The support designs of the triply even codes of length 48

Tsuyoshi Miezaki * and Hiroyuki Nakasora†

Abstract

A triply even code is a binary linear code in which the weight of every codeword is divisible by 8. The triply even codes of length 48 have been classified by Betsumiya and Munemasa. Herein we study the support designs of triply even codes of length 48 and present the complete list of triply even codes of length 48 with the support 1-designs obtained from the Assmus–Mattson theorem. Moreover, we show that some of such codes have the support 2-designs. This is the first example of a code having the support $t$-designs for all weights obtained from the Assmus–Mattson theorem and has the support $t'$-designs for some weight with some $t' > t$.

Key Words and Phrases. support designs, triply even codes, harmonic weight enumerator.

2010 Mathematics Subject Classification. Primary 05B05; Secondary 94B05, 20B25.

1 Introduction

The present paper is a sequel to the previous papers [10, 11].
Let $\mathbb{F}_q$ be the finite field of $q$ elements. A binary linear code $C$ of length $n$ is a subspace of $\mathbb{F}_2^n$. For $x := (x_1, \ldots, x_n) \in C$, the weight of $x$ is defined as follows: $\text{wt}(x) := \sharp\{i \mid x_i \neq 0\}$. The minimum distance of a code $C$ is

$$\min\{\text{wt}(x) \mid x \in C, x \neq 0\}.$$ 

A linear code of length $n$, dimension $k$, and minimum distance $d$ is called an $[n, k, d]$ code (or $[n, k]$ code for short).

A $t$-$(v, k, \lambda)$ design (or $t$-design for short) is a pair $D = (X, \mathcal{B})$, where $X$ is a set of points of cardinality $v$, and $\mathcal{B}$ a collection of $k$-element subsets of $X$ called blocks, with the property that any $t$ points are contained in precisely $\lambda$ blocks.

The support of a nonzero vector $x := (x_1, \ldots, x_n), x_i \in \mathbb{F}_q = \{0, 1, \ldots, q-1\}$ is the set of indices of its nonzero coordinates: $\text{supp}(x) = \{i \mid x_i \neq 0\}$. The support design of a code of length $n$ for a given nonzero weight $w$ is the design with points $n$ of coordinate indices, and blocks the supports of all codewords of weight $w$.

A triply even code is a binary linear code in which the weight of every codeword is divisible by 8; they have previously been classified by Betsumiya and Munemasa [5, 4]. Herein we study the support designs of triply even codes of length 48 and present the complete list of triply even codes of length 48 which have the support 1-designs obtained from the Assmus–Mattson theorem. It is interesting to note that some of such codes have the support 2-designs. This is the first example of code having the support $t$-designs for all weights obtained from the Assmus–Mattson theorem and has the support $t'$-designs for some weights with some $t' > t$.

This paper is organized as follows. In Section 2, we review the concept of harmonic weight enumerators which are used in the proof of the main results. In Section 3, we present the main result, namely, we study the support designs of the triply even codes of length 48. Finally, in Section 4, we give some remarks.

## 2 The harmonic weight enumerators

In this section, we review the concept of the harmonic weight enumerators.

Let $C$ be a code of length $n$. The weight distribution of a code $C$ is the sequence $\{A_i \mid i = 0, 1, \ldots, n\}$, where $A_i$ is the number of codewords of
weight $i$. The polynomial

$$W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$$

is called the weight enumerator of $C$. The weight enumerator of a code $C$ and its dual $C^\perp$ are related. The following theorem, due to MacWilliams, is called the MacWilliams identity:

**Theorem 2.1** ([6]). Let $W_C(x, y)$ be the weight enumerator of an $[n, k]$ code $C$ over $\mathbb{F}_q$ and let $W_{C^\perp}(x, y)$ be the weight enumerator of the dual code $C^\perp$. Then

$$W_{C^\perp}(x, y) = q^{-k} W_C(x + (q - 1)y, x - y).$$

A striking generalization of the MacWilliams identity was obtained by Bachoc [2], who gave the concept of harmonic weight enumerators and a generalization of the MacWilliams identity. The harmonic weight enumerators have many applications; particularly, the relations between coding theory and design theory are reinterpreted and progressed by the harmonic weight enumerators [2, 3]. For the reader’s convenience, we quote the definitions and properties of discrete harmonic functions from [2, 9].

Let $\Omega = \{1, 2, \ldots, n\}$ be a finite set (which will be the set of coordinates of the code) and let $X$ be the set of its subsets, while, for all $k = 0, 1, \ldots, n$, $X_k$ is the set of its $k$-subsets. We denote by $\mathbb{R}X$, $\mathbb{R}X_k$ the free real vector spaces spanned by respectively the elements of $X$, $X_k$. An element of $\mathbb{R}X_k$ is denoted by

$$f = \sum_{z \in X_k} f(z) z$$

and is identified with the real-valued function on $X_k$ given by $z \mapsto f(z)$.

Such an element $f \in \mathbb{R}X_k$ can be extended to an element $\tilde{f} \in \mathbb{R}X$ by setting, for all $u \in X$,

$$\tilde{f}(u) = \sum_{z \in X_k, z \subseteq u} f(z).$$

If an element $g \in \mathbb{R}X$ is equal to some $\tilde{f}$, for $f \in \mathbb{R}X_k$, we say that $g$ has degree $k$. The differentiation $\gamma$ is the operator defined by the linear form

$$\gamma(z) = \sum_{y \in X_{k-1}, y \subseteq z} y$$
for all $z \in X_k$ and for all $k = 0, 1, \ldots, n$, and $\text{Harm}_k$ is the kernel of $\gamma$:

$$\text{Harm}_k = \ker(\gamma|_{\mathbb{R}X_k}).$$

**Theorem 2.2** ([9]). A set $B \subset X_k$ of blocks is a $t$-design if and only if $
\sum_{b \in B} \tilde{f}(b) = 0$ for all $f \in \text{Harm}_k$, $1 \leq k \leq t$.

In [2], the harmonic weight enumerator associated with a binary linear code $C$ was defined as follows:

**Definition 2.3.** Let $C$ be a binary code of length $n$ and let $f \in \text{Harm}_k$. The harmonic weight enumerator associated with $C$ and $f$ is

$$W_{C,f}(x, y) = \sum_{c \in C} \tilde{f}(c)x^{n - \text{wt}(c)}y^{\text{wt}(c)}.$$  

Bachoc proved the following MacWilliams-type equality:

**Theorem 2.4** ([2]). Let $W_{C,f}(x, y)$ be the harmonic weight enumerator associated with the code $C$ and the harmonic function $f$ of degree $k$. Then

$$W_{C,f}(x, y) = (xy)^k Z_{C,f}(x, y)$$

where $Z_{C,f}$ is a homogeneous polynomial of degree $n - 2k$, and satisfies

$$Z_{C,f}(x, y) = (-1)^k \frac{2^{n/2}}{|C|} Z_{C,f} \left( \frac{x + y}{\sqrt{2}}, \frac{x - y}{\sqrt{2}} \right).$$

### 3 The support designs of triply even codes of length 48

In this section, we study the designs of triple even codes of length 48.

The following theorem is due to Assmus and Mattson [1]. It is one of the most important theorems in coding theory and design theory:

**Theorem 3.1** ([1]). Let $C$ be an $[n, k, d]$ linear code over $\mathbb{F}_q$ and $C^\perp$ be the dual $[n, n - k, d^\perp]$ code. Denote by $n_0$ the largest integer $\leq n$ such that $n_0 - \frac{n_0 + q - 2}{q-1} < d$, and define $n_0^\perp$ similarly for the dual code $C^\perp$. Suppose that for some integer $t$, $0 < t < d$, there are at most $d - t$ non-zero weights $w$ in $C^\perp$ such that $w \leq n - t$. Then:
the support design for any weight \( u, \ d \leq u \leq n_0 \) in \( C \) is a \( t \)-design;

(2) the support design for any weight \( w, \ d^\perp \leq w \leq \min\{n - t, n_0^\perp\} \) in \( C^\perp \) is a \( t \)-design.

If a \( t \)-design \( (t > 0) \) is obtained from some linear code by this theorem, the code is said to be applicable to the Assmus–Mattson theorem.

In \[4, 5\], triply even codes of length 48 are classified where each code is described by the form \( \langle \text{Dimension, Code Id, [ Generators ]} \rangle \). In this paper, we use the form \( \langle \text{Dimension, [Code Id]} \rangle \).

**Proposition 3.2.** If a triply even code length 48 is applicable to the Assmus–Mattson theorem, then one of the following:

\((A)\) \( \langle 7, [144] \rangle, \langle 8, [129, 130, 131, 132, 133] \rangle, \langle 9, [59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 1109, 1712, 1714, 1716, 1960] \rangle, \langle 10, [16, 17, 18, 19, 20, 21, 22, 549, 550, 554, 1001, 1245, 1246, 1247] \rangle, \langle 11, [6, 7, 154, 520] \rangle, \langle 12, [3] \rangle, \langle 13, [1] \rangle \).

\((B)\) \( \langle 2, [1] \rangle, \langle 3, [4] \rangle, \langle 4, [7] \rangle, \langle 5, [12] \rangle \)

**Proof.** Let \( C \) be a triply even code length 48 in \((A)\). Then \( C \) have weight 0,16,24,32,48 and the dual code of \( C \) have the minimum weight 4. Since there are 4 – \( t = 3 \) non-zero weights \( w \) with \( w \leq 48 - t \) in \( C \), we can take \( t = 1 \).

In the case \((B)\), these codes have weight 0, 24, 48 and the dual codes have the minimum weight 2. Since there are 2 – \( t = 1 \) non-zero weights \( w \) with \( w \leq 48 - t \), we can take \( t = 1 \).

\( \square \)

The following Lemma is easily seen:

**Lemma 3.3** ([Page 3, Proposition 1.4]). (1) Let \( B_1 \) be the block set of a 2-(48, \( k, \lambda_2 \)) design. Then \( |B_1| \) is divisible by 47.

(2) Let \( B_2 \) be the block set of a 3-(48, 6, \( \lambda_3 \)) design. Then \( |B_2| \) is divisible by 47 \( \cdot \) 23.

We now present the main result:

**Theorem 3.4.** Let \( C \) be a triply even code length 48 in Proposition 3.2. Let \( D_w \) and \( D_w^\perp \) be the support \( t \)-design of weight \( w \) of \( C \) and \( C^\perp \).
(1) For all $w$, $D_w$ and $D_w^\perp$ are 1-designs.

(2) If $C$ is a code in Proposition 3.2 (A) except for $\langle 13, [1] \rangle$, $D_6^\perp$ is a 2-design but is not a 3-design. For the other cases, $D_w$ and $D_w^\perp$ are not 2-designs.

**Proof.** By Proposition 3.2, we have (1).

Next, we show that if $C$ is a code according to Proposition 3.2 (A) except for $\langle 13, [1] \rangle$, $D_6^\perp$ is a 2-design but is not a 3-design. Let $W_{C,f}(x, y)$ be the harmonic weight enumerator associated with the code $C$ in Proposition 3.2 (A) and the harmonic function $f$ of degree 2. Then we have

$$W_{C,f}(x, y) = \sum_{c \in C} \tilde{f}(c)x^{48 - wt(c)}y^{wt(c)}$$

$$= ax^{32}y^{16} + bx^{24}y^{24} + ax^{16}y^{32}$$

$$= (xy)^2(ax^{30}y^{14} + bx^{22}y^{22} + ax^{14}y^{30})$$

$$= (xy)^2Z_{C,f}(x, y),$$

where $a, b$ and $c$ are not equal to 0.

By Theorem 2.4, there exists coefficients $a', b'$ such that

$$Z_{C^\perp,f}(x, y) = (-1)^2\frac{2^{24}}{|C|}Z_{C,f}\left(\frac{x + y}{\sqrt{2}}, \frac{x - y}{\sqrt{2}}\right)$$

$$= a'(x + y)^{30}(x - y)^{14} + b'(x + y)^{22}(x - y)^{22} + a'(x + y)^{14}(x - y)^{30}$$

Since $C^\perp$ has minimum weight 4, the coefficient of $x^{44}$ in $Z_{C^\perp,f}$ is equal to 0. Then we have $b' = -2a'$. Hence we have

$$W_{C^\perp,f}(x, y)$$

$$= (xy)^2\left(a'(x + y)^{30}(x - y)^{14} - 2a'(x + y)^{22}(x - y)^{22} + a'(x + y)^{14}(x - y)^{30}\right).$$

By a direct computation, the coefficient of $x^{42}y^6$ in $W_{C^\perp,f}$ is equal to 0. Hence $D_6^\perp$ is a 2-design. We have checked numerically that the number of blocks of $D_6^\perp$ is not divisible by $47 \cdot 23$. Therefore, $D_6^\perp$ is not a 3-design by Lemma 3.3 (2).

In the case $\langle 13, [1] \rangle$, this code has the weight enumerator

$$x^{48} + 759x^{32}y^{16} + 6672x^{24}y^{24} + 759x^{16}y^{32} + y^{48}.$$
By Theorem 2.1, we have \( A^\perp_6 = 0 \), where \( A^\perp_6 \) is the number of weight 6 of the dual code. Hence there are no blocks of \( D^\perp_6 \).

Next, we show that for the other cases, \( D_w \) and \( D^\perp_w \) are not 2-designs. We have checked numerically that the number of the blocks of \( D_w \) and \( D^\perp_w \) except for \( D^\perp_6 \) is not divisible by 47. Therefore, \( D_w \) and \( D^\perp_w \) except for \( D^\perp_6 \) are not 2-designs by Lemma 3.3 (1).

Let \( \hat{C} \) be a triply even code of length 48 in Proposition 3.2 (B). If \( \hat{D}_w \) and \( \hat{D}^\perp_w \) is the support design for all weight \( w \) of \( \hat{C} \) and \( \hat{C}^\perp \), we have checked numerically that the number of the blocks of \( \hat{D}_w \) and \( \hat{D}^\perp_w \) is not divisible by 47. Therefore, \( \hat{D}_w \) and \( \hat{D}^\perp_w \) are not 2-designs by Lemma 3.3 (1). The numbers of the blocks are listed in one of the author’s homepage [12]. This completes the proof of Theorem 3.4.

Remark 3.5. Three codes \( \langle 1, [1] \rangle \), \( \langle 3, [5] \rangle \) and \( \langle 6, [363] \rangle \) are not applicable to the Assmus–Mattson theorem, but their support designs for all weight are 1-designs since their codes are generated by the minimum weights which divide 48 coordinates into equal parts. Their codes have a transitive automorphism group.

4 2-designs from triple even codes of length 48

We list 2-designs obtained from Theorem 3.4 in Table 1. In this section, we give the concluding remarks related to 2-designs of triple even codes of length 48 discussed in Section 3.

Remark 4.1. It is interesting to note that the dual code of the first triply even code \( \langle 7, [144] \rangle \) is called Miyamoto’s moonshine code [13]. This triply even code has the weight enumerator

\[
x^{48} + 3x^{32}y^{16} + 120x^{24}y^{24} + 3x^{16}y^{32} + y^{48}.
\]

Using Theorem 2.1, we obtained the weight enumerator of the dual code. Then we have \( A^\perp_6 = 189504 \), where \( A^\perp_6 \) is the number of weight 6 of the dual code. By Theorem 3.4, \( D^\perp_6 \) is a 2-(48, 6, 2520) design.

In the case dimension of 8, there are five triply even codes [129,130,131,132,133]. We have checked by Magma [7] that their codes give the five non isomorphic
2-(48, 6, 1240) designs. Similarly, each triply even code in dimension 9–12 gives a different 2-design.

In the case of triply even code \( \langle 13, [1] \rangle \), there are no code words of weight 6 of the dual code.

**Remark 4.2.** For the 2-design \( D_{6}^{\perp} \) obtained from Theorem 3.4 in Table 1, we calculated the automorphism groups of the 2-designs. The data of the automorphism groups are listed in one of the author’s homepage [12].

**Remark 4.3.** We have checked by Magma [7] that for the 2-design \( D_{6}^{\perp} \) obtained from Theorem 3.4 in Table 1 the codewords of weight 6 generate the code \( C^{\perp} \). This gives rise to a natural problem:

**Problem 4.4.** Let \( C \) be a linear code. Characterize the weight \( w \) such that the codewords of weight \( w \) generate the code \( C \). Moreover, do we characterize such weight \( w \) from the point of view of design theory?

**Remark 4.5.** One of the main result of the present paper is to give the first example that a code has the support \( t \)-designs for all weights obtained from the Assmus–Mattson theorem and has the support \( t' \)-designs for some of the weights with some \( t' > t \). We conclude the present paper to with the following problem:

**Problem 4.6.** For \( t \geq 2 \), find an example that a code has the support \( t \)-designs for all weight obtained from the Assmus–Mattson theorem and has the support \( t' \)-designs for some weight with some \( t' > t \).
Table 1: support 2-designs of weight 6

| Dimension | Code Id | Weight distribution \((i, A_i)\) for \(A_i \neq 0\) | \(2-(v, k, \lambda)\) Numbers |
|-----------|--------|---------------------------------|-------------------|
| 7         | [144](0, 1), (16, 3), (24, 120), (32, 3), (48, 1) | 2-(48, 6, 2520)   |
| 8         | [129,130,131,132,133](0, 1), (16, 15), (24, 224), (32, 15), (48, 1) | 2-(48, 6, 1240)   |
| 9         | [59,60,61,62,63,64,65,66,67,68,69,1109,1712,1714,1716,1960](0, 1), (16, 39), (24, 432), (32, 39), (48, 1) | 2-(48, 6, 600) |
| 10        | [16,17,18,19,20,21,22,54,55,56,57,1001,1245,1246,1247](0, 1), (16, 87), (24, 848), (32, 87), (48, 1) | 2-(48, 6, 280) |
| 11        | [6,7,154,520](0, 1), (16, 183), (24, 1680), (32, 183), (48, 1) | 2-(48, 6, 120) |
| 12        | [3](0, 1), (16, 375), (24, 3344), (32, 375), (48, 1) | 2-(48, 6, 40) |
| 13        | [1](0, 1), (16, 759), (24, 6672), (32, 759), (48, 1) | - |

Acknowledgments

The authors thank Koichi Betsumiya and Akihiro Munemasa for his helpful discussions and computations on this research. The first author is supported by JSPS KAKENHI (15K04775, 17K05164, 18K03217).

References

[1] E. F. Assmus, Jr. and H. F. Mattson, Jr., New 5-designs, *J. Combinatorial Theory* 6 (1969), 122-151.

[2] C. Bachoc, On harmonic weight enumerators of binary codes, *Des. Codes Cryptogr.* 18 (1999), no. 1-3, 11-28.

[3] E. Bannai, M. Koike, M. Shinohara and M. Tagami, Spherical designs attached to extremal lattices and the modulo p property of Fourier
coefficients of extremal modular forms, *Mosc. Math. J.* 6 (2006), 225-264.

[4] K. Betsumiya, DATABASE: Triply even codes of length 48, [http://www.st.hirosaki-u.ac.jp/~betsumi/triply-even/](http://www.st.hirosaki-u.ac.jp/~betsumi/triply-even/)

[5] K. Betsumiya and A. Munemasa, On triply even binary codes, *J. London Math. Soc.* 86 (1) (2012), 1-16.

[6] J. Macwilliams, A theorem on the distribution of weights in a systematic code, *Bell System Tech. J.* 42 (1963), 79–84.

[7] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* 24 (1997), 235–265.

[8] P.J. Cameron, J.H. van Lint, *Designs, graphs, codes and their links*, London Mathematical Society Student Texts, 22. Cambridge University Press, Cambridge, 1991.

[9] P. Delsarte, Hahn polynomials, discrete harmonics, and $t$-designs, *SIAM J. Appl. Math.* 34 (1978), no. 1, 157-166.

[10] N. Horiguchi, T. Miezaki and H. Nakasora, On the support designs of extremal binary doubly even self-dual codes, *Des. Codes Cryptogr.*, 72 (2014), 529-537.

[11] T. Miezaki and H. Nakasora, An upper bound of the value of $t$ of the support $t$-designs of extremal binary doubly even self-dual codes, *Des. Codes Cryptogr.*, 79 (2016), 37-46.

[12] T. Miezaki, Tsuyoshi Miezaki’s website: https://sites.google.com/site/tmiezaki/data

[13] M. Miyamoto, A new construction of the Moonshine vertex operator algebras over the real number field, *Ann. of Math.*, 159 (2004), 535–596.