BIRATIONAL ISOMORPHISMS BETWEEN TWISTED GROUP ACTIONS

ZINOVY REICHSTEIN AND ANGELO VISTOLI

Abstract. Let $X$ be an algebraic variety with a generically free action of a connected algebraic group $G$. Given an automorphism $\phi: G \to G$, we will denote by $X^\phi$ the same variety $X$ with the $G$-action given by $g \cdot x \mapsto \phi(g) \cdot x$.

V. L. Popov asked if $X$ and $X^\phi$ are always $G$-equivariantly birationally isomorphic. We construct examples to show that this is not the case in general. The problem of whether or not such examples can exist in the case where $X$ is a vector space with a generically free linear action, remains open. On the other hand, we prove that $X$ and $X^\phi$ are always stably birationally isomorphic, i.e., $X \times \mathbb{A}^m$ and $X^\phi \times \mathbb{A}^m$ are $G$-equivariantly birationally isomorphic for a suitable $m \geq 0$.

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1. Introduction

Throughout this note all algebraic varieties, algebraic groups, group actions and maps between them will be defined over a fixed base field $k$. By a $G$-variety $X$ we shall mean an algebraic variety with a (regular) action of a linear algebraic group $G$. A morphism (respectively, rational map, birational isomorphism, etc.) of $G$-varieties is a $G$-equivariant morphism (respectively, rational map, birational isomorphism).

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Given an automorphism $\phi$ of $G$, we can “twist” a group action $\alpha: G \times X \to X$ by $\phi$ to obtain a new $G$-action $\alpha^\phi$ on $X$ as follows:

$$\alpha^\phi: G \times X \xrightarrow{\phi \cdot \text{id}} G \times X \xrightarrow{\alpha} X.$$ 

Note that the new action has the same orbits as the old one. If $X$ is a $G$-variety (via $\alpha$) then we will denote the “twisted” $G$-variety (i.e., $X$ with the action given by $\alpha^\phi$) by $X^\phi$.

Now suppose that the action $\alpha$ of $G$ on $X$ is generically free, i.e., that there exists a $G$-invariant open dense subset $U$ of $X$ such that the stabilizer of every geometric point of $U$ is trivial. V. L. Popov asked if $X$ and $X^\phi$ are always birationally isomorphic as $G$-varieties. Of particular interest to him was the case where $X$ is a linear representation of $G$.

Katsylo’s conjecture is closely related to a conjecture of Katsylo [Ka], which says that generically free linear $G$-representations $V$ and $W$ are ($G$-equivariantly) birationally isomorphic if and only if $\dim(V) = \dim(W)$. Katsylo’s conjecture is known to be false for some finite groups $G$; in particular, there are counterexamples, where $W = V^\phi$ for an automorphism $\phi$ of $G$; see [RY, Section 7]. On the other hand, Katsylo’s conjecture remains open for many finite groups (e.g., for the symmetric groups $G = S_n$, $n \geq 5$) and for all connected semisimple groups.

Two simple observations are now in order. First note that only the class of $\phi$ in the group of outer automorphisms of $G$ matters here. Indeed, suppose $\phi' = \phi \circ \text{inn}_h$, where $\text{inn}_h: G \to G$ is conjugation by $h \in G$, i.e., $\phi(g) = hgh^{-1}$. Then $X^\phi$ and $X^{\phi'}$ are isomorphic via $x \mapsto hx$. In particular, $X$ and $X^\phi$ are always isomorphic if $G$ has no outer automorphisms, e.g., if $G$ is the full symmetric group $S_n$ ($n \neq 6$) or if $G$ is semisimple algebraic group whose Dynkin diagram has no non-trivial automorphisms; cf. [KMRT, Theorem 25.16]. The latter class of groups includes every (almost) simple algebraic group, other than those of type $A_n$, $D_n$ and $E_6$; cf. [KMRT, pp. 354 – 355].

Secondly, the $G$-invariant rational functions for $X$ and $X^\phi$ are exactly the same, i.e.,

$$k(X)^G = k(X^\phi)^G \subset k(X).$$

Recall that the inclusion $k(X) \subset k(X)^G$ induces a dominant rational map $\pi: X \dashrightarrow X/G$, which is called the rational quotient map. The $k$-variety $X/G$ is defined (up to birational isomorphism) by $k(X/G) = k(X)^G$. Thus [1] can be rephrased by saying that a rational quotient map $\pi$ for $X$ is also a rational quotient map for $X^\phi$. Note that by a theorem of Rosenlicht [Ro1, Ro2], $\pi^{-1}(x)$ is a single $G$-orbit for $x \in X/G$ in general position; cf. also [PV, Section 2.4].

The main results of this note are Theorems 1 and 2 below.

**Theorem 1.** Let $X$ be a generically free $G$-variety and $\phi: G \to G$ be an automorphism of $G$. Then the $G$-varieties $X$ and $X^\phi$ are stably birationally isomorphic. More precisely there exists an integer $m \geq 0$ and a birational...
isomorphism of
\[ f : X \times \mathbb{A}^m \rightarrow X^\phi \times \mathbb{A}^m \]
such that the diagram
\[
\begin{array}{ccc}
X \times \mathbb{A}^m & \xrightarrow{f} & X^\phi \times \mathbb{A}^m \\
\downarrow & & \downarrow \\
X/G & \xrightarrow{\phi} & X^\phi/G,
\end{array}
\]
commutes.

Here \( \mathbb{A}^m \) is the \( m \)-dimensional affine space with trivial \( G \)-action, and the vertical map on the left is the composition of the projection \( X \times \mathbb{A}^m \rightarrow X \) with the rational quotient map \( X \rightarrow X/G \) (similarly for the vertical map on the right).

**Theorem 2.** Let \( n \geq 3 \) and let \( \phi \) be the (outer) automorphism of \( G = \text{PGL}_n \) given by \( g \mapsto (g^{-1})^{\text{transpose}} \). Assume the base field \( k \) is infinite and contains a primitive \( n \)-th root of unity. Then there exists a generically free \( \text{PGL}_n \)-variety \( X \) such that \( X \) and \( X^\phi \) are not birationally isomorphic over \( k \).

The problem of whether or not such examples can exist in the case where \( X \) is a vector space with a generically free linear action of a connected linear algebraic group \( G \), remains open.

### 2. The no-name lemma

Recall that \( G \)-bundle \( \pi : E \rightarrow X \) is an algebraic vector bundle with a \( G \)-action on \( E \) and \( X \) such that \( \pi \) is \( G \)-equivariant and the action of every \( g \in G \) restricts to a linear map \( \pi^{-1}(x) \rightarrow \pi^{-1}(gx) \) for every \( x \in X \).

Our proof of Theorem 1 in the next section will heavily rely on the following result.

**Lemma 3** (No-name Lemma). Let \( \pi : E \rightarrow X \) be a \( G \)-bundle of rank \( r \). Assume that the \( G \)-action on \( X \) is generically free. Then there exists a birational isomorphism \( \pi : E \rightarrow X \times \mathbb{A}^r \) of \( G \)-varieties such that the following diagram commutes
\[
\begin{array}{ccc}
E & \xrightarrow{\phi} & X \times \mathbb{A}^r \\
\downarrow & \pi & \downarrow \\
X \end{array}
\]
Here \( G \) is assumed to act trivially on \( \mathbb{A}^r \), and \( \text{pr}_1 \) denotes the projection to the first factor.

The term “no-name lemma” is due to Dolgachev \[Do\]. In the case where \( G \) is a finite group, a proof can be found, e.g., in \[EM\] Proposition 1.1], \[L\] Proposition 1.3] or \[Sh\] Appendix 3]. For a proof in the case where the
base field $k$ is algebraically closed, $\text{char}(k) = 0$, and $G$ is an arbitrary linear algebraic group, see [BK], [Ka], [CGR, Section 4].

In the sequel we would like to use Lemma 3 in the case where $k$ is not necessarily algebraically closed. With this in mind, we will prove a more general variant of this result (Proposition 5 below). For the rest of this section we will work over an arbitrary base field $k$.

**Remark 4.** Suppose $G$ is a group scheme of finite type over $k$, $X$ is an arbitrary quasi-separated scheme (or algebraic space) over $k$, on which $G$ acts (quasi-separated means that the diagonal embedding $X \hookrightarrow X \times \text{Spec} \ k X$ is quasi-compact; this is automatically satisfied when $X$ is of finite type over $k$). We say that the action is *free* when the stabilizers of all geometric point of $X$ are trivial (as group schemes). This is equivalent to saying that the morphism $G \times \text{Spec} \ k X \rightarrow X \times \text{Spec} \ k X$ defined in functorial terms by $(g, x) \mapsto (gx, x)$ is categorically injective (equivalently, it is injective on geometric points and unramified). Then, by a result of Artin ([LMR, Corollaire 10.4]), the quotient sheaf $X/G$ in the fppf topology is an algebraic space, and $G \times \text{Spec} \ k X = X \times_{X/G} X$. There is a Zariski open dense subspace $V \subseteq X/G$ that is a scheme ([Kn, Proposition 6.7]); if $U$ is the inverse image of $V$ in $X$, then the restriction $U \rightarrow U/G = V$ is a $G$-torsor (i.e. a principal $G$-bundle in the fppf topology, cf. [DG]). In the case where $X$ is a $k$-variety, this is precisely the rational quotient map we discussed in the introduction, i.e., $k(U/G) = k(X)^G$.

**Proposition 5.** Assume that $G$ is a group scheme of finite type over $k$, acting on a quasi-separated $k$-scheme $X$, with a non-empty invariant open subscheme on which the action is free. Let $\mathcal{E}$ be a $G$-equivariant locally free sheaf of rank $r$ on $X$. Then there exists a non-empty open $G$-invariant subscheme $U$ of $X$, such that the restriction $\mathcal{E}|_U$ is isomorphic to the trivial $G$-equivariant sheaf $\mathcal{O}_U^r$.

To see that Lemma 3 (over an arbitrary base field $k$) follows from Proposition 5, recall the the well-known equivalence between the category of $G$-equivariant vector bundles on $X$ and the category of $G$-equivariant locally free sheaves on $X$. One passes from a $G$-bundle $V \rightarrow X$ to the $G$-equivariant locally free sheaf of sections of $V$; conversely, to each $G$-equivariant locally free sheaf $\mathcal{E}$ on $X$ one associates the spectrum of the sheaf of symmetric algebras of the dual $\mathcal{E}^\vee$ over $X$.

Note also that in the course of proving Lemma 3 we may assume without loss of generality that $X$ is *primitive*, i.e., $G$ transitively permutes the irreducible components of $X$ (equivalently, $k(X)^G$ is a field). Indeed, an arbitrary $G$-variety $X$ is easily seen to be birationally isomorphic to a disjoint union of primitive $G$-varieties $X_1, \ldots, X_r$, and it suffices to prove Lemma 3 for each $X_i$. On the other hand, if $X$ is primitive, then every non-empty $G$-invariant open subset is dense in $X$. This shows that Lemma 3 follows from Proposition 5 as claimed.
Proof of Proposition 5. After replacing $X$ by a non-empty open subscheme we may assume that the action of $G$ on $X$ is free. By passing to a dense invariant subscheme of $X$, we may assume that the action of $G$ on $X$ is free. By passing to a dense invariant subscheme of $X$, we may assume that $X/G$ is a scheme, and $X \to X/G$ is a $G$-torsor; see Remark 4. By descent theory, the $G$-equivariant sheaf $\mathcal{E}$ comes from a locally free sheaf $\mathcal{F}$ on $X/G$; see, for example, [V, Theorem 4.46]. By restricting to a non-empty open subscheme of $X/G$ once again, we may assume that $\mathcal{F}$ is isomorphic to $\mathcal{O}_{X/G}$. Then $\mathcal{E}$ is $G$-equivariantly isomorphic to $\mathcal{O}_{X/G}$, as claimed. □

Remark 6. The same argument goes through if the base field $k$ (or, equivalently, the base scheme $\text{Spec}(k)$) is replaced by an algebraic space $B$, so that $X$ is defined over $B$, and the group scheme $G$ is assumed to be flat and finitely presented over $B$.

3. Proof of Theorem 1

We will prove Theorem 1 in two steps: first in the case where $X = V$ is a generically free linear representation of $G$, then for arbitrary $X$.

Step 1: Suppose $X = V$ is a generically free linear representation of $G$.

Let $m = \dim(V)$. By the no-name lemma, there exist $G$-equivariant birational isomorphisms $\alpha$ and $\beta$ such that the diagram

\[
\begin{array}{ccc}
W \times \mathbb{A}^m & \overset{\alpha}{\longrightarrow} & W \times W^\phi \\
\downarrow \text{pr}_1 & & \downarrow \text{pr}_1 \\
W & \longrightarrow & W \\
\end{array}
\quad
\begin{array}{ccc}
W \times W^\phi & \overset{\beta}{\longrightarrow} & W^\phi \times \mathbb{A}^m \\
\downarrow \text{pr}_2 & & \downarrow \text{pr}_1 \\
W^\phi & \longrightarrow & W^\phi \\
\end{array}
\]

commutes. Now we can take $f = \beta \circ \alpha: V \times \mathbb{A}^m \dashrightarrow V^\phi \times \mathbb{A}^m$.

Step 2: Suppose $X$ is an arbitrary generically free $G$-variety.

Let $V$ be a generically free linear representation of $G$ and $p: X \times V \dashrightarrow V$ be the projection onto the second factor. By the no-name lemma, $X \times V$ is birationally isomorphic to $X \times \mathbb{A}^m$; this yields a dominant rational map of $G$-varieties $X \times \mathbb{A}^m \dashrightarrow V$, which we will continue to denote by $p$. After replacing $X$ by $X \times \mathbb{A}^m$, we may assume that there exists a dominant rational map $p: X \dashrightarrow V$. We now consider the commutative diagram

\[
\begin{array}{ccc}
X & \overset{p}{\longrightarrow} & V \\
\downarrow & & \downarrow \\
X/G & \overset{p/G}{\dashrightarrow} & V/G, \\
\end{array}
\]

where the vertical arrows are rational quotient maps. We claim that $X$ is birationally isomorphic to the fiber product $X/G \times_{V/G} V$, where the $G$-action
on this fiber product is induced from the $G$-action on $V$ (in other words, $G$ acts trivially on $X/G$ and on $V/G$). In the case where $k$ is an algebraically closed field of characteristic zero, this is proved in [Re, Lemma 2.16]. For general $k$, choose an open invariant subscheme $U$ of $V$ such that $G$ acts freely over $U$, the quotient $U/G$ is a scheme, and the projection $U \to U/G$ is a $G$-torsor; see Remark 4. By restricting $X$, we may assume that the morphism $X \to X/G$ is also a $G$-torsor, and that $X$ maps into $U$. Then we get a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p} & U \\
\downarrow & & \downarrow \\
X/G & \xrightarrow{p/G} & U/G \\
\end{array}
\]

where the columns are $G$-torsors and the top row is $G$-equivariant. Any such diagram is well known to be cartesian; this proves our claim.

Similarly, $X^\phi \simeq X^\phi/G \times_{V^\phi/G} V^\phi$. By Step 1 there is a $G$-equivariant birational isomorphism $f: V \times \mathbb{A}^m \dashrightarrow V^\phi \times \mathbb{A}^m$ which makes the diagram

\[
\begin{array}{ccc}
V \times \mathbb{A}^m & \xrightarrow{f} & V^\phi \times \mathbb{A}^m \\
\downarrow \pi_V & & \downarrow \pi_{V^\phi} \\
X/G & \xrightarrow{p/G} & V^\phi/G \\
\end{array}
\]

commute. Consequently, $f$ induces a $G$-equivariant birational isomorphism between the fiber products $X/G \times_{V/G} (V \times \mathbb{A}^m)$ and $X^\phi/G \times_{V^\phi/G} (V^\phi \times \mathbb{A}^m)$ i.e., between $X \times \mathbb{A}^m$ and $X^\phi \times \mathbb{A}^m$. This completes the proof of Theorem 1.

Remark 7. Our proof shows that the integer $m$ in the statement of Theorem 1 can be taken to be the minimal value of $2 \dim(W)$, as $W$ ranges over the generically free linear representations of $G$. If $X$ is itself a generically free linear representation of $G$ then we can take $m$ to be the minimal value of $\dim(W)$, where again $W$ ranges over the generically free linear representations of $G$; see the proof of Step 1.

4. Examples where $X$ and $X^\phi$ are birationally isomorphic

The question of whether or not $X$ and $X^\phi$ are birationally isomorphic over $k$ is delicate in general. Birational isomorphism over $K = k(X)^G$ is more accessible because it can be restated in terms of Galois cohomology. In this section we will to show that in many cases $X$ and $X^\phi$ are, indeed, birationally isomorphic over $K$ (and thus over $k$).

Let $X$ be a primitive $G$-variety. Recall that $X$ is called primitive if $G$ transitively permutes the irreducible components of $X$; see Section 2. That
is, $X$ is primitive if $K = k(X)^G$ is a field or equivalently, if the rational quotient variety $X/G$ is irreducible. As we saw in Remark 4, the rational quotient map $\pi : X \dashrightarrow X/G$ is a $G$-torsor over a non-empty open subscheme of $X/G$; hence, the $G$-action on $X$ gives rise to a Galois cohomology class in $H^1(K, G)$, which we shall denote by $[X]$; cf., [Se1], [Po].

An automorphism $\phi$ of $G$ induces an automorphism $\phi^*$ of the (pointed) cohomology set $H^1(K, G)$, where $[X^\phi] = \phi^*([X])$. In particular, the $G$-varieties $X$ and $X^\phi$ are birationally isomorphic over $K$ if and only if $\phi^*([X]) = [X]$ in $H^1(K, G)$.

**Example 8.** If $[X] = 1$ then $\phi^*([X]) = 1$; hence, $X$ and $X^\phi$ are birationally isomorphic. Explicitly, in this case $X$ is birationally isomorphic to the “split” $G$-variety $Y \times G$, with $G$ acting on the second component by left translations, and a birational isomorphism between $Y \times G$ and $(Y \times G)^\phi$ is given by $(y, g) \mapsto (y, \phi(g))$.

In particular, if $G$ is a special group, i.e., $H^1(K, G) = \{1\}$ for every $K/k$, then $X$ and $X^\phi$ are birationally isomorphic for every generically free $G$-variety $X$. Examples of special groups are $G = \text{GL}_n, \text{SL}_n, \text{Sp}_{2n}$; see [Se2], Chapter X.

The following lemma extends this simple argument a bit further.

**Lemma 9.** Let $X$ be a primitive generically free $G$-variety and $K = k(X)^G$. Suppose $[X]$ lies in the image of the natural map $H^1(K, G_0) \rightarrow H^1(K, G)$, where $G_0$ is a closed subgroup of $G$ such that $\phi|_{G_0} = \text{id}: G_0 \rightarrow G_0$. Then $X$ and $X^\phi$ are birationally isomorphic as $K$-varieties.

**Proof.** Let $i : G_0 \hookrightarrow G$ be the inclusion map. The commutative diagram

$$
\begin{array}{ccc}
G_0 & \rightarrow & G \\
\downarrow{id} & & \downarrow{\phi} \\
G_0 & \rightarrow & G
\end{array}
$$

of groups induces a commutative diagram of cohomology sets

$$
\begin{array}{ccc}
H^1(K, G_0) & \rightarrow & H^1(K, G) \\
\downarrow{id} & & \downarrow{\phi^*} \\
H^1(K, G_0) & \rightarrow & H^1(K, G)
\end{array}
$$

Since $[X]$ is in the image of $i^*$, this diagram shows that $\phi^*([X]) = [X]$. □

Note that if $[X] = 1$ then $[X]$ is the image of the trivial element of $H^1(K, G_0)$, where $G_0 = \{1\}$. Example 8 is thus a special case of Lemma 9.

We now turn to a more sophisticated application of Lemma 9 (with non-trivial $G_0$).
Proposition 10. Let $G$ be the special orthogonal group $SO(q)$, where $q$ is a non-degenerate isotropic $n$-dimensional quadratic form defined over $k$, and let $X$ be an irreducible generically free $G$-variety. Assume $n \neq 4$ and $\text{char}(k) \neq 2$. Then $X$ and $X^\phi$ are birationally isomorphic over $K = k(X)^{SO(q)}$ (and hence, over $k$) for any automorphism $\phi$ of $SO(q)$.

Proof. As we remarked in the introduction, we are allowed to replace $\phi$ by $\phi' = \phi \circ \text{inn}_h$, where $\text{inn}_h$ denotes conjugation by $h \in SO(q)$; indeed, $X^\phi$ and $X^{\phi'}$ are isomorphic (over $K$) via $x \mapsto h \cdot x$. In particular, the proposition is true if $\phi$ is an inner automorphism of $SO(q)$.

It is well known that every automorphism $\phi: SO(q) \to SO(q)$ has the form $g \mapsto hgh^{-1}$, where $h \in O(q)$; cf., e.g., [Die] [Section XI]. If $n$ is odd then, after replacing $h$ by $\det(h)h$, we may assume that $h \in SO(q)$, i.e., $\phi$ is inner. This proves the proposition for odd $n$.

From now on we will assume that $n \geq 6$ is even and $\det(h) = -1$. We may also assume that $q(x_1, \ldots, x_n) = a_1x_1^2 + \cdots + a_nx_n^2$ for some $a_1, \ldots, a_n \in k^*$ and, after composing $\phi$ with an inner automorphism of $SO(q)$, that $h = \text{diag}(-1, 1, \ldots, q)$. Let $D_0 \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$, $D \simeq (\mathbb{Z}/2\mathbb{Z})^n$ be the subgroups of diagonal matrices in $SO(q)$, $O(q)$ respectively, and $i: D_0 \hookrightarrow SO(q)$, $j: D \hookrightarrow SO(q)$ be the natural inclusion maps. Note that $\phi$ restricts to a trivial automorphism of $D_0$. Thus, in view of Lemma 9, it suffices to show that $i_*: H^1(K, D_0) \to H^1(K, SO(q))$ is surjective.

Consider the commutative diagram

$$
\begin{array}{ccc}
1 & \to & D_0 \\
\downarrow i & & \downarrow j \\
SO(q) & \to & O(q)
\end{array}
$$

of algebraic groups and the induced commutative diagram

$$
\begin{array}{ccc}
1 & \to & H^1(K, D_0) \\
\downarrow i_* & & \downarrow j_* \\
H^1(K, SO(q)) & \to & H^1(K, O(q))
\end{array}
$$

in cohomology. The top row is an exact sequence of abelian groups; $H^1(K, D_0)$ can thus be identified with the kernel of the product map $p: (K^*/(K^*)^2)^n \to K^*/(K^*)^2$, where $p(b_1, \ldots, b_n) = b_1 \cdots b_n \pmod{(K^*)^2}$.

Now recall that $H^1(K, O(q))$ is in a natural 1-1 correspondence with isometry classes of $n$-dimensional quadratic forms $q'$ and that $j_*$ takes $(b_1, \ldots, b_n) \in (K^*/K^*)^n$ to the quadratic form $q' = a_1b_1x_1^2 + \cdots + a_nb_nx_n^2$. Similarly, $H^1(K, SO(q))$ is in a natural 1-1 correspondence with isometry classes of $n$-dimensional quadratic forms $q'$ such that $q'$ has the same discriminant as $q$, and $i_*$ takes $(b_1, \ldots, b_n) \in (K^*/K^*)^n$, with $b_1 \cdots b_n = 1$ in $K^*/K^*$, to
\[ q' = a_1 b_1 x_1^2 + \cdots + a_n b_n x_n^2; \text{ cf., e.g., } \text{Se} \text{I, Appendix 2, §2}. \] It is clear from this description that both \( i_\ast \) and \( j_\ast \) are surjective. \( \square \)

5. Proof of Theorem 2

Recall that elements of \( H^1(K, \text{PGL}_n) \) are in a natural 1-1 correspondence with

(i) generically free \( \text{PGL}_n \)-varieties \( X \), with \( k(X)^{\text{PGL}_n} = K \), up to birational isomorphism over \( K \), or alternatively, with

(ii) central simple algebras \( A/K \) of degree \( n \), up to \( K \)-isomorphism; see \[ \text{Se} \text{I}, \text{Po}, \text{RV} \]. We will denote the central simple algebra corresponding to an irreducible generically free \( \text{PGL}_n \)-variety \( X \) (respectively, to an element \( \alpha \in H^1(K, \text{PGL}_n) \)) by \( A_X \) (respectively, by \( A_\alpha \)). If \( \phi: \text{PGL}_n \rightarrow \text{PGL}_n \) is the automorphism given by \( g \rightarrow (g^{-1})^{\text{transpose}} \) then \( A_{\phi_\ast(\alpha)} \) is the opposite algebra \( A_\alpha^{\text{op}} \); cf. e.g., \[ \text{Se} \text{II}, \text{pp. 152-153} \]. In other words, \( A_X^\phi = A_X^{\text{op}} \). The following lemma gives a necessary and sufficient conditions for \( X \) and \( X^\phi \) to be birationally isomorphic over \( K \).

**Lemma 11.** Let \( X \) be an irreducible generically free \( \text{PGL}_n \)-variety. Then the following conditions are equivalent.

(a) \( X \) and \( X^\phi \) are birationally isomorphic over \( K = k(X)^{\text{PGL}_n} \),

(b) \( A_X \) is \( K \)-isomorphic to \( A_X^{\text{op}} \),

(c) \( A_X \) has exponent 1 or 2 in the Brauer group \( \text{Br}(K) \).

**Proof.** (a) \( \Leftrightarrow \) (b): Let \( \alpha = [X] \in H^1(K, \text{PGL}_n) \). Then as we observed in Section 4 \[ [X^\phi] = \phi_\ast(\alpha) \]. Thus \( A_{X^\phi} = A_{\phi_\ast(\alpha)} = A_\alpha^{\text{op}} = A_X^{\text{op}} \), so that \( X \) and \( X^\phi \) are birationally isomorphic over \( K \) if and only if \( A_X \) is \( K \)-isomorphic to \( A_X^{\text{op}} \).

The equivalence of (b) and (c) is obvious, since \( A^{\text{op}} \) is the inverse of \( A \) in \( \text{Br}(K) \). \( \square \)

Lemma 11 does not directly address the question we are interested in, namely, the question of whether \( X \) and \( X^\phi \) are birationally isomorphic over the base field \( k \). Note however, that a birational isomorphism \( \alpha: X \rightarrow X^\phi \) defined over \( k \), restricts to a \( k \)-automorphism of the field of invariants \( K = k(X)^{\text{PGL}_n} = k(X^\phi)^{\text{PGL}_n} \). Our proof of Theorem 2 is based on the observation that if \( \text{Aut}_k(K) = \{1\} \), then

\( X \) and \( X^\phi \) are isomorphic over \( k \) \( \iff \)

\( X \) and \( X^\phi \) are isomorphic over \( K \) \( \iff \)

\( A_X \) has exponent 1 or 2 in the Brauer group of \( K \).

Thus in order to prove Theorem 2 it suffices to construct (i) a finitely generated field extension \( K/k \) such that \( \text{Aut}_k(K) = \{1\} \) and (ii) a central simple algebra \( A/K \) of degree \( n \) and exponent \( n \). These constructions are carried out in Lemmas 12 and 13 below. To simplify the exposition, we will state Lemmas 12 and 13 for an algebraically closed ground field \( k \). We
will then explain how to modify our construction to make it work over any infinite field $k$ containing a primitive $n$th root of unity.

**Lemma 12.** Suppose $k$ is an algebraically closed field. Then there exists an algebraic surface $S/k$ which admits no non-trivial birational automorphisms. In other words, $\text{Aut}_k k(S) = \{1\}$.

**Proof.** Choose two smooth non-isomorphic curves $C_1$ and $C_2$, of genus $g_1$ and $g_2$ respectively (say, $g_1 \geq g_2$), with no non-trivial automorphisms, and set $S = C_1 \times C_2$.

We claim that every birational automorphism $f: S \to S$ is trivial. To prove this, note that $f$ restricts to a regular map $C \to C$ for every smooth curve $C$ in $S$. Taking $C = C_1 \times \{y\}$ for some $y \in C_2$, we see that $pr_2 \circ f$ is a regular map $C \to C_2$. (Here $pr_2: S \to C_2$ is the projection to the second factor.) Since $g_1 \geq g_2$, the Hurwitz formula tells us that this map cannot be dominant, i.e., it sends $C$ to a single point. In other words, $f(C_1 \times \{y\}) \subset C_1 \times \{y'\}$ for some $y' \in C_2$. Since $f$ is a birational automorphism, $f(C_1 \times \{y\})$ can be a single point for only finitely many $y \in C_2$. For every other $y \in C_2$ there exists a $y' \in C_2$ such that

$$f(C_1 \times \{y\}) = C_1 \times \{y'\}.$$ 

Applying the Hurwitz formula once again, we see that $f$ induces an isomorphism between $C_1 \times \{y\}$ and $C_1 \times \{y'\}$. Since $C_1$ has no non-trivial automorphisms, this isomorphism is given by $f(x, y) = (x, y')$. Equivalently, for $x \in C_1$, $f$ restricts to a morphism $\{x\} \times C_2 \to \{x\} \times C_2$. The Hurwitz formula now tells us that this map is an automorphism. Since $C_2$ has no non-trivial automorphisms, we conclude that $f(x, y) = (x, y)$, for every $x, y \in S$. \hfill $\square$

**Lemma 13.** Assume $k$ is an algebraically closed field, $\text{char}(k)$ does not divide $n$ and $K/k$ be a finitely generated field extension of transcendence degree $\geq 2$. Then for every $n \geq 3$ there exists a division algebra $D/K$ of degree $n$ and exponent $n$.

**Proof.** Consider a model $X$ for $K$, i.e., an algebraic variety $X$ with function field $k(X) = K$. Choose a smooth point $x \in X$ and a system of local parameters $t_1, \ldots, t_d$ in the local ring $O_x(X)$; here $d = \dim(X) = \text{trdeg}_K(K) \geq 2$.

We claim that the symbol algebra $D = (t_1, t_2)_n$, i.e., the $K$-algebra given by generators $x_1, x_2$ and relations $x_1^n = t_1$, $x_2^n = t_2$, $x_1 x_2 = \zeta x_2 x_1$ has exponent $n$ in $\text{Br}(K)$. (Here $\zeta_n$ is a primitive $n$th root of unity in $k$.)

To prove this, consider the completion $\hat{O}_x(X) = k[[t_1, \ldots, t_d]]$ of the local ring $O_x(X)$, where $k[[t_1, \ldots, t_d]]$ denotes the ring of formal power series in the variables $t_1, \ldots, t_d$. Note that $O_x(X) \subset \hat{O}_x(X)$ and thus, after passing to the fields of fractions, $K \subset k((t_1, \ldots, t_d))$. The image of $D$ under the restriction map $\text{Br}(K) \to \text{Br}(k((t_1, \ldots, t_d)))$ is the symbol algebra $D' = (t_1, t_2)_n$ over $k((t_1, \ldots, t_d))$. A simple valuation-theoretic argument shows that $D'$ has exponent $n$; cf. [Ro, Proposition 3.3.26]. Hence, so does $D$. \hfill $\square$
Lemmas 12 and 13 complete the proof of Theorem 2 in the case where the base field $k$ is algebraically closed and $\text{char}(k)$ does not divide $n$. Then the same argument will work over a non-closed field $k$ with a primitive $n$th root of unity, if we can choose the curves $C_i$ ($i = 1, 2$) in Lemma 12 so that $C_i$ each is defined over $k$ and has a $k$-point $p_i$. (The primitive $n$th root of unity is needed to define the symbol algebra $D$ in the proof of Lemma 13.) Taking the $k$-variety $X$ in Lemma 13 to be $C_1 \times C_2$, we see that $x = (p_1, p_2)$ is a smooth $k$-point of $X$, and the rest of the proof of Lemma 13 goes through unchanged.

To construct the curve $C_i$ as above (for $i = 1, 2$), fix a $k$-point $p_i$ in $\mathbb{P}^2$ and $C_i$ to be the general element of the $k$-linear system of degree $d_i$ curves passing through $p_i$. If $k$ is an infinite field then $C_i$ is a smooth curve of genus $G_i = \frac{1}{2}(d_i - 1)(d_i - 2)$. By construction, $C$ has a $k$-point $p_i$. Moreover, if $d_i \geq 4$, then $C_i$ has no non-trivial automorphisms.

This completes the proof of Theorem 2.

Remark 14. Theorem 2 remains valid if the condition that $k$ contains a primitive $n$th root of unity is replaced by the (weaker) condition that $k$ contains a primitive $m$th root of unity for some divisor $m$ of $n$ such that $m \geq 3$. The proof is the same, except that instead of the symbol algebra $D = (t_1, t_2)$ of degree $n$ and exponent $n$, we use the algebra $M_{n/m}(E)$ of degree $n$ and exponent $m$, where $E = (t_1, t_2)$.

Remark 15. Theorem 2 fails for $n = 2$; indeed, $A$ and $A^{\text{op}}$ are isomorphic over $K$ for any central simple algebra $A/K$ of degree 2. Alternatively, $\text{PGL}_2 \simeq \text{SO}_3$, so if Theorem 2 were true for $n = 2$, it would contradict Proposition 10.

6. Further examples

In this section we will assume that $G$ is a finite group and $k$ is an algebraically closed field of characteristic zero.

Proposition 16. (a) For every finitely generated field extension $K/k$ and every finite group $G$ there exists a $G$-Galois extension $L/K$. Equivalently, there exists an irreducible $G$-variety $X$ such that $k(X)^G = K$.

(b) Suppose $\text{Aut}_k(K) = \{1\}$ and $\phi: G \to G$ is an outer automorphism of a finite group $G$. Then for every $X$, as in (a), the $G$-varieties $X$ and $X^\phi$ are not birationally isomorphic.

Proof. (a) By the Riemann existence theorem there exists a $G$-Galois extension $L_0/k(t)$, where $t$ is an independent variable. Hence, there exists a $G$-Galois extension $L_1/K(t)$, where $L_1 = L \otimes_{k(t)} K(t)$. The Hilbert irreducibility theorem now allows to construct a $G$-Galois extension $L/K$ by suitably specializing $t$ in $K$.

(b) Irreducible $G$-varieties $X$ (up to birational isomorphism) such that $k(X)^G = K$, are in 1-1 correspondence with $G$-Galois field extensions $L/K$. 
A birational isomorphism $\alpha: X \dasharrow X^\phi$ of $G$-varieties induces an isomorphism

$$
L = k(X) \xrightarrow{\alpha} k(X^\phi) = L
$$

Then $\alpha \in \text{Gal}(L/K) = G$, and since the above diagram commutes, we have $\alpha g(l) = \phi(g)\alpha(l)$ for every $g \in G$ and $l \in L$. In other words, $\phi(g) = \alpha g \alpha^{-1}$, contradicting our assumption that $\phi$ is an outer automorphism. \qed

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