LEPAGE EQUIVALENTS OF SECOND-ORDER EULER–LAGRANGE FORMS AND THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

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In the calculus of variations, Lepage \((n+1)\)-forms are closed differential forms, representing Euler–Lagrange equations. They are fundamental for investigation of variational equations by means of exterior differential systems methods, with important applications in Hamilton and Hamilton–Jacobi theory and theory of integration of variational equations. In this paper, Lepage equivalents of second-order Euler–Lagrange quasi-linear PDE's are characterised explicitly. A closed \((n+1)\)-form uniquely determined by the Euler–Lagrange form is constructed, and used to find a geometric solution of the inverse problem of the calculus of variations.

Keywords: Second-order Euler–Lagrange equations; Euler–Lagrange form; Lepage form; Lepage equivalent of a Lagrangian; Lepage equivalent of an Euler–Lagrange form; inverse problem of the calculus of variations.

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1. Introduction

The inverse problem of the calculus of variations is concerned with the question when a system of ordinary or partial differential equations of order \(r \geq 1\) identifies with Euler–Lagrange equations, i.e., equations for extremals of a variational functional. This problem was first considered by Helmholtz in 1887, for a system of second order ordinary differential equations \([11]\). In his seminal paper, Helmholtz found necessary conditions for variationality, now called Helmholtz conditions (Mayer \([23]\) later proved that the conditions are also sufficient). Since that time, questions on existence, multiplicity and construction of Lagrangians to differential equations have been investigated by many authors. A remarkable progress in the solution of the inverse variational problem was achieved around 1980 by methods of differential geometry and global analysis, in connection with new developments of the calculus of variations on fibre manifolds and the theory of variational bicomplexes: Helmholtz conditions were generalised to PDE’s of an arbitrary order by Anderson and Duchamp \([1]\) and Krupka \([14]\), and the inverse problem was extended to study conditions for existence of a global Lagrangian. Nowadays, geometric and global rather than analytical aspects of the inverse variational problem are of main interest, and relations between variationality and geometry of differential equations are intensively studied and explored (see e.g. \([4, 21, 22]\)).
One of the most significant advances in this direction was the discovery of an intimate relationship between variational equations and closed forms, i.e., between the Euler–Lagrange operator and the operator of the exterior derivative of differential forms (Crampin, Prince and Thompson [3], DeDecker and Tulczyjew [5], Krupka [14, 15], Tonti [25]). Due to this relationship, the local version of the inverse variational problem is transferred to an application of the Poincaré Lemma, and global existence results follow from De Rham Theorem. A direct geometric expression of this property is realised within the concept of Lepage \((n+1)\)-form, where \(n\) is the number of independent variables (Krupková [17–19]).

Lepage \((n+1)\)-forms are closed differential forms, exclusively representing variational equations; they are also called Lepage equivalents of variational equations. Besides questions connected with the inverse variational problem, they are used to study variational equations and their solutions with exterior differential systems methods. Most important applications arise in investigations of symmetries and conservation laws, Hamilton and Hamilton–Jacobi theory, and integration of variational equations. Techniques using Lepage forms also hold the promise of a natural extension of methods and results from the calculus of variations to the class of differential equations for which no Lagrangian exists. For ordinary variational equations (in physical terminology “higher-order mechanics”), the theory of Lepage \((n+1)\)-forms is well-established. For an exposition of results and applications with stress on the geometry of ordinary differential equations and the inverse problem of the calculus of variations we refer the reader to the book [20] and recent survey papers [21, 22]. On the other hand, for partial differential equations (“field theory”) results achieved so far are by no means complete.

The aim of this paper is to investigate Lepage \((n+1)\)-forms associated with second-order Euler–Lagrange quasi-linear PDE’s. After a brief survey of the current status of the subject in Sec. 2, new results are reported in Sec. 3 and proved in Sec. 4.

While for ordinary differential equations there is a one-to-one correspondence between Lepage 2-forms and variational equations, for partial differential equations the situation is more complicated. We study the structure of Lepage \((n+1)\)-forms, and provide an explicit characterisation of all corresponding Lepage equivalents. It turns out, however, that the class of Lepage equivalents contains a distinguished (local) “fundamental equivalent” uniquely determined by the Euler–Lagrange expressions. We discuss global existence of such an \((n+1)\)-form, and find its relationship with the so-called “fundamental Lepage equivalent of a Lagrangian” (Krupka \(n\)-form) [2, 13]. Finally, the closed \((n+1)\)-form uniquely determined by the Euler–Lagrange form is used to obtain the variationality conditions in an intrinsic form.

Proofs of the theorems are quite straightforward, however, sometimes require long and difficult calculations. To make the article easily accessible to different readers, we decided to divide the exposition into two parts: Results are summarised in Sec. 3, and for interested readers, complete proofs are included in Sec. 4.

Finally, we note that analogous differential forms were considered on Grassmann bundles by Grigore and Popp, and Grigore [8, 9]. They introduced closed \((n+1)\)-forms representing variational equations (“Lagrange–Souriau forms”), and used them to study Noether symmetries.

2. Lepage Forms

Throughout the paper, all manifolds and mappings are assumed smooth, and the summation over repeated indices is used whenever appropriate. The background for our considerations is the theory of jet bundles and the calculus of variations on fibred manifolds (see e.g. [16, 20, 24]). We consider a fibred manifold \(\pi: Y \to X\), \(\dim X = n\), \(\dim Y = n+m\), where \(n, m \geq 1\). For \(r = 1, 2\) we denote by \(\pi_r: J^rY \to X\) the \(r\)-jet prolongation of \(\pi\), and by \(\pi_{r,k}: J^rY \to J^kY\), \(0 \leq k \leq r\), the canonical projections (here \(J^1Y = Y\)). A section of \(\pi\) is a mapping \(\gamma: U \to Y\), where \(U \subset X\) is an open set.
such that $\pi \circ \gamma = id_Y$. The $r$-jet prolongation of $\gamma$ is denoted by $J^r\gamma$; it is a section of the fibred manifold $\pi_r$.

On jet bundles it is convenient to use vector fields and differential forms adapted to the fibred structure [12]: A vector field $\xi$ on $J^rY$ is called $\pi_r$-vertical if it projects onto the zero vector field on $X$. A differential $q$-form $\eta$ on $J^rY$ is called horizontal (or $0$-contact) with respect to the projection $\pi_r$ if $i_\eta \xi = 0$ for every $\pi_r$-vertical vector field $\xi$ on $J^rY$; $\eta$ is called contact if $J^r\gamma \eta = 0$ for every section $\gamma$ of $\pi_r$.

A contact form is said to be $1$-contact if for every vertical vector field $\xi$ on $J^rY$, the contraction $i_\xi \eta$ is horizontal. Recursively, $\eta$ is said to be $k$-contact if for every vertical vector field $\xi$, $i_\xi \eta$ is $(k-1)$-contact. We have a useful Structure Theorem due to Krupa [12], stating that every $q$-form $\eta$ on $J^rY$ admits a unique decomposition as a sum of forms on $J^{r+1}Y$ as follows:

$$
\pi^*_{r+1} \eta = h\eta + \sum_{k=1}^{\lfloor q/k \rfloor} p_k\eta_i,
$$

where $h\eta$ is a horizontal form (called the horizontal part of $\eta$); and $p_k\eta_i$, $1 \leq k \leq q$, is a $k$-contact form (called the $k$-contact part of $\eta$).

Let $x_i, y^i, 1 \leq i \leq n$, be fibred coordinates on $Y$, defined on an open set $V \subset Y$; we denote by $(x^i, y^i, y_j^i)$, and $(x^i, y^i, y_j^i, y_{ij}^i)$, $1 \leq j \leq 2$, the associated coordinates on $J^1Y$ and $J^2Y$, respectively. Next, we write

$$
\omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \quad \omega_{1, i_1} = \omega_{1, j\ldots i_n} = \omega_{1, j\ldots i_n} = \omega_{1, j\ldots i_n}, \quad 1 \leq k \leq q,
$$

for the local volume on $X$ and its contractions, and

$$
\omega^n = dy^n - y^n dx^j, \quad \omega^n_j = dy^n - y^n dx^j,
$$

for the associated basis of contact 1-forms on $\pi^*_{n,1}(V) \subset J^rY$.

A dynamical form $E$ of order $r$ is a $(n+1)$-form on $J^rY$, horizontal with respect to the projection onto $Y$. In fibered coordinates,

$$
E = E_0 \omega^n \wedge \omega_0,
$$

where $E_0$ are local functions on $J^rY$.

Let $\lambda$ be a first-order Lagrangian, i.e., a horizontal $n$-form on $J^rY$. A differential $n$-form $\eta$ is called Lepage equivalent of $\lambda$ (see [12]) if in the decomposition (2.1), $h\eta = \lambda$, and $p_1\eta_1$ is a dynamical form; the $(n+1)$-form $E_0 = p_1\eta_1$ is then called the Euler–Lagrange form of $\lambda$. For a first-order Lagrangian $\lambda$ the Euler–Lagrange form is defined on $J^rY$ and in every fibered chart reads

$$
E_0 = \frac{\partial L}{\partial y^n} - d_j \frac{\partial L}{\partial y^j} \omega^n \wedge \omega_0.
$$

Note that components $E_0$ of $E_0$ are functions affine in the second derivatives, since

$$
\frac{\partial E_0}{\partial y^n} = -\frac{1}{2} \left( \frac{\partial^2 L}{\partial y^n \partial y^i} + \frac{\partial^2 L}{\partial y^n \partial y^j} \right),
$$

are defined on an open subset of $J^rY$.

As proved in [12], to every first-order Lagrangian a (global) Lepage equivalent exists and is non-unique. The family of Lepage equivalents of $\lambda$ contains distinguished representatives that are
completely determined by the Lagrangian: here we mention the famous Poincaré–Cartan form [6, 7, 12],
\[ \Theta_\lambda = L\omega_\lambda + \frac{\partial L}{\partial y_j} \omega_j, \]
and the Krupka form [13] (see also [2]),
\[ p_\lambda = L\lambda_\lambda + \sum_{k=1}^{n} \frac{1}{k!} \sum_{j_1, \ldots, j_k} \frac{\delta^k L}{\partial y_{j_1} \cdots \partial y_{j_k}} \omega_{j_1} \wedge \cdots \wedge \omega_{j_k} \wedge \omega_{j_{k+1}}. \] (2.2)
The latter Lepage equivalent of \( \lambda \) has the following important property (not possessed by the Poincaré–Cartan form): \( dp_\lambda = 0 \Leftrightarrow E_\lambda = 0 \).

Consider a dynamical form \( E \) on \( J^r Y \). \( E \) is said to be locally variational if to every point in \( J^r Y \) one has a neighbourhood \( U \), and a Lagrangian \( \lambda \) on \( U \), such that \( E|_U = E_\lambda \). It is known that \( E \) is locally variational if and only if the components of \( E \) satisfy the following identities:
\[ \frac{\partial E_\epsilon}{\partial y_j} \wedge \frac{\partial E_\nu}{\partial y_j} = 0, \] (2.3)
\[ \frac{\partial E_\epsilon}{\partial y_j} \wedge \frac{\partial E_\nu}{\partial y_j} - 2\lambda \frac{\partial E_\epsilon}{\partial y_j \partial y_j} = 0, \] (2.4)
\[ \frac{\partial E_\epsilon}{\partial y_j} + \frac{\partial E_\nu}{\partial y_j} + d_i \frac{\partial E_\epsilon}{\partial y_j} \wedge \sigma \frac{\partial E_\nu}{\partial y_j \partial y_j} = 0. \] (2.5)

We recall a fundamental theorem due to Krupka [15], relating locally variational forms with closed forms:

**Theorem 1.** A dynamical form \( E \) is locally variational if and only if to every point in the domain of \( E \) there exists a neighbourhood \( W \) and an at least 2-contact form \( F_\epsilon \) on \( W \) such that the form \( \alpha_\epsilon = E + F_\epsilon \) is closed.

\( (n+1) \)-form \( \alpha \) is called Lepage equivalent of \( E \) [17] if \( p\alpha = E \) and \( d\alpha = 0 \). One can see immediately that if \( \alpha \) is a Lepage equivalent of \( E \) then, around every point, \( \alpha = \eta \) where \( \eta \) is a Lepage equivalent of a local Lagrangian for \( E \).

The above theorem guarantees local existence of Lepage equivalents; it does not provide us with explicit formulas for \( \alpha_\epsilon \) by means of the components of \( E \).

The problem of (global) existence and multiplicity of Lepage equivalents has been completely solved for locally variational forms on \( J^r Y \) in [10].

**Theorem 2.** Every first-order locally variational form \( E \) has a unique Lepage equivalent defined on \( Y \). It is denoted by \( \alpha_E \) and takes the form
\[ \alpha_E = E\omega + \sum_{k=1}^{n} \frac{1}{k!(k+1)!} \frac{\delta^k E}{\partial y_{j_1} \cdots \partial y_{j_k}} \omega_{j_1} \wedge \cdots \wedge \omega_{j_k} \wedge \omega_{j_{k+1}}. \] (2.6)
Moreover, in a neighbourhood \( U \) of every point in \( Y \),
\[ \alpha_E|_U = dp_\lambda, \] (2.7)
where \( \lambda \) is a local first order Lagrangian for \( E \). All Lepage equivalents of \( E \) are then described by the formula \( \alpha = \alpha_E + \phi \) where \( \phi \) is an arbitrary closed at least 2-contact form defined on \( J^r Y \), \( r \geq 1 \).
In the next section we shall be interested in similar questions for second-order Euler–Lagrange equations. To this end we shall use the following result [10]:

**Lemma 1.** Let \( \alpha \) be a Lepage equivalent of a locally variational form \( E \) on \( J^2Y \). Then
\[
\frac{\partial^2 E_2}{\partial y_j^p \partial y_k^q} = 0,
\]
(3.1)
meaning that the components \( E_2 \) of \( E \) are affine functions in the second derivatives.

In what follows, we shall study the structure of Lepage equivalents of Euler–Lagrange forms the components of which are affine in the second derivatives. In this section we summarise the results of the paper, complete proofs are postponed to the next section.

The problem is to find all closed \((n+1)\)-forms \( \alpha \) such that \( p_2 \alpha = E \). The closedness condition on \( \alpha \) means that at least some of the components of the higher-degree contact parts of \( \alpha \) depend upon the Euler–Lagrange expressions \( E_\sigma, 1 \leq \sigma \leq m \). This means that \( \alpha \) splits into a (not necessarily invariant) sum
\[
\alpha = \alpha_E + \phi,
\]
(3.2)
where \( \alpha_E \) is completely determined by the Euler–Lagrange expressions, while \( \phi \) does not depend upon \( E \). Hence, the first step to solve the structure problem is to find the form \( \alpha_E \).

**Theorem 3.** Let \( E \) be a locally variational form on \( J^2Y \). If the condition (3.1) is satisfied then \( \alpha_E \) is affine in the \( \omega^\nu_j \)'s, and takes the form
\[
\alpha_E = E_2 \omega^\nu_j \wedge \omega^\nu_{j+1}
+ \sum_{k=1}^n \frac{1}{k!(k+1)!} \frac{\partial^k E_2}{\partial y_{j_1}^p \ldots \partial y_{j_k}^q} \omega^\nu_{j_1} \wedge \cdots \wedge \omega^\nu_{j_k} \wedge \omega^\nu_{j_{k+1}},
\]
(3.3)
Moreover, \( \alpha_E \) is \( \pi_{2,1} \)-projectible.

In view of the preceding theorem we obtain the following solution to the problem of the structure of Lepage equivalents of “quasi-linear” second-order Euler–Lagrange equations:

**Theorem 4.** Let \( E \) be a locally variational form on \( J^2Y \) satisfying (3.1). Every Lepage equivalent \( \alpha \) of \( E \) takes the form
\[
\alpha = \alpha_E + \phi,
\]
(3.4)
where \( \alpha_E \) is a closed (first-order) form, given by (3.3), and \( \phi \) is a closed, at least 2-contact form.

**Corollary 1.** Let \( E \) be a locally variational form on \( J^2Y \), satisfying condition (3.1). Given a fibred chart \((V, \psi)\) on \( Y \) with coordinates \((x^i, y^j)\), the \((n+1)\)-form \( \alpha_E \) determined by the Euler–Lagrange expressions of \( E \) and defined on \( \pi_{2,1}^{-1}(V) \subset J^2Y \) by (3.3) is a Lepage equivalent of \( E \).
It remains to illuminate transformation properties of the form \( \alpha_K \) with respect to fibred coordinates. Consider two overlapping charts \((V, \psi)\), \((\overline{V}, \overline{\psi})\), and \((\overline{V}, \overline{\psi}) = (x^i, \theta^i)\) on \(Y\). Using in the expression for \( \alpha_K \) transformation formulas

\[
\tilde{\omega}_{j_n \ldots j_1} = \det \left( \frac{\partial x^k}{\partial x_j} \right) \left( \frac{\partial x^l}{\partial x_\rho} \right) \ldots \left( \frac{\partial x^p}{\partial x_i} \right),
\]

\[
\tilde{\omega}^p = \frac{\partial \theta^p}{\partial \theta^j}, \quad \tilde{\omega}^j = \frac{\partial \theta^j}{\partial \theta^p} \frac{\partial \theta^p}{\partial \theta^l} + \frac{\partial \theta^j}{\partial \theta^l},
\]

\[
\tilde{\omega}^j_{\alpha} = \frac{1}{\partial x^k} \left( \frac{\partial \theta^m}{\partial \theta^l} \frac{\partial \theta^l}{\partial \theta^p} \frac{\partial \theta^p}{\partial \theta^j} - \frac{\partial \theta^m}{\partial \theta^p} \frac{\partial \theta^p}{\partial \theta^l} \frac{\partial \theta^l}{\partial \theta^j} \right),
\]

we obtain the following result:

- The at most 2-contact part (also called principal part) of \( \alpha_K \)

\[
\tilde{\alpha}_K = \frac{E_\rho}{\partial y^p} \omega^p \wedge \omega_j + \frac{1}{2} \frac{\partial E_\rho}{\partial y^p} \omega^p \wedge \omega_j \wedge \omega_j + \frac{\partial E_\rho}{\partial y^p} \omega^p \wedge \omega_j \wedge \omega_j
\]

is invariant with respect to fibred coordinate transformations. This means that formula (3.7) defines a global differential form. The form \( \tilde{\alpha}_K \), however, is in general not closed.

- For \( k \geq 3 \) the form \( p_2 \alpha_K \) is generally not invariant. Consequently, \( \alpha_K \) is not invariant, i.e., formula (3.3) does not define a global differential form.

**Remark.** We have seen that the \((n + 1)\)-form \( \alpha_K \) is global for \( r = 1 \) but no longer for \( r \geq 2 \). We remind the reader that this situation is analogous to the case of the well-known Poincaré-Cartan form \( \Theta \) that is global for \( r \leq 2 \) but not for higher order Lagrangians (cf., e.g., [15]). An important case when \( \alpha_K \) for second order \( E \) is global is when \( E \) arises from a global Lagrangian (see Theorem 6 below).

The next results clarify the meaning of the Lepage equivalent \( \alpha_K \) of a locally variational form \( E \).

**Theorem 5.** Let \( E \) be a dynamical form on \( J^2Y \), satisfying (3.1). The following conditions are equivalent:

1. \( E \) is locally variational.
2. \( \alpha_K \) is closed.
3. \( p_2 \alpha_K = 0 \).
4. Components \( E_\rho \) of \( E \) satisfy conditions (4.23).

The above theorem provides us with a geometric meaning of the variationality conditions, as conditions, under which the \((n + 1)\)-form \( \alpha_K \) is closed. Otherwise speaking,

\[
p_2 \alpha_K = 0
\]

is an intrinsic expression of variationality conditions (2.3)–(2.5) (respectively, (4.23) below).

Since the form \( \alpha_K \) is a Lepage equivalent of a locally variational form \( E \), around every point it equals \( dp_\rho \), where \( p_\rho \) is a Lepage equivalent of a Lagrangian of \( E \). We shall answer the question which of the Lepage equivalents of \( \lambda \) corresponds to \( \alpha_K \).
Theorem 6. Let \( \lambda \) be a Lagrangian on \( J^1Y \). Then

\[ d\rho_\lambda = \alpha_{E_0}, \]

where \( \rho_\lambda \) is the Krupka form (2.2) of \( \lambda \).

4. Lepage Equivalents of Second Order Euler–Lagrange Forms: Proofs and Computations

Proof of Theorem 3. If \( \alpha \) is an \((n + 1)\)-form such that \( p_1\alpha = E \) then in fibred coordinates

\[ \alpha = E_{alt} \wedge \omega_0 + \sum_{r+s=2}^{s+1} \left( \partial F_{x_1...x_{i-1}p_{i+1}...p_s} \right) \wedge \omega_0 + \cdots + \omega_{n-s} \wedge \omega_{s-r-1} \]

where the components \( F_{x_1...x_{i-1}p_{i+1}...p_s} \) are completely skew-symmetric in lower indices \( s_1...s_r \), completely skew-symmetric in upper indices \( s_1...s_{n-s-1} \) and completely skew-symmetric in pairs of indices \((p_1,p_s)\) and \((p_2,p_{s-1})\).

In what follows, let us denote the components of \( \alpha_{E_0} \), i.e., the part of the components of \( \alpha \), completely determined by \( E \), by \( \tilde{F}_{x_1...x_{i-1}p_{i+1}...p_s}^0 \).

Lemma 1 provides us with the following components of the \((n + 1)\)-form \( \alpha_{E_0}^0 \):

\[ \tilde{F}_{x_1...x_{i-1}p_{i+1}...p_s}^0 = \frac{1}{4} \left( \frac{\partial E_r}{\partial y_j} \frac{\partial E_s}{\partial y_j} \right), \quad \tilde{F}_{x_1...x_{i-1}p_{i+1}...p_s}^0 = \frac{\partial E_r}{\partial \rho_p} \frac{\partial E_s}{\partial \rho_p} \frac{\partial E_{\rho_{p+1}}}{\partial \rho_{p+1}} = 0. \]

Computing \( d\alpha = 0 \) we obtain the following relations:

(A) \( r, s \geq 0, r + s = 2, \ldots, n \):

\[ \frac{\partial F_{x_1...x_{i-1}p_{i+1}...p_s}^0}{\partial y_j} = \left[ \partial F_{x_1...x_{i-1}p_{i+1}...p_s}^0 \right]_{\partial y_j}, \]

and

\[ (-1)^r \frac{1}{(r+1)^{n-s}} \frac{\partial F_{x_1...x_{i-1}p_{i+1}...p_s}^0}{\partial y_{j+p}} \] \[ \left|_{\partial y_j} \right| = 0. \]

(B) \( r, s \geq 0, r + s = n + 1 \):

\[ (-1)^{r} \frac{\partial F_{x_1...x_{i-1}p_{i+1}...p_s}^0}{\partial y_{j+p}} \] \[ \left|_{\partial y_j} \right| = 0. \]

that yield, on one hand, recurrence formulas for components of \( \alpha \), and, on the other hand, relations between derivatives of the components of \( \alpha \). We note that if expressed by means of the components of the form \( \alpha_{E_0} \), these relations contain variationality conditions (2.3–2.5).
Above and in what follows, sym\{\} and alt\{\} means complete symmetrisation and skew-symmetrisation in the indicated indices (pairs of indices), respectively.

For the explicit computation of the form $\alpha_E$ we shall explore formulas (4.3) and (4.4). The desired recurrence formulas are obtained with help of the skew-symmetry conditions for the components of $\alpha$ following from (4.1). Using (4.3) we easily obtain

$$F^{\rho_1\ldots\rho_j}\big|_{\text{sym}(j+\rho_{j+1})} = \frac{1}{(s+1)(r+s)} \frac{\partial F^{\rho_1\ldots\rho_j\ldots\rho_{j+1}}}{\partial y_{j+\rho_{j+1}}}.$$ (4.7)

Working with (4.4) we must be more careful: First of all, if $s = 0$ we simply get for $r \geq 2$

$$F^{\rho_1\ldots\rho_j}\big|_{\text{alt}(\rho_{j+1}\ldots\rho_n))} = \frac{1}{r(r+1)} \left( \frac{\partial F^{\rho_1\ldots\rho_j\ldots\rho_{j+1}}}{\partial y_{j+\rho_{j+1}}} - \frac{\partial F^{\rho_1\ldots\rho_j\ldots\rho_{j+1}}}{\partial y_{j+1}} + r\frac{\partial F^{\rho_1\ldots\rho_j\ldots\rho_{j+1}}}{\partial y_{j+\rho_{j+1}}} \right).$$ (4.8)

Let $s \geq 1$. Accounting skew-symmetry conditions for the components of $\alpha$ we notice that

$$F^{\rho_1\ldots\rho_{j+1},\ldots\rho_n\ldots\rho_{j+1}}|_{\text{alt}(\rho_{j+1}\ldots\rho_n))} = \frac{1}{(r+1)(r+s)} \left( (-1) \frac{\partial F^{\rho_1\ldots\rho_j\ldots\rho_{j+1}}}{\partial y_{j+\rho_{j+1}}} \bigg|_{\text{alt}(\rho_{j+1}\ldots\rho_n))} + \frac{\partial F^{\rho_1\ldots\rho_j\ldots\rho_{j+1}}}{\partial y_{j+1}} \bigg|_{\text{alt}(\rho_{j+1}\ldots\rho_n))} \right) + (-1)^r(r+s)\frac{\partial F^{\rho_1\ldots\rho_{j+1},\ldots\rho_n\ldots\rho_{j+1}}}{\partial y_{j+\rho_{j+1}}} \bigg|_{\text{alt}(\rho_{j+1}\ldots\rho_n))}$$ (4.9)

since

$$\frac{\partial F^{\rho_1\ldots\rho_{j+1},\ldots\rho_n\ldots\rho_{j+1}}}{\partial y_{j+\rho_{j+1}}} \bigg|_{\text{alt}(\rho_{j+1}\ldots\rho_n))} = \frac{\partial F^{\rho_1\ldots\rho_{j+1},\ldots\rho_n\ldots\rho_{j+1}}}{\partial y_{j+1}} \bigg|_{\text{alt}(\rho_{j+1}\ldots\rho_n))} \bigg|_{\text{alt}(\rho_{j+1}\ldots\rho_n))} \bigg|_{\text{alt}(\rho_{j+1}\ldots\rho_n))},$$

and similarly for $\partial F^{\rho_1\ldots\rho_{j+1},\ldots\rho_n\ldots\rho_{j+1}}$. This means, however, that formula (4.9) splits into two parts: one completely skew-symmetrised both in the lower indices $\sigma_1\ldots\sigma_{j+1}$ and the upper indices $\rho_{j+2}\ldots\rho_n$; and the complementary part. Applying the complete skew-symmetrisation, we notice that

$$F^{\rho_1\ldots\rho_{j+1},\ldots\rho_n\ldots\rho_{j+1}}|_{\text{alt}(\rho_{j+1}\ldots\rho_n))} \bigg|_{\text{alt}(\rho_{j+1}\ldots\rho_n))} = 0,$$

being symmetric in the pairs of indices $(\rho_{j+1}\rho_{j+2})\ldots(\rho_{j+1}\rho_{n+1})$, and at the same time by definition, skew-symmetric in the pairs of indices $(\rho_{j+1}\rho_{j+2})\ldots(\rho_{j+1}\rho_{n+1})$. Summarising, the completely
skew-symmetrised part of the splitting of (4.9) provides only another relation between derivatives of components of \( \alpha \), while the recurrence formulas for the \( F \)'s are provided by the complementary part of the splitting, and read as follows:

\[
F^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s} \equiv \mathcal{alt}(\sigma_{r+1} \ldots \sigma_s | \sigma_1 \ldots \sigma_r)
\]

\[
= (-1)^r \frac{1}{(r+1)(r+s)} \frac{\partial F^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s}}{\partial y_{r+1}}
\]

\[
= (-1)^r \frac{1}{(r+1)(r+s)} \frac{\partial F^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s}}{\partial y_{r+1}}
\]

\[
\text{Let us solve the recurrence formulas (4.7), (4.8) and (4.10) explicitly:}
\]

(i) Consider (4.8). First notice that the last two terms entering in this formula,

\[
F^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s} \equiv \mathcal{alt}(\sigma_{r+1} \ldots \sigma_s | \sigma_1 \ldots \sigma_r)
\]

\[
F^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s} \equiv \mathcal{alt}(\sigma_{r+1} \ldots \sigma_s | \sigma_1 \ldots \sigma_r)
\]

are completely symmetric in the pairs of indices \((\sigma_1 j_1) \ldots (\sigma_s j_s)\) and \((\sigma_1 \ldots \sigma_r)\), respectively, and completely skew-symmetric in the upper indices. This means that these functions cannot be obtained from the recurrence formulas (4.7) and (4.10), i.e., in particular, they are independent upon a choice of \( E \), and hence do not enter into \( \alpha_E \). In this way we obtain

\[
F^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s} = \mathcal{alt}(\sigma_{r+1} \ldots \sigma_s | \sigma_1 \ldots \sigma_r)
\]

\[
= \frac{1}{r(r+1)} \frac{\partial E^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s}}{\partial y_{r+1}}
\]

\[
= \frac{1}{r(r+1)} \frac{\partial E^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s}}{\partial y_{r+1}}
\]

(ii) Using (4.10) for \( r = 1 \) and \( s = 1 \) we can see that

\[
F^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s} \equiv \mathcal{alt}(\sigma_{r+1} \ldots \sigma_s | \sigma_1 \ldots \sigma_r)
\]

\[
= \frac{1}{r^s} \frac{\partial F^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s}}{\partial y_{j+1}}
\]

\[
= \frac{1}{r^s} \frac{\partial F^4_{\sigma_1 \ldots \sigma_r \sigma_{r+1} \ldots \sigma_s}}{\partial y_{j+1}}
\]

and since \( F^4_{\sigma_1 \ldots \sigma_r} \) do not depend upon a choice of \( E \), the \( F \)'s above do not enter into \( \alpha_E \). In order to compute components at \( \alpha_E \), we have to use (4.7) for \( s = 0 \). Then with help of (4.12) we get
for \( r \geq 2 \)

\[
\hat{F}_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)} = \frac{1}{s(\sigma + 1)} \frac{\partial F_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)}}{\partial y_{\rho}^{\sigma + 1}},
\]

(4.14)

Since by Lemma 1 the functions \( F_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)} \) do not depend upon the Euler–Lagrange expressions, we get that in (4.1) the \( F_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)} \) are independent of the choice of \( E \). Moreover, condition \( d\sigma = 0 \) gives no formulas for the skew-symmetric parts of these functions. Hence, all components of \( \alpha_E \) for \( r = 0 \) are equal to zero. Note that this means that the form \( \alpha_E \) belongs to the ideal generated by the one-forms \( \omega^\sigma, 1 \leq \sigma \leq m \).

(iv) Finally, we shall show that the remaining components of \( \alpha_E \) are equal to zero. To this end we first notice that (4.10) gives \( \hat{F}_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)} \) expressed by means of derivatives of \( F_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)} \) and assumed (3.1). We have at disposal formulas for the symmetrical part of the latter functions in \( p_{\rho} \). The remaining parts are left arbitrary, i.e. do not contribute to \( \alpha_E \). Thus, next we have to consider (4.7) for \( r = 1 \). Substituting \( \hat{F}_{\delta, j}^{(r, j)} \) from (4.2) and using assumption (3.1) we get for \( s \geq 1 \)

\[
\hat{F}_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)} = \frac{1}{s(\sigma + 1)} \frac{\partial F_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)}}{\partial y_{\rho}^{\sigma + 1}},
\]

(4.15)

so that also \( \hat{F}_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)} \) are equal to zero. Hence

\[
\hat{F}_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)} = \frac{1}{s(\sigma + 1)} \frac{\partial F_{\delta, j, \rho \cdots}^{(r, j, \rho \cdots)}}{\partial y_{\rho}^{\sigma + 1}},
\]

(4.16)

in view of (4.7), (4.14), and assumption (3.1).

It remains to prove that \( \alpha_E \) is projectable onto an open subset of \( J^YY \). To this end let us write

\[
E_\sigma = A_\sigma + B_\sigma^\rho \partial y_\rho,
\]

(4.18)
where we may assume $B_{pq}^{ik}$ symmetric in $j, k$. Then
\[
\alpha_E = \cdots + B_{pq}^{ik} y_p^r \omega^r \wedge \omega^s \wedge \cdots \wedge \omega^s \wedge \omega_{jk...j_k}
\]
where the dots indicate first order terms. We shall show that the last term above vanishes. It is easy
and have to satisfy conditions $p_2 \alpha_E = 0$, that is,
\[
\begin{align*}
\frac{\partial E_x}{\partial y^p} + \frac{\partial E_x}{\partial y^p} & - 2 \xi \frac{\partial E_x}{\partial y^{pq}} = 0, \\
\frac{\partial E_x}{\partial y^{pq}} & = 0,
\end{align*}
\]
and $p_3 \alpha_E = 0$, $k \geq 3$, i.e., conditions (4.3)–(4.6).
Conditions (4.23) are obviously equivalent with the variationality conditions (2.3–2.5); this means that they express the fact that \( E \) is locally variational.

Let us turn to relations (4.3–4.6). Let \( s = 0 \), (4.3) read

\[
\dot{F}^0_{r_1 \ldots r_m} = \frac{1}{r} \frac{\partial \dot{F}^0_{r_1 \ldots r_m}}{\partial y_{p_r}}
\]

Accounting symmetries of the \( \dot{F} \)'s, they split into two parts:

\[
\dot{F}^0_{r_1 \ldots r_m} = \frac{1}{r} \frac{\partial \dot{F}^0_{r_1 \ldots r_m}}{\partial y_{p_r}} = \frac{1}{|\text{sym}(j_1, \ldots, j_r)|} \biggr|_{\text{alt}(j_1, \ldots, j_r)}
\]

that are obviously satisfied with (4.22), and

\[
\frac{\partial \dot{F}^0_{r_1 \ldots r_m}}{\partial y_{p_r}} \biggr|_{\text{sym}(j_1, \ldots, j_r)} = 0, \quad \frac{\partial \dot{E}_q}{\partial y_{p_r}} \biggr|_{\text{alt}(x_1, \ldots, x_n)} = 0.
\]

It is easy to show that due to the variationality conditions, the latter relations are identities: indeed, differentiating the first of (4.23) with respect to \( y_{p_r} \) we obtain

\[
\frac{\partial^2 E_q}{\partial y_{p_r} \partial y_{p_j}} \biggr|_{\text{sym}(x_1), \text{sym}(x_2)} = 0, \quad (4.24)
\]

proving our assertion. Next, relations (4.4) for \( s = 0 \) give us the following conditions on components of \( \alpha E \):

\[
\dot{F}^0_{r_1 \ldots r_m} = \frac{1}{(r + 1)!} \frac{\partial \dot{F}^0_{r_1 \ldots r_m}}{\partial y_{p_r}} \biggr|_{\text{sym}(j_1, \ldots, j_r)} + r \frac{\dot{E}_q}{\partial y_{p_r}} \biggr|_{\text{alt}(x_1, \ldots, x_n)} = 0,
\]

where \( 2 \leq r \leq n \). Substituting from (4.22), the former conditions are apparently satisfied. The latter ones become

\[
\left( \frac{\partial \dot{E}_q}{\partial y_{p_1} \ldots \partial y_{p_r}} \biggr|_{\text{sym}(j_1, \ldots, j_r)} \right) + r \frac{\partial \dot{E}_q}{\partial y_{p_r}} - \frac{\partial \dot{E}_q}{\partial y_{p_r}} \biggr|_{\text{alt}(x_1, \ldots, x_n)} = 0, \quad (4.25)
\]

where we have used that, in view of (4.24), the \( \frac{\partial \dot{E}_q}{\partial y_{p_r}} \biggr|_{\text{sym}(j_1, \ldots, j_r)} \) are skew-symmetric in \( j \). We show that (4.25) are again identities as a consequence of the variationality conditions. Indeed, for \( r = 2 \) they read

\[
\left( \frac{\partial^2 E_q}{\partial y_{p_r} \partial y_{p_j}} \biggr|_{\text{sym}(j_1, j_2)} - 2 \frac{\partial^2 E_q}{\partial y_{p_r} \partial y_{p_j}} \biggr|_{\text{sym}(j_1, j_2)} - \frac{\partial^2 E_q}{\partial y_{p_r} \partial y_{p_j}} \biggr|_{\text{alt}(x_1, x_2)} \right) = 0, \quad (4.26)
\]
Proof of Theorem 5. However, this is nothing but the derivative of the first of the variationality conditions (4.23) by \( y'_p \). Relations (4.25) for \( r = 3, \ldots, n - 1 \) are then apparently obtained by consecutive differentiation of those for \( r = 2 \). It remains to check (4.5) and (4.6), which for \( s = 0 \) yield

\[
\frac{\partial F_{i}^{j_1 \ldots j_r p_k}}{\partial y'_{j_r}} \bigg|_{\text{alt}(r_i, \ldots, n_i, \text{alt}(l_j, r_j))} = 0, \quad \frac{\partial F_{i}^{j_1 \ldots j_r p_k}}{\partial y'_{j_r}} \bigg|_{\text{alt}(r_i, \ldots, n_i, \text{alt}(l_j, r_j))} = 0,
\]

and

\[
\frac{\partial F_{i}^{j_1 \ldots j_r p_k}}{\partial y'_{j_r}} = 0,
\]

respectively. Substituting for the \( F \)'s and taking into account that \( 1 \leq j_1, \ldots, j_n \leq n \), we can see immediately that the second of (4.27) are identities, and (4.28) are consequences of the variationality conditions, more precisely, of (4.24). Finally, we notice that the first set of relations in (4.27) arises in the same way as (4.25) by one more differentiation, hence these relations are identities due to the variationality conditions, as well.

To finish the proof we have to consider relations (4.3–4.6) for the case \( s = 1 \). In view of our assumption (3.1), (4.3) and (4.6) are satisfied trivially. (4.5) read

\[
\frac{\partial^{s+1} E_{p}^{\nu_{1} \ldots \nu_{s+1}}}{\partial y'^{\nu_{1}} \cdots \partial y'^{\nu_{s+1}} \partial y''_{(p)}} \bigg|_{\text{alt}(\nu_{1}, \ldots, \nu_{s+1} ; y''_{(p)})} = 0.
\]

This is an identity: indeed, in the sum, terms where \( p_2 = j_k \) are 0 due to (4.24), and terms where \( p_2 = j_l \neq j_k \) vanish due to skew-symmetry in \( \{(\nu_{1}p_1) / (\nu_{2}p_2)\} \) and in \( \{j_k, j_l\} \). Finally, (4.4) split to

\[
\frac{\partial F_{i}^{j_1 \ldots j_r p_k p_l}}{\partial y'_{j_r} \partial y'_{j_l} \partial y''_{(p)}} \bigg|_{\text{alt}(j_r, j_l, p_1, p_2 ; \text{alt}(\nu_{1}, \nu_{2} ; (p)))} = \frac{1}{(r + 1)!} \frac{\partial F_{i}^{j_1 \ldots j_r p_k p_l}}{\partial y'_{j_r} \partial y'_{j_l} \partial y''_{(p)}} \bigg|_{\text{alt}(j_r, j_l, p_1, p_2 ; \text{alt}(\nu_{1}, \nu_{2} ; (p)))},
\]

1 \( \leq r \leq n - 1 \), which are identities due to (4.24), and

\[
\frac{\partial F_{i}^{j_1 \ldots j_r p_k p_l}}{\partial y'_{j_r} \partial y'_{j_l} \partial y''_{(p)}} = 0,
\]

1 \( \leq r \leq n - 1 \); the left-hand sides, however, vanish identically, since the \( F \)'s are symmetric in \( j_r, p_1 \).

This completes the proof. □

Proof of Theorem 5. With help of Theorem 4, the proof is easy. (1) ⇒ (2) was proved above.

(2) ⇒ (3) follows from the Structure Theorem (formula (2.1)). (3) ⇒ (4) was shown in the proof of Theorem 4. Finally, (4) ⇒ (1) was proved in [15]: one has to show that if \( E_{\nu} \) satisfies (4.23) then

\[
L = y^\nu \int_0^1 E_{\nu}(x^l, y'^{\nu}, u'y'^{\nu}, u''y'^{\nu}) \, du
\]

is a local Lagrangian for \( E \). This is done by a direct computation, showing that the Euler–Lagrange expressions of \( L \) are equal to the given functions \( E_{\nu} \). □
Thus, components of $d\rho$ take the following form:

$$ dp_j = E_j + \sum_{k=1}^n \frac{1}{k!(k-1)!} \sum_{\omega_1, \ldots, \omega_k} \partial_l L_{\omega_1 \cdots \omega_k} \left( \frac{\partial^{k+1} L}{\partial y_j \partial y_{\omega_1} \cdots \partial y_{\omega_k}} \right)_{alt(\omega_1 \cdots \omega_k)} $$

This, components of $d\rho_j$ take the following form:

$$ F_{k, j}^{i_1 \cdots i_k} = \frac{1}{k!(k+1)!} \left( \frac{\partial^{k+2} L}{\prod_{\omega_1} \partial y_j} \right)_{alt(\omega_1 \cdots \omega_k)} $$

and

$$ F_{k, j}^{i_1 \cdots i_k \rho} = \frac{1}{k! \sum_{\omega_1} \partial y_j} \left( \frac{\partial^{k+2} L}{\prod_{\omega_1} \partial y_j} \right)_{alt(\omega_1 \cdots \omega_k, \omega_\rho)} $$

for $1 \leq k \leq n$.
\[ F^{\nu_1 \ldots \nu_n}_{\rho_1 \ldots \rho_m} = \frac{1}{n!} \frac{\partial^{n+1} L}{\partial y_{\nu_1} \ldots \partial y_{\nu_n} \partial \rho_1 \ldots \partial \rho_m} |_{\alpha(\nu_1 \ldots \nu_n) = 0} \] 

(4.34)

Hence, \( dp_{\alpha} = \alpha E \), as desired.

Note that in formula (4.32),

\[ \frac{d}{\partial y_{\nu_1} \ldots \partial y_{\nu_n} \partial \rho_1 \ldots \partial \rho_m} \Bigg|_{\alpha(\nu_1 \ldots \nu_n) = 0} = 0. \]  

(4.35)

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