Formulae for calculating Hurwitz numbers

Jared ONGARO∗

∗School of Mathematics, University of Nairobi, Riverside Drive P.O. Box 30197, 00100-Nairobi, Kenya.

Abstract
In this paper, we provide various formulae for calculating single Hurwitz numbers. Single Hurwitz numbers count certain classes of meromorphic functions on complex algebraic curves and have a rich geometric structure behind them which has attracted many mathematicians and physicists. Formulation of the enumeration problem is purely of topological nature, but with connections to several modern areas of mathematics and physics.

Keywords: Hurwitz numbers, branched coverings, Hurwitz spaces.
2010 MSC: 14H30, 14H52.

1. Introduction

The number of non-equivalent branched coverings with a given set of branch points and branched profile is called the Hurwitz number. The question of determining the Hurwitz number for a given branch profile is called the Hurwitz enumeration problem. Hurwitz numbers count the branched coverings between complex projective curves with specified branch profile. Branched coverings were first described in the famous paper [25] by Riemann who developed the idea of representing nonsingular curves as branched coverings of \( \mathbb{P}^1 \) in order to study their moduli. However, systematic investigation of branched coverings was initiated by Hurwitz in [16, 17] more than thirty years later. Hurwitz numbers can be computed explicitly for non-complicated branched profiles thanks to the nice combinatorial interpretations they posses that they can be interpreted in terms of factorization permutations as first observed by Adolf Hurwitz in [16, 17].

Hurwitz numbers have a rich geometric structure behind them which has attracted many mathematicians and physicists. The effect is that formulae for computing Hurwitz numbers arise from different branches of mathematics starting from algebraic geometry, combinatorics, representation of symmetric groups, tropical geometry and theoretical phyiscs. Explicit answers to the Hurwitz enumeration problem are usually difficult to obtain. One important case when this problem has a rather explicit answer, is when at most one branch point has an arbitrary branch type while all the others are simple. In case of \( Y = \mathbb{P}^1 \), we usually suppose that the degenerate branch point is at \( \infty \in \mathbb{P}^1 \) and we call its preimages \( f^{-1}(\infty) \) poles. In other
words, we are in the situation where all the branch points in \( C \) are simple, i.e. correspond to transpositions while the permutation at infinity can be described by some partition \( \mu = (\mu_1, \ldots, \mu_n) \).

Hurwitz enumeration problem is an old but still active research problem due to its connections to several modern areas of mathematics and physics. Below we walk you through the journey from Topology to Physics in presenting various formulae and connections in calculating single Hurwitz number.

2. Partitions and Irreducible representations

We work over the field of complex numbers. The cardinality of a set \( S \) will be denoted by \( |S| \). All definitions and results on the symmetric group represented below are classical, and can be found in most standard texts.

**Definition 2.1.** A partition of a positive integer \( d \), is a finite, weakly decreasing sequence of positive integers \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) called parts of \( \mu \) such that \( \mu_1 + \mu_2 + \ldots + \mu_n = d \).

Denote a partition by \( \mu \vdash d \) and refer to \( d \) as the size of \( \mu \). The number \( n \) of parts of \( \mu \) is called length of \( \mu \) and is denoted by \( \ell(\mu) \).

**Example 2.2.** There are 5 integer partitions of \( d = 4 \), namely

\[(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).\]

Denote the set consisting of the first \( d \) positive integers \( \{1, 2, \ldots, d\} \) by \( [d] \). Let \( i \) be an integer in the set \( \{1, 2, \ldots, d\} \), the multiplicity of \( i \) in \( \mu \) which we shall denote by \( m_i(\mu) \) is the number of parts \( \mu_j \) equaling \( i \). We often use exponents to indicate repeated parts, whence a partition \( \mu \) can be written multiplicatively as \( \mu = 1^{m_1(\mu)} \cdot 2^{m_2(\mu)} \ldots k^{m_k(\mu)} \) with \( |\mu| = \sum_{i=1}^{k} m_i(\mu) \). For instance, the partition \( (2, 1, 1) = 1^2 \cdot 2 \). The number of permutations of the parts of \( \mu \) is the quantity

\[|Aut(\mu)| = \prod_{i=1}^{k} m_i(\mu)! .\]

We can also represent partitions pictorially using Young diagrams.

**Definition 2.3.** A Young diagram is an array of left and top-justified boxes, such that the row sizes are weakly decreasing. The Young diagram corresponding to \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) is the one that has \( n \) rows, and \( \mu_i \) boxes in the \( i \)th row.

For instance, the Young diagrams corresponding to the above mentioned partitions of 4 are given below.

\[
\begin{align*}
(4) & \quad (3, 1) & \quad (2, 2) & \quad (2, 1, 1) & \quad (1, 1, 1, 1) \\
\end{align*}
\]

The conjugate of the Young tableau \( \lambda \) is the reflection of the tableau \( \lambda \) along the main diagonal. This is also a standard Young tableau. We will write \( \lambda^t \) to denote the conjugate partition of \( \lambda \).

\[\text{Conjugate of } \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} \]
Let $S_d$ be the group of all permutations on $[d]$, we make the convention that permutations are multiplied from right to left. A permutation $\alpha \in S_d$ is a cycle of length $k$ or a $k$–cycle if there exist numbers $i_1, i_2, \ldots, i_k \in [d]$ such that

$$\alpha(i_1) = i_2, \quad \alpha(i_2) = i_3, \quad \ldots, \quad \alpha(i_k) = i_1.$$ 

Thus, we can write $\alpha$ in the form $(i_1, i_2, \ldots, i_k)$. A cycle of length two is called a transposition. If we fix $\sigma \in S_d$, then $\sigma$ can be uniquely decomposed into a product of disjoint cycles. The sum of the cycle lengths of $\sigma$ is equal to $d$, so the lengths form a partition of $d$. The cycle type of $\sigma$ is an expression of the form

$$1^{m_1} \cdot 2^{m_2} \cdots d^{m_d},$$

where the $m_i$ is the number of $i$–cycles in $\sigma$. We denote the set of all elements conjugate to $\sigma$ in the symmetric group $S_d$ by $C_\sigma$, that is

$$C_\sigma = \{ \pi \sigma \pi^{-1} : \pi \in S_d \}.$$

Recall that two permutations are conjugate if and only if they have the same cycle type. Each conjugacy class of $S_d$ corresponds to a partition of $d$ and we can use the combinatorial properties of these partitions to explicitly construct the irreducible representations $S^\lambda$, from which we can compute the irreducible characters.

Young tableaux and symmetric functions [19] provide not only a straight-forward way of constructing irreducible representations of $S_d$, but also an explicit formula for computing the corresponding characters. Denote by $\chi^\lambda(C)$ the character of $S^\lambda$ on the conjugacy class $C$. Since a conjugacy class $C$ of an element in $S_d$ consists of all permutations of the same cycle type, we use the notation $\chi^\lambda_\mu$ to represent the character of $S^\lambda$ at the conjugacy class of the cycle type $\mu$. It can be shown that the dimension of the irreducible representation corresponding to $\lambda$ is given by the hook formula

$$f^\lambda = \frac{d!}{\prod_{(i,j) \in \lambda} h_{ij}}.$$ 

The hooklength $h_{ij}$ is the number of boxes directly to the right and directly below $(i, j)$ including the box $(i, j)$. In particular, $h_{ij} = \lambda_i - j + \lambda'_j - i + 1$. For instance, if $\lambda = (3, 1)$, the hook length $h_{(2, 1)}$ is 2.

**Example 2.4.** The degree of the irreducible representation of $S_4$ corresponding to partition $\lambda = (3, 1)$ is 4 is the number of standard tableaux which can be calculated as

$$f^{(3,1)} = \frac{4!}{4 \cdot 2 \cdot 1 \cdot 1} = 3.$$

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \vdash d$ and consider the independent formal variables $x = (x_1, x_2, \ldots, x_m)$. The power sum function $p_\mu(x)$ is defined as

$$p_\mu(x) = \prod_{i=1}^n (x_1^{\mu_1} + \cdots + x_m^{\mu_m}).$$

**Theorem 2.5** (Frobenius Character Formula). Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \vdash d$. The character $\chi^\lambda_\mu$ is equal to the coefficient of $\prod_{i=1}^n x_i^{\lambda_i + m - i}$ in $\Delta(x)p_\mu(x)$ where $\Delta(x)$ is the Vandermonde determinant

$$\prod_{i < j} (x_i - x_j) = \det \begin{pmatrix} x_1^{\mu_1-1} & x_2^{\mu_2-1} & \cdots & x_m^{\mu_m-1} \\ x_1 & x_2 & \cdots & x_m \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_m \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$
Recall that both the conjugacy classes and irreducible representations of \( S_d \) are in one to one correspondence with partitions of \( d \). By \( \chi^\lambda_{\mu} \), we denote the character of any permutation of cycle type \( \mu \) in the representation \( \lambda \) of \( S_d \).

**Theorem 2.6** (The Burnside formula). The number of tuples of permutations \((\sigma_1, \ldots, \sigma_m)\) in \( S_d \) such that:

1. \( \sigma_i \) has cycle type \( \mu_i \)
2. \( \sigma_1 \ldots \sigma_m = 1 \)

is given by the expression

\[
\sum_{\lambda \vdash d} \left( \frac{\dim \lambda}{d!} \right)^2 \prod_{\mu \vdash d} f^\lambda(C_{\mu})
\]

where,

\[
f^\lambda(C_{\mu}) = \frac{|C_{\mu}|}{\dim \lambda} \chi^\lambda(\mu').
\]

In what follows, a curve, always means a smooth complex projective algebraic curve.

3. Hurwitz Numbers and Hurwitz Spaces

Let \( X \) be a complex nonsingular curve of genus \( g \) (Note that we impose further conditions on \( X \), see for example in [22, 23]). A single Hurwitz numbers \( h_{g,\mu} \) enumerate Hurwitz covering arising from meromorphic functions on \( X \). A Hurwitz covering of type \((g, \mu)\) is a meromorphic function \( f : X \to \mathbb{C} \) on \( X \) with labelled poles \( \{p_1, \ldots, p_n\} \) given by a divisor \( \mu_1 p_1 + \ldots + \mu_n p_n \), and except for these poles, the holomorphic 1-form of \( f \) has simple zeros on \( X \setminus \{p_1, \ldots, p_n\} \) with distinct critical values of \( f \). A meromorphic function \( f \) gives a finite morphism to the complex projective line \( \mathbb{P}^1 \) whose degree \( d \) by definition is the degree of the morphism \( f : X \to \mathbb{P}^1 \). Observe, we call a meromorphic function a covering because if we remove the critical values of \( f \) (including \( \infty \)) from \( \mathbb{P}^1 \), then on this open set \( f \) becomes a topological covering. More precisely, let the branched locus \( B = \{z_1, \ldots, z_w, \infty\} \) denote the set of distinct critical values of \( f \). Then

\[
f_0 : X \setminus f^{-1}(B) \to \mathbb{P}^1 \setminus B
\]

is a topological covering of degree \( d \).

A holomorphic map \( f : X \to \mathbb{P}^1 \) is called a meromorphic function. Thus, given a meromorphic function \( f \), for \( \infty \in \mathbb{P}^1 \) we have the polar divisor \( f^{-1}(\infty) = \mu_1 p_1 + \ldots + \mu_n p_n \), where \( p_1, \ldots, p_n \) are distinct points on \( X \) and \( \mu = (\mu_1, \ldots, \mu_n) + d \) the branch type of \( f \) at a point \( \infty \) gives a partition of \( d \). For instance, the branch type for a simple branch point is \((2)\) or \((2, 1, \ldots, 1)\).

**Example 3.1.** Let \( X \) be the cubic curve in \( \mathbb{P}^2 \) defined by \( y^2 z = x(x + z)(x - z) \), where \([x, y, z]\) are homogeneous coordinates in \( \mathbb{P}^2 \) as discussed in [24]. Let \( f \) be the linear projection of \( X \) from \( p = [0, 1, 0] \in \mathbb{P}^2 \setminus X \) onto \( \mathbb{P}^1 \). It defines a 2-sheeted branched covering of \( \mathbb{P}^1 \). All the 4 branch points namely;

\[
[0, 1], [1, 0], [-1, 1] \text{ and } [1, 1] \in \mathbb{P}^1
\]

are simple implying that the meromorphic function on the linear projection of \( X \) to \( \mathbb{P}^1 \) from a point \( p = [0, 1, 0] \) is a simple branch covering.

The set of all branch points \( B \) is called the branching locus of \( f \). In this way, every non-constant meromorphic function on a curve \( X \) is a branched covering. In Hurwitz covering, we consider the case where branch type at \( \infty \) is given by the partition \( \mu = (\mu_1, \ldots, \mu_n) + d \) and there are exactly \( w = 2g - 2 + n + d \) simple branch points by Riemann-Hurwitz formula. The basic problem is then the enumeration of such maps \( f : X \to \mathbb{P}^1 \) for a given \( g \) and \( d \) for a prescribed branch type over each branch point of \( f \).
Definition 3.2. Two Hurwitz covering $f_1 : X_1 \rightarrow \mathbb{P}^1$, covering with poles at $\{p_1, \ldots, p_n\}$ and $f_2 : X_2 \rightarrow \mathbb{P}^1$ with poles at $\{q_1, \ldots, q_n\}$ are called equivalent if there exists an isomorphism $\phi : X_1 \xrightarrow{\sim} X_2$ such that $\phi(p_i) = q_i$, for all $i = 1, \ldots, n$ and the diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\sim} & X_2 \\
\downarrow_{f_1} & & \downarrow_{f_2} \\
\mathbb{P}^1 & & \\
\end{array}
\]

commutes.

If $X_1 = X_2$, $p_i = q_i$, and $f_1 = f_2 = \phi$, then $\phi$ is an automorphism of a Hurwitz covering $f$. Thus, we count a Hurwitz covering with the automorphism factor $\frac{1}{|\text{Aut}(f)|}$ to compensate for the relabeling. If $\phi : X_1 \rightarrow X_2$ is just a homeomorphism, we say $f_1$ and $f_2$ have the same topological type.

Throughout we let the integers $d \geq 1$, $g \geq 0$ and a partition $\mu = (\mu_1, \ldots, \mu_n) \vdash d$.

3.1. Topological Definition of Hurwitz Numbers

Definition 3.3. The connected single Hurwitz number is

\[
h_{g, \mu} = \sum_{\{f\} / |\text{Aut}(f)|'} \frac{1}{|\text{Aut}(f)|'},
\]

where the sum runs all topological equivalence classes of Hurwitz coverings of type $(g, \mu)$ for $g \geq 0$ and a partition $\mu = (\mu_1, \ldots, \mu_n) \vdash d$ for connected complex nonsingular curves $X$ of genus $g$.

It is convenient to define the disconnected Hurwitz numbers $h_{g, \mu}^*$ as the connected Hurwitz numbers can be computed from the disconnected Hurwitz via the inclusion-exclusion formula.

Definition 3.4. The disconnected single Hurwitz number of type $(g, \mu)$ for $g \geq 0$ and a partition $\mu = (\mu_1, \ldots, \mu_n) \vdash d$ is

\[
h_{g, \mu}^* = \sum_{\{f\} / |\text{Aut}(f)|'} \frac{1}{|\text{Aut}(f)|'},
\]

where the sum runs all topological equivalence classes of Hurwitz coverings $f : X \rightarrow \mathbb{P}^1$ for possibly a disconnected complex nonsingular curves $X$ of genus $g$. 
3.2. Geometric Formulation of Single Hurwitz Numbers

Fix \( g \geq 0 \) and a partition \( \mu \vdash d \) on branched coverings \( f : X \to \mathbb{P}^1 \) and the number \( w \) of branch points, then equivalence classes of branched coverings form a moduli space called single Hurwitz space of type \((g, \mu)\) Hurwitz coverings denoted by

\[
\mathcal{H}_{g,\mu} = \left\{ f : X \to \mathbb{P}^1 \mid \mu \vdash d, X \text{ has genus } g \text{ and } f \text{ is a Hurwitz covering of type } (g, \mu) \right\}/\sim. \tag{3.3}
\]

\( \mathcal{H}_{g,\mu} \) possess the structure of an irreducible smooth algebraic variety (see §21 of [1] or [11]) of dimension equal to \( w = 2g + 2d - 2 \). The fundamental group of the configuration space of \( w \) branch points in \( \mathbb{P}^1 \) acts on the fibers of \( \mathcal{H}_{g,\mu} \) and the orbits of this action are in one to one correspondence with the connected components of \( \mathcal{H}_{g,\mu} \). Furthermore, \( \mathcal{H}_{g,\mu} \) comes with a natural finite \( \acute{e}tale \) covering

\[
\Phi : \mathcal{H}_{g,\mu} \to \text{Sym}^w \mathbb{P}^1 \backslash \Delta \tag{3.4}
\]

\[(f : X \to \mathbb{P}^1) \mapsto \{\text{branch locus of } f\} \]

where \( \text{Sym}^w \mathbb{P}^1 \) is the space of unordered \( w \)-tuples of points in \( \mathbb{P}^1 \) and \( \Delta \) is the discriminant hypersurface corresponding to sets of cardinality less than \( w \). The morphism \( \Phi \) is called the branching morphism and its degree is the single Hurwitz number \( h_{g,\mu} = \Phi^{-1}(B) \) for a fixed branch locus \( B \).

3.3. Group Theoretic Formulation of Hurwitz Numbers

To every meromorphic function \( f : X \to \mathbb{P}^1 \) of degree \( d \) we can associate its monodromy data and we obtain a formulation of single Hurwitz numbers in terms of counting sequences of factorization of a permutation.

**Definition 3.5.** Fix \( \sigma \in S_{d^r} \) a sequence \((a_1b_1), (a_2b_2), \ldots, (a_nb_n)\) such that the product

\[
(a_1b_1)(a_2b_2) \cdots (a_nb_n) = \sigma
\]

is called a transposition factorization of \( \sigma \) of length \( m \).

The factorization is not unique, for instance \((123) = (12)(13) = (13)(23) \). However, the number of transpositions in the factorization depends on the cycle type of the permutation \( \sigma \) rather than the permutation itself.

**Definition 3.6.** Let \( \mu \vdash d \) for \( d \geq 1 \). Consider an ordered sequence of permutations \((\tau_1, \ldots, \tau_w, \sigma) \in (S_d)^{w+1} \) where \( w = 2g - 2 + n + d \). The single Hurwitz number \( h_{g,\mu} \) is given by

\[
h_{g,\mu} = \frac{1}{d!} \times \text{number of ordered } w\text{-tuples } (\tau_1, \ldots, \tau_w) \in (S_d)^w
\]

such that:
1. \((\tau_1, \ldots, \tau_w)\) are transpositions in \( S_{d^r} \)
2. the product \( \tau_1 \circ \cdots \circ \tau_w = \sigma \) in \( S_d \) whose cycle type is \( \mu \)
3. the subgroup \( \langle \tau_1, \ldots, \tau_w \rangle \subset S_d \) acts transitively on the set \{1, 2, \ldots, d\}.

Observe that the third condition is equivalent to requiring that the covering surface be connected. So we can define the disconnected Hurwitz numbers by relaxing the third condition.

**Definition 3.7.** The disconnected single Hurwitz number is

\[
h^\bullet_{g,\mu} = \text{number of } \left\{ (\tau_1, \ldots, \tau_w) \in (S_d)^w \mid \tau_i \text{ are transpositions with } \tau_1 \circ \cdots \circ \tau_w = \sigma \in S_d \right\} / d! \tag{3.6}
\]

such that \( \sigma \) in \( S_d \) has cycle type \( \mu \).

In this form the problem was for the first time formulated by A. Hurwitz.
3.4. The Hurwitz Formula

In several specific cases A. Hurwitz calculated $h_{g,\mu}$ using purely combinatorial methods in 1891 and in terms of irreducible characters of $S_n$ in 1902. In [16] he also observed that the calculation $h_{g,\mu}$ is a purely group-theoretic problem, but its solution is complicated for arbitrary $\mu$ and $d$. On page 17 of [16], Hurwitz found answers for calculating the degree of the map (3.4) for small $d \leq 6$ and any $g \geq 0$. Namely,

$$h_{g, \infty} = \frac{1}{2},$$

$$h_{g, \infty} = \frac{1}{3!}(3^{2g+3} - 3),$$

$$h_{g, 0} = \frac{1}{4!}(2^{2g+4} - 4)(3^{2g+5} - 3),$$

$$h_{g, 0} = \frac{10^{2g+8}}{7200} - \frac{2^{2g+8}}{288} + \frac{5^{2g+8}}{450} - \frac{4^{2g+8}}{72} + \frac{3^{2g+8}}{18} + \frac{2^{2g+8}}{12} - \frac{5}{9},$$

$$h_{g, 0} = \frac{15^{2g+10}}{2 \cdot (360)^2} - \frac{10^{2g+10}}{7200} + \frac{9^{2g+10}}{2 \cdot (72)^2} - \frac{7^{2g+10}}{2 \cdot (24)^2} + \frac{6^{2g+10}}{2 \cdot (36)^2} - \frac{5^{2g+10}}{360} +$$

$$+ \frac{4^{2g+10}}{36} - \frac{19^{2g+10}}{324} - \frac{19^{2g+10}}{144} + \frac{7^{2g+10}}{1152}.$$ (3.7)

For instance, it is immediate to enumerate all degree 3 single Hurwitz numbers for all $g \geq 0$.

Figure 2: Associated to counting permutations up to conjugation in $S_d$

**Example 3.8.** Let $\mu = (2, 1) \vdash 3$ and $g \geq 0$. To compute, $h_{g, \mu}$, all we need, is to count sequences of $w = 2g+4$ transpositions with the above properties. That is, we need to count sequences of $w$ transpositions which...
generate a transitive subgroup of \( S_d \) whose product is identity. Notice that we are free to choose \( 2q + 3 \) elements of the sequence as the last of them is determined by the requirement that the product must be identity as the product of \( 2q + 3 \) transpositions as the same parity as one transposition in \( S_3 \). Also, to avoid disconnected coverings we have to avoid always choosing the same transpositions \( 2q + 3 \) times. Thus, we immediately find the number of simple branched coverings of degree 3 is

\[
h_{g, \mathbb{P}^1} = \frac{3^{2q+3} - 3}{6}
\]

for all \( g \geq 0 \) as found by A. Hurwitz in Equation (3.7).

Similarly, \( h_{g, \mathbb{P}^1} \), the number for non-isomorphic branched coverings of degree 3 over \( \mathbb{P}^1 \) with one complicated branch point can easily be calculated.

**Example 3.9.** Indeed we establish that the single Hurwitz number \( h_{g, \mathbb{P}^1} = 3^{2q} \) as follows. Notice that for complicated branch point we can choose freely any 3-cycle in \( S_3 \). The 3-cycle guarantee that we generate \( S_3 \). Then we are free to choose cycle for the next \( 2q + 1 \) simple branch points, the last is uniquely determined by the fact that the multiplication is identity. So we get \( 2 \cdot 3^{2q+1} \) elements of \( S_3 \). We divide by \( 3! \) to account for relabelling of the sheets of the branched coverings.

### 3.5. Minimal Transposition Factorisation

For genus \( g = 0 \), the single Hurwitz number \( h_{0, \mathbb{P}^1} \) is equivalent to counting factorizations of a permutation \( \sigma \in S_d \) of cycle type \( \mu \vdash d \) into a product of transpositions of minimal length divided by \( d! \), a result known and published by Hurwitz.

**Definition 3.10.** Let \( \sigma \in S_d \) be a fixed permutation of length \( m \). The sequence \( (\tau_1, \ldots, \tau_n) \) is called a minimal transitive factorization of \( \sigma \) into transpositions if the following 3 conditions are satisfied:

1. **Product cycle type condition:** \( \tau_1 \ldots \tau_n = \sigma \),
2. **Minimality condition:** \( n := m + d - 2 \),
3. **Transitivity condition:** The graph \( G_{\sigma} \) is connected, where \( G_\mu \) is the graph corresponding to factorization \( \sigma \) into a product of \( n \) transpositions.

Note that, one needs at least \( d - 1 \) transpositions to build a cycle of length \( d \). Then \( n \geq d - 1 \).

**Example 3.11.**

1. If \( \mu = (2) \vdash 2 \) and \( m = 1 \), the only transposition is \( (12) = (21) \). Therefore

\[
h_{0, \mathbb{P}^1} = \frac{1}{2} \cdot 1 = \frac{1}{2}.
\]

This example also shows that Hurwitz numbers can be rational and not always a positive integer.

2. If \( \mu = (3) \vdash 3 \), \( m = 2 \) there exist 3 transposition factorizations of the three-cycle \( (123) = (12)(13) := (23)(21) := (31)(32) \) and we have \( 3 \cdot 2 \) three-cycles in \( S_3 \) corresponding to connected trees. Thus

\[
h_{0, \mathbb{P}^1} = \frac{1}{6} (3 \cdot 2) = 1.
\]

3. If \( \mu = (2, 1) \vdash 3 \) and \( m = 3 \) we have \( 3^3 \) triples of transpositions but 3 of the triples consists of coinciding transpositions and thus the corresponding covering surface is not connected. This implies that the single Hurwitz number

\[
h_{0, \mathbb{P}^1} = \frac{1}{6} (3^3 - 3) = 4.
\]
Now, since for $\mu = (d)$ the graph $G_\mu$ is a tree, assuming bijective results [20] the corresponding Hurwitz number follows immediately from Cayley’s formula of 1860 for enumeration of trees. (Observe, the Cayley formula in the language of transpositions, is attributed to the Hungarian mathematician Dénes [6]).

**Theorem 3.12** (Dénes). There exist $d^{d-2}$ transposition factorization of an $d$-cycle into $d-1$ distinct transpositions.

In the case $m = 2$, V.I. Arnol’d [2] found the corresponding Hurwitz number by using the notion of complex trigonometric polynomials.

**Theorem 3.13** (Arnol’d). For a partition $\mu = (\mu_1, \mu_2) \vdash d$ the number of distinct minimal transitive transposition factorizations of $\sigma$ whose cycle type equals $\mu$ is

$$h_{\mu_1, \mu_2}^\mu = \frac{(\mu_1 + \mu_2 - 1)!}{(\mu_1 - 1)! (\mu_2 - 1)!} \tag{3.8}$$

Still another case was settled not that long ago by two physicists M. Crescimanno and W. Taylor.

**Theorem 3.14** (Crescimanno-Taylor). If $m = d$ means $\mu = (1^d)$ i.e. the factorization of the identity, then the number of distinct minimal transitive factorizations into transpositions

$$(2d - 2)! d^{d-3} \tag{3.9}$$

was discovered in [5], who apparently asked the combinatorialist Richard Stanley who consulted Goulden-Jackson about the result. Finally, Goulden-Jackson also independently [14, 13] discovered and proved the Hurwitz formula in its complete generality.

4. Hurwitz numbers and the symmetric groups

Let $\mathbb{C}[S_d]$ be the group algebra of $S_d$. See more details in [4]. The group algebra $\mathbb{C}[S_d]$ is $d!$ dimensional over $\mathbb{C}$. For each partition $\mu$ of $d$, denote by $C_\mu \in \mathbb{C}[S_d]$ the basis elements in $S_d$, i.e. the sum of all permutations in $S_d$ of cyclic type $\mu$. We will denote by $C_e$ the class $C_{(1,1,...,1)}$ of the identity permutation, which is the unit of the algebra $\mathbb{C}[S_d]$, and $C_2$ for the sum of $C_{(2,1,...,1)}$ of all transpositions.

**Proposition 4.1.** Let $S_d$ denote the symmetric group of permutations of $d$ elements. For each partition $\mu$ of $d$, the sum of all elements in $S_d$ of cyclic type $\mu$ span the center of $\mathbb{C}[S_d]$.

**Example 4.2.** The center of the group algebra $\mathbb{C}[S_d]$ is spanned by the three elements

$$C_e = 1, \quad C_2 = (12) + (23) + (13), \quad C_{(3)} = (123) + (132).$$

The disconnected simple Hurwitz numbers possess the following natural interpretation.

**Theorem 4.3.** The product of the class $C_\mu$ with the $w$th power of the class $C_2$. Then

$$h_{g,\mu} = \frac{1}{d!} [C_e] C_\mu C_2^w \tag{4.1}$$

where $[C_e] C_\mu C_2^w$ is the coefficient of $C_e$ in the product $C_\mu \circ C_2^w$. 

Example 4.4. Given \( d = 3, \, g = 1 \) and \( \mu = (3) \) and \( w = 2g - 2 + \ell(\mu) + d = 2 \cdot 1 - 2 + 1 + 3 = 4 \) we have

\[
\begin{align*}
C_{(3)}C_2^4 &= ((123) + (132)) ((12) + (23) + (13)) \\
&= 54e + 27 ((123) + (132)) \\
&= 54C_e + 27C_{(3)}.
\end{align*}
\]

Thus

\[
h_{g, \mu}^* = h_{g, \mu} = \frac{1}{3!} |C_\mu| C_{(3)}C^4_2 = \frac{54}{6} = 9.
\]

Note that in this case, the connected and disconnected cases we have identical calculations as the special branch point has exactly one part i.e. \((3) + 3.\)

4.1. Burnside character formula

Calculating Hurwitz numbers is multiplication problem in conjugacy class basis on the center of the group algebra \( \mathbf{C}[S_d] \). Recall that both the conjugacy classes and irreducible representations of \( S_d \) are in one to one correspondance with partitions of \( d \). Using Burnside formula in (2.1), we obtain a closed formula by involving a basis form irreducible representation of \( S_d \).

**Theorem 4.5** (Burnside character formula). Let \( \rho \) be an irreducible representation of \( S_d \). Denote the character of \( \rho \) by \( \chi^\lambda \). The single Hurwitz numbers

\[
h_{g, \mu}^* = \sum_{\lambda \models d} \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 f^\lambda(C_\mu)f^\lambda(C_2)^w
\]

where,

\[
f^\lambda(C_\mu) = \frac{|C_\mu|}{\dim \lambda} \chi^\lambda(\mu').
\]

This connection was already known to A. Hurwitz in 1902 and has provided a rich interplay between geometry and combinatorics for a long time.

4.2. Hurwitz Monodromy Group

Recall that the single Hurwitz number corresponding to a fixed branching data is given by the degree of the covering map

\[
\Phi : \mathcal{H}_{g, \mu} \longrightarrow \text{Sym}^w \mathbb{P}^1 \backslash \Delta
\]

It is an unsolved problem to determine the image, that is the monodromy group for branching morphism as described in (3.4) called the **Hurwitz monodromy group**. However, in special cases see [7] a good description can be obtained. This cases include the Hurwitz spaces \( \mathcal{H}_{g, \mu} \). The image of the fundamental group \( \pi_1(\text{Sym}^w \mathbb{P}^1 \backslash \Delta_{g,w}) \) to the symmetric group \( S_{h_{g,\mu}} \) (where \( h_{g,\mu} \) is single hurwitz number) is the Hurwitz monodromy group. Directly from the Hurwitz formulae in (3.7) we have an intuitive indication, that the Hurwitz monodromy groups are less than the full symmetric group at least for the first nontrivial cases \( d = 3 \) and 4, but nothing much we can say for \( d > 4 \) from the shape of the formulae seen earlier. Thus for \( d = 3 \) or 4, the Hurwitz monodromy groups can be anticipated to have a structure which heavily reflects the geometrical structure of \( \mathbb{F}_2 \) and \( \mathbb{F}_3 \) vector spaces.
Indeed, as given in (3.7) the single Hurwitz numbers $h_{g_1,0}$ and $h_{g_1,0}$ for degree 3 and 4 consists of the factors $\frac{3^n-1}{2}$ and $2^n - 1$. Recall that $\frac{3^n-1}{2}$ is the number of points in the $n - 1$ dimensional projective space over a field with 3 elements and $2^n - 1$ is the number of points in a $n$ dimensional projective space over a field with 2 elements.

In fact, one way to compute the single Hurwitz numbers $h_{g_1,0}$ and $h_{g_1,0}$ is via a bijection between transpositions and elements of finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$ respectively.

**Example 4.6.** To compute the $h_{1,0}$ for degree 3 Hurwitz coverings, we establish a bijection between transpositions $t_1 \ldots, t_w$ in $S_3$ specifying a covering curve $X$ with $w = 2g + 4$ branch points and the projective space of dimension $w - 3$ over $\mathbb{F}_3$. This is easily obtained. Up to conjugation we can assume $t_1 = (1, 2)$ and consider the assignment

$$\mu : (12) \mapsto 0 \quad (13) \mapsto 1, (23) \mapsto 2.$$ (4.4)

Let $f^*(12)t_2 \ldots t_w = (\mu(t_2), \ldots, \mu(t_{w-1}))$ we define the map $f$ from the projective points via

$$f(X) = f^*(t_2), \ldots, \mu(t_{w-1})$$

As an example, if $(12)(13)(12)(23)(13))$ represent $X$, $f(X) = (1, 0, 2, 1)$. One then can easily show the map $f$ is well defined from the requirement that the product of the transpositions must be identity, moreover its a bijection. Thus,

$$h_{g_1,0} = \frac{1}{2}(3^{w-2} - 1)$$

which is the number of points in the projective space of dimension $w - 1$ over a field $\mathbb{F}_3$ with three elements.

**5. Hurwitz numbers in terms monodromy graphs**

We can now compute single Hurwitz numbers in terms of monodromy graphs. This is motivated from the definition of single Hurwitz numbers as equivalent to counting permutation factorizations into transpositions, we have another algebraic definition of single Hurwitz numbers via enumeration of graphs. In the presentation, we follow [3].

The core behind the derivation of this special case is the fact that multiplication of permutation by a transposition $\tau = (ab)$ can be easily understood; it either *cuts* or *joins* cycles of the permutation. Namely, if $\sigma \in S_d$ has $m$ cycles then the product $\tau \circ \sigma$ has either

1. **Cut:** $m - 1$ cycles if $a$ and $b$ are in different cycles of $\sigma$.
2. **Join:** $m + 1$ cycles if $a$ and $b$ are in same cycle of $\sigma$.

![Figure 3: Effects on multiplicities of a cycle type of $\sigma \in S_d$ in the composition $\tau_m \circ \sigma$.](image)

Example 5.1. The multiplication of permutation \((12345) \in S_5\) on the left by \((14)\) gives \((15)(234)\). In other words, cuts it into two cycles. On the other hand, multiplication of the permutation \((15)(234)\) on the left by \((14)\) joins the two cycles together.

We associate a graph called a monodromy graphs which project to the segment \([\infty, 1, \ldots, w]\) with labeled edges as follows:

Definition 5.2. Constructing the monodromy graph \(\Gamma\), project to the segment \([0, w]\) marked and labelled \(\infty, 1, \ldots, w\) as follows:

1) Start with \(n\) small strands over \(\infty\) decorated by weights labels \(\mu_1, \ldots, \mu_n\).

2) Over the point \(1\) create a three-valent vertex by either joining two strands or splitting one with weight strictly greater than \(1\).

Join: If a join, label the new strand with the sum of the weights of the edges joined.

Cut: If a cut, label the two new strands in all possible (positive) ways adding to the weight of the split edge,

3) In each case, consider a unique representative for any isomorphism class of labeled graphs.

4) Repeat (5.2) and (5.2) for all successive integers up to \(w\),

5) Retain all connected graphs that terminate with weight \((2)\) or \((2, 1, \ldots, 1)\) over \(w\).

We obtain a connected graph \(\Gamma\) of genus \(g\) with a map to \([0, w]\). We call \(\Gamma\) together with the map the monodromy graph of type \((g, \mu)\) corresponding to \((\sigma, \tau_1, \ldots, \tau_w)\).

Definition 5.3. Given a monodromy graph, a balanced fork is a tripod with weights \(n, n, 2n\) such that the vertices of weight \(n\) lie over \(\infty\) or \(w\). A wiener consists of a strand of weight \(2n\) splitting into two strands of weight \(n\) and then re-joining

![Figure 4: Local structure for balanced left pointing forks, wiener and balanced right pointing forks](image_url)

The map \(\Gamma \to [0, w]\) can be viewed as a tropical cover of degree \(d\), where the edges adjacent to vertices over \(\infty\) yield the profile \(\mu\). The balancing condition for monodromy graphs, comes from the observation that by definition a monodromy graph is a combinatorial type of a tropical morphism see [3] for more details.
Definition 5.4. An isomorphism of monodromy graphs $\Gamma_1 \rightarrow [0, w]$ and $\Gamma_2 \rightarrow [0, w]$ of type $(g, \mu)$ is a graph isomorphism $f : \Gamma_1 \rightarrow \Gamma_2$, such that

$$\Gamma_1 \xrightarrow{f} \Gamma_2$$

commutes.

Definition 5.5. Let $B_\mu$ denote the set of all isomorphism classes of monodromy graphs. Then the single Hurwitz number

$$h_{g,\mu} = \sum_{[\Gamma] \in B_\mu} \frac{1}{|\text{Aut}(\Gamma)|}$$

is the number of monodromy graphs $\Gamma$ in $B_\mu$ divided by $|\text{Aut}(\Gamma)|$.

We simplify the definition 5.6 above to give the Cavalieri-Johnson-Markwig formula in [3].

Proposition 5.6. The Hurwitz number $h_{g,\mu}$ is computed as a weighted sum over monodromy graphs.

$$h_{g,\mu} = \sum_{[\Gamma] \in B_\mu} \frac{1}{|\text{Aut}(\Gamma)|} \prod \omega(e)$$

where we take the product of all the interior edge weights $\omega(e)$. The automorphism group $\text{Aut}(\Gamma)$ involve factors of $1/2$ coming from the balanced forks and wieners of $\Gamma$.

Example 5.7. If $g = 1, d = 3$ and $\mu = (2, 1) + 3$, we will show that

$$h_{1,3} = 40.$$

That is there is a family of 40 non-isomorphic cubics over 6 points in $\mathbb{P}^1$. Observe that in this case $g = 1, d = 3$, then by Riemann-Hurwitz formula give us

$$w = 2g - 2 + \ell(\mu) + d = 2 \cdot 1 - 2 + 2 + 3 = 5.$$

Note that in the computation of the total contribution, graphs which have a vertical symmetry will yield another representative. Thus we multiply the factor $\prod \omega(e)/\text{Aut}(\Gamma)$ by 2 to factor for this contribution. Refer to Table 1 for the specific monodromy graphs.

$$h_{1,3} = 6 + 2 + 12 + 2 + 18 = 40.$$

Single Hurwitz numbers turn out to be closely related to the intersection theory on the moduli space of stable curves. We formulate remarkable ELSV formula [8, 9] following a result of Ekedahl-Lando-Shapiro-Vainshtein. It provides a strong connection between geometry of moduli spaces and the Hurwitz numbers. In practice it is very difficult to use but it remains one of the most striking results related to Hurwitz enumeration problem.
Table 1: Illustration of Hurwitz numbers using Monodromy graphs for degree 3.

| Graph type | $\prod_{\omega} \left( e \right)$ | $\frac{\prod_{\omega}(\omega)}{\text{Aut}(\Gamma)}$ | Contribution |
|------------|-----------------------------------|--------------------------------------------------|--------------|
| ![Graph 1](image1) | $1 \cdot 1 \cdot 2 \cdot 3$ | $\frac{2}{2}$ | $3 \cdot 2 = 6$ |
| ![Graph 2](image2) | $1 \cdot 1 \cdot 2 \cdot 1$ | $\frac{1}{2}$ | $1 \cdot 2 = 2$ |
| ![Graph 3](image3) | $1 \cdot 1 \cdot 2 \cdot 3$ | $\frac{1}{6}$ | $6 \cdot 2 = 12$ |
| ![Graph 4](image4) | $1 \cdot 1 \cdot 2 \cdot 3$ | $\frac{1}{2}$ | $2 \cdot 1 = 2$ |
| ![Graph 5](image5) | $3 \cdot 2 \cdot 1 \cdot 3$ | $\frac{1}{18}$ | $18 \cdot 1 = 18$ |
6. The ELSV Formula

Recall that the Hurwitz number \( h_{g,\mu} \) is the number of branched coverings of degree \( d \) from smooth curves of genus \( g \) to \( \mathbb{P}^1 \) with one branch point (usually taken to be \( \infty \in \mathbb{P}^1 \)) of branched type \( \mu = (\mu_1, \ldots, \mu_n) \) and \( \omega = d + n + 2g - 2 \) other simple branch points.

**Theorem 6.1** (The ELSV formula). Suppose that \( g, n \) are integers \((g \geq 0, n \geq 1)\) such that \( 2g - 2 + n > 0 \). Let \( \mu = (\mu_1, \ldots, \mu_n) \vdash d \) and \( \text{Aut}(\mu) \) denote the automorphism group of the partition \( \mu \). Then,

\[
h_{g,\mu} = \frac{\omega!}{|\text{Aut}(\mu)|} \prod_{i=1}^{\mu_1} \frac{1}{\mu_i!} \int_{\overline{M}_{g,n}} \frac{1 - \lambda_1 + \ldots + (-1)^g \lambda_g}{(1 - \mu_1 \psi_1) \ldots (1 - \mu_n \psi_n)}
\]  

(6.1)

where \( \psi_i = c_1(L_i) \in H^{2i}(\overline{M}_{g,n}, \mathbb{Q}) \) is the first Chern class of the tangent line bundle \( L_i \to \overline{M}_{g,n} \) and \( \lambda_j = c_j(E) \in H^{2j}(\overline{M}_{g,n}, \mathbb{Q}) \) is the \( j \)-th Chern class of the Hodge bundle \( E \to \overline{M}_{g,n} \).

\[
\frac{1}{1 - \mu_i \psi_i} = 1 + \mu_1 \psi_1 + \ldots + \mu_1^i \psi_i^i + \ldots
\]

(Observe that the above expansion terminates because \( \psi_i \in H^{2i}(\overline{M}_{g,n}, \mathbb{Q}) \) is nilpotent).

Notice that the ELSV formula is a polynomial in the variables \( \mu_1, \ldots, \mu_n \). This fact is stated in the Golden-Jackson polynomiality conjecture [12] which this formula settles.

**Remark 6.2.** The ELSV formula is not applicable to coverings of genus 0 with 1 and 2 marked points since the stability condition \( 2g - 2 + n > 0 \) is violated. However, the ELSV formula remains true for these two cases as well

\[
\int_{\overline{M}_{0,1}} \frac{1}{(1 - \mu_1 \psi_1)} = \frac{1}{\mu_1^2} \quad \text{and} \quad \int_{\overline{M}_{0,2}} \frac{1}{(1 - \mu_1 \psi_1)(1 - \mu_2 \psi_2)} = \frac{1}{\mu_1 + \mu_2}
\]

(6.2)

Apart from the easy combinatorial factor, the ELSV formula involves the integrals of the form

\[
\int_{\overline{M}_{g,n}} \psi_1^{m_1} \ldots \psi_n^{m_n} \lambda_1^{k_1} \ldots \lambda_g^{k_g}
\]

(6.3)

called the Hodge integrals which can be reduced to other integrals only involving the \( \psi \)-classes. The latter integral are called descendant integrals [10]. The explicit evaluation of these integrals or computation of the intersection numbers is a difficult task. On the other hand, we can see that using the ELSV formula (6.1) makes it possible to calculate the intersection numbers on \( \overline{M}_{g,n} \) once the single Hurwitz numbers are known.

6.1. Hurwitz Formula via the ELSV formula

Although, the ELSV formula (6.1) is hard to use, there is a couple of very well-known cases. These cases are related to Witten conjecture [26] now known as the Kontsevich’s theorem [18] which gives a recursive relation for Hodge integrals involving \( \psi \)-classes only. In return some of Hodge integrals can be evaluated recursively through string equation and the KdV hierarchy. In particular, we can recover the following well-known cases.

**Theorem 6.3** (Hurwitz Formula [16]). The single Hurwitz Number formula \( h_{0,\mu} \) is given by,

\[
h_{0,\mu} = \frac{(n + d - 2)!}{|\text{Aut}(\mu)|} \prod_{i=1}^{\mu_1} \frac{\mu_i!}{\mu_i^{d-3}}
\]

(6.4)

where \( n + d - 2 \) is the number of simple branch points.
Proof. By the ELSV formula and string equation

\[ h_{0,\mu} = \frac{(d + n - 2)!}{|\text{Aut}(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\mathcal{M}_{0,n}} \frac{1}{(1 - \mu_1 \psi_1) \cdots (1 - \mu_n \psi_n)} \]

\[ = \frac{(d + n - 2)!}{|\text{Aut}(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \sum_{m_1 + \cdots + m_n = n-3} (\tau_{m_1} \cdots \tau_{m_n})_0 \cdot \mu_1^{m_1} \cdots \mu_n^{m_n} \]

\[ = \frac{(d + n - 2)!}{|\text{Aut}(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \sum_{m_1 + \cdots + m_n = n-3} \frac{(n-3)!}{m_1! \cdots m_n!} \cdot \mu_1^{m_1} \cdots \mu_n^{m_n} \]

\[ = \frac{(d + n - 2)!}{|\text{Aut}(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} d^{n-3}. \]

Moreover, we can recover the classical formulas of Denes, Arnol’d and Crescimano-Taylor, cf. (3.12), (3.8) and (3.9) respectively:

**Corollary 6.4** (Polynomial case). If \( \mu = (d) \) then

\[ h_{0,\mu} = (d - 1)! \frac{d^d}{d!} d^{-2} = d^{d-3}. \]

**Corollary 6.5** (Rational case). If \( g = 0 \) and \( \mu = (1^d) \) then

\[ h_{0,\mu} = \frac{(2d - 2)!}{d!} d^{d-3}. \]

**Corollary 6.6** (Arnol’d Case). If \( g = 0 \) and \( \mu = (\mu_1, \mu_2) \vdash d \) then

\[ h_{0,\mu_1,\mu_2} = \frac{\mu_1^{\mu_1}}{\mu_1!} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!} \cdot (\mu_1 + \mu_2 - 1)! \cdot \]

Another well-known case with an explicit generating formula occur in the computation of genus 1 Hurwitz numbers \( h_{1,\mu} \). The details can be found in [15]. There has been some progress in calculation of more generalized Hurwitz numbers.

### 7. Single Hurwitz Numbers and Fock space operator techniques

Characters of the symmetric group can be easily expressed in the infinite wedge space, and so we can use Fock space techniques to study the Hurwitz numbers. Fock space techniques arose in physics, and were introduced into Hurwitz theory in [21]. Translating Hurwitz enumeration problem into a question of operators on the Fock space gives access to the structure of generating functions for enumerative invariants.

Let \( V \) be a vector space over \( \mathbb{C} \) with a basis indexed by half-integers

\[ V = \bigoplus_{i \in \mathbb{Z}^{+}_{1/2}} \mathbb{C} \cdot i. \]

Write \( \mathbb{Z}^{+}_{1/2} \) for the positive half integers, and \( \mathbb{Z}^{-}_{1/2} \) for the negative half integers. A state \( S \) is a subset of half integers \( S = \{s_1 < s_2 < \ldots\} \subset \mathbb{Z} + \frac{1}{2} \) such that both \( S \setminus \mathbb{Z}^{-}_{1/2} \) and \( \mathbb{Z}^{-}_{1/2} \setminus S \) are finite. The fermionic Fock space is the vector space

\[ \wedge^{\mathbb{Z}^{-}_{1/2}} V = \bigoplus_{S} \mathbb{C} v_S. \]
with a basis \( \{v_S\} \) spanned by all formal symbols in the wedge product
\[
v_S = s_1 \wedge s_2 \wedge \cdots .
\]
Denote by \( (\cdot, \cdot) \) the unique inner product on \( \bigwedge^\infty V \) for which our basis \( \{v_S\} \) is orthonormal. Observe that, the wedge product is associative, bilinear, and anticommutative, that is \( a \wedge b = -b \wedge a \) for any half integers \( a, b \). Assign an integer \( c \) to each semi-infinite wedge \( v_S \) of \( \bigwedge^\infty V \) called the charge of \( S \) defined by
\[
c = |S \cap \mathbb{Z}_{1/2}^-| - |S \cap \mathbb{Z}_{1/2}^+|.
\]
Let \( \bigwedge^\infty_c V \) denotes the subspace generated by semi-infinite wedges of charge \( c \). Then the Fock space is decomposable by the charge:
\[
\bigwedge^\infty V = \bigoplus_{c \in \mathbb{Z}} \bigwedge^\infty_c V.
\]
We will mostly be concerned with the charge zero subspace of Fock space \( \bigwedge^\infty_0 V \subset \bigwedge^\infty V \) – which is the subspace spanned by all basis elements with charge 0. Remarkably, the charge zero subspace has a basis indexed by integer partitions \( \lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \) of all integers \( P \):
\[
v_S := v_\lambda = \lambda_1 - \frac{1}{2} \wedge \lambda_2 - \frac{1}{2} \wedge \lambda_3 - \frac{1}{2} \wedge \cdots \lambda_i - \frac{1}{2} \wedge \cdots .
\]
Note that the state \( S \) is given by \( S = \{\lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \cdots, \lambda_i - \frac{1}{2} \} \) for some unique integer partition \( \lambda \).

7.1. Representation of a state in a Maya diagram

A useful way to represent a state or rather a basis element of the Fock Space is through a Maya diagram: a sequence of circles centered at \( \mathbb{Z} + \frac{1}{2} \) on the real line, with the positive entries going to the left and the negative entries to the right. A black bead is placed at each position \( i \) where the corresponding vector \( s_i \) appears in the wedge of the entries of \( S \). For example, the partition \( \lambda = (4, 3, 1, 1) \) corresponds to the Maya diagram and wedge product below.

This notation is a gateway to an intuitive bijection between partitions \( P \) and basis elements of the 0-charge subspace \( \bigwedge^\infty_0 V \). To see this, draw partitions rotated \( \pi/4 \) radians counterclockwise and scaled up by a factor of \( \sqrt{2} \), so that each segment of the border path of \( \lambda \) is centered above a half integer on the x-axis, with origin above the square 0. Placing a black bead for every line segment in \( \lambda \) in the direction \((1, 1)\) (an upstep) above each half integer \( s \in S \). For instance, the partition \( \lambda = (4, 3, 1, 1) \) corresponds to the Maya diagram shown.
The charge 0 state corresponds to the empty partition represented by a special vector $v_{\emptyset}$ called the vacuum vector,

$$v_{\emptyset} = -\frac{1}{2} \land -\frac{3}{2} \land -\frac{5}{2} \land -\frac{7}{2} \land -\frac{9}{2} \cdots.$$ 

Namely, the Maya diagram for the vacuum vector has a black bead for every negative half-integer.

### 7.2. Operators on the fermionic Fock spaces

We now define interesting natural operators among Fock spaces by their action on the basis element.

I). Basic operators on Fock spaces $\bigwedge^c_\mathbb{F} V$ for any charge $c$.

(a) The **wedging operator** $\psi_k : \bigwedge^c_\mathbb{F} V \rightarrow \bigwedge^{c+1}_\mathbb{F} V$ indexed by the half integers is defined by

$$\psi_k(v_S) = k \land v_S = \begin{cases} 0, & k \in S \\ \pm v_{SU[k]}, & k \notin S. \end{cases}$$

The sign is obtained by applying the anticommutativity of the wedge product until the sequence is decreasing. For example,

$$\psi_{\frac{3}{2}}(v_{(3,1)}) = \frac{3}{2} \land \left( \frac{5}{2} \land -\frac{1}{2} \land -\frac{3}{2} \land -\frac{7}{2} \land \cdots \right)$$

$$= -\left( \frac{5}{2} \land \frac{3}{2} \land -\frac{1}{2} \land -\frac{3}{2} \land -\frac{7}{2} \land \cdots \right)$$

$$= -v_{(2,2,1)}.$$

(b) The **contracting operator** $\psi_k^* : \bigwedge^{c+1}_\mathbb{F} V \rightarrow \bigwedge^c_\mathbb{F} V$ is the adjoint of $\psi_k$ with respect to our inner product given by

$$\psi_k^*(v_S) = k \land v_S = \begin{cases} \pm v_{SU[k]}, & k \in S \\ 0, & k \notin S. \end{cases}$$

These operators satisfy the anti-commutation relations:

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}, \quad \psi_i \psi_j + \psi_j \psi_i = 0, \quad \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0.$$
The operator $\psi_k$ increases the charge by 1, and $\psi_k^*$ reduces the charge by 1. We can apply them in sequence and preserve charge which yield interesting operators on $\bigwedge^\infty_0 V$.

II). To keep track of the convergence of infinite sums of products of the wedging and contraction operators, we define the normally ordered products of $\psi_i$ and $\psi_i^*$ by

$$ : \psi_i \psi_i^* : = \begin{cases} \psi_i \psi_i^*, & j > 0 \\ -\psi_i^* \psi_i, & j < 0 \end{cases} $$

(a) The operator $E_{ij} := : \psi_i \psi_j^* :$ is attempting to move a bead from position $j$ to position $i$ if it is possible. The normal ordering puts a minus sign if $j$ is negative.

If $i < j$, the result of applying $E_{ij}$ to a vector $v_\lambda$, where $\lambda$ is some partition, is to remove a ribbon to $\lambda$ if possible and a sign is added according to the parity of the height of the rim; if the ribbon cannot be removed, the result is zero. For example,

$$ E_{-\frac{1}{2}, \frac{3}{2}} (v_{(4,3,1)}) = -v_{(2,1,1)}. $$

This is illustrated in the following diagram below.

If $i > j$, the result of applying $E_{ij}$ to a vector $v_\lambda$ is to add a ribbon to $\lambda$ if possible, and to add a sign according to the parity of the height of the rim. If the addition of the ribbon is not possible, the result is zero.

The case when $i = j$, anti-commutation relations and the normal ordering of the product takes effect.

(b) The bosonic operator $\alpha_n$. These operators are constructed from the fermionic operators as follows.

$$ \alpha_n := \sum_{k \in \mathbb{Z} + \frac{1}{2}} E_{k-n,k}. $$

If $n > 0$, the operator attempts to remove ribbons of length $n$ from the Maya diagram of $\lambda$ in $v_\lambda$. If there are multiple ribbons that can be removed, the operator returns the sum of all contributions, weighted by this sign. The sign is $(-1)^{h-1}$, where $h$ is the height $h$ of the ribbon removed and it is defined as the number of rows it occupies.

For example, it follows that $\alpha_3 v_{(5,4,3)} = v_{(3,1,1)} - v_{(5,2,2)} - v_{(5,4)}$. This is be illustrated using Maya diagrams below.
If $n < 0$, the operator acts in a similar way. It attempts to move downsteps $n$ places to the right, which graphically corresponds to adding a ribbon of length $n$. The sign can be calculated in the same way from the height of the added ribbon.

For example, consider the effect of $\alpha_{-3}$ on $v_{(3,1)}$. We depict the action in the following diagrams:
It follows that \( \alpha_{-3}v_{(3,1)} = v_{(6,1)} - v_{(3,2,2)} - v_{(3,1,1,1,1)} \).

(c) The adjoint of this operator \( \alpha_n \) can be found from the adjoint of the \( \psi_k \) operators as follows:

\[
\alpha_n^* = \left( \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k \psi_k^* \right)^* = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_{k-n} \psi_k^* = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k \psi_{k+n}^* = \alpha_{-n}
\]

(d) The operator, \( E_n(z) \)–a weighted version of the \( \alpha_n \). We denote by \( \zeta(z) \) the function \( e^{z/2} - e^{-z/2} \)

\[
E_n(z) := \sum_{k \in \mathbb{Z} + \frac{1}{2}} e^{i(k-2)z} E_{k-n,k} + \frac{\delta_{n,0}}{\zeta(z)}
\]

This operator obeys the commutation relation below

\[
[E_a(z), E_b(w)] = \zeta(aw - bz)E_{a+b}(z + w).
\]

(e) The content operator \( F_2 \), defined as

\[
F_2 := \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{k^2}{2} E_{k,k}.
\]

We define the vacuum expectation value of an operator \( \mathcal{P} \) to be \( \langle \mathcal{P} \rangle := \langle 0 | \mathcal{P} | 0 \rangle \), where \( | 0 \rangle \) is the dual of \( | 0 \rangle \) with respect to the inner product. The single Hurwitz numbers admit the following immediate expression in the infinite wedge space:
Theorem 7.1. The disconnected single Hurwitz numbers can be computed as expectation value in the infinite wedge given by

$$ h_{g,\mu}^* = \frac{|C_\mu|}{d!} \left( e^{a_1 f_{2}} \prod_{i=1}^{n} \alpha_{-\mu_i} \right). $$

7.3. Generating functions of Hurwitz Numbers

In this section, we want to obtain the generating series for the single Hurwitz numbers, giving a recursion for a single Hurwitz number in terms of single Hurwitz numbers of lower genera. In the generating function, we consider both connected and disconnected coverings.

Let $p_1, p_2, p_3, \ldots$ be formal commuting variables and set $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ for $\mu = (\mu_1, \ldots, \mu_n) \vdash d$ and also $p_\mu = p_{\mu_1} \cdots p_{\mu_n}$. Now we introduce the generating functions for connected and disconnected single Hurwitz numbers as

$$ H(t, \mathbf{p}) = \sum_{g \geq 0} \sum_{l(\mu) = n} \sum_{d, n \geq 1} \sum_{\mu \vdash d} h_{g,\mu} P_\mu \frac{t^w}{w!} $$

(7.1)

$$ H^*(t, \mathbf{p}) = \sum_{g \geq 0} \sum_{l(\mu) = n} \sum_{d, n \geq 1} \sum_{\mu \vdash d} h_{g,\mu}^* P_\mu \frac{t^w}{w!} $$

(7.2)

where in each case the summation is over all partitions of length $n$ and $w = 2g - 2 + d + n$ is the number of simple branch points. The $\mathbf{p} = p_1, p_2, p_3, \ldots$ are parameters that encodes the cycle type of $\sigma$. The parameter $t$ counts the number of simple branch points. Since $w$ and $\mu$ recover the genus $g$, $t$ is thus a topological parameter.

7.4. The Cut-and-Join Equation

Hurwitz numbers satisfy combinatorial conditions of partial differential equations (PDEs) called the cut-and-join equation. These PDEs are only useful for very specific branched covering with a given branch profile. In particular, single hurwitz numbers satisfy a cut-and-join equation of Goulden-Jackson in [14]. Namely,

$$ H^* = \exp(H) $$

(7.3)

where the exponential generating function for single Hurwitz numbers is defined to be

$$ \exp(H(t, \mathbf{p})) = 1 + H(t, \mathbf{p}) + \frac{H(t, \mathbf{p})^2}{2!} + \frac{H(t, \mathbf{p})^3}{3!} + \cdots $$

and counts disconnected single branched coverings and the power of $H(t, \mathbf{p})$ is the number of connected components. Then the cut and join recursion takes the following form:

Lemma 7.2.

$$ \frac{\partial H^*}{\partial t} = \left[ \frac{1}{2} \sum_{i,j \geq 1} \left( p_{i+j} \cdot (i \cdot j) \cdot \frac{\partial}{\partial p_i} \cdot \frac{\partial}{\partial p_j} + p_i \cdot p_j \cdot (i + j) \cdot \frac{\partial}{\partial p_{i+j}} \right) \right] H^* $$

(7.4)

We immediately deduce the cut-and-join equation of Goulden-Jackson for the generating function $H(t, \mathbf{p})$ of the number of connected single Hurwitz numbers.
Theorem 7.3 (Cut and Join equation, [14]). The generating function $H$ satisfy the following partial differential equation

$$\frac{\partial H}{\partial t} = \frac{1}{2} \sum_{i,j} p_{i+j} \cdot (i \cdot j) \cdot \frac{\partial H}{\partial p_i} + (i \cdot j) p_{i+j} \cdot \frac{\partial^2 H}{\partial p_i \partial p_j} + p_i \cdot p_j \cdot (i + j) \cdot \frac{\partial H}{\partial p_{i+j}}$$

In particular, $H$ is the unique formal power series solution of the cut and join partial differential equation.

Remark 7.4. The fact that $H$ satisfies a second order partial equation, is not surprising as more is known to hold. Namely, the KP (Kadomtsev-Petviashvili) Hierarchy for Hurwitz numbers. The KP (Kadomtsev-Petviashvili) Hierarchy is a completely integrable system of partial differential equations originating from mathematical physics.

8. Acknowledgment

This paper was written during my fellowship period at Department of Mathematics, University of Stockholm with funding from International Science Programme, Uppsala University, Sweden. My advisor Boris Shapiro has always been invaluable to me. I also want to thank S. Shadrin, J. Bergström, O. Bergvall and Balázs Szendrői for being available for many discussions. Many thanks to M. Shapiro for explaining nagging details of the ELSV formula.

References

[1] E. Arbarello, M. Cornalba, and P. A. Griffiths, *Geometry of algebraic curves. Volume II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 268, Springer, Heidelberg, 2011, With a contribution by Joseph Daniel Harris.

[2] V. I. Arnol’d, *Topological classification of complex trigonometric polynomials and the combinatorics of graphs with an identical number of vertices and edges*, Funktsional. Anal. i Prilozhen. 30 (1996), no. 1, 1–17, 96.

[3] Renzo Cavalieri, Paul Johnson, and Hannah Markwig, *Tropical Hurwitz numbers*, J. Algebraic Combin. 32 (2010), no. 2, 241–265.

[4] Renzo Cavalieri and Eric Miles, *Riemann surfaces and algebraic curves*, vol. 87, Cambridge University Press, 2016.

[5] Michael Crescimanno and Washington Taylor, *Large $N$ phases of chiral QCD$_2$, Nuclear Phys. B* 437 (1995), no. 1, 3–24.

[6] Jozsef Déné, *The representation of a permutation as the product of a minimal number of transpositions, and its connection with the theory of graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl. 4 (1959), 63–71.

[7] David Eisenbud, Noam Elkies, Joe Harris, and Robert Speiser, *On the Hurwitz scheme and its monodromy*, Compositio Math. 77 (1991), no. 1, 95–117.

[8] Torsten Ekedahl, Sergei Lando, Michael Shapiro, and Alek Vainshtein, *On Hurwitz numbers and Hodge integrals*, C. R. Acad. Sci. Paris Sér. I Math. 328 (1999), no. 12, 1175–1180.

[9] ______, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. 146 (2001), no. 2, 297–327.

[10] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. 139 (2000), no. 1, 173–199.

[11] William Fulton, *Hurwitz schemes and irreducibility of moduli of algebraic curves*, Ann. of Math. (2) 90 (1969), 542–575.

[12] I. P. Goulden and D. M. Jackson, *The number of ramified coverings of the sphere by the double torus, and a general form for higher genera*, J. Combin. Theory Ser. A 88 (1999), no. 2, 259–275.

[13] ______, *A proof of a conjecture for the number of ramified coverings of the sphere by the torus*, J. Combin. Theory Ser. A 88 (1999), no. 2, 246–258.

[14] I. P. Goulden, D. M. Jackson, and F. G. Latour, *Inequivalent transitive factorizations into transpositions*, Canad. J. Math. 53 (2001), no. 4, 758–779.

[15] I. P. Goulden, D. M. Jackson, and A. Vainshtein, *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. Comb. 4 (2000), no. 1, 27–46.

[16] A. Hurwitz, *Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. 39 (1891), no. 1, 1–60.

[17] ______, *Ueber die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. 55 (1901), no. 1, 53–66.

[18] Maxim Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. 147 (1992), no. 1, 1–23.

[19] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015, With contribution by A. V. Zelevinsky and a foreword by Richard Stanley.

[20] Paul Moszkowski, *A solution to a problem of Déné: a bijection between trees and factorizations of cyclic permutations*, European J. Combin. 10 (1989), no. 1, 13–16.
A. Okounkov and R. Pandharipande, *Gromov-witten theory, hurwitz theory, and completed cycles*, Annals of Mathematics 163 (2006), no. 2, 517–560.

Jared Ongaro, *Plane hurwitz numbers*, Ph.D. thesis, Stockholm University, Department of mathematics, 2014.

Jared Ongaro, *A note on planarity stratification of Hurwitz spaces*, Canad. Math. Bull. 58 (2015), no. 3, 596–609.

Jared Ongaro, *On a zeuthen-type problem*, arXiv preprint arXiv:1903.11135 (2019).

Bernhard Riemann, *Theorie der abelschen functionen*, Journal für die reine und angwandte Mathematik 54 (1857), 115–155.

Edward Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310.