Helicity in Classical Electrodynamics and Turbulence

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Abstract

For the electromagnetic fields, hydrodynamic media and turbulent flows we consider the relationship between a topological characteristic of vector fields known as helicity and the angular momentum of the medium, and discuss, in this respect, the problem of helicity and angular momentum transfer from the electromagnetic field to a medium.

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1 Introduction

Taking into account topology is one of the most important aspects in studying various physical objects. Nontrivial topological properties guaranty the stability of classical and quantum states of a physical system, cause the possibility of existing in nature such stable point-like objects as vortices, solitons, monopoles and instantons, and may be a cause and explanation of several physically observed phenomena (Abricosov vortices, Aharonov–Bohm effect, quantum Hall effect etc.) ([1, 2] and references therein). The topological characteristics of the system are topological invariants conserved under continuous deformations of system parameters. From physical point of view these topological invariants should correspond to physical observables, i.e. integrals of motion or conserved charges, characterizing the dynamics of the system. In this respect establishing the relationship between topological and physical characteristics has proved to be important for deeper understanding of physical properties and the evolution of the system.

Among the topological invariants having been engaged in various fields of modern physics the Hopf invariant [3] takes one of the first places. In 3–dimensional space it characterizes, in particular, the linking and the knot number of the integral lines of a vector field $A_\alpha(x)$ ($\alpha, \beta = 1, 2, 3$) describing a physical system. The helicity of $A_\alpha(x)$, determined by the integral

$$h = \frac{1}{4\pi c} \int d^3 x \varepsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma,$$  

(1)

(where $\varepsilon^{\alpha\beta\gamma}$ is the unit skew-symmetric tensor with $\varepsilon^{123} = 1$), has the direct relation to the Hopf invariant [3]. The integral (1) does not depend on the (local) metric properties of the 3–dimensional space and, hence, characterizes global properties of the space and the vector field.

The properties of helical fields have been studied in detail, for example, in the theory of magnetic dynamo [4] and hydrodynamics [5]. For instance, it is well known that for being able to generate a large-scale magnetic field by a small-scale motion of conducting fluid it is necessary for the turbulence to possess helicity. One more example of helicity manifestation is the generation of large-scale structures in hydrodynamic turbulence [6]. If a small-scale turbulence of a fluid or gas possess helicity, then large-scale instability leading to the generation of the vortex structures arises. The generation of the large-scale structures is caused by the mirror noninvariance of the turbulence.

In plasmas the problem of current maintenance stimulated great interest in the possibility of generating the current by injecting helical electromagnetic waves [7], the problem being mainly considered in a way similar to the dynamo effect and examined with magneto-hydrodynamic (MHD) equations.
In quantum field theory formulated in $D = 2 + 1$ space-time Eq. (1), called the Chern–Simons term (or action), is the basis for developing so called topologic quantum field theories [8]. Being invariant under (abelian) gauge transformations of the vector field $(A_\alpha(x) \rightarrow A_\alpha(x) + \partial_\alpha \varphi(x))$, where $\varphi(x)$ is an arbitrary function which at the infinity tends quickly enough to zero) the Chern–Simons term allows one in $D = 2 + 1$ to consider massive vector fields without breaking the gauge invariance of the theory [9]. An approach to the description of particles with fractional statistics and spin [10], which in their turn may occur to be relevant to the problem of high–$T_c$ superconductivity [11], is also based on the interaction of matter fields with gauge fields described by the Chern–Simons action (1) (see [12] and references therein).

In view of the variety of ways the helicity manifests itself in physics and for deeper understanding of this feature it seems of interest to study the meaning of the invariant (1) for such well-studied physical objects as electromagnetic fields in 4-dimensional space-time, hydrodynamic media and turbulence, and to find the connection of $h$ with observable physical characteristics of the system, thus establishing the role of conservation laws in helicity transfer from the field to the medium. Since the helicity characterizes the 'twisting' of the vector fields it is natural to assume that the helicity is connected with the angular momentum of the system. For $D = 2 + 1$ Chern-Simons field theory this is well understood, but for ordinary electromagnetic fields and turbulent flows, as we are aware, these problems have not been discussed in detail yet.

In the present paper we consider the relationship between the helicity (1) and the angular momentum of the electromagnetic field, hydrodynamic and turbulent flows, and discuss the problems of angular momentum and helicity transfer from the electromagnetic field to the medium. It is shown (Sections 2, 3) that the trace of the tensor of the spin momentum flow density of the electromagnetic field is equivalent to the helicity density of the electromagnetic field. Physically it means that helicity density characterizes the value of the spin component of the angular momentum flow passing per unit of time through the unit surface normal to the direction of electromagnetic field propagation (characterized by the Poynting vector) in the observation point. As is well-known, the projection of the angular momentum onto the momentum of the electromagnetic field is an invariant of the Poincaré group called the helicity (not to be confused with (1)!1) of the photon. To distinguish these two helicities we denote the first of them by $h$ and the second one by $s$. In classical electrodynamics $s$–helicity corresponds to the degree of electromagnetic wave polarization. As we shall see, Eq. (1) determines the degree of the circular polarization of the electromagnetic wave and is proportional to the mean value of angular momentum projection on to the Poynting vector. In the case of the plane wave Eq. (1) indeed coincides

\[ 2 \]
with the expression for the classical analogue of the photon helicity \( s \), and, therefore, is conserved (physically observable) quantity.

In Section 4 by analogy with the electromagnetic case we consider the relationship between the helicity and angular momentum for turbulent flows, and introduce the kinetic definition of the helicity of particle systems. This kinetic definition is similar to the hydrodynamic one, but it allows us to get deeper insight into the microscopic nature of the \( h \)-helicity.

In Section 5 a problem of angular momentum and helicity transfer from the field to the medium is discussed by use of a simple example of a charged particle propagating in a circular polarized plane electromagnetic wave. It is shown that the particle can acquire angular momentum from electromagnetic waves with nonzero helicity.

2 The angular momentum of the electromagnetic field

A density tensor of the angular momentum of the electromagnetic field may be constructed by the analogy with the orbital momentum tensor of the classical particle [13]:

\[
M_{ij}^k = x_i T_j^k - x_j T_i^k, \tag{2}
\]

where

\[
T_{ij} = \frac{1}{4\pi c} (F_{ik} F_{kj} - \frac{1}{4} g_{ij} F_{kl} F^{kl}) \tag{3}
\]
is the symmetric energy-momentum tensor and \( F_{ik} = \partial_i A_k - \partial_k A_i \) is the electromagnetic field stress tensor, \( g_{ik} \) is the Minkowski metric with a signature chosen to be \((+,-,-,-)\), and \( i, j, k, ... = 0, 1, 2, 3 \).

However, more natural and favorable (for our consideration) is to write down a density tensor of the angular momentum from fundamental principles of the symmetry of electrodynamics under the Lorentz group transformations i.e. by varying the electromagnetic field Lagrangian in compliance with the Noether theorem (see, for example, [14]). In this case the angular momentum density tensor has the form

\[
M_{ij}^{(N)k} = [M_{ij}^k + \frac{1}{4\pi c} (x_i (\partial_j A_j) F^{lk} - x_j (\partial_i A_i) F^{lk})] + \frac{1}{4\pi c} (-A_i F_{jk}^k + A_j F_{ik}^k), \tag{4}
\]

where the first two terms correspond to the density of the orbital momentum and the last two terms describe the spin momentum of the electromagnetic field. It is just the manifest representation of the angular momentum as the sum of the orbital and the spin part enables one, as we will see below, to understand the physical meaning of Eq.(1).

These two definitions, (2) and (4), of the angular momentum differ by a total derivative

\[
M_{ij}^{(N)k} - M_{ij}^k = \frac{1}{4\pi c} \partial_t (x_i A_j F^{lk} - x_j A_i F^{lk}), \tag{5}
\]
and since physical observables are spatial integrals of the conserved current densities and under the condition that the fields tend quickly enough to zero at infinity, the integrals of (2) and (4) taken over the volume of 3-dimensional space coincide. However, if the fields tend to zero rather slow or spread to infinity (as, for example, the plane waves), then taking into account boundary conditions becomes essential for the computation of the integrals of (2) and (4). In overwhelming majority of cases this leads to interesting physical observations concerning (global) properties of the physical system. On the other hand the neglect of boundary conditions may result in incorrect physical conclusions.

The simplest example is the calculation of the projection of the angular momentum on to the momentum of a plane electromagnetic wave. If for this calculation we use Eq. (2), then the result will be identically zero. Indeed, the Poynting vector of the plane wave has the form

\[ P_\alpha = \int d^3x T^{\alpha 0}(x) = \frac{n_\alpha}{8\pi c} \int d^3x (E^2 + H^2), \]  

where \( n_\alpha \) is the unit vector along the direction of wave propagation, and \( E_\alpha = F_{\alpha 0} \) and \( H_\alpha = \frac{1}{2} \varepsilon_{\alpha\beta\gamma} F^{\alpha\beta} \) are electric and magnetic field strength, respectively, satisfying (for the plane wave) the relation

\[ H = n \times E. \]  

The angular momentum calculated with the Eq. (2) has the form

\[ M_\alpha = -\frac{1}{8\pi c} \varepsilon_{\alpha\beta\gamma} n_\beta \int d^3x x_\gamma (E^2 + H^2), \]  

Thus, the helicity

\[ s = \frac{M \cdot P}{|P|} \]  

of any plane electromagnetic wave identically turns to zero if one substitutes into (9) Eq. (6) and (8), which runs counter the observation: in general the plane wave has nonzero circular polarization whose degree is characterized just by the magnitude of \( s \) (or, another words, it is well-known that the photon has nonzero helicity).

On the other hand the computation of \( s \) with the use of Eq. (4) of the angular momentum density tensor gives the correct result and allows one to find the connection between \( h \) and \( s \) helicity of the electromagnetic wave.

It is convenient to carry out the computation in the Coulomb gauge

\[ A_0 = 0, \quad \frac{\partial}{\partial x^\alpha} A^\alpha(x) = 0, \]  

then the only contribution into the value of \( s \) (3) is given by the spinning part of the angular momentum (4), and Eq. (3) (with taking into account (4)) takes the form

\[ s = -\frac{1}{4\pi c} \varepsilon_{\alpha\beta\gamma} \int d^3x n_\alpha A_\beta E_\gamma = \frac{1}{4\pi c} \int d^3x A_\alpha H_\alpha = \frac{1}{4\pi c} \int d^3x A(\nabla \times A), \]  

4
which coincides with the $h$–helicity (Eq. (1)) of the electromagnetic field.

Thus, the $s$–helicity (or circular polarization) of the plane electromagnetic wave is determined by Eq. (1), which allows one to make a conclusion, that the integral lines of electromagnetic field with nonzero circular polarization have the nontrivial topological structure characterized by a linking number.

### 3 Angular momentum flow and $h$–helicity

Let us show that the $h$–helicity of an arbitrary electromagnetic field is equal to the trace of the spin component of the angular momentum flow.

It follows from the conservation law for the angular momentum (12)

$$
\partial_0 M^{(N)}_{\alpha\beta,0} - \partial_\gamma M^{(N)}_{\alpha\beta,\gamma} = 0
$$

that the density of the spin momentum flow has the form

$$
G_{\alpha\beta} = -\frac{1}{2\pi c} \varepsilon_{\alpha\gamma\delta} A_\gamma F_{\delta\beta},
$$

and

$$
G \equiv Tr G_{\alpha\beta} = -\frac{1}{2\pi c} \varepsilon_{\alpha\gamma\delta} A_\gamma F_{\delta\beta} = \frac{1}{4\pi c} \mathbf{A}(\nabla \times \mathbf{A}).
$$

Thus, the density of the spin momentum flow along the direction of electromagnetic field propagation (i.e. the density of $s$–helicity flow) is determined by the $h$–helicity density, and Eq. (1) characterizes the average value, over spatial volume, of the $s$–helicity flow density. The topologically nontrivial configurations of the electromagnetic field manifest themselves through the nonzero helicity values.

### 4 Helicity in turbulence

Let us now proceed with studying the relationship between the helicity and angular momentum of turbulent flows. As we have already mentioned in the Introduction the notion of the helicity of random fields plays an important role in the theory of turbulence; thus, establishing the connection between these two physical notions may occur to be useful for better understanding the turbulence phenomenon.

We shall consider this problem by the example of the most simple type of turbulent motion being homogeneous in space and time and isotropic in space. Just that every case has been considered for studying problems of magnetic field generation ($\alpha$–effect) \[4\] and the generation of large scale structures by small scale helical fields \[6\].
The pair correlation tensor of the velocity field $V(r, t)$

$$\langle V_i(r, t)V_j(r', t') \rangle \equiv B_{ij}(r, r', t, t'). \quad (15)$$

($\langle \rangle$ denotes statistical averaging) is the most important characteristic of turbulence. So let us discuss this characteristic in more detail. The requirement for the tensor (15) to conserve the form under rotations and shifts in the space leads to the following most general form of the correlation tensor [15]

$$\langle V_i V_j \rangle_{R, \tau} = A(R, \tau)\delta_{ij} + B(R, \tau)R_iR_j + G(R, \tau)\varepsilon_{ijl}R_l, \quad (16)$$

where $R = r - r'$ and $\tau = t - t'$, $A(R, \tau)$ and $B(R, \tau)$ are scalar functions and $G(R, \tau)$ is a pseudoscalar function. All this functions are expressed through the corresponding correlation functions of $V(r, t)$. For example,

$$A(0, 0) = \frac{1}{3} \langle V_i(r, t)V_i(r, t) \rangle$$

has the physical meaning of the average energy density of the fluid, and

$$G(0, 0) = \frac{1}{6} \langle V(r, t)(\nabla \times V(r, t)) \rangle \quad (17)$$

determines the helicity density of $V$.

The physical meaning of $B(R, \tau)$ is more complicated and we will not discuss it here.

One can easily see that the form of the right hand side of (16) is conserved not only under the spatial rotations and shifts but under the reflections as well. But since the statement that turbulence is invariant under some group transformations means that all correlation characteristics of turbulence (and not only, for example, (16)) are form-invariant under this transformations, and (17) is indeed reflection noninvariant, it follows that turbulence with nonzero $G(0, 0)$ is reflection noninvariant.

Let us show now that in agreement with the results of the previous sections concerning the helicity of the electromagnetic field the helicity of the turbulent velocity field also has (up to a dimensional factor) the “dynamical” meaning of the projection of the total orbital momentum onto the total momentum of the turbulent medium.

Assuming for simplicity that the density $\rho$ of the fluid is constant we write down the projection in the form

$$s = \frac{\rho}{V}(\int dr[r \times V(r, t)] \int dr'V(r', t)), \quad (18)$$

where $V$ is the volume of the medium. Then, using (16), the equation (18) may be reduced to the form

$$s = \rho \int dRR^2G(R, 0) = \rho \int dRR^2 \langle V(R, 0)\nabla \times V(R, 0) \rangle. \quad (19)$$
If for $R$ larger then a correlation radius $G(R,0)$ tends to zero quick enough we get from (19) that

$$ s \sim \langle \mathbf{V}(r,t) \nabla \times \mathbf{V}(r,t) \rangle. \quad (20) $$

Eqs. (19), (20) represent the relationship we have been looking for. The observation of this fact points to the close connection between topological properties and dynamical characteristics of turbulence. Such a connection plays an important role for understanding the processes of helicity (or angular momentum) exchange between turbulent flows and external fields acting on the former, the generation of various turbulence structures possessing helicity, etc. \[4, 5, 6\]. A simple example of helicity and angular momentum transfer is considered in the next section.

Using the results obtained we may go even further and introduce the notion of helicity for a system of particles by defining a microscopic helicity as

$$ s^M = \sum_a \left[ \mathbf{r}_a \times \mathbf{p}_a \right] \sum_b \mathbf{p}_b, \quad (21) $$

where $\mathbf{r}_a$ and $\mathbf{p}_a$ are the radius vectors and the momentums of the particles, $a, b = 1, ..., N$ and $N$ is the number of particles. Using the microscopic phase density \[16\]

$$ \mathcal{N}(\mathbf{r}, \mathbf{p}, t) = \sum_a \delta (\mathbf{r} - \mathbf{r}(t)) \delta (\mathbf{p} - \mathbf{p}(t)) \quad (22) $$

we can rewrite (21) as

$$ s^M = \frac{1}{2} \int d\Omega d\Omega' [\mathbf{p} \times \mathbf{p}'](\mathbf{r} - \mathbf{r}') \mathcal{N}(\mathbf{r}, \mathbf{p}, t) \mathcal{N}(\mathbf{r}', \mathbf{p}', t), \quad (23) $$

where $d\Omega$ is the measure of integration over the phase space.

Taking into account the fluctuation of the microscopic phase density

$$ \delta \mathcal{N} = \mathcal{N} - \overline{\mathcal{N}}, $$

where $\overline{\mathcal{N}} = nf(\mathbf{r}, \mathbf{p}, t)$, $f(\mathbf{r}, \mathbf{p}, t)$ is the one particle distribution function and $n = \mathcal{N}/V$ is the average particle density in the system, we write down the average value of the microscopic helicity as follows:

$$ s = \langle s^M \rangle = \frac{1}{2} \int d\Omega d\Omega' [\mathbf{p} \times \mathbf{p}'](\mathbf{r} - \mathbf{r}') f(\mathbf{r}, \mathbf{p}, t) f(\mathbf{r}', \mathbf{p}', t) $$

$$ + \frac{1}{2} \int d\Omega d\Omega' [\mathbf{p} \times \mathbf{p}'](\mathbf{r} - \mathbf{r}') \langle \delta \mathcal{N}(\mathbf{r}, \mathbf{p}, t) \delta \mathcal{N}(\mathbf{r}', \mathbf{p}', t) \rangle. \quad (24) $$

The first term in (24) describes a “hydrodynamic” part of the average helicity and can be rewritten in the form (compare with (18))

$$ n^2m^2 \int d\mathbf{r} d\mathbf{r}' [\mathbf{V}(\mathbf{r}, t) \times \mathbf{V}(\mathbf{r}', t)](\mathbf{r} - \mathbf{r}') \quad (25) $$
where $V(r, t) = \frac{1}{m} \int dpf(r, p, t)$ and $m$ is the particle mass.

The second term characterizes a “correlation” part of the average helicity. If the correlation function of the phase density fluctuation is isotropic the correlation part of the average helicity turns to zero. One may also show that the correlation function of a weak interacting homogeneous classical gas does not contribute, in the Landau approximation, into the helicity, and, hence, an inhomogeneous state should be taken into account. Thus, for these cases the kinematic definition of the helicity of a particle system introduced above coincides with the hydrodynamic definition of the helicity. In general case the correlation part of the helicity may cause new correlation effects, but this point requires additional studies.

5 Angular momentum transfer from field to medium

In previous sections we have elucidated the connection between the angular momentum and the $h$–helicity of the electromagnetic field and medium.

In many physical applications a problem arises to determine the value of energy, momentum and angular momentum transferred from the field to the medium. This problem is important for understanding and predicting the behavior of the system.

However, while, for example, in plasmas and turbulence the energy and momentum transfer have been studied in detail [17], the problem of angular momentum transfer has not been developed in the same extent, though it has been under consideration since Sadovski found the effect of turning a small plate by an electromagnetic wave [18].

We see that the problem of angular momentum transfer is related to the problem of the helicity transfer from fields to the medium discussed, for example, in [7].

The problem of angular momentum transfer from electromagnetic fields to particles and consequences of this effect for the electromagnetic pumping up of the angular momentum into plasmas were discussed, in particular, in Ref. [19]. Here we consider this question from somewhat different point of view based on the relationship between the angular momentum and the $h$–helicity, and, in contrast to [19], demonstrate that the charged particles may acquire orbital momentum from plane electromagnetic waves with nonzero helicity and that this effect may result in pumping up the angular momentum into plasma.

We restrict ourselves by the consideration of the simplest possible model of a system of non-interacting charged particles, so that the problem is actually reduced to considering the motion of a single particle.

Let a particle with a charge $e$ and a mass $m$ be placed into the field of a circular
polarized plane wave propagating along the axis $x$ and having electric strength

$$\mathbf{E} = e_y E_0 \cos \omega (t - \frac{x}{c}) + \alpha e_z E_0 \sin \omega (t - \frac{x}{c}),$$  \hspace{1cm} (26)$$

where $E_0$ is the wave amplitude, $\omega$ is the electromagnetic field frequency and $\alpha = \pm 1$ denotes left or right polarization.

Note that due to Eqs. (3), (11) the angular momentum of the plane wave is directed along the wave momentum (axis $x$), and its value coincides with the helicity of the electromagnetic wave, the helicity density of the latter being

$$G = \frac{\alpha}{4\pi c} \mathbf{A} \nabla \times \mathbf{A} = \frac{\alpha E_0^2}{4\pi \omega} = \frac{\alpha \varepsilon}{\omega},$$  \hspace{1cm} (27)$$

where $\varepsilon = \frac{E_0^2}{4\pi}$ is the energy density of the wave. If we considered a wave with arbitrary polarization the value of $\alpha$ in (27) would have determined the corresponding polarization properties of the wave. For instance, if the wave is linear polarized, $\alpha = 0$, and, hence $h = s = 0$, which testifies to the triviality of the corresponding topological characteristics of the linear polarized wave.

The equation of motion of the particle in the electromagnetic wave is nonlinear, and, in general, finding an exact solution of the equation is nontrivial problem. In view of this, and since here we do not aim at carrying out the rigorous derivation of the expression for the particle orbital momentum acquired, we shall consider the process of angular momentum transfer from the wave to the particle using semiquantitative consideration analogous to that used, for example, in the computation of the strength of high frequency pressure [20].

We look for the solution as a series expansion in the field amplitude and restrict ourselves to the nonrelativistic approximation. Particle motion consists in quick rotation in $(y, z)$ plane and slow motion averaged over the oscillations. Let us further assume that the particle velocity $\langle \mathbf{v} \rangle$ averaged over the oscillations is directed along wave propagation (axis $x$). We shall see below that for particle to get an angular momentum from the wave it is crucial that the former accelerates along the wave. Slow variation of $\langle \mathbf{v} \rangle$ with time may be caused, in the first place, by the momentum transfer from the scattered wave to the particle, or by a weak constant electric field.

Then from the equation

$$m \frac{d^2 \mathbf{r}}{dt^2} = e \mathbf{E}(\langle \mathbf{r} \rangle, t) + \frac{e}{c} \langle \mathbf{v} \rangle \times \mathbf{H}(\langle \mathbf{r} \rangle, t)$$  \hspace{1cm} (28)$$

\footnote{Note that the well known particular solution of the equation, which describes particle motion along a circle perpendicular to the wave momentum, implies that the particle had the angular momentum \textit{a priori}, and the latter does not vary in the course of motion. Hence, this solution is not one we are looking for.}
for the oscillating part \( \mathbf{r} \) of the particle radius vector we get

\[
\mathbf{r} = -\frac{eE_0}{m\omega^2(1 - \frac{\langle v \rangle}{c})}(\mathbf{e}_yE_0 \cos (1 - \frac{\langle v \rangle}{c})\omega t + \alpha \mathbf{e}_zE_0 \sin (1 - \frac{\langle v \rangle}{c})\omega t).
\]  

(\langle v \rangle \equiv |\langle v \rangle|)

Eq. (29) may be directly obtained by solving Eq. (28) in the coordinate system where \( \langle v \rangle \) is equal to zero in a given time moment and by passing to the laboratory system with making use of the Lorentz transformation (Doppler effect) in the nonrelativistic approximation.

Momentum variation with time is expressed by the formula

\[
\frac{d\mathbf{M}}{dt} = [\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}]  
\]

(30)

Taking into account slow dependence of \( \langle v \rangle \) on time we get from (29) and (30) that

\[
\frac{d\mathbf{M}}{dt} = \frac{4\pi e^2}{mc\omega^2} \frac{d\langle v \rangle}{dt} G
\]

(31)

We see that the rate of momentum variation is proportional to the helicity density \( G \) and particle’s slow acceleration along the wave. This testifies to the existence of the effect of momentum transfer from an electromagnetic wave to a charged particle, and indicates that such a transfer is possible if only the wave possesses the helicity, the process being governed by the angular momentum conservation law. The simple example having been considered shows a way the topologically nontrivial fields affect the dynamics of the charged particles.

For plasmas this effect means the possibility of pumping up the angular momentum into the plasmas by the helical fields. The angular momentum of the plasma will increase until the particle acceleration along the wave is compensated by the friction force, and until the increase is terminated because of the angular momentum loss caused by particle radiation.

6 Conclusion

For electromagnetic fields, turbulent media and particle systems we have established the relationship between the \( h \)–helicity (1), being a characteristics of topologically nontrivial vector fields, and the \( s \)–helicity (3) which characterizes the value of the projection of the angular momentum onto the momentum of the medium. This testifies to the close connection between corresponding topological and dynamical properties of the physical objects, which should be taken into account when studying the effects of angular momentum and helicity exchange between fields and media.
For hydrodynamics and turbulence the momentum transfer between fields and media points to the “intertwining” of the helicity of the field with that of the medium velocity field, which may point to the existence of a conservation law for a topological charge characterizing the field-medium system as a whole.

**Note added.** When this work was completed the authors became aware of Ref. [21], where the connection between the two types of helicity of the electromagnetic field was pointed out in the framework of a topological electromagnetism.

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