Some properties of substochastic matrices

A. Puhalskii

August 30, 2018

Abstract

In this note we establish some properties of matrices that we haven’t been able to find in the literature.

Let $I$ represent the $n \times n$–identity matrix and let $P = (p_{lm})$ represent an $n \times n$ row substochastic matrix of spectral radius less than unity. The next two assertions seem to concern new properties of substochastic matrices, cf., Bellman [1], Gantmacher [2] and Lancaster [3].

**Theorem 1.** The diagonal elements of $(I - P^T)^{-1}$ are maximal elements of their respective rows.

**Proof.** Let $P$ be of size $n \times n$. Let $C = (c_{ml})$ be defined by $C = (I - P^T)^{-1}$. We have that

$$c_{ml} = \frac{1}{\det(I - P^T)} (-1)^{m+l} M_{lm},$$

where $M_{lm}$ represents the $(l, m)$–minor of $I - P^T$. We note that $\det(I - P^T) > 0$. Indeed, $\det I = 1$ and $\det(I - \lambda P^T) \neq 0$ for $\lambda \in [0, 1]$ by $P$ being of spectral radius less than one. By continuity, $\det(I - P^T) > 0$. Thus, one needs to prove that $(-1)^{m+1} M_{lm} \leq M_{nm}$. Suppose that $l = m + 1$. Then one needs that $M_{nm} + M_{m+1,m} \geq 0$. By the determinant being multilinear, $M_{nm} + M_{m+1,m}$ is the determinant of matrix $\tilde{I} - \tilde{P}$, where $\tilde{I}$ is the identity $(n - 1) \times (n - 1)$–matrix and $\tilde{P}$ is the $(n - 1) \times (n - 1)$–matrix that is obtained from $P^T$ by adding up rows $m$ and $m + 1$ and deleting the $m$th column. As $P^T$ is column substochastic, $\tilde{P}$ is column substochastic as well. Hence, it’s of spectral radius less than or equal to one. It follows that $\det(\tilde{I} - \tilde{P}) \geq 0$, so, $M_{m+1,m} + M_{m,m} \geq 0$. Suppose that $l > m + 1$. By transposing adjacent rows and columns one can move row $l$ of $\tilde{I} - \tilde{P}$ into the position of row $m + 1$ and move column $l$ into the position of column $m + 1$, respectively, without disturbing the order in which the other rows and
columns are arranged. The matrix thus obtained is of the form $I - \tilde{P}$. For this matrix, $\tilde{M}_{m+1,m} = (-1)^{l-m-1}M_{lm}$, as the minor sign will flip only when the columns are transposed. Also $\tilde{M}_{mm} = (-1)^{2(l-m-1)}M_{mm}$. Since $\tilde{M}_{mm} \geq (-1)^{l-m-1}M_{lm}$, we conclude that $M_{mm} \geq (-1)^{l-m-2}M_{lm} = (-1)^{l+m}M_{lm}$. The case where $l < m$ is dealt with similarly.

For square matrix $B$, let $B(i|j)$ denote the matrix that is obtained from $B$ by deleting the $i$th row and the $j$th column. Let $b_l$ represent the $l$th column of $B$ with the $l$th entry deleted and let $b_l$ represent the $l$th row of $B$ with $l$th entry deleted. Let $e_i$ represent the $i$th element of the standard basis in $\mathbb{R}^{n-1}$ and let for $l \neq m$

$$f_{ml} = \begin{cases} e_m^T, & m < l, \\ e_{m-1}^T, & m > l, \end{cases}$$

**Theorem 2.** The following identities hold:

$$\frac{p_m((I - P)(m|m))^{-1}p_m}{1 - p_{mm} - p_m((I - P)(m|m))^{-1}p_m} = \sum_{k \neq m} p_{km}f_{mk}((I - P)(ll))^{-1}p_k$$

and

$$\frac{(1 - p_{mm})f_{lm}((I - P)(m|m))^{-1}p_m}{1 - p_{mm} - p_m((I - P)(m|m))^{-1}p_m} = \frac{p_{lm}}{1 - p_{ll} - p_l((I - P)(ll))^{-1}p_l} + \sum_{k \neq l, k \neq m} p_{km}f_{lk}((I - P)(k|k))^{-1}p_k$$

The assertion of the theorem is a special case of the following result.

**Theorem 3.** Let $B = (b_{ij})$ be an $n \times n$ matrix with nonzero principal minors. The following identities hold:

$$\frac{b_{mm}(B(m|m))^{-1}b_m}{b_{mm} - b_m(B(m|m))^{-1}b_m} = \sum_{l \neq m} b_{lm}f_{ml}(B(ll))^{-1}b_l$$

(1)

and

$$\frac{-b_{mm}f_{lm}(B(m|m))^{-1}b_m}{b_{mm} - b_m(B(m|m))^{-1}b_m} = -\frac{b_{lm}}{b_{ll} - b_l(B(ll))^{-1}b_l} + \sum_{k \neq l, k \neq m} b_{km}f_{lk}(B(kk))^{-1}b_k.$$

(2)
We precede the proof with two lemmas. Let adj be used to denote the adjoint matrix and let $M_{ij}(l|m)$ denote the $(i,j)$–minor of the matrix $B(l|m)$.

**Lemma 1.** If $l \neq m$, then 

$$f_{ml} \ adj(B(l|m)) b_{l} = (-1)^{m+l+1} \ det(B(l|m)).$$

**Proof.** Suppose that $l > m$. We have that 

$$e_m^T \ adj(B(l|m)) b_{l} = \sum_{j=1}^{l-1} (-1)^{m+j} M_{jm}(l|m) b_{jl} + \sum_{j=l+1}^{n} (-1)^{m+j-1} M_{j-1,m}(l|m) b_{jl}.$$ 

Since $M_{jm}(l|m) = M_{j,l-1}(l|m)$ when $j \leq l-1$ and $M_{j-1,m}(l|m) = M_{j-1,l-1}(l|m)$ when $j \geq l+1$, we have that $e_m^T \ adj(B(l|m)) b_{l} = (-1)^{m+l+1} \ det(B(l|m))$.

Suppose that $l < m$. Similarly to the above, 

$$e_{m-1}^T \ adj(B(l|m)) b_{l} = \sum_{j=1}^{l-1} (-1)^{m+j-1} M_{jm-1}(l|m) b_{jl} + \sum_{j=l+1}^{n} (-1)^{m+j} M_{j-1,m-1}(l|m) b_{jl}$$

$$= \sum_{j=1}^{l-1} (-1)^{m+j-1} M_{jl}(l|m) b_{jl} + \sum_{j=l+1}^{n} (-1)^{m+j} M_{j-1,l}(l|m) b_{jl} = (-1)^{m+l+1} \ det(B(l|m)).$$

\[\square\]

**Lemma 2.** For arbitrary $l = 1,2,\ldots,n$, 

$$b_{ll} \ det(B(l|l)) - b_{l} \ adj(B(l|l)) b_{l} = det(B).$$

**Proof.** Since 

$$b_{l} \ adj(B(l|l)) b_{l} = \sum_{j \neq l} b_{lj} f_{jl} \ adj(B(l|l)) b_{l},$$

an application of Lemma [1] yields 

$$b_{l} \ adj(B(l|l)) b_{l} = \sum_{j \neq l} b_{lj} (-1)^{j+l+1} \ det(B(l|j)) = -det(B) + b_{ll} \ det(B(l|l)).$$

\[\square\]

**Proof of Theorem 3.** Equation (1) holds if and only if 

$$\frac{b_{mm} \ adj(B(m|m)) b_{m}}{b_{mm} \ det(B(m|m)) - b_{m} \ adj(B(m|m)) b_{m}} = \sum_{l \neq m} \frac{b_{lm} f_{ml} \ adj(B(l|l)) b_{l}}{b_{ll} \ det(B(l|l)) - b_{l} \ adj(B(l|l)) b_{l}}.$$ 

(3)
By Lemma 2, the denominators in (3) equal \(\det(B)\). One thus needs to prove that
\[
b_m \cdot \text{adj}(B(m|m))b_m = \sum_{l \neq m} b_{lm} f_{ml} \text{adj}(B(l|l))b_l.
\]

By Lemma 1,
\[
\sum_{l \neq m} b_{lm} f_{ml} \text{adj}(B(l|l))b_l = \sum_{l \neq m} b_{lm} (-1)^{m+l+1} \det(B(l|m)) = -\det(B) + b_{mm} \det(B(m|m)),
\]
which concludes the proof of (1) by Lemma 2.

Multiplying the numerators and denominators in (2) with the determinants of the matrices being inverted and applying Lemma 2 obtain that (2) is equivalent to the equation
\[
-b_{mm} f_{lm} \text{adj}(B(m|m))b_m = -b_{lm} \det(B(l|l)) + \sum_{k \neq l, k \neq m} b_{km} f_{lk} \text{adj}(B(k|k))b_k.
\]

By Lemma 1,
\[
\sum_{k \neq l} b_{km} f_{lk} \text{adj}(B(k|k))b_k = \sum_{k \neq l} b_{km} (-1)^{k+l+1} \det(B(k|l)).
\]

Since \(\sum_{k=1}^n b_{km} (-1)^{k+l} \det(B(k|l)) = 0\) because the lefthand side is the determinant of the matrix that is obtained from the matrix \(B\) by replacing the \(l\)th column with the \(m\)th column,
\[
\sum_{k \neq l} b_{km} f_{lk} \text{adj}(B(k|k))b_k = b_{lm} \det(B(l|l)).
\]

Therefore,
\[
\sum_{k \neq l, k \neq m} b_{km} f_{lk} \text{adj}(B(k|k))b_k = b_{lm} \det(B(l|l)) - b_{mm} f_{lm} \text{adj}(B(m|m))b_m.
\]

Equation (4) is proved.

References

[1] R. Bellman. *Introduction to matrix analysis*. Second edition. McGraw-Hill Book Co., New York-Düsseldorf-London, 1970.
[2] F. R. Gantmacher. *The theory of matrices. Vol. 1*. AMS Chelsea Publishing, Providence, RI, 1998. Translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation.

[3] P. Lancaster. *Theory of matrices*. Academic Press, New York-London, 1969.