Gromov’s Problem: Bound the Expansion Coefficient from below in terms of the Observable Diameter of a Metric Measure Space, and its Diameter Bounds

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September 19, 2018

Abstract

In the celebrated book entitled ‘Metric Structures for Riemannian and Non-Riemannian Spaces’, so-called ‘Green Book’, Gromov presented a problem regarding a metric measure space. Gromov posed the question ‘Bound the expansion coefficient from below in terms of the observable diameter’. The overall aim of the current study is to demonstrate the answer to this problem. To begin solving this problem, the concentration of measure phenomenon on the metric measure space must be considered. The concentration function to evaluate the measure phenomenon is connected by the observable diameter and the expansion coefficient. Furthermore, the procedure for our answer gives us the upper bound for the expansion coefficient in terms of the observable diameter. Combining the desired lower bound for the expansion coefficient with its upper bound, we eventually obtain the upper bound for the observable diameter. Simultaneously, this reasoning has enabled us to obtain the upper bound for the diameter of a bounded metric measure space in terms of the expansion coefficient. We will apply the above-mentioned results to a compact connected Riemannian manifold with non-negative Ricci curvature, which makes the bounds more explicit. More precisely, they are in terms of the doubling constant of the Riemannian measure and the first non-trivial eigenvalue of the Laplacian on the Riemannian manifold.

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The author was supported by Grant-in-Aid for Young Scientists (B), Japan Society for the Promotion of Science(JSPS), Grant Number 25730022.
1 Introduction

Mikhail Gromov has, in his celebrated ‘Green Book’, proposed the following problem, which concerns the expansion coefficient and observable diameter of a metric measure space:
Exercise 1 (3.1.35 of [15]). Bound the expansion coefficient from below in terms of the observable diameter.

This paper aims to obtain an answer to the above problem and determine the novel bounds for the diameter of a bounded metric measure space. More precisely, we will show that the upper and lower bounds for the diameter are in terms of the expansion coefficient and Laplace functional, respectively.

To the best of our knowledge, the expansion coefficient has two proposals: one by Mikhail Gromov and the other by Michel Ledoux. Needless to say, the expansion coefficient stated in Gromov’s problem has been proposed by Gromov. However, in the procedure for our calculations, we will exploit Ledoux’s expansion coefficient. Therefore, our answer to Gromov’s problem is in terms of not only the observable diameter but also Ledoux’s expansion coefficient (Theorem 6.1).

When we apply our answer to a compact connected Riemannian manifold, Ledoux’s expansion coefficient makes the lower bound for Gromov’s expansion coefficient of the manifold explicit. Explicitly, its lower bound is in terms of the doubling constant of the Riemannian measure and the first non-trivial eigenvalue of the Laplacian on the Riemannian manifold, as well as the observable diameter (Example 6.2).

In order to undertake Gromov’s problem, we have paid attention to the concentration of measure phenomenon on a metric measure space. The concentration function enables us to describe the concentration phenomenon. For this reason, previous studies of the concentration phenomenon on the metric measure space have centred on evaluation of the concentration function. Remarkably, the concentration function plays an implicit, although pivotal, role to bridge Gromov’s and Ledoux’s expansion coefficients and the observable diameter.

Furthermore, the procedure (Theorems 5.5 and 5.6) for our answer yields additional by-products. These include, for a metric measure space, the lower bound for Ledoux’s expansion coefficient of the Riemannian manifold (Corollary 6.3), the upper bounds for Gromov’s expansion coefficient (Corollary 6.4) and for the observable diameter (Corollary 6.5) and its application to the Riemannian manifold (Example 6.6) and the upper bound for the diameter (Theorem 7.1) and its application to the Riemannian manifold (Example 7.3).

Acknowledgements

The author would like to thank Shin-ichi Ohta, an associate professor of Kyoto University: Takumi Yokota, an assistant professor of Research Institute for Mathematical Sciences: Dr’s. Yu Kitabeppu and Ryunosuke Ozawa and Mr. Kohei Suzuki of Kyoto University for their fruitful comments and suggestions.
2 Review of materials

In this section, for the convenience of the reader, we will briefly summarize some pre-requisite materials of the metric measure spaces for a statement on Gromov’s problem and its answer to the problem: the concentration function, the expansion coefficient and the observable diameter. It is the crux of the problem to observe the relation between these concepts. The concentration function is a device to connect the expansion coefficient with the observable diameter. These three materials play a significant role in grasping the structure of ambient sources, namely metric measure spaces, when accompanying the concentration of the measure phenomenon. See [21] and the references therein for further accounts.

2.1 Concentration function

The concentration function on the metric measure spaces describes the concentration of the measure phenomenon on these said spaces. Loosely speaking, the concentration phenomenon occurs when a set of spaces has a sufficiently large measure, where ‘most’ of the points in the spaces get ‘close’ to the neighbourhood of the set, which is referred to as an isoperimetric enlargement (called isoperimetric neighbourhood as well; see Definition 3.2). The classical isoperimetric inequality in Euclidean space is in terms of isoperimetric enlargement (see [19, p. 170] for detailed accounts).

This notion of utilizing the concentration function to evaluate the concentration of the measure phenomenon was first introduced in [2], which has been formalized in [16] and further analysed in [30]. The function relies on two main entities: a normalized measure (i.e. probability measure) and a notion concerning isoperimetric enlargement, with respect to which concentration is evaluated.

The most significant outcome for the function is that the measure permits a very small concentration function as the isoperimetric enlargement tends to be ‘large’. As mentioned in [29, Section 4], the chief problem in the investigation of the concentration phenomenon is estimating the concentration function; accordingly Milman shows three techniques including an isoperimetric inequalities approach. It is remarkable that the concentration function may be controlled in a rather large number of cases, especially its central classes—which are those that decay exponentially or Gaussian, which we call exponential concentration or Gaussian (or normal) concentration. We refer the reader to [19], [20] and [21] for detailed accounts.

2.2 Observable diameter

The current expository accounts for the observable diameter are principally due to [5, p. 336] and [21, Section 1.4].

The idea of the observable diameter is to introduce notions corresponding to physical reality and physical experiments, and as mentioned in [12, p. 50] and [13, p. 105], due to the quantum and statistical mechanics. Physical reality is
defined as a metric space. An object can be observed only by the signals we can perceive. The signals are Lipschitz functions. What we perceive, due to the lack of accuracy of our instruments, results in only a small error, and the observable diameter is intended to capture this variability.

The notion concerning the observable diameter can be defined for any geometric concept such as the central radius (the minimal radius of a ball covering the whole metric space) and the centre of mass (or barycenter). A metric and a measure are enough to define such notions.

Historically, the first estimation of the observable diameter was for standard spheres. As early as 1919, Paul Lévy studied the concentration phenomenon for the spheres, which could be described in terms of the observable diameter; see Example 4.6 below for detailed accounts. It is true for Gromov’s result for Riemannian manifolds with positive Ricci curvature as well; see Example 4.7 below for detailed accounts. The contemporary treatment of the observable diameter by Gromov is found to be a ‘visual’ description of the concentration phenomenon.

The observable diameter may be viewed as a dual component of the concentration function. It describes the diameter of a metric space viewed through a given probability measure on the Borel sets of metric space.

2.3 Expansion coefficient

The notion concerning the expansion coefficient implies a volume ratio of a set of metric measure spaces to its isoperimetric enlargement. The expansion coefficient has two proponents: Gromov (see [15, Section 3.1], [35]) and Ledoux (see [21, Section 1.5]); accordingly, in the present paper, each of these will be referred to as the “Gromov’s and Ledoux’s expansion coefficients,” respectively. These two expansion coefficients may be thought of as working side-by-side.

It can be inferred that Ledoux’s expansion coefficient is analogous to the so-called Cheeger isoperimetric constant; see [21, p.32] and the references therein for detailed accounts. As we will discuss, it is a conspicuous property that when Ledoux’s expansion coefficient is greater than 1, it gives rise to the above-mentioned exponential concentration.

3 Concentration of measure phenomenon on metric measure spaces

In this section, using the concentration function, we will work over the concentration of the measure phenomenon on metric measure spaces.

3.1 Setup

We now define the metric measure space in the sense of [21, Section 1.2]; also refer to [15] for the pioneering work on the present topic by Gromov.
**Definition 3.1.** A metric measure space is defined to be a metric space \((X, d_X)\) equipped with a finite Borel measure \(\mu_X\) on \((X, d_X)\). We denote it by a triplet \((X, d_X, \mu_X)\), which is often referred to as mm space as well.

In all what follows, we will focus on \(\mu_X(X) = 1\), which permits a stochastic structure of \((X, d_X, \mu_X)\). We will employ the same letter \(X\) briefly to designate \((X, d_X, \mu_X)\).

Let us now discuss the concentration phenomenon on the metric measure space. Historically, the concentration of the measure phenomenon was most vigorously put forward by V.D. Milman in the local theory of Banach spaces in the study of Dvoretzky’s theorem on almost Euclidean sections of convex bodies [19, p. 178] and its references and [23, p. 14]. To oscillate the measure on the space dynamically makes it capable of grasping the space structure. The dynamic association is exactly the concentration of the measure phenomenon on the space. In fact, the pertinence of the appellation for the concentration phenomenon is due to a concentration inequality on the space; see Proposition 3.5 and Remark 3.3 below.

As mentioned in [19, p. 170], the concentration phenomenon is rather concerned with the behaviour of ‘large’ isoperimetric enlargement. In fact, the so-called Lévy-Gromov isoperimetric inequality (see Theorem 3.1) in which the isoperimetric enlargement has a large measure establishes the concentration function.

**Definition 3.2** (Isoperimetric enlargement, isoperimetric neighbourhood). For all \(A \subset X\) and for all \(r \geq 0\), we define the **isoperimetric enlargement** or **isoperimetric neighbourhood** of order \(r\) to be

\[
A_r = \{ x \in X; d_X(x, A) \leq r \},
\]

which is referred to as the \(r\)-inflation or \(r\)-extension of \(A\) with respect to \(d_X\) as well.

### 3.2 The Lévy-Gromov isoperimetric inequality inspires the concentration function

This subsection aims to find the so-called Lévy-Gromov isoperimetric inequality, which is a generalization of Lévy’s isoperimetric inequality; see [36, Theorem 2.3] for detailed account. We refer the reader to [15, Appendix C] for the Lévy-Gromov isoperimetric inequality as well as its original issue [14] by Gromov.

**Theorem 3.1** (Lévy-Gromov isoperimetric inequality; e.g., p. 362 of [17]). Let \(M\) be an \(n\) \((\geq 2)\)-dimensional compact connected Riemannian manifold with Ricci curvature \(\text{Ric}(M)\) bounded below in terms of a positive constant. Let \(\delta > 0\) be a radius of the \(n\)-dimensional Euclidean sphere \(\mathbb{S}^n(\delta)\) relative to its intrinsic Riemannian metric such that

\[
\text{Ric}(M) = \text{Ric}(\mathbb{S}^n(\delta)) = (n - 1)/\delta^2.
\]

\[1\]
Write $\sigma^n_\delta$ for a normalized rotation-invariant measure, namely a normalized Haar measure on $S^n(\delta)$. Then, for all Borel subset $A$ in $M$ and for all $r > 0$,

$$\mu_M(A_r) \geq \sigma^n_\delta(B_r),$$

where $B$ is a spherical cap of $S^n(\delta)$ such that $\mu_M(A) = \sigma^n_\delta(B)$.

**Corollary 3.2.** In particular, if $A \subset M$ has a sufficiently large measure, say $\mu_M(A) \geq 1/2$, then we have, by Lévy-Gromov isoperimetric inequality,

$$\mu_M(A_r) \geq 1 - \sqrt{\frac{\pi}{8}} \exp\left(\frac{-(n-1)r^2}{2\delta^2}\right).$$

(2)

See [28] for detailed account.

Combining (2) with (1) yields

$$\mu_M(A_r) \geq 1 - \sqrt{\frac{\pi}{8}} \exp\left(\frac{-\text{Ric}(M)r^2}{2}\right).$$

(3)

**Remark 3.1.** We employ the constant $\sqrt{\frac{\pi}{8}}$, which modifies $\sqrt{2}$ that appeared in the original article [16, p. 844]; [17, eq. (2)]. See [17] for the account on the Ricci curvature condition. Historically, Corollary 3.2 goes back as far as the work of Lévy [24]; also refer to Poincaré [35]. A simple and heuristic proof of the result of Corollary 3.2 by Lévy is given by Ledoux [18], to which we refer the reader for the argument due to the heat semigroup and Bochner’s formula.

Put this way, henceforth, we will regard the sphere and Rimannian manifold as a metric measure space naturally.

Due to the result in (3), the measure of the complement of $A_r$ is bounded above by the Gaussian kernel (Gaussian density) with the Ricci curvature as a coefficient, which will be referred to as the Gaussian (or normal) concentration; see Definition 3.6 below. The Gaussian concentration is a significant class of the concentration phenomenon. Therefore, the Lévy-Gromov isoperimetric inequality enables us to inspire the concept of the concentration function, which describes the concentration phenomenon; see Definitions 3.3 and 3.4 below.

As a result, the concentration of the measure phenomenon for a metric measure space is concerned with two main components: a finite measure, such as a probability measure on metric measure spaces, and the above-mentioned isoperimetric enlargement, with respect to which the measure concentration is evaluated.

**Definition 3.3** (e.g., Section 1.2 of [21]). We define the concentration function $\alpha_{(X,d_X,\mu_X)}$ of metric measure spaces $(X,d_X,\mu_X)$ by

$$\alpha_{(X,d_X,\mu_X)}(r) := \sup\{ 1 - \mu_X(A_r); X \supset A : \text{Borel set}, \mu_X(A) \geq 1/2 \}$$

for all $r \geq 0$.

In what follows, to exhibit an answer to Gromov’s problem, we are mostly concerned with the generalization of Definition 3.3 with respect to the lower bound for $A$ as above; see also [21, Section 1.3] for the generalization.
**Definition 3.4.** Let $0 < \varepsilon < 1$. We define the *concentration function* $\alpha^\varepsilon(X,d_X,\mu_X)$

$$\alpha^\varepsilon(X,d_X,\mu_X)(r) := \sup\{1 - \mu_X(A_r); X \supset A : \text{Borel set }, \mu_X(A) \geq \varepsilon\}$$

for all $r \geq 0$.

Therefore, we see that

**Proposition 3.3.**

$$\alpha^\varepsilon(X,d_X,\mu_X)(r) \leq \alpha^{1-\varepsilon}(X,d_X,\mu_X)(r) \quad \text{for all } r \geq 0,$$

provided $\varepsilon \geq 1/2$, and vice versa.

Since the concept of the concentration phenomenon is attributed to the isoperimetric inequality, as described above, the concentration function is also referred to as an ‘isoperimetric constant’. For further references in this paper, we will write $\alpha^\varepsilon(X,d_X,\mu_X)$ with $\varepsilon = 1/2$ as $\alpha(X,d_X,\mu_X)$ solely.

As mentioned in Subsection 2.1, two significant classes of metric measure spaces share the exponential and Gaussian upper bounds for the concentration function, each of which is defined as follows:

**Definition 3.5 (cf., [21], especially Section 1.2).** Let $0 < \varepsilon < 1$. A metric measure space $(X,d_X,\mu_X)$ has *exponential concentration* if there exist universal numeric constants $C_i$, $i = 1, 2$ such that

$$\alpha^\varepsilon(X,d_X,\mu_X)(r) \leq C_1 \exp(-C_2 r) \quad \text{for all } r \geq 0. \quad (4)$$

Milman has obtained necessary and sufficient conditions for Cheeger’s isoperimetric and Poincaré inequalities on a metric measure space in terms of the exponential concentration; see [27, Theorem 1.5] for further accounts.

**Definition 3.6 (cf., Section 1.2 of [21]).** Let $0 < \varepsilon < 1$. A metric measure space $(X,d_X,\mu_X)$ has *Gaussian (normal) concentration* if there exist universal numeric constants $C_i$, $i = 1, 2$ such that

$$\alpha^\varepsilon(X,d_X,\mu_X)(r) \leq C_1 \exp(-C_2 r^2) \quad \text{for all } r \geq 0. \quad (5)$$

**Remark 3.2.** Gaussian concentration yields a sharper estimation than exponential concentration: If an arbitrary metric measure space has Gaussian concentration, then it has exponential concentration. Indeed, for each universal numeric constant $C_i > 0$, $i = 1, 2$ there exists a constant $C'_i > 0$ such that

$$C_1 \exp(-C_2 r^2) \leq C'_1 \exp(-C_2 r) \quad \text{for all } r \geq 0.$$
Example 3.4 (e.g., p. 274 of [29] and pp. 362–363 of [17]). Let $S^n(\delta)$ be the $n$ $(\geq 2)$-dimensional Euclidean sphere of radius $\delta > 0$ equipped with the geodesic distance $d_{S^n(\delta)}$ and the rotation-invariant normalized measure $\mu_{S^n(\delta)}$. Then, $(S^n(\delta), d_{S^n(\delta)}, \mu_{S^n(\delta)})$ has Gaussian concentration as follows:

$$\alpha_{(S^n(\delta), d_{S^n(\delta)}, \mu_{S^n(\delta)})}(r) \leq C_1 \exp(-C_2 r^2) \quad \text{for all } r \geq 0,$$

with $C_1 = \sqrt{\pi/8}$ and $C_2 = \text{Ric}(S^n(\delta))/2 = (n-1)/2\delta^2$, the latter due to (1).

Next, we are concerned with the concept concerning the diameter of bounded metric measure spaces. When $X$ is bounded, it is noteworthy for the range of isoperimetric enlargement $r$ in the concentration function to range up to the diameter of $X$, which is denoted by

$$\text{diam}(X) := \sup\{ d_X(x, y); x, y \in X \}.$$  

In Theorem 7.1, we will address the estimate for diam$(X)$.

It is conspicuous that the concentration function tends towards 0 as the isoperimetric enlargement is close to diam$(X)$. As mentioned in [21], this, however, will not usually be specified. The function result will decrease rapidly as the enlargement, or the dimension of $X$, is extremely large, and this reflects the concentration phenomenon.

### 3.3 Concentration inequality

In this subsection, we will establish the concentration inequality for $\alpha^\varepsilon(X, d_X, \mu_X)$. Please note the concepts mentioned below.

**Definition 3.7.** Set a positive real number $\varepsilon < 1$. Let $f$ be a measurable real-valued function on $(X, d_X, \mu_X)$. Define a real number $m_f$ of $f$ for $\mu_X$ such that

$$\mu_X(\{ f \leq m_f \}) \geq \varepsilon, \quad \mu_X(\{ f \geq m_f \}) \geq 1 - \varepsilon.$$

If one regards $f$ as a random variable, then $m_f$ is referred to as the *quantile of order* $\varepsilon$ of $f$ for $\mu_X$ or the *100th percentile* of $f$ for $\mu_X$. In particular, if $\varepsilon = 1/2$, then $m_f$ exactly coincides with the so-called *Lévy mean* or *median* of $f$ for $\mu_X$. Note that $m_f$ exists and may not be unique. Nevertheless [23, p. 21] shows that the median of the Gaussian kernel (density) for the canonical Gaussian measure on an $n$-dimensional Euclidean space is uniquely determined.

As will be shown below, the Lipschitz property on metric spaces $(X, d_X)$, involving Lipschitz function and its Lipschitz constant, enables us to observe the concentration phenomenon on $(X, d_X, \mu_X)$ and to introduce the concept of the observable diameter of $(X, d_X, \mu_X)$; see Section 4.

**Definition 3.8.** We call a real-valued function $f$ on $(X, d_X)$ *Lipschitz* if

$$\|f\|_{\text{Lip}} := \sup_{x, y \in X; x \neq y} \frac{|f(x) - f(y)|}{d_X(x, y)} < \infty.$$
$\|f\|_{\text{Lip}}$ is referred to as the Lipschitz constant of $f$. In particular, we say that $f$ is 1-Lipschitz if $\|f\|_{\text{Lip}} \leq 1$.

We are now ready to state the concentration inequality for $\alpha^\varepsilon_{(X,d_X,\mu_X)}$:

**Proposition 3.5** (Concentration inequality). Set a positive real number $\varepsilon < 1$. Let $f$ be a Lipschitz function on $(X,d_X)$, and let $m_f$ be the quantile of order $\varepsilon$. We have

$$
\mu_X(\{|f - m_f| > r\}) \leq \alpha^\varepsilon_{(X,d_X,\mu_X)}(r/\|f\|_{\text{Lip}}) + \alpha^{1-\varepsilon}_{(X,d_X,\mu_X)}(r/\|f\|_{\text{Lip}})
$$

for all $r \geq 0$. In particular, if $f$ is 1-Lipschitz, then (6) is given by

$$
\mu_X(\{|f - m_f| > r\}) \leq \alpha^\varepsilon_{(X,d_X,\mu_X)}(r) + \alpha^{1-\varepsilon}_{(X,d_X,\mu_X)}(r) \quad \text{for all } r \geq 0.
$$

**Proof.** Set $A := \{f \leq m_f\}$. Then $\mu_X(A) \geq \varepsilon$ follows from definition $m_f$. For all $r \geq 0$, fix $x \in A_r$. One can see that

$$
\mu_X(A_r) \leq \mu_X(\{f \leq m_f + \|f\|_{\text{Lip}} r\}) \quad \text{for all } r \geq 0.
$$

Indeed, it follows immediately from Definition 3.8 that

$$
f(x) \leq f(a) + \|f\|_{\text{Lip}} d_X(x, a) \quad \text{for all } x, a \in X.
$$

Hence, especially for $a \in A$, we actually have $f(x) \leq m_f + \|f\|_{\text{Lip}} d_X(x, a)$. By taking the infimum over $a \in A$, we have, from the definition of $x \in A_r$, $f(x) \leq m_f + \|f\|_{\text{Lip}} r$. Therefore, we have $x \in \{f \leq m_f + \|f\|_{\text{Lip}} r\}$. This implies (8), namely

$$
\mu_X(\{f > m_f + \|f\|_{\text{Lip}} r\}) \leq 1 - \mu_X(A_r) \quad \text{for all } r \geq 0.
$$

Hence,

$$
\mu_X(\{f > m_f + r\}) \leq \alpha^\varepsilon_{(X,d_X,\mu_X)}(r/\|f\|_{\text{Lip}}) \quad \text{for all } r \geq 0.
$$

We call (9) a deviation inequality; see [21, p. 6].

We now apply this argument again, with $A$ replaced by $\{-f \leq -m_f\}$, to obtain

$$
\mu_X(A_r) \leq \mu_X(\{f \geq m_f - \|f\|_{\text{Lip}} r\}).
$$

Similarly, we can see that

$$
\mu_X(\{f < m_f - \|f\|_{\text{Lip}} r\}) \leq 1 - \mu_X(A_r) \quad \text{for all } r \geq 0.
$$

Hence,

$$
\mu_X(\{f < m_f - r\}) \leq \alpha^{1-\varepsilon}_{(X,d_X,\mu_X)}(r/\|f\|_{\text{Lip}}) \quad \text{for all } r \geq 0,
$$

where $1 - \varepsilon$ is due to $\mu_X(A) = \mu_X(\{f \geq m_f\}) \geq 1 - \varepsilon$.  

We conclude from (9) and (10) that
\[
\mu_X(\{ |f - m_f| > r \}) \leq \alpha^\varepsilon_{(X,d_X,\mu_X)}(r/\|f\|_{\text{Lip}}) + \alpha^{1-\varepsilon_{(X,d_X,\mu_X)}}(r/\|f\|_{\text{Lip}}).
\]

In particular, if \(\|f\|_{\text{Lip}} \leq 1\) in (6), then we see from the fact that \(\alpha^{1-\varepsilon_{(X,d_X,\mu_X)}}\) decreases such that
\[
\alpha^{1-\varepsilon_{(X,d_X,\mu_X)}}(r/\|f\|_{\text{Lip}}) \leq \alpha^{1-\varepsilon_{(X,d_X,\mu_X)}}(r) \quad \text{for all } r \geq 0.
\]
Hence, we obtain (7). This proves the proposition.

\[\square\]

**Proposition 3.6 (Concentration inequality).** Under the hypotheses of Propositions 3.5, we get
\[
\mu_X(\{ |f - m_f| > r \}) \leq 2\alpha^{1-\varepsilon_{(X,d_X,\mu_X)}}(r/\|f\|_{\text{Lip}}) \quad \text{if } \varepsilon \geq 1/2. \quad (11)
\]

In particular, if \(f\) is 1-Lipschitz, then (11) is given by
\[
\mu_X(\{ |f - m_f| > r \}) \leq 2\alpha^{1-\varepsilon_{(X,d_X,\mu_X)}}(r) \quad \text{if } \varepsilon \geq 1/2. \quad (12)
\]

**Proof.** Combining (9) and (10) with Proposition 3.3, (11) readily follows. The verification for the case that \(f\) is 1-Lipschitz coincides with that of Proposition 3.5. Thus, we have (12). This completes the proof. \[\square\]

**Remark 3.3.** One can see from the two aforementioned concentration inequalities that the Lipschitz function is concentrated around its Lévy mean, with the rate given by the concentration function.

### 4 Observable diameter

In this section, we will focus on the observable diameter of metric measure spaces. As mentioned in [21, Section 1.4], the observable diameter might work in conjunction with the concentration function; see e.g., Propositions 4.3 and 4.4 below.

#### 4.1 Partial diameter

To introduce the observable diameter of metric measure spaces \((X, d_X, \mu_X)\), we first need to define the partial diameter. For a thorough discussion on the observable diameter, we refer the reader to [15, 34–20] and [5, 336–337].

**Definition 4.1 (Partial diameter).** Let \(\kappa > 0\). We call the infimal \(D\) such that there exists a subset \(A\) of \(X\) with \(\text{diam}(A) \leq D\) and \(\mu_X(A) \geq 1 - \kappa\) the partial diameter of \(X\) with respect to \(\mu_X\). We denote by \(\text{PartDiam}_{\mu_X}(X; 1 - \kappa)\) the partial diameter.
**Definition 4.2** (Lipschitz dominate; Definition 2.10 of [36]). Let \((X, d_X, \mu_X)\) and \((Y, d_Y, \mu_Y)\) be the metric measure spaces, respectively. We say that \(X\) **Lipschitz dominates** \(Y\) if there exists a 1-Lipschitz map \(f : X \to Y\) such that

\[ f_* \mu_X = \mu_Y, \]

where \(f_* \mu_X\) stands for the **push-forward measure** of \(\mu_X\) by \(f\).

The following asserts that the partial diameter is monotone for the aforementioned Lipschitz domination:

**Proposition 4.1** (3.12 of [15] and Section 1.4 of [21]). Suppose that \(X\) Lipschitz dominates \(Y\). Then, it follows readily that

\[ \text{PartDiam}_{\mu_Y}(Y; 1 - \kappa) \leq \text{PartDiam}_{\mu_X}(X; 1 - \kappa). \]

### 4.2 Observation device for diameter

What is not obvious is that the partial diameter may dramatically decrease under all 1-Lipschitz maps from a metric measure space to a certain metric space. We will now call the target metric space the **screen**; see [15, 3.20] and [21, Section 1.4]. Denoting the screen set as a 1-dimensional Euclidean space \(\mathbb{R}\) provides us with more geometric view to concentration. The geometric observation device itself can also be the observable diameter. In actuality, the observable diameter permits us to describe the diameter of a metric measure space viewed through a given Borel probability measure on the space.

Incidentally, Naor et al. have discussed a class of metric measure space whose observable diameter is much smaller than its diameter, which is sometimes (following Milman) referred to as a “small isoperimetric constant”; see [33] for detailed accounts.

**Definition 4.3** (Observable diameter). We define the \((\kappa-)\text{observable diameter}\) of \((X, d_X, \mu_X)\) with respect to \(\mu_X\), denoted by \(\text{ObsDiam}(X; -\kappa)\), to be the supremum of \(\text{PartDiam}_{f_* \mu_X}(\mathbb{R}; 1 - \kappa)\) over each \(f_* \mu_X\), namely

\[ \text{ObsDiam}(X; -\kappa) := \sup\{\text{PartDiam}_{f_* \mu_X}(\mathbb{R}; 1 - \kappa); 1\text{-Lipschitz function } f : X \to \mathbb{R}\}, \]

where the supremum is taken over all \(f\).

**Remark 4.1.** According to [5, p. 336] and [15, 3.20], the observable diameter is usually rather insensitive to a positive real number \(\kappa < 1\). Actually, Gromov suggests setting \(\kappa = 10^{-10}\). Therefore, one may employ the notation \(\text{ObsDiam}(X)\) simply for the observable diameter \(\text{ObsDiam}(X; -\kappa)\).

To facilitate access to the observable diameter, we shall briefly review [5, pp. 336–337], [15, Section 3.20] and [21, Section 1.4]. Taken from a physical point of view, one ascribes the idea of observable diameter to the notions corresponding to ‘physical reality’ and ‘physical experiments’. Indeed, we may
actually take the physical reality, namely configuration space to be a metric space \((X, d_X)\). We think of \(\mu_X\) on \((X, d_X)\) as a 'state' on the configuration space. 1-Lipschitz functions \(f\) on a metric measure space \((X, d_X, \mu_X)\) behave like the signals; more precisely, 'observable', namely an observation device giving us the visual (tomographic) image on the screen \(\mathbb{R}\); see Corollary 4.5. Thus, with the naked eye, one can view the state via the observable \(f_\ast \mu_X\) on the screen and cannot identify a part of the screen of measure (luminosity) less than a positive real number \(\kappa < 1\).

**Proposition 4.2.** Suppose that \(X\) Lipschitz dominates \(Y\). Then, it follows that

\[
\text{ObsDiam}(Y; -\kappa) \leq \text{ObsDiam}(X; -\kappa).
\]

*Proof.* See [36, Proposition 2.18].

### 4.3 Duality between the concentration function and the observable diameter

In this subsection, we will discuss the duality between the concentration function and the observable diameter as follows:

**Proposition 4.3** (Proposition 1.12 of [21]). Let \(\kappa > 0\) be small. Then, we have

\[
\text{ObsDiam}(X; -\kappa) \leq 2 \inf \{r > 0; \alpha_{(X, d_X, \mu_X)}(r) \leq \kappa/2\}.
\]

The following assertion is still true:

**Proposition 4.4.** Set a positive real number \(\varepsilon < 1\). We have for a positive real number \(\kappa < 1\)

\[
\text{ObsDiam}(X; -\kappa) \leq 2 \inf \{r > 0; \alpha^\varepsilon_{(X, d_X, \mu_X)}(r) + \alpha^{1-\varepsilon}_{(X, d_X, \mu_X)}(r) \leq \kappa\}. \tag{13}
\]

In particular,

\[
\text{ObsDiam}(X; -\kappa) \leq 2 \inf \{r > 0; \alpha^\varepsilon_{(X, d_X, \mu_X)}(r) \leq \kappa/2\}, \quad \varepsilon \leq 1/2. \tag{14}
\]

The argument of the current proof is due to that of Proposition 1.12 of [21]:

*Proof.* For a small \(\kappa > 0\), pick \(r > 0\) such that

\[
\alpha^\varepsilon_{(X, d_X, \mu_X)}(r) + \alpha^{1-\varepsilon}_{(X, d_X, \mu_X)}(r) \leq \kappa. \tag{15}
\]

Let \(f\) be a 1-Lipschitz function on \(X\). Set \(A := f(\{x \in X; |f(x) - m_f| \leq r\})\), where \(m_f\) is the quantile of order \(\varepsilon\) of \(f\) for \(\mu_X\), i.e. it satisfies \(\mu_X(\{f \leq m_f\}) \geq \varepsilon\) and \(\mu_X(\{f \geq m_f\}) \geq 1 - \varepsilon\). To see the observable diameter, observe that

\[
f_\ast \mu_X(A) \geq \mu_X(\{x \in X; |f(x) - m_f| \leq r\})
\]

\[
\geq 1 - \left(\alpha^\varepsilon_{(X, d_X, \mu_X)}(r) + \alpha^{1-\varepsilon}_{(X, d_X, \mu_X)}(r)\right),
\]

\[
\geq 1 - \left(\frac{\alpha^\varepsilon_{(X, d_X, \mu_X)}(r) + \alpha^{1-\varepsilon}_{(X, d_X, \mu_X)}(r)}{2}\right) = 1 - \frac{\kappa}{2}.
\]
where we have used (7) in the last inequality. Further, from (15), we have
\[
f_{*,\mu_X}(A) \geq 1 - \kappa.
\] (16)
Furthermore, it turns out that
\[
diam(A) \leq (m_f + r) - (m_f - r) = 2r.
\] (17)
Thereby, adding (16) and (17), we obtain
\[
PartDiam_{f,*\mu_X}(\mathbb{R}; 1 - \kappa) \leq 2r,
\]
from which (13) follows.

For the remainder of this paper, on account of Proposition 3.3, we can select
\[\frac{1}{2}\] as a threshold for \(\varepsilon > 0\), which appeared in Definition 3.4. By combining
the aforementioned argument with Proposition 3.3, we eventually see that
\[
\text{ObsDiam}(X; -\kappa) \leq 2\inf\{r > 0; \alpha^{1-\varepsilon}(X,d_{X},\mu_X)(r) \leq \kappa/2\} \quad \text{if } \varepsilon \geq 1/2;
\]
\[
\text{ObsDiam}(X; -\kappa) \leq 2\inf\{r > 0; \alpha^{\varepsilon}(X,d_{X},\mu_X)(r) \leq \kappa/2\} \quad \text{if } \varepsilon \leq 1/2.
\]
In consequence, we obtain (14) as desired.

On account of Propositions 4.3 and 4.4, which imply the duality between
the concentration function and the observable diameter, the upper bound for
the concentration function enables us to control the observable diameter. While
the following corollary is fairly straightforward, it plays a crucial role in giving
the answer to the current Gromov’s problem.

**Corollary 4.5.** If \(X\) has exponential concentration (4), then we have by (14)
\[
\text{ObsDiam}(X; -\kappa) \leq \frac{2}{C_2} \ln \frac{2C_1\kappa}{\kappa}, \quad \kappa > 0,
\]
where each universal numeric constant \(C_i > 0, i = 1, 2\) has already appeared in
(4).

If \(X\) has Gaussian concentration (5), then we have by (14)
\[
\text{ObsDiam}(X; -\kappa) \leq 2\sqrt{\frac{1}{C_2} \ln \frac{2C_1\kappa}{\kappa}}, \quad \kappa > 0,
\]
where each universal numeric constant \(C_i > 0, i = 1, 2\) has already appeared in
(5).

**Remark 4.2 (cf. p. 15 of [21]).** The significant parameter \(C_2\) that appeared in
Corollary 4.5 implies the exponential decay of the concentration function.

Corollary 4.5 further facilitates access to the observable diameter. Examples
follow below:
Example 4.6 (cf. Section 1.1 and p. 15 of [21]). Combining Corollary 4.5 with Example 3.4 shows that the observable diameter of the $n$ ($\geq 2$)-dimensional Euclidean sphere of radius $\delta$, namely $S^n(\delta)$ is of the order $n^{-1/2}$; more precisely,

$$\text{ObsDiam}(S^n(\delta); -\kappa) \leq 2\delta \sqrt{\frac{2}{n-1}} \ln \sqrt{\frac{\pi}{2\kappa}}, \quad \kappa > 0,$$

from which it follows that

$$\text{ObsDiam}(S^n(\delta); -\kappa) = O(n^{-1/2}), \quad n \to \infty.$$

Specifically, the observable diameter of the unit sphere is given explicitly; see [36] for further accounts.

Example 4.7. Let $M$ be an $n$ ($\geq 2$)-dimensional compact connected Riemannian manifold with Ricci curvature $\text{Ric}(M) \geq K$ for some constant $K > 0$. Applying Corollary 4.5 to Corollary 3.2, we get

$$\text{ObsDiam}(M; -\kappa) \leq 2\sqrt{\frac{(n-1)}{Kn}} \ln \sqrt{\frac{\pi}{2\kappa}}, \quad \kappa > 0.$$

Now, in Example 6.6, we will derive the upper bound for $\text{ObsDiam}(M; -\kappa)$ in terms of the first non-trivial eigenvalue of the Laplacian on $M$ to be stated in Subsection 5.3 later, and the doubling constant of the Riemannian measure of $M$ to be stated in Appendix A later.

Example 4.8. Let $M$ be a compact connected Riemannian manifold. Applying Corollary 4.5 to Theorem 5.3 to be shown later, we get

$$\text{ObsDiam}(M; -\kappa) \leq \frac{2\ln(3/2\kappa)}{\ln(3/2)\sqrt{\lambda_1(M)}}, \quad \kappa > 0.$$

5 Expansion coefficients

The expansion coefficient of metric measure spaces is proposed by Gromov and Ledoux independently; see [15, 31.35] and [21, Section 1.5], respectively. Before stating the proof results on the expansion coefficient, let us correct their statements of its definition.

Gromov has defined the expansion coefficient to be the infimum of real numbers $e \geq 1$ such that, if $\mu_X(A) \geq \varepsilon$ for all $A \subset X$, then it follows that $\mu_X(A_p) \geq e\varepsilon$ for $\rho > 0$. In contrast, Ledoux has defined it to be the infimum of real numbers $e \geq 1$ such that, if $\mu_X(B_p) \leq 1/2$ for all $B \subset X$, then it follows that $\mu_X(B_p) \geq e\mu_X(B)$ for $\rho > 0$.

In the work that follows, utilizing the proposal by both Gromov and Ledoux, we shall distinguish these two expansion coefficients: $\text{Exp}^{\text{Gromov}}$ and $\text{Exp}^{\text{Ledoux}}$; see below-mentioned Definitions 5.1 and 5.2 for their definitions. Nevertheless, inf in both the above definitions is not correct because $\text{Exp}^{\text{Gromov}} = \text{Exp}^{\text{Ledoux}} = 1$. Therefore, inf should be substituted by sup; see Definitions 5.1 and 5.2.
5.1 Gromov’s expansion coefficient and its properties

**Definition 5.1** (Erratum for Gromov’s Expansion coefficient; see 3.35 of [15]).

Set a positive real number \( \varepsilon < 1 \). We define *Gromov’s expansion coefficient* of \( \mu_X \) on \((X, d_X)\) of order \( \rho > 0 \) to be

\[
\text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho) := \sup\{ e \geq 1; \mu_X(A_{\rho}) \geq e\varepsilon, X \supset A : \text{Borel set}, \mu_X(A) \geq \varepsilon \}.
\]

(18)

It turns out from (18) that

\[
\mu_X(A_{\rho}) \geq \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho)\varepsilon.
\]

(19)

The following asserts that Gromov’s expansion coefficient is monotone for Lipschitz maps:

**Proposition 5.1** (cf. 3.35 of [15]). Let \( f \) be a Lipschitz map between \((X, d_X, \mu_X)\) and \((Y, d_Y, f_*\mu_X)\). Then, we have

\[
\text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho/\|f\|_{\text{Lip}}) \leq \text{Exp}_{\text{Gromov}}(Y; \varepsilon, \rho).
\]

In particular, if \( X \) Lipschitz dominates \( Y \), then it instantly follows that

\[
\text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho) \leq \text{Exp}_{\text{Gromov}}(Y; \varepsilon, \rho).
\]

Proof. The following claim makes it allowable to evaluate the inequalities above.

The verification of the claim is straightforward:

**Claim 1.** If \( f \) is a Lipschitz map, then

\[
A_{r/\|f\|_{\text{Lip}}} \subset f^{-1}(f(A))_r \quad \text{for all } r \geq 0.
\]

We have

\[
\begin{align*}
\text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho/\|f\|_{\text{Lip}}) \\
\leq \sup\{ e \geq 1; f_*\mu_X((f(A))_\rho) \geq e\varepsilon, \mu_X(A) \geq \varepsilon \} \quad \text{by Claim 1} \\
\leq \sup\{ e \geq 1; f_*\mu_X((f(A))_\rho) \geq e\varepsilon, f_*\mu_X(f(A)) \geq \varepsilon \} \\
= \text{Exp}_{\text{Gromov}}(Y; \varepsilon, \rho),
\end{align*}
\]

as required. \( \Box \)

5.2 Ledoux’s expansion coefficient and its properties

**Definition 5.2** (Erratum for Ledoux’s Expansion coefficient; see Remark 5.1).

Set a positive real number \( \varepsilon < 1 \). We define *Ledoux’s expansion coefficient* of \( \mu_X \) on \((X, d_X)\) of order \( \rho > 0 \) to be

\[
\text{Exp}_{\text{Ledoux}}(X; \varepsilon, \rho) := \sup\{ e \geq 1; \mu_X(B_{\rho}) \geq e\mu_X(B), X \supset B : \text{Borel set}, \mu_X(B_{\rho}) \leq \varepsilon \}.
\]

(20)
Remark 5.1. Ledoux has originally proposed the expansion coefficient with $\varepsilon = 1/2$; see Section 1.5 of [21]. As we mentioned previously in Section 1, if $M$ is a compact Riemannian manifold, then $\text{Exp}_{\text{Ledoux}}(M; 1/2, \rho)$ is analogous to Cheeger’s isoperimetric constant; see [7] for further accounts. In fact, from the viewpoint of expander graphs, Ledoux discusses the relation between Ledoux’s expansion coefficient with $\varepsilon = 1/2$ and Cheeger’s isoperimetric constant; see [21, pp. 31–32] and the reference therein.

It turns out from (20) that

$$\mu_X(B_\rho) \geq \text{Exp}_{\text{Ledoux}}(X; \varepsilon, \rho) \mu_X(B).$$

(21)

If $B$ is such that $\mu_X(B_{k\rho}) \leq \varepsilon$ for some integer $k \geq 1$, then (21) inductively yields

$$(\text{Exp}_{\text{Ledoux}}(X; \varepsilon, \rho))^k \mu_X(B) \leq \mu_X(B_{k\rho}) \leq \varepsilon.$$  

(22)

Remark 5.2. One sees immediately from (22) that if $\text{Exp}_{\text{Ledoux}}(X; \varepsilon, \rho) > 1$, then $B$ has an extremely small measure. In what follows, we shall principally concern ourselves with metric measure spaces with $\infty > \text{Exp}_{\text{Ledoux}}(X; \varepsilon, \rho) > 1$; see Appendix A for its observation.

5.3 Application to a Riemannian manifold

Throughout this subsection, let $M$ be a compact connected Riemannian manifold and $\Delta$ the Laplacian (Laplace-Beltrami operator) on $M$. As mentioned in [17, p. 363], what can be said if the concentration function on a compact Riemannian manifold when no lower bound for the Ricci curvature is available is unclear? Concerning this problem, Gromov and Milman have observed the isoperimetric enlargement on a compact connected Riemannian manifold; see [16, Theorem 4.1]. Their proof is provided by the so-called Poincaré inequality:

Theorem 5.2 (Poincaré inequality; e.g., [29]). It is well-known that $-\Delta$ has its discrete spectrum consisting of the eigenvalues $0 = \lambda_0 < \lambda_1(M) \leq \lambda_2(M) \ldots$. The mini-max principle characterizes the first non-trivial eigenvalue denoted by $\lambda_1(M)$ as the largest constant in the Poincaré inequality

$$\lambda_1(M) \text{Var}_{\mu_M}(f) \leq \int_M |\nabla f|^2 \, d\mu_M$$

for each smooth real-valued function $f$ on $M$, where $\text{Var}_{\mu_M}(f)$ stands for the variance of $f$ with respect to $\mu_M$, namely

$$\text{Var}_{\mu_M}(f) := \int_M \left| f - \int_M f \, d\mu_M \right|^2 \, d\mu_M,$$

and where $|\nabla f|$ stands for the Riemannian length of the gradient of $f$.

Now, using the argument of Ledoux’s expansion coefficient [21, Proposition 1.13], Ledoux has re-stated the result by Gromov and Milman in terms of the concentration function:
Theorem 5.3 (e.g., p. 364 of [17]). $M$ has exponential concentration:

$$\alpha_{(M,d_M,\mu_M)}(r) \leq C_1 \exp(-C_2r) \text{ for all } r \geq 0,$$

where $C_1 = 3/4$ and $C_2 = \sqrt{\lambda_1(M)} \ln(3/2)$.

Theorem 5.4 (Cf. p. 48 of [21]). We have

$$\text{Exp}_{\text{Ledoux}}(M; 1-\epsilon, \rho) \geq 1 + \lambda_1(M) \epsilon \rho^2 \text{ for some } \rho > 0.$$

Proof. We omit the details because the assertion actually follows from a slight change in the proof by Ledoux. 

We will address Gromov’s expansion coefficient of $M$; see Example 6.2.

5.4 Exponential concentration in terms of expansion coefficients

In this subsection, we will show that Gromov’s and Ledoux’s expansion coefficients give rise to exponential concentration.

5.4.1 Ledoux’s expansion coefficient

The following proposition provides the upper bound for $\alpha_{(X,d_X,\mu_X)}(r)$ in terms of $\text{Exp}_{\text{Ledoux}}(X; 1-\epsilon, \rho)$ only; cf. Proposition 1.13 of [21], in which Ledoux has discussed the case where $\epsilon = 1/2$ especially, although his result is not in terms of the expansion coefficient $\text{Exp}_{\text{Ledoux}}(X; 1/2, \rho)$ but $e \leq \text{Exp}_{\text{Ledoux}}(X; 1/2, \rho)$.

Theorem 5.5. Set a positive real number $\epsilon < 1$. For each $r > 0$, select $\rho > 0$ such that $\rho \leq r$. Then, we have

$$\alpha_{(X,d_X,\mu_X)}(r) \leq (1-\epsilon) \text{Exp}_{\text{Ledoux}}(X; 1-\epsilon, \rho) \cdot (\text{Exp}_{\text{Ledoux}}(X; 1-\epsilon, \rho))^{-r/\rho}. \quad (23)$$

In particular, if $\text{Exp}_{\text{Ledoux}}(X; 1-\epsilon, \rho) > 1$, then $X$ has exponential concentration as follows:

$$\alpha_{(X,d_X,\mu_X)}(r) \leq (1-\epsilon) \text{Exp}_{\text{Ledoux}}(X; 1-\epsilon, \rho) \cdot \exp\left(-(\ln \text{Exp}_{\text{Ledoux}}(X; 1-\epsilon, \rho))r/\rho \right). \quad (24)$$

Proof. We first interpolate each $r > 0$ between $k \rho$ and $(k+1) \rho$ for some $k \in \mathbb{N}$. Let $A \subset X$ with $\mu_X(A) \geq \epsilon$. Put $B := A_{k\rho}^c$. We see that $B_{k\rho} \setminus B \subset A_{k\rho} \setminus A$, namely $B_{k\rho} \subset A^c$. Hence,

$$\mu_X(B_{k\rho}) \leq 1 - \mu_X(A) \leq 1 - \epsilon. \quad (25)$$

Furthermore, combining (25) with (22) yields

$$\mu_X(B_{k\rho}) \geq (\text{Exp}_{\text{Ledoux}}(X; 1-\epsilon, \rho))^k \mu_X(B)$$

$$= (\text{Exp}_{\text{Ledoux}}(X; 1-\epsilon, \rho))^k(1 - \mu_X(A_{k\rho})). \quad (26)$$
Hence, by adding (25) and (26), we obtain

\[
\mu_X(A_{k\rho}) \geq \frac{\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho)^k - (1 - \varepsilon)}{\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho)^k}.
\] (27)

Consequently, we deduce from the interpolation of \(k\) and (27) that

\[
1 - \mu_X(A_r) \leq 1 - \mu_X(A_{k\rho}) \\
\leq \frac{1 - \varepsilon}{(\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^k} \\
\leq \frac{1 - \varepsilon}{(\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^{\frac{r}{\rho} - 1}}.
\]

Therefore, we obtain (23), from which (24) follows readily whenever \(\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1\). This completes the proof. 

5.4.2 Gromov’s and Ledoux’s expansion coefficients: The key to Gromov’s problem

The following theorem plays a key role in Gromov’s problem and a by-product; see Corollary 6.4.

**Theorem 5.6.** Set a positive real number \(\varepsilon < 1\). For each \(r > 0\), select \(\rho > 0\) such that \(\rho \leq r\). Then, we have

\[
\alpha^\varepsilon_{(X,d_{X},\mu_X)}(r) \leq (1 - \varepsilon) \exp_{\text{Gromov}}(X; \varepsilon, \rho)(\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^2 \\
\quad \cdot (\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^{-r/\rho}.
\]

In particular, if \(\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1\), then \(X\) has exponential concentration as follows:

\[
\alpha^\varepsilon_{(X,d_{X},\mu_X)}(r) \leq (1 - \varepsilon) \exp_{\text{Gromov}}(X; \varepsilon, \rho)(\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^2 \\
\quad \cdot \exp(-\ln(\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))r/\rho). \quad (28)
\]

**Proof.** The strategy of the current proof is similar to that of the proof of Theorem 5.5. Interpolate each \(r > 0\) between \(k\rho\) and \((k + 1)\rho\) for some \(k \in \mathbb{N}\). Let \(A \subset X\) with \(\mu_X(A) \geq \varepsilon\). Put \(B := A_{(k-1)\rho}^c\), where \(A_0 := A\) if \(k = 1\). Likewise,

\[
\mu_X(B_{(k-1)\rho}) \leq 1 - \mu_X(A) \leq 1 - \varepsilon.
\]

Under the same reasoning as that of the proof of Theorem 5.5, we have

\[
\mu_X(B_{(k-1)\rho}) \geq (\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^{k-1} \mu_X(B) \\
= (\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^{k-1}(1 - \mu_X(A_{(k-1)\rho})),
\]

In particular, if \(\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1\), then \(X\) has exponential concentration as follows:

\[
\alpha^\varepsilon_{(X,d_{X},\mu_X)}(r) \leq (1 - \varepsilon) \exp_{\text{Gromov}}(X; \varepsilon, \rho)(\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^2 \\
\quad \cdot \exp(-\ln(\exp_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))r/\rho). \quad (28)
\]
where replacing $A$ with $A_\rho$ and by using $A_\ell \supset (A_{(\ell - 1)\rho})_\rho$ for each $\ell \in \mathbb{N}$, we get

$$1 - (\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^{k-1} (1 - \mu_X(A_{k\rho})) \geq \mu_X(A_\rho) \geq \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho),$$

where we have used (19) in the last inequality. Hence, we obtain

$$\mu_X(A_{k\rho}) \geq \frac{(\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^{k-1} - (1 - \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho)) \varepsilon}{(\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^{k-1}} > 0,$$

where the last inequality is due to (19) and Definition 5.2.

Hence, we conclude from the interpolation of $k$ and (29) that

$$1 - \mu_X(A_r) \leq 1 - \mu_X(A_{k\rho}) \leq \frac{1 - \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho) \varepsilon}{(\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^{k-1}} \leq \frac{1 - \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho) \varepsilon}{(\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^{k-1}} \leq \frac{(1 - \varepsilon) \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho)}{(\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^{k-2}}. \quad (30)$$

Following the proof of the conclusion of Theorem 5.5, we obtain the desired result.

Remark 5.3. One sees immediately that the upper bounds for $\alpha^\varepsilon(X_\rho, d_X, \mu_X)$ in Theorem 5.5 are more sharper than those in Theorem 5.6.

Remark 5.4. The procedure for the current proof gives more, namely (30) permits us to obtain the upper bound for the expansion coefficient in terms of the observable diameter; see Corollary 6.4.

Now, as we have assumed that $\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1$ in Theorems 5.5 and 5.6, it is reasonable to ask when $\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1$. Thus, in Appendix A, we will address a sufficient condition for the assumption.

6 Observation for Gromov’s problem, and by-products

In this section, we will state the main result on Gromov’s problem, which gives the answer to Gromov’s problem; see Exercise 1 given in Section 1. Furthermore, the proof of the procedure for the answer enables one to observe the upper bounds for Gromov’s expansion coefficient and the observable diameter in terms of Ledoux’s expansion coefficient.
6.1 Lower bounds for the expansion coefficients: An answer to Gromov’s problem and its application to a Riemannian manifold

The following theorem is our answer to Gromov’s problem:

**Theorem 6.1.** Let \( \varepsilon \leq 1/2 \). Assume that \( \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1 \) for some \( \rho > 0 \). Then, \( \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho) \) is bounded from below in terms of \( \text{ObsDiam}(X; -\kappa), 0 < \kappa < 1 \), and \( \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) \) as follows:

\[
\text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho) \geq \frac{\kappa \exp \left( \text{ObsDiam}(X; -\kappa) \ln \left( \frac{\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho)}{\kappa} \right) \right)}{2(1 - \varepsilon) (\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^2}.
\]  

(31)

**Proof.** Let \( \rho > 0 \) satisfy the hypotheses of Theorem 5.6. To establish the current theorem, we first apply (14) to (28); accordingly,

\[
\text{ObsDiam}(X; -\kappa) \leq 2 \rho \ln \left( \frac{2(1 - \varepsilon) \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho) (\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))^2}{\kappa} \right),
\]

from which the desired result follows computationally. \( \square \)

**Example 6.2.** Let us now return to the context of Subsection 5.3. Under the hypotheses of Theorem 6.1, we will be concerned with \( M \) having \( \text{Ric}(M) \geq 0 \).

Now, the following claim permits us to estimate Ledoux’s expansion coefficient in terms of \( \lambda_1(M) \) and the doubling constant of the Riemannian measure of \( M \):

**Claim 2.** Let \( A \) and \( B \) be subsets of \( M \) such that \( d_M(A, B) > 0 \) and \( \mu_M(A) \geq \varepsilon \). Now, we deduce that for each \( \rho > 0 \)

\[
\mu_M(B(x, 2\rho)) \leq \mu_M(B_{\rho}) \leq 1 - \varepsilon \quad \text{for some } x \in M.
\]

Here, taking \( \rho > 0 \) such that \( \mu_M(B_{\rho}) = 1 - \mu_M(A) \), we have

\[
1 + \lambda_1(M) \varepsilon \rho^2 \leq \text{Exp}_{\text{Ledoux}}(M; 1 - \varepsilon, \rho) \leq 2^n,
\]  

(32)

where the first and second inequalities are given by Theorem 5.4 and because the doubling constant of the Riemannian measure of \( M \), which is the upper bound for \( \text{Exp}_{\text{Ledoux}}(M; 1 - \varepsilon, \rho) \), is equal to \( 2^n \). The doubling constant is given by the Bishop-Gromov volume comparison theorem (also called Riemannian volume comparison theorem); see e.g., [4] and pp. 377–378 of [39].

For such a \( \rho > 0 \), combining (32) with (31), we have

\[
\text{Exp}_{\text{Gromov}}(M; \varepsilon, \rho) \geq \frac{\kappa \exp \left( \text{ObsDiam}(M; -\kappa) \ln(1 + \lambda_1(M) \varepsilon \rho^2) / 2\rho \right)}{2^{2n+1}(1 - \varepsilon)}.
\]  

(33)
Remark 6.1. Let $M$ be a compact Riemannian manifold. [36] has shown that
\[ \text{Exp}_{\text{Gromov}}(M; \varepsilon, \rho) \geq \min\{1 + \lambda_1(M)\rho^2/4, 2\} \]
for all $\rho > 0$, provided $0 < \varepsilon \leq 1/4$.

The current subsection will end up with showing the relation between Ledoux’s expansion coefficient and the observable diameter:

Corollary 6.3. Let $M$ be an $n$-dimensional compact connected Riemannian manifold. Under the hypotheses of Theorem 6.1, we have, for some $\rho > 0$ that appeared in the context of Example 6.2,
\[ \text{Exp}_{\text{Ledoux}}(M; 1 - \varepsilon, \rho) \geq \kappa \exp \left( \frac{\text{ObsDiam}(M; -\kappa) \ln(1 + \lambda_1(M)\varepsilon \rho^2)/2\rho}{2(1 - \varepsilon)} \right). \]

Proof. The same reasoning as that utilized in (33) applies to (24), hence the corollary.

6.2 By-products: Upper bounds for Gromov’s expansion coefficient and for the observable diameter

As mentioned in Remark 5.4, we will show the upper bound for Gromov’s expansion coefficient in terms of the observable diameter and Ledoux’s expansion coefficient; accordingly, Gromov’s expansion coefficient is bounded from above and below by the two geometric quantities. Ultimately, one can derive the upper bound for the observable diameter in terms of Ledoux’s expansion coefficient.

The procedure for the proof of Theorem 5.6 implies the upper bound for Ledoux’s expansion coefficient:

Corollary 6.4. Under the hypotheses of Theorem 6.1, assume further that
\[ 2(1 - \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho)) \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) \geq \kappa \quad \text{for some } \rho > 0. \]

Then, the upper bound for $\text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho)$ is given by
\[ \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho) \leq \frac{2 - \kappa \exp((\text{ObsDiam}(X; -\kappa) - 4\rho) \ln(\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))/2\rho)}{2\varepsilon}. \]

Proof. For each $r > 0$, select $\rho > 0$ such that $\rho \leq r$. Then, one sees immediately from (30) that
\[ \alpha^\varepsilon(X, d_X, \mu, \lambda)(r) \leq (1 - \text{Exp}_{\text{Gromov}}(X; \varepsilon, \rho)) \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho)^2 \cdot \exp(-((\ln(\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho)))r/\rho). \]

On account of (34), one can see that the procedure for the proof of Theorem 6.1 works for (36) as well. We leave it to the reader to verify its computation.
Consequently,

\[
\text{ObsDiam}(X; -\kappa) \leq \frac{2\rho \ln \left( 2(1 - \text{Exp}_\text{Ledoux}(X; 1 - \varepsilon, \rho)) \frac{(1 - \varepsilon)^2}{\text{Exp}_\text{Ledoux}(X; 1 - \varepsilon, \rho)^2} \right)}{\ln \text{Exp}_\text{Ledoux}(X; 1 - \varepsilon, \rho)},
\]

whence the current corollary establishes computationally.

Theorem 6.1 and Corollary 6.4 yield the upper bound for the observable diameter:

**Corollary 6.5.** Under the hypotheses of Theorem 6.1 and Corollary 6.4, the upper bound for the observable diameter is in terms of Ledoux’s expansion coefficient:

\[
\text{ObsDiam}(X; -\kappa) \leq \frac{2\rho}{\ln \text{Exp}_\text{Ledoux}(X; 1 - \varepsilon, \rho)} \ln \left( \frac{2(\text{Exp}_\text{Ledoux}(X; 1 - \varepsilon, \rho))^2}{(1 + (\text{Exp}_\text{Ledoux}(X; 1 - \varepsilon, \rho))^2)^{\varepsilon \kappa}} \right).
\]

**Proof.** By combining (31) and (35), we have

\[
\kappa \exp \left( \text{ObsDiam}(X; -\kappa) \ln(\text{Exp}_\text{Ledoux}(X; 1 - \varepsilon, \rho))/2\rho \right) \leq 2 - \kappa \exp \left( ((\text{ObsDiam}(X; -\kappa) - 4\rho) \ln(\text{Exp}_\text{Ledoux}(X; 1 - \varepsilon, \rho))/2\rho) \right),
\]

from which the desired result follows computationally.

**Example 6.6.** Let \( M \) be an \( n \)-dimensional compact connected Riemannian manifold with \( \text{Ric}(M) \geq 0 \). Corollary 6.5 yields the upper bound for the observable diameter of \( M \) in terms of the doubling constant of the Riemannian measure of \( M \), which is equal to \( 2^n \), and \( \lambda_1(M) \). We have for some \( \rho > 0 \), which appeared in the context of Example 6.2,

\[
\text{ObsDiam}(M; -\kappa) \leq \frac{2\rho}{\ln(1 + \lambda_1(M)\varepsilon \rho^2)} \min \left\{ \ln \frac{2^{2n+1}(1 - \varepsilon)}{(1 + (1 + \lambda_1(M)\varepsilon \rho^2)^2)^{\varepsilon \kappa}}, \ln \frac{2(1 - \varepsilon)}{\varepsilon \kappa} \right\};
\]

cf. Example 4.7.

### 7 Estimate for the diameter of a bounded metric measure space

The intent of the present appendix is to give an upper bound and a lower bound for the diameter of certain metric measure spaces.
7.1 Upper bound for the diameter

The exponential concentration obtained in (24) and Theorem A.2 are clues to the proof of Theorem 7.1.

**Theorem 7.1.** Let $(X, d_X, \mu_X)$ be a bounded metric measure space with a doubling measure, whose doubling constant is denoted by $C$, and satisfy the requirement $\min\{\exp_{\text{Ledoux}}(X; \varepsilon, \rho), \exp_{\text{Ledoux}}(X; 1-\varepsilon, \rho)\} > 1$ with $\varepsilon \leq 1/2$. Then, its diameter is bounded from above in terms of Ledoux’s expansion coefficient and $C$ as follows:

$$\text{diam}(X) \leq 3\rho \max\left\{ \ln\left(\frac{C^4(1-\varepsilon)\varepsilon^{-1} \exp_{\text{Ledoux}}(X; 1-\varepsilon, \rho)}{\ln \exp_{\text{Ledoux}}(X; 1-\varepsilon, \rho)} \right), \frac{2\ln\left(\frac{C^3\ln C/\ln 2 \exp_{\text{Ledoux}}(X; \varepsilon, \rho)}{\ln \exp_{\text{Ledoux}}(X; \varepsilon, \rho)} \right)}{\ln \exp_{\text{Ledoux}}(X; \varepsilon, \rho)} \right\}, \quad \rho > 0.$$  

*Proof.* As shown in Appendix A, the assumption on the doubling measure implies that Ledoux’s expansion coefficient is finite; see (64). Theorem 5.6 is a crux to prove this theorem. It follows from (24) that for all $r \geq 0$ and for all $A \subset X$, such that $\mu_X(A) \geq \varepsilon$,

$$1 - \mu_X(A_r) \leq (1-\varepsilon) \exp_{\text{Ledoux}}(X; 1-\varepsilon, \rho) \exp(-\ln \exp_{\text{Ledoux}}(X; 1-\varepsilon, \rho))r/\rho). \quad (37)$$

Let us regard $r$ as being sufficiently small and fixed. To estimate $\text{diam}(X)$, we consider a ball with radius $\tau \text{diam}(X)$ centred at $x \in X$ attaining $\text{diam}(X)$, where $\tau \leq 1$ is a positive parameter of $\text{diam}(X)$. We need to observe whether $\tau$ is reasonable. Then, for the desired upper bound to be sharp, note that such a point $x$ makes it allowable, for which we refer the reader to Remark 7.1. Take now a distinct point $z \in X$ of $x$ such that

$$d_X(x, z) = 2r + \tau \text{diam}(X) \leq \text{diam}(X)$$

for some $\tau$ such that $\tau \text{diam}(X) \leq r$. Hence,

$$\tau \leq 1/3. \quad (38)$$

Hereafter, we will evaluate the measure of the ball $B(x, \tau \text{diam}(X))$ separately by means of $\varepsilon$. It follows that

$$\mu_X(B(z, r)) \leq 1 - \mu_X(B(x, r + \tau \text{diam}(X))) \leq 1 - \mu_X(B(x, \tau \text{diam}(X))), \quad (39)$$

$$B(x, \tau \text{diam}(X)) \subset B(z, 2(r + \tau \text{diam}(X))). \quad (40)$$

We now apply Theorem A.2 to $B(z, 2(r + \tau \text{diam}(X)))$ and $B(z, r)$ to obtain

$$\frac{\mu_X(B(z, 2(r + \tau \text{diam}(X))))}{\mu_X(B(z, r))} \leq C^2 \left( \frac{2(r + \tau \text{diam}(X))}{r} \right)^{\frac{\ln C/\ln 2}{r}},$$

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from which it follows that
\[
\mu_X(B(z, r)) \geq C^{-2} \left( \frac{2(r + \tau \text{diam}(X))}{r} \right)^{-\ln C/\ln 2} \mu_X(B(z, 2(r + \tau \text{diam}(X))).
\]  
(41)

Combining (41) with (40), we have
\[
\mu_X(B(z, r)) \geq C^{-2} \left( \frac{2(r + \tau \text{diam}(X))}{r} \right)^{-\ln C/\ln 2} \mu_X(B(x, \tau \text{diam}(X))).
\]  
(42)

We begin by letting \( A = B(x, \tau \text{diam}(X)) \) with \( \mu_X(B(x, \tau \text{diam}(X)) \geq \varepsilon \).

Then, it follows from (39) and (42) that
\[
C^{-2} \left( \frac{2(r + \tau \text{diam}(X))}{r} \right)^{-\ln C/\ln 2} \varepsilon \leq 1 - \mu_X(B(x, \tau \text{diam}(X))) = 1 - \mu_X(A_r).
\]  
(43)

By virtue of (37), plugging it with (43), we obtain
\[
C^{-2} \left( \frac{2(r + \tau \text{diam}(X))}{r} \right)^{-\ln C/\ln 2} \varepsilon \leq (1 - \varepsilon) \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) \exp(-\ln(\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho)r/\rho)).
\]  
(44)

Now, \( r \) being arbitrary, especially letting \( r = \tau \text{diam}(X) \) in (44), gives
\[
\text{diam}(X) \leq \frac{\rho \ln(C^4(1 - \varepsilon)e^{-1} \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho))}{\tau \ln \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho)}
\]  
(45)

whenever \( \varepsilon \leq 1/2 \) for the upper bound of (45) to be legitimate.

We next consider the case \( \mu_X(B(x, \tau \text{diam}(X)) < \varepsilon \).

Then, letting \( A \) be the complement of \( B(x, \tau \text{diam}(X)) \) furnishes that
\[
\mu_X(A) \geq 1 - \varepsilon,
\]  
(46)

\[
B(x, \tau \text{diam}(X)/2) \subset (A_{\tau \text{diam}(X)/2})^c
\]  
(47)

because
\[
A_{\tau \text{diam}(X)/2} \subset (B(x, \tau \text{diam}(X) - \tau \text{diam}(X)/2))^c = (B(x, \tau \text{diam}(X)/2))^c.
\]

Similarly to the case where \( \mu_X(B(x, \tau \text{diam}(X)) \geq \varepsilon \), Theorem A.2 allows us to deduce that
\[
\frac{\mu_X(B(x, \tau \text{diam}(X)))}{\mu_X(B(x, \tau \text{diam}(X)/2))} = \frac{1}{\mu_X(B(x, \tau \text{diam}(X)/2))} \leq C^2 \left( \frac{\text{diam}(X)}{\tau \text{diam}(X)/2} \right)^{\ln C/\ln 2} \leq C^2 \left( \frac{2}{\tau} \right)^{\ln C/\ln 2},
\]  

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which indicates that
\[
C^{-2} \left( \frac{2}{\tau} \right)^{-\ln C/\ln 2} \leq \mu_X(B(x, \tau \text{diam}(X)/2)).
\] (48)

Combining (48) with (47), we have
\[
C^{-2} \left( \frac{2}{\tau} \right)^{-\ln C/\ln 2} \leq 1 - \mu_X(A_\tau \text{diam}(X)/2).
\]

Furthermore, under (46), (37) leads to
\[
C^{-2} \left( \frac{2}{\tau} \right)^{-\ln C/\ln 2} \leq \varepsilon \text{ExpLedoux}(X; \varepsilon, \rho) \exp(-\ln \text{ExpLedoux}(X; \varepsilon, \rho) \tau \text{diam}(X)/2\rho).
\] (49)

It follows from (49) that
\[
diam(X) \leq \frac{2\rho \ln \left( C^3 \tau^{-\ln C/\ln 2} \text{ExpLedoux}(X; \varepsilon, \rho) \right)}{\tau \ln \text{ExpLedoux}(X; \varepsilon, \rho)}.
\] (50)

Finally, we note that (38) gives more, namely one can eventually conclude from (50) that letting \( \tau = 1/3 \) attains the upper bound for the diameter appropriately, hence the theorem.

Remark 7.1. One sees immediately that letting the parameter \( \tau \) be maximal makes it allowable that the upper bound for diam(X) of (50) is sharp. In fact, distinct points attaining diam(X) make \( \tau \) be so.

Remark 7.2. To make (45) legitimate, with an assumption on \( \varepsilon \), it is sufficient to assume that \( \varepsilon \leq 1/2 \) for (45) only. In fact, if (50) does not appear correct, the desired upper bound will be taken as (45).

Remark 7.3. In order to get the sharper upper bound, it is adequate in the proof of Theorem 7.1 to adopt not (28) but (24).

Remark 7.4. To the best of our knowledge, Naor et al. were the first ones to show that the upper bound for the diameter of a certain bounded metric measure space with doubling constant \( C > 3 \) is in terms of the observable diameter; see Theorem 1.7 of [33] for more rigorous treatments.

The current subsection will end up with discussing the upper bounds for the diameters of some metric measure spaces and applying Theorem 7.1 to a Riemannian manifold. The following is the most well-known diameter estimate and control theorem for a Riemannian manifold, which goes back as far as Myers [31] and [32], and is currently called '(Bonnet-)Myers theorem'; see e.g., Section 1 of [3] and p. 378 of [39] for a modern treatment:
Theorem 7.2 ((Bonnet-)Myers theorem). Let $M$ be an $n \geq 2$-dimensional complete connected Riemannian manifold with $\text{Ric}(M) \geq K > 0$. Then, we have

$$\text{diam}(M) \leq \pi \sqrt{\frac{n-1}{K}}.$$  \hfill (51)

Furthermore, $M$ is compact.

Thereafter, Cheng has shown that the equality of (51) holds if and only if $M$ is isometric to an $n$-dimensional Euclidean sphere $\mathbb{S}^n(\delta)$ of radius $\delta > 0$ with constant sectional curvature $K$ given by $(n-1)/\delta^2$, which is referred to as the ‘generalized Toponogov sphere theorem’; see Theorem 3.1 of [8].

The (Bonnet-)Myers theorem for a metric measure space has been established by J. Lott and C. Villani, K.-T. Sturm, and S. Ohta; see [25] and [26], [37] and [38], and [34] for detailed accounts.

Combining Theorem 7.1 with Claim 2 provides the upper bound for the diameter of a compact connected Riemannian manifold in terms of the doubling constant of the Riemannian measure of the manifold and the first non-trivial eigenvalue of the Laplacian on the manifold:

Example 7.3. Let $M$ be an $n$-dimensional compact connected Riemannian manifold with $\text{Ric}(M) \geq 0$. Let $\rho > 0$ be in the context of Claim 2. Applying Theorem 7.1 to $M$, we computationally derive by Claim 2:

$$\text{diam}(M) \leq 3\rho \max \left\{ \frac{\ln \left(2^{5n}(1-\varepsilon)\varepsilon^{-1}\right)}{\ln (1 + \lambda_1(M)\varepsilon \rho^2)} \cdot \frac{2\ln \left(2^{4n}3n\varepsilon\right)}{\ln (1 + \lambda_1(M)(1-\varepsilon)\rho^2)} \right\}.$$

As the preceding study on the current subsection, the diameter upper bounds with spectral (see [22, Section 3]) or with logarithmic Sobolev constant (see [22, Section 4]) are discussed in the references therein.

7.2 Lower bound for the diameter

In the present subsection, we will show that the lower bound for the diameter of a bounded metric measure space is in terms of the Laplace functional on metric measure spaces. As with the preceding study on the current subsection, the diameter lower bounds with spectral gap (see [22, Section 3]) or with a logarithmic Sobolev constant (see [22, Section 4]) are discussed in the references therein.

Definition 7.1 (Laplace functional; cf. Section 1.6 of [21] and Section 1 of [22]).

$$E_{(X, d_X, \mu_X)}(\lambda) = \sup \int_X \exp \left( \lambda f(x) \right) \, d\mu_X \quad \text{for all } \lambda > 0,$$

where the supremum runs over all bounded 1-Lipschitz functions with mean zero on $X$. We call $E_{(X, d_X, \mu_X)}$ the Laplace functional of $\mu_X$ on $X$.

In [21, Section 1.6], the Laplace functional is defined for $\lambda = 0$ as well, whereas we will be concerned only with $\lambda > 0$.
The Laplace functional allows us to establish the concentration measure phenomenon on a bounded Cartesian metric measure space; see [21, Section 1.6].

**Proposition 7.4** (Proposition of [18]). Let \( M \) be a compact Riemannian manifold with Ricci curvature \( \text{Ric}(M) \) bounded below from a positive constant. Under the hypotheses of Definition 7.1, Ledoux shows that

\[
\int_M \exp (\lambda f(x)) \ d\mu_M \leq \exp(\lambda^2 / 2 \text{Ric}(M)).
\]

It follows immediately from Proposition 7.4 that

**Corollary 7.5.**

\[
\mathbb{E}_{(M,d_M,\mu_M)}(\lambda) \leq \exp(\lambda^2 / 2 \text{Ric}(M)).
\]

**Theorem 7.6.** Let \( X \) be a bounded metric measure space. For each \( \lambda > 0 \): If \( f \) is a bounded Lipschitz function with mean zero on \( X \), then

\[
diam(X) \geq \frac{\ln \int_X \exp (\lambda f(x)) \ d\mu_X}{\lambda \| f \|_{\text{Lip}}}. \tag{52}
\]

In particular, if \( f \) is a bounded 1-Lipschitz function with mean zero on \( X \), then

\[
diam(X) \geq \sqrt{2 \ln \mathbb{E}_{(X,d_X,\mu_X)}(\lambda) \lambda}. \tag{53}
\]

Thus under the hypothesis same as (53), we deduce that

\[
diam(X) \geq \frac{1}{\lambda} \min \left\{ \ln \mathbb{E}_{(X,d_X,\mu_X)}(\lambda), \sqrt{2 \ln \mathbb{E}_{(X,d_X,\mu_X)}(\lambda) \lambda} \right\}. \tag{54}
\]

**Proof.** Our proof starts with estimating an exponential integral of a bounded Lipschitz function \( f \) with mean zero on \( X \). The strategy of the proof adopts that of Ledoux [21, Proposition 1.16]. However, the argument of the proof by Ledoux is not given in detail. Thus we will explain the argument in detail. Thus

\[
\int_X \exp (\lambda f(x)) \ d\mu_X
\]

\[
= \int_X \exp (\lambda f(x)) \ d\mu_X \exp \left( -\lambda \int_X f(y) \ d\mu_X \right) \tag{55}
\]

\[
\leq \int_X \exp (\lambda f(x)) \ d\mu_X \int_X \exp (-\lambda f(y)) \ d\mu_X \quad \text{by Jensen’s inequality}
\]

\[
= \iint_{X \times X} \exp (\lambda (f(x) - f(y))) \ d\mu_X d\mu_X \quad \text{by Fubini’s inequality,} \tag{56}
\]

where, in (55), we have used the standing assumption that the mean of \( f \) is equal to zero, and hence

\[
\int_X \exp (\lambda f(x)) \ d\mu_X \leq \iint_{X \times X} \exp (\lambda (f(x) - f(y))) \ d\mu_X d\mu_X. \tag{57}
\]
We see that
\[ \int_{X \times X} \exp(\lambda (f(x) - f(y))) \, d\mu_X d\mu_X \leq \exp(\lambda \| f \|_{Lip} \text{diam}(X)) \]
because \( f \) is Lipschitz on \( X \). Consequently,
\[ \int_X \exp(\lambda f(x)) \, d\mu_X \leq \exp(\lambda \| f \|_{Lip} \text{diam}(X)), \] (58)
from which (52) follows. In particular, if \( f \) is 1-Lipschitz, then by (58), we get
\[ E_{(X,d_{X,\mu_X})}(\lambda) \leq \exp(\lambda \text{diam}(X)). \]
Hence one sees that
\[ \text{diam}(X) \geq \frac{\ln E_{(X,d_{X,\mu_X})}(\lambda)}{\lambda}. \] (59)

Next, we consider the case where \( f \) is a bounded 1-Lipschitz with mean zero on \( X \). The subsequent claim thus is the following:

**Claim 3.** Under the hypotheses of Theorem 7.6,
\[ \exp(\lambda (f(x) - f(y))) \leq \cosh(\lambda \text{diam}(X)) + \frac{f(x) - f(y)}{\text{diam}(X)} \sinh(\lambda \text{diam}(X)). \]

**Proof.** From convexity of the exponential function, it follows for all \( \tau \in \mathbb{R} \) and for all \( x \in \mathbb{R} \) such that \( |x| \leq 1 \) that
\[
\exp(\tau x) \leq \frac{1 + x}{2} \exp(\tau) + \frac{1 - x}{2} \exp(-\tau)
= \frac{1 + x}{2} \left( \frac{\exp(\tau) + \exp(-\tau)}{2} + \frac{\exp(\tau) - \exp(-\tau)}{2} \right)
+ \frac{1 - x}{2} \left( \frac{\exp(\tau) + \exp(-\tau)}{2} - \frac{\exp(\tau) - \exp(-\tau)}{2} \right)
= \frac{1 + x}{2} \cosh(\tau) + \frac{1 - x}{2} \sinh(\tau)
+ \frac{1 - x}{2} \left( \cosh(\tau) - \sinh(\tau) \right)
= \cosh(\tau) + x \sinh(\tau),
\]
namely
\[ \exp(\tau x) \leq \cosh(\tau) + x \sinh(\tau). \] (60)
Since \( f \) is 1-Lipschitz, we readily see that
\[ \frac{f(x) - f(y)}{\text{diam}(X)} \leq \| f \|_{Lip} \leq 1. \] (61)
From (61), we see that the current argument is in agreement with (60). Therefore, Claim 3 holds.
On account of Claim 3, we can now continue estimating (56) as follows:

\[
\iint_{X \times X} \exp (\lambda (f(x) - f(y))) \, d\mu_X d\mu_X \\
\leq \iint_{X \times X} \cosh (\lambda \text{diam}(X)) \, d\mu_X d\mu_X \\
+ \frac{\sinh (\lambda \text{diam}(X))}{\text{diam}(X)} \iint_{X \times X} (f(x) - f(y)) \, d\mu_X d\mu_X \\
= \iint_{X \times X} \cosh (\lambda \text{diam}(X)) \, d\mu_X d\mu_X \\
= \cosh (\lambda \text{diam}(X)) \\
\leq \sum_{i=0}^{\infty} \frac{(\lambda \text{diam}(X))^2 i}{2^i!} \\
= \sum_{i=0}^{\infty} \frac{((\lambda \text{diam}(X))^2/2)^i}{i!} \\
= \exp \left( (\lambda \text{diam}(X))^2/2 \right),
\]

consequently,

\[
\iint_{X \times X} \exp (\lambda (f(x) - f(y))) \, d\mu_X d\mu_X \leq \exp \left( (\lambda \text{diam}(X))^2/2 \right).
\]  

(62)

For this reason, we conclude by adding (57) and (62) that

\[
\int_X \exp (\lambda f(x)) \, d\mu_X \leq \exp \left( (\lambda \text{diam}(X))^2/2 \right),
\]

from which (53) follows immediately. By (53) and (59), one can arrive at (54). This therefore proves the theorem.

\[\square\]

8 Conclusions

We will conclude the body of the paper by mentioning the work in progress. The overall aim of this advanced study is to evaluate Gromov’s and Ledoux’s expansion coefficients in terms of distances and measures on a metric measure space. The study has been motivated by the work of [6] and [9], [10] and [11], who have actually derived some bounds for the spectrum of the Laplacian on a compact connected Riemannian manifold (as a continuous space) and a graph (as a discrete space).

The above-mentioned concentration inequality stated in Propositions 3.5 and 3.6 and the Laplace functional stated in Definition 7.1 will play a pivotal role in our advanced work, which will be discussed elsewhere.
A Sufficient condition for Ledoux’s expansion coefficient to be \( \infty > \text{Exp}_{\text{Ledoux}} > 1 \)

Up to Subsection 7.1 insomuch as we have concerned ourselves with Ledoux’s expansion coefficient to be \( \infty > \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1 \), in the present section, we shall address the sufficient condition for the quantity to be so.

Before the discussion, let us set up the doubling measure, which is often assumed in geometry and analysis on metric measure spaces. We will denote by \( B(x, r) \) a ball in \( X \) with centre \( x \in X \) and radius \( r > 0 \).

**Definition A.1** (Definition 5.2.1 of [1]). Let \( \mathcal{B}(X) \) denote a \( \sigma \)-algebra of all Borel subsets of \( X \). A measure \( \mu_X : \mathcal{B}(X) \to [0, +\infty] \) is said to be doubling if \( \mu_X \) is finite on bounded sets and there exists a constant \( C_{\mu_X} \) such that

\[
\mu_X(B(x, 2r)) \leq C_{\mu_X} \mu_X(B(x, r)) \quad \text{for all } x \in X \text{ and } r > 0. \tag{63}
\]

The best constant \( C_{\mu_X}(\geq 1) \) in (63) is called a doubling constant, which will be briefly written by \( C \).

It follows from iteration of (63) that for an arbitrary integer \( k \geq 0 \)

\[
\mu_X(B(x, 2^k r)) \leq C^k \mu_X(B(x, r)).
\]

**Example A.1.** A typical example of the doubling measure is Lebesgue measure on a Euclidean space.

The following theorem characterizes the doubling measure \( \mu_X \) on metric space \((X, d_X)\) by providing a lower bound for the decay of \( r \mapsto \mu_X(B(x, r)) \) for \( \mu_X \). The characterization will play a crucial role in the estimate for the diameter of \((X, d_X, \mu_X)\) with \( \infty > \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1 \), see Theorem 7.1.

**Theorem A.2** (Theorem 5.2.2 of [1]). Let a measure \( \mu_X : \mathcal{B}(X) \to [0, +\infty] \) be finite on bounded sets. Then, \( \mu_X \) is doubling if and only if there exists a constant \( C > 0 \) such that

\[
\frac{\mu_X(B(y, r_2))}{\mu_X(B(x, r_1))} \leq C^2 \left( \frac{r_2}{r_1} \right)^{\ln C / \ln 2}
\]

for each \( r_i, i = 1, 2 \) such that \( 0 < r_1 \leq r_2 \) and all \( x, y \in X \) such that \( x \in B(y, r_2) \).

**Proposition A.3.** We first observe the assumption that \( \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) < \infty \). We will verify that the doubling measure makes \( \text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) \) finite. Let \( \mu_X \) be a doubling measure on \((X, d_X)\), whose doubling constant is denoted by \( C \). Fix a positive numerical parameter \( \varepsilon < 1 \) arbitrarily. For some \( \rho > 0 \) such that \( \mu_X(B(x, 2\rho)) \leq 1 - \varepsilon \) for all \( x \in X \), combining the context regarding the doubling constant with (21), we deduce that

\[
\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) \leq \frac{\mu_X(B(x, 2\rho))}{\mu_X(B(x, \rho))} \leq C, \tag{64}
\]

as claimed.
Our next claim is to discuss the sufficient condition on $\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1$, for which the Poincaré inequality (see (65) below) on metric measure spaces plays a crucial role. Before stating the condition to be observed, we give the following two quantities: For all local Lipschitz real-valued functions $f$ on $(X, d_X)$,

$$\text{Var}_{\mu_X}(f) := \int_X f^2 \, d\mu_X - \left( \int_X f \, d\mu_X \right)^2$$

and

$$|\nabla f|(x) := \limsup_{y \to x} \frac{|f(x) - f(y)|}{d_X(x, y)},$$

which we call the variance of $f$ with respect to $\mu_X$ and the length of the gradient of $f$ at the point $x \in X$, respectively.

The Poincaré inequality to be stated ensures that $\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1$:

**Theorem A.4** (Corollary 3.2 of [21]). Let $(X, d_X, \mu_X)$ satisfy the Poincaré inequality with respect to the generalized length of gradient $|\nabla f|$ for all locally Lipschitz real-valued functions $f$ on $(X, d_X)$:

$$\text{Var}_{\mu_X}(f) \leq C \int_X |\nabla f|^2 \, d\mu_X$$

(65)

for some universal numerical constant $C > 0$. Then we have $\text{Exp}_{\text{Ledoux}}(X; 1 - \varepsilon, \rho) > 1$.

**Proof.** We omit the proof because its scenario runs almost parallel to that of Theorem 3.1 of [21].

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