On the $c_0$-extension property

by

CLAUDIA CORREA (Santo André)

Abstract. In this work we investigate the $c_0$-extension property. This property generalizes Sobczyk’s theorem in the context of nonseparable Banach spaces. We prove that a sufficient condition for a Banach space to have this property is that its closed dual unit ball is weak-star monolithic. We also present several results about the $c_0$-extension property in the context of $C(K)$ Banach spaces. An interesting result in the realm of $C(K)$ spaces is that the existence of a Corson compactum $K$ such that $C(K)$ does not have the $c_0$-extension property is independent of the axioms of ZFC.

1. Introduction. Sobczyk’s theorem [24] is a classical result about the structure of separable Banach spaces. It states that if $X$ is a separable Banach space, then every $c_0$-valued bounded operator defined on a closed subspace of $X$ admits a $c_0$-valued bounded extension defined on $X$. The search for generalizations of Sobczyk’s theorem in the context of nonseparable Banach spaces has attracted a lot of attention in the last decades [6, 5, 7, 8, 14, 21, 22]. In [6], the $c_0$-extension property was introduced. We say that a Banach space $X$ has the $c_0$-extension property ($c_0$EP) if every $c_0$-valued bounded operator defined on a closed subspace of $X$ admits a $c_0$-valued bounded extension defined on $X$. Clearly Sobczyk’s theorem implies that every separable Banach space has the $c_0$EP.

Let us recall the main results known about the $c_0$EP. An adaptation of Veech’s proof of Sobczyk’s theorem [25] shows that every weakly compactly generated Banach space has the $c_0$EP [5 Proposition 2.2]. Recall that a Banach space is said to be weakly compactly generated (WCG) if it contains a weakly compact subset that is linearly dense. Clearly, every separable and every reflexive Banach space is WCG and the canonical example of a neither separable nor reflexive WCG space is given by $c_0(I)$, for any uncountable...
In [6, Theorem 2.2], it was shown that if $K$ is a compact line, then $C(K)$ has the $c_0$EP if and only if $K$ is monolithic. Here, as usual, $C(K)$ denotes the Banach space of real-valued continuous functions defined on a compact Hausdorff space $K$, endowed with the supremum norm. Recall that a compact line is a totally ordered set that is compact when endowed with the order topology. The notion of monolithicity plays a central role in this work. We say that a compact Hausdorff space $K$ is monolithic if every separable subspace of $K$ is second countable.

One of the main results of this work is Theorem 2.5, where we show that if $X$ is a Banach space with closed dual unit ball weak-star monolithic, then $X$ has the $c_0$EP. We denote the closed dual unit ball of a Banach space $X$ by $B_X^*$, and the weak-star topology by $w^*$-topology. Observe that all the spaces with the $c_0$EP mentioned above have closed dual unit ball $w^*$-monolithic. Indeed, it is well-known that if $X$ is a separable Banach space, then $B_X^*$ is $w^*$-metrizable [10, Proposition 3.103] and therefore $B_X^*$ is $w^*$-monolithic. Moreover, if $X$ is a WCG Banach space, then $(B_X^*, w^*)$ is an Eberlein compactum [10, Theorem 13.20] and therefore $B_X^*$ is $w^*$-monolithic. Recall that a compact space is an Eberlein compactum if it is homeomorphic to a weakly compact subset of a Banach space, endowed with the weak topology. Finally, it follows from Lemma 3.1 that if $K$ is a monolithic compact line, then $B_{C(K)^*}$ is $w^*$-monolithic.

A really interesting class of monolithic compact spaces is the class of Corson compacta. We say that a compact space $K$ is a Corson compactum if there exists a set $I$ such that $K$ is homeomorphic to a closed subset of $\Sigma(I)$, endowed with the product topology. By $\Sigma(I)$ we denote the subset of $\mathbb{R}^I$ formed by functions with countable support. Since every Corson compactum is monolithic, it follows from Theorem 2.5 that a Banach space $X$ has the $c_0$EP whenever $(B_X^*, w^*)$ is a Corson compactum. Those are precisely the weakly Lindelöf determined Banach spaces (WLD). Therefore the class of Banach spaces with the $c_0$EP contains the class of WLD spaces. Note that there exist Banach spaces with the $c_0$EP that are not WLD. For instance, the space of continuous functions $C[0, \omega_1]$ defined on the ordinal segment $[0, \omega_1]$ has the $c_0$EP, since $[0, \omega_1]$ is a monolithic compact line, but $C[0, \omega_1]$ is not WLD. To see that $C[0, \omega_1]$ is not WLD, note that a necessary condition for a $C(K)$ space to be WLD is that $K$ is a Corson compactum and $[0, \omega_1]$ is not a Corson compactum. A well-known class of Banach spaces that contains properly the WLD spaces is the class of 1-Plichko spaces. We say that a Banach space $X$ is 1-Plichko if $X^*$ contains a 1-norming $\Sigma$-subspace (see [15] for more information on those spaces). Recall that a Banach space $X$ is WLD if and only if $X^*$ is a $\Sigma$-subspace of itself [15, Theorem 5.37] and therefore every WLD space is 1-Plichko. An example of a non-WLD space
that is 1-Plichko is given by the space $\ell_1(I)$, for any uncountable set $I$ \cite{16} Example 6.9. A natural question at this point is whether every 1-Plichko space has the $c_0$EP. In Proposition 2.8, we answer this question negatively by showing that if $I$ is an uncountable set, then the space $\ell_1(I)$ does not have the $c_0$EP.

In Section 3, we investigate the $c_0$EP in the context of $C(K)$ spaces. It is well-known that if $K$ is a compact Hausdorff space, then $C(K)$ is separable if and only if $K$ is metrizable \cite{10} Lemma 3.102. Therefore, it follows from Sobczyk’s theorem that $C(K)$ has the $c_0$EP, for every metrizable compact space $K$. This result can be extended to the class of Eberlein compacta that generalizes the class of metrizable compact spaces. Recall that if $K$ is a compact Hausdorff space, then $K$ is an Eberlein compactum if and only if $C(K)$ is WCG \cite{10} Theorem 14.9. Thus $C(K)$ has the $c_0$EP whenever $K$ is an Eberlein compactum. Note that the class of Corson compacta generalizes properly the class of Eberlein compacta. In this work we address the problem of determining whether $C(K)$ has the $c_0$EP, for every Corson compactum $K$.

In Theorem 3.5, we establish that the existence of a Corson compactum $K$ such that $C(K)$ does not have the $c_0$EP is independent of the axioms of ZFC. Another interesting result presented here is the characterization of the $c_0$EP for spaces of continuous functions on scattered compact spaces with small height. More precisely, in Theorem 3.15 we show that if $K$ is a scattered compact space with height at most $\omega + 1$, then $C(K)$ has the $c_0$EP if and only if $K$ is monolithic. It is worthwhile to compare this result with a similar characterization of the $c_0$EP in the context of compact lines obtained in \cite{6} Theorem 2.2.

2. General results. In this work we consider only real Banach spaces. An apparently stronger property than the $c_0$EP is the $c_0$-extension property with some constant.

Definition 2.1. Let $X$ be a Banach space and let $\lambda \geq 1$. We say that $X$ has the $c_0$-extension property with constant $\lambda$ ($\lambda$-$c_0$EP) if any bounded operator $T : Y \to c_0$, defined on a closed subspace $Y$ of $X$, admits a bounded extension $\tilde{T} : X \to c_0$ with $\|\tilde{T}\| \leq \lambda\|T\|$.

We observe that all known proofs of $c_0$EP are actually proofs of $2$-$c_0$EP. It was shown in \cite{5} Proposition 2.2 that every WCG Banach space has the $2$-$c_0$EP. Moreover, the proof of \cite{6} Theorem 2.2 establishes that if $K$ is a monolithic compact line, then $C(K)$ has the $2$-$c_0$EP.

In what follows we devote ourselves to the proof of Theorem 2.5 which states that if $X$ is a Banach space such that $B_{X^*}$ is $w^*$-monolithic, then $X$ has the $2$-$c_0$EP. This proof relies essentially on Proposition 2.4 that establishes an interesting characterization of the monolithicity of the dual unit ball of a
Banach space in terms of $\ell_\infty$-valued bounded operators. The key ingredient in the proof of Proposition 2.4 is a translation of monolithicity in terms of absolute bipolar sets presented in Lemma 2.2. Let us recall some terminology and facts. Given a real normed space $X$, there is a natural bilinear pairing of $X$ and $X^*$ given by

$$X \times X^* \ni (x, \alpha) \mapsto \alpha(x) \in \mathbb{R}.$$  

Given a subset $A$ of $X^*$, the absolute polar $A^0$ of $A$, with respect to this bilinear pairing, is defined as

$$A^0 = \{ x \in X : |\alpha(x)| \leq 1, \forall \alpha \in A \}$$

and given a subset $B$ of $X$, the absolute polar $B^0$ of $B$, with respect to this bilinear pairing, is defined as

$$B^0 = \{ \alpha \in X^* : |\alpha(x)| \leq 1, \forall x \in B \}.$$  

If $A$ is a subset of $X^*$, then the absolute bipolar $A^{00}$ of $A$ is defined as $A^{00} = (A^0)^0$. Here we are considering only this natural bilinear pairing. The proof of Lemma 2.2 relies on the deep relationship between the absolutely convex hull of subsets of $X^*$ and their absolute bipolars that is given by the absolute bipolar theorem [3, Theorem 3.1.1]. For this particular bilinear pairing, the absolute bipolar theorem ensures that $A^{00}$ coincides with the $w^*$-closure of the absolutely convex hull of $A$, for every subset $A$ of $X^*$. Recall that if $A$ is a subset of $X^*$, then the absolutely convex hull of $A$, denoted here by $\text{absconv}(A)$, is the smallest absolutely convex subset of $X^*$ containing $A$. It is easy to see that

$$\text{absconv}(A) = \left\{ \sum_{i=1}^{n} t_i a_i : n \geq 1, a_i \in A, t_i \in \mathbb{R} \text{ and } \sum_{i=1}^{n} |t_i| \leq 1 \right\}.$$  

**Lemma 2.2.** Let $X$ be a Banach space. The following conditions are equivalent:

(i) $B_{X^*}$ is $w^*$-monolithic.

(ii) $\overline{\text{absconv}(A)}^{w^*}$ is $w^*$-metrizable for every countable subset $A$ of $B_{X^*}$.

(iii) $A^{00}$ is $w^*$-metrizable for every countable subset $A$ of $B_{X^*}$.

**Proof.** The equivalence of (ii) and (iii) follows directly from the absolute bipolar theorem. Observe that Urysohn’s metrization theorem implies that a compact space $K$ is monolithic if and only if the closure of any countable subset of $K$ is metrizable. Clearly (ii) implies (i). To see that (i) implies (ii), note that if $A$ is a countable subset of $X^*$, then $\overline{\text{absconv}(A)}^{w^*}$ is $w^*$-separable. This separability follows from the fact that the countable set

$$\left\{ \sum_{i=1}^{n} q_i a_i : n \geq 1, a_i \in A, q_i \in \mathbb{Q} \text{ and } \sum_{i=1}^{n} |q_i| \leq 1 \right\}$$

is $w^*$-dense in $\overline{\text{absconv}(A)}^{w^*}$.  

Let $X$ be a real normed space and $A$ be a bounded subset of $X^*$. We denote by $\rho_A : X \to [0, +\infty[$ the seminorm defined as

$$\rho_A(x) = \sup_{\alpha \in A} |\alpha(x)|, \quad \forall x \in X.$$ 

**Lemma 2.3.** Let $X$ be a Banach space. The following conditions are equivalent:

(a) $B_{X^*}$ is $w^*$-monolithic.

(b) $(X, \rho_A)$ is separable for every countable subset $A$ of $B_{X^*}$.

**Proof.** To prove that (a) and (b) are equivalent it suffices to show the equivalence of (b) and condition (iii) of Lemma 2.2. To do this, we will show that $A^{00}$ coincides with the closed unit ball of the dual space $(X, \rho_A)^*$, for any subset $A$ of $B_{X^*}$. The result will then follow from the fact that $(X, \rho_A)$ is separable if and only if the closed unit ball of its dual space is $w^*$-metrizable [10, Proposition 3.103]. Let $A$ be a subset of $B_{X^*}$ and note that $A^{00} = \{\alpha \in X^* : |\alpha(x)| \leq 1, \forall x \in B_{(X, \rho_A)}\}$, where $B_{(X, \rho_A)}$ denotes the closed unit ball of $(X, \rho_A)$. Finally, observe that $A^{00} \subset (X, \rho_A)^*$ and that $(X, \rho_A)^* \subset X^*$. ■

Recall that if $X$ is a real normed space, then there exists a bijective correspondence between bounded operators $X \to \ell_\infty$ and bounded sequences in $X^*$. More precisely, if $S : X \to \ell_\infty$ is a bounded operator, then the sequence associated to $S$ is $(\pi_n \circ S)_{n \geq 1}$, where $\pi_n : \ell_\infty \to \mathbb{R}$ denotes the $n$th projection. Conversely, if $(\alpha_n)_{n \geq 1}$ is a bounded sequence in $X^*$, then the operator associated to this sequence is defined as $S(x) = (\alpha_n(x))_{n \geq 1}$ for every $x \in X$. Note that if $(\alpha_n)_{n \geq 1}$ is the sequence associated to a bounded operator $S : X \to \ell_\infty$, then $\|S\| = \sup_{n \geq 1} \|\alpha_n\|$. It follows from this correspondence and Hahn–Banach’s theorem that every bounded $\ell_\infty$-valued operator defined on a subspace of $X$ admits a bounded $\ell_\infty$-valued extension defined on $X$ with the same norm. Any of those extensions is called a Hahn–Banach extension of the original operator. Given an operator $S$, we denote its image by $\text{Im} S$.

**Proposition 2.4.** Let $X$ be a Banach space. The following conditions are equivalent:

(1) $B_{X^*}$ is $w^*$-monolithic.

(2) $\text{Im} S$ is separable for every bounded operator $S : X \to \ell_\infty$.

**Proof.** Assume (1) and let $S : X \to \ell_\infty$ be a bounded operator. Without loss of generality, we may assume that $\|S\| = 1$. Denote by $(\alpha_n)_{n \geq 1}$ the sequence associated to $S$ and set $A = \{\alpha_n : n \geq 1\}$. Since $A \subset B_{X^*}$, Lemma 2.3 ensures that $(X, \rho_A)$ is separable. The separability of $\text{Im} S$ follows from the fact that the onto operator $S : (X, \rho_A) \to \text{Im} S$ is bounded and therefore continuous. Now let us prove that (2) implies condition (b) of
Lemma 2.3. Let $A$ be a countable subset of $B_{X^*}$ and enumerate $A = \{\alpha_n : n \geq 1\}$. Denote by $S : X \to \ell_\infty$ the bounded operator associated to the sequence $(\alpha_n)_{n \geq 1}$. Let $E$ be a countable and dense subset of $\text{Im} S$ guaranteed by (2) and denote by $D$ a countable subset of $X$ such that $S[D] = E$. It is easy to see that $D$ is dense in $(X, \rho_A)$. Indeed, for fixed $x \in X$ and $\epsilon > 0$, the density of $E$ in $\text{Im} S$ implies that there exists $e \in E$ with $\|S(x) - e\| < \epsilon$. If $d \in D$ satisfies $S(d) = e$, then

$$\rho_A(x - d) = \sup_{n \geq 1} |\alpha_n(x - d)| = \|S(x - d)\| = \|S(x) - e\| < \epsilon. \blacksquare$$

**Theorem 2.5.** Let $X$ be a Banach space. If $B_{X^*}$ is $w^*$-monolithic, then $X$ has the $2$-$c_0\text{EP}$.

**Proof.** Let $Y$ be a closed subspace of $X$ and $T : Y \to c_0$ be a bounded operator. If $S : X \to \ell_\infty$ denotes a Hahn–Banach extension of $T$, then Proposition 2.4 ensures that $\text{Im} S$ is a separable subspace of $\ell_\infty$. Therefore $\overline{\text{Im} T}$ is a closed subspace of the separable Banach space $\overline{\text{Im} S}$. It follows from Sobczyk’s theorem that the inclusion map $i : \overline{\text{Im} T} \to c_0$ admits a bounded extension $L : \overline{\text{Im} S} \to c_0$ with $\|L\| \leq 2\|i\| \leq 2$. The map $L \circ S : X \to c_0$ is a bounded extension of $T$ and $\|L \circ S\| \leq \|S\|\|L\| \leq 2\|T\|$. \blacksquare

We do not know if the converse of Theorem 2.5 holds.

**Corollary 2.6.** If $X$ is a WLD Banach space, then $X$ has the $2$-$c_0\text{EP}$.

**Proof.** This follows from Theorem 2.5 and the fact that every Corson compactum is monolithic. \blacksquare

Now let us see some stability properties of the class of Banach spaces with the $c_0\text{EP}$.

**Proposition 2.7.** Let $X$ and $Z$ be Banach spaces and assume $X$ has the $c_0\text{EP}$.

(a) If $Z$ is a subspace of $X$, then $Z$ has the $c_0\text{EP}$.
(b) If $Q : X \to Z$ is a bounded and onto operator, then $Z$ has the $c_0\text{EP}$.

**Proof.** The proof of (a) is straightforward. To prove (b), let $Y$ be a closed subspace of $Z$ and let $T : Y \to c_0$ be a bounded operator. Since $X$ has the $c_0\text{EP}$, the bounded operator $T \circ Q|_{Q^{-1}[Y]}$ admits a bounded extension $S : X \to c_0$. Define $\tilde{T} : Z \to c_0$ as $\tilde{T}(z) = S(x)$, where $x$ is any element of $X$ satisfying $Q(x) = z$. The fact that $\tilde{T}$ is well-defined follows from the fact that $\text{Ker} Q$ is contained in $Q^{-1}[Y]$. Indeed, if $x_1$ and $x_2$ are elements of $X$ with $Q(x_1) = Q(x_2)$, then $x_1 - x_2 \in \text{Ker} Q \subset Q^{-1}[Y]$. Therefore,

$$S(x_1 - x_2) = T(Q(x_1 - x_2)) = 0.$$  

It is easy to see that the map $\tilde{T}$ is a bounded extension of $T$. \blacksquare
The next proposition states that the 1-Plichko space $\ell_1(I)$ does not have the $c_0$EP, for any uncountable set $I$. We present its proof after Remark 3.17 since it depends on the results developed in Section 3.

**Proposition 2.8.** If $I$ is an uncountable set, then $\ell_1(I)$ does not have the $c_0$EP.

**Corollary 2.9.** Let $X$ be a Banach space and $I$ be an uncountable set. If $X$ contains an isomorphic copy of $\ell_1(I)$, then $X$ does not have the $c_0$EP.

**Proof.** This follows from Propositions 2.7(a) and 2.8. 

It is interesting to observe that even though not every 1-Plichko space has the $c_0$EP, these spaces have a weaker property that was also introduced in [6]: the separable $c_0$-extension property. We say that a Banach space $X$ has the separable $c_0$-extension property (separable $c_0$EP) if every $c_0$-valued bounded operator defined on a closed and separable subspace of $X$ admits a $c_0$-valued bounded extension defined on $X$. The fact that every 1-Plichko space has the separable $c_0$EP follows from the fact that such spaces have the separable complementation property (SCP) [15, p. 105] and clearly the SCP implies the separable $c_0$EP. Note that Proposition 2.8 provides an example of a space with the SCP but not the $c_0$EP. A stronger property than the SCP is the controlled separable complementation property (CSCP) that was introduced in [26] and studied in a series of papers [12, 11, 13]. A surprising consequence of Theorem 2.5 is that the CSCP implies the $c_0$EP.

**Proposition 2.10.** Let $X$ be a Banach space. If $X$ has the CSCP, then $X$ has the 2-$c_0$EP.

**Proof.** It follows from Theorem 2.5 and the fact that if $X$ has the CSCP, then $B_{X^*}$ is $w^*$-monolithic [17, Proposition 1.5].

### 3. The $c_0$EP for $C(K)$ spaces.

In this section, we develop the theory of the $c_0$EP in the context of Banach spaces of the form $C(K)$. Throughout, we will refer to a compact and Hausdorff space just as a compact space. For a compact space $K$, we identify the dual space $C(K)^*$ with the space $M(K)$ of regular signed finite Borel measures on $K$, endowed with the total variation norm. In Lemma 3.1, we establish conditions on a compact space $K$ that ensure that $B_{M(K)}$ is $w^*$-monolithic. Clearly, if $B_{M(K)}$ is $w^*$-monolithic, then $K$ is monolithic, since $K$ is homeomorphic to a closed subspace of $(B_{M(K)},w^*)$. The additional condition on monolithic compacta used in Lemma 3.1 is property (M). We say that a compact space $K$ has property (M) if the support of every measure in $M(K)$ is separable. Recall that if $\mu$ is an element of $M(K)$, then its support is defined as

$$\text{supp } \mu = \{ p \in K : |\mu|(U) > 0 \text{ for every open neighborhood } U \text{ of } p \},$$

where $|\mu|$ denotes the total variation of $\mu$. 


Lemma 3.1. If $K$ is a monolithic compact space with property (M), then $B_{M(K)}$ is $w^*$-monolithic.

Proof. Let $\{\mu_n : n \geq 1\}$ be a countable subset of $B_{M(K)}$ and denote by $L$ the closed subset of $K$ defined as $L = \bigcup_{n \geq 1} \text{supp} \mu_n$. Since $K$ has property (M) and is monolithic, it follows that $L$ is a compact metric space and therefore $B_{M(L)}$ is $w^*$-metrizable. To show that $\{\mu_n : n \geq 1\}$ is $w^*$-metrizable, we will prove that this space is a subspace of an homeomorphic image of $(B_{M(L)}, w^*)$. Denote by $i_* : M(L) \to M(K)$ the operator defined as $i_*(\nu)(B) = \nu(B \cap L)$ for every Borel subset $B$ of $K$ and every $\nu \in M(L)$. Note that $i_*$ is injective and $w^*$-continuous. Therefore the $w^*$-compactness of $B_{M(L)}$ ensures that $(i_*[B_{M(L)}], w^*)$ is homeomorphic to $(B_{M(L)}, w^*)$. Finally, observe that if $\mu \in B_{M(K)}$ and $\text{supp} \mu \subset L$, then $\mu \in i_*[B_{M(L)}]$. This implies that $\{\mu_n : n \geq 1\}^{w^*} \subset i_*[B_{M(L)}]$. ■

Theorem 3.2. If $K$ is a monolithic compact space with property (M), then $C(K)$ has the 2-$c_0$EP.

Proof. This follows from Theorem 2.5 and Lemma 3.1. ■

Clearly every metrizable compact space has property (M) and it is easy to see that every scattered compact space has property (M). We say that a topological space $X$ is scattered if for every closed subspace $Y$ of $X$, the set of isolated points of $Y$ is dense in $Y$. Other examples of spaces with property (M) are compact lines (see [17, Lemma 2.1]), Eberlein compacta and Rosenthal compacta (see [1, Remark 3.2]). Recall that a Rosenthal compactum is a pointwise compact subset of the space of Baire class one real-valued functions on a Polish space.

Corollary 3.3. If $K$ is a monolithic compact space belonging to any of the following classes, then $C(K)$ has the 2-$c_0$EP:

(a) compact lines,
(b) scattered spaces,
(c) Rosenthal compacta.

Proof. This follows from Theorem 3.2 and the fact that those spaces have property (M). ■

It is well-known that the existence of a Corson compactum without property (M) is independent of the axioms of ZFC. It was shown in [11, Remark 3.2.3] that if we assume Martin’s axiom and the negation of the continuum hypothesis ($MA + \neg CH$), then every Corson compactum has property (M). Therefore Theorem 3.2 implies that, under $MA + \neg CH$, $C(K)$ has the 2-$c_0$EP, for every Corson compactum $K$. On the other hand, if we assume CH, then there exists a Corson compactum without property (M).
Theorem 3.12]. It turns out that the space of continuous functions on the Corson compactum constructed in [1, Theorem 3.12] does not have the $c_0$EP.

**Proposition 3.4.** Assume CH. There exists a Corson compactum $K$ such that $C(K)$ does not have the $c_0$EP.

**Proof.** Let $K$ denote the Corson compactum without property (M) constructed in [1, Theorem 3.12]. It follows from [1, Theorem 3.13] that $C(K)$ contains an isomorphic copy of $\ell_1(\omega_1)$. Therefore the result follows from Corollary 2.9.

From what was discussed above we obtain the following result.

**Theorem 3.5.** The existence of a Corson compactum $K$ such that $C(K)$ does not have the $c_0$EP is independent of the axioms of ZFC.

**Remark 3.6.** The Corson compactum without property (M) constructed in [1, Theorem 3.12] is defined by an adequate family of sets. In [1, Theorem 3.13], it was shown that if $K$ is a Corson compactum defined by an adequate family of sets, then $K$ has property (M) if and only if $\ell_1(\omega_1)$ does not embed isomorphically in $C(K)$. Thus Corollary 2.9 and Theorem 3.2 imply that if $K$ is a Corson compactum defined by an adequate family of sets, then $C(K)$ has the $c_0$EP if and only if $K$ has property (M). We do not know if this equivalence holds for any Corson compactum. For instance, it is unclear whether $C(K)$ has the $c_0$EP for the Corson compactum $K$ without property (M) constructed in [20].

Note that Proposition 3.4 also shows that if we assume CH, then there exists a monolithic compact space $K$ such that $C(K)$ does not have the $c_0$EP. However this is not possible under $MA + \neg CH$.

**Proposition 3.7.** Assume $MA + \neg CH$. If $K$ is a monolithic compact space, then $C(K)$ has the $2-c_0$EP.

**Proof.** It was shown in [2, Theorem 3] that if we assume $MA + \neg CH$, then every monolithic compact space with the countable chain condition is separable. It is easy to see that this implies that, under $MA + \neg CH$, every monolithic compact space has property (M). Thus the result follows from Theorem 3.2.

From what was discussed above we obtain the following result.

**Theorem 3.8.** The existence of a monolithic compact space $K$ such that $C(K)$ does not have the $c_0$EP is independent of the axioms of ZFC.

Now we present some useful stability results for the $c_0$EP in the context of $C(K)$ spaces.
Proposition 3.9. Let $K$ and $L$ be compact spaces and assume that $C(K)$ has the $c_0$EP.

(1) If $L$ is a subspace of $K$, then $C(L)$ has the $c_0$EP.
(2) If $L$ is a quotient of $K$, then $C(L)$ has the $c_0$EP.

Proof. (1) Note that the restriction operator $C(K) \to C(L)$ is a bounded and onto operator. Therefore the result follows from Proposition 2.7(b).

(2) It is easy to see that if $\phi : K \to L$ is a continuous and onto map, then the composition operator $\phi^* : C(L) \to C(K)$ defined as $\phi^*(f) = f \circ \phi$ for all $f \in C(L)$, is an isometric embedding. Proposition 2.7(a) ensures that $\phi^*[C(L)]$ has the $c_0$EP and therefore $C(L)$ has the $c_0$EP. □

In what follows, we investigate the $c_0$EP in the class of scattered compact spaces. Let us recall some basic definitions and facts. Given a topological space $X$ and an ordinal number $\alpha$, we denote by $X^{(\alpha)}$ the $\alpha$th Cantor–Bendixson derivative of $X$. It is well-known that a topological space $X$ is scattered if and only if there exists an ordinal $\alpha$ such that $X^{(\alpha)} = \emptyset$ [19 Proposition 17.8]. The height of a scattered topological space $X$, denoted here by $\text{ht}(X)$, is defined as the least ordinal $\alpha$ such that $X^{(\alpha)} = \emptyset$. Finally, if the height of $X$ is a natural number, then we say that $X$ has finite height.

In Corollary 3.3(b), we proved that if $K$ is a monolithic compact scattered space, then $C(K)$ has the $c_0$EP. In Theorem 3.13 and Corollary 3.14, we will show that if $K$ is a scattered compact space of height at most $\omega + 1$ such that $C(K)$ has the $c_0$EP, then $K$ is monolithic.

Lemma 3.10. Let $K$ be a scattered compact space of height 3. If $K$ is separable and nonmetrizable, then $C(K)$ does not have the $c_0$EP.

Proof. First, let us prove that we can assume that $K^{(2)}$ has only one point. Write $K^{(2)} = \{p_1, \ldots, p_k\}$, where $k = |K^{(2)}|$. Clearly, there exist disjoint clopen subsets $C_1, \ldots, C_k$ of $K$ such that $K = \bigcup_{i=1}^k C_i$ and $p_i \in C_i$, for every $i = 1, \ldots, k$. It is easy to see that $C_i$ is separable and $C_i^{(2)} = \{p_i\}$, for every $i = 1, \ldots, k$, and that there exists an index $i_0$ such that $C_{i_0}$ is nonmetrizable. Thus Proposition 3.9(1) ensures that the result follows if we show that $C(C_{i_0})$ does not have the $c_0$EP. Note that the restriction operator $R : C(K) \to C(K^{(1)})$ provides the following short exact sequence:

$$0 \to \text{Ker}(R) \to C(K) \xrightarrow{R} C(K^{(1)}) \to 0.$$

It is well-known that $\text{Ker}(R)$ is isomorphic to $c_0$ and that $C(K^{(1)})$ is isomorphic to $c_0(K^{(1)})$. Therefore we have an exact sequence of the form

(3.1) $$0 \to c_0 \to C(K) \to c_0(K^{(1)}) \to 0.$$

In order to conclude that $C(K)$ does not have the $c_0$EP, we will show that the isomorphic copy of $c_0$ inside $C(K)$ given by (3.1) is not complemented.
in $C(K)$. Assume for contradiction that this copy of $c_0$ is complemented in $C(K)$. In this case $C(K)$ is isomorphic to $c_0 \oplus c_0(K^{(1)})$ and therefore $C(K)$ is a WCG space. This implies that $K$ is an Eberlein compactum. But this is not possible, since $K$ is not monolithic.

An interesting consequence of Lemma 3.10 is that the $c_0$EP is not a three-space property. We say that a property $\mathcal{P}$ is a three-space property if every twisted sum of Banach spaces with property $\mathcal{P}$ also has property $\mathcal{P}$. Classical examples of three-space properties for Banach spaces are separability and reflexivity [4].

**Corollary 3.11.** The $c_0$EP is not a three-space property.

**Proof.** Consider the exact sequence (3.1) described in the proof of Lemma 3.10. The spaces $c_0$ and $c_0(K^{(1)})$ have the $c_0$EP, since they are WCG spaces, but $C(K)$ does not have the $c_0$EP.

The strategy to prove Theorem 3.13 is to proceed by induction on the height of $K$. In order to decrease the height and use the induction hypothesis we will use a quotient of $K$. Given a compact space $K$ and a closed subset $F$ of $K$, we define the following equivalence relation on $K$:

$$x \sim_F y \iff x = y \text{ or } x, y \in F.$$  

Clearly the quotient space $K/\sim_F$ is compact and it is easy to see that it is also Hausdorff.

**Lemma 3.12.** Given an ordinal $\alpha$, if $F = K^{(\alpha)}$, then $(K/\sim_F)^{(\alpha+1)} = \emptyset$.

**Proof.** It follows from the fact that if $K$ and $L$ are compact spaces and $q : K \rightarrow L$ is a continuous and onto map, then $L^{(\alpha)} \subset q[K^{(\alpha)}]$ for every ordinal $\alpha$ [18, Lemma 1.8 and Theorem 1.9].

**Theorem 3.13.** Let $K$ be a compact space of finite height. If $C(K)$ has the $c_0$EP, then $K$ is monolithic.

**Proof.** By Proposition 3.9(1), it suffices to prove that if $K$ is a separable compact space of finite height such that $C(K)$ has the $c_0$EP, then $K$ is metrizable. Let us proceed by induction on the height of $K$. The case $\text{ht}(K) \leq 2$ is trivial and the case $\text{ht}(K) = 3$ is established in Lemma 3.10.

Now assume that $\text{ht}(K) = N + 1$ with $N \geq 3$ and denote by $F$ the closed subset $K^{(N-1)}$ of $K$. Note that Lemma 3.12 implies that $K/\sim_F$ is a scattered space with $\text{ht}(K/\sim_F) \leq N$. Since Proposition 3.9(2) ensures that $C(K/\sim_F)$ also has the $c_0$EP, it follows from the induction hypothesis that $K/\sim_F$ is metrizable. Recall that if an infinite compact space is scattered, then its weight coincides with its cardinality [19, Proposition 17.10]. Thus $K/\sim_F$ is
countable and consequently \( K \setminus F \) is countable. This implies that \( K^{(1)} \setminus K^{(2)} \) is countable and therefore \( K^{(1)} \) is separable. Note that \( \text{ht}(K^{(1)}) = N \) and that Proposition 3.9(1) ensures that \( C(K^{(1)}) \) has the \( c_0 \)EP. Thus it follows from the induction hypothesis that \( K^{(1)} \) is metrizable and therefore \( K^{(1)} \) is countable, since it is scattered. Finally, the metrizability of \( K \) follows from the fact that \( K \setminus K^{(1)} \) is also countable, since \( K \) is separable. □

**Corollary 3.14.** Let \( K \) be a compact space of height \( \omega + 1 \). If \( C(K) \) has the \( c_0 \)EP, then \( K \) is monolithic.

**Proof.** By Proposition 3.9(1), it suffices to prove that if \( K \) is a separable compact space of height \( \omega + 1 \) such that \( C(K) \) has the \( c_0 \)EP, then \( K \) is metrizable. To do so, we will prove that \( K \setminus K^{(N+1)} \) is countable for every \( N \in \omega \). This will imply that \( K \) is countable and therefore metrizable, since \( K = K^{(\omega)} \cup \bigcup_{N \in \omega} K \setminus K^{(N+1)} \) and \( K^{(\omega)} \) is finite. Given \( N \in \omega \), denote by \( F \) the closed subset \( K^{(N+1)} \) of \( K \) and note that Lemma 3.12 implies that \( K/\sim_F \) is a scattered compact space with \( \text{ht}(K/\sim_F) \leq N + 2 \). Since Proposition 3.9(2) ensures that \( C(K/\sim_F) \) also has the \( c_0 \)EP, it follows from Theorem 3.13 that \( K/\sim_F \) is metrizable. This implies that \( K/\sim_F \) is countable and thus \( K \setminus F \) is countable. □

**Theorem 3.15.** If \( K \) is a scattered compact space of height at most \( \omega + 1 \), then \( C(K) \) has the \( c_0 \)EP if and only if \( K \) is monolithic.

**Proof.** This follows from Corollary 3.3(b), Theorem 3.13 and Corollary 3.14. □

An interesting consequence of Lemma 3.10 is the existence of Valdivia compacta whose spaces of continuous functions do not have the \( c_0 \)EP. Recall that a compact space \( K \) is said to be a Valdivia compactum if there exists a set \( I \) and a homeomorphic embedding \( \varphi : K \to \mathbb{R}^I \) such that \( \varphi^{-1}([\Sigma(I)]) \) is dense in \( K \). Clearly, every Corson compactum is Valdivia and it is easy to see that if \( \kappa \) is an uncountable cardinal, then the Cantor cube \( 2^\kappa \) is a non-Corson Valdivia compactum [16, Theorem 3.29].

**Proposition 3.16.** If \( \kappa \) is an uncountable cardinal, then the space \( C(2^\kappa) \) does not have the \( c_0 \)EP.

**Proof.** Proposition 3.9(1) ensures that it suffices to show that \( C(2^{\omega_1}) \) does not have the \( c_0 \)EP, since \( 2^{\omega_1} \) embeds homeomorphically in \( 2^\kappa \), for any uncountable cardinal \( \kappa \). Let \( K \) be a separable scattered compact space of height 3 and weight \( \omega_1 \). Since \( K \) is zero-dimensional, \( K \) embeds homeomorphically in \( 2^{\omega_1} \). Therefore the result follows from Proposition 3.9(1) and Lemma 3.10. □
Remark 3.17. Recall that if $K$ is a Valdivia compactum, then $C(K)$ is a 1-Plichko space \cite[Theorem 5.55]{15}. Thus it follows from what was discussed in the previous section that $C(K)$ has the separable $c_0$EP for every Valdivia compactum $K$.

Proof of Proposition 2.8. According to Proposition 2.7(b), it suffices to show that if $I$ is an uncountable set, then $\ell_1(I)$ has a Banach quotient without the $c_0$EP. It is well-known that every Banach space with density $|I|$ is a quotient of $\ell_1(I)$. The result follows from Proposition 3.16 since the density of $C(2^I)$ is $|I|$.

We can generalize Proposition 3.16 to the class of dyadic compacta. A compact space is said to be a \textit{dyadic compactum} if it is a continuous image of the Cantor cube $2^\kappa$, for some cardinal $\kappa$.

Corollary 3.18. If $K$ is a nonmetrizable dyadic compactum, then $C(K)$ does not have the $c_0$EP.

Proof. Gerlits and Efimov showed that every nonmetrizable dyadic compactum contains a homeomorphic copy of the Cantor cube $2^{\omega_1}$ (see \cite[3.12.12]{9}). Therefore the result follows from Propositions 3.9(1) and 3.16.

We conclude this section by showing that if $K$ is a compact space such that $C(K)$ has the $c_0$EP, then $K$ admits only measures with small Maharam type. Given a compact space $K$ and a cardinal $\kappa$, we say that a nonnegative measure $\mu \in M(K)$ has \textit{Maharam type} $\kappa$ if $\kappa$ is the density of the Banach space $L_1(\mu)$.

Proposition 3.19. Let $K$ be a compact space. If $C(K)$ has the $c_0$EP, then every nonnegative measure in $M(K)$ has Maharam type at most $\omega_1$.

Proof. If there exists a nonnegative measure in $M(K)$ with Maharam type strictly greater than $\omega_1$, then $C(K)$ contains an isomorphic copy of $\ell_1(\omega_1)$ \cite{23}, and Corollary 2.9 implies that $C(K)$ does not have the $c_0$EP.

Assuming $\text{MA} + \neg\text{CH}$, we obtain the following stronger result.

Proposition 3.20. Assume $\text{MA} + \neg\text{CH}$. Let $K$ be a compact space. If $C(K)$ has the $c_0$EP, then every nonnegative measure in $M(K)$ has countable Maharam type.

Proof. Under $\text{MA} + \neg\text{CH}$, if there exists a nonnegative measure in $M(K)$ with uncountable Maharam type, then $C(K)$ contains an isomorphic copy of $\ell_1(\omega_1)$ \cite{23}, and Corollary 2.9 implies that $C(K)$ does not have the $c_0$EP.

Acknowledgements. The author was partially supported by FAPESP grant 2018/09797-2.
References

[1] A. A. Argyros, S. Mercourakis and S. Negrepontis, *Functional-analytic properties of Corson-compact spaces*, Studia Math. 89 (1988), 197–229.

[2] A. V. Arkhangel’skii and B. E. Shapirovskii, *On the structure of monolithic spaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1987, no. 4, 72–74 (in Russian).

[3] V. I. Bogachev and O. G. Smolyanov, *Topological Vector Spaces and Their Applications*, Springer, 2017.

[4] J. M. F. Castillo and M. González, *Three-Space Problems in Banach Space Theory*, Springer, 1997.

[5] C. Correa and D. V. Tausk, *On extensions of c₀-valued operators*, J. Math. Anal. Appl. 405 (2013), 400–408.

[6] C. Correa and D. V. Tausk, *Compact lines and the Sobczyk property*, J. Funct. Anal. 266 (2014), 5765–5778.

[7] C. Correa and D. V. Tausk, *On the c₀-extension property for compact lines*, J. Math. Anal. Appl. 428 (2015), 165–186.

[8] P. Drygier and G. Plebanek, *Compactifications of ω and the Banach space c₀*, Fund. Math. 237 (2017), 1252–1266.

[9] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[10] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach Space Theory: The Basis for Linear and Nonlinear Analysis*, Springer, 2011.

[11] J. Ferrer, *The controlled separable complementation property and monolithic compacta*, Banach J. Math. Anal. 8 (2014), no. 2, 67–78.

[12] J. Ferrer, *A note on monolithic scattered compacta*, J. Math. Anal. Appl. 421 (2015), 950–954.

[13] J. Ferrer and M. Wójtowicz, *The controlled separable projection property for Banach spaces*, Cent. Eur. J. Math. 9 (2011), 1252–1266.

[14] E. M. Galego and A. Plichko, *On Banach spaces containing complemented and uncomplemented subspaces isomorphic to c₀*, Extracta Math. 18 (2003), 315–319.

[15] P. Hájek, V. M. Santalucía, J. Vanderwerff and V. Zizler, *Biorthogonal Systems in Banach Spaces*, Springer, 2008.

[16] O. Kalenda, *Valdivia compact spaces in topology and Banach space theory*, Extracta Math. 15 (2000), 1–85.

[17] O. Kalenda and W. Kubiś, *Complementation in spaces of continuous functions on compact lines*, J. Math. Anal. Appl. 386 (2012), 241–257.

[18] V. Kannan and M. Rajagopalan, *Scattered spaces II*, Illinois J. Math. 21 (1977), 735–751.

[19] S. Koppelberg, *Handbook of Boolean Algebras*, Vol. 1, North-Holland, 1989.

[20] K. Kunen, *A compact L-space under CH*, Topology Appl. 12 (1981), 283–287.

[21] A. Moltó, *On a theorem of Sobczyk*, Bull. Austral. Math. Soc. 43 (1991), 123–130.

[22] W. M. Patterson, *Complemented c₀-subspaces of a non-separable C(K)-space*, Canad. Math. Bull. 36 (1993), 351–357.

[23] G. Plebanek, *On compact spaces carrying random measures of large Maharam type*, Acta Univ. Carolin. Math. Phys. 43 (2002), no. 2, 87–99.

[24] A. Sobczyk, *Projection of the space (m) on its subspace (c₀)*, Bull. Amer. Math. Soc. 47 (1941), 938–947.

[25] W. A. Veech, *Short proof of Sobczyk’s theorem*, Proc. Amer. Math. Soc. 28 (1971), 627–628.

[26] M. Wójtowicz, *Generalizations of the c₀-ℓ₁-ℓ∞ theorem of Bessaga and Pelczyński*, Bull. Polish Acad. Sci. Math. 50 (2002), 373–382.
