A FAMILY OF IRREDUCIBLE FREE DIVISORS IN $\mathbb{P}^2$

RAMAKRISHNA NANDURI

ABSTRACT. An infinite family of irreducible homogeneous free divisors in $K[x, y, z]$ is constructed. Indeed, we identify sets of monomials $X$ such that the general polynomial supported on $X$ is a free divisor.

1. Introduction

The goal of this note is to construct a family of irreducible homogeneous free divisors in $K[x, y, z]$, where $K$ is any field.

Free divisors are originally associated with hyperplane arrangement theory. Also they play an important role in singularity theory, see [Sa80], [BEG09] and references therein. Here we give the definition of free divisors in the polynomial ring $K[x_1, \ldots, x_n]$.

Definition 1.1. A (formal) free divisor is a reduced polynomial $F \in K[x_1, \ldots, x_n]$ such that its Jacobian ideal $J(F) = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, F)$ is perfect of codimension 2 in $K[x_1, \ldots, x_n]$.

In $K[x_1, x_2]$, every reduced homogeneous polynomial is a free divisor. In [Si06], A. Simis constructed a family of irreducible homogeneous free divisors of degree 6 in $K[x, y, z]$ called Cayley sextics, by studying the depth of the Jacobian ideal, [Si06, Proposition 4.4]. Later Simis and Tohaneanu constructed a family of irreducible homogeneous free divisors in $K[x, y, z]$ of degree at least 5, [ST12, Proposition 2.2]. Also [NS11, ST09, TI2] and [Si05] give some more insights on free divisors in three variables. Recently R.-O. Buchweitz and A. Conca studied the free divisors annihilated by many Euler vector fields and they gave a chain rule and classification of toric free surfaces and binomial quasi-homogeneous free divisors, see [BC12]. We study the following question. Describe sets of monomials $X$ in $K[x, y, z]$ of the same degree such that the sum of monomials in $X$ with general scalar coefficients, is a free divisor.

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Motivated by Simis-Tohaneanu example, in this note we construct a family of irreducible free divisors in \( \mathbb{P}^2 \). For any \( d \geq 5 \), we show that

\[
x^{d-\alpha}F_1(x, y) + y^\left\lfloor \frac{d}{2} \right\rfloor + \alpha + 1 F_2(x, y) + x^\beta y^{d-\beta-1}z
\]

is an irreducible homogeneous free divisor in \( K[x, y, z] \), if \( 0 \leq \alpha, \beta \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 3 \), \( 0 \leq \alpha + \beta \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 3 \) and any homogeneous polynomials \( F_1, F_2 \in K[x, y] \) of degrees \( \alpha \) and \( (d - \left\lfloor \frac{d}{2} \right\rfloor - \alpha - 1) \) respectively such that \( y \nmid F_1 \) and \( x \nmid F_2 \), see Theorem 2.2 below.

This family of free divisors is larger than that of Simis-Tohaneanu’s family. If we take \( \alpha = 0 = \beta, F_1 = a \) and \( F_2 = a_1 y^{d - \left\lfloor \frac{d}{2} \right\rfloor - \alpha - 1} + a_2 x y^{d - \left\lfloor \frac{d}{2} \right\rfloor - \alpha - 2} + a_3 x^2 y^{d - \left\lfloor \frac{d}{2} \right\rfloor - \alpha - 3} \), where \( a, a_1, a_2, a_3 \in K \) such that \( a \neq 0, a_1 \neq 0 \), then we get the Simis-Tohaneanu’s example. We prove our main theorem by describing the Saito matrix and the structure of syzygies of the Jacobian ideal. The structure of the syzygies that we present here has been identified by using the software CoCoA, [Co].

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2. Main Results

For any real number \( r \), let \( \lfloor r \rfloor \) denote the largest integer less than or equal to \( r \) and \( \lceil r \rceil \) denote the smallest integer bigger than or equal to \( r \). For any polynomial \( F \in K[x_1, \ldots, x_n] \), the vector \( \nabla F = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}) \), is called the gradient of \( F \). We state a theorem of K. Saito on free divisors, for the polynomial algebra \( K[x_1, \ldots, x_n] \):

**Theorem 2.1** (Saito’s criterion, [Sa80]). A reduced polynomial \( F \in K[x_1, \ldots, x_n] \) is a free divisor if and only if there exists a \( n \times n \) matrix \( A \) with entries in \( K[x_1, \ldots, x_n] \) such that \( \det(A) = F \) and \( (\nabla F)A \equiv 0 \mod (F) \).

The matrix appearing in the Saito’s criterion, is called a discriminant matrix or Saito matrix of the polynomial. Now we prove the main theorem.

**Theorem 2.2.** Let \( d \geq 5 \) be an integer and \( v = \left\lfloor \frac{d}{2} \right\rfloor \). Then the homogeneous polynomial of degree \( d \)

\[
F = x^{d-\alpha}F_1(x, y) + y^{v+\alpha+1}F_2(x, y) + x^\beta y^{d-\beta-1}z
\]

is an irreducible free divisor in \( K[x, y, z] \) provided:
(1) $\alpha, \beta \geq 0$ and $0 \leq \alpha + \beta \leq \lfloor \frac{d+1}{2} \rfloor - 3$.

(2) $F_1(x, y)$ is homogeneous of degree $\alpha$ and $y \nmid F_1$.

(3) $F_2(x, y)$ is homogeneous of degree $d - v - \alpha - 1$ and $x \nmid F_2$.

In the proof of the Theorem we will show also how to construct the Saito matrix of $F$. For instance, for $\beta \neq 0$ we will show that the Saito matrix of $F$ has the following shape:

$$\frac{1}{d\mu} \begin{bmatrix} x & x^{\beta} y^{\gamma+2} \frac{\partial F_1}{\partial y} & H_1 + x^{\beta} y^{\gamma+1} \frac{\partial F_1}{\partial y} E \\ y & -x^{\beta} y^{\gamma+2} \left(x \frac{\partial F_1}{\partial x} + (d - \alpha)F_1\right) & H_3 - x^{\beta-1} y^{\gamma+1} z \left(x \frac{\partial F_1}{\partial x} + (d - \alpha)F_1\right) E \\ z & G_3 - x^{\beta} y^{\gamma+1} z W & H_5 + H_6 z - x^{\beta-1} y^{\gamma} z^2 E W \end{bmatrix},$$

where

(i) $\gamma = d - v - 3 - \alpha - \beta$,

(ii) $H_1, H_3, H_5 \in K[x, y]$ are homogeneous of degree $v$, $H_6 \in K[x, y]$ is homogeneous of degree $v - 1$, $E \in K[x, y]$ is linear if $d$ is odd and quadratic if $d$ is even. The nature of these polynomials is described in the proof.

(iii) $W = \beta y \frac{\partial F_1}{\partial y} - (d - \beta - 1) \left(x \frac{\partial F_1}{\partial x} + (d - \alpha)F_1\right)$,

and

$$G_3 = \begin{vmatrix} 1 & -\left(\frac{2d-v}{\alpha}\right) y \frac{\partial F_1}{\partial y} & \frac{d(d-v+\alpha)}{\alpha(d-\alpha)} x \frac{\partial F_1}{\partial x} \\ 1 & \frac{d(2d-v)}{\alpha(d-\alpha)} x \frac{\partial F_1}{\partial x} & -\frac{d-\alpha}{\alpha} y \frac{\partial F_1}{\partial y} \\ 0 & x \frac{\partial F_2}{\partial x} & y \frac{\partial F_2}{\partial y} \end{vmatrix}, \quad \text{if } \alpha \neq 0,$$

$$-d \left(y \frac{\partial F_2}{\partial y} + (d - v) F_2\right) F_1, \quad \text{if } \alpha = 0.$$

Proof. Let $F = x^{d-\alpha} F_1 + y^{\alpha+1} F_2 + x^{\beta} y^{d-\beta-1} z$, where $F_1, F_2 \in K[x, y]$ are homogeneous polynomials of degrees $\alpha, d - v - \alpha - 1$ respectively such that $y \nmid F_1$ and $x \nmid F_2$. Since $F$ is linear in $z$ and by the assumptions on $F_1$ and $F_2$, forces that $F$ is irreducible. We prove the theorem for the case $\beta \neq 0$ and the proof is similar for the case $\beta = 0$. We prove $F$ has a Saito matrix say $A$, as stated above. That is, we need to show the product

$$\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{bmatrix} A \equiv 0 \mod (F) \text{ and } \det(A) = F.$$

Since $F$ is homogeneous, by Euler’s
equation we have $x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} = d F$. Therefore the first entry of the above product matrix is congruent to zero modulo $(F)$. Now it suffices to show that the last two entries of the above product matrix are congruent to zero modulo $(F)$ and $\det(A) = F$. In fact, we show that the second column of $A$ is a syzygy of $J(F)$, and there exist homogeneous polynomials $H_1, H_3, H_5 \in K[x, y]$ of degree $v$, $H_6 \in K[x, y]$ of degree $v - 1$, such that the third column of $A$ is a syzygy of $J(F)$. Then by Saito’s criterion, Theorem 2.1, $F$ is a free divisor. We have

$$
\frac{\partial F}{\partial x} = x^{d-\alpha} \left( x \frac{\partial F_1}{\partial x} + (d - \alpha)F_1 \right) + y^{d-v+\alpha} \frac{\partial F_2}{\partial x} + \beta x^{\beta-1} y^{d-\beta-1} z,
$$

$$
\frac{\partial F}{\partial y} = x^{d-\alpha} \frac{\partial F_1}{\partial y} + y^{d-v+\alpha-1} \left( y \frac{\partial F_2}{\partial y} + (d - v + \alpha)F_2 \right) + (d - \beta - 1)x^{\beta} y^{d-\beta-2} z,
$$

$$
\frac{\partial F}{\partial z} = x^{\beta} y^{d-\beta-1}.
$$

Now we show that the second column of $A$ is a syzygy of $J(F)$. That is, we show that

$$
(1) \quad x^2 y^2 M G_1 \frac{\partial F}{\partial x} - x y^2 M G_2 \frac{\partial F}{\partial y} + (G_3 - x y z M(\beta y G_1 + (d - \beta - 1)G_2)) \frac{\partial F}{\partial z} = 0
$$

where $M = x^{\beta-1} y^{\gamma}$, $G_1 = \frac{\partial F_1}{\partial y}$ and $G_2 = - \left( x \frac{\partial F_1}{\partial x} + (d - \alpha)F_1 \right)$. Consider

$$
= x^2 y^2 M \frac{\partial F_1}{\partial y} \left( (d - \alpha)x^{d-\alpha-1} F_1 + x^{d-\alpha} \frac{\partial F_1}{\partial x} + y^{d-v+\alpha} \frac{\partial F_2}{\partial x} + \beta x^{\beta-1} y^{d-\beta-1} z \right)
\quad + x y^2 M G_2 \left( x^{d-\alpha} \frac{\partial F_1}{\partial y} + (d - v + \alpha) y^{d-v+\alpha-1} F_2 \right)
\quad + x y^2 M G_2 \left( y^{d-v+\alpha} \frac{\partial F_2}{\partial y} + (d - \beta - 1)x^{\beta} y^{d-\beta-2} z \right)
$$
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$$-xy^{v+\alpha+2}M \left( xy \frac{\partial F_2}{\partial x} \frac{\partial F_1}{\partial y} + G_2(y \frac{\partial F_2}{\partial y} + (d-v+\alpha)F_2) \right)$$
$$-x^2y^{v+\alpha+3}zM^2 \left( \beta y \frac{\partial F_1}{\partial y} + (d-\beta-1)G_2 \right)$$
$$= x^2y^2M \frac{\partial F_1}{\partial y} \left( (d-\alpha)x^{d-\alpha-1}F_1 + x^{d-\alpha} \frac{\partial F_1}{\partial x} + y^{d-v+\alpha} \frac{\partial F_2}{\partial x} \right)$$
$$+ xy^2MG_2 \left( x^{d-\alpha} \frac{\partial F_1}{\partial y} + (d-v+\alpha)y^{d-v+\alpha-1}F_2 + y^{d-v+\alpha} \frac{\partial F_2}{\partial y} \right)$$
$$-xy^{v+\alpha+2}M \left( xy \frac{\partial F_2}{\partial x} \frac{\partial F_1}{\partial y} + G_2(y \frac{\partial F_2}{\partial y} + (d-v+\alpha)F_2) \right)$$
$$+ z \left( \beta x^{\beta+1}y^{d-\beta+1}M \frac{\partial F_1}{\partial y} + (d-\beta-1)x^{\beta+1}y^{d-\beta}G_2 \right)$$
$$-x^{\beta+1}y^{d-\beta}zM \left( \beta y \frac{\partial F_1}{\partial y} + (d-\beta-1)G_2 \right)$$
$$= 0 + z 0 \text{ (by substituting } G_2 \text{ value)}$$
$$= 0.$$

Thus (1) is a syzygy of $J(F)$. Now we show that the third column of $A$ is a syzygy of $J(F)$. We prove this for the case $d = 2v + 1$, is odd. Assume $E = ax + by$. We show that

$$\left( H_1 + xyzMH_2 \right) \frac{\partial F}{\partial x} + \left( H_3 + yzMH_4 \right) \frac{\partial F}{\partial y} + \left( H_5 + H_6z - Mz^2 (\beta yH_2 + (d-\beta-1)H_4) \right) \frac{\partial F}{\partial z} = 0$$

where $H_2 = G_1(ax + by), H_4 = G_2(ax + by)$ and $H_1, H_3, H_5 \in K[x, y]$ are homogeneous polynomials of degree $v$ satisfying:

$$\begin{bmatrix}
-G_2 & xG_1 & 0 \\
\frac{\partial F_2}{\partial x} (y \frac{\partial F_2}{\partial y} + (d-v+\alpha)F_2) & x^\alpha y^{v-\alpha-\beta} & \frac{\partial F_1}{\partial y} + (d-\alpha)F_1
\end{bmatrix}
\begin{bmatrix}
H_1 \\
H_3 \\
H_5
\end{bmatrix}
= \begin{bmatrix}
\mu y^{v+\alpha} \\
-\mu x^{2v-\alpha}
\end{bmatrix}$$

and

$$H_6 = -\beta V_1 - (d-\beta-1)U_1 - \frac{\partial F_2}{\partial x} \frac{\partial F_1}{\partial y} (ax + by) + \frac{\partial F_2}{\partial y} \left( x \frac{\partial F_1}{\partial x} + (d-\alpha)F_1 \right)$$
$$+ b \frac{\partial F_1}{\partial x} \left( y \frac{\partial F_2}{\partial y} + (d-v+\alpha)F_2 \right) + W_1 + W_2,$$
where
\[ H_1 = xV_1 + \ast y^v, \]
\[ H_3 = yU_1 + \ast x^v, \]
\[ a(d - v + \alpha)F_2(x \frac{\partial F_1}{\partial x} + (d - \alpha)F_1) = yW_1 + \ast x^v, \]
\[ b(d - \alpha)F_1(y \frac{\partial F_2}{\partial y} + (d - v + \alpha)F_2) = xW_2 + \ast y^v, \]

where * are elements of K. From the expression of \( H_6 \), we have
\[ (4) \quad \beta yH_1 + xy \frac{\partial F_2}{\partial x}H_2 + (d - \beta - 1)xH_3 + \left( y \frac{\partial F_2}{\partial y} + (d - v + \alpha)F_2 \right) H_4 + xyH_6 = 0. \]

Now by comparing the pure powers of x and y in system (3) and equation (4), we get
\[ a = \frac{-\mu(d - \beta - 1)}{(d - v + \alpha)^2[F_2][x^{v-\alpha}][\beta + (d - \alpha)F_1][x^\alpha]}, \]
\[ \mu = \frac{b(d - \alpha)^2 \beta}{\beta}[F_1][y^v][\alpha + (d - v + \alpha)F_2][y^{v-\alpha}], \]

where \([H|x^s]\) denote the coefficient of \( x^s \) of \( H \), for any polynomial \( H \). Now consider,
\[ (H_1 + xyzMH_2) \frac{\partial F}{\partial x} + (H_3 + yzMH_4) \frac{\partial F}{\partial y} + (H_5 + H_6z - Mz^2(\beta yH_2 + (d - \beta - 1)H_4)) \frac{\partial F}{\partial z} \]

it is clear that its coefficient of \( z^2 \) is zero. Now its coefficient of \( z = y^{v+\alpha+1} M(\beta yH_1 + xy \frac{\partial F_2}{\partial x}(ax + by) + (d - \beta - 1)xH_3 - (y \frac{\partial F_2}{\partial y} + (d - v + \alpha)F_2)(x \frac{\partial F_1}{\partial x} + (d - \alpha)F_1)(ax + by) + xyH_6) + x^{d-\alpha}yM(-G_2H_2 + G_1H_4). \) This is zero by equation (4) and the fact \( G_1H_4 = G_2H_2. \)

Now the coefficient of \( z^0 \) in the expression that we considered is equal to,
\[ x^{d-\alpha-1} \left( (x \frac{\partial F_1}{\partial x} + (d - \alpha)F_1)H_1 + x \frac{\partial F_2}{\partial y}H_3 \right) \]
\[ + y^{d-\alpha-1} \left( y \frac{\partial F_1}{\partial y}H_1 + ((d - v + \alpha)F_2 + y \frac{\partial F_2}{\partial y})H_3 + x^\beta y^{v-\alpha-\beta}H_5 \right) \]
\[ = x^{d-\alpha-1} (-G_2H_1 + xG_1H_3) + y^{d-\alpha-1} \left( y \frac{\partial F_1}{\partial y}H_1 + ((d - v + \alpha)F_2 + y \frac{\partial F_2}{\partial y})H_3 + x^\beta y^{v-\alpha-\beta}H_5 \right). \]

Since \( x^{d-\alpha-1}, y^{d-\alpha-1} \) is a regular sequence, therefore this expression is equal to zero if and only if the system (3) has a solution. Now we prove the system (3), has a solution. Let \( S = K[x, y] \) and \( N \) be the graded \( S \)-submodule of \( S(-v + \alpha) \oplus S(-\alpha) \) generated by the columns of the matrix of the system (3). That is, \( N \) is generated by \[ \left[ \begin{array}{c} -G_2 \\ y \frac{\partial F_1}{\partial x} \end{array} \right], \left[ \begin{array}{c} xG_1 \\ y \frac{\partial F_2}{\partial y} + (d - v + \alpha)F_2 \end{array} \right] \]
and \[ \left[ \begin{array}{c} 0 \\ x^\beta y^{v-\alpha-\beta} \end{array} \right]. \] Then \( N \) has the graded free resolution of the
form:

\[ 0 \rightarrow S(-2v) \rightarrow S^3(-v) \rightarrow S(-(v - \alpha)) \oplus S(-\alpha) \rightarrow 0. \]

Where the first map is given by the matrix

\[
\begin{bmatrix}
  x^{\beta+1}y^{\alpha-\beta}G_1 \\
  x^{\beta}y^{\alpha-\beta}G_2 \\
  -xy^2F_2 G_1 + G_2((d - v + \alpha)F_2 + y^2F_{\alpha y})
\end{bmatrix}
\]

Therefore the Hilbert series of \( S(-(v - \alpha)) \oplus S(-\alpha) / N \)

\[
\frac{z^\alpha + z^{v-\alpha} - 3z^v + z^{2v}}{(1 - z)^2}.
\]

This is equal to the polynomial

\[ z^\alpha (1 + z + \cdots + z^{v-\alpha-1})^2 + z^{v-\alpha} (1 + z + \cdots + z^{\alpha-1})(1 + z + \cdots + z^{v-1}), \]

of degree \( 2v - 2 \). This implies that \([S(-(v - \alpha)) \oplus S(-\alpha) / N]_i = 0\), for all \( i \geq 2v - 1 \).

Since the degree of \( \begin{bmatrix} \mu y^{\nu + \alpha} \\ \mu x^{2v - \alpha} \end{bmatrix} \), is \( 2v \) in \( S(-(v - \alpha)) \oplus S(-\alpha) \), therefore the system (3) has a solution. Therefore (2) is a syzygy of \( J(F) \). In a similar manner one can prove the even case also. Hence \( \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \end{bmatrix} A \equiv 0 \mod (F) \). If we know, the syzygies (1) and (2) are not multiple of a syzygy, then by [ST12 Lemma 1.1], \( J(F) \) is perfect. But by looking at the structure of the syzygies (1) and (2) it is difficult to say that one is not a multiple of another or both are not multiples of a syzygy.

Now we prove \( \det(A) = F \). We have

\[ d\mu \det(A) = x^2y^2MG_2U - x(G_3 - xyzM(\beta yG_1 + (d - \beta - 1)G_2))(H_3 + yzMH_4) \]

\[-x^2y^2M^2 \frac{\partial F_1}{\partial y} [yH_5 + yzH_6 - H_3z - yz^2M(\beta yH_2 + (d - \beta)H_4)]\]

\[ +x^2y^2z^2M^2 \frac{\partial F_1}{\partial y} \left( \beta y \frac{\partial F_1}{\partial x} - (d - \beta) \left( x \frac{\partial F_1}{\partial x} + (d - \alpha)F_1 \right) \right) \]

\[ + (H_1 + xyzMH_2)(yG_3 - xy^2zM(\beta yG_1 + (d - \beta)G_2)). \]

Now, the coefficient of \( z^2 \) in \( d\mu \det(A) = x^2y^2M^2(\beta yG_1 + (d - \beta - 1)G_2)(H_4 - yH_2) + x^2y^2M(\beta yH_2 + (d - \beta - 1)H_4)(-G_2 + yG_1). \]

This is equal to zero, because \( H_2 = G_1E, H_4 = G_2E \) and \(-G_2 + yG_1 = dF \). Now the coefficient of \( z \) in \( d\mu \det(A) = x^2y^2MG_2H_6 - xyMG_3H_4 + x^2yMH_3(\beta yG_1 + (d - \beta - 1)G_2) - x^2y^3MG_1H_6 \]

\[ + x^2y^2MG_1H_3 - xy^2MH_1(\beta yG_1 + (d - \beta - 1)G_2) - xy^2MG_2H_1 + xy^2MG_3H_2 \]
Now substituting the system (3) in this, then this is equal to
\[ xyM[\mu(\beta + 1)y^{v+\alpha+1} + (\beta + 1 - d)yG_2H_1 - G_3H_4 + yG_3H_2 - xy\frac{\partial F_2}{\partial y}G_2H_2 - G_2H_4((d - v + \alpha)F_2 + y\frac{\partial G}{\partial y}) + xy^2\frac{\partial G}{\partial x}G_1H_2 + (d - \beta - 1)xyG_1H_3 + yG_1H_4((d - v + \alpha)F_2 + y\frac{\partial G}{\partial y})]. \]
Now by using the fact, \( G_2H_2 = G_1H_4 \) and system (3), this expression is equal to
\[ xyM[\mu dy^{v+\alpha+1} + xy\frac{\partial F_2}{\partial y}G_1H_4 - yG_1H_4((d - v + \alpha)F_2 + y\frac{\partial G}{\partial y}) + G_1H_4(-2x\frac{\partial F_2}{\partial x} + dF_2)]. \]
Now by using Euler’s equation this is equal to \( \mu dx^\beta y^{2v-\beta} \). Thus we showed that the coefficient of \( z \) in \( d\mu \det(A) \) is equal to \( \mu dx^\beta y^{2v-\beta} \). Now consider

the coefficient of \( z^0 \) in \( d\mu \det(A) \) = \[ \begin{align*}
& x^2y^2MG_2H_5 - xG_3H_3 - x^2y^3MG_1H_5 + yH_1G_3 \\
& = -xG_3H_3 + yG_3H_1 - dxF_1[\mu x^{2v-\alpha} - y\frac{\partial F_2}{\partial x} H_1] \\
& -H_3(y\frac{\partial F_2}{\partial y} + (d - v + \alpha)F_2)] \\
& \text{(by using the system (3))} \\
& = d\mu F_1x^{2v-\alpha+1} + dxyF_2G_1H_3 - dyF_2G_2H_1 \\
& \text{(by substituting G3 and Euler’s equation )} \\
& = d\mu F_1x^{2v-\alpha+1} + d\mu F_2y^{v+\alpha+1} \\
& \text{(by using the system (3)).}
\end{align*} \]

Therefore
\[ \det(A) = \frac{1}{d\mu} \left[ d\mu F_1x^{2v-\alpha+1} + d\mu F_2y^{v+\alpha+1} + d\mu x^\beta y^{2v-\beta}z \right] \]
\[ = x^{2v-\alpha+1}F_1 + y^{v+\alpha+1}F_2 + x^\beta y^{2v-\beta}z \]
\[ = F. \]

Thus \( A \) is a discriminant matrix of \( F \). If \( \beta = 0 \), then the Saito matrix is same as \( A \) except its last column. The last column is the syzygy of \( J(F) \) of the form:
\[ \begin{align*}
& \left( H_1 - \lambda xy^{v-\alpha-1}z\frac{\partial F_1}{\partial y} \right) \frac{\partial F}{\partial x} + \left( H_3 + \lambda y^{v-\alpha-1}z((d - \alpha)F_1 + \frac{\partial F_1}{\partial x}) \right) \frac{\partial F}{\partial y} \\
& + \left( H_5 + Uz - \lambda(d - 1)y^{v-\alpha-2}z^2((d - \alpha)F_1 + \frac{\partial F_1}{\partial x}) \right) \frac{\partial F}{\partial z} = 0,
\end{align*} \]
where \( H_1, H_3, H_5 \in K[x, y] \) are as above, satisfying the system (3) with \( \beta = 0 \) and
\[ U = ((d - \alpha)F_1 + \frac{\partial F_1}{\partial x}) \left( V_3 - \frac{\lambda(d - v + \alpha)}{d - 1}W_3 - \frac{\lambda}{d - 1}x\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y} \right). \]
\[ + \frac{\lambda}{x} x^2 \left( \frac{\partial F_1}{\partial y} \right)^2 \frac{\partial F_2}{\partial x} + \mu y^{\nu+\alpha-1}, \text{ where } \lambda \in K, \ x \frac{\partial F_1}{\partial y} F_2 = W_3 y + *x^v, \ H_1 = y V_3 + *x^v, * \text{ are elements of } K. \]

We end this article, with a remark.

**Remark 2.3.** The support of the free divisors in the Theorem 2.2 contains two intervals \( \{ x^d, \ldots, x^{d-\alpha} y^\alpha \}, \ \{ x^{v/2} y^{d-[v/2]} + \alpha, \ldots, y^d \} \) and it is maximal. There is a computational evidence that there are maximal supports for irreducible free divisors in \( \mathbb{P}^2 \), for instance

\[
\{ x^d \} \cup \{ x^{d-v+2} y^{-2} \} \cup \{ x^{v/2} + y^{d-[v/2]-2}, \ldots, x^5 y^{d-5} \} \cup \{ x^2 y^{d-2}, xy^{d-1}, y^d \} \cup \{ y^{d-1} z \}
\]

and different from the family presented in the Theorem 2.2, for \( d = 2v + 1 \).

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**Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, I-16146, Italia.**

E-mail address: nandurirk@gmail.com