^0_s \} \).
Let $\mu \in E^* \setminus E^0$ and put $K := \lambda^{-1}_{\mu_1} \Theta((s_{\mu s_{\mu}^*} \otimes 1), (s_{\mu s_{\mu}^*} \otimes 1))$ where $\mu_1 \in E^1$ is such that $\mu_1 \lambda \mu = \mu$ for $\lambda \in E^*$ (we recall that $a \otimes 1 \in X_L$, for $a \in \mathcal{D}_E$, is given by (11)). We claim that $\phi(s_{\mu s_{\mu}^*}) = K$. Indeed, for any $a, b \in \mathcal{D}_E$ we have

$$K(a \otimes b) = K(a \otimes 1)b = \lambda^{-1}_{\mu_1} (s_{\mu s_{\mu}^*} \otimes \mathcal{L}(s_{\mu s_{\mu}^*} a)) b = (s_{\mu s_{\mu}^*} \otimes s_{\lambda \mu s_{\mu}^*} s_{\mu_1}) b.$$

Moreover, for any $x, y \in \mathcal{D}_E$ we have

$$\langle s_{\mu s_{\mu}^*} \otimes s_{\lambda \mu s_{\mu}^*} s_{\mu_1}, x \otimes y \rangle_L = \lambda^{-1}_{\mu_1} (s_{\mu s_{\mu}^*} \otimes \mathcal{L}(s_{\mu s_{\mu}^*} x) y = \lambda_{\mu_1} s_{\lambda \mu s_{\mu}^*} s_{\mu_1} x s_{\mu_1} y = \mathcal{L}(s_{\mu s_{\mu}^*} x) y = \langle s_{\lambda \mu s_{\mu}^*} (a \otimes 1), x \otimes y \rangle_L.$$

Hence $s_{\mu s_{\mu}^*} \otimes s_{\lambda \mu s_{\mu}^*} s_{\mu_1} = s_{\mu s_{\mu}^*} (a \otimes 1)$. Thus in view of (50) we get $K(a \otimes b) = (s_{\mu s_{\mu}^*} a \otimes 1)b = \phi(s_{\mu s_{\mu}^*})(a \otimes b)$, which proves our claim. If $v \in E^0 \setminus E^0_{in,f}$, then using what we have just shown we get

$$\phi(p_v) = \phi(\sum_{f \in r^{-1}(v_0)} s_f s_{\mu}^*) = \sum_{f \in r^{-1}(v_0)} \lambda_f^{-1} \Theta((s_{\mu} s_{\mu}^* \otimes 1), (s_f s_{\mu}^* \otimes 1)) \in \mathcal{K}(X_L).$$

This shows that $\text{span}\{s_{\mu s_{\mu}^*} : \mu \in E^* \setminus E^0_{in,f}\} \subseteq J(X_L)$. Suppose, on the contrary, that this inclusion is proper. Then there exists an element in $J(X_L)$ of the form $a = \sum_{\mu \in E^* \setminus E^0_{in,f}} c_{\mu} s_{\mu s_{\mu}^*} + \sum_{v \in E^0_{in,f}} c_v p_v$ where $c_{\mu}, c_v$ are complex numbers and there is $v_0 \in E^0_{in,f}$ such that $c_{v_0} \neq 0$. Then $\sum_{v \in E^0_{in,f}} c_v p_v = a - \sum_{\mu \in E^* \setminus E^0_{in,f}} c_{\mu} s_{\mu s_{\mu}^*}$ is in $J(X_L)$, and accordingly $p_{v_0} = c_{v_0}^{-1} c_{v_0} \sum_{v \in E^0_{in,f}} c_v p_v$ is in $J(X_L)$. We show that the latter is impossible. Indeed, any operator in $\mathcal{K}(X_L)$ can be approximated by $K \in \mathcal{K}(X_L)$ given by a finite linear combination of the form

$$K = \sum_{\mu, \nu, \eta, \tau \in F} \lambda_{\mu, \nu, \eta, \tau} \Theta((s_{\nu} s_{\mu}^* \otimes s_{\eta} s_{\tau}^*), (s_{\eta} s_{\tau}^* \otimes s_{\nu} s_{\mu}^*))$$

where $F \subseteq E^*$ is a finite set. For any such combination we can find an edge $g \in r^{-1}(v_0)$ such that the projection $p_{s_{(g)}}$ is orthogonal to every projection $s_{\mu s_{\mu}^*}$, $\mu \in F$. Then for $\tau \in F$, and any $\eta \in E^*$, we have

$$\langle s_{\eta} s_{\eta}^* \otimes s_{\tau} s_{\tau}^*, s_{\eta} s_{\eta}^* \otimes 1 \rangle_L = s_{\tau} s_{\tau}^* \mathcal{L}(s_{\eta} s_{\eta}^* s_{\eta} s_{\eta}^*) = s_{\tau} s_{\tau}^* p_{s_{(g)}}(\lambda_{\eta} s_{\eta}^* s_{\eta} s_{\eta}^*) = 0.$$ 

This implies that $K(s_{\eta} s_{\eta}^* \otimes 1) = 0$. Thus, as $\lambda_{\eta}^{-1} \|s_{\eta} s_{\eta}^* \otimes 1\| = 1$, we get

$$\|\phi(p_{v_0}) - K\| \geq \lambda_{\eta}^{-1} \|\phi(p_{v_0})(s_{\eta} s_{\eta}^* \otimes 1) - K(s_{\eta} s_{\eta}^* \otimes 1)\| = \lambda_{\eta}^{-1} \|s_{\eta} s_{\eta}^* \otimes 1\| = 1.$$ 

Accordingly, $\phi(p_{v_0}) \notin \mathcal{K}(X_L)$ which is a contradiction. Thus $\text{span}\{s_{\mu s_{\mu}^*} : \mu \in E^* \setminus E^0_{in,f}\} = J(X_L)$. □

By Proposition 5.3 if $E$ has no infinite emitters then (49) defines an endomorphism $\alpha : \mathcal{D}_E \to \mathcal{D}_E$.

**Lemma 5.9.** Suppose $E$ has no infinite emitters and $\alpha$ is given by (49). Then

$$\alpha(\mathcal{D}_E) \mathcal{D}_E = N^1_L \cap J(X_L).$$

**Proof.** Let $= \mu_1 \lambda \mu \in E^* \setminus E^0$ where $\mu_1 \in E^1$. Since

$$s_{\mu} s_{\mu}^* = s_{\mu_1} \lambda \mu s_{\mu}^* = s_{\mu_1} s_{\mu}^* \lambda \mu \alpha(s_{\mu} s_{\mu}^*) \in \alpha(\mathcal{D}_E) \mathcal{D}_E,$$

it follows from Lemma 5.8 that $N^1_L \cap J(X_L) \subseteq \alpha(\mathcal{D}_E) \mathcal{D}_E$. For the reverse inclusion it suffices to show that for any $a \in \mathcal{D}_E$ we have $\alpha(a) \in N^1_L \cap J(X_L)$. To this end, consider a net $\mu F := \sum_{e \in F} s_{\mu} s_{\mu}^* \in N^1_L \cap J(X_L) = \text{span}\{s_{\mu} s_{\mu}^* : \mu \in E^* \setminus E^0\}$ indexed by finite sets $F \subseteq E^1$ ordered by inclusion. Clearly, $\mu F \alpha(a)$ converges to $\alpha(a)$. Hence $\alpha(a) \in N^1_L \cap J(X_L)$. □
Thus applying Proposition 3.18 we see that the assertion.

By virtue of Lemma 5.8, we have the characteristic function of a non-compact set). Thus for all \( s \in D \) holds. The crossed products \( L^*(D_E, L) \) in \( B(H) \). We claim that it is covariant, in the sense of Definition 3.16 i.e. \( N^*_E \cap J(X) \subseteq D_{E}u_\alpha D_{E}u_\lambda \). Indeed, taking \( s \mu s_\mu^* \) where \( \mu \in E^* \setminus E^0 \), and writing \( \mu = \mu_1 \mu_2 \) where \( \mu_1 \in E^1 \) and \( \mu_2 = \mu_1 \mu_2 \) we get

\[
s_\mu s_\mu^* = s_\mu \mu_2 s_\mu^* = \lambda_1^{-1} s_\mu_1 s_\mu^* \ u_\lambda(s_\mu_2 s_\mu_2^*) u_\lambda^* s_\mu_1 s_\mu_1^* = D_{E}u_\lambda D_{E}u_\lambda^*.
\]

By virtue of Lemma 5.8 this proves our claim. Hence by Proposition 3.17 the mapping \( j_\lambda(a) \mapsto a, j_\lambda(a)s \mapsto au_\lambda, a \in D_E \), gives rise to a homomorphism from \( C^*(D_E, L) \) into \( C^*(E) \). Let us denote it by \( id \times u_\lambda \) and note that it is actually an epimorphism because we have

\[
s_e := (\sqrt{\lambda_e})^{-1}(s_e s_e^*) u_\lambda p s(e), \quad \text{for all } e \in E^1.
\]

Moreover, for the canonical gauge circle action \( \gamma \) on \( C^*(E) \) we have

\[
\gamma_z(a) = a, \quad \gamma_z(au_\lambda) = zau_\lambda, \quad \text{for all } a \in D_E, z \in \mathbb{T}.
\]

Thus applying Proposition 3.18 we see that \( id \times u_\lambda \) is an isomorphism. This proves the main part of the assertion.

i). Suppose now that \( E \) has no infinite emitters. Then (19) converges in norm by Proposition 5.3. Since

\[
L(\alpha(a)b) = \sum_{e,f \in E^1} \lambda_f s_e^* a s_e^* b s_f = \sum_{e \in E^1} \lambda_e p s(e) a s_e^* b s_e = a \sum_{e \in E^1} \lambda_e s_e^* b s_e = a L(b),
\]

for all \( a, b \in D_E \), the triple \( (D_E, \alpha, L) \) is an Exel system. Similar calculations show that

\[
\alpha(L(\alpha(a))) = \sum_{e \in E^1} \left( \sum_{f \in \mathcal{s}^{-1}(s(e))} \lambda_f \right) s_e a s_e^*.
\]

Hence, in view of Proposition 4.11, \( L \) is a regular transfer operator for \( \alpha \) if and only if (16) holds. The crossed products \( C^*(D_E, L) \) and \( D_E \times_{\alpha, L} \mathbb{N} \) coincide by Lemma 5.9 and Theorem 4.11.

ii). Suppose \( \alpha \) is an endomorphism such that \( (D_E, \alpha, L) \) is an Exel system. Putting \( b = s_e s_e^* \), \( e \in E^1 \), in the equation \( L(\alpha(a)b) = a L(b) \) we get \( s_e^* \alpha(a) s_e = a s_e^* s_e \). This in turn implies that

\[
\alpha(a) s_e s_e^* = a s_e a s_e^*, \quad e \in E^1.
\]

Lack of infinite receivers in \( E \) implies that the projections \( s_e s_e^* \) sum up strictly to a projection in \( M(C^*(E)) \). Let us denote it by \( p \). It follows that \( \alpha(a)p = \sum_{e \in E^1} s_e a s_e^* \) is in \( D_E \) for any \( a \in D_E \). If there would be an infinite emitter \( v \in E^0 \), then \( \alpha(p_v) = \sum_{e \in s^{-1}(v)} s_e s_e^* \) would not be an element of \( D_E \) (otherwise it would correspond via the isomorphism \( D_E \cong C_0(\partial E) \) to a characteristic function of a non-compact set). Thus \( E \) must be locally finite. Furthermore, in view of Lemma 5.8 we have \( p D_E = N^*_E \). Therefore if \( \alpha(D_E)D_E \subseteq N^*_E \) then \( \alpha \) has to be given by (19).

Item iii) follows from item ii) because for a locally finite graph without sources we have \( N^*_E = D_E \) by Lemma 5.8.

\[\square\]

Acknowledgments

The authors are grateful to the referee for valuable comments.

\[\text{References}\]

[1] S. ADJ, M. LACA, M. NILSEN and I. RAEBURN. Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups, Proc. Amer. Math. Soc. 122 (1994), 1133-1141.

[2] A. B. ANTOPOK, V. I. BAKHTIN and A. V. LEBEDEV. Crossed product of a \( C^* \)-algebra by an endomorphism, coefficient algebras and transfer operators. Math. Sb. (9) 202 (2011), 1253-1283.

[3] J. ARLEDGE, M. LACA and I. RAEBURN. Semigroup crossed products and Hecke algebras arising from number fields, Doc. Math. 2 (1997), 115-138.
[4] B. Brenken. C∗-algebras associated with topological relations. J. Ramanujan Math. Soc. 19.1 (2004), 35-55.
[5] N. Brown, N. Ozawa. C∗-algebras and Finite-Dimensional Approximations, Grad. Stud. Math. 88, Amer. Math. Soc. Providence, RI 2008.
[6] N. Brownlowe, Realising the C∗-algebra of a higher-rank graph as an Exel’s crossed product. J. Operator Theory (1) 68 (2012), 101-130.
[7] N. Brownlowe, N. Larsen and S. Stammeier. C∗-algebras of algebraic dynamical systems and right LCM semigroup. arXiv:1503.01599.
[8] N. Brownlowe and I. Raeburn. Exel’s crossed product and relative Cuntz-Pimsner algebras. Math. Proc. Camb. Phil. Soc. 141 (2006), 497-508.
[9] N. Brownlowe, I. Raeburn and S. T. Vittadello. Exel’s crossed product for non-unital C∗-algebras. Math. Proc. Camb. Phil. Soc. 149 (2010), 423-444.
[10] J. Cuntz. Simple C∗-algebras generated by isometries. Commun. Math. Phys. 57 (1977), 173-185.
[11] J. Cuntz and W. Krieger. A class of C∗-algebras and topological Markov chains. Inventiones Math. 56 (1980), 256-268.
[12] R. Exel. A new look at the crossed-product of a C∗-algebra by an endomorphism. Ergodic Theory Dyn. Syst. 23 (2003), 1733-1750.
[13] R. Exel. Interactions. J. Funct. Analysis, 244 (2007), 26-62.
[14] R. Exel and D. Royer. The crossed product by a partial endomorphism. Bull. Braz. Math. Soc. 38 (2007), 219-261.
[15] R. Exel and A. Vershik. C∗-algebras of irreversible dynamical systems. Canadian J. Math. 58 (2006), 39-63.
[16] C. Farthing, P. Muhly and T. Yeend. Higher-rank graph C∗-algebras: an inverse semigroup approach. Semigroup Forum. 71 (2005), 159-187.
[17] A. An Huef and I. Raeburn. Stacey crossed products associated to Exel systems. Integr. Equ. Oper. Theory 72 (2012), 537-561.
[18] M. Ionescu, P. Muhly and V. Vega. Markov operators and C∗-algebras. Houston J. Math. (3) 38 (2012), 775-798.
[19] JA. A. Jeong and Gi Hyun Park. Topological entropy for the canonical completely positive maps on graphs C∗-algebras. Bull. Austral. Math. Soc. 70 (2004), 101-116.
[20] T. Katsura. A construction of C∗-algebras from C∗-correspondences. Contemp. Math. vol. 335, pp. 173-182, Amer. Math. Soc. Providence (2003)
[21] T. Katsura. On C∗-algebras associated with C∗-correspondences. J. Funct. Anal. (2) 217 (2004), 366-401.
[22] E. T. A. Kakariadis and J. R. Peters. Representations of C∗-dynamical systems implemented by Cuntz families. Münster J. Math. 6 (2013), 383-411.
[23] B. G. Kwaśniewski. On transfer operators for C∗-dynamical systems. Rocky J. Math. (3) 42 (2012), 919-938.
[24] B. K. Kwaśniewski. C∗-algebras generalizing both relative Cuntz-Pimsner and Doplicher-Roberts algebras. Trans. Amer. Math. Soc. 365 (2013), 1809–1873.
[25] B. K. Kwaśniewski. Crossed products by interactions and graph algebras. Integral Equation Operator Theory 80(3) (2014), 415-451.
[26] B. K. Kwaśniewski. Ideal structure of crossed products by endomorphisms via reversible extensions of C∗-dynamical systems. accepted to Int. J. Math
[27] B. K. Kwaśniewski. Pure infiniteness and ideal structure of crossed products by endomorphisms of C0(X)-algebras. arXiv:1412.8240
[28] B. K. Kwaśniewski and A. V. Lebedev. Crossed products by endomorphisms and reduction of relations in relative Cuntz-Pimsner algebras. J. Funct. Anal. 264 (2013), 1806-1847.
[29] E. C. Lance. Hilbert C∗-modules: A toolkit for operator algebraists. London Math. Soc. Lecture Note Series. vol. 210. (Cambridge Univ. Press, 1994).
[30] M. Laca. Semigroups of ∗-endomorphisms, Dirichlet series and phase transitions, J. Funct. Anal. 152 (1998), 330–378.
[31] M. Laca and I. Raeburn. A semigroup crossed product arising in number theory, J. London Math. Soc. 59 (1999), 330-344.
[32] N. S. Larsen. Nonunital semigroup crossed products, Math. Proc. Roy. Irish. Acad. 100A(2) (2000), 205-218.
[33] N. Larsen. Crossed products by abelian semigroups via transfer operators. Ergodic Theory Dynam. Systems 30 (2010), 1147-1164.
[34] J. Lindiarni and I. Raeburn, Partial-isometric crossed products by semigroups of endomorphisms, J. Operator Theory, 52 (2004), 61-87.
[35] P. S. Muhly and B. Solel. Tensor algebras over C∗-correspondences (representations, dilations, and C∗-envelopes). J. Funct. Anal. 158 (1998), 389-457.
[36] P. S. Muhly and B. Solel. On the Morita equivalence of Tensor algebras. Proc. Lond. Math. Soc. 81 (2000), 113-168.

[37] P. S. Muhly and M. Tomforde. Topological quivers. Internat. J. Math. 16 (2005), 693-756.

[38] G. J. Murphy. Crossed products of $C^\ast$-algebras by endomorphisms. Integral Equations Operator Theory 24 (1996), 298-319.

[39] V. Paulsen. Completely Bounded Maps and Operator Algebras. Cambridge Studies in Advanced Mathematics (Cambridge University Press, Cambridge, 2002)

[40] W. L. Paschke. Inner product modules over $B^\ast$-algebras. Trans. Amer. Math. Soc. 182 (1973), 443-464.

[41] W. L. Paschke. The crossed product of a $C^\ast$-algebra by an endomorphism", Proc. Amer. math. Soc. 80 (1980), 113-118.

[42] A. Pelczyński. Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions, Dissertationes Math. (Rozprawy Mat.) 58 (1968), 1-92.

[43] I. Raeburn. Graph Algebras. CBMS Regional Conference Series in Math., vol. 103 (Amer. Math. Soc., Providence, 2005).

[44] M. Rørdam. Classification of certain infinite simple $C^\ast$-algebras. J. Funct. Anal. 131 (1995), 415-458.

[45] J. Schweizer. Crossed products by $C^\ast$-correspondences and Cuntz-Pimsner algebras. $C^\ast$-Algebras: Proceedings of the SFB-Workshop on $C^\ast$-Algebras, Münster, 1999 (Springer-Verlag, Berlin, 2000), pages 203-226.

[46] P. J. Stacey. Crossed products of $C^\ast$-algebras by $^\ast$-endomorphisms. J. Austral. Math. Soc. Ser. A 54 (1993), 204-212.

[47] M. Takesaki. Theory of operator algebras I. EMS on Operator Algebras and Non-Commutative Geometry vol. V (Springer-Verlag, 2002).

[48] S. B. G. Webster. The path space of a directed graph. Proc. Amer. Math. Soc., 142(1) (2014), 213-225.

Department of Mathematics and Computer Science, The University of Southern Denmark, Campusvej 55, DK–5230 Odense M, Denmark

Institute of Mathematics, University of Bialystok, ul. Akademicka 2, PL-15-267 Bialystok, Poland

Institute of Mathematics, Polish Academy of Science, ul. Śniadeckich 8, PL-00-956 Warszawa, Poland

E-mail address: bartoszk@math.uwb.edu.pl
URL: http://math.uwb.edu.pl/~zaf/kwasniewski