Geometric Correspondence Controlled by Fermionic Vertex Algebras

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Abstract. We review our recent results and point out further directions in understanding of the vertex algebra approach in the geometric Langlands correspondence.

Correlation Functions in Conformal Field Theory

Vertex algebras [1] give rise to non-trivial relations among the representation theory, analytic number theory, and algebraic geometry. In particular they serve as algebraic tool in generating modular forms as higher characters for vertex operator algebra modules. In physics language this corresponds to multiple-point correlation functions for appropriate conformal field theories defined on compact Riemann surfaces.

A complete description of all n-point functions for the Heisenberg bosonic and lattice VOAs in [2] and for fermionic vertex operator superalgebras in [3] has been given. In a rigorous and constructive manner we introduce the partition functions (graded dimensions) and correlations n-point functions (higher characters) for a the twisted module Mf of a vertex operator (super) algebra V with the formal parameter associated to a complex parameter on a Riemann surface Σ(0) of genus g greater or equal to one obtained by the surface surgery. A modification of the Zhu reduction procedure expressing a correlation n+1-point function via a finite sum of n-point functions with twisted higher Weierstrass functions as coefficients is introduced. In this algebraic approach we can find closed formulas and explicitly establish modular properties of bosonic and fermionic characters on the torus, q = e(2πiτ), zi ∈ Σ(0), vi ∈ V [3, 8]:

\[ Z^{(1)}_{Mf}(v_1; z_1; \ldots; v_n; z_n; q) = \text{STr}_{Mf} Yf(v_1; z_1) Yf(v_n; z_n) q^{L(0)} C_{/24}. \]  

(1)

and at genus two [4, 5]. Modularity properties of n-point functions with respect to appropriate group follows explicitly from the vertex operator derivation. The final expressions are given by the determinants of the matrices with elements being coefficients in the expansions of the regular parts of corresponding differentials (Bergman (bosons) Aa, a = 1, 2 or Szegő (fermions) Q kernels) [3-6]. Deep combinatorial properties of these formulas are established by means of the Principal MacMahon Theorem [8]. The 1-point function for the Virasoro vector satisfies the genus two Ward identity [6, 7]. The partition functions on Riemann surfaces of arbitrary genuses formed in the Schottky uniformization were calculated [8]. In [8] we consider the definition and computation of the genus g partition function (in the multiple torus sewing and Schottky formations of a genus g Riemann surface) for bosonic and fermionic vertex models. We define a genus g partition function by inductively sewing together lower genus 1-point functions \(Z^{(g)}_{M}(u, z_i), i = 1, 2; \ zi ∈ Σ(0), u \) belongs to n-th grading subspace of V, for a V -module M:

\[ Z^{(g)}_{M}(ε) = \sum_{n \geq 0} ε^{(1)} \sum u ∈ V[n] \ Z^{(g)}_{M}(u, z_1) Z^{(g)}_{M}(u, z_2). \]  

(2)

where g = g1 + g2 and z1, z2 are insertion points, ε is the sewing parameter, and ü, u are related by a non-degenerate bilinear invariant form on V.

For instance, with suitable local coordinates in the neighborhood of these points we find for the Heisenberg VOA M2,

\[ Z^{(g)}_{M2}(ε) = \text{det}^{-1} (1 - Q^{(g1; g2)}) Z^{(g1)}_{M} Z^{(g2)}_{M}. \]  

(3)
where $Z^{(0)}M^2(\epsilon)$ is a non-vanishing holomorphic function on the sewing domain and is automorphic with respect to $\text{SL}(2; Z) \times \ldots \times \text{SL}(2; Z) \in \text{Sp}(2g; Z)$ with automorphy factor $\det(\Omega^{(0)} + D)^{-1}$, and a multiplier system, $Q(g1; g2)$ is a moment matrix containing genus $g$ sewing data, and $\Omega^{(0)}$ is genus $g$ period matrix. For further discussion of computation of vertex operator superalgebra n-point functions and their modular properties see [9, 10]

**Automorphic Counterpart**

Previously, (see, e.g., [11-16]) formal expressions for VOA correlation functions were usually derived using their general, in particular, analytic properties, while expressions for the partition functions were simply postulated. At the same time, modular properties of correlations function were assumed from the very beginning. It is the algebraic approach that made possible rigorous mathematical computation of the partition and correlations functions simultaneously establishing their automorphic properties. In many examples of vertex operator algebras, modular properties for these functions are proven [17, 18, 20, 21, 3] to follow from algebraic properties of vertex algebra modules. Appropriate sets of correlation (including the partition) functions constitute an automorphic representation for corresponding modular group. This provides one side of the geometric correspondence for vertex algebras. In [21-26, 3-6] we prove automorphic properties of the partition and n-point functions for several classes of vertex operator algebras (in particular, for the Heisenberg and free fermion VOAs) on Riemann surfaces of genus $g \leq 2$. These considerations can be extended to any genus [4]. Using the above mentioned computational achievements, we are able to construct in two above mentioned VOA cases the Hecke eigensheaves as D-modules on the moduli stack of G-bundles over X in a similar way as in [27].

**Geometric Counterpart**

Using methods of algebraic geometry one is able [12, 28, 11] to determine formally general differential structure of correlation functions in certain bundles over Riemann surfaces. Alternatively, the algebraic technique used in explicit computations of the partition and n-point function for important classes of vertex operator algebra modules allows us to express correlation functions in terms of data describing meromorphic differentials on Riemann surfaces. n-point functions can be represented as holomorphic functions multiplied by the partition functions. For major cases, such as the Heisenberg vertex operator algebras [22, 23, 25, 26, 8, 4], or the free fermion vertex operator superalgebras [3, 4, 7, 5], final expressions for n-point functions can be written as matrices involving coefficients of the regular parts of expansions of classical (Bergman, for bosons, and Szegö, for fermions) kernels [3-6]. In order to compute correlation functions we introduce and explicitly calculate [4] special differentials on Riemann surfaces which are generating functions for all n-point functions. The term "generating" means that all n-point functions can obtain by expanding such function.

Let $X$ be a compact connected simply connected algebraic curve. As it is explained in [27, 30, 32], twisted D-modules on the moduli stack BunG of G-bundles over X arise in conformal field theories as sheaves of conformal blocks on the moduli space $M_{g,n}$ of pointed complex curves of genus $g$. These D-modules encode chiral correlation functions of a model. For CFT's with Lie algebraic symmetries [27], correlation functions satisfy Ward identities which involve [33, 34] projectively at connections on a bundle of conformal blocks. For sheaves of conformal blocks on BunG, one obtains a projective connection in the bundle or the structure of a twisted D-module on the sheaf of conformal blocks.

Based on algebraic properties of VOA modules, we show [7] that certain normalized (i.e., divided by appropriate Heisenberg VOA partition function), generating differentials for n-point functions can be represented as differential operators acting on functions with nice automorphic properties (such as $\theta$-functions). Then such differential operators turn out to be connections in bundles over Riemann surfaces. This construction can be extended to differentials associated to arbitrary n-point functions even on higher genus Riemann surfaces. This construction represents another side of the geometric correspondence for vertex algebras.
Geometric Correspondence for Fermionic Vertex Operator Algebras

Results we obtained stimulate further interest in application of algebraic technology of vertex algebras to number theory. We work now on procedures described above for super Riemann surfaces, on the identities for modular forms, \( 0 \)-functions and primary forms (in particular, generalizations of Macdonald-Kac identities [4, 5]) at genuses \( g \geq 2 \), the partition and \( n \)-point correlation functions, the generation of twisted Jacobi forms, relations with co-operads, cluster algebras and structures generalizing vertex algebras [7], and applications of the theory of vertex algebras to Langlands program.

For a reductive group \( G \), let \( (L)^G \) be the Langlands dual group. Then the categorical version of the geometric Langlands conjecture establishes a correspondence between a derived category of \( \mathbb{O} \)-modules on \( \text{Loc}^{(L)^G} \) (i.e., the moduli stack of at \( (L)^G \)-bundles (local systems) on a curve \( X \)) to a derived category of \( \mathbb{D} \)-modules on \( \text{BunG} \) (an algebraic moduli stack (a set of isomorphism classes of \( G \)-bundles on \( X \))), [29]. The geometric Langlands correspondence has been proven for a number of cases (see [29] for details). A significant progress has been achieved in the last few years in works summarized in [29]. Recall that in [29] the geometric Langlands correspondence is described for the reductive groups with corresponding Kac-Moody algebras at critical level (see details in [29]). Those considerations were based on algebraic properties of the center of an appropriate vertex operator algebra. Note that at critical level the Virasoro algebra is then missing in considerations.

One is able to develop further applications of the vertex algebra machinery to various problems accompanying the geometric Langlands correspondence. The algebraic approach allows us to compute explicitly the partition and \( n \)-point functions for a number vertex operator superalgebras. Moreover, modular properties for these functions immediately follow from algebraic properties of vertex algebra modules. Previously, explicit expressions for the correlation functions were derived using their general properties, in particular, singularity properties. At the same time, expressions for the partition functions were simply postulated, while in our approach we compute the partition functions explicitly. All this encourages us to make a connection to the geometric Langlands correspondence. Let us elaborate more specifically on this topic. In [4] we gave an example of the construction of a connection over corresponding bundle (related to appropriate conformal blocks). This is the basis for further construction of the local systems associate to a curve, and the space of parameters, which should provide us with one side of the geometric Langlands correspondence.

Namely the space of parameters for local systems \( \text{Loc}^{(L)^G} \) can in turn be related to the space parameterizing generating functions for correlations functions [4]. Since we are able to describe modularity properties for the partition and correlation functions (at any genus of Riemann surface) for bosonic/fermionic model, then we are able to reproduce the other side of the desired correspondence, i.e., the space of parameters related to automorphic representations, and describing Hecke eigensheaves on \( \text{BunG} \). In contrast to [29], our examples of bosonic and fermionic vertex operator (super) algebras possess Virasoro subalgebras. Thus we should be able to prove the geometric Langlands conjecture formulated at non-critical level of corresponding Kac-Moody Lie algebra.

This gives us a hope to promote in the proof of the geometric Langlands conjecture at non-critical level (see also [25]). In particular, the structure of the "spectral decomposition" for the space of local Langlands parameters for the torus in the case of the Heisenberg modules at integral negative level [Be] is quite similar to the structure of the parameterization of the generalized vertex operator algebras constructed on the base of the extension of Heisenberg vertex operator algebra extended by its twisted modules [7]. Thus we hope to use properties of the generalized vertex algebras to prove the Beilinson's conjecture for groups other than torus.

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