ON THE BOUNDEDNESS OF SOLUTIONS OF SPDES

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Abstract. In this paper estimates for the uniform norm of solutions of parabolic SPDEs are derived. The result is obtained through iteration techniques, motivated by the work of Moser in deterministic settings. As an application of the main result, solvability of a class of semilinear SPDEs is derived.

1. Introduction

In the present work we consider the following stochastic partial differential equation (SPDE) on $[0, T] \times Q$,

$$du_t = (L_t u_t + f_t)dt + (M_t^k u_t + g_t^k)dw_t^k, \quad u_0 = \psi,$$

where the operators $L_t$, and $M_t^k$ are given by

$$L_t u = \partial_j (a_{ij} \partial_i u) + b_i \partial_i u + c_t u, \quad M_t^k u = \sigma_{ik} \partial_i u_t + \mu_t^k u,$$

and $Q$ is a bounded Lipschitz domain in $\mathbb{R}^d$. We are interested in boundedness properties of weak solutions. The corresponding problem in the deterministic case, has been extensively studied. The first results are due to [3] and [9]. Boundedness of solutions of SPDEs usually is obtained through embedding theorems of Sobolev spaces. Such results can be obtained from $L_p$–theory, see e.g. [6], for equations considered on the whole space. This approach requires some smoothness of the coefficients. In [2], through the technique of Moser’s iteration, introduced in [9], boundedness results are derived without posing smoothness assumption on the coefficients, by staying in the $L_2$–framework. This served as a main motivation to our work. However, in [2], it is assumed that there exist constants $\lambda > \beta > 0$, such that for any $\xi \in \mathbb{R}^d$, one has $a^{ij} \xi_i \xi_j \geq \lambda |\xi|^2$ and $(72+1/2)\sigma^{ik} \sigma^{jk} \xi_i \xi_j \leq \beta |\xi|^2$. In the present paper only the classical stochastic parabolicity condition will be assumed in order to get estimates for the uniform bound of solution.

With the use of our main theorem, existence and uniqueness results for semilinear SPDEs are derived, under a weak condition on the growth of the non-linear term. We construct the solutions, by using comparison techniques, adapted from [4].

Let us introduce some of the notation that will be used through the paper (for general notions on SPDEs we refer to [12] and [10]). We consider a

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complete probability space \((\Omega, \mathcal{F}, P)\). It is equipped with a right-continuous filtration \((\mathcal{F}_t)_{t\geq 0}\), such that \(\mathcal{F}_0\) contains all \(P\)-zero sets, and \(\{w^k_i\}_{k=1}^\infty\) is a sequence of independent real valued \(\mathcal{F}_t\)-Wiener processes. We denote with \(H^0_0(Q)\) the space of all measurable functions \(v\) on \(Q\), vanishing on the boundary, such that \(v\) and its generalized derivatives of first order lie in \(L_2(Q)\). The norm on \(H^0_0(Q)\) will be denoted by \(|\cdot|\). For \(p \in [1, \infty]\), the norm in \(L_p(Q)\) will be denoted by \(|\cdot|_p\), while the norm in \(L_p([0,T] \times Q)\) will be denoted by \(|\cdot|_p\). The same notation will be used also for the norm in spaces of \(l_2\)-valued functions. We set \(L_p := L_p(\Omega, \mathcal{F}_0; L_p(Q))\), \(L_p := L_p(\Omega \times [0,T], \mathcal{P}; L_p(Q))\), and \(L_p(l_2) := L_p(\Omega \times [0,T], \mathcal{P}; L_p(Q; l_2))\) where \(\mathcal{P}\) denotes the predictable \(\sigma\)-algebra. To ease the notation, we use the summation convention with respect to repeated indices, and when it does not cause confusion, for integrals of the type \(\int_Q f \, dx\) and \(\int_0^t \int_Q f \, dxdts\), we drop the measures and we write \(\int_Q f\) and \(\int_0^t \int_Q f\), respectively. The constants in the calculations, usually denoted by \(N\), may change from line to line, but, unless otherwise noted, they always depend only on the structure constants of the equation (see Section 2).

The rest of the paper is organized as follows. In Section 2 the assumptions are formulated and the main theorems are stated. In Section 3 preliminary results are collected, which are then used in the proof of the main theorem in Section 4. In Section 5 we apply our result, in combination with the comparison principle, to construct solutions for a class of semilinear SPDEs.

2. Formulation and Main Results

We pose the following conditions on equation (1.1).

**Assumption 2.1.** i) The coefficients \(a^i, b^i\) and \(c\) are real-valued \(\mathcal{P} \times \mathcal{B}(Q)\) measurable functions on \(\Omega \times [0,T] \times Q\) and are bounded by a constant \(K \geq 0\), for any \(i, j = 1, ..., d\). The coefficients \(\sigma^i = (\sigma^{ik})_{k=1}^\infty\) and \(\mu = (\mu^k)_{k=1}^\infty\) are \(l_2\)-valued \(\mathcal{P} \times Q\)-measurable functions on \(\Omega \times [0,T] \times Q\) such that

\[
\sum_i \sum_k |\sigma_t^{ik}(x)|^2 + \sum_k |\mu_t^k(x)|^2 \leq K \quad \text{for all } \omega, t \text{ and } x,
\]

ii) \(f\) and \(g = (g^k)_{k=1}^\infty\) are \(\mathcal{P} \times \mathcal{B}(Q)\)-measurable functions on \(\Omega \times [0,T] \times Q\) with values in \(\mathbb{R}\) and \(l_2\), respectively, such that \(\mathbb{E}(\|f\|_2^2 + \|g\|_2^2) < \infty\)

iii) \(\psi\) is an \(\mathcal{F}_0\)-measurable random variable in \(L_2(Q)\) such that \(\mathbb{E}|\psi|_2^2 < \infty\)

**Assumption 2.2.** There exists a constant \(\lambda > 0\) such that for all \(\omega, t, x\) and for all \(\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d\) we have

\[
a_t^{ij}(x)\xi_i\xi_j - \frac{1}{2}\sigma_t^{ik}(x)\sigma_t^{kj}(x)\xi_i\xi_j \geq \lambda|\xi|^2,
\]

We will refer to the constants \(K, T, \lambda, d\) and \(|Q|\), where the later is the Lebesgue measure of \(Q\), as structure constants.
Definition 2.1. An $L_2$–solution of equation (1.1) is understood to be an $L_2(Q)$-valued, $\mathcal{F}_t$-adapted, strongly continuous process $(u_t)_{t \in [0,T]}$, such that

(i) $u_t \in H^1$, for $dP \times dt$ almost every $(\omega, t) \in \Omega \times [0,T]$

(ii) $\mathbb{E} \int_0^T |u_t|^q_2 \, dt < \infty$

(iii) for all $\phi \in C_c^\infty(Q)$ we have with probability one

$$
(u_t, \phi) = (\psi, \phi) + \int_0^t - \left( a_s^{ij} \partial_j u_s, \partial_i \phi \right) + \left( b_s^i \partial_i u_s + c_s u_s, \phi \right) + \left( f_s(u_s), \phi \right) \, ds \\
+ \int_0^t \left( M_s^k u_s + g_s^k, \phi \right) \, dw^k_s,
$$

for all $t \in [0,T]$.

By [8], under these assumptions (1.1) admits a unique $L_2$–solution. Moreover, the following estimate holds for $q \leq 2$,

$$
\mathbb{E} \sup_{0 \leq t \leq T} |u_t|^q_2 \leq N \mathbb{E} (|\psi|^q_2 + \|f\|^q_2 + \|g\|^q_2),
$$

(2.2)

where $N = N(q,d,K,\lambda,T)$.

The following is our main result.

Theorem 2.1. Suppose that Assumptions 2.1 and 2.2 hold, and let $u$ be the unique $L_2$–solution of equation (1.1). Then for any $q > 0$

$$
\mathbb{E} \|u\|^q_\infty \leq N \mathbb{E} (|\psi|^q_\infty + \|f\|^q_\infty + \|g\|^q_\infty),
$$

(2.3)

where $N = N(q,d,K,\lambda,|Q|,T)$.

Remark 2.1. Notice that by interpolating between (2.2) and (2.3), for any $p \geq 2$ and $q \leq 2$, one obtains

$$
\mathbb{E} \sup_{0 \leq t \leq T} |u_t|^q_p \leq N \mathbb{E} (|\psi|^q_p + \|f\|^q_p + \|g\|^q_p),
$$

where $N$ can be chosen to be independent of $p$. In fact, such a uniform estimate for the $L_p$-norms of the solutions is equivalent to Theorem 2.1 for $q \leq 2$.

The arguments in Sections 3 and 4 can be easily applied to equations of other types, even when uniqueness of solutions is not known. In [2], for instance, the following quasilinear equation is considered:

$$
du_t = (L_tu_t + f_t(u, \nabla u) + \partial_i h^i_t(u, \nabla u)) \, dt + (g^k_t(u, \nabla u)) \, dw^k_t, \quad u_0 = \psi. \quad (2.4)
$$

Here in addition to Assumption 2.1 we assume that the functions $f$, $g$, and $h$ are $\mathcal{P} \times \mathcal{B}(Q \times \mathbb{R} \times \mathbb{R}^d)$-measurable, with values in $\mathbb{R}$, $\mathbb{R}^d$, and $l_2$, respectively. Moreover, they are supposed to satisfy the following growth conditions uniformly in $\omega, t, x$:

$$
|f_t(x, r, z)| \leq |f_t(x)| + K|r| + K|z|,
$$

$$
|h_t(x, r, z)| \leq |f_t(x)| + K|r| + \alpha|z|,
$$

$$
|g_t(x, r, z)| \leq |g_t(x)| + K|r| + \beta|z|.
$$
for some $\alpha, \beta$ constants, and Assumption 2.2 is modified to

$$a^ij(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \alpha + \beta^2/2 < \lambda.$$ 

**Theorem 2.2.** Let the assumptions formulated above hold, and let $u$ be an $L_2$-solution of equation (2.4). Then for any $q > 0$

$$E\|u\|^q_{\infty} \leq N E(|\psi|_{\infty}^q + \|f\|_{\infty}^q + \|g\|_{\infty}^q),$$

where $N = N(q, d, K, \lambda, |Q|, T).$

Notice that Theorem 2.1 is not a consequence of Theorem 2.2 and vice versa. However, as mentioned before, Theorem 2.2 can be proved by following the same steps as in the linear case, and therefore the proof is omitted.

3. Preliminaries

We set $\gamma = (d + 2)/d$. The following embedding theorem is an additive version of Theorem 6.9, [5].

**Lemma 3.1.** Suppose that $v \in L_2([0, T], H_0^1(Q)) \cap L_\infty([0, T], L_2(Q)).$ Then $v \in L_{2\gamma}([0, T] \times Q)$ and

$$\left(\int_0^T \int_Q |v_t|^2 \gamma \right)^{1/\gamma} \leq N \left( \sup_{0 \leq t \leq T} \int_Q |v_t|^2 + \int_0^T \int_Q |\nabla v_t|^2 \right)$$

with $N = N(d, |Q|, T)$

We are also going to use the following result (see Proposition IV.4.7 and Exercise IV.4.31/1, [11]).

**Proposition 3.2.** Let $X$ be a non-negative, adapted, right-continuous process, and let $A$ be a non-decreasing, continuous process such that

$$E X_\tau \leq E A_\tau$$

for any bounded stopping time $\tau$. Then for any $\sigma \in (0, 1)$

$$E \sup_{t \leq T} X_\sigma^t \leq \sigma^{-\sigma}(1 - \sigma)^{-1} E A_\tau^T.$$ 

The difference between the next lemma and Lemma 8 in [2], is that we obtain supremum estimates, that are essential for having (3.7) almost surely, for all $t \in [0, T]$. Therefore, we give a whole proof for the sake of completeness.

**Lemma 3.3.** Let $\tau \leq T$ be a stopping time, and suppose that $u$ satisfies equation (1.1), for $t \leq \tau$. Suppose that $f \in L_p$, $g \in L_p(l_2)$, and $\psi \in L_p$ for some $p \geq 2$. Then for any $0 < q \leq p$ there exists a constant $N = N(q, d, K, \lambda, p)$, such that

$$E \left( \sup_{t \leq \tau} |u_t|^p + E \int_0^\tau \int_Q |\nabla u_s|^2 |u_s|^{p-2} \right)^{q/p} \leq N E(|\psi|_{p}^q + \|f\|_{p}^q + \|g\|_{p}^q).$$

(3.6)
Moreover, almost surely
\[\int_{Q} |u_t|^p = \int_{Q} |u_0|^p + p \int_{0}^{t} \int_{Q} (\sigma_s^{ik} \partial_i u_s + \mu^k u_s + g^k) u_s |u_s|^{p-2} dx dw^k_s\]

\[+ \int_{0}^{t} \int_{Q} -p(p-1) a_{ij}^{s} \partial_i u_s |u_s|^{p-2} \partial_j u_s + p(b_s^{i} \partial_i u_s + c_s u_s + f_s) u_s |u_s|^{p-2}\]

\[+ \frac{1}{2} p(p-1) \int_{0}^{t} \int_{Q} |\sigma_s^{ik} \partial_i u_s + \mu^k u_s + g_s|^2 |u_s|^{p-2}, \quad (3.7)\]
for any \(t \leq \tau\).

Remark 3.1. The reason why \((3.7)\), and many subsequent calculations are done up to stopping times is to avoid using uniqueness of solutions. This is necessary, for instance, when one wishes to apply these arguments for the proof of Theorem 2.2.

Proof. Consider the functions

\[\phi_n(r) = \begin{cases} 
|r|^p & \text{if } |r| < n \\
\frac{n^{p-2} p(p-1)}{2} |r|^2 - n^2 + pn^{p-1}(|r| - n) + n^p & \text{if } |r| \geq n.
\end{cases}\]

Then one can see that \(\phi_n\) are twice continuously differentiable, and satisfy

\[|\phi_n(x)| \leq N|x|^2, \quad |\phi'_n(x)| \leq N|x|, \quad |\phi''_n(x)| \leq N,\]

where \(N\) depends only on \(p\) and \(n \in \mathbb{N}\). Let \(\tau^1 \leq T\) be a stopping time and set \(\tau^2 = \tau^1 \wedge \tau\). Then for each \(n \in \mathbb{N}\) we have almost surely

\[\int_{Q} \phi_n(u_{\tau^2 \wedge t}) = \int_{Q} \phi_n(u_0) + \int_{0}^{\tau^2 \wedge t} \int_{Q} (\sigma_s^{ik} \partial_i u_s + \mu^k u_s + g^k) \phi_n(u_s) dx dw^k_s\]

\[+ \int_{0}^{\tau^2 \wedge t} \int_{Q} -a_{ij}^{s} \partial_i u_s \phi'_n(u_s) \partial_j u_s + b_s^{i} \partial_i u_s \phi'_n(u_s) + c_s u_s \phi'_n(u_s) + f_s \phi'_n(u_s)\]

\[\quad + \frac{1}{2} \int_{0}^{\tau^2 \wedge t} \int_{Q} |\sigma_s^{ik} \partial_i u_s + \mu^k u_s + g_s|^2 \phi''_n(u_s), \quad (3.8)\]
for any \(t \in [0, T]\) (see for example, Section 3 in [7]). For the functions \(\phi_n\) we have that for any \(r \in \mathbb{R}, \phi_n(r) \to |r|^p, \phi'_n(r) \to p|r|^{p-2}r, \phi''_n(r) \to p(p-1) |r|^{p-2}\), as \(n \to \infty\), and

\[\phi_n(r) \leq N|r|^p, \quad \phi'_n(r) \leq N|r|^{p-1}, \quad \phi''_n(r) \leq N|r|^{p-2}, \quad (3.9)\]

where \(N\) depends only on \(p\). Also the following inequalities hold:

i) \(|\phi'_n(r)| \leq p\phi_n(r)\)
ii) \(|r^2 \phi''_n(r)| \leq p(p-1)\phi_n(r)\)
iii) \(|\phi'_n(r)|^2 \leq 4p \phi''_n(r)\phi_n(r)\)
iv) $|\phi''_n(r)|^{p/(p-2)} \leq |p(p-1)|^{p/(p-2)} \phi_n(r)$

By Young’s inequality, and the parabolicity condition we have

$$
\int_Q \phi_n(u_{t_2}) \leq m_{t_2} + \int_Q \phi_n(u_0) + \int_0^{t_2} \int_Q (\epsilon |\nabla u|_s^2 + N|u_s|^2 + N|g_s^k|^2) \phi''_n(u_s)
$$

$$
\int_0^{t_2} \int_Q -\lambda |\nabla u_s|^2 \phi''_n(u_s) + b^i_s \partial_i u_s \phi'_n(u_s) + c_s u_s \phi'_n(u_s) + f_s \phi'_n(u_s), \quad (3.10)
$$

where $N = N(d, K, \epsilon)$, and $m^{(n)}_t$ is the martingale from (3.8). By the properties of $\phi_n$ and its derivatives, the following hold:

i) $\partial_t u_s \phi'_n(u_s) \leq \epsilon \phi''_n(u_s)|\partial_t u_s|^2 + N \phi_n(u_s)$

ii) $|u_s \phi'_n(u_s)| \leq p \phi_n$

iii) $|f_s \phi'_n(u_s)| \leq |f_s| \phi''_n(u_s)^{1/2} \phi_n(u_s)^{1/2} \leq N|f_s|^p + N \phi_n(u_s)$

iv) $|u_s|^2 \phi''_n(u_s) \leq N \phi_n(u_s)$

v) $\sum_k |g_s^k|^2 \phi''(u_s) \leq N \phi_n(u_s) + N \left( \sum_k |g_s^k|^2 \right)^{p/2}$

where $N$ depends only on $p$ and $\epsilon$.

Hence by taking expectations we obtain

$$
\mathbb{E} \int_Q \phi_n(u_{t_2}) + \mathbb{E} \int_0^{t_2} \int_Q |\nabla u_s|^2 \phi''_n(u_s) \leq N \mathbb{E} \mathcal{K}_{\tau_1} + N \int_0^{\tau_2} \mathbb{E} \int_Q \phi_n(u_{s \land \tau_2}),
$$

where $N = N(d, p, K, \lambda)$ and

$$
\mathcal{K}_s = |\psi|^p + \int_0^s |f_s|^p + |g_s|^p.
$$

By Gronwall’s lemma we get

$$
\mathbb{E} \int_Q \phi_n(u_{t_1}) + \mathbb{E} \int_0^{t_2} \int_Q |\nabla u_s|^2 \phi''_n(u_s) \leq N \mathbb{E} \mathcal{K}_{\tau_1}
$$

for any $t \in [0, T]$, with $N = N(T, d, p, K, \lambda)$. Going back to (3.10), using the same estimates, and the above relation, by taking suprema up to $\tau_1$ we have

$$
\mathbb{E} \sup_{t \leq \tau_1} \int_Q \phi_n(u_{t \land \tau_1}) \leq N \mathbb{E} \mathcal{K}_{\tau_1} + \mathbb{E} \sup_{t \leq \tau_2} |m^{(n)}_t|.
$$

$$
\leq N \mathbb{E} \mathcal{K}_{\tau_1} + N \mathbb{E} \left( \int_0^{t_2} \sum_k \left( \int_Q |\sigma^{i_k} \partial_i u_s + \mu^k u_s + g^k |\phi''_n(u_s) \phi_n(u_s)|^{1/2} \right)^2 \right)^{1/2}
$$

$$
\leq N \mathbb{E} \mathcal{K}_{\tau_1} + N \mathbb{E} \left( \int_0^{t_2} \int_Q (|\nabla u_s|^2 + |u_s|^2 + |g_s|^2) \phi''_n(u_s) dx \int_Q \phi_n(u_s) dx ds \right)^{1/2}
$$
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\[ \leq N\mathbb{E}K_{\tau_1} + \frac{1}{2}\mathbb{E} \sup_{t \leq \tau_1} \int_Q \phi_n(u_{t \wedge \tau}) < \infty, \]

where \( N = N(T, d, p, K, \lambda). \) Hence,

\[ \mathbb{E} \sup_{t \leq \tau_1} \int_Q \phi_n(u_{t \wedge \tau}) + \mathbb{E} \int_{\tau \wedge \tau_1} \int_Q |\nabla u_s|^2 \phi_n'(u_s) \leq N\mathbb{E}K_{\tau_1}, \]

and by Fatou’s lemma we get (3.6) for \( q = p. \) The \( q < p \) case can be covered by applying Proposition 3.2 with \( \sigma = q/p. \) For (3.7), we go back to (3.8), and by letting a subsequence \( n(k) \to \infty \) and using the dominated convergence theorem, we see that each term converges to the corresponding one in (3.7) almost surely, for all \( t \leq \tau. \) This finishes the proof.

\[ \square \]

**Corollary 3.4.** Let \( \tau \leq T \) be a stopping time, and suppose that \( u \) satisfies equation (1.1) for \( t \leq \tau \) Set \( p = 2\gamma^n \) with \( n \in \mathbb{N}. \) Then, almost surely, for all \( t \leq \tau \)

\[ \int_Q |u_t|^p + \frac{p^2}{4} \int_0^t \int_Q |\nabla u|^2 |u|^{p-2} \]

\[ \leq N \left( \int_Q |\psi|^p + p^k \int_0^t \int_Q |u_s|^p + p^{-p} \int_0^t \int_Q |f_s|^p + |g_s|^p \right) + \mathbb{N}m_t, \]

where \( m_t \) is the martingale from (3.7), \( \kappa = 4\gamma/(\gamma - 1), \) and \( N, \mathbb{N} \) are constants depending only on \( K, d, T, \lambda. \)

**Proof.** By Lemma 3.3 the parabolicity condition, and Young’s inequality we have

\[ \int_Q |u_t|^p + \frac{p^2}{4} \int_0^t \int_Q |\nabla u|^2 |u|^{p-2} \]

\[ \leq N'' \left( \int_Q |\psi|^p + p^2 \int_0^t \int_Q |u_s|^p + \int_0^t \int_Q p|f_s||u_s|^{p-1} + p^2|g_s|^2|u_s|^{p-2} \right) \]

\[ + \mathbb{N}m_t. \]

Then by Young’s inequality we have

\[ p|f_s||u_s|^{p-1} \leq p^{-p}|f_s|^p + p^{2p/(p-1)}|u_s|^p \leq p^{-p}|f_s|^p + p^\kappa |u_s|^p, \]

and for \( n \geq 1 \)

\[ p^2|g_s|^2|u_s|^{p-2} \leq p^{-p}|g_s|^p + p^{4p/(p-2)}|u_s|^p \leq p^{-p}|g_s|^p + p^\kappa |u_s|^p. \]

The case \( n = 0 \) can be covered by taking \( N = 16N'' \).

\[ \square \]
4. Proof of Theorem 2.1

Proof. Throughout the proof, the constants $N$ in our calculations will be allowed to depend on $q$ as well as on the structure constants. Without loss of generality we assume that the right hand side in (2.3) is finite. In addition, first suppose that for every $p \geq 2$,

$$E|\psi|_{\infty}^p + E \int_0^T |f_t|_{\infty}^p + E \int_0^T |g_t|_{\infty}^p < \infty. \quad (4.11)$$

Let $\tau \leq T$ be a stopping time, and let $v$ satisfy equation (1.1), for $t \leq \tau$. Recall that $\gamma = (d + 2)/d > 1$. By applying Lemma 3.1 to $\bar{v} = |v|^{p/2}$, we have, for any $p \geq 2$

$$E \left[|\psi|_{\infty}^{q/p} \vee \left( \int_0^\tau \int_Q |v_t|^{p\gamma} \right)^{q/p} \right] \leq Nq/p \left( \sup_{0 \leq t \leq \tau} \int_Q |v_t|^p + \frac{p^2}{4} \int_0^\tau \int_Q |\nabla v_t|^2 |v_t|^{p-2} \right)^{q/p}. \quad (4.12)$$

To estimate the right-hand side above, first notice that, if $p = 2\gamma^n$ for some $n$, then by taking supremum in the inequality from Corollary 3.4, we have for any stopping time $\tau_1 \leq T$

$$\sup_{0 \leq s \leq \tau_2} \int_Q |v_s|^p \leq N \left( |\psi|_{\infty}^{p^\prime} + p^\prime \int_0^{\tau_2} \int_Q |v_s|^p + \frac{p^\prime}{4} \int_0^{\tau_2} \int_Q |\nabla v_s|^2 |v_s|^{p^\prime-2} + \sup_{0 \leq s \leq \tau_2} |m_s| \right) \quad (4.13)$$

where $\tau_2 = \tau_1 \wedge \tau$. By the Davis inequality we can write

$$E \sup_{0 \leq s \leq \tau_2} |m_s| \leq N E \left( \int_0^{\tau_2} \left( \int_Q p(\sigma_1^{ik} \partial_i v_s + \mu^k v_s + g^k v_s)|v_s|^{p-2} \right)^2 \right)^{1/2} \leq N E \left( \sup_{0 \leq s \leq \tau_2} \int_Q |v_s|^p \right)^{1/2} \left( \int_0^{\tau_2} \int_Q p^2 |\sigma_1^{ik} \partial_i v_s + \mu^k v_s + g^k|^2 |v_s|^{p-2} \right)^{1/2}. \quad (4.14)$$

Applying Young’s inequality and recalling the already seen estimates for the second term yields

$$E \sup_{0 \leq s \leq \tau_2} |m_s| \leq \varepsilon E \sup_{0 \leq s \leq \tau_2} \int_Q |v_s|^p \quad (4.15)$$

$$+ \frac{N}{\varepsilon} E \left( p^2 \int_0^{\tau_2} \int_Q |\nabla v_s|^2 |v_s|^{p-2} + \frac{p^2}{4} \int_0^{\tau_2} \int_Q |v_s|^p + \frac{p}{d} \int_0^{\tau_2} |g_{\tau_2}| \right)$$

for any $\varepsilon > 0$. With the appropriate choice of $\varepsilon$, combining this with (4.13) and using Corollary 4.4 once again, now without taking supremum, we get

$$E \left( \sup_{0 \leq s \leq \tau_2} \int_Q |v_s|^p + \frac{p^2}{4} \int_0^{\tau_2} \int_Q |\nabla v_s|^2 |v_s|^{p-2} \right) \leq N E |\psi|_{\infty}^p$$
and the last expectation vanishes. Now consider $C$ for a large enough, but fixed $n$.

Consider the minimal $\kappa \leq Np \leq \frac{1}{2} + 1$.

Upon combining the above with (4.12), for $q \leq \infty \leq p\leq \frac{1}{2} + 1$.

Taking any integer $m \geq n_0 = n_0(d, q)$ such that $p_0 > 2q$. Taking any integer $m \geq n_0$ we have

$$\prod_{n=n_0}^{m-1} c_n \leq \prod_{n=n_0}^{m-1} (N\gamma^{\kappa+1})^{q_n/2\gamma^n} c_{2q/2\gamma^n}$$

$$\leq \exp \left[ \log(N\gamma^{\kappa+1}) \sum_{n=n_0}^{m-1} \frac{q^n}{2\gamma^n} + \frac{m}{n_0} \frac{q}{\gamma^n} \right] \leq N_0,$$
where \( N_0 \) does not depend on \( m \). Also,
\[
N \sum_{n=n_0}^{m-1} p_n^{-q} \leq N_1,
\]
where \( N_1 \) does not depend on \( m \). Therefore, by iterating (4.14) we get
\[
\liminf_{m \to \infty} \mathbb{E}|\psi|^q_{\infty} \vee \|1_{t \leq \tau} v\|^q_{p_m} \leq N_0(\mathbb{E}N_1(\|f\|^q_{\infty} + \|g\|^q_{\infty}) + \mathbb{E}|\psi|^q_{\infty} \vee \|1_{t \leq \tau} v\|^q_{p_{n_0}}),
\]
and thus by Fatou’s lemma and (3.6)
\[
\mathbb{E}\|1_{t \leq \tau} v\|^q_{\infty} \leq N\mathbb{E}(\|f\|^q_{\infty} + \|g\|^q_{\infty} + |\psi|^q_{\infty} + \sup_{0 \leq t \leq \tau} |v_t|^q_{p_{n_0}})
\]
\[
\leq N\mathbb{E}(\|f\|^q_{\infty} + \|g\|^q_{\infty} + |\psi|^q_{\infty} + \|f\|^q_{p_{n_0}} + \|g\|^q_{p_{n_0}} + |\psi|^q_{p_{n_0}})
\]
\[
\leq N\mathbb{E}(\|f\|^q_{\infty} + \|g\|^q_{\infty} + |\psi|^q_{\infty}).
\] (4.15)

As for the general case, define
\[
\tau_n = \inf\{t \geq 0 : |\psi|_{\infty} + \int_0^t e^{|f_s|_{\infty} + |g_s|_{\infty}} \geq n\} \wedge T,
\]
and
\[
\psi^{(n)} = \psi 1_{\tau_n > 0}, f_s^{(n)} = f_s 1_{\tau_n \geq s}, g_s^{(n)} = g_s 1_{\tau_n \geq s}.
\]

Notice that for almost all \( \omega \), \( \tau_n(\omega) = T \) for large enough \( n = n(\omega) \). Notice that if \( u \) satisfies (1.1) for \( t \leq T \), then \( u_t^{(n)} = 1_{t \leq \tau_n} u_t \) satisfies
\[
du_t^{(n)} = (Lu_t^{(n)} + f_t^{(n)}) dt + (M_t^{k} u_t^{(n)} + g_t^{(k)}) dw_t^{k}, u_0^{(n)} = \psi^{(n)}
\]
for \( t \leq \tau_n \). The new data now satisfy (1.1), therefore by (4.15)
\[
\mathbb{E}\|1_{t \leq \tau_n} u^{(n)}\|^q_{\infty} \leq N\mathbb{E}(\|f^{(n)}\|^q_{\infty} + \|g^{(n)}\|^q_{\infty} + |\psi^{(n)}|_{\infty}^q)
\]
\[
\leq N\mathbb{E}(\|f\|^q_{\infty} + \|g\|^q_{\infty} + |\psi|_{\infty}^q).
\]

Applying Fatou’s lemma once more yields
\[
\mathbb{E}\|u\|^q_{\infty} \leq N\mathbb{E}(\|f\|^q_{\infty} + \|g\|^q_{\infty} + |\psi|_{\infty}^q),
\]
and the proof is complete.

\[\square\]

5. Semilinear SPDEs

In this section, we will use the uniform estimates obtained in the previous section, to construct solutions for the following equation
\[
du_t = (L_t u_t + f_t(u_t)) dt + (M_t^{k} u_t + g_t^{k}) dw_t^{k}, u_0 = \psi \quad (5.16)
\]
for \( (t, x) \in [0, T] \times Q \).
Theorem 5.2. Suppose that Assumptions 2.2, 2.1 and 5.2 hold. Let \( f \) and there exists
\[
(5.1)
\]
\( h \) and for any \( r \), there exists a function \( h^N \in L_2 \) with \( \mathbb{E}\|h^N\|_\infty < \infty \), such that
\[
|f_t(x, r)| \leq |h_t^N(x)|,
\]
whenever \( |r| \leq N \).

Definition 5.1. Solution of equation (5.16) is an \( \mathcal{F}_t \)-adapted, strongly continuous process \((u_t)_{t \in [0,T]} \) with values in \( L_2(Q) \) such that

(i) \( u_t \in H^1_0 \), for \( dP \times dt \) almost every \((\omega, t) \in \Omega \times [0, T] \)

(ii) \( \int_0^T |u_t|^2 dt < \infty \) (a.s.)

(iii) almost surely, \( u \) is essentially bounded in \((t, x)\)

(iv) for all \( \phi \in C^\infty_c(Q) \) we have with probability one
\[
(u_t, \phi) = (\psi, \phi) + \int_0^t -(a s \partial_j u_s \partial_j \phi) + (b s \partial_i u_s + c s u_s, \phi) + (f_s(u_s), \phi)ds
\]
\[
+ \int_0^t (M_s^k u_s + g_s^k, \phi)dw_s^k,
\]
for all \( t \in [0, T] \).

Notice that by Assumption 5.1(iii), and (iii) from Definition 5.1 the term \( \int_0^t (f_s(u_s), \phi)ds \) is meaningful.

Theorem 5.1. Under Assumptions 2.1, 2.2, and 5.1, there exists a unique solution of equation (5.16).

Remark 5.1. From now on we can and we will assume that the function \( f \) is decreasing in \( r \) or else, by virtue of Assumption 5.1 we can replace \( f_t(x, r) \) by \( \tilde{f}_t(x, r) := f_t(x, r) - Kr \) and \( \tilde{c}_t(x) \) with \( \tilde{c}_t(x) := c_t(x) + K \).

We will need the following particular case from [1]. We consider two equations
\[
du^i_t = (L^i u^i_t + f^i_t(u^i_t))dt + (M^k u^i_t + g^k_t)dw^k_t, \ u^i_0 = \psi^i, \tag{5.17}
\]
for \( i = 1, 2 \).

Assumption 5.2. The functions \( f^i, i = 1, 2 \), are appropriately measurable, and there exists \( h \in L_2 \) and a constant \( C > 0 \), such that for any \( \omega, t, x \), and for any \( r \in \mathbb{R} \) we have
\[
|f^1_t(x, r)|^2 + |f^2_t(x, r)|^2 \leq C|r|^2 + |h_t(x)|^2.
\]

Theorem 5.2. Suppose that Assumptions 2.2, 2.1, and 5.1 hold. Let \( u^i, i = 1, 2 \) be the \( L_2 \)-solutions of the equations in (5.17), for \( i = 1, 2 \) respectively. Suppose that \( f^1 \leq f^2, \psi^1 \leq \psi^2 \) and assume that either \( f^1 \) or \( f^2 \) satisfy Assumption 5.1. Then, almost surely and for any \( t \in [0, T] \), \( u^1_t \leq u^2_t \) for almost every \( x \in Q \).
Proof of Theorem 5.1. We truncate the function $f$ by setting

$$
 f^{n,m}_t(x,r) = \begin{cases} 
 f_t(x,m) & \text{if } r > m \\
 f_t(x,r) & \text{if } -n \leq r \leq m \\
 f_t(x,-n) & \text{if } r < -n, 
\end{cases}
$$

for $n, m \in \mathbb{N}$ we consider the equation

$$
 du^{n,m}_t = (L_t u^{n,m}_t + f^{n,m}_t(u^{n,m}_t))dt + (M^k_t u^{n,m}_t + g^k_t)dw^k_t, 
$$

with $u^{0,m}_0 = \psi$.

We first fix $m \in \mathbb{N}$. One can easily check that the conditions from (5.19) are satisfied, and therefore equation (5.18) has a unique $L_2$--solution $(u^{n,m}_t)_{t \in [0,T]}$. We also have that for $n' \geq n$, $f^{n',m} \geq f^{n,m}$. By the comparison principle we get that almost surely, for all $t \in [0,T]$

$$
 u^{n',m}_t(x) \geq u^{n,m}_t(x), \text{ for almost every } x. \tag{5.19}
$$

We define now the stopping time

$$
 \tau^{R,m} := \inf\{ t \geq 0 : \int_Q (u^{1,m}_t + R)^2 dx > 0 \} \wedge T.
$$

We claim that for each $R \in \mathbb{N}$, there exists a set $\Omega_R$ of full probability, such that for each $\omega \in \Omega_R$, and for all $n \geq R$ we have that

$$
 u^{n,m}_t = u^{R,m}_t, \text{ for } t \in [0, \tau^{R,m}]. \tag{5.20}
$$

Notice that by (5.19) and the definition of $\tau^{R,m}$, for all $n \geq R$

$$
 f^{n,m}_t(x, u^{n,m}_t(x)) = f^{R,m}_t(x, u^{n,m}_t(x)), \text{ for } t \in [0, \tau^{R,m}].
$$

This means that for all $n \geq R$ the processes $u^{n,m}_t$ satisfies

$$
 dv_t = (L_t v_t + f^{R,m}_t(v_t))dt + (M^k_t v_t + g^k_t)dw^k_t, 
$$

with $v_0 = \psi$.

We use the notation $\tilde{\Omega}$ for the set $\Omega := \{ \omega : \tau^{R,m} = T \}$ for all $R$ large enough. On the set $\tilde{\Omega}$ we define $u^{\infty,m}_t := \lim_{n \to \infty} u^{n,m}_t$, where the limit is in the sense of $L_2(Q)$. Since for each $\omega \in \tilde{\Omega}$, we have $u^{\infty,m}_t = u^{n,m}_t$ for all $t \leq \tau^{R,m}$, and for any $n \geq R$, it follows that the process $(u^{\infty,m}_t)_{t \in [0,T]}$ is an adapted continuous $L_2(Q)$--valued process such that

(i) $u^{\infty,m}_t \in H^1_0$, for $dP \times dt$ almost every $(\omega, t) \in \Omega \times [0,T]$

(ii) $\int_0^T |u^{\infty,m}_t|_{L^2(Q)}^2 dt < \infty$ (a.s.)

(iii) $u^{\infty,m}_t$ is almost surely essentially bounded in $(t,x)$

(iv) for all $\phi \in C_c^\infty(Q)$ we have with probability one

$$
 (u^{\infty,m}_t, \phi) = \int_0^t (a^{ij}_s \partial_{ij} u^{\infty,m}_s, \phi) + (b^i_s \partial_i u^{\infty,m}_s + c_s u^{\infty,m}_s, \phi) + (f^m_s(u^{\infty,m}_s), \phi) ds
$$

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\[ + \int_0^t (\sigma^i_k \partial_i u^\infty_m + \nu_s^k u^\infty_m + g_s^k \phi)dw_s^k + (\psi, \phi), \]

for all \( t \in [0, T] \), where

\[ f^m_t(x, r) = \begin{cases} f_t(x, m) & \text{if } r > m, \\ f_t(x, r) & \text{if } r \leq m. \end{cases} \]

Now we will let \( m \to \infty \). Let us define the stopping time

\[ \tau^R := \inf \{ t \geq 0 : \int_Q (u^\infty_t - R)^2 dx > 0 \} \wedge T. \]

As before we claim that for any \( R > 0 \), there exists a set \( \Omega'_R \) of full probability, such that for any \( \omega \in \Omega'_R \) and any \( m, m' \geq R \),

\[ u^\infty_t = u^\infty_{t, m'} \quad \text{on } [0, \tau^R]. \quad (5.22) \]

To show this it suffices to show that for each \( R \in \mathbb{N} \), almost surely, for all \( m \geq R \), we have \( u^{n, m}_n = u^{n, R}_n \) on \([0, \tau^R] \) for all \( n \in \mathbb{N} \). To show this we set

\[ \tau^R_n := \inf \{ t \geq 0 : \int_Q (u^{n, 1}_t - R)^2 dx > 0 \} \wedge T. \]

For all \( m \geq R \) we have that the processes \( u^{n, m}_t \) satisfy the equation

\[ dv_t = (L_t v_t + f^{n, R}_t(v_t))dt + (M^k_t v_t + g^k_t)dw^k_t, \]

\[ v_0(x) = \psi(x), \quad (5.23) \]

for \( t \leq \tau^R_n \). It follows that almost surely, \( u^{n, m}_t = u^{n, R}_n \) for \( t \leq \tau^R_n \) for all \( n \).

We just note here that by the comparison principle again, we have \( \tau^R \leq \tau^R_n \) and this shows \( (5.22) \). Also for almost every \( \omega \in \Omega \), we have \( \tau^R = T \) for \( R \) large enough. Hence we can define \( u_t = \lim_{m \to \infty} u^\infty_{t, m} \), and then one can easily see that \( u_t \) has the desired properties.

For the uniqueness, let \( u^{(1)} \) and \( u^{(2)} \) be solutions of \( (5.16) \). Then one can define the stopping time

\[ \tau_N = \inf \{ t \geq 0 : \int_Q (|u^{(1)} - N|)^2 dx \vee \int_Q (|u^{(2)} - N|)^2 dx > 0 \}, \]

to see that for \( t \leq \tau_N \), the two solutions satisfy equation \( (5.18) \) with \( n = m = N \), and the claim follows, since \( \tau_N = T \) almost surely, for large enough \( N \).

\[ \square \]

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REFERENCES

[1] K. Dareiotis, I. Gyöngy, A comparison principle for stochastic integro-differential equations, arXiv:1210.5926

[2] L. Denis, A. Matoussi, L. Stoica, $L^p$ estimates for the uniform norm of solutions of quasilinear SPDE's, Probability Theory and Related Fields, 2005, Volume 133, Issue 4, pp 437-463

[3] E. Di Giorgi, Sulla differenziabilità e analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Math. Nat., 3, 1957, 2543.

[4] I. Gyöngy and E. Pardoux, Weak and strong solutions of white noise driven parabolic SPDEs. Preprint No. 22/92, Laboratoire de Mathématiques Marseille, Université de Provence.

[5] G. M. Lieberman, Second order parabolic differential equations. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[6] N. V. Krylov, An analytic approach to SPDEs, Stochastic Partial Differential Equations: Six Perspectives, AMS Mathematical surveys and Monographs, 64, 185-242.

[7] N. V. Krylov, A relatively short proof of Itô’s formula for SPDEs and its applications, arXiv:1208.3709

[8] Krylov, N. V.; Rozovski, B. L. Stochastic evolution equations. (Russian) Current problems in mathematics, Vol. 14 (Russian), pp. 71147, 256, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979.

[9] J. Moser, On Harnack's theorem for elliptic differential equations. Comm. Pure Appl. Math. 14 1961 577591.

[10] C. Prevôt, M. Röckner, A concise course on stochastic partial differential equations. Lecture Notes in Mathematics, 1905. Springer, Berlin, 2007. vi+144 pp.

[11] D. Revuz, M. Yor, Continuous martingales and Brownian motion. Third edition. Grundlehren der Mathematischen Wissenschaften, 293. Springer-Verlag, Berlin, 1999.

[12] B.L. Rozovski, Stochastic evolution systems. Linear theory and applications to nonlinear filtering. Mathematics and its Applications (Soviet Series), 35. Kluwer Academic Publishers Group, Dordrecht, 1990. xviii+315 pp

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