On Almost-Global Tracking for a Certain Class of Simple Mechanical Systems

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Abstract—In this paper we present a tracking controller for a fully actuated simple mechanical system (SMS) on a compact Riemannian manifold. The controller is derived from an error function which is the composition of a configuration error and a navigation function, and depends on the choice of these two maps. Error functions present in literature have been used for the purpose of stabilization and tracking for an SMS on a Riemannian manifold. However, these control laws are local and require the initial conditions of the system trajectory to lie in a neighborhood of the reference trajectory. The control law we propose achieves almost-global asymptotic tracking of any bounded, smooth reference trajectory where "almost-global asymptotic tracking" means tracking from almost all initial conditions with asymptotic convergence. We demonstrate the results of this tracking controller for a pendulum on $S^2$ and an externally actuated rigid body which is an SMS on $SO(3)$.

I. INTRODUCTION

The problem of stabilization of an equilibrium point on an SMS on a Lie group and a Riemannian manifold has been well studied in the literature in a geometric framework \cite{5}, \cite{15}, \cite{1}, \cite{4}, \cite{3}. Further extensions of these results to the problem of tracking smooth and bounded trajectories in SMSs can be found in \cite{5}, \cite{7}, and \cite{10}. A SMS is completely specified by a manifold, the kinetic energy which defines the Riemannian metric, the potential forces and the external forces or one forms on the manifold. In \cite{5}, proportional and derivative plus feed forward (PD+FF) feedback control laws for tracking using error functions for configuration error and transport maps for velocity error are designed to achieve stability with exponential convergence for a fully actuated SMS for certain local initial conditions. However, all these results are local with respect to the choice of initial conditions for an SMS intended to track the desired trajectory. As pointed out in \cite{9} and \cite{7}, global stabilization and hence tracking for an SMS on a manifold possible only if the configuration manifold is homeomorphic to $\mathbb{R}^n$. This leads us to the question of whether almost-global asymptotic stabilization (AGAS) of an equilibrium point and, subsequently, almost-global asymptotic tracking (AGAT) of a suitable class of reference trajectories is possible on an SMS on a Riemannian manifold. Different approaches to this problem for intrinsic and coordinate free control on an SMS are available in \cite{16}, \cite{17} and \cite{20}. AGAS problems trace their origin to an early work by Koditschek \cite{19}. In \cite{9}, the notion of a navigation function on the configuration space or manifold is introduced and the flow of the negative gradient vector field defined by this navigation function is analysed. A navigation function is a polar Morse function, or a Morse function with a unique minimum \cite{13}. It is observed that there is dense set from which the trajectories of the gradient system converge to the minimum of the navigation function. By lifting the gradient field to an equivalent dissipative SMS on the tangent bundle, the AGAT property is demonstrated.

An immediate application of this idea is to the problem of tracking on Lie groups where the notion of configuration error is naturally defined using the Lie group structure and thereafter the error dynamics are defined using this configuration error. In \cite{9}, a control law is proposed for AGAT of a rigid body by stabilizing the error dynamics about identity in $SO(3)$ using a simple proportional derivative (PD) control. The gradient vector field is generated by a navigation function, which leads to the AGAT property of the tracking controller. This idea is generalized in \cite{7}, where the authors propose a control law to almost-globally asymptotically track a reference trajectory on a compact Lie group or, on a group which is a direct product of a compact Lie group and $\mathbb{R}^n$. It is observed that in case the Riemannian manifold is a Lie group, the problem of tracking the reference trajectory reduces to stabilizing the error dynamics about the group identity as the error is defined naturally by the group action. The authors use a separation principle so that the error dynamics is a SMS with dissipation and the fact that Lie group identity is AGAS follows from Koditschek’s theorem. Specific problems of almost-global tracking and stabilization in Lie groups have been studied in the literature as well. In \cite{19}, \cite{11}, control laws for AGAT are proposed for an autonomous underwater vehicle (AUV) on $SE(3)$ using Morse functions.

In all the existing tracking control laws for an SMS, the velocity error is defined along the system trajectory or the controlled trajectory. In this paper, we define the notion of a generalized configuration error on a Riemannian manifold and the velocity error along the error trajectory. We then stabilize the error dynamics about the minimum of the navigation function and using a separation principle leads us to the controller for the tracking problem we were set to solve. The first section is a brief introduction to relevant terminology in associated literature and the second section is a review of existing local PD+FF control law. In the third section we introduce the notion of almost-global stabilization and state Koditschek’s theorem for AGAS of certain equilibrium points of a dissipative SMS. In the following section we state our control law for AGAT for a fully actuated SMS on a compact Riemannian manifold. In the two subsequent sections we implement this control law for AGAT for a spherical pendulum which is an SMS on $S^2$ and AGAT for Lie groups, in particular for an externally actuated rigid body which is an SMS on $SO(3)$.

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II. PRELIMINARIES

This section introduces known mathematical notions to describe simple mechanical systems which can be found in [5], [12], [2]. A Riemannian manifold is denoted by the 2-tuple \((Q, G)\), where \(Q\) is a smooth connected manifold and \(G\) is the metric on \(Q\). \(\nabla\) denotes the Riemannian connection on \((Q, G)\) ([18], [21]). Let \(\Psi : Q \to \mathbb{R}\) be a twice differentiable function on \(Q\). The hessian \(Hess \Psi\) of \(\Psi\) is the symmetric \((0,2)\) tensor field on \(Q\) defined by \(Hess \Psi(q)(v_q, w_q) = \{\langle v_q, \nabla_{w_q} grad \Psi \rangle \}\), where \(v_q, w_q \in T_Q Q\). If \(x_0\) is a critical point of \(\Psi\), and \((U, \phi)\) is a local chart on \(Q\) about \(x_0\) with coordinates \(\{x^1, \ldots, x^n\}\) then the hessian at \(x_0\) is \(\langle Hess \Psi(x_0) \rangle_{ij} = \partial^2 \Psi / \partial x^i \partial x^j(x_0)\) which is independent of the coordinates. The flat map \(G^1 : T_Q Q \to T_Q^* Q\) is given by \(G^1(x,q) = \{G^1_i(x,q)\} = \langle \nabla_i(x,q) \rangle\) for \(x,q \in Q\) and the sharp map \(\sharp \) is its dual \(\sharp^1 : T_Q^* Q \to T_Q Q\). The flat map \(G^1(x,q) = \{G^1_i(x,q)\} = \langle \nabla_i(x,q) \rangle\) for \(x,q \in Q\) and the sharp map \(\sharp \) is its dual \(\sharp^1 : T_Q^* Q \to T_Q Q\). The governing equations for the above SMS considering no control inputs is given by

\[
\nabla \gamma(t) \dot{\gamma}(t) = -\nabla V(\gamma(t)) + \nabla V(F(\gamma(t))) \quad (1)
\]

where \(\nabla V(\gamma(t)) = \nabla V(\gamma(t))\).

B. Error function, tracking error function, and transport map

This section briefly goes over the existing notions of error maps and velocity error maps to compare configurations and velocities in a Riemannian manifold. A smooth function \(\Psi : Q \to \mathbb{R}\) is an error function on \(Q\) about an element \(q_0\) if \(\Psi\) is smooth, proper, bordered from below and satisfies (i) \(d\Psi(q_0) = 0\) and, (ii) \(Hess \Psi(q_0)\) is positive definite. A smooth symmetric function \(\Psi : Q \times Q \to \mathbb{R}\) is a tracking error function if for every \(r \in Q\), \(\Psi_r : Q \to \mathbb{R}\) is a configuration error function about \(r\), where \(\Psi_r\) is defined as \(\Psi_r(q) = \Psi(q,r)\). Therefore for all \(r \in Q\), \(\Psi_r\) satisfies (i) \(d_1 \Psi_r(q,r) = d_1 \Psi_r(q,0) = 0\), and (ii) \(Hess \Psi_r(q,r)\) is positive definite.

A transport map \(T : Q \times T_Q Q \to T_Q Q\) is smooth such that \(T(q,v_q) = v_q \forall q \in Q, v_q \in T_Q Q\). Equivalently, \(T(q,v_q) = (T_q(q), v_q)\). The pair \((\Psi, T)\) is said to be compatible if the following holds for any two parameterized curves \(\gamma : \mathbb{R} \to Q\) and \(\gamma_{ref} : \mathbb{R} \to Q\).

\[
\frac{d}{dt} \Psi(\gamma(t), \eta(t)) = \langle d_1 \Psi(\gamma(t), \eta(t)) \rangle_v v(t), \quad (2)
\]

where \(v(t) : \mathbb{R} \to T_Q Q\) is the velocity error vector field along \(\gamma\) given by \(v(t) := \dot{\gamma}(t) - T(\gamma(t), \eta(t)), \eta(t) \in T_{\gamma(t)} Q\).

III. PD+FF CONTROL LAW FOR LOCAL TRACKING FOR AN SMS ON A RIEMANNIAN MANIFOLD

In this section we state the well known proportional derivative plus feed forward (PD+FF) control for stable tracking for an SMS on a Riemannian manifold using an error function \(\Psi\) and a compatible transport map \(T\).

Definition 1. Given a reference trajectory \(\gamma_{ref}\), a controlled trajectory \(\dot{\gamma}\) and a function \(E_{cl} : T_Q Q \times T_Q Q \to \mathbb{R}\), we say that the curve \(\gamma_{ref}\) is stable with respect to the error function \(t \to E_{cl}(\gamma(t), \gamma_{ref}(t))\) if there exists a neighborhood \(U\) of \(\gamma_{ref}(0)\) such that, for all initial conditions \(\gamma(0) \in U\), the error function \(t \to E_{cl}(\gamma(t))\) is nonincreasing.

Lemma 1. ([5]) For a fully actuated SMS \((Q, G, V = 0, F, T)\), a given bounded smooth reference trajectory \(\gamma_{ref} : \mathbb{R} \to Q\) and the controlled trajectory \(\gamma : \mathbb{R} \to Q\), under the following PD+FF control law for the compatible pair \((\Psi, T)\), \(\gamma_{ref}\) is stable with respect to \(E_{cl}(v_q, v_{ref}) := \Psi(q,r) + \frac{1}{2} \|v_q - T(q,r).w_r\|^2\).

\[
F_{PD}(t, v_q) = -d_1 \Psi(q, \gamma_{ref}(t)) \quad (3a)
\]

\[
+ F_{diss}(v_q - T(q, \gamma_{ref}(t)).\gamma_{ref}(t)),
\]

\[
F_{FF}(t, v_q) = G^1 \nabla \Psi(T(q, \gamma_{ref}(t)).\gamma_{ref}(t)), \gamma_{ref}(t)) \quad (3b)
\]

where \(F_{diss}\) is any dissipative force with \(\langle F_{diss}(v_q), v_q \rangle \leq 0\) for all \(v_q \in T_q Q\).

IV. ALMOST-GLOBAL TRACKING

It is well known ([3], [4]) that the global asymptotic behaviour of an SMS with dissipation \((Q, G, V, F_{diss})\) is related to the global asymptotic behaviour of the differential equation

\[
\dot{\gamma}(t) = -\nabla V(\gamma(t)).
\]

The integral curves (or flows) of both the dynamical systems converge to the critical points of \(V\) if the sublevel sets of \(V\) given by \(\{x : x \in V^{-1}(L), L \in \mathbb{R}\}\) are compact. A dynamical system is said to be AGAS about an equilibrium point if all integral curves with initial conditions in an open dense set from the phase space converge to the equilibrium point. It is shown by Koditschek that a dissipative SMS on a compact...
Riemannian manifold (with and without boundary) is AGAS about a certain equilibrium when the potential function is a navigation function.

Since the local control law in Lemma 1 relies on the choice of a tracking error function $\Psi : Q \times Q \to \mathbb{R}$, we modify the error function by introducing a generalized configuration error map $E : Q \times Q \to Q$ and defining $\Psi = \psi \circ E$ where $\psi$ is a navigation function. This allows us to express the tracking error dynamics for the SMS and we derive the AGAT control by stabilizing the error dynamics. In the following subsections we define navigation functions, the AGAS result for a dissipative SMS (from [9]) and AGAT for an SMS, both on a compact Riemannian manifold.

A. AGAS for a dissipative SMS

A dissipative SMS is represented by the 3-tuple $(Q, M, f_d)$ where $Q$ is a Riemannian manifold with metric $M$ and $f_d$ is a dissipative force. An elaborate study of stability in dissipative mechanical systems can be found in [14]. A Morse function is real valued and has all non degenerate, hence isolated critical points.

**Definition 2 (9).** A polar Morse function $\psi \in C^2[q, [0, 1]]$ on a compact manifold $Q$ which takes unique minimum at $q_d \in Q$ is called a navigation function. (Note that $[0, 1]$ here is any arbitrary closed interval.)

**Lemma 2 (9).** Let $(Q, M, f_d)$ be a SMS where $Q$ is a smooth, compact, connected Riemannian manifold with metric $M$ and $f_d$ is a dissipative force and let $E_\Delta$ be the flow describing solution trajectories. Suppose $\psi : Q \to [0, 1]$ is a navigation function for $Q$ and $\eta$ is the total energy (kinetic due to $M$ plus potential due to $\psi$). Then the entire zero section of $TQ$ is included in the closed subset $\eta^{-1}[0, 1] \subseteq TQ$. Also, $\eta^{-1}[0, 1]$ is positive invariant with respect to $E_\Delta$ and includes the entirety of $TQ$ if $Q$ has no boundary. Moreover, there exists an open dense set in $\eta^{-1}[0, 1]$ whose limit set under $E_\Delta$ is the desired configuration $q_d$ at zero velocity, where $q_d$ is the unique minimum of $\psi$.

**Remark:** The negative gradient flow $-\nabla \psi$ on $Q$ determines local behavior of the SMS around equilibrium states in $TQ$. This fact is referred to as the lifting property of dissipative mechanical systems.

B. AGAT for an SMS on a compact Riemannian manifold

The configuration error map $E$ on a Riemannian manifold $(Q, G)$ is a function $E : Q \times Q \to Q$. Let $\gamma : \mathbb{R} \to Q$ and $\gamma_{ref} : \mathbb{R} \to Q$ be two parameterized curves on $Q$ and $E$ be a smooth map in both arguments. The derivative of $E(\gamma(t), \gamma_{ref}(t))$ is

$$\dot{E}(\gamma(t), \gamma_{ref}(t)) = d_1 E_{\gamma} + d_2 E_{\gamma_{ref}}$$

where the first differential is $d_1 E(\gamma, \gamma_{ref}) : T_\gamma Q \to T_{E(\gamma, \gamma_{ref})} Q$, the second differential is $d_2 E(\gamma, \gamma_{ref}) : T_{\gamma_{ref}} Q \to T_{E(\gamma, \gamma_{ref})} Q$ and $E(\gamma, \gamma_{ref}) \in T_{E(\gamma, \gamma_{ref})} Q$. We define $E_r(q) = E(q, r)$ for a fixed $r \in Q$. Let $f_r(q) : E_r(q) \to q$ be a local diffeomorphism about all $q \in Q$. Then $d_1 E(q, r) = dE_r(q)$ is invertible at all $q \in Q$.

**Definition 3.** Consider a manifold $Q$, a function $\psi : Q \to \mathbb{R}$ and a smooth configuration error $E : Q \times Q \to Q$. The pair $(\psi, E)$ is an appropriate choice for the tracking problem if $\psi$ is a navigation function, $d_1 E(q, r)$ is invertible for all $(q, r) \in Q \times Q$, $||\psi(E(q, r))||_R = ||\psi(E(r, q))||_R$ and $E(q, q) = q_0$ where $q_0$ is the (unique or global) minimum of $\psi$.

**Theorem 1.** (AGAT) Consider a smooth simple mechanical system on a compact, connected $n$-dimensional Riemannian manifold without boundary given by $(Q, G, \mathbb{R}^n)$ and an at least twice differentiable trajectory $\gamma_{ref} : \mathbb{R} \to Q$ with bounded velocity. If there exists a navigation function $\psi$ on $Q$, a configuration error map $E$ so that $(\psi, E)$ is an appropriate choice for the tracking problem (as in Definition 3), then there exists an open dense set $S$ in $TQ$, such that for all $(\gamma(0), \gamma(0)) \in S$, $\gamma_{ref}(t)$ can be tracked with the following control law:

$$u = (d_1 E)^{-1}[−G \nabla \psi(E) + \mathcal{G}F_{diss}(v_e)$$

$$− (d_1(d_1 E))(\gamma, \dot{\gamma}) − (d_2(d_1 E))(\dot{\gamma}_{ref}, \dot{\gamma}) − C(d_1 E_{\gamma}, v_e)$$

$$− \nabla v_e (d_2 E_{\gamma_{ref}})]$$

where $v_e := \dot{E}$, $F_{diss}$ is a dissipative force and $C(X, Y) = \Gamma^k_{ij} X^i Y^j$ for vector fields $X, Y : Q \to TQ$.

**Proof:** Let the closed loop error dynamics be given by $(E, v_e)$ as follows

$$\dot{E} = v_e$$

$$\nabla_E v_e = u_1$$

where $u_1$ is an intermediate control that is to be designed. From the expression for $\dot{E}(\gamma, \gamma_{ref})$ in [4],

$$\nabla_E v_e = \nabla_E (d_1 E_{\gamma} + d_2 E_{\gamma_{ref}})$$

$$= \nabla_E (d_1 E_{\gamma}) + \nabla_E (d_2 E_{\gamma_{ref}})$$

$$= \frac{d}{dt}(d_1 E_{\gamma}) + C(d_1 E_{\gamma}, \dot{E}) + \nabla_E (d_2 E_{\gamma_{ref}})$$

$$= (d_1(d_1 E))(\gamma, \dot{\gamma}) + (d_2(d_1 E))(\dot{\gamma}_{ref}, \dot{\gamma}) + d_1 E \nabla_{\gamma}\dot{\gamma}$$

$$+ C(d_1 E_{\gamma}, \dot{E}) + \nabla_E (d_2 E_{\gamma_{ref}})$$

where $C(X, Y) = \Gamma^k_{ij} X^i Y^j$ for vector fields $X$ and $Y$ on $Q$. We have used the following identity

$$\frac{d}{dt}(d_1 E(\gamma, \gamma_{ref}))$$

$$= d_1(d_1 E)(\gamma, \dot{\gamma}) + d_2(d_1 E)(\dot{\gamma}_{ref}, \dot{\gamma}) + (d_1 E) \nabla_{\gamma}\dot{\gamma}$$

as $d_1 E(\gamma, \gamma_{ref})$ is dependent on both $\gamma$ and $\gamma_{ref}$. The quantities $d_1(d_1 E)$ and $d_2(d_1 E)$ are tensors having two vectors as input since $d_1 E(q, r) : T_q Q \to T_{E(q, r)} Q$ and $d_2 E(q, r) : T_q Q \to T_{E(q, r)} Q$ both have one vector input argument. Note that the last term is the covariant derivative $\nabla_{\gamma}\dot{\gamma}$ since $d_1 E(\gamma, \gamma_{ref}) : T_q Q \to T_{E(q, r)} Q$.

By the assumption in the theorem, $d_1 E$ is invertible everywhere. So

$$\nabla_{\gamma}\dot{\gamma} = (d_1 E)^{-1}[u_1 − (d_1(d_1 E))(\gamma, \dot{\gamma}) − (d_2(d_1 E))(\dot{\gamma}_{ref}, \dot{\gamma})$$

$$− C(d_1 E_{\gamma}, \dot{E}) − \nabla_E (d_2 E_{\gamma_{ref}})]$$
The closed loop dynamics for the SMS are
\[
\nabla \gamma \dot{\gamma} = u
\]
where \( u \) is the control for the tracking problem and depends on the intermediate control law \( u_1 \) for the error dynamics through the equation (8). As the error dynamics in (6) are defined for the controlled trajectory \( \gamma(t) \) and the reference trajectory \( \gamma_{ref}(t) \), stability properties of (9) will follow from the stability of error dynamics. Let us now define \( u_1 = -G^j d\psi(E) + G^j F_{diss}(v_e) \). The closed loop error dynamics then are
\[
\dot{E} = v_e
\]
\[
\nabla E v_e = -G^j d\psi(E) + G^j F_{diss}(v_e)
\]
We define a Lyapunov function as \( E_{cl} = \psi(E) + \frac{1}{2}||v_e||^2 \). Then
\[
\frac{d}{dt} E_{cl}(t) = (d\psi(E), v_e) + (v_e, \nabla E v_e)
\]
\[
= (d\psi(E), v_e) + G(v_e, -G^j(d\psi(E) - F_{diss}(v_e)))
\]
\[
= (d\psi(E), v_e) - (d\psi(E), v_e) + (F_{diss}(v_e), v_e) \leq 0
\]
as \( F_{diss} \) is dissipative. Therefore the error dynamics in (6) are locally stable about \( (q_e, 0) \) where \( q_e \) is the minimum of \( \psi \).

As \( \Gamma = \bar{\Gamma}_{ij} \) and without boundary \( TQ \) is a positively invariant set. The equilibria of (6) are \( (\bar{q}, 0) \) where \( \bar{q} \) are the critical points of \( \psi \). So by LaSalle’s invariance theorem, the limit set of all solution trajectories \( f_\Delta \) originating in the positive invariant set \( TQ \) is the set \( \{(\bar{q}, 0)\} \). Linearizing (6) around an equilibrium point \( (q_e, 0) \), we have
\[
\begin{pmatrix}
\dot{E} \\
\dot{v}_e
\end{pmatrix} =
\begin{pmatrix}
-G^j(q_e) d^2 \psi(q_e) & G^j(q_e) \\
I_n & 0
\end{pmatrix}
\begin{pmatrix}
E \\
v_e
\end{pmatrix}.
\]
This linearization characterizes the local behaviour of the error dynamics around \( (q_e, 0) \). This behaviour is the same as that of the negative gradient vector field \( -G^j d\psi \) from a result in [6].

Lemma 2 can be applied as \( \psi \) is a navigation function and \( \Gamma \) is a compact Riemannian manifold. So there is an open dense set \( S \in TQ \) from which all trajectories along \( q_m \) converge to \( (q_m, 0) \) where \( q_m \) is the unique minimum of \( \psi \). On substituting the almost global intermediate control \( u_1 \) in (8) we obtain (5) which is the almost- global tracking control for the actual problem.

Remark 1: The essential fact that the proof uses is the lifting property of dissipative mechanical systems. The controller \( u_1 \) in (5) is chosen so that the closed loop error dynamics are that of a dissipative SMS.

Remark 2: Theorem 1 reduces the problem of almost-global tracking of a given reference trajectory to finding a navigation function \( \psi \) on the manifold, a configuration error \( E \) so that \( (\psi, E) \) is an appropriate choice for the tracking problem.

Remark 3: The control law given in [5] uses a single transport map \( \gamma \) to transport \( \gamma_{ref} \) to \( TQ \). Instead of this, we have two transport maps \( d_1E \) and \( d_2E \) to transport \( \gamma \) and \( \gamma_{ref} \) respectively to \( T_E(\gamma, \gamma_{ref})Q \) along the error trajectory \( E \). The position error is given by \( \psi(E) \) which is called the tracking error function and denoted by \( \psi \). However, instead of the velocity error along \( \gamma(t) \), we consider the velocity error \( E \) along \( E(t) \).

Remark 4: If \( \Psi \) is chosen as \( \psi \circ E \) for an appropriate choice of \( (\psi, E) \) and the PD+FF controller in (3a) (as in [5]) is used to track \( \gamma_{ref} \) with the velocity error defined along \( \gamma(t) \) as \( v_e = \dot{\gamma} - \dot{T}_{\gamma_{ref}} \), then
\[
\nabla \gamma(t) v_e = \nabla \gamma \left( \dot{\gamma} - \dot{T}(\gamma, \gamma_{ref}) \gamma_{ref}(t) \right)
\]
\[
= G^j(F_{PD} + F_{FF}) - G^j F_{FF}
\]
\[
= -G^j(d_1\Psi(\gamma, \gamma_{ref})) + G^j(F_{diss}(v_e))
\]
The error dynamics are given by the following equations
\[
\dot{E} = d_1E, v_e
\]
\[
\dot{v}_e = G^j(\nabla \psi(d_1E, F_{diss}(v'_e)) - I(v_e', T(\gamma_{ref})) - C(v'_e)
\]
where \( I(v'_e, T(\gamma_{ref})) = I_{ij} v'_e \) and \( C(v'_e) = I_{ij} v'_e \). \( 12a \) comes from the equivalent compatibility condition
\[
d_2E(\gamma, \gamma_{ref}) = -d_1E(\gamma, \gamma_{ref})T((\gamma, \gamma_{ref})).
\]
Linearizing (12a)+(12b) about \( (q_e, 0) \) where \( q_e \in \bar{q} \), we get
\[
\begin{pmatrix}
\dot{E} \\
\dot{v}_e
\end{pmatrix} =
\begin{pmatrix}
0 & -G^j d\psi(q_e) d_1E \\
G^j F_{diss}(q_e, 0) & -I(\dot{T}_{\gamma_{ref}})
\end{pmatrix}
\begin{pmatrix}
E \\
v_e
\end{pmatrix}.
\]
As \( P(\dot{T}_{\gamma_{ref}}) \) is a time dependent term, the flow of error dynamics around \( (q_e, 0) \) cannot be approximated by the flow of \( -G^j d\psi \). As a result, the lifting property of dissipative systems cannot be used for the error dynamics and hence AGAS cannot be established.

Remark 5: Koditschek (in [9]) considers only proportional derivative controller, whereas the feed forward terms in (5) give additional information about acceleration of the desired trajectory. Moreover, the problem of tracking is considered only on \( SO(3) \). Theorem 1 holds for the more general problem of tracking for a SMS on a compact Riemannian manifold.

Remark 6: Definition 2 enforces that \( d_1E(q, r) \) is invertible at all points on \( Q \times Q \). However it sufficient to impose the invertibility condition in \( S \times Q \) where \( S \) is dense in \( Q \) and defined in Theorem 1 for a given \( \psi \).

Remark 7: Existence of Morse functions on compact manifolds without boundary is well addressed in literature [13].
C. AGAT on $S^2$

The stereographic coordinates $(x, y) \in \mathbb{R}^2$ are given by the parameterization $f: \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ for all points in $S^2 \setminus \{(0, 0, 1)\}$ as

$$f(x, y) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right).$$

The height function $\psi(x_1, y_1, z_1) = z_1, (x_1, y_1, z_1) \in S^2 \subset \mathbb{R}^3$ is a navigation function on the sphere. In local stereographic projection coordinates,

$$\psi(x, y) = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.$$  

It is observed that $\psi$ has a unique minimum at $(0, 0, -1) \in \mathbb{R}^3$ which corresponds to $(0, 0)$ in the stereographic coordinates and the Hessian at $(0, 0, -1)$ is positive definite. In coordinates,

$$d\psi (x, y) = \frac{2x(1 + y^2)}{1 + x^2 + y^2} - 2y(1 + x^2) d\epsilon_1 + \frac{2y(1 + x^2)}{1 + x^2 + y^2} d\epsilon_2.$$  

Remark: The stereographic coordinate chart is chosen as it leaves out just the maximum of the navigation function. Therefore the navigation function, configuration error map and subsequently the control law is defined for all points in $S^2 \setminus \{(0, 0, 1)\}$. We define the configuration error in local coordinates of two configurations $(x, y)$ and $(x_{ref}, y_{ref})$ as the Riemannian distance,

$$E((x, y), (x_{ref}, y_{ref})) = \sqrt{(x - x_{ref})^2 + (y - y_{ref})^2}.$$  

As $\psi(E((x, y), (x_{ref}, y_{ref}))) = \psi(E((x_{ref}, y_{ref}), (x, y)))$ and,  $E((x, y), (x, y)) = (0, 0)$, $(0, 0)$ being the unique minimum of $\psi$, $(\psi, E)$ is an appropriate choice for any tracking problem on $S^2$. Also, $d_1E = I_2$ and $d_2E = -I_2$. $\dot{E} = v_e$ is given as $v_e = (\dot{x} - x_{ref}, \dot{y} - y_{ref})$. The differential of the error function $\psi \circ E$ is given as

$$d\psi(E((x, y), (x_{ref}, y_{ref}))) = \begin{bmatrix} 2(x-x_{ref})(1+y-y_{ref}) \frac{1+x^2+y^2}{(1+x^2+y^2)^2} + (y-y_{ref}) \frac{2y(y-y_{ref})}{(1+x^2+y^2)^2}(1+x-x_{ref}) \frac{2y(y-y_{ref})}{(1+x^2+y^2)^2} - 2y \frac{2y(y-y_{ref})}{(1+x^2+y^2)^2} & \frac{2y(y-y_{ref})}{(1+x^2+y^2)^2} \end{bmatrix}.$$  

We consider a spherical pendulum of unit mass and unit length without potential forces. The kinetic energy in local coordinates is $E((x, y, \dot{x}, \dot{y})) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$. Therefore the kinetic energy metric $G$ and its inverse in $(x, y)$ coordinates are

$$G(x, y) = \frac{1}{2} \begin{bmatrix} 1 & \dot{x}^2 + \dot{y}^2 \end{bmatrix} I_2, G^{ij}(x, y) = \frac{(1 + x^2 + y^2)^2}{2}.$$  

The Christoffel symbols of second kind in local coordinates $(x, y)$ are computed to be

$$\Gamma^{11}_{11} = \frac{-2x}{(1 + x^2 + y^2)^2}, \Gamma^{12}_{11} = \frac{2x}{2(1 + x^2 + y^2)}, \Gamma^{12}_{12} = \frac{2y}{(1 + x^2 + y^2)}.$$  

$$\Gamma^{12}_{12} = \frac{2y}{(1 + x^2 + y^2)}, \Gamma^{11}_{12} = \frac{-2y}{2(1 + x^2 + y^2)}, \Gamma^{22}_{12} = \frac{-2y}{(1 + x^2 + y^2)}.$$  

As $S^2$ is a compact Riemannian manifold and $(\psi, E)$ is an appropriate choice for the tracking problem we apply Theorem 1 to write the feedback control. Let $(x(t), y(t))$ be local coordinates of the control trajectory $\gamma(t)$ and $(x_{ref}(t), y_{ref}(t))$ be local coordinates of the reference trajectory $\gamma_{ref}(t)$. We now derive the control law in the case $d_1E((x, y), (x_{ref}, y_{ref})) = I_2$, $d_2E((x, y), (x_{ref}, y_{ref})) = -I_2$. Therefore,

$$\dot{E} = d_1E\dot{\gamma} + d_2E\dot{\gamma}_{ref} = \dot{\gamma} - \dot{\gamma}_{ref}$$

where $\dot{\gamma} = (\dot{x}, \dot{y})$ and $\dot{\gamma}_{ref} = (\dot{x}_{ref}, \dot{y}_{ref})$. 

$$\nabla_E v_e = \nabla_E \dot{E} = \nabla_E (d_1E\dot{\gamma} + d_2E\dot{\gamma}_{ref}) = \nabla_E (d_1E\dot{\gamma}) + \nabla_E (d_2E\dot{\gamma}_{ref}) = \frac{d}{dt} (d_1E\dot{\gamma}) + C(\dot{\gamma}, \dot{\gamma}) + \frac{d}{dt} (d_2E\dot{\gamma}_{ref}) = \frac{d}{dt} (d_1E\dot{\gamma}) + C(\dot{\gamma} - \dot{\gamma}_{ref}, \dot{\gamma} - \dot{\gamma}_{ref}) + \frac{d}{dt} (d_2E\dot{\gamma}_{ref})$$

where $C(X, Y) = \Gamma^k_{ij}X^iY^j$ and $X, Y$ are vectors fields on $S^2$. We have,

$$\frac{d}{dt} (d_1E\dot{\gamma}) = d_1(d_1E)(\dot{\gamma}, \dot{\gamma}) + d_2(d_1E)(\dot{\gamma}_{ref}, \dot{\gamma}) + d_1E\nabla_\gamma \dot{\gamma} = \nabla_\gamma \dot{\gamma}$$

as $d_1(d_1E) = d_2(d_1E) = 0$. Similarly,

$$\frac{d}{dt} (d_2E\dot{\gamma}_{ref}) = (d_2E)\nabla_{\gamma_{ref}} \dot{\gamma}_{ref} - \nabla_{\gamma_{ref}} \dot{\gamma}_{ref}.$$  

Therefore,

$$\nabla_E \dot{E} = \nabla_E \dot{\gamma} + C(\dot{\gamma} - \dot{\gamma}_{ref}, \dot{\gamma} - \dot{\gamma}_{ref}) - \nabla_{\gamma_{ref}} \dot{\gamma}_{ref} = \nabla_\gamma \dot{\gamma} + C(\dot{\gamma}, \dot{\gamma}) - 2C(\dot{\gamma}, \dot{\gamma}_{ref}) + C(\dot{\gamma}_{ref}, \dot{\gamma}_{ref})$$

$$- \dot{\gamma}_{ref} = C(\dot{\gamma}_{ref}, \dot{\gamma}_{ref})$$

$$= u - \dot{\gamma}_{ref} + C(\dot{\gamma}, \dot{\gamma}) - 2C(\dot{\gamma}, \dot{\gamma}_{ref})$$

As $\nabla_E \dot{E} = u_1$ is the intermediate controller for the error dynamics and given as

$$u_1 = -G^{-1}(d_1E) + k(\dot{\gamma} - \dot{\gamma}_{ref})$$

with $k > 0$, therefore, the tracking control $u$ is given as

$$u = u_1 + \dot{\gamma}_{ref} - C(\dot{\gamma}, \dot{\gamma}) + 2C(\dot{\gamma}, \dot{\gamma}_{ref}).$$

In the stereographic coordinates,

$$C(\dot{\gamma}, \dot{\gamma}_{ref}) = \left(\frac{-2x\dot{x}^2 + 2y\dot{y}^2}{1 + x^2 + y^2} \right) \frac{2y\dot{x}^2 + 2y\dot{y}^2}{1 + x^2 + y^2} 

u_1 = -\frac{1 + x^2 + y^2}{2(1 + x - x_{ref})^2 + y - y_{ref})^2} \frac{2(2(x - x_{ref})(1 + (y - y_{ref})^2) - k(1 + x^2 + y^2)}{2(1 + x - x_{ref})^2 + y - y_{ref})^2} \left(\dot{x} - \dot{x}_{ref} \right)$$

and $\dot{\gamma}_{ref} = (\dot{x}_{ref}, \dot{y}_{ref})$. We consider the reference trajectory to be generated by a dummy spherical pendulum of unit mass and length with initial conditions in stereographic projection coordinates as
\begin{align*}
(x_{\text{ref}}(0), \dot{x}_{\text{ref}}(0), y_{\text{ref}}(0), \dot{y}_{\text{ref}}(0)) &= (3, 1, 4, 2).
\end{align*}

The spherical pendulum which needs to follow the motion of this reference using the almost global tracking control is given by initial conditions \((x(0), \dot{x}(0), y(0), \dot{y}(0)) = (20, 20, 30, 20)\). The tracking results shown below were simulated using MATLAB®R2015b using an ode45 solver for generating for reference (in orange) and actual (in blue) trajectories.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{(a)\(x(t), x_{\text{ref}}(t)\), (b)\(\dot{x}(t), \dot{x}_{\text{ref}}(t)\), (c)\(y(t), y_{\text{ref}}(t)\), (d)\(\dot{y}(t), \dot{y}_{\text{ref}}(t)\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{(a)Control effort \(\|u(t)\|_2\) for \(S^2\), (b) Control effort \(\|u(t)\|_2\) for \(SO(3)\).}
\end{figure}

\section{AGAT on Lie groups}

\subsection{Almost-global tracking law}

In this section, we specialize the AGAT control law in \([5]\) to an SMS on a Lie group with a particular choice of the configuration error map \(E\). Therefore the problem of AGAT is reduced to finding an appropriate navigation function according to definition 3. This also enables us to express the control law in a coordinate free or intrinsic manner which might be useful in many contexts. The key difference in what follows compared to existing literature in AGAT on Lie groups is the consideration of velocity error along the error trajectory. Let \(G\) be a Lie group and let \(g\) denote its Lie algebra. Let \(\phi : G \times \mathbb{R} \to G\) be the left group action in the first argument defined as \(\phi(g, h) := L_g(h) = gh\) for all \(g, h \in G\). The Lie bracket on \(g\) is \([\cdot, \cdot]\). Let \(Ad_g : g \to g\) be defined as \(Ad_g(\xi) := T_e L_g R_g^{-1} \xi\). The tangent map to \(Ad_g\) is called adjoint map, \(ad_g : g \to g\) for \(\xi \in g\) and defined as \(ad_g \xi := \frac{d}{dt}|_{t=0} Ad_{exp(t\xi)}\). Let \(I : g \to g^*\) be an isomorphism on the Lie algebra to its dual and the inverse is denoted by \(I^\dagger : g^* \to g\). \(I\) induces a left invariant metric on \(G\) (\([5]\)), which we denote by \(G_1\) and define by the following.

\[G_1(g, (X_g, Y_g)) = \langle (T_g L_g^{-1}(X_g)), T_g L_g^{-1}(Y_g) \rangle\]

for all \(g \in G\) and \(X_g, Y_g \in T_g G\).

\begin{lemma}
Given a differentiable parameterized curve \(\gamma : \mathbb{R} \to G\) and a vector field \(X\) along \(\gamma(t)\) we have the following equality
\begin{align*}
\nabla_{\dot{\gamma}} X &= T_e L_{\dot{\gamma}} \left( \frac{d}{dt} \left( T_{\gamma} L_{\gamma^{-1}} X(t) + \frac{\partial}{\partial T_{\gamma} L_{\gamma^{-1}} \gamma} T_{\gamma} L_{\gamma^{-1}} X(t) \right) \right) \\
\text{where } \nabla \text{ is the bilinear map defined as }
\nabla_{\eta} \xi &= \frac{1}{2} [\eta, \xi] - \frac{1}{2} \bar{\xi} (ad^*_\eta \xi + ad^*_\xi \eta)
\end{align*}

for \(\xi, \eta \in g\).
\end{lemma}

\begin{proof}
Let \(\{e_1, \ldots, e_n\}\) be a basis for \(g\). Let,
\[X(\gamma(t)) = T_e L_{\gamma}(\sum_{i=1}^{n} v_i^X(\gamma(t)) e_i) = \sum_{i=1}^{n} v_i^X(\gamma(t)) (e_i)_L\gamma(t)\]
where \(v_X = T_{\gamma} L_{\gamma^{-1}} X(\gamma(t))\) and \((e_i)_L\gamma(g) = T_e L_g e_i\) and therefore \((e_i)_L\gamma(g)\) is a basis for \(T_g G\) for all \(g \in G\). Similarly let
\[\dot{\gamma}(t) = T_e L_{\gamma}(\sum_{i=1}^{n} v_i^\gamma(t) e_i) = (\sum_{i=1}^{n} v_i^\gamma(t) (e_i)_L\gamma(t))\]
where \(v_\gamma(t) = T_{\gamma} L_{\gamma^{-1}} \dot{\gamma}(t)\). Using properties of affine connection we have,
\begin{align*}
\nabla_{\dot{\gamma}} X &= \nabla_{v_\gamma(t)} (v_i^X) (e_i) = \frac{d}{dt} \left( v_i^X (e_i)_L\gamma(t) + v_k^X v_i^X \nabla_{(e_i)_L\gamma} (e_k)_L\gamma(t) \right) \\
&= T_e L_{\gamma}(\frac{d}{dt} v_i^X) e_i + v_k^X v_i^X \nabla_{(e_i)_L\gamma} (e_k)_L \\
&= T_e L_{\gamma}(\frac{d}{dt} \left( T_{\gamma} L_{\gamma^{-1}} X(t) + \nabla_{T_{\gamma} L_{\gamma^{-1}} \gamma} T_{\gamma} L_{\gamma^{-1}} X(t) \right) )
\end{align*}
\end{proof}

Let the configuration error be defined as \(E(g, g_r) = g_r g^{-1}\). For parameterized curves \(g : \mathbb{R} \to G\) and \(g_r : \mathbb{R} \to G\) we define the error curve is \(E(t) = g(t) g_r^{-1}(t)\). The derivative is
\[\dot{E} = -g_r g^{-1} \dot{g} g^{-1} + \dot{g}_r g^{-1}\]

Hence, \(d_1 E(\dot{g}) = -g_r g^{-1} \dot{g} g^{-1} = T_{g_r} L_{g^{-1}} R_{g^{-1}}\) and \(d_2 E(\dot{g}_r) = \dot{g}_r g^{-1} = T_{g_r} R_{g^{-1}} \dot{g}_r\). The derivative of the error trajectory is given by
\[\nabla_{\dot{E}} \dot{E} = \nabla_{\dot{E}} \left( d_1 E(\dot{g}) + d_2 E(\dot{g}_r) \right)
\]
where $\xi := T_g L_{g^{-1}} \dot{g}$ and similarly $\nabla_E (d_2 E(\dot{g}_r))$ is

$$\nabla_E (d_2 E(\dot{g}_r)) = \frac{d}{dt} (E^{-1} d_2 E(\dot{g}_r)) + \frac{\partial}{\partial \dot{g}_r} (E^{-1} d_2 E(\dot{g}_r))$$

Hence from (16),

$$\hat{G} \dot{E} = E(g \xi g^{-1} + \frac{\partial}{\partial \dot{g}_r} E^{-1} d_1 E(\dot{g}))$$

(17)

$$= E(g \xi g^{-1} + \frac{\partial}{\partial \dot{g}_r} E^{-1} d_1 E(\dot{g})) + \frac{d}{dt} (E^{-1} d_2 E(\dot{g}_r))$$

Let $(E, \psi)$ be an appropriate choice for the tracking problem. We define the intermediate control $u_1 \in T_{E} G$ as

$$\hat{G} \dot{E} = E^{-1} d_1 E(\dot{g})$$

(18)

where $u_1 = \hat{G} (\nabla_E \dot{E} - d_\psi(E) + F_{diss} \dot{E})$. Further the actual dynamics of the SMS with control vector field $u \in \mathfrak{g}$ are,

$$\dot{\xi} = I^\xi a_d^{\xi} I \xi = u$$

Therefore $\dot{\xi} = u + I^\xi a_d^{\xi} I \xi$ and putting these expressions in (17) and substituting $\eta = E^{-1} \dot{E}$ we get

$$u_1 = E(-g(u + I^\xi a_d^{\xi} I \xi) g^{-1} + \frac{\partial}{\partial \dot{g}_r} (E^{-1} d_2 E(\dot{g}_r)))$$

which gives

$$u = g^{-1}(-E^{-1} u_1 + \frac{\partial}{\partial \dot{g}_r} (E^{-1} d_2 E(\dot{g}_r)))$$

(19)

$$= g^{-1}(-E^{-1} u_1 + \frac{\partial}{\partial \dot{g}_r} (E^{-1} d_2 E(\dot{g}_r))) + I^\xi a_d^{\xi} I \xi$$

There,fore,

$$g^{-1} \left( \frac{d}{dt} E^{-1} d_2 E(\dot{g}_r) \right) = g^{-1} \left( \dot{g} \Omega_d g^{-1} + g \Omega_d g^{-1} \right)$$

(20)

$$- g \Omega_d g^{-1} (g g^{-1} \dot{g} g^{-1}) = g^{-1} \dot{g} \Omega_d + \dot{\Omega}_d - \Omega_d \dot{g}^{-1} \dot{g}$$

$$= \dot{\Omega}_d + \dot{\Omega} - \Omega_d \dot{\Omega} = \dot{\Omega}_d + [\Omega, \Omega_d]$$

where $[\cdot, \cdot]$ is the vector cross product because of isomorphism $\sim$.

The modified trace function $\psi(R) = trace(P(I - R))$ where $P$ is a positive definite symmetric matrix is a navigation function on $\text{SO}(3)$ as shown in [8] and used for AGAS tracking of the rigid body in [7] and [10]. As $T_R L_{R} d\psi(R) = \text{skew}(PR)$, the intermediate control $u_1 \in T_E G$ in (18) is $u_1 = \hat{G} (\nabla_E \dot{E} + F_{diss} \dot{E})$.

$$\hat{G} u_1 = -d_\psi(E) + F_{diss} \dot{E}$$

By definition of $\hat{G}$,

$$T_e L_{E^{-1}} (T_e L_{E^{-1}} u_1) = -d_\psi(E) + F_{diss} \dot{E}$$

Therefore,

$$E^{-1} u_1 = \nabla_E \dot{E}$$

(21)

The bilinear operator $\nabla_\eta \eta$ defined in (14) for $\eta = E^{-1} \dot{E}$ is

$$\nabla_\eta \eta = \frac{1}{2} \nabla_\eta (2a_d^{\eta} \eta) = -\nabla_\eta a_d^{\eta} \eta.$$  

(22)

Further, considering the isomorphism $\sim$ and $\eta \in \mathbb{R}^3$, $a_d^{\eta} \eta = [\eta, \eta]$. Therefore, from (20), (21) and (22), the control $u$ in (19) is

$$u = -g^{-1} (E^{-1} u_1 - \nabla_\eta \eta) + \dot{\Omega}_d[\Omega, \Omega_d] - \nabla_\eta a_d^{\eta} \eta$$

(23)

The consolidated closed loop dynamical equations for the externally actuated rigid body are

$$\dot{R} = R \dot{\Omega}$$

(24a)

$$\dot{\Omega} = I^\xi a_d^{\xi} \Omega = \dot{u}$$

(24b)

and substituting for $u$ from (23) in (24b),

$$\dot{\Omega} = Ad_{R^{-1}} (E^{-1} u_1 - \nabla_\eta \eta) + \dot{\Omega}_d + [\Omega, \Omega_d]$$

(25)

where given a smooth bounded reference trajectory $R_d : \mathbb{R} \rightarrow \text{SO}(3)$, we define $E = R_d R^{-1}$,

$$\eta = E^{-1} \dot{E} = Ad_{R}(\dot{\Omega}_d - \dot{\Omega})$$

$\text{Remark 1:}$ The key difference in our AGAS tracking law for Lie groups and those in [3], [7] and [5] is consideration of the velocity error $\dot{E}$ in $T_E G$ instead of $\dot{g}$. The group identity is not considered as an equilibrium point for error dynamics (as in [7]), before choosing the navigation function.

$\text{Remark 2:}$ The acceleration of the error trajectory $E(g(t), g_r(t))$ in $\mathfrak{g}$ is given as $Ad_{g^{-1}} (\nabla_\eta \eta)$, where $\eta = E^{-1} \dot{E}$. This term appears in addition to the usual terms seen in [5], [7] and [3].

$\text{Remark 3:}$ It is observed that the AGAT control law for an SMS on a Lie group in [19] is intrinsic or coordinate free whereas the AGAT control law for an SMS in general (5) is not. This is a direct consequence of use of Lemma 1 to simplify (5).
C. Simulations

We simulate the AGAS tracking result for a rigid body actuated by external thrusters. The thrust input is given by (25) and the dynamical equations are (24). The reference trajectory is generated by a dummy rigid body with inertia matrix

\[ \mathbb{I}_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \]

The actual rigid body has an inertia matrix given by \( \mathbb{I} = \begin{pmatrix} 4 & 1 & 1 \\ 15 & 0.2 & 2 \\ 1 & 2 & 6.3 \end{pmatrix} \). The initial conditions are \( R(0) = \begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0 \\ 0.48 & 0.64 & 0.60 \end{pmatrix} \) and \( \Omega = \begin{pmatrix} 1 & 2.2 & 5.1 \end{pmatrix} \). The actual rigid body is generated by a dummy rigid body with inertia matrix \( \mathbb{I}_d(3,3) \) for almost-global stabilization of the error dynamics.

The simulations are performed using MATLAB® R2015b and ode45 solver is used to generate the reference (in orange) and actual (in blue) trajectories.

Fig. 6: (a) \( R(1,1) \) and \( R_d(1,1) \), (b) \( R(1,2) \) and \( R_d(1,2) \), \( R(1,3) \) and \( R_d(1,3) \)

Fig. 7: (a) \( R(2,1) \) and \( R_d(2,1) \), (b) \( R(2,2) \) and \( R_d(2,2) \), \( R(2,3) \) and \( R_d(2,3) \)

Fig. 8: (a) \( R(3,1) \) and \( R_d(3,1) \), (b) \( R(3,2) \) and \( R_d(3,2) \), \( R(3,3) \) and \( R_d(3,3) \)

VI. CONCLUSIONS AND FUTURE WORK

In this note we have established AGAT of a bounded reference trajectory on a compact Riemannian manifold by introducing a configuration error map and using a navigation function for almost-global stabilization of the error dynamics. The results have been demonstrated for an SMS on \( SO(3) \) and \( S^2 \). However, construction of such an appropriate configuration error map for any arbitrary, compact Riemannian manifold is not addressed. This is a possible area of investigation in the future.

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