Research article

Fekete-Szegö problem for Bi-Bazilevič functions related to Shell-like curves

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Abstract: In the present investigation, we define a subclass of bi-univalent functions related to shell-like curves connected with Fibonacci numbers to find the estimates of second, third Taylor-Maclaurin coefficients and Fekete-Szegö inequalities. Further, certain special cases are also discussed.

Keywords: univalent functions; bi-univalent functions; shell-like function; Bazilevič function; Fibonacci number; Fekete-Szegö inequality

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $S$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$.

For analytic functions $f$ and $g$ in $\mathbb{D}$, $f$ is said to be subordinate to $g$ if there exists an analytic function $w$ such that

$$w(0) = 0, \quad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)), \quad z \in \mathbb{D}.$$  

This subordination will be denoted here by

$$f \prec g, \quad z \in \mathbb{D}.$$
or, conventionally, by
\[ f(z) < g(z), \quad z \in \mathbb{D}. \]

In particular, when \( g \) is univalent in \( \mathbb{D} \),
\[ f < g \quad (z \in \mathbb{D}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}). \]

Let \( \mathcal{P} \) denote the class of functions of the form
\[ p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots, \quad z \in \mathbb{D} \tag{1.2} \]
which are analytic with \( \Re \{ p(z) \} > 0 \). Here \( p(z) \) is called as Caratheodory functions [1]. It is well known that the following correspondence between the class \( \mathcal{P} \) and the class of Schwarz functions \( w \) exists: \( p \in \mathcal{P} \) if and only if \( p(z) = 1 + w(z)/1 - w(z) \). Let \( \mathcal{P}(\beta), 0 \leq \beta < 1 \), denote the class of analytic functions \( p \) in \( \mathbb{D} \) with \( p(0) = 1 \) and \( \Re \{ p(z) \} > \beta \).

Recently, Sokół [2] and Dziok et al. [3] studied the classes \( \mathcal{SL}(\overline{p}) \) and \( \mathcal{KSL}(\overline{p}) \) of shell-like functions and convex shell-like functions which are characterized by \( z f''/f^\prime(z) < \overline{p}(z) \) or \( 1 + z^2 f''/f^\prime(z) < \overline{p}(z) \), where \( \overline{p}(z) = (1 + \tau^2z^2)/(1 - \tau^2 - \tau^2z^2) \), \( \tau = (1 - \sqrt{5})/2 \approx -0.618 \) [4, 5] and the function \( \overline{p} \) is not univalent in \( \mathbb{D} \), but it is univalent in the disc \( |z| < (3 - \sqrt{5})/2 \approx 0.38 \). For example, \( \overline{p}(0) = \overline{p}(-1/2\tau) = 1 \) and \( \overline{p}(e^{\pi} \arccos(1/4)) = 1/\sqrt{5} \) and it may also be noticed that \( 1/|\tau| = |\tau|/1 - |\tau| \) which shows that the number \( |\tau| \) divides \( 0, 1 \) such that it fulfills the golden section. The image of the unit circle \( |z| = 1 \) under \( \overline{p} \) is a curve described by the equation given by
\[ (10x - \sqrt{5})^2 = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2, \]
which is translated and revolved trisectrix of Maclaurin. The curve \( \overline{p}(re^{it}) \) is a closed curve without any loops for \( 0 < r \leq r_0 = (3 - \sqrt{5})/2 \approx 0.38 \). For \( r_0 < r < 1 \), it has a loop and for \( r = 1 \), it has a vertical asymptote. Since \( \tau \) satisfies the equation \( \tau^2 = 1 + \tau \), this expression can be used to obtain higher powers \( \tau^n \) as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of \( \tau \) and 1. The resulting recurrence relationships yield Fibonacci numbers \( u_n \) as
\[ \tau^n = u_n\tau + u_{n-1}. \]

Also, Raina and Sokol [5] proved that
\[
\overline{p}(z) = \frac{1 + \tau^2z^2}{1 - \tau^2 - \tau^2z^2} = \left( t + \frac{1}{t} \right) \frac{t}{1 - t - t^2} = \frac{1}{\sqrt{5}} \left( t + \frac{1}{t} \right) \left( \frac{1}{1 - (1 - \tau)t} - \frac{1}{1 - \tau t} \right) = \left( t + \frac{1}{t} \right) \sum_{n=2}^{\infty} \frac{(1 - \tau)^n - \tau^n}{\sqrt{5}} t^n = \left( t + \frac{1}{t} \right) \sum_{n=2}^{\infty} u_n t^n
\]
\[
= 1 + \sum_{n=2}^{\infty} (u_{n-1} + u_{n+1}) \tau^n z^n,
\]

where
\[
u_n = \frac{(1 - \tau)n - \tau^n}{\sqrt{5}}, \quad \tau = \frac{1 - \sqrt{5}}{2}, \quad n = 1, 2, \ldots.
\]

This shows that the relevant connection of \(\tilde{p}\) with the sequence of Fibonacci numbers \(u_n\), such that
\[
u_0 = 0, \quad \nu_1 = 1, \quad \nu_{n+2} = \nu_n + \nu_{n+1}, \quad n = 0, 1, 2, 3, \ldots.
\]

Hence
\[
\tilde{p}(z) = 1 + \sum_{n=1}^{\infty} p_n z^n
\]
\[
= 1 + (u_0 + u_2) \tau z + (u_1 + u_3) \tau^2 z^2 + \sum_{n=3}^{\infty} (u_{n-3} + u_{n-2} + u_{n-1} + u_n) \tau^n z^n
\]
\[
= 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + \cdots
\]

(1.3)

We note that the function \(\tilde{p}\) belongs to the class \(P(\beta)\) with \(\beta = \frac{\sqrt{5}}{10} \approx 0.2236\) [5].

It is well known that every function \(f \in S\) has an inverse \(f^{-1}\), defined by
\[
(f^{-1}(f(z))) = z, \quad z \in \mathbb{D}
\]

and
\[
f(f^{-1}(w)) = w, \quad |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4},
\]

where
\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots
\]

(1.4)

A function \(f \in A\) is said to be bi-univalent in \(\mathbb{D}\) if both \(f(z)\) and \(f^{-1}(z)\) are univalent in \(\mathbb{D}\). Let \(\Sigma\) denote the class of bi-univalent functions in \(\mathbb{D}\) given by (1.1) for more details one can refer [6–13] and references therein. Also the various subclasses of bi-univalent functions related to shell-like curves were studied in [14–16].

Recently, the initial coefficient estimates are found for functions in the class of bi-univalent functions defined through certain polynomials like the Faber polynomial, the Lucas polynomial, the Chebyshev polynomial, the Gegenbauer polynomial and the Meixner-Pollaczek polynomial. Motivated in this line, in the present work, we introduce the following new subclass of bi-univalent function, as follows:

**Definition 1.1.** A function \(f \in \Sigma\) of the form (1.1) belongs to the class \(BSL^\mu_{\Sigma} \lambda (\tilde{p})\), \(\mu \geq 0, \lambda \geq 1, \delta \geq 0\), if the following conditions are satisfied:
\[
(1 - \lambda) \left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z}\right)^{\mu-1} + \xi z f''(z) < \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau^2 z - \tau^2 z^2}, \quad z \in \mathbb{D}
\]
and for \( g(w) = f^{-1}(w) \)

\[
(1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} + \xi \delta g''(w) < \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \quad w \in \mathbb{D},
\]

where \( \tau = \frac{1 - \sqrt{5}}{2} \approx -0.618 \) and \( \xi = \frac{2 + \mu}{2 + \tau}. \)

By suitably specializing the values of \( \mu, \lambda \) and \( \delta \), the class \( \mathcal{B} \mathcal{S} \mathcal{L}_\Sigma^{\mu, \delta, \lambda}(\tilde{p}) \) reduces to various new subclasses, we illustrate the following subclasses:

1. For \( \delta = 0, \) we get the class \( \mathcal{B} \mathcal{S} \mathcal{L}_\Sigma^{\mu, 0, \lambda}(\tilde{p}) \equiv \mathcal{N} \mathcal{S} \mathcal{L}_\Sigma^{\mu, \lambda}(\tilde{p}) \). A function \( f \in \Sigma \) of the form (1.1) is said to be in \( \mathcal{N} \mathcal{S} \mathcal{L}_\Sigma^{\mu, \lambda}(\tilde{p}(z)), \) if the following conditions

\[
(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} < \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \quad z \in \mathbb{D}
\]

and for \( g(w) = f^{-1}(w) \)

\[
(1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} < \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \quad w \in \mathbb{D},
\]

hold, where \( \tau = \frac{1 - \sqrt{5}}{2} \approx -0.618. \)

2. For \( \lambda = 1 \) and \( \delta = 0, \) we get the class \( \mathcal{B} \mathcal{S} \mathcal{L}_\Sigma^{1, 0, \lambda}(\tilde{p}) \equiv \mathcal{R} \mathcal{S} \mathcal{L}_\Sigma^{\mu, \lambda}(\tilde{p}) \). A function \( f \in \Sigma \) of the form (1.1) is said to be in \( \mathcal{R} \mathcal{S} \mathcal{L}_\Sigma^{\mu, \lambda}(\tilde{p}), \) if the following conditions

\[
f''(z) \left( \frac{f(z)}{z} \right)^{\mu-1} < \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \quad z \in \mathbb{D}
\]

and for \( g(w) = f^{-1}(w) \)

\[
g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} < \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \quad w \in \mathbb{D},
\]

hold, where \( \tau = \frac{1 - \sqrt{5}}{2} \approx -0.618. \)

3. For \( \mu = 1, \) we get the class \( \mathcal{B} \mathcal{S} \mathcal{L}_\Sigma^{1, \delta, \lambda}(\tilde{p}) \equiv \mathcal{W} \mathcal{S} \mathcal{L}_\Sigma^{\delta, \lambda}(\tilde{p}) \). A function \( f \in \Sigma \) of the form (1.1) is said to be in \( \mathcal{W} \mathcal{S} \mathcal{L}_\Sigma^{\delta, \lambda}(\tilde{p}), \) if the following conditions

\[
(1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) < \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \quad z \in \mathbb{D}
\]

and for \( g(w) = f^{-1}(w) \)

\[
(1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) < \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \quad w \in \mathbb{D},
\]

hold, where \( \tau = \frac{1 - \sqrt{5}}{2} \approx -0.618. \)
4. For \( \lambda = \mu = 1 \), we get the class \( BSL_{\Sigma}^{1,1,1}(\overline{p}) \equiv FSL_{\Sigma}(\delta, \overline{p}) \). A function \( f \in \Sigma \) of the form (1.1) is said to be in \( FSL_{\Sigma}(\delta, \overline{p}) \), if the following conditions

\[
f'(z) + \delta zf''(z) < \overline{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \quad z \in \mathbb{D}
\]

and for \( g(w) = f^{-1}(w) \)

\[
g'(w) + \delta wg''(w) < \overline{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \quad w \in \mathbb{D},
\]

hold, where \( \tau = \frac{1 - \sqrt{5}}{2} \approx -0.618 \).

5. For \( \mu = 1 \) and \( \delta = 0 \), we obtain the class \( BSL_{\Sigma}^{1,0,1}(\overline{p}) \equiv BSL_{\Sigma}(\lambda, \overline{p}) \). A function \( f \in \Sigma \) of the form (1.1) is said to be in \( BSL_{\Sigma}(\lambda, \overline{p}(z)) \), if the following conditions

\[
(1 - \lambda)f(z) + \lambda f'(z) < \overline{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \quad z \in \mathbb{D}
\]

and for \( g(w) = f^{-1}(w) \)

\[
(1 - \lambda)g(w) + \lambda g'(w) < \overline{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \quad w \in \mathbb{D},
\]

hold, where \( \tau = \frac{1 - \sqrt{5}}{2} \approx -0.618 \).

6. For \( \lambda = 1, \mu = 1 \) and \( \delta = 0 \), we have the class \( BSL_{\Sigma}^{1,0,0}(\overline{p}) \equiv HSL_{\Sigma}(\overline{p}) \). A function \( f \in \Sigma \) of the form (1.1) is said to be in \( HSL_{\Sigma}(\overline{p}) \), if the following conditions

\[
f'(z) < \overline{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \quad z \in \mathbb{D}
\]

and for \( g(w) = f^{-1}(w) \)

\[
g'(w) < \overline{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \quad w \in \mathbb{D},
\]

hold, where \( \tau = \frac{1 - \sqrt{5}}{2} \approx -0.618 \).

In order to prove our results for the functions in the class \( BSL_{\Sigma}^{\mu,\delta,\lambda}(\overline{p}) \), we need the following lemma.

**Lemma 1.1.** [10] If \( p \in \mathcal{P} \), then \( |p_i| \leq 2 \) for each \( i \), where \( \mathcal{P} \) is the family of all functions \( p \), analytic in \( \mathbb{D} \), for which

\[
\Re(p(z)) > 0 \quad (z \in \mathbb{D}),
\]

where

\[
p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{D}).
\]

In this investigation, we find the estimates for the coefficients \( |a_2| \) and \( |a_3| \) for functions in the class \( BSL_{\Sigma}^{\mu,\delta,\lambda}(\overline{p}) \) and its special cases. Also, Fekete-Szegö inequality for functions in this subclass.
2. Coefficient estimates and Fekete-Szegő inequality

In the following theorem, we discuss coefficient estimates and Fekete-Szegő inequality for functions in the class $f \in \mathcal{BSL}_\Sigma^{\mu, \delta, \lambda} (\tilde{p})$.

**Theorem 2.1.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{BSL}_\Sigma^{\mu, \delta, \lambda} (\tilde{p})$. Then

$$|a_2| \leq |\tau| \frac{2}{M}, \quad |a_3| \leq \frac{(2\lambda + \mu - 1) \tau + 2(1-3\tau)(\lambda + \mu + 2\xi \delta)^2}{M(2\lambda + \mu + 6\xi \delta)} |\tau|$$

and for $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\tau|}{2\lambda + \mu + 6\xi \delta} : 0 \leq |\nu - 1| \leq \frac{M}{2(2\lambda + \mu + 6\xi \delta)|\tau|} \\ \frac{M}{2(2\lambda + \mu + 6\xi \delta)|\tau|} : |\nu - 1| \geq \frac{M}{2(2\lambda + \mu + 6\xi \delta)|\tau|} \end{cases}$$

where

$$M = \left[ \tau [(2\lambda + \mu)(\mu + 1) + 12\xi \delta] + 2(1-3\tau)(\lambda + \mu + 2\xi \delta)^2 \right].$$

**Proof.** Since $f \in \mathcal{BSL}_\Sigma^{\mu, \delta, \lambda} (\tilde{p})$, from the Definition 1.1, we have

$$\left( 1 - \lambda \right) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} + \xi \delta z f''(z) = \tilde{p}(p(z)) \quad (2.1)$$

and for $g = f^{-1}$

$$\left( 1 - \lambda \right) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} + \xi \delta w g''(w) = \tilde{p}(q(w)), \quad (2.2)$$

where $z, w \in \mathbb{D}$. Using the fact the function $p$ of the form (1.2) and $p < \tilde{p}$. Then there exists an analytic function $p$ such that $|p(z)| < 1$ in $\mathbb{D}$ and $p(z) = \tilde{p}(p(z))$. Therefore, define the function

$$h(z) = \frac{1 + p(z)}{1 - p(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

is in the class $\mathcal{P}$. It follows that

$$p(z) = \frac{h(z) - 1}{h(z) + 1} = \frac{p_1}{2} z + \left( p_2 - \frac{p_1^2}{2} \right) \frac{z^2}{2} + \left( p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) \frac{z^3}{2} + \cdots$$

and

$$\tilde{p}(p(z)) = 1 + \tilde{p}_1 \left( \frac{p_1}{2} z + \left( p_2 - \frac{p_1^2}{2} \right) \frac{z^2}{2} + \left( p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) \frac{z^3}{2} + \cdots \right)$$

$$+ \tilde{p}_2 \left( \frac{p_1}{2} z + \left( p_2 - \frac{p_1^2}{2} \right) \frac{z^2}{2} + \left( p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) \frac{z^3}{2} + \cdots \right)^2$$

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\[ + \bar{p}_3 \left( \frac{p_1}{2} z + \left( p_2 - \frac{p_1^2}{2} \right) \frac{z^2}{2} + \left( p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) \frac{z^3}{2} + \cdots \right)^3 + \cdots \]
\[ = 1 + \frac{\bar{p}_1 p_1}{2} z + \left( \frac{1}{2} \left( p_2 - \frac{p_1^2}{2} \right) \bar{p}_1 + \frac{p_1^2}{4} \bar{p}_2 \right) z^2 \]
\[ + \left( \frac{1}{2} \left( p_3 - p_1 p_2 + \frac{p_1^3}{4} \right) \bar{p}_1 + \frac{1}{2} p_1 \left( p_2 - \frac{p_1^2}{2} \right) \bar{p}_2 + \frac{p_1^3}{8} \bar{p}_3 \right) z^3 + \cdots . \] (2.3)

Similarly, there exists an analytic function \( v \) such that \( |q(w)| < 1 \) in \( \mathbb{D} \) and \( p(w) = \bar{p}(q(w)) \). Therefore, the function
\[ k(w) = \frac{1 + q(w)}{1 - q(w)} = 1 + q_1 w + q_2 w^2 + \cdots \]
is in the class \( \mathcal{P} \). It follows that
\[ q(w) = \frac{k(w) - 1}{k(w) + 1} = \frac{q_1}{2} w + \left( q_2 - \frac{q_1^2}{2} \right) \frac{w^2}{2} + \left( q_3 - q_1 q_2 + \frac{q_1^3}{4} \right) \frac{w^3}{2} + \cdots \]
and
\[ \bar{p}(q(w)) = 1 + \bar{p}_1 \left( \frac{q_1}{2} w + \left( q_2 - \frac{q_1^2}{2} \right) \frac{w^2}{2} + \left( q_3 - q_1 q_2 + \frac{q_1^3}{4} \right) \frac{w^3}{2} + \cdots \right) \]
\[ + \bar{p}_2 \left( \frac{q_2}{2} w + \left( q_3 - q_1 q_2 + \frac{q_1^3}{4} \right) \frac{w^3}{2} + \cdots \right)^2 \]
\[ + \bar{p}_3 \left( \frac{q_3}{2} w + \left( q_3 - q_1 q_2 + \frac{q_1^3}{4} \right) \frac{w^3}{2} + \cdots \right)^3 + \cdots \]
\[ = 1 + \frac{\bar{p}_1 q_1}{2} w + \left( \frac{1}{2} \left( q_2 - \frac{q_1^2}{2} \right) \bar{p}_1 + \frac{q_1^3}{4} \bar{p}_2 \right) w^2 \]
\[ + \left( \frac{1}{2} \left( q_3 - q_1 q_2 + \frac{q_1^3}{4} \right) \bar{p}_1 + \frac{1}{2} q_1 \left( q_2 - \frac{q_1^2}{2} \right) \bar{p}_2 + \frac{q_1^3}{8} \bar{p}_3 \right) w^3 + \cdots . \] (2.4)

By virtue of (2.1), (2.2), (2.3) and (2.4), we have

\[ (\lambda + \mu + 2\xi \delta) a_2 = \frac{p_1 t}{2}, \] (2.5)

\[ (2\lambda + \mu) \left[ \left( \frac{\mu - 1}{2} \right) a_2^2 + \left(1 + \frac{6\delta \xi}{2\lambda + \mu} \right) a_3 \right] = \frac{1}{2} \left( p_2 - \frac{p_1^2}{2} \right) \tau + \frac{3p_1^2}{4} \tau^2, \] (2.6)

\[ - (\lambda + \mu + 2\xi \delta) a_2 = \frac{q_1 t}{2}, \] (2.7)

and

\[ (2\lambda + \mu) \left[ \left( \frac{\mu + 3}{2} + \frac{12\delta \xi}{2\lambda + \mu} \right) a_2^2 - \left(1 + \frac{6\delta \xi}{2\lambda + \mu} \right) a_3 \right] = \frac{1}{2} \left( q_2 - \frac{q_1^2}{2} \right) \tau + \frac{3q_1^2}{4} \tau^2. \] (2.8)

From (2.5) and (2.7), we obtain

\[ p_1 = -q_1. \]
and

\[ 2(\lambda + \mu + 2\xi \delta)^2 a^2_2 = \frac{(p_1^2 + q_1^2)\tau^2}{4}, \]
\[ a^2_2 = \frac{(p_1^2 + q_1^2)\tau^2}{8(\lambda + \mu + 2\xi \delta)^2}. \]  
(2.9)

By adding (2.6) and (2.8), we have

\[ [(2\lambda + \mu)(\mu + 1) + 12\xi \delta]a^2_2 = \frac{1}{2}(p_2 + q_2)\tau - \frac{1}{4}(p_1^2 + q_1^2)\tau + \frac{3}{4}(p_1^2 + q_1^2)\tau^2. \]  
(2.10)

By substituting (2.9) in (2.10), we reduce that

\[ a^2_2 = \frac{(p_2 + q_2)\tau^2}{2\tau [(2\lambda + \mu)(\mu + 1) + 12\xi \delta] + 2(1 - 3\tau)(\lambda + \mu + 2\xi \delta)^2}. \]  
(2.11)

Now, applying Lemma 1.1, we obtain

\[ |a_2| \leq \sqrt{2|\tau|}. \]  
(2.12)

By subtracting (2.8) from (2.6), we obtain

\[ a^3 = \frac{(p_2 - q_2)\tau}{4(2\lambda + \mu + 6\xi \delta)} + a^2_2. \]  
(2.13)

Hence by Lemma 1.1, we have

\[ |a_3| \leq \frac{(|p_2| + |q_2|)|\tau|}{4(2\lambda + \mu + 6\xi \delta)} + |a_2|^2 \leq \frac{|\tau|}{2\lambda + \mu + 6\xi \delta} + |a_2|^2. \]

Then in view of (2.12), we obtain

\[ |a_3| \leq \frac{|\tau|(2\lambda + \mu)(\mu - 1)\tau + 2(1 - 3\tau)(\lambda + \mu + 2\xi \delta)^2)}{(2\lambda + \mu + 6\delta \xi)[(2\lambda + \mu)(\mu + 1) + 12\delta \xi] + 2(1 - 3\tau)(2\delta \xi + \lambda + \mu)} \]

From (2.13), we have

\[ a_3 - \nu a^2_2 = \frac{(p_2 - q_2)\tau}{4(2\lambda + \mu + 6\xi \delta)} + (1 - \nu) a^2_2. \]  
(2.14)

By substituting (2.11) in (2.14), we have

\[ a_3 - \nu a^2_2 = \frac{(p_2 - q_2)\tau}{4(2\lambda + \mu + 6\xi \delta)} + \frac{(1 - \nu)(p_2 + q_2)\tau^2}{2\tau [(2\lambda + \mu)(\mu + 1) + 12\xi \delta] + 2(1 - 3\tau)(\lambda + \mu + 2\xi \delta)^2}] \]
\[ = \left( h(\nu) + \frac{|\tau|}{4(2\lambda + \mu + 6\xi \delta)} \right) p_2 + \left( h(\nu) - \frac{|\tau|}{4(2\lambda + \mu + 6\xi \delta)} \right) q_2. \]  
(2.15)
where
\[ h(\nu) = \frac{(1 - \nu) \tau^2}{2 \left[ \tau \left( (2\lambda + \mu)(\mu + 1) + 12\xi\delta \right) + 2(1 - 3\tau)(\lambda + \mu + 2\xi\delta)^2 \right]}. \]

Thus by taking modulus of (2.15), we conclude that
\[
|a_3 - \nu a_2^2| \leq \begin{cases} 
\left| \frac{\tau}{2 \lambda + \mu + 6\xi\delta} \right| & 0 \leq |h(\nu)| \leq \frac{|\tau|}{4 (2\lambda + \mu + 6\xi\delta)} \\
\left| \frac{2 |1 - \nu| \tau^2}{4 |h(\nu)|} \right| & |h(\nu)| \geq \frac{|\tau|}{4 (2\lambda + \mu + 6\xi\delta)}. 
\end{cases}
\]

3. Corollaries and consequences

In this section, we give coefficient estimates and Fekete-Szegő inequalities for the subclasses of \( BSL^\mu,\delta,\lambda \{\tilde{p}\} \).

**Corollary 3.1.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be in the class \( NSL^\mu,\lambda \{\tilde{p}\} \). Then
\[
|a_2| \leq |\tau| \sqrt{\frac{2}{M_1}}, \quad |a_3| \leq \frac{|\tau| \left\{ (2\lambda + \mu)(\mu - 1) \tau + 2(1 - 3\tau)(\lambda + \mu)^2 \right\}}{M_1 (2\lambda + \mu)}
\]
and for \( \nu \in \mathbb{R} \),
\[
|a_3 - \nu a_2^2| \leq \begin{cases} 
\left| \frac{\tau}{2 \lambda + \mu} \right| & 0 \leq |\nu - 1| \leq \frac{M_1}{2 (2\lambda + \mu)|\tau|} \\
\left| \frac{2 |1 - \nu| \tau^2}{M_1} \right| & |\nu - 1| \geq \frac{2 (2\lambda + \mu)|\tau|}{M_1} 
\end{cases}
\]
where
\[
M_1 = \tau(2\lambda + \mu)(\mu + 1) + 2(1 - 3\tau)(\lambda + \mu)^2.
\]

**Corollary 3.2.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be in the class \( RSL^\mu \{\tilde{p}\} \). Then
\[
|a_2| \leq |\tau| \sqrt{\frac{2}{M_2}}, \quad |a_3| \leq \frac{|\tau| \left\{ (\mu + 2)(\mu - 1) \tau + 2(1 - 3\tau)(1 + \mu)^2 \right\}}{M_2 (\mu + 2)}
\]
and for \( \nu \in \mathbb{R} \),
\[
|a_3 - \nu a_2^2| \leq \begin{cases} 
\left| \frac{\tau}{2 + \mu} \right| & 0 \leq |\nu - 1| \leq \frac{M_2}{2 (2 + \mu)|\tau|} \\
\left| \frac{2 |1 - \nu| \tau^2}{M_2} \right| & |\nu - 1| \geq \frac{2 (2 + \mu)|\tau|}{M_2} 
\end{cases}
\]
where
\[
M_2 = 2(1 + \mu)^2 - (1 + \mu)(4 + 5\mu)\tau.
\]
Remark 3.1. For $\mu = 1$, results discussed in Corollaries 3.2 is coincides with bounds obtained in [14, Corollary 1, p.78].

Corollary 3.3. Let $f(z) = z + \sum_{n=2}^\infty a_nz^n$ be in the class $\mathcal{WSL}_\delta^{\pm,\lambda}(\tilde{p})$. Then

$$|a_2| \leq \frac{|\tau|}{\sqrt{M_3}}, \quad |a_3| \leq \frac{2|\tau|(1 - 3\tau)(1 + \lambda + 2\delta)^2}{2M_3(1 + 2\lambda + 6\delta)}$$

and for $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\tau|}{1 + 2\lambda + 6\delta} & ; 0 \leq |\nu| - 1 \leq \frac{M_3}{(1 + 2\lambda + 6\delta)|\tau|} \\ \frac{|\tau|}{M_3} & ; |h(\nu)| \geq \frac{M_3}{(1 + 2\lambda + 6\delta)|\tau|}, \end{cases}$$

where

$$M_3 = \tau(1 + 2\lambda + 6\delta) + (1 - 3\tau)(1 + \lambda + 2\delta)^2.$$

Corollary 3.4. Let $f(z) = z + \sum_{n=2}^\infty a_nz^n$ be in the class $\mathcal{FSL}_\Sigma(\delta, \tilde{p})$. Then

$$|a_2| \leq \frac{|\tau|}{\sqrt{M_4}}, \quad |a_3| \leq \frac{8|\tau|(1 - 3\tau)(1 + \delta)^2}{6M_4(1 + 2\delta)}$$

and for $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\tau|}{3 + 6\delta} & ; 0 \leq |\nu| - 1 \leq \frac{M_4}{(3 + 6\delta)|\tau|} \\ \frac{|\tau|}{M_4} & ; |\nu| - 1 \geq \frac{M_4}{(3 + 6\delta)|\tau|}, \end{cases}$$

where

$$M_4 = 3\tau(1 + 2\delta) + 4(1 - 3\tau)(1 + \delta)^2.$$

Corollary 3.5. Let $f(z) = z + \sum_{n=2}^\infty a_nz^n$ be in the class $\mathcal{BSL}_\lambda^{\pm,\lambda}(\tilde{p})$. Then

$$|a_2| \leq \frac{|\tau|}{\sqrt{M_5}}, \quad |a_3| \leq \frac{|\tau|(1 - 3\tau)(1 + \lambda)^2}{(1 + 2\lambda)M_5}$$

and for $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\tau|}{1 + 2\lambda} & ; 0 \leq |\nu| - 1 \leq \frac{M_5}{(1 + 2\lambda)|\tau|} \\ \frac{|\tau|}{M_5} & ; |\nu| - 1 \geq \frac{M_5}{(1 + 2\lambda)|\tau|}, \end{cases}$$

where

$$M_5 = \tau(1 + 2\lambda) + (1 - 3\tau)(1 + \lambda)^2.$$
Corollary 3.6. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be in the class \( \mathcal{HSL}_\Sigma(\tilde{p}) \). Then

\[
|a_2| \leq \frac{|\tau|}{\sqrt{4 - 9\tau}}, \quad |a_3| \leq \frac{|\tau|(4 - 12\tau)}{3(4 - 9\tau)}
\]

and for \( \nu \in \mathbb{R} \),

\[
|a_3 - \nu a_2^2| \leq \begin{cases} 
\frac{|\tau|}{3} ; 0 \leq |\nu - 1| \leq \frac{4 - 9\tau}{3 |\tau|} \\
\frac{|1 - \nu|\tau^2}{4 - 9\tau} ; |\nu - 1| \geq \frac{4 - 9\tau}{3 |\tau|}.
\end{cases}
\]

4. Conclusions

In this investigation, we obtain upper bounds for the coefficients \( |a_2| \), \( |a_3| \) and Fekete-Szegő inequality \( |a_3 - \nu a_2^2| \) for functions in the class \( \mathcal{BSL}_\Sigma^{\mu, \delta, \lambda}(\tilde{p}) \). Also, certain special cases are also discussed.

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Conflict of interest

The authors declare that they have no conflict of interest.

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