Convergent Expansions of Eigenvalues of the Generalized Friedrichs Model with a Rank-One Perturbation

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Received: 10 April 2021 / Accepted: 23 August 2021 / Published online: 23 October 2021
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Abstract
We study analytic behavior of eigenvalues of the generalized Friedrichs model $H_\mu(p)$, with a rank-one perturbation, depending on parameters $\mu > 0$ and $p \in \mathbb{T}^2$. Under certain conditions, the existence of a unique eigenvalue lying below the essential spectrum has been shown in Lakaev (Abstract Appl Anal 2012, 2012). Here, we obtain an absolutely convergent expansion for that eigenvalue at $\mu(p)$, the coupling constant threshold. The expansion is dependent to a large extent on whether the lower bound of the essential spectrum is a threshold resonance, a threshold eigenvalue or neither of them.

Keywords Generalized Friedrichs model · Coupling constant threshold · Hamiltonian · Dispersion relation · Threshold resonance · Threshold eigenvalue

Mathematics Subject Classification Primary 81Q10; Secondary 47A10

1 Introduction

The generalized Friedrichs model $H_\mu(p) := H_0(p) - \mu V$, depending on the parameters $p \in \mathbb{T}^2$ and $\mu > 0$, is investigated for the perturbation of rank one (see also [2] and the references therein). These are generalizations of the two-particle Schrödinger...
operators \( H_\mu(k) = H_0(k) + \mu V \) with a fixed quasi-momentum \( k \in \mathbb{T}^d = (-\pi, \pi]^d \) considered in [3] and [4].

The Friedrichs model [5], being mathematically solvable, is one of the best tools to describe quantum decay in physics (see [6,7] and [8] for detailed reviews of applications of the Friedrichs models in various physical and mathematical problems).

Another important aspect of studying the generalized Friedrichs models is that, they describe the Hamiltonians for systems of both bosons and fermions (see, e.g., [9,10]). The works [11,12] and [3] on the Efimov effect for three quantum mechanical particles on lattices reveal the existence of the super Efimov effect for a system of three spinless fermions on \( \mathbb{Z}^2 \). There is an interesting feature of the super Efimov effect, that a system of three spinless fermions on the lattice \( \mathbb{Z}^2 \) depending on the quasi-momentum may have an infinite number of bound states, however there are at most a finite number negative energy levels for the same system in the three dimensional lattice (see, e.g., [13]).

We study the behavior of eigenvalues of the generalized Friedrichs model \( H_\mu(p), \mu > 0 \) in the important case of \( d = 2 \) (see, e.g., [14,15]) at \( \mu(p) \), the coupling constant threshold (c.c.th.). The phenomenon c.c.th. is used for \( \mu_0 \geq 0 \), which as \( \mu \) tends to \( \mu_0 \) the corresponding eigenvalue is absorbed into the threshold of continuum (in our case the bottom of the essential spectrum), and conversely, as \( \mu \) seeks to \( \mu_0 + \epsilon, \epsilon > 0 \), the continuum gives birth to a new eigenvalue (see [16,17] and [18]).

Absolutely convergent expansions (asymptotics) of the eigenvalues are found explicitly for each of the following cases: the threshold \( m(p) \) is a threshold resonance, a threshold eigenvalue or neither a threshold resonance nor a threshold eigenvalue (see Theorem 2.7). Note that expansions of the eigenvalues \( E(\mu, p) \) lying above the essential spectrum can be derived analogously at the coupling constant threshold \( \mu(p) \geq 0 \), for the operator, which is associated to a system of two fermions (see, (ii) and (iii) of Theorem 2.7) [10].

The paper is organized as follows: Sect. 2 is devoted to preliminaries and main results. The proofs of the main results are given in Sect. 3.

### 2 Preliminaries and Main Results

We use the following notations throughout the paper: \( \mathbb{Z}^2 \) is the two-dimensional hypercubic lattice, \( \mathbb{T}^2 = (\mathbb{R}/2\pi \mathbb{Z})^2 = (-\pi, \pi]^2 \) is the two-dimensional torus (Brillion zone), the dual group of \( \mathbb{Z}^2 \), \( L^2(\mathbb{T}^2) \) is the Hilbert space of square-integrable functions defined on the torus \( \mathbb{T}^2 \).

Let \( w \) be a real-valued analytical function on \( (\mathbb{T}^2)^2 \), and \( \varphi \in L^2(\mathbb{T}^2) \).
Consider the generalized Friedrichs model $H_\mu(p)$, $p \in \mathbb{T}^2$ acting in $L^2(\mathbb{T}^2)$ defined as

$$H_\mu(p) = H_0(p) - \mu V, \quad \mu > 0,$$

where $H_0(p)$, $p \in \mathbb{T}^2$ is a multiplication operator by the function $w_p(\cdot) := w(p, \cdot)$:

$$(H_0(p)f)(q) = w_p(q)f(q), \quad f \in L^2(\mathbb{T}^2).$$

and $V : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ is the perturbation operator of the form

$$(Vf)(q) = \varphi(q)(f, \varphi),$$

where $(\cdot, \cdot)$ stands for the inner product in $L^2(\mathbb{T}^2)$.

The perturbation $V$ of the non-perturbed operator $H_0(p)$, $p \in \mathbb{T}^2$ is a positive operator of rank one. Consequently, by the well-known Weyl theorem [19, Theorem XIII.14] on the compact perturbations

$$\sigma_{\text{ess}}(H_\mu(p)) = \sigma_{\text{ess}}(H_0(p)) = \sigma(H_0(p)) = [m(p), M(p)],$$

where

$$m(p) = \min_{q \in \mathbb{T}^2} w_p(q), \quad M(p) = \max_{q \in \mathbb{T}^2} w_p(q).$$

**Remark 2.1** The positivity of the perturbation operator $V$ yields the absence of eigenvalues of the operator $H_\mu(p)$ lying above $M(p)$, the upper bound of the essential spectrum.

Let us introduce a hypothesis, which we always assume to hold in presenting our results.

**Hypothesis 2.2** The functions $\varphi(\cdot)$ and $w(\cdot, \cdot)$, used in the definition of the operator $H_\mu(p)$ satisfy the following conditions:

(i) $\varphi(\cdot)$ is nontrivial and real-analytic on $\mathbb{T}^2$;
(ii) $w(\cdot, \cdot)$ is real-analytic on $(\mathbb{T}^2)^2 = \mathbb{T}^2 \times \mathbb{T}^2$ and has a unique non-degenerate minimum at $(0, 0) \in (\mathbb{T}^2)^2$.

Hypothesis 2.2 implies the existence of a $\delta$-neighborhood $U_\delta(0) \subset \mathbb{T}^2$ of the point $0 \in \mathbb{T}^2$ and of an analytic function $q_0 : U_\delta(0) \to \mathbb{T}^2$ that for any $p \in U_\delta(0)$ the point $q_0(p) \in \mathbb{T}^2$ is a unique non-degenerate minimum of the function $w_p(\cdot)$.

For any $\mu > 0$ and $p \in \mathbb{T}^2$, we define an analytic function $\Delta(\mu, p; \cdot)$ (the Fredholm determinant, associated to the operator $H_\mu(p)$) in $\mathbb{C}\backslash[m(p), M(p)]$ as

$$\Delta(\mu, p; \cdot) = 1 - \mu \Omega(p; \cdot),$$
where
\[
\Omega(p; z) = \int_{\mathbb{T}^2} \frac{\varphi^2(q) dq}{w_p(q) - z}, \quad p \in \mathbb{T}^2, \quad z \in \mathbb{C}\setminus[m(p), M(p)].
\] (2.1)

In our proofs we apply the results of Lemmas 3.1 and 3.7 of [1] and that is why, for readers’ convenience, we recall these results as a lemma.

**Lemma 2.3** Assume Hypothesis 2.2.

(i) For any \( \mu > 0 \) and \( p \in \mathbb{T}^2 \), a number \( z \in \mathbb{C}\setminus\sigma_{ess}(H_\mu(p)) \) is an eigenvalue of the operator \( H_\mu(p) \) if and only if
\[
\Delta(\mu, p; z) = 0.
\]
The corresponding eigenfunction \( f \) is of the form
\[
f_{\mu, p}(q) = \frac{C \mu \varphi(q)}{w_p(q) - z},
\]
and is analytic on \( \mathbb{T}^2 \), where \( C = C(p) > 0 \) is the normalizing constant.

(ii) Let \( q_0(p), \ p \in U_\delta(0) \) be a unique non-degenerate minimum point of the function \( w_p \), and let \( \varphi(q_0(p)) = 0 \) and \( \nabla \varphi(q_0(p)) \neq 0 \) (resp. \( \nabla \varphi(q_0(p)) = 0 \)). Then
\[
\Delta(\mu, p; m(p)) = 0
\]
if and only if \( z = m(p) \) is a threshold resonance (resp. an eigenvalue) for the operator \( H_\mu(p) \), \( \mu > 0 \), i.e., the equation
\[
H_\mu(p)f = m(p)f
\]
has a nonzero solution
\[
f_{\mu, p}(\cdot) = \frac{C \mu \varphi(\cdot)}{w_p(\cdot) - m(p)},
\]
which belongs to \( L_1(\mathbb{T}^2) \setminus L_2(\mathbb{T}^2) \) (resp. \( L_2(\mathbb{T}^2) \)), where \( C = C(p) > 0 \).

**Definition 2.4** For \( p \in U_\delta(0) \), we define the number \( \mu(p) > 0 \) as
\[
\mu(p) = \left( \int_{\mathbb{T}^2} \frac{\varphi^2(q) dq}{w_p(q) - m(p)} \right)^{-1} > 0 \quad \text{if} \quad \varphi(q_0(p)) = 0
\]
and \( \mu(p) = 0 \) if \( \varphi(q_0(p)) \neq 0 \).
Remark 2.5 Note that in the case of \( \varphi(q_0(p)) = 0 \), the existence of the integral in (2.2) is proven in [1]. The defined number \( \mu(p) > 0 \) is called the coupling threshold constant.

In the next theorem we recall, for readers’ convenience, the existence criterion of a unique eigenvalue of \( H_\mu(p) \) below \( m(p) \), \( p \in U_\delta(0) \) ([1, Theorem 2.3]).

**Theorem 2.6** Assume Hypothesis 2.2. Then for any fixed \( p \in U_\delta(0) \), the operator \( H_\mu(p) \) has a unique eigenvalue \( E(\mu, p) \) below \( m(p) \) if and only if \( \mu > \mu(p) \). Moreover, if \( \mu = \mu(p) \), \( \varphi(q_0(p)) = 0 \) and \( \nabla \varphi(q_0(p)) \neq 0 \) (resp. \( \nabla \varphi(q_0(p)) = 0 \)), then the threshold \( m(p) \) is a virtual level (resp. an eigenvalue) of the operator \( H_\mu(p) \).

Next, we present the main result of the current paper, where an absolutely convergent expansion for the eigenvalue \( E(\mu, p) \) at the coupling constant threshold \( \mu(p) \) defined in (2.2) is obtained in the cases when the threshold \( m(p) \) is a threshold resonance, a threshold eigenvalue or neither of them.

**Theorem 2.7** Assume Hypothesis 2.2. Then for any fixed \( p \in U_\delta(0) \), \( \mu \) tends to \( \mu(p) \) if and only if \( E(\mu, p) \) tends to the threshold \( m(p) \). Moreover, for any fixed \( p \in U_\delta(0) \) and sufficiently small positive \( \mu - \mu(p) \), the unique eigenvalue \( E(\mu, p) \) of the operator \( H_\mu(p) \) has the following absolutely convergent expansions, which are of the different forms depending on the values of \( \varphi(q_0(p)) \) and \( \nabla \varphi(q_0(p)) \):

Case (i). If \( \varphi(q_0(p)) \neq 0 \), then

\[
E(\mu, p) = m(p) - a(p)e^{(\alpha_0(p)\mu)^{-1}} - \sum_{m \geq 0, n \geq 1, m+n \geq 2} c(m, n)(p)\mu^m \tau^n,
\]

where

\[
\tau = e^{(\alpha_0(p)\mu)^{-1}}, \quad a(p) = e^{-c_0(p)/\alpha_0(p)}, \quad \alpha_0(p) < 0
\]

and \( c_0(p) \), \( c(n, m)(p) \), \( m, n = 0, 1, 2, ... \) are real numbers.

Case (ii) If \( \varphi(q_0(p)) = 0 \) and \( \nabla \varphi(q_0(p)) \neq 0 \), then

\[
E(\mu, p) = m(p) - [\hat{\alpha}_1(p)\mu^2(p)]^{-1}\frac{\hat{\mu}}{\ln \hat{\mu}^{-1}} - \sum_{n \geq 1, r \geq 1, s \geq 2, n+r+s \geq 3} c(n, s)(p)\tau^n \hat{\mu}^r \omega^s,
\]

where \( \hat{\alpha}_1(p) > 0 \) is defined in (3.3), \( c(n, r, s)(p) \), \( n, r, s = 0, 1, 2, ... \) are real numbers and

\[
\tau = \frac{1}{\ln \hat{\mu}^{-1}}, \quad \omega = \frac{\ln \ln \hat{\mu}^{-1}}{\ln \hat{\mu}^{-1}}, \quad \hat{\mu} = \mu - \mu(p).
\]
Case (iii) If $\varphi(q_0(p)) = 0$ and $\nabla\varphi(q_0(p)) = 0$, then

$$E(\mu, p) = m(p) - a(p)\hat{\mu} - \sum_{l \geq 0, s \geq 1, l+s \geq 2} \hat{c}(l, s)(p) \tau^l \hat{\mu}^s$$

(2.3)

$$\tau = \hat{\mu} \ln \hat{\mu}, \quad \hat{\mu} = \mu - \mu(p),$$

where $a(p) > 0$ and $\hat{c}(l, s)(p), l, s = 0, 1, 2, ...$ are real numbers.

The asymptotics of the eigenvalue $E(\mu, p)$ at the coupling constant threshold $\mu(p) \geq 0$ is given in the following corollary.

**Corollary 2.8** Assume Hypothesis 2.2. For any fixed $p \in U_\delta(0)$, the following asymptotics relation holds for the difference $m(p) - E(\mu, p)$:

(i) If $\varphi(q_0(p)) \neq 0$, then

$$m(p) - E(\mu, p) = a(p)e^{(\alpha_0(p)\mu)^{-1}} + O([\mu \tau]) \quad \text{as} \quad \mu \to 0,$$

$$\tau = e^{(\alpha_0(p)\mu)^{-1}}, \quad a(p) = e^{-c_0(p)/\alpha_0(p)}, \quad \alpha_0(p) < 0.$$

(ii) If $\varphi(q_0(p)) = 0$ and $\nabla\varphi(q_0(p)) \neq 0$, then

$$m(p) - E(\mu, p) = [\hat{\alpha}_1(p)\mu^2(p)]^{-1} \frac{\hat{\mu}}{\ln \hat{\mu}^{-1}} + O([\hat{\mu} \tau]) \quad \text{as} \quad \mu \to \mu(p),$$

where $\hat{\alpha}_1(p) > 0$ and

$$\tau = \frac{1}{\ln \hat{\mu}^{-1}}, \quad \omega = \frac{\ln \hat{\mu}^{-1}}{\ln \hat{\mu}^{-1}}, \quad \hat{\mu} = \mu - \mu(p).$$

(iii) If $\varphi(q_0(p)) = 0$ and $\nabla\varphi(q_0(p)) = 0$, then

$$m(p) - E(\mu, p) = a(p)\hat{\mu} + O([\hat{\mu} \tau]) \quad \text{as} \quad \mu \to \mu(p),$$

$$\tau = \hat{\mu} \ln \hat{\mu}, \quad \hat{\mu} = \mu - \mu(p),$$

where $a(p) > 0$.

### 3 Proof of the Results

Hypothesis 2.2 and the parametric Morse lemma yield the existence, for each $p \in U_\delta(0)$, of the map $\psi(y, p)$ of the sphere $W_\gamma(0) \subset \mathbb{R}^2$ to a neighborhood $U(q_0(p))$ of the point $q_0(p) = (q_1^0(p), q_2^0(p)) \in \mathbb{T}^2$ such that the function $w_p(\psi(y, p))$ can be represented as

$$w_p(\psi(y, p)) = m(p) + y^2 = m(p) + y_1^2 + y_2^2.$$
The functions $\psi(y, \cdot)$ and $\psi(\cdot, p)$ are holomorphic in $U_{\delta}(0)$ and $W_\gamma(0)$, respectively, and $\psi(0, p) = q_0(p)$. Moreover, the Jacobian $J(\psi(y, p))$ of the mapping $\psi(y, p)$ is positive, i.e.,

$$J(\psi(y, p)) = \left| \begin{array}{cc} \frac{\partial \psi_1}{\partial y_1}(y, p) & \frac{\partial \psi_1}{\partial y_2}(y, p) \\ \frac{\partial \psi_2}{\partial y_1}(y, p) & \frac{\partial \psi_2}{\partial y_2}(y, p) \end{array} \right| > 0 \tag{3.1}$$

for all $p \in U_{\delta}(0)$ and $y \in W_\gamma(0)$.

Now we establish an expansion for $\Delta(\mu, p; z)$ in the half-neighborhood $(m(p) - \varepsilon, m(p))$ of the point $m(p)$, which plays an important role in the proof of the main results.

**Lemma 3.1** Assume Hypothesis 2.2. Then, for a sufficiently small $m(p) - z > 0$, the function $\Delta(\mu, p; \cdot), \mu > 0, p \in U_{\delta}(0)$ can be represented as the following convergent series:

(i) if $\varphi(q_0(p)) \neq 0$, then

$$\Delta(\mu, p; z) = 1 - \mu \alpha_0(p) \ln(m(p) - z) - \mu \ln(m(p) - z)$$

$$\times \sum_{n=1}^{\infty} \alpha_n(p)(m(p) - z)^n - \mu F(p, z),$$

where $\ln(\cdot)$ is the branch of function $\ln$ assuming the real values for $m(p) - z > 0$, $\alpha_0(p) = -\frac{1}{2} \varphi^2(q_0(p)) J(q_0(p))$, the coefficients $\alpha_1(p), \alpha_2(p), \ldots$ are real numbers and

$$F(\mu, z) = \sum_{n=0}^{\infty} c_n(p)(m(p) - z)^n,$$

with real coefficients $c_0(p), c_1(p), c_2(p), \ldots$.

(ii) if $\varphi(q_0(p)) = 0$ and $\nabla \varphi(q_0(p)) = \left( \frac{\partial \varphi}{\partial q_1}(q_0(p)), \frac{\partial \varphi}{\partial q_2}(q_0(p)) \right) \neq 0$, then

$$\Delta(\mu, p; z) = 1 - \frac{\mu}{\mu(p)} - \mu \ln(m(p) - z) \sum_{n=1}^{\infty} \hat{\alpha}_n(p)(m(p) - z)^n - \mu \hat{F}(p, z), \tag{3.2}$$

where

$$\hat{\alpha}_1(p) = \frac{\pi}{2} J(q_0(p)) \left\{ \left[ \frac{\partial \varphi}{\partial q_1}(q_0(p)) \frac{\partial \psi_1}{\partial y_1}(0, p) + \frac{\partial \varphi}{\partial q_2}(q_0(p)) \frac{\partial \psi_2}{\partial y_1}(0, p) \right]^2 \\
+ \left[ \frac{\partial \varphi}{\partial q_1}(q_0(p)) \frac{\partial \psi_1}{\partial y_2}(0, p) + \frac{\partial \varphi}{\partial q_2}(q_0(p)) \frac{\partial \psi_2}{\partial y_2}(0, p) \right]^2 \right\} > 0, \tag{3.3}$$
\[ \hat{\alpha}_n(p), \ n = 2, 3, \ldots \text{ are real numbers and} \]
\[ \hat{F}(p, z) = \sum_{n=1}^{\infty} \hat{c}_n(p) (m(p) - z)^n \]

with real coefficients \( \hat{c}_n(p), \ n = 1, 2, \ldots \)

\[ \text{Remark 3.2} \] We remark that, if we assume that \( \varphi(q_0(p)) = 0 \) and \( \nabla \varphi(q_0(p)) = 0 \) in the part (ii), then all assertions of Lemma 3.1, except for the inequality \( \hat{\alpha}_1(p) > 0 \), hold.

\[ \text{Proof} \] The part (i) can be proven as Lemma 3.6 in [1]. Therefore, we only prove the part (ii).

We represent the function (2.1) as a sum of functions

\[ \Omega(p, z) = \Omega_1(p, z) + \Omega_2(p, z), \quad (3.4) \]

where

\[ \Omega_1(p, z) = \int_{U(q_0(p))} \frac{\varphi^2(q) dq}{w_p(q) - z}, \quad \Omega_2(p, z) = \int_{\mathbb{T}^2 \setminus U(q_0(p))} \frac{\varphi^2(q) dq}{w_p(q) - z}, \quad (3.5) \]

and \( U(q_0(p)) \in \mathbb{T}^2 \) is a sufficiently small neighborhood of the minimum point \( q_0(p) \in \mathbb{T}^2 \).

Since \( m(p) \) is a unique minimum of the function \( w_p \), one concludes that for any \( p \in U_\delta(0) \) the function \( \Omega_2(p, \cdot) \) is analytic in some neighborhood of the point \( z = m(p) \).

Taylor’s expansion of the function \( \varphi^2 \) in the neighborhood \( U(q_0(p)) \) of \( q_0(p) = (q_1^0(p), q_2^0(p)) \) can be written as

\[ \varphi^2(q) = a_1^2(p)(q_1 - q_1^0(p))^2 + a_2^2(p)(q_2 - q_2^0(p))^2 + 2a_1(p)a_2(p)(q_1 - q_1^0(p))(q_2 - q_2^0(p)) \]
\[ + \sum_{n=3}^{\infty} \sum_{i_1, i_2, \ldots, i_n = 1}^{2} a_{i_1i_2\ldots i_n}(p) \prod_{k=1}^{n}(q_{i_k} - q_{i_k}^0(p)), \]

where \( a_1(p) = \frac{\partial \varphi}{\partial q_1}(q_0(p)), \quad a_2(p) = \frac{\partial \varphi}{\partial q_2}(q_0(p)) \) and \( a_{i_1i_2\ldots i_n}(p), \ i_1, i_2, \ldots, i_n = 1, 2, \ldots \text{ are real numbers.} \)

Then, (3.5) can be represented as

\[ \Omega_1(p, z) = \Omega_{11}(p, z) + \Omega_{12}(p, z) + \Omega_{13}(p, z) + \Omega_{14}(p, z), \quad (3.6) \]
where

\[ \Omega_{11}(p, z) = a_1^2(p) \int_{U(q_0(p))} \frac{(q_1 - q_1^0(p))^2 dq}{w_p(q) - z}, \quad \Omega_{12}(p, z) = a_2^2(p) \int_{U(q_0(p))} \frac{(q_2 - q_2^0(p))^2 dq}{w_p(q) - z}, \quad (3.7) \]

\[ \Omega_{13}(p, z) = 2a_1(p)a_2(p) \int_{U(q_0(p))} \frac{(q_1 - q_1^0(p))(q_2 - q_2^0(p))dq}{w_p(q) - z} , \]

\[ \Omega_{14}(p, z) = \sum_{n=3}^{\infty} \sum_{i_1, i_2, \ldots, i_n=1}^2 a_{i_1i_2\ldots i_n}(p) \int_{U(q_0(p))} \frac{\prod_{k=1}^{n} (q_i - q_i^0(p))}{w_p(q) - z} dq. \quad (3.8) \]

Making a change of variables as \( q = \psi(y, p) \) in the first integral of (3.7) yields

\[ \Omega_{11}(p, z) = a_1^2(p) \int_{W_{\psi}(0)} \frac{(\psi_1(y, p) - q_1^0(p))^2 J(\psi(y, p))dy}{y^2 + m(p) - z}. \quad (3.9) \]

The holomorphicity of the functions \( \psi(y, p) \) and \( J(\psi(y, p)) \) in \( W_{\psi}(0) \) imply the following expansions

\[ \psi_1(y, p) = q_1^0(p) + b_1(p)y_1 + b_2(p)y_2 + \sum_{k,l \in \mathbb{N}, k+l=4} b_{kl}(p)y_1^{k-1}y_2^{l-1}, \]

\[ J(\psi(y, p)) = J(q_0(p)) + \sum_{k,l \in \mathbb{N}, k+l=3} d_{kl}(p)y_1^{k-1}y_2^{l-1}, \]

and consequently

\[ (\psi_1(y, p) - q_1^0(p))^2 J(\psi(y, p)) = b_1^2(p)J(q_0(p))y_1^2 + b_2^2(p)J(q_0(p))y_2^2 \]

\[ + 2b_1(p)b_2(p)J(q_0(p))y_1y_2 + \sum_{k,l \in \mathbb{N}, k+l=5} g_{kl}(p)y_1^{k-1}y_2^{l-1}, \quad (3.10) \]

where

\[ b_1(p) = \frac{\partial \psi_1}{\partial y_1}(0, p), \quad b_2(p) = \frac{\partial \psi_1}{\partial y_2}(0, p), \]
and \( g_{kl}(p) \), \( k, l \in \mathbb{N} \) are real numbers. By (3.9), for \( \Omega_{11}(p, z) \) we get the representation

\[
\Omega_{11}(p, z) = \Omega_{11}^{(1)}(p, z) + \Omega_{11}^{(2)}(p, z) + \Omega_{11}^{(3)}(p, z) + \Omega_{11}^{(4)}(p, z),
\]

(3.11)

where

\[
\begin{align*}
\Omega_{11}^{(1)}(p, z) &= a_1^2(p)b_1^2(p)J(q_0(p)) \int_{W_{\gamma}(0)} \frac{y_1^2 dy}{y^2 + m(p) - z}, \\
\Omega_{11}^{(2)}(p, z) &= a_1^2(p)b_2^2(p)J(q_0(p)) \int_{W_{\gamma}(0)} \frac{y_2^2 dy}{y^2 + m(p) - z}, \\
\Omega_{11}^{(3)}(p, z) &= 2a_1^2(p)b_1(p)b_2(p)J(q_0(p)) \int_{W_{\gamma}(0)} \frac{y_1y_2 dy}{y^2 + m(p) - z},
\end{align*}
\]

(3.12)

and

\[
\Omega_{11}^{(4)}(p, z) = \sum_{k, l \in \mathbb{N}, k + l = 5} g_{kl}(p) \int_{W_{\gamma}(0)} \frac{y_1^{k-1}y_2^{l-1} dy}{y^2 + m(p) - z}.
\]

(3.13)

In polar coordinates with \( y_1 = r \cos \alpha, y_2 = r \sin \alpha, 0 \leq r \leq \gamma, 0 \leq \alpha \leq 2\pi \), (3.12) can be rewritten as

\[
\Omega_{11}^{(1)}(p, z) = \pi a_1^2(p)b_1^2(p)J(q_0(p)) \int_{0}^{\gamma} \frac{r^3 dr}{r^2 + m(p) - z}.
\]

(3.14)

Recall that for any \( \xi < 0 \) and \( n \in \mathbb{N} \) the following equality holds

\[
I_n(\xi) = \int_{0}^{\delta} \frac{r^{2n+1} dr}{r^2 - \xi} = -\frac{1}{2} \xi^n \ln(-\xi) + \hat{I}_n(\xi),
\]

(3.15)

where \( \hat{I}_n(\xi) \) is a regular function in some neighborhood of the origin [20, Lemma 5]. Therefore,

\[
\begin{align*}
\Omega_{11}^{(1)}(p, z) &= \pi a_1^2(p)b_1^2(p)J(q_0(p))(m(p) - z) \ln(m(p) - z) \\
&\quad + \sum_{n=0}^{\infty} \xi_n(p)(m(p) - z)^n,
\end{align*}
\]

(3.16)
where $\xi_n(p), n = 1, 2, \ldots$ are real numbers. Analogously, for $\Omega_{11}^{(2)}(p, z)$, we have the expansion

$$
\Omega_{11}^{(2)}(p, z) = \frac{\pi}{2} a_1^2(p) b_2^2(p) J(q_0(p))(m(p) - z) \ln(m(p) - z)
$$

$$
+ \sum_{n=0}^{\infty} \eta_n(p)(m(p) - z)^n
$$

(3.17)

and the equality $\Omega_{11}^{(3)}(p, z) = 0$. In polar coordinates with $y_1 = r \cos \alpha, y_2 = r \sin \alpha,$
$0 \leq r \leq \gamma, 0 \leq \alpha \leq 2\pi$, we write (3.13) as

$$
\Omega_{11}^{(4)}(p, z) = \sum_{k,l \in \mathbb{N}, k+l=5} g_{kl}(p) \int_0^{\gamma} \frac{r^{k+l-1} dr}{r^2 + m(p) - z} \int_0^{2\pi} \cos^{k-1} \alpha \sin^{l-1} \alpha d\alpha.
$$

Note that the integral

$$
\int_0^{2\pi} \cos^{k-1} \alpha \sin^{l-1} \alpha d\alpha
$$

is nonzero if both $k$ and $l$ are odd, otherwise it turns to zero. Therefore, the function $\Omega_{11}^{(4)}(p, z)$ is represented as

$$
\Omega_{11}^{(4)}(p, z) = \sum_{n=2}^{\infty} \hat{g}_n(p) \int_0^{\gamma} \frac{r^{2n+1} dr}{r^2 + m(p) - z}.
$$

The equality (3.15) yields

$$
\Omega_{11}^{(4)}(p, z) = \ln(m(p) - z) \sum_{n=2}^{\infty} q_n(p)(m(p) - z)^n + \sum_{n=0}^{\infty} \hat{\theta}_n(p)(m(p) - z)^n,
$$

(3.18)

where $q_n(p)$ and $\theta_n(p)$ are real numbers. Taking into account the expressions (3.16), (3.17), (3.18) and (3.11), we have the following expansion

$$
\Omega_{11}(p, z) = \ln(m(p) - z) \sum_{n=1}^{\infty} d_n(p)(m(p) - z)^n + \sum_{n=0}^{\infty} \hat{d}_n(p)(m(p) - z)^n,
$$

where

$$
d_1(p) = \frac{\pi}{2} a_1^2(p) J(q_0(p))(b_1^2(p) + b_2^2(p)).
$$
Analogously can be found the expansions

\[
\Omega_{12}(p, z) = \ln(m(p) - z) \sum_{n=1}^{\infty} e_n(p)(m(p) - z)^n + \sum_{n=0}^{\infty} \hat{e}_n(p)(m(p) - z)^n,
\]

\[
\Omega_{13}(p, z) = \ln(m(p) - z) \sum_{n=1}^{\infty} f_n(p)(m(p) - z)^n + \sum_{n=0}^{\infty} \hat{f}_n(p)(m(p) - z)^n.
\]

Here \(e_n(p), \hat{e}_n(p), f_n(p)\) and \(\hat{f}_n(p)\) are real numbers with

\[
e_1(p) = \frac{\pi}{2} a_2^2(p) J(q_0(p))(l_1^2(p) + l_2^2(p)),
\]

\[
f_1(p) = \pi a_1(p) a_2(p) J(q_0(p))(b_1(p) l_1(p) + b_2(p) l_2(p)),
\]

where \(l_1(p) = \frac{\partial \psi}{\partial y_1}(0, p)\) and \(l_2(p) = \frac{\partial \psi}{\partial y_2}(0, p)\).

Making a change of variables \(q = \psi(y, p)\) in (3.8) we find

\[
\Omega_{14}(p, z) = \sum_{n=3}^{\infty} \sum_{i_1, i_2, \ldots, i_n=1}^{2} a_{i_1 i_2 \ldots i_n}(p) \int_{W_p(0)}^{\infty} \prod_{k=1}^{n} \frac{(\psi_{ik}(y, p) - q_{ik}^0(p)) J(\psi(y, p))}{y^2 + m(p) - z} dy.
\]

Expanding the function \((\psi_{ik}(y, p) - q_{ik}^0(p)) J(\psi(y, p))\) at the point \(y = 0\), as in (3.10) allows us to write the last equation as

\[
\Omega_{14}(p, z) = \sum_{k, l \in \mathbb{N}, k+l=5}^{\infty} \hat{a}_{kl}(p) \int_{W_p(0)}^{\infty} \frac{\gamma_1^{k-1} \gamma_2^{l-1}}{y^2 + m(p) - z} dy,
\]

\(\hat{a}_{kl}(p), k, l \in \mathbb{N}\) are real numbers. Then we obtain the following expansion

\[
\Omega_{14}(p, z) = \ln(m(p) - z) \sum_{n=2}^{\infty} g_n(p)(m(p) - z)^n + \sum_{n=0}^{\infty} \hat{g}_n(p)(m(p) - z)^n.
\]

where \(g_n(p)\) and \(\hat{g}_n(p)\) are real numbers.

The expansions of the functions \(\Omega_{11}(p, z), \Omega_{12}(p, z), \Omega_{13}(p, z), \Omega_{14}(p, z),\) and the equations (3.4) and (3.6) give

\[
\Omega(p, z) = \ln(m(p) - z) \sum_{n=1}^{\infty} \hat{a}_n(p)(m(p) - z)^n + \sum_{n=0}^{\infty} \hat{c}_n(p)(m(p) - z)^n,
\]

(3.19)
Theorem 1]. As every concave function on $\mathbb{R}$, where

Next we show that $\hat{a}(p) \neq 0$. Assume the converse, i.e.,

$$
\begin{cases}
a_1(p)b_1(p) + a_2(p)l_1(p) = 0 \\
a_1(p)b_2(p) + a_2(p)l_2(p) = 0.
\end{cases}
$$

(3.20)

Since, $a_1(p) = \frac{\partial \varphi}{\partial q_1}(q_0(p))$ and $a_2(p) = \frac{\partial \varphi}{\partial q_2}(q_0(p))$ by the assumption (ii) of Lemma 3.1, at least one of these two numbers $a_1(p)$ or $a_2(p)$ is not zero. Assume $a_1(p) \neq 0$. Then, multiplying the first equation of the system (3.20) by $l_2(p)$, the second one by $l_1(p)$ and subtracting them term by term we obtain the equality

$$
b_1(p)l_2(p) = b_2(p)l_1(p),$$

which contradicts to the inequality (3.1). Thus $\hat{a}(p) > 0$. In the case $z \to m(p)$, from (3.19), we get $\hat{c}(0) = 1/\mu(p)$, which completes the proof of Lemma 3.1.

The part (ii) of Lemma 3.1 yields

Corollary 3.3 If $\varphi(q_0(p)) = \nabla \varphi(q_0(p)) = 0$, then $\hat{a}(p) = 0$.

Lemma 3.4 Assume Hypothesis 2.2, $p \in U_\delta(0)$ and $\varphi(q_0(p)) = 0$. Then for the coefficients $\hat{a}(p)$ and $\hat{c}(p)$ in the expansion (3.2), the following relation holds

$$
|\hat{a}(p)| + |\hat{c}(p)| \neq 0.
$$

Proof Assume the converse, that is $\hat{a}(p) = \hat{c}(p) = 0$. Then Lemma 2.3 and the equality (3.2) imply

$$
-\frac{\mu - \mu(p)}{\mu(p)} - \mu \ln(m(p) - E(\mu, p)) \sum_{n=2}^{\infty} \hat{a}_n(p) (m(p) - E(\mu, p))^n
$$

$$
= -\mu \sum_{n=2}^{\infty} \hat{c}_n(p) (m(p) - E(\mu, p))^n = 0. \quad (3.21)
$$

For each $p \in \mathbb{T}^2$, the eigenvalue $E(\mu, p)$ is a concave function of $\mu \geq 0$ (see [21, Theorem 1]). As every concave function on $\mathbb{R}$ has a finite right derivatives, the limit

$$
\lim_{\mu \to \mu(p)^+} \frac{E(\mu, p) - m(p)}{\mu - \mu(p)}
$$

exists and is finite. Therefore,

$$
m(p) - E(\mu, p) = C(\mu - \mu(p)) + o(\mu - \mu(p)), \quad \mu \to \mu(p), \quad 0 \leq C < \infty.
Consequently, using (3.21) we have

\[-\mu \ln[C(\mu - \mu(p)) + o(\mu - \mu(p))] \sum_{n=2}^{\infty} \hat{\alpha}_n(p) [C(\mu - \mu(p)) + o(\mu - \mu(p))]^{n-1}
- \mu \sum_{n=2}^{\infty} \hat{c}_n(p) [C(\mu - \mu(p)) + o(\mu - \mu(p))]^{n-1} = \frac{1}{\mu(p)} \quad \text{as} \quad \mu \to \mu(p).\]

When \(\mu \to \mu(p)\), left hand side of the equation turns to zero and we obtain \(1/\mu(p) = 0\). This contradiction shows that \(|\hat{\alpha}_1(p)| + |\hat{c}_1(p)| \neq 0\).

We are now ready to prove the main results.

**Proof of Theorem 2.7** For convenience, we introduce the notation \(\mu(p, z) = \Omega^{-1}(p, z) > 0, p \in U_b(0), z \in (-\infty, m(p))\). The function \(\mu(p, z) : z \in (-\infty, m(p)) \mapsto p \in (\mu(p), +\infty)\) is continuous and monotone decreasing in \(z \in (-\infty, m(p))\). Then

\[\lim_{z \to m(p)-0} \mu(p, z) = \mu(p) \geq 0.\]

Therefore \(\mu(p, z)\) has a continuous inverse \(E(\cdot, p) : \mu \in (\mu(p), +\infty) \mapsto z \in (-\infty, m(p))\). Clearly, \(\Delta(\mu, p : E(\mu, p)) \equiv 0\). These arguments lead us to a logical conclusion that \(E(\cdot, p) \to m(p) - 0\), if and only if \(\mu \to \mu(p) + 0\). The proofs of the parts (i) and (ii) of Theorem 2.7 can be reduced to the proof of the implicit function theorem for several variables by applying appropriate changes of variables (see, e.g., [2,9]).

Therefore, we prove only the part (iii) of Theorem 2.7. Denote \(\hat{\mu} = \mu - \mu(p)\) and \(\alpha = m(p) - E(p, z)\), where \(\mu(p) > 0\).

(iii) Let \(\varphi(q_0(p)) = 0\) and \(\nabla \varphi(q_0(p)) = 0\). Then Lemma 3.1 implies that \(\hat{\alpha}_1(p) = 0\) and Lemma 3.4 gives \(\hat{c}_1(p) \neq 0\). Hence, Lemmas 3.1 and 2.3 yield the equation

\[-\frac{\hat{\mu}}{\hat{\mu}(p) + \mu^2(p)} = \ln \alpha \sum_{n=2}^{\infty} \hat{\alpha}_n(p) \alpha^n + \sum_{n=1}^{\infty} \hat{c}_n(p) \alpha^n,\]

(3.22)

reveals that \(\hat{c}_1(p) < 0\).

Introducing the variables

\[\alpha = \hat{\mu} (a(p) + u), \quad a(p) = [-\hat{\alpha}_1(p) \mu^2(p)]^{-1},\]

(3.23)

implies that in the region, where \(|\alpha|\) is small enough, the equation (3.22) is equivalent to

\[F(u, \tau, \hat{\mu}) = [\tau + \hat{\mu} \ln(a(p) + u)] \times \sum_{n=2}^{\infty} \hat{\alpha}_n(p) \hat{\mu}^{n-2}(a(p) + u)^n
+ \sum_{n=1}^{\infty} \hat{c}_n(p) \hat{\mu}^{n-1}(a(p) + u)^n + \frac{1}{\hat{\mu}(p) + \mu^2(p)} = 0\]

(3.24)
with \( \tau = \hat{\mu} \ln \hat{\mu} \). The function \( F \) satisfies the following conditions:

(i) \( u = 0, \tau = 0, \hat{\mu} = 0 \) is a solution of \( F(\cdot, \cdot, \cdot) = 0 \);

(ii) \( F \) is analytic for small \( |u|, |\tau|, |\hat{\mu}| \);

(iii) \( \partial F / \partial u(0, 0, 0) = \hat{c}_1(p) \neq 0 \).

The implicit function theorem then yields that for sufficiently small \( \tau \) and \( \hat{\mu} \), the equation (3.24) has a unique solution \( u(\hat{\mu}, \tau) \), which is given by the absolutely convergent series

\[
\sum_{l,s \geq 0} c(l, s) \tau^l \hat{\mu}^s.
\]

The condition \( \tau = \hat{\mu} = 0 \) is equivalent to the relation \( c(0, 0) = 0 \). Thus, from (3.23) we obtain the expansion

\[
\alpha = a(p)\hat{\mu} + \sum_{l,s \geq 0, l+s \geq 1} c(l, s) \tau^l \hat{\mu}^{s+1} = a(p)\hat{\mu} + \sum_{l \geq 0, s \geq 1, l+s \geq 2} \hat{c}(l, s) \tau^l \hat{\mu}^{s+1},
\]

which yields (2.3).

\[\square\]

Acknowledgements The authors acknowledge support from the Foundation for Basic Research of the Republic of Uzbekistan (Grant No.OT-F4-66). We also thank the referees for their valuable comments and remarks which helped to improve the manuscript.

Availability of data and material Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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