IRREDUCIBILITY AND SMOOTHNESS OF THE MODULI SPACE OF MATHEMATICAL 5–INSTANTONS OVER \( \mathbb{P}_3 \)

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Abstract. We prove that the space of mathematical instantons with second Chern class 5 over \( \mathbb{P}_3 \) is smooth and irreducible. Unified and simple proofs for the same statements in case of second Chern class \( \leq 4 \) are contained.

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Introduction

A mathematical instanton bundle is an algebraic vector bundle \( \mathcal{E} \) over \( \mathbb{P}_3(k) \), \( k \) an algebraically closed field of characteristic 0, if it has rank 2, Chern classes \( c_1 = 0, c_2 = n > 0 \), and if it satisfies the vanishing conditions \( H^0\mathcal{E} = 0 \) and \( H^1\mathcal{E}(-2) = 0 \). The name was chosen in twistor theory in the 1970’s, when holomorphic bundles on \( \mathbb{P}_3(\mathbb{C}) \) with the

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same properties were considered as counterparts of (anti-)self-dual Yang–Mills fields on the 4–sphere, see [1], [27], [12] for reference. Let a mathematical instanton bundle with \( c_2 = n \) be called \( n \)--instanton or \((n,2)\)--instanton for short. The isomorphism classes of \( n \)--instantons are the closed points of a coarse moduli scheme \( MI(n) \). Since its first consideration, it is an open problem whether \( MI(n) \) is smooth and irreducible for any \( n \). An affirmative answer for \( n \leq 4 \) has been given in several papers, for each \( n \) separately, and recently Katsylo and Ottaviani proved smoothness for \( n = 5 \), see historical remarks in [13].

The main result of this paper is that also \( MI(5) \) is irreducible, with a new proof of smoothness included. The method used also enables a simple and unified proof for all the previous cases \( n \leq 4 \), see Section 5.

It is well–known that any \( n \)--instanton is the cohomology of a short complex \( H_n \otimes \mathcal{O}(-1) \to N \otimes \mathcal{O} \to H_n^* \otimes \mathcal{O}(1) \), where \( H_n \) and \( N \) are \( k \)--vector spaces of dimensions \( n \) and \( 2n + 2 \), respectively, also called the Horrocks construction. We consider also higher rank instanton bundles, called \((n,r)\)--instantons, which can be constructed from the same type of complexes with the same \( H_n \) but with \( \dim N = 2n + r \), \( 2 \leq r \leq 2n \). We prove that, given an \((n,r)\)--instanton \( \mathcal{E} \), one can choose a linear form \( \xi \in H_n^* \) such that with \( H_{n-1} \cong \ker(\xi) \) the induced complex \( H_{n-1} \otimes \mathcal{O}(-1) \to N \otimes \mathcal{O} \to H_{n-1}^* \otimes \mathcal{O}(1) \) defines an \((n-1,r+2)\)--instanton \( \mathcal{E}_\xi \), together with a complex \( \mathcal{O}(-1) \to \mathcal{E}_\xi \to \mathcal{O}(1) \), whose cohomology is the original \( \mathcal{E} \), see Section 3. Together with technical details, this observation enables us to perform induction steps \((n-1,r+2) \sim (n,r)\) for \( n \leq 5 \) for irreducibility and smoothness. These induction steps are short for \( n \leq 4 \), see Section 5, while the case \((4,4) \sim (5,2)\) is more elaborate. In Section 3 the induction step is explained in more details. The proof of smoothness in each case is achieved by proving that \( H^2S^2\mathcal{E}_\xi = 0 \) implies \( H^2S^2\mathcal{E} = 0 \) for suitable \( \xi \).

**Notations**

- Throughout the paper \( k \) will be an algebraically closed field of characteristic zero
- If \( E \) is a finite dimensional \( k \)--vector space, \( \mathbb{P}E \) will denote the projective space of 1–dimensional and \( G_mE \) the Grassmannian of \( m \)--dimensional subspaces.
- The invertible sheaf of degree \( d \) over \( \mathbb{P}E \) is denoted \( \mathcal{O}_{\mathbb{P}E}(d) \) such that \( E^* = H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(1)) \).
- \( \mathbb{P} = \mathbb{P}V \cong \mathbb{P}_3 \) will denote the projective 3–space over \( k \) for a fixed vector space \( V \) of dimension 4. We will omit the index \( \mathbb{P} \) at the structure sheaf \( \mathcal{O} = \mathcal{O}_\mathbb{P} \) and at the invertible sheaves \( \mathcal{O}(d) = \mathcal{O}_\mathbb{P}(d) \).
- For an \( \mathcal{O}_\mathbb{P} \)--module \( \mathcal{F} \) we use the abbreviations \( \mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_\mathbb{P}(d) \), \( H^i\mathcal{F} = H^i(\mathbb{P}, \mathcal{F}) \) and \( h^i\mathcal{F}(d) = \dim_k H^i\mathcal{F}(d) \), and \( \text{ext}^i(\mathcal{F}, \mathcal{G}) = \dim_k \text{Ext}^i(\mathcal{F}, \mathcal{G}) \).
- If \( E \) is a finite dimensional \( k \)--vector space, the sheaf of sections of the trivial vector bundle over a scheme \( X \) will be denoted \( E \otimes \mathcal{O}_X = E \otimes_k \mathcal{O}_X \) and \( E \otimes \mathcal{F} \) is written for \( (E \otimes \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \).
If \( F \) respectively \( G \) are coherent sheaves on schemes \( X \) resp. \( Y \), \( F \boxtimes G \) denotes the sheaf \( p^*F \otimes q^*G \) on \( X \times_k Y \) where \( p \) respectively \( q \) are the projections.

If \( F \) is a coherent sheaf on the scheme \( X \) and \( Y \) a closed subscheme of \( X \) we simply write \( F_Y \) for the restriction on \( F \otimes_{O_X} O_Y = F|Y \).

Throughout the paper a vector bundle is a locally free sheaf of finite rank. If necessary, we will refer specifically to its bundle space as a fibration of vector spaces.

The Chern classes \( c_i(F) \) of a coherent sheaf \( F \) over \( \mathbb{P}^3 \) are considered as integers and we also call the triple \( (c_1(F), c_2(F), c_3(F)) \) the Chern class of \( F \).

1. Instanton bundles

1.1. Let \( M(2; 0, n, 0) \) denote the moduli space of semistable coherent sheaves on \( \mathbb{P}^3 \) of rank 2 with Chern class \( (c_1, c_2, c_3) = (0, n, 0) \), which had been constructed by M. Maruyama, [17], [18]. It contains the open set \( M^b(2; 0, n) \) of stable rank 2 vector bundles on \( \mathbb{P}^3 \) with Chern class \( (c_1, c_2) = (0, n) \). Recall that a rank 2 vector bundle \( E \) on \( \mathbb{P}^3 \) with Chern class \( c_1 = 0 \) is stable if and only if \( H^0E = 0 \). Then it is also simple, i.e. \( \text{hom}(E, E) = 1 \), see [21]. In this case the Riemann-Roch formula becomes

\[
\text{ext}^1(E, E) - \text{ext}^2(E, E) = 8n - 3,
\]

whereas \( \text{Ext}^1(E, E) \) is isomorphic to the tangent space of \( M^b(2; 0, n) \) at \([E]\), the isomorphism class of \( E \). For large \( n \) the spaces \( M^b(2; 0, n) \) have many irreducible components and some of them have a much bigger dimension than \( 8n - 3 \).

1.2. Definition: A symplectic mathematical instanton bundle with second Chern class \( n \geq 0 \) and of rank \( r \), or an \((n, r)\)-instanton for short, is a locally free sheaf \( E \) over \( \mathbb{P} = \mathbb{P}V \cong \mathbb{P}^3 \) with the following properties

(i) \( E \) has Chern class \((0, n, 0)\) and \( 2 \leq r = rk(E) \leq 2n \)
(ii) \( H^0E = 0 \) and \( H^1E(-2) = 0 \)
(iii) \( E \) admits a symplectic isomorphism \( E \xrightarrow{\varphi} E^*, \varphi^* = -\varphi \).

Then \( E \) must have even rank by (iii). In case \( rk(E) = 2 \) condition (iii) can be dropped because then the non–degenerate pairing \( E \otimes E \rightarrow \Lambda^2E \cong O \) determines a symplectic form, which then is unique up to a scalar.

If \( rk(E) = 2 \), then \( E \) is stable by condition (ii). Therefore, the open part

\[
MI(n) \subset M^b(2; 0, n)
\]

defined by the condition \( h^1E(-2) = 0 \) is the set of isomorphism classes of \((n, 2)\)-instantons, also called \( n \)-instantons.
1.3. Conjecture: \( MI(n) \) is smooth and irreducible of dimension \( 8n - 3 \) for \( n \geq 1 \).

This conjecture emerged in the late 1970’s as \( n \)–instantons were considered as counterparts of self–dual Yang–Mills fields on the 4–sphere, see [1], [27], in twistor theory.

The conjecture has been proved for \( n \leq 4 \). For \( n = 1 \) the space \( MI(1) \) is the complement of the Grassmannian \( G(2, V) \) in \( \mathbb{P}(\Lambda^2 V) \). For \( n = 2 \) the space \( MI(2) \) had been described by R. Hartshorne in [12] as a smooth and irreducible fibration. The case \( n = 3 \) was proved by G. Ellingsrud–S.A. Strømme in [10]. W. Barth proved in [3] that \( MI(4) \) is irreducible and J. LePotier in [21] that \( MI(4) \) is smooth. Recently P.I. Katsylo and G. Ottaviani proved that also \( MI(5) \) is smooth, see [15]. In this paper we prove that \( MI(5) \) is both smooth and irreducible. The method also yields simple and unified proofs for the previous results for \( n \leq 4 \).

1.4. Further results on \( n \)–instantons:

(1) Any \( n \)–instanton \( E \) is stable because \( H^0 E = 0 \) and \( \text{rk}(E) = 2 \). Then the Grauert–Mülich theorem states that \( E \) has trivial splitting type, i.e. for a general line \( L \) in \( \mathbb{P}_3 \) the restricted bundle \( E_L \) is isomorphic to \( 2O_L \), see [2].

(2) A line \( L \) is called a jumping line of the \( n \)–instanton \( E \) if \( E_L \cong O_L(-a) \oplus O_L(a) \) with \( a \neq 0 \), and this number is called the order of the jumping line. It is an easy consequence of the monad representation of \( E \), see [23] that the set \( J(E) \) of all jumping lines of \( E \) is a hypersurface of degree \( n \) in the Grassmannian \( G \) of lines in \( P_3 \). Moreover, \( n \) is the highest order possible for a jumping line.

(3) For any \( n \)–instanton, \( h^0 E(1) \leq 2 \), see [3]. The \( n \)–instantons with \( h^0 E(1) = 2 \) are called special ‘tHooft bundles. They can be presented as extensions

\[
0 \to 2O_{P_3}(-1) \to E \to O_Q(-n, 1) \to 0,
\]

where \( Q \) is a smooth quadric in \( P_3 \), see [13]. It was shown there that \( MI(n) \) is smooth along the locus of special ‘tHooft bundles, which has dimension \( 2n + 9 \).

(4) A plane \( P \) in \( P_3 \) is called unstable for an \( n \)–instanton \( E \) if the restricted bundle \( E_P \) has sections, otherwise stable. If \( E \) is special ‘tHooft, the unstable planes form a smooth quadric surface in \( P_3^* \), the dual of \( Q \) in (3). In [3] it was proved that an \( n \)–instanton \( E \) is already special ‘tHooft if its variety of unstable planes has dimension \( \geq 2 \).

(5) In [19] it was shown that \( MI(n) \) is smooth at any \( E \) with \( h^0 E(1) \neq 0 \) and that \( \text{Ext}^2(E, E(-1)) = 0 \) for such a bundle. The locus of these bundles has dimension \( 5n + 4 \).

(6) P. Rao, [22], and M. Skiti, [24], proved independently that \( MI(n) \) is even smooth along the locus of bundles \( E \) which admit jumping lines of highest order \( n \). These bundles form a subvariety of dimension \( 6n + 2 \). Moreover, in [22] it is proved that \( MI(5) \) is smooth at bundles which have jumping lines of order 4.
(7) L. Costa and G. Ottaviani, [1], proved that $M/I(5)$ is an affine scheme by describing the non-degeneracy condition for the monads, see [2,3], as the non-vanishing of a hyperdeterminant.

2. Instanton Bundles and Multilinear Algebra

In this section we describe the monad construction of instantons from hypernets of quadrics. Throughout this paper

$$H = H_n$$

denotes an $n$-dimensional $k$-vector space, $n \geq 1$. We identify $\Lambda^2(H^* \otimes V^*)$ with the space of anti-selfdual $k$–linear maps

$$H \otimes V \rightarrow H^* \otimes V^*.$$

Each $\omega \in \Lambda^2(H^* \otimes V^*)$ gives rise to a diagram

$$
\begin{array}{ccc}
H \otimes V & \xrightarrow{\omega} & H^* \otimes V^* \\
\downarrow u^* & & \downarrow u \\
N^* & \xrightarrow{\varphi \sim} & N \\
\end{array}
$$

where $N = N_\omega$ is the image and $Q = Q_\omega$ the cokernel of $\omega$, and $\varphi = \varphi_\omega$ is the canonically induced symplectic isomorphism. Note that the rank of $\omega$ is always even. $\omega$ is called non-degenerate if $\omega(h \otimes v) \neq 0$ for any non-zero decomposable tensor $h \otimes v$ in $H \otimes V$.

2.1. Rank Stratification: The space $\Lambda^2(H^* \otimes V^*)$ comes with the rank stratification

$$\Lambda^2(H^* \otimes V^*) = \Omega_{4n} \supset \Omega_{4n-2} \supset \ldots \supset \Omega_2,$$

where $\Omega_{2m} = \Omega_{2m}(H) = \{\omega \in \Lambda^2(H^* \otimes V^*) \mid \text{rk}(\omega) \leq 2m\}$. This had been considered already in [25] by A. Tyurin.

It is easy to prove by standard arguments that each $\Omega_{2m}$ is irreducible and smooth outside $\Omega_{2m-2}$ of codimension $\binom{4n-2m}{2}$. The tangent space at $\omega \in \Omega_{2m} \setminus \Omega_{2m-2}$ is the kernel in

$$0 \rightarrow T_\omega \Omega_{2m} \rightarrow \Lambda^2(H^* \otimes V^*) \rightarrow \Lambda^2 Q \rightarrow 0$$

which gives the dimension formula. There is the canonical decomposition

$$\Lambda^2(H^* \otimes V^*) = (S^2H^* \otimes \Lambda^2 V^*) \oplus (\Lambda^2 H^* \otimes S^2V^*),$$

and on the first summand we have the (induced) rank stratification

$$S^2H^* \otimes \Lambda^2 V^* = M_{4n} \supset M_{4n-2} \supset \ldots \supset M_2$$
with \( M_{2m} = M_{2m}(H) = \Omega_{2m} \cap (S^2H^* \otimes \Lambda^2V^*) \). We let
\[
\Delta = \Delta(H) = \{ \omega \in S^2H^* \otimes \Lambda^2V^* \mid \omega \text{ is degenerate} \}
\]
be the subset of degenerate tensors.

**Remark:** One can prove that \( \Delta \) is a closed and irreducible subvariety of \( S^2H^* \otimes \Lambda^2V^* \) of codimension \( 2(n-1) \) for \( n \geq 2 \) and of codimension 1 for \( n = 1 \).

### 2.2. The bundles \( E_\omega \).

To each \( \omega \in S^2H^* \otimes \Lambda^2V^* \) one can associate the complex
\[
H \otimes O_P(-1) \xrightarrow{\varphi \circ \alpha^*} N \otimes O_P \xrightarrow{\alpha} H^* \otimes O_P(1),
\]
where \( \alpha \) is the composition of \( N \otimes O_P \xrightarrow{u} H^* \otimes V^* \otimes O_P \) and \( H^* \otimes V^* \otimes O_P \xrightarrow{id \otimes ev} H^* \otimes O_P(1) \).

Then the following are equivalent:

(i) \( \alpha \) is surjective
(ii) \( \varphi \circ \alpha^* \) is a subbundle
(iii) \( \omega \) is non-degenerate.

Clearly (i) and (ii) are equivalent by definition. Now \( \alpha \) is surjective if and only if for any \( \langle v \rangle \in \mathbb{P}V \) the induced homomorphism \( N \to H^* \otimes \langle v \rangle^* \) on the fibre is surjective or equivalently, that \( H \otimes \langle v \rangle \xrightarrow{\omega} H^* \otimes V^* \) is injective, which is the non-degeneracy.

By this observation each \( \omega \in M_{2m} \setminus (M_{2m-2} \cup \Delta) \), \( n < m \leq 2n \), gives rise to an associated locally free sheaf
\[
E_\omega = \text{Ker}(\alpha)/\text{Im}(\varphi \circ \alpha^*).
\]

This bundle is in fact an instanton bundle of rank \( 2m - 2n \) and with Chern class \((c_1, c_2, c_3) = (0, n, 0)\). This follows directly from the defining complex, also called the **monad** of \( E_\omega \). Rank and Chern classes can be computed from those of the sheaves of the complex as well as \( H^0E_\omega = 0 \) and \( H^1E_\omega(-2) = 0 \). Because \( E^*_\omega \) is the cohomology of the dual complex, it follows that the symplectic isomorphism \( N^* \xrightarrow{\mathcal{E}} N \) induces a symplectic isomorphism \( E^*_\omega \xrightarrow{j} E_\omega \).

**Remark:** \( M_{2n} \subset \Delta \) because otherwise a non-degenerate \( \omega \in M_{2n} \setminus M_{2n-2} \) would define the bundle \( E_\omega = 0 \) with non-trivial second Chern class. Therefore, \( 2n + 2 \) is the lowest possible rank for a tensor \( \omega \in S^2H^* \otimes \Lambda^2V^* \), which is non-degenerate, and which then defines an instanton of rank 2.

In the following lemma and proposition it is proved that any \((n, r)\)-instanton arises by the above construction and that the isomorphism classes of \((n, r)\)-instantons are in 1 : 1 correspondence with the equivalence classes of the operators \( \omega \in M_{2n+r} \setminus (M_{2n+r-2} \cup \Delta) \).

We use the following **notation** for \( 2 \leq r \leq 2n \).
\[
M(n, r) := M_{2n+r} \setminus (M_{2n+r-2} \cup \Delta)
\]
2.2.1. **Lemma:** For any \( \omega \in M(n, r) \), there are isomorphisms

\[
H \cong H^2(\mathcal{E}_\omega \otimes \Omega^3(1)) \quad N \cong H^1(\mathcal{E}_\omega \otimes \Omega^1) \quad Q \cong H^1\mathcal{E}_\omega
\]

\[
H^* \cong H^1\mathcal{E}_\omega(-1) \quad N^* \cong H^2(\mathcal{E}_\omega \otimes \Omega^2)
\]

which are compatible with Serre duality and the symplectic isomorphism \( \mathcal{E}_\omega^* \cong \mathcal{E}_\omega \), making the following diagram commute

\[
\begin{array}{ccccccc}
H \otimes V & \to & N^* & \cong & N & \to & H^* \otimes V^* & \to & Q & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^2(\mathcal{E}_\omega \otimes \Omega^3(1)) \otimes V & \to & H^2(\mathcal{E}_\omega \otimes \Omega^2) & \cong & H^1(\mathcal{E}_\omega \otimes \Omega^1) & \to & H^1(\mathcal{E}_\omega(-1)) \otimes V^* & \to & H^1\mathcal{E}_\omega & \to & 0
\end{array}
\]

**Proof.** The homomorphisms of the bottom row of the diagram are induced by the natural homomorphism \( \Omega^3(1) \otimes V \to \Omega^2 \) and the exact sequences \( 0 \to \Omega^2 \to \Lambda^2V^* \otimes \mathcal{O}(-2) \to \Omega^1 \to 0 \) and \( 0 \to \Omega^1 \to V^* \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0 \) of the Koszul complex of \( V^* \otimes \mathcal{O}(-1) \to \mathcal{O} \). Then the isomorphisms of the lemma and the commutativity of the diagram follow by tracing all data from the defining complex of \( \mathcal{E}_\omega \), using the functoriality of Serre-duality. See also [4] for monads of vector bundles on projective spaces. \( \square \)

2.3. **Proposition:** (a) Let \( \mathcal{E} \) be an \((n, r)\)–instanton with symplectic isomorphism \( \mathcal{E}^* \xrightarrow{j} \mathcal{E}, 2 \leq r \leq 2n \). Then there exists a non–degenerate \( \omega \in S^2H^* \otimes \Lambda^2V^* \) of rank \( 2n + r \) such that \((\mathcal{E}, j) \cong (\mathcal{E}_\omega, j_\omega)\).

(b) \((\mathcal{E}_\omega, j_\omega) \cong (\mathcal{E}_{\omega'}, j_{\omega'})\) if and only if \( \omega \) and \( \omega' \) are in the same \( GL(H) \)–orbit in \( S^2H^* \otimes \Lambda^2V^* \).

(c) The stabilizer \( \text{Stab}(\omega) \) of \( \omega \) in \( GL(H) \) is isomorphic to the automorphism group \( \text{Aut}(\mathcal{E}_\omega, j_\omega) \).

**Proof.** (a) By [2.2.1] it is enough to show that the Beilinson spectral sequence of \( \mathcal{E} \) results in the complex

\[
0 \to H^2(\mathcal{E} \otimes \Omega^3(1)) \otimes \mathcal{O}(-1) \xrightarrow{\beta} H^1(\mathcal{E} \otimes \Omega^1) \otimes \mathcal{O} \xrightarrow{\alpha} H^1(\mathcal{E}(-1)) \otimes \mathcal{O}(1) \to 0,
\]

where \( \alpha \) resp. \( \beta \) correspond to the natural homomorphisms

\[
H^2(\mathcal{E} \otimes \Omega^3(1)) \otimes V \to H^2(\mathcal{E} \otimes \Omega^2) \leftrightarrow H^1(\mathcal{E} \otimes \Omega^1)
\]

resp.

\[
H^1(\mathcal{E} \otimes \Omega^1) \to H^1\mathcal{E}(-1) \otimes V^*.
\]
which are Serre–dual to each other. Now for any coherent sheaf \( F \) on \( \mathbb{P}_n \) there is the Beilinson I complex
\[
0 \to C^{-n}(F) \to \cdots \to C^0(F) \to \cdots \to C^n(F) \to 0
\]
with terms
\[
C^p(F) = \bigoplus_{p=i-j} H^i(F \otimes \Omega^j) \otimes \mathcal{O}(-j),
\]
which is exact except at \( C^0(F) \) and which has \( F \) as its cohomology at \( C^0(F) \). In our case for \( F = E(-1) \) this complex reduces in fact to
\[
0 \to H^1(E \otimes \Omega^3(1)) \otimes \mathcal{O}(-2) \to H^1(E \otimes \Omega^1) \otimes \mathcal{O}(-1) \to H^1(E(-1)) \otimes \mathcal{O} \to 0
\]
by verifying that the instanton conditions imply the vanishing of the other terms of the Beilinson complex. Moreover, the Koszul sequence \( 0 \to \Omega^3 \to \Lambda^3 V^* \otimes \mathcal{O}(-3) \to \Omega^2 \to 0 \) induces the isomorphism \( H^1(E \otimes \Omega^3(1)) \cong H^2(E \otimes \Omega^3(1)) \) because \( H^i(E(-2)) = 0 \) for \( i = 1, 2 \). Now \( E \) is the cohomology of the complex
\[
0 \to H^2(E \otimes \Omega^3(1)) \otimes \mathcal{O}(-1) \overset{\beta}{\to} H^1(E \otimes \Omega^1) \otimes \mathcal{O} \overset{\alpha}{\cong} H^1(E(-1)) \otimes \mathcal{O}(1) \to 0.
\]
It follows from the Riemann–Roch formula that
\[
\chi(E(d)) = r \left( \frac{d+3}{3} \right) - n(d+2)
\]
and from the instanton conditions that
\[
h^1(E(-1)) = n \quad \text{and} \quad h^1(E \otimes \Omega^1) = 2n + r.
\]
Thus \( E \) defines a non–degenerate \( \omega \in S^2 H^* \otimes \Lambda^2 V^* \) via
\[
H^2(E \otimes \Omega^3(1)) \otimes V \to H^2(E \otimes \Omega^2) \leftarrow H^1(E \otimes \Omega^1) \to H^1(E(-1)) \otimes V^*
\]
together with an isomorphism \( H^* \cong H^1(E(-1)) \), such that \( (E, j) \cong (E_\omega, j_\omega) \).

(b) and (c) follow from (a) and 2.2.1 by the isomorphisms of the complexes. \( \square \)

2.4. Corollary: If \( E_\omega \) is simple, then \( \text{Stab}_{GL(H)}(\omega) = \{ \pm \text{id}_H \} \).

2.5. Remark: There is also a Beilinson II monad for an instanton bundle \( E_\omega \). This can be treated in the same way as the above Beilinson I monad. Any \( \omega \in S^2 H^* \otimes \Lambda^2 V^* \) defines an operator \( H \to H^* \otimes \Lambda^2 V^* \) which is symmetric with respect to \( H \) and the exact sequence
\[
0 \to N \overset{u}{\to} H^* \otimes V^* \to Q \to 0
\]
as above. Now combined with the Koszul homomorphisms
\[
H^* \otimes \mathcal{O}(-1) \to \Omega^1(1) \quad \text{and} \quad \Omega^1(1) \to V^* \otimes \mathcal{O}
\]
we obtain the complex
\[
0 \to H \otimes \mathcal{O}(-1) \overset{u^*}{\to} H^* \otimes \Omega^1(1) \overset{u}{\to} Q \otimes \mathcal{O} \to 0.
\]
If \( \omega \) is non-degenerate, this complex defines the \((n, r)\)-instanton \( E_\omega \), where \( \text{rk}(\omega) = 2n + r \).

This could be shown as in \[2.2.1\] \[2.3\], but follows already from the following commutative diagram induced by \( \omega \), which is a direct transformation between the two monad types.

\[
\begin{array}{cccc}
0 & \rightarrow & H \otimes \mathcal{O}(-1) & \rightarrow & H^* \otimes \Omega^1(1) & \rightarrow & Q \otimes \mathcal{O} & \rightarrow & 0 \\
\downarrow & & \downarrow & \mu & \downarrow & & \downarrow & & \\
0 & \rightarrow & N \otimes \mathcal{O} & \rightarrow & H^* \otimes V^* \otimes \mathcal{O} & \rightarrow & Q \otimes \mathcal{O} & \rightarrow & 0 \\
\downarrow & & \downarrow & \nu & \downarrow & & \downarrow & & \\
H^* \otimes \mathcal{O}(1) & \rightarrow & H^* \otimes \mathcal{O}(1) & \rightarrow & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & 
\end{array}
\]

2.6. The morphism \( M_{2n+2 \setminus \Delta} \overset{b}{\rightarrow} MI(n) \)

According to Proposition \[2.3\] we are given a surjective map \( \omega \overset{b}{\mapsto} [E_\omega] \) from \( M_{2n+2 \setminus \Delta} \) to \( MI(n) \) whose fibres are the orbits under the action of \( \text{GL}(H) \). The map \( b \) is the underlying map of a morphism, because there is a universal monad over \( (M_{2n+2 \setminus \Delta}) \times \mathbb{P} \) with the universal family of \( n \)-instantons as cohomology. Then \( b \) is the modular morphism onto the open part \( MI(n) \) of the Maruyama scheme. It can be shown that \( b \) is a geometric quotient and a principal \( GL(H)/\{\pm 1\} \)–bundle in the etale topology. However the latter fact will not be used in this paper.

2.7. Criterion for irreducibility

Since \( \Omega_{2n+2} \) is irreducible of codimension \( \binom{2n-2}{2} \) in \( \Lambda^2(H^* \otimes V^*) \), it follows that every irreducible component of \( M_{2n+2} \) has codimension \( \leq \binom{2n-2}{2} \) in \( S^2H^* \otimes \Lambda^2V^* \) or dimension \( \geq 3n(n+1) - \binom{2n-2}{2} = n^2 + 8n - 3 \). From this observation we obtain

2.7.1. Lemma: Any component of \( MI(n) \) has dimension \( \geq 8n - 3 \) and \( MI(n) \) is irreducible of dimension \( 8n - 3 \) if and only if \( M_{2n+2 \setminus \Delta} \) is irreducible of the expected codimension \( \binom{2n-2}{2} \) in \( S^2H^* \otimes \Lambda^2V^* \).

For the proof use the following lemma.
2.7.2. Lemma: Let \( X \xrightarrow{f} Y \) be a morphism of reduced schemes, let \( Y \) be irreducible and assume that each fibre \( f^{-1}(f(x)), \) \( x \in X, \) is irreducible of constant dimension \( e, \) and that each irreducible component of \( X \) is of dimension \( \geq \dim Y + e. \) Then \( X \) is irreducible.

Proof. Let \( X_1, \ldots, X_p \) be the irreducible components of \( X. \) According to the theorem of the dimension of the fibres, \( f|X_i \) is dominant over \( Y \) for any \( i. \) Let \( X'_i \subset X_i \) be the complement of the other components and let \( y \in \cap f(X'_i). \) Then

\[
f^{-1}(y) = \cup (f^{-1}(y) \cap X_i)
\]

and the intersections \( f^{-1}(y) \cap X_i \) are distinct closed subsets of \( f^{-1}(y). \) As \( f^{-1}(y) \) is irreducible, it follows that \( p = 1. \)

2.7.3. Remark: In [26], Example 1, A. Tyurin showed that \( M_{2n} \) and (consequently) \( M_{2n+2} \) have not the expected codimension \( \left( \begin{array}{c} 2n \end{array} \right) \) respectively \( \left( \begin{array}{c} 2n-2 \end{array} \right) \) in \( S^2 H^* \otimes \Lambda^2 V^*. \)

2.8. Criterion for smoothness

Concerning the smoothness of \( MI(n), \) let \( E \) be an \( n \)-instanton (of rank 2). Because \([E] \) is a stable point of \( MI(n), \) this is a smooth point if \( \text{Ext}^2(E, E) = 0. \) But since \( \text{ext}^1(E, E) - \text{ext}^2(E, E) = 8n - 3 \) the vanishing of \( \text{Ext}^2(E, E) \) is also necessary if \([E] \) is a smooth point and \( MI(n) \) has dimension \( 8n - 3 \) at \([E]. \) On the other hand, we have

\[
\text{Ext}^2(E, E) \cong H^2(E^\vee \otimes E) \cong H^2(E \otimes E) \cong H^2(S^2 E).
\]

Now for any \( (n, r) \)-instanton we have the following

2.8.1. Lemma: Let \( \omega \in S^2 H^* \otimes \Lambda^2 V^* \) be non–degenerate of rank \( 2m, \) \( n < m \leq 2n, \) and let \( E = E_\omega. \) Then the following conditions are equivalent.

(i) \( H^2(S^2 E) = 0 \)

(ii) \( \Omega_{2m} \) and \( S^2 H^* \otimes \Lambda^2 V^* \) intersect transversally in \( \omega \) inside \( \Lambda^2(H^* \otimes V^*). \)

(iii) \( M_{2m} \) is smooth at \( \omega \) of expected dimension \( 5m - 5n^2 + 8mn - 2m^2 - m. \)

Proof. Transversality in (ii) means that the tangent space \( T_\omega \Omega_{2m} \) and \( S^2 H^* \otimes \Lambda^2 V^* \) span \( \Lambda^2(H^* \otimes V^*). \) Then (ii) and (iii) are equivalent by standard dimension counts, because \( \Omega_{2m} \) has codimension \( \left( \begin{array}{c} 4n-2m \end{array} \right) \) and is smooth at \( \omega. \)

Next, let us recall a general fact. To any bounded complex \( K^* \) of vector bundles on a variety one can associate its second symmetric power \( S^2 K^* \) by decomposing \( K^* \otimes K^* \) according to the eigenspaces of the canonical involution on \( K^* \otimes K^*. \) If \( K^* \) has only three non–zero terms \( K^{-1} \rightarrow K^0 \rightarrow K^1, \) then \( S^2 K^* \) is isomorphic to

\[
0 \rightarrow \Lambda^2 K^{-1} \rightarrow K^{-1} \otimes K^0 \rightarrow S^2 K^0 \oplus (K^{-1} \otimes K^1) \rightarrow K^0 \otimes K^1 \rightarrow \Lambda^2 K^1 \rightarrow 0
\]
whose differentials are naturally deduced from those of $K^\bullet$. Moreover, if $H^i(K^\bullet) = 0$ for $i \neq 0$ and $H^0(K^\bullet) = F$, then $H^i(S^2K^\bullet) = 0$ for $i \neq 0$ and $H^0(S^2K^\bullet) \cong S^2F$. Hence, in our case $S^2\mathcal{E}$ is the degree 0 cohomology of the derived complex

$$\Lambda^2 H \otimes \mathcal{O}(-2) \to H \otimes N \otimes \mathcal{O}(-1) \to (S^2 N \oplus H \otimes H^*) \otimes \mathcal{O} \to N \otimes H^* \otimes \mathcal{O}(1) \to \Lambda^2 H^* \otimes \mathcal{O}(2).$$

The terms of this monad are cohomologically acyclic. Hence one can compute the cohomology of $S^2\mathcal{E}$ just by passing to global sections and then taking cohomology. Particularly, one gets an exact sequence

$$N \otimes H^* \otimes V^* \to \Lambda^2 H^* \otimes S^2V^* \to H^2(S^2\mathcal{E}) \to 0.$$

Now condition (ii) is equivalent to

$$T_\omega \Omega_{2m} + (S^2H^* \otimes \Lambda^2V^*) = \Lambda^2(H^* \otimes V^*),$$

noting that $\omega$ is a smooth point of $\Omega_{2m}$. Taking into account the exact sequences

$$N \otimes (H^* \otimes V^*) \to \Lambda^2(H^* \otimes V^*) \to \Lambda^2Q \to 0$$

and

$$0 \to S^2H^* \otimes \Lambda^2V^* \to \Lambda^2(H^* \otimes V^*) \to \Lambda^2H^* \otimes S^2V^* \to 0,$$

one finds that (ii) is equivalent to the surjectivity of the composed map

$$N \otimes H^* \otimes V^* \to \Lambda^2(H^* \otimes V^*) \to \Lambda^2H^* \otimes S^2V^*,$$

which is the map whose cokernel is $H^2(S^2\mathcal{E})$. This proves the equivalence of (i) and (ii).

Now, by the criteria for irreducibility and smoothness, it is clear that the conjecture 1.3 on $MI(n)$ is equivalent to

2.9. Transcribed Conjecture: $\Omega_{2n+2}$ and $S^2H^* \otimes \Lambda^2V^*$ intersect transversally inside $\Lambda^2(H^* \otimes V^*)$ along $M_{2n+2} \setminus \Delta$ and this intersection is irreducible.

Note, that in this conjecture, with $\Delta$ also $M_{2n}$ has been extracted from $M_{2n+2}$ because $M_{2n} \subset \Delta$. According to Tyurin’s example the whole of $M_{2n+2}$ has components of excessive dimension for large $n$.

2.10. Remark: Katsylo–Ottaviani gave in [15] the following interpretation of the transversality condition. Dualizing the sequence with $H^2(S^2\mathcal{E})$ as cokernel in the proof of Lemma 2.8.1, we have the exact sequence

$$0 \to H^2(S^2\mathcal{E})^* \to \Lambda^2H \otimes S^2V \to N^* \otimes H \otimes V.$$
Interpreting the elements of $\Lambda^2 H \otimes S^2 V$ as anti–selfdual linear maps $H^* \otimes V^* \to H \otimes V$ and the elements of $N^* \otimes H \otimes V$ as linear maps $N \to H \otimes V$, the morphism in the above sequence can be described by $\sigma \mapsto \sigma \circ u$, where $u$ is the inclusion of $\text{Im}(\omega) = N \subset H^* \otimes V^*$. Therefore,

$$H^2(S^2\mathcal{E})^* \cong \{ \sigma \in \Lambda^2 H \otimes S^2 V \mid \sigma \circ \omega = 0 \}$$

and the three conditions of [2.8.1] are equivalent to

(iv) if $\sigma \circ \omega = 0$ for $\sigma \in \Lambda^2 H \otimes S^2 V$, then $\sigma = 0$.

In order to illustrate this point of view and for later use, we prove

2.11. Proposition: $M(n, 2n - 2) = M_{4n-2} \setminus (\Delta \cup M_{4n-4})$ is smooth and (obviously) of codimension 1 in $S^2 H^* \otimes \Lambda^2 V^*$ for $n \geq 2$.

Proof. Let $\omega \in M_{4n-2} \setminus (\Delta \cup M_{4n-4})$ and $\sigma \in \Lambda^2 H \otimes S^2 V$ such that $\sigma \circ \omega = 0$. Since both $\omega$ and $\sigma$ are anti–selfdual, it follows that $\omega \circ \sigma = 0$. Now $\text{rk}(\omega) = 4n - 2$ implies $\text{rk}(\sigma) \leq 2$. One can now easily show (see e.g. [25], Proposition 2.1.1) that in this case $\sigma$ is decomposable, $\sigma = \eta \otimes f$, with $\eta \in \Lambda^2 H$ of rank $\leq 2$ and $f \in S^2 V$ of rank $\leq 1$. Consequently, if $\sigma \neq 0$, then $\text{Im}(\sigma) \subset H \otimes V$ contains decomposable vectors, contradicting $\omega \notin \Delta$. \qed

3. The Method

Let $\mathcal{E}$ be an $(n, r)$–instanton and $H^* = H_n^* \cong H^1 Hendr(-1)$ as above with $\omega \in S^2 H^* \otimes \Lambda^2 V^* \setminus \Delta$ of rank $2n + r$. To any $\xi \in H^*$ with kernel $\tilde{H}$ we can associate the restriction $\tilde{\omega} = \omega_{\xi} = \text{res}_{\xi}(\omega)$ of $\omega$ to $S^2 \tilde{H}^* \otimes \Lambda^2 V^*$ such that we have the diagram

$$\begin{array}{ccc}
\tilde{H} \otimes V & \xrightarrow{\tilde{\omega}} & \tilde{H}^* \otimes V^* \\
\downarrow & & \downarrow \\
H \otimes V & \xrightarrow{\omega} & H^* \otimes V^*
\end{array}$$

Then $\text{rk}(\tilde{\omega}) \leq \text{rk}(\omega)$. 
The choice of $\xi$ gives rise to a diagram

\[
\begin{array}{cccccc}
0 & \downarrow & 0 \\
\hat{H} \otimes O(-1) & \downarrow \beta & H \otimes O(-1) & N \otimes O & H^* \otimes O(1) & 0 \\
0 & \downarrow & 0 & \downarrow \alpha & \downarrow \bar{\alpha} \\
O(-1) & \downarrow & 0 & \downarrow & \bar{H}^* \otimes O(1) \\
0 & \downarrow & 0
\end{array}
\]

such that $\bar{\beta}, \bar{\alpha}$ constitute a monad for a bundle $\bar{E}$ with an induced symplectic isomorphism, and such that $\beta, \bar{\alpha}$ are induced by $\bar{\omega}$. Then $\bar{E}$ has rank $r + 2$ and second Chern class $n - 1$. Any splitting homomorphism of $H \otimes O(-1) \to O(-1)$ then induces a unique homomorphism $O(-1) \to \bar{E}$ and by duality a monad

\[
0 \to O(-1) \to E \to O(1) \to 0
\]

whose cohomology is again $E$. This observation will be used to perform an induction $(n - 1, r + 2) \leadsto (n, r)$ for irreducibility and smoothness of the spaces of $(n, r)$–instantons for $n \leq 5$.

Immediate relations between the tensors $\omega$ and $\bar{\omega}$ and the corresponding sheaves $E$ and $\bar{E}$ are stated in the next two lemmata.

3.1. Lemma: Let $\bar{\omega}$ be the restriction of $\omega$ for $\xi \in H^*$. The following conditions are equivalent:

(i) $rk(\bar{\omega}) = rk(\omega)$
(ii) $Im(\omega) \cap (\xi \otimes V^*) = 0$
(iii) The multiplication map $H^1\mathcal{E}(-1) \otimes V^* \to H^1\mathcal{E}$ restricts to an injection $\xi \otimes V^* \hookrightarrow H^1\mathcal{E}$.
(iv) $H^0\bar{E} = 0$
Proof. We have the diagram
\[
\begin{array}{ccc}
\bar{H} \otimes V & \xrightarrow{\bar{\omega}} & \bar{H}^* \otimes V^* \\
\downarrow j & & \downarrow j^* \\
H \otimes V & \xrightarrow{\omega} & H^* \otimes V^* \\
\downarrow u^* & & \downarrow u \\
N^* & \xrightarrow{\phi} & N
\end{array}
\]
where \( N \) is the image of \( \omega \), \( \text{Ker}(j^*) = \xi \otimes V^* \), and \( \mu \) is isomorphic to the multiplication map \( H^1\mathcal{E}(-1) \otimes V^* \rightarrow H^1\mathcal{E} \). It follows that \( j^* \circ u \) is injective if and only if the image of \( \bar{\omega} \) has the same dimension as \( N \), which proves the equivalence of (i) and (ii). The other statements are immediately seen to be equivalent to the injectivity of \( j^* \circ u \). For (iv), note, that \( N \rightarrow \bar{H}^* \otimes V^* \) corresponds to the right part of the monad of \( \mathcal{E} \).

\[\square\]

If the conditions of the lemma are satisfied, then \( \mathcal{E} \) is an \((n-1, r+2)\)-instanton, provided \( r < 2n \). In this case we have

**3.2. Lemma:** Let \( \omega \in S^2H^* \otimes \Lambda^2V^* \) be non-degenerate and \( \text{rk}(\bar{\omega}) = \text{rk}(\omega) \). If \( H^2(S^2\mathcal{E}) = 0 \) and \( H^1\mathcal{E}(1) = 0 \), then \( H^2(S^2\mathcal{E}) = 0 \).

**Proof.** Because \( \mathcal{E} \) is the cohomology of the monad \( \mathcal{O}(-1) \rightarrow \bar{\mathcal{E}} \rightarrow \mathcal{O}(1) \), \( S^2\mathcal{E} \) is the cohomology of the induced monad, see proof of Lemma [2.8.1](#).

\[ \mathcal{E}(-1) \rightarrow \mathcal{O} \oplus S^2\bar{\mathcal{E}} \rightarrow \bar{\mathcal{E}}(1). \]

The computation of \( H^2(S^2\mathcal{E}) \) from this monad gives the vanishing. \[\square\]

**3.3. Construction lemma**

For \( n \geq 3 \) and \( 2 \leq r \leq 2n - 2 \) and for any \( \xi \in H^* \) we consider the open subset
\[ M(n, r)_\xi := \{ \omega \in M(n, r) \mid \text{Im}(\omega) \cap (\xi \otimes V^*) = 0 \}. \]

This is exactly the open subset of \( M(n, r) \) which is mapped to \( M(n-1, r+2) \) under the restriction map
\[ \text{res}_\xi: \omega \mapsto \omega_\xi = \omega \mid S^2H^* \otimes \Lambda^2V^*. \]

We also use the notation
\[ M^0(n, r) = \{ \omega \in M(n, r) \mid H^1\mathcal{E}_\omega(1) = 0 \}. \]
Note that this set might be empty if $h^0\mathcal{E}_\omega(1) - h^1\mathcal{E}_\omega(1) = 4r - 3n < 0$, while $h^2\mathcal{E}_\omega(1) = h^3\mathcal{E}_\omega(1) = 0$. Next we let

$$M(n, r)_{\xi}' = \{ \omega \in M(n, r)_{\xi} | \text{res}_{\xi}(\omega) \in M^0(n - 1, r + 2) \}$$

be the inverse image. For the fibres of res$_{\xi}$ we have the following lemma, which enables the induction. Note, that in this lemma the fibre $R(\bar{\omega})$ may contain degenerate $\omega$’s.

### 3.3.1. Lemma

1. For any $\bar{\omega} \in M(n - 1, r + 2)$ there is an isomorphism

$$R(\bar{\omega}) := \{ \omega \in S^2H^*_n \otimes \Lambda^2V^* | \text{res}_\xi(\omega) = \bar{\omega}, \ rk(\omega) = rk(\bar{\omega}) \} \cong \text{Hom}_\mathcal{O}_\mathbb{P}(\mathcal{C}_\omega, \mathcal{O}_\mathbb{P}(1))$$

where $\mathcal{C}_\omega$ denotes the cokernel of the left part of the monad of $\mathcal{E}_\omega$, and

$$\dim \text{Hom}_\mathcal{O}_\mathbb{P}(\mathcal{C}_\omega, \mathcal{O}_\mathbb{P}(1)) = n - 1 + h^0\mathcal{E}_\omega(1).$$

2. $\omega \in R(\bar{\omega})$ is non–degenerate if and only if the corresponding homomorphism induces an epimorphism $\mathcal{E}_\omega \rightarrow \mathcal{O}_\mathbb{P}(1)$. In that case $\omega$ defines an $(n, r)$–instanton bundle $\mathcal{E}_\omega$ which is also the cohomology of the self–dual monad $\mathcal{O}_\mathbb{P}(-1) \rightarrow \mathcal{E}_\omega \rightarrow \mathcal{O}_\mathbb{P}(1)$ defined by the epimorphism.

**Proof.** a) For $\xi$ fixed we may choose a decomposition $H = H_1 \oplus \bar{H}$ such that $\xi$ induces an isomorphism $H_1 \cong k$. Let $\bar{\omega}$ decompose into

$$\bar{H} \otimes V \xrightarrow{\bar{u}^*} N^* \xrightarrow{\phi} N \xrightarrow{\bar{u}} \bar{H}^* \otimes V^*$$

with associated monad

$$\bar{H} \otimes \mathcal{O}_\mathbb{P}(-1) \xrightarrow{\bar{\beta}} N \otimes \mathcal{O}_\mathbb{P} \xrightarrow{\bar{u}} \bar{H}^* \otimes \mathcal{O}_\mathbb{P}(1).$$

With $\mathcal{C}_\omega = \text{coker}(\bar{\beta})$ we have the exact sequence

$$0 \rightarrow \mathcal{E}_\omega \rightarrow \mathcal{C}_\omega \rightarrow \bar{H}^* \otimes \mathcal{O}_\mathbb{P}(1) \rightarrow 0.$$

Any $u_1 \in \text{Hom}(N, H_1^* \otimes V^*)$ gives rise to a skew–symmetric operator

$$\omega = \left( \begin{array}{c|c} u_1 \circ \varphi \circ u_1^* & u_1 \circ \varphi \circ \bar{u}^* \\ \bar{u} \circ \varphi \circ u_1^* & \bar{\omega} \end{array} \right)$$

$$(H_1 \oplus \bar{H}) \otimes V \xrightarrow{\omega} (H_1^* \oplus \bar{H}^*) \otimes V^*$$

which factors also through $\varphi$ by its definition, so that $\text{rk}(\omega) = \text{rk}(\bar{\omega})$ and $\omega$ is symplectic. However, the component $u_1 \circ \varphi \circ \bar{u}^* : \bar{H} \otimes V \rightarrow H_1^* \otimes V^*$ is not necessarily skew with respect to $V$. This is the case if and only if $\omega \in R(\bar{\omega})$ or if and only if the composition

$$\bar{H} \otimes \mathcal{O}_\mathbb{P}(-1) \xrightarrow{\bar{\beta}} N \otimes \mathcal{O}_\mathbb{P} \xrightarrow{\bar{u}_1} H_1^* \otimes \mathcal{O}_\mathbb{P}(1)$$
of the associated sheaf homomorphisms is zero, or, if and only if $\tilde{u}_1$ factors through $\mathcal{C}_\omega$. Let $\text{Hom}(N, H_1^* \otimes V^*)'$ denote the subspace of $\text{Hom}(N, H_1^* \otimes V^*)$ defined by this condition. Then $u_1 \mapsto \omega$ defines an isomorphism

$$\text{Hom}(N, H_1^* \otimes V^*)' \cong R(\bar{\omega})$$

because $u_1 \mapsto \omega$ is injective since $\varphi$ and $\bar{u}$ are surjective and because any $\omega \in R(\bar{\omega})$ arises in this way. On the other hand, the factorization of $\bar{u}_1 : N \otimes O_P \to \mathcal{C}_\omega \xrightarrow{\gamma} H_1^* \otimes O_P(1)$ defines an isomorphism $u_1 \leftrightarrow \gamma$ between $\text{Hom}(N, H_1^* \otimes V^*)'$ and $\text{Hom}(\mathcal{C}_\omega, H_1^* \otimes O_P(1))$.

b) By a) any $\omega \in R(\bar{\omega})$ gives rise to a diagram

![Diagram]

with $\pi$ induced by $u_1$ or $\alpha_1$. It follows that $\alpha = \alpha_1 + \bar{\alpha} : N \otimes O_P \to (H_1^* \oplus \bar{H}^*) \otimes O_P(1)$ is surjective if and only if $\pi$ is surjective. This proves b), because the surjectivity of $\alpha$ (as right part of the monad of $\omega$) is equivalent to the non–degeneracy of $\omega$. In that case the induced sequence

$$H_1 \otimes O_P(-1) \to \mathcal{E}_\omega \xrightarrow{\pi} H_1^* \otimes O_P(1)$$

is also a monad for the bundle $\mathcal{E}_\omega$.

**3.3.2. Corollary:** If $M^0(n-1, r+2)$ is irreducible of the expected dimension $3(n-1)n - \binom{2n-4-r}{2}$, then also $M(n, r)'_\xi$ is irreducible of the expected dimension

$$3n(n+1) - \binom{2n-r}{2},$$

if it is not empty.

**Proof.** The dimension of any component of $M(n, r)$ is $\geq$ then the expected dimension by Lemma 2.7.2. Because the fibres of res$_\xi$ have constant dimension by 3.3.1, the corollary follows from Lemma 2.7.2. \qed
3.4. Lemma: If $M(n,r)_{\xi} \neq \emptyset$, then for any other $\eta \neq 0$,

$$M(n,r)_{\xi} \cap M(n,r)_{\eta} \neq \emptyset.$$

Proof. For $\omega_0 \in M(n,r)_{\xi}$ we consider the set

$$U_0 = \{ \eta \in H^* | \text{Im}(\omega_0) \cap (\eta \otimes V^*) = 0, \ h^1\mathcal{E}_{\eta}(1) = 0 \}$$

where $\mathcal{E}_{\eta}$ denotes the bundle obtained from $\omega^0$ by restriction to the kernel of $\eta$. Because $(\mathcal{E}_{\eta})$ is a flat family on the open set of $\eta$’s defined by $\text{Im}(\omega_0) \cap (\eta \otimes V^*) = 0$, the semicontinuity theorem implies that also $U_0$ is open. Because $\xi \in U_0$, there is a $\xi' \in U_0$ which is independent of $\xi$. Then $\omega_0 \in M(n,r)_{\xi} \cap M(n,r)_{\xi'}$.

Let now $\eta \neq 0$ be arbitrary in $H^*$. There is a transformation $g \in GL(H)$ such that $g*\xi = \xi$ and $g*\xi' = \eta$. Then

$$(g^* \otimes \text{id}_{V^*}) \circ \omega_0 \circ (g \otimes \text{id}_V) \in M(n,r)_{\xi} \cap M(n,r)_{\eta},$$

3.5. The induction: Suppose now that $M^0(n-1,r+2)$ is irreducible and that also $M(n,r)_{\xi} \neq \emptyset$ for some $\xi \neq 0$. Then by Lemma 3.4 the union

$$\bigcup_{\xi \neq 0} M(n,r)_{\xi}$$

is an irreducible open subset of $M(n,r)$ of the expected dimension. If, in addition, $M^0(n-1,r+2)$ is smooth (as transversal intersection), then by Lemma 3.2 also $H^2S^2\mathcal{E}_\omega = 0$ for any $\omega$ in the above union. Concerning rank 2 instantons, we shall prove that

$$M(n,2) = \bigcup_{\xi \neq 0} M(n,2)_{\xi}$$

for $n \geq 3$. For $n = 3, 4$ we shall even prove that

$$M(n,2) = \bigcup_{\xi \neq 0} M(n,2)_{\xi}.$$

This will give a unified proof of the instanton conjecture for $n \leq 4$.

For $n = 5$, however, we are at present not able to prove that for any $\omega \in M(5,2)$ there exists a $\xi \neq 0$ such that $H^1\mathcal{E}_\omega(1) = 0$. But we shall prove the weaker result, that there exists a $\xi \neq 0$ with $h^1\mathcal{E}_\omega(1) \leq 1$. This is already sufficient to prove the conjecture for $n = 5$ in the sequel.

For $n = 6$ one might hope that $\bigcup M(6,2)_{\xi}^\prime$ is the complement of the subvariety of ’tHooft bundles (defined by $h^0\mathcal{E}(1) \neq 0$). However, for $n > 6$ the present induction method doesn’t seem to work anymore.
4. On Jumping Lines of $n$–Instantons

In this section we are going to prove the following Proposition which enables us to choose $\xi \in H^*_n$ for any $n$–instanton $E_\omega$, $n \geq 3$, such that $E_\xi := E_\omega$ is again an $(n - 1, 4)$–instanton, i.e. $H^0 E_\xi = 0$. For $n \geq 5$ the proposition allows us to choose a second $\eta \in H^*_{n-1}$ such that also $H^0 (E_\xi) \eta = 0$.

In order to prepare the proof we include the following lemmata on jumping lines of arbitrary instantons.

4.1. Lemma: Let $E$ be a stable rank 2 vector bundle on $\mathbb{P}_2$ with Chern class $(c_1, c_2) = (0, n)$, $n \geq 2$, let $L_1, L_2$ be distinct lines and let $a_1, a_2 \geq 0$ be defined by $E_{L_\nu} \cong O_{L_\nu}(a_\nu) \oplus O_{L_\nu}(-a_\nu)$. Then

(a) $a_1 + a_2 \leq n$
(b) If $a_1 + a_2 = n$, and $a_1 \geq 2$, $a_2 \geq 2$, then $E$ can be realized as an extension

$$0 \to O_{\mathbb{P}_2}(-1) \to E \to \mathcal{I}_Z(1) \to 0$$

with $Z \subset L_1 \cup L_2$, a 0–dimensional subscheme of length $n + 1$.

Proof. (a) There is the natural exact sequence

$$0 \to O_{L_1 \cup L_2} \to O_{L_1} \oplus O_{L_2} \to O_{L_1 \cap L_2} \to 0. \quad (1)$$

Tensoring this with $E$ and taking sections, this implies

$$a_1 + a_2 \leq h^0 E_{L_1 \cup L_2}.$$

On the other hand, the sequence

$$0 \to E(-2) \to E \to E_{L_1 \cup L_2} \to 0 \quad (2)$$

implies the exact sequence

$$0 \to H^0 E \to H^0 E_{L_1 \cup L_2} \to H^1 E(-2) \to H^1 E.$$

Because $E$ is stable, $h^0 E = 0$ and $h^2 E(-2) = h^0 E(-1) = 0$, so that $h^1 E(-2) = n$ by the Riemann–Roch formula, and then $h^0 E_{L_1 \cup L_2} \leq n$.

(b) According to the proof of (a), if $a_1 + a_2 = n$, then $a_1 + a_2 = h^0 E_{L_1 \cup L_2} = n$ and

$$H^0 E_{L_1} \oplus H^0 E_{L_2} \to H^0 E_{L_1 \cap L_2}$$

is surjective. It follows that then also

$$H^0 E_{L_1}(1) \oplus H^0 E_{L_2}(1) \to H^0 E_{L_1 \cap L_2}(1)$$

is surjective and from this, that

$$h^0 E_{L_1 \cup L_2}(1) = n + 2.$$

Using sequence (2) with $E(1)$, we obtain $h^0 E(1) \geq 2$. Then there are exact sequences

$$0 \to O_{\mathbb{P}_2}(-1) \to E \to \mathcal{I}_Z(1) \to 0$$
with 0-dimensional subschemes $Z$ of $\mathbb{P}_2$ of length $n+1$. Tensoring (1) with $\mathcal{O}_Z$ and taking lengths, one obtains

$$\text{length}(Z \cap (L_1 \cup L_2)) \geq \text{length}(Z \cap L_1) + \text{length}(Z \cap L_2) - 1 = (a_1 + 1) + (a_2 + 1) - 1 = n + 1 = \text{length}(Z).$$

Then $Z \subset L_1 \cup L_2$ as schemes.

4.2. Lemma: Let $\mathcal{E}$ be a semistable rank 2 vector bundle on $\mathbb{P}_2$ with Chern class $(c_1, c_2) = (0, n)$, $n \geq 2$. Then

(a) For odd $n \geq 3$ or even $n \geq 6$, $\mathcal{E}$ has only finitely many jumping lines of order $\geq n/2$.

(b) For $n = 2$ or 4, $\mathcal{E}$ has at most one jumping line of order $> n/2$.

Proof. Note, that semistability in this case means that $H^0\mathcal{E}(-1) = 0$, see [20]. Let firstly $\mathcal{E}$ be stable with $H^0\mathcal{E} = 0$. Then [4.1], (a), implies that $\mathcal{E}$ has at most one jumping line of order $> n/2$. Suppose $n \geq 4$ is even. According to [4.1], (b), if $\mathcal{E}$ has two jumping lines $L_1, L_2$ of order $\geq n/2$, then there is an extension

$$0 \to \mathcal{O}_{\mathbb{P}_2}(-1) \to \mathcal{E} \to \mathcal{I}_Z(1) \to 0$$

with $Z \subset L_1 \cup L_2$ of length $n+1$. If $L$ is any other line, then

$$\text{length } (Z \cap L) \leq \text{length } (L \cap (L_1 \cup L_2)) = 2,$$

and it follows that $\mathcal{E}_L \cong \mathcal{O}_L(a) \oplus \mathcal{O}_L(-a)$ with $0 \leq a \leq 1$.

Now assume that $\mathcal{E}$ is properly semistable. In this case we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_2} \to \mathcal{E} \to \mathcal{I}_Z \to 0$$

with a 0-dimensional scheme $Z$ of length $n$. If $L$ is a jumping line of $\mathcal{E}$ order $\geq n/2$, then also length $(Z \cap L) \geq n/2$ by the exact sequence tensored with $\mathcal{O}_L$. If $\mathcal{E}$ should have infinitely many jumping lines of order $\geq n/2$, then there is a point $x \in Z$ such that infinitely many such jumping lines meet $Z$ exactly in $x$. Then length $(\mathcal{O}_{Z,x} \otimes \mathcal{O}_{L,x}) \geq n/2$ for infinitely many lines. It follows that the germ $(Z, x)$ is defined by two equations $f, g \in \mathcal{O}_{\mathbb{P}_2,x}$ ($Z$ is a locally complete intersection) with $\text{mult}(f), \text{mult}(g) \geq n/2$. Then

$$n = \text{length}(Z) \geq \text{length}(\mathcal{O}_{Z,x}) \geq \text{mult}(f)\text{mult}(g).$$

This is not possible for odd $n \geq 3$ or even $n \geq 6$. For $n = 2$ or 4 the same kind of argument as in the last part of the proof of [4.1], (b), shows that $\mathcal{E}$ has at most one jumping line of order $> n/2$. 

□
4.3. Proposition: Let $E$ be an $n$-instanton on $\mathbb{P}^3$, $n \geq 2$. Then the set of jumping lines of $E$ of order $> n/2$ has dimension $\leq 1$.

Proof. Let $G$ denote the Grassmannian of lines in $\mathbb{P}^3$ and consider the incidence diagram

$$G \xleftarrow{p} F \xrightarrow{q} \mathbb{P}^3$$

of lines in planes. Let $\Sigma$ be the set of jumping lines of order $> n/2$. According to 4.2 the projection $q|p^{-1}(\Sigma)$ has finite fibres. Suppose $\dim \Sigma \geq 2$. Then $q(p^{-1}(\Sigma)) = \mathbb{P}^3$. Let then $P_0$ be a stable plane for $E$. It must contain a jumping line $L_0$ of order $> n/2$. For a general plane $P$ containing $L_0$, $E_P$ is stable and, according to 4.1, (a), $L_0$ is the only jumping line of order $> n/2$ contained in $P$. On the other hand, for any $P$ containing $L_0$ the bundle $E_P$ is semistable. Hence, by 4.2, $P$ contains at most finitely many jumping lines of order $> n/2$. Consequently, there are only finitely many jumping lines $L$ of order $> n/2$ meeting $L_0$. This means that $T_{L_0}(G) \cap \Sigma$ is a finite set, where $T_{L_0}(G)$ denotes the geometric tangent hyperplane to $G$ at $L_0$ in $\mathbb{P}^5$, whose intersection with $G$ is the cone of lines meeting $L_0$. This contradicts $\dim \Sigma \geq 2$. \qed

4.4. Remark: One can show, using the method of R. Strano and M. Green, as in [8], that a non special 'tHooft $n$-instanton $E$, $n \geq 5$, satisfies $h^0E_P(1) \leq 1$ for a general plane $P$. Then, using the above arguments, one deduces that the set of jumping lines of $E$ of order $\geq n/2$ is at most 1-dimensional. But this improves 4.2 only for even $n \geq 6$.

From the above statements on jumping lines we can now derive the following proposition, which is the key of the induction process of this paper.

4.5. Proposition: a) Let $E$ be any $n$–instanton. If $n \geq 3$, then for a general $\xi \in H^1E(-1)$ the multiplication map $\xi \otimes V^* \to H^1E$ is injective.

b) If $n \geq 5$, then for a general 2–dimensional subspace $U \subset H^1E(-1)$ the multiplication map $U \otimes V^* \to H^1E$ is injective.

Note, that the condition in a) means that $\text{Im}(\omega) \cap (\xi \otimes V^*) = 0$, see [3] because the multiplication map is isomorphic to $H \otimes V^* \to Q$.

Proof. a) The right part $N \otimes O \to H^* \otimes O(1)$ of the monad of $E$ corresponds to the exact sequence $0 \to N \to H^* \otimes V^* \xrightarrow{\mu} Q \to 0$, where $\mu$ is isomorphic to the multiplication map. If $P \subset \mathbb{P}$ is a plane with equation $z \in V^*$, we obtain the exact sequence

$$0 \to H^0E_P \to N \to H^* \otimes V^*/\langle z \rangle$$

and, therefore,

$$H^0E_P \cong N \cap (H^* \otimes z). \quad (1)$$
On the other hand, if $h^0\mathcal{E}_P \neq 0$, we are given an exact sequence

$$0 \to \mathcal{O}_P \to \mathcal{E}_P \to \mathcal{I}_Z \to 0$$

where $Z$ is a 0–dimensional subscheme of $P$ of length $n$, because $h^0\mathcal{E}_P(-1) = 0$. It follows that

$$0 \leq h^0\mathcal{E}_P \leq 1. \tag{2}$$

We let now $\mathbb{P}(N)_1$ denote the set of decomposable classes $\langle \xi \otimes z \rangle \in \mathbb{P}(H^* \otimes V^*)$ which are contained in $P(N)$, so that

$$\mathbb{P}(N)_1 = \mathbb{P}(N) \cap S_1$$

where $S_1$ is the image of the Segre embedding $\mathbb{P}H^* \times \mathbb{P}V^* \subset \mathbb{P}(H^* \otimes V^*)$. There are the two projections

$$\mathbb{P}H^* \xleftarrow{\pi_1} \mathbb{P}(N)_1 \xrightarrow{q_1} \mathbb{P}V^*.$$

The isomorphism (1) implies that for any $z \neq 0$ we have

$$\mathbb{P}(H^0\mathcal{E}_P) \cong q_1^{-1}(\langle z \rangle),$$

where $P$ is the plane with equation $z$. Therefore, the image of $q_1$ is contained in the subvariety $J\mathcal{P}(E) \subset \mathbb{P}V^*$ of unstable planes of $\mathcal{E}$.

Now (2) implies that $\mathbb{P}(N)_1$ is isomorphic to $J\mathcal{P}(E)$. Because $\dim J\mathcal{P}(E) \leq 2$, see [2], it follows that $p_1(\mathbb{P}(N)_1)$ has dimensions $\leq 2$. This proves a) in case $n \geq 4$. If $n = 3$ and $\mathcal{E}$ is not special 'tHooft, then $\dim J\mathcal{P}(E) \leq 1$ by [4], (4), so that a) is true in that case, too. But in case $n = 3$ and $\mathcal{E}$ is special 'tHooft, the claim follows from remark [10] below.

b) For the proof of part b) we consider the intersection

$$\mathbb{P}(N)_2 = \mathbb{P}(N) \cap S_2$$

with the secant variety of $S_1$ in $\mathbb{P}(H^* \otimes V^*)$, such that $\mathbb{P}(N)_2$ consists of all elements of type $\langle \xi_1 \otimes z_1 + \xi_2 \otimes z_2 \rangle$ which are contained in $\mathbb{P}(N)$. Then we have the two projections

$$G(2, H^*) \xleftarrow{\pi_2} \mathbb{P}(N)_2 \xrightarrow{q_2} G(2, V^*).$$

We have to show that a general 2–dimensional subspace $K \subset H^*$ satisfies $N \cap (K \otimes V^*) = 0$.

Now $N \cap (K \otimes V^*) \neq 0$ if and only if either $\mathbb{P}(K) \cap p_1(\mathbb{P}(N)_1) \neq \emptyset$ or $K \in \text{Im}(p_2)$.

Because $\dim p_1(\mathbb{P}(N)) \leq 2$ by part a), it follows that the set of $K$ with $\mathbb{P}(K) \cap p_1(\mathbb{P}(N)_1) \neq \emptyset$ has dimension $\leq 2 + (n - 2) = n$. By assumption, $\dim G(2, H^*) = 2(n - 2) > n + 1$ for $n \geq 6$ and $= n + 1$ for $n = 5$. Therefore, it remains to prove that $\dim \text{Im}(p_2) \leq n + 1$ for $n \geq 6$ and $\dim \text{Im}(p_2) \leq 5$ for $n = 5$.

In order to derive the estimates, we consider the fibres of $q_2$. Let $W \subset V^*$ be any 2–dimensional subspace with the dual line $L \subset \mathbb{P}V$. Then

$$N \cap (H^* \otimes W) \cong H^0\mathcal{E}_L$$
as can easily be derived from the monad description of $\mathcal{E}$. On the other hand,

$$q_2^{-1}(W) = \mathbb{P}(N \cap H^* \otimes W) \setminus \mathbb{P}(N)_1$$

by definition of $\mathbb{P}(N)_2$. Therefore,

$$\dim q_2^{-1}(W) = h^0\mathcal{E}_L - 1.$$  \hfill (3)

Let $G(2, V^*) \supset J_1 \supset J_2 \supset \cdots \supset J_n$ be the filtration by the sets $J_k$ of jumping lines $L$ of order $h^0\mathcal{E}_L - 1 \geq k$. It follows that

$$\dim q_2^{-1}(J_k \setminus J_{k+1}) \leq 3 + k \leq n$$

for $k \leq n - 3$. For $k = n - 2$

$$\dim q_2^{-1}(J_{n-2} \setminus J_{n-1}) \leq 1 + (n - 2) = n - 1$$

because of Lemma 4.3 since $n - 2 > n/2$, and finally

$$\dim q_2^{-1}(J_{n-1} \setminus J_n) \leq n \quad \text{and} \quad \dim q_2^{-1}(J_n) \leq n + 1.$$  

Totally we have $\dim (\mathbb{P}(N)_2 \setminus \mathbb{P}(N)_1) \leq n + 1$, which is sufficient for $n \geq 6$. In case $n = 5$, $\dim (\mathbb{P}(N)_2 \setminus \mathbb{P}(N)_1) = 6$ is only possible if $\dim J_5 = 1$. In that case $\mathcal{E}$ is a special 'tHooft bundle, and it follows by the direct argument in remark 4.6, that there are 2–dimensional subspaces $K \subset H^*$ with $N \cap (K \otimes V^*) = 0$. This proves b).  

4.6. Remark: Let $\mathcal{E}$ be an $(n, 2)$–instanton on $\mathbb{P}^3$ with $n \geq 2m + 2 \geq 3$. One can prove, along the same lines, that then, for a general $m$–dimensional subspace $K \subset H^1\mathcal{E}(-1)$ the multiplication map $K \otimes V^* \to H^1\mathcal{E}$ is injective. If $\mathcal{E}$ is a special 'tHooft bundle, this can be verified by the following direct argument. Let $z_0, \ldots, z_3$ be a basis of $V^*$. Then bases of $N$ and $H^*$ can be chosen such that the operator $N \to H^* \otimes V^*$, i.e. the right part of the monad of $\mathcal{E}$, can be represented by the matrix

$$\begin{pmatrix}
  z_0 & z_1 & z_2 & z_3 \\
  z_0 & z_1 & z_2 & z_3 \\
  \vdots & \vdots & \vdots & \vdots \\
  z_0 & z_1 & z_2 & z_3
\end{pmatrix}.$$  

If $n = 2m + 1$ or $2m + 2$ and $e_1, \ldots, e_n$ is the corresponding basis of $H^*$, then $K = \text{span}(e_2, e_4, \ldots, e_{2m})$ satisfies $N \cap (K \otimes V^*) = 0$.  

For later use, we consider the space $\Pi(\xi) \subset G(2, H^*)$ of all 2–dimensional subspaces $K \subset H^*$ which contain $\xi$. We have:

4.7. Lemma: Let $n = 5$ and $\omega \in M(5, 2)$ such that $\mathcal{E}_\omega$ is not special 'tHooft. Then for a general $\xi \in H^*$ there are closed subsets $T_1 \subset T_2 \subset \Pi(\xi)$, $\dim T_1 \leq 1$ and $T_2$ a surface, such that
(i) \( N \cap (K \otimes V^*) = 0 \) for \( K \in \Pi(\xi) \setminus T_2 \);
(ii) \( \dim N \cap (K \otimes V^*) = 1 \) and \( N \cap (K \otimes V^*) \) contains no non–zero decomposable vector of \( H^* \otimes V^* \) for \( K \in T_2 \setminus T_1 \).

Proof. We use the previous notation and consider the morphisms

\[
P(N)_1 \xrightarrow{p_1} \mathbb{P}(H^*) \quad \text{and} \quad \mathbb{P}(N)_2 \setminus \mathbb{P}(N)_1 \xrightarrow{p_2} G(2, H^*) =: G.
\]

a) Let \( Y_1 \subset G \) denote the subset of those \( K \in G \) for which \( N \cap (K \otimes V^*) \) contains a non–zero decomposable vector or, equivalently, \( \mathbb{P}(K) \cap p_1(\mathbb{P}(N)_1) \neq \emptyset \). Because \( \dim p_1(\mathbb{P}(N)_1) \leq 1 \), the set of lines \( \mathbb{P}(K) \) in \( \mathbb{P}(H^*) \) with this condition is closed and of dimension \( \leq 4 \).

b) Let \( Y_2 \subset G \) be the set of all \( K \subset G \) for which \( N \cap (K \otimes V^*) \neq 0 \). If \( K \subset H^* \otimes \mathcal{O}_G \) denotes the universal subbundle, we consider the homomorphism

\[
K \otimes V^* \xrightarrow{\phi} Q \otimes \mathcal{O}_G,
\]

obtained as the composition of \( K \otimes V^* \subset H^* \otimes V^* \otimes \mathcal{O}_G \) and the multiplication map \( H^* \otimes V^* \rightarrow Q \) with kernel \( N \). Then \( Y_2 \) is the determinantal subvariety of \( \phi \). Because, for a general \( K \) we have \( N \cap (K \otimes V^*) = 0 \) by Proposition 4.5, \( Y_2 \) is a hypersurface in \( G \), \( \dim Y_2 = 5 \). By definition of \( Y_2 \) we have \( \text{Im}(p_2) \subset Y_2 \), and, furthermore,

\[
Y_2 = Y_1 \cup \text{Im}(p_2).
\]

c) For \( K \in Y_2 \setminus Y_1 \) the fibre \( p_2^{-1}(K) \) consists of vectors \( \xi_1 \otimes z_1 + \xi_2 \otimes z_2 \in N \), where \( \xi_1, \xi_2 \) is a basis of \( K \) and \( z_1, z_2 \in V^* \), i.e.

\[
p_2^{-1}(K) \cong \mathbb{P}(N \cap (K \otimes V^*)).
\]

Now, by the proof of Proposition 4.3, \( p_2(\mathbb{P}(N)_2 \setminus \mathbb{P}(N)_1) \) has dimension \( \leq 5 \). It follows that, for a general \( K \) in each component of \( Y_2 \setminus Y_1 \), the fibre \( p_2^{-1}(K) \) is a point, or, equivalently, \( N \cap (K \otimes V^*) \) is 1–dimensional. Then the subvariety \( Y'_2 \subset Y_2 \), defined by \( \dim N \cap (K \otimes V^*) \geq 2 \), has dimensional \( \leq 4 \). Let \( Z_1 = Y_1 \cup Y'_2 \). Then \( \dim Z_1 \leq 4 \) and for \( K \in Y_2 \setminus Z_1 \) the intersection \( N \cap (K \otimes V^*) \) is 1–dimensional and contains no non–zero decomposable vector.

d) Because \( \dim Z_1 \leq 4 \), the general 3–space \( \Pi(\xi) \) meets \( Z_1 \) at most in dimension 1. Let, now, \( T_1 = \Pi(\xi) \cap Z_1 \) and \( T_2 = \Pi(\xi) \cap Y_2 \). We may assume that \( \Pi(\xi) \not\subset Y_2 \) by Proposition 4.3, b), so that \( T_2 \) is a surface in \( \Pi(\xi) \). Then \( T_1 \subset T_2 \) satisfy the properties of the lemma. \( \square \)
5. The varieties $M(n, r)$ for $n \leq 4$

We first note that $M(n, 2n)$ is an open subset of the affine space $S^2 H^* \otimes \Lambda^2 V^*$. In fact, it is the complement of the hypersurface $M_{4n-2}$. In this case, if $\omega \in M(n, 2n)$, the bundle $E_\omega$ is the cokernel in

$$0 \to H \otimes O_P(-1) \to H^* \otimes \Omega^1(1) \to E_\omega \to 0,$$

see 2.5 with $Q = 0$. Then $H^1 E_\omega(i) = 0$ for $i \geq 0$ and we have

$$M^0(n, 2n) = M(n, 2n).$$

5.1. Proposition: For $n = 2$, $M_6 = M_{4n-2}(H)$ is an irreducible hypersurface in $S^2 H^* \otimes \Lambda^2 V^*$.

Proof. Note that $M_6 \setminus (M_4 \cup \Delta)$ had been shown in 2.11 to be a smooth hypersurface, and $M_6$ is a homogeneous hypersurface in $S^2 H^* \otimes \Lambda^2 V^*$. In order to prove that it is irreducible, it suffices to prove that it is non–singular in codimension 1. Now $\Delta = \Delta(H_2)$ is closed and irreducible in $S^2 H^*_2 \otimes \Lambda^2 V^*$ of codimension 2. We have $M_4 \subset \Delta$ and hence $M_6 \setminus \Delta$ is smooth. It is therefore sufficient to find an $\omega \in \Delta \setminus M_4$ which is a smooth point of $M_6$. For that choose an isomorphism $H_2 \cong k^2$ and let $H_2 \twoheadrightarrow H_2^* \otimes \Lambda^2 V^*$ be presented by the matrix

$$\begin{pmatrix} x_2 \wedge x_3 & x_2 \wedge x_4 \\ x_2 \wedge x_4 & x_1 \wedge x_2 + x_3 \wedge x_4 \end{pmatrix}$$

where $x_1, \ldots, x_4$ is a basis of $V^*$. One can easily check that this $\omega$ is degenerate at the point $x_2 = x_3 = x_4 = 0$ and that $\text{rk}(\omega) = 6$. Moreover, $\omega$ is a smooth point of $M_6$, using the argument of Katsylo–Ottaviani as in the proof of Proposition 2.11.

5.2. Corollary 1: (Hartshorne) $M(I(2))$ is smooth and irreducible of the expected dimension 13.

Proof. $M(2, 2) = M_6 \setminus (M_4 \cup \Delta)$ is a smooth transversal intersection and irreducible. By 2.9 the results follows.

5.3. Corollary 2: (Ellingsrud - Strømme) $M(I(3))$ is smooth and irreducible of the expected dimension 21.

Proof. As noted above $M^0(2, 4) = M(2, 4)$, and this is irreducible. By Proposition 4.5 $M(3, 2)$ is the union of the open sets $M(3, 2)_\xi$ which are equal to $M(3, 2)'_\xi$ and which are transversal intersections and irreducible. The result follows now as in the previous case.
In order to treat the case $MI(4)$, we prove the following

5.4. Lemma: For $n = 3$, $M_{10} = M_{4n-2}(H)$ is an irreducible hypersurface in $S^2H^* \otimes \Lambda^2V^*$.

Proof. By 2.11 $M_{10}$ is a hypersurface in $S^2H^*_3 \otimes \Lambda^2V^*$ and $M(3,4) = M_{10} \setminus (\Delta \cup M_8)$ is smooth. Because for $n = 3$ we have $M_6 \subset \Delta$, $M(3,2) = M_8 \setminus \Delta$ and hence $\Delta \cup M_8 = \Delta \cup M(3,2)$. Now $\Delta$ is irreducible in $S^2H^*_3 \otimes \Lambda^2V^*$ of codimension 4, see remark in 2.1, and $M(3,2)$ has codimension 6 in $S^2H^*_3 \otimes \Lambda^2V^*$ by corollary 5.3. It follows that $M_{10}$ is smooth in codimension 1 and so $M_{10}$ is irreducible.

5.5. Corollary: $M(3,4)$ is irreducible and a (smooth) transversal intersection of expected dimension 45.

5.6. Proposition: (Barth, LePotier) $MI(4)$ is smooth and irreducible of the expected dimension 29.

Proof. 1) Recall that the open subsets $M(4,2)_{\xi}' \subset M(4,2)$ are defined by

$$M(4,2)_{\xi}' = \{ \omega \in M(4,2) \mid \bar{\omega} \in M(3,4) \text{ and } H^1E_\omega(1) = 0 \},$$

where $\bar{\omega} = \text{res}_\xi(\omega)$. In this proof we write $F = E_\omega$, which depends on the choice of $\xi$. By 3.3 $M(4,2)_{\xi}'$ is irreducible. It is also smooth: by 2.8.1 $H^2S^2F = 0$ for $\bar{\omega} \in M(3,4)$ because $M(3,4)$ is a smooth transversal intersection of the expected (co)dimension. It follows from Lemma 3.2 that also $H^2S^2E_\omega = 0$ for $\omega \in M(4,2)_{\xi}'$, hence again by 2.8.1 $M(4,2)_{\xi}'$ is smooth at any of its points and of expected dimension.

2) It suffices now to prove that

$$M(4,2) = \bigcup_{\xi \neq 0} M(4,2)_{\xi}'.$$

Let $\omega \in M(4,2)$. By 3.3 there is $\xi \in H^1E_\omega(-1)$ such that $H^1F_P(1) = 0$ for any plane $P$ in $\mathbb{P}^3$, provided $E_\omega$ is not special 'tHooft. In case $n = 4$ we obtain $h^1F = 2$. Then the operator $H^1F \otimes V^* \rightarrow H^1F(1)$ can be presented by a matrix

$$A = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ v_{21} & \cdots & v_{2m} \end{pmatrix}$$

with $m = h^1F(1)$ if $h^1F(1) \neq 0$. Let now $P$ be a plane with equation $z = 0$ and such that $z(v_{11}) = z(v_{21}) = 0$. Then

$$H^1F \xrightarrow{A(z)} H^1F(1)$$
cannot be surjective and it would follow that $H^1 \mathcal{F}_P(1) = \text{coker} A(z) \neq 0$, a contradiction. This proves that $\omega \in M(4,2|z)$ for the chosen $\xi$. If, however, $\mathcal{E}_\omega$ is special ’tHooft, the pairing $H^1 \mathcal{E}_\omega(-1) \otimes V^* \to H^1 \mathcal{E}_\omega$ can be presented by a $4 \times 6$ matrix

$$
\begin{pmatrix}
 v_1 v_2 \\
 v_3 v_4 & v_1 v_2 \\
 v_3 v_4 & v_1 v_2 \\
 v_3 v_4 \\
\end{pmatrix},
$$

where $v_1, \ldots, v_4$ is a basis of $V$, see \[\text{3}\]. Then, choosing $\xi = (0, 1, 0, 0) \in k^4 \cong H^1 \mathcal{E}_\omega(-1)$, the resulting homomorphism $H^1 \mathcal{F}(-1) \otimes V^* \to H^1 \mathcal{F}$ is presented by the matrix

$$
A = \begin{pmatrix}
 0 & 0 \\
 v_1 & v_2 \\
 v_3 & v_4 \\
\end{pmatrix}.
$$

It follows that $H^1 \mathcal{F}(1) = 0$ because of the exact sequence $H^1 \mathcal{F}(-1) \otimes \Lambda^2 V^* \to H^1 \mathcal{F} \otimes V^* \to H^1 \mathcal{F}(1) \to 0$.

5.7. Remark: Concerning $MI(5)$ and $M(4, 4)$, it follows from \[\text{13}\] that the open set

$$
\bigcup_{\xi \neq 0} M(4, 4|\xi)
$$

is irreducible and a (smooth) transversal intersection of expected dimension, because $M^0(3, 6) = M(3, 6)$ is smooth of expected dimension. This will suffice to prove that $MI(5)$ is smooth and irreducible of dimension 37. Using the method of Katsylo–Ottaviani as in the proof of \[\text{2.11}\], one can show that $M(4, 4)$ is smooth of the expected dimension. It is an open question whether it is also irreducible.

6. A Technical Result about 5–Instantons

6.1. Proposition: Let $\mathcal{E}_\omega$ be a 5–instanton on $\mathbb{P}_3$. Then for a general $\xi \in H^1 \mathcal{E}_\omega(-1)$ the associated rank–4 bundle $\mathcal{F} = \mathcal{E}_\omega$ satisfies $h^1 \mathcal{F}(1) \leq 1$.

For the proof we need the following lemmata on the vanishing of $H^1 \mathcal{F}_L(1)$ and $H^1 \mathcal{F}_P(1)$ for lines and planes.

6.2. Lemma: Let $\mathcal{E}_\omega$ be a n–instanton, $n = 4$ or 5, and assume that $\mathcal{E}_\omega$ is not a special ’tHooft bundle ($h^0 \mathcal{E}_\omega(1) < 2$). Then, for a general $\xi \in H^1 \mathcal{E}_\omega(-1)$, the bundle $\mathcal{F} = \overline{\mathcal{E}}_\omega$ has the property that $H^1 \mathcal{F}_L(1) = 0$ for any line $L$ in $\mathbb{P}_3$. 
Proof. Let $\mathcal{E} = \mathcal{E}_\omega$. For any plane $P$ in $\mathbb{P}_3$ we have $H^1\mathcal{E}(-1) \xrightarrow{\sim} H^1\mathcal{E}_P(-1)$. Then for any line $L \subset P$ the restriction $H^1\mathcal{E}(-1) \to H^1\mathcal{E}_L(-1)$ is surjective because $H^1\mathcal{E}_P(-1) \to H^1\mathcal{E}_L(-1)$ is surjective, since $H^1\mathcal{E}_P(-2) = 0$. Because $\mathcal{E}_L$ is the cohomology of the monad

$$0 \to \mathcal{O}_L(-1) \to \mathcal{F}_L \to \mathcal{O}_L(1) \to 0,$$

we obtain

$$H^1\mathcal{F}_L(1) \cong H^1\mathcal{E}_L(1)/\xi_L \cdot H^0\mathcal{O}_L(2),$$

where $\xi_L$ denotes the restriction of $\xi$ in $H^1\mathcal{E}_L(-1)$. On the other hand, $\mathcal{E}_L \cong \mathcal{O}_L(a) \oplus \mathcal{O}_L(-a)$, $0 \leq a \leq n$. If $0 \leq a \leq 2$, then $H^1\mathcal{E}_L(1) = 0$. If $a = 3$, $H^1\mathcal{E}_L(-1) \cong H^1\mathcal{O}_L(-4)$ and $H^1\mathcal{E}_L(1) \cong H^1\mathcal{O}_L(-2)$. Then for $\eta \in H^1\mathcal{O}_L(-4)$ we have $\eta \cdot H^0\mathcal{O}_L(2) \neq H^1\mathcal{O}_L(-2)$ if and only if $\eta = 0$. Consequently, for the vanishing of $H^1\mathcal{F}_L(1)$ we have only to assume that $\xi_L \neq 0$ or equivalently that

$$\xi \not\in \text{Ker}(H^1\mathcal{E}(-1) \to H^1\mathcal{E}_L(-1)),$$

which is 1– or 2–dimensional. Since by [4,3] the set of jumping lines of $\mathcal{E}$ of order $\geq 3$ is at most 1–dimensional, $\xi$ has to avoid a subvariety of dimension $\leq 2$ or 3. If $a = 4$, the elements $\xi$ or $\eta \in H^1\mathcal{O}_L(-5)$ should avoid the condition $\eta H^0\mathcal{O}_L(2) \neq H^1\mathcal{O}_L(-3)$. The set of these $\eta$ is the affine cone over the rational normal curve in $\mathbb{P}H^1\mathcal{O}_L(-5)$. Namely, if $s, t$ are homogeneous coordinates on $L$, we have

$$\eta = \sum_{\nu=1}^{4} \frac{a_{\nu}}{s^{5-4\nu}} \quad \text{in} \quad H^1\mathcal{O}_L(-5)$$

and

$$s^2\eta = \frac{a_1}{s^2t} + \frac{a_2}{st^2}, \quad st\eta = \frac{a_2}{s^2t} + \frac{a_3}{st^3}, \quad t^2\eta = \frac{a_3}{s^2t} + \frac{a_4}{st^2}.$$ 

Then the condition $\eta H^0\mathcal{O}_L(2) \neq H^1\mathcal{O}_L(-3)$ becomes

$$\text{rk} \left( \begin{array}{ccc} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{array} \right) \leq 1.$$ 

So $\xi$ has to avoid another 3–dimensional subvariety in $H^1\mathcal{E}(-1)$. Finally, if $a = 5$, in case $n = 5$ only, the set of $\eta \in H^1\mathcal{O}_L(-6)$ with $\eta H^0\mathcal{O}_L(2) \neq H^1\mathcal{O}_L(-4)$ is a cubic hypersurface in $H^1\mathcal{O}_L(-6)$, as can be seen by a similar argument. So $\xi$ has to avoid this hypersurface in $H^1\mathcal{E}(-1) \cong H^1\mathcal{E}_L(-1)$ in case $a = 5$. But $\mathcal{E}$ has only finitely many jumping lines of order 5 because it is not special 'tHooft, see also [24], [22]. Totally $\xi$ has to avoid two 3–dimensional and finitely many 4–dimensional subvarieties in $H^1\mathcal{E}(-1)$, in case $n = 5$, in order to satisfy the condition of the lemma. \hfill $\square$

6.3. Lemma: Let $\mathcal{E} = \mathcal{E}_\omega$ be as in the previous lemma. Then, for a general $\xi \in H^1\mathcal{E}_\omega(-1)$, the bundle $\mathcal{F} = \mathcal{E}_\omega$ has the property that $H^1\mathcal{F}_P(1) = 0$ for any plane $P$ in $\mathbb{P}_3$. 
Proof. By 12 we may assume that $H^1(\mathcal{F}_L)(1) = 0$ for any line $L$ if $\xi$ is in a fixed open set of $H^1(\mathcal{E})(-1)$. We let $\xi_P$ denote the element corresponding to $\xi$ under the isomorphism $H^1(\mathcal{E})(-1) \cong H^1(\mathcal{E}_P)(-1)$ for a plane $P$. We are going to show that $H^1(\mathcal{F}_P)(1) = 0$ for any plane if $\xi$ avoids some additional subvarieties of $H^1(\mathcal{E})(-1)$. These subvarieties will be estimated in dimension in the following cases.

**case 1:** $P$ is a stable plane of $\mathcal{E}$. In this case there is no condition on $\xi$ because we show that then already $H^1(\mathcal{F}_P)(1) = 0$.

Proof of case 1: We prove first that $h^1(\mathcal{F}_P) \leq 2$. As in the previous proof we have the exact sequence

$$\xi_P \otimes H^0(\mathcal{O}_P)(1) \xrightarrow{m(\xi)} H^1(\mathcal{E}_P) \rightarrow H^1(\mathcal{F}_P) \rightarrow 0.$$ 

Because $h^0(\mathcal{E}_P) = 0$, we have $h^1(\mathcal{E}_P) = n - 2$. So, if $n = 4$, then $h^1(\mathcal{F}_P) \leq 2$. If $n = 5$, the homomorphism $m(\xi)$ is zero if and only if $\xi_P$ is mapped to zero under the map $H^1(\mathcal{E}_P)(-1) \rightarrow W \otimes H^1(\mathcal{E}_P)$, where $W^* = H^0(\mathcal{O}_P)(1)$, and the kernel of this map is the image of $\text{Hom}(\Omega^1_P(1), \mathcal{E}_P)$ in $\text{Ext}^1(\mathcal{O}_P(1), \mathcal{E}_P) \cong H^1(\mathcal{E}_P)(-1)$. Therefore, $\xi_P$ is induced by a non–zero homomorphism $\Omega^1_P(1) \rightarrow \mathcal{E}_P$. Because both bundles are stable, $\varphi$ is generically and then globally injective. Then $\mathcal{C} = \text{coker}(\varphi)$ has Hilbert polynomial $\chi(\mathcal{C}) = m - 3$ and is Cohen–Macaulay. It follows that $\mathcal{C} = \mathcal{O}_L(-4)$ for a line $L \subset P$. If follows that $L$ is a jumping line of order 4 and that the restricted homomorphism $\Omega^1_P \otimes \mathcal{O}_L \rightarrow \mathcal{E}_L(-1)$ factors through $\mathcal{O}_L(3)$. Then the diagram

$$\begin{array}{ccc}
H^1(\Omega^1_P) & \longrightarrow & H^1(\mathcal{E}_L)(-1) \\
\downarrow & & \downarrow \\
H^1(\Omega^1_P \otimes \mathcal{O}_L) & \longrightarrow & H^1(\mathcal{E}_L)(-1)
\end{array}$$

implies that the image $\xi_L$ of $\xi$ in $H^1(\mathcal{E}_L)(-1)$ is zero, and $H^1(\mathcal{F}_L)(1) \cong H^1(\mathcal{F}_L)(1)/\xi_LH^0(\mathcal{O}_L)(2) \cong H^1(\mathcal{O}_L)(-3) \neq 0$ contradicting Lemma 12. This proves that $h^1(\mathcal{F}_P) \leq 2$ if $\xi$ is general.

Let now $L \subset P$ be any line with equation $z$. By the assumption on $\xi$ the multiplication map $H^1(\mathcal{F}_P) \rightarrow H^1(\mathcal{F}_P)(1)$ is surjective. Applying the bilinear map lemma of H. Hopf to

$$H^1(\mathcal{F}_P)(1)^* \otimes H^0(\mathcal{O}_P)(1) \rightarrow H^1(\mathcal{F}_P)^*,$$

we deduce that

$$h^1(\mathcal{F}_P)(1) \leq h^1(\mathcal{F}_P) - h^0(\mathcal{O}_P)(1) + 1 \leq 0.$$ 

(when $K$ denotes the kernel of the map, the above surjectivity condition implies that $\mathbb{P}H^1(\mathcal{F}_P)(1)^* \times \mathbb{P}H^0(\mathcal{O}_P)(1)$ has an empty intersection with $\mathbb{P}K$ in $\mathbb{P}(H^1(\mathcal{F}_P)(1)^* \otimes H^0(\mathcal{O}_P)(1))$, which implies the estimate).

**case 2:** $P$ is an unstable plane of $\mathcal{E}$ but contains no jumping line of order 5.
In this case we have an exact sequence
\[ 0 \to \mathcal{O}_P \to \mathcal{E}_P \to \mathcal{I}_{Z,P} \to 0 \]
where \( Z \) is a 0–dimensional subscheme of \( P \) of length \( n \) with \( H^0\mathcal{I}_{Z,P}(1) = 0 \), because \( P \) contains no jumping line of order \( n \). It follows that \( h^0\mathcal{E}_P = 1 \), \( h^0\mathcal{E}_P(1) = 3 \) and then \( h^1\mathcal{E}_P = n - 1 \), \( h^1\mathcal{E}_P(1) = n - 3 \). If \( h^1\mathcal{F}_P \leq 2 \), then, as in case 1, \( h^1\mathcal{F}_P(1) = 0 \). If \( h^1\mathcal{F}_P \geq 3 \), then the multiplication map \( m(\xi) \), see case 1, has rank \( n - 4 \). If \( n = 4 \), then \( m(\xi) = 0 \), and we get a contradiction by the argument in case 1, which leads to \( h^1\mathcal{F}_P \leq 2 \). If \( n = 5 \), then \( m(\xi) \) has rank 1 and \( \xi_P \) is annihilated by two linear forms \( z_0, z_1 \in H^0\mathcal{O}_P(1) \). Let \( x \in P \) be the point determined by \( z_0, z_1 \) and let \( \mathcal{I}_{(x)}, P \) be its ideal sheaf. The standard resolution of this sheaf yields the exact sequence
\[ 0 \to H^0(\mathcal{E}_P)^2 \to H^0(\mathcal{I}_{(x)}, P \otimes \mathcal{E}_P(1)) \to H^1\mathcal{E}_P(1) \xrightarrow{z_0, z_1} H^1(\mathcal{E}_P)^2. \]
On the other hand, the defining sequence of \( \mathcal{I}_{Z,P} \) and \( H^0\mathcal{O}_P(1) \cong H^0\mathcal{E}_P(1) \) implies
\[ h^0(\mathcal{I}_{(x)}, P \otimes \mathcal{E}_P(1)) = \begin{cases} 2 & \text{if } x \in P \setminus Z \\ 3 & \text{if } x \in Z \end{cases}. \]
Therefore, \( h^1\mathcal{F}_P \geq 3 \) can only occur if \( \xi_P \) avoids at most five 1–dimensional vector subspaces of \( H^1\mathcal{E}_P(-1) \). Because \( \mathcal{E} \) has only a 1–dimensional variety of unstable planes, it follows that \( h^1\mathcal{F}_P \leq 2 \) and \( h^1\mathcal{F}_P(1) = 0 \) for any unstable plane, if \( \xi \) avoids an at most 2–dimensional subvariety of \( H^1\mathcal{E}(-1) \).

**case 3:** \( P \) is an unstable plane of \( \mathcal{E} \) and contains a jumping line \( L \) of order \( n \).

In this case \( H^1\mathcal{E}_P(1) \xrightarrow{\sim} H^1\mathcal{E}_L(1) \) and \( h^1\mathcal{E}_P(1) = n - 2 = h^1\mathcal{E}_L(1) \), such that \( H^1\mathcal{E}_P(1) \xrightarrow{\sim} H^1\mathcal{E}_L(1) \). It follows that also \( H^1\mathcal{F}_P(1) \xrightarrow{\sim} H^1\mathcal{F}_L(1) = 0 \).

**Proof of proposition 6.1:** Let us assume, firstly, that \( \mathcal{E} = \mathcal{E}_\omega \) is not special ’tHooft. By 4.3 we may assume that the multiplication map \( \xi \otimes V^* \to H^1\mathcal{E} \) is injective, hence \( h^1\mathcal{F} = 4 \), \( \mathcal{F} = \mathcal{E}_\omega \). By 6.3 we may assume that \( H^1\mathcal{F}_P(1) = 0 \) for any plane. Therefore, the multiplication map \( H^1\mathcal{F} \to H^1\mathcal{F}(1) \) is surjective for any linear form. Now an application of the bilinear lemma of H. Hopf to
\[ H^1(\mathcal{F}(1))^* \otimes V^* \to H^1(\mathcal{F})^* \]
implies \( h^1(\mathcal{F}(1)) \leq h^1\mathcal{F} - 4 + 1 = 1 \). If \( \mathcal{E} \) is a special ’tHooft bundle, it can even be shown that for a general \( \xi \in H^1\mathcal{E}(-1) \) we have \( H^1\mathcal{F}(1) = 0 \). If \( \mathcal{E} \) is special ’tHooft, then the pairing \( V^* \otimes H^1\mathcal{E}(-1) \to H^1\mathcal{E} \) can be defined by a \( 5 \times 8 \) matrix
\[
\begin{pmatrix}
v_1 & v_2 & v_3 & v_4 & v_1 & v_2 & v_3 & v_4 \\
v_1 & v_2 & v_3 & v_4 & v_1 & v_2 & v_3 & v_4
\end{pmatrix}
\]
where \( v_1, \ldots, v_4 \) is a basis of \( V \), after choosing suitable bases of the cohomology spaces, see [6]. Then, choosing \( \xi = (0, 0, 1, 0, 0) \) in \( k^5 \cong H^1E(-1) \), the resulting homomorphism \( V^* \otimes H^1F(-1) \to H^1F \) is described by the matrix

\[
\begin{pmatrix}
v_1v_2 & 0 \\
v_3v_4 & v_1v_2 \\
0 & v_3v_4
\end{pmatrix}
\]

which describes at the same time the right part \( 4\Omega^1(1) \to 4\mathcal{O} \) of the Beilinson II monad of \( F \). It follows that \( h^1F(1) = 0 \) in this case.

7. Irreducibility of \( MI(5) \)

In this section the irreducibility of \( MI(5) \) is proved. The proof is mainly based on the properties of the sets \( M(4, 4) \) and the fact that for any \( E_\omega \in MI(5) \) there is a \( \xi \in H^1E_\omega(-1) \) such that \( (4, 4) \)-instanton \( E_\omega \) satisfies \( h^1E_\omega(1) \leq 1 \). The plan of the proof is the following. Let us recall the notations

\[
M(4, 4) = \{ \omega \in S^2H_4^* \otimes \Lambda^2V^* \mid \text{rk}(\omega) = 12, \ \omega \text{ non–degenerate} \}
\]

and

\[
M^0(4, 4) = \{ \omega \in M(4, 4) \mid H^1E_\omega(1) = 0 \}.
\]

In addition we need a partial completion of \( M(4, 4) \), by forgetting the non–degeneracy,

\[
\widetilde{M}(4, 4) := \{ \omega \in S^2H_4^* \otimes \Lambda^2V^* \mid \text{rk}(\omega) = 12 \}
\]

and for \( \xi \in H^*_4 \), consistent with the earlier notation,

\[
\widetilde{M}(4, 4)_\xi := \{ \omega \in \widetilde{M}(4, 4) \mid \text{Im}(\omega) \cap (\xi \otimes V^*) = 0 \}.
\]

We have

\[
M(4, 4)_\xi = M(4, 4) \cap \widetilde{M}(4, 4)_\xi \quad M^0(4, 4)_\xi = M^0(4, 4) \cap \widetilde{M}(4, 4)_\xi
\]

The space \( \widetilde{M}(4, 4) \) is introduced for technical reasons which become apparent in Lemma 7.1. For an element \( \omega \in \widetilde{M}(4, 4) \) we only get a sheaf \( E_\omega \) from the monad construction 2.2. There is the morphism

\[
\rho = \text{res}_\xi : \widetilde{M}(4, 4)_\xi \to M(3, 6) = M^0(3, 6)
\]

assigning to \( \omega \) the map \( \tilde{H}_3 \otimes V \xrightarrow{\rho} \tilde{H}_3^* \otimes V^* \), where \( \tilde{H}_3 = \ker(\xi) \). Because \( \text{rk}(\omega) = 12 \), this map is an isomorphism and hence non–degenerate. We are going to prove
7.1. Lemma: \( \widetilde{M}(4,4)_\xi \xrightarrow{\xi} M(3,6) \) is an affine bundle of dimension 54 and of fibre dimension 18.

Because \( M(3,6) \) is an open part of \( S^2 \hat{H}_2^* \otimes \Lambda^2 V^* \), it is smooth and irreducible of dimension 36, and therefore \( \widetilde{M}(4,4)_\xi \) is smooth and irreducible of dimension 54. Then also the open subset \( M(4,4)_\xi \) of \( \widetilde{M}(4,4)_\xi \) is smooth and irreducible of dimension 54.

Remark: Irreducibility and smoothness of \( M(4,4)_\xi \) is already proved \( \text{[5.7]} \). The bundle structure and the partial completion will be used to prove

7.2. Lemma: For any \( \xi \in H^*_1 \setminus \{0\} \), \( \dim(M(4,4)_\xi \setminus M^0(4,4)_\xi) \leq \dim M(4,4)_\xi - 2 \).

Assuming the two lemmata, the irreducibility of \( MI(5) \) is achieved with the following arguments. The open part

\[
W := \bigcup_{\eta \neq 0} M(4,4)_\eta
\]

of \( M(4,4) \) is smooth and irreducible by \( \text{[3.4]} \). We let

\[
W^0 := \{ \omega \in W \mid H^1 E_\omega(1) = 0 \} = \bigcup_{\eta \neq 0} M^0(4,4)_\eta.
\]

and

\[
W^1 := \{ \omega \in W \mid h^1 E_\omega(1) \leq 1 \}.
\]

Then \( W^0 \subset W^1 \subset W \) are open subsets. It follows from Lemma \( \text{[7.2]} \), that

\[
\text{codim}(W^1 \setminus W^0) \leq \dim W^1 - 2. \tag{1}
\]

Now we consider the morphisms \( \omega \mapsto \bar{\omega} = \text{res}_\xi \omega \)

\[
M(5,2)_\xi \to M(4,4)
\]

and the inverse images under \( r = \text{res}_\xi \),

\[
U^1_\xi \xrightarrow{r_1} W^1 \quad \bigcup \quad U^0_\xi \xrightarrow{r_0} W^0.
\]

By Proposition \( \text{[6.1]} \), Lemma \( \text{[3.1]} \), and Proposition \( \text{[4.5]} \), any element \( \omega \in M(5,2) \) is contained in one of the open sets \( U^1_\xi \), i.e.

\[
M(5,2) = \bigcup_{\xi \neq 0} U^1_\xi,
\]

but we don’t know whether the same holds true for the sets \( U^0_\xi \). By \( \text{[3.3]} \) the fibres of \( r_1 \) and \( r_0 \) are open sets of a linear space and their dimension are

\[
\dim r_1^{-1}(\bar{\omega}) = 4 + h^0 E_\omega(1) \leq 9 \\
\dim r_0^{-1}(\bar{\omega}) = 4 + h^0 E_\omega(1) = 8.
\]
Because the difference of the fibre dimensions is at most 1, the estimate (1) implies
\[ \dim(U^1_\xi \setminus U^0_\xi) < \dim U^0_\xi \] (2)
for any \( \xi \neq 0 \). It follows that
\[ \dim(M(5, 2) \setminus \bigcup_{\xi \neq 0} U^0_\xi) < \dim \bigcup_{\xi \neq 0} U^0_\xi = 62. \]
Because any component of \( M(5, 2) \) has dimension \( \geq 25 + 40 - 3 = 62 \) and because \( \bigcup_{\xi \neq 0} U^0_\xi \) is irreducible, it follows that \( M(5, 2) \) is irreducible of the expected dimension 62.
It follows now from Lemma 2.7.1 that also \( MI(5) \) is irreducible of the expected dimension 37. We thus have

**7.3. Theorem:** \( MI(5) \) is irreducible of dimension 37.

**7.4. Proof of Lemma 7.1:** We show in fact that \( \tilde{M}(4, 4)_\xi \cong M(3, 6) \times \text{Hom}_k(H_3, \Lambda^2 V^*) \).
For the proof we need to distinguish between the linear maps \( A \xrightarrow{\phi} B \otimes \Lambda^2 V^* \) and the corresponding operators \( A \otimes V \xrightarrow{\alpha} B \otimes V^* \) which are skew with respect to \( V \) for any two vector spaces \( A \) and \( B \). Because \( \xi \) is fixed, we can choose a decomposition
\[ H_1 \oplus \tilde{H}_3 = H_4, \]
where \( \tilde{H}_3 \) is the kernel of \( \xi \). Given \( \bar{\omega} \in M(3, 6) \) and \( \tilde{H}_3 \xrightarrow{\alpha} \Lambda^2 V^* \cong H_1^* \otimes \Lambda^2 V^* \), we obtain the operator
\[ \bar{\omega} = \left( \begin{array}{c|c} \alpha & \bar{\alpha} \\ \hline \bar{\alpha}^* & \bar{\omega} \end{array} \right) : (H_1 \oplus \tilde{H}_3) \otimes V \rightarrow (H_1^* \oplus \tilde{H}_3^*) \otimes V^* \]
where \( \alpha^* \) denotes the dual of \( \alpha \) with respect to \( \tilde{H}_3 \) and \( H_1 \). Because the upper row of \( \bar{\omega} \) is a combination of the lower row, we have \( \text{rk}(\bar{\omega}) = \text{rk}(\tilde{\omega}) = 12 \). It is clear that \( \tilde{\omega} \) is skew with respect to \( V \). Therefore \( \tilde{\omega} \) defines an element \( \omega \in \tilde{M}(4, 4)_\xi \). We thus have a morphism
\[ M(3, 6) \times \text{Hom}(\tilde{H}_3, \Lambda^2 V^*) \rightarrow \tilde{M}(4, 4)_\xi. \]
This is even an isomorphism over \( M(3, 6) \), because if
\[ \omega = (\tilde{\omega}, \tilde{\alpha}) : (H_1 \oplus \tilde{H}_3) \rightarrow (H_1^* \oplus \tilde{H}_3^*) \otimes \Lambda^2 V^* \]
is in \( M(4, 4)_\xi \), we have \( 12 = \text{rk}(\tilde{\omega}) = \text{rk}(\tilde{\omega}) \) and it follows that
\[ (\tilde{\varphi}, \tilde{\alpha}) = \tilde{\alpha}^{-1}(\tilde{\alpha}^*, \tilde{\omega}). \]
7.5. Proof of Lemma 7.2: For fixed $\eta \in H^*_4 \setminus \{0\}$ there is an isomorphism
\[ \tilde{M}(4,4)_\eta \cong M(3,6) \times \text{Hom}(H_3, \Lambda^2 V^*) =: \tilde{X} \]
by Lemma 7.1. To each pair $(\bar{\omega}, \alpha) \in \tilde{X}$ we have the simplified Beilinson II presentation of $E_{\bar{\omega}}$ together with a homomorphism $\sigma(\alpha)$ induced by the diagram
\[
\begin{array}{ccccccccc}
0 & \rightarrow & H_3 \otimes \mathcal{O}(-1) & \xrightarrow{a(\bar{\omega})} & H_3^* \otimes \Omega^1(1) & \xrightarrow{\varepsilon} & \mathcal{E}_{\bar{\omega}} & \rightarrow & 0 \\
& & \downarrow{\varepsilon} & & \downarrow{j} & & \downarrow{\sigma(\alpha)} & & \\
& & H_3^* \otimes V^* \otimes \mathcal{O} & \xrightarrow{\tilde{\alpha}} & H_3 \otimes V \otimes \mathcal{O} & \xrightarrow{\tilde{\alpha}} & \mathcal{O}(1) & & \\
& & \approx \mathcal{E}_{\bar{\omega}} & & & & & & \\
\end{array}
\]
Note here that $\tilde{\alpha} \circ \varepsilon = 0$ because $\alpha$ is skew with respect to $V$, such that $\tilde{\alpha} \circ \bar{\omega}^{-1} \circ j$ factors through $\mathcal{E}_{\bar{\omega}}$. Note further that in case $(\bar{\omega}, \alpha)$ corresponds to an $\omega \in M(4,4)_\eta$, then $E_{\omega}$ is the cohomology of the monad
\[
\mathcal{O}(-1) \xrightarrow{\sigma(\alpha)^*} E_{\bar{\omega}}^* \xrightarrow{\sim} E_{\bar{\omega}} \xrightarrow{\sigma(\alpha)} \mathcal{O}(1)
\]
in which case $\sigma(\alpha)$ is surjective, see beginning of Section 3.
We let $X \subset \tilde{X}$ be the open subset corresponding to $M(4,4)_\eta$, or defined by the surjectivity of $\sigma(\alpha)$. In addition we let $\tilde{X}^0 \subset \tilde{X}$ denote the open part of $\tilde{X}$ where
\[ H^0\sigma(\alpha)(1) : H^0E_{\bar{\omega}}(1) \rightarrow H^0\mathcal{O}(2) \]
is surjective. Then
\[ M^0(4,4)_\eta \cong X^0 = X \cap \tilde{X}^0 \]
under the above isomorphism because the cokernel of $H^0\sigma(\alpha)(1)$ is then isomorphic to $H^1E_{\bar{\omega}}(1)$. To prove Lemma 7.2, it is sufficient to prove that
\[
\text{codim}(\tilde{X} \setminus \tilde{X}^0, \tilde{X}) \geq 2. \quad (*)
\]
Now $(*)$ will follow from the following two statements.

Claim F: There exists an $\bar{\omega} \in M(3,6)$ such that for the fibres, $\tilde{X}_\omega^0 \subset \tilde{X}_\omega$ of $\tilde{X}^0 \subset \tilde{X}$ over $\bar{\omega}$,
\[
\text{codim}(\tilde{X}_\omega \setminus \tilde{X}_\omega^0, \tilde{X}_\omega) \geq 2.
\]

Claim B: Let $\Sigma \subset M(3,6)$ be the closed subvariety of points $\bar{\omega}$ for which $\tilde{X}_\omega^0 = \emptyset$, i.e. the set of points $\bar{\omega}$ for which $H^0\sigma(\alpha)(1)$ is not surjective for any $\alpha$. Then
\[
\text{codim}(\Sigma, M(3,6)) \geq 2.
\]
7.5.1. Proof of claim B:

a) In order to incorporate a projective curve which doesn’t meet Σ, we enlarge $M(3, 6)$ and $\Sigma$ as follows. Let

$$\tilde{M}(3, 6) := M(3, 6) \cup M^0(3, 4).$$

Then $\tilde{M}(3, 6) \subset S^2H_3^* \otimes \Lambda^2V^*$ consists of non-degenerate $\tilde{\omega}$ of rank 10 if $\tilde{\omega} \in M^0(3, 4)$ and of rank 12 if $\tilde{\omega} \in M(3, 6)$, and such that $H^1\mathcal{E}_{\tilde{\omega}}(1) = 0$. If $\tilde{\omega} \in M^0(3, 4)$, the Beilinson II monad of $\mathcal{E}_{\tilde{\omega}}$ is of the type

$$0 \to H_3 \otimes \mathcal{O}(−1) \xrightarrow{a(\tilde{\omega})} H_3^* \otimes \Omega^1(1) \to Q \otimes \mathcal{O} \to 0$$

where $Q$ is the cokernel of $\tilde{\omega}$.

If $\mathcal{F}_{\tilde{\omega}}$ is the cokernel of $a(\tilde{\omega})$, we have the exact sequence

$$0 \to \mathcal{E}_{\tilde{\omega}} \to \mathcal{F}_{\tilde{\omega}} \to Q \otimes \mathcal{O} \to 0.$$ 

Given a second component $\alpha \in \text{Hom}(H_3, \Lambda^2V^*)$, we also obtain a homomorphism $\mathcal{F}_{\tilde{\omega}} \overset{\sigma(\alpha)}{\longrightarrow} \mathcal{O}(1)$ by the diagram

$$\begin{array}{ccc}
H_3 \otimes V \otimes \mathcal{O} & \xrightarrow{\tilde{\alpha}} & \mathcal{O}(1) \\
\downarrow \tilde{\omega} & & \\
H_3^* \otimes Vx \otimes \mathcal{O} & \xrightarrow{\sigma(\alpha)} & \\
\downarrow & & \\
0 & \to & H_3 \otimes \mathcal{O}(−1) \xrightarrow{a(\tilde{\omega})} H_3^* \otimes \Omega^1(1) \to \mathcal{F}_{\tilde{\omega}} \to 0
\end{array}$$

which in case $\tilde{\omega} \in M(3, 6)$ coincides with $\mathcal{E}_{\tilde{\omega}} \overset{\sigma(\alpha)}{\longrightarrow} \mathcal{O}(1)$. The surjectivity of $H^0\sigma(\alpha)(1)$ does not depend on the choice of the factorization of $\tilde{\alpha}$.

We let now $\tilde{\Sigma} \subset \tilde{M}(3, 6)$ be the locus of points $\tilde{\omega}$ for which $H^0\sigma(\alpha)(1)$ is not surjective for any $\alpha$. By this definition we have

$$\Sigma = \tilde{\Sigma} \cap M(3, 6).$$

b) Because $M_{10}(H_3)$ had been shown to be an irreducible hypersurface in $S^2H_3^* \otimes \Lambda^2V^*$ whose complement is $M(3, 6)$, see [4], and because $M^0(3, 4)$ is an open part of it, the complement of $\tilde{M}(3, 6)$ in $S^2H_3^* \otimes \Lambda^2V^*$ has codimension $\geq 2$. In order to show that codim$(\tilde{\Sigma}, \tilde{M}(3, 6)) \geq 2$, we first construct an embedding $k^2 \setminus \{0\} \to \tilde{M}(3, 6) \setminus \tilde{\Sigma}$.

c) This embedding is defined as follows. We let $e_1, \ldots, e_4$ be the standard basis of $H_4 = k^4$ and $e_1^*, \ldots, e_4^*$ be its dual basis. We let

$$H_4 \xrightarrow{\tilde{\omega}} H_4^* \otimes \Lambda^2V^*$$
be given by the matrix

$$\omega = \begin{pmatrix} \omega'_{11} & \omega'_{12} & 0 & 0 \\ \omega'_{12} & \omega'_{22} & 0 & 0 \\ 0 & 0 & \omega''_{11} & \omega''_{12} \\ 0 & 0 & \omega''_{12} & \omega''_{22} \end{pmatrix}$$

with $\omega'_{ij}, \omega''_{ij} \in \Lambda^2 V^*$ which represents the direct sum of two 2–instantons $\mathcal{E}'$ and $\mathcal{E}'', \mathcal{E}_\omega = \mathcal{E}' \oplus \mathcal{E}''$. Then $H^1 \mathcal{E}_\omega(1) = 0$. For $t = (t_0, t_1) \neq 0$ we consider

$$\xi_t = (-t_1, 0, t_0, 0) \in H^*_4$$

and its kernel

$$0 \rightarrow k^3 \xrightarrow{f_t} k^4 \xrightarrow{\xi_t} k \rightarrow 0$$

defined by the matrix

$$f_t = \begin{pmatrix} t_0 & 0 & 0 \\ 0 & 1 & 0 \\ t_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

We let $\bar{\omega}_t = (f_t^* \otimes \text{id}) \circ \omega \circ f_t$ such that

$$\bar{\omega}_t = \begin{pmatrix} t_0^2 \omega'_{11} + t_1^2 \omega''_{11} & t_0^2 \omega'_{12} & t_1^2 \omega''_{12} \\ t_0 \omega'_{12} & \omega'_{22} & 0 \\ t_1 \omega''_{12} & 0 & \omega''_{22} \end{pmatrix}.$$ 

We may assume that both $\omega''_{22}$ and $\omega''_{12}$ have rank 4 as operators $V \rightarrow V^*$. Then by an elementary matrix operation we can kill $t_0 \omega'_{12}$ and $t_1 \omega''_{12}$ in the first row and obtain a matrix

$$\begin{pmatrix} t_0^2 \eta'_{11} + t_1^2 \eta''_{11} & 0 & 0 \\ t_0 \omega'_{12} & \omega'_{22} & 0 \\ t_1 \omega''_{12} & 0 & \omega''_{22} \end{pmatrix}$$

with $\eta'_{11} = \omega'_{11} - \omega'_{12} \omega''_{22} \omega'_{12}$ and $\eta''_{11} = \omega''_{11} - \omega''_{12} \omega''_{22} \omega''_{12}$.

Because $\omega'$ and $\omega''$ represent 2–instantons, $\text{rk}(\omega') = \text{rk}(\omega'') = 6$ and therefore

$$\text{rk } \eta'_{11} = \text{rk } \eta''_{11} = 2.$$ 

But we can choose $\omega'$ and $\omega''$ such that $\text{rk } \omega''_{22} = \text{rk } \omega''_{12} = 4$ and in addition

$$\text{Im } \eta'_{11} + \text{Im } \eta''_{11} = V^*.$$ 

Then $t_0^2 \eta'_{11} + t_1^2 \eta''_{11}$ is an isomorphism $V \rightarrow V^*$ for $t_0 t_1 \neq 0$. Therefore, with this choice of $\omega'$ and $\omega''$, we have

$$\bar{\omega}_t \in M(3, 6) \text{ for } t_0 t_1 \neq 0$$

$$\bar{\omega}_t \in M^0(3, 4) \text{ if } t_0 = 0 \text{ or } t_1 = 0.$$
The first statement follows directly from the fact that \( \text{rk} \bar{\omega} = 12 \) if \( t_0 t_1 \neq 0 \). If \( t_0 = 0 \) or \( t_1 = 0 \), then \( \text{rk} \bar{\omega} = 10 \). In that case, e.g., \( t_1 = 0 \),

\[
\bar{\omega} = \begin{pmatrix}
t_0 \omega'_{11} & t_0 \omega'_{12} & 0 \\
t_0 \omega'_{21} & \omega''_{22} & 0 \\
0 & 0 & \omega''_{22}
\end{pmatrix}
\]

and then the bundle of \( \bar{\omega} \) is the direct sum

\[
\mathcal{E}_{\bar{\omega}} \cong \mathcal{E}_{\omega'} \oplus \mathcal{E}_{\omega''_{22}}
\]

with \( \mathcal{E}' = \mathcal{E}_{\omega'} \) a 2–instanton and \( \mathcal{E}_{\omega''_{22}} \) a 1–instanton (null–correlation bundle). So \( \bar{\omega} \) is non–degenerate and \( H^1 \mathcal{E}_{\bar{\omega}}(1) = 0 \). It follows that \( t \mapsto \bar{\omega} \) is a morphism

\[
k^2 \setminus \{0\} \to \widetilde{M}(3, 6)
\]

which is an embedding by the shape of the matrix \( \bar{\omega} \). Moreover, \( \bar{\omega} \not\in \tilde{\Sigma} \) for any \( t \). To see this, we consider the case \( t_1 \neq 0 \) first. In that case the linear embedding \( k^3 \mathbb{f}_{g_{t}} \to k^4 \) can be considered as the kernel of a fixed \( \xi = (1, 0, 0, 0) \) by the following diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & k^3 & f_{t} & k^4 & \xi_{t} & k & \rightarrow & 0 \\
0 & \rightarrow & k^3 & g & k^4 & \xi & k & \rightarrow & 0
\end{array}
\]

where \( \xi_{t} = (-t_1, 0, t_0, 0) \) and \( \xi = (1, 0, 0, 0) \) with

\[
g = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 1 & 0
\end{pmatrix}
\]

and

\[
a_{t} = \begin{pmatrix}
-1 & 0 & t_0/t_1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

If we denote

\[
\omega_{t} := a_{t}^{-1} \ast \omega_{a_{t}^{-1}}
\]

we have

\[
\bar{\omega}_{t} = g_{t} \ast \omega_{t} = \text{res}_{\xi_{t}} \omega_{t}.
\]

Because \( \omega_{t} \sim \omega \), we have \( \mathcal{E}_{\omega_{t}} \cong \mathcal{E}_{\omega} \) and so \( h^{1} \mathcal{E}_{\omega_{t}}(1) = 0 \). If \( \bar{\omega}_{t} \in M(3, 6) \), then \( \bar{\omega}_{t} \not\in \Sigma \). If, however, \( \bar{\omega}_{t} \in M^{0}(3, 4) \), then \( \mathcal{E}_{\omega_{t}} \) is the cohomology of the complex

\[
\mathcal{O}(-1) \rightarrow \mathcal{F}_{\omega_{t}}^{*} \rightarrow \mathcal{F}_{\omega_{t}} \rightarrow \mathcal{O}(1).
\]

Then also in this case \( H^{0} \mathcal{F}_{\omega_{t}}(1) \rightarrow H^{0} \mathcal{O}(2) \) has cokernel \( H^{1} \mathcal{E}_{\omega_{t}}(1) = 0 \). Therefore \( \bar{\omega}_{t} \not\in \tilde{\Sigma} \) for \( t_1 \neq 0 \). By symmetry in \( t_0, t_1 \), also \( \bar{\omega}_{t} \not\in \tilde{\Sigma} \) for \( t_0 \neq 0 \). Therefore we have an embedding

\[
k^2 \setminus \{0\} \hookrightarrow \tilde{M}(3, 6) \setminus \tilde{\Sigma}.
\]
d) The map \( t \mapsto \bar{\omega}_t \) cannot directly be used to define a line in \( \mathbb{P}S^2H_3^* \otimes \Lambda^2V^* \) because \( \bar{\omega}_t \) is not homogeneous in \( t \). But it can be used to construct a projective line in \( \mathbb{P}H_4^* \) which doesn’t intersect a transformation of \( \Sigma \) so that we can conclude from that that \( \Sigma \) has codimension \( \geq 2 \). To do this we consider the following transformation diagram

\[
\mathbb{P}H_4^* \leftarrow \pi \mathbb{P}\text{Hom}(H_3, H_4)^0 \theta \rightarrow \mathbb{P}S^2H_3^* \otimes \Lambda^2V^*.
\]

In this diagram \( \mathbb{P}\text{Hom}(H_3, H_4)^0 \) denotes that open set of injective maps and \( \pi \) is the principal \( \text{PGL}(H_3) \)-bundle over the Grassmannian \( G(3, H_4) = \mathbb{P}H_4^* \). The morphism \( \theta \) is defined by

\[
f \mapsto (f^* \otimes \text{id}) \circ \omega \circ (f \otimes \text{id})
\]

using the form of \( \omega \) in c). We let

\[
\widehat{\Sigma} \subset \widehat{M}(3, 6) \subset \mathbb{P}S^2H_3^* \otimes \Lambda^2V^*
\]

denote the subvarieties of which \( \tilde{\Sigma} \) and \( \tilde{M}(3, 4) \) are the affine cones. So \( \widehat{\Sigma} \) is closed in the open set \( \widehat{M}(3, 6) \) of codimension \( \geq 2 \). Now

\[
\theta^{-1}\widehat{\Sigma} \subset \theta^{-1}\widehat{M}(3, 6) \subset \mathbb{P}\text{Hom}(H_3, H_4)^0
\]

are \( \text{PGL}(H_3) \)-invariant and therefore there are an open subset \( U \subset \mathbb{P}H_4^* \) and a closed subscheme \( Z \subset U \) such that

\[
\theta^{-1}\widehat{\Sigma} = \pi^{-1}Z \quad \text{and} \quad \theta^{-1}\widehat{M}(3, 6) = \pi^{-1}U.
\]

Moreover, there is the irreducible hypersurface \( \widehat{M}_{10}(H_3) \subset \mathbb{P}S^2H_3^* \otimes \Lambda^2V^* \) defined by \( \text{rk} \bar{\omega} \leq 10 \). Its inverse image \( \theta^{-1}\widehat{M}_{10}(H_3) \) is the subvariety of those \( \langle f \rangle \in \mathbb{P}\text{Hom}(H_3, H_4)^0 \), for which \( (H_4/\text{Im}(f))^* \otimes V^* \rightarrow H^1E_\omega \) is not an isomorphism, see \[3.1\], because then \( \bar{\omega} = \theta(f) \) has rank 10. It is then easily seen that this condition defines an effective divisor, which is \( \theta^{-1}\widehat{M}_{10}(H_3) \). Because this is also \( \text{PGL}(H_3) \)-invariant, there is a divisor \( D \subset \mathbb{P}H_4^* \) such that

\[
\theta^{-1}\widehat{M}_{10}(H_3) = \pi^{-1}D.
\]

Now the family \( (f_t) \) of c) defines a linear embedding \( \langle t \rangle \mapsto \pi(\langle f_t \rangle) = \langle \xi_t \rangle \)

\[
\mathbb{P}1 \rightarrow L \subset \mathbb{P}H_4^*
\]

and by the result of c) we have

\[
L \subset U \setminus Z \quad \text{and} \quad L \not\subset D.
\]

Let \( Y \subset \mathbb{P}H_4^* \) be the complement of \( U \). Then \( L \) doesn’t intersect \( Y \cup Z = Y \cup \bar{Z} \) and so \( Z \) is of codimension \( \geq 2 \). Then also \( \theta^{-1}\widehat{\Sigma} \) has codimension \( \geq 2 \) in \( \theta^{-1}\widehat{M}(3, 6) \). But since also the complement of \( \theta^{-1}\widehat{M}(3, 6) \) has codimension \( \geq 2 \) in \( \mathbb{P}\text{Hom}(H_3, H_4) \), there is a line \( L' \subset \mathbb{P}\text{Hom}(H_3, H_4) \) such that

\[
L' \subset \theta^{-1}\widehat{M}(3, 6) \setminus \theta^{-1}\widehat{M} \quad \text{and} \quad L' \not\subset \theta^{-1}\widehat{M}_{10}(H_3).
\]
We let now $\Gamma = \theta(L')$. This is a complete curve in $\mathbb{P}S^2H_3^* \otimes S^2V^*$ (not contracted to a point because $L'$ intersects the divisor $\theta^{-1}\widehat{M}_{10}(H_3)$ and is not contained in it) and such that

$$\Gamma \subset \widehat{M}(3,6) \setminus \widehat{\Sigma}.$$  

Then $\Gamma$ cannot intersect the closure of $\widehat{\Sigma}$ in $\mathbb{P}S^2H_3^* \otimes \Lambda^2V^*$. So we have shown that $\text{codim}(\widehat{\Sigma}, \widehat{M}(3,6)) \geq 2$. This is equivalent to claim B.

### 7.5.2. Remarks on null–correlation bundles:

The tensors $\bar{\omega}$ provided for claim F are diagonal matrices of elements $\eta \in \Lambda^2V^*$ of rank 4 (or indecomposable) which correspond to 1–instantons or null-correlation bundles (nc–bundles for short). Given $\eta$ of rank 4, we have the defining sequence $0 \to \mathcal{O}(\mathcal{O}(\eta)) \to \Omega^1(1) \to E_{\eta} \to 0$. Then for any $\alpha \in \Lambda^2V^*$ we obtain a homomorphism $E_{\eta} \to \mathcal{O}(1)$ as in

$$0 \to \mathcal{O}(\mathcal{O}(\eta)) \to \Omega^1(1) \to E_{\eta} \to \mathcal{O}(1).$$

One can easily see that $\sigma(\alpha) = 0$ if and only if $\alpha$ and $\eta$ are proportional in $\Lambda^2V^*$ or $\langle \alpha \rangle = \langle \eta \rangle$. Therefore, a non–zero image $\mathcal{I}$ of $E_{\eta}(\mathcal{O}(\eta)) \to \mathcal{O}$ is determined by a line $g = (x, y)$ in $\mathbb{P}\Lambda^2V^*$ and is independent of $\langle \alpha \rangle$ on $g$. We therefore write $\mathcal{I}_g$ for the image of $E_{\eta}(\mathcal{O}(\eta))$ under $\sigma(\alpha)$.

In this section we identify the Grassmannians $G = G(2, V)$ and $G(2, V^*)$ via the Plücker embeddings in $\mathbb{P}\Lambda^2V^* \cong \mathbb{P}\Lambda^2V^* \otimes \Lambda^4V$ canonically. Then for a line $l$ in $\mathbb{P}V$ through $\langle x \rangle, \langle y \rangle$ with equations $z, w$ we write $l = \langle x \wedge y \rangle = \langle z \wedge w \rangle$.

**Lemma 1**: Let $\langle \eta \rangle \in \mathbb{P}\Lambda^2V^* \setminus G$ and let $g$ be a line in $\mathbb{P}\Lambda^2V^*$ through $\langle \eta \rangle$ with associated ideal sheaf $\mathcal{I}_g$. Then

a) If $g$ meets the Grassmannian $G$ in two points $l_1 \neq l_2$, then $l_1 \cap l_2 = \emptyset$ in $\mathbb{P}V$ and $\mathcal{I}_g = \mathcal{I}_{l_1 \cup l_2}$.

b) If $g$ is tangent to $G$ at $l$, then $\mathcal{I}_g$ is the twisted double structure on the line $l$ given by the tangent direction along $g$.

In both cases $h^0\mathcal{I}_g(2) = 4$.

**Proof.** Let $Z$ be the zero scheme of $\mathcal{I}_g$. We have the exact sequence $0 \to \mathcal{O}(\mathcal{O}(\eta)) \to \mathcal{E}_{\eta} \to \mathcal{O}(1) \to \mathcal{O}_Z(1) \to 0$ and by that for the Hilbert polynomial $\chi\mathcal{O}_Z(m) = 2m + 2$. 


Lemma 3: (i) either $Z \in H^\sigma$ under order to prove that $I$ and ideal sheaves $E$ We consider first two nc–bundles

\[ 7.5.3. \text{Three nc–bundles} : \]

(1) Let $\alpha \in H^0(I(1)) \to O(1)$, and ideal sheaves $I_1, I_2$ respectively. Then the image of $H^0(E_1(1) \oplus E_2(1)) \to H^0(O(2))$

under $\sigma_1 + \sigma_2$ is $H^0(I_1(2)) + H^0(I_2(2))$, and similarly for three such data $(I_i, \sigma_i)$. We let $Z_1, Z_2$ be the zero schemes of $I_1, I_2$ respectively.

**Lemma 2:** (1) Let $p_1, p_2$ be two different points on a line $l$ in $P_3$ and let $l_1, l_2$ be two skew lines not meeting $p_1, p_2$. Then the space of quadrics containing $p_1, p_2, l_1, l_2$ is 2–dimensional if and only if $l$ doesn’t meet both $l_1$ and $l_2$, and otherwise this space is 3–dimensional.

(2) If $l_0$ is a twisted double line as in Lemma 1 above, which doesn’t meet $l$, then the space of quadrics containing $p_1, p_2$ and $l_0$ is 2–dimensional, too.

**Proof.** By elementary computation using suitable coordinates. □

7.5.3. Three nc–bundles:

We consider first two nc–bundles $E_1 = E_{\eta_1}, E_2 = E_{\eta_2}$ with homomorphisms $E_i \xrightarrow{\sigma_i} O(1)$, and ideal sheaves $I_1, I_2$ respectively. Then the image of $H^0(E_1(1) \oplus E_2(1)) \to H^0(O(2))$

under $\sigma_1 + \sigma_2$ is $H^0(I_1(2)) + H^0(I_2(2))$, and similarly for three such data $(I_i, \sigma_i)$. We let $Z_1, Z_2$ be the zero schemes of $I_1, I_2$ respectively.

**Lemma 3:** Let $Z_1 = l_1 \cup l_{12}$ and $Z_2 = l_2 \cup l_{22}$ with all 4 lines different. Suppose that either

(i) $Z_1 \cap Z_2 = \emptyset$ and the 4 lines do not belong to the same ruling of a quadric, or
Z₁ ∩ Z₂ = \{x₁, x₂\} ⊂ l₁₁ and Z₂ ∩ l₁₁ = ∅. Then

\[ H^0\mathcal{I}_1(2) \cap H^0\mathcal{I}_2(2) = 0. \]

**Proof.** (i) well–known. (ii) Let Q be a smooth quadric containing l₁₁, l₁₂ (in one ruling). Because l₂ᵥ ∩ l₁₂ = ∅ and l₂ᵥ ∩ l₁₁ = \{xᵥ\}, we have l₂₁, l₂₂ ⊄ Q. Therefore the intersection of the \( H^0\mathcal{I}_v(2) \) does not contain a smooth quadric. Then any \( Q ∈ H^0\mathcal{I}_1(2) \cap H^0\mathcal{I}_2(2) \) is not smooth and contains the skew lines l₁₁ and l₁₂. So Q = P₁ ∪ P₂ is a pair of planes. Let l₁₁ ⊂ P₁. Then l₁₂ ⊄ P₁ because l₁₁ ∩ l₁₂ = ∅ and so l₁₂ ⊂ P₂. Then l₂₁, l₂₂ ⊄ P₂ and l₂₁, l₂₂ ⊂ P₁. But l₂₁ ∩ l₂₂ = ∅, a contradiction. □

**Remark:** In case (ii) the 8–dimensional space \( H^0\mathcal{I}_1(2) + H^0\mathcal{I}_2(2) \) equals \( H^0\mathcal{I}_\{x₁, x₂\}(2) \).

**Corollary:** Let the situation be as in Lemma 3 and let \( E_3 \xrightarrow{σ₃} \mathcal{O}(1) \) be a third homomorphism of an nc–bundle with ideal \( \mathcal{I}_3 \) and zero scheme \( Z₃ \). If \( Z₃ \) consists of two (then skew) lines \( l₃₁, l₃₂ \) or of a twisted double line \( l₃ \), such that the line \( l₁₁ = \overline{x₁, x₂} \)
doesn’t meet both of \( l₃₁ \) and \( l₃₂ \), then

\[ H^0\mathcal{I}_1(2) + H^0\mathcal{I}_2(2) + H^0\mathcal{I}_3(2) = H^0\mathcal{O}(2). \]

**Proof.** By Lemma 3 the sum of the first two is \( H^0\mathcal{I}_{\{x₁, x₂\}}(2) \). By Lemma 2 of 7.5.2

\[ \dim H^0\mathcal{I}_{\{x₁, x₂\}}(2) \cap H^0\mathcal{I}_3(2) = 2. \] □

### 7.5.4. Proof of claim F:

a) We choose an isomorphism \( \overline{H}_₃ \cong k^3 \) and

\[ \overline{ω} = \begin{pmatrix} η₁ & 0 & 0 \\ 0 & η₂ & 0 \\ 0 & 0 & η₃ \end{pmatrix} \]

where \( ηᵢ ∈ Λ²V^∗ \) are indecomposable. Let \( E_i = E_{ηᵢ} \) be the associated nc–bundles such that

\[ E = E_{\overline{ω}} = E_1 ⊕ E_2 ⊕ E_3. \]

By the definition of \( \overline{X}_{\overline{ω}} \) in 7.5.3 this fibre is isomorphic to \( \text{Hom}(k^3, Λ²V^*) \) and we have a surjective linear map

\[ \overline{X}_{\overline{ω}} \rightarrow \text{Hom}(E_1 ⊕ E_2 ⊕ E_3, \mathcal{O}(1)) \]
by
\[(\alpha_1, \alpha_2, \alpha_3) \mapsto (\sigma_1(\alpha_1), \sigma_2(\alpha_2), \sigma_3(\alpha_3)).\]
The open set \(\tilde{X}_0^i\) in \(\tilde{X}_0\) is then the inverse image of the set of those \((\sigma_1, \sigma_2, \sigma_3)\), \(\mathcal{E}_i \rightarrow \mathcal{O}(1)\), such that for the corresponding ideal sheaves \(\mathcal{I}_i\) we have
\[H^0\mathcal{I}_1(2) + H^0\mathcal{I}_2(2) + H^0\mathcal{I}_3(2) = H^0\mathcal{O}(2).\]
(\*)
We put \(W_i = \text{Hom}(\mathcal{E}_i, \mathcal{O}(1))\) and let
\[Z \subset \mathbb{P}(W_1 \oplus W_2 \oplus W_3)\]
be the closed subvariety of points \(\langle \sigma_1, \sigma_2, \sigma_3 \rangle\) for which (\*) is not satisfied. Then codim \(Z \geq 2\) implies claim \(F\) for the chosen \(\omega\).

b) In order to prove \(\text{codim} Z \geq 2\) we consider the natural projection
\[\mathbb{P}(W_1 \oplus W_2 \oplus W_3) \setminus Z' \xrightarrow{\pi} \mathbb{P}W_1 \times \mathbb{P}W_2 \times \mathbb{P}W_3\]
where \(Z'\) is the subvariety of points \(\langle \sigma_1, \sigma_2, \sigma_3 \rangle\) with at least one component equal to 0. It is easy to see that \(\pi\) is a principal bundle with fibre and group \((k^*)^3/k^*\). We have \(Z' \subset Z\) and \(Z \setminus Z'\) is invariant under \((k^*)^3/k^*\). Then \(Y = \pi(Z \setminus Z')\) is closed and \(Z \setminus Z' = \pi^{-1}(Y)\). Now \(Y\) is the subvariety of triples \(\langle \langle \sigma_1, \langle \sigma_2, \langle \sigma_3 \rangle \rangle \rangle\) for which (\*) is not satisfied for the images \(\mathcal{I}_\nu(1) = \text{Im} \sigma_\nu\). Claim \(F\) will be proved if \(\text{codim} Y \geq 2\). This follows now from

c) Proposition: Let \(\langle \eta_1 \rangle, \langle \eta_2 \rangle, \langle \eta_3 \rangle \in \mathbb{P} \Lambda^2 V^* \setminus G\) be in general position (not co-linear). Then the subvariety \(Y\) has codimension \(\geq 2\).

Proof. We let \(y_i\) denote the points \(\langle \eta_i \rangle\) in \(\mathbb{P} \Lambda^2 V\).

(i) Let \(H(y_2)\) be the polar hyperplane of \(y_2\) with respect to the quadric \(G\). There is point \(l \in H(y_2) \cap G\) such that the tangent hyperplane \(T_l G\) doesn’t contain \(y_3\). Denote by
\[C(l) = G \cap T_l(G)\]
the cone of lines in \(\mathbb{P} V\) meeting \(l\). Then \(C(l) \cap H(y_2)\) has codimension 2.

(ii) Choose any \(l_{11} \in H(y_2) \cap G\) with \(y_3 \not\in T_{l_{11}} G\) and such that the line \(g_1 = \overline{y_1, l_{11}}\) meets \(G\) in two different points \(l_{12} \neq l_{11}\). We have \(l_{11}, y_2 \in T_{l_{11}} G\) because \(l_{11} \in H(y_2)\). Next we choose a line \(g_2\) in \(T_{l_{11}} G\) through \(y_2\) which meets \(G\) in two different points \(l_{21}, l_{22}\). These lines belong to the cone \(C(l_{11})\) and thus each meets \(l_{11}\) in a point \(x_1\) and \(x_2\). We may assume that \(g_2 \cap C(l_{12}) = \emptyset\), such that \(l_{21}\) and \(l_{22}\) don’t meet \(l_{12}\). By this choice the conditions of Lemma 3, are satisfied, and thus
\[H^0\mathcal{I}_{g_1}(2) + H^0\mathcal{I}_{g_2}(2) = H^0\mathcal{I}_{\langle x_1, x_2 \rangle}(2)\]
is 8–dimensional, where \(\mathcal{I}_{g_\nu}\) is the ideal corresponding to the line \(g_\nu\) through \(y_\nu\).

(iii) Let now \(P(x_\nu) \subset G\) be the \(\alpha\)–plane of all lines through \(x_\nu\) and let
\[S = P(x_1) \cup P(x_2) \cup (C(l_{11}) \cap H(y_3)).\]
Because \( y_3 \notin T_{l_1G}, \) \( C(l_{11}) \cap H(y_3) \) is 2–dimensional and hence \( S \) is 2–dimensional. Let

\[ \mathbb{P} \Lambda^2 V \setminus \{ y_3 \} \xrightarrow{\pi} \mathbb{P}(\Lambda^2 V/y_3) \]

be the central projection. We have \( y_3 \notin S \). Then \( \pi(S) \) has dimension 2 and there is a line \( L \subset \mathbb{P}(\Lambda^2 V/y_3) \) which doesn’t meet \( \pi(S) \). Let \( a_3 \in L \) be any point and let \( g_3 \) denote the line through \( y_3 \) in \( \Lambda^2 V \) given by \( \pi^{-1}(a_3) \). If \( a_3 \notin \pi(H(y_3) \cap G) \), the branch locus of \( \pi|G \), then \( g_3 \cap G \) consists of two different points \( l_{31}, l_{32} \) which don’t meet \( x_1, x_2 \). Suppose that \( l_{31} \) meets \( l_{11} \), i.e. \( l_{31} \in C(l_{11}) \subset T_{l_{11}G} \). Then \( l_{32} \notin T_{l_{11}G} \) because otherwise also \( y_3 \in T_{l_{11}G} \), which had been excluded by the choice of \( l_{11} \). By the corollary in \( \text{7.5.3} \) \( H^0 \mathcal{I}_{y_3}(2) \) intersects \( H^0 \mathcal{I}_{(x_1, x_2)}(2) \) in dimension 2. Then for \( \langle \alpha_\nu \rangle \in g_\nu, \nu = 1, 2, 3 \), we have a point \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \in \mathbb{P}W_1 \times \mathbb{P}W_2 \times \mathbb{P}W_3 \setminus Y \). If however, \( a_3 \in \pi(H(y_3) \cap G) \), then \( g_3 \) is tangent to \( G \) at some \( l_0 \) and \( \mathcal{I}_{g_3} \) is a twisted double structure on \( l_0 \). In this case \( l_0 \notin C(l_{11}) \) because \( l_0 \in H(y_3) \), and so \( l_0 \cap l_{11} = \emptyset \). Again by the corollary \( H^0 \mathcal{I}_{g_3}(2) \) and \( H^0 \mathcal{I}_{(x_1, x_2)}(2) \) intersect in dimension 2.

(iv) Let now \( a'_3, a''_3 \in L \) be any two points, \( g'_3, g''_3 \) the lines in \( \Lambda^2 V \) through \( y_3 \), defined by \( \pi^{-1}(a'_3), \pi^{-1}(a''_3) \) and let \( \langle \alpha_1 \rangle \in g_1 \setminus \{ y_1 \}, \langle \alpha_2 \rangle \in g_2 \setminus \{ y_2 \}, \langle \alpha'_3 \rangle \in g'_3 \setminus \{ y_3 \}, \langle \alpha''_3 \rangle \in g''_3 \setminus \{ y_3 \} \). Then the points

\[ \langle \alpha_1, \alpha_2 \rangle, \langle s\alpha'_3 + t\alpha''_3 \rangle \in \mathbb{P}W_1 \times \mathbb{P}W_2 \times \mathbb{P}W_3 \]

do not belong to \( Y \) for any \( (s, t) \neq (0, 0) \). We thus have a complete curve in the product space not meeting \( Y \), i.e. \( \text{codim} Y \geq 2 \). This completes the proof of claim F and with that the proof of Lemma \( \text{7.2} \) and finally the proof of theorem \( \text{7.3} \).

8. Smoothness of \( MI(5) \)

The induction step \( \mathcal{E}_\omega \rightsquigarrow \mathcal{E}_\omega \) will be used in order to prove that \( H^2 S^2 \mathcal{E}_\omega = 0 \) follows from \( H^2 S^2 \mathcal{E}_\omega = 0 \) even if \( h^1 \mathcal{E}_\omega(1) = 1 \).

As before we use the following notation. \( \mathcal{E} \) will denote a 5–instanton and \( \mathcal{E} \) the (4,4)–instanton obtained from a general \( \xi \in H^*_5 \cong H^1 \mathcal{E}(-1) \) by the process \( \omega \mapsto \bar{\omega} = \text{res}_\xi \omega \), such that \( \mathcal{E} = \mathcal{E}_\omega \) and \( \bar{\mathcal{E}} = \mathcal{E}_\omega \), see \( \text{1.3} \).

By Proposition \( \text{4.3} \) we may assume that there is a second element \( \eta \in H^1 \mathcal{E}(-1) \), linear independent of \( \xi \), such that the multiplication map \( \langle \xi, \eta \rangle \otimes V^* \rightarrow H^1 \mathcal{E} \) is injective (in fact an isomorphism). Let \( \bar{\mathcal{H}} = \text{Ker}(\xi) \subset H_5 \) and let \( \bar{\eta} \in \bar{\mathcal{H}}^*_4 \) be the image of \( \eta \). If \( \bar{\omega} = \text{res}_\xi \omega \), then \( \bar{\omega} \in M(4,4)_{\bar{\eta}} \), and we have \( H^2(S^2 \bar{\mathcal{E}}) = 0 \), see \( \text{3.2} \) in case \( n = 4 \). Now \( \mathcal{E} \) appears as the cohomology of the monad

\[ 0 \rightarrow \mathcal{O}_F(-1) \rightarrow \bar{\mathcal{E}} \rightarrow \mathcal{O}_F(1) \rightarrow 0 \]

resulting from \( \xi \). Then \( S^2 \mathcal{E} \) appears as the cohomology of the derived monad

\[ 0 \rightarrow \bar{\mathcal{E}}(-1) \rightarrow S^2 \bar{\mathcal{E}} \oplus \mathcal{O} \rightarrow \bar{\mathcal{E}}(1) \rightarrow 0, \]
Because of the vanishing conditions for instantons, we obtain an exact sequence

$$H^1(S^2\mathcal{E}) \to H^1(\mathcal{E}(1)) \to H^2(S^2\mathcal{E}) \to H^2(S^2\mathcal{E}) = 0.$$ 

By Proposition 6.1 we may assume that $h^1(\mathcal{E}(1)) \leq 1$. In what follows, we shall, practically, describe the kernel of the map $H^2(\mathcal{E}(1))^* \to H^1(S^2\mathcal{E})^*$ and show that it is zero. Then it follows that $H^2S^2\mathcal{E} = 0$. In order to do that, we first describe the dual of $H^1(\mathcal{E}(1))$ in terms of its defining tensor $\bar{\omega}$, which can be done for arbitrary $n$.

8.1. Lemma: Let $H$ be an $n$–dimensional vector space, let $\omega \in S^2H^* \otimes \Lambda^2V^*$ be non–degenerate and let $\mathcal{E} = \mathcal{E}_\omega$, and interpret the elements of $H \otimes S^2V$ as linear maps $H^* \otimes V^* \to V$ which are selfdual with respect to $V$. Then

(a) $H^1(\mathcal{E}(1))^* \cong \{ \gamma \in H \otimes S^2V \mid \gamma \circ \omega = 0 \}$

(b) If $P = \mathbb{P}W$ is a plane in $\mathbb{P}_3$, $W \subset V$, then $H^1(\mathcal{E}_P(1))^* \cong \{ \gamma \in H \otimes S^2V \mid \gamma \circ \omega = 0 \text{ and } \text{Im} \gamma \subset W \}$

Proof. a) From the selfdual monad associated to $\omega$, one derives the exact sequence

$$N \otimes V^* \to H^* \otimes S^2V^* \to H^1(\mathcal{E}(1)) \to 0$$

and its dual

$$0 \to H^1(\mathcal{E}(1))^* \to H \otimes S^2V \to N^* \otimes V.$$ 

Interpreting the elements of $N^* \otimes V$ as linear maps $N \to V$ and the elements of $H \otimes S^2V$ as linear maps $H^* \otimes V^* \longrightarrow V \cong H_1 \otimes V$, the homomorphism in the last sequence can be described by $\gamma \mapsto \gamma \circ u$, where $u : N \hookrightarrow H^* \otimes V^*$.

Since $N = \text{Im}(\omega)$, it follows that $\gamma \circ u = 0$ if and only if $\gamma \circ \omega = 0$.

b) If $\gamma \in H \otimes S^2V$ and $\text{Im}(\gamma) \subset W$, then $\gamma \in H \otimes S^2W$, i.e. $\gamma$ factorizes as

$$H^* \otimes V^* \xrightarrow{\gamma} V$$

$$\downarrow$$

$$H^* \otimes W^* \xrightarrow{\gamma_0} W$$

for some $\gamma_0$ which is selfdual with respect to $W$. Now restrict the monad of $\omega$ to $P$ and use the same argument as in a).

8.2. Proposition: Any $(5, 2)$–instanton $\mathcal{E}$ on $\mathbb{P}_3$ satisfies $H^2S^2\mathcal{E} = 0$. 

$\square$
Proof. a) Let $\omega \in M(5,2)$ define $\mathcal{E} = \mathcal{E}_{\omega}$. As before, we denote by $\bar{\mathcal{E}}$ the bundle $\mathcal{E}_{\omega}$ defined by $\bar{\omega} = \text{res}_\xi(\omega)$ for a general $\xi \in H^*$. We may assume that $h^1\bar{\mathcal{E}}(1) \leq 1$ by Proposition 6.1. As mentioned above, we may also assume that $H^2S^2\mathcal{E} = 0$.

b) Let $H_1 \oplus \bar{H} = H$ be a decomposition defined by $\xi$ with $\bar{H} = \text{Ker}(\xi)$. Then

$$(H_1 \oplus \bar{H}) \otimes V \overset{\omega}{\to} (H_1^* \oplus \bar{H}^*) \otimes V^*$$

decomposes as

$$\omega = \begin{pmatrix} e \circ \bar{\varphi} \circ e^* & e \circ \bar{\varphi} \circ \bar{u}^* \bar{\omega} \\ \bar{u} \circ \bar{\varphi} \circ \bar{e}^* \end{pmatrix}$$

where $\bar{N} \overset{e}{\to} H_1^* \otimes V^*$ is a linear map and where

$$\begin{array}{ccc}
\bar{H} \otimes V & \overset{\bar{\omega}}{\to} & \bar{H}^* \otimes V^* \\
\downarrow \bar{u}^* & & \uparrow \bar{u} \\
\bar{N} & \overset{\bar{\varphi}}{\to} & \bar{N}
\end{array}$$

is the decomposition of $\bar{\omega} = \bar{u} \circ \bar{\varphi} \circ \bar{u}^*$. Note that $\omega$ is symmetric, resp. skew with respect to $H$, resp. $V$. Similarly, any $\sigma \in \Lambda^2 H \otimes S^2V$ can be written as a linear operator

$$(H_1^* \oplus \bar{H}^*) \otimes V^* \overset{\sigma}{\to} (H_1 \oplus \bar{H}) \otimes V$$

and decomposed as

$$\sigma = \begin{pmatrix} 0 & \gamma \\ -\gamma^* & \bar{\sigma} \end{pmatrix}$$

with $\gamma \in \bar{H} \otimes S^2V$ because $\sigma$ is skew with respect to $H$.

c) We are now using the exact sequence

$$0 \to H^2(S^2\mathcal{E})^* \to \Lambda^2 H \otimes S^2V \to N^* \otimes H \otimes V$$

of Remark 2.10, the homomorphism being $\sigma \mapsto \sigma \circ \omega$. By the above decomposition the condition $\sigma \circ \omega = 0$ is equivalent to the two conditions

$$\gamma \circ \bar{u} = 0 \text{ and } \gamma^* \circ e = \bar{\sigma} \circ \bar{u}.$$ 

In order to show that $\sigma \circ \omega = 0$ implies $\sigma = 0$, it is now sufficient to prove that the two conditions on $\gamma$ imply that $\gamma = 0$. For then, also, $\bar{\sigma} \circ \bar{u} = 0$ or $\bar{\sigma} \circ \bar{\omega} = 0$, and then $\bar{\sigma} = 0$, because $H^2S^2\bar{\mathcal{E}} = 0$, using the same sequence for $\bar{\mathcal{E}}$.

d) The first condition $\gamma \circ \bar{u} = 0$ or $\gamma \circ \bar{\omega} = 0$ means that

$$\gamma \in H^1(\bar{\mathcal{E}}(1))^* \subset \bar{H} \otimes S^2V$$

by Lemma 8.1, (a). If $H^1\bar{\mathcal{E}}(1) = 0$, there is nothing to prove. So we may assume that $\gamma \neq 0$. By Lemma 5.3 we may assume that $H^1\bar{\mathcal{E}}_P(1) = 0$ for any plane $P = \mathbb{P}W$ in
\[ PV. \] Then, by Lemma 8.1, (b), \( \text{Im}(\gamma) \not\subset W \) for any 3–dimensional subspace \( W \subset V \). Therefore,

\[ \tilde{H}^* \otimes V^* \xrightarrow{\gamma} H_1 \otimes V \]

is surjective. Because \( \gamma \circ \bar{u} = 0 \) and \( \dim \bar{N} = 12 \), we obtain the exact diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & \bar{N} & \xrightarrow{\cong} & \tilde{H}^* \otimes V^* & \xrightarrow{\gamma} & H_1 \otimes V & \rightarrow & 0 \\
& & \downarrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & N & \rightarrow & H^* \otimes V^* & \rightarrow & Q & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
\xi \otimes V^* & \rightarrow & 0 \\
\end{array}
\]

Note that the induced projection induces an isomorphism between \( N \) and \( \bar{N} \), because \( \omega \) and \( \bar{\omega} \) have the same rank 12. It follows from the decomposition of \( \omega \) that any vector of \( N \) can be written uniquely as

\[ e(\theta) + \theta \]

with \( \theta \in \bar{N} \subset \tilde{H}^* \otimes V^* \), with the isomorphism given by \( e(\theta) + \theta \leftrightarrow \theta \), and \( \text{Im}(e) \subset \xi \otimes V^* \).

e) We are now going to show that the condition \( \gamma^* \circ e = \bar{\sigma} \circ \bar{u} \) implies a contradiction. For that we choose a 2–dimensional subspace \( K \subset H^* \) containing \( \xi \), using Lemma 4.7, such that \( N \cap (K \otimes V^*) \) contains no non–zero decomposable vector. Let \( \xi, \xi' \) be the basis of \( K \), \( \xi' \in \tilde{H}^* \). Then there are two independent linear forms \( z, z' \in V^* \) such that \( N \cap (K \otimes V^*) \) is spanned by

\[ \xi \otimes z + \xi' \otimes z'. \]

Because \( \xi \otimes z \mapsto 0 \) under \( N \rightarrow \bar{N} \), we have with \( \theta = \xi' \otimes z' \) that

\[ \xi \otimes z + \xi' \otimes z' = e(\xi' \otimes z') + \xi' \otimes z' \]

or \( e(\xi' \otimes z') = \xi \otimes z \). Therefore, \( \gamma^*(\xi \otimes z) = \bar{\sigma}(\xi' \otimes z') \).

Because \( \bar{\sigma} \in \Lambda^2 \tilde{H} \otimes S^2 V \) is skew with respect to \( \tilde{H} \), it follows \( \bar{\sigma}(\eta \otimes w) \in \ker(\eta) \otimes V \) under \( \tilde{H}^* \otimes V^* \xrightarrow{\bar{\gamma}} \tilde{H} \otimes V \) for any \( \eta \in \tilde{H}^* \) and any \( w \in V^* \). It follows that \( (\xi' \otimes \text{id}) \circ \gamma^*(\xi' \otimes z) = 0 \), and dually, that \( \gamma(\xi' \otimes z) = 0 \), by considering the diagrams

\[
\begin{array}{cccccc}
H^* \otimes V^* & \xrightarrow{\gamma^*} & \tilde{H} \otimes V \\
\downarrow & & \downarrow \xi' \otimes \text{id} \\
\langle \xi \otimes z \rangle & \xrightarrow{0} & V \\
\end{array}
\quad
\begin{array}{cccccc}
H_1 \otimes V & \xleftarrow{\gamma^*} & \tilde{H} \otimes V^* \\
\uparrow \xi \otimes z & & \uparrow k \xi \otimes V^* \\
0 & \xleftarrow{0} & \xi' \otimes V^* \\
\end{array}
\]

and using that \( \gamma \) is symmetric with respect to \( V \).
Now \( \xi' \otimes z \in \bar{N} \) and, because \( e \) has its image in \( \xi \otimes V^* \), there is a form \( z'' \in V^* \) with \( e(\xi' \otimes z) = \xi \otimes z'' \). Then

\[
\xi \otimes z'' + \xi' \otimes z \in N \cap (K \otimes V^*)
\]

is a second vector which is independent of \( \xi \otimes z + \xi' \otimes z' \), a contradiction to the choice of \( K \). Therefore, the condition \( \gamma^* \circ e = \sigma \circ \bar{u} \) for \( \gamma \in H^1(\bar{E}(1))^\ast \) implies \( \gamma = 0 \), which proves Proposition 8.2.

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