KdV Charges and the Generalized Torus Partition Sum in $T\bar{T}$ deformation

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We consider KdV currents in a quantum field theory obtained by deforming a two dimensional conformal field theory on a cylinder via the irrelevant operator $T\bar{T}$. In this paper we determine their one–point functions modular properties. We find that the one–point functions factorize into two components each with a definite modular weight. We also obtain a general differential equation that the generalized torus partition sum satisfies.
1. Introduction

Conformal field theories in two dimensions can be characterized by two copies of an infinite dimensional symmetry algebra known as the Virasoro algebra $\text{Vir}$. The two copies are generated by the moments of the left–moving (holomorphic) $T$ and right–moving (antiholomorphic) $\bar{T}$ components of the energy momentum tensor.

The universal covering algebra $U\text{Vir}$ generated by polynomials in the Virasoro generators contains an infinite dimensional abelian subalgebra $[2, 3, 4]$. The generators of this subalgebra are obtained by integrating conserved currents built from positive powers of the energy momentum tensor and its derivatives. These generators are known as the (quantum) KdV charges. We denote the left–moving (holomorphic) charges, following the notation used in $[5, 6]$, as $P_s$. They have the form

$$P_s = \frac{1}{2\pi} \oint dz T_{s+1}(z), \quad T_{s+1}(z) := T^{s+1} : + \cdots, \quad \partial T = 0,$$

where $\cdots$ represents normal ordered lower polynomial in $T$ and its derivatives. The first left–moving (holomorphic) KdV charge $P_1$, for example, is the zero mode of the left–moving (holomorphic) component of the energy momentum tensor $T$. The subscript $s$ indicates the charge’s spin, and it always takes odd values $[7]$. For convenience, we denote the right–moving (antiholomorphic) counterparts as $P_{-s}$.

The mutually commuting KdV charges can be used to define a generalized partition function $[8]$. For a conformal field theory on a cylinder the generalized torus partition sum takes the form

$$Z (\{\nu_s\}) = \text{Tr} \left[ e^{-2\pi i \sum_s \nu_s P_s} \right],$$

where $\nu_s$ is the chemical potential for the charge $P_s$.

Under $TT$ deformation $[3, 10]$ it is argued that the KdV charges can be adjusted to remain conserved, and mutually commuting along the deformation. That is, $TT$ deformation preserves the infinite dimensional subalgebra $[2, 6]$. The (adjusted) left–moving KdV charges $P_s$ in the resulting quantum field theory take the form

$$P_s = \frac{1}{2\pi} \oint (dz T_{s+1}(z) + d\sigma \Theta_{s-1}), \quad \partial T_{s+1} - \partial \Theta_{s-1} = 0.$$  

Analogous expression hold for the adjusted right–moving KdV charges $P_{-s}$.

In this paper we consider $TT$ deformation of a conformal field theory on an infinite cylinder. We derive a recurrence relation for the KdV charges one–point functions using
their flow equations and we determine their modular transformation properties. We also obtain a differential equation that the generalized torus partition sum obeys.

The rest of the paper is organized as follows. In section 2 we obtain a recurrence relation for the KdV charges one-point functions at zero chemical potentials in the $T\overline{T}$ deformed quantum field theory on a cylinder. We discuss also their modular transformation properties. In section 3 we derive the flow equations for the KdV charges as a consistency check from the recurrence relation obtained in the previous section. In section 4 we derive a differential equation for the generalized torus partition sum with all the chemical potentials turned on. In section 5 we discuss the main results and future research directions.

2. KdV charges one–point functions

We consider deforming a two dimensional conformal field theory on an infinite cylinder of radius $R$ via the irrelevant operator $T\overline{T}$. We first consider the torus partition sum with only the chemical potential $\nu_0 := \nu$ corresponding to a $U(1)$ current is turned on. We denote the charge corresponding to this conserved current $P_0 := Q$.

Under the deformation it is argued that the charge $Q$ does not deform. Thus, the torus partition sum in the deformed theory is given by

$$Z(\tau, \overline{\tau}, \nu|\lambda) = \sum_n e^{2\pi i \tau_1 R P_n - 2\pi \tau_2 R E_n - 2\pi i \nu Q_n} = \sum_n e^{2\pi i \overline{\tau} R \langle P_{-1} \rangle_n - 2\pi i \tau R \langle P_{+1} \rangle_n - 2\pi i \nu Q_n},$$

(2.1)

where $\tau = \tau_1 + i\tau_2$ is the complex modulus of the torus, $\lambda$ is the dimensionless coupling of the deformation, and the sum runs over all the mutual eigenstates $|n\rangle_\lambda$ of the Hamiltonian $H$, spatial momentum $P$ and $U(1)$ charge $Q$.

We denote respectively the energy, momentum and $U(1)$ charge of the state $|n\rangle_\lambda$ in the deformed theory as $E_n(\lambda)$, $P_n$ and $Q_n$. The left and right–moving components of the Hamiltonian $H$ in the deformed state $|n\rangle_\lambda$ are given in terms of the KdV charges $P_{\pm 1}$ by

$$\langle P_{\pm 1} \rangle_n = -\frac{E_n \pm P_n}{2}.$$

(2.2)

The flow equations for the energy $E_n(\lambda)$, momentum $P_n$, and $U(1)$ charge $Q_n$ are given by

$$\pi R E_n(\lambda) = \frac{1}{\lambda} \left( \sqrt{1 + 2\lambda \cdot \pi R E_n + \lambda^2 \cdot (\pi R P_n)^2 - 1} \right), \quad \partial_\lambda P_n = 0, \quad \partial_\lambda Q_n = 0, \quad (2.3)$$

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where $E_n$ is the energy of the undeformed state $|n\rangle$ in the original conformal field theory.

The torus partition sum (2.1) can be shown to satisfy a recurrence relation among its expansion coefficients in the dimensionless coupling $\lambda$. We next obtain this relation.

We first write

$$R E_n := \sum_p \lambda^p e^{(p)}_n,$$

and we define the coefficients $Z_p$ as

$$Z := \sum_p \lambda^p Z_p.$$

At each order in the dimensionless coupling $\lambda$ in (2.5) we make the following trades

$$\pi RE_n \to -\frac{1}{2} \partial_{\tau_2}, \quad \pi RP_n \to -\frac{i}{2} \partial_{\tau_1}.$$

By demanding a recurrence relation that involves only positive powers of $\tau_2$ and $\nu$ so that it is well–behaved, we find, after rewriting, the following recurrence relation for the expansion coefficients $Z_p$

$$Z_{p+1} = \frac{1}{p+1} \left[ \tau_2 \left( \mathcal{M}^{(p,p)}_{(\tau,\tau)} + \frac{p}{2\tau_2} \partial_{\tau_2} \right) Z_p - \frac{1}{2} \left( \partial_{\tau_2} \tau_2 \mathcal{M}^{(p-1,p-1)}_{(\tau,\tau)} + \partial_{\tau} \partial_{\tau} \right) Z_{p-1} \right],$$  

(2.7)

where

$$\mathcal{M}^{(p,p)}_{(\tau,\tau)} = D^{(p)}_{\tau} D^{(p)}_{\tau} - \frac{p(p-1)}{4\tau_2^2},$$

(2.8)

and the modular covariant derivates are

$$D^{(p)}_{\tau} = \partial_{\tau} - \frac{ip}{2\tau_2}, \quad D^{(p)}_{\tau} = \partial_{\tau} + \frac{ip}{2\tau_2}.$$  

(2.9)

Acting with $D^{(p)}_{\tau}$ on a modular function or form of weight $(p, q)$ gives a modular function or form of weight $(p + 2, q)$. Similarly, acting with $D^{(q)}_{\tau}$ gives a modular function or form of weight $(p, q + 2)$. To show that indeed the second–order linear homogeneous recurrence relation (2.7) is correct, we obtain from it the flow equations for the energy $E_n$, momentum $P_n$, and $U(1)$ charge $Q_n$. We first rewrite (2.8) using (2.9) as

$$\mathcal{M}^{(p,p)}_{(\tau,\tau)} = \partial_{\tau} \partial_{\tau} + \frac{p}{2\tau_2} \partial_{\tau_2}.$$  

(2.10)

\footnote{Note that $Z_{-1} = 0$.}
After substituting (2.10) in (2.7) and using (2.5) we get
\[ \partial_\lambda Z = \sum_p \lambda^p \cdot (p + 1) Z_{p+1} \]
\[ = \sum_p \left[ \left( \tau_2 \partial_\tau \partial_\tau + \frac{1}{2} \cdot d_\lambda \cdot \partial_\tau \right) + \frac{1}{2} \cdot d_\lambda \cdot \partial_\tau \right] \lambda^p Z_p \]
\[ - \frac{1}{2} \sum_p \lambda \left\{ \partial_\tau \left[ \tau_2 \partial_\tau \partial_\tau + \frac{1}{2} \cdot d_\lambda \cdot \partial_\tau \right] + \partial_\tau \partial_\tau \right\} \lambda^p Z_p, \] (2.11)
which we write as
\[ \left[ \left( 1 - \frac{\lambda}{2} \partial_\tau_2 \right) \cdot \left( \tau_2 \partial_\tau \partial_\tau + \frac{1}{2} \cdot d_\lambda \cdot \partial_\tau \right) - \frac{\lambda}{2} \partial_\tau \partial_\tau \right] Z = 0, \] (2.12)
where \( d_\lambda = \lambda \partial_\lambda \). Note that the power of \( \tau_2 \) is positive. We have from (2.1)
\[ Z = \sum_n e^{2\pi i \tau R(P_1)_{n} - 2\pi i \tau R(P_{+1})_{n} - 2\pi i \nu Q_{n}}. \] (2.13)
Using this in (2.12) we get the correct flow equations
\[ \partial_\lambda e_n + \pi e_n d_\lambda e_n + \frac{\pi}{2} (e_n^2 - p_n^2) = 0, \quad \partial_\lambda p_n = 0, \quad \partial_\lambda Q_n = 0. \] (2.14)
where \( e_n := RE_n \) and \( p_n := RP_n \). This form of the flow equation of the energy is known as Abel equation of the second kind.

We next consider turning on all the chemical potentials \( \nu_s \) that couple with the KdV currents. We write the generalized torus partition sum (2.1) as
\[ Z(\tau, \bar{\tau}, \nu_s | \lambda) = \sum_n e^{2\pi i \tau R(P_1)_{n} - 2\pi i \tau R(P_{+1})_{n} - 2\pi i \nu Q_{n} - 2\pi i \sum_{|s| = 3} \nu_s \langle P_{s} \rangle_{n}}. \] (2.15)
We now generalize the recurrence relation (2.7) for the one–point functions of the KdV charges at zero chemical potentials. We have from their flow equations [3]
\[ \langle P_{s} \rangle_{n} = \langle c_{|s|} \rangle_{n} \langle P_{1} \rangle_{n}^{|s|}, \quad \langle P_{-|s|} \rangle_{n} = \langle c_{-|s|} \rangle_{n} \langle P_{-1} \rangle_{n}^{|s|}. \] (2.16)
where \( \langle c_s \rangle_n \) are independent of \( \lambda \). We look for a recurrence relation that involves only positive powers of the torus modulus \( \tau_2 \) so that it is well–behaved. We first write
\[ \langle P_{s} \rangle_{n} := \sum_p \lambda^p \langle P_{s} \rangle_{n}^{(p)}. \] (2.17)
We expand the one-point function of $P_s$ in powers of the dimensionless coupling $\lambda$ as
\[
Z' := - \partial_{\nu_s} Z|_{\nu_t=0} := \sum_p \lambda^p Z'_p, \quad t = \pm 3, \pm 5, \ldots. \tag{2.18}
\]

In what follows we assume $s$ is positive. For negative $s$ the analysis is exactly identical. Note that we set all the chemical potentials $\nu_t$ to zero. Thus, $Z'_0$ is a modular function of weight $(s + 1, 0)$ under modular transformation \[8\].

To begin with, we assume a recurrence relation of the form \(2.7\) since it reproduces the correct flow equation for the energy, that is, we start with
\[
Z'_{p+1} = \frac{1}{p+1} \left[ \tau_2 \left( M^{(p,p)}(\tau,\tau) + \frac{p}{2 \tau_2} \partial_{\tau_2} \right) Z'_p \right. \left. - \frac{1}{2} \left( \partial_{\tau_2} \tau_2 M^{(p-1,p-1)}(\tau,\tau) + \partial_{\tau} \partial_{\tau} \right) Z'_{p-1} \right], \tag{2.19}
\]

where $M^{(p,p)}(\tau,\tau)$ is given in \(2.8\). We now consider the terms in the sequence to determine $M^{(p,p)}(\tau,\tau)$ for the current case. We have from \(2.18\)
\[
Z'_1 = \sum_n 2\pi i \left( R \langle P_s \rangle^{(1)}_n + R \langle P_s \rangle^{(0)}_n \right) e^{2\pi i R \langle P_{-1} \rangle^{(0)}_n - 2\pi i R \langle P_{+1} \rangle^{(0)}_n}, \tag{2.20}
\]
and from \(2.16\) and \(2.6\)
\[
R \langle P_s \rangle^{(1)}_n = s \cdot (\langle c_s \rangle_n + \langle P_{+1} \rangle^{(0)}_n)^{s-1} \cdot \frac{e^{(1)}_n}{2} \quad \Rightarrow \quad R \langle P_s \rangle^{(1)}_n \rightarrow R \langle P_s \rangle^{(0)}_n \cdot s \cdot -\frac{i}{2} \partial_{\tau}. \tag{2.21}
\]

Using this in \(2.20\) we get
\[
Z'_1 = \left( \tau_2 \partial_\tau - \frac{i}{2} \cdot s \cdot \partial_{\tau} \right) \sum_n 2\pi i R \langle P_{+1} \rangle^{(0)}_n e^{2\pi i R \langle P_{-1} \rangle^{(0)}_n - 2\pi i R \langle P_{+1} \rangle^{(0)}_n}, \tag{2.22}
\]
and since $Z'_0$ has modular weight $(s + 1, 0)$, we rewrite it as
\[
Z'_1 = \tau_2 \left( D^{(s+1)}_{\tau} D^{(0)}_{\tau} + \frac{i}{2 \tau_2} D^{(0)}_{\tau} \right) Z'_0. \tag{2.23}
\]

From \(2.19\), on the other hand, we find
\[
Z'_1 = \tau_2 \partial_\tau \partial_{\tau} Z'_0, \tag{2.24}
\]
and since $Z'_0$ is a modular function of weight $(s + 1, 0)$, we rewrite this as
\[
Z'_1 = \tau_2 \partial_\tau \partial_{\tau} Z'_0 = \left( \tau_2 D^{(s+1)}_{\tau} D^{(0)}_{\tau} + \frac{i}{2} D^{(0)}_{\tau} + \frac{i}{2} s \partial_{\tau} \right) Z'_0. \tag{2.25}
\]
This suggests that we need to adjust $\mathcal{M}_{(\tau,\overline{\tau})}^{(p,p)}$ in (2.7) by adding (2.21) as

$$
\mathcal{M}_{(\tau,\overline{\tau})}^{(p+s+1,p)} := \mathcal{M}_{(\tau,\overline{\tau})}^{(p,p)} + s \cdot \frac{i}{2\tau_2} \partial_\tau = D_\tau^{(p+s+1)}D_\tau^{(p)} - \frac{p(p + s - 1)}{4\tau_2^2} + \frac{i}{2\tau_2}D_\tau^{(p)}.
$$

(2.26)

With this (2.19) becomes

$$
Z'_{p+1} = \frac{1}{p + 1} \left[ \tau_2 \left( \mathcal{M}_{(\tau,\overline{\tau})}^{(p+s+1,p)} + \frac{p}{2\tau_2} \partial_\tau \right) Z'_p - \frac{1}{2} \left( \partial_\tau \tau_2 \mathcal{M}_{(\tau,\overline{\tau})}^{(p+s,p-1)} + \partial_\tau \partial_\tau \right) Z'_{p-1} \right],
$$

(2.27)

where now

$$
\mathcal{M}_{(\tau,\overline{\tau})}^{(p+s+1,p)} = D_\tau^{(p+s+1)}D_\tau^{(p)} - \frac{p(p + s - 1)}{4\tau_2^2} + \frac{i}{2\tau_2}D_\tau^{(p)}.
$$

(2.28)

The recurrence relation (2.27) gives to all orders the correct terms for the one–point functions. We show this in the next section by reproducing from it the flow equations [3] for the KdV charges. Note that $Z'_{1}$ (2.24) has two terms with modular weights $(p+s+1, p)$ and $(p + s, p + 1)$ with $p = 1$. The factorization appears to be generic to all orders in the coupling $\lambda$. We give two more examples.

From (2.27) with (2.28) we find that

$$
Z'_2 = \frac{1}{2} \left( \tau_2 \mathcal{M}_{(\tau,\overline{\tau})}^{(s+2,1)} Z'_1 - \frac{1}{2} \frac{m^{(s+1)}}{\tau_2^2} \partial_\tau Z'_0 - \frac{i}{4\tau_2} \partial_\tau Z'_0 \right).
$$

(2.29)

We showed that $Z'_1$ has a term with modular wight $(s + 2, 1)$, this leads, thus, to terms in $Z'_2$ with modular weights $(s + 3, 2)$ and $(s + 2, 3)$. We note that (2.28) can also be put into the form

$$
\mathcal{M}_{(\tau,\overline{\tau})}^{(p+s+1,p)} = D_\tau^{(p+s)}D_\tau^{(p+1)} - \frac{p(p + s - 2)}{4\tau_2^2} - \frac{i}{2\tau_2}D_\tau^{(p+s)},
$$

(2.30)

thus, the remaining term in $Z'_1$ which has the modular weight $(s + 1, 2)$ gives terms in $Z'_2$ with modular weights $(s + 2, 3)$ and $(s + 3, 2)$. Therefore, $Z'_2$ consists of only terms with modular wights $(p + s + 1, p)$ and $(p + s, p + 1)$ with $p = 2$. We next consider $Z'_3$.

From (2.27) we get

$$
Z'_3 = \frac{1}{3} \left( \tau_2 \mathcal{M}_{(\tau,\overline{\tau})}^{(s+3,2)} Z'_2 - \frac{1}{4} \partial_\tau \partial_\tau \partial_\tau Z'_0 - \frac{1}{2} \partial_\tau \partial_\tau Z'_0 \right).
$$

(2.31)

Using the expression given in (2.23) for $Z'_1$ which we rewrite it here as

$$
Z'_1 = \tau_2 \partial_\tau \partial_\tau Z'_0 - \frac{i}{2} s \partial_\tau Z'_0.
$$

(2.32)
we rewrite the last two terms in (2.31) as
\[
\frac{1}{3} \left[ \frac{i(s + 2)}{4} \partial_\tau \partial_\tau^2 - \frac{1}{2} \tau_2 \partial_\tau^2 D_\tau^{(2)} \partial_\tau - \frac{i}{4\tau_2} \partial_\tau - \frac{1}{2\tau_2} \partial_\tau \partial_\tau \right] Z'_0, \tag{2.33}
\]
This can be put into the natural form
\[
\frac{1}{3} \left[ -\frac{i\tau_2}{2} D_\tau^{(s+3)} D_\tau^{(s+1)} D_\tau^{(2)} \partial_\tau - \frac{i(s + 2)}{4} \partial_\tau^2 + \frac{s}{4\tau_2} D_\tau^{(s+1)} \partial_\tau + \frac{is(s + 2)}{8\tau_2} \partial_\tau \right] Z'_0. \tag{2.34}
\]
Thus, we note using (2.34) in (2.31) that $Z'_3$ involves only terms with modular weights $(p + s + 1, p)$ and $(p + s, p + 1)$ with $p = 3$.

This leads naturally to conjecture that $Z'_{p+1}$ consists of only terms with modular weights $(p + s + 2, p + 1)$ and $(p + s + 1, p + 2)$. Since the coupling has modular weight $(-1, -1)$, this implies that the one–point function factorizes into two components with modular weights $(s + 1, 0)$ and $(s, 1)$. In particular, for $s = 0$, we only have terms of modular weights $(1, 0)$ and $(0, 1)$.

Similarly, by considering the one–point function of $P_{-|s|}$, we find that for negative $s$ we need to make the following adjustment
\[
\mathcal{M}_{(\tau, \overline{\tau})}^{(p, p + |s| + 1)} := \mathcal{M}_{(\tau, \overline{\tau})}^{(p, p)} - \frac{i}{2\tau_2} \partial_\tau = D_\tau^{(p)} D_\overline{\tau}^{(p + |s| + 1)} - \frac{p(p + |s| - 1)}{4\tau_2^2} - \frac{i}{2\tau_2} D_\tau^{(p)} + \frac{|s| + 1}{4\tau_2^2}, \tag{2.35}
\]
which is the complex conjugate of (2.26). In this case $Z'_1$ has two terms with modular weights $(p, p + |s| + 1)$ and $(p + 1, p + |s|)$ with $p = 1$. Similarly, $Z'_{p+1}$ involves only terms with modular weights $(p + 1, p + |s| + 2)$ and $(p + 2, p + |s| + 1)$.

In what follows we use the above recurrence relations to derive the flow equations \[8\] for the KdV charges and from the resulting flow equations a general differential equation that the generalized torus partition sum satisfies.

3. KdV charges spectrums

In this section we obtain the flow equations for the KdV charges using the recurrence relation (2.27) that we found in the previous section. We first consider the case in which $s$ is positive.

We first rewrite (2.28) as
\[
\mathcal{M}_{(\tau, \overline{\tau})}^{(p + s + 1, p)} = \partial_\tau \partial_\tau + \frac{p}{2\tau_2} \partial_\tau \partial_\tau - \frac{i}{2\tau_2} s \partial_\tau. \tag{3.1}
\]
Using this (2.27) becomes
\[(p+1)Z'_{p+1} = \left(\tau_2 \partial_\tau \partial_\tau + p\partial_\tau - \frac{is}{2} \partial_\tau\right) Z'_{p} - \frac{1}{2} \left[ \partial_\tau_2 \left(\tau_2 \partial_\tau \partial_\tau + \frac{p-1}{2} \partial_\tau - \frac{is}{2} \partial_\tau\right) + \partial_\tau \partial_\tau\right] Z'_{p-1}.\]

Now using (3.2) in (2.18) we get
\[\partial_\lambda Z' = \sum_p \lambda^p \cdot (p + 1) Z'_{p+1},\]
\[= \sum_p \left[ \left(\tau_2 \partial_\tau \partial_\tau + \frac{1}{2} d_\lambda \partial_\tau - \frac{is}{2} \partial_\tau\right) + \frac{1}{2} d_\lambda \partial_\tau_2 \right] \lambda^p Z'_p\]
\[- \frac{1}{2} \sum_p \lambda \left\{ \left[ \partial_\tau_2 \left(\tau_2 \partial_\tau \partial_\tau + \frac{1}{2} d_\lambda \partial_\tau - \frac{is}{2} \partial_\tau\right) + \partial_\tau \partial_\tau\right] \right\} \lambda^p Z'_p,
\]
which we rewrite as
\[\left[ \left(1 - \frac{\lambda}{2} \partial_\tau_2\right) \cdot \left(\tau_2 \partial_\tau \partial_\tau + \frac{1}{2} d_\lambda \cdot \partial_\tau - \frac{is}{2} \partial_\tau - \partial_\lambda\right) - \frac{\lambda}{2} \partial_\tau \partial_\tau\right] Z' = 0.\]

We recall (2.18) which is
\[Z' = 2\pi i \sum_n \langle P_s \rangle_n e^{2\pi i \tau R(P_{-1})_n - 2\pi i \tau R(P_1)_n},\]
using (3.3) in (3.4) we find
\[\partial_\lambda \langle P_s \rangle_n + \pi e_\lambda \langle P_s \rangle_n + \frac{\pi s \langle P_s \rangle_n}{2} (e_n - p_n) = 0, \quad \partial_\lambda e_n + \pi e_\lambda \partial_\lambda e_n + \frac{\pi}{2} (e_n^2 - p_n^2) = 0, \quad \partial_\lambda p_n = 0,\]
where \(d_\lambda = \lambda \partial_\lambda, e_n = E_n R,\) and \(p_n = P_n R.\)

For negative \(s\) we get from (2.18) with (2.35)
\[Z'_{p+1} = \left(\tau_2 \partial_\tau \partial_\tau + p\partial_\tau - \frac{is}{2} \partial_\tau\right) Z'_{p} - \frac{1}{2} \left[ \partial_\tau_2 \left(\tau_2 \partial_\tau \partial_\tau + \frac{p-1}{2} \partial_\tau - \frac{is}{2} \partial_\tau\right) + \partial_\tau \partial_\tau\right] Z'_{p-1},\]
from this it follows that
\[\partial_\lambda \langle P_s \rangle_n + \pi e_n \partial_\lambda \langle P_s \rangle_n - \frac{\pi s \langle P_s \rangle_n}{2} (e_n + p_n) = 0, \quad \partial_\lambda e_n + \pi e_n \partial_\lambda e_n + \frac{\pi}{2} (e_n^2 - p_n^2) = 0, \quad \partial_\lambda p_n = 0.\]

Equations (3.6) and (3.8) can be put into the compact form
\[\partial_\lambda \langle P_s \rangle_n + \pi e_n \partial_\lambda \langle P_s \rangle_n + \frac{\pi}{2} \langle P_s \rangle_n (|s| e_n - sp_n) = 0, \quad \partial_\lambda e_n + \pi e_n \partial_\lambda e_n + \frac{\pi}{2} (e_n^2 - p_n^2) = 0, \quad \partial_\lambda p_n = 0.\]

These equations are in agreement with the results obtained in [9, 5, 10, 6]. This form of the differential equations is known as Abel equation of the second kind. In the next section we derive a differential equation that the generalized partition sum satisfies.
4. The generalized torus partition sum

In this section using the flow equations we obtain a differential equation that the partition sum satisfies with all the chemical potentials $\nu_s$ turned on. The generalized torus partition sum is given in (2.15) with the charges running with the deformation coupling $\lambda$. We first note that

$$
\left( -\frac{1}{2} d_\lambda \partial_{\tau_2} + \partial_\lambda \right) Z = \sum_{n} \left[ -2\pi \tau_2 \left( \partial_\lambda e_n + \pi e_n d_\lambda e_n \right) - 2\pi i \sum_{s} \nu_s \left( \partial_\lambda \langle P_s \rangle_n + \pi e_n d_\lambda \langle P_s \rangle_n \right) \right]
\times e^{2\pi i \tau R(P_{-1})_n - 2\pi i \tau R(P_{+1})_n - 2\pi i \sum_{s=3} \nu_s \langle P_s \rangle_n}
+ \sum_{n} \pi d_\lambda e_n e^{2\pi i \tau R(P_{-1})_n - 2\pi i \tau R(P_{+1})_n - 2\pi i \sum_{s=3} \nu_s \langle P_s \rangle_n}.
$$

(4.1)

Making use of this we also note that the expression

$$
-\frac{\lambda}{2\pi} \partial_{\tau_2} \left( -\frac{1}{2} d_\lambda \partial_{\tau_2} + \partial_\lambda \right) Z + \frac{1}{\pi} \left( -\frac{1}{2} d_\lambda \partial_{\tau_2} + \partial_\lambda \right) Z,
$$

(4.2)

can be written as

$$
\left( -\frac{\lambda}{2\pi} \partial_{\tau_2} + \frac{1}{\pi} \right) \sum_{n} \left[ -2\pi \tau_2 \left( \partial_\lambda e_n + \pi e_n d_\lambda e_n \right) - 2\pi i \sum_{s} \nu_s \left( \partial_\lambda \langle P_s \rangle_n + \pi e_n d_\lambda \langle P_s \rangle_n \right) \right]
\times e^{2\pi i \tau R(P_{-1})_n - 2\pi i \tau R(P_{+1})_n - 2\pi i \sum_{s=3} \nu_s \langle P_s \rangle_n}
+ \sum_{n} d_\lambda e_n (1 + \pi \lambda e_n) e^{2\pi i \tau R(P_{-1})_n - 2\pi i \tau R(P_{+1})_n - 2\pi i \sum_{s=3} \nu_s \langle P_s \rangle_n}.
$$

(4.3)

Now using the flow equations for the charges in (4.3) we find

$$
\left( -\frac{\lambda}{2\pi} \partial_{\tau_2} + \frac{1}{\pi} \right) \sum_{n} \left[ \pi \tau_2 \left( e_n^2 - p_n^2 \right) + i\pi^2 \sum_{s} \nu_s \langle P_s \rangle_n \right]
\times e^{2\pi i \tau R(P_{-1})_n - 2\pi i \tau R(P_{+1})_n - 2\pi i \sum_{s=3} \nu_s \langle P_s \rangle_n}
+ \frac{\pi^2}{2} \sum_{n} (e_n^2 - p_n^2) e^{2\pi i \tau R(P_{-1})_n - 2\pi i \tau R(P_{+1})_n - 2\pi i \sum_{s=3} \nu_s \langle P_s \rangle_n}.
$$

(4.4)

It follows from (4.4) and (4.2) upon using (2.6) that the generalized torus partition sum with all the chemical potentials turned on satisfies the following differential equation

$$
\left( \frac{\lambda}{2} \partial_{\tau_2} - 1 \right) \left( -\frac{1}{2} d_\lambda \partial_{\tau_2} + \partial_\lambda - \tau_2 \partial_\tau \partial_\tau + \frac{i}{2} \sum_{s} \frac{1}{2} (s \partial_{\tau_1} + i|s| \partial_{\tau_2}) d_\nu_s \right) Z = \frac{\lambda}{2} \partial_\tau Z,
$$

(4.5)
where \( d_{\nu_s} = \nu_s \partial_{\nu_s} \). Note that the powers of \( \tau_2 \) and \( \lambda \) are positive.

We consider the case in which only the charge corresponding to \( s = 0 \) is turned on. From the above differential equation we have

\[
\left( \frac{\lambda}{2} \partial_{\tau_2} - 1 \right) \left( -\frac{1}{2} d_\lambda \partial_{\tau_2} + \partial_\lambda - \tau_2 \partial_\tau \partial_{\tau} \right) Z = \frac{\lambda}{2} \partial_\tau \partial_{\tau} Z. \tag{4.6}
\]

The \( s = 0 \) case is studied in the papers [12, 13] and the partition sum is shown to satisfy the following differential equation

\[
\left( -\frac{1}{2} d_\lambda \partial_{\tau_2} + \partial_\lambda - \tau_2 \partial_\tau \partial_{\tau} \right) Z = -\frac{1}{2 \tau_2} d_\lambda Z. \tag{4.7}
\]

Using equation (4.7) in (4.6) we find that

\[
\frac{\lambda}{2 \tau_2} \left( -\frac{1}{2} d_\lambda \partial_{\tau_2} + \partial_\lambda - \tau_2 \partial_\tau \partial_{\tau} + \frac{1}{2 \tau_2} d_\lambda \right) Z = 0. \tag{4.8}
\]

thus, our result is consistent with [12, 13].

5. Discussion

In this paper we first considered the tours partition sum of a \( T \bar{T} \) deformed theory on a cylinder with only a \( U(1) \) current turned on. We obtained the recurrence relation

\[
Z_{p+1} = \frac{1}{p+1} \left[ \tau_2 \left( \mathcal{M}^{(p,p)}_{(\tau,\bar{\tau})} + \frac{p}{2 \tau_2} \partial_{\tau_2} \right) Z_p - \frac{1}{2} \left( \partial_{\tau_2} \tau_2 \mathcal{M}^{(p-1,p-1)}_{(\tau,\bar{\tau})} + \partial_\tau \partial_{\bar{\tau}} \right) Z_{p-1} \right], \tag{5.1}
\]

where

\[
\mathcal{M}^{(p,p)}_{(\tau,\bar{\tau})} = D^{(p)}_\tau D^{(p)}_{\bar{\tau}} - \frac{p(p-1)}{4 \tau_2^2}, \tag{5.2}
\]

consistent with the flow equations of the energy, momentum and \( U(1) \) charge. Note that the recurrence relation is independent of the chemical potential that couples to the current; the chemical potential does not appear explicitly. Thus, in the case in which there are no \( U(1) \) currents it can be readily used to show that \( Z_{p+1} \) is a modular function of weight \((p+1,p+1)\) consistent with modular invariance of the partition sum.

We next turned on the chemical potentials that couple to higher spin quantum KdV charges. In this case we obtained a recurrence relation for the one–point functions of the KdV charges \( P_s \) at zero chemical potentials. For the left–moving charges we found that

\[
Z'_{p+1} = \frac{1}{p+1} \left[ \tau_2 \left( \mathcal{M}'^{(p+s+1,p)}_{(\tau,\bar{\tau})} + \frac{p}{2 \tau_2} \partial_{\tau_2} \right) Z'_p - \frac{1}{2} \left( \partial_{\tau_2} \tau_2 \mathcal{M}'^{(p+s,p-1)}_{(\tau,\bar{\tau})} + \partial_\tau \partial_{\bar{\tau}} \right) Z'_{p-1} \right], \tag{5.3}
\]
where
\[
M_{p+s+1, p}(\tau, \bar{\tau}) = D_{p+s+1}(\tau)D(p) - \frac{p(p + s - 1)}{4\tau^2} + \frac{i}{2\tau^2}D(p). \tag{5.4}
\]

For the right-moving charges we show that the corresponding recurrence relation is obtained by taking the complex conjugate of (5.3).

By studying their modular properties order by order in the coupling we found that the one-point functions factorize into two components with modular weights \((s + 1, 0)\) and \((s, 1)\) for the left-moving charges, and \((0, |s| + 1)\) and \((1, |s|)\) for the right-moving charges.

We also obtained as a consistency check the flow equations for the charges \([6]\) using the recurrence relation (5.3),
\[
\partial_\lambda \langle P_s \rangle_n + \pi e_n d_\lambda \langle P_s \rangle_n + \frac{\pi}{2} \langle P_s \rangle_n (|s|e_n - sp_n) = 0, \tag{5.5}
\]
where the energy \(E_n\) and momentum \(P_n\) are given by
\[
e_n = E_n R = -R (\langle P_{+1} \rangle_n + \langle P_{-1} \rangle_n), \quad p_n = P_n R = -R (\langle P_{+1} \rangle_n - \langle P_{-1} \rangle_n), \tag{5.6}
\]
which using (5.5) give \([6, 11]\)
\[
\partial_\lambda e_n + \pi e_n d_\lambda e_n + \frac{\pi}{2} (e_n^2 - p_n^2) = 0, \quad \partial_\lambda p_n = 0. \tag{5.7}
\]

We obtained also a general differential equation that the generalized torus partition sum satisfies. The generalized torus partition sum with all the chemical potentials turned on,
\[
Z(\{\nu_s\}|\lambda) = \sum_n e^{-2\pi i \sum_s \nu_s \langle P_s \rangle_n}, \tag{5.8}
\]
satisfies the differential equation
\[
\left[ \frac{i}{2\lambda} (\partial_{\nu_1} + \partial_{\nu_{-1}}) - 1 \right] \left\{ -\frac{i}{2} d_\lambda (\partial_{\nu_1} + \partial_{\nu_{-1}}) + \partial_\lambda - \frac{i}{2} (\nu_1 + \nu_{-1}) \partial_{\nu_1} \partial_{\nu_{-1}} \right. - \frac{i}{2} \sum_s d_{\nu_s} \left[ \left( \frac{s - |s|}{2} \right) \partial_{\nu_1} - \left( \frac{s + |s|}{2} \right) \partial_{\nu_{-1}} \right] \right\} Z = -\frac{\lambda}{2} \partial_{\nu_1} \partial_{\nu_{-1}} Z, \tag{5.9}
\]
where \(\nu_1 = \tau, \nu_{-1} = -\bar{\tau}, d_\lambda = \lambda \partial_\lambda, d_{\nu_s} = \nu_s \partial_{\nu_s}\). Here \(s\) takes the values \(0, \pm 1, \pm 3, \cdots\).

The chemical potential \(\nu_0\) couples to a \(U(1)\) charge. The particular appearances of \(\nu_1\) and \(\nu_{-1}\) is due to the fact that we are deforming the conformal field theory with product of the KdV currents corresponding to \(\nu_1\) and \(\nu_{-1}\). The differential equation (5.3) can be thought of another non-perturbative description of the \(TT\) deformed theory.
In this work, we considered the KdV charges in the case in which only the $T\bar{T}$ coupling turned on. However, in general, one has to also turn on $J\bar{T}$ and $T\bar{J}$ couplings \[14, 13\]. It would be nice to derive the flow equations for the KdV charges and also obtain the generalized torus partition sum from holography. We leave these for future work.

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