ALGEBRAIC GROUPS OVER A 2-DIMENSIONAL LOCAL FIELD:
IRREDUCIBILITY OF CERTAIN INDUCED REPRESENTATIONS

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To Raoul Bott, with admiration

Abstract. Let $G$ be a split reductive group over a local field $K$, and let $G((t))$ be the corresponding loop group. In [1] we have introduced the notion of a representation of (the group of $K$-points) of $G((t))$ on a pro-vector space. In addition, we have defined an induction procedure, which produced $G((t))$-representations from usual smooth representations of $G$. We have conjectured that the induction of a cuspidal irreducible representation of $G$ is irreducible. In this paper we prove this conjecture for $G = SL_2$.

1. The result

1.1. Notation. The notation in this paper follows closely that of [1]. Let remind the main characters. We denote by $Set_0$ the category of finite sets, and $Set := \text{Ind}(\text{Pro}(Set_0))$, $Set = \text{Ind}(\text{Pro}(Set))$. By $Vect_0$ we denote the category of finite-dimensional vector spaces over $\mathbb{C}$, $Vect = \text{Ind}(Vect_0)$ is the category of all vector spaces, and $Vect$ is the category $\text{Pro}(Vect)$ of pro-vector spaces.

Let $G$ be a split reductive group over $K$, $G$ the corresponding group-object of $Set$. We have the pro-algebraic group of arcs $G[[t]]$ and for $n \in \mathbb{N}$ we denote by $G^n \subset G[[t]]$ the corresponding congruence subgroup. By $G[[t]]$ (resp., $G^n \subset G[[t]]$) we denote the corresponding group-objects of $\text{Pro}(Set)$.

Finally $G = G((t))$ is the group-object of $Set$, which is our main object of study. We denote by $\text{Rep}(G)$ the category of representations of $G$ on $Vect$, cf. [1], Sect. 2.

1.2. Let us recall the formulation of Conjecture 4.7 of [1]. Recall that we have an exact functor $r^G_G : \text{Rep}(G) \to \text{Rep}(G, Vect)$, and its right adjoint, denoted $i^G_G$ and called the induction functor.

The functors $r^G_G$ and $i^G_G$ are direct loop-group analogs of the Jacquet and induction functors for usual reductive groups over $K$.

Let $\pi$ be an irreducible cuspidal representation of $G$, and set $\Pi := i^G_G(\pi)$. In [1], Sect. 4.5 it was shown that the cuspidality assumption on $\pi$ implies that the natural map

$$r^G_G(\Pi) = r^G_G \circ i^G_G(\pi) \to \pi$$

is an isomorphism. In particular, this implies that

$$\text{End}_{\text{Rep}(G)}(\Pi) \simeq \text{Hom}_{\text{Rep}(G, Vect)}(r^G_G(\Pi), \pi) \simeq \text{Hom}_{\text{Rep}(G, Vect)}(\pi, \pi) \simeq \mathbb{C}.$$ 

We have formulated:
Conjecture 1.3. The object $\Pi \in \text{Rep}(G)$ is irreducible.

In this paper we will prove:

Theorem 1.4. Conjecture 1.3 holds for $G = SL_2$.

Note that in [1] Conjecture 1.3 was stated slightly more generally, when we allow representations of a central extension $\hat{G}$ with a given central charge. The proof of Theorem 1.4 generalizes to this set-up in a straightforward way.

It should be remarked that from the definition of the category of representations of $G((t))$, it is not at all clear that $G((t))$ admits any non-trivial irreducible representations is non-obvious. Therefore, the fact that the above-mentioned irreducibility conjecture holds is somewhat surprising.

1.5. We will now consider a functor $\text{Rep}(G, \mathcal{V}ect) \rightarrow \text{Rep}(G)$, which will be the left adjoint of the functor $r_G^G$.

First, recall from [2], Proposition 2.7, that the functor $\text{Co inv}^G_1 : \text{Rep}(G^1, \mathcal{V}ect) \rightarrow \mathcal{V}ect$ does admit a left adjoint, denoted $\text{Inf}^G_1$.

Proposition 1.6. The functor $\text{Co inv}^G_1 : \text{Rep}(G[[t]], \mathcal{V}ect) \rightarrow \text{Rep}(G, \mathcal{V}ect)$ admits a left adjoint.

Proof. For $\pi = (\mathcal{V}, \rho) \in \text{Rep}(G, \mathcal{V}ect)$, consider the functor $\text{Rep}(G[[t]], \mathcal{V}ect) \rightarrow \mathcal{V}ect$ given by

$$\Pi \mapsto \text{Hom}_{\text{Rep}(G, \mathcal{V}ect)}(\pi, \text{Co inv}^G_1(\Pi)).$$

We claim that it is enough to show that this functor is pro-representable. Indeed, this follows by combining Lemma 1.2, Proposition 2.5 and Lemma 2.7 of [1].

Consider the object $\text{Inf}^G_1(\mathcal{V}) \in \text{Rep}(G^1, \mathcal{V}ect)$, where $\mathcal{V}$ is regarded just as a pro-vector space, and

$$\text{Co ind}^G_{G^1}(\text{Inf}^G_1(\mathcal{V})) \in \text{Rep}(G[[t]], \mathcal{V}ect),$$

where $\text{Co ind}^G_{G^1}$ is as in [2], Corollary 2.34.

Evidently,

$$\text{Hom}_{\text{Rep}(G, \mathcal{V}ect)}(\pi, \text{Co inv}^G_1(\Pi)) \hookrightarrow \text{Hom}_{\mathcal{V}ect}((\mathcal{V}), \text{Co inv}^G_1(\Pi)),$$

and the latter, in turn, identifies with

$$\text{Hom}_{\text{Rep}(G^1, \mathcal{V}ect)}(\text{Inf}^G_1(\mathcal{V}), \Pi) \simeq \text{Hom}_{\text{Rep}(G[[t]], \mathcal{V}ect)}(\text{Co ind}^G_{G^1}(\text{Inf}^G_1(\mathcal{V})), \Pi).$$

Hence, the pro-representability follows from Proposition 1.4 of [1].

We will denote the resulting functor by $\text{Inf}^G_{G[[t]]}$. Note that by construction, for a representation $\pi$ of $G$ we have a surjection

$$\text{Co ind}^G_{G^1}(\text{Inf}^G_1(\pi)) \hookrightarrow \text{Inf}^G_{G[[t]]}(\pi).$$

By composing $\text{Inf}^G_{G[[t]]}$ with the functor $\text{Co ind}^G_{G[[t]]} : \text{Rep}(G[[t]], \mathcal{V}ect) \rightarrow \text{Rep}(G)$ we obtain a functor, left adjoint to $r_G^G$. 
We will now formulate the main step in the proof of Theorem 1.4. Note that if \( \pi \) is a cuspidal representation of \( G \), isomorphism (1) implies that we have a canonical map

\[
(2) \quad \text{Coind}_{G[[t]]}^G (\text{Inf}_G^{G[[t]]}(\pi)) \to \Pi.
\]

We will deduce Theorem 1.4 from the following one:

**Theorem 1.7.** If \( G = SL_2 \), the map of (2) is surjective.

Of course, we conjecture that the map (2) is surjective for any \( G \), but we are unable to prove that at the moment.

1.8. Let us show how Theorem 1.7 implies Theorem 1.4. Suppose that \( \Pi' \) is a non-zero sub-object of \( \Pi \) and let \( \Pi'' := \Pi / \Pi' \) be the quotient. By definition of the induction functor, we have a map in \( \text{Rep}(G, \text{Vect}) \).

\[
r_G^G(\Pi') \to \pi.
\]

Using Proposition 2.4. of \([1]\), we obtain that \( r_G^G(\Pi') \) must surject onto \( \pi \), since the latter was assumed irreducible. Since the functor \( r_G^G \) is exact (cf. Lemma 2.6. of loc.cit.), this implies that \( r_G^G(\Pi'') = 0 \).

However, \( \text{Hom}_{\text{Rep}(G)}(\text{Coind}_{G[[t]]}^G (\text{Inf}_G^{G[[t]]}(\pi)), \Pi'') \simeq \text{Hom}_{\text{Rep}(G, \text{Vect})}(\pi, r_G^G(\Pi'')) \). By Theorem 1.7, this implies that \( \Pi'' = 0 \).

2. The key lemma

2.1. The rest of the paper is devoted to the proof of Theorem 1.7. We will slightly abuse the notation, and for a scheme \( Y \) over \( K \) we will make no distinction between the corresponding object \( Y \in \text{Set} \) and \( Y(K) \), regarded as a topological space.

Recall the affine Grassmannian \( \text{Gr}_G = G((t))/G[[t]] \) of \( G \), and the corresponding object \( \text{Gr}_G \in \text{Ind} (\text{Set}) \). Let us represent \( \text{Gr}_G \) as the direct limit of closures of \( G[[t]] \)-orbits, \( \text{Gr}_G^\lambda \), with respect to the natural partial ordering on the set of dominant coweights.

Let us also denote by \( \text{Gr}_G^\lambda \) the ind-scheme \( G((t))/G^\lambda \), which is a principal \( G \)-bundle over \( \text{Gr}_G \). Let \( \text{Gr}_G^\lambda \) and \( \text{Gr}_G^\lambda \) denote the preimages in \( \text{Gr}_G \) of the \( G[[t]] \)-orbit \( \text{Gr}_G^\lambda \) and its closure, respectively. Let \( \text{Gr}_G^\lambda \) and \( \text{Gr}_G^\lambda \) denote the corresponding objects of \( \text{Set} \).

By construction (cf. \([1]\), Sect. 3.9), as a \( G[[t]] \)-representation, \( \Pi \) is the inverse limit of \( \Pi^\lambda \), where each \( \Pi^\lambda \) is the vector space consisting of locally constant \( G \)-equivariant functions on \( \text{Gr}_G^\lambda \) with values in \( \pi \).

Set \( \Pi^\lambda \) be the kernel of \( \Pi^\lambda \to \bigoplus_{\lambda < \lambda} \Pi^\lambda \). Let \( ev \) denote the natural evaluation map \( \Pi \to \Pi^0 \simeq \pi \), which sends a function \( f \in \text{Funct}_{LC} (\text{Gr}_G^\lambda, \pi) \) to \( f(1) \). More generally, for \( \tilde{g} \in \text{Gr}_G^\lambda \), we will denote by \( ev_{\tilde{g}} \) the map \( \Pi \to \pi \), corresponding to evaluation at \( \tilde{g} \).

To prove Theorem 1.7, we must show that the composition

\[
(3) \quad \text{Coind}_{G[[t]]}^G (\text{Inf}_G^{G[[t]]}(\pi)) \to \Pi \to \Pi^\lambda
\]
is surjective for every \( \lambda \). We will argue by induction. Therefore, let us first check that the map of (3) is indeed surjective for \( \lambda = 0 \).

We have a natural map

\[
\text{Inf}_{G_1}^G(\pi) \to \text{Inf}_G^G[[t]](\pi) \to \text{Coind}_G^G[[t]] \left( \text{Inf}_G^G[[t]](\pi) \right),
\]
and its composition with

\[
\text{Coind}_G^G[[t]] \left( \text{Inf}_G^G[[t]](\pi) \right) \to \Pi^\text{ev} \to \Pi^\lambda
\]
is the natural surjection \( \text{Inf}_{G_1}^G(\pi) \to \pi \).

Thus, we have to carry out the induction step. We will suppose that the composition

\[
\text{Coind}_G^G[[t]] \left( \text{Inf}_G^G[[t]](\pi) \right) \to \Pi \to \Pi^\lambda
\]
is surjective for \( \lambda' < \lambda \), and we must show that

\[
\text{Coind}_G^G[[t]] \left( \text{Inf}_G^G[[t]](\pi) \right) \times \Pi^\lambda \to \Pi^\lambda
\]
is surjective as well.

2.2. For \( \lambda \) as above let \( t^\lambda \) be the corresponding point in \( G((t)) \). By a slight abuse of notation we will denote by the same symbol its image in \( \text{Gr}_G \) and \( \tilde{\text{Gr}}_G \).

Consider the action of \( G^1 \subset G((t)) \) on \( \text{Gr}_G \) given by

\[
g \times x = \text{Ad}_{t^\lambda}(g) \cdot x.
\]
Let \( Y \subset \text{Gr}_G \) be the closure of \( \text{Ad}_{t^\lambda}(G^1) \cdot \tilde{\text{Gr}}_G^\lambda \). Let \( G_\lambda \) be a finite-dimensional quotient of \( G^1 \), through which it acts on \( Y \).

We will denote by \( Y \) and \( G_\lambda \), respectively, the corresponding objects of \( \text{Set} \). Let \( \Pi_Y \) denote the quotient of \( \Pi \), equal to the space of \( G \)-equivariant locally constant \( \pi \)-valued functions on the set of \( K \)-points of the preimage of \( Y \) in \( \tilde{\text{Gr}}_G \).

Let \( N \subset G \) be the maximal unipotent subgroup. Since \( \lambda \) is dominant, \( \text{Ad}_{t^\lambda}(N[[t]]) \) is a subgroup of \( N[[t]] \). Let \( N^\lambda \subset N[[t]] \) be any normal subgroup of finite codimension, contained in \( \text{Ad}_{t^\lambda}(N[[t]]) \). (Later we will specify to the case when \( G = SL_2 \); then \( N \simeq G_a \) and is abelian, and we will take \( N^\lambda = \text{Ad}_{t^\lambda}(N[[t]]) \).) Let \( N_\lambda \) denote the quotient \( N[[t]]/N^\lambda \), and let \( N_\lambda \) be the corresponding group-object in \( \text{Set} \).

Let now \( K_N \) be an open compact subgroup in \( N_\lambda \), and \( K_{G_\lambda} \) an open compact subgroup in \( G_\lambda \).

Now we are ready to state our main technical claim, Main Lemma 2.4. However, before doing that, let us explain the idea behind this lemma:

From the isomorphism (1), we will obtain that for any \( f \in \Pi_Y \) and a large compact subgroup \( K_{G_\lambda} \) as above, the integral \( f' := \int_{k \in K_{G_\lambda}} f^k \) “localizes” near \( t^\lambda \), i.e., \( f' \) will be 0 outside a ”small” ball around \( t^\lambda \). We will then average \( f' \) with respect to a fixed open subgroup \( K_N \) of \( N_\lambda \), and obtain a new element, denoted \( f'' \in \Pi^\lambda \).
Main Lemma 2.4 will insure that the compact subgroup $K_{G, \lambda}$ can be chosen so that $f''$ will still be localized near $t^\lambda$, and such the resulting elements $f''$ for various subgroups $K_N$, and their translations by elements of $G((t))$, span $\Pi^\lambda$.

2.3. In precise terms, we proceed as follows. Consider the operator $A_{K_N, K_{G, \lambda}} : \Pi \to \pi$ given by

$$f \mapsto \int_{n \in K_N} \int_{k \in K_{G, \lambda}} ev_{t^\lambda}(f^n k),$$

where the integral is taken with respect to the Haar measures on both groups. (In the above formula $f \mapsto f^x$ denotes the action of $x \in G((t))$ on $\Pi$.) By the definition of $\Pi_Y$, the above map factors through $\Pi_Y \to \Pi^\lambda$.

For a point $\tilde{g} \in \tilde{\Gr}_G^\lambda$ we have a map $A_{\tilde{g}, K_{G, \lambda}} : \Pi \to \pi$ given by

$$f \mapsto \int_{k \in K_{G, \lambda}} ev_{\tilde{g}}(f^k).$$

This map also factors through $\Pi_Y$.

Our main technical claim, which we prove for $G = SL_2$ is the following. (We do not know whether an analogous statement holds for groups $G$ of higher rank.)

**Main Lemma 2.4.** For $v \in \pi$, an open compact subgroup $K_N \subset N_\lambda$ and open compact subset $X \subset \Gr_G^\lambda$ containing $t^\lambda$, there exists a finite-dimensional subspace $F(v) \subset \Pi_Y$ and an increasing exhausting family of compact subgroups $K_{G, \lambda}(v) \subset G_\lambda$ such that:

1. For all sufficiently large indices $\alpha$ the vector $v$ would belong to the image of $A_{K_N, K_{G, \lambda}(v)}(F(v))$.
2. For every $f \in F(v)$ and for all sufficiently large indices $\alpha$, the vector $A_{\tilde{g}, K_{G, \lambda}(v)}(f)$ will vanish, unless the image of $\tilde{g}$ under $\tilde{\Gr}_G^\lambda \to \tilde{\Gr}_G^\lambda$ belongs to $X$.

2.5. Let us show how Lemma 2.4 implies the induction step in the proof of Theorem 1.7.

Recall that the orbit of the point $t^\lambda$ under the action of $N[[t]]$ is open in $\Gr_G^\lambda$. For an open compact subgroup $K_N \subset N_\lambda$, let $X \subset \Gr_G^\lambda$ be its orbit under $K_N$. Let $(\Pi^\lambda)^K N \subset \Pi^\lambda$ be the subspace of $K_N$-invariants. We have a direct sum decomposition

$$(\Pi^\lambda)^K N = V_1 \oplus V_2,$$

where the first direct summand consists of functions that vanish on the preimage of $X$, and the second one functions of sections that vanish outside the preimage of $X$. We have $V_2 \subset \Pi^\lambda$ and the map $ev_{t^\lambda}$ identifies $V_2$ with $\pi$.

We claim that it suffices to show that the image of the map

$$(6) \quad \text{Coind}^G_{G[[t]]} \left( \text{Inf}^G_{G[[t]]}(\pi) \right) \to \Pi \to \Pi^\lambda \to (\Pi^\lambda)^K N,$$

where the last arrow is given by averaging with respect to $K_N$, contains $V_2$. 
Indeed, let $G[[t]]_\lambda$ be a finite-dimensional quotient through which $G[[t]]$ acts on $\text{Gr}_G^\lambda$, and let $G[[t]]_\lambda$ be the corresponding group-object of $\text{Set}$. The vector space $\Pi^\lambda$ is spanned by elements of the following form. Each is invariant under some (small) open compact subgroup $K_{G[[t]]_\lambda} \subset G[[t]]_\lambda$, and is supported on a preimage in $\text{Gr}_G^\lambda$ of a single $K_{G[[t]]_\lambda}$-orbit on $\text{Gr}_G^\lambda$. By $G[[t]]$-invariance, we can assume that the orbit in question is that of the element $t^\lambda \in \text{Gr}_G^\lambda$.

By setting $K_N := N_\lambda \cap K_{G[[t]]_\lambda}$, we obtain that any element of the form specified above is contained in the corresponding $V_2$.

We will show that Main Lemma 2.4 implies that $V_2$ belongs to the image of the map

$$\text{Inf}^{G^1}(\pi) \to \text{Coinv}_{G[[t]]}^G \left( \text{Inf}_{G[[t]]}^G(\pi) \right) \to (\Pi^\lambda)^{K_N},$$

where first the arrow is the composition of the map of (4), followed by the action of $t^\lambda$.

For that let us write down in explicit terms the composition

(7) \[ \text{Inf}^{G^1}(\pi) \to \text{Coinv}_{G[[t]]}^G \left( \text{Inf}_{G[[t]]}^G(\pi) \right) \to \Pi \to \Pi_Y. \]

First, let us observe that the resulting map factors through the surjection $\text{Inf}^{G^1}(\pi) \to \text{Inf}^{G^\lambda}(\pi)$. Secondly, let us recall (cf. 2, Sect. 2.8) that $\text{Inf}^{G^\lambda}(\pi)$ is the inductive limit, taken in $\text{Vect}$, over finite-dimensional subspaces $F' \subset \pi$ of

$$\text{"lim" } \bigwedge_{\alpha} \text{Distr}_c(G_\lambda/K_{G_\lambda}^\alpha) \otimes F',$$

where $K_{G_\lambda}^\alpha$ runs through any exhausting family of open compact subgroups of $G_\lambda$.

By (1), the map $\Pi_Y \to \pi$ induces an isomorphism $\text{Coinv}_{G_\lambda}(\Pi_Y) \cong \pi$. For a given finite-dimensional subspace $F'$ let us choose a finite-dimensional subspace $F \subset \Pi_Y$ which projects surjectively onto $F'$, and for every index $\alpha$ consider the map

$$\text{Distr}_c(G_\lambda/K_{G_\lambda}^\alpha) \otimes F \to \Pi_Y$$

given by

$$\mu \otimes f \mapsto \mu \ast f,$$

where $f \in F$ and $\mu \in \text{Distr}_c(G_\lambda/K_{G_\lambda}^\alpha)$ is regarded as an element of the Hecke algebra.

The resulting system of maps (eventually in $\alpha$) factors through $\text{Distr}_c(G_\lambda/K_{G_\lambda}^\alpha) \otimes F \to \text{Distr}_c(G_\lambda/K_{G_\lambda}^\alpha) \otimes F'$, and defines the map in (7).

Let us now recall that if $W = \text{lim } W_\alpha$ is a pro-vector space mapping to a vector space $V$, the surjectivity of this map means that the eventually defined maps $W_\alpha \to V$ are all surjective, or, which is the same, that $\forall v \in V$, $v \in \text{Im}(W_\alpha)$ for those indices $\alpha$, for which the map $W \to V$ factors through $W_\alpha \to V$.

For a vector $v \in \pi$, let $F(v)$ be the finite-dimensional subspace of $\Pi_Y$, given by Lemma 2.4 and let $K_{G_\lambda}^\alpha(v)$ be the corresponding system of subgroups. Let $F'(v)$ denote the image of $F(v)$ in $\pi$.

Consider the composition:

$$\text{Distr}_c(G_\lambda/K_{G_\lambda}^\alpha(v)) \otimes F(v) \to \Pi_Y \to \Pi^\lambda \to (\Pi^\lambda)^{K_N}.$$
Let us take the unit element in \( \text{Distr}_c(\mathbb{G}_\lambda/K_{\mathbb{G}_\lambda}^G(v)) \), corresponding to the Haar measure on \( K_{\mathbb{G}_\lambda}^G(v) \). We obtain a map \( F(v) \to (\Pi^\lambda)^{K_N} \).

By Lemma 2.4(2), the image of this map is contained in \( V_2 \). When we further compose it with the evaluation map \( V_2 \hookrightarrow \Pi_Y \to \pi \) we obtain a map \( F(v) \to \pi \) equal to \( A_{K_N, K_{\mathbb{G}_\lambda}^G}(v) \), whose image contains \( v \), by Lemma 2.4(1).

This establishes the required surjectivity.

3. Proof of Main Lemma 2.4

3.1. For a given subgroup \( K_N \subset N_\lambda \), a subset \( X \subset \text{Gr}_G^\lambda \) and an arbitrary finite-dimensional subspace \( F \subset \Pi_Y \) we will construct a family of open compact subgroups \( K_{\mathbb{G}_\lambda}^G \subset G_\lambda \), such that the expressions \( A_{K_N, K_{\mathbb{G}_\lambda}^G}(f) \) and \( A_{\tilde{g}, K_{\mathbb{G}_\lambda}^G}(f) \) for \( f \in F \) can be evaluated explicitly.

From now on we will fix \( G = SL_2 \). We will change the notation slightly, and identify the set of dominant coweights with \( N_\lambda \); in which case we will replace \( \lambda \) by \( l \) and \( t^\lambda \in G((t)) \) becomes the matrix

\[
\begin{pmatrix}
t^l & 0 \\
0 & t^{-l}
\end{pmatrix}.
\]

Let us translate our initial subscheme \( Y \) by \( t^{-\lambda} \), in which case the point \( t^\lambda \) itself will go over to the unit point \( 1_{\text{Gr}_G} \in \text{Gr}_G \), and \( t^{-\lambda} \cdot Y \) will be contained in \( \overline{\text{Gr}_G^{2l}} \). (We denote by \( \text{Gr}_G^r \) the \( G[[t]] \)-orbit of the point \( \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix} \) in \( \text{Gr}_G \), and by \( \overline{\text{Gr}_G^r} \) its closure.) For the purposes of Lemma 2.4 we can replace \( t^{-\lambda} \cdot Y \) by the entire \( \overline{\text{Gr}_G^{2l}} \), with the standard action of the congruence subgroup \( G^1 \).

Note that the action of \( G^1 \) on \( \overline{\text{Gr}_G^{2l}} \) (resp., \( \overline{\text{Gr}_G^{2l}} \)) factors through \( G^1/G^{2l+1} \) (resp., \( G^1/G^{2l+1} \)).

For an integer \( r \) let us denote by \( G_r \) the quotient \( G^1/G^{2r+1} \), and by \( N_r \) the quotient \( t^{-r} \cdot N[[t]]/N[[t]] \). We will write elements of \( N_r \) as \( \sum_{1 \leq i \leq r} t^{-i} \cdot n_i \) with \( n_i \in K \), and thus think of it as an \( r \)-dimensional vector space over \( K \).

Similarly, we will identify \( G_r := G^1/G^{2r+1} \) with an \( 6r \)-dimensional vector space over \( K \), by writing its elements as matrices:

\[
\text{Id} + \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}
\]

and \( k_{lm} = \sum_{1 \leq i \leq 2r} t^i \cdot (k_{lm})_i \). In particular, we can speak of \( O_K \)-lattices in \( G_r \), where \( O_K \subset K \) is the ring of integers.
3.2. In what follows, for a point \( g \in G((t)) \), we will denote by \( \tilde{g} \) (resp., \( g \)) its image in \( \tilde{Gr}_G \) (resp., \( Gr_G \)).

Thus, we are interested in computing the integral

\[
\int_{k \in K_{Gr}} \text{ev}(f^{g,k}),
\]

when \( g \) is such that either \( g \in N_r \), or the corresponding point \( \tilde{g} \in Gr_G \) lies in \( \tilde{Gr}_G - X \), where \( X \) is a fixed open compact subset of \( \tilde{Gr}_G \) containing \( 1_{Gr_G} \), and \( f \in F \), where \( F \) is a fixed finite-dimensional subspace of \( \Pi' \).

Let \( p \) denote the projection \( \tilde{Gr}_G \to Gr_G \). Let \( s \) be a continuous section \( Gr_G \to \tilde{Gr}_G \), such that \( s(1_{Gr_G}) = 1_{\tilde{Gr}_G} \). A choice of such section defines an isomorphism

\[ \tilde{Gr}_G \cong Gr_G \times G. \]

We will denote by \( q \) the resulting map \( \tilde{Gr}_G \to G \).

Let us fix an open neighbourhood \( Z \) of \( 1_{Gr_G} \) in \( Gr_G \) small enough so that

\[ f(s(x)) = \text{ev}(f) \]

for \( x \in Z \) and \( f \in F \). Let \( K_{G}(F) \) be an open compact subgroup of \( G \), such that \( \text{ev}(f) \in \pi \) is \( K_{G}(F) \)-invariant for \( f \in F \).

Let \( K_{N_r} \) be an open compact subgroup of \( N_r \).

**Proposition 3.3.** There exists an \( O_K \)-lattice \( K_{Gr} \subset Gr_G \), which contains any given open subgroup of \( G_r \), such that the following is satisfied:

1. There exists an open compact subgroup \( K_{N_r}^{sm} \subset K_{N_r} \) such that:
   - (1a) For \( g = k \cdot n \in G((t)) \) with \( k \in K_{Gr} \) and \( n \in K_{N_r}^{sm} \), the corresponding point \( \tilde{g} \in \tilde{Gr}_G \) belongs to \( Z \).
   - (1b) For \( g \) as above, the left coset of \( q(\tilde{g}) \in G \) with respect to \( K_{G}(F) \subset G \) equals that of
     \[
     \left( 1 - \sum_{1 \leq i \leq r} (k_{12})_{2i} \cdot n_i^2 \right) / 1.
     \]
   - (1c) The integral \( \int_{k \in K_{Gr}} \text{ev}(f^{n_k}) \) vanishes if \( n \in K_{N_r} - K_{N_r}^{sm} \) and \( f \in F \).

2. If \( g \in G((t)) \), such that \( \tilde{g} \in \tilde{Gr}_G - X \), the integral \( \int_{k \in K_{Gr}} \text{ev}(f^{\tilde{g},k}) \) vanishes.

3.4. Let us deduce Main Lemma \( \ref{main-lemma} \) from Proposition \( \ref{proposition-3.3} \). Given a vector \( v \in \pi \) let us first define the subspace \( F(v) \in \Pi' \).

Recall that \( N \simeq K \) is the maximal unipotent subgroup of \( G = SL_2(K) \), and let \( N^* \) denote the Pontriagin dual group. Since \( N^* \) is also (non-canonically) isomorphic to \( K \), we have a valuation map \( \nu : N^* \to Z \), defined up to a shift. In particular, we can consider the subalgebra \( \text{Funct}_{val}(N^*) \simeq \text{Funct}(Z) \) inside the algebra \( \text{Funct}_{lc}(N^*) \) of all locally constant functions on \( N^* \).
Any smooth representation of $N$, and in particular $\pi$, can be thought of as a module over the algebra of $\text{Funct}_{\text{val}}(N^*)$, such that every element of this module has compact support. If a representation is cuspidal, this means that the support of every section is disjoint from $0 \in N^*$.

Therefore, if $v$ is a vector in a cuspidal representation $\pi$, the vector space $\text{Funct}_{\text{val}}(N^*) \cdot v \subset \pi$ is finite-dimensional. We denote this vector subspace by $F'(v)$ and let $F(v) \subset \Pi'$ to be any subspace surjecting onto $F'(v)$ by means of $ev$. We claim that $F(v)$ satisfies the requirements of Main Lemma 2.4.

Property (2) in the lemma is satisfied due to Proposition 3.3(2). To check property (1) we will rewrite $A_{KN,KG}(f)$ more explicitly in terms of the action of $G$ on $\pi$.

Note that by Proposition 3.3(1c), the integral
\[
\int_{n \in K_{N_r}K_{G_r}} \int_{k \in K_{G_r}} ev(f^{n-k})
\]
equals the integral over a smaller domain, namely,
\[
\int_{n \in K_{N_r}^m} \int_{k \in K_{G_r}} ev(f^{n-k}).
\]

By Proposition 3.3(1a) and (1b), the latter can be rewritten as
\[
(8) \quad \int_{n \in K_{N_r}^m} \int_{k \in K_{G_r}} (1)_{0 \leq i \leq l} (k_{12i} \cdot n_i^2) \cdot ev(f).
\]

For $n = \sum t^{-i} \cdot n_i \in N_r$, consider the map $\phi_n : G_r \to N_r$ given by
\[
k = \begin{pmatrix}
1 + \sum_i t^i \cdot (k_{11i}) & \sum_i t^i \cdot (k_{12i}) \\
\sum_i t^i \cdot (k_{21i}) & 1 + \sum_i t^i \cdot (k_{22i})
\end{pmatrix} \mapsto \begin{pmatrix}1 & \sum_{1 \leq i \leq r} (k_{12i}) \cdot n_i^2 \\
0 & 1.
\end{pmatrix}
\]

Thus, the expression in (8) can be rewritten as
\[
(9) \quad \int_{n \in K_{N_r}^m} (\phi_n)_* (\mu(K_{G_r})) \cdot ev(f),
\]
where $\mu(K_{G_r})$ denotes the Haar measure of $K_{G_r}$, and $(\phi_n)_* (\mu(K_{G_r}))$ is its push-forward under $\phi_n$, regarded as a distribution on $N_r$.

Note, however, that when we identify $G_r \simeq G^{1}/G^{2r+1}$ with a linear space over $K$, the Haar measure on this group goes over to a linear Haar measure. From this we obtain that for each $n \in N_r$, the distribution $(\phi_n)_* (\mu(K_{G_r}))$, thought of as a function on $N^*$, is the characteristic function of some $O_K$-lattice in $N^*$. Moreover, this lattice grows as $n \to 0$.

In particular, $(\phi_n)_* (\mu(K_{G_r}))$, as a function on $N^*$, belongs to $\text{Funct}_{\text{val}}(N^*)$, and the integral
\[
\int_{n \in K_{N_r}^m} (\phi_n)_* (\mu(K_{G_r})),
\]
being positive at every point of $N^*$, defines an invertible element of $\text{Funct}_{\text{val}}(N^*)$. 

Hence
\[ v \in (\phi_n)_* (\mu(K_G)) \cdot (\text{Funct}_{\text{val}}(N^*) \cdot v) = \text{Im} \left( \int_{n \in K_{N_r}} (\phi_n)_* (\mu(K_G)) \cdot ev(F(v)) \right), \]
which is what we had to show.

4. Proof of Proposition 3.3

4.1. We will construct the subgroup $K_{G_r}$ by induction with respect to the parameter $r$. (For property (2) we take $X \cap \overline{\text{Gr}_G}^{-1}$ as the corresponding open compact subset of $\overline{\text{Gr}_G}^{-1}$.)

When $r = 0$ all the subgroups in question are trivial. So, we can assume having constructed the subgroups $K_{G_{r-1}}$ and $K_{N_{r-1}}^{\text{sm}}$, and let us perform the induction step.

The key observation is provided by the following lemma:

**Lemma 4.2.** Let $X$ be a compact subset of $\text{Gr}_G^r$ and $f \in \Pi$. Then the integral
\[ \int_{k \in K_{G^{2r}/G^{2r+1}}} ev_{\tilde{g}}(f^k) = 0 \]
if $K_{G^{2r}/G^{2r+1}}$ is a sufficiently large compact subgroup of $G^{2r}/G^{2r+1}$, and $p(\tilde{g}) \in X$.

**Proof.** Since $\text{Gr}_G^r$ is a $G[[t]]$-orbit of $t^\lambda := \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}$ and since $G^{2r}$ is normalized by $G[[t]]$ and acts trivially on $\overline{\text{Gr}_G}$, by the compactness of $X$, the assertion of the lemma reduces to the fact that
\[ \int_{k \in K_{G^{2r}/G^{2r+1}}} ev_{t^\lambda}(f^k) = 0 \]
for every sufficiently large subgroup $K_{G^{2r}/G^{2r+1}}$.

Note that for $k \in G^{2r}$ written as
\[ \begin{pmatrix} 1 + t^{2r} \cdot k_{11} & t^{2r} \cdot k_{12} \\ t^{2r} \cdot k_{21} & 1 + t^{2r} \cdot k_{22} \end{pmatrix}, \]
$\text{Ad}_{t^\lambda}(k) \in G[[t]]$ projects to the element
\[ \begin{pmatrix} 1 & k_{12} \\ 0 & 1 \end{pmatrix} \in G = G[[t]]/G^1. \]

We have
\[ ev_{t^\lambda}(f^k) = \text{Ad}_{t^\lambda}(k) \cdot ev_{t^\lambda}(f). \]

Therefore, the integral in question equals the averaging of the vector $ev_{t^\lambda}(f) \in \pi$ over a compact subgroup of the maximal unipotent subgroup of $G$. Moreover, this subgroup grows together with $K_{G^{2r}/G^{2r+1}}$. Hence, our assertion follows from the cuspidality of $\pi$. \qed
4.3. To carry out the induction step we first choose $K'_{G_r} \subset G_r$ to be any $O_K$-lattice, which projects onto $K_{G_{r-1}} \subset G_{r-1}$.

By continuity and the compactness of $K'_{G_r}$, there exists an $O_K$-lattice $L \subset K$, such that for $K_{N_r}^{\text{sm}} = K_{N_r-1}^{\text{sm}} + t^{-r}L$ the following holds:

- (a') For $g = k' \cdot n \in G((t))/G^1$ with $k \in K'_{G_r}$, $n \in K_{N_r}^{\text{sm}}$, the point $\bar{g} \in \overline{G}_{r_G}$ belongs to $Z$.
- (b') For $g$ as above, the left coset of $q(\bar{g}) \in G$ with respect to $K_G(F) \subset G$ equals that of
  \[
  \begin{pmatrix}
  1 & -\sum_{1 \leq i \leq r} (k_{12})_{2i} \cdot n_i^2 \\
  0 & 1
  \end{pmatrix}
  \]

- (c') For $f \in F$, $f(k' \cdot (n' + t^{-r}n_r)) = f(k' \cdot n')$ for $n' \in K_{N_r-1}^{\text{sm}}$, $k' \in K'_{G_r}$, and $n_r \in L$.

Note that for any $n \in N_r$, $k' \in G_r$ and
\[
k = \begin{pmatrix}
1 + t^{2r} \cdot k_{11} & t^{2r} \cdot k_{12} \\
t^{2r} \cdot k_{21} & 1 + t^{2r} \cdot k_{22}
\end{pmatrix} \in G^{2r},
\]
we have:
\[
k' \cdot n = k' \cdot n \cdot \begin{pmatrix}
1 & -k_{12} \cdot n_r^2 \\
0 & 1
\end{pmatrix} \mod G^1.
\]

The group $K_{G_r}$ will be obtained from $K'_{G_r}$ by adding to it an (arbitrarily large) lattice in $G^{2r}/G^{2r+1}$.

Note that since $G^{2r}$ acts trivially on $\overline{G}_{r_G}$, any such subgroup would satisfy condition (1a) of Proposition 3.3 because $K_{G_r}$ satisfies (a') above. It will also automatically satisfy (1b) in view of (11) and (b') above. Thus, we have to arrange so that $K_{G_r}$ satisfies conditions (1c) and (2) of Proposition 3.3

4.4. By Lemma 1.2, we can find an open compact subgroup $K_{G_r}^{2r}/G^{2r+1} \subset G^{2r}/G^{2r+1}$, such that the integrals
\[
\int_{k \in K_{G_r}^{2r}/G^{2r+1}} e_v(f^{n \cdot k'} \cdot k)
\]
would vanish for $f \in F$, $k' \in K'_{G_r}$ and $n \in K_{N_r}$ is such that $n_r \notin L$.

Let us enlarge the initial $K'_{G_r}$ by adding to it any $O_K$-lattice in $G^{2r}/G^{2r+1}$ containing the above $K_{G_r}^{2r}/G^{2r+1}$. We claim that the resulting subgroup satisfies condition (1c) of Proposition 3.3

Indeed, let $n = n' + t^{-r}n_r$, $n' \in N_{r-1}$, $n_r \in K$ be an element in $K_{N_r} - K_{N_r}^{\text{sm}}$. If $n_r \notin L$, the integral vanishes by the choice of $K_{G_r}^{2r}/G^{2r+1}$. Thus, we can assume that $n_r \in L$, but $n' \notin K_{N_{r-1}}^{\text{sm}}$. But then the integral vanishes by (c') and the induction hypothesis.
Now, let us deal with condition (2) of Proposition 3.3. By the induction hypothesis, the integrals
\begin{equation}
\int_{k \in K'_r} ev(f^{g \cdot k})
\end{equation}

vanish when \( \overline{\mathcal{G}} \in \overline{\text{Gr}}_G^{r-1} - (\overline{\text{Gr}}_G^{r-1} \cap X) \).

Hence, by continuity and since \( K'_r \) is compact, there exists a neighbourhood \( X_1 \) of \( \overline{\text{Gr}}_G^{r-1} \) such that the integral \( \text{(11)} \) will vanish for the same subgroup \( K'_r \) and all \( g \) for which \( \overline{\mathcal{G}} \in X_1 \).

The sought-for subgroup \( K_G \) will be again obtained from the initial \( K'_r \) by adding to it an arbitrarily large open compact subgroup of \( G^{2r}/G^{2r+1} \). We claim that for any such \( K'_G \), the integral
\begin{equation}
\int_{k \in K_G} ev(f^{g \cdot k})
\end{equation}

will still vanish for \( \overline{\mathcal{G}} \in X_1 \).

This follows from the fact that the \( G^{2r} \)-action on \( \text{Gr}_G^r \) is trivial, and hence for \( k \in G^{2r}, k' \in G_r \) and \( g \in G((t)) \) projecting to \( \overline{\mathcal{G}} \in \overline{\text{Gr}}_G^r \),
\[ f(k \cdot k' \cdot g) = k_1 \cdot f(k' \cdot g) \]
for some \( k_1 \in G \).

We choose the suitable subgroup in \( G^{2r}/G^{2r+1} \) as follows. Set \( X_2 = (\overline{\text{Gr}}_G^r - X) - X_1 \). This is a compact subset of \( \text{Gr}_G^r \), and let us apply Lemma 4.2 to the compact set \( K'_r \cdot X_2 \subset \text{Gr}_G^r \).

We obtain that there exists an open compact subgroup \( K_G \subset G^{2r}/G^{2r+1} \), such that
\[ \int_{k \in K'_G} ev(f^{g \cdot k \cdot k'}) = 0 \]
for \( \overline{\mathcal{G}} \in X_2, k' \in K'_G \).

Let \( K'_G \) be the resulting subgroup of \( G_r \). We claim that it does satisfy condition (2) of Proposition 3.3. Indeed, consider again the integral \( \text{(12)} \) for \( \overline{\mathcal{G}} \in \overline{\text{Gr}}_G^r - X = X_1 \cup X_2 \).

We already know that it vanishes for \( \overline{\mathcal{G}} \in X_1 \). And if \( \overline{\mathcal{G}} \in X_2 \), it vanishes by the choice of \( K'_G \).

This completes the induction step in the proof of Proposition 3.3.

**References**

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