Exact, Rotational, Infinite Energy, Blowup Solutions to the 3-Dimensional Euler Equations

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Abstract

In this paper, we construct a new class of blowup solutions with elementary functions to the 3-dimensional compressible or incompressible Euler and Navier-Stokes equations. In detail, we obtain a class of global rotational exact solutions for the compressible fluids with $\gamma > 1$:

\[
\begin{align*}
&\rho = \max \left\{ \frac{\gamma}{\gamma-1} \left[ C^2 \left[ x^2 + y^2 + z^2 - (xy + yz + xz) \right] - \dot{a}(t)(x + y + z) + b(t) \right], 0 \right\}^{\frac{1}{\gamma-1}} \\
u_1 &= a(t) + C(y - z) \\
u_2 &= a(t) + C(-x + z) \\
u_3 &= a(t) + C(x - y)
\end{align*}
\]

where

\[
a(t) = c_0 + c_1 t
\]

and

\[
b(t) = 3c_0c_1 t + \frac{3}{2}c_1^2 t^2 + c_2
\]

with $C$, $c_0$, $c_1$ and $c_2$ are arbitrary constants;

And the corresponding blowup or global solutions for the incompressible Euler equations are also given. Our constructed solutions are similar to the famous Arnold-Beltrami-Childress (ABC) flow. The solutions with infinite energy can exhibit the interesting behaviors locally. Besides, the corresponding global solutions are also given for the compressible Euler equations. Furthermore, due to $\text{div} \, \vec{u} = 0$ for the solutions, the solutions also work for the 3-dimensional incompressible Euler and Navier-Stokes equations.

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1 Introduction

The $N$-dimensional Euler equations can be formulated as the follows:

\[
\begin{align*}
&\rho_t + \nabla \cdot (\rho \vec{u}) = 0 \\
&\rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] + \nabla P(\rho) = 0
\end{align*}
\]

where $\vec{x} = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$, $\rho = \rho(t, \vec{x})$ and $\vec{u} = \vec{u}(t, \vec{x}) = (u_1, u_2, ..., u_N) \in \mathbb{R}^N$ are the density and the velocity respectively. The $\gamma$-law could be applied to the pressure function, i.e.

\[
P(\rho) = K \rho^\gamma
\]
with \( K \geq 0 \) and \( \gamma \geq 1 \). The Euler equations (4) can be rewritten by the scalar form,

\[
\rho \left( \frac{\partial u}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u}{\partial x_k} \right) + \rho \sum_{k=1}^{N} \frac{\partial u_k}{\partial x_k} = 0,
\]

\[
\rho \left( \frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial P}{\partial x_i} = 0, \text{ for } i = 1, 2, \ldots N. \tag{6}
\]

The Euler equations (4) are the fundamental model in fluid mechanics [2] and [5].

Constructing exact solutions is a very important part in mathematical physics to understand the nonlinear behaviors of the system. For the pressureless fluids \( K = 0 \), a class of exact blowup solutions were given in Yuen and Yeung’s papers [14] and [11], [15] and [16]. In addition, there are two papers to investigate for a special structural analytical solutions for the compressible Euler or Navier-Stokes equations in [9], [8], [6], [12], [13], [14], [11], [17] and [16]. In addition, there are two papers to investigate for a special structural form which generalizes the exact solutions for Burgers’ vortices in [7] and [3].

In this paper, we manipulate the elementary functions to construct some exact rotational solutions for the 3-dimensional compressible Euler equations (4):

**Theorem 1** For the 3-dimensional compressible Euler equations (4), there exists a class of rotational solutions:

for \( \gamma > 1 \):

\[
\rho = \max\left\{ \frac{\gamma-1}{2} \left[ C^2 \left[ x^2 + y^2 + z^2 - (xy + yz + xz) \right] - \dot{a}(t)(x + y + z) + b(t) \right] , 0 \right\}^{1/\gamma},
\]

\[
u_1 = a(t) + C (y - z)
\]

\[
u_2 = a(t) + C (-x + z)
\]

\[
u_3 = a(t) + C (x - y)
\]

where

\[
\dot{a}(t) = c_0 + c_1 t
\]

and

\[
b(t) = 3c_0 c_1 t + \frac{3}{2} c_1^2 t^2 + c_2
\]

with \( C, c_0, c_1 \) and \( c_2 \) are arbitrary constants;

for \( \gamma = 1 \):

\[
\rho = e^{C^2 \left[ x^2 + y^2 + z^2 - (xy + yz + xz) \right] - \dot{a}(t)(x + y + z) + b(t)}
\]

\[
u_1 = a(t) + C (y - z)
\]

\[
u_2 = a(t) + C (-x + z)
\]

\[
u_3 = a(t) + C (x - y)
\]

The solutions (8) and (II) globally exist.

**Remark 2** In 1965, Arnold first introduced the famous Arnold-Beltrami-Childress (ABC) flow

\[
\begin{align*}
\nu_1 &= A \sin z + C \cos y \\
\nu_2 &= B \sin x + A \cos z \\
\nu_3 &= C \sin y + B \cos x
\end{align*}
\]

with constants \( A, B, C \) and a suitable pressure function \( P \) only for the incompressible Euler equations in [I]. We observe that our solutions (8) and (II) are similar to the ABC flow.
Remark 3  The solutions with infinite energy can exhibit the interesting behaviors locally. The exact solutions with infinite energy of the systems may be regionally applicable to understand the great complexity that exists in turbulent phenomena.

Remark 4  We notice that the rational functional form with $a(t) = 0$ and $C = 1$ in solutions (5) and (11) for the velocity $\vec{u}$ comes from Senba and Suzuki’s book [10]. The velocities $\vec{u}$ in solutions (5) and (11) are not spherically symmetric.

Remark 5  The exact rotational solutions (8) and (11) could be good examples for testing numerical methods for fluid dynamics.

2  Compressible Rotational Fluids

The main technique of this article is just to use the primary assumption about the velocities $\vec{u}$:

\[
\begin{align*}
&u_1 = a(t) + C (y - z) \\
&u_2 = a(t) + C (-x + z) \\
&u_3 = a(t) + C (x - y)
\end{align*}
\]

(13)

which we could require the condition for the density function:

\[
\rho = \max \left\{ \frac{\gamma - 1}{K \gamma} \left[ C^2 \left[ x^2 + y^2 + z^2 - (xy + yz + xz) \right] - \dot{a}(t)(x + y + z) + b(t) \right], 0 \right\}
\]

(17)

to have

\[
= \dot{a}(t) + [a(t) + C(-x + z)] \frac{\partial}{\partial y} [a(t) + C(y - z)] + [a(t) + C(x - y)] \frac{\partial}{\partial z} [a(t) + C(y - z)]
\]

\[
+ \frac{\partial}{\partial x} \left[ C^2 \left[ x^2 + y^2 + z^2 - (xy + yz + xz) \right] - \dot{a}(t)(x + y + z) + b(t) \right]
\]

(18)

\[
= \dot{a}(t) - 2C^2 x + C^2 z + C^2 y + 2C^2 x - C^2 y - C^2 z - \dot{a}(t)
\]

(19)

Similarly by the nice symmetry of the functions [5], we could balance the second momentum equation (12,2):

\[
\frac{\partial}{\partial t} u_2 + u_1 \frac{\partial}{\partial x} u_2 + u_3 \frac{\partial}{\partial z} u_2 = \frac{K \gamma}{\gamma - 1} \frac{\partial}{\partial y} \rho \gamma^{-1}
\]

(22)
\[\frac{\partial}{\partial t} \left[ a(t) + C (y-z) \right] \frac{\partial}{\partial x} \left[ a(t) + C (-x+z) \right] + \frac{\partial}{\partial z} \left[ a(t) + C (-x+z) \right] \]

\[+ \frac{\partial}{\partial y} \left[ C^2 \left[ (x^2 + y^2 + z^2) - (xy + yz + xz) \right] - \dot{a}(t)(x+y+z) + b(t) \right] \]

\[= \dot{a}(t) + [a(t) + C (y-z)]((C) + [a(t) + C (x-y)] C \]

\[+ 2C^2 x - C^2 x - C^2 z - \dot{a}(t) \]

\[= 0. \quad (23) \]

For the last equation \([4]_{2,3}\), we have

\[\frac{\partial}{\partial t} u_3 + u_1 \frac{\partial}{\partial x} u_3 + u_2 \frac{\partial}{\partial y} u_3 + \frac{K\gamma}{\gamma - 1} \frac{\partial}{\partial z} \rho^{-1} \]

\[= \dot{a}(t) + [a(t) + C (y-z)](a(t) + C (x-y)] \]

\[+ \frac{\partial}{\partial x} \left[ C^2 \left[ (x^2 + y^2 + z^2) - (xy + yz + xz) \right] - \dot{a}(t)(x+y+z) + b(t) \right] \]

\[= \dot{a}(t) + [a(t) + C (y-z)] C + [a(t) + C (-x+z)] (-C) + 2C^2 z - C^2 y - C^2 x - \dot{a}(t) \]

\[= 0. \quad (28) \]

For the mass equation \([4]_{1}\), we have with the density \([17]\) to obtain:

\[\rho_t + \nabla \cdot (\rho \bar{v}) \]

\[= \frac{\gamma - 1}{K\gamma} \left( C^2 (x^2 + y^2 + z^2 - (xy + yz + xz)) - \dot{a}(t)(x+y+z) + b(t) \right) \]

\[+ \frac{\partial}{\partial x} \left[ \gamma - 1 \left( C^2 (x^2 + y^2 + z^2 - (xy + yz + xz)) - \dot{a}(t)(x+y+z) + b(t) \right) \right] \]

\[= a(t) + C (y-z) \]

\[+ \frac{\partial}{\partial y} \left[ \gamma - 1 \left( C^2 (x^2 + y^2 + z^2 - (xy + yz + xz)) - \dot{a}(t)(x+y+z) + b(t) \right) \right] \]

\[= a(t) + C (-x+z) \]

\[+ \frac{\partial}{\partial z} \left[ \gamma - 1 \left( C^2 (x^2 + y^2 + z^2 - (xy + yz + xz)) - \dot{a}(t)(x+y+z) + b(t) \right) \right] \]

\[= a(t) + C (x-y) \]

\[= \frac{1}{\gamma - 1} \left[ \gamma - 1 \left( C^2 (x^2 + y^2 + z^2 - (xy + yz + xz)) - \dot{a}(t)(x+y+z) + b(t) \right) \right] \]

\[+ \frac{\gamma - 1}{K\gamma} \left[ \left( C^2 x - C^2 y - C^2 z - \dot{a}(t) \right) \left( a(t) + C (y-z) \right) \right] \]

\[+ \frac{\gamma - 1}{K\gamma} \left[ \left( C^2 y - C^2 x - C^2 z - \dot{a}(t) \right) \left( a(t) + C (-x+z) \right) \right] \]

\[+ \frac{\gamma - 1}{K\gamma} \left[ \left( C^2 z - C^2 x - C^2 y - \dot{a}(t) \right) \left( a(t) + C (x-y) \right) \right] \]

\[= \frac{1}{K\gamma} \left[ \gamma - 1 \left( C^2 (x^2 + y^2 + z^2 - (xy + yz + xz)) - \dot{a}(t)(x+y+z) + b(t) \right) \right] \]

\[+ \frac{\gamma - 1}{K\gamma} \left[ \left( C^2 x - C^2 y - C^2 z - \dot{a}(t) \right) \left( a(t) + C (y-z) \right) \right] \]

\[+ \frac{\gamma - 1}{K\gamma} \left[ \left( C^2 y - C^2 x - C^2 z - \dot{a}(t) \right) \left( a(t) + C (-x+z) \right) \right] \]

\[+ \frac{\gamma - 1}{K\gamma} \left[ \left( C^2 z - C^2 x - C^2 y - \dot{a}(t) \right) \left( a(t) + C (x-y) \right) \right] \]

\[= \frac{1}{K\gamma} \left[ \gamma - 1 \left( C^2 (x^2 + y^2 + z^2 - (xy + yz + xz)) - \dot{a}(t)(x+y+z) + b(t) \right) \right] \]

\[+ \frac{\gamma - 1}{K\gamma} \left[ \left( C^2 x - C^2 y - C^2 z - \dot{a}(t) \right) \left( a(t) + C (y-z) \right) \right] \]

\[+ \frac{\gamma - 1}{K\gamma} \left[ \left( C^2 y - C^2 x - C^2 z - \dot{a}(t) \right) \left( a(t) + C (-x+z) \right) \right] \]

\[+ \frac{\gamma - 1}{K\gamma} \left[ \left( C^2 z - C^2 x - C^2 y - \dot{a}(t) \right) \left( a(t) + C (x-y) \right) \right] \]
Exact Solutions to the 3-dim Euler Equations

\[ \frac{1}{K\gamma} \left[ \frac{\gamma - 1}{K\gamma} \left( C^2 (x^2 + y^2 + z^2 - (xy + yz + xz)) - \dot{a}(t)(x + y + z) + b(t) \right) \right]^{\frac{1}{\gamma - 1}} \cdot \left\{ -\ddot{a}(t)(x + y + z) + \dot{b}(t) - 3\dot{a}(t)a(t) \right\} \]

(34)

\[ = 0 \]

(35)

by choosing

\[ a(t) = c_0 + c_1 t \]

(36)

and

\[ \dot{b}(t) = 3\dot{a}(t)a(t) \]

(37)

\[ \dot{b}(t) = 3c_1 (c_0 + c_1 t) \]

(38)

\[ b(t) = 3c_0c_1 t + \frac{3}{2}c_1^2 t^2 + c_2 \]

(39)

where \( c_2 \) is the arbitrary constant.

As for \( \gamma = 1 \), the proof is similar, we may skip the details here.

It is clear to see that the solutions (8) and (11) globally exist.

The proof is completed. ■

Here the masses of the solutions (8) and (11) are infinite:

\[ \int_{R^3} \rho dx = +\infty. \]

(40)

3 Incompressible Rotational Fluids

For the 3-dimensional incompressible Euler equations:

\[ \begin{cases} \text{div} \vec{u} & = 0 \\ \rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] + \nabla P(\rho) & = 0, \end{cases} \]

(41)

the solutions with elliptical symmetric velocity in [9], [8] and [16] for the compressible Euler equations (4), could not be applied to the incompressible Euler equations (41) because of the linear functional form:

\[ \begin{cases} u_1 = c_1(t)x + d_1(t) \\ u_2 = c_2(t)y + d_2(t) \\ u_3 = c_3(t)z + d_3(t) \end{cases} \]

(42)

with some functions \( c_i(t) \) and \( d_i(t) \) to have

\[ \text{div} \vec{u} \neq 0 \]

(43)

for non-trivial solutions \( c_1(t) + c_2(t) + c_3(t) \neq 0 \). However, our solutions (8) and (11) could be applied to the incompressible Euler equations (41). The set of the corresponding solutions for the incompressible Euler equations (41) could be larger than the compressible ones (4):

**Theorem 6** For the 3-dimensional incompressible Euler equations (41), there exists a class of rotational solutions:

\[ \begin{cases} \frac{\rho(\rho)}{\rho} = \left[ C^2 \left[ x^2 + y^2 + z^2 - (xy + yz + xz) \right] - \left[ \dot{a}_1(t) + C(a_2(t) + a_3(t)) \right] \right] x \\ - \left[ \dot{a}_2(t) + C(a_1(t) + a_3(t)) \right] y - \left[ \dot{a}_3(t) + C(a_1(t) + a_2(t)) \right] z + b(t) \end{cases} \]

\[ \begin{cases} u_1 = a_1(t) + C(y - z) \\ u_2 = a_2(t) + C(-x + z) \\ u_3 = a_3(t) + C(x - y) \end{cases} \]

(44)
and

\[
\frac{P(\omega)}{\rho} = \begin{cases}
-C^2 (x^2 + y^2 + z^2 + xy + yz + xz) \\
-\dot{a}_1(t) + C(a_2(t) + a_3(t)) x \\
-\dot{a}_2(t) + C(a_1(t) + a_3(t)) y \\
-\dot{a}_3(t) + C(a_1(t) + a_2(t)) z + b(t)
\end{cases}
\]

(45)

where \(a_i(t)\) is an arbitrary local \(C^1\) function for \(i = 1, 2\) or \(3\), \(b(t)\) is an arbitrary function, \(C\) is an arbitrary constant.

In particular,

1. if \(|a_i(T)| = \infty\) or \(|\dot{a}_i(T)| = \infty\) with the first finite positive constant \(T\), the solutions (44) and (45) blow up in the finite time \(T\);

2. For global \(C^1\) functions \(a_i(t)\), the solutions (44) and (45) globally exist.

\textbf{Proof.} For the checking of the functions (44), the detail is similar to the proof of Theorem 1:

For the checking of the functions (45), we have for the first momentum equation (41)2,2:

\[
\frac{\partial}{\partial t} u_1 + u_1 \frac{\partial}{\partial x} u_1 + u_2 \frac{\partial}{\partial y} u_1 + u_3 \frac{\partial}{\partial z} u_1 + \frac{\partial}{\partial x} \left( \frac{P}{\rho} \right) = 0
\]

(46)

by requiring the pressure function:

\[
P \frac{1}{\rho} = \begin{cases}
C^2 \left[ x^2 + y^2 + z^2 - (xy + yz + xz) \right] - [\dot{a}_1(t) + C(a_2(t) + a_3(t))] x \\
- [\dot{a}_2(t) + C(a_1(t) + a_3(t))] y - [\dot{a}_3(t) + C(a_1(t) + a_2(t))] z + b(t)
\end{cases}
\]

(47)

(48)

\[
\begin{align*}
\dot{a}_1(t) + [a_2(t) + C(-x + z)] \frac{\partial}{\partial y} [a_1(t) + C(y - z)] + [a_3(t) + C(x - y)] \frac{\partial}{\partial z} [a_1(t) + C(y - z)] \\
+ \frac{\partial}{\partial x} \left[ C^2 \left[ x^2 + y^2 + z^2 - (xy + yz + xz) \right] - [\dot{a}_1(t) + C(a_2(t) + a_3(t))] x \\
- [\dot{a}_2(t) + C(a_1(t) + a_3(t))] y - [\dot{a}_3(t) + C(a_1(t) + a_2(t))] z + b(t) \right]
\end{align*}
\]

(49)

(50)

\[
\begin{align*}
= \dot{a}_1(t) + [a_2(t) + C(-x + z)] C + [a_3(t) + C(x - y)] (-C) \\
+ [2C^2 x - C^2 y - C^2 z - \dot{a}(t) - C(a_2(t) + a_3(t))]
\end{align*}
\]

Similarly by the nice symmetry of the functions (44), we could balance the second momentum equation (41)2,2:

\[
\frac{\partial}{\partial t} u_2 + u_1 \frac{\partial}{\partial x} u_2 + u_3 \frac{\partial}{\partial y} u_2 + \frac{\partial}{\partial y} \left( \frac{P}{\rho} \right) = 0
\]

(53)

(54)

(55)
Similarly, we can balance the second momentum equation (41) with the pressure function

\[ P \]

For the last equation (41), we have

\begin{align*}
\frac{\partial}{\partial t} u_3 + u_1 \frac{\partial}{\partial x} u_3 + u_2 \frac{\partial}{\partial y} u_3 + \frac{\partial}{\partial z} \left( \frac{P}{\rho} \right)
\end{align*}

\[ (57) \]

\[ = 0. \]

(56)

For the last equation (41)_{2,3}, we have

\[ \frac{\partial}{\partial t} u_3 + u_1 \frac{\partial}{\partial x} u_3 + u_2 \frac{\partial}{\partial y} u_3 + \frac{\partial}{\partial z} \left( \frac{P}{\rho} \right) \]

\[ = \dot{a}_3(t) + [a_1(t) + C (y - z)] \frac{\partial}{\partial x} [a_3(t) + C (x - y)] + [a_2(t) + C (-x + z)] \frac{\partial}{\partial y} [a_3(t) + C (x - y)] \]

\[ + \frac{\partial}{\partial z} \left[ C^2 \left( x^2 + y^2 + z^2 - (xy + yz + xz) \right) - [\dot{a}_1(t) + C(a_2(t) + a_3(t))] x \right] \]

\[ = \dot{a}_3(t) + [a_1(t) + C (y - z)] C + [a_2(t) + C (-x + z)] (-C) + 2C^2 z - C^2 y - C^2 x - \dot{a}_3(t) - C(a_1(t) + a_2(t)) \]

\[ = 0. \]

Next, for the checking of the solutions (41), we have for the first momentum equation (41)_{2,1}:

\[ \frac{\partial}{\partial t} u_1 + u_2 \frac{\partial}{\partial y} u_1 + u_3 \frac{\partial}{\partial z} u_1 + \frac{\partial}{\partial x} \left( \frac{P}{\rho} \right) \]

\[ (61) \]

with the pressure function

\[ \frac{P}{\rho} = \left[ -C^2 \left( x^2 + y^2 + z^2 + xy + yz + xz \right) - [\dot{a}_1(t) + C(a_2(t) + a_3(t))] x \right] \]

\[ - [\dot{a}_2(t) + C(a_1(t) + a_3(t))] y - [\dot{a}_3(t) + C(a_1(t) + a_2(t))] z + b(t) \]

\[ (62) \]

to have

\[ = \dot{a}_1(t) + [a_2(t) + C(x + z)] \frac{\partial}{\partial y} [a_1(t) + C (y + z)] + [a_3(t) + C (x + y)] \frac{\partial}{\partial z} [a_1(t) + C (y + z)] \]

\[ + \frac{\partial}{\partial x} \left[ -C^2 \left( x^2 + y^2 + z^2 + xy + yz + xz \right) - [\dot{a}_1(t) + C(a_2(t) + a_3(t))] x \right] \]

\[ = \dot{a}_1(t) + [a_2(t) + C(x + z)] C + [a_3(t) + C (x + y)] C - 2C^2 z - C^2 z - \dot{a}_3(t) - C(a_2(t) + a_3(t)) \]

\[ = 0. \]

(63)

(64)

Similarly, we can balance the second momentum equation (41)_{2,2} with the symmetric form of the functions (45):

\[ \frac{\partial}{\partial t} u_2 + u_1 \frac{\partial}{\partial x} u_2 + u_3 \frac{\partial}{\partial z} u_2 + \frac{\partial}{\partial y} \left( \frac{P}{\rho} \right) \]

\[ (67) \]

\[ = \dot{a}_2(t) + [a_1(t) + C (y + z)] \frac{\partial}{\partial z} [a_2(t) + C (x + z)] + [a_3(t) + C (x + y)] \frac{\partial}{\partial x} [a_2(t) + C (x + z)] + \frac{\partial}{\partial y} \left( \frac{P}{\rho} \right) \]

\[ (68) \]

\[ = \dot{a}_2(t) + [a_1(t) + C (y + z)] C + [a_3(t) + C (x + y)] C - 2C^2 y - C^2 z - \dot{a}_2(t) - C(a_1(t) + a_3(t)) \]

\[ = 0. \]

(69)

For the last equation (41)_{2,3}, we could get

\[ \frac{\partial}{\partial t} u_3 + u_1 \frac{\partial}{\partial x} u_3 + u_2 \frac{\partial}{\partial y} u_3 + \frac{\partial}{\partial z} \left( \frac{P}{\rho} \right) \]

\[ (70) \]

\[ = \dot{a}_3(t) + [a_1(t) + C (y + z)] \frac{\partial}{\partial x} [a_3(t) + C (x + y)] + [a_2(t) + C (x + z)] \frac{\partial}{\partial y} [a_3(t) + C (x + y)] + \frac{\partial}{\partial z} \left( \frac{P}{\rho} \right) \]

\[ = 0. \]
\[ \dot{a}_3(t) + [a_1(t) + C(y + z)]C + [a_2(t) + C(x + z)]C - 2C^2z - C^2y - C^2x - \dot{a}_3(t) - C(a_1(t) + a_2(t)) \]
\[ = 0. \]
\[ (71) \]

It is clear to see that
1. if \( |a_i(T)| = \infty \) or \( |\dot{a}_i(T)| = \infty \) for \( i = 1, 2 \) or 3, with the first finite positive constant \( T \), the solutions \((44)\) and \((45)\) blow up in the finite time \( T \);
2. for all global \( C^1 \) functions \( a_i(t) \), the solutions \((44)\) and \((45)\) globally exist.
The proof is completed. □

Here the kinetic energy of the solutions \((44)\) and \((45)\) for the incompressible fluids are infinite:
\[ \frac{1}{2} \int_{R^3} \vec{v}^2 = +\infty. \]
\[ (73) \]

And the blowup solutions \((44)\) and \((45)\) are the other examples with infinite energy for the incompressible Euler equations \((41)\), with respect to the blowup cylindrical ones in Gibbon, Moore and Stuart’s work \([4]\).

Additionally it is because
\[ \Delta \vec{u} = \vec{0} \]
\[ (74) \]
in the solutions \((8)\) and \((11)\) to be the corresponding solutions for the 3-dimensional compressible Navier-Stokes equations:
\[ \begin{cases} \rho_t + \nabla \cdot (\rho \vec{u}) = 0 \\ \rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] + \nabla P(\rho) = \mu \Delta \vec{u} \end{cases} \]
\[ (75) \]
with \( \mu > 0 \).

And the solutions \((44)\) and \((45)\) are the corresponding ones for the 3-dimensional incompressible Navier-Stokes equations:
\[ \begin{cases} \text{div} \vec{u} = 0 \\ \rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] + \nabla P(\rho) = \mu \Delta \vec{u} \end{cases} \]
\[ (76) \]
with \( \mu > 0 \).

## 4 Discussion

1. We observe that the constructed density functions \( \rho \) \((8)\) and \((11)\), and the pressure functions \((44)\) and \((45)\) share some nice algebraic geometry, but we do not understand for the implication of this geometric structure for the fluids. Could the solutions with small perturbations of these solutions converge to the original ones?

2. Could we just cut a finite partial volume of the solutions to construct new weak solutions? If we could, what happens for the corresponding weak solutions of the classical ones with infinite energy?

3. By taking consideration with Makino’s blowup solutions form \((42)\), we could guess that the more general functional form for the 3-dimensional velocities \( \vec{u} \) in Cartesian coordinate:
\[ \begin{cases} u_1 = a_1(t) + b_{11}(t)x + b_{12}(t)y + b_{13}(t)z \\ u_2 = a_2(t) + b_{21}(t)x + b_{22}(t)y + b_{23}(t)z \\ u_3 = a_3(t) + b_{31}(t)x + b_{32}(t)y + b_{33}(t)z \end{cases} \]
\[ (77) \]
with the local smooth time \( C^1 \) functions \( a_i(t) \) and \( b_{ij}(t) \) for \( i, j = 1, 2 \) and 3, could be the more general solutions structure for the systems. Then, it is more difficult to handle the linear functional structure for the higher dimensional cases \( R^N \) with \( N \geq 4 \). Could we construct a blowup example with rotation for the incompressible fluids? In principle, we could adopt the guessing approach to construct the density functions. Some trials with luck are needed to obtain a right novel functional form to construct solutions for the systems, but it costs a lot of time for computation. Thus, we hope to arouse the readers’ interests to investigate more systematic
approaches to calculate the possible particular solutions with the linear functional velocities $\vec{u}$ (77) in Cartesian coordinate or other coordinates.

Finally, further researches are needed for understanding this class (77) of blowup or global solutions for the systems in the future.

References

[1] V. I. Arnold, Sur la Topologie des Ecoulements Stationnaires des Fluides Parfaits, (French) C. R. Acad. Sci. Paris 261 (1965), 17–20.

[2] G.Q. Chen and D.H. Wang, The Cauchy Problem for the Euler Equations for Compressible Fluids, Handbook of Mathematical Fluid Dynamics, Vol. I, 421-543, North-Holland, Amsterdam, 2002.

[3] P. Constantin, The Euler Equations and Nonlocal Conservative Riccati Equations, Internat. Math. Res. Notices 2000, 455–465

[4] J. D. Gibbon, D. R. Moore and J. T. Stuart, Exact, Infinite Energy, Blow-up Solutions of the Three-dimensional Euler Equations, Nonlinearity 16 (2003), 1823–1831.

[5] P.L. Lions, Mathematical Topics in Fluid Mechanics, Vols. 1, 2, 1998, Oxford: Clarendon Press.

[6] T. Makino, Exact Solutions for the Compressible Euler equation, Journal of Osaka Sangyo University Natural sciences 95 (1993), 21–35.

[7] K. Ohkitani and J. D. Gibbon, Numerical Study of Singularity Formation in a Class of Euler and Navier-Stokes Flows, Phys. Fluids 12 (2000), 3181–3194.

[8] L. V. Ovsiannikov, Dokl. Akad. Nauk SSSR 111 (1965), 47.

[9] L. I. Sedov, Dokl. Akad. Nauk SSSR 40 (1953), 753.

[10] T. Senba and T. Suzuki, Applied Analysis, Mathematical Methods In Natural Science, Imperial College Press, London, 2004. xiv+378 pp.

[11] L.H. Yeung and M.W. Yuen, Note for "Some Exact Blowup Solutions to the Pressureless Euler Equations in $\mathbb{R}^N$ " [Commun. Nonlinear Sci. Numer. Simul. 16 (2011), 2993-2998], Commun. in Nonlinear Sci. and Numer. Simul., In Press, DOI: 10.1016/j.cnsns.2011.04.016.

[12] M.W. Yuen, Blowup Solutions for a Class of Fluid Dynamical Equations in $\mathbb{R}^N$, J. Math. Anal. Appl. 329 (2007), 1064–1079.

[13] M.W. Yuen, Analytical Solutions to the Navier-Stokes Equations, J. Math. Phys., 49 (2008), 113102, 10pp.

[14] M.W. Yuen, Some Exact Solutions to the Pressureless Euler Equations in $\mathbb{R}^N$, Commun. Nonlinear Sci. Numer. Simul. 16 (2011), 2993–2998.

[15] M.W. Yuen, Perturbational Blowup Solutions to the 1-dimensional Compressible Euler Equations, Pre-print, arXiv:1012.2033.

[16] M.W. Yuen, Self-Similar Solutions with Elliptic Symmetry for the Compressible Euler and Navier-Stokes Equations in $\mathbb{R}^N$, Pre-print, arXiv:1104.3687