The Logarithmic Contributions to the $O(\alpha_s^3)$ Asymptotic Massive Wilson Coefficients and Operator Matrix Elements in Deeply Inelastic Scattering

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Abstract

We calculate the logarithmic contributions to the massive Wilson coefficients for deep-inelastic scattering in the asymptotic region $Q^2 \gg m^2$ to 3-loop order in the fixed-flavor number scheme and present the corresponding expressions for the massive operator matrix elements needed in the variable flavor number scheme. Explicit expressions are given both in Mellin-$N$ space and $z$-space.

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1 Introduction

The heavy flavor corrections to deep-inelastic structure functions amount to sizeable contributions, in particular in the region of small values of the Bjorken variable $x$. Starting from lower values of the virtuality, over a rather wide kinematic range, their scaling violations are very different from those of the massless contributions. Currently the heavy flavor corrections are known in semi-analytic form to 2–loop (NLO) order \cite{1}. The present accuracy of the deep-inelastic data reaches the order of 1\% \cite{2}. It therefore requires the next-to-next-to-leading order (NNLO) corrections for precision determinations of both the strong coupling constant $\alpha_s(M_Z^2)$ \cite{3} and the parton distribution functions (PDFs) \cite{4, 5}, as well as the detailed understanding of the heavy flavor production cross sections in lepton–nucleon scattering \cite{6}. The precise knowledge of these quantities is also of central importance for the interpretation of the physics results at the Large Hadron Collider, LHC, \cite{7}.

In the kinematic region at HERA, where the twist-2 contributions to the deep-inelastic scattering (DIS) dominate cf. \cite{8}, i.e. $Q^2/m^2 \gtrsim 10$, with $m = m_c$ the charm quark mass, it has been proven in Ref. \cite{10} that the heavy flavor Wilson coefficients factorize into massive operator matrix elements (OMEs) and the massless Wilson coefficients. The massless Wilson coefficients for the structure function $F_2(x, Q^2)$ are known to 3-loop order \cite{11}. In the region $Q^2 \gg m^2$, where $Q^2 = -q^2$, with $q$ the space-like 4–momentum transfer and $m$ the heavy quark mass, the power corrections $O((m^2/Q^2)^k), k \geq 1$ to the heavy quark structure functions become very small.

In Ref. \cite{12} a series of fixed Mellin moments $N$ up to $N = 10, \ldots, 14$, depending on the respective transition, has been calculated for all the OMEs at 3–loop order \cite{3}. Also the moments of the transition coefficients needed in the variable flavor scheme (VFNS) have been calculated. Here, the massive OMEs for given total spin $N$ were mapped onto massive tadpoles which have been computed using \textsc{MATAD} \cite{14}.

In the present paper, we calculate the logarithmic contributions to the unpolarized massive Wilson coefficients in the asymptotic region $Q^2 \gg m^2$ to 3–loop order and the massive OMEs needed in the VFNS. These include the logarithmic terms $\log(Q^2/m^2)$. In the following, we set the factorization and renormalization scales equal $\mu_F = \mu_R \equiv \mu$ and exhibit the $\log(m^2/\mu^2)$ dependence on the Wilson coefficients, besides their dependence on the virtuality $Q^2$. The logarithmic contributions are determined by the lower order massive OMEs \cite{15–19}, the mass- and coupling constant renormalization constants, and the anomalous dimensions \cite{20, 21}, as has been worked out in Ref. \cite{12}. For the structure function $F_L(x, Q^2)$ the asymptotic heavy flavor Wilson coefficients at $O(\alpha_s^3)$ were calculated in \cite{22}. They are also presented here, for inclusive hadronic final states. In this case the corrections, however, become effective only at much higher scales of $Q^2$ \cite{10} compared to the case of $F_2(x, Q^2)$. We first choose the fixed flavor number scheme to express the heavy flavor contributions to the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$. This scheme has to be considered as the genuine scheme in quantum field theoretic calculations since the initial states, the twist–2 massless partons can, at least to a good approximation, be considered as LSZ-states. The representations in the VFNS can be obtained using the respective transition coefficients within the appropriate regions, where one single heavy quark flavor becomes effectively massless. Here, appropriate matching scales have to be applied, which vary in dependence on the observable considered, cf. \cite{23}.

Two of the OMEs, $A_{qq,Q}^{PS}(N)$ and $A_{qq,Q}^{PS}(N)$, have already been calculated completely including the constant contribution in Ref. \cite{24}. They and the corresponding massive Wilson coefficients

\footnotetext{\textsuperscript{2}For higher order corrections to the gluonic contributions in the threshold region, cf. \cite{9}.}

\footnotetext{\textsuperscript{3}For the corresponding contributions in case of transversity see \cite{13}.}
contribute first at 2– and 3–loop order, respectively. For these quantities we also derive numerical results. The quantities being presented in the present paper derive from OMEs which were computed in terms of generalized hypergeometric functions \[25\] and sums thereof, prior to the expansion in the dimensional variable \(\varepsilon = D - 4\), cf. \[26, 28\]. Finally, they are represented in terms of nested sums over products of hypergeometric terms and harmonic sums, which can be calculated using modern summation techniques \[29–33\]. They are based on a refined difference field of \[34\] and generalize the summation paradigms presented in \[35\] to multi-summation. The results of this computation can be expressed in terms of nested harmonic sums \[36, 37\]. The corresponding representations in \(z\)-space are obtained in terms of harmonic polylogarithms \[38\].

Here, the variable \(z\) denotes the partonic momentum fraction. The results in Mellin \(N\)-space can be continued to complex values of \(N\) as has been described in Refs. \[26, 39\].

It is the aim of the present paper to provide a detailed documentation of formulae both in \(N\)- and \(z\)-space for all logarithmic contributions to the heavy flavor Wilson coefficients of the structure functions \(F_2(x, Q^2)\) and \(F_L(x, Q^2)\) and the massive OMEs needed in the variable flavor number scheme up to \(O(\alpha_s^3)\). Here, we refer to a minimal representation, i.e. we use all the algebraic relations between the harmonic sums and the harmonic polylogarithms, respectively, leading to a minimal number of basic functions. Based on the known Mellin moments \[12\] we also perform numerical comparisons between the different contributions to the Wilson coefficients and massive OMEs at \(O(\alpha_s^3)\) referring to the parton distributions \[5\].

The paper is organized as follows. In Section 2, we summarize the basic formalism. The Wilson coefficients \(L_{q,2}^{PS}\) and \(L_{g,2}^{S}\) are discussed in Section 3. As they are known in complete form we also present numerical results. In Section 4, the logarithmic contributions to the Wilson coefficients \(H_{q,2}^{PS}\) and \(H_{g,2}^{S}\) are derived. The corresponding Wilson coefficients for the longitudinal structure function \(F_L(x, Q^2)\) in the asymptotic region are presented in Section 5. In Section 6, we compare the different loop contributions to the massive Wilson coefficients and OMEs for a series of Mellin moments in dependence on the virtuality \(Q^2\). Section 7 contains the conclusions. In Appendix A, the massive OMEs needed in the VFNS are given in Mellin \(N\)-space. The asymptotic heavy flavor Wilson coefficients contributing to the structure function \(F_2(x, Q^2)\) are presented in \(z\)-space in Appendix B, retaining all contributions except for the 3-loop constant part of the unrenormalized OMEs \(a_{ij}^{(3)}\) being not yet known. Likewise, in Appendix C and D, the asymptotic heavy flavor Wilson coefficients for the structure function \(F_L(x, Q^2)\) and the massive OMEs are given in \(z\)-space.

# 2 The heavy flavor Wilson coefficients in the asymptotic region

We consider the heavy flavor contributions to the inclusive unpolarized structure functions \(F_2(x, Q^2)\) and \(F_L(x, Q^2)\) in deep-inelastic scattering, cf. \[41, 42\], in case of single electro-weak gauge-boson exchange at large virtualities \(Q^2\). At higher orders in the strong coupling constant these corrections receive both contributions from massive and massless partons in the hadronic final state, which is summed over completely. In the latter case, the heavy flavor corrections are also due to virtual contributions. We consider the situation in which the contributions to the twist-2 operators dominate in the Bjorken limit. Here, no transverse momentum effects of the initial state contribute. In the present paper, we consider only heavy flavor contributions.

\[4\] The expressions for the non-singlet Wilson-coefficient, are presented elsewhere together with the OME for transversity \[40\].
due to $N_F$ massless and one massive flavor of mass $m$. The Wilson coefficients are calculable perturbatively and are denoted by

$$C_{i,(2,L)}^{S,PS,NS} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right).$$  \hfill (1)

Here, $x$ denotes the Bjorken variable, the index $i$ refers to the respective initial state on-shell parton $i = q, g$ being a quark or gluon, and $S, PS, NS$ label the flavor singlet, pure-singlet and non-singlet contributions, respectively. In the twist-2 approximation the Bjorken variable $x$ and the parton momentum fraction $z$ are identical. Representations in momentum fraction space are therefore also called $z$-space representation in what follows.

The massless flavor contributions in (1) may be identified and separated in the Wilson coefficients into a purely light part $C_{i,(2,L)}$, and a heavy part by:

$$C_{i,(2,L)}^{S,PS,NS} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = C_{i,(2,L)}^{S,PS,NS} \left( x, N_F, \frac{Q^2}{\mu^2} \right) + H_{i,(2,L)}^{S,PS} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) + L_{i,(2,L)}^{S,PS,NS} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right).$$  \hfill (2)

The heavy flavor Wilson coefficients are defined by $L_{i,j}$ and $H_{i,j}$, depending on whether the exchanged electro-weak gauge boson couples to a light ($L$) or heavy ($H$) quark line. From this it follows that the light flavor Wilson coefficients $C_{i,j}$ depend on $N_F$ light flavors only, whereas $H_{i,j}$ and $L_{i,j}$ may contain light flavors in addition to the heavy quark, indicated by the argument $N_F + 1$. The perturbative series of the heavy flavor Wilson coefficients read

$$H_{g,(2,L)}^S \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_{i=1}^{\infty} a_s^i H_{g,(2,L)}^{(i),S} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right),$$  \hfill (3)

$$H_{q,(2,L)}^{PS} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_{i=1}^{\infty} a_s^i H_{q,(2,L)}^{(i),PS} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right),$$  \hfill (4)

$$L_{g,(2,L)}^S \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_{i=2}^{\infty} a_s^i L_{g,(2,L)}^{(i),S} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right),$$  \hfill (5)

$$L_{q,(2,L)}^S \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_{i=2}^{\infty} a_s^i L_{q,(2,L)}^{(i),S} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right).$$  \hfill (6)

Here, we defined $a_s = \alpha_s/(4\pi)$. At leading order, only the term $H_{g,(2,L)}$ contributes via the photon–gluon fusion process, \textsuperscript{44}\textsuperscript{49}

$$\gamma^* + g \rightarrow Q + \overline{Q}.$$  \hfill (7)

At $O(a_s^2)$, the terms $H_{q,(2,L)}^{PS}, L_{q,(2,L)}^S$ and $L_{g,(2,L)}^S$ contribute as well. They result from the processes

$$\gamma^* + q(\overline{q}) \rightarrow q(\overline{q}) + X,$$  \hfill (8)

$$\gamma^* + g \rightarrow q(\overline{q}) + X,$$  \hfill (9)

\textsuperscript{5}At 3–loop order there are also contributions by graphs carrying heavy quark lines of different mass. These are dealt with elsewhere \textsuperscript{43}. 

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where $X$ may contain heavy flavor contributions. $L^S_{q_i(2, L)}$ can be split into the flavor non-singlet and pure-singlet contributions

$$L^S_{q_i(2, L)} = L^{\text{NS}}_{q_i(2, L)} + L^{\text{PS}}_{q_i(2, L)},$$

and at $O(a_s^2)$ only the non-singlet term contributes. The pure-singlet term emerges at 3-loop order.

The heavy quark contribution to the structure functions $F_{(2, L)}(x, Q^2)$ for one heavy quark of mass $m$ and $N_F$ light flavors is then given by, cf. [15], in case of pure photon exchange\(^6\)

$$\frac{1}{x} F^{Q\gamma}_{(2, L)}(x, N_F + 1, Q^2, m^2) =$$

$$\sum_{k=1}^{N_F} e_k^2 \left\{ L^{\text{NS}}_{q_i(2, L)} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \left[ f_k(x, \mu^2, N_F) + f^*_k(x, \mu^2, N_F) \right] + \frac{1}{N_F} L^{\text{PS}}_{q_i(2, L)} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, N_F) \right\}$$

$$+ e_Q^2 \left[ H^{PS}_{q_i(2, L)} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, N_F) + H^S_{q_i(2, L)} \left( x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, N_F) \right],$$

(11)

The meaning of the argument $(N_F + 1)$ in Eqs. (11) in the massive Wilson coefficients shall be interpreted as $N_F$ massless and one massive flavor. $N_F$ denotes the number of massless flavors. The symbol $\otimes$ denotes the Mellin convolution\(^7\)

$$[A \otimes B](x) = \int_0^1 \int_0^1 dx_1 dx_2 \, \delta(x - x_1 x_2) A(x_1) B(x_2).$$

(12)

The charges of the light quarks are denoted by $e_k$ and that of the heavy quark by $e_Q$. The scale $\mu^2$ is the factorization scale, and $f_k, f^*_k, \Sigma$ and $G$ are the quark, anti-quark, flavor singlet and gluon distribution functions, with

$$\Sigma(x, \mu^2, N_F) = \sum_{k=1}^{N_F} \left[ f_k(x, \mu^2, N_F) + f^*_k(x, \mu^2, N_F) \right].$$

(13)

An important part of the kinematic region in case of heavy flavor production in DIS is located at larger values of $Q^2$, cf. e.g. [54, 55]. As has been shown in Ref. [10], the heavy flavor Wilson coefficients $H_{i,j}$, $L_{i,j}$ factorize in the limit $Q^2 \gg m^2$ into massive operator matrix elements $A_{ki}$ and the massless Wilson coefficients $C_{i,j}$, if one heavy quark flavor and $N_F$ light flavors are considered. The massive OMEs are process independent quantities and contain all the mass dependence except for the power corrections $\propto (m^2/Q^2)^k$, $k \geq 1$. The process dependence is implied by the massless Wilson coefficients. This allows the analytic calculation of the NLO

\(^6\)For the heavy flavor corrections in case of $W^\pm$-boson exchange up to $O(a_s^2)$ see [50–53].

\(^7\)Note that the heavy flavor threshold in the limit $Q^2 \gg m^2$ is again $x$ and not $x(1 + 4m^2/Q^2)$, which is the case retaining also power corrections.
heavy flavor Wilson coefficients, [10,17]. Comparing these asymptotic expressions with the exact LO and NLO results obtained in Refs. [44–47,49] and [1], respectively, one finds that this approximation becomes valid in case of \( F_2^{Q^2} \) for \( Q^2/m^2 > 10 \). These scales are sufficiently low and match with the region analyzed in deeply inelastic scattering for precision measurements. In case of \( F_2^{Q^2} \), this approximation is only valid for \( Q^2/m^2 > 800 \), [10]. For the latter case, the 3–loop corrections were calculated in Ref. [22]. This difference is due to the emergence of terms \( \propto (m^2/Q^2) \ln(m^2/Q^2) \), which only vanish slowly in the limit \( Q^2/m^2 \to \infty \).

In order to derive the factorization formula, one considers the inclusive Wilson coefficients \( C_{i,j}^{S,PS,NS} \), which have been defined in Eq. (1). After applying the light cone expansion (LCE) [56] to the partonic tensor, or the forward Compton amplitude, corresponding to the respective Wilson coefficients, one arrives at the factorization relation,

\[
C_{j,(2,L)}^{S,PS,NS,\text{asymp}}(N, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}) = \sum_i A_{ij}^{S,PS,NS}(N, N_F + 1, \frac{m^2}{\mu^2}) C_{i,(2,L)}^{S,PS,NS}(N, N_F + 1, \frac{Q^2}{\mu^2}) + O(\frac{m^2}{Q^2}).
\] (14)

Here, \( \mu \) refers to the factorization scale between the heavy and light contributions in \( C_{j,i} \) and \( '\text{asymp}' \) denotes the limit \( Q^2 \gg m^2 \). The \( C_{i,j} \) are the light Wilson coefficients, cf. [11], taken at \( N_F + 1 \) flavors. This can be inferred from the fact that in the LCE the Wilson coefficients describe the singularities for very large values of \( Q^2 \), which can not depend on the presence of a quark mass. The mass dependence is given by the OMEs \( A_{ij} \), between partonic states. Eq. (14) accounts for all mass effects but corrections which are power suppressed, \( (m^2/Q^2)^k, k \geq 1 \). This factorization is only valid if the heavy quark coefficient functions are defined in such a way that all radiative corrections containing heavy quark loops are included. Otherwise, (14) would not show the correct asymptotic \( Q^2 \)–behavior, [15,19]. An equivalent way of describing Eq. (14) is obtained by considering the calculation of the massless Wilson coefficients. Here, the initial state collinear singularities are given by evaluating the massless OMEs between off–shell partons, leading to transition functions \( \Gamma_{ij} \). The \( \Gamma_{ij} \) are given in terms of the anomalous dimensions of the twist–2 operators and transfer the initial state singularities to the bare parton–densities due to mass factorization, cf. e.g. [10,15]. In the case at hand, something similar happens: The initial state collinear singularities are transferred to the parton densities except for those which are regulated by the quark mass and described by the OMEs. Instead of absorbing these terms into the parton densities as well, they are used to reconstruct the asymptotic behavior of the heavy flavor Wilson coefficients. Here,

\[
A_{ij}^{S,NS}(N, N_F + 1, \frac{m^2}{\mu^2}) = \langle j|O^{S,NS}_i|j \rangle = \delta_{ij} + \sum_{i=k}^\infty a^{(k)}_{ij} = \sum_{i=k}^\infty a^{(k)}_{ij} \]

are the operator matrix elements of the local twist–2 operators between on–shell partonic states \( |j \rangle, \ j = q, g \).

Let us now derive the explicit expressions for the massive Wilson coefficients in the asymptotic region. One may split Eq. (14) into parts by considering the different \( N_F \) contributions. We define

\[
\tilde{f}(N_F) \equiv \frac{f(N_F)}{N_F}.
\] (16)

This is necessary in order to separate the different types of contributions in Eq. (11), weighted by the electric charges of the light and heavy flavors, respectively. Since we would like to derive
the heavy flavor part, we define as well for later use

\[ \hat{f}(N_F) \equiv f(N_F + 1) - f(N_F) , \]  

where \( \hat{f}(N_F) \equiv \overline{\hat{f}(N_F)} \). The following Eqs. (18)–(22) are the same as Eqs. (2.31)–(2.35) in Ref. [15]. We present these terms here again, however, since Ref. [15] contains a few inconsistencies regarding the \( \hat{f} \)-description. Contrary to the latter reference, the argument corresponding to the number of flavors stands for all flavors, light or heavy. The separation for the \( \text{NS} \)-term is obtained by

\[
C_{q,(2,L)}^{\text{NS}}(N, N_F, \frac{Q^2}{\mu^2}) + I_{q,(2,L)}^{\text{NS}}(N, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}) = \]

\[
A_{qq,Q}^{\text{NS}}(N, N_F + 1, \frac{m^2}{\mu^2}) C_{q,(2,L)}^{\text{NS}}(N, N_F + 1, \frac{Q^2}{\mu^2}) .
\]

Here and in the following, we omit the index "asympt" to denote the asymptotic heavy flavor Wilson coefficients. For the remaining terms, we suppress the arguments \( N, Q^2/\mu^2 \) and \( m^2/\mu^2 \) for brevity, all of which can be inferred from Eqs. (2, 14). Additionally, we will suppress from now on the index \( S \) and label only the \( \text{NS} \) and \( \text{PS} \) terms explicitly. The contributions to \( L_{i,j} \) read

\[
C_{q,(2,L)}^{\text{PS}}(N_F) + I_{q,(2,L)}^{\text{PS}}(N_F + 1) = \left[ A_{qq,Q}^{\text{NS}}(N_F + 1) + A_{qq,Q}^{\text{PS}}(N_F + 1) + A_{Qq}^{\text{PS}}(N_F + 1) \right]
\]

\[
\times N_F \hat{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) + A_{qg,Q}^{\text{PS}}(N_F + 1) C_{q,(2,L)}^{\text{NS}}(N_F + 1)
\]

\[
+ A_{qg,Q}^{\text{NS}}(N_F + 1) N_F \hat{C}_{g,(2,L)}^{\text{NS}}(N_F + 1) ,
\]

\[
C_{g,(2,L)}(N_F) + L_{g,(2,L)}(N_F + 1) = A_{gg,Q}^{\text{NS}}(N_F + 1) N_F \hat{C}_{g,(2,L)}^{\text{NS}}(N_F + 1)
\]

\[
+ A_{qg,Q}^{\text{NS}}(N_F + 1) C_{q,(2,L)}^{\text{NS}}(N_F + 1)
\]

\[
+ \left[ A_{qg,Q}^{\text{NS}}(N_F + 1) + A_{Qg}^{\text{PS}}(N_F + 1) \right] N_F \hat{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) .
\]

The terms \( H_{i,j} \) are given by

\[
H_{q,(2,L)}^{\text{PS}}(N_F + 1) = A_{Qq}^{\text{PS}}(N_F + 1) \left[ C_{q,(2,L)}^{\text{NS}}(N_F + 1) + \hat{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) \right]
\]

\[
+ \left[ A_{qg,Q}^{\text{NS}}(N_F + 1) + A_{qg,Q}^{\text{PS}}(N_F + 1) \right] \hat{C}_{q,(2,L)}^{\text{PS}}(N_F + 1)
\]

\[
+ A_{qg,Q}^{\text{NS}}(N_F + 1) \hat{C}_{g,(2,L)}^{\text{NS}}(N_F + 1) ,
\]

\[
H_{g,(2,L)}(N_F + 1) = A_{gg,Q}^{\text{NS}}(N_F + 1) \hat{C}_{g,(2,L)}^{\text{NS}}(N_F + 1) + A_{qg,Q}^{\text{NS}}(N_F + 1) \hat{C}_{q,(2,L)}^{\text{PS}}(N_F + 1)
\]

\[
+ A_{Qg}^{\text{NS}}(N_F + 1) \left[ C_{q,(2,L)}^{\text{NS}}(N_F + 1) + \hat{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) \right] .
\]

Expanding the above relations up to \( O(a_s^2) \), we obtain, using Eqs. (16, 17), the heavy flavor Wilson coefficients in the asymptotic limit, cf. [12] :

\[
L_{q,(2,L)}^{\text{NS}}(N_F + 1) = a_s^2 \left[ A_{qq,Q}^{(2),\text{NS}}(N_F + 1) \delta_2 + \hat{C}_{q,(2,L)}^{(2),\text{NS}}(N_F) \right]
\]

\[
+ a_s^3 \left[ A_{qq,Q}^{(3),\text{NS}}(N_F + 1) \delta_2 + A_{qg,Q}^{(2),\text{NS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \right] .
\]
Wilson coefficients in the asymptotic region derives from the unrenormalized massive OMEs \( \mu \) massless Wilson coefficients is

\[ L_{q,(2,L)}^{PS}(N_F + 1) = a_s^2 \left[ A_{gq}^{(1),PS}(N_F + 1) \delta_2 + A_{gq}^{(2)}(N_F + 1) \right. \]

\[ + N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \]

\[ + N_F \tilde{C}_{q,(2,L)}^{(3),PS}(N_F) \],

(24)

The renormalized massive OMEs depend on the ratio \( m^2/\mu^2 \), while the scale ratio in the massless Wilson coefficients is \( \mu^2/Q^2 \). The latter are pure functions of the momentum fraction \( z \), or the Mellin variable \( N \), if one sets \( \mu^2 = Q^2 \). The mass dependence on the heavy flavor Wilson coefficients in the asymptotic region derives from the unrenormalized massive OMEs

\[ \tilde{A}^{(3)}_{ij}(\varepsilon) = \frac{1}{\varepsilon^3} \tilde{a}^{(3),3}_{ij} + \frac{1}{\varepsilon^2} \tilde{a}^{(3),2}_{ij} + \frac{1}{\varepsilon} \tilde{a}^{(3),1}_{ij} + \tilde{a}^{(3),0}_{ij}, \]

(28)

applying mass, coupling constant, and operator-renormalization, as well as mass factorization.
The renormalized massive OMEs obey then the general structure

\[ A_{ij}^{(3)} \left( \frac{m^2}{Q^2} \right) = a_{ij}^{(3),3} \ln^3 \left( \frac{m^2}{Q^2} \right) + a_{ij}^{(3),2} \ln^2 \left( \frac{m^2}{Q^2} \right) + a_{ij}^{(3),1} \ln \left( \frac{m^2}{Q^2} \right) + a_{ij}^{(3),0} . \]  

(29)

The subsequent calculations will be performed in the $\overline{\text{MS}}$ scheme for the coupling constant and the on-shell scheme for the heavy quark mass $m$. The transition to the scheme in which $m$ is renormalized in the $\overline{\text{MS}}$-scheme is described in Ref. [12]. The strong coupling constant is obtained as the perturbative solution of the equation

\[ \frac{d \alpha_s(\mu^2)}{d \ln(\mu^2)} = -\sum_{l=0}^{\infty} \beta_l \alpha_s^{l+2}(\mu^2) \]  

(30)

to 3-loop order, where $\beta_k$ are the expansion coefficients of the QCD $\beta$-function and $\mu^2$ denotes the renormalization scale. For simplicity we identify the factorization ($\mu_F$) and renormalization ($\mu_R$) scales from now on. In the subsequent sections we present explicit expressions of the asymptotic heavy flavor Wilson coefficients in Mellin-$N$ space. They depend on the logarithms

\[ L_Q = \ln \left( \frac{Q^2}{\mu^2} \right) \quad \text{and} \quad L_M = \ln \left( \frac{m^2}{\mu^2} \right) , \]  

(31)

where $\mu \equiv \mu_F = \mu_R$.

Besides the Wilson coefficients, the massive OMEs are important themselves to establish the matching conditions in the variable flavor number scheme in describing the process of a single massive quark becoming massless\footnote{For the VFNS in case of both the bottom and charm quarks transmuting into massless states, see [43].} at large enough scales $\mu^2$; [12,15]. Here, the PDFs for $N_F + 1$ massless quarks are related to the former $N_F$ massless quarks process independently. The corresponding relations to 3-loop order read, cf. also [15]:

\[ f_k(N_F + 1, \mu^2) + f_\bar{k}(N_F + 1, \mu^2) = A_{q_q,Q}^{\text{NS}} \left( N_F, \frac{\mu^2}{m^2} \right) \otimes \left[ f_k(N_F, \mu^2) + f_\bar{k}(N_F, \mu^2) \right] \]

\[ + A_{q_g,Q}^{\text{PS}} \left( N_F, \frac{\mu^2}{m^2} \right) \otimes \Sigma(N_F, \mu^2) \]

\[ + A_{g_g,Q}^{\text{S}} \left( N_F, \frac{\mu^2}{m^2} \right) \otimes G(N_F, \mu^2) \]  

(32)

\[ f_{Q+\bar{Q}}(N_F + 1, \mu^2) = A_{Q_q}^{\text{PS}} \left( N_F, \frac{\mu^2}{m^2} \right) \otimes \Sigma(N_F, \mu^2) + A_{\bar{Q}_q}^{\text{S}} \left( N_F, \frac{\mu^2}{m^2} \right) \otimes G(N_F, \mu^2) \]  

(33)

\[ G(N_F + 1, \mu^2) = A_{g_q,Q}^{\text{S}} \left( N_F, \frac{\mu^2}{m^2} \right) \otimes \Sigma(N_F, \mu^2) + A_{g_g,Q}^{\text{S}} \left( N_F, \frac{\mu^2}{m^2} \right) \otimes G(N_F, \mu^2) . \]  

(34)

\[ \Sigma(N_F + 1, \mu^2) = \left[ A_{q_q,Q}^{\text{NS}} \left( N_F, \frac{\mu^2}{m^2} \right) + N_F A_{q_q,Q}^{\text{PS}} \left( N_F, \frac{\mu^2}{m^2} \right) + A_{Q_q}^{\text{PS}} \left( N_F, \frac{\mu^2}{m^2} \right) \right] \]

\[ \otimes \Sigma(N_F, \mu^2) \]

\[ + \left[ N_F A_{g_q,Q}^{\text{S}} \left( N_F, \frac{\mu^2}{m^2} \right) + A_{\bar{Q}_q}^{\text{S}} \left( N_F, \frac{\mu^2}{m^2} \right) \right] \otimes G(N_F, \mu^2) \]  

(35)
Here, the $N_F$-dependence of the OMEs is understood as functional and $\mu^2$ denotes the matching scale, which for the heavy-to-light transitions is normally much larger than mass scale $m^2$, [23]. We will present the corresponding OMEs in Appendix A. The results of the calculations being presented in the subsequent sections have been obtained making mutual use of the packages HarmonicSums.m [58] and Sigma.m [29].

3 The Wilson Coefficients $L_{q,2}^{PS}$ and $L_{g,2}^{S}$

The OMEs for these Wilson coefficients have been calculated in [24]. They contribute for the first time at 3– and 2–loop order, respectively, and stem from processes in which the virtual electro-weak gauge boson couples to a massless quark. As a shorthand notation we also define the function

$$\gamma_{qq}^0 = -4 \frac{N^2 + N + 2}{N(N + 1)(N + 2)}$$

(36)

denoting the kinetic part of the leading order anomalous dimensions separating off the corresponding color factor.

In Mellin-$N$ space the Wilson coefficient $L_{q,2}^{PS}$ reads:

$$L_{q,2}^{PS} = \frac{1}{2} \left[ 1 + (-1)^N \right]$$

$$\times a_s^3 \left\{ C_F N_F T_F^2 \left[ -\frac{32 P_4 L_Q^2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + L_Q \left[ \frac{64 P_6}{27(N - 1)N^4(N + 1)^4(N + 2)^3} 

- \frac{256 P_1 (-1)^N}{9(N - 1)N^2(N + 1)^3(N + 2)^3} + \frac{2(\gamma_{qq}^0)^2(N + 2)L_M^2}{3(N - 1)} 

+ \left[ \frac{64(N^2 + N + 2)(8N^3 + 13N^2 + 27N + 16)}{9(N - 1)N^2(N + 1)^3(N + 2)} - \frac{64(N^2 + N + 2)^2S_1}{3(N - 1)N^2(N + 1)^2(N + 2)} \right] L_M 

+ \frac{512S_{-2}}{3(N - 1)N(N + 1)(N + 2)} \right] - \frac{32 P_4 L_M^2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} 

+ \left[ -\frac{32 P_7}{27(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{64 P_2 S_1}{3(N - 1)N^3(N + 1)^3(N + 2)^2} 

+ \frac{(N^2 + N + 2)^2}{(N - 1)N^2(N + 1)^2(N + 2)} \frac{32}{3} (S_1^2 - S_2) \right] L_M 

- \frac{32 P_9}{243(N - 1)N^5(N + 1)^5(N + 2)^4} + \frac{16 P_3 S_1^2}{16(N - 1)N^4(N + 1)^4(N + 2)^3} 

- \frac{27(N - 1)N^3(N + 1)^3(N + 2)^2}{27(N - 1)N^3(N + 1)^3(N + 2)^2} - \frac{32 L_Q^3(N^2 + N + 2)^2}{9(N - 1)N^2(N + 1)^2(N + 2)^3} + \frac{32(N^2 + N + 2)^2 L_M^3}{9(N - 1)N^2(N + 1)^2(N + 2)} 

+ \left[ -\frac{64}{27} S_1^3 + \frac{32}{9} S_2 S_1 + \frac{160 S_3}{27} + \frac{256 \zeta_3}{9} \right] \right] \right\} ,$$

(37)
with the polynomials

\[
\begin{align*}
P_1 &= 4N^6 + 22N^5 + 48N^4 + 53N^3 + 45N^2 + 36N + 8 \\
P_2 &= N^7 - 15N^5 - 58N^4 - 92N^3 - 76N^2 - 48N - 16 \\
P_3 &= N^7 - 37N^6 - 248N^5 - 799N^4 - 1183N^3 - 970N^2 - 580N - 168 \\
P_4 &= 11N^7 + 37N^6 + 53N^5 + 7N^4 - 68N^3 - 56N^2 - 80N - 48 \\
P_5 &= 49N^7 + 185N^6 + 340N^5 + 287N^4 + 65N^3 + 62N^2 - 196N - 168 \\
P_6 &= 85N^{10} + 530N^9 + 1458N^8 + 2112N^7 + 1744N^6 + 2016N^5 + 3399N^4 + 2968N^3 \\
&\quad + 1864N^2 + 1248N + 432 \\
P_7 &= 143N^{10} + 838N^9 + 1995N^8 + 1833N^7 - 1609N^6 - 5961N^5 - 7503N^4 - 6928N^3 \\
&\quad - 4024N^2 - 816N + 144 \\
P_8 &= 176N^{10} + 973N^9 + 1824N^8 - 948N^7 - 10192N^6 - 19173N^5 - 20424N^4 - 16036N^3 \\
&\quad - 7816N^2 - 1248N + 288 \\
P_9 &= 1717N^{13} + 16037N^{12} + 66983N^{11} + 161797N^{10} + 241447N^9 + 216696N^8 + 86480N^7 \\
&\quad - 67484N^6 - 170003N^5 - 165454N^4 - 81976N^3 - 15792N^2 - 1008N - 864. \\
\end{align*}
\]

For the massless 3-loop Wilson coefficients \( C_i^k \) we refer to Ref. [11]. Here and in the following, their expression will be kept symbolically. The corresponding \( z \)-space expressions are given in Appendix \( \text{B} \).

Likewise the Wilson coefficient \( L_{g,2}^5 \) is given by :

\[
L_{g,2}^5 = \frac{1}{2} \left[ 1 + (-1)^N \right] \left\{ a_s^3 T_F^2 N_f \left\{ L_M \left[ \frac{4}{3} \gamma_{qq}^0 S_1 - \frac{16(N^3 - 4N^2 - N - 2)}{3N^2(N+1)(N+2)} \right] \\
- \frac{4}{3} \gamma_{qq}^0 L_Q L_M \right\} + a_s^3 \left\{ N_f T_F^2 \left[ \frac{16}{9} \gamma_{qq}^0 S_1 - \frac{64(N^3 - 4N^2 - N - 2)}{9N^2(N+1)(N+2)} \right] - \frac{16}{9} \gamma_{qq}^0 L_Q L_M \right\} \\
+ C_A N_f T_F^2 \left[ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)(N+2)^2} + \frac{8}{3} \gamma_{qq}^0 S_1 \right] L_Q^3 + \left[ \frac{64(-1)^N(N^3 - 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right] \left[ \frac{8P_{25}}{9(N-1)N^2(N+1)^2(N+2)^3} + \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1}{9(N-1)N(N+1)(N+2)^2} \right] \right. \\
+ L_M \left\{ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{3(N-1)N^2(N+1)(N+2)^2} + \frac{8}{3} \gamma_{qq}^0 S_1 \right\} + \frac{8}{3} \gamma_{qq}^0 \left[ \frac{4}{3} S_1^2 + \frac{4S_2}{3} + \frac{8}{3} S_{-2} \right] \right\} \right. \\
\left. + \left[ \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1^2}{9(N-1)N(N+1)^2(N+2)} + \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{3(N+1)^3(N+2)^3} \right] \right. \\
- \frac{32P_{24}S_1}{27(N-1)N^2(N+1)^3(N+2)^3} + \frac{64(-1)^NP_{18}}{9(N-1)N^2(N+1)^4(N+2)^4} \\
- \frac{16P_{32}}{27(N-1)N^3(N+1)^4(N+2)^4} + L_M^2 \left[ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)^2} \right] \\
+ \frac{8}{3} \gamma_{qq}^0 S_1 + \frac{32(8N^4 + 13N^3 - 22N^2 - 9N - 26)S_2}{9(N-1)N(N+1)(N+2)^2} + \frac{128(N^2 + N - 1)S_3}{9(N(N+1)(N+2)} \\
+ \frac{64(8N^5 + 15N^4 + 6N^3 + 11N^2 + 16N + 16)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2} + L_M \left[ \frac{32P_{26}}{9(N-1)N^3(N+1)^3(N+2)^3} \right] \\
- \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right. \\
- \frac{64(2N - 1)(N^3 + 9N^2 + 7N + 7)S_1}{9(N-1)N(N+1)^2(N+2)} \right. \\
\left. \right. \]
\[\begin{align*}
&\frac{8}{3}S_2 + \frac{16}{3}S_{-2} \quad \frac{8}{3}S_2 + \frac{16}{3}S_{-2} \quad \frac{8}{3}S_2 + \frac{16}{3}S_{-2} \\
+ \frac{256S_{-2,1}}{3N(N + 1)(N + 2)} + \frac{(N - 1)\left[\frac{64}{3}S_{-2}S_1 - 32\zeta_3\right]}{N(N + 1)} L_Q + \frac{16P_{12S_1}}{81N(N + 1)^3(N + 2)^3} \\
+ \frac{8P_{39}}{243(N - 1)N^5(N + 1)^5(N + 2)^5} + \frac{512}{9} \left(\frac{N^2 + N + 1}{N - 1}N^2\right)^2(N + 2) \\
+ \frac{8P_{36S_1}}{243(N - 1)N^4(N + 1)^4(N + 2)^4} + \frac{L^3_M}{9(N - 1)N^2(N + 1)^2(N + 2)^2} \\
- \frac{8}{9}\zeta_9 S_1 \quad - \frac{16P_{13S_2}}{81N(N + 1)^3(N + 2)^3} + \frac{64(5N^4 + 38N^3 + 59N^2 + 31N + 20)S_3}{81N(N + 1)^2(N + 2)^2} \\
- \frac{32(121N^3 + 293N^2 + 414N + 224)S_{-2}}{81N(N + 1)^2(N + 2)} + \frac{L^2_M}{3(N + 1)^3(N + 2)^3} \\
+ \frac{8S_{35}}{3} + \frac{8}{3}S_{-2} \quad + \frac{128(5N^2 + 8N + 10)S_{-3}}{27N(N + 1)(N + 2)} \\
+ \frac{(5N^4 + 20N^3 + 41N^2 + 49N + 20)\left[\frac{32S_1}{81S_1} - \frac{32}{27}S_2S_1 + \frac{128}{27}S_{2,1}\right]}{N(N + 1)^2(N + 2)^2} \\
+ \frac{L_M}{9(N - 1)N(N + 1)^2(N + 2)^2} + \frac{16P_{27S_1}}{27(N - 1)N^3(N + 1)^3(N + 2)^3} \\
+ \frac{128(-1)^N\left(\frac{N^3 + 4N^2 + 7N + 5}{3(N + 1)^3(N + 2)^3}\right)S_1}{16P_{34}} - \frac{64(-1)^N\gamma_{14}}{9(N - 1)N^2(N + 1)^3(N + 2)^4} \\
+ \frac{16P_{34}}{27(N - 1)N^4(N + 1)^4(N + 2)^4} - \frac{32(2N^5 + 21N^4 + 51N^3 + 23N^2 - 11N - 14)S_2}{9(N - 1)N^2(N + 1)^2(N + 2)^2} \\
+ \frac{64S_3}{3(N + 2)} - \frac{64(2N^5 + 21N^4 + 36N^3 - 7N^2 - 68N - 56)S_{-2}}{9(N - 1)N(N + 1)^2(N + 2)^2} \\
+ \frac{(N - 1)\left(\frac{64}{3}S_{-2}S_1 - 32\zeta_3\right)}{N(N + 1)} + \zeta_9 \left[\frac{1}{27}S_4^2 + \frac{2}{9}S_2S_1 + \left[\frac{16}{9}S_{2,1} - \frac{40S_3}{27}\right]S_1\right] \\
+ \frac{64}{9}\zeta_3S_1 + \frac{1}{9}S_2 + \frac{14S_4}{9} + \frac{32}{9}S_{-4} + \frac{32}{9}S_{3,1} + \frac{16}{9}S_{2,1,1}\right] \\
+ C_F N_F T_F^2 \left[\frac{16(N^2 + N + 1)(N^2 + N + 2)(3N^4 + 6N^3 - N^2 - 4N + 12)}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \zeta_9 S_1\right] L^3_Q \\
+ \left[-\frac{4P_{31}}{9(9N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{16P_{21S_1}}{9(9N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{20S_2}{3} - 4S^2_{1,1}\right] \\
+ \frac{16P_{22S_2}}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{16P_{22S_2}}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{16P_{21S_1}}{9(9N - 1)N^3(N + 1)^3(N + 2)^2} \\
- \frac{64(-1)^N P_{37}}{16} \left[\frac{8(N^2 + N + 2)P_{10}}{3(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{8}{3}\zeta_9 S_1\right] L^3_Q + \frac{16L^3_M(N^2 + N + 2)^3}{(N - 1)N^3(N + 1)^3(N + 2)^2} \\
- \frac{45(N - 2)(N - 1)^2N^3(N + 1)^4(N + 2)^4(N + 3)^3}{4P_{42}} + \frac{64(-1)^N P_{37}}{8P_{30S_1}} \\
+ \frac{45(N - 1)^2N^3(N + 1)^3(N + 2)^4(N + 3)^3 - 9(N - 1)N^4(N + 1)^3(N + 2)^3}{4P_{42}} \\
\end{align*}\]
\[ + \frac{16P_{23}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} + L_M \left[ -\frac{16P_{28}}{3(N-1)N^4(N+1)^4(N+2)^3} \right] \]

\[ + \frac{16P_{17}S_1}{3(N-1)N^3(N+1)^3(N+2)^2} + \gamma_{qq}^0 \left( \frac{16S_2}{3} - \frac{16}{3} S_1^2 \right) - \frac{256(N^2 + N + 1)S_3}{3N(N+1)(N+2)} \]

\[ + \frac{64P_{16}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)} + \gamma_{qq}^0 \left[ \frac{8}{3} S_1^3 - 8S_2S_1 - \frac{32}{3} S_2 \right] \]

\[ + \frac{\frac{512}{3} S_1 S_{-2} + \frac{256}{3} S_{-3} - \frac{512}{3} S_{-2,1}}{N(N+1)(N+2)} + \frac{64(N-1)\zeta_3}{N(N+1)} \]

\[ + \frac{8 \left( 215N^4 + 481N^3 + 930N^2 + 748N + 120 \right) S_2^2}{243(N-1)N^6(N+1)^6(N+2)^2} + \frac{\frac{4P_{35}S_1}{243(N-1)N^5(N+1)^5(N+2)^2} + L_M^3 \left[ \frac{8(N^2 + N + 2) P_{10}}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{9} \gamma_{qq}^0 S_1 \right] } {N^2(N+1)N} \]

\[ + \frac{8 \left( 109N^4 + 291N^3 + 478N^2 + 324N + 40 \right) S_2}{27N^2(N+1)^2(N+2)} + \frac{\frac{16P_{20}S_1}{9(N-1)N^4(N+1)^4(N+2)^4} + \gamma_{qq}^0 \left[ \frac{4}{3} S_1^2 - \frac{4S_2}{3} \right] } {N^4(N+1)(N+2)} + \frac{\frac{8(N^2 + N + 2) P_{10}}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{9} \gamma_{qq}^0 S_1 } {N^2(N+1)N} \]

\[ + \frac{8 \left( 5N^3 - 16N^2 + 9 \right) S_3^5}{81N^2(N+1)(N+2)} + \gamma_{qq}^0 \left[ \frac{2}{27} S_1^4 - \frac{2}{9} S_2 S_1^3 - \frac{8}{27} S_3 S_1^2 - \frac{64}{9} \zeta_3 S_1 - \frac{1}{9} S_2^2 + \frac{14S_4}{9} \right] \]

\[ + \frac{8 \left( 109N^4 + 291N^3 + 478N^2 + 324N + 40 \right) S_2}{27N^2(N+1)^2(N+2)} + \frac{\frac{16P_{20}S_1}{9(N-1)N^4(N+1)^4(N+2)^4} + \gamma_{qq}^0 \left[ \frac{4}{3} S_1^2 - \frac{4S_2}{3} \right] } {N^4(N+1)(N+2)} + \frac{\frac{8(N^2 + N + 2) P_{10}}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{9} \gamma_{qq}^0 S_1 } {N^2(N+1)N} \]

\[ + L_M^3 \left[ \frac{8P_{19}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{27(N-1)N^4(N+1)^4(N+2)^3}{45(N-2)(N-1)^2 N^3(N+1)^3(N+2)^2 + \frac{4P_{41}}{45(N-2)(N-1)^2 N^3(N+1)^3(N+2)^2} + \frac{8(N^2 + N + 2) P_{11}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{64P_{15}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)} + \frac{325(N-1)^2 N^5(N+1)^5(N+2)^4(N+3)^3}{N^2(N+1)(N+2)} \right] \]

\[ + \frac{\gamma_{qq}^0 \left[ \frac{8}{3} S_1^3 - \frac{8}{3} S_2 S_1^2 - \frac{16}{3} S_{2,1} \right] + \frac{512}{3} S_1 S_{-2} + \frac{256}{3} S_{-3} - \frac{512}{3} S_{-2,1}}{N(N+1)(N+2)} + \frac{(N-1)(64\zeta_3 - \frac{64S_4}{3})}{N(N+1)} \]
\[ P_{22} = 38N^8 + 146N^7 + 177N^6 + 35N^5 - 249N^4 - 373N^3 - 218N^2 - 60N - 72 \]  
\[ P_{23} = 56N^8 + 194N^7 + 213N^6 + 83N^5 - 231N^4 - 469N^3 - 290N^2 - 60N - 72 \]  
\[ P_{24} = 113N^8 + 348N^7 + 109N^6 - 289N^5 - 272N^4 - 859N^3 - 778N^2 - 172N + 72 \]  
\[ P_{25} = 9N^9 + 54N^8 + 56N^7 - 110N^6 - 381N^5 - 568N^4 - 364N^3 - 72N^2 + 128N + 96 \]  
\[ P_{26} = 9N^9 + 54N^8 + 167N^7 + 397N^6 + 780N^5 + 1241N^4 + 1448N^3 + 1200N^2 + 608N + 144 \]  
\[ P_{27} = 55N^9 + 336N^8 + 218N^7 - 2180N^6 - 6529N^5 - 9764N^4 - 9368N^3 - 6032N^2 - 2448N - 576 \]  
\[ P_{28} = N^{11} - 56N^9 - 236N^8 - 373N^7 + 82N^6 + 1244N^5 + 2330N^4 + 2560N^3 + 1712N^2 + 896N + 288 \]  
\[ P_{29} = 33N^{11} + 231N^{10} + 662N^9 + 1254N^8 + 1801N^7 + 2759N^6 + 5440N^5 + 9884N^4 + 12512N^3 + 9200N^2 + 5184N + 1728 \]  
\[ P_{30} = 45N^{11} + 383N^{10} + 958N^9 + 526N^8 - 763N^7 + 1375N^6 + 7808N^5 + 13028N^4 + 12976N^3 + 8016N^2 + 4608N + 1728 \]  
\[ P_{31} = 81N^{11} + 483N^{10} + 1142N^9 + 1086N^8 - 767N^7 + 4645N^6 - 8936N^5 - 11980N^4 - 12352N^3 - 8272N^2 - 4800N - 1728 \]  
\[ P_{32} = 120N^{11} + 1017N^{10} + 2737N^9 + 1292N^8 - 8086N^7 - 20743N^6 - 24563N^5 - 16702N^4 - 6840N^3 + 120N^2 + 2432N + 960 \]  
\[ P_{33} = 121N^{11} + 988N^{10} + 3554N^9 + 6972N^8 + 7131N^7 - 846N^6 - 14806N^5 - 25354N^4 - 26096N^3 - 16752N^2 - 8352N - 2592 \]  
\[ P_{34} = 27N^{12} + 441N^{11} + 2206N^{10} + 5360N^9 + 7445N^8 + 8555N^7 + 18766N^6 + 44852N^5 + 67572N^4 + 63960N^3 + 39632N^2 + 15648N + 2880 \]  
\[ P_{35} = 2447N^{12} + 16902N^{11} + 59649N^{10} + 125860N^9 + 128761N^8 - 36530N^7 - 248341N^6 - 304460N^5 - 162188N^4 - 11724N^3 + 29160N^2 + 19440N + 7766 \]  
\[ P_{36} = 3361N^{12} + 23769N^{11} + 62338N^{10} + 59992N^9 - 63303N^8 - 317823N^7 - 585520N^6 - 640602N^5 - 430132N^4 - 167536N^3 - 27648N^2 + 9504N + 5184 \]  
\[ P_{37} = 76N^{14} + 802N^{13} + 2979N^{12} + 1847N^{11} - 19377N^{10} - 58253N^9 + 26543N^8 + 170601N^7 + 362177N^6 + 225119N^5 - 103240N^4 - 193092N^3 - 137160N^2 - 11702N - 25920 \]  
\[ P_{38} = 76N^{14} + 1042N^{13} + 5979N^{12} + 16367N^{11} + 11883N^{10} + 47693N^9 + 125723N^8 - 86079N^7 + 36437N^6 + 22559N^5 + 51700N^4 + 24828N^3 + 132840N^2 + 116208N + 25920 \]  
\[ P_{39} = 3180N^{15} + 38835N^{14} + 188728N^{13} + 456665N^{12} + 460954N^{11} - 406761N^{10} - 1972948N^9 - 2827653N^8 - 1857970N^7 + 109786N^6 + 1302824N^5 + 1092456N^4 + 265888N^3 - 227616N^2 - 194688N - 44928 \]  
\[ P_{40} = 28503N^{17} + 2973639N^{16} + 1232041N^{15} + 2461407N^{14} + 2169615N^{13} + 662941N^{12} + 2110979N^{11} + 5346653N^{10} + 2021366N^9 - 7290864N^8 - 11721384N^7 - 3689680N^6 + 15676192N^5 + 32276800N^4 + 31869312N^3 + 18809856N^2 + 6856704N + 124160 \]  
\[ P_{41} = 75N^{18} + 3330N^{17} + 35497N^{16} + 175010N^{15} + 486862N^{14} + 966996N^{13} + 2037362N^{12} + 3604404N^{11} + 1625689N^{10} - 29506022N^9 + 78753403N^8 - 107977014N^7 + 71548880N^6 + 18344016N^5 + 89016048N^4 + 92657952N^3 + 58942080N^2 + 25505280N + 5598720 \]  
\[ P_{42} = 325N^{18} + 4280N^{17} + 17759N^{16} - 14880N^{15} - 412326N^{14} - 1696848N^{13} - 3216546N^{12} - 1169232N^{11} + 8956857N^{10} + 23914216N^9 + 31536899N^8 + 25361392N^7 + 9982840N^6 - 10154128N^5 - 26098704N^4 - 26761536N^3 - 17642880N^2 - 8087040N - 1866240 \]
In all the representations of the massive Wilson coefficients and OMEs in \( N \)-space we apply algebraic reduction \[59\]. The 2–loop term in \([47]\) is purely multiplicative and induced by renormalization only, while the 3–loop contributions require the calculation of massive OMEs. The above Wilson coefficients depend on the harmonic sums

\[ S_1, S_{-2}, S_2, S_{-3}, S_3, S_{-4}, S_4, S_{-2,1}, S_{2,1}, S_{3,1}, S_{2,1,1}, \]

apart of those defining the massless 3–loop Wilson coefficients \([11]\). The harmonic sums are defined recursively by, cf. \([36,37]\),

\[ S_{h,d}(N) = \sum_{k=1}^{N} \frac{\text{sign}(b)^k}{k^{|b|}} S_d(k), \quad b, a_i \in \mathbb{Z}\setminus\{0\}, N \in \mathbb{N}, N \geq 1, S_0 = 1. \]

In the above \( \zeta_l = \sum_{k=1}^{\infty} 1/k^l, l \in \mathbb{N}, l \geq 2 \) denote the Riemann \( \zeta \)-values, which are convergent harmonic sums in the limit \( N \to \infty \). In the constant part of the other Wilson coefficients it is expected that more complicated multiple zeta values emerge, which have been dealt with in \([61]\).

In Eq. \([47]\) denominator terms \( \propto 1/(N-2) \) occur. They cancel in the complete expression and the rightmost singularity is located at \( N = 1 \) as expected for this Wilson coefficient. Let us now consider both the small- and large-\( x \) dominant terms for both Wilson coefficients. Those of the massless parts were given in \([11]\) before. Both Wilson coefficients contain terms \( \propto 1/(N-1) \).

For simplicity we consider the choice of scale \( Q^2 = \mu^2 \) here. The expansion of the heavy flavor contribution, subtracting the massless 3–loop Wilson coefficients, denoted by \( \bar{L}_i \), around \( N = 1(x \to 0) \) and in the limit \( N \to \infty(x \to 1) \), setting \( Q^2 = \mu^2 \), are given by

\[ \begin{align*}
\hat{L}_{q,2}^{PS}(N \to 1) & \propto \frac{1}{N-1} C_F T_F^2 N_F \left\{ \frac{1024}{27} \zeta_3 - \frac{64}{729} \left( 54 \zeta_3^2 - 81 \zeta_3 \right) + 342 \zeta_3 + 500 \right\} \\
\hat{L}_{g,2}^{PS}(N \to 1) & \propto \frac{1}{N-1} C_F T_F^2 N_F \left\{ \frac{512}{27} \zeta_3 - \frac{16}{729} \left( 108 \zeta_3^2 + 54 \zeta_3 \right) + 509 \zeta_3 + 3091 \right\} \\
\hat{L}_{q,2}^{PS}(N \to \infty) & \propto -\frac{64}{27} C_F T_F^2 N_F \ln^3(\bar{N}) \\
\hat{L}_{g,2}^{PS}(N \to \infty) & \propto -\frac{4}{27} \left( \frac{N-2}{N^2} \right) \ln^4(\bar{N}) (C_A - C_F) T_F^2 N_F .
\end{align*} \]

The corresponding limits for the contributions of the massless Wilson coefficients behave like

\[ \begin{align*}
N_F \hat{C}_{2,q}^{PS,(3)}(N_F)(N \to 1) & \propto \frac{4}{N-1} C_F T_F^2 N_F \left\{ \frac{22112}{729} - \frac{32}{9} \zeta_2 + \frac{128}{27} \zeta_3 \right\} \\
N_F \hat{C}_{2,q}^{PS,(3)}(N_F)(N \to 1) & \propto \frac{4}{N-1} C_F T_F^2 N_F \left\{ -\frac{572}{729} + \frac{160}{27} \zeta_2 + \frac{64}{27} \zeta_3 \right\} \\
N_F \hat{C}_{2,q}^{PS,(3)}(N_F)(N \to \infty) & \propto \frac{\ln^3(\bar{N})}{N^3} C_F T_F^2 N_F \frac{64}{27} .
\end{align*} \]

\[10\] For the algebraically reduced representations see \([60]\).
\[ N_F \hat{\hat{S}}^{(3)}_{2,g} (N_F) (N \to \infty) \propto \ln^4(\bar{N}) \left[ \frac{68}{27} C_F T_F^2 N_F + \frac{28}{27} C_A T_F^2 N_F \right] , \tag{90} \]

where \( \bar{N} = N \exp(\gamma_E) \) and \( \gamma_E \) denotes the Euler-Mascheroni constant.

Figure 1: The \( O(a_s^2) \) contribution by \( L_{g,2}^S \) to the structure function \( F_2(x, Q^2) \).

Figure 2: The \( O(a_s^3) \) contribution by \( L_{g,2}^S \) to the structure function \( F_2(x, Q^2) \).

While the expression for \( L_{g,2}^{PS} \) is the same in the \( \overline{\text{MS}} \)- and on-mass-shell scheme to \( O(a_s^3) \), \( L_{g,2}^S \), in its 3-loop contribution, changes by the term

\[ L_{g,2}^{S,(3),\overline{\text{MS}}} (N) = L_{g,2}^{S,(3),\text{OMS}} (N) + a_s^3 \frac{32}{3} C_F T_F^2 N_F [3L_M - 4] \]
setting $Q^2 = \mu^2$. Here, we have identified the logarithms $L_M$ in both schemes symbolically. In applications, either the on-shell or the $\overline{\text{MS}}$ mass has to be used here. The corresponding expression in $z$–space reads

$$L_{g,2}^{S,(3),\overline{\text{MS}}} (z) = L_{g,2}^{S,(3),\text{OMS}} (z) + \frac{a_s^3}{3} \frac{32}{3} C_F T_F N_F [3L_M - 4]$$

$$\times \left[ (2z^2 - 2z + 1) [H_0(z) + H_1(z)] + 8z^2 - 8z + 1 \right],$$

with $H_a(z)$ harmonic polylogarithms, see Eq. (592).

Figure 3: The $O(a_s^3)$ contribution by $L_{q,2}^{PS}$ to the structure function $F_2(x, Q^2)$.

| $x$   | $Q^2 = 20\text{GeV}^2$ | $Q^2 = 100\text{GeV}^2$ | $Q^2 = 1000\text{GeV}^2$ |
|-------|------------------------|--------------------------|---------------------------|
| $10^{-4}$ | 1.946                  | 3.200                    | 5.340                     |
| $10^{-3}$ | 1.141                  | 1.702                    | 2.526                     |
| $10^{-2}$ | 0.641                  | 0.825                    | 1.040                     |
| $10^{-1}$ | 0.400                  | 0.409                    | 0.412                     |

Table 1: Values of the structure function $F_2(x, Q^2)$ in the low $x$ region using the PDF-parameterization [5].
In Figure 1 we illustrate the contribution of $L_{g,2}^{PS}$ to the structure function $F_2(x, Q^2)$ using the PDFs of Ref. [5], cf. Eq. (11). Likewise Figures 2 and 3 show the corresponding contributions by $L_{g,2}^{S}$ at $O(a_s^2)$ and $O(a_s^3)$, respectively. Note that the $O(a_s^2)$-terms, cf. also Ref. [19] are smaller than those at $O(a_s^3)$, which is caused by terms $\propto 1/z$ in the 3-loop contribution to $L_{g,2}^{S}$, which are absent at 2-loop order.

These contributions emerging on the 2- and 3-loop level are minor compared to the values of the structure function $F_2(x, Q^2)$, for which typical values are given in Table 1. A global comparison of all heavy flavor contributions up to 3-loop order can presently only be performed using the known number of Mellin moments, cf. [12], given in Section 6.

4 The Logarithmic Contributions to $H_{g,2}^{PS}$ and $H_{g,2}^{S}$ to $O(a_s^3)$

The pure-singlet Wilson coefficient $H_{g,2}^{PS}$, except for the constant part $a_{Qg}^{PS,(3)}$ of the unrenormalized operator matrix element in the on-shell scheme can be expressed by harmonic sums and rational functions in $N$ only. As before we reduce to a basis eliminating the algebraic relations [59]. It is given by:

$$H_{g,2}^{PS} = \frac{1}{2}[1 + (-1)^N] \times \left\{ a^2_s \left\{ C_F T_F \left[ -\frac{4L_M^2(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} + \frac{(4S_1^2 - 12S_2)(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \right.$$  
$$+ \frac{4L_M^2(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left. \right. - \frac{32(-1)^NP_{45}}{3(N-1)N^2(N+1)^3(N+2)^3} \right. \right.$$  
$$+ \frac{8P_{75}S_1}{8P_{57}} \left. \right. \right.$$  
$$+ \left. \left. \frac{3(N-1)N^4(N+1)^4(N+2)^3}{(N-1)N^3(N+1)^3(N+2)^3} + \frac{(N-1)N^3(N+1)^3(N+2)^2}{(N-1)N^3(N+1)^3(N+2)^3} \right. \right.$$  
$$+ \left. \left. \frac{8S_1(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{8P_{57}}{3(N-1)N^4(N+1)^4(N+2)^2} \right. \right.$$  
$$+ \frac{64S_{-2}}{(N-1)N^2(N+1)(N+2)} - \frac{8(N^2 + 5N + 2)(5N^3 + 7N^2 + 4N + 4)L_M}{(N-1)N^4(N+1)^4(N+2)^2} \right\} \right\} \right\} \right\}$$

$$+ a^3_s \left\{ C_F^2 T_F \left[ L_0^3 \left[ \frac{8(N^2 + N + 2)^2(3N^2 + 3N + 2)}{3(N-1)N^3(N+1)^3(N+2)} - \frac{32(N^2 + N + 2)^2S_1}{3(N-1)N^2(N+1)^2(N+2)} \right. \right. \right.$$  
$$+ \left. \left. L_2^2 \left[ \frac{(24S_1^2 - 24S_2)(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{4P_{73}}{(N-1)N^4(N+1)^4(N+2)^2} \right. \right. \right.$$  
$$+ \left. \left. \frac{8P_{62}S_1}{(N-1)N^3(N+1)^3(N+2)^2} \right. \right.$$  
$$+ \left. \left. \frac{(104S_1S_2 - \frac{50}{3}S_3^3)(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right.$$  
$$- \frac{16(N^2 + N - 22)S_3(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)} + \frac{(128S_{-2} - 256S_{-2,1} - 384\zeta_3)(N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)} \right. \right.$$  
$$- \frac{4P_{07}S_1^2}{(N-1)N^3(N+1)^3(N+2)^2} - \frac{15(N-2)(N-1)N^3(N+1)^3(N+2)^4(N+3)^3}{128(-1)^NP_{98}} \right.$$  
$$+ \frac{4P_{100}}{15(N-1)^3N^5(N+1)^5(N+2)^4(N+3)^3 + 3(N-1)N^2(N+1)^3(N+2)^3} \right\} \right\} \right\}$$

Note that the kinematic region at small $x$ probed at HERA is limited to values $x \geq Q^2/(yS)$, with $S \simeq 10^5$GeV$^2$ and $y \in [0,1]$. 

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\[\begin{align*}
&- 8 P_{79} S_1 + \frac{32 N^2 + N + 2}{3(N - 1) N^2 (N + 1)^2 (N + 2)} S_1 + \frac{512 S_{-2} S_1}{(N - 1) N^2 (N + 1)^2 (N + 2)} + \frac{4 P_{65} S_2}{(N - 1) N^3 (N + 1)^3 (N + 2)^2} \\
&+ L_M^2 \left[ \frac{16 (N^2 + N + 2)^2}{(N - 1) N^2 (N + 1)^2 (N + 2)} - \frac{4 (N^2 + N + 2)^2}{(N - 1) N^3 (N + 1)^3 (N + 2)} \right] \\
&+ L_M \left[ \frac{32 (N^2 + 5 N + 2) (5 N^3 + 7 N^2 + 4 N + 4)}{(N - 1) N^3 (N + 1)^3 (N + 2)^2} \right] \\
&- \frac{8 (N^2 + 5 N + 2) (3 N^2 + 3 N + 2) (5 N^3 + 7 N^2 + 4 N + 4)}{(N - 1) N^4 (N + 1)^4 (N + 2)^2} \\
&+ \frac{32 P_{69} S_{-2}}{(N - 2)(N - 1) N^3 (N + 1)^3 (N + 2)/(N + 3)} - \frac{8 (N^2 + N + 2) S_1^3}{3(N - 1) N^3 (N + 1)^3 (N + 2)} \\
&+ \frac{4 P_{74} S_2^2}{(N - 1) N^4 (N + 1)^4 (N + 2)^2} - \frac{2 (N^2 + N + 2) \zeta_2}{(N - 1) N^4 (N + 1)^4 (N + 2)} P_{85} - \frac{4 P_{94}}{(N - 1) N^5 (N + 1)^6 (N + 2)^3} \\
&- \frac{4}{3} \left( \frac{N^2 + N + 2}{(N - 1) N^4 (N + 1)^3 (N + 2)} \right)^2 + \frac{4}{3} \left( \frac{N^2 + N + 2}{(N - 1) N^4 (N + 1)^3 (N + 2)} \right) S_1 \\
&+ \frac{4 P_{91} S_3}{(N - 1) N^4 (N + 1)^3 (N + 2)^3} + L_M^3 \left[ \frac{4 (N^2 + N + 2)^2}{3(N - 1) N^3 (N + 1)^3 (N + 2)} \right] \\
&- \frac{8 (N^2 + N + 2) (3 N^4 + 9 N^3 + 15 N^2 + 11 N - 2)}{(N - 1) N^3 (N + 1)^3 (N + 2)} S_1 S_2 \\
&+ \frac{4 P_{76} S_2}{(N - 1) N^4 (N + 1)^4 (N + 2)^2} + L_M^2 \left[ - \frac{4 (13 N^2 + 5 N - 6) S_1}{(N - 1) N^3 (N + 1)^3 (N + 2)} \right] \\
&+ \frac{4 P_{71} (N^2 + N + 2)}{(N - 1) N^4 (N + 1)^4 (N + 2)} \\
&\frac{8 (N^2 + N + 2) (3 N^4 + 48 N^3 + 43 N^2 - 22 N - 8)}{(N - 1) N^3 (N + 1)^2 (N + 2)} S_3 + \frac{32 (N^2 - 3 N - 2) (N^2 + N + 2) S_{2,1}}{(N - 1) N^3 (N + 1)^2 (N + 2)} \\
&+ \frac{(N^2 + N + 2)^2}{(N - 1) N^2 (N + 1)^2 (N + 2)} \left[ \frac{2}{3} S_1 - 12 S_2 S_2^2 + \left( \frac{16 S_3}{3} + 32 S_{2,1} \right) S_1 + \frac{16}{3} \zeta_3 S_1 + 18 S_2 - 12 S_4 \right] \\
&+ 32 S_{3,1} - 64 S_{2,1,1} + (4 S_1^2 - 12 S_2) \zeta_2 + L_M \left[ \frac{(N^2 + N + 2)^2}{(N - 1) N^2 (N + 1)^2 (N + 2)} \left[ \frac{8}{3} S_3^3 - 24 S_2 S_1 \right] \right] \\
&- \frac{80 S_3}{3} + 32 S_{2,1} + 96 \zeta_3 \\
&+ \frac{8 P_{77} S_1}{(N - 1) N^4 (N + 1)^3 (N + 2)^2} - \frac{4 P_{69} S_1^2}{(N - 1) N^3 (N + 1)^3 (N + 2)^2} \\
&+ C_F T^2 \left[ \frac{32 (N^2 + N + 2)^2}{9 (N - 1) N^2 (N + 1)^2 (N + 2)} L_Q^3 - \frac{32 P_{69} L_Q}{9 (N - 1) N^3 (N + 1)^3 (N + 2)^2} \right] \\
&+ L_Q \left[ \frac{32 (N^2 + N + 2)^2 L_M^3}{3(N - 1) N^2 (N + 1)^2 (N + 2)} + \frac{64 (N^2 + N + 2) (8 N^3 + 13 N^2 + 27 N + 16)}{9 (N - 1) N^2 (N + 1)^3 (N + 2)} \right] \\
&- \frac{64 (N^2 + N + 2)^2 S_1}{3(N - 1) N^2 (N + 1)^2 (N + 2)} L_M - \frac{256 (-1)^N P_{45}}{9 (N - 1) N^2 (N + 1)^3 (N + 2)^3} \\
\end{align*}\]
\[ + \frac{64P_{52}}{27(N - 1)N^4(N + 1)^3(N + 2)^3} + \frac{512S_{-2}}{3(N - 1)N(N + 1)(N + 2)} \]

\[ - \frac{128(N^2 + N + 2)^2 L_M^3}{9(N - 1)N^2(N + 1)^2(N + 2)} + \frac{16(N^2 + N + 2)(7N^4 + 16N^3 + 32N^2 + 19N + 2) S_{-1}^2}{3(N - 1)N^3(N + 1)^3(N + 2)} \]

\[ - \frac{32(11N^5 + 26N^4 + 57N^3 + 142N^2 + 84N + 88)L_M^2}{9(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{32C_2 P_{58}}{9(N - 1)N^3(N + 1)^2(N + 2)^2} \]

\[ + \frac{91(N - 1)N^5(N + 1)^5(N + 2)^4}{32P_{58} S_1} - \frac{32P_{58} S_1}{27(N - 1)N^3(N + 1)^4(N + 2)} \]

\[ + L_M \left[ \frac{32S_1^2 - 32S_2}{9(N - 1)N^2(N + 1)^2(N + 2)} \right] - \frac{64P_{80}}{27(N - 1)N^4(N + 1)^4(N + 2)^3} \]

\[ + \frac{64P_{57} S_1}{3(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{16P_{61} S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \]

\[ + \frac{(N^2 + N + 2)^2}{(N - 1)N^2(N + 1)^2(N + 2)^2} \left[ - \frac{64S_1^3}{9} - \frac{32S_2 S_1}{3} - \frac{32C_2 S_1}{9} + \frac{160S_3}{9} + \frac{128S_3^2}{9} \right] \]

\[ + NFT_C F \left[ \frac{32(N^2 + N + 2)^2 L_Q^3}{9(N - 1)N^2(N + 1)^2(N + 2)} - \frac{32P_{66} L_Q^2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \right] \]

\[ + \left[ - \frac{16S_1^2}{3} - \frac{16S_2}{3} \right] \frac{(N^2 + N + 2)^2}{(N - 1)N^2(N + 1)^2(N + 2)} + \frac{32(8N^3 + 13N^2 + 27N + 16) S_1 (N^2 + N + 2)}{9(N - 1)N^2(N + 1)^3(N + 2)} \]

\[ - \frac{256(-1)^N P_{45}}{9(N - 1)N^2(N + 1)^2(N + 2)^3} + \frac{32P_{84}}{27(N - 1)N^4(N + 1)^4(N + 2)^3} \]

\[ + \frac{512S_{-2}}{3(N - 1)N(N + 1)(N + 2)} L_Q - \frac{32(N^2 + N + 2)^2 L_M^3}{9(N - 1)N^2(N + 1)^2(N + 2)} \]

\[ - \frac{16}{9} \zeta_2 \frac{(N - 1)N^3(N + 1)^3(N + 2)^2 P_{63}}{3(N - 1)N^3(N + 1)^4(N + 2)^4} - \frac{32P_{87}}{32P_{84}} \]

\[ + L_M^2 \left[ \frac{32P_{64}}{9(N - 1)N^3(N + 1)^3(N + 2)^3} - \frac{32(N^2 + N + 2)^2 S_1}{3(N - 1)N^2(N + 1)^2(N + 2)} \right] \]

\[ + L_M \left[ \frac{- \frac{16S_1^2}{3} - \frac{80S_2}{3}}{(N - 1)N^2(N + 1)^2(N + 2)} (N^2 + N + 2)^2 - \frac{32P_{78}}{27(N - 1)N^4(N + 1)^4(N + 2)^3} \right] \]

\[ + \frac{32P_{68} S_1}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{64(N^2 + 5N + 2)(5N^3 + 7N^2 + 4N + 4) S_2}{3(N - 1)N^3(N + 1)^3(N + 2)^2} \]

\[ + \frac{(N^2 + N + 2)^2}{(N - 1)N^2(N + 1)^2(N + 2)^2} \left[ \frac{64S_3}{3} + \frac{16S_3 S_2}{9} + \frac{32S_3}{9} \right] \]

\[ + C_A C_F T_F \left[ L_Q^3 \left[ - \frac{16S_1(N^2 + N + 2)^2}{3(N - 1)N^2(N + 1)^2(N + 2)} \right] \right. \]

\[ - \frac{8(11N^4 + 22N^3 - 23N^2 - 34N - 12)(N^2 + N + 2)^2}{9(N - 1)N^2N^3(N + 1)^3(N + 2)^2} \]

\[ + L_Q^2 \left[ - \frac{16(5N^2 - 1)S_1(N^2 + N + 2)^2}{(N - 1)N^2N^3(N + 1)^3(N + 2)^2} + \frac{(16S_2^2 - 16S_2 - 32S_{-2})(N^2 + N + 2)^2}{(N - 1)N^2(N + 1)^2(N + 2)} \right] \]
\[+40S_2S_1 + 32(1 + (-1)^N)S_{-2}S_1 + 16(-1)^NS_{-3} - 32S_{2,1} + 12(-9 + (-1)^N)\xi_3\]
\[+ \frac{4(17N^4 - 6N^3 + 41N^2 - 16N - 12)S_2^2(N^2 + N + 2)}{3(N - 1)^2N^3(N + 1)(N + 2)} + \frac{4P_{56}S_2(N^2 + N + 2)}{3(N - 1)^2N^3(N + 1)^3(N + 2)^2} \]
\[+ \frac{8(31N^2 + 31N + 74)S_3(N^2 + N + 2)}{3(N - 1)N^2(N + 1)^2(N + 2)} + \frac{16(7N^2 + 7N + 10)S_{-3}(N^2 + N + 2)}{(N - 1)N^2(N + 1)^2(N + 2)} \]
\[- \frac{128(N^2 + N + 1)S_{-2,1}(N^2 + N + 2)}{(N - 1)N^2(N + 1)^2(N + 2)} + \frac{(N^2 - N - 4)(N^2 + N + 2)32(-1)^NS_{-2}}{(N - 1)N(N + 1)^3(N + 2)^2} \]
\[- \frac{64(-1)^NP_{81}}{9(N - 1)N^3(N + 1)^5(N + 2)^4} + \frac{27(N - 1)^2N^5(N + 1)^5(N + 2)^4}{8P_{90}S_1} + \frac{3(N - 1)N^2(N + 1)^3(N + 2)^3}{16P_{58}S_{-2}} \]
\[- \frac{16P_{58}S_{-2}}{(N - 1)N^3(N + 1)^3(N + 2)^2} \]}
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\[ P_{71} = 6N^8 - 42N^7 - 241N^6 - 579N^5 - 307N^4 + 477N^3 + 602N^2 + 492N + 168 \] (122)
\[ P_{72} = 10N^8 + 71N^7 + 244N^6 + 497N^5 + 698N^4 + 720N^3 + 512N^2 + 248N + 48 \] (123)
\[ P_{73} = 19N^9 + 86N^8 + 144N^7 - 38N^6 - 535N^5 - 1016N^4 - 1180N^3 - 872N^2 - 416N - 96 \] (124)
\[ P_{74} = N^{10} + 15N^9 + 105N^8 + 361N^7 + 660N^6 + 828N^5 + 814N^4 + 384N^3 - 112N^2 - 128N - 32 \] (125)
\[ P_{75} = 6N^{10} + 49N^9 + 197N^8 + 472N^7 + 833N^6 + 1469N^5 + 2142N^4 + 1904N^3 + 1040N^2 + 432N + 96 \] (126)
\[ P_{76} = 11N^{10} + 123N^9 + 541N^8 + 1273N^7 + 1806N^6 + 1672N^5 + 1006N^4 + 320N^3 - 16N^2 - 64N - 32 \] (127)
\[ P_{77} = 19N^{10} + 143N^9 + 412N^8 + 426N^7 - N^6 + 159N^5 + 1066N^4 + 1552N^3 + 1456N^2 + 848N + 224 \] (128)
\[ P_{78} = 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3 + 16N^2 + 672N + 288 \] (129)
\[ P_{79} = 60N^{10} + 397N^9 + 1073N^8 + 1111N^7 + 623N^6 + 4328N^5 + 12432N^4 + 15944N^3 + 12704N^2 + 6816N + 1728 \] (130)
\[ P_{80} = 67N^{10} + 383N^9 + 867N^8 + 696N^7 - 755N^6 - 2391N^5 - 3027N^4 - 2744N^3 - 1256N^2 - 48N + 144 \] (131)
\[ P_{81} = 77N^{10} + 646N^9 + 2553N^8 + 6903N^7 + 14498N^6 + 22898N^5 + 24861N^4 + 17068N^3 + 7040N^2 + 1760N + 192 \] (132)
\[ P_{82} = 85N^{10} + 530N^9 + 1458N^8 + 2112N^7 + 1744N^6 + 2016N^5 + 3399N^4 + 2968N^3 + 1864N^2 + 1248N + 432 \] (133)
\[ P_{83} = 118N^{10} + 675N^9 + 1588N^8 + 1652N^7 + 326N^6 + 357N^5 + 876N^4 + 1672N^3 + 3440N^2 + 2544N + 576 \] (134)
\[ P_{84} = 127N^{10} + 740N^9 + 1737N^8 + 1308N^7 - 1592N^6 - 2226N^5 + 1386N^4 + 3064N^3 + 3040N^2 - 2496N + 864 \] (135)
\[ P_{85} = 151N^{10} + 708N^9 + 1156N^8 + 464N^7 - 967N^6 + 372N^5 + 3672N^4 + 5236N^3 + 6152N^2 + 3792N + 864 \] (136)
\[ P_{86} = 3N^{11} + 66N^{10} + 104N^9 - 1152N^8 - 3801N^7 - 2510N^6 + 3318N^5 + 8076N^4 + 9608N^3 + 6512N^2 + 2432N + 384 \] (137)
\[ P_{87} = 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 + 473N^5 - 682N^4 - 904N^3 + 592N^2 - 208N + 32 \] (138)
\[ P_{88} = 118N^{11} + 529N^{10} + 1264N^9 + 3846N^8 + 11353N^7 + 23684N^6 + 32793N^5 + 31801N^4 + 22836N^3 + 10448N^2 + 2592N + 432 \] (139)
\[ P_{89} = 127N^{11} + 820N^{10} + 2197N^9 + 1890N^8 - 1847N^7 + 1906N^6 + 3843N^5 + 9730N^4 + 13632N^3 + 10688N^2 + 4944N + 864 \] (140)
\[ P_{90} = 136N^{11} + 1039N^{10} + 3100N^9 + 3534N^8 - 1295N^7 - 6352N^6 - 8421N^5 - 11729N^4 - 7644N^3 + 1376N^2 + 1920N + 144 \] (141)
\[ P_{91} = 7N^{12} + 47N^{11} + 123N^{10} + 76N^9 - 598N^8 - 2178N^7 - 3626N^6 - 3933N^5 - 3254N^4 - 1608N^3 - 144N^2 + 112N + 32 \] (142)
\[ P_{92} = 37N^{12} + 305N^{11} + 1017N^{10} + 1462N^9 + 592N^8 + 408N^7 + 4064N^6 + 9645N^5 + 12222N^4 + 10280N^3 + 6064N^2 + 2192N + 352 \] (143)
\[ P_{93} = 242N^{12} + 1853N^{11} + 6173N^{10} + 12711N^9 + 18608N^8 + 17040N^7 - 302N^6 - 24986N^5 - 32225N^4 - 20010N^3 - 7904N^2 - 2016N - 288 \]  
\[ (144) \]

\[ P_{94} = 5N^{13} + 27N^{12} - 97N^{11} - 1410N^{10} - 5754N^9 - 12428N^8 - 16530N^7 - 14531N^6 - 7956N^5 - 1038N^4 + 2176N^3 + 1632N^2 + 448N + 32 \]  
\[ (145) \]

\[ P_{95} = 119N^{13} + 1897N^{12} + 12595N^{11} + 48221N^{10} + 124877N^9 + 239946N^8 + 345670N^7 + 356234N^6 + 253043N^5 + 129982N^4 + 55768N^3 + 20112N^2 + 5616N + 864 \]  
\[ (146) \]

\[ P_{96} = 686N^{14} + 8408N^{13} + 39228N^{12} + 89257N^{11} + 113445N^{10} + 109336N^9 + 76360N^8 - 109649N^7 - 393915N^6 - 482272N^5 - 376932N^4 - 263440N^3 - 155472N^2 - 56448N - 8640 \]  
\[ (147) \]

\[ P_{97} = 1790N^{14} + 15938N^{13} + 56250N^{12} + 90805N^{11} + 43917N^{10} - 38450N^9 - 42314N^8 - 169217N^7 - 616623N^6 - 992860N^5 - 964980N^4 - 697072N^3 - 376464N^2 - 127872N - 19008 \]  
\[ (148) \]

\[ P_{98} = 30N^{16} + 397N^{15} + 1996N^{14} + 3786N^{13} - 3905N^{12} - 30084N^{11} - 44372N^{10} + 5100N^9 + 71344N^8 + 27709N^7 - 104744N^6 - 146534N^5 - 30293N^4 + 77346N^3 + 33768N^2 - 23544N - 3888 \]  
\[ (149) \]

\[ P_{99} = 12N^{17} + 162N^{16} + 1030N^{15} + 4188N^{14} + 11527N^{13} + 19051N^{12} + 11176N^{11} - 17182N^{10} - 36527N^9 - 27469N^8 - 11770N^7 + 5554N^6 + 32640N^5 + 46456N^4 + 34528N^3 + 14816N^2 + 3584N + 384 \]  
\[ (150) \]

\[ P_{100} = 1245N^{18} + 19980N^{17} + 133282N^{16} + 461805N^{15} + 787161N^{14} + 185392N^{13} - 1368400N^{12} - 225082N^{11} + 6978631N^{10} + 13143336N^9 + 5808466N^8 - 11433627N^7 - 19928573N^6 - 12013164N^5 + 1462668N^4 + 8209584N^3 + 6906384N^2 + 2980800N + 544320 \]  
\[ (151) \]

The Wilson coefficient \( H_{g,2}^S \), except for the constant contribution \( a_{gq}^{(3)} \), has a similar structure. It is given by:

\[ H_{g,2}^S = \frac{1}{2}[1 + (-1)^N] \times \left\{ a_{TF} \left\{ -\delta_{qg} L_Q - \frac{4(N^3 - 4N^2 - N - 2)}{N^2(N + 1)(N + 2)} + \delta_{qg} S_1 + \delta_{qg} L_M \right\} \right. \\
\left. + \frac{a_2^2}{2} \left[ \frac{4\delta_{qg} L_Q^2}{3} - \frac{4\delta_{qg} L_Q L_M}{3} + \frac{4\delta_{qg} S_1}{3} - \frac{16(N^3 - 4N^2 - N - 2)}{N^2(N + 1)(N + 2)} \right] L_M \right\} \\
+ C_{TF} \left[ \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^2(N + 2)} + 2\delta_{qg} S_1 \right] L_Q - \frac{4P_{111}}{N^3(N + 1)^3(N + 2)} - \frac{4(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^2(N + 2)} \\
+ \frac{4(3N^4 + 2N^3 - 9N^2 - 16N - 12) S_1}{N^2(N + 1)^2(N + 2)} \right\} L_M - \frac{4(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^2(N + 2)} \\
+ \frac{16(-1)^N P_{252}}{15(N - 2)(N - 1)^2 N^2(N + 1)^4(N + 2)^4(N + 3)^3} + \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^2(N + 2)} \]
\[
\begin{align*}
&= \frac{16P_{233}}{27(N-1)N^3(N+1)^4(N+2)^4} + L^2_M \left[ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{3}qgS_1 \right] \\
&+ \frac{32(8N^4 + 13N^3 - 22N^2 - 9N - 26)S_2}{9(N-1)N(N+1)(N+2)^2} + \frac{64(8N^5 + 15N^4 + 6N^3 + 11N^2 + 16N + 16)S_{-2}}{9(N-1)N^3(N+1)^2(N+2)^2} \\
&+ \frac{128(N^2 + N - 1)S_3}{9(N+1)(N+2)} + L^2_M \left[ \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^4(N+2)^3} + \frac{32P_{195}}{9(N-1)N^3(N+1)^3(N+2)^3} \\
&- \frac{64(2N-1)(N^3 + 9N^2 + 7N + 7)S_1}{9(N-1)N(N+1)^2(N+2)} + \frac{\gamma_{qg}^0}{3} \left[ \frac{8 S_2^1 + 8 S_2}{3} + \frac{16}{3} S_{-2} \right] \right] - \frac{128(N^2 + N + 3)S_{-3}}{3(N+1)(N+2)} \\
&+ \frac{\gamma_{qg}^0}{9} \left[ \frac{8 S_1^1 - 8 S_2 S_1 + \frac{32}{3} S_2,1}{3(N+1)(N+2)} \right] + \frac{256S_{-1,2}}{3(N+1)(N+2)} + \frac{(N-1) \left[ \frac{64}{3} S_{-2} S_1 - \frac{32}{3} \zeta_3 \right]}{N(N+1)} \\
&+ \frac{256}{8P_{249}} \left[ \frac{9(N^3 + 8N^2 + 11N + 2)S_1^3}{9(NN+1)^4(N+2)^3} + \frac{8P_{152}S_1^2}{27N(N+1)^3(N+2)^3} + \frac{4}{9(N-1)N^3(N+1)^3(N+2)^3} \right] \\
&+ \frac{8P_{249}S_1}{3(N-1)N^3(N+1)^3(N+2)^3} - \frac{8P_{192}S_1}{3(N+1)(N+2)^2} - \frac{32(3N^3 - 12N^2 - 27N - 2)S_1 S_2}{3(N+1)^2(N+2)^2} \\
&+ \frac{256(N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6)S_3}{9(N-1)N^4(N+1)^2(N+2)^2} + \frac{L^2_M}{3(N+1)^2(N+2)^3} \left[ \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{(N+1)^3(N+2)^3} \right] \\
&- \frac{8P_{199}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{32(8N^5 + 9N^4 + 57N^3 - 31N^2 + 25N - 26)S_1}{9(N-1)N(N+1)^2(N+2)^2} \\
&+ \frac{\gamma_{qg}^0}{9} \left[ \frac{-20 S_2^1 - 4 S_2 - 8 S_{-2}}{3} \right] + \frac{128(-1)^N(N^4 + 2N^3 + 7N^2 + 22N + 20)S_{-2}}{3(N+1)^2(N+2)^3} \\
&+ \frac{(N^2 - N - 4)}{(N+1)^2(N+2)^2} \left[ \frac{-256}{3}(-1)^N S_1 S_{-2} - \frac{128}{3}(-1)^N S_{-3} - \frac{256}{3} S_{-2,1} - \frac{128}{3}(-1)^N S_1 \zeta_2 \right] \\
&- \frac{32(-1)^N \zeta_4}{3} + \frac{\gamma_{qg}^0}{9} \left[ \frac{2 S_1^1 + 20 S_2^1}{3} + \frac{32}{3}(-1)^Ns_{-3} S_1 + \frac{160S_3}{3} - \frac{64}{3} S_{-2,1} \right] S_1 + \frac{2}{3} S_2 \\
&+ \frac{8}{9} \left[ -2 + 9(-1)^N \zeta_4 S_1 + \frac{32}{3}(-1)^N S_2 \right] + \frac{32}{3}(-1)^N S_2 + 12 S_4 \\
&+ \frac{16}{9}(-1)^N S_{-4} - \frac{16}{3} S_{-3,1} - \frac{32}{3} S_{-2,1} - \frac{32}{3} S_{-3,1} - \frac{16}{3} S_{-2,1,1} + \frac{64}{3} S_{-2,1,1} + \frac{2}{3} \left[ -3 + 8(-1)^N \right] S_2^2 \\
&+ \frac{2}{3} \left[ -3 + 8(-1)^N \right] S_2 + \frac{4}{3} \left( 1 + 4(-1)^N \right) S_2 \zeta_2 \right] + L^2_M \left[ \frac{64(N^5 + 9N^4 + 3N^3 + N^2 + 26N - 4)S_2^2}{9(N-1)N(N+1)^2(N+2)^2} \right] \\
&+ \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{3(N+1)^3(N+2)^3} + \frac{16P_{198}S_1}{27(N-1)N^3(N+1)^2(N+2)^3} + \frac{64(N-1)S_{-2} S_1}{3N(N+1)} \\
&- \frac{9(N-1)N^2(N+1)^3(N+2)^4}{16P_{242}} + \frac{27(N-1)N^4(N+1)^4(N+2)^4}{9(N+1)(N+2)} + \frac{64(7N^2 + 7N + 8)S_3}{64P_{104} S_2} \\
&- \frac{9(N-1)N^2(N+1)^2(N+2)^2}{9(N-1)N(N+1)^2(N+2)^2} \\
&= 25,
\end{align*}
\]
\[
\begin{align*}
&- \frac{128S_{-3}}{3N(N+1)(N+2)} - \frac{128S_{-2,1}}{3(N+2)} + \frac{(N^2 - N - 4)\frac{128}{3}(-1)^N S_{-2}}{(N+1)^2(N+2)^2} - \frac{16(3N^2 + 3N - 2)\zeta_3}{N(N+1)(N+2)} \\
&+ \gamma_9 \left[ \frac{8}{9} S_1 - \frac{40}{3} S_2 S_1 - \frac{32}{3} (-1)^N S_{-2} S_1 - \frac{16}{3} (-1)^N S_{-3} - 4(-1)^N \zeta_3 \right] \\
&+ C_{T_F}^2 N_F \left[ \left( \frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8\gamma_9 S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \right) L_Q^2 + \left( -\frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right) \right] \\
&+ \frac{8\gamma_9}{9(N-1)N^3(N+1)^3(N+2)^3} \left[ \frac{32(N^4 - 7N^3 + 5N^2 - 17N - 13)S_1}{9(N-1)N(N+1)^2(N+2)^2} + \frac{\gamma_9}{-\frac{4}{3} S_1^2 + \frac{4S_2}{3}} \right] \\
&+ \frac{8}{3} S_{-2} \right] L_Q^2 + \left[ -\frac{32(N^4 - 7N^3 + 5N^2 - 17N - 13)S_1}{9(N-1)N(N+1)^2(N+2)^2} + \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{3(N+1)^3(N+2)^3} \right] \\
&- \frac{16P_{187} S_1}{27(N-1)N^2(N+1)^3(N+2)^3} + \frac{64(-1)^NP_{170}}{(N-1)N^2(N+1)^4(N+2)^4} + \frac{128(N^2 + N - 1)S_3}{9(N-1)N(N+1)(N+2)} \\
&+ \frac{8P_{248}}{27(N-1)N^4(N+1)^4(N+2)^4} + \frac{32(N^4 + 13N^3 - 22N^2 - 9N - 26)S_2}{9(N-1)N(N+1)^2(N+2)^2} \\
&+ \frac{64(N^5 + 15N^4 + 6N^3 + 11N^2 + 16N + 16)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2} - \frac{128(N^2 + N + 3)S_{-3}}{3(N+1)(N+2)} \\
&+ \gamma_9 \left[ \frac{8}{9} S_1^2 - 8S_2 S_1 + \frac{32}{3} S_{2,1} \right] + \frac{256S_{-2,1}}{3N(N+1)(N+2)} + \frac{(N-1)\left[ \frac{64}{3} S_{-2} S_1 - 32\zeta_3 \right]}{N(N+1)} L_Q \\
&+ \frac{16(N^3 + 8N^2 + 11N + 2)S_3}{9(N+1)^2(N+2)^2} + \frac{8P_{102} S_1}{3(N+1)^3(N+2)^3} + \frac{4}{9} \frac{P_{170} \zeta_2}{(N-1)N^3(N+1)^3(N+2)^2} \\
&+ \frac{16P_{201}}{3(N-1)N^5(N+1)^5(N+2)^5} + \frac{16(N^5 - 14N^4 - 19N^3 + 52N^2 + 12N + 8)\zeta_3}{9(N-1)N^2(N+1)^2(N+2)^2} \\
&- \frac{16P_{166} S_1}{3N(N+1)^4(N+2)^4} + \frac{16(N^5 + 32N^3 + 47N^2 + 28N + 20)\zeta_2}{9(N+1)^2(N+2)^2} S_1 \\
&+ \frac{L_M^3 \left[ \frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{8\gamma_9 S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \right]}{3(N+1)^3(N+2)^3} \\
&+ \frac{8P_{191} S_2}{3(N-1)N^3(N+1)^3(N+2)^3} \\
&- \frac{16(3N^3 - 12N^2 - 27N - 2)S_1 S_2}{3(N+1)^2(N+2)^2} + \frac{128(N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6)S_3}{9(N-1)N^2(N+1)^2(N+2)^2} \\
&+ \frac{L_M^2 \left[ \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} - \frac{8P_{172}}{9(N-1)N^2(N+1)^2(N+2)^2} \right]}{3(N+1)^3(N+2)^3} \\
&+ \frac{32(5N^4 + 20N^3 + 47N^2 + 58N + 20)S_1}{9(N+1)^2(N+2)^2} + \frac{\gamma_9 \left[ -\frac{4}{3} S_1^2 - \frac{4S_2}{3} - \frac{8}{3} S_{-2} \right]}{3(N+1)^3(N+2)^3} \\
&+ \frac{(N^4 + 2N^3 + 7N^2 + 22N + 20) \left[ \frac{64}{3}(-1)^N S_{-2} + \frac{32}{3}(-1)^N \zeta_2 \right]}{3(N+1)^3(N+2)^3} \\
&+ \frac{(N^2 - N - 4) \left[ -\frac{128}{3}(-1)^N S_1 S_{-2} - \frac{64}{3}(-1)^N S_{-3} + \frac{128}{3}(-1)^N S_{-2,1} - \frac{64}{3}(-1)^N S_1 \zeta_2 - 16(-1)^N \zeta_3 \right]}{(N+1)^2(N+2)^2} \\
&+ \frac{\gamma_9 \left[ \frac{1}{9} S_1^2 + \frac{10}{3} S_2 S_1^2 + \frac{16}{3}(-1)^N S_{-3} S_1 + \frac{80S_3}{9} - \frac{32}{3} S_{-2,1} \right]}{3(N+1)^3(N+2)^3} S_1 + \frac{4}{9} \frac{(-7 + 9(-1)^N) \zeta_3 S_1}{3S_2} + \frac{1}{3} S_2^2 \\
&+ \frac{S_{-2} \left[ \frac{16}{3}(-1)^N S_{-2} + \frac{16}{3}(-1)^N S_2 \right] + 6S_4 + \frac{8}{3}(-1)^N S_{-4} - \frac{8}{3} S_{3,1} - \frac{16}{3} S_{-2,2} - \frac{16}{3} S_{-3,1} - \frac{8}{3} S_{2,1,1}}{3(N+1)^3(N+2)^3} \\
&+ \frac{32}{3} S_{-2,1,1} + \left[ \frac{4}{3}(-1 + 2(-1)^N) S_2^2 + \frac{4}{3}(-1 + 2(-1)^N) S_2 + \frac{8}{3}(-1)^N S_{-2} \right] \zeta_2 \\
\end{align*}
\]
\[ + L_M \left[ \frac{-16(10N^4 + 43N^3 + 106N^2 + 131N + 46)S_1^2}{9(N + 1)^2(N + 2)^2} + \frac{16P_{140}S_1}{27N(N + 1)^3(N + 2)^3} \right. \\
\left. - \frac{64(-1)^N(7N^5 + 43N^3 + 117N^2 + 166N^2 + 107N + 16)}{9(N + 1)^4(N + 2)^4} + \frac{8P_{246}}{27(N - 1)N^4(N + 1)^4(N + 2)^4} \right. \\
\left. - \frac{16P_{122}S_2}{9(N - 1)N^2(N + 1)^2(N + 2)^2} - \frac{64(5N^2 + 8N + 10)S_{-2}}{9N(N + 1)(N + 2)} + \frac{(N^2 - N - 4)\frac{1}{2}(1)^N S_{-2}}{(N + 1)^2(N + 2)^2} \right. \\
\left. + \gamma_{99}^2 \left[ -\frac{8}{9}S_1^2 - \frac{8}{3}S_2S_1 - \frac{16}{3}(-1)^N S_{-2}S_1 - \frac{40S_3}{9} - \frac{8}{3}(2 + (-1)^N)S_{-3} - \frac{16}{3}S_{2,1} + \frac{16}{3}S_{-2,1} \right] \right] \\
+ C^2_T F_T \left[ \frac{(15N^4 + 6N^3 - 25N^2 - 32N - 28)S_1^4}{N^3(N + 1)^3(N + 2)} + \frac{2P_{159}S_1^2}{N^3(N + 1)^4(N + 2)} \right. \\
\left. - \frac{4(3N^5 - 47N^4 - 147N^3 - 93N^2 + 8N + 12)S_2^2}{2(5N^4 - 14N^3 + 53N^2 + 120N + 28)S_2S_1^2} - \frac{2P_{211}S_1}{N^3(N + 1)^5(N + 2)} - \frac{4P_{118}S_2S_1}{N^3(N + 1)^3(N + 2)} \right. \\
\left. + \frac{8(3N^4 + 90N^3 + 83N^2 - 44N - 4)S_3S_1}{3N^2(N + 1)^2(N + 2)} - \frac{16(3N^4 + 2N^3 + 19N^2 + 28N + 12)S_{2,1}S_1}{N^2(N + 1)^2(N + 2)} \right. \\
\left. - \frac{8P_{332}\zeta_2S_1}{N^3(N + 1)^3(N + 2)} - \frac{(11N^4 + 142N^3 + 147N^2 - 32N - 12)S_2^2}{N^2(N + 1)^3(N + 2)} + \frac{P_{247}}{N^6(N + 1)^6(N + 2)} \right. \\
\left. + \gamma_3^0 \zeta_S^2 \right] + L_3^\tau \left[ \frac{-\frac{2(N^2 + N + 2)(3N^2 + 3N + 2)^2}{3N^3(N + 1)^3(N + 2)}}{16(N^2 + N + 2)S_1(3N^2 + 3N + 2)} + \frac{16(N^2 + N + 2)S_1(3N^2 + 3N + 2)}{3N^2(N + 1)^3(N + 2)} \right. \\
\left. - \frac{\frac{2P_{182}S_2}{N^4(N + 1)^4(N + 2)} + \frac{4(21N^5 + 217N^4 + 415N^3 + 351N^2 + 152N - 4)S_3}{3N^2(N + 1)^3(N + 2)}}{16(N^2 - 3N - 2)(3N^2 + 3N + 2)S_{2,1}} + \frac{L^2}{3N^3(N + 1)^2(N + 2)} \right. \\
\left. - \frac{\frac{2P_{338}S_1}{N^3(N + 1)^4(N + 2)} - \frac{32(-1)^N(N^2 + N + 2)}{N(N + 1)^4(N + 2)} - \frac{P_{178}}{N^4(N + 1)^4(N + 2)} \right. \\
\left. - \frac{16(N^2 + N + 2)(3N^2 + 3N + 2)S_2}{N^2(N + 1)^2(N + 2)} - \frac{32(N^2 + N + 2)S_{-2}}{N^2(N + 1)^2(N + 2)} + \gamma_{99}^0 \left[ \frac{8S_3}{3} - 16S_{2,1} \right] \right] \\
\left. \frac{-16S_{-2}S_1 - 8S_3 - 8S_{-3} + 16S_{-2,1}}{16S_{-2}S_1 - 8S_3 - 8S_{-3} + 16S_{-2,1}} \right] + L_3^\tau \left[ \frac{32(N^3 + 5N^2 + 6N + 4)S_1^2}{N^2(N + 1)^2(N + 2)} + \frac{2P_{338}S_1}{N^3(N + 1)^3(N + 2)} \right. \\
\left. - \frac{-\frac{32(-1)^N(N^2 + N + 2)}{N(N + 1)^4(N + 2)} - \frac{P_{178}}{N^4(N + 1)^4(N + 2)}}{16(N^2 + N + 2)S_1(3N^2 + 3N + 2)} + \frac{L^2}{3N^2(N + 1)^2(N + 2)} \right. \\
\left. - \frac{-\frac{32(-1)^N(N^2 + N + 2)}{N(N + 1)^4(N + 2)} - \frac{P_{178}}{N^4(N + 1)^4(N + 2)}}{16(N^2 + N + 2)S_1(3N^2 + 3N + 2)} + \frac{L^2}{3N^2(N + 1)^2(N + 2)} \right. \\
\left. - \frac{32(N^2 + N + 2)S_{-2}}{N^2(N + 1)^2(N + 2)} + \gamma_{99}^0 \left[ \frac{8S_3}{3} - 16S_{2,1} - 16S_{-2}S_1 - 8S_3 - 8S_{-3} + 16S_{-2,1} \right] \]
\[
\frac{P_{190} \zeta_2}{2N^4(N+1)^4(N+2)} + \frac{16(N^2 + N + 2)S_{-2} \zeta_2}{N^2(N+1)^2(N+2)} + 96 \gamma_{0g}^0 \log(2) \zeta_2 \\
\frac{(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} \left[ 6S_4 - 16S_{3,1} + 32S_{2,1,1} + 8S_{2} \zeta_2 - \frac{16}{3} S_1 \zeta_3 \right] + \gamma_{0g}^0 \left[ \frac{1}{3} S_1^5 \right] \\
\frac{2}{3} S_2 S_3^3 + \left( \frac{16}{3} S_3 - 16S_{2,1} \right) S_1^2 - \frac{8}{3} S_3 S_2^2 + \left[ -3S_2^2 + 6S_4 - 16S_{3,1} + 32S_{2,1,1} \right] S_1 \\
\frac{8}{3} S_2 S_3 + \left[ -4S_3^3 + 8S_2 S_1 + 8S_{-2} S_1 + 4S_3 + 4S_{-3} - 8S_{-2,1} \right] \zeta_2 \\
+ L_Q \left[ \frac{16(3N^4 - 13N^2 - 18N - 12)S_1^3}{N^2(N+1)^2(N+2)} - \frac{2P_{136} S_1^2}{N^3(N+1)^3(N+2)} + \frac{64(1)N(N^2 + N + 2)S_1}{N(N+1)^4(N+2)} \\
- \frac{2P_{139} S_1}{N(N+1)^3} - \frac{16(7N^4 + 20N^3 + 7N^2 - 22N - 20)S_2 S_1}{N^2(N+1)^2(N+2)} - \frac{64(2N^3 + N^2 + 3N - 10)S_{-2} S_1}{N^2(N+1)^2(N+2)} \right] \\
- \frac{32(-1)^N P_{266}}{5(N-N)(N-1)N^2(N+1)^3(N+2)^5(N+3)^3} + \frac{5(N-1)^2 N^2(N+1)^6(N+2)^6(N+3)^4}{N^2(N+1)^3(N+2)} \\
- \frac{(9N^4 + 10N^3 + 9N^2 - 8N + 12) \zeta_3}{N^2(N+1)^2(N+2)} + L_M \left[ \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^3(N+1)^3(N+2)} \right] \\
- \frac{16(N^2 + N + 2)S_1(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} - \frac{8 \gamma_{0g}^0 S_1^2}{N^3(N+1)^3(N+2)} + \frac{2P_{147} S_2}{N^3(N+1)^3(N+2)} \\
- \frac{(N^2 + N + 2)(3N^2 + 3N + 2)S_2}{N^2(N+1)^2(N+2)} + L_M \left[ \frac{64(N^2 + 5N^2 + 6N + 4)S_1^2}{N^2(N+1)^2(N+2)} \right] \\
- \frac{4P_{138} S_1}{N^2(N+1)^3(N+2)} + \frac{64(-1)^N(N^2 + N + 2)}{N(N+1)^4(N+2)} + \frac{2P_{178}}{N^4(N+1)^4(N+2)} \\
+ \frac{32(N^2 + N + 2)(3N^2 + 3N + 2)S_2}{N^2(N+1)^2(N+2)} + \frac{64(N^2 + N + 2)S_{-2}}{N^2(N+1)^3(N+2)} + \gamma_{0g}^0 \left[ -16S_1^3 + 32S_2 S_1 + 32S_{-2} S_1 \right. \\
\left. + 16S_3 + 16S_{-3} - 32S_{-2,1} \right] \\
+ \left[ 16S_{-2,1} - 32S_{-2,1} \right] S_1 - 48 \zeta_3 S_1 - 16S_2^2 - 32S_{-2}^2 + S_{-2} \left[ 32S_1^2 - 32S_2 \right] - 32S_1 - 80S_{-4} - 32S_{3,1} \\
+ \left[ 32S_{-2,2} + 64S_{-3,1} \right] \right] + L_M \left[ \frac{-16(3N^4 - 13N^2 - 18N - 12)S_1^3}{N^2(N+1)^3(N+2)} + \frac{2P_{136} S_1^2}{N^3(N+1)^3(N+2)} \right] \\
- \frac{64(-1)^N(N^2 + N + 2)S_1}{N(N+1)^4(N+2)} + \frac{2P_{139} S_1}{N^3(N+1)^3} + \frac{16(7N^4 + 20N^3 + 7N^2 - 22N - 20)S_2 S_1}{N^2(N+1)^2(N+2)} \\
- \frac{64(2N^3 + N^2 + 3N - 10)S_{-2} S_1}{N^2(N+1)^2(N+2)} + \frac{32(-1)^N P_{266}}{5(N-N)(N-1)N^2(N+1)^3(N+2)^5(N+3)^3} \\
- \frac{5(N-1)^2 N^2(N+1)^6(N+2)^6(N+3)^4}{N^2(N+1)^3(N+2)} + 48 \left( 9N^4 + 10N^3 + 9N^2 - 8N + 12 \right) \zeta_3 \\
+ \frac{2P_{147} S_2}{N(N+1)^3(N+2)} + \frac{16(3N^4 + 10N^3 + 15N^2 + 16N - 12)S_3}{N^2(N+1)^2(N+2)} \right]
\]
\[
\frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_{-2}}{(N + 1)^3(N + 2)^3} + \frac{64P_{212}S_{-2}}{(N - 2)N^3(N + 1)^3(N + 2)^3(N + 3)} + \\
\frac{64(N^2 + 3N + 4)S_{-3}}{N(N + 1)^2(N + 2)} - \frac{16(N^2 + N + 2)(3N^2 + 3N + 2)S_{2, 1}}{N^2(N + 1)^2(N + 2)} - \frac{128(N - 1)(N^2 + 2N + 4)S_{-2, 1}}{N^2(N + 1)^2(N + 2)}\
+ \gamma_{q9}^0 \left[ 8S_1^4 - 24S_2S_1^2 + 48S_{-3}S_1 + \left[ 32S_{-2, 1} - 16S_{2, 1} \right] S_1 + 48S_3S_1 + 16S_2^2 + 32S_{-2}^2 \right. \right. \\
+ S_{-2} \left[ 32S_2 - 32S_1^2 \right] + 32S_4 + 80S_{-4} + 32S_{3, 1} - 32S_{-2, 2} - 64S_{-3, 1} \left. \right] \right] \\
+ C_F^2 T_F \left[ \frac{P_{123}S_1^4}{9(N - 1)N^2(N + 1)^2(N + 2)^2} - \frac{4P_{176}S_1^3}{9(N - 1)N^2(N + 1)^2(N + 2)^3} \right. \\
\frac{4}{3} \frac{P_{131}S_1^4}{(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{2P_{215}S_1^2}{3(N - 1)N^2(N + 1)^2(N + 2)^4} \\
\frac{2(N - 2)(55N^5 + 347N^4 + 379N^3 + 213N^2 + 326N + 120)S_2S_1^2}{3(N - 1)N^2(N + 1)^2(N + 2)^2} \\
\frac{4}{9} \frac{P_{217}S_1^4}{(N - 1)^2N^3(N + 1)^3(N + 2)^3} - \frac{4P_{262}S_1}{3(N - 1)N^5(N + 1)^5(N + 2)^5} \\
\frac{3(N - 1)N^3(N + 1)^3(N + 2)^3}{4P_{193}S_2S_1} + \frac{16P_{144}S_3S_1}{9(N - 1)N^2(N + 1)^2(N + 2)^2} \\
\frac{8}{3} \frac{(-1)^NP_{213}S_2}{(N - 1)^2N^3(N + 1)^4(N + 2)^4} - \frac{2}{9} \frac{P_{251}S_2}{(N - 1)^2N^4(N + 1)^4(N + 2)^4} \\
\frac{4(11N^4 + 22N^3 - 35N^2 - 46N - 24)P_{261}}{3(N - 1)^2N^6(N + 1)^6(N + 2)^6} - \frac{4}{9} \frac{\zeta_3P_{148}S_1}{(N - 1)N^2(N + 1)^2(N + 2)^2} \\
\frac{4}{9} \frac{(11N^4 + 22N^3 - 35N^2 - 46N - 24)(9N^5 - 14N^4 - 19N^3 + 52N^2 + 12N + 8)\zeta_3}{(N - 1)^2N^3(N + 1)^3(N + 2)^3} \\
\frac{4L_2}{L_{q9}^0} \left[ \frac{8(N^2 + 2N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_1}{9(N - 1)N^2(N + 1)^2(N + 2)^2} \right. \\
\frac{16(N^2 + N + 1)(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} \\
\frac{8(N^2 + N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_1}{9(N - 1)N^2(N + 1)^2(N + 2)^2} \\
\frac{16(N^2 + N + 1)(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} \\
\frac{2(11N^4 + 22N^3 - 35N^2 - 46N - 24)P_{191}S_2}{3(N - 1)^2N^4(N + 1)^4(N + 2)^4} \\
\frac{32(11N^4 + 22N^3 - 35N^2 - 46N - 24)(N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6)S_3}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} \\
\frac{16(-1)^N(N^4 + 2N^3 + 7N^2 + 22N + 20)(11N^4 + 22N^3 - 35N^2 - 46N - 24)S_{-2}}{3(N - 1)N(N + 1)^4(N + 2)^4} \\
\frac{-32(N^2 + N + 1)(N^2 + N + 2)\zeta_2}{(N - 1)N^2(N + 1)^2(N + 2)^2} S_{-2} + \frac{L_2^2}{3} \frac{4P_{127}S_1^2}{(N - 1)N^2(N + 1)^2(N + 2)^2} \\
\frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{(N + 1)^3(N + 2)^3} + \frac{8P_{216}S_1}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} \]
\[\begin{align*}
&= \frac{16(-1)^N P_{196}}{3(N-1)N^3(N+1)^4(N+2)^4} + \frac{16P_{241}}{9(N-1)^2N^4(N+1)^3(N+2)^4} + \frac{4(N^2 + N + 2)(11N^4 + 22N^3 - 83N^2 - 94N - 72)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} \\
&= \frac{8(N^2 + N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_{-2}}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{\gamma_{99}^0}{4} \left[ 4S_3^2 + 12S_2S_1 + 16S_{-2}S_1 \right] \\
&= \frac{64(-1)^N (N^3 + 4N^2 + 7N + 5)S_1}{(N + 1)^3(N + 2)^3} - \frac{8P_{218}S_1}{9(N-1)^2N^3(N+1)^3(N+2)^3} + \frac{3(N-1)N^3(N+1)^4(N+2)^4}{16(-1)^NP_{196}} \\
&= \frac{4P_{41}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{4(N^2 + N + 2)(11N^4 + 22N^3 - 83N^2 - 94N - 72)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} \\
&= \frac{8(N^2 + N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_{-2}}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{\gamma_{99}^0}{4} \left[ 4S_3^2 + 12S_2S_1 + 16S_{-2}S_1 \right] \\
&= \frac{(5N^5 - 131N^3 - 58N^2 + 232N + 96)}{(N-1)N(N+1)^2(N+2)^3} \frac{32}{3} \left[ (-1)^N S_1 S_{-2} + \frac{16}{3} (-1)^N S_1 \zeta_2 \right] \\
&= \frac{(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{(N-1)N^2(N+1)^2(N+2)^2} \frac{1}{3} S_2^2 + \frac{16}{3} (-1)^N S_{-2} S_2 \\
&= \frac{6S_4 + \frac{8}{3} (-1)^N S_{-4} - \frac{8}{3} S_{3,1} - \frac{16}{3} S_{-2,2} - \frac{16}{3} S_{-3,1} - \frac{8}{3} S_{2,1,1} + \frac{32}{3} S_{-2,1,1}}{12} \\
&= \frac{\frac{8}{3} (-1)^N S_2 + \frac{8}{3} (-1)^N S_{-2} \zeta_2}{2} + \frac{(N^2 - N - 4)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{(N-1)N(N+1)^3(N+2)^3} \\
&\times \left[ \frac{4(-1)^N \zeta_3 + \frac{16}{3} (-1)^N S_{-3} - \frac{32}{3} S_{-2,1}}{2} + \frac{P_{30}}{16} \frac{(16 \frac{1}{3} (-1)^N S_{-2} S_1^2 + \frac{8}{3} (-1)^N \zeta_2 S_1^2)}{(N-1)N^2(N+1)^2(N+2)^2} \right] \\
&= \frac{(11N^5 + 34N^4 - 49N^3 - 24N^2 - 68N - 48)}{(N-1)N^2(N+1)^2(N+2)^2} \frac{16}{3} \left[ (-1)^N S_1 S_{-3} - \frac{32}{3} S_1 S_{-2,1} + 4(-1)^N S_1 \zeta_3 \right] \\
&= \frac{32(N-2)(N+3)S_{-2}S_1^2}{N(N+1)(N+2)} - \frac{32(-1)^NP_{197}S_1}{3(N-1)N^3(N+1)^4(N+2)^4} + \frac{8P_{257}S_1}{27(N-1)^2N^4(N+1)^4(N+2)^4} \\
&= \frac{8P_{37}S_2S_1}{N(N+1)(N+2)} - \frac{32(N^2 + N + 6)S_3S_1}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{16P_{42}S_{-2}S_1}{3(N-1)N^2(N+1)^2(N+2)^2} \\
&= \frac{32(9N^2 + 9N + 22)S_{-3}S_1}{N(N+1)(N+2)} - \frac{64(3N^2 + 3N + 10)S_{-2,1}S_1}{N(N+1)(N+2)} - \frac{384\zeta_3 S_1}{N(N+1)(N+2)} \\
&= \frac{8\gamma_{99}^0}{3} \left[ 4S_3^2 + 12S_2S_1 + 16S_{-2}S_1 \right] + \frac{16(-1)^NP_{256}}{9(N-1)^2N^4(N+1)^5(N+2)^5} + \frac{8P_{219}S_2}{9(N-1)^2N^3(N+1)^3(N+2)^3} \\
&= \frac{9(N-1)^2N^2(N+1)^2(N+2)^2 - \frac{27(N-1)^2N^5(N+1)^5(N+2)^5}{3(N-1)N^2(N+1)^2(N+2)^2}}{8P_{269}S_3} - \frac{16P_{220}S_2}{9(N-1)^2N^3(N+1)^3(N+2)^3}
\end{align*}\]
\[ + \left( \frac{N^3 + 4N^2 + 7N + 5}{(N + 1)^3(N + 2)^3} \right) \left( -64(-1)^N S_1^2 + 64(-1)^N S_2 + 192(-1)^N S_{-2} \right) \]
\[ + \frac{16P_{133} S_{-3}}{3(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{32P_{112} S_{-2,1}}{3(N - 1)N^2(N + 1)^2(N + 2)^2} \]
\[ + \gamma_{qq}^0 \left[ -2S_1^4 + 16S_2 S_1^2 - 2S_1^2 - 12S_2^2 - 16S_{-2} S_2 - 4S_4 - 44S_{-4} - \frac{88}{3} S_{2,1} - 16S_{3,1} + 56S_{-2,1} \right] + 64S_{-3,1} - 96S_{-2,1,1} \]
\[-\frac{8P_{229}S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{45(N - 2)(N - 1)^2N^3(N + 1)^4(N + 2)^4(N + 3)^3}{3(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{64(-1)^NP_{259}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} + L_M^2 \left[ -\frac{16(N^2 + N + 2)P_{109}}{3(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{32P_{173}S_1}{3(N - 1)N^3(N + 1)^3(N + 2)^2} \right] + \frac{8P_{231}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} - \frac{32P_{173}S_1}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{16P_{183}S_2}{N(N + 1)(N + 2)} S_3 \right] \]
\[+L_M \left[ -\frac{8(N^2 + N + 2)(3N^2 + 3N + 2)}{3N^2(N + 1)^2(N + 2)} - \frac{8}{3} \tilde{\gamma}^0_{qg} S_1 \right] + \tilde{\gamma}^0_{qg} \left[ \frac{20S_2}{3} - 4S_1^2 \right] L_Q^2\]

\[+ \left[ -\frac{16P_{181}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{8P_{229}S_1}{9(N-1)N^4(N+1)^4(N+2)^3} \right]
+ \frac{64(-1)^NP_{259}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3} + \frac{64(N-1)^N}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{8P_{276}}{45(N-1)N^5(N+1)^5(N+2)^4(N+3)^3}

\[+ L_M \left[ \frac{8(N^2 + N + 2)(57N^4 + 72N^3 + 29N^2 - 22N - 24)}{9N^2(N + 1)^3(N + 2)} + \tilde{\gamma}^0_{qg} \left[ \frac{8}{3}S_1^2 - 8S_2 \right] \right]
- \frac{16(N^2 + N + 2)(29N^2 + 29N - 6)S_1}{9N^2(N + 1)^2(N + 2)} - \frac{256(N^2 + N + 1)S_3}{3N^2(N + 1)(N + 2)} + \frac{16P_{183}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}

\[+ \frac{64P_{163}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)^3} + \tilde{\gamma}^0_{qg} \frac{8}{3}S_1^2 - 8S_2 S_1 - \frac{32}{3} S_{2,1} \]

\[+ \frac{\frac{312}{3} S_{1,2} - \frac{256}{3} S_{3,2} - \frac{512}{3} S_{2,1}}{N(N+1)(N+2)} + \frac{64(N-1)\zeta_3}{N(N+1)} \right] L_Q - \frac{2}{9} \frac{P_{235} \zeta_2}{(N-1)N^3(N+1)^3(N+2)^3}

\[+ \left[ \frac{8(N^4 - 5N^3 - 32N^2 - 18N - 4)S_2^2}{3N^2(N + 1)^2(N + 2)} - \frac{8}{9} \left( \frac{N^2 + N + 2}{N^3(N + 1)^3(N + 2)^2} \right) P_{108} \right]
- \frac{4P_{271}}{3(N-1)N^6(N+1)^6(N+2)^5} + \frac{3N^2(N+1)^3(N+2)}{3(N-1)N^4(N+1)^4(N+2)^3}

\[+ \frac{16 \left( 5N^3 + 11N^2 + 28N + 12 \right) \zeta_2 S_1}{N^2(N+1)(N+2)} + \frac{3}{9} \left( \frac{N^2 + N + 2}{N^3(N+1)^3(N+2)^2} \right) P_{108} \right]
+ \frac{8P_{225}S_2}{3(N-1)N^4(N+1)^4(N+2)^3} + \frac{(3N + 2)}{N^2(N+2)} \left[ \frac{-16}{9} S_1^2 - \frac{16}{3} S_2 S_1 \right]

\[+ \left[ \frac{16P_{169}S_3}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{32(5N^3 + 8N^2 + 19N + 6)S_1}{9N^2(N+1)(N+2)} + \tilde{\gamma}^0_{qg} \left[ \frac{4}{3} S_1^2 + \frac{4S_2}{3} \right] \right]

\[+ \frac{8P_{258}}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{64(N^2 - 3N - 2)S_{2,1}}{3N^2(N+1)(N+2)} \]

\[+ \frac{8P_{146}S_1}{9N^3(N+1)^3(N+2)} - \frac{8P_{184}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} + \tilde{\gamma}^0_{qg} \left[ \frac{16}{3} S_1 S_2 - 16S_3 + \frac{16}{3} S_{2,1} \right] \]

\[+ \tilde{\gamma}^0_{qg} \left[ -\frac{1}{9} S_1^4 - \frac{2}{3} S_2 S_1^2 - \frac{4}{3} S_2^2 \right] + \left[ -\frac{8}{9} S_3 - \frac{16}{3} S_{2,1} \right] S_1 - \frac{8}{9} \zeta S_3 S_1 - \frac{1}{3} S_2^2 + 2S_4 - \frac{16}{3} S_{3,1} + \frac{32}{3} S_{2,1,1} \]
\[\begin{align*}
\frac{4P_{206}S_2S_1}{3(N-1)N^3(N+1)^2(N+2)^2} &+ \frac{8P_{153}S_3S_1}{16(11N^5 + 45N^4 - 3N^3 - 145N^2 - 176N - 20)S_{2,1}S_1} - \frac{2P_{101}S_2^2}{3(N-1)N^2(N+1)^2(N+2)^2} \\
\frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{(N+1)^3(N+2)^3} &+ \frac{8(-1)^N\zeta_2}{N^2(N+1)^4(N+2)^3}P_{115} - \frac{2P_{188}\zeta_3}{9(N-1)N^2(N+1)^3(N+2)^2} \\
\frac{1}{18(N-1)N^3(N+1)^4(N+2)^2} &+ \frac{3(N-1)N^6(N+1)^6(N+2)^5}{P_{208}\zeta_2} - \frac{P_{273}}{P_{273}} \\
+L_M^3 &\left[ -\frac{16}{3}\gamma_{qg}^0S_1^2 - \frac{8(N^2 + N + 2)\left(7N^2 + 7N + 4\right)S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \right] + L_Q^3 \left[ -\frac{8}{3}\gamma_{qg}^0S_1^2 \right] \\
\frac{2(N^2 + N + 2)\left(3N^2 + 3N + 2\right)\left(11N^2 + 22N^3 - 59N^2 - 70N - 48\right)}{9(N-1)N^2(N+1)^3(N+2)^2} &+ \frac{8(N^2 + N + 2)\left(13N^2 + 26N^3 - 43N^2 - 56N - 12\right)S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \\
-\frac{4(N^2 + N + 2)\left(3N^2 + 3N + 2\right)\left(11N^2 + 22N^3 - 23N^2 - 34N - 12\right)}{9(N-1)N^2(N+1)^3(N+2)^2} &- \frac{\zeta_2P_{107}S_2}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{4P_{243}S_2}{3(N-1)N^4(N+1)^4(N+2)^4} + \frac{4P_{185}S_3}{9(N-1)N^3(N+1)^3(N+2)^2} \\
+\frac{4(N^2 + N + 2)\left(19N^2 + 38N^3 - 22N^2 - 41N - 30\right)S_4}{(N-1)N^2(N+1)^2(N+2)^2} &+ \frac{16(-1)^NP_{16}S_{2,1}}{N^2(N+1)^4(N+2)^3} \\
-\frac{16(N^2 - 3N - 2)\left(11N^2 + 22N^3 - 35N^2 - 46N - 24\right)S_{2,1}}{3(N-1)N^3(N+1)^2(N+2)^2} &- \frac{8(N^2 + N + 2)\left(31N^2 + 62N^3 - 73N^2 - 104N - 60\right)S_{3,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
+L_M^2 &\left[ -\frac{8\left(10N^5 + 6N^4 + 5N^3 - 38N^2 - 17N + 2\right)S_1^2}{(N-1)N^2(N+1)^2(N+2)} \right] + \frac{64(-1)^N\left(3N^3 + 4N^2 + 7N + 5\right)S_1}{(N+1)^3(N+2)^3} \\
-\frac{4P_{204}S_1}{9(N-1)N^3(N+1)^3(N+2)^3} &- \frac{16(-1)^N\left(3N^4 + 11N^3 + 19N^2 + 15N + 2\right)}{N(N+1)^3(N+2)^3} \\
+\frac{P_{238}}{9(N-1)N^4(N+1)^4(N+2)^3} + L_M &\left[ \frac{22(N^4 + N + 2)\left(3N^2 + 3N + 2\right)}{3N^2(N+1)^2(N+2)} \right] \\
+\frac{22}{3}\gamma_{qg}^0S_1 &+ \frac{8\left(N^2 + N + 2\right)\left(23N^4 + 46N^3 - 50N^2 - 73N - 18\right)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} + \gamma_{qg}^0\left[ 4S_1^2 - 12S_2S_1 \right] \\
+4S_3 + 6S_{2,-2} + 4S_{3,-1} - 8S_{2,-1} &\right] + L_M^2 \left[ \frac{8P_{121}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} \right] \\
+\frac{64(-1)^N\left(3N^3 + 4N^2 + 7N + 5\right)S_1}{(N+1)^3(N+2)^3} &- \frac{8P_{203}S_1}{9(N-1)N^3(N+1)^3(N+2)^3} \\
-\frac{16(-1)^N\left(3N^4 + 11N^3 + 19N^2 + 15N + 2\right)}{N(N+1)^3(N+2)^3} &+ \frac{P_{224}}{9(N-1)N^4(N+1)^3(N+2)^3} \\
+\frac{8(N^2 + N + 2)\left(N^4 + 2N^3 + 20N^2 + 19N + 30\right)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} &+ \gamma_{qg}^0\left[ -12S_1^3 + 4S_2S_1 + 4S_3 + 6S_{2,-2} \right] \\
+4S_{3,-1} - 8S_{2,-1} &\right] + \frac{8\left(N^2 + N + 2\right)\left(35N^4 + 70N^3 - 137N^2 - 172N - 84\right)S_{2,1,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
\end{align*}\]
\[
\frac{8(N^2 + N + 2)S_{-2} \zeta_2}{N^2(N + 1)^2(N + 2)} - 48\gamma_0 \log(2) \zeta_2 + \frac{P_{168}}{N^4(N + 1)^3(N + 2)^3} \left[ 16(-1)^N S_1 S_{-2} + 8(-1)^N S_1 \zeta_2 \right]
\]

\[
(N^4 + 3N^3 + 8N^2 + 12N + 4) \left[ 96(-1)^N S_{-2} S_1^2 + 48(-1)^N \zeta_2 S_1^2 \right]
\]

\[
\frac{3N^5 + 10N^4 + 25N^3 + 38N^2 + 20N + 8}{N(N + 1)^2(N + 2)^2} \left[ 16(-1)^N S_2 S_{-2} + 8(-1)^N S_2 \zeta_2 \right]
\]

\[
\frac{N^2(N + 1)^2(N + 2)^2}{N^2(N + 1)^2(N + 2)^2} \left[ 8(-1)^N S_{-2} \zeta_2 + 8(-1)^N S_{-4} - 16S_{-2,2} - 16S_{-3,1} + 32S_{-2,1,1} \right]
\]

\[
P_{156} \left[ 8(-1)^N S_{-3} - 16S_{-2,1} + 6(-1)^N \zeta_3 \right]
\]

\[
\frac{9N^5 + 28N^4 + 73N^3 + 110N^2 + 44N + 8}{N(N + 1)^3(N + 2)^2} \left[ 8(-1)^N S_1 S_{-3} - 16S_1 S_{-2,1} + 6(-1)^N S_1 \zeta_3 \right]
\]

\[
\gamma_0 \frac{20}{3} S_2 S_1^3 + \left[ 24S_3 + 16S_{2,1} - 24S_{-2,1} \right] S_1^2 + 8(-1)^N S_{-4} S_1 + \left[ 8S_2^2 + 12S_1 + 8S_{3,1} - 16S_{-2,2} \right] S_1^2 - 16S_{-3,1} - 40S_{2,1,1} + 32S_{-2,1,1} S_1 + S_{-3} \left[ 12(-1)^N S_1^2 + 4(-1)^N \zeta_2 \right] + S_2 \left[ \frac{16S_3}{3} - 8S_{-2,1} \right] + S_{-2} \left[ 8(-1)^N S_1^2 + 24(-1)^N S_2 S_1 + 32 \right] + \left[ 4(1 + (-1)^N) S_1^3 + 4(-1 + 2(-1)^N) S_{-2} S_1 - 25S_3 \right] - 2S_{-3} + 4S_{-2,1} \zeta_2 + \frac{1}{3} \left( -11 + 27(-1)^N \right) S_1^2 \zeta_3 \right]
\]

\[
L_Q \left[ \frac{8(29N^5 + 81N^4 + 117N^3 - 49N^2 - 362N - 104) S_1^3}{3(N - 1)N(N + 1)^2(N + 2)^2} + \frac{4P_{206} S_1^2}{9(N - 1)N(N + 1)^3(N + 2)^3} \right]
\]

\[
\frac{15(N - 2)(N - 1)^2 N^2(N + 1)^4(N + 2)^4(N + 3)^3}{8(17N^5 + 33N^4 - 43N^3 - 153N^2 - 254N - 80) S_2 S_1} + \frac{2P_{270} S_1}{32(N - 2)(N + 3) S_3 S_1} + \frac{32(N - 2)(N + 3) S_3 S_1}{3N - 2)(N - 1)N(N + 1)^2(N + 2)^2} + \frac{512S_{-2,1} S_1}{N(N + 1)(N + 2)} - \frac{24(11N^2 + 11N - 10) \zeta_3 S_1}{N(N + 1)(N + 2)} + \frac{16(-1)^N P_{272}}{2P_{279}}
\]

\[
\frac{45(N - 1)^3 N^5(N + 1)^5(N + 2)^5(N + 3)^3}{54(N - 1)N(N + 1)^2(N + 2)^2} + \frac{8(N^2 + N + 2)(3N^4 + 6N^3 + 7N^2 + 4N + 4) S_1}{(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{16(N^2 + N + 1)(N^2 + N + 2)(3N^2 + 3N + 2)}{(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{16(-1)^N \zeta_3}{N(N + 1)(N + 2)} + \frac{4P_{207} S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^3} + \frac{(N^3 + 4N^2 + 7N + 5)(128(-1)^N S_2 - 128(-1)^N S_1^2)}{(N + 1)^3(N + 2)^3} + \frac{32P_{355} S_3}{16(-1)^N(3N^5 - 6N^4 - 61N^3 - 124N^2 - 96N - 16) S_2} + \frac{3(N - 2)(N - 1)N^3(N + 1)^3(N + 2)^3(N + 3)}{32P_{228} S_2} - \frac{3(N - 1)N^2(N + 1)^2(N + 2)^2}{3(N - 1)N^2(N + 1)^2(N + 2)^2}
\]
\[
\begin{align*}
&\frac{16(N^2 + N + 2)(31N^2 + 31N + 6)S_{2,1}}{3N^3(N + 1)^2(N + 2)} - \frac{16P_{120}S_{-2,1}}{3(N - 1)N^2(N + 1)^2(N + 2)^2} + L_M \left[ \frac{16(10N^5 + 40N^4 + 121N^3 + 161N^2 + 52N + 12)S^2_1}{3N^3(N + 1)^2(N + 2)^2} + \frac{4P_{205}S_1}{9(N - 1)N^3(N + 1)^3(N + 2)^3} \right. \\
&\left. - \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{(N + 1)^3(N + 2)^3} \right] + \frac{32(-1)^N(3N^4 + 11N^3 + 19N^2 + 15N + 2)}{N(N + 1)^3(N + 2)^3} \\
&\frac{2P_{236}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} - \frac{16(N^2 + N + 2)(4N^2 + 4N - 1)S_2}{N^2(N + 1)^2(N + 2)} + \gamma_{qq}^0 \left[ 8S^4_1 + 8S_2S_1 - 8S_3 ight. \\
&\left. - 12S_{-2} - 8S_{-3} + 16S_{-2,1} \right] \left[ \frac{3N^5 + 8N^4 + 27N^3 + 46N^2 + 20N + 8}{N^2(N + 1)^2(N + 2)} \right] 16(-1)^NS_1S_{-2} \\
+ \frac{(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^2(N + 2)} \left[ (-1)^NS_{-3} + 6(-1)^N\zeta_3 \right] + \gamma_{qq}^0 \left[ 40S_2S^2_1 + 16(-1)^NS_{-2}S_1^2 \right. \\
&\left. + 8(-1)^NS_{-3}S_1 - 16S_{-2,1}S_1 + 6(-1)^N\zeta_3S_1 - 8S^2_2 - 24S^2_2 + 8S_{-4} + 40S_{-4} \right. \\
&\left. + 32S_{-3,1} - 16S_{-3,1} - 32S_{-2,1,1} \right] \left[ \frac{8(3N^5 + 2N^4 - 61N^3 - 112N^2 - 56N - 24)}{N^2(N + 1)^2(N + 2)^2} \right] \\
&\frac{2P_{209}S^2_1}{9(N - 1)N^3(N + 1)^3(N + 2)^3} + \frac{32(-1)^N(15N^5 + 97N^4 + 260N^3 + 328N^2 + 158N - 4)}{N(N + 1)^3(N + 2)^4} \\
&\left. \frac{2P_{250}S_1}{8P_{114}S_2S_1} \right] - \frac{16(-1)^N}{N(N + 1)^2(N + 2)} \\
&\left. \frac{16P_{267}}{5(N - 2)(N - 1)^2N^3(N + 1)^5(N + 2)^5(N + 3)^3} + \frac{45(N - 1)^2N^5(N + 1)^5(N + 2)^5(N + 3)^3}{N^3(N + 1)^5(N + 2)^5(N + 3)^3} \right. \\
&\left. \frac{2P_{210}S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^3} + \frac{(N^3 + 4N^2 + 7N + 5)(128(-1)^NS^2_1 - 128(-1)^NS_2)}{(N + 1)^3(N + 2)^3} \right. \\
&\left. \frac{16P_{128}S_3}{16(-1)^N(3N^5 - 6N^4 - 61N^3 - 124N^2 - 96N - 16)} \right. \\
&\left. \frac{16P_{214}S_{-2}}{16(-1)^N(3N^5 + 4N^3 - 9N^2 - 14N + 8)} - \frac{32(N^2 + N + 2)(10N^4 + 20N^3 + 5N^2 - 5N + 6)}{N^2(N + 1)^2(N + 2)} \right. \\
&\left. \frac{16(3N^4 + 10N^3 + 43N^2 + 44N - 20)}{3(N - 1)N^2(N + 1)^2(N + 2)^2} \right. \\
&\left. \frac{N^2(N + 1)^2(N + 2)}{N^2(N + 1)^2(N + 2)} \right. \\
&\left. \frac{(3N^5 + 8N^4 + 27N^3 + 46N^2 + 20N + 8)}{N^2(N + 1)^2(N + 2)^2} \right. \\
&\left. \frac{2P_{278}}{(N^2 + N + 2)(3N^2 + 3N + 2)} \right. \\
&\left. \frac{8S^4_1 - 32S_2S^2_1}{(N^2 + N + 2)} \right. \\
&\left. \frac{16(1 + (-1)^N)S_{-2}S^2_1 - 8(1 + (-1)^N)S_{-3}S_1 + 32S_{-2,1} - 8S_3}{32S_{-3,1} + 16S_{-3,1} + 32S_{-2,1,1}} \right] \right]
\end{align*}
\]
\[ + a_Q^{(3)} + C_{2g}^{(3)} (N_F + 1) \]

with the polynomials

\[
P_{101} = N^6 - 81N^5 - 264N^4 - 185N^3 - 307N^2 - 256N - 204
\]
\[
P_{102} = N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8
\]
\[
P_{103} = N^6 + 7N^5 - 7N^4 - 39N^3 + 14N^2 + 40N + 48
\]
\[
P_{104} = N^6 + 21N^5 + 57N^4 + 31N^3 + 26N^2 + 20N + 24
\]
\[
P_{105} = 2N^6 - 7N^5 - 41N^4 - 31N^3 - 29N^2 - 22N - 16
\]
\[
P_{106} = 2N^6 - 7N^5 - 24N^4 - 35N^3 - 44N^2 - 44N - 16
\]
\[
P_{107} = 3N^6 + 5N^5 + 27N^4 + 35N^3 + 6N^2 + 12N + 8
\]
\[
P_{108} = 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 + 28N - 24
\]
\[
P_{109} = 3N^6 + 9N^5 + 2N^4 - 11N^3 - 23N^2 - 16N - 12
\]
\[
P_{110} = 3N^6 + 30N^5 + 15N^4 - 64N^3 - 56N^2 - 20N - 8
\]
\[
P_{111} = 4N^6 + 5N^5 - 10N^4 - 39N^3 - 40N^2 - 24N - 8
\]
\[
P_{112} = 6N^6 - 12N^5 + 17N^4 + 106N^3 + 127N^2 + 104N + 84
\]
\[
P_{113} = 6N^6 + 18N^5 + 7N^4 - 16N^3 - 31N^2 - 20N - 12
\]
\[
P_{114} = 7N^6 - 93N^5 - 327N^4 - 287N^3 - 316N^2 - 112N - 24
\]
\[
P_{115} = 7N^6 - 20N^5 - 176N^4 - 335N^3 - 276N^2 - 116N - 16
\]
\[
P_{116} = 7N^6 - 19N^5 - 171N^4 - 325N^3 - 264N^2 - 108N - 16
\]
\[
P_{117} = 7N^6 + 21N^5 + 5N^4 - 25N^3 - 204N^2 - 188N - 192
\]
\[
P_{118} = 8N^6 + 13N^5 - 111N^4 - 193N^3 - 89N^2 - 56N - 20
\]
\[
P_{119} = 9N^6 + 21N^5 + 11N^4 - 5N^3 - 104N^2 - 76N - 144
\]
\[
P_{120} = 9N^6 + 39N^5 + 53N^4 + 25N^3 + 94N^2 + 44N + 312
\]
\[
P_{121} = 10N^6 + 18N^5 - 111N^4 - 164N^3 - 61N^2 - 16N + 36
\]
\[
P_{122} = 10N^6 + 63N^5 + 105N^4 + 31N^3 + 17N^2 + 14N + 48
\]
\[
P_{123} = 11N^6 - 15N^5 - 327N^4 - 181N^3 - 292N^2 - 20N - 48
\]
\[
P_{124} = 11N^6 + 15N^5 - 285N^4 - 319N^3 - 254N^2 - 368N - 240
\]
\[
P_{125} = 11N^6 + 33N^5 - 189N^4 - 361N^3 - 194N^2 - 92N - 72
\]
\[
P_{126} = 11N^6 + 33N^5 - 114N^4 - 247N^3 - 263N^2 - 176N - 108
\]
\[
P_{127} = 11N^6 + 33N^5 - 87N^4 - 85N^3 + 4N^2 - 116N - 48
\]
\[
P_{128} = 11N^6 + 35N^5 + 59N^4 + 57N^3 - 38N^2 - 68N + 40
\]
\[
P_{129} = 11N^6 + 47N^5 + 7N^4 + 9N^3 + 90N^2 + 28N + 96
\]
\[
P_{130} = 11N^6 + 57N^5 - 39N^4 - 109N^3 - 44N^2 - 116N - 48
\]
\[
P_{131} = 11N^6 + 81N^5 + 9N^4 - 133N^3 - 92N^2 - 116N - 48
\]
\[
P_{132} = 13N^6 + 36N^5 + 39N^4 + 8N^3 - 21N^2 - 29N - 10
\]
\[
P_{133} = 16N^6 + 78N^5 - 23N^4 - 228N^3 - 503N^2 - 408N - 228
\]
\[
P_{134} = 17N^6 + 111N^5 + 234N^4 + 203N^3 - 89N^2 - 296N - 36
\]
\[
P_{135} = 22N^6 + 69N^5 + 71N^4 + 23N^3 - 57N^2 - 68N + 84
\]
\[
P_{136} = 23N^6 - 7N^5 - 237N^4 - 593N^3 - 678N^2 - 548N - 200
\]
\[
P_{137} = 23N^6 + 9N^5 - 71N^4 - 53N^3 - 184N^2 - 92N - 16
\]
\[
P_{138} = 25N^6 + 35N^5 - 55N^4 - 243N^3 - 286N^2 - 204N - 72
\]
\[
P_{139} = 29N^6 + 91N^5 + 235N^4 + 405N^3 + 272N^2 + 288N + 120
\]
\[ P_{140} = 29N^6 + 176N^5 + 777N^4 + 1820N^3 + 1878N^2 + 776N + 232 \]

\[ P_{141} = 35N^6 - 15N^5 - 183N^4 - 133N^3 - 356N^2 - 164N - 48 \]

\[ P_{142} = 35N^6 - 15N^5 - 101N^4 + 31N^3 + 54N^2 + 164N + 120 \]

\[ P_{143} = 44N^6 + 96N^5 + 369N^4 + 290N^3 - 695N^2 - 428N - 108 \]

\[ P_{144} = 55N^6 + 141N^5 - 195N^4 - 401N^3 - 772N^2 - 748N - 384 \]

\[ P_{145} = 55N^6 + 165N^5 - 420N^4 - 899N^3 - 1561N^2 - 1336N - 1188 \]

\[ P_{146} = 57N^6 + 161N^5 - 25N^4 - 193N^3 - 172N^2 - 36N + 48 \]

\[ P_{147} = 65N^6 + 199N^5 + 197N^4 - 143N^3 - 330N^2 - 316N - 120 \]

\[ P_{148} = 77N^6 + 339N^5 - 105N^4 - 487N^3 - 356N^2 - 668N - 240 \]

\[ P_{149} = 80N^6 + 60N^5 + 9N^4 + 230N^3 + 901N^2 + 988N + 1188 \]

\[ P_{150} = 81N^6 + 211N^5 - 23N^4 - 355N^3 - 334N^2 - 4N - 344 \]

\[ P_{151} = 83N^6 + 249N^5 - 111N^4 - 637N^3 - 956N^2 - 596N - 624 \]

\[ P_{152} = 130N^6 + 865N^5 + 2316N^4 + 3811N^3 + 4434N^2 + 2884N + 536 \]

\[ P_{153} = 133N^6 + 699N^5 + 1395N^4 + 2173N^3 - 880N^2 + 164N + 288 \]

\[ P_{154} = 155N^6 + 369N^5 + 211N^4 - 65N^3 - 1002N^2 - 556N - 1416 \]

\[ P_{155} = 215N^6 + 429N^5 + 891N^4 + 491N^3 - 2486N^2 - 1436N - 408 \]

\[ P_{156} = 3N^7 + 28N^6 + 66N^5 + 90N^4 + 107N^3 + 78N^2 + 36N + 8 \]

\[ P_{157} = 9N^7 + 71N^6 + 214N^5 + 320N^4 + 275N^3 + 215N^2 + 160N + 32 \]

\[ P_{158} = 21N^7 + 120N^6 - 128N^5 - 1038N^4 - 89N^3 + 2382N^2 + 1636N - 600 \]

\[ P_{159} = 81N^7 + 247N^6 + 291N^5 + 277N^4 + 108N^3 - 56N^2 + 20N + 24 \]

\[ P_{160} = N^8 + 5N^7 + 10N^6 + 27N^5 + 65N^4 + 124N^3 + 80N^2 + 80N + 32 \]

\[ P_{161} = N^8 + 5N^7 + 14N^6 + 23N^5 + 25N^4 + 52N^3 + 56N^2 + 48N + 16 \]

\[ P_{162} = N^8 + 8N^7 - 2N^6 - 60N^5 - 23N^4 + 108N^3 + 96N^2 + 16N + 48 \]

\[ P_{163} = N^8 + 8N^7 - 2N^6 - 60N^5 + N^4 + 156N^3 + 24N^2 - 80N - 240 \]

\[ P_{164} = N^8 + 22N^7 + 111N^6 + 211N^5 + 42N^4 - 281N^3 - 406N^2 - 204N - 72 \]

\[ P_{165} = 2N^8 + N^7 - 6N^6 + 26N^5 + 64N^4 + 51N^3 + 54N^2 + 28N + 8 \]

\[ P_{166} = 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \]

\[ P_{167} = 2N^8 + 44N^7 + 211N^6 + 485N^5 + 654N^4 + 581N^3 + 391N^2 + 192N + 32 \]

\[ P_{168} = 3N^8 + 41N^7 + 136N^6 + 233N^5 + 331N^4 + 360N^3 + 208N^2 + 80N + 16 \]

\[ P_{169} = 3N^8 + 54N^7 + 118N^6 - 44N^5 - 353N^4 - 314N^3 - 272N^2 - 200N - 144 \]

\[ P_{170} = 5N^8 - 8N^7 - 13N^6 - 436N^5 - 713N^4 - 672N^3 - 407N^2 - 192N - 32 \]

\[ P_{171} = 7N^8 + 40N^7 + 110N^6 + 193N^5 + 261N^4 + 313N^3 + 260N^2 + 96N + 16 \]

\[ P_{172} = 9N^8 + 54N^7 + 80N^6 - 110N^5 - 645N^4 - 1168N^3 - 1132N^2 - 672N - 160 \]

\[ P_{173} = 10N^8 + 46N^7 + 87N^6 + 85N^5 - 75N^4 - 251N^3 - 274N^2 - 132N - 72 \]

\[ P_{174} = 11N^8 + 74N^7 + 213N^6 + 281N^5 - 30N^4 - 427N^3 - 446N^2 - 180N - 72 \]

\[ P_{175} = 15N^8 + 36N^7 + 50N^6 - 252N^5 - 357N^4 + 152N^3 - 68N^2 + 88N + 48 \]

\[ P_{176} = 18N^8 + 101N^7 + 128N^6 + 208N^5 + 190N^4 - 769N^3 - 1200N^2 - 212N - 48 \]

\[ P_{177} = 19N^8 + 70N^7 + 63N^6 - 41N^5 - 192N^4 - 221N^3 - 142N^2 - 60N - 72 \]

\[ P_{178} = 21N^8 + 42N^7 - 38N^6 - 360N^5 - 631N^4 - 730N^3 - 472N^2 - 216N - 48 \]

\[ P_{179} = 23N^8 + 2N^7 - 135N^6 + 29N^5 + 210N^4 - 151N^3 - 350N^2 - 132N - 72 \]

\[ P_{180} = 27N^8 - 36N^7 - 956N^6 - 1724N^5 + 187N^4 + 1288N^3 + 70N^2 - 224N - 72 \]

\[ P_{181} = 38N^8 + 146N^7 + 177N^6 + 35N^5 - 249N^4 - 373N^3 - 218N^2 - 60N - 72 \]
\[
\begin{align*}
P_{182} &= 41N^8 + 5N^7 - 195N^6 - 97N^5 + 326N^4 + 424N^3 + 208N^2 + 72N + 16 \\
439P_{183} &= 56N^8 + 194N^7 + 213N^6 + 83N^5 - 231N^4 - 469N^3 - 290N^2 - 60N - 72 \\
421P_{184} &= 79N^8 + 196N^7 + 132N^6 + 274N^5 + 465N^4 + 82N^3 + 332N^2 + 456N + 288 \\
439P_{185} &= 105N^8 + 978N^7 + 1688N^6 - 1330N^5 - 5245N^4 - 4672N^3 - 2212N^2 - 544N - 288 \\
421P_{186} &= 113N^8 + 348N^7 + 109N^6 - 289N^5 - 272N^4 - 859N^3 - 778N^2 - 172N + 72 \\
439P_{187} &= 170N^8 + 369N^7 - 521N^6 - 1393N^5 - 761N^4 - 952N^3 - 544N^2 + 32N + 144 \\
421P_{188} &= 264N^8 + 1407N^7 + 2246N^6 + 1746N^5 + 804N^4 - 1069N^3 - 674N^2 - 92N - 24 \\
439P_{189} &= 283N^8 + 838N^7 + 1482N^6 + 628N^5 - 1497N^4 - 1130N^3 - 772N^2 + 456N + 288 \\
421P_{190} &= 633N^8 + 2532N^7 + 5036N^6 + 6142N^5 + 4275N^4 + 1118N^3 - 176N^2 - 184N - 48 \\
439P_{191} &= N^9 + 21N^8 + 85N^7 + 105N^6 + 42N^5 + 290N^4 + 600N^3 + 456N^2 + 256N + 64 \\
421P_{192} &= 4N^9 + 53N^8 + 193N^7 + 233N^6 + 87N^5 + 554N^4 + 117N^3 + 904N^2 + 512N + 128 \\
439P_{193} &= 6N^9 + 93N^8 + 576N^7 + 1296N^6 + 586N^5 + 359N^4 + 2000N^3 + 1996N^2 \\
&\quad + 1488N + 384 \\
421P_{194} &= 9N^9 + 54N^8 + 56N^7 - 110N^6 - 381N^5 - 568N^4 - 364N^3 - 72N^2 + 128N + 96 \\
439P_{195} &= 9N^9 + 54N^8 + 167N^7 + 397N^6 + 780N^5 + 1241N^4 + 1448N^3 + 1200N^2 + 608N + 144 \\
421P_{196} &= 11N^9 + 78N^8 + 214N^7 + 335N^6 + 383N^5 + 571N^4 + 916N^3 + 876N^2 + 480N + 96 \\
439P_{197} &= 35N^9 + 150N^8 + 232N^7 + 137N^6 + 119N^5 + 661N^4 + 1174N^3 + 876N^2 + 480N + 96 \\
421P_{198} &= 37N^9 + 210N^8 - 52N^7 - 2738N^6 - 7249N^5 - 9368N^4 - 8216N^3 - 5888N^2 \\
&\quad - 2448N - 576 \\
439P_{199} &= 45N^9 + 270N^8 + 820N^7 + 1478N^6 + 1683N^5 + 1996N^4 + 2356N^3 + 2328N^2 \\
&\quad + 1408N + 288 \\
421P_{200} &= 57N^9 + 624N^8 + 1756N^7 + 1092N^6 - 1803N^5 - 1512N^4 + 966N^3 + 1116N^2 \\
&\quad + 920N + 528 \\
439P_{201} &= 69N^9 + 366N^8 + 1124N^7 + 1966N^6 + 2523N^5 + 5228N^4 + 7340N^3 + 5352N^2 \\
&\quad + 3008N + 672 \\
421P_{202} &= 94N^9 + 597N^8 + 1616N^7 + 2410N^6 + 1841N^5 + 1165N^4 + 2191N^3 + 3802N^2 \\
&\quad + 2916N + 648 \\
439P_{203} &= 121N^9 + 696N^8 + 1535N^7 + 1585N^6 + 416N^5 - 749N^4 - 836N^3 + 16N^2 \\
&\quad + 528N + 144 \\
421P_{204} &= 197N^9 + 1242N^8 + 2938N^7 + 3524N^6 + 2713N^5 + 2234N^4 + 3680N^3 + 6176N^2 \\
&\quad + 4080N + 864 \\
439P_{205} &= 439N^9 + 2634N^8 + 6008N^7 + 6694N^6 + 3545N^5 + 736N^4 + 2008N^3 + 6208N^2 \\
&\quad + 5136N + 1152 \\
421P_{206} &= 538N^9 + 3333N^8 + 7802N^7 + 7630N^6 + 458N^5 - 1415N^4 + 7786N^3 + 1234N^2 \\
&\quad + 5592N + 864 \\
439P_{207} &= 664N^9 + 3861N^8 + 9038N^7 + 11830N^6 + 9344N^5 + 3793N^4 + 3874N^3 + 1104N^2 \\
&\quad + 9624N + 2592 \\
421P_{208} &= 891N^9 + 4455N^8 + 16078N^7 + 28774N^6 + 37047N^5 + 45835N^4 + 42192N^3 + 28888N^2 \\
&\quad + 10640N + 1776 \\
439P_{209} &= 923N^9 + 5208N^8 + 11824N^7 + 12854N^6 + 2185N^5 - 7030N^4 + 1436N^3 + 15032N^2 \\
&\quad + 12864N + 3456 \\
421P_{210} &= 965N^9 + 4884N^8 + 10816N^7 + 20810N^6 + 36895N^5 + 40442N^4 + 27692N^3 + 22712N^2 \\
\end{align*}
\]
\[ P_{211} = 2N^{10} - 46N^9 - 98N^8 + 282N^7 + 1063N^6 + 1569N^5 + 1275N^4 + 403N^3 - 94N^2 - 108N - 24 \]  
\[ P_{212} = 2N^{10} + 12N^9 + 24N^8 + 11N^7 - 48N^6 - 151N^5 - 282N^4 - 480N^3 - 664N^2 - 576N - 288 \]  
\[ P_{213} = 11N^{10} + 44N^9 + 74N^8 + 196N^7 + 31N^6 - 1426N^5 - 3044N^4 - 2762N^3 - 1476N^2 - 480N - 96 \]  
\[ P_{214} = 11N^{10} + 76N^9 + 138N^8 - 204N^7 - 1041N^6 - 988N^5 + 752N^4 + 1896N^3 + 944N^2 - 384N - 576 \]  
\[ P_{215} = 37N^{10} + 392N^9 + 2106N^8 + 6514N^7 + 9211N^6 + 1258N^5 - 9218N^4 - 6116N^3 - 72N^2 - 752N - 192 \]  
\[ P_{216} = 85N^{10} + 425N^9 + 902N^8 + 932N^7 - 521N^6 - 685N^5 + 2022N^4 + 2928N^3 + 968N^2 - 1296N - 576 \]  
\[ P_{217} = 103N^{10} + 575N^9 + 1124N^8 - 334N^7 - 1505N^6 + 3755N^5 + 4926N^4 + 36N^3 - 472N^2 - 2160N - 864 \]  
\[ P_{218} = 118N^{10} + 425N^9 + 197N^8 + 86N^7 + 1240N^6 + 2489N^5 + 4401N^4 + 3480N^3 + 524N^2 - 1728N - 864 \]  
\[ P_{219} = 118N^{10} + 557N^9 + 461N^8 - 94N^7 + 1300N^6 + 3521N^5 + 4509N^4 + 1920N^3 - 1132N^2 - 2376N - 1008 \]  
\[ P_{220} = 127N^{10} + 536N^9 + 611N^8 + 602N^7 + 1474N^6 + 2099N^5 + 798N^4 - 2301N^3 - 4486N^2 - 3708N - 936 \]  
\[ P_{221} = 170N^{10} + 883N^9 + 2041N^8 + 2998N^7 - 448N^6 - 5465N^5 + 129N^4 + 6624N^3 + 1132N^2 - 2016N - 864 \]  
\[ P_{222} = 170N^{10} + 1213N^9 + 3235N^8 + 2794N^7 - 2692N^6 - 3767N^5 - 1293N^4 - 1632N^3 - 5324N^2 - 6240N - 2016 \]  
\[ P_{223} = 226N^{10} + 317N^9 - 811N^8 + 662N^7 + 4552N^6 + 3857N^5 + 3933N^4 + 2364N^3 + 236N^2 - 1656N - 720 \]  
\[ P_{224} = 489N^{10} + 2934N^9 + 9364N^8 + 18830N^7 + 18627N^6 + 124N^5 - 19856N^4 - 19296N^3 - 10640N^2 - 2880N - 1152 \]  
\[ P_{225} = 3N^{11} + 42N^{10} + 144N^9 + 74N^8 - 459N^7 - 1060N^6 - 1152N^5 - 1424N^4 - 1688N^3 - 1232N^2 - 736N - 192 \]  
\[ P_{226} = 11N^{11} + 37N^{10} - 27N^9 - 118N^8 + 21N^7 - 249N^6 - 1097N^5 - 1138N^4 + 552N^3 + 3448N^2 + 3456N + 2016 \]  
\[ P_{227} = 21N^{11} + 231N^{10} + 1334N^9 + 4086N^8 + 6277N^7 + 1775N^6 - 9488N^5 - 18076N^4 - 18208N^3 - 11344N^2 - 5568N - 1728 \]  
\[ P_{228} = 33N^{11} + 231N^{10} + 698N^9 + 1290N^8 + 1513N^7 + 1463N^6 + 2236N^5 + 5096N^4 + 7328N^3 + 5456N^2 + 3456N + 1152 \]  
\[ P_{229} = 45N^{11} + 383N^{10} + 958N^9 + 526N^8 - 763N^7 + 1375N^6 + 7808N^5 + 13028N^4 + 12976N^3 + 8016N^2 + 4608N + 1728 \]  
\[ P_{230} = 51N^{11} + 269N^{10} + 46N^9 - 1934N^8 - 3973N^7 - 875N^6 + 7364N^5 + 14972N^4 + 16768N^3 + 10896N^2 + 5376N + 1728 \]  
\[ P_{231} = 51N^{11} + 357N^{10} + 1238N^9 + 2586N^8 + 2755N^7 - 1435N^6 - 9212N^5 - 15028N^4 \]
\[ P_{232} = 81N^{11} + 483N^{10} + 1142N^9 + 1086N^8 - 767N^7 - 4645N^6 - 8936N^5 - 11980N^4 + 12352N^3 - 8272N^2 - 4800N - 1728 \]  
\[ P_{233} = 120N^{11} + 1017N^{10} + 2737N^9 + 1292N^8 - 8086N^7 - 20743N^6 - 24563N^5 - 16702N^4 - 6840N^3 + 120N^2 + 2432N + 960 \]  
\[ P_{234} = 243N^{11} + 1701N^{10} + 5378N^9 + 10350N^8 + 11479N^7 + 1193N^6 - 14684N^5 - 20572N^4 - 16288N^3 - 8944N^2 - 4992N - 1728 \]  
\[ P_{235} = 333N^{11} + 2331N^{10} + 6556N^9 + 9270N^8 + 5081N^7 - 6701N^6 - 17554N^5 - 20036N^4 - 15680N^3 - 9200N^2 - 5664N - 1728 \]  
\[ P_{236} = 753N^{11} + 4809N^{10} + 13174N^9 + 20466N^8 + 17717N^7 + 8629N^6 + 3908N^5 + 15304N^4 + 25408N^3 + 20272N^2 + 8448N + 1152 \]  
\[ P_{237} = 837N^{11} + 7757N^{10} + 30120N^9 + 68575N^8 + 119176N^7 + 191350N^6 + 262979N^5 + 258308N^4 + 163106N^3 + 63360N^2 + 14848N + 1536 \]  
\[ P_{238} = 1017N^{11} + 6195N^{10} + 14050N^9 + 12738N^8 - 2023N^7 - 5093N^6 + 27548N^5 + 69760N^4 + 80752N^3 + 54064N^2 + 20928N + 3456 \]  
\[ P_{239} = 3N^{12} + 21N^{11} + 17N^{10} - 202N^9 - 842N^8 - 1924N^7 - 3378N^6 - 5059N^5 - 6008N^4 - 4860N^3 - 2536N^2 - 960N - 192 \]  
\[ P_{240} = 9N^{12} + 63N^{11} + 38N^{10} - 414N^9 - 1035N^8 - 1341N^7 - 1511N^6 - 2972N^5 - 6011N^4 - 8038N^3 - 6892N^2 - 3432N - 864 \]  
\[ P_{241} = 9N^{12} + 63N^{11} + 71N^{10} - 381N^9 - 1536N^8 - 2529N^7 - 1946N^6 - 1331N^5 - 2096N^4 - 4036N^3 - 4144N^2 - 2304N - 576 \]  
\[ P_{242} = 39N^{12} + 585N^{11} + 2938N^{10} + 7136N^9 + 9083N^8 + 7745N^7 + 14668N^6 + 38246N^5 + 59856N^4 + 55560N^3 + 32144N^2 + 12480N + 2304 \]  
\[ P_{243} = 48N^{12} + 459N^{11} + 2322N^{10} + 8290N^9 + 20159N^8 + 30862N^7 + 28247N^6 + 16109N^5 + 9312N^4 + 7488N^3 + 4064N^2 + 1328N + 192 \]  
\[ P_{244} = 61N^{12} + 302N^{11} + 531N^{10} + 348N^9 - 349N^8 - 786N^7 + 457N^6 + 2524N^5 + 2012N^4 + 204N^3 - 360N^2 - 240N - 96 \]  
\[ P_{245} = 92N^{12} + 796N^{11} + 3089N^{10} + 7550N^9 + 10547N^8 + 1029N^7 - 19496N^6 - 24199N^5 - 8960N^4 + 736N^3 + 1744N^2 + 816N + 192 \]  
\[ P_{246} = 201N^{12} + 1845N^{11} + 6910N^{10} + 12854N^9 + 8915N^8 - 7741N^7 - 17126N^6 - 4294N^5 + 16260N^4 + 22080N^3 + 12416N^2 + 4128N + 576 \]  
\[ P_{247} = 239N^{12} + 1338N^{11} + 3137N^{10} + 3164N^9 - 983N^8 - 6640N^7 - 8123N^6 - 4526N^5 - 342N^4 + 1232N^3 + 848N^2 + 256N + 32 \]  
\[ P_{248} = 255N^{12} + 2169N^{11} + 6496N^{10} + 7694N^9 - 127N^8 - 6973N^7 + 4132N^6 + 25502N^5 + 31956N^4 + 22656N^3 + 9632N^2 + 864N - 576 \]  
\[ P_{249} = 581N^{12} + 7035N^{11} + 37826N^{10} + 112904N^9 + 190293N^8 + 174327N^7 + 92032N^6 + 69438N^5 + 78364N^4 + 44464N^3 + 11520N^2 - 3168N - 1728 \]  
\[ P_{250} = 825N^{12} + 7363N^{11} + 25396N^{10} + 40686N^9 + 26213N^8 - 12749N^7 - 55498N^6 - 89796N^5 - 110552N^4 - 134960N^3 - 127584N^2 - 64704N - 12672 \]  
\[ P_{251} = 69N^{13} + 420N^{12} + 794N^{11} - 1357N^{10} - 10401N^9 - 15678N^8 + 532N^7 + 239N^6 - 40018N^5 - 69432N^4 - 69152N^3 - 43792N^2 - 18336N - 3456 \]  
\[ P_{252} = 76N^{13} + 922N^{12} + 4479N^{11} + 9107N^{10} - 3747N^9 - 52973N^8 - 76133N^7 + 42261N^6 \]
\[ P_{253} = 295N^{13} + 2387N^{12} + 8005N^{11} + 13687N^{10} + 10883A^9 + 389N^8 - 2641N^7 + 6029N^6 + 11034N^5 + 6644N^4 + 1384N^3 + 80N^2 + 128N + 64 \]  
\[ P_{254} = 296N^{13} + 2368N^{12} + 10916N^{11} + 27006N^{10} + 23644N^9 - 19764N^8 - 61931N^7 - 63733N^6 - 52001N^5 - 56865N^4 - 38104N^3 - 26644N^2 + 7344N + 432 \]  
\[ P_{255} = 377N^{13} + 4649N^{12} + 21813N^{11} + 38539N^{10} - 39339N^9 - 272611N^8 - 332971N^7 + 220377N^6 + 801934N^5 + 384958N^4 - 362030N^3 - 297864N^2 - 1080N - 864 \]  
\[ P_{256} = 859N^{13} + 7376N^{12} + 25294N^{11} + 47088N^{10} + 63868N^9 + 80876N^8 + 63648N^7 - 35856N^6 - 146967N^5 - 157168N^4 - 91320N^3 - 34800N^2 - 8640N - 1152 \]  
\[ P_{257} = 1211N^{13} + 5680N^{12} + 3338N^{11} - 17355N^{10} - 31517N^9 - 48486N^8 - 139667N^7 - 278026N^6 - 340745N^5 - 138568N^4 - 34632N^2 + 9072N + 3888 \]  
\[ P_{258} = 70N^{14} + 555N^{13} + 1599N^{12} + 1192N^{11} - 4430N^{10} - 13305N^9 - 11835N^8 + 8440N^7 + 35816N^6 + 57126N^5 + 60340N^4 + 44646N^3 + 27808N^2 + 12768N + 2880 \]  
\[ P_{259} = 76N^{14} + 802N^{13} + 2979N^{12} + 1847N^{11} - 19377N^{10} - 58253N^9 - 26543N^8 + 170601N^7 + 362177N^6 + 225119N^5 - 103240N^4 - 193092N^3 - 137160N^2 - 117072N - 25920 \]  
\[ P_{260} = 76N^{14} + 1042N^{13} + 5979N^{12} + 16367N^{11} + 11883N^{10} - 47963N^9 - 126247N^8 - 86739N^7 + 36437N^6 + 22559N^5 - 51700N^4 + 24828N^3 + 132840N^2 + 116208N + 25920 \]  
\[ P_{261} = 4N^{15} + 50N^{14} + 267N^{13} + 1183N^{12} + 682N^{10} - 826N^9 - 1858N^8 - 1116N^7 + 457N^6 + 1500N^5 + 2268N^4 + 2400N^3 + 1392N^2 + 448N + 64 \]  
\[ P_{262} = 26N^{15} + 314N^{14} + 1503N^{13} + 3222N^{12} + 2510N^{11} + 1996N^{10} + 15041N^9 + 40728N^8 + 54008N^7 + 44956N^6 + 31936N^5 + 30416N^4 + 29568N^3 + 16704N^2 + 5376N + 768 \]  
\[ P_{263} = 101N^{15} + 1234N^{14} + 6867N^{13} + 21904N^{12} + 40098N^{11} + 32226N^{10} - 22057N^9 - 86972N^8 - 114557N^7 - 111416N^6 - 89204N^5 - 37312N^4 + 13392N^3 + 23040N^2 + 9792N + 1536 \]  
\[ P_{264} = 390N^{15} + 5121N^{14} + 30556N^{13} + 114173N^{12} + 321958N^{11} + 771597N^{10} + 1583594A^9 + 2637549N^8 + 3381542N^7 + 3199120N^6 + 2183360N^5 + 1123200N^4 + 489952N^3 + 178176N^2 + 48384N + 6912 \]  
\[ P_{265} = 75N^{16} + 1245N^{15} + 8291N^{14} + 27609N^{13} + 43437N^{12} + 14221N^{11} - 5995N^{10} + 182937N^9 + 488696N^8 + 296818N^7 - 452292N^6 - 730430N^5 - 186180N^4 + 259728N^3 + 241056N^2 + 116640N + 25920 \]  
\[ P_{266} = 115N^{16} + 1838N^{15} + 11820N^{14} + 36114N^{13} + 30900N^{12} - 133946N^{11} - 454068N^{10} - 457420N^9 + 249211N^8 + 864716N^7 + 312979N^6 - 634466N^5 - 587862N^4 - 19556N^3 + 104832N^2 + 9504N + 1728 \]  
\[ P_{267} = 185N^{16} + 2988N^{15} + 19694N^{14} + 62954N^{13} + 64470N^{12} - 207876N^{11} - 792388N^{10} - 861230N^9 + 437231N^8 + 1750616N^7 + 869954N^6 - 1016136N^5 - 1130122N^4 - 96596N^3 + 199872N^2 + 31104N + 1728 \]  
\[ P_{268} = 939N^{16} + 10527N^{15} + 37207N^{14} + 18679N^{13} - 202006N^{12} - 617170N^{11} - 930025N^{10} - 882917N^9 - 157123N^8 + 1388549N^7 + 2739376N^6 + 2837500N^5 + 2088640N^4 + 1259696N^3 + 622464N^2 + 211392N + 34560 \]  
\[ P_{269} = 1155N^{16} + 12417N^{15} + 37693N^{14} - 12293N^{13} - 285754N^{12} - 613900N^{11} - 571735N^{10} - 134309N^9 + 778901N^8 + 2698745N^7 + 4995724N^6 + 5915740N^5 + 4978144N^4 + 3161840N^3 + 1498752N^2 + 479808N + 76032 \]
The term $z$ differs from the one given in Eq. (B.7) in $z$-space in Ref. [9] by the term

$$C_F T_F^2 N_F \frac{4(N^2 + N + 2)}{N(N + 1)(N + 2)} [28\zeta_2 - 69]$$

in $N$-space. This result of [9] is based on the calculation carried out in Ref. [12], including the renormalization formulae derived there. We have checked, however, that our result Eq. (329) is in full agreement with Eq. (27) and the moments having been calculated by part of the present authors in Ref. [12]. The corresponding expression in $z$-space is presented in Appendix B.
5 The Asymptotic Wilson Coefficients for the Longitudinal Structure Function

The Wilson coefficients have been calculated in Ref. [22] for exclusive heavy flavor production, retaining three contributions only. In total also here five Wilson coefficients contribute and the expressions are slightly modified in the inclusive case of the complete structure function $F_L(x, Q^2)$, cf. [12]. In Mellin-$N$ space they read:

\[
L_{q,L}^{PS,(3)} = \frac{1}{2}[1 + (-1)^N]
\]

\[
\times \left\{ a_3^5 \left[ C_T A T^2 \left[ \frac{128N^2(N^2 + N + 2)}{3(N-1)N(N+1)^2(N+2)^2} + \frac{128(N^2 + N + 2)N^2}{3(N-1)N(N+1)^2(N+2)} \right] 
- \frac{256L_Q(11N^5 + 35N^4 + 59N^3 + 55N^2 - 4N - 12)}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{256(8N^3 + 13N^2 + 27N + 16)}{9(N-1)N(N+1)^3(N+2)} \right]
- \frac{256(N^2 + N + 2)S_1}{3(N-1)N(N+1)^2(N+2)^2} \right] L_M + \frac{64(N^2 + N + 2)(S_1^2 + S_2)}{3(N-1)N(N+1)^2(N+2)}
\]

\[
- \frac{128(8N^3 + 13N^2 + 27N + 16)S_1}{9(N-1)N(N+1)^3(N+2)} + \frac{128(43N^4 + 105N^3 + 224N^2 + 230N + 86)}{27(N-1)N(N+1)^4(N+2)} \right]
\]

\[
+ N_F^2 \hat{C}_{L,q}^{PS,(3)}(N_F) \right\},
\]

(333)

\[
L_{q,L}^S = \frac{1}{2}[1 + (-1)^N]
\]

\[
\times \left\{ a_3^2 \left[ \frac{64N^2T^2}{3(N+1)(N+2)} + a_3^5 \left[ \frac{256L^2}{9(N+1)(N+2)} + C_AT^2 \right] \right]
\left[ \frac{256(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128S_1}{3(N+1)(N+2)} \] L_Q + \frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3}
\right],
\]

\[
+ \frac{64Q_2}{9(N-1)N(N+1)^2(N+2)^2} + \frac{256(11N^3 - 6N^2 - 8N - 3)S_1}{9(N-1)N(N+1)^2(N+2)}
\]

\[
+ \frac{512(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{256S_1}{3(N+1)(N+2)}
\]

\[
+ \frac{1}{(N+1)(N+2)} \left[ \frac{128}{3} S_1^2 - \frac{128S_2}{3} - \frac{256}{3} S_{-2} \right] L_Q + \frac{32Q_5}{27(N-1)N^3(N+1)^2(N+2)^2}
\]

\[
- \frac{64(56N + 47)S_1}{27(N+1)^2(N+2)} + L_M \left[ \frac{256(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128S_1}{3(N+1)(N+2)} \right]
\]

\[
+ L_M \left[ \frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} + \frac{128Q_4}{9(N-1)N^2(N+1)^3(N+2)^3} \right]
\]

\[
+ \frac{256(N^3 - 6N^2 + 2N - 3)S_1}{9(N-1)N(N+1)^2(N+2)} + \frac{128}{3} S_1^2 - \frac{128S_2}{3} - \frac{256}{3} S_{-2} \right] L_Q + \frac{64(N^2 + N + 2)(N^4 + 2N^3 + 2N^2 + N + 6)}{3(N-1)N^2(N+1)^3(N+2)^2}
\]

\[
+ \frac{256(-1)^NQ_6}{45(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3}
\]

44
\[ L_{Q, L} = \frac{1}{2} \left[ 1 + (-1)^N \right] \]
\[ \times \left\{ a_2^2 C_F T_F \left[ \frac{16 L_Q}{3(N+1)} - \frac{8(19 N^2 + 7 N - 6)}{9 N(N+1)^2} - \frac{16 S_1}{3(N+1)} \right] \right. \]
\[ + a_3^2 C_F^2 T_F \left[ \frac{8(3 N^2 + 3 N + 2)}{N(N+1)^2} - \frac{32 S_1}{N+1} \right] \frac{L_Q^2}{N} \]
\[ + \left[ 256(-1)^N Q_{11} - \frac{32 Q_{12}}{45(N-2)(N-1)^2 N^2(N+1)^4(N+2)^2(N+3)^3} \right] \]
\[ - \frac{512(N^4 + 2 N^3 - N^2 - 2 N - 6) S_{-2}}{3(N-2)(N+1)^2(N+3)} + \frac{1}{N+1} \left[ \frac{128 S_3^2}{3} - \frac{512}{3} S_{-2} - \frac{128 S_2}{3} - \frac{256 S_3}{3} \right] \]
\[ - \frac{256}{3} S_{-3} + \frac{512}{3} S_{-2,1} + 256 \zeta_3 \right] L_Q + \frac{2 Q_{10}}{27 N^3(N+1)^4} + L_M^2 \left[ \frac{8(3 N^2 + 3 N + 2)}{3 N(N+1)^2} \right] \]

with

\[ Q_1 = 2 N^6 + 6 N^5 + 7 N^4 + 4 N^3 + 9 N^2 + 8 N + 12 \]
\[ Q_2 = 3 N^6 + 3 N^5 - 121 N^4 - 391 N^3 - 474 N^2 - 308 N - 80 \]
\[ Q_3 = 3 N^6 + 9 N^5 - N^4 - 17 N^3 - 38 N^2 - 28 N - 24 \]
\[ Q_4 = 6 N^7 + 24 N^6 + 47 N^5 + 104 N^4 + 219 N^3 + 316 N^2 + 208 N + 48 \]
\[ Q_5 = 15 N^8 + 60 N^7 + 572 N^6 + 1470 N^5 + 2135 N^4 + 1794 N^3 + 722 N^2 - 24 N - 72 \]
\[ Q_6 = N^{10} - 13 N^9 - 39 N^8 + 222 N^7 + 1132 N^6 + 1787 N^5 + 913 N^4 + 392 N^3 + 645 N^2 - 324 N - 108 \]
\[ Q_7 = 15 N^{10} + 75 N^9 + 112 N^8 + 14 N^7 - 61 N^6 + 107 N^5 + 170 N^4 + 36 N^3 - 36 N^2 - 32 N - 16 \]
\[ Q_8 = 45 N^{13} + 656 N^{12} + 4397 N^{11} + 17513 N^{10} + 43665 N^9 + 63005 N^8 + 27977 N^7 - 71993 N^6 - 140386 N^5 - 78985 N^4 + 25350 N^3 + 80460 N^2 + 100008 N + 38880 \]
\[ Q_9 = 95 N^{13} + 1218 N^{12} + 6096 N^{11} + 14484 N^{10} + 11570 N^9 - 28440 N^8 - 117844 N^7 - 225884 N^6 - 238953 N^5 - 83290 N^4 + 57660 N^3 + 122040 N^2 + 182304 N + 77760 \]
\[-32S_1 \left(\frac{1}{3(N+1)}\right) + L_M \left[\frac{8(3N^4 + 6N^3 + 47N^2 + 20N - 12)}{9N^2(N+1)^3} + \frac{64S_2 - 320S_1}{3N+1}\right] + \frac{896}{27} + \frac{160S_2 - 32S_1}{9N+1}\]  
\[+ C_F T_F^2 N_F \left[\frac{128L^2_Q}{9(N+1)} + \frac{128(19N^2 + 7N - 6)}{27N(N+1)^2} - \frac{256S_1}{9(N+1)}\right] + \frac{64(19N^2 + 7N - 6)}{27N(N+1)^2} - \frac{128S_1}{9(N+1)}\]  
\[= \frac{128(-1)^NQ_{11}}{15(N-2)(N+1)^2} + \frac{1}{15(N-2)(N+1)^2} + \frac{16Q_{13}}{135(N-1)^2} + \frac{1}{135(N-1)^2} + \frac{1}{N+1}\]  
\[+ \frac{128}{3}S_{-3} - \frac{256}{3}S_{-2,1} - 128\zeta_3\right] - \frac{352L^2_Q}{9(N+1)} + C_{L,q}^{NS,3}(N_F)\right\} \right),
\end{equation}

with

\begin{align}
Q_{10} & = 219N^6 + 657N^5 + 1193N^4 + 763N^3 - 40N^2 - 48N + 72 \\
Q_{11} & = 2N^{11} + 41N^{10} + 226N^9 + 556N^8 + 963N^7 + 2733N^6 + 7160N^5 + 8610N^4 + 1969N^3 \\
& - 2748N^2 - 864N - 216 \\
Q_{12} & = 180N^{12} + 2385N^{11} + 11798N^{10} + 23030N^9 - 10466N^8 - 131068N^7 - 245294N^6 \\
& - 196786N^5 - 22282N^4 + 86571N^3 + 50688N^2 - 7236N - 3888 \\
Q_{13} & = 2345N^{13} + 31510N^{11} + 163614N^{10} + 380250N^9 + 208092N^8 - 794874N^7 - 1604762N^6 \\
& - 833938N^5 + 451419N^4 + 584028N^3 + 113724N^2 - 36288N + 7776,
\end{align}

\begin{equation}
H_{q,\ell}^{PS} = \frac{1}{2}[1 + (-1)^N]\left\{a^2 C_F T_F \left[\frac{32S_1(N^2 + N + 2)}{(N-1)N(N+1)^2(N+2)} + \frac{32L_Q(N^2 + N + 2)}{(N-1)N(N+1)^2(N+2)}\right]
\right\} + \frac{32(N^5 + 2N^4 + 2N^3 - 5N^2 - 12N - 4)}{(N-1)N^2(N+1)^2(N+2)^2}\left\{C_F^2 T_F \left[\frac{64(N^2 + N + 1)(N^2 + N + 2)}{N-1)N^2(N+1)^3(N+2)}\right]
\right\} + \frac{64(N^2 + N + 2)S_1}{(N-1)N(N+1)^2(N+2)}\left[\frac{128(-1)^N(N^2 + N + 2)}{15(N-2)(N-1)^2N^3(N+1)^4(N+2)^2(N+3)^3}\right]
\right\} + \frac{32Q_{18}}{15(N-1)^3N^3(N+1)^4(N+2)^2(N+3)^3}\left[\frac{128(2N^5 + 5N^4 + 7N^3 + 2N^2 - 12N - 8)S_1}{(N-1)N^2(N+1)^3(N+2)^2}\right]
\right\} + \frac{(N^2 + N + 2)(64S^2_1 - 64S_2)}{(N-1)N(N+1)^2(N+2)}\left[\frac{128(N^2 + N + 2)S_{-2}}{(N-2)N(N+1)^2(N+3)}\right] + \frac{16(N^2 + N + 2)^2L_M^2}{(N-1)N^2(N+1)^3(N+2)^3}\left[\frac{16Q_{17}}{(N-1)N^4(N+1)^5(N+2)^3}\right] + \frac{32(N^5 + 5N^4 + 7N^3 - 4N + 4)L_M}{(N-1)N^3(N+1)^4(N+2)^2}\left[\frac{C_F T_F^2 N_F}{3(N-1)N(N+1)^2(N+2)}\right] + \frac{256L_Q(11N^5 + 35N^4 + 59N^3 + 55N^2 - 4N - 12)}{9(N-1)N^2(N+1)^3(N+2)^2}\left[\frac{128(N^2 + N + 2)}{3(N-1)N(N+1)^2(N+2)}\right] + \frac{256(11N^5 + 35N^4 + 59N^3 + 55N^2 - 4N - 12)L_Q}{9(N-1)N^2(N+1)^3(N+2)^2}\left[\frac{128(N^2 + N + 2)}{3(N-1)N(N+1)^2(N+2)}\right] + \frac{256(11N^5 + 35N^4 + 59N^3 + 55N^2 - 4N - 12)L_Q}{9(N-1)N^2(N+1)^3(N+2)^2}\left[\frac{128(N^2 + N + 2)}{3(N-1)N(N+1)^2(N+2)}\right] + \frac{256(11N^5 + 35N^4 + 59N^3 + 55N^2 - 4N - 12)L_Q}{9(N-1)N^2(N+1)^3(N+2)^2}\left[\frac{128(N^2 + N + 2)}{3(N-1)N(N+1)^2(N+2)}\right]
\right\}.
\end{equation}
\[
\begin{align*}
&\frac{128(43N^4 + 105N^3 + 224N^2 + 230N + 86)}{27(N - 1)N(N + 1)^4(N + 2)} - \frac{128(8N^3 + 13N^2 + 27N + 16)S_1}{9(N - 1)N(N + 1)^3(N + 2)} \\
&+ L_M\left[\frac{256(8N^3 + 13N^2 + 27N + 16)}{9(N - 1)N(N + 1)^3(N + 2)} - \frac{256(N^2 + N + 2)S_1}{3(N - 1)N(N + 1)^2(N + 2)}\right] \\
&+ \left(\frac{N^2 + N + 2}{(N - 1)N(N + 1)^2(N + 2)} \left[ \frac{64S_1^2}{3} + \frac{64S_3}{3} \right] \right) \\
&+ C_F C_A T_F \left[ \left( - \frac{32(N^2 + N + 2)(11N^4 + 22N^3 - 23N^2 - 34N - 12)}{3(N - 1)^2N^2(N + 1)^3(N + 2)^2} \right) - \frac{64(N^2 + N + 2)S_1}{(N - 1)N(N + 1)^2(N + 2)} \right] L_Q^2 + \left[ \frac{128(-1)^NQ_{14}}{(N - 1)N^2(N + 1)^4(N + 2)^3} \right] \\
&+ \left( \frac{64Q_{15}}{9(N - 1)^2N^3(N + 1)^3(N + 2)^3} + \frac{128(N^4 - N^3 - 4N^2 - 11N - 1)S_1}{(N - 1)^2N(N + 1)^3(N + 2)} \right) \\
&+ \left( \frac{(N^2 + N + 2)[128S_1^2 - 128S_2 - 256S_{-2}]}{(N - 1)N(N + 1)^2(N + 2)} \right) L_Q \right] + C_{L_q}^{PS,(3)}(N_F + 1) \right\}, \tag{349}
\end{align*}
\]

with
\[
Q_{14} = N^6 + 8N^5 + 30N^4 + 58N^3 + 65N^2 + 42N + 8 \tag{350}
\]
\[
Q_{15} = 142N^8 + 593N^7 + 801N^6 + 199N^5 - 1067N^4 - 900N^3 + 976N^2 + 1128N + 288 \tag{351}
\]
\[
Q_{16} = N^{10} - 13N^9 - 39N^8 + 222N^7 + 1132N^6 + 1787N^5 + 913N^4 + 392N^3 + 645N^2 - 324N - 108 \tag{352}
\]
\[
Q_{17} = N^{10} + 8N^9 + 29N^8 + 49N^7 - 11N^6 - 131N^5 - 161N^4 - 160N^3 - 168N^2 - 80N - 16 \tag{353}
\]
\[
Q_{18} = 225N^{12} + 2494N^{11} + 9980N^{10} + 14480N^9 - 11602N^8 - 68380N^7 - 86828N^6 - 15080N^5 + 67401N^4 + 60334N^3 - 312N^2 - 33912N - 12528, \tag{354}
\]

and
\[
H^S_{g,L} = \frac{1}{2}[1 + (-1)^N] \left\{ C_{TF}^{16} \left( \frac{64L_M T_F^2}{3(N + 1)(N + 2)} \right) + C_A T_F \left[ \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{(N + 1)^3(N + 2)^3} \right] \right\}
\]
\[
- \frac{32(2N^5 + 9N^4 + 5N^3 - 12N^2 - 20N - 8)}{(N - 1)N^2(N + 1)^2(N + 2)^3} + \frac{64(2N^3 - 2N^2 - N - 1)S_1}{(N - 1)N(N + 1)^2(N + 2)} \\
+ \left[ \frac{128(N^2 + N + 1)}{(N - 1)N(N + 1)^2(N + 2)^2} - \frac{64S_1}{(N + 1)(N + 2)} \right] + \frac{32S_1^2 - 32S_2 - 64S_{-2}}{(N + 1)(N + 2)} \right\} \\
+ \left[ \frac{16L_M(N^2 + N + 2)}{N(N + 1)^2(N + 2)} + \frac{16L_Q(N^2 + N + 2)}{N(N + 1)^2(N + 2)} \right] \\
+ \left[ \frac{16Q_{29}}{15(N - 1)^2N^2(N + 1)^3(N + 2)^3} + \frac{64(-1)^NQ_{30}}{15(N - 2)(N - 1)^2N^2(N + 1)^3(N + 2)^2(N + 3)^3} \right] \\
- \left[ \frac{16(3N^2 + 3N + 2)S_1}{N(N + 1)^2(N + 2)} + \frac{64(N - 1)S_{-2}}{(N - 2)(N + 1)(N + 3)} \right] \right\}
\]
\[
+ \left\{ a_0^{16} \left[ \frac{1}{9(N + 1)(N + 2)} + \frac{C_A T_F}{3(N - 1)N(N + 1)^2(N + 2)^2} \right. \right\} \left[ \frac{256(N^2 + N + 1)}{3(N - 1)N(N + 1)^2(N + 2)^2} - \frac{128S_1}{3(N + 1)(N + 2)} \right] L_Q^2 \tag{47}
\]
\[
\begin{align*}
&+ 256(-1)^N \left( N^3 + 4N^2 + 7N + 5 \right) + 64Q_{22} \\
&+ \frac{256(11N^3 - 6N^2 - 8N - 3)S_1}{9(N - 1)N(N + 1)^2(N + 2)} + L_M \left[ \frac{512(N^2 + N + 1)}{3(N - 1)N(N + 1)^2(N + 2)^2} - \frac{256S_1}{3(N + 1)(N + 2)} \right] \\
&+ \frac{128S_2^3 - \frac{128S_2}{3} - \frac{256}{3}S_{-2}}{(N + 1)(N + 2)} \left[ L_Q + \frac{256(N^2 + N + 1)}{3(N - 1)N(N + 1)^2(N + 2)^2} - \frac{128S_1}{3(N + 1)(N + 2)} + L_M \left[ \frac{256(-1)^N (N^3 + 4N^2 + 7N + 5)}{3(N + 1)^3(N + 2)^3} \right] \right] \\
&+ L_M^2 \left[ \frac{256(N^2 + N + 1)}{3(N - 1)N(N + 1)^2(N + 2)^2} - \frac{128S_1}{3(N + 1)(N + 2)} + L_M \left[ \frac{256(-1)^N (N^3 + 4N^2 + 7N + 5)}{3(N + 1)^3(N + 2)^3} \right] \right] \\
&+ \frac{128Q_{24}}{9(N - 1)N^2(N + 1)^3(N + 2)^3} + \frac{256(N^3 - 6N^2 + 2N - 3)S_1}{9(N - 1)N(N + 1)^2(N + 2)} + \frac{128S_2^3 - \frac{128S_2}{3} - \frac{256}{3}S_{-2}}{(N + 1)(N + 2)} \left[ L_Q \right] \\
&+ C_{AT}^2N_F \left[ \frac{256(N^2 + N + 1)}{3(N - 1)N(N + 1)^2(N + 2)^2} - \frac{128S_1}{3(N + 1)(N + 2)} \right] L_Q^2 \\
&+ \frac{256(-1)^N (N^3 + 4N^2 + 7N + 5)}{3(N + 1)^3(N + 2)^3} + 64Q_{22} \\
&+ \frac{256(11N^3 - 6N^2 - 8N - 3)S_1}{9(N - 1)N(N + 1)^2(N + 2)} + \frac{128S_2^3 - \frac{128S_2}{3} - \frac{256}{3}S_{-2}}{(N + 1)(N + 2)} \left[ L_Q \right] \\
&+ \frac{128Q_{24}}{3(N - 1)^2N^2(N + 1)^3(N + 2)^3} + \frac{32(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_1}{3(N - 1)N(N + 1)^2(N + 2)^2} \\
&+ \frac{32(59N^4 + 70N^3 - 155N^2 - 118N - 72)S_2^3}{3(N - 1)N(N + 1)^2(N + 2)^2} - \frac{256(-1)^N (N^3 + 4N^2 + 7N + 5)S_1}{(N + 1)^3(N + 2)^3} \\
&+ \frac{64Q_{28}S_1}{9(N - 1)^2N^2(N + 1)^3(N + 2)^3} - \frac{64(-1)^N Q_{25}}{3(N - 1)^2N^2(N + 1)^4(N + 2)^4} \\
&+ \frac{32Q_{34}}{9(N - 1)^2N^3(N + 1)^3(N + 2)^4} + \frac{32(11N^4 + 22N^3 - 83N^2 - 94N - 72)S_2}{3(N - 1)N(N + 1)^2(N + 2)^2} \\
&+ \frac{64(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_{-2}}{3(N - 1)N(N + 1)^2(N + 2)^2} + \frac{1}{(N + 1)(N + 2)} \left[ -128S_1^3 \right] \\
&+ 384S_2S_1 + 512S_{-2}S_1 + 128S_3 + 128S_{-3} - 256S_{-2,1} \right] L_Q \\
&+ C_{AT}^2T_F \left[ \frac{8(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^3(N + 2)} - \frac{32(N^2 + N + 2)S_1}{N(N + 1)^2(N + 2)} \right] L_Q^2 \\
&+ \frac{128(-1)^N (N^2 + N + 2)Q_{34}}{5(N - 2)(N - 1)^2N^3(N + 1)^5(N + 2)^3(N + 3)^3} - \frac{8Q_{39}}{5(N - 1)^2N^3(N + 1)^5(N + 2)^3(N + 3)^3} \\
&+ \frac{16(9N^4 + 26N^3 + 49N^2 + 48N + 12)S_1}{N^2(N + 1)^3(N + 2)} + L_M \left[ \frac{64(N^2 + N + 2)S_1}{N(N + 1)^2(N + 2)} \right] \\
&+ \frac{16(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N + 1)^3(N + 2)} + \frac{256(N^2 + N + 2)(N^4 + 2N^3 - N^2 - 2N - 6)S_{-2}}{(N - 2)N^2(N + 1)^3(N + 2)(N + 3)} \right]
\end{align*}
\]
\begin{align}
&+ \frac{128 Q_{33} S_1}{15(N-1)^2 N^2 (N+1)^3 (N+2)^2 (N+3)^3} - \frac{128 (N^4 + 2 N^3 + N^2 + 12) S_{-2} S_1}{(N-2) N (N+1)^2 (N+2) (N+3)} \\
&- \frac{64 (-1)^N Q_{41}}{45 (N-2) (N-1)^3 N^3 (N+1) (N+2)^3 (N+3)^3} + \frac{8 Q_{42}}{45 (N-1)^3 N^3 (N+1) (N+2)^3 (N+3)^3} \\
&- \frac{128 Q_{23} S_{-2}}{3 (N-2) N^2 (N+1)^3 (N+2) (N+3)} + \frac{176 (N^2 + N+2) L_M}{3 N (N+1)^2 (N+2)} \\
&+ \left( \frac{N^2 + N+2}{N (N+1)^2 (N+2)} \right) \left[ -32 S_2 + 64 S_3 + 64 S_{-3} - 128 S_{-2} - 192 \zeta_3 \right] L_Q \\
&+ \frac{16 (N^3 + 8 N^2 + 11 N + 2) S_1^2}{N (N+1)^2 (N+2)} - \frac{16 Q_{35}}{(N-1) N^4 (N+1)^5 (N+2)^4} - \frac{16 Q_{20} S_1}{N (N+1)^4 (N+2)^3} \\
&+ L_M \left[ \frac{32 (N^2 + N+2) S_1}{N (N+1)^2 (N+2)} - \frac{64 (N^2 + N+1) (N^2 + N+2)}{(N-1) N^2 (N+1)^3 (N+2)^2} \right] \\
&- \frac{16 (N^3 + 2 N^2 + 13 N^2 + 21 N^2 + 18 N + 16) S_2}{(N-1) N^2 (N+1)^3 (N+2)^2} + \frac{(N^2 - N - 4) 64 (-1)^N S_{-2}}{(N+1)^3 (N+2)^2} \\
&+ \frac{(N^2 + N+2)}{N (N+1)^2 (N+2)} \left[ \frac{16}{3} S_1^3 + 48 S_2 S_1 + 64 (-1)^N S_{-2} S_1 + \frac{128 S_3}{3} + 32 (-1)^N S_{-3} - 64 S_{-2,1} \right] \\
&+ \left( \frac{N^2 + N+2}{N (N+1)^2 (N+2)} \right) \frac{64 (-1)^N Q_{36}}{5 (N-2) (N-1)^2 N^3 (N+1)^5 (N+2)^3 (N+3)^3} \\
&+ \frac{8 Q_{40}}{45 (N-1)^2 N^3 (N+1)^5 (N+2)^3 (N+3)^3} - \frac{16 (23 N^4 + 92 N^3 + 209 N^2 + 256 N + 92) S_1}{3 N (N+1)^3 (N+2)^2} \\
&+ \frac{64 (N^2 + N+2) (3 N^4 + 6 N^3 - 7 N^2 - 10 N - 12) S_{-2}}{(N-2) N^2 (N+1)^3 (N+2) (N+3)} \\
&+ \left( \frac{N^2 + N+2}{N (N+1)^2 (N+2)} \right) \frac{32 S_1^2 - 128 S_{-2} S_1 + 32 S_2 - 64 S_3 - 64 S_{-3} + 128 S_{-2,1} + 192 \zeta_3}{N (N+1)^2 (N+2)} \right) \bigg] \bigg] \\
&+ C_{L,g} (N_F + 1) \bigg\} \bigg\} \bigg\} \\
\end{align}

with

\begin{align}
Q_{19} &= N^6 + 3 N^5 - 2 N^4 - 9 N^3 - 17 N^2 - 12 N - 12 \\
Q_{20} &= N^6 + 8 N^5 + 23 N^4 + 54 N^3 + 94 N^2 + 72 N + 8 \\
Q_{21} &= 2 N^6 + 6 N^5 + 7 N^4 + 4 N^3 + 9 N^2 + 8 N + 12 \\
Q_{22} &= 3 N^6 + 3 N^5 - 121 N^4 - 391 N^3 - 474 N^2 - 308 N - 80 \\
Q_{23} &= 10 N^6 + 30 N^5 + N^4 - 48 N^3 - 89 N^2 - 60 N - 36 \\
Q_{24} &= 6 N^7 + 24 N^6 + 47 N^5 + 104 N^4 + 219 N^3 + 316 N^2 + 208 N + 48 \\
Q_{25} &= 11 N^8 + 66 N^7 + 106 N^6 - 121 N^5 - 775 N^4 - 1325 N^3 - 1130 N^2 - 552 N - 96 \\
Q_{26} &= 12 N^8 + 52 N^7 + 132 N^6 + 216 N^5 + 191 N^4 + 54 N^3 - 25 N^2 - 20 N - 4 \\
Q_{27} &= 15 N^8 + 60 N^7 + 572 N^6 + 1470 N^5 + 2135 N^4 + 1794 N^3 + 722 N^2 - 24 N - 72 \\
Q_{28} &= 133 N^8 + 430 N^7 - 271 N^6 - 1361 N^5 + 110 N^4 + 2023 N^3 + 1684 N^2 - 12 N - 144 \\
Q_{29} &= 26 N^9 + 539 N^8 + 3244 N^7 + 8465 N^6 + 9342 N^5 + 841 N^4 - 5720 N^3 - 2193 N^2 \\
&+ 2484 N + 1404 \\
Q_{30} &= N^{10} - 13 N^9 - 39 N^8 + 222 N^7 + 1132 N^6 + 1787 N^5 + 913 N^4 + 392 N^3 \\
&+ 645 N^2 - 324 N - 108
\end{align}
\[
Q_{31} = 3N^{10} - 48N^9 - 856N^8 - 2702N^7 - 1961N^6 + 2142N^5 + 3122N^4 - 1924N^3 - 5552N^2 - 4032N - 1152
\]

\[
Q_{32} = 15N^{10} + 75N^9 + 112N^8 + 14N^7 - 61N^6 + 107N^5 + 170N^4 + 36N^3 - 36N^2 - 32N - 16
\]

\[
Q_{33} = 35N^{10} + 372N^9 + 1263N^8 + 673N^7 - 5090N^6 - 11596N^5 - 8413N^4 + 2305N^3 + 8049N^2 + 3078N + 108
\]

\[
Q_{34} = 2N^{11} + 41N^{10} + 226N^9 + 556N^8 + 963N^7 + 2733N^6 + 7160N^5 + 8610N^4 + 1969N^3 - 2748N^2 - 864N - 216
\]

\[
Q_{35} = 2N^{12} + 20N^{11} + 86N^{10} + 192N^9 + 199N^8 - N^7 - 297N^6 - 495N^5 - 514N^4 - 488N^3 - 416N^2 - 176N - 32
\]

\[
Q_{36} = 12N^{13} + 143N^{12} + 591N^{11} + 954N^{10} + 371N^9 + 1658N^8 + 11559N^7 + 26626N^6 + 29129N^5 + 14011N^4 - 2374N^3 - 6576N^2 - 1944N - 432
\]

\[
Q_{37} = 95N^{13} + 1218N^{12} + 6096N^{11} + 14484N^{10} + 11570N^9 - 28440N^8 - 117844N^7 - 225884N^6 - 238953N^5 - 83290N^4 + 57660N^3 + 122040N^2 + 182304N + 77760
\]

\[
Q_{38} = 185N^{13} + 2582N^{12} + 15584N^{11} + 53036N^{10} + 109190N^9 + 124040N^8 + 12604N^7 - 200836N^6 - 294247N^5 - 116270N^4 + 85260N^3 + 158760N^2 + 193536N + 77760
\]

\[
Q_{39} = 35N^{14} + 465N^{13} + 1962N^{12} - 348N^{11} - 32130N^{10} - 131686N^9 - 280396N^8 - 363984N^7 - 290209N^6 - 122547N^5 + 6730N^4 + 47316N^3 + 11928N^2 - 21600N - 5184
\]

\[
Q_{40} = 1255N^{14} + 18165N^{13} + 107824N^{12} + 331744N^{11} + 515430N^{10} + 132498N^9 - 1057432N^8 - 2202648N^7 - 1979173N^6 - 534079N^5 + 350880N^4 - 29088N^3 - 519264N^2 - 382320N - 62208
\]

\[
Q_{41} = 11N^{15} - 2N^{14} + 308N^{13} + 5275N^{12} + 24535N^{11} + 52925N^{10} + 50941N^9 - 5977N^8 - 85550N^7 - 191059N^6 - 294877N^5 - 248414N^4 - 64728N^3 + 57636N^2 + 28944N + 6480
\]

\[
Q_{42} = 1255N^{15} + 16338N^{14} + 76085N^{13} + 117654N^{12} - 198422N^{11} - 971844N^{10} - 1002678N^9 + 1019372N^8 + 352532N^7 + 3236906N^6 + 272625N^5 - 1523746N^4 - 632844N^3 + 606888N^2 + 635904N + 129600
\]

The expressions in \( z \)-space are presented in Appendix C.

As has been outlined for the 2–loop results in Ref. [10] already, the scales at which the asymptotic expressions are dominating are estimated to be \( Q^2/m^2 \approx 800 \). They are far outside the kinematic region in which the structure function \( F_L(x, Q^2) \) can presently be measured in deep-inelastic scattering. The corresponding expressions are therefore of merely theoretical character and cannot be used in current phenomenological analyses.

6 Comparison of Mellin Moments for the Wilson Coefficients and OMEs

In order to compare the relative impact of the different Wilson coefficients on the structure function \( F_2(x, Q^2) \) we will consider the Mellin moments for \( N = 2 \) to 10 in the following, folded
with the moments of the respective parton distribution functions in the flavor singlet case, i.e. the gluon \( G(x, Q^2) \) and quark–singlet density \( \Sigma(x, Q^2) \) for \( N_F = 3 \) and characteristic values of \( Q^2 \). Since only a series of Mellin moments has been calculated at large momentum transfer \( Q^2 \) in Ref. [12], a detailed numerical comparison is only possible in this way at the moment. The numerical results for the moments of the contributing parton densities are given in Table 2. Note that for \( N \geq 2 \) the moments for the singlet-distribution are mostly larger than those of the gluon. We apply these parton densities to study the relative contributions of the different Wilson coefficients, normalizing to \( H_{g,2}^S \) within the respective order in \( a_s \) using the following ratios:

\[
R(L_{g,2}^S, H_{g,2}^S) = \frac{c_{N_F} L_{g,2}^S G}{c_Q H_{g,2}^S G},
\]

\[
R(L_{q,2}^{PS}, H_{g,2}^S) = \frac{c_{N_F} L_{q,2}^{PS} \Sigma}{c_Q H_{g,2}^S G},
\]

\[
R(H_{q,2}^{PS}, H_{g,2}^S) = \frac{c_Q H_{q,2}^{PS} \Sigma}{c_Q H_{g,2}^S G},
\]

where

\[
c_{N_F} = \frac{1}{N_F} \sum_{k=0}^{N_F} e_k^2, \quad c_Q = e_Q^2. \tag{380}
\]

In the numerical examples we set \( e_Q = e_c = 2/3 \).

| \( Q^2 \) | \( 20 \text{ GeV}^2 \) | \( 100 \text{ GeV}^2 \) | \( 1000 \text{ GeV}^2 \) |
|---|---|---|---|
| \( N \) | 2 | 4 | 6 | 8 | 10 |
| \( G \) | 0.4583 | 0.0044 | 0.0003 | \( 3.62 \times 10^{-5} \) | \( 7.78 \times 10^{-6} \) |
| \( \Sigma \) | 0.5417 | 0.0353 | 0.0070 | \( 2.10 \times 10^{-3} \) | \( 8.01 \times 10^{-4} \) |
| \( N \) | 2 | 4 | 6 | 8 | 10 |
| \( G \) | 0.4819 | 0.0038 | 0.0002 | \( 3.60 \times 10^{-5} \) | \( 8.55 \times 10^{-6} \) |
| \( \Sigma \) | 0.5181 | 0.0296 | 0.0056 | \( 1.61 \times 10^{-3} \) | \( 5.97 \times 10^{-4} \) |
| \( N \) | 2 | 4 | 6 | 8 | 10 |
| \( G \) | 0.5042 | 0.0032 | 0.0002 | \( 3.14 \times 10^{-5} \) | \( 7.60 \times 10^{-6} \) |
| \( \Sigma \) | 0.4958 | 0.0244 | 0.0043 | \( 1.20 \times 10^{-3} \) | \( 4.32 \times 10^{-3} \) |

Table 2: The moments \( N = 2, \ldots, 10 \) of the gluon and quark-singlet momentum density using the parton distribution functions [5].
Before we discuss quantitative results, a remark on the contributions by the color factor $d_{abc}d_{abc}$ to the massless Wilson coefficients Refs. [62,64] and [11,57] used in the present analysis, is in order. For $SU(N)$ one obtains

$$d_{abc}d_{abc} = \frac{(N^2 - 1)(N^2 - 4)}{N}.$$

It emerges weighted by $1/N_c$ and $1/N_A$ for external quark and gluon lines, respectively, with $N_c = \bar{N}$ and $N_A = N^2 - 1$. In Refs. [62,64] this group-theoretic expression has been used, while in [11,57] a factor of 16 has been taken out and was absorbed into the Lorentz structure of the corresponding contribution to the Wilson coefficient. We agree with the $N_f$-dependence as given in Refs. [62,64]. Furthermore, we note a typographical error in Eq. (4.13) of [11]. Here,

| $Q^2$ | 20 GeV$^2$ | 100 GeV$^2$ | 1000 GeV$^2$ |
|-------|-----------------|-----------------|-----------------|
| $\Sigma/G$ | | | |
| $O(a_s^2)$ | $R(L_{g,2}^S, H_{g,2}^S)$ | $R(H_{g,2}^PS, H_{g,2}^S)$ | $R(L_{g,2}^PS, H_{g,2}^S)$ | $R(H_{g,2}^PS, H_{g,2}^S)$ |
| $O(a_s^3)$ | $R(L_{g,2}^S, H_{g,2}^S)$ | $R(H_{g,2}^PS, H_{g,2}^S)$ | $R(L_{g,2}^PS, H_{g,2}^S)$ | $R(H_{g,2}^PS, H_{g,2}^S)$ |
| $O(a_s^2)$ | $R(L_{g,2}^S, H_{g,2}^S)$ | $R(H_{g,2}^PS, H_{g,2}^S)$ | $R(L_{g,2}^PS, H_{g,2}^S)$ | $R(H_{g,2}^PS, H_{g,2}^S)$ |
| $O(a_s^3)$ | $R(L_{g,2}^S, H_{g,2}^S)$ | $R(H_{g,2}^PS, H_{g,2}^S)$ | $R(L_{g,2}^PS, H_{g,2}^S)$ | $R(H_{g,2}^PS, H_{g,2}^S)$ |

Table 3: Relative impact of the moments $N = 2, ..., 10$ of the individual massive Wilson coefficients, weighted by moments of the corresponding parton distributions [5], at $O(a_s^2)$ and $O(a_s^3)$ normalized to the contribution to $H_{g,2}^S$ for $Q^2 = 20, 100$ and 1000 GeV$^2$.  

...
the corresponding term reads correctly\footnote{12}

\[
\begin{equation}
\tag{382}
c_{2,q}^{(3)}(x) \simeq -932.089 N_F \frac{L_0}{x} \ldots \quad \text{with} \quad L_0 = \ln(x).
\end{equation}
\]

Also in the pure-singlet case the massless Wilson coefficients contain terms \( \propto d_{abc}d_{abc} \), although with a generally different charge-weight factor, cf. \([62–64]\).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
Q^2 & 20 \text{GeV}^2 \\
N & 2 & 4 & 6 & 8 & 10 \\
\Sigma/G & 1.1821 & 7.9967 & 25.847 & 57.965 & 103.06 \\
O(a_s) : R(A_{gg,Q}, A_{Qg}) & -1.0000 & -1.8182 & -2.5455 & -3.2432 & -3.9286 \\
O(a_s^2) : R(A_{gg,Q}, A_{Qg}) & -1.0000 & -1.6395 & -2.3808 & -3.1781 & -4.0262 \\
& R(A_{Qg}, A_{Qg}) & -0.1259 & -0.3656 & -0.7822 & -1.3339 & -1.9352 \\
& R(A_{Qg,Q}, A_{Qg}) & -0.0584 & -1.1306 & -6.3206 & -20.735 & -49.508 \\
& R(A_{gg,Q}, A_{Qg}) & 0.1843 & 1.1422 & 3.9565 & 9.8073 & 18.995 \\
\hline
O(a_s^2) : R(A_{gg,Q}, A_{Qg}) & -1.0051 & -1.3397 & -1.8466 & -2.4306 & -3.0890 \\
& R(A_{Qg}, A_{Qg}) & -0.1604 & -0.4838 & -0.9635 & -1.5449 & -2.1295 \\
& R(A_{Qg,Q}, A_{Qg}) & -0.0404 & -0.5832 & -2.9406 & -9.1817 & -21.375 \\
& R(A_{gg,Q}, A_{Qg}) & 0.1473 & 1.1265 & 3.7972 & 8.9925 & 16.961 \\
& R(A_{gg,Q}, A_{Qg}) & 0.0051 & -0.0202 & -0.0326 & -0.0445 & -0.0567 \\
& R(A_{ps,Q}, A_{Qg}) & 0.0534 & 0.1093 & 0.2460 & 0.4678 & 0.7619 \\
\hline
Q^2 & 100 \text{GeV}^2 \\
\Sigma/G & 1.0753 & 7.7514 & 22.797 & 44.660 & 69.888 \\
O(a_s) : R(A_{gg,Q}, A_{Qg}) & -1.0000 & -1.8182 & -2.5455 & -3.2432 & -3.9286 \\
O(a_s^2) : R(A_{gg,Q}, A_{Qg}) & -1.0000 & -1.7746 & -2.6448 & -3.5805 & -4.5771 \\
& R(A_{Qg}, A_{Qg}) & -0.1884 & -0.4246 & -0.7627 & -1.0887 & -1.3514 \\
& R(A_{Qg,Q}, A_{Qg}) & -0.1247 & -2.4385 & -12.377 & -35.460 & -74.510 \\
& R(A_{gg,Q}, A_{Qg}) & 0.3131 & 1.3950 & 3.9075 & 7.7692 & 12.560 \\
\hline
O(a_s^2) : R(A_{gg,Q}, A_{Qg}) & -1.0048 & -1.6120 & -2.3201 & -3.0734 & -3.8667 \\
& R(A_{Qg}, A_{Qg}) & -0.2799 & -0.5808 & -0.9473 & -1.2540 & -1.4615 \\
& R(A_{Qg,Q}, A_{Qg}) & -0.1772 & -2.8698 & -13.694 & -37.928 & -77.874 \\
& R(A_{gg,Q}, A_{Qg}) & 0.3924 & 1.4140 & 3.4078 & 6.0520 & 8.9253 \\
& R(A_{gg,Q}, A_{Qg}) & 0.0048 & -0.0375 & -0.0491 & -0.0580 & -0.0657 \\
& R(A_{ps,Q}, A_{Qg}) & 0.0647 & 0.0984 & 0.1726 & 0.2553 & 0.3319 \\
\hline
\end{array}
\]

Table 4: Relative impact of the moments \(N = 2, \ldots, 10\) of the individual massive OMEs, weighted by moments of the corresponding parton distributions \([5]\), at the different orders in \(a_s\) normalized to the contribution to \(A_{Qg}\) for \(Q^2 = 20\) and 100 GeV^2.

\footnote{12}The expression in the parameterization given at \url{http://www.liv.ac.uk/~avogt/} is correct, however.
Let us now consider the relative impact of the individual massive Wilson coefficients. The ratios at $O(a_s^2)$ and $O(a_s^3)$ for different values of $Q^2$ and the moments $N = 2$ to 10 are given in Table 3. One first notes that at low values of $Q^2$ the moments of $L_{g,2}^S$ change sign, which is also the case for $H_{g,2}^{PS}$ in the whole region up to $Q^2 = 1000\text{ GeV}^2$. At $O(a_s^2)$ $L_{g,2}^S$ is small for low moments and grows 24% for $N = 10$ compared to $H_{g,2}^{PS}$ at $Q^2 = 100\text{ GeV}^2$, with lower values at higher $Q^2$. A comparable tendency is observed at $O(a_s^3)$. The fraction $|R(H_{g,2}^{PS}, H_{g,2}^S)|$ moves between 25% and 170% comparing the moments $N = 2$ to 10 at $Q^2 = 20\text{ GeV}^2$ and upper values of $\sim 70\%$ at $Q^2 = 1000\text{ GeV}^2$.

In the case of the comparison of the massive OMEs we normalize to $A_{Qg}$ with PDFs according to their appearance in the singlet and gluon transitions from $N_F \rightarrow N_F + 1$ massless flavors in the variable flavor number scheme, cf. Eqs. (33–35):

$$R(A_{gg,Q}, A_{Qg}) = \frac{A_{gg,Q} G}{A_{Qg} G} \quad R(A_{Qg}^{PS}, A_{Qg}) = \frac{A_{Qg}^{PS} G}{A_{Qg} G}$$

$$R(A_{gg,Q}^{NS}, A_{Qg}) = \frac{A_{gg,Q}^{NS} G}{A_{Qg} G} \quad R(A_{Qg,Q}^{Qg}, A_{Qg}) = \frac{A_{Qg,Q}^{Qg} G}{A_{Qg} G}$$

These ratios describe the relative impact, within the corresponding order in $a_s$, of the massive OMEs in the variable flavor number scheme for the flavor singlet contributions.

| $Q^2$ | 1000 GeV$^2$ |
|---|---|
| | $N$ | 2 | 4 | 6 | 8 | 10 |
| | $\Sigma/G$ | 0.9833 | 7.5948 | 20.958 | 38.236 | 56.876 |
| $O(a_s)$: | $R(A_{gg,Q}, A_{Qg})$ | −1.0000 | −1.8182 | −2.5455 | −3.2432 | −3.9286 |
| | $R(A_{gg,Q}^{NS}, A_{Qg})$ | −1.0000 | −2.0101 | −3.0170 | −4.0596 | −5.1403 |
| | $R(A_{Qg,Q}^{Qg}, A_{Qg})$ | −0.2555 | −0.4521 | −0.6997 | −0.8850 | −1.0072 |
| | $R(A_{Qg,Q}^{Qg}, A_{Qg})$ | −0.2048 | −3.7111 | −16.978 | −44.037 | −85.945 |
| | $R(A_{Qg,Q}^{Qg}, A_{Qg})$ | 0.4603 | 1.5427 | 3.5816 | 6.1302 | 8.8857 |
| $O(a_s^2)$: | $R(A_{gg,Q}, A_{Qg})$ | −1.0054 | −1.7515 | −2.4980 | −3.2560 | −4.0291 |
| | $R(A_{Qg,Q}^{PS}, A_{Qg})$ | −0.3731 | −0.6275 | −0.9067 | −1.0906 | −1.1928 |
| | $R(A_{Qg,Q}^{NS}, A_{Qg})$ | −0.2930 | −4.1599 | −17.532 | −43.424 | −82.110 |
| | $R(A_{Qg,Q}^{Qg}, A_{Qg})$ | 0.5934 | 1.5060 | 2.9692 | 4.5100 | 5.9397 |
| | $R(A_{Qg,Q}^{Qg}, A_{Qg})$ | 0.0054 | −0.0469 | −0.0561 | −0.0624 | −0.0675 |
| | $R(A_{Qg,Q}^{Qg}, A_{Qg})$ | 0.0727 | 0.0902 | 0.1341 | 0.1721 | 0.2011 |

Table 5: The same as Table 4 for $Q^2 = 1000\text{ GeV}^2$.

The numerical values for different scales of $Q^2$ are given in Tables 4 and 5. $|A_{gg,Q}/A_{Qg}|$ rises from about 1 to higher values from $N = 2$ to 10, irrespectively of the values of $Q^2$ and the order in $a_s$. The smallest contributions are $|A_{gg,Q}|$ and $A_{Qg,Q}^{PS}$ contributing the ratios $R$ by $\sim 0.5\%$ to $5\%$ and
form $\sim 5$ to $\sim 10\%$, respectively, for $N = 2$ and $4$, i.e. in the region dominated by lower values of the Bjorken variable $x$. The OMEs $|A_{Q, q}^\text{PS}|$ and $A_{gq, Q}$ have contributions of $16$–$62\%$ and $14$–$150\%$, respectively, for $N = 2$ and $4$. Also the flavor non-singlet Wilson coefficient $|A_{qq, Q}^\text{NS}|$ contributes in the flavor singlet transitions and is weighted by the distribution $\Sigma$ here. Its relative impact rises with $Q^2$ and amounts from $\sim 4\%$ to $370\%$ for the $R$-ratio considering the lower moments $N = 2$ and $N = 4$ only.

Right after having obtained a series of moments for the massive OMEs at 3–loops in [12], it became clear that the logarithmic contributions are of comparable order to the constant term. Moreover, there is a strong functional dependence w.r.t. $N$, as displayed in Tables [2–5]. To obtain a definite answer, the calculation of the constant parts of the unrenormalized OMEs $a_{ij}^{(3)}$ as a function of $N \in \mathbb{C}$ is necessary. In particular predictions for the range of small values $x \simeq 10^{-4}$ appear to be rather difficult otherwise.

7 Conclusions

We have derived the contributions of the massive Wilson coefficients to the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ in deep-inelastic scattering and the corresponding massive OMEs to 3–loop order in the asymptotic region $Q^2 \gg m^2$ both in Mellin–$N$ and $z$–space except for the constant parts $a_{ij}$ of the unrenormalized OMEs, which are not known for all quantities yet. Here, we retained both the scale-dependence due to the virtuality $Q^2$ and the factorization and renormalization scales $\mu^2$, which were set equal. This allows for scale variation studies in applications. Two of the Wilson coefficients, $L_{q, q}^\text{PS}$ and $L_{g, q}^\text{S}$, are known in complete form, and the corresponding results for $L_{g, q}^\text{NS}$ will be given in [40]. In the variable flavor number scheme being applied to describe the process through which an initially massive quark transmutes into a massless one at high momentum scales, moreover, the matching coefficients $A_{ij}$ are needed. Here, $A_{q, q}^\text{PS}$ and $A_{gq, Q}$ are known in complete form to 3–loop order and the results for $A_{gq}$ and $A_{gq}^\text{NS}$ are given in [65] and [40], respectively.

We have given numerical results for the Wilson coefficients $L_{q, q}^\text{PS}$ and $L_{g, q}^\text{S}$. Using the available Mellin moments we have performed a numerical comparison of the different Wilson coefficients and operator matrix elements inside the respective order in the coupling constant for the moments $N = 2$ to 10 and in the $Q^2$ range between 20 and 1000 GeV$^2$. While some of the quantities studied are of minor importance, several others of the Wilson coefficients and OMEs are of similar size, which is varying in the kinematic range of experimental interest for present and future precision measurements. Even in case of the charm-quark contributions the logarithmic terms are not dominant over the constant contributions in wide kinematic ranges, as. e.g. at HERA.

The expression which were derived in the present paper are available in form of computer-readable files on request via e-mail to Johannes.Bluemlein@desy.de.

A The massive operator matrix elements in $N$-space

In this appendix we present the massive OMEs in Mellin–space to be used in the matching coefficients in the variable flavor number scheme Eqs. (32–35). The corresponding representations in $z$–space are given in Appendix D. Thus far the OMEs $A_{q, q}^\text{PS}$ and $A_{gq, Q}$ are known completely. The other OMEs are presented except for the 3–loop constant part $a_{ij}$ in the unrenormalized OMEs. The OMEs $A_{q, q}^\text{NS}$ and $A_{gq, Q}^\text{S}$ are presented elsewhere [40,65].
The transition matrix elements are given by $A_{99,Q}^{PS}$ and $A_{qg,Q}^{PS}$:

$$
A_{99,Q}^{PS} = \frac{1}{2} [1 + (-1)^N] \times \left\{ a_3^2 C_F N_F T_F^2 \left[ L_M^3 \left( \frac{32(N^2 + N + 2)^2 S_1}{3(N - 1)N^3(N + 1)^3(N + 2)^2} - \frac{32P_{280}}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \right) + \frac{32P_{282}}{27(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{64P_{280}S_1}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \right] + \frac{(N^2 + N + 2)^2 \left[ -\frac{32}{3} S_1^2 - \frac{32S_2}{3} \right]}{(N - 1)N^3(N + 1)^3(N + 2)^2} \right\},
$$

(383)

with

$$
P_{280} = 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24
$$

(384)

$$
P_{281} = 40N^7 + 185N^6 + 430N^5 + 521N^4 + 452N^3 + 404N^2 - 16N - 96
$$

(385)

$$
P_{282} = 95N^{10} + 712N^9 + 2379N^8 + 4269N^7 + 4763N^6 + 4569N^5 + 3309N^4 + 200N^3
$$

$$
- 808N^2 - 48N + 144
$$

(386)

$$
P_{283} = 233N^{10} + 1744N^9 + 5937N^8 + 11454N^7 + 14606N^6 + 15396N^5 + 12030N^4 + 3272N^3
$$

$$
- 928N^2 - 96N + 288
$$

(387)

$$
P_{284} = 1330N^{13} + 13931N^{12} + 66389N^{11} + 187681N^{10} + 354532N^9 + 492456N^8 + 532664N^7
$$

$$
+ 423970N^6 + 204541N^5 + 34274N^4 - 11704N^3 - 3408N^2 - 1008N - 864
$$

(388)

and

$$
A_{qg,Q}^{PS} = \frac{1}{2} [1 + (-1)^N] \times \left\{ a_3^2 C_F N_F T_F^2 \left[ L_M^3 \left( \frac{8(N^2 + N + 2)P_{285}}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{8}{9}\gamma_{qg}^0 S_1 \right) \right] + \frac{4P_{291}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} - \frac{32(5N^3 + 8N^2 + 19N + 6)S_1}{9N^2(N + 1)(N + 2)} + \gamma_{qg}^0 \left[ -\frac{4}{3} S_1^2 - \frac{4S_2}{3} \right] \right\},
$$

(389)

$$
P_{291} = 16(10N^3 + 13N^2 + 29N + 6)S_1^2
$$

$$
\frac{9N^2(N + 1)(N + 2)}{27N^2(N + 1)^2(N + 2)} - \frac{16(103N^4 + 257N^3 + 594N^2 + 524N + 120)S_1}{27N^2(N + 1)^2(N + 2)}
$$

$$
\frac{4P_{293}}{27(N - 1)N^5(N + 1)^5(N + 2)^4} + \frac{16(10N^3 + 25N^2 + 29N + 6)S_2}{9N^2(N + 1)(N + 2)}
$$

$$
+ \gamma_{qg}^0 \left[ \frac{4}{9} S_1^3 + \frac{4}{3} S_2 S_1 - \frac{16S_3}{9} \right] L_M + \frac{8(215N^4 + 481N^3 + 930N^2 + 748N + 120)S_2^2}{81N^2(N + 1)^2(N + 2)}
$$

$$
\frac{64}{9} \frac{(N^2 + N + 2)P_{285}S_3}{(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{P_{285}}{243(N - 1)N^6(N + 1)^6(N + 2)^5}
$$

$$
\frac{16(1523N^5 + 5124N^4 + 11200N^3 + 14077N^2 + 7930N + 1344)S_1}{243N^2(N + 1)^2(N + 2)}
$$

$$
\frac{8(109N^4 + 291N^3 + 478N^2 + 324N + 40)S_2}{27N^2(N + 1)^2(N + 2)} + \frac{(10N^3 + 13N^2 + 29N + 6)\left[ -\frac{16}{81} S_1^3 - \frac{16}{27} S_2 S_1 \right]}{N^2(N + 1)(N + 2)}
$$

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\[
+ \frac{32(5N^3 - 16N^2 + N - 6)S_3}{81N^2(N + 1)(N + 2)} + \gamma_0^{(q)} \left[ - \frac{1}{27} S_1^4 - \frac{2}{9} S_2 S_1^2 - \frac{8}{27} S_3 S_1 - \frac{64}{9} \zeta_3 S_1 - \frac{1}{9} S_2^2 + \frac{14}{9} S_4 \right] \\
+ C_{ANF} T_F \left[ L_M^3 \left( - \frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N - 1)N^2(N + 1)^2(N + 2)^2} - \frac{8}{9} \gamma_0^{(q)} S_1 \right) \right] \\
+ \left[ \frac{8P_{289}}{9(N - 1)N^2(N + 1)^3(N + 2)^3} + \frac{32(5N^4 + 20N^3 + 47N^2 + 58N + 20)S_1}{9N(N + 1)^2(N + 2)^2} \right] \\
+ \left[ \frac{16P_{292}}{27(N - 1)N^4(N + 1)^4(N + 2)^4} - \frac{32(5N^4 + 26N^3 + 47N^2 + 43N + 20)S_2}{9N(N + 1)^2(N + 2)^2} \right] \\
+ \left[ \frac{16P_{289}S_1}{27N(N + 1)^3(N + 2)^3} - \frac{64(5N^2 + 8N + 10)S_2}{9N(N + 1)(N + 2)} + \gamma_0^{(q)} \left[ - \frac{4}{9} S_1^3 + \frac{4}{3} S_2 S_1 - \frac{8}{9} S_3 S_1 - \frac{16}{3} S_3 \right] \\
+ \left[ \frac{512}{9} (N^2 + N + 1)(N^2 + N + 2) \right] \left[ \frac{1}{N(N - 1)^3}(N + 1)^2(N + 2)^2 + \frac{16P_{290}S_1}{243(N - 1)N^2(N + 1)^4(N + 2)^4} \right] \\
+ \left[ \frac{16P_{288}S_2}{81N(N + 1)^3(N + 2)^3} + \frac{64(5N^4 + 38N^3 + 59N^2 + 31N + 20)S_3}{81N(N + 1)^2(N + 2)^2} \right] \\
+ \left[ \frac{32(21N^2 + 293N^2 + 414N + 224)S_2}{27N(N + 1)(N + 2)} - \frac{128(5N^2 + 8N + 10)S_3}{27N(N + 1)(N + 2)} \right] \\
+ \frac{(5N^4 + 20N^3 + 41N^2 + 49N + 20) \left[ \frac{32}{81} S_1^3 - \frac{32}{27} S_2 S_1 + \frac{128}{27} S_2 S_1 \right]}{N(N + 1)^2(N + 2)^2} + \gamma_0^{(q)} \left[ \frac{1}{27} S_1^4 - \frac{2}{9} S_2 S_1^2 \right] \\
+ \left[ \frac{16}{9} S_2 S_1 - \frac{40S_3}{27} \right] \left[ S_1 + \frac{64}{9} \zeta_3 S_1 + \frac{1}{9} S_2^2 + \frac{14}{9} S_4 + \frac{32}{9} S_3 - \frac{32}{9} S_3 S_1 - \frac{16}{9} S_{2,1,1} \right] \right),
\]
with the polynomials

\[P_{285} = 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24\]
\[P_{286} = 94N^6 + 631N^5 + 2106N^4 + 4243N^3 + 4878N^2 + 2812N + 680\]
\[P_{287} = 103N^6 + 694N^5 + 2148N^4 + 3991N^3 + 4494N^2 + 2704N + 680\]
\[P_{288} = 139N^6 + 1093N^5 + 3438N^4 + 5776N^3 + 5724N^2 + 3220N + 752\]
\[P_{289} = 9N^8 + 54N^7 + 56N^6 - 182N^5 - 717N^4 - 1120N^3 - 1012N^2 - 672N - 160\]
\[P_{290} = 1244N^{10} + 10557N^9 + 40547N^8 + 90323N^7 + 114495N^6 + 49344N^5 - 69902N^4 - 115200N^3 - 64352N^2 - 11264N + 864\]
\[P_{291} = 33N^{11} + 231N^{10} + 698N^9 + 1290N^8 + 1513N^7 + 1463N^6 + 2236N^5 + 5096N^4 + 7328N^3 + 5456N^2 + 3456N + 1152\]
\[P_{292} = 99N^{12} + 891N^{11} + 2902N^{10} + 3392N^9 + 4300N^8 - 20914N^7 - 33059N^6 - 28357N^5 - 114064N^4 + 3840N^3 + 7568N^2 + 4176N + 864\]
\[P_{293} = 159N^{14} + 1590N^{13} + 7223N^{12} + 20982N^{11} + 43703N^{10} + 65162N^9 + 62553N^8 + 30282N^7 - 28286N^6 - 145968N^5 - 257720N^4 - 241760N^3 - 158112N^2 - 73728N - 17280\]
\[P_{294} = 3315N^{15} + 39780N^{14} + 194011N^{13} + 471164N^{12} + 416251N^{11} + 860568N^{10} - 3525799N^9 - 6015120N^8 - 6333994N^7 - 4373672N^6 - 1907512N^5 - 499824N^4 - 217952N^3 - 264192N^2 - 160128N - 34560\]
Next we present the OMEs, which are known except for the constant term in the unrenormalized massive OME at 3-loop order, $a^{(3)}_{ij}$. The matrix element $A_{qq}^{PS}$ is given by:

\[
P_{295} = 13923N^{17} + 180999N^{16} + 1064857N^{15} + 3812487N^{14} + 9348807N^{13} + 16391845N^{12} + 20248499N^{11} + 17070917N^{10} + 11536274N^9 + 11303496N^8 + 13846104N^7 + 16104128N^6 + 22643488N^5 + 29337472N^4 + 26395008N^3 + 15388416N^2 + 5612544N + 995328.
\]
\[
\begin{align*}
&\frac{16(N^2 + N + 2)(8N^3 + 13N^2 + 27N + 16)S_1^2}{9(N - 1)N^2(N + 1)^3(N + 2)} + \frac{32P_{298}\zeta_2}{9(N - 1)N^3(N + 1)^2(N + 2)^2} + \frac{32P_{232}}{81(N - 1)N^5(N + 1)^5(N + 2)} - \frac{32(N^2 + N + 2)(43N^4 + 105N^3 + 224N^2 + 230N + 86)S_1}{27(N - 1)N^2(N + 1)^4(N + 2)} + \\
&\frac{16P_{310}S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{(N^2 + N + 2)^2 \left[ -\frac{16}{9}S_1^3 - \frac{16}{3}S_2S_1 - \frac{32}{3}S_2^2S_1 + \frac{2S_3}{9} + \frac{128}{9}\zeta_3 \right]}{(N - 1)N^2(N + 1)^2(N + 2)} + \\
&C_F T_F^2 N_F \left[ -\frac{16}{9} \frac{P_{320}\zeta_2}{(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{L_M^2}{32P_{328}} \left[ \frac{2N^2 + N + 2}{-27(N - 1)N^4(N + 1)^4(N + 2)^3} \left[ \frac{16}{3}S_1^3 - \frac{80S_3}{9}\right] \right] \right] + \\
&\frac{32P_{329}}{3(N - 1)N^5(N + 1)^5(N + 2)} + \frac{32(N^2 + N + 2)^2 L_M^2}{64(N^2 + 5N + 2)(5N^3 + 7N^2 + 4N + 4)S_2} + \frac{32P_{326}S_1}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \\
&\frac{4(\frac{N^2 + N + 2}{P_{301}}P_{300}\zeta_2)}{3(N - 1)^2N^3(N + 1)^4(N + 2)^2S_1 - \frac{8(N^2 + N + 2)}{(N - 1)N^2(N + 1)^5(N + 2)^4}S_3} + \frac{8(N^2 + N + 2)P_{300}S_3}{8(N^2 + N + 2)P_{300}S_3} + \\
&\frac{4P_{317}S_2}{9(N - 1)N^2(N + 1)^3(N + 2)^2} + \frac{16(N^2 + N + 2)P_{332}S_2}{3(N - 1)^2N^3(N + 1)^3(N + 2)^2} + \\
&\frac{16S_2 + 32S_{-2}}{(N - 1)N^2(N + 1)^2(N + 2)} + \frac{8P_{302}S_1}{3(N - 1)^2N^3(N + 1)^3(N + 2)^2} + \\
&\frac{4P_{317}S_2}{9(N - 1)^2N^4(N + 1)^4(N + 2)^3} - \frac{16(N^2 + N + 2)P_{332}S_2}{3(N - 1)^2N^3(N + 1)^3(N + 2)^2} + \\
&\frac{16S_2 + 32S_{-2}}{(N - 1)N^2(N + 1)^2(N + 2)} + \frac{8P_{302}S_1}{3(N - 1)^2N^3(N + 1)^3(N + 2)^2} - \\
&\frac{8P_{319}}{9(N - 1)^2N^4(N + 1)^4(N + 2)^3} \left[ (N^2 + N + 2)(N^4 + 2N^3 + 7N^2 + 22N + 20) [32(-1)^NS_{-2} + 16(-1)^N\zeta_2] \right] + \\
&\frac{(N^2 - N - 4)(N^2 + N + 2)}{(N - 1)N^2(N + 1)^3(N + 2)^2} \left[ -64(-1)^NS_{-1}S_{-2} - 32(-1)^NS_{-3} + 64S_{-1} - 32(-1)^NS_1 \zeta_2 \right] - \\
&\frac{24(-1)^N\zeta_3}{(N - 1)N^2(N + 1)^2(N + 2)} \left[ -\frac{2}{3}S_1^3 - 20S_2S_1^2 - 32(-1)^NS_{-3}S_1 + (64S_{-1} - \\
&\frac{160S_5}{3})S_1 - \frac{8}{3} \left[ 7 + 9(-1)^N\right] \zeta_3S_1 - 2S_2^2 - S_{-2}(-32(-1)^N) \right] \right]
\end{align*}
\]
with the polynomials

\begin{align*}
P_{296} &= \ N^6 + 6N^5 + 77N^4 + 24N^3 + 18N^2 + 16N - 8 \\
P_{297} &= \ 7N^6 + 15N^5 + 77N^4 - 23N^3 - 26N^2 - 20N - 8 \\
P_{298} &= \ 8N^6 + 29N^5 + 84N^4 + 193N^3 + 162N^2 + 124N + 24 \\
P_{299} &= \ 11N^6 + 6N^5 + 75N^4 + 68N^3 - 200N^2 - 80N - 24 \\
P_{300} &= \ 11N^6 + 29N^5 - 7N^4 - 25N^3 - 56N^2 - 72N - 24 \\
P_{301} &= \ 17N^6 + 27N^5 + 75N^4 + 149N^3 - 20N^2 - 80N - 24 \\
P_{302} &= \ 17N^6 + 51N^5 + 51N^4 + 89N^3 + 40N^2 - 80N - 24 \\
P_{303} &= \ 27N^6 + 102N^5 + 135N^4 + 56N^3 - 8N^2 - 20N - 8 \\
P_{304} &= \ 38N^6 + 108N^5 + 151N^4 + 106N^3 + 21N^2 - 28N - 12 \\
P_{305} &= \ 73N^6 + 189N^5 + 45N^4 + 31N^3 - 238N^2 - 412N - 120 \\
P_{306} &= \ 2N^7 + 14N^6 + 37N^5 + 102N^4 + 155N^3 + 158N^2 + 132N + 40 \\
P_{307} &= \ 3N^7 - 15N^6 - 133N^5 - 449N^4 - 658N^3 - 500N^2 - 296N - 96 \\
P_{308} &= \ 8N^7 + 37N^6 + 68N^5 - 11N^4 - 86N^3 - 56N^2 - 104N - 48 \\
P_{309} &= \ 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24 \\
P_{310} &= \ 8N^7 + 37N^6 + 158N^5 + 565N^4 + 796N^3 + 628N^2 + 400N + 96 \\
P_{311} &= \ 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \\
P_{312} &= \ N^{10} + 8N^9 + 29N^8 + 49N^7 - 11N^6 - 131N^5 - 161N^4 - 160N^3 - 168N^2 - 80N - 16 \\
P_{313} &= \ 19N^{10} + 143N^9 + 427N^8 + 567N^7 + 454N^6 + 822N^5 + 1560N^4 + 1784N^3 + 1488N^2 + 768N + 192 \\
P_{314} &= \ 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3 + 16N^2 + 672N + 288 \\
P_{315} &= \ 43N^{10} + 320N^9 + 1059N^8 + 1914N^7 + 2431N^6 + 2874N^5 + 2379N^4 + 820N^3 + 352N^2 + 336N + 144 \\
P_{316} &= \ 136N^{10} + 647N^9 + 1110N^8 - 438N^7 - 2555N^6 - 2106N^5 - 3105N^4 - 3167N^3 + 418N^2 + 924N + 72 \\
P_{317} &= \ 3N^{11} + 66N^{10} + 104N^9 - 1152N^8 - 3801N^7 - 2510N^6 + 3318N^5 + 8076N^4 + 9608N^3 \end{align*}
\[
P_{318} = 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 - 473N^5 - 682N^4 - 904N^3 - 592N^2 - 208N - 32
\]

\[
P_{319} = 118N^{11} + 793N^{10} + 2281N^9 + 3402N^8 + 2428N^7 + 1457N^6 + 1917N^5 + 2476N^4 + 4392N^3 + 4976N^2 + 2832N + 576
\]

\[
P_{320} = 127N^{11} + 820N^{10} + 2251N^9 + 2196N^8 - 1109N^7 - 934N^6 + 4491N^5 + 9334N^4 + 12552N^3 + 9680N^2 + 4656N + 864
\]

\[
P_{321} = 37N^{12} + 305N^{11} + 1107N^{10} + 2328N^9 + 3520N^8 + 5020N^7 + 7642N^6 + 10519N^5 + 10938N^4 + 8248N^3 + 4656N^2 + 1712N + 288
\]

\[
P_{322} = 248N^{13} + 2599N^{12} + 12793N^{11} + 39593N^{10} + 87182N^9 + 148026N^8 + 196942N^7 + 192416N^6 + 128195N^5 + 63406N^4 + 32344N^3 + 15984N^2 + 5616N + 864
\]

\[
P_{323} = 4N^{14} + 56N^{13} + 443N^{12} + 2139N^{11} + 6049N^{10} + 10762N^9 + 13272N^8 + 11692N^7 + 6106N^6 + 339N^5 - 1254N^4 - 72N^3 + 496N^2 + 240N + 32
\]

\[
P_{324} = 686N^{14} + 6560N^{13} + 25572N^{12} + 43489N^{11} + 9045N^{10} - 72944N^9 - 125240N^8 - 156761N^7 - 206883N^6 - 241600N^5 - 250212N^4 - 225808N^3 - 150864N^2 - 56448N - 8640
\]

\[
P_{325} = 12N^{17} + 162N^{16} + 1030N^{15} + 4188N^{14} + 11527N^{13} + 19051N^{12} + 11176N^{11} + 17182N^{10} - 36527N^9 - 27469N^8 - 111770N^7 + 5554N^6 + 32640N^5 + 4656N^4 + 34528N^3 + 14816N^2 + 3584N + 384
\]

\[
P_{326} = 8N^7 + 37N^6 + 68N^5 - 11N^4 - 86N^3 - 56N^2 - 104N - 48
\]

\[
P_{327} = 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24
\]

\[
P_{328} = 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3 + 16N^2 + 672N + 288
\]

\[
P_{329} = 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 - 473N^5 - 682N^4 - 904N^3 - 592N^2 - 208N - 32.
\]

The OME \( A_{Qg} \) except for the term \( a_{Qg}^{(3)} \) reads:

\[
A_{Qg} = \frac{1}{2} [1 + (-1)^N] \times \left\{ a_g \gamma_{gq} T_F L_M + a_s \left\{ \frac{4}{3} \gamma_{gq} T_F^2 L_M + C_F T_F \left[ \frac{4(3N + 2) S_1^2}{N^2(N + 2)} + \frac{4(N^4 - N^3 - 20N^2 - 10N - 4) S_1}{N^2(N + 1)^2(N + 2)} \right] \right\} \right. \\
+ \left. \frac{2P_{375}}{N^4(N + 1)^4(N + 2)} + \frac{16S_1}{N^2} + \gamma_{gq} \left[ \frac{2S_1^2 - 2S_2}{N(N + 1)^3(N + 2)} + C_A T_F \left[ \frac{4(N^3 + 8N^2 + 11N + 2) S_1^2}{N(N + 1)^2(N + 2)^2} - \frac{4P_{333} S_1}{N(N + 1)^3(N + 2)^3} \right] \right]\right\} \\
+ \left. \left\{ \frac{4P_{413}}{(N - 1) N^4(N + 1)^4(N + 2)^4} + \frac{L_2^M}{8P_{383}} \right. \\
- \left. \frac{4(7N^5 + 21N^4 + 13N^3 + 21N^2 + 18N + 16) S_2}{(N - 1) N^2(N + 1)^2(N + 2)^2} \right\} + \frac{8P_{383}}{(N - 1) N^3(N + 1)^3(N + 2)^3} \right\}
\]
\[ -\frac{32(2N + 3)S_1}{(N + 1)^2(N + 2)^2} + \gamma_0^q \left[ -2S_1^2 - 2S_2 - 4S_{-2} \right] + \frac{(N^2 - N - 4)}{(N + 1)^2(N + 2)^2} 16(-1)^N S_{-2} \]

\[ + \gamma_0^q \left[ \frac{1}{3} S_1^3 - 3S_2 S_1 - 4(-1)^N S_{-2} S_1 - \frac{8S_3}{3} - 2(-1)^N S_{-3} + 4S_{-2,1} \right] \}

\[ + a_3^q \left\{ T_M^3 \left[ \frac{16}{9} \gamma_0^q S_1 - \frac{16\gamma_0^q S_3}{9} \right] + C_{AT}^T \left[ \frac{32(N^3 + 8N^2 + 11N + 2)S_1^3}{9N(N + 1)^2(N + 2)^2} + \frac{8P_{334} S_1^2}{3N(N + 1)^3(N + 2)^3} \right. \]

\[ - \frac{16P_{106} S_1}{81(N - 1)N^3(N + 1)^4(N + 2)^4} - \frac{32(3N^3 - 12N^2 - 27N - 2)S_2 S_1}{3N(N + 1)^2(N + 2)^2} + \frac{P_{392} S_2}{9(N - 1)N^2(N + 1)^2(N + 2)^2} \]

\[ + \frac{8P_{427} S_1}{81(N - 1)N^3(N + 1)^5(N + 2)^5} \]
\[+L_M^3 \left[ -\frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N + 1)^2(N + 2)^2} - \frac{8}{9} \gamma_{qq} S_1 \right] + \frac{8P_{384}}{3(N-1)N^3(N + 1)^3(N + 2)^3} + \frac{128(N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6)S_3}{9(N-1)N^2(N + 1)^2(N + 2)^2} + \frac{8P_{374}}{3(N-1)N^3(N + 1)^3(N + 2)^3} - \frac{32(5N^4 + 20N^3 + 47N^2 + 58N + 20)S_1}{9N(N + 1)^2(N + 2)^2} + \frac{7_0^0 \left[ \frac{4}{3}S_1 \frac{2}{3} - \frac{8}{3} S_{-2} \right]}{3} + \left( N^4 + 2N^3 + 7N^2 + 22N + 20 \right) \left[ \frac{64}{3}(-1)^N S_{-2} + \frac{32}{3}(-1)^N \zeta_2 \right] \]

\[+ (N^2 - N - 4) \frac{S_1^2 + 16}{9(N+1)^2(N + 2)^2} - \frac{8}{3} \gamma_{qq} \left[ \frac{1}{9} S_1^2 + \frac{10}{3} S_2 S_1 + \frac{16}{3} (-1)^N S_{-2} S_1 + \left[ \left( \frac{80S_3}{9} - \frac{32}{3} S_{-2} \right) S_1 + \frac{4}{9} \left( -7 + 9(-1)^N \right) \zeta_3 S_1 + \frac{1}{3} S_2 \right] + \frac{16}{3} \left( -1 \right)^N S_2 \right] + \frac{8}{3} S_{-2} + \frac{4}{3} (-1 + 2(-1)^N) S_3^2 + \frac{4}{3} \left( -1 + 2(-1)^N \right) S_2 + \frac{8}{3} \left( -1 \right)^N S_{-2} \zeta_2 \]

\[+L_M^2 \left[ -\frac{16P_{357} S_1}{9N(N + 1)^2(N + 2)^2} + \frac{27N^2(N + 1)^3(N + 2)^3}{36(N-1)N^2(N + 1)^2(N + 2)^2} - \frac{16P_{345} S_2}{9N(N+1)(N+2)} - \frac{8(N^2 + N - 4) 64}{3(N+1)^2(N+2)^2} \left[ \frac{8}{9} S_1^3 - \frac{8}{3} S_2 S_1 - \frac{16}{3} (-1)^N S_{-2} S_1 - \frac{40S_3}{9} \right] - \frac{8}{3} \left( 2(-1)^N \right) S_{-3} - \frac{16}{3} S_{-2,1} + \frac{16}{3} S_{-2,1} \right] + \frac{C_T^2 T_F}{3} \left[ -\frac{3N^4 + 54N^3 + 139N^2 + 120N + 36}{3N^2(N+1)^2(N+2)} S_1^4 \right]

\[+\frac{4(18N^5 - 15N^4 - 180N^3 - 111N^2 - 40N - 4) S_3^3}{27N^3(N + 1)^3(N + 2)} - \frac{4(3N^4 + 14N^3 + 43N^2 + 48N + 20) \zeta_2 S_2}{N^2(N + 1)^2(N + 2)} + \frac{2P_{359} S_1^2}{N^3(N + 1)^3(N + 2)} - \frac{8(9N^4 + 102N^3 + 97N^2 - 36N - 12) S_3 S_1}{3N^2(N + 1)^2(N + 2)} - \frac{8P_{353} S_1 S_2}{N^3(N + 1)^3(N + 2)} - \frac{P_{414}}{N^6(N + 1)^6(N + 2)} \]

\[+\frac{2}{3} \left( N^2 + N + 2 \right) (153N^4 + 306N^3 + 165N^2 + 12N + 4) \zeta_3 \left[ \frac{2}{3} \left( 2N^2 + 2N \right) S_1 \left( 3N^2 + 3N + 2 \right) \right] + \frac{16(N^2 + N + 2)(3N^2 + 3N + 2)^2}{3N^2(N + 1)^2(N + 2)} + \frac{8}{3} \gamma_{qq} S_2^3 \]

\[+L_M^3 \left[ -\frac{2P_{370} S_2}{N^4(N + 1)^4(N + 2)} + \frac{4P_{362} S_3}{3N^3(N + 1)^3(N + 2)} \right] + \frac{16(N^2 - 3N - 2)(3N^2 + 3N + 2) S_2}{N^4(N + 1)^4(N + 2)} + \frac{P_{378}}{N^4(N + 1)^4(N + 2) + \frac{4P_{388} S_1}{N^3(N + 1)^3(N + 2)} - \frac{8P_{338} S_1}{N^3(N + 1)^3(N + 2)} + \frac{P_{378}}{N^4(N + 1)^4(N + 2)} \]

\[-\frac{12(N^2 + N + 2)(3N^2 + 3N + 2) S_2}{N^2(N + 1)^2(N + 2)} - \frac{32(N^2 + N + 2) S_{-2}}{N^2(N + 1)^2(N + 2)} + \frac{8}{3} \gamma_{qq} S_1^3 - 12S_2 S_1 - 16S_{-2} S_1 \]
\[ -8S_3 - 8S_{-3} + 16S_{-2,1} \] + \frac{P_{382}\zeta_2}{2N^4(N+1)^4(N+2)} + \frac{16(N^2 + N + 2)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)} + 96q^0_0 \log(2)\zeta_2

\[ \begin{aligned}
+ L_M & \left[ \frac{64(N+1)S_5^3}{N^2(N+2)} - \frac{8P_{339}S_1^2}{N^3(N+1)^3(N+2)} + \frac{16(N^4 + 10N^3 + 3N^2 - 18N - 4)S_2S_1}{N^2(N+1)^2(N+2)} \right] \\
& + \frac{8P_{372}S_1}{N^4(N+1)^4(N+2)} - \frac{P_{303}}{2N^5(N+1)^5(N+2)} - \frac{8(N^2 + 5N + 2)(3N^2 - N + 2)S_3}{N^2(N+1)^2(N+2)} \\
& + \frac{4P_{350}S_2}{N^3(N+1)^3(N+2)} + \frac{32(2N^5 + 4N^4 - N^3 + 2N^2 - 2N + 8)S_2}{(N+1)^2(N+2)} - \frac{64(N-1)S_{-3}}{(N+1)^2(N+2)} \\
& - \frac{64(N^2 + N + 2)S_{2,1}}{N^2(N+1)^2(N+2)} + \frac{(N^2 - N + 2)\left[128S_{-2,1} - 128S_{-2,1}\right]}{(N+1)^2(N+2)} \\
& + \frac{2S_1^4 + 4S_2S_1^2 + 32S_{-3}S_1 + (24S_3 - 32S_{2,1})S_1 + 6S_2^2 + 8S_{-2}^2 + 16S_{-2}S_2 + 20S_4 + 40S_{-4}}{(N+1)^2(N+2)} \\
& - \frac{8S_{3,1} - 16S_{-2,2} - 32S_{-3,1} + 24S_{2,1,1}}{N^2(N+1)^2(N+2)} + \frac{48(N-1)(3N^2 + 3N - 2)\zeta_3}{N^2(N+1)^2} \\
& + \frac{(N^2 + N + 2)(3N^2 + 3N + 2)\left[-S_2^2 + 8\zeta_2S_2 + 6S_4 - 16S_{3,1} + 32S_{2,1,1} - \frac{16}{7}S_1\zeta_3\right]}{N^2(N+1)^2(N+2)} \\
& + \frac{1}{3}S_1^5 - 2S_2S_1^3 + \frac{8}{3}S_3 - 16S_{1,2}\left[S_2^2 - \frac{8}{3}\zeta_4S_2^2 + \left[-S_2^2 + 6S_4 - 16S_{3,1} + 32S_{2,1,1}\right]S_1 \right] \\
& + \left[-4S_1^3 + 8S_2S_1 + 8S_{-2}S_1 + 4S_3 + 4S_{-3} - 8S_{-2,1}\right]S_2\zeta_2 \right] + C_{TF}^2 \left[ \frac{P_{346}S_1^4}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{2P_{400}S_1^2}{3(N-1)N^2(N+1)^4(N+2)^4} \right]
\end{aligned} \]
\[\begin{align*}
&+ \frac{16 P_{418}}{9(N - 1)^2 N^4(N + 1)^4(N + 2)^4} - \frac{4(N^2 + N + 2)(11N^4 + 22N^3 - 83N^2 - 94N - 72)S_2}{3(N - 1)^2 N^2(N + 1)^2(N + 2)^2} \\
&- \frac{8(N^2 + N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_2}{3(N - 1)^2 N^2(N + 1)^2(N + 2)^2} \\
&+ 5_{7q}^0 \left[ 45S_1^3 + 12S_2S_1 + 16S_{-2}S_1 + 4S_3 + 4S_{-3} - 8S_{-2,1} \right] \\
&+ \frac{(N^4 + 2N^3 + 7N^2 + 22N + 20)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{(N - 1)N^2(N + 1)^2(N + 2)^2} \left[ -\frac{16}{3}(-1)^N S_{-2} \right] \\
&- \frac{8}{3}(-1)^NS_2 \right] + \frac{(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{(N - 1)N^2(N + 1)^2(N + 2)^2} \\
&\times \left[ \frac{1}{3}S_2^2 + \frac{16}{3}(-1)^NS_{-2}S_2 + 6S_4 + \frac{8}{3}(-1)^NS_{-4} - \frac{8}{3}S_{3,1} - \frac{16}{3}S_{-2,2} \right] \\
&- \frac{16}{3}S_{-3,1} - \frac{8}{3}S_{2,1,1} + \frac{32}{3}S_{-2,1,1} + \left( \frac{8}{3}(-1)^NS_2 + \frac{8}{3}(-1)^NS_{-2} \right) \left[ (-1)^NS_{-3,1} - \frac{16}{3}S_{-2,1,1} + 4(-1)^NS_3 \right] \\
&+ \frac{(11N^5 + 34N^4 - 49N^3 - 24N^2 - 68N - 48)}{(N - 1)N^2(N + 1)^3(N + 2)^3} \left[ (-1)^NS_1S_{-3} - \frac{32}{3}S_1S_{-2,1} + 4(-1)^NS_3 \right] \\
&+ L_M \left[ -\frac{8(11N^4 + 26N^3 - 139N^2 - 218N + 8)S_3^3}{9N(N + 1)^2(N + 2)^2} + \frac{4P_{404}S_2^2}{8P_{420}S_1} \right] \\
&- \frac{8P_{367}S_2S_1}{27(N - 1)^2N^4(N + 1)^2(N + 2)^2} + \frac{3(N - 1)^2N^2(N + 1)^2(N + 2)^2}{32(2N^5 - 23N^4 - 32N^3 + 13N^2 + 4N - 12)S_{-2}S_1} + \frac{\zeta_3 P_{367}}{4P_{329}} \\
&- \frac{27(N - 1)^2N^5(N + 1)^5(N + 2)^5}{9(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{16P_{394}S_{-2}}{16P_{348}S_{-2}} + \frac{16P_{348}S_{-2}}{16P_{349}S_{-3}} \\
&\frac{16P_{349}S_{-3}}{3(N - 1)N^2(N + 1)^2(N + 2)^2} - \frac{3(N - 1)N^2(N + 1)^2(N + 2)^2}{16(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)} \left[ (-1)^NS_{-2} \right] \\
&\frac{(11N^5 + 34N^4 - 49N^3 - 24N^2 - 68N - 48)}{(N - 1)N^2(N + 1)^2(N + 2)^2}
\end{align*}\]
\[
\begin{align*}
&\frac{(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{(N - 1)N^2(N + 1)^2(N + 2)^2} \left[-\frac{5}{6}(-1)^N S_{-3} - 2(-1)^N \zeta_3 \right] + \gamma_{qq} \left[2S_1^4 + 2S_2^2\right] \\
&+ 32S_2 S_1^2 + 8(8 + (-1)^N) S_{-3} S_1 + \left[40S_3 - 16S_{-2} - 80S_{-2,1}\right] S_1 + 6\left(-5 + (-1)^N\right) \zeta_3 S_1 + 12S_2^2 \\
&+ S_{-2}\left[8(3 + 2(-1)^N) S_1^2 + 16S_2 + 4S_4 + 44S_{-4} + 16S_{-2,1} - 56S_{-2,2} - 64S_{-3,1} + 96S_{-3,1,1} \right] \\
&+ C_{FT} \left[ \frac{16(N^2 + N + 2) P_{341}}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{32}{9} \gamma_{qq} S_1 \right] L_M^3 + \frac{4P_{410}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \\
&+ \frac{32(5N^3 + 14N^2 + 37N + 18) S_1}{9N^2(N + 1)(N + 2)} + \gamma_{qq} \left[4S_1^2 - 4S_2^2 + 4S_4 \right] L_M^3 + \frac{16(10N^3 + 31N^2 + 59N + 18) S_1^2}{9N^2(N + 1)(N + 2)} \\
&+ 16(29N^4 + 163N^3 + 786N^2 + 592N + 192) S_1 \\
&+ \frac{16(2N^4 + 39N^3 + 42N^2 - 5N - 2) S_2}{3N^2(N + 1)^2(N + 2)^3} + \gamma_{qq} \left[\frac{4}{3}S_1^3 + 4S_2 S_1 - 8S_3 \right] L_M \\
&+ \frac{16(N^4 - 5N^3 - 32N^2 - 18N - 4) S_1^2}{3N^2(N + 1)^2(N + 2)} - \frac{16}{9} \left(N^2 + N + 2\right) P_{341} \zeta_3 \\
&+ \frac{2}{9(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{32(2N^5 - 2N^4 - 11N^3 - 19N^2 - 44N - 12) S_1}{3N^2(N + 1)^2(N + 2)} \\
&+ \frac{2P_{421}}{3(N - 1)N^6(N + 1)^6(N + 2)^2} - \frac{80}{9} \left[N^3 + 4N^2 + 11N + 6\right] \zeta_2 S_1 - \frac{16P_{336} S_2}{3N^3(N + 1)^3(N + 2)} \\
&+ \frac{128(N^2 - 3N - 2) S_{1,1}}{3N^2(N + 1)(N + 2)} + \gamma_{qq} \left[\frac{2}{3}S_1^4 - \frac{4}{3}S_2 S_1^2 + \left(-\frac{16}{9} S_3 - \frac{32}{3} S_{1,1}\right) S_1 - \frac{32}{9} \zeta_3 S_1 - \frac{2}{3} S_2^2 + 4S_4 \right] \\
&+ \frac{32}{3} S_{3,1} + \frac{64}{3} S_{2,1,1} + (2S_2 - \frac{10}{3} S_1^2) \zeta_2 \right] \\
&+ C_{FT} \left[ \frac{8(N^2 + N + 2) P_{335}}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{8}{9} \gamma_{qq} S_1 \right] L_M^3 + \frac{4P_{408}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \\
&+ \frac{32(5N^3 + 8N^2 + 19N + 6) S_1}{9N^2(N + 1)(N + 2)} + \gamma_{qq} \left[\frac{4}{3}S_1^3 + \frac{4S_2^2}{3} \right] L_M^2 + L_M \left[\frac{32(5N^3 + 11N^2 + 22N + 6) S_1^2}{9N^2(N + 1)(N + 2)} \right] \\
&+ \frac{32(19N^4 + 77N^3 + 303N^2 + 251N + 78) S_1}{27N^2(N + 1)^2(N + 2)} + \frac{2P_{422}}{27(N - 1)N^5(N + 1)^5(N + 2)^4} \\
&+ \frac{16P_{369} S_2}{3(N - 1)N^3(N + 1)^3(N + 2)^2} + \gamma_{qq} \left[\frac{8}{9} S_1^3 + \frac{8}{3} S_2 S_1^2 - \frac{56S_3}{9}\right] - \frac{8}{9} \left(N^2 + N + 2\right) P_{335} \zeta_3 \\
&+ \frac{8(N^4 - 5N^3 - 32N^2 - 18N - 4) S_1^2}{3N^2(N + 1)^2(N + 2)} - \frac{2}{9} \left(N^2 + N + 2\right) P_{412} \zeta_2 \\
&+ \frac{16(2N^5 - 2N^4 - 11N^3 - 19N^2 - 44N - 12) S_1}{3N^2(N + 1)^3(N + 2)} - \frac{16}{9} \left(5N^3 + 11N^2 + 28N + 12\right) \zeta_2 S_1 \\
&+ \frac{4P_{430}}{3(N - 1)N^6(N + 1)^6(N + 2)^5} - \frac{8P_{407} S_2}{3(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{(3N + 2)\left[-\frac{16}{3} S_1^3 - \frac{16}{3} S_2 S_1^2 \right]}{N^2(N + 2)} \\
&- \frac{16P_{373} S_3}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{64(N^2 - 3N - 2) S_{1,1}}{3N^2(N + 1)(N + 2)} + \gamma_{qq} \left[-\frac{1}{9} S_1^4 - \frac{2}{3} S_2 S_1^2 \right]
\end{align*}
\]
\[-\frac{4}{3} \zeta_2 S_1^2 + \left[ \frac{8}{9} S_3 - \frac{16}{3} S_{2,1} \right] S_1 + \frac{8}{9} S_3 S_1 - \frac{1}{3} S_2^2 + 2 S_4 - \frac{16}{3} S_{3,1} + \frac{32}{3} S_{2,1,1} \]

\[+ C_{FT}\text{CF}_{ATL} \left[ \frac{2 P_{330} S^4_1}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8 P_{388} S^3_1}{9(N-1)N^3(N+1)^3(N+2)^3} \right] \]

\[- \frac{8}{3} (N-1) N^3 S_2^2 + \frac{4 P_{344} S_1^2}{P_{344} S_2^2} - \frac{3(N-1) N^3(N+1)^4(N+2)}{P_{354} S_2^2} \]

\[+ \frac{4}{3} \left( \frac{3(N-1) N^3(N+1)^4(N+2)}{P_{375} S_1 \zeta_3} + \frac{8 P_{391} S_2 S_1}{P_{391} S_2 S_1} \right) \]

\[+ \frac{9}{N-1)N^3(N+1)^4(N+2)^2} + \frac{3(N-1) N^3 S_2^2}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8 P_{426} S_1}{8 P_{426} S_1} \]

\[- \frac{16(11N^5 + 45N^4 - 3N^3 - 145N^2 - 176N - 20) S_{2,1}}{16(N^2 + N + 2)(N^4 + 2N^3 - 16N^2 - 17N - 6) S^2_2} \]

\[- \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)}{(N-1)N^3(N+1)^3(N+2)^2} \]

\[- \frac{4 P_{399} S_2}{3(N-1) N^3(N+1)^4(N+2)^3} - \frac{4(N^2 + N + 2) (3N^4 + 6N^3 + 7N^2 + 4N + 4) \zeta_2}{9(N-1)N^2(N+1)^2(N+2)^2} \]

\[- \frac{8(N^2 + N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48) S_{2,1}}{3(N-1) N^3(N+1)^2(N+2)^2} \]

\[- \frac{8(N^2 + N + 2)(31N^4 + 62N^3 - 73N^2 - 104N - 60) S_{3,1}}{3(N-1) N^2(N+1)^2(N+2)^2} \]

\[+ L^2_M \left[ \frac{8 P_{331} S^2_1}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{8 P_{389} S_1}{9(N-1)N^3(N+1)^3(N+2)^3} \right] \]

\[- \frac{P_{396}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{8(N^2 + N + 2)(3N^4 + 6N^3 + 7N^2 + 4N + 4) \zeta_2}{3(N-1) N^2(N+1)^2(N+2)^2} \]

\[- 8 S^3_1 + 4 S_3 + 6 S_{-2} + 4 S_{-3} - 8 S_{-2,1} \]
\[-32S_{-2,1} + 12(-1)^N\zeta_3 + \frac{3N^5 + 8N^4 + 27N^3 + 46N^2 + 20N + 8}{N^2(N + 1)^2(N + 2)^2}\left[16(-1)^NS_{-3} - 32S_1S_{-2,1}\right]
+ 12(-1)^NS_1\zeta_3 + \hat{\gamma}_{19}^0\left[\frac{2N^5 + 12S_2S_3}{3} + 16(-1)^NS_{-3}S_1^2 + \left[\frac{88S_3}{3} + 16S_{-2,1} - 32S_{-2,1}\right]S_1^2\right]
+ \frac{4}{3}( -5 + 9(-1)^N)\zeta_3S_2^2 + (8(-1)^NS_{-3}S_1 + 2S_2^2 + 12S_4 + 8S_{3,1} - 16S_{-2,2} - 16S_{-3,1} - 40S_{2,1,1} + 32S_{-2,1,1})S_1 + S_2\left[16(-1)^NS_1^2 + 16(-1)^NS_2S_1 + 32\right] + \left[8(-1)^NS_1^3 + 4(-1 + 2(-1)^NS_2S_1\right]
- 2S_3 - 2S_{-3} + 4S_{-2,1}\zeta_2 + L_M\left[\frac{8(11N^5 - 46N^4 - 499N^3 - 866N^2 - 496N - 144)S_1^3}{9N^2(N + 1)^2(N + 2)^2}\right] - \frac{4P_{393}S_1^2}{9(N - 1)N^3(N + 1)^3(N + 2)^3}
+ \frac{8P_{305}S_1}{27(N - 1)N^4(N + 1)^3(N + 2)^2}
- \frac{(N - 1)N^2(N + 1)^2(N + 2)^2}{8P_{305}S_3}
- \frac{9(N - 1)^2N^2(N + 1)^3(N + 2)^2}{16\left(3N^4 + 8N^3 - 5N^2 - 6N + 8\right)S_{-3}^2 + 16(3N^4 + 42N^3 + 39N^2 + 32N - 12)S_{-2,1}}
+ \frac{48(N^2 + N + 2)^2S_{2,1}}{(N - 1)N(N + 1)(N + 2)^2}
- \frac{(N^2 - N - 4)(3N^2 + 3N + 2)16(-1)^NS_{-2}}{N(N + 1)^3(N + 2)^2}
+ \frac{(N^2 + N + 2)(3N^2 + 3N + 2)( -8(-1)^NS_{-3} - 6(-1)^N\zeta_3)}{N^2(N + 1)^2(N + 2)^2}
- \frac{8(3 + 2(-1)^NS_{-2}S_1^2 - 8(-1)^NS_{-3}S_1 + ( -32S_3 + 48S_{2,1} + 16S_{-2,1})S_1 - 6(-1 + 2(-1)^NS_2S_1\right]
+ 8S_2^2 - 12S_{-2}^2 - 20S_{-4} - 8S_{3,1} - 8S_{-2,2} - 24S_{2,1,1} + 32S_{-2,1,1}\right]\right) + a_{Qg}^{(3)}\right\} \right] \right] (436)

where

\[P_{330} = N^6 - 93N^5 - 444N^4 - 317N^3 + 329N^2 + 296N + 84 \] (437)
\[P_{331} = N^6 - 9N^5 - 120N^4 - 137N^3 + 29N^2 + 56N + 36 \] (438)
\[P_{332} = N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \] (439)
\[P_{333} = N^6 + 8N^5 + 23N^4 + 54N^3 + 94N^2 + 72N + 8 \] (440)
\[P_{334} = 2N^6 + 11N^5 + 8N^4 - 7N^3 + 14N^2 + 12N - 24 \] (441)
\[P_{335} = 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 \] (442)
\[P_{336} = 3N^6 + 30N^5 + 15N^4 - 64N^3 - 56N^2 - 20N - 8 \] (443)
\[P_{337} = 5N^6 + 15N^5 + 36N^4 + 51N^3 + 25N^2 + 8N + 4 \] (444)
\[P_{338} = 5N^6 + 18N^5 + 51N^4 + 84N^3 + 60N^2 + 34N + 12 \] (445)
\[P_{339} = 5N^6 + 26N^5 + 97N^4 + 160N^3 + 135N^2 + 79N + 22 \] (446)
\[P_{340} = 5N^6 + 42N^5 + 84N^4 + 35N^3 + 40N^2 + 34N + 48 \] (447)
\[P_{341} = 6N^6 + 18N^5 + 7N^4 - 16N^3 - 31N^2 - 20N - 12 \] (448)
\[P_{342} = 6N^6 + 47N^5 + 136N^4 + 223N^3 + 256N^2 + 172N + 32 \] (449)
\[P_{343} = 9N^6 + 27N^5 - 65N^4 - 319N^3 - 404N^2 - 200N - 40 \] (450)
\[ P_{344} = 10N^6 - 6N^5 - 39N^4 - 44N^3 - 97N^2 + 20N + 12 \]

\[ P_{345} = 10N^6 + 63N^5 + 105N^4 + 31N^3 + 17N^2 + 14N + 48 \]

\[ P_{346} = 11N^6 - 15N^5 - 327N^4 - 181N^3 + 292N^2 - 20N - 48 \]

\[ P_{347} = 11N^6 + 15N^5 - 285N^4 - 319N^3 - 254N^2 - 368N - 240 \]

\[ P_{348} = 11N^6 + 33N^5 - 189N^4 - 361N^3 - 194N^2 - 92N - 72 \]

\[ P_{349} = 11N^6 + 33N^5 - 114N^4 - 247N^3 - 263N^2 - 176N - 108 \]

\[ P_{350} = 11N^6 + 33N^5 - 87N^4 - 85N^3 + 4N^2 - 116N - 48 \]

\[ P_{351} = 11N^6 + 57N^5 - 39N^4 - 109N^3 - 44N^2 - 116N - 48 \]

\[ P_{352} = 11N^6 + 81N^5 + 9N^4 - 133N^3 - 92N^2 - 116N - 48 \]

\[ P_{353} = 13N^6 + 36N^5 + 39N^4 + 8N^3 - 21N^2 - 29N - 10 \]

\[ P_{354} = 17N^6 + 51N^5 + 390N^4 + 359N^3 - 389N^2 - 200N - 84 \]

\[ P_{355} = 18N^6 + 87N^5 + 57N^4 - 119N^3 - 131N^2 - 60N - 20 \]

\[ P_{356} = 23N^6 + 39N^5 + 75N^4 + 157N^3 + 96N^2 + 70N + 28 \]

\[ P_{357} = 29N^6 + 176N^5 + 777N^4 + 1820N^3 + 1878N^2 + 776N + 232 \]

\[ P_{358} = 45N^6 + 135N^5 - 91N^4 - 407N^3 - 214N^2 + 12N - 248 \]

\[ P_{359} = 51N^6 + 140N^5 + 227N^4 + 208N^3 - 202N^2 - 96N - 8 \]

\[ P_{360} = 55N^6 + 141N^5 - 195N^4 - 401N^3 - 772N^2 - 748N - 384 \]

\[ P_{361} = 55N^6 + 165N^5 - 420N^4 - 899N^3 - 1561N^2 - 1336N - 1188 \]

\[ P_{362} = 57N^6 + 297N^5 + 519N^4 + 399N^3 + 92N^2 - 68N - 16 \]

\[ P_{363} = 67N^6 + 93N^5 + 351N^4 + 259N^3 - 1054N^2 - 556N - 312 \]

\[ P_{364} = 76N^6 + 487N^5 + 1692N^4 + 3271N^3 + 3186N^2 + 1516N + 536 \]

\[ P_{365} = 77N^6 + 195N^5 + 627N^4 + 977N^3 - 128N^2 - 452N + 432 \]

\[ P_{366} = 77N^6 + 339N^5 - 105N^4 - 487N^3 - 356N^2 - 668N - 240 \]

\[ P_{367} = 83N^6 + 249N^5 - 111N^4 - 637N^3 - 956N^2 - 596N - 624 \]

\[ P_{368} = 97N^6 + 591N^5 + 1311N^4 + 229N^3 - 712N^2 + 308N + 192 \]

\[ P_{369} = N^8 + 24N^7 + 62N^6 + 8N^5 - 123N^4 - 128N^3 - 108N^2 - 72N - 48 \]

\[ P_{370} = N^8 + 427N^7 + 1133N^6 + 697N^5 - 434N^4 - 636N^3 - 244N^2 - 64N - 16 \]

\[ P_{371} = 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \]

\[ P_{372} = 2N^8 + 29N^7 + 179N^6 + 441N^5 + 529N^4 + 332N^3 + 172N^2 + 92N + 24 \]

\[ P_{373} = 3N^8 + 54N^7 + 118N^6 - 44N^5 - 353N^4 - 314N^3 - 272N^2 - 200N - 144 \]

\[ P_{374} = 9N^8 + 54N^7 + 56N^6 - 182N^5 - 717N^4 - 1120N^3 - 1012N^2 - 672N - 160 \]

\[ P_{375} = 12N^8 + 52N^7 + 132N^6 + 216N^5 + 191N^4 + 54N^3 - 25N^2 - 20N - 4 \]

\[ P_{376} = 15N^8 + 36N^7 + 50N^6 - 252N^5 - 357N^4 + 152N^3 - 68N^2 + 88N + 48 \]

\[ P_{377} = 18N^8 + 101N^7 + 128N^6 + 208N^5 + 190N^4 - 769N^3 - 1200N^2 - 212N - 48 \]

\[ P_{378} = 33N^8 + 132N^7 + 350N^6 + 636N^5 + 685N^4 + 528N^3 + 292N^2 + 128N + 32 \]

\[ P_{379} = 121N^8 + 370N^7 + 924N^6 + 358N^5 - 381N^4 + 184N^3 - 1096N^2 - 48N + 144 \]

\[ P_{380} = 321N^8 + 1674N^7 + 2360N^6 - 1378N^5 - 6565N^4 - 5992N^3 - 1972N^2 + 128N - 96 \]

\[ P_{381} = 507N^8 + 2190N^7 + 3002N^6 + 1692N^5 - 681N^4 - 2554N^3 - 404N^2 + 664N + 192 \]

\[ P_{382} = 633N^8 + 2532N^7 + 5036N^6 + 6174N^5 + 4307N^4 + 1182N^3 - 176N^2 - 184N - 48 \]

\[ P_{383} = N^9 + 6N^8 + 15N^7 + 25N^6 + 36N^5 + 85N^4 + 128N^3 + 104N^2 + 64N + 16 \]

\[ P_{384} = N^9 + 21N^8 + 85N^7 + 105N^6 + 42N^5 + 290N^4 + 600N^3 + 456N^2 + 256N + 64 \]

\[ P_{385} = 4N^9 + 53N^8 + 193N^7 + 233N^6 + 87N^5 + 554N^4 + 1172N^3 + 904N^2 + 512N + 128 \]
\begin{align*}
P_{386} &= 6N^9 + 93N^8 + 576N^7 + 1296N^6 + 586N^5 + 359N^4 + 2000N^3 + 1996N^2 \\
&+ 1488N + 384 \\
(493)\\
P_{387} &= 25N^9 - 43N^8 - 424N^7 + 462N^6 + 4345N^5 + 7513N^4 + 6446N^3 + 4020N^2 \\
&+ 1944N + 480 \\
(494)\\
P_{388} &= 36N^9 + 156N^8 - 115N^7 - 1116N^6 - 1251N^5 - 78N^4 + 300N^3 + 84N^2 - 128N - 48 \\
(495)\\
P_{389} &= 40N^9 + 273N^8 + 635N^7 + 613N^6 + 119N^5 - 2N^4 - 314N^3 - 668N^2 + 24N + 144 \\
(496)\\
P_{390} &= 45N^9 + 270N^8 + 724N^7 + 1262N^6 + 1731N^5 + 2740N^4 + 3484N^3 + 2928N^2 \\
&+ 1696N + 384 \\
(497)\\
P_{391} &= 66N^9 + 534N^8 + 1409N^7 + 1080N^6 - 933N^5 - 1116N^4 + 588N^3 + 996N^2 \\
&+ 736N + 240 \\
(498)\\
P_{392} &= 69N^9 + 366N^8 + 1100N^7 + 1894N^6 + 2451N^5 + 5276N^4 + 7460N^3 + 5352N^2 \\
&+ 3008N + 672 \\
(499)\\
P_{393} &= 80N^9 + 441N^8 + 568N^7 - 502N^6 - 1202N^5 + 2003N^4 + 4106N^3 + 3116N^2 \\
&+ 2712N + 864 \\
(500)\\
P_{394} &= 94N^9 + 597N^8 + 1508N^7 + 2086N^6 + 1517N^5 + 1381N^4 + 2731N^3 + 3802N^2 \\
&+ 2916N + 648 \\
(501)\\
P_{395} &= 251N^9 + 1586N^8 + 4206N^7 + 6764N^6 + 4008N^5 - 2242N^4 + 13N^3 + 7122N^2 \\
&+ 6156N + 1944 \\
(502)\\
P_{396} &= 489N^9 + 2934N^8 + 7636N^7 + 12206N^6 + 6675N^5 - 12692N^4 - 24608N^3 + 65242N^2 \\
&- 2864N + 2304 \\
(503)\\
P_{397} &= 891N^9 + 5751N^8 + 15070N^7 + 21430N^6 + 37623N^5 + 55339N^4 + 44064N^3 + 25144N^2 \\
&+ 9488N + 1776 \\
(504)\\
P_{398} &= 10N^{10} + 62N^9 + 407N^8 + 1119N^7 + 1405N^6 + 889N^5 + 240N^4 - 90N^3 - 114N^2 \\
&- 48N - 8 \\
(505)\\
P_{399} &= 36N^{10} + 456N^9 + 2448N^8 + 7171N^7 + 12399N^6 + 13213N^5 + 8997N^4 + 5000N^3 + 2888N^2 \\
&+ 992N + 112 \\
(506)\\
P_{400} &= 37N^{10} + 392N^9 + 2106N^8 + 6514N^7 + 9211N^6 + 1258N^5 - 918N^4 - 6116N^3 - 72N^2 \\
&- 752N - 192 \\
(507)\\
P_{401} &= 85N^{10} + 425N^9 + 830N^8 + 788N^7 - 521N^6 - 325N^5 + 2238N^4 + 2568N^3 + 968N^2 \\
&- 1296N - 576 \\
(508)\\
P_{402} &= 103N^{10} + 575N^9 + 1124N^8 - 334N^7 - 1505N^6 + 3755N^5 + 4926N^4 + 36N^3 - 472N^2 \\
&- 2160N - 864 \\
(509)\\
P_{403} &= 149N^{10} + 793N^9 + 2368N^8 + 5026N^7 + 6853N^6 + 6277N^5 + 5062N^4 + 3168N^3 + 1296N^2 \\
&+ 400N + 96 \\
(510)\\
P_{404} &= 170N^{10} + 883N^9 + 1897N^8 + 2710N^7 - 448N^6 - 4745N^5 + 561N^4 + 5904N^3 + 1132N^2 \\
&- 2016N - 864 \\
(511)\\
P_{405} &= 170N^{10} + 1213N^9 + 3091N^8 + 2506N^7 - 2692N^6 - 3047N^5 - 861N^4 - 2352N^3 - 5324N^2 \\
&- 6240N - 2016 \\
(512)\\
P_{406} &= 436N^{10} + 3960N^9 + 15787N^8 + 36343N^7 + 46431N^6 + 17745N^5 - 28270N^4 - 33648N^3 \\
&- 11056N^2 - 1936N + 864 \\
(513)\\
P_{407} &= 3N^{11} + 42N^{10} + 144N^9 + 74N^8 - 459N^7 - 1060N^6 - 1152N^5 - 1424N^4 - 1688N^3 \\
&- 1232N^2 - 736N - 192 \\
(514)\end{align*}
\[ P_{408} = 33N^{11} + 231N^{10} + 698N^{9} + 1290N^{8} + 1513N^{7} + 1463N^{6} + 2236N^{5} + 5096N^{4} + 7328N^{3} + 5456N^2 + 3456N + 1152 \]  
\[ P_{409} = 95N^{11} + 853N^{10} + 3599N^{9} + 9245N^{8} + 12320N^{7} - 282N^6 - 23342N^5 - 26920N^4 - 10832N^3 - 1712N^2 - 416N - 192 \]  
\[ P_{410} = 129N^{11} + 903N^{10} + 2894N^{9} + 5730N^{8} + 6505N^{7} + 383N^6 - 9646N^5 - 13912N^4 - 11680N^3 - 6640N^2 - 3648N - 1152 \]  
\[ P_{411} = 243N^{11} + 1701N^{10} + 5378N^{9} + 10350N^{8} + 11479N^{7} + 1193N^6 - 14684N^5 - 20572N^4 - 16288N^3 - 8944N^2 - 4992N - 1728 \]  
\[ P_{412} = 333N^{11} + 2331N^{10} + 6556N^{9} + 9270N^{8} + 5081N^{7} - 6701N^6 - 17554N^5 - 20036N^4 - 15680N^3 - 9200N^2 - 5664N - 1728 \]  
\[ P_{413} = 2N^{12} + 20N^{11} + 86N^{10} + 192N^9 + 199N^8 - N^7 - 297N^6 - 495N^5 - 514N^4 - 488N^3 - 416N^2 - 176N - 32 \]  
\[ P_{414} = 23N^{12} + 138N^{11} - 311N^{10} - 3148N^9 - 7605N^8 - 8462N^7 - 4163N^6 + 246N^5 + 1540N^4 + 1066N^3 + 444N^2 + 120N + 16 \]  
\[ P_{415} = 111N^{12} + 1035N^{11} + 3634N^{10} + 5168N^9 - 2662N^8 - 21724N^7 - 37157N^6 - 34963N^5 - 19122N^4 - 4560N^3 + 80N^2 + 1008N + 288 \]  
\[ P_{416} = 201N^{12} + 1845N^{11} + 6742N^{10} + 11990N^9 + 7139N^8 - 8917N^7 - 15710N^6 - 2110N^5 + 16644N^4 + 22080N^3 + 12416N^2 + 4128N + 576 \]  
\[ P_{417} = 7299N^{12} + 53973N^{11} + 206656N^{10} + 532170N^9 + 820775N^8 + 650149N^7 + 204230N^6 + 189820N^5 + 606016N^4 + 664624N^3 + 372192N^2 + 143424N + 27648 \]  
\[ P_{418} = 9N^{13} + 72N^{12} + 101N^{11} - 511N^{10} - 2325N^9 - 4428N^8 - 4619N^7 - 3841N^6 - 4462N^5 - 6012N^4 - 6992N^3 - 5296N^2 - 2592N - 576 \]  
\[ P_{419} = 69N^{13} + 420N^{12} + 794N^{11} - 1501N^{10} - 11265N^9 - 18414N^8 - 4436N^7 - 5017N^6 - 41818N^5 - 65616N^4 - 62960N^3 - 39184N^2 - 17184N - 3456 \]  
\[ P_{420} = 296N^{13} + 2368N^{12} + 9908N^{11} + 22254N^{10} + 13564N^9 - 31716N^8 - 71723N^7 - 71221N^6 - 44369N^5 - 33249N^4 - 26584N^3 + 4968N^2 + 7344N + 432 \]  
\[ P_{421} = 385N^{14} + 2567N^{13} + 6877N^{12} + 9235N^{11} + 5375N^{10} - 1207N^9 - 3313N^8 + 905N^7 + 3876N^6 + 1676N^5 + 256N^4 + 154N^3 + 161N^2 + 736N + 192 \]  
\[ P_{422} = 531N^{14} + 5454N^{13} + 25877N^{12} + 77604N^{11} + 159437N^{10} + 205070N^9 + 82971N^8 - 207408N^7 - 490544N^6 - 694320N^5 - 735104N^4 - 562304N^3 - 355854N^2 - 158976N - 34560 \]  
\[ P_{423} = 1773N^{14} + 18018N^{13} + 80795N^{12} + 214620N^{11} + 371423N^{10} + 398930N^9 + 154773N^8 - 228072N^7 - 435356N^6 - 492936N^5 - 534656N^4 - 453440N^3 - 299712N^2 - 144000N - 3456 \]  
\[ P_{424} = 4N^{15} + 50N^{14} + 267N^{13} + 765N^{12} + 1183N^{11} + 682N^{10} - 826N^9 - 1858N^8 - 1116N^7 + 457N^6 + 1500N^5 + 2268N^4 + 2400N^3 + 1392N^2 + 448N + 64 \]  
\[ P_{425} = 26N^{15} + 314N^{14} + 1503N^{13} + 3222N^{12} + 2510N^{11} + 1996N^{10} + 15041N^9 + 40728N^8 + 54008N^7 + 44956N^6 + 31936N^5 + 30416N^4 + 29568N^3 + 16704N^2 + 5376N + 768 \]  
\[ P_{426} = 28N^{15} + 335N^{14} + 1953N^{13} + 6497N^{12} + 11508N^{11} + 6624N^{10} - 11753N^9 - 27541N^8 - 33352N^7 - 40915N^6 - 40468N^5 - 16628N^4 + 10416N^2 + 4032N + 576 \]  
\[ P_{427} = 435N^{15} + 5436N^{14} + 32317N^{13} + 119006N^{12} + 307057N^{11} + 620328N^{10} + 1065977N^9 + 1575060N^8 + 1889534N^7 + 1704634N^6 + 1113248N^5 + 592440N^4 + 328672N^3 \]
\[ P_{428} = 939N^{16} + 10527N^{15} + 47251N^{14} + 101719N^{13} + 66350N^{12} - 155710N^{11} - 322813N^{10} - 16829N^9 + 702425N^8 + 1332497N^7 + 1596952N^6 + 1640548N^5 + 1506496N^4 + 1099952N^3 + 604032N^2 + 211392N + 34560 \]  
\[ P_{429} = 1623N^{16} + 20963N^{15} + 119399N^{14} + 394315N^{13} + 831483N^{12} + 1160715N^{11} + 1086519N^{10} + 1172841N^9 + 425270N^8 + 337718N^7 + 207634N^6 - 73752N^5 - 261272N^4 - 200160N^3 - 64672N^2 - 4480N + 1408 \]  
\[ P_{430} = 87N^{17} + 1099N^{16} + 6055N^{15} + 19019N^{14} + 37119N^{13} + 45159N^{12} + 29583N^{11} - 2639N^{10} - 30218N^9 - 40778N^8 - 39994N^7 - 35844N^6 - 30808N^5 - 30384N^4 - 28256N^3 - 16064N^2 - 5248N - 768. \]  

The operator matrix element \( A_{g9, Q} \) except for the term \( a_{g9, Q}^{(3)} \) is given by:

\[
A_{g9, Q} = \frac{1}{2}[1 + (-1)^N] \times 
\left\{ \begin{array}{c}
\frac{4}{3}T_F L_M + a_s^2 \left[ \frac{16}{9} T_F^2 L_M + C_A T_F \left[ \frac{16(N^2 + N + 1)}{3(N - 1)N(N + 1)(N + 2)} - \frac{8S_1}{3} \right] \right] L_M^2 \\
\frac{16P_{433}}{9(N - 1)N^2(N + 1)^2(N + 2)} - \frac{80S_1}{9} L_M + \frac{2P_{451}}{27(N - 1)N^3(N + 1)^3(N + 2)} - \frac{4(56N + 47)S_1}{27(N + 1)} \\
+ C_F T_F \frac{4(N^2 + N + 2)^2}{(N - 1)N^2(N + 1)^2(N + 2)} L_M^2 + \frac{4P_{445}}{(N - 1)N^3(N + 1)^3(N + 2)} L_M \\
- \frac{P_{461}}{(N - 1)N^4(N + 1)^4(N + 2)} \right\}
\end{array} \right. 
\]

\[
+ a_s^2 \left[ \frac{64}{27} L_M^3 - \frac{64C_3}{27} \right] + C_A T_F^2 \left[ \frac{448(N^2 + N + 1)}{27(N - 1)N(N + 1)(N + 2)} - \frac{224S_1}{27} \right] L_M^3 \\
- \frac{8P_{441}}{27(N - 1)N^2(N + 1)^2(N + 2)} L_M + \frac{8P_{440}S_1}{9(N - 1)N^2(N + 1)^2(N + 2)} \frac{8S_1^2}{3(N + 1)} - \frac{P_{442}S_2}{4(27(N - 1)N^2(N + 1)^2(N + 2))} \\
- \frac{8P_{460}}{81(N - 1)N^4(N + 1)^4(N + 2)} + 16(328N^4 + 256N^3 - 247N^2 - 175N + 54)S_1 \\
- \frac{448}{27} \frac{(N^2 + N + 1)^2}{3(N + 1)} - \frac{8(2N + 1)S_2}{3(N + 1)} + \frac{560}{27} S_1 S_2 + \frac{224}{27} S_1 S_3 \\
+ NF \left[ \frac{128(N^2 + N + 1)}{27(N - 1)N(N + 1)(N + 2)} - \frac{64S_1}{27} \right] L_M^3 + \frac{4P_{457}}{81(N - 1)N^3(N + 1)^3(N + 2)} \\
- \frac{16P_{443}S_1}{81(N - 1)N^2(N + 1)^2(N + 2)} L_M + \frac{16S_1^2}{9(N + 1)} - \frac{P_{437}S_2}{4(27(N - 1)N^2(N + 1)^2(N + 2))} \\
+ \frac{32P_{465}}{243(N - 1)N^4(N + 1)^4(N + 2)} + \frac{32(328N^4 + 256N^3 - 247N^2 - 175N + 54)S_1}{243(N - 1)N(N + 1)^2} \\
- \frac{128}{27} \frac{(N^2 + N + 1)\zeta_3}{(N - 1)N(N + 1)(N + 2)} - \frac{16(2N + 1)S_2}{9(N + 1)} + \frac{160}{27} S_1 S_2 + \frac{64}{27} S_1 S_3 \right] \right] 
\]
\[+ L_M^2 \left[ \frac{-16S_2 - 32S_{-2}}{(N - 1)^2N^2(N + 1)^2(N + 2)^2} - \frac{2P_{478}}{9(N - 1)^2N^4(N + 1)^4(N + 2)^3} \right.\]
\[- \left. \frac{8P_{159}S_1}{3(N - 1)^2N^3(N + 1)^3(N + 2)^2} \right] - 64\log(2)\zeta_2\]
\[+ \left( N^2 + N + 2 \right) \left( N^4 + 2N^3 + 7N^2 + 22N + 20 \right) \left[ -32(-1)^NS_{-2} - 16(-1)^N\zeta_2 \right] \]
\[+ \frac{N^2 - N - 4}{(N - 1)N(N + 1)^4(N + 2)^3} \left( 64(-1)^NS_{-2} + 32(-1)^NS_3 - 64S_{-2,1} + 32(-1)^NS_1\zeta_2 \right) \]
\[+ 24(-1)^N\zeta_3 + \frac{(N^2 + N + 2)^2}{(N - 1)N(N + 1)^4(N + 2)^5} \left[ \frac{2S_1^4 + 20S_2S_1^2 + 32(-1)^NS_{-3}S_1}{3} \right.\]
\[+ \left. \frac{160S_3}{3} - 64S_{-2,1} \right] S_1 + \frac{8}{3} \left( -7 + 9(-1)^N \right) \zeta_3 S_1 + 2S_2^2 + S_{-2} \left[ 32(-1)^N S_1^2 + 32(-1)^NS_2 \right] \]
\[+ 36S_4 + 16(-1)^NS_{-4} - 16S_{3,1} - 32S_{-2,2} - 32S_{-3,1} - 16S_{2,1,1} + 64S_{-2,1,1} + \left[ 4(-3 + 4(-1)^N)S^2_2 \right.\]
\[+ 4(-1 + 4(-1)^N)S_2 + 8(1 + 2(-1)^N)S_{-2} \left. \right] \zeta_2 \]
\[+ L_M \left[ \frac{(N^2 + N + 2)^2}{(N - 1)N^2(N + 1)^2(N + 2)^2} \right.\]
\[\left. \frac{8S^2_1 - 40S_2S_1}{3} \right] - 32(-1)^NS_{-2}S_1 - 16(-1)^NS_3 + 32S_{2,1} - 12(-1)^N\zeta_3 \]
\[+ \frac{4(17N^4 - 6N^3 + 41N^2 - 16N - 12)S_2^2(N^2 + N + 2)}{(N - 1)^2N^3(N + 1)^2(N + 2)^2} + \frac{96(N^2 + N + 4)S_{-3}(N^2 + N + 2)}{(N - 1)N^2(N + 1)^2(N + 2)^2} \]
\[- \frac{32(N^2 + N + 14)S_{-2,1}(N^2 + N + 2)}{(N - 1)^2N^2(N + 1)^2(N + 2)^2} - \frac{(N^2 - N - 4)32(-1)^NS_{-2}(N^2 + N + 2)}{(N - 1)N^2(N + 1)^3(N + 2)^2} \]
\[+ \frac{8P_{481}}{27(N - 1)^2N^5(N + 1)^5(N + 2)^4} - \frac{4(N^2 + N + 10)(5N^2 + 5N + 18)\zeta_3}{(N - 1)N^2(N + 1)^2(N + 2)^2} \]
\[- \frac{8P_{472}S_1}{9(N - 1)^2N^4(N + 1)^4(N + 2)^2} + \frac{64(N^4 + 2N^3 + 7N^2 + 6N + 16)S_{-2}S_1}{(N - 1)^2N^2(N + 1)^4(N + 2)^2} \]
\[- \frac{4P_{446}S_2}{(N - 1)^2N^3(N + 1)^3(N + 2)^2} - \frac{32P_{447}S_{-2}}{(N - 1)^2N^3(N + 1)^4(N + 2)^2} + 64\zeta_3 \zeta_3 \Bigg\} \right] + a^{(3)}_{gg,Q}, \quad (538) \]

with

\[
P_{431} = N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \quad (539)\]
\[
P_{432} = 3N^6 + 9N^5 - 113N^4 - 241N^3 - 274N^2 - 152N - 24 \quad (540)\]
\[
P_{433} = 3N^6 + 9N^5 + 22N^4 + 29N^3 + 41N^2 + 28N + 6 \quad (541)\]
\[
P_{434} = 3N^6 + 30N^5 + 15N^4 - 64N^3 - 56N^2 - 20N - 8 \quad (542)\]
\[
P_{435} = 4N^6 + 3N^5 - 50N^4 - 129N^3 - 100N^2 - 56N - 24 \quad (543)\]
\[
P_{436} = 7N^6 + 15N^5 + 7N^4 - 23N^3 - 26N^2 - 20N - 8 \quad (544)\]
\[
P_{437} = 9N^6 + 27N^5 + 161N^4 + 277N^3 + 358N^2 + 224N + 48 \quad (545)\]
\[
P_{438} = 20N^6 + 60N^5 + 11N^4 - 78N^3 - 121N^2 - 72N - 108 \quad (546)\]
\[
P_{439} = 20N^6 + 60N^5 + 11N^4 - 78N^3 - 85N^2 - 36N - 108 \quad (547)\]
\[
P_{440} = 40N^6 + 114N^5 + 19N^4 - 132N^3 - 147N^2 - 70N - 32 \quad (548)\]
\[
P_{441} = 63N^6 + 189N^5 + 367N^4 + 419N^3 + 626N^2 + 448N + 96 \quad (549)\]
\[
P_{442} = 99N^6 + 297N^5 + 631N^4 + 767N^3 + 1118N^2 + 784N + 168 \quad (550)\]
\[
P_{443} = 136N^6 + 390N^5 + 19N^4 - 552N^3 - 947N^2 - 630N - 288 \quad (551)\]
\[ P_{444} = N^8 + 4N^7 + 2N^6 + 64N^5 + 173N^4 + 292N^3 + 256N^2 - 72N - 72 \]  
\[ P_{445} = N^8 + 4N^7 + 8N^6 + 6N^5 - 3N^4 - 22N^3 - 10N^2 - 8N - 8 \]  
\[ P_{446} = 3N^8 - 14N^7 - 164N^6 - 454N^5 - 527N^4 - 204N^3 - 112N^2 + 80N + 48 \]  
\[ P_{447} = 3N^8 + 10N^7 + 13N^6 + N^5 + 28N^4 + 81N^3 + 4N^2 - 12N - 32 \]  
\[ P_{448} = 3N^8 + 23N^7 + 51N^6 + 95N^5 + 142N^4 + 158N^3 + 56N^2 - 32N - 16 \]  
\[ P_{449} = 15N^8 + 60N^7 + 76N^6 - 18N^5 - 275N^4 - 546N^3 - 400N^2 - 224N - 96 \]  
\[ P_{450} = 15N^8 + 60N^7 + 86N^6 + 12N^5 - 166N^4 - 378N^3 - 245N^2 - 148N - 84 \]  
\[ P_{451} = 15N^8 + 60N^7 + 572N^6 + 1470N^5 + 2135N^4 + 1794N^3 + 722N^2 - 24N - 72 \]  
\[ P_{452} = 23N^8 + 92N^7 + 46N^6 - 88N^5 + 79N^4 + 476N^3 + 428N^2 - 96N - 96 \]  
\[ P_{453} = 24N^8 + 96N^7 + 93N^6 - 57N^5 - 143N^4 - 79N^3 - 34N^2 - 20N - 8 \]  
\[ P_{454} = 27N^8 + 108N^7 - 1440N^6 - 4554N^5 - 5931N^4 - 3762N^3 - 256N^2 + 1184N + 480 \]  
\[ P_{455} = 63N^8 + 252N^7 + 196N^6 - 258N^5 - 551N^4 - 282N^3 - 220N^2 - 80N + 48 \]  
\[ P_{456} = 131N^8 + 524N^7 + 691N^6 + 239N^5 - 848N^4 - 1483N^3 - 586N^2 + 108N + 360 \]  
\[ P_{457} = 297N^8 + 1188N^7 + 640N^6 - 2094N^5 - 1193N^4 + 2874N^3 + 5008N^2 + 3360N + 864 \]  
\[ P_{458} = N^9 + 21N^8 + 85N^7 + 105N^6 + 42N^5 + 290N^4 + 600N^3 + 456N^2 + 256N + 64 \]  
\[ P_{459} = 3N^{10} + 15N^9 + 35N^8 + 50N^7 + 91N^6 + 233N^5 + 255N^4 + 150N^3 - 24N^2 \]  
\[ -184N - 48 \]  
\[ P_{460} = 3N^{10} + 15N^9 + 3316N^8 + 12778N^7 + 22951N^6 + 23815N^5 + 14212N^4 + 3556N^3 \]  
\[ -30N^2 + 288N + 216 \]  
\[ P_{461} = 15N^{10} + 75N^9 + 112N^8 + 14N^7 - 61N^6 + 107N^5 + 170N^4 + 36N^3 - 36N^2 \]  
\[ -32N - 16 \]  
\[ P_{462} = 18N^{10} + 90N^9 + 119N^8 - 91N^7 - 167N^6 + 101N^5 + 162N^4 - 72N^3 - 504N^2 \]  
\[ -184N - 48 \]  
\[ P_{463} = 30N^{10} + 150N^9 + 163N^8 - 224N^7 - 586N^6 - 368N^5 - 39N^4 - 78N^3 + 144N^2 \]  
\[ +184N + 48 \]  
\[ P_{464} = 40N^{10} + 200N^9 + 282N^8 - 66N^7 - 615N^6 - 753N^5 - 509N^4 - 205N^3 - 2N^2 \]  
\[ +68N + 24 \]  
\[ P_{465} = 63N^{10} + 315N^9 - 1142N^8 - 6260N^7 - 11927N^6 - 12359N^5 - 7235N^4 - 1778N^3 \]  
\[ +15N^2 - 144N - 108 \]  
\[ P_{466} = 67N^{10} + 335N^9 + 368N^8 - 762N^7 - 3349N^6 - 6669N^5 - 8310N^4 - 7656N^3 \]  
\[ -4648N^2 - 1600N - 288 \]  
\[ P_{467} = 219N^{10} + 1095N^9 + 1640N^8 - 82N^7 - 2467N^6 - 2947N^5 - 3242N^4 - 4326N^3 \]  
\[ -3466N^2 - 1488N - 360 \]  
\[ P_{468} = 693N^{10} + 3465N^9 - 11014N^8 - 62668N^7 - 120361N^6 - 125113N^5 - 73393N^4 \]  
\[ -18010N^3 + 165N^2 - 1584N - 1188 \]  
\[ P_{469} = 3N^{11} + 21N^{10} - 124N^9 - 1014N^8 - 2185N^7 - 2099N^6 - 934N^5 - 2060N^4 - 4632N^3 \]  
\[ -4256N^2 - 2688N - 768 \]  
\[ P_{470} = 27N^{11} + 189N^{10} + 631N^9 + 1356N^8 + 2155N^7 + 2207N^6 + 211N^5 - 4984N^4 - 8400N^3 \]  
\[ -5824N^2 - 2544N - 576 \]  
\[ P_{471} = N^{12} + 6N^{11} - 5N^{10} - 80N^9 - 379N^8 - 846N^7 - 1057N^6 - 786N^5 + 84N^4 + 490N^3 \]  
\[ +324N^2 + 152N + 48 \]  
\[ P_{472} = 15N^{12} + 90N^{11} + 80N^{10} - 452N^9 - 1401N^8 - 2298N^7 - 5002N^6 - 6516N^5 - 1116N^4 \]
\[ P_{473} = 233N^{12} + 2724N^{11} + 13349N^{10} + 34680N^9 + 46703N^8 + 12096N^7 - 69461N^6 \]
\[-137724N^5 - 141176N^4 - 91776N^3 - 34832N^2 - 6336N - 2304 \]  
\[ P_{474} = 310N^{12} + 2058N^{11} + 5939N^{10} + 17235N^9 + 44700N^8 + 93240N^7 + 140861N^6 \]
\[ +113169N^5 + 145784N^4 - 40374N^3 - 33372N^2 - 12312N - 3888 \]  
\[ P_{475} = 391N^{12} + 2346N^{11} + 4795N^{10} + 2758N^9 - 2243N^8 + 1150N^7 + 7713N^6 \]
\[ +4546N^5 - 792N^4 + 1224N^2 + 864N + 288 \]  
\[ P_{476} = 1593N^{12} + 9558N^{11} + 15013N^{10} - 8758N^9 - 62269N^8 - 82318N^7 - 79041N^6 \]
\[-90898N^5 - 70928N^4 - 15872N^3 + 7344N^2 + 5184N + 1728 \]  
\[ P_{477} = 15N^{13} + 120N^{12} + 530N^{11} + 1562N^{10} + 2042N^9 - 1680N^8 + 9220N^7 - 12524N^6 \]
\[-7911N^5 - 5230N^4 - 5880N^3 - 3344N^2 - 2544N - 864 \]  
\[ P_{478} = 33N^{13} + 264N^{12} + 479N^{11} - 1366N^{10} - 8809N^9 - 23124N^8 - 34351N^7 - 26198N^6 \]
\[-3624N^5 + 5240N^4 - 2496N^3 - 7232N^2 - 7104N - 2304 \]  
\[ P_{479} = 2493N^{13} + 19944N^{12} + 79295N^{11} + 208394N^{10} + 375431N^9 + 531516N^8 + 623697N^7 \]
\[ +733338N^6 + 963340N^5 + 1047352N^4 + 895648N^3 + 559488N^2 + 222336N + 41472 \]  
\[ P_{480} = 39N^{14} + 273N^{13} + 741N^{12} + 1025N^{11} + 1343N^{10} + 3479N^9 + 6707N^8 + 6555N^7 \]
\[ +2258N^6 - 1520N^5 - 1944N^4 - 532N^3 + 280N^2 + 208N + 32 \]  
\[ P_{481} = 276N^{16} + 3036N^{15} + 13660N^{14} + 30172N^{13} + 22464N^{12} - 50653N^{11} - 171627N^{10} \]
\[ -246412N^9 - 204934N^8 - 83791N^7 + 28263N^6 + 43144N^5 - 33372N^4 - 82640N^3 \]
\[-79152N^2 + 47232N - 12096 \]  
\[ P_{482} = 3135N^{19} + 43890N^{18} + 257636N^{17} + 794084N^{16} + 1224418N^{15} + 244448N^{14} \]
\[-2371724N^{13} - 3594388N^{12} - 792201N^{11} + 2719198N^{10} + 2284064N^9 - 85568N^8 \]
\[ -227344N^7 + 952768N^6 + 1160704N^5 + 807552N^4 + 574464N^3 + 305664N^2 \]
\[ +104448N + 18432 \].

B The asymptotic Heavy Flavor Wilson Coefficients contributing to $F_2(x, Q^2)$ in $z$-space

The representation of the Wilson coefficients in momentum fraction or $z$-space of the contributions being known at present can be obtained in terms of harmonic polylogarithms \cite{38} over the alphabet

\[ \mathcal{A} = \left\{ \frac{dz}{z}, \frac{dz}{1-z}, \frac{dz}{1+z} \right\} \equiv \{ f_0(z), f_1(z), f_{-1}(z) \} \]

as iterated integrals

\[ H_{b,\vec{a}}(z) = \int_0^z f_b(y) H_{\vec{a}}(y), \quad f_b \in \mathcal{A}, \quad H_{\vec{0}} = 1, H_{\vec{a},...,\vec{0}}(z) := \frac{1}{k!} \ln^k(z). \]

For brevity we will drop the argument $z$ of the harmonic polylogarithms in the following. As a shorthand notation we introduce the Mellin-inversion of \cite{36}

\[ \gamma_{aq} = -4 \left[ z^2 + (1-z)^2 \right] , \]
where the Mellin transform is defined by

$$ M[f(x)](N) = \int_0^1 dx \, x^{N-1} \, f(x) \, . $$

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The Wilson coefficients \( L_{q,2}^{PS} \) and \( L_{g,2}^S \) are given in \( z \)-space by:

$$ L_{q,2}^{PS} = a_s^3 \left\{ \right. \begin{array} {l}
\frac{C_F N_F T_F^2}{64} \left[ \frac{64}{9} (z + 1) H_0 - \frac{32(z - 1)(4z^2 + 7z + 4)}{27z} \right] L_Q^3 + \left[ -\frac{64}{3} (z + 1) H_0^2 \right] \\
+ \frac{64}{9} (4z^2 - 11z - 8) H_0 + \frac{32(z - 1)(10z^2 + 33z - 2)}{9z} \right] L_Q^2 + L_Q \left[ \frac{64}{3} H_0^3 (z + 1) \right] \\
+ \frac{256}{9} H_{0,0,1} - \frac{256}{9} H_{1,0,0} \left( z + 1 \right)^3 - \frac{32}{9} (12z^2 - 59z - 29) H_0^2 \\
+ \frac{64}{3} (z + 1) H_0 - \frac{32(z - 1)(4z^2 + 7z + 4)}{9z} \right] L_M^2 - \frac{64(z - 1)(304z^2 + 811z + 124)}{81z} \\
- \frac{64}{27} (60z^2 - 155z - 233) H_0 - \frac{256(3z^2 + 1) \zeta_2}{9z} + L_M \left[ -\frac{64(z - 1)(38z^2 + 47z + 20)}{27z} \right] \\
+ \frac{128}{9} (2z^2 + 11z + 8) H_0 + \frac{64(z - 1)(4z^2 + 7z + 4) H_1}{9z} + (z + 1) \left[ \frac{128 \zeta_2}{3} - \frac{128}{3} H_{0,1} \right] \right\} \\
+ \frac{32(z - 1)(4z^2 + 7z + 4)}{27z} \left[ -\frac{64(z - 1) H_0^2}{9z} + \frac{64}{9} (4z^2 - 11z - 8) H_0 \right] \\
- \frac{32}{27} \left( 68z^3 + 53z^2 + 5z - 32 \right) \frac{\zeta_3}{z} - \frac{32}{243} (48z^2 + 5087z + 2783) H_0 \\
- \frac{32(z - 1)(256z^2 + 521z - 14) H_1}{81z} - \frac{32(z - 1)(38z^2 + 47z + 20) H_0 H_1}{81z} \\
- \frac{64(42z^3 - 227z^2 - 74z + 30) H_{0,0,1}}{81z} + \frac{64}{9} (2z^2 + 11z + 8) H_0 H_{0,0,1} \right\} \\
+ \frac{32}{27} (6z^2 + 4z - 5) H_{0,0,1} + \frac{128(9z^3 + 7z^2 + 7z + 3) H_{0,0,1}}{27z} \\
+ \frac{16}{9} H_0 H_{1,1}^2 - \frac{32}{9} \zeta_2 H_{1,1} \right] + L_M \left[ \frac{256 \zeta_2 z^2}{3} - \frac{16}{9} (8z^2 + 73z + 61) H_0^2 \right] \\
+ \frac{64}{27} (28z^2 - 221z - 107) H_0 + (z + 1) \left[ -\frac{32}{3} H_0^3 + \frac{128}{3} H_{0,0,1} - \frac{256}{3} \zeta_2 H_0 + \frac{128}{3} H_{0,0,1} \right] \\
- \frac{128 \zeta_3}{3} \right\} + \frac{32(z - 1)(383z^2 + 1213z + 76)}{81z} + \frac{128(z - 1)(4z^2 - 26z + 13) H_1}{27z} \\
+ \frac{(z - 1)(4z^2 + 7z + 4)}{9z} \left[ -\frac{32}{9} H_0^3 - \frac{64}{9} H_0 H_1 \right] - \frac{64(8z^3 - 3z^2 + 3z + 4) H_{0,0,1}}{9z} \right\}$$

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\[
+z + 1 \left[ \frac{8}{27} H_0^4 - \frac{464}{81} H_0^3 - \frac{64}{3} H_{0,1,1} H_0 + \frac{832}{9} \zeta_3 H_0 - \frac{224 \zeta_2^2}{15} + \frac{128}{9} H_{0,0,0,1} \right] - \frac{64}{9} H_{0,0,1,1} - \frac{256}{9} H_{0,1,1,1} + \left( \frac{64}{9} H_{0,1} - \frac{64}{9} H_0^2 \right) \Omega_2 \right] + N_F \tilde{\xi}_{2q}^{PS,(3)} (N_F) \right) \right) 
\]

and

\[
I_{g,2}^5 = \nonumber \\
\sum_{qg}^5 \left[ \frac{4}{9} \left( 8z^2 + 7z + 5 \right) H_{-1} H_0^2 + \frac{16}{27} (2z - 1) \zeta_2 H_0^2 + \frac{8}{27} (4641z^2 - 67330z + 3473) H_0 \right] - \frac{16}{81} (532z^2 + 2586z - 193) H_0^3
\]

\[
+ \frac{16}{81} (5z^2 - 50z + 25) H_1 H_0 + \frac{16}{27} (14z^2 + 11z + 10) H_{0,-1} H_0 - \frac{8}{27} (14z^2 + 11z + 10) H_{0,1} H_0
\]

\[
+ \frac{16}{27} (14z^2 + 11z + 10) H_{0,-1} H_0 - \frac{4}{27} (30z - 13) H_0^4 - \frac{8}{81} (32z^2 + 628z - 169) H_0^3 - \frac{16}{81} (532z^2 + 2586z - 193) H_0^2
\]

\[
- \frac{64}{27} (172z^2 - 163z + 56) H_1^2 - \frac{64}{49} (8z + 3) \zeta_2^2
\]

\[
+ \frac{8}{27} (276317z^3 - 271875z^2 + 11280z - 6182) - \frac{224}{27} (248z^3 - 438z^2 + 33z - 32) \frac{\zeta_3}{z}
\]

\[
+ \frac{8}{27} (28805z^3 - 28460z^2 + 4612z - 1596) H_1 + \frac{4}{27} (16(z - 1) (31z^2 + 7z + 4) H_0
\]

\[
- \frac{16}{9} H_0^2 - \frac{8}{9} H_0 H_1 + \frac{16}{9} (200z^2 + 191z + 112) [32 \frac{81}{81} H_{0,-1} - \frac{32}{81} H_{0,-1} H_0
\]

\[
- \frac{32}{27} (98z^2 + 347z + 16) H_{0,1} - \frac{128}{27} (7z^2 + 4z + 5) H_{0,0,-1} + \frac{16}{27} (28z^2 - 202z + 77) H_{0,0,1}
\]

\[
+ \frac{32}{9} (14z^2 - 15z + 10) H_{0,1,1} + (2z^2 + 2z + 1) \left[ \frac{64}{27} H_{-1} H_0^3 + \frac{64}{9} H_{0,-1} H_0^2 - \frac{128}{9} H_{0,0,0,1} H_0
\]

\[
+ \frac{16}{9} (49z^2 - 136z - 13) H_0 + \frac{8}{27} (1048z^3 - 894z^2 - 87z - 40) + \frac{16}{9} (8z^3 - 8z^2 + 13z - 4) H_1
\]

\[
+ \frac{4}{3} H_1^2 \right] + (2z^2 + 2z + 1) \left[ \frac{32}{3} H_{0,-1} - \frac{32}{3} H_{-1} H_0 \right] - \frac{32}{3} (4z + 1) H_{0,0,1}
\]

\[
\frac{8}{3} H_0 H_1 + (2z^2 + 2z + 1) \left[ \frac{32}{3} H_{0,-1} - \frac{32}{3} H_{-1} H_0 \right] - \frac{32}{3} (4z + 1) H_{0,0,1}
\]

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\[
\begin{align*}
&\left[ -\frac{16(z-1)(31z^2 + 7z + 4)}{9z} + \frac{32}{3} (4z + 1)H_0 + \frac{8}{3} \gamma q H_1 \right] L_M - \frac{64}{3} (z - 2)z \zeta_2 \\
&+ (7z^2 - 7z + 5) \left[ \frac{32}{81} H_1^3 + \frac{32}{27} H_0 H_1^2 + \frac{32}{27} H_0^2 H_1 - \frac{128}{27} H_0 H_1 + \frac{64}{27} \zeta_q H_1 \right] \\
&+ L_Q \left[ \frac{448}{9} z H_0^3 - \frac{16}{9} (63z^2 - 296z - 10) H_0^2 + \frac{32}{3} (5z^2 + 4z + 2) H_{-1} H_0^2 \right] \\
&- \frac{32}{3} (3z^2 - 4z + 2) H_1 H_0^2 - \frac{64}{27} (325z^2 - 832z - 31) H_0 - \frac{64}{3} (3z^2 + 4z + 2) H_{0,-1} H_0 \\
&+ \frac{64}{3} (z^2 + 2z + 2) H_{0,1} H_0 + \frac{128}{3} (z - 5)z \zeta_2 H_0 + \left[ -\frac{16(z-1)(31z^2 + 7z + 4)}{9z} \\
&+ \frac{32}{3} (4z + 1)H_0 + \frac{8}{3} \gamma q H_1 \right] L_M - \frac{8(25657z^3 - 23556z^2 - 969z - 412)}{81z} \\
&+ \frac{32}{9} (105z^3 - 184z^2 + 3z - 4) \frac{z^2}{z} - \frac{32}{9} (103z^2 - 964z^2 + 65z - 20) H_1 \\
&+ \frac{(87z^3 - 80z^2 + 13z - 4) \left[ -\frac{16}{9} H_1^3 - \frac{32}{9} H_0 H_1 \right]}{\left[ (17z^3 + 7z^2 - 4z - 2) \left[ \frac{64}{9} H_{-1} H_0 - \frac{64}{9} H_{0,-1} \right] \right]}
\end{align*}
\]
\[
+ \frac{32}{3} (6z^2 - 2z - 1) \zeta_3 + \gamma_{qg}^0 \left[ \frac{1}{27} H_1^4 - \frac{8}{9} H_{0,1} H_1^2 - \frac{4}{9} H_0^3 H_1 + \left[ \frac{16}{9} H_{0,1,1} + \frac{16}{9} H_{0,1,1} \right] H_1 \right]
+ \frac{16}{3} \zeta_3 H_1 - \frac{8}{9} H_{0,1}^2 + H_0^2 \left[ \frac{4}{3} H_{0,1} - \frac{2}{9} H_1^2 \right] + H_0 \left[ \frac{4}{27} H_1^3 - \frac{8}{3} H_{0,0,1} + \frac{8}{9} H_{0,1,1} \right] - \frac{8}{9} H_{0,0,1,1} + \left[ \frac{4}{9} H_1^2 - \frac{8}{9} H_0 H_1 + \frac{8}{9} H_{0,1,1} \right] \zeta_2 \right] + C_F N_F T_F^2 \left[ -\frac{2}{27} (56z^2 + 448z - 179) H_1^4 \right]
+ \frac{4}{81} (288z^2 - 6524z + 3259) H_0^3 - \frac{4}{81} \left( 4096z^2 + 23771z - 21328 \right) H_0^2 + 8(12z - 5) \zeta_2 H_1^2
+ \frac{56}{243} \left( 1491z^2 - 4715z + 17578 \right) H_0 - \frac{448}{81} (z^2 - z + 2) H_1 H_0 + 112(5z - 2) \zeta_2 H_0
+ \frac{16}{9} \left( 96z^2 + 92z - 109 \right) \zeta_3 H_0 + \left[ -\frac{16}{3} (2z - 1) H_0^2 - \frac{32}{9} (6z^2 - z - 4) H_0 \right]
+ \left[ \frac{8}{27}(124z^3 - 258z^2 + 159z - 16) \right]
+ \left[ \frac{8}{27}(124z^3 - 258z^2 + 159z - 16) \right] \times \frac{8}{9} \gamma_{qg}^0 H_1
\]
\[
\begin{align*}
&= \frac{16(144z^3 - 5632z^2 + 9213z + 4)H_0}{45z} + \frac{32(198z^3 - 283z^2 + 140z - 8)H_1H_0}{9z} \\
&+ \frac{128}{3}(z - 2)(3z - 2)H_{0,-1}H_0 + \frac{32}{3}(16z^2 - 8z - 5)H_{0,1}H_0 - \frac{32}{3}(4z^2 + 84z - 33)\zeta_2H_0 \\
&- \frac{16(168z^3 - 253z^2 + 131z - 8)H_1^2}{9z} + \left[16(2z - 1)H_0^2 + \frac{16}{3}(8z^2 - 9)H_0 \right] \\
&- \frac{16(z - 1)(62z^2 - 73z + 8)}{9z} \left[ H_1 \right] + 4 \left( \frac{269954z^3 - 828996z^2 + 567861z - 11744}{405z} \right) \\
&+ \frac{16}{45} \left( \frac{144z^4 + 1600z^3 - 1400z^2 + 3045z - 80}{z} \right) \left( \frac{\zeta_2}{z} - \frac{8(3080z^2 - 8448z^2 + 5247z + 256)H_1}{27z} \right) \\
&+ \frac{16(76z^3 - 254z^2 - 329z - 16)}{9z} H_{0,1} + \frac{\zeta_0}{9q} \left[ \frac{8}{3}H_1^3 + 8H_0H_1^2 + \frac{32}{3}H_{0,1}H_1 \right] \\
&- \frac{128}{3} \left( \frac{7z^2 - 14z + 9}{z - 1} \right)H_{0,0,-1} - \frac{32}{3} \left( \frac{8z^2 - 60z + 15}{z - 1} \right)H_{0,0,1} + \frac{32}{3} \left( \frac{24z^2 - 20z + 1}{z - 1} \right)H_{0,0,1} \\
&+ \frac{64}{3} \left( \frac{8z^2 - 6z + 3}{z} \right)H_1 \zeta_2 + \left( \frac{z + 2}{z} \right) \left[ \frac{-128}{3}H_0H_2 + \left( \frac{64}{3}H_0^2 + \frac{256}{3}H_{0,-1} \right)H_{-1} - \frac{128}{3} \zeta_2 H_{-1} \right] \\
&- \frac{256}{3} \left( \frac{H_{0,-1,-1}}{3} \right) + L_M \left[ \frac{16}{3} \left( \frac{32z^2 + 44z - 25}{z} \right) H_0^2 - \frac{32}{3} \left( \frac{8z^2 - 21z + 89}{z} \right) H_0 \right] \\
&- \frac{16(1006z^3 - 3333z^2 + 2208z + 128)}{27z} - \frac{16(64z^3 - 198z^2 + 141z - 16)H_1}{9z} + \frac{\zeta_0}{9q} \left[ \frac{16}{3}H_1^2 \right] \\
&+ \frac{32}{3} \left( \frac{H_0H_1}{3} \right) + \frac{64}{3} \left( \frac{z - 1}{z} \right) \left( \frac{4z + 5}{z} \right)H_{0,1} - \frac{64}{3} \left( \frac{8z^2 - 6z - 3}{z} \right) \zeta_2 + \left( \frac{z - 1}{z} \right) \left[ 32H_0^3 - 64 \zeta_2 H_0 + 64H_{0,0,1} \right] \\
&- \frac{64}{3} \left( \frac{H_{0,0,1}}{3} \right) + L_M \left[ \frac{-64}{9} \left( \frac{14z^2 + 41z - 13}{z} \right) H_0^2 - \frac{16}{45} \left( \frac{72z^3 + 360z^2 + 2390z - 2835}{z} \right) H_0 \right] \\
&+ \frac{64}{3} \left( \frac{z^2}{135z} \right)H_1H_0^2 - \frac{8(524z^3 + 29468z^2 - 50797z + 24)H_0}{135z} + \frac{16(28z^3 - 162z^2 + 123z - 16)H_1H_0}{9z} \\
&- \frac{128}{3} \left( \frac{z^2 - 4z + 2}{z} \right)H_{0,-1}H_0 - \frac{64}{3} \left( \frac{z^2 - 4z - 1}{z} \right) \left( \frac{2z + 1}{z} \right)H_{0,1}H_0 + \frac{32}{3} \left( \frac{36z^2 + 64z - 23}{z} \right) \zeta_2 H_0 \\
&+ \frac{8(44z^3 - 178z^2 + 143z - 16)}{9z} H_1^2 + \frac{4(259856z^3 - 763164z^2 + 514989z - 11456)}{405z} \\
&+ \frac{32}{45} \left( \frac{72z^4 - 70z^3 + 345z^2 - 1340z + 40}{z} \right) \zeta_2 + \frac{16(794z^3 - 3157z^2 + 2114z + 128)H_1}{27z} \\
&\times \left( \frac{36z^5 + 155z^4 + 40z^3 - 45z^2 + 20z + 1}{z} \right) \left[ \frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1} \right] + \frac{16(56z^2 + 413z + 16)}{9z} H_{0,1} \\
&+ \zeta_0 \left[ \frac{8}{3}H_1^3 + \frac{16}{3}H_0H_1^2 + \frac{16}{3}H_{0,1}H_1 \right] + \frac{128}{3} \left( \frac{z^2 - 10z + 3}{z} \right) \zeta_2 H_{0,-1} - \frac{32}{3} \zeta_2 \left( \frac{8z^2 + 52z - 11}{z} \right) H_{0,0,1} \\
&+ \frac{64}{3} \left( \frac{2z^2 + 3z - 6}{z} \right)H_{0,1,1} + \frac{64}{3} \left( \frac{4z^2 - 2z + 1}{z} \right)H_1 \zeta_2 + \left( \frac{z + 1}{z} \right) \left[ \frac{-128}{3}H_0H_2 + \left( \frac{64}{3}H_0^2 \right) \right] \\
&+ \frac{256}{3} \left( \frac{H_{0,-1}}{3} \right)H_{-1} - \frac{128}{3} \zeta_2 H_{-1} - \frac{256}{3} \left( \frac{H_{0,-1}}{3} \right) + \frac{32}{3} \left( \frac{24z^2 + 8z - 27}{z} \right) \zeta_3 \right] \\
&(2z - 1) \left[ -16H_0^4 + 96 \zeta_2 H_0^2 - 64H_{0,0,1}H_0 + 128 \zeta_3 H_0 + \frac{32 \zeta_2^2}{5} - 64H_{0,0,1,1} \right] \\
\end{align*}
\]
The pure-singlet Wilson coefficient $H_{q,2}^{PS}$ reads:

$$
H_{q,2}^{PS} = a_s^2 C_F T_F \left\{ \left[ \frac{32}{3} H_{0,-1} - \frac{32}{3} H_{-1} H_0 \right] (z + 1)^3 + \left[ \frac{16}{3} H_0^3 + 32 H_{0,1} H_0 - 32 \zeta_2 H_0 - 32 H_{0,0,0,1} + 32 \zeta_5 \right] + N_F \hat{C}^{S,(3)}_{2,g} (N_F) \right\}
$$

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The pure-singlet Wilson coefficient $H_{q,2}^{PS}$ reads:
\[+480H_{0,0,0,1} + 192H_{0,0,1,1} - 224H_{0,1,1,1} + \left\{ 288H_{0}^{2} + 192H_{0,1} \right\} \zeta_{2} (z + 1)\]
\[+\frac{16}{9} (52z^{2} - 22z + 15) H_{0}^{3} + \frac{2}{45} (288z^{3} + 3680z^{2} + 1525z - 7695) H_{0}^{2}\]
\[-\frac{4}{3} (z - 1) (8z^{2} - 305z + 20) H_{1}^{2} - \frac{32}{5} (3z + 23) \zeta_{2}^{2} - \frac{4}{3} (z - 1) (19536z^{2} - 1103z + 4056) \]
\[-\frac{8 (6298z^{3} - 32859z^{2} + 606z + 48) H_{0}}{135z} + \frac{64 (z - 1) (z^{2} + z + 1) H_{0}^{2} H_{1}}{z}\]
\[-\frac{8(z - 1) (986z^{2} - 4747z - 292) H_{1}}{27z} - 32(z - 1) (6z^{2} + 5z + 6) \frac{\zeta_{2}}{z} H_{1}\]
\[-\frac{8(z - 1) (32z^{2} - 299z + 32) H_{0} H_{1}}{3z} + \frac{(z - 1) (4z^{2} + 7z + 4) \left[ \frac{56}{7} H_{1}^{3} + \frac{80}{7} H_{0} H_{1}^{2} \right]}{z}\]
\[-\frac{64}{3} (2z^{2} + 7z - 3) H_{0} H_{0,-1} + \frac{8 (536z^{3} - 327z^{2} - 453z - 96) H_{0,1}}{9z}\]
\[+\frac{32 (32z^{3} + 6z^{2} - 21z + 12) H_{0} H_{0,1}}{3z} + \frac{128}{3} (2z^{2} + z - 9) H_{0,0,-1} + 256z^{2} H_{0,0,0,-1}\]
\[-\frac{32 (20z^{3} + 63z^{2} - 12z + 12) H_{0,0,1}}{3z} + \frac{16 (36z^{3} - 45z^{2} - 27z + 40) H_{0,1,1}}{3z}\]
\[+L_{M}^{2} \left[ \frac{-16 (4z^{2} + 7z + 4) H_{1} (z - 1)}{3z} - \frac{92 (z - 1)}{3} - \frac{16}{3} (4z + 3) H_{0}\right]\]
\[+(z + 1) (8H_{0}^{2} + 32H_{0,1} - 32 \zeta_{2}) \right] - \frac{16}{45} (72z^{3} + 1100z^{2} - 545z - 3615) \zeta_{2}\]
\[-\frac{32}{3} (56z^{2} - 61z - 30) H_{0} \zeta_{2} + (z - 1) \left[ -64H_{0,-1} H_{0}^{2} + 256H_{0,-1,1} H_{0} - 128H_{0,1}^{2}\right]\]
\[+128H_{0,-1} \zeta_{2} + \frac{16 (219z^{2} + 51z - 16) \zeta_{3}}{3z} - 64 (3z - 1) H_{0} \zeta_{3}\]
\[+L_{M} \left[ \frac{16}{3} (8z^{2} + 9z + 3) H_{0}^{2} - \frac{8}{9} (224z^{2} - 99z + 81) H_{0} - \frac{16}{9} (z - 1) (30z + 23)\right]\]
\[-\frac{64 (z - 1) (28z^{2} + z + 10) H_{1}}{9z} + \frac{(8z^{2} + 15z + 3) \left[ \frac{32}{3} H_{0,1} - \frac{32 \zeta_{2}^{2}}{3} \right] + (z + 1) \left[ -\frac{16}{3} H_{0}^{3}}{9z} + 64 \zeta_{2} H_{0} - 64 H_{0,0,1} + 64 \zeta_{3}\right] \left[ L_{Q} - \frac{4}{27} (212z^{2} + 225z + 207) H_{0}^{3}\right]\]
\[-\frac{8(z - 1) (6z^{2} - z - 6) H_{1}^{3}}{9z} + \frac{2}{27} (88z^{2} - 1899z - 1548) H_{0}^{2}\]
\[+\frac{4(z - 1) (584z^{2} + 359z + 71) H_{1}^{2}}{27z} - \frac{4(z - 1) (24z^{2} + 65z + 24) H_{0} H_{1}^{2}}{3z}\]
\[+16 \frac{74z^{2} + 69z - 60}{15} \zeta_{2}^{2} + \frac{4(z - 1) (1312z^{2} + 3535z + 699)}{27z}\]
\[+\frac{4}{27} (1532z^{3} + 3033z^{2} + 1305z - 540) \frac{\zeta_{2}}{z} + \frac{4}{27} (2040z^{2} + 2015z - 2297) H_{0}\]
\[-\frac{8(z - 1) (22z^{2} + 53z + 10) H_{0}^{2} H_{1}}{3z} + \frac{4(z - 1) (2844z^{2} + 3572z + 753) H_{1}}{27z}\]
\[+\frac{4}{3} (z - 1) (8z^{2} + 97z + 44) \frac{\zeta_{2}}{z} H_{1} + \frac{8(z - 1) (944z^{2} + 809z - 370) H_{0} H_{1}}{27z}\]
\[+\frac{8(4z^{3} + 45z^{2} + 15z - 12) H_{0}^{3} H_{0,1}}{3z} - \frac{8(1548z^{3} + 1395z^{2} - 648z + 370) H_{0,1}}{27z}\]
\[
-\frac{8}{3}(12z^3 + 51z^2 + 33z - 8)H_{0,1} + \frac{16}{9}(82z^3 + 135z^2 + 180z + 30)H_0H_{0,1}
\]
\[
-\frac{32}{3}(z - 1)H_1H_{0,1} + \frac{2}{9}(4z^2 + 9z + 3)H_0^4 + \frac{16}{3}H_0^2H_{0,1}
\]
\[
+ \frac{4}{9}(16z^3 + 1917z^2 + 1017z + 120)H_{0,0,1} + \frac{32}{3}(9z^2 + 6z - 4)H_0H_{0,1,1}
\]
\[
+ \frac{32}{3}(z - 1)H_1H_{0,1} + \frac{32}{3}(14z^3 - 9z^2 - 42z + 6)H_{0,0,0,1}
\]
\[
- \frac{8}{9}(188z^3 - 60z^2 + 141z + 72)H_{0,1,1} + \frac{8}{3}(33z^2 + 45z - 8)H_{0,0,1,1}
\]
\[
+ \frac{16}{9}(28z^3 + 21z^2 - 33z - 32)H_{0,1,1,1}
\]
\[
- \frac{8}{9}(z - 1)(4z + 3)H_0^2\zeta_2 + \frac{2}{9}(368z^2 - 1401z - 417)H_0\zeta_2 + \frac{16}{3} \left( \frac{16(4z^2 + 7z + 4)H_1(z - 1)}{9z} \right)
\]
\[
+ \frac{92}{9}(z - 1) + \frac{16}{9}(4z + 3)H_0 + (z + 1)\left[ -\frac{8}{3}H_0^2 - \frac{32}{3}H_{0,1} + \frac{32\zeta_2}{3} \right]
\]
\[
+ \frac{4(z - 1)(88z^2 + 135z + 40)H_1}{3z} + \frac{(z - 1)(4z^2 + 7z + 4)}{9z} \left[ \frac{8}{3}H_0^2 + \frac{32}{3}H_0H_1 \right]
\]
\[
+ \frac{8}{3}(8z^2 + 91z - 37)H_0 + \frac{4(z - 1)(61z + 12)}{3z}
\]
\[
+ \frac{16}{3}(4z^3 + 30z^2 + 15z - 8)H_{0,1} - \frac{16}{3}(4z - 24z - 21)\zeta_2 + (z + 1)\left[ -8H_0^2 - 64H_{0,1}H_0 \right]
\]
\[
+ 48\zeta_2 H_0 + 80H_{0,0,1} - 32H_{0,1,1} - 48\zeta_3 \right] - \frac{8}{9}(180z^2 + 1069z + 245)\zeta_3
\]
\[
- \frac{16}{9}(4z^2 - 96z + 27)H_0\zeta_3 + \frac{2}{9}(4z^2 + 7z + 4)\left[ \frac{2}{9}H_1^4 - \frac{16}{9}H_0H_1^3 - \frac{8}{3}H_0^2H_1^2 \right]
\]
\[
+ \frac{16}{9}H_{0,1}H_1^2 - \frac{8}{3}H_0^3H_1 - \frac{64}{3}H_{0,1,1}H_1 - \frac{16}{9}\zeta_3 H_1 + \frac{16}{3}H_0H_1 - \frac{4}{3}H_1^2\zeta_2
\]
\[
+ \frac{4(z - 1)(112z^2 + 1655z + 180)}{27z} - \frac{16}{9}(823z^2 - 1088z - 59)H_1 - \frac{16}{9}(12z^3 + 9z^2 + 7z - 8)\zeta_3
\]
\[
+ \frac{(z - 1)(4z^2 + 7z + 4)}{9z} \left[ -\frac{8}{9}H_1^2 - \frac{16}{3}H_0H_1^2 + \frac{32}{3}H_{0,1}H_1 \right]
\]
\[
+ \frac{32}{3}(16z^3 + 30z + 27)H_{0,0,1} - \frac{32}{3}(28z^3 + 57z^2 + 9z - 8)H_{0,1,1}
\]
\[
- \frac{8}{3}(76z^2 - 111z + 207)\zeta_2 + (z + 1)\left[ \frac{22}{3}H_0^2 - 112\zeta_2 H_0^2 + (160H_{0,0,1} + 64H_{0,1,1})H_0 \right]
\]
\[
+ 224\zeta_3 H_0 - 32H_0^2 + \frac{576\zeta_3^2}{5} - 256H_{0,0,0,1} + 64H_{0,0,1,1} + 32H_{0,1,1,1} \right]
\]
\[
+ (z + 1)\left[ 16H_{0,1}H_0^3 + \left[ 32H_{0,1,1} - 96H_{0,0,1} \right] H_0^2 - 80\zeta_2^2 H_0 + \left[ 256H_{0,0,0,1} - 96H_{0,0,1,1} \right] \right]
\]
\[ +64 H_{0,1,1,1} \left[ H_0 - 64 H_{0,1} H_{0,1,1} - 272 H_{0,0,0,0,0,1} + 64 H_{0,0,0,1,1,1} - 64 H_{0,0,1,0,1} + 288 H_{0,0,1,1,1} \right] + 192 H_{0,1,1,1,1} + 32 H_{0,1,1,1,1} + \zeta_2 \left[ \frac{20}{3} H_0^3 - 32 H_{0,1} H_0 + 128 H_{0,1,1} + 16 H_{0,1,1,1} - \frac{368 \zeta_3}{3} \right] + \left[ \frac{32}{3} H_{0,1} - \frac{16}{3} H_0^2 \zeta_3 + 240 \zeta_5 \right] + C_F T_F^2 \left[ \left( \frac{64}{9} (z + 1) H_0 - \frac{32 (z - 1)(4z^2 + 7z + 4)}{27z} \right) L_Q^3 \right. \\
+ \left[ - \frac{64}{3} (z + 1) H_0^2 + \frac{64}{9} (4z^2 - 11z - 8) H_0 + \frac{32 (z - 1)(10z^2 + 33z - 2)}{9z} \right] L_Q^3 \\
+ \left[ \frac{256}{9} H_{0,1} - \frac{256}{9} H_{0,0,1} (z + 1)^3 + \frac{64}{3} H_0^3 (z + 1) - \frac{32}{9} (12z^2 - 59z - 29) H_0^2 \right] L_Q^2 \\
+ \left[ \frac{64}{3} (z + 1) H_0 - \frac{32 (z - 1)(4z^2 + 7z + 4)}{9z} \right] L_Q^2 - \frac{128}{9} (2z^2 + 11z + 8) H_0 + \frac{64 (z - 1)(4z^2 + 7z + 4) H_1}{9z} + (z + 1) \left[ \frac{128 \zeta_2}{3} - \frac{128}{3} H_{0,1} \right] \right] L_Q \\
+ \left[ \frac{128(z - 1)(4z^2 + 7z + 4)}{27z} - \frac{256}{9} (z + 1) H_0 \right] L_M^2 - \frac{32}{27} (75z^2 + 100z + 64) H_0^2 \left[ \frac{16}{27} (22z^2 + 29z + 22) H_0^2 + \frac{32 (z - 1)(142z^2 + 103z + 34)}{27z} \right] \\
- \frac{64}{9} (4z^2 + 26z + 11) H_0 \left( L_M^2 - \frac{64 (z - 1)(461z^2 - 460z + 740)}{243z} \right) \\
- \frac{64}{27} (14z^3 - 62z^2 - 77z - 20) \frac{12}{z} - \frac{32}{27} (148z^3 + 279z^2 + 111z - 16) \frac{12}{z} + \frac{32}{81} (784z^2 - 473z + 463) H_0 - \frac{32 (z - 1)(16z^2 - 161z - 254) H_1}{81z} \\
+ \frac{32(z - 1)(74z^2 - 43z + 20) H_0 H_1}{27z} + \frac{(z - 1)(4z^2 + 7z + 4) \left[ \frac{64}{27} H_0^3 + \frac{16}{9} H_0 H_1^2 - \frac{32}{9} H_0^2 H_1 \right]}{9z} - \frac{64(2z^3 + z^2 - 2z + 4) H_0 H_{0,1}}{9z} \\
+ \frac{64(10z^3 + 16z^2 + z + 4) H_{0,0,1}}{9z} + \frac{128(3z^3 + 12z^2 + 12z + 1) H_{0,1,1}}{9z} \\
+ \frac{(8z^2 + 15z + 3) \left[ \frac{16}{27} H_0^6 + \frac{32}{9} \zeta_2 H_0 \right] + (z + 1) \left[ - \frac{8}{9} H_0^4 + \frac{64}{3} H_{0,1} H_0^2 - \frac{32}{3} \zeta_2 H_0^2 \right]}{9z} \\
+ L_M \left[ \frac{256 \zeta_2 z^2}{3} - \frac{128}{27} (14z^2 + 127z + 64) H_0 + (z + 1) \left[ - \frac{64}{9} H_0^3 - \frac{928}{9} H_0^2 \right] \right] \\
+ \frac{256}{3} H_{0,1} H_0 - \frac{256}{3} \zeta_2 H_0 - \frac{256}{3} H_{0,0,1} + \frac{128}{3} H_{0,1,1} + \frac{128 \zeta_3}{3} \right] + \frac{64(z - 1)(616z^2 + 667z + 94)}{81z} \\
+ \frac{128(z - 1)(4z^2 - 26z + 13) H_1}{27z} + \frac{(z - 1)(4z^2 + 7z + 4) \left[ - \frac{32}{9} H_1^2 - \frac{128}{9} H_0 H_1 \right]}{9z} \\
- \frac{128(2z^3 - 3z^2 + 3z + 4) H_{0,1}}{9z} \right) ]]}
\[ + C_F N_F T_F^2 \left[ \left( \frac{64}{9} (z + 1) H_0 - \frac{32 (z - 1)(4 z^2 + 7 z + 4)}{27 z} \right) L_Q^3 \right. \\
+ \left[ - \frac{64}{3} (z + 1) H_0^2 + \frac{64}{9} (4 z^2 - 11 z - 8) H_0 + \frac{32 (z - 1)(10 z^2 + 33 z - 2)}{9 z} \right] L_Q^2 \right. \\
+ \left( \frac{256}{9} H_{-1} - \frac{256}{9} H_{-1} H_0 \right) (z + 1)^3 + \left[ \frac{64}{3} H_0^3 - \frac{64}{3} H_{0,1,1} + \frac{64 \zeta_3}{3} \right] (z + 1) \\
- \frac{32}{9} \left( 12 z^2 - 59 z - 29 \right) H_0^2 + \frac{16 (z - 1)(4 z^2 + 7 z + 4) H_1^2}{9 z} - \frac{64 (z - 1)(194 z^2 + 683 z + 68)}{81 z} \\
- \frac{64}{9} \left( 2 z^3 + 23 z^2 + 8 z + 4 \right) \frac{\zeta_2}{z} - \frac{64}{27} \left( 79 z^2 - 88 z - 190 \right) H_0 - \frac{32 (z - 1)(38 z^2 + 47 z + 20) H_1}{27 z} \\
+ \frac{64}{9} \left( 2 z^2 + 11 z + 8 \right) H_{0,1} \right] L_Q + \frac{16}{27} \left( 8 z^2 + 15 z + 3 \right) H_0^3 + \left[ \frac{32 (z - 1)(4 z^2 + 7 z + 4)}{27 z} \right. \\
\left. - \frac{64}{9} (z + 1) H_0 \right] L_M^3 - \frac{32}{27} \left( 56 z^2 + 33 z + 21 \right) H_0^2 - \frac{64 (z - 1)(1156 z^2 - 203 z + 328)}{243 z} \\
- \frac{16}{27} \left( z - 1 \right) \left( 74 z^2 - 43 z + 20 \right) \frac{\zeta_2}{z} - \frac{32}{27} \left( 100 z^3 + 183 z^2 + 33 z - 4 \right) \frac{\zeta_3}{z} \\
+ \frac{32}{81} \left( 800 z^2 - 57 z + 111 \right) H_0 + \frac{(z - 1) \left( 28 z^2 + z + 10 \right) \left[ \frac{128}{27} H_0 H_1 - \frac{128}{27} H_{0,1} \right]}{z} \\
- \frac{128}{9 z} \left( 2 z^3 + 6 z^2 + 3 z + 2 \right) H_0 H_{0,1} + \frac{64 \left( 12 z^3 + 27 z^2 + 9 z + 4 \right) H_{0,0,1}}{9 z} \\
+ \frac{32}{9} \left( 6 z^2 + 4 z - 5 \right) H_0 \zeta_2 + \frac{(z - 1)(4 z^2 + 7 z + 4) \left[ - \frac{32}{9} H_1 H_0^2 - \frac{16}{9} H_1 \zeta_2 \right]}{z} \\
+ L_M^2 \left[ \frac{32}{3} (z - 1)(2 z - 5) - \frac{32}{9} (4 z^2 - 7 z - 13) H_0 + \frac{32 (z - 1)(4 z^2 + 7 z + 4) H_1}{9 z} \right] \\
+ (z + 1) \left[ \frac{32}{3} H_0^2 - \frac{64}{3} H_{0,1} + \frac{64 \zeta_3}{3} \right] \right] + (z + 1) \left[ - \frac{8}{9} H_0^3 + \frac{64}{3} H_{0,1} H_0^2 - \frac{128}{3} H_{0,0,1} H_0 \right. \\
+ \frac{832}{9} \zeta_3 H_0 - \frac{32 \zeta_2^2}{3} \left[ \frac{32}{3} H_{0,1} - \frac{32}{3} H_0^2 \right] \zeta_2 + L_M \left[ \frac{32}{9} (4 z^2 - 7 z - 13) H_0^2 \right. \\
+ \frac{64}{27} \left( 2 z^2 + 2 z - 58 \right) H_0 + \frac{128(z - 1)(25 z^2 + 94 z + 34)}{81 z} + \frac{32(z - 1)(74 z^2 - 43 z + 20) H_1}{27 z} \left] \right. \\
+ \frac{(z - 1)(4 z^2 + 7 z + 4) \left[ \frac{16}{9} H_1^2 - \frac{64}{9} H_0 H_1 \right]}{81 z} - \frac{64(2 z^3 + z^2 - 2 z + 4) H_{0,1}}{9 z} \\
- \frac{64}{9} \left( 6 z^2 + 4 z - 5 \right) \zeta_2 + (z + 1) \left[ - \frac{64}{9} H_0^3 + \frac{128}{3} H_{0,1} H_0 - \frac{128}{3} \zeta_2 H_0 - \frac{128}{3} H_{0,0,1} \right. \\
- \frac{64}{3} H_{0,1,1} + \frac{64 \zeta_3}{3} \right] \left] + C_F C_A T_F \left[ - \frac{2}{9} (4 z - 17) H_1^4 - \frac{4}{9} (36 z^2 + 47 z + 36) H_0^3 \right. \\
+ \frac{4}{27} (2132 z^2 - 681 z + 855) H_0^2 + \frac{8(z - 1)(122 z^2 - 19 z + 113) H_1 H_0^2}{9 z} \right. \\
- \frac{8(19 z^2 + 19 z + 8) H_{0,1} H_0^2}{3 z} - \frac{32(2 z - 1) H_{0,0,1} H_0^2 - \frac{8}{3}(10 z - 17) \zeta_2 H_0^2 - \frac{16}{3} (14 z - 13) \zeta_3 H_0^2}{9 z} \\
- \frac{16(z - 1)(19 z^2 + 16 z + 10) H_1 H_0^2}{9 z} + \frac{16}{3} (38 z - 5) \zeta_2 H_0 + \frac{16}{9} (13 z^2 - 215 z - 4) \frac{\zeta_3}{z} H_0 \right] \]
\[
\begin{align*}
- & \frac{4(54204z^3 + 9339z^2 + 17082z + 2624)H_0}{81z} - \frac{4}{9}(474z^3 + 121z^2 + 295z + 80)\frac{\zeta_2}{z}H_0 \\
- & \frac{32(z - 1)(769z^2 - 62z + 337)H_1H_0}{27z} - \frac{32(19z^3 - 24z^2 - 6z + 10)H_{0,-1}H_0}{9z} \\
+ & \frac{16(18z^3 + 127z^2 + 6z + 51)H_{0,1}H_0}{3z} + \frac{64(10z^2 + z + 4)H_{0,0,1}H_0}{3z} + 64(5z - 2)H_{0,0,0,1}H_0 \\
+ & \frac{8(z - 1)(2z + 1)(14z + 1)H^3_1}{27z} + \frac{4(z - 1)(328z^2 + 313z + 67)H^2_1}{27z} \\
+ & \frac{4(z - 1)(75516z^2 - 7654z + 21765)}{81z} + \frac{8z}{27}(1841z^3 - 1719z^2 + 1230z - 515)\frac{\zeta_2}{z} \\
+ & \frac{8}{15}(20z^3 + 340z^2 - 137z + 152)\frac{\zeta_2}{z} + \frac{8}{9}(768z^3 + 1158z^2 + 687z + 76)\frac{\zeta_3}{z} \\
+ & \frac{4(z - 1)(2500z^2 + 2755z + 1771)H_1}{81z} + \frac{4}{9}(z - 1)(154z^2 + 163z + 46)\frac{\zeta_2}{z}H_1 \\
+ & \frac{(z + 1)(182z^2 - 122z + 47)}{z}\left[\frac{32}{9}H_{-1}H_0 - \frac{32}{27}H_{0,-1}\right] \\
+ & \frac{8(2964z^3 - 3393z^2 + 1368z - 1348)H_{0,1}}{27z} + \frac{32(19z^3 - 51z^2 - 6z + 10)H_{0,0,-1}}{9z} \\
- & \frac{16(230z^3 + 621z^2 + 168z + 193)H_{0,0,1}}{9z} + \frac{8(56z^3 - 105z^2 - 66z - 40)H_{0,1,1}}{9z} \\
- & \frac{16(21z^2 - 15z + 8)H_{0,0,0,1}}{3z} - \frac{16(20z^3 + 18z^2 - 15z - 20)H_{0,0,1,1}}{3z} \\
- & \frac{32(z - 1)(z + 2)(2z + 1)H_{0,1,1,1}}{3z} - 128(4z - 1)H_{0,0,0,0,1} + L_M^3\left[\frac{16}{3}(2z - 1)H_0^2\right] \\
+ & \frac{16(8z^2 + 11z + 4)H_0}{9z} - \frac{8(z - 1)(44z^2 - z + 44)}{9z} - \frac{16(z - 1)(4z^2 + 7z + 4)H_1}{9z} \\
+ & \frac{(z + 1)\left[\frac{32}{3}H_{0,1} - \frac{32\zeta_2}{3}\right]}{z} - \frac{8}{3}(23z + 14)H_{0,1}\zeta_2 + \frac{(z + 1)(19z^2 - 16z + 10)}{z}\left[-\frac{32}{9}H_0H_{-1}^2\right] \\
+ & \left(\frac{16}{9}H_0^2 + \frac{64}{9}H_{0,-1}\right)H_{-1} - \frac{32}{9}\zeta_2H_{-1} - \frac{64}{9}H_{0,-1,-1}\right] + L_Q^3\left[-\frac{16}{3}(2z - 1)H_0^2\right] \\
- & \frac{16(8z^2 + 11z + 4)H_0}{9z} + \frac{8(z - 1)(44z^2 - z + 44)}{9z} + \frac{16(z - 1)(4z^2 + 7z + 4)H_1}{9z} \\
+ & \left(\frac{16}{3}H_0^3 - \frac{32}{3}H_{0,1}\right) + \frac{8}{3}(43z + 37)\zeta_2\zeta_3 + L_Q^2\left[\frac{16}{3}(4z - 3)H_0^3\right] \\
- & \frac{32(101z^3 - 53z^2 + 82z + 13)H_0}{9z} - \frac{32(2z + 1)\zeta_2H_0}{9z} + \frac{(z - 1)(4z^2 + 7z + 4)}{27z}\left[-\frac{16}{3}H_1^2 - \frac{32}{3}H_0H_1\right] \\
- & \frac{(z + 1)(4z^2 - 7z + 4)\left[\frac{32}{3}H_{-1}H_0 - \frac{32}{3}H_{0,-1}\right]}{z} + 16(2z - 1)H_{0,1} + (z - 1)\left[64H_0H_{0,-1}\right] \\
- & 128H_{0,0,-1} - 96H_{0,0,1} + (z + 1)\left[\frac{224}{3}H_0^2 + 64H_{0,1}H_0 + 64H_{0,1,1}\right] + \frac{16}{3}(8z^2 - 6z - 3)\zeta_2 \\
+ & 32(z - 2)\zeta_3 + L_M^2\left[\frac{8}{3}(16z - 17)H_0^2 + \frac{8(272z^3 + 103z^2 + 139z + 40)}{9z}H_0\right]
\end{align*}
\]
\[ + a_{PS,(3)} + C_{\pi,q}^{PS,(3)}(N_F + 1) \tag{597} \]

The Wilson coefficient \( H_{g,2}^S \) is given by:

\[
H_{g,2}^S = a_s T_F \left\{ - \gamma_{qq}^0 L_Q - 4(8z^2 - 8z + 1) + \gamma_{qq}^0 \left[ H_0 + H_1 \right] + \gamma_{qq}^0 L_M \right\} \\
+ a_s^2 \left\{ T_F^2 \left[ \frac{\gamma_{qq}^0 L_M^2 - \frac{4}{3} \gamma_{qq}^0 L_Q L_M}{3} + \left[ \frac{4}{3} H_0^2 - \frac{4}{3} \gamma_{qq}^0 H_0 \right] - \frac{16}{3} (8z^2 - 8z + 1) \right] L_M \right\} \\
+ C_A T_F \left[ \frac{16}{3} (3z + 1) H_0^3 - 2z(57z - 92) H_0^2 + 4(8z^2 + 6z + 3) H_{-1} H_0^2 - 4(4z^2 - 6z + 3) H_1 H_0^2 \\
- \frac{4}{9} (1445z^3 - 747z - 219) H_0 - \frac{4(199z^3 - 168z^2 - 3z - 16) H_1 H_0}{3z} - 8(4z^2 + 6z + 3) H_{0,-1} H_0 \right. \\
+ 8(14z + 5) H_{0,1} H_0 + 16(2z^2 - 8z - 1) \zeta H_0 - \frac{2(107z^3 - 96z^2 + 9z - 8) H_1}{3z} \\
+ \left. \left[ \frac{4(z - 1)(3z^2 + 7z + 4)}{3z} - \frac{8(4z + 1) H_0 - 2 \gamma_{qq}^0 H_1}{9z} \right] L_M^2 + \frac{2(439z^3 - 130z^2 - 233z - 40)}{9z} \right\} \\
+ \frac{4}{3} (219z^3 - 204z^2 + 12z - 8) \zeta H_0 - \frac{4(749z^3 - 645z^2 - 84z + 52)}{9z} H_1 \\
+ L_Q \left[ - \frac{4(z - 1)(3z^2 + 7z + 4)}{3z} + \frac{8(4z + 1) H_0 + 2 \gamma_{qq}^0 H_1}{9z} \right] + \frac{16}{3} \zeta H_{-1} H_0 \right. \\
- \frac{16}{3} H_{0,-1} - \frac{4(20z^3 - 48z^2 + 15z + 16) H_{0,1}}{3z} + \gamma_{qq}^0 \left[ 4H_0 H_1^2 - 4H_1 H_{0,1} \right] + z^2 \left[ -16H_0 H_2 \right. \\
+ 32H_{0,-1} H_{-1} - 32H_{0,-1,-1} \right] + 24(2z + 1) H_{0,0,-1} - 8(18z + 5) H_{0,0,1} + (2z^2 + 2z + 1) \left[ 16H_{-1} H_{0,1} \right. \\
- 16H_{0,-1} - 16H_{0,-1,-1} \right] - 32(z - 3) \zeta H_{0,1,1} - 16(3z^2 + 2z + 1) H_{-1} \zeta + 16(1 - 1) H_2 \zeta \right. \\
+ L_M \left[ 8(2z + 1) H_0^2 - \frac{8}{3} (44z^2 + 24z + 3) H_0 - 2 \gamma_{qq}^0 H_1^2 + \frac{8(218z^3 - 225z^2 + 18z - 20)}{9z} \right. \\
+ 32(z - 1) \zeta H_1 + (2z^2 + 2z + 1) \left[ 16H_{-1} H_0 - 16H_{0,-1} \right] + 32 \zeta \zeta \right. \\
+ L_Q \left[ -16(3z + 1) H_0^2 \right. \\
+ 8z(25z - 24) H_0 + \frac{4(407z^3 - 276z^2 - 165z + 52)}{9z} + \frac{8(67z^3 - 60z^2 + 3z - 4) H_1}{3z} + \gamma_{qq}^0 \left[ -2H_1^2 \right. \\
- 4H_0 H_1 \right] + (2z^2 + 2z + 1) \left[ 16H_{0,-1} - 16H_{-1} H_0 \right] - 16(4z + 1) H_{0,1} - 32(z - 2) \zeta \zeta \right. \\
+ 8(6z^2 + 2z + 5) \zeta \zeta \right. \\
+ C_F T_F \left[ -2(4z - 1) - 4(4z^2 - 2z + 1) H_0 + 2 \gamma_{qq}^0 H_1 \right] L_Q^2 \\
+ \left[ 8(4z^2 - 2z + 1) H_0^2 + 8(10z^2 - 6z + 1) H_0 + 4(4z^2 - 17z + 9) + 4(2z - 1)(10z - 7) H_1 \right. \\
+ \gamma_{qq}^0 \left[ -4H_1^2 - 8H_0 H_1 \right] + 8(2z - 1) H_{0,1} + \left[ -4(4z - 1) + 8(4z^2 - 2z + 1) H_0 - 4 \gamma_{qq}^0 H_1 \right] L_M \\
- \left. 8(8z^2 - 6z + 3) \zeta \right] L_Q - \frac{4}{15} (72z^3 + 195z^2 - 10z + 15) H_0^2 - 2(42z^2 - 44z + 11) H_1^2 \right. \]

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\[
+ \left[2(4z - 1) - 4(4z^2 - 2z + 1)H_0 + 2\gamma_{qq}^0 H_1 \right] L_M^2 + \frac{4(246z^3 + 51z^2 - 226z + 4)}{15z} \\
- \frac{8(252z^3 - 51z^2 + 89z + 2)H_0}{15z} - 16z^2 H_0^2 H_1 - 4 \left[24z^2 - 33z + 7\right] H_1 \\
- 8(13z^2 - 14z + 6) H_0 H_1 + \frac{(36z^5 + 40z^3 + 90z^2 + 1)}{z^2} \left[\frac{16}{15} H_{-1} H_0 - \frac{16}{15} H_{0,-1}\right] \\
+(z - 1)^2 H_0 \left[32 H_{0,-1} - 32 H_{0,1}\right] - 8(8z^2 - 6z - 3) H_{0,1} + \gamma_{qq}^0 \left[2H_1^3 + 4H_0 H_1^2 + 4H_{0,1} H_1\right] \\
- 32(3z^2 - 2z + 3) H_{0,0,-1} - 32(2z - 1) H_{0,0,1} + 8(4z^2 - 6z + 3) H_{0,1,1} \\
+ \frac{8}{15} \left(72z^3 + 315z^2 - 220z + 45\right) \zeta_2 + 32(3z^2 - 2z + 1) H_0 \zeta_2 + L_M \left[-8(4z^2 - 2z + 1) H_0^2 \\
- 8(10z^2 - 6z + 1) H_0 - 4(4z^2 - 17z + 9) - 4(2z - 1)(10z - 7) H_1 + \gamma_{qq}^0 \left[4H_1^2 + 8H_0 H_1\right] \\
- 8(2z - 1) H_{0,1} + 8(8z^2 - 6z + 3) \zeta_2\right] + (z + 1)^2 \left[-32 H_0 H_{-1}^2 + \left[16H_0^2 + 64H_{0,-1}\right] H_{-1} \\
- 32 \zeta_2 H_{-1} - 64 H_{0,-1,-1}\right] + (4z^2 - 2z + 1) \left[16H_1 \zeta_2 - \frac{8}{3} H_0^3\right] + 8(20z^2 + 2z + 7) \zeta_3\right]\right) \\
+ a_s^3 T_F^3 \left[\frac{16}{9} \gamma_{qq}^0 L_M^3 - \frac{16}{9} \gamma_{qq}^0 L_Q L_M^2 + \left[\gamma_{qq}^0 \left[16H_0^3 \frac{9}{9} + 16H_1^2 \frac{9}{9}\right] - \frac{64}{9} (8z^2 - 8z + 1) L_M^2 - \frac{16\gamma_{qq}^0 \zeta_3}{9}\right]\right] \\
+C_A T_F^3 \left[-\frac{8}{9} (2z + 1) H_0^3 + \frac{16}{27} (23z^2 - 6z + 3) H_0^3 - \frac{8}{27} (822z^2 + 592z + 229) H_0^2 \\
- \frac{16(z - 1)(65z^2 + 17z + 8) H_1 H_0^2}{9z} + \frac{64}{3} (4z + 1) H_{0,1} H_0^2 - \frac{8}{3} (2z + 9) \zeta_2 H_0^2 + \frac{64}{3} (z - 1) z H_1 H_0 \\
+ \frac{8}{81} (16303z^3 - 4390z^2 + 20000) H_0 + \frac{16(1128z^3 - 1147z^2 + 146z - 80) H_1 H_0}{27z} \\
- \frac{32(3z^2 - 9z^2 + 15z + 8) H_{0,1} H_0}{9z} - \frac{128}{3} (6z + 1) H_{0,0,1} H_0 + \frac{8}{9} (272z^2 + 360z + 3) \zeta_2 H_0 \\
+ \frac{64}{9} (41z + 14) \zeta_3 H_0 - \frac{16}{9} (z - 1)(5z + 1) H_1^2 + L_M^3 \left[\frac{112(z - 1)(31z^2 + 7z + 4)}{27z} - \frac{56}{9} \gamma_{qq}^0 H_1\right] \\
- \frac{224}{9} (4z + 1) H_0 \right] + \frac{8}{27} (206z^2 - 143z + 67) H_1^2 - \frac{4(30335z^3 - 36798z^2 + 17367z - 13244)}{243z} \\
- \frac{32}{15} (77z + 2) \zeta_2^2 - \frac{4}{9} (776z^3 - 1628z^2 + 59z - 128) \zeta_2 \frac{2}{z} - \frac{16}{27} (1255z^3 + 150z^2 + 108z - 28) \zeta_3 \frac{3}{z} \\
+ \frac{8(6819z^3 - 6878z^2 + 106z - 628) H_1}{81z} + L_Q^3 \left[-\frac{16(z - 1)(31z^2 + 7z + 4)}{27z}\right] \\
- \frac{32}{9} (4z + 1) H_0 + \frac{8}{9} \gamma_{qq}^0 H_1 \right] - \frac{32(672z^3 - 257z^2 + 76z - 40) H_{0,1}}{27z} \\
+ \frac{16(222z^3 + 244z^2 + 61z + 16) H_{0,0,1}}{9z} - \frac{32}{3} z(5z - 1) H_{0,1,1} + \frac{32}{3} (22z + 1) H_{0,0,0,1} \\
- \frac{160}{9} (5z^2 - 5z + 1) H_1 \zeta_2 + L_M^2 \left[16H_0^2 - \frac{16}{9} (127z^2 + 232z + 25) H_0 \\
+ \frac{8(2612z^3 - 2514z^2 - 33z - 200)}{27z} + \frac{16(135z^3 - 128z^2 + 13z - 4) H_1}{9z} + \gamma_{qq}^0 \left[-\frac{20}{3} H_1^2\right}\right]
\]
\[-\frac{8}{3}H_0H_1 \right) + (2z^2 + 2z + 1) \left[ 32H_{-1}H_0 - 32H_{0,-1} \right] - \frac{32}{3} (4z + 1)H_{0,1} - \frac{64}{3} (z - 6)z\zeta_2 \]

\[+ L^2 Q \left[ \frac{-\frac{16}{3} (8z + 1)H_0^2 + \frac{16}{9} (49z^2 - 136z - 13)H_0 + \frac{8}{3} (1048z^3 - 894z^2 - 87z - 40)}{27z} \right] \]

\[+ \frac{16(87z^3 - 80z^2 + 13z - 4)H_0}{9z} + \gamma^0_{qq} \left[ -\frac{4}{3} H_1^2 - \frac{8}{3} H_0H_1 \right] \]

\[+ (2z^2 + 2z + 1) \left[ \frac{32}{3} H_{0,-1} - \frac{32}{3} H_{-1}H_0 \right] - \frac{32}{3} (4z + 1)H_{0,1} + L_M \left[ \frac{-\frac{16}{3} (z - 1) (31z^2 + 7z + 4)}{9z} \right] \]

\[+ \frac{32}{3} (4z + 1)H_0 - \frac{8}{3} \gamma^0_{qq} H_1 \right) - \frac{64}{3} (z - 2)z\zeta_2 \right] + z(z + 1) \left[ \frac{128}{3} H_0H_1^2 \right] \]

\[\left[ -\frac{64}{3} H_0^2 - 128H_0 - \frac{256}{3} H_{0,-1} \right] H_{-1} + \frac{128}{3} \zeta_2 H_{-1} + \frac{128}{3} H_0H_{0,-1} + 128H_{0,-1} \]

\[+ \frac{256}{3} H_{0,-1,-1} - \frac{128}{3} H_{0,0,0,1} \right] + L^2 Q \left[ \frac{448}{9} (z^2 - 96z - 412) \right] \]

\[- \frac{16(z - 1) (31z^2 + 7z + 4)}{9z} \left[ \frac{32}{3} (4z + 1)H_0 + \frac{8}{3} \gamma^0_{qq} H_1 \right] \]

\[+ \frac{32}{3} (5z^2 + 4z + 2) H_{-1}H_0^2 + \frac{32}{3} (3z^2 - 4z + 2) H_1H_0^2 - \frac{64}{27} (325z^2 - 832z - 31)H_0 \]

\[- \frac{64}{3} (3z^2 + 4z + 2) H_{0,0,-1} + \frac{64}{3} (z^2 + 2z + 2) H_{0,1}H_0 + \frac{128}{3} (z - 5)z\zeta_2 H_0 \]

\[+ \left[ \frac{-16(z - 1) (31z^2 + 7z + 4)}{9z} \right] \]

\[+ \left[ \frac{81z}{27z} + \frac{32}{9} \left( 105z^3 - 184z^2 + 3z - 4 \right) \frac{\zeta_2}{z} \right] \]

\[\frac{32 (1032z^3 - 964z^2 + 65z - 20) H_1}{27z} + \left[ \frac{17z^3 + 7z^2 - 4z - 2}{z^2} \right] \left[ \frac{64}{9} H_{-1}H_0 - \frac{64}{9} H_{0,-1} \right] - \frac{64 (9z^3 - 59z^2 - 5z + 2)}{9z} H_{0,1} \]

\[+ \gamma^0_{qq} \left[ \frac{8}{9} H_1^2 + \frac{16}{3} H_0H_1^2 - \frac{32}{3} H_{0,1}H_1 \right] + (2z + 1) \left[ \frac{32}{3} H_0H_1^2 + \frac{64}{3} H_{0,-1}H_{1,-1} + \frac{64}{3} H_{0,-1,-1} \right] \]

\[= \frac{64}{3} (z^2 + 4z + 2) H_{0,0,-1} - \frac{64}{3} (3z^2 - 5z + 1) H_{0,1,1} - \frac{32}{3} (4z^2 + 2z + 1) H_{-1}H_1^2 - \frac{32}{3} (2z - 1) H_{1}H_1 \]

\[+ L_M \left[ \frac{-32}{3} (4z + 3) H_1^2 + \frac{32}{9} (101z^2 - 8z + 13) H_0 - \frac{32 (8z^3 - 14z^2 + 11z - 8)}{3z} \right] \]

\[+ \frac{32 (47z^3 - 40z^2 - 7z - 4) H_1}{9z} + \gamma^0_{qq} \left[ \frac{8}{9} H_1^2 - \frac{16}{3} H_0H_1 \right] + (2z^2 + 2z + 1) \left[ \frac{64}{3} H_{0,-1} \right] \]

\[- \frac{64}{3} H_{-1}H_0 - \frac{64}{3} (4z + 1) H_{0,1} - \frac{128}{3} (z - 2)z\zeta_2 \right] + \frac{32}{3} (6z^2 - 22z + 3) \zeta_3 \]

\[+ L_M \left[ \frac{-64}{9} (z - 2) H_1^3 + \frac{32}{9} (54z^2 + 10z + 5) H_0 + \frac{32}{3} (3z^2 + 2z + 1) H_{-1}H_0^2 \right] \]

\[\frac{32}{3} (z - 1) H_1 H_0^2 - \frac{16}{27} (1680z^2 + 1744z - 269) H_0 - \frac{128 (28z^3 - 22z^2 - 4z - 3) H_1H_0}{9z} \]

\[- \frac{64}{3} (z + 1)^2 H_{0,-1}H_0 - \frac{64}{3} (z^2 - 12z - 3) H_{0,1}H_0 + \frac{128}{3} (z^2 - 3z - 1) \zeta_2 H_0 \]

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$$- \frac{64(5z^3 - 4z^2 - z - 1)H_1^2}{9z} + \frac{8(20863z^3 - 20616z^2 + 159z - 172)}{81z}$$

$$+ \frac{32}{9} \left( 114z^3 - 20z^2 + 9z - 4 \right) \frac{\zeta_2}{z} + \frac{16(476z^3 - 548z^2 + 253z - 144)H_1}{27z}$$

$$+ \gamma_{qg}^0 \left[ \frac{16}{3} H_0 H_1^2 - \frac{8}{9} H_3^3 \right] \left[ \frac{(9z^3 - z^2 - 14z - 2)}{z} \left[ \frac{64}{9} H_{-1} H_0 - \frac{64}{9} H_{0,-1} \right] \right]$$

$$- \frac{32(2z^3 + 70z^2 + 25z + 12)H_{0,1}}{9z} + \left( 4z^2 + 2z + 1 \right) \left[ - \frac{32}{3} H_0 H_1^2 + \frac{64}{3} H_{0,-1} H_1 \right]$$

$$- \frac{64}{3} H_{0,-1,1} \left[ z - \frac{64}{3} (z^2 - 2z - 1)H_{0,0,-1} + \frac{64}{3} (z^2 - 20z - 3)H_{0,1,1} + \left( 2z^2 + 2z + 1 \right) \left[ \frac{64}{3} H_{-1} H_{0,0} \right] \right]$$

$$- \frac{64}{3} H_{0,-1,1} - \frac{64}{3} H_{0,1,1} + \frac{128}{3} (z^2 + z + 1)H_{0,1,1} - \frac{32}{3} (8z^2 + 6z + 3) H_{-1} \zeta_2$$

$$+ \frac{32}{3} \left( 4z^2 - 6z + 3 \right) H_1 \zeta_2 + \frac{32}{3} \left( 6z^2 + 26z + 7 \right) \zeta_3 \right] + \left( 2z^2 + 2z + 1 \right) \left[ - \frac{128}{9} H_0 H_1 \right]$$

$$+ \frac{\left[ \frac{32}{3} H_0^2 + \frac{128}{3} H_{0,1} \right] H_1}{H_0} \zeta_3 + \left[ \frac{32}{3} H_3^3 + \left[ \frac{128}{3} H_{0,1} - \frac{128}{3} H_{0,-1} \right] H_0 \right] \left[ \frac{256}{3} H_{0,1} \right] - \frac{64}{3} \left( H_1 + 64 H_{0,1} \right) - \frac{32}{3} \left( \frac{H_0}{H_{0,1}} \right) \zeta_3$$

$$+ \frac{128}{3} \left( \frac{H_0}{H_{0,1}} \right) + \frac{256}{3} \left( \frac{H_0}{H_{0,1}} \right) - \frac{64}{3} \left( H_0 - \frac{128}{3} H_{0,1} \right) - \frac{64}{3} \left( H_{0,1} \right) + \frac{256}{3} \left( H_0 \right) - \frac{128}{3} \left( H_{0,1} \right) - \frac{256}{3} \left( H_0 \right)$$

$$+ \frac{\left[ 80 H_{0,1} + \frac{16}{3} H_{0,1,1} \right] + \zeta_2 + \frac{\gamma_{qg}^0 \left[ \frac{2}{9} H_1^4 + \frac{4}{3} \frac{H_0}{H_1} + \frac{10}{3} \zeta_2 H_1 \right]}{C_{AN} T_F^2} \left[ \frac{-8}{9} (2z + 1) H_0 + \frac{8}{27} (23z^2 + 12z + 3) H_3^3 \right] + \frac{8}{27} \left( 292z^2 + 39z + 42 \right) H_3^2$$

$$- \frac{8}{27} \left( 292z^2 + 39z + 42 \right) H_3^2 - \frac{8(z - 1) (65z^2 + 17z + 8) H_1 H_0^2}{9z} + \frac{32}{3} (4z + 1) H_{0,1} H_0^2$$

$$- \frac{32}{3} (z - 1) \frac{H_2^2}{H_0} + \frac{16}{81} \left( 3392z^2 + 645z + 111 \right) H_0$$

$$+ \frac{32}{3} (z - 1) \left( 254z^2 - 7z + 20 \right) H_1 H_0$$

$$- \frac{16}{27} \left( 23z^3 + 96z^2 + 15z + 8 \right) H_{0,1} H_0$$

$$+ \frac{8}{9} (62z^2 - 16z - 7) \zeta_2 H_0 + \frac{32}{9} (28z + 13) \zeta_3 H_0$$

$$- \frac{32}{9} (z - 1) (5z + 1) H_3^3 + \left[ \frac{16}{9} (z - 1) \frac{(31z^2 + 7z + 4)}{27z} - \frac{32}{9} (4z + 1) H_0 - \frac{8}{9} \gamma_{qg}^0 \frac{H_1}{L_M} \right]$$

$$+ \frac{8}{3} (4z^2 - 6z + 1) H_1^2 - \frac{32}{243} \left( 5854z^3 - 6219z^2 + 531z - 328 \right) + \frac{4}{9} \left( 19z^3 - 50z^2 + 2z - 4 \right) \frac{\zeta_2}{z}$$

$$- \frac{16}{27} \left( 550z^3 + 228z^2 + 33z - 4 \right) \frac{\zeta_3}{z} + \frac{32}{3} (2z^2 - 2z - 1) H_1 + \frac{L_Q}{z} $$

$$- \frac{16}{27} \left( 290z^3 - 261z^2 + 27z - 20 \right) H_{0,1}$$

$$+ \frac{16}{9} (111z^3 + 144z^2 + 21z + 8) H_{0,0,1} - \frac{16}{3} \frac{H_3 (5z - 4) H_{0,1,1}}{H_{0,1,1}} - \frac{16}{9} \left( 2z^2 - 2z - 5 \right) H_1 \zeta_2$$

$$+ \frac{32}{9} (4z + 1) H_0 + \frac{8}{9} \gamma_{qg}^0 \frac{H_1}{L_Q}$$

$$+ \frac{16}{9} \left( 87z^3 - 80z^2 + 13z - 4 \right) H_1 + \frac{L_Q}{z} \left[ \frac{16}{9} (8z + 1) H_0^2 + \frac{16}{9} \left( 49z^2 - 136z - 13 \right) H_0 + \frac{8}{27} \left( 1048z^3 - 894z^2 - 87z - 40 \right) \right]$$

$$+ \frac{16}{9} \left( 87z^3 - 80z^2 + 13z - 4 \right) H_1 + \gamma_{qg}^0 \left[ - \frac{4}{3} H_1^2 - \frac{8}{3} H_0 H_1 \right] + \left( 2z^2 + 2z + 1 \right) \left[ \frac{32}{3} \right] H_{0,-1}$$
\[-\frac{32}{3} H_{-1} H_0 \left\{ \frac{32}{3} (4z + 1) H_{0,1} - \frac{64}{3} (z - 2) z \zeta_2 \right\} + z (z + 1) \left( \frac{64}{3} H_0 H_1^2 \right) + \left[ \frac{32}{3} H_0^2 - 64 H_0 \right] \]

\[-\frac{128}{3} H_{0,1} H_{-1} + \frac{64}{3} \zeta_2 H_1 + \frac{64}{3} H_0 H_{0,-1} + 64 H_{0,-1} + \frac{128}{3} (H_{0,-1} - 1) = 64 H_{0,0,-1} \]

\[+ (z + 1) \left[ -\frac{8}{3} \zeta_2 H_0^2 - \frac{64}{3} H_{0,0,1} H_0 \right] + z \left[ 128 H_{0,0,0,1} - \frac{1216 \zeta_2^2}{15} \right] + L_M^2 \left[ -\frac{4}{3} \zeta_2 H_1^2 \right] \]

\[-\frac{32}{9} (4z^2 - 4z + 5) H_1 + \frac{8}{27} (205z^3 - 168z^2 + 42z - 52) - \frac{32}{9} (9z^2 - 20z - 5) H_0 \]

\[+ (z^2 + 2z + 1) \left[ \frac{32}{3} H_{-1} H_0 - \frac{32}{3} H_{0,-1} \right] + z \left[ \frac{64}{3} H_0^2 + \frac{64 \zeta_2}{3} \right] + L_M \left[ -\frac{16}{9} (10z - 1) H_0^3 \right] \]

\[+ \frac{8}{9} (7z^2 - 66z - 13) H_0^2 + \frac{16}{27} (58z^2 - 269z - 2) H_0 - \frac{16 (z - 1) (65z^2 + 17z + 8) H_1 H_0}{9z} \]

\[-\frac{32}{3} (2z^2 - 10z - 1) H_{0,1} H_0 - \frac{8}{9} (13z^2 - 16z + 23) H_1^2 + \frac{8 (592z^3 - 268z^2 - 119z - 4)}{27z} \]

\[+ \frac{16}{27} (64z^2 - 64z + 29) H_1 + (4z^2 + 4z + 5) \left[ \frac{64}{9} H_{0,1} \right. - \frac{64}{9} \frac{H_{0,1} - 1}{H_1} \right] \]

\[+ 16 (68z^3 - 54z^2 - 9z - 8) H_{0,1} + \frac{32}{3} (2z^2 - 18z - 3) H_{0,0,1} - \frac{16}{9} z (3z + 10) \zeta_2 \]

\[+ (2z^2 + 2z + 1) \left[ -\frac{32}{3} H_{0,0,1} + \left( \frac{64}{3} H_{0,1} - \frac{16}{3} H_0^2 \right) H_1 - \frac{32}{3} \zeta_2 H_1 + \frac{32}{3} H_0 H_{0,1} - \frac{64}{3} H_{0,-1} \right] \]

\[+ \frac{32}{3} \left( H_{0,0,1} \right. - \left. \frac{16}{3} H_1^2 \right) \zeta_2 + \frac{32 (5z^2 + 4z + 2) H_{-1} H_0^2}{9z} \]

\[-\frac{32}{3} (3z^2 - 4z + 2) H_1 H_0^2 - \frac{16}{27} (1520z^2 - 2828z - 89) H_0 - \frac{64}{3} (3z^2 + 4z + 2) H_{0,1} H_0 \]

\[+ \frac{64}{3} (z^2 + 2z + 2) H_0,1 H_0 + \frac{128}{3} (z - 5) z \zeta_2 H_0 - \frac{8 (2306z^3 - 2056z^2 - 181z + 16)}{9z} \]

\[-\frac{32}{9} (105z^3 - 181z^2 + 3z - 4) \zeta_2^2 + \frac{16 (1934z^3 - 1807z^2 + 83z - 40) H_1}{27z} \]

\[+ (87z^3 - 80z^2 + 13z - 4) \left[ -\frac{16}{9} H_1^2 - \frac{32}{9} H_0 H_1 \right] + \frac{27z}{z} \left( 17z^3 + 7z^2 - 4z - 2 \right) \left[ \frac{64}{9} H_{-1} H_0 - \frac{64}{9} H_{0,1} \right] \]

\[-\frac{32z (18z^3 - 115z^2 - 10z + 4) H_{0,1}}{9z} + \frac{\zeta_2}{z} \left[ \frac{8}{9} H_1^3 + \frac{16}{3} H_0 H_1^2 - \frac{32}{3} H_{0,1} H_1 \right] + (2z + 1) \left[ \frac{32}{3} H_0 H_2^2 \right] \]

\[-\frac{64}{3} H_{0,-1} H_{-1} + \frac{64}{3} H_{0,-1,1} \right) + \frac{64}{3} \left[ z^2 - 2z + 2 \right] H_{0,0,1} - \frac{64}{3} (z^2 - 2z + 2) H_{0,0,1} \]

\[+ (2z^2 + 2z + 1) \left[ \frac{64}{3} H_{-1} H_1 - \frac{64}{3} H_{0,1} - \frac{64}{3} H_{0,1,1} \right] - \frac{128}{3} (3z^2 - 5z + 1) H_{0,1,1} \]

\[-\frac{32}{3} (4z^2 + 2z + 1) H_{-1} \zeta_2 - \frac{32}{3} (2z - 1) H_1 \zeta_2 + \frac{32}{3} (6z^2 - 22z + 3) \zeta_3 \]

\[+ (2z^2 + 2z + 1) \left[ -\frac{64}{9} H_0 H_3^1 + \left[ \frac{16}{3} H_0 + \frac{64}{3} H_{0,-1} \right] H_1 - \frac{16}{3} H_0 H_{0,-1} + \frac{16}{3} H_{0,0,-1} \right] H_0 \]

\[-\frac{128}{3} H_{0,-1,1} + \frac{64}{3} H_{0,0,1} - \frac{128}{3} H_{0,0,1} \right] H_1 + \frac{32}{3} \zeta_3 H_{0,1} - \frac{16}{3} H_0^2 H_{0,-1} + H_0 \left[ \frac{64}{3} H_{0,-1,1} \right] \]

\[-\frac{64}{3} H_{0,-1,1} + \frac{32}{3} H_{0,0,1} - \frac{64}{3} H_{0,1,1} \right] + \frac{128}{3} H_{0,-1,1} - \frac{64}{3} H_{0,0,1} - \frac{64}{3} H_{0,0,1} \]
\[
\begin{align*}
&+ \frac{128}{3} H_{0,0,-1,1} - \frac{32}{3} H_{0,0,0,-1} + \frac{128}{3} H_{0,0,1,-1} + \left[ -\frac{32}{3} H_{2,1}^2 - \frac{32}{3} H_0 H_1 - \frac{32}{3} H_{0,-1} \right] \zeta_2 + \gamma_4^0 \left[ \frac{1}{9} H_1^4 \right. \\
&+ \frac{2}{3} H_0^2 H_1^3 + \frac{4}{3} \zeta_2 H_1^3 + \left[ \frac{4}{3} H_0,0,1 \right] H_1 - \frac{40}{9} \zeta_3 H_1 - \frac{8}{3} H_{0,1}^2 + H_0 \left[ -\frac{8}{9} H_1^3 - 8 H_{0,1} H_1 \right. \\
&+ \frac{40}{3} H_{0,1,1} \right] - \frac{40}{3} H_{0,0,1,-1} - \frac{8}{3} H_{0,1,1,1} \right) + C_4^2 T_F \left[ \frac{1}{9} (18 z^2 - 26 z + 23) H_1^4 + \frac{2}{9} (225 z^2 - 12 z - 35) H_0^3 \\
&- \frac{4 (60 z^3 + 46 z^2 + 5 z + 8) H_1 H_0^2}{9 z} + \frac{8 (z - 1) (65 z^2 + 17 z + 8) H_1 H_0^3}{9 z} - \frac{32}{3} (4 z + 1) H_0 H_1 H_0^2 \\
&+ \frac{8}{3} (8 z - 5) \zeta_2 H_0^3 - \frac{4 (120 z^3 + 106 z^2 + 5 z + 8) H_1 H_0^2}{3 z} + \frac{2 (190 z^3 - 142 z^2 - 13 z - 24) H_1^2 H_0^2}{3 z} \\
&+ \frac{2}{27} (5428 z^2 - 6573 z + 738) H_0^3 + \frac{8 (z + 1) (359 z^2 - 32 z + 20) H_1 H_0^2}{3 z} \\
&+ \frac{2}{9} (32 z^2 - 62 z + 7) H_0,0,1 H_0^2 - 16 (2 z^2 - 6 z + 3) H_{0,0,-1} H_0^2 - 64 (z - 1) H_{0,0,1} H_0^2 \\
&+ 96 (z - 3) z H_{0,1,1} H_0^2 + \frac{2}{3} (38 z^2 - 6 z + 47) \zeta_2 H_0^2 - \frac{32}{3} (31 z - 5) \zeta_3 H_0^2 \\
&+ \frac{8 (93 z^3 - 82 z^2 + 8 z - 8) H_0^3}{9 z} - \frac{8 (241 z^3 - 229 z^2 + 11 z - 20) H_1 H_0^2}{9 z} + 32 (z^2 + 9 z + 3) H_0 H_1 H_0 \\
&+ \frac{16}{5} (124 z - 19) \zeta_2 H_0 - \frac{4 (22596 z^3 + 100242 z^2 + 16377 z + 2624) H_0}{81 z} \\
&+ \frac{2}{9} (6474 z^3 + 1864 z^2 + 1165 z + 160) \zeta_2 H_0 + \frac{8}{9} (210 z^3 + 340 z^2 - 269 z - 8) \zeta_3 H_0 \\
&+ \frac{8}{3} (67 z^3 + 46 z^2 + 2 z + 12) \zeta_2 \frac{H_1 H_0}{z} - \frac{8 (16040 z^3 - 16335 z^2 + 1575 z - 1388) H_1 H_0}{27 z} \\
&+ \frac{16}{3} (137 z^2 + 41 z + 20) \zeta_2 H_0 H_1 - \frac{16 (323 z^3 + 111 z^2 - 12 z + 20) H_0,0,-1 H_0}{9 z} \\
&+ \frac{4 (1501 z^3 + 5892 z^2 + 93 z + 468) H_0,0,1 H_0}{9 z} - \frac{8 (406 z^3 - 330 z^2 - 3 z - 40) H_1 H_0,0,1 H_0}{3 z} \\
&+ \frac{16 (108 z^3 + 82 z^2 + 5 z + 8) H_0,0,-1 H_0}{3 z} - \frac{16 (129 z^3 - 210 z^2 + 28 z - 8) H_0,0,0,1 H_0}{3 z} \\
&+ \frac{16 (129 z^3 - 210 z^2 + 28 z - 8) H_0,0,0,1 H_0}{3 z} + \frac{8 (768 z^3 - 398 z^2 + 19 z - 56) H_0,0,1 H_0}{3 z} \\
&- 128 (4 z^2 + 3) H_0,0,0,1 H_0 - 128 (7 z - 1) H_0,0,0,1 H_0 - 32 (6 z^2 + 14 z + 9) H_0,0,1 H_0 \\
&- 32 (6 z^2 - 14 z + 1) H_0,0,1,1 H_0 + 8 (2 z^2 + 38 z + 11) H_1 H_0 \zeta_3 H_0 + \frac{(204 z^3 - 166 z^2 - 19 z - 8) H_1^4}{9 z} \\
&+ \frac{2 (1321 z^3 - 1440 z^2 + 15 z - 4) H_1^3}{27 z} + \frac{8}{3} (132 z^3 + 118 z^2 + 5 z + 8) \zeta_2 H_0^2 \\
&+ \frac{2 (3536 z^3 - 3108 z^2 + 39 z - 134) H_1^2}{27 z} + \frac{4}{3} (102 z^3 - 94 z^2 + 11 z - 8) \zeta_2 H_1^2 + 32 z^2 H_0^2 \\
&- \frac{8 (161 z^3 - 130 z^2 - 4 z - 16) H_0^3}{3 z} + \frac{4 (115880 z^3 - 117883 z^2 + 83079 z - 66007)}{243 z} \\
&+ \frac{2}{27} (57548 z^3 - 10209 z^2 + 4590 z - 2176) \zeta_2 + \frac{16}{15} (86 z^3 + 1673 z^2 - 90 z + 76) \zeta_2 \\
&+ \frac{4}{27} (16364 z^3 + 25128 z^2 + 3537 z + 448) \zeta_3 - \frac{16}{9} (z + 1) (413 z^2 - 32 z + 20) \zeta_2 H_1 \\
&- 8 (112 z^3 + 98 z^2 + 5 z + 8) \zeta_2 H_0 \zeta_1 + \frac{4 (18046 z^3 - 18101 z^2 + 128 z + 1465) H_1}{27 z}.
\end{align*}
\]
\[
\begin{align*}
&\frac{4}{9}(3111z^3 - 3052z^2 + 116z - 242)\frac{\zeta_2}{z} H_1 - \frac{8}{9}(874z^3 - 698z^2 - 41z - 80)\frac{\zeta_2}{z} H_1 \\
&+ \frac{(z + 1)(2443z^2 - 244z + 94)}{z} \left[ \frac{16}{27} H_{-1} H_0 - \frac{16}{27} H_{-1,0,-1} \right] - \frac{8}{3} \left( 103z^3 + 70z^2 + 2z + 12 \right) \frac{\zeta_2}{z} H_{0,-1} \\
&+ \frac{8(9594z^3 - 17343z^2 + 1125z - 1388)H_{0,0,1}}{27z} - \frac{8}{3} \left( 124z^3 + 258z^2 + 33z + 16 \right) \frac{\zeta_2}{z} H_{0,1} \\
&+ \frac{(z + 1)(305z^2 - 32z + 20)}{z} \left[ -\frac{16}{9} H_0 H_{-1}^2 + \frac{32}{9} H_{0,-1} H_{-1} - \frac{32}{9} H_{0,-1,-1} \right] \\
&+ \frac{16(287z^3 - 105z^2 - 12z + 20)H_{0,0,-1}}{9z} - \frac{4(3525z^3 + 9840z^2 + 495z + 628)H_{0,0,1}}{9z} \\
&+ \frac{8(728z^3 - 590z^2 - 11z - 72)H_1 H_{0,0,1}}{3z} - 192(z^2 + 5z + 2)H_{0,1}H_{0,0,1} \\
&+ \left( 84z^3 + 70z^2 + 5z + 8 \right) \left[ \frac{16}{3} H_{0,0,-1} H_{0,1} - \frac{16}{3} H_{-1,0,1} \right] - \frac{16}{3} H_{-1} H_{0,0,1} \\
&+ \frac{4(3425z^3 - 3312z^2 + 48z - 240)H_{0,1,1}}{9z} + \frac{8(84z^3 - 70z^2 + 5z - 8)H_1 H_{0,1,1}}{3z} \\
&+ \left( 108z^3 + 94z^2 + 5z + 8 \right) \left[ \frac{16}{3} H_{0,1} H_{0,1} - \frac{16}{3} H_{0,1,0} - \frac{16}{3} H_{0,-1,1} - \frac{16}{3} H_{0,-1,-1,1} \right] \\
&+ \frac{(96z^3 + 82z^2 + 5z + 8)}{z} \left[ \frac{16}{3} H_{-1} H_{0,0,-1} - \frac{16}{3} H_{0,-1,0,1} \right] + (4z - 1) \left[ \frac{4}{15} H_0^5 - 64H_{0,0,-1,0,1} \right] \\
&+ \frac{(72z^3 + 58z^2 + 5z + 8)}{z} \left[ \frac{90z^3 + 76z^2 + 5z + 8}{3z} \frac{32}{3} H_{-1} H_{0,0,1} - \frac{32}{3} H_{0,0,1,1} \right] + \frac{8(168z^3 + 118z^2 + 5z + 8)H_{0,0,0,1}}{3z} \\
&- \frac{32}{3} H_{0,0,1} - \frac{8(932z^3 - 266z^2 + 31z - 56)H_{0,0,1,1}}{3z} + z(z + 1) \left[ -64H_{0,1} H_{0,1}^2 \right] \\
&+ \left[ 192H_{0,1} + 128H_{0,-1,1} + 128H_{0,1,-1} \right] H_{-1} - 192H_{0,-1,1} - 192H_{0,1,-1} - 128H_{0,-1,-1} \\
&- 128H_{0,-1,-1,1} + 128H_{0,1,-1,1} - \frac{8(30z^3 - 10z^2 - 7z - 8)H_{0,1,1,1}}{3z} + (z^2 - z + 1) \left[ \frac{32}{3} H_{0,-1} H_0^3 \right] \\
&- 64H_0^2 H_{0,0,1} + H_{0,1} \left[ 256H_{0,-1,1} + 128H_{0,0,0,1} - 512H_{0,-1,0,1,1} - 1024H_{0,0,0,0,1} \right] \\
&- 128H_{0,0,0,1,1,1} - 32(2z^2 - 2z + 7)H_{0,0,0,1,1,1} - 32(54z - 1)H_{0,0,0,0,1,1} \\
&+ (2z - 5z + 2) \left[ 64H_{0,0,0,0,1,1} + 256H_{0,0,0,0,-1,1} + 256H_{0,0,0,0,1,-1} \right] + 64(13z^2 + 83z + 32)H_{0,0,0,1,1} \\
&+ (2z^2 - 14z + 5) \left[ -32H_{0,-1,0,0,1} + 32H_{0,0,0,-1,0,1} + 32H_{0,0,0,0,-1,1} + 32H_{0,0,0,0,1,0,1} \right] \\
&+ 64(5z^2 + 29z + 11)H_{0,0,0,1,0,1} - 128(12z^2 - 8z + 7)H_{0,0,1,1,1} - 64(9z^2 - 7z + 5)H_{0,0,0,1,1} \\
&- 64(z^2 + z + 1)H_{0,1,1,1,1} + (6z^2 - 10z + 7)(24H_{0,0,-1} - 16H_{0,-1,1}) \zeta_2 \\
&- 8(2z^2 + 46z + 7)H_{0,0,1,1} \zeta_2 - 16(30z^2 - 22z + 17)H_{0,0,1,1} \zeta_2 + L^3_M \left[ \frac{16}{3} (8z - 1) H_0^2 \right] \\
&- \frac{8(18z^3 - 152z^2 - 11z - 8)H_0}{9z} - \frac{4(z - 1)(1883z^2 - 97z + 272)}{27z} - \frac{8(146z^3 - 118z^2 - z - 16)H_1}{9z} \\
&+ \frac{\gamma_2 (z)}{3} + \frac{8}{3} H_0^2 \right] + \frac{32}{3} (2z^2 + 6z + 3)H_{0,1} - \frac{64}{3} (4z + 1) \zeta_2 \right] + L^3 \left[ -\frac{16}{3} (8z - 1) H_0^2 \right]
\end{align*}
\]
\[
\begin{align*}
&+ \frac{8(18z^3 - 152z^2 - 11z - 8)H_0}{9z} + \frac{4(z - 1)(1883z^2 - 97z + 272)}{27z} + \frac{8(146z^3 - 118z^2 - z - 16)H_1}{9z} \\
&+ \gamma_0 \left[ -\frac{8}{3} H_1^3 + \frac{8}{3} H_0 H_1 \right] - \frac{32}{3} \left( 2z^2 + 6z + 3 \right) H_{0,1} + \frac{64}{3} (4z + 1) \zeta_2 \right] + \left( 6z^2 - 2z + 5 \right) \left[ 16H_0^3 H_{0,-1,-1} - 24H_0 H_{0,-1} \zeta_2 \right] + 32 \left( 4z^2 - 8z + 5 \right) H_{0,-1} \zeta_3 + \frac{160}{3} \left( 2z^2 + 6z + 3 \right) H_{0,1} \zeta_3 \\
&+ \frac{8}{3} (278z + 83) \zeta_2 \zeta_3 + L_2^2 \left[ \frac{16}{3} (16z - 3) H_0^3 - \frac{4}{3} (54z^2 - 436z - 23) H_0^2 \right] \\
&\frac{4(5183z^3 - 3512z^2 + 1009z + 104)H_0}{9z} - \frac{8(280z^3 - 238z^2 + 5z - 24)H_1 H_0}{3z} \\
&\frac{16(6z^2 - 10z + 7) H_{0,-1} H_0 + 16(2z^2 + 22z + 7) H_{0,1} H_0 - 32(14z + 3) \zeta_2 H_0}{4(352z^3 - 310z^2 + 17z - 24) H_1^2} - \frac{2(6639z^3 - 7318z^2 - 509z + 1260)}{9z} \\
&\frac{4(4549z^3 - 4204z^2 + 59z - 168)H_1}{3z} + \left( \frac{16z^3 + 118z^2 - z + 16}{16z^2 - 26z + 11} \right) \left[ \frac{8}{3} H_{-1} H_0 - \frac{8}{3} H_{0,-1} \right] \\
&+ 16 \left( 2z^2 - 6z - 5 \right) H_{0,0,1} \\
+ &64 \left( z^2 + 5z + 2 \right) H_{0,1,1} + \frac{16}{3} \left( 126z^2 - 196z - 9 \right) \zeta_2 + \left( 2z^2 + 2z + 1 \right) \left[ -32H_0 H_{-1}^2 \right] \\
&+ \left[ 24H_0^2 + 64H_{0,-1} + 32H_{0,1} \right] H_{-1} - 64 \zeta_2 H_{-1} - 64H_{0,-1,-1} - 32H_{0,-1,1} \\
&- 32H_{0,1,-1} \right] + \gamma_0 \left[ 4H_1^3 + 12H_0 H_1^2 + 6H_0^2 H_1 - 16 \zeta_2 H_1 \right] + 32 \left( z - 3 \right) \zeta_3 \\
&+ L_2^2 \left[ -\frac{32}{3} \left( 4z - 1 \right) H_0^3 + \frac{16}{3} (9z^2 + z - 3) H_0^2 \right] + \frac{8(1335z^3 + 710z^2 + 266z + 40) H_0}{9z} \\
&+ \frac{16(z - 1)(19z^2 + 7z + 4) H_1 H_0}{3z} + \frac{16(2z^2 - 6z + 7) H_{0,-1} H_0 - 16(2z^2 + 6z + 3) H_{0,1} H_0}{9z} \\
&- 64 \left( z - 1 \right) \zeta_2 H_0 - \frac{4(132z^3 - 118z^2 + 5z - 8) H_1^2}{3z} - \frac{8(2695z^3 - 2661z^2 + 192z - 244)}{9z} \\
&- \frac{16}{3} \left( 44z^3 + 83z^2 - 3z + 4 \right) \zeta_2 \zeta_3 - \frac{8(617z^3 - 604z^2 - 70z - 28) H_1}{9z} \\
&\frac{16}{3} \left( 146z^3 + 118z^2 - z + 16 \right) \left[ \frac{8}{3} H_{0,-1} - \frac{8}{3} H_{-1} H_0 \right] + \frac{16}{3} \left( 25z + 36 \right) H_{0,1} \\
&- 16 \left( 6z^2 - 26z + 11 \right) H_{0,0,-1} + 32 \left( 4z + 1 \right) H_{0,1,1} + \left( 2z^2 + 2z + 1 \right) \left[ 32H_0 H_{-1}^2 + \left[ -24H_0^2 \right. \right. \\
&- 64H_{0,-1} - 32H_{0,1} \right] H_{-1} + 64 \zeta_2 H_{-1} + 64H_{0,-1,-1} + 32H_{0,-1,1} + 16H_{0,0,1} + 32H_{0,1,-1} \\
&+ \gamma_0 \left[ 4H_1^3 - 2H_0^3 H_1 - 8 \zeta_2 H_1 \right] - 32 \left( 17z - 2 \right) \zeta_3 \right] + \left( 2z^2 + 2z + 1 \right) \left[ -16H_0 H_{-1}^2 + \left[ \frac{64}{3} H_0^2 \right. \right. \\
&+ 64H_{0,-1} + \frac{64}{3} H_{0,1} \right] H_{-1}^2 + \left[ -\frac{8}{3} H_0^3 + \left[ 32H_0 - 64H_{0,-1} \right] H_0 - 192H_{0,-1,-1} - 64H_{0,-1,1} \\
&- 96H_{0,0,1} - 64H_{0,1,-1} \right] H_{-1}^2 - \frac{376}{5} \zeta_2^2 H_{-1} + \left[ \frac{4}{3} H_0^3 + \left[ -16H_{0,-1} - 32H_{0,1} \right] H_0^2 \right. \\
&+ \left[ 128H_{0,-1,-1} - 64H_{0,-1,1} + 96H_{0,0,-1} + 64H_{0,0,1} - 64H_{0,1,-1} - 64H_{0,1,1} \right] H_0 \\
&+ 384H_{0,-1,-1} + 128H_{0,-1,1} + 128H_{0,0,-1,1} + 128H_{0,0,1,-1} + 192H_{0,0,-1,1} - 160H_{0,0,1,1} - 32H_{0,0,0,1} + 192H_{0,0,1,-1} + 128H_{0,0,1,1} + 128H_{0,1,-1,1} - 32H_{0,0,0,1} \\
&- 32H_{0,0,1} + 192H_{0,0,1,-1} + 128H_{0,0,1,1} + 128H_{0,1,-1,1} \right] H_{-1} + H_0^2 \left[ 32H_{-1,1} + 32H_{0,-1,1} \right] 
\end{align*}
\]
+32(12z^2 + 4z + 9)H_{0,0,1,1} + \frac{8}{3}(136z^2 + 40z - 85)H_{-1}\zeta_2 - 16(16z^2 - 2z + 13)H_{0,-1}\zeta_2

-16(20z^2 - 26z + 9)H_{0,1}\zeta_2 + (2z^2 + 2z + 1)\left[-\frac{160}{3}H_0H_{-3}^2 + \left[8H_0^2 + 160H_{0,-1}\right]H_{-1}^2\right]

+\left[\frac{8}{3}H_0^3 + 64H_{0,-1}H_0 - 320H_{0,-1,-1} - 160H_{0,0,1} + 32H_{0,0,1,1} - 64H_{0,1,1}\right]H_{-1}

+192\zeta_3H_{-1} + 320H_{0,-1,-1} + 160H_{0,0,1}H_{0,-1} - 320H_{0,0,1,1} - 32H_{0,0,1}H_{-1}

+64H_{0,1,1,1} + 64H_{0,0,1,1} + \left[64H_{-1}H_0 - 80H_{-1}^2\right]\zeta_2 + 7_{qg}\left[2H_4^4 - 4H_0^2H_1^2 + 8H_0H_1^2\right]

+\frac{10}{3}H_0^3H_1 + \left[48H_{0,0,1} - 64H_{0,1,1}\right]H_1 - 136\zeta_3H_1 - 24H_{0,-1}H_{0,1} + H_0\left[-\frac{20}{3}H_3^3\right]

+\left[-24H_{0,-1} - 16H_{0,1}\right]H_1 + 24H_{0,0,1,1} + 10H_{0,1,1,1} - 8H_{0,0,1,1} + \left[-8H_1^2 - 40H_0H_1\right]\zeta_2\right]

+L_Q\left[\frac{-4}{3}(11z - 11)\frac{H_0^4}{9} + \frac{4}{9}(198z^2 - 1648z - 83)H_0^3 - \frac{8}{3}(22z^2 + 18z + 9)H_{-1}H_0^3\right]

+\frac{8}{3}(22z^2 - 26z + 13)H_1H_0^3 + 8(16z^2 + 10z + 5)H_1^2H_0^2 + 16(7z^2 - 8z + 4)H_1^2H_0^2

+\frac{2}{9}(18688z^2 - 14584z + 3849)H_0^3 - \frac{8(389z^3 + 317z^2 + 7z + 28)H_{-1}H_0^2}{3z}

+\frac{4(822z^3 - 631z^2 - 61z - 72)H_0H_0^2}{3z} + 16(3z^2 - 22z + 9)H_{0,-1}H_0^2

-4(36z^2 + 134z + 63)H_{0,0}H_0^2 - 4(24z^2 - 230z - 9)\zeta_2H_0^2

+\frac{8(477z^3 - 428z^2 + 37z - 28)H_1^2H_0}{3z} + \frac{4(37209z^3 - 68257z^2 - 7837z - 1136)H_0}{27z}

-\frac{8(708z^3 - 1801z^2 - 21z - 16)\frac{\zeta_2}{z}H_0}{3z} + \frac{16(3243z^3 - 3023z^2 + 10z - 58)H_1H_0}{9z}

+\frac{16(129z^3 + 194z^2 - 50z + 36)H_{0,0,1}H_0}{3z} - 64(7z^2 + 6z + 3)H_{-1}H_{0,-1}H_0

-\frac{8(42z^3 + 833z^2 + 49z - 40)H_{0,0,1}H_0}{3z} - 64(z - 1)^2H_1H_{0,1}H_0 + 128(5z^2 + 2z + 3)H_{0,-1,-1}H_0

+32(5z^2 + 44z - 7)H_{0,0,1}H_0 + 32(11z^2 + 24z + 19)H_{0,0,1}H_0 + (10z^2 - 2z + 5)\left[32H_{0,0,1,1}\right]

+32H_{0,1,1}H_0 + 32(2z^2 - 32z - 5)H_{0,0,1,1}H_0 + 32(14z^2 + 10z + 5)H_{-1}\zeta_2H_0

-128(5z^2 + 6z + 3)H_1\zeta_2H_0 + 32(64z + 17)\zeta_3H_0 + \frac{8(424z^3 - 382z^2 + 29z - 24)H_1^3}{9z}

+\frac{8(3661z^3 - 3443z^2 + 79z - 71)H_1^2}{9z} - 32(3z^2 - 2z + 3)H_{0,-1}^2 - 32(3z^2 - 4z + 2)H_{0,1}^2

+\frac{4}{5}(352z^2 - 18z + 157)\zeta_2^2 + \frac{2(503837z^3 - 529368z^2 + 89781z - 57320)}{81z}

-\frac{16}{9}(4600z^3 - 3718z^2 + 405z - 136)\frac{\zeta_2}{z} - \frac{8}{3}(1252z^3 - 1106z^2 + 15z - 96)\frac{\zeta_3}{z}

+\frac{8}{3}(536z^3 + 424z^2 - 7z + 24)\frac{\zeta_2}{z}H_{-1} + 4\frac{(52691z^3 - 54734z^2 + 469z + 3996)H_1}{27z}

-\frac{8}{3}(766z^3 - 676z^2 + 11z - 64)\frac{\zeta_2}{z}H_1 + \frac{(272z^3 + 257z^2 - 128z - 240)\left[\frac{16}{9}H_{0,-1} - \frac{16}{9}H_{-1}H_0\right]}{z}

+\frac{16(1357z^3 - 952z^2 + 395z + 162)H_{0,1}}{9z} + \frac{(100z^3 + 92z^2 + 19z - 8)\left[\frac{8}{3}H_0H_{-1}^2 - \frac{16}{3}H_{0,-1}H_{-1}\right]}{z}
\[+ \frac{16}{3} H_{0,1,-1} + \gamma_{0y}^0 \left[ -2H_1^4 + \left( \frac{88}{3} H_{0,1} - 48H_{0,0,-1} \right) H_1 + H_0 \left( 24H_1 H_{0,1} - \frac{28}{3} H_0^3 \right) + 24H_{0,1} H_{0,1} \right] \]
\[+ \frac{16}{3} (131z^3 - 71z^2 + 107z - 44) H_{0,0,-1} - \frac{16}{3} (15z^3 - 125z^2 + 29z + 4) H_{0,0,1} \]
\[+ \frac{64}{3} (4z^2 - 6z + 3) H_1 H_{0,0,1} + \left( \frac{218z^3 + 166z^2 - 13z + 16}{z} \right) \left[ - \frac{16}{3} H_{1} H_{0,1} + \frac{16}{3} H_{0,1,1} \right] \]
\[+ \left( \frac{16}{3} H_{0,1,-1} \right) + \frac{16}{3} (60z^3 - 488z^2 + z + 8) H_{0,1,1} + (2z^2 + 6z + 3) \left[ \frac{32}{3} H_0 H_{0,1} - 32H_{0,-1} H_2^2 \right] \]
\[+ \frac{16}{3} H_1 H_{0,1,-1} - 64H_{0,1,-1,1} + 64(5z^2 - 8z + 5) H_{0,-1,0,1} + (12z^2 + 14z + 7) \left[ 32H_{-1} H_{0,0,-1} \right] \]
\[+ 4 \left( 2z^2 + 2z + 1 \right) H_{-1} + 128H_{0,1,1,1} + 128H_{0,-1,1,1} \]
\[+ 128H_{0,1,-1,1} + \left( 2z^2 + 2z + 1 \right) H_{-1} \left[ -64H_0 H_{0,1} - 64H_{0,1,1} \right] + 64H_{0,1,1,1} + 64H_{0,1,1,1} \]
\[+ 64H_{0,1,1,1} - 32(2z^2 + 22z + 7) H_{0,1,1,1} \]
\[+ 16(24z^2 + 38z + 31) H_{0,1} \cdot H_{0,1} + 16(20z^2 + 6z + 3) H_{-1} \cdot \zeta_3 + (8z^2 - 10z + 5) \left[ -32 \zeta_2 H_1^2 - 112 \zeta_3 H_1 \right] \]
\[+ 272(8z - 1) \zeta_5 + C^2_{TF} \left[ \frac{2}{15} (8z^2 - 2z + 1) H_0^5 + \frac{1}{3} (52z^2 - 16z - 3) H_0^4 - \frac{2}{3} (36z^2 + 2z + 5) H_0^3 \right] \]
\[+ \frac{16}{3} (13z^2 - 17z + 3) H_1 H_0^3 + 16(2z - 1) H_0 H_{0,1} \cdot H_0^3 - \frac{2}{3} (2z - 3)(4z - 1) \zeta_2 H_0^3 \]
\[+ 2(40z^2 - 56z + 15) H_1 H_0^2 + (374z^2 - 196z + 63) H_0^2 - 4(18z^2 - 18z - 7) H_1 H_0^2 \]
\[+ 4(12z^2 - 60z + 17) H_0 H_{0,1} - 8(12z^2 + 10z - 5) H_{0,0,1} H_0^2 + 8(16z^2 - 6z + 3) H_{0,1,1} H_0^2 \]
\[+ 2(52z^2 - 37z + 4) \zeta_2 H_0^2 + \frac{16}{3} (22z^2 - 10z + 5) \zeta_3 H_0^2 - \frac{4}{3} (4z - 3)(4z + 1) H_0^3 \]
\[+ 2(88z^2 - 86z - 7) H_2^4 - 8(20z^2 - 14z + 7) H_{0,1} H_0 - \frac{8}{5} (284z^2 - 94z + 57) \zeta_2 H_0^2 \]
\[+ 4(136z^2 - 91z - 13) H_0 + 4(192z^2 - 156z + 25) H_1 H_0 - 16(13z^2 + 7z + 3) H_0 H_{0,1} \]
\[+ 8(64z - 25) H_{0,0,1} H_0 - 16(z - 1)(16z - 3) H_{0,1,1} + 32(12z^2 + 2z - 1) H_{0,0,1} H_0 \]
\[+ 192(2z - 1)^3 \zeta_2 \zeta_0 H_{0,1,1,1} H_0 + 16(36z^2 - 38z + 19) \zeta_0 H_{1,1,1,1} H_0 + 2(12z^2 + 38z + 29) \zeta_2 \zeta_0 H_0 \]
\[+ 4(4z - 3)(16z - 9) H_1 \zeta_2 H_0 - 32z^2 H_{0,1} \zeta_2 H_0 + 8(44z^2 - 58z + 29) H_{0,1} \zeta_2 H_0 \]
\[+ 8(4z - 3)(8z + 1) \zeta_3 H_0 - \frac{1}{3} (72z^2 - 76z + 19) H_1^4 - \frac{4}{3} (62z^2 - 28z - 31) H_1^3 \]
\[+ 78z^2 + 2(136z^2 - 129z + 74) H_1^2 - 4(12z^2 - 24z + 7) H_0^2 - \frac{8}{5} (176z^2 - 79z + 1) \zeta_2^2 \]
\[+ 84z + 2(272z^2 - 313z + 39) H_1 - 2(112z^2 - 7z + 22) H_0 + 2(104z^2 + 30z - 17) H_{0,1} \]
\[+ 64(5z^2 - 2z + 1) H_{0,1} H_{0,0,1} + (2z - 3)(2z - 1) \left[ 48 H_1 H_{0,0,1} - 24 H_0 H_1 H_0 \right] \]
\[+ 4(36z^2 + 102z - 37) H_{0,1,1,1} + 64(3z^2 - 4z + 2) H_{0,1} H_{0,0,1} + (4z - 1) \left[ 8 H_1^2 H_{0,1} - 32 H_1 H_{0,1,1} \right] \]
\[+ 8(24z^2 + 67z - 33) H_{0,0,0,1} + 8(32z^2 - 60z + 17) H_{0,0,0,1} - 32(2z - 1)(7z - 5) H_{0,1,1,1} \]
\[+ 8(48z^2 - 2z + 1) H_{0,0,0,1} - 16(92z^2 - 52z + 31) H_{0,0,0,1} - 32(20z^2 - 6z + 3) H_{0,0,1,1,1} \]
\[+ 16(36z^2 - 50z + 25) H_{0,0,1,1,1} - 36(4z^2 - 6z + 3) H_{0,1,1,1,1} - 16(40z^2 - 34z + 17) H_{0,1,1,1,1} \]
\[+ 16(z - 8) H_1^2 \zeta_2 - \frac{1}{2} (232z^2 - 510z - 375) \zeta_2 + 4(61z^2 - 82z - 10) H_1 \zeta_2 \]
\[(z + 1)(3z + 2) \left[ 16H_{0,-1} - 16H_{-1}H_0 \right] \zeta_2 + 4(128z^2 - 166z + 47)H_{0,1} \zeta_2 \]
\[+16(6z^2 + 2z + 1) H_{0,0,-1} \zeta_2 - 8(20z^2 - 74z + 37) H_{0,0,1} \zeta_2 - 16(2z^2 - 6z + 3) H_{0,1,1} \zeta_2 \]
\[+96\gamma_{qg}^0 \log(2) \zeta_2 + L_Q^3 \left[ \frac{64}{3} H_{0,1} z^2 - 8H_0 z + \frac{4}{3} (8z^2 - 2z + 1) H_0^2 - \frac{2}{3} (2z - 11) - \frac{16}{3} (4z - 1) H_1 \right] \]
\[+\gamma_{qg}^0 \left[ -\frac{8}{3} H_1^2 - \frac{8}{3} H_0 H_1 \right] - \frac{32}{3} (4z^2 - 2z + 1) \zeta_2 \right] + L_M^3 \left[ -\frac{64}{3} H_{0,1} z^2 + 8H_0 z \right] \]
\[-\frac{4}{3} (8z^2 - 2z + 1) H_0^2 + \frac{2}{3} (2z - 11) + \frac{16}{3} (4z - 1) H_1 + \gamma_{qg}^0 \left[ \frac{8}{3} H_1^2 + \frac{8}{3} H_0 H_1 \right] \]
\[+\frac{32}{3} (4z^2 - 2z + 1) \zeta_2 \right] + (2z^2 + 2z + 1) \left[ 16H_{-1} \zeta_2^2 + 16H_0 H_1^2 + \left[ -8H_0^2 - 32H_{0,-1} \right] H_{-1} \right] \]
\[+32H_{0,-1} \zeta_2 \right] + \frac{4}{3} (252z^2 - 34z + 13) \zeta_3 + \frac{64}{3} (12z^2 - 7z - 2) H_1 \zeta_3 \]
\[+\frac{128}{3} (5z^2 - 6z + 3) H_{0,1} \zeta_3 - \frac{16}{3} (20z^2 + 14z - 1) \zeta_2 \zeta_3 + L_M^3 \left[ -\frac{4}{3} (24z^2 + 2z + 3) H_0^3 \right] \]
\[-4z(32 + 9) H_0^2 - 4(18z^2 - 6z + 11) H_0 - 16(10z^2 - 18z + 5) H_1 H_0 + 64z^2 H_{0,-1} H_0 \]
\[-32(6z^2 - 2z + 1) H_{0,1} H_0 + 8(40z^2 - 10z + 9) \zeta_2 H_0 + 104z^2 - 32(z - 1)(5z - 2) H_1^2 \]
\[-92z - 2(36z^2 - 126z + 65) H_1 + (z + 1)(3z + 2) \left[ 32H_{-1} H_0 - 32H_{0,-1} \right] - 8(20z^2 + 5z - 4) H_{0,1} \]
\[-32(6z^2 + 2z + 1) H_{0,0,-1} + 8(24z^2 - 14z + 7) H_{0,0,1} - 32(2z^2 + 2z - 1) H_{0,1,1} \]
\[+8(40z^2 - 11z + 6) \zeta_2 + (2z^2 + 2z + 1) \left[ -32H_0 H_1^2 + 16H_0^2 + 64H_{0,-1} \right] H_{-1} \]
\[-32H_0 H_{-1} - 64H_{0,-1} \zeta_1 \right] + \gamma_{qg}^0 \left[ 8H_1^3 + 20H_0 H_1^2 + 8H_0^2 H_1 - 32 \zeta_2 H_1 \right] + 8(32z^2 + 2z + 7) \zeta_3 - 33 \right] \]
\[+L_Q^2 \left[ -\frac{4}{3} (24z^2 + 2z + 3) H_0^3 - 4z(32 + 9) H_0^2 - 4(18z^2 - 6z + 11) H_0 - 16(10z^2 - 18z + 5) H_1 H_0 \right] \]
\[+64z^2 H_{0,-1} H_0 - 32(6z^2 - 2z + 1) H_{0,1} H_0 + 8(40z^2 - 10z + 9) \zeta_2 H_0 + 104z^2 \]
\[-32(z - 1)(5z - 2) H_1^2 - 92z - 2(36z^2 - 126z + 65) H_1 + (z + 1)(3z + 2) \left[ 32H_{-1} H_0 - 32H_{0,-1} \right] \]
\[-8(20z^2 + 5z - 4) H_{0,1} - 32(6z^2 + 2z + 1) H_{0,0,-1} + 8(24z^2 - 14z + 7) H_{0,0,1} \]
\[-32(2z^2 + 2z - 1) H_{0,1,1} + 8(40z^2 - 11z + 6) \zeta_2 + L_M \left[ -64H_{0,1} z^2 + 24H_0 z - 4(8z^2 - 2z + 1) H_0^2 \right] \]
\[+2(2z - 11) + 16(4z - 1) H_1 + \gamma_{qg}^0 \left[ 8H_1^2 + 8H_0 H_1 \right] + 32(4z^2 - 2z + 1) \zeta_2 \right] \]
\[(2z^2 + 2z + 1) \left[ -32H_0 H_1^2 + 16H_0^2 + 64H_{0,-1} \right] H_{-1} - 32 \zeta_2 H_{-1} - 64H_{0,-1} \zeta_1 \right] \]
\[+\gamma_{qg}^0 \left[ 8H_1^3 + 20H_0 H_1^2 + 8H_0^2 H_1 - 32 \zeta_2 H_1 \right] + 8(32z^2 + 2z + 7) \zeta_3 - 33 \right] \]
\[+\gamma_{qg}^0 \left[ 8H_1^3 + 20H_0 H_1^2 + 8H_0^2 H_1 - 32 \zeta_2 H_1 \right] + 8(32z^2 + 2z + 7) \zeta_3 - 33 \right] + \gamma_{qg}^0 \left[ \frac{1}{3} H_1^5 + \frac{16}{3} H_{0,1} H_1^3 \right] \]
\[-\frac{4}{3} H_1^3 H_1^2 + \left[ -32H_{0,0,1} - 16H_{0,1,1} \right] H_1^2 - \frac{2}{3} H_0^4 H_1 + 20H_0^2 H_1 + \frac{264}{5} \zeta_2^2 H_1 + 12H_0 H_{0,1} H_1 \]
\[+H_0 \left[ H_1^4 + 20H_0 H_1^2 + \left[ -24H_{0,0,1} - 56H_{0,1,1} \right] H_1 \right] + \left[ -\frac{40}{3} H_1^3 - 18H_1^2 \right] \]
\[+4H_0^2 H_1 - 24H_{0,1} H_1 \zeta_2 + \left[ \frac{4}{3} H_1^2 + \frac{88}{3} H_0 H_1 \right] \zeta_3 + L_M \left[ -\frac{2}{3} (40z^2 + 6z + 5) H_0^4 \right] \]
\[-\frac{8}{3}(8z + 29)H_0^3 + \frac{2}{15}(576z^3 - 4620z^2 - 5510z - 585)H_0^2 + 128(z + 1)(2z + 3)H_{-1}H_0^2 \\
-8(50z^2 - 40z - 3)H_1H_0^2 + 32(10z^2 - 2z + 3)H_{0,-1}H_0^2 - 16(8z^2 - 14z - 1)H_{0,1}H_0^2 \\
+ 8(72z^2 - 22z + 15)\zeta_2H_0^2 - 64\left[(11z^2 - 13z + 4)H_1^2H_0 - \frac{4(1464z^3 + 1593z^2 + 308z - 16)H_0}{15z}\right] \\
-16(55z^2 - 95z + 29)H_1H_0 - 64(72z^2 + 18z - 2)H_{0,-1}H_0 - 128(4z^2 + 1)H_{0,0,-1}H_0 \\
-64(4z^2 + 10z + 3)H_{0,0,1}H_0 + 128(4z^2 - 6z + 3)H_{0,1,1}H_0 + 16(100z^2 - 31z + 9)\zeta_2H_0 \\
+ 32(44z^2 - 30z + 9)\zeta_3H_0 - 48(72z^2 - 8z + 2)H_1^2 - 2(368z^2 - 546z + 155)H_1^2 \\
-32(14z - 1)H_{0,0}^2 - 16(8z^2 - 6z + 3)H_{0,1}^2 - \frac{8}{5}(192z^2 - 346z + 57)\zeta_2^2 \\
+ \frac{2(1092z^3 - 1893z^2 + 383z - 32)}{15z} - 2(272z^2 - 216z - 85)H_1 - 16(37z^2 - 6z - 13)H_{0,0} \\
+ (z + 1)(z + 14)\left[-32H_0H_{-1}^2 + 64H_{0,-1}H_{-1} - 64H_{0,-1,-1}\right] + 128(3z^2 + 8z - 8)H_{0,0,0} \\
+ 16(26z^2 - 77z + 26)H_{0,0,0} + (z + 1)(3z + 2)\left[64H_{-1}H_{0,0} - 64H_{0,-1,1} - 64H_{0,1,1}\right] \\
-16(38z^2 - 3z - 15)H_{0,1,1} + (3z^2 - 4z + 2)\left[-128H_{0,0,-1,1} - 128H_{0,-1,0,1}\right] \\
-64(10z^2 - 2z + 5)H_{0,0,0,1} + 16(48z^2 + 34z + 27)H_{0,0,0,1} - 16(32z^2 - 34z + 17)H_{0,0,1,1} \\
-192(2z - 1)H_{0,1,1,1} - \frac{16}{15}(144z^3 - 1380z^2 - 665z - 240)\zeta_2 - 32(z + 1)(7z + 18)H_{-1}\zeta_2 \\
+ 16(86z^2 - 78z + 5)H_{0,1}\zeta_2 - 32(12z^2 - 6z + 5)H_{0,1}\zeta_2 - 64(2z + 3)H_{0,1}\zeta_2 \\
+ 16(128z^2 - 54z + 23)\zeta_3 + (2z^2 + 2z + 1)\left[\frac{128}{3}H_0H_{-1}^2 + \left[-112H_0^2 - 128H_{0,-1} - 64H_{0,1}\right]H_{-1}^2 \\
+ \left[\frac{64}{3}H_0^3 + 192H_{0,-1}H_0 + 256H_{0,-1,-1} + 128H_{0,-1,-1} + 64H_{0,0,-1} + 64H_{0,0,1} + 128H_{0,1,-1}\right]H_{-1} \\
-224\zeta_3H_{-1} - 256H_{0,-1,-1,-1} - 128H_{0,-1,-1,-1} - 128H_{0,-1,-1,-1} - 128H_{0,0,-1,-1} - 128H_{0,0,-1,-1} - 128H_{0,0,-1,-1} \right]H_{-1} \\
-64H_{0,0,0,1} - 128H_{0,0,1,1} - 128H_{0,0,1,1} + \left[128H_{-1}^2 - 64H_{-1}H_0\right]\zeta_2 + \frac{80}{27}(8H_1^4 + 2H_0^2H_1^2) \\
+ 8H_0H_{1}^2 + 8H_0^2H_1 + \left[128H_{0,0,-1} - 80H_{0,0,1}\right]H_1 - 120\zeta_3H_{1,1} - 64H_{0,0,-1}H_{0,0} \\
+ H_0\left[24H_1^2 + \left(64H_{0,1} - 64H_{0,-1}\right)H_1 + 76H_{0,1} + 64H_{0,-1,1} + 64H_{0,1,-1}\right] \\
+ \left[-64H_{1}^2 - 128H_0H_{1}\right]\zeta_2\right] + L_Q\left[\frac{2}{3}(40z^2 + 6z + 5)H_0^4 + \frac{8}{3}(82z + 29)H_0^3 \\
- \frac{2}{15}(576z^3 - 4620z^2 - 5510z - 585)H_0^2 - 128(z + 1)(2z + 3)H_{-1}H_0^2 + 8(50z^2 - 40z - 3)H_1H_0^2 \\
-32(10z^2 - 2z + 3)H_{0,-1}H_0^2 + 16(8z^2 - 14z - 1)H_{0,1}H_0^2 - 8(72z^2 - 22z + 15)\zeta_2H_0^2 \\
+ 64(11z^2 - 13z + 4)H_1^2H_0 + \frac{4(1464z^3 + 1593z^2 + 308z - 16)H_0}{15z} + 16(55z^2 - 95z + 29)H_1H_0 \\
+ 64(7z^2 + 18z - 2)H_{0,-1}H_0 + 128(4z^2 + 1)H_{0,0,-1}H_0 + 64(4z^2 + 10z + 3)H_{0,0,1}H_0 \\
- 128(4z^2 - 6z + 3)H_{0,1,1}H_0 - 16(100z^2 - 31z + 9)\zeta_2H_0 - 32(44z^2 - 30z + 9)\zeta_3H_0 \\
+ 48(7z^2 - 8z + 2)H_1^2 + 2(368z^2 - 546z + 155)H_1^2 + 32(14z - 1)H_{0,-1}^2 + 16(8z^2 - 6z + 3)H_{0,1}^2 \\
+ \frac{8}{5}(192z^2 - 346z + 57)\zeta_2^2 - \frac{2(1092z^3 - 1893z^2 + 383z - 32)}{15z} + 2(272z^2 - 216z - 85)H_1 \\
+ \frac{(72z^5 - 165z^4 - 1090z^3 - 915z^2 + 2)\left[\frac{32}{15}H_{-1}H_0 - \frac{32}{15}H_{0,-1}\right]}{z^2} + 16(37z^2 - 6z - 13)H_{0,1}\right]

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\[-16(4z - 1)H_1 H_{0,1} + (z + 1)(z + 14) \left[ 32H_0 H_1^2 - 64H_{0,-1} H_{-1} + 64H_{0,-1,-1} \right] \]
\[-128(3z^2 + 8z - 8) H_{0,0,-1} - 16(26z^2 - 77z + 26) H_{0,0,1} + (z + 1)(3z + 2) \left[ -64H_1 H_{0,-1} + 64H_{0,-1,1} + 64H_{0,1,-1} \right] + 16(38z^2 - 3z - 15) H_{0,1,1} + (3z^2 - 4z + 2) \left[ 128H_0 H_{0,-1,-1} + 128H_{0,-1,0} \right] \]
\[+ 64(10z^2 - 2z + 5) H_{0,0,0,-1} - 16(48z^2 + 34z + 27) H_{0,0,0,1} + 16(32z^2 - 34z + 17) H_{0,0,1,1} + 192(2z - 1) H_{0,1,1,1} + \frac{16}{15} (144z^3 - 1380z^2 - 665z - 240) \zeta_2 + 32(z + 1)(7z + 18) H_{-1} \zeta_2 \]
\[-16(86z^2 - 78z + 5) H_{1} \zeta_2 + 32(12z^2 - 6z + 5) H_{0,-1} \zeta_2 + 64(2z + 3) H_{0,1} \zeta_2 \]
\[+ L_M^2 \left[ 64H_{0,1} z^2 - 24H_0 z + 4(8z^2 - 2z + 1) H_0^2 - 2(2z - 11) - 16(4z - 1) H_1 \right] \]
\[+ \gamma_0^0 \left[ -8H_1^2 + 8H_0 H_1 \right] - 32(4z^2 - 2z + 1) \zeta_2 \right] - 16(128z^2 - 54z + 23) \zeta_3 \]
\[+ L_M \left[ \frac{8}{3} (24z^2 + 2z + 3) H_0^3 + 8z(32z + 9) H_0^2 + 8(18z^2 - 6z + 11) H_0 \right] \]
\[+ 32(10z^2 - 18z + 5) H_1 H_0 - 128z^2 H_{0,-1} H_0 + 64(6z^2 - 2z + 1) H_{0,1} H_0 - 16(40z^2 - 10z + 9) \zeta_2 H_0 \]
\[+ 64(z - 1)(5z - 2) H_1 H_0^2 - 2(104z^2 - 92z - 33) + 4(36z^2 - 126z + 65) H_1 \]
\[+ (z + 1)(3z + 2) \left[ 64H_{0,-1} - 64H_{-1} H_0 \right] + 16(20z^2 + 5z - 4) H_{0,1} + 64(6z^2 + 2z + 1) H_{0,0,-1} \]
\[-16(24z^2 - 14z + 7) H_{0,0,1} + 64(2z^2 + 2z - 1) H_{0,1,1} - 16(40z^2 - 11z + 6) \zeta_2 \]
\[+ (2z^2 + 2z + 1) \left[ \frac{128H_0^2 - 128H_{0,-1}}{H_{-1}} H_{-1} + 64\zeta_2 H_{-1} + 128H_{0,-1,-1} \right] \]
\[+ \gamma_0^0 \left[ -16H_1^3 - 40H_0 H_1^2 - 16H_0^2 H_1 + 64\zeta_2 H_1 \right] - 16(32z^2 + 2z + 7) \zeta_3 \]
\[+ (2z^2 + 2z + 1) \left[ -\frac{128}{3} H_0 H_1^2 + \left[ 112H_0^2 + 128H_{0,1} - 64H_{0,1} \right] H_1 \right] + \left[ -\frac{64}{3} H_0^3 \right. \]
\[-192H_{0,-1} H_0 - 256H_{0,-1,-1} - 128H_{0,0,-1} - 64H_{0,0,1} - 128H_{0,-1,1} - 64H_{0,0,1} \]
\[+ 224\zeta_2 H_{-1} + 256H_{0,-1,-1,1} + 128H_{0,0,-1,1} + 128H_{0,0,-1,-1} + 64H_{0,0,0,1} + 64H_{0,0,1,1} \]
\[+ 64H_{0,0,1,1} + 128H_{0,1,1,-1} + \left[ 64H_{-1} H_0 - 128H_{0,-1} \right] \zeta_2 \right] + \gamma_0^0 \left[ -8H_1^4 - 20H_0^2 H_1^2 - 8H_0^4 H_1^2 \right. \]
\[-8H_0^3 H_1 + \left[ 80H_{0,0,1} - 128H_{0,0,0} \right] H_1 + 120\zeta_3 H_1 + 64H_{0,-1} H_0 + H_0 \left[ -24H_1^3 \right. \]
\[+ \left[ 64H_{0,-1} - 64H_{0,1} \right] H_1 - 76H_{0,1} - 64H_{0,-1,-1} - 64H_{0,0,-1,1} \right] + \left[ 64H_1^2 + 128H_{0,1} \right] \zeta_2 \right] \]
\[-8(56z^2 - 2z + 1) \zeta_5 - 77 \right] + C_F T_F^2 \left[ \frac{4}{9} (4z^2 - 20z + 1) H_0^4 \right. \]
\[+ \frac{16}{9} (10z^2 - 63z + 4) H_0^3 - \frac{8}{3} (68z^2 + 75z - 22) H_0^2 + \frac{16}{3} (10z^2 - 12z + 1) H_1 H_0^2 \]
\[-\frac{64}{3} (z - 1)^2 H_{0,1} H_0^2 + \frac{4}{3} (32z^2 + 112z - 59) \zeta_2 H_0^2 + \frac{4}{3} (538z^2 - 1578z + 21) H_0 \]
\[-\frac{32}{3} (16z^2 - 27z + 10) H_1 H_0 - \frac{64}{3} (5z^2 - 4z + 2) H_{0,1} H_0 - \frac{128}{3} (z^2 + 2z - 1) H_{0,0,1} H_0 \]
\[+ \frac{4}{9} (24z^2 + 112z - 1063) \zeta_2 H_0 - \frac{32}{9} (36z^2 - 74z + 37) \zeta_3 H_0 + \frac{32}{9} (z - 1)(3z + 1) H_1^3 \]
\[
\begin{align*}
&+ \left[ -\frac{16}{3} (2z - 1) H_0^2 - \frac{16}{9} (24z^2 - 8z - 5) H_0 + \frac{16}{27} (62z^3 - 111z^2 + 75z - 8) \right] \\
&+ \frac{32}{9} \gamma_0^0 H_1 \left( L_M + \frac{16}{3} (12z^2 - 17z + 4) H_1^2 - \frac{16}{15} (218z^2 - 194z + 91) \right) \\
&+ \frac{2}{27} (699z^3 - 8610z^2 + 939z - 208) + \frac{2}{27} (1984z^3 + 7716z^2 - 12243z - 256) \frac{\zeta_2}{z} \\
&- \frac{16}{27} (170z^3 - 1128z^2 + 525z - 8) \frac{\zeta_4}{z} + \frac{4}{3} (810z^3 - 1558z^2 + 793z + 16) H_1 \\
&+ L_M^3 \left[ \frac{16}{3} (2z - 1) H_0^2 + \frac{16}{9} (4z - 11) H_0 - \frac{16}{27} (62z^3 - 147z^2 + 84z - 8) + \frac{16}{9} \gamma_0^0 H_1 \right] \\
&+ \frac{16}{3} (20z^2 - 113z + 140) H_{0,1} + \frac{16}{3} (20z^2 - 113z + 56) H_{0,0,1} + \frac{64}{3} (3z^2 - 6z - 1) H_{0,1,1} \\
&+ \frac{8}{9} H_0^3 H_1 + \frac{8}{9} (20z^2 - 28 + 11) H_{0,0,0,1} + \frac{80}{9} (4z^2 - 4z - 1) H_1 \zeta_2 + \frac{2}{9} H_1^4 + \frac{16}{9} \zeta_2 H_0 \zeta_3 + \frac{8}{3} H_1 H_0 \zeta_1 \\
&+ \frac{8}{9} \left[ - \frac{64}{3} H_{0,0,1} - \frac{64}{3} H_{0,1,1} \frac{H_1}{H_0} + \frac{160}{9} \zeta_3 H_1 + \frac{32}{3} H_0^2 + H_0 \left[ \frac{32}{3} H_1 H_0 \right] \right] \\
&+ \frac{8}{9} \left[ \frac{80}{3} H_{0,0,1,1,1} + \left[ - \frac{26}{3} H_1^2 - \frac{16}{3} H_0 H_1 - 12 H_{0,1} \right] \zeta_2 \right] + L_Q^2 \left[ \frac{8}{3} (8z^2 + 56z - 31) H_0 \right] \\
&+ \frac{8}{9} \left( 244z^2 - 236z + 112 - 9 \right) H_0^2 + \frac{8}{9} \left( 244z^2 - 236z + 112 - 9 \right) H_0^2 + \frac{4}{27} \left( 2156z^3 - 6732z^2 + 9977z + 256 \right) H_0 \\
&+ \left[ - 16 (2z - 1) H_0^2 - \frac{16}{3} (8z^2 - 9) H_0 + \frac{16}{9} (z - 1) (62z^2 - 73z + 8) \right] L_M \\
&- \frac{16}{3} (20z^2 - 113z + 140) \zeta_2 + (2z - 1) \left[ - 16 H_0^3 + 32 \zeta_2 H_0 - 32 H_{0,0,1} + 32 \zeta_3 \right] \\&+ L_M^2 \left[ - \frac{8}{3} (40z^2 + 40z - 23) H_0^2 + \frac{8}{9} (4z^2 - 92z + 547) H_0 + \frac{4}{27} \left( 1652z^3 - 6192z^2 + 4221z + 256 \right) \right] \\
&+ \left[ - 16 (2z - 1) H_0^2 - \frac{16}{3} (8z^2 - 9) H_0 + \frac{16}{9} (z - 1) (62z^2 - 73z + 8) \right] L_M \\
&- \frac{16}{3} (20z^2 - 113z + 140) \zeta_2 + (2z - 1) \left[ - 16 H_0^3 + 32 \zeta_2 H_0 - 32 H_{0,0,1} + 32 \zeta_3 \right] \\
&+ L_M \left[ \frac{64}{3} (2z^2 + 15z - 5) H_0^3 - \frac{16}{45} (72z^3 + 560z^2 - 2660z + 2975) H_0^2 \right] \\
&- \frac{64}{3} (5z^2 - 4z + 2) H_1 H_0^2 - \frac{16}{45} (144z^3 - 5632z^2 + 9213z + 4) H_0 \\
&- \frac{32}{9} (198z^3 - 283z^2 + 140z - 8) H_1 H_0 + \frac{128}{3} (z - 2) (3z - 2) H_{0,-1} H_0 \\
&+ \frac{32}{3} (16z^2 - 8z - 5) H_{0,1} H_0 - \frac{32}{3} (4z^2 + 84z - 33) \zeta_2 H_0 - \frac{16}{9} (168z^3 - 253z^2 + 131z - 8) H_1^2 \\
&+ \left[ - \frac{16}{9} (16z^2 - 8z - 5) H_{0,1} H_0 - \frac{32}{3} (4z^2 + 84z - 33) \zeta_2 H_0 - \frac{16}{9} (168z^3 - 253z^2 + 131z - 8) H_1^2 \right] L_M \\
&- \frac{4}{405z} (269954z^3 - 828996z^2 + 567861z - 11744) + \frac{16}{45} \left( 144z^4 + 1600z^3 - 1400z^2 + 3045z - 80 \right) \frac{\zeta_2}{z}
\end{align*}
\]
\[
\begin{align*}
&-\frac{8(3080z^3 - 8448z^2 + 5247z + 256)H_1}{27z} - \frac{128}{3}(7z^2 - 14z + 9)H_{0,0,-1} - \\
&\frac{(36z^5 - 155z^4 + 40z^3 + 225z^2 - 20z + 1)\left[\frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1}\right]}{z^2} + \\
&\frac{16(76z^3 - 254z^2 - 329z - 16)H_{0,1}}{9z} + \gamma_{qq}^0\left[\frac{8}{3}H_1^3 + 8H_0H_1^2 + \frac{32}{3}H_{0,1}H_1\right] - \\
&\frac{32}{3}(8z^2 - 60z + 15)H_{0,0,1} + \frac{32}{3}(24z^2 - 20z + 1)H_{0,1,1} + \frac{64}{3}(8z^2 - 6z + 3)H_1\zeta_2 + \\
&(z + 1)^2\left[-\frac{128}{3}H_0H_{2,1}^2 + \left[\frac{64}{3}H_0^2 + \frac{256}{3}H_{0,-1}\right]H_{-1} - \frac{128}{3}\zeta_2H_{-1} - \frac{256}{3}H_{0,-1,1}\right] + \\
&LM\left[16(8z^2 + 16z - 9)H_0^2 - \frac{16}{9}(124z^2 - 164z + 559)H_0 - \frac{8}{3}\gamma_{qq}^0H_1^2 - \\
&\frac{8(1904z^3 - 6912z^2 + 4599z + 256)}{27z} - \frac{32(70z^3 - 143z^2 + 91z - 8)H_1}{9z} + (12z^2 - 4z - 7)\left[\frac{32}{3}H_{0,1}^2 - \\
&\frac{32\zeta_2}{3}\right] + (2z - 1)\left[32H_0^3 - 64\zeta_2H_0 + 64H_{0,0,1} - 64\zeta_3\right] + \frac{128}{3}(4z^2 - 21z + 11)\zeta_3 + \\
&(2z - 1)\left[16H_0^4 - 96\zeta_2H_0^2 + 64H_{0,0,1}H_0 - 128\zeta_3H_0 - \frac{32\zeta_2^2}{5} + 64H_{0,0,1,1}\right] + \\
&LM\left[-\frac{64}{9}(10z^2 + 43z - 14)H_0^3 - \frac{16}{45}(72z^3 - 170z^2 + 2640z - 2945)H_0^2 + \\
&\frac{64}{3}(3z^2 - 4z + 2)H_1H_0^2 + \frac{32(173z^3 - 2669z^2 + 4406z - 2)H_0}{45z} + \\
&\frac{32(120z^3 - 199z^2 + 104z - 8)H_1H_0}{9z} - \frac{128}{3}(z^2 - 4z + 2)H_{0,-1}H_0 - 32(8z^2 - 8z + 1)H_{0,1}H_0 - \\
&\frac{32}{3}(28z^2 + 68z - 25)\zeta_2H_0 + \frac{16(42z^3 - 121z^2 + 98z - 8)H_1^2}{9z} + \\
&\frac{4(274706z^3 - 812364z^2 + 549969z - 11456)}{405z} + \frac{16}{45}(144z^4 - 340z^3 + 520z^2 - 2865z + 80)\zeta_2^3 + \\
&\frac{8(2108z^3 - 7152z^2 + 4941z + 256)H_1}{27z} + \gamma_{qq}^0\left[\frac{8}{3}H_1^3 + \frac{8}{3}H_0H_1^2\right] + \\
&\frac{(36z^5 + 155z^4 + 40z^3 + 45z^2 + 20z + 1)\left[\frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1}\right]}{z^2} + \\
&\frac{16(172z^3 - 326z^2 - 365z - 16)H_{0,1}}{9z} + \frac{128}{3}(z^2 - 10z + 3)H_{0,0,-1} + \\
&\frac{32}{3}(8z^2 - 76z + 23)H_{0,0,1} - \frac{32}{3}(16z^2 - 8z - 5)H_{0,1,1} + (z + 1)^2\left[-\frac{128}{3}H_0H_1^2 + \\
&\left[\frac{64}{3}H_0^2 + \frac{256}{3}H_{0,-1}\right]H_{-1} - \frac{128}{3}\zeta_2H_{-1} - \frac{256}{3}H_{0,-1,1}\right] + \frac{64}{3}(12z^2 + 44z - 15)\zeta_3 + \\
&(2z - 1)\left[-16H_0^4 - 64H_{0,0,1}H_0 + 128\zeta_3H_0 + \frac{32\zeta_2^2}{5} - 64H_{0,0,1,1} + \left[96H_0^2 + \frac{64H_1}{3}\right]\zeta_3\right] + \\
&(2z - 1)\left[\frac{40}{3}\zeta_2H_0^3 + \frac{64}{3}\zeta_3H_0^2 + \frac{64}{5}\zeta_2H_0 - 32H_{0,0,0,1} + 32\zeta_3\right] + \\
&CNF_T^2\left[-\frac{4}{9}(z - 2)(6z + 1)H_0^4 + \frac{4}{27}(404z^2 - 54z + 69)H_0^3 - \frac{4}{27}(3112z^2 - 1329z - 210)H_0^2\right]
\end{align*}
\]
\[-\frac{8}{9}(94z^3 - 234z^2 + 159z - 16)H_1H_0^2 + \frac{16}{3}(6z^2 + 4z - 11)H_{0,1}H_0^2 \]
\[-\frac{8}{3}(14z^2 - 5z - 11)\zeta_2H_0^2 + \frac{8}{81}(27824z^2 + 4929z + 2631)H_0 - \frac{32}{3}(2z^2 + 28z - 17)H_{0,0,1}H_0 \]
\[+ \frac{16}{3}(1556z^3 - 1539z^2 + 72z - 80)H_1H_0 - \frac{32}{27}(101z^3 + 222z^2 - 39z + 8)H_{0,1}H_0 \]
\[+ \frac{8}{9}(240z^2 + 166z + 193)\zeta_2H_0 + \frac{32}{9}(78z^2 - 133z - 28)\zeta_3H_0 + \frac{16}{9}(z - 1)(3z + 1)H_1^3 \]
\[+ \left[-\frac{16}{3}(2z - 1)H_0^2 - \frac{32}{9}(6z^2 - z - 4)H_0 + \frac{8}{27}(124z^3 - 258z^2 + 159z - 16) + \frac{8}{9}\gamma^0_{9}H_1 \right]L_3^3 \]
\[+ \frac{8}{3}(12z^2 - 17z + 4)H_1^2 - \frac{64}{15}(2z^2 + 58z - 11)\zeta_2^2 - \frac{4}{27}(221158z^3 - 226026z^2 + 17163z - 5248) \]
\[+ \frac{2}{27}(7280z^3 - 5646z^2 - 555z - 368)\zeta_2^3 - \frac{8}{27}(3784z^3 + 2046z^2 + 1095z - 16)\zeta_3^3 \]
\[+ \frac{16}{3}(20z^2 - 17z - 1)H_1L_3^3 \left[\frac{16}{3}(2z - 1)H_0^2 + \frac{16}{9}(4z - 11)H_0 - \frac{16}{27}(62z^3 - 147z^2 + 84z - 8) \right] \]
\[+ \frac{16}{9}\gamma^0_{9}H_1 \]
\[- \frac{16}{27}(1448z^3 - 1341z^2 + 18z - 80)H_{0,1}L_3^3 + \frac{16}{9}(498z^3 + 654z^2 + 3z + 16)H_{0,0,1} \]
\[+ \frac{32}{3}(3z^2 - 6z - 1)H_{0,1,1} - \frac{32}{3}(14z^2 - 74z + 19)H_{0,0,0,1} \]
\[+ L_3^3 \left[\frac{32}{3}(2z - 1)H_0^3 \right] \]
\[+ \frac{16}{3}(2z + 3)(4z - 5)H_0^2 - \frac{8}{9}(160z^2 + 146z + 305)H_0 + \frac{4}{27}(1000z^3 + 1356z^2 - 2247z - 208) \]
\[+ \frac{32}{9}(4z^2 - 4z + 5)H_1 + \gamma^0_{9}H_1 \left[\frac{4}{3}H_1^2 + \frac{8}{3}H_{0,1} - \frac{8}{27}\zeta_2^2 \right] \]
\[+ \frac{8}{9}H_{0,1}H_0^2 - \frac{32}{9}H_0^3H_1 + \left[-\frac{32}{3}H_{0,0,1} - \frac{32}{3}H_{0,1,1} \right]H_1 + \frac{88}{9}\zeta_3H_1 + \frac{16}{3}H_1H_0 \]
\[+ \frac{16}{3}H_0^2 + H_0 \left[\frac{16}{3}H_1H_0 \right] \]
\[+ \frac{32}{3}H_{0,1,1} \]
\[+ \frac{40}{3}H_{0,1,1,1} + \left[-4H_1^2 - \frac{4}{3}H_0H_1 - \frac{20}{3}H_{0,1} \right] \zeta_2 + L_3^3 \left[-\frac{8}{3}(8z^2 + 56z - 81)H_0^3 \right] \]
\[+ \frac{8}{9}(244z^2 - 236z + 571)H_0 + \frac{4}{27}(2156z^3 - 7632z^2 + 4977z + 256) + \gamma^0_{9} \left[-4H_1^2 - \frac{32}{3}H_0H_1 \right] \]
\[+ \frac{1}{9}(130z^3 - 215z^2 + 112z - 8)H_1 - \frac{16}{3}(12z^2 - 8z - 5)H_{0,1} + \left[-\frac{8}{3}(4z - 1) \right] \]
\[+ \frac{16}{3}(4z^2 - 2z + 1)H_0 - \frac{8}{3}\gamma^0_{9}H_1 \]
\[L_3^3 \left[-\frac{8}{3}(16z^2 + 108z - 95)H_0^3 \right] \]
\[+ \frac{32}{3}(32H_0 - 32H_{0,0,1} + 32\zeta_3) \]
\[+ L_3^3 \left[\frac{8}{3}(16z^2 + 108z - 95)H_0^3 \right] \]
\[+ \frac{8}{45}(144z^3 + 1120z^2 - 3745z + 5320)H_0^2 - \frac{16}{45}(1984z^3 - 5407z^2 + 7593z + 4)H_0 \]
\[+ \frac{64}{3}(5z^2 - 4z + 2)H_1H_0^2 - \frac{32}{9}(198z^3 - 283z^2 + 140z - 8)H_1H_0 + \frac{128}{3}(z - 2)(3z - 2)H_{0,-1}H_0 \]
\[+ \frac{32}{3}(16z^2 - 8z - 5)H_{0,1}H_0 - \frac{32}{3}(4z^2 + 8z - 33)\zeta_2H_0 - \frac{16}{9}(168z^3 - 253z^2 + 131z - 8)H_1^2 \]
\[+ \frac{8}{405z^2}(31851z^3 + 229863z - 6952) + \frac{16}{45}(144z^4 + 1600z^3 - 1400z^2 + 3045z - 80)\zeta_2^2 \]
\[
- \frac{8(3080z^3 - 8448z^2 + 5247z + 256)H_1}{27z} + \frac{16(76z^3 - 254z^2 - 329z - 16)H_{0,1}}{9z} + \frac{(36z^5 - 155z^4 + 40z^3 + 225z^2 - 20z + 1)}{z^2} \frac{64}{45} \left[H_{-1}H_0 - \frac{64}{45} H_{0,-1}\right] + \gamma_0^0 \frac{8}{3} H_1^3 + 8H_0H_1^2 \\
+ \frac{32}{3} H_{0,1}^1 \frac{128}{3} (7z^2 - 14z + 9) H_{0,0,-1} - \frac{32}{3} (8z^2 - 60z + 15) H_{0,0,1} \\
+ \frac{32}{3} (24z^2 - 20z + 1) H_{0,1,1} + \frac{64}{3} (8z^2 - 6z + 3) H_1 \zeta_2 + (z + 1)^2 \left[- \frac{128}{3} H_0H_{-1} + \frac{64}{3} H_0^2 \right] \\
+ \frac{256}{3} H_{0,-1,1} H_{-1} - \frac{128}{3} \zeta_2 H_{-1} - \frac{256}{3} H_{0,-1,1} \right] + \mathcal{L}_M \left[\frac{8}{9} (36z^2 + 82z - 61) \\
- \frac{16}{9} (76z^2 - 38z + 25) H_0 - \frac{16}{9} (76z^2 - 88z + 41) H_1 + \gamma_0^0 \frac{8}{3} H_1^2 + \frac{32}{3} H_0 H_1 \right] \\
+ \frac{32}{3} (4z^2 - 6z + 3) H_{0,1} + (4z^2 - 2z + 1) \left[\frac{32\zeta_3}{3} - \frac{32}{3} H_0^2 \right] + \frac{128}{3} (4z^2 - 21z + 11) \zeta_3 \\
+ (2z - 1) \left[\frac{44}{3} H_0^4 - 96\zeta_2 H_2^2 + 64H_{0,0,1} H_0 - 128\zeta_3 H_0 - \frac{32\zeta_2^2}{5} + 64H_{0,0,1,1} \right] \\
+ \mathcal{L}_M \left[- \frac{8}{9} (16z^2 + 28z - 47) H_0^2 + \frac{8}{9} (422z^2 + 137z + 410) H_0^2 - \frac{80}{27} (176z^2 - 213z - 297) H_0 \\
+ \frac{32}{9} (35z^3 + 29z^2 - 40z + 8) H_1 H_0 - \frac{32}{3} (4z^2 - 16z + 17) H_{0,1} H_0 - \frac{64}{3} (4z^2 - 2z + 1) \zeta_2 H_0 \\
+ \frac{8}{9} (86z^2 - 104z + 67) H_1^2 + \frac{8}{9} (3224z^3 - 19683z^2 + 16893z - 1064) \right] + \frac{8}{9} (152z^2 - 258z + 163) H_1 \\
+ \frac{16}{9} (16z^3 + 88z^2 + 131z - 16) H_{0,1,1} - \frac{128}{3} (5z - 7) H_{0,0,1} - \frac{32}{3} (12z^2 - 14z + 7) H_{0,1,1} \\
- \frac{16}{9} (86z^2 - 30z + 51) \zeta_2 + \gamma_0^0 \left[\frac{32}{3} H_1 H_0^2 - \frac{8}{3} H_1^2 H_0 - \frac{16}{3} H_1 H_{0,1} + \frac{32}{3} H_1 \zeta_2 \right] \\
+ \frac{32}{3} (16z^2 - 2z - 19) \zeta_3 + (2z - 1) \left[- \frac{20}{3} H_0^4 + 64H_{0,0,1} H_0 + 128\zeta_3 H_0 + \frac{384\zeta_2^2}{5} - 192H_{0,0,0,1} \right] \\
+ (2z - 1) \left[- \frac{4}{15} H_0^5 - 8\zeta_2 H_3^3 + 32H_{0,0,1} H_2^3 + \frac{208}{3} \zeta_3 H_2^3 - 128H_{0,0,0,1} H_0 + 128 H_{0,0,0,0,1} - 128\zeta_5 \right] \\
+ C_{FCAT} \left[- \frac{2}{15} H_0^5 + \frac{1}{18} (16z^2 + 178z + 13) H_0^4 - \frac{1}{27} (7624z^2 + 1752z + 113) H_0^3 \\
+ \frac{16}{3} (z + 1)(3z + 2) H_{-1} H_0^3 - \frac{4(404z^3 - 334z^2 - 25z - 40) H_1 H_0^3}{9z} + \frac{16}{3} (2z^2 + 16z + 7) H_{0,1} H_0^3 \\
+ \frac{8}{3} (4z^2 + 5z + 4) \zeta_2 H_3^3 - 2(20z^2 + 24z + 19) H_2^2 H_0^2 - \frac{4(151z^3 - 138z^2 - 16) H_1^2 H_0^2}{3z} \\
+ \frac{1}{9} (10622z^2 - 2987z - 346) H_2^2 - 4(4z^2 - 7) H_{-1} H_0^3 + \frac{2(124z^3 - 1392z^2 + 1269z - 40) H_1 H_0^2}{9z} \\
- 12(4z + 3) H_{0,0,1,1} H_0^2 - \frac{8(13z^3 + 46z^2 - 23z + 20) H_{0,1} H_0^2}{3z} + 16(2z^2 - 28z - 9) H_{0,0,1} H_0^2 \\
- 16(14z^2 - 26z + 1) H_{0,0,1} H_0^2 - \frac{1}{3} (128z^2 - 106z - 49) \zeta_2 H_0^2 + \frac{16}{3} (6z^2 + 13z + 10) H_3 H_0^2 \\
- \frac{4(94z^3 - 48z^2 - 3z - 16) H_3 H_0^2}{9z} + \frac{4(728z^3 - 1068z^2 + 303z - 8) H_1^2 H_0}{9z} \\
+ 16(6z^2 - 10z - 1) H_{0,1} H_0 - \frac{16}{5} (103z + 11) \zeta_2 H_0 - \frac{2}{81} (9514z^2 - 34311z + 17499) H_0 \right]
\]
\[
+ \frac{4(12769z^3 - 12504z^2 - 1401z + 740)H_1H_0}{27z} + \frac{4}{3}(314z^3 - 242z^2 - 47z - 48)\frac{G_2}{z}H_1H_0 \\
+ 8(2z^2 + 14z - 3)H_{0,-1}H_0 + 8(4z^2 + 5)H_{-1}H_{0,-1}H_0 - \frac{8(782z^3 + 387z^2 + 705z - 20)}{9z}H_{0,0,1}H_0 \\
+ \frac{32(54z^3 - 56z^2 + z - 4)H_1H_{0,1}H_0}{3z} - \frac{8(4z^2 - 16z - 11)}{3z}H_{0,-1,-1}H_0 - 16(2z^2 - 6z - 5)H_{0,0,-1}H_0 \\
- \frac{8}{3}(448z^2 - 328z - 49)H_{0,1}H_0 + \frac{8(370z^3 - 88z^2 - 133z + 40)}{3z}H_{0,0,1}H_0 \\
- 32(10z^2 - 34z - 7)H_{0,0,1}H_0 + 16(8z^2 - 22z + 1)H_{0,0,1}H_0 + 16(40z^2 - 22z + 23)H_{0,1,1}H_0 \\
+ \frac{2}{9}(1180z^2 - 1409z - 347)\zeta_2H_0 + 8(17z^2 + 27z + 13)H_{-1}\zeta_2H_0 + 80(z + 1)^2H_{0,-1}\zeta_2H_0 \\
- 8(26z^2 + 14z + 29)H_{0,1}\zeta_2H_0 - \frac{4}{9}(1828z^2 + 134z + 131)\zeta_3H_0 - \frac{2(12z^3 + 31z^2 - 56z - 4)}{9z}H_1^4 \\
- \frac{4(57z^3 - 94z^2 + 24z + 4)H_1^3}{3z} - \frac{2(2369z^3 - 3096z^2 + 2241z + 142)}{9z}H_1^2 \\
- \frac{4}{3}(290z^3 - 306z^2 + 57z - 8)\zeta_2H_1^2 - 64(z + 1)H_{0,-1}H_0 + \frac{4(384z^3 - 256z^2 + 35z - 16)}{3z}H_{0,1}^2 \\
+ \frac{4}{15}(2354z^2 - 3310z + 101)\zeta_2^2 - \frac{54588z^3 - 98666z^2 - 2299z + 8388}{81z} - \frac{1}{18}(11000z^3 + 11030z^2 \\
- 1283 + 1440)\zeta_2H_1 + \frac{4}{9}(1088z^3 - 1036z^2 + 65z - 64)\zeta_3H_1 + \frac{35z^2 + 40z + 21}{81z}H_{0,1}H_0 \\
- \frac{4(9292z^3 - 15039z^2 - 1215z + 740)}{H_0,0,1} - \frac{8}{27z} \\
- \frac{4}{3}(150z^3 - 338z^2 + 169z - 48)\zeta_2H_{0,1} + \frac{16}{3}(z - 1)(15z + 1)H_{0,1}H_{0,1} + (3z^2 + z - 5)H_0H_{-1} \\
- 16H_{0,-1}H_{-1} + 16H_{0,-1,-1} + 8(28z + 1)H_{0,0,1} - \frac{8(568z^3 + 2052z^2 + 801z - 20)}{9z}H_{0,0,1} \\
- \frac{8(432^3 - 436z^2 + 5z - 32)}{3z}H_1H_{0,0,1} + 48(6z + 5)H_{0,0,1}H_{0,0,1} + (4z^2 + 8z + 1)H_1H_{0,0,1} \\
+ H_0[H_{-16H_{0,-1,1} - 16H_{0,1,1}]} + (z - 2)(z + 1)[H_{-16H_{-1}H_{0,1} + 16H_{0,-1,1} + 16H_{0,1,1}} \\
- \frac{8(2599z^3 - 2553z^2 + 459z - 8)}{9z}H_{0,1,1}^2 - \frac{8(336z^3 - 292z^2 + 23z - 32)}{3z}H_{0,1,1}^2 \\
- \frac{32(14z^2 - 6z + 9)}{9z}H_{0,1}H_{0,1,1} + (z^2 + z + 1)H_0H_{0,1}^2 - 64H_{0,1}H_{0,1}^2 \\
+ 192H_{0,-1,1}H_{-1} - 192H_{0,0,1}H_{0,0,1} + 8(2z - 1)(2z + 5)H_{0,-1,0,1} \\
+(2z + 1)(2z + 3)[24H_{-1}H_{0,0,-1} - 24H_{0,0,-1,1}] + (4z^2 + 2z + 1)H_0H_{0,0,-1,1} \\
- 8H_0H_{0,0,1}^3 - \frac{8(590z^3 - 248z^2 - 233z + 40)}{3z}H_{0,0,0,1} \\
+ (4z^2 + 16z - 3)[H_{-1H_{0,0,1} + 8H_{0,0,1} + 8H_{0,0,1,1} + 8H_0H_{0,0,1,1}] + \frac{8}{3}(512z^2 - 240z - 183)H_{0,0,1,1} \\
+(8z^2 + 12z + 7)[H_{-1}H_{1}^2 + 16H_{0,-1,1} + 16H_{0,1,-1}H_{-1} - 16H_{0,-1,-1} + 16H_{0,1,-1} - 16H_{0,1,-1,1} + z(z + 1)H_{-64H_{-1}H_{0,0,1} - 64H_{0,0,1,1} + 64H_{0,0,1,1} - 64H_{0,1,1,1}} \\
+ 64(5z^2 + 6z + 3)H_{0,-1,1,1} + \frac{16(251z^3 - 139z^2 + 17z - 32)}{3z}H_{0,1,1,1} + z^2H_{0,0,1}H_{-1,0,1,1} \\
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\[-20H_0 H_1^2 - 6H_0^2 H_1 + 40\zeta_1 H_1\] - 96(2z^2 + 1)\zeta_3 + (2z^2 + 2z + 1)\left[8H_0 H_1^4 + (-8H_0^2ight.

\[-32H_{0,-1})H_3^\prime + \left[-\frac{16}{3}H_0^3 + (48H_{0,-1} - 64H_{0,1})H_0 + 96H_{0,-1,-1} - 48H_{0,0,-1} + 112H_{0,0,1}\right.\n
\[-32H_{0,1,1})H_3^\prime + 72\zeta_2^2 H_1 + \left[\frac{8}{3}H_0^3 + (16H_{0,-1} + 32H_{0,1})H_0^2 + [-96H_{0,-1,-1}\n
+128H_{0,-1,1} - 64H_{0,0,-1} - 32H_{0,0,1} + 128H_{0,1,-1} + 64H_{0,1,1}\right]H_0 - 192H_{0,-1,1,-1}\n
-160H_{0,-1,1} + 64H_{0,0,-1,1} + 96H_{0,0,0,-1} - 224H_{0,0,0,1} + 160H_{0,0,0,1} - 32H_{0,0,0,1} - 224H_{0,0,1,1}\n
-96H_{0,0,1,1} + 64H_{0,1,0,1,1} + 64H_{0,1,1,0,1,1}\left]H_{-1} + H_0^2 \left(32H_{0,-1,1} - 32H_{0,1,1}\right) + H_0 \left[96H_{0,-1,1,-1}\n
-128H_{0,-1,1,1} - 128H_{0,1,1,1} - 64H_{0,0,1,1} + 32H_{0,0,0,1} + 32H_{0,0,0,1} - 128H_{0,1,1,1}\n
-64H_{0,1,1,1} - 64H_{0,1,1,1}\right] + 192H_{0,-1,1,1,-1} - 64H_{0,-1,1,1,1} + 160H_{0,-1,0,1,1}\n
+160H_{0,-1,0,1,1}\n
+32H_{0,-1,0,1,1} + 64H_{0,-1,1,1,1} - 64H_{0,-1,1,1,1} - 96H_{0,0,-1,1,1} + 224H_{0,0,0,1,-1}\n
+224H_{0,0,0,1,-1}\n
+96H_{0,0,0,1,1,1} + 224H_{0,0,0,1,1,1} + 96H_{0,0,0,1,1,1} - 64H_{0,1,1,1,1} - 64H_{0,1,1,1,1}\n
-64H_{0,1,1,1,1} - 1\left] + \left[16H_3^\prime + 24H_0 H_1^2 - \left[-60H_0^2 + 16H_{0,-1} - 96H_1\right]H_{-1} + 96H_0 \n
+96H_{0,1,1,1} - 1\right] \zeta_2 + \left[-56H_0^2 - 16H_0 H_{-1}\right] \zeta_3 + \gamma_{qg}^0 \left[\frac{1}{3}H_1 H_0^4 + 2H_1^2 H_0^2\right] \n
+\left[\frac{2}{3}H_1^3 - 16H_1 H_0,1\right] H_0^2 + \left[\frac{5}{3}H_1^4 - 32H_{0,1} H_0^2 + \left[24H_{0,0,1} + 88H_{0,0,1}\right] H_1\right] \n
-\frac{424}{5} H_1 \zeta_2 - \frac{16}{3} H_1^2 H_0,1 + H_1^2 \left[52H_{0,0,1} + 20H_{0,1,1}\right] + H_1 \left[-28H_{0,1} + 16H_{0,0,0,1} - 24H_{0,0,1,1}\right]\n
+ \left[16H_1^3 + 20H_0 H_1^2 - H_0^2 H_1 + 32H_{0,1} H_1\right] \zeta_2 + \left[-\frac{14}{3} H_1^2 - \frac{116}{3} H_0 H_1\right] \zeta_3\n
+(2z + 1) \left[-\frac{16}{3} H_{0,-1} H_0^3 + 8H_{0,0,-1} H_0 - 48H_{0,0,-1} H_{0,0,-1} + 80H_{0,0,0,-1} + 32H_{0,0,0,0,-1} + 8H_{0,-1} \zeta_3\right]\n
+ \left[\frac{4}{3} (18z + 5) H_0^4 + \frac{8}{45} (144z^3 + 2305z^2 - 610z + 225) H_0^3 - \frac{16}{3} (18z^2 + 22z + 11) H_{-1} H_0^3\right]\n
+ \frac{16}{3} (6z^2 - 2z + 1) H_1 H_0^3 + 8(18z^2 - 14z + 7) H_1^2 H_0^2 + 8(4z^2 - 6z + 9) H_{-1} H_0^3\n
+ \frac{12(160z^4 + 16274z^3 - 10977z^2 - 602z + 24) H_0^2}{3z} + \frac{8(303z^3 - 364z^2 + 161z - 16) H_1 H_0^2}{3z}\n
- \frac{4288z^5 + 1140z^4 + 1000z^3 + 295z^2 + 80z + 48}{15z^2} H_{-1} H_0^3 - 24(4z^4 - 18z^3 - 5) \zeta_2 H_0^2\n
+ \frac{8(398z^3 - 366z^2 + 69z - 32) H_0^2 H_0}{3z} + \frac{4(242z^3 + 11758z^2 + 3445z + 64) H_0}{45z}\n
+ \frac{8(1834z^3 - 1520z^2 + 223z + 64) H_1 H_0}{3z} - \frac{8(1260z^4 - 300z^3 - 1655z^2 - 80z - 8) H_{0,-1} H_0}{15z^2}\n
+ \frac{8(518z^3 + 172z^2 - 257z + 32) H_{0,1} H_0}{9z} + 2(2z^2 - 6z + 3) H_0 H_{0,1} H_0\n
+ \frac{12(12z^2 + 38z - 5) H_{0,-1} H_0 + 64(11z^2 + 12z + 8) H_{0,0,-1} H_0 - 32(18z^2 + 14z + 19) H_{0,0,1} H_0}{3z}\n
+ \frac{32(2z^2 - 10z + 1) \left[-64H_{0,-1,1} - 64H_{0,1,-1}\right] H_0 - 32(2z^2 + 22z + 13) H_{0,1} H_0}{3z}\n
- \frac{8}{5} \left(192z^2 + 2090z^2 - 320z + 85\right) \zeta_2 H_0 + 32\left(34z^2 + 42z + 1\right) H_{-1} \zeta_2 H_0 + 256(z - 1)^2 H_1 \zeta_2 H_0\n
- \frac{32(4z^2 + 58z + 3) \zeta_3 H_0 + \frac{8(93z^3 - 82z^2 + 26z - 8) H_1^3}{3z} + \frac{8(518z^3 - 346z^2 + 65z + 32) H_1^2}{9z}}{112}
\[-16(4z^2 + 26z - 11) H_{0,-1}^2 + 32(5z^2 - 8z + 4) H_{0,1}^2 + \frac{8}{5}(188z^2 - 234z + 39) \zeta_2^2\]
\[-2(67432z^3 - 39737z^2 - 18123z - 2512) \zeta_2 \quad \frac{8}{45} \left( 4632z^4 + 29968z^3 - 22628z^2 + 1731z + 96 \right) \zeta_2 \]
\[\frac{8}{3} \left( 144z^4 - 6z^3 + 464z^2 + 49z + 32 \right) \zeta_3 \quad - \frac{4}{45} \left( 2(32084z^3 - 22856z^2 - 10349z - 544) H_1 \right) \zeta_2 \]
\[\frac{16}{15} \left( 216z^5 + 900z^4 + 1075z^3 + 560z^2 + 40z + 6 \right) \zeta_2 \quad H_{-1} \]
\[\frac{16}{15} \left( 72z^5 + 1610z^4 - 1585z^3 + 465z^2 - 120z - 2 \right) \zeta_2 \quad H_1 \]
\[\left( 1008z^5 - 558z^4 - 671z^3 - 4851z^2 + 12z + 28 \right) \zeta_2 \quad \frac{16}{45} \left( H_{0,-1} - \frac{16}{45} H_{-1} H_0 \right) \]
\[\frac{4}{45} \left( 216z^5 + 900z^4 + 1075z^3 + 560z^2 + 40z + 6 \right) \zeta_2 \quad H_{-1} \]
\[\frac{8}{3} \left( 144z^4 - 6z^3 + 464z^2 + 49z + 32 \right) \zeta_3 \quad - \frac{4}{45} \left( 2(32084z^3 - 22856z^2 - 10349z - 544) H_1 \right) \zeta_2 \]
\[\frac{16}{15} \left( 216z^5 + 900z^4 + 1075z^3 + 560z^2 + 40z + 6 \right) \zeta_2 \quad H_{-1} \]
\[\frac{16}{15} \left( 72z^5 + 1610z^4 - 1585z^3 + 465z^2 - 120z - 2 \right) \zeta_2 \quad H_1 \]
\[\left( 1008z^5 - 558z^4 - 671z^3 - 4851z^2 + 12z + 28 \right) \zeta_2 \quad \frac{16}{45} \left( H_{0,-1} - \frac{16}{45} H_{-1} H_0 \right) \]
\[\frac{4}{45} \left( 216z^5 + 900z^4 + 1075z^3 + 560z^2 + 40z + 6 \right) \zeta_2 \quad H_{-1} \]
\[\frac{16}{15} \left( 72z^5 + 1610z^4 - 1585z^3 + 465z^2 - 120z - 2 \right) \zeta_2 \quad H_1 \]
\[\left( 1008z^5 - 558z^4 - 671z^3 - 4851z^2 + 12z + 28 \right) \zeta_2 \quad \frac{16}{45} \left( H_{0,-1} - \frac{16}{45} H_{-1} H_0 \right) \]
\[\frac{4}{45} \left( 216z^5 + 900z^4 + 1075z^3 + 560z^2 + 40z + 6 \right) \zeta_2 \quad H_{-1} \]
\[\frac{16}{15} \left( 72z^5 + 1610z^4 - 1585z^3 + 465z^2 - 120z - 2 \right) \zeta_2 \quad H_1 \]
\[\left( 1008z^5 - 558z^4 - 671z^3 - 4851z^2 + 12z + 28 \right) \zeta_2 \quad \frac{16}{45} \left( H_{0,-1} - \frac{16}{45} H_{-1} H_0 \right) \]
\[\frac{4}{45} \left( 216z^5 + 900z^4 + 1075z^3 + 560z^2 + 40z + 6 \right) \zeta_2 \quad H_{-1} \]
\[\frac{16}{15} \left( 72z^5 + 1610z^4 - 1585z^3 + 465z^2 - 120z - 2 \right) \zeta_2 \quad H_1 \]
\[\left( 1008z^5 - 558z^4 - 671z^3 - 4851z^2 + 12z + 28 \right) \zeta_2 \quad \frac{16}{45} \left( H_{0,-1} - \frac{16}{45} H_{-1} H_0 \right) \]
\[\frac{4}{45} \left( 216z^5 + 900z^4 + 1075z^3 + 560z^2 + 40z + 6 \right) \zeta_2 \quad H_{-1} \]
\[\frac{16}{15} \left( 72z^5 + 1610z^4 - 1585z^3 + 465z^2 - 120z - 2 \right) \zeta_2 \quad H_1 \]
\[\left( 1008z^5 - 558z^4 - 671z^3 - 4851z^2 + 12z + 28 \right) \zeta_2 \quad \frac{16}{45} \left( H_{0,-1} - \frac{16}{45} H_{-1} H_0 \right) \]
The Wilson coefficients for the longitudinal structure function \( F_L(x, Q^2) \) in the asymptotic region in \( z \)-space are presented in the following. \( L_{q,L}^{PS} \) and \( L_{q,L}^{S} \) read:

\[
L_{q,L}^{PS} = a_3^q \left\{ C_F N_F T_F^2 \left[ L_M \left[ \frac{256}{9} z H_{0,1} - \frac{256 \zeta_2}{3} - \frac{256(z-1)(2z^2 + 2z - 1)}{9z} H_1 \right] \right] - \frac{256}{9} z(2z + 11) H_0 + \frac{256(z-1)(19z^2 + 16z - 5)}{27z} \right] + z \left[ \zeta_3 - \frac{128 \zeta_2}{9z} H_{0,1,1} \right] + L_Q^2 \left[ \frac{128(z-1)(2z^2 + 2z - 1)}{9z} - \frac{128}{3} zH_0 \right] \right. \\
\left. + L_Q \left[ - \frac{256}{9} (4z^2 - 8z - 3) H_0 + \frac{256}{3} zH_0^2 - \frac{256(z-1)(3z^2 + 6z - 2)}{9z} \right] + \left[ \frac{128(z-1)(2z^2 + 2z - 1)}{9z} - \frac{128}{3} zH_0 \right] L_M^2 + \frac{64(z-1)(2z^2 + 2z - 1)}{9z} H_1^2 \\
- \frac{128(z-1)(19z^2 + 16z - 5)}{27z} H_1 \right] \frac{128}{27} z(19z + 67) H_0 + \frac{256(z-1)(55z^2 + 43z - 14)}{81z} \right) + N_F C_L^{PS,(3)} (N_F) \right\}, \tag{598}
\]

\[
L_{q,L}^{S} = - \frac{64}{3} a_3^2 N_F T_F^2 (z-1) z L_M + a_3^q \left\{ - N_F T_F^2 \frac{256}{9} (z-1) z L_M^2 \right. \\
+ C_A T_F^2 N_F \left[ \frac{64(z-1)(17z^2 + 2z - 1)}{9z} - \frac{256}{3} zH_0 + \frac{128}{3} (z-1) zH_1 \right] L_Q^2 \\
+ \left[ - \frac{64(z-1)(461z^2 + 11z - 25)}{27z} - \frac{128}{9} z(26z - 59) H_0 \right] - \frac{128(z-1)(39z^2 + 2z - 1)}{9z} H_1 \right] + (z-1) z \left[ - \frac{128}{3} H_1^2 - \frac{256}{3} H_0 H_1 \right] \\
+ z(z + 1) \left[ \frac{256}{3} H_{-1} H_0 - \frac{256}{3} H_{0,-1} \right] + z \left[ \frac{512}{3} H_0^2 + \frac{512}{3} H_{0,1} \right] \\
+ \left[ \frac{128(z-1)(17z^2 + 2z - 1)}{9z} - \frac{512}{3} zH_0 + \frac{256}{3} (z-1) zH_1 \right] L_M \right\} \tag{599}
\]
The flavor non-singlet Wilson coefficient is given by:

\[ L_{q,L}^{\text{NS}} = + \frac{256}{3} (z - 2) z \zeta_2 \left[ L_Q - \frac{32}{9} (28 z - 3) H_0^2 + \left[ \frac{64(z - 1)(17z^2 + 2z - 1)}{9z} - \frac{256}{3} z H_0 \right] \right] 
+ \frac{128}{3} (z - 1) z H_1 L_M^2 + \frac{32(z - 1)(2714z^2 - 106z - 139)}{81z} - \frac{64}{27} (110z^2 + 277z - 33) H_0 H_1 
+ \frac{4160}{27} (z - 1) z H_1 + z \left[ \frac{64}{9} H_0^3 - \frac{64}{3} H_{0,1} + \frac{64 z \zeta_2}{3} \right] + L_M \left[ \frac{64(z - 1)(68z^2 + z - 7)}{9z} \right] 
\]

\[ - \frac{128}{9} (4z - 1)(13z + 6) H_0 - \frac{128(z - 1)(19z^2 + 2z - 1)}{9z} H_1 + (z - 1) z \left[ - \frac{128}{3} H_1^2 \right] 
- \frac{256}{3} H_0 H_1 + z (z + 1) \left[ \frac{256}{3} H_{1,1} H_0 - \frac{256}{3} H_{0,1} \right] + z \left[ \frac{256}{3} H_0^3 \right] 
\]

\[ + \frac{512}{3} H_{0,1} + \frac{256}{3} (z - 2) z \zeta_2 \right] + C_F T_F N_F \left[ - \frac{16}{3} z H_1^0 - \frac{32}{3} (z - 1) H_0^3 - 32(19z - 3) H_0^3 \right] 
- 64(6z^2 + 7z - 8) H_0 + \left[ -64 z H_0^2 - \frac{64}{3} (4z^2 + 3z - 3) H_0 \right] 
\]

\[ + \frac{128(z - 1)(17z^2 - 10z - 1)}{9z} L_M^2 + \frac{16(z - 1)(343z^2 - 242z + 4)}{3z} \]

\[ + L_Q \left[ - \frac{64}{9} z H_0^2 - \frac{64}{3} (z + 1)(4z - 3) H_0 + \frac{64(z - 1)(28z^2 - 23z - 2)}{9z} \right] 
+ L_Q \left[ - \frac{128(25z + 2) H_1(z - 1)^2}{9z} - \frac{32(2474z^2 - 4897z + 44)(z - 1)}{135z} \right] 
\]

\[ - \frac{64}{45} (12z^3 - 180z^2 - 265z + 90) H_0^2 - \frac{64(354z^3 - 397z^2 + 388z + 4) H_0}{45z} \]

\[ + \frac{(z + 1)(6z^4 - 6z^3 + z^2 - z + 1) \left( \frac{256}{45} H_{1,1} H_0 - \frac{256}{45} H_{0,1} \right)}{z^2} + \frac{128}{3} (z + 1)(4z - 3) H_{0,1} \]

\[ + \left[ 128 z H_0^2 + \frac{128}{3} (2z - 1)(2z + 3) H_0 - \frac{64(z - 1)(74z^2 - 37z - 4)}{9z} \right] L_M \]

\[ + \frac{128}{45} (12z^3 - 60z^2 - 25z + 45) \zeta_2 + z \left[ 128 H_0^3 - 256 \zeta_2 H_0 + 256 H_{0,1} - 256 \zeta_2 \right] \]

\[ + L_M \left[ - \frac{64}{45} (12z^3 + 180z^2 + 335z - 90) H_0^2 + \frac{64(456z^3 - 708z^2 + 347z - 4) H_0}{45z} \right] 
\]

\[ + \frac{64(z - 1)(2368z^2 - 2189z - 2)}{135z} + \frac{64(z - 1)(80z^2 - 37z - 4)}{9z} H_1 \]

\[ + \frac{(z + 1)(6z^4 - 6z^3 + z^2 - z + 1) \left( \frac{256}{45} H_{1,1} H_0 - \frac{256}{45} H_{0,1} \right)}{z^2} - \frac{128}{3} (2z - 1)(2z + 3) H_{0,1} \]

\[ + \frac{128}{45} (12z^3 + 60z^2 + 50z - 45) \zeta_2 + z \left[ -128 H_0^3 + 256 \zeta_2 H_0 - 256 H_{0,1} + 256 \zeta_2 \right] \]

\[ + N_F \tilde{C}_{L,g}^{5,(3)} (N_F) \right] \right). \]  

The flavor non-singlet Wilson coefficient is given by:

\[ L_{q,L}^{\text{NS}} = \]
\[ a_5^2 C_F T_F \left\{ \frac{16L_Q z}{3} + \left[ -\frac{32}{3} H_0 - \frac{16}{3} H_1 \right] z - \frac{8}{9}(25z - 6) \right\} \]

+ \[ a_3^2 C_F^2 T_F \left[ \left[ 8(z + 2) + z \left[ -16H_0 - 32H_1 \right] \right] L_Q^2 + \frac{16}{15} z(24z^2 - 5)H_0^2 + \frac{80}{9}(5z - 6)H_1 \right. \]

+ \[ 16(144z^3 - 227z^2 - 72z - 96)H_0 + 32(72z^3 - 223z^2 - 77z + 48) \]

+ \[ \frac{(3z^5 - 5z^3 - 10z^2 - 2)(256 \frac{15}{15} H_{0,-1} - 256 \frac{15}{15} H_{1,0})}{z^2} - \frac{32}{5} z(8z^2 + 5) \zeta_2 + z \left[ -\frac{256}{3} H_0 H_{1,0} \right. \]

+ \[ \left[ \frac{128}{3} H_0^2 + \frac{512}{3} H_{0,-1} \right] H_{1,0} + \frac{128}{3} H_1^2 - \frac{128}{3} H_0^2 H_1 + 32H_{0,1} + H_0 \left[ \frac{256H_1}{3} - \frac{256}{3} H_{0,1} \right. \]

+ \[ \left[ \frac{256}{3} H_{0,1} \right] - \frac{512}{3} H_{0,0,-1} - \frac{256}{3} H_{0,0,-1} - \frac{256}{3} H_{0,0,1} + \left[ \frac{256H_1}{3} - \frac{256}{3} H_{1,0} \right] \zeta_2 + \frac{512 \zeta_3}{3} \right] L_Q \]

+ \[ \frac{8}{9}(z + 3)H_0^2 + \left[ \frac{8(z + 2)}{3} + z \left[ -\frac{16}{3} H_0 - \frac{32H_1}{3} \right] \right] L_M^2 - \frac{2}{27}(653z - 872) + \frac{16}{27}(11z + 42)H_0 \]

+ \[ z \left[ -\frac{8}{9} H_0^3 - \frac{16}{3} H_1 H_0^2 + \frac{32}{3} H_{0,1} - \frac{160H_1}{9} \right] H_0 - \frac{896H_1}{27} + \frac{160}{9} H_{0,1} - \frac{32}{3} H_{0,0,1} \]

+ \[ \left[ \frac{8}{9}(53z - 56) + \frac{32}{9} (z + 3)H_0 + z \left[ -\frac{16}{3} H_0^2 - \frac{64}{3} H_1 H_0 - \frac{320H_1}{9} + \frac{64}{3} H_{0,1} \right] \right] L_M \]

+ \[ \left[ \frac{64zL_Q^2}{9} + \left[ z \left[ -\frac{256}{9} H_0 - \frac{128H_1}{9} \right] - \frac{64}{27}(25z - 6) \right] L_Q + N_F \left[ \frac{128zL_Q^2}{9} + \left[ z \left[ -\frac{512}{9} H_0 \right. \right. \]

- \[ \frac{256H_1}{9} \right] - \frac{128}{27}(25z - 6) L_Q \right] T^2_F + C_A \left[ \frac{-16(216z^3 - 3329z^2 + 624z + 144)}{135z} \right. \]

- \[ 64(18z^3 - 149z^2 + 6z - 12)H_0 + \frac{(3z^5 - 5z^3 - 10z^2 - 2)(128 \frac{15}{15} H_{1,0} - 128 \frac{15}{15} H_{0,-1})}{z^2} \]

+ \[ z(3z^2 - 5) \left[ \frac{128z}{15} - \frac{64}{15} H_0^2 \right] + z \left[ \frac{128}{3} H_0 H_{2,1} + \left[ -\frac{64}{3} H_0^2 - \frac{256}{3} H_{0,-1} \right] H_{1,0} + \frac{64}{3} H_0^2 H_1 \right. \]

+ \[ \frac{1088H_1}{9} + H_0 \left[ \frac{128}{3} H_{0,1} - \frac{128}{3} H_{0,0,1} \right] + \frac{256}{3} H_{0,0,-1} - \frac{128}{3} H_{0,0,-1} + \frac{128}{3} H_{0,0,1} + \left[ \frac{128}{3} H_{1,-1} \right. \]

- \[ \frac{128H_1}{3} \right] \zeta_2 - \frac{256 \zeta_3}{3} \right] - \frac{352L_Q^2 z}{9} \left] T_F \right[ \right. \]

C_F + C_{\text{NS},(3)}^L \left( N_F \right) \right\} .

The pure-singlet Wilson coefficient \( H_{q,L}^{PS} \) reads:

\[ H_{q,L}^{PS} = \]

\[ a_5^2 C_F T_F \left\{ \frac{32(z + 1)(10z^2 - 2z + 1)}{9z} - \frac{32(z + 1)(2z + 1)}{3z} \right\} - 32(z - 1)(2z - 1)H_0 - \frac{32(z - 1)(2z^2 + 2z - 1)}{3z} \]

+ \[ L_Q \left[ \frac{32(z - 1)(2z^2 + 2z - 1)}{3z} - 32zH_0 \right] + z \left[ 32H_0^2 + 32H_{0,1} - 32\zeta_2 \right] \]

+ \[ a_3^2 C_F^2 T_F \left\{ -\frac{8}{3}(5z + 2)H_0^3 - \frac{8}{3}(8z^2 + 3)H_0^2 + \frac{16}{9}(160z^2 + 93z - 39)H_0 \right\} \]

+ \[ 16zH_0^2 - 16(z + 2)H_0 - \frac{16(z - 1)(4z^2 - 11z - 2)}{3z} \]

\[ L_M^2 - \frac{32(z - 1)(440z^2 - 91z - 28)}{27z} \]
\[
+ \frac{(z-1)(4z^2 - 11z - 2)}{z} \left( \frac{32}{3} H_0 H_1 - \frac{32}{3} H_0 \right) + \left[ - \frac{32}{3} z H_0^3 + 16(z + 2) H_0^2 \right] + \frac{32}{3} \left( 8z^2 + 18z + 3 \right) H_0 - \frac{32(z-1)(80z^2 + 17z - 10)}{9z} \right] L_M + L_Q^2 \left[ - \frac{64}{3} (2z^2 - 3) H_0 \right] \] 
\[
+ \frac{32(z-1)(2z^2 - 9z - 1)}{3z} - \frac{64(z-1)(2z^2 + 2z - 1) H_1}{3z} + \left[ 64H_{0,1} - 64\zeta_2 \right] \] 
\[
+(z + 2) \left[ 32 H_0 H_{0,1} - 64 H_{0,0,1} + 64\zeta_3 \right] + z \left[ \frac{4}{3} H_0^3 - 64 H_{0,0,1} - 128\zeta_3 \right] + \frac{384\zeta_2}{5} \] 
\[
+ 192 H_{0,0,1} \right] + L_Q \left[ - \frac{32(z-1)(86z^2 - 33z + 6)}{45z} - \frac{32(56z^3 - 813z^2 + 142z + 16) H_0}{45z} \right] 
\[
+ \frac{64(z-1)(4z^2 + 55z - 5) H_1}{9z} + \frac{(z-1)(2z^2 + 2z - 1)}{z} \left[ - \frac{64}{3} H_1 + \frac{128}{3} H_0 H_1 \right] \] 
\[
+(z + 1) (6z^4 - 6z^3 + 11z^2 + 4z - 4) \left[ \frac{128}{45} H_{0,-1} - \frac{128}{45} H_{-1} H_0 \right] \] 
\[
+ \frac{128(4z^3 - 6z^2 - 3z - 1) H_{0,1}}{3z} + (6z^3 + 90z^2 - 85z - 90) \left[ \frac{64}{45} H_0^2 - \frac{128\zeta_2}{45} \right] \] 
\[
+z \left[ - \frac{64}{9} H_0^3 + \left[ \frac{128}{3} H_{0,-1} - 128 H_{0,1} \right] H_0 + \frac{896}{3} \zeta_2 H_0 - \frac{256}{3} H_{0,0,-1} - 128 H_{0,1,1} + 192\zeta_3 \right] \] 
\[
+ C_F T_F^2 N_F \left[ \left[ \frac{128(z-1)(2z^2 + 2z - 1)}{9z} - \frac{128}{3} z H_0 \right] L_Q^2 \right] + \left[ \frac{256}{3} z H_0^3 \right] \] 
\[
- \frac{256}{9} (4z^2 - 8z - 3) H_0 - \frac{256(z-1)(3z^2 + 6z - 2)}{9z} \right] L_Q \] 
\[
+ C_F T_F^2 \left[ \left[ \frac{128(z-1)(2z^2 + 2z - 1)}{9z} - \frac{128}{3} z H_0 \right] L_Q^2 + \left[ \frac{256}{3} z H_0^2 \right] - \frac{256}{9} (4z^2 - 8z - 3) H_0 \right] \] 
\[
- \frac{256(z-1)(3z^2 + 6z - 2)}{9z} \right] L_Q + \frac{64(z-1)(2z^2 + 2z - 1) H_1^2}{9z} + \left[ \frac{128(z-1)(2z^2 + 2z - 1)}{9z} \right] \] 
\[
- \frac{128}{3} z H_0 \right] L_Q^2 + \frac{256(z-1)(55z^2 + 43z - 14)}{81z} - \frac{128}{27} z (19z + 67) H_0 \] 
\[
- \frac{128(z-1)(19z^2 + 16z - 5) H_1}{27z} + L_M \left[ \frac{256(z-1)(19z^2 + 16z - 5)}{27z} - \frac{256}{9} z (2z + 11) H_0 \right] \] 
\[
- \frac{256(z-1)(2z^2 + 2z - 1) H_1}{9z} + z \left[ \frac{256}{3} H_{0,1} - \frac{256\zeta_2}{3} \right] + z (2z + 11) \left[ \frac{128}{9} H_{0,1} - \frac{128\zeta_2}{9} \right] \] 
\[
+z \left[ \frac{128\zeta_3}{3} - \frac{128}{3} H_{0,1,1} \right] + C_F C_T_F \left[ \left[ - \frac{16(z-1)(46z^2 - z - 21)}{3z} + \frac{64(7z^2 - 3z - 1) H_0}{3z} \right] \right] 
\[
- \frac{64(z-1)(2z^2 + 2z - 1) H_1}{3z} + z \left[ 64 H_0^2 + 64 H_{0,1} - 64\zeta_2 \right] \right] L_Q^2 + \left[ - \frac{32}{3} (19z - 12) H_0^2 \right] \] 
\[
+ \frac{32(122z^3 - 137z^2 - 114z + 4) H_0}{9z} - \frac{32(z-1)(670z^2 - 245z + 46)}{27z} \] 
\[
+ \frac{32(z-1)(106z^2 - 23z - 65) H_1}{9z} + \frac{(z-1)(2z^2 + 2z - 1)}{z} \left[ \frac{128}{3} H_1^2 + \frac{256}{3} H_0 H_1 \right] \right] \]
Finally, the gluonic Wilson coefficient is given by:

\[
H_{g,L}^5 = 
\begin{align*}
& -16 T_F(z-1) a_s + a_s^2 \left\{ -\frac{64}{3} (z-1) z L_M T_F^2 + C_A T_F \left[ -\frac{32(z-1)(53z^2 + 2z - 1)}{9z} \right. \right. \\
&\left. \left. + \frac{32(13z^2 - 8z - 1)}{3z} H_0 - \frac{32(z-1)(29z^2 + 2z - 1)}{3z} H_1 \right] + L_Q \left[ \frac{32(z-1)(17z^2 + 2z - 1)}{3z} \right] \right] \\
&\left[ 16(z-1)(6z^4 - 6z^3 + 2z - 1) H_0 - \frac{64}{15} \left( 3z^2 + 5 \right) H_0^2 \right] \\
&+ \left[ \frac{16(36z^3 - 78z^2 - 13z - 4)}{15z} H_0 + \frac{32(z-1)(63z^2 + 6z - 2)}{15z} \right] \\
&+ L_Q \left[ \frac{32z H_0}{15} - 16(z-1)(2z+1) \right] + 16(z-1)(4z+1) H_1 \\
&+ \left[ \frac{64}{15} H_{-1} H_0 - \frac{64}{15} H_{-1} H_0 \right] - 32 z H_{0,1} \\
&+ \left[ 16(z-1)(2z+1) - 32 z H_0 \right] L_M + \frac{32}{15} z (12 z^2 + 5) \zeta_2 \right] \\
&+ a_s^3 \left\{ -T_F^2 \left[ \frac{256}{9} (z-1) z L_M^2 + C_A T_F^2 \left[ \left[ -\frac{64(z-1)(17z^2 + 2z - 1)}{9z} \right. \right. \\
&\left. \left. - \frac{256}{3} z H_0 \right. \right] \\
&+ \frac{128}{3} (z-1) z H_1 \right] L_Q + \left[ -\frac{64(z-1)(461z^2 + 11z - 25)}{27z} - \frac{128}{9} z (26z - 59) H_0 \right] \\
&- \frac{128(z-1)(39z^2 + 2z - 1)}{9z} H_1 \right] + (z-1) z \left[ -\frac{128}{3} H_1^2 - \frac{256}{3} H_0 H_1 \right] \\
&+ \frac{512}{3} z H_0 + \frac{256}{3} (z-1) z H_1 \right] L_M + \frac{256}{3} (z-2) z \zeta_2 \right] L_Q - \frac{32}{9} (28z - 3) H_0^2 \\
&+ \left[ \frac{64(z-1)(17z^2 + 2z - 1)}{9z} \right] L_M + \frac{256}{3} (z-1) z H_1 \right] L_M^2 \\
&+ \frac{256}{3} (z-1) (2714z^2 - 106z - 139) \right] - \frac{64}{27} (110z^2 + 277z - 33) H_0 + \frac{4160}{27} (z-1) z H_1 \\
&+ \left[ \frac{64}{9} H_0^3 - \frac{64}{3} H_{0,1} + \frac{64}{3} \zeta_2 \right] + L_M \left[ \frac{64(z-1)(68z^2 + z - 7)}{9z} \right. \\
&- \frac{128}{9} (4z-1)(13z + 6) H_0 \\
\right\}.
\end{align*}
\]
\[-64H_{0,1} + 64\zeta_2\] - 32(2z^2 - 19z + 1)\zeta_3 + z \left[ \frac{4}{3}H_0^4 + 32H_{0,1}H_0^2 - 192H_{0,0,1}H_0 + 192\zeta_2H_0 \right]

\[-64\zeta_3H_0 - \frac{608\zeta_2}{5} + 384H_{0,0,0,1} - 64H_{0,0,1,1} - 64H_{0,1,1,1} \right] + L_M \left[ \frac{32}{15}(24z^3 + 90z^2 - 95z - 15)H_0^2 \right]

+ \frac{32(78z^3 + 141z^2 - 34z + 8)H_0}{15z} + 128(2z^2 - 3z - 1)H_{0,1}H_0 - \frac{8(z - 1)(6z + 1)(153z - 32)}{15z}

+ 16(z - 1)(4z - 3)H_1 + \frac{(z + 1)(12z^4 + 3z^3 - 73z^2 - 2z + 2)}{z^2} \left[ \frac{128}{15}H_{0,-1} - \frac{128}{15}H_{-1}H_0 \right]

+ 32(6z + 1)H_{0,1} - 64(4z^2 - 5z - 2)H_{0,0,1} - \frac{32}{15}(48z^3 + 120z^2 - 250z - 45)\zeta_2

+ (z + 1)(2z - 1) \left[ 128H_0H_{1,1} + \left[ -64H_0^2 - 256H_{0,-1} \right] H_{-1} + 128\zeta_2H_{-1} - 128H_{0,0,1} \right]

+ 256H_{0,-1,-1} + 384H_{0,0,0,1} + (z - 1)(2z + 1) \left[ -64H_1H_{0}^2 + 128H_1H_0 + 64H_1^2 + 128H_1\zeta_2 \right]

+ z \left[ \frac{32}{3}H_0^3 + \left[ 128H_{0,1} - 128H_{0,-1} \right] H_0^2 + \left[ 512H_{0,-1,-1} + 512H_{0,0,0,1} - 512H_{0,0,1} \right] H_0 \right]

- 256H_{0,-1,1} + \frac{768\zeta_2^2}{5} - 256H_{0,1,1} - 768H_{0,0,0,1} + 768H_{0,0,0,1} + \left[ 64H_0 + 256H_{0,-1} - 256H_{0,1} \right] \zeta_2

+ \left[ 512H_0 - 576\zeta_3 \right] + L_Q \left[ \frac{32}{15}(24z^3 + 90z^2 - 95z - 15)H_0^2 \right]

- \frac{32(78z^3 + 141z^2 - 34z + 8)H_0}{15z} - 128(2z^2 - 3z - 1)H_{0,1}H_0 + \frac{8(z - 1)(6z + 1)(153z - 32)}{15z}

- 16(z - 1)(4z - 3)H_1 + \frac{(z + 1)(12z^4 + 3z^3 - 73z^2 - 2z + 2)}{z^2} \left[ \frac{128}{15}H_{-1}H_0 - \frac{128}{15}H_{0,-1} \right]

- 32(6z + 1)H_{0,1} + 64(4z^2 - 5z - 2)H_{0,0,1} + L_M \left[ -64(2z + 1)H_1(z - 1) - 48(z - 1) \right]

- 32(2z - 1)(2z + 1)H_0 + z \left[ 32H_0^2 + 128H_{0,1} - 128\zeta_2 \right] + \frac{32}{15}(48z^3 + 120z^2 - 250z - 45)\zeta_2

+ (z + 1)(2z - 1) \left[ -128H_0H_{1,1} + \left[ 64H_0^2 + 256H_{0,-1} \right] H_{-1} - 128\zeta_2H_{-1} + 128H_{0,0,1} \right]

- 256H_{0,-1,-1} + 384H_{0,0,0,1} + (z - 1)(2z + 1) \left[ 64H_1H_{0}^2 - 128H_1H_0 - 64H_1^2 - 128H_1\zeta_2 \right]

+ z \left[ -\frac{32}{3}H_0^3 + \left[ 128H_{0,1} - 128H_{0,-1} \right] H_0^2 + \left[ -512H_{0,-1,-1} + 512H_{0,0,0,1} + 512H_{0,0,1} \right] H_0 + 256H_{0,-1}^2 \right]

- \frac{768\zeta_2^2}{5} + 256H_{0,1,1} + 768H_{0,0,0,-1} - 768H_{0,0,0,1} + \left[ -64H_0 - 256H_{0,-1} + 256H_{0,1} \right] \zeta_2

+ \left[ 512H_0 + 576\zeta_3 \right] + C_{FTH}^2 \left[ -\frac{16}{3}zH_0^4 - \frac{32}{3}(7z - 1)H_0^3 \right]

- 32(19z - 3)H_0^2 - 64(6z^2 + 7z - 8)H_0 + \left[ -64zH_0^2 - \frac{64}{3}(4z^2 + 5z - 3)H_0 \right]

+ \frac{64(z - 1)(40z^2 - 17z - 2)}{9z}L_M^2 + \frac{16(z - 1)(343z^2 - 242z + 4)}{3z} + L_Q^2 \left[ -64zH_0^2 \right]

- \frac{64}{3}(z + 1)(4z - 3)H_0 + \frac{64(z - 1)(28z^2 - 23z - 2)}{9z} + L_Q \left[ \frac{128(25z + 2)H_1(z - 1)^2}{9z} \right]
\[-128H_{0,-1,-1} + \left( z - 1 \right) \left( 2z + 1 \right) \left[ 32H_1H_0^2 + \left( 64H_{0,-1} - 64H_{0,1} \right) H_0 - 32H_1^2 + 64H_{0,0,1} \right] - 64H_1\zeta_2 \right) + \frac{128}{3} H_0^3 + \left( 64H_{0,-1} - 64H_{0,1} \right) H_0^2 + \left[ -256H_{0,-1,-1} - 256H_{0,0,1} \right] H_0 + 256\zeta_3 H_0 + 128H_{0,-1}^2 - \frac{384\zeta_2^2}{5} - \frac{544}{3} H_{0,1} + 128H_{0,1,1} + 384H_{0,0,0,1} \right] - 384H_{0,0,0,1} + \left[ 128H_{0,1} - 128H_{0,-1} \zeta_2 \right] \right]\right]\right] + \tilde{C}_{L,g}^{S,(3)} \left( N_F + 1 \right) \right\}. \quad (603)

D The OMEs in z-Space

In the following, we present the massive operator matrix elements in z-space. They are given by:

\[ A_{qg,\bar{q}g}^{PS} = \]

\[ a_3^3 \left\{ C_N T_F^2 \left[ \frac{L_2}{L} \left[ (z + 1) \left[ \frac{64}{9} \right] H_{0,1} - \frac{32}{3} H_0^2 - \frac{64\zeta_2}{3} \right] + \frac{32}{9} \left( 4z^2 - 7z - 13 \right) H_0 \right] \right\} \]

\[ - \frac{32(z - 1)(4z^2 + 7z + 4)H_1}{9z} - \frac{32}{3} \left( z - 1 \right) \left( 2z - 5 \right) + L_M \left[ \left( 4z^2 - 7z - 13 \right) \left[ \frac{64\zeta_2}{9} - \frac{64}{9} H_{0,1} \right] \right] \]

\[ + \left( z + 1 \right) \left[ \frac{128}{3} H_{0,0,1} - \frac{128}{3} H_{0,1,1} - \frac{128}{3} H_0\zeta_2 - \frac{32}{9} H_0^3 - \frac{464}{9} H_0^2 \right] + \frac{32(z - 1)(4z^2 + 7z + 4)H_1^2}{9z} \]

\[ + \frac{128}{27} \left[ 19z^2 - 16z - 40 \right] + \frac{64}{3} \left( z - 1 \right) \left( 2z - 5 \right) H_1 + \frac{32(z - 1)(8z^2 - 51z - 136)}{81z} \]

\[ + \frac{57z^2 - 131z - 203}{9z} \left[ \frac{64\zeta_2}{81} - \frac{64}{81} H_{0,1} \right] + \frac{1856}{27} \frac{H_{0,0,1}}{H_{0,0,1}} + \frac{128}{9} H_{0,0,0,1} - \frac{256}{9} H_{0,0,1,1} \]

\[ + \frac{320}{9} H_{0,1,1,1} + H_0^2 \left[ - \frac{64}{9} \zeta_2 - \frac{5312}{81} \right] + H_0 \left[ \frac{64\zeta_3}{9} - \frac{1856\zeta_2}{27} \right] - \frac{8}{27} H_0^4 + \frac{464}{81} H_0^3 \]

\[ - \frac{256}{15} \zeta_2^2 + \frac{64}{27} \left( 6z^2 - 25z - 34 \right) + L_M \left[ \frac{32(z - 1)(4z^2 + 7z + 4)}{27z} - \frac{64}{9} \left( z + 1 \right) H_0 \right] \]

\[ - \frac{80(z - 1)(4z^2 + 7z + 4)H_1^3}{81z} + \frac{16(z - 1)(34z^2 - 227z - 20)H_1^2}{81z} \]

\[ - \frac{64(z - 1)(10z^2 + 213z + 64)H_1}{81z} + \frac{32}{243} \left( 660z^2 - 1577z - 2351 \right) H_0 \]

\[ - \frac{64(2z^3 + 16z^2 - 17z - 16)\zeta_3}{27z} - \frac{64(z - 1)(139z^2 - 3701z - 752)}{729z} \right\}, \quad (604)

\[ A_{qg,\bar{q}g} = \]

\[ a_3^3 \left\{ C_N T_F^2 \left[ - \frac{8}{27} \left( 4z^2 + 16z - 5 \right) H_0^3 + \frac{32}{81} \left( 14z + 29 \right) H_0^3 \right] \right\} \]

\[ - \frac{16}{81} \left( 44z^2 + 243z - 56 \right) H_0^3 + \frac{16}{81} \left( 200z + 347 \right) H_0^2 - \frac{16}{81} \left( 402z^2 + 1472z - 205 \right) H_0^2 \]

\[ - \frac{64}{27} \left( 7z^2 + 7z + 5 \right) H_1^2 + \frac{8}{243} \left( 5772z^2 - 27934z - 451 \right) H_0 + z \left[ \frac{96}{9} \zeta_2 \right] \]

\[ + \frac{11392}{81} \left[ H_0 + \frac{256}{9} \left( z + 1 \right) \zeta_3 H_0 - \frac{16}{9} \left( 4z^2 - 7z - 1 \right) H_1 H_0 \right] \]
\[-\frac{32}{9} \left(6z^2 - z - 4\right)H_0 + \frac{8(124z^3 - 258z^2 + 159z - 16)}{27z} + \frac{8}{9} \gamma_9 H_1 \]
\[+ \frac{8}{81} \left(364z^2 - 373z + 224\right)H_0^2 - \frac{179524z^3 + 2535258z^2 - 2173863z - 42688}{729z} \]
\[-\frac{448}{81} \left(13z^2 - 13z + 8\right) \zeta_2 - \frac{64(117z^3 - 251z^2 + 154z - 16)}{27z} \zeta_3 + (2z - 1)\left[\frac{128}{3} H_0^2 \zeta_3 \right] \]
\[-\frac{16}{15} \left[H_0^3\right] - \frac{16}{243} \left(2188z^2 - 2278z + 1613\right)H_1 + L_M^2 \left[ -\frac{32}{3} (2z - 1)H_0^3 \right] \]
\[-\frac{16}{3} (2z + 3)(4z - 3)H_0^2 + \frac{8}{9} (160z^2 + 146z + 305)H_0 \]
\[-\frac{8}{27} (1000z^3 + 1356z^2 - 2247z - 208) - \frac{32}{9} (4z^2 - 4z + 5)H_1 \]
\[+ \gamma_9 \left[ -\frac{4}{3} H_1^2 + \frac{8\zeta_2}{3} - \frac{8}{3} H_0 \right] + (16z^2 - 16z + 5)\left[ \frac{16}{27} H_1 H_0^2 - \frac{32}{27} H_0, H_0 + \frac{32}{27} H_0, 0, 1 \right] \]
\[+ (7z^2 - 7z + 5) \left[ -\frac{32}{81} H_1^3 + \frac{896}{81} H_0, 1 - \frac{64}{27} H_0, 1, 1 \right] + L_M \left[ -\frac{20}{3} (2z - 1)H_0^4 \right] \]
\[-\frac{16}{9} (14z^2 + 37z - 26)H_0^3 + \frac{4}{9} (136z^2 - 174z + 909)H_0^2 - \frac{8}{27} (32z^2 - 2393z - 4145)H_0 \]
\[+ \frac{64}{3} (z - 1)zH_1 H_0 - \frac{4}{9} (3556z^3 + 33342z^2 - 38175z + 800) - \frac{16}{27} (140z^2 - 149z + 112)H_1 \]
\[+ (7z^2 - 7z + 5) \left[ \frac{32}{9} H_1^2 - \frac{64\zeta_2}{9} \right] + \frac{64}{9} (4z^2 - 4z + 5)H_0, 0 + \gamma_9 \left[ \frac{4}{9} H_1^3 - \frac{4}{3} H_0^2 H_1 - \frac{8\zeta_3}{3} \right] \]
\[+ \frac{8}{3} H_0, 0, 1 - \frac{8}{9} H_0, 0, 1, 1 \] + \gamma_9 \left[ -\frac{1}{27} H_1^4 - \frac{8}{27} H_0^3 H_1 - \frac{64}{9} \zeta_3 H_1 + \frac{16}{45} \zeta_2 \right] \]
\[+ \frac{8}{9} H_0^2 H_0, 0, 1 - \frac{16}{9} H_0 H_0, 0, 0, 1 + \frac{16}{9} H_0, 0, 0, 0, 1 - \frac{8}{9} H_0, 0, 1, 1, 1 \right] \}

\( A^{P5}_{Q_9} = \)
\[a_5^2 C_F T_F \left[ \frac{2}{3} (8z^2 + 15z + 3) H_0^2 - \frac{8}{9} (56z^2 + 33z + 21) H_0 \right] \]
\[+ L_M^2 \left[ \frac{4(z - 1)(4z^2 + 7z + 4)}{3z} - 8(z + 1)H_0 \right] + \frac{4(z - 1)(400z^2 + 121z + 112)}{27z} \]
\[+ \frac{(z - 1)(4z^2 + 7z + 4)}{z} \left[ \frac{8}{3} H_0, 1 - \frac{8}{3} H_0 H_1 \right] + (z + 1) \left[ -\frac{4}{3} H_0^3 + 16H_0, 1, 0 + 32\zeta_3 - 32H_0, 0, 1 \right] \]
\[+ L_M \left[ 8(z + 1)H_0^2 - \frac{8}{3} (8z^2 + 15z + 3) H_0 + \frac{16(z - 1)(28z^2 + z + 10)}{9z} \right] \}
\[+ a_5^3 \left[ a_{PS, (3)}^2 + C_F^2 T_F \left[ -\frac{2}{9} (4z^2 - 3z + 3) H_0^4 - \frac{2}{9} (400z^2 + 149z + 115) H_0^3 \right] \]
\[+ \frac{2}{3} (160z^2 + 191z - 117) H_0^3 - \frac{8}{3} (4z^2 - 3z - 3) \zeta_2 H_0^2 - \frac{4(z - 1)(20z^2 + 41z - 4) H_1 H_0^2}{3z} \]
\[+ \frac{8}{3} (4z^3 + 27z^2 + 3z - 4) H_0, 0, 1 H_0^2 - \frac{4}{3} (400z^2 - 135z + 222) H_0 - \frac{2}{3} (80z^2 + 469z + 221) \zeta_2 H_0 \]
\[\begin{align*}
&+ \frac{16}{9} (44z^2 + 51z - 18) \zeta_3 H_0 - \frac{8(z - 1)(16z^2 - 43z + 66) H_1 H_0}{3z} \\
&+ \frac{16(z - 1)(10z^2 + 11z - 2) H_{0,1} H_0}{3z} + 16(4z^3 - 33z^2 - 15z + 4) H_{0,0,1} H_0 \\
&- \frac{8(z - 1)(6z^2 - z - 6) H_1^3}{9z} + \frac{8}{15} (188z^2 - 27z - 105) \zeta_2^2 - \frac{4(z - 1)(24z^2 - 13z + 17) H_1^2}{3z} \\
&+ \frac{64(z + 1)^2(2z - 1) H_0^2}{3z} + \frac{4(z - 1)(2z + 1)(352z + 233)}{9z} + \frac{4(84z^3 + 79z^2 + 75z - 60) \zeta_2}{3z} \\
&- \frac{4(z - 1)(40z^2 + 33z + 4) H_1 \zeta_2}{3z} - \frac{8(12z^3 + 15z^2 + 9z + 8) H_{0,1} \zeta_2}{3z} \\
&+ \frac{4}{9} (72z^2 - 809z - 145) \zeta_3 - \frac{4(z - 1)(80z^2 - 181z - 9) H_1}{3z} + L^3_M \left[ \frac{16(4z^2 + 7z + 4) H_1(z - 1)}{9z} \right] \\
&+ \frac{92(z - 1)}{9} + \frac{16}{9} z(4z + 3) H_0 + (z + 1) \left[ -\frac{8}{3} H_0^2 + \frac{32 \zeta_2}{3} - \frac{32}{3} H_{0,1} \right] \\
&- \frac{8(8z^3 + 100z^2 - 85z + 66) H_{0,1}}{3z} - \frac{32}{3} (z - 1) H_1 H_{0,1} - \frac{8(20z^3 - 95z^2 - 43z + 4) H_{0,0,1}}{3z} \\
&+ L^2_M \left[ \frac{8}{3} (4z^2 - 9z - 3) H_0^2 + \frac{8}{3} (z + 1)(32z - 31) H_0 + \frac{16(z - 1)(4z^2 + 7z + 4) H_1 H_0}{3z} \right] \\
&+ \frac{4(z - 1)(32z^2 + 81z + 12)}{3z} + 32(3z + 2) \zeta_2 + \frac{8(z - 1)(32z^2 + 35z + 8) H_1}{3z} \\
&- \frac{16(4z^3 + 21z^2 + 9z - 4) H_{0,1}}{3z} + (z + 1) \left[ -\frac{16}{3} H_0^3 + \left[ 32 \zeta_2 - 32 H_{0,1} \right] H_0 - 32 \zeta_3 + 32 H_{0,0,1} \right] \\
&+ \frac{(z - 1)(4z^2 + 7z + 4)}{z} \left[ -\frac{2}{9} H_1^4 + \left[ \frac{16}{3} H_{0,1} - \frac{20 \zeta_2}{3} \right] H_1^2 - \frac{8}{9} H_0^3 H_1 \right] \\
&+ \left[ \frac{176 \zeta_3}{9} - \frac{64}{3} H_{0,0,1} - \frac{64}{3} H_{0,1,1} \right] H_1 + H_0 \left[ H_1 \left[ \frac{32}{3} H_{0,1} - \frac{16 \zeta_2}{3} \right] - \frac{64}{3} H_{0,1,1} \right] \\
&- \frac{8}{3} (12z^2 - 23z - 22) H_{0,1,1} - \frac{16(20z^3 - 21z^2 - 33z + 4) H_{0,0,0,1}}{3z} - \frac{32(3z^2 + 15z + 8) H_{0,0,1,1}}{3z} \\
&+ \frac{16(20z^3 + 15z^2 - 27z - 24) H_{0,1,1,1}}{3z} + L_M \left[ -\frac{32}{3} (4z^2 + z + 1) H_0^3 \right] \\
&+ \frac{2}{3} (136z^2 - 111z + 213) H_0^2 + \frac{8}{27} (242z^2 - 3984z - 633) H_0 + \frac{32}{3} (2z + 3) (8z - 3) \zeta_2 H_0 \\
&+ \frac{8(z - 1)(140z^2 - 127z + 104) H_1 H_0}{9z} - 32(4z^3 + 3z + 1) H_{0,1} H_0 \\
&+ \frac{4(z - 1)(28z^2 + 21z + 4) H_1^2}{3z} + \frac{4(z - 1)(3204z^2 + 1625z + 180)}{27z} - \frac{16}{9} (74z^2 + 18z + 297) \zeta_2 \\
&+ \frac{16(12z^3 - z^2 + z - 8) \zeta_3}{z} + \frac{16(z - 1)(229z^2 - 1175z - 239) H_1}{27z} \\
&+ \frac{8(8z^3 + 303z^2 + 363z + 104) H_{0,1}}{9z} + \frac{(z - 1)(4z^2 + 7z + 4)}{z} \left[ -\frac{8}{9} H_1^3 - \frac{16}{3} H_0 H_1^2 + \frac{32}{3} H_{0,1} H_1 \right] \\
&+ \frac{32}{3} (8z^2 + 15) H_{0,0,1} - \frac{16(12z^3 + 27z^2 + 3z - 8) H_{0,1,1}}{3z} + (z + 1) \left[ 6H_0^4 - 96 \zeta_3 H_0^2 \right] \\
&+ \left[ 192 \zeta_3 + 96 H_{0,0,1} + 64 H_{0,1,1} \right] H_0 + \frac{288}{5} \zeta_2^2 - 32 H_{0,1}^2 - 96 H_{0,0,0,1} + 32 H_{0,1,1,1} \right] \\
&+(z + 1) \left[ \frac{2}{15} H_0^5 + \left[ 4 \zeta_2 + \frac{16}{3} H_{0,1} \right] H_0^2 + \left[ \frac{88}{3} H_{0,0,1} - 48 H_{0,0,0,1} \right] H_0^2 + \left[ -\frac{448}{5} \zeta_2^2 + 32 H_{0,1} \zeta_2 \right] \right]
\end{align*}\]
\[-32H_{0,1}^3 + 160H_{0,0,0,1} + 128H_{0,0,1,1} \left[ H_0 + 32H_{0,0,1}\zeta_2 \right] + 80H_{0,1,1}\zeta_2 - \frac{80}{3}\zeta_2\zeta_3 + 160\zeta_5 \]
\[+ H_{0,1} \left[ -\frac{352}{3}\zeta_3 + 128H_{0,0,1} - 64H_{0,1,1} \right] - 192H_{0,0,0,1,1} - 768H_{0,0,0,1,1} - 320H_{0,0,1,0,1} \]
\[+ 416H_{0,0,1,1,1} + 192H_{0,1,0,1,1} + 32H_{0,1,1,1,1} \right] \] + C_F T_F^2 \left[ \frac{16}{27} (8z^2 + 15z + 3) H_0^3 \right]

\[\frac{32}{27} \left( 56z^2 + 33z + 21 \right) H_0^3 + \frac{32}{81} \left( 1020z^2 + 697z + 607 \right) H_0 + \frac{32}{9} \left( 12z^2 + 37z + 19 \right) \zeta_2 H_0 \]
\[+ \frac{128(z - 1)(28z^2 + z + 10) H_1 H_0}{27z} - \frac{128(2z^3 + 6z^2 + 3z + 2) H_0 H_1}{9z} \]
\[+ L_M^2 \left[ \frac{128(z - 1)(4z^2 + 7z + 4)}{27z} \right] - \frac{256}{9} (z + 1) H_0 \]
\[+ \frac{16(z - 1)(38z^2 + 47z + 20) H_1^2}{27z} + \frac{243z}{64(2z^3 + 129z^2 + 36z - 8) \zeta_3} \]
\[+ \frac{128(z - 1)(55z^2 + 64z + 28) H_1}{81z} + \frac{(z - 1)(4z^2 + 7z + 4)}{27z} \left[ \frac{16}{27} H_1^3 - \frac{32}{9} H_0^2 H_1 + \frac{32}{9} \zeta_2 H_1 \right] \]
\[- \frac{64(z - 1)(1781z^2 + 539z + 656)}{27z} - \frac{128(5z^2 + 5z - 2) H_0 H_1}{9z} \]
\[+ \frac{32(56z^2 - 179z^2 - 95z - 40) \zeta_2}{27z} + \frac{32(94z^2 + 49z + 40)}{27z} \]
\[- \frac{64(75z^3 + 13z^2 + 61z - 20) H_0,1}{27z} + \frac{32(z - 1)(94z^2 + 49z + 40)}{9z} \]
\[+ \frac{256(z - 1)(40z^2 + 79z + 31)}{81z} \]
\[- \frac{64(12z^3 + 27z^2 + 9z + 4) H_0,0,1}{9z} + \frac{128(5z^2 + 5z - 2) H_0,1}{9z} \]
\[+ (z + 1) \left[ -\frac{64}{9} H_0^3 + \left[ \frac{128}{3} H_0,1 - \frac{128\zeta_2}{3} \right] H_0 + \frac{256\zeta_3}{3} - \frac{128}{3} H_{0,0,1} - \frac{128}{3} H_{0,1,1}\right] \]
\[+ \frac{1024\zeta_4}{9} - \frac{128}{3} H_{0,0,1} \right] H_0 + \frac{448}{15} \zeta_2^2 - \frac{64}{3} H_{0,1}\zeta_2 + \frac{64}{3} H_{0,1,1,1} \right] \]
\[+ C_F T_F^2 N_F \left[ L_M^2 \left[ (z + 1) \left[ -\frac{64}{9} H_0,1 + \frac{32}{3} H_2 + \frac{64\zeta_2}{3} \right] \right] - \frac{32}{9} (4z^2 + 7z - 13) H_0 \]
\[+ \frac{32(z - 1)(4z^2 + 7z + 4) H_1}{9z} + \frac{32}{3} (z - 1)(2z - 5) \right] + L_M \left[ (z + 1) \left[ \frac{128}{3} H_0 H_1 \right] \right] \]
\[+ \frac{32(5z^2 + 5z - 2) H_0 H_1}{9z} - \frac{128}{3} H_{0,0,1} - \frac{64}{9} H_{0,1,1} \]
\[+ \frac{448}{15} \zeta_2^2 - \frac{64}{3} H_{0,1}\zeta_2 + \frac{64}{3} H_{0,1,1,1} \right] \]
\[+ \frac{1024\zeta_4}{9} - \frac{128}{3} H_{0,0,1} \right] H_0 + \frac{448}{15} \zeta_2^2 - \frac{64}{3} H_{0,1}\zeta_2 + \frac{64}{3} H_{0,1,1,1} \right] \]
\[+ \frac{32(z - 1)(4z^2 + 7z + 4) H_1}{9z} + \frac{32}{3} (z - 1)(2z - 5) \right] + L_M \left[ (z + 1) \left[ \frac{128}{3} H_0 H_1 \right] \right] \]
\[- \frac{128}{3} H_{0,0,1} - \frac{64}{9} H_{0,1,1} \]
\[+ \frac{32}{3} (4z^2 + 7z - 13) H_0 + \frac{32}{27} (z^2 + 2z - 58) H_0 \]
\[+ \frac{128(z - 1)(4z^2 + 43z + 20) H_1}{27z} + \frac{(z - 1)(4z^2 + 7z + 4)}{27z} \left[ \frac{16}{9} H_1^2 - \frac{64}{9} H_0 H_1 \right] \]
\[\begin{align*}
+ \frac{64}{9} \zeta_2 (6z^2 + 4z - 5) + \frac{128(z - 1)(25z^2 + 94z + 34)}{81z} + (z + 1) \left[ \zeta_2 \left( \frac{32}{3} H_{0,1} - \frac{32}{3} H_0^2 \right) \right] \\
+ \frac{64}{3} H_0^2 H_{0,1} - \frac{128}{3} H_0 H_{0,0,1} + \frac{832}{9} \zeta_3 H_0 - \frac{8}{9} H_0^4 - \frac{32 \zeta_2^2}{3} + \left( z - 1 \right) \left( 28z^2 + z + 10 \right) \\
\left[ \frac{128}{27} H_0 H_{1,0} - \frac{128}{27} H_0 \right] - \frac{16}{27} (z - 1) (74z^2 - 43z + 20) \zeta_2 - \frac{32}{27} (100z^3 + 183z^2 + 33z - 4) \zeta_3 \\
+ L^M_3 \left[ \frac{32(z - 1)(4z^2 + 7z + 4)}{27z} - \frac{64}{9} (z + 1) H_0 \right] + \frac{32}{9} \zeta_2 (6z^2 + 4z - 5) H_0 \\
\left( z - 1 \right) (4z^2 + 7z + 4) \left[ - \frac{16}{9} \zeta_2 H_1 - \frac{32}{9} H_1 H_0^2 \right] + \frac{16}{27} (8z^2 + 15z + 3) H_0^3 \\
\frac{32}{27} (56z^2 + 33z + 21) H_0^2 + \frac{32}{81} (800z^2 - 57z + 111) H_0 \\
\left[ \frac{64}{9} (z - 1)(1156z^2 - 203z + 328) \right] + C_A C_F T_F \left[ - \frac{2}{9} (4z - 17) H_0^4 - \frac{4}{9} (36z^2 + 47z + 36) H_0^3 \right] \\
- \frac{8}{3} (z + 3) \zeta_2 H_0^3 + \frac{64}{3} z^2 H_0^2 + \frac{4}{27} \left( 1988z^2 - 681z + 855 \right) H_0^4 + 8(z - 1)(2z + 1) \zeta_2 H_0^2 \\
\frac{8}{3} \left( 2z + 5 \right)(3z - 4) \zeta_2 H_0^2 - \frac{16}{3} (20z - 13) \zeta_3 H_0^2 + \frac{8(z - 1) \left( 122z^2 - 19z + 113 \right) H_1 H_0^2}{9z} \\
\frac{8(19z^2 + 19z + 8)}{3z} H_{0,1} H_0^2 + \frac{16}{5} (9z - 4) \zeta_2^2 H_0 + \frac{16}{5} (29z - 1) \zeta_2^2 H_0 \\
\frac{16(z - 1)(19z^2 + 16z + 10)}{9z} H_1 H_0^2 - \frac{64}{9} (37z^2 + 6) H_0 \\
- \frac{4(48876z^3 + 9339z^2 + 16218z + 2624)}{81z} H_0 - \frac{8}{9} (152z^2 - 39z + 60) \zeta_2 H_0 \\
- \frac{4(170z^3 + 199z^2 + 175z + 80)}{9z} \zeta_2 H_0 - \frac{16(24z^3 - 31z^2 + 215z + 4)}{9z} \zeta_3 H_0 \\
\frac{32(z - 1)(733z^2 - 62z + 301)}{27z} H_{1,0} H_0 - \frac{32(19z^2 - 24z^2 - 6z + 10)}{9z} H_{0,0,1} H_0 \\
\frac{16(18z^3 + 119z^2 - 2z + 51)}{9z} H_{0,1} H_0 - \frac{32(4z^3 - 23z^2 - 2z - 8)}{9z} H_{0,0,1} H_0 \\
\frac{8(z - 1)(2z + 1)(14z + 1)}{27z} H_1^3 - \frac{8(96z^3 - 427z^2 + 134z - 148)}{15z} \zeta_2^2 \\
\frac{8(116z^3 - 87z^2 - 3z + 4)}{27z} \zeta_2^2 + \frac{27z}{4(z - 1)(616z^2 + 313z + 355)} H_1^2 \\
\frac{15z}{4(z - 1)(75516z^2 - 7654z + 23205)} - \frac{8}{9} (9z^2 + 185z + 38) \zeta_2 \\
+ \frac{8(1868z^3 - 1164z^2 + 1344z - 515) \zeta_2}{27z} + \frac{4(z - 1)(154z^2 + 163z + 46)}{9z} H_1 \zeta_2 \\
\frac{8}{3} (23z + 14) H_{0,1} \zeta_2 - \frac{8(247z^3 - 9z^2 + 18z + 50) \zeta_3}{9z} \\
+ \frac{8(1015z^3 + 1149z^2 + 705z + 126) \zeta_3}{9z} - \frac{256}{3} (z - 2) \zeta_2 \zeta_3 + 8(25z - 9) \zeta_2 \zeta_3 \\
+ 8(3z + 5) \zeta_5 + 8(67z - 35) \zeta_5 - \frac{64(z - 1)(37z^2 + 16)}{9z} H_1 
\end{align*}\]
\[
\begin{aligned}
&+ \frac{4(z-1)(7828z^2 + 2755z + 4075)H_1}{81z} + \frac{(z-1)(z^2 + 1)\left[-\frac{128}{3}H_1^2 - \frac{128}{3}H_0H_1\right]}{z} \\
&+ \frac{(z+1)(182z^2 - 122z + 47)\left[\frac{32}{27}H_1H_0 - \frac{32}{27}H_{0,-1}\right]}{27z} + \frac{64(6z^3 + 19z^2 + 10z - 6)H_{0,1}}{9z} \\
&+ \frac{8(2820z^3 - 3849z^2 + 1128z - 1204)H_{0,1}}{27z} + L_M^3 \left[\frac{16}{3}(2z - 1)H_0^2 + \frac{16(8z^2 + 11z + 4)H_0}{9z}\right] \\
&- \frac{8(z-1)(44z^2 - z + 44)}{9z} \frac{16(z-1)(4z^2 + 7z + 4)H_1}{9z} + (z+1)\left[\frac{32}{3}H_{0,1} - \frac{32\zeta_2}{3}\right] \\
&+ \frac{32(19z^3 - 51z^2 - 6z + 10)H_{0,0,-1}}{9z} + \frac{64}{9}(19z^2 - 15z - 6)H_{0,0,1} \\
&- \frac{16(306z^3 + 561z^2 + 144z + 193)H_{0,0,1}}{9z} + L_M^2 \left[-\frac{8}{3}(4z^2 - 25z + 23)H_0^2\right] \\
&+ \frac{8}{3}(4z^2 - 9z + 6)H_0^2 + \frac{16}{9}(13z^2 - 30z + 24)H_0 + \frac{8(246z^3 + 163z^2 + 91z + 40)H_0}{9z} \\
&+ \frac{16(z-1)(4z^2 + 7z + 4)H_1H_0}{27z} - \frac{8(z-1)(35z^2 - 82z + 89)}{8(z-1)(104z^2 + 119z + 32)H_1} \\
&+ \frac{3z}{27z} \frac{8(z-1)(1829z^2 - 403z + 605)}{9z} - \frac{8(z-1)(104z^2 + 119z + 32)H_1}{9z} \\
&+ \frac{(z+1)(4z^2 - 7z + 4)\left[-\frac{16}{3}\zeta_2 - \frac{32}{3}H_1H_0 + \frac{32}{3}H_{0,-1}\right]}{z} + (10z + 7)\left[\frac{32}{3}H_{0,1} - \frac{32\zeta_2}{3}\right] \\
&+ (z-1)\left[-\frac{32}{3}H_0^3 + \left[-32\zeta_2 - 64H_{0,-1}\right]H_0 - 96\zeta_3 + 128H_{0,0,-1}\right] \\
&+ (z+1)\left[-32\zeta_3 - 32H_{0,0,1} + 32H_{0,0,1}\right] + \frac{(z+1)(19z^2 - 16z + 10)}{z} \left[-\frac{32}{9}H_0H_1^2\right] \\
&+ \left[\frac{16}{9}H_0^2 - \frac{32}{9}\zeta_2 + \frac{64}{9}H_{0,-1}\right]H_{-1} - \frac{64}{9}H_{0,0,-1} + \frac{8(56z^3 - 201z^2 - 162z - 40)H_{0,1,1}}{9z} \\
&+ \frac{(z-1)(4z^2 + 7z + 4)}{z} \left[\frac{2}{9}H_1 + \frac{4}{3}H_0^2H_1^2 + \frac{4}{3}\zeta_2H_1^2 + \left[-\frac{8}{3}\zeta_3 + \frac{8}{3}H_{0,0,1} + \frac{16}{3}H_{0,1,1}\right]H_1\right] \\
&- \frac{16}{3}H_{0,1} + H_0\left[-\frac{16}{9}H_0^3 + \left[-\frac{8}{3}\zeta_2 - 16H_{0,1}\right]H_1 + \frac{80}{3}H_{0,1,1}\right] \\
&+ \frac{16(32z^3 - 87z^2 + 45z - 24)}{3z}H_{0,0,0,0,1} + (2-z)\left[128H_0H_{0,0,0,0,1} - 32H_0^2H_{0,0,0,0,1}\right] \\
&- \frac{16(36z^3 + 62z^2 - 15z - 20)}{3z}H_{0,0,0,1} + z(4z - 3)H_0\left[\frac{32}{3}\zeta_3 + \frac{32}{3}H_{0,0,1}\right] \\
&+ \frac{(z+1)(4z^2 - 7z + 4)}{z} \left[\frac{32}{9}H_0H_1^3\right] \\
&+ \left[-\frac{8}{3}H_0^2 - \frac{16}{3}\zeta_2 - \frac{32}{3}H_{0,-1}\right]H_1^2 + \left[-\frac{8}{3}H_0 + \frac{24}{3}\zeta_2 + \frac{32}{3}H_{0,-1} - \frac{32}{3}H_{0,1}\right]H_0 \\
&- 16\zeta_3 + \frac{64}{3}H_{0,-1} - \frac{32}{3}H_{0,0,-1} + \frac{64}{3}H_{0,0,1}H_{-1} - \frac{24}{3}H_{0,-1}\zeta_2 + \frac{8}{3}H_0^2H_{0,-1} \\
&+ H_0\left[-\frac{32}{3}H_{0,-1} + \frac{32}{3}H_{0,0,-1} - \frac{16}{3}H_{0,0,-1} + \frac{32}{3}H_{0,1,1}\right] - \frac{64}{3}H_{0,-1,1,1} \\
&- \frac{32}{3}H_{0,-1,0,1} + \frac{32}{3}H_{0,0,-1} - \frac{64}{3}H_{0,0,-1} + \frac{16}{3}H_{0,0,0,-1} - \frac{64}{3}H_{0,0,0,1,1} \\
&+ L_M^4 \left[\frac{4}{3}(4z - 5)H_0^4 - \frac{8}{9}(35z - 46)H_0^4 - \frac{4}{9}(606z^2 - 346z + 377)H_0^2\right].
\end{aligned}
\]
\[
\begin{align*}
&+ \frac{8(z + 1)(16z^2 - 19z + 16)H_{-1}H_0^2}{3z} + \frac{128}{3z^2}H_0 + \frac{16(3575z^3 + 2093z^2 + 2330z + 224)H_0}{27z} - \frac{16}{3}(16z^2 - 19z - 25)\zeta_2H_0 \\
&+ \frac{8}{3}(40z^2 - 51z + 9)\zeta_2H_0 + 8(9z - 25)\zeta_3H_0 - 8(73z + 11)\zeta_3H_0 \\
&+ \frac{16(z - 1)(203z^2 + 47z + 140)H_1H_0}{9z} + \frac{64(2z^3 - 9z^2 + 3z - 4)H_{0,-1}H_0}{3z} \\
&+ \frac{8(16z^3 - 41z^2 - 77z - 40)H_{0,1}H_0}{9z} - \frac{32(5z - 1)H_{0,0,-1}H_0}{3z} - \frac{32(11z + 5)H_{0,0,1}H_0}{3z} \\
&+ \frac{16}{5}(7z + 27)\zeta_2^2 - \frac{24}{5}(65z + 11)\zeta_2^2 - \frac{4(z - 1)(20z^2 + 21z + 2)H_1^2}{3z} \\
&+ \frac{8(z - 1)(11542z^2 + 399z + 4036)}{9z} - \frac{32(z + 1)(53z^2 - 14z + 26)\zeta_2}{9z} \\
&+ \frac{8(80z^3 - 157z^2 + 521z - 64)\zeta_2}{9z} + \frac{16(z + 1)(4z^2 - z + 4)H_{-1}\zeta_2}{9z} \\
&+ \frac{4(44z^3 - 81z^2 - 213z - 180)\zeta_3}{3z} + \frac{4(164z^3 - 231z^2 + 81z - 12)\zeta_3}{3z} \\
&+ \frac{128(z - 1)(z^2 + 1)H_1}{3z} - \frac{8(z - 1)(258z^2 - 559z - 138)H_1}{9z} \\
&+ \frac{(z + 1)(19z^2 - 16z + 10)}{z} \left[ \frac{32}{9}H_{0,-1} - \frac{32}{9}H_{-1}H_0 \right] \\
&+ \frac{(z + 1)(53z^2 - 2z + 26)}{z} \left[ \frac{64}{9}H_{0,-1} - \frac{64}{9}H_{-1}H_0 \right] \\
&+ \frac{(z - 1)(4z^2 + 7z + 4)}{z} \left[ \frac{8}{9}H_1^3 + 8H_0H_1^2 + \left[ \frac{16\zeta_2}{3} - \frac{32}{3}H_{0,1}\right]H_1 \right] \\
&- \frac{8(274z^3 - 253z^2 + 611z - 200)H_{0,1}}{9z} - \frac{16(32z^3 - 75z^2 + 21z - 16)H_{0,0,-1}}{3z} \\
&- \frac{32}{3}(4z - 3)H_{0,0,1} + \frac{32(8z^2 + 17z + 14)H_{0,0,1}}{3z} \\
&+ \frac{(z + 1)(4z^2 - 7z + 4)}{z} \left[ \frac{32}{3}H_0H_1^2 + \left[ -\frac{8}{3}H_0^2 + 16\zeta_2 - \frac{64}{3}H_{0,-1} \right]H_1 \right] \\
&- \frac{32}{3}H_{0,1}\left[ H_{-1} + \frac{16}{3}H_0H_{0,-1} + \frac{64}{3}H_{0,-1,-1} + \frac{32}{3}H_{0,-1,-1} + \frac{16}{3}H_{0,0,-1} + \frac{32}{3}H_{0,1,-1} \right] \\
&+ \frac{(z + 1)(8z^2 - 5z + 8)}{z} \left[ \frac{32}{3}H_{-1}H_{0,1} - \frac{32}{3}H_{0,-1,1} - \frac{32}{3}H_{0,1,-1} \right] \\
&+ \frac{16(8z^3 + 23z^2 + 5z - 8)H_{0,1,1}}{3z} + \frac{(z - 1)}{z} \left[ 8\zeta_2 + 32H_{0,-1} \right]H_0^2 \\
&+ \left[ 128H_{0,-1,-1} + 64H_{0,0,-1} \right]H_0 - 64H_{0,-1}^2 + 64H_{0,-1}H_2 \\
&- 96H_{0,0,0,-1} - 64zH_{0,0,0,1} + 160(5z + 1)H_{0,0,0,1} \\
&+ (z + 1) \left[ 56H_{0,1}H_0^2 - 96H_{0,1,1}H_0 + 32H_{0,1}^2 + H_{0,1} \left[ -32\zeta_2 - \frac{128}{3} \right] \right] \\
&+ 192H_{0,0,0,-1} + 32H_{0,0,1,1} - 32H_{0,1,1,1} \right] + \frac{128(z - 1)}{3z} \\
&- \frac{32(z - 1)(z + 2)(2z + 1)H_{0,1,1,1}}{3z} - 64(3z - 2)H_{0,0,0,0,1} + 32(23z + 27)H_{0,0,0,1,1}
\end{align*}
\]
\[ A_{qg} = \]
\[
 \alpha_s \gamma_{qg}^0 T_F L_M + a_s^2 \left\{ \frac{4}{3} \gamma_{qg}^0 T_F^2 L_M^2 \right\} + C_A T_F \left[ \frac{4}{3} (2z + 1) H_0^3 + \frac{2}{3} (23z^2 + 12z + 3) H_0^2 - 32z^2 H_0 - \frac{4}{9} (328z^2 + 129z + 42) H_0 \right] \\
- \frac{4(z - 1) (5z + 1)}{3z} (2(2z + 1) H_0^3 + 16(z + 1) H_{0,1} H_0 + 2(z - 1)(5z + 1) H_1^2) \\
+ L_M^2 \left[ \frac{4(z - 1) (31z^2 + 7z + 4)}{3z} - 8(z + 1) H_0 - 2 \gamma_{qg}^0 H_1 \right] + \frac{4(1588z^3 - 1413z^2 - 9z - 112)}{27z} \\
- 4(z - 12)z \zeta_2 - 2(26z^2 + 26z + 5) \zeta_3 + 2(26z^2 + 82z + 21) \zeta_3 - 32(z - 1)z H_1 \\
+ 4(12z^2 - 12z - 1) H_1 + L_M \left[ 8(2z + 1) H_0^2 - \frac{8}{3} (44z^2 + 24z + 3) H_0 + 32(z - 1)z H_1 \right] \\
+ \frac{8(218z^3 - 225z^2 + 18z - 20)}{9z} + \gamma_{qg}^0 \left[ 2 \zeta_2 - 2 H_1^2 \right] + (2z^2 + 2z + 1) \left[ 8 \zeta_2 + 16 H_{-1} H_0 - 16 H_{0,-1} \right] \\
+ \frac{4(68z^3 - 72z^2 - 9z - 8) H_{0,1}}{3z} + z \left[ 32 H_{0,1} - 32 \zeta_2 \right] - 32(z^2 + 5z + 1) H_{0,0,1} \\
+ z(z + 1) \left[ 16 H_{-1} H_0 - 16 H_{0,-1} + 32 H_{0,0,1} \right] + (2z^2 + 2z + 1) \left[ -8 H_0 H_{2,-1} + 4 H_0^2 - 8 \zeta_2 \right] \\
+ 16 H_{0,-1} \left[ H_{-1} - 8 H_0 H_{0,-1} - 16 H_{0,-1,-1} + 8 H_{0,0,-1} \right] + \gamma_{qg}^0 \left[ -\frac{1}{3} H_1^3 + H_0 H_{1,1}^2 - 2 H_{0,1,1} \right] \\
+ C_F T_F \left[ \frac{2}{3} (4z^2 - 2z + 1) H_0^3 + (20z^2 - 12z - 1) H_0^2 - 2(24z^2 + 9z + 8) H_0 \right] \\
+ 4(10z^2 - 12z + 1) H_1 H_0 - 16(z - 1)^2 H_{0,1} H_0 - 4(z - 1)(3z + 1) H_1^2 \\
+ L_M^2 \left[ 2(4z - 1) - 4(4z^2 - 2z + 1) H_0 + 2 \gamma_{qg}^0 H_1 \right] + 2(40z^2 - 41z + 13) + 8(3z^2 - 6z - 1) \zeta_2 \\
+ 8(2z^2 + 2z - 1) \zeta_3 - 4z(12z - 13) H_1 + L_M \left[ -4(4z^2 - 2z + 1) H_0^2 - 4(8z^2 - 4z + 3) H_0 \right] \\
\]
\[-4(20z^2 - 29z + 14) - 32(z - 1)zH_1 + \gamma_{qg}\left[2H_1^3 + 4H_0H_1 - 4\zeta_2\right]\] 
- 4(16z^2 - 24z - 1)H_{0,1}

+ 8(2z^2 - 6z + 3)H_{0,0,1} + \gamma_{qg}\left[\frac{1}{3}H_1^3 - H_0^2H_1 + 2H_{0,1}\right]\}

+ a_3^3\{a_0(3) + T_F^2\left[\frac{16}{9}\gamma_{qg}L_M^3 - \frac{16}{9}\gamma_{qg}\zeta_3\right] + C_A T_F^2\left[-\frac{4}{9}(2z + 3)H_0^4\right]

+ \frac{8}{27} \left(46z^2 + 74z - 13\right)H_0^3 - \frac{8}{27} \left(458z^2 - 382z + 221\right)H_0^3 + \frac{8}{3} \left(16z^2 + 10z - 7\right)\zeta_2 H_0^2

- \frac{16(z - 1)(65z^2 + 17z + 8)}{9z} H_1^2 H_0^2 + \frac{64}{9} (4z + 1)H_0H_1^2 + \frac{16}{81} \left(6612z^2 + 5083z + 346\right)H_0

+ \frac{8}{9} \left(176z^2 + 2048z + 41\right)\zeta_2 H_0 + \frac{32}{9} \left(24z^2 + 100z + 31\right)\zeta_3 H_0

+ \frac{16(854z^3 - 882z^2 + 99z - 80)H_1 H_0}{27z} - \frac{32(23z^3 + 72z^2 + 15z + 8)H_{0,1} H_0}{9z}

+ \frac{128}{9} \left(2z^2 - 4z - 1\right)H_{0,0,1} H_0 - \frac{16}{9} \left(z - 1\right)(5z + 1)H_0^3 + L_M^3\left[\frac{112(z - 1)(31z^2 + 7z + 4)}{27z}\right]

- \frac{224}{9} \left(4z + 1\right)H_0 - \frac{56}{9} \gamma_{qg} H_1\left\[\right\] + \frac{8}{15} \left(222z^2 - 102z - 1\right)\zeta_2^2 - \frac{8}{15} \left(222z^2 + 222z - 1\right)\zeta_2^2

- \frac{8}{3} \left(38z^2 - 43z + 3\right)H_1^2 - \frac{8}{243z} \left(82666z^3 - 87018z^2 + 8835z - 5788\right) - \frac{160}{9} \left(5z^2 - 5z + 1\right)H_1\zeta_2

- \frac{4(896z^3 - 336z^2 + 43z - 128)\zeta_2}{9z} + \frac{32}{3} \left(z(13z + 5)\zeta_3\right) - \frac{16(1489z^3 + 408z^2 + 51z - 28)}{27z} \zeta_3

+ z^2\left[H_0\left[\frac{256}{3} \zeta_2 - \frac{1024}{3}\right] - \frac{128}{3} H_0^2\right] - \frac{16(2330z^3 - 2321z^2 + 391z + 36)}{81z} H_1

+ (z - 1)z\left[\frac{256}{3} H_1^2 + \frac{1024}{3} H_1 + H_0\left[\frac{64}{3} H_1^2 + \frac{256}{3} H_1\right]\right]

+ L_M^2\left[\frac{64}{3} \left(z + 1\right)H_0^3 - \frac{32}{9} \left(9z^2 + 68z + 11\right)H_0 + \frac{8(1769z^3 - 1788z^2 + 96z - 212)}{27z}\right]

+ \frac{32}{9} \left(28z^2 - 28z + 5\right)H_1 + \gamma_{qg}\left[4\zeta_2 - 4H_0^2\right]\right\[\] + 2z^2 + 2z + 1\left[\frac{16}{3} \zeta_2 + 32H_{-1} H_0\right]

- 32H_{0,1}\right]\] - \frac{16(980z^3 - 1218z^2 + 117z - 80)}{27z} H_{0,1} - \frac{256}{3} (z - 2)zH_{0,0,1}

+ \frac{32(135z^3 + 96z^2 + 21z + 8))}{9z} H_{0,0,1} + L_M^3\left[\frac{16}{9} \left(14z - 1\right)H_0^3 + \frac{8}{9} \left(30z^2 - 104z + 9\right) H_0^2\right]

- \frac{256}{9} z^2 H_0 - \frac{16}{27} \left(418z^2 + 904z + 79\right) H_0 - \frac{32(z - 1)(65z^2 + 17z + 8)H_1 H_0}{9z}

- \frac{32}{9} \left(2z^2 - 18z - 3\right)H_{0,1} H_0 + \frac{16}{9} \left(z^2 + 2z - 13\right)H_1^2 + \frac{8(3285z^3 - 2894z^2 + 95z - 264)}{27z}

+ \frac{32}{9} \left(4z^2 + 35z + 5\right)\zeta_2 - \frac{32}{9} \left(7z^2 + 31z + 5\right)\zeta_2 - \frac{32}{3} \left(10z^2 + 10z + 1\right)\zeta_3

+ \frac{32}{3} \left(10z^2 + 44z + 9\right)\zeta_3 - \frac{256}{3} \zeta_2 H_1 + \frac{16}{27} \left(374z^2 - 365z + 67\right) H_1

+ \frac{16(z^2 + 7z + 5)}{9z}\left[H_0_{-1} - \frac{64}{9} H_{-1} H_0\right] + \frac{256}{3} zH_{0,1} + \frac{32(68z^3 - 78z^2 - 9z - 8) H_{0,1}}{9z}

- \frac{32}{3} \left(6z^2 + 42z + 7\right) H_{0,0,1} + z(z + 1)\left[\frac{128}{3} H_{-1} H_0 - \frac{128}{3} H_{-1} H_0 + \frac{256}{3} H_{0,0,1}\right]
\[ + (2z^2 + 2z + 1) \left[ - \frac{64}{3} H_0 H_1^2 + \left[ - \frac{64}{3} \zeta_2 + \frac{128}{3} H_0, -1 \right] H_{-1} - \frac{128}{3} H_{0, -1, -1} \right] \\
+ \gamma_{qq}^0 \left[ - \frac{4}{3} H_1^2 + \frac{4}{3} H_0 H_1^2 - \frac{4}{3} H_0^2 H_1 + \left[ \frac{16}{3} H_0, 1 - \frac{8 \zeta_2}{3} \right] H_1 - \frac{40}{3} H_{0, 1, 1} \right] \right] - \frac{32}{3} z (5z - 19) H_{0, 0, 1, 1} \\
+ z \left[ \frac{512}{3} \zeta_2 - \frac{256}{3} H_0 H_{0, 1} - \frac{512}{3} H_{0, 1, 1} \right] - \frac{256}{3} z (4z + 1) H_{0, 0, 0, 0, 1} \\
+ \frac{64}{3} \left( 18z^2 - 2z + 5 \right) H_{0, 0, 1, 1} + z(z + 1) \left[ \frac{128}{3} H_0 H_0^2 + \left[ - \frac{64}{3} H_0^2 - 128 H_0 - \frac{128}{3} \zeta_2 \right. \right. \\
- \frac{256}{3} H_{0, -1} \right] H_{-1} - \frac{128}{3} H_0 \zeta_2 + 128 H_0, -1 - \frac{256}{3} H_{0, 1} + \frac{256}{3} H_{0, -1, -1} - \frac{128}{3} H_{0, 0, -1} \\
+ H_0 \left[ - \frac{256}{3} \zeta_3 + \frac{128}{3} H_{0, -1} - \frac{256}{3} H_{0, 1, 1} \right] + \frac{1024}{3} H_{0, 0, 0, 1} - \frac{512}{3} H_{0, 0, 1, 1} \\
+ (2z^2 + 2z + 1) \left[ - \frac{128}{9} H_0 H_1^3 + \left[ \frac{32}{3} H_0^2 - \frac{64}{3} \zeta_2 + \frac{128}{3} H_{0, -1} \right] H_1^2 \\
+ \left[ \frac{32}{9} H_0^3 + \left[ - \frac{80}{3} \zeta_2 - \frac{128}{3} H_{0, -1} + \frac{128}{3} H_{0, 1, 1} \right] H_0 + \frac{64 \zeta_3 - \frac{256}{3} H_{0, -1, -1} + \frac{128}{3} H_{0, 0, -1} \\
- \frac{256}{3} H_{0, 0, 1, 1} \right] H_{-1} + \frac{80}{3} \zeta_2 - \frac{32}{3} H_0^2 H_{0, -1} + H_0 \left[ \frac{128}{3} H_{0, -1, -1} - \frac{128}{3} H_{0, 0, -1} + \frac{64}{3} H_{0, 0, -1} \right] \\
- \frac{128}{3} H_{0, 1, -1} \right] - \frac{256}{3} H_{0, 1, -1, -1} + \frac{128}{3} H_{0, 0, -1, 0} - \frac{128}{3} H_{0, 0, -1, 1} + \frac{256}{3} H_{0, 0, -1, -1} - \frac{64}{3} H_{0, 0, 0, 0, -1} \\
+ \frac{256}{3} H_{0, 0, 1, -1} + \gamma_{qq}^0 \left[ \frac{2}{9} H_1^4 + \frac{4}{3} H_0^2 H_1^2 + \frac{10}{3} \zeta_2 H_1^2 + \left[ - \frac{40}{9} \zeta_3 + \frac{80}{3} H_{0, 0, 1} + \frac{16}{3} H_{0, 1, 1} \right] H_1 \\
- \frac{16}{3} H_{0, 1} + H_0 \left[ - \frac{16}{9} H_1^3 - 16 H_{0, 1} H_1 + \frac{80}{3} H_{0, 1, 1} \right] - \frac{16}{3} H_{0, 1, 1, 1} \right] \right] \\
+ C \mathcal{T}_F^2 N_F \left[ \frac{4}{9} (2z + 1) H_1^4 + \frac{8}{27} (23z^2 + 12z + 3) H_0^3 - \frac{8}{27} (292z^2 + 39z + 42) H_0^2 \\
- \frac{8}{9} (z - 1) (65z^2 + 17z + 8) H_1 H_0^2 + \frac{32}{3} (4z + 1) H_{0, 1} H_0^2 + \frac{32}{3} (z - 1) z H_0^3 H_0 \\
+ \frac{16}{81} (3392z^2 + 645z + 111) H_0 + \frac{32}{3} (z - 1) (254z^2 - 7z + 20) H_1 H_0 \\
- \frac{16 (23z^3 + 96z^2 + 15z + 8) H_{0, 1} H_0}{9z} + \frac{8}{9} (62z^2 - 16z - 7) \zeta_2 H_0 + \frac{32}{9} (28z + 13) \zeta_3 H_0 \\
- \frac{8}{9} (z - 1) (5z + 1) H_1^3 + L_3^M \left[ \frac{16(z - 1) (31z^2 + 7z + 4)}{27z} - \frac{32}{9} (4z + 1) H_0 \right] \\
- \frac{8}{9} \gamma_{qq}^0 \left[ \frac{4}{9} H_1^2 - \frac{8}{3} (4z^2 - 6z + 1) H_1^2 - \frac{32}{243z} (5854z^3 - 6219z^2 + 531z - 328) \right] \\
+ \frac{4}{9} (19z^3 - 50z^2 + 2z - 4) \zeta_2 - \frac{16}{27} (550z^3 + 228z^2 + 33z - 4) \zeta_3 \right] \right] \\
+ \frac{32}{3} \left( 2z^2 - 2z - 1 \right) H_1 - \frac{32}{27} (290z^3 - 261z^2 + 27z - 20) H_{0, 1} - \frac{16}{3} z (5z - 4) H_{0, 1, 1} \\
+ \frac{16 (111z^3 + 144z^2 + 21z + 8) H_{0, 0, 1}}{9z} - \frac{16}{9} (2z^2 - 2z - 5) H_1 \zeta_2 + (z + 1) \left[ \frac{64}{3} H_0 H_1^2 - \frac{16}{3} \right] \\
+ \left[ \frac{32}{3} H_0^2 - \frac{128}{3} H_0, -1 \right] H_{-1} + \frac{64}{3} \zeta_2 H_{-1} + \frac{64}{3} H_0 H_{0, -1} + \frac{128}{3} H_{0, -1, -1} \\
- \frac{64}{3} H_{0, 0, -1} \right] + (6z + 1) \left[ - \frac{8}{3} \zeta_2 H_0^3 - \frac{64}{3} H_{0, 0, 1} H_0 \right] + z \left[ 128 H_{0, 0, 0, 1} - \frac{1216 \zeta_2}{15} \right] \\
+ L_3^2 \left[ - \frac{4}{3} \gamma_{qq}^0 H_1^2 - \frac{32}{9} (4z^2 - 4z + 5) H_1 - \frac{8}{27z} (205z^3 - 168z^2 + 42z - 52) \right] \\
+ \text{134} \right]
\[-\frac{32}{9}(9z^2 - 20z - 5)H_0 + (2z^2 + 2z + 1)\left[\frac{32}{3}H_{-1}H_0 - \frac{32}{3}H_{0,-1}\right] + z\left[\frac{64}{3}H_0^2 + \frac{64\zeta_2}{3}\right]\]

\[+ LM\left[\frac{-16}{9}(10z - 1)H_0^3 + \frac{8}{9}(7z^2 - 66z - 13)H_0^2 + \frac{16}{27}(58z^2 - 269z - 2)H_0\right]\]

\[-\frac{16}{9}(z - 1)(65z^2 + 17z + 8)H_1H_0 - \frac{32}{3}(2z^2 - 10z - 1)H_{0,1}H_0 - \frac{8}{9}(13z^2 - 16z + 23)H_1^2\]

\[+ \frac{8}{27}(592z^3 - 268z^2 - 119z - 4) + \frac{16}{27}(64z^2 - 64z + 29)H_1 + (4z^2 + 4z + 5)\left[\frac{64}{9}H_0,1\right]\]

\[-\frac{64}{9}H_{-1}H_0 + \frac{16}{9}(68z^3 - 54z^2 - 9z - 8)H_0,1 + \frac{32}{3}(2z^2 - 18z - 3)H_{0,0,1} - \frac{16}{9}z(3z + 10)\zeta_2\]

\[+ (2z^2 + 2z + 1)\left[-\frac{32}{3}H_0H_1^2 H_0,1 + \left[\frac{64}{3}H_{0,-1} - \frac{16}{3}H_0\right]H_1 - \frac{32}{3}\zeta_2 H_1 + \frac{32}{3}H_0H_0,1\right]\]

\[+ \frac{64}{3}H_{0,-1} - \frac{32}{3}H_{0,0,-1}\right]\]

\[+ \gamma_0\left[\frac{-8}{9}H_1^3 - \frac{4}{3}H_0^2 H_1 + \frac{16}{3}H_0,1H_1 - \frac{8}{3}\zeta_2 H_1 - \frac{32}{3}H_{0,1,1}\right]\]

\[+ \frac{128}{3}(5z + 1)\zeta_3\]

\[+ (2z^2 + 2z + 1)\left[\frac{-64}{9}H_0H_0,1 + \left[\frac{16}{3}H_0 + \frac{64}{3}H_{0,-1}\right]H_1^2 + \frac{16}{9}H_0^3\right]\]

\[+ \gamma_0\left[\frac{-8}{9}H_1^3 - \frac{4}{3}H_0^2 H_1 + \frac{16}{3}H_0,1H_1 - \frac{8}{3}\zeta_2 H_1 - \frac{32}{3}H_{0,1,1}\right]\]

\[+ \gamma_0\left[\frac{-8}{9}H_1^3 - \frac{4}{3}H_0^2 H_1 + \frac{16}{3}H_0,1H_1 - \frac{8}{3}\zeta_2 H_1 - \frac{32}{3}H_{0,1,1}\right]\]

\[+ C_{A T F}\left[\frac{1}{9}(18z^2 - 26z + 23)H_0^4 + \frac{2}{9}(225z^2 - 12z - 35)H_0^3 - \frac{8}{3}(z - 15)\zeta_2 H_0^3\right]\]

\[+ \frac{8}{3}(z^2 - 7z - 5)\zeta_2 H_0^3 - \frac{4}{3}(60z^3 + 46z^2 + 5z + 8)H_{-1}H_0^3 + \frac{8}{3}(z - 1)(65z^2 + 17z + 8)H_1H_0^3\]

\[+ \frac{32}{3}(4z + 1)H_0,H_0^3 - \frac{4}{3}(120z^3 + 106z^2 + 5z + 8)H_{0,H}^3 + \frac{9}{z}\left(2(190z^3 - 142z^2 - 13z - 24)H_0^2 H_0^2\right)\]

\[+ \frac{2}{27}(5248z^2 - 6573z + 738)H_0^2 + \frac{8}{3}(37z^2 + 46z - 3)\zeta_2 H_0^2\]

\[+ \frac{2}{3}(110z^2 + 190z - 59)\zeta_2 H_0^3 - \frac{32}{3}(43z - 5)\zeta_3 H_0^2 + \frac{8}{9}(z + 1)(359z^2 - 32z + 20)H_{-1}H_0^2\]

\[+ \frac{2}{3}(z - 1)(1907z^2 - 169z + 308)H_1^2 H_0^2 + \frac{4}{3}(96z^3 + 70z^2 + 5z + 8)H_{0,-1}H_0^2\]

\[+ \frac{8}{3}(32z^2 - 86z + 7)H_0,H_0^3 + 16(62^2 - 2z + 5)H_{0,-1}H_0^3 - 64(z^2 + 2z - 1)H_{0,0,1}H_0^2\]

\[+ 96(z - 3)zH_{0,1,1}H_0^2 - \frac{8}{9}(93z^3 - 82z^2 + 8z - 8)H_1^2 H_0 + \frac{32}{5}(2z + 3)(10z - 3)\zeta_2^2 H_0\]

\[+ \frac{16}{5}(40z^2 - 76z + 1)\zeta_2^2 H_0 - \frac{8}{9}(169z^3 - 157z^2 + 11z - 20)H_2^3 H_0 + 32(z^2 + 9z + 3)H_{0,1}H_0\]

\[+ \frac{32}{9}(701z^2 + 12)H_0 - \frac{4}{3}(175490z^3 + 100242z^2 + 15513z + 2624)H_0\]

\[+ \frac{4}{9}(2359z^2 + 462z + 120)\zeta_2 H_0 - \frac{2}{9}(1756z^3 + 940z^2 + 925z + 160)\zeta_2 H_0\]

\[+ \frac{81z}{9}\]
\[-\frac{8(180z^3 + 166z^2 + 5z + 8)}{3z} H_{-1}\zeta_2 H_0 + \frac{8(247z^3 + 212z^2 + 7z + 20)}{3z} H_{-1}\zeta_2 H_0 - \frac{8(\zeta_2 z - 1)(137z^2 + 41z + 20)}{3z} H_1\zeta_2 H_0 - \frac{8(18z^2 - 22z + 19)}{3z} H_0, -1 \zeta_2 H_0 - 8(2z^2 + 38z + 11) H_{0,1}\zeta_2 H_0 + \frac{8}{3z(51z + 62)}\zeta_3 H_0 - 8(57z^3 + 154z^2 - 269z - 8)\zeta_3 H_0 + \frac{8(11936z^3 - 12231z^2 + 1431z - 1244)}{3z} H_1 H_0 - \frac{16(323z^3 + 111z^2 - 12z + 20)}{9z} H_{0, -1} H_0 - \frac{64}{3}(6z^2 + 33z + 2) H_{0,1} H_0 - \frac{4(1213z^3 + 4308z^2 - 3z + 468)}{9z} H_{0,1} H_0 - \frac{8(406z^3 - 330z^2 - 3z - 40)}{3z} H_1 H_{0,1} H_0 - \frac{16(108z^3 + 82z^2 + 5z + 8)}{9z} H_{0, -1, -1} H_0 - \frac{8(132z^3 + 94z^2 + 5z + 8)}{3z} H_{0,0, -1} H_0 - \frac{32}{3}(54z + 11) H_{0,0,1} H_0 - \frac{16(237z^3 - 188z^2 + 28z - 8)}{3z} H_{0,0,1} H_0 - \frac{8(768z^3 - 542z^2 + 19z - 56)}{3z} H_{0,1,1} H_0 - \frac{64(z^2 - 5z + 2)}{3z} H_{0,0,0, -1} H_0 - 256z(2z + 1) H_{0,0,0,1} H_0 + 128(4z^2 + 9z - 1) H_{0,0,0,1} H_0 - 32(18z^2 + 26z + 9) H_{0,0,1,1} H_0 - 32(6z^2 - 14z + 1) H_{0,1,1,1} H_0 + \frac{(204z^3 - 166z^2 - 19z - 8)}{9z} H^4_1 + \frac{(3049z^3 - 3168z^2 + 15z - 4)}{27z} H^3_1 - \frac{2(3846z^3 - 10226z^2 + 675z - 592)}{15z} \zeta_2^3 + \frac{2(4534z^3 + 3158z^2 - 45z + 16)}{27z} \zeta_2^2 - \frac{2(19952z^3 - 19524z^2 + 615z - 710)}{15z} H^2_1 - \frac{8(161z^3 - 130z^2 - 4z - 16)}{247z} \zeta_2^2 - \frac{4(115880z^3 - 1178838z^2 + 87399z - 70927)}{27z} + \frac{4(102z^3 - 94z^2 + 11z - 8)}{3z} H^2_1 \zeta_2 - \frac{2}{9}(5643z^2 - 5900z - 413) \zeta_2 + \frac{2(40619z^3 + 7491z^2 + 5019z - 2176)}{27z} \zeta_2 - \frac{4(2247z^3 - 2188z^2 + 116z - 242)}{9z} H_1 \zeta_2 + \frac{8(72z^3 + 94z^2 + 5z + 8)}{3z} H_{0, -1} \zeta_2 - \frac{8(17z^3 + 164z^2 + 7z + 20)}{9z} H_{0, -1} \zeta_2 - \frac{8(124z^3 + 114z^2 + 33z + 16)}{3z} H_{0,1} \zeta_2 - \frac{16(6z^2 - 26z + 11)}{3z} H_{0, -1, -1} \zeta_2 + 8(18z^2 - 62z + 29) H_{0,0, -1} \zeta_2 - 8(46z^2 + 2z - 7) H_{0,0,1} \zeta_2 - 16(30z^2 - 22z + 17) H_{0,1,1} \zeta_2 - \frac{4(3551z^3 + 1551z^2 + 36z + 100)}{9z} \zeta_3 + \frac{4(27017z^3 + 29781z^2 + 3645z + 748)}{27z} \zeta_3 - \frac{4(66z^2 - 134z + 19)}{9z} \zeta_2 \zeta_3 + \frac{4}{3}(198z^2 + 154z + 223) \zeta_2 \zeta_3 - \frac{4(304z^3 + 262z^2 + 15z + 24)}{z} H_{-1} \zeta_3 - \frac{4(528z^3 + 458z^2 + 25z + 40)}{z} H_{-1} \zeta_3 - \frac{8(874z^3 - 698z^2 - 41z - 80)}{9z} H_1 \zeta_3 + 32(13z^2 - 17z + 14) H_{0, -1} \zeta_3 - \frac{160}{3}(2z^2 + 6z + 3) H_{0,1} \zeta_3 + 8(12z + 5) \zeta_5 + 104(20z - 3) \zeta_5 + (4z - 1) \left[ \frac{4}{15} H^5_0 - 32 H_{0, -1} \zeta_2 H_0 + 64 H_{0,0, -1} \zeta_2 \right] - \frac{32(z - 1)(701z^2 + 32)}{9z} H_1 \]
\[
\begin{align*}
&4(34870z^3 - 34925z^2 + 2054z - 2233)H_1 + \frac{(z - 1)(57z^2 + 2)}{z} \left[-\frac{64}{3}H_1^2 - \frac{64}{3}H_0H_1\right] + 32(17z^2 + 20z - 12)H_{0,1} \\
&+ (z - 1)\left[-128H_1^2 - 192H_0H_1^2 - 64H_0^2H_1 + 384\zeta_2H_1\right] \\
&+ \frac{(z + 1)(2443z^2 - 244z + 94)}{z} \left[\frac{17}{27}H_1H_0 - \frac{16}{27}H_{0,-1}\right] + 8\left(\frac{84z^3 - 70z^2 + 5z - 8}{9z}\right)H_1H_{0,1,1} \\
&+ \frac{8(9594z^3 - 17547z^2 + 885z - 1244)H_{0,1}}{27z} + \frac{8(84z^3 - 70z^2 + 5z - 8)}{3z}H_1H_{0,1,1} \\
&+ L_M^3 \left[\frac{16}{3}(8z - 1)H_0^2 - \frac{8(18z^3 - 152z^2 - 11z - 8)}{9z}H_0 - \frac{4(z - 1)(1883z^2 - 97z + 272)}{27z}\right] \\
&- \frac{64}{3}(4z + 1)\zeta_2 - \frac{8\left(146z^3 - 118z^2 - z - 16\right)}{9z}H_1 + \gamma_9^0 \left[\frac{8}{3}H_1^2 + \frac{8}{3}H_0H_1\right] + \frac{32}{3}(2z^2 + 6z + 3)H_{0,1} \\
&+ \frac{16(287z^3 - 105z^2 - 12z + 20)H_{0,0,-1}}{9z} - \frac{32}{9}(251z^2 - 375z - 12)H_{0,0,1} \\
&- \frac{4(553z^3 + 6840z^2 + 399z + 628)H_{0,0,1}}{9z} + \frac{8(728z^3 - 590z^2 - 11z - 72)}{3z}H_1H_{0,0,1} \\
&+ \frac{(z + 1)(90z^2 + 12z + 20)}{9z}H_{0,0,1} - \frac{32}{9}H_{0,-1,-1} - \frac{32}{9}H_{0,-1,1} - \frac{32}{9}H_{0,1,-1} \\
&+ \frac{(z + 1)(251z^2 - 32z + 20)}{9z} \left[H_{-1} - \frac{32}{9}\zeta_2 - \frac{32}{9}H_{0,1}\right] + \frac{32}{9}H_{0,-1,1} + \frac{32}{9}H_{0,1,-1} \\
&+ \frac{128}{3}(9z^2 + 30z + 2)H_{0,1,1} + \frac{4(2561z^3 - 6192z^2 - 144z - 240)H_{0,1,1}}{9z} \\
&+ L_M^2 \left[\frac{16}{3}z(z + 1)H_0^3 - \frac{16}{3}(z^2 + 9z - 2)H_0^3 - \frac{8}{3}(13z^2 - 8z + 12)H_0^2\right] \\
&+ \frac{8}{3}(31z^2 - 6z + 6)H_0^2 + \frac{8}{9}(197z^2 - 42z + 48)H_0 + \frac{16(569z^3 + 376z^2 + 109z + 20)H_0}{9z} \\
&+ 32(2z^2 + 1)\zeta_2H_0 - 32(4z - 1)\zeta_2H_0 - \frac{16(z - 1)(19z^2 + 7z + 4)H_1H_0}{3z} \\
&+ 16(6z^2 - 10z + 7)H_{0,-1}H_0 - 16(2z^2 + 6z + 3)H_{0,1}H_0 - \frac{4(132z^3 - 118z^2 + 5z - 8)H_1^2}{3z} \\
&- \frac{4}{3}(30z^2 + 214z - 11)\zeta_2 + 8(2z^2 - 46z + 13)\zeta_3 \\
&- \frac{8}{3}(2z^2 + 22z + 5)\zeta_3 - \frac{8\left(617z^3 - 604z^2 - 70z - 28\right)}{9z}H_1 + \gamma_9^0 \left[4H_1^3 - 2H_0^2H_1 - 8\zeta_2H_1\right] \\
&+ \frac{146z^3 + 118z^2 - z + 16}{z} \left[-\frac{4}{3}\zeta_2 - \frac{8}{3}H_{-1}H_0 + \frac{8}{3}H_{0,-1}\right] + \frac{16}{3}z(25z + 36)H_{0,1} \\
&- 16(6z^2 - 26z + 11)H_{0,0,-1} - (2z^2 + 2z + 1) \left[32H_0H_2^2 + \left[-24H_0^2\right.\right] \\
&+ 64\zeta_2 - 64H_{0,-1} - 32H_{0,1} \left[H_{-1} + 64H_{0,-1,-1} + 32H_{0,-1,1}\right] \\
&+ 16H_{0,0,1} + 32H_{0,1,-1} + 32(4z + 1)H_{0,1,1} - 64z(3z + 2)H_{0,-1,0,1}
\end{align*}
\]
\[-\frac{16(60z^3 + 58z^2 + 5z + 8)H_{0,-1,0,1}}{3z} + \frac{8(168z^3 + 118z^2 + 5z + 8)}{3z} H_{0,0,0,-1} - \frac{64}{3} z(93z + 46)H_{0,0,0,1} + \frac{16(575z^3 - 224z^2 + 69z - 16)}{3z} H_{0,0,0,1} \]
\[\left(\frac{72z^3 + 58z^2 + 5z + 8}{z}\right) \left[H_{-1} \left[\frac{16}{3} H_{0,0,1} - \frac{16}{3} H_{0,0,-1}\right]\right] + \frac{16}{3} H_{0,0,-1,1} - \frac{16}{3} H_{0,0,-1,1} + \frac{64}{3} z(54z - 1) H_{0,0,1,1} - \frac{8}{3} \left(1364z^3 - 274z^2 + 31z - 56\right) H_{0,0,1,1} \]
\[\gamma_0 \left[-\frac{1}{3} H_1^2 + \frac{8}{3} \zeta H_1^3 - \frac{2}{3} H_0^3 H_1^2 + \left[\frac{52z^3}{3} - 44H_{0,0,1} - 4H_{0,1,1}\right] H_1^2 + H_0^2 \left[\zeta_2 + 8H_{0,1}\right] H_1 \right] + \left[46\zeta_2 - 32H_{0,1}\zeta_2 + 20H_0^2 - 16H_{0,0,0,1} + 24H_{0,0,1,1}\right] H_1 + H_0 \left[\frac{5}{3} H_1^3 + \left[\zeta_2 + 24H_{0,1}\right] H_1^2 \right] + \left(\frac{40\zeta_3 - 8H_{0,0,1} - 56H_{0,1,1}}{3z}\right) H_1 - 40H_{0,1,0,1} \]
\[\frac{16}{3} H_{0,0,1,1} + \frac{32}{3} H_{0,0,-1,1} + \frac{32}{3} H_{0,0,-1,1,1} + \frac{32}{3} H_{0,0,1,1,1} \]
\[\frac{108z^3 + 94z^2 + 5z + 8}{z} \left[H_{0,0,1,1} - \frac{8}{3} \zeta_2 H_1^3 + \left[\frac{16}{3} H_{0,1,1} - \frac{16}{3} H_{0,0,0,1}\right] H_1^2 + \left[-\frac{16}{3} H_0 H_1\right] \right] \]
\[\frac{32}{3} H_{0,0,1,1} + \frac{32}{3} H_{0,0,-1,1} + \frac{16}{3} H_{0,0,1,1} + \frac{32}{3} H_{0,0,1,1,1} \]
\[H_0 \left[\frac{16}{3} H_{0,0,1,1} + \frac{16}{3} H_{0,0,1,1,1} - \frac{32}{3} H_{0,0,1,1,1} - \frac{32}{3} H_{0,0,1,1,1}\right] \]
\[-\frac{16}{3} H_{0,0,-1,1} - \frac{16}{3} H_{0,0,-1,1} - \frac{32}{3} H_{0,0,-1,1} - \frac{32}{3} H_{0,0,-1,1} + L_M \left[\frac{1}{3} (19z - 5) H_0^3 - \frac{4}{9} (54z^2) \right] \]
\[\frac{1}{3} (19z - 5) H_0^3 - \frac{4}{9} (54z^2) \]
\[\frac{40z - 37}{3} H_0^3 - \frac{2}{9} (4303z^2 + 216z + 1303) H_0^2 + 4(34z - 9) \zeta_2 H_0^2 + 32(3z^2 - 7z + 1) \]
\[\times \zeta_2 H_0^2 - \frac{4}{3z} (108z^3 + 94z^2 + 5z + 8) H_{-1} H_0^2 + \frac{8}{3z} (122z^3 + 127z^2 + 32z + 16) H_{-1} H_0^2 \]
\[-\frac{4}{3z} (18z^3 + 59z^2 - 58z - 8) H_1 H_0^2 - 16(z^2 + 10z + 3) H_{0,-1} H_0^2 \]
\[\frac{4(36z^2 - 2z + 23)}{3z} H_0,1 H_0^2 + \frac{4(36z^2 - 2z + 23)}{3z} H_0,1 H_0^2 + \frac{8(1487z^2 - 5z + 192)}{3z} H_1 H_0 \]
\[\frac{8(108z^3 + 13409z^2 + 3023z + 448)}{3z} H_0 + \frac{4(31z^2 + 7z + 4)}{3z} H_0^2 \]
\[-\frac{32}{3} (57z^2 + 38z - 3) \zeta_2 H_0^2 + \frac{8}{3} (318z^2 + 59z + 27) \zeta_2 H_0 + 9(4z^2 + 24z - 21) \zeta_3 H_0 \]
\[-8(4z^2 + 28z + 15) \zeta_3 H_0 + \frac{16(28z^3 - 208z^2 - 5z - 16)}{3z} H_{0,-1} H_0 \]
\[\frac{8(72z^3 + 70z^2 + 5z + 8)}{3z} H_{0,-1} H_0 + 8(40z^3 - 281z^2 - 86z - 24) H_{0,1} H_0 \]
\[-\frac{32}{3} (8z^2 - 2z + 7) H_{0,-1} - H_0 + (2z^2 - 6z + 3) \left[-32H_{0,-1,1} - 32H_{0,0,1}\right] H_0 \]
\[-32(z + 1)(9z + 7) H_{0,0,1,1} + (2z^2 - 10z + 1) \left[-32H_{0,-1,1} - 32H_{0,1,1}\right] H_0 \]
\[-32(12z^2 + 7z + 1) H_{0,0,1} H_0 - \frac{8}{9} (103z^2 - 94z + 2) H_1 H_0^3 + \frac{8}{5} (24z^2 - 696z - 71) \zeta_2 \]
\[-\frac{4}{5} (160z^2 + 366z + 103) \zeta_2^2 - \frac{2(2599z^3 - 2608z^2 - 247z - 84)}{9z} H_1^2 - 32(z + 22) H_{0,0,-1} H_0 \]
\[ +32(z^2 - 4z + 2)H_{0,-1}^2 + 64(z^2 - 2)H_{0,1}^2 - \frac{4(290560z^3 - 295527z^2 + 32808z - 25024)}{81z} \\
+ 4(617z^3 - 2690z^2 + 490z - 128)\zeta_2 - \frac{8(975z^3 + 85z^2 - 136z + 104)\zeta_2}{9z} \\
- \frac{16(76z^3 + 103z^2 + 50z + 12)H_{-1}\zeta_2}{3z} - \frac{8(54z^3 - 76z^2 + 41z - 8)H_1\zeta_2}{3z} \\
- 16(6z - 1)H_{0,-1}\zeta_2 - 32(8z^2 - 4z + 7)H_{0,-1}\zeta_2 - 16(20z^2 - 26z + 9)H_{0,1}\zeta_2 \\
+ \frac{2(640z^3 - 4874z^2 - 543z - 360)\zeta_3}{3z} + \frac{2(2240z^3 - 614z^2 + 319z - 24)\zeta_3}{3z} \\
+ \frac{64(z - 1)(33z^2 + 2)H_1}{3z} - 4(14636z^3 - 18161z^2 + 3913z + 204)H_1 \\
+(z - 1)z\left[128H_1^2 + 128H_0H_1\right] + \frac{(z + 1)(251z^2 - 32z + 20)}{z} \left[\frac{16}{9}H_{0,-1} - \frac{16}{9}H_{-1}H_0\right] \\
+ \frac{(831z^3 + 733z^2 - 88z + 104)}{z} \left[\frac{16}{9}H_{0,-1} - \frac{16}{9}H_{-1}H_0\right] + (z^2 - z + 1) \cdot 32H_{0,-1}H_0' \\
+ 64H_{0,-1}' - \frac{64}{3}(21z + 2)H_0 - 4(3416z^3 - 4824z^2 + 1257z - 416)H_{0,-1} \\
- 16(40z^3 - 26z^2 - 17z - 8)H_1H_{0,1} + \frac{(z + 1)(40z^2 - 25z + 4)}{z} \left[\frac{16}{3}H_0H_{-1}^2 - \frac{32}{3}H_{0,-1}H_{-1}\right] \\
+ \frac{32}{3}H_{0,-1}' \cdot \frac{8(36z^3 + 46z^2 + 5z + 8)}{3z}H_{0,0,-1} - 16(178z^3 - 289z^2 + 22z - 16)H_{0,0,0,1} \\
- \frac{32}{3}H_{0,0,-1} \cdot \frac{8(16z^3 + 668z^2 + 75z + 40)}{3z}H_{0,0,0,1} \\
+ \frac{16}{3}H_{0,-1,1} + \frac{16}{3}H_{0,1,-1} \cdot \frac{(116z^3 + 118z^2 + 29z + 16)}{z} \left[\frac{16}{3}H_{-1}H_0 - \frac{16}{3}H_{0,-1} - \frac{16}{3}H_{0,-1,1}\right] \\
+ \frac{32}{3}(8z^2 + 73z - 11)H_{0,1,1,1} + z\left[-128H_0H_{0,1,1} - 256H_{0,1,1,1}\right] + z^2\left[704H_0 \\
- 128H_{0,0,0,1}\right] + 64(4z - 1)H_{0,-1,0,1} + 64(z^2 - 5z + 2)H_{0,0,0,0,-1} \\
+ 48(2z^2 + 22z + 5)H_{0,0,0,-1} + 128(z^2 + 1)H_{0,0,0,1} - 16(2z^2 - 118z - 13)H_{0,0,0,1} \\
+ 32(20z^2 + 12z + 9)H_{0,0,1,1} + z(z + 1)\left[-128H_0H_{0,0,1,1} - 256H_{0,0,1,1}\right] \\
+ (2z^2 + 2z + 1)\left[-\frac{160}{3}H_0H_{-1}^3 + \left[8H_0^2 - 80\zeta_2 + 160H_{0,-1}\right]H_1^2 + \left[\frac{8}{3}H_0^3 + 64\zeta_2 + 64H_{0,-1}\right]H_0 + 192\zeta_3 - 320H_{0,-1,-1} - 160H_{0,0,1} + 32H_{0,0,1} - 64H_{0,1,1}\right]H_{-1} + H_0\left[-64H_{0,-1,1} - 64H_{0,1,1}\right] \\
+ 320H_{0,-1,-1,1} + 64H_{0,-1,1,1} + 160H_{0,0,-1,-1} - 32H_{0,0,-1,1} - 32H_{0,0,1,1} \\
+ 64H_{0,0,1,1} + 64H_{0,1,1,1} \cdot \gamma_0^9 \left[2H_1^4 - 4H_0^2H_1^2 + 8H_{0,1} - 8\zeta_2\right]H_1^2 \\
+ \frac{10}{3}H_0^3H_1 + \left[-136\zeta_3 + 48H_{0,0,-1} + 64H_{0,0,1}\right]H_1 + H_0\left[H_1 \cdot \left[-40\zeta_2 - 24H_{0,-1} - 16H_0\right] - \frac{20}{3}H_1^2\right] \\
- 128(z - 1)\left[3\zeta_3 + 278z^2 - 7z - 8\right]H_{0,1,1,1} \\
+z\left[128\zeta_3 + 64H_{0,1,1}\right]H_0^2 + 384H_{0,1,1}H_0 - 384H_{0,1,1}\zeta_2 \\
+ 768H_{0,1,1,1} + (2z^2 - 6z + 3)\left[-16H_{0,0,-1}H_0^2 - 96H_{0,0,-1}\zeta_2 + 64H_{0,0,0,1}\right] \\
+ (z^2 - z + 1)\left[\frac{32}{3}H_{0,-1}H_0^3 + 128H_{0,-1,0,1} - 64H_{0,-1}^2\right]H_0 + H_{0,-1}\left[-288\zeta_3 + 256H_{0,-1,-1}\right]
\]
\[+128H_{0,0,1} + 256H_{0,-1,0,1} - 512H_{0,-1,0,1,1} - 256H_{0,-1,0,1,1,1} - 1024H_{0,0,0,1,1,1}
-128H_{0,0,1,1,1} - 32(2z^2 + 18z - 3)H_{0,0,0,1,1,1} - 32(2z^2 - 22z + 7)H_{0,0,0,0,0,1}
-32(2z^2 - 14z + 5)H_{0,0,0,0,0,1} - 32(40z^2 + 54z - 1)H_{0,0,0,0,1,1,1} + z^2 [32H_{0,-1}^2 + 1280H_{0,0,0,0,0,1}]
+(4z^2 + 16z - 1) \left[-64H_{0,0,0,1,1,1} - 64H_{0,0,0,1,1,1} \right] + (8z^2 - 4z + 7) \left[64H_{0,0,0,0,0,1} \right]
+64H_{0,0,0,0,1,1,1} - 128z(11z + 7)H_{0,0,0,1,1,1} + 64(35z^2 + 97z + 32)H_{0,0,0,1,1,1}
+(6z^2 + 14z + 1) \left[32H_{0,-1}H_{0,0,1,1} - 32H_{0,0,1,1,0} \right] + (4z^2 + 3) \left[-64H_{0,-1}H_{0,0,1} \right]
-128H_{0,-1,1}H_{0,-1,1,0,0} + 64(5z^2 + 9z + 11)H_{0,0,1,1,1}
-128(18z^2 - 2z + 7)H_{0,0,1,1,1,1} + z(z + 1) \left[64H_{0,0,1}H_{0}^2 + 128H_{-1}H_{0,1}H_{0} + \left(-128H_{0,-1,1} \right)
-128H_{0,1,1} + 384H_{0,0,1,1}H_{0} - 384H_{0,0,1,1}H_{0,1}\right] \right] \]
\[-64(9z^2 - 7z + 5)H_{0,1,0,1,1,1} + (2z^2 + 2z + 1) \left[-16H_{0}H_{1,1}^4 + \left[\frac{64}{3} H_{0}^2 - \frac{160}{3} \zeta_2 \right]
+64H_{0,-1} - \frac{64}{3} H_{0,1} \right] H_{1,1}^3 + \left[-\frac{8}{3} H_{0,1}^3 + \left[40 \zeta_2 - 64H_{0,-1} \right]
+32H_{0,1} \right] H_{0} + 160\zeta_3 - 192H_{0,-1,1} - 64H_{0,-1,1} - 96H_{0,0,0,1,1} - 64H_{0,0,1,1,1} \right] H_{1,1}^2 + \left[-\frac{4}{3} H_{0,1}^4 \right] \]
\[-36\zeta_2 - 16H_{0,-1} - 32H_{0,1} \right] H_{0}^2 + \left[-64\zeta_3 + 128H_{0,-1,1} - 64H_{0,-1,1} + 96H_{0,0,0,1,1} \right]
+64H_{0,0,1,1} - 64H_{0,0,1,1,1} \right] H_{0} - \frac{376}{5} \zeta_2 - 16H_{0,-1,1} \zeta_2 + 96H_{0,1,1} \zeta_2 + 384H_{0,0,1,1,1} \zeta_2 + 128H_{0,-1,1,1} + 128H_{0,0,-1,1,1} + 192H_{0,0,0,-1,1} - 160H_{0,0,0,0,1,1} \]
\[-32H_{0,0,0,1,1} + 192H_{0,0,1,1} + 128H_{0,0,1,1,1} + 128H_{0,0,1,1,1,1} \right] H_{1,1} - 96H_{0,-1,1,1} \zeta_2 - 96H_{0,1,1} \zeta_2 \]
\[+H_{0} \left[32H_{0,-1,1} + 32H_{0,-1,1,1} \right] \right] + H_{0} \left[64H_{0,-1,1,1} - 64H_{0,0,1,1,1} \right] \]
\[-64H_{0,0,1,1} - 64H_{0,0,1,1,1} \right] - 64H_{0,0,0,1,1} - 64H_{0,0,0,1,1,1} \right] - 64H_{0,-1,1,1} - 64H_{0,-1,1,1,1} \right] - 128H_{0,-1,0,1,1,1} - 128H_{0,-1,0,1,1,1,1} - 128H_{0,-1,0,1,1,1,1,1} \]
\[-192H_{0,0,0,-1,1} + 192H_{0,0,0,-1,1,1} - 128H_{0,0,0,-1,1,1} \right] \]
\[-128H_{0,0,0,1,1} - 128H_{0,0,0,1,1,1} - 128H_{0,0,0,1,1,1} \right] - 64(z^2 + z + 1)H_{0,1,1,1,1,1} - \frac{40(z - 1)}{9z} \]
\[+C_T^2 \left[\frac{1}{3}(20z^2 - 7z - 1)H_0^4 + \frac{3}{3}(32z^2 + 2z - 13)H_0^3 + \frac{16}{3}z(z + 1) \zeta_2 H_0^2 \right]
\[+\frac{4}{3}(20z^2 - 28z + 3)H_1H_0^3 + (2z - 1) \left[\frac{2z^2}{3} + 8H_{0,1} \right] H_0^3 + 4(10z^2 - 12z + 1)H_1H_0^3 \]
\[+3z(176z - 67)H_0^2 + 8(3z + 4) \zeta_2 H_0^2 - 2(24z^2 - 19z - 4) \zeta_2 H_0^2 \]
\[\frac{4}{3}(112z^2 - 34z + 17) \zeta_2 H_0^2 - 2(16z^2 - 46z - 1)H_1H_0^2 + 4(6z - 7)H_{0,1}H_0^2 \]
\[-8(12z^2 + 2z - 1)H_{0,0,1}H_0^2 + 32(5z^2 - 4z + 2)H_{0,0,1}H_0^2 \]
\[-8(4z^2 + 2z + 1) \zeta_2 H_0^2 - \frac{48}{5}(44z^2 - 14z + 7) \zeta_2 H_0 - 16(z - 1)(2z - 3)H_0^2 H_0 \]
\[-32(4z^2 - 2z + 1)H_{0,1}H_0 - 2(10z^2 + 105z + 59)H_0 + 64z \zeta_2 H_0 - 4z(11z - 4) \zeta_2 H_0 \]
\[-4(24z^2 - 20z + 1)H_1 \zeta_2 H_0 + 8(36z^2 - 38z + 19)H_{0,1} \zeta_2 H_0 - 8(32z^2 - 19z + 1) \zeta_3 H_0 \]
\[
+4(264z^2 - 233z - 6) H_1 H_0 - 8(14z + 11) H_{0,1} H_0 - 8(20z^2 - 14z - 9) H_{0,0,1} H_0 \\
+32(8z^2 - 18z - 1) H_{0,1} H_0 + z^2 \left[ 64H_{0,0,1,1} - 32H_{0,-1} \zeta_2 \right] H_0 + (16z^2 - 12z - 5) H_1^4 \\
+\frac{8}{3}(24z^2 - 40z + 7) H_1^4 + 562z^2 - \frac{8}{5}(255z^2 - 140z + 13) \zeta_2^2 + 2(152z^2 - 137z + 36) H_1^2 \\
-8(10z^2 - 16z + 1) H_{2,1}^2 - 718z + 4(4z^2 - 20z + 7) H_1^2 \zeta_2 - \frac{1}{2}(264z^2 + 1078z - 311) \zeta_2 \\
-4(11z^2 + 36z - 21) H_1 \zeta_2 + 4(32z^2 + 22z + 23) H_{0,1} \zeta_2 + 16(6z^2 + 2z + 1) H_{0,0,-1} \zeta_2 \\
-40(4z^2 - 10z + 5) H_{0,0,1} \zeta_2 - 32(5z^2 - 8z + 4) H_{0,1,1} \zeta_2 + (z + 1)(3z + 2) \left[ -8 \zeta_2^2 - 16H_{-1} H_0 \zeta_2 \\
+16H_{0,-1} \zeta_2 \right] + (2z^2 + 2z + 1) \left[ 16H_0 \zeta_2 H_{2,1}^2 + \left[ 16z^2 - 8H_0^2 \zeta_2 - 32H_{0,-1} \zeta_2 \right] H_{-1} + 32H_{0,-1,-1} \zeta_2 \right] \\
-\frac{2}{3}(384z^2 - 862z + 277) \zeta_3 - 16(5z^2 + 2z + 1) \zeta_2 \zeta_3 - 4(112z^2 - 271z - 24) H_{0,1} \\
+\frac{16}{3}(19z^2 - 8z + 4) \zeta_2 \zeta_3 - \frac{32}{3}(24z^2 - 34z + 1) H_1 \zeta_3 + \frac{32}{3}(14z^2 - 30z + 15) H_{0,1} \zeta_3 \\
-4(80z^2 - 111z + 41) H_1 + (z - 1)(3z + 1) \left[ \frac{16}{3} H_0 H_1^3 - 16 \zeta_2 \right] \\
+L_M^3 \left[ -\frac{64}{3} H_{0,1} z^2 + 8H_0 z - \frac{4}{3}(8z^2 - 2z + 1) H_0^2 + \frac{2}{3}(2z - 11) + \frac{32}{3}(4z^2 - 2z + 1) \zeta_2 \\
+\frac{16}{3}(4z - 1) H_1 + \gamma_0 \left[ \frac{8}{3} H_1^2 + \frac{8}{3} H_0 H_1 \right] \right] + 4(4z + 39) H_{0,0,1} + (5z^2 - 8z + 1) \left[ 64H_1 H_{0,0,1} \right] \\
-32H_0 H_1 H_{0,1} \right] + (6z^2 + 2z - 1) \left[ 32H_{0,1} H_{0,0,1} - 64 \zeta_5 \right] + 8(56z^2 - 95z + 29) H_{0,1,1} \\
+32(4z^2 - 6z + 3) H_{0,1,0,1} + L_M^2 \left[ -\frac{32}{3}(z + 1) H_0^3 - 16z(3z + 4) H_0^2 - 4(8z^2 - 2z + 1) H_0^2 \\
-128z H_0 - 2(100z^2 - 90z + 27) H_0 + 16(4z^2 + 2z + 1) \zeta_2 H_0 - 16(4z^2 - 8z + 1) H_1 H_0 \\
+64z_2 H_{0,-1} H_0 + 8z^2 - 4(16z^2 - 20z + 1) H_1^2 + 20z + 32(z - 1)(3z + 1) \\
+16z(5z - 2) \zeta_2 + 32(5z^2 + 2z + 1) \zeta_3 - 8(25z^2 - 37z + 17) H_1 \\
+\gamma_0 \left[ 4H_1^2 - 12H_0 H_1^2 + 4H_0^2 H_1 - 16 \zeta_2 H_1 - 4 \zeta_5 \right] + (z + 1)(3z + 2) \left[ 16 \zeta_2 \\
+32H_{-1} H_0 - 32H_{0,-1} \right] + (4z^2 - 2z + 1) \left[ H_0 \left[ 32 \zeta_2 - 32 H_{0,1} \right] - \frac{8}{3} H_0^3 \right] \\
-16(4z^2 + z + 1) H_{0,1} + (2z^2 + 2z + 1) \left[ -32H_0 H_{-1}^2 + \left[ 16H_0^2 - 32 \zeta_2 \right] \\
+64H_{0,-1} \right] H_{-1} - 64H_{0,-1,1} \right] - 32(6z^2 + 2z + 1) H_{0,0,-1} + 16(8z^2 - 6z + 3) H_{0,0,1} \\
-16(2z - 1) H_{0,0,1,1} + 5 \right] + (4z - 1) \left[ 8H_0^2 H_{0,1} - 32H_1 H_{0,1,1} \right] + 16(30z^2 - 22z - 5) H_{0,0,0,1} \\
+(8z^2 - 2z + 1) \left[ \frac{1}{15} H_0^5 + 48H_{0,0,0,1} H_0 \right] - 32(8z^2 - 23z - 2) H_{0,0,1,1} + 8(36z^2 - 44z \\
-17) H_{0,0,1,1} + L_M \left[ -\frac{1}{3}(24z^2 + 2z + 3) H_0^4 - \frac{4}{3}(40z^2 + 23z + 4) H_0^3 - 4(112z^2 \\
+32z + 25) H_0^2 + 16(2z - 1)(4z - 1) \zeta_2 H_0^2 + 32(2z^2 + 1) \zeta_2 H_0^2 + 32(z + 1)(5z + 4) H_{-1} H_0^2 \\
-4(16z - 11) H_1 H_0^2 + 16(6z^2 + 2z + 3) H_{0,-1} H_0^2 - 8(4z^2 - 6z - 1) H_{0,1} H_0^2 \\
-8(6z^2 + 2z - 7) H_1^2 H_0 - 4(94z^2 + 79z + 67) H_0 + 16(8z^2 + 2z + 11) \zeta_2 H_0 \\
+16(10z^2 - 18z - 7) \zeta_2 H_0 + 32(8z^2 - 8z - 1) \zeta_3 H_0 + 16(12z^2 - 6z + 7) \zeta_3 H_0 \\
-4(184z^2 - 214z + 63) H_1 H_0 - 32(2z^2 + 16z - 3) H_{0,-1} H_0 - 8(12z^2 - 46z + 17) H_{0,1} H_0 \\
\right] \]
\[ +L_M^2 \left[ - \frac{32}{3} (2z - 1) H_0^3 - \frac{64}{3} (z + 1) (3z - 2) H_0^2 + \frac{8}{9} (144z^2 + 166z + 325) H_0 \\
- \frac{4(2296z^3 - 24z^2 - 1677z - 208)}{27z} - \frac{32}{9} (8z^2 - 8z - 5) H_1 + \gamma_{qq}^0 \left[ 4H_1^2 + \frac{16}{3} H_0 H_1 - 8\zeta_2 \right] \\
+ \frac{8}{3} H_{0,1} \right] + \frac{32}{3} (10z^2 - 4z + 7) H_{0,0,1} + \frac{64}{3} (3z^2 - 6z - 1) H_{0,1,1} \]

\[ +L_M \left[ - \frac{20}{3} (2z - 1) H_0^3 - \frac{8}{9} (16z^2 + 80z - 55) H_0^3 + \frac{4}{9} (296z^2 - 260z + 931) H_0^2 \\
- \frac{8}{27} (240z^2 - 2203z - 4228) H_0 + \frac{64}{9} (23z^2 - 26z + 4) H_1 H_0 - \frac{64}{9} (4z^2 - 6z + 3) H_{0,1} H_0 \\
- \frac{32}{9} (2z^2 + z - 1) H_1^2 + \frac{8(6892z^3 - 85110z^2 + 85137z - 1600)}{81z} + \frac{64}{9} (2z^2 - 11z - 8) \zeta_2 \\
+ \frac{32}{3} (6z^2 + 2z - 1) \zeta_3 - \frac{16}{27} (244z^2 - 271z + 56) H_1 - \frac{64}{9} (25z^2 - 37z - 4) H_{0,1} \\
+ \frac{256}{3} (z - 1)^2 H_{0,0,1} + \gamma_{qq}^0 \left[ \frac{4}{3} H_1^2 - \frac{16}{3} H_0^3 H_1 + 8H_{0,1,1} \right] + \frac{64}{3} (10z^2 + 2z - 1) H_{0,0,0,1} \]

\[ +\gamma_{qq}^0 \left[ - \frac{2}{9} H_1^4 + \left[ \frac{16}{3} H_0, - \frac{26\zeta_2}{3} \right] H_1 + \frac{8}{9} H_0^3 H_1 + \left[ \frac{160\zeta_3}{9} - \frac{64}{3} H_0, - \frac{64}{3} H_{0,1,1} \right] H_1 \\
+ \frac{32}{3} H_0^2 - 12H_{0,1} \zeta_2 + H_0 \left[ \frac{32}{3} H_0, - \frac{16\zeta_2}{3} \right] - \frac{64}{3} H_{0,1,1} \right] + \frac{80}{3} H_{0,1,1,1} \]

\[ +C_F T_F^2 N_F \left[ - \frac{4}{9} (z - 2) (6z + 1) H_0^4 + \frac{4}{27} (404z^2 - 54z + 69) H_0^3 \\
- \frac{4}{27} (3112z^2 - 1329z - 210) H_0^2 - \frac{8(94z^3 - 234z^2 + 159z - 16) H_1 H_0^2}{9z} \\
+ \frac{16}{3} (6z^2 + 4z - 11) H_{0,1} H_0^2 - \frac{8}{3} (14z^2 - 5z - 11) \zeta_2 H_0^3 \\
+ \frac{8}{81} (27824z^2 + 4929z + 2631) H_0 + \frac{16(1556z^3 - 1539z^2 + 72z - 80) H_1 H_0}{27z} \\
- \frac{32(101z^3 + 222z^2 - 39z + 8) H_{0,1} H_0}{9z} - \frac{32}{3} (2z^2 + 28z - 17) H_{0,0,1} H_0 \\
+ \frac{8}{9} (240z^2 + 166z + 193) \zeta_2 H_0 + \frac{32}{9} (78z^2 - 133z - 28) \zeta_3 H_0 + \frac{16}{9} (z - 1)(3z + 1) H_1^3 \\
+ L_M^3 \left[ - \frac{16}{3} (2z - 1) H_0^3 - \frac{32}{9} (6z^2 - z - 4) H_0 + \frac{8(124z^3 - 258z^2 + 159z - 16)}{27z} + \frac{8}{9} \gamma_{qq}^0 H_1 \right] \\
+ \frac{8}{3} (12z^2 - 17z + 4) H_1^2 - \frac{64}{15} (2z^2 + 58z - 11) \zeta_2 - \frac{2}{27} (7280z^3 - 5646z^2 - 555z - 368) \zeta_3^2 \\
- \frac{4(221158z^3 - 226026z^2 + 17163z - 5248)}{243z} - \frac{8}{27} (3784z^3 + 2046z^2 + 1095z - 16) \zeta_3 \\
+ \frac{16}{3} (20z^2 - 17z - 1) H_1 - \frac{16(1448z^3 - 1341z^2 + 18z - 80) H_{0,1}}{27z} \\
+ \frac{16(948z^3 + 654z^2 + 3z + 16) H_{0,0,1}}{27z} + \frac{32}{3} (3z^2 - 6z - 1) H_{0,1,1} \\
- \frac{32}{3} (14z^2 - 74z + 19) H_{0,0,0,1} + L_M^2 \left[ \frac{32}{3} (2z - 1) H_0^2 + \frac{16}{3} (2z - 3)(4z - 3) H_0^2 \\
- \frac{8}{9} (160z^2 + 146z + 305) H_0 + \frac{4(1000z^3 + 1356z^2 - 2247z - 208)}{27z} + \frac{32}{9} (4z^2 - 4z + 5) H_1 \right] \right]
\[+\gamma_0 g \left( \frac{4}{3} H_1^3 + \frac{8}{3} H_{0,1} - \frac{8 \gamma_0}{3} \right) + \frac{16}{9} (2z^2 - 2z - 5) H_1 \gamma_0 - \frac{1}{9} H_1^4 + \frac{8}{3} H_{0,1} H_1^2\]

\[-\frac{4}{9} H_0^3 H_1 + \left[ -\frac{32}{3} H_{0,1,1} \right] H_1 + \frac{88}{9} \gamma_3 H_0 + \frac{16}{3} H_{0,1} + H_0 \left[ \frac{16}{3} H_{0,1} - \frac{32}{3} H_{0,1,1} \right] \]

\[+ \frac{40}{3} H_{0,1,1,1} + \left[ -4 H_1^2 - \frac{4}{3} H_0 H_1 - \frac{20}{3} H_0 \right] \gamma_2 + L_M \left[ \frac{1}{3} (6z^2 + 3z - 7) H_0^3 \right] \]

\[+ \frac{8}{9} (290z^2 + 194z + 377) H_0^2 - \frac{16}{27} (1244z^2 - 1226z - 1319) H_0 + \frac{32}{3} (2z^2 + 8z - 13) H_{0,1} H_0 \]

\[-\frac{16}{9} (62z^3 - 202z^2 + 149z - 16) H_1 H_0 + \frac{2}{9} (20960z^3 - 77478z^2 + 62367z - 4256) \]

\[+ \frac{16}{9} (5z^2 - 8z + 13) H_1^2 - \frac{32}{27} (68z^2 - 77z + 28) H_1 + \frac{16}{9} (72z^3 - 194z^2 + 175z - 16) H_{0,1} \]

\[-\frac{32}{3} (10z^2 + 10z - 23) H_{0,0,1} + \gamma_0 \left[ \frac{8}{9} H_1^3 + 4 H_0^3 H_1 + \frac{16}{3} H_{0,0,1} \right] - \frac{32}{9} (5z^2 + 4z + 13) H_2 \]

\[+ \frac{64}{3} (10z^2 - 9) \gamma_3 + (2z - 1) \left[ -\frac{20}{3} H_1^4 + 64 H_{0,0,0,1} H_0 + 128 \gamma_3 H_0 + \frac{384 \gamma_2^2}{5} - 192 H_{0,0,0,0,1} \right] \]

\[+ (2z - 1) \left[ -\frac{4}{15} H_0^5 - 8 \gamma_2 H_0^3 + 32 H_{0,0,1} H_0^2 + \frac{208}{3} \gamma_3 H_0^2 - 128 H_{0,0,0,0,1} H_0 + 128 H_{0,0,0,0,1} - 128 \gamma_5 \right] \]

\[+ C_{AC} T_F \left[ -\frac{1}{18} (76z^2 - 178z + 11) H_1^4 - \frac{1}{27} (4496z^2 + 1656z + 993) H_0^3 - \frac{8}{3} z (9z + 5) \gamma_2 H_0^3 \right] \]

\[+ \frac{8}{3} (13z^2 + 8z + 3) \gamma_3 H_0^3 - \frac{4}{3} (8z^2 + 4z - 1) H_{-1} H_0^3 - \frac{4}{3} (274z^3 - 238z^2 - 7z - 24) H_1 H_0^3 \]

\[+ \frac{16}{3} (2z^2 + 8z + 5) H_{0,1} H_0^3 + 4 (12z^2 + 16z + 1) H_{-1} H_0^3 - \frac{2}{3} (166z^3 - 156z^2 - 9z - 16) H_1 H_0^3 \]

\[+ \frac{32}{27} (3653z^2 + 3459z - 48) H_0^3 + 4z (13z + 4) \gamma_3 H_0^2 - \frac{1}{3} (76z^2 + 110z - 1) \gamma_2 H_0^2 \]

\[-32z (2z + 1) \gamma_3 H_0^2 + \frac{8}{3} (36z^2 + 2z + 11) \gamma_3 H_0^2 - 8 (5z^2 + 7z - 4) H_{-1} H_0^2 \]

\[+ \frac{2}{9} (952z^3 - 1740z^2 + 969z - 184) H_1 H_0^2 \]

\[+ 4 (32z^3 + 122z^2 - 25z + 24) H_{0,1} H_0^2 \]

\[-4 (32z^3 - 25z^2 + 24) H_{0,1} H_0^2 \]

\[-32 (z + 1)^2 (z + 1)^2 H_{0,1} H_0^2 + 96 (z^2 + 2z - 1) H_{0,0,1} H_0^2 \]

\[-32 (z + 1)^2 (z + 2)^2 (3z - 2) H_1^3 H_0^2 - \frac{32}{5} (4z^2 - 8z - 1) \gamma_3 H_0^2 + \frac{48}{5} (12z^2 \]

\[-49z + 3) \gamma_3 H_0^2 + \frac{8 (229z^3 - 297z^2 + 63z - 40) H_1^2 H_0^2}{9z} \]

\[+ 8 (2z - 5) (8z + 1) H_{0,0,1} H_0 \]

\[-\frac{12248z^2 - 6749z + 15057}{8} H_0 - 96 (z + 2) \gamma_3 H_0 \]

\[-\frac{2}{9} (72z^2 + 1757 z + 629) \gamma_2 H_0 \]

\[-8 (z + 19) H_{-1} H_0^2 + 8 (32z^2 + 36z + 1) H_{-1} H_0^2 \]

\[+ \frac{4 (138z^3 - 122z^2 - 23z - 16) H_1 H_0^2}{3z} \]

\[+ 16 (z^2 + 4z + 2) H_{0,1} H_0^2 - 8 (26z^2 - 18z + 21) H_{0,0,1} H_0 \]

\[-\frac{4}{9} (736z^2 + 1182z + 149) \gamma_3 H_0 \]

\[+ 16 (z^2 + 2z + 2) H_{0,1} H_0^2 - 8 (26z^2 - 18z + 21) H_{0,0,1} H_0 \]

\[-\frac{4}{9} (736z^2 + 1182z + 149) \gamma_3 H_0 \]

\[+ 16 (z^2 + 2z + 2) H_{0,1} H_0^2 - 8 (26z^2 - 18z + 21) H_{0,0,1} H_0 \]

\[+ 16 (z^2 + 2z + 2) H_{0,1} H_0^2 - 8 (26z^2 - 18z + 21) H_{0,0,1} H_0 \]

\[+ \frac{32 (9z^3 + 231z^2 - 99z + 23) H_{0,1} H_0^2}{9z} \]

\[+ \frac{8 (382z^3 - 368z^2 + 7z - 32) H_1 H_{0,1} H_0}{3z} \]

\[-64 (z + 19) H_{0,0,1} H_0 + \frac{16 (189z^3 - 8z^2 - 35z + 12) H_{0,0,1} H_0}{3z} \]
\[-16(4z^2 - 6z + 1)\zeta_2 H_0 + \frac{16(z - 1)(55z^2 + 7z + 4)H_1 H_0}{3z} + 16(2z^2 - 10z - 1)H_{0,1} H_0 \]
\[+ \frac{8(90z^3 - 89z^2 + 4z - 4)H_1^2}{3z} - 4(z - 1)(3z - 1) + \frac{2380z^3 - 1636z^2 - 533z - 48}{3z} \]
\[-\frac{4}{3}(70z^2 - 278z + 31)\zeta_2 - 8(18z^2 - 2z + 9)\zeta_3 + \frac{8(842z^3 - 937z^2 + 83z - 28)H_1}{9z} \]
\[+ 24(z + 1)H_0^2 + \gamma_9^0 \left[-8H_1^3 - 12H_0 H_1^2 - 2H_0^2 H_1 + 24\zeta_2 H_1 \right] - \frac{16(33z^3 + 26z^2 - 13z - 4)H_0,1}{3z} \]
\[-32(z^2 - 8z - 1)H_{0,0,1} + (2z^2 + 2z + 1) \left[-12\zeta_2 + 8\zeta_3 + 24H_{0,-1} - 16H_0 H_{0,-1} \right] \]
\[+ H_{-1} \left[8H_0^2 - 24H_0 - 32\zeta_2 + 32H_0,1 \right] - 32H_{0,-1,1} + 16H_{0,0,-1} - 32H_{0,1,-1} \right] - 48(2z + 1)H_{0,1,1} \]
\[-8(16z^2 + 20z + 1)H_{0,0,0,-1} - \frac{8(730z^3 - 260z^2 - 137z + 24)H_{0,0,0,1}}{3z} + (z - 1)z \left[128H_1^3 \right] \]
\[+ 512H_1^2 + 64H_0^2 H_1 + \left[-384\zeta_2 - 224 \right] H_1 - 24H_{0,-1,1} + \left[192H_1^2 + 512H_1 \right] + 256H_{0,0,0,1} \]
\[(2z - 1)(2z + 1) \left[H_{-1} \left[16H_{0,0,1} - 16H_{0,0,-1} \right] - 16H_{0,-1,0,1} + 16H_{0,0,-1,1} \right] \]
\[-16H_{0,0,-1,1} - 16H_{0,0,1,-1} + \gamma_9^0 \left[\frac{2}{3}H_1^5 + \left[\frac{32\zeta_2}{3} - \frac{16}{3}H_{0,1} \right]H_1^3 \right] \]
\[+ \frac{4}{3}H_0^3 H_1^2 + \left[-\frac{80}{3}\zeta_3 + 68H_{0,0,1} + 20H_{0,1,1} \right]H_1^2 + \frac{1}{3}H_0^4 H_1 + H_0^2 \left[-\zeta_2 - 20H_{0,1} \right]H_1 \]
\[+ \left[-\frac{494}{5}\zeta_2 + 48H_{0,1}\zeta_2 - 36H_{0,1}^2 + 16H_{0,0,0,1} - 24H_{0,0,1,1} \right]H_1 - 48\ln(2)\zeta_2 \]
\[+ H_0 \left[\frac{4}{3}H_1^4 + \left[16\zeta_2 - 40H_{0,1} \right]H_1^2 + \left[-\frac{152}{3}\zeta_3 + 32H_{0,0,1} + 104H_{0,1,1} \right]H_1 \right] \]
\[-128z(2z - 5)H_{0,0,1,1} + \frac{8(740z^3 - 546z^2 - 93z - 48)H_{0,0,1,1}}{3z} \]
\[+ (4z^2 + 8z + 1) \left[16\zeta_2 - 16H_{0,1} \right]H_{0,1}^2 + \left[-16H_0 H_{0,-1} + 32H_{0,-1,1} + 32H_{0,1,-1} \right]H_{-1} \]
\[+ 16H_0 H_{0,-1,1} - 32H_{0,-1,1,-1} - 32H_{0,0,-1,1} + 32H_{0,0,0,1} \right] + (8z^2 + 12z + 1) \left[-\frac{16}{3}H_0^3 \right] \]
\[+ (24\zeta_2 + 16H_{0,-1} + 16H_{0,1})H_{0,1}^2 + \left[16H_0 H_{0,1} - 32H_{0,-1,1} - 32H_{0,0,-1} - 16H_{0,0,1} \right] \]
\[-32H_{0,1,-1} \right]H_{-1} - 8H_{0,-1,1} + H_0 \left[-16H_{0,-1,1} + 8H_{0,0,-1} - 16H_{0,-1} \right] \]
\[+ 32H_{0,-1,1} + 32H_{0,-1,1,1} + 32H_{0,0,-1,1} + 16H_{0,0,0,1} + 16H_{0,0,1,1} + 32H_{0,1,-1,1} \]
\[+ L_M \left[\frac{4}{3}(5z + 2)H_0^4 - \frac{2}{3}(236z^2 - 2z + 11)H_0^3 - \frac{2}{9}(1578z^2 - 1883z - 353)H_0^2 \right] \]
\[+ 16(2z^2 - 4z - 3)\zeta_2 H_0^2 - 8(8z^2 + 23z + 3)\zeta_2 H_0^2 - 8(6z^2 + 8z + 5)H_{-1} H_0^2 \]
\[+ 4(8z^2 + 12z + 1)H_{-1} H_0^2 - \frac{4(27z^3 - 360z^2 - 111z - 16)H_1 H_0^2}{3z} \]
\[+ 8(2z^2 - 10z - 3)H_{0,0,0}^2 - 16(7z^2 - 10z + 2)H_{0,1} H_0^2 - \frac{8(97z^3 - 99z^2 + 6z - 16)H_0^2}{3z} \]
\[+ \frac{4}{27}(432z^2 - 10879z + 200)H_0 - 16(5z^2 + 13)\zeta_2 H_0 + 8(128z^2 + 29z - 49)\zeta_2 H_0 \]
\[-4(52z^2 + 10z + 29)\zeta_3 H_0 + 4(52z^2 + 370z + 97)\zeta_3 H_0 \]
\[+ \frac{4(450z^3 - 812z^2 + 415z - 208)H_1 H_0}{9z} - 8(4z + 1)H_{0,-1} H_0 - 16(4z^2 - 16z + 21)H_{0,-1} H_0 \]

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\[
-\frac{8(108z^3 + 394z^2 - 101z + 16)H_{0,1}H_0}{3z} - 64(6z^2 + 4z + 1)H_{0,-1,-1}H_0
-16(26z^2 - 2z + 11)H_{0,0,-1}H_0 + 16(18z^2 - 22z + 15)H_{0,0,1}H_0
+(2z^2 - 10z + 1)\left[32H_{0,-1,-1} + 32H_{0,1,-1}\right]H_0 - 16(8z^2 - 30z - 9)H_{0,1,1}H_0
+\frac{8}{9}(193z^2 - 166z - 16)H_3^2 - \frac{12}{5}(88z^2 + 6z + 75)\zeta_2^2 + \frac{12}{5}(88z^2 + 106z + 113)\zeta_2^2
+\frac{4(2365z^3 - 2510z^2 + 121z - 56)H_1^3}{9z} + 16(6z^2 + 2z - 1)H_{0,-1}^2 - 32(2z^2 + 3)H_{0,1}^2
+\frac{122748z^3 - 97910z^2 - 30697z - 1440}{54z} + 8(50z^2 - 19z - 37)\zeta_2
-\frac{8}{9}(1207z^2 - 1190z - 233)\zeta_2 - 16(5z^2 + 6z + 4)H_{-1}\zeta_2
+\frac{8(52z^3 - 78z^2 + 45z - 16)H_1\zeta_2}{3z} + 32(8z + 1)H_{0,-1}\zeta_2
+16(4z^2 + 10z + 5)H_{0,-1}\zeta_2 + 32(6z^2 - 16z + 3)H_{0,1}\zeta_2 - 2(4z^2 - 184z + 235)\zeta_3
-\frac{2(3428z^3 - 3120z^2 - 825z - 192)\zeta_3}{3z} + \frac{8(8065z^3 - 9010z^2 + 1094z + 102)H_1}{27z}
+(z - 1)\left[-128H_1^2 - 128H_0H_1\right] + (z + 1)(5z + 2)\left[-16H_{-1}H_0 - 16H_{0,-1}\right]
+(z + 1)(18z - 37)\left[16H_{-1}H_0 - 16H_{0,-1}\right] + (2z + 1)\left[-8H_{0,-1}H_0^2 - 16H_{0,-1}^2\right]
+\frac{4(1640z^3 - 1658z^2 - 215z + 208)H_{0,1}}{9z} + \frac{16(80z^3 - 90z^2 + 9z - 8)H_1H_{0,1}}{3z}
+(z + 1)(5z + 9)\left[16H_0H_{1,1}^2 - 32H_{0,1}H_{-1} + 32H_{0,-1,1}\right]
+z^2\left[H_0\left[-64H_{0,-1,1} - 64H_{0,0,1}\right] - 64H_0^2\right] - 8(8z^2 + 4z - 1)H_{0,0,1}
+16(14z^2 - 24z + 47)H_{0,0,0} - 32(17z - 4)(2z^2 + 2z - 1)H_{0,0,1}
+(4z^2 + 8z + 1)\left[-8H_0H_{2,1} + \left[-24\zeta_2 + 16H_{0,-1} + 16H_{0,1}\right]H_{-1} - 16H_{0,-1,1} - 16H_{0,-1,1}\right]
-16H_{0,1,1}\right] + (10z^2 + 20z + 13)\left[16H_{-1}H_{0,1} - 16H_{0,-1,1} - 16H_{0,1,1}\right]
-16\left[72z^2 + 290z + 11\right]H_{0,1,1} + z\left[128H_0H_{0,1} - 192H_{0,0,1} + 256H_{0,1,1}\right]
+\frac{8}{3}\left[72z^2 + 290z + 11\right]H_{0,1,1} + z\left[128H_0H_{0,1} - 192H_{0,0,1} + 256H_{0,1,1}\right]
+\frac{\zeta_{gg}}{104}\left[-4H_1^4 + 6H_0^2H_1^2 + \left[32\zeta_2 - 24H_{0,1}\right]H_1^2 - \frac{2}{3}H_0^2H_1 + \left[168\zeta_3 - 48H_{0,0,1} - 40H_{0,0,1}\right]
+24H_{0,1,1}\right]H_1 + H_0\left[4H_1^4 + \left[64\zeta_2 + 24H_{0,-1} - 8H_{0,1}\right]H_{1,1}\right] - 24H_{0,-1}H_{0,1}\right]H_1
-32(6z - 1)H_{0,-1,0,1} + 32(8z^2 - 6z + 3)H_{0,-1,0,1} + 16(8z^2 + 2z + 1)H_{0,0,0,1}
+16(50z^2 + 2z + 31)H_{0,0,0,1} - 64(4z^2 + 5)H_{0,0,0,1} - 16(18z^2 - 18z + 25)H_{0,0,0,1}
-32(10z^2 + 14z + 5)H_{0,0,1,1} + z(z + 1)\left[-128H_{0,1} + 128H_{0,0,1} + 256H_{0,1,1}\right]
+(2z^2 + 2z + 1)\left[\frac{160}{3}H_{0,1}H_3^2 + \left(-72H_0^2 + 208\zeta_2 - 160H_{0,-1} - 128H_{0,1}\right)H_2^2 + \left[\frac{104}{3}H_0^3\right.
+\left[-352\zeta_2 + 128H_{0,-1} + 64H_{0,1}\right]H_0 - 384\zeta_3 + 320H_{0,-1,1} + 256H_{0,-1,1} + 32H_{0,0,-1} + 96H_{0,0,1}
+256H_{0,1,-1} + 64H_{0,1,1}\right]H_{-1} - 320H_{0,-1,1,1} - 256H_{0,-1,1,1} - 256H_{0,-1,1,1} - 64H_{0,1,1,1}
-32H_{0,0,-1,1} - 96H_{0,0,1,1} - 256H_{0,1,-1,1} - 64H_{0,1,1,1} - 64H_{0,1,1,1}\right]}
\]
\[
+ 8(282z^3 + 160z^2 + 25z - 48)H_{0,1,1,1} + z\left[ \frac{4}{15} H_0^5 - 64H_{0,1}H_0^2 + \left[ -128H_{0,1} - 384H_{0,1,1,1} \right] H_0 \\
+ 384H_{0,1}\zeta_2 - 768H_{0,1,1,1} \right] + z^2\left[ -\frac{64}{3} H_0^3 + \left[ 256 - 32H_{0,0,-1} \right] H_0^2 - 224H_0 - 192H_{0,-1,-1}\zeta_2 \\
+ 128H_{0,-1,0,1} \right] + (2z + 1)\left[ -\frac{8}{3} H_{0,-1}H_0^2 + \left[ 16H_{0,-1}^2 - 32H_{0,-1,0,1} \right] H_0 \\
+ H_{0,-1}\left[ 72\zeta_3 - 64H_{0,-1,1} - 32H_{0,0,-1} \right] - 64H_{0,-1,-1,1} + 128H_{0,-1,0,-1} \\
+ 64H_{0,0,0,-1,1} + 256H_{0,0,-1,1,1} + 32H_{0,0,1,0,-1} \right] + 32(10z^2 + 6z + 3)H_{0,0,0,1} \\
- 64(5z^2 + 2z + 1)H_{0,0,0,-1} - 32(6z^2 + 2z + 1)H_{0,0,0,0,-1} - 320z(2z + 3)H_{0,0,0,0,1} \\
+ 16(80z^2 + 22z - 1)H_{0,0,0,0,1} + (20z^2 + 14z + 7)\left[ 32H_{0,0,0,0,1} + 32H_{0,0,0,0,0} \right] \\
+ (4z^2 + 10z + 5)\left[ -32H_{0,0,0,0,1} - 32H_{0,0,0,0,1} \right] + 128z(9z + 10)H_{0,0,0,0,1} \\
- 16(60z^2 + 374z + 129)H_{0,0,0,0,1} + (5z^2 + 4z + 2)\left[ 64H_{0,0,0,0,0} - 64H_{0,0,0,0,1} \right] \\
+ (4z^2 + 6z + 3)\left[ 32H_{0,0,0,0,0} + 64H_{0,0,1,0,0} + 64H_{0,0,0,1,1} - 32H_{0,0,0,0,0} \right] \\
- 16(110z + 53)H_{0,0,0,0,1} + 32(104z^2 - 8z + 55)H_{0,0,1,1,1} + z(z + 1)\left[ -64H_{0,0,0,0,1} \right] \\
- 128H_{0,0,1,0} + \left[ 128H_{0,0,1,1} + 128H_{0,0,1,1} - 384H_{0,0,0,1} \right] H_0 + 384H_{0,0,1}\zeta_2 \\
- 128H_{0,0,1} + 128H_{0,0,1,0} - 768H_{0,0,1,1,1} \right] + 16(72z^2 - 26z + 49)H_{0,0,1,1,1} \\
+ (2z^2 + 2z + 1)\left[ 16H_0H_{0,0}^4 + \left[ -\frac{64}{3} H_0^3 - \frac{160}{3} \zeta_2 - 64H_{0,0,1} \right] \\
- \frac{64}{3} H_0H_{0,0}^3 + \frac{8}{3} H_0^3 + \left[ -56\zeta_2 + 64H_{0,0,1} - 32H_{0,0,1} \right] H_0 - 160\zeta_3 + 192H_{0,0,1} \right] \\
+ 64H_{0,0,1} + 96H_{0,0,1,0} + 64H_{0,0,1,1} \right] H_0^2 + \left[ \frac{4}{3} H_0^4 + \left[ -28\zeta_2 + 16H_{0,0,1} + 32H_0 \right] \right] H_0^2 \\
+ \left[ 64\zeta_3 - 128 - 128H_{0,0,1,1} - 64H_{0,0,1,0} - 96H_{0,0,1,0} - 64H_{0,0,0,1} + 64H_{0,0,0,1} + 64H_{0,0,1,1} \right] H_0 \\
+ \frac{296}{5} \zeta_2^2 + 48H_{0,0,0,0,1} - 96H_{0,0,0,0,1} - 384H_{0,0,0,1,1,1} - 128H_{0,0,0,1,1,1} - 128H_{0,0,0,1,1,1} \right] H_1 \\
- 192H_{0,0,0,0,1,1} + 128H_{0,0,0,0,0,1} - 192H_{0,0,0,0,1,1} - 128H_{0,0,0,0,1,1} - 128H_{0,0,0,1,1,1} \right] H_1 \\
+ 96H_{0,0,0,0,0,1} + 96H_{0,0,0,1,1} \right] + H_0^2\left[ -32H_{0,0,1,1} - 32H_{0,0,1,1} \right] + H_0\left[ -64H_{0,0,0,1,1} \\
+ 64H_{0,0,0,1,1} - 64H_{0,0,0,1,1} + 96H_{0,0,0,0,1,1} + 64H_{0,0,0,0,1,1} - 64H_{0,0,0,0,1,1} \right] \\
- 64H_{0,0,0,0,1,1} + 64H_{0,0,0,0,1,1} + 384H_{0,0,0,0,0,1} + 128H_{0,0,0,0,0,1} \right] + 128H_{0,0,0,0,0,1} + 192H_{0,0,0,0,0,1} \right] \\
- 64H_{0,0,0,0,0,1} + 192H_{0,0,0,0,0,1} + 128H_{0,0,0,0,0,1} + 128H_{0,0,0,0,0,1} + 128H_{0,0,0,0,0,1} \right] + 16(20z^2 - 14z + 13)H_{0,1,1,1,1} \right] \right) \right). \tag{607}
\]

The OME $A_{gg,q}(z)$ as a diagonal element in the singlet-gluon matrix has distribution-valued ($+$, $\delta(1 - z)$) and regular (reg) contributions:

$$A_{gg,q}(z) = [A_{gg,q,+}(z)] + A_{gg,q,\text{reg}}(z) + C_{gg,q}\delta(1 - z), \tag{608}$$

with

$$\int_0^1 dz f(z) [A_{gg,q}(z)]_+ = \int_0^1 dz [f(z) - f(1)] A_{gg,q,+}(z) \tag{609}$$
\[
\int_0^1 dz C_{gq,q} \delta(1 - z) = C_{gq,q}.
\]

The different parts are given by:

\[
A_{gq,Q,+} =
\frac{a_s^2}{z - 1} \left\{ C_A T_F \left[ -\frac{8}{3} L_M^2 - \frac{80}{9} L_M - \frac{224}{27} \right] \right\}
+ \frac{a_s^3}{z - 1} \left\{ C_A^2 T_F \left[ L_M \left[ -\frac{64}{3} H_{0,-1} + \frac{32}{3} H_{0,1} - \frac{640}{9} H_1 - \frac{16}{3} \right] + \frac{128}{3} H_{0,0,-1} - \frac{64}{3} H_{0,0,1} \right)
+ \left[ -\frac{16}{3} H_1 - \frac{160}{9} \right] H_0^2 + \frac{32 \zeta_2}{9} - \frac{256 \zeta_3}{3} - \frac{1240}{81} \right] \right\}
+ L_M^2 \left[ -\frac{16}{3} H_0^2 - \frac{64}{3} H_1 H_0 + \frac{32 \zeta_2}{3} - \frac{184}{9} \right)
+ \left[ \zeta_2 \left[ \frac{8}{3} H_0^2 + \frac{32}{3} H_1 H_0 + \frac{16}{27} \right] - \frac{88 H_0}{9} - \frac{16 \zeta_2^2}{3} - \frac{176 \zeta_3}{27} - \frac{2267}{243} \right] + L_M^3 \frac{176}{27} \right\}
+ C_A C_F T_F \left[ L_M \left[ 64 \zeta_3 - \frac{40}{3} \right] - 8 L_M^2 \right]
+ C_A T_F^2 \left[ \left[ \frac{16 H_0}{3} + \frac{560 \zeta_2}{27} + \frac{224 \zeta_3}{27} + \frac{5248}{81} \right] - L_M^3 \frac{224}{27} - L_M^2 \frac{640}{27} - L_M \frac{320}{9} \right]
+ C_A T_F^2 N_F \left[ \left[ \frac{32 H_0}{9} + \frac{160 \zeta_2}{27} + \frac{64 \zeta_3}{27} - \frac{10496}{243} \right] - L_M^3 \frac{64}{27} - L_M \frac{2176}{81} \right]
+ a_{gq,q,+}^{(3)} \right\}
\]

\[
C_{gq,q} =
\frac{4}{3} a_s T_F L_M + a_s^2 \left\{ C_A T_F \left[ \frac{16}{3} L_M + \frac{10}{9} \right] + C_F T_F \left[ 4 L_M - 15 \right] + \frac{16}{9} T_F^2 L_M^2 \right\}
+ a_s^3 \left\{ C_A^2 T_F \left[ \left[ \frac{16 \zeta_3}{3} - \frac{2}{3} \right] L_M^2 + \left[ \frac{16 \zeta_2^2}{3} + \frac{160 \zeta_3}{9} + \frac{277}{9} \right] L_M + \zeta_2 \left( 4 - \frac{8 \zeta_3}{3} \right) - \frac{616}{27} \right\}
+ C_F C_A T_F \left\{ \frac{22}{3} L_M^3 + \frac{736}{9} L_M + \frac{20 \zeta_2}{3} + \frac{16 \zeta_3}{3} - \frac{1045}{6} - 64 \zeta_2 \log(2) \right\}
+ C_F T_F^2 N_F \left[ 28 \zeta_2 + \frac{118}{3} - \frac{268}{9} L_M \right]
+ C_F T_F^2 \left[ \frac{40}{3} L_M^3 - \frac{584}{9} L_M + \frac{782}{9} - \frac{40 \zeta_2}{3} \right]
+ C_A T_F^2 N_F \left[ \frac{224}{27} - \frac{4 \zeta_2}{3} - \frac{44}{3} L_M \right]
+ C_A T_F^2 \left[ \frac{56}{3} L_M^2 - 2 L_M - \frac{44 \zeta_2}{3} - \frac{8}{27} \right]
+ C_F^2 T_F \left[ -2 L_M + -80 \zeta_2 - 32 \zeta_3 - 39 + 128 \zeta_2 \log(2) \right]
+ T_F^3 \left[ \frac{64}{27} L_M^3 - \frac{64 \zeta_3}{27} \right]
+ a_{gq,q,\delta}^{(3)} \right\}
\]

and

\[
A_{gq,Q,\text{reg}} =
\frac{a_s^2}{z + 1} \left\{ C_A T_F \left[ 4 \left(z + 1\right) H_0^2 + \frac{4}{9}(22z + 13) H_0 - \frac{8 \left( z^3 - z^2 + 2z - 1 \right) L_M^2}{3z} \right] \right\}
\]
\[-\frac{4(175z^3 - 137z^2 + 157z - 139)}{27z} + \frac{4}{3}zH_1 + L_M \left[ \frac{16}{3}(z+1)H_0 - \frac{8(23z^3 - 19z^2 + 29z - 23)}{9z} \right] \]

\[+ CF_T \left[ \frac{4}{3}(z+1)H_0^3 + 2(5z + 3)H_0^2 + 16(3z + 2)H_0 + \frac{L_M^2}{3z} \left[ \frac{8(z+1)H_0 - \frac{4(z-1)(4z^2 + 7z + 4)}{3z} }{z} \right] \right] \]

\[-\frac{8(z-1)(3z^2 + 9z - 1)}{z} + L_M \left[ 8(z+1)H_0^2 + 8(5z + 3)H_0 - \frac{16(z-1)(5z^2 + 11z - 1)}{3z} \right] \]

\[+ a_2^2 \left( C_5^2 \right) \left[ \frac{8}{27} \left( z+1 \right) H_0^3 + \frac{44}{27} \left( 22z + 13 \right) H_0^2 + \frac{8}{3} \left( 2z^3 - 2z^2 - 4z - 1 \right) \frac{\zeta_2}{z+1} H_0^2 \]

\[+ \frac{8}{81} \left( 161z + 62 \right) H_0 + \frac{8}{3} \left( 44z^2 + 11z + 47 \right) \zeta_2 H_0 + \frac{176(z^3 - z^2 + 2z - 1)}{27z} L_M^3 - \frac{44}{9} z H_1^2 \]

\[+ L_M \left[ \frac{(z^2 + z + 1)^2}{z(z+1)} \left[ \frac{64}{3} H_{-1} H_0 - \frac{64}{3} H_{0,-1} \right] - \frac{16(z^2 + 2z^2 - 4z - 1)}{3(z+1)} \right] \]

\[-\frac{8(208z^3 - 273z^2 + 204z - 208)}{27z} + \frac{32}{3} \left( 2z^3 + 2z^2 + 4z + 3 \right) \frac{\zeta_2}{z+1} - \frac{64}{9} \left( 11z^2 + 9 \right) H_0 \]

\[-\frac{64(z^3 - z^2 + 2z - 1)}{3z} H_{0,H_1} \left( \frac{8(12283z^3 - 9665z^2 + 8143z - 7927)}{243z} \right) \]

\[-\frac{4}{27} \left( 416z^3 - 437z^2 + 433z - 416 \right) \frac{\zeta_2}{z} - \frac{16}{3} \left( 2z^3 + 2z^2 + 4z + 3 \right) \frac{\zeta_2^2}{z+1} + \frac{176(4z^2 + 3z - 3)}{27z} H_1 \]

\[+ \frac{(z^2 + z + 1)^2}{z(z+1)} \left[ \frac{32}{3} \left( H_{-1} H_0 - \frac{32}{3} H_{-1} H_0 \right) + \frac{(z^3 - z^2 + 2z - 1)}{z} \left[ \frac{32}{3} H_{0,H_1} \zeta_2 - \frac{176z \zeta_3}{27} \right] \right] \]

\[+ L_M \left[ \frac{(z+1)^2}{z} \left[ \frac{64}{3} H_{-1} H_{0,1} + \frac{64}{3} H_{0,-1,1} + \frac{64}{3} H_{0,1,-1} \right] + \frac{8(4z^2 - 31z + 4)}{3z} H_{-1} H_0^2(z+1) \right] \]

\[-\frac{16}{3} \left( 2z^2 + 35z - 2 \right) \frac{\zeta_2}{z} H_{-1} H_0(z+1) + \frac{(2z^2 - 9z + 2)}{z} \left( z+1 \right) \left[ -16H_0 H_0^2 + 32H_{0,-1} H_{-1} \right] \]

\[-32H_{0,-1,-1} \right] + \left[ -24H_{0,1} H_0^2 + 96H_{0,0,0} H_0 - \frac{1832}{9} H_{0,1} + 192H_{0,0,0,-1} - 192H_{0,0,0,1} \right] \]

\[+ 96H_{0,1} \zeta_2 \left( z+1 \right) + \frac{8}{9} \left( 19z - 12 \right) H_0^3 - \frac{24}{5} \left( 13z + 23 \right) \zeta_2^2 + \frac{4(13194z^3 - 13763z^2 + 12661z - 12402)}{81z} \]

\[+ \frac{8}{9} \left( 132z^4 - 137z^3 + 156z^2 + 481z + 96 \right) \frac{\zeta_2}{z(z+1)} - \frac{32}{3} \left( 8z^3 + 7z^2 + 31z - 11 \right) \frac{\zeta_3}{z} \]

\[-\frac{4(3918z^3 - 1495z^2 + 4022z + 568)}{27z} H_0 - \frac{8}{3} \left( 8z^4 + 43z^3 + 34z^2 - 5z - 8 \right) \frac{\zeta_2}{z(z+1)} H_0 \]

\[+ \frac{4}{3} \left( 25z - 29 \right) H_0^2 H_1 + \frac{4(410z^3 + 1965z^2 - 2031z - 410)}{27z} - \frac{16(z-1)(2z^2 + 9z + 2)}{z} \frac{\zeta_2}{z} H_1 \]

\[-\frac{32(33z^3 - 29z^2 + 49z - 33)}{9z} H_0 H_1 \right] + \frac{64}{3} \left( z^2 + 9z - 7 \right) H_0 H_{0,-1} - \frac{8}{3} \left( 17z - 37 \right) H_0 H_{0,1} \]

\[-\frac{16(12z^3 + 45z^2 - 83z + 4)}{3z} H_{0,0,-1} - 32(5z - 1) H_{0,0,-1} + \frac{16}{3} \left( 4z^2 + 31z - 23 \right) H_{0,0,1} \]

\[+ 8(5z + 3) H_0^2 \zeta_2 + (z-1) \left[ 48H_{0,-1} H_0^2 - 192H_{0,-1,-1} H_0 + 96 H_{0,-1}^2 - 96 H_{0,-1} \zeta_2 \right] \]

\[+ z(192H_0 \zeta_3 - \frac{4}{3} H_0^3) - \frac{8(124z^3 - 459z^2 - 613z - 70)}{9z} H_0 \]

\[+ \frac{114z^4 - 77z^3 - 342z^2 - 77z + 114}{z(z+1)} \left[ \frac{16}{9} H_{-1} H_0 - \frac{16}{9} H_{0,-1} \right] \]
\[ +C_A T_F^2 \left[ -L_M^3 \frac{224(z^3 - z^2 + 2z - 1)}{27z} + L_M^2 \left[ \frac{128}{9}(z + 1)H_0 - \frac{64(23z^3 - 19z^2 + 29z - 23)}{27z} \right] \right]
\[ +L_M \left[ -\frac{8(814z^3 - 633z^2 + 777z - 598)}{81z} - \frac{8}{27}(52z^2 - 89z - 2)H_0 - \frac{8(52z^3 - 51z^2 + 33z - 52)}{27z} H_1 \right]
\[ +(z + 1) \left[ \frac{80}{9}H_0^2 + \frac{128}{9}H_0, - \frac{128\zeta_2}{9} \right] \right] - \frac{8}{9}(22z + 13)H_0^2 + \frac{8}{3}zH_1 - \frac{16}{27}(161z + 62)H_0
\[ +\frac{16(187z^3 - 949z^2 + 881z - 791)}{81z} \right) + \frac{56}{27}(23z^3 - 19z^2 + 29z - 23)\frac{\zeta_2}{z} + \frac{224}{27}(z^3 - z^2 + 2z - 1)\frac{\zeta_3}{z} \right] + \frac{16(187z^3 - 949z^2 + 881z - 791)}{243z}
\[ +\frac{16}{9}zH_1^2 + \frac{16}{27}(23z^3 - 19z^2 + 29z - 23)\frac{\zeta_2}{z} + \frac{64}{27}(z^3 - z^2 + 2z - 1)\frac{\zeta_3}{z} - \frac{32}{81}(161z + 62)H_0
\[ -\frac{64(4z^2 + 3z - 3)H_1}{27z} + (z + 1) \left[ \frac{32}{27}H_0^3 - \frac{32}{9}\zeta_2H_0 \right] \right]
\[ +C_T F^2 \left[ \frac{2}{9}(4z^2 - 3z + 3)H_0^4 + \frac{2}{9}(40z^2 + 149z + 115)H_0^3 - \frac{2}{3}(64z^2 + 23z - 69)H_0^2 \right]
\[ +\frac{4(z - 1)(20z^2 + 41z - 4)H_1H_0^3 - \frac{8}{3}(4z^3 + 27z^2 + 3z - 4)H_0, H_0^2 + \frac{8}{3}(4z^2 - 3z - 3)\zeta_2H_0^2 \right]
\[ +\frac{4}{3}(80z^2 + 33z + 246)H_0 - \frac{8}{3}(80z^2 + 469z + 221)\zeta_2H_0 - \frac{16}{9}(44z^2 + 51z - 18)\zeta_3H_0
\[ +\frac{8(z - 1)(6z^2 - z - 6)H_1^3}{3z} + \frac{4(z - 1)(24z^2 - 13z + 17)H_1^2}{3z} - \frac{64(z + 1)^2(2z - 1)H_0,}{3z}
\[ -\frac{8}{15}(188z^2 - 27z - 105)\zeta_2^2 - \frac{4}{3}(z - 1)(56z^2 + 418z - 5)\zeta_2H_1
\[ +\frac{4}{3}(z - 1)(80z^2 - 181z - 9)H_1 + \frac{4}{3}(z - 1)(40z^2 + 33z + 4)\zeta_2H_1
\[ +\frac{8}{3}(56z^3 + 136z^2 - 121z + 18)H_0, + \frac{8}{3}(12z^3 + 15z^2 + 9z + 8)\zeta_2H_0, \frac{32}{3}(z - 1)H_1H_0, + \frac{8}{3}(20z^3 - 239z^2 - 187z + 4)H_0, + \frac{8}{3}(12z^2 - 23z - 22)H_0, + \frac{16}{3}(20z^3 - 21z^2 - 33z + 4)H_0, + \frac{32}{3}(3z^2 + 15z + 8)H_0, + \frac{16}{3}(20z^3 + 15z^2 - 27z - 24)H_0, + \frac{L_M^3 \left[ -\frac{16}{9}(4z^2 + 7z + 4)H_1(z - 1) - \frac{92(z - 1)}{9} \right] \right] - \frac{4}{9}(72z^2 - 1673z - 1009)\zeta_3 \]
\[\begin{align*}
& + L_M \left[ \frac{16}{3} (5z + 2) H_0^3 - \frac{2}{3} (7z^2 - 599z - 207) H_0^2 - \frac{8}{3} (7z^2 - 256z - 135) H_0 \right] \\
& - \frac{8(z - 1)(36z^2 + 25z + 24) H_1 H_0}{3z} + \frac{128(6z + 1) H_{0,1} H_0}{3z} - 256z H_{0,0,-1} H_0 \\
& - 32(8z + 3) \zeta_2 H_0 + 64(z - 3) \zeta_3 H_0 - \frac{4(z - 1)(28z^2 + 21z + 4)}{3z} H_1^2 - \frac{32}{5} (13z + 23) \zeta_2^2 \\
& - \frac{4(z - 1)(268z^2 + 377z - 68)}{3z} + \frac{16}{3} (36z^3 - 51z^2 + 3z - 8) \frac{\zeta_3}{z} - \frac{16(z - 1)(39z^2 - 45z - 13)}{3z} H_1 \\
& + \frac{8(8z^3 - 23z^2 - 43z - 24)}{3z} H_{0,1} + \frac{(z - 1)(4z^2 + 7z + 4)}{z} \left[ \frac{8}{9} H_1^3 + \frac{16}{3} H_0 H_1^2 - \frac{32}{3} H_{0,1} H_1 \right] \\
& - \frac{128(3z^3 + 3z^2 - 9z - 1)}{3z} H_{0,0,-1} - \frac{32}{3} (4z^3 - 12z^2 + 27z + 4) H_{0,0,1} \\
& + \frac{16(12z^3 + 27z^2 + 3z - 8)}{3z} H_{0,1,1,1} + \frac{16}{3} (14z^2 - 128z + 21) \zeta_2 \\
& + \frac{(z + 1)(z^2 - 4z + 1)}{z} \left[ - \frac{128}{3} H_0 H_{0,1}^2 + \frac{(6z + 1) H_0^3 + 256}{3} H_{0,0,-1} H_{0,1} - \frac{128}{3} \zeta_2 H_{0,1} - \frac{256}{3} H_{0,0,-1,1,1} \right] \\
& + \frac{(z - 1)(z^2 + 4z + 1)}{z} \left[ \frac{64}{3} H_1 H_0^2 + \frac{128}{3} H_{0,1} H_{0,0,-1} H_0 - \frac{128}{3} H_1 \zeta_2 \right] + (z - 1) \frac{64}{3} H_{0,0,-1} H_0^2 \\
& - 256 H_{0,0,-1,1} H_0 + 128 H_{0,1,1} - 128 H_{0,1} \zeta_2 \right] + (z + 1) \left[ \frac{2}{9} H_1^4 - \frac{64}{3} H_{0,1} H_0^2 - \frac{2144}{3} H_{1,1} H_0 \\
& + \frac{288}{3} H_{0,0,1} H_0 + \frac{2144}{3} H_{0,0,-1} + 384 H_{0,0,0,-1} - 416 H_{0,0,0,1} - 32 H_{0,1,1,1} \\
& + \frac{128}{3} H_{0,1,1} - 32 H_{0,0,1}^2 \right] \zeta_2 \right] + \frac{(z - 1)(4z^2 + 7z + 4)}{z} \left[ \frac{2}{9} H_1^4 - \frac{16}{3} H_0 H_1^2 + \frac{8}{9} H_0^3 H_1 \\
& + \frac{64}{3} H_{0,0,1} + \frac{64}{3} H_{0,1,1} H_1 - \frac{176}{9} \zeta_3 H_1 + H_0 \left[ \frac{64}{3} H_{0,1,1} - \frac{32}{3} H_1 H_0 \right] \right] + \frac{20}{3} H_1^2 + \frac{16}{3} H_0 H_1 \right] \zeta_2 \right] \\
& + L_M^2 \left[ \frac{8}{3} (4z^2 - 9z - 3) H_0^2 - \frac{8}{3} (z + 1)(32z - 31) H_0 - 16(z - 1)(4z^2 + 7z + 4) H_1 H_0 \right] \\
& - \frac{4(z - 1)(32z^2 + 81z + 12)}{3z} - \frac{8(z - 1)(32z^2 + 35z + 8)}{3z} H_1 + \frac{16(4z^3 + 21z^2 + 9z - 4)}{3z} H_{0,1} \\
& - \frac{32(3z + 2) \zeta_2 + (z + 1) \left[ \frac{16}{3} H_0^3 + 32 H_{0,1} H_0 - 32 \zeta_2 H_0 - 32 H_{0,0,1} + 32 \zeta_3 \right] + (z + 1) \left[ \frac{2}{15} H_0^5 \\
& - \frac{16}{3} H_0 H_0^3 + 48 H_{0,0,1} H_0^2 + \frac{448}{5} \zeta_2 H_0 + \left[ 32 H_{0,1} - 160 H_{0,0,0,1} - 128 H_{0,0,1,1} \right] \right] H_0 \\
& + H_0 \left[ 64 H_{0,0,1} - 128 H_{0,0,1} \right] + 192 H_{0,0,0,1} + 768 H_{0,0,0,1} + 320 H_{0,0,1,0,1} - 416 H_{0,0,1,1,1} \\
& - 192 H_{0,0,1,1} - 32 H_{0,0,1,1,1} + \left[ \frac{88}{3} H_0^3 + \frac{352}{3} H_0 \right] \zeta_3 + \zeta_2 \left[ -4 H_0^3 - 32 H_{0,1} H_0 \right] \\
& - 32 H_{0,0,1} - 80 H_{0,1,1} + \frac{80 \zeta_3}{3} - 160 \zeta_5 \right] \right] \\
& + C_F T_F^2 \left[ - \frac{8}{3} (5z + 3) H_0^3 - 32(3z + 2) H_0^2 - 64(7z + 5) H_0 + \frac{8}{9} (8z^2 - 61z - 31) \zeta_2 H_0 \right] \\
& + L_M^3 \left[ \frac{160}{9} (z + 1) H_0 - \frac{80(z - 1)(4z^2 + 7z + 4)}{27z} \right] + \frac{32(z - 1)(19z^2 + 64z - 5)}{3z} \\
& + \frac{16}{27} (z - 1)(67z^2 + 184z - 41) \frac{\zeta_2}{z} + L_M^2 \left[ - \frac{32(z - 1)(22z^2 + 85z - 32)}{27z} \right]
\end{align*}\]
\[-\frac{64}{9}(2z^2 - 4z - 1)H_0 - \frac{32(z - 1)(4z^2 + 7z + 4)H_1}{9z} + (z + 1)\left[\frac{64}{3}H_0^2 + \frac{64}{3}H_{0,1} - \frac{64\zeta_2}{3}\right]\]
\[+ \frac{(z - 1)(4z^2 + 7z + 4)}{z} \left[\frac{16}{9}H_1\zeta_2 + \frac{80\zeta_3}{27}\right] + (z + 1)\left[-\frac{4}{3}H_0^4 - \frac{160}{9}\zeta_3H_0 + \frac{32\zeta_2^2}{3}\right]\]
\[+ \left[-\frac{56}{3}H_0^2 - \frac{32}{3}H_{0,1}\right]\zeta_2 + L_M \left[-\frac{8}{9}(16z^2 + 13z + 19)H_0^2 - \frac{64}{27}(11z^2 - 13z - 76)H_0\right]
\[= \frac{32(z - 1)(73z^2 + 631z + 73)}{81z} - \frac{32(z - 1)(22z^2 + 85z - 32)H_1}{z} + \frac{(z - 1)(4z^2 + 7z + 4)}{z} \left[-\frac{16}{9}H_1^2 - \frac{64}{9}H_0H_1\right] + \frac{27z}{9z} \left[\frac{64(2z^3 + 7z^2 - 2z - 4)H_0}{11z^2 - 13z - 76}\right]
\[+ \frac{64}{9}(2z^2 - 4z - 1)\zeta_2 + (z + 1)\left[\frac{80}{9}H_0^2 + \frac{128}{3}H_{0,1}H_0 - \frac{128}{3}\zeta_2H_0 - \frac{128}{3}H_{0,0,1} + \frac{64}{3}H_{0,1,1} + \frac{64\zeta_3}{3}\right]\]
\[+ C_F N_F T_F^2 \left[-\frac{16}{9}(5z + 3)H_0^3 - \frac{64}{3}(3z + 2)H_0^2 - \frac{64}{81}(110z^2 + 755z + 518)H_0\right]
\[+ \frac{16}{9}(4z^2 + 37z + 25)\zeta_2H_0 + L_M \left[\frac{128}{9}(z + 1)H_0 - \frac{64(z - 1)(4z^2 + 7z + 4)}{27z}\right]
\[= \frac{16}{27}(8z^3 + 223z^2 + 271z + 14)\zeta_2\zeta_3 + \frac{48}{27}(10z^3 + 36z^2 + 21z - 4)\zeta_3\zeta_2\zeta_3\]
\[= \frac{128(z - 1)(55z^2 + 64z + 28)H_1}{z} - \frac{64}{27}(19z^2 + 67z + 43)H_{0,1}H_0 - \frac{64}{9}(2z^2 + 11z + 8)H_{0,0,1,1}\]
\[+ \frac{(z - 1)(4z^2 + 7z + 4)}{z} \left[-\frac{16}{27}H_1^3 - \frac{16}{9}H_2H_1\right] + \frac{243z}{27z} \left[\frac{64(z^2 + 5z - 41)H_0}{27z}\right]
\[+ \frac{32}{3}H_{0,1}^3 - \frac{16}{3}H_0^2\right]\zeta_2 + L_M \left[-\frac{16}{9}(16z^2 + 43z + 37)H_0^2 - \frac{64}{9}(z^2 + 5z - 41)H_0\right]
\[= \frac{64(z - 1)(7z^2 + 91z + 61)}{27z} - \frac{32(z - 1)(22z^2 + 41z - 28)H_1}{9z} + \frac{64(2z^3 + 3z^2 - 12z - 8)H_{0,0,1,1}}{9z}\]
\[+ \frac{(z - 1)(4z^2 + 7z + 4)}{z} \left[-\frac{16}{3}H_1^3 - \frac{128}{9}H_0H_1\right] + \frac{64}{3}(2z^2 + z + 2)\zeta_2\zeta_3\]
\[+ (z + 1)\left[\frac{32}{3}H_0^3 + \frac{256}{3}H_{0,1}H_0 - \frac{256}{3}\zeta_2H_0 - \frac{256}{3}H_{0,0,1} + \frac{64\zeta_3}{3}\right]\]
\[+ C_A C_F T_F^2 \left[\frac{9}{2}(26z + 5)H_0^4 + \frac{4}{27}(20z^2 + 141z + 174)H_0^3 - \frac{4}{9}(300z^2 - 901z - 169)H_0^2\right]
\[= \frac{8(z - 1)(26z^2 - 32z + 23)H_1H_0^2}{3z} - \frac{8(z^3 + 3z^2 - 8)H_{0,1}H_0^2}{3z} + 32(2z - 1)H_{0,0,1,1}\]
\[+ \frac{8}{3}(32z + 5)\zeta_2H_0^2 + \frac{16}{3}(14z - 13)\zeta_3H_0^2 + \frac{16(z - 1)(19z^2 + 16z + 10)H_0^2}{9z} - \frac{16}{5}(38z - 5)\zeta_2H_0\]
\[+ \frac{4(36604z^3 + 28737z^2 + 2760z + 2624)H_0}{81z} + \frac{4}{9}(298z^3 + 121z^2 + 427z + 80)\zeta_2\zeta_3\zeta_3\]
\[= \frac{16}{9}(145z^2 - 83z - 4)\zeta_3H_0 + \frac{32(z - 1)(461z^2 - 73z + 227)H_1H_0}{27z}
\[= \frac{32(19z^3 - 24z^2 - 6z + 10)H_{0,-1}H_0}{9z} - \frac{16(10z^3 + 249z^2 - 48z + 109)H_{0,1}H_0}{9z}\]

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\[-32(9z^2 - 9z + 8) H_{0,0,0,1} H_0 - \frac{64(5z - 2) H_{0,0,0,1} H_0}{27z} - \frac{8(z - 1)(2z + 1)(14z + 1) H_1^3}{27z}\]
\[-\frac{4(z - 1)(328z^2 + 313z + 67) H_1^2}{27z} - \frac{8 (1732z^3 - 2205z^2 + 972z - 908) H_{0,1}}{27z} - \frac{32(z - 1)^2(z + 2)(2z + 1) H_{0,0,1,1}}{3z} + 128(4z - 1) H_{0,0,0,0,1} + \frac{8}{3}(23z + 14) H_{0,1} \zeta_2\]
\[-\frac{(z + 1)(19z^2 - 16z + 10) H_{0,1}}{3z}\]
\[-\frac{16}{9} H_{0,1} H_{-1}^2 + \left[ \frac{16}{9} H_{0,1}^2 + \frac{64}{9} H_{0,-1} \right] H_{-1} + \frac{32}{9} \zeta_2 H_{-1} + \frac{64}{9} H_{0,-1,-1}\]
\[-\frac{16}{9} (2z - 1) H_0^2 + \frac{16}{9} (8z^2 + 11z + 4) H_0 + \frac{8(z - 1) H_{0,1}}{9z}\]
\[-\frac{8}{3} (38z + 5) H_0^2 - 8\left(\frac{184z^3 + 103z^2 + 205z + 40}{9z}\right) H_0\]
\[-\frac{16(z - 1)(4z^2 + 7z + 4) H_1 H_0}{3z} + \frac{8(517z^3 - 444z^2 + 45z - 127)}{9z} + \frac{16}{3} (4z^3 + 17z^2 + 11z + 4) \zeta_2\]
\[-\frac{8(z - 1)(104z^2 + 119z + 32) H_1}{9z} + \frac{(z + 1)(4z^2 - 7z + 4) \left[ \frac{32}{9} H_{-1} H_0 - \frac{32}{9} H_{0,-1}\right]}{z}\]
\[-\frac{32}{3} (10z + 7) H_{0,1} + \frac{(z + 1) \left[ 32 H_{0,1} H_{0,0,1} - 32 H_{0,0,0,1} \right]}{z} + \frac{(z - 1) \left[ \frac{32}{9} H_0^3 + 64 H_{0,-1} H_0 + 32 \zeta_2 H_0\right]}{z}\]
\[-128 H_{0,0,-1}\]
\[
\frac{16(z - 1)(115z^2 - 17z + 52)H_1H_0}{9z} - \frac{16(32z^3 + 9z^2 - 27z + 16)H_{0,-1}H_0}{3z}
\]
\[
+ \frac{16}{3}(12z^2 + 35z - 40)H_{0,1}H_0 + 64(3z + 1)H_{0,0,-1}H_0 - 192(z + 2)H_{0,0,1}H_0 - 32(2z + 3)\zeta_3H_0
\]
\[
+ \frac{4(z - 1)(20z^2 + 21z + 2)H_1^2}{3z} + \frac{16}{5}(89z + 73)\zeta_2^2 + \frac{8(2021z^3 - 1164z^2 - 1313z + 441)}{9z}
\]
\[
+ \frac{16}{9}(30z^3 + 539z^2 - 334z - 56)\frac{\zeta_2}{z} - \frac{16}{3}(4z^3 - 154z^2 - 49z + 16)\frac{\zeta_3}{z}
\]
\[
+ \frac{16}{3}(z + 1)(4z^2 - 79z + 4)\frac{\zeta_2}{z}H_{-1} + \frac{16(z - 1)(340z^2 - 1007z - 254)H_1}{27z}
\]
\[
+ \frac{32(z - 1)(8z^2 + 35z + 8)\frac{\zeta_2}{z}H_1 + (z + 1)(29z^2 + 259z - 34)}{z}
\]
\[
\left[\frac{32}{9}H_{-1}H_0 - \frac{32}{9}H_{0,-1}\right]
\]
\[
+ \frac{16(85z^3 - 95z^2 + 403z - 64)H_{0,0,1}}{9z}
\]
\[
+ \frac{(z + 1)(4z^2 - 21z + 4)}{z}
\]
\[
\left[16H_0H_{-1}^2 - 32H_{0,-1}H_{-1} + 32H_{0,-1,-1}\right]
\]
\[
+ \frac{16(56z^3 - 33z^2 - 105z + 24)H_{0,0,-1}}{3z}
\]
\[
+ \frac{(z + 1)^3}{z}
\]
\[
\left[\frac{128}{3}H_{-1}H_{0,0,1} - \frac{128}{3}H_{0,-1,-1} - \frac{128}{3}H_{0,1,-1}\right] - \frac{16(8z^3 + 23z^2 + 5z - 8)H_{0,0,1}}{3z}
\]
\[
- 96(3z + 5)H_{0,0,0,-1} + 96(z + 7)H_{0,0,0,1} + (z - 1)\left[-48H_{0,-1}H_0^2 + 448H_{0,-1,-1}H_0 - 224H_{0,-1}\right]
\]
\[
+ 224H_{0,-1}\zeta_2\right) + (z + 1)\left[112H_{0,1}H_0^2 + 96H_{0,1,1}H_0 - 32H_{0,1}^2 - 32H_{0,0,1,1} + 32H_{0,1,1,1} + \left[-96H_0^2 - 256H_{0,1}\right]\zeta_2\right)
\]
\[
+ (z - 1)\left(4z^2 + 7z + 4\right)\left[-\frac{2}{9}H_1^4 - \frac{4}{3}H_1^2H_2^2 + \left[-\frac{80}{3}H_{0,0,1} - \frac{16}{3}H_{0,1,1}\right]H_1 + \frac{80}{9}\zeta_3H_1\right)
\]
\[
+ \frac{16}{3}H_1^3 + \frac{16}{9}H_1^3 + 16H_{0,1}H_1 - \frac{80}{3}H_{0,1,1}\right] + \left[\frac{8}{3}H_0H_1 - \frac{4}{3}H_1^2\zeta_2\right]
\]
\[
+ (z - 1)\left[-\frac{4}{15}H_{0,1}^5\right]
\]
\[
+ \frac{16}{3}H_{0,-1}H_0^3 + \left[32H_{0,-1,-1} - 32H_{0,0,0,1}\right]H_0^2 + \left[-32H_{0,-1}^2 - 128H_{0,-1,-1,-1} + 64H_{0,-1,1}\right]
\]
\[
+ 96H_{0,0,0,-1}\right]H_0 + H_{0,-1}\left[128H_{0,-1,-1} + 64H_{0,0,0,-1} - 128H_{0,0,1}\right] - 256H_{0,-1,0,-1}\]
\[
- 512H_{0,0,0,-1,-1} - 64H_{0,0,0,-1,0} + 128H_{0,0,0,-1,1} - 192H_{0,0,0,-1,1} - 384H_{0,0,0,-1,1}\]
\[
- 128H_{0,0,0,0,1} + 384H_{0,0,0,1,0,1} + 128H_{0,0,0,1,0,1}\right] + \left[-8H_0^3 - 48H_{0,-1}H_0 - 64H_{0,-1}\right]
\]
\[
+ 96H_{0,0,-1}\zeta_3 + (z + 1)\left[16H_{0,1,1}H_0^2 + \left[-48H_0^2 + 128H_{0,0,1} - 64H_{0,1,1}\right]H_0
\]
\[
+ 160H_{0,1}H_{0,0,1} - 864H_{0,0,0,1,1} + 128H_{0,0,1,0,1} + 32H_{0,1,0,1,1} + 32H_{0,1,1,1}\right]
\]
\[
+ \left[-16H_{0,0,1} + 16H_{0,0,1} + 16H_{0,1,1}\right]\zeta_2 - \frac{160}{3}H_{0,1}\zeta_3 - 80(7z - 3)\zeta_5 + a_{gg,Q,reg}^{(3)}\right\}.
\]

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