LENGTH OF A CURVE IS QUASI-CONVEX ALONG A
TEICHMÜLLER GEODESIC

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Abstract. We show that for every simple closed curve \( \alpha \), the extremal
length and the hyperbolic length of \( \alpha \) are quasi-convex functions along
any Teichmüller geodesic. As a corollary, we conclude that, in Teichmüller
space equipped with the Teichmüller metric, balls are quasi-
convex.

1. Introduction
In this paper we examine how the extremal length and the hyperbolic
length of a measured lamination change along a Teichmüller geodesic. We
prove that these lengths are quasi-convex functions of time. The convexity
issues in Teichmüller space equipped with Teichmüller metric are hard to
approach and are largely unresolved. For example it is not known whether
it is possible for the convex hall of three points in Teichmüller space to be
the entire space. (This is an open question of Masur.)

Let \( S \) be a surface of finite topological type. Denote the Teichmüller space
of \( S \) equipped with the Teichmüller metric by \( \mathcal{T}(S) \). For a Riemann surface
\( x \) and a measured lamination \( \mu \), we denote the extremal length of \( \mu \) in \( x \) by
\( \text{Ext}_x(\mu) \) and the hyperbolic length of \( \mu \) in \( x \) by \( \text{Hyp}_x(\mu) \).

Theorem A. There exists a constant \( K \), such that for every measured lam-
ination \( \mu \), any Teichmüller geodesic \( \mathcal{G} \) and points \( x, y, z \in \mathcal{T}(S) \) appearing
in that order along \( \mathcal{G} \) we have
\[
\text{Ext}_y(\mu) \leq K \max\{ \text{Ext}_x(\mu), \text{Ext}_z(\mu) \},
\]
and
\[
\text{Hyp}_y(\mu) \leq K \max\{ \text{Hyp}_x(\mu), \text{Hyp}_z(\mu) \}.
\]

In sec \[7\], we provide some examples showing that the quasi-convexity is
the strongest statement one can hope for:

Theorem B. The hyperbolic length and the extremal length of a curve are
in general not convex functions of time along a Teichmüller geodesic.

This contrasts with the results of Kerckhoff [Ker83], Wolpert [Wol87] and
Bestvina-Bromberg-Fujiwara-Souto [BBFS09]. They proved, respectively,
that the hyperbolic length functions are convex along earthquake paths,
Weil-Petersson geodesics and a certain shearing paths.

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As a consequence of Theorem A, we show that balls in Teichmüller space are quasi-convex.

**Theorem C.** There exists a constant $c$ so that, for every $x \in T(S)$, every radius $r$ and points $y$ and $z$ in the ball $B(x, r)$, the geodesic segment $[y, z]$ connecting $y$ to $z$ is contained in $B(x, r + c)$.

We also construct an example of a long geodesic that stays near the boundary of a ball, suggestion that balls in $T(S)$ may not be convex.

A Teichmüller geodesic can be described very explicitly as a deformation of a flat structure on $S$, namely, by stretching the horizontal direction and contracting the vertical direction. Much is known about the behavior of a Teichmüller geodesic. Our proof consists of combining the length estimates given in [Min96, Raf05b, CR05] with the descriptions of the behavior of a Teichmüller geodesic developed in [Raf05a, Raf07, CRS06].

As a first step, for a curve $\gamma$ and a quadratic differential $q$, we provide an estimate for the extremal length of $\gamma$ in the underlying conformal structure of $q$ (Theorem 7) by describing what are the contributions to the extremal length of $\gamma$ from the restriction of $\gamma$ to various pieces of the flat surface associated to $q$. These pieces are either thick sub-surfaces or annuli with large moduli. We then introduce the notions of essentially horizontal and essentially vertical (Corollary 9 and Definition 10). Roughly speaking, a curve $\gamma$ is essentially horizontal in $q$ if the restriction of $\gamma$ to some piece of $q$ contributes a definite portion of the total extremal length of $\gamma$ and if $\gamma$ is mostly horizontal in that piece. We show that, while $\gamma$ is essentially vertical, its extremal length is essentially decreasing and while $\gamma$ is essentially horizontal its extremal length is essentially increasing (Theorem 13). This is because the flat length of the portion of $\gamma$ that is mostly horizontal grows exponentially fast and becomes more and more horizontal. The difficulty with making this argument work is that the thick-thin decomposition of $q$ changes as time goes by and the portion of $\gamma$ that is horizontal and has a significant extremal length can spread onto several thick pieces. That is why we need to talk about the contribution to the extremal length of $\gamma$ from every sub-arc of $\gamma$ (Lemma 11). The Theorem then follows from careful analysis of various possible situations. The proof for the hyperbolic length follows a similar path and is presented in sec 6.

1.1. **Notation.** The notation $A \asymp B$ means that the ratio $A/B$ is bounded both above and below by constants depending on the topology of $S$ only. When this is true we say $A$ is comparable with $B$ or $A$ and $B$ are comparable. The notation $A \preceq B$ means that $A/B$ is bounded above by a constant depending on the topology of $S$.

2. **Background**

2.1. **Hyperbolic metric.** Let $x$ be a Riemann surface or equivalently (using uniformization) a complete hyperbolic metric on $S$. By a curve on $S$
we always mean a free homotopy class of non-trivial non-peripheral simple closed curve. Every curve \( \gamma \) has a unique geodesic representative in the hyperbolic metric \( x \) which we call the \( x \)-\emph{geodesic representative} of \( \gamma \). We denote the hyperbolic length of the \( x \)-geodesic representative of \( \gamma \) by \( \ell_x(\gamma) \) and refer to it as the \( x \)-length \( \gamma \).

For a small positive constant \( \epsilon_1 \), the thick-think decomposition of \( x \) is a pair \( (\mathcal{A}, \mathcal{Y}) \), where \( \mathcal{A} \) is the set of curves in \( x \) that have hyperbolic length less than \( \epsilon_1 \) and \( \mathcal{Y} \) is the set of components of \( S \setminus (\cup_{\alpha \in \mathcal{A}} \alpha) \). Note that, so far, we are only recording the topological information. One can make this to a geometric decomposition as follows: for each \( \alpha \in \mathcal{A} \), consider the annulus that is a regular neighborhood of the \( x \)-geodesic representative of \( \alpha \) and has boundary length of \( \epsilon_0 \). For \( \epsilon_0 > \epsilon_1 > 0 \) small enough, these annuli are disjoint (the Margulis Lemma) and their complement is a union of subsurfaces with horocycle boundaries of length \( \epsilon_0 \). For each \( Y \in \mathcal{Y} \) we denote this representative of the homotopy class of \( Y \) by \( Y_x \).

If \( \mu \) is a set of curves, then \( \ell_x(\mu) \) is the sum of the lengths of the \( x \)-geodesic representatives of the curves in \( \mu \). A short marking in \( Y_x \) is a set \( \mu_Y \) of curves in \( Y \) so that \( \ell_x(\mu_Y) = O(1) \) and \( \mu_Y \) fills the surface \( Y \) (that is, every curve intersecting \( Y \) intersects some curve in \( \mu_Y \)).

If \( \gamma \) is a curve and \( Y \in \mathcal{Y} \), the \emph{restriction} \( \gamma|_{Y_x} \) of \( \gamma \) to \( Y_x \) is the union of arcs obtained by taking the intersection of the \( x \)-geodesic representative of \( \gamma \) with \( Y_x \). Let \( \gamma|_Y \) be the set of homotopy classes (rel \( \partial Y \)) of arcs in \( Y \) with end points on \( \partial Y \). We think of \( \gamma|_Y \) as a set of weighted arcs to keep track of multiplicity. Note that \( \gamma|_Y \) has only topological information while \( \gamma|_{Y_x} \) is a set of geodesic arcs. An alternate way of defining \( \gamma|_Y \) is to consider the cover \( \tilde{Y} \to S \) corresponding to \( Y \); that is, the cover where \( \tilde{Y} \) is homeomorphic to \( Y \) and such that \( \pi_1(\tilde{Y}) \) projects to a subgroup of \( \pi_1(S) \) that is conjugate to \( \pi_1(Y) \). Use the hyperbolic metric to construct a boundary at infinity for \( \tilde{Y} \). Then \( \gamma|_{\tilde{Y}} \) is the homotopy class of arcs in \( \tilde{Y} \) that are lifts of \( \gamma \) and are not boundary parallel. Now the natural homeomorphism from \( \tilde{Y} \) to \( Y \) sends \( \gamma|_{\tilde{Y}} \) to \( \gamma|_Y \).

By \( \ell_x(\gamma|_Y) \), we mean the \( x \)-length of the shortest representatives of \( \gamma|_Y \) in \( Y_x \). It is well known that (see, for example, [CR05])

\begin{equation}
\ell_x(\gamma|_Y) = \ell_x(\gamma|_{Y_x}) \lesssim i(\gamma, \mu_Y),
\end{equation}

where \( i(\cdot, \cdot) \) is the geometric intersection number and \( i(\gamma, \mu_Y) \) is the sum of the geometric intersection numbers between \( \gamma \) and curves in \( \mu_Y \).

**Euclidean metric.** Let \( q \) be a quadratic differential on \( x \). In a local coordinate \( z \), \( q \) can be represented as \( q(z)dz^2 \) where \( q(z) \) is holomorphic (when \( x \) has punctures, \( q \) is allowed to have poles of degree one at punctures). We call the metric \( |q| = |q(z)|(dx^2 + dy^2) \) the flat structure of \( q \). This is a locally flat metric with singularities at zeros of \( q(z) \) (see [Str80] for an introduction to the geometry of \( q \)). The \( q \)-geodesic representative of a curve is not always unique; there may be a family of parallel copies of geodesics foliating
a flat cylinder. For a curve \( \alpha \), we denote this flat cylinder of all \( q \)-geodesic representatives of \( \alpha \) by \( F_\alpha^q \) or \( F_\alpha \) if \( q \) is fixed.

Consider again the thick-thin decomposition \((A, Y)\) of \( x \). (If \( q \) is a quadratic differential on \( x \), we sometimes call this the thick-thin decomposition of \( q \). Note that \((A, Y)\) depends only on the underlying conformal structure.) For \( Y \in Y \), the homotopy class of \( Y \) has a representative with \( q \)-geodesic boundaries that is disjoint from the interior of the flat cylinders \( F_\alpha^q \), for every \( \alpha \in A \). We denote this subsurface by \( Y_q \). Note that \( Y_q \) may be degenerate and have no interior (see [Raf05a] for a more careful discussion).

Let \( \text{diam}_q(Y) \) denote the \( q \)-diameter of \( Y \). We recall the following theorem relating the hyperbolic and flat length of a curve in \( Y \).

**Theorem 1** ([Raf05b]). For every curve \( \gamma \) in \( Y \)
\[
\ell_q(\gamma) \asymp \ell_x(\gamma) \text{ diam}_q(Y).
\]

Since \( Y_q \) can be degenerate, one has to be more careful in defining \( \ell_q(\gamma|_Y) \). Again we consider the cover \( \tilde{Y} \to S \) corresponding to \( Y \) and this time we equip \( \tilde{Y} \) with the locally Euclidean metric \( \tilde{q} \) that is the pullback of \( q \). The subsurface \( Y_q \) lifts isometrically to a subsurface \( \tilde{Y}_q \) in \( \tilde{Y} \). Consider the lift \( \tilde{\gamma} \) of the \( q \)-geodesic representative of \( \gamma \) to \( \tilde{Y} \) and the restriction of \( \tilde{\gamma} \) to \( Y_q \). We define \( \ell_q(\gamma|_Y) \) to be the \( \tilde{q} \)-length of this restriction. Note that \( \ell_q(\gamma|_Y) \) may equal zero. (See the example at the end of [Raf05b].) However, a modified version of Equation (1) still holds true for \( \ell_q(\gamma|_Y) \):

**Proposition 2.** For every curve \( \gamma \) in \( Y \)
\[
\frac{\ell_q(\gamma|_Y)}{\text{diam}_q(Y)} + i(\gamma, \partial Y) \asymp i(\gamma, \mu_Y).
\]

**Proof.** As above, consider the cover \( \tilde{Y} \to S \), the subsurface \( \tilde{Y}_q \) that is the isometric lift of \( Y_q \) and the lift \( \tilde{\gamma} \) of the \( q \)-geodesic representative of \( \gamma \). For every curve \( \alpha \in \mu_Y \), there is a lift of \( \alpha \) that is a simple closed curve. To simplify notation, we denote this lift again by \( \alpha \) and the collection of these curves by \( \mu_Y \). Let \( d = \text{diam}_q(Y) \), let \( Z \) be the \( d \)-neighborhood of \( \tilde{Y}_q \) in \( \tilde{Y} \) and let \( \omega \) be an arc in \( Z \) constructed as follows: Choose and arc of \( \gamma|_{\tilde{Y}_q} \) (which is potentially just a point) and at each end point \( p \), extend this arc perpendicular to \( \partial \tilde{Y}_q \) until it hits \( \partial Z \) at a point \( p' \) (see Fig. 2.1).

From the construction we have
\[
\ell_q(\omega) = \ell_q(\omega|_{\tilde{Y}_q}) + 2d.
\]
Summing over all such arcs, we have:
\[
\sum \ell_q(\omega) = \ell_q(\gamma|_Y) + d i(\gamma, \partial Y).
\]
Also,
\[
\sum i(\omega, \mu_Y) = i(\gamma, \mu_Y).
\]
Hence, to prove the lemma, we only need to show
\[ d(i(\omega, \mu_Y)) \asymp \ell_q(\omega). \]

The arguments needed here are fairly standard. In the interest of brevity, we point the reader to some references instead of repeating the arguments. Let \( \alpha \) be a curve in \( \mu_Y \). By Theorem 1, the \( q \)-length of the shortest essential curve in \( Z \) (which has hyperbolic length comparable with 1) is comparable with \( d \), hence the argument in the proof of [Raf05b, Lemma 5] also implies
\[ \ell_q(\alpha) \ell_q(\omega) \asymp d^2 i(\omega, \alpha). \]

Therefore, \( l_q(\omega) \asymp d(i(\omega, \alpha)) \). Summing over curves \( \alpha \in \mu_Y \) (the number of which depends on the topology of \( S \) only), we have
\[ \ell_q(\omega) \asymp d(i(\omega, \mu_Y)). \]

It remains to show the other direction of Equation (2). Here, one needs to construct paths in \( Y_q \) (traveling along the geodesics in \( \mu_Y \)) representing arcs in \( \gamma\vert_{\tilde{Y}_q} \) whose lengths are of order \( d^2 i(\gamma, \mu_Y) \). This is done in the proof of [Raf05b, Theorem 6]). \( \square \)

**Regular and expanding annuli.** Let \((A, Y)\) be the thick-thin decomposition of \( q \) and let \( \alpha \in A \). Consider the \( q \)-geodesic representative of \( \alpha \) and the family of regular neighborhoods of this geodesic in \( q \). Denote the largest regular neighborhood that is still homeomorphic to an annulus by \( A_\alpha \). The annulus \( A_\alpha \) contains the flat cylinder \( F_\alpha \) in the middle and two expanding
annuli on each end which we denote by $E_\alpha$ and $G_\alpha$:

$$A_\alpha = E_\alpha \cup F_\alpha \cup G_\alpha.$$ 

We call $E_\alpha$ and $G_\alpha$ expanding because if one considers the foliation of these annuli by curves that are equidistant to the geodesic representative of $\alpha$, the length of these curves increases as one moves away from the $q$–geodesic representative of $\alpha$. This is in contrast with $F_\alpha$ where all the equidistance curves have the same length. (See [Min92] for precise definition and discussion.) We denote the $q$–distance between the boundaries of $A_\alpha$ by $d_q(\alpha)$ and $q$–distance between the boundaries of $E_\alpha$, $F_\alpha$ and $G_\alpha$ by $e_q(\alpha)$, $f_q(\alpha)$ and $g_q(\alpha)$ respectively. When $\alpha$ and $q$ are fixed, we simply use $e$, $f$ and $g$.

Lemma 3 ([CRS06]). For $\alpha \in A$,

$$\frac{1}{\text{Ext}_x(\alpha)} \asymp \text{Mod}_x(E_\alpha) + \text{Mod}_x(F_\alpha) + \text{Mod}_x(G_\alpha).$$

Furthermore,

$$\text{Mod}_x(E_\alpha) \asymp \log \left( \frac{e}{\ell_q(\alpha)} \right), \quad \text{Mod}_x(G_\alpha) \asymp \log \left( \frac{g}{\ell_q(\alpha)} \right),$$

and

$$\text{Mod}_x(F_\alpha) \asymp \frac{f}{\ell_q(\alpha)}.$$

Let $\gamma$ be a curve. The restriction $\gamma|_{A_\alpha}$ is the set of arcs obtained from restricting the $q$–geodesic representative of $\gamma$ to $A_\alpha$, and $\ell_q(\gamma|_{A_\alpha})$ is the sum of the $q$–lengths of these curves.

Lemma 4. For the thick-thin decomposition $(A, \mathcal{Y})$ of $q$, we have

$$\ell_q(\gamma) \asymp \sum_{Y \in \mathcal{Y}} \ell_q(\gamma|Y) + \sum_{\alpha \in A} \ell_q(\gamma|_{A_\alpha}).$$

Proof. The annuli $A_\alpha$ are not necessarily disjoint. But, the size of $A$ is uniformly bounded and $\ell_q(\gamma) \geq \ell_q(\gamma|_{A_\alpha})$. Similarly, the size of $\mathcal{Y}$ is uniformly bounded and $\ell_q(\gamma) \geq \ell_q(\gamma|_{Y})$. Hence

$$\ell_q(\gamma) \asymp \sum_{Y \in \mathcal{Y}} \ell_q(\gamma|Y) + \sum_{\alpha \in A} \ell_q(\gamma|_{A_\alpha}).$$

To see the inequality in the other direction, we note that every segment in the $q$–geodesic representative of $\gamma$ is either contained in some $A_\alpha$, $\alpha \in A$ or in some $Y_q$, $Y \in \mathcal{Y}$. $\square$

2.2. Teichmüller geodesics. Let $q = q(z)dz^2$ be a quadratic differential on $x$. It is more convenient to use the natural parameter $\zeta = \xi + i\eta$, which is defined as

$$\zeta(w) = \int_{z_0}^w \sqrt{q(z)} \, dz.$$
In these coordinates, we have \( q = d\xi^2 \). The lines \( \xi = \text{const} \) with transverse measure \( |d\xi| \) define the \textit{vertical} measured foliation, associated to \( q \). Similarly, the \textit{horizontal} measured foliation is defined by \( \eta = \text{const} \) and \( |d\eta| \). The transverse measure of an arc \( \alpha \) with respect to \( |d\xi| \), denoted by \( h_q(\alpha) \), is called the \textit{horizontal length} of \( \alpha \). Similarly, the \textit{vertical length} \( v_q(\alpha) \) is the measure of \( \alpha \) with respect to \( |d\eta| \).

A Teichmüller geodesic can be described as follows. Given a Riemann surface \( x \) and a quadratic differential \( q \) on \( x \), we can obtain a 1–parameter family of quadratic differentials \( q_t \) from \( q \) so that, for \( t \in \mathbb{R} \), if \( \zeta = \xi + i\eta \) are natural coordinates for \( q \), then \( \zeta_t = e^{-t}\xi + ie^t\eta \) are natural coordinates for \( q_t \). Let \( x_t \) be the conformal structure associated to \( q_t \). Then \( G : \mathbb{R} \to \mathcal{T}(S) \) which sends \( t \) to \( x_t \), is a Teichmüller geodesic.

Let \( G : [a, b] \to \mathcal{T}(S) \) be a Teichmüller geodesic and \( q_a \) and \( q_b \) be the initial and terminal quadratic differentials. We use \( \ell_a(\cdot) \) for \( q_a \)–length of a curve, we use \( \text{Ext}_a(\cdot) \) for the extremal length of a curve in \( q_a \). Similarly, we denote by \( \text{Mod}_a(\cdot) \) the modulus of an annulus in \( q_a \). We denote the thick thin decomposition of \( q_a \) by \((A_a, \gamma_b)\). We also write \( e_a(\alpha), d_a(\alpha), f_a(\alpha) \) and \( \ell_a(\alpha) \) in place of \( e_{q_a}(\alpha), d_{q_a}(\alpha), f_{q_a}(\alpha) \) and \( \ell_{q_a}(\alpha) \). When the curve \( \alpha \) is fixed, we simplify notation even further and use \( e_a, d_a, f_a \) and \( \ell_a \). Also, we denote the flat annulus and the expanding annuli corresponding to \( \alpha \) in \( q_a \) by \( F^a, E^a \) and \( G^a \), or by \( F^a, E^a \), and \( G^a \) when \( \alpha \) is fixed. Similar notation applies to \( q_b \). The following technical statement will be useful later.

**Corollary 5.** Let \( \alpha \) be a curve in the intersection of \( A_a \) and \( A_b \). Then

\[
\frac{\text{Ext}_a(\alpha)}{\ell_a(\alpha)} \lesssim e^{(b-a)} \frac{\text{Ext}_b(\alpha)}{\ell_b(\alpha)}
\]

**Proof.** The length of an arc along a Teichmüller geodesic changes at most exponentially fast. That is, \( e^{b-a} \) is and upper-bound for \( \frac{\ell_a}{e_a}, \frac{f_a}{e_a}, \frac{b_a}{\gamma_a} \) and \( \frac{\ell_a}{e_a} \). Let \( k = \frac{\ell_b}{\ell_a} \). Then

\[
\frac{\ell_b \text{Mod}_b(E^b)}{\ell_a \text{Mod}_a(E^a)} \approx k \frac{\log e_a}{\log e_a} \leq k \frac{\log \left( \frac{e^{b-a}e_a}{k \ell_a} \right)}{\log \left( \frac{e^{b-a}e_a}{k \ell_a} \right)} \leq k \frac{\log e_a}{\log \left( \frac{e^{b-a}e_a}{k \ell_a} \right)} \leq e^{b-a}.
\]

By a similar argument,

\[
\frac{\ell_b \text{Mod}_b(G^b)}{\ell_a \text{Mod}_a(G^a)} \lesssim e^{b-a}
\]

We also have

\[
\frac{\ell_b \text{Mod}_b(F^b)}{\ell_a \text{Mod}_a(F^a)} \lesssim \frac{f_b}{f_a} \leq e^{b-a}.
\]

Then, by Lemma 3 and the estimates above,

\[
\frac{\text{Ext}_a}{\ell_a} \geq \frac{\text{Ext}_b \left( \text{Mod}_b(E^b) + \text{Mod}_b(F^b) + \text{Mod}_b(G^b) \right)}{\ell_b} \lesssim \frac{\ell_a \left( \text{Mod}_a(E^a) + \text{Mod}_a(F^a) + \text{Mod}_a(G^a) \right)}{\ell_a} \lesssim e^{b-a},
\]

which is the desired inequality. \( \square \)
2.3. Twisting. In this section we define several notions of twisting and discuss how they relate to each other. First, consider an annulus $A$ with core curve $\alpha$ and let $\tilde{\beta}$ and $\tilde{\gamma}$ be homotopy classes of arcs connecting the boundaries of $A$ (here, homotopy is relative to the end points of an arc). The relative twisting of $\tilde{\beta}$ and $\tilde{\gamma}$ around $\alpha$, $\text{tw}_{\alpha}(\tilde{\beta}, \tilde{\gamma})$, is defined to be the geometric intersection number between $\tilde{\beta}$ and $\tilde{\gamma}$. If $\alpha$ is a curve on a surface $S$ and $\beta$ and $\gamma$ are two transverse curves to $\alpha$ we lift $\beta$ and $\gamma$ to the annular cover $\tilde{S}_\alpha$ of $S$ corresponding to $\alpha$. The curve $\beta$ (resp., $\gamma$) has at least one lift $\tilde{\beta}$ (resp., $\tilde{\gamma}$) that connects the boundaries of $\tilde{S}_\alpha$. We define $\text{tw}_{\alpha}(\beta, \gamma)$ to be $\text{tw}(\tilde{\beta}, \tilde{\gamma})$. This is well defined up to a small additive error ([Min96, §3]).

When the surface $S$ is equipped with a metric, one can ask how many times does the geodesic representative of $\gamma$ twist around a curve $\alpha$. However, this needs to be made precise. When $x$ is a Riemann surface we define $\text{tw}_{\alpha}(x, \gamma)$ to be equal to $\text{tw}_{\alpha}(\beta, \gamma)$ where $\beta$ is the shortest geodesic in $x$ intersecting $\alpha$. For a quadratic differential $q$, the definition is slightly different. We first consider $F_\alpha$ and let $\beta$ be an arc connecting the boundaries of $F_\alpha$ that is perpendicular to the boundaries. We then define $\text{tw}_{\alpha}(q, \gamma)$ to be the geometric intersection number between $\beta$ and $\gamma|_{F_\alpha}$. These two notions of twisting are related as follows:

**Theorem 6** (Theorem 4.3 in [Raf07]). Let $q$ be a quadratic differential in the conformal class of $x$, and let $\alpha$ and $\gamma$ be two intersecting curves, then

$$\left| \text{tw}_{\alpha}(q, \gamma) - \text{tw}_{\alpha}(x, \gamma) \right| \preceq \frac{1}{\text{Ext}_x(\alpha)}.$$

3. An Estimate for the Extremal Length

In [Min96], Minsky has shown that the extremal length of a curve is comparable to the maximum of the contributions to the extremal length from the pieces of the thick-thin decomposition of the surface. Using this fact and some results in [Raf05b] and [Raf07] we can state a similar result relating the flat length of a curve $\gamma$ to its extremal length.

**Theorem 7.** For a quadratic differential $q$ on a Riemann surface $x$, the corresponding thick-thin decomposition $(A, Y)$ and a curve $\gamma$ on $x$, we have

$$\text{Ext}_x(\gamma) \approx \sum_{Y \in Y} \frac{\ell_q(\gamma|_Y)^2}{\text{diam}_q(Y)^2} + \sum_{\alpha \in A} \left( \frac{1}{\text{Ext}_x(\alpha)} + \text{tw}_{\alpha}(q, \gamma) \text{Ext}_x(\alpha) \right) i(\alpha, \gamma)^2.$$

**Proof.** First we recall [Min96, Theorem 5.1] where Minsky states that the extremal length of a curve $\gamma$ in $x$ is the maximum of the contributions to the extremal length from each thick subsurface and from crossing each short curve. The contribution from each curve $\alpha \in A$ is given by an expression [Min96, Equation (4.3)] involving the $i(\alpha, \gamma)$, $\text{tw}_{\alpha}(x, \gamma)$ and $\text{Ext}_x(\alpha)$. For each subsurface $Y \in Y$, the contribution to the extremal length from $\gamma|_Y$ is shown to be [Min96, Theorem 4.3] the square of the hyperbolic length of $\gamma$ restricted to a representative of $Y$ with a horocycle boundaries of a fixed
length in $x$. This is known to be comparable to the square of the intersection number of $\gamma$ with a short marking $\mu_Y$ for $Y$.

To be more precise, let $\mu_Y$ be a set of curves in $Y$ that fill $Y$ so that $\ell_x(\mu_Y) = O(1)$. Then, Minsky’s estimate can be written as

$$\text{Ext}_x(\gamma) \asymp \sum_{Y \in \mathcal{Y}} i(\gamma, \mu_Y)^2 + \sum_{\alpha \in A} \left( \frac{1}{\text{Ext}_x(\alpha)} + tw_x^2(x, \gamma) \text{Ext}_x(\alpha) \right) i(\alpha, \gamma)^2.$$  \hspace{1cm} (4)

From Theorem 6, \[ |tw_\alpha(x, \gamma) - tw_\alpha(q, \gamma)| = O\left( \frac{1}{\text{Ext}_x(\alpha)} \right), \]

and hence,

$$1 + tw_\alpha(x, \gamma) \text{Ext}_x(\alpha) \asymp 1 + tw_\alpha(q, \gamma) \text{Ext}_x(\alpha).$$

Squaring both sides, and using $(a + b)^2 \asymp a^2 + b^2$, we get

$$1 + tw_\alpha^2(x, \gamma) \text{Ext}_x(\alpha)^2 \asymp 1 + tw_\alpha^2(q, \gamma) \text{Ext}_x(\alpha)^2.$$

We know divide both sides by $\text{Ext}_x(\alpha)$ to obtain

$$\left( \frac{1}{\text{Ext}_x(\alpha)} + tw_\alpha^2(x, \gamma) \text{Ext}_x(\alpha) \right) \asymp \left( \frac{1}{\text{Ext}_x(\alpha)} + tw_\alpha^2(q, \gamma) \text{Ext}_x(\alpha) \right).$$

That is, the second sum in Minsky’s estimate is comparable to the second sum in the statement of our Proposition.

Now consider the inequality in Proposition 2 for every $Y \in \mathcal{Y}$. After taking the square and adding up we get

$$\sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|Y)^2}{\text{diam}_q(Y)^2} + \sum_{\alpha \in A} i(\gamma, \alpha)^2 \asymp \sum_{Y \in \mathcal{Y}} i(\gamma, \mu_Y)^2.$$

But, the term $\sum_{\alpha \in A} i(\gamma, \alpha)^2$ is insignificant compared with the term $\frac{i(\gamma, \alpha)^2}{\text{Ext}_x(\alpha)}$ appearing in right side of Equation (4). Therefore, we can replace the term $\sum_{Y \in \mathcal{Y}} i(\gamma, \mu_Y)^2$ in Equation (4) with $\sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|Y)^2}{\text{diam}_q(Y)^2}$ and obtain the desired inequality. \hfill \Box

To simplify the situation, one can provide a lower bound for extremal length using the $q$–length of $\gamma$ and the sizes of the subsurface $Y_q$, $Y \in \mathcal{Y}$, and $A_\alpha, \alpha \in \mathcal{A}$.

**Corollary 8.** For any curve $\gamma$, the contribution to the extremal length of $\gamma$ from $A_\alpha, \alpha \in \mathcal{A}$, is bounded below by $\frac{\ell_q(\gamma|A_\alpha)^2}{d_q(\alpha)^2}$. In other words,

$$\text{Ext}_x(\gamma) \asymp \sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|Y)^2}{\text{diam}_q(Y)^2} + \sum_{\alpha \in A} \frac{\ell_q(\gamma|A_\alpha)^2}{d_q(\alpha)^2}. $$
Proof. Recall the notations \( E_\alpha, F_\alpha, G_\alpha, e f \) and \( g \) from the background section. Denote the \( q \)-length of \( \alpha \) by \( a \). Every arc of \( \gamma \mid A_\alpha \) has to cross \( A_\alpha \) and twist around \( \alpha \), \( \text{tw}_\alpha(q, \gamma) \)-times. Hence, its length is less than \( d_\alpha(\alpha) + \text{tw}_\alpha(q, \gamma) a \). Therefore,

\[
\ell_q(\gamma \mid A_\alpha)^2 \preceq i(\alpha, \gamma)^2 (d_\alpha(\alpha)^2 + \text{tw}_\alpha^2(q, \gamma) a^2).
\]

Thus

\[
\left( \frac{\ell_q(\gamma \mid A_\alpha)}{d_\alpha(\alpha) i(\alpha, \gamma)} \right)^2 \preceq \frac{d_\alpha(\alpha)^2 + \text{tw}_\alpha^2(q, \gamma) a^2}{d_\alpha(\alpha)^2} \times 1 + \frac{\text{tw}_\alpha^2(q, \gamma)}{d_\alpha(\alpha)^2 / a^2}.
\]

\[
\preceq \frac{1}{\text{Ext}_x(\alpha)} + \frac{\text{tw}_\alpha^2(q, \gamma)}{\log \frac{\xi}{a} \log \frac{\xi}{a} + \log \frac{2}{a}}.
\]

\[
\preceq \frac{1}{\text{Ext}_x(\alpha)} + \text{tw}_\alpha^2(x, \gamma) \text{Ext}_x(\alpha).
\]

We now multiply both sides by \( i^2(\alpha, \gamma) \) and replace the second term of estimate in Theorem 7 to obtain the corollary.

The estimate here seems excessively generous, but there is a case where the two estimates are comparable. This happens when \( \alpha \) is not very short, the twisting parameter is zero and \( \gamma \mid A_\alpha \) is a set of \( i(\gamma, \alpha) \)-many arcs of length comparable to one. \( \square \)

Essentially horizontal curves. Roughly speaking, a curve \( \gamma \) is essentially horizontal if there is an element in the thick-thin decomposition where \( \gamma \) is mostly horizontal, and which contributes a definite portion of the extremal length of \( \gamma \). One can always find a piece of a surface which satisfies the latter, according to the following corollary.

Corollary 9. Let \((\mathcal{A}, \mathcal{Y})\) be a thick-thin decomposition for \( q \) and let \( \gamma \) be a curve that is not in \( \mathcal{A} \). Then

1. For every \( Y \in \mathcal{Y} \)

\[
\text{Ext}_x(\gamma) \succ \frac{\ell_q(\gamma \mid Y)^2}{\text{diam}_q(Y)^2}.
\]

2. For \( \alpha \in \mathcal{A} \) and a flat annulus \( F_\alpha \) whose core curve is \( \alpha \),

\[
\text{Ext}_x(\gamma) \succ \frac{\ell_q(\gamma \mid F_\alpha)^2 \text{Ext}_x(\alpha)}{\ell_q(\alpha)^2}.
\]

3. For \( \alpha \in \mathcal{A} \) and an expanding annulus \( E_\alpha \) whose core curve is \( \alpha \),

\[
\text{Ext}_x(\gamma) \succ i(\alpha, \gamma)^2 \text{Mod}_x(E_\alpha).
\]

Furthermore, at least one of these inequalities is an equality up to a multiplicative error.

Proof. The parts one and three follow immediately from Theorem 7. We prove part two. For \( \alpha \in \mathcal{A} \), let \( a = \ell_q(\alpha) \) and let \( f = f_q(\alpha) \). As before,

\[
\ell_q(\gamma \mid F_\alpha)^2 \preceq (\text{tw}_\alpha(q, \gamma)^2 a^2 + f^2) i(\alpha, \gamma)^2.
\]
Hence
\[
\frac{\ell_q(\gamma|F_\alpha)^2 \text{Ext}_x(\alpha)}{\ell_q(\alpha)^2} \leq \frac{\ell_q(q, \gamma)^2 a^2 + f^2}{a^2} \text{Ext}_x(\alpha) \frac{i(\alpha, \gamma)^2}{\text{Ext}_x(\alpha)} \\
\leq \ell_q(q, \gamma)^2 \text{Ext}_x(\alpha) i(\alpha, \gamma)^2 + \text{Ext}_x(\alpha) \text{Mod}_x(F_\alpha)^2 i(\alpha, \gamma)^2.
\]

But \(\text{Ext}_x(\alpha) \text{Mod}_x(F_\alpha)^2 \leq \frac{1}{\text{Ext}_x(\alpha)}\) and thus, by Theorem 7, the above expression is bounded above by a multiple of \(\text{Ext}_x(\gamma)\).

To see that one of the inequalities have to be an equality, we observe that the number of pieces in the thick-thin decomposition \((A, \mathcal{Y})\) is uniformly bounded. Therefore, some term in Theorem 7 is comparable to \(\text{Ext}_x(\gamma)\). If this is a term in the first sum then the inequality in part one is an equality. Assume for \(\alpha \in A\) that
\[
\text{Ext}_x(\gamma) \geq \frac{i(\alpha, \gamma)^2}{\text{Ext}_x(\alpha)}.
\]

We either have \(\text{Ext}_x(E_\alpha) \asymp \text{Ext}_x(\alpha)\) or \(\text{Ext}_x(F_\alpha) \asymp \text{Ext}_x(\alpha)\). In the first case, the estimate in part three is comparable to \(\text{Ext}_x(\gamma)\). In the second case,
\[
\text{Ext}_x(\gamma) \geq \frac{i(\alpha, \gamma)^2}{\text{Ext}_x(F_\alpha)} \geq \left(\frac{i(\alpha, \gamma)^2 f^2}{a^2}\right) \left(\frac{a}{f}\right) \geq \frac{\ell_q(q|F_\alpha)^2}{\ell_q(\alpha)^2} \text{Ext}_x(\alpha),
\]

which means the inequality in part two is an equality.

The only remaining case is when
\[
\text{Ext}_x(\gamma) \asymp tw_\alpha(q, \gamma)^2 \text{Ext}_x(\alpha) i(\alpha, \gamma)^2.
\]

In this case, the estimate in part two is comparable to \(\text{Ext}_x(\gamma)\). This follows from \(\ell_q(\gamma|F_\alpha) \asymp tw_\alpha(q, \gamma)\ell_q(\alpha) i(\alpha, \gamma)\).

\[\square\]

**Definition 10.** We say that \(\gamma\) is *essentially horizontal*, if at least one of the following holds

1. \(\text{Ext}_x(\gamma) \asymp \frac{\ell_q(\gamma|Y)^2}{\text{diam}_q(Y)^2}\) and \(\gamma|_Y\) is mostly horizontal (i.e., its horizontal length is larger than its vertical length) for some \(Y \in \mathcal{Y}\).

2. \(\text{Ext}_x(\gamma) \asymp \frac{\ell_q(\gamma|F_\alpha)^2 \text{Ext}_x(\alpha)}{\ell_q(\alpha)^2}\) and \(\gamma|_{F_\alpha}\) is mostly horizontal for some flat annulus \(F_\alpha\) whose core curve is \(\alpha \in A\).

3. \(\text{Ext}_x(\gamma) \asymp i(\alpha, \gamma)^2 \text{Mod}_x(E_\alpha)\) for some expanding annulus \(E_\alpha\) whose core curve is \(\alpha \in A\).

**Extremal length of a geodesic arcs.** Consider the \(q\)-geodesic representative of a curve \(\gamma\) and let \(\omega\) be an arc of this geodesic. We would like to estimate the contribution that \(\omega\) makes to the extremal length of \(\gamma\) in \(q\). Let \((A, \mathcal{Y})\) be the thick-thin decomposition of \(q\). Let \(\lambda_\omega\) be the maximum over \(\text{diam}_q(Y)\) for subsurfaces \(Y \in \mathcal{Y}\) that \(\omega\) intersects and over all \(d_q(\alpha)\) for curves \(\alpha \in A\) that \(\omega\) crosses. Let \(\sigma_\omega\) be the \(q\)-length of the shortest curve \(\beta\)
that \( \omega \) intersects. We claim the contribution from \( \omega \) to the extremal length of \( \gamma \) is at least

\[
X(\omega) = \frac{\ell_q(\omega)^2}{\lambda_\omega^2} + \log \frac{\lambda_\omega}{\sigma_\omega}.
\]

This is stated in the following lemma:

**Lemma 11.** Let \( \Omega \) be a set of disjoints sub-arcs of \( \gamma \). Then

\[
\text{Ext}_q(\gamma) \gg |\Omega|^2 \min_{\omega \in \Omega} X(\omega).
\]

**Proof.** Let \((A, \mathcal{Y})\) be the thick-thin decomposition of \( q \). We have

\[
\text{Ext}_x(\gamma) \geq \sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|Y)^2}{\text{diam}_q(Y)^2} + \sum_{\alpha \in A} \frac{\ell_q(\gamma|\alpha_\omega)^2}{d_q(\alpha)^2}
\]

(6)

\[
\gg \left( \sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|Y)}{\text{diam}_q(Y)} + \sum_{\alpha \in A} \frac{\ell_q(\gamma|\alpha_\omega)}{d_q(\alpha)} \right)^2
\]

(7)

\[
\geq \left( \sum_{Y \in \mathcal{Y}, \omega \in \Omega} \frac{\ell_q(\omega|Y)}{\text{diam}_q(Y)} + \sum_{\alpha \in A, \omega \in \Omega} \frac{\ell_q(\omega|\alpha_\omega)}{d_q(\alpha)} \right)^2
\]

(8)

\[
\gg \left( \sum_{\omega \in \Omega} \left( \sum_{Y \in \mathcal{Y}} \frac{\ell_q(\omega|Y)}{\lambda_\omega} + \sum_{\alpha \in A} \frac{\ell_q(\omega|\alpha_\omega)}{\lambda_\omega} \right) \right)^2 \geq \left( \sum_{\omega \in \Omega} \frac{\ell_q(\omega)}{\lambda_\omega} \right)^2.
\]

(9)

Inequality (6) follows from Corollary 8. To obtain (7), we are using

\[
\sum_{i=1}^n x_i^2 \geq \frac{1}{n} \left( \sum_{i=1}^n x_i \right)^2
\]

and the fact that the number of components in \( \mathcal{Y} \) and in \( A \) are uniformly bounded. Line (8) follows from the fact that arcs in \( \Omega \) are disjoint sub-arcs of \( \gamma \). To get (9), we then rearrange terms and use the fact that for all non-zero terms, \( \text{diam}_q(Y) \) and \( d_q(\alpha) \) are less than \( \lambda_\omega \).

Now let \( \alpha_1, \ldots, \alpha_k \) be the sequence of curves in \( A \) that \( \omega \) intersects as it travels from the shortest curve \( \beta \) to the largest subsurface it intersects, which has size of at most \( \lambda_\omega \). Note that either \( \alpha_1 = \beta \) or \( \alpha_1 \) is the boundary of the thick subsurface containing \( \beta_\omega \). Either way, \( \ell_q(\alpha_1) \gg \sigma_\omega \). Also, \( d_q(\alpha_i) \gg \ell_q(\alpha_{i+1}) \). This is because \( \alpha_i \) and \( \alpha_{i+1} \) are boundaries of some subsurface \( Y \in \mathcal{Y} \). Finally, \( d_q(\alpha_k) \gg \lambda_\omega \). Therefore,

\[
\sum_{i=1}^k \log \frac{d_q(\alpha_i)}{\ell_q(\alpha_i)} = \log \prod_{i=1}^k \frac{d_q(\alpha_i)}{\ell_q(\alpha_i)} \gg \log \frac{d_q(\alpha_k)}{\ell_q(\alpha_1)} \gg \log \frac{\lambda_\omega}{\sigma_\omega}.
\]
Since $|A| \asymp 1$, we can conclude that, for each $\omega$, there is curve $\alpha$ so that $i(\alpha, \omega) \geq 1$ and $\log \frac{d_q(\alpha)}{\ell_q(\alpha)} \asymp \log \frac{\lambda_\omega}{\sigma_\omega}$. Using Theorem 7

\[ \text{Ext}_q(\gamma) \asymp \left( \sum_{\alpha \in A} \frac{i(\alpha, \gamma)^2}{\text{Ext}_q(\alpha)} \right)^2 \asymp \left( \sum_{\omega \in \Omega} \sum_{\alpha \in A} i(\alpha, \omega) \sqrt{\log \frac{d_q(\alpha)}{\ell_q(\alpha)}} \right)^2 \asymp \left( \sum_{\omega \in \Omega} \sqrt{\log \frac{\lambda_\omega}{\sigma_\omega}} \right)^2. \]

Combining the above two inequalities, we get

\[ \text{Ext}_q(\gamma) \asymp \left( \sum_{\omega \in \Omega} \frac{\ell_q(\omega)}{\lambda_\omega} \right)^2 + \left( \sum_{\omega \in \Omega} \sqrt{\log \frac{\lambda_\omega}{\sigma_\omega}} \right)^2 \asymp \left( \sum_{\omega \in \Omega} \sqrt{X(\omega)} \right)^2 \geq |\Omega|^2 \min_{\omega} X(\omega). \]

We also need the following technical lemma.

**Lemma 12.** Let $q_a$ and $q_b$ be two points along a Teichmüller geodesic and let $(A_a, Y_a)$ and $(A_b, Y_b)$ be their thick-thin decompositions respectively. Let $Y \in Y_a$,

- If $\beta \in A_b$ intersects $Y$, then $d_b(\beta) \leq e^{(b-a)} \text{diam}_a(Y)$.
- If $Z \in Y_b$ intersects $Y$, then $\text{diam}_b(Z) \leq e^{(b-a)} \text{diam}_a(Y)$.

Similarly, if $\alpha \in A_a$,

- If $\beta \in A_b$ intersects $\alpha$, then $d_b(\beta) \leq e^{(b-a)} \ell_a(\alpha)$.
- If $Z \in Y_b$ intersects $\alpha$, then $\text{diam}_b(Z) \leq e^{(b-a)} \ell_a(\alpha)$.

**Proof.** Let $\gamma$ be the shortest curve system in $q_a$ that fills $Y$. Then $l_b(\gamma) \asymp e^{(b-a)} \text{diam}_a(Y)$. If $Y$ intersects $\beta \in A_b$ then some curve in $\gamma$ has to intersect $A_\beta$ essentially and we have

\[ d_b(\alpha) \leq l_b(\gamma) \asymp e^{(b-a)} \text{diam}_a(Y). \]

If $Y$ intersects some subsurface $Z \in Y_b$, then $Z$ has an essential arc $\omega$ in $Z$ whose $q_b$–length is less than the $q_b$–length of $\gamma$. Also, if $Y$ intersects a boundary component $\delta$ of $Z$,

\[ \ell_b(\gamma) \asymp d_b(\delta) \geq \ell_b(\delta). \]

By doing a surgery between $\omega$ and $\delta$, one obtains an essential curve in $Z$ whose $q_b$–length is less than a fixed multiple of $\gamma$. Hence

\[ \text{diam}_b(Z) \asymp l_b(\gamma) \asymp e^{(b-a)} \text{diam}_a(Y). \]
Which is as we desired. The argument for \( \alpha \in A \) is similar.

4. The main theorem

Let \( G : \mathbb{R} \to \mathcal{T}(S) \) be a Teichmüller geodesic. We denote the Riemann surface \( G(t) \) by \( G_t \) and the corresponding quadratic differential in \( G_t \) by \( q_t \). For a curve \( \gamma \), denote the extremal length of \( \gamma \) on \( G_t \) by \( \text{Ext}_t(\gamma) \) and the thick-thin decomposition of \( q_t \) simply by \((A_t, Y_t)\).

**Theorem 13.** There exists a constant \( K \), such that for every measured foliation \( \mu \), any Teichmüller geodesic \( G \) and points \( x, y, z \in \mathcal{T}(S) \) appearing in that order along \( G \) we have

\[
\text{Ext}_x(\mu) \leq K \max \left( \text{Ext}_y(\mu), \text{Ext}_z(\mu) \right).
\]

**Proof.** Let the times \( a < b < c \in \mathbb{R} \) be such that \( x = G_a \), \( y = G_b \) and \( z = G_c \). Recall that the extremal length

\[
\text{Ext} : \mathcal{MF}(S) \times \mathcal{T}(S) \to \mathbb{R}
\]

is a continuous function, and that the weighted simple closed curves are dense in \( \mathcal{MF}(S) \). Since the limit of quasi-convex functions is itself quasi-convex and a multiple of a quasi-convex function is also quasi-convex (with the same constant in both cases), it is sufficient to prove the theorem for simple closed curves only. That is, we can assume that every leaf of \( \mu \) is homotopic to a curve \( \gamma \) and the transverse measure is one.

If the extremal length of \( \gamma \) is very short at \( b \) but not very short at either \( a \) or \( c \), the statement is clearly true. If \( \gamma \) is short at times \( a, b \) and \( c \), the statement is already known; in [Raf07], it is shown that the interval where \( \gamma \) is “short” is connected [Raf07, Corollary 3.4] and along this interval the extremal length (which comparable with hyperbolic length [Mas85]) is quasi-decreasing until the balanced time and is quasi-increasing afterwards [Raf07, Theorem 1.2]. Therefore, we can assume there is a lower bound on the length of \( \gamma \) at \( b \), where the lower bound depends on the topology of \( x \) only.

By Corollary 9, there is a subsurface of \( q_b \) with significant contribution and the restriction of \( \gamma \) to this subsurface is either mostly horizontal or mostly vertical. That is, \( \gamma \) is either essentially horizontal or essentially vertical. If \( \gamma \) is essentially horizontal, Proposition 14 implies \( \text{Ext}_b(\gamma) \prec \text{Ext}_c(\gamma) \) and we are done. Otherwise, \( \gamma \) is essentially vertical. In this case, we can reverse time, changing the role of the horizontal and vertical foliations, and using Proposition 14 again conclude \( \text{Ext}_b(\gamma) \prec \text{Ext}_a(\gamma) \). This finishes the proof.

**Proposition 14.** If \( \gamma \) is essentially horizontal for the quadratic differential \( q_a \), then for every \( b > a \) we have

\[
\text{Ext}_b(\gamma) \succ \text{Ext}_a(\gamma).
\]

**Proof.** We argue in 3 cases according to which inequality in Corollary 9 is an equality up a multiplicative error.
Case 1. Assume there is a subsurface $Y \in \mathcal{Y}_a$ so that

$$\text{Ext}_a(\gamma) \lesssim \frac{\ell_a(|Y|)^2}{\text{diam}_a(Y)^2}$$

such that $|\gamma|_Y$ is mostly horizontal. We have $\ell_b(|\gamma|) \gtrsim e^{(b-a)}\ell_a(|\gamma|)$. Let $\mathcal{Z}$ be the set of subsurfaces in $\mathcal{Y}_b$ that intersect $Y$ and let $\mathcal{B}$ be a set of annuli $A_\alpha$, where $\alpha \in \mathcal{A}_b$ and $\alpha$ intersects $Y$. Then $Y_b$ is contained in the union of $\bigcup_{Z \in \mathcal{Z}} Z_b$ and $\bigcup_{\beta \in \mathcal{B}} A_b(\beta)$. Therefore,

$$\ell_b(|Y|) \leq \sum_{Z \in \mathcal{Z}} \ell_b(|Z|) + \sum_{\beta \in \mathcal{B}} \ell_b(|A_b(\beta)|).$$

We also know that

$$\text{diam}_b(Z) \leq e^{(b-a)} \text{diam}_a(Y) \quad \text{and} \quad d_b(\beta) \leq e^{(b-a)} \text{diam}_a(Y).$$

Therefore,

$$\text{Ext}_b(\gamma) \gtrsim \frac{\sum_{Z \in \mathcal{Z}} \ell_b(|Z|)^2}{\text{diam}_b(Z)^2} + \frac{\sum_{\beta \in \mathcal{B}} \ell_b(|A_b(\beta)|)^2}{(d_b(\beta))^2}$$

$$\gtrsim \frac{\sum_{Z \in \mathcal{Z}} \ell_b(|Z|)^2 + \sum_{\beta \in \mathcal{B}} \ell_b(|A_b(\beta)|)^2}{e^{2(b-a)} \text{diam}_a(Y)^2}$$

$$\gtrsim \left( \frac{e^{(b-a)} \ell_a(|Y|)}{e^{(b-a)} \text{diam}_a(Y)} \right)^2 \gtrsim \text{Ext}_a(\gamma).$$

Case 2. Assume that there is a curve $\alpha \in \mathcal{A}$ so that

$$\text{Ext}_a(\gamma) \lesssim \frac{\ell_a(|F_\alpha|)^2}{\ell_a(\alpha)^2} \text{Ext}_a(\alpha),$$

and $|\gamma|_{F_\alpha}$ is mostly horizontal. Then $\ell_b(|F_\alpha|) \gtrsim e^{(b-a)}\ell_a(|F_\alpha|)$. If $\alpha$ is still short in $q_b$, then the proposition follows from Corollary [3].

Otherwise, let $\mathcal{Z}$ be the set of sub-surfaces in $\mathcal{Y}_b$ that intersect $\alpha$ and let $\mathcal{B}$ be the set of curves in $\mathcal{A}_b$ that intersect $\alpha$. Since $F_\alpha$ has geodesic boundaries, it is contained in the union of $\bigcup_{Z \in \mathcal{Z}} Z_b$ and $\bigcup_{\beta \in \mathcal{B}} A_b(\beta)$. The rest of the proof is exactly as in the previous case with the additional observation that $\text{Ext}_b(\alpha) \geq \epsilon_1 \geq \text{Ext}_a(\alpha)$.

Case 3. Assume there an expanding annulus $E$ with large modulus and the core curve $\alpha$ such that

$$\text{Ext}_a(\gamma) \lesssim i(\alpha, \gamma)^2 \text{Mod}_a(E)$$

and $|\gamma|_E$ is mostly horizontal. Let $\Omega$ be the set of sub-arcs of $\gamma$ that start and end in $\alpha$ and whose restriction to $E$ is at least $(1/3)$–horizontal (that is, the ratio of the horizontal length to the vertical length is bigger than $1/3$–times). We have

$$|\Omega| \geq (1/4) i(\alpha, \gamma).$$
Otherwise, \((3/4)\) of arc are 3-vertical, which implies that the total vertical length is larger than the total horizontal length. Recall that

\[
\text{Mod}_a(E) \asymp \log \left( \frac{d_a(\alpha)}{\ell_a(\alpha)} \right).
\]

For \(\omega \in \Omega\) we have \(\ell_a(\omega) \geq 2d_a(\alpha)\). Since, the restriction of \(\omega\) to \(E\) is mostly horizontal, we have

\[
\ell_b(\omega) \succ e^{(b-a)d_a(\alpha)}
\]

The arc \(\omega\) intersects \(\alpha\), so \(\sigma_\omega \leq \ell_b(\alpha) \leq e^{(b-a)\ell_a(\alpha)}\). Therefore,

\[
X(\omega) \succ \frac{e^{2(b-a)d_a(\alpha)}}{\lambda_\omega^2} + \log \frac{\lambda_\omega}{e^{(b-a)\ell_a(\alpha)}}
\]

This expression is minimum when \(\lambda_\omega = \sqrt{2}e^{(b-a)d_a(\alpha)}\). That is,

\[
X(\omega) \succ \log \frac{d_a(\alpha)}{\ell_a(\alpha)} \asymp \frac{1}{\text{Ext}_a(\alpha)}
\]

Using Lemma 11 we have

\[
\text{Ext}_b(\gamma) \succ |\Omega|^2 \min_{\omega \in \Omega} X(\omega) \succ \frac{i(\alpha, \gamma)^2}{\text{Ext}_a(\alpha)} \succ \text{Ext}_a(\gamma).
\]

This finishes the proof in this case. \(\square\)

5. QUASI-CONVEXITY OF A BALL IN TEICHMÜLLER SPACE

Consider a Riemann surface \(x\). Let \(B(x, r)\) denote the ball of radius \(r\) in \(T(S)\) centered at \(x\).

**Theorem 15.** There exists a constant \(c\) such that, for ever \(x \in T(S)\), every radius \(r\) and point \(y\) and \(z\) in the ball \(B(x, r)\), the geodesic segment \([y, z]\) connecting \(y\) to \(z\) is contained in \(B(x, r + c)\).

**Proof.** Let \(u\) be a point on the segment \([y, z]\). It is sufficient to show that

\[
d_T(x, u) \leq \max \left( d_T(x, y), d_T(x, z) \right) + c.
\]

There is a measured foliation \(\mu\) such that

\[
d_T(x, u) = \frac{1}{2} \log \frac{\text{Ext}_u(\gamma)}{\text{Ext}_x(\gamma)}.
\]

Also, from the convexity of extremal lengths Theorem 13 we have

\[
\text{Ext}_u(\mu) \leq K \max \left( \text{Ext}_y(\mu), \text{Ext}_z(\mu) \right)
\]

Therefore,

\[
d_T(x, u) \leq \frac{1}{2} \log \left( K \max \left( \frac{\text{Ext}_y(\gamma), \text{Ext}_z(\gamma)}{\text{Ext}_x(\gamma)} \right) \right)
\]

\[
\leq \max \left( d_T(x, y), d_T(x, z) \right) + c. \quad \square
\]
6. **Quasi-convexity of Hyperbolic Length**

In this section, we prove the analogue of Theorem 13 for the hyperbolic length:

**Theorem 16.** There exists a constant $K'$, such that for every measured foliation $\mu$, any Teichmüller geodesic $G$ and times $a < b < c \in \mathbb{R}$, we have

$$\text{Hyp}_b(\mu) \leq K' \max \left( \text{Hyp}_a(\mu), \text{Hyp}_c(\mu) \right)$$

**Proof.** The argument is identical to the one for Theorem 13, with Corollary 18 and Proposition 21 being the key ingredients. They are stated and proved below. \[\square\]

Our main goal for the rest of this section is the Proposition 21. To make the reading easier, we often take note of the similarities and skip some arguments when they are nearly identical to those for the extremal length case.

In place of Theorem 7 we have

**Theorem 17.** For a quadratic differential $q$ on a Riemann surface $x$, the corresponding thick-thin decomposition $(\mathcal{A}, \mathcal{Y})$ and a curve $\gamma$ on $x$, we have

$$\text{Hyp}_x(\gamma) \approx \sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|_Y)}{\text{diam}_q(Y)} + \sum_{\alpha \in \mathcal{A}} \left[ \log \frac{1}{\text{Ext}_x(\alpha)} + \text{tw}_x(q, \gamma) \text{Ext}_x(\alpha) \right] i(\alpha, \gamma).$$

**Proof.** The hyperbolic length of a curve $\gamma$ is, up to a universal multiplicative constant, the sum of the lengths of $\gamma$ restricted to the pieces of the thick-thin decomposition of the surface. The hyperbolic length of $\gamma|_Y$ is comparable to the intersection number of $\gamma$ with a short marking $\mu_Y$ of $Y$, which is, by the Proposition 2, up to a multiplicative error,

$$\frac{\ell_q(\gamma|_Y)}{\text{diam}_q(Y)} + \sum_{\alpha \in \partial Y} i(\gamma, \alpha).$$

The contribution from each curve $\alpha \in \mathcal{A}$ is (see, for example, [CRS06, Corollary 3.2]),

$$\left[ \log \frac{1}{\text{Hyp}_x(\alpha)} + \text{Hyp}_x(\alpha) \text{tw}_x(x, \gamma) \right] i(\alpha, \gamma).$$

Thus, we can write an estimate for the hyperbolic length of $\gamma$ as

$$\text{Hyp}_x(\gamma) \approx \sum_{Y \in \mathcal{Y}} \frac{\ell_q(\gamma|_Y)}{\text{diam}_q(Y)} + \sum_{\alpha \in \mathcal{A}} \left[ \log \frac{1}{\text{Hyp}_x(\alpha)} + \text{Hyp}_x(\alpha) \text{tw}_x(x, \gamma) \right] i(\alpha, \gamma).$$

Note that we are not adding 1 to the sum in the parenthesis above since the sum is actually substantially greater.
To finish the proof, we need to replace $\text{Hyp}_x(\alpha)$ with $\text{Ext}_x(\alpha)$ and $\text{tw}_\alpha(x, \gamma)$ with $\text{tw}_\alpha(q, \gamma)$. Maskit has shown [Mas85] that, when $\text{Hyp}_x(\alpha)$ is small, 
\[
\frac{\text{Hyp}_x(\alpha)}{\text{Ext}_x(\alpha)} \approx 1.
\]
Hence, we can replace $\text{Hyp}_x(\alpha)$ with $\text{Ext}_x(\alpha)$. Further, it follows from Theorem 6 that
\[
|\text{tw}_\alpha(q, \gamma) \text{Ext}_x(\alpha) - \text{tw}_\alpha(x, \gamma) \text{Ext}_x(\alpha)| = O(1).
\]
Since $\log \frac{1}{\text{Ext}_x(\alpha)}$ is at least 1 for $\alpha \in A$, we have
\[
\log \frac{1}{\text{Ext}_x(\alpha)} + \text{tw}_\alpha(q, \gamma) \text{Ext}_x(\alpha) \approx \log \frac{1}{\text{Ext}_x(\alpha)} + \text{tw}_\alpha(x, \gamma) \text{Ext}_x(\alpha),
\]
which means that we can replace $\text{tw}_\alpha(x, \gamma)$ with $\text{tw}_\alpha(q, \gamma)$. \hfill \Box

We almost immediately have:

**Corollary 18.** Let $(A, Y)$ be a thick-thin decomposition for $q$ and let $\gamma$ be a curve that is not in $A$. Then

1. For every $Y \in Y$
   \[
   \text{Hyp}_x(\gamma) \approx \frac{L_q(\gamma|Y)}{\text{diam}_q(Y)}.
   \]
2. For every $\alpha \in A$ and the flat annulus $F_\alpha$ whose core curve is $\alpha$,
   \[
   \text{Hyp}_x(\gamma) \approx \log \text{Mod}_x(F_\alpha) i(\alpha, \gamma)
   \]
3. For every $\alpha \in A$
   \[
   \text{Hyp}_x(\gamma) \approx \text{tw}_\alpha(q, \gamma) \text{Ext}_x(\alpha) i(\alpha, \gamma).
   \]
4. For every $\alpha \in A$ and an expanding annulus $E_\alpha$ whose core curve is $\alpha$,
   \[
   \text{Hyp}_x(\gamma) \approx \log \text{Mod}_x(E_\alpha) i(\alpha, \gamma).
   \]

Furthermore, at least one of these inequalities is an equality up to a multiplicative error.

**Proof.** The parts (1) – (4) follow immediately from Theorem 17 and the fact that the reciprocal of the extremal length of a curve $\alpha$ is bounded below by the modulus of any annulus homotopic to $\alpha$. Further, since the number of pieces in the thick-thin decomposition $(A, Y)$ is uniformly bounded, some term in Theorem 17 has to be comparable with $\text{Hyp}_x(\gamma)$. The only non-trivial case is when that term is $\log \frac{1}{\text{Ext}_x(\alpha)} i(\alpha, \gamma)$ for some $\alpha \in A$. But by Lemma 3 either
\[
\frac{1}{\text{Ext}_x(\alpha)} \approx \text{Mod}_x(F_\alpha),
\]
or
\[
\frac{1}{\text{Ext}_x(\alpha)} \approx \text{Mod}_x(E_\alpha),
\]
and the Lemma holds. \hfill \Box
As in the section §3, we need a notion of *essentially horizontal* for hyperbolic length. We say that \( \gamma \) is *essentially horizontal*, if at least one of the following holds

1. \( \text{Hyp}_x(\gamma) \asymp \frac{\ell_q(\gamma|_{Y})}{\text{diam}_q(Y)} \) and \( \gamma|_{Y} \) is mostly horizontal (i.e., its horizontal length is larger than its vertical length) for some \( Y \in \mathcal{Y} \).

2. \( \text{Hyp}_x(\gamma) \asymp \log \text{Mod}_x(F_\alpha) i(\alpha, \gamma) \) and \( \gamma|_{F_\alpha} \) is mostly horizontal for some flat annulus \( F_\alpha \) whose core curve is \( \alpha \in \mathcal{A} \).

3. \( \text{Hyp}_x(\gamma) \asymp t_{\omega}(q, \gamma) \text{Ext}_x(\alpha) i(\alpha, \gamma) \) and \( \gamma|_{F_\alpha} \) is mostly horizontal for some flat annulus \( F_\alpha \) whose core curve is \( \alpha \in \mathcal{A} \).

4. \( \text{Hyp}_x(\gamma) \asymp \log \text{Mod}_x(E_\alpha) i(\alpha, \gamma) \) for some expanding annulus \( E_\alpha \) whose core curve is \( \alpha \in \mathcal{A} \).

Further, Corollary 8 is replaced with

**Corollary 19.** For any curve \( \gamma \), the contribution to the hyperbolic length of \( \gamma \) from \( A_\alpha, \alpha \in \mathcal{A} \), is bounded below by \( \ell_{q}(\gamma|_{A_\alpha}) \). In other words,

\[
\text{Hyp}_x(\gamma) \asymp \ell_q(\gamma|_{A_\alpha}) + \log \text{diam}_q(\alpha) + \sum_{\alpha \in \mathcal{A}} \ell_q(\gamma|_{A_\alpha}).
\]

**Proof.** Identical to the proof of Corollary 8 after removing the squares and taking log when necessary. \( \square \)

Instead of the function \( X(\omega) \), to estimate the hyperbolic length of an arc, we define

\[
H(\omega) = \frac{\ell_q(\omega)}{\lambda_\omega} + \log \max \left\{ \log \frac{\lambda_\omega}{\sigma_\omega}, 1 \right\}.
\]

In place of Lemma 11 we get

**Lemma 20.** Let \( \Omega \) be a set of disjoints sub-arcs of \( \gamma \). Then

\[
\text{Hyp}_x(\gamma) \asymp \min_{\omega \in \Omega} H(\omega).
\]

**Proof.** Identical to the proof of Lemma 11 after removing the squares and taking log when necessary. \( \square \)

Finally, we have the analog of Proposition 14.

**Proposition 21.** If \( \gamma \) is essentially horizontal for the quadratic differential \( q_\alpha \), then for every \( b > a \) we have

\[
\text{Hyp}_b(\gamma) \asymp \text{Hyp}_a(\gamma).
\]

**Proof.** By the definition of essentially horizontal, there are four cases to consider. We deal with two of them, the flat annulus case and the twisting case, at once in Case 2.
Case 1. There is a thick subsurface $Y$ where $\gamma$ is mostly horizontal and such that
\[ \text{Hyp}_a(\gamma) \gtrsim \frac{\ell_a(\gamma|Y)}{\text{diam}_a(Y)} \]
The proof is as in the extremal length case after removing the squares.

Case 2. There exists a curve $\alpha \in A$ so that
\[ (11) \quad \text{Hyp}_a(\gamma) \gtrsim \log \text{Mod}_a(F_\alpha) i(\alpha, \gamma), \]
or
\[ (12) \quad \text{Hyp}_a(\gamma) \gtrsim \text{tw}_a(\alpha, \gamma) \text{Ext}_a(\alpha) i(\alpha, \gamma), \]
and $\gamma|_{F_\alpha}$ is mostly horizontal. We argue in three sub-cases.

Case 2.1. Suppose first that $\alpha$ is no longer short at $t = b$ and either (11) or (12) holds. Let $Z$ be the set of sub-surfaces in $Y_b$ that intersect $\alpha$ and let $B$ be the set of curves in $A_b$ that intersect $\alpha$. Then, by Corollary 19 and Lemma 12,
\[ \text{Hyp}_b(\gamma) \gtrsim \sum_{Z \in Z} \frac{\ell_b(\gamma|Z)}{\text{diam}_b(Z)} + \sum_{\beta \in B} \frac{\ell_b(\gamma|A_\beta)}{d_b(\beta)} \]
\[ \gtrsim \sum_{Z \in Z} \frac{\ell_b(\gamma|Z)}{e^{b-a} \ell_a(\alpha)} + \sum_{\beta \in B} \frac{\ell_b(\gamma|A_\beta)}{e^{b-a} \ell_a(\alpha)} \]
But $F_\alpha$ is contained in $(\bigcup_{Z \in Z} Z) \cup \left( \bigcup_{\beta \in B} A_\beta \right)$.
\[ \gtrsim \frac{\ell_b(\gamma|F_\alpha)}{e^{b-a} \ell_a(\alpha)} \gtrsim \frac{\ell_a(\gamma|F_\alpha)}{\ell_a(\alpha)} \]
\[ \gtrsim \max \{ \log \text{Mod}_a(F_\alpha) i(\alpha, \gamma), \text{tw}_a(\alpha, \gamma) \text{Ext}_a(\alpha) i(\alpha, \gamma) \} \]
\[ \overset{\sim}{\gtrsim} \text{Hyp}_a(\gamma). \]

Case 2.2. Suppose now that $\alpha \in A_b$ and that (11) holds. If $\alpha$ is mostly vertical at time $a$, the extremal length of $\alpha$ is decreasing exponentially fast for some interval $[a, c]$. That is, $\text{Mod}_c(F_\alpha) \overset{\sim}{\gtrsim} \text{Mod}_a(F_\alpha)$. It is sufficient to show that for $b \geq c$,
\[ \text{Hyp}_b(\gamma) \overset{\sim}{\gtrsim} \log \text{Mod}_c(F_\alpha) i(\alpha, \gamma). \]

Our plan is to argue that, while the modulus of $F_\alpha$ is decreasing, the hyperbolic length of $\gamma$ is not decreasing by much because the curve is twisting very fast around $\alpha$. We need to estimate the twisting of $\gamma$ around $\alpha$. Let $\omega$ be one of the arcs of $\gamma|_{F_\alpha}$. Note that $\omega$ is mostly horizontal at $c$ (since it was at $a$) and its length is larger than $f_c(\alpha)$. Also, since $\alpha$ is mostly horizontal at $c$, $f_t(\alpha)$ is decreasing exponentially fast at $t = c$. Hence, after replacing $c$ with a slightly larger constant, we can assume $\omega$ is significantly larger than $f_a(\alpha)$ and therefore, the number of times $\omega$ twists around $\alpha$ is approximately the length ratio of $\omega$ and $\alpha$ (see Equation 15 and 16 in
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and the related discussion for more details). That is, for \( c \leq t \leq b \),
\[
\text{tw}_a(q_t, \gamma) \text{ is essentially constant:}
\]
\[
\text{tw}_a(q_t, \gamma) \approx \frac{\ell_t(\omega)}{\ell_t(\alpha)} \cdot e^{(t-a)^{\alpha}_a(\omega)} = \frac{\ell_a(\omega)}{\ell_a(\alpha)}.
\]
Therefore,
\[
\text{Mod}_c(F_\alpha) = f_c(\alpha, \gamma) \leq \ell_t(\omega) \approx \text{tw}_a(q_c, \gamma).
\]
Keeping in mind that, for \( k \geq 0 \), the function \( f(x) = -\log x + kx > \log k \),
we have
\[
\text{Hyp}_b(\gamma) \gtrsim \left[ \log \frac{1}{\text{Ext}_b(\alpha)} + \text{tw}_a(b, \gamma) \text{Ext}_b(\alpha) \right] i(\alpha, \gamma)
\]
\[
\gtrsim \log \left( \text{tw}_a(q_b, \gamma) \right) i(\alpha, \gamma) \gtrsim \log \text{Mod}_c(F_a) i(\alpha, \gamma).
\]

**Case 2.3.** Suppose that \( \alpha \in A_b \) and that (12) holds. Since \( \gamma \) crosses \( \alpha \),
\[
\text{Hyp}_a(\gamma) \gtrsim \log \text{Mod}_a(E_\alpha) i(\alpha, \gamma).
\]
implies that \( \text{tw}_a(a, \gamma) \) is much larger than \( \text{Mod}_a(F_\alpha) \). That is, the angle
between \( \gamma \) and \( \alpha \) is small. Therefore, after perhaps replacing \( a \) with a
slightly larger number, we can assume that \( \alpha \) is mostly horizontal and that,
for \( a \leq t \leq b \),
\[
\text{tw}_a(t, \gamma) \approx \frac{\ell_t(\omega)}{\ell_t(\alpha)}.
\]
Applying Theorem 17, Equation (13), Corollary 5, Equation (12) in that
order we obtain:
\[
\text{Hyp}_b(\gamma) \gtrsim \left[ \text{tw}_a(b, \gamma) \text{Ext}_b(\alpha) \right] i(\alpha, \gamma)
\]
\[
\gtrsim \text{Ext}_b(\alpha) \ell_b(\omega) i(\alpha, \gamma)
\]
\[
\gtrsim \frac{e^{b-a}_a}{\ell_a(\alpha)} \ell_a(\omega) i(\alpha, \gamma)
\]
\[
\gtrsim \left[ \text{tw}_a(a, \gamma) \text{Ext}_a(\alpha) \right] i(\alpha, \gamma).
\]

**Case 3.** There is a curve \( \alpha \in A \) with expanding annulus \( E_\alpha \) such that \( \gamma|_{E_\alpha} \)
is mostly horizontal with
\[
\text{Hyp}_a(\gamma) \gtrsim \log \text{Mod}_a(E_\alpha) i(\alpha, \gamma)
\]
Following the proof for the corresponding case for extremal length, we have
\[
\text{H}(\omega) \gtrsim \frac{e^{b-a}_a d_a(\alpha)}{\lambda_\omega} + \log \max \left\{ \log \frac{\lambda_\omega}{e^{b-a}_a \ell_a(\alpha)}, 1 \right\}
\]
\[
\gtrsim \log \log \frac{d_a(\alpha)}{\ell_a(\alpha)} \approx \log \text{Mod}_a(E_\alpha).
One can verify the second inequality as follows. If \( \frac{e^{b-a}d_a(\alpha)}{\lambda_\omega} \leq \sqrt{d_a(\alpha)\ell_a(\alpha)} \), then
\[
\frac{e^{b-a}d_a(\alpha)}{\lambda_\omega} \geq \sqrt{d_a(\alpha)\ell_a(\alpha)} \sim \log \log \frac{d_a(\alpha)}{\ell_a(\alpha)}.
\]
Otherwise,
\[
\log \log \frac{\lambda_\omega}{e^{b-a}\ell_a(\alpha)} \geq \log \log \sqrt{d_a(\alpha)\ell_a(\alpha)} \sim \log \log \frac{d_a(\alpha)}{\ell_a(\alpha)}.
\]
and applying Lemma 20, we have
\[
\text{Hyp}_a(\gamma) \sim |\Omega| \min_{\omega \in \Omega} H(\omega) \sim i(\alpha, \gamma) \log \text{Mod}_a(E_\alpha) \sim \text{Hyp}_a(\gamma).
\]
This finishes the proof. \(\square\)

7. Examples

This section contains two examples. In the first example we describe a Teichmüller geodesic and a curve whose length is not convex along this geodesic. The second example is of a very long geodesic that spends its entire length near the boundary of a round ball.

Example 22 (Extremal length and hyperbolic length are not convex). To prove that the extremal and the hyperbolic lengths are not convex, we construct a quadratic differential and analyze these two lengths for a specific curve along the geodesic associated to this quadratic differential. We show that on some interval the average slope (in both cases) is some positive number and on some later interval the average slope is near zero. This shows that the two length functions are not convex. Note that, scaling the weight of a curve by a factor \( k \) increases the hyperbolic and the extremal length of that curve by factors of \( k \) and \( k^2 \), respectively. Thus, after scaling, one can produce examples where the average slope is very large on some interval and near zero on some later interval.

Let \( 0 < a \ll 1 \). Let \( T \) be rectangular torus obtained from identifying the opposite sides of the rectangle \( [0, a] \times [0, \frac{1}{a}] \). Also, let \( C \) be a euclidean cylinder obtained by identifying vertical sides of \( [0, a] \times [0, a] \). Take two copies \( T_1 \) and \( T_2 \) of \( T \), each cut along a horizontal segment of length \( a/2 \) (call it a slit), and join them by gluing \( C \) to the slits. This defines a quadratic differential \( q \) on a genus two surface \( x_0 \). The horizontal and the vertical trajectories of \( q \) are those obtained from the horizontal and the vertical foliation of \( \mathbb{R}^2 \) by lines parallel to the \( x \)-axis and \( y \)-axis respectively. We now consider the Teichmüller geodesic based at \( x_0 \) in the direction of \( q \). Let \( \alpha \) be a core curve of cylinder \( C \). We will show that, for small enough \( a \), \( \text{Ext}_{x_0}(\alpha) \) and \( \text{Hyp}_{x_0}(\alpha) \) are not convex along \( x_t \).

Let \( \rho \) be the metric which coincides with the flat metric of \( q \) on \( C \) and on the two horizontal bands in \( T_i \) of width and height \( a \) with the slit in the
middle, and is 0 otherwise. The shortest curve in the homotopy class of \( \alpha \) has length \( a \) in this metric. Then we have

\[
\text{Ext}_{x_0}(\alpha) \geq \frac{a^2}{3a^2} = \frac{1}{3}.
\]

Also, at time \( t < 0 \), we have \( \text{Mod}_{x_t}(C) = \frac{ae^{-t}}{ae^t} = e^{-2t} \) and, therefore,

\[
\text{Ext}_{x_t}(\alpha) \leq e^{2t}.
\]

Hence we see that the extremal length of \( \alpha \) grows exponentially on \( (-\infty, 0) \).

In particular, the average slope on the interval \( J = (-2, 0) \) is more than \( \frac{1}{8} \).

By Proposition 1 and Corollary 3 in [Mas85],

\[
2e^{-\frac{1}{2}} \text{Hyp}_{x_t}(\alpha) \leq \frac{\text{Hyp}_{x_t}(\alpha)}{\text{Ext}_{x_t}(\alpha)} \leq \pi.
\]

and it is easy to see that the slope of \( \text{Hyp}_{x_t}(\alpha) \) on this interval is also greater than \( \frac{1}{8} \).

![Figure 2. Metric \( \rho_t \) on \( x_t \) when \( t > 0 \).](image)

Further along the ray, when \( t > 0 \), the modulus of \( C \) is decreasing exponentially. We estimate \( \text{Ext}_{x_t}(\alpha) \) for \( t \in I = (0, \frac{1}{2} \log \frac{1}{a^2}) \). For the lower bound, consider the cylinder \( A \) which is the union of \( C \) and the maximal annuli in \( T_i \) whose boundary is a round circle centered at the middle of the slit. Then \( A \) contains two disjoint copies of annuli of inner radius \( (ae^t/4) \)
and outer radius \((ae^t/2)\) (the condition on \(t\) guarantees that these annuli do not touch the top or the bottom edges of \(T_1\) and \(T_2\).) Both of these annuli have modulus of \(\frac{1}{2\pi} \log 2\), and therefore \(\text{Mod}_{x_t}(A) \geq \frac{1}{\pi} \log 2\). Hence

\[
\text{Ext}_{x_t}(\alpha) \leq \frac{1}{\text{Mod}_{x_t}(A)} \leq \frac{\pi}{\log 2}.
\]

For the upper bound, we use the metric \(\rho_t\) defined as follows (see Fig. 2): Let \(A_i\) be the annuli in \(T_i\) centered at the midpoints of the corresponding slits with inner radius \(\frac{ae^t+\delta}{2}\) and outer radius \(\frac{ae^t-\delta}{2}\) for a very small \(\delta\). Let \(\rho_t\) be \(\frac{1}{|z|} |dz|\) on \(A_i\), and the flat metric \(|dz|\) on \(C\) scaled so that the circumference is \(2\pi\). The complement of \(A_i\) and \(C\) consists of two annuli \(B_1, B_2\) and two once-holed tori \(C_1\) and \(C_2\) with \(B_i, C_i \in T_i\). On each of these components, we will define \(\rho_t\) so that the shortest representative of \(\alpha\) has length at least \(2\pi\) and the area is bounded above. More precisely, let \(\rho_t = \frac{2\pi}{ae^t}|dz|\) on \(B_i\). On \(C_i\), let \(\rho_t\) be \(\frac{2}{ae^{t-\delta}}|dz|\) if \(|\text{Im } z| < \frac{1}{2} (\pi + 1)(ae^t - \delta)\), and zero otherwise.

The area of \(C\) in this metric is \((2\pi ae^t)(2\pi ae^{-t}) = O(1)\). The pieces \(B_i\) and \(C_i\) have diameters of order \(O(ae^t)\) in \(\rho_t\) and hence have area of order \(O(1)\).

The annuli \(A_i\) in this metric are isometric to flat cylinders of circumference \(2\pi\) and width less than \(\log 2\), which also has area one. Thus,

\[
\text{Area}_{\rho_t}(\mathcal{S}) = O(1).
\]

Also, the \(\rho_t\)-length of any curve \(\alpha'\) homotopic to \(\alpha\) is \(\ell_{\rho_t}(\alpha') \geq 2\pi\). Indeed, any curve contained in one of the annuli, has \(\rho_t\)-length at least \(2\pi\). Morever, any subarc of \(\alpha'\) with endpoints on a boundary component of an annulus can be homotoped relative to the endpoints to the boundary without increasing the length.

Since the area of \(\rho_t\) is uniformly bounded above (independent of \(a\) and \(t\)) and the length of \(\alpha\) in \(\rho_t\) is larger \(2\pi\), the extremal length

\[
\text{Ext}_{x_t}(\alpha) \geq \frac{\inf \ell_{\rho_t}(\alpha')^2}{\text{Area}_{\rho_t}(\mathcal{S})},
\]

is bounded below on \(I\) by a constant independent of \(t\) and \(a\). Combining this with (18) we see that, as \(a \to 0\) (and hence the size of \(I\) goes to \(\infty\)), the average slope of \(\text{Ext}_{x_t}(\alpha)\) on \(I\) is near zero. In particular, the average slope on \(I\) can be made smaller than \(\frac{1}{2}\) which implies that the function \(\text{Ext}_{x_t}(\alpha)\) is not convex. Combining (16) and the estimates of the extremal length above, we come to the same conclusion about \(\text{Hyp}_{x_t}(\alpha)\).

**Example 23** (Geodesics near the boundary). Here we describe how, for any \(R > 0\), a geodesic segment of length comparable to \(R\) can stay near the boundary of a ball of radius \(R\). This example suggests that metric balls in \(\mathcal{T}(S)\) might not be convex.

Let \(x\) be a point in the thick part of \(\mathcal{T}(S)\) and \(\mu_x\) be the short marking of \(x\). Pick any two disjoint curves \(\alpha, \beta\) in \(\mu_x\). Let \(y = \mathcal{D}^n_{(\alpha)} x\), and \(z = \mathcal{D}^n_{(\alpha,\beta)} x\),
where $D_\ast$ is the Dehn twist around a multicurve $(\ast)$. The intersection numbers between the short markings of $x, y, z$ satisfy

$$i(\mu_x, \mu_y) \asymp i(\mu_x, \mu_z) \asymp i(\mu_y, \mu_z) \asymp n.$$  

Hence, by Theorem 2.2 in [CR05], we have

$$d_T(x, y) \asymp d_T(x, z) \asymp d_T(y, z) \asymp \log n.$$  

That is, $[y, z]$ is a segment of length $\log n$ whose end points are near the boundary of the ball $B(x, \log n)$. We will show, for $w \in [y, z]$, that $d_T(x, w) \asymp \log n$, which means the entire geodesic $[y, z]$ stays near the boundary of the ball $B(x, \log n)$. Let $\alpha'$ be a curve that intersects $\alpha$, is disjoint from $\beta$, and $\text{Ext}_y(\alpha') = O(1)$. Since $\alpha'$ intersect $\alpha$,

$$\text{Ext}_x(\alpha') \asymp n^2,$$

and since $\alpha'$ is disjoint from $\beta$,

$$\text{Ext}_z(\alpha') = O(1).$$  

By Theorem 13,

$$\text{Ext}_w(\alpha') \leq K \max \{ \text{Ext}_y(\alpha'), \text{Ext}_z(\alpha) \} = O(1).$$  

We now have

$$d_T(S)(w, x) \geq \frac{1}{2} \log \frac{\text{Ext}_x(\alpha')}{\text{Ext}_w(\alpha')} \asymp \log n.$$  

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