In this work we present an analytical model, based on the path-integral formalism of Statistical Mechanics, for pricing options using first-passage time problems involving both fixed and deterministically moving absorbing barriers under possible non-gaussian distributions of the underlying object. We adapt to our problem a model originally proposed to describe the formation of galaxies in the universe of De Simone et al. (2011), which uses cumulant expansions in terms of the Gaussian distribution, and we generalize it to take into account drift and cumulants of orders higher than three. From the probability density function, we obtain an analytical pricing model, not only for vanilla options (thus removing the need of volatility smile inherent to the Black & Scholes (1973) model), but also for fixed or deterministically moving barrier options. Market prices of vanilla options are used to calibrate the model, and barrier option pricing arising from the model is compared to the price resulted from the relative entropy model.

**Keywords:** non-gaussian distribution; stochastic processes; first-passage time; moving barrier, Black and Scholes model; cumulant expansion; path integral; Breeden-Litzenberger theorem; relative entropy.
1. Introduction

In Stochastic Processes, the first passage time $\tau_f$, defined as the time a system takes to cross a barrier for the first time - usually associated to survival analysis - appears in several branches of science: from Biology, in cell transport phenomena, to Economics, in credit default events; Sociology, in group decisions; Physics, in Statistical Mechanics, Optics, Solid State; Chemistry, in reactions and corrosion; and Cosmology. In this latter case, to form a galaxy, the concentration of mass needs to reach a critical value, which can be seen as a barrier, not necessarily fixed, and possibly subject to a stochastic process.

The problem of finding the probability distribution of first passage time was first studied by Schrödinger (1915) in the context of a Brownian motion in a physical medium, and it was shown to be given by the Inverse Normal distribution. In Statistics, the distribution was first obtained by Wald (1947) in likelihood-ratio tests. In stochastic calculus, the problem can be studied in terms of the transition probability distribution between states emerging from boundary conditions imposed to the Fokker-Planck equation (Gardiner (2004), Risken (1989)), from which the cumulative probability distribution that the first passage time occurs after a given instant $T$, that is, $P(\tau_f > T)$, is derived.

The study of first passage time depends on the distribution of the underlying process that is assumed. In the case of galaxy formation, Maggiore & Riotto (2010a) and Maggiore & Riotto (2010b) discuss the treatment of Gaussian distributions, while Maggiore & Riotto (2010c) and De Simone et al. (2011) treat non-Gaussian diffusion, the latter including the case of moving barriers. Usually, the non-Gaussian approach is developed in terms of expansions based on a benchmark distribution, which is commonly taken as the Gaussian one.

In Finance, the first passage time problem may arise in derivative contracts that establish deactivation or activation conditions, upon the passage of a time dependent variable $P_t$ through a barrier $B$:

$$\tau_f = \min \left\{ t \mid P_t > B \right\}.$$  \hspace{1cm} (1.1)

The most frequently used distribution in Finance is the lognormal distribution for prices, in the context of the Black & Scholes (1973) model hypothesis. A single barrier knock-up-and-out european call option (KUO european call) is then a contract that enables its holder to buy a certain asset $S_T$, the underlying asset, at maturity date $T$, paying the contract strike price $K$, as long as the underlying asset does not cross a contractual barrier level $B$. Mathematically, the payoff of such contract is

$$\text{Call}_{KUO} (T) = 1_{S_t < B, \forall t \in [0, T]} \max (S_T - K, 0),$$  \hspace{1cm} (1.2)
Analytical Path-Integral Pricing of Moving-Barrier Options under non-Gaussian Distributions

and its value at any time under the Black & Scholes (1973) hypothesis has closed-form analytical solution, as shown in Shreve (2004). However, although the lognormal distribution requires a single parameter of volatility, evidence is that vanilla (non-barrier) options of different strikes demand different implied volatilities, given their market prices. This structure of volatility, dependent on the strike values, is called volatility smile, and through time it generates the volatility surface. An additional issue is related to barrier clauses in the contract: should we select the volatility according to the strike, to the barrier, or both? Furthermore, in the special case of moving barrier options, Kunitomo & Ikeda (1992) provide analytical solution to the pricing problem, but in a Black-Scholes setup of one volatility.

Several approaches have been proposed in order to explain the volatility smile and, subsequently, to allow exotic options’ pricing, such as barrier options. Among them we may cite the local volatility model (Dupire (1994)), the stochastic volatility model (Heston (1993)), the jump model (Merton (1976)), the relative entropy model (Avellaneda et al. (2001)); and also non-Gaussian distribution models based on expansions, an example of which is the Edgeworth expansion (Rubinstein (1998) and Balieiro & Rosenfeld (2004)). While some of them pose difficulties related to the need of a complete set of market prices to build the volatility surface, others that rely on numerical or simulation implementations demand careful attention regarding the behaviour of the process surrounding the barrier region.

In this paper we adapt the non-Gaussian model of galaxy formation, based on cumulant expansion, developed by De Simone et al. (2011), to the pricing of both fixed and deterministically moving up-and-out barrier options. By doing so, in the limiting case of infinite barrier values, we also obtain a non-Gaussian vanilla pricing model. Our adaptation consists on introducing a drift term in the expansion and also extending it to an arbitrary number of cumulants. In addition, we derive the martingale condition for risk-neutral pricing. The methodology employs the path integral formalism of Statistical Mechanics (Risken (1989)), and results in closed-form expressions for vanillas, fixed and deterministically moving barrier options. The development also takes into account the behaviour of the expansion in the neighbourhood of the barrier.

The model parameters, which are the cumulants, are calibrated with vanilla options and are afterwards used to price barrier options. As long as data for barrier options almost always refer to market-to-model quotes, we compare our results to the ones delivered by the relative entropy model (Avellaneda et al. (2001)).

With respect to the organization of the paper, we begin by describing the theoretical framework, which encompasses the development of Maggiore & Riotto (2010a) to De Simone et al. (2011). We first present the general formulation of cumulant expansion in terms of path integrals and recover the result for the Gaussian fluctuation and fixed barrier. Then we generalize to the case of non-Gaussian fluctuation and moving barrier. Next, we describe the calibration procedure, followed by the barrier options pricing. Finally, we present our
conclusions. We collect in the appendices some results used in the text.

2. Cumulant expansion and the path integral formalism

In this section, we present the cumulant expansion, connecting it to the path integral formalism. Let \( \omega = \omega(t) = \omega_t \) be the stochastic variable whose distribution we wish to model. In our case, \( \omega = \frac{1}{\sigma} \ln \frac{S_t}{S_0} \), where \( \sigma \) is the volatility parameter, \( S_0 \) and \( S_t \) are the underlying values at \( t = 0 \) and \( t \), respectively. A path begins at \( t = 0 \), with \( \omega_0 = 0 \), and evolves until the final instant of time \( t_n \), where \( \omega_n = \omega(t_n) = \omega(T) \). We assume the time discretization \( \Delta t = \epsilon \), with \( t_k = k\epsilon \). A price path is a collection \( \{\omega_1, \ldots, \omega_n\} \), such that \( \omega(t_k) = \omega_k \). If there is no absorbing barrier, \( \omega_t \in (-\infty, \infty) \). The probability density in the space of trajectories can be described by the expected value of a product of Dirac delta functions:

\[
W_n = W(\omega_0, \omega_1, \ldots, \omega_n; t_n) = \langle \delta(\omega(t_1) - \omega_1) \ldots \delta(\omega(t_n) - \omega_n) \rangle, \tag{2.1}
\]

which follows from

\[
\langle \delta(x_1 - \bar{x}_1) \delta(x_2 - \bar{x}_2) \ldots \delta(x_n - \bar{x}_n) \rangle =
\]

\[
\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} p(x_1, x_2, \ldots, x_n) \delta(x_1 - \bar{x}_1) \delta(x_2 - \bar{x}_2) \ldots \delta(x_n - \bar{x}_n) \, dx_1 \, dx_2 \ldots dx_n =
\]

\[
p(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n), \tag{2.2}
\]

which is the probability density. In terms of \( W \), the probability that the variable assumes the value \( \omega_n \) at instant \( t_n \), from \( \omega_0 \), at \( t = 0 \), in trajectories that never exceed \( \omega_c \), is given by:

\[
\Pi(\omega_0, \omega_n; t_n) = \int_{-\infty}^{\omega_c} d\omega_1 \ldots \int_{-\infty}^{\omega_n} d\omega_{n-1} W(\omega_0, \omega_1, \ldots, \omega_n; t_n). \tag{2.3}
\]

And the probability that the path remains in the region \( \omega < \omega_c \), for all instants lower than \( t_n \), is:

\[
\Pi(\omega_0; t_n) = \int_{-\infty}^{\omega_c} d\omega_n \Pi(\omega_0, \omega_n; t_n). \tag{2.4}
\]

\[\text{See Risken (1989), section 2.4.}\]
This equation represents the sum over all possible paths, thus representing the path integral that computes the probability function. We will express it in terms of the cumulants of the distribution. The characteristic function is the Fourier transform of the distribution:

\[ C_n (u_1, ..., u_n) = \langle e^{iu_1\omega_1 + \cdots + iu_n\omega_n} \rangle = \]

\[ \int \cdots \int e^{iu_1\omega_1 + \cdots + iu_n\omega_n} W (\omega_0, \omega_1, ..., \omega_n; t_n) d\omega_1 ... d\omega_n. \quad (2.5) \]

The joint moment function is defined by

\[ M_{m_1, ..., m_n} = (\omega_1^{m_1} ... \omega_n^{m_n}) \]

\[ = \left( \frac{\partial}{\partial (iu_1)} \right)^{m_1} \cdots \left( \frac{\partial}{\partial (iu_n)} \right)^{m_n} C_n (u_1, ..., u_n) \big|_{u_1 = \cdots = u_n = 0}. \quad (2.6) \]

The joint moments are the coefficients of the Taylor expansion of the characteristic function:

\[ C_n (u_1, ..., u_n) = \sum_{m_1, ..., m_n} M_{m_1, ..., m_n} \frac{(iu_1)^{m_1}}{m_1!} \cdots \frac{(iu_n)^{m_n}}{m_n!}. \quad (2.7) \]

The joint cumulants \( \kappa_{m_1, ..., m_n} \) of a distribution are related to the characteristic function by

\[ C_n (u_1, ..., u_n) = \exp \left( \sum_{m_1, ..., m_n} \kappa_{m_1, ..., m_n} \frac{(iu_1)^{m_1}}{m_1!} \cdots \frac{(iu_n)^{m_n}}{m_n!} \right) \quad (2.8) \]

\[ \kappa_{m_1, ..., m_n} = \left( \frac{\partial}{\partial (iu_1)} \right)^{m_1} \cdots \left( \frac{\partial}{\partial (iu_n)} \right)^{m_n} \ln |C_n (u_1, ..., u_n)|_{u_1 = \cdots = u_n = 0}. \quad (2.9) \]

\( \Pi_\epsilon (\omega_0, \omega_n; t_n) \) can be expressed in terms of the cumulants. To see this we use the following representation of the Dirac delta function

\[ \delta (\omega) = \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{-iu\omega}. \quad (2.10) \]

\( ^b \)See Risken (1989), section 2.3.
Substituting in (2.1),

\[ W(\omega_0, \omega_1, ..., \omega_n; t_n) = \left\langle \int_{-\infty}^{\infty} \frac{du_1}{2\pi} \cdots \frac{du_n}{2\pi} e^{-i \sum_{j=1}^{n} u_j(\omega(t_j) - \omega_j)} \right\rangle \]

\[ = \int_{-\infty}^{\infty} \frac{du_1}{2\pi} \cdots \frac{du_n}{2\pi} \cdot e^{-i \sum_{j=1}^{n} u_j \omega_j} \left\langle e^{-i \sum_{j=1}^{n} u_j \omega(t_j)} \right\rangle. \quad (2.11) \]

We define the integration measure

\[ \hat{\int}_{-\infty}^{\infty} Du \equiv \int_{-\infty}^{\infty} \frac{du_1}{2\pi} \cdots \frac{du_n}{2\pi}. \quad (2.12) \]

Therefore, using the definition (2.5) in (2.11) we can write

\[ W(\omega_0, \omega_1, ..., \omega_n; t_n) = \hat{\int}_{-\infty}^{\infty} Du \cdot \exp \left( i \sum_{j=1}^{n} u_j \omega_j + \sum_{m_1, ..., m_n} \kappa_{m_1, ..., m_n} \frac{(-iu_1)^{m_1}}{m_1!} \cdots \frac{(-iu_n)^{m_n}}{m_n!} \right). \quad (2.13) \]

This is the expansion of the probability of a given path in terms of the joint cumulants.

Keeping only terms with \( m_i \) is equal to 0 or 1, that is, \( m_1 = 0, 1; m_2 = 0, 1; ... m_n = 0, 1 \) (we will justify this shortly) results in:

\[ W(\omega_0, \omega_1, ..., \omega_n; t_n) = \hat{\int}_{-\infty}^{\infty} Du \cdot \exp \left( i \sum_{j=1}^{n} u_j \omega_j - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} \right. \]

\[ \left. + \frac{(-i)^3}{3!} \sum_{i,j,k=1}^{n} u_i u_j u_k \kappa_{ijk} + \frac{(-i)^4}{4!} \sum_{i,j,k,l=1}^{n} u_i u_j u_k u_l \kappa_{ijkl} + \ldots \right). \quad (2.14) \]

In this notation, \( \kappa_1 \equiv \kappa_{m_1=1,m_2=0,...,m_n=0}, \kappa_2 \equiv \kappa_{m_1=0,m_2=1,m_3=0,...,m_n=0}, \kappa_12 \equiv \kappa_{m_1=1,m_2=1,m_3=0,...,m_n=0}, \kappa_13 \equiv \kappa_{m_1=1,m_2=0,m_3=1,m_4=0,...,m_n=0}, \) and so on. (2.14) will be the version of equation (2.13) that we will use.

Had we considered other values of \( m_1, m_2, etc \) different from zero and one, we would have included generalized moments, beyond the usual covariance between two variables. For instance, the covariance between the fourth power of a variable \( i \)
and the cube of another variable $j$, in the case of $m_i = 2$ and $m_j = 3$, etc, and they contribute at higher orders. In our case, where we seek to calibrate market data, it will be enough to consider just the usual moments (variance, kurtosis, etc) and, thus, we will not consider covariances and its generalizations in the combinations of the several orders of the variables. In this notation, for example, in $\kappa_{ij}$, when $i = j$, we have the cumulant linked to the variance; in $\kappa_{ijk}$, when $i = j = k$, the cumulant related to asymmetry, etc.

Using this expansion in (2.3) one obtains

$$
\Pi_{\epsilon} (\omega_0, \omega_n; t_n) = \int_{-\infty}^{\omega_c} d\omega_1 \cdots \int_{-\infty}^{\omega_c} d\omega_{n-1} \int_{-\infty}^{\infty} Du \cdot \exp \left( i \sum_{j=1}^{n} u_j \omega_j - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} + \frac{i^3}{3!} \sum_{i,j,k=1}^{n} u_i u_j u_k \kappa_{ijk} + \frac{-i^4}{4!} \sum_{i,j,k,l=1}^{n} u_i u_j u_k u_l \kappa_{ijkl} + \cdots \right).
$$

(2.15)

This equation is the path integral representation of the probability distribution in terms of the cumulants.

3. Gaussian Fluctuations

In this Section we show that our formalism reproduces the well-known formulae for the price of barrier options for Gaussian fluctuations as a sanity check. In the case of Gaussian fluctuations, the cumulants are zero, except those satisfying $m_1 + m_2 + \ldots + m_r \leq 4$.

$$
\left\langle e^{\sum_{j=1}^{n} (-u_j)\omega_j(t_j)} \right\rangle = \exp \left( -i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} \right).
$$

(3.1)

In this case equations (2.14) and (2.15) assume the form (we put a superscript "g" to indicate Gaussian)

$$
W^g (\omega_0, \omega_1, \ldots, \omega_n; t_n) = \int_{-\infty}^{\infty} Du \cdot \exp \left( i \sum_{j=1}^{n} u_j \omega_j - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} \right); \tag{3.2}
$$

[Risken 1989], section 2.3.3.
\[ \Pi_g(\omega_0, \omega_n; t_n) = \int_{-\infty}^{c} d\omega_1 \cdots \int_{-\infty}^{c} d\omega_{n-1} \int_{-\infty}^{c} Du \cdot \exp \left( i \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} \right). \]  

\[ (3.3) \]

Besides, \( \kappa_{ij} = \sigma_{ij} \), where \( \sigma_{ij} \) is the covariance between \( i \) and \( j \).

Consider the case of Markovian processes, where just the previous state of the variable influences the present state:

\[ \Pi(\omega(t_n) \leq \omega_n | \omega(t_{n-1}), \ldots, \omega(t_1)) = \Pi(\omega(t_n) \leq \omega_n | \omega(t_{n-1})). \]  

\[ (3.4) \]

We will denote \( \Pi_{gm} \) and \( W_{gm} \) the Gaussian probability and probability density under the Markov hypothesis, that is, when the particle executes a Markovian Gaussian Brownian motion. In a stationary stochastic process, the moments are constant along time, and their values only depend on the least instant between periods. If the variable \( \omega \) is standard Gaussian, as in a Wiener process,

\[ \sigma_{ij} = <\omega_i \omega_j> = \epsilon \min(i, j) \equiv \epsilon A_{ij}. \]  

\[ (3.5) \]

The probability density \[3.2\] becomes:

\[ W_{gm}(\omega_0, \omega_1, \ldots, \omega_n; t_n) = \int_{-\infty}^{\infty} Du \cdot \exp \left( i \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{\epsilon}{2} \sum_{i,j=1}^{n} u_i u_j A_{ij} \right) \]

\[ = \int_{-\infty}^{\infty} Du \cdot \exp \left( i \sum_{i=1}^{n} u_i (\omega_i - \kappa_i) - \frac{\epsilon}{2} \sum_{i,j=1}^{n} u_i u_j A_{ij} \right). \]  

\[ (3.6) \]

To illustrate, consider one variable \( \omega_i = \omega \). Then,

\[ W^g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \cdot e^{iu(\omega-\kappa_1) - \frac{1}{2}u^2\kappa_2} \]

\[ = \frac{1}{\sqrt{2\pi \cdot \kappa_2}} e^{-\frac{(\omega-\kappa_1)^2}{2\kappa_2}}, \]  

\[ (3.7) \]

where \( \kappa_2 = <\omega^2> = \epsilon. \)
In the case of \( n \) Gaussian Markovian variables, with \( \kappa_2 = \epsilon \) and \( \kappa_1 = \epsilon \alpha \), where \( \alpha \) is the drift:

\[
W^{gm}(\omega_0, \omega_1, \ldots, \omega_n; t_n) = \frac{1}{(2\pi\epsilon)^{n/2}} e^{-\sum_{i=0}^{n-1} \frac{(\omega_{i+1} - \omega_i + \alpha \epsilon)^2}{2\epsilon}}.
\]  (3.8)

Therefore we can write

\[
W^{gm}(\omega_0, \omega_1, \ldots, \omega_n; t_n) = \Psi_\epsilon (\omega_n - \omega_{n-1}) W^{gm}(\omega_0, \omega_1, \ldots, \omega_{n-1}; t_{n-1})
\]  (3.9)

\[
\Psi_\epsilon (\Delta \omega) \equiv \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{(\Delta \omega + \alpha \epsilon)^2}{2\epsilon}}
\]  (3.10)

\[
\Delta \omega = \omega_n - \omega_{n-1}.
\]  (3.11)

Thus,

\[
\Pi^{gm}_\epsilon (\omega_0, \omega_n; t_n) = \int_{-\infty}^{\omega_n} d\omega_{n-1} \Psi_\epsilon (\omega_n - \omega_{n-1}) \Pi^{gm}_\epsilon (\omega_0, \omega_{n-1}; t_{n-1}).
\]  (3.12)

In the presence of a fixed barrier, the probability density in the case of and up absorbing barrier \( B \), to be used in the call KOU pricing, under the Black-Scholes assumptions, is (Shreve (2004)):

\[
\Pi^{gm}_{\epsilon \to 0} (\omega_0, \omega_n; t_n) = \frac{1}{\sqrt{2\pi t_n}} e^{\alpha (\omega_n - \omega_0) - \frac{1}{2} \alpha^2 t_n} \left[ e^{-\frac{(\omega_n - \omega_0)^2}{2t_n}} - e^{-\frac{(2\omega_n - \omega_0 - \omega_c)^2}{2t_n}} \right].
\]  (3.13)

\[
\omega_n = \omega(t_n) = \frac{1}{\sigma} \ln \frac{S_t}{S_0}; \quad \omega_c = b = \frac{1}{\sigma} \ln \frac{B}{S_0}.
\]  (3.14)
4. Analytical expansion for non-Gaussian distributions with moving barrier in the path-integral formalism

In this Section, the non-Gaussian distribution with absorbing moving barrier is obtained from the path integral formulation. As in the work of De Simone et al. (2011), we present two alternative approaches in the expansion: (i) first, the hypothesis of Sheth & Tormen (2002), which states that instants $t_i < t_n$ are insignificant compared to $t_n$ in derivatives higher than the first order and (ii) second, barrier moves slowly. The latter we call “adiabatic barriers”.

The accomplishment of this task involves expanding the non-Gaussian distribution with moving barrier, $\Pi_{\epsilon \rightarrow 0} (\omega_0, \omega_n; t_n)$, in an expression of the form:

$$\Pi_{\epsilon \rightarrow 0} (\omega_0, \omega_n; t_n) = \Pi_{mb}^{\epsilon \rightarrow 0} (\omega_0, \omega_n; t_n) + \text{derivatives of } \Pi_{mb}^{\epsilon \rightarrow 0} (\omega_0, \omega_n; t_n),$$  \hspace{1cm} (4.1)

where, in each approach (Sheth-Tormen and adiabatic barriers), $\Pi_{mb}^{\epsilon \rightarrow 0} (\omega_0, \omega_n; t_n)$ assumes different formats, both involving the Gaussian distribution with fixed barrier (3.13), plus terms regarding moving barriers.

4.1. The Sheth-Tormen approach

Consider the expansion in cumulants (2.15), in the case of a barrier that moves according to a deterministic rule $B_{\epsilon}(t_i)$, $i = 1, \ldots, n - 1$:

$$\Pi_{\epsilon \rightarrow 0} (\omega_0, \omega_n; t_n) = \Pi_{\epsilon \rightarrow 0}^{mb} (\omega_0, \omega_n; t_n) + \text{derivatives of } \Pi_{\epsilon \rightarrow 0}^{mb} (\omega_0, \omega_n; t_n),$$  \hspace{1cm} (4.1)

where, in each approach (Sheth-Tormen and adiabatic barriers), $\Pi_{\epsilon \rightarrow 0}^{mb} (\omega_0, \omega_n; t_n)$ assumes different formats, both involving the Gaussian distribution with fixed barrier (3.13), plus terms regarding moving barriers.

4.1. The Sheth-Tormen approach

Consider the expansion in cumulants (2.15), in the case of a barrier that moves according to a deterministic rule $B(t_i)$, $i = 1, \ldots, n - 1$:

$$\Pi_{\epsilon \rightarrow 0} (\omega_0, \omega_n; t_n) = \int_{-\infty}^{B(t_1)} d\omega_1 \cdots \int_{-\infty}^{B(t_{n-1})} d\omega_{n-1} W (\omega_0, \omega_1, \ldots, \omega_n; t_n),$$  \hspace{1cm} (4.2)

with $W (\omega_0, \omega_1, \ldots, \omega_n; t_n)$ given by (2.14).

Next, we assume that the barrier does not change significantly and expand in a Taylor series around $B(t_n) \equiv B_n$. Therefore,

$$B(t_i) = B(t_n) + \sum_{p=1}^{\infty} \frac{B_n^{(p)} (t_i - t_n)^p}{p!},$$  \hspace{1cm} (4.3)

$$B_n^{(p)} = \frac{d^p B(t_n)}{dt_n^p}.$$  \hspace{1cm} (4.4)

Redefining the variables $\omega_i$, $i = 1, \ldots, n - 1$:

$$\varpi_i \equiv \omega_i - \sum_{p=1}^{\infty} \frac{B_n^{(p)} (t_i - t_n)^p}{p!}$$
Thus, \( \varpi_1 = \omega - (B(t_i) - B(t_n)) \)

\[
d\varpi_1 = d\omega_1. \tag{4.5}
\]

Thus,

\[
\Pi_{\epsilon \to 0} (\omega_0 = 0, \varpi_n; t_n) = \int_{-\infty}^{B_n} d\varpi_1 \ldots \int_{-\infty}^{B_n} d\varpi_{n-1} \int_{-\infty}^\infty Du \cdot e^Z \tag{4.6}
\]

\[
Z = i \sum_{i=1}^n u_i \varpi_i + i \sum_{i=1}^{n-1} u_i \varpi_i \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} (t_i - t_n)^{p} - i \sum_{i=1}^n u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^n u_i u_j \kappa_{ij} + \frac{(-i)^3}{3!} \sum_{i,j,k=1}^n u_i u_j u_k \kappa_{ijk} + \frac{(-i)^4}{4!} \sum_{i,j,k,l=1}^n u_i u_j u_k u_l \kappa_{ijkl} + \ldots \tag{4.7}
\]

Since \( \varpi_1 \) is a dummy variable, we will use the notation \( \omega_i \) again. We work with the expansion until the 5th order, generalizing it later.

\[
Z = i \sum_{i=1}^n u_i \omega_i - i \sum_{i=1}^{n-1} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^n u_i u_j \kappa_{ij} + \frac{(-i)^3}{3!} \sum_{i,j,k=1}^n u_i u_j u_k \kappa_{ijk} + \frac{(-i)^4}{4!} \sum_{i,j,k,l=1}^n u_i u_j u_k u_l \kappa_{ijkl}
\]

\[
+ \frac{(-i)^5}{5!} \sum_{i,j,k,l,m=1}^n u_i u_j u_k u_m \kappa_{ijklm} + \ldots + i \sum_{i=1}^{n-1} u_i \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} (t_i - t_n)^{p}. \tag{4.8}
\]

The first line of this equation is the Gaussian term. Applying the Taylor expansion to the exponential term of the non-Gaussian part (2nd and 3rd lines of (4.8)), one can write:

\[
\Pi_{\epsilon \to 0} (\omega_0 = 0, \omega_n; t_n) = \int_{-\infty}^{B_n} d\omega_1 \ldots \int_{-\infty}^{B_n} d\omega_{n-1} \int_{-\infty}^\infty Du.
\]
The summation term involving the barrier can also be expanded in Taylor series.

We also consider up to second order:

\[
\exp \left( i \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} + i \sum_{i=1}^{n-1} u_i \sum_{p=1}^{\infty} \frac{B_p^{(p)}}{p!} (t_i - t_n)^p \right)
\]

\[
+ \int_{-\infty}^{B_n} d\omega_1 \int_{-\infty}^{B_n} d\omega_{n-1} \int_{-\infty}^{\infty} D u.
\]

\[
\exp \left( i \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} + i \sum_{i=1}^{n-1} u_i \sum_{p=1}^{\infty} \frac{B_p^{(p)}}{p!} (t_i - t_n)^p \right) \cdot
\]

\[
\left( \frac{(-i)^3}{3!} \sum_{i,j,k=1}^{n} u_i u_j u_k \kappa_{ijk} + \frac{(-i)^4}{4!} \sum_{i,j,k,l=1}^{n} u_i u_j u_k u_l \kappa_{ijkl} \right.
\]

\[
+ \frac{(-i)^5}{5!} \sum_{i,j,k,l,m=1}^{n} u_i u_j u_k u_l u_m \kappa_{ijklm} + \ldots \right) . \tag{4.9}
\]

The summation term involving the barrier can also be expanded in Taylor series.

We also consider up to second order:

\[
\exp \left( i \sum_{i=1}^{n-1} u_i \sum_{p=1}^{\infty} \frac{B_p^{(p)}}{p!} (t_i - t_n)^p \right) \simeq 1 + i \sum_{i=1}^{n-1} u_i \sum_{p=1}^{\infty} \frac{B_p^{(p)}}{p!} (t_i - t_n)^p
\]

\[
- \frac{1}{2} \sum_{i,j=1}^{n-1} u_i u_j \sum_{p,q=1}^{\infty} \frac{B_p^{(p)} B_q^{(q)}}{p! q!} (t_i - t_n)^p (t_j - t_n)^q + \ldots \tag{4.10}
\]

[4.9] can be rewritten as:

\[
\Pi_{\epsilon \to 0} (\omega_0 = 0, \omega_n; t_n) = \int_{-\infty}^{B_n} d\omega_1 \ldots \int_{-\infty}^{B_n} d\omega_{n-1} \int_{-\infty}^{\infty} D u.
\]

\[
\exp \left( i \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} \right)
\]

\[
+ \int_{-\infty}^{B_n} d\omega_1 \ldots \int_{-\infty}^{B_n} d\omega_{n-1} \int_{-\infty}^{\infty} D u.
\]
\[
\begin{align*}
\exp \left( i \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} \right) \cdot \\
\left( i \sum_{i=1}^{n-1} u_i \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} (t_i - t_n)^p \right) \\
\left( \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \sum_{p,q=1}^{\infty} \frac{B_n^{(p)} B_n^{(q)}}{p! q!} (t_i - t_n)^p (t_j - t_n)^q \right) \\
\left( \frac{-i}{3!} \sum_{i,j,k=1}^{n} u_i u_j u_k \kappa_{ijk} + \frac{-i}{4!} \sum_{i,j,k,l=1}^{n} u_i u_j u_k u_l \kappa_{ijkl} \\
+ \frac{-i}{5!} \sum_{i,j,k,l,m=1}^{n} u_i u_j u_k u_l u_m \kappa_{ijklm} + \ldots \right). 
\end{align*}
\]
\[
\exp \left( i \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} + i \sum_{i=1}^{n-1} u_i \sum_{p=1}^{\infty} \frac{B_p}{p!} (t_i - t_{n})^p \right)
\]

\[
\left( \frac{-i}{3!} \sum_{i,j,k=1}^{n} u_i u_j u_k \kappa_{ijk} + \frac{(-i)^4}{4!} \sum_{i,j,k,l=1}^{n} u_i u_j u_k u_l \kappa_{ijkl} + \cdots \right)
\]

where the Gaussian Markovian piece \( \Pi_{\epsilon \to 0}^{gm}(\omega_0, \omega_n; t_n) \) was already given in Eq. (3.13).

The remainder of this Section is devoted to the computation of the different terms in Equation (4.12).

Consider (3.2):

\[
W_{gm}(\omega_0, \omega_1, ..., \omega_n; t_n) = \int_{-\infty}^{\infty} \exp \left( i \sum_{j=1}^{n} u_j \omega_j - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} \right)
\]

\[
\equiv \int_{-\infty}^{\infty} \exp (Z_{gm})
\]

with

\[
Z_{gm} = i \sum_{j=1}^{n} u_j \omega_j - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij}.
\]

Defining \( \partial / \partial \omega_i = \partial_i \), we note that

\[
iu_i e^{iu_i \omega_i} = \partial_i e^{iu_i \omega_i}.
\]

Then, the second term of (4.10), in the first term of (4.9) can be rewritten as

\[
\Pi_{\epsilon \to 0}^{(1)}(\omega_n, t_n) = \int_{-\infty}^{\infty} d\omega_1 \cdots \int_{-\infty}^{\infty} d\omega_{n-1}
\]
\[\left[\int_{-\infty}^{\infty} Du \cdot \left( \sum_{i=1}^{n-1} u_i \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} (t_i - t_n)^p \cdot (t_i - t_n)^q \right) \right] \]

\[\exp \left( i \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} \right) \right] \]

\[\int_{-\infty}^{B_n} d\omega_1 \cdots \int_{-\infty}^{B_n} d\omega_{n-1} \left[ \sum_{i=1}^{n-1} \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} (t_i - t_n)^p \cdot \partial_i W^g (\omega_0, \omega_1, \ldots, \omega_n; t_n) \right]. \tag{4.16} \]

The third term of (4.10) also in the first term of (4.9), observing the rule (4.15).

\[\Pi_{\epsilon \to 0}^{(2)} (\omega_n, t_n) = \int_{-\infty}^{B_n} d\omega_1 \cdots \int_{-\infty}^{B_n} d\omega_{n-1} \]

\[\left[\int_{-\infty}^{\infty} Du \cdot \left( -\frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \sum_{p,q=1}^{\infty} \frac{B_n^{(p)} B_n^{(q)}}{p! q!} (t_i - t_n)^p (t_j - t_n)^q \right) \right] \]

\[\exp \left( i \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} \right) \right] \]

\[\int_{-\infty}^{B_n} d\omega_1 \cdots \int_{-\infty}^{B_n} d\omega_{n-1} \cdot \]

\[\left[ \left(\frac{1}{2} \sum_{i,j=1}^{n-1} \sum_{p,q=1}^{\infty} \frac{B_n^{(p)} B_n^{(q)}}{p! q!} (t_i - t_n)^p (t_j - t_n)^q \cdot \partial_i \partial_j W^g (\omega_0, \omega_1, \ldots, \omega_n; t_n) \right) \right]. \tag{4.17} \]

To evaluate (4.16), we use (A.5) and the transformation (B.30):

\[\Pi_{\epsilon \to 0}^{(1)} (\omega_n, t_n) = \frac{1}{\epsilon} \int_{0}^{t_n} dt_i \left[ \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} (t_i - t_n)^p \cdot \left( \Pi_{\epsilon}^g (\omega_0, \omega_i; t_i) \Pi_{\epsilon}^g (\omega_i, \omega_n; t_n - t_i) \right) \right]. \tag{4.18} \]

Now we use (B.41) and (B.42), with \(\omega_0 = 0\):
\[ \Pi_{\epsilon \to 0}^{(1)} (\omega_n, t_n) = \frac{1}{\epsilon} \int_0^{t_n} dt_i \left[ \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} (t_i - t_n)^p \right], \]

\[ \left( \sqrt{\frac{\epsilon}{\pi}} e^{\alpha (\omega_n - B_n) \frac{B_n - \omega_0}{t_i^{3/2}} e^{-\frac{[\alpha (\omega_n - B_n) - \alpha t_i]^2}{2 t_i}} \right), \]

\[ \left( \sqrt{\frac{\epsilon}{\pi}} e^{\alpha (\omega_n - B_n) \frac{B_n - \omega_0}{(t_n - t_i)^{3/2}} e^{-\frac{[\alpha (\omega_n - B_n) - \alpha (t_n - t_i)]^2}{2(t_n - t_i)}} \right) \]

\[ = \left( \frac{B_n - \omega_n}{\pi} \right) e^{2\alpha (\omega_n - B_n)} \sum_{p=1}^{\infty} \frac{(-1)^p B_n^{(p)}}{p!}. \]

\[ \cdot \int_0^{t_n} dt_i \left( \frac{t_n - t_i}{t_i^{3/2}} e^{-\frac{[\alpha (\omega_n - B_n) - \alpha t_i]^2}{2 t_i}} e^{-\frac{[\alpha (\omega_n - B_n) - \alpha (t_n - t_i)]^2}{2(t_n - t_i)}} \right) \]

(4.19)

To solve \( \Pi_{\epsilon \to 0}^{(1)} (\omega_n, t_n) \) and \( \Pi_{\epsilon \to 0}^{(2)} (\omega_n, t_n) \), we adopt at this point an approximation due to Sheth & Tormen (2002), abbreviated by “ST”, which implies that \( t_n \gg t_i \) in higher than first order derivatives in (4.3):

\[ (t_n - t_i)^{p-1} \simeq (t_n)^{p-1} \quad (4.20) \]

to obtain

\[ \Pi_{\epsilon \to 0}^{(1)} (\omega_n, t_n) = \frac{2}{\sqrt{2\pi t_i^{3/2}}} e^{-\frac{1}{2} \alpha^2 (2B_n - \omega_0 - \omega_n)^2} e^{2\alpha (\omega_n - B_n)} \sum_{p=1}^{\infty} \frac{(-1)^p B_n^{(p)}}{p!}. \]

(4.21)

We develop now (4.17), \( \Pi_{\epsilon \to 0}^{(2)} (\omega_n, t_n) \), also using (ST), (4.20). To do so, we will use (A.14), with (A.16):

\[ \sum_{i,j=1}^{n-1} \partial_i \partial_j = 2 \sum_{i<j} \partial_i \partial_j + \sum_{i=1}^{n-1} \partial_i^2 \]

\[ = 2 \sum_{j=2}^{n-1} \sum_{i=1}^{n-1-j} \partial_i \partial_j + \sum_{i=1}^{n-1} \partial_i^2 \]

(4.22)

As demonstrated in B.2, there are divergent terms in (A.16), which cancel with the second term of RHS (A.14). Specifically, there is a divergent term, when \( t_i = t_j \),
at the very beginning of the summation $\sum_{j=1}^{n-1}$, making the denominator zero, but its contribution is canceled by the second term of the RHS (A.14), which is fully divergent, that is, it is not just one part of it that diverges. As a consequence, we can write:

$$\sum_{i,j=1}^{n-1} \partial_i \partial_j \rightarrow 2 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \rightarrow 2 \frac{1}{\epsilon^2} \int_0^{t_n} \int_0^{t_n} dt_i \int_{t_i}^{t_n} dt_j. \quad (4.23)$$

Using (A.8), (B.48), (B.42) and (B.41) to compute (4.17):

\[
\int_{-\infty}^{B_n} d\omega_1 \cdots \int_{-\infty}^{B_n} d\omega_{n-1} \left[ \frac{1}{2} \sum_{i,j=1}^{n-1} \sum_{p,q=1}^{\infty} \frac{B_n^{(p)} B_n^{(q)}}{plq!} (t_i - t_n)^p (t_j - t_n)^q \cdot \partial_i \partial_j W^{gm} (\omega_0, \omega_1, \ldots, \omega_n; t_n) \right] \\
= \int_0^{t_n} dt_i \int_0^{t_n} dt_j \sum_{p,q=1}^{\infty} \frac{B_n^{(p)} B_n^{(q)}}{plq!} (t_i - t_n)^p (t_j - t_n)^q \frac{1}{\epsilon^2} \int_0^{t_n} \int_{t_i}^{t_n} dt_i \int_{t_i}^{t_n} dt_j e^{-\frac{(t_i - t_n)^2}{2\epsilon^2}} \frac{2}{(t_i - t_j)^{3/2}} \frac{e^{2\alpha(\omega_n - B_n)} e^{-\frac{1}{2}(t_n - t_j)^2}}{2(t_n - t_j)^2} \\
= \frac{(B_n - \omega_0)(B_n - \omega_n)}{\pi \sqrt{2\pi}} \sum_{p,q=1}^{\infty} \frac{B_n^{(p)} B_n^{(q)}}{plq!} (-t_n)^{p-1} (-t_n)^{q-1} e^{2\alpha(\omega_n - B_n)} \\
= \frac{(B_n - \omega_0)(B_n - \omega_n)}{\pi \sqrt{2\pi}} \sum_{p,q=1}^{\infty} \frac{B_n^{(p)} B_n^{(q)}}{plq!} (-t_n)^{p-1} (-t_n)^{q-1} e^{2\alpha(\omega_n - B_n)} e^{-\frac{1}{2}t_n^2} \\
= \int_0^{t_n} dt_i \int_{t_i}^{t_n} dt_i e^{-\frac{(t_i - t_n)^2}{2\epsilon^2}} \frac{e^{-\frac{1}{2}(t_i - t_n)^2}}{t_i^{3/2}} \frac{2}{(t_i - t_j)^{3/2}} \frac{e^{2\alpha(\omega_n - B_n)} e^{-\frac{1}{2}(t_n - t_j)^2}}{2(t_n - t_j)^2}. \quad (4.24)
\[ \Pi_{\epsilon \to 0}^{(2)} (\omega_n, t_n) = \]
\[ -\frac{2 (B_n - \omega_n)^2}{2^{5/2} \pi^{5/2} t_n^{5/2}} e^{-\frac{1}{2} \frac{(B_n - \omega_n - \omega_0)^2}{3 B_n}} e^{2 \alpha (\omega_n - B_n)} \left[ \sum_{p=1}^{\infty} \frac{(-t_n)^p}{p!} B^{(p)}_n \right]^2 . \] (4.25)

Denoting by \( W^{mb} \) the following expression, where \( "mb" \) refers to moving barrier:
\[ W^{mb} (\omega_0, \omega_1, ..., \omega_n; t_n) = \int_{-\infty}^{\infty} Du. \]
\[ \exp \left( \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} + i \sum_{i=1}^{n-1} u_i \sum_{p=1}^{\infty} \frac{B^{(p)}_n}{p!} (t_i - t_n)^p \right). \] (4.26)

(4.26) is in the second term \( \Pi_{\epsilon \to 0}^{(1)} (\omega_n, t_n) \) of (4.11), that we have just computed.

We denote the first line of (4.12) by:
\[ \Pi_{\epsilon \to 0}^{mb} = \int_{-\infty}^{B_n} d\omega_1 ... \int_{-\infty}^{B_n} d\omega_{n-1} W^{mb} (\omega_0, \omega_1, ..., \omega_n; t_n) = \Pi_{\epsilon \to 0}^{mb} (\omega_0, \omega_n; t_n) + \]
\[ \Pi_{\epsilon \to 0}^{(1)} (\omega_n, t_n) + \Pi_{\epsilon \to 0}^{(2)} (\omega_n, t_n) . \] (4.27)

Now consider the remaining term of (4.12):
\[ \int_{-\infty}^{B_n} d\omega_1 ... \int_{-\infty}^{B_n} d\omega_{n-1} \int_{-\infty}^{\infty} Du. \]
\[ \exp \left( \sum_{i=1}^{n} u_i \omega_i - i \sum_{i=1}^{n} u_i \kappa_i - \frac{1}{2} \sum_{i,j=1}^{n} u_i u_j \kappa_{ij} + i \sum_{i=1}^{n-1} u_i \sum_{p=1}^{\infty} \frac{B^{(p)}_n}{p!} (t_i - t_n)^p \right) \left( \sum_{i,j,k=1}^{n} u_i u_j u_k \kappa_{ijk} + \sum_{i,j,k,l=1}^{n} \frac{(-i)^4}{4!} u_i u_j u_k u_l \kappa_{ijkl} \right. \]
\[ + \left. \sum_{i,j,k,l,m=1}^{n} \frac{(-i)^5}{5!} u_i u_j u_k u_l u_m \kappa_{ijklm} + \ldots \right) \] (4.28)
Analytical Path-Integral Pricing of Moving-Barrier Options under non-Gaussian Distributions

with the relations

\[
(i)^3 u_i u_j u_k \exp \left( i \sum_{i=1}^{n} u_i \omega_i \right) = \partial_i \partial_j \partial_k \exp \left( i \sum_{i=1}^{n} u_i \omega_i \right)
\]

\[
(i)^4 u_i u_j u_k u_l \exp \left( i \sum_{i=1}^{n} u_i \omega_i \right) = \partial_i \partial_j \partial_k \partial_l \exp \left( i \sum_{i=1}^{n} u_i \omega_i \right)
\]

\[
(i)^5 u_i u_j u_k u_l u_m \exp \left( i \sum_{i=1}^{n} u_i \omega_i \right) = \partial_i \partial_j \partial_k \partial_l \partial_m \exp \left( i \sum_{i=1}^{n} u_i \omega_i \right),
\]

(4.29)

remembering that \( \partial_i = \partial / \partial \omega_i \).

At this point, we will take a step that is compatible with the stationarity of the time series. The cumulants \( \kappa_{ijk} \), \( \kappa_{ijkl} \), \( \ldots \) refer to time-dependent variables and, without the stationarity hypothesis, change over the interval \([0, t_n]\). If \( t_n \) is small, we can expand the cumulant in a Taylor series around \( t_i = t_j = \ldots = t_n \). For instance, \( \kappa_{ijk} \) is expanded around \( \kappa_{nnn} \), which we denote by \( \kappa_3 \). Thus, we define the derivative in the Taylor expansion of the cumulant (here we show the 5th one):

\[
G^{(p,q,r,s,t)}_5 (t_n) = \left. \frac{d^p}{dt^p} \frac{d^q}{dt^q} \frac{d^r}{dt^r} \frac{d^s}{dt^s} \frac{d^t}{dt^t} \kappa_{ijklm} \right|_{i=j=k=l=m=n},
\]

(4.31)

and write:
\[ \kappa_{ijklm} = \sum_{p,q,r,s,t=0}^{\infty} \frac{(-1)^p q^p r^q s^r t^t}{p!q!r!s!t!} (t_n - t_i)^p (t_n - t_j)^q (t_n - t_k)^r . \]

\[ (t_n - t_i)^n (t_n - t_m)^t G_5^{(p,q,r,s,t)} (t_n) . \]  

(4.32)

The relevant contribution is \( p = q = r = s = t = 0 \). Then, the summation of (4.32) reduces to \( \kappa_5 \) and the summation

\[ \sum_{i,j,k,l,m=1}^{n} \kappa_{ijklm} \partial_i \partial_j \partial_k \partial_l \partial_m \]  

(4.33)

becomes

\[ \kappa_5 \sum_{i,j,k,l,m=1}^{n} \partial_i \partial_j \partial_k \partial_l \partial_m. \]  

(4.34)

The other summation terms in cumulants in (4.30) have analogous expressions to (4.34). To sum up, what we have done is to approximate time-varying cumulants to their final values. This converges to the stationary case, where moments (cumulants) are constant, just depending on the size of the analyzed period.

As (4.34) suggests, we need to value objects such as:

- **cumulant object**: $3 \sum_{i,j,k=1}^{n} \partial_i \partial_j \partial_k$
- **cumulant object**: $4 \sum_{i,j,l=1}^{n} \partial_i \partial_j \partial_k \partial_l$
- **cumulant object**: $5 \sum_{i,j,k,l,m=1}^{n} \partial_i \partial_j \partial_k \partial_l \partial_m$
- : :

These summations can be expressed by components whose coefficients are given by the Pascal triangle. For the cumulants we have made explicit, we have:

\[ \sum_{i,j,k=1}^{n} \partial_i \partial_j \partial_k = \partial_n^3 + 3 \sum_{i,j=1}^{n-1} \partial_i \partial_j \partial_n + 3 \sum_{i=1}^{n-1} \partial_i \partial_n^2 + \sum_{i,j,k=1}^{n-1} \partial_i \partial_j \partial_k \]  

(4.35)
Analytical Path-Integral Pricing of Moving-Barrier Options under non-Gaussian Distributions

\[ \sum_{i,j,k,l=1}^{n} \partial_i \partial_j \partial_k \partial_l = \partial_n^4 + 4 \sum_{i,j=k=1}^{n-1} \partial_i \partial_j \partial_k \partial_n \]
\[ + 6 \sum_{i,j=1}^{n-1} \partial_i \partial_j \partial_n^2 + 4 \sum_{i,j}=1^{n-1} \partial_i \partial_n^4 + \sum_{i,j,k,l=1}^{n-1} \partial_i \partial_j \partial_k \partial_l \]  
(4.37)

\[ \sum_{i,j,k,l,m=1}^{n} \partial_i \partial_j \partial_k \partial_m = \partial_n^5 + 5 \sum_{i,j,k,l=1}^{n-1} \partial_i \partial_j \partial_k \partial_n + 10 \sum_{i,j,k,l=1}^{n-1} \partial_i \partial_j \partial_k \partial_n^2 \]
\[ + 10 \sum_{i,j,k,l,m=1}^{n-1} \partial_i \partial_j \partial_k \partial_m + 5 \sum_{i,j,k}=1^{n-1} \partial_i \partial_n^4 + \sum_{i,j,k,l,m=1}^{n-1} \partial_i \partial_j \partial_k \partial_n \partial_m \]  
(4.38)

As in (A.12) e (A.18), we get:

\[ \sum_{i=1}^{n-1} \int_{-\infty}^{B_n} d\omega_1 ... d\omega_{n-1} \partial_i W_{mb} (\omega_0, \omega_1, ..., \omega_n; t_n) = \frac{\partial \Pi_{e \rightarrow 0}^{mb}}{\partial B_n} (\omega_0 = 0, \omega_n; t_n) \]  
(4.39)

\[ : \]

\[ \sum_{i,j,k,l,m=1}^{n-1} \int_{-\infty}^{B_n} d\omega_1 ... d\omega_{n-1} \partial_i \partial_j \partial_k \partial_l \partial_m W_{mb} (\omega_0, \omega_1, ..., \omega_n; t_n) = \frac{\partial^5 \Pi_{e \rightarrow 0}^{mb}}{\partial B_n^5} (\omega_0 = 0, \omega_n; t_n) . \]  
(4.40)

where \( \Pi_{e \rightarrow 0}^{mb} \) is given by (4.27). Returning to (4.30),

\[ \Pi_{e \rightarrow 0} (\omega_0 = 0, \omega_n; t_n) = \Pi_{e \rightarrow 0}^{gm} (\omega_0, \omega_n; t_n) + \Pi_{e \rightarrow 0}^{(1)} (\omega_n, t_n) + \Pi_{e \rightarrow 0}^{(2)} (\omega_n, t_n) \]

\[ - \frac{1}{3!} \int_{-\infty}^{B_n} d\omega_1 ... d\omega_{n-1} \left\{ \left[ \partial_n^3 + 3 \sum_{i,j=1}^{n-1} \partial_i \partial_j \partial_n \right] \right. \]
\[ + 3 \sum_{i=1}^{n-1} \partial_i \partial_n^2 + \sum_{i,j,k=1}^{n-1} \partial_i \partial_j \partial_k \right\} W_{mb} (\omega_0, \omega_1, ..., \omega_n; t_n) \]
\begin{align*}
&+ \frac{1}{4!} \kappa_4 \int_{-\infty}^{B_n} d\omega_1 \ldots \int_{-\infty}^{B_n} d\omega_{n-1} \left\{ \left[ \partial_4^4 + \frac{n-1}{4} \sum_{i,j,k=1}^{n-1} \partial_i \partial_j \partial_k \partial_n \right] W^{mb}(\omega_0, \omega_1, \ldots, \omega_n; t_n) \right\} \\
&+ 6 \sum_{i,j=1}^{n-1} \partial_i \partial_j \partial_n^2 + 4 \sum_{i=1}^{n-1} \partial_i \partial_n^3 + \sum_{i,j,k,l=1}^{n-1} \partial_i \partial_j \partial_k \partial_l \left\{ \frac{1}{3!} \kappa_3 \left\{ \partial_3^3 \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_0, \omega_n; t_n) + 3 \frac{\partial^2}{\partial \omega_n^2} \frac{\partial}{\partial B_n} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) \right\} \right. \\
&\left. + 3 \frac{\partial}{\partial \omega_n} \frac{\partial^2}{\partial B_n^2} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) + \frac{\partial^3}{\partial B_n^3} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) \right\} \\
&+ \frac{1}{4!} \kappa_4 \left\{ \partial_4^4 \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) + 4 \frac{\partial^3}{\partial \omega_n^3} \frac{\partial}{\partial B_n^2} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) + 6 \frac{\partial^2}{\partial \omega_n^2} \frac{\partial^2}{\partial B_n^2} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) \right. \\
&\left. + 4 \frac{\partial}{\partial \omega_n} \frac{\partial^3}{\partial B_n^3} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) + \frac{\partial^4}{\partial B_n^2} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) \right\} \
\end{align*}

The derivative operators acting in $W^{mb}(\omega_0, \omega_1, \ldots, \omega_n; t_n)$ follow the relations (4.39) and their analogues, noticing that the operator $\partial_n^m$ exits the integral because $\omega_n$ is not part of the integration. Thus, making explicit some second order crossed terms, recovering (4.9),

$$
\Pi_{\epsilon \rightarrow 0}(\omega_0 = 0, \omega_n; t_n) = \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n)
$$

$$
- \frac{1}{3!} \kappa_3 \left\{ \partial_3^3 \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) + 3 \frac{\partial^2}{\partial \omega_n^2} \frac{\partial}{\partial B_n} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) \right\} \\
+ 3 \frac{\partial}{\partial \omega_n} \frac{\partial^2}{\partial B_n^2} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) + \frac{\partial^3}{\partial B_n^3} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) \\
+ \frac{1}{4!} \kappa_4 \left\{ \partial_4^4 \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) + 4 \frac{\partial^3}{\partial \omega_n^3} \frac{\partial}{\partial B_n^2} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) + 6 \frac{\partial^2}{\partial \omega_n^2} \frac{\partial^2}{\partial B_n^2} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) \right. \\
\left. + 4 \frac{\partial}{\partial \omega_n} \frac{\partial^3}{\partial B_n^3} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) + \frac{\partial^4}{\partial B_n^2} \Pi^{mb}_{\epsilon \rightarrow 0}(\omega_n, t_n) \right\}
$$
Analytical Path-Integral Pricing of Moving-Barrier Options under non-Gaussian Distributions

\[- \frac{1}{5!} \kappa_5 \left\{ \frac{\partial^5}{\partial \omega_n^5} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) + 5 \frac{\partial^4}{\partial \omega_n^4} \frac{\partial}{\partial B_n} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) + 10 \frac{\partial^3}{\partial \omega_n^3} \frac{\partial^2}{\partial B_n^2} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) \right\} + 10 \frac{\partial^2}{\partial \omega_n^2} \frac{\partial^3}{\partial B_n^3} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) + 5 \frac{\partial}{\partial \omega_n} \frac{\partial^4}{\partial B_n^4} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) + \frac{\partial^5}{\partial B_n^5} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) \right\} + \ldots \]

\[+ \left\{ \frac{1}{2} \left( \frac{1}{3!} \kappa_3 \right)^2 + \frac{1}{6!} \kappa_6 \right\} \left\{ \frac{\partial^6}{\partial \omega_n^6} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) \right\} \]

\[+ 6 \frac{\partial^5}{\partial \omega_n^5} \frac{\partial}{\partial B_n} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) + \ldots + \frac{\partial^6}{\partial B_n^6} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) \right\} \]

\[- \left\{ \left( \frac{1}{3!} \kappa_3 \right) \left( \frac{1}{4!} \kappa_4 \right) + \frac{1}{7!} \kappa_7 \right\} \left\{ \frac{\partial^7}{\partial \omega_n^7} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) \right\} \]

\[+ 7 \frac{\partial^6}{\partial \omega_n^6} \frac{\partial}{\partial B_n} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) + \ldots + \frac{\partial^7}{\partial B_n^7} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) \right\} \]

\[+ \left\{ \frac{1}{2} \left( \frac{1}{4!} \kappa_4 \right)^2 - \left( \frac{1}{3!} \kappa_3 \right) \left( \frac{1}{5!} \kappa_5 \right) + \frac{1}{8!} \kappa_8 \right\} \left\{ \frac{\partial^8}{\partial \omega_n^8} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) \right\} \]

\[+ 8 \frac{\partial^7}{\partial \omega_n^7} \frac{\partial}{\partial B_n} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) + \ldots + \frac{\partial^8}{\partial B_n^8} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) \right\} \]

\[- \left\{ \left( \frac{1}{4!} \kappa_4 \right) \left( \frac{1}{5!} \kappa_5 \right) + \left( \frac{1}{3!} \kappa_3 \right) \left( \frac{1}{6!} \kappa_6 \right) + \frac{1}{9!} \kappa_9 \right\} \left\{ \frac{\partial^9}{\partial \omega_n^9} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) \right\} \]

\[+ 9 \frac{\partial^8}{\partial \omega_n^8} \frac{\partial}{\partial B_n} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) + \ldots + \frac{\partial^9}{\partial B_n^9} \Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n) \right\} + \ldots \quad (4.42)\]

We remember that \(\Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n)\) and is given by (4.27) and depends on \(\Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n)\), \(\Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n)\) e \(\Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n)\), computed according to (B.22), (4.21) and (4.25), respectively.

If we keep up to the \(\kappa_3\) term in (4.42), and remove the drift, that is, impose \(\alpha = 0\) in the components \(\Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n)\), we recover the results of De Simone et al. (2011).

Instead of the (ST) hypothesis, next we develop alternative forms for \(\Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n)\) and \(\Pi_{\varepsilon \to 0}^{mb}(\omega_n, t_n)\), under the alternative hypothesis that the moving barrier evolves slowly through time.
4.2. Slowly moving barrier

In this section, the hypothesis is that the barrier moves slowly with $t$. For future references we may call it \textit{abiabatic barrier}. Under this assumption, we may discuss which terms in (4.3) are relevant. We will work with up to second order terms in the barrier time derivatives: $\frac{\partial^2 B_n}{\partial t^2_n} e (\partial B_n/\partial t_n)^2$. In a certain way, this hypothesis and the ST one work in the same direction: in (4.3), the (ST) acts in the terms $(t_i - t_n)^p$, while the slow varying barrier is related to the derivatives.

Under these conditions, (4.16) breaks into two terms, denoted by the following expressions:

$$\Pi^{(a)}_{\epsilon \to 0} (\omega_n, t_n) = \sum_{i=1}^{n-1} B_n' (t_i - t_n) \int_{-\infty}^{B_n} d\omega_1... \int_{-\infty}^{B_n} d\omega_{n-1} \partial_i W^{gm} (\omega_0, \omega_1, ..., \omega_n; t_n)$$

(4.43)

$$\Pi^{(b)}_{\epsilon \to 0} (\omega_n, t_n) = \frac{1}{2} \sum_{i=1}^{n-1} B_n'' (t_i - t_n)^2 \int_{-\infty}^{B_n} d\omega_1... \int_{-\infty}^{B_n} d\omega_{n-1} \partial_i \partial_j W^{gm} (\omega_0, \omega_1, ..., \omega_n; t_n).$$

(4.44)

Equation (4.17) enables us to write:

$$\Pi^{(c)}_{\epsilon \to 0} (\omega_n, t_n) = \frac{1}{2} \sum_{i=1}^{n-1} \left( B_n' \right)^2 (t_i - t_n) (t_j - t_n),$$

$$\int_{-\infty}^{B_n} d\omega_1... \int_{-\infty}^{B_n} d\omega_{n-1} \partial_i \partial_j W^{gm} (\omega_0, \omega_1, ..., \omega_n; t_n),$$

(4.45)

where $B_n' = \partial B (t_n) / \partial t_n$ and $B_n'' = \partial^2 B (t_n) / \partial t_n^2$.

Starting by $\Pi^{(a)}_{\epsilon \to 0} (\omega_n, t_n)$, from (4.19), with $p = 1$,

$$\Pi^{(a)}_{\epsilon \to 0} (\omega_n, t_n) =$$

$$= \frac{\left( B_n - \omega_0 \right)}{\pi} e^{2\imath (\omega_n - B_n)} \frac{dB_n}{dt_n}$$

$$\cdot \int_0^{t_n} dt_i \frac{(t_n - t_i)^{-\frac{3}{2}}}{t_i^{3/2}} e^{-\frac{\left| B_{\epsilon_0} - \omega_0 \right|^2}{2t_i}} e^{-\frac{\left| B_n - \omega_n - \alpha (t_n - t_i) \right|^2}{2(t_n - t_i)}}. \quad (4.46)$$
Analytical Path-Integral Pricing of Moving-Barrier Options under non-Gaussian Distributions

\[
\Pi^{(a)}_{\epsilon \to 0}(\omega_n, t_n) = -\frac{\sqrt{2}}{\pi} \frac{dB_n}{dt_n} \frac{(B_n - \omega_n)}{t_n^{1/2}} e^{2\alpha(\omega_n - B_n)} e^{-\frac{(2B_n - \omega_n - \omega_n)^2}{2\epsilon}}.
\] (4.47)

We now value \(\Pi^{(b)}_{\epsilon \to 0}(\omega_n, t_n)\). In (4.19), with \(p = 2\),

\[
\Pi^{(b)}_{\epsilon \to 0}(\omega_n, t_n) = \frac{(B_n - \omega_n)(B_n - \omega_0)}{\pi} e^{2\alpha(\omega_n - B_n)} \frac{1}{2} \frac{d^2 B_n}{dt_n^2}
\]

\[
\cdot \int_0^{t_n} dt_i \frac{(t_n - t_i)^{3/2}}{t_i^{3/2}} e^{-\frac{(B_n - \omega_n - \omega_0)^2}{2t_n}} e^{-\frac{(B_n - \omega_n - \omega_0)(t_n - t_i)}{2(t_n - t_i)}}.
\] (4.48)

Finally, now computing \(\Pi^{(c)}_{\epsilon \to 0}(\omega_n, t_n)\), in (4.45), which refers to \(\Pi^{(2)}_{\epsilon \to 0}(\omega_n, t_n)\), from (4.24), with \(p = q = 1\):

\[
\Pi^{(c)}_{\epsilon \to 0}(\omega_n, t_n) = \frac{(B_n - \omega_0)(B_n - \omega_n)}{\pi \sqrt{2\pi}} \left( \frac{dB_n}{dt_n} \right)^2 e^{\alpha(\omega_n - B_n)} e^{\alpha(\omega_n - B_0)} e^{-\frac{\omega_n^2}{2}}.
\]

(4.49)

Therefore, in the hypothesis of this section, the equivalent to (4.27) is
\[
\Pi_{\epsilon \to 0}^{mb} (\omega_n, t_n) = \int_{-\infty}^{B_n} d\omega_1 \ldots \int_{-\infty}^{B_n} d\omega_{n-1} W^{mb} (\omega_0, \omega_1, \ldots, \omega_n; t_n) = \Pi_{\epsilon \to 0}^{gm} (\omega_n; t_n) + 
\]

\[
\Pi_{\epsilon \to 0}^{(a)} (\omega_n, t_n) + \Pi_{\epsilon \to 0}^{(b)} (\omega_n, t_n) + \Pi_{\epsilon \to 0}^{(c)} (\omega_n, t_n),
\]

(4.52)

with \(\Pi_{\epsilon \to 0}^{gm} (\omega_n; t_n)\), \(\Pi_{\epsilon \to 0}^{(a)} (\omega_n, t_n)\), \(\Pi_{\epsilon \to 0}^{(b)} (\omega_n, t_n)\) and \(\Pi_{\epsilon \to 0}^{(c)} (\omega_n, t_n)\) given by (B.22), (4.47), (4.49) and (4.51), respectively, taking \(\omega_0 = 0\). (4.42) is still valid, with this specification for \(\Pi_{\epsilon \to 0}^{mb} (\omega_n, t_n)\).

Therefore, in the presence of fixed or moving barriers, we have expressed a distribution in terms of a cumulant expansion based on the Gaussian distribution with fixed barrier.

### 4.3. Non-Gaussian distribution with constant barrier

The constant barrier \(B_n = b = \frac{1}{\sigma} \ln \frac{B}{S_0}\), in a non-Gaussian distribution implies time derivatives of the barrier equal to zero, in both (ST) and adiabatic barrier hypothesis:

\[
\lim_{\epsilon \to 0} \frac{d}{dt_n} \Pi_{\epsilon \to 0}^{mb} (\omega_n, t_n) = \Pi_{\epsilon \to 0}^{gm} (\omega_n; t_n),
\]

(4.53)

where \(\Pi_{\epsilon \to 0}^{gm} (\omega_n; t_n)\) is given by (B.22). This distribution is used in (4.42) to get \(\Pi_{\epsilon \to 0}^{gm} (\omega_0 = 0, \omega_n; t_n)\) related to this case. Non-Gaussian corrections emerge from the derivatives related to \(\omega_n\) and \(B_n\).

### 4.4. Linearly moving barrier

The application to the moving barrier case depends on derivatives, which appear in \(\Pi_{\epsilon \to 0}^{mb} (\omega_n, t_n)\) components, with respect to the value of the barrier at the final instant \(t_n\). We will consider the case of a barrier that evolves linearly, to exemplify the use of the notation:

\[
B_i = B(t_i) = B_0 + \xi t_i.
\]

(4.54)

We first find the value at \(t_n\):

\[
B_n = B_0 + \xi t_n
\]

(4.55)
and write the derivatives. In our case, only the first order one takes non-null value:

\[ B_n^{(p)} = \frac{d^p B_n}{d (t_n)^p} = \begin{cases} \xi & p = 1 \\ 0 & p > 1 \end{cases} \tag{4.56} \]

### 4.5. Non-Gaussian distribution in the absence of barriers

The absence of barriers is a particular case of the probability densities of sections (4.1) or (4.2). Specifically, add an extra limit in (4.53), setting the barrier at infinity:

\[ \lim_{B_n \to \infty} \lim_{\frac{d B_n}{d t_n} \to 0} \lim_{\frac{d^2 B_n}{d t_n^2} \to 0} \Pi_{\varepsilon \to 0}^{mb} (\omega_n, t_n) = \Pi_{\varepsilon \to 0}^{gm} (\omega_n, t_n). \tag{4.57} \]

\[ \Pi_{\varepsilon \to 0}^{gm} (\omega_n, t_n) = \frac{1}{\sqrt{2\pi t_n}} e^{\alpha \omega_n - \frac{1}{2} \alpha^2 t_n} e^{-\frac{\omega^2}{2t_n}}. \tag{4.58} \]

Second, in (4.42), since the derivative related to the barrier \((\partial^i / \partial B_n^i)\) are limit operations, they can be applied after those of (4.57). At the infinity, the distribution converges to the Gaussian density, without barrier, according to (4.57). As a consequence, the derivative operator \(\partial^i / \partial B_n^i\) nullifies the term. In the case of first order derivative,

\[ \lim_{B_n \to \infty} \lim_{\frac{d B_n}{d t_n} \to 0} \lim_{\frac{d^2 B_n}{d t_n^2} \to 0} \Pi_{\varepsilon \to 0}^{mb} (\omega_n, t_n) = \lim_{B_n \to \infty} \lim_{\frac{d B_n}{d t_n} \to 0} \lim_{\frac{d^2 B_n}{d t_n^2} \to 0} \frac{\Delta \Pi_{\varepsilon \to 0}^{mb} (\omega_n, t_n)}{\Delta B_n} \]

\[ = \lim_{\Delta B_n \to \infty} \left[ \lim_{B_n \to \infty} \lim_{\frac{d B_n}{d t_n} \to 0} \lim_{\frac{d^2 B_n}{d t_n^2} \to 0} \Pi_{\varepsilon \to 0}^{mb} (\omega_n, t_n) \right] \]

\[ = \lim_{\Delta B_n \to \infty} \frac{\Delta \Pi_{\varepsilon \to 0}^{gm} (\omega_n, t_n)}{\Delta B_n} = 0; \tag{4.59} \]

and so on, for higher order derivatives in the barrier. Nevertheless, in (4.42) remain the derivative terms \(\partial^i / \partial \omega_i\), which assure the survival of non-Gaussian terms of the expansion.

In a nutshell, the non-barrier case of non-Gaussian distribution is given by substituting \(\Pi_{\varepsilon \to 0}^{mb} (\omega_n, t_n) \to \Pi_{\varepsilon \to 0}^{gm} (\omega_n, t_n)\) in (4.42), and excluding barrier derivatives of the distribution. We rewrite (4.42) in this case:
\[ \Pi_{\epsilon \to 0}^{\infty} (\omega_0 = 0, \omega_n; t_n) = \Pi_{\epsilon \to 0}^{gm} (\omega_n, t_n) \]

\[ -\frac{1}{3!} \kappa_3 \frac{\partial^3}{\partial \omega_n^3} \Pi_{\epsilon \to 0}^{gm} (\omega_n, t_n) + \frac{1}{4!} \kappa_4 \frac{\partial^4}{\partial \omega_n^4} \Pi_{\epsilon \to 0}^{gm} (\omega_n, t_n) \]

\[ -\frac{1}{3!} \kappa_5 \frac{\partial^5}{\partial \omega_n^5} \Pi_{\epsilon \to 0}^{gm} (\omega_n, t_n) + \left[ \frac{1}{2} \cdot \left( \frac{1}{3!} \kappa_3 \right)^2 + \frac{1}{6!} \kappa_6 \right] \frac{\partial^6}{\partial \omega_n^6} \Pi_{\epsilon \to 0}^{gm} (\omega_n, t_n) \]

\[ - \left[ \left( \frac{1}{3!} \kappa_3 \right) \left( \frac{1}{4!} \kappa_4 \right) + \frac{1}{7!} \kappa_7 \right] \frac{\partial^7}{\partial \omega_n^7} \Pi_{\epsilon \to 0}^{gm} (\omega_n, t_n) \]

\[ + \left[ \frac{1}{2} \cdot \left( \frac{1}{4!} \kappa_4 \right)^2 - \left( \frac{1}{3!} \kappa_3 \right) \left( \frac{1}{5!} \kappa_5 \right) + \frac{1}{8!} \kappa_8 \right] \frac{\partial^8}{\partial \omega_n^8} \Pi_{\epsilon \to 0}^{gm} (\omega_n, t_n) \]

\[ - \left[ \left( \frac{1}{4!} \kappa_4 \right) \left( \frac{1}{5!} \kappa_5 \right) + \left( \frac{1}{3!} \kappa_3 \right) \left( \frac{1}{6!} \kappa_6 \right) + \frac{1}{9!} \kappa_9 \right] \frac{\partial^9}{\partial \omega_n^9} \Pi_{\epsilon \to 0}^{gm} (\omega_n, t_n) + \ldots \] (4.60)

where

\[ \Pi_{\epsilon \to 0}^{gm} (\omega_0, \omega_n; t_n) = \frac{1}{\sqrt{2\pi t_n}} e^{\alpha (\omega_0 - \omega_n) - \frac{1}{2} \alpha^2 t_n} e^{-\frac{(\omega - \omega_0)^2}{2t_n}}. \] (4.61)

From now on, when we refer to the vanilla case under non-Gaussian distribution, we will deal with (4.60) and (4.61), the infinite barrier case.

### 4.6. Martingale condition for drift

Under the risk-neutral measure \( Q \), the martingale condition establishes the non-arbitrage drift condition

\[ e^{-r(t_n - t_{n-1})} E^Q \left[ S_n \mid \mathcal{F}_{t_{n-1}} \right] = S_{n-1}. \] (4.62)

In terms of the distribution (4.60),

\[ S_0 = e^{-rt_n} \int_{-\infty}^{\infty} S_0 e^{\sigma \omega_n} \Pi_{\epsilon \to 0}^{inf} (\omega_n, t_n) d\omega_n, \]

\[ \therefore 1 = e^{-rt_n} \int_{-\infty}^{\infty} e^{\sigma \omega_n} \Pi_{\epsilon \to 0}^{inf} (\omega_n, t_n) d\omega_n. \] (4.63)
4.7. Analytical option pricing

To price call vanilla european options, under the probability density $\Pi_{\epsilon \rightarrow 0}^{inf}(\omega_n, t_n)$,

$$P(S_0, K, t_n, \alpha, \kappa_3, \kappa_4, \ldots) = \int_{-\infty}^{\infty} \max \left[ (S_0 e^{\sigma \omega_n} - K), 0 \right] \Pi_{\epsilon \rightarrow 0}^{inf}(\omega_n, t_n) \, d\omega_n, \quad (4.64)$$

Barrier options are priced with $\Pi_{\epsilon \rightarrow 0}^{mb}(\omega_n, t_n)$ in (4.42), specifying $\Pi_{\epsilon \rightarrow 0}^{mb}(\omega_n, t_n)$ for fixed (4.53) or moving barriers, either in (ST), or in adiabatic barrier approximations. The price of a knock-up-and-out call is given by:

$$P = \int_{-\infty}^{b} (S_0 e^{\sigma \omega_n} - K) \Pi_{\epsilon \rightarrow 0}(\omega_n, t_n) \, d\omega_n. \quad (4.65)$$

and that of a knock-up-and-out put by:

$$P = \int_{-\infty}^{k} (K - S_0 e^{\sigma \omega_n}) \Pi_{\epsilon \rightarrow 0}(\omega_n, t_n) \, d\omega_n. \quad (4.66)$$

5. Calibration

In order to price barrier options, the parameters $\sigma$ and $\kappa_i$ are calibrated with european vanilla call options, for each maturity, meaning a piecewise constant set. In our example, the daily data consist of foreign exchange (FX) european call options of Brazilian Real against US dollar (BRL/USD), ranging from 05/2009 to 05/2014. Each day includes implied volatility rates corresponding to five standard delta values $\{10\%, 25\%, 50\%, 75\%, 90\%\}$, for twenty-four months $\{1, 2, 3, \ldots, 24\}$. Therefore, a volatility $\sigma_{ijk}$ is indexed by the date $t_i$, delta $\Delta_j$ and maturity $T_k$: $\sigma_{ijk} = \sigma(t_i, \Delta_j, T_k)$; $i = 1, \ldots, 1295$; $j = 1, \ldots, 5$; $k = 1, \ldots, 24$. Deltas are converted to strikes in the usual manner.

The valuation of these integrals result in closed-form expressions. However, if a high number of cumulants is included in the specification, numerical valuation may become less costly.
Concerning the model specification, the number of parameters in the case of vanilla options depends on the ability to fit the smile delta range. On the left side of figure 1, the maximum order was in $\kappa_7$, while, on the right side of the figure, the maximum was $\kappa_{15}$. However, higher order corrections gained with higher derivative orders in the expansion are done at the expense of adding more parameters to the model. Such improvements are necessary near the barrier region in pricing barrier options. Further, just including higher derivative orders is not enough if combinations of parameters are not considered (we took up to second order combinations, $\kappa_i \kappa_j$, with $i + j = n$, $n = $ order of derivative correction in the term). For instance, in the upper plot in figure 2, we used 18th order in derivatives for different values of the barrier, but didn’t include the second order combinations. When we included the second order combinations of parameters, the 15th order was enough to improve precision in the barrier region, as shown in the lower plot.

Figure 1. Smile calibration for vanillas. The first graph uses up to the 7th cumulant, while the second uses up to the 15th one.
Figure 2. Behaviour of non-Gaussian distribution in the barrier region under inclusion of second order combinations of parameters. Different values of the barrier are shown in the boxes.
Thus, in order to price knock-up-and-out calls, according to the integration limits in the pricing equation (4.65), one should guarantee a good distribution fitting in the interval \([k, b]\). Therefore, although price calibration is possible, because of the large amount of terms in the pricing formula, we calibrated the parameters of the model density (4.60) by fitting the probability density retrieved from the Breeden & Litzenberger (1978) theorem, taking into account the smile, as in Shimko (1993):

\[
\Pi_{\epsilon \to 0}^{inf} (\omega_n = k, t_n \equiv T) = e^{rT} K \sigma \frac{d^2 C}{dK^2}.
\]

The total derivative \(d^2 C/dK^2\) was computed numerically and analytically, using the parameters of cubic spline interpolation, both producing the same results. An example of calibration is given in figure 3.

The fitting of model prices \(P_{\text{model}}\) to 300 vanilla call option market prices \(P_{\text{market}}\)
can be summarized by fitting the linear regression ($e$ is the residual):

$$P_{Model} = a_p \cdot P_{market} + b_p + e,$$

where we hope to get $a_p = 1$ and $b_p = 0$. The result is in figure 4, which displays $R^2 \sim 1$, and $a_p = 1, b_p = -4 \times 10^{-5}$.

![Figure 4. Price regression: dependent variable is the model price and the independent one is the market prices of vanilla call options.](image)

6. Barrier option pricing

To analyze in a standardized way the effect of an absorbing up barrier on the call option price, according to its proximity to the initial underlying price or to the strike, we set fixed barrier levels defined by a group of multiplicative factors $\Theta = \{1.1; 1.2; 1.3; 1.5\}$, to be applied to data base strikes, which cover the deltas from 10...
to 90 in each maturity. In order to assure that the price does not start deactivated by the barrier, when multiplication results in a barrier below initial underlying future price, we change the rule, setting the fixed barrier to a multiplicative factor regarding the future price, given maturity. Thus, to each strike $i$ and maturity $j$, the barrier $B_k$ is defined, resulting in price

$$P_{ijk} = P_{ijk}(B_k, K_i, T_j); B_k = \begin{cases} \Theta_k \cdot K_i, & \Theta_k \cdot K_i > F_j \\ \Theta_k \cdot F_j, & \Theta_k \cdot K_i \leq F_j \end{cases} \quad (6.1)$$

where $k = 1, \ldots, 4$; and $F_j$ is the future value related to the maturity $T_j$.

Usually, data providers of barrier option prices rely on market-to-model values. Therefore, we have chosen to compare ours results to the relative entropy model of Avellaneda et al. (2001). When the absorbing up barrier is near the strike of the call or the initial underlying value, that is, when $\Theta_k = 1.1$, prices are closer to zero. In this situation, in longer maturities, we have found greater divergence comparing the path-integral approach, the lognormal model (Black-Scholes, with strike-related volatility) and the relative entropy model, as in figure 5. In figure 6, in the 18-month case, we notice that, although the average underlying price is higher in the path-integral model, the Black-Scholes model presents a greater dispersion, meaning that the barrier region has greater chance of being reached, thus displaying lower prices in the case of barrier close to initial underlying value. In shorter maturities, such as 1-month, the underlying process has less time to reach, the closer the barrier; consequently, prices are closer in the model comparison. Therefore, difference between path-integral and entropy models is expected, since the first approach includes the density behaviour in the vicinity of the barrier.
Analytical Path-Integral Pricing of Moving-Barrier Options under non-Gaussian Distributions
Figure 5. 18-month model comparison: path integral, relative entropy and Black-Scholes.
Figure 6. Comparison between non-Gaussian (path integral model) and Gaussian (Black-Scholes model) densities with absorbing barrier.
In the same way as vanilla pricing, we summarize pricing differences between the path integral and the relative entropy models by fitting a linear regression:

\[ P_{Path \text{ integral}} = a_p \cdot P_{Entropy} + b_p + c, \]  

(6.2)

In figure 8, we see that the higher divergence in lower barrier levels at longer maturities corresponds to the dispersion in the region of low prices in the graph (target < 0.05 and output < 0.05).
7. Conclusion

In this article we presented a non-gaussian probability distribution model, based on cumulant expansion in the well-known path integral formalism of Statistical Mechanics, including an absorbing, deterministically moving, barrier. The idea relies on the work of De Simone et al. (2011) for galaxy formation, in Cosmology, which we extend to include drift and more cumulants than the original authors do. In applying the model to option pricing in Finance, we find the condition for a risk-neutral drift, and present an analytical method to price deterministically moving absorbing barrier options. The development encompasses analysis of the behaviour of the distribution in the vicinity of the barrier. Usually, general distribution models used in such products’ pricing demand numerical and simulation methods; the work of Kunitomo & Ikeda (1992) being a case of closed-form solution, but under the lognormal distribution hypothesis of Black & Scholes (1973).

In the case of constant barrier, we obtain an analytic non-gaussian pricing model to price standard knock-up-and-out (KOU) barrier call options. And, in the limit of
infinite barrier, it becomes a non-gaussian probability distribution model to price vanilla options. Since the model parameters, volatility and cumulants, belong to both barrier and vanilla versions of the model, we calibrate them with vanilla option data and then price constant barrier KUO calls. Given that barrier option pricing data contributors often provide market-to-model values, we compare the constant barrier option prices of the path integral model with the ones obtained from the relative entropy model. We adapted the approach of Avellaneda et al. (2001) to analytical constant barrier option pricing.

The results demonstrate that our model reproduces those obtained by the Entropy model. The KUO barrier call option pricing presents larger differences when, in long maturity contracts, the barrier is set close the initial underlying price, and the delta is near 90% (low strikes). However, in such situation, the underlying process has enough time to reach the barrier, deactivating the contract and, therefore, prices are expected to be rather small in such combination, even more when there is the additional condition that the call strike is high. So, these larger differences between models refer to small prices; and it does not happen, for instance, when the barrier, in a short maturity contract, is close to the initial underlying price, since there is not enough time to reach the barrier. In addition, we notice that such discrepancies between the relative entropy and the path integral model we presented also happened in the region of higher deltas, where the relative entropy model did not fit properly to vanillas and, as we have emphasized, it is important to have a good calibration in the delta region associated to the barrier position.

Another point is that barrier option pricing requires the choice of a larger number of cumulants than vanilla option pricing do, because polynomial fine-tune corrections are important near the barrier.

Finally, as a future development, the model might include double barrier contracts, stochastic barriers, and might also be extended to other classes of products, such as interest rates.
Appendix A. Useful relations involving the path integral

Let \( F(t_i) \) be a function. Consider objects such as

\[
\sum_{i=1}^{n-1} F(t_i) \int_{-\infty}^{\omega_c} d\omega_1 ... d\omega_{n-1} \partial_i W^{gm}(\omega_0, \omega_1, ..., \omega_n; t_n),
\]

(A.1)

where \( F \) is a generic function and \( \partial_i \equiv \partial/\partial \omega_i \). Because \( W^{gm} \) is Gaussian, it is limited and tends to zero at \(-\infty\). Integrating,

\[
\int_{-\infty}^{\omega_c} d\omega_1 ... d\omega_{n-1} \partial_i W^{gm}(\omega_0, \omega_1, ..., \omega_n; t_n) =
\]

(A.2)

in which \( \hat{\omega}_i \) denotes a variable that is no longer in the integral, because it was integrated. The density \( W^{gm} \), by (3.9), satisfies

\[
W^{gm}(\omega_0, \omega_1, ..., \omega_i = \omega_c, ..., \omega_{n-1}, \omega_n; t_n) =
\]

(A.3)

In (A.2),

\[
\int_{-\infty}^{\omega_c} d\omega_1 ... d\omega_{i-1} \int_{-\infty}^{\omega_c} d\omega_{i+1} ... d\omega_{n-1} W^{gm}(\omega_0, \omega_1, ..., \omega_{i-1}, \omega_i = \omega_c; t_i)
\]

(A.4)

In (A.1),

\[
\sum_{i=1}^{n-1} F(t_i) \int_{-\infty}^{\omega_c} d\omega_1 ... d\omega_{n-1} \partial_i W^{gm}(\omega_0, \omega_1, ..., \omega_n; t_n)
\]

(A.5)
Another relationship, analogous to (A.2), is

$$\int_{-\infty}^{\omega_c} d\omega_1...d\omega_{n-1} \partial_{ij} W^{gm} (\omega_0, \omega_1, ..., \omega_n; t_n)$$

$$= \int_{-\infty}^{\omega_c} d\omega_1...d\omega_{i-1} \int_{-\infty}^{\omega_i} d\omega_{i+1} d\hat{\omega}_1...d\hat{\omega}_j...d\omega_{n-1}.$$  

$$W^{gm} (\omega_0, \omega_1, ..., \omega_{i-1}, \omega_i = \omega_c, ..., \omega_{j-1}, \omega_j = \omega_c, ..., \omega_{n-1}, \omega_n; t_n).$$ \hspace{1cm} (A.6)

Because

$$W^{gm} (\omega_0, \omega_1, ..., \omega_{i-1}, \omega_i = \omega_c, ..., \omega_{j-1}, \omega_j = \omega_c, ..., \omega_{n-1}, \omega_n; t_n)$$

$$= W^{gm} (\omega_0, \omega_1, ..., \omega_{i-1}, \omega_i = \omega_c; t_i) \cdot W^{gm} (\omega_c, \omega_{i+1}, ..., \omega_{j-1}, \omega_j = \omega_c; t_j - t_i).$$

$$W^{gm} (\omega_c, \omega_{j+1}, ..., \omega_{n-1}, \omega_n; t_n - t_j),$$ \hspace{1cm} (A.7)

we have:

$$\int_{-\infty}^{\omega_c} d\omega_1...d\omega_{n-1} \partial_{ij} W^{gm} (\omega_0, \omega_1, ..., \omega_n; t_n)$$

$$= \int_{-\infty}^{\omega_c} d\omega_1...d\omega_{i-1} \int_{-\infty}^{\omega_i} d\omega_{i+1} d\hat{\omega}_i...d\hat{\omega}_j...d\omega_{n-1} W^{gm} (\omega_0, \omega_1, ..., \omega_{i-1}, \omega_i = \omega_c; t_i).$$

$$W^{gm} (\omega_c, \omega_{i+1}, ..., \omega_{i-1}, \omega_i = \omega_c; t_j - t_i) \cdot W^{gm} (\omega_c, \omega_{j+1}, ..., \omega_{n-1}, \omega_n; t_n - t_j)$$

$$= \Pi^{gm}_2 (\omega_0, \omega_c; t_i) \Pi^{gm}_2 (\omega_c, \omega_c; t_j - t_i) \Pi^{gm}_2 (\omega_c, \omega_n; t_n - t_j).$$ \hspace{1cm} (A.8)

Besides, as in (A.2), we calculate the following integral:

$$\int_{-\infty}^{\omega_c} d\omega_1...d\omega_{n-1} \partial_i^2 W^{gm} (\omega_0, \omega_1, ..., \omega_n; t_n)$$
\[ A. \text{Catalão \& R. Rosenfeld} \]

\[ = \int_{-\infty}^{\omega_c} d\omega_1 \cdots d\omega_{n-1} \partial_i W^{gm}(\omega_0, \omega_1, \ldots, \omega_i = \omega_c, \ldots, \omega_n; t_n) \]

\[ = \partial_i \int_{-\infty}^{\omega_c} d\omega_1 \cdots d\omega_{n-1} W^{gm}(\omega_0, \omega_1, \ldots, \omega_i = \omega_c, \ldots, \omega_n; t_n) \]

\[ = \partial_i [\Pi^{gm}_{\omega} (\omega_0, \omega_i; t_i) \Pi^{gm}_{\omega} (\omega_i, \omega_n; t_n - t_i)]_{\omega_i = \omega_c}. \quad (A.9) \]

We also want to analyze the derivatives with respect to the barrier \( \omega_c \). Consider the derivative \( \frac{\partial \Pi^{gm}_{\omega}}{\partial \omega_c} (\omega_0 = 0, \omega_n; t_n) \):

\[ \frac{\partial \Pi^{gm}_{\omega}}{\partial \omega_c} (\omega_0 = 0, \omega_n; t_n) = \frac{\partial}{\partial \omega_c} \left( \int_{-\infty}^{\omega_c} d\omega_1 \cdots \int_{-\infty}^{\omega_n} W^{gm}(\omega_0, \omega_1, \ldots, \omega_n; t_n) \right), \quad (A.10) \]

where we used the definition \( \text{[2.3]} \). The limits of the integral depend on the variable with respect to which we derive, \( \omega_c \). By Leibniz rule:

\[ \frac{d}{dx} \int_{a(x)}^{b(x)} dt f(x, t) = f(b(x)) \frac{db(x)}{dx} - f(a(x)) \frac{da(x)}{dx} + \int_{a(x)}^{b(x)} dt \frac{df(x, t)}{dx}, \quad (A.11) \]

\[ \frac{\partial \Pi^{gm}_{\omega}}{\partial \omega_c} (\omega_0 = 0, \omega_n; t_n) = \Pi^{gm}_{\omega} (\omega_n = \omega_c, t_n) \frac{\partial \omega_c}{\partial \omega_c} \]

\[ + \sum_{i=1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 \cdots d\omega_{n-1} \partial_i W^{gm}(\omega_0, \omega_1, \ldots, \omega_i = \omega_c, \ldots, \omega_n; t_n), \quad (A.12) \]

where \( \Pi^{gm}_{\omega} (\omega_n = \omega_c, t_n) \) is the Gaussian density with barrier, \( \text{[3.13]} \), which becomes zero at the barrier. The first term on the RHS of \( \text{[A.12]} \) is zero, therefore. Then,

\[ \frac{\partial \Pi^{gm}_{\omega}}{\partial \omega_c} (\omega_0 = 0, \omega_n; t_n) = \sum_{i=1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 \cdots d\omega_{n-1} \partial_i W^{gm}(\omega_0, \omega_1, \ldots, \omega_i = \omega_c, \ldots, \omega_n; t_n) \]

\[ = \sum_{i=1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 \cdots d\omega_{n-1} W^{gm}(\omega_0, \omega_1, \ldots, \omega_i = \omega_c, \ldots, \omega_n; t_n) \]

\[ = \sum_{i=1}^{n-1} \Pi^{gm}_{\omega} (\omega_0, \omega_c; t_i) \Pi^{gm}_{\omega} (\omega_c, \omega_n; t_n - t_i), \quad (A.13) \]
where we have used (A.2), (A.4).

The second derivative with respect to the barrier, \( \frac{\partial^2 \Pi_{gm}}{\partial \omega_c^2} (\omega_0 = 0, \omega_n; t_n) \), can also be computed. Initially, we note that

\[
\sum_{i,j=1}^{n-1} \partial_i \partial_j = 2 \sum_{i<j} \partial_i \partial_j + \sum_{i=1}^{n-1} \partial_i^2. \tag{A.14}
\]

The first term in RHS (A.14) is equivalent to

\[
\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \partial_i \partial_j \tag{A.15}
\]

or

\[
\sum_{j=2}^{n-1} \sum_{i=1}^{n-j-1} \partial_i \partial_j, \tag{A.16}
\]

Thus, comparing to (A.8) and (A.9),

\[
\frac{\partial^2 \Pi_{gm}}{\partial \omega_c^2} (\omega_0 = 0, \omega_n; t_n) = \sum_{i,j=1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 \cdots d\omega_{i-1} \int_{-\infty}^{\omega_c} d\omega_i+1 d\omega_i \cdots d\omega_{j-1} d\omega_j \cdots d\omega_{n-1}.
\]

\[
W^{gm} (\omega_0, \omega_1, ..., \omega_{i-1}, \omega_i = \omega_c, ..., \omega_{j-1}, \omega_j = \omega_c, ..., \omega_{n-1}, \omega_n; t_n)
\]

\[
= 2 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 \cdots d\omega_{i-1} \int_{-\infty}^{\omega_c} d\omega_i+1 d\omega_i \cdots d\omega_{j-1} d\omega_j \cdots d\omega_{n-1}.
\]

\[
W^{gm} (\omega_0, \omega_1, ..., \omega_{i-1}, \omega_i = \omega_c, ..., \omega_{j-1}, \omega_j = \omega_c, ..., \omega_{n-1}, \omega_n; t_n)
\]

\[
+ \sum_{i=1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 \cdots d\omega_{i-1} d\omega_i+1 \frac{\partial}{\partial \omega_c} W^{gm} (\omega_0, \omega_1, ..., \omega_i = \omega_c, ..., \omega_{n-1}, \omega_n; t_n)
\]

\[
= 2 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 \cdots d\omega_{i-1} d\omega_i+1 \partial_i \partial_j W^{gm} + \sum_{i=1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 \cdots d\omega_{n-1} \partial_i^2 W^{gm}
\]
\[
\sum_{i,j=1}^{n-1} \int_{-\infty}^{\infty} d\omega_1 \cdots d\omega_{n-1} \partial_i \partial_j W^{gm}
\]  
(A.17)

Higher order derivatives follow analogous expressions.

**Appendix B. Analysis of the Gaussian distribution near the barrier**

In this appendix we analyze, in the presence of barrier \(\omega_c\), the behaviour of the Gaussian density of probability, given by (3.3), when the variable \(\omega_n\) approximates \(\omega_c\). The first discussed situation is when the variable starts at \(\omega_0\), reaching the barrier \(\omega_c\), in \(t_n\), \(\Pi^{gm}_{\epsilon \rightarrow 0} (\omega_0, \omega_n = \omega_c; t_n)\). The second situation is when the variables starts in the barrier vicinity, in \(t_0\), and arrives at \(\omega_n \neq \omega_c\) in \(t_n\), \(\Pi^{gm}_{\epsilon \rightarrow 0} (\omega_0 = \omega_c, \omega_n = \omega_c; t_n)\). The third situation corresponds to the case in which the variable starts in the barrier vicinity and stays near it, \(\Pi^{gm}_{\epsilon \rightarrow 0} (\omega_0 = \omega_c, \omega_n = \omega_c; t_n)\).

To do so, the behaviour of the Gaussian distribution near the barrier, as the variable approaches it, is obtained by the Taylor expansion in the element \(\Delta \omega = \omega_n - \Delta \omega\). We will see that there is a regime change in the relation of the probability with time discretization \(\epsilon\) as the variable tends to the barrier: \(\omega_n \rightarrow \omega_c\).

To arrive at the expressions \(\Pi^{gm}_{\epsilon \rightarrow 0} (\omega_0, \omega_n = \omega_c; t_n)\), \(\Pi^{gm}_{\epsilon \rightarrow 0} (\omega_0 = \omega_c, \omega_n; t_n)\) and \(\Pi^{gm}_{\epsilon \rightarrow 0} (\omega_0 = \omega_c, \omega_n = \omega_c; t_n)\), we expand the Gaussian density in powers of \(\sqrt{\epsilon}\).

**B.1. Behaviour of the Gaussian distribution near the barrier**

We start by getting some relations from \(\Pi^{gm}_{\epsilon} (\omega_0, \omega_n; t_n) \) (3.12). From (3.11), \(\omega_{n-1} = \omega_n - \Delta \omega\). In (3.12), changing the variable to \(\Delta \omega\), \(d (\Delta \omega) = -d\omega_{n-1}\), for a given \(\omega_n\), fixed. Thus, the limits of integration are \(\Delta \omega_1 = \omega_n - \omega_{n-1} |_{\omega_C} = \omega_n - \omega_c\) and \(\Delta \omega_2 = \omega_n - \omega_{n-1} |_{\infty} = \infty\); inverting them, due to the negative sign coming from \(d (\Delta \omega)\), we have:

\[
\Pi^{gm}_{\epsilon} (\omega_0, \omega_n; t_n = t_{n-1} + \epsilon) = - \int_{\infty}^{\omega_n - \omega_c} d (\Delta \omega) \Psi_{\epsilon} (\Delta \omega) \Pi^{gm}_{\epsilon} (\omega_0, \omega_{n-1}; t_{n-1})
\]

\[
= \int_{\omega_n - \omega_c}^{\infty} d (\Delta \omega) \Psi_{\epsilon} (\Delta \omega) \Pi^{gm}_{\epsilon} (\omega_0, \omega_{n-1}; t_{n-1})
\]
\[ \int_{\omega_n - \omega_c}^{\infty} d(\Delta \omega) \Psi_\epsilon (\Delta \omega) \Pi_{\epsilon}^{gm} (\omega_0, \omega_n - \Delta \omega; t_{n-1}) . \quad (B.1) \]

Notice that

\[ \lim_{\epsilon \to 0} \Psi_\epsilon (\Delta \omega) = \delta (\Delta \omega) . \quad (B.2) \]

If \( \omega_n - \omega_c < 0 \) (that is, if the variable crosses the barrier after \( t_n \)), it includes the support of the Dirac \( \delta \) function. If \( \omega_n - \omega_c > 0 \), the integral is zero because it is outside the support. And, if \( \omega_n = \omega_c \), with the condition of the continuum limit \( \epsilon \to 0 \), it goes to zero because of the initial condition. Therefore,

\[ \Pi_{\epsilon}^{gm} (\omega_0, \omega_n; t_n) = 0, \text{ if } \omega_n \geq \omega_c . \quad (B.3) \]

In the case \( \omega_n < \omega_c \), we analyze (B.1), starting by expanding its LHS, in terms of \( t_{n-1} \):

\[ \Pi_{\epsilon}^{gm} (\omega_0, \omega_n; t_{n-1} + \epsilon) = \Pi_{\epsilon}^{gm} (\omega_0, \omega_n; t_{n-1}) \]
\[ + \epsilon \frac{\partial \Pi_{\epsilon}^{gm} (\omega_0, \omega_n; t_{n-1})}{\partial t_{n-1}} + \frac{\epsilon^2}{2} \frac{\partial^2 \Pi_{\epsilon}^{gm} (\omega_0, \omega_n; t_{n-1})}{\partial t_{n-1}^2} + ... \quad (B.4) \]

Expanding in Taylor series the RHS (B.1) in terms of \( \Delta \omega \), in the region of \( \Delta \omega = 0 \), that is, when \( \omega_n \to \omega_c \):

\[ \int_{\omega_n - \omega_c}^{\infty} d(\Delta \omega) \Psi_\epsilon (\Delta \omega) \Pi_{\epsilon}^{gm} (\omega_0, \omega_n - \Delta \omega; t_{n-1}) = \]
\[ = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i \Pi_{\epsilon}^{gm} (\omega_0, \omega_n; t_{n-1})}{\partial \omega_n^i} \int_{\omega_n - \omega_c}^{\infty} d(\Delta \omega) (\Delta \omega)^i \Psi_\epsilon (\Delta \omega) \quad (B.5) \]

where, when expanding in \( \Delta \omega \), which involves \( \frac{\partial \Pi_{\epsilon}^{gm} (\omega_0, \omega_n - \Delta \omega; t_{n-1})}{\partial \Delta \omega} \), we applied the chain rule (also considering the expansion in the region \( \Delta \omega = 0 \)):

\[ \frac{\partial}{\partial \Delta \omega} = \frac{\partial}{\partial (\omega_n - \Delta \omega)} \frac{\partial (\omega_n - \Delta \omega)}{\partial \Delta \omega} = \frac{\partial}{\partial (\omega_n - \Delta \omega)} \cdot (-1) \bigg|_{\Delta \omega = 0} = \frac{\partial}{\partial \omega_n} \cdot (-1) . \quad (B.6) \]
For second order,
\[
\frac{\partial^2}{\partial \Delta \omega^2} = \frac{\partial}{\partial \Delta \omega} \left( \frac{\partial}{\partial \Delta \omega} \right) = \frac{\partial}{\partial (\omega_n - \Delta \omega)} \cdot \left( \frac{\partial}{\partial (\omega_n - \Delta \omega)} \right) \cdot (-1) \cdot \frac{\partial (\omega_n - \Delta \omega)}{\partial \Delta \omega}
\]
\[
= \frac{\partial^2}{\partial (\omega_n - \Delta \omega)^2} \cdot (-1) \bigg|_{\Delta \omega = 0} = \frac{\partial^2}{\partial \omega_n^2} \cdot (-1)^2.
\]
(B.7)

Then, for \(i\)th order,
\[
\frac{\partial^i}{\partial \Delta \omega^i} = \frac{\partial^i}{\partial \omega_n^i} \cdot (-1)^i.
\]
(B.8)

Back to (B.5), we see that, changing to the variable \(y = \frac{\Delta \omega}{\sqrt{2 \epsilon}}\),
\[
\int_{\omega_n - \omega_c}^{\infty} d (\Delta \omega) (\Delta \omega)^i \Psi_{\epsilon} (\Delta \omega) = \frac{(2 \epsilon)^{1/2}}{\sqrt{\pi}} \int_{-\frac{\sqrt{2 \epsilon} (\omega_n - \omega_c)}{2 \epsilon}}^{\infty} dy \cdot y^i e^{-y^2}.
\]
(B.9)

Consider the cases \(i = 0\) and \(i = 1\), with \(\omega_n \to \omega_c\):
\[
\int_{\omega_n - \omega_c}^{\infty} d (\Delta \omega) (\Delta \omega)^i \Psi_{\epsilon} (\Delta \omega) \bigg|_{i=0, \omega_n \to \omega_c} \implies \int_{0}^{\infty} d (\Delta \omega) \Psi_{\epsilon} (\Delta \omega) = \frac{1}{2}
\]
(B.10)
\[
\int_{\omega_n - \omega_c}^{\infty} d (\Delta \omega) (\Delta \omega)^i \Psi_{\epsilon} (\Delta \omega) \bigg|_{i=1, \omega_n \to \omega_c} \implies \int_{0}^{\infty} d (\Delta \omega) (\Delta \omega) \Psi_{\epsilon} (\Delta \omega) = \frac{(2 \epsilon)^{1/2}}{\sqrt{\pi}} \int_{0}^{\infty} dy \cdot y \cdot e^{-y^2} = \left(\frac{\epsilon}{2 \pi}\right)^{1/2}
\]
(B.11)

where we integrated by parts. Back to (B.5), taking, as mentioned, the terms \(i = 0\) and \(i = 1\),
\[
\int_{0}^{\infty} d (\Delta \omega) \Psi_{\epsilon} (\Delta \omega) \Pi_{\epsilon}^{gm} (\omega_0, \omega_n - \Delta \omega; t_{n-1})
\]
\[
= \frac{1}{2} \Pi_{\epsilon}^{gm} (\omega_0, \omega_c; t_{n-1}) - \left(\frac{\epsilon}{2 \pi}\right)^{1/2} \frac{\partial}{\partial \omega_n} \Pi_{\epsilon}^{gm} (\omega_0, \omega_n; t_{n-1}) \bigg|_{\omega_n = \omega_c} + ...
\]
(B.12)
Therefore, when \( \omega_n - \omega_c < 0 \), the behaviour when \( \omega_n \to \omega_c \), with \( \epsilon \to 0 \), depends on \( \sqrt{\epsilon} \). Consider the case \( \omega_n - \omega_c < 0 \), but when we are not in the situation \( \omega_n \to \omega_c \). In this case, in the continuum \( \epsilon \to 0 \), the inferior limit \( -\frac{\omega_c - \omega_n}{\sqrt{2\epsilon}} \) of the integral (B.9) becomes \(-\infty\). Then, the new integral (B.9), with new limits, is

\[
\int_{\omega_n - \omega_c}^{\infty} d(\Delta \omega) (\Delta \omega)^i \Psi_\epsilon(\Delta \omega) = \left( \frac{2\epsilon}{\sqrt{\pi}} \right)^{1/2} \int_{-\infty}^{\infty} dy \cdot y^i e^{-y^2} \to \left( \frac{2\epsilon}{\sqrt{\pi}} \right)^{1/2} \int_{-\infty}^{\infty} dy \cdot y^i e^{-y^2}.
\]

Because

\[
\int_{-\infty}^{\infty} dy \cdot y^n e^{-y^2} \cong \frac{1 + (-1)^n}{2} \frac{\sqrt{\pi}}{2^{n/2}} (n - 1)!!,
\]

then

\[
\int_{\omega_n - \omega_c}^{\infty} d(\Delta \omega) (\Delta \omega)^i \Psi_\epsilon(\Delta \omega) \to \begin{cases} 
\epsilon^{1/2} (n - 1)!! & i \text{ even} \\
0 & i \text{ odd} 
\end{cases}.
\]

In this case, the expansion (B.5) with the analogous terms to (B.10) and (B.11), respectively, for \( i = 0 \) and \( i = 2 \) are:

\[
\left. \int_{\omega_n - \omega_c}^{\infty} d(\Delta \omega) (\Delta \omega)^i \Psi_\epsilon(\Delta \omega) \right|_{i=0, \omega_n < \omega_c} \implies \int_{-\infty}^{\infty} d(\Delta \omega) \Psi_\epsilon(\Delta \omega) = 1
\]

\[
\left. \int_{\omega_n - \omega_c}^{\infty} d(\Delta \omega) (\Delta \omega)^i \Psi_\epsilon(\Delta \omega) \right|_{i=2, \omega_n < \omega_c} = \epsilon.
\]

Then, in the case \( \omega_n < \omega_c, \epsilon \to 0 \), but \( \omega_n \) not near \( \omega_c \),

\[
\left. \int_{-\infty}^{\infty} d(\Delta \omega) \Psi_\epsilon(\Delta \omega) \Pi^gm(\omega_0, \omega_n - \Delta \omega; t_{n-1}) \right|_{\omega_n = \omega_c} = \Pi^gm(\omega_0, \omega_c; t_{n-1}) + \frac{\epsilon}{2} \left. \frac{\partial^2}{\partial \omega_n^2} \Pi^gm(\omega_0, \omega_n; t_{n-1}) \right|_{\omega_n = \omega_c} + ...
\]

Thus, there is a regime change with respect to \( \epsilon \) in the passage from \( \omega_n < \omega_c \) to the case \( \omega_n < \omega_c \), with the extra condition \( \omega_n \to \omega_c \); in the first case, the next
leading term of the expansion (B.18) behaves as $\epsilon$, while in the second, the next leading term dominating the expansion (B.12) follows $\sqrt{\epsilon}$. Such transition is ruled by the inferior limit $(\omega_c - \omega_n) \sqrt{2 \epsilon}$ in integral (B.9). To tackle this, we define

$$\eta = \frac{\omega_c - \omega_n}{\sqrt{2 \epsilon}}. \quad (B.19)$$

We write $\Pi_{g\epsilon}^{gm}$ in the form

$$\Pi_{g\epsilon}^{gm} (\omega_0, \omega_n; t_n) = C_\epsilon (\omega_0, \omega_n; t_n) v (\eta). \quad (B.20)$$

Here, $C$ is the smooth part of the function, while $v(\eta)$ is responsible for the regime transition of the function $\Pi_{g\epsilon}^{gm}$. We must impose

$$\lim_{\eta \to \infty} v(\eta) = 1 \quad (B.21)$$

so that $C$ is the solution of $\Pi_{g\epsilon}^{gm}$ when $\omega_c - \omega_n$ is finite and positive. Consider the Gaussian probability density, which, in the continuum limit ($\epsilon \to 0$), in the presence of a barrier, is the solution (3.13):

$$\Pi_{g\epsilon}^{gm} \to 0 (\omega_0, \omega_n; t_n) = \frac{1}{\sqrt{2\pi t_n}} e^{\alpha (\omega_0 - \omega_n) - \frac{1}{2} \alpha^2 t_n} \left[ e^{-\frac{(\omega_c - \omega_n)^2}{2t_n}} - e^{-\frac{(2\omega_c - \omega_n)^2}{2t_n}} \right]. \quad (B.22)$$

Isolating $\omega_n$ in (B.19),

$$\Pi_{g\epsilon}^{gm} \to 0 (\omega_0, \omega_n; t_n) = \frac{1}{\sqrt{2\pi t_n}} e^{\alpha (\omega_n - \eta \sqrt{2 \epsilon} - \omega_0) - \frac{1}{2} \alpha^2 t_n} \left[ e^{-\frac{(\omega_c - \eta \sqrt{2 \epsilon} - \omega_0)^2}{2t_n}} - e^{-\frac{(\omega_c + \eta \sqrt{2 \epsilon} - \omega_0)^2}{2t_n}} \right]. \quad (B.23)$$

Expanding in powers of $\sqrt{\epsilon}$, which means that $\eta \to \infty$ and, consequently, (B.21):

$$\exp \left( -\frac{1}{2t_n} (\omega_c \mp \eta \sqrt{2 \epsilon} - \omega_0)^2 \right) = e^{-\frac{1}{2t_n} (\omega_c - \omega_0)^2} \left[ 1 \pm \frac{\eta \sqrt{2 \epsilon}}{2t_n} 2 (\omega_c - \omega_0) \sqrt{\epsilon} + \ldots \right] \quad (B.24)$$
\[ \exp \left( \alpha \left( \omega_c - \eta \sqrt{2 \epsilon} - \omega_0 \right) \right) = e^{\alpha (\omega_c - \omega_0)} \left[ 1 - \eta \sqrt{2} \alpha \sqrt{\epsilon} + \ldots \right] \quad (B.25) \]

\[ \therefore C_x (\omega_0 = 0, \omega_n, T) = \frac{1}{\sqrt{2 \pi t_n}} e^{-\frac{1}{2} \alpha^2 t_n e^{\alpha (\omega_n - \omega_0)} e^{-\frac{1}{2 t_n} (\omega_n - \omega_0)^2} \left[ 1 - \eta \sqrt{2} \alpha \sqrt{\epsilon} + \ldots \right]}. \]

\[ \cdot \left[ 1 + \frac{\eta \sqrt{2}}{t_n} (\omega_c - \omega_0) \sqrt{\epsilon} - 1 + \frac{\eta \sqrt{2}}{t_n} (\omega_c - \omega_0) \sqrt{\epsilon} + \ldots \right] \]

\[ = \frac{1}{\sqrt{2 \pi t_n}} e^{-\frac{1}{2} \alpha^2 t_n e^{\alpha (\omega_n - \omega_0)} e^{-\frac{1}{2 t_n} (\omega_n - \omega_0)^2} \left[ 1 - \eta \sqrt{2} \alpha \sqrt{\epsilon} + \ldots \right] \left[ 2 \frac{\eta \sqrt{2}}{t_n} (\omega_c - \omega_0) \sqrt{\epsilon} + \ldots \right] \]

\[ = \sqrt{\epsilon} \frac{2 \eta}{\sqrt{\pi}} \frac{(\omega_c - \omega_0)}{t_n^{3/2}} e^{-\frac{1}{2} \alpha^2 t_n e^{\alpha (\omega_n - \omega_0)} e^{-\frac{1}{2 t_n} (\omega_n - \omega_0)^2}} + \mathcal{O} (\epsilon). \quad (B.26) \]

In (B.20), when \( \omega_n \to \omega_c \),

\[ \Pi_{\epsilon}^{\alpha m} (\omega_0 = 0, \omega_n; t_n = T) = \sqrt{\epsilon} \gamma \frac{(\omega_c - \omega_0)}{t_n^{3/2}} e^{-\frac{1}{2} \alpha^2 t_n e^{\alpha (\omega_n - \omega_0)} e^{-\frac{1}{2 t_n} (\omega_n - \omega_0)^2}} + \mathcal{O} (\epsilon) \quad (B.27) \]

\[ \gamma = \frac{2}{\sqrt{\pi}} \lim_{\eta \to 0} \eta v (\eta). \quad (B.28) \]

In this equation, the limit is taken as \( \eta \to 0 \), because it corresponds to \( \omega_n \to \omega_c \) in definition (B.19). Next, we will show that

\[ \gamma = \frac{1}{\sqrt{\pi}} e^{\alpha (\omega_n - \omega_c)}. \quad (B.29) \]

Now consider the derivative with respect to the barrier, given by (A.13). In the limit \( \epsilon \to 0 \),

\[ \sum_{i=1}^{n-1} \frac{1}{\epsilon} \int_0^{t_n} dt_i. \quad (B.30) \]

In (A.13),
\[ \frac{\partial \Pi_{\epsilon \to 0}^{gm}}{\partial \omega_c} (\omega_0 = 0, \omega_n; t_n) = \sum_{i=1}^{n-1} \Pi_{\epsilon}^{gm} (\omega_i, \omega_c; t_i) \Pi_{\epsilon}^{gm} (\omega_c, \omega_n; t_n - t_i) \]

\[ = \int_0^{t_n} dt \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Pi_{\epsilon}^{gm} (\omega_0, \omega_c; t_i) \Pi_{\epsilon}^{gm} (\omega_c, \omega_n; t_n - t_i). \]  

(B.31)

The LHS of (B.31) is computed by (B.22). In the case \(\omega_0 \neq 0\),

\[ \frac{\partial \Pi_{\epsilon \to 0}^{gm}}{\partial \omega_c} (\omega_0, \omega_n; t_n) = \frac{2}{\sqrt{2\pi t_n}} e^{\alpha (\omega_n - \omega_0) - \frac{1}{2} \alpha^2 t_n} \left[ \frac{(2\omega_c - \omega_n - \omega_0)}{t_n} e^{-\frac{(2\omega_c - \omega_n - \omega_0)^2}{4t_n}} \right] \]

\[ = \left( \frac{2}{\pi} \right)^{1/2} \frac{(2\omega_c - \omega_n - \omega_0)}{t_n^{3/2}} e^{\alpha (\omega_n - \omega_0) - \frac{1}{2} \alpha^2 t_n} e^{-\frac{(2\omega_c - \omega_n - \omega_0)^2}{4t_n}}. \]  

(B.32)

The RHS (B.31) is given by (B.27). We notice that

\[ \Pi_{\epsilon \to 0}^{gm} (\omega_c, \omega_n; t_n) = \Pi_{\epsilon \to 0}^{gm} (\omega_n, \omega_c; t_n). \]  

(B.33)

This can be seen by (2.3) and (3.8):

\[ \Pi_{\epsilon \to 0}^{gm} (\omega_c, \omega_n; t_n) \Rightarrow \]

\[ W_{\epsilon \to 0}^{gm} (\omega_c, \omega_1, ..., \omega_n; t_n) = \frac{1}{(2\pi \epsilon)^{n/2}} e^{-\frac{(\omega_1 - \omega_n)^2}{2\epsilon} - \sum_{i=1}^{n-2} \frac{(\omega_{i+1} - \omega_i)^2}{2\epsilon}} \]  

\[ \times \frac{(\omega_n - \omega_{n-1})^2}{2\epsilon}. \]  

(B.34)

\[ \Pi_{\epsilon \to 0}^{gm} (\omega_n, \omega_c; t_n) \Rightarrow \]

\[ W_{\epsilon \to 0}^{gm} (\omega_n, \omega_1, ..., \omega_c; t_n) = \frac{1}{(2\pi \epsilon)^{n/2}} e^{-\frac{(\omega_1 - \omega_n)^2}{2\epsilon} - \sum_{i=1}^{n-2} \frac{(\omega_{i+1} - \omega_i)^2}{2\epsilon}} \]  

\[ \times \frac{(\omega_c - \omega_{c-1})^2}{2\epsilon}. \]  

(B.35)

This happens because \(\omega_0 (= \omega_c)\) and \(\omega_n\) are not integration variables in (2.3), that is, \(\omega_c\) to be with \(\omega_{n-1}\) or \(\omega_1\) does not matter, because they are integrated in
the same integration limits. Therefore, (B.33) is valid. Equation (B.27), with \( \omega_0 \neq 0 \) is written as:

\[
\Pi_{\epsilon \to 0}^{gm} (\omega_0, \omega_e; t_n) = \sqrt{\frac{\omega_e - \omega_0}{t_n^{3/2}}} e^{-\frac{1}{2} \alpha^2 t_n \epsilon \alpha (\omega_e - \omega_0)} e^{-\frac{1}{2 \pi} (\omega_e - \omega_0)^2}.
\] (B.36)

Then,

\[
\Pi_{\epsilon \to 0}^{gm} (\omega_e, \omega_H; t_n) = \Pi_{\epsilon \to 0}^{gm} (\omega_n, \omega_e; t_n)
\]

\[
= \sqrt{\frac{\omega_e - \omega_n}{t_n^{3/2}}} e^{-\frac{1}{2} \alpha^2 t_n \epsilon \alpha (\omega_e - \omega_n)} e^{-\frac{1}{2 \pi} (\omega_e - \omega_n)^2}.
\] (B.37)

Returning to (B.31), in \( \Pi_{\epsilon}^{gm} (\omega_0, \omega_e; t_i) \) we use (B.27), and, in \( \Pi_{\epsilon}^{gm} (\omega_e, \omega_H; t_n - t_i) \), (B.37).

We also use

\[
\int_0^c dx \frac{1}{x^{3/2}} (e^x)^{3/2} = \frac{\pi^{1/2}}{2a} e^{-\frac{(a + b)^2}{2c}},
\] (B.38)

so,

\[
\int_0^{t_n} dt_i \text{lim}_{\epsilon \to 0} \frac{1}{\epsilon} \Pi_{\epsilon}^{gm} (\omega_0, \omega_e; t_i) \Pi_{\epsilon}^{gm} (\omega_e, \omega_H; t_n - t_i)
\]

\[
= \int_0^{t_n} dt_i \text{lim}_{\epsilon \to 0} \frac{1}{\epsilon} \left( \sqrt{\frac{\omega_e - \omega_0}{t_i^{3/2}}} e^{-\frac{1}{2} \alpha^2 t_i \epsilon \alpha (\omega_e - \omega_0)} e^{-\frac{1}{2 \pi} (\omega_e - \omega_0)^2} \right).
\]

\[
\cdot \left( \sqrt{\frac{\omega_e - \omega_n}{(t_n - t_i)^{3/2}}} e^{-\frac{1}{2} \alpha^2 (t_n - t_i) \epsilon \alpha (\omega_e - \omega_n)} e^{-\frac{1}{2 \pi} (\omega_e - \omega_n)^2} \right)
\]

\[
= \int_0^{t_n} dt_i \gamma^2 \frac{(\omega_e - \omega_0) (\omega_e - \omega_n)}{(t_n - t_i)^{3/2} t_i^{3/2}} e^{-\frac{1}{2} \alpha^2 t_i \epsilon \alpha (\omega_e - \omega_0)} e^{-\frac{1}{2 \pi} (\omega_e - \omega_0)^2} e^{-\frac{1}{2 \pi} (\omega_e - \omega_n)^2} e^{-\frac{1}{2} \alpha^2 (t_n - t_i) \epsilon \alpha (\omega_e - \omega_n)} e^{-\frac{1}{2 \pi} (\omega_e - \omega_n)^2}.
\]

\[
= \sqrt{2\pi} \frac{(2\omega_e - \omega_n - \omega_0)}{t_n^{3/2}} \gamma^2 e^{-\frac{(2\omega_e - \omega_n - \omega_0)^2}{2t_n}}.
\] (B.39)

Equating with (B.32),
\[ \gamma = \frac{1}{\sqrt{\pi}} e^{\alpha(\omega_n - \omega_c)}. \quad (B.40) \]

Back to (B.36), with \( \omega_0 \neq 0 \),

\[ \Pi^{gm}_\varepsilon (\omega_0, \omega_c, t_n = T) = \sqrt{\varepsilon} \frac{1}{\sqrt{\pi}} e^{\alpha(\omega_n - \omega_0)} \frac{\omega_c - \omega_0}{t^{3/2}_n} e^{-\frac{1}{2} \alpha^2 t_n} e^{\alpha(\omega_c - \omega_0)} e^{-\frac{1}{\pi t_n} (\omega_c - \omega_0)^2} \]

\[ = \sqrt{\varepsilon} \frac{1}{\sqrt{\pi}} e^{\alpha(\omega_n - \omega_0)} \frac{\omega_c - \omega_0}{t^{3/2}_n} e^{-\frac{1}{2} \alpha^2 t_n} e^{-\frac{1}{\pi t_n} (\omega_c - \omega_0)^2}. \]

\[ \Rightarrow \Pi^{gm}_\varepsilon (\omega_0, \omega_c, t_n = T) = \sqrt{\varepsilon} \frac{1}{\sqrt{\pi}} e^{\alpha(\omega_n - \omega_0)} \frac{\omega_c - \omega_0}{t^{3/2}_n} e^{-\frac{(\omega_c - \omega_0)^2 + (\omega_c - \omega_0)^2}{2t_n}}. \quad (B.41) \]

In the case of \( \omega_n < \omega_c \), we use (B.33) and (B.41) to analyze the situation where the process starts near the barrier and finishes at \( \omega_n \):

\[ \Pi^{gm}_\varepsilon (\omega_c, \omega_n, t_n = T) = \sqrt{\varepsilon} \frac{1}{\sqrt{\pi}} e^{\alpha(\omega_n - \omega_c)} \frac{\omega_n - \omega_c}{t^{3/2}_n} e^{-\frac{1}{2} \alpha^2 t_n} e^{-\frac{(\omega_n - \omega_c)^2}{2t_n}}; \quad \omega_n < \omega_c. \quad (B.42) \]

In the case of \( \Pi^{gm}_\varepsilon (\omega_c, \omega_c, t_n = T) \), we must have translation invariance, and \( \Pi^{gm}_\varepsilon (\omega_0, \omega_c, t_n = T) \) can only depend on \( \omega_0 \) and \( \omega_c \) with \( \omega_c - \omega_0 \). The expansion (B.27), already developed in the case of (B.36), with \( \omega_0 \rightarrow \omega_c \) (or \( \eta \rightarrow 0 \)), tends to zero, as in (B.41). The next term of the expansion (B.26) must be proportional to \( \varepsilon/t^{3/2}_n \) and to the drift term:

\[ \Pi^{gm}_\varepsilon (\omega_c, \omega_c, t_n = T) = c \varepsilon \frac{e^{\alpha(\omega_n - \omega_c)}}{t^{3/2}_n} e^{-\frac{\omega_n^2}{2t_n}}. \quad (B.43) \]

In [Maggiore & Riotto (2010a)], the identification of the constant is done by (3.12), and it is sufficient to study the case of \( n = 2 \) variables.

\[ \Pi^{gm}_\varepsilon (\omega_0, \omega_2, t_2) = \int_{-\infty}^{\omega_2} d\omega_1 \frac{1}{2\pi \varepsilon} e^{-\frac{1}{2} \left[ (\omega_1 - \omega_0 - \alpha t)^2 + (\omega_2 - \omega_1 - \alpha t)^2 \right]}. \quad (B.44) \]

We can solve this integral in Mathematica, taking into account that in two steps \( t_n = 2 \varepsilon \):

\[ \Pi^{gm}_\varepsilon (\omega_0, \omega_2; t_2) = \frac{1}{2\pi \varepsilon} \frac{1}{2} e^{-\frac{((\omega_0 + \omega_2) - 2\alpha t)^2}{2(2\alpha t)^2}} \sqrt{\pi} \sqrt{\varepsilon} \left( 1 + Erf \left[ \frac{\omega_0 - \omega_2}{2\sqrt{\varepsilon}} \right] \right). \quad (B.45) \]
We make $\omega_2 - \omega_0 \sim 0$, because we are in the case $\omega_0 \to \omega_c$ and $\omega_n \to \omega_c$:

\[ e^{-\frac{\omega_0 - \omega_2}{2(2\epsilon)}} \sim 1 + \left[ \frac{2(\omega_0 - \omega_2)}{2(2\epsilon)} e^{-\frac{\omega_0 - \omega_2}{2(2\epsilon)}} \right] \quad (\omega_2 = \omega_0 = 0) \]

\[ + \frac{1}{2} \left[ \frac{2}{2(2\epsilon)} e^{-\frac{\omega_0 - \omega_2}{2(2\epsilon)}} - \left( \frac{2(\omega_0 - \omega_2)}{2(2\epsilon)} \right)^2 e^{-\frac{\omega_0 - \omega_2}{2(2\epsilon)}} \right] \quad (\omega_2 = \omega_0 = 0) \]

\[ = 1 + \frac{1}{2t_n} (\omega_2 - \omega_0)^2 \to 1 \]

Keeping the term in $\epsilon$ and, given that $Erf(\frac{\omega_0 - \omega_2}{2\sqrt{\epsilon}} \sim 0)$, we can write:

\[ \Pi^{gm}_\epsilon (\omega_0 = \omega_c, \omega_2 = \omega_c, t_n = T) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\epsilon}} \frac{1}{2\epsilon} e^{-\frac{t_n \alpha^2}{2}} = \frac{1}{\sqrt{2\pi}} \frac{1}{t_n^{3/2}} e^{-\frac{\omega_2^2 t_n}{2}}. \quad (B.46) \]

Hence,

\[ c = \frac{1}{\sqrt{2\pi}} \quad (B.47) \]

\[ \Pi^{gm}_\epsilon (\omega_c, \omega_c, t_n = T) = \frac{1}{\sqrt{2\pi}} \frac{1}{t_n^{3/2}} e^{-\frac{\omega_2^2 t_n}{2}}. \quad (B.48) \]

In a nutshell, we analyzed the behaviour near the barrier: $\Pi^{gm}_\epsilon (\omega_0, \omega_c, t_n = T)$, $\Pi^{gm}_\epsilon (\omega_c, \omega_n, t_n = T)$ and $\Pi^{gm}_\epsilon (\omega_c, \omega_c, t_n = T)$, described by (B.41), (B.42) and (B.48), respectively.
B.2. Analysis of divergent and finite terms in $\sum_{i,j=1}^{n-1} \partial_i \partial_j$

In equation (4.17) there is the term $\sum_{i,j=1}^{n-1} \partial_i \partial_j$, which was used in equation (A.14) in the following way:

$$\sum_{i,j=1}^{n-1} \partial_i \partial_j = 2 \sum_{i<j} \partial_i \partial_j + \sum_{i=1}^{n-1} \partial_i^2,$$  \hspace{1cm} (B.49)

where the first term of the RHS can be expressed as

$$\sum_{i<j} \partial_i \partial_j = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \partial_i \partial_j$$  \hspace{1cm} (B.50)

or

$$\sum_{j=2}^{n-1-j-1} \sum_{i=1}^{n-1} \partial_i \partial_j.$$  \hspace{1cm} (B.51)

By equation (4.17), when $W_{gm} (\omega_0, \omega_1, ..., \omega_n; t_n)$ receives the application of (i) the summation operator, according to (A.8), and also (ii) the second term of RHS (B.49), according to (A.9), we have

$$2 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \Pi_{\varepsilon}^{gm} (\omega_0, \omega_i; t_i) \Pi_{\varepsilon}^{gm} (\omega_i, \omega_c; t_j - t_i) \Pi_{\varepsilon}^{gm} (\omega_c, \omega_n; t_n - t_j)$$

$$+ \sum_{i=1}^{n-1} \partial_i [\Pi_{\varepsilon}^{gm} (\omega_0, \omega_i; t_i) \Pi_{\varepsilon}^{gm} (\omega_i, \omega_n; t_n - t_i)]_{\omega_i = \omega_c}.$$  \hspace{1cm} (B.52)

Given the form of (3.8),

$$\partial_i [\Pi_{\varepsilon}^{gm} (\omega_0, \omega_i; t_i)] = \partial_i [\Pi_{\varepsilon}^{gm} (\omega_i, \omega_n; t_n - t_i)].$$  \hspace{1cm} (B.53)

Thus, the second term of (B.52) becomes
\[ I_1 \equiv \sum_{i=1}^{n-1} 2 \partial_i \left[ \Pi^m_{\omega_i} (\omega_0; \omega_i; t_i) \right]_{\omega_i=\omega_{n}} \Pi^m_{\omega_n} (\omega_{n}; t_n - t_i). \]  
(B.54)

Let us compute this term. Regarding \( \Pi^m_{\omega_0} (\omega_0; \omega_i; t_i) \), we saw in (B.28) that in the limit \( \eta \to 0 \), the expression \( v(\eta) \) of (B.20) leads to \( \gamma \), given by (B.40):

\[ \gamma = \frac{1}{\sqrt{\pi}} e^{\alpha (\omega_n - \omega_c)} \]

More broadly, in (B.28):

\[ v(\eta) \propto \frac{\gamma \sqrt{\pi}}{2 \eta} = \frac{1}{2 \eta} e^{\alpha (\omega_n - \omega_c)} \]

we can propose that in this limit, still respecting \( \gamma \), \( v(\eta) \) is given by

\[ v(\eta) = e^{\alpha (\omega_n - \omega_c)} \left( \frac{1}{2 \eta} + \nu_0 + \nu_1 \eta + \mathcal{O}(\eta^2) \right). \]  
(B.55)

In (B.36),

\[ \Pi^m_{\omega_0} (\omega_0; \omega_c; t_n) = \sqrt{\epsilon} \gamma \frac{\omega_c - \omega_0}{t_n^{3/2}} e^{-\frac{1}{2} \alpha^2 t_n} e^{\alpha (\omega_c - \omega_0)} e^{-\frac{1}{2} \pi (\omega_c - \omega_0)^2} \]

\[ = \sqrt{\epsilon} \frac{2 \eta}{\sqrt{\pi}} v(\eta) \frac{\omega_c - \omega_0}{t_n^{3/2}} e^{-\frac{1}{2} \alpha^2 t_n} e^{\alpha (\omega_c - \omega_0)} e^{-\frac{1}{2} \pi (\omega_c - \omega_0)^2} \]

\[ = \sqrt{\epsilon} \frac{2 \eta}{\sqrt{\pi}} e^{\alpha (\omega_n - \omega_c)} \left( \frac{1}{2 \eta} + \nu_0 + \nu_1 \eta + \ldots \right) \frac{\omega_c - \omega_0}{t_n^{3/2}} e^{-\frac{1}{2} \alpha^2 t_n} e^{\alpha (\omega_c - \omega_0)} e^{-\frac{1}{2} \pi (\omega_c - \omega_0)^2} \]

\[ = \sqrt{\epsilon} \frac{1}{\sqrt{\pi}} e^{\alpha (\omega_n - \omega_c)} \frac{\omega_c - \omega_0}{t_n^{3/2}} e^{-\frac{1}{2} \alpha^2 t_n} e^{\alpha (\omega_c - \omega_0)} e^{-\frac{1}{2} \pi (\omega_c - \omega_0)^2} \]

\[ + \sqrt{\epsilon} \frac{2}{\sqrt{\pi}} e^{\alpha (\omega_n - \omega_c)} (\nu_0 \eta + \nu_1 \eta^2 + \ldots) \frac{\omega_c - \omega_0}{t_n^{3/2}} e^{-\frac{1}{2} \alpha^2 t_n} e^{\alpha (\omega_c - \omega_0)} e^{-\frac{1}{2} \pi (\omega_c - \omega_0)^2}. \]  
(B.56)

By the definition of (B.19),
\[ \partial_i \equiv \partial / \partial \omega_i = \frac{d\eta}{d\omega_i} \frac{\partial}{\partial \eta} \] (B.57)

\[ \frac{d\eta}{d\omega_i} = -\frac{1}{\sqrt{2\epsilon}} \] (B.58)

then

\[ \lim_{\epsilon \to 0} \lim_{\omega \to \omega^-} \frac{\partial}{\partial \omega} \Pi_{\omega}^m (\omega_0, \omega; t_n) \]

\[ = -v_0 \left( \frac{2}{\pi} \right)^{1/2} \frac{\omega_c - \omega_0}{t_n^{3/2}} e^{-\frac{1}{2} \alpha^2 t_n e^{\alpha (\omega_c - \omega_0)}} e^{-\frac{t_n}{2} (\omega_c - \omega_0)^2} e^{\alpha (\omega_n - \omega_c)}. \] (B.59)

There is not the term \( u_1 \eta + ... \) after the derivative because we have taken the limit \( \omega \to \omega^- \) and, in this case, \( \eta \to 0 \), annulling this term. Back to (B.54), also using (B.42), with \( t_n \) replaced by \( t_n - t_i \), and (B.30):

\[ I_1 = \sum_{i=1}^{n-1} \left( -v_0 \left( \frac{2}{\pi} \right)^{1/2} \frac{\omega_c - \omega_0}{t_n^{3/2}} e^{-\frac{1}{2} \alpha^2 t_n e^{\alpha (\omega_c - \omega_0)}} e^{-\frac{t_n}{2} (\omega_c - \omega_0)^2} e^{\alpha (\omega_n - \omega_c)} \right) \]

\[ = \sum_{i=1}^{n-1} \left( -v_0 \frac{\sqrt{2}}{\pi \sqrt{\epsilon}} \frac{\omega_c - \omega_0}{t_n^{3/2}} \frac{\omega_c - \omega_n}{(t_n - t_i)^{3/2}} e^{-\frac{[\omega_c - \omega_0 - \alpha (t_n - t_i)]^2}{2(t_n - t_i)}} e^{-\frac{[\omega_c - \omega_0 - \alpha (t_n - t_i)]^2}{2(t_n - t_i)}} e^{2\alpha (\omega_n - \omega_c)} \right) \]

\[ \Rightarrow -v_0 \frac{\sqrt{2}}{\pi \sqrt{\epsilon}} (\omega_c - \omega_0) (\omega_c - \omega_n) e^{2\alpha (\omega_n - \omega_c)} \int_0^{t_n} dt_i e^{-\frac{[\omega_c - \omega_0 - \alpha t_i]^2}{2t_i}} e^{-\frac{[\omega_c - \omega_0 - \alpha t_i]^2}{2(t_n - t_i)}} e^{2\alpha (\omega_n - \omega_c)}. \] (B.60)

\[ \therefore I_1 = -\frac{2v_0}{\sqrt{\pi \sqrt{\epsilon}}} (2\omega_c - \omega_n - \omega_0) \frac{1}{t_n^{3/2}} e^{-\frac{[2\omega_c - \omega_n - \omega_0 - \alpha t_n]^2}{2t_n}} e^{2\alpha (\omega_n - \omega_c)}. \] (B.61)

Therefore, \( I_1 \) diverges with \( 1/\sqrt{\epsilon} \). Now we analyze the first term of B.52.
\[ I_2 = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \Pi^{gm}_e(\omega_0, \omega_c; t_i) \Pi^{gm}_e(\omega_c, \omega_n; t_j) \Pi^{gm}_e(\omega_n; t_n - t_j). \] (B.62)

By (B.41), (B.48) and (B.42) we write

\[ I_2 = \sum_{i=1}^{n-2} \Pi^{gm}_e(\omega_0, \omega_c; t_i) \sum_{j=i+1}^{n-1} \Pi^{gm}_e(\omega_c, \omega_n; t_j) \Pi^{gm}_e(\omega_n; t_n - t_j) \]

\[ = \sum_{i=1}^{n-1} \frac{\sqrt{\epsilon}}{\sqrt{2\pi}} \epsilon^{\alpha(\omega_0 - \omega_c)} \frac{\omega_c - \omega_0}{t_i^{3/2}} e^{-\frac{B(\omega_0 - \omega_c)^2}{2t_i}}. \]

\[ \sum_{j=i+1}^{n-1} \frac{\epsilon}{\sqrt{2\pi}} \frac{1}{(t_j - t_j)^{3/2}} e^{\frac{\alpha^2}{4}(t_j - t_i)} \frac{\sqrt{\epsilon}}{\sqrt{2\pi}} \epsilon^{\alpha(\omega_0 - \omega_c)} \frac{\omega_c - \omega_n}{(t_n - t_j)^{3/2}} e^{-\frac{B(\omega_0 - \omega_c)^2}{2(t_n - t_j)}}. \] (B.63)

In (B.63) it is not possible to use \( \epsilon \sum_{j=i+1}^{n-1} = \int_{t_i}^{t_n} dt_j \), since it is only valid when the sum and the integral are finite when \( \epsilon \to 0 \). Here this does not happen, because the integrand in

\[ \int_{t_i}^{t_n} dt_j \frac{1}{(t_j - t_i)^{3/2}(t_n - t_j)^{3/2}} e^{-\frac{B(\omega_0 - \omega_c)^2}{2(t_n - t_j)}} e^{-\frac{\alpha^2(t_j - t_i)}{2}} \] (B.64)

diverges due to the annullment of \( (t_j - t_i)^{3/2} \) when the sum initiates, that is, when \( t_j = t_i \). We add the exponential term in \( \beta\epsilon \), which cuts the expression in \( t_j = t_i \):

\[ \sum_{j=i+1}^{n-1} \frac{\epsilon}{(t_j - t_i)^{3/2}(t_n - t_j)^{3/2}} e^{-\frac{B(\omega_0 - \omega_c)^2}{2(t_n - t_j)}} e^{-\frac{\alpha^2(t_j - t_i)}{2}} = \]

\[ = \int_{t_i}^{t_n} dt_j e^{-\frac{\beta\epsilon}{2(t_j - t_i)}} e^{-\frac{B(\omega_0 - \omega_c)^2}{2(t_n - t_j)}} e^{-\frac{\alpha^2(t_j - t_i)}{2}}. \] (B.65)

This factor still keeps the singular part of \( 1/\sqrt{\beta\epsilon} \) and the finite part. To see that there is a singular part in \( 1/\sqrt{\epsilon} \), we can open the integral (B.64) in the divergent term, integrating from \( t_i \) to \( t_i + \epsilon \), and another from \( t_i + 2\epsilon \) to \( t_n \) (finite term). To
A. Catalão & R. Rosenfeld

make it easier, we notice that the integral is dominated by \( t_j = t_i \), which can be taken out of the expression.

\[
I_3 = \frac{1}{(t_n - t_i)^{3/2}} e^{-\frac{(\omega_c - \omega_n - \alpha (t_n - t_i))^2}{2(t_n - t_i)}} \int_{t_i}^{t_{i+\epsilon}} \frac{e^{-\alpha^2 (t_j - t_i)}}{(t_j - t_i)^{3/2}} dt_j + \int_{t_{i+\epsilon}}^{t_n} \frac{e^{-\alpha^2 (t_j - t_i)}}{(t_j - t_i)^{3/2}} \frac{e^{-\frac{\alpha^2 (t_j - t_i)}{2}}}{(t_n - t_j)^{3/2}}. \tag{B.66}
\]

The term

\[
e^{-\alpha^2 (t_j - t_i)}
\]

in the first line of the integral converges to 1. Continuing,

\[
I_3 = \frac{2}{\sqrt{\epsilon}} \frac{1}{(t_n - t_i)^{3/2}} e^{-\frac{(\omega_c - \omega_n - \alpha (t_n - t_i))^2}{2(t_n - t_i)}} + \text{finite term.} \tag{B.67}
\]

Back to \( (B.65) \) and using \( (B.38) \),

\[
- \frac{(\omega_c - \omega_n)^2}{2(t_n - t_j)} + (\omega_c - \omega_n) \alpha - \frac{\alpha^2}{2} (t_n - t_i) = - \frac{(\omega_c - \omega_n)^2}{2(t_n - t_j)} + (\omega_c - \omega_n) \alpha - \frac{\alpha^2}{2} (t_n - t_i) \]

\[
I_3 = \int_{t_i}^{t_n} \frac{e^{-\frac{\alpha^2}{2(t_n - t_j)} (t_j - t_i)}}{(t_j - t_i)^{3/2} (t_n - t_j)^{3/2}} e^{-\frac{(\omega_c - \omega_n)^2}{2(t_n - t_j)} + (\omega_c - \omega_n) \alpha - \frac{\alpha^2}{2} (t_n - t_i)}
\]

\[
= \frac{\sqrt{2\pi}}{\sqrt{\beta \epsilon}} \left( \frac{1}{\omega_c - \omega_n} \right) \frac{1}{(t_n - t_i)^{3/2}} e^{-\frac{(\omega_c - \omega_n)^2}{2(t_n - t_i)}} e^{(\omega_c - \omega_n) \alpha - \frac{\alpha^2}{2} (t_n - t_i)}. \tag{B.68}
\]
Analytical Path-Integral Pricing of Moving-Barrier Options under non-Gaussian Distributions

We take $I_3$ to (B.63) to evaluate $I_2$:

$$I_2 = \frac{\epsilon}{\pi} (\omega_c - \omega_0) (\omega_c - \omega_n) \sum_{i=1}^{n-2} \frac{1}{t_i^{3/2}} e^{-\frac{[\omega_c-\omega_n]^{2}}{2t_i}} e^{2\alpha(\omega_n - \omega_c)}.$$

$$(\frac{1}{\sqrt{\beta \epsilon}} + \frac{1}{\omega_c - \omega_n}) \frac{1}{(t_n - t_i)^{3/2}} e^{-\frac{[\omega_c-\omega_n+\sqrt{\beta \epsilon}]^{2}}{2(t_n - t_i)}} e^{(\omega_c - \omega_n)\alpha} e^{-\frac{\omega_n^2}{\beta \epsilon}(t_n - t_i)}.$$

(B.69)

Now we use $\sum_{i=1}^{n-2} \frac{1}{t^3} \int_0^t dt_i$ and (B.38), regrouping terms:

$$I_2 = \frac{1}{\pi} (\omega_c - \omega_0) (\omega_c - \omega_n) \left(\frac{1}{\sqrt{\beta \epsilon}} + \frac{1}{\omega_c - \omega_n}\right) e^{\alpha(\omega_n - \omega_c)}.$$

$$\int_0^{t_n} dt_i \frac{1}{t_i^{3/2}} \frac{1}{(t_n - t_i)^{3/2}} e^{-\frac{[\omega_c-\omega_n+\sqrt{\beta \epsilon}]^{2}}{2(t_n - t_i)}} e^{(\omega_c - \omega_n)\alpha} e^{-\frac{\omega_n^2}{\beta \epsilon}(t_n - t_i)}.$$

(B.70)

$$= \frac{1}{\pi} (\omega_c - \omega_0) (\omega_c - \omega_n) \left(\frac{1}{\sqrt{\beta \epsilon}} + \frac{1}{\omega_c - \omega_n}\right) e^{\alpha(\omega_n - \omega_c)}.$$

$$\int_0^{t_n} dt_i \frac{1}{t_i^{3/2}} \frac{1}{(t_n - t_i)^{3/2}} e^{-\frac{[\omega_c-\omega_n+\sqrt{\beta \epsilon}]^{2}}{2(t_n - t_i)}} e^{\alpha(\omega_c - \omega_n)\alpha} e^{-\frac{\omega_n^2}{\beta \epsilon}(t_n - t_i)}.$$

(B.71)

$. \ I_2 = \sqrt{\frac{1}{\sqrt{\beta \epsilon}}} (\omega_c - \omega_0) (\omega_c - \omega_n) \left(\frac{1}{\sqrt{\beta \epsilon}} + \frac{1}{\omega_c - \omega_n}\right) e^{\alpha(\omega_n - \omega_c)}.$

$$e^{\alpha(\omega_c - \omega_n)\alpha} e^{-\frac{\omega_n^2}{\beta \epsilon}(t_n - t_i)}.$$

(B.72)

Expanding the exponential of $\sqrt{\beta \epsilon}$ around zero:

$$e^{-\frac{[\omega_c-\omega_n+\sqrt{\beta \epsilon}]^{2}}{2t_n}} \approx e^{-\frac{[\omega_c-\omega_n+\sqrt{\beta \epsilon}]^{2}}{4t_n}} - \frac{1}{t_n} (2\omega_c - \omega_0 - \omega_n) \sqrt{\beta \epsilon} e^{-\frac{[\omega_c-\omega_n+\sqrt{\beta \epsilon}]^{2}}{4t_n}} + ...$$

(B.73)
\[ I_2 = \left( \frac{2}{\pi} \right) \left[ \frac{2(\omega_c - \omega_0 - \omega_n)}{t_n^{\frac{3}{2}}} + \frac{\sqrt{\beta \epsilon}}{t_n} \left( \omega_c - \omega_n \right) e^{\alpha(\omega_n - \omega_c)} e^{\alpha(\omega_c - \omega_0)} e^{-\frac{2}{\beta t_n}} \right. \]

\[ e^{-\frac{(2\omega_c - \omega_0 - \omega_n)^2}{2t_n^{3/2}}} \left( 1 - \frac{\sqrt{\beta \epsilon}}{t_n} \left( 2\omega_c - \omega_0 - \omega_n \right) \right) \frac{1}{\omega_c - \omega_n + \sqrt{\beta \epsilon} (\omega_c - \omega_n) \sqrt{\beta \epsilon}} \]

\[ = \sqrt{\frac{2}{\pi}} \left[ \left( \frac{2(\omega_c - \omega_0 - \omega_n)}{\sqrt{\beta \epsilon}} \right) + 1 \right] \frac{1}{t_n^{\frac{3}{2}}} e^{-\frac{(2\omega_c - \omega_0 - \omega_n - \omega_n)^2}{2t_n^{3/2}}} e^{2\alpha(\omega_n - \omega_c)} \]

\[ \left( 1 - \frac{\sqrt{\beta \epsilon}}{t_n} \left( 2\omega_c - \omega_0 - \omega_n \right) \right) \]

\[ = \sqrt{\frac{2}{\pi}} \left[ \left( \frac{2(\omega_c - \omega_0 - \omega_n)}{\sqrt{\beta \epsilon}} \right) - \frac{(2\omega_c - \omega_0 - \omega_n)^2}{t_n} + 1 - \frac{\sqrt{\beta \epsilon}}{t_n} (2\omega_c - \omega_0 - \omega_n) \right] \]

\[ \frac{1}{t_n^{\frac{3}{2}}} e^{-\frac{(2\omega_c - \omega_0 - \omega_n - \omega_n)^2}{2t_n^{3/2}}} e^{2\alpha(\omega_n - \omega_c)}. \]  

(B.74)

The last term in the brackets becomes zero as \( \epsilon \to 0 \).

\[ \therefore I_2 = \sqrt{\frac{2}{\pi}} \left[ \left( \frac{2(\omega_c - \omega_0 - \omega_n)}{\sqrt{\beta \epsilon}} \right) + \left( 1 - \frac{(2\omega_c - \omega_0 - \omega_n)^2}{t_n} \right) \right]. \]

\[ \frac{1}{t_n^{\frac{3}{2}}} e^{-\frac{(2\omega_c - \omega_0 - \omega_n - \omega_n)^2}{2t_n^{3/2}}} e^{2\alpha(\omega_n - \omega_c)}. \]  

(B.75)

Therefore, in (B.49):

\[ \sum_{i,j=1}^{n-1} \partial_i \partial_j = \]

\[ 2 \left\{ \frac{1}{\sqrt{\epsilon}} \left( 1 - v_0 \sqrt{2} \right) \sqrt{\frac{2}{\pi}} \left( 2\omega_c - \omega_n - \omega_0 \right) \frac{1}{t_n^{\frac{3}{2}}} e^{-\frac{(2\omega_c - \omega_0 - \omega_n)^2}{2t_n^{3/2}}} e^{2\alpha(\omega_n - \omega_c)} \right\} \]

\[ + \sqrt{\frac{2}{\pi}} \left[ 1 - \frac{(2\omega_c - \omega_0 - \omega_n)^2}{t_n} \right] \frac{1}{t_n^{\frac{3}{2}}} e^{-\frac{(2\omega_c - \omega_0 - \omega_n - \omega_n)^2}{2t_n^{3/2}}} e^{2\alpha(\omega_n - \omega_c)} \right\}. \]  

(B.76)
The LHS (B.49) is applied to $W^{gm}(\omega_0, \omega_1, ..., \omega_n; t_n)$, according to (4.17). In (A.13), we saw that
\[
\frac{\partial \Pi^{gm}}{\partial \omega_c} (\omega_0 = 0, \omega_n; t_n) = \sum_{i=1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 ... d\omega_{i-1}d\omega_{i+1} ... d\omega_n W^{gm}(\omega_0, \omega_1, ..., \omega_i = \omega_c, ..., \omega_{n-1}, \omega_n; t_n). \tag{B.77}
\]

Now,
\[
\frac{\partial^2 \Pi^{gm}}{\partial \omega_c^2} (\omega_0 = 0, \omega_n; t_n) = \sum_{i,j=1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 ... d\omega_i d\omega_j ... d\omega_n W^{gm}(\omega_0, \omega_1, ..., \omega_i = \omega_c, ..., \omega_{n-1}, \omega_n; t_n)
\]

Using (B.49), (A.6) and (A.9):
\[
\frac{\partial^2 \Pi^{gm}}{\partial \omega_c^2} (\omega_0 = 0, \omega_n; t_n) = 2 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 ... d\omega_i d\omega_j ... d\omega_n W^{gm}(\omega_0, \omega_1, ..., \omega_i = \omega_c, ..., \omega_{n-1}, \omega_n; t_n)
\]
\[
+ \sum_{i=1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 ... d\omega_i d\omega_{n-1} W^{gm}(\omega_0, \omega_1, ..., \omega_i = \omega_c, ..., \omega_{n-1}, \omega_n; t_n)
\]
\[
= 2 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 ... d\omega_n \partial_i \partial_j W^{gm}(\omega_0, \omega_1, ..., \omega_n; t_n)
\]
\[
+ \sum_{i=1}^{n-1} \int_{-\infty}^{\omega_c} d\omega_1 ... d\omega_n \partial_i^2 W^{gm}(\omega_0, \omega_1, ..., \omega_n; t_n)
\]
\[
= \sum_{i,j=1}^{n-1} \int_{-\infty}^{\infty} d\omega_1 \cdots d\omega_{n-1} \partial_i \partial_j W_{gm}(\omega_0, \omega_1, \ldots, \omega_{n-1}, \omega_n; t_n). \tag{B.78}
\]

Then, the RHS (B.78), which corresponds to the LHS (B.49), can be evaluated by the LHS of (B.78). In the continuum, \( \epsilon \to 0 \), with \( \omega_0 \neq 0 \), we compute the second derivative of \( \Pi^\text{gm}_{\epsilon \to 0}(\omega_0, \omega_n, t_n) \) with respect to \( \omega_c \), by (B.22). The first derivative was computed in (B.32). Thus,

\[
\frac{\partial^2 \Pi^\text{gm}_{\epsilon \to 0}(\omega_0, \omega_n, t_n)}{\partial \omega_c^2} = \frac{\partial}{\partial \omega_c} \left( \frac{2}{\pi} \right)^{1/2} \frac{2 \omega_c - \omega_n - \omega_0}{t_n^{3/2}} e^{2\alpha(\omega_n - \omega_0 - \omega_c)} e^{-\frac{(2\omega_c - \omega_n - \omega_0 - \omega_c)^2}{2t_n}}
\]

\[
- \frac{2(2\omega_c - \omega_n - \omega_0)}{t_n^{5/2}} \left[ \frac{2 \omega_c - \omega_n - \omega_0}{t_n^{3/2}} - \alpha t_n \right] e^{2\alpha(\omega_n - \omega_0 - \omega_c)} e^{-\frac{(2\omega_c - \omega_n - \omega_0 - \omega_c)^2}{2t_n}}
\]

\[
- \frac{2\alpha t_n}{t_n^{5/2}} \frac{2 \omega_c - \omega_n - \omega_0}{t_n^{3/2}} e^{2\alpha(\omega_n - \omega_0 - \omega_c)} e^{-\frac{(2\omega_c - \omega_n - \omega_0 - \omega_c)^2}{2t_n}}
\]

\[
\therefore \frac{\partial^2 \Pi^\text{gm}_{\epsilon \to 0}(\omega_0, \omega_n, t_n)}{\partial \omega_c^2} = \frac{2}{\pi} \left[ 1 - \frac{(2\omega_c - \omega_n - \omega_0)^2}{t_n} \right] \frac{1}{t_n^{3/2}} e^{-\frac{(2\omega_c - \omega_n - \omega_0 - \omega_c)^2}{2t_n}} e^{2\alpha(\omega_n - \omega_0 - \omega_c)}. \tag{B.79}
\]

Now we equate (B.79) to (B.76). The second term of this equation, that is, of (B.79), is just equal to (B.76). Therefore, canceling both sides, we conclude that the first term of the RHS (B.76) must be null. In order to nullify it, we must have

\[
\left( \frac{1}{\sqrt{\beta}} - \nu_0 \sqrt{2} \right) = 0. \tag{B.80}
\]

This means that \( I_1 \) (term in \( \nu_0 \)), which diverges (with \( 1/\sqrt{\epsilon} \), cancels with the divergent terms of \( I_2 \) (term in \( 1/\sqrt{\beta} \)), which were separated when the exponential
term in $\beta \epsilon$ was introduced. It still remains the non-divergent part of the first term of RHS (B.49). Concluding, the summation

$$\sum_{i,j=1}^{n-1} \partial_i \partial_j$$

has two divergent terms, whose effects cancel mutually. Thus, non-divergent terms remain in the first summation of the RHS (B.49), and we can transform summations into integrals:

$$2 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \to 2 \frac{1}{\epsilon^2} \int_0^{t_n} \int_0^{t_n} dt_i \int_0^{t_n} dt_j$$

(B.81)

Appendix C. Drift of the non-gaussian distribution: the specific case of $\kappa_{15}$

In the case of the density of probability (4.60) be expanded up to the 15th order in derivatives, we evaluate the integral (4.63).

$$\alpha = \frac{1}{t_n \sigma} \left\{ (r - r_f) - \frac{1}{2} t_n \sigma^2 \right\} +$$

$$\{1, 307, 674, 368, 000 \cdot [1, 307, 674, 368, 000$$

$$+ \sigma^3 [217, 945, 728, 000 \kappa_3 + 1, 816, 214, 400 \cdot (10 \kappa_3^2 + \kappa_6) \sigma^3$$

$$+ 259, 459, 200 \cdot (35 \kappa_3 \kappa_4 + \kappa_7) \sigma^4$$

$$+ 32, 432, 400 \cdot (35 \kappa_4^2 + 56 \kappa_3 \kappa_5 + \kappa_8) \sigma^5 + 3, 603, 600 \cdot (126 \kappa_4 \kappa_5 + 84 \kappa_3 \kappa_6 + \kappa_9) \sigma^6$$

$$+ 360, 360 \cdot (\kappa_{10} + 6 \cdot (21 \kappa_5^2 + 35 \kappa_4 \kappa_6 + 20 \kappa_3 \kappa_7)) \sigma^7$$

$$+ 32.76 \cdot (\kappa_{11} + 33 \cdot (14 \kappa_5 \kappa_6 + 10 \kappa_4 \kappa_7 + 5 \kappa_3 \kappa_8)) \sigma^8$$

}
+ 2,730 \cdot (\kappa_{12} + 11 \cdot (42\kappa_{6}^2 + 72\kappa_{5}\kappa_{7} + 45\kappa_{4}\kappa_{8} + 20\kappa_{3}\kappa_{9})) \cdot \sigma^9 \\
+ 210 \cdot (\kappa_{13} + 143 \cdot (2\kappa_{10}\kappa_{3} + 12\kappa_{6}\kappa_{7} + 9\kappa_{5}\kappa_{8} + 5\kappa_{4}\kappa_{9})) \cdot \sigma^{10} \\
+ 15 \cdot (\kappa_{14} + 13 \cdot (28\kappa_{11}\kappa_{3} + 11 \cdot (7\kappa_{10}\kappa_{4} + 12\kappa_{7}^2 + 21\kappa_{8}\kappa_{6} + 14\kappa_{5}\kappa_{9}))) \cdot \sigma^{11} \\
+ (\kappa_{15} + 13 \cdot (35\kappa_{12}\kappa_{3} + 105\kappa_{11}\kappa_{4} + 231\kappa_{10}\kappa_{5} + 495\kappa_{7}\kappa_{8} + 385\kappa_{6}\kappa_{9}))) \cdot \sigma^{12} \\
+\{10,897, 286, 400 \cdot \sigma \cdot (5\kappa_{4} + \kappa_{5}\sigma)\}^{-1}\}.

\text{(C.1)}

References

M. Avellaneda, R. Buff, C. Friedman, N. Grandechamp, L. Kruk & J. Newman (2001) Weighted Monte Carlo: a new technique for calibrating asset-pricing models, International Journal of Theoretical and Applied Finance 4 (01), 91-119.

R. Balieiro & R. Rosenfeld (2004) Testing Option Pricing with Edgeworth Expansion, Physica A 344, 484-490.

F. Black & M. Scholes (1973) The pricing of options and corporate liabilities, Journal of Political Economy 81 (3), 637–654.

D. Breeden & R. Litzenberger (1978) Prices of state-contingent claims implicit in option prices, Journal of Business 51 (4), 621-651.

A. Catalão & R. Rosenfeld (2010). Modelagem Estocástica de Opções de Câmbio no Brasil: Aplicação da Transformada Rápida de Fourier e Expansão Asintótica ao Modelo de Heston. Master thesis. Instituto de Física Teórica (IFT), Universidade Estadual Paulista Júlio de Mesquita Filho (UNESP), São Paulo, SP, Brasil.

A. De Simone, M. Maggiore & A. Riotto (2011) Excursion Set Theory for generic moving barriers and non-Gaussian initial conditions, Monthly Notices of the Royal Astronomical Society 412 (4), 2587-2602.

D. Bruno (1994) Pricing with a smile, Risk 7 (1), 18-20.

C. Gardiner (2004) Handbook of Stochastic Methods for Physics, Chemistry and Natural Sciences, third edition. Berlin, Berlin: Springer-Verlag.

J. Gatheral (2006) The Volatility Surface: A Practitioner’s Guide, Vol. 357, first edition. N.J., United States: John Wiley & Sons.

S. Heston (1993) A closed-form solution for options with stochastic volatility with applications to bond and currency options, The Review of Financial Studies 6 (2), 327-343.

J. Hull (1993) Options, Futures, and Others Derivatives, second edition. Englewood Cliffs, New Jersey, United States: Prentice-Hall, Inc.

S. Iyer-Biswas & A. Zilman (2015) First Passage Processes in Cellular Biology, arXiv: 1503.00291v1.

N. Kunitomo & M. Ikeda (1992) Pricing options with curved boundaries, Mathematical finance 2 (4), 275-298.
M. Maggiore & A. Riotto (2010) The Halo Mass Function from Excursion Set Theory. I. Gaussian fluctuations with non-Markovian dependence on the smoothing scale, The Astrophysical Journal 711 (2), 907.
M. Maggiore & A. Riotto (2010) The Halo Mass Function from excursion set theory. II. The diffusing barrier, The Astrophysical Journal 717 (1), 515.
M. Maggiore & A. Riotto (2010) The Halo Mass Function from excursion set theory. III. Non-Gaussian fluctuations, The Astrophysical Journal 717 (1), 526.
R. Merton (1976) Option pricing when underlying stock returns are discontinuous, Journal of Financial Economics 3 (1-2), 125-144.
H. Risken (1989) The Fokker-Planck Equation - Methods of Solution and Applications, second edition. Berlin, Berlin: Springer-Verlag.
M. Rubinstein (1998) Edgeworth binomial trees, Journal of Derivatives 5 (3), 20-27.
E. Schrödinger (1915) Zur theorie der fallund steigversuche an teilchen mit Brownscher bewegung, Physikalische Zeitschrift 16, 289-295.
S. Shreve (2004) Stochastic Calculus for Finance II: Continuous-Time Models, Vol. 11, first edition. N.Y., United States: Springer Science & Business Media.
R. Sheth & G. Tormen (2002) An excursion set model of hierarchical clustering; ellipsoidal collapse and the moving barrier, Monthly Notices of the Royal Astronomical Society 329 (1), 61-75.
D. Shimko (1993) Bounds on probability, Risk 6, 33-47.
A. Wald (1947) Sequential Analysis, first edition. New York, NY, United States: Wiley.
P. Wilmott (1998) Derivatives: The Theory and Practice of Financial Engineering, first edition. Chichester: John Wiley & Sons Ltd.