CRITICAL BEHAVIOUR OF THE 3D GROSS-NEVEU 
AND HIGGS-YUKAWA MODELS

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Abstract

We measure the critical exponents of the three dimensional Gross-Neveu model with 
two four-component fermions. The exponents are inferred from the scaling behaviour 
of observables on lattice sizes $8^3$, $12^3$, $16^3$, $24^3$, and $32^3$. We find that the model has a 
second order phase transition with $\nu = 1.00(4)$ and $2 - \eta = \gamma/\nu = 1.246(8)$. We also 
calculate these exponents, through a second order $\epsilon$-expansion around four dimensions, 
for the three dimensional Higgs-Yukawa model, which is expected to be in the same 
universality class, and obtain $\gamma/\nu = 1.237$ and $\nu = 0.948$, while recent second order 
$1/N_f$-expansion calculations give $\gamma/\nu = 1.256$ and $\nu = 0.903$. We conclude that the 
equivalence of the two models remains valid in 3 dimensions at fixed small $N_f$ values.
1 Introduction

The Gross-Neveu (GN) model describes fermions with four-fermion interaction \[1, 2\]. It has a global discrete chiral symmetry, which can break down spontaneously to form a chiral condensate. This can be seen as a composite scalar particle that gives a non-zero mass for the fermions. Due to its simplicity, the GN model has been studied extensively. In two dimensions it is perturbatively renormalisable and asymptotically free. In addition, the chiral symmetry is broken.

In three dimensions (3d) it is renormalisable if the number of flavours, \(N_f\), is large enough. Hence, it is also the first renormalisable model known not to be perturbatively renormalisable. It has been proven to be renormalisable and non-trivial in dimensions between 2 and 4 by means of a \(1/N_f\)-expansion \[2\].

It has been suggested that a model with four-fermion interactions, which leads to a spontaneously broken global chiral symmetry with a chiral condensate, could be a candidate for the Higgs particle of the standard model \[3\]. Near four dimensions (4d) it has been related to Higgs-Yukawa (HY) type models \[4, 5\]. In fact, in 4d both the standard electroweak and the GN model (with certain modifications) are trivial and can be mapped onto each other \[4\] and in dimensions \(2 \leq d \leq 4\) the GN model and the HY model are equivalent in the framework of \(1/N_f\)-expansion \[5\].

The purpose of this work is to enlighten the connection between the composite and fundamental Higgs scenarios in a case where the models are not trivial, namely in 3d. The \(1/N_f\) expansions for the HY model and the GN model are order by order equivalent \[5\]. In order to go beyond the \(1/N_f\) perturbation theory, we ask whether this equivalence persists at small \(N_f\)’s. To answer this we study the critical properties of GN model at small \(N_f\) by means of a Monte Carlo (MC) simulation. To preserve discrete chiral symmetry, the restoration of which we are interested in, we use staggered fermions on the lattice and choose the smallest possible value of \(N_f\), that is \(N_f = 2\). A finite-size analysis of the numerical simulation gives the critical coupling and critical exponent values to be compared to the exponents of the Higgs-Yukawa model. In order to obtain a meaningful comparison, one has to go beyond the order \(\epsilon\) results in the HY model \[5\]. We have therefore extended the calculation and obtain the critical exponents to order \(\epsilon^2\). The relevance of the comparison can furthermore be evaluated by using recent results from a \(1/N_f\)-expansion of the GN model \[6, 7, 8, 9, 10\] to order \(1/N_f^2\).

The paper is divided into two major parts: section 2 is devoted to the analytical results of the Gross-Neveu and Higgs-Yukawa models with the \(\epsilon\)-expansion while in section 3 we describe the Monte Carlo runs performed. Section 4 is devoted to the comparison of numerical and analytical results. Within the uncertainties associated on the one hand with the statistics of the numerical simulations and on the other hand with the still short series expansion in \(\epsilon\) and \(1/N_f\), we confirm the equivalence of the two 3d models at small \(N_f\).
2 Analytical Results

Here we recall a few known properties of the models under study, and present our new analytical calculations, in particular the fixed $N_f \epsilon^2$ expansion of the HY model exponents. The $1/N_f$ expansions of the latter are then compared with the $1/N_f$ expansions of the GN model.

2.1 Continuum Gross-Neveu model

The continuum GN model with $N_f$ fermion flavours is defined by the Lagrangian

$$\mathcal{L} = \sum_{\alpha=1}^{N_f} \bar{\psi}^\alpha(x) \left[ \partial \psi^\alpha(x) + \frac{g^2}{2} \left( \sum_{\alpha=1}^{N_f} \bar{\psi}^\alpha(x) \psi^\alpha(x) \right)^2 \right].$$  \hspace{1cm} (1)

Usually an auxiliary scalar field $\sigma$ is introduced

$$\mathcal{L} = \sum_{\alpha=1}^{N_f} \bar{\psi}^\alpha(x) \left[ \partial - g\sigma(x) \right] \psi^\alpha(x) - \frac{1}{2} \sigma^2(x),$$  \hspace{1cm} (2)

which is formally equivalent to (1) upon integration over $\sigma$ field.

The GN model has a discrete chiral symmetry

$$\psi \rightarrow \gamma_5 \psi, \quad \bar{\psi} \rightarrow -\bar{\psi} \gamma_5, \quad \sigma \rightarrow -\sigma,$$  \hspace{1cm} (3)

which in 3d is spontaneously broken at small couplings.

The critical exponents for $d = 3$ to order $1/N_f^2$ are $[5, 10, 11, 11]$

$$\frac{1}{\nu} = 1 - \frac{32}{3\pi^2 N} + \frac{64(27\pi^2 + 632)}{27\pi^4 N^2},$$  \hspace{1cm} (4)

$$\frac{\gamma}{\nu} = 1 + \frac{64}{3\pi^2 N} + \frac{64(27\pi^2 - 304)}{27\pi^4 N^2},$$  \hspace{1cm} (5)

where $N = N_f \text{ Tr } 1$ is the total number of fermionic variables.

2.2 The discretised Gross-Neveu model

We consider the GN model defined on a 3d symmetric lattice. The discretisation of the continuum Lagrangian has to be implemented in such a way as to reproduce the correct symmetries in the continuum limit $[11]$. We use staggered fermions, which means that in 3d there are 8 doublers. Using 4 component spinors, we can assign the components to the corners of the cube, leaving $8/4 = 2$ continuum flavours from 1 staggered lattice fermion per site. The discretised action reads

$$S = N_h \sum_n \frac{c_n^2}{2\lambda} + \sum_{n,m} \sum_{\alpha=1}^{N_L} \chi^\alpha_n (D_{nm} + \Sigma_{nm}) \chi^\alpha_m.$$  \hspace{1cm} (6)
The staggered fermion matrix $D$ is given by

$$D_{nm} = \frac{1}{2} \sum_{j} \eta_{n,j} (\delta_{n,m+j} - \delta_{n,m-j}),$$

(7)

where the sum $j$ is over the directions ($j = 1,2,3$) and $\eta_{n,j}$ is the staggered fermion phase factor

$$\eta_{n,j} = (-1)^{n_1 + \ldots + n_j - 1}.$$  

(8)

with periodic boundary condition in $d-1$ dimensions and antiperiodic in the last one for a finite lattice.

The mass matrix $\Sigma$ is diagonal

$$\Sigma_{nm} = \bar{\sigma} \delta_{nm}$$

(9)

and depends on $\sigma$ field. We choose a discretisation in which $\bar{\sigma}$ is the average of $\sigma$ at the six nearest neighbours of the lattice site $n$.

The coupling $\lambda$ is connected to the continuum coupling $g$ by $\lambda = g^2 N_f$, with $N_f = 2N_L$ as explained.

One may integrate over the Grassmann variables $\chi^\alpha_x, \bar{\chi}^\alpha_x$ to express the partition function in terms of an effective action

$$S_{\text{eff}} = N_L \left[ \sum_x \frac{\sigma_x^2}{2\lambda} - \text{Tr} \ln(D + \Sigma) \right].$$

(10)

The $N_L = \infty$ critical value $\lambda^0_c$, where chiral symmetry is restored, can be obtained from the saddle point equation at $\sigma = 0$ [6] as

$$\lambda^0_c = 0.989.$$  

(11)

Taking into account the quadratic fluctuations [12] we can obtain the one loop value of $\lambda_c$ including $1/N_L$ corrections. As we are interested to rather small $N_L$ values, we choose to solve the gap equation of the one-loop effective potential. Solving this gap equation on the lattice with linear sizes $L = 32$ and 64 we estimate

$$\lambda^1_c(N_L = 1) = 0.800$$

(12)

$$\lambda^1_c(N_L = 6) = 0.938$$

(13)

This last value agrees with a direct calculation of the perturbative correction in the infinite lattice limit.

### 2.3 The Higgs-Yukawa model

The Lagrangian $\mathcal{L}'$ of the HY model has a fourth order interaction term and a kinetic term added for the $\sigma$ field,

$$\mathcal{L}' = \sum_{\alpha=1}^{N_f} \bar{\psi}^\alpha (\not{\partial} + g\sigma) \psi^\alpha + \frac{1}{2} m^2 \sigma^2 + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{\lambda_4}{4!} \sigma^4.$$  

(14)
This model becomes renormalisable in four dimensions. It has been argued that the terms added are irrelevant for the number of dimensions less than 4 and marginal at d=4. Hence, the critical behaviour of HY model and GN model should be identical.

Due to its renormalisability, the HY model can be studied by means of an $\epsilon$-expansion. The two-loop $\beta$ functions and anomalous dimension $\eta_\sigma$ can be obtained from [13] and we have computed the mass anomalous dimension $\eta_m$.

\[
\beta_{\lambda_4} = -\epsilon \lambda_4 + \frac{1}{(4\pi)^2} (3\lambda_4^2 + 2N\lambda_4 g^2 - 12Ng^4) + \frac{1}{(4\pi)^4} (-\frac{17}{3} \lambda_4^3 - 3N\lambda_4^2 g^2 + 7N\lambda_4 g^4 + 96Ng^6),
\]

\[
\beta_g = -\frac{\epsilon}{2} g + \frac{1}{(4\pi)^2} (N/2 + 3) g^3 + \frac{1}{(4\pi)^4} (-\frac{9 + 12N}{4} g^5 - 2\lambda_4 g^3 + \frac{1}{12} \lambda_4^2 g),
\]

\[
\eta_\sigma = \frac{1}{(4\pi)^2} Ng^2 + \frac{1}{(4\pi)^4} (\frac{1}{6} \lambda_4^2 - \frac{5}{2} Ng^4),
\]

\[
\eta_m = -\frac{1}{(4\pi)^2} \lambda_4 + \frac{1}{(4\pi)^4} (\lambda_4^2 + \lambda_4 Ng^2 - 2Ng^4) - \eta_\sigma,
\]

with $d = 4 - \epsilon$ and $N = N_f \text{ Tr } 1$.

The corresponding fix points to order $\epsilon^2$ are

\[
\frac{g^*}{(4\pi)^2} = \frac{1}{(N + 6)} \epsilon + \frac{(N + 66)\sqrt{N^2 + 132N + 36} - N^2 + 516N + 882}{108(N + 6)^3} \epsilon^2
\]

\[
\frac{\lambda_4^*}{(4\pi)^2} = \frac{-N + 6 + \sqrt{N^2 + 132N + 36}}{6(N + 6)} \epsilon + \left[ -\sqrt{N^2 + 132N + 36} (3N^3 - 43N^2 - 1545N - 1224) + 3N^4 + 155N^3 + 2745N^2 - 2538N + 7344 \right] \frac{1}{54(N + 6)^3\sqrt{N^2 + 132N + 36}} \epsilon^2.
\]

The anomalous dimensions at these fix points give the critical exponents $1/\nu = 2 + \eta_m(g^*, \lambda_4^*)$ and $\gamma/\nu = 2 - \eta_\sigma(g^*, \lambda_4^*)$

\[
\frac{1}{\nu} = 2 - \frac{5N + 6 + \sqrt{N^2 + 132N + 36}}{6(N + 6)} \epsilon - \left[ \frac{\sqrt{N^2 + 132N + 36} (3N^3 + 109N^2 + 510N + 684)}{54(N + 6)^3\sqrt{N^2 + 132N + 36}} - 3N^4 - 658N^3 - 333N^2 - 15174N + 4104 \right] \epsilon^2,
\]

\[
\frac{\gamma}{\nu} = 2 - \frac{N}{N + 6} \epsilon - \frac{(11N + 6)\sqrt{N^2 + 132N + 36} + 52N^2 - 57N + 36}{18(N + 6)^3} \epsilon^2.
\]
2.4 Gross-Neveu and Higgs-Yukawa Comparison

The $\epsilon = 4 - d$ expansions of the Gross-Neveu exponents \[8, 9, 10\] are

\[
\frac{1}{\nu}|_{\text{GN}} = 2 - \epsilon + (-6\epsilon + \frac{13}{2} \epsilon^2 - \frac{3}{8} \epsilon^3 + \cdots) \frac{1}{N} \\
+ (396\epsilon - \frac{1125}{2} \epsilon^2 - \frac{1140\zeta(3) - 401}{8} \epsilon^3 + \cdots) \frac{1}{N^2},
\]

(23)

\[
\frac{\gamma}{\nu}|_{\text{GN}} = 2 - \epsilon + (6\epsilon - \frac{7}{2} \epsilon^2 - \frac{11}{8} \epsilon^3 + \cdots) \frac{1}{N} \\
+ (-36\epsilon + \frac{51}{2} \epsilon^2 + \frac{192\zeta(3) + 281}{8} \epsilon^3 + \cdots) \frac{1}{N^2},
\]

(24)

while the $1/N$ expansion of Eqs. (21 and 22) gives

\[
\frac{1}{\nu}|_{\text{HY}} = 2 - \epsilon + (6\epsilon - \frac{7}{2} \epsilon^2 - \frac{11}{8} \epsilon^3 + \cdots) \frac{1}{N} \\
+ (36\epsilon - \frac{1125}{2} \epsilon^2 - \frac{26136}{N^3} + \cdots) \frac{1}{N^2},
\]

(25)

\[
\frac{\gamma}{\nu}|_{\text{HY}} = 2 - \epsilon + (\frac{6}{N} - \frac{36}{N^2} + \frac{216}{N^3} + \cdots) \epsilon \\
+ (-\frac{7}{2N} + \frac{51}{2N^2} + \frac{1215}{N^3} + \cdots) \epsilon^2.
\]

(26)

Up to order $\epsilon^2$ and $1/N^2$ the two models agree as expected. In the GN $1/N$-expansion, the $\epsilon^2$ terms are comparable to the $\epsilon$ ones and the $\epsilon^3$ is relatively small in $1/\nu$ and of the same magnitude in $\gamma/\nu$. In contrast, the HY $\epsilon$-expansion shows that the coefficients of the $1/N$ expansion are always rapidly increasing, in particular in the case of $1/\nu$. Thus a resummation for the GN $\nu$ and the HY $\gamma/\nu$ has to be made to improve the corresponding estimates. The necessity of such a resummation is also manifest from the importance, at low $N$, of the $1/N^2$ contribution for the GN $\nu$, and the $\epsilon^2$ one for the HY $\gamma/\nu$ (about 20% for $N=8$).

Because of the lack of information on asymptotic behaviour, we use a simple Padé-Borel resummation \[14\] with arbitrary choice of function, instead of the more sophisticated Borel resummation \[15\]. For an expansion

\[
A(x) = 1 + a_1 x + a_2 x^2 + O(x^3),
\]

(27)

we write

\[
A(x) = \frac{1}{x} \int_0^\infty dt e^{-t/x} [1 - a_1 t - (a_2/2 - a_1^2) t^2]^{-1}.
\]

(28)
These formulae are directly used for the GN $\nu$ obtained from Eq. (4) [10], while we first expand the HY $\gamma/\nu$ of Eq. (22) around the $N = \infty$ point as

$$
\frac{\gamma}{\nu} \big|_{HY} = (2 - \epsilon)(1 + a_1\epsilon + a_2\epsilon^2) + O(\epsilon^3),
$$

and resum with Eq. (28) only the second bracket.

The comparison of the resulting critical exponents for the two 3d models as a function of the fermion number $N$ is summarised in Fig. 1, where the dotted lines represent the computation of the GN $1/\nu$ and the HY $\gamma/\nu$ without the resummation procedure described above. The difference between the two models is small except for $\nu$ at low $N$. The data point at $N = 48$ comes from Ref. [4], while those at $N = 8$ result from the simulation described in the next section.

3 Numerical Results

Here we present our simulation and the analysis leading to estimates of the critical indices for the 3d Gross-Neveu model at $N = 8$.

3.1 Simulation of the lattice Gross-Neveu model

For the numerical simulation we consider the effective action Eq. (10) with $N_L = 1$ which corresponds to $N = 8$, and use an exact Hybrid Monte Carlo algorithm. It has a point update of $8\mu s - 14\mu s$ on a Cray Y-MP, increasing with lattice size. This is due to the fact that more conjugate gradient steps are needed to invert the fermion matrix for large lattices. We perform runs on lattice sizes $8^3$, $12^3$, $16^3$, $24^3$ and $32^3$. We use 20 time steps of length 0.2, except for the largest lattice where the time step is reduced to 0.05. As a rule, measurements are carried out every 5th trajectory. Details regarding the runs are listed in Table 1. The integrated autocorrelation time $\tau_{int}$ quoted is that for $\langle \sigma^2 \rangle$.

To analyse the data we use a variant of multihistogram reweighting analysis which does not require the binning of data [16]. This is used to obtain the values of observables in between the simulated data points. These points are close enough, and the simulations long enough, to produce bosonic energy distributions that fill the whole coupling range of interest.

3.2 The scaling and critical exponents from MC data

Let us first consider the critical coupling and the critical exponent $\nu$, which describes the behaviour of the correlation length near the phase transition. We define the renormalised coupling $g_R$ as

$$
g_R \equiv \frac{\langle \sigma^2 \rangle^2}{\langle \sigma^4 \rangle}.
$$
Table 1: Statistics of the simulations.

| Size | λ    | Trajectories | \( \tau_{int} \) |
|------|------|---------------|------------------|
| 8\(^3\) | 0.7875000 | 110000 | 8 |
| 8\(^3\) | 0.8156250 | 110000 | 11 |
| 8\(^3\) | 0.8437500 | 190000 | 18 |
| 12\(^3\) | 0.7875000 | 90000 | 8 |
| 12\(^3\) | 0.8156250 | 430000 | 11 |
| 12\(^3\) | 0.8437500 | 90000 | 15 |
| 16\(^3\) | 0.7875000 | 90000 | 9 |
| 16\(^3\) | 0.8156250 | 282000 | 11 |
| 16\(^3\) | 0.8437500 | 100000 | 17 |
| 24\(^3\) | 0.8156250 | 432240 | 14 |
| 24\(^3\) | 0.8184375 | 190000 | 4 |
| 32\(^3\) | 0.8167500 | 267160 | 5 |

This expression for \( g_R \) lacks a constant factor and is the inverse of the usual definition, but this does not affect its scaling properties. The scaling of \( g_R \) is extremely simple:

\[
g_R = f(L^{1/\nu} t),
\]

where \( f \) denotes a universal scaling function, \( L \) is the linear extent of the lattice and \( t = (\lambda^{-1} - \lambda_c^{-1}) \lambda_c \) is the reduced coupling (“temperature”). The subscript \( c \) refers to the infinite volume critical coupling.

We can determine \( \lambda_c \) by noting that, according to Eq. (31), the curves of \( g_R \) for different lattice sizes should cross at \( \lambda_c \), up to scaling violations visible on too small lattices. The simulation results are shown in Fig. 2. The reweighting analysis gives crossings at \( \lambda_c = 0.820(2) \) for \( 8^3 \) and \( 12^3 \), \( \lambda_c = 0.817(1) \) for \( 12^3 \) and \( 16^3 \), \( \lambda_c = 0.815(1) \) for \( 16^3 \) and \( 24^3 \) and \( \lambda_c = 0.817(3) \) for \( 24^3 \) and \( 32^3 \). We thus conclude that the GN model has a second order phase transition \( \lambda_c = 0.815(3) \). The renormalised coupling at \( \lambda_c \) is \( (g_R)_c = 0.473(4) \).

The usual way to avoid referring to \( \lambda_c \) in the critical exponents determination is to use thermodynamic quantities that peak in the scaling region. This is possible since, according to the scaling ansatz, \( L^{1/\nu} t \) is constant at the maxima. Unfortunately, there are no quantities whose scaling behaviour provide a direct estimate of the exponent \( \nu \). Therefore, if one tries to measure \( \nu \) one also has to specify the value of \( t \), and thus \( \lambda_c \).

We can relax this requirement by noting that the scaling formula derived above is independent of the critical coupling and is valid in the whole critical region as long as the scaling violations can be neglected. Hence, we can perform a scaling analysis to quantities which do not necessarily peak at the critical coupling. Moreover, the latter need not even be specified: inverting Eq. (31), \( L^{1/\nu} t \) can be expressed as a function of \( g_R \) and the finite size analysis can be made at constant value of \( g_R \) [17]. We have to pay
a price, though, since measuring $g_R$ can be demanding. It is also crucial to eliminate $t$, since its uncertainty contributes a lot to the errors in the exponents.

In order to extract the critical exponent $\nu$, we consider the logarithmic derivative of $g_R$ with respect to the reduced coupling

$$D \equiv \frac{\partial \ln(g_R)}{\partial t} = L^{1/\nu} \frac{f'(L^{1/\nu} t)}{f(L^{1/\nu} t)} \equiv L^{1/\nu} F(L^{1/\nu} t),$$  

with $F$ a new universal scaling function. Inverting the scaling equation of $g_R$ for $L^{1/\nu} t$ we obtain

$$D = L^{1/\nu} G(g_R),$$

where $G$ is a scaling function.

On the other hand, $D$ is a correlator of powers of the $\sigma$-field and the bosonic energy $S = 1/2 \sum_n \sigma_n^2$ from the definition of average quantities:

$$D = \langle S \rangle + \frac{\langle S \sigma^4 \rangle}{\langle \sigma^4 \rangle} - 2 \frac{\langle S \sigma^2 \rangle}{\langle \sigma^2 \rangle},$$

which can thus be measured numerically (i.e. by means of MC simulations).

From this measurement, we determine $\nu$ from a fit at constant $g_R$ to

$$\ln D|_{g_R=\text{const}} = 1/\nu \ln [L] + \text{const}$$

In Fig. 3, the value of $\nu$ is shown as a function $g_R$, together with the corresponding $\chi^2$ value of the fit. The scaling behaviour is realized for the entire fitting range

$$\chi^2 < 0.5$$

for three degrees of freedom. The errors on $\nu$ are coming from a fit which uses the errors of the original data. These were obtained by a jackknifed reweighting analysis. The estimate obtained with $g_R = (g_R)_c = 0.473$ is

$$\nu = 1.00(4).$$

Notice that the dependence on $(g_R)$ is indeed weak, and that we did not have to specify $\lambda_c$ in our fits. To see how well our data is actually scaling we display $g_R$ versus $tL^{1/\nu}$ with our MC estimates of $\nu$ and $\lambda_c$ in Fig. 4.

Using the hyperscaling relations, only two exponents are independent. As a second exponent we choose $2 - \eta = \gamma/\nu$. This governs the behaviour of the susceptibility $\chi$ near the critical point

$$\chi = \langle \sigma^2 \rangle - \langle \sigma \rangle^2 = L^{\gamma/\nu} g(L^{1/\nu} t),$$

where $g$ denotes a scaling function. However, in numerical simulations there is a problem concerning the susceptibility: on finite lattices the average of the $\sigma$ field is always zero.
The use of absolute values in definition (38) could distort the scaling behaviour and may lead to wrong exponents. To overcome this we used the susceptibility on the symmetric side \( \lambda > \lambda_c \) where

\[
\chi = \langle \sigma^2 \rangle = L^{\gamma/\nu} g(L^{1/\nu} t). \tag{39}
\]

Eliminating again \( t \) as in the case for \( D \) (and \( g_R \)), we obtain

\[
\ln \langle \sigma^2 \rangle_{|g_R=\text{const}} = \frac{\gamma}{\nu} \ln L + \text{const}. \tag{40}
\]

and get from a fit to the measured \( L \) dependence

\[
\frac{\gamma}{\nu} = 1.246(8). \tag{41}
\]

Fig. 5 shows the results of the fit. Notice that the fit is valid for the whole critical range \( (\chi^2 < 0.4 \text{ for } 3 \text{ d.o.f}) \) and the deviation as a function of \( g_R \) is very small as expected. The value we quote is taken at \( (g_R)_c = 0.473 \). From Fig. 6 one can see that the scaling of the data is excellent with our values of exponents: within error bars all the data from different lattice sizes lie on the same curve.

As a check of consistency with hyperscaling, we can measure other critical exponents. The expectation value of the sigma field acts as an order parameter for the discrete chiral symmetry Eq. (3), which is preserved on the lattice. On a finite lattice, the absolute value of \( \sigma \) yields an estimate for the combination \( \beta/\nu \), which is shown in Fig. 7. At \( g_R = (g_R)_c = 0.473 \) we get

\[
\frac{\beta}{\nu} = 0.877(4), \tag{42}
\]

with \( \chi^2 < 0.3 \text{ for } 3 \text{ d.o.f.} \) With this value of exponents the quality of scaling is again excellent, as one can see from Fig. 8.

The combination \( \beta/\nu \) is connected to \( \gamma/\nu \) through the hyperscaling relation

\[
\frac{\beta}{\nu} = \frac{1}{2}(d - \frac{\gamma}{\nu}). \tag{43}
\]

Using the value of \( \gamma/\nu \) given in (41) this gives \( \beta/\nu = 0.877(4) \), which is in complete agreement with estimate obtained above (42). This agreement is noteworthy since we used the absolute value of \( \sigma \), which can lead to a distortion of the scaling relation. At least in our case we see that it does not. Also, the definition of the susceptibility with the absolute value of \( \sigma \) leads to identical results. However, for the standard method of measuring the critical exponents from the scaling behaviour of thermodynamic quantities at their peak values, the usage of the absolute value \( \langle \sigma \rangle \) may result in a change in the position of the peaks and thus make the scaling analysis dubious.

The heat capacity should give the combination \( \alpha/\nu \). The hyperscaling relation predicts a value of \(-1\). This means that heat capacity does not diverge at the critical
Table 2: The critical exponents obtained from different methods.
The numbers with a star are obtained with resummation.

| Exponent | MC   | $\epsilon$ | $\epsilon^2$ | $1/N_f$ | $1/N_f^2$ |
|----------|------|------------|--------------|---------|-----------|
| $\nu$    | 1.00(4) | 0.9545   | 0.9480      | 1.135   | 0.903*    |
| $\gamma/\nu$ | 1.246(8) | 1.4285   | 1.237*       | 1.270   | 1.2559    |

point. In fact, it is dominated by its regular part which makes it impossible to extract $\alpha/\nu$. In order to do this we would need the second derivative of the heat capacity with respect to $t$. This quantity would diverge as $t^{-(\alpha/\nu+2)} \sim t^{-1}$. Unfortunately the quality of the MC data deteriorates as higher order derivatives of the free energy are taken: the 4th derivative is out of reach in the present simulation.

All of the previous analysis relied heavily on reweighting the data from a finite set of couplings to a very dense set of couplings. This enabled us to accurately explore the dependence of the variables on the renormalised coupling $g_R$. We note that both our method of analysis and the number of trajectories used allow us to achieve a better determination of the critical exponents than was achieved in a comparable analysis for $N_f = 12$ [6].

4 Conclusions

We have performed a high statistics simulation of the 3d GN model with two flavours of 4-spinors. We show that it has a second order phase transition at $\lambda = 0.815(3)$. Hence, it leads to a continuum field theory, which is characterised by critical exponents which we have measured. This proves numerically that the GN model is renormalisable in three dimensions, even for a small number of flavours. The transition point is close to the $1/N_f$ expectation $\lambda_1^c = 0.80$, but the $1/N_f^2$ correction can be as significant as in 2d calculation [2]. Table 2 displays the results from our simulations together with estimates obtained by other methods: the $\epsilon$-expansion is for the HY model to first order by Zinn-Justin [5], and to second order as presented above, the $1/N_f$-expansion for the GN model calculated to one loop by Hands et al. [6] and to order $1/N^2$ by Gracey [1, 8, 10] and by Derkachov et al. [9]. The second order contributions of the HY $\gamma/\nu$ and the GN $\nu$ are large and the corresponding expressions have been resummed as explained in Sect. 2.4.

The striking feature of the data is that the HY $\epsilon$-expansion results at the two-loop level are in very good agreement with the simulation values. Even without resummation, which can be found quite arbitrary, the direct result of $\gamma/\nu$ is not very far from the data point as seen in the Fig. 1. The GN second order $1/N_f$-expansion works very well for $\gamma/\nu$ which has a small $1/N_f^2$ correction. Concerning the GN $\nu$, even though the resummed value is not too far off from our numerical result, the discrepancy does
suggest that higher order terms may be important. However, the agreement with the HY $\nu$ shows that no new phenomenon appears at small $N$.

As a whole, our results strongly support the conjecture that these models are equivalent even in three dimensions, where they are not trivial, and that the properties inferred from perturbation theory are valid at low fermion number.

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References

[1] D.J. Gross and A. Neveu, Phys. Rev D 10 (1974) 3235.
[2] K. Wilson, Phys. Rev. D 7 (1973) 2911.
[3] W. A. Bardeen, C.N. Leung and S.T. Love, Phys. Rev. Lett. 56 (1986) 1230.
[4] A. Hasenfratz, P. Hasenfratz, K. Jansen, J. Kuti and Y. Shen, Nucl. Phys B 365 (1991) 79.
[5] J. Zinn-Zustin, Nucl. Phys B 367 (1991) 105.
[6] S. Hands, A. Kocic and J.B. Kogut, Four-Fermi theories in Fewer than Four Dimensions, CERN-TH 6557/92, ILL-(TH)-92 #19.
[7] J.A. Gracey, Int. J. Mod. Phys. A 6 (1991) 395.
[8] J.A. Gracey, Phys. Lett. B297 (1992) 293 and private communication.
[9] S.E. Derkachov, N.A. Kivel, A.S. Stepanenko and A.N. Vasil’ev, On calculation of $1/n$ Expansions of Critical Exponents in the Gross-Neveu Model with the Conformal Technique, SPhT-93/016.
[10] J.A. Gracey Computation of $\beta'(g_c)$ at $O(1/N^2)$ in the $O(N)$ Gross Neveu model in arbitrary dimensions, Liverpool preprint LTH-312.
[11] Th. Jolicœur, Phys. Lett. 171B (1986) 431; Th. Jolicœur, A. Morel and B. Petersson, Nucl. Phys. B274 (1986) 225.
[12] L. Bélanger, R. Lacaze, A. Morel, N. Attig, B. Petersson, and M. Wolff, Nucl. Phys. B340 (1990) 245.

[13] M.E. Machacek and M.T. Vaughn, Nucl. Phys. B222 (1983) 83, B222 (1983) 221, B249 (1983) 70.

[14] S. Hikami and E. Brézin, J. Phys. A11 (1978) 1141.

[15] J.C. Leguillou and J. Zinn-Justin, Phys. Rev B21 (1980) 3976.

[16] K. Kajantie, L. Kärkkäinen and K. Rummukainen, Nucl. Phys. B (1991) 693.

[17] J. Engels, J. Fingberg and D.E. Miller, Nucl. Phys B 387 (1992) 501.

[18] J. Engels, Nucl. Phys B (Proc. Suppl.) 30 (1993) 347.
Figure captions

Figure 1. Comparison of critical exponents obtained with different means as function of the effective fermion number $N$. Solid lines for the $\epsilon^2$ Higgs-Yukawa model, dashed lines for the $1/N^2$ Gross-Neveu model, dotted lines for HY $\gamma/\nu$ and GN $\nu$ direct results (without resummation). The data point at $N = 48$ is for $\nu$ from Ref. [6], those for $N = 8$ result from our simulation.

Figure 2. The renormalised coupling $g_R$ as function of the coupling $\lambda$ for lattice sizes $8^3$, $12^3$, $16^3$, $24^3$ and $32^3$ in order of increasing slopes. The results of simulations without the reweighting are shown as circles.

Figure 3. The critical exponent $\nu$ as function of the value of $g_R$. The corresponding $\chi^2$ plot gives the quality of the fit.

Figure 4. The renormalised coupling $g_R$ as function of $tL^{1/\nu}$ for different lattice sizes ($8^3$ is labeled with plus, $12^3$ with octagons, $16^3$ with squares, $24^3$ with circles and $32^3$ with diamonds). The $\nu$ and $\lambda_c$ have the measured MC values.

Figure 5. The critical exponent $\gamma/\nu$ as function of the critical value of $g_R$. The $\chi^2$ gives the quality of the fit.

Figure 6. The combination $\sigma^2L^{3-\gamma/\nu}$ and $g_R$ as a function of $tL^{1/\nu}$ for different lattice sizes ($8^3$ is labeled with plus, $12^3$ with octagons, $16^3$ with squares, $24^3$ with circles and $32^3$ with diamonds). The $\nu$, $\gamma/\nu$ and $\lambda_c$ have the measured MC values.

Figure 7. As in Fig. 5, but for the critical exponent $\beta/\nu$.

Figure 8. As in Fig. 6, but for the combination $|\sigma|L^{\beta/\nu}$.
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