Invertibility In the Sense of Ehrenpreis

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In this paper we consider zero sets of entire functions belonging to the Schwartz algebra. This algebra is defined as the Fourier-Laplace transform image of the space of all distributions compactly supported on the real line.

We study the conditions under which given complex sequence forms zero set of some invertible in the sense of Ehrenpreis element of the Schwartz algebra (slowly decreasing function).

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1 Introduction

We will use the following common notations. \( \mathcal{D} = C_0^\infty(\mathbb{R}) \) denotes the space of test functions, \( \mathcal{D}' = (C_0^\infty(\mathbb{R}))' \). \( \mathcal{E} = C^\infty(\mathbb{R}) \) is the Schwartz space equipped with its standard metrizable topology, \( \mathcal{E}' = (C^\infty(\mathbb{R}))' \) is its strong dual space consisting of all distributions compactly supported on the real line.

Given \( S \in \mathcal{E}' \), its Fourier-Laplace transform \( \mathcal{F}(S) \) is defined by the formula

\[
\mathcal{F}(S) = S(e^{-itz}).
\]

It is well-known that the image \( \mathcal{P} = \mathcal{F}(\mathcal{E}') \) is the linear space of all entire functions of exponential type having polynomial growth along the real axis [1, Theorem 7.3.1]. Being equipped with the topology induced from \( \mathcal{E}' \), \( \mathcal{P} \) becomes a locally convex space of the type \( (LN^\ast) \); and \( \mathcal{P} \) is also the topological algebra (Schwartz algebra).

Recall that \( S \in \mathcal{E}' \) is invertible distribution [2] if

\[
S \ast \mathcal{E} = \mathcal{E},
\]

\[
S \ast \mathcal{D}' = \mathcal{D}',
\]

where \( \ast \) denotes the convolution.
S is invertible if and only if the primary ideal algebraically generated by \( \varphi = \mathcal{F}(S) \) in \( \mathcal{P} \) is closed. It is also equivalent to the \( i \)division theorem\( i \): \( \Phi \in \mathcal{P} \) and \( \Phi/\varphi \in \text{Hol}(\mathbb{C}) \) imply \( \Phi/\varphi \in \mathcal{P} \).

We call the function \( \varphi \in \mathcal{P} \) invertible in the sense of Ehrenpreis if \( S = \mathcal{F}^{-1}(\varphi) \) is invertible.

Due to the results of the paper [2] (Theorem I, Theorem 2.2, Proposition 2.7), \( \varphi \in \mathcal{P} \) is invertible in the sense of Ehrenpreis if and only if this function is slowly decreasing. It means that there exists \( a > 0 \) with the property

\[
\forall x \in \mathbb{R} \ \exists x' \in \mathbb{R} : |x - x'| \leq a \ln (a + |x|), \ |\varphi(x')| \geq (a + |x'|)^{-a}. \quad (1)
\]

Below, we formulate two other equivalent definitions of slow decrease property for function \( \varphi \in \mathcal{P} \) ([2, Section 3]):

1) there exists \( a > 0 \) such that

\[
\forall x \in \mathbb{R} \ \exists z' \in \mathbb{C} : |x - z'| \leq a \ln (a + |x|), \ |\varphi(z')| \geq (a + |z'|)^{-a}; \quad (2)
\]

2) there exists \( a > 0 \) such that \( \forall x \in \mathbb{R} \) we can find a circumference \( C_x \) of radius not greater than \( a \ln (a + |x|) \) with \( x \) laying inside \( C_x \) and

\[
|\varphi(z)| \geq (a + |z|)^{-a} \ \forall z \in C_x.
\]

As a rule, given a class of entire functions defined by some growth restrictions, it is interesting and useful to study the properties of their zero sets of these functions. In particular, it is related to investigating the necessary conditions, the sufficient conditions and the criteria of invertibility in the sense of Ehrenpreis for the function \( \varphi \in \mathcal{P} \) in terms of some characteristics of its zero set.

There are well-known connections of invertible in the sense of Ehrenpreis functions with question of surjectivity for the convolution operator in \( \mathcal{E} \) and \( \text{if fundamental principle} \) for the set of solutions of homogeneous convolution equation (or system of such equations) (see survey [4]).

For instance, if \( S \in \mathcal{E}' \) then the zero set of \( \varphi = \mathcal{F}(S) \) up to the multiplier \((-i)\) equals the spectrum of the solution subspace for the homogeneous convolution equation \( S * f = 0 \). This subspace admits spectral synthesis [3]. Moreover, if \( \varphi \) is invertible in the sense of Ehrenpreis then each solution \( f \) is represented as a series of exponential monomials converging with respect to the topology of \( \mathcal{E} \) after some groupings of its members [2, Theorem 3.1].

It is natural to expect the similar role of invertible in the sense of Ehrenpreis functions in questions of representation and \( \text{if fundamental principle} \).
for differential-invariant subspaces of the space $\mathcal{E}(a; b) := C^\infty(a; b)$, where $(a; b)$ is finite or infinite interval of the real axis. Spectral synthesis problem for the differentiation operator in $\mathcal{E}(a; b)$ has studied since 2008. Most of the known results are contained in [5]–[10].

In this paper, we obtain some necessary conditions and common criteria of invertibility in the sense of Ehrenpreis for functions $\varphi \in \mathcal{P}$ which zero sets $\{\lambda_j\}$ satisfy the relation $\text{Im } \lambda_j = O(\ln |\lambda_j|)$ as $j \to \infty$.

In Section 2, we prove Lemma 1. This lemma allows to study only functions which zeros are real. Then, we prove two assertions on the necessary conditions of invertibility in the sense of Ehrenpreis (Lemma 2 and Theorem 1).

Section 3 contains the necessary and sufficient conditions under which given even entire function of exponential type with real zeros is an invertible in the sense of Ehrenpreis element of the algebra $\mathcal{P}$ (Theorem 2). We also prove the common criterion of invertibility in the sense of Ehrenpreis for an arbitrary function $\varphi \in \mathcal{P}$ which zeros are real (Theorem 3).

2 Necessary conditions of invertibility in the sense of Ehrenpreis

Let $\mathcal{M} = \{\mu_j\}$, $\mu_j = \alpha_j + i\beta_j \subset \mathbb{C}$, $0 < |\mu_1| \leq |\mu_2| \leq \ldots$, be such that $\beta_j = O(\ln |\mu_j|)$ as $j \to \infty$, and

$$
\psi(z) = \lim_{R \to \infty} \prod_{|\mu_j| \leq R} \left( 1 - \frac{z}{\mu_j} \right)
$$

is an entire function of exponential type.

**Lemma 1** The function $\psi \in \mathcal{P}$ and it is invertible in the sense of Ehrenpreis if and only if the function

$$
\psi_1(z) = \lim_{R \to \infty} \prod_{|\alpha_j| \leq R} \left( 1 - \frac{z}{\alpha_j} \right)
$$

is invertible in the sense of Ehrenpreis element of the algebra $\mathcal{P}$.

**Proof.** First, notice that either both sequences, $\{\mu_j\}$ and $\{\alpha_j\}$, have densities or not; and either both series, $\sum \mu_j^{-1}$ and $\sum \alpha_j^{-1}$, converge or not.
It follows that either both formulas, (3) and (4), define entire functions of the same exponential type or not.

For each multiplier in the right-hand side of (4), we have

$$\left| 1 - \frac{x}{\alpha_j} \right| \leq \left| 1 - \frac{x}{\mu_j} \right| \left( 1 + \frac{\beta_j^2}{\alpha_j^2} \right)^{1/2}, \quad x \in \mathbb{R}. $$

Hence, if $\psi$ is an entire function then

$$\ln |\psi_1(x)| \leq \ln |\psi(x)| + O(1), \quad x \in \mathbb{R}. \quad (5)$$

By this estimate, we get

$$\psi \in \mathcal{P} \implies \psi_1 \in \mathcal{P}, \quad (6)$$

and if $\psi \in \mathcal{P}$ then

$\psi_1$ is invertible in the sense of Ehrenpreis in $\mathcal{P} \implies$

$$\implies \psi \text{ is invertible in the sense of Ehrenpreis in } \mathcal{P}. \quad (7)$$

Further, set

$$\mathcal{M}^+ = \{ \mu_j : \beta_j \geq 0 \}, \quad \mathcal{M}^- = \mathcal{M} \setminus \mathcal{M}^+, $$

$$\psi^+(z) = \lim_{R \to \infty} \prod_{|\mu_j| \leq R} \left( 1 - \frac{z}{\mu_j} \right) \cdot \prod_{|\mu_j| \leq R} \left( 1 - \frac{z}{\alpha_j} \right).$$

Without loss of the generality (wlog), we may assume that $|\alpha_j| > 1, j \in \mathbb{N}$. Let $M_0 > 0$ be such that

$$|\beta_j| \leq M_0 \ln |\alpha_j|, \ j = 1, 2, \ldots$$

It is easy to see that

$$\left| 1 - \frac{z}{\mu_j} \right| \leq \left| 1 - \frac{z}{\alpha_j} \right|, $$

where $z = x + 2iM_0 \ln |x|, \ |x| > 2, \mu_j \in \mathcal{M}^+$ and $|\alpha_j| \leq x^4$. It is also true that

$$\left| 1 - \frac{z}{\mu_j} \right| \leq \left| 1 - \frac{z}{\alpha_j} \right| \cdot \left( 1 + \frac{4M_0^2 \ln^2 |\alpha_j|}{|\alpha_j|^2} \right)^{1/2}$$
where $z = x + 2iM_0 \ln |x|, |x| > 2, \mu_j \in \mathcal{M}^+ \text{ and } |\alpha_j| > x^4$.

From these inequalities we derive the estimate

$$\ln |\psi(z)| \leq \ln |\psi^+(z)| + O(1), \quad z = x + 2iM_0 \ln |x| \text{ as } |x| \to \infty. \quad (8)$$

By the same argument, we obtain

$$\ln |\psi^+(z)| \leq \ln |\psi_1(z)| + O(1), \quad z = x - 2iM_0 \ln |x| \text{ as } |x| \to \infty. \quad (9)$$

By (6)–(9), taking into account that the functions $\psi, \psi_1, \psi^+$ have the same exponential type and applying the Phragmen-Lindelöf principle, we get

$$\psi_1 \in \mathcal{P} \Longrightarrow \psi^+ \in \mathcal{P} \Longrightarrow \psi \in \mathcal{P} \quad (10)$$

$$\psi \in \mathcal{P} \text{ is invertible in the sense of Ehrenpreis} \Longrightarrow \quad (11)$$

$$\Longrightarrow \psi^+ \in \mathcal{P} \text{ is invertible in the sense of Ehrenpreis} \Longrightarrow \quad (12)$$

$$\Longrightarrow \psi_1 \in \mathcal{P} \text{ is invertible in the sense of Ehrenpreis}. \quad (13)$$

To obtain (13) we have to use the version (2) of the definition of slowly decreasing function because of the estimates of the functions $\psi^+$ and $\psi_1$ hold for non-real $z = x + 2iM_0 \ln |x|$, and $z = x - 2iM_0 \ln |x|$, correspondingly.

The implications (6), (7), (10), (13) lead to the required assertion.

Q.E.D.

**Remark 1.** Further, we will formulate and prove all assertions for entire functions with only real zeros. By Lemma 1, it is clear that each of them has natural generalization for the case of entire functions which zeros $\{\lambda_j\}$ are not necessarily real, but satisfy the relation $\Im \lambda_j = O(\ln |\lambda_j|), \ j \to \infty$.

Let $\psi \in \mathcal{P}$ and $\{(a_j; m_j)\}$ denote its zero set, where $m_j$ denotes the multiplicity of the zero $a_j \in \mathbb{C}$. In the paper [2, Proposition 6.1] L. Ehrenpreis shows that if this function is is slowly decreasing function then the inequality

$$\lim_{j \to \infty} \frac{m_j}{\ln |\Im a_j| + \ln |\Re a_j|} < \infty \quad (14)$$

holds.

Given sequence $\mathcal{M} = \{\mu_k\} \subset \mathbb{C}, |\mu_1| \leq |\mu_2| \leq \ldots$, we denote by $m(z,t)$ the number of its points $\mu_k$ in the closed disc of radius $t$ centered at $z$.

Assuming that the zero set $\mathcal{M} \subset \mathbb{R}$, we improve the cited result due to L. Ehrenpreis.
Lemma 2 Let \( \psi \in P \) be invertible in the sense of Ehrenpreis (equivalently, \( \psi \) is slowly decreasing) with zeros \( \mathcal{M} = \{ \mu_k \} \subset \mathbb{R} \).

Then,

\[
\lim_{|x| \to \infty} \frac{m(x, 1)}{\ln |x|} < \infty.
\]  

(15)

Proof.

Wlog, we assume that \( \psi \) is bounded on the real axis and its exponential type equals 1.

Suppose that (15) fails, that is

\[
\lim_{j \to \infty} \frac{m(x_j, 1)}{\ln |x_j|} = \infty
\]

(16)

is true for some \( x_j, |x_j| \to \infty \). For clarity, assume that \( x_j > 0 \); and set

\[
\psi_j(z) = \psi(z)(z - x_j)^{m_j} \cdot \prod_{k: |\mu_k - x_j| \leq 1} (z - \mu_k)^{-1},
\]

where \( m_j = m(x_j, 1), \ j = 1, 2, \ldots \) It is easy to check that \( \psi_j \) are the entire functions of the exponential type 1, and the estimates

\[
\sup_{x \in \mathbb{R}} |\psi_j(x)| \leq C_0 2^{m_j}, \quad j = 1, 2, \ldots,
\]

hold with \( C_0 = \sup_{t \in \mathbb{R}} |\psi(t)| \). By well-known theorem due to S. Bernstein [11, Chapter 11], the following estimates

\[
\sup_{x \in \mathbb{R}} |\psi_j^{(n)}(x)| \leq C_0 2^{m_j}
\]

(17)

are also true for all \( n, j \in \mathbb{N} \).

Further, we modify L.Ehrenpreis’ argument which he used in [2, Proposition 6.1].

By the Taylor expansion of the function \( \psi_j \) at \( x_j \) and the estimates (17), we derive that

\[
|\psi_j(z)| \leq C_0 2^{m_j} (m_j!)^{-1} |z - x_j|^{m_j} e^{z-x_j}, \quad z \in \mathbb{C}.
\]

Hence, for all \( x \in \mathbb{R} \) satisfying the condition

\[
\ln C_0 + m_j + |x - x_j| + m_j \ln |x - x_j| - \ln (m_j!) \leq -\ln x_j - m_j \ln 2
\]

(18)
we have the inequality
\[ |\psi_j(x)| \leq x^{-l} \cdot 2^{-m_j}; \quad (19) \]
here, \( l \in \mathbb{N}. \)

By Stirling's formula, the relation \( (18) \) will follow from the inequality
\[ |x - x_j| + m_j \ln |x - x_j| - m_j \ln m_j \leq -l \ln |x_j| - C_1 m_j, \]
where \( C_1 \) is an absolute constant.

Because of \( (16) \), for each \( l \in \mathbb{N} \) we can find \( j_l \) such that
\[ -l \ln x_j \geq -m_j, \quad j = j_l, j_l + 1, \ldots \quad (20) \]
Let \( b \in (0; 1) \) and \( b < e^{-C_1^{-2}}. \) The estimates \( (19) \) hold for \( j \geq j_l \) and all \( x \in \mathbb{R} \) such that \( |x - x_j| \leq b m_j. \) From this fact, inequalities \( (20) \) and the relations
\[ |\psi(z)| \leq 2^{m_j} |\psi_j(z)|, \quad z \in \mathbb{C}, \quad j = 1, 2, \ldots, \]
we derive that
\[ |\psi(x)| \leq |x_j|^{-l}, \quad \text{for} \quad x \in \mathbb{R} \quad |x - x_j| \leq b l \ln x_j, \quad j \geq j_l, \quad l \in \mathbb{N}. \]
It means that \( \psi \) is not slowly decreasing and leads to the contradiction.

Q.E.D.

Let \( \varphi \in \mathcal{P} \) and \( \Lambda = \{ \lambda_j \} \subset \mathbb{R} \setminus \{0\} \) be its zero set. We introduce the following notations. \( n(z, t) \) denotes the number of points \( \lambda_j \) in the closed disc of radius \( t \) centered at \( z; \nu(t) \) is the number of points \( \lambda_j \) in \( (0; t] \) as \( t > 0, \) and \((-\nu(t))\) is the number of points \( \lambda_j \) in \( [t; 0) \) as \( t < 0; \) \( 2\Delta = \lim_{j \to \infty} \frac{\lambda_j}{|\lambda_j|}. \)

**Theorem 1** If \( \varphi \in \mathcal{P} \) is invertible in the sense of Ehrenpreis and has only real zeros then
\[ \nu(t) - \Delta t = O(\ln^2 |t|), \quad \text{as} \quad |t| \to \infty. \quad (21) \]

**Proof.**

Below, we list some auxiliary facts.

F1) Invertibility in the sense of Ehrenpreis for the function \( \varphi \in \mathcal{P} \) with real zeros is equivalent to the existence of constants \( M_0 > 0 \) and \( r_0 > 1 \) such that
\[ \ln |\varphi(z)| \geq -M_0 \ln |x|, \quad z = x + iy, \quad |x| \geq r_0, \quad |y| \geq M_0 \ln |x|. \quad (22) \]
F2) [12, Section 3]. If $\varphi \in \mathcal{P}$ is invertible in the sense of Ehrenpreis and all its zeros are real then the set

$$\{ z : \ln |\varphi(z)| < -M_0 \ln |z|, \; |x| \geq r_0, \; |y| \leq M_0 \ln |x| \}$$

(23)

consists of relatively compact components $G$ of the diameter

$$d_G \leq M \ln |z|, \; \forall z \in G.$$ (24)

F3) By Lemma 2 and (24), the number of points $\lambda_j \in G$ is $O(\ln^2 |z|)$ $\forall z \in G$, for each relatively compact component $G$ of the set (23).

F4) Let $\varphi \in \mathcal{P}$ be invertible in the sense of Ehrenpreis and have only real zeros. By standard technique of the theory of entire functions, from F3), we derive the estimate

$$\ln |\varphi(z)| \geq -C \ln^2 |z| \ln \ln |z|$$

(25)

if $z = x + iy, \; |x| \geq r_0, \; |y| \geq r_0$. Here, $C > 0$ depends only on $M_0, \; r_0$ and the density of $\Lambda$.

F5) Let $\varphi \in \mathcal{P}$ be such as above. By Theorem III.G.1 [13],

$$\ln |\varphi(z)| = \pi \Delta \text{Im} z + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} z \ln |\varphi(t)|}{|z - t|^2} dt, \; \text{Im} z > 0,$$

(26)

$$\ln |\varphi(z)| = -\pi \Delta \text{Im} z - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} z \ln |\varphi(t)|}{|z - t|^2} dt, \; \text{Im} z < 0.$$ (27)

Now, we estimate the difference $(\nu(x_0) - \Delta x_0)$ for an arbitrary $x_0 > r_0, \; x_0 \not\in \Lambda$. According to the well-known formula, we have

$$\nu(x_0) = \frac{1}{2\pi i} \int_{\Gamma^+} \frac{\varphi'(z)}{\varphi(z)} dz,$$ (28)

where $\Gamma^+$ is the boundary of the rectangle

$$\{ z = x + iy : \; 0 \leq x \leq x_0, \; |y| \leq 3M_0 \ln x_0 \}.$$
Here, arg $\varphi$ is the imaginary part of some branch (not necessarily, the main one) of the function $(\ln \varphi)$ which is analytic in each of the half-planes $\text{Im} \, z < 0$ and $\text{Im} \, z > 0$. Generally speaking, we choose different branches for each half-plane.

By (26) and (27), we see that

$$\ln |\varphi(z)| = \text{Re} \left( -i\pi \Delta z + \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{z-t} + \frac{t}{t^2+1} \right) \ln |\varphi(t)| \, dt \right), \quad \text{Im} \, z > 0,$$

$$\ln |\varphi(z)| = \text{Re} \left( i\pi \Delta z - \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{z-t} + \frac{t}{t^2+1} \right) \ln |\varphi(t)| \, dt \right), \quad \text{Im} \, z < 0.$$  

Hence,

$$\arg |\varphi(z)| = \text{Im} \left( -i\pi \Delta z + \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{z-t} + \frac{t}{t^2+1} \right) \ln |\varphi(t)| \, dt \right) + \text{const},$$

as $\text{Im} \, z > 0$;

$$\arg |\varphi(z)| = \text{Im} \left( i\pi \Delta z - \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{z-t} + \frac{t}{t^2+1} \right) \ln |\varphi(t)| \, dt \right) + \text{const},$$

as $\text{Im} \, z < 0$ ([13, III.H.2]).

Generally speaking, there are different constants in the right-hand sides of two last formulas.

From the above, taking into account the relation $\overline{\varphi(z)} = \varphi(z)$, we conclude that

$$\nu(x_0) = -\frac{1}{\pi} \left( \arg \varphi(x_0 + 3iM_0 \ln x_0) - \arg \varphi(3iM_0 \ln x_0) \right) + \text{const} =$$

$$= \Delta x_0 - \int_{-\infty}^{\infty} \left( \frac{x_0 - t}{(x_0 - t)^2 + 9M_0^2 \ln^2 x_0} + \frac{t}{t^2 + 1} \right) \ln |\varphi(t)| \, dt +$$

$$\int_{-\infty}^{\infty} \left( \frac{-t}{t^2 + 9M_0^2 \ln^2 x_0} + \frac{t}{t^2 + 1} \right) \ln |\varphi(t)| \, dt. \quad (29)$$

Last formula does not completely suit for estimating the expression $(\nu(x_0) - \Delta x_0)$, because there is no finite estimates from below for the function $\ln |\varphi(t)|$ on the whole real line. To workaround this issue, we consider the function $\psi(\tilde{z}) = \varphi(z)$, where $\tilde{z} = z - 2iM_0 \ln x_0$. The function $\psi$ is analytic and non-vanishing in the closed upper half-plane $\text{Im} \, \tilde{z} \geq 0$. And we have the representation similar to (26) for $\ln |\psi(\tilde{z})|$:
\[ \ln |\psi(\tilde{z})| = \pi \Delta \text{Im } \tilde{z} + \frac{1}{\pi} \int_{-\infty}^{\infty} \text{Im } \tilde{z} \ln |\psi(t)| \, dt = \]
\[ = \text{Re} \left( -i\pi \Delta \tilde{z} + \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{\tilde{z} - t} + \frac{t}{t^2 + 1} \right) \ln |\psi(t)| \, dt \right), \quad \text{Im } \tilde{z} > 0. \]

Now, we rewrite this representation in terms of \( \varphi \) and \( z \):

\[ \ln |\varphi(z)| = \text{Re} \left( -i\pi \Delta (z - 2iM_0 \ln x_0) + \right. \]
\[ \left. + \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{z - 2iM_0 \ln x_0 - t} + \frac{t}{t^2 + 1} \right) \ln |\varphi(t + 2iM_0 \ln x_0)| \, dt \right) \]
if \( \text{Im } z > 2M_0 \ln x_0 \). By the same way as it has been done for (29), we obtain

\[ \nu(x_0) = \Delta x_0 - \int_{-\infty}^{\infty} \left( \frac{x_0 - t}{(x_0 - t)^2 + M_0^2 \ln^2 x_0} + \frac{t}{t^2 + 1} \right) \ln |\varphi(t + 2iM_0 \ln x_0)| \, dt + \]
\[ + \int_{-\infty}^{\infty} \left( \frac{-t}{t^2 + M_0^2 \ln^2 x_0} + \frac{t}{t^2 + 1} \right) \ln |\varphi(2iM_0 \ln x_0)| \, dt. \quad (30) \]

It follows that

\[ |\nu_0(x_0) - \Delta x_0| \leq |I_1| + |I_2|, \quad (31) \]

where

\[ I_1 = \int_{-\infty}^{\infty} \left( \frac{x_0 - t}{(x_0 - t)^2 + M_0^2 \ln^2 x_0} + \frac{t}{t^2 + 1} \right) \ln |\varphi(t + 2iM_0 \ln x_0)| \, dt, \]
\[ I_2 = \int_{-\infty}^{\infty} \left( \frac{-t}{t^2 + M_0^2 \ln^2 x_0} + \frac{t}{t^2 + 1} \right) \ln |\varphi(2iM_0 \ln x_0)| \, dt. \]

Wlog, we assume that

\[ \ln |\varphi(z)| \leq \pi \Delta |\text{Im } z|, \quad z \in \mathbb{C}. \]

Taking into account (22), we have

\[ |\ln |\varphi(z)|| \leq C_1 \ln x_0, \quad (32) \]
for all $z = t + 2iM_0 \ln x_0$ with $|t| \leq x_0^2$; here, the positive constant $C_1$ depends only on $r_0$, $M_0$, $\Delta$.

Further, by (25), we derive that

$$|\ln |\varphi(z)|| \leq C_2 \ln^2 |t| \ln \ln |t|$$

(33)

if $z = t + 2iM_0 \ln x_0$ and $|t| \geq x_0^2$, where the constant $C_2 > 0$ depends only on $r_0$, $M_0$, $\Delta$.

Now, we estimate $|I_1|$ by help of (32) and (33):

$$|I_1| \leq \left| \int_{|t| \leq x_0^2} \left( \frac{x_0 - t}{(x_0 - t)^2 + M_0^2 \ln^2 x_0} + \frac{t}{t^2 + 1} \right) \ln |\varphi(t + 2iM_0 \ln x_0)| dt \right| +$$

$$+ \left| \int_{|t| \geq x_0^2} \left( \frac{x_0 - t}{(x_0 - t)^2 + M_0^2 \ln^2 x_0} + \frac{t}{t^2 + 1} \right) \ln |\varphi(t + 2iM_0 \ln x_0)| dt \right| =$$

$$= |I_{11}| + |I_{12}|.$$ (34)

Taking into account (32), we get

$$|I_{11}| \leq C_1 \ln x_0 \left( \int_{-x_0^2}^{x_0^2} \frac{|x_0 - t|}{(x_0 - t)^2 + M_0^2 \ln^2 x_0} dt + \int_{-x_0^2}^{x_0^2} \frac{|t|}{t^2 + 1} dt \right) \leq$$

$$\leq \tilde{C}_1 \ln^2 x_0,$$

where $\tilde{C}_1 > 0$ depends only on $r_0$, $M_0$, $\Delta$.

By simple transformations, we see that

$$\left| \frac{x_0 - t}{(x_0 - t)^2 + M_0^2 \ln^2 x_0} + \frac{t}{t^2 + 1} \right| \leq \frac{C_3}{|t|^{3/2}}, \quad |t| \geq x_0^2,$$

where $C_3$ depends only on $r_0$, $M_0$, $\Delta$.

(33) means that

$$|\ln |\varphi(t + 2iM_0 \ln x_0)|| \leq C_2 \ln^2 |t| \ln \ln |t|.$$ (33)

Summarizing the above, we obtain that

$$|I_{12}| \leq C_4,$$

where $C_4 > 0$ depends only on $r_0$, $M_0$, $\Delta$. 

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From the representation (34) and the estimates for $I_{11}$ and $I_{12}$ it follows that
\[ I_1 = O(\ln^2 x_0^2), \quad x_0 \to +\infty. \]
Similarly, we get
\[ I_2 = O(\ln^2 x_0^2), \quad x_0 \to +\infty. \]
Now, the inequality (31) and two last estimates lead to the required relation
\[ \nu(x) - \Delta x = O(\ln x), \quad x \to +\infty, \quad x \notin \Lambda. \tag{35} \]
By the same way, we manage with the case $x < 0$.
At last, for $x \in \Lambda$, the asymptotic relation (21) follows from (35) and Lemma 2. Q.E.D.

Remark 2. Set $l(t) = \ln (1 + t^2)$, $t \in \mathbb{R}$, $\lambda_j = j + l(|j|)$, $j = \pm 1, \pm 2, \ldots$. It is known that the function
\[ \varphi(z) = \lim_{R \to +\infty} \prod_{|\lambda_j| < R} \left( 1 - \frac{z}{\lambda_j} \right) \]
is invertible in the sense of Ehrenpreis in the algebra $\mathcal{P}$ [14, Theorem 1]; and, at the same time,
\[ \nu(x) = [x - \ln (1 + x^2) + o(1)], \quad x \to +\infty \]
(see [14, Lemma 1]). Hence, the necessary condition (21) cannot be improved.

Remark 3. Generally speaking, the necessary condition of invertibility in the sense of Ehrenpreis (21) is not the sufficient one. Let us demonstrate it. Set
\[ \varphi_0(z) = \frac{\sin \pi z}{\pi z s_0(z)}, \]
where $s_0(z) = \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{\pi^2} \right)$. By well-known estimates for the functions $\sin \pi z$ and $s_0$ [11, Chapter 3], we see that
\[ \ln |\varphi_0(x)| \leq -C \ln^2 (2 + |x|), \quad x \in \mathbb{R}, \]
where $C > 0$. Hence, $\varphi_0$ is not slowly decreasing function; equivalently, it is not invertible in the sense of Ehrenpreis. Nevertheless, the corresponding function $\nu$ satisfies the condition
\[ \nu(x) - x = O(\ln |x|) \quad \text{as } |x| \to \infty. \]
3 Criteria of invertibility in the sense of Ehrenpreis

3.1 The zero set is an even sequence.

We will use the following lemma due to S. Favorov [15, Lemma 1].

**Lemma B.** Let $A = \{a_j\} \subset \mathbb{C} \setminus \{0\}$ be the sequence satisfying the conditions

\[\exists \lim_{R \to \infty} \sum_{|a_j| < R} a_j^{-1},\]  
\[n_A(0, t) = O(t), \quad t \to \infty,\]  
\[n_A(0, t + 1) - n_A(0, t) = o(t), \quad t \to \infty,\]

where $n_A(z, t)$ denotes the number of points $a_j$ in the closed disc of radius $t$ centered at $z$. Then,

\[g(z) = \lim_{R \to \infty} \prod_{|a_j| \leq R} \left(1 - \frac{z}{a_j}\right),\]

is the entire function of exponential type, and

\[\ln |g(z)| = \int_0^\infty \frac{n_A(0, t) - n_A(z, t)}{t} dt\]

holds for all $z \in \mathbb{C}$.

For $\Lambda^+ = \{\lambda_j\}, \ 0 < \lambda_1 \leq \lambda_2 \leq \ldots$, such that

\[\exists \lim_{j \to \infty} \frac{j}{\lambda_j} = \Delta,\]

the corresponding even sequence $\Lambda = \Lambda^+ \cup (-\Lambda^+)$ satisfies the assumptions of Lemma B. Hence, the formula

\[\varphi(z) = \prod_{j=1}^\infty \left(1 - \frac{z^2}{\lambda_j^2}\right)\]

defines an entire function of exponential type $\pi \Delta$. 

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By help of Theorem 1, we obtain the necessary and sufficient conditions under which $\varphi$ is invertible in the sense of Ehrenpreis element of the algebra $\mathcal{P}$. To arrive to this result, first, we prove some auxiliary assertions.

For $x \in \mathbb{R}$ and $t > 0$, we denote by $n^+(x; t)$ and $n^-(x; t)$ the numbers of points of $\Lambda$ laying in the intervals $(x; x + t]$ and $(x - t; x]$, correspondingly.

Lemma 3 If
\[
n^+(0; x) - \Delta x = O(\ln^2 x) \quad \text{as} \quad x \to +\infty
\]
then
\[
\int_{x \ln x}^{+\infty} \frac{n^+(t; x) - n^-(t; x)}{t} \, dt = O(\ln x) \quad \text{as} \quad x \to +\infty.
\] (42)

Proof. We fix an arbitrary $\sigma > 2$ and write the representation
\[
\int_{x \ln x}^{+\infty} \frac{n^+(t; x) - n^-(t; x)}{t} \, dt = \int_{x \ln x}^{x^\sigma} \frac{n^+(t; x) - n^-(t; x)}{t} \, dt + \int_{x^\sigma}^{+\infty} \frac{n^+(t; x) - n^-(t; x)}{t} \, dt = J_1 + J_2. \quad (43)
\]

By the evenness of $\Lambda$ and Theorem 1, we get
\[
J_2 = \int_{x^\sigma}^{+\infty} \frac{x n^+(t; x)}{t(t + x)} \, dt - \int_{x^\sigma}^{x^\sigma - x} \frac{n^+(t; x)}{t + x} \, dt =
\]
\[
= \int_{x^\sigma}^{+\infty} \frac{O(x^2)}{t(t + x)} \, dt + O \left( x \ln \left( 1 + \frac{x}{x^\sigma - x} \right) \right) =
\]
\[
= O(1), \quad x \to +\infty.
\]

Taking into account the evenness of $\Lambda$ and Theorem 1 one more time, we
estimate $J_1$:

\[
\int_{x \ln x}^{x^\sigma} \frac{n^+(t; x) - n^-(t; x)}{t} \, dt = - \int_{x \ln x}^{x^\sigma - x} \frac{n^+(t; x) \pm \Delta x}{t + x} \, dt + \int_{x \ln x}^{x^\sigma - x} n^+(t; x) \left( \frac{1}{t} - \frac{1}{t + x} \right) \, dt + \int_{x \ln x}^{x^\sigma} \frac{n^+(t; x)}{t} \, dt =
\]

\[
= O(\ln x) - \Delta x \ln \left( 1 + \frac{1}{\ln x} \right) + \int_{x \ln x}^{x^\sigma - x} (O(\ln^2 t) + \Delta x) \left( \frac{1}{t} - \frac{1}{t + x} \right) \, dt + \int_{x^\sigma - x}^{x^\sigma} (O(\ln^2 t) + \Delta x) \left( \frac{1}{t} - \frac{1}{t + x} \right) \, dt =
\]

\[
= O(\ln x) - \Delta x \ln \left( 1 + \frac{1}{\ln x} \right) + O(\ln x) + \Delta x \ln \frac{x^\sigma - x}{x} - \Delta x \ln \left( 1 - \frac{1}{\ln x} + 1 \right) + O(1) + \Delta x \ln \frac{x^\sigma}{x^\sigma - x} = O(\ln x), \quad x \to +\infty.
\]

From (43) and the estimates for $J_1$ and $J_2$, it follows that (42) holds.

Q.E.D.

**Lemma 4** Let $\Lambda = \Lambda^+ \cup \Lambda^-$, where $\Lambda^+ = \{ \lambda_j \}$ is positive sequence of density $\Delta$.

Then, for a fixed $A > 0$, the relation

\[
I := \int_{|\ln|x|}^{\infty} \frac{n(x, t) - n(x + iA\ln|x|; t)}{t} \, dt = O(A^2), \quad |x| \to \infty,
\]

holds; here $n(z, t)$ denotes the number of points $\pm \lambda_j$ in the closed disc of radius $t$ centered at $z$.

**Proof.**

Assume that $x > 0$ (for $x < 0$ the same argument works).

We have

\[
0 \leq I = \int_{|\ln|x|}^{\infty} \frac{n^+(t + x - r_t; r_t)}{t} \, dt + \int_{|\ln|x|}^{\infty} \frac{n^+(-t + x; r_t)}{t} \, dt = I_1 + I_2,
\]

where $r_t = \sqrt{t^2 - A^2\ln^2 x}$. 

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First, we estimate $I_1$:

$$0 \leq I_1 = \sum_{\lambda_j \geq x + x\ln x} \int_{\lambda_j}^{x} \frac{\sqrt{\lambda_j^2 + A^2\ln^2 x - x}}{t} \, dt \leq$$

$$\leq \text{const} \sum_{\lambda_j \geq x + x\ln x} \frac{A^2\ln^2 x}{(\lambda_j - x)^2} \leq C_0 A^2, \quad x \geq x_0 > 0;$$

here, the positive constant $C_0$ depends only on $\{\lambda_j\}$.

Second integral $I_2$ is estimated by the similar way.

From the estimates for $I_1$, $I_2$, we obtain the required one for $I = I_1 + I_2$.

Q.E.D.

Now, we are ready to formulate and prove the main result.

**Theorem 2** Let $\Lambda = \{\pm \lambda_j\}$, $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, and the finite limit exists

$$\lim_{j \to \infty} \frac{1}{\lambda_j} =: \Delta.$$

The entire function

$$\varphi(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{\lambda^2_j}\right)$$

belongs to the algebra $\mathcal{P}$ and is invertible in the sense of Ehrenpreis if and only if the following two conditions hold:

$$n^+(0, x) - \Delta x = O(\ln^2 x), \quad x \to \infty,$$

$$\lim_{A \to \infty} \lim_{x \to \infty} \frac{1}{A\ln x} \left| \int_{A\ln x}^{x\ln x} \frac{n(0, t) - n(x + iA\ln x, t)}{t} \, dt \right| < +\infty. (47)$$

**Proof.**

**Necessity.**

Assume that the function $\varphi$ defined by (45) is invertible in the sense of Ehrenpreis element of the algebra $\mathcal{P}$. From Theorem 1, it follows that the relation (46) is true.

Wlog, we may assume that $|\varphi(x)| \leq 1$ and

$$\ln |\varphi(z)| \leq \pi \Delta |\text{Im} z|, \quad z \in \mathbb{C}.$$
With Lemma B, it leads to the inequality

\[ \int_0^\infty \frac{n(0, t) - n(x + i\ln x, t)}{t} \, dt \leq \pi\Delta \ln x \]

for all \( A > 0 \) and \( x \geq x_0 > 0 \).

Hence, we have

\[ \int_{\ln x}^{\ln x} \frac{n(0, t) - n(x + i\ln x, t)}{t} \, dt \leq \pi\Delta \ln x + \int_{\ln x}^{\ln x} \frac{n(x, t) - n(0, t)}{t} \, dt \leq \pi\Delta \ln x + \text{const} \ln x \]

for all \( x \geq x_0 \); and in this chain of inequalities, the last one is true because of Lemma 3. The above estimates give us the following inequality

\[ \lim_{A \to \infty} \lim_{x \to \infty} \frac{1}{\ln x} \int_{\ln x}^{\ln x} \frac{n(0, t) - n(x + i\ln x, t)}{t} \, dt \leq \pi\Delta. \]  

(48)

Further, there exists \( A_0 > 0 \) such that

\[ \ln |\varphi(x + iA_0 \ln x)| \geq -A_0 \ln x, \quad x \geq x_0. \]

All zeros of \( \varphi \) are real. It implies the estimate

\[ \ln |\varphi(x + iA \ln x)| \geq -A_0 \ln x, \quad A \geq A_0, \quad x \geq x_0. \]

This estimate, together with Lemma B, Lemma 3, Lemma 4 lead to

\[ \int_{\ln x}^{\ln x} \frac{n(x + i\ln x, t) - n(0, t)}{t} \, dt \leq A_0 \ln x + \text{const} \ln x + \text{const} \ln x + \text{const} A^2. \]

Finally,

\[ \lim_{A \to \infty} \lim_{x \to \infty} \frac{1}{\ln x} \int_{\ln x}^{\ln x} \frac{n(x + i\ln x, t) - n(0, t)}{t} \, dt \leq \pi\Delta. \]  

(49)
The inequalities (48), (49) imply (47).

Sufficiency.

From the relation (46) it follows that \( \varphi \) (defined by (45)) belongs to the class \( C \) (Cartwright class of entire functions).

The condition (47), Lemma B and Lemma 3 give us the estimate

\[
\ln |\varphi(x)| = \int_0^{A_0 \ln x} \frac{n(0,t) - n(x,t)}{t} dt + \int_{A_0 \ln x}^{x \ln x} \frac{n(0,t) - n(x,t)}{t} dt + \int_{x \ln x}^{\infty} \frac{n(0,t) - n(x,t)}{t} dt \leq
\]

\[
\leq \text{const} \ln x + \int_{A_0 \ln x}^{x \ln x} \frac{n(0,t) - n(x,\im i A_0 \ln x, t)}{t} dt \leq \text{const} \ln x, \quad x \geq x_0,
\]

where \( A_0 > 0, x_0 > 1 \). Hence, \( \varphi \in \mathcal{P} \).

To prove that \( \varphi \) is invertible in the sense of Ehrenpreis, we again use Lemma B, Lemma 3 and the relation (47). It allows us to derive the following estimate

\[
\ln |\varphi(x + iA_0 \ln x)| \geq \int_{A_0 \ln x}^{x \ln x} \frac{n(0,t) - n(x,\im i A_0 \ln x, t)}{t} dt + \int_{x \ln x}^{\infty} \frac{n(0,t) - n(x,\im i A_0 \ln x, t)}{t} dt \geq \text{const} \ln x, \quad x \geq x_0,
\]

where \( A_0 > 0, x_0 > 1 \). This estimate means that \( \varphi \) is invertible in the sense of Ehrenpreis function.

Q.E.D.

3.2 Criterion of invertibility in the sense of Ehrenpreis for an arbitrary function in \( \mathcal{P} \) which zeros are real.

Assume that \( \psi \in \mathcal{P} \) and its zero set \( \Lambda \subset \mathbb{R} \setminus \{0\} \) has density \( 2\Delta \). Set

\[
L(t) = \nu(t) - \Delta t, \quad L^*(t) = L(t) - L(-t), \quad t \in \mathbb{R};
\]

here, as above, \( \nu(t) \) denotes the number of points \( \lambda_j \in \Lambda \) in the interval \((0; t]\) as \( t > 0 \), and \((-\nu(t))\) is the number of points \( \lambda_j \in \Lambda \) in the interval \([-t; 0)\) as \( t < 0 \).
Theorem 3  The function $\psi \in \mathcal{P}$ with zero set $\Lambda \subset \mathbb{R} \setminus \{0\}$ of density $2\Delta$ is invertible in the sense of Ehrenpreis if and only if the relations

\begin{equation}
L(x) = O(\ln^2 |x|), \quad |x| \to \infty, \tag{50}
\end{equation}

\begin{equation}
\lim_{A \to \infty} \lim_{x \to \infty} \frac{1}{A \ln x} \left| \int_{A \ln x}^{x \ln x} \frac{2L^*(t) - L^*(x + r_{t,A}) + L^*(x - r_{t,A})}{t} \, dt \right| < +\infty, \tag{51}
\end{equation}

hold. Here, $r_{t,A} = \sqrt{t^2 - A^2 \ln^2 x}$.

Proof. First of all, we notice that because of Theorem 2.2 [2], either both functions, $\psi(z) \in \mathcal{P}$ and $\varphi(z) := \psi(z)\psi(-z)$ are invertible in the sense of Ehrenpreis or not.

Further, it is not difficult to see that (50) is valid if and only if (46) hold for both functions, $\psi$ and $\varphi$. And the validity of (51) for $\psi$ is equivalent to the validity of (47) for even function $\varphi$.

From the above, we can conclude that (50) and (51) imply invertibility in the sense of Ehrenpreis of the function $\varphi(z) = \psi(z)\psi(-z)$ and, consequently, the same property for $\psi$.

Conversely, if $\psi \in \mathcal{P}$ is invertible in the sense of Ehrenpreis then the same is true for the function $\varphi(z) = \psi(z)\psi(-z)$. The relation (50) holds because of Theorem [1].

Applying Theorem 2 to the function $\varphi$, we see that (47) is true for the counting function of its zeros. It implies (51).

Q.E.D.

Remark 4. It is well-known that entire function $\varphi$ belongs to the algebra $\mathcal{P}$ and it is slowly decreasing if its zero set $\mathcal{M} = \{\mu_k\}, k \in \mathbb{Z}$, formed by bounded perturbations of the sequence of integers:

\begin{equation}
|\mu_k - k| \leq L, \quad k \in \mathbb{Z}, \tag{52}
\end{equation}

for some $L > 0$ ([16, Theorem XXXIII]).

We can derive more general and convenient sufficient conditions of invertibility in the sense of Ehrenpreis from Theorems 2 and 3.

In particular, assume that for a sequence $\Lambda = \{\lambda_j\}, 0 < \lambda_1 \leq \lambda_2 \leq \ldots$, the following relation

\begin{equation}
\nu(t) - \Delta t = O(1) \tag{53}
\end{equation}
holds as $t \to \infty$. Evidently, it is less restricted than (52). At the same time, we can easily check that the conditions of the criterion in Theorem 2 are satisfied for the entire function

$$\varphi(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\lambda_j^2}\right).$$

Hence, $\varphi \in \mathcal{P}$ and it is invertible in the sense of Ehrenpreis.

For example, both sequences

1) $\Lambda = \{\pm \lambda_j\} \cup \{\pm e^{\sqrt{j}}\}_{j=1}^{\infty}$ where $\lambda_j = j + \ln^2 j$, $j = 1, 2, \ldots$,

2) $\mathcal{M} = \{\mu_j\}_{j \in \mathbb{Z}'}$, where $\mu_j = j + \ln^2 |j|$, $j \in \mathbb{Z}' = \mathbb{Z} \setminus \{0\}$, satisfy (53). But (52) fails for each of them.

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