Orbifolds and minimal modular extensions

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Abstract

Let $V$ be a simple, rational, $C_2$-cofinite vertex operator algebra and $G$ a finite group acting faithfully on $V$ as automorphisms, which is simply called a rational vertex operator algebra with a $G$-action. It is shown that the category $E_{V,G}$ generated by the $V_G$-submodules of $V$ is a symmetric fusion category braided equivalent to the $G$-module category $E = \text{Rep}(G)$. If $V$ is holomorphic, then the $V^G$-module category $C_{V,G}$ is a minimal modular extension of $E$, and is equivalent to the Drinfeld center $Z(\text{Vec}_G^G)$ as modular tensor categories for some $\alpha \in H^3(G, S^1)$ with a canonical embedding of $E$. Moreover, the collection $M_v(E)$ of equivalence classes of the minimal modular extensions $C_{V,G}$ of $E$ for holomorphic vertex operator algebras $V$ with a $G$-action forms a group, which is isomorphic to a subgroup of $H^3(G, S^1)$. Furthermore, any pointed modular category $Z(\text{Vec}_G^G)$ is equivalent to $C_{V,G}$ for some positive definite even unimodular lattice $L$. In general, for any rational vertex operator algebra $U$ with a $G$-action, $C_{U,G}$ is a minimal modular extension of the braided fusion subcategory $F$ generated by the $U^G$-submodules of $U$-modules. Furthermore, the group $M_v(F)$ acts freely on the set of equivalence classes $M_v(F)$ of the minimal modular extensions $C_{W,G}$ of $F$ for any rational vertex operators algebra $W$ with a $G$-action.

1 Introduction

This paper is a continuation of our study of $V^G$-module category $C_{V,G}$ for a regular (rational and $C_2$-cofinite) vertex operator algebra $V$ with a finite automorphism group isomorphic to $G$ of $V$ (cf. [DNR]). It is established that if $V^G$ is regular, the category $E_{V,G}$ generated by the $V^G$-submodules of $V$ is a symmetric fusion category braided equivalent to the $G$-module category $E = \text{Rep}(G)$. If $V$ is holomorphic, then the $V^G$-module category $C_{V,G}$ is a minimal modular extension of $E$, and is equivalent to the Drinfeld center $Z(\text{Vec}_G^G)$ or to the module category of twisted Drinfeld double $D^\alpha(G)$ for some $\alpha \in H^3(G, S^1)$ with a canonical embedding of $E$. This result has been conjectured in [DPR] where the $D^\alpha(G)$ was introduced and studied. Moreover, the collection $M_v(E)$ of equivalence classes of the minimal modular extensions $C_{V,G}$ of $E$ for some holomorphic vertex operator algebra

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thermore, any pointed modular category $\mathcal{C}_{V_G}$ is equivalent to $\mathcal{C}_{V_G}$ for some positive definite even unimodular lattice $L$. For any rational vertex operator algebra $U$ with a $G$-action, $\mathcal{C}_{V_G}$ is a minimal modular extension of the braided fusion subcategory $\mathcal{F}$ generated by the $U^G$-submodules of $U$-modules. Furthermore, the group $\mathcal{M}_u(\mathcal{E})$ acts freely on the set of equivalence classes $\mathcal{M}_u(\mathcal{F})$ of the minimal modular extensions $\mathcal{C}_{W_G}$ of $\mathcal{F}$ for any rational vertex operators algebra $W$ with a $G$-action.

It was proved in [CM] that if $G$ is solvable, then the regularity of $V$ implies the regularity of $V^G$. So we only need to assume $V$ is regular in this case. More recently, the regularity of $V$ together with the $C_2$-cofiniteness of $V^G$ implies the rationality of $V^G$ [MC].

We now give a detail discussion on this paper. A braided fusion category $\mathcal{C}$ over $E = \text{Rep}(G)$, simply called a braided $\mathcal{E}$-category, is a pair $(\mathcal{C}, \eta)$ where $\mathcal{C}$ is a braided fusion category and $\eta : \mathcal{E} \to \mathcal{C}$ is a full and faithful braided tensor functor. Throughout this paper, we call any full and faithful braided tensor functor an embedding. A braided $\mathcal{E}$-category $(\mathcal{C}, \eta)$ is said to be nondegenerate if $\eta : \mathcal{E} \to \mathcal{C}'$ is an equivalence, where $\mathcal{C}'$ denotes the Müger center of $\mathcal{C}$. Note that $\mathcal{E}$ is a nondegenerate braided $\mathcal{E}$-category. We may simply write $\mathcal{C}$ for the braided $\mathcal{E}$-category $(\mathcal{C}, \eta)$ when there is no ambiguity.

An equivalence of braided $\mathcal{E}$-categories $(\mathcal{C}_1, \eta_1)$ and $(\mathcal{C}_2, \eta_2)$ is a braided tensor equivalence $F : \mathcal{C}_1 \to \mathcal{C}_2$ such that $\eta_2 \cong F \circ \eta_1$ as braided tensor functors.

A modular extension of a braided $\mathcal{E}$-category $\mathcal{C}$ is a pair $(\mathcal{D}, j_\mathcal{D})$ in which $\mathcal{D}$ is a modular tensor category and $j_\mathcal{D} : \mathcal{C} \to \mathcal{D}$ is an embedding. Similar to the equivalence of two braided $\mathcal{E}$-categories, two modular extensions $(\mathcal{D}_1, j_1), (\mathcal{D}_2, j_2)$ of $\mathcal{C}$ said to be equivalent if there exists a braided tensor equivalence $F : \mathcal{D}_1 \to \mathcal{D}_2$ such that $j_2 \cong F \circ j_1$ as braided tensor functors. We may simply write $\mathcal{D}$ for the modular extension $(\mathcal{D}, j_\mathcal{D})$ of $\mathcal{C}$ if there is no ambiguity, and $[\mathcal{D}]$ for the equivalence class of $(\mathcal{D}, j_\mathcal{D})$.

A modular extension $\mathcal{D}$ of a nondegenerate braided $\mathcal{E}$-category $\mathcal{C}$ is called minimal if $\text{FPdim}(\mathcal{D}) = o(G) \cdot \text{FPdim}(\mathcal{C})$. According to [BNRW] there are only finitely many inequivalent minimal modular extensions of $\mathcal{C}$ if there is one. Moreover, by [LKW1], the collection $\mathcal{M}(\mathcal{E})$ of equivalent classes $[\mathcal{C}]$ of minimal modular extensions of $\mathcal{E}$ forms a finite group isomorphic to $H^3(G, S^1)$ under the relative Deligne tensor product $\mathcal{C} \otimes_\mathcal{E} \mathcal{D}$ for $[\mathcal{C}], [\mathcal{D}] \in \mathcal{M}(\mathcal{E})$. In fact, any minimal modular extension of $\mathcal{E}$ is braided equivalent to the Drinfeld center $\mathcal{Z}(\text{Vec}_G^\alpha)$ where $\text{Vec}_G^\alpha$ is the fusion category of $G$-graded vector spaces over $\mathbb{C}$ whose associativity isomorphism is given by the 3-cocycle $\alpha$. Note that $\mathcal{Z}(\text{Vec}_G^\alpha)$ is braided equivalent to the module category of the twisted Drinfeld double $D^\alpha(G)$.

For any pseudounitary nondegenerate braided $\mathcal{E}$-category $\mathcal{F}$, if $\mathcal{F}$ has a minimal modular extension, then the collection $\mathcal{M}(\mathcal{F})$ of equivalence classes of minimal modular extensions of $\mathcal{F}$ admits a natural action of $\mathcal{M}(\mathcal{E})$ via the relative Deligne tensor product. Moreover, $\mathcal{M}(\mathcal{F})$ is a $\mathcal{M}(\mathcal{E})$-torsor [LKW1].

Our investigation of the $V^G$-module category $\mathcal{C}_{V_G}$ in terms of the minimal modular extensions of certain braided fusion category is influenced greatly by the work of [LKW1]. Note from [Hu] that $\mathcal{C}_{V_G}$ is a modular tensor category. Associated to a rational vertex operator algebra $V$ with a $G$-action are two more braided fusion subcategories $\mathcal{E}_{V_G}$ and $\mathcal{F}_{V_G}$ of $\mathcal{C}_{V_G}$. Here, $\mathcal{E}_{V_G}$ is the full subcategory of $\mathcal{C}_{V_G}$ generated by the $V^G$-submodules

\[ V \]
of $V$, and $\mathcal{F}_{VG}$ is the full subcategory of $\mathcal{C}_{VG}$ generated by $V^G$-submodules of $V$-modules. $\mathcal{F}_{VG}$ is a nondegenerate braided $\mathcal{E}$-category which satisfies

$$C_{\mathcal{C}_{VG}}(\mathcal{E}_{VG}) = \mathcal{F}_{VG} \quad \text{and} \quad C_{\mathcal{C}_{VG}}(\mathcal{F}_{VG}) = \mathcal{E}_{VG},$$

where $C_{\mathcal{C}}(\mathcal{B})$ denotes the Müger centralizer of the subcategory $\mathcal{B}$ in the braided fusion category $\mathcal{C}$. These two categories are the same if and only if $V$ is holomorphic. The main idea is to put these categories in the context of the minimal modular extensions. Recall from [DLM1] a Schur-Weyl type duality decomposition

$$V = \bigoplus_{\lambda \in \text{Irr}(G)} W_{\lambda} \otimes V_{\lambda}$$

where $W_{\lambda}$ is the irreducible $G$-module with character $\lambda$ and the multiplicity spaces $V_{\lambda}$ are inequivalent irreducible $V^G$-modules. Then $\mathcal{E}_{VG}$ is generated by $V_{\lambda}$ for $\lambda \in \text{Irr}(G)$. Our first result asserts that $\mathcal{E}_{VG}$ is a symmetric fusion category braided equivalent to $\mathcal{E}$ for any rational vertex operator algebra $V$ via an embedding $F^{V,G} : \mathcal{E} \rightarrow \mathcal{C}_{VG}$ (also see [Ki]). In particular, $(\mathcal{C}_{VG}, F^{V,G})$ is a braided $\mathcal{E}$-category.

In the case when $V$ is holomorphic, $(\mathcal{C}_{VG}, F^{V,G})$ is a minimal modular extension of $\mathcal{E}$. So $\mathcal{C}_{VG}$ is braided equivalent to $Z(\text{Vec}_G^\alpha)$ for some $\alpha \in H^3(G, S^1)$. Let $H_G$ be the collection holomorphic vertex operator algebras with a $G$-action. Then $\mathcal{M}_v(\mathcal{E})$ consisting of equivalence classes of $(\mathcal{C}_{VG}, F^{V,G})$ for some $V \in H_G$ is a subgroup of $\mathcal{M}(\mathcal{E})$. We certainly believe that $\mathcal{M}_v(\mathcal{E}) = \mathcal{M}(\mathcal{E})$. If $G$ is an abelian group generated by less than 3 elements or an odd dihedral group, we show that $\mathcal{M}_v(\mathcal{E}) = \mathcal{M}(\mathcal{E})$. We also show that the group operation given in [LKW1] can be realized from the tensor product of vertex operator algebras, i.e. $\mathcal{C}_{VG} \otimes_{\mathcal{E}} F^{V,G}, F^{U,G} \mathcal{C}_{UG} \cong (\mathcal{C}_{U \otimes U,G}, F^{V \otimes U,G})$ for $V, U \in H_G$, where the $G$-action on $V \otimes W$ is the diagonal action of $G \times G$.

Now we assume that $\mathcal{F}$ is an arbitrary pseudounitary nondegenerate braided $\mathcal{E}$-category. Let $R^G_{\mathcal{F}}$ be the collection of rational vertex operator algebras $W$ with a $G$-action such that $\mathcal{F}_{W,G}$ is equivalent to $\mathcal{F}$ as braided $\mathcal{E}$-categories. If $R^G_{\mathcal{F}}$ is not empty, we establish that $\mathcal{M}_v(\mathcal{E})$ acts freely on the set of equivalence classes $\mathcal{M}_v(\mathcal{F}) = \{ [\mathcal{C}_{W,G}] | W \in R^{V,G}_G \}$ such that $\mathcal{C}_{VG} \otimes_{\mathcal{E}} F^{V,G}, F^{W,G} \mathcal{C}_{WG} \cong (\mathcal{C}_{(V \otimes V,G),(F^{V \otimes W,G})} \mathcal{C}_{U,G})$ for $V, W \in R^G_{\mathcal{F}}$ where the $G$-action of $V \otimes W$ is the diagonal action of $G \times G$. Again, it is desirable that $\mathcal{M}_v(\mathcal{F}) = \mathcal{M}(\mathcal{F})$ and $\mathcal{M}_v(\mathcal{F})$ is a $\mathcal{M}_v(\mathcal{E})$-torsor whenever $\mathcal{M}_v(\mathcal{F}) \neq \emptyset$.

For any braided fusion category $\mathcal{C}$ with braiding isomorphism $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$, $\overline{c}_{X,Y} = c^{-1}_{Y,X}$ also defines a braiding on $\mathcal{C}$. We denote by $\overline{\mathcal{C}}$ the braided fusion category $\mathcal{C}$ equipped with the braiding $\overline{c}$. Note that $\overline{\mathcal{E}} = \mathcal{E}$ as braided tensor categories. Thus, if $(\mathcal{C}, j)$ is a braided $\mathcal{E}$-category, then so is $(\overline{\mathcal{C}}, j)$. Again, we will simply write $\overline{\mathcal{C}}$ for the braided $\mathcal{E}$-category $(\mathcal{C}, j)$. The braided $\mathcal{E}$-category $\overline{\mathcal{C}}$ plays an essential role in the group structure of $\mathcal{M}(\mathcal{E})$ and the $\mathcal{M}(\mathcal{E})$-torsor structure on $\mathcal{M}(\mathcal{F})$. In fact, if $\mathcal{M} \in \mathcal{M}(\mathcal{E})$, then its inverse is exactly $\overline{\mathcal{M}}$. The proof of free and transitive action of $\mathcal{M}(\mathcal{E})$ on $\mathcal{M}(\mathcal{F})$ in [LKW2] also uses $\overline{\mathcal{N}}$ for $\mathcal{N} \in \mathcal{M}(\mathcal{F})$. So it is necessary and important to understand $\overline{\mathcal{C}}$ for a rational vertex operator algebra $V$. For such $V$ we now have two modular tensor categories $\mathcal{C}_V$ and $\overline{\mathcal{C}_V}$. In the setting of braiding isomorphism, one needs to define $(-1)^n$ for rational number $n$. The braiding $c_{X,Y}$ and $c^{-1}_{Y,X}$ correspond to the choices $(-1)^n = e^{\pi i n}$ and $(-1)^n = e^{-\pi i n}$. 

3
From the point of view of vertex operator algebra, we conjecture that the modular tensor category $C_V$ is braided equivalent to $C_U$ for some rational vertex operator algebra $U$. This is consistent with the reconstruction program. That is, any modular tensor category $C$ can be realized as $C_W$ for some rational vertex operator algebra. Using the language of vertex operator algebra, the conjecture is equivalent to the statement: If $V$ is a rational vertex operator algebra then there exists a holomorphic vertex operator algebra $H$ containing $V$ as sub-VOA such that the double commutant $C_H(C_H(V)) = V$. Then $C_V$ and $C_{C_H(V)}$ are braided equivalent. We prove this conjecture for lattice vertex operator algebra $V_L$, affine vertex operator algebras associated to the integrable highest weight representations and the Virasoro vertex operator algebras associated to the discrete series.

This paper is organized as follows: We review the twisted modules and $g$-rationality of vertex operator algebras following [DLM3] in Section 2. Section 3 is a review of basics on the fusion categories, braided fusion categories, modular tensor categories and minimal modular extensions of a fusion category over $E$ [ENO, EGNO, KO]. We also present the main results concerning $M(E)$ and its torsor $M(E)$ from [LKW1]. Section 4 is a review of the modular tensor category $C_V$ associated to a rational, $C_2$-cofinite vertex operator algebra $V$ [HL1, HL2, HL3, Hu]. We discuss how to realize $C_V$ as $C_U$ with some conjecture and examples in the last half of this section. We also gives a necessary and sufficient condition for $C_V$ and $C_{C_H(V)}$ being braided equivalent. In Section 5, we recall from [DRX, DLXY] the classification of irreducible $V^G$-modules and related results. In Section 6 we prove that for any rational, $C_2$-cofinite vertex operator algebra $V$, $E$ and $E_{V^G}$ are braided equivalent, and the regular commutative algebra $C[G]^*$ in $E$ corresponds to commutative algebra $V$ in $E_{V^G}$ for any rational vertex operator algebra $V$ under the braided equivalence. Furthermore, $C_{V^G}$ is a minimal modular extension of $F_{V^G}$. In particular, if $V$ is holomorphic, then $C_{V^G}$ is a minimal modular extension of $E$. Section 7 is devoted to the proof that $M_v(E)$ is a finite abelian group under the product $C_{V^G} \cdot C_{U^G} = C_{(V \otimes U)^G}$ and $M_v(E)$ acts on $M_v(F)$ by $C_{V^G} \cdot C_{W^G} = C_{(V \otimes W)^G}$. We prove in Section 8 that if $Z(Vec^0_G)$ is pointed, then $Z(Vec^0_G)$ is equivalent to $C_{V^G}$ for some positive definite even unimodular lattice $L$.

2 Twisted modules

Let $V$ be a vertex operator algebra and $g$ an automorphism of $V$ of finite order $T$. Then $V$ is a direct sum of eigenspaces of $g: V = \bigoplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r$ where $V^r = \{v \in V | gv = e^{-2\pi ir/T}v\}$. We use $r$ to denote both an integer between 0 and $T - 1$ and its residue class mod $T$ in this situation.

A weak $g$-twisted $V$-module $M$ is a vector space equipped with a linear map

\[ Y_M : V \to (\text{End } M)[[z^{1/T}, z^{-1/T}]] \]

\[ v \mapsto Y_M(v, z) = \sum_{n \in \frac{1}{T}\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End } M), \]
which satisfies the following: for all 0 ≤ r ≤ T − 1, u ∈ V^r, v ∈ V, w ∈ M,
\[ Y_M(u, z) = \sum_{n \in \frac{r}{T} + Z} u_n z^{-n-1}, \]
\[ u_l w = 0 \quad \text{for} \quad l \gg 0, \]
\[ Y_M(1, z) = \text{Id}_M, \]
\[ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(v, z_2)Y_M(u, z_1) \]
\[ = z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2), \]
where \( \delta(z) = \sum_{n \in Z} z^n \) and all binomial expressions (here and below) are to be expanded in nonnegative integral powers of the second variable.

A \textit{g-twisted V-module} is a \( \mathbb{C} \)-graded weak \( g \)-twisted \( V \)-module \( M \) :
\[ M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda \]
where \( M_\lambda = \{ w \in M | L(0)w = \lambda w \} \) and \( L(0) \) is the component operator of \( Y(\omega, z) = \sum_{n \in Z} L(n)z^{-n-2} \). We also require that \( \dim M_\lambda \) is finite and for fixed \( \lambda, M_{\frac{r}{T}+\lambda} = 0 \) for all small enough integers \( n \). If \( w \in M_\lambda \), we refer to \( \lambda \) as the \textit{weight} of \( w \) and write \( \lambda = wt w. \)

We use \( Z_+ \) to denote the set of nonnegative integers. An \textit{admissible g-twisted V-module} is a \( \frac{1}{T} Z_+ \)-graded weak \( g \)-twisted \( V \)-module \( M \) :
\[ M = \bigoplus_{n \in \frac{1}{T} Z_+} M(n) \]
satisfying
\[ v_m M(n) \subseteq M(n + wt v - m - 1) \]
for homogeneous \( v \in V, m, n \in \frac{1}{T} Z. \)

If \( g = \text{Id}_V \), we have the notions of weak, ordinary and admissible \( V \)-modules [DLM3].

If \( M = \bigoplus_{n \in \frac{1}{T} Z_+} M(n) \) is an admissible \( g \)-twisted \( V \)-module, the contragredient module \( M' \) is defined as follows:
\[ M' = \bigoplus_{n \in \frac{1}{T} Z_+} M(n)^*, \]
where \( M(n)^* = \text{Hom}_\mathbb{C}(M(n), \mathbb{C}) \). The vertex operator \( Y_{M'}(a, z) \) is defined for \( a \in V \) via
\[ \langle Y_{M'}(a, z)f, w \rangle = \langle f, Y_M(e^z L(1)(-z^{-2})L(0)a, z^{-1})w \rangle, \]
where \( \langle f, w \rangle = f(w) \) is the natural paring \( M' \times M \to \mathbb{C} \). It follows from [FHL] and [X] that \( (M', Y_{M'}) \) is an admissible \( g^{-1} \)-twisted \( V \)-module. The \( g^{-1} \)-twisted \( V \)-module \( M' = (M', Y_{M'}) \) is called the contragredient module of the \( g \)-twisted \( V \)-module \( M \). Moreover, \( M \) is irreducible if and only if \( M' \) is irreducible.
A vertex operator algebra $V$ is called $g$-rational, if the admissible $g$-twisted module category is semisimple. $V$ is called rational if $V$ is 1-rational. A vertex operator algebra $V$ is $C_2$-cofinite if $V/C_2(V)$ is finite dimensional, where $C_2(V) = \langle v_{-2}u | v, u \in V \rangle \subset \mathbb{Z}$. A vertex operator algebra $V$ is called regular if every weak $V$-module is a direct sum of irreducible $V$-modules \cite{DLM2}. It is proved in \cite{ABD} that if $V$ is of CFT type, then regularity is equivalent to rationality and $C_2$-cofiniteness. Also $V$ is regular if and only if the weak module category is semisimple \cite{DYu}.

The following results about $g$-rational vertex operator algebras are well-known \cite{DLM3}, \cite{DLM4}.

**Theorem 2.1.** If $V$ is $g$-rational, then:

1. Any irreducible admissible $g$-twisted $V$-module $M$ is a $g$-twisted $V$-module. Moreover, there exists a number $\lambda \in \mathbb{C}$ such that $M = \oplus_{n \in \mathbb{Z}+} M_{\lambda+n}$ where $M_{\lambda} \neq 0$. The $\lambda$ is called the conformal weight of $M$;

2. There are only finitely many irreducible admissible $g$-twisted $V$-modules up to isomorphism.

3. If $V$ is also $C_2$-cofinite and $g^i$-rational for all $i \geq 0$ then the central charge $c$ and the conformal weight $\lambda$ of any irreducible $g$-twisted $V$-module $M$ are rational numbers.

A vertex operator algebra $V = \oplus_{n \in \mathbb{Z}} V_n$ is said to be of CFT type if $V_n = 0$ for negative $n$ and $V_0 = \mathbb{C}1$.

### 3 Fusion categories

In this section we will review fusion categories and modular tensor categories following \cite{ENO}, \cite{EGNO}, \cite{KO}. A fusion category $C$ is a $\mathbb{C}$-linear abelian semisimple, rigid monoidal category with finitely many inequivalent simple objects, and finite dimensional morphism spaces, together with a tensor product functor $\boxtimes : C \times C \rightarrow C$, a unit object $1_C$ satisfying certain axioms. We use $X'$ to denote the (left) dual object of $X \in C$, and $\mathcal{O}(C)$ denotes the set of equivalence classes of the simple objects. Throughout this paper, subcategories of $C$ are always assumed to be full. A fusion subcategory of $C$ is defined as expected.

A very useful concept is so called the Frobenius-Perron dimension. Let $K_0(C)$ be the Grothendieck ring of a fusion category $C$. Then there is a unique ring homomorphism $\text{FPdim} : K_0(C) \rightarrow \mathbb{R}$ satisfying $\text{FPdim}(M) \geq 1$ for any nonzero object $M$. The Frobenius-Perron dimension of $C$ is defined to be $\text{FPdim}(C) = \sum_{M \in \mathcal{O}(C)} \text{FPdim}(M)^2$. In the case $C$ is a fusion subcategory of the module category for a vertex operator algebra $V$, the Frobenius-Perron dimension $\text{FPdim}(M)$ is exactly the quantum dimension $qdim_V(M)$ studied in \cite{DJX} and \cite{DRX}.

A braided fusion category is a fusion category $C$ with a natural isomorphism $c_{X,Y} : X \boxtimes Y \rightarrow Y \boxtimes X$, called a braiding, which satisfies some compatible conditions. Associated
to a braided fusion category $\mathcal{C}$ is another braided fusion category $\overline{\mathcal{C}}$ which has the same fusion category as $\mathcal{C}$ with a new braiding $\tau_{X,Y} = c_{Y,X}^{-1}$. A braided fusion category $\mathcal{C}$ is called symmetric if $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ or $\mathcal{C} = \overline{\mathcal{C}}$ as braided fusion categories. For any collection $\mathcal{D}$ of objects in $\mathcal{C}$, the M"uger centralizer $C_\mathcal{C}(\mathcal{D})$ is the subcategory of $\mathcal{C}$ consisting of the objects $Y$ in $\mathcal{C}$ such that $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ for all $X$ in $\mathcal{D}$. The subcategory $C_\mathcal{C}(\mathcal{D})$ is closed under the tensor product of $\mathcal{C}$ and hence a braided fusion subcategory of $\mathcal{C}$. The symmetric fusion category $C_\mathcal{C}(\mathcal{C})$ is called the M"uger center of $\mathcal{C}$, and denoted by $\mathcal{C}'$. For example, for any finite group $G$, the finite dimensional $\mathbb{C}[G]$-module category $\text{Rep}(G)$ is a symmetric fusion category with the usual tensor product and braiding of $\mathbb{C}$-linear spaces. A symmetric fusion category $\mathcal{C}$ is called Tannakian if there is a finite group $G$ such that $\mathcal{C}$ is equivalent to $\text{Rep}(G)$ as braided fusion categories. According to [De], the braided fusion category $\mathcal{C}$ is Tannakian if and only if there exists a faithful braided tensor functor from $\mathcal{C}$ to Vec, where Vec denotes the category of finite dimensional $\mathbb{C}$-linear spaces with the usual tensor product and braiding.

A braided fusion category $\mathcal{C}$ is called nondegenerate if $\mathcal{C} \cong \text{Vec}$. A nondegenerate spherical braided fusion category is called a modular tensor category. This definition is equivalent to that the corresponding $S$-matrix is nonsingular [Mu2]. From [DGNO] we know that if $\mathcal{C}$ is a braided fusion category and $\mathcal{B}$ is a fusion subcategory then

$$\text{FPdim}(\mathcal{B}) \cdot \text{FPdim}(C_\mathcal{C}(\mathcal{B})) = \text{FPdim}(\mathcal{C}) \cdot \text{FPdim}(\mathcal{C}' \cap \mathcal{B}).$$

(3.1)

The Drinfeld center $Z(\mathcal{C})$ of a fusion category $\mathcal{C}$ is a braided fusion category whose objects are pairs $(X, z_{X,-})$ in which $X \in \mathcal{C}$ and $z_{X,-} : X \boxtimes (-) \to (-) \boxtimes X$ a natural isomorphism, called a half-braiding, satisfying certain conditions. Moreover, $\text{FPdim}(Z(\mathcal{C})) = \text{FPdim}(\mathcal{C})^2$. If $\mathcal{C}$ is a spherical fusion category then $Z(\mathcal{C})$ is a modular tensor category [Mu2]. In particular, $Z(\text{Rep}(G))$ is a modular tensor category.

Let $\mathcal{C}$ be a braided fusion category. Then $\mathcal{E} = \mathcal{C}'$ is a symmetric fusion category. A modular extension (ME) of $\mathcal{C}$ is a pair $(\mathcal{D}, \iota_D)$ where $\mathcal{D}$ is modular tensor category and $\iota_D : \mathcal{C} \to \mathcal{D}$ is a full and faithful braided tensor (and simply called an embedding in the sequel). The ME $(\mathcal{D}, j_D)$ is called minimal (MME) if $C_\mathcal{D}(\mathcal{E}) = \mathcal{C}$ under the identification of $\iota_D(\mathcal{C})$ with $\mathcal{C}$. We will simply write $\mathcal{D}$ for an ME $(\mathcal{D}, \iota_D)$ of $\mathcal{C}$ and identify $\mathcal{C}$ with $\iota_D(\mathcal{C})$ when the context is clear.

**Lemma 3.1.** A modular extension $\mathcal{D}$ of a braided fusion category $\mathcal{C}$ over $\mathcal{E}$ is minimal if and only if

$$\text{FPdim}(\mathcal{D}) = \text{FPdim}(\mathcal{C}) \cdot \text{FPdim}(\mathcal{E}).$$

**Proof.** Since the modular tensor category $\mathcal{D}$ is nondegenerate, we see that $\mathcal{D}' = \text{Vec}$ and $\text{FPdim}(\mathcal{D}' \cap \mathcal{E}) = 1$. It follows immediately from equation (3.1) since $\mathcal{C}$ is always a fusion subcategory of $C_D(\mathcal{E})$. □

Let $\mathcal{C}$ be any braided fusion category over $\mathcal{E}$. Two modular extensions $(\mathcal{D}_1, \iota_1)$ and $(\mathcal{D}_2, \iota_2)$ of $\mathcal{C}$ are equivalent if there is a braided equivalence $F : \mathcal{D}_1 \to \mathcal{D}_2$ such that $F \circ \iota_1 \cong \iota_2$ as braided tensor functors. Let $\mathcal{M}(\mathcal{C})$ be the set of equivalence classes of
We denote the local $A$-modules of $\mathcal{C}$ by $\text{MMEs of } \mathcal{C}$. Then $\mathcal{M}(\mathcal{C})$ is a finite set as every MME of $\mathcal{C}$ has the same Frobenius-Perron dimension $\text{FPdim}(\mathcal{C}) \cdot \text{FPdim}(\mathcal{E})$ and there are only finitely many modular tensor categories up to equivalence for any fixed Frobenius-Perron dimension $[\text{BNRW}]$. The following important result was obtained in $[\text{LKWI}]$.

**Theorem 3.2.** Let $\mathcal{C}$ be a braided $\mathcal{E}$-category. Then $\mathcal{M}(\mathcal{E})$ is a finite abelian group and $\mathcal{M}(\mathcal{E})$ acts on $\mathcal{M}(\mathcal{C})$ freely and transitively provided $\mathcal{M}(\mathcal{C}) \neq \emptyset$. In particular, the cardinality of $\mathcal{M}(\mathcal{C})$ equals to the order of $\mathcal{M}(\mathcal{E})$ if $\mathcal{M}(\mathcal{C}) \neq \emptyset$.

The definition of the product on $\mathcal{M}(\mathcal{E})$ and the action of $\mathcal{M}(\mathcal{E})$ on $\mathcal{M}(\mathcal{C})$ are quite complicated, and we will discuss later in Section 6 in details.

An object $A$ in a braided fusion category $\mathcal{C}$ is called a **commutative fusion algebra** if there are morphisms $\mu : A \boxtimes A \to A$ and $\eta : 1_\mathcal{C} \to A$ such that

$$
\mu \circ (\mu \boxtimes \text{id}_A) \circ \alpha_{A,A,A} = \mu \circ (\text{id}_A \boxtimes \mu), \quad \mu = \mu \circ c_{A,A}
$$

$$
\mu \circ (\eta \boxtimes \text{id}_A) \circ l_A^{-1} = \text{id}_A = \mu \circ (\text{id}_A \boxtimes \eta) \circ r_A^{-1}
$$

where $\alpha_{A,A,A} : A \boxtimes (A \boxtimes A) \to (A \boxtimes A) \boxtimes A$ is the associativity isomorphism, and $l_A : 1_\mathcal{C} \boxtimes A \to A$ and $r_A : A \boxtimes 1_\mathcal{C} \to A$ are respectively the left and the right unit isomorphisms. A commutative algebra $A$ in $\mathcal{C}$ is called **connected** if $\dim \text{Hom}_\mathcal{C}(1_\mathcal{C}, A) = 1$.

Let $A$ be a connected commutative algebra in $\mathcal{C}$. A right $A$-module $M$ is an object in $\mathcal{C}$ with a morphism $\mu_M : M \boxtimes A \to M$ such that $\mu_M \circ (\text{id}_M \boxtimes \mu) = \mu_M \circ (\mu_M \boxtimes \text{id}_A) \circ \alpha_{M,A,A}$ and $\mu_M \circ (\text{id}_M \boxtimes \eta) = r_M$ where $r_M : M \boxtimes 1_\mathcal{C} \to M$ is the right unit isomorphism. If $M, N$ are right $A$-modules, a morphism $f : M \to N$ in $\mathcal{C}$ is called an $A$-module morphism if $\mu_M \circ (f \boxtimes \text{id}_A) = \mu_N$. We denote the category of right $A$-modules by $\mathcal{C}_A$.

The left $A$-modules are defined similarly. Since $A$ is commutative, a right $A$-module $M$ admits two natural left $A$-module structure on $M$, namely $m, \overline{m} : A \boxtimes M \to M$ defined by

$$
m = \mu_M \circ c_{A,M} \quad \text{and} \quad \overline{m} = \mu_M \circ \overline{c}_{A,M}.
$$

These left $A$-module structures on $M$ defines two $A$-bimodule structures on $M$, and they coincide when $M \in C_{\mathcal{C}}(A)$. We will consider any right $A$-module as an $A$-bimodule under the left $A$-action $\overline{m}$ as discussed. An $A$-module $M$ is called **local** if $\mu_M \circ c_{M,A} \circ c_{A,M} = \mu_M$. We denote the local $A$-module category by $\mathcal{C}_A^0$. It is immediate to see that the $A$-modules in $C_{\mathcal{C}}(A)$ are local $A$-modules.

An algebra $A$ in $\mathcal{C}$ is an $A$-bimodule under product map $\mu$ of an algebra $A$. If $\mu : A \boxtimes A \to A$ splits as $A$-bimodule morphism in $\mathcal{C}$, then $A$ is called **separable**. Following the terminology in $[\text{LKWI}]$, an algebra $A$ in ribbon tensor category $\mathcal{C}$ is said to be **condensable** if $A$ is a commutative, separable and connected algebra in $\mathcal{C}$ with $\dim(A) \neq 0$ and $\theta_A = \text{id}_A$.

The $A$-module category, $\mathcal{C}_A$, is a fusion category with the tensor product $M \boxtimes_A N$, where $N$ is considered as an $A$-bimodule under the preceding convention. Moreover, the category $\mathcal{C}_A^0$ of local $A$-modules is a braided fusion category. Moreover, if $\mathcal{C}$ is modular tensor category and $A$ is a condensable algebra in $\mathcal{C}$, then $\mathcal{C}_A^0$ is modular $[\text{KO}]$. We will
only consider pseudounitary fusion category \( \mathcal{C} \) with \( \dim(X) = \text{FPdim}(X) \) for all object \( X \in \mathcal{C} \). In the case, the condition \( \dim(A) \neq 0 \) is satisfied automatically. Moreover, an embedding of (pseudounitary) braided fusion categories preserves the canonical pivotal structures and hence their ribbon structures.

The following identities give relations among dimensions of relevant categories \( \mathcal{C} \) and condensable algebras \( A \) in \( \mathcal{C} \):

\[
\text{FPdim}(\mathcal{C}_A) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)}, \quad \text{FPdim}(\mathcal{C}_A^0) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)^2}
\]

where the first identity holds for any braided fusion category \( \mathcal{C} \) and the second identity requires that \( \mathcal{C} \) is modular [DMNO].

We have mentioned that for a finite group \( G \), \( \text{Rep}(G) \) is a symmetric fusion category. Then \( A = \mathbb{C}[G]^\ast \) is a condensable in \( \text{Rep}(G) \), called the regular algebra of \( \text{Rep}(G) \). Then, \( \text{Rep}(G)_A = \text{Rep}(G)_A^0 \) is equivalent to the category \( \text{Vec} \) of finite dimensional vector spaces. Furthermore, any condensable algebra in \( \text{Rep}(G) \) is given by \( \mathbb{C}[G/H]^\ast \) where \( H \) is a subgroup of \( G \) [KO].

We now discuss the modular extension of \( \mathcal{E} = \text{Rep}(G) \). Let \( (\mathcal{M}, \iota_{\mathcal{M}}) \) be an MME of \( \mathcal{E} \). Then the regular algebra \( A \) of \( \mathcal{E} \) is a condensable algebra in \( \mathcal{M} \). Following [DGNO], \( \mathcal{M}_A \) is a pointed fusion category equivalent to \( \text{Vec}_G^\alpha \) for some \( \alpha \in H^3(G,S^1) \). That is, \( \mathcal{M}_A = \bigoplus_{g \in G}(\mathcal{M}_A)_g \) and each \( (\mathcal{M}_A)_g \cong \text{Vec} \) as \( \mathbb{C} \)-linear categories and

\[
(\mathcal{M}_A)_g \boxtimes_A (\mathcal{M}_A)_h \cong (\mathcal{M}_A)_{gh}.
\]

We denote the simple object of \( (\mathcal{M}_A)_g \) by \( e(g) \) up to isomorphism. Then

\[
\alpha : (e(g) \boxtimes_A (e(h)) \boxtimes_A e(k) \to e(g) \boxtimes_A (e(h) \boxtimes_A e(k))
\]

gives the associativity isomorphism of \( \mathcal{M}_A \). Moreover, \( \mathcal{M} \) and \( \mathcal{Z}(\text{Vec}_G^\alpha) \) are braided equivalent. Furthermore, \( \mathcal{Z}(\text{Vec}_G^\alpha) \) is an MME of \( \text{Rep}(G) \) for any \( \alpha \in H^3(G,S^1) \). It is well known that \( \mathcal{Z}(\text{Vec}_G^\alpha) \) is braided equivalent to the representation category of the twisted Drinfeld double \( D^\alpha(G) \) of \( G \) [DPR]. It was proved in [LKWI] that \( (\mathcal{M}, \iota_{\mathcal{M}}) \) is equivalent to \( (\mathcal{Z}(\text{Vec}_G^\alpha), \iota_{\alpha}) \) where \( \iota_{\alpha} : \mathcal{E} \to \mathcal{Z}(\text{Vec}_G^\alpha) \) is the canonical embedding, which can be described as follows: Recall that the center \( \mathcal{Z}(\text{Vec}_G^\alpha) \) of \( \text{Vec}_G^\alpha \) consists of the pairs \( (X, c_{X,-}) \), in which \( X \in \text{Vec}_G^\alpha \) and \( c_{X,Y} : X \otimes Y \to Y \otimes X \), called an half-braiding, is a natural isomorphism for \( Y \in \text{Vec}_G^\alpha \) satisfying compatibility conditions (cf. [EGNO] for the center construction). For each \( X \in \text{Rep}(G) \), we consider \( X \) as a homogeneous vector space of grading 1, and we define the half-braiding \( c_{X,-} \) by setting \( c_{X,e(g)} : X \otimes e(g) \to e(g) \otimes X \), \( x \otimes 1 \mapsto 1 \otimes g^{-1}x \) for \( x \in X \). This assignment \( X \mapsto (X, c_{X,-}) \) can be extended to an embedding, i.e., a faithful and full braided tensor functor, \( \iota_{\alpha} : \mathcal{E} \to \mathcal{Z}(\text{Vec}_G^\alpha) \).

By [LKWI], \( (\mathcal{M}, \iota_{\mathcal{M}}) \cong (\mathcal{Z}(\text{Vec}_G^\alpha), \iota_{\alpha}) \) as braided \( \mathcal{E} \)-categories, and the map \( \Phi_G : \alpha \mapsto (\mathcal{Z}(\text{Vec}_G^\alpha), \iota_{\alpha}) \) defines a group isomorphism from \( H^3(G,S^1) \) to \( \mathcal{M}(\mathcal{E}) \).
4 Modular categories associated to vertex operator algebras

Let $V$ be a rational, $C_2$-cofinite vertex operator algebra of CFT type such that the conformal weight of any irreducible $V$-module $M$ is nonnegative, and is zero if and only if $M = V$. Then the $V$-module category $\mathcal{C}_V$ is a modular tensor category [Hu]. For the purpose of later discussion we need details on the tensor product of two modules explicitly. So we first give a brief review on the construction of the tensor product of modules for $V$-modules from [HL1] [HL2] [HL3] [Hu].

For any complex number $z \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, there is a tensor product $W_1 \boxtimes_{P(z)} W_2$ for any $V$-modules $W_1, W_2$ together with a canonical intertwining operator $I = I_{\otimes P(z)}$ of type $(W_1 \otimes_{P(z)} W_2)$ satisfying some universal property such that for $w_i \in W_i$, there is a tensor element

$$w_1 \boxtimes_{P(z)} w_2 = I(w_1, z) w_2 \in W_1 \boxtimes_{P(z)} W_2$$

where $W_1 \boxtimes_{P(z)} W_2$ is the formal completion of $W_1 \boxtimes_{P(z)} W_2$. The operator $I(w_1, z)$ is understood to be $\sum_{n \in \mathbb{R}} (w_1)_n z^{-n-1}$ where $(w_1)_n \in \text{Hom}(W_2, W_1 \boxtimes_{P(z)} W_2)$, and $z^n = e^{n \log z}$ for any $n \in \mathbb{R}$ with $\log z = \log |z| + i \arg z$ and $0 \leq \arg z < 2\pi$. Moreover, $W_1 \boxtimes_{P(z)} W_2$ is spanned by the coefficients of $z^n$ for all $w_i$ and $n \in \mathbb{R}$. If we have three $V$-modules $W_i$ for $i = 1, 2, 3$ there is an associativity isomorphism

$$A_{z_1, z_2} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \to (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

characterized by

$$w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3) \mapsto (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3$$

for $|z_1| > |z_2| > |z_1 - z_2| > 0$. The tensor product in the modular tensor category $\mathcal{C}_V$ is given by $\otimes = \otimes_{P(1)}$. We will simply denote the corresponding intertwining operator $I_{\otimes_{P(1)}}$ by $I$.

To discuss the braiding and associativity isomorphism in $\mathcal{C}_V$ we also need the natural parallel transport isomorphisms. Fix $V$-modules $W_1, W_2$ and nonzero complex numbers $z_1, z_2$, for any continuous path $\gamma$ in $\mathbb{C}^\times$ from $z_1$ to $z_2$, the parallel transport isomorphisms $T_\gamma : W_1 \boxtimes_{P(z_1)} W_2 \to W_1 \boxtimes_{P(z_2)} W_2$ is determined by the extension

$$T_\gamma : \quad W_1 \boxtimes_{P(z_1)} W_2 \to W_1 \boxtimes_{P(z_2)} W_2$$

$$w_1 \boxtimes_{P(z_1)} w_2 \mapsto I_{\boxtimes_{P(z_2)}}(w_1, e^{l(z_1)}) w_2$$

where $l(z_1)$ is the value of $\log z_1$ determined uniquely by $\log z_2$ with $\arg z_2 \in [0, 2\pi)$ and the path. The braiding isomorphism $c_{W_1, W_2} : W_1 \boxtimes W_2 \to W_2 \boxtimes W_1$ is determined by

$$c_{W_1, W_2}(w_1 \boxtimes w_2) = e^{L(-1)} T_{\gamma^{-}}(w_2 \boxtimes_{P(-1)} w_1) = e^{L(-1)} I(w_2, e^{\pi i}) w_1$$

where $\gamma^{-}$ is a path on the upper half plane without 0 from $-1$ to $1$. Then $c_{W_2, W_1}^{-1}$ is determined by

$$c_{W_2, W_1}^{-1}(w_1 \boxtimes w_2) = e^{L(-1)} T_{\gamma^{+}}(w_2 \boxtimes_{P(-1)} w_1) = e^{L(-1)} I(w_2, e^{-\pi i}) w_1$$

10
\( \gamma^+ \) is a path on the lower half plane without 0 from -1 to 1. The associativity isomorphism

\[
A_{W_1, W_2, W_3}: W_1 \boxtimes (W_2 \boxtimes W_3) \to (W_1 \boxtimes W_2) \boxtimes W_3
\]
is given by

\[
A_{W_1, W_2, W_3} = T_{\gamma_3} \circ (T_{\gamma_4} \boxtimes p(z_2)) \circ A_{z_1, z_2} \circ (\text{id}_{W_1} \boxtimes p(z_1)) T_{\gamma_2} \circ T_{\gamma_1}
\]
where \( z_1 > z_2 > z_1 - z_2 > 0 \), \( \gamma_1, \gamma_2 \) are paths in \( \mathbb{R}^2 = \mathbb{R} \setminus \{0\} \) from 1 to \( z_1, z_2 \), respectively, and \( \gamma_3, \gamma_4 \) are paths in the real line with 0 from \( z_2, z_1 - z_2 \) to 1, respectively.

We now investigate more on the modular tensor category \( \mathcal{C}_V \) with braiding \( c_{W_2, W_1}^{-1} \). The difference between these two braidings is how we choose \((-1)^n\) for any rational number \( n \). For braiding \( c_{W_1, W_2} \), \((-1)^n\) is understood to be \( e^{\pi i n} \) and for braiding \( c_{W_2, W_1}^{-1} \), \((-1)^n\) is understood to be \( e^{-\pi i n} \) (see the proof of Theorem 6.2). So the two different braidings in the \( V \)-module category really comes from the two different ways of choosing the skew symmetry for intertwining operators which define the tensor product.

There is a twist \( \theta_W = e^{2\pi i \Delta W(0)}: W \to W \) for any \( W \in \mathcal{C}_V \). If \( W \) is irreducible then \( \theta_W = e^{2\pi i \Delta_W} \) where \( \Delta_W \) is the weight of \( W \). Then the twist in the modular tensor category \( \mathcal{C}_V \) is given by \( \overline{\theta}_W = e^{-2\pi i \Delta W(0)} \). If \( W \) is irreducible, \( \overline{\theta}_W \) is exactly the complex conjugation of \( \theta_W \) as \( \Delta_W \) is a rational number. The relation \( \overline{\theta}_{W_1 \boxtimes W_2} = \overline{\tau}_{W_2, W_1} \circ \tau_{W_1, W_2} \circ (\overline{\theta}_{W_1} \boxtimes \overline{\theta}_{W_2}) \) is immediate by taking the inverse of the relation \( \theta_{W_1 \boxtimes W_2} = c_{W_2, W_1} \circ c_{W_1, W_2} \circ (\theta_{W_1} \boxtimes \theta_{W_2}) \).

Here is a natural question: Assume that \( V \) is a rational, \( C_2 \)-cofinite vertex operator algebra. Is there a rational vertex operator algebra \( U \) such that \( \mathcal{C}_V \) and \( \mathcal{C}_U \) are braided equivalent?

**Conjecture 4.1.** Assume that \( V \) is a rational, \( C_2 \)-cofinite vertex operator algebra. Then there is a rational, \( C_2 \)-cofinite vertex operator algebra \( U \) such that \( \mathcal{C}_V \) and \( \mathcal{C}_U \) are braided equivalent.

Let \( W = (W, Y, 1, \omega) \) be a vertex operator algebra and \( U = (U, Y, 1, \omega^1) \) be a vertex operator subalgebra of \( W \) such that \( L(1)\omega^1 = 0 \). The commutant \( C_W(U) \) of \( U \) is defined to be

\[
C_W(U) = \{ w \in W | u_n w = 0, u \in U, n \geq 0 \}.
\]

Set \( \omega^2 = \omega - \omega^1 \). Then \( (C_W(U), Y, 1, \omega^2) \) is also a vertex operator subalgebra of \( V \) [GKO, FZ]. Here is a characterization of \( \mathcal{C}_V \):

**Theorem 4.2.** Let \( V, U \) be as before. Then \( \mathcal{C}_V \) and \( \mathcal{C}_U \) are braided equivalent if and only if there exists a holomorphic vertex operator algebra \( W \) such that \( V \otimes U \) is a conformal subalgebra of \( W \) satisfying \( C_W(V) = U \) and \( C_W(U) = V \).

**Proof.** It is well known that \( \bigoplus_{X \in \mathcal{O}(C_V)} X \otimes X' \) is a condensable algebra in the modular tensor category \( \mathcal{C}_V \otimes \mathcal{C}_V \). If \( \mathcal{C}_V \) and \( \mathcal{C}_U \) are braided equivalent, let \( \mathcal{F} : \mathcal{C}_V \to \mathcal{C}_U \) be a braided tensor functor giving the equivalence. Then \( \mathcal{O}(\mathcal{C}_U) = \{ \mathcal{F}(X') | X \in \mathcal{O}(\mathcal{C}_V) \} \) and \( W = \bigoplus_{X \in \mathcal{O}(\mathcal{C}_V)} X \otimes \mathcal{F}(X') \) is a condensable algebra in \( \mathcal{C}_V \otimes \mathcal{C}_U \), which is equivalent to \( \mathcal{C}_V \otimes \mathcal{C}_U \) as braided tensor categories. It follows from [HKL] that \( W \) is a vertex operator algebra.
which is an extension of $V \otimes U$. Since $\text{FPdim}((\mathcal{C}_{V \otimes U})^\text{W}) = 1$, one concludes immediately that $W$ is a holomorphic vertex operator algebra. It is clear that $C_W(V) = U$ and $C_W(U) = V$ from the construction of $W$.

Now assume that there exists a holomorphic vertex operator algebra $W$ such that $V \otimes U$ is a conformal subalgebra of $W$ satisfying $C_W(V) = U$ and $C_W(U) = V$. Using a result in [KM] we know that every irreducible $V$-module and $U$-module appear in $W$ as $W$ is holomorphic. By Theorem 3.3 of [Lin] we conclude that $\mathcal{C}_V$ and $\mathcal{C}_U$ are braided equivalent. □

The following result asserts that Conjecture 4.1 holds for lattice vertex operator algebra, which can also be obtained from Proposition 8.2.

**Proposition 4.3.** Let $L$ be a positive definite even lattice, then $\mathcal{C}_{V_L}$ is braided equivalent to $\mathcal{C}_{V_K}$ for some positive definite even lattice $K$.

**Proof.** First, the lattice vertex operator algebra $V_L$ [B] [FLM] is rational [D1] [DLM2] and $C_2$-cofinite [Z] [DLM4]. So $\mathcal{C}_{V_L}$ is a modular tensor category. If $L$ is also unimodular, $V_L$ is holomorphic and $\mathcal{C}_{V_L}$ is braided equivalent to Vec and $\mathcal{C}_{V_L} = \mathcal{C}_{V_L}$. Now we assume that $L$ is not unimodular. By Theorem 5.5 of [GH], $L$ can be embedded in a positive definite even unimodular lattice $E$ and $L$ is a direct summand of $E$ as abelian groups and $O(L)$ embeds in $O(E)$ where $O(L)$ is the isometry group of $L$. Then $V_E$ is a holomorphic vertex operator algebra. Let $K = L^\perp$ be the orthogonal complement of $L$ in $E$. Then $V_{L \otimes K} = V_L \otimes V_K$ is a conformal subalgebra of $V_E$ in the sense that $V_{L \otimes K}$ and $V_E$ have the same Virasoro element.

Let $C_{V_E}(V_L)$ be the commutant of $V_L$ in $V_E$. We claim that $C_{V_E}(V_L) = V_K$ and $C_{V_E}(V_K) = V_L$. Clearly, $V_K$ is a subalgebra of $C_{V_E}(V_L)$. Recall from [D1] that the irreducible $V_K$-modules are given by $\{V_{K+\alpha}| \alpha \in K^\circ/K\}$ where $K^\circ$ is the dual lattice of $K$ and $\alpha$ are the representatives of cosets of $K$ in $K^\circ$. So $C_{V_E}(V_L)$ is a simple current extension of $V_K$ and there is an even sublattice $K_1$ of $E$ containing $K$ such that $K_1$ and $K$ have the same rank, and $C_{V_E}(V_L) = V_{K_1}$. This implies that $K_1$ is orthogonal to $L$, $K_1 = K$ and $C_{V_E}(V_L) = V_K$. Similarly, $C_{V_E}(V_K) = V_{L_1}$ for an even sublattice $L_1$ of $E$ such that $L_1$ contains $L$ and $L_1$, $L$ have the same rank. Thus $L_1/L$ is a finite abelian group. The fact that $L$ is a direct summand of $E$ as abelian groups forces $L_1 = L$. The result now follows from Theorem 1.2 □

**Example 4.4.** Let $L_{A_1} = \mathbb{Z}\alpha$ with $(\alpha, \alpha) = 2$. Then $V_{L_{A_1}}$ has two irreducible modules $V_{L_{A_1}}, V_{L_{A_1}+\alpha/2}$. Let $K = L_{E_7}$ be the root lattice of type $E_7$. Recall from [Hum] that the root lattice $L_{E_8}$ of type $E_8$ is spanned by $\alpha_i$ for $i = 1, \ldots, 8$ where $\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8 - (\epsilon_3 + \cdots + \epsilon_7))$, $\alpha_2 = \epsilon_1 + \epsilon_2$, $\alpha_3 = \epsilon_{i-1} - \epsilon_{i-2}$ for $i = 3, \ldots, 8$ and $\{\epsilon_i|i = 1, \ldots, 8\}$ is the standard orthonormal basis of $\mathbb{R}^8$. Then $L_{E_7}$ can be identified with the sublattice $\oplus_{i=1}^7 \mathbb{Z}\alpha_i$ of $L_{E_8}$ and $L_{A_1}$ can be identified with sublattice $\mathbb{Z}(\epsilon_7 + \epsilon_8)$ of $L_{E_8}$. It is easy to see that $\oplus_{i=1}^7 \mathbb{Z}\alpha_i$ and $\mathbb{Z}(\epsilon_7 + \epsilon_8)$ are orthogonal.

We claim that $L_{E_7} + L_{A_1}$ has index 2 in $L_{E_8}$. Clearly, $\alpha_8 = \epsilon_6 - \epsilon_7$ does not lie in $L_{E_7} + L_{A_1}$. So it is good enough to show that $2\alpha_8$ lies in $L_{E_7} + L_{A_1}$. Observe that $2\epsilon_i$ belongs to $L_{E_7} + L_{A_1}$ for all $i$. Thus $L_{E_8} = (L_{E_7} + L_{A_1}) \cup (L_{E_7} + L_{A_1} + \alpha_8)$, as claimed.
One can verify that $C_{V_{L_8}}(V_{L_7}) = V_{L_{A_1}}$ and $C_{V_{L_8}}(V_{L_{A_1}}) = V_{L_7}$ by noting that $\alpha_8$ is not orthogonal to $L_{E_7}$ and $L_{A_1}$. It is immediate from Theorem 4.2 that $C_{V_{L_{A_1}}, V_{L_7}}$ and $C_{V_{L_{E_7}}, V_{L_7}}$ are braided equivalent.

## 5 $V^G$-modules

In the rest of this paper we assume the following:

(V1) $V = \oplus_{n \geq 0} V_n$ is a simple vertex operator algebra of CFT type,

(V2) $G$ acts faithfully on $V$ as automorphisms such that $V^G$ is regular,

(V3) The conformal weight of any irreducible $V^G$-module $N$ is nonnegative and is zero if and only if $N = V^G$.

If $\sigma \in \text{Aut}(G)$, then the action of $G$ on $V$ can be twisted by $\sigma$, i.e., $g \cdot v = \sigma(g)v$. This twisted $G$ action on $V$ defines another automorphism group of $V$. In general, if $G$ is an abstract group, it is possible to embed $G$ into $\text{Aut}(V)$ in different ways.

Assumption (V2) implies that $V$ is $C_2$-cofinite [ABD] and $V$ is $g$-rational for all $g \in G$ [ADJR]. The assumption (V3) implies that the conformal weight of any irreducible $g$-twisted $V$-module except $V$ is positive, and that both $V^G$ and $V$ are selfdual.

We remark that if $G$ is solvable, then $V^G$ is regular if and only $V$ is regular. For arbitrary $G$, the regularity of $V$ together with the $C_2$-cofiniteness of $V^G$ implies the rationality of $V^G$ [Mc].

From our assumptions, both $C_V$ and $C_{V^G}$ are modular tensor category. Moreover, $V$ is a condensable algebra in $C_{V^G}$ [HKL]. To simplify the notation, we use $\text{Rep}(V)$ to denote the $V$-module category $(C_{V^G})_V$ in $C_{V^G}$ [KO]. Then $\text{Rep}(V)$ consists of every $V^G$-intertwining operator $Y_W(\cdot, z)$ of type $(W, V^G, W)$ such that the following conditions are satisfied:

1. ( Associativity) For any $u, v \in V$, $w \in W$ and $w' \in W'$, the formal series
   \[
   \langle w', Y_W(u, z_1)Y_W(v, z_2)w' \rangle
   \]
   and
   \[
   \langle w', Y_W(Y(u, z_1 - z_2)v, z_2))w' \rangle
   \]
   converge on the domains $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to multivalued analytic functions which coincide on their common domain.

2. (Unit) $Y_W(1, z) = \text{Id}_W$.

It is proved in [DLXY] that if $V^G$ satisfies conditions (V1)-(V3) then

\[
\text{Rep}(V) = \bigoplus_{g \in G} \text{Rep}(V)_g
\]

where $\text{Rep}(V)_g$ is the $g$-twisted $V$-module category. Moreover, $\text{Rep}(V)$ is a fusion category [KO], [CKM], [EGNO] with tensor product $\boxtimes_{\text{Rep}(V)}$. Furthermore, $\text{Rep}(V)_1$ which is denoted by $\text{Rep}(V)^0$ is exactly the modular tensor category $C_V$ by [HKL].
We now discuss the connection between the Frobenius-Perron dimension and the quantum dimension defined in [DJX], [DRX] for a $g$-twisted $V$-module $M = \oplus_{n \in \frac{1}{2}\mathbb{Z}_+} M_{\lambda+n}$ where $T$ is the order of $g$. Define the character of $M$ by
\[
\chi_M(\tau) = q^{-c/24} \sum_{n \in \frac{1}{2}\mathbb{Z}_+} \dim M_{\lambda+n} q^{\lambda+n}
\]
where $q = e^{2\pi i \tau}$ and $\tau$ lies in the upper half plane. Then $\chi_M(\tau)$ is a modular function on a congruence subgroup $\mathcal{Z}$, [DLN], [DR]. The quantum dimension of $M$ over $V$ is defined as
\[
\text{qdim}_V(M) = \lim_{\tau \to \infty} \frac{\chi_M(\tau)}{\chi_V(\tau)}
\]
which is always a positive algebraic number greater than or equal to 1. Note that $M$ is an object in the fusion category $\text{Rep}(V)$. It turns out that $\text{qdim}_V(M) = \text{FPdim}(M)$.

Our goal is to understand various fusion categories associated to $V^G$. We first present a result on the classification of irreducible $V^G$-modules or determine $\mathcal{O}(C_V^G)$ from [DRX]. For this purpose, we need the action of $G$ on $\text{Rep}(V)$ [DLM4]. Let $g,h$ be two automorphisms of $V$ with $g$ of finite order. If $(M,Y_M)$ is a $g$-twisted $V$-module, there is a $h^{-1}gh$-twisted $V$-module $(M \circ h, Y_{M\circ h})$ where $M \circ h \cong M$ as vector spaces and
\[
Y_{M\circ h}(v,z) = Y_M(hv,z)
\]
for $v \in V$. This defines a right action of $G$ on the twisted $V$-modules and on isomorphism classes of twisted $V$-modules. Similarly, we can define a left action of $G$ on the twisted $V$-modules and on isomorphism classes of twisted $V$-modules such that $h \circ M = M$ as vector spaces and $Y_{h \circ M}(v,z) = Y_M(h^{-1}v,z)$ for $v \in V$. Then $G$ acts on $\text{Rep}(V)$ as monoidal functors and $\text{Rep}(V)$ is a braided $G$-crossed category [MC].

If $g,h$ commute, $h$ clearly acts on the $g$-twisted modules. Denote by $\mathcal{M}(g)$ the equivalence classes of irreducible $g$-twisted $V$-modules and set $\mathcal{M}(g,h) = \{M \in \mathcal{M}(g) | M \circ h \cong M\}$. Note from Theorem [2.1] that if $V$ is $g$-rational, both $\mathcal{M}(g)$ and $\mathcal{M}(g,h)$ are finite sets. For any $M \in \mathcal{M}(g,h)$, there is a $g$-twisted $V$-module isomorphism
\[
\phi(h) : M \to M \circ h.
\]
The linear map $\phi(h)$ is unique up to a nonzero scalar. If $h = 1$ we simply take $\phi(1) = \text{Id}_M$.

Let $M = (M,Y_M)$ be an irreducible $g$-twisted $V$-module. We define a subgroup $G_M$ of $G$ consisting of $h \in G$ such that $M \circ h$ and $M$ are isomorphic. As we mentioned in Section 2 there is a projective representation $h \mapsto \phi(h)$ of $G_M$ on $M$ such that
\[
\phi(h)Y_M(v,z)\phi(h)^{-1} = Y_M(hv,z)
\]
for $h \in G_M$ and $v \in V$. Let $\alpha_M$ be the corresponding 2-cocycle in $C^2(G, \mathbb{C}^\times)$. Then $\phi(h)\phi(k) = \alpha_M(h,k)\phi(hk)$ for all $h,k \in G_M$. We may assume $\alpha_M$ has finite order. That is, there is a fixed positive integer $n$ such that $\alpha_M(h,k)^n = 1$ for all $h,k \in G_M$. Let $\mathbb{C}^{\alpha_M}[G_M] = \oplus_{h \in G_M} \mathbb{C}h$ be the twisted group algebra with product $hk = \alpha_M(h,k)hk$. It
is well known that $\mathbb{C}^{\alpha M}[G_M]$ is a semisimple associative algebra. It follows that $M$ is a $\mathbb{C}^{\alpha M}[G_M]$-module.

Let $\Lambda_M$ (which was denoted by $\Lambda_{G_M,\alpha M}$ in [DRX]) be the set of all irreducible characters $\lambda$ of $\mathbb{C}^{\alpha M}[G_M]$. Denote the corresponding simple module by $W_\lambda$. Using the fact that $M$ is a semisimple $\mathbb{C}^{\alpha M}[G_M]$-module, we let $M^\lambda$ be the sum of simple $\mathbb{C}^{\alpha M}[G_M]$-submodules of $M$ isomorphic to $W_\lambda$. Then

$$M = \bigoplus_{\lambda \in \Lambda_M} M^\lambda.$$  

Moreover, $M^\lambda = W_\lambda \otimes M_\lambda$ where $M_\lambda = \text{Hom}_{\mathbb{C}^{\alpha M}[G_M]}(W_\lambda, M)$ is the multiplicity of $W_\lambda$ in $M$. As in [DLM1], we can, in fact, realize $M_\lambda$ as a subspace of $M$ in the following way. Let $w \in W_\lambda$ be a fixed nonzero vector. Then we can identify $\text{Hom}_{\mathbb{C}^{\alpha M}[G_M]}(W_\lambda, M)$ with the subspace

$$\{ f(w) | f \in \text{Hom}_{\mathbb{C}^{\alpha M}[G_M]}(W_\lambda, M) \}$$

of $M^\lambda$. This gives a decomposition

$$M = \bigoplus_{\lambda \in \Lambda_M} W_\lambda \otimes M_\lambda \quad (5.1)$$

and each $M_\lambda$ is a module for vertex operator subalgebra $V^{G_M}$-module. Recall that the group $G$ acts on the set $S = \bigcup_{g \in G} M(g)$ and $M \circ h$ and $M$ are isomorphic $V^G$-modules for any $h \in G$ and $M \in S$. It is clear that the cardinality of the $G$-orbit $|M \circ G|$ of $M$ is $|G : G_M|$. Let $J$ be the orbit representatives of $S$. Then we have the following results [DRX], [DJX].

**Theorem 5.1.** Assume that $V$ satisfies (V1)-(V3).

1. The set

$$\{ M_\lambda | M \in J, \lambda \in \Lambda_M \}$$

     gives a complete list of inequivalent irreducible $V^G$-modules. That is, any irreducible $V^G$-module is isomorphic to an irreducible $V^G$-submodule $M_\lambda$ for some $M \in J$ and $\lambda \in \Lambda_M$.

2. We have a relation between quantum dimensions

$$\text{qdim}_{V^G}(M_\lambda) = \dim W_\lambda \cdot |G : G_M| \cdot \text{qdim}_V(M)$$

where $M$ is an irreducible $g$-twisted $V$-module and $\lambda \in \Lambda_M$. In particular, $\Lambda_V = \text{Irr}(G)$ is the set of irreducible characters of $G$ and $\text{qdim}_{V^G}(V_\lambda) = \dim W_\lambda$ for $\lambda \in \text{Irr}(G)$.

6 Fusion categories $\mathcal{F}_{V^G}$ and $\mathcal{E}_{V^G}$

Let $\mathcal{F}_{V^G}$ be the subcategory of $\mathcal{C}_{V^G}$ generated by $V^G$-submodules of $V$-modules and $\mathcal{E}_{V^G}$ the subcategory of $\mathcal{C}_{V^G}$ generated by $V^G$-submodules of $V$. We prove in this section that
\(\mathcal{E}_{V,G}\) is equivalent to \(\text{Rep}(G)\) as braided tensor categories, and \(\mathcal{F}_{V,G}\) is a braided fusion subcategory of \(\mathcal{C}_{V,G}\) such that the Mußer center \(\mathcal{F}_{V,G}\) of \(\mathcal{F}_{V,G}\) is exactly \(\mathcal{E}_{V,G}\).

**Theorem 6.1.** \(\mathcal{E}_{V,G}\) is a fusion subcategory of \(\mathcal{C}_{V,G}\) equivalent to the symmetric fusion category \(\text{Rep}(G)\) via a canonical braided tensor functor \(F^{V,G} : \text{Rep}(G) \to \mathcal{E}_{V,G}\). In particular, \(\mathcal{E}_{V,G}\) is a symmetric fusion category.

**Proof.** First we prove that \(\mathcal{E}_{V,G}\) is a braided fusion subcategory of \(\mathcal{C}_{V,G}\). Since \(\mathcal{E}_{V,G}\) is a semisimple category and each simple object is isomorphic to \(V_\lambda\) for some \(\lambda \in \text{Irr}(G)\), it suffices to show that \(V_\lambda \otimes V_\mu\) lies in \(\mathcal{E}_{V,G}\) for any \(\lambda, \mu \in \text{Irr}(G)\). From [T], [DRX] we know that

\[
V_\lambda \otimes V_\mu = \sum_{\nu \in \text{Irr}(G)} N^\nu_{\lambda,\mu} V_\nu
\]

where the fusion rules \(N^\nu_{\lambda,\mu}\) is given by the tensor product decomposition of \(G\)-module

\[
W_\lambda \otimes W_\mu = \sum_{\nu \in \text{Irr}(G)} N^\nu_{\lambda,\mu} W_\nu.
\]

Thus, \(\mathcal{E}_{V,G}\) is closed under the tensor product and is a braided fusion subcategory of \(\mathcal{C}_{V,G}\).

Second, we establish that \(\mathcal{E}_{V,G}\) is a symmetric braided fusion category. Equivalently we need to show that

\[
c_{V_\nu,V_\lambda} \circ c_{V_\lambda,V_\mu} = \text{id}_{V_\lambda \otimes V_\mu}
\]

for any \(\lambda, \mu\). Since \(\theta_{V_\lambda} = 1\) for all \(\nu \in \text{Irr}(G)\) we see that

\[
\text{id}_{V_\lambda \otimes V_\mu} = \theta_{V_\lambda \otimes V_\mu} = c_{V_\nu,V_\lambda} \circ c_{V_\lambda,V_\mu} \circ (\theta_{V_\lambda} \otimes \theta_{V_\mu}) = c_{V_\nu,V_\lambda} \circ c_{V_\lambda,V_\mu}.
\]

Finally, we show that \(\mathcal{E}_{V,G}\) is braided equivalent to the symmetric braided fusion category \(\text{Rep}(H)\). The categorical dimension \(\dim(X)\) for an object \(X\) in a spherical fusion category defined as the trace of the identity morphism of \(X\). Under our assumption, \(\dim(X) = \text{FPdim}(X)\) always positive for any object \(X\) in \(\mathcal{C}_{V,G}\) or \(\text{Rep}(V)\). The positivity of \(\dim(V_\lambda)\) together with the fact \(\theta_{V_\lambda} = 1\) implies that \(\mathcal{E}_{V,G}\) is a Tannakian category. By [De], \(\mathcal{E}_{V,G}\) is braided equivalent to \(\text{Rep}(H)\) for a finite group \(H\). The problem is we do not know why \(H\) is isomorphic to \(G\).

We now prove that \(\text{Rep}(G)\) is braided equivalent to \(\mathcal{E}_{V,G}\) directly. For this purpose we define a functor \(F = F^{V,G}\) from \(\text{Rep}(G)\) to \(\mathcal{E}_{V,G}\) to be the composition functor

\[
F(X) = \text{Hom}_G(X^*, V) = \bigoplus_{n \geq 0} \text{Hom}_G(X^*, V_n)
\]

where \(X^*\) is the dual of \(X\). It is easy to see that \(F(X)\) is a \(V^G\)-module such that for \(a \in V^G\) and \(f \in F(X)\), \((Y(a,x)f)(u) = Y(a,x)f(u)\) for \(u \in X^*\) as \(X\) is a \(G\)-module. Now we prove that \(F(X)\) lies in category \(\mathcal{E}_{V,G}\). It is good enough to assume \(X = W_\lambda\) for some \(\lambda \in \text{Irr}(G)\). Note that \(W_\lambda^*\) is isomorphic to \(W_\lambda^*\) where \(\lambda^*\) is the dual character of \(\lambda\). One can easily see that \(F(X)\) is isomorphic to \(V_\lambda^*\) and lies in \(\mathcal{E}_{V,G}\). Moreover, \(\text{Hom}_G(X^*, V_n)\) is the eigenspace of \(L(0)\) with eigenvalue \(n\).
We also need to deal with morphisms. Let $X,Y$ be two $G$-modules and $\alpha : X \to Y$ be a $G$-module morphism. Let $\alpha' : Y^* \to X^*$ be the adjoint map. For $f \in F(X)$ define $F(\alpha)(f) = \alpha' \circ F(X) \to F(Y)$. We assert that $F(\alpha)$ is a $V^G$-module homomorphism. Let $f \in F(X)$. For any $v \in Y^*$ and $a \in V^G$ we see that

$$F(\alpha)(Y(a,x)f)(v) = Y(a,x)f \alpha'(v) = Y(a,x)F(\alpha)(f)(v).$$

So $F$ is a functor from $\text{Rep}(G)$ to $\mathcal{E}_{Y^G}$.

Next we show that $F$ is a braided tensor functor. Let $X,Y$ be $G$-modules. We use the natural identification between $(X \otimes Y)^*$ with $Y^* \otimes X^*$. For $f \in F(X)$ and $g \in F(Y)$, one can show that

$$\mathcal{Y}(f,x)g(v \otimes u) = Y(fu,x)gv$$

for $v \otimes u \in Y^* \otimes X^*$ defines an intertwining operator of type $\left(F(X \otimes Y) \otimes F(X), F(Y) \otimes F(X)\right)$. In fact, for any $a \in V^G$ and formal variables $x_0, x_1, x_2$ we have to prove that

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0}\right) Y(a,x_1)\mathcal{Y}(f,x_2)g - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(f,x_2)Y(a,x_1)g$$

$$= x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0}\right) Y(a,x_0) f(x_2)g$$

or

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0}\right) Y(a,x_1)Y(fu,x_2)gv - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0}\right) Y(fu,x_2)Y(a,x_1)gv$$

$$= x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0}\right) Y(a,x_0) fu(x_2)gv.$$

This is obvious as the last identity is just the Jacobi identity in $V$. Thus $\mathcal{Y}(f,z)g$ is an intertwining map of type $\left(F(X \otimes Y) \otimes F(X), F(Y) \otimes F(X)\right)$ for any $z \in \mathbb{C}^\times$. By the universal mapping property of the tensor product $\boxtimes_{P(z)}$ we have a unique $V^G$-module homomorphism

$$J_{X,Y} : F(X) \boxtimes_{P(z)} F(Y) \to F(X \otimes Y) = \text{Hom}_G(Y^* \otimes X^*, V)$$

defined by

$$J_{X,Y}(f \boxtimes_{P(z)} g)(v \otimes u) = Y(fu,z)gv$$

for any $u \in X^*$, $v \in Y^*$. Then we have

$$J_{X,Y}(f \boxtimes_{P(z)} g)(v \otimes u) = Y(fu,z)gv$$

$$= e^{L(-1)} Y(gv,-z)fu$$

$$= e^{L(-1)} J_{Y,X}(g \boxtimes_{P(-z)} f)(u \otimes v)$$

$$= \overline{J_{Y,X}(e^{L(-1)} z g \boxtimes_{P(-z)} f)(u \otimes v)}$$

$$= \overline{J_{Y,X} e^{L(X,F(Y))(f \boxtimes_{P(z)} g)}(u \otimes v)}.$$

17
In particular,
\[ J_{X,Y}(f \boxtimes g)(v \otimes u) = J_{Y,X}(F(f),F(g))(u \otimes v). \]

Let \( \pi_{X,Y} : X \otimes Y \to Y \otimes X \) be the natural braiding of the vector space tensor product. Then \( F(\pi_{X,Y})J_{X,Y}(f \boxtimes g)(u \otimes v) = J_{X,Y}(f \boxtimes g)(v \otimes u) \). That is, \( J_{X,Y} \) is a natural isomorphism such that the following commuting diagram holds for objects \( X, Y \) in \( \text{Rep}(G) \):

\[
\begin{array}{ccc}
F(X) \boxtimes F(Y) & \xrightarrow{\epsilon \otimes \text{id}} & F(Y) \boxtimes F(X) \\
\downarrow J_{X,Y} & & \downarrow J_{Y,X} \\
F(X \otimes Y) & \xrightarrow{F(\pi_{X,Y})} & F(Y \otimes X).
\end{array}
\]

Let \( \epsilon : V^G \to \text{F}^{V^G}(\mathbb{1}) \) be the natural map \( a \mapsto f_a \) where \( f_a(1) = a \) for \( a \in V^G \), where \( \mathbb{1} \) denote the trivial \( G \)-module \( \mathbb{C} \). Then \( \epsilon \) is clearly a \( V^G \)-module isomorphism. Now, we prove that \((F,J,\epsilon)\) is a monoidal functor. That is, we need to verify the following commuting diagrams

\[
\begin{array}{ccc}
F(X) \boxtimes (F(Y) \boxtimes F(Z)) & \xrightarrow{\text{id} \otimes \text{id} \otimes \epsilon} & F(Y) \boxtimes F(Z) \\
\downarrow A_{F(X),F(Y),F(Z)} & & \downarrow F(a_{X,Y,Z}) \\
(F(X \boxtimes F(Y)) \boxtimes F(Z)) & \xrightarrow{J_{X,Y} \otimes \text{id}} & F(X \otimes Y) \boxtimes F(Z) \\
\downarrow J_{X,Y,Z} & & \downarrow F(a_{X,Y,Z}) \\
F(X \otimes Y) \boxtimes F(Z) & \xrightarrow{J_{Y,Z} \otimes \epsilon} & F((X \otimes Y) \otimes Z)
\end{array}
\]  
(6.1)

\[
\begin{array}{ccc}
V^G \boxtimes F(X) & \xrightarrow{\epsilon \otimes \text{id}} & F(1) \boxtimes F(X) \\
\downarrow l_{F(X)} & & \downarrow j_{F(X)} \\
F(X) & \xrightarrow{F(l_X)} & F(1 \otimes X)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
F(X) & \xrightarrow{F(r_X)} & F(X \otimes 1) \\
\downarrow r_{F(X)} & & \downarrow J_{X,1} \\
F(1) \boxtimes F(X) & \xrightarrow{\text{id} \otimes \epsilon} & F(X) \boxtimes F(1)
\end{array}
\]  
(6.2)

where \( a_{X,Y,Z}, r_X, l_X \) are respectively the associativity, left and right unit isomorphisms of vector spaces, and \( l_{F(X)}, r_{F(X)} \) respectively denote the left and the right unit isomorphisms of \( V^G \)-mod.

Since the proofs of two commuting diagrams in (6.2) are similar, we only give a proof of the first commuting diagram. Note that \( l_{F(X)} : V^G \boxtimes F(X) \to F(X) \) is characterized by \( l_{F(X)}(1 \boxtimes g) = g \) for \( g \in F(X) \). That is, \( l_{F(X)}(1 \boxtimes g)(u) = g(u) \) for \( u \in X^* \). On the other hand,

\[
(F(l_X) \circ J_{1X} \circ \epsilon \otimes \text{id})(1 \boxtimes g)(u) = (J_{1X} \circ \epsilon \otimes \text{id})(1 \boxtimes g)(u \otimes 1)
\]

\[
= J_{1X}(f_1 \boxtimes g)(u \otimes 1)
\]

\[
= Y(1,1)g(u)
\]

\[
= g(u).
\]

That is, \( F(l_X) \circ J_{1X} \circ \epsilon \otimes \text{id} = l_{F(X)} \).
We now prove (6.3). Let \( z_1, z_2 > 0 \) and \( \gamma \) be a path in \( \mathbb{R}^+ \) from \( z_1 \) to \( z_2 \), then we have the following commuting diagram

\[
\begin{array}{ccc}
F(X) \boxtimes P(z_1) F(Y) & \xrightarrow{J_{X,Y}} & F(X \otimes Y) \\
\downarrow T_\gamma & & \downarrow \text{id} \\
F(X) \boxtimes P(z_2) F(Y) & \xrightarrow{J_{X,Y}} & F(X \otimes Y)
\end{array}
\]

by noting that

\[
(J_{X,Y} \circ T_\gamma)(f \boxtimes P(z_1) g)(v \otimes u) = J_{X,Y} I_{\otimes P(z_2)}(f, e^{\gamma(z_1)})g(v \otimes u) = J_{X,Y} I_{\otimes P(z_2)}(f, z_1)g(v \otimes u) = Y(fu, z_1)gv
\]

where we have used the fact that \( I_{\otimes P(z_2)} \) only involves with integral powers of \( z \).

Let \( z_1 > z_2 > z_1 - z_2 > 0 \) and \( \gamma_i \) be as before for \( i = 1, 2, 3, 4 \). So it is good enough to show the following diagram is commutative:

\[
\begin{array}{ccc}
F(X) \boxtimes (F(Y) \boxtimes F(Z)) & \xrightarrow{\text{id} \boxtimes J_{Y,Z}} & F(X) \boxtimes F(Y \otimes Z) \xrightarrow{J_{X,Y \otimes Z}} F(X \otimes (Y \otimes Z)) \\
\downarrow (\text{id} \boxtimes P(z_1) T_{\gamma_1}) & & \downarrow T_{\gamma_1} & & \downarrow \text{id} \\
F(X) \boxtimes P(z_1) (F(Y) \boxtimes P(z_2) F(Z)) & \xrightarrow{\text{id} \boxtimes J_{Y,Z}} & F(X) \boxtimes P(z_1) F(Y \otimes Z) \xrightarrow{J_{X,Y \otimes Z}} F(X \otimes (Y \otimes Z)) \\
\downarrow A_{z_1, z_2} & & \downarrow & & \downarrow \\
(F(X) \boxtimes P(z_1 - z_2) F(Y)) \boxtimes P(z_2) F(Z) & \xrightarrow{J_{X,Y \otimes Z}} & F(X \otimes Y) \boxtimes P(z_2) F(Z) \xrightarrow{J_{X,Y \otimes Z}} F((X \otimes Y) \otimes Z) \\
\downarrow & & \downarrow & & \downarrow \text{id} \\
(F(X) \boxtimes F(Y)) \boxtimes F(Z) & \xrightarrow{J_{X,Y \otimes Z}} & F(X \otimes Y) \boxtimes F(Z) \xrightarrow{J_{X,Y \otimes Z}} F((X \otimes Y) \otimes Z)
\end{array}
\]

From the discussion above, the sub-diagrams involving the first two rows and the last two rows are commutative.

Now, we discuss the commutativity of the sub-diagram involving the second and third rows. Let \( f \in F(X), g \in F(Y), h \in F(Z) \) and \( u \in X^*, v \in Y^*, w \in Z^* \). Then

\[
(F(a_{X,Y,Z}) \circ J_{X,Y \otimes Z} \circ \text{id} \otimes J_{Y,Z})(f \boxtimes P(z_1) (g \boxtimes P(z_2) h))(w \otimes (v \otimes u)) = Y(fu, z_1)Y(gv, z_2)hw, \quad \text{and}
\]

\[
(J_{X,Y \otimes Z} \circ J_{X,Y} \otimes \text{id} \circ A_{z_1, z_2})(f \boxtimes P(z_1) (g \boxtimes P(z_2) h))(w \otimes (v \otimes u)) = (J_{X,Y \otimes Z} \circ J_{X,Y} \otimes \text{id})(f \boxtimes P(z_1 - z_2) g \boxtimes P(z_2) h)(w \otimes (v \otimes u)) = Y(Y(fu, z_1 - z_2)gv, z_2)hw.
\]

That is the commutativity of the diagram

\[
\begin{array}{ccc}
F(X) \boxtimes P(z_1) (F(Y) \boxtimes P(z_2) F(Z)) & \xrightarrow{\text{id} \boxtimes J_{Y,Z}} & F(X) \boxtimes P(z_1) F(Y \otimes Z) \xrightarrow{J_{X,Y \otimes Z}} F(X \otimes (Y \otimes Z)) \\
\downarrow A_{z_1, z_2} & & \downarrow \\
(F(X) \boxtimes P(z_1 - z_2) F(Y)) \boxtimes P(z_2) F(Z) & \xrightarrow{J_{X,Y \otimes Z}} & F(X \otimes Y) \boxtimes P(z_2) F(Z) \xrightarrow{J_{X,Y \otimes Z}} F((X \otimes Y) \otimes Z)
\end{array}
\]
Finally we prove that $F$ is an equivalence. It is clear that $F$ is injective on morphisms. Since $F(W_\lambda) \cong V_\lambda$ for any irreducible character $\lambda$, $F$ is essentially surjective and $\dim \text{Hom}_G(X, Y) = \dim \text{Hom}_{V^G}(F(X), F(Y))$. Therefore, $F$ is bijective on morphism spaces, and $F$ is an equivalence. □

From Theorem 6.1, $F_{V,G} : \text{Rep}(G) \to \mathcal{C}_{V^G}$ is an embedding for any vertex operator algebras $V$ satisfying the assumptions (V1)-(V3), its image is equivalent to $\mathcal{E}_{V^G}$. We may simply denote $\text{Rep}(G)$ by $\mathcal{E}$ in the sequel, and the pair $(\mathcal{C}_{V^G}, F_{V,G})$ defines a braided $\mathcal{E}$-category.

**Theorem 6.2.** With the assumptions (V1)-(V3), we have

1. $\text{FPdim}(\mathcal{F}_{V^G}) = o(G) \cdot \text{FPdim}(\mathcal{C}_V)$ and $\text{FPdim}(\mathcal{C}_{V^G}) = o(G) \cdot \text{FPdim}(\mathcal{F}_{V^G})$.

2. $\mathcal{F}_{V^G}$ is a braided fusion category.

3. $\mathcal{F}_{V^G} = \mathcal{E}_{V^G}$. That is the Müger center of $\mathcal{F}_{V^G}$ is the symmetric fusion category $\mathcal{E}_{V^G}$, and $(\mathcal{F}_{V^G}, F_{V,G}^G)$ is a nondegenerate braided $\mathcal{E}$-category.

4. $\mathcal{C}_{V^G}$ is a minimal modular extension of $\mathcal{F}_{V^G}$. If $V$ is holomorphic, $\mathcal{C}_{V^G}$ is a minimal modular extension of $\mathcal{E}_{V^G}$ and is braided equivalent to $\mathcal{Z}(\text{Vec}_G^\alpha)$ for some $\alpha \in H^3(G, S^1)$.

5. $\overline{\mathcal{C}_{V^G}}$ is a minimal modular extension of $\mathcal{F}_{V^G}$. If $V$ is holomorphic, $\overline{\mathcal{C}_{V^G}}$ is a minimal modular extension of $\mathcal{E}_{V^G}$ and is braided equivalent to $\mathcal{Z}(\text{Vec}_G^\overline{\alpha})$ where $\overline{\alpha} = \alpha^{-1}$.

**Proof.** (1) Let $J_0$ be the orbit representatives consisting of irreducible $V$-modules. Then by Theorem 5.1

$$\text{FPdim}(\mathcal{F}_{V^G}) = \sum_{M \in J_0} \sum_{\lambda \in \Lambda_M} \text{qdim}_{V^G}(M_\lambda)^2.$$ 

and

$$\text{qdim}_{V^G}(M_\lambda) = [G : G_M] \dim(W_\lambda) \cdot \text{qdim}_V(M),$$

$$o(G_M) = \sum_{\lambda \in \Lambda_M} \dim(W_\lambda)^2.$$ 

Thus,

$$\text{FPdim}(\mathcal{F}_{V^G}) = \sum_{M \in J_0} \sum_{\lambda \in \Lambda_M} [G : G_M]^2 \dim(W_\lambda)^2 \cdot \text{qdim}_V(M)^2$$

$$= \sum_{M \in J_0} [G : G_M]^2 o(G_M) \text{qdim}_V(M)^2$$

$$= o(G) \sum_{M \in J_0} [G : G_M] \text{qdim}_V(M)^2$$

$$= o(G) \sum_{M \in J_0} \sum_{N \in G \cdot M} \text{qdim}_V(N)^2$$

$$= o(G) \cdot \text{FPdim}(\mathcal{C}_V).$$
The identity \( \text{FPdim}(C_{V^G}) = o(G) \cdot \text{FPdim}(\mathcal{F}_{V^G}) \) follows from \( \text{FPdim}(C_{V^G}) = o(G)^2 \cdot \text{FPdim}(C_V) \) \[\text{DRX}\].

(2) Since \( \mathcal{F}_{V^G} \) is a subcategory of the modular tensor category \( C_{V^G} \), it suffices to show that for any \( X, Y \in \mathcal{F}_{V^G} \), \( X \boxtimes_{V^G} Y \) is also in \( \mathcal{F}_{V^G} \).

Recall the fusion category \( \text{Rep}(V) = \bigoplus_{g \in G} \text{Rep}(V)_g \) from Section 4. There is an induction functor

\[
\text{Ind}^V_{V^G} : C_{V^G} \rightarrow \text{Rep}(V)
\]

such that \( \text{Ind}^V_{V^G}(X) = V \boxtimes_{V^G} X \) for any object \( X \) in \( C_{V^G} \). It follows from [KO] that \( \text{Ind}^V_{V^G} \) is a tensor functor, and it has a right adjoint \( \text{Res}^V_{V^G} : \text{Rep}(V) \rightarrow C_{V^G} \), which is the restriction functor. In particular, the following holds:

(i) \( \text{Hom}_V(\text{Ind}^V_{V^G}(X), M) \) and \( \text{Hom}_{V^G}(X, \text{Res}^V_{V^G}(M)) \) are naturally isomorphic for any \( V^G \)-module \( X \) and \( M \in \text{Rep}(V) \), and

(ii) \( \text{Ind}^V_{V^G}(W_1 \boxtimes_{V^G} W_2) \) and \( \text{Ind}^V_{V^G}(W_1) \boxtimes_{\text{Rep}(V)} \text{Ind}^V_{V^G}(W_2) \) are naturally isomorphic for any \( V^G \)-modules \( W_1, W_2 \).

If \( M \) is an irreducible \( g \)-twisted \( V \)-module and \( \lambda \in \Lambda_M \) we claim that

\[
\text{Ind}^V_{V^G}(M_{\lambda}) \cong \bigoplus_{N \in M \circ G} W_{\lambda} \otimes N = \bigoplus_{i=1}^{[G:G_M]} W_{\lambda} \otimes M \circ g_i \tag{6.3}
\]

as \( V \)-modules, where \( \{g_1, \ldots, g_{[G:G_M]}\} \) is a set of representatives of the right cosets of \( G_M \) in \( G \). Noting from Theorem 5.1, for any irreducible twisted \( V \)-module \( N \), we have \( \text{Hom}_{V^G}(M_{\lambda}, N) = 0 \) if \( N \notin M \circ G \), and \( \text{Hom}_{V^G}(M_{\lambda}, N) \cong W_{\lambda} \) if \( N \in M \circ G \). Using (i) immediately concludes the isomorphism in (6.3).

Let \( M, N \) be two irreducible \( V \)-modules and \( \lambda \in \Lambda_M, \mu \in \Lambda_N \) we claim that \( M_{\lambda} \boxtimes_{V^G} N_{\mu} \) lies in \( \mathcal{F}_{V^G} \). First, for any \( \lambda \in \Lambda_M \) and irreducible \( g \)-twisted \( V \)-module \( X \) with \( g \neq 1 \),

\[
\text{Hom}_{\text{Rep}(V)}(\text{Ind}^V_{V^G} M_{\lambda}, X) \cong \text{Hom}_{V^G}(M_{\lambda}, \text{Res}^V_{V^G} X) = 0
\]

by (i) and Theorem 5.1. Therefore, \( \text{Ind}^V_{V^G} M_{\lambda} \in \mathcal{C}_{V^G} \). By (ii), for any \( \mu \in \Lambda_N \), we have

\[
\text{Ind}^V_{V^G}(M_{\lambda} \boxtimes_{V^G} N_{\mu}) \cong \text{Ind}^V_{V^G}(M_{\lambda}) \boxtimes_{\text{Rep}(V)} \text{Ind}^V_{V^G}(N_{\mu}) \in \mathcal{C}_{V^G}.
\]

It follows from (i) that \( \text{Hom}_{V^G}(M_{\lambda} \boxtimes_{V^G} N_{\mu}, X) = 0 \) for any \( g \)-twisted \( V \)-module \( X \) and \( 1 \neq g \in G \). This implies that \( M_{\lambda} \boxtimes_{V^G} N_{\mu} \) lies in \( \mathcal{F}_{V^G} \). Thus \( \mathcal{F}_{V^G} \) is a fusion subcategory of \( \mathcal{C}_{V^G} \).

(3) We first prove that for any \( \lambda \in \text{Irr}(G) \), \( V_{\lambda} \) lies in \( \mathcal{F}_{V^G} \). and hence \( \text{FPdim}(\mathcal{F}_{V^G}) \geq o(G) \). Equivalently, we need to show that

\[
c_{V_{\lambda}, M_{\mu}} \circ c_{M_{\mu}, V_{\lambda}} = \text{id}_{M_{\mu} \boxtimes V_{\lambda}}
\]

for any irreducible \( V \)-module \( M, \lambda \in \text{Irr}(G) \) and \( \mu \in \Lambda_M \). It follows from (6.3) that

\[
\text{Ind}^V_{V^G}(V_{\lambda}) \cong W_{\lambda} \otimes V
\]
as \(V\)-modules. Therefore,

\[
\text{Ind}^V_{V,G}(M_\mu \boxtimes V_\lambda) \cong \text{Ind}^V_{V,G}(M_\mu) \boxtimes_{\text{Rep}(V)} \text{Ind}^V_{V,G}(V_\lambda)
\]

\[
\cong W_\lambda \otimes \text{Ind}^V_{V,G}(M_\mu) \cong W_\lambda \otimes W_\mu \otimes \bigoplus_{i=1}^{[G:G_M]} M \circ g_i
\]

By (i) and Theorem 5.1, we find \(M_\mu \boxtimes V_\lambda\) is isomorphic to a sum of some irreducible \(V^G\)-submodules of \(M\) as \(M \circ g_i\) and \(M\) are isomorphic \(V^G\)-modules. This implies that \(\theta_M = \theta_{M_\mu} = \theta_{M_\mu \boxtimes V_\lambda}\) as complex numbers. Using the fact that \(\theta_{V_\lambda} = 1\) and relation

\[
\theta_{M_\mu \boxtimes V_\lambda} = c_{V_\lambda,M_\mu} \circ c_{M_\mu,V_\lambda} \circ (\theta_{M_\mu} \boxtimes \theta_{V_\lambda}),
\]

we conclude \(c_{V_\lambda,M_\mu} \circ c_{M_\mu,V_\lambda} = \text{id}_{M_\mu \boxtimes V_\lambda}\).

As \(C_{V,G}\) is modular, it follows from (3.1) that

\[
\text{FPdim}(C_{V,G}) = \text{FPdim}(\mathcal{F}_{V,G}) \cdot \text{FPdim}(C_{\mathcal{F}_{V,G}}(\mathcal{F}_{V,G})).
\]

From (1) we know that

\[
\text{FPdim}(C_{V,G}) = \text{FPdim}(\mathcal{F}_{V,G}) \cdot o(G)
\]

which forces \(\text{FPdim}(C_{\mathcal{F}_{V,G}}(\mathcal{F}_{V,G})) = o(G)\). The fact that \(\mathcal{E}_{V,G}\) is a full subcategory of \(C_{\mathcal{F}_{V,G}}(\mathcal{F}_{V,G})\) and they have the same dimension immediately implies that

\[
\mathcal{F}_{V,G} = C_{\mathcal{F}_{V,G}}(\mathcal{F}_{V,G}) = \mathcal{E}_{V,G}.
\]

(4) By (1)-(3) and Lemma 3.1, \(C_{V,G}\) is a minimal modular extension of \(\mathcal{F}_{V,G}\). If \(V\) is holomorphic, then \(\mathcal{F}_{V,G} = \mathcal{E}_{V,G}\) and the statement follows from [LKWi] Thm. 4.22.

(5) By Theorem 6.1, \(\mathcal{E}_{V,G} = \mathcal{E}_{V,G}\) is a subcategory of \(\mathcal{F}_{V,G}\). It follows from (1)-(4) that \(\mathcal{C}_{V,G}\) is a minimal modular extension of \(\mathcal{F}_{V,G}\). If \(V\) is holomorphic, then \(\mathcal{F}_{V,G} = \mathcal{E}_{V,G}\) and the statement follows from [LKWi] Thm. 4.22. \(\square\)

Note that \(\mathbb{C}[G]^*\) is a regular algebra in \(\text{Rep}(G)\), which is the dual \(G\)-module of \(\mathbb{C}[G]\), and is a commutative associative algebra over \(\mathbb{C}\) with a basis \(\{e_a \mid a \in G\}\) of complete orthogonal idempotents given by \(e_a(b) = \delta_{a,b}\). It is easy to see that \(a \cdot e_b = e_{ab}\), and the product \(\mu : \mathbb{C}[G]^* \otimes \mathbb{C}[G]^* \to \mathbb{C}[G]^*\) defined by \(e_a e_b = \delta_{a,b} e_a\) is a \(G\)-module homomorphism. The unit map of this algebra is given by \(i_{\mathbb{C}[G]^*} : \mathbb{1} \to \mathbb{C}[G]^*\), \(1 \mapsto 1_{\mathbb{C}[G]^*} = \sum_{a \in G} e_a\). Now, we can prove the following result:

**Theorem 6.3.** Let \(V\) and \(G\) be as before. Then

1. \(F_{V,G}(\mathbb{C}[G]^*)\) is an algebra isomorphic to \(V\) in category \(\mathcal{E}_{V,G}\).

2. For any subgroup \(H\) of \(G\), \(F_{V,G}(\mathbb{C}[G/H]^*)\) is a subalgebra of \(F_{V,G}(\mathbb{C}[G]^*)\) isomorphic to \(V^H\) in category \(\mathcal{E}_{V,G}\).
Proof. (1) For short we set $F = F^{V,G}$ in this proof. We identify $\mathbb{C}[G]^{**}$ with $\mathbb{C}[G]$ in the usual way as $G$-modules, which means $b(e_a) = e_a(b) = \delta_{a,b}$. Since $\mathbb{C}[G]$ is a free $G$-module generated by 1, for any $G$-module $W$, we have the natural isomorphism of vector spaces $\text{Hom}_G(\mathbb{C}[G], W) \cong W$, which implies $\text{Hom}_G(\mathbb{C}[G], W) = \{f_w \mid w \in W\}$ where $f_w(a) = aw$ for $a \in G$. In particular, we have $F(\mathbb{C}[G]^*) = \{f_v \mid v \in V\}$.

Let $U_n = \text{Hom}(\mathbb{C}[G], V_n) = \{f_v \mid v \in V_n\}$ for $n \geq 0$. We now show that the algebra $U = \bigoplus_{n \geq 0} U_n = F(\mathbb{C}[G]^*)$ with product map $\mu_U = F(\mu) \circ J_{\mathbb{C}[G]^*, \mathbb{C}[G]^*}: U \otimes U \to U$ and unit map $i_U := F(i_{\mathbb{C}[G]^*}) \circ \epsilon : V^G \to U$ is isomorphic to $V$ in $\mathcal{E}_{V,G}$. Note that the adjoint map $\mu' : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ of $\mu$ is determined by $\mu'(a) = a \otimes a$ for any $a \in G$. Thus $F(\mu) : F(\mathbb{C}[G]^* \otimes \mathbb{C}[G]^*) \to F(\mathbb{C}[G]^*)$ is given by $F(\mu)(f) = f \mu'$ for $f \in F(\mathbb{C}[G]^* \otimes \mathbb{C}[G]^*)$. It follows from the braided tensor equivalence $F$ that $i_U := F(i_{\mathbb{C}[G]^*}) \circ \epsilon$ is the unit map of the algebra $U$ in $\mathcal{E}_{V,G}$. For any $u, v \in V$, and $a, G$,

$$\mu_U(f_u \otimes f_v)(a) = (F(\mu) \circ J_{\mathbb{C}[G]^*, \mathbb{C}[G]^*})(f_u \otimes f_v)(a)$$

$$= Y(au, 1)v = aY(u, 1)v = f_{Y(u,1)v}(a)$$

where $f_{Y(u,1)v}$ is understood to be $\sum_{n \in \mathbb{Z}} f_{u_n v}$. Therefore, $\mu_U(f_u \otimes f_v) = f_{Y(u,1)v}$. Recall from [HKL] that $V$ is also algebra in $\mathcal{E}_{V,G}$ with the algebra product map

$$\mu_V(u \boxtimes v) = Y(u, 1)v.$$ 

One can define the $\mathbb{C}$-linear isomorphism $\phi : v \mapsto f_v$ for $v \in V$ from $V$ to $U$. Then $\phi$ is a $V^G$-module map by the definition of $U$ which satisfies $\mu_U \circ (\phi \boxtimes \phi) = \phi \circ \mu_V$ and $i_U = \phi \circ i_V$, where unit map $i_V : V^G \to V$ of $V$ is the inclusion map. In particular, $U$ is a vertex operator algebra isomorphic to $V$. In fact, this can be seen directly that the vertex operator algebra structure on $U$ is given by $Y(f_u, x)f_v = f_{Y(u,x)v} = \sum_{n \in \mathbb{Z}} f_{u_n v} x^{-n-1}$ for $u, v \in V$. Since $F$ is a braided tensor equivalence, $U = F(\mathbb{C}[G]^*)$ is a regular algebra of $\mathcal{E}_{V,G}$ isomorphic to $V$.

(2) For any subgroup $H$ of $G$, $i : \mathbb{C}[G/H]^* \to \mathbb{C}[G]^*$, $e_{aH} \mapsto \sum_{h \in H} e_{ah}$ is an algebra embedding in $\mathcal{E}$, where $e_{aH}(bH) = \delta_{aH,bH}$. Therefore, $F(\mathbb{C}[G/H]^*) \xrightarrow{F(i)} F(\mathbb{C}[G]^*)$ is an algebra embedding in $\mathcal{E}_{V,G}$. We also identify $\mathbb{C}[G/H]^{**}$ with $\mathbb{C}[G/H]$ as $G$-modules. From (1) we see that

$$F(\mathbb{C}[G/H]^*) = \text{Hom}_G(\mathbb{C}[G/H], V) = \{f_v \mid v \in V, f_v(ah) = f_v(a) \forall a \in G, h \in H\}.$$ 

So $f_v \in F(\mathbb{C}[G/H]^*)$ if and only if $ahv = av$ for any $a \in G, h \in H$. This forces $v \in V^H$, and hence $F(\mathbb{C}[G/H]^*) = \{f_v \mid v \in V^H\}$. Recall from (1) that $\phi(v) = f_v$ for $v \in V$ is an algebra isomorphism from $V$ to $F(\mathbb{C}[G]^*)$. It is clear now that the restriction of $\phi$ to $V^H$ gives an algebra isomorphism from $V^H$ to $F(\mathbb{C}[G/H]^*)$, as desired. □

Remark 6.4. Theorem 6.3 gives a categorical interpretation of the Galois correspondence given in [DM2], [HMT] that there is a bijection between the subgroups $H$ of $G$ and vertex operator subalgebras of $V$ containing $V^G$ by sending $H$ to $V^H$. Combining with a result in [HKL] we know that the condensable algebras in $\mathcal{E}_{V,G}$ are exactly $V^H$ for subgroups $H$ of $G$.

On the other hand, the condensable algebras in $\text{Rep}(G)$ are given by $\mathbb{C}[G/H]^*$ for subgroups $H$ of $G$ [KQ]. It is easy from Theorem 6.3 to see that $F(\mathbb{C}[G/H]^*)$ is isomorphic to $V^H$. 

23
7 The group $\mathcal{M}_v(\mathcal{E})$ and $\mathcal{M}_v(\mathcal{E})$-sets

It has already been known from [LKW1, LKW2] that $\mathcal{M}(\mathcal{E})$ is an abelian group. Our goal is to understand this group structure from the point of view of vertex operator algebra.

We need more notations and results on the braided fusion category. Let $\mathcal{C}$ and $\mathcal{D}$ be braided fusion categories. We denote by $\mathcal{C} \otimes \mathcal{D}$ the Deligne tensor product of $\mathcal{C}$ and $\mathcal{D}$. Then $L_C = \oplus_{X \in \mathcal{O}(\mathcal{C})} X \otimes X^*$ is a contestable algebra in $\mathcal{C} \otimes \mathcal{C}$ [DMNO]. We also need a fact from [KO] that the right adjoint of the tensor functor $\mathcal{E} \otimes \mathcal{E} \overset{\otimes}{\rightarrow} \mathcal{E}$ defines a braided tensor equivalence $R : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{E} = (\mathcal{E} \otimes \mathcal{E})_{L_E}$ and $R(1) \cong L_E$ as algebras in $\mathcal{E} \otimes \mathcal{E}$. Now let $(\mathcal{C}, \iota_C)$ and $(\mathcal{D}, \iota_D)$ be braided $\mathcal{E}$-categories with embeddings $\iota_C : \mathcal{E} \rightarrow \mathcal{C}$ and $\iota_D : \mathcal{E} \rightarrow \mathcal{D}$. Then

$$\iota_C \otimes \iota_D : \mathcal{E} \xrightarrow{R} (\mathcal{E} \otimes \mathcal{E})_{L_E} \xrightarrow{\iota_C \otimes \iota_D} (\mathcal{C} \otimes \mathcal{D})^0_A$$

defines an embedding of $\mathcal{E}$ into $(\mathcal{C} \otimes \mathcal{D})^0_A$, where $A = (\iota_C \otimes \iota_D)R(1) \cong (\iota_C \otimes \iota_D)(L_E)$.

Following [DNO, LKW1] one can define the tensor product of braided $\mathcal{E}$-categories as

$$\mathcal{C} \otimes_{\mathcal{E}}^\mathcal{E} \mathcal{D} := ((\mathcal{C} \otimes \mathcal{D})^0_A, \iota_C \otimes \iota_D).$$ (7.1)

Let $G$ be a finite group and $V$ a vertex operator algebras satisfying conditions (V1)-(V3) such that $G$ acts faithfully on $V$ as automorphisms of $V$. We say that two such vertex operator algebras $V_1, V_2$ are equivalent if there exists an isomorphism $\sigma : V_1 \rightarrow V_2$ of vertex operator algebras which commutes with their $G$-actions. In this case, it is easy to see that $V_1^G$ and $V_2^G$ are isomorphic and their module categories are braided equivalent. So the number of inequivalent faithful $G$-action on $V$ as automorphisms is bounded by the cardinality of the conjugacy classes of $G$ in $\text{Aut}(V)$. For example, there are two inequivalent faithful $\mathbb{Z}_2$ actions on the Moonshine vertex operator algebra $V_2$ [FLM].

We denote by $\mathcal{R}_G$ for the collection of vertex operators algebras satisfying conditions (V1)-(V3) with $G$ acting faithfully as automorphisms. The subcollection of $\mathcal{R}_G$ consisting of holomorphic vertex operators algebras is denoted by $\mathcal{H}_G$.

The collection $\mathcal{H}_G$ of holomorphic vertex operators algebras could be generalized to non-holomorphic ones as follows: Fix a nondegenerate pseudounitary braided $\mathcal{E}$-category $\mathcal{F}$. Let $\mathcal{R}_G^\mathcal{F}$ be the collection of vertex operator algebras $V \in \mathcal{R}_G$ such that $\mathcal{F}_{V,G}$ is braided equivalent to $\mathcal{F}$. The underlying braided equivalence $j^{V,G} : \mathcal{F}_{V,G} \rightarrow \mathcal{F}$ induces an $\mathcal{E}$-braided equivalence $j^{V,G} : (\mathcal{F}_{V,G}, F_{V,G}) \rightarrow (\mathcal{F}, j^{V,G} \circ F_{V,G})$. In particular, if $\mathcal{F} = \mathcal{E}$, $\mathcal{R}_G^\mathcal{F} = \mathcal{H}_G$. We will use the notation $[\mathcal{C}_{V,G}]$ to denote the equivalence class of the braided $\mathcal{E}$-category $(\mathcal{C}_{V,G}, F_{V,G})$ for any $V \in \mathcal{R}_G^\mathcal{F}$. By Theorem 6.2, $\mathcal{C}_{V,G}$ is a minimal modular extension of $\mathcal{E}$ and for $V \in \mathcal{H}_G$ and $\mathcal{C}_{V,G}$ is a minimal modular extension of $\mathcal{F}$ for $U \in \mathcal{R}_G^\mathcal{F}$.

Set $\mathcal{M}_v(\mathcal{E}) = \{[\mathcal{C}_{V,G}] \mid V \in \mathcal{H}_G\}$, and $\mathcal{M}_v(\mathcal{F}) = \{[\mathcal{C}_{U,G}] \mid U \in \mathcal{R}_G^\mathcal{F}\}$. By Theorem 6.2 we have the inclusions $\mathcal{M}_v(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E})$ and $\mathcal{M}_v(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{F})$.

Theorem 7.1. Fix a finite group $G$ and let $\mathcal{E} = \text{Rep}(G)$. Then:

1. The product

$$[\mathcal{C}_{V,G}] \cdot [\mathcal{C}_{U,G}] = [\mathcal{C}_{(V \otimes U),G}]$$
for $U,V \in \mathbf{H}_G$ on $\mathcal{M}_v(\mathcal{E})$ coincides with the product of $\mathcal{M}(\mathcal{E})$ given by (7.1), and hence $\mathcal{M}_v(\mathcal{E})$ is a subgroup of $\mathcal{M}(\mathcal{E})$. Moreover, $\mathcal{M}_v(\mathcal{E})$ is isomorphic to a subgroup of $H^3(G, S^1)$, and $\mathcal{C}_{V,G}$ is braided equivalent to $\mathcal{Z}(\text{Vec}_G^0)$ for some $\alpha \in H^3(G, S^1)$ if $V \in \mathbf{H}_G$.

(2) For any (pseudounitary) braided $\mathcal{E}$-category $\mathcal{F}$, if $\mathbf{R}_G^F$ is not an empty collection, then the product $[\mathcal{C}_{V,G}] \cdot [\mathcal{C}_{W,G}] := [\mathcal{C}_{(V \otimes W),G}]$ for $V \in \mathbf{H}_G$ and $W \in \mathbf{R}_G^F$ defines a free action of $\mathcal{M}_v(\mathcal{E})$ on $\mathcal{M}_v(\mathcal{F})$ with at most $[\mathcal{M}(\mathcal{E}) : \mathcal{M}_v(\mathcal{E})]$ many orbits.

(3) The group $\mathcal{M}_v(\mathcal{E})$ acts freely on $\{[\mathcal{C}_{W,G}] | W \in \mathbf{R}_G\}$ such that $[\mathcal{C}_{V,G}] \cdot [\mathcal{C}_{W,G}] := [\mathcal{C}_{(V \otimes W),G}]$.

Proof. (1) Let $W$ be any holomorphic vertex operator algebra. Then there exists a positive integer $n$ such that $G$ can be realized as a subgroup of the symmetric group $S_n$. Thus $G$ is an automorphism group of holomorphic vertex operator algebra $V = W^\otimes n$. That means $\mathcal{C}_{V,G}$ lies in $\mathcal{M}_v(\mathcal{E})$ and so $\mathcal{M}_v(\mathcal{E})$ is not empty.

All the statements in the theorem is a consequence of the equality:

$$[\mathcal{C}_{V,G}] \cdot [\mathcal{C}_{U,G}] = [\mathcal{C}_{(V \otimes U),G}]$$

for any $V \in \mathbf{H}_G$ and $U \in \mathbf{R}_G$. It amounts to prove that

$$(\mathcal{C}_{(V \otimes U),G}, F^{V \otimes U,G}) \cong \mathcal{C}_{V,G} \otimes \mathcal{C}_{U,G} \otimes \mathcal{C}_{(V \otimes U),G},$$

(7.2)

as braided $\mathcal{E}$-categories, which means we need to find a braided tensor equivalence $\tilde{\phi} : (\mathcal{C}_{V,G} \otimes \mathcal{C}_{U,G})^0_A \to \mathcal{C}_{(V \otimes U),G}$ such that $\tilde{\phi} \circ (F^{V,G} \otimes F^{U,G}) \cong F^{V \otimes U,G}$ as tensor functors, where $A = (F^{V,G} \otimes F^{U,G})(L_\mathcal{E})$.

To find such a braided tensor equivalence $\tilde{\phi}$, we first show that $(V \otimes U)^G$ and $A$ are isomorphic algebras in $\mathcal{C}_{V,G} \otimes \mathcal{C}_{U,G}$ under an isomorphism $\phi$. This algebra isomorphism induces a (strict) tensor equivalence $\phi : (\mathcal{C}_{V,G} \otimes \mathcal{C}_{U,G})_A \to (\mathcal{C}_{V,G} \otimes \mathcal{C}_{U,G})_{(V \otimes U),G}$ and hence a braided tensor equivalence $\tilde{\phi} : (\mathcal{C}_{V,G} \otimes \mathcal{C}_{U,G})_A \to (\mathcal{C}_{V,G} \otimes \mathcal{C}_{U,G})_{(V \otimes U),G} = \mathcal{C}_{(V \otimes U),G}$.

By Theorem 6.3, there exists an algebra isomorphism

$$V \otimes U \xrightarrow{\tilde{\phi}} F^{V \otimes U,G} \otimes (\mathbb{C}|G \times G|^*)$$

in $\mathcal{E}_{(V \otimes U),G \times G}$. Consider $G$ as the diagonal subgroup of $G \times G$. Then $\mathbb{C}|(G \times G)/G|^*$ is a subalgebra of $\mathbb{C}|G \times G|^*$ in $\text{Rep}(G \times G)$. It follows from Theorem 6.3 (2) that the restriction of $\phi$ defines an isomorphism

$$(V \otimes U)^G \xrightarrow{\tilde{\phi}} F^{V \otimes U,G} \otimes (\mathbb{C}|(G \times G)/G|^*)$$

in $\mathcal{E}_{(V \otimes U),G \times G}$. Note that if one identifies $\text{Rep}(G \times G)$ with $\mathcal{E} \otimes \mathcal{E}$, then $\mathbb{C}|(G \times G)/G|^* = L_\mathcal{E}$ and $F^{V \otimes U,G} \otimes G = F^{V,G} \otimes F^{U,G}$. Therefore, $\phi$ defines an isomorphism of algebras in $\mathcal{C}_{V,G} \otimes \mathcal{C}_{U,G}$ from $(V \otimes U)^G$ to $A$. Since $A$ is an algebra in $\mathcal{E}_{V,G} \otimes \mathcal{E}_{U,G}$, so is $(V \otimes U)^G$. 25
Now, the algebra isomorphism $\phi$ in $C_{V,G} \otimes C_{U,G}$ induces a braided tensor equivalence $\tilde{\phi} : (C_{V,G} \otimes C_{U,G})_0^\sim \to C_{(V \otimes U)G}$, and $\phi : (V \otimes U)^G$ is an isomorphism of $(U \otimes V)^G$-modules, and $(\tilde{\phi} \circ (F^{V,G} \otimes E^{U,G}), J', \phi)$ is a braided tensor functor, where $J' = J_{V \otimes U,G \otimes G}$. To complete the proof of [7,2], we need to show that $(E_{V,G} \otimes E_{U,G})_{(V \otimes U)G} = E_{(V \otimes U)G}$ and the equivalence of the two braided tensor functors:

$$
(\tilde{\phi} \circ (F^{V,G} \otimes E^{U,G}), J', \phi) \cong (F^{V \otimes U,G}, J_{V \otimes U,G}, \epsilon).
$$

where $J_{V \otimes U,G}$ and $\epsilon$ are the monoidal structure of the functor $F^{V \otimes U,G}$ defined in Theorem 6.1.

Note that the simple $(V \otimes U)^G$-submodules $(V \otimes U)_\lambda \in (E_{V,G} \otimes E_{U,G})_{(V \otimes U)G}$ for $\lambda \in \text{Irr}(G)$. Since $\text{FPdim}((E_{V,G} \otimes E_{U,G})_{(V \otimes U)G}) = |G|$, we find $(E_{V,G} \otimes E_{U,G})_{(V \otimes U)G} = E_{(V \otimes U)G}$.

Using the identification $E \otimes E = \text{Rep}(G \times G)$, we can write each object $X \in (V \otimes U)G$ as a $G \times G$-module with a decomposition $X = \oplus_{x \in G \times G} X_x$ such that $(g, h)X_x = X_{(g,h)x}$ for $(g, h) \in G \times G$. In particular, $X_1$ is a $G$-module, where 1 denotes the diagonal subgroup of $G \times G$. Moreover, the induction functor $\text{Ind}_{G \times G}^G : E \rightarrow (V \otimes U)G$. The corresponding right adjoint of the braided equivalence $E \otimes E \cong E$. In this convention, $\tilde{\phi} \circ (F^{V,G} \otimes F^{U,G}) \circ \text{Ind}_{G \times G}^G \cong F^{V \otimes U,G}$, where $\psi$ is given by the natural isomorphism,

$$
(\tilde{\phi} \circ (F^{V,G} \otimes F^{U,G}) \circ \text{Ind}_{G \times G}^G)(X) = \text{Hom}_{G \times G}((\text{Ind}_{G \times G}^G(X))^*, V \otimes U)
$$

$$
\cong \text{Hom}_{G \times G}(\text{Ind}_{G \times G}^G(X^*), V \otimes U)
$$

$$
\cong \text{Hom}_G(X^*, V \otimes U)
$$

$$
= F^{V \otimes U,G}(X)
$$

for $X \in E$. With the identification of $X^*$ and $(\text{Ind}_{G \times G}^G(X))^*$, the inverse $\psi^{-1} : F^{V \otimes U,G}(X) \rightarrow \text{Hom}_{G \times G}((\text{Ind}_{G \times G}^G(X))^*, V \otimes U)$ is given by

$$
\psi^{-1}(f)(yu) = yf(u)
$$

for any $u \in X^*$, $y \in G \times G$ and $f \in F^{V \otimes U,G}(X)$. In particular, $\psi^{-1}(f)(yu) = f(yu)$ if $y \in G$. Now we need to show $\psi$ is a isomorphism of the tensor functors which requires to prove the commutativity of following diagrams for $X,Y \in E$:

$$
\begin{array}{ccc}
F^{V \otimes U,G}(X) \otimes F^{V \otimes U,G}(Y) & \xrightarrow{J'_{X,Y}} & F^{V \otimes U,G}(X \otimes Y) \\
\downarrow{\psi^{-1} \otimes \psi^{-1}} & & \downarrow{\psi^{-1}} \\
F'(X) \otimes F'(Y) & \xrightarrow{J'_{X,Y}} & F'(X \otimes Y)
\end{array}
$$

and

$$
\begin{array}{ccc}
(F \otimes U)^G & \xrightarrow{\epsilon} & F^{V \otimes U,G}(1) \\
\downarrow{\phi} & & \downarrow{\psi^{-1}} \\
F'(1) & & F'(1)
\end{array}
$$

where $F' = \tilde{\phi} \circ (F^{V,G} \otimes F^{U,G}) \circ \text{Ind}_{G \times G}^G$. Let $f \in F^{V \otimes U,G}(X), g \in F^{V \otimes U,G}(Y)$ and $u \in X^*, v \in Y^*$. We know from the proof of Theorem 6.1 that $\psi^{-1} \circ J'_{X,Y}$ is characterized by

$$
(\psi^{-1} \circ J'_{X,Y})(f \otimes g)(v \otimes u) = Y(fu, 1)gv.
$$
Since \( J' = J^{V \otimes U, G} = J^{V, G} \otimes J^{U, G} \) we immediately see that \( J'_{X,Y} \) is characterized by

\[
\overline{J'}_{X,Y}(\psi^{-1}(f) \boxtimes \psi^{-1}(g))(v \otimes u) = Y(\psi^{-1}(f)u, 1)\psi^{-1}(g)v = Y(fu, 1)gv,
\]

which proves the commutativity of the first diagram. Note that the \((V \otimes U)^G\)-module isomorphism \( \phi : (V \otimes U)^G \to F(1) \) is unique up to a scalar. Since \( \psi^{-1}(\epsilon(x \otimes y))(1) = x \otimes y = \phi(x \otimes y)(1) \) for any \( x \in V^G \) and \( y \in U^G \), the second commutativity follows.

Therefore, \( M_v(\mathcal{E}) \) is closed under the product of \( M(\mathcal{E}) \), and hence \( M_v(\mathcal{E}) \) is a subgroup of \( M(\mathcal{E}) \). Following the preceding discussion, there exists a unique \( \alpha \in H^3(G, S^1) \) such that \((C_{V\alpha}, F^{V,G})\) is equivalent to \((\mathcal{Z}(\text{Vec}_G^\alpha), \iota_\alpha)\). In particular, \( C_{V\alpha} \) is equivalent to \( \mathcal{Z}(\text{Vec}_G^\alpha) \) as modular tensor categories.

(2) From the proof of (1), for \( V \in H_G \) and \( W \in R^E_G \) for some (pseudounitary) nondegenerate braided \( \mathcal{E} \)-category \( \mathcal{F} \), \([C_{W\alpha}] \in M_v(\mathcal{F})\) and the pair \((C_{(V \otimes W)^G}, F^{V \otimes W,G}) \cong C_{V\alpha} \otimes_{\mathcal{E}} (F^{V,G}, F^{W,G}) C_{W\alpha}\). According \cite{LKW1}, \( C_{V \alpha} \otimes_{\mathcal{E}} (F^{V,G}, F^{W,G}) \) is a minimal modular extension of \( \mathcal{F} \), so is \((C_{(V \otimes W)^G}, F^{V \otimes W,G}) \). Therefore, \( V \otimes W \in R^F_G \) and \( M_v(\mathcal{F}) \) admits a \( M_v(\mathcal{E}) \)-action defined by \([C_{W\alpha}]|_{C_{V\alpha}} := [C_{(V \otimes W)^G}]\) which coincides with \( M(\mathcal{E}) \)-action on \( M(\mathcal{F}) \). By \cite{LKW1}, \( M(\mathcal{F}) \) is an \( M(\mathcal{E}) \)-torsor. So, the action of \( M_v(\mathcal{F}) \) on \( M_v(\mathcal{F}) \) is free. Since the cardinality \(|M(\mathcal{F})|\) is equal to \( o(M(\mathcal{E})) \), it is immediate that the number of \( M_v(\mathcal{E}) \)-orbits on \( M_v(\mathcal{F}) \) is less than or equal to the index \([M(\mathcal{E}) : M_v(\mathcal{E})]\).

(3) follows directly from (2). 

\[\square\]

**Remark 7.2.** We now explain how to associate 3-cocycle \( \alpha \in Z^3(G, S^1) \) to a holomorphic vertex operator algebra \( V \) satisfying conditions (VI)-(V3) such that \( C_{V\alpha} \) and \( \mathcal{Z}(\text{Vec}_G^\alpha) \) are braided equivalent. By Theorem 6.3, \( V \) is a condensable algebra in \( C_{V\alpha} \). Since \( V \) is holomorphic, for every \( g \in G \) there is a unique irreducible \( g \)-twisted \( V \)-module \( V(g) \) up to equivalence \cite{DLM3}. According \cite{DLXY}, every simple object in \((C_{V\alpha})_V\) is isomorphic to \( V(g) \) for some \( g \in G \). From the discussion in Section 4, \((C_{V\alpha})_V\) is \( G \)-graded fusion category such that \([(C_{V\alpha})_V]_g \) is generated by \( V(g) \). The associativity isomorphism

\[ (V(g) \boxtimes_{\text{Rep}(V)} V(h)) \boxtimes_{\text{Rep}(V)} V(k) \to V(g) \boxtimes_{\text{Rep}(V)} (V(h) \boxtimes_{\text{Rep}(V)} V(k)) \]

determines an \( \alpha \in H^3(G, S^1) \).

**Remark 7.3.** It is definitely desirable that \( M(\mathcal{E}) = M_v(\mathcal{E}) \), but we could not supply a proof to this claim. If this is true and \( M_v(\mathcal{F}) \neq \emptyset \) then \( M_v(\mathcal{F}) = M(\mathcal{F}) \) is an \( M_v(\mathcal{E}) \)-torsor.

**Remark 7.4.** It is worthy to mention that for any \( V \in H_G \), the inverse of \([C_{V\alpha}] \) in \( M_v(\mathcal{E}) \) is \([C_{V\alpha}]^{-1}\). By Theorem 7.1, \( C_{V\alpha} \) is braided equivalent to \( C_{(V \otimes m^{-1})^G} \) where \( m \) is the order of \([C_{V\alpha}] \). So Conjecture 7.1 holds for rational vertex operator algebra \( V^G \) for \( V \in H_G \).

It is proved in \cite{EG} that if \( G \) is solvable, then for any \( \alpha \in H^3(G, S^1) \) there is a regular vertex operator algebra \( V \) such that \( C_V \) is braided equivalent to \( \mathcal{Z}(\text{Vec}_G^\alpha) \). But it is not clear to us that this \( V \) can be realized as \( U^G \) for some holomorphic vertex operator algebra \( U \) with a faithful \( G \)-action. In the next section we give a proof when \( G \) is a dihedral group or abelian group with at most two generators.
8 Lattice vertex operator algebras and pointed minimal extensions

We explain in this section how to use lattice vertex operator algebras to realize the pointed modular categories. In particular, if \( \mathcal{Z}(\text{Vec}_L^G) \) is pointed for some \( \alpha \in H^3(G, S^1) \), we prove that there exists a positive definite even unimodular lattice \( L \) such that \( G \) can be realized as an automorphism group of the lattice vertex operator algebra \( V_L \), \( V_L^G \) is also a lattice vertex operator algebra and \( (\mathcal{Z}(\text{Vec}_L^G), \iota_\alpha) \cong (\mathcal{C}_{V_L^G}, F_{V_L^G}) \) as minimal modular extension of \( \text{Rep}(G) \).

For this purpose we need to recall the construction of the vertex operator algebra \( V_L \) associated to a positive definite even lattice \( L \) with a bilinear form \( (\cdot, \cdot) \) and its irreducible modules following [DL1, FL]. As usual we denote by \( L^0 \) the dual lattice of \( L \). Then there exists a positive even integer \( m \) and an alternating \( \mathbb{Z} \)-bilinear function

\[
c : L^0 \times L^0 \to \langle \zeta_m \rangle
\]

such that \( c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle} \) for \( \alpha, \beta \in L \) where \( \zeta_m = e^{2\pi i/m} \) (see [DL1, Remark 12.18]). In fact, \( c(\alpha, \beta) = \epsilon(\alpha, \beta)\epsilon(\beta, \alpha)^{-1} \) for some \( \mathbb{Z} \)-bilinear function \( \epsilon : L^0 \times L^0 \to \langle \zeta_m \rangle \) for \( \alpha, \beta \in L^0 \). Consider the corresponding central extension \( \widehat{L^0} \) of \( L^0 \) by the cyclic group \( \langle \zeta_m \rangle \):

\[
1 \to \langle \zeta_m \rangle \to \widehat{L^0} \to L^0 \to 0
\]

with commutator map \( c(\cdot, \cdot) \). Let \( e : L^0 \to \widehat{L^0} \), \( \lambda \mapsto e_\lambda \) be a section such that \( e_0 = 1 \) and \( e_\alpha e_\beta = \epsilon(\alpha, \beta)e_{\alpha + \beta} \) for any \( \alpha, \beta \in L \). Then the twisted group algebra \( \mathcal{C}[L^0] = \sum_{\alpha \in L^0} \mathbb{C} e^\alpha \) with product \( e^\alpha \cdot e^\beta = \epsilon(\alpha, \beta)e^{\alpha + \beta} \) for \( \alpha, \beta \in L^0 \) is a quotient of the group algebra \( \mathcal{C}[\widehat{L^0}] \) by identifying \( \zeta_m \in \widehat{L^0} \) with \( \zeta_m \in \mathbb{C} \).

It is easy to see that \( \mathcal{C}[L^0] = \oplus_{i \in L^0/L} \mathbb{C}[L + \lambda_i] \) where \( \mathcal{C}[L + \lambda_i] = \oplus_{\alpha \in L} \mathbb{C} e^{\lambda_i + \alpha} \) and \( L^0/L = \{ L + \lambda_i \mid i \in L^0/L \} \).

Let \( \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L \) be an abelian Lie algebra and extend the form \( (\cdot, \cdot) \) to \( \mathfrak{h} \) by \( \mathbb{C} \)-linearity. Let \( \{ h_1, ..., h_d \} \) be an orthonormal basis of \( \mathfrak{h} \) where \( d \) is the rank of \( L \). Then the affine Lie algebra

\[
\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} k
\]

has a unique irreducible module

\[
M(1) = [h_i(-n)|i = 1, ..., d, n > 0]
\]

such that \( h_i(n) \) acts as \( n \frac{\partial}{\partial h_i(-n)} \) if \( n > 0 \), as multiplication operator \( h_i(n) \) if \( n < 0 \) and \( 0 \) if \( n = 0 \), and \( k \) acts as \( 1 \) where \( h(n) = h \otimes t^n \) for \( h \in \mathfrak{h} \) and \( n \in \mathbb{Z} \). Set

\[
V_{L^0} = M(1) \otimes \mathcal{C}[L^0] = \bigoplus_{i \in L^0/L} V_{L + \lambda_i}
\]

where \( V_{L + \lambda_i} = M(1) \otimes \mathcal{C}[L + \lambda_i] \). Then \( V_L \) is a vertex operator algebra and \( \{ V_{L + \lambda_i} \mid i \in L^0/L \} \) is a complete list of inequivalent irreducible \( V_L \)-modules. Moreover, the ribbon
structure of $\mathcal{C}_{V_L}$ is given by $\theta_{\nu_{L+\lambda}} = e^{\pi i (\lambda, \lambda)}$. For any $\alpha \in \mathbb{Q} \otimes \mathbb{Z} L$, one can define an automorphism $\sigma_\alpha$ of finite order of $V_L$ by setting

$$\sigma_\alpha(u \otimes e^\beta) = e^{2\pi i (\alpha, \beta)}u \otimes e^\beta$$

for $u \in M(1)$ and $\beta \in L$. This type of automorphism will be useful in the following discussions.

We first show that $\mathcal{M}_v(\mathcal{E}) \cong H^3(G, S^1)$ when $G$ is a cyclic group or a dihedral group by using concrete lattice vertex operator algebras associated to the Niemeier lattice $L$ of type $A_2^{24}$.

**Proposition 8.1.** Let $V, G, F, \mathcal{E}$ be as before. If $G \cong \mathbb{Z}_n$ is a cyclic group or $G \cong D_{2m}$ is a dihedral group of order $2m$ with $m$ being odd, then $\mathcal{M}_v(\mathcal{E}) \cong M(\mathcal{E}) \cong H^3(G, S^1)$ and $\mathcal{M}_v(F)$ is a $\mathcal{M}_v(E)$-torsor if $\mathcal{M}_v(F)$ is not empty.

**Proof.** (1) $G \cong \mathbb{Z}_n$. In this case $H^3(G, S^1) \cong \mathbb{Z}_n$. Therefore, it suffices to show that $\mathcal{M}_v(\mathcal{E})$ has an element of order $n$. Consider the holomorphic lattice vertex operator algebra $V_L$ associated to the Niemeier lattice $L$ of type $A_2^{24}$. [FLM]. Then

$$L = \sum_{C \in G_{24}} \frac{1}{2} C + Q$$

where $G_{24}$ is the Golay code based on the set $\Omega = \{1, \ldots, 24\}$, $Q = \sum_{i=1}^{24} \mathbb{Z} \alpha_i$ is a positive definite lattice with $(\alpha_i, \alpha_j) = 2\delta_{i,j}$ and $\alpha_C = \sum_{i \in C} \alpha_i$. Let $G$ be the cyclic group generated by $\sigma = \sigma_{\alpha_1/n}$. Then $V_L^G = V_E$ where $E$ is the sublattice of $L$ given by $E = \{\alpha \in L \mid (\alpha_1, \alpha) \in n\mathbb{Z}\}$. Moreover, there is a unique irreducible $\sigma$-twisted module $V_L(\sigma) = M(1) \otimes \mathbb{C}[E] e^{-\alpha_1/n} [DM1]$. Note that $V_{E-\alpha_1/n} = M(1) \otimes \mathbb{C}[E] e^{-\alpha_1/n}$ is an irreducible $V_E$-module with $\theta_{V_{E-\alpha_1/n}} = e^{2\pi i/n^2}$. In particular, the order of $\tilde{t} = \text{diag}(\theta_M \mid M \in \mathcal{O}(C_{V_L}))$, denoted by $\text{FSexp}(C_{V_L})$, is a multiple of $n^2$.

By Theorem [28] $(C_{V_L}, F^{V_L,G}) \cong (\mathbb{Z}(\text{Vec}_{V_L}^G), t_\omega)$ as braided $\mathcal{E}$-categories for some $\omega \in H^3(G, S^1)$. By [NS2], $\text{FSexp}(\mathbb{Z}(\text{Vec}_{V_L}^G))$ is equal to least common multiple of $\text{ord}(\omega|_H) \cdot o(H)$, where $\omega|_H$ is the restriction of $\omega$ on $H$, and $H$ runs through the maximal cyclic subgroups of $G$. It follows from the preceding paragraph that $\text{ord}(\omega) = n$. Therefore, $[C_{V_L}] \in \mathcal{M}_v(\mathcal{E})$ is of order $n$, as desired.

(2) $G \cong D_{2m}$ for some odd integer $m$. Then $H^3(G, S^1) \cong \mathbb{Z}_{2m}$. Recall from [FLM] Theorems 10.1.2, 10.1.5 that the Golay code $G_{24}$ is built from the Hamming codes $C_1$ and $C_2$. In fact,

$$G_{24} = \langle (S, S, \emptyset), (S, \emptyset, S), (T, T, T) \mid S \in C_1, T \in C_2 \rangle.$$

Let $\Omega = \{1, \ldots, 24\}$ and $w = \{1, 2, 11, 12\}$. Then $|w \cap C|$ is always even for any $C \in G_{24}$. Define an isometry $\nu$ of $L$ such that

$$\nu\left(\sum_{i=1}^{24} k_i \alpha_i\right) = -\sum_{i \in w} k_i \alpha_i + \sum_{i \not\in w} k_i \alpha_i$$

for $u \in M(1)$ and $\beta \in L$. This type of automorphism will be useful in the following discussions.
where \( k_i \in \frac{1}{2} \mathbb{Z} \). Then \( (\nu(\alpha), \alpha) \in 2\mathbb{Z} \) for all \( \alpha \in \Lambda \). So \( \nu \) satisfies the assumption in [Le] and can be lifted to an automorphism \( \tau \) of \( V_L \) of order 2. Let \( V_L(\tau) \) be the unique irreducible \( \tau \)-twisted \( V_L \)-module. Then \( V_L(\tau) \) has a gradation \( V_L(\tau) = \oplus_{n \geq 0} V_L(\tau)_{\frac{1}{2} + \frac{1}{2} n} \) as the eigenspace of \( \nu \) on \( h \) with eigenvalue \(-1\) has dimension 4 [DL2]. Recall that \( \sigma_m = \sigma_{\alpha_1/m} \) is an automorphism of \( V_L \) of order \( m \). It is easy to see that the group \( G \) generated by \( \sigma_m \) and \( \tau \) is isomorphic to \( D_{2m} \). Note that for any \( h = \sigma_m^s \) with \( s \neq 0 \mod m \),

\[
V_L(h) = \oplus_{n \geq 0} V_L(h)_{\frac{1}{2} + \frac{1}{2} n},
\]

\[
V_L(h\tau) = \oplus_{n \geq 0} V_L(h\tau)_{\frac{1}{2} n}
\]

by [EMS] Theorem 5.11. In particular, \( \theta_{V_E - \alpha_1/m} = e^{2\pi i/m^2} \) and \( \theta_{V_L(\sigma_m \tau)} = i \) in \( C_{V_L^G} \) where \( E \) is defined as in (1) and \( V_L(\sigma_m \tau) = \oplus_{n \geq 0} V_L(\sigma_m \tau)_{\frac{1}{2} + \frac{1}{2} n} \). Therefore, \( \text{FSexp}(C_{V_L^G}) \) is a multiple of \( 4m^2 \).

By Theorem 7.1 again, \( (C_{V_L^G}, F_{V_L^G}) \cong (\mathbb{Z}(\text{Vec}_{V_L^G}), \iota_{\omega}) \) as braided \( \mathcal{E} \)-categories for some \( \omega \in H^3(G, S^1) \). Therefore, \( \text{FSexp}(\mathbb{Z}(\text{Vec}_{V_L^G})) = \text{FSexp}(C_{V_L^G}) \) is a multiple of \( 4m^2 \).

By [NS2], \( \text{FSexp}(\mathbb{Z}(\text{Vec}_{V_L^G})) \) is the least common multiple of \( \text{ord}(\omega|_H) \cdot o(H) \) where \( H \) runs through all the maximal cyclic subgroups of \( G \). Therefore, \( \text{ord}(\omega|_{(\sigma_m \tau)^i}) = m \) and \( \text{ord}(\omega|_{(\sigma_m \tau)^i}) = 2 \) for some \( i \). Since \( m \) is odd, \( \text{ord}(\omega) = 2m \) and hence \( [C_{V_L^G}] \in \mathcal{M}(\mathcal{E}) \) is of order \( 2m \). The proof is complete. \( \square \)

A fusion category \( \mathcal{C} \) is called pointed if \( \text{FPdim}(V) = 1 \) for any simple object \( V \) (cf. [ECNO] for more details). For any pointed fusion category, there exists a canonical spherical structure on \( \mathcal{C} \) such that \( \text{dim}(V) > 0 \) for each nonzero object \( V \), and this implies \( \text{dim}(V) = 1 \) if \( V \) is simple. This condition on the positivity of categorical dimensions has been assumed throughout this paper. The set \( A = \text{Irr}(\mathcal{C}) \) forms a group under the tensor product of simple objects, and \( \mathcal{C} \) is equivalent to \( \text{Vec}_{A}^{\omega} \) for some \( \omega \in Z^3(A, \mathbb{C}^\times) \) as fusion categories. If, in addition, \( \mathcal{C} \) is braided, then \( A \) is abelian and there exists a normalized 2-cochain \( c : A \times A \to \mathbb{C}^\times \) such that the scalar \( c(g, h) : e(g) \otimes e(h) \to e(h) \otimes e(g) \) defines a braiding on \( \text{Vec}_{A}^{\omega} \). Let \( \text{Vec}_{A}^{(\omega, c)} \) denote this braided fusion category and hence a ribbon category with the underlying spherical structure. The pair \( (\omega, c) \) also defines an Eilenberg-MacLane abelian 3-cocycle, and the cohomology class \([[(\omega, c)]\) in the corresponding cohomology group \( H^3_A(A, \mathbb{C}^\times) \) uniquely determines the braided equivalence class of \( \text{Vec}_{A}^{(\omega, c)} \) [JS].

Let \( (A, q) \) denote the quadratic form \( q : A \to \mathbb{C}^\times \) on \( A \). Then the set \( \text{Quad}(A) \) of quadratic forms on \( A \) forms a group under the pointwise multiplication. The cohomology group \( H^3_{ab}(A, \mathbb{C}^\times) \) is isomorphic to \( \text{Quad}(A) \) via the trace map \([[(\omega, c)]\) \mapsto q_c \) [PAL, EAP2], where \( q_c(a) = c(a, a) \) for \( a \in A \). The ribbon category \( \text{Vec}_{A}^{(\omega, c)} \) is modular if and only if the quadratic form \( (A, q_c) \) is nondegenerate. For any quadratic form \( (A, q) \), we denote by \( \mathcal{C}(A, q) \) a ribbon category \( \text{Vec}_{A}^{(\omega, c)} \) with \( q_c = q \) and so \( \theta_a := \theta_{c(a)} = q(a) \). In particular, \( \mathcal{C}(A, q_0) \), where \( q_0(a) = 1 \) for all \( a \in A \), is equivalent to the Tannakian category \( \text{Rep}(\hat{A}) \).

By [NS1], tensor equivalences of pseudounitary fusion categories preserve the canonical spherical structures. So, for any quadratic forms \( (A, q) \) and \( (A', q') \), the ribbon categories
$C(A, q)$ and $C(A', q')$ are equivalent if and only if $(A, q)$ and $(A', q')$ are equivalent quadratic forms, i.e., there exists an isomorphism $f : A \to A'$ of groups such that $q' \circ f = q$.

We call a rational vertex operator algebra $V$ pointed if every simple module of $V$ is simple current or $C_V$ is pointed. In general, every lattice vertex operator algebra $V_L$ of a positive definite even lattice $L$ is a pointed modular category given by the quadratic form $(L^o/L, q_L)$ where $q_L(L + \lambda) = \theta_{V_L + \lambda} = e^{\pi i(h, \lambda)}$. The converse is stated in the following proposition.

**Proposition 8.2.** Let $C$ be a pointed modular category (with positive dimensions). Then $C \cong C_{V_L}$ as modular categories for some positive definite even lattice $L$. In particular, for any pointed vertex operator algebra $V$, $C_V \cong C_{V_L}$ as modular categories for some positive definite even lattice $L$.

**Proof.** If $C$ is a pointed modular category with positive dimensions and ribbon structure $\theta$, then $C \cong C(A, q)$ where $A$ is the abelian group $\text{Irr}(C)$ under the tensor product, and a nondegenerate quadratic form $q : A \to \mathbb{C}^\times$ is given by $q(a) = \theta_a$. The first multiplicative central charge $c$ of $C$ is an 8-th root of unity (cf. [DLN, Prop.6.7 (ii)]). Therefore, $c = e^{2\pi i/8}$ for some unique element $n \in \mathbb{Z}_8$, called the signature of $(A, q)$. Note that $q(A) \subset S^1$, where $S^1$ is the group of all roots of unity in $\mathbb{C}$. Let $\ell$ be the minimal number of generators of $A$. It follows from [Ni, Corollary 1.10.2] that there exists a positive definite even lattice $L$ with $\ell < \text{rank}(L) \equiv n \mod 8$, and a group isomorphism $j : A \to L^o/L$ such that $q(a) = e^{\pi i(j(a), j(a))} = \theta_{V_L + j(a)} = q_L(j(a))$. Therefore, $(A, q)$ and $(L^o/L, q_L)$ are equivalent quadratic forms, and

$C \cong C(A, q) \cong C(L^o/L, q_L) \cong C_{V_L}$

as modular categories. □

**Remark 8.3.** Proposition 4.3 now follows from Proposition 8.2 as $C_{V_L}$ is pointed.

We now turn our attention to the case when $G$ is a finite abelian group and will prove that Proposition 8.1 holds if $G$ is generated by two elements. Set

$H_G^{\text{pt}} = \{ V \in H_G \mid V^G \text{ is pointed} \},$

$\mathcal{M}^{\text{pt}}(\mathcal{E}) = \{ [\mathcal{C}] \in \mathcal{M}(\mathcal{E}) \mid \mathcal{C} \text{ is pointed} \},$

$\mathcal{M}_v^{\text{pt}}(\mathcal{E}) = \{ [\mathcal{C}_V] \in \mathcal{M}_v(\mathcal{E}) \mid V \in H_G^{\text{pt}} \}.$

We first observe that $H_G^{\text{pt}}$ is closed under the tensor product of vertex operator algebras:

**Lemma 8.4.** Let $G$ be a finite abelian group and $\mathcal{E} = \text{Rep}(G)$. For any $U, V \in H_G^{\text{pt}}$, $U \otimes V \in H_G^{\text{pt}}$. Hence, $\mathcal{M}_v^{\text{pt}}(\mathcal{E})$ is a subgroup of $\mathcal{M}_v(\mathcal{E})$. 

31
Proof. Since $V$ is holomorphic, there is a unique irreducible $g$-twisted $V$-module $V(g)$ up to isomorphism. Recall from Section 5 that $V(g) \circ h \cong V(g)$ for any $h \in G$ as $G$ is abelian. That is, $G_{V(g)} = G$ and $V(g)$ is a $\mathcal{C}^{\alpha V(g)}[G]$-module with decomposition

$$V(g) = \oplus_{\lambda \in \Lambda_{V(g)}} W_\lambda \otimes V(g)_\lambda.$$ From Theorem 5.1 we know that \{\(V(g)_\lambda | g \in G, \lambda \in \Lambda\} gives a complete list of irreducible $V^G$-modules and \(q\dim_{V^G}(V(g)_\lambda) = \dim(W_\lambda) \cdot [G : G_{V(g)}] \cdot \qdim(V(g))\). Since $V^G$ is pointed, \(q\dim_{V^G}(V(g)_\lambda) = 1\). This implies \(q\dim_{V}(V(g)) = 1\), \([G : G_{V(g)}] = 1\) and \(\dim(W_\lambda) = 1\) for all \(\lambda\). Thus, $\mathcal{C}^{\alpha V(g)}[G]$ is a commutative semisimple algebra for $g \in G$. Similarly, $\mathcal{C}^{\alpha V(g)}[G]$ is a commutative semisimple algebra.

Identify $G \times G$ as a subgroup of $\text{Aut}(U \otimes V)$. Then $(U \otimes V)((g, g)) \cong U(g) \otimes V(g)$ and $\mathcal{C}^{\alpha(V \otimes V)((g, g))}[G \times G] = \mathcal{C}^{\alpha U(g)}[G] \otimes \mathcal{C}^{\alpha V(g)}[G]$ is a commutative semisimple algebra for $g \in G$. Regarding $G$ as a diagonal subgroup of $G \times G$ we can realize $\mathcal{C}^{\alpha(U \otimes V)((g, g))}[G]$ as a subalgebra of $\mathcal{C}^{\alpha(U \otimes V)((g, g))}[G \times G]$. Thus, $\mathcal{C}^{\alpha(U \otimes V)((g, g))}[G]$ is a commutative semisimple algebra of dimension $\alpha(g)$. Using Theorem 5.1 again gives

$$q\dim_{\mathcal{C}^{\alpha(V \otimes V)}}(U \otimes V)(g)_{\lambda} = \dim(W_{\lambda}[G : G_{(U \otimes V)((g, g))}]) q\dim_{\mathcal{C}^{\alpha(U \otimes V)}}(U \otimes V)((g, g)) = 1$$

for $\lambda \in \Lambda_{(U \otimes V)((g, g))}$. That is, $(U \otimes V)(g)_{\lambda} = (U \otimes V)((g, g))_{\lambda}$ is a simple current and $\mathcal{C}_{(U \otimes V)}^{\alpha(g)}$ is pointed, as desired. \(\square\)

We turn to understand the group $\mathcal{M}_v^\text{pt}(\mathcal{E})$ for any finite abelian group $G$. Let

$$H^3(G, S^1)_{\text{pt}} = \{ \alpha \in H^3(G, S^1) | \mathcal{Z}(\text{Vec}^\alpha_G) \text{ is pointed} \}.$$ The subgroup $H^3(G, S^1)_{\text{ab}}$ of $H^3(G, S^1)$ defined in [MNI p3471] is shown to be the same as $H^3(G, S^1)_{\text{pt}}$ by [MNI Corollary 3.6] with slightly different terminology. Therefore, $H^3(G, S^1)_{\text{pt}}$ is a subgroup of $H^3(G, S^1)$ isomorphic to $\mathcal{M}_v^\text{pt}(\mathcal{E})$.

**Lemma 8.5.** Let $G$ be a finite abelian group and $\mathcal{E} = \text{Rep}(G)$. Then

$$H^3(G, S^1)_{\text{pt}} \xrightarrow{\Phi_G} \mathcal{M}_v^\text{pt}(\mathcal{E}).$$

**Proof.** Recall the isomorphism $\Phi_G : H^3(G, S^1) \rightarrow \mathcal{M}(\mathcal{E})$ from Section 3. By definition, $\Phi_G(H^3(G, S^1)_{\text{pt}})$ is a subgroup of $\mathcal{M}_v^\text{pt}(\mathcal{E})$. Conversely, suppose $(\mathcal{Z}^{\alpha}_G, \iota)$ is a minimal modular extension of $\mathcal{E}$ such that $\mathcal{Z}^{\alpha}_G$ is pointed. There exists $\alpha' \in H^3(G, S^1)$ such that $(\mathcal{Z}^{\alpha'}_G, \iota)$ is equivalent to $(\mathcal{Z}^{\alpha}_G, \iota \alpha')$. In particular, $\mathcal{Z}^{\alpha'}_G$ is pointed and hence $\alpha' \in H^3(G, S^1)_{\text{pt}}$. Now, we have

$$\Phi_G(\alpha') = [(\mathcal{Z}^{\alpha'}_G, \iota \alpha')] = [\mathcal{Z}^{\alpha}_G, \iota],$$

and so $\Phi_G(H^3(G, S^1)_{\text{pt}}) = \mathcal{M}_v^\text{pt}(\mathcal{E})$. \(\square\)
Since $G$ is abelian, for any $\alpha = [\omega] \in H^3(G, S^1)_{\text{pt}}$, the embedding $i_\omega : \mathcal{E} \to \mathcal{Z}(\text{Vec}_G)$ is equivalent to the inclusion of quadratic forms $i_\omega : (\hat{G}, q_0) \to (\Gamma^\omega, q_\omega)$ where $\Gamma^\omega = \text{Irr}(\mathcal{Z}(\text{Vec}_G))$ and $q_\omega(x) = \theta_x$ for $x \in \Gamma^\omega$ (cf. [JS Thm. 3.3]). By [MN2 Prop. 5.2] or [MNT Prop. 3.5 and Cor. 3.6], we have an exact sequence of abelian groups

$$1 \to \hat{G} \xrightarrow{i} \Gamma^\omega \to G \to 1,$$

and its corresponding cohomology class in $H^2(\hat{G}, G)$ is determined by $\alpha$. The triple $(\Gamma^\omega, q_\omega, i_\omega)$, in fact, depends on the choice of representative $\omega$ of $\alpha$, but its equivalence class does not. This will be explained in the following discussion.

We denote by a triple $(\Gamma, q, i)$ for any nondegenerate quadratic form $q : \Gamma \to \mathbb{C}^\times$ on a finite abelian group $\Gamma$ of order $|G|^2$ containing an isotropic subgroup isomorphic to $G$ under a group monomorphism $i$. Let $b_\chi$ be the associated nondegenerate symmetric bicharacter of $\Gamma$. For any coset $\overline{x} = b_\chi(i(\chi), x) = b_\chi(i(\chi), x')$ for any $x' \in \overline{x}$ and $\chi \in \hat{G}$. There exists a unique element $g \in G$ such that $\chi(g) = b_\chi(i(\chi), x)$ for all $\chi \in \hat{G}$. The assignment $p : \Gamma/i(\hat{G}) \to G$, $\overline{x} \mapsto g$, defines a group monomorphism. Since $o(\Gamma/i(\hat{G})) = o(G)$, $p$ is an isomorphism and we have an exact sequence of abelian groups:

$$1 \to \hat{G} \xrightarrow{i} \Gamma \to G \to 1.$$

Two such triples $(\Gamma, q, i), (\Gamma', q', i')$ are called equivalent if there exists a group isomorphism $j : \Gamma \to \Gamma'$ such that $q' \circ j = q$ and $i' = j \circ i$. Let $Q(G)$ be the set of equivalence classes $[\Gamma, q, i]$ of the triples $(\Gamma, q, i)$. One can define the product of two such triples $(\Gamma, q, i), (\Gamma', q', i')$ as follows: Let $H$ be the subgroup $\{(i(\chi), i'(\overline{\chi})) \mid \chi \in \hat{G}\}$ of $\Gamma \times \Gamma'$, and

$$H^\perp = \{(x, y) \in \Gamma \times \Gamma' \mid b(x, i(\chi))b'(y, i'(\overline{\chi})) = 1 \text{ for all } \chi \in \hat{G}\}$$

where $b$ and $b'$ are nondegenerate bicharacters associated with $q, q'$ respectively. Then the quadratic form $q \perp q'$ on $\Gamma \times \Gamma'$ induces a nondegenerate quadratic form $q''$ on $H^\perp/H$, and $i''(\chi) = (i(\chi), 1)H$ for $\chi \in \hat{G}$ defines an embedding of quadratic form from $\hat{G}$ into $H^\perp/H$. Then $Q(G)$ forms an abelian group with the multiplication given by

$$[\Gamma, q, i] \cdot [\Gamma', q', i'] = [H^\perp/H, q'', i''],$$

and we have the exact sequence

$$1 \to \hat{G} \xrightarrow{i''} H^\perp/H \to G \to 1.$$

The quadratic form $ev : \hat{G} \times G \to \mathbb{C}^\times$, given by the evaluation map $ev$, with the embedding $i_0 : \hat{G} \to \hat{G} \times G, \chi \mapsto (\chi, 1)$ defines the identity class $[\hat{G} \times G, ev, i_0]$ of $Q(G)$.

For any $\omega, \omega' \in Z^3(G, S^1)_{\text{pt}}$ such that $\alpha = [\omega] = [\omega']$, the triples $(\Gamma^\omega, q_\omega, i_\omega)$ and $(\Gamma'^{\omega'}, q_{\omega'}, i_{\omega'})$ are equivalent [MNT Prop. 3.5], and its equivalence class will be denoted by $[\Gamma^\omega, q_\omega, i_\omega]$. The function $\phi_G : H^3(G, S^1)_{\text{pt}} \to Q(G), \alpha \mapsto [\Gamma^\omega, q_\omega, i_\omega]$, will be shown to be an isomorphism of groups.
Note that $Q(G)$ is a finite group. Any class $[\Gamma, q, i]$ determines an equivalent class of pointed minimal modular extension $[(\mathcal{C}(\Gamma), q), i]$ of $\mathcal{E}$ with the embedding $\iota : \mathcal{E} \to \mathcal{C}(\Gamma)$ given by $i$. Let $\psi : Q(G) \to \mathcal{M}^{pt}(\mathcal{E})$ be the mapping $[\Gamma, q, i] \mapsto [(\mathcal{C}(\Gamma), q), i]$ (cf. [JS Thm. 3.3]). Since every pointed minimal modular extension of $\mathcal{E}$ is an image of $\psi$, $\psi$ is a bijection. It is easy to see that $\psi$ preserves the products of these groups, and so we have proved the first assertion of following proposition.

**Proposition 8.6.** The map $\psi : Q(G) \to \mathcal{M}^{pt}(\mathcal{E})$ is an isomorphism of groups, and we have the commutative diagram:

$$
\begin{array}{ccc}
H^3(G, S^1)_{pt} & \xrightarrow{\Phi_G} & \mathcal{M}^{pt}(\mathcal{E}) \\
\downarrow{\phi_G} & & \downarrow{\psi^{-1}} \\
Q(G) & & 
\end{array}
$$

Hence, $\phi_G$ is an isomorphism of groups. In particular, for any $\alpha, \alpha' \in H^3(G, S^1)_{pt}$, we have

$$
[\Gamma^\alpha, q_\alpha, i_\alpha] \cdot [\Gamma^{\alpha'}, q_{\alpha'}, i_{\alpha'}] = [\Gamma^{\alpha\alpha'}, q_{\alpha\alpha'}, i_{\alpha\alpha'}].
$$

**Proof.** The equality $\psi^{-1} \circ \Phi_G = \phi_G$ follows directly from the definitions of $\phi_G$, $\Phi_G$ and $\psi$, and the fact that $\psi^{-1}([Z(\text{Vec}_G^\alpha), i_\alpha]) = [\Gamma^\alpha, q_\alpha, i_\alpha]$ for $\alpha \in H^3(G, S^1)_{pt}$. Since $\Phi_G$ and $\psi$ are isomorphisms of groups, and so is $\phi_G$. The last statement is a consequence of the commutative diagram. □

**Theorem 8.7.** Let $G$ be a finite abelian group and $\mathcal{E} = \text{Rep}(G)$. Then for any $\alpha \in H^3(G, S^1)_{pt}$, there exists a positive definite even unimodular lattice $E$ such that $V_E$ admits an automorphism group isomorphic to $G$ and $(\mathcal{C}_E \circ F_{E,G}) \cong (Z(\text{Vec}_G^\alpha), i_\alpha)$ as minimal modular extensions of $\mathcal{E}$. Moreover, $\mathcal{M}^{pt}_v(\mathcal{E}) = \mathcal{M}^{pt}(\mathcal{E}) \cong H^3(G, S^1)_{pt}$.

**Proof.** Let $[\Gamma, q, i] \in Q(G)$. By [Ni Cor. 1.10.2], there exists a positive definite even lattice $L$ such that $(\Gamma, q) \cong (L^o/L, q_L)$ for some isomorphism $j$ of quadratic forms. Let $E$ be the subgroup of $L^o$ containing $L$ such that $j \circ i(G) = E/L$. Then we have the following row exact commutative diagram:

$$
\begin{array}{ccc}
1 & \longrightarrow & \hat{G} \xrightarrow{i} \Gamma \\
& & \downarrow{j} \\
1 & \longrightarrow & \hat{G} \xrightarrow{j \circ i} L^o/L
\end{array}
$$

In particular, $[\Gamma, q, i] = [L^o/L, q_L, j \circ i]$ in $Q(G)$.

For any $x \in E$, $q_L(L + x) = 1$ or $(x, x)$ is a positive even integer. Thus, $E$ is a positive definite even lattice. Since $E/L \cong \hat{G}$, $[E : L] = o(G)$ and so

$$
|\det(E)| = |\det(L)|/[E : L]^2 = o(G)^2/o(G)^2 = 1.
$$
Therefore, $E$ is unimodular.

Now, we identify $L^o/E$ with $G$ via the isomorphism $p : L^o/E \to G$ with $p(E + x) = \pi(E + x)$ given by $b_L(j \circ i(\chi), L + x) = \chi(g)$ for all $\chi \in \hat{G}$ where $b_L$ is the associated bicharacter of $q_L$. We consider $G$ as an automorphism group of the lattice vertex operator algebra $V_E = M(1) \otimes \mathbb{C}[E]$ via $p$, namely $g = \sigma_{x_g}$ where $p(E + x_g) = g$. Then $V^G_E = V_L$, and

$$V_E = \bigoplus_{L + x \in E/L} V_{L+x} = \bigoplus_{\chi \in \hat{G}} V_{j \circ i(\chi)}$$

as a $V_L$-module. So, $\text{Irr}(\mathcal{E}_{V_E}) = \{V_{j \circ i(\chi)} \mid \chi \in \hat{G}\}$ and $F^{V_E,G}(\chi) = V_{j \circ i(\chi)}$. Thus, $\psi^{-1}([\mathcal{C}_{V_E}, F^{V_E,G}]) = [L^o/L, q_L, i']$ where $i'(\chi) = -j \circ i(\chi)$ for $\chi \in \hat{G}$. However, $(L^o/L, q_L, i') \cong (L^o/L, q_L, j \circ i)$ under the automorphism $x \mapsto -x$ in $\text{Aut}(L^o/L, q_L)$. Therefore,

$$\psi^{-1}([\mathcal{C}_{V_E}, F^{V_E,G}]) = [L^o/L, q_L, i'] = [L^o/L, q_L, j \circ i] = [\Gamma, q, i],$$

and hence $\psi^{-1}(M^\text{pt}_v(\mathcal{E})) = Q(G)$. The remaining statement follows immediately from Lemma 8.5 and Proposition 8.6. □

Recall from [MNI p3480] that the group epimorphism $\varphi^* : H^3(G, S^1) \to \text{Hom}(\wedge^3 G, S^1)$ defined by

$$\varphi^*([\omega])(a, b, c) = \frac{\omega(a, b, c)\omega(b, c, a)\omega(c, a, b)}{\omega(b, a, c)\omega(a, c, b)\omega(c, b, a)}$$

for $a, b, c \in G$ and $\omega \in Z^3(G, S^1)$. The definition of $\varphi^*$ is independent of the choice of representatives of the cohomology class $[\omega]$, and its kernel was characterized in [MNI Lem. 7.4] as

$$H^3(G, S^1)_{\text{pt}} = \ker \varphi^*.$$ 

This gives us the following corollary.

**Corollary 8.8.** If $G$ is a finite abelian group generated by two elements, then $M_v(\mathcal{E}) \cong H^3(G, S^1)$.

**Proof.** Suppose $G$ is generated by $a, b$. Then, for any $x, y, z \in G$ and $\alpha \in H^3(G, S^1)$, we have $\varphi^*(\alpha)(x, y, z)$ is a product of the values of $\varphi^*(\alpha)$ at the following triples:

$$(a, a, a), (a, a, b), (a, b, a), (b, a, a), (b, b, a), (b, a, b), (a, b, b), (b, b, b).$$

Since they are all equal to 1, $\alpha \in H^3(G, S^1)_{\text{pt}}$ by [MNI Lem. 7.4]. So $H^3(G, S^1)_{\text{pt}} = H^3(G, S^1)$. Now, the result follows from Theorem 8.7. □

**Remark 8.9.** The embedding $F^{V, G} : \mathcal{E} \hookrightarrow \mathcal{C}_{V_G}$ in $(\mathcal{C}_{V_G}, F^{V, G})$ plays an essential role in identifying $(\mathcal{C}_{V_G}, F^{V, G})$ with the corresponding $\alpha \in H^3(G, S^1)$. It is possible that the modular tensor categories $\mathcal{C}_{V_G}$ and $\mathcal{C}_{U_G}$ are braided equivalent but $(\mathcal{C}_{V_G}, F^{V, G})$ and $(\mathcal{C}_{U_G}, F^{U, G})$ give two different elements in group $\mathcal{M}_v(\mathcal{E})$. Or equivalently, there are two inequivalent embeddings $\mathcal{E} \hookrightarrow \mathcal{C}_{V_G}$. The following example explains this in details.
Example 8.10. Let $L = \Gamma_{16}$ be the spin lattice of rank 16. We now give an automorphism group $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ of $V_L$ such that there are three inequivalent embeddings: $\mathcal{E} \hookrightarrow \mathcal{C}_{V_L}$. The main idea is to find an even lattice $K$ of $L$ such that $L/K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $K^o/K \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ where $K^o$ is the dual lattice of $K$ as usual. We thank Griess Jr. for providing us with such $K$.

Let $\{e_1, \ldots, e_{16}\}$ be the standard orthonormal basis of $\mathbb{R}^{16}$. Recall that root lattice $L_{D_{16}} = \sum_{i=1}^{16} \mathbb{Z}e_i$ of type $D_{16}$ where $e_i = e_i - e_{i+1}$ for $i = 1, \ldots, 15$ and $e_{16} = e_{15} + e_{16}$. Then $L = L_{D_{16}} + \mathbb{Z}w = L_{D_{16}} \cup (L_{D_{16}} + w)$ where $w = \frac{1}{4}(e_1 + \cdots + e_{16})$. Also let $u = \frac{1}{2}(e_1 + \cdots + e_8)$ and $v = \frac{1}{2}(e_8 - e_{16})$. Then $(u, u) = 2$, $(v, v) = \frac{1}{2}$ and $(u, v) = \frac{1}{4}$. Let

$$K = \{\alpha \in L \mid (u, \alpha) \in \mathbb{Z} \text{ and } (v, \alpha) \in \mathbb{Z}\}$$

be a sublattice of $L$. It is easy to see that

$$L/K = \{K, K + 2u, K + 2v, K + 2u + 2v\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

and

$$K^o = \bigcup_{i,j=0}^3 (K + iu + jv), \quad K^o/K = \langle u + K, v + K \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4.$$

Recall that $V_L = M(1) \otimes \mathbb{C}[L]$. We have automorphisms $\sigma_u, \sigma_v \in \text{Aut}(V_L)$. Then $G = \langle \sigma_u, \sigma_v \rangle$ is isomorphic to $L/K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, $V^G_L = V_K$ and the irreducible $V_K$-modules are $V_{K+z}$, and

$$\theta_{V_{K+z}} = e^{2\pi i(s^2 + \frac{1}{4}(t^2 + st))} = i^{s+t}$$

for $z = su + tv$ with $s, t = 0, \ldots, 3$. The pair $(K^o/K, q)$ defines a quadratic form with

$$q(K + z) = \theta_{V_{K+z}}$$

and $C_{V^G_L}$ is equivalent to the pointed modular category $\mathcal{C}(K^o/K, q)$. Consider the generating set $\{x, y\}$ of $K^o/K$ where $x = u + K$ and $y = -u + v + K$. For any $K + z \in K^o/K$, $K + z = sx + ty = (s - t)u + tv + K$ for some $s, t = 0, \ldots, 3$, and so

$$q(K + z) = \theta_{V_{K+z}} = i^{st}.$$ 

By direct computation, the automorphisms of the quadratic form $(K^o/K, q)$ are given by

$$\text{Aut}(K^o/K, q) = \{f \in \text{Aut}(K^o/K) \mid q = q \circ f\} = \{\pm id, \pm \kappa\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \quad (8.1)$$

where $\kappa(sx + ty) = tx + sy$.

Recall that the Tannakian category $\mathcal{E} = \text{Rep}(G)$ is equivalent to $\mathcal{C}(\hat{G}, q_0)$ where $q_0$ is the trivial quadratic form given by $q_0 = 1$. Let $\psi_1, \psi_2 \in G$ such that

$$\psi_1(\sigma_u) = -1, \quad \psi_1(\sigma_v) = 1 \quad \text{and} \quad \psi_2(\sigma_u) = 1, \quad \psi_2(\sigma_v) = -1.$$ 

Then $F^{V_L,G}(\psi_1) = 2(x + y)$ and $F^{V_L,G}(\psi_2) = 2x$ and $F^{V_L,G}(\psi_1 \psi_2) = 2y$ and $F^{V_L,G}$ induces an embedding $F^{V_L,G} : (\hat{G}, q_0) \rightarrow (K^o/K, q)$ of quadratic forms.
Now, we twist the $G$ action on $V_L$ by an automorphism $\gamma$ of $G$, that means $g \cdot a = \gamma(g)(a)$ for $g \in G$ and $a \in V_L$, and we denote this new $G$-module by $V^G_L$. Note that $(V^G_L)^G = V^G_L$ as vertex operator algebras. The automorphism $\gamma$ also acts on $\hat{G}$ by composition, and $F^{V^G_L,G}(\psi) = F^{V_L,G}(\psi \circ \gamma^{-1})$ for $\psi \in \hat{G}$. Thus, the corresponding embedding of quadratic forms $F^{V^G_L,G} : (\hat{G}, q_0) \to (K^0/K, q)$ can be expressed as $F^{V^G_L,G}(\psi) = F^{V_L,G}(\psi \circ \gamma^{-1})$ for $\psi \in \hat{G}$.

The automorphism group of $G$ is isomorphic to $S_3$ and each automorphism $\gamma$ is completely determined by its images of $\sigma_u$ and $\sigma_v$. The equivalence $(\mathcal{C}_{V^G_L}, F^{V^G_L,G}) \cong (\mathcal{C}_{V_L^G}, F^{V_L^G,G})$ implies the embeddings of quadratic forms $F^{V^G_L,G}, F^{V_L^G,G} : (\hat{G}, q_0) \to (K^0/K, q)$ are equivalent, i.e., there exists automorphism $f \in \text{Aut}(K^0/K, q)$ such that $f \circ F^{V_L,G} = F^{V^G_L,G}$. By (8.1), $\gamma = \text{id}_G$ or

$$\gamma : \sigma_v \mapsto \sigma_v, \quad \sigma_u \mapsto \sigma_u \sigma_v.$$

Let $\delta$ be the automorphism of $G$ given by the 3-cycle $(\sigma_u, \sigma_u \sigma_v, \sigma_v)$ in $S_3$. Then $(\mathcal{C}_{V^G_L}, F^{V^G_L,G})$ and $(\mathcal{C}_{V^G_L}, F^{V^G_L^2,G})$ are not equivalent to $(\mathcal{C}_{V_L^G}, F^{V_L^G,G})$. One can further show directly from (8.1) that $(\mathcal{C}_{V^G_L}, F^{V^G_L,G}) \ncong (\mathcal{C}_{V^G_L}, F^{V^G_L^2,G})$. These three inequivalent embedding of $\mathcal{E}$ correspond to three different cohomology classes $\alpha \in H^3(G, S^1)$ such that $\mathcal{Z}(\text{Vec}_G^\alpha)$ are equivalent modular tensor categories.

Example 8.10 also gives the following result.

**Proposition 8.11.** If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ then $\mathcal{M}_v(\mathcal{E}) = \mathcal{M}(\mathcal{E}) \cong H^3(G, S^1)$.

**Proof.** From the discussion before Corollary 8.8 we see that $H^3(G, S^1)/H^3(G, S^1)_{\text{pt}}$ is isomorphic to $\text{Hom}(\wedge^3 G, S^1) \cong \mathbb{Z}_2$, and so $[H^3(G, S^1) : H^3(G, S^1)_{\text{pt}}] = 2$. By Theorem 8.7 $\mathcal{M}_v^\text{pt}(\mathcal{E}) \cong H^3(G, S^1)_{\text{pt}}$. It is enough to show that there exists $[\mathcal{C}_{V^G_L}] \in \mathcal{M}_v(\mathcal{E})$ such that $\mathcal{C}_{V^G_L}$ is not pointed. Let $V_L$ be the lattice vertex operator algebra defined in Example 8.10. Let $\tau \in \text{Aut}(V_L)$ such that

$$\tau(h_{i_1}(n_1) \cdots h_{i_k}(n_k) \otimes e^\alpha) = (-1)^k h_{i_1}(n_1) \cdots h_{i_k}(n_k) \otimes e^{-\alpha}$$

for $i_1, \ldots, i_k \in \{1, \ldots, 16\}$, $n_i < 0$ and $\alpha \in L$. Then $G \cong \langle \sigma_u, \sigma_v, \tau \rangle$ and

$$V^G_L = V^+_K = \{ a \in V_K \mid \tau(a) = a \}.$$  

Since $V_{K+u}$ is an irreducible $V^+_K$-module (see [ADL]) and $	ext{FPdim}(V_{K+u}) = \text{qdim}(V^+_K) = 2$, we conclude that $V_{K+u}$ is not a simple current and so $\mathcal{C}_{V^G_L}$ is not pointed. □

It is worth noting that for any nonabelian group $H$ of order 8, $\mathcal{Z}(\text{Rep}(H)) \cong \mathcal{Z}(\text{Vec}_H)$ is braided equivalent to some nonpointed $\mathcal{Z}(\text{Vec}_G^\alpha)$ where $G = \mathbb{Z}_2^3$ (cf. [GMN]). Thus, there exists an embedding $\iota : \text{Rep}(H) \to \mathcal{Z}(\text{Vec}_G^\alpha)$ so that $(\mathcal{Z}(\text{Vec}_G^\alpha), \iota)$ is a minimal modular extension of $\text{Rep}(H)$. In general, for any finite group $A$, $\mathcal{Z}(\text{Vec}_G^\alpha)$ is a minimal modular extension of any symmetric fusion subcategory $\mathcal{E}$ of $\mathcal{Z}(\text{Vec}_G^\alpha)$ with $\text{dim}(\mathcal{E}) = |A|$,
i.e., a Lagrangian subcategory of $\mathcal{Z}(\text{Vec}_A^\alpha)$. If $\mathcal{E}$ is Tannakian, then $\mathcal{E}$ is braided equivalent to $\text{Rep}(B)$ for some uniquely determined group $B$, and $\mathcal{Z}(\text{Vec}_A^\alpha)$ is braided equivalent to $\mathcal{Z}(\text{Vec}_B^\alpha)$ for some $\alpha' \in H^3(B, S^1)$.

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