Duality in G. Saccomandi’s Challenge on Analytical Solutions to Anti-plane Shear Problem in Finite Elasticity

David Yang Gao

Federation University Australia, Mt Helen, VIC 3353, Australia
Research School of Engineering, Australian National University, Canberra, Australia

Abstract

This note is a response to G. Saccomandi’s recent challenge by showing basic mistakes in his conclusions. The proof is elementary, but leads to some fundamental results in correctly understanding an extensively studied problem in continuum mechanics.

1 Problem and Arguments

The so-called anti-plane shear deformation in the literature \cite{1, 10} is simply defined by

\[ \chi = \{x_1, x_2, x_3 + u(x_1, x_2)\} : \Omega \subset \mathbb{R}^2 \rightarrow B \subset \mathbb{R}^3. \]  

(1)

The only displacement is \( u(x_1, x_2) \) in \( x_3 \) direction, which is also the only unknown to be determined for any given boundary value problems. The deformation gradient is

\[ F = \nabla \chi = \left\{ \frac{\partial \chi_i}{\partial x_j} \right\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_1 & u_2 & 1 \end{pmatrix} \]  

(2)

The principal invariants of \( B = FF^T \) are \( I_1 = I_2 = 3 + \gamma^2, \ I_3 = 1 \), where \( \gamma = |\gamma| \) and \( \gamma = \nabla u = \{u_\alpha\} (\alpha = 1, 2) \) is the shear strain. Thus, the strain energy \( W(F) \) can be equivalently written in the following different forms

\[ W(F) = \bar{W}(I_1, I_2) = \hat{W}(\gamma). \]  

(3)

Simply following the local analysis adopted first by Knowles in \cite{1}, Saccomandi claimed in \cite{2} that the anti-plane shear deformation for a homogeneous, isotropic, incompressible material should be governed by an over-determined system:

\[ \text{div} S(u, p) = 0 \ \forall x \in B \subset \mathbb{R}^3, \]  

(4)

where \( S = \nabla W(F) \) is the first Piola-Kirchhoff stress defined by (Equation (10) in \cite{10})

\[ S = 2\bar{W}_1 F + 2\bar{W}_2 (I_1 F - BF) - pF^{-T}, \ \bar{W}_\alpha = \partial \bar{W}/\partial I_\alpha, \ \alpha = 1, 2, \]  

(5)

\text{“while} p = p(x_1, x_2, x_3) \text{ is an arbitrary function (the Lagrange multiplier associated with the incompressibility constraint), which must be of the form} \]  

\[ p = cx_3 + \bar{p}(x_1, x_2), \]  

(6)

\footnote{See also Equation (2.22) in \cite{1}.}
where \( c \) is a constant \( [2] \). Due to three equations but two unknowns (\( u, p \)), it was proved in \( [1] \) that the strain energy must satisfy additional constraints.

**Theorem (Knowles, 1976 \([1]\))** If the strain energy \( \bar{W}(I_1, I_2) \) is such that the ellipticity condition (i.e. the equation (3.5) in \([1]\) for \( \lambda = 1 \))

\[
[2\gamma(\bar{W}_1 + \bar{W}_2)]_\gamma > 0
\]

holds, then the associated incompressible elastic material admits nontrivial states of anti-plane shear if and only if \( \bar{W}(I_1, I_2) \) also satisfies the following constitutive constraint (i.e. equation (3.22) in \([1]\) for \( \lambda = 1 \))

\[
b\bar{W}_1 + (b - 1)\bar{W}_2 = 0 \text{ for some constant } b, \quad \forall I_1 = I_2 = 3 + \gamma^2, \quad \gamma = |\gamma| \geq 0.
\]  

As highly cited papers \([1, 10]\), Knowles’ over-determined system has been extensively applied to many anti-plane shear deformation problems in literature, see G. Saccomandi’s recent papers \([2, 12, 13, 14]\) and references cited therein.

Dual to the local analysis on the strong form of the equilibrium problem \([4]\), Gao’s approach \([3]\) is based on minimum total potential principle, i.e.

\[
(P) : \min \left\{ \Pi(u) = \int_\Omega \bar{W}(\nabla u) d\Omega - \int_{\Gamma_t} utd\Gamma \quad | \quad u \in \mathcal{U}_c \right\}
\]

where \( t : \Gamma_t \subset \partial\Omega \rightarrow \mathbb{R} \) is a given boundary shear force, \( \mathcal{U}_c \) is the *kinetically admissible space*, in which, homogeneous boundary condition is given. Since the incompressible condition \( \det F(u) \equiv 1 \) is trivially satisfied, \( \mathcal{U}_c \) is a convex set, the weak variation \( \delta \Pi(u) = 0 \) leads to only one equilibrium equation in \( x_3 \) direction:

\[
\text{div}\tau(\nabla u) = 0 \quad \text{in } \Omega, \quad n \cdot \tau(u) = t \quad \text{on } \Gamma_t,
\]

where \( \tau(\gamma) = \nabla \bar{W}(\gamma) \) is the shear stress, \( n = \{n_\alpha\} \) is a unit norm vector on \( \partial\Omega \). The author proved in \([3]\) that for any given \( t(x) \neq 0 \) on \( \Gamma_t \), this well-defined fully nonlinear PDE has at least one nontrivial solution \( u \), which can be obtained analytically by a canonical duality theory developed in \([4]\). Additionally, both global minimizer and local extrema (i.e. local min and local max) can be identified by an associated triality theory \([3]\).

In G. Saccomandi’s recent paper \([2]\), instead of finding directly any possible mistake in \([3]\), his arguments are based on Knowles’s over-determined system. He claimed “a major mistake contained in the paper \([3]\)” due to the following two issues in his conclusions:

- “The results contained in \([3]\) are only valid
  - for a very special class of unconstrained (or compressible) materials;
  - for a very special class of incompressible materials and in the special case that the gradient of the pressure in the \( x_3 \) direction is null (i.e., \( c = 0 \)).”

## 2 Duality in Arguments

It is interesting to see a multi-level duality pattern in the arguments:

**Level 1**: By using opposite approaches (local vs. global), Knowles and Gao obtained two different systems (over-determined vs. determined).
**Level 2:** If G. Saccomandi’s conclusions in [2] hold, then his related works are correct, but Gao’s paper contained “a major mistake”. Dually, if Gao is correct, then Saccomandi’s both conclusions and his related works (such as [13]) must be wrong.

The prove of the dual statement in Level 2 is elementary.

First, by the facts that for any given anti-plane shear deformation problems, the incompressibility condition \( \det \mathbf{F}(u) \equiv 1 \) is trivially satisfied and the equivalent forms [3] hold without any additional constitutive constraints, any critical solution to the variational problem (\( P \)) must satisfy the condition \( \det \mathbf{F}(u) \equiv 1 \). Also, the author never claimed that the results in [3] are valid for general constrained materials. Therefore, Saccomandi’s first conclusion is incorrect.

Second, by the fact that the strain energy \( \bar{W}(I_1, I_2) \) depends only on \((x_1, x_2) \in \Omega, \) the stress \( \mathbf{S} \) is independent of \( x_3 \). Also, since the condition \( \det \mathbf{F} = 1 \) is independent of \( x_3 \), its Lagrange multiplier \( p \) should be independent of \( x_3 \) and we must have \( c \equiv 0 \) in [10]. Therefore, the Saccomandi’s second conclusion is wrong.

Indeed, even Knowles himself found in his second paper that “all quantities are independent of \( x_3 \)” (the first line on page 4 [10]). Also, in [2] Saccomandi let \( p = -2W_3(I_1, I_2, 1) \). If he is careful, he should know that the gradient of the pressure in the \( x_3 \) direction must be null. Unfortunately, this obvious mistake happened in his related works, see page 168 in [13].

It is well-known that the equilibrium equations obtained (under certain regularity conditions) by potential variational principle are naturally compatible regardless of any possible constitutive laws. Since \( \det \mathbf{F}(u) \equiv 1 \) is trivially satisfied, the incompressibility condition is not a variational constraint. According to the complementarity condition \( p (\det \mathbf{F} - 1) = 0 \) in KKT theory [11], the Lagrange multiplier \( p \) in [2] could be any arbitrary nonzero parameter, but can’t be considered as an unknown variable. By the fact that

\[
\tau = \nabla \hat{W}(\gamma) = \frac{\partial \hat{W}(I_1, I_2)}{\partial \gamma} = \hat{W}_1 \frac{\partial I_1}{\partial \gamma} + \hat{W}_2 \frac{\partial I_2}{\partial \gamma} = 2\gamma[\hat{W}_1 + \hat{W}_2] \tag{11}
\]

Saccomandi’s equation (1.7) in [2] is exactly the same as Gao’s equation (10) in \( x_3 \) direction. Since there is no displacement in both \( x_1 \) and \( x_2 \) directions, Saccomandi’s two extra equilibrium equations (1.6) in [2] can’t be obtained via either potential variational principle or the virtual-work principle. Therefore, these two equations are not needed for the problem considered. Indeed, due to the arbitrary Lagrange multiplier \( p \) in these two equations, they could be self-balanced, but are useless for the anti-plane shear deformation problem.

Finally, let us check the constitutive constraints Saccomandi emphasized in his challenge. By the condition \( I_1 = I_2 \) and chain rule in calculus, we must have \( \hat{W}_1 = \hat{W}_2 (\partial I_2 / \partial I_1) = \hat{W}_2 \). Thus, Knowles’ constitutive constraint [8] is naturally satisfied for \( b = 1/2 \). Also by [11] one can easily check that Knowles’ ellipticity condition [11] is just a special case of the strong Legendre condition \( \nabla^2 \hat{W}(\gamma) > 0 \) \( \forall \gamma \), which can only guarantee the convexity of \( W(\mathbf{F}) = \hat{W}(\gamma) \). Therefore, Knowles’ constitutive constraints are neither necessary nor sufficient for an incompressible homogeneous elastic material to admit nontrivial states of anti-plane shear. Indeed, by the canonical duality theory, complete sets of analytical solutions have been obtained for general nonconvex finite deformation problems, and the existence of these solutions depends mainly on boundary conditions and external forces [3, 5, 6, 7].

Ellipticity in PDEs is a classical concept originally from linear systems where the stored energy is a quadratic function \( \hat{W}(\nabla u) = \frac{1}{2}H_{\alpha\beta}u_{\alpha\beta} \) and the linear operator \( L[u] = -[H_{\alpha\beta}u_{\beta}]_{,\alpha} \)

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is elliptic if the Hessian matrix $\{H_{\alpha\beta}\} \succ 0$. However, this definition is only for convex systems. It is shown in the recent paper [9] that for nonconvex systems, the ellipticity depends not only on the stored energy $W(F)$, but also on the external force field. For any given nonconvex $W(F)$, the problem can have unique solution as long as the external force is bigger enough. This result is naturally included in the triality theory with extensive applications in multidisciplinary fields of nonconvex analysis and global optimization [8].

3 Conclusions

The conclusions contained in [2] are wrong. The proof is truly elementary.

This mistake leads to many other problems in G. Saccomandi’s related papers. By the fact that $I_1 = I_2$, $I_3 = 1$, the anti-plane shear deformation of a homogeneous elastic body must be governed by a generalized neo-Hookean model, i.e. $W(F) = \bar{W}(I_1)$. The proof is trivial [9].

The constitutive constraint in [1] has been reconsidered, which is automatically satisfied for $b = \lambda/2$. The ellipticity condition in [1, 10] is neither necessary nor sufficient for an isotropic homogeneous elastic material to admit nontrivial states of anti-plane shear deformation. For nonconvex systems, the ellipticity of a fully nonlinear boundary value problem depends not only on the stored energy, but also on the external force. The triality theory can be used to identify both global minimizer and local extrema. Detailed study is given in recent paper [9].

The results presented in [3] are valid for general nonconvex anti-plane shear deformation problems as long as the equivalent form (3) holds for the strain energy functions $\bar{W}$ and $\hat{W}$. Also Gao’s analytical solutions are obtained from global analysis in Banach space, the nonsmoothness is allowed. Dually, even if the overdetermined problem addressed in Saccomandi’s papers is correct, which allows only unique smooth solution due to regularity and ellipticity constraints. By the fact that the mistakes in [2] are elementary and also contained in [13], readers must be careful with entire results presented in [12, 13, 14].

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