Finiteness of Stationary Configurations of the Planar Four-vortex Problem

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Abstract

For the planar four-vortex problem, we show that there are finitely many stationary configurations consisting of equilibria, rigidly translating configurations, relative equilibria (uniformly rotating configurations) and collapse configurations. We also provide upper bounds for these classes of stationary configurations.

Key Words: Point vortices; Relative equilibrium; Finiteness.

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1 Introduction

We consider the motion of \( N \) point vortices on a plane. This problem has been investigated by many researchers for a long time and can be dated back to Helmholtz’s work on hydrodynamics in 1858 [11]. Let the \( n \)-th point vortex have vortex strength (or vorticity) \( \Gamma_n \in \mathbb{R}\setminus\{0\} \) and position \( r_n \in \mathbb{R}^2 \) (\( n = 1, 2, \ldots, N \)), then the motion of the \( N \)-vortex problem is governed by

\[
\Gamma_n \dot{r}_n = J \nabla_k H = J \sum_{1 \leq j \leq N, j \neq n} \frac{\Gamma_n \Gamma_j (r_j - r_n)}{|r_j - r_n|^2}, \quad n = 1, 2, \ldots, N. \tag{1.1}
\]

where \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), \( H = -\sum_{1 \leq j < k \leq N} \ln |r_j - r_k| \), \( \nabla_n \) denotes the two-dimensional partial gradient vector with respect to \( r_n \) and \( |\cdot| \) denotes the Euclidean norm in \( \mathbb{R}^2 \).

The system (1.1) (the \( N \)-vortex problem) was introduced by Helmholtz [11], then Kirchhoff [12] first found its Hamiltonian structure.

The \( N \)-vortex problem is formally derived from the planar Euler equations and is a widely used model for providing finite-dimensional approximations to vorticity evolution.

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in fluid dynamics. It is well known that the analysis of the \( N \)-vortex problem could lead to the understanding of vortex dynamics in inviscid flows \[3\]. For example, similar to the Newtonian \( N \)-body problem in celestial mechanics, one of the notable features of the \( N \)-vortex problem is the existence of self-similar collapsing solutions, and the mechanism of vortex collapse plays an important role to understand fluid phenomena. Indeed it has been pointed out that the vortex collapse is related to the loss of the uniqueness of solutions to the Euler equations \[15\].

For considerable physical interest, it is natural to study stationary configurations which produce self-similar solutions of the \( N \)-vortex problem. In \[16\] it is shown that the only stationary configurations of vortices are equilibria, rigidly translating configurations, relative equilibria (uniformly rotating configurations) and collapse configurations. It turns out that \( L = \sum_{1 \leq j<k \leq N} \Gamma_j \Gamma_k = 0 \) is a necessary condition on the vorticities for the existence of equilibria, and \( \Gamma = \sum_{j=1}^N \Gamma_j = 0 \) is a necessary condition on the vorticities for the existence of rigidly translating solutions.

Although a great amount of work on stationary configurations of the \( N \)-vortex problem has been done (a review can be found in \[4\]), in general there is nothing known about stationary configurations besides three-vortex case and some special cases of \( N \geq 4 \).

In this study, we are interested in the number of stationary configurations for giving vorticities. All stationary configurations for two and three point vortices are known explicitly \[8, 21, 14, 15, 2, 4\]. In \[16\] O’Neil proved that for almost every choice of vorticities of the \( N \)-vortex problem, there are finite equilibria, rigidly translating configurations and collinear relative equilibria. In \[18, 16\] lower bounds on the number of relative equilibria have been established for the \( N \)-vortex problem with all positive or large positive and small negative vorticities by the use of topological arguments. Aside from special cases with certain symmetries \[5\], the only general work on stationary configurations are for the four-vortex problem: O’Neil \[17\] and Hampton and Moeckel \[9\] independently proved that for almost every choice of vorticities of the four-vortex problem, there are finite equilibria, rigidly translating configurations and relative equilibria. In particular, little is known for collapse configurations. More specifically, O’Neil proved

**Theorem 1.1 (O’Neil)** If the vorticities \( \Gamma_n (n \in \{1, 2, 3, 4\}) \) are nonzero then the four-vortex problem has:

- at most 6 collinear relative equilibria when \( \Gamma = 0 \)
- at most 14 strictly planar relative equilibria when \( \Gamma = 0 \)
- at most 56 planar relative equilibria when \( \Gamma \neq 0 \) provided \( L \neq 0 \), \( \Gamma_1 \Gamma_3 - \Gamma_2 \Gamma_4 \neq 0 \), \( \Gamma_1 \Gamma_4 - \Gamma_2 \Gamma_3 \neq 0 \), \( \Gamma_j + \Gamma_k \neq 0 \) (\( j, k \in \{1, 2, 3, 4\} \)), \( \Gamma_1 + \Gamma_2 + \Gamma_1 \neq 0 \) (\( l \in \{3, 4\} \)) and \( \Gamma_1 \Gamma_2 + \Gamma_1 (\Gamma_1 + \Gamma_2) \neq 0 \) (\( l \in \{3, 4\} \));

and Hampton and Moeckel proved

**Theorem 1.2 (Hampton-Moeckel)** If the vorticities \( \Gamma_n (n \in \{1, 2, 3, 4\}) \) are nonzero then the four-vortex problem has:
• exactly 2 equilibria when the necessary condition \( L = 0 \) holds

• at most 6 rigidly translating configurations the necessary condition \( \Gamma = 0 \) holds

• at most 12 collinear relative equilibria

• at most 14 strictly planar relative equilibria when \( \Gamma = 0 \)

• at most 74 strictly planar relative equilibria when \( \Gamma \neq 0 \) provided \( \Gamma_j + \Gamma_k + \Gamma_l \neq 0 \) for all distinct indices \( j, k, l \in \{1, 2, 3, 4\} \).

Here a configuration is called strictly planar if it is planar but not collinear, and planar configurations include collinear and strictly planar configurations.

It is well known that a continuum of relative equilibria can exist for the five-vortex problem [20], thus the finiteness of relative equilibria and/or stationary configurations would be expected only for generic in general. However the main purposes of the present study is to show that there are finitely many stationary configurations for the four-vortex problem:

**Theorem 1.3** If the vorticities \( \Gamma_n \ (n \in \{1, 2, 3, 4\}) \) are nonzero then the four-vortex problem has finitely many stationary configurations.

It suffices to focus on the relative equilibria and collapse configurations of the four-vortex problem. The proof of Theorem 1.3 is motivated by the elegant method of Albouy and Kaloshin for celestial mechanics [1]. The principle of the method is to follow a possible continuum of central configurations in the complex domain and to study its possible singularities there.

Both proofs of Theorem 1.2 and Theorem 1.3 borrow the similarity between the four-vortex problem and the Newtonian four-body problem in relative equilibria, and in both proofs one follows a continuum of relative equilibria in the complex domain until it reaches a singularity. Our analysis of the singularities is different from Hampton and Moeckel’s. In contrast to the method of BKK theory by Hampton and Moeckel [9], our proof of Theorem 1.3 does not require any difficult computation.

We embed equations of relative equilibria and collapse configurations into a polynomial system (see the following (2.7)), then except two cases, a continuum of relative equilibria and collapse configurations is excluded by analysis of the singularities. For the two exceptional cases, we directly solve a polynomial system equivalent to the system (2.7) and prove the corresponding finiteness. The two exceptional cases are directly checked by simply solving an equivalent polynomial system of (2.7) by employing standard commands in Mathematica on a desktop computer, and this computation takes less than ten seconds.

In general the \( N \)-vortex problem is simpler than the Newtonian \( N \)-body problem for a given \( N \), especially when problem of relative equilibria is concerned. For example, the way that the equations of relative equilibria reduce to polynomial systems is simpler for the \( N \)-vortex problem than for the Newtonian \( N \)-body problem [10]. On the other hand, the Newtonian \( N \)-body problem is simpler than the \( N \)-vortex problem for a given \( N \). For example, the Hamiltonian \( H \) of the \( N \)-vortex problem is a transcendental
function rather than that of the Newtonian $N$-body problem; in particular, the larger set of parameter values ($\Gamma_n < 0$ is allowed) yields that there is a continuum of relative equilibria in a certain $N$-vortex problem \([20]\).

Once the finiteness is proved, an explicit upper bound on the number of relative equilibria and collapse configurations is obtained by direct application of Bézout Theorems. However, such a bound is not optimal for relative equilibria or collapse configurations. Nevertheless, we provide upper bounds in the following summary result:

**Corollary 1.4** If the vorticities $\Gamma_n$ ($n \in \{1, 2, 3, 4\}$) are nonzero, then the four-vortex problem has:

- exactly 2 equilibria when the necessary condition $L = 0$ holds
- at most 6 rigidly translating configurations the necessary condition $\Gamma = 0$ holds
- at most 12 collinear relative equilibria, more precisely,
  i. at most 12 collinear relative equilibria when $L \neq 0$
  ii. at most 10 collinear relative equilibria when $L = 0$
  iii. at most 6 collinear relative equilibria when $L \neq 0$ and $\Gamma = 0$
- at most 74 strictly planar relative equilibria, furthermore, at most 14 strictly planar relative equilibria when $L \neq 0$ and $\Gamma = 0$
- at most 130 collapse configurations when the necessary condition $L = 0$ holds.

Note that there is no collinear collapse configuration for the general $N$-vortex problem.

The paper is structured as follows. In **Section 2**, we give some notations and definitions. In particular, following Albouy and Kaloshin [1], we introduce singular sequences of normalized central configurations. In **Section 3**, we discuss some tools to classify the singular sequences. In **Section 4**, we study all possibilities for a singular sequence and reduce the problem to the ten diagrams in Figure [11]. In **Section 5**, we obtain the constraints on the vorticities corresponding to each of the ten diagrams. In **Section 6**, based upon the prior work, we prove the main result on finiteness. Finally, in **Section 7**, we investigate upper bounds on the number of relative equilibria and collapse configurations.

# 2 Preliminaries

In this section we give some notations and definitions that will be needed later.

## 2.1 Stationary configurations

First it is more convenient to consider the vortex positions $r_n \in \mathbb{R}^2$ as complex numbers $z_n \in \mathbb{C}$ for us, in which case the dynamics are given by $\dot{z}_n = -iV_n$ where

$$V_n = \sum_{1 \leq j \leq N, j \neq n} \frac{\Gamma_j z_{jn}}{r_{jn}^2} = \sum_{j \neq n} \frac{\Gamma_j}{z_{jn}}. \quad (2.2)$$
\[ z_{jn} = z_n - z_j, \quad r_{jn} = |z_{jn}| = \sqrt{z_{jn} \overline{z_{jn}}}, \quad i = \sqrt{-1} \] and the overbar denotes complex conjugation.

Let \( \mathbb{C}^N = \{ z = (z_1, \cdots, z_N) : z_j \in \mathbb{C}, j = 1, \cdots, N \} \) denote the space of configurations for \( N \) point vortices. Let \( \Delta = \{ z \in \mathbb{C}^N : z_j = z_k \text{ for some } j \neq k \} \) be the collision set in \( \mathbb{C}^N \). Then the set \( \mathbb{C}^N \setminus \Delta \) is the space of collision-free configurations.

**Definition 2.1** The following quantities are defined:

| Total vorticity | \( \Gamma = \sum_{j=1}^{N} \Gamma_j \) |
|------------------|-------------------------------------|
| Total vortex angular momentum | \( L = \sum_{1 \leq j < k \leq N} \Gamma_j \Gamma_k \) |
| Moment of vorticity | \( M = \sum_{j=1}^{N} \Gamma_j z_j \) |
| Angular impulse | \( I = \sum_{j=1}^{N} \Gamma_j |z_j|^2 = \sum_{j=1}^{N} \Gamma_j z_j \overline{z}_j \) |

Then it is easy to see that

\[ \Gamma I - M M = \sum_{1 \leq j < k \leq N} \Gamma_j \Gamma_k z_{jk} \overline{z}_{jk} = \sum_{1 \leq j < k \leq N} \Gamma_j \Gamma_k r_{jk}^2 \triangleq S, \quad (2.3) \]

and

\[ \Gamma^2 - 2L > 0. \quad (2.4) \]

Following O’Neil [16] we will call a configuration stationary if the relative shape remains constant, i.e., if the ratios of intervortex distances \( r_{jk}/r_{lm} \) remain constant (such solutions are often called homographic). More precisely,

**Definition 2.2** A configuration \( z \in \mathbb{C}^N \setminus \Delta \) is stationary if there exists a constant \( \Lambda \in \mathbb{C} \) such that

\[ V_j - V_k = \Lambda(z_j - z_k), \quad 1 \leq j, k \leq N. \quad (2.5) \]

It is shown that the only stationary configurations of vortices are equilibria, rigidly translating configurations, relative equilibria (uniformly rotating configurations) and collapse configurations [16].

**Definition 2.3**

i. \( z \in \mathbb{C}^N \setminus \Delta \) is an equilibrium if \( V_1 = \cdots = V_N = 0 \).

ii. \( z \in \mathbb{C}^N \setminus \Delta \) is rigidly translating if \( V_1 = \cdots = V_N = V \) for some \( V \in \mathbb{C} \setminus \{0\} \). (The vortices are said to move with common velocity \( V \).)

iii. \( z \in \mathbb{C}^N \setminus \Delta \) is a relative equilibrium if there exist constants \( \lambda \in \mathbb{R} \setminus \{0\}, z_0 \in \mathbb{C} \) such that \( V_n = \lambda(z_n - z_0), \quad 1 \leq n \leq N \).

iv. \( z \in \mathbb{C}^N \setminus \Delta \) is a collapse configuration if there exist constants \( \Lambda, z_0 \in \mathbb{C} \) with \( \text{Im}(\Lambda) \neq 0 \) such that \( V_n = \Lambda(z_n - z_0), \quad 1 \leq n \leq N \).

It is easy to see that \( L = 0 \) is a necessary condition for the existence of equilibria, and \( \Gamma = 0 \) is a necessary condition for the existence of configurations [16].

**Proposition 2.1** Every equilibrium has vorticities satisfying \( L = 0 \); every rigidly translating configuration has vorticities satisfying \( \Gamma = 0 \).
If $z'$ differs from $z$ only by translation, rotation, and change of scale (dilation) in the plane, then it is easy to see that $z$ is stationary if and only if $z'$ is stationary.

**Definition 2.4** A configuration $z$ is equivalent to a configuration $z'$ if for some $a, b \in \mathbb{C}$ with $b \neq 0$, $z'_n = b(z_n + a), \quad 1 \leq n \leq N$.

$z \in \mathbb{C}^N \setminus \Delta$ is a translation-normalized configuration if $M = 0$; $z \in \mathbb{C}^N \setminus \Delta$ is a rotation-normalized configuration if $z_{12} \in \mathbb{R}$. Fixing the scale of a configuration, we can give the definition of dilation-normalized configuration, however, we do not specify the scale here.

A configuration, which is translation-normalized, rotation-normalized and dilation-normalized, is called a normalized configuration.

Note that for a given configuration $z$, there always is certain rotation-normalized configuration and dilation-normalized configuration being equivalent to $z$; if $\Gamma \neq 0$, then there always is certain translation-normalized configuration being equivalent to $z$ too, at this time, there is exactly one normalized configuration being equivalent to $z$.

### 2.2 Central configurations

In this paper we consider only relative equilibria and collapse configurations, then it is easy to see that there always is certain translation-normalized configuration being equivalent to a given configuration.

The equations of relative equilibria and collapse configurations can be unified into the following formula

$$V_n = \Lambda(z_n - z_0), \quad 1 \leq n \leq N, \quad (2.6)$$

$\Lambda \in \mathbb{R} \setminus \{0\}$ corresponds to relative equilibria and $\Lambda \in \mathbb{C} \setminus \mathbb{R}$ corresponds to collapse configurations.

**Definition 2.5** Relative equilibria and collapse configurations are both called central configurations.

The equations (2.6) can be reducible to

$$\Lambda z_n = V_n, \quad 1 \leq n \leq N, \quad (2.7)$$

if we replace $z_n$ by $z_n + z_0$, i.e., the solutions of equations (2.7) have been removed the translation freedoms. In fact, it is easy to see that the solutions of equations (2.7) satisfy

$$M = 0, \quad (2.8)$$

$$\Lambda I = L. \quad (2.9)$$

To remove the dilation freedoms, we set $|\Lambda| = 1$. Following Albouy and Kaloshin we introduce

**Definition 2.6** A real normalized central configuration of the planar $N$-vortex problem is a solution of (2.7) satisfying $z_{12} \in \mathbb{R}$ and $|\Lambda| = 1$. 

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Remark 2.1  1. Note that solutions \( z \) (real normalized central configurations) of (2.6) come in a pair: \( -z \mapsto z \) sends solution on solution, that is, central configurations is determined up to a common factor \( \pm 1 \) by normalizing here. Thus we count the total central configurations up to a common factor \( \pm 1 \) below.

2. The word “real” refers to the reality hypothesis, in general it is omitted in a real context. However, we will study complex central configurations and establish in Section 6 strong statements about their finiteness. Note that the distances \( r_{jk} = \sqrt{z_{jk} \bar{z}_{jk}} \) are now bi-valued, but we do not need the positivity condition of the distances \( r_{jk} \) at all, so it is unnecessary to introduce the notion “positive normalized central configuration” as in [1].

We conclude this subsection with the following simple fact.

Proposition 2.2  Collapse configurations satisfy \( \Gamma \neq 0 \) and

\[
S = I = L = 0. \tag{2.10}
\]

For relative equilibria we have \( S = 0 \iff \begin{cases} \Gamma \neq 0 \\ I = L = 0 \end{cases} \) or \( \begin{cases} \Gamma = 0 \\ I \neq 0, L \neq 0 \end{cases} \)

Proof.  The proof is trivial by (2.3), (2.4) and (2.9).

\[\square\]

2.3 Complex central configurations

To eliminate complex conjugation in (2.7) we introduce a new set of variables \( w_n \) and a “conjugate” relation:

\[
\Lambda z_n = \sum_{j \neq n} \frac{\Gamma_j}{w_{jn}}, \quad 1 \leq n \leq N,
\]

\[
\Lambda w_n = \sum_{j \neq n} \frac{\Gamma_j}{z_{jn}}, \quad 1 \leq n \leq N,
\]

(2.11)

where \( z_{jn} = z_n - z_j \) and \( w_{jn} = w_n - w_j \).

The rotation freedom is expressed in \( z_n, w_n \) variables as the invariance of (2.11) by the map \( R_a : (z_n, w_n) \mapsto ( a z_n, a^{-1} w_n ) \) for any \( a \in \mathbb{C} \setminus \{0\} \) and any \( n = 1, 2, \ldots, N \). The condition \( z_{12} \in \mathbb{R} \) we proposed to remove this rotation freedom becomes \( z_{12} = w_{12} \).

To the variables \( z_n, w_n \in \mathbb{C} \) we add the variables \( Z_{jk}, W_{jk} \in \mathbb{C} \) \( (1 \leq j < k \leq N) \) such that \( Z_{jk} = 1/w_{jk}, W_{jk} = 1/z_{jk} \). For \( 1 \leq k < j \leq N \) we set \( Z_{jk} = -Z_{kj}, W_{jk} = -W_{kj} \). Then equations (2.7) together with the condition \( z_{12} \in \mathbb{R} \) and \(|\Lambda| = 1 \) becomes

\[
\Lambda z_n = \sum_{j \neq n} \Gamma_j Z_{jn}, \quad 1 \leq n \leq N,
\]

\[
\Lambda w_n = \Lambda^{-1} w_n = \sum_{j \neq n} \Gamma_j W_{jn}, \quad 1 \leq n \leq N,
\]

\[
Z_{jk} w_{jk} = 1, \quad 1 \leq j < k \leq N,
\]

\[
W_{jk} z_{jk} = 1, \quad 1 \leq j < k \leq N,
\]

\[
z_{jk} = z_k - z_j, \quad w_{jk} = w_k - w_j, \quad 1 \leq j, k \leq N,
\]

\[
Z_{jk} = -Z_{kj}, \quad W_{jk} = -W_{kj}, \quad 1 \leq k < j \leq N,
\]

\[
z_{12} = w_{12}.
\]

(2.12)
This is a polynomial system in the variables $Q = (Z, W) \in (\mathbb{C}^N \times \mathbb{C}^{N(N-1)/2})^2$, here

$$
Z = (z_1, z_2, \cdots, z_N, Z_{12}, Z_{13}, \cdots, Z_{(N-1)N}),
W = (w_1, w_2, \cdots, w_N, W_{12}, W_{13}, \cdots, W_{(N-1)N}).
$$

It is easy to see that a positive normalized central configuration of (2.7) is a solution $Q = (Z, W)$ of (2.12) such that $z_n = \overline{w}_n$ and vice versa.

Following Albouy and Kaloshin [1] we introduce

**Definition 2.7 (Normalized central configuration)** A normalized central configuration is a solution $Q = (Z, W)$ of (2.12). A real normalized central configuration is a normalized central configuration such that $z_n = \overline{w}_n$ for any $n = 1, 2, \cdots, N$.

Definition 2.7 of a real normalized central configuration coincides with Definition 2.6.

**Definition 2.8** We will use the name “distance” for the $r_{jk} = \sqrt{|z_{jk}w_{jk}|}$. We will use the name $z$-separation (respectively $w$-separation) for the $z_{jk}$’s (respectively the $w_{jk}$’s) in the complex plane.

Note that solutions $Q = (Z, W)$ of (2.12) come in a pair: $(-Z, -W) \mapsto (Z, W)$ sends solution on solution, that is, a solution of (2.12) is determined up to a common factor $\pm 1$.

### 2.4 Elimination theory

Let $m$ be a positive integer. Following [13], we define a closed algebraic subset of the affine space $\mathbb{C}^m$ as the set of common zeroes of a system of polynomials on $\mathbb{C}^m$.

The polynomial system (2.12) defines a closed algebraic subset $A \subset (\mathbb{C}^N \times \mathbb{C}^{N(N-1)/2})^2$. For the planar four-vortex problem, we will prove that this subset is finite, then positive normalized central configurations is finite. To distinguish the two possibilities, finitely many or infinitely many points, we will only use the following result (see [1]) from elimination theory.

**Lemma 2.1** Let $\mathcal{X}$ be a closed algebraic subset of $\mathbb{C}^m$ and $f : \mathbb{C}^m \to \mathbb{C}$ be a polynomial. Either the image $F(\mathcal{X}) \subset \mathbb{C}$ is a finite set, or it is the complement of a finite set. In the second case one says that $f$ is dominating.

### 2.5 Singular sequences of normalized central configurations

We consider a solution $Q = (Z, W)$ of (2.12), this is a normalized central configuration. Let $N = N(N + 1)/2$. Set

$$
Z = (Z_1, Z_2, \cdots, Z_{N^2}) = (z_1, z_2, \cdots, z_N, Z_{12}, Z_{13}, \cdots, Z_{(N-1)N}),
W = (W_1, W_2, \cdots, W_{N^2}) = (w_1, w_2, \cdots, w_N, W_{12}, W_{13}, \cdots, W_{(N-1)N}).
$$
Let $\|z\| = \max_{j=1,2,\ldots,N} |z_j|$ be the modulus of the maximal component of the vector $z \in \mathbb{C}^N$. Similarly, set $\|w\| = \max_{k=1,2,\ldots,N} |w_k|$.

Consider a sequence $Q(n)$, $n = 1, 2, \ldots$, of normalized central configurations. Extract a sub-sequence such that the maximal component of $z(n)$ is always the same, i.e., $\|z(n)\| = |z_j(n)|$ for a $j \in \{1, 2, \ldots, N\}$ that does not depend on $n$. Extract again in such a way that the vector sequence $z(n)/\|z(n)\|$ converges. Extract again in such a way that there is similarly an integer $k \in \{1, 2, \ldots, N\}$ such that $\|w(n)\| = |w_k(n)|$ for all $n$.

If the initial sequence is such that $z(n)$ or $w(n)$ is unbounded, so is the extracted sequence. Note that $\|z(n)\|$ and $\|w(n)\|$ are bounded away from zero: if the first $N$ components of the vector $z(n)$ or $\|w(n)\|$ all go to zero, then the denominator $z_{12} = w_{12}$ of the component $z_{12} = W_{12}$ go to zero and $z(n)$ and $w(n)$ are unbounded. There are two possibilities for the extracted sub-sequences above:

- $z(n)$ and $w(n)$ are bounded,
- at least one of $z(n)$ and $w(n)$ is unbounded.

**Definition 2.9 (Singular sequence)** Consider a sequence of normalized central configurations. A sub-sequence extracted by the above process, in the unbounded case, is called a singular sequence.

Our method to prove the finiteness of the central configurations consists essentially of two steps. First, we study all possibilities for a singular sequence. We show that such an unbounded sequence is impossible for the planar four-vortex problem. Second, we use Lemma 2.1 to prove that if there are infinitely many normalized central configurations, there exist singular sequences, and even singular sequences where some distance goes to zero or to infinity. Consequently there are finitely many normalized central configurations for the planar four-vortex problem.

### 3 Rules of colored diagram

**Definition 3.1 (Notation of asymptotic estimates)** $a \sim b$ means $a/b \rightarrow 1$

$a \prec b$ means $a/b \rightarrow 0$

$a \preceq b$ means $a/b$ is bounded

$a \approx b$ means $a \preceq b$ and $a \succeq b$

**Definition 3.2 (Strokes and circles.)** We pick a singular sequence. We write the indices of the bodies in a figure and use two colors for edges and vertices.

The first color, the $z$-color, is used to mark the maximal order components of

$$z = (z_1, z_2, \ldots, z_N, Z_{12}, Z_{13}, \ldots, Z_{(N-1)N}).$$
They correspond to the components of the converging vector sequence \( Z^{(n)} / \| Z^{(n)} \| \) that do not tend to zero. We draw a circle around the name of vertex \( j \) if the term \( z_j^{(n)} \) is of maximal order among all the components of \( Z^{(n)} \). We draw a stroke between the names \( k \) and \( l \) if the term \( Z_{kl}^{(n)} \) is of maximal order among all the components of \( Z^{(n)} \).

The following rules mainly concern \( z \)-diagram, but they apply as well to the \( w \)-diagram.

If there is a maximal order term in an equation, there should be another one. This gives immediately the following Rule I.

**Rule I** There is something at each end of any \( z \)-stroke: another \( z \)-stroke or/and a \( z \)-circle drawn around the name of the vertex. A \( z \)-circle cannot be isolated; there must be a \( z \)-stroke emanating from it. There is at least one \( z \)-stroke in the \( z \)-diagram.

**Definition 3.3 (\( z \)-close)** Consider a singular sequence. We say that bodies \( k \) and \( l \) are close in \( z \)-coordinate, or \( z \)-close, or that \( z_k \) and \( z_l \) are close, if \( z_{kl}^{(n)} \prec \| Z^{(n)} \| \).

The following statement is obvious.

**Rule II** If bodies \( k \) and \( l \) are \( z \)-close, they are both \( z \)-circled or both not \( z \)-circled.

**Definition 3.4 (Isolated component)** An isolated component of the \( z \)-diagram is a subset of vertices such that no \( z \)-stroke is joining a vertex of this subset to a vertex of the complement.

**Rule III** The moment of vorticity of a set of bodies forming an isolated component of the \( z \)-diagram is \( z \)-close to the origin.

**Rule IV** Consider the \( z \)-diagram or an isolated component of it. If there is a \( z \)-circled vertex, there is another one. The \( z \)-circled vertices can all be \( z \)-close together only if the total vorticity of these vertices is zero.

**Definition 3.5 (Maximal \( z \)-stroke)** Consider a \( z \)-stroke from vertex \( k \) to vertex \( l \). We say it is a maximal \( z \)-stroke if \( k \) and \( l \) are not \( z \)-close.

**Rule V** There is at least one \( z \)-circle at certain end of any maximal \( z \)-stroke. As a result, if an isolated component of the \( z \)-diagram has no \( z \)-circled vertex, then it has no maximal \( z \)-stroke.

On the same diagram we also draw \( w \)-strokes and \( w \)-circles. Graphically we use another color. The previous rules and definitions apply to \( w \)-strokes and \( w \)-circles. What we will call simply the diagram is the superposition of the \( z \)-diagram and the \( w \)-diagram. We will, for example, adapt Definition 3.4 of an isolated component: a subset of bodies forms an isolated component of the diagram if and only if it forms an isolated component of the \( z \)-diagram and an isolated component of the \( w \)-diagram.

**Definition 3.6 (Edges and strokes)** There is an edge between vertex \( k \) and vertex \( l \) if there is either a \( z \)-stroke, or a \( w \)-stroke, or both. There are three types of edges, \( z \)-edges, \( w \)-edges and \( zw \)-edges, and only two types of strokes, represented with two different colors.
3.1 New normalization. Main estimates.

One does not change a central configuration by multiplying the z coordinates by \( a \in \mathbb{C}\setminus\{0\} \) and the w coordinates by \( a^{-1} \). Our diagram is invariant by such an operation, as it considers the z-coordinates and the w-coordinates separately.

We used the normalization \( z_{12} = w_{12} \) in the previous considerations. In the following we will normalize instead with \( \|Z\| = \|W\| \). We start with a central configuration normalized with the condition \( z_{12} = w_{12} \), then multiply the z-coordinates by \( a > 0 \), the w-coordinates by \( a^{-1} \), in such a way that the maximal component of \( Z \) and the maximal component of \( W \) have the same modulus, i.e., \( \|Z\| = \|W\| \).

A singular sequence was defined by the condition either \( \|Z^{(n)}\| \to \infty \) or \( \|W^{(n)}\| \to \infty \). We also remarked that both \( \|Z^{(n)}\| \) and \( \|W^{(n)}\| \) were bounded away from zero. With the new normalization, a singular sequence is simply characterized by \( \|Z^{(n)}\| = \|W^{(n)}\| \to \infty \). From now on we only discuss singular sequences.

Set \( \|Z^{(n)}\| = \|W^{(n)}\| = 1/\epsilon^2 \), then \( \epsilon \to 0 \).

**Proposition 3.1 (Estimate)** For any \((k, l)\), \(1 \leq k < l \leq N\), we have \( \epsilon^2 \leq z_{kl} \leq \epsilon^{-2}\), \( \epsilon^2 \leq w_{kl} \leq \epsilon^{-2}\) and \( \epsilon^2 \leq r_{kl} \leq \epsilon^{-2}\).

- There is a z-stroke between \( k \) and \( l \) if and only if \( w_{kl} \approx \epsilon^2 \), then \( r_{kl} \approx 1 \).
- There is a maximal z-stroke between \( k \) and \( l \) if and only if \( z_{kl} \approx \epsilon^{-2}, w_{kl} \approx \epsilon^2 \), then \( r_{kl} \approx 1 \).
- There is a z-edge between \( k \) and \( l \) if and only if \( z_{kl} \approx \epsilon^{-2}, w_{kl} \approx \epsilon^2 \), then \( \epsilon^2 \prec r_{kl} \leq 1 \).
- There is a maximal z-edge between \( k \) and \( l \) if and only if \( z_{kl} \approx \epsilon^{-2}, w_{kl} \approx \epsilon^2 \), then \( r_{kl} \approx 1 \).
- There is a zw-edge between \( k \) and \( l \) if and only if \( z_{kl}, w_{kl} \approx \epsilon^2 \), this can be characterized as \( r_{kl} \approx \epsilon^2 \).

**Proof.**
The proof is trivial, so it is omitted.

**Remark 3.1** By the estimates above, the strokes in a zw-edge are not maximal. A maximal z-stroke is exactly a maximal z-edge.

**Rule VI** If there are two consecutive z-stroke, there is a third z-stroke closing the triangle.

**Proof.**
Suppose \( Z_{12} \approx \epsilon^{-2} \) and \( Z_{13} \approx \epsilon^{-2} \), then \( w_{12} \approx \epsilon^2 \) and \( w_{13} \approx \epsilon^2 \).
Therefore, \( w_{23} = w_{13} - w_{12} \leq \epsilon^2 \), but \( w_{23} \geq \epsilon^2 \), thus \( w_{23} \approx \epsilon^2 \), or \( Z_{23} \approx \epsilon^{-2} \).
4 Systematic exclusion of four-vortex diagrams

4.1 Possible diagrams

We call a bicolored vertex of the diagram a vertex which connects at least a stroke of $z$-color with at least a stroke of $w$-color. The number of edges from a bicolored vertex is at least 1 and at most $N - 1$. The number of strokes from a bicolored vertex is at least 2 and at most $2N - 2$. Given a diagram, we define $C$ as the maximal number of strokes from a bicolored vertex. We use this number to classify all possible diagrams.

Recall that the $z$-diagram indicates the maximal terms among a finite set of terms. It is nonempty. If there is a circle, there is an edge of the same color emanating from it. So there is at least a $z$-stroke, and similarly, at least a $w$-stroke.

4.1.1 $C = 0$

If there is no bicolored vertex, then $C$ is not defined or we can say $C = 0$, there are at most two strokes and they are parallel by Rule I, VI and $C = 0$. Thus the only possible diagram is the following Figure 2.

![Figure 2: $C = 0$](image1)

4.1.2 $C = 2$

There are two cases: a $zw$-edge exists or not.

If it is present, it should be isolated. Thus the only possible diagrams are that in the following Figure 3.

![Figure 3: $C = 2$, $zw$-edge appears](image2)

If it is not present, there are adjacent $z$-edges and $w$-edges. From any such adjacency there is no other edge by Rule VI. By trying to continue it, we see that the only diagram is the following Figure 4.
4.1.3 $C = 3$

Consider a bicolored vertex (let us say, 1) with three strokes. There are two cases: a \(zw\)-edge exists or not.

If it is not present, suppose two \(z\)-edges respectively connected vertex 2 and vertex 3, with a \(w\)-edge connected vertex 4. Then vertices 1, 2, 3 formed a \(z\)-color triangle and there are no other strokes between vertices 1, 2, 3 by Rule VI and \(zw\)-edge is not present. So vertices 1, 2, 3 are all \(w\)-circled by Rule I, II and Estimate 3.1. Then by Rule I again, vertex 2 and vertex 3 are both formed \(w\)-strokes with vertex 4, but it follows that vertex 2 and vertex 3 formed a \(w\)-stroke, this is a contradiction.

If it is present, suppose the \(zw\)-edge connected vertex 2, with a \(z\)-edge connected vertex 3. Then vertices 1, 2, 3 formed a \(z\)-color triangle and there is no any other stroke between vertices 1, 2, 3 by Rule VI and $C = 3$. So vertices 1, 2, 3 are all \(w\)-circled by Rule I, II and Estimate 3.1, thus vertex 3 and vertex 4 formed a \(w\)-edge and vertex 4 is \(w\)-circled. Hence the only possible diagrams are that in the following Figure 5.

In the second case, vertex 1 has one adjacent \(zw\)-edge connected vertex 2, and two \(z\)-edges connected vertex 3 and vertex 4 separately. Then vertices 1, 2, 3, 4 form a fully \(z\)-stroked diagram by Rule VI, and there is no any \(w\)-stroke between vertex j and vertex k ($j = 1, 2, k = 3, 4$) by $C = 4$. By Rule I there are \(w\)-circle at vertex 1 and vertex 2, then there are \(w\)-circle at vertex 3 and vertex 4 too. Thus vertex 3 and vertex 4 form a \(w\)-stroke by Rule I. According to if there are \(z\)-circle at vertices, the only possible diagrams are that in the following Figure 7.

\[\text{Figure 4: } C = 2, \text{ no } zw\text{-edge}\]

\[\text{Figure 5: } C = 3, \text{ } zw\text{-edge appears}\]

4.1.4 $C = 4$

Consider a bicolored vertex (let us say, 1 ) with four strokes.

In the first case, vertex 1 has two adjacent \(zw\)-edges connected vertex 2 and vertex 3 separately. A third \(zw\)-edge closes the triangles by Rule VI. As $C = 4$ there is no any stroke connected vertex 4, thus vertex 4 is neither \(z\)-circled nor \(w\)-circled. Hence the only possible diagrams are that in the following Figure 6.

In the second case, vertex 1 has one adjacent \(zw\)-edge connected vertex 2, and two \(z\)-edges connected vertex 3 and vertex 4 separately. Then vertices 1, 2, 3, 4 form a fully \(z\)-stroked diagram by Rule VI, and there is no any \(w\)-stroke between vertex j and vertex k ($j = 1, 2, k = 3, 4$) by $C = 4$. By Rule I there are \(w\)-circle at vertex 1 and vertex 2, then there are \(w\)-circle at vertex 3 and vertex 4 too. Thus vertex 3 and vertex 4 form a \(w\)-stroke by Rule I. According to if there are \(z\)-circle at vertices, the only possible diagrams are that in the following Figure 7.

\[\text{Figure 5: } C = 3, \text{ } zw\text{-edge appears}\]
In the last case, vertex 1 has one adjacent $zw$-edge connected vertex 2, one $z$-edges connected vertex 3 and one $w$-edges connected vertex 4 separately. Then vertices $1,2,3$ formed a $z$-color triangle and $1,2,4$ formed a $w$-color triangle by Rule VI. It is easy to see that there is no any other stroke between vertices by Rule VI and $C = 4$, furthermore, there is no any circle at vertices, otherwise, there will be new stroke between vertices by Rule IV, I and Estimate 3.1 Thus the only possible diagram is the following Figure 8.

4.1.5 $C = 5$

Consider a bicolored vertex (let us say, 1) with five strokes.
Suppose vertex 1 has one adjacent $z$-edge connected vertex 4, and two adjacent $zw$-edges connected vertex 2 and vertex 3 separately. Then vertices $1,2,3$ formed a $w$-color triangle and vertices $1,2,3,4$ form a fully $z$-stroked diagram by Rule VI. As $C = 5$ there is no any $w$-stroke connected vertex 4. Thus there is no any $w$-circle at vertices by Rule II and Estimate 3.1. According to if there are $z$-circle at vertices, the only possible diagrams are that in the following Figure 9.
4.1.6 \( C = 6 \)

Consider a bicolored vertex with six strokes. Then vertices 1, 2, 3, 4 form a fully \( zw \)-edged diagram by Rule VI. According to if there are \( z \)-circle or \( w \)-circle at vertices, the only possible diagrams are that in the following Figure 10.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
4 & & & 4 \\
3 & & & 3 \\
\end{array}
\]

Figure 10: \( C = 6 \)

4.2 Exclusion of diagrams

We derive a list of problematic diagrams which we cannot exclude without further hypotheses on the vorticities, are incorporated into the list of diagrams in Figure 11.

**Proposition 4.1** Suppose a diagram has two \( z \)-circled vertices (say 1 and 2) which are also \( z \)-close, if none of all the other vertices is \( z \)-close with them, then \( \Gamma_1 + \Gamma_2 \neq 0 \) and \( \overline{\Lambda}z_{12}w_{12} \sim \Gamma_1 + \Gamma_2 \). In particular, vertices 1 and 2 cannot form a \( z \)-stroke.

**Proof.** According to \( \overline{\Lambda}w_1 = \sum_{j \neq 1} \Gamma_j W_{j1} \) and \( \overline{\Lambda}w_2 = \sum_{j \neq 2} \Gamma_j W_{j2} \), it follows that

\[
\overline{\Lambda}w_{12} = (\Gamma_1 + \Gamma_2)W_{12} + \sum_{j > 2} \Gamma_j (W_{j2} - W_{j1}).
\]  

(4.13)

It is easy to see that \( W_{j2} - W_{j1} = \frac{1}{z_{j2}} - \frac{1}{z_{j1}} < \epsilon^2 \) for any \( j > 2 \), therefore,

\[
\overline{\Lambda}w_{12} \sim (\Gamma_1 + \Gamma_2)/z_{12}.
\]

Note that \( \epsilon^2 \leq z_{kl} < \epsilon^{-2} \), \( \epsilon^2 \leq w_{kl} \leq \epsilon^{-2} \), the proof is trivial now. \( \square \)
Proposition 4.2 Suppose a diagram has an isolated $z$-stroke, then vertices of it are both $z$-circled; if the two vertices are $z$-close (for example, provided the two vertices are connected by $w$-stroke), then the total vorticity of them is zero.

Proof.

The proof is trivial by Rule I, IV and Estimate 3.1. □

Proposition 4.3 Suppose a diagram has an isolated $z$-color triangle, and none of vertices (say $1, 2, 3$) of it are $z$-circled, then

\[ Z_{12} \approx \frac{Z_{13}}{\Gamma_2} \approx \frac{Z_{31}}{\Gamma_1}, \]

or

\[ w_{12} \Gamma_3 \approx w_{23} \Gamma_1 \approx w_{31} \Gamma_2. \]

(4.14)

According to the fact that $w_{12} + w_{23} + w_{31} = 0$, it follows that $\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0$. □

Proposition 4.4 Suppose a fully $z$-stroked sub-diagram with four vertices exists in isolation in a diagram, and none of vertices (say $1, 2, 3, 4$) of it are $z$-circled, then

\[ L_{1234} = \Gamma_1 \Gamma_2 + \Gamma_2 \Gamma_3 + \Gamma_3 \Gamma_4 + \Gamma_4 (\Gamma_1 + \Gamma_2 + \Gamma_3) = 0. \]

(4.15)

Proof.

According to $\Lambda z_n = \sum_{j \neq n} \Gamma_j Z_{jn}$, it follows that

\[ \Gamma_2 Z_{12} + \Gamma_3 Z_{13} + \Gamma_4 Z_{14} \approx \epsilon^{-2}, \quad \Gamma_1 Z_{21} + \Gamma_3 Z_{23} + \Gamma_4 Z_{24} \approx \epsilon^{-2}, \quad \Gamma_1 Z_{31} + \Gamma_2 Z_{32} + \Gamma_4 Z_{34} \approx \epsilon^{-2}. \]

Thus

\[ Z_{13} \sim -\frac{\Gamma_2 Z_{12} - \Gamma_4 Z_{14}}{\Gamma_3}, \quad Z_{24} \sim \frac{\Gamma_1 Z_{21} - \Gamma_1 Z_{23} + \Gamma_3 Z_{23}}{\Gamma_4}. \]

According to the fact that $w_{12} + w_{23} + w_{31} = 0$, it follows that

\[ \Gamma_2 Z_{12} + (\Gamma_2 + \Gamma_3) Z_{23} Z_{12} \sim -\Gamma_4 Z_{14} (Z_{12} + Z_{23}). \]

Then $Z_{12} + Z_{23} \approx \epsilon^{-2}$ and

\[ Z_{14} \sim -\frac{\Gamma_2 Z_{12}^2 + (\Gamma_2 + \Gamma_3) Z_{23} Z_{12}}{\Gamma_4 (Z_{12} + Z_{23})}. \]
Set $Z_{12} \sim \epsilon^{-2}$ and $Z_{24} \sim a\epsilon^{-2}$, where $a \in \mathbb{C}\{0,-1\}$, then

\[
Z_{13} \sim \frac{a}{a+1}\epsilon^{-2}, \quad Z_{34} \sim \frac{a((a+1)\Gamma_2+\Gamma_1)}{(a+1)\Gamma_4} \epsilon^{-2}, \quad Z_{24} \sim \frac{\Gamma_1-a\Gamma_3}{\Gamma_4} \epsilon^{-2}, \quad Z_{14} \sim \frac{\frac{\epsilon^2 \Gamma_1}{\Gamma_4}}{\frac{\epsilon^2 (a+1)\Gamma_2+\Gamma_1}{(a+1)\Gamma_4}} \epsilon^{-2}.
\]

According to the facts that $w_{12} + w_{24} + w_{41} = 0$, $w_{13} + w_{34} + w_{41} = 0$ and $w_{23} + w_{34} + w_{42} = 0$, it follows that

\[
-a^2 \Gamma_3(\Gamma_3 + \Gamma_4) + (a + 1) \Gamma_2(\Gamma_4 - a \Gamma_3) + \Gamma_1((a + 1) \Gamma_2 + a(\Gamma_3 + \Gamma_4) + \Gamma_4) = 0, \quad (4.16)
\]

\[
(a + 1)^2 \Gamma_2^2 + (a + 1) \Gamma_2(a(\Gamma_3 + \Gamma_4) + \Gamma_4) + a \Gamma_3 \Gamma_4 + \Gamma_1((a + 1) \Gamma_2 + a(\Gamma_3 + \Gamma_4)) = 0, \quad (4.17)
\]

\[
- \Gamma_1((a + 1) \Gamma_2 - a \Gamma_3 + \Gamma_4) + a(a + 1)(\Gamma_3 \Gamma_4 + \Gamma_2(\Gamma_3 + \Gamma_4)) - \Gamma_1^2 = 0. \quad (4.18)
\]

A straightforward computation of Groebner basis for (4.16), (4.17) and (4.18) shows that

\[
(\Gamma_2 \Gamma_3 + \Gamma_1(\Gamma_2 + \Gamma_3))(\Gamma_3 \Gamma_4 + \Gamma_2(\Gamma_3 + \Gamma_4) + \Gamma_1(\Gamma_2 + \Gamma_3 + \Gamma_4)) = 0.
\]

By the symmetry of vertices 1, 2, 3, 4, the following relations also hold.

\[
(\Gamma_2 \Gamma_4 + \Gamma_1(\Gamma_2 + \Gamma_4))(\Gamma_2 \Gamma_3 + \Gamma_4) + \Gamma_1(\Gamma_2 + \Gamma_3 + \Gamma_4) + \Gamma_3 \Gamma_4) = 0.
\]

\[
(\Gamma_3 \Gamma_4 + \Gamma_1(\Gamma_3 + \Gamma_4))(\Gamma_1(\Gamma_2 + \Gamma_3 + \Gamma_4) + (\Gamma_2 + \Gamma_4) \Gamma_3 + \Gamma_2 \Gamma_4) = 0.
\]

\[
(\Gamma_3 \Gamma_4 + \Gamma_2(\Gamma_3 + \Gamma_4))(\Gamma_1 + \Gamma_3 + \Gamma_4) \Gamma_2 + (\Gamma_1 + \Gamma_4) \Gamma_3 + \Gamma_1 \Gamma_4) = 0.
\]

It is easy to see that there must be $L_{1234} = 0$.

\[\square\]

**Remark 4.1** In general, we claim that

**Proposition 4.5** Suppose a fully z-stroked sub-diagram with $n$ vertices exists in isolation in a diagram, and none of vertices (say 1, 2, ···, n) of it are z-circled, then

\[L_{1\cdots n} = \sum_{1 \leq j < k \leq n} \Gamma_j \Gamma_k = 0.\]

However it is not easy to prove this result for general $n$.

### 4.2.1 Problematic diagrams

First, we give a simple result for the following discussions.

**Proposition 4.6** Suppose $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are all nonzero, then

\[
\begin{cases}
\Gamma_1 + \Gamma_2 + \Gamma_3 = 0 \\
\frac{1}{\Gamma_1} = \frac{1}{\Gamma_2} = \frac{1}{\Gamma_3} = 0
\end{cases} \iff \begin{cases}
\Gamma_1 + \Gamma_2 + \Gamma_3 = 0 \\
L_{1234} = 0
\end{cases} \iff \begin{cases}
L_{1234} = 0 \\
\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0
\end{cases}
\]

And in this case, $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ cannot all be real.
Proof.
The proof is simple, and we only point out that the inequality
\[(\Gamma_1 + \Gamma_2 + \Gamma_3)^2 - 2(\Gamma_1 \Gamma_2 + \Gamma_2 \Gamma_3 + \Gamma_3 \Gamma_1) > 0\]
for real \(\Gamma_1, \Gamma_2, \Gamma_3\).
\[\blacksquare\]

As a result of Proposition 4.1 and Proposition 4.2, it is easy to see that the first two
diagrams in Figure 3, the second diagram in Figure 5 and the second diagram in Figure 7 are impossible.
The first diagram in Figure 5 is impossible by Proposition 4.2 and Proposition 4.3.
The second diagram in Figure 6 and the second diagram in Figure 9 are impossible
by Rule IV and Proposition 4.3.
The first diagram in Figure 7 is impossible by Proposition 4.2 and Proposition 4.4.
The first diagram in Figure 9 is impossible by Proposition 4.3 and Proposition 4.4.
The second diagram in Figure 10 is impossible by Rule IV and Proposition 4.4.
The conclusion of this section is that any singular sequence should converge to one
of the ten diagrams in Figure 11.

5 Problematic diagrams

We could not eliminate the diagrams in Figure 11. Some singular sequence could
still exist and approach any of these diagrams. In this section we obtain the constraints
on the vorticities corresponding to each of the ten diagrams from Figure 11.

5.1 Diagram I

First, if \(\Gamma_1 + \Gamma_2 = 0\), then two z-circled vertices 1 and 2 are z-close, however this
contradicts Proposition 4.1. Thus \(\Gamma_1 + \Gamma_2 \neq 0\). Similarly, we have also \(\Gamma_3 + \Gamma_4 \neq 0\).

Without loss of generality, assume \(z_1 \sim -\Gamma_2 \epsilon^{-2}\) and \(z_2 \sim \Gamma_1 \epsilon^{-2}\).

According to (4.13), it follows that
\[
\Lambda w_{12} = (\Gamma_1 + \Gamma_2)W_{12} + \Gamma_3(W_{32} - W_{31}) + \Gamma_4(W_{42} - W_{41}).
\]

(5.19)

It is easy to see that
\[
W_{12} = \frac{1}{z_{12}} \sim \frac{\epsilon^2}{\Gamma_1 + \Gamma_2}
\]

and
\[
W_{j2} - W_{j1} = \frac{1}{z_{j2}} - \frac{1}{z_{j1}} \sim \frac{1}{z_2} - \frac{1}{z_1} \sim (\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2}) \epsilon^2
\]

for any \(j \in \{3, 4\}\).

By \(\Lambda z_n = \sum_{j \neq n} \Gamma_j Z_{jn}\) it follows that
\[
\Lambda z_2 \sim \Gamma_1 Z_{12}, \quad \text{or} \quad w_{12} \sim \frac{\epsilon^2}{\Lambda}.
\]
Therefore, 
\[ \Lambda / \Lambda = 1 + (\Gamma_3 + \Gamma_4)\left(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2}\right). \]

First, it follows that \( \Lambda \in \mathbb{C} \setminus \mathbb{R} \), otherwise, we have \((\Gamma_1 + \Gamma_2)(\Gamma_3 + \Gamma_4) = 0\), this gives a contradiction.

As a result, we have \( \Lambda = \pm i \) and
\[ (\Gamma_1 + \Gamma_2)(\Gamma_3 + \Gamma_4) + 2\Gamma_1 \Gamma_2 = 0. \quad (5.20) \]

On the other hand, it follows from \( \Lambda \in \mathbb{C} \setminus \mathbb{R} \) that
\[ L = \sum_{1 \leq j < k \leq 4} \Gamma_j \Gamma_k = 0 \quad (5.21) \]
and the total vorticity \( \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \neq 0 \).

By (5.20) and (5.21), it follows that
\[ \Gamma_1 \Gamma_2 - \Gamma_3 \Gamma_4 = 0. \]

To summarize, the following relations on the four vorticities should be satisfied if the Diagram I is approached by a singular sequence:
\[ \Lambda = \pm i, \]
\[ \Gamma_1 \Gamma_2 = \Gamma_3 \Gamma_4, \]
\[ L = 0, \]
\[ \Gamma \neq 0, \]
\[ \Gamma_1 + \Gamma_2 \neq 0, \Gamma_3 + \Gamma_4 \neq 0. \quad (5.22) \]
Moreover, for Diagram I it is noteworthy that
\[ r_{12}, r_{34} \approx 1, \quad r_{13}, r_{23}, r_{14}, r_{24} \approx \epsilon^{-2}. \] (5.23)

The proof is straightforward and we omit it.

5.2 Diagram II

By Proposition 4.4 it follows that
\[ L = \sum_{1 \leq j < k \leq 4} \Gamma_j \Gamma_k = 0, \] (5.24)
thus the total vorticity \( \Gamma \neq 0 \).

5.3 Diagram III

For Diagram III we can assume that
\[ z_1 \sim z_4 \sim -\Gamma_2 a \epsilon^{-2}, \quad z_2 \sim z_3 \sim \Gamma_1 a \epsilon^{-2}, \]
\[ w_1 \sim w_2 \sim -\Gamma_4 b \epsilon^{-2}, \quad w_3 \sim w_4 \sim \Gamma_1 b \epsilon^{-2}. \]

It follows that
\[ \Gamma_1 \Gamma_3 = \Gamma_2 \Gamma_4. \] (5.25)

**Case A:** \( \Gamma \neq 0 \).
We claim that
\[ (\Gamma_1 + \Gamma_2)(\Gamma_2 + \Gamma_3)(\Gamma_3 + \Gamma_4)(\Gamma_1 + \Gamma_4) \neq 0. \]

Otherwise, without loss of generality, assume \( \Gamma_1 + \Gamma_2 = 0 \), by (5.25) it follows that \( \Gamma_3 + \Gamma_4 = 0 \), this contradicts \( \Gamma \neq 0 \).

In this case it is easy to see that
\[ r_{12}^2 \sim \frac{\Gamma_1 + \Gamma_3}{\Lambda}, \quad r_{23}^2 \sim \frac{\Gamma_2 + \Gamma_3}{\Lambda}, \quad r_{34}^2 \sim \frac{\Gamma_3 + \Gamma_4}{\Lambda}, \quad r_{14}^2 \sim \frac{\Gamma_1 + \Gamma_4}{\Lambda}, \]
\[ r_{13}^2 \sim (\Gamma_1 + \Gamma_2)(\Gamma_1 + \Gamma_4)ab\epsilon^{-4}, \quad r_{24}^2 \sim -(\Gamma_1 + \Gamma_2)(\Gamma_1 + \Gamma_4)ab\epsilon^{-4}. \] (5.26)

**Case B:** \( \Gamma = 0 \).
We claim that
\[ (\Gamma_1 + \Gamma_2)(\Gamma_2 + \Gamma_3)(\Gamma_3 + \Gamma_4)(\Gamma_1 + \Gamma_4) = 0. \]

Indeed, by (5.25) and \( \Gamma = 0 \) it follows that
\[ (\Gamma_1 + \Gamma_2)(\Gamma_1 + \Gamma_4) = 0. \]
The claim above is obvious now.

Furthermore, we claim that
\[ \Gamma_1 + \Gamma_2 = \Gamma_3 + \Gamma_4 = \Gamma_1 + \Gamma_4 = 0. \quad (5.27) \]

Without loss of generality, assume \( \Gamma_1 + \Gamma_2 = 0 \), by (5.25) it follows that \( \Gamma_3 + \Gamma_4 = 0 \).
If \( \Gamma_2 + \Gamma_3 \neq 0 \), by (5.25) it follows that \( \Gamma_1 + \Gamma_4 \neq 0 \). Then
\[ w_{32} \sim w_{31} \sim w_{42} \sim w_{41} \sim -(\Gamma_1 + \Gamma_4)b\epsilon^{-2}. \]

According to \( \Lambda z_1 = \sum_{j\neq1} \Gamma_j Z_{j1} \) and \( \Lambda z_2 = \sum_{j\neq2} \Gamma_j Z_{j2} \), it follows that
\[ \Lambda z_{12} = (\Gamma_1 + \Gamma_2)Z_{12} + \Gamma_3(Z_{32} - Z_{31}) + \Gamma_4(Z_{42} - Z_{41}). \]

Thus
\[ \epsilon^2 \prec \Lambda z_{12} = \Gamma_3\left(\frac{1}{w_{32}} - \frac{1}{w_{31}}\right) + \Gamma_4\left(\frac{1}{w_{42}} - \frac{1}{w_{41}}\right) \prec \epsilon^2, \]
this is a contradiction.

It follows that
\[ \Gamma_1 = \Gamma_3, \quad \Gamma_2 = \Gamma_4, \quad \Gamma = 0. \]

Then \( L \neq 0 \) and \( \Lambda = \pm 1 \).

Moreover, in this case it is noteworthy that
\[ r_{12}, r_{23}, r_{34}, r_{41} \prec 1. \quad (5.28) \]

The rest \( r_{24} \) and \( r_{31} \) maybe any case in \( < 1, \approx 1 \) or \( > 1 \), but by \( S = \sum_{1 \leq j < k \leq 4} \Gamma_j \Gamma_k r_{jk}^2 = 0 \) it follows that \( r_{24} \) and \( r_{31} \) are both \( > 1 \) or not.

5.4 Diagram III’

By Proposition 4.2 it follows that
\[ \Gamma_1 + \Gamma_2 = 0, \quad \Gamma_3 + \Gamma_4 = 0. \]

Then \( \Gamma = 0 \implies L \neq 0 \) and \( \Lambda = \pm 1 \).

By Proposition 4.1 it follows that
\[ z_1 \sim z_2 \sim z_3 \sim z_4 \sim a\epsilon^{-2}, \]
\[ w_1 \sim w_2 \sim w_3 \sim w_4 \sim b\epsilon^{-2}. \]

It is easy to see that
\[ z_{12} \sim z_{34} \sim w_{12} \sim z_{34} \approx \epsilon^2, \]
\[ \epsilon^2 \prec z_{32} \sim z_{31} \sim z_{42} \sim z_{41} \prec \epsilon^{-2}, \]
\[ \epsilon^2 \prec w_{32} \sim w_{31} \sim w_{42} \sim w_{41} \prec \epsilon^{-2}. \]

By
\[ \Lambda z_{12} = (\Gamma_1 + \Gamma_2)Z_{12} + \Gamma_3(Z_{32} - Z_{31}) + \Gamma_4(Z_{42} - Z_{41}), \]
it follows that
\[ \frac{\Lambda z_{12}}{w_{21}} = \frac{\Gamma_3}{w_{13}w_{23}} + \frac{\Gamma_4}{w_{14}w_{24}} < \frac{1}{w_{13}^2}, \]
thus \( w_{13} < 1 \). Similarly, \( z_{13} < 1 \). Hence
\[ \epsilon^4 < r_{32}^2 \approx r_{31}^2 \approx r_{42}^2 \approx r_{41}^2 < 1. \]

To summarize, the following relations on the four vorticities should be satisfied if the Diagram \( III' \) is approached by a singular sequence:

\[ \Lambda = \pm 1, \quad \Gamma_1 + \Gamma_2 = 0, \quad \Gamma_3 + \Gamma_4 = 0, \quad \Gamma = 0, \quad L \neq 0, \quad r_{12}^2 \approx r_{34}^2 \approx \epsilon^4 < r_{32}^2 \approx r_{31}^2 \approx r_{42}^2 \approx r_{41}^2 < 1. \quad (5.29) \]

### 5.5 Diagram IV

By Proposition 4.2 it follows that
\[ \Gamma_1 + \Gamma_2 = 0, \quad \Gamma_3 + \Gamma_4 = 0. \]

Then \( \Gamma = 0 \implies L \neq 0 \) and \( \Lambda = \pm 1 \).

Moreover, for Diagram IV it is noteworthy that
\[ r_{14}, r_{24}, r_{34}, r_{12}, r_{23}, r_{31} \leq 1. \quad (5.30) \]

Indeed, let us just notice that \( w_{jk} \approx \epsilon^2, z_{jk} \leq \epsilon^{-2} \) for any \((j, k), 1 \leq j < k \leq 4\).

To summarize, the following relations on the four vorticities should be satisfied if the Diagram IV is approached by a singular sequence:

\[ \Lambda = \pm 1, \quad \Gamma_1 + \Gamma_2 = 0, \quad \Gamma_3 + \Gamma_4 = 0, \quad \Gamma = 0, \quad L \neq 0. \quad (5.31) \]

### 5.6 Diagram V

By Rule IV and Estimate 3.1 it follows that
\[ \Gamma = 0. \quad (5.32) \]

Then \( L \neq 0 \) and \( \Lambda = \pm 1 \).

Moreover, for Diagram V
\[ r_{14}, r_{24}, r_{34}, r_{12}, r_{23}, r_{31} \approx \epsilon^2. \quad (5.33) \]
5.7 Diagram VI

By Proposition 4.3 it follows that

\[ \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0. \]  \hspace{1cm} (5.34)

Then it is easy to see that the total vortex angular momentum \( L \neq 0 \), as a result, \( \Lambda = \pm 1 \).

We claim that the total vorticity \( \Gamma \neq 0 \). If not, by (2.8)

\[
M = \sum_{j=1}^{4} \Gamma_j z_j = 0,
\]

it follows that

\[
\Gamma_1 z_{14} + \Gamma_2 z_{24} + \Gamma_3 z_{34} = \Gamma z_4 = 0.
\]  \hspace{1cm} (5.35)

For Diagram VI it is easy to see that

\[ z_{14}, z_{24}, z_{34} \succ \epsilon^2; \quad z_{12}, z_{23}, z_{31} \approx \epsilon^2. \]

Thus

\[ z_{14} \sim z_{24} \sim z_{34} \succ \epsilon^2. \]

Then by (5.35) it follows that

\[ \Gamma_1 + \Gamma_2 + \Gamma_3 = 0, \]

this contradicts (5.34).

We also claim that

\[ r_{14} \sim r_{24} \sim r_{34} \preceq 1. \]

Indeed, by \( z_{14} \sim z_{24} \sim z_{34} \) and \( w_{14} \sim w_{24} \sim w_{34}, r_{14} \sim r_{24} \sim r_{34} \) holds. If \( r_{34} \succ 1 \), then by \( S = \sum_{1 \leq j < k \leq 4} \Gamma_j \Gamma_k r_{jk}^2 \) it follows that

\[(\Gamma_1 + \Gamma_2 + \Gamma_3) \Gamma_4 = 0.\]

this contradicts (5.34).

To summarize, the following relations on the four vorticities should be satisfied if the Diagram VI is approached by a singular sequence:

\[ \Lambda = \pm 1, \]

\[ \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0, \]

\[ L \neq 0, \Gamma \neq 0. \]  \hspace{1cm} (5.36)

Moreover, for Diagram VI

\[ r_{12}, r_{23}, r_{31} \approx \epsilon^2; \quad \epsilon^2 \prec r_{14}, r_{24}, r_{34} \preceq 1 \quad \text{and} \quad r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23} \prec 1. \]  \hspace{1cm} (5.37)
5.8 Diagram VII

By Rule IV and Estimate \[3.1\] it follows that
\[
\Gamma_1 + \Gamma_2 + \Gamma_3 = 0. \tag{5.38}
\]
Then it is easy to see that the total vortex angular momentum \(L \neq 0\) and the total vorticity \(\Gamma \neq 0\). As a result, \(\Lambda \in \mathbb{R}\) or \(\Lambda = \pm 1\).

Moreover, for Diagram VII
\[
\begin{align*}
  r_{12}, r_{23}, r_{31} &\approx \varepsilon^2, & r_{14}, r_{24}, r_{34} &\approx \varepsilon^{-2} & \text{and} & r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23} &\approx 1. \tag{5.39}
\end{align*}
\]

5.9 Diagram VIII

By Proposition \[4.3\] it follows that
\[
\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0.
\]
Then it is easy to see that the total vortex angular momentum \(L \neq 0\). As a result, \(\Lambda \in \mathbb{R}\) or \(\Lambda = \pm 1\).

We claim that the total vorticity \(\Gamma \neq 0\). If not, we have
\[
\begin{align*}
  \Gamma_1 + \Gamma_2 &= -2\Gamma_3, \\
  \Gamma_1\Gamma_2 &= -\Gamma_3(\Gamma_1 + \Gamma_2) = 2\Gamma_3^2.
\end{align*}
\]
But this contradicts \((\Gamma_1 + \Gamma_2)^2 \geq 4\Gamma_1\Gamma_2\).

Moreover, for Diagram VIII it is noteworthy that
\[
r_{34} \approx 1 \tag{5.40}
\]
and all the other \(r_{jk} \prec 1\). Indeed, it is easy to see that all \(r_{jk} \prec 1\) hold except \(r_{34}\). And \(r_{34} \approx 1\) holds because \(S = \Gamma I = \frac{L}{\Lambda} \neq 0\). Thus \(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23} \prec 1\)

To summarize, the following relations on the four vorticities should be satisfied if the Diagram VIII is approached by a singular sequence:
\[
\begin{align*}
  \Lambda &= \pm 1, \\
  \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} &= 0, \\
  \Gamma_3 &= \Gamma_4, \\
  L &\neq 0, \Gamma \neq 0. \tag{5.41}
\end{align*}
\]

5.10 Diagram IX

By Proposition \[4.3\] it follows that
\[
\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0.
\]
Then \(L \neq 0\) and \(\Lambda = \pm 1\).
We claim that the total vorticity $\Gamma \neq 0$. If not, we have
\[ \Gamma_1 z_{14} + \Gamma_2 z_{24} + \Gamma_3 z_{34} = \Gamma z_4 = 0. \] (5.42)
For Diagram IX it is easy to see that
\[ z_{14}, z_{24}, z_{34} \gg \epsilon^2; \quad z_{12}, z_{23}, z_{31} \approx \epsilon^2. \]
Thus
\[ z_{14} \sim z_{24} \sim z_{34} \gg \epsilon^2. \] (5.43)
Then by (5.42) it follows that
\[ \Gamma_1 + \Gamma_2 + \Gamma_3 = 0, \]
this contradicts (5.42).

Moreover, by (5.42) and (5.43) it follows that
\[ z_{14} \sim z_{24} \sim z_{34} \approx \epsilon^{-2}. \]

Note that $w_{14}, w_{24}, w_{34} \approx \epsilon^2$, thus for Diagram IX we have
\[ r_{14}, r_{24}, r_{34} \approx 1, \quad r_{12}, r_{23}, r_{31} \approx \epsilon^2 \quad \text{and} \quad r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23} \ll 1. \] (5.44)

To summarize, the following relations on the four vorticities should be satisfied if the Diagram IX is approached by a singular sequence:
\[ \Lambda = \pm 1, \]
\[ \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0, \]
\[ L \neq 0, \Gamma \neq 0. \] (5.45)

6 Finiteness results

The main Theorem 1.3 in this paper is an obvious inference of the following Theorem 6.1, 6.2, 6.3, 6.5 and 6.6. And we remark that the following results of finiteness are all on normalized central configurations in the complex domain, more than real configurations.

6.1 Central configurations with $L = 0$

First we consider the finiteness of central configurations with vanishing total vortex angular momentum, i.e., $L = 0$. In this case only Diagram I, Diagram II and Diagram III in Case A are possible.

**Theorem 6.1** If the vorticities $\Gamma_n \ (n \in \{1, 2, 3, 4\})$ are nonzero such that $L = 0$, then the four-vortex problem has finitely many relative equilibria; and except perhaps for two pairs of equal vorticities with ratios $(\sqrt{3} - 2)^{\pm 1}$ (for example, $\Gamma_3 = \Gamma_4 = (\sqrt{3} - 2)^{\pm 1}\Gamma_1 = (\sqrt{3} - 2)^{\pm 1}\Gamma_2$) with $\Lambda = \pm i$, the four-vortex problem has finitely many collapse configurations.
Proof.

Elementary geometry shows that giving five of the \( r_{jk}^2 \)'s \((1 \leq j < k \leq 4)\), the six squares of the mutual distances, determines finitely many geometrical configurations up to rotation. If there are infinitely many solutions of (2.12), at least two of the \( r_{jk}^2 \)'s should take infinitely many values. We suppose \( r_{12}^2 \) does, and we take it as the polynomial function in Lemma 2.1, thus \( r_{12}^2 \) is dominating. There is a sequence of normalized central configurations such that \( r_{12}^2 \rightarrow 0 \), i.e., \( z_1^{(n)}w_1^{(n)} \rightarrow 0 \). Whatever the renormalization is, \( Z^{(n)} \) or \( W^{(n)} \) is unbounded on this sequence. We extract a singular sequence. It corresponds to one of the diagrams in Figure 11, in fact, only Diagram II is possible. Therefore, all of the \( r_{jk}^2 \)'s should go to zero. Thus all of them take infinitely many values. As a result, the \( r_{jk}^2 \)'s are all dominating. Similarly, \( r_{12}^2r_{34}^2 \), \( r_{13}^2r_{24}^2 \) and \( r_{14}^2r_{23}^2 \) are all dominating.

Diagram II and Diagram III may correspond to relative equilibria or collapse configurations.

Case 1: When corresponding to relative equilibria.

Note that in this case only Diagram II and Diagram III in Figure 11 are possible for \( L = 0 \).

By considering \( r_{12}^2 \rightarrow \infty \), we are in Diagram III, then

\[
\Gamma_1\Gamma_2 = \Gamma_3\Gamma_4. \tag{6.46}
\]

Similarly, push \( r_{13}^2 \) and \( r_{14}^2 \) to infinity, we have

\[
\Gamma_1\Gamma_3 = \Gamma_2\Gamma_4, \\
\Gamma_1\Gamma_4 = \Gamma_2\Gamma_3. \tag{6.47}
\]

Note that \( \Gamma \neq 0 \), by (6.46) and (6.47) it follows that

\[
\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4.
\]

However, this contradicts \( L = 0 \).

Case 2: When corresponding to collapse configurations.

In this case Diagram I, Diagram II and Diagram III in Figure 11 are all possible for \( L = 0 \).

If Diagram I does not occur, a similar discussion as above shows the Theorem. Thus, without lose of generality, we assume that there always exists a singular sequence of collapse configurations approaching Diagram I below. Then \( \Lambda = \pm i \).

Consider \( r_{12}^2r_{34}^2 \). Push it to infinity. We are in Diagram I or Diagram III. It is easy to see that the constraint on the vorticities in Diagram I is \( \Gamma_1\Gamma_3 = \Gamma_2\Gamma_4 \) or \( \Gamma_1\Gamma_4 = \Gamma_2\Gamma_3 \); the constraint on the vorticities in Diagram III is \( \Gamma_1\Gamma_2 = \Gamma_3\Gamma_4 \). So we have

\[(\Gamma_1\Gamma_2 - \Gamma_3\Gamma_4)(\Gamma_1\Gamma_3 - \Gamma_2\Gamma_4)(\Gamma_1\Gamma_4 - \Gamma_2\Gamma_3) = 0.\]

We claim that at least two of \( \Gamma_1\Gamma_2 - \Gamma_3\Gamma_4, \Gamma_1\Gamma_3 - \Gamma_2\Gamma_4 \) and \( \Gamma_1\Gamma_4 - \Gamma_2\Gamma_3 \) are zero. If the claim is true, without lose of generality, assume

\[
\Gamma_1\Gamma_3 - \Gamma_2\Gamma_4 = 0, \quad \Gamma_1\Gamma_4 - \Gamma_2\Gamma_3 = 0. \tag{6.48}
\]
In addition to these equations, we also have
\[ L = \Gamma_1 \Gamma_2 + \Gamma_2 \Gamma_3 + \Gamma_3 \Gamma_1 + (\Gamma_1 + \Gamma_2 + \Gamma_3)\Gamma_4 = 0. \quad (6.49) \]

A straightforward computation shows that the solutions of the equations (6.48) and (6.49) are
\[ \Gamma_1 = \Gamma_2, \quad \Gamma_3 = \Gamma_4 = (-\sqrt{3} - 2)\Gamma_1 \text{ or } \Gamma_3 = \Gamma_4 = (\sqrt{3} - 2)\Gamma_1. \]

**By reduction to absurdity to prove the claim:** If the claim is not true, without lose of generality, assume
\[ \Gamma_1 \Gamma_2 - \Gamma_3 \Gamma_4 = 0, \quad (\Gamma_1 \Gamma_3 - \Gamma_2 \Gamma_4)(\Gamma_1 \Gamma_4 - \Gamma_2 \Gamma_3) \neq 0. \quad (6.50) \]
Then the complete diagrams may be approached by some singular sequence are the Figure 12.

We take \( r_{12}^2 + r_{34}^2 \) as the polynomial function in Lemma (2.1) and claim that it is not dominating. Otherwise, by considering \( r_{12}^2 + r_{34}^2 \to \infty \), it is only possible that we are in Diagram III of Figure 12. However, by
\[ S - \Gamma_1 \Gamma_3 r_{13}^2 - \Gamma_1 \Gamma_4 r_{14}^2 - \Gamma_2 \Gamma_3 r_{23}^2 - \Gamma_2 \Gamma_4 r_{24}^2 = \Gamma_1 \Gamma_2 (r_{12}^2 + r_{34}^2) \]
and the results in (5.26) we know that this is a contradiction.

Thus we consider the system (2.12) by restricting to some level set \( r_{12}^2 + r_{34}^2 \equiv \text{const} \) in the following. It is easy to see that it suffices to consider the level set \( r_{12}^2 + r_{34}^2 \equiv 0 \). That is, if system (2.12) possesses infinitely many solutions, then some level set \( r_{12}^2 + r_{34}^2 \equiv 0 \) also includes infinitely many solutions of system (2.12).
By considering $r_{12}^2 \to \infty$, we are in Diagram III of Figure 12. Recall that
\[
S = \sum_{1 \leq i < k \leq 4} \Gamma_i \Gamma_k r_{jk}^2 = 0, \quad r_{14}^2 \sim \frac{\Gamma_1 + \Gamma_4}{\pm 1}, \quad r_{42}^2 \sim \frac{\Gamma_4 + \Gamma_2}{\pm 1}, \quad r_{23}^2 \sim \frac{\Gamma_2 + \Gamma_3}{\pm 1}, \quad r_{31}^2 \sim \frac{\Gamma_3 + \Gamma_1}{\pm 1}.
\]
It follows that
\[
\Gamma_1 \Gamma_3 (\Gamma_1 + \Gamma_3) - \Gamma_2 \Gamma_3 (\Gamma_2 + \Gamma_3) - \Gamma_1 \Gamma_4 (\Gamma_1 + \Gamma_4) + \Gamma_2 \Gamma_4 (\Gamma_2 + \Gamma_4) = 0
\]
or
\[
(\Gamma_1 - \Gamma_2) (\Gamma_3 - \Gamma_4) \Gamma = 0. \tag{6.51}
\]
However, there is no solution for the equations (6.51), (6.50) and (6.49). This proves the claim.

\[\square\]

**Theorem 6.2** Suppose $\Gamma_3 = \Gamma_4 = (\sqrt{3} - 2)^{\pm 1} \Gamma_1 = (\sqrt{3} - 2)^{\pm 1} \Gamma_2$, then the four-vortex problem with $\Lambda = \pm i$ does not have any collapse configurations.

**Proof.** We consider the system (7.57) with $\Lambda = \pm i$.

Without loss of generality, suppose $\Gamma_1 = \Gamma_2 = 1, \Gamma_3 = \Gamma_4 = \kappa$, where $\kappa = (\sqrt{3} - 2)^{\pm 1}$ are the roots of the polynomial $f = \kappa^2 + 4\kappa + 1$. Then a simple computation by Mathematica shows that the system (7.57) combined the equation $f = 0$ does not have any solution.

\[\square\]

### 6.2 Central configurations with $\Gamma = 0$

Next we consider the finiteness of central configurations with vanishing total vorticity, i.e., $\Gamma = 0$. In this case only Diagram III in Case B, Diagram III', Diagram IV and Diagram V are possible. These Diagrams can only correspond to relative equilibria, so the following result of finiteness has been proved essentially by O’Neil [17] and Hampton and Moeckel [9] independently.

**Theorem 6.3** If the vorticities $\Gamma_n \ (n \in \{1, 2, 3, 4\})$ are nonzero such that $\Gamma = 0$, then the four-vortex problem has finitely many central configurations.

**Proof.** A similar argument as in the proof of Theorem 6.1 shows that at least two of the $r_{jk}^2$’s should take infinitely many values. We suppose $r_{13}^2$ does, then $r_{13}^2$ is dominating.

There is a sequence of normalized central configurations such that $r_{13}^{(n)} \to \infty$, i.e., $z_{13}^{(n)} w_{13}^{(n)} \to \infty$. Then we can extract a singular sequence. It corresponds to one of the diagrams in Figure 11 in fact, only Diagram III is possible. In this case, it follows that $r_{13}^2 \to \infty, r_{24}^2 \to \infty$ and all of the rest of $r_{jk}^2$’s should go to zero. Thus all of them
take infinitely many values. As a result, the $r^2_{jk}$'s are all dominating. Similarly, $r^2_{12}r^2_{34}$, $r^2_{13}r^2_{24}$ and $r^2_{14}r^2_{23}$ are all dominating.

Consider $r^2_{13}r^2_{24}$. Push it to infinity. We are in Diagram III. The constraints on the vorticities in Diagram III are

$$
\Gamma_1 = \Gamma_3, \quad \Gamma_2 = \Gamma_4, \quad \Gamma = 0.
$$

Similarly, push $r^2_{12}r^2_{34}$ to infinity, we have

$$
\Gamma_1 = \Gamma_2, \quad \Gamma_3 = \Gamma_4, \quad \Gamma = 0;
$$

push $r^2_{14}r^2_{23}$ to infinity, we have

$$
\Gamma_1 = \Gamma_4, \quad \Gamma_2 = \Gamma_3, \quad \Gamma = 0.
$$

However it is easy to see that the equations above have only zero solution: $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = 0$. This is a contradiction.

\[\square\]

### 6.3 Central configurations with $\Gamma \neq 0$ and $L \neq 0$.

At last we consider the finiteness of central configurations with nonvanishing total vorticity and total vortex angular momentum, i.e., $\Gamma \neq 0$ and $L \neq 0$. In this case only Diagram III in Case A, Diagram VI, Diagram VII, Diagram VIII and Diagram IX are possible.

First, we establish the following result:

**Lemma 6.4** If some product $r^2_{jk}r^2_{lm}$ of two nonadjacent distances’ squares is not dominating on the closed algebraic subset $\mathcal{A}$. Then system (2.12) possesses finitely many solutions.

**Proof of Lemma 6.4**

If system (2.12) possesses infinitely many solutions, without lose of generality, assume that $r^2_{12}r^2_{34}$ is not dominating on the closed algebraic subset $\mathcal{A}$. Then some level set $r^2_{12}r^2_{34} \equiv const \neq 0$ also includes infinitely many solutions of system (2.12).

In this level set the complete diagrams may be approached by some singular sequence are the Figure 13.

Similar as previous discussion, at least two of $r^2_{kl}$’s must take infinitely many values and thus are dominating. Suppose that $r^2_{kl}$ is dominating for some $1 \leq k < l \leq 4$. There must exist a singular sequence of central configurations with $r^2_{kl} \to 0$, which happens only in the Diagram VII. In each case, note that all of $r^2_{kl}$’s are dominating.

Considering $r^2_{12} \to \infty$, then $r^2_{34} \to 0$, we are in Diagram VIII1 or Diagram VII2. Without lose of generality, assume that we are in Diagram VIII1, thus

$$
\Gamma_2 + \Gamma_3 + \Gamma_4 = 0.
$$
Similarly, considering $r_{12}^2 \to 0$, then $r_{34}^2 \to \infty$, we are in Diagram VII3 or Diagram VII4, assume that we are in Diagram VII3, then

$$\Gamma_1 + \Gamma_2 + \Gamma_4 = 0.$$ 

By further considering $r_{13}^2 \to 0$, we are in Diagram VII2 or Diagram VII4. Note that we can not be both in Diagram VII2 or Diagram VII4 now. Without lose of generality, assume that we are in Diagram VII2, thus

$$\Gamma_1 + \Gamma_3 + \Gamma_4 = 0.$$ 

As a result,

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = -\frac{1}{2} \Gamma_4.$$ 

These vorticities are not compatible with Diagram III. So only Diagram VII1, Diagram VII2 and Diagram VII3 are possible for us. However, by considering $r_{14}^2 r_{24}^2 r_{34}^2$, we know $r_{14}^2 r_{24}^2 r_{34}^2 \to 0$ and $r_{14}^2 r_{24}^2 r_{34}^2$ is dominating. But we will be out of Diagram VII1, Diagram VII2 and Diagram VII3 if $r_{14}^2 r_{24}^2 r_{34}^2 \to \infty$, this is a contradiction.

So we consider all of $r_{jk}^2 r_{lm}^2$’s are dominating from now on.

**Theorem 6.5** If the vorticities $\Gamma_n$ ($n \in \{1, 2, 3, 4\}$) are nonzero such that $\Gamma \neq 0$ and $L \neq 0$, then the four-vortex problem has finitely many central configurations except perhaps if three of the vorticities are all negatively a half the rest vorticity, (for example, $-\frac{1}{2} \Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4$).

**Proof.**

Case 1: If Diagram III does not occur.
Since at least two of the $r_{jk}^2$'s should take infinitely many values. We suppose $r_{14}^2$ does, then $r_{14}^2$ is dominating. There is a sequence of normalized central configurations such that $r_{14}^2 \to \infty$. Then we can extract a singular sequence corresponding to one of the diagrams in Figure 11. In Diagram VI, Diagram VIII and Diagram IX in Figure 11 no distance is going to infinity, only Diagram VII is possible. In this case, without loss of generality, we consider

$$r_{12}, r_{23}, r_{31} \approx \epsilon^2; \quad r_{14}, r_{24}, r_{34} \approx \epsilon^{-2}.$$  

As a result, the $r_{jk}^2$'s are all dominating and

$$\Gamma_1 + \Gamma_2 + \Gamma_3 = 0.$$  

Consider $r_{12}^2$. Push it to infinity. We are in Diagram VII again. The constraint on the vorticities this time is $\Gamma_4 + \Gamma_2 + \Gamma_3 = 0$ or $\Gamma_1 + \Gamma_4 + \Gamma_3 = 0$. In other words

$$(\Gamma_1 - \Gamma_4)(\Gamma_2 - \Gamma_4) = 0.$$  

Similarly, push $r_{23}^2$ to infinity, we have

$$(\Gamma_3 - \Gamma_4)(\Gamma_2 - \Gamma_4) = 0;$$  

push $r_{13}^2$ to infinity, we have

$$(\Gamma_1 - \Gamma_4)(\Gamma_3 - \Gamma_4) = 0.$$  

It is easy to see that the equations above have only the untrivial solutions of the following type:

$$-\frac{1}{2}\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4.$$  

**Case 2: If Diagram III occurs.**

Without lose of generality, we assume that there always exists a singular sequence of central configurations approaching Diagram III such that $r_{13} \to \infty$ below. Then

$$\Gamma_1 \Gamma_3 - \Gamma_2 \Gamma_4 = 0,$$  

and $r_{13}^2, r_{24}^2$ and $r_{13}^2r_{24}^2$ are all dominating.

Push $r_{13}^2 \to 0$, then we are in Diagram VI, Diagram VII, Diagram VIII and Diagram IX. Thus other distances, say $r_{12}$ and $r_{23}$, should also go to zero. They also take infinitely many values. So $r_{12}^2$ and $r_{23}^2$ dominating. It is easy to see that in these diagrams the product of any two nonadjacent distances tends to zero except in Diagram VII. However by Lemma 6.4 we can consider that $r_{24}^2r_{13}^2$ and $r_{14}^2r_{23}^2$ are both dominating.

Considering $r_{12}^2r_{34}^2 \to \infty$ we are in Diagram III and the constraint on the vorticities are

$$\Gamma_1 \Gamma_2 - \Gamma_3 \Gamma_4 = 0;$$  

similarly, considering $r_{14}^2r_{23}^2 \to \infty$ we are in Diagram III and the constraint on the vorticities are

$$\Gamma_1 \Gamma_4 - \Gamma_2 \Gamma_3 = 0.$$  

Note that $\Gamma \neq 0$ it follows that
\[ \Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4. \]

However, these vorticities are not compatible with Diagram VI, Diagram VII, Diagram VIII and Diagram IX. This is a contradiction. 

\[ \blacksquare \]

**Theorem 6.6** If the vorticities $\Gamma_n$ ($n \in \{1, 2, 3, 4\}$) are nonzero such that $-\frac{1}{2} \Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4$, then the four-vortex problem has exactly 44 central configurations and six of them are real central configurations.

**Proof.**
We consider the system (7.57) with $\Lambda = \pm 1$.

Without loss of generality, suppose $\Gamma_1 = -2, \Gamma_2 = \Gamma_3 = \Gamma_4 = 1$. Then a simple computation by Mathematica shows that the system (7.57) has exactly 88 solutions and 12 of them satisfy conjugate conditions $Z_{jk} = W_{jk}$.

\[ \blacksquare \]

7 Upper bounds

7.1 An equivalent form of (2.12) and a coarse upper bound by Bézout Theorem

If $\Gamma \neq 0$, the system (2.12) is equivalent to

\[
\begin{align*}
\Gamma z_n &= \sum_{j \neq n} \Gamma_j z_{jn}, \\
\Lambda z_n &= (\Gamma_n + \Gamma_N) z_n + \sum_{j < n, j \neq n} \Gamma_j (Z_{jn} - Z_{jn}), \\
\Lambda^{-1} w_n &= (\Gamma_n + \Gamma_N) w_n + \sum_{j < n, j \neq n} \Gamma_j (W_{jn} - W_{jn}), \\
Z_{jk} &= z_{jN} - z_{kN}, \\
w_{jk} &= w_{jN} - w_{kN}, \\
1 \leq j < k \leq N,
\end{align*}
\]

or the following closed system in the variables $z_{jk}, w_{jk}, Z_{jk}, W_{jk}$ ($1 \leq j < k \leq N$):

\[
\begin{align*}
\Lambda z_n &= (\Gamma_n + \Gamma_N) z_n + \sum_{j < n, j \neq n} \Gamma_j (Z_{jn} - Z_{jn}), \\
\Lambda^{-1} w_n &= (\Gamma_n + \Gamma_N) w_n + \sum_{j < n, j \neq n} \Gamma_j (W_{jn} - W_{jn}), \\
Z_{jk} &= z_{jN} - z_{kN}, \\
w_{jk} &= w_{jN} - w_{kN}, \\
1 \leq j < k \leq N,
\end{align*}
\]
The system (7.56) above can essentially be regarded as a closed system in the variables $Z_{jk}, W_{jk}$ (1 ≤ j < k ≤ N), for example, when N = 4, the system (7.56) is equivalent to

\[
\begin{align*}
Z_{12} ((\Gamma_1 + \Gamma_2) W_{12} + \Gamma_3 W_{13} - \Gamma_3 W_{23} + \Gamma_4 W_{14} - \Gamma_4 W_{24}) &= \Lambda^{-1}, \\
Z_{13} (\Gamma_2 W_{12} + \Gamma_2 W_{23} + (\Gamma_1 + \Gamma_3) W_{13} + \Gamma_4 W_{14} - \Gamma_4 W_{34}) &= \Lambda^{-1}, \\
Z_{14} (\Gamma_2 W_{12} + \Gamma_2 W_{24} + \Gamma_3 W_{13} + \Gamma_3 W_{34} + (\Gamma_1 + \Gamma_4) W_{14}) &= \Lambda^{-1}, \\
Z_{23} (\Gamma_1 (- W_{12}) + \Gamma_1 W_{13} + \Gamma_2 W_{23} + \Gamma_3 W_{23} + \Gamma_4 W_{24} - \Gamma_4 W_{34}) &= \Lambda^{-1}, \\
Z_{24} (\Gamma_1 (- W_{12}) + \Gamma_1 W_{14} + \Gamma_2 W_{24} + \Gamma_3 W_{23} + \Gamma_3 W_{34} + \Gamma_4 W_{24}) &= \Lambda^{-1}, \\
Z_{34} (\Gamma_1 (- W_{13}) + \Gamma_1 W_{14} - \Gamma_2 W_{23} + \Gamma_2 W_{24} + \Gamma_3 W_{34} + \Gamma_4 W_{34}) &= \Lambda^{-1}.
\end{align*}
\]

We embed the system (7.57) above into a polynomial system in the projective space $\mathbb{P}_C^4$:

\[
\begin{align*}
Z_{12} ((\Gamma_1 + \Gamma_2) W_{12} + \Gamma_3 W_{13} - \Gamma_3 W_{23} + \Gamma_4 W_{14} - \Gamma_4 W_{24}) &= \Lambda^{-1} T^2, \\
Z_{13} (\Gamma_2 W_{12} + \Gamma_2 W_{23} + (\Gamma_1 + \Gamma_3) W_{13} + \Gamma_4 W_{14} - \Gamma_4 W_{34}) &= \Lambda^{-1} T^2, \\
Z_{14} (\Gamma_2 W_{12} + \Gamma_2 W_{24} + \Gamma_3 W_{13} + \Gamma_3 W_{34} + (\Gamma_1 + \Gamma_4) W_{14}) &= \Lambda^{-1} T^2, \\
Z_{23} (\Gamma_1 (- W_{12}) + \Gamma_1 W_{13} + \Gamma_2 W_{23} + \Gamma_3 W_{23} + \Gamma_4 W_{24} - \Gamma_4 W_{34}) &= \Lambda^{-1} T^2, \\
Z_{24} (\Gamma_1 (- W_{12}) + \Gamma_1 W_{14} + \Gamma_2 W_{24} + \Gamma_3 W_{23} + \Gamma_3 W_{34} + \Gamma_4 W_{24}) &= \Lambda^{-1} T^2, \\
Z_{34} (\Gamma_1 (- W_{13}) + \Gamma_1 W_{14} - \Gamma_2 W_{23} + \Gamma_2 W_{24} + \Gamma_3 W_{34} + \Gamma_3 W_{34}) &= \Lambda^{-1} T^2, \\
W_{12} ((\Gamma_1 + \Gamma_2) Z_{12} + \Gamma_3 Z_{13} - \Gamma_3 Z_{23} + \Gamma_4 Z_{14} - \Gamma_4 Z_{24}) &= \Lambda T^2, \\
W_{13} (\Gamma_2 Z_{12} + \Gamma_2 Z_{23} + (\Gamma_1 + \Gamma_3) Z_{13} + \Gamma_4 Z_{14} - \Gamma_4 Z_{34}) &= \Lambda T^2, \\
W_{14} (\Gamma_2 Z_{12} + \Gamma_2 Z_{24} + \Gamma_3 Z_{13} + \Gamma_3 Z_{34} + (\Gamma_1 + \Gamma_4) Z_{14}) &= \Lambda T^2, \\
W_{23} (\Gamma_1 (- Z_{12}) + \Gamma_1 Z_{13} + \Gamma_2 Z_{23} + \Gamma_3 Z_{23} + \Gamma_4 Z_{24} - \Gamma_4 Z_{34}) &= \Lambda T^2, \\
W_{24} (\Gamma_1 (- Z_{12}) + \Gamma_1 Z_{14} + \Gamma_2 Z_{24} + \Gamma_3 Z_{23} + \Gamma_3 Z_{34} + \Gamma_4 Z_{24}) &= \Lambda T^2, \\
W_{34} (\Gamma_1 (- Z_{13}) + \Gamma_1 Z_{14} - \Gamma_2 Z_{23} + \Gamma_2 Z_{24} + \Gamma_3 Z_{34} + \Gamma_4 Z_{34}) &= \Lambda T^2, \\
((\Gamma_1 + \Gamma_2) Z_{12} + \Gamma_3 Z_{13} - \Gamma_3 Z_{23} + \Gamma_4 Z_{14} - \Gamma_4 Z_{24}) &= \Lambda^2 (\Gamma_1 + \Gamma_2) W_{12} + \Gamma_3 W_{13} - \Gamma_3 W_{23} + \Gamma_4 W_{14} - \Gamma_4 W_{24}), \\
Z_{12} &= W_{12}.
\end{align*}
\]

Here the system (7.57) is just an affine piece of the system (7.58) for $T \neq 0$.

After deleting the first or the seventh equation from the system (7.58) above, it is easy to see that the degree of the system (7.58) is no more than $2^{11}$. We remark that by direct application of the following Bézout Theorem it follows that the number of central configurations for the four-vortex problem is no more than $2^{10} = 1024$. 

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Lemma 7.1 \([\textbf{[7]}]\) Let \(\mathcal{V}_1, \ldots, \mathcal{V}_m\) be subvarieties of \(\mathbb{P}^N\). Let \(\mathcal{U}_1, \ldots, \mathcal{U}_n\) be the irreducible components of \(\mathcal{V}_1 \cap \cdots \cap \mathcal{V}_m\). Then

\[
\sum_{j=1}^n \deg(\mathcal{U}_j) \leq \prod_{j=1}^m \deg(\mathcal{V}_j). \tag{7.59}
\]

To obtain a better upper bound, we estimate the number of the irreducible components of the system \((7.58)\) for \(T = 0\). This is a linear variety in \(\mathbb{P}^n\):

\[
\begin{align*}
Z_{12} ((\Gamma_1 + \Gamma_2) W_{12} + \Gamma_3 W_{13} - \Gamma_3 W_{23} + \Gamma_4 W_{14} - \Gamma_4 W_{24}) &= 0, \\
Z_{13} (\Gamma_3 W_{12} + \Gamma_3 W_{23} + (\Gamma_1 + \Gamma_3) W_{13} + \Gamma_4 W_{14} - \Gamma_4 W_{34}) &= 0, \\
Z_{14} (\Gamma_3 W_{12} + \Gamma_3 W_{24} + \Gamma_3 W_{13} + \Gamma_3 W_{34} + (\Gamma_1 + \Gamma_4) W_{14}) &= 0, \\
Z_{23} (\Gamma_1 (-W_{12}) + \Gamma_1 W_{13} + \Gamma_2 W_{23} + \Gamma_3 W_{23} + \Gamma_4 W_{24} - \Gamma_4 W_{34}) &= 0, \\
Z_{24} (\Gamma_1 (-W_{12}) + \Gamma_1 W_{14} + \Gamma_2 W_{24} + \Gamma_3 W_{23} + \Gamma_3 W_{34} + \Gamma_4 W_{24}) &= 0, \\
Z_{34} (\Gamma_1 (-W_{13}) + \Gamma_1 W_{14} - \Gamma_2 W_{23} + \Gamma_2 W_{24} + \Gamma_3 W_{34} + \Gamma_4 W_{34}) &= 0, \\
W_{12} ((\Gamma_1 + \Gamma_2) Z_{12} + \Gamma_3 Z_{13} - \Gamma_3 Z_{23} + \Gamma_4 Z_{14} - \Gamma_4 Z_{24}) &= 0, \\
W_{13} (\Gamma_2 Z_{12} + \Gamma_2 Z_{23} + (\Gamma_1 + \Gamma_3) Z_{13} + \Gamma_4 Z_{14} - \Gamma_4 Z_{34}) &= 0, \\
W_{14} (\Gamma_2 Z_{12} + \Gamma_2 Z_{24} + \Gamma_3 Z_{13} + \Gamma_3 Z_{34} + (\Gamma_1 + \Gamma_4) Z_{14}) &= 0, \\
W_{23} (\Gamma_1 (-Z_{12}) + \Gamma_1 Z_{13} + \Gamma_2 Z_{23} + \Gamma_3 Z_{23} + \Gamma_4 Z_{24} - \Gamma_4 Z_{34}) &= 0, \\
W_{24} (\Gamma_1 (-Z_{12}) + \Gamma_1 Z_{14} + \Gamma_2 Z_{24} + \Gamma_3 Z_{23} + \Gamma_3 Z_{34} + \Gamma_4 Z_{24}) &= 0, \\
W_{34} (\Gamma_1 (-Z_{13}) + \Gamma_1 Z_{14} - \Gamma_2 Z_{23} + \Gamma_2 Z_{24} + \Gamma_3 Z_{34} + \Gamma_4 Z_{34}) &= 0, \\
((\Gamma_1 + \Gamma_2) Z_{12} + \Gamma_3 Z_{13} - \Gamma_3 Z_{23} + \Gamma_4 Z_{14} - \Gamma_4 Z_{24}) &= 0, \\
Z_{12} &= W_{12}. 
\end{align*}
\]

A straightforward computation shows that the greatest lower bound of the number of irreducible components of the linear variety \((7.60)\) above is no more than 441 in \(\mathbb{P}^n\). Therefore, the upper bound of the number of central configurations for the four-vortex problem is at least \(\frac{2^{12} - 441}{2} = 803\) by direct application of the Bézout Theorem above.

7.2 A new equivalent form of (2.6) and a better upper bound by Bézout Theorem

We use Bézout Theorem again, but transform the system (2.6) or (2.11) into a new equivalent form.

Following O’Neil \([17]\) we introduce the relations

\[
\begin{align*}
\frac{1}{2} \sum_{1 \leq j, k \leq N, j \neq k} \frac{\Gamma_{j z}}{(\zeta - z_j)(\zeta - z_k)} &= \Lambda \sum_{1 \leq k \leq N} \frac{\Gamma_{j z}}{\zeta - z_k}, \\
\frac{1}{2} \sum_{1 \leq j, k \leq N, j \neq k} \frac{\Gamma_{j z}}{(\zeta - z_j)(\zeta - z_k)} &= \Lambda \sum_{1 \leq k \leq N} \frac{\Gamma_{j z}}{\zeta - z_k}, \tag{7.61}
\end{align*}
\]

by the identity

\[
\frac{1}{(\zeta - z_j)(\zeta - z_k)} = \frac{1}{z_j - z_k} \left( \frac{1}{\zeta - z_j} - \frac{1}{\zeta - z_k} \right) \tag{7.62}
\]

and (2.6).
It is easy to see that (2.6) is equivalent to (7.61) provided \( z_j - z_k \neq 0 \) for any \( 1 \leq j < k \leq N \). Indeed, by (7.62) it follows that

\[
\frac{1}{2} \sum_{1 \leq j, k \leq N, j \neq k} \frac{\Gamma_j \Gamma_k}{(\zeta - z_j)(\zeta - z_k)} = \sum_{1 \leq j, k \leq N, j \neq k} \frac{\Gamma_j \Gamma_k}{(\zeta - z_k)(z_k - z_j)} = \sum_{1 \leq k \leq N} \frac{\Gamma_k}{\zeta - z_k} \sum_{1 \leq j \leq N, j \neq k} \frac{\Gamma_j}{(z_k - z_j)} = \sum_{1 \leq k \leq N} \frac{\Gamma_k}{\zeta - z_k} \nabla_k.
\]

Therefore, (7.61) holds if and only if \( V_k = \Lambda z_k \). That is to say, provided \( z_j - z_k \neq 0 \) for any \( 1 \leq j < k \leq N \), \( z \) is a central configuration if and only if (7.61) holds for any \( \zeta \in \mathbb{C} \).

Both sides of (7.61) are rational functions of \( \zeta \); by eliminating denominators of both sides one gets two polynomial equations in \( \zeta \), with coefficients that are expressions with \( z_k, \Lambda \) and their conjugations. (7.61) holds exactly when all coefficients of the two polynomials are zero. According to these facts, we can transform the system (2.6) into a new equivalent form by the relations of their coefficients.

In a similar way, we can transform the system (2.61) into a new equivalent form:

\[
\begin{align*}
\frac{1}{2} \sum_{1 \leq j, k \leq N, j \neq k} \frac{\Gamma_j \Gamma_k}{(\zeta - z_j)(\zeta - z_k)} &= \frac{\Lambda}{N} \sum_{1 \leq k \leq N} \frac{\Gamma_k \omega_k}{\zeta - z_k}, \\
\frac{1}{2} \sum_{1 \leq j, k \leq N, j \neq k} \frac{\Gamma_j \Gamma_k}{(\zeta - w_j)(\zeta - w_k)} &= \Lambda \sum_{1 \leq k \leq N} \frac{\Gamma_k \omega_k}{\zeta - w_k},
\end{align*}
\]

(7.63)

or the relations of their coefficients (from now on we consider only \( N = 4 \)):

\[
\begin{align*}
\Lambda M_z = 0, & \quad \Lambda M_w = 0, \\
L - \Lambda I + \Lambda M_w \sum_{j=1}^{4} w_j = 0, & \quad L - \Lambda I + \Lambda M_z \sum_{j=1}^{4} w_j = 0, \\
\Lambda(M_w \sum_{1 \leq j < k \leq 4} z_j z_k + F_z - I \sum_{j=1}^{4} z_j) - \sum_{1 \leq j < k \leq 4} \Gamma_j \Gamma_k (z_j + z_k) + L \sum_{j=1}^{4} z_j = 0, \\
\Lambda(M_z \sum_{1 \leq j \leq N} w_j w_k + F_w - I \sum_{j=1}^{4} w_j) - \sum_{1 \leq j < k \leq N} \Gamma_j \Gamma_k (w_j + w_k) + L \sum_{j=1}^{4} w_j = 0, \\
\sum_{1 \leq j < k \leq 4, i < k, i, m = \{1,2,3,4\}} \Gamma_j \Gamma_k z_i z_m + \Lambda G_z = 0, \\
\sum_{1 \leq j < k \leq 4, i < k, i, m = \{1,2,3,4\}} \Gamma_j \Gamma_k w_i w_m + \Lambda G_w = 0,
\end{align*}
\]

(7.64)

or

\[
\begin{align*}
M_z &= 0, & \quad M_w &= 0, \\
L - \Lambda I &= 0, & \quad L - \Lambda I &= 0, \\
\Lambda F_z - f_z &= 0, & \quad \Lambda F_w - f_w &= 0, \\
\Lambda G_z + g_z &= 0, & \quad \Lambda G_w + g_w &= 0,
\end{align*}
\]

where

\[
M_z = \sum_{j=1}^{4} \Gamma_j z_j, \quad M_w = \sum_{j=1}^{4} \Gamma_j w_j, \\
I = \sum_{j=1}^{4} \Gamma_j z_j w_j, \\
F_z = \sum_{j=1}^{4} \Gamma_j z_j^2 w_j, \quad f_z = \sum_{1 \leq j < k \leq 4} \Gamma_j \Gamma_k (z_j + z_k), \\
F_w = \sum_{j=1}^{4} \Gamma_j z_j^2 w_j, \quad f_w = \sum_{1 \leq j < k \leq 4} \Gamma_j \Gamma_k (w_j + w_k), \\
G_z = \Gamma_1 w_1 z_2 z_3 z_4 + \Gamma_2 w_2 z_1 z_3 z_4 + \Gamma_3 w_3 z_1 z_2 z_4 + \Gamma_4 w_4 z_1 z_2 z_3, \\
g_z = \sum_{1 \leq j < k \leq 4, i < k, i, m = \{1,2,3,4\}} \Gamma_j \Gamma_k z_i z_m, \\
G_w = \Gamma_1 z_1 w_2 w_3 w_4 + \Gamma_2 z_2 w_1 w_3 w_4 + \Gamma_3 z_3 w_1 w_2 w_4 + \Gamma_4 z_4 w_1 w_2 w_3, \\
g_w = \sum_{1 \leq j < k \leq 4, i < k, i, m = \{1,2,3,4\}} \Gamma_j \Gamma_k w_i w_m.
\]

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It follows that, in the case of relative equilibria, provided \( z_j - z_k \neq 0 \) for any \( 1 \leq j < k \leq 4 \), normalized central configurations are characterized by

\[
\begin{align*}
&M_z = 0, \\
&L - \Lambda I = 0, \\
&\Lambda F_z - f_z = 0, \\
&\Lambda G_z + g_z = 0,
\end{align*}
\]

and

\[
\begin{align*}
&M_w = 0, \\
&z_2 - z_1 = w_2 - w_1, \\
&\Lambda F_w - f_w = 0, \\
&\Lambda G_w + g_w = 0,
\end{align*}
\]

(7.65)

\[
\begin{align*}
&M_z = 0, \\
&L - \Lambda I = 0, \\
&\Lambda F_z - f_z = 0, \\
&\Lambda G_z + g_z = 0,
\end{align*}
\]

and

\[
\begin{align*}
&M_w = 0, \\
&z_2 - z_1 = w_2 - w_1, \\
&\Lambda F_w - f_w = 0, \\
&\Lambda G_w + g_w = 0.
\end{align*}
\]

(7.66)

Here, without loss of generality, one can further assume that \( \Lambda = 1 \); in the case of collapse configurations, normalized central configurations are characterized by

\[
\begin{align*}
&M_z = 0, \\
&L - \Lambda I = 0, \\
&\Lambda F_z - f_z = 0, \\
&\Lambda G_z + g_z = 0,
\end{align*}
\]

and

\[
\begin{align*}
&M_w = 0, \\
&z_2 - z_1 = w_2 - w_1, \\
&\Lambda F_w - f_w = 0, \\
&\Lambda G_w + g_w = 0.
\end{align*}
\]

(7.66)

After embedding the system (7.65) or (7.66) above into a system in the projective space \( \mathbb{P}_C^8 \), it is easy to see that the degree of the systems is no more than \( 2 \times 3^2 \times 4^2 = 288 \).

By direct application of the following Bézout Theorem it follows that the number of central configurations for the four-vortex problem is no more than 144.

We remark that

**Proposition 7.1** If \( L = 0 \) and \( z_j = z_k \) (or \( w_j = w_k \)) for some \( j \neq k \), then there is only trivial solution in the system (7.64) above.

**Proof.**

If \( z_j = z_k \) for some \( j \neq k \), assume that \( \Xi \) is the subset of the index set \( \{1, 2, 3, 4\} \) such that \( z_j = z_k \triangleq z_\ast \) for any \( j, k \in \Xi \). According to (7.63), it follows that

\[
\sum_{j<k,j,k \in \Xi} \Gamma_j \Gamma_k = 0.
\]

Otherwise, the right side of (7.63) has no double poles, while the left side will have a double pole at \( z = z_\ast \). By \( L = 0 \) it is easy to see that \( \Xi = \{1, 2, 3, 4\} \). Thus

\[
\sum_{1 \leq k \leq 4} \frac{\Gamma_k w_k}{\zeta - z_\ast} = 0 \quad \text{or} \quad \sum_{1 \leq k \leq 4} \Gamma_k w_k = 0.
\]

Since \( w_2 - w_1 = z_2 - z_1 = 0 \), a similar argument shows that

\[
\begin{align*}
w_1 &= w_2 = w_3 = w_4, \\
\sum_{1 \leq k \leq 4} \Gamma_k z_k &= 0.
\end{align*}
\]

Note that \( \Gamma \neq 0 \) by \( L = 0 \). As a result, there is only trivial solution in the system (7.64) above.

\( \square \)
7.2.1 An upper bound for collinear central configurations

If \( w_j = z_j \) (\( j = 1, 2, 3, 4 \)), the systems (7.65) and (7.66) respectively reduce to

\[
\begin{align*}
M_z &= 0, \quad L - I = 0, \\
F_z - f_z &= 0, \quad G_z + g_z = 0;
\end{align*}
\]

(7.67)

and

\[
\begin{align*}
M_z &= 0, \quad L = 0, \quad I = 0, \\
F_z = f_z &= 0, \quad G_z = g_z = 0.
\end{align*}
\]

(7.68)

It is easy to see that collinear normalized central configurations are characterized by the systems (7.65) or (7.66) respectively. On the other hand, a straightforward computation shows that the system (7.68) has only trivial solution, thus there is no any collinear collapse configuration in the four-vortex problem. In fact, it is also easy to see that there is no collinear collapse configuration for the general \( N \)-vortex problem.

An upper bound for collinear relative equilibria. In this case, we embed the system (7.67) into a system in the projective space \( \mathbb{P}^4 \):

\[
M_z = 0, \quad L t^2 - I = 0, \quad F_z - f_z t^2 = 0, \quad G_z + g_z t^2 = 0.
\]

(7.69)

Then the system (7.67) is just an affine piece of the system (7.69) for \( t \neq 0 \). And the algebraic variety (7.69) is a disjoint union of (7.67) and the variety

\[
M_z = 0, \quad I = 0, \quad F_z = 0, \quad G_z = 0.
\]

(7.70)

Since a straightforward computation shows that the system (7.70) has only trivial solution, it follows that the variety (7.69) is equal to (7.67) and has exactly 24 points (counting with the appropriate multiplicity) in \( \mathbb{P}^4 \). As a result, there are at most 12 collinear relative equilibria for the four-vortex problem.

Remark 7.1 The result, that there are at most 12 collinear relative equilibria for the four-vortex problem, has been proved by Hampton and Moeckel [9].

Refined Bézout Theorem. In the following we will frequently employ a refined version of Bézout Theorem

**Lemma 7.2** ([19]) Let \( \mathcal{V}_1, \ldots, \mathcal{V}_m \) be pure dimensional subvarieties of \( \mathbb{P}^N \). Let \( \mathcal{U}_1, \ldots, \mathcal{U}_n \) be the irreducible components of \( \mathcal{X} \triangleq \mathcal{V}_1 \cap \cdots \cap \mathcal{V}_m \). Then

\[
\sum_{j=1}^n l(\mathcal{X}; \mathcal{U}_j) \deg(\mathcal{U}_j) \leq \prod_{j=1}^m \deg(\mathcal{V}_j),
\]

(7.71)

where \( l(\mathcal{X}; \mathcal{U}_j) \) is the length of well-defined primary ideals, i.e., the multiplicity of \( \mathcal{X} \) along \( \mathcal{U}_j \).

Recall that
Definition 7.1 (See [6]) The multiplicity $l(X; P)$ of $X$ at a point $P \in X$ is the degree of the projectivized tangent cone $\mathbb{T}_C P X$. The multiplicity $l(X; U)$ of a scheme $X$ along an irreducible component $U$, is equal to the multiplicity of $X$ at a general point of $U$.

Lemma 7.3 ([7]) Let $V_1, \cdots, V_m$ be pure-dimensional subschemes of $\mathbb{P}^N$, with

$$\sum_{j=1}^{m} \dim(V_j) = (m - 1)N.$$ 

Assume $P$ is an isolated point of $X \triangleq V_1 \cap \cdots \cap V_m$. Then

$$l(X; P) \geq \prod_{j=1}^{m} l(V_j; U) + \sum_{j=1}^{n} \deg(U_j), \quad (7.72)$$

where $U_1, \cdots, U_n$ are the irreducible components of $\mathbb{T}_C P V_1 \cap \cdots \cap \mathbb{T}_C P V_m$. In particular,

$$l(X; P) \geq \prod_{j=1}^{m} l(V_j; U) \quad (7.73)$$

with equality if and only if $\mathbb{T}_C P V_1 \cap \cdots \cap \mathbb{T}_C P V_m = \emptyset$.

An upper bound for collinear relative equilibria with $L = 0$. In this case, the system (7.67) becomes:

$$M_z = 0, \quad I = 0, \quad F_z - f_z = 0, \quad G_z + g_z = 0. \quad (7.74)$$

The zero point $O$ is a trivial solution of system (7.74), indeed, an isolated solution by Proposition 7.1. Then by Lemma 7.3 it follows that

$$l(X; O) = 4.$$ 

As a result, the system (7.74) has exactly 20 points (counting with the appropriate multiplicity) in $\mathbb{P}^4$ such that $z_j - z_k \neq 0$ for any $1 \leq j < k \leq 4$. Thus there are at most 10 collinear relative equilibria for the four-vortex problem with $L = 0$.

7.2.2 A better upper bound for relative equilibria by Bézout Theorem

We embed the system (7.65) above with $\Lambda = 1$ into a polynomial system in the projective space $\mathbb{P}_C^8$:

$$\begin{cases} 
M_z = 0, & M_w = 0, \\
L I^2 - I = 0, & z_2 - z_1 = w_2 - w_1, \\
F_z - f_z t^2 = 0, & F_w - f_w t^2 = 0, \\
G_z + g_z t^2 = 0, & G_w + g_w t^2 = 0;
\end{cases} \quad (7.75)$$
It is easy to see that the algebraic variety (7.75) is a disjoint union of (7.65) and the algebraic variety

\[
\begin{aligned}
  M_z = 0, & \quad M_w = 0, \\
  I = 0, & \quad z_2 - z_1 = w_2 - w_1, \\
  F_z = 0, & \quad F_w = 0, \\
  G_z = 0, & \quad G_w = 0.
\end{aligned}
\] (7.76)

First, we remark that, a straightforward computation shows that the algebraic variety (7.76) is one-dimensional. Indeed, it is easy to see that (7.76) at least contains two one-dimensional irreducible components (i.e., two one-dimensional lines):

\[
\begin{aligned}
  w_1 = w_2 = w_3 = w_4 = 0, & \quad M_z = 0, \quad z_2 - z_1 = 0; \\
  z_1 = z_2 = z_3 = z_4 = 0, & \quad M_w = 0, \quad w_2 - w_1 = 0;
\end{aligned}
\] (7.77)

and four isolated points:

\[
\begin{aligned}
  z_2 = 0, & \quad \Gamma_1 z_1 + \Gamma_3 z_3 = 0, \quad z_4 = 0, \quad w_1 = 0, \quad w_2 + z_1 = 0, \quad w_3 = 0, \quad \Gamma_4 w_4 - \Gamma_1 z_1 = 0; \\
  z_2 = 0, & \quad z_3 = 0, \quad \Gamma_1 z_1 + \Gamma_4 z_4 = 0, \quad w_1 = 0, \quad w_2 + z_1 = 0, \quad \Gamma_3 z_3 - \Gamma_2 z_1 = 0, \quad w_4 = 0; \\
  z_1 = 0, & \quad \Gamma_2 z_2 + \Gamma_3 z_3 = 0, \quad z_4 = 0, \quad w_1 + z_2 = 0, \quad w_2 = 0, \quad w_3 = 0, \quad \Gamma_4 w_4 - \Gamma_1 z_2 = 0; \\
  z_1 = 0, & \quad z_3 = 0, \quad \Gamma_2 z_2 + \Gamma_4 z_4 = 0, \quad w_1 + z_2 = 0, \quad w_2 = 0, \quad \Gamma_3 z_3 - \Gamma_1 z_2 = 0, \quad w_4 = 0.
\end{aligned}
\] (7.78)

A straightforward computation shows that both multiplicities of two one-dimensional irreducible components are at least 6 and all multiplicities of four isolated points are at least 2. It follows that the degree of algebraic subset (7.65) is no more than \(288 - 6 \times 2 - 4 \times 2 = 268\).

**An upper bound for strictly planar relative equilibria with \(L \neq 0\).** In this case, by the result for collinear central configurations above, it is easy to see that the degree of (7.65) corresponding to strictly planar relative equilibria is no more than \(268 - 24 = 244\). Therefore, the number of strictly planar relative equilibria for the four-vortex problem with \(L \neq 0\) is no more than \(\left\lceil \frac{244}{2} \right\rceil = 122\).

**An upper bound for strictly planar relative equilibria with \(L = 0\).** In this case, firstly, by the result for collinear central configurations above, it is easy to see that the degree of (7.65) corresponding to collinear relative equilibria is 20. Secondly, note that there is an isolated trivial solution of system (7.65) by Proposition 7.1. And it is easy to see that the multiplicity of this trivial solution is 8.

Therefore, the number of strictly planar relative equilibria for the four-vortex problem with \(L = 0\) is no more than \(\left\lceil \frac{268 - 8 - 20}{2} \right\rceil = 120\).

**A better upper bound by computation of mixed volumes.** On the other hand, by the method of computation of mixed volumes of Newton polytope the system (7.56), we can obtain a better upper bound than the numbers above. The method is introduced by Hampton and Moeckel in [10, 9] for relative equilibria. We remark that the computation by Hampton and Moeckel in [9] showed that the number of the strictly planar relative equilibria is no more than 74.
7.2.3 A better upper bound for collapse configurations by Bézout Theorem

In this case, we embed the system (7.66) above into a polynomial system in the projective space $\mathbb{P}_C^8$:

$$
\begin{aligned}
M_z &= 0, & M_w &= 0, \\
I &= 0, & z_2 - z_1 &= w_2 - w_1, \\
\Lambda F_z - t^2 f_z &= 0, & \Lambda F_w - t^2 f_w &= 0, \\
\Lambda G_z + t^2 g_z &= 0, & \Lambda G_w + t^2 g_w &= 0.
\end{aligned}
$$

(7.79)

To obtain an upper bound of collapse configurations, we estimate the number of the irreducible components of the system (7.79) for $t = 0$. This is an algebraic variety in $\mathbb{P}_C^7$ which is the same as (7.76) in form (note that $L = 0$ here):

$$
\begin{aligned}
M_z &= 0, & M_w &= 0, \\
I &= 0, & z_2 - z_1 &= w_2 - w_1, \\
F_z &= 0, & F_w &= 0, \\
G_z &= 0, & G_w &= 0.
\end{aligned}
$$

(7.80)

A straightforward computation shows that the variety (7.80) above is one-dimensional and at least contains two one-dimensional irreducible components (7.77) and four isolated points (7.78). Similarly, both multiplicities of two one-dimensional irreducible components are at least 6 and all multiplicities of four isolated points are at least 2. It follows that the degree of algebraic subset (7.66) is no more than $288 - 6 \times 2 - 4 \times 2 = 268$.

On the other hand, note that there is an isolated trivial solution of system (7.65) by Proposition 7.1. And it is easy to see that the multiplicity of this trivial solution is 8.

Therefore, the number of collapse configurations for the four-vortex problem is no more than \( \frac{268 - 8}{2} = 130 \).

7.3 Conclusion on upper bounds

To summarize, by the results in this section and Theorem 1.1 and 1.2, we get Corollary 1.4.

8 Conclusion

Inspired by the elegant method of Albouy and Kaloshin for celestial mechanics, which provides an effective way to analyse the singularities, we develop a novel analysis of the singularities for a possible continuum of central configurations of the N-vortex problem. We proved that there are finitely many complex central configurations in the planar four-vortex problem. As a result, there are finitely many stationary configurations consisting of equilibria, rigidly translating configurations, relative equilibria (uniformly rotating configurations) and collapse configurations.

Once the finiteness is proved, an explicit upper bound on the number of relative equilibria and collapse configurations is obtained by direct application of Bézout Theorems. However, to obtain good upper bounds, it is necessary to transform the system
into an available equivalent form, which is based on an observation of O’Neil for the planar $N$-vortex problem, and to estimate the multiplicity of the irreducible components not corresponding to central configurations. To obtain better upper bounds, the method of mixed volumes for the system (7.56) is employed. The method is introduced by Hampton and Moeckel. Unfortunately, the computation is not easy, thus we simply apply the data of Hampton and Moeckel in [9] to show that the number of the strictly planar relative equilibria is no more than 74.
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