Adaptive control the fractional unified chaotic system based on the estimated eigenvalue theory

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Abstract A simple but efficient method for controlling a fractional chaotic system is proposed based on the estimated eigenvalue theory. Using the new method, the fractional chaotic system with known or unknown parameters can be controlled. Numerical simulation results are presented to show the effectiveness of the proposed scheme.

1. Introduction

Fractional calculus is a 300-year-old mathematical topic. Although it has long history, applications are only recently focus of interest. It is found that many systems display fractional order dynamics, such as viscoelastic system [1], dielectric polarization, electrode-electrolyte polarization, and electromagnetic waves. Some systems also behaves chaotically, for example, the fractional Chua system[2], the fractional Duffing system[3], the fractional Lorenz system[4], the fractional order Rössler system[5], and the fractional Chen system[6,7].

There are many control methods to control chaotic systems such as the adaptive control, the sliding mode control, the observer-based design method, the impulsive control. A new control method based on the estimate eigenvalue theory is presented in this paper. The method can be applied to control chaotic systems with integral or fractional order, and with known or unknown parameters. It is simple and can be easily realized.

This paper is organized as follows. In Section 2, the fractional calculus and the control method is presented. In Section 3, the control method for fractional unified chaotic system with known parameter is given. In Section 4, Adaptive control fractional unified chaotic system with unknown parameter is introduced. In Section 5, conclusions are drawn.

2. Fractional calculus and introduction of the control method

Fractional calculus is a generalization of ordinary (integer order) integration and differentiation to its non-integer (fractional) order counterpart. There are several basic definitions of the non-integer integration and differentiation. Among them, the most popular two definitions are the Grünwald-Letnikov (GL) definition and the Riemann-Liouville (RL) definition. The GL definition is [8-14]
\[ D^\alpha f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[t-j/h]} (-1)^j \left( \frac{a}{j!} \right) f(t-jh) \]  

(1)

While the RL definition is

\[ \frac{d^a f(t)}{dt^a} = \frac{1}{\Gamma(n-a)} \frac{d^n f(\tau)}{d\tau^n} \int_{0}^{t} (t-\tau)^{n-a-1} d\tau \quad (n \leq a < n+1) \]  

(2)

Where \( \Gamma(\cdot) \) is a gamma function, and the operator \( D^\alpha \) is generally called “\( a \)-order Caputo differential operator” [15-21].

2.1 Stability theory for linear fractional system

Suppose \( n \) order linear fractional system can be written as:

\[
\begin{pmatrix}
\frac{d^\alpha x_1}{dt^\alpha} \\
\frac{d^\alpha x_2}{dt^\alpha} \\
\vdots \\
\frac{d^\alpha x_n}{dt^\alpha}
\end{pmatrix} = A(x)
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]

(3)

Here \( A(x) \) can be regarded as the Jacobian of chaotic system at the origin, \( 0 < q_1, q_1, \cdots, q_n < 1 \). Its order is denoted by \( q = q_1, q_1, \cdots, q_n \). To make system (3) stable, we introduce feedback item \( u(t) = (k_1 x_1(t), k_2 x_2(t), \cdots, k_n x_n(t))^T \). Then the controlled system (3) can be written as:

\[
\begin{pmatrix}
\frac{d^\alpha x_1}{dt^\alpha} \\
\frac{d^\alpha x_2}{dt^\alpha} \\
\vdots \\
\frac{d^\alpha x_n}{dt^\alpha}
\end{pmatrix} = (A(x) - u)
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} = \begin{pmatrix}
a_{11} - k_1 & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} - k_2 & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} - k_n
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]

(4)

Lemma 1. The autonomous system (3) is stable if \( |\arg(spec(A(x) - u))| > a\pi / 2 \), where \( a = \max(q_1, q_1, \cdots, q_n) \).

According as estimate eigenvalue theory, all eigenvalues of matrix \( A(x) - u \) are located in the merge set of the \( n \) circles:

\[ |z - (a_{ii} - k_i)| \leq |a_{ii}| + |a_{i1}| + \cdots |a_{i,i-1}| + a_{i,i+1} + \cdots + |a_{ii}| \quad (i = 1, 2, \cdots, n) \]  

(5)
So controlled system (4) can be controlled if control item $u$ makes every circle denoted by (5) located in stable area as shown in Fig. 1.

![Stability region of linear fractional system with order $a$](image)

**Fig. 1 Stability region of linear fractional system with order $a$**

3. **Control fractional unified chaotic system with known parameter**

The fractional unified chaotic system can be written as:

\[
\begin{align*}
\frac{d^{\sigma}x_1(t)}{dt^{\sigma}} &= (28 - 35a - x_3) \quad \frac{d^{\sigma}x_2(t)}{dt^{\sigma}} = (25a + 10) - (25a + 10) \\
\frac{d^{\sigma}x_3(t)}{dt^{\sigma}} &= (29a - 1) - (29a - 1) \\
\end{align*}
\]

where the parameter $a \in [0, 1]$ and $0 < q_1, q_2, q_3 < 1$. When $a \in [0, 0.8)$, formula (6) represents Lorenz fractional chaotic system; Its strange attractors is shown in fig.2. When $a = 0.8$, it represents L"u
Fig. 2 The strange attractors of (6) \((a = 0.2, q_1 = 0.8, q_2 = 0.8, q_3 = 0.9)\)
fractional chaotic system; when \(a \in (0.8, 1]\), it represents Chen fractional chaotic system.

To make system (6) stable, we introduce feedback item \(u(t) = (k_1 x_1(t), k_2 x_2(t), k_3 x_3(t))^T\). Then the controlled system (6) can be written as:

\[
\begin{pmatrix}
\frac{d^q x_1(t)}{dt^q} \\
\frac{d^q x_2(t)}{dt^q} \\
\frac{d^q x_3(t)}{dt^q}
\end{pmatrix} =
\begin{pmatrix}
-(25a+10) - k_1 & -(25a+10) & 0 \\
(28 - 35a - x_3) & (29a-1) - k_2 & 0 \\
0 & 0 & -\frac{a+8}{3} - k_3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

(7)

It is obvious that \(\lambda_1 = -\frac{a+8}{3} - k_3\) is an eigenvalue of matrix B. According to formula (5), the other two eigenvalues of matrix B is:

\[
\lambda_2, \lambda_3 \in \left\{ y \left| y - (29a - 1 - k_2) \leq |28 - 35a| - x_3 \right| \subseteq (28 - 35a) - x_3 \right\} \cup \left\{ y \left| y + (25a + 10 - k_1) \leq |-(25a + 10)| \right| \subseteq -(25a + 10) \right\}
\]

(9)

Because \(a \in [0, 1]\), \(\lambda_1, \lambda_2, \lambda_3\) are located in the stable area if

\[
k_1 \geq (29a - 1 - \max \left| (28 - 35a) - x_3 \right|), \quad k_2 \geq 0, \quad k_3 \geq -\frac{a+8}{3}.
\]

Because parameters \(a, q_1, q_2, q_3\) and the initial value of system all affect the range of chaotic attractor, the range of \(x_3\) depends on the specific \(a, q_1, q_2, q_3\) and the initial value.
Fig. 3 The evolution of $x_1$ with time $t$ $(a = 0.2, q_1 = q_2 = 0.8, q_3 = 0.9, x_1(0) = x_2(0) = x_3(0))$

If $a = 0.2, q_1 = 0.8, q_2 = 0.8, q_3 = 0.9$ and $x_1(0) = 0, x_2(0) = 0, x_3(0) = 0$, $x_3$ fluctuates in the area $(0, 60)$, and then $k_i > 35.2$ meets the control demand. Numerical simulation verifies the validity of the method, as shown in Fig. 3.

4. **Adaptive control fractional unified chaotic system with unknown parameter**

The control method proposed need design a feedback $u(t)$ according to the range of $x_3$ which depends on the specific $a, q_1, q_2, q_3$, and the initial value. But the parameters and the initial value are usually unknown or all unknown. We propose an adaptive control method to deal with the problem.

Formula (5) shows that all eigenvalues of matrix $A(x) - u$ are located in the merge set of the $n$ circles, and the centers of circles can be moved through changing $u(t)$. So chaotic system (7) can be controlled if the circle

$$\{ y \mid y - (29a - 1 - k_2) \leq |(28 - 35a) - x_3| \}$$

is located in stable area as shown in Fig. 3. So design the adaptive control law:

$$\frac{dk_2}{dt} = |x_2|$$

(11) indicates that circle (10) moves toward left if only $x_2 \neq 0$. If circle (10) is located in stable area, $x_2$ gradually decline to zero. Control of the fractional chaotic system (7) is realized. The numerical simulation result is shown in Fig. 4. The result certificate the rationality of the adaptive law proposed.

Fig. 4 The evolution of $x_2$ with time $t$ $(a = 0.2, q_1 = q_2 = 0.8, q_3 = 0.9, x_1(0) = x_2(0) = x_3(0))$

5. **Conclusion**

In this paper, we propose a scheme for control the fractional unified chaotic system with known parameter and unknown parameter. The results of numerical simulation validate the efficiency of the
scheme. We emphasize that this scheme can straightforwardly be extended to cases of control of other similar chaotic fractional-order differential systems.

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