THETA FUNCTIONS FOR HOLOMORPHIC TRIPLES

GEORG HEIN AND THANG QUYET TRUONG

Abstract. We introduce an generalization of the theta divisor to the theory of holomorphic triples on a smooth projective curve $X$. We show that a given triple $T = (E_1 \to E_0)$ is $\alpha$-semistable iff there exists an orthogonal triple $S = (F_1 \to F_0)$ with given numerical invariants. This yields globally generated theta line bundles on the moduli space of semistable triples.

1. Introduction

We fix a smooth projective curve $X$ of genus $g$ over an algebraically closed ground field $k$. When investigating the coarse moduli space $U_X(r,d)$ of $S$-equivalence classes of semistable bundles on $X$, an ample Cartier divisor which allows a geometric interpretation facilitates its study. Drezet and Narasimhan defined in [4] with the generalized theta line bundle $\mathcal{O}_{U_X(r,d)}(\Theta)$ such a line bundle. The generalized theta divisor has nice sections $\theta_F$ associated to vector bundles $F$ of slope $\mu(F) = (g-1) - \frac{d}{r}$ with vanishing divisor $\Theta_F$. The $k$ points of this divisor are

$$\Theta_F(k) = \{[E] \in U_X(r,d)(k) \mid \text{such that } H^*(X, E \otimes F) \neq 0 \}.$$  

The result of Faltings’ about the existence of orthogonal objects is basic for the theory of the generalized theta divisor:

Theorem 1.1. (Theorem [5, Theorem 1.2]) For a vector bundle $E$ on $X$ we have the following equivalence:

$$E \text{ is semistable } \iff H^*(X, E \otimes F) = 0 \text{ for a vector bundle } F \neq 0.$$  

This result is a key step in showing that the generalized theta line bundle is ample. Indeed, it shows that a certain power of the generalized theta line bundle is globally generated. Furthermore, as shown by Popa, the rank and the determinant of $F$ can be fixed a priori:

Theorem 1.2. (Theorem [9, Theorem 5.3]) For a vector bundle $E$ of rank $r$ and degree $d$ on $X$ we have the following equivalence for any fixed line bundle $L$ of degree $r^2(g-1) - r \cdot d$:

$$E \text{ is semistable } \iff H^*(X, E \otimes F) = 0 \text{ for a } F \text{ with } \text{rk}(F) = r^2, \text{det}(F) \cong L.$$  

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The result of Popa gives a concrete bound for the generatedness of the generalized theta line bundle.

On the other side there is a coarse moduli space of semistable triples introduced by Bradlow and García-Prada in [2], further studied together with Gothen in [3]. This space parameterizes pairs \((E_1 \xrightarrow{\varphi} E_0)\) of vector bundles \(E_i\) on \(X\) together with a homomorphism \(\varphi\) between them. The construction of triples was extended to the case of arbitrary characteristic by Álvarez-Cónsul in [1]. The aim of this article is to introduce a generalized theta divisor for holomorphic triples which also possesses a nice geometric description of its vanishing divisor, as well as versions of the above theorems 1.1 and 1.2.

Indeed, for a triple \(T = (E_1 \rightarrow E_0)\), we show in Theorem 5.1, that the \(\alpha\)-semistability of \(T\) is equivalent to the existence of an orthogonal triple \(S = (F_1 \rightarrow F_0)\) satisfying certain numerical conditions which involve the parameter \(\alpha\). Orthogonality means, that the morphism

\[
(E_1 \otimes F_0) \oplus (E_0 \otimes F_1) \xrightarrow{-\varphi \otimes \text{id}_{F_0} + \text{id}_{E_0} \otimes \psi} E_0 \otimes F_0
\]

is surjective and its kernel \(K\) satisfies \(H^*(X, K) = 0\).

This allows to define the set of all triples \(T\) such that \(T\) is not orthogonal to \(S\). As we see in Section 3 this set is a Cartier divisor \(\Theta_S\) corresponding to the generalized theta line bundle on the moduli space of holomorphic triples. We see in Proposition 5.6 that we can bound the ranks in the triple \(S\) independent of \(T\). This result is the equivalent to Popa’s base point free theorem 1.2.

**The plan of the article** is as follows: We start in Section 2 with the equivalence of triples on \(X\) and certain short exact sequences on \(X \times \mathbb{P}^1\). This is similar to the approach in [2]. Next in Section 3 we show the Bogomolov inequality for \(X \times \mathbb{P}^1\). Since, we show it in any characteristic this result is new in positive characteristic. In Section 4 we explain how this induces effective restriction theorems on \(X \times \mathbb{P}^1\). Indeed, we almost follow the book [7] of Huybrechts and Lehn in doing so. Section 5 contains the main result of this article the equivalence of semistability and orthogonality for triples. Having done this, we define in Section 6 the generalized theta divisor \(\Theta_R\) for a triple. Here we follow the exposition in the article [4] of Drezet and Narasimhan. In the last Section we present a simple example of a theta divisor \(\Theta_R\) on a specific moduli space of triples such that \(\Theta_R\) is ample and the linear system \(|2 \cdot \Theta_R|\) is globally generated.

**Notation.** We fix a smooth projective curve \(X\) of genus \(g\) over an algebraically closed field \(k\). For a triple \(T = (E_1 \xrightarrow{\varphi} E_0)\) we denote the ranks by \(r_1\) and \(r_0\), and analogously the degrees by \(d_1\) and \(d_0\), respectively. The \(\mathbb{P}^1\) is the space \(\mathbb{P}(V)\) for the two dimensional \(k\) vector space \(V\). We denote the projections from \(X \times \mathbb{P}^1\) as follows: \(X \xrightarrow{p} X \times \mathbb{P}^1 \xrightarrow{q} \mathbb{P}^1\). We use \(F_p\) and \(F_q\) for the classes of the fibers of points with respect to \(p\) and \(q\) in the numerical Chow group \(\text{CH}^1(X \times \mathbb{P}^1)\), \(\text{pt}\) for the class of a \(k\)-point in the numerical Chow group \(\text{CH}^2(X \times \mathbb{P}^1)\). For
a fixed positive rational number \( \alpha \) we denote by \( H_\alpha \) the rational polarization \( H_\alpha = F_q + \alpha \cdot F_p \) on \( X \times \mathbb{P}^1 \).

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### 2. From Triples to Sheaves on \( X \times \mathbb{P}^1 \)

#### 2.1. From \( T \) to \( E_T \)

We fix a triple \( T = (E_1 \xrightarrow{\varphi} E_0) \) on the curve \( X \). We have on \( \mathbb{P}^1 \) the Euler sequence

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0.
\]

So we have on \( X \times \mathbb{P}^1 \) two morphisms to \( p^* E_0 \) as indicated in the next diagram.

The pull back of both morphisms we name \( E_T \).

\[
\begin{array}{ccc}
E_T & \xrightarrow{\beta} & p^* E_1 \\
\downarrow & & \downarrow \\
p^* E_0 \otimes q^*(V \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) & \rightarrow & p^* E_0
\end{array}
\]

We obtain a short exact sequence on \( X \times \mathbb{P}^1 \)

\[
0 \rightarrow E_0 \otimes \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow E_T \rightarrow p^* E_1 \rightarrow 0.
\]

So we compute the (numerical) Chern character

\[
\text{ch}(E_T) = (r_0 + r_1) + ((d_0 + d_1)F_p - 2r_0F_q) - 2d_0 \text{pt}.
\]

To obtain the numerical invariants of \( E_T \), as

\[
\text{rk}(E_T) = r_0 + r_1, \quad \text{c}_1(E_T) = (d_0 + d_1)F_p - 2r_0F_q, \quad \text{and} \quad \text{c}_2(E_T) = 2d_0 - 2r_0(d_0 + d_1).
\]

The degree with respect to the polarization \( H_\alpha \) is

\[
\text{deg}(E_T) = \text{deg}_{H_\alpha}(E_T) = c_1(E_T).H_\alpha = d_0 + d_1 - 2\alpha r_0.
\]

Thus, we have motivated the definition of the rank, \( \alpha \)-degree, and \( \alpha \)-slope of the triple \( T = (E_1 \xrightarrow{\varphi} E_0) \)

\[
\text{rk}(T) = r_0 + r_1, \quad \text{deg}_\alpha(T) = d_0 + d_1 - 2\alpha r_0, \quad \mu_\alpha(T) = \frac{\text{deg}_\alpha(T)}{\text{rk}(T)}.
\]

Once this is done we proceed to the definition of (semi)stability.

**Definition:** The triple \( T = (E_1 \xrightarrow{\varphi} E_0) \) is \( \alpha \)-(semi)stable \iff the sheaf \( E_T \) is (semi)stable with respect to the polarization \( H_\alpha \) on \( X \times \mathbb{P}^1 \).

**Lemma 2.2.** For a triple \( T = (E_1 \xrightarrow{\varphi} E_0) \) the following is equivalent:

1. The triple \( T \) is \( \alpha \)-semistable.
2. For all sub triples \( T'' \rightarrow T \) we have the inequality \( \mu_\alpha(T') \leq \mu_\alpha(T) \).
3. For all quotient triples \( T \rightarrow T'' \) we have the inequality \( \mu_\alpha(T) \leq \mu_\alpha(T'') \).
Proof. (1) $\implies$ (2). This direction is easy. If $T' \rightarrow T$ destabilizes $T$, then $E_{T'} \subset E_T$ destabilizes $E_T$. The equivalence of (2) and (3) is standard.

(2) $\implies$ (1). To see this we consider $X \times \mathbb{P}^1$ with the group action of $\text{SL}_2$ where the action on $X$ is trivial, and the action on $\mathbb{P}^1$ comes from an identifications $\text{SL}_2 = \text{SL}(V)$ and $\mathbb{P}^1 = \mathbb{P}(V)$. Since the Euler sequence on $\mathbb{P}^1$ is a sequence of $\text{SL}_2$-bundles, we obtain that $E_T$ is also a $\text{SL}_2$-bundle, and the short exact sequence

$$0 \rightarrow p^*E_0 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow E_T \rightarrow p^*E_1 \rightarrow 0$$

is a sequence of $\text{SL}_2$-bundles. Assume that $E_T$ is not semistable, the there exists a unique destabilizing subsheaf $F \subset E_T$. The uniqueness implies that $F$ is $\text{SL}_2$-invariant, or $F$ is a $\text{SL}_2$-sub bundle of $E_T$. We conclude that we have the following diagram where all sheaves are $\text{SL}_2$-bundles, and all morphisms $\text{SL}_2$-morphisms:

$$
\begin{array}{cccccc}
0 & \rightarrow & F' & \rightarrow & F & \rightarrow & F'' & \rightarrow & 0 \\
0 & \rightarrow & p^*E_0 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) & \rightarrow & E_T & \rightarrow & p^*E_1 & \rightarrow & 0 \\
\end{array}
$$

We deduce that $F'' = p^*F_1$ and $F' = p^*F_0 \otimes \mathcal{O}_{\mathbb{P}^1}(-2)$. The long exact sequence for $p_*$ yields the diagram

$$
\begin{array}{cccc}
F_1 & \rightarrow & F_0 & \\
\downarrow & & \downarrow & \\
E_1 & & \varphi & \rightarrow & E_0
\end{array}
$$

with injective vertical morphisms. Thus, $(F_1 \rightarrow F_0)$ destabilizes $(E_1 \rightarrow E_0)$. \qed

3. The Bogomolov inequality for $X \times \mathbb{P}^1$

3.1. The Harder-Narasimhan functor $E \mapsto E(\mu)$. We recall that for any vector bundle $E$ on $X$ we have the unique Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_k$$

which satisfies $E_i/E_{i-1}$ is semistable for all $i = 1, \ldots, k$, and the rational numbers $\mu_i = \mu(E_i/E_{i-1})$ are strictly decreasing. The set $\{\mu_1, \mu_2, \ldots, \mu_k\}$, is the set of slopes appearing in the associated graded object $\text{gr}(E) = \bigoplus_{i=1}^k E_i/E_{i-1}$. For a rational number $\mu$ we define

$$E(\mu) := E_j \quad \text{where} \quad j = \max\{i = 1, \ldots, k \mid \mu_i \geq \mu\}.$$

The advantage of the notation is, that we obtain a functor $\text{Coh}(X) \rightarrow \text{Coh}(X)$ sending $E \mapsto E(\mu)$ for all $\mu \in \mathbb{Q}$. This follows from the next
Lemma 3.2. Let $E \xrightarrow{\varphi} F$ be a morphism of vector bundles, and $\mu \in \mathbb{Q}$, then we obtain a natural morphism $E_{(\mu)} \xrightarrow{\varphi_{(\mu)}} F_{(\mu)}$ making the diagram

$$
\begin{array}{ccc}
E_{(\mu)} & \xrightarrow{\varphi_{(\mu)}} & F_{(\mu)} \\
\downarrow & & \downarrow \\
E & \xrightarrow{\varphi} & F
\end{array}
$$

commutative. The vertical arrows are the natural inclusions.

Proof. The natural inclusions give a composite homomorphisms

$$
\alpha : E_{(\mu)} \to E \xrightarrow{\varphi} F \to F/F_{(\mu)}.
$$

Since there are no homomorphism of semistable sheaves of slope $\geq \mu$ to those of slope $< \mu$ we conclude that $\alpha$ is zero. This induces the assertion. \qed

Lemma 3.3. Let $E$ and $F$ two vector bundles on $X$. If $\mu(E) > \mu(F)$, then there exists a rational number $\mu$ such that

$$
\text{rk}(E_{(\mu)}) > \text{rk}(F_{(\mu)}).
$$

Proof. Let $\{\mu_1 > \mu_2 > \ldots > \mu_k\}$ be the union of the slopes appearing in the associated graded objects gr($E$) and gr($F$). We see that

$$
\mu(E) = \frac{1}{\text{rk}(E)} \sum_{i=1}^{k} (\text{rk}(E_{(\mu_i)}) - \text{rk}(E_{(\mu_i-1)})) \mu_i = \sum_{i=1}^{k-1} \frac{\text{rk}(E_{(\mu_i)})}{\text{rk}(E)} (\mu_i - \mu_{i+1}) + \mu_k,
$$

and obtain a similar formula for $\mu(F)$. From both formulas we deduce that

$$
0 < \mu(E) - \mu(F) = \sum_{i=1}^{k-1} \left( \frac{\text{rk}(E_{(\mu_i)})}{\text{rk}(E)} - \frac{\text{rk}(F_{(\mu_i)})}{\text{rk}(F)} \right) (\mu_i - \mu_{i+1}).
$$

Since the differences $(\mu_i - \mu_{i+1})$ are all positive, the statement follows. \qed

Theorem 3.4. (Bogomolov inequality for curves times $\mathbb{P}^1$ in any characteristic) Let $X$ be a smooth projective curve over some algebraically closed field $k$. Let $E$ be a vector bundle on $X \times \mathbb{P}^1$. We consider the discriminant $\Delta(E)$, which is the number

$$
\Delta(E) = \int_{X \times \mathbb{P}^1} (\text{rk}(E) - 1)c_1^2(E) - 2\text{rk}(E)c_2(E).
$$

If $E$ is semistable with respect to one polarization $H$ on $X \times \mathbb{P}^1$, then we must have

$$
\Delta(E) \leq 0.
$$

Indeed, if $\Delta(E) > 0$, then there exists a subsheaf $E' \subset E$ such that $E'$ contradicts the semistability, that means $\mu_H(E') > \mu_H(E)$, with respect to any polarization $H$ on $X$. 

Proof. We assume that $E$ is a vector bundle satisfying $\Delta(E) > 0$ and construct a subsheaf $E'$ of $E$ contradicting the semistability for all polarizations $H$. To make the proof more accessible, we divide it into steps.

**Step 1: Notation.** We consider the morphisms $X \xrightarrow{p} X \times \mathbb{P}^1 \xrightarrow{q} \mathbb{P}^1$. First we remark, that the discriminant is unchanged when we twist $E$ with any line bundle $L$. Since semistability also remains unchanged when passing from $E$ to $E \otimes L$, we may assume that $\text{R}^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. The resolution of the diagonal

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \rightarrow \mathcal{O}_{\Delta_{\mathbb{P}^1}} \rightarrow 0$$

when pulled back to $X$ yields a resolution

$$0 \rightarrow p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow p^*E_0 \rightarrow E \rightarrow 0.$$

Using this short exact sequence, and using the numbers $r_i = \text{rk}(E_i)$ and $d_i = \deg(E_i)$ for $i \in \{0, 1\}$, we find that

$$\Delta(E) = 2(r_0d_1 - r_1d_0) = 2r_0r_1(\mu(E_1) - \mu(E_0)).$$

Thus, we conclude from $\Delta(E) > 0$ that $2r_0r_1 > 0$ and $\mu(E_1) > \mu(E_0)$ hold.

**Step 2: Choice of $\mu$.** Since $\mu(E_1) > \mu(E_0)$ we conclude from Lemma 3.3 that there exists a rational number $\mu$ such that $\frac{\text{rk}(E_{1(\mu)})}{\text{rk}(E_1)} > \frac{\text{rk}(E_{0(\mu)})}{\text{rk}(E_0)}$ holds. Now we choose $\mu$ in such a way that the quotient $\frac{\text{rk}(E_{1(\mu)})}{\text{rk}(E_{0(\mu)})}$ becomes maximal. This implies that $\mu(E_{1(\mu)}) \leq \mu(E_{0(\mu)})$. Indeed, if $\mu(E_{1(\mu)}) > \mu(E_{0(\mu)})$, then we could apply Lemma 3.3 again, to deduce that for a rational number $\nu$ we have $\frac{\text{rk}(E_{1(\mu)(\nu)})}{\text{rk}(E_{0(\mu)(\nu)})} \leq \mu(E_{1(\mu)}) \leq \mu(E_{0(\mu)})$.

We have the two inequalities:

(1) $\frac{\text{rk}(E_{0(\mu)})}{\text{rk}(E_{1(\mu)})} < \frac{\text{rk}(E_0)}{\text{rk}(E_1)}$ and $\mu(E_{1(\mu)}) \leq \mu(E_{0(\mu)})$.

**Step 3: The subsheaf $E' \subset E$.**

When restricting to a section of $q$, we see that $E_1$ is a subsheaf of $E_0$. We deduce from Lemma 3.2 that $E_{1(\mu)}(-1)$ is also a subsheaf of $E_{0(\mu)}$.

We obtain the following commutative diagram with exact rows

$$\begin{array}{ccc}
0 & \rightarrow & p^*E_{1\mu} \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \\
& \downarrow & \downarrow \alpha \\
n & \rightarrow & p^*E_0 \rightarrow E'' \rightarrow 0
\end{array}$$

Now we define $E'$ to be the image of $\alpha$. From the long kernel cokernel sequence we obtain

$$0 \rightarrow \ker(\alpha) \rightarrow E'' \rightarrow E' \rightarrow 0 \quad \text{and} \quad \ker(\alpha) \subset p^*(E_1/E_{1(\mu)}) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1).$$
When we restrict $E''$ to a fiber $F_p$ of $p$ we obtain that $E''|_{F_p}$ is globally generated. Thus, the kernel of $\alpha$ which has no global sections is strictly smaller. We deduce that $E'$ is not the zero subsheaf of $E$. In particular, its rank is positive. To show that $E'$ is destabilizing, we use that up to numerical equivalence every ample class on $X$ is of type $aF_q + bF_p$ with $a$ and $b$ positive integers and $F_q$, $F_p$ the fibers of $q$ and $p$. Thus, to show that $E'$ is destabilizing with respect to any polarization it suffices to show that it destabilizes when restricted to $F_q$ and $F_p$.

**Step 4:** The subsheaf $E'$ destabilizes $E$ when restricted to $F_p$. We use the short exact sequences of (2) and (3) to compute the ranks and degrees of $E|_{F_p}$, $E''|_{F_p}$, and ker($\alpha$)|$_{F_p}$.

\[
\begin{align*}
\text{rk}(E|_{F_p}) &= \text{rk}(E_0) - \text{rk}(E_1) \\
\text{rk}(E''|_{F_p}) &= \text{rk}(E_0(\mu)) - \text{rk}(E_1(\mu)) \\
\text{rk}(\text{ker}(\alpha)|_{F_p}) &= r_{\alpha}
\end{align*}
\]

\[
\begin{align*}
\text{deg}(E|_{F_p}) &= \text{rk}(E_1) \\
\text{deg}(E''|_{F_p}) &= \text{rk}(E_1(\mu)) \\
\text{deg}(\text{ker}(\alpha)|_{F_p}) &\leq -r_{\alpha}.
\end{align*}
\]

We deduce that the slope of $E'|_{F_p}$ is given by

\[
\mu(E'|_{F_p}) = \frac{-\text{deg}(\text{ker}(\alpha)|_{F_p}) + \text{rk}(E_1(\mu))}{\text{rk}(E_0(\mu)) - \text{rk}(E_1(\mu)) - r_{\alpha}} = \frac{r_{\alpha} + \text{rk}(E_1(\mu))a}{\text{rk}(E_0(\mu)) - \text{rk}(E_1(\mu)) - r_{\alpha}}.
\]

The function on the left is strictly increasing in $r_{\alpha}$, which satisfies $r_{\alpha} \geq 0$. Thus, we conclude that

\[
\mu(E'|_{F_p}) \geq \frac{\text{rk}(E_1(\mu))}{\text{rk}(E_0(\mu)) - \text{rk}(E_1(\mu))} = \frac{1}{\frac{\text{rk}(E_0(\mu))}{\text{rk}(E_1(\mu))} - 1}.
\]

The last inequality together with the inequality (1) gives

\[
\mu(E'|_{F_p}) > \frac{1}{\frac{\text{rk}(E_0)}{\text{rk}(E_1)} - 1} = \mu(E|_{F_p}).
\]

Therefore, the subsheaf $E'$ is destabilizing with respect to the fibers of $p$.

**Step 5:** The subsheaf $E'$ destabilizes $E$ when restricted to $F_q$. We will repeatedly use the following formula for a short exact sequence of vector bundles

\[
0 \to G' \to G \to G'' \to 0
\]

on $X$ which gives the slope of $G$ in terms of the slopes of $G'$ and $G''$:

\[
\mu(G) = \frac{\text{rk}(G')}{\text{rk}(G') + \text{rk}(G'')}\mu(G') + \frac{\text{rk}(G'')}{\text{rk}(G') + \text{rk}(G'')}\mu(G'').
\]

Thus, $\mu(G)$ is a weighted average of $\mu(G')$ and $\mu(G'')$. We conclude that any relation ($\prec, \preceq, =, \succeq, \succ$) between $\mu(G')$ and $\mu(G)$ implies the same relation between $\mu(G)$ and $\mu(G'')$.

The short exact sequence from (2) restricted to $F_q$ yields

\[
(4) \quad \mu(E_0) > \mu(E|_{F_q}).
\]
since we have $\mu(E_1) > \mu(E_0)$. Since $E_{0(\mu)}$ appears in the Harder-Narasimhan filtration of $E_0$ we have

\[ (5) \quad \mu(E_{0(\mu)}) \geq \mu(E_0). \]

By the definition of $E_{0(\mu)}$ we conclude that

\[ (6) \quad \mu(E_{0(\mu)}) \geq \mu. \]

The second inequality in (1) with the short exact sequence from (2) restricted to $F_q$ yields

\[ (7) \quad \mu(E_{0(\mu)}) \leq \mu(E'|_{F_q}). \]

The inclusion $\ker(\alpha) \subset p^*(E_1/E_{1(\mu)}) \otimes q^*O_{\mathbb{P}^1}(-1)$ from (3) when restricted to $F_q$ yields $\ker(\alpha)|_{F_q} \subset E_1/E_{1(\mu)}$. The graded object of $E_1/E_{1(\mu)}$ consists of semistable sheaves of slope strictly smaller than $\mu$. So it follows, that $\mu(\ker(\alpha)|_{F_q}) < \mu$. This gives together with $\mu(E''|_{F_q}) \geq \mu$ which is a conclusion of inequalities (6) and (7) that

\[ (8) \quad \mu(E'|_{F_q}) \geq \mu(E''|_{F_q}). \]

The inequalities (4), (5), (7), and (8) yield $\mu(E'|_{F_q}) > \mu(E|_{F_q})$. This finishes the proof. \qed

4. Bogomolov’s restriction theorem for $X \times \mathbb{P}^1$

The next result is a standard conclusion how Bogomolov’s inequality (Theorem 3.4) induces effective restriction theorems. We follow the presentation of Section 7.3 in the book [7] of Huybrechts and Lehn.

**Proposition 4.1.** (cf. [7, Theorem 7.3.5]) Let $E$ be a stable vector bundle of rank $r \geq 2$ on $X \times \mathbb{P}^1$. Let $H$ be an ample divisor on $X \times \mathbb{P}^1$ and $k \in \mathbb{N}$ a number such that

1. the linear system $|k \cdot H|$ contains smooth curves, and
2. we have $k > \frac{C^2}{r}(-\Delta(E)) + \frac{1}{(r-1)r \cdot H^2}$.

Then the restriction of $E$ to any smooth curve $C$ in the linear system $|k \cdot H|$ is stable.

**Proof.** Assume that $C \in |k \cdot H|$ is a smooth divisor such that $E|_C$ is not stable. We have a short exact sequence of vector bundles on $C$

\[ 0 \to G_1 \to E|_C \to G_2 \to 0 \]

with $\mu(G_2) \leq \mu(E|_C)$. We let $E'$ be the kernel of $E \to E|_C \to G_2$ and compute

\[ \Delta(E') = \Delta(E) + C^2(r \cdot \rk_C(G_2) - \rk_C(G_2)^2) + 2r \rk_C(G_2)(\mu(E|_C) - \mu(G_2)) \]

Since $(r \cdot \rk_C(G_2) - \rk_C(G_2)^2) \geq (r - 1)$ and the last summand is not negative we deduce that

\[ \Delta(E') \geq \Delta(E) + (r - 1)C^2. \]
From assumption (2) it follows that $k^2H^2(r-1) + \Delta(E) > 0$. We deduce that $E'$ has a positive discriminant and is therefore unstable by Bogomolov’s Theorem. We consider the Harder-Narasimhan filtration of $E'$ with respect to the polarization $H$.

$$0 = E'_0 \subset E'_1 \subset E'_2 \subset \ldots \subset E'_k = E'.$$

The graded objects $E'_i/E'_{i-1}$ we denote by $F_i$. We have the equality

$$\Delta(E') = \frac{\Delta(E'_i)}{r} - \sum_{i=1}^{k} \frac{\Delta(F_i)}{\text{rk}(F_i)} = \frac{1}{r} \sum_{1 \leq i < j \leq k} \text{rk}(F_i) \text{rk}(F_j) \left( \frac{c_1(F_i)}{\text{rk}(F_i)} - \frac{c_1(F_j)}{\text{rk}(F_j)} \right)^2.$$

Since the $F_i$ are semistable, we have that $\Delta(F_i) \leq 0$, so we conclude

$$\Delta(E') \leq \sum_{1 \leq i < j \leq k} \text{rk}(F_i) \text{rk}(F_j) \left( \frac{c_1(F_i)}{\text{rk}(F_i)} - \frac{c_1(F_j)}{\text{rk}(F_j)} \right)^2.$$

Now multiplying both sides with $H^2$ and using the Hodge index Theorem yields

$$H^2 \cdot \Delta(E') \leq \sum_{1 \leq i < j \leq k} \text{rk}(F_i) \text{rk}(F_j) (\mu(F_i) - \mu(F_j))^2 \quad \text{with} \quad \mu(F_i) = \frac{c_1(F_i) \cdot H}{\text{rk}(F_i)}.$$

Now Lemma 1.4 of [8] gives an upper bound of the right hand side of that inequality. So we obtain

$$H^2 \cdot \Delta(E') \leq r^2 (\mu(F_1) - \mu(E')) (\mu(E') - \mu(F_k)).$$

From the short exact sequence $0 \to E' \to E \to G_2 \to 0$ we deduce that $F_1$ is a subsheaf of the stable sheaf $E$. Hence we have

$$\mu(F_1) - \mu(E') \leq \frac{\text{rk}(G_2)}{r} C.H - \frac{1}{r(r-1)}.$$

From the short exact sequence $0 \to E(-C) \to E' \to G_1 \to 0$ we see that the quotient $F_k$ of $E'$ contains a quotient of the stable sheaf $E(-C)$ of rank $\text{rk}(F_k)$. Therefore, we have

$$\mu(E') - \mu(F_k) \leq \frac{\text{rk}(G_1)}{r} C.H - \frac{1}{r(r-1)}.$$

From the last three inequalities we deduce, using $\text{rk}(G_1) + \text{rk}(G_2) = r$ that

$$H^2 \cdot \Delta(E') \leq (r - \text{rk}(G_2)) \text{rk}(G_2)(C.H)^2 - \frac{r}{r-1} C.H + \frac{1}{(r-1)^2}.$$

Multiplying the equality for $\Delta(E')$ with $H^2$, and using $\mu(E|C) - \mu(G_2) \geq 0$ we get the inequality:

$$H^2 \cdot \Delta(E') \geq H^2 \cdot \Delta(E) + H^2 \cdot C^2 (r - \text{rk}(G_2)) \text{rk}(G_2).$$

Putting both together gives

$$H^2 \cdot (-\Delta(E)) \geq \frac{r}{r-1} C.H - \frac{1}{(r-1)^2}.$$
Dividing both sides by $H^2$ gives
\[-\Delta(E) \geq \frac{r \cdot k}{r - 1} - \frac{1}{H^2(r - 1)^2}.
\]
This violates assumption (2). This contradiction shows that $E|_C$ is stable. □

**Theorem 4.2.** Let $E$ be a semistable coherent sheaf of rank $r \geq 2$ on $X \times \mathbb{P}^1$. Let $H$ be an ample divisor on $X \times \mathbb{P}^1$ and $k$ be an integer such that

1. the linear system $|k \cdot H|$ contains smooth curves, and
2. we have $k > \frac{r - 1}{r}(\Delta(E)) + \frac{1}{(r - 1) \cdot H^2}$.

If $0 = E_0 \subset E_1 \subset \ldots \subset E_k = E$ is the Jordan-Hölder filtration of $E$, then for a general smooth curves $C \in |k \cdot H|$ the sheaf $E_i/E_{i-1}$ restricted to $C$ is a vector bundle for all $i = 1, \ldots, k$, and the Jordan-Hölder filtration of $E|_C$ is given by $0 = E_0 \subset E_1|_C \subset \ldots \subset E_k|_C = E|_C$. In particular: $E|_C$ is semistable for a general curve in $|k \cdot H|$.

**Proof.** We start with the remark, that for a coherent torsion free sheaf $E$, the double dual $E^{\vee \vee}$ satisfies $\Delta(E^{\vee \vee}) \geq \Delta(E)$, and the injection $E \to E^{\vee \vee}$ has cokernel $T$ of dimension zero. Let $C$ be a smooth curve in $|k \cdot H|$. The action of $\text{SL}_2$ on the homogeneous space $\mathbb{P}^1$ allows to move $C$ on $X \times \mathbb{P}^1$ to a curve which does not contain any of the points of the support of $T$. So we may assume that all coherent sheaves which appear in the proof are vector bundles.

Denoting the graded objects $E_i/E_{i-1}$ by $F_i$ we obtain as in the proof of Proposition 4.1 that
\[
\sum_{i=1}^{k} \frac{-\Delta(F_i)}{\text{rk}(F_i)} = \frac{-\Delta(E)}{\text{rk}(E)} + \frac{1}{\text{rk}(E)} \sum_{1 \leq i < j \leq k} \left( \frac{c_1(F_i)}{\text{rk}F_i} - \frac{c_1(F_j)}{\text{rk}F_j} \right)^2.
\]

Since we have a Jordan-Hölder filtration we have that the slope of the $F_i$ with respect to $H$ are all the same, or $\left( \frac{c_1(F_i)}{\text{rk}F_i} - \frac{c_1(F_j)}{\text{rk}F_j} \right) \cdot H = 0$. From the Hodge index theorem we get that $\left( \frac{c_1(F_i)}{\text{rk}F_i} - \frac{c_1(F_j)}{\text{rk}F_j} \right)^2 \leq 0$. We get the inequality
\[
\sum_{i=1}^{k} \frac{-\Delta(F_i)}{\text{rk}(F_i)} \leq \frac{-\Delta(E)}{\text{rk}(E)}.
\]

Multiplying by $(r - 1)$ this yields
\[
\sum_{i=1}^{k} \frac{r - 1}{\text{rk}(F_i)} (-\Delta(F_i)) \leq \frac{(r - 1)}{r} (-\Delta(E)).
\]

The summands on the left hand side are all non negative by Proposition 4.1. The number $k$ is greater than the right hand side by assumption, and so we have that $k$ is greater than each summand, that is
\[
k > \frac{r - 1}{\text{rk}(F_i)} (-\Delta(F_i)) > \frac{\text{rk}(F_i) - 1}{\text{rk}(F_i)} (-\Delta(F_i)) + \frac{1}{\text{rk}(F_i)(\text{rk}(F_i) - 1)H^2}.
\]
By Proposition 4.1 all the $F_i$ are stable when restricted to $C \in |k \cdot H|$.

5. Existence of orthogonal triples

Let $\alpha \in \mathbb{Q}$ be a rational number. We say that a triple $F_1 \rightarrow F_0$ is of $\alpha$-orthogonal-type (in short is of type $\alpha^\perp$), if the following three conditions hold

- the morphism $\psi$ is surjective,
- for the ranks we have the relation $\text{rk}(F_1) = 2 \cdot \text{rk}(F_0)$, and
- the slopes differ by $\alpha$, i.e. $\mu(F_0) - \mu(F_1) = \alpha$.

We say that two triples $(E_1 \rightarrow E_0)$ and $(F_1 \rightarrow F_0)$ are orthogonal when the morphism

$$(E_1 \otimes F_0) \oplus (E_0 \otimes F_1) \xrightarrow{\pi = -\varphi \otimes \text{id}_{F_0} + \text{id}_{E_0} \otimes \psi} E_0 \otimes F_0$$

is surjective, and we have $H^*(X, \ker(\pi)) = 0$.

**Theorem 5.1.** For any $\alpha \in \mathbb{Q}_+$ we have the following two equivalent conditions for any triple $T = (E_1 \rightarrow E_0)$ on our curve $X$:

1. $T$ is $\alpha$-semistable.
2. there exists an orthogonal triple $S = (F_1 \rightarrow F_0)$ of type $\alpha^\perp$.

The proof of this theorem will occupy this section. However the implication (2) $\implies$ (1) in Theorem 5.1 is not so hard to see. We show it in Corollary 5.3. The harder inclusion (1) $\implies$ (2) is Proposition 5.4 which relies on results of all previous sections.

**Lemma 5.2.** Let $S = (F_1 \rightarrow F_0)$ be a triple of type $\alpha^\perp$. Then for any triple $T = (E_1 \rightarrow E_0)$ the morphism

$$(E_1 \otimes F_0) \oplus (E_0 \otimes F_1) \xrightarrow{\pi = -\varphi \otimes \text{id}_{F_0} + \text{id}_{E_0} \otimes \psi} E_0 \otimes F_0$$

is surjective, and we have $\mu(\ker(\pi)) = \mu_\alpha(T) + \mu(F_0)$.

**Proof.** The surjectivity of $F_1 \rightarrow F_0$ implies that $E_0 \otimes F_1 \rightarrow E_0 \otimes F_0$ is surjective. Hence $\pi$ is surjective. Now the computation of the numerical invariants of $\ker(\pi)$ is straightforward and yields

$$\text{rk}(\ker \pi) = \text{rk}(F_0) \cdot (\text{rk}(E_1) + \text{rk}(E_0))$$
$$\text{deg}(\ker \pi) = \text{rk}(F_0) \cdot (\text{deg}(E_1) + \deg(E_0) - 2\alpha \cdot \text{rk}(E_0)) + (\text{rk}(E_1) + \text{rk}(E_0)) \cdot \deg(F_0)$$

$$= \text{rk}(\ker \pi) \cdot (\mu_\alpha(T) + \mu(F_0)) .$$

This is the statement of the lemma. \qed

**Corollary 5.3.** Let $T = (E_1 \rightarrow E_0)$ be a given triple. If $T$ is orthogonal to the triple $S = (F_1 \rightarrow F_0)$ of type $\alpha^\perp$, then $T$ is $\alpha$-semistable.
Proof. Assume the triple $S$ is of type $\alpha^\perp$ and orthogonal to $T$. For any sub object $T' = (E_1' \xrightarrow{\varphi'} E_0') \subset T$ we obtain the following commutative diagram with injective vertical arrows:
\[
\begin{array}{ccccc}
0 & \to & \ker(\pi') & \xrightarrow{\pi' = -\varphi' \otimes \text{id}_{E_0} + \text{id}_{E_0'} \otimes \psi} & (E_1' \otimes F_0) \oplus (E_0 \otimes F_1) \\
& & \downarrow & & \downarrow \\
0 & \to & \ker(\pi) & \xrightarrow{\pi = -\varphi \otimes \text{id}_{E_0} + \text{id}_{E_0} \otimes \psi} & (E_1 \otimes F_0) \oplus (E_0 \otimes F_1)
\end{array}
\]

We study now the short exact sequence $0 \to \ker \pi' \to \ker \pi \to Q \to 0$ on the curve $X$. Since $S$ is orthogonal to $T$, we have $H^0(X, \ker \pi) = 0$. Thus $\mu(\ker \pi) = g - 1$ by the Riemann-Roch Theorem. Since $H^0(X, \ker \pi')$ is a subspace of $H^0(X, \ker \pi)$ this vector space is zero. Hence, $\chi(\ker \pi') \leq 0$. We deduce that $\mu(\ker \pi') \leq g - 1$. The inequality $\mu(\ker \pi') \leq \mu(\ker \pi)$ translates as $\mu_\alpha(T') \leq \mu_\alpha(T)$ by Lemma 5.2.

Proposition 5.4. Let $T = (E_1 \xrightarrow{\varphi} E_0)$ be a $\alpha$-semistable triple. There exists a triple $S = (F_1 \xrightarrow{\psi} F_0)$ of type $\alpha^\perp$ which is orthogonal to $T$.

Proof. Let $T = (E_1 \xrightarrow{\varphi} E_0)$ be a $\alpha$-semistable triple. We deduce by Lemma 5.2 that there exists a vector bundle $E_T$ on the surface $X \times \mathbb{P}^1$ which is $H_\alpha$-semistable and in the following pull back diagram.

\[
\begin{array}{ccc}
E_T & \xrightarrow{p^*} & E_1 \\
\downarrow & & \downarrow p^* \varphi \\
p^* E_0 \otimes q^*(V \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) & \xrightarrow{\gamma} & p^* E_0
\end{array}
\]

Having in mind that $\gamma$ is surjective, we may rewrite this as a short exact sequence
\[
0 \to E_T \to p^* E_1 \oplus p^* E_0 \otimes q^*(V \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \xrightarrow{-p^* \varphi + \gamma} p^* E_0 \to 0.
\]

Now since $E_T$ is $H_\alpha$-semistable, by Theorem 4.2 the restriction of $E_T$ to a smooth curve $C$ in the linear system $|m \cdot H|$ is also semistable for $m \gg 0$. Denoting the restriction of $p$, and $q$ to $C$ by $\tilde{p}$, and $\tilde{q}$, respectively. We remark that $\tilde{p}$ is an affine morphism.

We obtain the short exact sequence
\[
0 \to E_T|_C \to \tilde{p}^* E_1 \oplus \tilde{p}^* E_0 \otimes \tilde{q}^*(V \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \xrightarrow{-\tilde{p}^* \varphi + \gamma} \tilde{p}^* E_0 \to 0.
\]

Now by Theorem 5.1 there exists a vector bundle $F$ on $C$ such that $H^*(X, E_T|_C \otimes F) = 0$. Tensoring the last exact sequence with $F$ gives after a push forward via
\[ \cdots \to 0 \to \tilde{p}_*(E_T|_C \otimes F) \to \tilde{p}_*(\tilde{p}^*E_1 \oplus \tilde{p}^*E_0 \otimes \tilde{q}^*(V \otimes O_{\mathbb{P}^1}(-1)) \otimes F) \sim \cdots \]

The horizontal isomorphisms follow from the projection formula. Thus, defining the triple \( S \) by \( S = (F_1 \xrightarrow{\psi} F_0) := \tilde{p}_*(\tilde{q}^*(V \otimes O_{\mathbb{P}^1}(-1)) \otimes F) \to F \) gives the desired orthogonal triple. The computation that \( S \) is of type \( \alpha^\perp \) is straightforward. \( \square \)

5.5. **Effective bounds for orthogonal triples.** In the proof of Proposition 5.4 we did not used that for the curve \( C \in |m \cdot H_a| \) the number \( m \) can be given explicitly, as well as the rank and the determinant of the vector bundle \( F \) on \( C \). Using these two facts which follow from Theorems 4.2 and 1.2 we obtain:

**Proposition 5.6.** Let \( \alpha = \frac{a}{b} \) be a positive rational number with coprime \( a, b \in \mathbb{N} \), and a line bundle \( L \) on \( X \) of degree one. Then for a holomorphic triple \( T = (E_1 \to E_0) \) of rank \( (r_1, r_0) \) and degree \( (d_1, d_0) \) on \( X \) we have the equivalence

1. \( T \) is \( \alpha \)-semistable.
2. there exists an orthogonal triple \( S = (F_1 \xrightarrow{\psi} F_0) \) of type \( \alpha^\perp \) with
   \[
   \begin{align*}
   \text{rk}(F_0) &= mb(r_0 + r_1)^2, \\
   \text{rk}(F_1) &= 2mb(r_0 + r_1)^2
   \end{align*}
   \]
   det(\( F_0 \)) \cong L^{\otimes d},\ det(\( F_1 \)) \cong L^{\otimes 2d - 2(r_0 + r_1)^2m}

for \( m = \max\{4(r_1d_0 - r_0d_1), \left[ \frac{2g}{a} \right] \} \), and

\[
\begin{align*}
   d &= m(r_0 + r_1)(b(r_0 + r_1)(g - 1) - b(d_0 + d_1) + 2ar_0).
\end{align*}
\]

**Proof.** We have only to show that (1) implies (2). The opposite implication follows from Theorem 5.1. So we assume that \( T \) is \( \alpha \)-semistable. We consider the ample line \( L_{a,b} = p^*L^{\otimes a} \otimes q^*O_{\mathbb{P}^1}(b) \). By Lemma 2.2. The sheaf \( E_T \) on \( X \times \mathbb{P}^1 \) is semistable with respect to the polarization \( L_{a,b} \). We computed the discriminant of \( E_T \) to be \( \Delta(E_T) = 4(r_0d_1 - r_1d_0) \). So our choice of \( m \) guarantees that \( L_{a,b}^{\otimes m} \) is globally generated, and the restriction of \( E_T \) to the restriction of a smooth divisor \( C \) in the associated linear system is also semistable by Theorem 1.2. By Theorem 1.2 there exists a vector bundle \( F \) on \( C \) of rank \( (r_0 + r_1)^2 \) and degree \( \deg(F) = (r_0 + r_1)^2(g - 1) - (r_0 + r_1) \deg(E_T) \). Now we proceed like in the proof of
Proposition 5.4. That is we have $F_0 = p_*(F)$ and $F_1 = p_*(F \otimes q*(V \otimes O_{\mathbb{P}^1}(-1)))$. Indeed, since we can choose the determinant of $F$ arbitrary, we obtain $\det(F_0) = L^\otimes d$.

5.7. A homological view on orthogonal triples. We follow here the notation of Weibel in [11]. Considering two holomorphic triples $T = (E_1 \overset{\varphi}{\to} E_0)$ and $S = (F_1 \overset{\psi}{\to} F_0)$ as chain complexes, then we see that the brutal truncation $\sigma_{<2}(T \otimes S)$ of the tensor product is the complex

$$(E_1 \otimes F_0) \oplus (E_0 \otimes F_1) \overset{\pi}{\longrightarrow} E_0 \otimes F_0$$

we investigate in this Section. Even though the brutal truncation is less common, we can describe the orthogonality by

$$H^*(X, H_1(\sigma_{<2}(T \otimes S))) = 0 \quad \text{and} \quad H_0(\sigma_{<2}(T \otimes S)) = 0.$$ 

When $S$ is of type $\alpha^\perp$ the description becomes

$$H^*(X, H_*(\sigma_{<2}(T \otimes S))) = 0.$$ 

6. The generalized theta line bundle for holomorphic triples

6.1. The theta line bundle on the stack. Let $T = (\mathcal{E}_1 \overset{\varphi}{\to} \mathcal{E}_0)$ be a triple on $X \times S$ with projections $pr_X$ and $pr_S$. For two fixed classes $c_1 = [F_1]$ and $c_0 = [F_0]$ in the Grothendieck group $K(X)$ we define the theta divisor $\mathcal{O}_{S,T}(\Theta_{c_1,c_0})$ to be the determinant of cohomology

$$\mathcal{O}_{S,T}(\Theta_{c_1,c_0}) = \det(R^*pr_{S*}(\mathcal{E}_0 \otimes pr_X^*F_0))) \otimes \det(R^*pr_{S*}(\mathcal{E}_1 \otimes pr_X^*F_1)))^{-1} \otimes \det(R^*pr_{S*}(\mathcal{E}_0 \otimes pr_X^*F_1)))^{-1}.$$

Indeed, this line bundle depends only on the classes $c_i$ of the two bundles $F_i$ in $K(X)$. Using the projection formula we see the following relation between the theta line bundle for $T$ and $pr^*L \otimes T = (pr^*_S L \otimes \mathcal{E}_1 \to pr^*_S L \otimes \mathcal{E}_0)$ for any line bundle $L$ on our base scheme $S$:

$$\mathcal{O}_{S,pr^*L\otimes T}(\Theta_{c_1,c_0}) = L^{\otimes \chi_{0,0} - \chi_{1,0} - \chi_{0,1}} \otimes \mathcal{O}_{S,T}(\Theta_{c_1,c_0})$$

where $\chi_{i,j} = \chi_{X_0}(\mathcal{E}_i \otimes pr_X^*F_j)$ is the Euler characteristic of the vector bundle $\mathcal{E}_i \otimes pr_X^*F_j$ on a fixed fiber $X_0$ of $pr_S$. If we fix $\lambda \in k^*$ and denote the multiplication by $\lambda$ with $m_\lambda : T \to T$, then the induced isomorphism of $\mathcal{O}_{S,T}(\Theta_{c_1,c_0}) \to \mathcal{O}_{S,T}(\Theta_{c_1,c_0})$ is given by multiplication with $\lambda^{\otimes \chi_{0,0} - \chi_{1,0} - \chi_{0,1}}$.

So by [4 Theorem 2.3] the line bundle $\mathcal{O}(\Theta_{c_1,c_0})$ descends to the stable locus of the moduli space of triples, when the number $\chi_{0,0} - \chi_{1,0} - \chi_{0,1}$ is zero. We will assume from now on that $\chi_{0,0} - \chi_{1,0} - \chi_{0,1} = 0$, and that we have an surjection $F_1 \overset{\psi}{\longrightarrow} F_0$.

To such a datum we will construct a theta divisor $\Theta_R$ where $R = (F_1 \overset{\psi}{\longrightarrow} F_0)$.

For any triple $T = (\mathcal{E}_1 \overset{\varphi}{\to} \mathcal{E}_0)$ on $X \times S$ we obtain a surjection

$$\pi : \mathcal{E}_1 \otimes pr^*_SF_0 \oplus \mathcal{E}_0 \otimes pr^*_SF_1 \overset{-\varphi \otimes id_{F_0} + id_{E_0} \otimes \psi}{\longrightarrow} \mathcal{E}_0 \otimes pr^*_SF_0$$

$$\Theta_R = (\mathcal{E}_0 \otimes pr^*_SF_0)$$

Indeed, since we can choose the determinant of $F$ arbitrary, we obtain $\det(F_0) = L^\otimes d$. 

\[\square\]
Now we define $\Theta_R$ to be the theta divisor associated to the vector bundle $\ker(\pi)$ as in the article \cite{4} of Drezet and Narasimhan. This way we obtain a Cartier divisor $\Theta_R$ on the moduli space of stable triples. The closed points of this divisor are give as

$$\Theta_R(k) = \{ S = (E_1 \xrightarrow{\varphi} E_0) | S \text{ is not orthogonal to } R \}.$$  

6.2. Base point freeness. Proposition \cite{5,6} yields a base point free result for these generalized theta divisors. Let $M = M^{c}_{(r_1, r_0, d_1, d_0)}$ be the moduli space of $c$-semistable triples $T = (E_1 \xrightarrow{\varphi} E_0)$ with $\text{rk}(E_i) = r_i$, and $\deg(E_i) = d_i$, then the theta divisor $\Theta_R$ is base point free for $R = (F_1 \xrightarrow{\psi} F_0)$ for sheaves $F_i$ as in Proposition \cite{5,6}.

6.3. A criterion for ampleness. We fix the numerical date $c = (r_0, r_1, d_0, d_1, \alpha)$ and obtain a map from semistable $c$-triples on $X$ to semistable $\bar{c}$-triples on the curve $C \subset X \times \mathbb{P}^1$ in the linear system $|m \cdot H_\alpha|$. Since the theta divisor on the moduli space of $S$-equivalence classes of rank $r_0 + r_1$ bundles on $C$ is ample, we can hope that this also holds for its pull back to the moduli space of $S$-equivalence classes of $c$-triples. However, it turns out that we need an additional condition to ensure this.

**Proposition 6.4.** Let $R = (F_1 \xrightarrow{\psi} F_0)$ be a triple satisfying:

1. $\psi$ is surjective,
2. $\mu(F_0) - \mu(F_1) = \alpha,$
3. $\text{rk}(F_1) = 2\text{rk}(F_0)$, and
4. $\mu(F_0) + \mu_\alpha = g - 1.$

If for all semistable triples $T = (E_1 \rightarrow E_0)$ we have $\text{Hom}(E_0, E_1) = 0$, then the divisor $\Theta_R$ is ample.

**Proof.** The construction of the theta divisor uses the morphism

$$\rho : \{ \text{semistable triples of type } c \text{ on } X \} \rightarrow \{ \text{semistable vector bundles on } C \}$$

where $C$ is a curve in the linear system $|m \cdot H_\alpha|$. Since the theta divisor is ample on the moduli space of semistable bundles on $C$, it suffices to show that $\rho$ is infinitesimal injective. The morphism $\rho$ can be decomposed into three steps:

1. We assign the triple $T = (E_1 \rightarrow E_0)$ on $X$ the short exact sequence $(0 \rightarrow E_0 \otimes \mathcal{O}_X(-2) \rightarrow E_T \rightarrow p^*E_1 \rightarrow 0)$ on $X \times \mathbb{P}^1$.
2. We go from $(0 \rightarrow E_0 \otimes \mathcal{O}_X(-2) \rightarrow E_T \rightarrow p^*E_1 \rightarrow 0)$ to the vector bundle $E_T$ on $X \times \mathbb{P}^1$.
3. We go to the restriction of $E_T$ on the curve $C$.

The first step is an equivalence as we have seen in part \cite{2}. To see that the third step is infinitesimal injective, we remark first that we can choose a curve $C$ in a linear system $|m \cdot H_\alpha|$ for $m \gg 0$. However, if $H^1(X \times \mathbb{P}^1, \text{End}(E_T)(-m)) = 0$, then the tangent map $\text{Ext}^1(E_T, E_T) \rightarrow \text{Ext}^1(E_T|C, E_T|C)$ is injective. Since the
semistable triples form a bounded family we can choose an integer $m$ which works for all triples of the fixed type $c$.

Now study infinitesimal injectivity of step (2). Assume that we have a deformation of the short exact sequence $(0 \to E_0 \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2) \to E_T \to p^*E_1 \to 0)$ over $X \times \mathbb{P}^1 \times \text{Spec}(k[\varepsilon])$ which is constant when we consider only the deformation of $E_T$. This defines a tangent vector in the Quot scheme in the point $[E_T \to p^*E_1]$.

The tangent space $T_{[E_T \to p^*E_1]}$ is given by

$$T_{[E_T \to p^*E_1]} = \text{Hom}(E_0 \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2), p^*E_1) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \otimes \text{Hom}(E_0, E_1).$$

Thus, the condition $\text{Hom}(E_0, E_1) = 0$ forces the infinitesimal injectivity of $\rho$ which gives the assertion. $\square$

7. An example

In this section $X$ denotes a curve of genus $g = 3$. We study the coarse moduli space $M = M_3^g(2,1,1,2)$ of semistable pairs $T = (E_1 \to p^*E_0)$ where

$$\text{rk}(E_1) = 2, \quad \text{deg}(E_1) = 1 \quad \text{rk}(E_0) = 1, \quad \text{deg}(E_0) = 2$$

for $\alpha = \frac{3}{2}$. Checking the few possible numerical types of sub triples we find that

**Proposition 7.1.** For a holomorphic triple $T = (E_1 \to p^*E_0)$ we have the following equivalence:

$$T \text{ is (semi)stable } \iff (1) \quad \varphi \neq 0, \text{ and } (2) \quad \deg(\ker(\varphi)) \leq 0, \text{ and } (3) \quad \mu_{\max}(E_1) \leq 1.$$

We conclude two corollaries:

**Corollary 7.2.** $T \text{ is semistable } \implies \text{Hom}(E_0, E_1) = 0.$

**Corollary 7.3.** $T \text{ is stable } \iff E_1 \text{ is stable, and } \varphi \text{ is surjective. Thus, any stable triple } T \text{ is of type } \alpha^\perp.$

**Proof.** The surjectivity and the stability of $E_1$ is a consequence of Proposition 7.1. The equalities $2\text{rk}(E_0) = \text{rk}(E_1)$ and $\mu(E_0) - \mu(E_1) = \alpha$ are obvious. $\square$

For a fixed triple $S = (F_1 \to p^*F_0) \in M(k)$ we defined the corresponding $\Theta$-divisor by

$$\Theta_S := \left\{(E_1 \to p^*E_0) \in M(k) \mid H^* \left( \ker \left( E_1 \otimes F_0 \oplus E_0 \otimes F_1 \varphi \boxtimes \text{id}_{F_0} \otimes \text{id}_{E_0} \otimes \psi \right) \right) \neq 0 \right\}.$$

In this special case we have the next

**Proposition 7.4.** For any $S \in M(k)$ the generalized theta divisor $\mathcal{O}_M(\Theta_S)$ is ample. The divisor $\Theta_S$ always contains the point $S$. 

Proof. We showed in Proposition 6.4 that Corollary 7.2 implies the ampleness of \( \Theta_S \).

The observation that the point \( S = (F_1 \xrightarrow{\psi} F_0) \) is always contained in \( \Theta_S \) follows from \( H^0(G) \neq 0 \) where \( G \) is the kernel of the morphism

\[
F_1 \otimes F_0 \oplus F_0 \otimes F_1 \xrightarrow{\psi \otimes \text{id}_{F_0} - \text{id}_{F_0} \otimes \psi} F_0 \otimes F_0.
\]

We have \( \mu(F_1 \otimes F_0) = \frac{2}{g} > g - 1 \). Therefore we have \( H^0(F_1 \otimes F_0) \) has a positive dimension. Let \( e \neq 0 \) be any element in \( H^0(F_0 \otimes F_1) \). Let \( \sigma : F_1 \otimes F_0 \xrightarrow{\sim} F_0 \otimes F_1 \) be the interchanging isomorphism. The element \( (e, H^0(\sigma)(e)) \) is a non trivial global section of \( G \).

\( \square \)

Proposition 5.6 implies that \( 216 \cdot \Theta_S \) is globally generated. However, this is far from a good bound.

Lemma 7.5. If the triple \( T = (E_1 \rightarrow E_0) \) is semistable but not stable, then there exists a triple \( S = (F_1 \rightarrow F_0) \in M(k) \) which is orthogonal to \( T \).

Proof. If \( T \) is not stable it contains a proper sub triple \( T' \) with \( \mu_a(T') = 0 \). There are two such possibilities: Either the triple \( T' = T_0 = (L_0 \rightarrow 0) \) with \( \deg(L_0) = 0 \), or we have \( T' = T_1 = (L_2(-Q) \rightarrow L_2) \) with \( \deg(L_2) = 2 \) and \( Q \in X(k) \). Since \( T \) sits in a short exact sequence

\[
0 \rightarrow T_i \rightarrow T \rightarrow T_{1-i} \rightarrow 0
\]

with \( i = 0 \) or \( i = 1 \), it is enough to find a triple \( S \) which is orthogonal to \( T_0 \) and \( T_1 \). Indeed, we can choose \( S = S_0 \oplus S_1 \) as follows:

\( S_0 = (M_0 \rightarrow 0) \) with \( M_0 \) a degree zero line bundle. For any \( M_0 \) we have that \( S_0 \) is orthogonal to \( T_0 = (L_0 \rightarrow 0) \). \( S_0 \) is orthogonal to \( T_1 = (L_2(-P) \rightarrow L_2) \) iff \( H^*(L_2 \otimes M_0) = 0 \). Thus, for a general \( M_0 \) the triple \( S_0 \) is orthogonal to \( T_0 \) and \( T_1 \).

\( S_1 = (M_2(-Q) \rightarrow M_2) \) with \( M_2 \) of degree two. We have that \( (L_0 \rightarrow 0) \) is orthogonal to \( S_1 \) iff \( H^*(L_0 \otimes M_2) = 0 \), again an open condition for \( M_2 \). For the orthogonality of \( S_1 \) to \( T_1 \) we need two things: First of all the morphism \( \pi : L_2(-P) \otimes M_2 \otimes L_2 \otimes M_2(-Q) \rightarrow L_2 \otimes M_2 \) must be surjective. This is the case whenever \( P \neq Q \). Under this condition, the kernel of \( \pi \) is the line bundle \( L_2 \otimes M_2(-P - Q) \). Again for a general \( M_2 \) we have \( H^*(\ker(\pi)) = 0 \).

\( \square \)

Lemma 7.6. If the triple \( T = (E_1 \xrightarrow{\varphi} E_0) \) is stable, then there exists a triple \( S = (F_1 \rightarrow F_0) \in M(k) \) which is orthogonal to \( T \).

Proof. By Corollary 7.3 we know that \( E_1 \) is stable and \( \varphi \) is surjective. Our strategy will be similar to the above proof. We show that \( T \) is orthogonal to a direct sum \( S = S_0 \oplus S_1 \). As before we set \( S_0 = (M_0 \rightarrow 0) \) and \( S_1 = (M_2(-Q) \rightarrow M_2) \).

The orthogonality of \( T \) to \( S_0 \) is equivalent to the vanishing of \( H^*(M_0 \otimes E_0) \) which
is true for a general $M_0$. Next we look for a triple $S_1$ which is orthogonal to $T$. We choose $M_2$ such that:

1. $h^0(M_2 \otimes E_0) = \chi(M_2 \otimes E_0) = 2$,
2. $h^0(M_2 \otimes \ker(\varphi)) = 0$, and
3. $h^0(M_2 \otimes E_1) = \chi(M_2 \otimes E_1) = 1$.

The first two conditions are obviously satisfied for a general $M_2$, and are open conditions on Pic$^2(X)$. For condition (3) we use any surjection $E \to k(P)$. The kernel of $\pi$ is semistable of rank two and degree zero. By Raynaud’s result in [10] for a general $M_2$ we have $H^*(M_2 \otimes \ker(\pi)) = 0$. This implies (3). For a line bundle $M_2$ satisfying (1)–(3) we have that the morphism $\sigma : H^0(M_2 \otimes E_1) \to H^0(M_2 \otimes E_0)$ is injective with image spanned by a global section $s \in H^0(M_2 \otimes E_0)$. Now we take a point $Q \in X(k)$ such that $s$ is not in the image of $H^0(M_2(-Q) \otimes E_0) \to H^0((M_2 \otimes E_0)$. It follows that the morphism

$$H^0(M_2 \otimes E_1) \oplus H^0(M_2(-Q) \otimes E_0) \to H^0((M_2 \otimes E_0)$$

is an isomorphism. Thus, $S_1 = (M(-Q) \to M)$ is orthogonal to $T$. □

References

[1] L. Álvarez-Cónsul: Some results on the moduli space of quiver bundles, Cent. Europ. Jour. of Math. 139(1) (2009) 99–120.
[2] S. B. Bradlow, O. García-Prada: Stable triples, equivariant bundles, and dimensional reduction, Math. Annalen 304(2) (1996) 225–252.
[3] S. B. Bradlow, O. García-Prada, P. Gothen: Moduli spaces of holomorphic triples over compact Riemann surfaces, Math. Annalen 328(1–2) (2004) 299–351.
[4] J.-M. Drezet, M. S. Narasimhan: Groupe de Picard des varits de modules de fibrs semi-stables sur les courbes algbriques, Invent. Math. 97 (1989), no. 1, 5394.
[5] G. Faltings: Stable G-bundles and projective connections, J. Algebraic Geom. 2 (1993), 507–568.
[6] R. Hartshorne: Algebraic Geometry, GTM 52, Springer New York, 1977.
[7] D. Huybrechts and M. Lehn: The geometry of moduli spaces of sheaves, Aspects of Mathematics, E31, Braunschweig 1997.
[8] A. Langer: Semistable sheaves in positive characteristic, Ann. of Math. 159 (2004), 251–276.
[9] M. Popa: Dimension estimates for Hilbert schemes and effective base point freeness on moduli spaces of vector bundles on curves, Duke Math. J. 107 (2001), 469–495.
[10] M. Raynaud: Section des fibrés vectoriels sur une courbe, Bull. soc. math. France 110 (1982), 103-125.
[11] C. Weibel: An Introduction to Homological Algebra, Cambridge Univ. Press, 1994.

E-mail address: georg.hein@uni-due.de
E-mail address: algebraic.geometry.2011@gmail.com

Fakultät für Mathematik, Universität Duisburg-Essen, 45117 Essen, Germany