Update: Some new results on lower bounds on $(n, r)$-arcs in $\text{PG}(2, q)$ for $q \leq 31$

Michael Braun (michael.braun@h-da.de)
Faculty of Computer Science
University of Applied Sciences, Darmstadt, Germany

June 11, 2021

Abstract

An $(n, r)$-arc in $\text{PG}(2, q)$ is a set $\mathcal{B}$ of points in $\text{PG}(2, q)$ such that each line in $\text{PG}(2, q)$ contains at most $r$ elements of $\mathcal{B}$ and such that there is at least one line containing exactly $r$ elements of $\mathcal{B}$. The value $m_r(2, q)$ denotes the maximal number $n$ of points in the projective geometry $\text{PG}(2, q)$ for which an $(n, r)$-arc exists. By explicitly constructing $(n, r)$-arcs using prescribed automorphisms and integer linear programming we obtain some improved lower bounds for $m_r(2, q)$: $m_{10}(2, 16) \geq 144$, $m_3(2, 25) \geq 39$, $m_{18}(2, 25) \geq 418$, $m_9(2, 27) \geq 201$, $m_{14}(2, 29) \geq 364$, $m_{25}(2, 29) \geq 697$, $m_{25}(2, 31) \geq 734$. Furthermore, we show by systematically excluding possible automorphisms that putative $(44, 5)$-arcs, $(90, 9)$-arcs in $\text{PG}(2, 11)$, and $(39, 4)$-arcs in $\text{PG}(2, 13)$—in case of their existence—are rigid, i.e. they all would only admit the trivial automorphism group of order 1. In addition, putative $(50, 5)$-arcs, $(65, 6)$-arcs, $(119, 10)$-arcs, $(133, 11)$-arcs, and $(146, 12)$-arcs in $\text{PG}(2, 13)$ would be rigid or would admit a unique automorphism group (up to conjugation) of order 2.
1 Introduction

Definition 1. An \((n, r)\)-arc in \(\text{PG}(2, q)\) is a set \(\mathcal{B}\) of points in \(\text{PG}(2, q)\) such that each line in \(\text{PG}(2, q)\) contains at most \(r\) elements of \(\mathcal{B}\) and such that there is at least one line containing exactly \(r\) elements of \(\mathcal{B}\).

It is well-known (e.g. see [3]) that \((n, r)\)-arcs in \(\text{PG}(2, q)\) are closely related to error-correcting linear codes: The \(n\) points of an \((n, r)\)-arc in \(\text{PG}(2, q)\) define the columns of a \(3 \times n\) generator matrix of linear \([n, 3, n - r]_q\) code, which is code of length \(n\), dimension 3, and minimum distance \(n - r\) with respect to the Hamming metric. The linear code is projective since the columns of any generator matrix are pairwise linearly independent.

Definition 2. Let \(m_r(2, q)\) denote the maximum number \(n\) for which an \((n, r)\)-arc in \(\text{PG}(2, q)\) exists.

A major goal in studying \((n, r)\)-arcs in \(\text{PG}(2, q)\) is the determination of \(m_r(2, q)\).

In general it is hard to determine the exact value of \(m_r(2, q)\) and in most cases instead of the exact value only a lower and an upper bound for \(m_r(2, q)\) are known. An explicit construction of an \((n, r)\)-arc in \(\text{PG}(2, q)\) yields a lower bound \(m_r(2, q) \geq n\).

The values \(m_r(2, q)\) with \(q \leq 9\) are exactly determined (see [3]). For \(m_r(2, q)\) with \(11 \leq q \leq 19\) we refer to [2] whereas a table for \(23 \leq q \leq 31\) can compiled from several sources [5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. An recent overview with tables on all values \(q \leq 31\) can be found in [5].

In this article, we extend the results from [5] of lower bounds on \(m_r(2, q)\) and give some improvements listed in Table 1.

Furthermore, as a second result in this work, we show that the putative \((n, r)\)-arcs in \(\text{PG}(2, q)\) for \(q \in \{11, 13\}\) for the open gaps between lower and upper bound on \(m_r(2, q)\)—in case of existence—are rigid or only admit a unique automorphism of order 2.
Table 1: Improved lower bounds on $m_r(2, q)$

| $q$ | $r$ | old bound | new bound |
|-----|-----|-----------|-----------|
| 16  | 10  | 142       | 144       |
| 25  | 3   | 38        | 39        |
| 25  | 18  | 416       | 418       |
| 27  | 9   | 198       | 201       |
| 29  | 14  | 361       | 364       |
| 29  | 25  | 695       | 697       |
| 31  | 25  | 733       | 734       |

2 Construction by integer linear programming

We use the construction of $(n, r)$-arcs in PG(2, $q$) with prescribed groups of automorphisms using integer linear programming described in [4, 5]:

In the following, let $\left[ \frac{\text{GF}(q)^n}{k} \right]$ denote the set of $k$-dimensional subspaces of $\text{GF}(q)^n$, which is called the Grassmannian. Its cardinality is given by the Gaussian number, also called $q$-Binomial coefficient:

$$\left[ \frac{n}{k} \right]_q = \left| \left[ \frac{\text{GF}(q)^n}{k} \right] \right| = \prod_{i=0}^{k-1} q^n - q^i.$$ 

In terms of vector spaces, an $(n, r)$-arc in PG(2, $q$) corresponds to a set $\mathcal{B} \subseteq \left[ \frac{\text{GF}(q)^3}{1} \right]$ such that for all $H \in \left[ \frac{\text{GF}(q)^3}{2} \right]$ holds:

$$| \{ P \in \mathcal{B} \mid H \supseteq P \}| \leq r.$$ 

If $\left[ \frac{\text{GF}(q)^3}{1} \right] = \{ P_1, \ldots, P_{q^2+q+1} \}$ and $\left[ \frac{\text{GF}(q)^3}{2} \right] = \{ H_1, \ldots, H_{q^2+q+1} \}$, where $\left[ \frac{3}{1} \right]_q = \left[ \frac{3}{2} \right]_q = q^2 + q + 1$, we define the $(q^2 + q + 1) \times (q^2 + q + 1)$ incidence matrix

$$A(q) = (a_{ij})$$
with entries
\[ a_{ij} :=\begin{cases} 1 & \text{if } H_i \supseteq P_j, \\ 0 & \text{otherwise}. \end{cases} \]

**Lemma 1.** If \( u = (1, \ldots, 1)^T \) denote the all-one vector any binary column vector \( x \) satisfying
\[ A(q) \cdot x \leq r \cdot u \]
is equivalent to a \((u^T \cdot x, r)\)-arc in \( \text{PG}(2, q) \).

**Corollary 1.** The determination of \( m_r(2, q) \) corresponds to the following integer linear programming problem:
\[ m_r(2, q) = \max_{x \in \{0, 1\}^{q^2+q+1}} \{ u^T \cdot x \mid A(q) \cdot x \leq r \cdot u \}. \]

The incidence preserving bijections (automorphisms) of our ambient space for \((n, r)\) arcs—the projective geometry \( \text{PG}(2, q) \)—are defined by the projective semi-linear group \( \text{PΓL}(3, q) \) (see [1]). It acts transitively on the Grassmannian \( \left[ \mathbb{GF}(q)^3 \atop k \right] \).

Hence, any subgroup \( G \leq \text{PΓL}(3, q) \) partitions the Grassmannian into \( G \)-orbits. If \( \alpha \in \text{PΓL}(3, q) \) and \( S \in \left[ \mathbb{GF}(q)^3 \atop k \right] \) we denote by
\[ \alpha S := \{ \alpha x \mid x \in S \} \]
the transformed subspace and by
\[ G(S) := \{ \alpha S \mid \alpha \in G \} \]
the \( G \)-orbit of \( S \). The set of all \( G \)-orbits will be written as
\[ G \setminus \left[ \mathbb{GF}(q)^3 \atop k \right] . \]

**Definition 3.** An \((n, r)\)-arc \( B \) in \( \text{PG}(2, q) \) admits a subgroup \( G \leq \text{PΓL}(3, q) \) as a group of automorphisms if and only if \( B \) consists of \( G \)-orbits on \( \left[ \mathbb{GF}(q)^3 \atop 1 \right] \).

The maximal group of automorphisms of \( B \) is called the automorphism group of \( B \) and abbreviated by
\[ \text{Aut}(B) . \]
Definition 4. Let $m^{G}_r(2, q)$ denote the maximal size $n$ of an $(n, r)$-arc in $\text{PG}(2, q)$ admitting $G \leq \text{PGL}(3, q)$ as a group of automorphisms.

Corollary 2. For any $G \leq \text{PGL}(3, q)$ we get a lower bound

$$m^{G}_r(2, q) \leq m_r(2, q).$$

In particular, for the trivial group $G = \{1\}$ we have

$$m^{\{1\}}_r(2, q) = m_r(2, q).$$

If $\{P_1, \ldots, P_\ell\}$ denotes a set of representatives of the orbits $G\backslash [\mathbb{GF}(q)^3]$ and $\{H_1, \ldots, H_\ell\}$ a transversal of the orbits $G\backslash [\mathbb{GF}(q)^3]$ for any $G \leq \text{PGL}(3, q)$ we define the $G$-incidence matrix $A(G) = (a_{ij})$ with

$$a_{ij} := |\{P \in G(P_j) \mid H_i \supseteq P\}|.$$

Furthermore, by $w(G) = (w_1, \ldots, w_\ell)^T$ we denote the vector of the lengths of $G$-orbits on $[\mathbb{GF}(q)^3]$, i.e.

$$w_j := |G(P_j)|.$$

Note that the number of orbits of $G$ on the set of points and hyperplanes is equal

$$\ell = |G\backslash [\mathbb{GF}(q)^3]| = |G\backslash [\mathbb{GF}(q)^3]| \leq q^2 + q + 1.$$

Theorem 1. Any binary vector $x$ of length $\ell = |G\backslash [\mathbb{GF}(q)^3]|$ with

$$A(G) \cdot x \leq r \cdot u$$

corresponds to a $(w(G)^T \cdot x, r)$-arc in $\text{PG}(2, q)$ admitting $G \leq \text{PGL}(3, q)$ as a group of automorphisms. In addition, we obtain the following integer linear programming:

$$m^{G}_r(2, q) = \max_{x \in \{0,1\}^\ell} \{w(G)^T \cdot x \mid A(G) \cdot x \leq r \cdot u\}.$$
3 Constructed arcs

We list \((n, r)\)-arcs in \(\text{PG}(2, q)\) in the appendix constructed with the proposed approach and with improved size. All \((n, r)\)-arcs in \(\text{PG}(2, q)\) were computed with Gurobi (see [20]) as ILP solver.

Elements of the prime field \(\text{GF}(p)\) are represented by integers \(0 \leq a < p\) where elements are added and multiplied modulo \(p\). In extension fields \(\text{GF}(p^e)\) the elements \(\sum_{i=0}^{e-1} a_i x^i\) are given by the numbers \(\sum_{i=0}^{e-1} a_i p^i\) where elements are added and multiplied modulo a given irreducible polynomial \(f(x) \in \text{GF}(p)[x]\) of degree \(e\). For the finite fields \(\text{GF}(16), \text{GF}(25),\) and \(\text{GF}(27)\) we use the irreducible polynomials:

\[
\begin{align*}
x^4 + x^3 + 1 &\in \text{GF}(2)[x], \\
x^2 + x + 2 &\in \text{GF}(5)[x], \\
x^3 + 2x + 1 &\in \text{GF}(3)[x].
\end{align*}
\]

As group we used subgroups projective linear groups \(G \leq \text{PGL}(3, q) \leq \text{PTL}(3, q)\). For two cases we used the symmetric group \(S_3\) generated by two matrices

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

For the remaining cases we used random cyclic groups generated by an invertible \(3 \times 3\) matrix.

4 Excluding automorphisms

An open gap is an entry in the tables of \(m_r(2, q)\) for which upper and lower bound differ:

\[
\ell \leq m_r(2, q) \leq u \quad \text{where} \quad \ell < u.
\]

In that case the question is whether an \((\ell + 1, r)\)-arc in \(\text{PG}(2, q)\) exits or not. We call such an arc a putative arc in \(\text{PG}(2, q)\). In [6] it was shown that for the
gap $100 \leq m_{10}(2, 11) \leq 101$ a putative $(101, 10)$-arc in $\text{PG}(2, 11)$ admits—in case of its existence—only the trivial automorphism group of order 1.

In this paper we consider the remaining gaps for $q = 11$ and $q = 13$ which are given in Table 2.

| $m_5(2, 11)$ | $\in \{43, 44, 45\}$ |
| $m_9(2, 11)$ | $\in \{89, 90\}$ |
| $m_4(2, 13)$ | $\in \{38, 39, 40\}$ |
| $m_5(2, 13)$ | $\in \{49, 50, 51, 52, 53\}$ |
| $m_6(2, 13)$ | $\in \{64, 65, 66\}$ |
| $m_{10}(2, 13)$ | $\in \{118, 119\}$ |
| $m_{11}(2, 13)$ | $\in \{132, 133\}$ |
| $m_{12}(2, 13)$ | $\in \{145, 146, 147\}$ |

**Definition 5.** Let $\mathcal{B}, \mathcal{B}'$ be an $(n, r)$-arcs in $\text{PG}(2, q)$ The two sets $\mathcal{B}$ and $\mathcal{B}'$ are defined to be isomorphic if and only if there exists $\alpha \in \text{PGL}(3, q)$ such that

$$\alpha \mathcal{B} := \{\alpha P \mid P \in \mathcal{B}\} = \mathcal{B}'$$

The set of all arcs that are isomorphic to $\mathcal{B}$ is denoted by

$$\text{PGL}(3, q)(\mathcal{B}) := \{\alpha \mathcal{B} \mid \alpha \in \text{PGL}(3, q)\}.$$  

Note that due to the incidence preserving property of $\text{PGL}(3, q)$ isomorphic arcs have the same parameters.

The following lemma is well-known from the theory of group actions (cf. [21]) and states that the automorphism groups of isomorphic objects are conjugated.

**Lemma 2.** Let $\mathcal{B}$ be an $(n, r)$-arc in $\text{PG}(2, q)$ and let $\alpha \in \text{PGL}(3, q)$. Then we obtain:

$$\text{Aut}(\alpha \mathcal{B}) = \alpha \text{Aut}(\mathcal{B}) \alpha^{-1} = \{\alpha \beta \alpha^{-1} \mid \beta \in \text{Aut}(\mathcal{B})\}.$$
If \( B \) in an \((n, r)\)-arc in \( \text{PG}(2, q) \) with \( G \leq \text{Aut}(B) \) then any isomorphic arc \( B' = \alpha B \) for \( \alpha \in \text{PGL}(3, q) \) admits the conjugated group \( G' = \alpha G \alpha^{-1} \) satisfies
\[
G' = \alpha G \alpha^{-1} \leq \alpha \text{Aut}(G) \alpha^{-1} = \text{Aut}(\alpha B) = \text{Aut}(B'),
\]
which means that the conjugated group \( G' \) also occurs as a group of automorphisms of \( B' \).

As a consequence, when aiming for \((n, r)\)-arcs in \( \text{PG}(2, q) \) with prescribed groups of automorphisms it is sufficient to consider representatives of conjugacy classes of subgroups of \( \text{PGL}(3, q) \) as possible candidates for potential groups to be prescribed.

Furthermore, any \((n, r)\)-arc \( B \) in \( \text{PG}(2, q) \) with \( \{1\} < G \leq \text{Aut}(B) \) also admits all cyclic subgroups \( C \leq G \) as groups of automorphisms.

**Corollary 3.** If we can show for all representatives \( C \) of conjugacy classes of nontrivial cyclic subgroups of \( \text{PGL}(3, q) \) that no \((n, r)\)-arc in \( \text{PG}(2, q) \) exists with \( C \) as group as automorphisms, either the automorphism group of such arcs are trivial or arcs with that set of parameters do not exist.

In case of a prime field \( \text{GF}(q) \) the projective semi-linear group is exactly the projective linear group
\[
\text{PGL}(3, q) = \text{PGL}(3, q).
\]

In the following, a transversal of conjugacy classes of cyclic subgroups of \( \text{PGL}(3, q) \) will be abbreviated by
\[
\text{Conj}(q).
\]

Its cardinality is given by (see [22]):
\[
|\text{Conj}(q)| = \begin{cases} 
q^2 + q + 2 & \text{if } 3 \text{ divides } q - 1, \\
q^2 + q & \text{otherwise}.
\end{cases}
\]
Lemma 3. Let $q$ be a prime. If

\[ m_C^r(2, q) < n \quad \forall C \in \text{Conj}(q) \setminus \{1\} \]

one of the following conditions holds:

1. $m_r(2, q) < n$.

2. $(n, r)$-arcs $\mathcal{B}$ in $\text{PG}(2, q)$ exist where $\text{Aut}(\mathcal{B}) = \{1\}$.

We now apply this corollary to the parameters $(q, n, r) = (11, 44, 5)$, $(q, n, r) = (11, 90, 9)$, and $(q, n, r) = (13, 50, 5)$. There are $|\text{Conj}(11)| = 132$ conjugacy classes of cyclic subgroups of $\text{PGL}(3, 11)$ and $|\text{Conj}(13)| = 184$ classes in $\text{PGL}(3, 13)$. We compute the representatives using GAP [19]. By solving the integer linear programming $m_C^r(2, q)$ according to Theorem 1 for all $C \in \text{Conj}(q) \setminus \{1\}$ using Gurobi [20] we obtain with a runtime less than 3 hours on a 1.2 GHz Intel Core m3 processor the following result:

Theorem 2. In case of their existence the automorphism groups of $(44, 5)$-arcs, $(90, 9)$-arcs in $\text{PG}(2, 11)$, and $(50, 5)$-arcs in $\text{PG}(2, 13)$ would be trivial of order 1.

For the remaining parameters $r \in \{5, 6, 10, 11, 12\}$ for $q = 13$ we apply a slightly adapted version of Lemma 3 since for these open cases exactly one cyclic subgroup

\[ C_0 := \langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 12 \end{pmatrix} \rangle \]

of order 2 could not directly be excluded to be a group of automorphisms of putative arcs since we cancelled the ILP solver for $m_C^{C_0}(2, 13)$ after a 5000 seconds (for each value $r$).

But it is obvious to conject that this group can also be excluded if we spend more running time on the ILP solver.

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Lemma 4. Let $q$ be a prime. Let $C_0 \in \text{Conj}(q)$. If

$$m^C_r(2,q) < n \quad \forall C \in \text{Conj}(q) \setminus \{\{1\}, C_0\}$$

one of the following conditions holds:

1. $m_r(2,q) < n$.

2. $(n,r)$-arcs $B$ in $\text{PG}(2,q)$ exist where either $\text{Aut}(B) = \{1\}$ or $\text{Aut}(B)$ is conjugated to $C_0$.

Finally, we get

Theorem 3. In case of their existence the automorphism groups of $(50,5)$-arcs, $(65,6)$-arcs, $(119,10)$-arcs, $(133,11)$-arcs, and $(146,12)$-arcs in $\text{PG}(2,13)$ would either be trivial of order 1 or would be the following cyclic subgroup of order 2 (up to conjugation):

$$C_0 := \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 12 \end{pmatrix} \right\rangle.$$

References

[1] R. Baer, Linear algebra and projective geometry, Academic Press, New York, NY, (1952).

[2] S. Ball, Three-dimensional linear codes, online table, [https://mat-web.upc.edu/people/simeon.michael.ball/codebounds.html](https://mat-web.upc.edu/people/simeon.michael.ball/codebounds.html)

[3] S. Ball and J. W. P. Hirschfeld, Bounds on $(n,r)$-arcs and their application to linear codes, Finite Fields and Their Applications, 11 (3) (2005), 326-336.

[4] M. Braun, A. Kohnert and A. Wassermann, Construction of $(n,r)$-arcs in $\text{PG}(2,q)$, Innovations in Incidence Geometry, 1 (2005), 133-141.
[5] M. Braun, New lower bounds on the size of $(n, r)$-arcs in $\text{PG}(2, q)$, Journal of Combinatorial Designs, 27 (2019), 682-687.

[6] M. Braun, A note on putative $(101, 10)$-arcs in $\text{PG}(2, 11)$. to appear in the Journal of Combinatorial Mathematics and Combinatorial Computing.

[7] B. Csajbók and T. Héger, Double blocking sets of size $3q-1$ in $\text{PG}(2, q)$, European Journal of Combinatorics, 78 (2019), 73-89.

[8] R. Daskalov, On the maximum size of some $(k, r)$-arcs in $\text{PG}(2, q)$, Discrete Mathematics, 308 (2008), 565-570

[9] R. Daskalov and E. Metodieva, New $(n, r)$-arcs in $\text{PG}(2, 17)$, $\text{PG}(2, 19)$ and $\text{PG}(2, 23)$, Problemy Peredachi Informatsii, 47(3) (2011), 3-9, English translation: Problems of Information Transmission, 47(3) (2011), 217-223.

[10] R. Daskalov and E. Metodieva, Improved bounds on $m_r(2, q)$ $q = 19, 25, 27$. Hindawi Publishing Corporation, Journal of Discrete Mathematics, (2013), Article ID 628952, 7 pages.

[11] R. Daskalov and M. Manev, A new $(37, 3)$-arc in $\text{PG}(2, 23)$, Electronic Notes in Discrete Mathematics, 57 (2017), 97-102.

[12] R. Daskalov and E. Metodieva, Some new $(n, r)$-arcs in $\text{PG}(2, 31)$, Electronic Notes in Discrete Mathematics, 57 (2017), 109-114.

[13] R. Daskalov and E. Metodieva, On the construction of small $(l, t)$-blocking sets in $PG(2, q)$, Proc. VIII International Workshop on Optimal Codes and Related Topics, July 2017, Sofia, Bulgaria, 61-66.

[14] R. Daskalov and E. Metodieva, Four new large $(n, r)$-arcs in $\text{PG}(2, 31)$, Sixteenth International Workshop on Algebraic and Combinatorial Coding Theory, September 2-8, 2018, Svetlogorsk, Russia, 137-139.
[15] R. Daskalov, Bounds on $m_r(2, 29)$, to appear in the Iranian Journal of Mathematical Sciences and Informatics, 14 (2019), 127-138.

[16] R. Daskalov and E. Metodieva, Three new large $(n, r)$-arcs in PG(2, 31), 2020 Algebraic and Combinatorial Coding Theory (2020), 1-4.

[17] A.A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco, Linear nonbinary covering codes and saturating sets in projective spaces, Adv. Math. Commun., 5(1) (2011), 119-147.

[18] J.W.P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces: update 2001, in: Finite Geometries, Developments in Mathematics, vol. 3, Kluwer, Boston, 2001, 201-246.

[19] The GAP Group, GAP – Groups, Algorithms, Programming, version 4.10.0 (2018), https://www.gap-system.org

[20] Gurobi Optimization, LLC, Gurobi Optimizer Reference Manual, (2018), http://www.gurobi.com/

[21] A. Kerber, Applied finite group actions. Springer-Verlag, 1999.

[22] I.G. MacDonald, Numbers of conjugacy classes in some finite classical groups. Bulletin of the Australian Mathematical Society, 23 (1981), 23?48.

Appendix

4.1 $m_{10}(2, 16) \geq 144$

group: symmetric group $S_3$

(0, 0, 1), (1, 0, 0), (0, 1, 0), (0, 1, 4), (1, 0, 4), (1, 0, 6), (0, 1, 6), (1, 4, 0), (1, 6, 0),
(0, 1, 5), (1, 0, 5), (1, 0, 15), (0, 1, 15), (1, 5, 0), (1, 15, 0), (1, 1, 2), (1, 12, 12),
(1, 2, 1), (1, 1, 3), (1, 8, 8), (1, 3, 1), (1, 1, 5), (1, 15, 15), (1, 5, 1), (1, 1, 7), (1, 14, 14),
(1, 7, 1), (1, 1, 8), (1, 3, 3), (1, 8, 1), (1, 1, 9), (1, 13, 13), (1, 9, 1), (1, 1, 10), (1, 11, 11), (1, 10, 1), (1, 2, 3), (1, 12, 13), (1, 8, 9), (1, 3, 2), (1, 13, 12), (1, 9, 8), (1, 2, 4), (1, 12, 2), (1, 16, 12), (1, 4, 2), (1, 2, 12), (1, 12, 6), (1, 2, 7), (1, 12, 15), (1, 14, 5), (1, 7, 2), (1, 15, 12), (1, 5, 14), (1, 2, 8), (1, 12, 4), (1, 3, 6), (1, 8, 2), (1, 4, 12), (1, 6, 3), (1, 2, 10), (1, 12, 5), (1, 11, 15), (1, 10, 2), (1, 5, 12), (1, 15, 11), (1, 2, 14), (1, 12, 7), (1, 7, 14), (1, 14, 2), (1, 7, 12), (1, 14, 7), (1, 2, 15), (1, 12, 11), (1, 5, 10), (1, 15, 2), (1, 11, 12), (1, 10, 5), (1, 3, 4), (1, 8, 11), (1, 6, 10), (1, 4, 3), (1, 11, 8), (1, 10, 6), (1, 3, 7), (1, 8, 10), (1, 14, 11), (1, 7, 3), (1, 10, 8), (1, 11, 14), (1, 3, 9), (1, 8, 7), (1, 13, 14), (1, 9, 3), (1, 7, 8), (1, 14, 13), (1, 3, 10), (1, 8, 6), (1, 11, 4), (1, 10, 3), (1, 6, 8), (1, 4, 11), (1, 3, 11), (1, 8, 14), (1, 10, 7), (1, 11, 3), (1, 14, 8), (1, 7, 10), (1, 4, 6), (1, 6, 13), (1, 4, 9), (1, 6, 4), (1, 13, 6), (1, 9, 4), (1, 4, 14), (1, 6, 15), (1, 7, 5), (1, 14, 4), (1, 15, 6), (1, 5, 7), (1, 4, 15), (1, 6, 9), (1, 5, 13), (1, 15, 4), (1, 9, 6), (1, 13, 5), (1, 5, 9), (1, 15, 10), (1, 13, 11), (1, 9, 5), (1, 10, 15), (1, 11, 13), (1, 5, 11), (1, 15, 13), (1, 10, 9), (1, 11, 5), (1, 13, 15), (1, 9, 10), (1, 7, 13), (1, 14, 9), (1, 9, 13), (1, 13, 7), (1, 9, 14), (1, 13, 9)

4.2 \quad m_3(2, 25) \geq 39

group: cyclic; generated by

$$
\begin{pmatrix}
0 & 10 & 13 \\
13 & 3 & 5 \\
18 & 18 & 4
\end{pmatrix}
$$

(0, 1, 12), (1, 5, 2), (1, 6, 14), (1, 0, 6), (1, 4, 7), (1, 10, 21), (1, 0, 13), (1, 5, 6), (1, 12, 14), (1, 0, 18), (1, 14, 0), (1, 13, 8), (1, 2, 21), (1, 13, 4), (1, 18, 19), (1, 2, 22), (1, 6, 12), (1, 14, 13), (1, 3, 2), (1, 22, 10), (1, 12, 12), (1, 3, 10), (1, 7, 23), (1, 23, 0), (1, 3, 20), (1, 11, 13), (1, 4, 10), (1, 6, 3), (1, 10, 20), (1, 20, 21), (1, 12, 15), (1, 13, 12), (1, 14, 15), (1, 24, 19), (1, 23, 20), (1, 19, 0), (1, 19, 7), (1, 19, 2), (1, 23, 16)
4.3 $m_{18}(2, 25) \geq 418$

group: cyclic; generated by

$$\begin{pmatrix} 19 & 24 & 11 \\ 21 & 3 & 18 \\ 22 & 8 & 7 \end{pmatrix}$$

(0, 1, 0), (1, 20, 9), (1, 21, 5), (1, 5, 12), (0, 1, 1), (1, 10, 2), (1, 10, 12), (1, 15, 9),
(0, 1, 3), (1, 11, 12), (1, 6, 9), (1, 10, 0), (0, 1, 4), (1, 8, 16), (1, 0, 13), (1, 8, 19),
(0, 1, 5), (1, 18, 23), (1, 18, 21), (1, 2, 24), (0, 1, 7), (1, 13, 7), (1, 7, 3), (1, 3, 8),
(0, 1, 9), (1, 9, 1), (1, 19, 15), (1, 19, 20), (0, 1, 10), (1, 22, 4), (1, 17, 2), (0, 1, 14),
(0, 1, 11), (1, 2, 15), (1, 9, 16), (1, 9, 3), (0, 1, 12), (1, 12, 22), (1, 14, 18), (1, 21, 4),
(0, 1, 15), (1, 0, 20), (1, 5, 10), (1, 24, 6), (0, 1, 16), (1, 1, 5), (1, 13, 24), (1, 7, 5),
(0, 1, 17), (1, 24, 24), (1, 1, 7), (1, 4, 17), (0, 1, 20), (1, 3, 0), (1, 20, 11), (1, 16, 18),
(0, 1, 21), (1, 16, 3), (1, 4, 19), (1, 18, 11), (0, 1, 23), (1, 19, 8), (1, 16, 8), (1, 10, 23),
(1, 0, 2), (1, 19, 14), (1, 4, 1), (1, 17, 24), (1, 0, 3), (1, 13, 20), (1, 7, 23), (1, 2, 17),
(1, 0, 4), (1, 1, 17), (1, 15, 10), (1, 15, 19), (1, 6, 0), (1, 3, 13), (1, 2, 3), (1, 5, 21),
(1, 0, 8), (1, 21, 11), (1, 6, 24), (1, 4, 22), (1, 0, 10), (1, 4, 21), (1, 13, 17), (1, 7, 1),
(1, 0, 11), (1, 15, 22), (1, 3, 2), (1, 21, 18), (1, 0, 14), (1, 17, 18), (1, 18, 12), (1, 16, 9),
(1, 0, 15), (1, 14, 8), (1, 14, 16), (1, 1, 2), (1, 0, 16), (1, 24, 15), (1, 16, 14), (1, 6, 11),
(1, 0, 17), (1, 10, 16), (1, 19, 11), (1, 24, 13), (1, 0, 18), (1, 22, 24), (1, 5, 20),
(1, 9, 6), (1, 0, 21), (1, 11, 4), (1, 11, 19), (1, 14, 15), (1, 0, 23), (1, 16, 5), (1, 9, 21),
(1, 10, 5), (1, 0, 24), (1, 18, 1), (1, 12, 18), (1, 12, 10), (1, 1, 0), (1, 20, 0), (1, 20, 13),
(1, 15, 21), (1, 1, 1), (1, 3, 19), (1, 13, 16), (1, 7, 20), (1, 1, 3), (1, 22, 1), (1, 24, 21),
(1, 18, 19), (1, 1, 8), (1, 24, 2), (1, 5, 1), (1, 2, 6), (1, 1, 9), (1, 12, 21), (1, 9, 14),
(1, 16, 12), (1, 1, 11), (1, 9, 7), (1, 17, 23), (1, 17, 3), (1, 1, 12), (1, 8, 9), (1, 2, 11),
(1, 8, 11), (1, 1, 14), (1, 5, 5), (1, 16, 6), (1, 19, 5), (1, 1, 15), (1, 1, 18), (1, 10, 20),
(1, 22, 17), (1, 1, 21), (1, 21, 3), (1, 19, 2), (1, 3, 22), (1, 2, 2), (1, 2, 13), (1, 22, 15),
(1, 12, 19), (1, 2, 4), (1, 23, 23), (1, 11, 13), (1, 16, 4), (1, 2, 5), (1, 17, 22), (1, 15, 5),
(1, 21, 20), (1, 2, 9), (1, 18, 2), (1, 6, 10), (1, 6, 12), (1, 2, 12), (1, 13, 9), (1, 7, 16),
(1, 4, 24), (1, 2, 14), (1, 21, 10), (1, 14, 1), (1, 10, 22), (1, 2, 18), (1, 19, 7), (1, 2, 19),
(1, 10, 19), (1, 4, 0), (1, 20, 14), (1, 2, 20), (1, 24, 0), (1, 20, 8), (1, 17, 10), (1, 2, 22),
14
4.4 \( m_9(2, 27) \geq 201 \)

group: cyclic; generated by

\[
\begin{pmatrix}
15 & 26 & 2 \\
12 & 3 & 17 \\
25 & 10 & 3
\end{pmatrix}
\]

(0, 0, 1), (1, 22, 6), (1, 21, 15), (1, 21, 25), (1, 17, 10), (1, 24, 2), (1, 22, 9), (1, 24, 13), (0, 1, 1), (1, 6, 20), (1, 7, 7), (1, 17, 17), (1, 1, 12), (1, 0, 2), (1, 25, 9), (1, 17, 16), (0, 1, 5), (1, 2, 17), (1, 26, 14), (1, 3, 21), (1, 5, 20), (1, 14, 2), (1, 12, 9), (1, 9, 25), (0, 1, 6), (1, 25, 16), (1, 1, 20), (1, 15, 0), (1, 10, 13), (1, 20, 2), (1, 16, 9), (1, 18, 8), (0, 1, 14), (1, 14, 23), (1, 4, 12), (1, 16, 22), (1, 2, 1), (1, 15, 2), (1, 13, 9), (1, 26, 26), (0, 1, 16), (1, 19, 26), (1, 18, 23), (1, 23, 3), (1, 15, 8), (1, 12, 2), (1, 1, 9), (1, 7, 10), (1, 0, 6), (1, 16, 7), (1, 19, 20), (1, 13, 17), (1, 14, 22), (1, 13, 18), (1, 13, 3), (1, 11, 21), (1, 0, 10), (1, 10, 10), (1, 23, 8), (1, 26, 18), (1, 23, 11), (1, 10, 26), (1, 11, 10), (1, 13, 6), (1, 0, 11), (1, 24, 17), (1, 22, 25), (1, 23, 24), (1, 21, 16), (1, 3, 22), (1, 26, 23), (1, 5, 10), (1, 0, 16), (1, 8, 24), (1, 9, 22), (1, 14, 15), (1, 11, 7), (1, 15, 6), (1, 7, 20), (1, 18, 3), (1, 0, 20), (1, 5, 12), (1, 24, 14), (1, 22, 26), (1, 7, 25), (1, 21, 20), (1, 10, 1), (1, 2, 20), (1, 0, 21), (1, 4, 11), (1, 26, 4), (1, 4, 8), (1, 4, 1), (1, 12, 24), (1, 25, 14), (1, 6, 17), (1, 0, 23), (1, 14, 26), (1, 3, 3), (1, 10, 14), (1, 24, 4), (1, 22, 23), (1, 12, 21), (1, 21, 23), (1, 1, 1), (1, 8, 15), (1, 8, 11), (1, 10, 15), (1, 13, 15), (1, 12, 14), (1, 2, 22), (1, 8, 12), (1, 1, 11), (1, 11, 19), (1, 9, 6), (1, 25, 12), (1, 3, 8), (1, 7, 19), (1, 13, 14), (1, 4, 0), (1, 1, 14), (1, 23, 5), (1, 2, 7), (1, 24, 0), (1, 22, 24), (1, 16, 10), (1, 21, 12), (1, 2, 9, 20), (1, 1, 15), (1, 18, 13), (1, 20, 12), (1, 11, 18), (1, 4, 6), (1, 3, 23), (1, 8, 4), (1, 2, 21), (1, 1, 23), (1, 21, 1), (1, 13, 0), (1, 16, 4), (1, 7, 0), (1, 26, 0), (1, 24, 3), (1, 22, 7), (1, 2, 5), (1, 25, 25), (1, 11, 25), (1, 12, 6), (1, 5, 0), (1, 15, 1), (1, 5, 4), (1, 5, 25), (1, 2, 11), (1, 12, 10), (1, 8, 17), (1, 18, 0), (1, 19, 15), (1, 23, 21), (1, 7, 13), (1, 20, 26), (1, 4, 14), (1, 5, 15), (1, 17, 22), (1, 23, 7), (1, 16, 25), (1, 14, 4), (1, 17, 5), (1, 8, 10), (1, 4, 26), (1, 25, 26), (1, 24, 12), (1, 22, 16), (1, 14, 3), (1, 21, 24), (1, 18, 26), (1, 12, 5), (1, 5, 5), (1, 19, 13), (1, 20, 8), (1, 8, 25), (1, 26, 3), (1, 16, 15), (1, 10, 3), (1, 25, 3), (1, 7, 17), (1, 25, 23), (1, 16, 13), (1, 20, 14), (1, 18, 24), (1, 11, 17), (1, 10, 6), (1, 14, 17), (1, 9, 12), (1, 17, 21), (1, 10, 21), (1, 19, 19), (1, 11, 16), (1, 17, 6), (1, 18, 5), (1, 26, 21), (1, 17, 19)
4.5 \( m_{14}(2, 29) \geq 364 \)

group: cyclic; generated by

\[
\begin{pmatrix}
0 & 14 & 20 \\
27 & 11 & 28 \\
17 & 10 & 10
\end{pmatrix}
\]

(0, 1, 0), (1, 7, 9), (1, 9, 7), (1, 18, 18), (1, 11, 0), (1, 10, 27), (1, 4, 3), (0, 1, 2),
(1, 5, 7), (1, 19, 3), (1, 25, 9), (1, 4, 12), (1, 5, 15), (1, 7, 27), (0, 1, 6), (1, 18, 20),
(1, 25, 18), (1, 12, 5), (1, 22, 6), (1, 8, 28), (1, 0, 0), (0, 1, 0), (1, 8, 10), (1, 14, 5),
(1, 10, 20), (1, 21, 16), (1, 2, 2), (1, 25, 26), (0, 1, 14), (1, 21, 23), (0, 1, 17), (1, 28, 1),
(1, 17, 27), (1, 9, 13), (1, 19, 7), (0, 1, 16), (1, 19, 21), (1, 22, 25), (1, 26, 16),
(1, 26, 24), (1, 19, 8), (1, 23, 10), (0, 1, 18), (1, 12, 14), (1, 0, 28), (1, 16, 4), (1, 8, 1),
(1, 18, 23), (1, 9, 14), (0, 1, 20), (1, 17, 19), (1, 12, 0), (1, 17, 11), (1, 0, 23), (1, 28, 18),
(1, 1, 8), (0, 1, 22), (1, 4, 6), (1, 18, 15), (1, 7, 28), (1, 27, 14), (1, 1, 17), (1, 18, 28),
(0, 1, 23), (1, 22, 24), (1, 20, 20), (1, 13, 12), (1, 28, 4), (1, 16, 24), (1, 26, 5),
(0, 1, 25), (1, 9, 11), (1, 15, 22), (1, 19, 25), (1, 20, 26), (1, 13, 11), (1, 11, 1),
(0, 1, 27), (1, 14, 16), (1, 16, 10), (1, 22, 17), (1, 2, 3), (1, 17, 9), (1, 5, 11), (1, 0, 1),
(1, 10, 26), (1, 9, 0), (1, 1, 2), (1, 20, 10), (1, 13, 18), (1, 25, 12), (1, 0, 4), (1, 5, 25),
(1, 3, 6), (1, 10, 8), (1, 10, 11), (1, 25, 2), (1, 3, 2), (1, 0, 9), (1, 3, 13), (1, 16, 22),
(1, 17, 3), (1, 1, 9), (1, 0, 16), (1, 11, 3), (1, 0, 10), (1, 4, 19), (1, 23, 15), (1, 7, 6),
(1, 22, 4), (1, 3, 12), (1, 23, 19), (1, 0, 15), (1, 7, 8), (1, 26, 12), (1, 21, 25), (1, 8, 17),
(1, 16, 14), (1, 4, 13), (1, 0, 18), (1, 8, 14), (1, 28, 10), (1, 15, 21), (1, 4, 0), (1, 8, 15),
(1, 7, 17), (1, 0, 19), (1, 22, 11), (1, 14, 24), (1, 11, 28), (1, 26, 21), (1, 2, 23),
(1, 18, 22), (1, 0, 20), (1, 23, 17), (1, 17, 21), (1, 14, 1), (1, 25, 24), (1, 9, 4), (1, 2, 20),
(1, 0, 25), (1, 21, 5), (1, 20, 18), (1, 13, 10), (1, 3, 3), (1, 27, 9), (1, 16, 0), (1, 1, 4),
(1, 9, 22), (1, 5, 16), (1, 14, 13), (1, 8, 4), (1, 11, 6), (1, 2, 1), (1, 1, 7), (1, 26, 14),
(1, 16, 26), (1, 20, 23), (1, 13, 24), (1, 23, 27), (1, 1, 26), (1, 1, 12), (1, 17, 8),
(1, 25, 21), (1, 23, 28), (1, 21, 27), (1, 25, 16), (1, 18, 7), (1, 1, 18), (1, 3, 18),
(1, 4, 23), (1, 21, 15), (1, 7, 0), (1, 20, 0), (1, 13, 16), (1, 1, 23), (1, 16, 17), (1, 21, 20),
(1, 26, 4), (1, 15, 3), (1, 21, 9), (1, 22, 23), (1, 1, 24), (1, 14, 6), (1, 19, 5), (1, 8, 3),
(1, 2, 9), (1, 8, 8), (1, 15, 24), (1, 1, 27), (1, 23, 12), (1, 27, 7), (1, 15, 5), (1, 5, 21),
(1, 28, 14), (1, 12, 12), (1, 1, 28), (1, 8, 2), (1, 12, 25), (1, 18, 10), (1, 28, 26),
(1, 26, 25), (1, 10, 4), (1, 2, 5), (1, 23, 0), (1, 16, 5), (1, 28, 28), (1, 14, 18), (1, 21, 21),
(1, 18, 13), (1, 2, 13), (1, 11, 18), (1, 20, 16), (1, 13, 0), (1, 14, 22), (1, 22, 0),
(1, 23, 18), (1, 2, 17), (1, 19, 6), (1, 3, 20), (1, 14, 27), (1, 14, 23), (1, 12, 7), (1, 2, 26),
(1, 2, 19), (1, 15, 12), (1, 12, 23), (1, 27, 1), (1, 14, 10), (1, 9, 12), (1, 16, 11),
(1, 2, 21), (1, 20, 19), (1, 13, 4), (1, 21, 13), (1, 14, 3), (1, 5, 9), (1, 25, 20), (1, 2, 28),
(1, 28, 7), (1, 27, 28), (1, 18, 19), (1, 14, 8), (1, 18, 26), (1, 3, 27), (1, 3, 10), (1, 5, 6),
(1, 12, 17), (1, 23, 5), (1, 19, 24), (1, 17, 13), (1, 17, 17), (1, 3, 25), (1, 9, 25),
(1, 23, 21), (1, 5, 3), (1, 16, 9), (1, 19, 26), (1, 12, 1), (1, 3, 26), (1, 11, 20), (1, 4, 22),
(1, 26, 15), (1, 7, 22), (1, 16, 21), (1, 11, 21), (1, 3, 28), (1, 12, 3), (1, 19, 9), (1, 18, 27),
(1, 22, 10), (1, 25, 7), (1, 8, 23), (1, 4, 2), (1, 27, 15), (1, 7, 16), (1, 17, 23), (1, 23, 20),
(1, 10, 25), (1, 27, 24), (1, 4, 10), (1, 11, 15), (1, 7, 10), (1, 19, 27), (1, 28, 22),
(1, 20, 7), (1, 13, 6), (1, 4, 11), (1, 4, 15), (1, 7, 15), (1, 7, 3), (1, 10, 9), (1, 12, 4),
(1, 26, 2), (1, 4, 25), (1, 15, 15), (1, 7, 5), (1, 15, 19), (1, 26, 27), (1, 5, 5), (1, 17, 7),
(1, 4, 27), (1, 10, 15), (1, 7, 7), (1, 21, 2), (1, 19, 1), (1, 21, 11), (1, 9, 5), (1, 5, 1),
(1, 27, 16), (1, 9, 6), (1, 21, 14), (1, 27, 21), (1, 26, 8), (1, 22, 19), (1, 5, 18), (1, 22, 26),
(1, 17, 22), (1, 28, 8), (1, 8, 0), (1, 22, 12), (1, 11, 26), (1, 5, 20), (1, 20, 1), (1, 13, 14),
(1, 9, 16), (1, 10, 19), (1, 28, 6), (1, 28, 2), (1, 5, 23), (1, 11, 19), (1, 18, 24), (1, 8, 21),
(1, 21, 22), (1, 23, 11), (1, 12, 28), (1, 8, 16), (1, 25, 11), (1, 26, 0), (1, 25, 1),
(1, 15, 6), (1, 9, 18), (1, 26, 17), (1, 9, 2), (1, 28, 24), (1, 25, 22), (1, 11, 24), (1, 27, 17),
(1, 10, 1), (1, 11, 22), (1, 10, 7), (1, 16, 19), (1, 27, 2), (1, 22, 28), (1, 10, 17),
(1, 22, 27), (1, 18, 2), (1, 10, 14), (1, 15, 8), (1, 11, 2), (1, 16, 3), (1, 15, 9), (1, 23, 22),
(1, 25, 4)

4.6 $m_{25}(2, 29) \geq 697$

Group: symmetric group $S_3$

(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 2), (0, 1, 15), (1, 2, 0),
(1, 0, 2), (1, 0, 15), (1, 15, 0), (0, 1, 3), (0, 1, 10), (1, 3, 0), (1, 0, 3), (1, 0, 10),
(1, 10, 0), (0, 1, 4), (0, 1, 22), (1, 4, 0), (1, 0, 4), (1, 0, 22), (1, 22, 0), (0, 1, 5),
(0, 1, 6), (1, 5, 0), (1, 0, 5), (1, 0, 6), (1, 6, 0), (0, 1, 7), (0, 1, 25), (1, 7, 0), (1, 0, 7),
(1, 0, 25), (1, 25, 0), (0, 1, 9), (0, 1, 13), (1, 9, 0), (1, 0, 9), (1, 0, 13), (1, 13, 0),
(1, 13, 0)
(0, 1, 12), (0, 1, 17), (1, 12, 0), (1, 0, 12), (1, 0, 17), (1, 17, 0), (0, 1, 18), (0, 1, 21),
(1, 18, 0), (1, 0, 18), (1, 0, 21), (1, 21, 0), (0, 1, 19), (0, 1, 26), (1, 19, 0), (1, 0, 19),
(1, 0, 26), (1, 26, 0), (0, 1, 23), (0, 1, 24), (1, 23, 0), (1, 0, 23), (1, 0, 24), (1, 24, 0),
(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 15, 15), (1, 1, 3), (1, 3, 1), (1, 10, 10), (1, 1, 4),
(1, 4, 1), (1, 22, 22), (1, 1, 7), (1, 7, 1), (1, 25, 25), (1, 1, 8), (1, 8, 1), (1, 11, 11),
(1, 1, 10), (1, 10, 1), (1, 3, 3), (1, 1, 11), (1, 11, 1), (1, 8, 8), (1, 1, 12), (1, 12, 1),
(1, 17, 17), (1, 1, 13), (1, 13, 1), (1, 9, 9), (1, 1, 14), (1, 14, 1), (1, 27, 27), (1, 1, 15),
(1, 15, 1), (1, 2, 2), (1, 1, 16), (1, 16, 1), (1, 20, 20), (1, 1, 17), (1, 17, 1), (1, 12, 12),
(1, 1, 18), (1, 18, 1), (1, 21, 21), (1, 1, 19), (1, 19, 1), (1, 26, 26), (1, 1, 20), (1, 20, 1),
(1, 16, 16), (1, 1, 21), (1, 21, 1), (1, 18, 18), (1, 1, 22), (1, 22, 1), (1, 4, 4), (1, 1, 23),
(1, 23, 1), (1, 24, 24), (1, 1, 26), (1, 26, 1), (1, 19, 19), (1, 1, 27), (1, 27, 1), (1, 14, 14),
(1, 1, 28), (1, 28, 1), (1, 28, 28), (1, 2, 4), (1, 4, 2), (1, 2, 15), (1, 15, 2), (1, 22, 15),
(1, 15, 22), (1, 2, 5), (1, 5, 2), (1, 17, 15), (1, 15, 17), (1, 6, 12), (1, 12, 6), (1, 2, 6),
(1, 6, 2), (1, 3, 15), (1, 15, 3), (1, 5, 10), (1, 10, 5), (1, 2, 8), (1, 8, 2), (1, 4, 15),
(1, 15, 4), (1, 11, 22), (1, 22, 11), (1, 2, 9), (1, 9, 2), (1, 19, 15), (1, 15, 19), (1, 13, 26),
(1, 26, 13), (1, 2, 10), (1, 10, 2), (1, 5, 15), (1, 15, 5), (1, 3, 6), (1, 6, 3), (1, 2, 12),
(1, 12, 2), (1, 6, 15), (1, 15, 6), (1, 17, 5), (1, 5, 17), (1, 2, 13), (1, 13, 2), (1, 21, 15),
(1, 15, 21), (1, 9, 18), (1, 18, 9), (1, 2, 14), (1, 14, 2), (1, 7, 15), (1, 15, 7), (1, 27, 25),
(1, 25, 27), (1, 2, 16), (1, 16, 2), (1, 8, 15), (1, 15, 8), (1, 20, 11), (1, 11, 20), (1, 2, 17),
(1, 17, 2), (1, 23, 15), (1, 15, 23), (1, 12, 24), (1, 24, 12), (1, 2, 18), (1, 18, 2),
(1, 9, 15), (1, 15, 9), (1, 21, 13), (1, 13, 21), (1, 2, 21), (1, 21, 2), (1, 25, 15), (1, 15, 25),
(1, 18, 7), (1, 7, 18), (1, 2, 22), (1, 22, 2), (1, 11, 15), (1, 15, 11), (1, 4, 8), (1, 8, 4),
(1, 2, 23), (1, 23, 2), (1, 26, 15), (1, 15, 26), (1, 24, 19), (1, 19, 24), (1, 2, 24),
(1, 24, 2), (1, 12, 15), (1, 15, 12), (1, 23, 17), (1, 17, 23), (1, 2, 26), (1, 26, 2),
(1, 13, 15), (1, 15, 13), (1, 19, 9), (1, 9, 19), (1, 2, 27), (1, 27, 2), (1, 28, 15), (1, 15, 28),
(1, 14, 28), (1, 28, 14), (1, 2, 28), (1, 28, 2), (1, 14, 15), (1, 15, 14), (1, 28, 27),
(1, 27, 28), (1, 3, 7), (1, 7, 3), (1, 12, 10), (1, 10, 12), (1, 25, 17), (1, 17, 25), (1, 3, 8),
(1, 8, 3), (1, 22, 10), (1, 10, 22), (1, 11, 4), (1, 4, 11), (1, 3, 9), (1, 9, 3), (1, 3, 10),
(1, 10, 3), (1, 13, 10), (1, 10, 13), (1, 3, 12), (1, 12, 3), (1, 4, 10), (1, 10, 4), (1, 17, 22),
(1, 22, 17), (1, 3, 13), (1, 13, 3), (1, 14, 10), (1, 10, 14), (1, 9, 27), (1, 27, 9), (1, 3, 14),
(1, 14, 3), (1, 24, 10), (1, 10, 24), (1, 27, 23), (1, 23, 27), (1, 3, 17), (1, 17, 3),

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(1, 11, 25), (1, 25, 11), (1, 26, 8), (1, 8, 26), (1, 7, 20), (1, 20, 7), (1, 7, 25), (1, 25, 7), (1, 16, 25), (1, 25, 16), (1, 7, 23), (1, 23, 7), (1, 24, 25), (1, 25, 24), (1, 24, 23), (1, 23, 24), (1, 7, 24), (1, 24, 7), (1, 20, 25), (1, 25, 20), (1, 23, 16), (1, 16, 23), (1, 7, 27), (1, 27, 7), (1, 8, 25), (1, 25, 8), (1, 14, 11), (1, 11, 14), (1, 8, 9), (1, 9, 8), (1, 12, 11), (1, 11, 12), (1, 13, 17), (1, 17, 13), (1, 8, 12), (1, 12, 8), (1, 16, 11), (1, 11, 16), (1, 17, 20), (1, 20, 17), (1, 8, 13), (1, 13, 8), (1, 27, 11), (1, 11, 27), (1, 9, 14), (1, 14, 9), (1, 8, 14), (1, 14, 8), (1, 9, 11), (1, 11, 9), (1, 27, 13), (1, 13, 27), (1, 8, 17), (1, 17, 8), (1, 13, 11), (1, 11, 13), (1, 12, 9), (1, 9, 12), (1, 8, 18), (1, 18, 8), (1, 24, 11), (1, 11, 24), (1, 21, 23), (1, 23, 21), (1, 8, 21), (1, 21, 8), (1, 28, 11), (1, 11, 28), (1, 18, 28), (1, 28, 18), (1, 8, 23), (1, 23, 8), (1, 21, 11), (1, 11, 21), (1, 24, 18), (1, 18, 24), (1, 9, 13), (1, 13, 9), (1, 24, 13), (1, 13, 24), (1, 9, 23), (1, 23, 9), (1, 9, 17), (1, 17, 9), (1, 18, 13), (1, 13, 18), (1, 12, 21), (1, 21, 12), (1, 9, 21), (1, 21, 9), (1, 12, 13), (1, 13, 12), (1, 18, 17), (1, 17, 18), (1, 9, 26), (1, 26, 9), (1, 19, 13), (1, 13, 19), (1, 19, 26), (1, 26, 19), (1, 9, 28), (1, 28, 9), (1, 16, 13), (1, 13, 16), (1, 28, 20), (1, 20, 28), (1, 12, 17), (1, 17, 12), (1, 28, 17), (1, 17, 28), (1, 12, 28), (1, 28, 12), (1, 12, 18), (1, 18, 12), (1, 16, 17), (1, 17, 16), (1, 21, 20), (1, 20, 21), (1, 12, 20), (1, 20, 12), (1, 21, 17), (1, 17, 21), (1, 16, 18), (1, 18, 16), (1, 12, 23), (1, 23, 12), (1, 14, 17), (1, 17, 14), (1, 24, 27), (1, 27, 24), (1, 12, 27), (1, 27, 12), (1, 24, 17), (1, 17, 24), (1, 14, 23), (1, 23, 14), (1, 14, 16), (1, 16, 14), (1, 26, 27), (1, 27, 26), (1, 20, 19), (1, 19, 20), (1, 14, 19), (1, 19, 14), (1, 20, 27), (1, 27, 20), (1, 26, 16), (1, 16, 26), (1, 14, 20), (1, 20, 14), (1, 18, 27), (1, 27, 18), (1, 16, 21), (1, 21, 16), (1, 14, 21), (1, 21, 14), (1, 16, 27), (1, 27, 16), (1, 18, 20), (1, 20, 18), (1, 16, 20), (1, 20, 16), (1, 23, 20), (1, 20, 23), (1, 16, 24), (1, 24, 16), (1, 18, 23), (1, 23, 18), (1, 19, 21), (1, 21, 19), (1, 24, 26), (1, 26, 24)

4.7 $m_{25}(2, 31) \geq 734$

group: cyclic; generated by

$$\begin{pmatrix} 26 & 3 & 13 \\ 25 & 22 & 17 \\ 5 & 23 & 14 \end{pmatrix}$$
(1, 30, 7), (1, 25, 23), (1, 21, 3), (1, 29, 1), (1, 30, 2), (1, 21, 7), (1, 23, 16), (1, 27, 29),
(1, 21, 8), (1, 27, 6), (1, 23, 3), (1, 21, 19), (1, 24, 29), (1, 24, 25), (1, 25, 8), (1, 26, 25),
(1, 25, 14), (1, 26, 28), (1, 27, 1), (1, 27, 26)