Determining the Rolle function in Lagrange interpolatory approximation

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Abstract

We determine the Rolle function in Lagrange polynomial approximation using a suitable differential equation. We then propose a device for improving the Lagrange approximation by exploiting our knowledge of the Rolle function.

1 Introduction

Approximation of nonlinear functions is of fundamental importance in applied mathematics, especially by means of polynomials. Those techniques for which the approximation error is well understood are particularly useful. In this paper, we describe how the error in Lagrange polynomial interpolation can be precisely determined, by solving an appropriate initial-value problem. We then consider using the knowledge so obtained to improve the quality of the original approximation.

2 Relevant Concepts, Terminology and Notation

Let \( f(x) \) be a real-valued univariate function. The \textit{Lagrange interpolating polynomial} \( P_n(x) \) of degree \( n \), at most, that interpolates the data \( \{f(x_0), \)
\( f(x_1), \ldots, f(x_n) \) at the nodes \( \{x_0, x_1, \ldots, x_n\} \), where \( x_0 < x_1 < \cdots < x_n \), has the property

\[
P_n(x_k) = f(x_k)
\]

for \( k = 0, 1, \ldots, n \). Naturally, we regard \( P_n(x) \) as an approximation to \( f(x) \).

The pointwise error in Lagrange interpolation, on \([x_0, x_n] \), is

\[
\Delta(x; P_n) \equiv f(x) - P_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{k=0}^{n} (x - x_k),
\]

(1)

where \( x_0 < \xi(x) < x_n \), and is derived by invoking Rolle’s Theorem [1, 2]. Clearly, we assume here that \( f(x) \) is \((n + 1)\)-times differentiable and, as will be seen, we must assume that \( f(x) \) is, in fact, \((n + 2)\)-times differentiable.

We refer to \( \xi \) generically as the Rolle number, and to \( \xi(x) \) as the Rolle function.

### 3 Determining the Rolle Function

Using the notation \( \pi(x) \equiv \prod_{k=0}^{n} (x - x_k) \) we have, by differentiating with respect to \( x \),

\[
(n + 1)! (f(x) - P_n(x)) = f^{(n+1)}(\xi(x)) \pi(x)
\]

\[
\Rightarrow (n + 1)! (f'(x) - P'_n(x)) = f^{(n+1)}(\xi(x)) \pi'(x) + \pi(x) \frac{df^{(n+1)}(\xi)}{d\xi} \frac{d\xi}{dx} \]

\[
= f^{(n+1)}(\xi(x)) \pi'(x) + \pi(x) f^{(n+2)}(\xi) \frac{d\xi}{dx}.
\]

We have used the well-known prime notation for differentiation with respect to \( x \). In this expression, the factor \( f^{(n+1)}(\xi) \) denotes the \((n + 1)\)th derivative of \( f(\xi) \) with respect to \( \xi \), and similarly for \( f^{(n+2)}(\xi) \). We now find

\[
\frac{d\xi}{dx} = \frac{(n + 1)! (f'(x) - P'_n(x)) - f^{(n+1)}(\xi(x)) \pi'(x)}{\pi(x) f^{(n+2)}(\xi)}
\]

and, if we have a particular value \( \xi_z = \xi(x_z) \) at our disposal, we then have an initial-value problem that can, in principle, be solved to yield the Rolle function \( \xi(x) \). Note the necessity of our assumption that \( f(x) \) is \((n + 2)\)-times differentiable.
4 Calculations

Consider the Lagrange interpolation of

\[ f(x) = e^x \sin x \]

over the nodes \( \{0, \frac{3\pi}{2}\} \). We have \( n = 1 \) so that the Lagrange polynomial is

\[ P_1(x) = \left( \frac{e^{\frac{3\pi}{2}} \sin \left( \frac{3\pi x}{2} \right)}{\frac{3\pi}{2}} \right) x = \left( \frac{2e^{\frac{3\pi}{2}}}{3\pi} \right) x. \]

Furthermore, we have

\[ \Delta (x; P_1) = e^x \sin x - \left( \frac{2e^{\frac{3\pi}{2}}}{3\pi} \right) x = \left( x^2 - \frac{3\pi x}{2} \right) e^{\xi(x)} \cos (\xi(x)) \]

(2)

and

\[ \frac{d\xi}{dx} = \frac{2 \left( e^x (\cos x + \sin x) - \frac{2e^{\frac{3\pi}{2}}}{3\pi} \right) - (2e^\xi \cos \xi) \left( 2x - \frac{3\pi}{2} \right)}{(x^2 - \frac{3\pi x}{2}) (2e^\xi (\cos \xi - \sin \xi))}. \]

(3)

We solve this differential equation using an initial value chosen close to the node \( x_0 = 0 \). Observe that it is not possible to find the Rolle number at any interpolation node, because the factor \( \prod_{k=0}^{n} (x - x_k) \) in (1) ensures that \( \Delta (x; P_n) = 0 \) at the interpolation nodes, irrespective of the value of \( \xi \). Hence, we choose here \( x_z = 10^{-5} \) and determine the corresponding Rolle number by applying Newton’s method to (2). Naturally, we compute \( \Delta (x_z; P_1) = e^{x_z} \sin x_z - \left( \frac{2e^{\frac{3\pi}{2}}}{3\pi} \right) x_z \) to facilitate this calculation. In fact, we find two values for \( \xi_z \), giving the initial values \((x_z, \xi_z) = (10^{-5}, 2.1931)\) and \((x_z, \xi_z) = (10^{-5}, 4.6631)\). (Note to the reader: we quote numerical values correct to no more than four decimal places throughout this paper, simply for ease of presentation, but all calculations are performed in double precision). Using these two initial values we solve (3) to find \( \xi(x) \), shown respectively in Figures 1(a) and 1(b). We are then able to compute the pointwise error \( \Delta (x; P_1) \) using (2), and this is shown in Figure 1(c) for \((x_z, \xi_z) = (10^{-5}, 2.1931)\). Of course, we can compare this error curve with the actual error, and the magnitude of the difference between the two - the ‘error in \( \Delta \)’, so to speak - is shown in Figure 1(d). Clearly, this error is small, indicating the accuracy of our numerical estimate of \( \xi(x) \). Similar results obtain for \((x_z, \xi_z) = (10^{-5}, 4.6631)\), with a maximum magnitude in the error in \( \Delta \) of \( \sim 2 \times 10^{-11} \). This accuracy is a consequence of using a fourth-order Runge-Kutta method [3, 4] to solve (3) with a small stepsize \( \sim 5 \times 10^{-5} \).
As a second example, let us use the same objective function $f(x)$ as above, but now with three interpolatory nodes $\{0, 2, \frac{3\pi}{2}\}$. So, we have $n = 2$ which gives

$$P_2(x) = ax^2 + bx + c,$$

where $a = -9.9476$, $b = 23.2546$ and $c = 0$. Also, with

$$\pi(x) = (x - 0)(x - 2)\left(x - \frac{3\pi}{2}\right) = \left(x^3 + \left(-\frac{3\pi}{2} - 2\right)x^2 + 3\pi x\right),$$

$$\pi'(x) = 3x^2 + (-3\pi - 4)x + 3\pi,$$

we have

$$\Delta(x; P_2) = \frac{e^x (\cos \xi - \sin \xi) \pi(x)}{3}$$

and

$$\frac{d\xi}{dx} = \frac{6(e^x (\cos x + \sin x) - (2ax + b)) - 2e^x (\cos \xi - \sin \xi) \pi'(x)}{\pi(x) (-4e^x \sin \xi)}. \quad (4)$$

We perform similar calculations as above, with results depicted in Figure 2. Again, we find two initial values $(x_z, \xi_z) = (10^{-5}, 1.7845)$ and $(x_z, \xi_z) = (10^{-5}, 3.8165)$, and the error plot in Figure 2(d) corresponds to $(x_z, \xi_z) = (10^{-5}, 1.7845)$. For $(x_z, \xi_z) = (10^{-5}, 3.8165)$ the maximum magnitude in the error in $\Delta$ is $\sim 3 \times 10^{-12}$.

Our second example allows us to make a point about the continuity of $\xi(x)$. Since $P_2(x)$ is continuous and the assumed $(n + 2)$-times differentiability of $f(x)$ implies the continuity of both $f(x)$ and $f^{(n+1)}(x)$, we will assume, from (1), that $\xi(x)$ is continuous. Hence, even though it is not really meaningful to ask for the value of $\xi(x)$ at an interpolatory node, the continuity of $\xi(x)$ allows us to infer a value at such a node. We refer to such a value as an implied Rolle number. In our second example we determine the implied values $\xi(2) = 2.0991$ from Figure 2(a) and $\xi(2) = 3.7381$ from Figure 2(b). Our technique for doing this is as follows: we ensure that $x = 2$ is not amongst the nodes used for the Runge-Kutta calculation, because this would lead to a zero in the denominator on the RHS of (4), but we use the values of $\xi(x)$ at the Runge-Kutta nodes on either side of $x = 2$ to estimate $\xi(2)$ using linear interpolation. Of course, we acknowledge that it is not necessary to know the Rolle number at any interpolatory node, because the pointwise error at interpolatory nodes is always zero. As such, $\xi(x)$ is actually arbitrary at interpolatory nodes, but our technique does allow a
sensible estimate to be made, if only for completeness’ sake. Note also that simple linear extrapolation will allow an estimate of \( \xi(x) \) to be made at the endpoints of the interval.

5 A Possible Application

Our ability to determine the Rolle function suggests an interesting possibility. If we know \( \xi(x) \) then we know \( f^{(n+1)}(\xi(x)) \). If we approximate \( f^{(n+1)}(\xi(x)) \) by means of a polynomial - a least-squares fit, or a truncated Taylor series, for example - then, using \( \Pi \), we find

\[
f(x) \approx P_n(x) + \frac{P_\xi(x)}{(n+1)!} \prod_{k=0}^{n} (x - x_k),
\]

where \( P_\xi(x) \) denotes the polynomial that approximates \( f^{(n+1)}(\xi(x)) \). Notice that the RHS of this expression is simply a polynomial, and so constitutes a polynomial approximation to \( f(x) \). Thus, our knowledge of \( \xi(x) \) allows us to improve the approximation \( P_n(x) \) by adding a polynomial term that approximates the error in \( P_n(x) \).

By way of example, we return to the first of our earlier calculations. Here, we have

\[
\begin{align*}
f(x) &= e^x \sin x \\
P_1(x) &= \left(\frac{2e^{\frac{3\pi}{2}}}{3\pi}\right) x \\
\Delta(x; P_1) &= \left(x^2 - \frac{3\pi x}{2}\right) \xi(x) \cos(\xi(x))
\end{align*}
\]

using the nodes \( \{0, \frac{3\pi}{2}\} \). Once we have found \( \xi(x) \) - using \((x_0, \xi_0) = (10^{-5}, 2.1931)\) and shown in Figure 1(a) - we determine a polynomial approximation to \( e^{\xi(x)} \cos(\xi(x)) \) using a least-squares fit. For the purpose of demonstration, we choose to fit a degree six polynomial so that \( \Delta(x; P_1) + P_1(x) \) becomes an eighth-degree polynomial approximation, which we denote \( \Delta_8(x) \). In Figure 3(a) we show the pointwise error in \( P_1(x) \), and observe it has a maximum magnitude of \( \sim 75 \). In Figure 3(b) we plot the pointwise error

\[
e^x \sin x - \Delta_8(x).
\]

This error is significantly smaller than that in Figure 3(a) and has maximum magnitude \( \sim 6 \times 10^{-3} \). In other words, \( \Delta_8(x) \) is an approximation more
than four orders of magnitude (!) more accurate than $P_1(x)$, and this was achieved only through our knowledge of the Rolle function $\xi(x)$. This example serves to illustrate the value of knowing $\xi(x)$, and certainly warrants further investigation, but we will reserve a detailed study thereof for future research.

6 Conclusion

We have described how the Rolle function in Lagrange interpolatory polynomial approximation can be determined by solving an initial-value problem. Knowledge of the Rolle function permits the calculation of the approximation error. In particular, the Rolle term in the expression for the approximation error can itself be approximated by means of a polynomial, once the Rolle function is known, and this can lead to a significant improvement in the quality of the Lagrange approximation overall. Of course, the ideas presented in this paper are not restricted to Lagrange approximation, but could also be applied to Hermite interpolation, for example, in which the error term is also derived through the use of Rolle’s Theorem.

References

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[4] Hairer, E., Norsett, S.P., and Wanner, G. (2000). *Solving Ordinary Differential Equations I: Nonstiff Problems*, Berlin: Springer.
Figure 1(a) - Rolle number vs. x

Figure 1(b) - Rolle number vs. x

Figure 1(c) - Delta vs. x

Figure 1(d) - Error in Delta vs. x
