Note on weighted proper orientations of outerplanar graphs

Ruijuan Gu\(^1\), Gregory Gutin\(^2\), Yongtang Shi\(^3\), Zhenyu Taoqiu\(^3\)

\(^1\) Sino-European Institute of Aviation Engineering
Civil Aviation University of China, Tianjin 300300, China
millet90@163.com

\(^2\) Department of Computer Science
Royal Holloway, University of London
Egham, Surrey, TW20 0EX, UK
g.gutin@rhul.ac.uk

\(^3\) Center for Combinatorics and LPMC
Nankai University, Tianjin 300071, China
shi@nankai.edu.cn, tochy@mail.nankai.edu.cn

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Abstract

A weighted proper orientation of a given graph \(G\), denoted by \((D, w)\), is an orientation \(D\) with a weight function \(w : E(G) \to \mathbb{Z}_+\), such that the in-weight of any adjacent vertices are distinct, where the in-weight of any vertex \(v\) in \(D\), denoted by \(w_D^-(v)\), is the sum of the weights of arcs towards \(v\). The weighted proper orientation number of a graph \(G\), denoted by \(\overrightarrow{\chi}_w(G)\), is the minimum of maximum in-weight of \(v\) in \(D\) over all weighted proper orientation \((D, w)\) of \(G\). This parameter was first introduced by Araújo et al. (2019). When the weights of all edges equal to one, this parameter is equal to the proper orientation number of \(G\). The optimal weighted proper orientation is a weighted proper orientation \((D, w)\) such that \(\max_{v \in V(G)} w_D^-(v) = \overrightarrow{\chi}_w(G)\).

Araújo et al. (2016) showed that \(\overrightarrow{\chi}(G) \leq 7\) for every cactus \(G\) and the bound is tight. We prove that for every cactus \(G\), \(\overrightarrow{\chi}_w(G) \leq 3\) and the bound is tight. Araújo et al. (2015) asked whether there is a constant \(c\) such that \(\overrightarrow{\chi}(G) \leq c\) for all outerplanar graphs \(G\). While this problem remains open, Araújo et al. (2019) asked to consider it in the weighted case. We prove that for every outerplanar graph \(G\), \(\overrightarrow{\chi}_w(G) \leq 4\) and the bound is tight.

Keywords: proper orientation number; weighted proper orientation number; outerplanar graph

1 Introduction

For basic notation in graph theory, the reader is referred to [8]. All graphs in this paper are considered to be simple. An orientation \(D\) of a graph \(G\) is a digraph obtained from \(G\) by replacing each edge by exactly one of two possible arcs with the same endvertices. The in-degree of \(v\) in \(D\), denoted by \(d_D^-(v)\), is the number of arcs towards \(v\) in \(D\) for each \(v \in V(G)\). We will use the notation without subscript when the orientation \(D\) is clear from context.

For a given undirected graph \(G\), an orientation \(D\) of \(G\) is proper if \(d^-(u) \neq d^-(v)\) for all \(uv \in E(G)\). An orientation with maximum in-degree at most \(k\) is called a \(k\)-orientation. The
proper orientation number of a graph $G$ is the minimum integer $k$ such that $G$ admits a proper $k$-orientation, denoted by $\chi^-(G)$. The existence of proper orientation was demonstrated by Borowiecki et al. in [7], where it was shown that every graph $G$ has a proper $\Delta(G)$-orientation, where $\Delta(G)$ is the maximum degree of $G$. Later, Ahadi and Dehghan [1] introduced the concept of the proper orientation number. This parameter was widely investigated recently, for more details, we refer the reader to [1, 2, 3, 4, 5, 10]. Note that every proper orientation of a graph $G$ induces a proper vertex coloring of $G$. Hence, we have the following sequences of inequalities:

$$\omega(G) - 1 \leq \chi(G) - 1 \leq \chi^-(G) \leq \Delta(G) \quad (1)$$

These inequalities are best possible since, for a complete graph $K_n$, $\omega(K_n) - 1 = \chi(K_n) - 1 = \chi^-(K_n) = \Delta(K_n) = n - 1$. Ahadi and Dehghan [1] proved that it is NP-complete to compute $\chi^-(G)$ even for planar graphs. Araujo et al. [4] strengthened this result by showing that it holds for bipartite planar graphs of maximum degree 5. The following two problems have received great attention by researchers.

**Problem 1** ([4]). Is there a constant $c$ such that $\chi^-(G) \leq c$ for every planar graph $G$?

**Problem 2** ([5]). Is there a constant $c$ such that $\chi^-(G) \leq c$ for every outerplanar graph $G$?

Knox et al. [10] proved that $\chi^-(G) \leq 5$ for a 3-connected planar bipartite graph $G$ and Noguci [11] showed that $\chi^-(G) \leq 3$ for any bipartite planar graph $G$ with $\delta(G) \geq 3$. Araujo et al. [5] proved $\chi^-(G) \leq 7$ for any cactus, i.e., an outerplanar graph with every 2-connected component being either an edge or a cycle and $\chi^-(T) \leq 4$ for any tree $T$ (see also [10] for a short algorithmic proof). Ai et al. [3] proved that $\chi^-(G) \leq 3$ for any triangle-free, 2-connected outerplanar graph $G$ and $\chi^-(G) \leq 4$ for any triangle-free, bridgless or tree-free outerplanar graph $G$.

Recently Araújo et al. [6] introduced the notion of a weighted proper orientation of graphs. A weighted proper orientation of a given graph $G$, denoted by $(D, w)$, is an orientation $D$ with a weight function $w : E(G) \rightarrow \mathbb{Z}_+$, such that the in-weight of any adjacent vertices are distinct, where the in-weight of $v$ in $D$, denoted by $w_D^-(v)$, is the sum of the weights of arcs towards $v$. Let $\mu^-(D, w)$ be the maximum of $w_D^-(v)$ over all vertices $v$ of $G$. We drop the subscript when the orientation and weight function are clear from the context. The weighted proper orientation number of a graph $G$, denoted by $\chi^w(G)$, is the minimum of $\mu^-(D, w)$ over all weighted proper orientations $(D, w)$ of $G$. An optimal weighted proper orientation is a weighted proper orientation $(D, w)$ such that $\mu^-(D, w) = \chi^w(G)$. Dehghan [9] also studied the weighted proper orientation number under the name “semi-proper orientation number”, and proved the following result.

**Theorem 1** ([9]). Every graph $G$ has an optimal weighted proper orientation $(D, w)$ such that the weight of each edge is one or two.
It is easy to see that $\chi_w(G) \leq \chi(G)$. Moreover, by the definition of a weighted proper orientation, the in-weights of adjacent vertices are different. Consequently, by (1), we have

$$\omega(G) - 1 \leq \chi(G) - 1 \leq \chi_w(G) \leq \chi(G) \leq \Delta(G)$$

(2)

Dehghan [9] observed that there exist graphs $G$ such that $\chi_w(G) < \chi(G)$. Indeed, while as observed in [9], we have $\chi_w(T) \leq 2$ for every three $T$, there are trees $T$ with $\chi(T) = 4$ [4]. Thus, one natural problem is to study the gap between these two parameters.

**Problem 3 ([9]).** Is there any constant $c_1$ such that $\chi(G) - \chi_w(G) \leq c_1$ for every graph $G$?

In this paper, we prove a sharp upper bound for the weighted proper orientation number of cacti in Theorem 2, which implies that $c_1 \geq 4$ if $c_1$ exist, due to the sharp upper bound $\chi(G) \leq 7$ for cacti proved in [5].

In [6], the authors proved that it is (weakly) NP-complete to determine whether $\chi_w(G) \leq k$ for trees, but can be solved by a pseudo-polynomial time algorithm. In [9], Dehghan showed that determining whether a given planar graph $G$ with $\chi_w(G) = 2$ has an optimal weighted proper orientation $(D,w)$ such that the weight of each edge is one is NP-complete. He also proved that the problem of determining the weighted proper orientation number of planar bipartite graphs is NP-hard.

We prove the following two results. Theorem 2 gives a tight bound for cacti in the weighted case. Note that this theorem and the tight bound on the proper orientation number of cacti imply that $c_1 \geq 4$ in Problem 3 (provided that $c_1$ exists). While Problem 2 remains open, Araújo et al. [6] asked to consider the problem in the weighted case. Theorem 3 solves this problem. Due to Theorem 1, the bounds in these theorems can be achieved for optimal weighted proper orientations where every edge weight is 1 or 2.

**Theorem 2.** For every cactus $G$, we have $\chi_w(G) \leq 3$ and this bound is tight.

**Theorem 3.** For every outerplanar graph $G$, we have $\chi_w(G) \leq 4$ and this bound is tight.

While the tightness proof of the bound in Theorem 2 is quite easy, that in Theorem 3 is more involved as an optimal weighted proper orientation of a significantly larger graph is considered.

The remainder of the paper is organized as follows. We provide some definitions and simple lemmas for orientations on paths in Section 2. Next, we study (weighted) proper orientations of cacti and outerplanar graphs, and prove Theorems 2 and 3 in Sections 3 and 4 respectively. We conclude the paper in Section 5.

## 2 Preliminaries

Let us consider briefly some graph theory terminology and notation used in this paper. For more information on blocks and ear decomposition, see e.g. [8].
We denote a path and cycle by $P$ and $C$, respectively, and the order of $P$ and $C$ by $|P|$ and $|C|$, respectively. We call an edge $e$ an $a$-$b$ edge if the end points of $e$ have in-weight $a$ and $b$, respectively.

A block of a graph $G$ is a maximal nonseparable subgraph of $G$ and a block of order $i$ is said to be an $i$-block. Note that every $i$-block with $i \geq 3$ is a 2-connected graph, 2-block is an edge (bridge) of $G$ and 1-block is an isolated vertex of $G$. Thus, if $G$ is connected, it has no 1-blocks.

A block tree associated to $G$ is the tree $T(G)$ with vertex set $V(T(G)) = \{v_i: B_i$ is block of $G}\cup S$, where $S$ is the set of cut vertices of $G$, and edge set $E(T(G)) = \{v_i s_j: s_j \in B_i\}$. Choose a block $B_0$ of $G$ as a root of $T(G)$, and run depth-first search (DFS) algorithm on $T(G)$ from $B_0$. Then we can get an ordering of blocks in $G$ as $B_0, B_1, \ldots, B_p$. If a cut vertex $s_i \in B_i \cap B_j$ and $j < i$, then say $s_i$ is the root of $B_i$.

For a subgraph $H$ of $G$, an ear of $H$ is a non-trivial path $P$ in $G$ with end-vertices in $H$ but internal vertices not. We say an ear is attached to the corresponding ends in $H$ and we call such pair of end-vertices active. Especially, if the vertices of an active pair are adjacent to each other, we call the pair active edge. It is well known that every 2-connected graph $G$ has an ear decomposition defined as follows.

- Choose a cycle $C_0$ of $G$ and let $G_0 = C_0$.
- Add an ear $P_i$ attached to an active pair $(a_i, b_i)$ of $G_i$, where $a_i \neq b_i$ and let $G_{i+1} = G_i \cup P_i$, $0 \leq i < k$.
- $G_k = G$.

Now we introduce a class of outerplanar graphs, called universal outerplanar graphs, which will be used in our proof. A universal outerplanar graph, denoted by UOP($n$), is defined as follows:

- UOP(1) is a triangle.
- Add 2-length ears to all edges of UOP(1), then we get UOP(2). The new added edges are called outeredges of UOP(2) and the new added vertices are called outervertices of UOP(2).
- UOP($k + 1$) is obtained from UOP($k$) by adding 2-length ears to all outeredges of UOP($k$).

We give some lemmas for orientations on paths below, which will be used later.

**Lemma 1** ([3]). Let $P = v_1 v_2 \ldots v_n$ be a path of length $n - 1$.

1. If $n \geq 7$, then there are three weighted proper orientations with weights of all edges one such that $w^-(v_1) = 0$ and $w^-(v_n) = 0$ and

   (a) $w^-(v_2) = 2$, $w^-(v_{n-2}) = 0$, $w^-(v_{n-1}) = 2$, and
(b) \( w^-(v_2) = 1, w^-(v_3) = 2, w^-(v_{n-3}) = 0, w^-(v_{n-2}) = 2, w^-(v_{n-1}) = 1 \), and
(c) \( w^-(v_2) = 1, w^-(v_{n-2}) = 0, w^-(v_{n-1}) = 2 \), respectively.

2. If \( n = 6 \), then there are two weighted proper orientations with weights of all edges one such that \( w^-(v_1) = 0 \) and \( w^-(v_6) = 0 \) and

(a) \( w^-(v_2) = 2, w^-(v_3) = 0, w^-(v_4) = 1, w^-(v_5) = 2 \), and
(b) \( w^-(v_2) = 1, w^-(v_3) = 2, w^-(v_4) = 0, w^-(v_5) = 2 \), respectively.

3. If \( n = 5 \), then there are two weighted proper orientations with weights of all edges one such that \( w^-(v_1) = 0 \) and \( w^-(v_5) = 0 \) and

(a) \( w^-(v_2) = 1, w^-(v_3) = 2, w^-(v_4) = 1 \), and
(b) \( w^-(v_2) = 2, w^-(v_3) = 0, w^-(v_4) = 2 \), respectively.

4. If \( n = 4 \), then there exists a weighted proper orientation with weights of all edges one such that \( w^-(v_1) = 0, w^-(v_2) = 1, w^-(v_3) = 2 \) and \( w^-(v_4) = 0 \).

Lemma 2. Let \( P = v_1v_2v_3 \) be a path with length two. Then there exist three weighted proper orientations with weights of all edges at most two such that \( w^-(v_1) = w^-(v_3) = 0 \),

1. weights of all edges are one and \( w^-(v_2) = 2 \).
2. \( w^-(v_2) = 3, w(v_1v_2) = 2 \) and \( w(v_3v_2) = 1 \).
3. \( w^-(v_2) = 4 \) and \( w(v_1v_2) = w(v_3v_2) = 2 \).

Proof. 1. 

\[ \begin{array}{c}
\text{v}_1 \rightarrow \text{v}_2 \rightarrow \text{v}_3 \\
\end{array} \]

2. 

\[ \begin{array}{c}
\text{v}_1 \rightarrow \text{v}_2 \rightarrow \text{v}_3 \\
\end{array} \]

3. 

\[ \begin{array}{c}
\text{v}_1 \rightarrow \text{v}_2 \rightarrow \text{v}_3 \end{array} \]

\[ \begin{array}{c}
\text{v}_1 \rightarrow \text{v}_2 \rightarrow \text{v}_3 \\
\end{array} \]

Lemma 3. Let \( P = v_1v_2v_3v_4 \) be a path of length 3. Then there exist two weighted proper orientations with weights of all edges at most two such that \( w^-(v_1) = w^-(v_4) = 0 \),

1. \( w^-(v_2) = 2, w^-(v_3) = 3, w(v_1v_2) = w(v_3v_4) = 2 \) and \( w(v_2v_3) = 1 \).
2. \( w^-(v_2) = 1, w^-(v_3) = 3, w(v_2v_3) = 2 \) and \( w(v_1v_2) = w(v_3v_4) = 1 \).

Proof. 1. 

\[ \begin{array}{c}
\text{v}_1 \rightarrow \text{v}_2 \rightarrow \text{v}_3 \rightarrow \text{v}_4 \\
\end{array} \]

\[ \begin{array}{c}
\text{v}_1 \rightarrow \text{v}_2 \rightarrow \text{v}_3 \rightarrow \text{v}_4 \\
\end{array} \]
Lemma 4. Let \( P = v_1v_2v_3v_4v_5 \) be a path with length 4. Then there exists a weighted proper orientation with weights of all edges at most two such that \( w^-(v_1) = w^-(v_3) = w^-(v_5) = 0, w^-(v_2) = 2, w^-(v_4) = 3, w(v_4v_5) = 2 \) and other edges with weight one.

Proof.

Lemma 5. Let \( P = v_1v_2v_3v_4v_5v_6 \) be a path with length 5. Then there exists a weighted proper orientation with weights of all edges at most two such that \( w^-(v_1) = w^-(v_6) = 0, w^-(v_2) = w^-(v_5) = 1, w^-(v_3) = 3, w^-(v_4) = 2, w(v_2v_3) = w(v_4v_5) = 2 \) and other edges with weight one.

Proof.

3 Proof of Theorem 2

Since \( G \) is a cactus, we can label blocks of \( G \) and construct a sequence of induced subgraphs of \( G \) as follows.

- Choose a block \( B_0 \) as the root of block tree \( T(G) \), i.e. \( G_0 = B_0 \).
- Run DFS algorithm on \( T(G) \) from \( B_0 \) to get an ordering of blocks in \( G \).
- Add block \( B_i \) to its root \( s_i \), i.e., \( G_i = G_{i-1} + B_i, 1 \leq i \leq k \).
- \( G_k = G \).

Note that \( B_i \) is either a 2-block or a cycle in \( G \).

We prove it by induction on \( k \). When \( k = 0 \), orient \( B_0 \) using Lemmas 1 and 2 or orient it arbitrarily if \( B_0 \) is 2-block. By the induction hypothesis, \( G_{k-1} \) has a desired orientation \( (D_{k-1}, w) \).

Consider the case when \( B_k \) is a cycle \( C \). If \( |C| = 3 \) we can apply Lemmas 3 and 1 by setting \( v_1 = v_4 = s_k \). If \( w^-(s_k) = 1 \) in \( (D_{k-1}, w) \), then we use Lemma 3 to orient \( C \) such that \( w^-(v_2) = 2 \) and \( w^-(v_3) = 3 \). If \( w^-(s_k) = 2 \) in \( (D_{k-1}, w) \), then use Lemma 3 to orient \( C \) such that \( w^-(v_2) = 1 \) and \( w^-(v_3) = 3 \). If \( w^-(s_k) \in \{0, 3\} \) in \( (D_{k-1}, w) \), then use Lemma 4 to orient \( C \) such that \( w^-(v_2) = 1 \) and \( w^-(v_3) = 2 \).

If \( |C| = 4 \), we can apply Lemma 1(b) by setting \( v_1 = v_5 = s_k \) and using (a) if \( w^-(s_k) \neq 1 \) in \( (D_{k-1}, w) \) and (b) otherwise. If \( |C| = 5 \) set \( v_1 = v_6 = s_k \). If \( w^-(s_k) = 2 \) in \( G_{k-1} \), then use Lemma 5. Otherwise, use Lemma 1(a). Now we assume \( |C| \geq 6 \) and set \( v_1 = v_n = s_k \). If \( w^-(s_k) = 1 \) in \( (D_{k-1}, w) \), then use Lemma 1(a) and otherwise Lemma 1(b).
Now consider the case when \(B_k\) is a 2-block \(s_kv\). Then orient it from \(s_k\) to \(v\). If \(w^-(s_k) = 1\) in \((D_{k-1},w)\), then let \(w(s_kv) = 2\) such that \(w^-(v) = 2\). Otherwise, let \(w(s_kv) = 1\) such that \(w^-(v) = 1\).

For both cases, we have \(w^-(D_{k,w})(s_k) = w^-(D_{k-1,w})(s_k)\). This implies that \(G_k\) has a desired orientation \((D_k,w)\).

![Figure 1: A tight example of Theorem 2.](image)

A tight example is given in Figure 1. Let \(G\) be a cactus with vertex set \(V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}\) and edge set \(E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_4v_5, v_4v_6, v_5v_6\}\). Suppose \(G\) has a weighted proper 2-orientation \(D\). Without loss of generality, we may assume that edge \(v_3v_4\) in \(G\) is oriented in \(D\) from \(v_4\) to \(v_3\). Hence, \(1 \leq w_D^-(v_3) \leq 2\). Since \(w_D^-(v_3) \leq 2\), without loss of generality, we may assume that edge \(v_2v_3\) is oriented from \(v_3\) to \(v_2\). Thus, \(1 \leq w_D^-(v_2) \leq 2\). We cannot have \(1 \leq w_D^-(v_1) \leq 2\) as well since that would imply that there are vertices \(v_i, v_j\) with \(1 \leq i < j \leq 3\) such that \(w_D^-(v_i) = w_D^-(v_j)\). Hence, \(w_D^-(v_1) = 0\), but then both \(v_1v_2\) and \(v_1v_3\) must be oriented from \(v_1\) implying that \(w_D^-(v_2) = w_D^-(v_3) = 2\), a contradiction.

\[\square\]

4 Proof of Theorem 3

We start from the following:

**Lemma 6.** Let \(G\) be a 2-connected outerplanar graph and let \(s\) be an arbitrary vertex of \(G\). Then there exists a weighted proper orientation \((D,w)\) such that \(\chi^*_w(G) \leq 4\) and \(w^-(s) = 0\).

**Proof.** Since \(G\) is 2-connected, recall that we can construct \(G\) by the process of ear decomposition as follows.

- Choose a cycle \(C_0\) containing \(s\) and let \(G_0 = C_0\).
- Add an ear \(P_i\) attached to an active pair \((a_i, b_i)\) of \(G_i\), where \(a_i \neq b_i\) and let \(G_{i+1} = G_i \cup P_i, 0 \leq i < k\).
- \(G_k = G\).

Note that \(a_i\) is adjacent to \(b_i\) as \(G\) is outerplanar.
We prove this lemma by induction on \( k \). When \( k = 0 \), orient \( G_0 \) using Lemma 1 or orient it in any other proper way such that \( w^-(s) = 0 \). By the induction hypothesis, \( G_{k-1} \) has a desired orientation \( (D_{k-1}, w) \). Assume that \( e = \{a_k, b_k\} \) is an active edge of \( P_k = v_1v_2\ldots v_n \) and assume without loss of generality that \( a_k = v_1, b_k = v_n, w^-(a_k) < w^-(b_k) \) in \( (D_{k-1}, w) \). We consider the following cases.

**Case 1** \(|P_k| = 3\). If \( e \) is a 2-3 edge in \( (D_{k-1}, w) \), then use Lemma 23 to orient \( P_k \) such that \( w^-(v_2) = 4 \). If neither \( a_k \) nor \( b_k \) has in-weight 2, then use Lemma 21 to orient \( P_k \) such that \( w^-(v_2) = 2 \). If neither \( a_k \) nor \( b_k \) has in-weight 3, then use Lemma 22 to orient \( P_k \) such that \( w^-(v_2) = 3 \).

**Case 2** \(|P_k| = 4\). If \( w^-(a_k) = 1 \) or \( w^-(b_k) = 2 \) in \( (D_{k-1}, w) \), then use Lemma 14 to orient \( P_k \) such that \( w^-(v_2) = 2 \) and \( w^-(v_3) = 1 \). If not, then use Lemma 11 to orient \( P_k \) such that \( w^-(v_2) = 1 \) and \( w^-(v_3) = 2 \).

**Case 3** \(|P_k| = 5\). If \( e \) is a 1-2 edge in \( (D_{k-1}, w) \), then use Lemma 4 to orient \( P_k \) such that \( w^-(v_2) = 2 \) and \( w^-(v_4) = 3 \). If neither \( a_k \) nor \( b_k \) has in-weight 1, then use Lemma 13(a) to orient \( P_k \) such that \( w^-(v_2) = w^-(v_4) = 1 \). If neither \( a_k \) nor \( b_k \) has in-weight 2, then use Lemma 13(b) to orient \( P_k \) such that \( w^-(v_2) = w^-(v_4) = 2 \).

**Case 4** \(|P_k| = 6\). If \( w^-(a_k) = 1 \) or \( w^-(b_k) = 2 \) in \( (D_{k-1}, w) \), then use Lemma 12(b) to orient \( P_k \) such that \( w^-(v_2) = 2 \) and \( w^-(v_5) = 1 \). If not, then use Lemma 12(b) to orient \( P_k \) such that \( w^-(v_2) = 1 \) and \( w^-(v_5) = 2 \).

**Case 5** \(|P_k| \geq 7\). If \( w^-(a_k) = 1 \) or \( w^-(b_k) = 2 \) in \( (D_{k-1}, w) \), then use Lemma 11(c) to orient \( P_k \) such that \( w^-(v_2) = 2 \) and \( w^-(v_{n-1}) = 1 \). If not, then use Lemma 11(c) to orient \( P_k \) such that \( w^-(v_2) = 1 \) and \( w^-(v_{n-1}) = 2 \).

For all cases above, the in-weights of \( a_k \) and \( b_k \) in \( (D_k, w) \) are the same as that in \( (D_{k-1}, w) \). This implies that \( G_k \) has a desired orientation \( (D_k, w) \), where in particular \( w^-(s) = 0 \). \( \square \)

To complete the proof of Theorem 3, it remains to consider the case when \( G \) is connected but not 2-connected. Let \( B_0, B_1, \ldots, B_k \) be a list of blocks of \( G \) such that for every \( i \in \{0, 1, 2, \ldots, k\} \), the subgraph \( G_i \) of \( G \) induced by the union of blocks \( B_0, B_1, \ldots, B_i \) is connected. Such a list can be obtained e.g. by using DFS on \( T(G) \) as described in the beginning of the previous section. Let \( s \) be the root of \( B_k \). We prove the following extension of the theorem by induction on \( i \in \{0, 1, \ldots, k\} \):

For every \( i \in \{0, 1, \ldots, k\} \), \( G_i \) has a weighted proper orientation \( (D_i, w) \) such that \( \mu^-(D_i, w) \leq 4 \) and if \( s \in V(G_i) \) then \( w^-(s) = 0 \).

If \( B_0 \) is a 2-connected outerplanar graph, then by Lemma 6 \( G_0 \) has a weighted proper orientation \( (D_0, w) \) such that \( \mu^-(D_0, w) \leq 4 \) and \( w^-(s) = 0 \) if \( s \in V(G_0) \). If \( B_0 \) is an edge, then we orient the edge from \( s \) to ensure that \( w^-(s) = 0 \) if \( s \in V(G_0) \) and arbitrarily, otherwise. By the induction hypothesis, let \( G_{i-1} \) have a desired orientation \( (D_{i-1}, w) \) such that \( w^-(s) = 0 \) if \( s \in V(G_{i-1}) \).

First consider the case when \( B_i \) is a 2-connected outerplanar graph. By Lemma 6, \( B_i \) has a
weighted proper orientation \((D', w)\) such that \(\mu^{-}(D', w) \leq 4\) and \(w^{-}(s) = 0\) if \(s \in V(B_i)\). Thus, \((D', w)\) does not add the in-weight of \(s\) and \(w^{-}(s) = 0\) in the resulting weighted proper orientation of \(G_i\) provided \(s \in V(G_i)\). If \(B_i\) is an edge \(e\) then orient it from \(s\) if \(s\) is an end-vertex of \(e\) and arbitrarily, otherwise. Then we obtain a desired orientation as above.

Now we show the tightness of the bound. We will have \(G=\text{UOP}(4)\) as a tight example, which is depicted in Figure 2. Suppose \(\chi^{-}_w(G) \leq 3\). Since \(G\) contains a \(K_3\)-subgraph, \(\chi^{-}_w(G) = 3\). Let \(D\) be an optimal weighted proper orientation of \(G\) and let \(V_i\) be the set of vertices in \(D\) with in-weight \(i \in \{0, 1, 2, 3\}\).

Note that the vertices of \(G\) can be partitioned into three size-8 sets \(A, B, C\) such that every \(K_3\)-subgraph of \(G\) has one vertex \(a \in A, b \in B\) and \(c \in C\) as depicted in Figure 2 (In other words, \(A, B, C\) is a proper 3-coloring of \(G\).) Let \(S = \sum_{v \in V(G)} w_D(v)\). We have \(S \geq \sum_{v \in V(G)} d_G(v) = 45\). For every \(K_3\)-subgraph of \(G\) with vertices \(a, b, c\) we have

\[
\{w^{-}(a), w^{-}(b), w^{-}(c)\} \in \{\{1, 2, 3\}, \{0, 2, 3\}, \{0, 1, 3\}, \{0, 1, 2\}\}.
\]

Thus, \(S \leq 8(1 + 2 + 3) = 48\) implying that the gap between the upper bound and lower bound of \(S\) is 3. Hence, \(G\) has at most three edges of weight 2 in \(D\). By the lower bound, \(|V_3| \geq 7\).

Suppose \(|V_3| = 8\). By propagation of in-weights from one \(K_3\)-subgraph to another \(K_3\)-subgraph sharing an edge with the former, we will get four outervertices of in-weight 3 implying that \(G\) has four edges of weight 2 in \(D\), a contradiction. Thus, \(|V_3| = 7\). If either \(|V_1| < 8\) or \(|V_2| < 8\) then \(S < 45\) implying that \(|V_1| = |V_2| = 8\) and \(S = 45\). Hence, all edges of \(G\) have weight 1 in \(D\). However, by the propagation of in-weights, we will conclude that at least three outervertices have in-weight 3 implying that \(G\) has three edges of weight 2 in \(D\), a contradiction. \(\square\)

![Figure 2: Optimal weighted proper orientation \(D\) of UOP(4).](image)
5 Conclusion

The main result of this paper solves the weighted version of Problem 2. If the answer is shown to be positive for the question of Problem 3 on outerplanar graphs, then our result solves Problem 2 too. The next natural research direction is to attack the weighted versions of other appropriate problems stated in [4, 5], in particular that of Problem 1.

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