APPROXIMATION OF THE SECOND EIGENVALUE
OF THE $p$-LAPLACE OPERATOR IN SYMMETRIC
DOMAINS

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ABSTRACT. A new idea to approximate the second eigenfunction
and the second eigenvalue of $p$-Laplace operator is given. In the
case of Dirichlet boundary condition, the scheme has the restriction
that the positive and the negative part of the second eigenfunction
have equal $L^p$-norm, however in the case of Neumann boundary
condition, our algorithm has not such restriction. Our algorithm
generates a descending sequence of positive numbers that converges
to the second eigenvalue. We give various examples and computa-
tional tests.

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1. Introduction

The $p$-Laplace operator is a homogeneous nonlinear operator which
arises frequently in various applications in physics, mechanics, and im-
age processing. Derivation of the $p$-Laplace operator from a nonlinear
Darcy law and the continuity equation has been described in [5]. The
eigenvalues and the corresponding eigenfunctions of the $p$-Laplace op-
erator have been much discussed in the literature, due to mathemat-
ical challenges, open questions cf. [3, 14, 24, 25], and regarding related
applications in image processing we refer to [15, 17]. In particular,
the second eigenvalue and eigenfunction have been studied extensivley,
see [1, 12, 27].

For dimension higher than one, except for the case $p = 2$, the struc-
ture of higher eigenfunctions are not well understood. For the one-
dimensional case see [13, 16, 26]. In this work, we are interested in
numerical approximation of the second eigenvalue $\lambda_2$ and the corre-
sponding eigenfunction of the $p$-Laplace operator. We treat the second
eigenvalue as an optimal bi-partition of the first eigenvalue. There are
various problems in mathematical physics and probability theory re-
lated to optimal partitioning of the first eigenvalue, see [4, 6, 7]. In [18]
investigated Constrained Descent Method and the Constrained Mountain Pass Algorithm to approximate the two smallest eigenvalue for $1.1 \leq p \leq 10$.

The structure of this paper is as follows. In Section 2, we review mathematical background and characterization of the second eigenvalue. Next in Section 3, we present our numerical approximation along the proof of convergence. In Section 4, as an application of our algorithm, we study the spectral clustering. Last section deals with the numerical implementation.

2. Mathematical Background

In this section, we briefly review some known results about the first and second eigenfunction of $p$-Laplace operator with zero Dirichlet boundary condition in bounded domain.

For $1 \leq p < \infty$ the first eigenvalue of the $p$-Laplace operator in $W^{1,p}_0(\Omega)$, denoted by $\lambda_{1,p}(\Omega)$ is given by

$$\lambda_{1,p}(\Omega) := \min_{\substack{u \in W^{1,p}_0(\Omega) \setminus \{0\}}} \frac{\int_\Omega |\nabla u(x)|^p dx}{\int_\Omega |u(x)|^p dx}.$$  \hfill (2.1)

For every $1 < p < \infty$, the first eigenvalue is simple and isolated and the first eigenfunction doesn’t change the sign. The corresponding minimizer satisfies the Euler-Lagrange equation

$$\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases} \hfill (2.2)$$

Here $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ which for $p = 2$, we have Laplace operator.

**Definition 2.1.** A non zero function $u \in W^{1,p}_0(\Omega) \cap C(\overline{\Omega})$, is called a $p$-eigenfunction in the weak sense if there exist a $\lambda \in \mathbb{R}$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx = \lambda \int_\Omega |u|^{p-2} u \phi dx, \quad \forall \phi \in W^{1,p}_0(\Omega).$$ \hfill (2.3)

The associated number $\lambda$ is called a $p$-eigenvalue. It is not, however, known whether every such quantity is a "variational eigenvalue" like for the case $p = 2$. In [1] Anane and Tsouli gave a characterization of the variational eigenvalues of Problem (2.2) by the following minimax principle. To do this, first define Krasnoselskii genus of a set $A \subseteq W^{1,p}_0(\Omega)$ by

$$\gamma(A) = \min\{k \in \mathbb{N} : \text{there exist } f : A \to \mathbb{R}^k \setminus 0, f \text{ continuous and odd}\}.$$ 

For $k \in \mathbb{N}$ define

$$\Gamma_k := \{A \subseteq W^{1,p}_0(\Omega), \text{symmetric, compact and } \gamma(A) \geq k\}.$$
Then the eigenvalues of the \( p \)-Laplace are
\[
\lambda_{k,p}(\Omega) = \min_{A \in \Gamma} \sup_{u \in A} \frac{\int_{\Omega} |\nabla u(x)|^p \, dx}{\int_{\Omega} |u(x)|^p \, dx},
\]
which satisfying
\[
0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \to \infty,
\]
as \( k \) tends to infinity, see [1, 22, 24, 25].

It is shown by Anane and Tsouli that \( \lambda_2 \) defined by (2.4) is essentially the second eigenvalue of the Dirichlet \( p \)-Laplace, means that the eigenvalue problem (2.2) has no other eigenvalue between \( \lambda_1 \) and \( \lambda_2 \).

In the one dimensional case, \( \Omega = (a,b) \subset \mathbb{R} \), it is known that all eigenvalues are simple and the eigenfunction corresponding to \( \lambda_n \) has exactly \( n + 1 \) zeros, counting the ending boundary points \( a, b \). The eigenvalues can be computed explicitly by variational formula and the corresponding eigenfunctions are obtained in terms of the Gaussian hypergeometric function, see [16, 26]).

Here, we consider another characterization of the second eigenvalue which we use for our numerical simulation.

**Definition 2.2.** Given a bounded open set \( \Omega \subset \mathbb{R}^d \), a class bi-partition of \( \Omega \) (or decomposition) is a family of pairwise disjoint, open and connected subsets \( \{\Omega_1, \Omega_2\} \) such that
\[
\Omega_1, \Omega_2 \subseteq \Omega, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \Omega = \Omega_1 \cup \Omega_2.
\]
By \( \mathcal{D}_2 \) we mean the set of all bi-partition of \( \Omega \).

For any arbitrary partition \( \mathcal{D} = (\Omega_1, \Omega_2) \in \mathcal{D}_2 \), we define
\[
\Lambda_2(\mathcal{D}) = \max (\lambda_1(\Omega_1), \lambda_1(\Omega_2)).
\]
Also let \( \mathcal{L}_2(\Omega) \) denote the infimum of \( \Lambda_2(\mathcal{D}) \), over all the bi-partition i.e.,
\[
(2.5) \quad \mathcal{L}_2(\Omega) = \inf_{\mathcal{D} \in \mathcal{D}_2} \Lambda_2(\mathcal{D}).
\]
An optimal bi-partition is a partition which realizes the infimum in (2.5). For \( p = 2 \) the optimal partition of the first eigenvalue has been studied extensively, see [6, 11].

We know that the second eigenfunction changes its sign on the domain, i.e., the second eigenfunction can be written as \( u = u_+ - u_- \), where
\[
u_+ = \max(u, 0), \quad u_- = \max(-u, 0).
\]
Obviously
\[
u_+, u_- \geq 0, \quad u_+ \cdot u_- = 0 \quad \text{in} \quad \Omega.
\]
Nodal domains of \( u \) denoted by \( \Omega_+ \) and \( \Omega_- \) are defined as the support of positive and negative part of \( u \)
\[
\Omega_+ = \{x \in \Omega : u(x) > 0\}, \quad \Omega_- = \{x \in \Omega : u(x) < 0\}.
\]
The following Lemma in [12] shows existence for minimal two partitions and implies that
\[ \lambda_2(\Omega) = \mathcal{L}_2(\Omega). \]

**Lemma 2.1.** There exists \( u \in W^{1,p}_0(\Omega) \) such that \( \{u_+ > 0\}, \{u_- > 0\} \) achieves infimum in (2.5). Furthermore,
\[ \lambda_1(\{u_+ > 0\}) = \lambda_1(\{u_- > 0\}). \]

It is also known that any eigenfunction associated to an eigenvalue different from \( \lambda_1 \) changes sign. The following properties for second eigenvalue hold:

- If \( \Omega_1 \subseteq \Omega_2 \subseteq \Omega \), then
  \[ \lambda_2(p, \Omega_2) \leq \lambda_2(p, \Omega_1). \]
- Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \), then the eigenfunction associated to \( \lambda_2(p, \Omega) \) admits exactly two nodal domains.
- The Courant theorem implies that in the linear case \( p = 2 \), the number of nodal domains of an eigenfunction associated to \( \lambda_2 \) is exactly 2.
- In [2, 3] is shown that the second eigenfunctions are not radial in ball.

The limiting cases \( p \to 1 \) and \( p \to \infty \) are more complicated and requires tools from non smooth critical point theory and the concept viscosity solutions. Note for the limiting case \( p \) tends to one, there are several ways to define the second eigenfunction of the 1-Laplace operator which it does not satisfy many of the properties of the second eigenfunction of the p-Laplace operator in general [27]. As \( p \) tends to one, in some cases the second eigenfunction takes the form
\[ u_2 = c_1 \chi_{C_1} - c_2 \chi_{C_2}. \]

Here \( \chi_A \) is the characteristic function of given set \( A \), the pair \( (C_1, C_2) \) is so called Cheeger-2-cluster of \( \Omega \), while in other cases functions of that type can’t be eigenfunctions at all.

In the limiting case as \( p \) tends to infinity, the second eigenvalue has a geometric characterization. Following Section 4 of [23] define
\[ \Lambda_2 = \frac{1}{r_2}, \]
where
\[ r_2 = \sup\{r \in \mathbb{R}^+ : \text{there are disjoint balls } B_1, B_2 \subset \Omega \text{ with radius } r\}. \]

Then the following lemma from [23] states that the second eigenvalue of infinity Laplace is \( \Lambda_2 \).
Lemma 2.2. Let $\lambda_2(p)$ be the second $p$-eigenvalue in $\Omega$. Then it holds that

$$\lim_{p \to \infty} \lambda_2(p)^{\frac{1}{p}} \to \Lambda_2.$$ 

$\Lambda_2 \in \mathbb{R}$ is the second eigenvalue of the infinity Laplace.

Furthermore, it is shown in [23] that the second eigenfunction of the infinity Laplace operator can be obtained as a viscosity solution of the following equation

$$F_{\Lambda}(x, u, \nabla u, D^2 u) = 0 \quad \text{for } x \in \Omega,$$

for which $F_{\Lambda}$ is given by

$$F_{\Lambda}(x, u, \nabla u, D^2 u) = \begin{cases} 
\min \{|\nabla u| - \Lambda u, -\Delta_{\infty} u\} & u(x) > 0, \\
-\Delta_{\infty} u & u(x) = 0, \\
\max \{-|\nabla u| - \Lambda u, -\Delta_{\infty} u\} & u(x) < 0.
\end{cases}$$

Here $\Lambda = \Lambda_2 \in \mathbb{R}$ denotes the second eigenvalue of the infinity Laplace given by (2.6).

3. An iterative scheme

In this section we discuss our algorithm to approximate the second eigenvalue $\lambda_2$ and corresponding eigenfunction denoted by $u$. For Dirichlet boundary condition our main assumption is that domain $\Omega$ has the following symmetric property

$$\|u_+\|_{L^p(\Omega)} = \|u_-\|_{L^p(\Omega)}.$$ 

In [10] the authors studied gradient flows of $p$-homogeneous functionals on a Hilbert space and proved that after suitable rescaling the flow always converges to a nonlinear eigenfunction of the associated subdifferential operator. They also gave conditions for convergence to the first eigenfunction.

The inverse power method is known to be an efficient method to approximate the first eigenvalue of given operator see [8, 9, 19]. Our aim here is to extend this method combining with Lemma 2.1 to approximate the second eigenvalue.

Following Lemma 2.1 and notation introduced before this Lema, we know that restriction of the second eigenfunction on each nodal domain is the first eigenfunction, i.e.,

$$\lambda_2(\Omega) = \lambda_1(\Omega_+) = \lambda_1(\Omega_-).$$

So the second eigenvalue problem can be written as

$$(3.1) \quad \begin{cases} 
-\Delta_p(u_+ - u_-) = \lambda_1(\Omega_+)u_+^{p-1} - \lambda_1(\Omega_-)u_-^{p-1} & \text{in } \Omega, \\
u_+ = u_- = 0 & \text{on } \partial \Omega.
\end{cases}$$

The main steps are as follows:
(1) Choose arbitrary initial bi-partition \( \Omega^0_+ \) and \( \Omega^0_- \). Since \( \Omega^0_+ \) and \( \Omega^0_- \) are disjoint and connected components of \( \Omega \) this prevent of obtaining higher eigenfunctions than second one.

(2) Given \( \Omega^k_+ \) and \( \Omega^k_- \), \( k = 0, 1, \cdots \), obtain the first eigenvalues denoted by \( \lambda^k_1(\Omega_+) \), \( \lambda^k_1(\Omega_-) \) and first eigenfunctions \( (u^k_+, u^-) \) normalized in \( L^p \) related to \( (\Omega^k_+, \Omega^k_-) \).

(3) Solve the following boundary value problem

\[
\begin{aligned}
-\Delta_p u &= \lambda^k_1(\Omega_+)(u^k_+)^{p-1} - \lambda_1(\Omega_-)(u^-)^{p-1} \quad &\text{in } \Omega, \\
\quad u &= 0 \quad &\text{on } \partial \Omega.
\end{aligned}
\]

(4) Update \( (\Omega_+, \Omega_-) \) as supports of positive part and negative part of the solution \( u \).

(5) Go to step (2).

Note that in step 2, given \( \Omega^k_+ \) and \( \Omega^k_- \), one needs to calculate \( \lambda_1(\Omega^k_+) \) and \( \lambda_1(\Omega^k_-) \) and corresponding first eigenvalues for each sub-domain. This step can be modified by implementing the inverse power method along our Algorithm. Assume \( u^k_+ \) and \( u^- \) are given normalized in \( L^p \) with disjoint supports, then define values \( \lambda^k_+ \) and \( \lambda^k_- \) by

\[
\begin{aligned}
\lambda^k_+ = \lambda^k_1(\Omega^k_+) &= \int_{\Omega^k_+} |\nabla u^k_+(x)|^p \, dx, \\
\lambda^k_- = \lambda^k_1(\Omega^k_-) &= \int_{\Omega^k_-} |\nabla u^-_+(x)|^p \, dx.
\end{aligned}
\]

The algorithm to approximate the second eigenvalue and the second eigenfunction are as follows:

\textbf{Algorithm 1: Second eigenvalue algorithm}

inputs : \( u^0 = u^0_+ - u^0_- , \epsilon \).

output : Approximation of second eigenvalue and second eigenfunction.

(1) Set \( k = 0 \), choose initial arbitrary guesses \( u^0_+ > 0, u^0_- > 0 \) having disjoint supports, normalized in \( L^p(\Omega) \) and vanishing on the boundary with \( \Omega_+ \) and \( \Omega_- \) as the supports of functions \( u^0_+ \) and \( u^0_- \) respectively.

(2) Given \( u^k_+ = u^k_+ - u^k_- \) where \( u^k_+ \) and \( u^k_- \) are normalized in \( L^p \), with disjoint supports, then obtain \( \lambda^k_+ \) and \( \lambda^k_- \) by (3.2).

(3) Solve

\[
\begin{aligned}
-\Delta_p u &= |u^k|^p-2 \left( \lambda^k_+ u^k_+ - \lambda^k_- u^k_- \right) \quad &\text{in } \Omega, \\
\quad u &= 0 \quad &\text{on } \partial \Omega.
\end{aligned}
\]

(4) Set \( u^{k+1}_+ \) and \( u^{k+1}_- \) as positive and negative part of the solution of (3.3).

(5) If \( |\lambda^{k+1}_1(\Omega_+)| - |\lambda^k(\Omega_+)| \geq \epsilon, \) \( |\lambda^{k+1}_1(\Omega_-)| - |\lambda^k(\Omega_-)| \geq \epsilon \) then

(6) Set \( k = k + 1 \) and go step (2).

\textbf{Remark 1.} Note that \( \Omega_+ \) and \( \Omega_- \) change in iterations but for simplicity we write \( \Omega_+ \) and \( \Omega_- \) instead of \( \Omega^k_+ \) and \( \Omega^k_- \).
Remark 2. Consider the case $p = 2$. Let denote the first eigenfunction by $w_1$. The second eigenvalue is given by

$$\lambda_2(\Omega) = \inf_{u \perp w_1} \frac{\int_{\Omega} |\nabla u(x)|^2 \, dx}{\int_{\Omega} |u(x)|^2 \, dx}.$$  

Assume the initial guess $u_0$ be chosen such that $(\lambda_+^0 u_+^0 - \lambda_-^0 u_-^0) \perp w_1$ then the Algorithm generates sequence $\{u_n\}$ which are orthogonal to $w_1$.

Multiply the equation

$$-\Delta u = \lambda_0^+ (\Omega_+) u_+^0 - \lambda_0^- (\Omega_-) u_-^0$$

by $w_1$ and integrating by parts two times on left side, implies

$$\int_{\Omega} u(\Delta w_1) \, dx = \int_{\Omega} (\lambda_+^0 u_+^0 - \lambda_-^0 u_-^0) w_1 \, dx = 0.$$  

This shows

$$\int_{\Omega} u \, w_1 \, dx = 0.$$  

3.1. Neumann case. In this part, we consider the $p$-Laplace eigenvalue problem with Neumann boundary.

$$\begin{cases} 
-\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\
|\nabla u|^{p-2} \nabla u \cdot \nu = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $1 < p < \infty$ and $\nu$ is the unit normal vector to $\partial \Omega$.

A non zero function $u \in W^{1,p}(\Omega)$ is called a $p$-eigenfunction of (3.5) in the weak sense, if there exist a $\lambda \geq 0$ such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \lambda \int_{\Omega} |u|^{p-2} u \phi \, dx, \quad \phi \in W^{1,p}(\Omega).$$

The number $\lambda$ is called a $p$-eigenvalue. The first eigenfunction is a constant function and $\lambda_1 = 0$. Note that by testing (3.6) with $\phi = 1$ we obtain that any eigenfunction with $\lambda > 0$ necessarily fulfills

$$\int_{\Omega} |u|^{p-2} u \, dx = 0.$$  

Equivalently,

$$\|u_+\|_{L^{p-1}(\Omega)} = \|u_-\|_{L^{p-1}(\Omega)}.$$  

This motivates the following definition:

The variational second $p$-eigenvalue is defined as

$$\lambda_{2,p}(\Omega) := \inf \left\{ \frac{\int_{\Omega} |\nabla u(x)|^p \, dx}{\int_{\Omega} |u(x)|^p \, dx} : \int_{\Omega} |u|^{p-2} u = 0 \right\}.$$  

Any $u \in W^{1,p}(\Omega) \setminus \{0\}$ realizing the infimum is called variational second $p$-eigenfunction.
Definition 3.1. The $p$-mean of a function $u \in L^p(\Omega)$ is defined as

\begin{equation}
\text{mean}_p(u) := \frac{1}{|\Omega|} \int_{\Omega} |u|^{p-2}u\,dx.
\end{equation}

Remark 3.1. The following $p$-Laplace problem

\begin{equation}
\begin{cases}
-\Delta_p u = f & \text{on } \Omega, \\
|\nabla u|^{p-2}\nabla u \cdot \nu = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

has a one parameter family of solutions; adding any constant to a solution will be a solution. Furthermore, the datum $f$ has to admit the compatibility condition $\text{mean}_2(f) = 0$.

The algorithm for Neumann case is as following.

Algorithm 2: Second eigenvalue for Neumann boundary

inputs : $u^0 = u^+_0 - u^-_0$, $\epsilon$.
output : Approximation of second eigenvalue and second eigenfunction.

while $|\lambda^1_+(\Omega_+) - \lambda^1_-(\Omega_-)| \geq \epsilon$ and $|\lambda^2_+(\Omega_+) - \lambda^2_-(\Omega_-)| \geq \epsilon$
do

(1) Set $k = 0$. Initialize with arbitrary guess $u^{(0)} \in W^{1,p}(\Omega)$ with $\text{mean}_p(u^{(0)}) = 0$.

(2) For $k \geq 0$, we set $u^+_k := \max(\pm u^+_k, 0)$ and define

$\lambda^+_k := \frac{\int_{\Omega} |\nabla u^+_k|^p \,dx}{\int_{\Omega} |u^+_k|^p \,dx}$,

$f^k := \lambda^+_k (u^+_k)^{p-1} - \lambda^-_k (u^-_k)^{p-1}$.

(3)

(4) Next define $u^{(k+1)}$ as the unique solution to the problem

\begin{equation}
\begin{cases}
-\Delta_p u = f^k & \text{in } \Omega \\
|\nabla u|^{p-2}\nabla u \cdot \nu = 0 & \text{on } \partial \Omega \\
\text{mean}_p(u) = 0.
\end{cases}
\end{equation}

(5) Set $k = k + 1$ and go back to (2).

Remark 3.2. To enforce the condition $\text{mean}_p(u) = 0$ numerically in (3.11), we fix the value of the solution at a single node of the grid to an arbitrary value, which yields a unique solution $\tilde{u}$. Then we set $u := \tilde{u} - c$ where $c$ is such that $\text{mean}_p(u) = 0$ holds.

3.2. Convergence of the Algorithm. In the sequel, for any $v \in W^{1,p}_0(\Omega)$ by $\tilde{v}$ we mean the normalized in $L^p(\Omega)$

$\tilde{v} = \frac{v}{\|v\|_{L^p(\Omega)}}$.

Furthermore, we define $u^+_k, u^-_k \in W^{1,p}_0(\Omega)$ as the positive and negative parts of the solution of the following Dirichlet problem inductively,

\begin{equation}
\begin{cases}
-\Delta_p u^+_k = \lambda^+_k (\tilde{u}^{k-1}_+)^{p-1} - \lambda^-_k (\tilde{u}^{k-1}_-)^{p-1} & \text{in } \Omega, \\
u^+_k = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}

Here $\tilde{u}^{k-1} = \tilde{u}^{k-1}_+ - \tilde{u}^{k-1}_-$. 

Proposition 3.1. With the introduced notations, the following facts hold:

\[ |\tilde{u}^{k-1}| = |\tilde{u}_+^{k-1} - \tilde{u}_-^{k-1}| = \tilde{u}_+^{k-1} + \tilde{u}_-^{k-1}, \]
\[ |\tilde{u}^{k-1}|^{p-2} = (\tilde{u}_+^{k-1})^{p-2} + (\tilde{u}_-^{k-1})^{p-2}, \]
\[ \|\lambda_+^{k-1}\tilde{u}_+^{k-1} - \lambda_-^{k-1}\tilde{u}_-^{k-1}\|^{p}_{L^p} = (\lambda_+^{k-1})^p + (\lambda_-^{k-1})^p, \]
\[ \|\lambda_+^{k-1}\nabla\tilde{u}_+^{k-1} - \lambda_-^{k-1}\nabla\tilde{u}_-^{k-1}\|^{p}_{L^p} = (\lambda_+^{k-1})^{p+1} + (\lambda_-^{k-1})^{p+1}. \]

The next Lemma shows the monotonicity of sequence of approximations of second eigenvalues as \( p \) tends to one.

Lemma 3.2. Let \( \lambda_+^k(\Omega_+) \) and \( \lambda_-^k(\Omega_-) \) be obtained by Algorithm 1. Then as \( p \to 1 \) the following holds

\[
\max \left( \lambda_+^k(\Omega_+), \lambda_-^k(\Omega_-) \right) \leq \max \left( \lambda_+^{k-1}(\Omega_+), \lambda_-^{k-1}(\Omega_-) \right),
\]

for every \( k \geq 1 \).

Proof. To start, multiply the first equation (3.12) by \( u_+^k \) and integrate over \( \Omega \) to deduce

\[
\int_{\Omega} |\nabla u_+^k|^p \, dx = \int_{\Omega} u_+^k |\tilde{u}^{k-1}|^{p-2} \left[ \lambda_+^{k-1}\tilde{u}_+^{k-1} - \lambda_-^{k-1}\tilde{u}_-^{k-1} \right] \, dx
\]

\[
= \int_{\Omega} u_+^k \left[ \lambda_+^{k-1}(\tilde{u}_+^{k-1})^{p-1} - \lambda_-^{k-1}(\tilde{u}_-^{k-1})^{p-1} \right] \, dx.
\]

Hölder inequality on right hand side gives

\[
\int_{\Omega} |\nabla u_+^k|^p \, dx \leq \|u_+^k\|_{L^p} \|\lambda_+^{k-1}(\tilde{u}_+^{k-1})^{p-1} - \lambda_-^{k-1}(\tilde{u}_-^{k-1})^{p-1}\|_{L^{\frac{p}{p-1}}}.
\]

By Proposition (3.1) we obtain

\[
\|\nabla u_+^k\|_{L^p} \leq \|u_+^k\|_{L^p} \left[ (\lambda_+^{k-1})^{\frac{p}{p-1}} + (\lambda_-^{k-1})^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}}.
\]

The same argument as above indicates

\[
\|\nabla u_-^k\|_{L^p} \leq \|u_-^k\|_{L^p} \left[ (\lambda_+^{k-1})^{\frac{p}{p-1}} + (\lambda_-^{k-1})^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}}.
\]

The inequalities (3.14) and (3.15) imply

\[
\max \left( \frac{\|\nabla u_+^k\|_{L^p}}{\|u_+^k\|_{L^p}}, \frac{\|\nabla u_-^k\|_{L^p}}{\|u_-^k\|_{L^p}} \right) \leq \left[ (\lambda_+^{k-1})^{\frac{p}{p-1}} + (\lambda_-^{k-1})^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}}.
\]

Let \( p \to 1^+ \) and using the fact

\[
\lim_{a \to \infty} (a^\alpha + b^\alpha)^{\frac{1}{\alpha}} = \max(a, b),
\]

complete the proof. \( \square \)

Lemma 3.3. With same assumptions as before the sequence \( u^k \) is bounded from below for every \( p \geq 1 \).
Proof. Multiply equation (3.12) by $u^k$, integrate over $\Omega$ and Hölder inequality give
\begin{equation}
\int_\Omega |\nabla u^k|^p \, dx \leq \|u^k\|_{L^p} \|\lambda_{+}^{-1}(u_+^{k-1})^{p-1} - \lambda_{-}^{-1}(\tilde{u}_-^{k-1})^{p-1}\|_{L^{\frac{p}{p-1}}}.
\end{equation}
\[ \|\nabla u^k\|_{L^p}^p \leq \|u^k\|_{L^p} \left[ (\lambda_{+}^{-1})^{\frac{p}{p-1}} + (\lambda_{-}^{-1})^{\frac{p}{p-1}} \right] \frac{p-1}{p}. \]

Note that $u^k$ satisfies
\[ \lambda_{2} \|u^k\|_{L^p}^p \leq \|\nabla u^k\|_{L^p}^p. \]

This shows
\[ \lambda_{2} \|u^k\|_{L^p}^p \leq \|u^k\|_{L^p} \left[ (\lambda_{+}^{-1})^{\frac{p}{p-1}} + (\lambda_{-}^{-1})^{\frac{p}{p-1}} \right] \frac{p-1}{p}, \]
which implies
\begin{equation}
\|u^k\|_{L^p} \leq \frac{1}{\lambda_{2}} \left[ (\lambda_{+}^{-1})^{\frac{p}{p-1}} + (\lambda_{-}^{-1})^{\frac{p}{p-1}} \right] \frac{1}{\frac{1}{p}}.
\end{equation}

Multiply (3.12) by $\tilde{u}_+^{k-1}$ and integrate over $\Omega$ to deduce
\[ \int_\Omega |\nabla u^k|^{p-2} \nabla u^k \cdot \nabla \tilde{u}_+^{k-1} \, dx = \lambda_{+}^{k-1}. \]

Here we used the facts that
\[ \tilde{u}_+^{k-1} \cdot \tilde{u}_-^{k-1} = 0, \quad \int_\Omega (\tilde{u}_+^{k-1})^p \, dx = 1. \]

Form here we get
\[ \lambda_{+}^{k-1} \leq \|\nabla u^k\|_{L^p}^{p-1} \|\nabla \tilde{u}_+^{k-1}\|_{L^p}, \]
the same argument shows
\[ \lambda_{-}^{k-1} \leq \|\nabla u^k\|_{L^p}^{p-1} \|\nabla \tilde{u}_-^{k-1}\|_{L^p}, \]

Considering $\|\nabla \tilde{u}_+^{k-1}\|_{L^p} = (\lambda_{+}^{k-1})^{\frac{1}{p}}$ and $\|\nabla \tilde{u}_-^{k-1}\|_{L^p} = (\lambda_{-}^{k-1})^{\frac{1}{p}}$ implies
\begin{equation}
\left\{ \begin{array}{l}
(\lambda_{+}^{k-1})^{\frac{p-1}{p}} \leq \|\nabla u^k\|_{L^p}^{p-1}, \\
(\lambda_{-}^{k-1})^{\frac{p-1}{p}} \leq \|\nabla u^k\|_{L^p}^{p-1}.
\end{array} \right\
\end{equation}

Inserting the inequalities (3.18) in (3.16) yields
\begin{equation}
1 \leq 2^{\frac{p-1}{p}} \|u^k\|_{L^p}.
\end{equation}
Also from inequalities (3.18) and (3.14) it follows
where

\begin{align}
\|\nabla u^k\|_{L^p}^p &\leq 2^{\frac{p-1}{p}} \|u^k\|_{L^p} \|\nabla u^k\|_{L^p}^p, \\
||\nabla u^k||_{L^p}^p &\leq 2^{\frac{p-1}{p}} \|u^k\|_{L^p} \|\nabla u^k\|_{L^p}^p.
\end{align}

(3.20)

\[ \lambda^k = \frac{\int_{\Omega} |\nabla u^k|^p \, dx}{\int_{\Omega} |u^k|^p \, dx}. \]

Proof. We rewrite the right hand side of (3.16) as

\[ ||\nabla u^k||_{L^p}^p \leq ||u^k||_{L^p}^{p-1} \left( \lambda^k - \lambda^{-1} \right). \]

Next by Proposition 3.1 we have

\[ \left\| \lambda^k - \lambda^{-1} \right\|_{L^p}^{p-1} = \left( \left( \lambda^k - \lambda^{-1} \right) \right)^{\frac{p}{p-1}}. \]

Also

\[ \|\lambda^k - \lambda^{-1} \|^p_{L^p} = \int_{\Omega} \left( \lambda^k - \lambda^{-1} \right) \cdot \left( \lambda^k - \lambda^{-1} \right) \, dx \]

\[ = \int_{\Omega} \left( \nabla u^k \cdot \nabla \lambda^k - \nabla u^{-k} \cdot \nabla \lambda^{-1} \right) \, dx \]

Inserting the last inequality in (3.21) and dividing by \|u^k||_{L^p} we obtain

\[ \frac{\int_{\Omega} |\nabla u^k|^p \, dx}{\int_{\Omega} |u^k|^p \, dx} \leq \frac{\left( \int_{\Omega} |\nabla u^k|^p \, dx \right)^{\frac{1}{p-1}}}{\left( \int_{\Omega} |u^k|^p \, dx \right)^{\frac{1}{p-1}}} \frac{\left( \lambda^k - \lambda^{-1} \right) \cdot \left( \lambda^k - \lambda^{-1} \right) \, dx}{\left( \lambda^k - \lambda^{-1} \right)^{\frac{p}{p-1}} + \left( \lambda^{-1} \right)^{\frac{p}{p-1}}}. \]

The inequality above yields

\[ \frac{\int_{\Omega} |\nabla u^k|^p \, dx}{\int_{\Omega} |u^k|^p \, dx} \leq \frac{\left( \lambda^k - \lambda^{-1} \right) \cdot \left( \lambda^k - \lambda^{-1} \right) \, dx}{\left( \lambda^k - \lambda^{-1} \right)^{\frac{p}{p-1}} + \left( \lambda^{-1} \right)^{\frac{p}{p-1}}}. \]
Thus
\begin{equation}
\int_{\Omega} |\nabla u_k|^p \, dx \leq \frac{(\lambda_{k-1}^+)^{\frac{p}{p-1}} + (\lambda_{k-1}^-)^{\frac{p}{p-1}}}{(\lambda_{k-1}^+)^{\frac{p}{p-1}} + (\lambda_{k-1}^-)^{\frac{p}{p-1}}}.
\end{equation}

Form (3.22) we infer
\begin{equation}
\lambda_k \leq \max (\lambda_{k-1}^+, \lambda_{k-1}^-).
\end{equation}

\square

4. Graph $p$-Laplacian

The aim of this part is to perform data clustering by using our algorithm. Given some data and a notion of similarity, we aim to partition the input data into maximally homogeneous groups (i.e. clusters). The main idea is to find a low-dimensional embedding and then to project data points to new space. Recursive bi-partitioning method is widely used in many clustering algorithm for the multi-class problem. As a basic idea, for given graph $G$ a traditional spectral clustering algorithm uses the first $k$ eigenvectors for $\Delta_G$ as a low-dimensional embedding of the graph. Different graph Laplacian and their basic properties, spectral clustering algorithms are described in [20,21].

Let $G = (V, E)$ be an undirected graph with vertex set $V = \{v_1, \ldots, v_n\}$ or simply $V = \{1, \ldots, n\}$, and $E$ is the set of edges. The weighted adjacency matrix $W$ encode the similarity of pairwise data points, or weight $w_{ij} \geq 0$ between two vertices $v_i$ and $v_j$;
\[ W = (w_{ij}) \quad i, j = 1, \ldots, n. \]

Note that $G$ being undirected means $w_{ij} = w_{ji}$. The degree of a vertex $i \in V$ denoted by $d_i$ is
\[ d_i = \sum_{j \in V} w_{ij}. \]

The degree matrix $D$ is defined as the diagonal matrix with the degrees $d_1, \ldots, d_n$ on the diagonal.

For given graph $(V, E)$ and a subset of vertex $C \subset V$ the $\text{Cut}(C, C^c)$ (or the perimeter $|\partial C|$) is defined by
\[ \text{Cut}(C, C^c) := \sum_{i \in C, j \in C^c} w_{ij}, \]

where $C^c$ is the complement of $C$. The ratio Cheeger cut $\text{RCC}(C, C^c)$ and normalized Cheeger cut $\text{NCC}(C, C^c)$ are defined respectively, by
\[ \text{RCC}(C, C^c) = \frac{\text{cut}(C, C^c)}{\min(|C|, |C^c|)}, \quad \text{NCC}(C, C^c) = \frac{\text{cut}(C, C^c)}{\min(\text{vol}|C|, \text{vol}|C^c|)}. \]

The minimum is achieved if $|C| = |C^c|$. 

For function $f : V \to \mathbb{R}$, the unnormalized $p$-Laplace operator $\Delta_p^u$ and normalized $p$-Laplace operator $\Delta_p^n$ are defined as follow (depends on the choice of inner product):

$$(\Delta_p^u f)_i = \sum_j \phi_p(f_i - f_j), \quad (\Delta_p^n f)_i = \frac{1}{d_i} \sum_j \phi_p(f_i - f_j),$$

where

$$\phi_p(\cdot) = |\cdot|^{p-1}\text{sign}(\cdot),$$

for $p = 2$ we get $\phi_2(x) = x$. Also for functions $f, g : V \to \mathbb{R}$ the inner product is defined by

$$<f, g> = \sum_{i \in V} d_i f_i g_i.$$

Next consider the following problem. Find $f : V \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that

$$(\Delta_p^u f)_i = \lambda \phi_p(f_i), \quad \forall i \in V.$$

We call $\lambda$ an eigenvalue of $\Delta_p^u$ associated with eigenvector $f$. In the case $p = 2$, the operator $\Delta_p^u$ is the regular graph Laplacian given by

$$\Delta_2^u = L = D - W.$$

The Rayleigh quotient is defined by

$$R_p(f) = \frac{Q_p(f)}{\|f\|_p^p} = \frac{\sum_{i,j} (f_i - f_j)^p}{2\|f\|_p^p},$$

with

$$Q_p(f) := <f, \Delta_p^u f> = \frac{1}{2} \sum_{i,j} w_{ij} |f_i - f_j|^p, \quad \|f\|_p^p = \sum_i |f_i|^p.$$

Similar to continuous case, $f$ is a $p$-eigenfunction of $\Delta_p^u$ if and only if the functional $R_p$ has a critical point at $f \in \mathbb{R}^{|V|}$. The corresponding eigenvalue $\lambda_p$ is given as

$$\lambda_p = R_p(f).$$

The first eigenvalue is zero and the first eigenvector is a constant vector. Then to find the second eigenvalue we minimize the Rayleigh quotient $R_p(f)$ over all $f$ such that

$$\sum_{i \in V} f_i = 0,$$

which is the requirement of Algorithm 2 presented in previous section.

We point out that in [11] is shown that the cut obtained by thresholding the second eigenvector of $p$-Laplace converges to optimal Cheeger cut as $p$ tends to 1 and

$$\lambda_2(\Delta_1) = \text{RCC}^*.$$
However, finding optimal ratio Cheeger cut $RCC^* = \min_{C \subset V} RCC$ is NP-hard problem. To see more about minimization of $R_p(f)$ with constraint $\sum_{i \in V} f_i = 0$, see [11].

5. NUMERICAL IMPLEMENTATION

In this section, we briefly explain about numerical approximation of the following problem

(5.1) \[
\begin{cases}
-\Delta_p u = f(x) & \text{in } \partial \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

For $p > 1$ the solution of Problem (5.1) is the unique minimizer of the following functional

$$E(u) = \min_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \int_\Omega \frac{1}{p} |\nabla u(x)|^p - f(x)u(x) \, dx.$$  

Equation in (5.1) is understood in weak sense:

(5.2) \[
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \int_\Omega f(x) \phi \, dx, \quad \forall \phi \in W_0^{1,p}(\Omega).
\]

The discretization of problem is as follows. Let $T_h$ be a regular triangulation of $\Omega_h$ which is composed of disjoint open regular triangles $T_i$, that is,

$$\overline{\Omega_h} = \bigcup_{i=1}^n T_i.$$  

Considering the regularity for the solution of the $p$-Laplace equation, we deal with continuous piecewise linear element. Consider a finite dimensional subspace $V_h$ of $C^0(\Omega_h)$, such that the restriction on elements of $T_h$, where $P_1$ is the linear function space:

$$V_0^h = \{v \in H^1_0 : v|_T \in P_1, \forall T \in T_h\}.$$  

Assume $u_n \in V_0^h$ be the current approximation the, we associate the residual denoted by $R_n$

$$R_n = f(x) - \nabla(|\nabla u_n|^{p-2} \nabla u_n).$$

Equivalently,

$$\int_\Omega R_n \phi \, dx = \int_\Omega (f(x)\phi - |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \phi) \, dx.$$  

Then to update $u_n$ and obtaining next approximation denoted by $u_{n+1}$

$$u_{n+1} = u_n + \alpha_n w_n,$$

where $w_n$ is determined solution of linearized $p$-Laplace and the source term residual, i.e.,
(5.3) \[
\begin{aligned}
-\Delta w_n &= R_n \quad \text{in } \Omega, \\
  w_n &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
Or
\[
\int_{\Omega} \left( \varepsilon + |\nabla u_n| \right)^{p-2} \nabla w_n \cdot \nabla \phi \, dx = \int_{\Omega} R_n \phi \, dx.
\]

The step length $\alpha_n$ in search direction $w_n$ can be obtained as
\[
E(u_n + \alpha_n w_n) = \min_{\alpha} E(u_n + \alpha w_n).
\]

5.1. Numerical Examples. This section provides some examples of numerical approximations to the $p$-Laplace eigenvalue problem for different values of $p$. For initial guess we use the second eigenfunction of Laplace operator.

Example 5.1. Here, we verify our algorithm by invoking it in dimension one. Let $\lambda_p$ denote the second eigenvalue in the interval $(a, b)$ then by result in [13, 26] we have
\[
\sqrt[p-1]{\lambda_p} = \frac{\sqrt{p-1} \pi}{(b-a) \sin(\frac{\pi}{p})}.
\]

Let $\Omega = (-2, 2)$ then for $p = 100$ the above formula gives:
\[
\sqrt[50]{\lambda_{100}} = 2.0944.
\]

For $p = 50, p = 100$ our approximation of second eigenvalue are
\[
\sqrt[50]{\lambda_{50}} = 2.1717 \quad \sqrt[100]{\lambda_{100}} = 2.0981,
\]
with asymptotic given by (2.6) or (5.4)
\[
\lim_{p \to \infty} \sqrt[p]{\lambda_p} = 2.
\]

Figure 1. The second eigenfunction for $p = 100$. 
Example 5.2. Let $p = 2$ and the domain be $\Omega = [-2 \ 2] \times [-2 \ 2]$. The second eigenvalue is $\lambda_2 = \frac{5\pi^2}{16}$. We set $\Delta x = \Delta y = .005$. The initial value is given in Figure 2. and our approximate value after 50 iterations is: $3.0843295$. with

$$|\lambda_2 - \lambda_2^{(50)}| \leq 0.002.$$

Figure 2. Initial guess

In Figure 3, and 4 the second eigenfunctions for Dirichlet and Neumann cases are shown.

Figure 3. Surface of $u_2$.

Example 5.3. Let domain be the square $[0 \ 2] \times [0 \ 2]$. Picture 5 shows the second eigenfunction for $p = 10$. 

Example 5.4. In this experiment, we use $\varepsilon$-graph. The points are generated by standard uniform distribution. We choose $n = 5000$, and $\varepsilon = .05$.

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Figure 6. Top figure $n = 5000$ points with $\varepsilon = 0.02$.
The second eigenvector for $p=1.25$.

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