Eigenfunctions of the Laplacian and associated Ruelle operator

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Abstract

Let $\Gamma$ be a co-compact Fuchsian group of isometries on the Poincaré disk $\mathbb{D}$ and $\Delta$ the corresponding hyperbolic Laplace operator. Any smooth eigenfunction $f$ of $\Delta$, equivariant by $\Gamma$ with real eigenvalue $\lambda = -s(1 - s)$, where $s = \frac{1}{2} + it$, admits an integral representation by a distribution $D_{f,s}$ (the Helgason distribution) which is equivariant by $\Gamma$ and supported at infinity $\partial \mathbb{D} = S^1$.

The geodesic flow on the compact surface $\mathbb{D}/\Gamma$ is conjugate to a suspension over a natural extension of a piecewise analytic map $T: S^1 \rightarrow S^1$, the so-called Bowen–Series transformation. Let $L_s$ be the complex Ruelle transfer operator associated with the Jacobian $-s \ln |T'|$. Pollicott showed that $D_{f,s}$ is an eigenfunction of the dual operator $L_s^*$ for the eigenvalue 1. Here we show the existence of a (nonzero) piecewise real analytic eigenfunction $\psi_{f,s}$ of $L_s$ for the eigenvalue 1, given by an integral formula

$$\psi_{f,s}(\xi) = \int \frac{J(\xi, \eta)}{|\xi - \eta|^{2s}} D_{f,s} (d\eta),$$

where $J(\xi, \eta)$ is a $\{0, 1\}$-valued piecewise constant function whose definition depends upon the geometry of the Dirichlet fundamental domain representing the surface $\mathbb{D}/\Gamma$.

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1. Introduction

Consider the Laplace operator $\Delta$ defined by

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$
on the Lobatchevskii upper half-plane $\mathbb{H} = \{ w = x + iy \in \mathbb{C}; y > 0 \}$, equipped with the hyperbolic metric $d\mathbb{H} = \frac{|dw|}{y}$, and the eigenvalue problem

$$\Delta f = -s(1-s)f,$$

where $s$ is of the form $s = \frac{1}{2} + i\tau$, with $\tau$ real. We shall also consider the same corresponding Laplace operator

$$\Delta = \frac{1}{4}(1-|z|^2)^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

and eigenvalue problem

$$\Delta f = -s(1-s)f,$$

defined on the Poincaré disk $\mathbb{D} = \{ z = x + iy \in \mathbb{C}; |z| < 1 \}$, equipped with the metric $ds_\mathbb{D} = \frac{1}{1-|z|^2} |dz|$.\n
Helgason showed in [11] and [12] that any eigenfunction $f$ associated with this eigenvalue problem can be obtained by means of a generalized Poisson representation

$$f(w) = \int_{-\infty}^{\infty} \left( \frac{1+i\tau}{x-i\tau} \right)^s D_{f,s}^\mathbb{D}(t),$$

or

$$f(z) = \int_{\mathbb{D}} \left( \frac{1-|z|^2}{|z-z'|^2} \right)^s D_{f,s}^\mathbb{D}(\xi),$$

where $D_{f,s}^\mathbb{D}$ or $D_{f,s}^\mathbb{H}$ are analytic distributions called from now on Helgason’s distributions. We have used the canonical isometry between $z \in \mathbb{D}$ and $w \in \mathbb{H}$, namely $w = \frac{i(1+z)}{1-z}$ or $z = \frac{i\tau w}{1+\tau^2w}$. The hyperbolic metric is given in $\mathbb{H}$ and in $\mathbb{D}$ by

$$ds_\mathbb{H}^2 = \frac{dx^2 + dy^2}{y^2}, \quad ds_\mathbb{D}^2 = \frac{4(dx^2 + dy^2)}{(1-|z|^2)^2}.$$

We shall be interested in a more restricted problem, where the eigenfunction $f$ is also automorphic with respect to a co-compact Fuchsian group $\Gamma$, i.e. a discrete subgroup of the group of Möbius transformations (see [5, 20, 25]) with compact fundamental domain. It is known that the eigenvalues $\lambda = s(1-s) = \frac{1}{2} + i\tau$ form a discrete set of positive real numbers with finite multiplicity and accumulating at $+\infty$ (see [13]).

Pollicott showed [21] that Helgason’s distribution can be seen as a generalized eigenmeasure of the dual complex Ruelle transfer operator associated with a subshift of the finite type defined at infinity. Let $T_L$ be the left Bowen–Series transformation that acts on the boundary $S^1 = \partial \mathbb{D}$ and is associated with a particular set of generators of $\Gamma$. The precise definition of $T_L$ has been given in [8, 22–24], and more geometrical descriptions have then been given in [1, 18]. Specific examples of the Bowen–Series transformation have been studied in [4, 17] for the modular surface and in [3] for a symmetric compact fundamental domain of genus two. The map $T_L$ is known to be piecewise $\Gamma$-Möbius constant, Markovian with respect to a partition $\{ L_k \}$ of intervals of $S^1$, on which the restriction of $T_L$ is constant and equal to an element $\gamma_k$ of $\Gamma$, transitive and orbit equivalent to $\Gamma$. Let $L^1_k$ be the complex Ruelle transfer operator associated with the map $T_L$ and the potential $A_L = -s \ln |T_L'|$, namely

$$(L^1_k \psi)(\xi') = \sum_{T_L(\xi) = \xi'} e^{A_L(\xi)} \psi(\xi) = \sum_{T_L(\xi) = \xi'} \frac{\psi(\xi)}{|T_L'|},$$

where the summation is taken over all preimages $\xi$ of $\xi'$ under $T_L$. Here $T_L'$ denotes the Jacobian of $T_L$ with respect to the canonical Lebesgue measure on $S^1$. In the case of an automorphic
eigenfunction $f$ of $\Delta$, Pollicott showed that the corresponding Helgason distribution $\mathcal{D}_{f,s}$ satisfies the dual functional equation
\[(\mathcal{L}^*_s)^*(\mathcal{D}_{f,s}) = \mathcal{D}_{f,s}\]
or, according to Pollicott’s terminology, the parameter $s$ is a (dual) Perron–Frobenius value, that is, $1$ is an eigenvalue for the dual Ruelle transfer operator.

Although suggested in [21], it is not clear whether $s$ could be a Perron–Frobenius value, that is, whether $1$ could also be an eigenvalue for $L^*_s$, not only for $(\mathcal{L}^*_s)^*$. Our goal in this paper is to show that this is actually the case.

The three main ingredients we use are the following:

• Otal’s proof of Helgason’s distribution in [19], giving more precise information on $\mathcal{D}_{f,s}$ and enabling us to integrate piecewise $C^1$ test functions, instead of real analytic globally defined test functions;

• a more careful reading of [1, 8, 18, 24], or a careful study of a particular example in [16], which enables us to construct a piecewise $\Gamma$-Möbius baker transformation (‘arithmetically’ conjugate to the geodesic billiard);

• the existence of a kernel that we introduced in [3], which enables us to permute past and future coordinates and transfer a dual eigendistribution to a piecewise real analytic eigenfunction. Haydn (in [10]) has introduced a similar kernel in a more abstract setting, without geometric considerations.

More precisely, we prove the following theorem:

**Theorem 1.** Let $\Gamma$ be a co-compact Fuchsian group of the hyperbolic disk $\mathbb{D}$ and $\Delta$ the corresponding hyperbolic Laplace operator. Let $\lambda = s(1 - s)$, with $s = \frac{1}{2} + it$, and let $f$ be an eigenfunction of $-\Delta$, automorphic with respect to $\Gamma$, that is, $\Delta f = -\lambda f$ and $f \circ \gamma = f$, for every $\gamma \in \Gamma$. Then there exists a (nonzero) piecewise real analytic eigenfunction $\psi_{f,s}$ on $S^1$ that is a solution of the functional equation
\[L^*_s(\psi_{f,s}) = \psi_{f,s},\]
where $L^*_s$ is the complex Ruelle transfer operator associated with the left Bowen–Series transformation $T_L : S^1 \to S^1$ and the potential $A_L = -s \ln |T_L'|$.

Moreover, $\psi_{f,s}$ admits an integral representation via Helgason’s distribution $\mathcal{D}_{f,s}$, representing $f$ at infinity, and a geometric positive kernel $k(\xi, \eta)$ defined on a finite set of disjoint rectangles $\cup_k I^L_k \times Q^R_k \subset S^1 \times S^1$, namely,
\[\psi_{f,s}(\xi) = \int_{Q^R_k} k^*(\xi, \eta) \mathcal{D}_{f,s}(\eta) = \int_{Q^R_k} \frac{1}{|\xi - \eta|^s} \mathcal{D}_{f,s}(\eta),\]
for every $\xi \in I^L_k$, where $I^L_k$ and $Q^R_k$ are intervals of $S^1$ with disjoint closure, and $\{I^L_k\}_k$ is a partition of $S^1$ where $T_L$ is injective, Markovian and piecewise $\Gamma$-Möbius constant.

Lewis [14] and, later, Lewis and Zagier [15], started a different approach to understand Maass wave forms. They were able to identify in a bijective way Maass wave forms of $PSL(2, \mathbb{Z})$ and solutions of a functional equation with three terms closely related to Mayer’s transfer operator. Their setting is strongly dependent on the modular group. Our theorem 1 may be viewed as part of their programme for co-compact Fuchsian groups. The Helgason distribution has been used by Zelditch in [26] to generalize microlocal analysis on hyperbolic surfaces, by Flaminio and Forni in [9], to study invariant distributions by the horocycle flow, and by Anantharaman and Zelditch in [2], to understand the ‘quantum unique ergodicity conjecture’.
2. Preliminary results

Let $\Gamma$ be a co-compact Fuchsian group of the Poincaré disk $\mathbb{D}$. We denote by $d(w, z)$ the hyperbolic distance between two points of $\mathbb{D}$, given by the Riemannian metric $d^2 = 4(dx^2 + dy^2)/(1 - |z|^2)^2$. Let $M = \mathbb{D}/\Gamma$ be the associated compact Riemann surface, $N = T^1M$ the unit tangent bundle and $\Delta$ the Laplace operator on $M$. Let $f : M \to \mathbb{R}$ be an eigenfunction of $-\Delta$ or, in other words, a $\Gamma$-automorphic function $f : \mathbb{D} \to \mathbb{R}$ satisfying $\Delta f = -s(1-s)f$ for the eigenvalue $\lambda = s(1-s) > 1/4$ and such that $f \circ \gamma = f$, for every $\gamma \in \Gamma$. We know that $f$ is $C^\infty$ and uniformly bounded on $\mathbb{D}$. Thanks to Helgason’s representation theorem, $f$ can be represented as a superposition of horocycle waves, given by the Poisson kernel

$$P(z, \xi) := e^{b\theta(O,z)} = \frac{1 - |z|^2}{|z - \xi|^2},$$

where $b\theta(w, z)$ is the Busemann cocycle between two points $w$ and $z$ inside the Poincaré disk, observed from a point at infinity $\xi \in S^1$, defined by

$$b\theta(w, z) := \langle d(w, \xi) - d(z, \xi) \rangle = \lim_{t \to \xi} d(w, t) - d(z, t),$$

where the limit is uniform in $t \to \xi$ in any hyperbolic cone at $\xi$. Helgason’s theorem states that

$$f(z) = \int_{\mathbb{D}} P'(z, \xi) D_{f,s}(\xi) = \langle D_{f,s}, P'(z, .) \rangle$$

for some analytic distribution $D_{f,s}$, acting on real analytic functions on $S^1$. Unfortunately, Helgason’s work is too general and is valid for any eigenfunction not necessarily equivariant by a group. For bounded $C^2$ functions $f$, Otal [19] has shown that the distribution $D_{f,s}$ has stronger properties and can be defined in a simpler manner.

We first recall some standard notation in hyperbolic geometry. We call $d(O, \theta) = \dfrac{1}{2} \log \left( \frac{1 + \sinh(\theta)}{1 - \sinh(\theta)} \right)$ the angular arc at the hyperbolic distance $r$ from the origin, $O$. Let $C(O, r)$ denote the set of points in $\mathbb{D}$ at hyperbolic distance $r$ from the origin,

$$C(O, r) = \{ z \in \mathbb{D} ; |z| = \tanh(\frac{r}{2}) \}$$

and, more generally, given any interval $I$ at infinity and any point $z_0 \in \mathbb{D}$, let $C(z_0, r, I)$ denote the angular arc at the hyperbolic distance $r$ from $z_0$ delimited at infinity by $I$, that is,

$$C(z_0, r, I) = \{ z \in \mathbb{D} ; z \in [z_0, \xi] \text{ for some } \xi \in I \text{ and } d(z, z_0) = r \},$$

where $[z_0, \xi]$ denotes the geodesic ray from $z_0$ to the point $\xi$ at infinity. Let $\omega = \frac{d}{dz}$ denote the exterior normal derivative to $C(O, r)$ and $|dz| = \sinh(r) d\theta$ the hyperbolic arc length on $C(O, r)$.

**Theorem 2 ([19]).** Let $f$ be a bounded $C^2$ eigenfunction satisfying $\Delta f = -s(1-s)f$. Then:

1. There exists a continuous linear functional $D_{f,s}$ acting on $C^1$ functions of $S^1$, defined by

$$\int_{C(O,r)} \langle D_{f,s}, P'(z, .) \rangle = \lim_{r \to \infty} \frac{1}{c(s)} \int_{C(O,r)} \psi(z)e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) |dz|,$$

where $c(s)$ is a nonzero normalizing constant such that $\langle D_{f,s}, 1 \rangle = f(0)$, and $\psi(z)$ is any $C^1$ extension of $\psi(\xi)$ to a neighbourhood of $S^1$.

2. $D_{f,s}$ represents $f$ in the following sense:

$$f(z) = \int_{\mathbb{D}} [P(z, \xi)]^s D_{f,s}(\xi), \quad \forall z \in \mathbb{D}.$$

$D_{f,s}$ is unique and is called the Helgason distribution of $f$. 
3. For all $0 \leq \alpha \leq 2\pi$, the following limit exists:

$$\tilde{D}_{f,s}(\alpha) := \lim_{r \to +\infty} \frac{1}{c(s)} \int_0^a e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) \left( \tanh \left( \frac{r}{2} \right) e^{it} \right) \sinh(r) \, d\theta.$$  

The convergence is uniform in $\alpha \in [0, 2\pi]$ and $\tilde{D}_{f,s}(0) = 0$.  

4. $\tilde{D}_{f,s}$ can be extended to $\mathbb{R}$ as a $\frac{1}{2}$-Hölder continuous function satisfying:  

(a) $\tilde{D}_{f,s}(\theta + 2\pi) = \tilde{D}_{f,s}(\theta) + f(0)$, for every $\theta \in \mathbb{R}$,  

(b) for any $C^1$ function $\psi : S^1 \to \mathbb{C}$, denoting $\tilde{\psi}(\theta) = \psi(e^{i\theta})$,  

$$\int \tilde{\psi}(\xi) \tilde{D}_{f,s}(\xi) = \tilde{\psi}(0) f(0) - \int_0^{2\pi} \frac{d\tilde{\psi}}{d\theta} \tilde{D}_{f,s}(\theta) \, d\theta.$$  

Using similar technical tools as Otal, one can prove the following extension of $\tilde{D}_{f,s}$ on piecewise $C^1$ functions, that is, on functions not necessarily continuous but which admit a $C^1$ extension on each interval $[\xi_k, \xi_{k+1}]$ of some finite and ordered subdivision $\{\xi_0, \xi_1, \ldots, \xi_{n-1}\}$ of $S^1$.

**Proposition 3.** Let $f$ and $D_{f,s}$ be as in theorem 2.

1. For any interval $I \subset S^1$ and any function $\psi : \{0\} \to \mathbb{C}$, which is $C^1$ on the closure of $I$ and null outside $I$, the following limit exists:

$$\int \psi(\xi) D_{f,s}(\xi) := \frac{1}{c(s)} \lim_{r \to +\infty} \int_{C(0,r,1)} \psi(z) e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) |dz|.$$  

where again $\psi(z)$ is any $C^1$ extension of $\psi(\xi)$ to a neighbourhood of $S^1$.

2. For any $0 < \alpha < \beta < 2\pi$ and any $C^1$ function $\psi$ on the interval $I = [\exp(\alpha), \exp(\beta)]$,  

$$\int \psi(\xi) D_{f,s}(\xi) = \tilde{\psi}(\beta) \tilde{D}_{f,s}(\beta) - \tilde{\psi}(\alpha) \tilde{D}_{f,s}(\alpha) - \int_\alpha^\beta \frac{d\tilde{\psi}}{d\theta} \tilde{D}_{f,s}(\theta) \, d\theta,$$

where $\tilde{D}_{f,s}$ and $\tilde{\psi}(\theta)$ have been defined in theorem 2.

**Proof.** Given $\alpha \in [0, 2\pi]$, let $I = \{e^{i\theta} \mid 0 \leq \theta \leq \alpha\}$ be an interval in $S^1$, and $\psi$ a $C^1$ function defined on a neighbourhood of $S^1$. Denote $\tilde{\psi}(r, \theta) = \psi(\tanh(\xi) e^{i\theta})$ and $K(r, \theta) = e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) \left( \tanh(\xi) e^{i\theta} \right) \sinh(r)$. Then

$$\frac{1}{c(s)} \int_{C(0,r,1)} \psi(z) e^{-sr} \left( \frac{\partial f}{\partial n} + sf \right) |dz|$$

$$= \int_0^\alpha \tilde{\psi}(r, \beta) K(r, \beta) \, d\beta$$

$$= \int_0^\alpha \left[ \tilde{\psi}(r, \alpha) + \int_\beta^\alpha - \frac{d\tilde{\psi}}{d\theta}(r, \theta) \, d\theta \right] K(r, \beta) \, d\beta$$

$$= \tilde{\psi}(r, \alpha) \int_\beta^\alpha K(r, \beta) \, d\beta - \int_0^\alpha \frac{d\tilde{\psi}}{d\theta}(r, \theta) \left[ \int_\beta^\alpha K(r, \beta) \, d\beta \right] \, d\theta.$$  

Since $\int_0^\alpha K(r, \beta) \, d\beta \to \tilde{D}_{f,s}(\alpha)$ uniformly in $\alpha \in [0, 2\pi]$, the left-hand side of the previous equality converges to

$$\int \tilde{\psi}(\xi) 1_{I \subset I} D_{f,s}(\xi) = \tilde{\psi}(\alpha) \tilde{D}_{f,s}(\alpha) - \int_0^\alpha \frac{d\tilde{\psi}}{d\theta}(\theta) \tilde{D}_{f,s}(\theta) \, d\theta.$$
The second part of the proposition follows subtracting such an expression from another one, such as:
\[
\int \psi(\xi) \mathbf{1}_{[\xi = \omega^\gamma_1, 0 \leq \gamma \leq \beta]} \tilde{D}_{f,s}(\xi) = \int \psi(\xi) \mathbf{1}_{[\xi = \omega^\gamma_1, 0 \leq \gamma \leq \alpha]} \tilde{D}_{f,s}(\xi).
\]

If, in addition, we assume that \( f \) is equivariant with respect to a co-compact Fuchsian group \( \Gamma \), Pollicott observed in [21] that \( D_{f,s} \), acting on real analytic functions, is equivariant by \( \Gamma \), that is, satisfies \( \gamma^* (D_{f,s})(\xi) = |\gamma'(\xi)|^2 D_{f,s}(\xi) \), for all \( \gamma \in \Gamma \). Because Otal’s construction is more precise and implies that Helgason’s distribution also acts on piecewise \( C^1 \) functions, the above equivariance property can be improved in the following way.

**Proposition 4.** Let \( f : \mathbb{D} \to \mathbb{R} \) be a \( C^2 \) function, \( I \subset S^1 \) an interval and \( \psi : I \to \mathbb{C} \) a \( C^1 \) function on the closure of \( I \). If \( f \) satisfies \( f \circ \gamma = f \), for some \( \gamma \in \Gamma \) (\( f \) is not necessarily automorphic), then
\[
\langle D_{f,s}, \psi \gamma^{-1} \rangle_{\gamma(\mathcal{I})} = \langle D_{f,s}, \psi \mathcal{I} \rangle.
\]

The main difficulty here is to transfer the equivariance property \( f \circ \gamma = f \) to an equivalent property for the extension of \( D_{f,s} \) to piecewise \( C^1 \) functions. If \( I = S^1 \) and \( \psi \) is real analytic, then, by uniqueness of the representation, proposition 4 is easily proved. It seems that just knowing the fact that \( D_{f,s} \) is the derivative of some Hölder function is not enough to reach a conclusion. The following proof uses Otal’s approach and, essentially, the extension of \( D_{f,s} \) described in part 1 of proposition 3.

**Proof of proposition 4.** First we prove the proposition for \( \psi = 1 \). Let \( g(z) = \exp(-s d(\mathcal{O}, z)) \).

By the definition of \( D_{f,s} \), we obtain
\[
\int \mathbf{1}_I(\xi) D_{f,s}(\xi) = \lim_{r \to +\infty} \frac{1}{c(s)} \int_{\mathcal{C}(O,r,I)} \left( \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right) \, dz_\mathcal{O}
\]
\[
= \lim_{r \to +\infty} \frac{1}{c(s)} \int_{\mathcal{C}(O',r',I')} \left( \frac{\partial f}{\partial n} - f \frac{\partial g'}{\partial n} \right) \, dz_\mathcal{O},
\]
where \( r' = r + d(O, O') \), \( O' = \gamma(O) \) and \( g' = g \circ \gamma^{-1} \). Notice that the domain bounded by the circle \( \mathcal{C}(O', r') \) contains the circle \( \mathcal{C}(O, r) \). Let \( PQ \) be the positively oriented arc \( \mathcal{C}(O, r, \gamma(I)) \) and \( P'Q' \) be the arc \( \mathcal{C}(O', r', \gamma(I)) \). Then the two geodesic segments \( [P, P'] \) and \( [Q, Q'] \) belong to the annulus \( r \leq d(z, \mathcal{O}) \leq r + 2d(O, O') \) and their length is uniformly bounded.

We now use Green’s formula to compute the right-hand side of the above expression. Let \( \Omega \) denote the domain delimited by \( P, P', Q', Q \) using the corresponding arcs and geodesic segments, and let \( dv = \sinh(r) \, dr \, d\theta \) be the hyperbolic volume element. We obtain
\[
\int_{\mathcal{C}(O')} \left( \frac{\partial f}{\partial n} - f \frac{\partial g'}{\partial n} \right) \, dz_\mathcal{O} = \int_{\mathcal{C}(O)} \left( \frac{\partial f}{\partial n} - f \frac{\partial g'}{\partial n} \right) \, dz_\mathcal{O}
\]
\[
- \int_{[P, P']} \cdots \, dz_\mathcal{O} - \int_{[Q, Q']} \cdots \, dz_\mathcal{O} + \int_{\Omega} (g' \Delta f - f \Delta g') \, dv.
\]
When \( r \) tends to infinity, the last three terms at the right-hand side tend to 0, since along the geodesic segments \( [P, P'] \) and \( [Q, Q'] \), the gradient \( \nabla g' \) is uniformly bounded by \( \exp(-\frac{1}{2}r) \) and
\[
g' \Delta f - f \Delta g' = sg' f \sinh(d(z, O'))^{-2} \quad \text{and} \quad \frac{\partial}{\partial n} g' + sg'...
\]
are uniformly bounded by a constant times \( \exp(-\frac{2}{3}r) \) in the domain \( \Omega \), for the first expression, and by a constant times \( \exp(-\frac{3}{2}r) \) on \( \mathcal{C}(O, r) \), for the second expression. It follows that

\[
\int 1_f(\xi) D_{f,s}(\xi) = \lim_{r \to \infty} \frac{1}{c(s)} \int_{C(O,r,\gamma(1))} g' \left( \frac{\partial f}{\partial \nu} + sf \right) |dz|_D
\]

\[
= \lim_{r \to \infty} \frac{1}{c(s)} \int_{C(O,r,\gamma(1))} \left[ \psi(z) \right]^2 e^{2\alpha} \left( \frac{\partial f}{\partial \nu} + sf \right) |dz|_D,
\]

where \( \psi(z) = \exp \left( \frac{d(O, z) - d(O, \gamma^{-1}(z))}{s} \right) \). Now we observe that

\[
\begin{cases}
\psi(z) = \exp s (d(O, z) - d(\gamma(1), z)), & \text{for } z \in \mathbb{D}, \\
\psi(\xi) = \exp S \gamma(1), & \text{for } \xi \in \partial \mathbb{D},
\end{cases}
\]

actually coincides with a real analytic function \( \Psi(z) \) defined in a neighbourhood of \( S^1 \), given explicitly by

\[
\Psi(z) = \left( \frac{(1 + |z|)^2}{(1 + |\gamma^{-1}(z)|)^2 |\gamma' \circ \gamma^{-1}(z)|} \right)^\frac{1}{2}.
\]

Thus we have proved that

\[
\int 1_f(\xi) D_{f,s}(\xi) = \int \frac{1_{\gamma(1)}(\xi)}{|\gamma' \circ \gamma^{-1}(\xi)|^2} D_{f,s}(\xi).
\]

Now we prove the general case. We use the same notation for the lifting \( \gamma : \mathbb{R} \mapsto \mathbb{R} \) of a Möbius transformation \( \gamma : S^1 \mapsto S^1 \). The lifting satisfies \( \gamma(\alpha + 2\pi) = \gamma(\alpha) + 2\pi \), \( \exp(i\gamma(\alpha)) = \gamma(\exp(i\alpha)) \) and \( \gamma'(\alpha) = |\gamma'(\alpha)| \), for all \( \alpha \in \mathbb{R} \). Using proposition 3, we obtain

\[
\tilde{D}_{f,s}(\beta) - \tilde{D}_{f,s}(\alpha) = \frac{\tilde{D}_{f,s} \circ \gamma(\beta)}{\gamma'(\beta)^2} - \frac{\tilde{D}_{f,s} \circ \gamma(\alpha)}{\gamma'(\alpha)^2} - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial}{\partial \theta} \left( \frac{1}{\gamma' \circ \gamma^{-1}(\theta)^2} \right) D_{f,s}(\theta) \, d\theta.
\]

For any \( C^1 \) function \( \psi(\xi) \) defined on \( I \), we denote \( \tilde{\psi}(\theta) = \psi(\exp(i\theta)) \), and obtain

\[
LHS := \int \psi(\xi) 1_f(\xi) D_{f,s}(\xi)
\]

\[
= \tilde{\psi}(\beta) \tilde{D}_{f,s}(\beta) - \tilde{\psi}(\alpha) \tilde{D}_{f,s}(\alpha) - \int_{\alpha}^{\beta} \frac{\partial}{\partial \theta} \tilde{\psi}(\theta) \tilde{D}_{f,s}(\theta) \, d\theta
\]

\[
= \tilde{\psi}(\beta) \tilde{D}_{f,s}(\beta) - \tilde{\psi}(\alpha) \tilde{D}_{f,s}(\alpha) - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial}{\partial \theta} \left( \tilde{\psi} \circ \gamma^{-1}(\theta) \right) \tilde{D}_{f,s}(\gamma^{-1}(\theta)) \, d\theta
\]

\[
= \tilde{\psi}(\beta) \left( \tilde{D}_{f,s}(\beta) - \tilde{D}_{f,s}(\alpha) \right) - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial}{\partial \theta} \tilde{\psi}(\gamma^{-1}(\theta)) \left( \tilde{D}_{f,s}(\gamma^{-1}(\theta)) - \tilde{D}_{f,s}(\alpha) \right) \, d\theta.
\]

We now use the above equivariance and replace both \( \tilde{D}_{f,s}(\beta) - \tilde{D}_{f,s}(\alpha) \) and \( \tilde{D}_{f,s}(\gamma^{-1}(\theta)) - \tilde{D}_{f,s}(\alpha) \) by the corresponding formula involving \( \tilde{D}_{f,s} \circ \gamma(\beta), \tilde{D}_{f,s} \circ \gamma(\alpha), \tilde{D}_{f,s}(\theta) \). Thus

\[
LHS = \frac{\tilde{\psi}(\beta) \tilde{D}_{f,s} \circ \gamma(\beta)}{\gamma'(\beta)^2} - \frac{\tilde{\psi}(\alpha) \tilde{D}_{f,s} \circ \gamma(\alpha)}{\gamma'(\alpha)^2} - \int_{\gamma(\alpha)}^{\gamma(\beta)} \frac{\partial}{\partial \theta} \left( \tilde{\psi}(\gamma^{-1}(\theta)) \right) D_{f,s}(\theta) \, d\theta
\]

\[
= \int \frac{\psi \circ \gamma^{-1}(\xi)}{|\gamma' \circ \gamma^{-1}(\xi)|^2} 1_{\gamma(1)} D_{f,s}(\xi).
\]
Following [1, 8, 18, 22–24] for the general case and [16] for a specific example we recall the definition of the left $T_L$ and right $T_R$ Bowen–Series transformations. The hyperbolic surface we are interested in is given by the quotient of the hyperbolic disk $\mathbb{D}$ by a co-compact Fuchsian group $\Gamma$. Given a point $O \in \mathbb{D}$, let

$$D_{\Gamma, O} = \{ z \in \mathbb{D}; d(z, O) < d(z, \gamma(O)), \quad \forall \gamma \in \Gamma \}$$

denote the corresponding Dirichlet domain, a convex fundamental domain with compact closure in $\mathbb{D}$, admitting an even number of geodesic sides and an even number of vertices, some of which may be elliptic. More precisely, the boundary of $D_{\Gamma, O}$ is a disjoint union of semi-closed geodesic segments $S^L_{1, r}, \ldots, S^L_{1, -1}, S^L_{1, 1}, \ldots, S^L_{1, r}$, closed to the left and open to the right, or, equivalently, to a union of semi-closed geodesic segments $S^R_{1, r}, S^R_{1, -1}, S^R_{1, 1}, \ldots, S^R_{1, r}$, closed to the right and open to the left; for each $k$, the intervals $S^L_{k}$ and $S^R_{k}$ have the same endpoints and $S^L_{k}$ is associated with $S^R_{k}$ by an element $a_k \in \Gamma$ satisfying $a_k(S^L_{k}) = S^R_{k}$. The elements $a_k$ generate $\Gamma$ and satisfy $a_k = a_k^{-1}$, for $k = \pm 1, \ldots, \pm r$.

To define the two Bowen–Series transformations $T_L$ and $T_R$ geometrically, we need to impose a geometric condition on $\Gamma$: following [8, 22, 24], we say that $\Gamma$ satisfies the even corner property if, for each $1 \leq |k| \leq r$, the complete geodesic line through $S^L_k$ is equal to a disjoint union of $\Gamma$-translates of the sides $S^L_k$, with $1 \leq |l| \leq r$. Some $\Gamma$ do not satisfy this geometric property. Nevertheless, any two co-compact Fuchsian groups $\Gamma$ and $\Gamma'$, with identical signature, are geometrically isomorphic, that is, there exists a group isomorphism $h_* : \Gamma \to \Gamma'$ and a quasi-conformal orientation preserving homeomorphism $h : \partial \mathbb{D} \to \partial \mathbb{D}$, admitting an extension to a conjugating homeomorphism $h : \mathbb{D} \to \mathbb{D}$, that is,

$$h(\gamma(z)) = h_*(\gamma)(h(z)), \quad \forall \gamma \in \Gamma.$$ 

An important observation in [8, 22, 24] is that any co-compact Fuchsian group is geometry isomorphic to a Fuchsian group with identical signature and satisfying the even corner property. We are going to recall the Bowen and Series construction in the case that $\Gamma$ possesses the even corner property and will show that their main conclusions remain valid under geometric isomorphisms.

The complete geodesic line associated with a side $S^L_k$ cuts the boundary at infinity $S^1$ at two points $s^L_k$ and $s^L_{-k}$, positively oriented with respect to $S^L_k$, the oriented geodesic line $]s^L_k, s^L_{-k}[$ seeing the origin $O$ to the left. Both end points $s^L_k$ and $s^L_{-k}$ are neutrally stable with respect to the associated generator $a_k$, that is, $|a'_k(s^L_k)| = |a'_k(s^L_{-k})| = 1$. The family of open intervals $]s^L_k, S^L_k[$ covers $S^1$, since these intervals $]s^L_k, s^L_{-k}[$ overlap each other, there is no canonical partition adapted to this covering. Nevertheless, we may associate two well-defined partitions, the left partition $\mathcal{A}_L$ and the right partition $\mathcal{A}_R$. The former consists of disjoint half-closed intervals,

$$\mathcal{A}_L = \{ A^L_{1, r}, \ldots, A^L_{1, -1}, A^L_{1, 1}, \ldots, A^L_{1, r} \},$$

given by $A^L_k = [s^L_k, s^L_{1,(k)}]$ where $s^L_{1,(k)}$ denotes the nearest point $s^L_k$ after $s^L_k$, according to a positive orientation. Each $A^L_k$ belongs to the unstable domain of the hyperbolic element $a_k$, that is, $|a'_k(\xi)| \geq 1$, for each $\xi \in A^L_k$. By definition, the left Bowen–Series transformation $T_L : S^1 \mapsto S^1$ is given by

$$T_L(\xi) = a_k(\xi), \quad \text{if } \xi \in A^L_k.$$
Analogously, $S^1$ can be partitioned into half-closed intervals

$$A_R = \{A^R_1, \ldots, A^R_k, A^R, \ldots, A^R_m\},$$

where $A^R_k = [s_{i(k)}^R, s_{i(k)}^R]$, and $s_{i(k)}^R$ denotes the nearest $s^R$ before $s^R_k$, according to a positive orientation. The right Bowen–Series transformation is given by

$$T_R(\eta) = a_k(\eta), \quad \text{if } \eta \in A^R_k.$$

The two partitions $A^L$ and $A^R$ generate two ways of coding a trajectory. Let $\gamma_L : S^1 \mapsto \Gamma$ and $\gamma_R : S^1 \mapsto \Gamma$ be the left and right symbolic coding defined by

$$\gamma_L[\xi] = a_k, \quad \text{if } \xi \in A^L_k, \quad \text{and} \quad \gamma_R[\eta] = a_k, \quad \text{if } \eta \in A^R_k.$$

In particular, $T_R(\eta) = \gamma_R[\eta](\eta)$ and $T_L(\xi) = \gamma_L[\xi](\xi)$, for each $\xi \in S^1$. Also, it is known that $T^R_L$ and $T^R_R$ are expanding. Series, in [22–24], and later, Adler and Flatto in [1], proved that $T_L$ (respectively, $T_R$) is Markov with respect to a partition of $T^L = \{I^L_k\}_{k=1}^m$ (respectively, $T^R = \{I^R_k\}_{k=1}^m$) that is finer than $A_L$ (respectively, $A_R$). The semi-closed intervals $I^L_k$ and $I^R_k$ are of the same kind as $A^L_k$ and $A^R_k$, and have the same closure.

**Definition 5.** A dynamical system $(S^1, T, \{I_k\})$ is said to be a piecewise $\Gamma$-Möbius Markov transformation if $T : S^1 \to S^1$ is a surjective map, and $\{I_k\}$ is a finite partition of $S^1$ into intervals such that:

1. for each $k$, $T(I_k)$ is a union of adjacent intervals $I_l$;
2. for each $k$, the restriction of $T$ to $I_k$ coincides with an element $\gamma_k \in \Gamma$;
3. some finite iterate of $T$ is uniformly expanding.

**Theorem 6 ([8, 24]).** For any co-compact Fuchsian group $\Gamma$, there exists a piecewise $\Gamma$-Möbius Markov transformation $(S^1, T, \{I_k\})$ which is transitive and orbit equivalent to $\Gamma$.

The Ruelle transfer operator can be defined for any piecewise $C^2$ Markov transformation $(S^1, T, \{I_k\})$ and any potential function $A$. Actually, we need a particular complex transfer operator given by the potential

$$A = -s \ln |T'|.$$ 

For any function $\psi : S^1 \to \mathbb{C}$, define

$$(L_\psi(\psi))(\xi') = \sum_{T(\xi) = \xi'} e^{-A(\xi)} \psi(\xi) = \sum_{T(\xi) = \xi'} \psi(\xi) |T'(\xi)|^s,$$

where the summation is taken over all preimages $\xi'$ of $\xi$ under $T$. We modify $L_\psi$ slightly, so that it acts on the space of piecewise $C^1$ functions. Let $\{I_k\}_{k=1}^m$ be a partition of $S^1$. Given a piecewise $C^1$ function and $\otimes_{k=1}^m \psi_k \in \otimes_{k=1}^m C^1(I_k)$ set

$$L^L_\psi \psi = \otimes_{k=1}^m \psi_k \theta_l,$$

where $\theta_l = \sum_{T(I_l) \cap T(I'_l) \neq \emptyset} \psi_k \circ T^{-1}_{I_l}$, and $T^{-1}_{I_l}$ denotes the restriction to $I_l$ of the inverse of $T : I_k \to T(I_k) \supset I_l$.

**Proposition 7.** Let $\Gamma$ be a co-compact Fuchsian group. Let $s = \frac{1}{2} + it$ and $f$ be an automorphic eigenfunction of $-\Delta$, that is, $\Delta f = -s(1-s) f$. Let $(S^1, T, \{I_k\})$ be a piecewise $\Gamma$-Möbius Markov transformation and $L_\psi$ be the Ruelle transfer operator corresponding to the observable $A = -s \ln |T'|$. Then the Helgason distribution $D_{f,s}$ satisfies

$$(L_\psi)^* D_{f,s} = D_{f,s}.$$
Proof. Let $\oplus_{k=1}^q \psi_k$ be a piecewise $C^1$ function in $\oplus_{k=1}^q C^1(T_k)$. Using proposition 4,

$$\int (L_\psi)(\xi) \ D_{f,s}(\xi) = \sum_{l=1}^b \int_{h} (L_\psi)(\xi) \ D_{f,s}(\xi) = \sum_{T(l)b} \int_{h} \psi_k \circ T_{k,l}^{-1}(\xi) \ D_{f,s}(\xi) = \sum_{T(l)b} \int_{h} \psi_k(\xi) \ D_{f,s}(\xi) = \sum_{k,l} \int \psi_k(\xi) \ D_{f,s}(\xi) = \int \psi(\xi) \ D_{f,s}(\xi). $$

Series in [24], Adler and Flatto in [1] and Morita in [18] noticed that $T_L$ admits a natural extension $\hat{T} : \hat{\Sigma} \to \hat{\Sigma}$ strongly related to $T_R$. We also showed the existence of such a $\hat{T}$ in [16], and it was an important step in the proof of theorem 3 of [16]. The following definition explains how the two maps $T_L$ and $T_R$ are glued together in an abstract way.

Definition 8. Let $\Gamma$ be a co-compact Fuchsian group. A dynamical system $(\hat{\Sigma}, \hat{T}, \{I^L_k\}, \{I^R_k\}, J)$ is said to be a piecewise $\Gamma$-Möbius baker transformation if it admits a description as follows.

1. $\{I^L_k\}$ and $\{I^R_k\}$ are finite partitions of $S^1$ into disjoint intervals; $J(k, l)$ is a $[0, 1]$-valued function, and $\hat{\Sigma}$ is the subset of $S^1 \times S^1$ defined by

$$\hat{\Sigma} = \bigsqcup_{J(k, l)=1} I^L_k \times I^R_l.$$

2. For each $k$, $Q^R_k = \bigsqcup_{J(k, l)=1} I^R_k$, is an interval whose closure is disjoint from $\overline{I^L_k}$.

For each $l$, $Q^L_l = \bigsqcup_{J(k, l)=1} I^L_k$, is an interval whose closure is disjoint from $\overline{I^R_l}$.

Let $I^L_k(\xi) = I^L_k$ and $Q^R_k(\xi) = Q^R_k$, for $\xi \in I^L_k$. Let $I^R_k(\eta) = I^R_k$ and $Q^L_k(\eta) = Q^L_k$, for $\eta \in I^R_k$.

3. $\hat{T} : \hat{\Sigma} \to \hat{\Sigma}$ is bijective and is given by

$$\hat{T}(\xi, \eta) = (T_L(\xi), S_R(\xi, \eta)), \quad \hat{T}^{-1}(\xi', \eta') = (S_L(\xi', \eta'), T_R(\eta')), $$

for certain maps $T_L, S_L : S^1 \to S^1$ and $S_R, T_R : \hat{\Sigma} \to \hat{\Sigma}$.

4. $(\hat{\Sigma}, T_L, \{I^L_k\})$ and $(\hat{\Sigma}, T_R, \{I^R_k\})$ are piecewise $\Gamma$-Möbius Markov transformations. There exist two functions $\gamma_L : S^1 \to \Gamma$, respectively $\gamma_R : S^1 \to \Gamma$, that are piecewise constant on each $I^L_k$, respectively $I^R_k$, and satisfying

$$\hat{T}(\xi, \eta) = (\gamma_L(\xi)(\xi), \gamma_L(\xi)(\eta)), \quad \hat{T}^{-1}(\xi', \eta') = (\gamma_R(\eta')(\xi'), \gamma_R(\eta')(\eta')).$$

The maps $T_L$ and $T_R$ are called the left and right Bowen–Series transformations, whereas $\gamma_L$ and $\gamma_R$ are the left and right Bowen–Series codings. Finally, we say that $J$ is the incidence matrix, which we extend as a function on $S^1 \times S^1$ defining

$$J(\xi, \eta) = 1, \quad \text{if } (\xi, \eta) \in \hat{\Sigma},$$

$$J(\xi, \eta) = 0, \quad \text{if } (\xi, \eta) \notin \hat{\Sigma}.$$
Notice that this definition is equivariant by geometric isomorphisms. For co-compact Fuchsian groups satisfying the even corner property, Adler and Flatto in [1], Series in [24] (and, for a particular example, in [16]) obtained geometrically the existence of a piecewise \( \Gamma \)-Möbius baker transformation with left \( T_L \) and right \( T_R \) maps orbit equivalent to \( \Gamma \). By geometric isomorphism considerations, we obtain more generally the following.

**Proposition 9 ([1, 16, 24]).** For any co-compact Fuchsian group \( \Gamma \), there exists a piecewise \( \Gamma \)-Möbius baker transformation with left and right Bowen–Series transformations that are transitive and orbit equivalent to \( \Gamma \).

The two maps \( T_L \) and \( T_R \) are related to the action of the group \( \Gamma \) on the boundary \( \mathbb{S}^1 \). The baker transformation \((\hat{\Sigma}, \hat{T})\) encodes this action into a unique dynamical system. For later reference, we state two further properties of this encoding.

**Remark 10.**

1. The two codings \( \gamma_L \) and \( \gamma_R \) are reciprocal, in the following sense:
   \[
   \gamma_L[\eta]^{-1} = \gamma_R^{-1}[\xi], \quad \text{whenever } (\xi', \eta') = \hat{T}(\xi, \eta).
   \]

2. For any \( \xi' \) and \( \eta \) in \( \mathbb{S}^1 \), there is a bijection between the two finite sets
   \[
   \{ \xi; (\xi', \eta) \in \hat{\Sigma} \text{ and } T_L(\xi) = \xi' \}, \quad \{ \eta'; (\xi', \eta') \in \hat{\Sigma} \text{ and } T_R(\eta') = \eta \}.
   \]

   In order to better understand this baker transformation, we briefly explain how \((\hat{\Sigma}, \hat{T})\) is conjugate to a specific Poincaré section of the geodesic flow on the surface \( N = T^1M \). We assume for the rest of this section that \( \Gamma \) satisfies the even corner property.

Since \( D_{T, O} \) is a convex fundamental domain, every geodesic (modulo \( \Gamma \)) cuts \( \partial D_{T, O} \) at two distinct points \( p \) and \( q \), unless the geodesic is tangent to one of the sides of \( D_{T, O} \). These tangent geodesics correspond to a finite union of closed geodesics. We could have parametrized the set of oriented geodesics by all pairs \((p, q) \in \partial D_{T, O} \times \partial D_{T, O}\), with \( p \) and \( q \) not belonging to the same side of \( D_{T, O} \), but we prefer to introduce the space \( X \) of all \((x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 \) oriented geodesics \([y, x]\), either cutting the interior of \( D_{T, O} \) or passing through one of the corners of \( D_{T, O} \) and seeing \( O \) to the left. Using this notation, we define the two intersection points \( p = p(x, y) \in \partial D_{T, O} \) and \( q = q(x, y) \in \partial D_{T, O} \) for every oriented geodesic \([y, x]\). \((x, y) \in X\), such that \([q, p] = [y, x] \cap D_{T, O}\) has the same orientation as \([y, x]\).

For a geodesic passing through a corner, \( p = q \), unless the geodesic is tangent to a side of \( D_{T, O} \). We are now in a position to define a geometric Poincaré section \( B : X \to X \). If \((x, y) \in X\), the geodesic \([y, x]\) leaves \( D_{T, O} \) at \( p = p(x, y) \in S_i\), for some side \( S_i^L\). Since \( S_i^L \) and \( S_i^R \) are permuted by the generator \( a_i \), the new geodesic \( a_i([y, x]) = [y', x']\) enters again the fundamental domain at a new point \( q' = q(x', y') \) with \( q' = a_i(p) \in S_i^R \). By definition, \( B(x, y) = (x', y') \) and the map \( B : X \to X \) is called a geodesic billiard, such as the codings for \( T_L \) and \( T_R \), we introduce two geometric codings \( \gamma_B : X \to \Gamma \) and \( \tilde{\gamma}_B : X \to \Gamma \) given by

\[
\gamma_B[x, y] = a_i \quad \text{if } p(x, y) \in S_i^L, \\
\tilde{\gamma}_B[x, y] = a_i \quad \text{if } q(x, y) \in S_i^R.
\]

Now the geodesic billiard can be defined by

\[
B(x, y) = (\gamma_B[x, y](x), \gamma_B[x, y](y)), \\
B^{-1}(x', y') = (\tilde{\gamma}_B[x', y'](x'), \tilde{\gamma}_B[x', y'](y')).
\]

Notice that \( \tilde{\gamma}_B \circ B = \gamma_B^{-1} \). The map \( B \) is very close to being a baker transformation: \( B \) and \( B^{-1} \) have the same structure as \( \hat{T} \) and \( \hat{T}^{-1} \), and \( \gamma_B \) (respectively, \( \tilde{\gamma}_B \)) plays the role of \( \gamma_L \) (respectively, \( \gamma_R \)). The main difference is that \( \gamma_B[x, y] \) depends on both \( x \) and \( y \), but \( \gamma_L[\xi] \) depends only on \( \xi \). Nevertheless, we have the following crucial result.
Theorem 11 ([1,16,24]). There exists a $\Gamma$-Möbius baker transformation $(\hat{\Sigma},\hat{T})$ conjugate to $(X,B)$. More precisely, there exists a map $\rho : X \to \Gamma$ such that $\pi(x,y) = (\rho[x,y](x),\rho[x,y](y))$, defines a conjugating map $\pi : X \to \hat{\Sigma}$ between $\hat{T}$ and $B$, such that $\hat{T} \circ \pi = \pi \circ B$. Equivalently, $\gamma_L \circ \pi$ and $\gamma_B$ are cohomologous over $(X,B)$, that is, $\gamma_L \circ \pi \rho = \rho \circ B \gamma_B$, and $\gamma_R \circ \pi$ and $\gamma_B$ are cohomologous over $(X,B)$, that is, $\gamma_R \circ \pi \rho = \rho \circ B^{-1} \gamma_B$.

3. Proof of theorem 1

We want to associate with any eigenfunction $f$ of the Laplace operator a nonzero piecewise real analytic function $\psi_{f,s}$ that is a solution of the functional equation

$$L_s^\Sigma(\psi_{f,s}) = \psi_{f,s}, \quad \text{where } L_s^\Sigma(\psi)(\xi') = \sum_{T_\Sigma(\xi)=\xi'} \frac{\psi(\xi)}{|T_\Sigma(\xi)|^s}.$$

The main idea is to use a kernel $k(\xi,\eta)$ introduced in theorem 7 of [3], as well by Haydn in [10], and by Bogomolny and Carioli in [6,7], in the context of double-sided subshifts of the finite type. We begin by extending this definition to include baker transformations.

Definition 12. Let $(\hat{\Sigma},\hat{T})$ be a piecewise $\Gamma$-Möbius baker transformation, with $T_L$ and $T_R$ the left and right Bowen–Series transformations. Let $A_L : S^1 \to \mathbb{C}$ and $A_R : S^1 \to \mathbb{C}$ be two potential functions. We say that $A_L$ and $A_R$ are in involution if there exists a nonzero kernel $k : \hat{\Sigma} \to \mathbb{C}^*$, called an involution kernel, such that

$$k(\xi,\eta)e^{k(\xi)} = k(\xi',\eta')e^{k(\xi')}, \quad \text{whenever } (\xi',\eta') = \tilde{T}(\xi,\eta) \in \hat{\Sigma}.$$

The kernel $k$ is extended to $S^1 \times S^1$ by $k(\xi,\eta) = 0$, for $(\xi,\eta) \notin \hat{\Sigma}$.

Remark 13.

1. Let $W(\xi,\eta) = \ln k(\xi,\eta)$, for $(\xi,\eta) \in \hat{\Sigma}$. Then $A_L$ and $A_R$ are cohomologous, that is $A_L - A_R \circ \tilde{T} = W \circ \tilde{T} - W$.
2. If $A_L(\xi)$ is Hölder, then there exists a Hölder function $A_R(\eta)$ (depending only on $\eta$) in involution with $A_L$ with a Hölder involution kernel.
3. If $L_L$ and $L_R$ are the two Ruelle transfer operators associated with $A_L$ and $A_R$, if $A_L$ and $A_R$ are in involution with respect to a kernel $k$, and if $\nu$ is an eigenmeasure of $L_R$, that is, $L_R^*\nu = \lambda \nu$, then $\psi(\xi) = \int k(\xi,\eta) d\nu(\eta)$ is an eigenfunction of $L_L$, that is, $L_L^*\psi = \lambda \psi$.

These remarks appeared first in [10] and were later rediscovered in [3], in the context of a subshift of the finite type. The proofs in this general context can be easily reproduced. The third remark suggests a strategy to obtain the eigenfunction $\psi_{f,s}$, by taking $A_L = -s \ln |T_L'|$, $A_R = -s \ln |T_R'|$ and replacing $\nu$ by the distribution $\mathcal{D}_{f,s}$. All there is left to prove is that $-\ln |T_L'|$ and $-\ln |T_R'|$ are in involution with respect to a piecewise $C^1$ involution kernel. It so happens that this involution kernel exists and is given by the Gromov distance.

Definition 14. The Gromov distance $d(\xi,\eta)$ between two points $\xi$ and $\eta$ at infinity is given by

$$d^2(\xi,\eta) = \exp \left( -b_L(O,z) - b_R(O,z) \right),$$

for any point $z$ on the geodesic line $[\xi,\eta]$. Notice that this definition depends on the choice of the origin $O$ (but not on $z \in [\xi,\eta]$).
In the Poincaré disk model, \((\xi, \eta) \in S^1 \times S^1\), or in the upper half-plane, \((s, t) \in \mathbb{R} \times \mathbb{R}\), the Gromov distance takes the simple form
\[
d^2(\xi, \eta) = \frac{1}{4} |\xi - \eta|^2, \quad \text{or} \quad d^2(s, t) = \frac{|s - t|^2}{(1 + s^2)(1 + t^2)}.
\]

**Lemma 15.** Let \(T_L : S^1 \to S^1\) and \(T_R : S^1 \to S^1\) be the two left and right Bowen–Series transformations of a \(\Gamma\)-Möbius Markov baker transformation \((\hat{\Sigma}, \hat{T})\). Then the two potential functions \(A_L(\xi) = -\ln |T_L'(\xi)|\) and \(A_R(\eta) = -\ln |T_R'(\eta)|\) are in involution and
\[
A_L(\xi) - A_R(\eta) = W(\xi', \eta') - W(\xi, \eta), \quad \text{for } (\xi', \eta') = \hat{T}(\xi, \eta) \in \hat{\Sigma},
\]
where \(W(\xi, \eta) = b_\xi(O, z) + b_\eta(O, z)\) and \(z\) is any point of the geodesic line \([[\xi, \eta]]\). In particular, \(k(\xi, \eta) = \exp(W(\xi, \eta)) = 4/d^2(\xi, \eta)\) is an involution kernel.

**Proof of lemma 15.** To simplify the notation, we call \((\xi', \eta') = \hat{T}(\xi, \eta), \gamma_L = \gamma_L[\xi]\), and \(\gamma_R = \gamma_R[\eta]\). We also recall the relation \(\gamma_R = \gamma_L^{-1}\). Then, choosing any point \(z \in [[\xi, \eta]]\), we get
\[
A_L(\xi) - A_R(\eta) = -b_\xi(O, \gamma_L^{-1}O) + b_\eta(O, \gamma_R^{-1}O) = -b_\xi(O, z) - b_\eta(O, \gamma_L(z)) + b_\eta(O, \gamma_R(z)) = W(\xi', \eta'),
\]
where \(W(\xi', \eta') = b_\eta(O, \gamma_L(z)) - b_\xi(O, \gamma_L^{-1}O)\) and \(W(\xi, \eta) = b_\xi(O, z) - b_\eta(O, \gamma_L(z))\).

Notice that if \(A(\xi)\) and \(\tilde{A}(\eta)\) are in involution by a positive kernel \(k(\xi, \eta)\), then \(sA(\xi)\) and \(s\tilde{A}(\eta)\) are in involution by \(k(\xi, \eta)^s\).

**Lemma 16.** Let \(T_L : S^1 \to S^1\) and \(T_R : S^1 \to S^1\) be the two left and right Bowen–Series transformations of a \(\Gamma\)-Möbius Markov baker transformation \((\hat{\Sigma}, \hat{T})\). Let \(A_L : S^1 \to \mathbb{R}\) and \(A_R : S^1 \to \mathbb{R}\) be two potential functions in involution with respect to a kernel \(k(\xi, \eta)\). Let \(\mathcal{L}_L\) and \(\mathcal{L}_R\) be the two Ruelle transfer operators associated with \(A_L\) and \(A_R\). Then, for any \(\xi' \in S^1\) and \(\eta \in S^1\),
\[
\mathcal{L}_R(k(\xi', \cdot))(\eta) = \mathcal{L}_L(k(\cdot, \eta))(\xi').
\]

**Proof.** Given \(\xi' \in S^1\) and \(\eta \in S^1\), the two finite sets
\[
\{\eta' \in S^1; T_R(\eta') = \eta, \quad J(\xi', \eta') = 1\}, \quad \{\xi \in S^1; T_L(\xi) = \xi', \quad J(\xi, \eta) = 1\}
\]
are in bijection. Thus, we obtain
\[
\mathcal{L}_R(k(\xi', \cdot))(\eta) = \sum_{\xi = T_L(\eta')} k(\xi, \eta) e^{A_L(\eta)} = \sum_{\xi = T_L(\eta')} k(\xi, \eta) e^{A_L(\xi)} = \mathcal{L}_L(k(\cdot, \eta))(\xi').
\]

Theorem 1 now follows immediately from lemmas 15 and 16.

**Proof of theorem 1.** We first prove that \(\psi_{f,s} : \xi' = \int k(\xi, \eta)^s \mathcal{D}_{f,s}(\xi, \eta)\), with \(k(\xi, \eta) = J(\xi, \eta)/d^2(\xi, \eta)\), is a solution of the equation \(\mathcal{L}_f^s \psi_f = \psi_f\). In fact, we have
\[
\psi_{f,s}(\xi') = \int k(\xi', \eta') \mathcal{D}_{f,s}(\eta') = \int \mathcal{L}_s^R(k(\xi', \cdot))(\eta) \mathcal{D}_{f,s}(\eta) = \mathcal{L}_s^L(\psi_{f,s})(\xi').
\]
We next prove that $\psi_{f,s} \neq 0$. Suppose on the contrary that $\psi_{f,s}(\xi') = 0$ for each $\xi' \in S^1$. Following Haydn [10], we introduce step functions of the form

$$\tilde{\chi}(\xi', \eta') = \chi \circ pr_1 \circ \hat{T}^{-1}(\xi', \eta'),$$

where $\chi = \chi(\xi)$ depends only on $\xi$. For instance, for some fixed $\xi'$, let $\chi$ be the characteristic function of the interval $I^L(n, \xi) = \cap_{k=0}^{n-1} T^{-k}_{L}((I^R \circ T^L_k(\xi)))$, for some $\xi$ such that $T^L_k(\xi) = \xi'$. Let $Q^R(\xi) = \{ \eta \in S^1; \beta(\eta) = 1 \}$ and write

$$\gamma_n[n, \xi] = \gamma_n[T^L_n^{-1}(\xi)] \cdots \gamma_n[T^L_1(\xi)] \gamma_n[\xi], \quad Q^K(n, \xi) = \gamma_n[n, \xi] Q^R(\xi).$$

Then $\tilde{\chi}$ equals the characteristic function of the rectangle $I^L(\xi') \times Q^K(n, \xi)$ and $Q^R(\xi')$ is equal to the disjoint union of the intervals $Q^R(n, \xi)$, for all $\xi$ such that $T^L_n(\xi) = \xi'$. We also denote by $\Delta(\xi')$ the set of endpoints of $Q^K(n, \xi)$, for all $T^L_n(\xi) = \xi'$, and observe that $\Delta(\xi')$ is a dense subset of $Q^R(\xi')$. Using the same ideas as in lemma 16, we obtain

$$\int \tilde{\chi}(\xi', \eta') k^L(\xi', \eta') D_{f,s}(\eta') = (L^L_n)^n(\chi \psi_{f,s})(\xi') = 0, \quad \forall \xi' \in S^1.$$

In particular, if $\tilde{\alpha}(\xi') < \tilde{\beta}(\xi') < \tilde{\alpha}(\xi') + 2\pi$ are chosen such that $\exp i\tilde{\alpha}(\xi')$ and $\exp i\tilde{\beta}(\xi')$ are the two endpoints of the interval $Q^R(\xi')$, if $\tilde{k}(\theta) = k(\xi', \exp i\theta)$, then

$$\tilde{k}(\beta) \tilde{D}_{f,s}(\beta) = \tilde{k}(\tilde{\alpha}(\xi')) \tilde{D}_{f,s}(\tilde{\alpha}(\xi')) + \int_{\tilde{\alpha}(\xi')}^{\beta} \frac{\partial \tilde{k}}{\partial \theta} \tilde{D}_{f,s}(\theta) d\theta$$

for every $\beta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')] \cap \Delta(\xi')$. Since $\tilde{k}(\theta) \neq 0$, for each $\theta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')]$, we conclude that the above equality applies to all $\beta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')]$, the two functions $\tilde{k}(\beta) \tilde{D}_{f,s}(\beta)$ and $\tilde{D}_{f,s}(\beta)$ are $C^1$, and

$$\int_{\tilde{\alpha}(\xi')}^{\beta} k(\theta) \frac{\partial \tilde{D}_{f,s}}{\partial \theta} d\theta = 0, \quad \forall \beta \in [\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')].$$

Therefore, $\tilde{D}_{f,s}(\theta)$ is a constant function on each $[\tilde{\alpha}(\xi'), \tilde{\beta}(\xi')]$, thus everywhere on $S^1$. It follows that the distribution $D_{f,s}$ would have to be equal to zero, which is impossible, because it represents a nonzero eigenfunction $f$. \hfill \blacksquare

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References

[1] Adler R and Flatto L 1991 Geodesic flows, interval maps and symbolic dynamics Bull. Am. Math. Soc. 25 229–334
[2] Anantharaman N and Zelditch S 2007 Patterson–Sullivan distributions and quantum ergodicity Ann. H Poincaré 8 361–426
[3] Baraviera A, Lopes A O and Thieullen Ph 2006 A large deviation principle for equilibrium states of Hölder potentials: the zero temperature case Stochastics Dyn. 6 77–96
[4] Bedford T, Keane M and Series C 1991 Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces (Oxford: Oxford University Press)
Eigenfunctions of the Laplacian and associated Ruelle operator

[5] Bekka M and Mayer M 2000 Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces (Oxford: Oxford University Press)

[6] Bogomolny E B and Caroli M 1992 Quantum maps of geodesic flows on surfaces of constant negative curvature 4th Int. Conf. ‘Path Integrals from meV to MeV’ (Tutzing, May 1992) pp 18–21

[7] Bogomolny E B and Caroli M 1993 Quantum maps from transfer operator Physica D 67 88–112

[8] Bowen R and Series C 1979 Markov maps associated to Fuchsian groups Publ. Math. IHES 50 153–70

[9] Flaminio L and Forni G 2003 Invariant distributions and the time averages for horocycle flows Duke Math. J. 119 465–526

[10] Haydn N 1990 Gibbs’ functionals on subshifts Commun. Math. Phys. 134 217–36

[11] Helgason S 1972 Analysis on Lie Groups and Homogeneous Spaces (Providence, RI: American Mathematical Society)

[12] Helgason S 1981 Topics in Harmonic Analysis on Homogeneous Spaces (Basel: Birkhauser)

[13] Iwaniec H 2002 Spectral Methods of Automorphic Forms (Providence, RI: American Mathematical Society)

[14] Lewis J B 1997 Spaces of holomorphic functions equivalent to the even Maass cusp forms Invent. Math. 127 271–306

[15] Lewis J and Zagier D 2001 Period functions for Maass wave forms Ann. Math. 153 191–258

[16] Lopes A O and Thieullen P 2006 Mather theory and the Bowen–Series transformation Ann. Inst. H Poincaré, Anal. Non-Linéaire 23 663–82

[17] Mayer D H 1991 Thermodynamic formalism approach to Selberg’s zeta function for PSL (2, Z) Bull. Am. Math. Soc. 25 55–60

[18] Morita T 1997 Markov systems and transfer operators associated with cofinite Fuchsian groups Ergod. Theory Dyn. Syst. 17 1147–81

[19] Otal J-P 1998 Sur les fonctions propres du laplacien du disque hyperbolique C. R. Acad. Sci. Paris Série I Math. 327 161–6

[20] Patterson S J 1976 The limit set of a Fuchsian group Acta Math. 136 241–73

[21] Pollicott M 1991 Some applications of thermodynamic formalism to manifolds with constant negative curvature Adv. Math. 85 161–92

[22] Series C 1981 Symbolic dynamics for geodesic flows Acta Math. 146 103–28

[23] Series C 1985 The modular surface and continuous fractions J. Lond. Math. Soc. 31 69–80

[24] Series C 1986 Geometrical Markov coding on surfaces of constant negative curvature Ergod. Theory Dyn. Syst. 6 601–25

[25] Sullivan D 1979 The density at infinity of a discrete group of hyperbolic motions Publ. Math. IHES 50 171–202

[26] Zelditch S 1987 Uniform distribution of eigenfunctions on compact hyperbolic surfaces Duke Math. J. 55 919–41