Finite time stabilization for nonlinear systems with unstructured uncertainties

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Abstract
This paper investigates the finite time stabilization problem for a class of nonlinear systems with unknown control directions and unstructured uncertainties. The unstructured uncertainties indicate that not only the parameters but also the structure of the system nonlinearities are uncertain. In order to solve the above problem, a new adaptive controller is proposed. Logic-based switching rules are utilized to tune the controller parameters online to stabilize the system in finite time. Different from the existing logic-based adaptive controller for structured/parametric uncertainties, new switching barrier Lyapunov method and supervisory functions are introduced to overcome the obstacles caused by unstructured uncertainties and unknown control directions. Moreover, an extension is made for the proposed method such that all the system states can be regulated to zero in prescribed finite time. Simulations are conducted to verify the effectiveness of the proposed design scheme.

Keywords: finite time stabilization, barrier Lyapunov functions, unknown control directions, backstepping, time-varying feedback

1. Introduction

1.1. Background and motivations
Finite time stabilization problem has attracted increasing attention in the past few years. Finite time stabilization means that by designing a proper feedback controller, all the states of the closed loop systems will become exact zero after a finite time (Zhao & Jiang, 2018). However, for asymptotic stabilization, the states will converge to zero in an infinite time. Lots of works (Li, Zhao, He, & Lu, 2019; Yu, Shi, & Zhao, 2018; Chen & Sun, 2020; Wang, Liu, & Niu, 2017) have shown that finite time control has some promising features in contrast with asymptotic control. These may lie in: 1) Faster convergence rate and higher precision; 2) possibility to decouple the stabilization problem from other control objectives (Cao, Ren, Casbeer, & Schumacher, 2016).

Many interesting results have been obtained for finite time control. The works of Zhao and Jiang (2018) have studied finite time output feedback stabilization for strict feedback nonlinear systems. Zhao, Li, and
Liu (2018) have extended the finite time control to high order stochastic nonlinear systems. Recently, the finite time control problem has been investigated for multi-agent and networked systems (Li & Ji, 2018). Moreover, several real practical applications, such as robot manipulators (Yang, Jiang, He, Na, Li, & Xu, 2018) and servo motor systems (Mishra, Wang, Zhu, Yu, & Jalili; Zheng & Li, 2018), have been considered for finite time control.

Unknown control directions are often encountered in real engineering world. It means that the sign of control coefficient is unknown. This will bring difficulties to the controller design because a control effort with wrong direction can drive the states away from the equilibrium point. Nussbaum-gain technique, which was originally introduced in Nussbaum (1983), is a common way to handle unknown control direction. Plentiful works (Chen, 2019; Li, & Liu, 2018; Liu, & Tong, 2017; Yin, Gao, Qiu, & Kaynak, 2017; Zhang, & Yang, 2017) have been done on the control of nonlinear systems by incorporating Nussbaum-gain function. Nevertheless, as discussed in Chen, Wen, and Wu (2018) and Wu, Chen, and Li (2016), the Nussbaum-gain technique could only achieve asymptotic stability because the constructed Lyapunov function cannot be negative definite.

In fact, there are very few works concentrating on finite time stabilization of nonlinear systems with unknown control directions. Lately, in the framework of backstepping method (Krstic, Kanellakopoulos, & Kokotovic, 1995; Krstic, Kanellakopoulos, & Kokotovic, 1992), a new adaptive control strategy is proposed in Chen et al. (2018), Wu, Li, Zong, and Chen (2017) and Wu et al. (2016) to solve this problem. The idea of the method is to adopt a logic-based switching rule to tune the controller parameters online according to a well-defined supervisory function. Finite time stability can then be achieved despite unknown control directions.

The aforementioned works (Chen et al., 2018; Wu et al., 2017), however, only consider the finite time stabilization problem for nonlinear systems suffering from structured/parametric uncertainties. This means that the structures of the nonlinear uncertain functions are available, but contain some unknown parameters. The structured uncertainties are mainly used to describe the parameter variations in the systems. However, the nonlinear uncertainties are often very complicated in practical systems. Hence, it may be difficult or impossible to obtain the exact form of the uncertainties, and express the uncertainties in a parametric way. This class of uncertainties is often referred to as unstructured/nonparametric uncertainties, which can represent those unknown nonlinearities caused by complex system dynamics and modeling errors. Therefore, a natural question arises:

How to solve the finite time stabilization problem for nonlinear systems with unknown control directions and unstructured uncertainties?

To the best of our knowledge, little effort has been made to answer the above issue. The main challenges may lie in the following two aspects:

1) Due to the structure of the nonlinearities is uncertain, the nonlinearities cannot be parameterized.
Hence, it is difficult to directly extend the adaptive control scheme presented in Chen et al. (2018) and Fu, Ma, and Chia (2017) to solve the above problem. Consequently, the design procedures become involved.

2) As previously mentioned, a logic-based switching mechanism has to be adopted to achieve finite time stability due to the possible limitations of the Nussbaum-gain technique. Thus, the entire closed-loop system will exhibit hybrid features, which introduce difficulties to the stability analysis.

1.2. Contributions

This paper proposes a new adaptive control strategy to solve the finite time stabilization problem for a class of nonlinear systems with unknown control directions and unstructured uncertainties. Logic-based switching rules are used to tune the controller parameters online. The main contributions are as follows:

- Novel switching barrier Lyapunov functions are constructed for the controller design. Thus, the unstructured uncertainties can be treated similar to structured uncertainties, which contributes to the feasibility of the adaptive control scheme.

- By constructing an auxiliary system, new supervisory functions are presented to guide the logic-based switching. Using these supervisory functions, the single/multiple unknown control directions can be well handled. Meanwhile, they can guarantee that switching times are only finite and all the states will reach exact zero in finite time.

- Several important extensions are given for the proposed results: 1) By introducing a new time-varying backstepping scheme, an improved method is presented such that all the states can be regulated to zero in prescribed finite time; 2) Systems with non-feedback linearizable and non-vanishing dynamics are considered.

- The proposed controller could achieve a higher control performance than Nussbaum-gain technique. Due to the finite time feature, a faster convergence speed could be obtained by the proposed method. Moreover, via the switching mechanism, our controller can quickly find the appropriate control directions. Thus, it could often result in a smaller control overshoot.

- The controller is in a simple structure. The “explosion of complexity” problem in traditional backstepping design is avoided. No approximation techniques are needed, i.e., fuzzy logic or neural networks will not be adopted.

1.3. Organizations

The organization of the paper is as follows. Problem formulation and preliminaries are presented in Section 2. Section 3 focuses on the finite time control for a first order nonlinear system. It gives the basic idea of the design procedure. Section 4 presents the main result and concentrates on the finite time controller
design for high order nonlinear systems. Section 5 extends the result to prescribed finite time control method. Some discussions and comparisons are provided in Section 6. Section 7 presents the simulations conducted. Section 8 concludes the paper. Some proofs are provided in the Appendices and supplementary file.

2. Preliminaries and problem formulation

2.1. Problem formulation

Consider the following system

\[ \begin{align*}
\dot{x}_i &= h_i(\mathbf{x}_i) x_{i+1} + f_i(\mathbf{x}_i), \quad i = 1, 2, \ldots, n - 1 \\
\dot{x}_n &= h_n(\mathbf{x}_n) u + f_n(\mathbf{x}_n), \\
y &= x_1
\end{align*} \]

where \( \mathbf{x}_i = (x_1, x_2, \ldots, x_i)^T \in \mathbb{R}^i, \ i = 1, 2, \ldots, n \) are the system states, \( y \) is the system output. \( f_i(\mathbf{x}_i), h_i(\mathbf{x}_i)(i = 1, \ldots, n) \) are all unknown continuously differentiable nonlinear functions. \( f_i(\mathbf{x}_i) \) represents the system nonlinearities and uncertainties such that \( f_i(0, 0, \ldots, 0) \equiv 0 \). \( h_i(\mathbf{x}_i) \) are the control gains such that their signs are unknown and satisfy \( |h_i(\mathbf{x}_i)| > 0 \). \( u \) denotes the control input.

Remark 1. Note that system 1 is more general than the existing works (Chen et al., 2018; Huang, Wen, Wang, & Song, 2016; Huang, & Xiang, 2016; Wu et al., 2016; Zhang & Li, 2019 etc.) due to the following reasons:

1) \( f_i(\mathbf{x}_i), h_i(\mathbf{x}_i)(i = 1, 2, \ldots, n) \) represent unstructured uncertainties, i.e., not only parameters in \( f_i(\mathbf{x}_i), h_i(\mathbf{x}_i) \) but also the form of \( f_i(\mathbf{x}_i), h_i(\mathbf{x}_i) \) are uncertain. In fact, the continuously differentiable nonlinearities \( f_i(\mathbf{x}_i), h_i(\mathbf{x}_i) \) only need to satisfy \( f_i(0, 0, \ldots, 0) \equiv 0, |h_i(\mathbf{x}_i)| > 0 \). This is much more general than structured uncertainties in Chen et al. (2018), Wu et al. (2017) and Fu et al. (2017). In these references, \( f_i(\mathbf{x}_i), h_i(\mathbf{x}_i) \) need to satisfy \( |f_i(\mathbf{x}_i)| \leq (|x_1| + \cdots + |x_i|) \psi_i(\mathbf{x}_i, \theta) \) and \( 0 < \psi_i(\mathbf{x}_i, \theta) \leq |h_i(\mathbf{x}_i)| \leq \psi_i(\mathbf{x}_i, \theta) \) where \( \psi_i(\mathbf{x}_i, \theta) \) and \( \overline{h}_i(\mathbf{x}_i, \theta) \) are known smooth functions, \( \theta, \overline{h}_i \) are unknown parameters.

2) System 1 contains multiple unknown control directions, i.e., for \( \forall i = 1, 2, \ldots, n \), the sign of \( h_i(\mathbf{x}_i) \) is unknown. Moreover, compared with Liu, Zhai, Li, and Zhang (2019), Rezaei, Kabiri, Menhaj, (2018) and Yin et al., (2017), \( |h_i(\mathbf{x}_i)| \) only needs to be larger than zero, not a positive constant.

According to the above analysis, we can see that very little information is needed for \( f_i(\mathbf{x}_i), h_i(\mathbf{x}_i) \). This will bring many difficulties to the controller design. In addition, note that by unstructured/nonparametric uncertainties, it means that it is difficult to obtain the exact form of the uncertainties, and the uncertain functions cannot be parameterized by unknown parameters. However, some crude information of the uncertainties may need to be known. For instance, the nonlinear function \( f_i(\mathbf{x}_i) \) needs to satisfy \( f_i(0, 0, \ldots, 0) \equiv 0. \)
Yet, we can see the structures of the nonlinearities are still uncertain because many kinds of nonlinear functions satisfy $f_i(0, 0, ..., 0) \equiv 0$. Moreover, in Section 6 we will discuss the situation where $f_i(0, 0, ..., 0) \neq 0$. That is $f_i(x_i)$ can be any continuously differentiable functions.

The definition of finite time stability is as follows.

**Definition 1.** (Bhat & Bernstein, 1998; Chen et al., 2018) Consider a nonlinear switched system

$$
\dot{x} = f_{\sigma(t)}(x), \quad f(0) = 0
$$

(2)

where $\sigma(t) : [0, +\infty) \rightarrow \{1, 2, ..., N\}$ is a piecewise constant function representing the switching signal, $N$ is a finite positive integer. For $\forall k \in \{1, 2, ..., N\}$, $f_k : U_0 \times [0, +\infty) \rightarrow \mathbb{R}^n$ is continuous on an open neighborhood $U_0 \subset \mathbb{R}^n$ of the origin $x = 0$. It is assumed that the system states do not jump at switching times.

If for any initial condition $x(0) \in U$ where $U \subseteq U_0$, every solution $x(t; x(0))$ of the system (2) is Lyapunov stable and satisfies

$$
\lim_{t \to T(x(0))} x(t; x(0)) = 0; \quad x(t; x(0)) = 0, \quad \forall t \geq T(x(0))
$$

where $T(x(0)) > 0$ is a finite settling time. Then the system is finite time stable.

Our aim in this paper is to develop an adaptive control method for system (1) such that

1) All the signals in the systems are bounded, and;
2) Finite time stability is achieved.

2.2. Technical lemmas

Some useful lemmas will be presented, which will be used in the controller design.

**Lemma 1.** (Huang & Xiang, 2016) Consider the following Young’s inequality

$$
|x|^a|y|^b \leq \frac{a}{a+b} \zeta(x,y)|x|^{a+b} + \frac{b}{a+b} \zeta^{-a/b}(x,y)|y|^{a+b}
$$

where $x,y \in \mathbb{R}$, $a,b$ are positive constants, $\zeta(x,y) > 0$ is any real value function.

**Lemma 2.** (Qian & Lin, 2001; Huang & Xiang, 2016) Given a real constant $p \geq 1$, we have:

$$
|x - y|^p \leq 2^{p-1}|x^p - y^p|;
$$

If $0 < p < 1$, we have

$$
|x^p - y^p| \leq 2^{1-p}|x - y|^p,
$$

$$
\left( \sum_{i=1}^{n} |z_i| \right)^p \leq \sum_{i=1}^{n} |z_i|^p \leq n^{1-p} \left( \sum_{i=1}^{n} |z_i| \right)^p
$$

where $x, y, z_i \in \mathbb{R}$. 

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Lemma 3. *(Lemma 11.1 in Chen & Huang, 2015; Lin & Gong, 2003)* Given a continuously differentiable nonlinear function \( f(\mathbf{x}_n) : \mathbb{R}^n \to \mathbb{R} \) where \( \mathbf{x}_n = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \) and \( f(0, 0, \ldots, 0) \equiv 0 \), then there exists a non-negative smooth function \( \psi(\mathbf{x}_n) : \mathbb{R}^n \to \mathbb{R} \) such that
\[
|f(\mathbf{x}_n)| \leq (|x_1| + \cdots + |x_n|)\psi(\mathbf{x}_n).
\] (3)

Lemma 4. Given four time-varying continuous functions \( x(t), y(t), a(t), b(t) : [0, +\infty) \to \mathbb{R} \) such that
\[
\dot{x}(t) = -a(t)x^\gamma(t) + b(t),
\]

\[
\dot{y}(t) \leq -a(t)y^\gamma(t) + b(t)
\]
for \( \forall t \in [t_0, t_1] \subseteq [0, +\infty) \) where \( x(t_0) = y(t_0) + \varepsilon, \varepsilon \geq 0 \) and \( 0 < \gamma \leq 1 \) are constants, \( a(t) > 0 \) on \( [0, +\infty) \). Then \( x(t) \geq y(t) \) for \( \forall t \in [t_0, t_1] \).

*Proof.* Please see Appendix A for detailed proof. \( \square \)

3. Finite time controller design for first order nonlinear system

This section will take the following first order nonlinear system as an example to explain the basic design idea of our proposed controller.

\[
\dot{x} = h(x)u + f(x)
\]

(6)
where \( x \in \mathbb{R} \) is the state variable, \( f(x), h(x) \) are unknown continuously differentiable nonlinear functions such that \( f(0) \equiv 0 \) and \( |h(x)| > 0 \). The sign of \( h(x) \) is also unknown and \( u \) is the control input. The control objective aims to make the state \( x \) converge to zero in finite time.

3.1. Controller design

First, consider the following Lyapunov function
\[
V = \frac{1}{2} \ln \left( \frac{\chi^2}{\chi^2 - s^2} \right)
\]

(7)
where \( \chi > 0 \) is a positive constant, \( s \triangleq x \) and \( |s(0)| < \chi \).

Notably, \( V \) is a barrier Lyapunov function (Liu, Lu, Tong, Chen, Chen, & Li, 2018; Zheng & Li, 2018) such that if \( |s| < \chi \), then \( V \to +\infty \) as \( |s| \to \chi \). The positive constant \( \chi \) acts as a barrier for \( s \). The purpose of adopting the barrier Lyapunov function is to constrain the state \( x \) in the interval \(( -\chi, \chi )\). Note that as long as \( V \) is bounded, \( s \in ( -\chi, \chi ) \). In the following design, we assume \( |s| < \chi \). This will be verified later in the stability analysis.
Differentiating $V$ with respect to time, using system dynamic (6) and Lemma 3, we have

$$\dot{V} \leq \frac{s(hu + f(x))}{\chi^2 - s^2} \leq \frac{shu + s^2\psi(x)}{\chi^2 - s^2} \leq \frac{shu}{\chi^2 - s^2} + \frac{Us^{1+q}F(x, \chi)}{(\chi^2 - s^2)^{2+2\alpha}}$$

(8)

where $q \triangleq \alpha \in (\frac{1}{2}, 1)$ is a ratio of odd integers, $U > 0$ is a design parameter, $\psi(x)$ and $F(x, \chi) = s^{1-q}\psi(x)(\chi^2 - s^2)^{1+2\alpha}/U$ are unknown functions that caused by unstructured uncertainties.

Then, we design the control effort $u$ as

$$u = \hat{\Theta}(\sigma(t)) \left[ -Ks^\alpha - \frac{Us^q}{(\chi^2 - s^2)^{1+2\alpha}} \right]$$

(9)

where $K$ is a design parameter. $\hat{\Theta}$ is an adaptive parameter. It will vary according to a switching signal $\sigma(t)$.

$$\hat{\Theta}(\sigma(t)) = (-1)^{\sigma(t)}\theta(\sigma(t))$$

(10)

where $\sigma(t) : [0, +\infty) \to \mathbb{N}$ is a non-decreasing piecewise constant switching signal. $\theta(\sigma) : \mathbb{N} \to \mathbb{R}$ is an increasing function with respect to $\sigma$ such that $\theta(0) > 0$ and $\theta(\sigma) \to +\infty$ as $\sigma \to +\infty$. A typical example of $\theta(\sigma)$ is $\theta(0) = 1, \theta(1) = 2, \theta(2) = 3, ...$. The idea of tuning rule (10) for $\hat{\Theta}$ is that by changing its sign repeatedly, one may expect to find a correct control direction.

Substituting (9) into (8), we get

$$\dot{V} \leq -K's^{1+\alpha} + \frac{(K' - hK\hat{\Theta})s^{1+\alpha}}{\chi^2 - s^2} + \frac{U(F(x, \chi) - h\hat{\Theta})s^{1+\alpha}}{(\chi^2 - s^2)^{2+2\alpha}}$$

(11)

where $K' > 0$ is positive constant.

From the above inequality, we can see that if $|s| = |x| < \chi$, then there exist unknown positive constants $\bar{F}, \bar{h}$ such that $F(x, \chi) \leq \bar{F}$ and $|h(x)| \geq \bar{h} > 0$. Thus, we can deal with the unstructured uncertainties similar to structured uncertainties. That is, (11) can be put in the following form when $|s| = |x| < \chi$.

$$\dot{V} \leq -K's^{1+\alpha} + \frac{(K' - hK\hat{\Theta})s^{1+\alpha}}{\chi^2 - s^2} + \frac{U(\bar{F} - h\hat{\Theta})s^{1+\alpha}}{(\chi^2 - s^2)^{2+2\alpha}}$$

(12)

with $|h| \geq \bar{h} > 0$. We can see that the unknown function $F(x, \chi)$ is replaced by an unknown parameter $\bar{F}$. Meanwhile, the unknown function $h(x)$ is larger than an unknown positive constant $\bar{h}$. This finding is quite similar to the work in Chen et al. (2018) and Fu et al. (2017) for parametric uncertainties.
Moreover, from (10), we find that there exists a sufficiently large integer \( \sigma \) such that

\[
\text{sgn}(\hat{\Theta}(\sigma)) = \text{sgn}(h(x)),
\]

\[
K' - hK\hat{\Theta}(\sigma) \leq K' - hK\hat{\Theta}(\sigma) < 0,
\]

\[
F - h\hat{\Theta}(\sigma) \leq F - h\hat{\Theta}(\sigma) < 0.
\]

Then (12) becomes

\[
\dot{V} \leq -K's^{1+\alpha} \leq -aV^\gamma
\]

(13)

where \( a > 0 \) is a positive constant. \( \gamma = \frac{1+\alpha}{2} \in (\frac{1}{2}, 1) \). That is we have (13) when \( \sigma \) is sufficiently large and \( |x(t)| < \chi \). This result will be used in the stability analysis.

By solving the above equation, we have

\[
0 \leq V \leq \left[ V^{1-\gamma}(t_0) - a(1 - \gamma)(t - t_0) \right]^\frac{1}{1-\gamma}
\]

(14)

where \( t_0 \) denotes the initial time instant when (13) holds. Hence, we can conclude that the system is finite time stable if \( V(t_0) \) is bounded.

According to the above analysis, we can see that the key for the controller design is to guarantee \( x \) vary in \((-\chi, \chi)\) and design an appropriate switching law for \( \sigma \). This will be explained in the following subsection.

### 3.2. Logic-based switching law

The logic-based switching mechanism is shown in Algorithm 1. The switching signal \( \sigma \) is guided by a supervisory function \( \mathcal{S}(\cdot) \) defined as:

\[
\mathcal{S}(\cdot) \triangleq V(\cdot) - \eta(t),
\]

(15)

\[
\dot{\eta}(t) = -a\eta^\gamma
\]

(16)
where \( \eta \) is an auxiliary variable, \( a \) is defined in [13] and [16] is called an auxiliary system.

The idea of the algorithm is as follows. As shown in Fig. [1] at each time instant, we verify whether or not the supervisory function \( \mathcal{S}(\cdot) \) is larger than zero. If not, then the switching signal \( \sigma \) remains constant; otherwise, the switching signal \( \sigma \) is increased by one and the adaptive parameter \( \hat{\Theta}(\sigma(t)) \) is updated by (10). Meanwhile, we reset \( \eta \) to make sure it is larger than \( V \). This will avoid the situation where the parameters are updated repeatedly.

In more detail, at each switching time \( t^m_s (m = 0, 1, 2...,) \), we reset \( \eta \) such that \( V < \eta \) and \( \mathcal{S}(\cdot) < 0 \). Since \( x, \eta \) are continuous if \( \eta \) is not reset, there exists a small time interval \([t^m_s, t^m_s + \epsilon)\) such that \( \mathcal{S}(\cdot) < 0 \) holds where \( \epsilon > 0 \) is a small constant. This means that the switching signal \( \sigma(t) \) is right continuous, i.e., \( \sigma(t) \) will not change on \([t^m_s, t^m_s + \epsilon)\). This will avoid the chattering phenomenon and guarantee that the switching times are strictly increasing.

The purpose of the algorithm is to let \( \mathcal{S}(\cdot) = V(\cdot) - \eta(t) \leq 0 \) hold forever after finite switching times. Given that \( \eta(t) \) will become zero in finite time by [16], so will \( V(\cdot) \) and \( x \).

Note that the finite switching times are possible because there exists a sufficiently large integer \( \sigma \) such that (13) holds when \( x \in (-\chi, \chi) \). Then by Lemma 4 and (16) with appropriate initial condition, we can show \( \mathcal{S}(\cdot) = V(\cdot) - \eta(t) \leq 0 \) holds after finite switching times (let \( x(t) = \eta(t), y(t) = V(t), b(t) = 0 \) in Lemma 4). We will prove the above idea in the following result.

**Theorem 1.** Consider the nonlinear systems in [6]. Then the controller [9] with Algorithm 1 can make the state \( x \) converge to zero in finite time.

**Proof.** Define a time sequence \( \{0 = t^0_s < t^1_s < ... < t^m_s < ... \} \) with \( m \in \{0, 1, 2, ...\} \). \( t^m_s \) denotes switching time, i.e.,

\[
t^{m+1}_s = \inf \{t | t \geq t^m_s, \mathcal{S}(\cdot) > 0 \}.
\]  

(17)

From Algorithm 1, we also know that this is the time instant when \( \sigma, \hat{\Theta} \) update their values. Meanwhile, during time interval \([t^m_s, t^{m+1}_s)\), no switching occurs and the supervisory function satisfies \( \mathcal{S}(\cdot) = V - \eta \leq 0 \). The proof is then divided into the following three claims. Fig. 4 shows one possible variation of \( \sigma, V, \eta \).

**Claim 1a.** Given any finite integer \( m \), on time interval \([t^m_s, t^{m+1}_s)\), \( |x(t)| < \chi \) and \( \eta, V \) are bounded.

**Proof.** \( \eta \) is non-increasing on \([t^m_s, t^{m+1}_s)\) because \( \dot{\eta}(t) = -a_\eta/(1+\alpha) \leq 0 \) where we have used the fact that \( 0 \leq V \leq \eta \) between two switching times. Moreover, from Algorithm 1-4) in Switching logic, at each switching time \( t^m_s \), \( \eta \) will be increased by a finite value. Hence, we can conclude that \( \eta(t) \) is bounded on \([t^m_s, t^{m+1}_s)\) for any finite \( m \). Since \( 0 \leq V \leq \eta \) on \([t^m_s, t^{m+1}_s)\), \( V \) is bounded. Finally, according to the barrier Lyapunov function [7], we can conclude that \( |x(t)| < \chi \).

**Claim 1b.** The switching times are finite. Moreover, \( |x(t)| < \chi \) and \( \eta, V \) are bounded on \([0, +\infty)\).
Algorithm 1 Logic-based switching law.

**Initialization**

At $t = 0$.

1. *Set initial design parameters.* Set $K, U, a, \varepsilon$ where $\varepsilon$ is a small positive constant;
2. *Set initial values.* Let $\sigma(0) = 1$, $\chi(0) > s(0)$, $\eta(0) = V(0) + \varepsilon$;
3. *Implement control effort $u(0)$ by (9).*

**Switching logic**

while $t > 0$

1. *Compute supervisory function.* Obtain the current states $x(t)$ and compute $V, \eta, S$ by (7), (16) and (15);
2. *Verify parameters update condition.* Check whether or not $S(\cdot) > 0$. If $S(\cdot) > 0$, go to 3); otherwise $\hat{\Theta}$ is not updated, i.e., $\sigma(t) = \sigma(t^-)$, go to 5);
3. *Update $\hat{\Theta}$. If $S(\cdot) > 0$, let $\sigma(t) = \sigma(t^-) + 1$ and compute $\hat{\Theta}$ by (10);
4. *Reset $\eta$. Update $\eta(t) = V + \varepsilon$ to make $S(\cdot) < 0$;
5. *Implement control effort $u(t)$.* Use the updated parameter $\hat{\Theta}$ to implement control effort $u$ by (9).

end

**Proof.** This is proved by contradiction. If the claim is not true, then the switching times can be infinite. Note that by Claim 1a, $|x(t)| < \chi$ on $[t^n_s, t^{m+1}_s)$ with finite integer $m$. Then there must exist a sufficiently large $m'$ such that (13) holds on $[t^n_s, t^{m'+1}_s)$. Since $\eta$ is determined by (16) with $\eta(t^n_s') > V(t^n_s')$, by Lemma 4 we can see $\eta$ will always be larger than $V$ when $t \geq t^n_s'$ without resetting $\eta(t)$. This indicates that the switching times cannot be infinite, thereby contradicting the assumption. Since the switching times are finite, we can conclude $|x(t)| < \chi$ and $\eta, V$ are both bounded on $[0, +\infty)$.

**Claim 1c.** The state $x$ converges to zero in finite time.

**Proof.** Since the switching times are finite, then there exists a switching time $t^n_s'$ such that $0 \leq V \leq \eta$ always holds on $[t^n_s', \infty)$. Solving (16) and noting $\eta(t^n_s')$ is bounded by Claim 1b, we can conclude $V$ will converge to zero in finite time.

4. Finite time controller design for high order nonlinear system

In this section, we will extend the result in Section 3 to the high order nonlinear system (1). Specifically, we will design a controller in the following form to achieve finite time stability.

$$u = \phi(x_n, \hat{\vartheta})$$
where $\phi(\cdot)$ is a known function to be designed, $\dot{\theta}$ are adaptive parameters.

The controller is constructed in two steps. In Section 4.1, we will mainly present the controller structure $\phi(\cdot)$, which contains adaptive parameters $\dot{\theta}$. Section 4.2 will focus on the logic-based switching law for the adaptive parameters. Note that all the adaptive parameters are updated in a discrete manner. Therefore, for the controller design phase, the adaptive parameters are regarded as constants.

### 4.1. Controller design

It is noted that the high order system [1] can be regarded as a cascade of $n$ first order subsystems. The controller design for this class of system is often based on backstepping or adding a power integrator technique (Fu et al., 2017). Let us consider the following change of coordinate:

\[ s_i \triangleq x_i, \]

\[ s_i \triangleq x_i^{1/q_2} - x_i^{1/q_1} \quad (i = 2, 3, \ldots, n) \tag{18} \]

where $q_2 \triangleq \alpha \in \left( \frac{1}{2}, 1 \right)$ is a ratio of odd integers, $q_{i+1}(i = 2, 3, \ldots, n)$ are determined by $q_{i+1} = \alpha - 1 + q_i (2 \leq i \leq n)$. These parameters are accounting for the power of Lyapunov function, which result in a form like [13].

Note that $x_{i+1}^*(i = 1, 2, \ldots, n)$ are the virtual or real control efforts with $x_{n+1}^* \triangleq u$. $x_{i+1}^*$ can be regarded as the control effort for the first order subsystem $\dot{x}_i = h_i(\pi_i)x_i + f_i(\pi_i)$ in [1]. They will be designed successively in the following $n$ steps. The purpose is to make all $s_i$ converge to zero in finite time.

**Step 1.** This step is similar to the controller design in Section 3. Take the following barrier Lyapunov function

\[ V_1 = \frac{1}{2} \ln \left( \frac{\hat{x}_1^2}{\hat{x}_1^2 - s_1^2} \right) \tag{19} \]

where $\hat{x}_1 > 0$ is a positive constant. We assume $|s_1| < \hat{x}_1$ in the following.

Differentiating $V_1$ with respect to time and using [1], Lemma [3] we have

\[ \dot{V}_1 \leq \frac{s_1 h_1 x_2^* + f_1(x_1)}{\hat{x}_1^2 - s_1^2} + \frac{s_1 h_1(x_2 - x_2^*)}{\hat{x}_1^2 - s_1^2} \]
\[ \leq \frac{s_1 h_1 x_2^* + s_1^2 \psi_1(x_1)}{\hat{x}_1^2 - s_1^2} + \frac{s_1 h_1(x_2 - x_2^*)}{\hat{x}_1^2 - s_1^2} \tag{20} \]

where $\psi_1(x_1)$ is an unknown function.

By the inequalities in Lemmas [12] and [18], it follows that

\[ \dot{V}_1 \leq \frac{s_1 h_1 x_2^*}{\hat{x}_1^2 - s_1^2} + \frac{U_1 s_1^2 \psi_1/U_1}{\hat{x}_1^2 - s_1^2} \]
\[ + \frac{h_1^{1+q_2} c_1 s_1^{1+q_2}}{(\hat{x}_1^2 - s_1^2)^{1+q_2}} + c_12 s_1^{1+q_2} \]
\[ = \frac{s_1 h_1 x_2^*}{\hat{x}_1^2 - s_1^2} + \frac{U_1 s_1^{1+q_2} F_1(x_1, \hat{x}_1)}{(\hat{x}_1^2 - s_1^2)^{2+2\alpha}} + c_12 s_1^{1+\alpha} \tag{21} \]
where \( c_{11}, c_{12} \) are positive known constants, \( F_1(x_1, \dot{x}_1) = s_1^{1-q_2} \dot{\psi}_1(\bar{\chi}_1^2 - s_1^2)^{1+2\alpha}/U_1 + h_1^{1+q_2} c_{11}(\bar{\chi}_1^2 - s_1^2)^{1+\alpha} \) is an unknown function.

Then, the virtual control effort \( x_2^* \) is in the form of (20) and designed as

\[
x_2^* = \dot{\Theta}_1 \left[ -K_1 s_1^{q_2} - \frac{U_1 s_1^{q_2}}{(\bar{\chi}_1^2 - s_1^2)^{1+2\alpha}} \right]
\]

(22)

where \( K_1, U_1 \) are positive constants, \( \dot{\Theta}_1(\sigma_1(t)) \) is an adaptive parameter. It will vary according to a switching signal \( \sigma_1(t) \).

\[
\dot{\Theta}_1 = (-1)^{\sigma_1(t)} \theta_1(\sigma_1(t))
\]

(23)

where \( \sigma_1(t) : [0, +\infty) \to \mathbb{N} \) is a piecewise non-decreasing switching signal, \( \theta_1(\sigma_1) : \mathbb{N} \to \mathbb{R} \) is an increasing function with respect to \( \sigma_1 \) such that \( \theta_1(0) > 0 \) and \( \theta_1(\sigma_1) \to +\infty \) as \( \sigma_1 \to +\infty \). The detail switching law for \( \sigma_1(t) \) will be given in Section 4.2. For now, one can assume \( \dot{\Theta}_1(\sigma_1(t)) \) is fixed and regarded it as a positive constant.

Substituting (22) into (21), we get

\[
\dot{V}_1 \leq -\frac{K_1^n s_1^{1+\alpha}}{\lambda_1^2 - s_1^2} + \frac{(K_1^n - h_1 K_1 \dot{\Theta}_1) s_1^{1+\alpha}}{\lambda_1^2 - s_1^2} + \frac{U_1 (F_1(x_1, \dot{x}_1) - h_1 \dot{\Theta}_1) s_1^{1+\alpha}}{(\bar{\chi}_1^2 - s_1^2)^{2+2\alpha}} + c_{12} s_2^{1+\alpha}
\]

(24)

where \( K_1^n \) is a positive constant.

**Remark 2.** (24) is similar to (11) in the first order case but with an extra term \( c_{12} s_2^{1+\alpha} \). This will result in a different \( \eta \)-auxiliary system in logic-based switching law (see Section 4.2).

**Step i** (2 \( \leq i \leq n \)). Consider the following Lyapunov function

\[
V_i = \int_{x_i^*}^{x_i} \frac{v_i^{2-q_i}(\tau)}{\lambda_i^2(\tau) - v_i^2(\tau)} d\tau
\]

(25)

where \( v_i(\tau) = \tau^{1/q_i} - x_i^{1/q_i}, \hat{\chi}_i(t) \in [0, +\infty) \to \mathbb{R} \) is a piecewise constant adaptive parameter. It will be updated according to the switching logic in Section 4.2. For now, one can regard it as a positive constant.

The above Lyapunov function is in fact a class of barrier Lyapunov function (Zheng & Li, 2018). Based on this Lyapunov function we try to make the virtual control error \( s_i \) converge to zero. Similar to the barrier Lyapunov function in (19), it has following properties.

**Proposition 1.** Suppose \( \hat{\chi}_i(t) \) is a positive constant and \( |s_i| < \hat{\chi}_i \). Then \( V_i \) has the following properties:

1) \( V_i \) is positive definite;

2) \( V_i \leq \frac{2 \hat{\chi}_i^2}{\lambda_i^2 - s_i^2} \);

(26)

3) If \( x_i^* \) is bounded, then \( |s_i| \to \hat{\chi}_i, V_i \to +\infty \).
Proof. The proof can be obtained by following the line of Zheng and Li (2018).

Next, similar to (8) and (21), we present the following property for $\dot{V}_i$.

**Proposition 2.** Suppose $\hat{\Theta}_i(t), \chi_i(t)$ are both positive constant vector and $|s_i| < \dot{\chi}_i$ where $\Theta_{i-1} \triangleq (\hat{\Theta}_1, \hat{\Theta}_2, ..., \hat{\Theta}_{i-1})^T$, $\dot{\chi}_i \triangleq (\dot{\chi}_1, \dot{\chi}_2, ..., \dot{\chi}_{i-1})^T$. Then the time derivative of $V_i$ is given by

$$
\dot{V}_i \leq \frac{s_i^2 - q_i x_{i+1}}{\lambda_i^2 - s_i^2} + \frac{U_i s_i^{1+\alpha} F_i(x_i, \Theta_{i-1}, \chi_{i-1})}{(\lambda_i^2 - s_i^2)^{2+2\alpha}} \\
+ \sum_{j=1}^{i-1} c_{ij} s_j^{1+\alpha} + c_{i,i+1} s_{i+1}^{1+\alpha}
$$

(27)

where $F_i(x_i, \Theta_{i-1}, \chi_{i-1})$ is an unknown non-negative function, $c_{ij} (j = 1, 2, ..., i + 1)$ and $U_i$ are all known positive constants, $s_{n+1} \triangleq 0$.

**Proof.** The proof can be obtained by following the line of Zheng and Li (2018). It is a direct computation of $\dot{V}_i$ and utilization of Lemmas 1-2.

Then, the control effort $x_{i+1}^*$ is in the form of (9) and designed as:

$$
x_{i+1}^* = \hat{\Theta}_i \left[ -K_i s_i^{q_i+1} - \frac{U_i s_i^{q_i+1} F_i(x_i, \Theta_{i-1}, \chi_{i-1})}{( \lambda_i^2 - s_i^2)^{1+2\alpha}} \right]
$$

(28)

where $K_i$ is a positive constant, $\hat{\Theta}_i(\sigma_i(t))$ is an adaptive parameter given by

$$
\hat{\Theta}_i = (-1)^{\sigma_i(t)} \theta_i(\sigma_i(t))
$$

(29)

$\sigma_i(t) \in [0, +\infty) \rightarrow N$ is a switching signal. $\theta_i(\sigma_i)$ is an increasing function with $\theta_i(0) > 0$ and $\theta_i(\sigma_i) \rightarrow +\infty$ as $\sigma_i \rightarrow +\infty$.

Substituting (28) into (27), we get

$$
\dot{V}_i \leq - \frac{K_i' s_i^{1+\alpha}}{\lambda_i^2 - s_i^2} + \frac{(K_i' - h_i K_i \hat{\Theta}_i) s_i^{1+\alpha}}{\lambda_i^2 - s_i^2} \\
+ \frac{U_i s_i^{1+\alpha} F_i(x_i, \Theta_{i-1}, \chi_{i-1}) - h_i \hat{\Theta}_i}{(\lambda_i^2 - s_i^2)^{2+2\alpha}} \\
+ \sum_{j=1}^{i-1} c_{ij} s_j^{1+\alpha} + c_{i,i+1} s_{i+1}^{1+\alpha}
$$

(30)

where $K_i'$ is a positive constant. The above inequality is similar to (24) and (11).

**Remark 3.** Note that $V_i (i = 1, 2, ..., n)$ are all barrier Lyapunov functions. By using these functions, we expect to constrain all the virtual control errors and states. Thus, similar to the first order case, there exist
unknown positive constants \( F_i, h_i \) such that \( F_i(x_i, \hat{\Theta}_{i-1}, \hat{\chi}_{i-1}) \leq F_i \) and \( |h_i| \geq h_i > 0 \). In this way, we can deal with the unstructured uncertainties like structured uncertainties. Namely, (30) can be written as

\[
\dot{V}_i \leq -K'_i s_i^{1+\alpha} + (K_i - h_i K_i \hat{\Theta}_i) s_i^{1+\alpha} + \sum_{j=1}^{i-1} c_{ij} s_j^{1+\alpha} + c_{i,i+1} s_{i+1}^{1+\alpha}
\]

where \( |h_i| \geq h_i > 0 \). The unknown functions/unstructured uncertainties \( F_i(x_i, \hat{\Theta}_{i-1}, \hat{\chi}_{i-1}) \) become an unknown parameter \( F_i \). Then according to (29), there exists a sufficiently large \( \sigma_j \) such that \( K'_i - h_i K_i \hat{\Theta}_i < 0 \) and \( F_i - h_i \hat{\Theta}_i < 0 \). Hence, the uncertainties can be canceled (see the proof of Claim 2b in Section 5 for details).

**Remark 4.** When the unknown function \( F_i(x_i, \bar{\Theta}_{i-1}, \bar{\chi}_{i-1}) \) is replaced by an unknown parameter \( F_i \), the complexity of the controller can be reduced. In fact, in traditional finite time control (Huang & Xiang, 2016), the form of the nonlinearities \( F_i(x_i, \bar{\Theta}_{i-1}, \bar{\chi}_{i-1}) \) is used to design the virtual control effort \( x_{i+1} \). Since \( F_i(x_i, \bar{\Theta}_{i-1}, \bar{\chi}_{i-1}) \) becomes very complicated as the order of the system increases, the complexity of the controller \( x_{i+1} \) will explode. Nevertheless, in our case, we only need to use an adaptive parameter \( \hat{\Theta}_i \) to cancel the nonlinearities \( F_i(x_i, \bar{\Theta}_{i-1}, \bar{\chi}_{i-1}) \) as discussed in Remark 3. This will result in a simple controller structure as (22) and (23). There is no need to compute some complex nonlinear functions (Huang et al., 2016) or adopt approximation methods, such as neural networks or fuzzy logic (Liu et al., 2019; Zheng, Shi, Wang, & Shi, 2019; Zhao, Li, Wang, & Fan, 2017). Note that though the parameters updating rules are a bit complicated, they can be implemented in a host computer like Lu, Zheng, Tang, and Song (2013). This will further reduce the computational burden of the proposed controller design.

**Remark 5.** Note that \( s_i \) in (18) is expressed as \( s_i \Delta x_i^{1/q_i} - x_i^{*1/q_i} \). Since \( x_i^* \) contains \( \hat{\Theta}_{i-1}(\sigma_{i-1}) \) by (28), it is discontinuous with respect to time. This implies that \( s_i \) is also discontinuous with respect to time. Therefore, it is possible that at some time instants, \( s_i \) will jump outside the barrier \( \chi_i \). Hence, \( \chi_i \) also needs to be updated. This is a key difference with the existing barrier Lyapunov methods (Liu et al., 2018; Tee, Ge, & Tay, 2009).

### 4.2 Logic-based switching law and main results

We will present the algorithm for updating adaptive parameters \( \hat{\Theta}_i \) and \( \hat{\chi}_i \) for \( i \in \{1, 2, ..., n\} \). First, for \( i \in \{1, 2, ..., n\} \), define the following new supervisory functions \( S_i(\cdot) \).

\[
S_i(\cdot) \triangleq V_i(\cdot) - \eta_i(t),
\]

\[
\eta_i(t) = -a_i \eta_i^{(1+\alpha)/2} - Q_i s_i^{1+\alpha} + \sum_{j=1}^{i-1} c_{ij} s_j^{1+\alpha} + c_i s_i^{1+\alpha}
\]
where $V_i$ is given by (19) and (25), $\eta_i$ is an auxiliary variable, $a_i, Q_i > 0$ are design parameters.

Based on the above supervisory functions, we present the parameters update algorithm. Please see Algorithm 2 in Table I and Fig. 2. Then we have the following main result.

**Theorem 2.** Consider the nonlinear system in (1). Then the controller (22), (28) with Algorithm 2 can guarantee that:

1) All the signals in the closed-loop system are bounded with the output $|y(t)| < \hat{\chi}_i$ for $\forall t \in [0, +\infty)$, and;

2) All the states will converge to zero in finite time.

The detail proof for the above result is put in the next subsection. For now, we will give some insights into this algorithm.

The idea of the algorithm is similar to Algorithm 1 for first order nonlinear system with additional adaptive parameters $\hat{\chi}_i$. First, we verify whether or not the supervisory functions $S_i(\cdot)$ are larger than zero. If yes, then parameters need to be updated. Second, note that as long as $\hat{\Theta}_i$ is updated, The virtual control error $s_i$ will change accordingly. This may make $s_i$ jump at the updating moment. Therefore, the barrier $\hat{\chi}_i$ also needs to be updated. In this way, we can guarantee that $\hat{\chi}_i$ is always larger than $s_i$ when $\hat{\Theta}_i$ finishes its updating. Third, we reset $\eta_i$ to make sure it is larger than $V_i$ when $\hat{\Theta}_i, \hat{\chi}_i$ have been updated. This will make the supervisory functions less than zero, thus avoiding the situations such that the parameters are updated repeatedly.

The purpose of Algorithm 2 is to let $S_i(\cdot) = V_i(\cdot) - \eta_i(t) \leq 0$ hold forever after finite switching times. In fact, when $S_i(\cdot) = V_i(\cdot) - \eta_i(t) \leq 0$ holds, let $\bar{\eta} \triangleq \sum_{i=1}^{n} \eta_i$, from (32) we have

\[
\bar{\eta} = -\sum_{i=1}^{n} a_i \eta_i^{(1+\alpha)/2} - \sum_{i=1}^{n} Q_i s_i^{1+\alpha} + \sum_{i=1}^{n} \sum_{j=1}^{i-1} c_{ij} s_j^{1+\alpha} + \sum_{i=1}^{n} c_{i,i+1} s_{i+1}^{1+\alpha}
\]

\[
= -\sum_{i=1}^{n} a_i \eta_i^{(1+\alpha)/2} - \sum_{i=1}^{n} \left( Q_i - \sum_{j=i+1}^{n} c_{ji} - c_{i-1,i} \right) s_i^{1+\alpha}
\]  

(33)

where $s_{n+1} \triangleq 0$, $\sum_{j=n+1}^{n} c_{ji} \triangleq 0$, $c_{0n} \triangleq 0$. It can be seen that when $Q_i$ is sufficiently large such that $Q_i - \sum_{j=i+1}^{n} c_{ji} - c_{i-1,i} > 0$, by Lemma 2, we have

\[
\bar{\eta} \leq -\sum_{i=1}^{n} a_i \eta_i^{(1+\alpha)/2} \leq -a' \bar{\eta}^{(1+\alpha)/2}
\]

(34)

where $a' > 0$ is a positive constant. This means $\bar{\eta}$ may converge to zero when the initial value is bounded. Due to $S_i(\cdot) = V_i(\cdot) - \eta_i(t) \leq 0$, $V_i, x_i$ will converge to zero in finite time.

Also note that the finite switching times are possible. Take (24) for example, when $x_1$ is bounded, similar to the first order case, there exists a sufficiently large integer $\sigma_1$ such that $\dot{V}_1 \leq -a_1 V_1^{1+\alpha} - Q_1 s_1^{1+\alpha} + c_{12} s_2^{1+\alpha}$.
Algorithm 2 Logic-based switching law

**Initialization**

At $t = 0$.

1. **Set initial design parameters.** For $i = 1, 2, ..., n$, set $K_i, U_i; r_i, t_i; a_i, Q_i, c_i (j = 1, 2, ..., i), \varsigma, \varepsilon$ where $\varsigma > 0$ and $\varepsilon$ is an arbitrary small positive constant;

2. **Set initial values.** For $i = 1, 2, ..., n$, let $\sigma_i(0) = 1; \chi_i(0) > s_i(0); \eta_i(0) = V_i(0) + \varepsilon$;

3. **Implement control effort** $u(0)$ by (22), (28).

**Switching logic**

while $t > 0$

1. **Compute supervisory functions.** Obtain the current states $\pi_n(t)$. For $i = 1, 2, ..., n$, compute $V_i, \eta_i, S_i$ by (19), (25), (32) and (31);

2. **Verify parameters update conditions.** For $i = 1, 2, ..., n$, check whether or not $S_i(\cdot) > 0$. If $S_i(\cdot) > 0$ for some $i \in \{1, 2, ..., n\}$, go to 3; otherwise $\hat{\Theta}_i, \hat{\chi}_i(i = 1, 2, ..., n)$ are not updated, i.e., $\sigma_i(t) = \sigma_i(t^-)$, $\hat{\chi}_i(t) = \hat{\chi}_i(t^-)$ go to 6;

3. **Update $\hat{\Theta}_i$.** For $i = 1, 2, ..., n$, if $S_i(\cdot) > 0$, let $\sigma_i(t) = \sigma_i(t^-) + 1$ and compute $\hat{\Theta}_i$ by (23) or (29); otherwise, $\sigma_i(t), \hat{\Theta}_i$ are not updated;

4. **Update $\hat{\chi}_i$.** For $i = 1, 2, ..., n$, recompute $s_i$, if $|s_i| \geq |\hat{\chi}_i|$, update $\hat{\chi}_i(t) = |s_i| + \varsigma$ to make $|s_i| < \hat{\chi}_i$; otherwise $\hat{\chi}_i$ is not updated;

5. **Reset $\eta_i$.** For $i = 1, 2, ..., n$, recompute $V_i$, if $V_i \geq \eta_i$, update $\eta_i(t) = V_i + \varepsilon$ to make $S_i(\cdot) < 0$; otherwise $\eta_i$ is not updated;

6. **Implement control effort** $u(t)$. Use updated parameters to implement control effort $u$ by (22), (28).

end

Then by Lemma 4 and (32) with appropriate initial condition, we can show $S_i(\cdot) = V_i(\cdot) - \eta_i(t) \leq 0$ holds after finite switching times.

4.3. Proof of Theorem 3

This subsection will focus on the proof of Theorem 2. Similar to the first order case in Section 3, we know at each switching time, $\eta_i$ is reset to make $S_i(\cdot) > 0$. Then we can define a switching time sequence \( \{0 = t_0^s < t_1^s < ... < t_m^s < ...\} \) such that

$$
t_{m+1}^s = \inf \{ t | t \geq t_m^s, S_i(\cdot) > 0, i \in \{1, 2, ..., n\} \}.
$$

From Algorithm 2, we also know that this is the time instant when some $\hat{\Theta}_i, \hat{\chi}_i(i = 1, 2, ..., n)$ update their values. Meanwhile, during time interval $[t_m^s, t_{m+1}^s)$, no switch occurs and the supervisory function satisfies...
$S_i(t) = V_i - \eta_i(t) \leq 0$ for $\forall i = 1, 2, ..., n$. This also implies that $\hat{\Theta}_i, \hat{\chi}_i (i = 1, 2, ..., n)$ are all constants on $[t^m_s, t^{m+1}_s)$.

Similar to the proof of Theorem 1, the proof will be obtained by proving the following three claims, i.e., Claims 2a, 2b and 2c. Claim 2a is like Claim 1a in the proof of Theorem 1. It tries to show the boundedness of signals in the system if the switching times are finite. Claims 2b-2c are similar to Claims 1b-1c. They attempt to show the switching times are finite and the system is finite time stable. Compared with the proof of Theorem 1 since a high order system is considered, it will become much more complicated.

Claim 2a. For a given finite integer $m$, we have

1) The closed loop nonlinear system admits a continuous solution $x_n(t)$ on $[t^0_s, t^{m+1}_s]$;

2) There exists a positive constant $\delta^m_i$ such that $|s_i(t)| \leq \hat{\chi}_i(t^m_s) - \delta^m_i$ for $\forall i = 1, 2, ..., n$ and all the signals in the system are bounded on $[t^m_s, t^{m+1}_s]$.

The detail proof for the above claim is put in Appendix B. Here, we give some intuition for the result. Similar to Claim 1a for the first order system, note that from (34), we can see the auxiliary system $\eta(t) = \sum_{i=1}^n \eta_i(t)$ is non-increasing during time interval $[t^m_s, t^{m+1}_s]$. Meanwhile, at each switching time $t^m_s$, the adaptive parameters and $\eta_i(t)$ are only increased by some finite values according to Algorithm 2. Therefore, $\eta(t)$ must be bounded on $[0, t^{m+1}_s]$. Due to $S_i(t) = V_i - \eta_i(t) \leq 0$, we can show $V_i(i = 1, 2, ..., n)$ are all bounded. Finally, according to the properties of barrier Lyapunov functions in (19) and (25), we have $|s_i(t)| < \hat{\chi}_i(t^m_s)$ and all the signals are bounded on $[t^m_s, t^{m+1}_s]$ with any finite integer $m$.

Claim 2b. 1) The switching times are finite;

2) The closed loop nonlinear system admits a continuous solution solution $x_n(t)$ on $[0, +\infty)$;

3) There exist positive constants $\chi_i, \delta_i$ such that $|s_i(t)| \leq \chi_i - \delta_i$ for $\forall i = 1, 2, ..., n$ and all the signals in the system are bounded on $[0, +\infty)$.
Proof. We will first show there are only finite switching times. We will successively prove \( \hat{\chi}_i, \hat{\Theta}_i (i = 1, 2, ..., n) \) only have finite switching times. The proof is divided into the following steps. Note that these steps correspond to the \( n \) steps controller design procedures in Section 4.1.

**Step 1.** Show \( \hat{\chi}_1, \hat{\Theta}_1 \) have finite switching times.

1) We will show \( \hat{\chi}_1 \) does not switch on \([0, t_{s1}^{m+1})\).

Note that according to Claim 2a, we have \(|s_1(t)| ≤ \hat{\chi}_1(t_{s1}^m) - \delta_1^m\) on \([t_{s1}^m, t_{s1}^{m+1})\) for any finite integer \( m \), and \( s_1(t) = x_1(t) \) is continuous on \([0, t_{s1}^{m+1})\). Then according to Algorithm 2-4) in Switching logic, \( \hat{\chi}_1 \) will not be updated since \( s_1(t) \) will never transgress the barrier \( \hat{\chi}_1(0) \). That is for \( \forall t \in [0, t_{s1}^{m+1}) \), \( \hat{\chi}_1(t) = \hat{\chi}_1(t_{s1}^m) = \hat{\chi}_1(t_{s1}^{m-1}) = \cdots = \hat{\chi}_1(0) \).

2) We will show \( \hat{\Theta}_1 \) has finite switching times.

This is proved by contradiction. If this is not true, then \( \hat{\Theta}_1 \) will switch infinite times.

From Claim 2a and the fact that \( \hat{\chi}_1(t) = \hat{\chi}_1(0) \), we have \(|s_1(t)| < \hat{\chi}_1(0)\) holds on \([0, t_{s1}^{m+1})\). It follows that on \([0, t_{s1}^{m+1})\), the unstructured uncertainties \( h_1(x_1) \) and \( F_1(x_1, \hat{\chi}_1) \) in (24) satisfy

\[
|h_1(x_1)| ≥ h_1 > 0, \quad 0 ≤ F_1(x_1, \hat{\chi}_1) ≤ F_1
\]

where \( h_1, F_1 \) are unknown constants irrelevant with switching times \( m \).

Then from the tuning rule (23), we can conclude that there exists a sufficiently large finite integer \( m_1 \) such that at time instant \( t_{s1}^{m_1} \), we have

\[
\text{sgn}(\hat{\Theta}_1(\sigma_1)) = \text{sgn}(h_1(x_1)), \quad K'_1 - h_1K_1\hat{\Theta}_1(\sigma_1) ≤ K'_1 - h_1K_1\hat{\Theta}_1(\sigma_1) < 0, \quad F_1(x_1, \hat{\chi}_1) - h_1\hat{\Theta}_1(\sigma_1) ≤ F_1 - h_1\hat{\Theta}_1(\sigma_1) < 0.
\]

This implies that (24) will become

\[
\dot{V}_1 ≤ -a_1V_1^{\frac{1+\alpha}{2}} - Q_1s_1^{1+\alpha} + c_{12}s_2^{1+\alpha}. \tag{36}
\]

On the other hand, the auxiliary variable \( \eta_1(t) \) in (32) satisfies

\[
\dot{\eta}_1 = -a_1\eta_1^{\frac{1+\alpha}{2}} - Q_1s_1^{1+\alpha} + c_{12}s_2^{1+\alpha} \tag{37}
\]

where \( \eta_1(t_{s1}^{m_1}) > V_1(t_{s1}^{m_1}) \) according to Algorithm 2-5) in Switching logic.

From Lemma [4] we know \( V_1(t) ≤ \eta_1(t) \) will hold on \([t_{s1}^{m_1}, t_{s1}^{m+1})\) for any \( m + 1 > m_1 \) without resetting \( \eta_1(t) \). This means that \( \hat{\Theta}_1 \) will not be updated after \( t_{s1}^{m_1} \), which contradicts the fact that \( \hat{\Theta}_1 \) has infinite switching times.

**Step 2.** Show \( \hat{\chi}_2, \hat{\Theta}_2 \) have finite switching times.
The proof will be conducted on $[t^{m_1}_s, t^{m+1}_s)$ such that $t^{m_1}_s$ denotes the time instant when $\hat{\chi}_1, \hat{\Theta}_1$ stop switching.

1) We will show $\hat{\chi}_2(t)$ does not switch on $[t^{m_1}_s, t^{m+1}_s)$. Note that $\hat{\Theta}_1$ is not updated on $[t^{m_1}_s, t^{m+1}_s)$. Meanwhile, according to Claim 2a, we know $x_2$ is continuous on $[t^{m_1}_s, t^{m+1}_s)$. Then by (15) and (22), we know $s_2$ is also continuous on $[t^{m_1}_s, t^{m+1}_s)$. Also by Claim 2a, we have $|s_2(t)| \leq \hat{\chi}_2(t^{m_1}_s) - \delta_2^m$ on $[t^{m_1}_s, t^{m+1}_s)$ for any finite integer $m$. Therefore, according to Algorithm 2-4) in Switching logic, $\hat{\chi}_2$ will not be updated on $[t^{m_1}_s, t^{m+1}_s)$ since $s_2(t)$ will never transgress the barrier $\hat{\chi}_2(t^{m_1}_s)$.

2) We will prove $\hat{\Theta}_2(t)$ has finite switching times. This is proved by contradiction. We suppose $\hat{\Theta}_2(t)$ has infinite switching times. First, since $\hat{\chi}_2$ will not be updated on $[t^{m_1}_s, t^{m+1}_s)$, we have $|s_2(t)| < \hat{\chi}_2(t^{m_1}_s)$. In addition, from Claim 2a, we know $\hat{\chi}_2(t^{m_1}_s)$ is bounded.

Using (37) in Step 1 we know $\eta_1$ satisfies
\[ \dot{\eta}_1 = -a_1\eta_1^{1+\alpha} - Q_1s_1^{1+\alpha} + c_1s_2^{1+\alpha} \leq -a_1\eta_1^{1+\alpha} + c_1s_2^{1+\alpha}(t^{m_1}_s) \]
on $[t^{m_1}_s, t^{m+1}_s)$. Therefore, it can be concluded that $V_1 \leq \eta_1$ is bounded by a constant irrelevant with $m$ (see Corollary 1 in Yu et al. (2018)). Then from the barrier Lyapunov function (19), we know $|s_1(t)| \leq \hat{\chi}_1(0) - \delta_1$ with a positive constant $\delta_1$ irrelevant with $m$. Also from (22) and (18), we know $x_2^* x_2$ are both bounded by constants irrelevant with $m$.

Hence, we conclude that on $[t^{m_1}_s, t^{m+1}_s)$, $h_2(\bar{x}_2)$ and $F_2(\bar{x}_2, \hat{\Theta}_1, \hat{\chi}_1)$ in (27) satisfy
\[ h_2(\bar{x}_2) \geq h_3 > 0, \]
\[ 0 \leq F_2(\bar{x}_2, \hat{\Theta}_1, \hat{\chi}_1) \leq \bar{F}_2 \]
where $h_2, \bar{F}_2$ are positive constants irrelevant with $m$.

Then from tuning rule (29), there exists a finite integer $m_2 \geq m_1$ such that at time instant $t^{m_2}_s$, we have
\[ \text{sgn}(\hat{\Theta}_2(\sigma_2)) = \text{sgn}(h_2(\bar{x}_2)), \]
\[ K_1' - h_2K_3\hat{\Theta}_2(\sigma_2) \leq K'_2 - h_2K_3\hat{\Theta}_2(\sigma_2) < 0, \]
\[ F_2(\bar{x}_2, \hat{\Theta}_1, \hat{\chi}_1) - h_2\hat{\Theta}_2(\sigma_2) \leq \bar{F}_2 - h_3\hat{\Theta}_2(\sigma_2) < 0. \]
This implies that (30) will become
\[ \dot{V}_2 \leq -a_2V_2 - Q_2s_2^{1+\alpha} + c_{21}s_1^{1+\alpha} + c_{23}s_3^{1+\alpha}. \]
(38)

On the other hand, the auxiliary variable $\eta_2(t)$ in (32) satisfies
\[ \dot{\eta}_2 \leq -a_2\eta_2 - Q_2s_2^{1+\alpha} + c_{21}s_1^{1+\alpha} + c_{23}s_3^{1+\alpha} \]
\[ \eta_2(t_m^2) > V(t_m^2) \] according to Algorithm 2-5 in Switching logic.

From Lemma [4] we know \( V_2(t) \leq \eta_2(t) \) will hold on \( [t_m^{m+1}, t_m^{m+2}] \) for any \( m + 1 > m_2 \) without resetting. This means that \( \hat{\Theta}_2 \) will not be updated after \( t_m^{m+2} \) which contradicts the fact that \( \hat{\Theta}_2 \) has infinite switching times.

Step \( i (3 \leq i \leq n) \). By repeating the above procedures, we can show all the parameters \( \hat{\Theta}_i, \hat{\chi}_i (i = 1, 2, ..., n) \) have finite switching times.

Next, for Statements 2) and 3) in Claim 2b, according to Claim 2a, they hold naturally when switching times are finite. The proof is completed.

Claim 2c. All the states will converge to zero in finite time.

Proof. From Claim 2b, we know the switching times are finite. This indicates that there exists a finite time instant \( t_s^{m_n} \) such that when \( t \in [t_s^{m_n}, +\infty) \), \( V_i \leq \eta_i(t) \) holds for \( \forall i \in \{1, 2, ..., n\} \). Then on \( [t_s^{m_n}, +\infty) \), by (32)-(34) we have

\[ \dot{\eta}(t) \leq -a' \gamma(t) \]

where \( a' > 0 \) is a positive parameter, \( \gamma = \frac{1+\alpha}{2} \).

Therefore, during time interval \( [t_s^{m_n}, +\infty) \), we have

\[
0 \leq \sum_{i=1}^{n} V_i \leq \eta(t) \\
\leq \frac{\eta(t)}{1 - \gamma} = -a'(t - t_s^{m_n}) \]

Note that by Claim 2b, all the signals including \( \eta(t_s^{m_n}) \) are bounded on \([0, +\infty)\). Hence, we can conclude that after a finite time \( t_s^{m_n} + \frac{\eta(t_s^{m_n})}{a'(1 - \gamma)} \), \( V_i (i = 1, 2, ..., n) \) must be equal to zero and stay there. The proof is completed.

5. Extension to prescribed finite time control

In this section, we will give an extension to the controller designed in Section 4. Inspired by Song, Wang, Holloway, and Krstic (2017), a time-varying gain will be integrated into the controller. In this way, the states will be regulated to zero in prescribed finite time. In the following, the initial time is set zero for simplicity.

5.1. Controller design

Consider the following change of coordinate:

\[
\begin{align*}
    s_1 & \triangleq x_1, \\
    s_i & \triangleq x_i - x_i^*(i = 2, 3, ..., n)
\end{align*}
\]
where $\mu$ is a time-varying function defined on $[0, T)$ such that
$$
\mu(t) = \frac{T}{T - t} \geq 1
$$
with a prescribed positive constant $T > 0$.

**Step 1.** Take the following Lyapunov function
$$
V_1 = \frac{1}{2} \ln \left( \frac{\hat{\chi}_1^2}{\hat{\chi}_1^2 - s_1^2} \right)
$$
where $\hat{\chi}_1 > 0$ is a positive constant.

Then, similar to (21) in Section 4.1, by Young’s inequality in Lemma 1, we have
$$
\dot{V}_1 \leq \sum_{i=1}^{n} h_i x_i^2 + \sum_{i=1}^{n} s_i^2 \psi_1(x_1) + \sum_{i=1}^{n} h_i s_1 s_2
$$
$$
\leq \sum_{i=1}^{n} s_i h_i x_i^2 + \sum_{i=1}^{n} \frac{\mu^\beta U_1 s_i^2 \mathcal{F}_1(x_i, \hat{\chi}_1)}{\hat{\chi}_1^2 - s_i^2} + c_{12} s_i^2
$$
where $U_1, c_{12}$ and $\beta_1 \in (1, +\infty)$ are positive design parameters, $\mathcal{F}_1(x_i, \hat{\chi}_1)$ is an unknown function.

The virtual control effort $x_i^2$ is designed as:
$$
x_i^2 = \hat{\Theta}_1 \left[ -K_1 s_1 + \mu^\beta U_1 s_1 \right]
$$
where $\hat{\Theta}_1$ is defined as (23), $K_1$ is a positive design parameter.

Substituting (42) into (41), we have
$$
\dot{V}_1 \leq \sum_{i=1}^{n} \frac{\mu^\beta K'_1 s_i^2}{\hat{\chi}_1^2 - s_i^2} - \frac{h_1 K_1 \hat{\Theta}_1 s_1^2}{\hat{\chi}_1^2 - s_i^2}
$$
$$
+ \frac{U_1 (\mu^\beta \mathcal{F}_1(x_1, \hat{\chi}_1) - \mu^\beta h_1 \hat{\Theta}_1) s_i^2}{(\hat{\chi}_1^2 - s_i^2)^2} + c_{12} s_i^2
$$
where $K'_1 > 0$ is a positive constant. $\mathcal{F}_1(x_1, \hat{\chi}_1) = \mathcal{F}_1(x_1, \hat{\chi}_1) + K'_1 (\hat{\chi}_1^2 - s_i^2)/U_1$ is an unknown function.

The above inequality is similar to (24) in Section 4.1 but with extra time-varying gain $\mu^\beta$, which is used to regulate the states to zero in prescribed finite time.

**Step i (2 ≤ i ≤ n).** Take the following Lyapunov function
$$
V_i = \frac{1}{2} \ln \left( \frac{\hat{\chi}_i^2}{\hat{\chi}_i^2 - s_i^2} \right)
$$
where $\hat{\chi}_i(t) \in [0, +\infty) \to \mathbb{R}$ is a piecewise constant adaptive parameter. It will be updated according to the switching logic law in Section 6.2.

Then similar to Proposition 2 we have following result:
Proposition 3. Suppose $\Theta_{i-1}(t), \hat{\chi}_i(t)$ are both positive constant vector and $|s_i| < \hat{\chi}_i$ where $\Theta_{i-1} \triangleq (\hat{\Theta}_1, \hat{\Theta}_2, ..., \hat{\Theta}_{i-1})^T$, $\hat{\chi}_i \triangleq (\hat{\chi}_1, \hat{\chi}_2, ..., \hat{\chi}_{i-1})^T$. Then the time derivative of $V_i$ in (44) is given by

$$\dot{V}_i \leq \frac{\beta_i s_i \hat{\Theta}_i}{\chi_i^2 - s_i^2} + \frac{\beta_i U_i s_i}{\chi_i^2 - s_i^2} + \sum_{j=1}^{i-1} c_{ij} \mu_j s_j^2 + \frac{c_{i,i+1} s_{i+1}^2}{\mu_i}$$

where $F_i(x_i, \Theta_{i-1}, \chi_{i-1})$ is an unknown non-negative function, $\beta_j, c_{ij} (j = 1, 2, ..., i - 1)$ and $\beta_i, c_{i,i+1}, U_i$ are all known positive constants, $s_{n+1} \triangleq 0$.

**Proof.** The detail proof is in Appendix C in the supplementary file. It is a direct computation of $\dot{V}_i$ and utilization of Lemma 1.

Then the virtual control effort $x_{i+1}^*$ is designed as:

$$x_{i+1}^* = \hat{\Theta}_i \left[-K_i s_i - \frac{\beta_i U_i s_i}{\chi_i^2 - s_i^2} \right]$$

where $\hat{\Theta}_i$ is defined as (29), $K_i$ and $\beta_i \in (1, +\infty)$ are positive design parameters.

Substituting (46) into (45), we obtain the following inequality like (30) in Section 4.1

$$\dot{V}_i \leq \frac{\beta_i K_i \hat{s}_i^2}{\chi_i^2 - s_i^2} - \frac{h_i K_i \hat{\Theta}_i \hat{s}_i^2}{\chi_i^2 - s_i^2} + \frac{\beta_i U_i s_i}{\chi_i^2 - s_i^2} \left(F_i(x_i, \Theta_{i-1}, \chi_{i-1}) - h_i \hat{\Theta}_i \right) + \sum_{j=1}^{i-1} c_{ij} \mu_j s_j^2 + \frac{c_{i,i+1} s_{i+1}^2}{\mu_i}$$

where $K_i^*$ is a positive constant, $F_i(\cdot)$ is an unknown function.

5.2. Logic-based switching law

We will present the algorithm for updating adaptive parameters $\hat{\Theta}_i$ and $\hat{\chi}_i$ for $i \in \{1, 2, ..., n\}$. It is very similar to the algorithm in Section 4.2 but with time-varying gains.

First, for $i \in \{1, 2, ..., n\}$, define the following new supervisory functions $S_i(\cdot)$.

$$S_i(\cdot) \triangleq V_i(\cdot) - \eta_i(\cdot)$$

$$\dot{\eta}_i = - a_i \mu^\beta \eta_i - Q_i \mu^\beta \hat{s}_i^2 + \sum_{j=1}^{i-1} c_{ij} \mu^\beta s_j^2 + \frac{c_{i,i+1} s_{i+1}^2}{\mu_i} (i = 1, 2, ..., n - 1),$$

$$\dot{\hat{s}_i} = \dot{\eta}_i$$

$$\hat{s}_i = \mu^\beta \hat{s}_i$$
\[ \dot{\eta}_n = -a_n \mu \beta_n \eta_n - Q_n \eta_n^2 + \sum_{j=1}^{n-1} c_{nj} \mu \beta_j \eta_j^2 \]  

(50)

where \( V_i \) is given by (19) and (25), \( \eta_i \) is an auxiliary variable, \( a_i, Q_i > 0 \) are design parameters.

Based on the above supervisory functions, we present the parameters update algorithm. The algorithm is also given by Table I and Fig. 2. The difference is the computation of control effort \( u \), Lyapunov function \( V_i \), \( \eta_i \) and \( S_i(\cdot) \). The control effort \( u \) is from (42) and (46). \( V_i, \eta_i \) and \( S_i(\cdot) \) are from (40), (44), (49), (50) and (48) separately. The algorithm is referred to as Algorithm 3. Then we have the following result.

**Theorem 3.** Consider the nonlinear system in (1). Then the controller (42), (46) with Algorithm 3 can guarantee that:

1) All the signals except for the time-varying gain \( \mu \) in the closed-loop system are bounded on \([0, T)\) with the output \(|y(t)| < \hat{\chi}_1 \) for \( \forall t \in [0, T) \), and;

2) All the states will be regulated to zero in prescribed finite time \( T \), i.e., there exists a time \( T_0 \leq T \) such that \( x_i(t) \to 0 \) as \( t \to T_0^- \) for \( i = 1, 2, ..., n \).

The detail proof for the above result is put in Appendix D in the supplementary file. It follows the line of Theorem 2 and is divided into three claims. Note that the purpose of the algorithm is also to let \( S_i(\cdot) = V_i(\cdot) - \eta_i(t) \leq 0 \) hold forever after finite switching times. In fact, when \( S_i(\cdot) = V_i(\cdot) - \eta_i(t) \leq 0 \) holds, let \( \bar{\eta} \triangleq \sum_{i=1}^{n} \eta_i \), from (49) and (50) we can conclude that with sufficiently large \( Q_i \),

\[ \bar{\eta} \leq - \sum_{i=1}^{n} a_i \mu \beta_i \eta_i \leq -\mu^{\beta'} a' \bar{\eta} \]  

(51)

where \( \beta' \in (1, +\infty) \), \( a' > 0 \) are positive constants. According to Song et al. (2017), it implies that \( \bar{\eta} \) may become zero in prescribed finite time. Due to \( S_i(\cdot) = V_i(\cdot) - \eta_i(t) \leq 0 \), \( V_i, x_i \) will also become zero in prescribed finite time.

**Remark 6.** It is noted that the above prescribed finite time performance should be understood in the sense of Song et al. (2017) instead of Definition 1. Meanwhile, note that though theoretically we can prove the state can reach zero in prescribed finite time, as stated in Song et al. (2017) the utilization of an unbounded time-varying gain \( \mu \) may result in some numerical issues in real practice. This issue can be avoided by setting \( T \) larger than the desired finite time but with a sacrifice of control accuracy. Also the complexity of the controller will increase when using a time-varying gain. This might bring some difficulties to the implementation of the controller. Please see Song et al. (2017) for detail information.

6. Discussions and comparisons

In this section, we will give some further extensions and remarks on the proposed method. Some comparisons with existing works will also be conducted.
6.1. More general systems

Our method can be naturally extended to the case when system (1) contains power orders, non-feedback linearizable dynamics and time-varying dynamics as in Fu et al. (2017) and Qian and Lin (2001). Specifically, (1) can be revised into

\[ \dot{x}_i = h_i(x_i, t)x_{i+1}^{p_i} + f_i(x_i, t), \quad i = 1, 2, ..., n - 1 \]
\[ \dot{x}_n = h_n(x_n, t)u + f_n(x_n, t) \]

where \( p_i \) is a ratio of positive odd integers, \( h_i(x_i, t) \) and \( f_i(x_i, t) \) are unknown functions such that \( |h_i(x_i, t)| > 0 \) and \( |f_i(x_i, t)| \leq (|x_1|^{\lambda_1} + \cdots + |x_i|^{\lambda_i})\psi_i(x_i) \) where \( \lambda_i > 0 \) are some specified positive constants and \( \psi_i(x_i) \) is unknown. However, it is difficult to extend the existing works (e.g., Yin et al., 2017; Li & Liu, 2018; Liu & Tong, 2017; Zhang & Yang, 2019) to this kind of systems.

Moreover, there are several ways to relax the condition \( f_i(0) = 0 \) in (1). The first way is to consider practical finite time stability, i.e., the tracking error will converge to a small neighborhood around origin in finite time (Yu et al, 2018). This can be done by replacing the auxiliary systems (32) with \( \dot{\eta}_i(t) = -a\eta_i^{(1+\alpha)/2} - Q_i s_i^2 + \sum_{j=1}^{i-1} c_{ij} s_j^{1+\alpha} + c_{i,i+1} s_{i+1}^{1+\alpha} + \zeta \) where \( a, \zeta \) are positive design parameters. Then Algorithm 2 in Section 4 is referred to as Algorithm 2’. We have the following result.

**Corollary 1.** Consider the nonlinear system in (1) with \( f_i(0) \neq 0 \) (\( i = 1, 2, ..., n \)). Then the controller (22), (28) with Algorithm 2’ can guarantee that:

1) All the signals in the closed-loop system are bounded, and;
2) Practical finite time stability is achieved, i.e., there exists a finite time \( T_0 \) such that \( \lim_{t \to T_0} |V_i(t)| \leq 2n\zeta/a \) where \( V_i(t) \) are given by (19) and (25).

Note that since \( \zeta, a \) are freely chosen design parameters, the states can be regulated to a very small neighborhood around zero in finite time.

The second way is to revise (50) into \( \dot{\eta}_n = -a_\alpha \eta_n^\beta - Q_n s_n^2 + \sum_{j=1}^{n-1} c_{ij} s_j^{1+\alpha} + \zeta_n/\mu^{\beta n} \) with a positive constant \( \zeta_n \). Then the controller designed in Section 5 can make the states to zero in prescribed finite time when \( f_n(0) \neq 0 \). To this end, Algorithm 3 in Section 5 is referred to as Algorithm 3’ and we have the following corollary.

**Corollary 2.** Consider the nonlinear system in (1) with \( f_n(0) \neq 0 \). Then the controller (42), (46) with Algorithm 3’ can guarantee that:

1) All the signals except for the time-varying gain \( \mu \) in the closed-loop system are bounded on \([0, T]\), and;
2) All the states will be regulated to zero in prescribed finite time \( T \).

At last, one can adopt the disturbance observer technique (Yang, Chen, Li, Guo, & Yan, 2017) to handle
the non-vanishing nonlinearities \( f_i(\pi_i) \). The controller in (42) and (46) is revised into

\[
\hat{d}_i = l_i(x_i + \hat{d}_{i-1}) + p_i, \tag{53}
\]

\[
\dot{p}_i = -l_i x_{i+1} - \hat{l}_i (x_i + \hat{d}_{i-1}) - l_i \hat{d}_i, \tag{54}
\]

\[
x_{i+1}^* = -K_i s_i - \frac{\mu_i^\beta U_i s_i}{\lambda_i^2 - s_i^2} - \hat{d}_i (i = 1, 2, \ldots, n-1), \tag{55}
\]

\[
u = \hat{\Theta}_n \left[ -K_n s_n - \frac{\mu_n^\beta U_n s_n}{\lambda_n^2 - s_n^2} \right], \tag{56}
\]

where \( l_i(t) = \mu_i^\beta(t) \) are time-varying gains with \( \beta_i \in (1, +\infty) \), \( \hat{d}_0 \triangleq 0 \). \( \hat{\Theta}_n \) is an adaptive parameter. The tuning algorithm for \( \hat{\Theta}_n \) is a revised form of Algorithm 3. Specifically, let \( S_i(\cdot) = -1 < 0(i = 1, 2, \ldots, n-1) \),

\[
\dot{\eta}_n = -a_n \mu_n^\beta \eta_n - Q_n s_n^2 + \sum_{j=1}^{n-1} c_{ij} \mu_j^\beta s_j^2 + \zeta_n / \mu_n^\beta \]

with positive constant \( \zeta_n \). This algorithm is referred to as Algorithm 3”. \( \hat{\Theta}_n \) in the above controller are disturbance observers which are used to compensate for the non-vanishing parts in \( f_i(\pi_i) \). Here we only consider the single unknown control direction, i.e., the sign of \( h_n(\pi_n) \) is unknown. Hence, there is only one adaptive parameter \( \hat{\Theta}_n \) in the last equation (56) and \( S_i(\cdot)(i = 1, 2, \ldots, n-1) \) is set to -1. We have the following result.

**Corollary 3.** Consider the nonlinear system in (1) with \( h_i(\pi_i) = 1(i = 1, 2, \ldots, n-1), f_i(0) \neq 0(i = 1, 2, \ldots, n) \). Then the controller (53)-(56) with Algorithm 3’ can guarantee that:

1) All the signals except for the time-varying gain \( \mu_0 \) in the closed-loop system are bounded on \([0, T])\), and;

2) The output \( y = x_1 \) will be regulated to zero in prescribed finite time \( T \).

**Proof.** The proof is put in Appendix E in the supplementary file.

6.2. Convergence speed

Finite time stability can be obtained by the proposed method despite unknown control directions. However, the works in Liu and Tong (2017), Li and Liu (2018), Yin et al. (2017) etc. can only achieve asymptotic stability meaning the convergence time is infinite. It has been shown in many references (e.g., Sun, Shao, & Chen, 2019; Chen & Sun, 2020; Yu et al., 2018) that finite time control has a smaller settling time, faster convergence rate and higher precision than infinite time control methods. The reasons are as follows:

1) According to Section 3, we know for our method the Lyapunov function \( V \leq \eta \) where

\[
\dot{\eta} \leq -a \eta^\gamma = -a \frac{1}{\eta^{1-\gamma}} \eta
\]

with \( \gamma \in (0, 1) \), \( a > 0 \). For exponential or asymptotic stability, the Lyapunov function \( V \) of the system may satisfy \( \dot{V} \leq -a'V \) with \( a' > 0 \). It can be seen that due to the additional fractional power \( \gamma \) in finite time control, the closer to the equilibrium point, the faster the convergence rate of the system states.
2) We have extended our method to obtain prescribed finite time control performance in Section 5. This will further reduce the settling time.

3) The auxiliary system in (1) or (32) can be easily modified into \( \dot{\eta} \leq -a'\eta - a\eta^\gamma \) with positive constant \( a' \). This modification can render fast finite time control as discussed in Sun et al. (2019), which can achieve a faster convergence rate than asymptotic/exponential control strategies.

4) Since the control gain is updated in a discrete manner, according to Hespanha, Liberzon, and Morse (2003), the proposed method could obtain a higher control performance than traditional continuous adaptive control method.

6.3. Control overshoot

The proposed method may obtain a smaller control overshoot than Nussbaum method. The large overshoot phenomenon of Nussbaum method has been recognized in many references (e.g., Chen, Liu, Zhang, Chen, & Xie, 2016; Scheinker & Krstic, 2013). Moreover, as stated in Huang and Yu (2018), the logic-based switching controller is more sensitive to the mismatch of the unknown control directions than Nussbaum method. Therefore, it may detect the right control directions more quickly than Nussbaum method, which will result in a smaller control overshoot. The simulations in Section 7 have verified the above claim.

Next, we take the following simple Nussbaum controller as an example to further explain this.

\[
\begin{align*}
    u &= \mathcal{N}(\xi)x, \\
    \dot{\xi} &= V(x)
\end{align*}
\]  

where \( \mathcal{N}(\xi) = \xi^2 \cos(\xi) \) or \( \mathcal{N}(\xi) = e^{\xi^2} \cos(\xi\pi/2) \). \( V(x) = x^2 \geq 0 \) is a Lyapunov function. The Nussbaum gain \( \mathcal{N}(\xi) \) will switch its sign when \( \xi \) increases to some threshold values, i.e., zeros of the cosine function in \( \mathcal{N}(\xi) \).

For the logic-based switching controller, the control direction is detected according to the supervisory function, which is selected as \( S(\cdot) = V(x) - \eta \) where \( \dot{\eta} = -a\eta^\gamma \) and \( \eta(0) = V(0) + \varepsilon \) with a small constant \( \varepsilon \). The control gain will be switched when \( S(\cdot) > 0 \). At the initial phase, if the states of the system are driven away by a wrong control direction, i.e., \( V(x) \) is increasing, then the supervisory function \( S(\cdot) \) will quickly become larger than zero and the control gain will be switched. Thus, the right control direction will be found. If the control direction is right, then \( V(x) \) will decrease and we can also identify a right control direction.

However, for Nussbaum gain, since \( \xi \) in (57) is an integration of the Lyapunov \( V(x) \), it will always increase whether the states are driven away or converge to zero. This implies that a longer time may be needed to find the right control direction.

In addition, the polynomial or exponential increasing terms \( \xi^2, e^{\xi^2} \) in \( \mathcal{N}(\xi) \) may lead to large control effort.
6.4. Control parameters selection

It is noted that theoretically the states will converge to zero in finite time if all the design parameters are positive and $Q_i - \sum_{j=i+1}^{n} c_{ji} - c_{i-1,i} > 0$ in (33). This is due to the adaptive feature of our proposed method. The theoretical results imply that the parameters selection can be very flexible and straightforward. This can be also seen from the design procedures in Sections 4.1 and 5.1 where Young’s inequalities are extensively used. In fact, from the simulations in Section 7, we can see that many of the design parameters are the same and set to one. It is also noted that appropriate design parameters can result in smaller settling time and better transient performance. Next, we present some guidelines for the parameters selections.

Basically, larger control gain $K_i, U_i$ and smaller $q_i$ may result in a faster convergence speed, but bigger overshoot and control effort. By increasing the value $a_i$ in the supervisory functions, we can obtain a smaller settling time. The initial value $\theta_i(0)$ should be set small to prevent the control effort from becoming very large in the beginning. Meanwhile, the initial value $\hat{\chi}_i(0)$ should be set slightly large to reduce the switching times, thus resulting in a quicker response.

Moreover, by resorting to the time-varying gain technique in Section 5, one can prescribe a convergence time or transient performance for the closed loop systems. In fact, from Theorems 2,3 we know the output satisfies $|s_1| < \hat{\chi}_1$ where $\hat{\chi}_1$ is a constant. One can replace this constant with some time-varying performance function, and then a transient performance can be prescribed. This will bring more convenience to the parameters selection.

7. Simulations

Example 1. Consider the following second order nonlinear system:

$$
\begin{align*}
\dot{x}_1 &= h_1 x_2 + f_1(x_1), \\
\dot{x}_2 &= h_2 u + f_2(x_2), \\
y &= x_1.
\end{align*}
$$

We consider the following six cases:

Case A: $h_1 = 1, h_2 = 0.8, f_1 = \overline{f}_1, f_2 = \overline{f}_2$;

Case B: $h_1 = 1, h_2 = -0.8, f_1 = \overline{f}_1, f_2 = \overline{f}_2$;

Case C: $h_1 = 1, h_2 = 0.8, f_1 = 5\overline{f}_1, f_2 = 5\overline{f}_2$;

Case D: $h_1 = -1, h_2 = 0.8, f_1 = \overline{f}_1, f_2 = \overline{f}_2$;

Case E: $h_1 = -1, h_2 = -0.8, f_1 = \overline{f}_1, f_2 = \overline{f}_2$;

Case F: $h_1 = -1, h_2 = 0.8, f_1 = f^*_1, f_2 = f^*_2$.
The controller parameters are set as:

\[
\begin{align*}
\mathcal{f}^* &\triangleq 0.1 \sin(x_1) x_1; \\
\mathcal{f}^2 &\triangleq -4.9 \sin(x_1) + 0.05 \sin(x_1) e^{-x_2} + 0.1 \sin(x_2) x_2^3; \\
f_1 &\triangleq -3x_1^2/2 - x_1^3/2 \text{ and } \\
f_2^* &\triangleq 0.1 \sin(x_2). 
\end{align*}
\]

Case A is the nominal case. It can be used to describe the dynamics of a single link robot manipulator (Xing, Wen, Liu, Su, & Cai, 2017). Other five situations represent the variation of control directions and modeling uncertainties. Specifically, Case F indicates that the whole system has been changed into another form. The initial conditions are \(x_1(0) = 0.1, x_2(0) = 0.2\). For the controller design, we assume \(h_1, h_2, f_1, f_2\) are all unknown.

The simulations will be conducted in the following three aspects.

1) Effectiveness of the logic-based switching

By (22) and (28), the controller is designed as:

\[
u = \hat{\Theta}_2 \left[ -K_2 s_2^q - \frac{U_2 s_2^q}{(\chi_2 - s_2^q)^{1+2\alpha}} \right] \tag{59}
\]

where \(s_2 = x_2^{1/q_2} - x_2^{1/q_2}\), \(x_2^n = \hat{\Theta}_1 \left[ -K_1 s_2^q - \frac{U_1 s_2^q}{(\chi_1 - s_2^q)^{1+2\alpha}} \right] \tag{60}
\]

The controller parameters are set as: \(K_1 = U_1 = K_2 = U_2 = 1, \alpha = 41/49, \chi_1 = \chi_2(0) = 2, \theta_1(0) = \theta_2(0) = 0.1 \text{ and for } \sigma_i(t) \geq 1(i = 1, 2), \text{ we have } \theta_i(\sigma_i) = (-1)^{\sigma_i(t)} (\iota_{i1} + \iota_{i2} \sigma_i) \text{ where } \iota_{i1} = \iota_{i2} = 1. \hat{\Theta}_1, \hat{\Theta}_2, \chi_1, \chi_2 \text{ are updated by Algorithm 1 with } \varsigma = 4, \varepsilon = 0.01, a_1 = a_2 = 0.2, Q_1 = Q_2 = c_{11} = c_{21} = 1.

The control performance is shown in Fig. 3. It can be seen that both states converge to zero in a very short time for the above six situations. This implies that the finite time stability has been achieved despite multiple unknown control directions and unstructured uncertainties.

Fig. 4 shows the variation of parameters \(\hat{\Theta}_1, \hat{\Theta}_2\). We can see that in this case \(\hat{\Theta}_1\) remains the same and \(\hat{\Theta}_2\) is updated around time 0.66s. It is noted that in this case \(x_1(i = 1, 2)\) both remain unchanged. Fig. 4 also demonstrates the variation of \(\eta_2\) and \(V_2\). It shows that \(V_2\) is always no more than \(\eta_2\). This indicates that the supervisory functions are guiding the switching signals. Meanwhile, it can be seen that at time instant 0.66s, \(\eta_2\) is reset. This is also the time instant when parameter \(\hat{\Theta}_2\) changes. All these verify the validity of the logic-based switching mechanism.

Next, we intentionally reduce the value of \(\chi_2(0)\) and set \(\chi_2(0) = 0.8, \chi_1(0) = 2\). The variations of \(s_i, \hat{\Theta}_i, \hat{\chi}_i(i = 1, 2)\) are shown in Fig. 5. Note that \(s_1\) is constrained into the tube \([-2, 2]\). This shows the validity of the barrier Lyapunov function \(V_1\) in (19). Also we can see \(s_2\) is constrained into the tube \([-0.8, 0.8]\) during time interval \([0, 1.1]\) and is larger than \(\chi_2(0) = 0.8\) at time instant 1.1s. Therefore, \(\hat{\chi}_2\) jumps to 4.3 to contain \(s_2\) in the later time. The reason for \(s_2\) jumping outside 0.8 is that \(\hat{\Theta}_1\) is updated at 1.1s. Note that though \(s_2\) transgresses over the barrier \(\chi_2(0) = 0.8, s_1, s_2\) still can converge to zero in finite time. All these show the effectiveness of the switching barrier Lyapunov function.

2) Comparison with Nussbaum-gain method

Given a finite time controller by (59) with the same parameters as before, and another controller designed
by Nussbaum-gain technique:

\[ u = N_2(\xi_2)v_2, \]
\[ v_2 = K_2s_2 + \frac{U_2s_2}{\lambda_2^2 - s_2^2}, \]
\[ \dot{s}_2 = \frac{s_2v_2}{\lambda_2^2 - s_2^2} \]

where

\[ s_2 = x_2 - x_2^*, \]
\[ x_2^* = N_1(\xi_1)v_1, \]
\[ v_1 = K_1s_1 + \frac{U_1s_1}{\lambda_1^2 - s_1^2}, \]
\[ \dot{s}_1 = \frac{s_1v_1}{\lambda_1^2 - s_1^2} \]

\( N_i(\xi_i)(i = 1, 2) \) are Nussbaum functions such that \( N_i(\xi_i) = \xi_i^2 \cos(\xi_i) \). \( K_1, U_1, K_2, U_2, \dot{\chi}_1, \dot{\chi}_2 \) are positive design parameters. The parameters are set as: \( K_1 = U_1 = 1, K_2 = U_2 = 4, \dot{\chi}_1 = 2, \dot{\chi}_2 = 1 \). Using these parameters, a satisfactory control performance can be obtained in the nominal case. This controller is a variation of the method in Liu and Tong (2017), which adapts to the unstructured uncertainties.

Fig. 6 shows the state trajectories and variation of control effort in Case A. We can see that the finite time control method has a faster convergence speed and higher precision than Nussbaum-gain method. In fact, the states by finite time control is around \( 3.75 \times 10^{-5} \) after 6.5s, while states by Nussbaum-gain method is 0.01. Moreover, the proposed method has a smaller overshoot and control effort than Nussbaum-gain method. The reason for this may be that the proposed method can find a proper control direction quickly by switching logic.

Next, with the same controller parameters, we consider the control performance in Cases B-F. Fig. 7 shows the state trajectories in Cases B and C. We can see that in both cases, the proposed method has a superior performance with smaller overshoot and faster convergence rate. Fig. 8 demonstrates the state trajectories in Cases D and E, it can be seen that control performance of Nussbaum-gain method deteriorates a lot. For Case F, the Nussbaum-gain method has become highly unstable. All these show the stronger robustness of the proposed method.

3) Time-varying feedback

Finally, we consider the control performance by the controller designed in Section 5. By \( \omega_2 \) and \( \omega_6 \), the controller is given by:

\[ u = \hat{\Theta}_2 \left[ -K_2s_2 - \frac{\mu^{s_2}U_2s_2}{\lambda_2^2 - s_2^2} \right] \tag{61} \]

where \( s_2 = x_2 - x_2^* \),

\[ x_2^* = \hat{\Theta}_1 \left[ -K_1s_1 - \frac{\mu^{s_1}U_1s_1}{\lambda_1^2 - s_1^2} \right]. \tag{62} \]
The controller parameters are selected the same as (59), (60) except $\beta_1 = 1.4$, $\beta_2 = 4.2$. The prescribed finite time $T = 4.5s$.

Fig. 9 shows the control performance for this controller. We can see that the states are regulated to zero before the prescribed time $4.5s$ for the considered six cases. This shows that the presented method can achieve prescribed finite time performance.

**Example 2.** Consider the following third order nonlinear system:

\[
\begin{align*}
\dot{x}_1 &= h_1 x_2 + f_1(x_1), \\
\dot{x}_2 &= h_2 x_3 + f_2(x_2), \\
x_3 &= h_3 u + f_3(x_3), \\
y &= x_1
\end{align*}
\]

where $h_1 = \pm 1$, $f_1 = -1.8x_1 + 0.15\sin(x_1)$, $h_2 = \pm 1$, $f_2 = 0.7x_2 - 4.4x_1 + 0.1\sin(x_2)x_1^2$, $h_3 = \pm 3$, $f_3 = -7x_2 + 0.1\sin(x_3)$. The above systems can be used to describe some circuit systems (Huang & Liu, 2019). The initial conditions are $x_1(0) = 0.1, x_2(0) = 0.2, x_3(0) = 0.5$. 

The controller parameters are selected the same as (59), (60) except $\beta_1 = 1.4$, $\beta_2 = 4.2$. The prescribed finite time $T = 4.5s$.

Fig. 9 shows the control performance for this controller. We can see that the states are regulated to zero before the prescribed time $4.5s$ for the considered six cases. This shows that the presented method can achieve prescribed finite time performance.

**Example 2.** Consider the following third order nonlinear system:

\[
\begin{align*}
\dot{x}_1 &= h_1 x_2 + f_1(x_1), \\
\dot{x}_2 &= h_2 x_3 + f_2(x_2), \\
x_3 &= h_3 u + f_3(x_3), \\
y &= x_1
\end{align*}
\]

where $h_1 = \pm 1$, $f_1 = -1.8x_1 + 0.15\sin(x_1)$, $h_2 = \pm 1$, $f_2 = 0.7x_2 - 4.4x_1 + 0.1\sin(x_2)x_1^2$, $h_3 = \pm 3$, $f_3 = -7x_2 + 0.1\sin(x_3)$. The above systems can be used to describe some circuit systems (Huang & Liu, 2019). The initial conditions are $x_1(0) = 0.1, x_2(0) = 0.2, x_3(0) = 0.5$. 

The controller parameters are selected the same as (59), (60) except $\beta_1 = 1.4$, $\beta_2 = 4.2$. The prescribed finite time $T = 4.5s$.

Fig. 9 shows the control performance for this controller. We can see that the states are regulated to zero before the prescribed time $4.5s$ for the considered six cases. This shows that the presented method can achieve prescribed finite time performance.

**Example 2.** Consider the following third order nonlinear system:

\[
\begin{align*}
\dot{x}_1 &= h_1 x_2 + f_1(x_1), \\
\dot{x}_2 &= h_2 x_3 + f_2(x_2), \\
x_3 &= h_3 u + f_3(x_3), \\
y &= x_1
\end{align*}
\]

where $h_1 = \pm 1$, $f_1 = -1.8x_1 + 0.15\sin(x_1)$, $h_2 = \pm 1$, $f_2 = 0.7x_2 - 4.4x_1 + 0.1\sin(x_2)x_1^2$, $h_3 = \pm 3$, $f_3 = -7x_2 + 0.1\sin(x_3)$. The above systems can be used to describe some circuit systems (Huang & Liu, 2019). The initial conditions are $x_1(0) = 0.1, x_2(0) = 0.2, x_3(0) = 0.5$. 

The controller parameters are selected the same as (59), (60) except $\beta_1 = 1.4$, $\beta_2 = 4.2$. The prescribed finite time $T = 4.5s$.

Fig. 9 shows the control performance for this controller. We can see that the states are regulated to zero before the prescribed time $4.5s$ for the considered six cases. This shows that the presented method can achieve prescribed finite time performance.
Figure 5: Variations of $s_i, \hat{\Theta}_i, \hat{\chi}_i (i = 1, 2)$.

Figure 6: Control performance comparison for Case A.

Figure 7: Control performance comparison for Cases B and C.
Figure 8: Control performance comparison for Cases D and E.

Figure 9: Control performance for time-varying feedback.

Figure 10: Control performance for third order system.
The controller is given by (22) and (28) with parameters selected as: $K_i = U_i = 1$, $\alpha = 41/49$; $\hat{\chi}_1 = 2$, $\hat{\chi}_2(0) = 5$, $\hat{\chi}_3(0) = 8$; $\theta_i(0) = 0.1 (i = 1, 2, 3)$. For $\sigma_i(t) \geq 1$, let $\theta_i(\sigma_i) = (-1)^{a_i(t_i)}(\iota_{i1} + i\varsigma_{i2})$ where $\iota_{i1} = 1$, $\iota_{i2} = 1$, $\varsigma = 5$, $\varsigma = 0.01$, $a_i = 0.2$, $Q_1 = Q_2 = 2$, $Q_3 = 1$, $c_{12} = c_{21} = c_{31} = c_{32} = 1$.

Fig. 10 shows the control performance for this controller. According to the sign of the control direction $h_i (i = 1, 2, 3)$, there are total $2 \times 2 \times 2 = 8$ situations. Each blue line in Fig. 10 represents one of these situations. We can see that all the states converge to zero in finite time for all these eight cases. This demonstrates the validity and robustness of the proposed method.

8. Conclusions

In this paper, a logic-based switching adaptive controller is designed, and the finite time stability can be guaranteed for the nonlinear systems suffering from multiple unknown control directions and unstructured uncertainties. Future work will be focused on extending the presented design approach to general hybrid systems.

Appendix A. Proof of Lemma 4

Proof. This is a variation of Comparison Principle (Khalil, 2002). Subtracting (4) from (5). We have:

$$\dot{e}(t) \leq -a[y^\gamma(t) - x^\gamma(t)] \quad (A.1)$$

where $e(t) \triangleq y(t) - x(t)$. Then, we only need to prove $e(t) \leq 0$ for $\forall t \in [t_0, t_1)$. In fact, if this is not true, there must exists a time instant $t^+ \in [t_0, +\infty)$ such that $e(t^+) > 0$. Since $e(t_0) = y(t_0) - y(t_0) - \varepsilon = -\varepsilon \leq 0$, we can conclude that there exists a time interval $[\tau_1, \tau_2] \subseteq [t_0, t^+]$ such that $e(\tau_1) = 0$, $e(t) > 0$ for $\forall t \in (\tau_1, \tau_2)$. By Mean Value Theorem, there exists a time instant $\tau_3 \in [\tau_1, \tau_2]$ such that $e(\tau_3) > 0$ and $\dot{e}(\tau_3) > 0$. However, this contradicts (A.1). It completes the proof. \phantom{.}

Appendix B. Proof of Claim 2a

The proof is divided into two parts. For the first part, we will show Claim 2a holds with $m = 0$. The second part will show Claim 2a is true for any finite integer $m$.

Part I

We will prove the following claim.

Claim 2a'. Suppose $|s_i(t^0_i)| < \hat{\chi}_i(t^0_i)$ and $\eta_i(t^0_i) > V_i(t^0_i)$ for $\forall i = 1, 2, ..., n$, then we have

1) The considered closed loop nonlinear system admits a continuous solution $\bar{x}_n(t)$ on $[t_i^0, t^1_i]$;

2) There exists a positive constant $\delta_i^0$ such that $|s_i(t)| \leq \hat{\chi}_i(t^0_i) - \delta_i^0$ for $\forall i = 1, 2, ..., n$ and all the signals in the system are bounded on $[t_i^0, t^1_i]$. 

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For Statement 1), according to the Initialization in Algorithm 2, we know $|s_i(t^0_i)| < \hat{\chi}_i(t^0_i)$ and $\eta_i(t^0_i) > V_i(t^0_i)$ indeed hold. Then since nonlinear functions $h_i(\cdot), f_i(\cdot)$ are $C^1$, according to Tee et al. (2009) we can conclude that the closed loop system has a continuous maximally extended solution on $[t^0_i, t_0) \subseteq [t^0_i, t^1_i)$ such that $|s_i(t)| < \hat{\chi}_i(t^0_i)$ for $i = 1, 2, \ldots, n$.

Next, we will prove $t_0$ can be selected to be $t^1_1$. This is proved by contradiction. If this is not true, then $t_0 < t^1_1$ and there exists an integer $i' \in \{1, 2, \ldots, n\}$ such that when $t \to t_0$, $|s_{i'}(t)| \to \hat{\chi}_{i'}(t^0_{i'})$. Meanwhile, $|s_i(t)| < \hat{\chi}_i(t^0_i)$ for $i = 1, 2, \ldots, n$ on $[0, t_0)$.

Note that from (34), we know $\hat{\eta} \leq -d_2\pi^{1/2} \leq 0$ where we have used the fact $\eta_i \geq V_i \geq 0$. This means that all the Lyapunov function $V_i$ are bounded. Next, analysis will be taken on each step in the controller design to seek a contradiction.

**Step 1.** Since $V_i$ is bounded, from (19) we conclude that there exists a positive constant $\delta_0^i$ such that $|s_i| \leq \hat{\chi}_i(0) - \delta_0^i$. Using (22) and (18), we know $x_2^i, x_2$ are both bounded.

**Step 2 ($2 \leq i \leq n$).** Due to $x^*_{i'}, V_i$ are bounded, from Proposition 1 we know there exists a positive constant $\delta_0^i$ such that $|s_i| \leq \hat{\chi}_i(0) - \delta_0^i$. Using (28) and (18), we conclude that $x_{i+1}^i, x_{i+1}$ are both bounded.

Therefore, we have $|s_i| \leq \hat{\chi}_i(0) - \delta_0^i$ for $i = 1, 2, \ldots, n$. This contradicts the fact that there exists an integer $i' \in \{1, 2, \ldots, n\}$ such that when $t \to t_0$, $|s_{i'}(t)| \to \hat{\chi}_{i'}(t^0_{i'})$. Then we can conclude that the system has a continuous solution on $[t^0_i, t^1_i)$ with $|s_i(t)| < \hat{\chi}_i(t^0_i)$.

For Statement 2), one can repeat the above procedures and use the fact $|s_i(t)| < \hat{\chi}_i(t^0_i)$. Then we can conclude that on time interval $[t^0_i, t^1_i)$, there exist positive constants $\delta_0^i$ such that $|s_i| \leq \hat{\chi}_i(0) - \delta_0^i$ for $i = 1, 2, \ldots, n$ and all the signals are bounded. The prove is completed.

**Part II**

We will prove Claim 2a holds for any finite integer $m$.

According to Claim 2a’, we know the system has a continuous solution on $[t^0_i, t^1_i)$. Then it is sufficient to show at switching time $t^1_i$, we have

1) $s_i(t^1_i), \hat{\chi}_i(t^1_i), \eta_i(t^1_i), V_i(t^1_i)$ are all bounded for $i = 1, 2, \ldots, n$;

2) $|s_i(t^1_i)| < |\hat{\chi}_i(t^1_i)|$ and $\eta_i(t^1_i) > V_i(t^1_i)$ for $i = 1, 2, \ldots, n$.

In fact, if the above statements are true, by regarding $t^1_i$ as a new initial time instant and repeating the procedures in Claim 2a’, we can prove Claim 2a holds with $m = 1$. Finally, repeating the the above procedures iteratively, we can prove the result for any finite $m$.

To prove 1), we only show $s_i(t^1_i)$ is bounded for $i = 1, 2, \ldots, n$. Other signals can be proved similarly. The following steps will be taken to show the boundedness of $s_i(t^1_i)$.

**Step 1.** For $s_1$, from Claim 2a’, we know there exists a positive constant $\delta_0^1$ such that $|s_1| \leq \hat{\chi}_1(0) - \delta_0^1$ on $[t^0_1, t^1_1)$ and there is no jump at time instant $t^1_1$, then $s_1(t^1_1)$ is bounded.

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Step 2. For \( s_2 \), it is expressed as \( s_2 = x_2^{1/q_2} - x_2^{1/q_2} \). It is sufficient to show \( x_2^1(t_s^1), x_2(t_s^1) \) are both bounded. First, consider \( x_2^1 \), it is given by (22), that is
\[
x_2^1 = \hat{\Theta}_1 \left[ -K_1 s_1^{q_2} - \frac{U_1 s_1^{q_2}}{(\hat{x}_1^1 - s_1^{q_2})^{1+2\alpha}} \right].
\] (B.1)
From (23), we know \( \hat{\Theta}_1(t_s^n) \) is bounded as long as \( m \) is finite. Meanwhile, by Algorithm 2-4) in Switching logic, using the updated parameter \( \hat{\chi}_1 \), we always have \( |\hat{\chi}_1(t_s^1)| > |s_1(t_s^1)| \). This implies \( \frac{1}{\hat{\chi}_1(t_s^1) - s_1(t_s^1)} \) is well defined and bounded. Hence, \( x_2^1 \) is bounded by (B.1) or (22). Next, for \( x_2 \), according to Claim 2a', it is bounded on \([t_s^0, t_s^1]\) and there is no jump at \( t_s^1 \). Therefore, \( x_2(t_s^1) \) and \( s_2(t_s^1) \) are both bounded.

Step \( i(3 \leq i \leq n) \). By repeating the above procedures, we can show \( s_i(t_s^1) \) are all bounded.

To prove 2), that is \( |s_i(t_s^1)| < |\hat{\chi}_i(t_s^1)| \) and \( \eta(t_s^1) > V_i(t_s^1) \) for \( i = 1, 2, ..., n \). This can be easily derived from Algorithm 2-4) and 5) in Switching logic. The proof is completed.

Appendix C. Proof of Proposition 3

Proof. We only show the proposition holds with \( i = 2 \). For \( i \geq 2 \), we can use an induction method to prove it similarly. From Lyapunov function (44) and coordinate transformation (39), we have
\[
\dot{V}_2 \leq \frac{s_2(h_2 x_2 + f_2 - \dot{x}_2^1)}{\hat{x}_2^1 - \hat{s}_2^1} + \frac{h_2 s_2^1}{\hat{x}_2^1 - \hat{s}_2^1}.
\] (C.1)

By Lemma 3 and coordinate transformation (39), we can obtain
\[
\frac{s_2 f_2(\pi_2)}{\hat{x}_2^1 - \hat{s}_2^1} \leq \frac{s_2(x_1 + x_2)}{\hat{x}_2^1 - \hat{s}_2^1} \leq \frac{|s_2 x_1| |\psi_2|}{\hat{x}_2^1 - \hat{s}_2^1} + \frac{|s_2 x_2| |\psi_2|}{\hat{x}_2^1 - \hat{s}_2^1} + \frac{s_2^2 |\psi_2|}{\hat{x}_2^1 - \hat{s}_2^1} \leq \frac{\mu^3 |s_2 x_1| |\psi_2|}{\hat{x}_2^1 - \hat{s}_2^1} + \frac{s_2^2 |\psi_2|}{\hat{x}_2^1 - \hat{s}_2^1}
\] (C.2)
where \( \psi_2 \) is an unknown function.

Also we have
\[
\frac{s_2 x_2^2}{\hat{x}_2^1 - \hat{s}_2^1} = \frac{s_2}{\hat{x}_2^1 - \hat{s}_2^1} \frac{\partial x_2^2}{\partial s_1} (h_1 x_2 + f_1(x_1)) + \frac{s_2}{\hat{x}_2^1 - \hat{s}_2^1} \frac{\partial x_2^2}{\partial \mu} \mu
\] (C.3)
where \( \psi_2, \psi_2^2 \) are unknown functions.

Substituting (C.2)-(C.3) into (C.1) and using Young’s inequality, we have
\[
\dot{V}_2 \leq \frac{s_2 h_2 x_2^2}{\hat{x}_2^1 - \hat{s}_2^1} + \frac{\mu^3 U_2 s_2^2 T^2(\pi_2, \hat{\Theta}_1, \hat{\chi}_1)}{(\hat{x}_2^1 - \hat{s}_2^1)^2}
\] (C.4)
\[+ e_{21} \mu^3 s_2^1 + \frac{c_{23} s_2^2}{\mu^3} \]
where $c_{21}, c_{23}$ are positive constants, $F_2(\cdot)$ is an unknown function. Let $\beta_2 = 3\beta_1$, it completes the proof. \hfill \Box

Appendix D. Proof of Theorem 3

Similar to Theorem 2, define a time sequence \( \{0 = t_0^s < t_1^s < \ldots < t_m^s < \ldots < T\} \) with \( m \in \{0, 1, 2, \ldots\} \). \( t_m^s \) denotes the time instant when switch happens. Then the proof will be divided into three claims.

Claim 3a. For a given finite integer \( m \), we have

1) The closed loop nonlinear system admits a continuous solution \( x_n(t) \) on \( [t_0^s, t_m^s + 1] \)

2) There exists a positive constant \( \delta^m_i \) such that \( |s_i(t)| \leq \overline{\chi}_i(t_m^s) - \delta^m_i \) for \( \forall i = 1, 2, \ldots, n \) and all the signals except for \( u \) in the system are bounded on \( [t_m^s, t_{m+1}^s) \).

Claim 3b. 1) The switching times are finite;

2) The closed loop nonlinear system admits a continuous solution \( x_n(t) \) on \( [0, T) \);

3) There exist a positive constant \( \delta_i \) such that \( |s_i(t)| \leq \chi_i - \delta_i \) for \( \forall i = 1, 2, \ldots, n \) and all the signals except for \( u \) in the system are bounded on \( [0, T) \).

Claim 3c. All the states will converge to zero in prescribed finite time \( T \).

The proof for the above three claims follows the line of the proof of Theorem 2. Yet, in order to handle the time-varying gain, an additional lemma will be used to show the boundedness of the signals and prescribed finite time convergence.

Lemma 5. Given two time-varying non-negative continuous functions \( \xi(t), b(t) : [0, T) \to \mathbb{R} \) such that on \( [t_1, T) \), we have

\[
\dot{\xi}(t) \leq -a \mu^\alpha(t) \xi(t) + b(t), \quad (D.1)
\]

\[
b(t) \leq \frac{\overline{x}}{\mu^\alpha(t)} \quad (D.2)
\]

where \( t_1, T, a, \chi, \alpha, \beta \) are positive constants with \( t_1 < T \), \( \alpha \in (1, +\infty) \) and \( \mu(t) = \frac{T}{t_1} \).

Meanwhile, there exists a time-varying continuous function \( x(t) : [0, T) \to \mathbb{R} \) such that

\[
\ln \left( \frac{\overline{x}^2}{\overline{x}^2 - x^2(t)} \right) \leq \xi(t) \quad (D.3)
\]

where \( \overline{x} \) is a positive constant and \( |x(t_1)| < \overline{x} \).

Then, we have \( \xi(t), x^2(t) \mu^{\alpha+\beta}(t) \) are both bounded on \( [t_1, T) \).

Particularly, if \( b(t) = 0 \), then \( \lim_{t \to T^-} x^2(t) \mu^\gamma(t) = 0 \) with a positive constant \( \gamma \).

Proof. The proof is put in Appendix F. \hfill \Box

Next, we will prove Claims 3b and 3c. Claim 3a can be proved similarly by resorting to Appendix B for Claim 2a.
We will first show there are only finite switching times. We will successively prove $\hat{\chi}_i, \hat{\Theta}_i (i = 1, 2, ..., n)$ only have finite switching times.

**Step 1.** Prove $\hat{\chi}_1, \hat{\Theta}_1$ have finite switching times.

1) We will show $\hat{\chi}_1$ does not switch on $[0, t_{s+1}^m)$.

This can be proved by following the line of Claim 2b and using Claim 3a.

2) We will show $\hat{\Theta}_1$ has finite switching times.

This is proved by contradiction. If this is not true, then $\hat{\Theta}_1$ will switch infinite times.

Due to $\hat{\chi}_1$ is not updated, the unstructured uncertainties $h_1(x_1)$ and $F_1(x_1, \hat{\chi}_1)$ in (43) satisfy

$$|h_1(x_1)| \geq h_1 > 0,$$

$$0 \leq F_1(x_1, \hat{\chi}_1) \leq F_1$$

where $h_1, F_1$ are positive unknown constants irrelevant with switching times $m$.

Then from the definition of $\hat{\Theta}_1$ and (43), we can conclude that there exists a sufficiently large finite integer $m_1$ such that at time instant $t_{s+1}^{m_1}$, we have

$$\dot{V}_1 \leq -\mu^s a_1 V_1 - Q_1 \mu^{s_1} s_1^2 + \frac{c_{12} s_2^2}{\mu^{s_1}}. \quad \text{(D.4)}$$

On the other hand, the auxiliary variable $\eta_1(t)$ in (49) satisfies

$$\dot{\eta}_1 \leq -\mu^s a_1 \eta_1 - Q_1 \mu^{s_1} s_1^2 + \frac{c_{12} s_2^2}{\mu^{s_1}} \quad \text{(D.5)}$$

where $\eta_1(t_{s+1}^{m_1}) > V_1(t_{s+1}^{m_1})$.

From Lemma 4 we know $V_1(t) \leq \eta_1(t)$ will hold on $[t_{s+1}^{m_1}, t_{s+1}^{m+1})$ for any $m + 1 > m_1$ without resetting $\eta_1(t)$. This means that $\hat{\Theta}_1$ will not be updated after $t_{s+1}^{m_1}$.

**Step 2.** Prove $\hat{\chi}_2, \hat{\Theta}_2$ have finite switching times.

The proof will be conducted on $[t_{s+1}^{m_1}, t_{s+1}^{m+1})$ such that $t_{s+1}^{m_1}$ denotes the time instant when $\hat{\chi}_1, \hat{\Theta}_1$ stop switching.

1) We will show $\hat{\chi}_2(t)$ does not switch on $[t_{s+1}^{m_1}, t_{s+1}^{m+1})$.

This can be proved by following the line of Claim 2b.

2) We will show $\hat{\Theta}_2$ has finite switching times.

This is proved by contradiction. If this is not true, then $\hat{\Theta}_2$ will switch infinite times.

Since $\hat{\chi}_2$ will not be updated on $[t_{s+1}^{m_1}, t_{s+1}^{m+1})$, we have $|s_2(t)| < \chi_2(t_{s+1}^{m_1})$ on $[t_{s+1}^{m_1}, t_{s+1}^{m+1})$. In addition, from Claim 3a, we know $\hat{\chi}_2(t_{s+1}^{m_1})$ is bounded.

Note that from (D.5) in Step 1, we know $\eta_1$ satisfies

$$\dot{\eta}_1 \leq -\mu^s a_1 \eta_1 - Q_1 \mu^{s_1} s_1^2 + \frac{c_{12} s_2^2}{\mu^{s_1}} \leq -\mu^s a_1 \eta_1 + \frac{c_{12} \chi_2^2(t_{s+1}^{m_1})}{\mu^{s_1}}$$
on \([t_s^{m_1}, t_s^{m_1 + 1})\). Based on Lemma 5 and (40), we know \(\eta_1\) and \(\mu^{2\beta_1} s_1^2\) are bounded by some constants irrelevant with switching times \(m\) (let \(\xi(t) = \eta_1, \chi = c_12\hat{x}_2(t_s^{m_1}), x(t) = s_1(t), \chi = \hat{\chi}_1\) in Lemma 5).

Since \(V_1 \leq \eta_1\) is bounded, by (40) we know \(|s_1(t)| \leq \chi_1(0) - \delta_1\) with a positive constant \(\delta_1\) irrelevant with \(m\). Meanwhile, due to \(\mu^{2\beta_1} s_1^2\) is bounded, by (42) and (39) we obtain \(x_2^*, x_2\) are both bounded by some constants irrelevant with \(m\).

Hence, we conclude that on \([t_s^{m_1}, t_s^{m_1 + 1}), h_2(\varphi_2)\) and \(F_2(\varphi_2, \hat{\Theta}_1, \hat{\chi}_1)\) in (45) satisfy
\[
h_2(\varphi_2) \geq h_2 > 0, \\
0 \leq F_2(\varphi_2, \hat{\Theta}_1, \hat{\chi}_1) \leq F_2
\]
where \(h_2, F_2\) are positive constants irrelevant with \(m\).

Then for (47) we can conclude that there exists a finite integer \(m_2 \geq m_1\) such that at time instant \(t_s^{m_2}\), we have
\[
\dot{V}_2 \leq -\mu^{\beta_2} a_2 V_2 - Q_2 \mu^{\beta_2} s_2^2 + c_21 \mu^{\beta_1} s_1^2 + \frac{c_23 s_3^{\beta_3}}{\mu^{\beta_2}}.
\]

On the other hand, the auxiliary variable \(\eta_2(t)\) in (32) satisfies
\[
\dot{\eta}_2 \leq -\mu^{\beta_2} a_2 \eta_2 - Q_2 \mu^{\beta_2} s_2^2 + c_21 \mu^{\beta_1} s_1^2 + \frac{c_23 s_3^{\beta_3}}{\mu^{\beta_2}}
\]
where \(\eta_2(t_s^{m_2}) > V(t_s^{m_2})\).

From Lemma 4 we know \(V_2(t) \leq \eta_2(t)\) will hold on \([t_s^{m_2}, t_s^{m_1 + 1})\) for any \(m + 1 > m_2\). This means that \(\hat{\Theta}_2\) will not be updated after \(t_s^{m_2}\).

Step \(i(3 \leq i \leq n)\). By repeating the above procedures, we can show all parameters \(\hat{\Theta}_i, \hat{\chi}_i(i = 1, 2, ..., n)\) have finite switching times. This proves Statement 1) in Claim 3b.

Next, for Statements 2) and 3), according to Claim 3a, they hold naturally when switching times are finite. Claim 3b is proved.

To prove Claim 3c, from Claim 3b, we know the switching times are finite. This indicates that there exists a switching time \(t_s^{m_2}\) such that when \(t \in [t_s^{m_2}, T)\), \(V_i \leq \eta_i(t)\) holds for \(\forall i \in \{1, 2, ..., n\}\). Then on \([t_s^{m_2}, T)\), by (51) and Lemma 5 we know all the states will converge to zero in prescribed finite time.

**Appendix E. Proof of Corollary 3**

**Proof.** We only show the case when \(n = 2\). For \(n \geq 2\), one can prove it similarly but with more complex derivation.

In this case, the disturbance observer in (53)-(56) is given by
\[
\dot{d}_1 = l_1 x_1 + p_1, \quad \text{(E.1)} \\
\dot{p}_1 = -l_1 u - l_1 x_1 - l_1 \dot{d}_1. \quad \text{(E.2)}
\]
Let \( d_1 \equiv f_1(0) \) in (1) which may not equal to zero. Then the first line of (1) can be re-expressed as:

\[
\dot{x}_1 = x_2 + g_1(x_1) + d_1
\]  

(E.3)

where \( g_1(x_1) \equiv f_1(x_1) - d_1 \) such that \( g_1(0) = 0 \).

We can see that (1) has been transformed into a system with a constant disturbance \( d_1 \) and an unknown nonlinear function \( g_1(x_1) \) vanishing at zero.

For the estimation error \( e_{d1} = \hat{d}_1 - d_1 \), by (E.1)-(E.3) we have

\[
\dot{e}_{d1} = l_1(e_{d1} + l_1 g_1(x_1)) + l_1 x_1 - l_1 x_1 - l_1 \hat{d}_1 
\]

\( = - l_1 e_{d1} + l_1 g_1(x_1). \)  

(E.4)

Considering the Lyapunov function \( V_{d1} = \frac{1}{2} e_{d1}^2 \) and using Young’s inequality, we get

\[
\dot{V}_{d1} = e_{d1}(-l_1 e_{d1} + l_1 g_1(\bar{t}_{1})) 
\]

\[
\leq -\frac{l_1}{2} e_{d1}^2 + \frac{l_1 g_1^2(\bar{t}_{1})}{2} 
\]

\[
\leq -\frac{l_1}{2} e_{d1}^2 + \frac{l_1 x_1^2 \psi_1^2(x_1)}{2} \]  

(E.5)

where we have used Lemma 3 for the last inequality due to \( g_1(0) = 0 \), and \( \psi_1(x_1) \) is an unknown function.

Based on (E.5) and Lemma 4, the rest of the proof is similar to the proof of Theorem 3 and is divided into three claims.

\[\square\]

Appendix F. Proof of Lemma 4

Proof. By solving the differential equation (D.1), we obtain

\[
\xi(t) \leq e^{-\alpha(t)} \xi(t_1) + e^{-\alpha(t)} \int_{t_1}^{t} e^{\alpha(t)} b(\tau)d\tau 
\]

where \( t \in [t_1, T) \),

\[
\bar{\mu}(t) = \int_{t_1}^{t} \mu^{\alpha(t)}d\tau = c_1 \mu^{\alpha(t)} - c_2 \geq 0 
\]

with positive constants \( c_1, c_2 \) satisfying \( c_1 \mu^{\alpha(t)}(t_1) = c_2 \).

Then, we have

\[
\mu^{\alpha+\beta}(t) \xi(t) \leq A_1(t) + A_2(t) 
\]

where

\[
A_1(t) = \mu^{\alpha+\beta} e^{-\alpha(t)} \xi(t_1), 
\]  

(F.1)

\[
A_2(t) = \int_{t_1}^{t} e^{\alpha(t)} b(\tau)d\tau 
\]

39
\[ A_2(t) = \mu^{\alpha+\beta} e^{-a\pi(t)} \int_{t_1}^t e^{a\pi(\tau)} \frac{\chi}{\mu^{\beta}(\tau)} d\tau. \]  \hspace{1cm} (F.2)

Next, we will show \( A_1(t) \) and \( A_2(t) \) are both bounded. For \( A_1(t) \), note that by Taylor expansion

\[ e^{a\pi(t)} = \sum_{n=0}^{+\infty} \frac{(a\pi)^n}{n!}. \]  \hspace{1cm} (F.3)

Hence, there exists a positive constant \( c_3 \) such that

\[ e^{a\pi(t)} \geq 1 + c_3(c_1\mu^{\alpha-1}(t) - c_2)^n \]  \hspace{1cm} (F.4)

where \( n \) can be any finite positive integers.

Using (F.4) for (F.1), we have

\[ A_1(t) \leq \xi(t_1) \frac{\mu^{\alpha+\beta}}{1 + c_3(c_1\mu^{\alpha-1} - c_2)^n} \]  \hspace{1cm} (F.5)

Hence, if \( n \) is large enough such that \( n(\alpha - 1) > \alpha + \beta \), then \( \mu^n(\alpha-1) \) will dominate other terms in (F.5) when \( t \rightarrow T^- \). As a result,

\[ \lim_{t \rightarrow T^-} \xi(t_1) \frac{\mu^{\alpha+\beta}}{1 + c_3(c_1\mu^{\alpha-1} - c_2)^n} = 0. \]

Then we can conclude that \( A_1(t) \) is bounded.

For \( A_2(t) \), note that it is continuous on \([0, T)\). Hence, we only need to show when \( t \rightarrow T^- \), \( A_2(t) \) is bounded. Then \( A_2(t) \) is bounded on \([0, T)\). By L’Hospital rule,

\[
\lim_{t \rightarrow T^-} A_2(t) = \lim_{t \rightarrow T^-} \frac{\int_{t_1}^t e^{a\pi(\tau)} d\tau}{\mu^{\alpha+\beta}(t)} = \lim_{t \rightarrow T^-} \frac{\chi e^{a\pi(t)} - c_4}{\mu^{\alpha+\beta}(t)}
\]

\[
= \lim_{t \rightarrow T^-} \frac{\chi e^{a\pi(t)} - c_4}{c_5 e^{a\pi(t)} \mu^{2\alpha+2\beta} - c_6 e^{a\pi(t)} \mu^{\alpha+2\beta} - c_7 e^{a\pi(t)} \mu^{\alpha+2\beta+1}}
\]

\hspace{1cm} (F.6)

where \( c_4, c_5, c_6, c_7 \) are positive constants.

According to (F.3), we know \( e^{a\pi(t)} \mu^{2\alpha+2\beta} \) will dominate the other terms in (F.6) when \( t \rightarrow T^- \). Hence

\[ \lim_{t \rightarrow T^-} A_2(t) = \frac{\chi}{c_5} \]

which is bounded. This indicates that \( \mu^{\alpha+\beta}(t) \xi(t) \) and \( \xi(t) \) are both bounded on \([0, T)\).

From (D.3), we know when \( \xi(t) \) is bounded, there exists a positive constant \( \delta \) such that \(|x^2(t)| \leq \chi^2 - \delta\). Let \( X = x^2 \), by Mean Value Theorem,

\[ \ln \left( \frac{\chi^2}{\chi^2 - X} \right) = \frac{\partial \ln \left( \frac{\chi}{\chi^2 - X} \right)}{\partial X} \bigg|_{X \in [\chi^2 - \delta]} X \geq c_8 x^2 \]

\[ 40 \]
where $c_8$ is a positive constant.

Therefore, using (D.3) we obtain $x^2(t)\mu^{\alpha+\beta}(t)$ is bounded on $[0, T)$.

Finally, when $b(t) = 0$, similar to (F.5), we know

$$\xi(t) \leq \xi(t_1) \frac{\mu^{\gamma}}{1 + c_3(c_1\mu^{\alpha-1} - c_2)^n}.$$  

Thus, if $n(\alpha - 1) > \gamma$, $\xi(t)$ will become zero when $t \to T^-$. This completes the proof.

\[\square\]

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