A General Formula for the Stationary Distribution of the Age of Information and Its Application to Single-Server Queues

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Abstract

This paper considers the stationary distribution of the age of information (AoI) in information update systems. We first derive a general formula for the stationary distribution of the AoI, which holds for a wide class of information update systems. The formula indicates that the stationary distribution of the AoI is given in terms of the stationary distributions of the system delay and the peak AoI. To demonstrate its applicability and usefulness, we analyze the AoI in single-server queues with four different service disciplines: first-come first-served (FCFS), preemptive last-come first-served (LCFS), and two variants of non-preemptive LCFS service disciplines. For the FCFS and the preemptive LCFS service disciplines, the GI/GI/1, M/GI/1, and GI/M/1 queues are considered, and for the non-preemptive LCFS service disciplines, the M/GI/1 and GI/M/1 queues are considered. With these results, we further show comparison results for the mean AoIs in the M/GI/1 and GI/M/1 queues under those service disciplines.

Keywords: Age of information, stationary distribution, peak AoI, single-server queues, FCFS, preemptive LCFS, non-preemptive LCFS.

1 Introduction

We consider an information update system composed of an information source equipped with a sensor, a processor (a server), and a monitor. The state of the information source changes over time, which is observed by the sensor occasionally. Whenever the state is sensed, the sensor generates a packet that contains the sensed data and its time-stamp, and sends the packet to the server. The server processes the received data, appends the result to log database, and updates information displayed on the monitor. The age of information (AoI) is a primary performance measure in information update systems, which is defined as the length of time elapsed from the time-stamp of information being displayed on the monitor.

The information update system described above is an abstraction of various situations where the freshness of data is of interest, e.g., status monitoring in manufacturing systems, satellite imagery for weather report, tracking trends in social networks, and so on. The AoI has recently attracted a considerable attention due to its applicability to a wide range of information and communication systems. Readers are referred to [1] for a detailed introduction and review of this subject.

Information update systems are usually modeled as queueing systems, where packets arriving at a queueing system correspond to information packets. In most previous work on the analysis of the AoI, the mean AoI is of primary concern, which is defined as the long-run time-average of the AoI. To be more specific, consider a stationary, ergodic queueing system, and let $A_t \ (t \geq 0)$ denote the AoI at time $t$:

$$A_t := t - \eta_t, \quad t \geq 0,$$

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where $\eta_t$ ($t \geq 0$) denotes the time-stamp of information being displayed on the monitor at time $t$. The mean AoI $E[A]$ is defined as

$$E[A] = \lim_{T \to \infty} \frac{1}{T} \int_0^T A_t dt,$$

and under a fairly general setting, $E[A]$ is given by

$$E[A] = \frac{E[G_n D_n] + E[(G_t)^2]/2}{E[G_t]},$$

where $E[G_t]$ and $E[(G_t)^2]$ denote the mean and the second moment of interarrival times and $E[G_n D_n]$ denotes the mean product of the interarrival time $G_n$ of the $(n-1)$st and the $n$th packets and the system delay $D_n$ of the $n$th packet. This formula has been the starting point in most previous work on the analysis of the AoI. As stated in [1, Page 170], however, the calculation of the mean AoI based on (2) is cumbersome because $G_n$ and $D_n$ are dependent in general and their joint distribution can take a complicated form.

The purpose of this paper is twofold. The first one is the derivation of a general formula for the stationary distribution $A(x)$ ($x \geq 0$) of the AoI in ergodic information update systems, which is defined as the long-run fraction of time in which the AoI is not greater than an arbitrarily fixed value $x$:

$$A(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{A_t \leq x\}} dt,$$

where $\mathbb{1}_{\{\cdot\}}$ denotes an indicator function. Although the mean AoI $E[A]$ is a primary performance metric, it alone is not sufficient to characterize the long-run behavior of the AoI and its related processes. First of all, if the stationary distribution $A(x)$ of the AoI is available, we can evaluate the deviation of the AoI from its mean value. To support our claim further, we provide two examples, which show that the stationary distribution of the AoI plays a central role in the analysis of AoI-related processes.

**Example 1.** In [4], a non-linear age penalty function is introduced to expand the concept of the AoI, which is also referred to as the Cost of Update Delay (CoUD) metric in [5]. For a non-negative and non-decreasing function $f(x)$ ($x \geq 0$) with $f(0) = 0$, CoUD $C_t$ at time $t$ is defined as $C_t = f(A_t)$ ($t \geq 0$). Clearly, the time-average of CoUD is given in terms of the stationary distribution $A(x)$ ($x \geq 0$) of the AoI:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T C_t dt = \int_0^\infty f(x) dA(x).$$

**Example 2.** Consider a remote estimation of a stationary Wiener process $\{X_t; t \geq 0\}$ via a channel modeled as a stationary queueing system [6]. We define $\{\hat{X}_t; t \geq 0\}$ as an estimator $\hat{X}_t = X_{\eta_t}$ of $X_t$ (see [4]). As shown in [6], if a sequence of sampling times is independent of $\{X_t; t \geq 0\}$,

$$E[(X_t - \hat{X}_t)^2] = E[(\text{Norm}(0, A_t))^2] = E[A],$$

where $\text{Norm}(\mu, \sigma^2)$ denotes a normal random variable with mean $\mu$ and variance $\sigma^2$. Therefore, the mean square error of the estimator $\{\hat{X}_t; t \geq 0\}$ is equal to the mean AoI for the Wiener process.

As an extension of this result, we consider the stationary distribution of the estimation error. It is readily verified that the characteristic function of the estimation error is given by

$$E[e^{i\omega(X_t - \hat{X}_t)}] = E[e^{i\omega\text{Norm}(0, A_t)}] = a^\ast \left( \frac{\omega^2}{2} \right),$$

where $a^\ast(s)$ ($s > 0$) denotes the Laplace-Stieltjes transform (LST) of the stationary distribution of the AoI.

In [3], a performance metric called the peak AoI is introduced, which is defined as the AoI immediately before information updates. The formulation of the peak AoI is simpler than that of the AoI, and it can be used as an alternative metric of the freshness of data. In particular, one of the primary motivations of introducing the peak AoI in [3] is that one can characterize its stationary distribution using standard methods in queueing theory; the stationary distribution of the peak AoI is given in terms of that of the system delay.

In this paper, we show that under a fairly general setting, the stationary distribution $A(x)$ of the AoI is given in terms of those of the system delay and the peak AoI. As we will see, this formula holds sample-path-wise, regardless of the service discipline or the distributions of interarrival and service times.
Table 1: Summary of known results for the mean AoI in standard queueing systems, where the fourth entry in Kendall’s notation indicates the capacity of the waiting room, and P and NP stand for preemptive and non-preemptive.

| Reference | Service discipline | System               |
|-----------|--------------------|----------------------|
| 2         | FCFS               | M/M/1/∞              |
| 3         | FCFS               | M/M/1/0              |
| 7         | FCFS               | M/GI/1/0             |
| 9         | P-LCFS (C)         | M/M/1/0              |
| 10        | P-LCFS             | M/Gamma/1/0          |
|           | NP-LCFS (C)        | M/M/1/1              |
|           | NP-LCFS (C)        | M/Erlang/1/1         |

Therefore, the analysis of the stationary distribution of the AoI in ergodic queueing systems is reduced to the analysis of those of the peak AoI and the system delay, which can be analyzed via standard techniques in queueing theory. An important consequence of this observation is that the peak AoI is not merely an alternative performance metric to the AoI but rather an essential quantity in elucidating properties of the AoI. Furthermore, this observation leads to an alternative formula for the mean AoI in terms of the second moments of the peak AoI and the system delay.

The second purpose of this paper is the derivation of various analytical formulas for the AoI in single-server queues, which demonstrates the wide applicability of our general formula. More specifically, we consider the following four service disciplines:

(A) First-come, first-served (FCFS),

(B) Preemptive last-come, first-served (LCFS),

(C) Non-preemptive LCFS with discarding, and

(D) Non-preemptive LCFS without discarding.

Under the FCFS service discipline, all packets are served in order of arrival, while under the LCFS service disciplines (B)–(D), the newest packet is given priority. In the preemptive LCFS discipline, newly arriving packets immediately start receiving their services on arrival, interrupting the ongoing service (if any). In the non-preemptive LCFS service discipline, on the other hand, arriving packets have to wait until the completion of the ongoing service, and waiting packets are also overtaken by those which arrive during their waiting times.

Note that the non-preemptive LCFS service discipline has two variants, (C) and (D): overtaken packets are discarded in the service discipline (C), while overtaken packets remain in the system and they are served eventually in the service discipline (D). Although (D) yields a larger AoI than (C), this service discipline is also of interest in evaluating the logging overhead caused by sending all generated packets to the database. Note that for the preemptive LCFS discipline, the treatment of overtaken packets (i.e., discarding them or not) does not affect the AoI performance.

Table 1 summarizes known results for the mean AoI in standard queueing systems. For the analysis of LCFS queues, all the previous works listed in Table 1 utilize the memoryless property of exponential interarrival times to simplify the derivation of the cross-term $E[G_i D_n]$ in (2). Also, to the best of our knowledge, no closed formula for the AoI under the service discipline (D) has been reported in the literature. As we will see, our general formula is readily applicable to non-Poisson arrival cases and the service discipline (D). We note that in addition to those listed in Table 1, there are analytical results for the mean AoI in queueing models with additional features in the literature, e.g., queues with packet deadline [11], multi-class queues [12], and priority queues [13].

Our contribution to the analysis of the AoI in single-server queues is summarized as follows.
(A): For the FCFS GI/GI/1 queue, we show that the stationary distribution of the AoI is given in terms of the system delay distribution. We also derive upper and lower bounds for the mean AoI in the FCFS GI/GI/1 queue. In addition, we derive explicit formulas for the LST of the stationary distribution of the AoI in the FCFS M/GI/1 and GI/M/1 queues.

(B): For the preemptive LCFS GI/GI/1, M/GI/1, and GI/M/1 queues, we derive explicit formulas for the LST of the stationary distribution of the AoI. In addition, for the preemptive M/GI/1 and GI/M/1 queues, we derive ordering properties of the AoI in terms of the service time and the interarrival time distributions.

(C) and (D): For the non-preemptive M/GI/1 and GI/M/1 queues with and without discarding, we derive explicit formulas for the LST of the stationary distribution of the AoI.

In Appendix A we also present specialized formulas for the M/M/1, M/D/1, and D/M/1 queues.

Remark 3. Throughout this paper, we strictly distinguish between the symbols ‘G’ and ‘GI’ in Kendall’s notation: ‘GI’ represents that interarrival or service times are independent and identically distributed (i.i.d.) random variables, while ‘G’ represents that there are no restrictions on the arrival or service processes.

Taking the derivative of the LST of the AoI, we can obtain the mean and higher moments of the AoI. In all of the above models, we provide formulas for the mean AoI. We also derive formulas for higher moments of the AoI when they take simple forms. Furthermore, we obtain comparison results for the mean AoI among the four service disciplines in the M/GI/1 and GI/M/1 queues. See Table 2 which summarizes our results for the AoI in standard queueing systems.

The rest of this paper is organized as follows. In Section 2 we derive a general formula for sample-paths of the AoI, and using it, we obtain various formulas for the AoI in a general information update system. In Section 3 we consider the applications of the general formula to single-server queues operated under the FCFS, preemptive LCFS, and non-preemptive LCFS service disciplines. Furthermore, we provide some comparison results for the mean AoI among these service disciplines in the M/GI/1 and GI/M/1 queues. Finally, the conclusion of this paper is provided in Section 4.

2 Sample-Path Analysis in a General Setting

We consider a sample path of the AoI process \( \{A_t\}_{t \geq 0} \), where \( A_t \) is defined in (1). Note that the process \( \{\eta_t\}_{t \geq 0} \) of the time-stamp of the information being displayed on the monitor is a step function of \( t \), i.e., it is piece-wise constant and has discontinuous points at which information is updated. Therefore, any sample path of the AoI process \( \{A_t\}_{t \geq 0} \) is piece-wise linear with slope one and it has (downward) jumps when information is updated. In what follows, we assume that \( \{A_t\}_{t \geq 0} \) is right-continuous, i.e., \( \lim_{t \to t_0^+} A_t = A_{t_0} \).

Any sample path of the AoI process \( \{A_t\}_{t \geq 0} \) can be constructed in the following way. Let \( \beta_n \) \((n = 1, 2, \ldots)\) denote the time instant of the \( n \)th information update, and let \( X_n := A_{\beta_n} \) \((n = 1, 2, \ldots)\) denote the AoI immediately after the \( n \)th update. Also, let \( \beta_0 := 0 \) and \( X_0 := A_0 \). For simplicity, we assume \( \beta_n < \beta_n \) \((n = 1, 2, \ldots)\). In these settings, \( A_t \) is given by

\[
A_t = X_{n-1} + (t - \beta_{n-1}), \quad t \in [\beta_{n-1}, \beta_n), n = 1, 2, \ldots.
\]  

(3)

The sample-path of the AoI process \( \{A_t\}_{t \geq 0} \) is thus determined completely by the deterministic marked point process \( \{ (\beta_n, X_n) \}_{n=0,1,\ldots} \). We define \( A_{\text{peak},n} \) \((n = 1, 2, \ldots)\) as the \( n \)th peak AoI (i.e., the AoI immediately before the \( n \)th update).

\[
A_{\text{peak},n} = \lim_{t \to \beta_{n-1}^-} A_t = X_{n-1} + (\beta_n - \beta_{n-1}).
\]

Figure 1 shows an example of sample paths of the AoI process \( \{A_t\}_{t \geq 0} \).

Remark 4. As we will see, \( X_n \) equals to the system delay of a packet in a queuing system when the time-stamps of packets are equal to their arrival times. At this moment, however, we do not make any assumption on \( X_n \) other than its non-negativity, so that the general formula for the AoI distribution to be obtained is applicable to various situations, e.g., a network of queues, time-stamps with noise or uncertainty, information updates obtained by sensor fusion techniques, and so on.
Table 2: Summary of our results for the AoI in standard queueing models, where the fourth entry in Kendall’s notation indicates the capacity of the waiting room, and P and NP stand for preemptive and non-preemptive.

| Model               | Results                                                                 |
|---------------------|-------------------------------------------------------------------------|
| FCFS GI/GI/1/∞      | LST (Lemma 16); bounds for E[A] (Theorem 19)                           |
| FCFS M/GI/1/∞       | LST, E[A], and E[A^2] (Theorem 17 (i))                                 |
| FCFS GI/M/1/∞       | LST, E[A], and E[A^2] (Theorem 17 (ii))                                |
| FCFS M/M/1/∞        | LST, dist. function, E[A] and E[A^2] (Appendix A.1)                    |
| FCFS M/D/1/∞        | E[A] and E[A^2] (Appendix A.1)                                         |
| FCFS D/M/1/∞        | E[A] and E[A^2] (Appendix A.1)                                         |
| P-LCFS GI/GI/1/0    | LST and E[A] (Theorem 20); Decomposition formula (Corollary 27)       |
| P-LCFS M/GI/1/0     | LST, E[A], and E[A^2] (Corollary 28 (i)); ordering of E[A] (Corollary 31 (i)) |
| P-LCFS GI/M/1/0     | LST, E[A], and E[A^2] (Corollary 28 (ii)); ordering of dist. function (Corollary 31 (ii)) |
| P-LCFS M/M/1/0      | LST, dist. function, E[A] and E[A^2] (Appendix A.2)                    |
| P-LCFS M/D/1/0      | E[A] and E[A^2] (Appendix A.2)                                         |
| P-LCFS D/M/1/0      | E[A] and E[A^2] (Appendix A.2)                                         |
| NP-LCFS (C) M/GI/1/1| LST (Theorem 35 (i)); E[A] (Corollary 36 (i))                          |
| NP-LCFS (C) GI/M/1/1| LST (Theorem 35 (ii)); E[A] (Corollary 36 (ii))                        |
| NP-LCFS (C) M/M/1/1 | LST and E[A] (Appendix A.3)                                            |
| NP-LCFS (C) M/D/1/1 | E[A] (Appendix A.3)                                                    |
| NP-LCFS (C) D/M/1/1 | E[A] (Appendix A.3)                                                    |
| NP-LCFS (D) M/GI/1/∞| LST (Theorem 35 (iii)); E[A] (Corollary 36 (iii))                      |
| NP-LCFS (D) GI/M/1/∞| LST (Theorem 35 (iv)); E[A] (Corollary 36 (iv))                        |
| NP-LCFS (D) M/M/1/∞ | LST, E[A], and E[A^2] (Appendix A.4)                                   |
| NP-LCFS (D) M/D/1/∞ | E[A] (Appendix A.4)                                                    |
| NP-LCFS (D) D/M/1/∞ | E[A] (Appendix A.4)                                                    |
| M/GI/1              | ordering of E[A] among FCFS, P-LCFS, NP-LCFS (C), and NP-LCFS (D)  (Theorems 37 (i) and 38) |
| GI/M/1              | ordering of E[A] among FCFS, P-LCFS, NP-LCFS (C), and NP-LCFS (D)  (Theorem 37 (ii)) |
In what follows, we first show a general formula for sample paths of the AoI process \( \{A_t\}_{t \geq 0} \), which is represented as a deterministic function in terms of the deterministic marked point process \( \{ (\beta_n, X_n) \}_{n=0,1,...} \). We then discuss the AoI in a general queueing system represented as a stationary, ergodic stochastic model, assuming that the time-stamps of packets are equal to their arrival times.

Suppose that the deterministic marked point process \( \{ (\beta_n, X_n) \}_{n=0,1,...} \) is given. Let \( A^\sharp(x) \), \( A^\sharp_{\text{peak}}(x) \), and \( X^\sharp(x) \) (\( x \geq 0 \)) denote the asymptotic frequency distributions (see, e.g., [14, Section 2.6]) of \( \{A_t\}_{t \geq 0} \), \( \{A_{\text{peak},n}\}_{n=1,2,...} \), and \( \{X_n\}_{n=1,2,...} \), respectively, on the sample path:

\[
A^\sharp(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{A_t \leq x\}} dt, \quad x \geq 0,
\]

\[
A^\sharp_{\text{peak}}(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \mathbb{1}_{\{A_{\text{peak},n} \leq x\}}, \quad x \geq 0,
\]

\[
X^\sharp(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \mathbb{1}_{\{X_n \leq x\}}, \quad x \geq 0,
\]

if the limits exist. Let \( M_t \) (\( t \geq 0 \)) denote the total number of information updates by time \( t \).

\[
M_t = \sup\{n \in \{0, 1, 2, \ldots\}; \beta_n \leq t\}.
\]

The following lemma is a sample-path analogue of the elementary renewal theorem.

**Lemma 5** ([14, Lemma 1.1]). For \( \lambda^\dagger \in (0, \infty) \),

\[
\lim_{t \to \infty} \frac{M_t}{t} = \lambda^\dagger \iff \lim_{n \to \infty} \frac{\beta_n}{n} = \frac{1}{\lambda^\dagger}.
\]

To proceed further, we make the following assumptions.

**Assumption 6.** (i) \( \lim_{n \to \infty} (\beta_n/n) = 1/\lambda^\dagger \) for some \( \lambda^\dagger \in (0, \infty) \).

(ii) The limits in (4) and (5) exist for each \( x \geq 0 \).

Assumption 6 (i) implies that \( \lim_{n \to \infty} \beta_n = \infty \), so that \( A_t \) is well-defined for \( t \in [0, \infty) \).

**Lemma 7.** Under Assumption 6, the limit (6) exists for each \( x \geq 0 \) and it is given by

\[
A^\sharp(x) = \lambda^\dagger \int_0^x (X^\sharp(y) - A^\sharp_{\text{peak}}(y)) dy.
\]

**Proof.** By definition, we have for \( T > \beta_1 \),

\[
\frac{1}{T} \int_0^T \mathbb{1}_{\{A_t \leq x\}} dt = \frac{1}{T} \sum_{n=1}^{M_T-1} S_n(x) + \frac{\epsilon(x; T)}{T},
\]

where

\[
\epsilon(x; T) = \mathbb{1}_{\{A_T \leq x\}} - \lambda^\dagger (X^\sharp(x) - A^\sharp_{\text{peak}}(x)).
\]
where \( S_n(x) \) \((n = 1, 2, \ldots)\) and \( \epsilon(x; T) \) are given by

\[
S_n(x) = \int_{\beta_n}^{\beta_{n+1}} \mathbb{1}_{\{A_t \leq x\}} \, dt,
\]
\[
\epsilon(x; T) = \int_0^{\beta_1} \mathbb{1}_{\{A_t \leq x\}} \, dt + \int_{\beta_M}^T \mathbb{1}_{\{A_t \leq x\}} \, dt.
\]

Thus, to prove the theorem, it suffices to show that

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^{M_T-1} S_n(x) = \lambda^* \int_0^x (X^+(u) - A^\#_{\text{peak}}(u)) \, du,
\]
\[
\lim_{T \to \infty} \frac{\epsilon(x; T)}{T} = 0.
\]

We first consider (12). From Lemma 5 and Assumption 6 (i), we have

\[
\lim_{T \to \infty} M_T = \lambda^*.
\]

and thus

\[
\lim_{T \to \infty} \frac{\beta_{M_T}}{T} = \lim_{T \to \infty} \left( \frac{M_T}{T}, \frac{\beta_{M_T}}{M_T} \right) = \lim_{T \to \infty} \frac{M_T}{T} \cdot \lim_{N \to \infty} \frac{\beta_N}{N} = 1.
\]

It then follows from Assumption 6 (i), (10), (13), and (14) that

\[
0 \leq \lim_{T \to \infty} \frac{\epsilon(x; T)}{T} \leq \lim_{T \to \infty} \left( \frac{\beta_1}{T} + \frac{\beta_{M_T+1} - \beta_{M_T}}{T} \right) = \lim_{T \to \infty} \frac{M_T + 1}{T} \cdot \frac{\beta_{M_T+1}}{M_T + 1} - \lim_{T \to \infty} \frac{\beta_{M_T}}{T} = 0,
\]

which proves (12).

Next we consider (11). Substituting (3) into (9) yields

\[
S_n(x) = \int_{\beta_n}^{\beta_{n+1}} \mathbb{1}_{\{X_n + (t-\beta_n) \leq x\}} \, dt
\]
\[
= \int_{X_n}^{A^\#_{\text{peak}, n+1}} \mathbb{1}_{\{u \leq x\}} \, du
\]
\[
= \int_0^\infty \mathbb{1}_{\{u \leq x\}} \mathbb{1}_{\{X_n \leq u\}} \mathbb{1}_{\{A^\#_{\text{peak}, n+1} \leq u\}} \, du
\]
\[
= \int_0^x \left( \mathbb{1}_{\{X_n \leq u\}} - \mathbb{1}_{\{A^\#_{\text{peak}, n+1} \leq u\}} \right) \, du,
\]

where we used \( \mathbb{1}_{\{A^\#_{\text{peak}, n+1} \leq u\}} = \mathbb{1}_{\{X_n \leq u\}} \mathbb{1}_{\{A^\#_{\text{peak}, n+1} \leq u\}} \) in the last equality. Applying the bounded convergence theorem to (16), and using (5) and (6), we obtain

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N S_n(x) = \int_0^x (X^+(u) - A^\#_{\text{peak}}(u)) \, du.
\]

Combining this with (13) yields

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^{M_T-1} S_n(x) = \lim_{T \to \infty} \frac{M_T}{T} \cdot \frac{M_T - 1}{M_T} \cdot \frac{1}{M_T - 1} \sum_{n=1}^{M_T-1} S_n(x)
\]
\[
= \lambda^* \int_0^x (X^+(u) - A^\#_{\text{peak}}(u)) \, du,
\]

which completes the proof.

\[\square\]

Remark 8. In the proof of Lemma 7 we do not use the inequality \( X_n \leq A^\#_{\text{peak}, n} \) \((n = 1, 2, \ldots)\) that should hold in the context of the AoI.
In the rest of this paper, we focus on queueing systems represented as stationary and ergodic stochastic models, where \( \{(\beta_n, X_n)\}_{n=0,1,...} \) is represented as a random marked point process. We further restrict our attention to the case that time-stamps of packets are identical to their arrival times. To make the notation simpler, we use the following conventions. For a non-negative random variable \( Y \), let \( Y(x) (x \geq 0) \) denote the probability distribution function (PDF) of \( Y \) and let \( y^*(s) (s > 0) \) denote the \( \ell \)th moment of \( Y^\ell \):

\[
Y(x) = \Pr(Y \leq x), \quad y^*(s) = E[\exp(-sY)].
\]

Furthermore, let \( y^{(n)}(s) (s > 0, n = 1, 2, \ldots) \) denote the \( n \)th derivative of \( y^*(s) \):

\[
y^{(n)}(s) = \frac{d^n}{ds^n} y^*(s) = (-1)^n E[Y^n \exp(-sY)]. \tag{17}
\]

Note that \( Y \) has a finite \( n \)th moment if and only if \( \lim_{s \to 0+} |y^{(n)}(s)| < \infty \), so that we can write \( E[Y^n] = \lim_{s \to 0+} (-1)^n y^{(n)}(s) \) even when \( \lim_{s \to 0+} (-1)^n y^{(n)}(s) \) diverges \cite[Page 435]{21}. For simplicity, however, we hereafter assume that all random variables under consideration have finite moments whenever results on their moments are stated.

Without loss of generality, we can restrict our attention to a general FIFO (first-in first-out) queueing system, owing to the following observation. In general, arriving packets are classified into two types: informative and non-informative packets. Informative packets update information displayed on the monitor, while non-informative packets do not. If arriving packets are processed on an FCFS basis, all arriving packets are informative. On the other hand, if the order of processing packets is controllable, it might be reasonable to give priority to the newest packet, because it is expected to improve the AoI performance. In such a case, older (overtaken) packets do not update information on the monitor, so that they are regarded as non-informative. If we ignore all non-informative packets and observe only the stream of informative packets, they are processed and depart from the system in a FIFO manner.

We thus consider a general stationary, ergodic FIFO queueing system, where only informative packets are visible. For each sample path, let \( \alpha_n (n = 1, 2, \ldots) \) denote the arrival time of the \( n \)th informative packet, and let \( \beta_n (n = 1, 2, \ldots) \) denote the departure (i.e., information update) time of the \( n \)th informative packet. We assume that \( \beta_1 > 0 = \beta_0, \alpha_n \leq \alpha_{n+1}, \text{ and } \alpha_n \leq \beta_n < \beta_{n+1} (n = 1, 2, \ldots) \). Let \( G^\dagger_n (n = 1, 2, \ldots) \) denote the interarrival time of the \((n-1)\)st and the \( n \)th informative packets and let \( D_n (n = 1, 2, \ldots) \) denote the system delay of the \( n \)th informative packet:

\[
G^\dagger_n = \alpha_n - \alpha_{n-1}, \quad D_n = \beta_n - \alpha_n. \tag{18}
\]

Under the assumption that the time-stamp of the \( n \)th informative packet is equal to its arrival time \( \alpha_n \), we have

\[
X_n = D_n, \quad n = 1, 2, \ldots, \tag{19}
\]

\[
A_{peak,n} = \beta_n - \alpha_{n-1} = G^\dagger_n + D_n, \quad n = 1, 2, \ldots \tag{20}
\]

For convenience, we define \( D_0 := X_0 = A_0 \).

**Assumption 9.** (i) The mean arrival rate of informative packets is positive and finite i.e.,

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^\infty \mathbb{1}_{\{\alpha_n \leq T\}} = \lambda^\dagger \in (0, \infty), \tag{21}
\]

with probability one.

(ii) The system is stable, i.e., the mean departure rate of informative packets is equal to the mean arrival rate of informative packets. More concretely,

\[
\lim_{T \to \infty} \frac{M_T}{T} = \lambda^\dagger, \tag{22}
\]

with probability one, where \( M_t \) \((t \geq 0)\) is defined in \cite{7}.

(iii) The marked point process \( \{(\beta_n, D_n)\}_{n=0,1,\ldots} \) is stationary and ergodic.
In the rest of this paper, we refer to $\lambda^\dagger$ as the mean arrival rate of informative packets. Let $E[G^\dagger]$ denote the mean interarrival time of informative packets.

$$E[G^\dagger] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} G_n^\dagger = \lim_{N \to \infty} \frac{\alpha N}{N}.$$  

It then follows from Lemma 5 that under Assumption 9 (i) and (ii), we have

$$E[G^\dagger] = \frac{1}{\lambda^\dagger}.$$  

(23)

Recall that $A^\sharp(x)$, $A^\sharp\text{peak}(x)$, and $X^\sharp(x)$ ($x \geq 0$) denote asymptotic frequency distributions defined on a sample path (see (4), (5), (6), and (19)). Under Assumption 9 (iii), it follows from the pointwise ergodic theorem [23, Page 50] that the asymptotic frequency distributions $A^\sharp\text{peak}(x)$, $X^\sharp(x)$, and $A^\sharp(x)$ ($x \geq 0$) exist and the following equations hold with probability one:

$$A^\sharp(x) = A(x), \quad A^\sharp\text{peak}(x) = A\text{peak}(x), \quad X^\sharp(x) = D(x),$$

where $A$, $A\text{peak}$, and $D$ denote generic random variables for the stationary AoI, peak AoI, and system delay, respectively. Note that by definition, we have for any $t$ and $n$,

$$A_t \overset{d}{=} A, \quad D_n \overset{d}{=} D, \quad A\text{peak}_n \overset{d}{=} A\text{peak},$$

where $\overset{d}{=}$ stands for the equality in distribution.

Theorem 10 presented below is thus immediate from Lemma 7 and basic properties of LST.

**Theorem 10.** In the general FIFO queueing system satisfying Assumption 9,

(i) the density function $a(x)$ ($x \geq 0$) of the AoI is given by

$$a(x) = \lambda^\dagger(D(x) - A\text{peak}(x)), \quad (24)$$

(ii) the LST $a^*(s)$ ($s > 0$) of the AoI is given by

$$a^*(s) = \lambda^\dagger \cdot \frac{d^*(s) - a^*_\text{peak}(s)}{s}, \quad (25)$$

and

(iii) the $k$th ($k = 1, 2, \ldots$) moment of the AoI is given by

$$E[A^k] = \lambda^\dagger \cdot \frac{E[(A\text{peak})^{k+1}] - E[D^{k+1}]}{k+1}. \quad (26)$$

**Remark 11.** Letting $k = 1$ in (26), we obtain

$$E[A] = \lambda^\dagger \cdot \frac{E[(A\text{peak})^2] - E[D^2]}{2}. \quad (27)$$

The formula for the mean AoI in (2) is thus reproduced from (20), (23), and (27).

**Remark 12.** (20) and (23) imply

$$\lambda^\dagger = \frac{1}{E[A\text{peak}] - E[D]} \quad (28)$$

so that $a^*(s)$ in (25) trivially satisfies $\lim_{s \to 0^+} a^*(s) = 1$.

**Remark 13.** Theorem 10 can be applied to information update systems with any service disciplines: Recall that we introduced the general FIFO queueing system as a virtual system where only informative packets are visible, and the original system (with non-informative packets) is not required to be FIFO.

In the following section, we present applications of Theorem 10 to single-server queues. As shown in (17), the mean AoI can also be obtained by taking the derivative of $a^*(s)$ and letting $s \to 0^+$, which we will use repeatedly in the rest of this paper.
3 Applications to Single-Server Queues

In this section, we present analytical results for single-server queues operated under four service disciplines: FCFS, preemptive LCFS, and non-preemptive LCFS with and without discarding. We start with summarizing symbols used in this section. Throughout this section, we assume that interarrival times of packets, which are possibly informative or non-informative, are i.i.d. We also assume that service times of packets are i.i.d. Let \( G \) (resp. \( H \)) denote a generic random variable for interarrival times (resp. service times). We define \( \lambda := 1/E[G] \) as the mean arrival rate of packets, and \( \rho := \lambda E[H] \) as the traffic intensity.

Let \( G \) denote a generic random variable for residual interarrival times, i.e., the time to the next arrival from a randomly chosen time instant. Similarly, let \( H \) denote a generic random variable for residual service times. By definition, their LSTs are given by

\[
\hat{t}^*(s) = \frac{1 - h^*(s)}{sE[H]}, \quad \hat{g}^*(s) = \frac{1 - g^*(s)}{sE[G]}, \quad s > 0.
\]

Let \( G^\dagger_n, \lambda^\dagger, D_n, \) and \( A_{\text{peak,n}} \) denote the interarrival times of the \((n-1)\)st and the \(n\)th informative packets, the mean arrival rate of informative packets, the system delay of the \(n\)th informative packet, and the \(n\)th peak AoI, respectively, as defined in the preceding section. In addition, let \( H^\dagger_n \) \((n = 1, 2, \ldots)\) denote the service time of the \(n\)th informative packet. We also define \( G^\dagger, H^\dagger, D, \) and \( A_{\text{peak}} \) as generic random variables for \( G^\dagger_n, H^\dagger_n, D_n, \) and \( A_{\text{peak,n}} \), respectively. Note that \( G^\dagger \) does not follow the same distribution as \( G \) (in particular, \( \lambda^\dagger \neq \lambda \)) unless the service discipline is FCFS (see Remark \[13\]). Similarly, we will see that \( H^\dagger_n \) follows a different distribution from \( H \) under the preemptive LCFS service discipline.

To avoid trivialities, we make the following assumption in the rest of this paper.

**Assumption 14.** At least one of \( G \) and \( H \) is non-deterministic, i.e., the system is not a D/D/1 queue.

The rest of this section is organized as follows. We analyze FCFS queues in Section \[3.1\] preemptive LCFS queues in Section \[3.2\] and non-preemptive LCFS queues with and without discarding in Section \[3.3\]. We then provide comparison results among these service disciplines in Section \[3.4\].

### 3.1 Applications to FCFS Queues

In this subsection, we consider the stationary FCFS GI/GI/1, M/GI/1, and GI/M/1 queues. Throughout this subsection, we assume \( \rho < 1 \), so that the system is stable.

**Remark 15.** Under the stability condition \( \rho < 1 \), the system delay \( \{D_n\}_{n=1,2,\ldots} \) of informative packets is shown to be a regenerative process with finite mean regeneration time \[22, \text{Chapter X, Proposition 1.3}]\, where the system delay of an informative packet which finds the system empty on arrival is a regeneration point. Under Assumption \[14\], it is then readily verified that \( \{(\beta_n, D_n)\}_{n=0,1,\ldots} \) is mixing \[23, \text{Page 49}\], so that it is also ergodic. Theorem \[10\] is thus applicable to FCFS single-server queues discussed below.

Under the FCFS service discipline, all arriving packets are informative, so that

\[
G^\dagger \overset{d}{=} G, \quad \lambda^\dagger = \lambda = \frac{1}{E[G]}.
\]

In addition, we have an alternative formula for the \(n\)th peak AoI \( A_{\text{peak,n}} = \beta_n - \alpha_{n-1} \) (cf. (20)) as depicted in Figure 2

\[
A_{\text{peak,n}} = \begin{cases} 
G^\dagger_n + H^\dagger_n, & G^\dagger_n > D_{n-1}, \\
D_{n-1} + H^\dagger_n, & G^\dagger_n \leq D_{n-1}.
\end{cases}
\]

(30)

(31)

Note here that \( G^\dagger_n \) is independent of \( D_{n-1} \). From (29), (31), and Theorem 10, we then obtain the following result.

**Lemma 16.** In the FCFS GI/GI/1 queue, the LST \( a^*(s) \) \((s > 0)\) of the AoI is given in terms of the system delay distribution \( D(x) \) \((x \geq 0)\) by

\[
a^*(s) = \frac{d^*(s) - a_{\text{peak}}^*(s)}{sE[G]},
\]

(31)

(32)
Figure 2: The relation of the peak AoI $A_{\text{peak},n}$, the system delay $D_{n-1}$, the interarrival time $G_n^\dagger$, and the service time $H_n^\dagger$ in the FCFS GI/GI/1 queue.

where

$$a_{\text{peak}}^*(s) = \left[ \int_0^\infty e^{-sx}G(x)dD(x) + \int_0^\infty e^{-sx}D(x)dG(x) - E[1_{(G>D)}e^{-sG}] \right]h^*(s).$$

For the FCFS M/GI/1 and GI/M/1 queues, expressions for the system delay distribution are well-known:

(i) In the FCFS M/GI/1 queue [17, Page 199],

$$d^*(s) = \frac{(1 - \rho)s}{s - \lambda + \lambda h^*(s)} \cdot h^*(s). \quad (32)$$

(ii) In the FCFS GI/M/1 queue [17, Page 252],

$$d^*(s) = \frac{\mu - \mu\gamma}{s + \mu - \mu\gamma}, \quad (33)$$

where $\gamma$ denotes the unique solution of

$$x = g^*(\mu - \mu x), \quad 0 < x < 1. \quad (34)$$

Therefore, we can obtain the LST $a^*(s)$ of $A$ by specializing Lemma [16] with these equations, and taking the derivatives of $a^*(s)$, we obtain moments of the AoI in these special cases.

**Theorem 17.** (i) In the FCFS M/GI/1 queue,

$$a^*(s) = \rho d^*(s)h^*(s) + d^*(s + \lambda) \cdot \frac{\lambda}{s + \lambda} \cdot h^*(s)$$
In Figure 3(a), the mean $E[A]$ and the standard deviation $SD[A]$ of the AoI in the FCFS D/M/1 queue with $E[H] = 1$ are plotted as functions of $\rho$. From this figure, we observe that $E[A]$ and $SD[A]$ take the minimum values at different values of $\rho$: $E[A]$ is minimal at $\rho = \rho^* \approx 0.516885$, while $SD[A]$ is minimal at $\rho \approx 0.408982$.

Figure 3(b) shows the parametric curve of $E[A]$ and $SD[A]$ with parameter $\rho$. We observe that given the same mean AoI $E[A]$, the smaller standard deviation $SD[A]$ of the AoI is achieved by the smaller traffic intensity, which implies that the fewer arrival rate is more effective than the excessive arrival rate.

Before closing this subsection, we derive upper and lower bounds of the mean AoI $E[A]$ in the general FCFS GI/GI/1 queue. We rewrite (2) to be

$$E[A] = E[D] + \frac{1 + (Cv[G])^2}{2} \cdot E[G] + \frac{Cov[G_n, D_n]}{E[G]},$$

where $Cv[\cdot]$ denotes the coefficient of variation (the standard deviation divided by the mean) and $Cov[G_n, D_n]$ denotes the covariance of $G_n$ and $D_n$. Note that $Cov[G_n, D_n]$ does not depend on $n$ because of the stationarity of the system.
Lemma 18. In the FCFS GI/GI/1 queue, \( \text{Cov}[G_n, D_n] \) is bounded as follows.

\[
-\mathbb{E}[G]\mathbb{E}[D] \Pr(G < \mathbb{E}[G]) \leq \text{Cov}[G_n, D_n] \leq 0.
\]

The proof of Lemma 18 is provided in Appendix B.

Theorem 19. In the FCFS GI/GI/1 queue, the mean AoI is bounded as follows.

\[
\begin{align*}
\mathbb{E}[A] &\geq \mathbb{E}[D] \Pr(G \geq \mathbb{E}[G]) + \frac{1 + (\text{Cv}[G])^2}{2} \cdot \mathbb{E}[G], \\
\mathbb{E}[A] &\leq \mathbb{E}[D] + \frac{1 + (\text{Cv}[G])^2}{2} \cdot \mathbb{E}[G].
\end{align*}
\]

(41) \hspace{1cm} (42)

Proof. Theorem 19 follows from (40) and Lemma 18.

Remark 20. The bounds in Theorem 19 are tight in the sense that both equalities hold in the D/GI/1 queue.

Remark 21. It is known that the mean delay \( \mathbb{E}[D] \) in the FCFS GI/GI/1 queue is bounded by \([15, 16]\)

\[
\begin{align*}
\mathbb{E}[D] &\geq \mathbb{E}[H] + \mathbb{E}\left[\max(0, H - G)\right], \\
\mathbb{E}[D] &\leq \mathbb{E}[H] + \frac{\mathbb{E}[G]}{2(1 - \rho)} \left(\rho(2 - \rho)(\text{Cv}[G])^2 + \rho^2(\text{Cv}[H])^2\right).
\end{align*}
\]

(43)

Bounding \( \mathbb{E}[D] \) in (41) and (42) by these inequalities, we can obtain a bound for \( \mathbb{E}[A] \) in terms of only the distributions of \( G \) and \( H \).

Because (20) implies

\[
\mathbb{E}[A_{\text{peak}}] = \mathbb{E}[D] + \mathbb{E}[G],
\]

the following corollary is immediate from (42).

Corollary 22. In the FCFS GI/GI/1 queue, if \( \text{Cv}[G] \leq 1 \), then \( \mathbb{E}[A] \leq \mathbb{E}[A_{\text{peak}}] \).

Remark 23. \( \mathbb{E}[A] \leq \mathbb{E}[A_{\text{peak}}] \) does not hold in general, which might sound counterintuitive. A simple counter example is the case that \( \mathbb{E}[H] = 0 \) and \( \text{Cv}[G] > 1 \). Because \( \mathbb{E}[H] = 0 \) leads to \( \mathbb{E}[D] = 0 \), we have \( \text{Cov}[G_n, D_n] = 0 \). The inequality \( \mathbb{E}[A] > \mathbb{E}[A_{\text{peak}}] \) then follows from (40), (44), and \( \text{Cv}[G] > 1 \).

Furthermore, we can derive a sufficient condition for \( \mathbb{E}[A] > \mathbb{E}[A_{\text{peak}}] \) in the stationary FCFS GI/M/1 queue. It follows from (38), (44), and \( \text{Cv}[H] = 1 \) that

\[
\mathbb{E}[A_{\text{peak}}] \leq \mathbb{E}[H] + \frac{\mathbb{E}[G]}{2(1 - \rho)} \left[\rho(2 - \rho)(\text{Cv}[G])^2 + \rho^2(\text{Cv}[H])^2\right] + \mathbb{E}[G].
\]

On the other hand, from (38) and \(-g(1)(\mu - \mu') > 0\), we have

\[
\mathbb{E}[A] \geq \frac{\mathbb{E}[G]}{2} \left((\text{Cv}[G])^2 + 1\right) + \mathbb{E}[H].
\]

Therefore,

\[
\mathbb{E}[A] - \mathbb{E}[A_{\text{peak}}] \geq \frac{\mathbb{E}[G]}{2(1 - \rho)} \left[\left(1 - 3\rho + \rho^2\right)(\text{Cv}[G])^2 - (1 - \rho + \rho^2)\right] - \mathbb{E}[A].
\]

Note here that \( 1 - 3\rho + \rho^2 > 0 \) if \( \rho < (3 - \sqrt{5})/2 \). We thus conclude that in the stationary FCFS GI/M/1 queue,

\[
\rho \in (0, (3 - \sqrt{5})/2) \text{ and } (\text{Cv}[G])^2 \geq 1 + \frac{2\rho}{1 - 3\rho + \rho^2} \Rightarrow \mathbb{E}[A] > \mathbb{E}[A_{\text{peak}}].
\]
3.2 Applications to preemptive LCFS queues

In this subsection, we consider the AoI in the stationary preemptive LCFS GI/GI/1, M/GI/1, and GI/M/1 queues. As stated in Section 2, even in LCFS systems, informative packets arrive and depart in a FIFO manner, so that Theorem 10 is applicable under Assumption 9. Recall that Assumption 9 is posed only on informative packets, and therefore the results in this subsection may hold even if \( \rho \geq 1 \).

We first consider a GI/GI/1 queue. We define \( \zeta = \Pr(G < H) \).

Note that if \( \zeta = 1 \), every service is interrupted by the next arrival with probability one, so that the AoI goes to infinity as time passes. On the other hand, if \( \zeta = 0 \), all packets are informative, i.e., \( d^*(s) = h^*(s) \) and \( a^*_{\text{peak}}(s) = g^*(s)h^*(s) \). To avoid trivialities and for simplicity, we assume the following:

**Assumption 24.** \( \Pr(H = G) = 0 \) and \( 0 < \zeta < 1 \).

**Remark 25.** We can verify that under Assumption 24, the system delay of informative packets \( \{D_n\}_{n=1,2,...} \) is a regenerative process with finite mean regeneration time, where the system delay of an informative packet before its service completion. We thus have

\[
\lim_{n \to \infty} E[D_n] = H < \infty.
\]

Under the preemptive LCFS service discipline, a packet becomes non-informative if the next packet arrives before its service completion. We have

\[
D_n = H_n^+ \overset{d}{=} H_n < G_n,
\]

where \( H_n < G_n \) denotes a generic conditional random variable for a service time given that it is smaller than interarrival time.

Note that the \( m \)th arriving packet after time \( \beta_n \) becomes the \( (n+1) \)st informative packet with probability \( \zeta_{m-1}(1-\zeta) \), and the peak AoI in this case is given by

\[
A_{\text{peak},n+1} \overset{d}{=} G^*_{>H} + \sum_{i=1}^{m-1} G^*_{<H} + H_{<G},
\]

where \( G^*_{>H} \) (resp. \( G^*_{<H} \)) denotes a generic conditional random variable for an interarrival time given that it is greater (resp. smaller) than a service time. We can verify that \( G^*_{>H} \) represents the interarrival time of the first packet arriving after the departure of the \( n \)th informative packet, \( G^*_{<H} \) represents the interarrival time of the \( n \)th non-informative packet and the next packet, and \( H_{<G} \) represents the service time of the \( (n+1) \)st informative packet. We define \( G_{<G} := G^*_{<H} \). Note that the LSTs of \( H_{<G}, G_{<H} \), and \( G_{>H} \) are given by

\[
h^*_{<G}(s) = \frac{1}{1-\zeta} \int_0^\infty e^{-sx}(1-G(x))dG(x),
\]

\[
g^*_{<H}(s) = \frac{1}{\zeta} \int_0^\infty e^{-sx}(1-H(x))dG(x),
\]

\[
g^*_{>H}(s) = \frac{1}{1-\zeta} \int_0^\infty e^{-sx}H(x)dG(x).
\]

By definition, we have

\[
g^*(s) = \zeta g^*_{>H}(s) + (1-\zeta)g^*_{<H}(s).
\]

We can obtain the following result from Theorem 10 noting that \( m+1 \) random variables on the right-hand side of (46) are mutually independent in the GI/GI/1 queue (see Appendix C for more details).

**Theorem 26.** In the preemptive LCFS GI/GI/1 queue,

\[
a^*(s) = h^*_{<G}(s) \cdot g^*(s) \cdot \frac{1-\zeta}{1-\zeta g^*_{<H}(s)},
\]

\[
E[A] = E[H_{<G}] + \frac{E[G^2]}{2E[G]} + \frac{\zeta}{1-\zeta}E[G_{<H}].
\]
Corollary 27. In the preemptive LCFS GI/GI/1 queue, the AoI is decomposed stochastically into three independent factors:

\[ A \overset{d}{=} H \_G + \bar{G} + Z, \]  

(49)

\( \bar{G} \) denotes a generic random variable for residual interarrival times, and \( Z \) denotes a random variable for an interval from the arrival of a randomly chosen packet to the arrival of the next informative packet, whose LST is given by

\[ z^*(s) = \frac{1 - \zeta}{1 - \zeta g_{<H}(s)}. \]

Proof. (49) immediately follows from Theorem 26. The probabilistic interpretation of \( Z \) comes from

\[ \frac{1 - \zeta}{1 - \zeta g_{<H}(s)} = \sum_{m=1}^{\infty} \zeta^{m-1}(1 - \zeta) \cdot (g_{<H}(s))^{m-1}. \]

Below, we consider two special cases: the preemptive LCFS M/GI/1 and GI/M/1 queues. Note that (47) in these cases takes simple forms, and the moments of the AoI are readily obtained.

Corollary 28. (i) In the preemptive LCFS M/GI/1 queue,

\[ a^*(s) = \frac{\lambda h^*(s + \lambda)}{s + \lambda h^*(s + \lambda)}, \]

(50)

\[ E[A] = \frac{E[H]}{\rho h^*(\lambda)}, \]

(51)

\[ E[A^2] = 2\left[1 - \lambda h^{(1)}(\lambda)\right] \left(\frac{E[H]}{\rho h^*(\lambda)}\right)^2. \]

(52)

(ii) In the preemptive LCFS GI/M/1 queue,

\[ a^*(s) = \bar{g}(s) \cdot \frac{\mu}{s + \mu}, \]

(53)

\[ E[A^n] = \sum_{m=0}^{n} \frac{n!E[G^{m+1}]}{(m+1)!E[G]} E[H]^{n-m}, \quad n = 1, 2, \ldots, \]

and in particular,

\[ E[A] = \frac{E[G^2]}{2E[G]} + E[H], \]

(54)

\[ E[A^2] = \frac{E[G^3]}{3E[G]} + 2E[H]E[A]. \]

Remark 29. (51) is identical to Eq. (15) in [7].

From Corollary 28, we can discuss the effect of the variability of service and interarrival times on the AoI. To this end, we introduce the convex order of random variables.

Definition 30 ([18, Page 109]). Consider two random variables \( X \) and \( Y \) with the same mean \( E[X] = E[Y] \). \( X \) is said to be smaller than or equal to \( Y \) in the convex order (denoted by \( X \leq_{cx} Y \)) if and only if

\[ E[\phi(X)] \leq E[\phi(Y)], \]

for all convex functions \( \phi \), provided the expectations exist.

By definition, the convex order is a partial order over the set of all real-valued random variables. Roughly speaking, \( X \leq_{cx} Y \) implies that \( Y \) is more variable than \( X \) [18]. In particular, it is readily verified that

\[ X \leq_{cx} Y \Rightarrow CV[X] \leq CV[Y]. \]

(55)

Consider two preemptive LCFS queues with the same mean interarrival time \( E[G] \) and the same mean service time \( E[H] \). Let \( G^{(k)} \), \( H^{(k)} \), and \( A^{(k)} \) \( (k = 1, 2) \) denote generic random variables for interarrival times, service times and the AoI in the \( k \)th queue.
Corollary 31.  (i) For two preemptive LCFS M/GI/1 queues,
\[
H^{(1)} \leq_{cx} H^{(2)} \Rightarrow E[A^{(1)}] \geq E[A^{(2)}].
\]  \tag{56}

(ii) For two preemptive LCFS GI/M/1 queues,
\[
G^{(1)} \leq_{cx} G^{(2)} \Rightarrow A^{(1)} \leq_{st} A^{(2)},
\]  \tag{57}
where \( \leq_{st} \) stands for the usual stochastic order \cite[Page 4]{Page_4}, i.e., \( A^{(1)} \leq_{st} A^{(2)} \Leftrightarrow E[\phi(A^{(1)})] \leq E[\phi(A^{(2)})] \) for all non-decreasing functions \( \phi \), provided that the expectations exist.

Remark 32. Clearly, (57) implies
\[
G^{(1)} \leq_{cx} G^{(2)} \Rightarrow E[A^{(1)}] \leq E[A^{(2)}].
\]

Proof. We have (56) from (51) because \( H \) is independent of \( G^{(k)} \) \( (k = 1, 2) \), it follows from \cite[Theorem 1.A.3, Eq. (3.A.7)]{Page_4} that
\[
G^{(1)} \leq_{cx} G^{(2)} \Rightarrow \tilde{G}^{(1)} \leq_{st} \tilde{G}^{(2)}
\]
\[
\Rightarrow \tilde{G}^{(1)} + H \leq_{st} \tilde{G}^{(2)} + H.
\]

Therefore, we obtain (57) from (58).

Corollary 31 (i) shows that the preemptive LCFS service discipline is particularly effective in terms of the mean AoI when service times are highly variable, given that packets arrive according to a Poisson process. On the other hand, Corollary 31 (ii) shows that lowering the variability of interarrival times reduces the AoI, when the service time distribution is exponential.

Note that Jensen’s inequality leads to \( E[X] \leq_{cx} X \) for any random variable \( X \), i.e., the deterministic distribution achieves the minimum in the convex order \cite[Theorem 3.A.24]{Page_4}. Therefore, the M/D/1 queue gives the maximum (i.e., the worst) mean AoI among all preemptive LCFS M/GI/1 queues with the same mean interarrival time \( E[G] \) and the same mean service time \( E[H] \). On the other hand, the D/M/1 queue gives the minimum (i.e., the best) mean AoI among all preemptive LCFS GI/M/1 queues with the same mean interarrival time \( E[G] \) and the same mean service time \( E[H] \).

We now present some numerical examples, where we fix \( E[H] = 1 \) and consider the following probability distributions for service times:

- Deterministic distribution \( \text{Cv}[H] = 0 \).
- Gamma distribution \( 0 < \text{Cv}[H] < 1 \).
- Exponential distribution \( \text{Cv}[H] = 1 \).
- Hyper-exponential distribution of order two with balanced means \cite[Page 359]{Page_359} \( \text{Cv}[H] > 1 \).

Note that this setting enables us to specify the service time distribution uniquely once we do \( \text{Cv}[H] \).

In Figure 1, the mean AoI \( E[A] \) in the preemptive LCFS M/GI/1 queue, given by (51), is plotted as a function of \( \rho \) for various values of \( \text{Cv}[H] \). This figure confirms Corollary 31 (i), and we observe that the variability of the service times has a significant impact on the mean AoI.

Next, using (58), we present Figures 2 and 3 for \( E[A] \) in the D/GI/1 queue, where \( \text{Cv}[H] \geq 1 \) in Figure 2 and \( 0 \leq \text{Cv}[H] \leq 1 \) in Figure 3. When \( \text{Cv}[H] \geq 1 \), \( E[A] \) decreases with an increase in \( \text{Cv}[H] \), as in the case of Poisson arrivals in Figure 1. When \( \text{Cv}[H] \leq 1 \), on the other hand, the mean AoI \( E[A] \) for \( \rho < 1 \) increases with an increase in the variability of service times.

Therefore, the preemptive LCFS service discipline is particularly effective when service times are highly variable. In addition, if arrival times of packets are deterministic, this service discipline is also effective for less variable service times. Finally, we observe from Figures 1-3 that for gamma service time distributions with \( \text{Cv}[H] < 1 \), the mean AoI \( E[A] \) diverges to infinity as the arrival rate goes to infinity, which is not the case for exponential and hyper-exponential service time distributions. These phenomena are in fact not dependent on the details of the service time distributions, but hold in general as stated in the following theorem.
Theorem 33. Consider the M/GI/1 and D/GI/1 queues with an absolutely continuous service time distribution. If the probability density function $h(x)$ ($x \geq 0$) of service times is bounded and continuous, we have for both of the M/GI/1 and D/GI/1 queues,

$$E[A] \to \frac{1}{h(0)} \quad (\lambda \to \infty), \quad (59)$$

in the sense that the limit value of $E[A]$ is finite if $h(0) \neq 0$, and otherwise $E[A]$ diverges to infinity as $\lambda \to \infty$.

The proof of Theorem 33 is provided in Appendix E.

3.3 Applications to non-preemptive LCFS queues with and without discarding

In this subsection, we apply Theorem 10 to the stationary non-preemptive LCFS M/GI/1 and GI/M/1 queues with and without discarding. In non-preemptive LCFS queues with discarding, non-informative packets are discarded without receiving their services. In non-preemptive LCFS queues without discarding, on the other hand, all packets are served eventually, whether they are informative or not.

Similarly to preemptive LCFS systems, Theorem 10 is applicable under Assumption 9, so that the results in this subsection may hold even if $\rho < 1$. For systems without discarding, however, we focus on the case of $\rho < 1$ because otherwise the mean number of waiting non-informative packets goes to infinity as time passes.

Remark 34. Because the non-preemptive LCFS service disciplines are work-conserving, the workload process is identical to that of the FCFS case. Similarly to Remark 15, we can thus show that under the stability condition $\rho < 1$, the system delay $\{D_n\}_{n=1,2,...}$ of informative packets is a regenerative process with finite mean regeneration time, where the system delay of an informative packet which finds the system empty on arrival is a regeneration point. Therefore, under Assumption 14, $\{(\beta_n, D_n)\}_{n=0,1,...}$ is mixing and ergodic (cf. Remark 15). Theorem 10 is thus applicable to the non-preemptive single-server queues discussed below.

Under the non-preemptive LCFS service disciplines, the waiting time $W_n$ ($n = 1, 2, \ldots$) of the $n$th informative packet is independent of its service time $H_n$. Note that the system delay $D_n$ of this packet is given by

$$D_n = W_n + H_n^\dagger. \quad (60)$$

Furthermore, on the service initiation of an informative packet, there exists no waiting packet which can be informative: no packets are waiting in the discarding case, while all waiting packets are non-informative in the non-discarding case. We thus always have $W_n < G_{n+1}^\dagger$. Furthermore, if $W_n + H_n^\dagger > G_{n+1}^\dagger$, i.e., at least one packet arrives in the service time of the $n$th informative packet, the service of the $(n+1)$st informative packet starts immediately after the service completion. If $W_n + H_n^\dagger \leq G_{n+1}^\dagger$, on the other hand, it takes $G_{n+1}^\dagger + W_{n+1}$ before the service initiation of the $(n+1)$st informative packet, where $G_{n+1} = G_{n+1}^\dagger - W_n - H_n^\dagger$.

With these observations, we obtain (cf. $\lambda$)

$$A_{\text{peak},n+1} = \begin{cases} W_n + H_n^\dagger + H_{n+1}^\dagger, & W_n + H_n^\dagger > G_{n+1}^\dagger, \\ W_n + H_n^\dagger + \hat{G}_{n+1}^\dagger + W_{n+1} + H_{n+1}^\dagger, & W_n + H_n^\dagger \leq G_{n+1}^\dagger \\ W_{n+1} + W_{n+1} + H_{n+1}^\dagger, & W_n + H_n^\dagger > G_{n+1}^\dagger, \\ G_{n+1}^\dagger + W_{n+1} + H_{n+1}^\dagger, & W_n + H_n^\dagger \leq G_{n+1}^\dagger. \end{cases} \quad (61)$$

Note that in non-preemptive LCFS queues with discarding,

$$W_n + H_n^\dagger \leq G_{n+1}^\dagger \Rightarrow W_{n+1} = 0, \quad (63)$$

so that $\lambda$ and $\lambda$ are simplified.

We can characterize the distributions of the stationary system delay $D$ and peak AoI $A_{\text{peak}}$ based on $\lambda$, $\lambda$, and $\lambda$, so that the LST $a^*(s)$ of the AoI is obtained from Theorem 10 (see Appendix D for the details).

Theorem 35. (i) In the non-preemptive LCFS M/GI/1 queue with discarding,

$$a^*(s) = \left(h^*(\lambda) + \tilde{h}^*(s + \lambda)\right) h^*(s) \cdot \frac{\rho \tilde{h}^*(s) + h^*(s + \lambda)}{\rho + h^*(\lambda)} \cdot \frac{\lambda}{s + \lambda}. \quad (64)$$
Figure 4: $E[A]$ in the preemptive LCFS $M/GI/1$ queue.

Figure 5: $E[A]$ in the preemptive LCFS $D/GI/1$ queue ($Cv[H] \geq 1$).

Figure 6: $E[A]$ in the preemptive LCFS $D/GI/1$ queue ($Cv[H] \leq 1$).
(ii) In the non-preemptive LCFS GI/M/1 queue with discarding,

\[ a^*(s) = \left[ \tilde{g}^*(s) + \rho \cdot \frac{\mu}{s + \mu} \left( g^*(s + \mu) - g^*(\mu) \cdot \frac{1 - \mu(-g^{(1)}(s + \mu))}{1 - \mu(-g^{(1)}(\mu))} \right) \right] \frac{\mu}{s + \mu}. \]  

(iii) In the non-preemptive LCFS M/GI/1 queue without discarding which satisfies \( \rho < 1 \),

\[ a^*(s) = \frac{\lambda}{s + \lambda} \cdot h^*(s) \left[ \rho h^*(s) + \frac{(1 - \rho)(s + \lambda)(1 - h^*(s) + h^*(s + \lambda))}{s + \lambda h^*(s + \lambda)} \right]. \]  

(iv) In the non-preemptive LCFS GI/M/1 queue without discarding which satisfies \( \rho < 1 \),

\[ a^*(s) = \left[ \tilde{g}^*(s) + \rho(g^*(s + \mu - \mu\gamma) - \gamma) \right] \frac{\mu}{s + \mu}. \]  

where \( \gamma \) denotes the unique solution of (64).

Taking the derivatives of \( a^*(s) \), we can obtain the moments of AoI. We provide only the mean AoI below, because formulas for higher moments are messy.

**Corollary 36.**

(i) In the non-preemptive LCFS M/GI/1 queue with discarding,

\[ E[A] = \frac{1}{\rho + h^*(\lambda)} \left( \frac{\lambda E[H^2]}{2} + \frac{h^*(\lambda)}{\lambda} + (-h^{(1)}(\lambda)) \right) + \frac{1}{\lambda} \frac{h^*(\lambda)}{\lambda} - (-h^{(1)}(\lambda)) + E[H]. \]  

(ii) In the non-preemptive LCFS GI/M/1 queue with discarding,

\[ E[A] = E[H] + \frac{E[G^2]}{2E[G]} + \rho \left( (-g^{(1)}(\mu)) + \frac{\mu g^*(\mu)g^{(2)}(\mu)}{1 - \mu(-g^{(1)}(\mu))} \right). \]  

(iii) In the non-preemptive LCFS M/GI/1 queue without discarding which satisfies \( \rho < 1 \),

\[ E[A] = \frac{\lambda E[H^2]}{2} + \left( \frac{(1 - \rho)^2}{\rho h^*(\lambda)} + 2 \right) E[H]. \]  

(iv) In the non-preemptive LCFS GI/M/1 queue without discarding which satisfies \( \rho < 1 \),

\[ E[A] = E[H] + \frac{E[G^2]}{2E[G]} + \rho(-g^{(1)}(\mu - \mu\gamma)). \]  

As mentioned in Section 1, we can evaluate the logging overhead by comparing the non-preemptive LCFS queues with and without discarding. In Figure 7 the mean AoI \( E[A] \) is plotted as a function of \( \rho \) for the non-preemptive LCFS M/M/1 and D/M/1 queues with and without discarding, where \( E[H] = 1 \). We observe that the logging overhead is relatively large in the M/M/1 queue, while in the D/M/1 queue, the minimum values of \( E[A] \) have little difference between the cases with and without discarding non-informative packets. Therefore, for the deterministic arrival case, one can collect all sampled data with a small effect on the mean AoI \( E[A] \), using the non-preemptive LCFS service discipline without discarding.

### 3.4 Mean AoI comparison among four service disciplines

In this subsection, we compare the mean AoI in the stationary M/GI/1 and GI/M/1 queues with four service disciplines. Let \( E[A_{\text{FCFS}}] \), \( E[A_{\text{LCFS}}] \), \( E[A_{\text{NP-M/W}}] \), and \( E[A_{\text{NP-LCFS}/O}] \) denote the mean AoI in the FCFS queue, preemptive LCFS queue, non-preemptive LCFS queue with discarding, and non-preemptive LCFS queue without discarding, respectively.

For the M/GI/1 and GI/M/1 queues, we obtain the following relations, whose proofs are given in Appendix.
Theorem 37. (i) In the stationary M/GI/1 queue, we have
\[ E[A_{NP-W}^{\text{LCFS}}] \leq E[A_{NP-W/O}^{\text{LCFS}}] \leq E[A_{FCFS}]. \]

(ii) In the stationary GI/M/1 queue, we have
\[ E[A_{P}^{\text{LCFS}}] \leq E[A_{NP-W}^{\text{LCFS}}] \leq E[A_{NP-W/O}^{\text{LCFS}}] \leq E[A_{FCFS}]. \]

Theorem 37 (ii) shows that for exponential service times, \( E[A_{P}^{\text{LCFS}}] \) is the smallest among four service disciplines, which is almost obvious from the memoryless property of the exponential distribution. Readers are referred to [20] for a detailed discussion on the optimality of the preemptive LCFS discipline in queues with exponential service times.

Figures 8 and 9 show the mean AoI in the M/M/1 and M/D/1 queues with \( E[H] = 1 \) as a function of the traffic intensity \( \rho \). We observe that when the traffic intensity \( \rho \) is large, \( E[A_{FCFS}] \) increases with \( \rho \) because it diverges as \( \rho \to 1 \) (cf. (36)). On the other hand, \( E[A_{NP-W/O}^{\text{LCFS}}] \) takes a moderate value for large \( \rho \), even though the system delay of non-informative packets diverges as \( \rho \to 1 \).

As discussed in Section 3.2, the mean AoI in the preemptive M/GI/1 queue is influenced strongly by the service time distribution. Particularly, in the M/D/1 queue, using the results in Appendix A, we can analytically show that \( E[A_{P}^{\text{LCFS}}] \geq E[A_{NP-W/O}^{\text{LCFS}}] \) for all \( \rho \in (0,1) \) and
\[ E[A_{P}^{\text{LCFS}}] \geq E[A_{FCFS}], \quad \rho \in (0,\hat{\rho}^*], \]
where \( \hat{\rho}^* \approx 0.643798 \) denotes the unique positive solution of \( 2(1-\rho)\exp(\rho) + \rho - 2 = 0 \).

We close this section with the following theorem whose proof is given in Appendix G.

Theorem 38. In the stationary M/GI/1 queue, we have
\[ \rho \in (0, 2 - \sqrt{2}) \text{ and } (Cv[H])^2 \leq v(\rho) \]
\[ \Rightarrow E[A_{P}^{\text{LCFS}}] \geq E[A_{FCFS}], \tag{72} \]
where \( v(\rho) \) is a decreasing function of \( \rho \), which is given by
\[ v(\rho) = \frac{1}{\rho^2} \left( \sqrt{1 + \rho^2((2 - \rho)^2 - 2)} - 1\right) < v(0^+) = 1. \]

Figure 10 depicts the region (72) of \( (\rho, Cv[H]) \) for \( E[A_{P}^{\text{LCFS}}] \geq E[A_{FCFS}] \).
4 Conclusion

We presented a general formula for the stationary distribution of the AoI, which holds for a wide class of information update systems. The formula shows that the stationary distribution of the AoI is given in terms of the stationary distributions of the system delay and the peak AoI. Therefore it provides a unified, efficient approach to the analysis of the AoI because the system delay and the peak AoI can be analyzed by standard techniques in queueing theory. To demonstrate this fact, we analyzed the stationary distributions of the AoI in single-server queues under four different service disciplines: first-come first-served (FCFS), preemptive last-come first-served (LCFS), and non-preemptive LCFS with and without discarding. Moreover, we showed comparison results for the mean AoI in the M/GI/1 and GI/M/1 queues under these service disciplines. The results in this paper will form a basis for the analysis of the AoI in developing sophisticated information update systems.

A Summary of Results for Special Cases

In this appendix, we summarize some simplified formulas for three special cases: The M/M/1, M/D/1, and D/M/1 queues. Note that formulas for the M/M/1 queue can be obtained from those in either the M/GI/1 or
GI/M/1 queues, and formulas for the M/D/1 (resp. D/M/1) queue can be obtained from those for the M/GI/1 (resp. GI/M/1) queue.

A.1 FCFS Queues

The FCFS M/M/1 queue

Consider the stationary FCFS M/M/1 queue. From (35), we have

\[ a^*(s) = \frac{(1 - \rho)\mu}{s + (1 - \rho)\mu} - \frac{(1 - \rho)(s + \lambda + \mu)}{(s + \mu)^2(s + \lambda)}, \]

and therefore,

\[ A(x) = 1 - e^{-(1 - \rho)x} + \left( \frac{1}{1 - \rho} + \rho px \right) e^{-\rho x} - \frac{1}{1 - \rho} \cdot e^{-\lambda x}. \]

Because \( E[H^2] = 2(E[H])^2 \) holds in this model, (36) and (37) are also reduced to

\[
E[A] = \left( \frac{1}{1 - \rho} + \frac{1}{\rho} - \rho \right) E[H],
\]

\[
E[A^2] = 2 \left( \frac{1}{(1 - \rho)^2} - 2\rho + \frac{1}{\rho} + \frac{1}{\rho^2} \right) (E[H])^2.
\]

Note that (73) is identical to Eq. (17) in [2].

The FCFS M/D/1 queue

Consider the stationary FCFS M/D/1 queue. From (36) and (37), we have

\[
E[A] = \left( \frac{1}{2(1 - \rho)} + \frac{1}{2} + \frac{(1 - \rho)\exp(\rho)}{\rho} \right) E[H],
\]

\[
E[A^2] = \left( \frac{1}{2(1 - \rho)^2} + \frac{1}{3(1 - \rho)} + \frac{1}{6} + \frac{2(1 - \rho)\exp(2\rho)}{\rho^2} \right) (E[H])^2.
\]

Although the FCFS M/D/1 queue is considered in [2], no explicit formula for \( E[A] \) is provided there. With the explicit formula (74), one can deduce that \( E[A] \) is minimized when \( \rho \approx 0.6291 \).

In addition, with (73) and (74), it is easy to verify that \( E[A] \) in the FCFS M/D/1 queue is strictly smaller than that in the FCFS M/M/1 queue with the same \( E[H] \) and \( \rho \) (0 < \( \rho < 1 \)).
The FCFS D/M/1 queue

Consider the stationary FCFS D/M/1 queue. From (38) and (39), we have

\[ E[A] = \left( \frac{1}{2\rho} + \frac{1}{1-\gamma} \right) E[H], \]
\[ E[A^2] = \left( 2 \left( \frac{1}{1-\gamma} \right)^2 + \frac{1}{(1-\gamma)\rho} + \frac{1}{3\rho^2} \right) (E[H])^2. \] (75)

We note that (75) is identical to Eq. (25) in [2].

A.2 Preemptive LCFS queues

The preemptive LCFS M/M/1 queue

Consider the preemptive LCFS M/M/1 queue. In this case, (50) is reduced to

\[ a^*(s) = \frac{\lambda}{s+\lambda} \cdot \frac{\mu}{s+\mu}, \]

so that the AoI is given by the sum of an interarrival time and a service time, which are independent. We thus have if \( \lambda \neq \mu \),

\[ \Pr(A \leq x) = 1 - \mu \cdot e^{-\lambda x} + \frac{\lambda}{\mu-\lambda} \cdot e^{-\mu x}, \quad x \geq 0, \]

and if \( \lambda = \mu \), the AoI \( A \) follows an Erlang distribution of the second order.

\[ \Pr(A \leq x) = 1 - e^{\lambda x} - \lambda xe^{-\lambda x}, \quad x \geq 0. \]

Furthermore,

\[ E[A] = \left( 1 + \frac{1}{\rho} \right) E[H], \]
\[ E[A^2] = 2 \left( 1 + \frac{1}{\rho} + \frac{1}{\rho^2} \right) (E[H])^2. \] (76)

Note that (76) is identical to Eq. (48) in [9].

The preemptive LCFS M/D/1 queue

Consider the preemptive LCFS M/D/1 queue. From Corollary 28, we have

\[ E[A] = \frac{\exp(\rho)}{\rho} \cdot E[H], \]
\[ E[A^2] = 2 \left( \frac{\exp(\rho)}{\rho} - 1 \right) \frac{\exp(\rho)}{\rho} (E[H])^2. \] (77)

The preemptive LCFS D/M/1 queue

Consider the preemptive LCFS D/M/1 queue. From (54), we have

\[ E[A] = \frac{E[G]}{2} + E[H], \]
\[ E[A^2] = \frac{(E[G])^2}{3} + E[H]E[G] + 2(E[H])^2. \]
A.3 Non-preemptive LCFS queues with discarding

The non-preemptive LCFS M/M/1 queue with discarding

Consider the non-preemptive LCFS M/M/1 queue with discarding. From Theorem 35 and Corollary 36 we have

\[ a^*(s) = \frac{1 + \rho}{1 + \rho + \rho^2} \left( \frac{\lambda}{s + \lambda} + \rho \cdot \frac{\mu}{s + \mu} - \rho \cdot \frac{\lambda + \mu}{s + \lambda + \mu} \right) \left( \frac{1}{1 + \rho} + \frac{\rho}{1 + \rho} \cdot \frac{\lambda + \mu}{s + \lambda + \mu} \right) \frac{\mu}{s + \mu}, \]

\[ E[A] = \frac{E[H]}{1 + \rho + \rho^2} \left[ 2\rho^2 + 3\rho + \frac{1}{\rho} + \frac{3}{1 + \rho} - \left( \frac{1}{1 + \rho} \right)^2 \right]. \] (78)

We can verify that (78) is identical to Eq. (65) in [3].

The non-preemptive LCFS M/D/1 queue with discarding

Consider the non-preemptive LCFS M/D/1 queue with discarding. From Corollary 36 we have

\[ E[A] = \frac{E[G]}{2} + \frac{E[H]}{1 - \rho^{-1} \exp(-\rho^{-1})}. \]

The non-preemptive LCFS D/M/1 queue with discarding

Consider the stationary non-preemptive LCFS D/M/1 queue with discarding. From Corollary 36 we have

\[ E[A] = \frac{E[G]}{2} + \frac{E[H]}{1 - \rho^{-1} \exp(-\rho^{-1})}. \]

A.4 Non-preemptive LCFS queues without discarding

The non-preemptive LCFS M/M/1 queue without discarding

Consider the stationary non-preemptive LCFS M/M/1 queue without discarding. From Theorem 35 and Corollary 36 we have

\[ a^*(s) = \left[ 1 - \rho + \rho(2 - \rho) \cdot \frac{\mu}{s + \mu} - \rho(1 - \rho) \left( \frac{\mu}{s + \mu} \right)^2 \right] \cdot \frac{\lambda}{s + \lambda} \cdot \frac{\mu}{s + \mu}, \]

\[ E[A] = \left( \rho^2 + 1 + \frac{1}{\rho} \right) E[H], \]

\[ E[A^2] = 2 \left( 3\rho^2 + 1 + \frac{1}{\rho} + \frac{1}{\rho^2} \right) (E[H])^2. \]

The non-preemptive LCFS M/D/1 queue without discarding

Consider the stationary non-preemptive LCFS M/D/1 queue without discarding. From Corollary 36 we have

\[ E[A] = \left( \frac{\rho}{2} + 2 + \frac{(1 - \rho)^2}{\rho} \cdot \exp(\rho) \right) E[H]. \] (79)

The non-preemptive LCFS D/M/1 queue without discarding

Consider the stationary non-preemptive LCFS D/M/1 queue without discarding. In this case, we have \(-g^{(1)}(\mu - \mu \gamma) = \gamma E[G]\). Corollary 36 then yields

\[ E[A] = \frac{E[G]}{2} + (1 + \gamma) E[H]. \]
We first prove
\[ \text{Cov}[G_n, D_n] = \int_0^\infty \mathbb{E}[(y - G)(G - \mathbb{E}[G]) \mid G \leq y] G(y) dD(y). \] (80)

It follows from Lindley’s recursion that
\[ D_n = \mathbb{1}_{\{G_n \leq D_{n-1}\}} (D_{n-1} - G_n^\dagger) + H_n^\dagger, \quad n = 1, 2, \ldots. \] (81)

Since both \( D_{n-1} \) and \( H_n^\dagger \) are independent of \( G_n^\dagger \), it follows from (81) and the stationarity of the system that
\[ \mathbb{E}[G_n^\dagger D_n] - \mathbb{E}[G] \mathbb{E}[H] = \mathbb{E}[G_n^\dagger (D_n - \mathbb{E}[H])]. \]
\begin{align*}
\int_0^\infty \mathbb{E} \left[ \mathbb{1}_{\{y \leq G\}} (y - G) \mathbb{1}_{\{G < \mathbb{E}[G]\}} \mathbb{1}_{\{G \leq y\}} \right] G(y) dD(y) \\
\geq \int_0^\infty \mathbb{E} \left[ (y - G)(G - \mathbb{E}[G]) \mathbb{1}_{\{G < \mathbb{E}[G]\}} \mathbb{1}_{\{G \leq y\}} \right] dD(y) \\
= \int_0^\infty \mathbb{E} \left[ -\mathbb{E}[G]y + G \cdot (y - G + \mathbb{E}[G]) \right] \mathbb{1}_{\{G < \mathbb{E}[G]\}} \mathbb{1}_{\{G \leq y\}} dD(y) \\
\geq -\mathbb{E}[G] \int_0^\infty \mathbb{E} \left[ \mathbb{1}_{\{G < \mathbb{E}[G]\}} \mathbb{1}_{\{G \leq y\}} \right] y dD(y) \\
= -\mathbb{E}[G] \mathbb{Pr}(G < \mathbb{E}[G]) \mathbb{E}[D].
\end{align*}

Finally, we prove \( \text{Cov}[G_n^\dagger, D_n] \leq 0 \). Consider an arbitrarily fixed \( y \) such that \( G(y) = \Pr(G \leq y) > 0 \). If \( \Pr(G > y) = 0 \), then \( \mathbb{E}[G] = \mathbb{E}[G \mid G \leq y] \), and otherwise
\begin{align*}
\mathbb{E}[G] &= \Pr(G \leq y) \mathbb{E}[G \mid G \leq y] + \Pr(G > y) \mathbb{E}[G \mid G > y] \\
&\geq \Pr(G \leq y) \mathbb{E}[G \mid G \leq y] + \Pr(G > y) \mathbb{E}[G \mid G \leq y] \\
&= \mathbb{E}[G \mid G \leq y],
\end{align*}

because \( \mathbb{E}[G \mid G \leq y] \leq y \leq \mathbb{E}[G \mid G > y] \) if \( \Pr(G > y) > 0 \). Note also that \((y - x)x\) is concave with respect to \( x \). Using these facts and Jensen’s inequality, we have
\begin{align*}
\mathbb{E}[G] - \mathbb{E}[G \mid G \leq y] &\leq \mathbb{E}[(y - G)G \mid G \leq y] \leq \mathbb{E}[(y - G)G \mid G \leq y] \\
&\leq \mathbb{E}[y - G \mid G \leq y] \mathbb{E}[G],
\end{align*}
(84)

for all \( y \geq 0 \) such that \( G(y) > 0 \). Combining (82), (83) and (84), we obtain \( \text{E}[G_n^\dagger D_n] \leq \text{E}[G_n^\dagger] \text{E}[D_n] \) and thus \( \text{Cov}[G_n^\dagger, D_n] \leq 0 \).
C Derivation of (47)

From (46), we obtain

\[
 a^{\text{peak}}(s) = g^*_H(s) \sum_{m=1}^{\infty} (1 - \zeta) \zeta^{m-1} \cdot (g^*_{<H}(s))^{m-1} h^*_{<G}(s)
\]

\[
= \frac{(1 - \zeta) g^*_H(s)}{1 - \zeta g^*_H(s)} \cdot h^*_{<G}(s).
\]

(85)

With straightforward calculations using (28), (45), and (85), we obtain

\[
\lambda^* = 1 - \zeta \frac{E[G]}{E[A]}.
\]

(86)

Therefore, (47) follows from Theorem 10, (45), (85), and (86). Furthermore, taking the derivative of \( a^*(s) \), we obtain \( E[A] \).

D Proof of Theorem 35

D.1 Derivation of (64)

It is readily seen that the waiting time \( W_n \) \((n = 2, 3, \ldots)\) of the \( n \)th informative packet is given by

\[
W_n = \begin{cases} 
X(H^\dagger_{n-1}), & \text{with probability } 1 - h^*(\lambda), \\
0, & \text{with probability } h^*(\lambda),
\end{cases}
\]

(87)

where \( X(H^\dagger_{n-1}) \) denotes the remaining service time seen by the last packet arrived in \( H^\dagger_{n-1} \). We can verify that the LST \( \psi^*(s) \) of \( X(H^\dagger_{n-1}) \) is given by

\[
\psi^*(s) = \frac{1}{1 - h^*(\lambda)} \int_{x=0}^{\infty} dH(x) \sum_{k=1}^{\infty} \int_{y=0}^{x} \frac{e^{-\lambda y} (\lambda y)^{k-1} \lambda}{(k-1)!} \cdot e^{-\lambda (x-y)} e^{-s(x-y)} dy \\
= \frac{\lambda}{s + \lambda} \cdot \frac{1 - h^*(s + \lambda)}{1 - h^*(\lambda)},
\]

(88)

so that we have from (87),

\[
w^*(s) = h^*(\lambda) + \frac{\lambda}{s + \lambda} (1 - h^*(s + \lambda)).
\]

Since service times are i.i.d. and services are non-preemptive, \( W_n \) and \( H^\dagger_{n-1} \) are independent. Therefore, we obtain from (60),

\[
d^*(s) = w^*(s) h^*(s) = \left(h^*(\lambda) + \frac{\lambda}{s + \lambda} h^*(s + \lambda)\right) h^*(s).
\]

(89)

Using (61) and (63), we can verify that

\[
a^{\text{peak}}_*(s) = (1 - h^*(\lambda)) \cdot w^*(s) \cdot \frac{h^*(s) - h^*(s + \lambda)}{1 - h^*(\lambda)} \cdot h^*(s) \\
= h^*(\lambda) \cdot w^*(s) \cdot \frac{h^*(s + \lambda)}{h^*(\lambda)} \cdot \frac{\lambda}{s + \lambda} \cdot h^*(s)
\]

\[
= d^*(s) \left[h^*(s) - h^*(s + \lambda) \left(1 - \frac{\lambda}{s + \lambda}\right)\right].
\]

(90)

We then obtain (64) with straight forward calculations based on (25), (28), (89), and (90).
D.2 Derivation of (65)

Note first that
\[
\Pr(W_{n+1} > 0 \mid W_n = 0) = \int_0^\infty e^{-\mu x}dG(x) = g^*(\mu),
\]
and
\[
\Pr(W_{n+1} > 0 \mid W_n > 0) = \frac{1}{1 - g^*(\mu)} \int_0^\infty dG(x) \int_{y=0}^x \mu e^{-\mu y} \cdot e^{-\mu(x-y)}dy
\]
\[
= \frac{\mu g^{(1)}(\mu)}{1 - g^*(\mu)}.
\]

Therefore, considering the stationary distribution of a discrete-time Markov chain with two states “no wait” and “wait”, we obtain
\[
\Pr(W = 0) = \frac{q}{q + g^*(\mu)}, \quad \Pr(W > 0) = \frac{g^*(\mu)}{q + g^*(\mu)}, \quad (91)
\]
where \(q\) is given by
\[
q = 1 - \frac{\mu(-g^{(1)}(\mu))}{1 - g^*(\mu)}.
\]

We then have from (60),
\[
d^*(s) = \frac{\mu}{s + \mu} \left[ \Pr(W = 0) + \frac{\Pr(W > 0)}{1 - g^*(\mu)} \int_0^\infty e^{-sx} \cdot \mu e^{-\mu x} (1 - G(x))dx \right]
\]
\[
= \frac{\mu}{s + \mu} \frac{\Pr(W = 0) + \Pr(W > 0) \cdot \frac{1 - g^*(s + \mu)}{1 - g^*(\mu)}}.
\quad (92)
\]

Let \(a^*_{\text{peak},0}(s)\) (resp. \(a^*_{\text{peak},+}(s)\)) denote the conditional LST of the \((n+1)\)st peak AoI \(A_{\text{peak},n+1}\) given \(W_n = 0\) (resp. \(W_n > 0\)):
\[
a^*_{\text{peak}}(s) = \Pr(W = 0)a^*_{\text{peak},0}(s) + \Pr(W > 0)a^*_{\text{peak},+}(s). \quad (93)
\]

When \(W_n = 0\), it follows from (62) and (63) that
\[
A_{\text{peak},n+1} = \max(H_{n+1}^\dagger, G_{n+1}^\dagger) + H_{n+1}^\dagger,
\]
which implies
\[
a^*_{\text{peak},0}(s) = g^*(s + \mu) \left( \frac{\mu}{s + \mu} \right)^2 + (g^*(s) - g^*(s + \mu)) \frac{\mu}{s + \mu}. \quad (94)
\]

On the other hand, when \(W_n > 0\), \(W_n\) has the same distribution as the conditional service time \(H_{<G}\) given that \(H < G\), where \(H\) is exponentially distributed with parameter \(\mu\). From (62) and (63), the peak AoI is then given by
\[
A_{\text{peak},n+1} \overset{\text{d}}{=} \max(H_{<G} + H_{n+1}^\dagger, G) + H_{n+1}^\dagger
\]
\[
= H_{<G} + \max(H_{n+1}^\dagger, G - H_{<G}) + H_{n+1}^\dagger.
\]

We define \(f^*(s, \omega) = E[e^{-sH_{<G}}e^{-\omega(G-H_{<G})}]\) as the joint LST of the waiting time \(H_{<G}\) and the remaining interarrival time \(G - H_{<G}\), which is given by
\[
f^*(s, \omega) = \int_0^\infty \frac{dG(x)}{1 - g^*(\mu)} \int_0^x e^{-sy}e^{-\omega(x-y)}\mu e^{-\mu y}dy
\]
\[
= \frac{\mu}{1 - g^*(\mu)} \frac{g^*(\omega) - g^*(s + \mu)} {s + \mu - \omega}.
\]

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It can be verified from the above observations that the conditional LST $a_{\text{peak},+}(s)$ of the peak AoI is given by
\[
a_{\text{peak},+}(s) = \left[ f^{**}(s, s + \mu) \frac{\mu}{s + \mu} + f^{**}(s, s) - f^{**}(s, s + \mu) \right] \frac{\mu}{s + \mu}
\]
\[
= \mu \left[ \frac{\mu(-g^{(1)}(s + \mu))}{1 - g^*(s + \mu)} \cdot \frac{\mu}{s + \mu} + \frac{g^*(s) - g^*(s + \mu) - \mu(-g^{(1)}(s + \mu))}{1 - g^*(\mu)} \right],
\]
where we used
\[
f^{**}(s, s + \mu) = \lim_{\omega \to s + \mu} f^{**}(s, \omega) = \frac{\mu \cdot (-g^{(1)}(s + \mu))}{1 - g^*(\mu)}.
\]
We then obtain (65) with some calculations based on (25), (28), (92), (93), (94), and (95).

**D.3 Derivation of (66)**

Because the non-preemptive LCFS service discipline is work-conserving, the stationary queue length distribution is identical to that in the FCFS M/GI/1 queue. Let $L$ denote the number of packets, and let $H$ denote the remaining service time in steady state provided that a packet is being served. Note that $\Pr(L \geq 1) = \rho$. Furthermore, it is known that [25,26]
\[
\Pi(z, s) := \mathbb{E}[z^L e^{-sH} | L \geq 1] = \frac{(1 - \rho) z(z - 1)}{z - h^*(\lambda \rho z)} \cdot \frac{h^*(\lambda - \lambda z) - h^*(s)}{E[H](s - \lambda + \lambda z)}.
\]

Let $W$ denote the waiting time of informative packets and $L^\dagger$ denote the number of waiting (non-informative) packets at the beginnings of services of the informative packets. Arriving packets become informative if (i) they arrive at the empty system or (ii) they arrive when a packet is being served and no subsequent packets arrive in the remaining service time. Therefore, owing to PASTA, we obtain
\[
w^{**}(z, s) := \mathbb{E}[z^{L^\dagger} e^{-sW}] = \frac{1}{1 - \rho + \rho h^*(\lambda)} \left[ 1 - \rho + \rho \cdot \frac{\Pi(z, s + \lambda)}{z} \right].
\]

It then follows from $d^*(s) = w^{**}(1, s)h^*(s)$ and (96) that
\[
d^*(s) = \frac{1 - \rho + \rho h^*(s + \lambda)}{1 - \rho + \rho h^*(\lambda)} \cdot h^*(s).
\]

Next, we consider the peak AoI. We define $w_k^+(s)$ ($k = 0, 1, \ldots$) as
\[
w_k^+(s) = \Pr(L^\dagger = k) \mathbb{E}[e^{-sW} | L^\dagger = k].
\]
By definition, $w^{**}(z, s) = \sum_{k=0}^{\infty} w_k^+(s) z^k$. With straightforward calculations based on (61), we can verify that
\[
a_{\text{peak}}^*(s) = \left( 1 - h^*(\lambda) \right) \cdot w^{**}(1, s) \cdot \frac{h^*(s) - h^*(s + \lambda)}{1 - h^*(\lambda)} \cdot h^*(s)
\]
\[
+ h^*(\lambda) \cdot \frac{1}{h^*(\lambda)} \sum_{k=0}^{\infty} w_k^+(s)h^*(s + \lambda) \left[ \sum_{\ell=1}^{k} (h^*(s + \lambda))^{\ell-1}(h^*(s) - h^*(s + \lambda))h^*(s) \right]
\]
\[
= d^*(s) \cdot \frac{h^*(s) - h^*(s + \lambda)}{1 - h^*(s + \lambda)} + w^{**}(h^*(s + \lambda), s)h^*(s + \lambda)h^*(s) \left( \frac{\lambda}{s + \lambda} - \frac{h^*(s) - h^*(s + \lambda)}{1 - h^*(s + \lambda)} \right).
\]

Therefore, we can obtain (66) with some calculations based on [25], [28], (96), (97), (98), and (99).
D.4 Derivation of (67)

Let $L^A$ denote the queue length seen by a randomly chosen packet on arrival in steady state. Since the non-preemptive LCFS service discipline without discarding is work-conserving, the queue length distribution immediately before arrivals is identical to that in the FCFS service discipline, which is given by [17, Page 251]

$$
\Pr(L^A = k) = (1 - \gamma)\gamma^k, \quad k = 0, 1, \ldots,
$$

where $\gamma$ denotes the unique solution of (34).

Let $W$ denote the waiting time of informative packets in steady state. Arriving packets become informative if (ii) they arrive at the empty system or (ii) they arrive when a packet is being served and no subsequent packets arrive in the exponentially distributed remaining service time. Note here that for a randomly chosen arrival, the event (i) happens with probability $1 - \gamma$ and the event (ii) happens with probability $\gamma(1 - g^*(\mu))$. From these observations, we have

$$
\Pr(W = 0) = \frac{1 - \gamma}{1 - \gamma g^*(\mu)}, \quad \Pr(W > 0) = \frac{\gamma(1 - g^*(\mu))}{1 - \gamma g^*(\mu)}.
$$

With those, we obtain

$$
d^*(s) = \left[ \Pr(W = 0) + \Pr(W > 0) \cdot \frac{1 - g^*(\mu)}{1 - g^*(\mu)} \int_0^\infty e^{-sx} e^{-\mu x} (1 - G(x)) dx \right] \frac{\mu}{s + \mu}
= \left[ \Pr(W = 0) + \Pr(W > 0) \cdot \frac{\mu}{s + \mu} \right] \frac{\mu}{s + \mu}.
$$

By considering two exclusive cases $W_n = 0$ and $W_n > 0$, we write the LST $a^*_{\text{peak}}(s)$ of the peak AoI $A_{\text{peak},n+1}$ in the form of (103). When $W_n = 0$, i.e., an informative packet finds the system empty on arrival, the packet served next to this informative packet is also informative and $W_{n+1} = 0$. We then have from (62),

$$
a^*_{\text{peak},0}(s) = \left[ g^*(s + \mu) \cdot \frac{\mu}{s + \mu} + (g^*(s) - g^*(s + \mu)) \right] \frac{\mu}{s + \mu}
= \left[ g^*(s) - \frac{s g^*(s + \mu)}{s + \mu} \right] \frac{\mu}{s + \mu}.
$$

When $W_n > 0$, on the other hand, the number of non-informative packets served before the arrival of the next informative packet need to be taken into account. We define $b_k(s) \ (s > 0, k = 0, 1, \ldots)$ as

$$
b^*_k(s) = \int_0^\infty e^{-sx} \frac{e^{-\mu x} (\mu x)^k}{k!} dG(x).
$$

If $L^A = k - 1$, the queue length immediately after the arrival is equal to $k$. Note here that

$$
\Pr(L^A \geq k \mid L^A > 0) = \gamma^{k-1}, \quad k = 1, 2, \ldots.
$$

We thus obtain from (62),

$$
a^*_{\text{peak},+}(s) = \sum_{k=1}^\infty b^*_k(s) \left[ 1 - \gamma^{k-1} + \gamma^{k-1} \cdot \frac{\mu}{s + \mu} \right] \frac{\mu}{s + \mu}
= \frac{1}{1 - g^*(\mu)} \left[ g^*(s) - g^*(s + \mu) - \frac{s [g^*(s + \mu - \mu \gamma) - g^*(s + \mu)]}{\gamma(s + \mu)} \right] \frac{\mu}{s + \mu}.
$$

We can obtain (67) with some calculations based on (25), (28), [93], (100), (101), (102), (103).

E Proof of Theorem 33

We first consider the M/GI/1 queue. We rewrite (51) as

$$
E[A] = \frac{1}{\lambda h^* (\lambda)}.
$$
Under the assumptions of Theorem 33, it follows from the initial value theorem [27] on Page 243 that

\[ \lim_{\lambda \to \infty} \lambda h^*(\lambda) = h(0), \]

which implies 59.

We then consider the D/GI/1 queue. Suppose that interarrival times are given by \( G_n = \tau \), where \( \tau > 0 \) is a real number. Because \( \lambda = 1/\tau \), we consider the limit \( \tau \to 0^+ \). Obviously, the first two terms on the right-hand side of (48) converge to zero as \( \tau \to 0^+ \). Furthermore, the third term is rewritten as

\[ \zeta E[G<H] = \frac{\tau \Pr(H > \tau)}{1 - \Pr(H > \tau)} = \frac{\int_0^\infty h(x)dx}{1 - \int_0^\tau h(x)dx}. \quad (104) \]

Because \( h(x) \) is assumed to be continuous, we have

\[ \lim_{\tau \to 0^+} \frac{1}{\tau} \int_0^\tau h(x)dx = h(0), \quad (105) \]

whether \( h(0) \neq 0 \) or not. Therefore, we obtain (59) from (104), (105), and \( \lim_{\tau \to 0^+} \int_0^\infty h(x)dx = 1 \).

**F Proof of Theorem 37**

**F.1 Proof of the M/GI/1 case**

We first show \( E[A_{\text{LCFS}}^{\text{NP-W/O}}] \leq E[A_{\text{FCFS}}] \). Because

\[ 1 - \lambda x \leq e^{-\lambda x} \leq 1 - \lambda x + \frac{\lambda^2 x^2}{2}, \quad \lambda \geq 0, \quad x \geq 0, \]

we have

\[ 1 - \rho \leq h^*(\lambda) \leq 1 - \rho + \frac{\lambda^2 E[H^2]}{2}. \quad (106) \]

It then follows from (36), (70), and (106) that

\[
E[A_{\text{FCFS}}] - E[A_{\text{LCFS}}^{\text{NP-W/O}}] = \frac{\lambda \rho E[H^2]}{2(1 - \rho)} + \frac{(1 - \rho) E[H]}{h^*(\lambda)} - E[H] \\
= \frac{E[H]}{h^*(\lambda)} \left( \frac{\lambda^2 E[H^2]}{2} \cdot h^*(\lambda) \right) \left( 1 - \rho + 1 - \rho - h^*(\lambda) \right) \\
\geq \frac{E[H]}{h^*(\lambda)} \left( \frac{\lambda^2 E[H^2]}{2} \right) \left( 1 - \rho - h^*(\lambda) \right) \\
\geq 0.
\]

Next we consider \( E[A_{\text{LCFS}}^{\text{NP-W}}] \leq E[A_{\text{LCFS}}^{\text{NP-W/O}}] \). It follows from Corollary 36 and (70) that

\[
E[A_{\text{LCFS}}^{\text{NP-W/O}}] - E[A_{\text{LCFS}}^{\text{NP-W}}] = \left( 1 - \frac{1}{\rho + h^*(\lambda)} \right) \left( \frac{\lambda E[H^2]}{2} + \frac{h^*(\lambda)}{h^*(\lambda)} - h^{(1)}(\lambda) \right) + E[H] + \frac{(1 - \rho)^2}{\lambda h^*(\lambda)} - \frac{1}{\lambda} \\
= \frac{h^*(\lambda) - (1 - \rho)}{\lambda h^*(\lambda)(\rho + h^*(\lambda))} \left[ h^*(\lambda) \left( \frac{\lambda^2 E[H^2]}{2} + h^*(\lambda) - h^{(1)}(\lambda) \right) - (1 - \rho)(\rho + h^*(\lambda)) \right] \\
= \frac{h^*(\lambda) - (1 - \rho)}{\lambda h^*(\lambda)(\rho + h^*(\lambda))} \left[ h^*(\lambda) \left( \frac{\lambda^2 E[H^2]}{2} + h^*(\lambda) - h^{(1)}(\lambda) - 1 \right) + \rho h^*(\lambda) - (1 - \rho) \right],
\]

which implies that

\[
\frac{\lambda^2 E[H^2]}{2} + h^*(\lambda) - \lambda h^{(1)}(\lambda) \geq 1 \quad \Rightarrow \quad E[A_{\text{LCFS}}^{\text{NP-W/O}}] \geq E[A_{\text{LCFS}}^{\text{NP-W}}].
\]
Note here that
\[
\frac{\lambda^2 E[H^2]}{2} + h^*(\lambda) - \lambda h^{(1)}(\lambda) = \int_0^\infty f(x) dH(x),
\]
where \( f(x) (x \geq 0) \) is given by
\[
f(x) = \frac{\lambda^2 x^2}{2} + e^{-\lambda x} + \lambda xe^{-\lambda x}.
\]
It is readily seen that \( f(x) \geq 1 (x \geq 0) \) because \( f(0) = 1 \) and for \( x \geq 0 \),
\[
\frac{d}{dx} f(x) = \lambda^2 x (1 - e^{-\lambda x}) \geq 0.
\]
We thus have
\[
\int_0^\infty f(x) dH(x) \geq \int_0^\infty dH(x) = 1,
\]
which completes the proof.

### F.2 Proof of the GI/M/1 case

Note that \( E[A_{\text{LCFS}}] \leq E[A_{\text{NP-W}}] \) and \( E[A_{\text{NP-W/O}}] \leq E[A_{\text{FCFS}}] \) are readily verified from (38), (54), (69), and (71). We thus prove only \( E[A_{\text{NP-W}}] \leq E[A_{\text{NP-W/O}}] \) below.

By definition (34), \( \gamma \) satisfies
\[
\gamma = \int_0^\infty e^{-\mu (\mu - \mu \gamma)x} dG(x) = \sum_{n=0}^\infty b_n \gamma^n,
\]
where \( b_n \) \((n = 0, 1, \ldots)\) is given by
\[
b_n = \int_0^\infty e^{-\mu x} \frac{(\mu x)^n}{n!} dG(x).
\]
We then have
\[
\gamma \geq b_0 + b_1 \gamma,
\]
so that
\[
\gamma \geq \frac{b_0}{1 - b_1} = \frac{g^*(\mu)}{1 - \mu (-g^{(1)}(\mu))}.
\]
Similarly, we can verify that
\[
-g^{(1)}(\mu - \mu \gamma) = \int_0^\infty xe^{-\mu (\mu - \mu \gamma)x} dG(x)
\]
\[
= \sum_{n=0}^\infty \frac{(n+1)b_{n+1}}{\mu} \gamma^n
\]
\[
\geq \frac{b_1}{\mu} + \frac{2b_2}{\mu} \cdot \gamma
\]
\[
= (-g^{(1)}(\mu)) + \gamma \mu g^{(2)}(\mu).
\]
It then follows from (69), (71), (107), and (108) that
\[
E[A_{\text{NP-W/O}}] \leq E[H] + \frac{E[G^2]}{2E[G]} + \rho \left( (-g^{(1)}(\mu)) + \gamma \mu g^{(2)}(\mu) \right)
\]
\[
\leq E[H] + \frac{E[G^2]}{2E[G]} + \rho \left( -g^{(1)}(\mu - \mu \gamma) \right)
\]
\[
= E[A_{\text{NP-W/O}}],
\]
which completes the proof.
G Proof of Theorem 38

We first consider (72). It follows from (36), (51), and (106) that

\[
E[A^p_{\text{LCFS}}] - E[A^p_{\text{FCFS}}] \geq \frac{1}{h^*(\lambda)} - \frac{\lambda E[H^2]}{2(1-\rho)E[H]} - 1
\]

\[
\geq \frac{1}{1-\rho + \lambda^2 E[H^2]/2} - \frac{\lambda E[H^2]}{2(1-\rho)E[H]} - 1
\]

\[
= \frac{2}{2(1-\rho) + \rho^2((Cv[H])^2 + 1)} - \frac{\rho}{2(1-\rho)((Cv[H])^2 + 1)} - 1
\]

where we use \(E[H^2] = (E[H])^2((Cv[H])^2 + 1)\). Straightforward calculations then yield

\[
E[A^p_{\text{LCFS}}] - E[A^p_{\text{FCFS}}] \geq -\frac{\rho^2(Cv[H])^2 - 2\rho^2(Cv[H])^2 + \rho^2((2-\rho)^2 - 2)}{2\rho(1-\rho)(1 + (1-\rho)^2 + \rho^2(Cv[H])^2)}
\]

(109)

Note that the denominator on the right-hand side of (109) is always positive for \(\rho \in (0, 1)\). We thus consider

\[ f_1(x) := -x^2 - 2x + \rho^2((2-\rho)^2 - 2) \quad (x \geq 0), \]

which corresponds to the numerator with \(x = \rho^2(Cv[H])^2 \geq 0\). It is easy to verify that \(f_1(x) \geq 0\) is equivalent to

\[ (2-\rho)^2 - 2 \geq 0 \quad \text{and} \quad x \leq -1 + \sqrt{1 + \rho^2((2-\rho)^2 - 2)}, \]

from which (72) follows. It can also be verified that for \(\rho \in (0, 2 - \sqrt{2})\), \(v(\rho)\) is a decreasing function of \(\rho\) and \(\lim_{\rho \to 0^+} v(\rho) = 1\).

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