Quantization of $(-1)$-Shifted Derived Poisson Manifolds

Kai Behrend$^1$, Matt Peddie$^2$, Ping Xu$^2$

$^1$ Department of Mathematics, University of British Columbia, Vancouver, Canada.
E-mail: behrend@math.ubc.ca

$^2$ Department of Mathematics, Pennsylvania State University, University Park, USA.
E-mail: matt.peddie11@gmail.com; ping@math.psu.edu

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Dedicated to Jean-Luc Brylinski on his 70th birthday

Abstract: We investigate the quantization problem of $(-1)$-shifted derived Poisson manifolds in terms of BV$_\infty$-operators on the space of Berezinian half-densities. We prove that quantizing such a $(-1)$-shifted derived Poisson manifold is equivalent to the lifting of a consecutive sequence of Maurer–Cartan elements, each obtained from a short exact sequence of differential graded Lie algebras. At each step, the obstruction is a certain class in the second Poisson cohomology. Consequently, a $(-1)$-shifted derived Poisson manifold is quantizable if the second Poisson cohomology group vanishes. We also prove that for any $L_\infty$-algebroid $\mathcal{A}$, its corresponding linear $(-1)$-shifted derived Poisson manifold $\mathcal{A}^\vee[-1]$ admits a canonical quantization. Finally, given a Lie algebroid $\mathcal{A}$ and a one-cocycle $s \in \Gamma(\mathcal{A}^\vee)$, the $(-1)$-shifted derived Poisson manifold corresponding to the derived intersection of coisotropic submanifolds determined by the graph of $s$ and the zero section of the Lie–Poisson $\mathcal{A}^\vee$ is shown to admit a canonical quantization in terms of Evens–Lu–Weinstein module.

Introduction

The notion of homotopy Schouten algebras was introduced by Khudaverdian–Voronov in their seminar paper [28] in 2008. In particular, they discovered a homotopy analogue of Koszul–Brylinski construction [13,35] that the space of differential forms on a $P_\infty$-manifold [15,46] admits a canonical homotopy Schouten algebra structure. Khudaverdian–Voronov’s construction is mainly in the context of supergeometry, i.e. $\mathbb{Z}_2$-grading. In the context of $\mathbb{Z}$-grading, following Pridham [47,49], the underlying geometric object in the $C^\infty$-context is called $(-1)$-shifted derived Poisson manifolds [3].

A $(-1)$-shifted derived Poisson manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ whose algebra of functions $C^\infty(\mathcal{M})$ is a homotopy Schouten algebra, called $(+1)$-shifted derived Poisson algebra in the paper. Equivalently, a $(-1)$-shifted derived Poisson manifold $\mathcal{M}$ is a differential graded (dg) manifold $(\mathcal{M}, Q)$ equipped with a degree $(+1)$ formal power
series \( \Pi \in \Gamma(\hat{S}T_M) \), where \( \Pi = \sum_{n \geq 2} \Pi_n \) with \( \Pi_n \in \Gamma(S^n T_M) \), satisfies the Maurer–Cartan equation

\[
\{ Q, \Pi \} + \frac{1}{2} \{ \Pi, \Pi \} = 0.
\]

Here the bracket \( \{-, -\} \) refers to the standard Poisson bracket on \( \Gamma(\hat{S}T_M) \cong C^\infty(T^*_M) \) corresponding to the canonical symplectic structure on \( T^*_M \). We often write \( \Pi_1 = Q \), and call \( Q + \Pi \) the Poisson tensor. A \((-1)\)-shifted derived Poisson manifold is usually denoted \((M, Q, \Pi)\).

This paper is devoted to the study of quantizations of \((-1)\)-shifted derived Poisson manifolds. Following Kravchenko [36], a quantization of a \((-1)\)-shifted derived Poisson manifold \((M, Q, \Pi)\) is a square-zero differential operator of degree \(+1\), defining a Batalin–Vilkovisky algebra up to homotopy, whose associated \((+1)\)-shifted derived Poisson algebra is the one on \(C^\infty(M)\) corresponding to \((M, Q, \Pi)\). For quantizations being intrinsic, instead of differential operators on \(C^\infty(M)\), we consider differential operators on \(\Gamma(\text{Dens}_{1/2}^M)\), the space of Berezinian half-densities on \(M\). More precisely, a quantization of a \((-1)\)-shifted derived Poisson manifold \((M, Q, \Pi)\) is a \(BV_{\infty}\)-operator \(\Delta \in hDO^+(\text{Dens}_{1/2}^M)\), i.e. a square-zero self-adjoint \(h\)-enhanced differential operator, whose extended principal symbol, evaluated at \(h = 1\), equals to \(Q + \Pi\). See Definition 3.4. In [25], Khudaverdian–Peddie constructed an example of non-quantizable \((-1)\)-shifted derived Poisson manifold, which contains only one term \(\Pi_2 \in \Gamma(S^2 T_M)\) with \(Q\) being zero, i.e. \((-1)\)-shifted Poisson manifold. See Example 3.19.

Our first main result is the following theorem describing the obstruction class to the quantizations.

**Theorem A (Theorem 3.10).** Let \((M, Q, \Pi)\) be a \((-1)\)-shifted derived Poisson manifold. Assume that the second Poisson cohomology group \(H^2(M, Q + \Pi)\) vanishes, then \((M, Q, \Pi)\) is quantizable.

Our strategy of proof is to convert the quantization problem into the problem of lifting Maurer–Cartan elements of a short exact sequence of differential graded Lie algebras (dglas). For a given dg manifold \((M, Q, \Pi)\), a \((-1)\)-shifted derived Poisson structure \(\Pi\) is equivalent to a Maurer–Cartan element of the dglas \((\Gamma(\hat{S}T_M), \{\cdot, \cdot\}, \{Q, \cdot\})\), while a \(BV_{\infty}\)-operator \(\Delta \in hDO^+(\text{Dens}_{1/2}^M)\), or more precisely \(\Delta = hL_Q\), corresponds to a Maurer–Cartan element of the dglas \((hDO^+(\text{Dens}_{1/2}^M), \{\cdot, \cdot\}_h, [hL_Q, \cdot\cdot]_h)\). These two dglas are related by the short exact sequence

\[
0 \longrightarrow hDO_2^+(\text{Dens}_{1/2}^M) \xrightarrow{i} hDO^+(\text{Dens}_{1/2}^M) \xrightarrow{\phi} \Gamma(\hat{S}T_M) \longrightarrow 0,
\]

where \(hDO_2^+(\text{Dens}_{1/2}^M)\) is considered as a dg Lie subalgebra of \(hDO^+(\text{Dens}_{1/2}^M)\), the morphism \(i\) is the inclusion map and \(\phi\) is the morphism defined by the extended principal symbol, evaluated at \(h = 1\). Here, for any integer \(t \geq 0\), \(hDO_t^+(\text{Dens}_{1/2}^M)\), denotes the space of those \(h\)-enhanced differential operators \(\Delta\) such that \(\frac{1}{\Pi^t} \Delta \in hDO^+(\text{Dens}_{1/2}^M)\). Therefore, the lifting of a Maurer–Cartan element in the short exact sequence (2) is equivalent to the lifting of a consecutive sequences of Maurer–Cartan elements, each obtained from the short exact sequences of dglas:

\[
0 \longrightarrow hDO_{2k}^+(\text{Dens}_{1/2}^M) \xrightarrow{i_k} hDO_2^+(\text{Dens}_{1/2}^M) \xrightarrow{\phi_k} hDO_{2k}^+(\text{Dens}_{1/2}^M) \longrightarrow 0,
\]

(3)
\( k = 1, 2, \ldots \). Here \( i_k \) is the natural inclusion and \( \phi_k \) is the natural projection. These are indeed square-zero extensions of dglas. The obstruction to lifting a Maurer–Cartan element is proved to be a certain class in the second Poisson cohomology \( \mathcal{H}^2(M, Q + \Pi) \).

Theorem A describes the obstruction to the existence of quantizations, which is a sufficient condition for quantizing \((-1\)-shifted derived Poisson manifolds. In many situation, however, although the second Poisson cohomology group \( \mathcal{H}^2(M, Q + \Pi) \) does not vanish, \( (M, Q, \Pi) \) may still be quantizable.

An important class of \((-1\)-shifted derived Poisson manifolds arises as linear Poisson structures on a vector bundle, which corresponds to Theorem C (Theorem 3.25). Theorem B (Theorem 3.22). For any \( L_\infty \)-algebroid \( \mathcal{A} \), its corresponding linear \((-1\)-shifted derived Poisson manifold \( \mathcal{A}^V[-1] \) admits a canonical quantization.

Our construction of the BV\( \infty \)-operator utilizes the fiberwise \( h \)-Fourier transform [52] by Voronov–Zorich [63]:

\[
\mathcal{F} : \Gamma(Dens^{1/2}_{A[1]}) \xrightarrow{\sim} \Gamma(Dens^{1/2}_{A^V[-1]}).
\]

To any \( L_\infty \)-algebroid \( \mathcal{A} \), there is a Chevalley–Eilenberg differential \( D \), which is a homological vector field on \( \mathcal{A}[1] \). Therefore, \( (\mathcal{A}[1], D) \) is a dg manifold. We prove that \( \Delta := \mathcal{F} \circ \mathcal{L}_hD \circ \mathcal{F}^{-1} \in \hbar D^{\infty}(Dens^{1/2}_{A^V[-1]}) \) is a BV\( \infty \)-operator quantizing the linear \((-1\)-shifted derived Poisson manifold on \( \mathcal{A}^V[-1] \). If \( \mathcal{M} \) is a \( P_\infty \)-manifold [15, 46], its cotangent bundle \( T^*\mathcal{M} \) naturally carries an \( L_\infty \)-algebroid structure [28], Thus \( T^*\mathcal{M}[-1] \) is a linear \((-1\)-shifted derived Lie–Poisson manifold, whose algebra of functions are differential forms on \( \mathcal{M} \). Its quantization has been studied by Khudaverdian–Voronov [28] and more recently by Shemyakova [52].

Another important class of \((-1\)-shifted derived Poisson manifolds are derived intersections of coisotropic submanifolds in a Poisson manifold. As a toy model, we consider the Lie–Poisson manifold corresponding to a Lie algebroid. Let \( \mathcal{A} \) be a Lie algebroid and \( s \in \Gamma(A^V) \) a Lie algebroid 1-cocycle, i.e. a smooth section satisfying the condition that \( d_{CE}s = 0 \), where \( d_{CE} \) is the Chevalley–Eilenberg differential of the Lie algebroid \( \mathcal{A} \). Both the graph of \( s \) and the zero section are coisotropic submanifolds of the Lie–Poisson manifold \( A^V \). Their derived intersection is a \((-1\)-shifted derived Poisson manifold \( (A^V)[-1], \iota_s, \Pi_2 \), where \( \Pi_2 \in \Gamma(S^2T^{\infty}_{A^V[-1]}) \) is the Poisson tensor determined by the Schouten algebra on \( C^\infty(A^V[-1]) \cong \Gamma(\Lambda^{*-\bullet}) \). In this situation, the BV\( \infty \)-operator is related to the Evens–Lu–Weinstein module [19].

Theorem C (Theorem 3.25). Let \( \mathcal{A} \) be a Lie algebroid and \( s \in \Gamma(A^V) \) a Lie algebroid 1-cocycle. Assume that \( \mathcal{M} \) is an orientable manifold and the vector bundle \( A \to \mathcal{M} \) is also orientable as well. Then the \((-1\)-shifted derived Poisson manifold \( (A^V)[-1], \iota_s, \Pi_2 \) admits a canonical quantization:

\[
\Delta = \hbar u_s + \hbar^2 \Phi \circ d_{CE}^{ELW} \circ \Phi^{-1} : \Gamma(\Lambda^{*-\bullet}A \otimes (\Lambda^{top}A^V \otimes \Lambda^{top}T^*_M)^{1/2}) \to \Gamma(\Lambda^{*-\bullet+1}A \otimes (\Lambda^{top}A^V \otimes \Lambda^{top}T^*_M)^{1/2}).
\]

Here \( \Phi \) is the canonical isomorphism [1, Section 5] [21, Section 6.3]:

\[
\Phi : \Gamma(\Lambda^kA^V \otimes (\Lambda^{top}A \otimes \Lambda^{top}T^*_M)^{1/2}) \xrightarrow{\sim} \Gamma(\Lambda^{top-k}A \otimes (\Lambda^{top}A^V \otimes \Lambda^{top}T^*_M)^{1/2}),
\]
for $k = 0, 1, \ldots$. In particular, when $A$ is $T_M$ and $s = df \in \Omega^1(M)$ an exact one-form, where $f \in C^\infty(M)$, the line bundle $(\Lambda^\text{top} A \otimes \Lambda^\text{top} T_M^\vee)^{1/2}$ is canonically isomorphic to the trivial line bundle $M \times \mathbb{R}$ and $\omega^\text{ELW}$ reduces to the ordinary de Rham differential. As an immediate consequence, we see that

$$h \text{d}f + h^2 \Phi \circ \text{d}_\text{DR} \circ \Phi^{-1} : \Gamma(\Lambda_{-\bullet}^\text{top} T_M \otimes (\Lambda^\text{top} T_M^\vee)) \to \Gamma((\Lambda_{-\bullet}^{\text{top}+1}) T_M \otimes (\Lambda^\text{top} T_M^\vee))$$

(7)

is a BV$_\infty$-operator quantizing the ($-1$)-shifted derived symplectic manifold $(T_M^\vee[-1], l\text{d}_f, \omega_{\text{can}})$, where $\Phi : \Omega^k(M) \to \Gamma(\Lambda^\text{top}^{k-1} T_M \otimes (\Lambda^\text{top} T_M^\vee)), \forall k = 0, 1, \ldots$ is the canonical isomorphism.

Finally, we note that several works on related subject have appeared recently in the literature. For instance, we refer the readers to [4, 10, 22, 25–27, 48, 49, 52] and references therein. In particular, we note that recently Bandiera proved the homotopy transfer theorem for BV$_\infty$-algebras and showed that the homotopy transfer is compatible with quantizations [2]. The work in this paper is presented in the context of $\mathbb{Z}$-graded manifolds. However, it works for supermanifolds (i.e. $\mathbb{Z}_2$-graded) as well.

**Notation.** Let $V = \oplus_{n \in \mathbb{Z}} V_n$ be a $\mathbb{Z}$-graded vector space over $\mathbb{R}$, where an element $v \in V_n$ has degree $n$, written $|v| = n$. For $k \in \mathbb{Z}$, the symbol $V[k]$ denotes the $k$-shifted space such that $(V[k])_n := V_{n+k}$. In particular, an element $v \in V$ of degree $n$ lies in $(V[k])_{n-k}$ if viewed as an element of $V[k]$. The dual vector space $V^\vee$ is graded according to $(V^\vee)_n := (V_{-n})^\vee$, which ensures the non-degenerate paring $V \times V^\vee \to \mathbb{R}$ carries degree zero.

Given a graded vector space $V$, let $S(V)$ denote the symmetric algebra of $V$, where $S(V) = \oplus_{m \geq 0} S^m(V)$ decomposes into homogeneous terms graded naturally by weight. A weight $m$, degree $l$ element of $S(V)$ is thus

$$v_1 \odot \cdots \odot v_m \in S^m(V), \quad |v_1| + \cdots + |v_m| = l.$$

The symbol $\hat{S}(V) = \prod_{m=0}^\infty S^m(V)$ denotes the $m$-adic completion of $S(V)$ with respect to the ideal $m$ generated by $V$.

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ is a smooth manifold $M$ (called the support), together with a sheaf of $\mathbb{Z}$-graded commutative $C^\infty$-algebras over $M$, isomorphic to $C^\infty(U) \otimes \mathbb{R} \hat{S}(V^\vee)$ for any sufficiently small open neighborhood $U \subset M$, where $V$ is a fixed $\mathbb{Z}$-graded vector space, and $C^\infty_M$ denotes the sheaf of $\mathbb{R}$-valued smooth functions over $M$. If $M$ and $V$ are both finite dimensional, then $\mathcal{M}$ is said to be of finite dimension. Throughout this work, we only deal with finite dimensional $\mathbb{Z}$-graded manifolds. A dg manifold is a $\mathbb{Z}$-graded manifold equipped with a homological vector field, i.e. a vector field $Q$ of degree $(+1)$ such that $Q^2 = 0$.

### 1. ($-1$)-Shifted Derived Poisson Manifolds

Our definitions and conventions follow those in [3], adopted from [47,49].
1.1. Definition.

**Definition 1.1.** A (+1)-shifted derived Poisson algebra is a $\mathbb{Z}$-graded commutative algebra $\mathcal{A}$ endowed with a sequence of degree (+1) multi-linear maps $\lambda_n : \mathcal{A}^\otimes n \rightarrow \mathcal{A}$, $n \geq 1$, called Poisson multi-brackets, defining a $L_\infty[1]$-algebra structure on $\mathcal{A}$, and such that for all $n \geq 1$, and homogeneous elements $a_1, \ldots, a_{n-1} \in \mathcal{A}$ the map

$$a \mapsto \lambda_n (a_1, \ldots, a_{n-1}, a)$$

is a graded derivation of degree $(1 + |a_1| + \cdots |a_n|)$.

**Remark 1.2.** In general, for any $k \in \mathbb{Z}$, one can consider $k$-shifted derived Poisson algebras [3]. The associated $\mathbb{Z}_2$-graded derived Poisson algebras are known as homotopy Poisson algebras for even $k$ and homotopy Schouten algebras for odd $k$ respectively, which are due to Khudaverdian–Voronov [28,60]. The general behavior of these derived Poisson structures depends on the parity of the shift, and not on the value of the integer itself. For $k = 0$, such algebras are known as $P_\infty$-algebras [15,46].

**Definition 1.3.** A morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ of (+1)-shifted derived Poisson algebras is a sequence $(\phi_l)_{l \geq 1}$ of degree zero linear maps $\phi_l : S^l \mathcal{A} \rightarrow \mathcal{B}$, which defines an $L_\infty[1]$-morphism from $\mathcal{A}$ to $\mathcal{B}$, and in addition satisfies the Leibniz property:

$$\phi_{n+1} (a_1, \ldots, a_n, bc) = \sum_{l=0}^{n} \sum_{\tau \in \text{Sh}(l,n-l)} \varepsilon(\tau) \phi_{l+1} (a_{\tau(1)}, \ldots, a_{\tau(l)}, b) \phi_{n-l+1} (a_{\tau(l+1)}, \ldots, a_{\tau(n)}, c),$$

where $a_1, \ldots, a_n, b, c \in \mathcal{A}$ are homogeneous elements, $\text{Sh}(l,n-l)$ denotes the $(l,n-l)$-shuffles, and $\varepsilon(\tau)$ is the Koszul sign.

**Definition 1.4.** A (−1)-shifted derived Poisson manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ whose algebra of functions $C^\infty(\mathcal{M})$ is a (+1)-shifted derived Poisson algebra.

Similar to an ordinary Poisson manifold, a (−1)-shifted derived Poisson structure can also be defined in terms of a formal power series of symmetric contravariant tensor fields on $\mathcal{M}$ of degree (+1). A degree $l$ vector field $X \in \Gamma(T\mathcal{M})$ as a section of the tangent bundle $T\mathcal{M}$ is a derivation $X : C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ of degree $l$. The completed symmetric algebra

$$\Gamma(\hat{T}\mathcal{M}) := \prod_{n \geq 0} \Gamma(S^n T\mathcal{M})$$

then consists of formal power series of symmetric contravariant tensor fields. It comes furnished with the degree 0 Poisson bracket:

$$\{ X_1 \circ \cdots \circ X_n, Y_1 \circ \cdots \circ Y_m \} = \sum_{k=1}^{n} \sum_{l=1}^{m} \delta_k \varepsilon_l X^{[k]} \circ [X_k, Y_l] \circ Y^{[l]},$$

where $X^{[k]} = X_1 \circ \cdots \circ X_{k-1} \circ X_{k+1} \circ \cdots \circ X_n$, $Y^{[l]} = Y_1 \circ \cdots \circ Y_{l-1} \circ Y_{l+1} \circ \cdots \circ Y_n$, and the graded signs are given by the Koszul rule $\delta_k = (-1)|X_k|(|X_{k+1}| + \cdots + |X_n|)$ and $\varepsilon_l = (-1)^{|Y_l|(|Y_1| + \cdots + |Y_{l-1}|)}$. 
Remark 1.5. The completed symmetric algebra $\Gamma(\hat{ST}_M)$ may be naturally identified with the algebra of fiberwise formal power series on $T^\vee_M$, whence the degree 0 Poisson bracket (8) coincides with the degree 0 Poisson bracket arising from the canonical symplectic structure on $T^\vee_M$.

Theorem 1.6. [3,47,49] A $(-1)$-shifted derived Poisson manifold $M$ is equivalent to a differential graded (dg) manifold $(M, Q)$ equipped with a degree $(+1)$ formal power series $\Pi \in \Gamma(\hat{ST}_M)$, where $\Pi = \sum_{n \geq 2} \Pi_n$ with $\Pi_n \in \Gamma(S^n T_M)$, satisfies the Maurer–Cartan equation

$$\{ Q, \Pi \} + \frac{1}{2} \{ \Pi, \Pi \} = 0. \quad (9)$$

Here $\{-,-\}$ stands for the canonical Poisson bracket (8).

As in classical Poisson geometry, $Q + \Pi$ is often called a Poisson tensor.

Let $M$ and $M'$ be $(-1)$-shifted derived Poisson manifolds with Poisson multi-brackets $\lambda_{l} : (C^\infty(M))^{\otimes l} \to C^\infty(M)$ and $\lambda'_{l} : (C^\infty(M'))^{\otimes l} \to C^\infty(M')$, $l \geq 1$, respectively. A morphism of $(-1)$-shifted derived Poisson manifolds from $M$ to $M'$ is a map of $\mathbb{Z}$-graded manifolds $\phi : M \to M'$ together with a collection of maps

$$\phi_n : (C^\infty(M'))^{\otimes n} \to C^\infty(M), \quad n = 2, 3, \ldots$$

such that $\phi_\infty = (\phi_1 = \phi^*, \phi_2, \phi_3, \ldots)$ is a morphism of degree $(+1)$ derived Poisson algebras from $(C^\infty(M'), \lambda'_1, \lambda'_2, \ldots, \lambda'_n, \ldots)$ to $(C^\infty(M), \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$ [3]. In particular, $\phi : M \to M'$ is a map of dg manifolds. Geometrically, these morphisms $\phi_\infty$ correspond to the thick morphisms of graded manifolds due to Voronov [30,31,61,62].

Example 1.7. For a finite dimensional $L_\infty$-algebra $g$, the multi-brackets $l_n : S^n(g[1]) \to g[1]$ on the $L_\infty[1]$-algebra $g[1]$ may be extended via the Leibniz rule to equip $g^\vee[-1] \cong (g[1])^\vee$ with the structure of a $(-1)$-shifted derived Poisson manifold. Define

$$\lambda_n(\xi_{a_1}, \ldots, \xi_{a_n}) := \pm c_{a_1 \ldots a_n}^{b} \xi_{b},$$

where $\xi_{a_1}, \ldots, \xi_{a_n}$ are the coordinate functions on $g^\vee[-1]$ corresponding to a basis $\{ e_a \}$ of $g[1]$, and $C_{a_1 \ldots a_n}^{b}$ are the degree $(+1)$ structure constants of the $n$th multi-bracket $l_n$.

Example 1.8. Let $A$ be a Lie algebroid and $s \in \Gamma(A^\vee)$ a Lie algebroid 1-cocycle, i.e. a smooth section satisfying the condition that $d_{CE}s = 0$, where $d_{CE}$ is the Chevalley–Eilenberg differential of the Lie algebroid $A$. Consider the dg manifold $(A^\vee[-1], Q)$, where the algebra of functions $C^\infty(A^\vee[-1]) \cong \Gamma(A^\vee A)$ and $Q = t_s$, the interior product with $s$. This dg manifold describes the derived intersection of the graph of $s$ with the zero section of $A^\vee$. Note that both the graph of $s$ and the zero section are coisotropic submanifolds of the Lie–Poisson manifold $A^\vee$. Let $\Pi_2 \in \Gamma(S^2(T_{A^\vee[-1]}))$ be the Poisson tensor defining the Schouten bracket on $C^\infty(A^\vee[-1]) \cong \Gamma(A^\vee A)$. Then $(A^\vee[-1], t_s, \Pi_2)$ is indeed a $(-1)$-shifted derived Poisson manifold, which describes the derived intersection of coisotropic submanifolds graph$(s)$ and the zero section of the Lie–Poisson manifold $A^\vee$.

In particular, when $A$ is $T_M$ and $s \in \Omega^1(M)$ is a closed one-form, we have a $(-1)$-shifted derived Poisson manifold $(T_M^\vee[-1], t_s, \Pi_2)$, which is indeed a $(-1)$-shifted derived symplectic manifold, denoted $(T_M^\vee[-1], t_s, \omega_{can})$. It is the derived intersection, in the symplectic manifold $T_M^\vee$, of two Lagrangian submanifolds: the graph of $s$ and the zero section.
The following example is due to Khudaverdian–Voronov [31]. See also [52].

Example 1.9. A homotopy Poisson structure on a $\mathbb{Z}$-graded manifold $\mathcal{M}$, according to Khudaverdian–Voronov [28], is defined by a degree $(+2)$ formal power series $P = \sum_{n \geq 1} P_n \in \Gamma(\hat{S}(T\mathcal{M}[-1]))$, satisfying the equation $[P, P] = 0$, where the canonical Schouten bracket $[\cdot, \cdot]$ carries degree $(-1)$. Thus $P$ corresponds to a degree $(+2)$ formal power series in $C^\infty(T\mathcal{M}^\vee[1])$, whose Hamiltonian vector field $d_P$ is a homological vector field on $T\mathcal{M}^\vee[1]$. It in turn determines a degree $(+1)$ fiberwise linear function on the cotangent bundle $h_P \in C^\infty(T\mathcal{M}^\vee[1])$ such that

$$\{h_P, -\} = d_P, \quad \{h_P, h_P\} = 0,$$

where the brackets stand for the canonical degree 0 Poisson bracket on $T\mathcal{M}^\vee[1]$. Via the canonical isomorphism of double vector bundles $T\mathcal{M}^\vee \cong T\mathcal{M}^\vee[-1]$, one obtains a degree $(+1)$ function on $T\mathcal{M}^\vee[-1]$ satisfying the Maurer–Cartan equation, and hence a $(-1)$-shifted derived Poisson structure on $T\mathcal{M}[-1]$.

Remark 1.10. In the $\mathbb{Z}_2$-grading case, the multi-brackets corresponding to the $(-1)$-shifted derived Poisson structure on $T\mathcal{M}[-1]$ in Example 1.9 are called the higher Koszul brackets [28,31,52]. In this case, if there is only a single binary bracket, it reduces to the classical Koszul bracket [13,35,64] defined on differential forms.

1.2. Poisson cohomology. Let $(\mathcal{M}, Q, \Pi)$ be a $(-1)$-shifted derived Poisson manifold. The Lichnerowicz differential $d_\Pi : \Gamma(\hat{S}T\mathcal{M}) \to \Gamma(\hat{S}T\mathcal{M})$ of degree $(+1)$ is defined by

$$d_\Pi = \mathcal{L}_Q + \{\Pi, -\} = \{Q + \Pi, -\}, \quad |d_\Pi| = 1.$$

That $d_\Pi^2 = 0$ is equivalent to the Maurer–Cartan equation (9). Then $(\Gamma(\hat{S}T\mathcal{M}), d_\Pi)$ is a cochain complex, whose cohomology groups form the Poisson cohomology of the $(-1)$-shifted derived Poisson manifold $\mathcal{M}$:

$$\mathcal{H}^k(\mathcal{M}, Q + \Pi) := \ker(d_\Pi : \Gamma(\hat{S}T\mathcal{M})_k \to \Gamma(\hat{S}T\mathcal{M})_{k+1}) \over \im(d_\Pi : \Gamma(\hat{S}T\mathcal{M})_{k-1} \to \Gamma(\hat{S}T\mathcal{M})_k), \quad k \in \mathbb{Z}.$$

Example 1.11. Let $\mathcal{M}$ be a $(-1)$-shifted derived Poisson manifold with a single binary bracket defined by the degree $(+1)$ symmetric tensor field $\Pi_2$. Assume that $\Pi_2$ is non-degenerate. Hence $\mathcal{M}$ is a $(-1)$-shifted symplectic manifold, and there exists a bundle isomorphism $T\mathcal{M}^\vee \sim T\mathcal{M}[1]$ defined by $\alpha \mapsto \Pi_2(\alpha, -)$, for any differential one form $\alpha \in \Gamma(T\mathcal{M}^\vee)$.

This isomorphism induces an isomorphism between Poisson and de Rham cohomologies in the usual sense.

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1 In this case, $C^\infty(\mathcal{M})$ is equipped with a $P\infty$-algebra structure [15,46].
2. $BV_\infty$-Operators on Berezinian Half-Densities

2.1. Differential operators on half-densities. In this section, we recall some basic properties regarding differential operators acting on half densities [17, 29, 52, 59].

Here for a $\mathbb{Z}$-graded manifold $M$, we always assume that the normal bundle $N_M/M_{\text{even}}$ to the even submanifold $M_{\text{even}} \subset M$ is orientable so that one can speak about half densities and the scalar product of half densities. See [29, Remark 2]. Let $\text{Dens}\frac{1}{2}_M$ denote the Berezinian half-density line bundle over a graded manifold $M$, and let $\Gamma(\text{Dens}\frac{1}{2}_M)$ be the space of global smooth sections, i.e. the space of Berezinian half-densities on $M$. For a fixed coordinate system $(x^a)$ on $M$, a local basis element will be represented by the symbol $\sqrt{Dx}$ and a Berezinian half-density takes the expression $s = s(x)\sqrt{Dx}$, where $s(x) \in C^\infty(M)$. As a geometric object, a half-density $s$ transforms according to the law

$$s = s(x)\sqrt{Dx} = s(x(x'))|\text{Ber}\left(\frac{\partial x}{\partial x'}\right)|^{\frac{1}{2}}\sqrt{Dx'}, \quad x = x(x'),$$

where $|\text{Ber}\left(\frac{\partial x}{\partial x'}\right)|$ is the absolute value of the Berezinian, or superdeterminant, of the Jacobian matrix $\left(\frac{\partial x}{\partial x'}\right)$ of the coordinate transformation $x = x(x')$ [29, 52, 59].

Example 2.1. Let $M$ be a $\mathbb{Z}$-graded manifold corresponding to a graded vector bundle $E \rightarrow M$. That is, the graded manifold with support $M$ of where the sheaf $A$ of $\mathbb{Z}$-graded commutative $C^\infty_M$-algebras over $M$ is given by $A|_U = \Gamma(U, \hat{S}(E^\vee))$, for any open neighborhood $U \subset M$. Then $M_{\text{even}}$ corresponds to the graded vector bundle $E_{\text{even}} \rightarrow M$ of even degrees. The condition of $N_M/M_{\text{even}}$ being orientable is equivalent to requiring that the vector bundle $E_{\text{odd}} \rightarrow M$ of odd degrees is orientable as an ordinary vector bundle by forgetting about the degrees. In this case, the Berezinian line bundle $\text{Ber}(M) \rightarrow M$ is isomorphic to the pull back bundle of the line bundle $\Lambda^{\text{top}}E_{\text{even}} \otimes \Lambda^{\text{top}}T^\vee_M \rightarrow M$ via the projection map $\pi : M \rightarrow M$ [14], while the Berezinian half-density line bundle $\text{Dens}\frac{1}{2}_M$ is isomorphic to the pull back bundle of the line bundle $(\Lambda^{\text{top}}E_{\text{odd}})^{\frac{1}{2}} \otimes |\Lambda^{\text{top}}E^\vee_{\text{even}} \otimes \Lambda^{\text{top}}T^\vee_M|^{\frac{1}{2}} \rightarrow M$ via the projection map $\pi : M \rightarrow M$. Here $|\Lambda^{\text{top}}E^\vee_{\text{even}} \otimes \Lambda^{\text{top}}T^\vee_M|$ denotes the line bundle $\Lambda^{\text{top}}E^\vee_{\text{even}} \otimes \Lambda^{\text{top}}T^\vee_M$ tensored by its orientation bundle. Since $E_{\text{odd}} \rightarrow M$ is orientable, $\Lambda^{\text{top}}E_{\text{odd}}$ can be identified with $|\Lambda^{\text{top}}E_{\text{odd}}|$ and hence $(\Lambda^{\text{top}}E_{\text{odd}})^{\frac{1}{2}}$ is defined, by abuse of notation.

Definition 2.2. A differential operator acting on Berezinian half-densities, of order $\leq n \in \mathbb{Z}$, is an endomorphism of sections

$$\Delta : \Gamma(\text{Dens}\frac{1}{2}_M) \rightarrow \Gamma(\text{Dens}\frac{1}{2}_M)$$

such that:

- for any $n < 0$, all operators are identically zero;
- for $n = 0$, any operator $\Delta$ is identified with a function $f \in C^\infty(M)$ which acts through the $C^\infty(M)$-module structure; and
- for all $n > 0$, the commutator $[\Delta, f]$ is a differential operator of order $\leq n - 1$ for any operator $f$ of zero order.
Under a chosen coordinate system \((x^a)\), a differential operator of order \(\leq n\) is determined by a basis of partial derivatives:

\[
\Delta s = \sum_{k=0}^{n} \frac{1}{k!} \Delta^{a_1 \cdots a_k} (x) \partial_{a_k} \cdots \partial_{a_1} s(x) \sqrt{\mathcal{D} x}, \quad s \in \Gamma (\text{Dens}^{1/2}_M).
\] (10)

The space of all differential operators on Berezinian half-densities is denoted by \(\text{DO}(\text{Dens}^{1/2}_M)\), and carries an increasing filtration determined by the order of differential operators:

\[
\text{DO}^0 (\text{Dens}^{1/2}_M) \subset \text{DO}^{\leq 1} (\text{Dens}^{1/2}_M) \subset \text{DO}^{\leq 2} (\text{Dens}^{1/2}_M) \subset \cdots ,
\] (11)

where \(\text{DO}^{\leq n} (\text{Dens}^{1/2}_M)\) denotes the space of differential operators of order at most \(n\). In particular, \(\text{DO}^0 (\text{Dens}^{1/2}_M) \cong C^\infty (\mathcal{M})\), where functions act by the \(C^\infty (\mathcal{M})\)-module structure, while \(\text{DO}^{\leq 1} (\text{Dens}^{1/2}_M)\) is naturally identified with \(\Gamma (T_M) \oplus C^\infty (\mathcal{M})\), where vector fields act by taking the Lie derivative.

It is clear that \(\text{DO}(\text{Dens}^{1/2}_M)\) is a filtered associated algebra when being equipped with the natural multiplication of composition of operators:

\[
\text{DO}^{\leq n} (\text{Dens}^{1/2}_M) \cdot \text{DO}^{\leq m} (\text{Dens}^{1/2}_M) \subseteq \text{DO}^{\leq n+m} (\text{Dens}^{1/2}_M).
\]

When being equipped with the commutator of endomorphisms, \(\text{DO}(\text{Dens}^{1/2}_M)\) is a filtered Lie algebra:

\[
\left[ \text{DO}^{\leq n} (\text{Dens}^{1/2}_M), \text{DO}^{\leq m} (\text{Dens}^{1/2}_M) \right] \subseteq \text{DO}^{\leq n+m-1} (\text{Dens}^{1/2}_M)
\] (12)

reducing the order by one. Define the associated graded algebras

\[
\text{GrDO}(\text{Dens}^{1/2}_M) = \bigoplus_{n \geq 0} \text{Gr}^n \text{DO}(\text{Dens}^{1/2}_M), \quad \text{and}
\]

\[
\hat{\text{GrDO}}(\text{Dens}^{1/2}_M) = \prod_{n \geq 0} \text{Gr}^n \text{DO}(\text{Dens}^{1/2}_M),
\]

where

\[
\text{Gr}^n \text{DO}(\text{Dens}^{1/2}_M) := \text{DO}^{\leq n} (\text{Dens}^{1/2}_M)/\text{DO}^{\leq n-1} (\text{Dens}^{1/2}_M).
\]

The following is an extension, in the graded context, of a well-known result for ordinary manifolds. For completeness, we will sketch a proof.

**Lemma 2.3.**

(1). As a \(C^\infty (\mathcal{M})\)-module, \(\text{Gr}^n \text{DO}(\text{Dens}^{1/2}_M)\) can be naturally identified with \(\Gamma (S^n T_M)\).

(2). Under the identification above, \(\text{GrDO}(\text{Dens}^{1/2}_M)\) and \(\hat{\text{GrDO}}(\text{Dens}^{1/2}_M)\) can be naturally identified, as \(C^\infty (\mathcal{M})\)-modules, with \(\Gamma (ST_M)\) and \(\Gamma (\hat{ST}_M)\), respectively.

**Proof.**
(1). Introduce the \( n \)th principal symbol map

\[ \sigma_n : \text{DO}^{\leq n} (\text{Dens}_{M}^{1/2}) \to \Gamma(S^n T_M) \]

as follows. Locally, write \( \Delta = \sum_{k=0}^{n} \frac{1}{k!} \Delta^{a_1 \cdots a_k} (x) \partial_{a_k} \cdots \partial_{a_1} \), as in Eq. (10). Then \( \sigma_n (\Delta) \) is a weight \( n \) symmetric contravariant tensor field on \( M \) of equal degree, given by

\[ \sigma_n (\Delta) = \sigma_n \left( \sum_{k=0}^{n} \frac{1}{k!} \Delta^{a_1 \cdots a_k} (x) \partial_{a_k} \cdots \partial_{a_1} \right) = \frac{1}{n!} \Delta^{a_1 \cdots a_n} (x) \partial_{a_n} \cdots \partial_{a_1}. \]

(13)

It is simple to check that \( \sigma_n \) is indeed a well-defined \( C^\infty (M) \)-module map, and moreover, any operator \( \Delta \) is of order \( < n \) if and only if \( \sigma_n (\Delta) \) vanishes identically. Thus it follows that \( \ker \sigma_n \cong \text{DO}^{\leq n-1} (\text{Dens}_{M}^{1/2}) \) and therefore the conclusion follows.

(2). This is an immediate consequence of (1).

\[ \square \]

Again the following lemma extends a standard result for ordinary differential operators to the graded context.

**Lemma 2.4.** Let \( \Delta \in \text{DO}^{\leq n} (\text{Dens}_{M}^{1/2}) \) and \( \Delta' \in \text{DO}^{\leq m} (\text{Dens}_{M}^{1/2}) \). Then

(1).

\[ \sigma_{n+m} (\Delta \cdot \Delta') = \sigma_n (\Delta) \sigma_m (\Delta') \quad \text{and} \]

(14)

(2).

\[ \sigma_{n+m-1} ([\Delta, \Delta']) = \{ \sigma_n (\Delta), \sigma_m (\Delta') \}, \]

(15)

where \( \{ \cdot, \cdot \} \) is the degree 0 Poisson bracket on \( \Gamma(ST_M) \) as given by Eq. (8).

**Proof.**

(1). It suffices to prove Eq. (14) locally. The operators \( \Delta \) and \( \Delta' \) can be written locally as follows:

\[ \Delta = \sum_{k=0}^{n} \frac{1}{k!} \Delta^{a_1 \cdots a_k} (x) \partial_{a_k} \cdots \partial_{a_1} \quad \text{and} \]

\[ \Delta' = \sum_{k=0}^{m} \frac{1}{k!} (\Delta')^{a_1 \cdots a_k} (x) \partial_{a_k} \cdots \partial_{a_1}. \]

Then \( \Delta \cdot \Delta' \in \text{DO}^{\leq n+m} (\text{Dens}_{M}^{1/2}) \) and its leading term is equal to

\[ \frac{1}{n! m!} (-1)^{(\Delta')^{b_1 \cdots b_m} (x) (\partial_{a_n} \cdots \partial_{a_1})} \Delta^{a_1 \cdots a_n} (x) (\Delta')^{b_1 \cdots b_m} (x) \partial_{a_n} \cdots \partial_{a_1} \partial_{b_m} \cdots \partial_{b_1}. \]
Therefore,
\[
\sigma_{n+m} \left( \Delta \cdot \Delta' \right) = \frac{1}{n!} (-1)^{|\Delta'|^{b_1 \cdots b_m} + |\sum_{b_{a_n} + \cdots + b_{a_1}}|} \Delta^{a_1 \cdots a_n} (x) \Delta'^{b_1 \cdots b_m} (x) \partial_{a_n} \otimes \cdots \otimes \partial_{a_1} \sigma_m (\Delta').
\]

(2) We prove it by induction on the pair \((n, m)\). It follows from a direct verification that Eq. (15) holds in the case that \(n = 0, 1\) and \(m = 0, 1\). Assume that Eq. (15) holds for \(n \leq N\) and \(m \leq M\) with \(N \geq 1\) and \(M \geq 1\). We will prove that Eq. (15) holds for \(n \leq N\) and \(m = M + 1\). Consider the case that \(\Delta' = \Delta'_1 \cdot \Delta'_2\), where \(\Delta'_1 \in DO^{\leq m_1} (\text{Dens}^{1/2}_{\mathcal{M}_1})\) and \(\Delta'_2 \in DO^{\leq m_2} (\text{Dens}^{1/2}_{\mathcal{M}_2})\) are of homogeneous degree, and \(m = M + 1 = m_1 + m_2\) with \(m_1 \leq M\) and \(m_2 \leq M\). Then
\[
\sigma_{n+m-1} ((\Delta, \Delta')) = \sigma_{n+m-1} ((\Delta, \Delta'_1 \cdot \Delta'_2)) = \sigma_{n+m-1} \left( (\Delta, \Delta'_1) \cdot \Delta'_2 + (-1)^{|\Delta| |\Delta'_1|} (\Delta'_1 \cdot [\Delta, \Delta'_2]) \right) \quad \text{(by Eq. (14))}
\]
\[
= \sigma_{n+m-1} ((\Delta, \Delta'_1)) \cdot \sigma_{m_2} (\Delta'_2)
\]
\[
+ (-1)^{|\Delta| |\Delta'_1|} \sigma_{m_1} (\Delta'_1) \sigma_{m_2} (\Delta'_2) - \sigma_{m_1} (\Delta'_1) \sigma_{m_2} (\Delta'_2) \quad \text{(by induction assumption)}
\]
\[
= \{ \sigma_n (\Delta), \sigma_{m_1} (\Delta'_1) \} \sigma_{m_2} (\Delta'_2) + (-1)^{|\Delta| |\Delta'_1|} \sigma_{m_1} (\Delta'_1) \{ \sigma_n (\Delta), \sigma_{m_2} (\Delta'_2) \}
\]
\[
= \{ \sigma_n (\Delta), \sigma_{m_1+m_2} (\Delta'_1 \cdot \Delta'_2) \} \quad \text{(by Eq. (14))}
\]
\[
= \{ \sigma_n (\Delta), \sigma_m (\Delta') \}.
\]

Since any operator in \(DO^{\leq M+1} (\text{Dens}^{1/2}_{\mathcal{M}})\) can always be written as such a product \(\Delta'_1 \cdot \Delta'_2\) module \(DO^{\leq M} (\text{Dens}^{1/2}_{\mathcal{M}})\), we conclude that Eq. (15) holds for any \(n \leq N\) and \(m = M+1\). By induction, it implies that Eq. (15) holds for any \(n \leq N\) and \(m \in \mathbb{N}\). Since both sides of Eq. (15) are skew-symmetric, the conclusion thus follows. \(\Box\)

As a consequence, the principal symbol maps establish an isomorphism of degree zero Poisson algebras
\[
\hat{\text{GrDO}} (\text{Dens}^{1/2}_{\mathcal{M}}) \cong \Gamma (\hat{\text{ST}}_{\mathcal{M}}),
\]
where the bracket on the completed graded algebra \(\hat{\text{GrDO}} (\text{Dens}^{1/2}_{\mathcal{M}})\) is induced from the commutator of differential operators, and the bracket on \(\Gamma (\hat{\text{ST}}_{\mathcal{M}})\) is the canonical Poisson bracket (8).

The existence of a canonical non-degenerate scalar product \(\langle \cdot, \cdot \rangle\) on half-densities allows us to speak naturally about the (formal) adjoint of a differential operator as in the classical case [9].

**Definition 2.5.** Let \(\Delta \in DO (\text{Dens}^{1/2}_{\mathcal{M}})\) be a differential operator of homogeneous degree. The (formal) adjoint operator \(\Delta^+ \in DO (\text{Dens}^{1/2}_{\mathcal{M}})\) is defined by the relation
\[
\langle \Delta s, t \rangle = (-1)^{|\Delta||s|} \langle s, \Delta^+ t \rangle.
\]
where \( s, t \in \Gamma(\text{Dens}_{1/2}^1) \) are compactly supported half-densities of homogeneous degrees.

As in the classical case of ordinary differential operators [9, Section 2.1] [50, Section VIII], one can see that for any operator \( \Delta \), the (formal) adjoint operator always exists and is unique.

The following lemma can be verified directly.

**Lemma 2.6.** (1) Taking the (formal) adjoint is a linear operation on \( \text{DO}(\text{Dens}_{1/2}^1) \).
(2) If \( \Delta \) is a differential operator of order \( n \) and of homogeneous degree, then the (formal) adjoint operator \( \Delta^+ \) is also of order \( n \) and carries the same degree.
(3) For any operator \( \Delta \), the (formal) adjoint is an involution, i.e. \( (\Delta^+)^+ = \Delta \).
(4) For any pair of operators \( \Delta \) and \( \Delta' \) of homogeneous degrees,
\[
(\Delta \cdot \Delta')^+ = (-1)^{1|\Delta||\Delta'|} \Delta^+ \cdot \Delta^+.
\]

**Lemma 2.7.** Let \( \Delta \) be a first order differential operator on \( \text{Dens}_{1/2}^1 \) of homogeneous degree, described locally by \( \Delta = X^a(x) \partial_a + X(x) \). The adjoint operator is given by
\[
\Delta^+ = -X^a(x) \partial_a - (-1)^{1|\partial_a||X^a|} \partial_a (X^a)(x) + X_0(x).
\]

**Proof.** By integration by parts, we have
\[
\langle \partial_a s, t \rangle = -(-1)^{1|\partial_a||s|} \langle s, \partial_a t \rangle.
\]
It thus follows that
\[
\partial_a^+ = -\partial_a. \tag{17}
\]
On the other hand,
\[
\langle X^a(x) \cdot s, t \rangle = (-1)^{1|X^a||s|} \langle s, X^a(x) \cdot t \rangle.
\]
Thus it follows that
\[
X^a(x)^+ = X^a(x), \tag{18}
\]
where, by abuse of notation, \( X^a(x) \) means the multiplication operator by \( X^a(x) \).

Therefore, we have
\[
\Delta^+ = \left( X^a(x) \partial_a + X_0(x) \right)^+ = (X^a(x) \partial_a)^+ + (X_0(x))^+ \quad \text{(by Lemma 2.6(4))}
= (-1)^{1|\partial_a||X^a|} \partial_a^+ \cdot X^a(x)^+ + X_0(x)^+ \quad \text{(by Eqs. (17)-(18))}
= -(-1)^{1|\partial_a||X^a|} \partial_a \cdot X^a(x) + X_0(x)
= -X^a(x) \partial_a - (-1)^{1|\partial_a||X^a|} \partial_a (X^a)(x) + X_0(x).
\]

**Definition 2.8.** An operator \( \Delta \in \text{DO}(\text{Dens}_{1/2}^1) \) is called self-adjoint if \( \Delta^+ = \Delta \), and anti-self-adjoint if \( \Delta^+ = -\Delta \).
Let $\text{DO}^+/(\text{Dens}_{\mathcal{M}}^{1/2}) \subset \text{DO}(\text{Dens}_{\mathcal{M}}^{1/2})$ denote the subset consisting of all self-adjoint and anti-self-adjoint differential operators on Berezinian half-densities. Note that although both the space of self-adjoint differential operators and the space of anti-self-adjoint differential operators are vector spaces, $\text{DO}^+/(\text{Dens}_{\mathcal{M}}^{1/2})$ is not a vector space. The sum of a self-adjoint and anti-self-adjoint operator is no longer self or anti-self adjoint.

**Lemma 2.9.** For any differential operator $\Delta \in \text{DO}(\text{Dens}_{\mathcal{M}}^{1/2})$ of order $n$, we have

$$\sigma_n(\Delta^+) = (-1)^n \sigma_n(\Delta). \quad (19)$$

**Proof.** Let $(x^a)$ be coordinates on $\mathcal{M}$. Assume that the leading term of $\Delta$ is $\frac{1}{n!} \Delta^{a_1 \cdots a_n}(x) \partial_{a_n} \cdots \partial_{a_1}$. According to Lemma 2.6(2) (4) and Eqs. (17)–(18), we have

$$\Delta^+ = (-1)^n \frac{1}{n!} \Delta^{a_1 \cdots a_n}(x) \partial_{a_n} \cdots \partial_{a_1} + \cdots.$$  

The conclusion thus follows immediately. \qed

**Corollary 2.10.** Let $\Delta \in \text{DO}^+/(\text{Dens}_{\mathcal{M}}^{1/2})$ be a differential operator of order $n$. If $\Delta$ is self-adjoint, then $n$ is necessarily even. Likewise if $\Delta$ is anti-self-adjoint, then $n$ is necessarily odd. Hence, we have

$$\Delta^+ = (-1)^n \Delta.$$  

**Proof.** Assume that $\Delta$ is a self-adjoint operator of order $n$. Then $\Delta^+ = \Delta$. Applying the principal symbol map, we have $(-1)^n \sigma_n(\Delta) = \sigma_n(\Delta)$ according to Lemma 2.9. Since $\sigma_n(\Delta) \neq 0$, thus it follows that $n$ must be even. Similarly we prove that if $\Delta$ is anti-self-adjoint, then $n$ is necessarily odd. \qed

**Proposition 2.11.** The set $\text{DO}^+/(\text{Dens}_{\mathcal{M}}^{1/2})$ is closed under the commutator (12), and inherits the increasing filtration (11) determined by the order.

**Proof.** That this set inherits the filtration is evident. Now let $\Delta, \Delta' \in \text{DO}^+/(\text{Dens}_{\mathcal{M}}^{1/2})$ be operators of order $n$ and $m$ respectively. Consider the commutator $[\Delta, \Delta']$. From Lemma 2.6 and Corollary 2.10, it follows that

$$[\Delta, \Delta']^+ = \left( \Delta \cdot \Delta' - (-1)^{|\Delta||\Delta'|} \Delta' \cdot \Delta \right)^+ = (-1)^{|\Delta||\Delta'|} \Delta'^+ \cdot \Delta^+ - \Delta^+ \cdot \Delta'^+ = (-1)^{|\Delta||\Delta'|+n+m} \Delta' \cdot \Delta - (-1)^{n+m} \Delta \cdot \Delta' = (-1)^{n+m-1} [\Delta, \Delta']. \quad (20)$$

Since $[\Delta, \Delta']$ is an operator of order at most $n + m - 1$, it must be either self-adjoint or anti-self-adjoint depending on the parity of $n + m - 1$. \qed

Note that although the set $\text{DO}^+/(\text{Dens}_{\mathcal{M}}^{1/2})$ is closed under the commutator, it does not form a Lie algebra since it is not a vector space.

**Corollary 2.12.** Let $\Delta, \Delta' \in \text{DO}^+/(\text{Dens}_{\mathcal{M}}^{1/2})$ be differential operators of order $n$ and $m$, respectively. Assume that the order of the commutator $[\Delta, \Delta']$ is strictly less than $n + m - 1$. Then $[\Delta, \Delta']$ must be of order strictly less than $n + m - 2$ also.
Note that the terms of order $n + m - 2$ must also vanish, since the parity of the order does not coincide with the adjoint condition. Hence $[\Delta, \Delta']$ is an operator of order at most $n + m - 3$.

\section*{2.2. ($h$)-enhanced differential operators.} Introduce a degree 0 parameter $h$, and define the space of $h$-enhanced differential operators $h\text{DO}(\text{Dens}^{1/2}_{\mathcal{M}})$ by

$$h\text{DO}(\text{Dens}^{1/2}_{\mathcal{M}}) := \text{DO}^{\leq 0}(\text{Dens}^{1/2}_{\mathcal{M}}) \oplus h\text{DO}^{\leq 1}(\text{Dens}^{1/2}_{\mathcal{M}}) \oplus \cdots = \bigoplus_{n \geq 0} h^n \text{DO}^{\leq n}(\text{Dens}^{1/2}_{\mathcal{M}}),$$

where $h^n \text{DO}^{\leq n}(\text{Dens}^{1/2}_{\mathcal{M}})$ denotes the space of differential operators of order at most $n$, acting on Berezian half-densities enhanced with coefficients in $\mathbb{R}[h]$. An arbitrary operator $\Delta \in h\text{DO}(\text{Dens}^{1/2}_{\mathcal{M}})$ is written as

$$\Delta = \sum_{n \geq 0} h^n \Delta_n, \quad \Delta_n \in \text{DO}^{\leq n}(\text{Dens}^{1/2}_{\mathcal{M}}), \quad (21)$$

where the maximum possible order of each operator $\Delta_n$ with coefficient $h^n$ is $n$.

The introduction of the parameter $h$ allows us to introduce a secondary filtration on $h\text{DO}(\text{Dens}^{1/2}_{\mathcal{M}})$. Let $\Delta \in h\text{DO}(\text{Dens}^{1/2}_{\mathcal{M}})$ be an arbitrary operator as defined in (21), and define the non-negative integer

$$t(\Delta) := \min_{n \geq 0} \left\{ n - \text{order}(\Delta_n) \mid \Delta = \sum_{n \geq 0} h^n \Delta_n \right\}. \quad (22)$$

For example, if any of the operators $\Delta_n$ in the expansion of $\Delta$ is of order $n$, then $t(\Delta) = 0$. For each $t \geq 0$, define the set of operators

$$h\text{DO}_t(\text{Dens}^{1/2}_{\mathcal{M}}) := \left\{ \Delta \in h\text{DO}(\text{Dens}^{1/2}_{\mathcal{M}}) \mid t(\Delta) \geq t \right\}.$$
In particular, $\hbar DO_0(\text{Dens}^{1/2}_\mathcal{M})$ is the whole set $\hbar DO(\text{Dens}^{1/2}_\mathcal{M})$, whilst for an arbitrary integer $t > 0$, $\hbar DO_t(\text{Dens}^{1/2}_\mathcal{M})$ consists of those operators $\Delta \in \hbar DO(\text{Dens}^{1/2}_\mathcal{M})$ such that $\hbar^{-t} \Delta \in \hbar DO_0(\text{Dens}^{1/2}_\mathcal{M})$. This defines a decreasing filtration on $\hbar DO(\text{Dens}^{1/2}_\mathcal{M})$ indexed by $t$:
\[
\cdots \subset \hbar DO_t(\text{Dens}^{1/2}_\mathcal{M}) \subset \cdots \subset \hbar DO_1(\text{Dens}^{1/2}_\mathcal{M}) \subset \hbar DO_0(\text{Dens}^{1/2}_\mathcal{M}) = \hbar DO(\text{Dens}^{1/2}_\mathcal{M}).
\]

(23)

It is beneficial to define a commutator modified by the introduced parameter. Set the modified commutator $[-, -]_h$ on $\hbar DO(\text{Dens}^{1/2}_\mathcal{M})$ as the degree 0 operation
\[
[\Delta, \Delta']_h := \frac{1}{\hbar} [\Delta, \Delta'] = \sum_{n \geq 0} h^n \sum_{i+j=n+1, \ i,j \geq 0} [\Delta_i, \Delta_j],
\]
with $\Delta = \sum_{i \geq 0} h^i \Delta_i$ and $\Delta' = \sum_{j \geq 0} h^j \Delta_j'$, and $[-, -]$ denotes the usual commutator.

**Remark 2.14.** The parameter $\hbar$ should serve as a counting variable, relating the order of an operator to the weight of the principal symbol as a contravariant tensor. Since the commutator of two operators reduces the order, it is natural to reduce the power of an operator to the weight of the principal symbol as a contravariant tensor. Since $\hbar$ compensates the degree shift between $(-1)$-shifted derived Poisson structures and $(+1)$-shifted derived Poisson structures. Recently, Shemyakova also systematically studied $\langle h \rangle$-enhanced differential operators in connection with the study of $\text{BV}_\infty$ operators generating higher Koszul brackets on differential forms [52].

The following lemma is quite obvious.

**Lemma 2.15.** (1) For any integer $t \geq 0$ and $\Delta \in \hbar DO_t(\text{Dens}^{1/2}_\mathcal{M})$, there exists a unique $\Delta' \in \hbar DO_0(\text{Dens}^{1/2}_\mathcal{M})$ such that $\Delta = \hbar^t \Delta'$.

(2) For any integers $s, t \geq 0$, the modified commutator respects the $\hbar$-induced filtration
\[
[\hbar DO_s(\text{Dens}^{1/2}_\mathcal{M}), \ hDO_t(\text{Dens}^{1/2}_\mathcal{M})]_h \subset \hbar DO_{s+t}(\text{Dens}^{1/2}_\mathcal{M}).
\]

**Corollary 2.16.** The space of $\hbar$-enhanced differential operators $\hbar DO(\text{Dens}^{1/2}_\mathcal{M})$ is a filtered Lie algebra under the modified commutator $[-, -]_h$ and the filtration (23).

**Definition 2.17.** The extended principal symbol map is defined as
\[
\sigma_h : \hbar DO(\text{Dens}^{1/2}_\mathcal{M}) \to \Gamma(\hat{ST}_\mathcal{M})[\hbar], \quad \sigma_h(\Delta) = \sum_{n \geq 0} h^n \sigma_n(\Delta_n),
\]
for any $\Delta = \sum_{n \geq 0} h^n \Delta_n \in \hbar DO(\text{Dens}^{1/2}_\mathcal{M})$, where $\sigma_n$ is the $n$th principal symbol map as in (13), and $\Gamma(\hat{ST}_\mathcal{M})[\hbar]$ denotes the space of formal power series in $\hbar$ with coefficients in the completed symmetric algebra.

The following lemma follows immediately from Eq. (24) and Lemma 2.4.

**Lemma 2.18.** For any pair of operators $\Delta, \Delta' \in \hbar DO(\text{Dens}^{1/2}_\mathcal{M})$,
\[
\sigma_h([\Delta, \Delta']_h) = \frac{1}{\hbar} \{\sigma_h(\Delta), \sigma_h(\Delta')\}.
\]
Remark 2.21. If we introduce an adjoint operation on the formal parameter \( \Delta = \sum_{n \geq 0} h^n \Delta_n \), the extended principal symbol map is non-zero only on those terms where the order of the operator \( \Delta_n \) is exactly equal to \( n \), and hence cannot see symbols of operators of lower orders. To remedy this, one can define a sequence of maps, viewing \( h\text{DO}(\text{Dens}^{1/2}_\mathcal{M}) \) as \( h\text{DO}_0(\text{Dens}^{1/2}_\mathcal{M}) \) in the decreasing \( h \)-induced filtration (23), which extend to each \( h\text{DO}_i(\text{Dens}^{1/2}_\mathcal{M}) \) in turn. For each \( i \geq 1 \), define the \( t \)-th extended principal symbol map

\[
\sigma^t_h : h\text{DO}_i(\text{Dens}^{1/2}_\mathcal{M}) \to \Gamma(\hat{ST}_\mathcal{M})[\|h\|], \quad \sigma^t_h(\Delta) = h^t \sigma_h(h^{-t} \Delta),
\]

for \( \Delta \in h\text{DO}_i(\text{Dens}^{1/2}_\mathcal{M}) \). The extended principal symbol map (25) fits into this sequence with \( t = 0 \).

**Corollary 2.19.** For operators \( \Delta \in h\text{DO}_i(\text{Dens}^{1/2}_\mathcal{M}) \), and \( \Delta' \in h\text{DO}_s(\text{Dens}^{1/2}_\mathcal{M}) \), the sequence of extended symbol maps preserve the modified commutator

\[
\sigma^t_h(\{[\Delta, \Delta']_h\}) = \frac{1}{h^t} \{\sigma^t_h(\Delta), \sigma^t_h(\Delta')\}. \tag{28}
\]

**Definition 2.20.** An operator \( \Delta = \sum_{n \geq 0} h^n \Delta_n \in h\text{DO}(\text{Dens}^{1/2}_\mathcal{M}) \) is called self-adjoint if \( \Delta_n \) is self-adjoint for all even \( n \), and anti-self-adjoint for all odd \( n \).

**Remark 2.21.** If we introduce an adjoint operation on the formal parameter \( h \) such that \( h^+ = -h \), then the self-adjointness of an operator \( \Delta \in h\text{DO}(\text{Dens}^{1/2}_\mathcal{M}) \) can be simply written as

\[ \Delta^+ = \Delta. \]

Denote the set of all \( h \)-enhanced self-adjoint operators by \( h\text{DO}^+((\text{Dens}^{1/2}_\mathcal{M}) \), which inherits both filtrations (11) defined by order, and (23) defined by the introduction of \( h \).

**Lemma 2.22.** The space of self-adjoint differential operators \( h\text{DO}^+((\text{Dens}^{1/2}_\mathcal{M}) \) is a Lie subalgebra of \( h\text{DO}(\text{Dens}^{1/2}_\mathcal{M}) \) under the modified commutator (24).

**Proof.** The space of self-adjoint operators is clearly a vector space. Assume that \( \Delta = \sum_{n \geq 0} h^n \Delta_n \) and \( \Delta' = \sum_{n \geq 0} h^n \Delta'_n \) are self-adjoint differential operators. By Eq. (24),

\[ [\Delta, \Delta']_h = \sum_{n \geq 0} h^n \tilde{\Delta}_n, \]

where \( \tilde{\Delta}_n = \sum_{i+j=n+1, i,j \geq 0} [\Delta_i, \Delta_j] \in \text{DO}^{\xi\xi}(\text{Dens}^{1/2}_\mathcal{M}) \). By assumption, for all \( n \geq 0 \), \( \Delta_n \) and \( \Delta'_n \) are operators satisfying \( \Delta^+_n = (-1)^n \Delta_n \) and \( (\Delta'_n)^+ = (-1)^n \Delta'_n \). It thus follows that

\[ \tilde{\Delta}_n^+ = \sum_{i+j=n+1, i,j \geq 0} [\Delta_i, \Delta_j]^+ = \sum_{i+j=n+1, i,j \geq 0} (-1)^n [\Delta_i, \Delta_j] = (-1)^n \tilde{\Delta}_n. \]

Here we used essentially the same computation as in (20) to prove that \( [\Delta_i, \Delta_j]^+ = (-1)^n [\Delta_i, \Delta_j] \). Therefore \( [\Delta, \Delta']_h \in h\text{DO}^+((\text{Dens}^{1/2}_\mathcal{M}) \). Hence, \( h\text{DO}^+((\text{Dens}^{1/2}_\mathcal{M}) \) is indeed a Lie subalgebra of \( h\text{DO}(\text{Dens}^{1/2}_\mathcal{M}) \). \( \square \)

**Lemma 2.23.** Let \( \Delta \in h\text{DO}^+_0((\text{Dens}^{1/2}_\mathcal{M}) \). Then \( \sigma_h(\Delta) = 0 \) is equivalent to that \( \Delta \in h\text{DO}^+_2((\text{Dens}^{1/2}_\mathcal{M}) \).
Proof. Let \( \Delta = \sum_{n \geq 0} \hbar^n \Delta_n \). Since \( \sigma_{\hbar} (\Delta) = 0 \), it follows from (25) that \( \sigma_{\hbar} (\Delta_n) = 0 \), for all \( n \geq 0 \). Hence the order of \( \Delta_n \) is at most \( (n - 1) \). By assumption \( \Delta_n \in \text{DO}^+ (\text{Den}_{1/2}^1 (\mathcal{M})) \) and \( \Delta_\hbar^+ = (-\hbar)^n \Delta_n \), so \( \Delta_n \) cannot be of order \( (n - 1) \) according to Corollary 2.20, and thus at most \( n - 2 \). Hence it follows that \( \Delta \) indeed belongs to \( \hbar \text{DO}^+_2 (\text{Den}_{1/2}^1 (\mathcal{M})) \). The converse is obvious. \( \square \)

**Corollary 2.24.** Let \( \Delta \) and \( \Delta' \in \text{hDO}^+_t (\text{Den}_{1/2}^1 (\mathcal{M}) \) be self-adjoint operators with \( t (\Delta) = t (\Delta') \) such that \( \sigma_{\hbar} (\Delta) = \sigma_{\hbar} (\Delta') \). Then \( \Delta - \Delta' \in \text{hDO}^+_t (\text{Den}_{1/2}^1 (\mathcal{M}) \).\( \square \)

**Proof.** By assumption, we have \( \sigma_{\hbar} (\frac{1}{\hbar t} (\Delta - \Delta')) = 0 \). According to Lemma 2.23, \( \frac{1}{\hbar t} (\Delta - \Delta') \in \text{hDO}^+_t (\text{Den}_{1/2}^1 (\mathcal{M}) \). Hence \( \Delta - \Delta' \in \text{hDO}^+_t (\text{Den}_{1/2}^1 (\mathcal{M}) \).\( \square \)

2.3. **BV\(_\infty\)-operators and \([-1\)-shifted derived Poisson manifolds.** Following Kravchenko [36], we introduce the following

**Definition 2.25.** A BV\(_\infty\)-operator acting on the line bundle \( \text{Den}_{1/2}^1 (\mathcal{M}) \) is a degree \((+1)\) differential operator \( \Delta \in \text{hDO}_0 (\text{Den}_{1/2}^1 (\mathcal{M}) \) such that \( \sigma_{\hbar} (\Delta) \neq 0 \), and:

1. \( \lim_{\hbar \to 0} \Delta = 0 \),
2. \( \Delta^2 = \frac{\hbar}{2} [\Delta, \Delta]_{\hbar} = 0 \).

Condition (1) ensures that \( \Delta \) can be expressed as

\[
\Delta = \sum_{n \geq 1} \hbar^n \Delta_n ,
\]

where each \( \Delta_n \in \text{DO}^{\leq n} (\text{Den}_{1/2}^1 (\mathcal{M})) \) is a differential operator of order at most \( n \).

**Remark 2.26.** Choose a nowhere vanishing Berezinian density \( \rho \in \Gamma (\text{Den}_{1/2}^1 (\mathcal{M}) \). With such a choice of Berezinian density, we may identify differential operators on \( \Gamma (\text{Den}_{1/2}^1 (\mathcal{M}) \) with differential operators on \( C^\infty (\mathcal{M}) \) by

\[
\Delta_{\rho} f := \frac{1}{\sqrt{\rho}} \Delta (f \sqrt{\rho}), \quad f \in C^\infty (\mathcal{M}), \quad \Delta \in \text{DO} (\text{Den}_{1/2}^1 (\mathcal{M}) \).
\]

(29)

In this case, our BV\(_\infty\)-operators coincide with the homotopy BV-operators of Kravchenko [36]. See also [2,4,10,28,41,43,48,52,60].

Since the Berezinian density line bundle over \( \mathcal{M} \) is always trivalizable, in fact, one can always work with differential operators on \( C^\infty (\mathcal{M}) \) instead of those on half densities \( \Gamma (\text{Den}_{1/2}^1 (\mathcal{M}) \). Here, however, we choose to work with the latter following [8,24,27,29]. This approach has the advantage of being intrinsic and allows us to speak about the (formal) adjoint operator of a differential operator naturally, which plays a crucial role in our construction. We refer the interested readers to [24] for details on justification regarding the half densities approach as the proper framework for the Batalin–Vilkovisky (BV) formalism in the classical case.

For any \( X \in \Gamma (\hat{S} T_{\mathcal{M}}) \), where \( X = \sum_{n \geq 0} X_n \) with \( X_n \in \Gamma (\hat{S}^n T_{\mathcal{M}}) \), define the map

\[
\cdot_\hbar : \Gamma (\hat{S} T_{\mathcal{M}}) \rightarrow \Gamma (\hat{S} T_{\mathcal{M}})[[\hbar]], \quad \cdot_\hbar (X) = X_\hbar := \sum_{n \geq 0} \hbar^n X_n .
\]

(30)
The degree zero Poisson bracket (8) on $\Gamma(\hat{S}T\mathcal{M})$ then extends to the algebra of formal power series $\Gamma(\hat{S}T\mathcal{M})[[h]]$ by $h$-linearity. The following lemma can be easily verified directly.

**Lemma 2.27.** For any $X, Y, Z \in \Gamma(\hat{S}T\mathcal{M})$,

$$Z = \{X, Y\} \Leftrightarrow Z_h = \frac{1}{h}\{X_h, Y_h\}.$$  

Essentially, Lemma 2.27 indicates that the map (30) is a morphism of Poisson algebras. Denote

$$\Gamma(\hat{S}T\mathcal{M})_h := \{X_h|X_h = \sum_{n \geq 0} h^n X_n, \text{ where } X_n \in \Gamma(S^n T\mathcal{M})\}.$$

Then $\Gamma(\hat{S}T\mathcal{M})_h$ is a Poisson subalgebra of $\Gamma(\hat{S}T\mathcal{M})[[h]]$ and the map (30) is an isomorphism of Poisson algebras from $\Gamma(\hat{S}T\mathcal{M})$ to $\Gamma(\hat{S}T\mathcal{M})_h$.

The theorem below indicates that a BV$_\infty$-operator induces a $(-1)$-shifted derived Poisson structure. This can be proved using Voronov’s derived bracket construction [60]. Below we will give an alternative proof using Theorem 1.6. Note that this result is known in the literature in various different forms. See [52, Theorem 3.2] and [10, Corollary], for instance.

**Theorem 2.28.** If $\Delta \in hDO_0(D\text{ens}^{1/2}_\mathcal{M})$ is a BV$_\infty$-operator, then $C^\infty(\mathcal{M})$ together with the sequence of multi-brackets $(\lambda_n)_{n \geq 1}$ defined by the following equation:

$$\lambda_n(f_1, \ldots, f_n)s := \lim_{h \to 0} [\cdots [\Delta, f_1]_h, \ldots, f_n]_h s, \quad (31)$$

for any functions $f_1, \ldots, f_n \in C^\infty(\mathcal{M})$, and any half-density $s \in \Gamma(D\text{ens}^{1/2}_\mathcal{M})$, is a $(-1)$-shifted derived Poisson algebra. Hence $\mathcal{M}$ is a $(-1)$-shifted derived Poisson manifold.

**Proof.** Write $\Delta = \sum_{n \geq 1} h^n \Delta_n$, where each $\Delta_n \in DO^{\leq n}(D\text{ens}^{1/2}_\mathcal{M})$ is of degree $(+1)$ and order at most $n$. The extended principal symbol

$$\hat{\Pi}_h := \sigma_h(\Delta) = \sum_{n \geq 1} h^n \Pi_n, \quad \Pi_n = \sigma_n(\Delta_n) \in \Gamma(S^n T\mathcal{M}),$$

determines a formal power series of degree $(+1)$ symmetric contravariant tensor fields dependent on $h$. Evaluating at $h = 1$, we obtain a degree $(+1)$ formal power series $\hat{\Pi} := \hat{\Pi}_1|_{h=1} = \sum_{n \geq 1} \Pi_n \in \Gamma(\hat{S}T\mathcal{M})$.

Since the operator $\Delta$ is of degree $(+1)$, its square is expressible via the commutator $\Delta^2 = \frac{h}{2}[\Delta, \Delta]_h$. Hence it follows that $[\Delta, \Delta]_h = 0$ by definition.

According to Lemma 2.18,

$$\sigma_h([\Delta, \Delta]_h) = \frac{1}{h}\{\sigma_h(\Delta), \sigma_h(\Delta)\} = \frac{1}{h}\{\hat{\Pi}_h, \hat{\Pi}_h\}.$$  

From Lemma 2.27, it follows that $\{\hat{\Pi}, \hat{\Pi}\} = 0$. One can write $\Pi_1 = Q$, whence $(\mathcal{M}, Q)$ is a dg manifold equipped with a formal power series $\Pi = \sum_{n \geq 2} \Pi_n$ satisfying the Maurer–Cartan equation (9). Hence by Theorem 1.6, $\mathcal{M}$ is equipped with a $(-1)$-shifted derived Poisson structure.

To write down the multi-brackets of the corresponding $(+1)$-shifted derived Poisson algebra, we note that the right hand side of (31) is exactly equal to $[\cdots [\Delta_n, f_1], \ldots, f_n]s$. Since $\Delta_n \in DO^{\leq n}(D\text{ens}^{1/2}_\mathcal{M})$ and $\sigma_n(\Delta_n) = \Pi_n$, it is standard [60] that the latter is exactly equal to $\Pi_n(df_1 \odot \cdots \odot df_n)s$. This concludes the proof of the theorem.  \(\square\)
We note that Eq. (31) is essentially Voronov’s derived bracket formula [60]. Recently Bandiera proved the homotopy transfer theorem for BV_{\infty}-algebras [2]. In particular, he showed that the homotopy transfer of a BV_{\infty}-algebra is compatible with its induced (+1)-shifted derived Poisson algebra in the sense of Theorem 2.28.

Remark 2.29. One may compare Eq. (31) with the multi-brackets constructed in [4,10,41,43,52]. As in some of these references [4,10,52], we use the \( \hbar \)-modification in order to absorb the modified commutator. Finally, note that to generate a \( +1 \)-shifted derived Poisson algebra structure, instead of requiring \( \Delta^2 = 0 \), it suffices to require that the extended principal symbol of \( \Delta^2 \) vanishes, i.e. \( \sigma_\hbar(\Delta^2) = 0 \).

Example 2.30. Consider the special case of BV_{\infty}-operators consisting of only one term

\[
\Delta = \hbar^2 \Delta_2,
\]

where \( \Delta_2 \) is a second order differential operator of degree (+1) satisfying the condition that \( \Delta_2 \cdot \Delta_2 = 0 \). In this case, Theorem 2.28 implies that \( \lambda_n = 0 \) for all \( n \neq 2 \), and the binary bracket \( \{ -,- \} := \lambda_2(\cdot,-) \) is given by,

\[
\{ f, g \}_s := \Delta_2(fg) - (-1)^{|f|+1}|g|\Delta_2(f) + (-1)^{|f|} |f| \Delta_2(g) + (-1)^{|f|+1}|g| f \Delta_2 g,
\]

for any \( f, g \in C^\infty(\mathcal{M}) \) of homogeneous degree and \( s \in \Gamma(\text{Dens}_{1/2}) \).

Assume that there exists a nowhere vanishing Berezinian density \( \rho \in \Gamma(\text{Dens}_\mathcal{M}) \) such that \( \Delta_2(\sqrt{\rho}) = 0 \). By \( \Delta_\rho \), we denote the corresponding second order differential operator on \( C^\infty(\mathcal{M}) \) as in (29) (with \( \Delta \) being replaced by \( \Delta_2 \)), then \( \Delta_\rho \) is of degree (+1) and \( \Delta^2_\rho = 0 \). Eq. (32) implies that

\[
\{ f, g \} = \Delta_\rho(fg) - \Delta_\rho(f)g - (-1)^{|f|} f \Delta_\rho(g).
\]

Thus \( \Delta_\rho \) is an ordinary BV-operator and \( \{ \cdot, \cdot \} \) is its corresponding Schouten bracket. See [20,26,27,34,35,54,64].

3. Quantization of \((-1)\)-Shifted Derived Poisson Manifolds

3.1. Quantization of graded manifolds. Recall that an (affine) connection on a graded manifold \( \mathcal{M} \) is a degree zero \( \mathbb{R} \)-bilinear map \( \nabla : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M}) \) such that

1. \( \nabla f \mathcal{X}Z = f \nabla \mathcal{X}Z \);
2. \( \nabla \mathcal{X}(f \mathcal{Y}) = \mathcal{X}(f \mathcal{Y}) + (-1)^{|f||\mathcal{X}|} f \nabla \mathcal{X} \mathcal{Y}, \)

for any vector fields \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(T\mathcal{M}) \) and smooth functions \( f, g \in C^\infty(\mathcal{M}) \), where \( \mathcal{X} \) and \( f \) are homogeneous.

We say that a connection \( \nabla \) on \( \mathcal{M} \) is torsion-free if its torsion vanishes. That is,

\[
\nabla \mathcal{X} \mathcal{Y} - (-1)^{|\mathcal{X}||\mathcal{Y}|} \nabla \mathcal{Y} \mathcal{X} - [\mathcal{X}, \mathcal{Y}] = 0
\]

for any homogeneous vector fields \( \mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{M}) \). It is well known that torsion-free connections always exist for any graded manifold \( \mathcal{M} \) [39].

Fix a torsion-free connection \( \nabla \) on \( \mathcal{M} \), and denote the induced linear connection on the line bundle \( \text{Dens}_{1/2}^\mathcal{M} \) by the same symbol \( \nabla \).
**Definition 3.1.** Let $\mathcal{M}$ be a graded manifold with a torsion-free connection $\nabla$. Define a $C^\infty(\mathcal{M})$-module map:

$$\mathcal{P}_\hbar : \Gamma(\hat{\mathcal{ST}}\mathcal{M}) \rightarrow \hbar DO(\text{Dens}^{1/2}_\mathcal{M})$$

by the inductive relations

$$\mathcal{P}_\hbar(f) = f, \quad f \in C^\infty(\mathcal{M}),$$
$$\mathcal{P}_\hbar(X) = \hbar \nabla X, \quad X \in \Gamma(T\mathcal{M}),$$

and for homogeneous elements $X_1, \ldots, X_n \in \Gamma(T\mathcal{M}),$

$$\mathcal{P}_\hbar(X_1 \odot \cdots \odot X_n) = \frac{\hbar}{n} \sum_{k=1}^{n} \varepsilon_k \left( \nabla_{X_k} \mathcal{P}_\hbar(X_1^{[k]}) - \mathcal{P}_\hbar(\nabla_{X_k} X_1^{[k]}) \right),$$

(35)

where $\varepsilon_k = (-1)^{|X_k|(|X_1|+\cdots+|X_{k-1}|)}$ and $X_1^{[k]} = X_1 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$.

**Proposition 3.2.**

1. $\mathcal{P}_\hbar$ is a well defined $C^\infty(\mathcal{M})$-module map;
2. For any $X \in \Gamma(\hat{\mathcal{ST}}\mathcal{M})$, we have

$$\sigma_\hbar(\mathcal{P}_\hbar(X)) = X_\hbar.$$ 

**Proof.** Note that $\text{Dens}^{1/2}_\mathcal{M}$ is a real line bundle over $\mathcal{M}$. Hence local flat non-zero sections always exist. Let $U \subset \mathcal{M}$ be any contractible open submanifold and $s \in \Gamma(\text{Dens}^{1/2}_\mathcal{M}|U)$ a non-zero flat section with respect to the connection $\nabla$, i.e. $\nabla_X s = 0$ for any $X \in \Gamma(T\mathcal{M}|U)$. Thus we have an isomorphism

$$\Phi : DO(\text{Dens}^{1/2}_\mathcal{M}|U) \xrightarrow{\cong} D(\mathcal{M}|U)$$

$$\Delta \quad \xrightarrow{\Delta(f)s}$$

for any $f \in C^\infty(\mathcal{M}|U)$. It is simple to check that under such an isomorphism, the map $\mathcal{P}_\hbar$ can be identified with the PBW map, i.e.

$$(\Phi \circ \mathcal{P}_\hbar)(X) = \hbar^n \text{pbw}(X), \quad \forall X \in \Gamma(S^n T\mathcal{M}|U),$$

where $\text{pbw} : \Gamma(ST\mathcal{M}) \rightarrow D(\mathcal{M})$ is the PBW map introduced by Liao-Stienon [39, Definition 3.1]. According to [39, Lemma 3.2], $\Phi \circ \mathcal{P}_\hbar$ is a well-defined $C^\infty(U)$-module map from $\Gamma(\hat{\mathcal{ST}}\mathcal{M}|U)$ to $D(\mathcal{M}|U)$, which implies that the inductive formula in Definition 3.1, when being restricted to $U$, gives rise to a well-defined $C^\infty(U)$-module map from $\Gamma(\hat{\mathcal{ST}}\mathcal{M}|U)$ to $DO(\text{Dens}^{1/2}_\mathcal{M}|U)$. Therefore it follows that $\mathcal{P}_\hbar$ is indeed a well-defined $C^\infty(\mathcal{M})$-module map.

To prove the second claim, we can write down explicitly the principal symbol of $(\Phi \circ \mathcal{P}_\hbar)(X)$ locally. Below we give a direct proof using induction. It is clear that the
claim is true for all \( X \in \Gamma(S^1 T_M) \). Assume that it is true for any \( X \in \Gamma(S^n T_M) \), where \( n \geq 2 \). According to Eq. (35), by induction assumption, we have

\[
\sigma_h \left( \mathcal{P}_h(X_1 \circ \cdots \circ X_n) \right) = \frac{\hbar}{n} \sum_{k=1}^{n} \varepsilon_k \left( X_k \circ (\sigma_h \mathcal{P}_h)(X^{[k]}) \right) = \left( \hbar^{n-1} X^{[k]} \right) = \hbar^{n} X_1 \circ \cdots \circ X_n.
\]

Hence, it follows that the claim holds for any \( X \in \Gamma(S^n T_M) \). This concludes the proof. \( \square \)

Note that \( \mathcal{P}_h(X) \) may not be self-adjoint. In order to obtain self-adjoint operators, we need to make some modification. Introduce a linear map

\[
Q_h : \Gamma(\hat{ST}_M) \to \hbar DO(\text{Dens}_{\hat{M}}^{1/2})
\]
as follows. For any \( X \in \Gamma(\hat{ST}_M) \), write

\[
\mathcal{P}_h(X) = \sum_{n \geq 0} \hbar^n \Delta_n \in \hbar DO(\text{Dens}_{\hat{M}}^{1/2}).
\]

Define

\[
Q_h(X) = \sum_{n \geq 0} \frac{\hbar^n}{2} \left( \Delta_n + (-1)^n \Delta_n^+ \right). \tag{36}
\]

**Proposition 3.3.** Let \( \mathcal{M} \) be a \( \mathbb{Z} \)-graded manifold with a torsion-free connection \( \nabla \). Then \( Q_h \), defined by (36), is a degree 0 map

\[
Q_h : \Gamma(\hat{ST}_M) \to \hbar DO^+(\text{Dens}_{\hat{M}}^{1/2}) \tag{37}
\]

satisfying the condition that, for any \( X \in \Gamma(\hat{ST}_M) \),

\[
\sigma_h(Q_h(X)) = X_h.
\]

**Proof.** We have, for any \( n \geq 0 \),

\[
(\Delta_n + (-1)^n \Delta_n^+)^+ = \Delta_n^+ + (-1)^n (\Delta_n^+)^+ = \Delta_n^+ + (-1)^n \Delta_n = (-1)^n (\Delta_n + (-1)^n \Delta_n^+).
\]
Hence, it follows that $Q_h(X) \in hDO^+(\text{Dens}_{1/2}/M)$. Moreover,

$$
\sigma_h(Q_h(X)) = \sigma_h\left( \sum_{n \geq 0} \frac{\hbar^n}{2} (\Delta_n + (-1)^n \Delta_n^+) \right) = \sum_{n \geq 0} \frac{\hbar^n}{2} (\sigma_n(\Delta_n) + (-1)^n \sigma_n(\Delta_n^+)) \quad \text{(by Eq. (19))}
$$

$$
= \sum_{n \geq 0} \frac{\hbar^n}{2} (\sigma_n(\Delta_n) + \sigma_n(\Delta_n)) = \sum_{n \geq 0} \hbar^n \sigma_n(\Delta_n) = \sigma_h(\mathcal{P}_h(X)) = X_h.
$$

This concludes the proof of the proposition.

3.2. Quantization of $(-1)$-shifted derived Poisson manifolds. If $\Gamma(\text{Dens}_{1/2}/M)$ carries a $\text{BV}_\infty$-operator as defined in Definition 2.25, Theorem 2.28 states that $M$ inherits a $(-1)$-shifted derived Poisson structure generated by the $\text{BV}_\infty$-operator. But one can ask for the converse: when does a given $(-1)$-shifted derived Poisson manifold admit a $\text{BV}_\infty$-operator which generates the derived Poisson structure? In this section, we will address this problem. First we will introduce the following (see also [16,48]):

**Definition 3.4.** A quantization of a $(-1)$-shifted derived Poisson manifold $(M, Q, \Pi)$ is a $\text{BV}_\infty$-operator $\Delta \in hDO^+(\text{Dens}_{1/2}/M)$ such that

$$
\sigma_h(\Delta)|_{\hbar=1} = Q + \Pi,
$$

where $\sigma_h$ is the extended principal symbol map as in (25).

**Remark 3.5.** Classical BV-operators appeared in the pioneering work of Batalin–Vilkovisky [5–7] in their study of quantization of gauge theories. The geometry and the quantization procedure of odd symplectic manifolds were studied by many authors, in particular, by Schwarz [51]. In a series of papers [8,24–27,29], it was pointed out that BV-operators can be intrinsically realized as odd Laplace operators acting on half densities, which are self-adjoint. Our definition above was motivated by the consideration to extend such an approach to the general $\text{BV}_\infty$-context. It would also be an interesting question to investigate the obstruction theory to quantization by relaxing the self-adjoint condition in the definition.

Note that, for a given dg manifold $(M, Q)$, a derived Poisson structure $\Pi$ is equivalent to a Maurer–Cartan element of the dgla

$$
(\Gamma(\hat{ST}_M), [\cdot, \cdot], [Q, \cdot]).
$$

(38)

On the other hand, an operator $\Delta \in hDO^+(\text{Dens}_{1/2}/M)$ is a $\text{BV}_\infty$-operator if and only if $\Delta - h\mathcal{L}_Q$ is a Maurer–Cartan element of the dgla

$$
(hDO^+(\text{Dens}_{1/2}/M), [\cdot, \cdot]_h, [h\mathcal{L}_Q, \cdot]_h).
$$

(39)

Summarizing the discussion earlier, we have the following
Theorem 3.6. We have the following short exact sequence of dglas

\[
0 \longrightarrow \hDO^+_2(Dens^{1/2}_M) \xrightarrow{i} \hDO^+(Dens^{1/2}_M) \xrightarrow{\phi} \Gamma(\hat{ST}_M) \longrightarrow 0,
\]

(40)

where \( \Gamma(\hat{ST}_M) \) and \( \hDO^+(Dens^{1/2}_M) \) stand for the dglas (38) and (39), respectively, \( \hDO^+_2(Dens^{1/2}_M) \) is considered as an ideal of the dgla \( \hDO^+(Dens^{1/2}_M) \), the morphism \( i \) is the inclusion map and \( \phi \) is the morphism defined by

\[
\phi(\Delta) = \sigma_h(\Delta)|_{\hbar = 1}, \quad \forall \Delta \in \hDO^+(Dens^{1/2}_M).
\]

(41)

Proof. From Lemmas 2.18 and 2.27, it follows that \( \phi \) is indeed a Lie algebra morphism and preserves the differentials, i.e. a morphism of dglas. Proposition 3.3 implies that \( \phi \) is indeed surjective. Finally, to identify the kernel of \( \phi \), note that, for any \( \Delta \in \hDO^+(Dens^{1/2}_M) \), the condition that \( \phi(\Delta) = 0 \) is equivalent to that \( \sigma_h(\Delta) = 0 \). The latter is equivalent to that \( \Delta \in \hDO^+_2(Dens^{1/2}_M) \) according to Lemma 2.23. Thus it follows that \( \ker \phi \cong \hDO^+_2(Dens^{1/2}_M) \). The proof is completed. \( \square \)

As a consequence, the quantization problem in Definition 3.4 can be reinterpreted as the lifting problem of a Maurer–Cartan element of the dgla (38) to a Maurer–Cartan element of the dgla (39), where both dglas are related by the short exact sequence (40). As is standard, we can find the solution to the latter by lifting a sequence of Maurer–Cartan elements as follows.

Note that we have the following sequence of short exact sequences of dglas, for all \( k = 1, 2, \ldots \)

\[
0 \longrightarrow \hDO^+_{2k}(Dens^{1/2}_M) \xrightarrow{i_k} \hDO^+_2(Dens^{1/2}_M) \xrightarrow{\phi_k} \hDO^+_2(Dens^{1/2}_M) \longrightarrow 0,
\]

(42)

where \( i_k \) is the natural inclusion and \( \phi_k \) is the natural projection.

As an immediate consequence of Theorem 3.6, the morphism \( \phi \) in (41) induces an isomorphism of dglas, denoted by the same symbol

\[
\phi : \frac{\hDO^+(Dens^{1/2}_M)}{\hDO^+_2(Dens^{1/2}_M)} \cong \Gamma(\hat{ST}_M).
\]

(43)

Note that the extended principal symbol map (25) descends to a well defined map, for any \( k = 1, 2, \ldots \), denoted by the same symbol, by abuse of notation

\[
\sigma_h : \frac{\hDO^+(Dens^{1/2}_M)}{\hDO^+_2(Dens^{1/2}_M)} \rightarrow \Gamma(\hat{ST}_M)[[\hbar]].
\]

The following lemma is thus obvious.

Lemma 3.7. A \((-1)-\)shifted derived Poisson manifold \((\mathcal{M}, Q, \Pi)\) is quantizable if and only if there exists a sequence of operators \( \hat{\Delta}^{(k)} \in \hDO^+(Dens^{1/2}_M) \), \( k = 1, 2, \ldots \), such that \( [\hat{\Delta}^{(k)}] \in \frac{\hDO^+(Dens^{1/2}_M)}{\hDO^+_2(Dens^{1/2}_M)} \) is a Maurer–Cartan element and

\[
\phi_k[\hat{\Delta}^{(k+1)}] = [\hat{\Delta}^{(k)}],
\]

(44)
where

\[ [\hat{\Lambda}^{(1)}] = \phi^{-1}(\Pi). \quad (45) \]

Here \( [\hat{\Lambda}^{(k)}] \in \frac{\mathcal{h}D\mathcal{O}^+(\text{Dens}_{1/2}^{\mathcal{M}})}{\mathcal{h}D\mathcal{O}_{2k}^+(\text{Dens}_{1/2}^{\mathcal{M}})} \) denotes the equivalent class of \( \hat{\Lambda}^{(k)} \in \mathcal{h}D\mathcal{O}^+(\text{Dens}_{1/2}^{\mathcal{M}}) \), and \( \phi \) is the isomorphism (43).

**Remark 3.8.** As is standard, the existence of the sequence of Maurer–Cartan elements \( [\hat{\Lambda}^{(k)}] \in \frac{\mathcal{h}D\mathcal{O}^+(\text{Dens}_{1/2}^{\mathcal{M}})}{\mathcal{h}D\mathcal{O}_{2k}^+(\text{Dens}_{1/2}^{\mathcal{M}})} \), where \( \hat{\Lambda}^{(k)} \in \mathcal{h}D\mathcal{O}^+(\text{Dens}_{1/2}^{\mathcal{M}}) \), \( k = 1, 2, \ldots \), is not canonical. It depends on how one chooses the sequence. In other words, for any \( n \), the existence of a Maurer–Cartan element \( [\hat{\Lambda}^{(n+1)}] \in \frac{\mathcal{h}D\mathcal{O}^+(\text{Dens}_{1/2}^{\mathcal{M}})}{\mathcal{h}D\mathcal{O}_{2n+2}^+(\text{Dens}_{1/2}^{\mathcal{M}})} \) depends on how \( [\hat{\Lambda}^{(n)}] \in \frac{\mathcal{h}D\mathcal{O}^+(\text{Dens}_{1/2}^{\mathcal{M}})}{\mathcal{h}D\mathcal{O}_{2n}^+(\text{Dens}_{1/2}^{\mathcal{M}})} \) is picked.

Note that (44) implies that \( \sigma_{\mathcal{h}}(\lfloor \hat{\Lambda}^{(k+1)} \rfloor) = \sigma_{\mathcal{h}}(\lfloor \hat{\Lambda}^{(k)} \rfloor) \), and therefore, by (45), we have

\[ \sigma_{\mathcal{h}}(\lfloor \hat{\Lambda}^{(k+1)} \rfloor) = \Pi_{\mathcal{h}}, \quad k = 1, 2, \ldots. \]

Thus we have the following

**Lemma 3.9.** Let \( \hat{\Lambda}^{(k)} \in \mathcal{h}D\mathcal{O}^+(\text{Dens}_{1/2}^{\mathcal{M}}) \), \( k = 1, 2, \ldots \) be operators as in Lemma 3.7. Then, for any \( k = 1, 2, \ldots \), we have

\[ \sigma_{\mathcal{h}}(\lfloor \hat{\Lambda}^{(k)} \rfloor) = \Pi_{\mathcal{h}}. \quad (46) \]

Finally, note that for any operator \( \hat{\Delta} \in \mathcal{h}D\mathcal{O}^+(\text{Dens}_{1/2}^{\mathcal{M}}) \), we have the following equivalent relation:

\[ [\hat{\Delta}] \in \text{MC} \left( \frac{\mathcal{h}D\mathcal{O}^+(\text{Dens}_{1/2}^{\mathcal{M}})}{\mathcal{h}D\mathcal{O}_{2k}^+(\text{Dens}_{1/2}^{\mathcal{M}})} \right) \leftrightarrow \Delta^2 \in \mathcal{h}D\mathcal{O}_{2k+4}^+(\text{Dens}_{1/2}^{\mathcal{M}}), \quad (47) \]

where

\[ \Delta = \mathcal{h}LQ + \hat{\Delta}. \]

In the sequel, we will use the interpretation of either side of (47).

### 3.3. Obstruction classes.

In what follows, we will investigate the obstruction class for quantizing a \((-1)\)-shifted derived Poisson manifold. We will prove the following main theorem of the paper.

**Theorem 3.10.** Let \((\mathcal{M}, Q, \Pi)\) be a \((-1)\)-shifted derived Poisson manifold. Assume that the second Poisson cohomology group \( \mathcal{H}^2(\mathcal{M}, Q + \Pi) \) vanishes, then \((\mathcal{M}, Q, \Pi)\) is quantizable.
According to Lemma 3.7, the quantization problem of a $(-1)$-shifted derived Poisson manifold $(\mathcal{M}, Q, \Pi)$ is equivalent to that of lifting a sequence of Maurer–Cartan elements in the short exact sequences of dglas (42). Our strategy is to find the obstruction class for the lifting in each such a short exact sequence, which should be of independent interest.

Assume that $\hat{\Delta}^{(k)} \in \hbar \text{DO}^+(\text{Dens}^{1/2}_{\mathcal{M}})$ such that $$[\hat{\Delta}^{(k)}] \in \text{MC} \left( \frac{\text{hDO}^+(\text{Dens}^{1/2}_{\mathcal{M}})}{\text{hDO}^+_2(\text{Dens}^{1/2}_{\mathcal{M}})} \right)$$

and satisfies the condition (46).

Let

$$\Delta^{(k)} = \hbar L_Q + \hat{\Delta}^{(k)}$$

and

$$\Omega^{(k)} := \left( \Delta^{(k)} \right)^2 = \frac{\hbar}{2} [\Delta^{(k)}, \Delta^{(k)}]_{\hbar}.$$ 

Then

$$\sigma_{\hbar}(\Delta^{(k)}) = \hbar Q + \Pi_{\hbar}. \quad (48)$$

According to (47), $\Omega^{(k)} \in \text{hDO}^+_2k+1(\text{Dens}^{1/2}_{\mathcal{M}})$ and therefore we have $\frac{\Omega^{(k)}}{\hbar^{2k+1}} \in \text{hDO}^+(\text{Dens}^{1/2}_{\mathcal{M}})$. Denote by

$$S_{\hbar}^{(k)} = \sigma_{\hbar}\left( \frac{\Omega^{(k)}}{\hbar^{2k+1}} \right), \quad \text{and} \quad S^{(k)} = S_{\hbar}^{(k)} |_{\hbar=1}.$$ 

By the graded Jacobi identity, we have

$$[\Delta^{(k)}, \Omega^{(k)}]_{\hbar} = [\Delta^{(k)}, \frac{\hbar}{2} [\Delta^{(k)}, \Delta^{(k)}]_{\hbar}]_{\hbar} = 0.$$ 

Therefore, by taking the extended principal symbol, we have

$$\frac{1}{\hbar} \left\{ \hbar Q + \Pi_{\hbar}, \ S_{\hbar}^{(k)} \right\} = \frac{1}{\hbar} \left\{ \hbar Q + \Pi_{\hbar}, \ \sigma_{\hbar}(\frac{\Omega^{(k)}}{\hbar^{2k+1}}) \right\} = \sigma_{\hbar}[\Delta^{(k)}, \frac{\Omega^{(k)}}{\hbar^{2k+1}}]_{\hbar} = 0.$$ 

Thus it follows that $\left\{ Q + \Pi, \ S^{(k)} \right\} = 0$, i.e. $S^{(k)}$ is a $d\Pi$-cocycle of degree 2.

Assume that $\hat{\Delta}'^{(k)} \in \text{hDO}^+ (\text{Dens}^{1/2}_{\mathcal{M}})$ is another operator representative of $[\hat{\Delta}^{(k)}]$ in $\frac{\text{hDO}^+(\text{Dens}^{1/2}_{\mathcal{M}})}{\text{hDO}^+_2(\text{Dens}^{1/2}_{\mathcal{M}})}$. Then there exists an operator $\Psi^{(k)} \in \text{hDO}^+(\text{Dens}^{1/2}_{\mathcal{M}})$ such that

$$\hat{\Delta}'^{(k)} - \hat{\Delta}^{(k)} = \frac{\hbar^2}{2} \Psi^{(k)}.$$ 

Let $\Delta'^{(k)} = \hbar L_Q + \hat{\Delta}'^{(k)}$. Therefore

$$\Delta'^{(k)} = \Delta^{(k)} + \frac{\hbar^2}{2} \Psi^{(k)}.$$ 

Hence we have
\[ \Omega^{(k)} := \left( \Delta^{(k)} \right)^2 = \frac{\hbar}{2} \left[ \Delta^{(k)}, \Delta^{(k)} \right]_{\hbar} = \Omega^{(k)} + \hbar^{2k+1} [\Delta^{(k)}, \Psi^{(k)}]_{\hbar} + \frac{\hbar^{4k+1}}{2} [\Psi^{(k)}, \Psi^{(k)}]_{\hbar}. \]

Then

\[ \frac{\Omega^{(k)}}{\hbar^{2k+1}} = \frac{\Omega^{(k)}}{\hbar^{2k+1}} + [\Delta^{(k)}, \Psi^{(k)}]_{\hbar} + \frac{\hbar^{2k}}{2} [\Psi^{(k)}, \Psi^{(k)}]_{\hbar}. \]

By applying the extended principal map \( \sigma_h \), we have

\[ S^{(k)}_{\hbar} = S^{(k)}_{\hbar} + \frac{1}{\hbar} \{ \hbar Q + \Pi_{\hbar}, \sigma_h(\Psi^{(k)}) \}. \]

Hence

\[ S^{(k)} = S^{(k)} + \{ Q + \Pi, Z^{(k)} \}, \]

where \( S^{(k)}_{\hbar} = \sigma_h \left( \frac{\Omega^{(k)}}{\hbar^{2k+1}} \right) \), \( S^{(k)} = S^{(k)}_{|h=1} \), and \( Z^{(k)} = \sigma_h(\Psi^{(k)})_{|h=1} \). As a result \( S^{(k)} \) and \( S^{(k)} \) define the same class in \( \mathcal{H}^2(\mathcal{M}, Q + \Pi) \), independent of the choice of the operator representative in the operator class of \( \frac{\hbar \DO^+(\Dens^1_{\mathcal{M}})}{\hbar \DO^+_{2k}(\Dens^1_{\mathcal{M}})} \).

**Definition 3.11.** Let \( \hat{\Delta}^{(k)} \) be a Maurer–Cartan element satisfying the condition (46). The class \( [\hat{\Delta}^{(k)}] \in \mathcal{H}^2(\mathcal{M}, Q + \Pi) \) is called the modular class of \( \hat{\Delta}^{(k)} \).

In particular, for \( k = 1 \), we have a canonical element \( [\hat{\Delta}^{(1)}] = \phi^{-1}(\Pi) \in \MC \left( \frac{\hbar \DO^+(\Dens^1_{\mathcal{M}})}{\hbar \DO^+_{2}(\Dens^1_{\mathcal{M}})} \right) \). Thus we are led to the following

**Definition 3.12.** For any \((-1)\)-shifted derived Poisson manifold \((\mathcal{M}, Q, \Pi)\), the modular class of \( [\hat{\Delta}^{(1)}] = \phi^{-1}(\Pi) \in \MC \left( \frac{\hbar \DO^+(\Dens^1_{\mathcal{M}})}{\hbar \DO^+_{2}(\Dens^1_{\mathcal{M}})} \right) \) is called the modular class of the derived Poisson manifold \((\mathcal{M}, Q, \Pi)\).

The modular class in Definition 3.12 is an intrinsic invariant of a \((-1)\)-shifted derived Poisson manifold. However, for \( k \geq 2 \), the modular class \( [\hat{\Delta}^{(k)}] \) is not an intrinsic invariant, it depends on the choice of the Maurer–Cartan element \( [\hat{\Delta}^{(k)}] \in \MC \left( \frac{\hbar \DO^+(\Dens^1_{\mathcal{M}})}{\hbar \DO^+_{2}(\Dens^1_{\mathcal{M}})} \right) \).

**Remark 3.13.** In the context of supergeometry, the modular class of an odd Poisson supermanifold was originally introduced by Peddie-Khudaverdian in [25] considering odd Laplace-type operators acting on functions via a choice of volume form. The class was further studied in [25], where the class was identified when considering nilpotency conditions of arbitrary second order degree \((+1)\) operators on odd Poisson supermanifolds.

**Proposition 3.14.** Consider the short exact sequence of dglas (42), \( \forall k = 1, 2, \ldots \). Given any Maurer–Cartan element \( [\hat{\Delta}^{(k)}] \in \MC \left( \frac{\hbar \DO^+(\Dens^1_{\mathcal{M}})}{\hbar \DO^+_{2k}(\Dens^1_{\mathcal{M}})} \right) \) satisfying the condition (46), the modular class \( [S^{(k)}] \in \mathcal{H}^2(\mathcal{M}, Q + \Pi) \) is the obstruction class of lifting \( [\hat{\Delta}^{(k)}] \) to a Maurer–Cartan element in \( \frac{\hbar \DO^+(\Dens^1_{\mathcal{M}})}{\hbar \DO^+_{2k+2}(\Dens^1_{\mathcal{M}})} \).
Assume that the modular class vanishes. That is, there exists a degree \((+1)\) element \(X^{(k)} \in \Gamma(\hat{S}T_M)\) such that \(S^{(k)}_h = \{ hQ + \Pi, \ X^{(k)} \} \). Hence, by Lemma 2.27, we have
\[
S^{(k)}_h = \frac{1}{\hbar}\{ hQ + \Pi_h, \ X^{(k)}_h \}.
\]

Choose a self-adjoint operator \(\Phi^{(k)} \in hDO^+(\text{Dens}^{1/2}_{\mathcal{M}})\) satisfying the condition that \(\sigma_h(\Phi^{(k)}) = X^{(k)}_h\). This is always possible according to Proposition 3.3. Let
\[
\hat{\Delta}^{(k+1)} = \hat{\Delta}^{(k)} - \hbar^2 k \Phi^{(k)}.
\]

It is clear that
\[
\phi_k[\hat{\Delta}^{(k+1)}] = [\hat{\Delta}^{(k)}] \in \frac{hDO^+(\text{Dens}^{1/2}_{\mathcal{M}})}{hDO^*_{2k}(\text{Dens}^{1/2}_{\mathcal{M}})}.
\]

Here \([\hat{\Delta}^{(k+1)}]\) denotes the class of \(\hat{\Delta}^{(k+1)}\) in \(\frac{hDO^+(\text{Dens}^{1/2}_{\mathcal{M}})}{hDO^*_{2k+2}(\text{Dens}^{1/2}_{\mathcal{M}})}\), while \([\hat{\Delta}^{(k)}]\) denotes the class of \(\hat{\Delta}^{(k)}\) in \(\frac{hDO^+(\text{Dens}^{1/2}_{\mathcal{M}})}{hDO^*_{2k}(\text{Dens}^{1/2}_{\mathcal{M}})}\). It remains to prove that \([\hat{\Delta}^{(k+1)}]\) is a Maurer–Cartan element in \(\frac{hDO^+(\text{Dens}^{1/2}_{\mathcal{M}})}{hDO^*_{2k+2}(\text{Dens}^{1/2}_{\mathcal{M}})}\). Let \(\Delta^{(k+1)} = h\mathcal{L}_Q + \hat{\Delta}^{(k+1)}\). Then
\[
\Delta^{(k+1)} = \Delta^{(k)} - \hbar^2 k \Phi^{(k)}.
\]

Now we have
\[
\Omega^{(k+1)} = \left(\Delta^{(k+1)}\right)^2 = \frac{\hbar}{2}\{ \Delta^{(k+1)}, \Delta^{(k+1)} \}
= \frac{\hbar}{2}\{ \Delta^{(k)} - \hbar^2 k \Phi^{(k)}, \Delta^{(k)} - \hbar^2 k \Phi^{(k)} \}
= \Omega^{(k)} - \hbar^2 k + 1\{ \Delta^{(k)}, \Phi^{(k)} \} + \frac{\hbar^4 k + 1}{2}\{ \Phi^{(k)}, \Phi^{(k)} \}.
\]

Thus it follows that
\[
\frac{\Omega^{(k+1)}}{\hbar^{2k+1}} = \frac{\Omega^{(k)}}{\hbar^{2k+1}} - \{ \Delta^{(k)}, \Phi^{(k)} \} + \frac{\hbar^2 k}{2}\{ \Phi^{(k)}, \Phi^{(k)} \}.
\]

Hence, by applying the extended principal map \(\sigma_h\), we have
\[
\sigma_h\left(\frac{\Omega^{(k+1)}}{\hbar^{2k+1}}\right) = \sigma_h\left(\frac{\Omega^{(k)}}{\hbar^{2k+1}}\right) - \frac{1}{\hbar}\{ \sigma_h(\Delta^{(k)}), \sigma_h(\Phi^{(k)}) \}
= S^{(k)}_h - \frac{1}{\hbar}\{ hQ + \Pi_h, \ X^{(k)}_h \}
= 0.
\]

Therefore it follows that \(\Omega^{(k+1)} \in D_0^{+}(\text{Dens}^{1/2}_{\mathcal{M}})\). According to the equivalent relation \((47)\), \([\Delta^{(k+1)}]\) is indeed a Maurer–Cartan element in \(\frac{hDO^+(\text{Dens}^{1/2}_{\mathcal{M}})}{hDO^*_{2k+2}(\text{Dens}^{1/2}_{\mathcal{M}})}\). This proves one direction of the proposition. The converse can be proved using the same argument by going backwards. \(\square\)
Remark 3.15. For any $k = 1, 2, \ldots$, the short exact sequence (42) is in fact a square-zero extension of dglas. Recall that a square-zero extension of dglas is a short exact sequence of dglas

$$0 \longrightarrow K \xrightarrow{i} L \xrightarrow{\pi} M \longrightarrow 0$$

such that $[K, K] = 0$. One can prove that for a square-zero extension of dglas (50), the obstruction to lifting a Maurer–Cartan element $x$ in $M$ is a well defined class in

$$\mathcal{H}^2(K, d_K + [y, -]),$$

where $y \in L$ is any lift of $x$, i.e. $\pi(y) = x$. Note that the differential $d_K + [y, -]$ and hence the cohomology group $\mathcal{H}^2(K, d_K + [y, -])$ is independent of the choice of the lift $y$. One can prove that for the square-zero extension of dglas (42), the corresponding cohomology group (51) is isomorphic to the second Poisson cohomology group $\mathcal{H}^2(M, Q + \Pi)$. See [42] and [11, Section 3.1.3] for references on a related problem (the case when $K$ is an abelian ideal of $L$).

For each $X \in \Gamma(\hat{S}^2 T_{\mathcal{M}})$, by $\text{Op}(X)$, we denote the set of all self-adjoint operators $\Delta \in h\mathcal{D}O^+(\text{Dens}^{1/2}_{\mathcal{M}})$ such that $\sigma_h(\Delta) = X_h$, where $\sigma_h$ is the extended principal symbol map (25), and $X_h$ is defined as in (30). The subset $\text{Op}(X)$ is called the operator class of the formal power series $X$.

By considering the particular case of $k = 1$, Proposition 3.14 implies the following

Corollary 3.16. For any $(-1)$-shifted derived Poisson manifold $(\mathcal{M}, Q, \Pi)$, its modular class is the obstruction class to the existence of an operator $\hat{\Delta} \in \text{Op}(\Pi)$ satisfying the condition that $\Delta^2 \in h\mathcal{D}O^+_{\mathcal{M}}(\text{Dens}^{1/2}_{\mathcal{M}})$, where $\Delta = h\mathcal{L}_Q + \hat{\Delta}$.

Now we are ready for the proof of the main theorem.

Proof of Theorem 3.10. It is an immediate consequence of Lemma 3.7, Definition 3.11, and Proposition 3.14. □
More generally, if $(\mathcal{M}, Q)$ is a dg manifold equipped with the degree $(+1)$-Poisson bracket determined by a degree $(+1)$ element $\Pi_2 \in \Gamma(S^2 T\mathcal{M})$ such that
\[
\{ \Pi_2, \Pi_2 \} = 0, \quad \{ Q, \Pi_2 \} = 0,
\]
then $(\mathcal{M}, Q, \Pi_2)$ is called a $(-1)$-shifted dg Poisson manifold. Let $\Delta_2 \in \text{DO}^+(\text{Dens}_{\mathcal{M}}^{1/2})$ be an operator as before, and $\Delta = hL_Q + \hbar^2 \Delta_2$. Then
\[
\Omega := \Delta^2 = h^3 [L_Q, \Delta_2] + h^4 \Delta_2 = h^3 [L_Q, \Delta_2] + h^4 L_{X_1},
\]
Therefore
\[
\sigma_h(\frac{\Omega}{h^3}) = X_0 + hX_1,
\]
where $X_0 = \sigma_h([L_Q, \Delta_2]) \in C^\infty(\mathcal{M})$. Thus $[X_0 + X_1] \in \mathcal{H}^2(\mathcal{M}, Q + \Pi_2)$ is the modular class of the $(-1)$-shifted dg Poisson manifold $(\mathcal{M}, Q, \Pi_2)$. According to Corollary 3.16, it is the obstruction class to the existence of $\Delta \in \text{Op}(\Pi)$ such that $(hL_Q + \hat{\Delta})^2 \in \text{hDO}_{\mathcal{M}}^2(\text{Dens}_{\mathcal{M}}^{1/2})$. Note that the vanishing of the modular class $[X_0 + X_1] \in \mathcal{H}^2(\mathcal{M}, Q + \Pi_2)$ does not necessarily mean that $(\mathcal{M}, Q, \Pi_2)$ is quantizable. However, if we assume furthermore that $X_0 + X_1$ can be expressed as a particular type of coboundary, i.e. there exists a function $f \in C^\infty(\mathcal{M})$ such that
\[
X_0 + X_1 = \{ Q + \Pi_2, f \}.
\]
Then
\[
\Delta' = hL_Q + \hbar^2 (\Delta_2 - f)
\]
is clearly a $\text{BV}_{\infty}$-quantization of $(\mathcal{M}, Q, \Pi_2)$.

**Proposition 3.17.** Let $(\mathcal{M}, Q, \Pi_2)$ be a $(-1)$-shifted dg Poisson manifold, $\Delta_2 \in \text{Op}(\Pi)$, and $\Delta = hL_Q + \hbar^2 \Delta_2$. Assume that there exists a function $f \in C^\infty(\mathcal{M})$ such that Eq. (54) holds where $X_0$ and $X_1$ are as in (53). Then $(\mathcal{M}, Q, \Pi_2)$ is quantizable.

In particular, as an immediate consequence, we recover the following theorem due to Khudaverdian–Peddie.

**Corollary 3.18.** [25] A $(-1)$-shifted Poisson manifold $(\mathcal{M}, \Pi_2)$ is quantizable if and only if its modular class vanishes.

**Proof.** If $(\mathcal{M}, \Pi_2)$ is quantizable, then its modular class must vanish according to Corollary 3.16. Conversely, assume that the modular class of $(\mathcal{M}, \Pi_2)$ vanishes. By Eq. (52), the modular cocycle is a degree 2 vector field $X_1 \in \Gamma(T\mathcal{M})$. Since it is a coboundary by assumption, by the weight counting, we have $X_1 = \{ \Pi_2, f \}$ for some function $f \in C^\infty(\mathcal{M})$. According to Proposition 3.17, $(\mathcal{M}, \Pi_2)$ is indeed quantizable. \qed

Below is an example of $(-1)$-shifted Poisson manifold with non-vanishing modular class, which is adapted from [25].
Example 3.19. Let $\mathcal{M} = \mathbb{R}^2[-1] \times \mathbb{R}[-2]$, and choose global coordinates $\xi, \tau, z$ with assigned degrees

$$|\xi| = |\tau| = -1, \quad |z| = -2.$$ 

Define two vector fields on $\mathcal{M}$:

$$P = \tau \frac{\partial}{\partial \xi} + \tau \xi \frac{\partial}{\partial z}, \quad Q = \frac{\partial}{\partial \tau}.$$ 

Then $|P| = 0$ and $|Q| = 1$. It is simple to check that

$$[Q, Q] = 0, \quad [P, P] = 0, \quad [P, Q] = 0.$$ 

Hence $\Pi_2 = P \circ Q$ is a degree (+1) contravariant tensor satisfying $\{ \Pi_2, \Pi_2 \} = 0$. Then

$$\Delta_2 = \frac{1}{2} (\mathcal{L}_P \circ \mathcal{L}_Q + \mathcal{L}_Q \circ \mathcal{L}_P) \in \text{DO}^+(\text{Dens}^{1/2}_\mathcal{M}) \quad \text{(55)}$$

is a degree (+1) self-adjoint second order differential operator such that $\sigma_2(\Delta_2) = \Pi_2$. The operator has the coordinate expression

$$\Delta_2 = \tau \frac{\partial}{\partial \tau} \tau \xi \frac{\partial}{\partial \tau} + \frac{1}{2} \xi \frac{\partial}{\partial z} + \frac{1}{2} \frac{\partial}{\partial \xi},$$

from which we can explicitly calculate the square:

$$\Delta_2 \circ \Delta_2 = \frac{1}{4} \frac{\partial}{\partial z}.$$ 

Hence $X_1 = \frac{1}{4} \frac{\partial}{\partial z}$. Note that $\frac{\partial}{\partial z}$ cannot be a Hamiltonian vector field, since the (+1)-shifted Poisson bracket on $\mathcal{M}$ reads

$$\{ \xi, \tau \} = \pm \tau, \quad \{ z, \tau \} = \pm \tau \xi,$$

which is always proportional to $\tau$. Hence $[\frac{\partial}{\partial z}] \in \mathcal{H}^2(\mathcal{M}, \Pi_2)$ is a non-trivial modular class of the $(-1)$-shifted Poisson manifold $(\mathcal{M}, \Pi_2)$. As a consequence, $(\mathcal{M}, \Pi_2)$ is not quantizable.

Example 3.20. As in Example 3.19, we consider the $(-1)$-shifted dg Poisson manifold $(\mathcal{M}, Q, \Pi_2)$. Let $\Delta = \hbar \mathcal{L}_Q + \hbar^2 \Delta_2$, where $\Delta_2$ is as in (55). Then

$$[\mathcal{L}_Q, \Delta_2] = \frac{1}{2} \left( \mathcal{L}_Q \circ \mathcal{L}_P + \mathcal{L}_P \circ \mathcal{L}_Q \right) = \mathcal{L}_Q \circ \mathcal{L}_P \circ \mathcal{L}_Q = (\mathcal{L}_Q \circ \mathcal{L}_P + \mathcal{L}_P \circ \mathcal{L}_Q) \circ \mathcal{L}_Q = 0.$$ 

Therefore $X_0 = \sigma_0([\mathcal{L}_Q, \Delta_2]) = 0$, and $[\frac{\partial}{\partial z}] \in \mathcal{H}^2(\mathcal{M}, Q + \Pi_2)$ is a non-trivial modular class of the $(-1)$-shifted dg Poisson manifold $(\mathcal{M}, Q, \Pi_2)$. 
3.5. \textit{(−1)-shifted derived Poisson manifolds and }\mathcal{L}_\infty\text{-algebroids.} Following the conventions of \cite{[3,12,37,60]}, an \mathcal{L}_\infty\text{-algebroid is a }\mathbb{Z}\text{-graded vector bundle }\mathcal{A} \rightarrow \mathcal{M} \text{ together with}

- a sequence of multi-liner maps } \lambda_n : \bigwedge^n \Gamma(\mathcal{A}) \rightarrow \Gamma(\mathcal{A}) \text{ of degree } 2 - n, n \geq 1, \text{ which determine an } \mathcal{L}_\infty\text{-structure on } \Gamma(\mathcal{A}); \text{ and }

- a sequence of bundle maps } \rho_n : \bigwedge^n \mathcal{A} \rightarrow T\mathcal{M} \text{ of degree } 1 - n, n \geq 0, \text{ called multi-anchor maps, such that }

  \begin{itemize}
  \item[(i)] its induced maps } \rho_n : \bigwedge^n \Gamma(\mathcal{A}) \rightarrow \mathfrak{X}(\mathcal{M}) \text{, } n \geq 1 \text{ define an } \mathcal{L}_\infty\text{-morphism from } \Gamma(\mathcal{A}) \text{ to the dgla } (\mathfrak{X}(\mathcal{M}), \rho_0), \text{ and moreover, }

  \begin{align*}
  \lambda_n(v_1, \ldots, v_{n-1}, f v_n) &= \rho_{n-1}(v_1, \ldots, v_{n-1})(f) v_n \\
  &+ (-1)^{|f|(|n|+|v_1|+\cdots+|v_{n-1}|)} f \lambda_n(v_1, \ldots, v_n), 
  \end{align*}

  \text{ for all } n \geq 1, v_1, \ldots, v_n \in \Gamma(\mathcal{A}) \text{ and } f \in C^\infty(\mathcal{M}).

  \end{itemize}

When } \mathcal{M} \text{ is an ordinary manifold } M \text{ (being considered of degree zero), and the vector bundle } \mathcal{A} = \bigoplus_{i \geq 0} \mathcal{A}^i \rightarrow \mathcal{M} \text{ is a non-negative graded vector bundle, this reduces to the } \mathcal{L}_\infty\text{-algebroids studied by Laurent-Gengoux et al.} \cite{[38]}. In particular, if } \mathcal{A} \text{ is a usual vector bundle concentrated in degree 0, it becomes an ordinary Lie algebroid over } M. \text{ Here, we adapt more general definition by allowing } \mathcal{M} \text{ to be } \mathbb{Z}\text{-graded as well. For references on } \mathcal{L}_\infty\text{ algebroids, see } \cite{[3,23,28,32,33,38,53,56–58]} \text{ and the references therein.}

Similar to the ordinary Lie–Poisson construction, any } \mathcal{L}_\infty\text{-algebroid gives rise to a } (−1)\text{-shifted derived Poisson manifold in a natural fashion.}

The following proposition is standard. See \cite{[3,31,52]}.

\textbf{Proposition 3.21.} An } \mathcal{L}_\infty\text{-algebroid structure on } \mathcal{A} \text{ is equivalent to a linear } (−1)\text{-shifted derived Poisson manifold structure on } \mathcal{A}^\vee[-1].

Let us explain the word \textit{linear}. Note that under the isomorphism:

\begin{equation}
C^\infty(\mathcal{A}^\vee[-1]) \cong \Gamma(\hat{\mathfrak{S}}^*(\mathcal{A}[1])),
\end{equation}

\(C^\infty(\mathcal{A}^\vee[-1]),\) besides the underlying } \mathbb{Z}\text{-grading, possesses a natural non-negative } \mathbb{N}\text{-grading by the weight, which we denote by } N\text{-degree. A } (−1)\text{-shifted derived Poisson manifold structure on } \mathcal{A}^\vee[-1] \text{ is said to be } \textit{linear}, \text{ if for each } n \geq 1, \text{ the Poisson multi-bracket } \check{\lambda}_n : C^\infty(\mathcal{A}^\vee[-1])^{\otimes n} \rightarrow C^\infty(\mathcal{A}^\vee[-1]) \text{ is of } N\text{-degree } 1 - n. \text{ More precisely, under the isomorphism } (57), \text{ }

\begin{equation}
\check{\lambda}_n : \Gamma(\mathfrak{S}^{k_1}(\mathcal{A}[1])) \times \cdots \times \Gamma(\mathfrak{S}^{k_n}(\mathcal{A}[1])) \rightarrow \Gamma(\mathfrak{S}^{k_1+\cdots+k_n+(1-n)}(\mathcal{A}[1])),
\end{equation}

\text{ for any non-negative integers } k_1, \ldots, k_n. \text{ In this case, we say that } \Pi_n \text{ is of } N\text{-degree } 1 - n.

Below we describe briefly one direction of the construction in Proposition 3.21. The construction of the other direction can be carried out by going backwards.

Let } \mathcal{A} \rightarrow \mathcal{M} \text{ be an } \mathcal{L}_\infty\text{-algebroid with the structure maps } (\lambda_n)_{n \geq 1} \text{ and } (\rho_n)_{n \geq 0} \text{ as in the beginning of the subsection. Then } \Gamma(\mathcal{A}[1]) \text{ is an } \mathcal{L}_\infty[1]\text{-algebra. The corresponding degree } (+1)\text{-Poisson multi-brackets } \check{\lambda}_n \text{ as in } (58) \text{ are essentially the natural extension of } (\lambda_n)_{n \geq 1} \text{ by the Leibniz rule in Definition 1.1 with the help of the multi-anchor maps.}
Note that the 0-th anchor map $\rho_0$ defines a homological vector field $Q_M \in \mathfrak{X}(\mathcal{M})$. The unary bracket $\lambda_1 : \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$ and $\rho_0$ are compatible:

$$\lambda_1(f a) = Q_M(f a) + (-1)^{|f|} f \lambda_1(a), \quad \forall a \in \Gamma(\mathcal{A}), f \in C^\infty(\mathcal{M}).$$

Hence $\Gamma(\mathcal{A})$ is a dg module over $(C^\infty(\mathcal{M}), Q_M)$. As a consequence, $\mathcal{A} \to \mathcal{M}$ is a dg vector bundle $[44, 45]$, which implies that $\mathcal{A}^\vee \to \mathcal{M}$ is a dg vector bundle as well. Hence $\mathcal{A}^\vee[-1]$ is, naturally, a dg manifold, whose corresponding homological vector field is denoted $Q$. As before, we write $\Pi_1 = Q$, and the corresponding degree $(+1)$ Poisson tensor $\Pi \in \Gamma(S(T\mathcal{A}^\vee[-1]))$ on $\mathcal{A}^\vee[-1]$ is

$$\Pi = \sum_{n \geq 1} \Pi_n, \quad |\Pi| = 1, \quad N(\Pi_n) = 1 - n.$$

Choose a local coordinate chart $(x^a)$ on $\mathcal{M}$ and a local frame $(e_1, \ldots, e_n)$ on $\mathcal{A}$. Let $(f^1, \ldots, f^n)$ be its corresponding dual local frame on $\mathcal{A}^\vee$. These data determine a coordinate chart on $\mathcal{A}^\vee[-1]$:

$$(x^a, \xi_i), \quad (59)$$

and as well as a coordinate chart on $\mathcal{A}[1]$:

$$(x^a, \eta^i). \quad (60)$$

Under the local coordinate chart $(59)$, $\Pi_n$ can be written explicitly as follows

$$\Pi_n = \frac{1}{(n-1)!} \rho^a_{i_1 \cdots i_{n-1}}(x) \frac{\partial}{\partial \xi_{i_1}} \cdots \odot \frac{\partial}{\partial \xi_{i_{n-1}}} \odot \frac{\partial}{\partial x^a} + \frac{1}{n!} C^j_{i_1 \cdots i_n}(x) \xi_j \frac{\partial}{\partial \xi_{i_1}} \odot \cdots \odot \frac{\partial}{\partial \xi_{i_n}}, \quad (61)$$

where the structure functions $(\rho^a_{i_1 \cdots i_{n-1}}(x))$ and $(C^j_{i_1 \cdots i_n}(x))$ are determined by the corresponding multi-anchors and multi-brackets:

$$\rho_{n-1}(e_{i_1} \wedge \cdots \wedge e_{i_{n-1}}) = \pm \rho^a_{i_1 \cdots i_{n-1}}(x) \frac{\partial}{\partial x^a}, \quad \lambda_n(e_{i_1}, \ldots, e_{i_n}) = \pm C^j_{i_1 \cdots i_n}(x) e_j.$$

**Theorem 3.22.** For any $L_\infty$-algebroid $\mathcal{A}$, its corresponding linear $(-1)$-shifted derived Poisson manifold $\mathcal{A}^\vee[-1]$ admits a canonical quantization.

**Proof.** To any $L_\infty$-algebroid $\mathcal{A}$, there is a Chevalley–Eilenberg differential $D$, which is a homological vector field on $\mathcal{A}[1]$. Indeed, $L_\infty$-algebroid structures on a graded vector bundle $\mathcal{A} \to \mathcal{M}$ are equivalent to homological vector fields on $\mathcal{A}[1]$ vanishing on the zero section $[3]$. The homological vector field $D$ on $\mathcal{A}[1]$ is necessarily of the form $D = \sum_{n=1}^{\infty} D_n$, where $D_n$ is a derivation on $C^\infty(\mathcal{A}[1])$ of weight $(n - 1)$:

$$D_n : \Gamma(S(\mathcal{A}^\vee[-1])) \to \Gamma(S^{n-1}(\mathcal{A}^\vee[-1])). \quad (62)$$

Under the local coordinate chart $(60)$, the derivation $D_n$ can be written as

$$D_n = \pm \frac{1}{(n-1)!} \rho^a_{i_1 \cdots i_{n-1}}(x) \eta^{i_1} \cdots \eta^{i_{n-1}} \frac{\partial}{\partial x^a} \pm \frac{1}{n!} C^j_{i_1 \cdots i_n}(x) \eta^{i_1} \cdots \eta^{i_n} \frac{\partial}{\partial \eta^j}. \quad (63)$$
According to Khudaverdian–Voronov [31], the homological vector field \( D \in \mathfrak{X}(\mathcal{A}[1]) \) and the degree (+1) Poisson tensor \( \Pi \in \Gamma(\hat{S}(T_{\mathcal{A}^\vee[1]})) \) are related as follows

\[
\Phi^* D = \Pi, \quad (64)
\]

where

\[
\Phi : T_{\mathcal{A}^\vee[1]} \xrightarrow{\sim} T_{\mathcal{A}[1]} \quad (65)
\]

is the canonical isomorphism [40], \( D \in \mathfrak{X}(\mathcal{A}[1]) \) is identified with its corresponding linear function on \( T_{\mathcal{A}[1]} \), while \( \Pi \) is considered as a formal polynomial on \( T_{\mathcal{A}^\vee[1]} \). Following Shemyakova [52, Equation (48)] (see also [62]), we denote by

\[
\mathcal{F} : \Gamma(\text{Dens}_{\mathcal{A}[1]}^{1/2}) \xrightarrow{\sim} \Gamma(\text{Dens}_{\mathcal{A}^\vee[1]}^{1/2}) \quad (66)
\]

the fiberwise \( h \)-Fourier transform\(^2\):

\[
\mathcal{F}[f(x, \eta) Dx^{1/2} D\eta^{1/2}] = \left( \int_{\mathcal{A}_x} e^{\frac{i}{\hbar} \langle \xi, \eta \rangle} f(x, \eta) \, D\eta \right) Dx^{1/2} D\xi^{1/2}, \quad (67)
\]

where \( (x, \eta) \in \mathcal{A}[1] \) and \( (x, \xi) \in \mathcal{A}^\vee[1] \) (we emphasize the base point \( x \in \mathcal{M} \)). Now set

\[
\Delta := \mathcal{F} \circ \mathcal{L}_{hD} \circ \mathcal{F}^{-1}. \quad (68)
\]

Since the fiberwise \( h \)-Fourier transform \( \mathcal{F} \) preserves the scalar products and \( \mathcal{L}_{hD} \) is self-adjoint, it follows that \( \Delta \) is self-adjoint. Moreover, since \( D \) is a homological vector field, it follows that \( \Delta^2 = 0 \). Finally, it is simple to check, by using local coordinates, and Eqs. (61) and (63), that \( \Delta \) is indeed a quantization of the linear \((-1)\)-shifted derived Poisson manifold \( \mathcal{A}^\vee[1] \). \hfill \Box

### 3.6. Quantization of the derived intersection of coisotropic submanifolds of a Lie–Poisson manifold.

Next, as a special case, we consider the case when \( \mathcal{A} \) is an ordinary Lie algebroid \( A \) over the base manifold \( M \). In this situation, \( A^\vee[1] \) is a \((-1)\)-shifted Poisson manifold, where the degree (+1)-Poisson bracket on \( C^\infty(A^\vee[1]) \cong \Gamma(\Lambda^{\bullet\cdot} A) \) is the Schouten bracket. For the simplicity of notation, assume that the rank of the vector bundle \( A \) is \( n \). By \( D : \Gamma(\Lambda^{\bullet} A^\vee) \to \Gamma(\Lambda^{\bullet+1} A^\vee) \), we denote the Chevalley–Eilenberg differential of the Lie algebroid. Thus \((\mathcal{A}[1], D)\) is a dg manifold. Assume, for the sake of simplicity, that \( M \) is an orientable manifold and the vector bundle \( A \to M \) is orientable as well. Then both vector bundles \( A \oplus T_M \) and \( A \oplus T_M^\vee \) are orientable. We have the following standard identifications [14,21].

\[
\Gamma(\text{Dens}_{\mathcal{A}[1]}^{1/2}) \cong \Gamma(\Lambda A^\vee \otimes (\Lambda^{\top} A \otimes \Lambda^{\top} T^\vee_M)^{1/2}) \quad (69)
\]

while

\[
\Gamma(\text{Dens}_{\mathcal{A}^\vee[1]}^{1/2}) \cong \Gamma(\Lambda A \otimes (\Lambda^{\top} A^\vee \otimes \Lambda^{\top} T^\vee_M)^{1/2}). \quad (70)
\]

\(^2\) Fiberwise Fourier transform was discovered by Voronov–Zorich [63] in their study of supermanifold integration theory. It can be considered as the quantum counterpart of the isomorphism \( \Phi \) (65).
Here, by abuse of notation, $\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T^\vee_M$ and $\Lambda^{\text{top}} A^\vee \otimes \Lambda^{\text{top}} T_M^\vee$ are identified with $|\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T^\vee_M|$ and $|\Lambda^{\text{top}} A^\vee \otimes \Lambda^{\text{top}} T_M^\vee|$, respectively, and therefore their square root make sense. On the other hand, it is standard [1, Section 5] [21, Section 6.3] that there is a canonical isomorphism

$$
\Phi : \Gamma\left(\Lambda^k A^\vee \otimes \left(\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T_M^\vee\right)^{\frac{1}{2}}\right) \xrightarrow{\sim} \Gamma\left(\Lambda^{\text{top} - k} A \otimes \left(\Lambda^{\text{top}} A^\vee \otimes \Lambda^{\text{top}} T_M^\vee\right)^{\frac{1}{2}}\right).
$$

(71)

It is simple to check that, under the identifications (69)–(70), the fiberwise $\hbar$-Fourier transform (66) and $\Phi$ are related as follows

$$
\mathcal{F}|_{\Gamma\left(\Lambda^k A^\vee \otimes \left(\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T_M^\vee\right)^{\frac{1}{2}}\right)} = \left(\frac{1}{\hbar}\right)^{n-k} \Phi,
$$

(72)

where $n$ is the rank of $A$. Thus, it follows that, under the identifications (69)–(70),

$$
\mathcal{F} \circ \mathcal{L}_{\hbar D} \circ \mathcal{F}^{-1} = \hbar^2 \Phi \circ \mathcal{L}_D \circ \Phi^{-1}.
$$

It is known [21, 34] that, under the identification (69), as a degree $(+1)$ operator on $\Gamma(\text{Dens}_{1/2} M)$, $\mathcal{L}_D$ coincides with the Chevalley–Eilenberg differential of the Lie algebroid $A$ with values in the Evens–Lu–Weinstein module [19] $(\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T_M^\vee)^{\frac{1}{2}}$:

$$
d_{\text{CE}}^{\text{ELW}} : \Gamma\left(\Lambda^* A^\vee \otimes \left(\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T_M^\vee\right)^{\frac{1}{2}}\right) \rightarrow \Gamma\left(\Lambda^{\text{top} + 1} A^\vee \otimes \left(\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T_M^\vee\right)^{\frac{1}{2}}\right).
$$

As a consequence, we have the following

**Proposition 3.23.** For any Lie algebroid $A$ over $M$, assuming that $M$ is an orientable manifold and the vector bundle $A \rightarrow M$ is orientable as well, the $(-1)$-shifted Poisson manifold $A^\vee[-1]$ admits a canonical quantization

$$
\Delta = \hbar^2 \Phi \circ d_{\text{CE}}^{\text{ELW}} \circ \Phi^{-1} : \Gamma\left(\Lambda^{-*} A \otimes \left(\Lambda^{\text{top}} A^\vee \otimes \Lambda^{\text{top}} T_M^\vee\right)^{\frac{1}{2}}\right) \rightarrow \Gamma\left(\Lambda^{\text{top} - (n+1)} A \otimes \left(\Lambda^{\text{top}} A^\vee \otimes \Lambda^{\text{top}} T_M^\vee\right)^{\frac{1}{2}}\right).
$$

(73)

**Remark 3.24.** Proposition 3.23, in different forms, have been known in the literature. When the Lie algebroid $A$ is the cotangent Lie algebroid of a Poisson manifold, this is essentially the Koszul–Brylinski operator [13, 35]. When choosing a flat $A$-connection on $\Lambda^{\text{top}} A$, $\Delta$ can be identified with an BV-operator on $\Gamma(\Lambda^{-*} A)$ as shown in [64]. See also [16, Section 9.4] for the approach from the viewpoint of factorization algebras.

Finally, we consider the $(-1)$-shifted derived Poisson manifold $(A^\vee[-1], \iota_\ast, \Pi_2)$ as in Example 1.8, which arises as the derived intersection of two coisotropic submanifolds of the Lie–Poisson manifold $A^\vee$: the graph of $s$ and the zero section.

Let $Q = \iota_\ast$. It is simple to see that, under the identification (70), the Lie derivative

$$
\mathcal{L}_Q : \Gamma(\text{Dens}_{A^\vee[-1]}) \rightarrow \Gamma(\text{Dens}_{A^\vee[-1]})
$$

becomes the contraction operator

$$
\iota_\ast : \Gamma\left(\Lambda A \otimes \left(\Lambda^{\text{top}} A^\vee \otimes \Lambda^{\text{top}} T_M^\vee\right)^{\frac{1}{2}}\right) \rightarrow \Gamma\left(\Lambda A \otimes \left(\Lambda^{\text{top}} A^\vee \otimes \Lambda^{\text{top}} T_M^\vee\right)^{\frac{1}{2}}\right).
$$
Thus it follows that $\Phi^{-1} \circ \iota_s \circ \Phi$, the conjugation by $\Phi$, coincides with the multiplication operator:

$$m_s : \Gamma\left(\Lambda^\bullet A^\vee \otimes \left(\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T^\vee_M\right)^{\frac{1}{2}}\right) \rightarrow \Gamma\left(\Lambda^{\bullet+1} A^\vee \otimes \left(\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T^\vee_M\right)^{\frac{1}{2}}\right).$$

Since $d_{\mathcal{C}ES} = 0$, it follows that 

$$[d_{\mathcal{E}LW}, m_s] = 0.$$

Therefore

$$[\Phi \circ d_{\mathcal{E}LW} \circ \Phi^{-1}, \iota_s] = 0.$$

Hence $\hbar \iota_s + \hbar^2 \Phi \circ d_{\mathcal{E}LW} \circ \Phi^{-1}$ is indeed a BV$_\infty$-operator quantizing the $(-1)$-shifted derived Poisson manifold $(A^\vee[-1], \iota_s, \Pi_2)$.

**Theorem 3.25.** Let $A$ be a Lie algebroid over $M$, and $s \in \Gamma(A^\vee)$ a Lie algebroid 1-cocycle. Assume that $M$ is an orientable manifold and the vector bundle $A \rightarrow M$ is orientable as well. Then the $(-1)$-shifted derived Poisson manifold $(A^\vee[-1], \iota_s, \Pi_2)$ as in Example 1.8 admits a canonical quantization

$$\Delta = \hbar \iota_s + \hbar^2 \Phi \circ d_{\mathcal{E}LW} \circ \Phi^{-1} : \Gamma\left(\Lambda^{-\bullet} A \otimes \left(\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T^\vee_M\right)^{\frac{1}{2}}\right) \rightarrow \Gamma\left(\Lambda^{-\bullet+1} A \otimes \left(\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T^\vee_M\right)^{\frac{1}{2}}\right).$$

In particular, when $A$ is $T_M$ and $s = df \in \Omega^1(M)$ an exact one-form, where $f \in C^\infty(M)$, then $\left(\Lambda^{\text{top}} A \otimes \Lambda^{\text{top}} T^\vee_M\right)^{\frac{1}{2}}$ is canonically isomorphic to the trivial line bundle $M \times \mathbb{R}$ and the operator $d_{\mathcal{E}LW}$ reduces to the ordinary de Rham differential. As an immediate consequence, we see that

$$\hbar \iota_{df} + \hbar^2 \Phi \circ d_{\text{DR}} \circ \Phi^{-1} : \Gamma\left(\Lambda^{-\bullet} T_M \otimes \left(\Lambda^{\text{top}} T^\vee_M\right)\right) \rightarrow \Gamma\left(\Lambda^{-\bullet+1} T_M \otimes \left(\Lambda^{\text{top}} T^\vee_M\right)\right)$$

is a BV$_\infty$-operator quantizing the $(-1)$-shifted derived symplectic manifold $(T^\vee_M[-1], \iota_{df}, \omega_{\text{can}})$. Here

$$\Phi : \Omega^k(M) \rightarrow \Gamma\left(\Lambda^{\text{top} - k} T_M \otimes \left(\Lambda^{\text{top}} T^\vee_M\right)\right), \quad \forall k = 0, 1, \ldots$$

is the canonical isomorphism.

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