From Liouville to Chern-Simons, Alternative Realization of Wilson Loop Operators in AGT Duality

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Abstract

We propose an $SL(2,\mathbb{R})$ Chern-Simons description of Liouville field theory (LFT), whose correlation function duals to partition function of $\mathcal{N} = 2$ $SU(2)$ gauge theories. We give the dual expressions for conformal blocks, fusion rules, and Wilson loop operators. By realizing Wilson loop operator in Liouville as a Hopf link in $S^3$ on which lives an $SL(2,\mathbb{R})$ Chern-Simons theory, we obtain an alternative description of monodromy of this loop operator in Liouville field theory as the ratio of link invariants. We show how to calculate t’Hooft loops in the simplest example – the $\mathcal{N} = 4$ super Yang-Mills theory. The results we obtained are consistent with those in 0909.0945 and 0909.1105.
1 Introduction

Recently, Alday, Gaiotto and Tachikawa (AGT) [5] established a new duality between Liouville theory and four dimensional $\mathcal{N} = 2$ gauge theories. These $\mathcal{N} = 2$ gauge theories can be obtained by compactifying coincided $M5$ branes on specified Riemann surfaces $C$ with punctures [2], which builds a bridge between 2d and 4d field theories, thus theories living on one side will obtain new information and then benefit from the other. Later after that, many people generalized this duality to various situations [10, 12]. Also there are some developments on this new duality from different features [11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. Dijkgraaf and Vafa [9] even proved this duality at a very general level in two different approaches. In [5], it was shown that the Nekrasov partition functions [38, 39, 40] of the generalized quiver gauge theories on $\mathbb{R}^4$ are identical to the correlation functions in Liouville field theory. In this duality, the Liouville momenta at the marked points specify the masses of the flavor multiplets, while the momenta in the intermediate channels are identified as the Coulomb branch parameters. Very recently, two groups (Gaiotto et al [6] and Drukker et al [8]) calculated Wilson loop operators in gauge theories using the dual Liouville language, the results perfectly match with those in gauge theories. This can also be seen as a strong test of the AGT duality.

Loop operators in $\mathcal{N} = 2$ $SU(2)$ 4d quiver gauge theories are believed to be monodromies of related conformal blocks in the dual Liouville theory. To be precise, a loop operator in $\mathcal{N} =$
2 $SU(2)$ gauge theories can be identified, in the Liouville theory side, as an insertion of Liouville loop operator in related conformal blocks of correlation functions which involve two chiral degenerated primary operators. In order to obtain the loop operator, we need a long complicated calculation in which fusion and braiding matrices are highly involved. To simplify this, we noticed that one can always realize a loop operator on Riemann surface $C$ as a world line of a charged particle and then gluing the time direction back to its origin. This actually visualizes the loop operator as a knot or link in the extended world-volume of the Riemann surface $C$. Using the standard surgery method, one can always glue this world-volume as $S^3$, the eigenvalue of a loop operator acting on a specific conformal block is identified to a correlation function of a related knot or link in $S^3$ on which there is a topological invariant theory. This surgery method is very similar with that used in Chern-Simons-Witten(CSW) \cite{1} theory. The CSW theory is a dual geometric description of Wess-Zumino-Witten(WZW) models which are rational conformal field theories(RCFT) \cite{44}. We propose that the 3D dual theory is exactly a Chern-Simons theory with gauge group $SL(2, \mathbb{R})$. This conjecture, however, is not arbitrary since there is a alternative realization of Liouville theory as a gauged $SL(2, \mathbb{R})$ WZW model \cite{45, 46, 47}, which is believed to dual to an $SL(2, \mathbb{R})$ Chern-Simons theory \cite{49, 50, 51, 52, 53}. In another way, to understand the physical origin of the Liouville filed on the modular geometry, we expect that this Liouville theory comes from a mother theory which has M branes source, like a Chern-Simons type theory of M2 branes \cite{59, 60}. Even though we have not known all details of the modular theory, we will give a direct derivation from Chern-Simons to Liouville theory as a realization on this proposal. The derivation is slightly different from the well known CSW-WZW correspondence. Surprisingly, the ingredients of Liouville theory have very simple expressions in Chern-Simons theory side. The holomorphic part of Liouville correlation function has been related to knot invariant; fusion rules correspond to loop algebras; braiding becomes passing through of two lines; sewing of conformal blocks becomes surgery operation; different normalization of Liouville partition function corresponds to different surgeries. These correspondences offer enough tools to deal with loop operators.

Using this proposal, realizing Wilson loop operator in Liouville as a Hopf link in $S^3$, we obtain dual descriptions of these monodromies in $SL(2, \mathbb{R})$ Chern-Simons theory which completely depend on the modular properties of the related affine algebra. They are ratios of corresponding correlation functions of links (links invariants) in three dimension, which can be easily calculated by standard surgery method. Now we have not to know details about monodromy matrices and braiding factors. The only ingredients are the modular S matrices of the affine algebra. This method simplifies the calculation a lot. In general, all monodromies which related to general loop operators have dual descriptions as ratios of links invariants in $S^3$ with an $SL(2, \mathbb{R})$ Chern-Simons action. If this conjecture is valid in more general situations, then one could expect this could be easily used in $\mathcal{N} = 2$ $SU(N)$ gauge theories and Toda field theories. Thus, the problem refers to the modular S matrices and their analytic continuations of characters of affine $sl(n)$ algebra. There remain further works on it.

The structure of the paper is as following. In section 2, we reviewed some important results for AGT duality and loop operators in this duality, including the dictionary, loops in gauge theory, loops in liouville theory. We also reviewed the modular bootstrap for Liouville field
theory in this section. In section 3, we showed that from an $SL(2, \mathbb{R})$ Chern-Simons on a manifold without boundary, one can extract a gauged $SL(2, \mathbb{R})$ WZW model which is exactly equivalent to a Liouville theory at the level of partition function. In section 4, we gave the equivalent relations for Liouville ingredients and we calculated Wilson loops in general case and checked the simplest t’Hooft loop. Section 5 leaves the conclusions.

2 Loop Operators in AGT Duality

In this section we review the relation between loop operators in $\mathcal{N} = 2$ gauge theories and Liouville theory briefly. For details, see original works by Gaiotto et al [5] [6] and Drukker et al [8].

2.1 AGT Duality

It was shown in [2] a large class of four dimensional $\mathcal{N} = 2$ $SU(2)$ SCFTs can be obtained by compactifying the six-dimensional $(2, 0)$ theory of type $A_1$ on a Riemann surface with punctures. Each puncture is associated to an $SU(2)$ flavor symmetry, which can be used to give mass to the hypermultiplets. Each SCFT in this class can be labeled by two integers $g, n$, which are the genus and the number of punctures of the Riemann surface $C_{g, n}$. The parameter space of the theory coincides with the complex moduli space of the punctured Riemann surface.

| Gauge Theory | Liouville Theory |
|--------------|------------------|
| Deformation parameters $\epsilon_1, \epsilon_2$ | Liouville parameters $\epsilon_1 : \epsilon_2 = b : 1/b$ |
| Mass parameter $m$ associated to an $SU(2)$ flavor | Insertion of a Liouville vortex operator $e^{2m\varphi}$ |
| one $SU(2)$ gauge group with UV coupling $\tau$ | a thin channel with sewing parameter $q = \exp(2\pi i \tau)$ |
| Vacuum expectation value $a$ of an $SU(2)$ gauge group | Primary $e^{2a\varphi}$ for the channel $\alpha = Q/2 + a$ |
| Instanton part of $Z$ | Conformal blocks |
| One-loop part of $Z$ | Product of DOZZ factors |
| Integral of $|Z_{full}^2|^2$ | Liouville correlator |

Table 1: Dictionary between the Liouville correlation functions and Nekrasov’s partition function $Z$. This table comes directly from [5].

Given a genus-$g$ Riemann surface with $n$ punctures and a particular sewing of the surface from three-punctured spheres, consider the generalized quiver gauge theory naturally associated to it. Then, the AGT duality [5] is as the following statement: the conformal block for
This sewing is the instanton part of Nekrasov’s partition function of this gauge theory [38]. Furthermore, the n-point function of the Liouville theory on this Riemann surface is equal to the integral of the absolute value squared of Nekrasov’s full partition function of this gauge theory. The dictionary of this duality is in table 1.

2.2 Surface and Loop Operators in $\mathcal{N} = 2$ $SU(2)$ Gauge Theories

There are three gauge invariant operators in gauge theories which can be obtained from compactifying of two M5 branes on Reimann surfaces: surface, line and point operators [3][6]. These operators are all connected to M2 branes which are attached to M5 branes. For the present aim, we will only review surface operators and loop operators.

The surface operators are defined by considering an M2 boundary surface $S$ to be embedded in the 4d space-time $\mathbb{R}^4$ and localized as a point $z$ on $C$. As in [3][6], the expectation value of the elementary surface operator in the $\mathcal{N} = 2$ $SU(2)$ gauge theory is related to an insertion of a degenerate primary operator $V_{1,2}(z) = e^{(b/2)\phi(z)}$ into Liouville correlation function. The notation of the degenerate primary operator will be clarified in the next subsection. It should be reminded that we are now considering gauge theory living on $\mathbb{R}^4$ which can be seen as a “chiral” part of the same gauge theory on $S^4$. By the hemispherical stereographic projection of $S^4$ onto two copies of $\mathbb{R}^4$, this scenario is shown clearly in Fig.3 of [6].

The line or loop operators are represented by M2 brane boundaries that wrap a circle $\gamma$ on $C$, and extend along an infinite line or closed loop $C$ in $\mathbb{R}^4$. A Wilson-t’Hooft loop operator is labeled by the circle $\gamma$ and can be computed in Liouville theory. It is proposed in [6] that the expectation values of loop operators are identical to associated monodromies in Liouville theory. The physical explanation is as following: consider the annihilation of two identical surface operators in $S^4$, both are at the same position in $S^4$ and $C$, except that one of them has traveled along a circle $\gamma$. Thus there exists a discontinuity between them. This discontinuity defect can be identified as a monodromy in Liouville language. Now recall that the Nekrasov partition function on $S^4$ is equal to the full Liouville correlation function, thus on $\mathbb{R}^4$, we are dealing with the chiral conformal block of the Liouville correlation function. Finally, the monodromy can be recognized as the effect that a chiral primary operator travels along a nontrivial circle $\gamma$ once.

2.3 Loop operators in Liouville theory

We will now review loop operators dual to those in 4d gauge theories. Modular bootstrap of Liouville theory will also be briefly reviewed for further usage.
2.3.1 Liouville Field Theory

The action describing Liouville theory on an arbitrary genus $g$ Riemann surface with $n$ punctures $C_{g,n}$ is given by:

$$S = \frac{1}{4\pi} \int d^2z (g^{ab} \partial_a \phi \partial_b \phi + Q R \phi + 4\pi \mu e^{2b\phi}),$$  \hfill (2.1)$$

where $Q$ is the background charge, $\mu$ is the cosmological coupling constant. Liouville field theory (LFT) is a conformal field theory if and only if $Q = b + 1/b$ is satisfied. The central charge is: $c = 1 + 6Q^2$ (See reference [29] for a review of LFT). Primary fields are of the form $V_\alpha = e^{2\alpha \phi}$, and have conformal weight

$$h_\alpha = \alpha (Q - \alpha).$$  \hfill (2.2)$$

Note that primaries $V_\alpha$ and $V_{Q-\alpha}$ have the same conformal weight and are closely related. This will bring some ambiguities in calculation of correlation functions. More precisely, the Liouville reflection amplitude reads [31]

$$\mathcal{R}_L(\alpha) = - (\pi \mu \gamma(b^2))^{(Q-2\alpha)/b} \frac{\Gamma(1 - (Q - 2\alpha)b) \Gamma(1 - (Q - 2\alpha)/b)}{\Gamma(1 + (Q - 2\alpha)b) \Gamma(1 + (Q - 2\alpha)/b)},$$  \hfill (2.3)$$

and allows us to write $V_\alpha = \mathcal{R}_L(\alpha) V_{Q-\alpha}$, a relation which holds in any correlation function. Here $\gamma(x) = \Gamma(x)/\Gamma(1-x)$.

Physical (unitary) representations are obtained for

$$2\alpha = Q + is,$$  \hfill (2.4)$$

with $s \in \mathbb{R}$, which can be further restricted to $s \in \mathbb{R}^+$ because of the reflection relation. They are the so-called non-degenerate spectrum (or continuum spectrum). There exist degenerate representations which can be labeled by two coprime numbers $(m, n)$. The charge (Liouville momentum) is given by

$$2\alpha = b^{-1}(1 - m) + b(1 - n).$$  \hfill (2.5)$$

We denote the corresponding vertex operator at $z$ as $V_{m,n}(z)$. However, degenerate spectrums are not unitary representations except the $(1, 1)$ state which corresponds to the basic vacuum in LFT [33] [34].

The conformal bootstrap allows us to compute n-point correlation functions which can be formally written as:

$$\mathcal{Z}_{S^4} = \langle \prod_{a=1}^n V_{m_a}(z_a) \rangle_{C_{g,n}} = \int d\nu(\alpha) \mathcal{F}^{(\sigma)}_{\alpha,E} \mathcal{F}^{(\sigma)}_{\alpha,E},$$  \hfill (2.6)$$

where $\mathcal{F}^{(\sigma)}_{\alpha,E}$ denotes conformal block associated to the sewing $\sigma$, $\alpha \equiv \{\alpha_1, \ldots, \alpha_{3g-3+n}\}$ label the internal Liouville momenta associated to the sewing of conformal blocks, while $E \equiv \{m_1, \ldots, m_n\}$ label the external Liouville momenta related to the masses of hypermultiplets in $\mathcal{N} = 2$ gauge theory. The measure $\nu(\alpha)$ includes for each trivalent graph dissection of the conformal block. The explicit expression for $\nu(\alpha)$ was introduced in [32], and derived in [37].
After absorbing the prefactor into the conformal block, one will arrive at the simple expression used in [6]:

\[ Z_{S^4} = \int \, da_i | \langle \prod_{a=1}^{n} V_{m_a}(z_a) \rangle_{\{a_i\}} |^2, \tag{2.7} \]

where the lower index \( \{a_i\} \) labels the channel of conformal blocks and \( 2a_i = 2\alpha_i - Q \). This partition function is non-chiral and invariant under the modular S transformation. However, as reviewed in last section, the loop operator can be seen as an action of a chiral degenerate vertex operator on a conformal block of the full partition function. Thus the main interesting thing is actually the block

\[ Z_{4d} = \langle \prod_{a=1}^{n} V_{m_a}(z_a) \rangle_{\{a_i\}} \tag{2.8} \]

with an insertion of chiral degenerate vertex operator.

For the convenience of computation, it is useful to introduce another normalized conformal block \( G_{\alpha,E}(\sigma) \) which was first proposed by Ponsot and Teschner in [36]. This conformal block can be related to \( F_{\alpha,E}(\sigma) \) by normalization, as in [8]. The partition function in \( S^4 \) now can be written as:

\[ \hat{Z}_{S^4} = \int d\mu(\alpha) \bar{G}_{\alpha,E}^{(\sigma)} G_{\alpha,E}^{(\sigma)}. \tag{2.9} \]

The partition function \( \hat{Z}_{S^4} \) is slightly different with the former one \( Z_{S^4} \) due to the normalization, where the measure \( d\mu(\alpha) \) is:

\[ d\mu(\alpha) = \prod_{i=1}^{3g-3+n} d\alpha_i(-4\sin(2\pi\alpha_i b)\sin(2\pi\alpha_i b^{-1})). \tag{2.10} \]

Later on, we will see that this is just a modular S transformation of the former up to an irrelevant normalization number. So conformal blocks \( Z_{4d} \) and \( G_{\alpha,E}^{(\sigma)} \) are related by an modular S transformation. This can be further clarified via the 3d surgery method which we will focus on in section 4.

2.3.2 Loop Operators in Liouville Field Theory

The computation strategy of loop operators in LFT is clear now. First, introduce an identity operator in the conformal block. Second, split it into two chiral degenerate operators \( V_{1,2}(z) \) \(^3\). Third, let one of these two operators round along a circle which labels the loop one time. Finally, glue the two operators back to identity. For Wilson line, this is very simple. One will only consider \( V_{1,2}(z) \) round another internal vertex operator which can be associated to the sewing of conformal blocks\(^4\) once [54]. For t’Hooft line, this becomes complicated because now \( V_{1,2}(z) \) should travel around all the background, including all handles and punctures. However, the computation of these quantities are all considered and finished in [6] and also [7, 8]. For details, one may refer to these two articles [7, 8].

\(^3\)The corresponding charged particles are in the spin 1/2 representation, in general case, the loop operator can be in arbitrary spin representation of SU(2). For a \( j/2 \) spin loop operator, the associate chiral degenerate operator is \( V_{j,l} \) with \( l = 2j + 1 \).

\(^4\)Or equivalently, the thin tube which connects two pants components of the Riemann surface \( C \).
### 2.3.3 Modular Bootstrap

Another way to invoke the same LFT is the so-called modular bootstrap. This method was introduced by Zamolodchikov brothers [31] and further developed by Jego and Troost [33] and Eguchi et al [34, 35]. The basic ingredients are characters of highest weight states including degenerate and non-degenerate ones. For non-degenerate representations $(2\alpha = Q + is, s \in \mathbb{R}^+)$, the character and conformal dimension are

$$
\chi_s(\tau) = \frac{q^{s^2/4}}{\eta(\tau)}, \quad h_s = \frac{1}{4}(Q^2 + s^2),
$$

where $\eta$ is the Dedekind function, $q = e^{2\pi \tau}$. For degenerate representations (which for non-rational $b$ have a single null vector at level $nm$), the character and conformal dimension are

$$
\chi_{m,n}(\tau) = \frac{q^{-(m/b+nb)^2/4} - q^{-(m/b-nb)^2/4}}{\eta(\tau)}, \quad h_{m,n} = \frac{1}{4}(Q^2 - (m/b + nb)^2).
$$

The modular transformations of the characters are [33, 34, 35]

$$
\chi_s(-\frac{1}{\tau}) = \int_0^{\infty} S_{s}^{s'}(\tau) ds', \quad S_{s}^{s'} = \sqrt{2}\cos(\pi ss'),
$$

$$
\chi_{m,n}(-\frac{1}{\tau}) = \int_0^{\infty} S_{m,n}^{s',n'}(\tau) ds', \quad S_{m,n}^{s',n'} = 2\sqrt{2}\sinh(\pi m s'/b)\sinh(\pi nb s'/n'),
$$

$$
S_{m,n}^{m',n'} = -2\sqrt{2}\sin((\pi mb^{-1}(m'b^{-1} + n'b))\sin((\pi nb(m'b^{-1} + n'b)).
$$

The third modular S transformation for degenerate states can be obtained from analytic continuation of the second one. However, as we mentioned before, these degenerate representations are not unitary. They become unitary only if there exists a bigger system in which LFT as a subsystem. The most possible system is the supersymmetric extension of LFT. Actually, it is well known [35] that supersymmetric Liouville theory does have unitary degenerate representations. This implies that the good dual theory for $\mathcal{N} = 2$ SCFTs is a supersymmetric version of LFT. We hope further studies will clarify this.

Notice that these modular S transformations come from the affine algebra. It is just another representation of CFT since there exists one to one correspondence between CFTs and quantum groups representations [44].

### 3 From Chern-Simons to Liouville

A Chern-Simons theory with compact gauge group on (2+1)d is an exact dual description of (1+1)d WZW model with the same gauge group. This is clarified in Witten’s illuminating work [1] two decades ago. However, for non-compact group, this duality is far from clear on both sides. Fortunately, for $SL(2, \mathbb{R})$ considered in present situation, there are some important...
developments\textsuperscript{[49, 50, 51, 52]} which we will briefly review in the following text. The action of $SL(2, \mathbb{R})$ Chern-Simons theory on 3d manifold $M$ can be written as

$$I_{CS}[A] = \frac{k}{2\pi} \int_M \text{Tr} \{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \},$$

(3.1)

with $A$ taking value in the Lie algebra space $\mathbb{H}$ of $SL(2, \mathbb{R})$. “Tr” is an invariant form on $\mathbb{H}$. If $M$ is compact, this action is gauge invariant, but this is not the case when $M$ is a 3-manifold with boundary $\Sigma$: under a gauge transformation

$$A = g^{-1}dg + g^{-1}A^g,$$

(3.2)

the action $I_{CS}$ transforms as

$$I_{CS}[A] = I_{CS}[\bar{A}] - \frac{k}{4\pi} \int_{\Sigma} \text{Tr} ((dgg^{-1}) \wedge \bar{A}) - \frac{k}{12\pi} \int_M \text{Tr}(g^{-1}dg)^3.$$

(3.3)

Naively, if one chooses the 3-manifold as the topologically trivial one: $\mathbb{R} \times D$, where $D$ is a disk, this means that gauge potential $A$ in the bulk is a “pure gauge”

$$A = g^{-1}dg.$$

(3.4)

Substituting this into (3.3), a straightforward computation shows that the resulting action for $g$ is a WZW action on the boundary $\partial M = \mathbb{R} \times S^1$\textsuperscript{[53]}. For a general 3-d manifold with boundary, it is necessary to introduce proper boundary condition and boundary counterterm if one would like to keep gauge invariance in the bulk. A proper boundary condition should constrain half degrees of the phase space. However, in Chern-Simons theory, it can be easily read off from the canonical quantization that the gauge potentials $A$ are both canonical positions and momenta. Thus the natural choice is to fix the value of boundary gauge potential, or equivalently, choose a complex structure on $\Sigma$. Without loss of generality, we fix the value for $A_z$ and the boundary term now can be written as

$$I_{bdry}[A] = \frac{k}{4\pi} \int_{\Sigma} \text{Tr} A_z A_{\bar{z}},$$

(3.5)

which transforms as

$$I_{bdry}[A] = I_{bdry}[\bar{A}] + \frac{k}{4\pi} \int_{\Sigma} \text{Tr}(\partial_z gg^{-1} \partial_{\bar{z}} gg^{-1} + \partial_z gg^{-1} A_{\bar{z}} + \partial_{\bar{z}} gg^{-1} A_z).$$

(3.6)

Now the full action transforms as

$$(I_{CS} + I_{bdry})[A] = (I_{CS} + I_{bdry})[\bar{A}] + k I_{WZW}^+[g^{-1}, \bar{A}],$$

(3.7)

with the chiral WZW action

$$I_{WZW}^+[g^{-1}, \bar{A}] = \frac{1}{4\pi} \int_{\Sigma} \text{Tr}(\partial_z gg^{-1} \partial_{\bar{z}} gg^{-1} - 2g^{-1} \partial_z g A_{\bar{z}}) + \frac{1}{12\pi} \int_M \text{Tr}(g^{-1}dg)^3.$$

(3.8)

Note that now $g$ is a dynamical field on $\Sigma$, which implies that pure gauge transformation in bulk becomes real symmetry on boundary. The additional degree of freedom\textsuperscript{b} is just a result from reducing the second class constraints to the first class constraints.

\textsuperscript{b}Here, we refer to the degrees of freedom of $g^{-1}$. 8
To get a CFT dual description for Chern-Simons on 3-manifold $M$ without boundary, one can cut $M$ into two pieces $M_1$ and $M_2$ with the same boundary $\Sigma$; on each piece there is a Chern-Simons. By gluing these two pieces back into $M$ carefully, one can get a Chern-Simons on $M$. From the CFT side of view, one just combines a chiral WZW theory and an anti-chiral WZW theory to a non-chiral WZW theory on $\Sigma$. This is feasible when a CFT is holomorphic factorizable [48]. The “sewing” of WZW [54] models with gauge fields $A_\bar{z}^+$ and $A_\bar{z}^-$ can be realized using Hamiltonian reduction method [45, 46, 47]. In the process of “sewing”, one should introduce additional constraints. The simplest case is $A_\bar{z}^+ = A_\bar{z}^- = 0$, say, the gauge fields vanish simultaneously. Now the “sewing” is trivial:

$$\partial_- J_+ = 0, \quad \partial_+ J_- = 0,$$

(3.9)

where we have changed the labels $\partial_+ \equiv \partial_{\bar{z}}, \partial_- \equiv \partial_z$ for further simplicity. $J_+$ and $J_-$ are left and right Kac-Moody currents respectively:

$$J_+ = (\partial_+ g) g^{-1}, \quad J_- = g^{-1} (\partial_- g).$$

(3.10)

These are just equations of motion for left and right $SL(2,\mathbb{R})$ invariant vector fields $g^{-1}dg$ and $dgg^{-1}$ respectively. Before rushing to the off-shell situation, we should keep under observation on gauge fields $A_+$ and $A_-$. These fields naturally introduce a complex structure and further define inner products on $M_1$ and $M_2$

$$\partial_+ \mapsto \partial_+ - A_+, \quad \partial_- \mapsto \partial_- + A_-,$$

(3.11)

where the different definition comes from the opposite chirality. If one drags $A_+$ to $M_2$ (or drags $A_-$ to $M_1$), it is necessary to change $A_+$ ($A_-$) to its right(left) invariant form $g^{-1}A_+g$ ($gA_-g^{-1}$). Now we can introduce the gauge-invariant action

$$I[g, A_+, A_-] = I_{WZW}[g] - \frac{k}{2\pi} \int_\Sigma \text{Tr} \{ (A_- (\partial_+ g) g^{-1} - \sqrt{\mu}) + (g^{-1} \partial_- g A_+ - \sqrt{\mu}) + A_- g A_+ g^{-1}) \},$$

(3.12)

where $\sqrt{\mu}$ is a constant valued in Cartan subalgebra $\mathcal{H}$ of $SL(2,\mathbb{R})$, whose meaning will be clarified in following text. The equations of motion are given by

$$[D_-, D_+^\prime - J_-] = 0, \quad [D_+, D_-^\prime + J_+] = 0,$$

(3.13)

$$J_+ + gA_+ g^{-1} - \sqrt{\mu} = 0, \quad J_- + gA_- g^{-1} - \sqrt{\mu} = 0,$$

(3.14)

where $D_- = \partial_- + A_-, D_+ = \partial_+ + A_+$ and

$$D_+^\prime = \partial_+ - gA_+ g^{-1}, \quad D_-^\prime = \partial_- + g^{-1}A_- g,$$

(3.15)

reflecting the transition of the connection due to the “sewing”. If one identifies

$$A_- = gA_+ g^{-1}, \quad A_+ = g^{-1}A_- g,$$

(3.16)

One can treat this as an on-shell constraint, or the classical constraint.
the first two equations of motion are just chiral anomaly equations:

\[ [D_-, J_+] = F_{-+}, \quad [D_+, J_-] = F_{+-}. \quad (3.17) \]

A good “sewing” should be anomaly free, or equivalently, identifying two patches only up to a pure gauge transformation. So the gauge strength \( F \) should vanish and the associated gauge connection is the flat connection. Because of this, one can at first set \( A_\pm = 0 \), then the theory will reduce to ordinary WZW action \( I_{WZW}[g] \) with constraints:

\[ J_+ - \sqrt{\mu} = 0, \quad J_- - \sqrt{\mu} = 0. \quad (3.18) \]

Now the derivation is straightforward, first let us parameterize \( g \in SL(2, \mathbb{R}) \) via the Gauss decomposition

\[ g = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\phi & 0 \\ 0 & e^{-\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{v} & 1 \end{pmatrix}. \]

In these coordinates, the action \( I_{WZW} \) becomes

\[ I = \frac{k}{8\pi} \int dz^2 \left\{ \partial_z \phi \partial_{\bar{z}} \bar{\phi} + e^{-2\phi} \partial_{\bar{z}} v \partial_z \bar{v} \right\}. \quad (3.19) \]

Second, substituting the constraints (3.18) into the action, one immediately obtain the Liouville action

\[ I_{Liou} = I = \frac{k}{8\pi} \int dz^2 \left\{ \partial_z \phi \partial_{\bar{z}} \bar{\phi} + \mu e^{2\phi} \right\}, \]

where the meaning of \( \mu \) is clear: though it is only an integral constant in WZW models, now it has a physical meaning as the cosmological constant.

This connection between Chern-Simons and Liouville theory is rather rough. One should expect there is a dictionary from Liouville to Chern-Simons. In Liouville side, given the knowledge of central charge, conformal dimensions of primary fields, fusion rules and conformal blocks, one can totally determine the whole theory. Now the central subject is to identify these objects in Chern-Simons theory.

### 4 Links, Surgery and Wilson Loop

In this section we define the monodromy of loop operator in Liouville theory as a ratio of link invariants in Chern-Simons theory which lives on \( S^3 \). We also consider t’Hooft loops in \( \mathcal{N} = 4 \) SYM theory.

#### 4.1 Assumptions

A Wilson line could also be regarded as the space-time trajectory of a charged particle, this point of view will immediately lift the theory under consideration to 3d. Alternatively, one could realize the effect of a loop operator in Liouville theory as a special knot or link in 3d. Since each 3d manifold could be obtained by chopping \( S^3 \) into pieces and then gluing them back
after several diffeomorphism [1], it is sufficient to consider $S^3$ only. However, it is important to bare in mind that Liouville theory is living on the boundary of one piece of chopped 3d manifold. Tracking the geometric diffeomorphism step by step, one could expect to obtain the dual description of Liouville theory. This is exactly what CSW-RCFT [1] means. We expect this kind of chopping and gluing back (surgery) should remain true in general situation even for noncompact groups and irrational CFTs. Also, we expect that there exist a one to one correspondence from Liouville conformal blocks to quantum Hilbert spaces obtained by quantizing a three dimensional theory. This is just a generalization of Witten’s statement on CSW-RCFT correspondence [1].

### 4.2 Surgery

The easiest way to construct a 3d manifold from 2d CFT is as following: let the 2d Riemann surface $\Sigma$ with $n$ marked points travel in 3d imaginary Euclidean space-time, then bend the “time” direction to a circle, which will identify the original surface and the final surface. The resulting manifold will be a trivial bundle: $\Sigma \times S^1$ with $n$ Wilson lines living on it. This process is shown in Fig.1. In general, VEVs of Wilson loops in this manifold are hard to calculate. However, Witten provided a very powerful method to deal with this problem [1]: the surgery method. This method admits to replace a 3d manifold with $n$ Wilson lines as $S^3$ with different numbers of Wilson lines. Moreover, one could also “erase” all the $n$ Wilson lines by $n$ times sequent surgeries, these operations will replace the original 3-manifold as a new manifold due to surgeries.
For a concrete view of the surgery, consider a very simple case. \( \Sigma \) is a Riemann sphere with no punctures. Now the resulting manifold is an \( S^2 \times S^1 \), next draw a mathematical loop \( C \) on it and then cut the neighborhood \( M_R \) of the loop, the boundary of \( M_R \) is simply a torus denoted as \( T \). We pick a basis of \( H^1(T; \mathbb{Z}) \) consisting of cycles \( a \) and \( b \) indicated in Fig. 2b, then choose a diffeomorphism \( S : T \rightarrow T \) as the map \( a \rightarrow b, b \rightarrow -a \). Then glue the changed \( M_R \) back to the remain \( M_L \), one will immediately obtain an \( S^3 \). This means that computations on \( S^3 \) are equivalent to computations on \( S^2 \times S^1 \) with a physical Wilson line where the surgery was made. We will call this surgery of \( S^3 \) as “modular surgery” hereafter.

One can also recognize the special diffeomorphism that exchanges \( a, b \) circles as nothing but the modular \( S \) transformation of the torus \( T \). Now we want to explain what this \( S \) transformation means in 2d CFT. To do this, let us look at the surgery more closely. It is easy to see that cutting the neighborhood of \( C \) corresponds to cutting \( S^2 \) into two disks \( D \) and \( D' \), then the partition function of the CFT on \( S^2 \) can be obtained by gluing two CFTs (holomorphic CFT and anti-holomorphic CFT, respectively) on different disks with a common boundary. Now the meaning of the modular \( S \) transformation is clear: it is just a change of boundary condition of the CFT on \( D' \). Recall that partition function of a holomorphic factorizable CFT can always be constructed by characters of the related affine algebra, the modular \( S \) transformation of the holomorphic character can be thought as the consequence of the changing of boundary condition. Using this surgery to get an \( S^3 \) corresponds to the conformal blocks used in \([8, 36]\). However, there is an easier way to obtain an \( S^3 \) from two 3-manifolds: picking two identical 3-balls \( B_1, B_2 \) with boundary 2-spheres \( S^2_1, S^2_2 \), gluing \( S^2_1, S^2_2 \) without diffeomorphisms, one will immediately get an \( S^3 \). This corresponds to the conformal blocks used in \([9]\). We will call this surgery of \( S^3 \) as “simple surgery” hereafter. Now we recognize the difference between these two kinds of blocks is nothing but a modular transformation \( S_1,1^{(-i2a)} \), the lower \((1,1)\) labels the basic vacuum, \( i = 2a \) can be read off immediately from the expression of conformal dimension \( h \) which shows in \([2, 4]\) and \( 2a = Q + 2a \). For further convenience, we do not distinguish the label \( S_{1,1}^{(-i2a)} \) and \( S_{1,1}^{2a} \).

We now consider the standard surgery for generic 3d manifold \( M \)(Fig.2a). To do so we first thicken \( C \) to a “tubular neighborhood”, a solid torus centered on \( C \). Removing this solid torus, \( M \) is split into two pieces; the solid torus is called \( M_R \)(Fig.2b), and the remainder is called \( M_L \). One then makes a diffeomorphism on the boundary of \( M_R \) and glues \( M_R \) and \( M_L \) back together to get a new three manifold \( \tilde{M} \)(Fig.2c). The canonical quantization of Chern-Simons theory on 3d manifold makes it clear that Hilbert spaces \( \mathcal{H}_L \) and \( \mathcal{H}_R \), canonically dual to one another, are associated with the boundaries of \( M_L \) and \( M_R \). The path integrals on \( M_L \) and \( M_R \) give vectors \( \psi \) and \( \chi \) in \( \mathcal{H}_L \) and \( \mathcal{H}_R \), and the partition function on \( M \) is just the natural pairing \( (\psi, \chi) \). If we act on the boundary \( M_R \) with a diffeomorphism \( K \) before gluing \( M_L \) and \( M_R \) together, then \( \chi \) is replaced by \( K\chi \) so \( (\psi, \chi) \) is replaced by \( (\psi, K\chi) \). So the partition function \( Z \) on \( \tilde{M} \) can be related to that on \( M \)(Fig.2d) as

\[
Z(\tilde{M}) = \sum_j K_\alpha^j \cdot Z(M; R_j),
\]

(4.1)

\(^7\)The process of this surgery is shown in Fig.2, where \( M \) stands for \( S^2 \times S^1 \), \( \tilde{M} \) stands for \( S^3 \).

\(^8\)For more details see the reference \([\_]\).
Figure 2: In a), an imaginary loop $C$ was drawn on $M$. In b), the neighborhood of $C$ was cut out, now the $M$ had been cut into $M_L$ and $M_R$ which is a solid torus. In c), the solid torus had been glued back with $M_L$ after a diffeomorphism $K$, and formed a new 3-manifold $\tilde{M}$. In d), It follows that partition function in $\tilde{M}$ can be obtained from that in $M$ with a Wilson loop $C$ and the knowledge of diffeomorphism $K$. 
where $K^j_i$ is the diffeomorphism associated with the surgery. It is clear that if $M$ is $S^2 \times S^1$ and $\tilde{M}$ is $S^3$, $K^j_i \equiv S^j_i$.

### 4.3 Generalized Surgery

Now we will consider the generalized surgery on 3d manifold $M$. In this situation, before the surgery a Wilson line in the $R_i$ representation was already present on the imaginary circle $C$. Surgery amounts to cutting out a neighborhood of $C$ and then gluing it back in, and after this process the $R_i$ Wilson line will still be present in $\tilde{M}$. Now the diffeomorphism also acts on the $R_i$ Wilson line, then detailed analysis gives \[ (4.2) \]

\[
Z(\tilde{M}; R_j) = \sum_j K^j_i \cdot Z(M; R_j).
\]

### 4.4 Path Integrals on $S^1 \times X$

The three manifolds whose partition functions can be computed in a particularly simple way, from the axioms of quantum field theory, are those of the form $X \times S^1$, for various $X$. $X \times S^1$ can have a “Hamiltonian” formalism if one treats the $S^1$ as the “bended” time direction. This can be realized as following: one constructs the Hilbert space $\mathcal{H}_X$ of $X$, then introduces a “time” direction represented by a unit interval $I = [0, 1]$, and then propagates the vector in $\mathcal{H}_X$ from “time” 0 to “time” 1. This operation is trivial, since the Chern-Simons theory is a topological field theory. It has a vanishing Hamiltonian. Finally, one obtains $X \times S^1$ by gluing $X \times \{0\}$ to $X \times \{1\}$; this identifies the initial and final states, giving a trace:

\[
Z(X \times S^1) = \text{Tr}_{\mathcal{H}_X}(1) = \dim \mathcal{H}_X. \tag{4.3}
\]

Now we would like to review the dimension of $\mathcal{H}_{S^2,n}$, the Riemann sphere with $n$ punctures. There are well-known results for this question \[43, 44\] and can be found explicitly in ref. \[1\]. We now copy these results as follows:

(I) For the Riemann sphere with no punctures (marked points), the Hilbert space is one dimensional.

(II) For the Riemann sphere with one puncture in a representation $R_i$, the Hilbert space is one dimensional only if $R_i$ is trivial, and zero dimensional otherwise.

(III) For the Riemann sphere with two punctures with representation $R_i$ and $R_j$, the Hilbert space is one dimensional if $R_j$ is the dual of $R_i$ and zero dimensional otherwise.

(IV) For the Riemann sphere with three punctures with representation $R_i$, $R_j$, and $R_k$, the dimension of $\mathcal{H}_{S^2,3}$ is the Verlinde number $N_{ijk}$.

(V) From the results of Verlinde \[43\], the dimension of the physical Hilbert spaces for an arbitrary collection of punctures on $S^2$ can be determined from a knowledge of the Verlinde number $N_{ijk}$. This refers to the fusion rules of the CFT.
Using these results, one can immediately obtain the partition function on $S^2 \times S^1$

$$Z(S^2 \times S^1) = 1. \quad (4.4)$$

However, we should note here that this result is obtained by a normalization, the partition function of $S^2 \times S^1$ can be strictly obtained either by quantization of Chern-Simons theory \[55\] or the operator formalism \[56\]. Another thing that should be clarified here is that this partition function is only the holomorphic part of the full partition function, say, the holomorphic character. As in \[56\], the states of a basis of the Hilbert space are in one to one correspondence with the characters of the CFT\[9\]. Moreover, if there exist $N$ unknotted and unlinked Wilson lines on the given 3-manifold $M$, then the wave function related to a surgery of $M$ is equivalent to a conformal block of the $2N$ correlation functions up to a normalization constant \[57\]. In this scenario, the Wilson line operators also had been identified as the Verlinde operators in the dual CFT since they satisfied the same fusion algebra. A geometric description on this fusion algebra can be found in \[28\].

If we are given a diffeomorphism $K : X \to X$, then one can form the mapping cylinder $X \times_K S^1$ by identifying $x \times \{1\}$ with $K(x) \times \{1\}$ for every $x \in X$. The initial and final states are identified via $K$, so the generalization of (4.3) is

$$Z(X \times_K S^1) = \text{Tr}_{\mathcal{H}_X} (K). \quad (4.5)$$

For $X$ is $S^2$ with some punctures $P_a, a = 1 \ldots s$ to which representations $R_{i(a)}$ are assigned, we can use the above results for Riemann sphere with punctures. We denote the Hilbert space as $\mathcal{H}_{S^2, <R>}$ for $< R >$ representing the collection of the punctures with representations. The partition function is:

$$Z(S^2 \times S^1; < R >) = \dim \mathcal{H}_{S^2, <R>}. \quad (4.6)$$

Then for one puncture with representation $R_a$

$$Z(S^2 \times S^1; R_a) = \delta_{a,0}. \quad (4.7)$$

For two punctures with representation $R_a$ and $R_b$

$$Z(S^2 \times S^1; R_a, R_b) = g_{ab}, \quad (4.8)$$

where $g_{ab}$ is defined as 1 if $R_b$ is the dual of $R_a$ and 0 otherwise. For three punctures with representation $R_a, R_b$ and $R_c$

$$Z(S^2 \times S^1; R_a, R_b, R_c) = N_{abc}. \quad (4.9)$$

The Verlinde number $N_{abc}$ can be obtained either from loop algebra of unknotted and unlinked loops in 3d \[56\] or from the fusion algebra in 2d CFT.

\[9\]There the authors considered the cases for compact gauge groups, we assume this is also true in the non-compact case. Actually, the derivation for the characters of degenerate fields are quite parallel. However, the generalization for non-degenerate fields still unclear and we hope further works will clarify this.
4.5 Hopf Links and Wilson Loop

Now we can examine the above things on $S^3$. Using the surgery from $S^2 \times S^1$ to $S^3$, one can easily get

\[ Z(S^3) = \sum_j S_0^j Z(S^2 \times S^1; R_j) = \sum_j S_0^j \delta_{j,0} = S_{0,0}, \]  

\[ Z(S^3; R_j) = \sum_i S_0^i Z(S^2 \times S^1; R_i, R_j) = \sum_i S_0^i g_{ij} = S_{0,j}, \]  

\[ Z(S^3, R_j, R_k) = \sum_i S_0^i Z(S^2 \times S^1; R_i, R_j, R_k) = \sum_i S_0^i N_{ijk}. \]  

The left-hand side of the last equation can be independently calculated by cutting and gluing as in ref [1], and the result is

\[ Z(S^3, R_j, R_k) = \frac{Z(S^3; R_j)Z(S^3; R_k)}{Z(S^3)} = \frac{S_{0,j}S_{0,k}}{S_{0,0}}. \]  

This is a special case of Verlinde formalism,

\[ \frac{S_{0,j}S_{0,k}}{S_{0,0}} = \sum_i S_0^i N_{ijk}. \]  

It has an obvious meaning that two knots have been fused to a single knot. One can recognize this as the basic fusion rule for the CFT. As a simple check, we consider the fusion of one degenerate state $V_{1,2}$ and one non-degenerate state $V_\alpha$. In Chern-Simons theory, this can be identified as the action of taking one unknotted loop to be close to another one with associated representations. The result is simply what we just obtained

\[ \frac{S_{1,1}^{1.2}S_{1,1}^{2a}}{S_{1,1}^{1.1}} = -2\cos(\pi b^2)(-2\sqrt{2}\sin(\pi b^{-1}2a)\sin(\pi b2a)) \]  

\[ = -2\sqrt{2}\sin(\pi b^{-1}(2a + b))\sin(\pi b(2a + b)) \]  

\[ - 2\sqrt{2}\sin(\pi b^{-1}(2a - b))\sin(\pi b(2a - b)) \]  

\[ = S_{1,1}^{2a+b} + S_{1,1}^{2a-b}, \]

which is the fusion rule

\[ [V_{1.2}] \times [V_\alpha] = [V_{\alpha+\frac{1}{2}}] + [V_{\alpha-\frac{1}{2}}], \]

where $[V_\alpha]$ denotes the whole Verma module for the primary field $V_\alpha$. More general fusion rules can be derived straightforwardly. One can also generalize this to the generalized surgery situations. Consider the case that there exist two braided loops in $S^2 \times S^1$ in representations $R_a$ and $R_b$ as showing in Fig.3a, making the generalized surgery on $R_b$ circle and one gets a Hopf link $L(R_a, R_b)$ on $S^3$ as shown in Fig.3b. The usage of the formula for generalized surgery \[4.2\] therefore determines the partition function of $S^3$ with a pair of linked Wilson lines:

\[ Z(S^3; L(R_i; R_j)) = \sum_k S_i^k Z(S^2 \times S^1; R_k, R_j) = S_{i,j}. \]
Figure 3: a) In $S^2 \times S^1$, two loops $C$ and $C'$ rounded the uncontractable circle and braided with each other, the generalized surgery was taken on $C$. b) After surgery, $C$ and $C'$ become a linked Hopf link in $S^3$.

Again, one can easily obtain the entire fusion algebra by gluing two Hopf links to a satellite (or connected sum) link \[1, 28\]

\[
\frac{S_{i,j}S_{i,k}}{S_{0,i}} = \sum_l S_{i}^l N_{ljk}.
\]

(4.16)

Since the Hopf link corresponds to the Wilson loop operator in LFT, we now compute it explicitly using the modular $S$ transformation of LFT. The holonomy of the Wilson loop associated with representation $1/2$ is the phase factor of taking $V_{1,2}$ around the internal vertex operator $V_\alpha$ exact once. This can be considered as the ratio of the final conformal block due to the operation and the initial conformal block. This process can be lifted to $S^2 \times S^1$ as two braided Wilson lines. So the holonomy can be obtained by comparing the “initial” partition function (without Wilson loop) with the “final” one (with Wilson loop). The “final” partition function corresponds to a Hopf link invariant in $S^3$

\[
S_{1,2}^{2a} = -2\sqrt{2}\sin(2\pi b^{-1}a)\sin(4\pi ba).
\]

(4.17)

It has the meaning of partition function only if one has normalized the vacuum partition function on $S^2 \times S^1$ to unity. It is easy to write down the “initial” partition function (two unknotted loops)

\[
\frac{S_{1,1}^{2a}S_{1,1}^{1,2}}{S_{1,1}^{1,1}} = -4\sqrt{2}\sin(2\pi b^{-1}a)\sin(2\pi ba)\cos(\pi bQ),
\]

(4.18)

where

\[
\frac{S_{1,1}^{1,1}}{S_{1,2}^{1,1}} = \frac{1}{2\cos(\pi bQ)},
\]

(4.19)
Figure 4: A torus can be cut into two identical spheres with two punctures which are attached with conjugated representations $R$ and $\bar{R}$.

is the quantum dimension of chiral degenerate vertex operator $V_{1,2}$. Now one arrives at the holonomy $h_{1,2;\alpha}$ of $V_{1,2}$ rounding a nontrivial loop

$$h_{1,2;\alpha} = \frac{S_{1,2}^{2a} S_{1,1}^{1,1}}{S_{1,2}^{2a} S_{1,1}^{1,1}} = \frac{\cos(2\pi ba)}{\cos(\pi bQ)}.$$  \hspace{1cm} (4.20)

This is exactly the result obtained in [8,11]. Now the generalization to spin $j$ particle is straightforward, as in [8]. One can replace the degenerate operator by $V_{1,2j+1}$, then the associated monodromy can be obtained from modular $S$ transformations as follows:

$$h_{1,2j+1;\alpha} = \frac{S_{1,2j+1}^{2a} S_{1,1}^{1,1}}{S_{1,1}^{2a} S_{1,2j+1}^{1,1}} = \frac{\sin(2\pi b(2j+1)a)}{\sin(2\pi ba)} \frac{\sin(\pi bQ)}{\sin(\pi(2j+1)bQ)}.$$  \hspace{1cm} (4.21)

Again, this agrees with the result in [8,11]. We now claim that this calculation is valid for all Riemann surfaces. The reason is that the Wilson loop rounds only on one tube of the Riemann surface, and one can cut the Riemann surface to its pants decompositions, draw a Wilson loop on the specified pants and glue it back. This process should not be interfered with other parts of the Riemann surface, so it is sufficient to compute the Wilson loop on Riemann sphere.

### 4.6 t’Hooft loop in $\mathcal{N} = 4$ SYM

So far we have considered the contribution of Wilson loop. The topology of the Riemann surface will be highly involved in the computation of t’Hooft loops or Dyonic loops. We now consider the simplest case: t’Hooft loop in $\mathcal{N} = 4$ super Yang-Mills theory (SYM).

The topology of the associated Riemann surface for $\mathcal{N} = 4$ is a torus. For a torus, one could not obtain a simple 3-manifold by rotating around a circle since $T^2 \times S^1$ is a little hard to deal with. The situation becomes more serious if one considers higher genus geometry. However,
Figure 5: a) The boundary of two balls $B_1$ and $B_2$ are identified with spheres with two conjugated punctures, these two punctures are endpoints of a Wilson lines in the bulk. b) By gluing $B_1$ and $B_2$ back into $S^3$, one gets an $S^3$ with a single Wilson loop. c) t’Hooft loop generated by $(1,2)$ now can be seen as the $(1,2)$ Wilson loop surrounding the $a$ loop.
there is a simple operation in CFT, the sewing operation [54]. A torus CFT can be sewed from the same CFT living on a two-punctured sphere. The correlation function on torus can also be obtained from the sphere CFT by the sewing procedure. The translation from Liouville to Chern-Simons for this sewing procedure is simple as we will clarify below. Cut a torus into two identical cylinders with the same topology of two-punctured spheres $S_1, S_2$ as in Fig.4. These two punctures have been added representations conjugate to each other. Instead of using the “modular surgery” method, we now prefer to the “simple surgery” method, say, we identify $S_1$ and $S_2$ as boundaries of two identical balls $B_1$ and $B_2$, then in each ball there is a half Wilson loop with two end points as the punctures on the boundary (Fig.5a). Finally, glue $B_1$ and $B_2$ to $S^3$ with a Wilson loop, as in Fig.5b. The result is really nontrivial since the topology of torus (the genus) now becomes a Wilson loop on $S^3$.

It may be a little subtle that we have used the “simple surgery”. However, one can also obtain the same result by the “modular surgery”. Let us show how this can be done. First, cut the torus into two pieces as in Fig. 4. Instead of treating a cylinder as two-punctured sphere, we let the cylinders deform to two one-punctured disks. Second, rounding each punctured disk to form a solid torus within which a Wilson loop rounds the $b$ circle of the solid torus. Third, making the modular $S$ transformation for one solid torus (leaving the other invariant), thus the Wilson loop within it now rounds the $a$ circle of the solid torus. Finally, gluing both tori, one can get an $S^3$ with a single Wilson loop since one should join two Wilson loops together following the spirit of surgery. This reconstructs the result we obtained using the “simple surgery”.

Now the t’Hooft loop can be lifted to $S^3$ easily. Since $V_{1,2}$ should round all the background once, which is equivalent to drawing a parallel Wilson loop of the one corresponding with the torus geometry (Fig.5c). If one denotes the holomorphic conformal block for the torus LFT without t’Hooft loop as $Z(a)$, where $a$ denotes the representation of the intermediate state of the sewing, then the monodromy of the t’Hooft loop generated by chiral degenerate operator $V_{1,2}$ can also be computed by the sewing procedure or directly by using fusion and braiding moves as in [6, 8]. The chiral degenerate operator will only affect the holomorphic part of partition functions which have a natural representation as a partition function of knots or links invariant on $S^3$, thus the holomorphic partition function $Z(a)$ should be given as

$$Z(a) = S_{1,1}^{2a},$$

(4.22)

up to a normalization factor. The action of the t’Hooft loop will be given by the fusion of two loops

$$S_{1,1}^{1/2} S_{1,1}^{2a} = S_{1,1}^{2a+b} + S_{1,1}^{2a-b} = Z(a + b/2) + Z(a - b/2).$$

(4.23)

However, the contribution from “zero mode” $V_\alpha = V_{1,1}$ should be normalized, this gives the same factor $N = \frac{1}{2 \cos(\pi b Q)}$, as in (4.19). The result also matches with that in [6, 8]. The

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10These two punctures are attached with conjugated representations.
generalization to high genus topology is straightforward: each genus gives a representation-attached circle on $S^3$, fusing these circles to the degenerate one. The fusion rules will give the correct contribution of the t’Hooft loop.

5 Conclusions and Discussion

In this note we have considered the three dimensional $SL(2, \mathbb{R})$ Chern-Simons description for Liouville field theory which duals to an $\mathcal{N} = 2$ SCFT in four dimensions by AGT conjecture. We have pointed out equivalence between Chern-Simons and LFT from several points of view: actions, Hilbert spaces, conformal blocks, loop algebra and fusion rules. Using these equivalent relations, we computed the contribution of Wilson loop in general case, also, we give a simplest check on t’Hooft loop for $\mathcal{N} = 4$ SYM, the results we obtained match with those in [6, 8].

The generalization of this Chern-Simons description of toda field theory just needs a change of gauge group, for $A_{N-1}$ toda theory which duals to $SU(N)$ quiver gauge theories in 4d, the gauge group is $SL(N, \mathbb{R})$ for Chern-Simons theory. Now loop operators in $SU(N)$ quiver gauge theories can be computed in Chern-Simons if given the modular properties of the affine algebra of $SL(N, \mathbb{R})$. We are now preparing for this work.

The Chern-Simons/Liouville duality itself is far from a completed one. There are many problems to be resolved. The first emergent problem is how to derive DOZZ [30, 31] formula in Chern-Simons theory, since this is the building block of the Liouville theory. In principle, this can be done by canonical quantization or using the operator formalism of Chern-Simons theory, but it still needs a concrete work.

Second, does this Chern-Simons theory have an specified physical origin instead of being a tool for calculations? One guess is there may exist an origin, the M2 branes. This is a natural guess since we are dealing with the systems which come from the configuration of M2 and M5 branes. Actually, Ooguri and Vafa had considered a similar configuration a decade ago [58]. Another evidence is that the Chern-Simons theory should be supersymmetric extended since the corresponding LFT should be supersymmetric, in order to cure the non-unitarity of degenerate representations which we used in the context. Thus the Chern-Simons theory can be supersymmetric, which could be related to the ABJM or BLG description [59, 60] for M2 branes.

Third, if one can extract Chern-Simons theory from LFT, then following the spirit of the AGT duality, there should be a direct path from $\mathcal{N} = 2$ SCFTs to the same Chern-Simons theory. Thus the Chern-Simons theory plays the role of a bridge which connects LFT with $\mathcal{N} = 2$ SCFTs. Furthermore, if this connection is found, it will strongly imply the duality between M2 and M5 branes in general construction.

Of course, there are other problems, for example, the relations between Chern-Simons and topological strings, also and matrix theories, which still need to be clarified. S-duality in Liouville or SCFTs also should have a cousin in Chern-Simons theory, which may be closely related to mirror symmetry in three dimension. We hope future works will clarify these problems.
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