A New Proof of Monjardet’s Median Theorem

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Dedicated to Bernard Monjardet

Abstract

New proofs are given for Monjardet’s theorem that all strong simple games (i.e., ipsodual elements of the free distributive lattice) can be generated by the median operation. Tighter limits are placed on the number of iterations necessary. Comparison is drawn with the $\chi$ function which also generates all strong simple games.

1 Introduction

In the preceding article, [3] we studied the composition of several games. In such a composition, the voters of one game are replaced by committees each voting according to the rules of the other games. Here, we remove the condition that the committees be disjoint. In fact, adding dummy voters when necessary, we may assume without loss of generality, that each of the committees is in fact the entire set of voters. Thus, in the generalized compound games considered here, the $n$ voters vote $k$ times according to the rules of various games. These $k$ results are combined via a $k$-voter game. This process defines a new game, preserving the strength and/or simplicity of the original games.

As opposed to the situation with disjoint committees, here there is no sense any irreducible strong simple games. Nevertheless, the three-voter democracy $Dem_3$ together with the $n$ dictatorships can be singled out as basic games, since when combined using this generalized composition, they generate all $n$-voter strong simple games. This fact was proven by Monjardet [4, 5]. However, a new proof is given here which makes no reference to games which are not both simple and strong. This new proof sets a tighter bound on the number of compositions required to generate all strong simple games, yet is completely elementary requiring no lattice theory.

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To generate all \( n \)-voter strong games, an additional basic game is needed. It is the winning game
\[
\hat{1} = \{ \text{all subsets of } V \} = (0 \ldots 0).
\]
The notation \([a_1a_2\cdots]/q\) means that each voter \( i \) is given \( a_i \) “votes.” A coalition wins if it attains the quota of \( q \) votes. Any game which can be expressed in this notation is called a quota game.

Similarly, to generate all \( n \)-voter simple games, the losing game \( \hat{0} \) is needed.
\[
\hat{0} = \emptyset = (0 \ldots 0).1
\]
To generate all games, both \( \hat{0} \) and \( \hat{1} \) are needed as additional basic games.

Monjardet’s proof makes use of the well-known identification of \( n \)-voter games with points in the free-distributive lattice on \( n \)-generators \( FD_n \). The generators \( i \) themselves represent dictatorial games in which \( C \) is a winning coalition if and only if the dictator \( i \in C \).

Given three elements of a distributive lattice \( S, T, U \), one defines their median to be the meet of their joins:
\[
m(S, T, U) = ST + SU + TU = (S + T)(S + U)(T + U).
\]
In terms of games, the median law is the composition of the three games with \( Dem_3 \).

\( FD_n \) is a self-dual lattice. (The dual of any given element can be found by expressing it in terms of the generators and exchanging “meet” with “join” throughout.) Many interesting properties of games can be expressed in terms of this duality. In a strong game \( S \) (resp. simple, resp. strong simple), we have \( S \geq S^* \) in \( FD_n \) (resp. \( S \leq S^* \), resp. \( S = S^* \)). In the language of free distributive lattices, this property is called supraduality (resp. infraduality, resp. ipsoduality).complement loses.

It is obvious that if \( S, T, U \in FD_n \) are all ipsodual (resp. infradual, resp. supradual), then so will be their median, since
\[
m(S, T, U)^* = m(S^*, T^*, U^*).
\]
However, less obvious is the result proven by Monjardet that all ipsodual elements can be generated from the \( n \) dictatorial games via repeated application of the median operation.

**Theorem 1 (Monjardet [6, 7])** Let \( FM_n \) be the smallest set containing the generators of \( FD_n \) and closed under the median operation. Given \( S \in FD_n \), \( S \) is ipsodual if and only if \( S \in FM_n \).

In other words, strong simple games can be identified with elements of the free median set. That is to say, all games can be thought of as compositions of triples of simpler games, where the simplest games of all are dictatorial.

See [3] for additional motivation and notation.
2 Quotient Games

Let $S$ be a strong simple game with player set $V$, and let $f : V \to W$ be some function. Define the quotient game $f(S) = \{A \subseteq W : f^{-1}(A) \in S\}$. $V$ can be thought of as a set of offices and $W$ as a set of voters. $f$ describes which offices are held by which voters.

Voters which appear in no minimal winning coalitions are called dummies, otherwise voters are called powerful. If $f$ is non-surjective, then certain voters will hold no office, and thus be dummies. If $f$ is non-injective, then certain voters will combine the functions of several offices. A single vote of a voter is taken into account as the vote of each of his offices. If $f$ is bijective, then $f(S)$ is isomorphic to $S$.

Note that the minimal winning coalitions of $f(S)$ are the images of certain minimal winning coalitions of $S$.

If $S$ is a quota game, then $f(S)$ will also be a quota game in which the weight of each voter is equal to the sum of the weights of his offices.

Our first proof of Theorem 1 depends on the following two lemmata.

Lemma 1 Let $S$ be a game with voter set of $V$, and let $f : V \to W$ be some function. Then $f(S)$ is a game on $W$. If $S$ is strong (resp. simple), then $f(S)$ is too.\[\square\]

Proof: $f^{-1}$ is a monotone function from $2^W$ to $2^V$, thus $f(S)$ is a game. Suppose $S$ is simple, and $B \in f(S)$. Then $f^{-1}(B) \in S$. Hence, $V - f^{-1}(B) = f^{-1}(W - B) \notin S$. Strength follows as $W - B \notin f(S)$.\[\square\]

Lemma 2 All nondictatorial strong simple games are the medians of strong simple games with a strictly greater number of dummies.

Proof: Let $x, y, z$ be distinct powerful voters. Let $f_{ij} : V \to V$ fix all members of $V$ with the exception of $i$ for which $f_{ij}(i) = j$. We will show that

$$S = m(f_{xy}(S), f_{yz}(S), f_{zx}(S)).$$

If so, then we have expressed $S$ in terms of strong simple games involving at most $n - 1$ powerful voters.

Let $T$ denote the right hand side of equation (1). Since $|S| = |T| = 2^{n-1}$, it will suffice to show that $S \subseteq T$. Suppose, $A \in S$. Consider then the possible values of $|A \cap \{x, y, z\}|$

0. If $x, y, z \notin A$. Then $A \in f_{xy}(S), f_{yz}(S), f_{zx}(S)$. Thus, $A \in T$.

1. Without loss of generality, suppose $x \in A$, and $y, z \notin A$. Then $A \in f_{yz}(S), f_{zx}(S)$, so $A \in T$.

2. Without loss of generality, suppose $x, y \in A$, and $z \notin A$. Then $A \in f_{xy}(S), f_{zx}(S)$, so $A \in T$.

3. If $x, y, z \in A$. Then $A \in f_{xy}(S), f_{yz}(S), f_{zx}(S)$. Thus, $A \in T$.\[\square\]
Proof of Theorem 1: Since the number of dummies is bounded by \( n - 1 \), Monjardet’s theorem now follows by induction. QED.

We have the following result more generally.

**Theorem 2** Let \( S \) be a game (resp. strong game, resp. simple game, resp. strong simple game) with at least two powerful voters. Then \( S \) is the median of three games (resp. strong games, resp. simple games, resp. strong simple games) with a strictly greater number of dummies.

**Proof:** If \( S \) is a strong simple game, then the lemma above suffices since all nondictatorial games have at least two powerful voters.

If \( S \) has three or more powerful voters, then the above reasoning is still valid.

If \( S \) has exactly two powerful voters (say \( x \) and \( y \)) then \( S \) must either be the simple majority game \( xy = (1,1,0,0,\ldots)_2 \) or the strong majority game \( x + y = (1,1,0,0,\ldots)_1 \). In the game \( xy \), the participation of both \( x \) and \( y \) are needed to win, whereas in \( x + y \) the participation of either is sufficient. Notice that \( xy = m(\text{Dict}_x, \text{Dict}_y, 1) \) and \( x + y = n(\text{Dict}_x, \text{Dict}_y, 0) \) where \( \text{Dict}_x \) and \( \text{Dict}_y \) denote dictatorship by \( x \) and \( y \) respectively. (Note that in the games \( \hat{0} \) and \( \hat{1} \), all voters are dummies.\( \square \)

**Corollary 1** All games (resp. strong games, resp. simple games) can be generated via generalized composition of various dictatorships, and the three-voter democracy \( \text{Dem}_3 = (1,1,1)_2 \) along with \( \hat{0} \) and \( \hat{1} \) (resp. \( \hat{0} \) alone, resp. \( \hat{1} \) alone).

**Proof:** By induction, all games can be reduced via \( \text{Dem}_3 \) to games of the same “type” having only 0 or 1 powerful voters. The only games with 1 powerful voter are the dictatorships. The only games with no powerful voters are the predetermined win \( \hat{1} \) and the predetermined loss \( \hat{0} \). The former is strong and the latter is simple.\( \square \)

Note that the dictatorships along with \( \hat{0}, \hat{1}, \text{or} = (1,1)_1 \), and \( \text{and} = (1,1)_2 \) also forms a minimal basis from which all games can be constructed. In other words, all monotonic boolean functions can be written as a for example conjunction of disjunctions.

### 3 Weight

Define the weight of a strong simple game \( S \), to be the number of times the median operation must be iterated in order to generate \( S \) from dictatorships. In other words, a game \( S \) is of weight 0 if it is a dictatorship. For \( k > 0 \), \( S \) is of weight \( k \) if it is the median of three strong simple games of weight \( k - 1 \) or less, but not the median of three strong simple games of weight \( k - 2 \) or less.

Define \( \text{Dem}_3^0 \) to be the unique one-voter strong simple game \( \text{Dem}_1 \), and \( \text{Dem}_3^{d+1} = \text{Dem}_3[\text{Dem}_3^d, \text{Dem}_3^d, \text{Dem}_3^d] \). \( \text{Dem}_3^d \) is clearly a transitive strong simple game on 3\(^d\) voters. A strong simple game has weight at most \( d \) if and only if it is quotient of \( \text{Dem}_3^d \).

Let \( W(n) \) be the largest weight of an \( n \)-voter strong simple game.
Proposition 1  (1) For \( n \leq 6 \), \( W(n) \) is given by the following table.

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 |
|---------|---|---|---|---|---|---|
| \( W(n) \) | 0 | 0 | 1 | 2 | 3 | 3 |

(2) For \( n \geq 6 \), \( W(n) \leq n - 3 \).

(3) \( W(n) \) is asymptotically greater than \( \frac{\ln^2 n}{\ln 3} \).

(4) \( W(n) \) is weakly increasing, restricted growth function. i.e. \( W(n+1) \) always equals either \( W(n) \) or \( W(n) + 1 \).

Proof: (1) See above.

(2) By the proof of theorem \[ \] above, the weight of a strong simple game on \( n \) voters is no more than one greater than the largest such weight on \( n - 1 \) voters.

(3) Every \( n \)-voter strong simple game is a quotient of \( \text{Dem}_3^{W(n)} \). There are \( n^{W(n)} \) such quotients. Whereas, (see \[ \]) for \( n \) odd, the number of strong simple games is asymptotically

\[
2^{(n-1)/2} \exp \left( \frac{n-1}{(n-1)/2} \left( 2^{-(n-1)/2} + 3n^{2-2^{-n-4}} - n^{2^{-n-2}} \right) + \right)
\]

and for \( n \) even,

\[
2^{(n-1)/2} \exp \left( \frac{n-1}{n/2-1} \left( 2^{-n/2-1} + n^{2^{-n-4}} \right) + \frac{n}{n/2+1} \left( 2^{-n/2-1} + n^{2^{-n-5}} - n^{2^{-n-4}} \right) \right).
\]

In either case, taking the double \( \ln \) of these asymptotic formulas yields

\[
W(n) \geq \frac{n \ln 2}{\ln 3} - \frac{\ln \ln n}{\ln 3} - \frac{\ln ((\ln 2)^2/2\pi)}{\ln 9} + O(n^{-1}).
\]

(4) \( W(n) \) is a restricted growth function by the proof of theorem \[ \]. Now, let \( S \) be an \( n \)-voter strong simple game of weight \( W(n) \). Let \( \iota(S) = S \cup \{ A \cup \{ n+1 \} : A \in S \} \) be the corresponding \( n+1 \)-voter game in which the additional voter is powerless. By hypothesis, \( \iota(S) \) is a quotient of \( \text{Dem}_3^{W(n+1)} \). However, \( S = \iota(S)_{n,n+1} \) is a quotient of \( \iota(S) \). Thus, \( S \) is a quotient of \( \text{Dem}_3^{W(n+1)} \). Hence, \( W(n+1) \geq W(n) \). \( \square \)

4 Choice Function

A finite multi-player deterministic sequential-move perfect-knowledge game can be represented by a labeled rooted tree. The label of an internal node indicates which player must move. His move consists of the selection of a child of that node. The label of a leaf indicates the winner of the game. Without loss of generality, we can consider only choice binary trees, since choices between a large number of possibilities can be made via iterated binary choices.
The win-type of a game-tree is the set of winning coalitions. A coalition is said to be winning if it has a combined strategy that ensures that the winner of the game will be one of its members.

Let \( \tau \) be the game-tree \((\tau_1 \leftarrow a \tau_2)\), where \( \tau_1 \) and \( \tau_2 \) are sub-game-trees with win-types \( S_1 \) and \( S_2 \) respectively, and \( a \in V \) is a player. What is the win-type \( S \) of the game \( \tau \)?

Obviously, \( A \subseteq V \) wins \( \tau \) if it wins both \( \tau_1 \) and \( \tau_2 \). Moreover, if \( a \in A \subseteq V \), then \( A \) can win \( \tau \) even if it wins only one of \( \tau_1 \) and \( \tau_2 \). Thus, \( S = (S_1 \cap S_2 \cup \{A \in S_1 \cup S_2 : a \in A\}\) We will denote this combination of \( S_1 \) and \( S_2 \) by \( \chi_a(S_1, S_2) \) and call it the choice by \( a \) between \( S_1 \) and \( S_2 \).

**Proposition 2**

1. The win-type of a game-tree is a strong simple game.
2. Conversely, all strong simple games are the win-type of some game-tree.

**Proof:**

(1) By induction on game-trees.

If the height is zero, then the tree is trivial. It contains one node. The game is an automatic win for some player. A coalition is winning if and only if it contains this player. The win-type is a dictatorship by this player.

Otherwise, \( \tau \) is of the form \( \chi_a(S_1, S_2) \) where \( S_1 \) and \( S_2 \) are strong simple games. Let \( a \in A \) and \( B = X \setminus A \). If \( A \notin \chi_a(S_1, S_2) \), then \( A \notin S_1 \) and \( A \notin S_2 \). Thus, \( B \in S_1 \) and \( B \in S_2 \). Thus, \( B \in \chi_a(S_1, S_2) \). Conversely, if \( A \in \chi_a(F,S_2) \), then \( A \in S_1 \) or \( A \in S_2 \). In either case, \( B \notin \chi_a(S_1, S_2) \).

(2) Consider the following game. Players take turns eliminating all but one of the coalitions in \( S \) to which they belong (if any). After each player has taken his turn there will be one set left. The first member of this set wins.\( \square \)

Thus, the functions \( \chi_a \) together with the dictatorships can be used to generate all strong simple games.

**Alternate Proof of Theorem 1:** Notice that \( \chi_a(S, T) = m(S, T, \text{Dict}_a) \);

\[
\begin{align*}
m(S, T, \text{Dict}_a) &= (S \cap T) \cup (S \cap \text{Dict}_a) \cup (T \cap \text{Dict}_a) \\
&= (S \cap T) \cup \{A : a \in A \in S\} \cup \{A : a \in A \in T\} \\
&= \chi_a(S, T).
\end{align*}
\]

Since \( \chi \) generates all strong simple games, a fortiori \( m \) does.\( \square \)

**5 Depth**

In analogy to Monjardet’s definition of weight, define the depth of a strong simple game \( S \), to be the number of times the choice functions must be iterated in order to generate \( S \) from dictatorships. In other words, a game \( S \) is of depth 0 if it is a dictatorship. \( S \) is of depth \( k > 0 \) if it is the choice by some player between two
strong simple games of depth $k-1$ or less, but not between two strong simple games of depth $k-2$ or less.

$S$ is of depth $k$ if it is the win-type of a game-tree of height $k$ and no such game-tree of lesser height. Let $B_k$ be the complete binary tree of height $k$. Label the nodes of $B_k$ consecutively with the integers 1 to $2^{n+1} - 1$. 

\[
B_0 = (1), \quad B_1 = \left( \frac{1}{2} \right) \quad \text{and} \quad B_2 = \left( \begin{array}{c}
\frac{1}{2} \\
\frac{3}{4} \\
\frac{5}{6} \\
\frac{7}{8}
\end{array} \right)
\]

$S$ is of depth $k$ if and only if it is a quotient of the $2^{n+1} - 1$ voter strong simple game $B_k$.

Let $D(n)$ be the largest depth of an $n$-voter strong simple game.

\begin{proposition}

(1) The depth of any strong simple game is at least as great as its weight. Thus, $D(n) \geq W(n)$.

(2) $D(n)$ is nondecreasing.

(3) $D(n)$ is asymptotically greater than $n$.

(4) $D(n)$ is asymptotically less than $\pi^2 n^3 / 72$.

(5) Let $S$ be a strong simple game of weight $w$. $S$ has depth at most $2^w - 1$. This bound can not be improved in general.
\end{proposition}

Proof: (1) $\chi_a(S, T) = m(S, T, Dict_a)$.

(2) Let $S$ be an $n$-voter strong simple game of depth $D(n)$. Let $\iota(S)$ be the corresponding $n+1$-voter game in which the additional voter is powerless. Let $\iota(S)$ be the win-type of an $n+1$-player game-tree of depth $d$. Since the additional player is powerless, we can arbitrarily relabel nodes of the game-tree to form an $n$-player game-tree of win-type $S$. Thus, $D(n+1) \leq d \leq D(n)$.

(3) There are $n^{2^d+1} - 1$ ways to label $B_d$ with $n$ players. Thus, there are at most $n^{2^d+1} - 1$ $n$-player strong simple games of depth $d$ or less. Compare with \[8\].

(4) Proof of theorem \[2\].

(5) Let $\tau$ (resp. $\tau'$, $\tau''$) be the game-tree of $\text{Dem}_3^{w-1}$ with voters labelled 1 to $3^{w-1}$ (resp. $3^{w-1} + 1$ to $2 \times 3^{w-1}$, $2 \times 3^{w-1}$ to $3^{w}$). Place a copy of $\tau'$ and $\tau''$ under each leaf of $\tau$. The resulting tree is the game-tree of $\text{Dem}_3^w$. By induction, the height of the tree is $2^w - 1$.

No smaller tree can represent $\text{Dem}_3^w$ since every vertical chain in a game-tree gives rise to a winning coalition and $\text{Dem}_3^w$ has no winning coalitions smaller than $2^w$. \[\Box\]

Given a strong simple game $S$, and two voters $x, y \in V$. Say that $x$ is at least as influential as $y$, written $x \geq y$ or $x \geq_S y$ if for all $A \in S$, we have $(A - \{y\}) \cup \{x\} \in S$.

\begin{proposition}

Influence is a pre-order on the set of voters.
\end{proposition}
Proof: (Reflexivity) Let \( x \in A \in S \). Then \((A - \{x\}) \cup \{x\} = A \in S\).

(Transitivity) Let \( x \leq y \leq z \). Let \( x \in A \in S \). Then \( y \in B = (A - \{x\}) \cup \{y\} \in S \). Hence, \( C = (B - \{y\}) \cup \{z\} \in S \). Note that \( C = (A - \{x\}) \cup \{z\} \).

If \( S \) is a quota game, then the influence relation is total. \( w_x \geq w_y \) implies \( x \geq y \).

In the proof of lemma\(^3\), we defined \( f_{xy}(S) \). Informally, \( S_{xy} = f_{xy}(S) \) is the voting scheme in which \( x \) “leaves the room” having left instructions to vote according to \( y \). Thus,

\[
S_{xy} = \{ A : y \notin A \text{ and } x \in A \} \in S_{x/y} \text{ for all } x/y.
\]

Similarly, we can imagine a voting scheme \( S_{x/y} \) in which \( x \) “leaves the room” having left instructions to vote against \( y \). Thus,

\[
S_{x/y} = \{ A : y \in A \text{ and } x \notin A \} \in S_{x/y} \text{ for all } x/y.
\]

Proposition 5 Let \( S \) be a game. \( S_{x/y} \) is a game if and only if \( x \) is less influential than \( y \), \( x \leq_S y \).

Proof: \( S_{x/y} \) is a game if and only if whenever \( A \in S_{x/y} \), we have \( A \cup \{a\} \in S_{x/y} \) for all \( a \in V \).

Suppose \( A \in S_{x/y} \). If \( a \neq y \), then \( A \cup \{a\} \in S_{x/y} \), since \( S \) is a game.

On the other hand, let \( a = y \). The condition \( A \cup \{y\} \in S_{x/y} \) is of interest only if \( y \notin A \). In that case, it reduces to the question of whether \( A \cup \{x\} \in S \) implies \( A \cup \{y\} \in S \). In other words, whether \( x \leq_S y \).

Let \( S = (w_1, w_2, \ldots, w_n)_q \) be a quota game. Without loss of generality, suppose \( w_1 \geq w_n \). Then

\[
S_{n/1} = \left( w_1 + w_n, w_2, w_3, \ldots, w_{n-1}, 0 \right)_q \quad \text{and} \quad S_{n/1} = \left( w_1 - w_n, w_2, w_3, \ldots, w_{n-1}, 0 \right)_{q-w_n}
\]

are both quota games. If \( S \) was a homogeneous or efficient quotient \( S_{xy} = f_{xy}(S) \) played in median decompositions (lemma\(^3\)).

The opposition \( S_{x/y} \) plays the same role in choice function \( \chi \) decompositions as the quotient \( S_{xy} = f_{xy}(S) \) played in median decompositions (lemma\(^3\)).

Proposition 6 Let \( S \) be a strong simple game, and suppose \( i \leq_S j \). Then

\[
S = \chi_i(S_{ij}, S_{ij}).
\]

Proof: Say \( A \) needs \( i \) if \( A \in S \) and \( A - \{i\} \notin S \). Clearly, if \( A \in S \) does not need \( i \), then \( A \in S_{ij} \) and \( A \in S_{ij} \). Thus, \( A \in \chi_i(S_{ij}, S_{ij}) \).

If \( A \in S \) needs \( i \), then either \( j \in A \) in which case \( A \in S_{ij} \) or \( j \notin A \) in which case \( A \in S_{ij} \). In either case, \( A \in \chi_i(S_{ij}, S_{ij}) \).

In formally, this means that \( S \) is a “choice” by player \( i \) of whether to join forces with player \( j \) or to oppose him.

\(^1\)Nevertheless, the influence relation on \( S_{6,23} \) is total \( 1 > 2 > 3 > 4 > 5 > 6 \), yet \( S_{6,23} \) is not a quota game. See §A.
Corollary 2  If $S$ is a $n$-voter quota game ($n \geq 3$) then $S$ has weight at most $n - 2$.

Proof: By induction, and equations (3) and (4). □

[2, Lemma 2] can be thought of as the converse of corollary 2 since it uses methods similar to equations (2) and (1) to enumerate all quota games.

Note however that for some strong simple games, no pair of distinct voters are comparable via influence. The only example with less than seven vertices is the icosahedral game $I = S_{6,30}$: Associate each of six voters with a pair of opposite vertices on an icosahedron. A coalition is winning if and only if it contains a face. (See §A.)

Such games prevent us from generalizing the reasoning of corollary 2.

A Classification of strong simple games

In Table 1, we classify strong simple games according to their height and weight. Table 1 includes all 30 isomorphism classes of strong simple games with up to six powerful voters, as well as those with a transitive automorphism group and up to eight powerful voters, and those of weight or height up to two. Each isomorphism class is listed only once.

Games $S$ are grouped according to their number $n$ of powerful voters as indicated in the first column. The second column provides the weights of a quota game if applicable, or otherwise identifies the game. The remaining columns give the weight $w$ and depth $d$ of the game along with some optimal decompositions. Those case where our methods do not give (or are not known to give) optimal decompositions are marked * (or *?). A question mark appears when optimality of $d$ or $w$ is uncertain. A ♥ appears in the first column if the game has a transitive automorphism group.

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2Proposition 2 does not apply to games whose influence relation is trivial such as the Icosahedral game $I$.

3The players in the Fano game are identified with points in the projective plane over the field $Z_2$. A coalition wins if it contains a line. Proposition 1 does not apply.
### Table 1: Classification of Strong Simple Games

| n | S | w | m-Median decomposition | d | χ-Choice decomposition |
|---|---|---|------------------------|---|-----------------------|
| 1 | (1) | 0 | 0 | 0 | 0 |
| 3 | (111) | 1 | m(100|.(010), (001) | 1 | χ((100), (010)) |
| 4 | (211) | 2 | m((110), (110), (101)) | 2 | χ((110), (110)) |
| 5 | (221) | 2 | m((110), (110), (101)) | 2 | χ((110), (110)) |
| 6 | (321) | 2 | m((110), (110), (101)) | 2 | χ((110), (110)) |
| 7 | (331) | 3 | m((2011), (2011), (0121)) | 3 | χ((011), (011)) |
| 8 | (4111) | 3 | m((10000), (10011), (00010)) | 3 | χ((10000), (11110)) |
| 9 | (5221) | 3 | m((10000), (10011), (00010)) | 3 | χ((10000), (11110)) |
| 10 | (4321) | 3 | m((10000), (10011), (00010)) | 3 | χ((10000), (11110)) |
| 11 | (5321) | 3 | m((10000), (10011), (00010)) | 3 | χ((10000), (11110)) |
| 12 | (6221) | 3 | m((10000), (10011), (00010)) | 3 | χ((10000), (11110)) |
| 13 | (6321) | 3 | m((10000), (10011), (00010)) | 3 | χ((10000), (11110)) |

**Note:** The table lists the classification of strong simple games based on their median and choice decompositions.