Are the String and Einstein Frames Equivalent?

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Abstract

The low energy physics as predicted by strings can be expressed in two (conformally related) different variables, usually called frames. The problem is raised as to whether it is physically possible in some situations to tell one from the other.
1 Introduction

In the low energy (that is, wavelengths much longer than the string scale, \( l_s \equiv (\alpha')^{1/2} \)) limit, all string theories predict at least a common sector of massless fields (in any dimension): one associated with the graviton \( g_{\mu\nu} \), another with a scalar field \( \phi \), called the dilaton\(^ 2 \), which is related to the string coupling constant, \( g_s \) in the sense that a constant variation of the dilaton, \( \delta \phi \) produces a corresponding variation in the coupling constant \( \frac{\delta g_s}{g_s} = \delta \phi \), and a two-index antisymmetric field, \( b_{\mu\nu} \), called the Kalb-Ramond field, which is often associated to axions in four dimensional compactifications.

The coupling of these fields to the embeddings of the two-dimensional world sheet of the string, \( \Sigma \) in the spacetime \( M, x^\mu(\sigma, \tau) \) is given by the two-dimensional nonlinear sigma model which in the conformal gauge reads:

\[
S_2 = \frac{1}{4\pi l_s^2} \int d^2 \sigma \left[ g_{\mu\nu} \partial_a x^\mu \partial_b x^\nu \eta^{ab} + i b_{\mu\nu} \partial_a x^\mu \partial_b x^\nu \epsilon^{ab} + l_s^2 R^{(2)}(x) \right]
\]

(1)

(where \( \sigma^a \equiv (\sigma^1, \sigma^2) \equiv (\sigma, \tau) \), \( \eta^{ab} \) represents the two-dimensional flat metric, and \( \epsilon^{ab} \) stands for the two-dimensional Levi-Civita symbol, whereas \( R^{(2)} \) stands for the two-dimensional scalar curvature).

In order for self-consistency, (i.e., two-dimensional conformal invariance, which entails the vanishing of the corresponding \( \beta \)-functions) these backgrounds have to obey some equations which, to the lowest order in \( l_s^2 \) can be derived from a \( n \)-dimensional action principle:

\[
S_n \equiv \int d(vol)_n \frac{1}{2\kappa_n^2} e^{-2\phi} \left[ R - \frac{n-26}{3l_s^2} - \frac{1}{12} H_{abc} H^{abc} + 4(\nabla \phi)^2 \right]
\]

(2)

(where \( H \equiv dB \) is the field strength of the Kalb-Ramond field, i.e., \( H_{abc} = \partial_{[a} b_{bc]} \), and \( d(vol)_n \equiv \sqrt{g} d^n x \), whereas \( \kappa_n \) stands for the \( n \)-dimensional Planck's constant).

The mixing between the dilaton and the graviton in the kinetic term can be avoided\(^ 2 \) Which we shall choose as dimensionless.
with a field redefinition consistent in a Weyl transformation:

$$g_{\mu\nu}^E \equiv e^{\frac{4(\phi_0 - \phi)}{n-2}} g_{\mu\nu}$$

(3)

(where $\phi_0 \equiv \langle \phi \rangle$ is the unknown dilaton vacuum expectation value, which is different in principle from its asymptotic value at infinity, $\phi_\infty$) leading to the spacetime effective action written in what is usually called the Einstein Frame (while we say that the former action (2) was written in the String Frame).

$$S^E_n \equiv \int d(vol)_n \frac{1}{2\kappa_n^2} \left[ R - \frac{n - 26}{3l_s^2} e^{\frac{4\phi}{n-2}} - \frac{1}{12} e^{\frac{8\phi}{n-2}} H_{abc} H^{abc} - \frac{4}{n - 2} (\nabla \phi_E)^2 \right]$$

(4)

and we have defined $\phi_E \equiv \phi - \phi_0$, in such a way that $\langle \phi_E \rangle = 0$.

There are other frames associated with topological defects, like D-branes, which differ in the power of the exponential of the dilaton (because they appear at different order in the string coupling constant $g_s$). They can be easily included in the framework of our discussion, however.

Although it is not known what is the general form of the higher order (sigma model) corrections, it has been conjectured in [6] that it would have the general form in the string frame:

$$S_n = \int d(vol)_n \left[ \frac{B_g(\phi)}{l_s^{n-2}} R + \frac{B_\phi(\phi)}{l_s^{n-2}} (4\nabla^2 \phi - 4(\nabla \phi)^2) - \frac{B_f(\phi) k}{l_s^{n-4}} \frac{k}{4} tr F^2 - B_\psi(\phi) \bar{\psi} \nabla \psi - B_m(\phi) m \bar{\psi} \psi \right]$$

(5)

The dilaton dressing functions $B_i(\phi)$ are unknown, and we have indicated some of the gauge fields and fermion fields present, as well as a typical mass term, which will be needed in further considerations. The Kac-Moody level $k$ is an numerical constant. The term involving the dilaton kinetic energy can be rewritten upon partial integration as:

$$- \frac{4}{l_s^{n-2}} (B_\phi'(\phi) + B_\phi(\phi)) (\nabla \phi)^2$$

(6)

3This form is in any case general enough to include all intermediate frames associated to different branes

4Which is only valid when the dilaton field vanishes at infinity fast enough
The Einstein frame will now be defined through the Weyl transformation:

$$g_{\mu\nu} \equiv \frac{(l_s m_p)^2}{B_g^{2/(n-2)}} g_{\mu\nu}^E$$

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The equations of motion for the metric and the dilaton in the String Frame read:

\[ \frac{\delta S}{\delta g_{\mu\nu}} = \frac{B_g}{l_s^{n-2}} R_{\mu\nu} - \frac{4}{l_s^{n-2}} (B_\phi + B_\phi') \nabla_\mu \phi \nabla_\nu \phi - \frac{B_f}{l_s^{n-4}} \frac{k}{2} tr F_{\mu\alpha} F^\alpha_{\nu} - B_\psi \bar{\psi} \gamma_\mu \nabla_\nu \psi \]
\[ - \frac{1}{2} g_{\mu\nu} \left( \frac{B_g}{l_s^{n-2}} R - 4 \frac{(B_\phi + B_\phi')}{l_s^{n-2}} (\nabla \phi)^2 - \frac{B_f}{l_s^{n-4}} \frac{k}{4} tr F^2 - B_\psi \bar{\psi} \nabla \psi - m B_m \bar{\psi} \psi \right) = 0 \]  

(11)

and

\[ \frac{\delta S}{\delta \phi} = \frac{B'_\phi}{l_s^{n-2}} R + \frac{4}{l_s^{n-2}} (B''_\phi + B'_\phi)(\nabla \phi)^2 - \frac{k B'_f}{4 l_s^{n-4}} tr F^2 \]
\[ - B'_\psi \bar{\psi} \nabla \psi + \frac{8}{l_s^{n-2}} (B'_\phi + B_\phi) \nabla^2 \phi - m B'_m \bar{\psi} \psi = 0. \]  

(12)

The equivalence of the equations of motion in the two frames has been worked out in detail in [5]. Let us now however discuss in turn some delicate issues.

## 2 Dualities

There is now a certain amount of evidence for different kinds of symmetries between different string theories (See for example [11]). The two more important ones are S-duality and T-duality. We shall say that two (not necessarily different) theories, \( T_1 \) and \( T_2 \) are T-dual, when \( T_1 \) compactified at large Kaluza-Klein volume is physically equivalent to \( T_2 \) at small Kaluza-Klein volume. If we call \( t \) the modulus associated to global variations of the Kaluza-Klein volume, by \( Vol \sim e^t \), this implies a relationship of the general form

\[ t(1) = -t(2). \]  

(13)

This symmetry can be proven true when there is an isometry in the spacetime manifold whose Killing vector is written in adapted coordinates as \( \frac{\partial}{\partial x^0} \) by several means (11,10) and is such that the string metric transforms in a simple way, to wit:

\[ \bar{g}_{00} = \frac{1}{g_{00}}, \]
\[ \bar{g}_{0i} = \frac{b_{0i}}{g_{00}}, \]
\[ \bar{b}_{0i} = \frac{g_{0i}}{g_{00}}. \]
\[
\begin{align*}
\tilde{g}_{ij} &= g_{ij} - \frac{g_{0i}g_{0j} - b_{0i}b_{0j}}{g_{00}}, \\
\tilde{b}_{ij} &= b_{ij} - \frac{g_{0i}b_{0j} - g_{0j}b_{0i}}{g_{00}},
\end{align*}
\] (14)

S-duality, on the other hand, refers to the equivalence of \(T_1\) at small coupling with \(T_2\) at large coupling. Given the relationship we already mentioned between the dilaton and the string coupling, it demands for the dilaton something like

\[
\phi(1) = -\phi(2),
\] (15)

and, by definition, lies beyond the possibilities of verification by means of perturbation theory. For IIB strings, the (modified, cf. ref. [12]) Einstein metric is inert under this transformation,

\[
\tilde{g}^E_{\mu\nu} = g^E_{\mu\nu}
\] (16)

We then see that each frame seems to be most appropriate depending on which symmetry we believe to be the most fundamental.

### 3 The definition of the vacuum state

Insofar as

\[
T^{(\text{matt})}_{\mu\nu} = 0
\] (17)

we always have a vacuum solution of the equations of motion:

\[
\begin{align*}
\phi &= 0 \\
g_{\mu\nu} &= \eta_{\mu\nu}
\end{align*}
\] (18)

On the other hand, it has been proven in ref. [3] that under certain hypothesis (mainly the equality of all dressing functions \(B_i\)), the cosmological evolution attracts the dilaton towards the point where the dressed masses are stationary

\[
\left. \frac{\partial m_E}{\partial \phi} \right|_{\phi = \phi_0} = 0
\] (19)
It has also been argued that the existence of several different minima is probably incompati-
ble with present bounds on the equivalence principle (namely a relative difference in
acceleration $\frac{\Delta a}{a} \leq 10^{-13}$ [6]). Although it is known that in general the hypothesis of
reference [6] are not fulfilled, there are actual string models where this mechanism is
automatic [2]).

4 The principle of equivalence

Were to be strings the probes of the metric, it is obvious that the most natural frame would
be the String Frame. But in most classical experiments, the metric is detected through
its effect on classical test particles, which describe geodesics of the spacetime metric. At
a higher level of precision, the geodesic deviation equation, gives direct information of the
Riemann tensor.

Let us now review how the concept of particle is recovered from the concept of field.
The latter is written as a formal (WKB) series

$$\phi = e^{\frac{1}{\epsilon} \sum \epsilon^n \phi_n}$$

(20)

The Klein-Gordon equation then gives, in the eikonal approximation (actually, to the
dominant order $\frac{1}{\epsilon^2}$)

$$k^2 = -m^2$$

(21)

where the mass has to be considered as $o(\epsilon^{-2})$, and

$$k_\mu \equiv \nabla_\mu \phi_0$$

(22)

This implies that the flow lines defined by the congruence $k^\mu$ are geodesic, since

$$0 = \nabla_\rho (k^2) = 2k^\rho \nabla_\rho k_\mu = 2k^\mu \nabla_\mu k_\rho$$

(23)

where the last step is justified since the vector $k^\rho$ is itself a gradient.
Let us now consider a scalar field other than the dilaton (which remains massless to all orders in perturbation theory), with Lagrange density:

\[ L \equiv -\frac{1}{2}(B_\chi (\nabla \chi)^2 + m^2 B_m \chi^2) \]  

The equations of motion in the eikonal approximation now yield

\[ B_\chi k^2 = -m^2 B_m \]  

which violates the principle of equivalence unless \( B_m = B_\chi \), but does not distinguish qualitatively (although it does it quantitatively) between different frames.

That is, particles will propagate along geodesics of that metric (if any) such that \( B_\chi = B_m \). In addition, the dressing factors \( B_\chi \) will depend generically on the particle considered, and so will depend the trajectories, and it is this fact which violates the equivalence principle.

5 The fluid approximation

The situation is perhaps less clear when both the metric and the dilaton are singular in one frame, but the metric is regular in the other. This clearly changes the physics of test particles propagating in the physical spacetime.

Let us now point out a related, but simpler, situation.

Imagine that matter (that is all fields except gravitation itself)is such that its energy-momentum tensor corresponds to a perfect fluid,

\[ T_{\mu \nu} \equiv (\rho + p) u_\mu u_\nu + pg_{\mu \nu} \]  

Then the question is: will the same matter still behave as a perfect fluid in the other frame? This is a meaningful question, because matter is almost always considered of such a form in cosmological investigations.
The general conditions for a fluid description to be valid consist in demanding that the wavelength (as measured in a Local Inertial Frame, LIF, defined by a vielbein, $e^{a}_{\mu} \partial_{\mu}$, $a = 0, \ldots, n - 1$) should be much less than both the macroscopical length of the wavepacket, $l$, and the scale of variation of Riemann’s tensor, $r$.

$$\lambda << \min (r, l) \quad (27)$$

Now, when changing frames, quantities in the LIF obviously do not change, (neglecting the new interactions introduced through the change of frame) whereas lengths scale as

$$l_S = l_E \frac{l_s m_p}{B_{g}^{1/(n-2)}} \quad (28)$$

(Where an average value for the dilaton field in the region considered is implicitly assumed). In the case of Riemann’s tensor, there are in addition extra terms proportional to the square of the first derivative and to the second derivative of the dilaton field, which could dominate for a rapidly fluctuating dilaton.

A sufficient condition for such a hydrodynamic fluid description is to be at thermodynamic equilibrium. It is in turn clear that a necessary condition for it (it is less clear whether it will be sufficient) is that the mean free time between collisions should be smaller than Hubble’s time

$$\tau << H^{-1} \quad (29)$$

The mean free time, in turn, is related to quantities computed in a LIF: basically $\tau \sim (\rho v \sigma)^{-1}$, where $\rho$ is the average density of particles, $v$ is the average velocity, and $\sigma$ is the total cross section (again, neglecting the new interactions).

The scale factor, on the other hand, is a global quantity, and, as such, scales as above

$$R_S = R_E \frac{l_s m_p}{B_{g}^{1/(n-2)}} \quad (30)$$

in such a way that

$$H_S \equiv \frac{\dot{R}_S}{R_S} = H_E - \frac{1}{n - 2} \frac{B_{g}'}{B_{g} \phi} \quad (31)$$
The crucial quantity now is the dimensionless quotient

$$\frac{B'_g \dot{\Phi}}{(n-2)B_g H_E} \quad (32)$$

If it is small, equilibrium in both frames is equivalent.

But if it is large (corresponding physically to a wildly fluctuating dilaton or to the vicinity of a place in which the dilaton itself is singular), then equilibrium in Einstein's frame does not guarantee equilibrium in the string frame.

It is worth remarking that even for a time independent dilaton (which does not spoil the equilibrium condition) the preceding formula (28) indicates that for large dilatons averaged (corresponding to large $g_s$), so that $B_g << 1$ and correspondingly, $l_s >> l_E$, matter enjoying a fluid description in the String Frame does not necessarily do so in the Einstein Frame. For large negative averaged dilaton couplings (corresponding to small $g_s$) the converse is true: a fluid description in the Einstein Frame does not guarantee a fluid description in String Frame).

6 Conclusions

There is in our opinion no doubt of the equivalence of all frames for the description of the gravitational effects of string theories at a basic level, at least when all functions involved are smooth.

When this is not the case, the solution depends on what is the quantum resolution of the classical gravitational singularities. The symmetries of string theory (T-duality, in particular) suggest that there is, in a sense, a minimal measurable length, but the issue is far from settled; it could be, in particular, that certain particle-like topological defects, known as $D0$ branes, could probe shorter lengths.

A different question is what is the classical metric felt by a particular probe (usually, a test particle). Here, again, strings give a unique answer (depending on the probe
used), which seems difficult to reconcile \textit{a priori} with existing bounds on violations of the equivalence principle. A detailed comparison is however difficult owing to our limited understanding of the dynamics of the dilaton as well as other scalars in the string spectrum.

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