UNIQUE RECOVERY OF ELECTRICAL CONDUCTIVITY AND MAGNETIC PERMEABILITY FROM MAGNETO-TELLURIC DATA

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Abstract. We present a comprehensive mathematical study of the Magneto-Telluric (MT) method, on bounded domain in $\mathbb{R}^3$. We show that electrical conductivity and magnetic permeability, assumed to be $C^2$, can be uniquely recovered from MT data measured on the boundary of the domain. The proof is based on the construction of complex geometric optics solutions. Furthermore, we obtain a unique determination result in the case when the MT data are measured only on an open subset of the boundary. Here, we assume that the part of the boundary inaccessible for measurements is a subset of a sphere.

1. Introduction

The magnetotelluric (MT) method uses electromagnetic passive sources in the magnetosphere and ionosphere to estimate electrical resistivity in Earth’s interior [10]. This method was introduced almost 70 years ago by Cagniard [4]. The passive sources excite a certain spectrum, that is, interval of frequencies. There is a long history of studies based on some form of data fitting including a Bayesian approach [23]. It has been understood that high frequency data enable determining the resistivity in the “near” subsurface, while low frequency ones enable obtaining an estimate of the resistivity in the “deep” interior. The MT method has been used to study melt and hydration in Earth’s crust and mantle [41, 21, 20, 40], and general electrical structure in the lithosphere [34, 50] and below the ocean floor [18]. Moreover, the MT method has been applied to earthquake prediction [12].

In this paper, we analyze the inverse problem for the MT method. We let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with $C^{1,1}$ boundary and let $\epsilon, \sigma, \mu \in C^2(\overline{\Omega})$ be non-negative. We consider the time-harmonic Maxwell equations for electromagnetic fields, $E$ and $H$,

\begin{equation}
\nabla \times E = i \omega \mu H \quad \text{and} \quad \nabla \times H = -i \omega \left( \epsilon + \frac{i \sigma}{\omega} \right) E \quad \text{in} \quad \Omega,
\end{equation}

where $\omega > 0$ is a fixed frequency. The functions $\epsilon, \sigma$ and $\mu$ represent the material parameters, namely, electrical permittivity, electrical conductivity and magnetic permeability, respectively. In the case of MT, one invokes the following

Assumption 1.1. The electrical permittivity vanishes, $\epsilon = 0$, and the electrical conductivity and magnetic permeability satisfy $\sigma \geq \sigma_0, \mu \geq \mu_0$ on $\overline{\Omega}$ for some constants $\sigma_0, \mu_0 > 0$.

This assumption has its origin in the low-frequency reduction of Maxwell’s equations.

We let $v$ be the outer unit normal to the boundary $\partial \Omega$, and define the trace operator $t : C^\infty(\overline{\Omega}; C^3) \to C^\infty(\partial \Omega; C^3)$ as

$$
t(u) := v \times u|_{\partial \Omega} \quad \text{for} \quad u \in C^\infty(\overline{\Omega}; C^3).
$$

The trace $t$ can be extended to a bounded linear operator from $H^1_{\text{Div}}(\Omega)$ into $TH^{1/2}_{\text{Div}}(\partial \Omega)$, where

$$
H^1_{\text{Div}}(\Omega) := \left\{ u \in H^1(\Omega; C^3) : t(u) \in TH^{1/2}_{\text{Div}}(\partial \Omega) \right\}
$$

and

$$
TH^{1/2}_{\text{Div}}(\partial \Omega) := \left\{ f \in H^{1/2}(\partial \Omega; C^3) : \nabla \cdot f = 0 \quad \text{and} \quad \text{Div}(f) \in H^{1/2}(\partial \Omega; C^3) \right\},
$$

and $\text{Div}$ denotes the surface divergence on $\partial \Omega$. We refer the reader to [38] for more details. Under Assumption 1.1, when $\omega > 0$ does not belong to a discrete set of magnetic resonant frequencies, the equation (1) with the boundary condition $t(H) = f \in TH^{1/2}_{\text{Div}}(\partial \Omega)$ has a unique solution $(H, E) \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\Omega)$; see Section 2. Then the impedance map $Z^0_{\sigma, \mu} : TH^{1/2}_{\text{Div}}(\partial \Omega) \to TH^{1/2}_{\text{Div}}(\partial \Omega)$ is defined as

$$
Z^0_{\sigma, \mu}(f) := t(E), \quad f \in TH^{1/2}_{\text{Div}}(\partial \Omega).
$$


The inverse MT problem (IMTP) is to determine \( \sigma \) and \( \mu \) from the knowledge \( Z_{\sigma, \mu}^w \). In the geophysics literature, one commonly considers the resistivity, that is, the reciprocal of conductivity \[52\].

**Remark 1.2.** In practice, the data are represented as the graph of \( Z_{\sigma, \mu}^w: TH_{\text{Div}}^{1/2}(\partial \Omega) \to TH_{\text{Div}}^{1/2}(\partial \Omega) \).

Our first main result is

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with \( C^{1,1} \) boundary and let \( \sigma_j, \mu_j \in C^2(\overline{\Omega}), \) \( j = 1, 2 \), be such that \( \sigma_j \geq \sigma_0 \) and \( \mu_j \geq \mu_0 \) for some constants \( \sigma_0, \mu_0 > 0 \). Suppose that \( \omega > 0 \) is not a resonant frequency for \( (\sigma_1, \mu_1) \) and \( (\sigma_2, \mu_2) \) and that

\[
\delta^a \sigma_1|_{\partial \Omega} = \delta^a \sigma_2|_{\partial \Omega} \quad \text{and} \quad \delta^a \mu_1|_{\partial \Omega} = \delta^a \mu_2|_{\partial \Omega} \quad \text{for} \quad |\alpha| \leq 2.
\]

Then \( Z_{\sigma_3, \mu_1}^w = Z_{\sigma_2, \mu_2}^w \) implies that \( \sigma_1 = \sigma_2 \) and \( \mu_1 = \mu_2 \).

Condition (2) is not important. We expect that a suitable boundary determination result would allow to remove it as in \[7, 26, 36\]. See also Remark B.3.

The inverse problem considered in the present paper formally looks like a standard inverse electromagnetic problem (IEMP) proposed in \[45\]: Determine \( \varepsilon, \mu \) and \( \sigma \), from the knowledge of the admittance map \( \Lambda_{\varepsilon, \mu, \sigma}^w: t(E) \to t(H) \) for all \( (E, H) \in H_{\text{Div}}^1(\Omega) \times H_{\text{Div}}^1(\Omega) \) solving \( \text{IEMP} \). However, the conditions in the IEMP do not allow the vanishing of \( \varepsilon \) as in our Assumption 1.1. To be precise, IEMP invokes

**Assumption 1.4.** The electrical permittivity, electrical conductivity and magnetic permeability satisfy \( \varepsilon \geq \varepsilon_0, \mu \geq \mu_0 \) and \( \sigma \geq 0 \) on \( \Omega \) for some constants \( \varepsilon_0, \mu_0 > 0 \).

As a consequence, the IMTP and IEMP problems are in fact different. For comparison, we also mention the EIT, or inverse conductivity problem, also known as complex geometric optics solutions, following the celebrated paper \[49\] on the inverse conductivity problem. One of the main challenges in adopting the method of \[49\] is the fact that \( \text{IEMP} \) with Assumption 1.4 is not elliptic. The linearized problem at constant material parameters was studied in \[45\]. For the nonlinear problem, a uniqueness result was given in \[48\] when the electromagnetic parameters are close to constants. In this paper, to get ellipticity, equation \( \text{IEMP} \) with Assumption 1.4 was reduced to a system whose principal part is the Laplacian. However, this reduction gives some first order terms. For material parameters that are nearly constant, the authors were able to manage the first-order terms and introduce complex geometrical solutions for \( \text{IEMP} \). The first global uniqueness result was proven in \[38\]. This proof was later simplified in \[39\]. The important point in the simplified proof is to augment \( \text{IEMP} \) with Assumption 1.4 to a certain \( 8 \times 8 \) Dirac equation and connect it via some other Dirac operator to an \( 8 \times 8 \) system whose principal part is the Laplacian while its remainder involves only zeroth-order terms. This allowed the authors to construct complex geometric optics solutions for the latter system and connect them to \( \text{IEMP} \) with Assumption 1.4 by applying the Dirac operator that was initially introduced. This technique became popular in the subsequent works on various aspects of IEMP \[3, 9, 27\].

In the setting of the IMTP, however, one cannot simply employ the complex geometric optics solutions constructed in \[38, 39\] for the IEMP. Moreover, the elliptization argument of \[39\], applied to \( \text{IEMP} \) with Assumption 1.1, does not help avoiding first-order terms. Instead of that, we follow \[48\] and reduce \( \text{IEMP} \) with Assumption 1.1 to a system whose principal part is the Laplacian. We then introduce novel complex geometrical optics solutions for the reduced system that are essentially solutions for \( \text{IEMP} \) with Assumption 1.1. Moreover, using this reduction gives an integral identity with a clear relation to \( \text{IEMP} \). To deal with the first-order terms, we use the ideas from \[13\] with substantial modifications since the latter paper assumes that \( \mu \) is constant.

In the MT method, performing measurements on the entire boundary (that is, the surface of the earth) is impossible. Therefore, the analysis of the inverse problem with local measurements is important. We can assume that the measurements are performed on a nonempty open subset \( \Gamma \) of \( \partial \Omega \) only and that the inaccessible part of the boundary \( \Gamma_0 = \partial \Omega \setminus \Gamma \) is a part of a sphere (our planet’s surface). Our second main result is the following.
Theorem 1.5. Let $\Omega \subset B_0$ be a bounded domain with $C^{1,1}$ boundary included in an open ball $B_0 \subset \mathbb{R}^3$ and let

$$\Gamma_0 = \partial \Omega \cap \partial B_0, \Gamma_0 \neq \partial B_0 \text{ and } \Gamma = \partial \Omega \setminus \Gamma_0.$$ 

Suppose that $\sigma_j, \mu_j \in C^2(\Omega)$, $j = 1, 2$, satisfy $\sigma_j \geq \sigma_0$ and $\mu_j \geq \mu_0$, for some constants $\sigma_0, \mu_0 > 0$, and

$$\partial^a \sigma_j |_{\Gamma} = \partial^a \sigma_2 |_{\Gamma} \quad \text{and} \quad \partial^a \mu_j |_{\Gamma} = \partial^a \mu_2 |_{\Gamma} \quad \text{for} \quad |a| \leq 2.$$

In addition, assume that $\sigma_j$ and $\mu_j$, $j = 1, 2$, can be extended to $\mathbb{R}^3$ as $C^2$ functions which are invariant under reflection across $\partial B_0$. Suppose that $\omega > 0$ is not a resonant frequency for $(\sigma_1, \mu_1)$ and $(\sigma_2, \mu_2)$. If

$$Z^{\omega}_{\sigma_1, \mu_1}(f)|_{\Gamma} = Z^{\omega}_{\sigma_2, \mu_2}(f)|_{\Gamma} \quad \text{for all} \quad f \in TH^{1/2}_{D0}(\partial \Omega) \quad \text{with} \quad \text{supp}(f) \subset \Gamma,$$

then $\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$.

For the proof of Theorem 1.3, we follow Isakov’s reflection approach [25] which was originally proposed for the inverse conductivity problem. An analogous result for IEMP was obtained in [8].

We briefly describe a connection of our results to the land-based CSEM (controlled source electromagnetic) method in geophysical exploration [47]. Contrary to the MT method, the CSEM method employs active sources. In recent work by Schaller et al. [44], a land-based CSEM survey was designed and performed at the Schoonebeek oil field. The application of land-based CSEM for low-cost CO$_2$ monitoring was studied in [35]. The marine CSEM method, which was introduced by Cox et al. [18], would require a careful incorporation of the ocean layer in the analysis, which we do not pursue in this paper. It has successful applications in direct identification of hydrocarbons [10, 19], and the study of the oceanic lithosphere and active spreading centers [11, 15, 17, 20, 33, 53]. For a more detailed exposition of progress made on the marine CSEM, we refer to a review paper by Constable [14]. Various basic data-fitting approaches have been developed for CSEM [1, 24, 32, 31]. From a mathematical point of view, the data for the land-based CSEM is modeled by point source measurements. More precisely, for an arbitrary unit vector $\alpha$ and $y \in \partial \Omega$, consider the equation

$$\nabla \times E\delta(x, y) = i \omega \mu \alpha H\alpha(x, y) + \delta(x - y) \nu(\nu' - \alpha) \quad \text{and} \quad \nabla \times H\alpha(x, y) = \sigma(\nu' - \alpha) E\alpha(x, y) \quad \text{in} \quad \mathbb{R}^3,$$

with the outgoing radiation condition. The equation governs the electromagnetic field of a magnetic dipole (active source) tangential to the boundary $\partial \Omega$. Then the inverse problem for the land-based CSEM is to determine $\sigma$ and $\mu$ from

$$\mathcal{A}_{\sigma, \mu} := \left\{(v(x) \times H\delta(x, y), v(x) \times E\delta(x, y)) : x, y \in \partial \Omega, \quad x \neq y, \quad j = 1, 2, 3\right\},$$

where $e_j$, $j = 1, 2, 3$, denote the Cartesian coordinate vectors. We expect that following the arguments similar to [39], one can show that the knowledge of $\mathcal{A}_{\sigma, \mu}$ is equivalent to the knowledge of the graph of $Z^{\omega}_{\sigma, \mu}$ via layer potentials. Then the land-based CSEM and MT would concern the same inverse problem with boundary data.

The paper is organized as follows. In Section 2, we prove the well-posedness of the direct problem using standard arguments. In Section 3, we first rewrite (1) as the curl-curl equation and then construct complex geometric optics solutions for it. We use these solutions to prove Theorem 1.3 in Section 4. Next, in Section 5 we perform the reflection approach of Isakov [25] and prove an analog of Theorem 1.5 but in the case when the part of the boundary inaccessible for measurements is a subset of the plane $\{x \in \mathbb{R}^3 : x_3 = 0\}$. Theorem 1.5 is then proved in Section 6 by analyzing the behavior of (1) under the Kelvin transform. Appendix A contains properties of pullbacks used in the main text. Finally, in Appendix B we show that the impedance map $Z^{\omega}_{\sigma, \mu}$ is a pseudodifferential operator of order 1 if $\sigma, \mu \in C^\infty(\Omega)$. Using this fact, we gain insight in the notion of apparent resistivity used in geophysics from a mathematical point of view.

2. Well-posedness of the direct problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{1,1}$ boundary as before, and let $\sigma, \mu \in C^1(\overline{\Omega})$ be such that $\sigma \geq \sigma_0$ and $\mu \geq \mu_0$ for some constants $\sigma_0, \mu_0 > 0$. Consider the following system of equations for electromagnetic fields $E$ and $H$:

$$\nabla \times E = i \omega \mu H \quad \text{and} \quad \nabla \times H = \sigma E \quad \text{in} \quad \Omega,$$

with the tangential boundary condition $t(H) = f$, where $\omega$ is a complex number. The main result of the present section is
Theorem 2.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{1,1}$ boundary and let $\sigma, \mu \in C^1(\overline{\Omega})$ be such that $\sigma \geq \sigma_0$ and $\mu \geq \mu_0$ for some constants $\sigma_0, \mu_0 > 0$. There is a discrete subset $\Sigma$ of $\mathbb{C}$ such that for all $\omega \in \Sigma$ and for a given $f \in TH^{1/2}_{\text{Div}}(\partial \Omega)$ the system (4) with $t(H) = f$ has a unique solution $(E, H) \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\Omega)$ satisfying

$$
\|E\|_{H^1_{\text{Div}}(\Omega)} + \|H\|_{H^1_{\text{Div}}(\Omega)} \leq C\|f\|_{TH^{1/2}_{\text{Div}}(\partial \Omega)}
$$

for some constant $C > 0$ independent of $f$.

For $\omega > 0$ with $\omega \in \Sigma$, we define the impedance map $Z_{\omega, \mu}^0$ as

$$
Z_{\omega, \mu}^0(f) := t(E), \quad f \in TH^{1/2}_{\text{Div}}(\partial \Omega),
$$

where $(E, H) \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\Omega)$ is the unique solution of the system (4) with $t(H) = f$, guaranteed by Theorem 2.1. Moreover, the estimate provided in Theorem 2.1 implies that the impedance map is a well-defined and bounded operator $Z_{\omega, \mu}^0 : TH^{1/2}_{\text{Div}}(\partial \Omega) \to TH^{1/2}_{\text{Div}}(\partial \Omega)$.

To prove Theorem 2.1, we consider the following non-homogeneous problem. Let $J_e$ and $J_m$ be vector fields defined in $\Omega$ representing current sources. We consider the non-homogenous time-harmonic Maxwell equations,

$$
\begin{align*}
\nabla \times E &= i \omega \mu H + J_m & \text{in } \Omega, \\
\nabla \times H &= \sigma E + J_e & \text{in } \Omega.
\end{align*}
$$

We have

Theorem 2.2. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{1,1}$ boundary and let $\sigma, \mu \in C^1(\overline{\Omega})$ be such that $\sigma \geq \sigma_0$ and $\mu \geq \mu_0$ for some constants $\sigma_0, \mu_0 > 0$. Suppose that $J_e, J_m \in L^2(\Omega; \mathbb{C}^3)$ and $\mu \mu_0 \in L^2(\partial \Omega)$. Then, there is a discrete subset $\Sigma$ of $\mathbb{C}$ such that for all $\omega \in \Sigma$ the boundary value problem (5) with $t(H) = f$ has a unique solution $(E, H) \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\partial \Omega)$ satisfying

$$
\begin{align*}
\|E\|_{H^1_{\text{Div}}(\Omega)} + \|H\|_{H^1_{\text{Div}}(\Omega)} &= C(\|J_e\|_{L^2(\Omega; \mathbb{C}^3)} + \|J_m\|_{L^2(\Omega; \mathbb{C}^3)} + \|
abla \cdot J_e\|_{L^2(\Omega; \mathbb{C}^3)} + \|
abla \cdot J_m\|_{L^2(\Omega; \mathbb{C}^3)} + \|
abla \times J_e\|_{H^{1/2}(\partial \Omega)} + \|
abla \times J_m\|_{H^{1/2}(\partial \Omega)}),
\end{align*}
$$

for some constant $C > 0$ independent of $J_e$ and $J_m$.

We first prove Theorem 2.2 and then show that it can be used to prove Theorem 2.1.

2.1. Proof of Theorem 2.2. We introduce some notion that will be used for the proof. We work with the following Hilbert space which is the largest domain of $\nabla \times$:

$$
H(\text{curl}; \Omega) := \{ w \in L^2(\Omega; \mathbb{C}^3) : \nabla \times w \in L^2(\Omega; \mathbb{C}^3) \}
$$

endowed with the norm $\|w\|_{H(\text{curl}; \Omega)} := \|w\|_{L^2(\Omega; \mathbb{C}^3)} + \|\nabla \times w\|_{L^2(\Omega; \mathbb{C}^3)}$. Then the tangential trace operator has its extensions to bounded operators $t : H(\text{curl}; \Omega) \to H^{-1/2}(\partial \Omega; \mathbb{C}^3)$. We also work with the space of vector fields in $H(\text{curl}; \Omega)$ having zero tangential trace

$$
H(\text{curl}, 0; \Omega) := \{ w \in H(\text{curl}; \Omega) : t(w) = 0 \}.
$$

For the short proof, we follow the standard variational methods used in [28, 37]. Substituting the first equation of (5) into the second one, we obtain

$$
\begin{align*}
\nabla \times (\sigma^{-1} \nabla \times H) - i \omega \mu H &= J_m + \nabla \times (\sigma^{-1} J_e) & \text{in } \Omega.
\end{align*}
$$

Our first step is to find a unique solution $H \in H(\text{curl}, 0; \Omega)$ of this equation satisfying

$$
\|H\|_{H(\text{curl}; \Omega)} \leq C(\|J_e\|_{L^2(\Omega; \mathbb{C}^3)} + \|J_m\|_{L^2(\Omega; \mathbb{C}^3)}).
$$

By Helmholtz type decompositions in [28, Section 4.1.3] or [37, Section 3.7], we can uniquely decompose

$$
H = H_0 + \nabla h, \quad H_0 \in H(\text{curl}, 0; \Omega) : \nabla \cdot (\mu \omega) = 0, \quad h \in H^1_0(\Omega; \mathbb{C}),
$$

$$
\mu^{-1} J_m = J_{m,0} + \nabla j_m, \quad J_{m,0} \in L^2(\Omega; \mathbb{C}^3) : \nabla \cdot (\mu \omega) = 0, \quad j_m \in H^1_0(\Omega; \mathbb{C}).
$$

We note here that

$$
\|j_m\|_{H^1(\Omega; \mathbb{C})} \leq C\|J_m\|_{L^2(\Omega; \mathbb{C}^3)}.
$$

Using these decompositions, (6) can be rewritten as

$$
\begin{align*}
\nabla \times (\sigma^{-1} \nabla \times H_0) - i \omega \mu H_0 - i \omega \mu \nabla h &= \mu J_{m,0} + \mu \nabla j_m + \nabla \times (\sigma^{-1} J_e) & \text{in } \Omega.
\end{align*}
$$

To extract $h$ from (9), we simply set $h = -(i\omega^{-1}) j_m$. Thus, we need to find a unique $H_0 \in H(\text{curl}, 0; \Omega)$ with $\nabla \cdot (\mu H_0) = 0$ satisfying

$$
\begin{align*}
\nabla \times (\sigma^{-1} \nabla \times H_0) - i \omega \mu H_0 &= \mu J_{m,0} + \nabla \times (\sigma^{-1} J_e) & \text{in } \Omega.
\end{align*}
$$
To solve this equation, we need the following result on existence of a solution operator

**Proposition 2.3.** There exist a constant \(\lambda > 0\) and a bounded linear map \(T_{\lambda} : H(\text{curl}, 0; \Omega') \to H(\text{curl}, 0; \Omega)\) such that

\[
\nabla \times (\sigma^{-1} \nabla \times T_{\lambda} u) + \lambda \mu T_{\lambda} u = u, \quad u \in H(\text{curl}, 0; \Omega')
\]

and

\[
T_{\lambda}(\nabla \times (\sigma^{-1} \nabla \times e) + \lambda \mu e) = e, \quad e \in H(\text{curl}, 0; \Omega).
\]

Furthermore, if \(\nabla \cdot u = 0\), then \(T_{\lambda} u \in H(\text{curl}, 0; \Omega)_{0, \mu}\).

Here and in what follows, \((\cdot, \cdot)_{\Omega}\) is the duality between \(H(\text{curl}, 0; \Omega')\) and \(H(\text{curl}, 0; \Omega)\) naturally extending the inner product of \(L^2(\Omega; \mathbb{C}^3)\).

**Proof.** The proof is similar to that of [2, Proposition 5.1] using Lax-Milgram's lemma. \(\square\)

Then \(H_0 \in H(\text{curl}, 0; \Omega)\) with \(\nabla \cdot (\mu H_0) = 0\) solves (10) if and only if

\[
H_0 - (i\omega + \lambda) \tilde{T}_{\lambda} H_0 = T_{\lambda} \left(\mu J_{m, 0} + \nabla \times (\sigma^{-1} J_e)\right)
\]

where \(\tilde{T}_{\lambda} = T_{\lambda} \circ m_\mu \circ D_{\mu}\), \(m_\mu\) is multiplication by \(\mu\), and \(D_{\mu}\) is the bounded orthogonal projection of \(L^2(\Omega; \mathbb{C}^3)\) onto \(L^2(\Omega; \mathbb{C}^3)_{0, \mu}\) constructed in [28, Section 4.1.3]. Since \(J_{\mu, 0} \in L^2(\Omega; \mathbb{C}^3)_{0, \mu}\) and \(\nabla \cdot \times = 0\), we then have \(\nabla \cdot (\mu J_{m, 0} + \nabla \times (\sigma^{-1} J_e)) = 0\). Therefore, by the second part of Proposition 2.3, this implies that \(T_{\lambda} (\mu J_{m, 0} + \nabla \times (\sigma^{-1} J_e))\) belongs to \(H(\text{curl}, 0; \Omega)_{0, \mu}\). The second part of Proposition 2.3 implies also that \(\tilde{T}_{\lambda}\) can be considered as a bounded linear operator

\[
\tilde{T}_{\lambda} : L^2(\Omega; \mathbb{C}^3)_{0, \mu} \rightarrow L^2(\Omega; \mathbb{C}^3)_{0, \mu},
\]

Using the compactness of the inclusion \(H(\text{curl}, 0; \Omega)_{0, \mu} \hookrightarrow L^2(\Omega; \mathbb{C}^3)\) [51] and following similar reasoning as at the end of [2, Section 5], one can show that for any \(\omega \in \Sigma\), where \(\Sigma = \{\omega \in \mathbb{C} \setminus (\pm \lambda) : (i\omega + \lambda)^{-1} \in \text{Spec}(\tilde{T}_{\lambda})\}\) which is discrete, (12) has a unique solution \(H_0 \in H(\text{curl}, 0; \Omega)_{0, \mu}\) satisfying

\[
\|H_0\|_{H(\text{curl}, \Omega)} \leq C(\|J_e\|_{L^2(\Omega; \mathbb{C}^3)} + \|J_{m, \beta}\|_{L^2(\Omega; \mathbb{C}^3)}).
\]

Next, setting \(H = H_0 - (i\omega)^{-1} J_m\), we obtain a unique \(H(\text{curl}, 0; \Omega)\) solution for (6) satisfying (7) thanks to (8). Defining \(E := \sigma^{-1}(\nabla \times H - J_e)\) we obtain a unique \((E, H) \in H(\text{curl}, 0; \Omega) \times H(\text{curl}, \Omega)\) solving (5) and satisfying

\[
\|E\|_{H(\text{curl}, \Omega)} + \|H\|_{H(\text{curl}, \Omega)} \leq C(\|J_e\|_{L^2(\Omega; \mathbb{C}^3)} + \|J_m\|_{L^2(\Omega; \mathbb{C}^3)}).
\]

To prove that \((E, H) \in H^1_{\text{div}}(\Omega) \times H^1_{\text{div}}(\Omega)\), apply \(\nabla\cdot\) to (5) and get \(\nabla \cdot (i\omega \mu H) = -\nabla \cdot J_m\) and \(\nabla \cdot (\sigma E) = -\nabla \cdot J_e\). Hence \(\nabla \cdot E, \nabla \cdot H \in L^2(\Omega)\), since \(\nabla \cdot J_m, \nabla \cdot J_m \in L^2(\Omega)\) by assumption. Then \(t(H) = 0\) and the results in [16] imply that \(H \in H^1_{\text{div}}(\Omega)\) and

\[
\|H\|_{H^1_{\text{div}}(\Omega)} \leq C(\|H\|_{H(\text{curl}, \Omega)} + \|J_e\|_{L^2(\Omega; \mathbb{C}^3)} + \|J_m\|_{L^2(\Omega; \mathbb{C}^3)} + \|\nabla \cdot J_m\|_{L^2(\Omega)} + \|\nabla \cdot J_e\|_{L^2(\Omega)}).
\]

To show that \(E \in H^1(\Omega; \mathbb{C}^3)\), observe that \(\nabla \cdot (\nabla \times H) \partial \Omega = -\text{Div}(t(H)) = 0\) by [28, Corollary A.20]. Then by (5), \(\nabla \cdot E \partial \Omega = \sigma^{-1} \nabla \cdot (\nabla \times H) \partial \Omega - \sigma^{-1} \nabla \cdot J_e \partial \Omega = -\sigma^{-1} \nabla \cdot J_e \partial \Omega \in H^{1/2}(\partial \Omega)\). According to the results in [16], this implies that \(E \in H^1(\Omega; \mathbb{C}^3)\) and

\[
\|E\|_{H^1_{\text{div}}(\Omega)} \leq C(\|E\|_{H(\text{curl}, \Omega)} + \|J_e\|_{L^2(\Omega; \mathbb{C}^3)} + \|\nabla \cdot J_m\|_{L^2(\Omega)} + \|\nabla \cdot J_e\|_{H^{1/2}(\partial \Omega)}).
\]

Next, using [28, Corollary A.20] and (5), we can show \(\text{Div}(t(E)) = -\nabla \cdot (\nabla \times E) \partial \Omega = -i\omega \mu \nu \cdot H \partial \Omega + \|J_m\|_{\partial \Omega} \in H^{1/2}(\partial \Omega)\). Thus, \(E \in H^1_{\text{div}}(\Omega)\). Finally, the estimate in the statement of the theorem follows by combining all the above estimates. The proof of Theorem 2.2 is thus complete.

### 2.2. Proof of Theorem 2.1.

First prove the uniqueness of the solution. For a fixed \(\omega \in \mathbb{C}\), suppose that \((E_j, H_j) \in H^1_{\text{div}}(\Omega) \times H^1_{\text{div}}(\Omega), j = 1, 2\), solve (4) and satisfy \(t(H_j) = t(H_2)\). Then \((E, H) \in H^1_{\text{div}}(\Omega) \times H^1_{\text{div}}(\Omega)\) also solve (4) and satisfy \(t(H) = 0\), where \(E := E_1 - E_2\) and \(H := H_1 - H_2\). The uniqueness part of Theorem 2.2 (with \(J_e = J_m = 0\)) gives that \(E = 0\) and \(H = 0\).

Next, prove existence of a solution. For a given \(f \in TH^{1/2}_{\text{div}}(\partial \Omega)\), there is \(H' \in H^1_{\text{div}}(\Omega)\) such that \(t(H') = f\) and 

\[
\|H'\|_{H^1_{\text{div}}(\Omega)} \leq C\|f\|_{TH^{1/2}_{\text{div}}(\partial \Omega)}.
\]

Applying Theorem 2.2 with \(J_e = -\nabla \times H'\) and \(J_m = i\omega \mu H'\), we obtain a unique \((E_0, H_0) \in H^1_{\text{div}}(\Omega) \times H^1_{\text{div}}(\Omega)\) solving

\[
\nabla \times E_0 = i\omega \mu H_0 + i\omega \mu H', \quad \nabla \times H_0 = \sigma E_0 - \nabla \times H', \quad t(H_0) = 0.
\]
and satisfying $\|E_0\|_{H^{1/2}_0(\partial\Omega)} + \|H_0\|_{H^{1/2}_0(\partial\Omega)} \leq C\|f\|_{H^{1/2}_0(\partial\Omega)}$. Here, we used the fact that $\nu \cdot (\nabla \times H')|_{\partial\Omega} = -\text{Div}(t(H')) \in H^{1/2}(\partial\Omega)$ by [28, Corollary A.20]. Then $(E, H) \in H^1_0(\Omega) \times H^1_0(\Omega)$ solves (4) with $t(E) = f$, where $E := E_0 + E'$ and $H := H_0$. The proof is complete.

3. Construction of complex geometric optics solutions

Throughout this section, we assume that $\sigma$ and $\mu$ can be extended to the whole $\mathbb{R}^3$ so that $\sigma \geq \sigma_0, \mu \geq \mu_0$ and

$$\sigma - \sigma_0, \mu - \mu_0 \in C^0(\mathbb{R}^3),$$

for some constants $\sigma_0, \mu_0 > 0$. We also let $R > 0$ be large enough (but fixed) so that $B_R(0)$ contains both $\text{supp}(\sigma - \sigma_0)$ and $\text{supp}(\mu - \mu_0)$.

Substituting the first equation of (1) into the second one, we obtain the following second-order equation

$$\nabla \times (\sigma^{-1} \nabla \times H) - i\omega \mu H = 0 \quad \text{in} \quad \Omega.$$

The aim of the present section is to construct a complex geometric optics solution in $H \in H^1_0(\Omega)$ for the above equation. Instead of working in $\Omega$, we conduct our analysis in the whole $\mathbb{R}^3$. Therefore, we consider

$$\nabla \times (\sigma^{-1} \nabla \times H) - i\omega \mu H = 0 \quad \text{in} \quad \mathbb{R}^3,$$

Taking the divergence of (15), it straightforwardly follows that $\nabla \cdot (\mu H) = 0$ in $\mathbb{R}^3$. Therefore, we obtain

$$\nabla \times \nabla \times = -\Delta H - \nabla (\nabla \beta \cdot H) \quad \text{in} \quad \mathbb{R}^3,$$

where $\beta := \log \mu$. Then, we use the latter identity in (15) to show that this equation is equivalent to the system

$$L_{\sigma, \mu} H := -\Delta H - \nabla (\nabla \beta \cdot H) - \nabla \alpha \times \nabla \times H - i\omega \mu H = 0 \quad \text{in} \quad \mathbb{R}^3,$$

$$\nabla \cdot (\mu H) = 0 \quad \text{in} \quad \mathbb{R}^3,$$

where $\alpha := \log \sigma$. We note that the derivatives $\partial^k \alpha$ and $\partial^k \beta$ are uniformly continuous on $\mathbb{R}^3$ for $|k| = 0, 1, 2$.

The complex geometric optics solutions we aim to construct are of the form

$$H(x; \zeta) = e^{i\zeta x}(a(x; \zeta) + r(x; \zeta)),$$

where $\zeta \in C^3 \setminus \{0\}$ such that $\zeta \cdot \zeta = i\omega \sigma_0 \mu_0$, $a$ is a specific complex-valued smooth vector field on $\mathbb{R}^3$ and $r$ is the correction term. Then (16) is equivalent to

$$e^{-i\zeta x} L_{\sigma, \mu} (e^{i\zeta x} r) = -f, \quad f := e^{-i\zeta x} L_{\sigma, \mu} (e^{i\zeta x} a).$$

3.1. Solution operator. For $\zeta \in C^3 \setminus \{0\}$ such that $\zeta \cdot \zeta = i\omega \sigma_0 \mu_0$, we define the operators

$$\nabla \zeta := \nabla + i\zeta, \quad \Delta \zeta := \Delta + 2i\zeta \cdot \nabla.$$

Then

$$e^{-i\zeta x} \circ \nabla \circ e^{i\zeta x} = \nabla \zeta \quad \text{and} \quad e^{-i\zeta x} \circ \Delta \circ e^{i\zeta x} = \Delta \zeta - \nabla \cdot (\mu \zeta \nabla).$$

For $\delta \in \mathbb{R}$, we define the $L^2$-based weighted space on $\mathbb{R}^3$

$$L^2_\delta := \left\{ f : \mathbb{R}^3 \to \mathbb{C}^3 : \|f\|_{L^2_\delta} := \left( \int_{\mathbb{R}^3} (1 + |x|^2)^{\delta} |f(x)|^2 \, dx \right)^{1/2} < \infty \right\},$$

and

$$H^1_\delta := \left\{ f \in L^2_\delta : \|f\|_{H^1_\delta} := \|f\|_{L^2_\delta} + \sum_{j=1}^3 \|\partial_j f\|_{L^2_\delta} < \infty \right\}.$$

Proposition 3.1. For $k \in \mathbb{C}$, suppose $\zeta \in C^3$ with $\zeta \cdot \zeta = k$, $-1 < \delta < 0$. Assume that $\gamma \in C^2(\mathbb{R}^3)$ is positive. Then for $f \in L^2_\delta$ there is a unique $u \in L^2_\delta$ solving

$$(-\Delta \zeta - \nabla \log \gamma \cdot \nabla \zeta) u = f \quad \text{in} \quad \mathbb{R}^3$$

such that

$$\|u\|_{L^2_\delta} \leq C \frac{1}{|k|} \|f\|_{L^2_\delta},$$

for some constant $C > 0$. Furthermore, $u$ belongs to $H^1_\delta$. 
Proof. It follows from the identity
\[ (-\Delta + \nabla \log \gamma \cdot \nabla \zeta) u = \gamma^{-1/2} (-\Delta + q) (\gamma^{1/2} u), \quad q := \gamma^{-1/2} \Delta \gamma^{1/2} \in C_0(\mathbb{R}^3), \]
that solving (20) is equivalent to solving
\[ (-\Delta + q) \tilde{u} = \gamma^{1/2} f, \quad \text{in} \quad \mathbb{R}^3, \]
where \( \tilde{u} := \gamma^{1/2} u. \) By [49, Theorem 1.6] there is a unique \( \tilde{u} \in L^2_\beta \) solving the above equation and satisfying
\[ \| \tilde{u} \|_{L^2_\beta} \leq \frac{C}{|\zeta|} \| \gamma^{1/2} f \|_{L^2_{\beta+1}}. \]
Next, [48, Lemma 1.15] implies that \( \tilde{u} \in H^1_\delta. \) The result now follows immediately by setting \( u = \gamma^{-1/2} \tilde{u}. \)

According to Proposition 3.1, for sufficiently large \( |\zeta|, \) there is a bounded inverse \( G_{\zeta,Y} : L^2_{\beta+1} \to L^2_\delta \) of \(-\Delta + \nabla \log \gamma \cdot \nabla \zeta\) such that
\[ \| G_{\zeta,Y} \|_{L^2_{\beta+1},L^2_\delta} = \Theta \left( \frac{1}{|\zeta|} \right) \quad \text{as} \quad |\zeta| \to \infty. \]
Moreover, \( G_{\zeta,Y} \) maps \( L^2_{\beta+1} \) into \( H^1_\delta. \)

3.2. Mollified \( \sigma \) and \( \mu. \) Let \( \Phi \in C_0^\infty(\mathbb{R}^3) \) with \( 0 \leq \Phi \leq 1 \) and \( \int_{\mathbb{R}^3} \Phi(x) \, dx = 1. \) For a fixed \( \epsilon \) with \( 0 < \epsilon < 1/8, \) we consider
\[ \Phi_\tau(x) := \left( \frac{1}{1 + \epsilon} \right)^3 \Phi \left( \frac{x}{\tau} \right) \quad \text{for large} \quad \tau > 0. \]
We define
\[ \alpha^\tau(x;\tau) := \alpha * \Phi_\tau(x), \quad \beta^\tau(x;\tau) := \beta * \Phi_\tau(x) \quad \text{for} \quad x \in \mathbb{R}^3. \]
Then \( \alpha^\tau(\cdot,\tau), \beta^\tau(\cdot,\tau) \in C_0^\infty(\mathbb{R}^3). \) From \( \partial^k \alpha^\tau = (\partial^k \alpha) * \Phi_\tau \) and \( \partial^k \beta^\tau = (\partial^k \beta) * \Phi_\tau, \) it follows that
\[ \text{supp}(\partial^k \alpha^\tau), \text{supp}(\partial^k \beta^\tau) \subset B_{R+2\tau-\epsilon}(0), \quad |\kappa| = 1, 2. \]
We also have
\[ \| \alpha - \alpha^\tau \|_{W^{2,\infty}(\mathbb{R}^3)} = o(1), \quad \| \beta - \beta^\tau \|_{W^{2,\infty}(\mathbb{R}^3)} = o(1) \quad \text{as} \quad \tau \to \infty. \]
Indeed,
\[ \partial^k \alpha(x) - \partial^k \alpha^\tau(x;\tau) = \int_{\mathbb{R}^3} \Phi(y) [\partial^k \alpha(x) - \partial^k \alpha(x - \tau^{-\epsilon} y)] \, dy, \quad |\kappa| = 0, 1, 2, \]
and uniform continuity of \( \partial^k \alpha \) on \( \mathbb{R}^3 \) gives the desired estimate using [22, Theorem 0.13]. A similar argument can be used for \( \beta. \)

Finally, using that \( \partial^k \alpha^\tau = \alpha * (\partial^k \Phi_\tau) \) and \( \partial^k \beta^\tau = \beta * (\partial^k \Phi_\tau), \) a direct calculation shows that
\[ \| \partial^k \alpha^\tau \|_{L^{\infty}(\mathbb{R}^3)} \leq \Theta_k (\tau^{|k|}) \quad \text{for} \quad |k| \geq 0 \quad \text{as} \quad \tau \to \infty, \]
which implies that
\[ \| \alpha^\tau \|_{W^{k,\infty}(\mathbb{R}^3)} \leq \Theta_k (\tau^{|k|}) \quad \text{for} \quad k = 0, 1, 2, \ldots \quad \text{as} \quad \tau \to \infty. \]
We note that stronger estimates follow from (22)
\[ \| \alpha^\tau \|_{W^{2,\infty}(\mathbb{R}^3)}, \quad \| \beta^\tau \|_{W^{2,\infty}(\mathbb{R}^3)} = \Theta(1) \quad \text{as} \quad \tau \to \infty. \]

3.3. Transport equation. Since \( \zeta \cdot \zeta = i \omega \sigma_0 \mu_0, \) using (19), we obtain (cf. (18))
\[ f = -\Delta a - \nabla (\nabla \zeta \cdot a) - \nabla a \times \zeta \times a - i \omega \sigma a + i \omega \sigma_0 a \]
\[ = -\Delta a - \nabla (\nabla \zeta \cdot a) - \nabla a \times \zeta \times a - i \omega (\sigma_0 \mu_0) a - 2i \zeta \cdot \nabla a \cdot (\nabla \zeta \cdot a) \zeta \times a \times (\zeta \times a). \]
We shall consider \( \zeta \) of the form \( \zeta = \tau \rho + \zeta_1 \) where \( \tau > 0 \) is a large parameter, \( \rho \in \mathbb{C}^3 \) is independent of \( \tau \) and satisfies \( \text{Re} \rho, \text{Im} \rho = 0 \) and \( |\text{Re} \rho| = |\text{Im} \rho| = 1, \) and \( \zeta_1 = \Theta(1) \) as \( \tau \to \infty. \) Then
\[ f = -\Delta a - \nabla (\nabla \zeta \cdot a) - \nabla a \times \zeta \times a - i \omega (\sigma_0 \mu_0) a - 2i \zeta_1 \cdot \nabla a \cdot (\nabla \zeta \cdot a) \zeta_1 \times a \times (\zeta_1 \times a) \]
\[ - i \tau (\Delta (\nabla \beta \cdot a) - \nabla (\nabla \beta \cdot a) \rho + (\alpha - \alpha^\tau) \rho \times (\rho \times a) + \Delta (\nabla \beta \cdot a) \rho + \varphi a^\tau \rho \times (\rho \times a)). \]
In order to get \( \| f \|_{L^2_\delta} = o(\tau) \) as \( \tau \to \infty, \) for \(-1 < \delta < 0, \) we should construct \( a \) satisfying the transport equation, that is,
\[ 2 \rho \cdot \nabla a + (\nabla \beta) \cdot a) \rho + \varphi a^\tau \rho \times (\rho \times a) = o(1) \quad \text{in} \quad L^2_{\delta+1} \quad \text{as} \quad \tau \to \infty. \]
We get
\[(26) \quad 2\rho \cdot \nabla a + (\nabla \rho^\delta \cdot a)\rho + (\nabla a^\delta \cdot a)\rho - (\rho \cdot \nabla a^\delta) a = o(1) \quad \text{in} \quad L^2_{\delta + 1} \quad \text{as} \quad \tau \to \infty.\]
For arbitrary \(s_0 \in \mathbb{R}\), we seek solutions of (26) in the form
\[a = e^{-a^{1/2} \rho} + s_0 e^{a^{1/2} \rho} \Psi^\rho \bar{\rho},\]
where \(\chi(x) := \chi(\rho^\theta x), 0 < \theta < 1/2\) with \(\chi \in C^\infty_0(\mathbb{R}^3)\) such that \(\chi(x) \equiv 1\) for \(|x| < 1/2\) and \(\chi(x) \equiv 0\) for \(|x| \geq 1\). Then
\[(27) \quad \partial_j a = -\frac{1}{2} \partial_j a^\rho e^{-a^{1/2} \rho} + \frac{s_0}{2} \partial_j a^\rho e^{a^{1/2} \rho} e^{\tau \psi^\rho} \frac{\bar{\rho}}{\rho} + \frac{s_0}{\tau^\theta} (\partial_j \chi)(\rho^\theta x) e^{\tau \psi^\rho} e^{a^{1/2} \rho} e^{\tau \psi^\rho} \frac{\bar{\rho}}{\rho} + s_0 \partial_j \psi^\rho e^{a^{1/2} \rho} e^{\tau \psi^\rho} \frac{\bar{\rho}}{\rho}.\]

Substituting these into (26), we come to
\[s_0 \left( \tau^{-\theta} \rho^\delta 2\rho \cdot (\nabla \chi)(\tau^{-\theta} \cdot) + \chi e^{\tau \psi^\rho} e^{a^{1/2} \rho} e^{\tau \psi^\rho} \frac{\bar{\rho}}{\rho} \right) = o(1) \quad \text{in} \quad L^2_{\delta + 1} \quad \text{as} \quad \tau \to \infty.\]

We observe that by (21), \(\nabla (\rho^\delta + \bar{\rho}^\delta) = \chi (\nabla (\rho^\delta + \bar{\rho}^\delta))\) for large enough \(\tau > 0\). Therefore, it is enough to find \(\Psi^\rho\) in \(C^\infty(\mathbb{R}^3)\) with a nice control on \(\|\Psi^\rho\|_{L^\infty(\mathbb{R}^3)}\) and satisfying
\[(28) \quad 2\rho \cdot \nabla \Psi^\rho + \bar{\rho} \cdot (\nabla \chi)(\rho^\delta + \bar{\rho}^\delta) = 0 \quad \text{in} \quad \mathbb{R}^3.\]
Since \(\rho \cdot \rho = 0\) and \(\Re \rho \cdot \Im \rho = 0\), the operator \(N^\rho := \rho \cdot \nabla\) is just the \(\bar{\rho}\)-operator in certain linear coordinates. Its inverse is defined as
\[N^{-1} \rho f(x) := \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{f(x - y_1 \Re \rho - y_2 \Im \rho)}{y_1 + iy_2} \, dy, \quad f \in C_0(\mathbb{R}^3).\]

Then by [42, Lemma 4.6],
\[\Psi^\rho(x, \rho; \tau) := -\frac{1}{2} N^{-1} \rho \left( \rho \cdot \nabla (\rho^\delta + \bar{\rho}^\delta) \right) \in C^\infty(\mathbb{R}^3)\]
satisfies equation (28). It also follows from [42, Lemma 4.6] and (22) that
\[(29) \quad \|\Psi^\rho\|_{W^{2,\infty}(\mathbb{R}^3)} \leq \Theta(1) \left\{ \|\rho^\delta\|_{W^{2,\infty}(\mathbb{R}^3)} + \|\bar{\rho}^\delta\|_{W^{2,\infty}(\mathbb{R}^3)} \right\} = \Theta(1) \quad \text{as} \quad \tau \to \infty.\]
Furthermore, [42, Lemma 4.6] and (23) imply that
\[(30) \quad \|\Psi^\rho\|_{W^{k,\infty}(\mathbb{R}^3)} \leq \Theta_k(1) \left\{ \|\rho^\delta\|_{W^{k+1,\infty}(\mathbb{R}^3)} + \|\bar{\rho}^\delta\|_{W^{k+1,\infty}(\mathbb{R}^3)} \right\} = \Theta(1) \quad \text{as} \quad \tau \to \infty.\]
We set
\[\Psi(x, \rho) := -\frac{1}{2} N^{-1} \rho \left( \rho \cdot \nabla (\rho^\delta + \bar{\rho}^\delta) \right) \in L^\infty(\mathbb{R}^3).\]

Then [49, Lemma 3.1] together with (21) and (22) implies that
\[\|\Psi^\rho(\cdot, \rho; \tau) - \Psi(\cdot, \rho)\|_{L^\infty_{\rho} (\mathbb{R}^3)} = o(1) \quad \text{as} \quad \tau \to \infty.\]
Finally, by [43, Lemma 3.1] together with (23) and (24),
\[(31) \quad |\partial^\kappa \Psi^\rho(x, \rho; \tau)| \leq \Theta_k(1) \left\{ \begin{array}{ll}
(1 + |x_T|^2)^{-1/2} & \text{if} \ |\kappa| = 0, 1,
(1 + |x_T|^2)^{-1/2} & \text{otherwise}
\end{array} \right. \quad \text{as} \quad \tau \to \infty,\]
where \(x_T\) is the projection of \(x\) onto \(\text{Span}(\rho_1, \rho_2)\) and \(x_\perp = x - x_T\), and \(R > 0\) is such that \(B(0, R)\) contains \(\text{supp}(\sigma - \sigma_0)\) and \(\text{supp}(\mu - \mu_0)\).

It follows that
\[(32) \quad \int_{|x| \leq R, |x_\perp| \leq R} (1 + |x_T|^2)^{-1/2} (1 + |x|^2)^{\delta + 1} \, dx \leq \Theta(1) \int_{|x_T| \leq R} (1 + |x_T|^2)^{\delta} \, dx_T = \Theta(\tau^{2\delta + 1}) \quad \text{as} \quad \tau \to \infty.\]
Similarly,
\[(33) \quad \int_{|x| \leq R, |x_\perp| \leq R} (1 + |x_T|^2)^{-j} (1 + |x|^2)^{\delta + 1} \, dx \leq \Theta(1) \left( 1 - \frac{1}{(1 + \tau^2)^{j+2-2\delta}} \right) = \Theta(1) \quad \text{as} \quad \tau \to \infty \quad \text{for} \quad j \geq 2.\]
We have
Lemma 3.2. Let $\chi_\tau$ and $\Psi^i$ be as above. Then

\begin{align}
(34) & \quad \|\partial_j (\chi_\tau \Psi^i)\|_{L^2_{\delta + 1}} = o(\tau^{\delta+1/\theta}), \\
(35) & \quad \|\partial_j (\chi_\tau \Psi^i) \partial_k (\chi_\tau \Psi^i)\|_{L^2_{\delta + 1}} = o(1), \\
(36) & \quad \|\partial_j (\chi_\tau \Psi^i) \partial_k (\chi_\tau \Psi^i) \partial_l (\chi_\tau \Psi^i)\|_{L^2_{\delta + 1}} = o(1), \\
(37) & \quad \|\partial_j \partial_k (\chi_\tau \Psi^i)\|_{L^2_{\delta + 1}} = o(\tau), \\
(38) & \quad \|\partial_j \partial_k (\chi_\tau \Psi^i)\|_{L^2_{\delta + 1}} = o(\tau^{3\delta}), \\
(39) & \quad \|\partial_j \partial_k (\chi_\tau \Psi^i)\|_{L^2_{\delta + 1}} = o(\tau^{1+\varepsilon})
\end{align}
as $\tau \to \infty$.

Proof. Using (31) and (32), we obtain

$$
\|\partial_j (\chi_\tau \Psi^i)\|_{L^2_{\delta + 1}} \leq \frac{1}{\tau^{\theta}} \|\partial_j \chi (\tau^{-\theta} \cdot) \Psi^i\|_{L^2_{\delta + 1}} + \|\chi_\tau \partial_j \Psi^i\|_{L^2_{\delta + 1}}
\leq \Theta \left( \frac{1}{\tau^{\theta}} + 1 \right) \int_{|x_\tau| \leq \tau^\theta, |x_\delta| \leq \tau} (1 + |x_\tau|^2)^{-1} (1 + |x|^2)^{\delta + 1} \, dx
= \Theta (\tau^{\delta+1/\theta}) \quad \text{as} \quad \tau \to \infty.
$$

The other estimates follow readily. \hfill \Box

3.4. Estimating $\|f\|_{L^2_{\delta + 1}}$. With our choice of $a$, we have

\begin{equation}
2\mu \cdot \nabla a + (\nabla \beta \cdot a) \rho + \nabla a \times (\rho \times a) = s_0 \tau^{-\theta} 2\mu \cdot (\nabla \chi)(\tau^{-\theta} \cdot) \Psi^i e^{\alpha t/2} e^{\chi_\tau \Psi^i} p.
\end{equation}

Then, as in the proof of (34), we use (24), (29), (31) and (32) to obtain

$$
\|i\tau (2\mu \cdot \nabla a + (\nabla \beta \cdot a) \rho + \nabla a \times (\rho \times a))\|_{L^2_{\delta + 1}} \leq \Theta (\tau^{1-\theta}) \int_{|x_\tau| \leq \tau^\theta, |x_\delta| \leq \tau} (1 + |x_\tau|^2)^{-1} (1 + |x|^2)^{\delta + 1} \, dx \right)^{1/2} = \Theta (\tau^{1+\delta \theta}) \quad \text{as} \quad \tau \to \infty.
$$

Using (13), (21), (22), (24) and (29), it is then straightforward to show that

$$
\|i\tau ((\nabla \beta \cdot a) \rho + \nabla(a - a^\theta) \times (\rho \times a))\|_{L^2_{\delta + 1}} = o(1) \quad \text{as} \quad \tau \to \infty,
$$

$$
\| (\nabla \beta \cdot a) i\xi_1 + \nabla a \times (i\xi_1 \times a)\|_{L^2_{\delta + 1}} = o(1) \quad \text{as} \quad \tau \to \infty,
$$

$$
\|i\omega (\sigma \mu - \sigma_0 \mu_0) a\|_{L^2_{\delta + 1}} = o(1) \quad \text{as} \quad \tau \to \infty.
$$

Now, using expressions (24) and (29), we find that

$$
\|\partial_j a\|_{L^2_{\delta + 1}} \leq \Theta (1) \|\partial_j a^\theta\|_{L^2_{\delta + 1}} + \Theta (1) \|\partial_j (\chi_\tau \Psi^i)\|_{L^2_{\delta + 1}}
$$

Thus with (21), (24) and (34), we obtain

\begin{equation}
\|\partial_j a\|_{L^2_{\delta + 1}} = o(\tau^\theta) \quad \text{as} \quad \tau \to \infty,
\end{equation}

and therefore,

$$
\|2i\xi_1 \cdot \nabla a\|_{L^2_{\delta + 1}} = o(\tau^\theta) \quad \text{as} \quad \tau \to \infty,
$$

$$
\|\nabla \times \nabla a\|_{L^2_{\delta + 1}} \leq \Theta (1) \|\nabla \times a\|_{L^2_{\delta + 1}} \leq \Theta (1) \sum_{j=1}^3 \|\partial_j a\|_{L^2_{\delta + 1}} = o(\tau^\theta) \quad \text{as} \quad \tau \to \infty
$$

and

$$
\|\nabla (\nabla \beta \cdot a)\|_{L^2_{\delta + 1}} \leq \|\nabla \nabla \beta\| \|a\|_{L^2_{\delta + 1}} + \Theta (1) \sum_{j=1}^3 \|\partial_j a\|_{L^2_{\delta + 1}} = o(\tau^\theta) \quad \text{as} \quad \tau \to \infty.
$$
In the last step, we used (13) and that supp$(\nabla \nabla \beta)$ is compact. Finally, by (24) and (29)
\[
\| \partial_j \partial_k a \|_2 \leq \Theta(1) \| \partial_j \partial_k a^2 \|_2 + \Theta(1) \| \partial_j a^2 \partial_j a^4 \|_2 + \Theta(1) \| \partial_j (\partial_k (\chi \psi)) \|_2 + \Theta(1) \| \partial_j (\chi \psi) \|_2 + \Theta(1) \| \partial_j (\chi \psi) \|_2
\]
whence, by (21), (24), (34), (35) and (37),
\[
(42) \quad \| \partial_j \partial_k a \|_2 = o(\tau) \quad \text{as} \quad \tau \to \infty.
\]
Thus,
\[
\| \Delta a \|_2 = o(\tau) \quad \text{as} \quad \tau \to \infty.
\]
Combining all of the above estimates, we come to
\[
f \|_{L^2_{\tau \wedge 1}} = o(\tau) \quad \text{as} \quad \tau \to \infty.
\]

3.5. Estimating $\| \nabla \times f \|_{L^2_{\tau \wedge 1}}$. Since
\[
\nabla \times f = \nabla \times f + i \mathcal{I} \times f \quad \text{and} \quad \| i \mathcal{I} \times f \|_{L^2_{\tau \wedge 1}} = o(\tau^2),
\]
we need to estimate $\| \nabla \times f \|_{L^2_{\tau \wedge 1}}$. By straightforward calculations,
\[
\nabla \times f = -\nabla \Delta a - \nabla \times (\nabla (\alpha \times \nabla a)) - i \omega \nabla \times (\sigma \mu - \sigma_0 \mu_0) \alpha - 2i \nabla \times (\zeta_1 \cdot \nabla a) - \nabla (\nabla \beta \cdot a) - \nabla \times (\nabla \alpha \times (i \zeta_1 \times a))
\]
\[- i \tau \nabla \times (\nabla (\beta - \beta^2) \cdot a) \times \rho - i \tau \nabla \times (\nabla (\alpha - \alpha^2) \times (\rho \times a)) - i \tau \nabla \times (2 \rho \cdot \nabla \alpha + (\nabla \beta \cdot a) \rho + \nabla \alpha^2 \times (\rho \times a)).
\]
By (40),
\[
\nabla \times (2 \rho \cdot \nabla \alpha + (\nabla \beta \cdot a) \rho + \nabla \alpha^2 \times (\rho \times a)) = \frac{s_0}{\tau^3} \left[ (2 \rho \cdot \nabla \chi)(\tau^{-\theta} \cdot \Psi^0 e^{2i/2e^{\tau/4}} \chi) \right] \times \rho.
\]
Then, as in the proof of (37), we use (24) and (29),
\[
\| i \tau \nabla \times (2 \rho \cdot \nabla \alpha + (\nabla \beta \cdot a) \rho + \nabla \alpha^2 \times (\rho \times a)) \|_{L^2_{\tau \wedge 1}} \leq \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j \partial_k \chi \tau^{-\theta} \cdot \Psi^0 e^{2i/2e^{\tau/4}} \chi \|_{L^2_{\tau \wedge 1}} + \Theta(1) \right)
\]
\[
\leq \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j \partial_k \chi \tau^{-\theta} \cdot \Psi^0 e^{2i/2e^{\tau/4}} \chi \|_{L^2_{\tau \wedge 1}} \right) + \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j \partial_k \chi \tau^{-\theta} \cdot \Psi^0 e^{2i/2e^{\tau/4}} \chi \|_{L^2_{\tau \wedge 1}} \right) = o(\tau) \quad \text{as} \quad \tau \to \infty,
\]
where in the last step, we also used (21), (31), (32) and (33). Next, using (13), (21), (22), (24), (29) and (41), we obtain
\[
\| i \tau \nabla \times (\nabla (\beta - \beta^2) \cdot a) \times \rho \|_{L^2_{\tau \wedge 1}} \leq \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j \partial_k (\beta - \beta^2) \partial_j a \|_{L^2_{\tau \wedge 1}} \right) = o(\tau^{1+\theta}) \quad \text{as} \quad \tau \to \infty.
\]
Similarly,
\[
\| i \tau \nabla \times (\nabla (\alpha - \alpha^2) \times (\rho \times a)) \|_{L^2_{\tau \wedge 1}} \leq \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j \partial_k (\alpha - \alpha^2) \partial_j a \|_{L^2_{\tau \wedge 1}} \right) + \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j (\alpha - \alpha^2) \partial_j a \|_{L^2_{\tau \wedge 1}} \right) = o(\tau^{1+\theta}),
\]
\[
\| \nabla (\nabla \beta \cdot a) \times \zeta_1 \|_{L^2_{\tau \wedge 1}} \leq \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j \partial_k \chi \beta a \|_{L^2_{\tau \wedge 1}} \right) + \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_k \beta \partial_j a \|_{L^2_{\tau \wedge 1}} \right) = o(\tau^{\theta}),
\]
\[
\| \nabla \times (\nabla \alpha \times (i \zeta_1 \times a)) \|_{L^2_{\tau \wedge 1}} \leq \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j \partial_k \alpha a \|_{L^2_{\tau \wedge 1}} \right) + \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_k \alpha \partial_j a \|_{L^2_{\tau \wedge 1}} \right) = o(\tau^{\theta}),
\]
and
\[
\| i \omega \nabla \times (\sigma \mu - \sigma_0 \mu_0) \alpha \|_{L^2_{\tau \wedge 1}} \leq \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j (\sigma \mu) a \|_{L^2_{\tau \wedge 1}} \right) + \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j (\sigma \mu - \sigma_0 \mu_0) \partial_j a \|_{L^2_{\tau \wedge 1}} \right) = o(\tau^{\theta})
\]
as $\tau \to \infty$. Using (42) we find that
\[
\| 2i \nabla \cdot (\zeta_1 \cdot \nabla a) \|_{L^2_{\tau \wedge 1}} \leq \Theta(1) \left( \sum_{j,k=1}^3 \| \partial_j \partial_k a \|_{L^2_{\tau \wedge 1}} \right) = o(\tau)
and
\[ \| \nabla \times (\nabla a \times \nabla a) \|_{L^2_{\delta + 1}} \leq \Theta(1) \sum_{j,k,l} \| \partial_j \partial_k a \partial_l a \|_{L^2_{\delta + 1}} \]

as \( \tau \to \infty \).

Using (24) and (29), we estimate
\[ \| \partial_l \partial_j \partial_k a \|_{L^2_{\delta + 1}} \leq \Theta(1) \| \partial_l \partial_j \partial_k a \|_{L^2_{\delta + 1}} + \Theta(1) \| \partial_l \partial_j \partial_k a \|_{L^2_{\delta + 1}} + \Theta(1) \| \partial_l \partial_j \partial_k a \|_{L^2_{\delta + 1}} \]

and conclude that
\[ \| \nabla \times (\nabla a \times \nabla a) \|_{L^2_{\delta + 1}} = o(\tau) \]

as \( \tau \to \infty \).

This implies that
\[ \| \nabla \times \Delta a \|_{L^2_{\delta + 1}} = o(\tau) \]

as \( \tau \to \infty \).

Combining all of these, we finally come to \( \| \nabla \times f \|_{L^2_{\delta + 1}} = o(\tau) \) and, hence, \( \| \nabla a \times f \|_{L^2_{\delta + 1}} = o(\tau) \) as \( \tau \to \infty \).

### 3.6. Construction of complex geometric optics solutions.

Now, we are ready to construct complex geometric optics solutions for the system (16) and (17) which is equivalent to

(43)
\[ e^{-i\kappa \cdot x} L_{\sigma, \mu}(e^{i\kappa \cdot r}) = -f, \]

(44)
\[ \nabla \kappa \cdot r + \nabla \log \mu \cdot r = -\nabla \kappa \cdot a + \nabla \log \mu \cdot a. \]

According to the discussion in Section 3.4, we have \( \| \nabla \times f \|_{L^2_{\delta + 1}} = o(\tau) \) as \( \tau \to \infty \).

First, we need to show that for sufficiently large \( |\kappa| \) there is \( r \in L^2_\delta \) solving (43). Using (19),

(45)
\[ e^{-i\kappa \cdot x} L_{\sigma, \mu}(e^{i\kappa \cdot r}) = -\Delta r - \nabla \kappa (\nabla \log \mu \cdot r) - \nabla \log \sigma \times \nabla \kappa \times r - i\omega (\sigma \mu - \sigma_0 \mu_0) r. \]

We have
\[ \nabla \kappa (\nabla \log \mu \cdot r) = \nabla \log \mu \times \nabla \kappa \times r + \nabla \log \mu \cdot \nabla \kappa \times r + \nabla \nabla (\log \mu) r \]

Therefore, (43) can be written as

(46)
\[ -\Delta r - \nabla \log \mu \cdot \nabla \kappa \times r - \nabla \log (\sigma \mu) \times \nabla \kappa \times r - V_1 r = -f, \]

where
\[ V_1 := \nabla \nabla (\log \mu) + i\omega (\sigma \mu - \sigma_0 \mu_0). \]

We need to deal with the third term on the left-hand-side of (46). We define \( Q := \nabla \kappa \times r \), and find that
\[ \Delta r = \nabla \kappa \cdot r - \nabla \nabla \kappa \times r. \]

Also, it follows from (44) that
\[ \nabla \kappa \nabla \kappa \cdot r = -\nabla \kappa \nabla \kappa \cdot a - \nabla \kappa (\nabla \log \mu \cdot a) - \nabla \kappa (\nabla \log \mu \cdot r) \]

Substituting these into (45), we come to
\[ e^{-i\kappa \cdot x} L_{\sigma, \mu}(e^{i\kappa \cdot r}) = \nabla \kappa \times r + \nabla \kappa \nabla \kappa \cdot a + \nabla \kappa (\nabla \log \mu \cdot a) - \nabla \log \sigma \times \nabla \kappa \times r - i\omega (\sigma \mu - \sigma_0 \mu_0) r. \]

Hence, (43) implies
\[ \nabla \kappa \times Q + \nabla \kappa \nabla \kappa \cdot a + \nabla \kappa (\nabla \log \mu \cdot a) - \nabla \log \sigma \times Q - i\omega (\sigma \mu - \sigma_0 \mu_0) r = -f. \]

Applying \( \nabla \kappa \times \) to it, we get
\[ \nabla \kappa \times \nabla \kappa \times Q - \nabla \kappa \times (\nabla \log \sigma \times Q) - i\omega \nabla (\sigma \mu) \times r - i\omega (\sigma \mu - \sigma_0 \mu_0) Q = -\nabla \kappa \times f \]
as
\[ \nabla_\xi \times (\nabla_\xi \nabla_\xi \cdot a) = 0 \quad \text{and} \quad \nabla_\xi \times (\nabla_\xi (\nabla \log \mu \cdot a)) = 0. \]

Using the fact that \( \nabla_\xi \cdot Q = 0 \), we can write
\[ \nabla_\xi \times (\nabla \log \sigma \times Q) = \nabla \nabla (\log \sigma) Q - (\Delta \log \sigma) Q - \nabla \log \sigma \cdot \nabla \xi Q. \]

Thus, we come to
\[ -(\nabla_\xi Q - \nabla \log \sigma \cdot \nabla_\xi Q - V_2 Q = i \omega \nabla (\sigma \mu) \times r - \nabla_\xi \times f, \]
where
\[ V_2 \defeq \nabla \nabla (\log \sigma) + i \omega (\sigma \mu - \sigma_0 \mu_0) - \Delta \log \sigma. \]

According to the discussion in Section 3.5, for sufficiently large \( \tau \), there are bounded inverses
\[ G_{\xi,\mu} : L^2_{\delta + 1} \rightarrow L^2_\delta \quad \text{and} \quad G_{\xi,\sigma} : L^2_{\delta + 1} \rightarrow L^2_\delta \]
of \( -\Delta_\xi - \nabla \log \mu \cdot \nabla_\xi \) and \( -\Delta_\xi - \nabla \log \sigma \cdot \nabla_\xi \), respectively, satisfying
\[ \| G_{\xi,\mu} \|_{L^2_{\delta + 1}; L^2_\delta} \leq \Theta \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to \infty \]
and
\[ \| G_{\xi,\sigma} \|_{L^2_{\delta + 1}; L^2_\delta} \leq \Theta \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to \infty. \]

Furthermore, \( G_{\xi,\mu} \) and \( G_{\xi,\sigma} \) map the space \( L^2_{\delta + 1} \) into \( H^1_\delta \).

If \( \tau \) is large enough, we apply \( G_{\xi,\sigma} \) to (47) and obtain the following identity
\[ \langle \nabla_\xi r - \nabla \log \mu \cdot \nabla_\xi r - i \omega \nabla_\xi \times f, \sigma \rangle = 0; \]
where
\[ W \defeq \nabla \nabla (\sigma \mu) \times (\id - G_{\xi,\sigma} V_1) - 1 \circ G_{\xi,\sigma} \circ \nabla (\sigma \mu) \times , \]
\[ F \defeq f + \nabla \nabla (\sigma \mu) \times (\id - G_{\xi,\sigma} V_1) - 1 \circ G_{\xi,\sigma} \circ \nabla_\xi \times f. \]

It follows from (49) and (13) that
\[ \| W \|_{L^2_{\delta + 1}; L^2_\delta} \leq \Theta \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to \infty. \]
and
\[ \| F \|_{L^2_{\delta + 1}} \leq \| f \|_{L^2_{\delta + 1}} + \Theta \left( \frac{1}{\tau} \right) \| \nabla_\xi \times f \|_{L^2_{\delta + 1}} = o(\tau) \quad \text{as} \quad \tau \to \infty. \]

Applying \( G_{\xi,\mu} \) to (52), we get
\[ \langle \nabla_\xi r - \nabla \log \mu \cdot \nabla_\xi r - i \omega \nabla_\xi \times f, \sigma \rangle = 0; \]
where
\[ r \defeq - (\id - i \omega G_{\xi,\mu} W - G_{\xi,\mu} V_1) r = - G_{\xi,\mu} F. \]

Since \( V_1 \) is compactly supported and \( W \) satisfies (53), the operator \( \id - i \omega G_{\xi,\mu} W - G_{\xi,\mu} V_1 \) is invertible in \( L^2_\delta \) for \( \tau \) sufficiently large. Therefore, one can solve the above identity for \( r \in L^2_\delta \) by
\[ r = -(\id - i \omega G_{\xi,\mu} W - G_{\xi,\mu} V_1) - 1 G_{\xi,\mu} F. \]

Finally, by (49) and (51), we can show that
\[ \| r \|_{L^2_\delta} \leq \Theta \left( \frac{1}{\tau} \right) \| F \|_{L^2_{\delta + 1}} = o(1) \quad \text{as} \quad \tau \to \infty \]
and
\[ \| \nabla_\xi \times r \|_{L^2_\delta} \leq \Theta \left( \frac{1}{\tau} \right) \{ \| r \|_{L^2_\delta} + \| \nabla_\xi \times f \|_{L^2_{\delta + 1}} \} = o(\tau) \quad \text{as} \quad \tau \to \infty. \]
It follows from (50) and (54) that

\[
\begin{align*}
    r &= i\omega G_{\zeta,\mu}(Wr) + G_{\zeta,\mu}(V_1 r) - G_{\zeta,\mu} F, \\
    Q &= G_{\zeta,\sigma}(V_2 Q) + G_{\zeta,\sigma} \left\{ i\omega V(\mu u) \times r - \nabla \zeta \times f \right\}.
\end{align*}
\]

Since \( V_1, V_2 \) and \( W \) are compactly supported and \( G_{\zeta,\mu} \) and \( G_{\zeta,\sigma} \) map the space \( L^2_{\delta+1} \) into \( H^1_\delta \), this implies that \( r, \nabla \zeta \times r \in H^1_\delta \). Thus, we have constructed the following complex geometric optics solution for (16)

\[
H(x;\zeta) = e^{icx} \left\{ e^{-a^2(x;\tau)/2} \rho + s_0 b \alpha \theta^2(x;\tau)/2 e^{\varphi^2(x;\rho)} \mathbf{P} + r(x;\zeta) \right\}.
\]

Our next step is to show that \( \nabla \cdot (\mu H) = 0 \). We observe that (16) is equivalent to

\[
\nabla \times (\sigma^{-1} \nabla \times H) - \sigma^{-1} \nabla (\mu^{-1} \nabla \cdot (\mu H)) - i\omega \mu H = 0.
\]

Applying the divergence to this identity and setting \( \nu = \mu^{-1} \nabla \cdot (\mu H) \), we get

\[
-\nabla \cdot (\sigma^{-1} \nabla \nu) - i\omega \mu \nu = 0
\]

which can be written as

\[
-\Delta \tilde{\nu} + \tilde{q} \tilde{\nu} = 0, \quad \tilde{\nu} := \sigma^{-1/2} \nu, \quad \tilde{q} := \sigma^{1/2} \Delta \sigma^{-1/2} - i\omega \sigma \mu.
\]

Straightforward calculations give

\[
\tilde{\nu} = e^{icx} u, \quad u := \sigma^{-1/2} \mu^{-1} \nabla \zeta \cdot (\sigma(a + r)).
\]

Then, by (19), \( u \) satisfies

\[
-\Delta u = qu, \quad q := -\sigma^{1/2} \Delta \sigma^{-1/2} + i\omega (\sigma \mu - \sigma_0 \mu_0).
\]

By [49, Theorem 1.6], again, there is a bounded inverse, \( G_\zeta : L^2_{\delta+1} \rightarrow L^2_\delta \) of \( -\Delta \zeta \) such that \( \| G_\zeta \|_{L^2_\delta \rightarrow L^2_\delta} \leq \Theta(|\zeta|^{-1}) \) as \( |\zeta| \rightarrow \infty \). Since \( u = G_\zeta (qu) \) for large enough \( |\zeta| \) and \( q \) is compactly supported,

\[
\| u \|_{L^2_\delta} \leq \| G_\zeta (q u) \|_{H^1_\delta} \leq \Theta \left( \frac{1}{|\zeta|} \right) \| u \|_{L^2_\delta} \quad \text{as} \quad |\zeta| \rightarrow \infty.
\]

This implies that \( u = 0 \) and hence \( \nabla \cdot (\mu H) = 0 \) for sufficiently large \( |\zeta| \). Thus, restricting \( H \) onto \( \Omega \), we get \( H \in H^1(\Omega; C^3) \) solving (14) and satisfying \( \nabla \times H \in H^1(\Omega; C^3) \). Clearly, \( \nabla \times H|_{\partial \Omega} \in H^{1/2}(\partial \Omega; C^3) \). Then by [28, Corollary A.20],

\[
\text{Div}(\nu \times H)|_{\partial \Omega} = -\nu \cdot (\nabla \times H)|_{\partial \Omega} \in H^{1/2}(\partial \Omega; C^3)
\]

Therefore, \( \nu \times H|_{\partial \Omega} \in TH^{1/2}(\partial \Omega) \) and hence \( H \in H^{1/2}(\partial \Omega; C^3) \). In a similar way, also using the fact that \( H \) is a solution of (14), one can easily show that \( \nu \times (\nabla \times H)|_{\partial \Omega} \in TH^{1/2}(\partial \Omega) \) and hence \( \nabla \times H \in H^1(\Omega; \Omega) \). Thus, we proved

**Proposition 3.3.** Let \( \Omega \subset \mathbb{R}^3 \) be an open bounded set with \( C^{1,1} \) boundary and \( \sigma, \mu \in C^2(\overline{\Omega}) \) with \( \sigma \leq \sigma_0, \mu \geq \mu_0 \) for some \( \sigma_0, \mu_0 > 0 \). Assume that \( \sigma \) can be extended positively to \( \mathbb{R}^3 \) so that \( \sigma - \sigma_0, \mu - \mu_0 \in C^0_\rho(\mathbb{R}^3) \). Let \( \zeta \in C^3 \) be such that \( \zeta \cdot \zeta = i\omega \sigma \mu_0 \), \( \zeta = \tau \rho + \zeta_1 \) where \( \tau > 0 \) is a large parameter, \( \rho \in C^3 \) is independent of \( \tau \) and satisfies \( \text{Re} \rho \cdot \text{Im} \rho = 0 \) and \( |\text{Re} \rho| = |\text{Im} \rho| = 1 \), and \( \zeta_1 = \Theta(1) \) as \( \tau \rightarrow \infty \). Then, for any \( s_0 \in \mathbb{R} \), there is a solution \( H \in H^1_\Omega(\Omega) \) for (14) of the form

\[
H(x;\zeta) = e^{icx} \left\{ e^{-a^2(x;\tau)/2} \rho + s_0 b \alpha \theta^2(x;\tau)/2 e^{\varphi^2(x;\rho)} \mathbf{P} + r(x;\zeta) \right\}.
\]

Furthermore, \( \nabla \times H \in H^1_\Omega(\Omega) \). The function \( \Psi^\zeta(\cdot, \rho; \tau) \in C^\infty(\mathbb{R}^3) \) satisfies \( \| \Psi^\zeta \|_{L^1(\mathbb{R}^3)} = o(1) \) as \( \tau \rightarrow \infty \) and converges to \( \Psi(\cdot, \rho) := -N^- \rho^{-1} \mathbf{P} \cdot \nabla \log(\sigma \mu)^{1/2} \in L^\infty(\mathbb{R}^3) \) in \( L^2_{\text{loc}}(\mathbb{R}^3) \) as \( \tau \rightarrow \infty \). The function \( a^2(\cdot; \tau) \in C^\infty(\mathbb{R}^3) \) satisfies \( a^2 \| \Psi^\zeta \|_{L^1(\mathbb{R}^3)} = o(1) \) as \( \tau \rightarrow \infty \) and \( a^2 \| \log(\sigma \mu)^{1/2} \|_{L^2_{\text{loc}}(\mathbb{R}^3)} = o(1) \) as \( \tau \rightarrow \infty \). The correction term, \( r \), satisfies \( \| r \|_{H^1(\mathbb{R}^3)} \) is \( o(1) \) and \( \| \nabla \zeta \times r \|_{L^2(\mathbb{R}^3)} \) is \( o(1) \) as \( \tau \rightarrow \infty \).

4. Proof of Theorem 1.3

Since we assume that \( \partial^\alpha \sigma_1|_{\partial \Omega} = \partial^\alpha \sigma_2|_{\partial \Omega} \) and \( \partial^\alpha \mu_1|_{\partial \Omega} = \partial^\alpha \mu_2|_{\partial \Omega} \) for \( |\alpha| \leq 2 \), we can extend \( \sigma_j \) and \( \mu_j, j = 1, 2 \), to \( C^2 \) functions defined on \( \mathbb{R}^3 \), still denoted by \( \sigma_j \) and \( \mu_j \), such that \( \sigma_j \geq \sigma_0, \mu_j \geq \mu_0 \) on \( \mathbb{R}^3 \), \( \sigma_j - \sigma_0, \mu_j - \mu_0 \in C^0_\rho(\mathbb{R}^3) \) and \( \sigma_j = \sigma_2 \) and \( \mu_j = \mu_2 \) on \( \mathbb{R}^3 \setminus \overline{\Omega} \). These kind of extensions (of Whitney type) hold for all functions defined on any closed subset of \( \mathbb{R}^3 \) that can be approximated by certain polynomials. The argument to prove the existence of such polynomials is similar to the one in [6, Section 2] for \( C^{1,\ell} \) functions on \( \Omega \). The only difference, here, is that the authors of [6] refer to [46, Section 2 of Chapter VI], while we refer to [46, Section 4.7 of Chapter VI]; see also [9, Section 3].
Proposition 4.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{1,1}$ boundary and $\sigma_j, \mu_j \in C^2(\bar{\Omega})$, $j = 1, 2$, with $\sigma_j \geq \sigma_0$, $\mu_j \geq \mu_0$ for some $\sigma_0, \mu_0 > 0$. Suppose that $Z^\omega_{\sigma_1, \mu_1} = Z^\omega_{\sigma_2, \mu_2}$; then

$$
\int_\Omega (\mu_1 - \mu_2) H_1 \cdot H_2 \, dx + \frac{1}{i \omega} \int_\Omega \frac{[\sigma_1 - \sigma_2]}{\sigma_1 \sigma_2} \nabla \times H_1 \cdot \nabla \times H_2 \, dx = 0
$$

for all $H_j \in H^1_{\text{Div}}(\Omega)$ with $\nabla \times H_j \in H^1_{\text{Div}}(\Omega)$ solving

$$
\nabla \times (\sigma_j^{-1} \nabla \times H_j) - i \omega \mu_j H_j = 0 \quad \text{in} \quad \Omega, \quad j = 1, 2.
$$

Proof. Define

$$
E_j := \sigma_j^{-1} \nabla \times H_j, \quad j = 1, 2.
$$

Then $E_j \in H^1_{\text{Div}}(\Omega)$ and $\nabla \times E_j = i \omega \mu_j H_j$. Hence $(H_j, E_j) \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\Omega)$, $j = 1, 2$, solve

$$
\nabla \times E_j = i \omega \mu_j H_j \quad \text{and} \quad \nabla \times H_j = \sigma_j E_j \quad \text{in} \quad \Omega, \quad j = 1, 2.
$$

Then the assumption $Z^\omega_{\sigma_1, \mu_1} = Z^\omega_{\sigma_2, \mu_2}$ implies existence of $(H', E') \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\Omega)$ satisfying

$$
\nabla \times E' = i \omega \mu_2 H' \quad \text{and} \quad \nabla \times H' = \sigma_2 E' \quad \text{in} \quad \Omega
$$

and

$$
t(H') = t(H_1) \quad \text{and} \quad t(E') = t(E_1) \quad \text{on} \quad \partial \Omega.
$$

Integrating by parts,

$$
\int_\Omega \nabla \times (H' - H_1) \cdot E_2 \, dx - \int_\Omega i \omega \mu_2 (H' - H_1) \cdot H_2 \, dx = \int_\Omega \nabla \times (H' - H_1) \cdot E_2 \, dx - \int_\Omega (H' - H_1) \cdot \nabla \times E_2 \, dx
$$

$$
= \int_{\partial \Omega} t(H' - H_1) \cdot E_2 \, dS(x) = 0,
$$

where $dS$ is the surface measure on $\partial \Omega$. Similarly,

$$
\int_\Omega \nabla \times (E' - E_1) \cdot H_2 \, dx - \int_\Omega \sigma_2 (E' - E_1) \cdot E_2 \, dx = 0.
$$

Adding these two identities, we obtain

$$
\int_\Omega [\nabla \times (H' - H_1) - \sigma_2 (E' - E_1)] \cdot E_2 \, dx + \int_\Omega [\nabla \times (E' - E_1) - i \omega \mu_2 (H' - H_1)] \cdot E_2 \, dx = 0.
$$

It is easy to show that

$$
\nabla \times (H' - H_1) - \sigma_2 (E' - E_1) = (\sigma_2 - \sigma_1) E_1, \quad \nabla \times (E' - E_1) - i \omega \mu_2 (H' - H_1) = i \omega (\mu_2 - \mu_1) H_1.
$$

Substituting these into the latter integral identity, we come to

$$
\int_\Omega (\sigma_2 - \sigma_1) E_1 \cdot E_2 \, dx + \int_\Omega i \omega (\mu_2 - \mu_1) H_1 \cdot H_2 \, dx = 0.
$$

This implies (55). $\square$

Let $\xi, \rho_1, \rho_2 \in \mathbb{R}^3$ be such that $|\rho_1| = |\rho_2| = 1$ and $\rho_1 \cdot \rho_2 = \rho_1 \cdot \xi = \rho_2 \cdot \xi = 0$. Consider

$$
\xi^1 = \frac{\xi}{2} + i \tau \rho_2 + \sqrt{1 - \frac{|\xi|^2}{4 \tau^2} + \frac{i \omega \sigma_0 \mu_0}{\tau^2}} \rho_1,
$$

$$
\xi^2 = \frac{\xi}{2} + i \tau \rho_2 + \sqrt{1 - \frac{|\xi|^2}{4 \tau^2} + \frac{i \omega \sigma_0 \mu_0}{\tau^2}} \rho_1.
$$

Here, by $\sqrt{\cdot}$ we mean its principal branch. Then

$$
\xi^1 = \tau \rho + \xi^1_1 \quad \text{with} \quad \xi^1_1 = \Theta(1) \quad \text{as} \quad \tau \to \infty \quad \text{and} \quad \xi^1 - \xi^2 = \xi, \quad \xi^1 \cdot \xi^2 = i \omega \sigma_0 \mu_0,
$$

where $\rho := \rho_1 + i \rho_2$. By Proposition 3.3, there are complex geometric optics solutions $H_1, H_2 \in H^1_{\text{Div}}(\Omega)$, with $\nabla \times H_1, \nabla \times H_2 \in H^1_{\text{Div}}(\Omega)$ satisfying

$$
\nabla \times (\sigma_1^{-1} \nabla \times H_1) - i \omega \mu_1 H_1 = 0 \quad \text{and} \quad \nabla \times (\sigma_2^{-1} \nabla \times H_2) - i \omega \mu_2 H_2 = 0 \quad \text{in} \quad \Omega,
$$

respectively, which have the following forms

$$
H_1(x; \xi^1) = e^{i \xi^1 \cdot x} \left( a_1 \rho + \frac{1}{2} b_1 \rho + r_1 \right), \quad H_2(x; \xi^2) = e^{-i \xi^2 \cdot x} \left( -a_2 \rho - \frac{1}{2} b_2 \rho + r_2 \right),
$$

where $a_1, a_2, b_1, b_2, r_1, r_2$ are constant vectors in $\mathbb{R}^3$. Therefore, we have

$$
\int_{\partial \Omega} t(H_1 - H_2) \cdot E_1 \, dS(x) = 0,
$$

which gives

$$
\int_{\partial \Omega} t(H_1) \cdot E_1 \, dS(x) - \int_{\partial \Omega} t(H_2) \cdot E_1 \, dS(x) = 0.
$$

By the divergence theorem,

$$
\int_\Omega \nabla \times (H_1 - H_2) \cdot E_1 \, dx = \int_{\partial \Omega} t(H_1 - H_2) \cdot E_1 \, dS(x) = 0.
$$

Hence

$$
\nabla \times (H_1 - H_2) \cdot E_1 = 0 \quad \text{in} \quad \Omega.
$$

By the divergence theorem again,

$$
\int_\Omega (\sigma_1^{-1} \nabla \times H_1 - \sigma_2^{-1} \nabla \times H_2) \cdot E_1 \, dx = 0,
$$

which gives

$$
\nabla \times (\sigma_1^{-1} \nabla \times H_1 - \sigma_2^{-1} \nabla \times H_2) = 0 \quad \text{in} \quad \Omega.
$$

Therefore, we have

$$
\int_\Omega (\sigma_1^{-1} \nabla \times H_1 - \sigma_2^{-1} \nabla \times H_2) \cdot E_1 \, dx = 0,
$$

where $E_1$ is the electric field in the complex geometric optics solutions $H_1, H_2 \in H^1_{\text{Div}}(\Omega)$.
where
\[
\begin{align*}
    a_1 &= e^{-a_1^1(x,t)/2}, \\    b_1 &= e^{a_1^1(x,t)/2} e^{\Psi_1^1(x,t,\rho)}, \\    a_2 &= e^{-a_2^2(x,t)/2}, \\    b_2 &= e^{a_2^2(x,t)/2} e^{\Psi_2^2(x,t,\rho)}.
\end{align*}
\]

The functions \( \Psi_1^1(\cdot, \rho, \tau), \Psi_2^2(\cdot, \rho, \tau) \in C^\infty(\mathbb{R}^3) \) satisfy
\[
\begin{align}
    &\| \Psi_1^1 \|_{W^{1,\infty}(\mathbb{R}^3)}, \| \Psi_2^2 \|_{W^{1,\infty}(\mathbb{R}^3)} = O(1) \quad \text{as} \quad \tau \to \infty, \\
    &\| \Psi_1^1 - \Psi_1 \|_{L^2_{\text{loc}}(\mathbb{R}^3)}, \| \Psi_2^2 - \Psi_2 \|_{L^2_{\text{loc}}(\mathbb{R}^3)} = o(1) \quad \text{as} \quad \tau \to \infty,
\end{align}
\]
where
\[
\Psi_j(\cdot, \rho) := -N_\rho^{-1} |\rho| \cdot \nabla \log(\sigma_j \mu_j)^{1/2} \in L^\infty(\mathbb{R}^3), \quad j = 1, 2.
\]

Furthermore, the functions \( a_1^1(\cdot; \tau), a_2^2(\cdot; \tau) \in C^\infty(\mathbb{R}^3) \) satisfy
\[
\begin{align}
    &\| a_1^1 \|_{W^{2,\infty}(\mathbb{R}^3)}, \| a_2^2 \|_{W^{2,\infty}(\mathbb{R}^3)} = O(1) \quad \text{as} \quad \tau \to \infty, \\
    &\| a_1^1 - \log \sigma_1 \|_{W^{2,\infty}(\mathbb{R}^3)}, \| a_2^2 - \log \sigma_2 \|_{W^{2,\infty}(\mathbb{R}^3)} = o(1) \quad \text{as} \quad \tau \to \infty.
\end{align}
\]

The correction terms, \( r_1, r_2 \in H^1_{\text{div}}(\Omega) \), satisfy
\[
\| r_j \|_{L^2(\Omega)} = o(1) \quad \text{and} \quad \| \nabla \xi \times r_j \|_{L^2(\Omega)} = o(\tau) \quad \text{as} \quad \tau \to \infty, \quad j = 1, 2.
\]

Then, using that \( \xi^j = \tau \rho + \xi^j \), we find that
\[
\begin{align*}
    \nabla \times E_1 &= e^{-ik^{1,2}(x, \tau)} \nabla \times (a_1 \rho + \frac{1}{2} b_1 \nabla + r_1), \\
    \nabla \times E_2 &= e^{-ik^{2,1}(x, \tau)} \left( -a_2 \rho - \frac{1}{2} b_2 \nabla + r_2 \right).
\end{align*}
\]

Substituting \( H_1, H_2, \nabla \times H_1 \) and \( \nabla \times H_2 \) into (55) and dividing the whole identity by \( \tau^2 \), we obtain
\[
\begin{align*}
    \frac{1}{\tau^2} \int_{\Omega} (\mu_1 - \mu_2) e^{ik^{1,2}(x, \tau)} \left( a_1 \rho + \frac{1}{2} b_1 \nabla + r_1 \right) \cdot \left( -a_2 \rho - \frac{1}{2} b_2 \nabla + r_2 \right) dx \\
    - \frac{1}{\tau^2} \int_{\Omega} i \omega \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{ik^{1,2}(x, \tau)} \left( \nabla \times (a_1 \rho + \frac{1}{2} b_1 \nabla + r_1) \right) \cdot \left( \nabla \times (a_2 \rho + \frac{1}{2} b_2 \nabla + r_2) \right) dx \\
    + \frac{1}{\tau^2} \int_{\Omega} i \omega \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{ik^{1,2}(x, \tau)} \left( b_1 \rho_1 + \rho_2 \right) \cdot \left( b_2 \rho_1 + \rho_2 \right) dx \\
    - \frac{1}{\tau^2} \int_{\Omega} i \omega \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{ik^{1,2}(x, \tau)} \left( \nabla \times (a_1 \rho + \frac{1}{2} b_1 \nabla + r_1) \right) \cdot \left( \nabla \times (a_2 \rho + \frac{1}{2} b_2 \nabla + r_2) \right) dx \\
    + \frac{1}{\tau^2} \int_{\Omega} i \omega \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{ik^{1,2}(x, \tau)} \left( b_1 \rho_1 + \rho_2 \right) \cdot \left( \nabla \times (a_2 \rho + \frac{1}{2} b_2 \nabla + r_2) \right) dx \\
    + \frac{1}{\tau^2} \int_{\Omega} i \omega \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{ik^{1,2}(x, \tau)} \left( \nabla \times (a_1 \rho + \frac{1}{2} b_1 \nabla + r_1) \right) \cdot \left( b_2 \rho_1 + \rho_2 \right) dx + \frac{1}{\tau^2} \int_{\Omega} i \omega \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{ik^{1,2}(x, \tau)} b_1 \rho_2 dx = 0.
\end{align*}
\]

For the last term on the left-hand side, we also used the property \(|\rho_1 \rho_2| = 1\) as \( \rho_1 \rho_2 = 0 \) and \(|\rho_1| = |\rho_2| = 1\). By the Cauchy-Schwartz inequality,
\[
\left| \int_{\Omega} (\mu_1 - \mu_2) e^{ik^{1,2}(x, \tau)} \left( a_1 \rho + \frac{1}{2} b_1 \nabla + r_1 \right) \cdot \left( -a_2 \rho - \frac{1}{2} b_2 \nabla + r_2 \right) dx \right| \leq O(1) \left\| (\mu_1 - \mu_2) \left( a_1 \rho + \frac{1}{2} b_1 \nabla + r_1 \right) \right\|_{L^2(\Omega)} \left\| -a_2 \rho - \frac{1}{2} b_2 \nabla + r_2 \right\|_{L^2(\Omega)} \quad \text{as} \quad \tau \to \infty.
\]

Then, by (56), (58) and (60),
\[
\left| \int_{\Omega} (\mu_1 - \mu_2) e^{ik^{1,2}(x, \tau)} \left( a_1 \rho + \frac{1}{2} b_1 \nabla + r_1 \right) \cdot \left( -a_2 \rho - \frac{1}{2} b_2 \nabla + r_2 \right) dx \right| = O(1) \quad \text{as} \quad \tau \to \infty.
\]
In a similar way, we obtain
\[
\left| \int_{\Omega} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{i k \cdot x} \left[ \nabla \cdot (\nabla_1 a_1 \times \rho + \frac{1}{2} \nabla \nabla_1 b_1 \times \nabla_2) \right] \cdot \left[ \nabla \cdot (\nabla_1 a_2 \times \rho + \frac{1}{2} \nabla \nabla_1 b_2 \times \nabla_2) \right] \, dx \right| = \mathcal{O}(1),
\]
\[
\left| \int_{\Omega} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{i k \cdot x} \left[ \nabla \cdot (\nabla_1 a_1 \times \rho + \frac{1}{2} \nabla \nabla_1 b_1 \times \nabla_2) \right] \cdot (b_2 \rho_1 \times \rho_2) \, dx \right| = \mathcal{O}(1),
\]
\[
\left| \int_{\Omega} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{i k \cdot x} (b_1 \rho_1 \times \rho_2) \cdot (\nabla \nabla_1 \cdot r_2) \, dx \right| = o(\tau),
\]
\[
\left| \int_{\Omega} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{i k \cdot x} (\nabla_1 \times a_1) \cdot (\nabla \nabla_1 \cdot r_1) \, dx \right| = o(\tau),
\]
\[
\left| \int_{\Omega} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{i k \cdot x} (b_1 \rho_1 \times \rho_2) \cdot (\nabla \nabla_1 \cdot r_2) \, dx \right| = o(\tau),
\]
as \(\tau \to \infty\). Here, we used again that \(\xi^j_1 = \mathcal{O}(1)\) as \(\tau \to \infty\), \(j = 1, 2\). Finally, we use (56)-(59), to show that
\[
\left| \int_{\Omega} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} e^{i k \cdot x} b_1 b_2 \, dx \right| \leq \mathcal{O}(1) \left\| e^{a_1^j/2 + a_2^j/2} e^{\Psi_1^j} - a_1^j/2 e^{\Psi_1^j} \right\|_{L^2(\Omega)},
\]
\[
\leq \mathcal{O}(1) \left\| e^{a_1^j/2 + a_2^j/2} e^{\Psi_1^j} - a_1^j/2 e^{\Psi_1^j} \right\|_{L^2(\Omega)},
\]
\[
\leq \mathcal{O}(1) \left\| a_1^j - \log \sigma_1 \right\|_{L^2(\Omega)} + \mathcal{O}(1) \left\| a_2^j - \log \sigma_2 \right\|_{L^2(\Omega)} + \mathcal{O}(1) \left\| \Psi_1^j - \Psi_1^j \right\|_{L^2(\Omega)} + \mathcal{O}(1) \left\| \Psi_2^j - \Psi_2^j \right\|_{L^2(\Omega)} = o(1) \quad \text{as} \quad \tau \to \infty,
\]
employing the basic inequality,
\[
|e^z - e^w| \leq |z - w| e^{\max(Rez,Rew)}, \quad z, w \in \mathbb{C}
\]
from [29]. According to these estimates, taking the limit as \(\tau \to \infty\) in (61), we come to
\[
\int_{\mathbb{R}^3} e^{i k \cdot x} \frac{(\sigma_1 - \sigma_2)}{\sigma_1^2 \sigma_2^2} e^{\Psi_1^j} \, dx = 0.
\]

Note that the integration is extended to all of \(\mathbb{R}^3\) since \(\sigma_1 - \sigma_2 = 0\) on \(\mathbb{R}^3 \setminus \Omega\). This implies that \(\sigma_1 = \sigma_2\).

Next, we set \(\sigma = \sigma_1 = \sigma_2\). By Proposition 3.3, there are complex geometric optics solutions \(H_1, H_2 \in H^1_{\text{Div}}(\Omega)\), with \(\nabla \times H_1, \nabla \times H_2 \in H^1_{\text{Div}}(\Omega)\) satisfying
\[
\nabla \times (\sigma^{-1} \nabla \times H_1) - i \omega \mu_1 H_1 = 0 \quad \text{and} \quad \nabla \times (\sigma^{-1} \nabla \times H_2) - i \omega \mu_2 H_2 = 0 \quad \text{in} \quad \Omega,
\]
respectively, which have the following forms
\[
H_1(x; \xi^1) = e^{i \xi^1 \cdot x} \left( a_1 \rho + r_1 \right), \quad H_2(x; \xi^2) = e^{i \xi^2 \cdot x} \left( - a_2 \rho - \frac{1}{2} b_2 \nabla_2 + r_2 \right),
\]
where
\[
a_1 = e^{-a_1^j(x;\tau)/2}, \quad a_2 = e^{-a_2^j(x;\tau)/2}, \quad b_2 = e^{a_2^j(x;\tau)/2} e^{\Psi^j(x;\rho;\tau)}.
\]
The function \(\Psi^j(\cdot; \rho; \tau) \in C^\infty(\mathbb{R}^3)\) satisfies
\[
\| \Psi^j \|_{W^{1,\infty}(\mathbb{R}^3)} = \mathcal{O}(1) \quad \text{and} \quad \| \Psi^j - \Psi \|_{L^1_{\text{loc}}(\mathbb{R}^3)} = o(1) \quad \text{as} \quad \tau \to \infty,
\]
where
\[
\Psi(\cdot; \rho) := - N_{\rho}^{-1} (\nabla \log(\sigma_2^{-1/2})) / \nabla \log(\sigma_2) \in L^\infty(\mathbb{R}^3), \quad j = 1, 2.
\]
Furthermore, the function \(\alpha^j(\cdot; \tau) \in C^\infty(\mathbb{R}^3)\) satisfies
\[
\| \alpha^j \|_{W^{2,\infty}(\mathbb{R}^3)} = \mathcal{O}(1) \quad \text{and} \quad \| \alpha^j - \log \sigma \|_{W^{2,\infty}(\mathbb{R}^3)} = o(1) \quad \text{as} \quad \tau \to \infty.
\]
The correction terms \( r_1, r_2 \in H^1_{\text{Div}}(\Omega) \) satisfy
\[
\|r_j\|_{L^2(\Omega)} = o(1) \quad \text{and} \quad \|\nabla r_j\|_{L^2(\Omega)} = o(\tau) \quad \text{as} \quad \tau \to \infty, \quad j = 1, 2.
\]
Substituting \( H_1, H_2 \) and \( \sigma = \sigma_1 = \sigma_2 \) into (55), we come to
\[
- \int_\Omega (\mu_1 - \mu_2) e^{i k x} a_1 b_2 \, dx + \int_\Omega (\mu_1 - \mu_2) e^{i k x} a_1 \cdot r_2 \, dx \\
- \int_\Omega (\mu_1 - \mu_2) e^{i k x} r_1 \cdot (a_2 \rho + \frac{1}{2} b_2 \rho) \, dx + \int_\Omega (\mu_1 - \mu_2) e^{i k x} r_1 \cdot r_2 \, dx = 0.
\]
By the Cauchy-Schwarz inequality together with (64) and (65),
\[
\int_\Omega (\mu_1 - \mu_2) e^{i k x} a_1 \cdot r_2 \, dx \leq \Theta(1) \int_\Omega |a_1 \cdot r_2| \, dx \leq \Theta(1) \|a_1\|_{L^2(\Omega)} \|r_2\|_{L^2(\Omega)} = o(1)
\]
as \( \tau \to \infty \). In a similar way, and also using (63), one can show that
\[
\int_\Omega (\mu_1 - \mu_2) e^{i k x} r_1 \cdot (a_2 \rho + \frac{1}{2} b_2 \rho) \, dx = o(1) \quad \text{and} \quad \int_\Omega (\mu_1 - \mu_2) e^{i k x} r_1 \cdot r_2 \, dx = o(1) \quad \text{as} \quad \tau \to \infty.
\]
Finally, using (63) and (64),
\[
\int_{\mathbb{R}^3} e^{i k x} (\mu_1 - \mu_2) a \cdot b \, dx = \int_{\mathbb{R}^3} (\mu_1 - \mu_2) e^{i k x} \Psi \, dx \leq \Theta(1) \|\Psi\|_{L^2(\Omega)} \leq \Theta(1) \|\Psi\|_{L^2(\Omega)} = o(1)
\]
as \( \tau \to \infty \). Here, we have again employed inequality (62). Thus, letting \( \tau \to \infty \), we obtain
\[
\int_{\mathbb{R}^3} e^{i k x} (\mu_1 - \mu_2) e^{i k x} \, dx = 0.
\]
The integration is extended to all of \( \mathbb{R}^3 \) since \( \mu_1 - \mu_2 = 0 \) on \( \mathbb{R}^3 \setminus \Omega \). This implies that \( \mu_1 = \mu_2 \) completing the proof of Theorem 1.3.

5. Reflection approach

In this section, we use Isakov’s reflection approach [25] to prove the following local uniqueness result where the region of the boundary that is inaccessible for measurements is a part of a plane. For a closed \( \Gamma \subset \partial \Omega \), define
\[
C_{\Gamma}(\sigma, \mu; \omega) := \{(t(H)|_\Gamma, t(E)|_\Gamma) : (H, E) \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\Omega) \text{ is a solution to (1) with supp}(t(H)) \subseteq \Gamma\}.
\]

**Theorem 5.1.** Let \( \Omega \subset \{x \in \mathbb{R}^3 : x_3 < 0\} \) be a bounded domain with \( C^{1,1} \) boundary and let \( \Gamma_0 = \partial \Omega \cap \{x \in \mathbb{R}^3 : x_3 = 0\} \) and \( \Gamma = \partial \Omega \setminus \Gamma_0 \). Suppose that \( \sigma_j, \mu_j \in C^2(\Omega), \ j = 1, 2, \) satisfy \( \sigma_j \geq \sigma_0 \) and \( \mu_j \geq \mu_0 \), for some constants \( \sigma_0, \mu_0 > 0 \), and
\[
\partial^a \sigma_j|_\Gamma = \partial^a \sigma_0|_\Gamma \quad \text{and} \quad \partial^a \mu_j|_\Gamma = \partial^a \mu_2|_\Gamma \quad \text{for} \quad |a| \leq 2.
\]
In addition, assume that \( \sigma_j \) and \( \mu_j, \ j = 1, 2, \) can be extended into \( \mathbb{R}^3 \) as \( C^2 \) functions which are invariant under reflection across the plane \( \{x \in \mathbb{R}^3 : x_3 = 0\} \). Then \( C_{\Gamma}(\sigma_1, \mu_1; \omega) = C_{\Gamma}(\sigma_2, \mu_2; \omega) \) implies \( \sigma_1 = \sigma_2 \) and \( \mu_1 = \mu_2 \).

Similar results were obtained for the inverse conductivity problem in [25] and for the IEEM in [8]. Consider the reflected domain
\[
\Omega^* := \{(x_1, x_2, -x_3) \in \mathbb{R}^3 : (x_1, x_2, x_3) \in \Omega\}
\]
and define
\[
\mathcal{U} := \Omega \cap \Gamma_0 \cap \Omega^*.
\]
By the assumptions in Theorem 5.1, we can extend the coefficients \( \sigma_j \) and \( \mu_j \) into \( \mathcal{U} \) as \( C^2 \) functions which are even with respect to \( x_j \) for \( j = 1, 2 \). Next, by the assumption (66), we can extend \( \sigma_j \) and \( \mu_j, \ j = 1, 2, \) to \( C^2 \) functions defined on \( \mathbb{R}^3 \), still denoted by \( \sigma_j \) and \( \mu_j \), such that \( \sigma_j \geq \sigma_0, \mu_j \geq \mu_0 \) on \( \mathbb{R}^3 \), \( \sigma_1 - \sigma_0, \sigma_2 \) in \( C^2(\mathbb{R}^3) \) and \( \sigma_1 = \sigma_2 \) and \( \mu_1 = \mu_2 \) on \( \mathbb{R}^3 \setminus \mathcal{U} \).

**Proposition 5.2.** Let \( \Omega \subset \{x \in \mathbb{R}^3 : x_3 < 0\} \) be a bounded domain with \( C^{1,1} \) boundary. Let \( \Gamma_0 := \partial \Omega \cap \{x \in \mathbb{R}^3 : x_3 = 0\} \) and \( \Gamma := \partial \Omega \setminus \Gamma_0 \). Suppose that
\[
Z^{\omega}_{\sigma_1, \mu_1}(f)|_\Gamma = Z^{\omega}_{\sigma_2, \mu_2}(f)|_\Gamma \quad \text{for all} \quad f \in TH^1_{\text{Div}}(\partial \Omega) \quad \text{with} \quad \text{supp}(f) \subseteq \Gamma;
\]
then
\[
\int_\Omega (\mu_1 - \mu_2) H_1 \cdot H_2 \, dx + \frac{1}{i \omega} \int_\Omega \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} (\nabla \times H_1) \cdot (\nabla \times H_2) \, dx = 0
\]
for all $H_j \in H^1_{\text{Div}}(\Omega)$ with $\nabla \times H_j \in H^1_{\text{Div}}(\Omega)$ solving
\[ \nabla \times (\sigma_j^{-1} \nabla \times H_j) - i \omega \mu_j H_j = 0 \quad \text{in} \quad \Omega, \quad j = 1, 2, \]
and satisfying $\text{supp}(t(H_j)) \subseteq \Gamma$.

**Proof.** Similarly as in the proof of (55), define
\[ E_j := \sigma_j^{-1} \nabla \times H_j, \quad j = 1, 2. \]
Then $E_j \in H^1_{\text{Div}}(\Omega)$ and $\nabla \times E_j = i \omega \mu_j H_j$. Hence $(H_j, E_j) \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\Omega)$, $j = 1, 2$, solve
\[ \nabla \times E_j = i \omega \mu_j H_j \quad \text{and} \quad \nabla \times H_j = \sigma_j^{-1} E_j \quad \text{in} \quad \Omega, \quad j = 1, 2. \]
Then by the assumption (67), there is $(H', E') \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\Omega)$ with $\text{supp}(t(H')) \subseteq \Gamma$ satisfying
\[ \nabla \times E' = i \omega \mu_2 H' \quad \text{and} \quad \nabla \times H' = \sigma_2 E' \quad \text{in} \quad \Omega \]
and
\[ t(H')|_{\Gamma} = t(H_1)|_{\Gamma} \quad \text{and} \quad t(E')|_{\Gamma} = t(E_1)|_{\Gamma}. \]
Integrating by parts, leads to
\[ \int_{\Omega} \nabla \times (H' - H_1) \cdot E_2 \, dx - \int_{\Omega} i \omega \mu_2 (H' - H_1) \cdot H_2 \, dx = \int_{\Omega} \nabla \times (H' - H_1) \cdot E_2 \, dx - \int_{\Omega} (H' - H_1) \cdot \nabla \times E_2 \, dx \]
\[ = \int_{\partial \Omega} t(H' - H_1) \cdot E_2 \, dS(x) + \int_{\Gamma} t(H' - H_1) \cdot t(E_2) \, dS(x) = 0, \]
since both $t(H')$ and $t(H_1)$ are supported on $\Gamma$. Similarly,
\[ \int_{\Omega} \nabla \times (E' - E_1) \cdot H_2 \, dx - \int_{\Omega} \sigma_2 (E' - E_1) \cdot E_2 \, dx = \int_{\Omega} \nabla \times (E' - E_1) \cdot \nabla \times H_2 \, dx = 0, \]
since $t(H_2)$ is supported on $\Gamma$. The remainder of the proof of (68) is similar to that of (55).

For $\beta : \mathbb{R}^3 \to \mathbb{C}$ and $X : \mathbb{R}^3 \to \mathbb{C}^3$, we define the reflections as
\[ \beta^*(x) := \beta(x_1, x_2, -x_3), \quad X^*(x) := \{(X_1(x_1, x_2, -x_3), X_2(x_1, x_2, -x_3), -X_3(x_1, x_2, -x_3)) \}
\]
with the properties,
\[ \nabla \beta^* = (\nabla \beta)^*, \quad (\beta X)^* = \beta^* X^*, \quad \nabla \times X^* = -(\nabla \times X)^*. \]

**Proof of Uniqueness.** Consider $\zeta^1$ and $\zeta^2$ defined as in the proof of Theorem 1.3. Then, by Proposition 3.3, there are complex geometric optics solutions $\tilde{H}_1, \tilde{H}_2 \in H^1_{\text{Div}}(\mathcal{U})$, with $\nabla \times \tilde{H}_j, \nabla \times \tilde{H}_2 \in H^1_{\text{Div}}(\mathcal{U})$, for
\[ \nabla \times (\sigma_j^{-1} \nabla \times \tilde{H}_j) - i \omega \mu_j \tilde{H}_j = 0 \quad \text{and} \quad \nabla \times (\sigma_j^{-1} \nabla \times \tilde{H}_2) - i \omega \mu_2 \tilde{H}_2 = 0 \quad \text{in} \quad \mathcal{U}, \]
respectively, which have the following forms
\[ \tilde{H}_1(x; \zeta^1) = e^{\ii \zeta^1 \cdot x} \left( a_1 \rho + \frac{1}{2} b_1 \overrightarrow{p} + r_1 \right), \quad \tilde{H}_2(x; \zeta^2) = e^{\ii \zeta^2 \cdot x} \left( -a_2 \rho - \frac{1}{2} b_2 \overrightarrow{p} + r_2 \right), \]
where
\[ a_1 = e^{-\alpha_1^2(x; \rho; \tau)/2}, \quad b_1 = e^{\alpha_1^2(x; \rho; \tau)/2} e^{\Psi_1^1(x, \rho; \tau)}, \]
\[ a_2 = e^{-\alpha_2^2(x; \rho; \tau)/2}, \quad b_2 = e^{\alpha_2^2(x; \rho; \tau)/2} e^{\Psi_2^2(x, \rho; \tau)}. \]
The functions $\Psi_1^1(\cdot, \rho; \tau), \Psi_2^2(\cdot, \rho; \tau) \in C^\infty(\mathbb{R}^3), \alpha_1^2(\cdot; \tau), \alpha_2^2(\cdot; \tau) \in C^\infty(\mathbb{R}^3)$ and the correction terms $r_1, r_2 \in H^1_{\text{Div}}(\mathcal{U})$ satisfy (57)-(60). Using (69), it follows that
\[ \nabla \times (\sigma_j^{-1} \nabla \times \tilde{H}_j^*) - i \omega \mu_j \tilde{H}_j^* = -\nabla \times (\sigma_j^{-1} \nabla \times \tilde{H}_j) - (i \omega \mu_j \tilde{H}_j)^* = (\nabla \times (\sigma_j^{-1} \nabla \times \tilde{H}_j) - i \omega \mu_j \tilde{H}_j)^* = 0 \quad \text{in} \quad \mathcal{U}. \]
Therefore,
\[ H_1(x; \rho) := \tilde{H}_1(x; \rho) - \tilde{H}_1^*(x; \rho), \quad H_2(x; \rho) := \tilde{H}_2(x; \rho) - \tilde{H}_2^*(x; \rho), \]
also satisfy
\[ \nabla \times (\sigma_j^{-1} \nabla \times \tilde{H}_1) - i \omega \mu_1 \tilde{H}_1 = 0 \quad \text{and} \quad \nabla \times (\sigma_j^{-1} \nabla \times \tilde{H}_2) - i \omega \mu_2 \tilde{H}_2 = 0 \quad \text{in} \quad \mathcal{U}. \]
As in the proof of Proposition 3.3, it is not difficult to see that $H_1|_{\Omega}$ and $H_2|_{\Omega}$, still denoted by $H_1$ and $H_2$, respectively, belong to $H^1_{\text{Div}}(\Omega)$. Moreover, $\nabla \times H_1, \nabla \times H_2 \in H^1_{\text{Div}}(\Omega)$. Using the fact that $\nu = (0,0,1)$ on $\Gamma$, we have
\( \nabla \times H_1 = \nabla \times H_1 + (\nabla \times \tilde{H}_1)^* \)

\[
= e^{i \xi^1 \cdot x} \left( \nabla_{\xi^1} a_1 \times \rho + \frac{1}{2} \nabla_{\xi^1} \mathbf{b}_1 \times \mathbf{p} + b_1 \tau \mathbf{p}_1 \times \mathbf{p}_2 + \nabla_{\xi^1} r_1 \right) \\
+ e^{i \xi^2 \cdot x} \left( \nabla_{\xi^2} a_1 \times \rho + \frac{1}{2} (\nabla_{\xi^2} \mathbf{b}_1 \times \mathbf{p})^* + b_2^* \tau (\mathbf{p}_1 \times \mathbf{p}_2)^* + (\nabla_{\xi^2} r_1)^* \right).
\]

Similarly,

\[
\nabla \times H_2 = e^{-i \xi^2 \cdot x} \left( -\nabla_{\xi^2} a_1 \times \rho - \frac{1}{2} \nabla_{\xi^2} \mathbf{b}_2 \times \mathbf{p} + b_2 \tau \mathbf{p}_1 \times \mathbf{p}_2 + \nabla_{\xi^2} r_2 \right) \\
+ e^{-i \xi^2 \cdot x} \left( -\nabla_{\xi^2} a_1 \times \rho - \frac{1}{2} (\nabla_{\xi^2} \mathbf{b}_2 \times \mathbf{p})^* + b_2^* \tau (\mathbf{p}_1 \times \mathbf{p}_2)^* + (\nabla_{\xi^2} r_2)^* \right).
\]

We take a closer look at the phases of products of these vector fields. We have

\[ i(\xi^1 - \xi^2) \cdot x = i \xi \cdot x, \quad i(\xi^1 - \xi^2)^* \cdot x = i \xi^* \cdot x, \]

where

\[
\hat{\xi}_\pm = \left( \xi^1 \pm 2 \sqrt{1 - \frac{|\xi|^2}{4 \tau^2}}, \rho_{1,3} \right), \quad |\hat{\xi}_\pm| \to \infty \quad \text{as} \quad \tau \to \infty
\]

and

\[
\eta_\pm := \pm \frac{2 \omega \sigma_0 \mu_0 \rho_{1,3} x_3}{\tau \left( \sqrt{1 - \frac{|\xi|^2}{4 \tau^2}} + \frac{2 \omega \sigma_0 \mu_0}{\tau^2} + \sqrt{1 - \frac{|\xi|^2}{4 \tau^2}} \right)}, \quad |\eta_\pm| = \mathcal{O}(\tau^{-1}) \quad \text{as} \quad \tau \to \infty.
\]

Furthermore, we assume that \( \rho_{1,3} \neq 0 \) and \( \rho_{2,3} = 0 \). Then

\[
\int_{\Omega} e^{i \xi \cdot x - \eta_1} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} \left( \nabla_{\xi^1} a_1 \times \rho + \frac{1}{2} \nabla_{\xi^1} \mathbf{b}_1 \times \mathbf{p} \right) \cdot \left( (\nabla_{\xi^2} a_1 \times \rho)^* + \frac{1}{2} (\nabla_{\xi^2} \mathbf{b}_1 \times \mathbf{p})^* \right) dx
\]

\[
= \int_{\Omega} e^{i \xi \cdot x} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} \left[ e^{-\eta_1} \left( \nabla_{\xi^1} a_1 \times \rho + \frac{1}{2} \nabla_{\xi^1} \mathbf{b}_1 \times \mathbf{p} \right) \cdot \left( (\nabla_{\xi^2} a_1 \times \rho)^* + \frac{1}{2} (\nabla_{\xi^2} \mathbf{b}_1 \times \mathbf{p})^* \right) \\
- \left( \nabla_{\xi^1} \mu_{1/2}^1 \times \rho + \frac{1}{2} \nabla_{\xi^1} (\mu_{1/2}^{-1} e^{\eta^2} \mathbf{y}_2^1) \times \mathbf{p} \right) \cdot \left( (\nabla_{\xi^2} \mu_{1/2}^1 \times \rho)^* + \frac{1}{2} (\nabla_{\xi^2} (\mu_{1/2}^{-1} e^{\eta^2} \mathbf{y}_2^1) \times \mathbf{p})^* \right) \right] dx
\]

\[
+ \int_{\Omega} e^{i \xi \cdot x} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} \left( \nabla_{\xi^1} \mu_{1/2}^1 \times \rho + \frac{1}{2} \nabla_{\xi^1} (\mu_{1/2}^{-1} e^{\eta^2} \mathbf{y}_2^1) \times \mathbf{p} \right) \cdot \left( (\nabla_{\xi^2} \mu_{1/2}^1 \times \rho)^* + \frac{1}{2} (\nabla_{\xi^2} (\mu_{1/2}^{-1} e^{\eta^2} \mathbf{y}_2^1) \times \mathbf{p})^* \right) dx
\]

The first integral on the right-side goes to zero as \( \tau \to \infty \) according to estimates (56)-(59) and the fact that \( |\eta_\pm| = \mathcal{O}(\tau^{-1}) \) as \( \tau \to \infty \). The second integral goes to zero as \( \tau \to \infty \) by the Riemann-Lebesgue lemma. Therefore,

\[
\left| \int_{\Omega} e^{i \xi \cdot x - \eta_1} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} \left( \nabla_{\xi^1} a_1 \times \rho + \frac{1}{2} \nabla_{\xi^1} \mathbf{b}_1 \times \mathbf{p} \right) \cdot \left( (\nabla_{\xi^2} a_1 \times \rho)^* + \frac{1}{2} (\nabla_{\xi^2} \mathbf{b}_1 \times \mathbf{p})^* \right) dx \right| = o(1) \quad \text{as} \quad \tau \to \infty.
\]
In a similar way, one can show that

\[ \left| \int_{\Omega} e^{i\xi \cdot x - \eta} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} \left( (\nabla \xi \cdot a_1) + \frac{1}{2}(\nabla \xi \cdot b_1 \times \nabla) \right) \cdot \left( (\nabla \xi \cdot a_2) + \frac{1}{2}(\nabla \xi \cdot b_2 \times \nabla) \right) dx \right| = o(1) \]

\[ \left| \int_{\Omega} e^{i\xi \cdot x - \eta} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} \left( \nabla \xi \cdot a_1 \times \rho + \frac{1}{2}(\nabla \xi \cdot b_1 \times \rho) \right) \cdot \left( \nabla \xi \cdot a_2 \times \rho + \frac{1}{2}(\nabla \xi \cdot b_2 \times \rho) \right) dx \right| = o(\tau) \]

\[ \left| \int_{\Omega} e^{i\xi \cdot x - \eta} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} \left( \nabla \xi \cdot a_1 \cdot \rho + \frac{1}{2}(\nabla \xi \cdot b_1 \cdot \rho) \right) \cdot \left( \nabla \xi \cdot a_2 \cdot \rho + \frac{1}{2}(\nabla \xi \cdot b_2 \cdot \rho) \right) dx \right| = o(\tau) \]

\[ \left| \int_{\Omega} e^{i\xi \cdot x - \eta} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} \left( \nabla \xi \cdot a_1 \cdot \rho + \frac{1}{2}(\nabla \xi \cdot b_1 \cdot \rho) \right) \cdot \left( \nabla \xi \cdot a_2 \cdot \rho + \frac{1}{2}(\nabla \xi \cdot b_2 \cdot \rho) \right) dx \right| = o(\tau) \]

as \( \tau \to \infty \). Now we substitute \( H_1, H_2, \nabla \times H_1 \) and \( \nabla \times H_2 \) into (68) and divide the whole identity by \( \tau^2 \). According to the estimates obtained above, the terms with phases \( i\xi \cdot x - \eta \) that do not involve the correction terms \( r_1 \) and \( r_2 \) go to zero as \( \tau \to \infty \). The terms with phases \( i\xi \cdot x - \eta \) that involve the correction terms \( r_1 \) and \( r_2 \) go to zero as \( \tau \to \infty \), because of the correction terms. Finally, the terms with phases \( i\xi \cdot x \) or \( i\xi \cdot x \) can be controlled exactly as in the proof of Theorem 1.3 using (56) - (60) and (62). Thus, letting \( \tau \to \infty \), we obtain

\[ \int_{\Omega} e^{i\xi \cdot x} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} \left( \frac{1}{2} \nabla \xi \cdot \rho + \frac{1}{2} \nabla \xi \cdot \rho \right) \cdot \left( \frac{1}{2} \nabla \xi \cdot \rho + \frac{1}{2} \nabla \xi \cdot \rho \right) dx = 0. \]

Making the change of the variables \((x_1, x_2, x_3) \rightarrow (x_1, x_2, -x_3)\) in the second integral, we come to

\[ \int_{\mathbb{R}^3} e^{i\xi \cdot x} \frac{(\sigma_1 - \sigma_2)}{\sigma_1 \sigma_2} \left( \frac{1}{2} \nabla \xi \cdot \rho + \frac{1}{2} \nabla \xi \cdot \rho \right) \cdot \left( \frac{1}{2} \nabla \xi \cdot \rho + \frac{1}{2} \nabla \xi \cdot \rho \right) dx = 0. \]

This integral can be extended to all of \( \mathbb{R}^3 \) since \( \sigma_1 - \sigma_2 = 0 \) on \( \mathbb{R}^3 \backslash \Omega \). Then this will imply that \( \sigma_1 = \sigma_2 \) in \( \mathbb{R}^3 \).

Next, we set \( \sigma = \sigma_1 = \sigma_2 \). By Proposition 3.3, there are complex geometric optics solutions \( H_1, H_2 \in H^1_{\operatorname{Div}}(\Omega) \), with \( \nabla \times H_1, \nabla \times H_2 \in H^1_{\operatorname{Div}}(\Omega) \), for

\[ \nabla \times (\sigma^{-1} \nabla \times H_1) - i \omega \mu_1 H_1 = 0 \quad \text{and} \quad \nabla \times (\sigma^{-1} \nabla \times H_2) - i \omega \mu_2 H_2 = 0 \quad \text{in} \quad \Omega, \]

respectively, which have the following forms

\[ \tilde{H}_1(x; \xi^1) = e^{-i\xi \cdot x} (a_1 \rho + r_1), \quad \tilde{H}_2(x; \xi^2) = e^{-i\xi \cdot x} \left( -a_2 \rho - \frac{1}{2} b_2 \nabla + r_2 \right), \]

where

\[ a_1 = e^{-a_1(x; \rho) / 2}, \quad a_2 = e^{-a_2(x; \rho) / 2}, \quad b_2 = e^{a_2(x; \rho) / 2} e^{p(x; \rho)} \cdot \]

The functions \( p(\cdot, \rho; \tau), a^2(\cdot; \tau) \in C^\infty(\mathbb{R}^3) \) and the correction terms \( r_1, r_2 \in H^1_{\operatorname{Div}}(\Omega) \) satisfy (63) - (65). In a similar way as before, one can show that

\[ H_1(x; \rho) := \tilde{H}_1(x; \rho) - \tilde{H}_2^*(x; \rho), \quad H_2(x; \rho) := \tilde{H}_2(x; \rho) - \tilde{H}_1^*(x; \rho). \]

satisfy

\[ \nabla \times (\sigma^{-1} \nabla \times H_1) - i \omega \mu_1 H_1 = 0 \quad \text{and} \quad \nabla \times (\sigma^{-1} \nabla \times H_2) - i \omega \mu_2 H_2 = 0 \quad \text{in} \quad \Omega. \]

Also, the restrictions of \( H_1 \) and \( H_2 \) onto \( \Omega \), still denoted by \( H_1 \) and \( H_2 \), respectively, belong to \( H^1_{\operatorname{Div}}(\Omega) \) and satisfy \( \nabla \times H_1 \mid \Omega = \nabla \times H_2 \mid \Omega = 0 \). Thus, \( H_1 \) and \( H_2 \) satisfy the hypotheses of Proposition 5.2. We substitute \( H_1, H_2 \) and
\( \sigma = \sigma_1 = \sigma_2 \) into (68). As before, we assume that \( \rho_{1,3} \neq 0 \) and \( \rho_{2,3} = 0 \). Therefore, we obtain
\[
0 = \int_{\Omega} \left( (\mu_1 - \mu_2) e^{i\xi \cdot x} (a_1 \rho + r_1) \left[ -a_2 \rho - \frac{1}{2} b_2 \bar{p} + r_2 \right] \right) dx
\]
\[
+ \int_{\Omega} \left( (\mu_1 - \mu_2) e^{i\xi \cdot x} (a_1^* \rho^* + r_1^*) \left[ -a_2^* \rho^* - \frac{1}{2} b_2^* \bar{p}^* + r_2^* \right] \right) dx
\]
\[
- \int_{\Omega} \left( (\mu_1 - \mu_2) e^{i\xi \cdot x - \eta_1} (a_1 \rho + r_1) \left[ -a_2 \rho - \frac{1}{2} b_2 \bar{p} + r_2 \right] \right) dx
\]
\[
- \int_{\Omega} \left( (\mu_1 - \mu_2) e^{i\xi \cdot x - \eta_1} (a_1^* \rho^* + r_1^*) \left[ -a_2^* \rho^* - \frac{1}{2} b_2^* \bar{p}^* + r_2^* \right] \right) dx.
\]
As before, we use (63) - (64), the fact that \( |\eta_\pm| = O(\tau^{-1}) \) as \( \tau \to \infty \) and the Riemann-Lebesgue lemma, to show that
\[
\left\| \int_{\Omega} \left( (\mu_1 - \mu_2) e^{i\xi \cdot x - \eta_1} a_1 \rho \left[ -a_2 \rho - \frac{1}{2} b_2 \bar{p} \right] \right) dx \right\| = o(1),
\]
\[
\left\| \int_{\Omega} \left( (\mu_1 - \mu_2) e^{i\xi \cdot x - \eta_1} a_1^* \rho^* \left[ -a_2^* \rho^* - \frac{1}{2} b_2^* \bar{p}^* \right] \right) dx \right\| = o(1)
\]
as \( \tau \to \infty \). These estimates guarantee that the terms with phases \( i\xi_B \cdot x - \eta_\pm \) that do not involve the correction terms \( r_1 \) and \( r_2 \) go to zero as \( \tau \to \infty \). The terms with phases \( i\xi_B \cdot x - \eta_\pm \) that involve the correction terms \( r_1 \) and \( r_2 \) also go to zero as \( \tau \to \infty \) by (65). Finally, the terms with phases \( i\xi \cdot x \) or \( i\xi^* \cdot x \) can be controlled exactly as in the proof of Theorem 1.3 using (63) - (65) and (62). Thus, letting \( \tau \to \infty \), we obtain
\[
\int_{\Omega} e^{i\xi \cdot x} (\mu_1 - \mu_2) e^y dx + \int_{\Omega} \left( (\mu_1 - \mu_2) e^y \right)^* dx = 0.
\]
Making the change of the variables \((x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)\) in the second integral, we come to
\[
\int_{\Omega} e^{i\xi \cdot x} (\mu_1 - \mu_2) e^y dx = 0.
\]
This integral can be extended to all of \( \mathbb{R}^3 \) since \( \mu_1 - \mu_2 = 0 \) on \( \mathbb{R}^3 \setminus \Omega \). This implies that \( \mu_1 = \mu_2 \) completing the proof of Theorem 5.1.

6. PROOF OF THEOREM 1.5

Without loss of generality, we can assume that \( B_0 \) is the open ball of radius 1/2 centered at \( x_0 = (0, 0, 1/2) \) and that \( 0 \in \Omega \). We recall that
\[
\Gamma_0 = \partial \Omega \cap \partial B_0, \quad \Gamma_0 \neq \partial B_0 \quad \text{and} \quad \Gamma = \overline{\partial \Omega \setminus \Gamma_0}.
\]
We define the map
\[
K : \Omega \to \mathbb{R}^3 \setminus \{0\}, \quad K(x) := |x|^{-2} x
\]
which is known as the Kelvin transform. One can easily verify that \( K^{-1}(y) = |y|^{-2} y \) for \( y \in K(\Omega) \). We let \( DK \) and \( DK^{-1} \) denote the Jacobian matrices of \( K \) and \( K^{-1} \), respectively.

Next, we define \( \Omega := \{-y + x_0 : y \in K(\Omega)\} \) and
\[
F : \Omega \to \Omega, \quad F(y) := -K^{-1}(y) + x_0, \quad y \in \Omega.
\]
Then
\[
F^{-1}(x) = -K(x - x_0), \quad x \in \Omega.
\]
It is not difficult to verify that \( \Omega \subset \{x \in \mathbb{R}^3 : x_3 < 0\} \) and \( \Gamma_0 := F^{-1}(\Gamma) \) is a subset of the plane \( \{x \in \mathbb{R}^3 : x_3 = 0\} \). We also write \( \Gamma := F^{-1}(\Gamma) \). Thus, we are in a situation when the inaccessible part of the boundary is part of a plane. A direct calculation gives
\[
DF^{-1} = -|x - x_0|^2 I + 2(x - x_0) (x - x_0)^T, \quad DF = -|y|^{-2} I + 2|y|^{-4} yy^T,
\]
where \( x - x_0 \) and \( y \) are considered as column vectors and \( I \) is the \( 3 \times 3 \) identity matrix. These identities can be used to show that
\[
DF = (DF)^T \quad \text{and} \quad DF(DF)^T = |y|^{-4} I
\]
and
\[
DF^{-1} \circ F = |y|^4 DF, \quad DF = (DF)^T \quad \text{and} \quad \det(DF) = |y|^{-6}.
\]
Lemma 6.1. Let $(H_j, E_j) \in H^1_{\text{Div}}(\Omega) \times H^1_{\text{Div}}(\Omega)$, $j = 1, 2$. Consider their pullbacks onto $\Omega$,

\[ \tilde{H}_j := F^* H_j, \quad \tilde{E}_j := F^* E_j, \quad \tilde{\sigma}_j = \sigma_j \circ F, \quad \tilde{\mu}_j = \mu_j \circ F, \quad j = 1, 2. \]

Then

\[ \nabla \times E_j = i\omega \mu_j H_j \quad \text{and} \quad \nabla \times H_j = \sigma_j E_j \quad \text{in} \quad \Omega \]

if and only if

\[ \tilde{\nabla} \times \tilde{E}_j = i\omega |\tilde{y}|^{-2} \tilde{\mu}_j \tilde{H}_j \quad \text{and} \quad \tilde{\nabla} \times \tilde{H}_j = |\tilde{y}|^{-2} \tilde{\sigma}_j \tilde{E}_j \quad \text{in} \quad \tilde{\Omega}. \]

Here and in what follows, $\tilde{\nabla}$ denotes the curl operator with respect to the coordinates in $\tilde{\Omega}$.

Proof. The claim of the lemma is easy to prove by straightforward calculations using (71) and the facts from Appendix A. \qed

Lemma 6.2. The following holds true,

\[ Z_{\omega}^{\omega}_{[y]^{-2} \tilde{\sigma}_j, [y]^{-2} \tilde{\mu}_j} (F^* f) = F^* (Z_{\sigma_j, \mu_j}^{\omega} (f)) \]

for all $f \in TH^1_{\text{Div}}(\partial \Omega), j = 1, 2$.

Proof. Consider a $C^{1,1}$ boundary defining function $\rho$ for $\partial \Omega$. Then using (70),

\[ \tilde{\nabla} = \frac{-\nabla (\rho \circ F)}{|\nabla (\rho \circ F)|} \bigg|_{\partial \Omega} = -\frac{(DF)^T (\nabla \rho) \circ F}{|(DF)^T (\nabla \rho) \circ F|} \bigg|_{\partial \Omega} = -\frac{|y|^2 F^* (\nabla \rho)}{|\nabla (\rho \circ F)|} \bigg|_{\partial \Omega} = |y|^2 F^* \nabla \rho. \]

Using Lemma A.1 and (71),

\[ \tilde{\nabla} \times \tilde{H}_j = |y|^2 F^* \nabla \times F^* H_j = |y|^{-4} ((DF)^{-1} (\nabla \times H_j)) \circ F = DF ((\nabla \times H_j) \circ F) = (DF)^T ((\nabla \times H_j) \circ F) = F^* (\nabla \times H_j). \]

Similarly, $\tilde{\nabla} \times \tilde{E}_j = F^* (\nabla \times E_j)$. \qed

With $Z_{\sigma_j, \mu_j}^{\omega} = Z_{\sigma_2, \mu_2}^{\omega}$, it follows from these lemmas that

\[ Z_{[y]^{-2} \tilde{\sigma}_j, [y]^{-2} \tilde{\mu}_j}^{\omega} = Z_{[y]^{-2} \tilde{\sigma}_2, [y]^{-2} \tilde{\mu}_2}^{\omega}. \]

Finally, by hypothesis, $\sigma_j$ and $\mu_j$, $j = 1, 2$, can be extended to $\mathbb{R}^3$ as $C^2$ functions which are invariant under reflection across $\partial \Omega_0$. This is equivalent to the invariance of such extensions of $\sigma_j$ and $\mu_j$, $j = 1, 2$, under the map $x \mapsto F \circ R \circ F^{-1}(x)$, where $R(y_1, y_2, y_3) = (y_1, y_2, -y_3)$. Therefore, $\tilde{\sigma}_j$ and $\tilde{\mu}_j$ can be extended into $\mathbb{R}^3$ as $C^2$ functions which are invariant under reflection across the plane $\{x \in \mathbb{R}^3 : x_3 = 0\}$. Thus, by Theorem 5.1, we get $|y|^{-2} \tilde{\sigma}_1 = |y|^{-2} \tilde{\sigma}_2$ and $|y|^{-2} \tilde{\mu}_1 = |y|^{-2} \tilde{\mu}_2$ in $\tilde{\Omega}$. Hence, $\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$ in $\Omega$ as desired.

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Appendix A. Identities with pullbacks

Suppose that $\Omega, \tilde{\Omega} \subset \mathbb{R}^3$ are bounded domains and $F : \tilde{\Omega} \rightarrow \Omega$ is a $C^1$ bijective map. For a given $u \in H^1_{\text{Div}}(\Omega)$, the pullback $F^* u \in H^1_{\text{Div}}(\tilde{\Omega})$ is defined as

\[ F^* u := (DF)^T (u \circ F), \]

where $DF$ is the Jacobi matrix of $F$. According to [37, Corollary 3.58],

\[ \tilde{\nabla} \times (F^* u) = \left( \det(DF) DF^{-1} \nabla \times u \right) \circ F. \]

Lemma A.1. Suppose $u, v \in H^1_{\text{Div}}(\Omega)$ and $F : \tilde{\Omega} \rightarrow \Omega$ is a $C^1$ bijective map as before. Then

\[ F^* u \times F^* v = \det(DF) (DF^{-1} (u \times v)) \circ F. \]
Proof. Let $x$ and $y$ be coordinate systems in $\Omega$ and $\tilde{\Omega}$, respectively. By definition,

$$u_i = \sum_{k=1}^{3} \frac{\partial v_k}{\partial x_i}(F^* u)_k \circ F^{-1} \quad \text{and} \quad v_j = \sum_{l=1}^{3} \frac{\partial y_l}{\partial x_j}(F^* v)_l \circ F^{-1}.$$ 

Then

$$u_i v_j - v_i u_j = \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial v_k}{\partial x_i}(F^* u)_k(F^* v)_l - (F^* v)_l(F^* u)_k \circ F^{-1} \frac{\partial y_l}{\partial x_j}.$$

If $A$ and $B$ are $3 \times 3$ matrices defined as

$$A_{ij} = \langle u_i v_j - v_i u_j \rangle \circ F \quad \text{and} \quad B_{kl} = \langle (F^* u)_k(F^* v)_l - (F^* v)_l(F^* u)_k \rangle,$$

then the above identity can be rewritten as

$$A = (DF)^{-T}B(DF)^{-1}.$$ 

Clearly, both $A$ and $B$ are skew symmetric. According to the statement right before [37, Corollary 3.58], this completes the proof.

Let $\rho \in C^{0,1}(\mathbb{R}^3; \mathbb{R})$ be a boundary defining function for $\partial \Omega$, i.e. $\Omega = \{x \in \mathbb{R}^3 : \rho(x) > 0\}$ and $\partial \Omega = \{x \in \mathbb{R}^3 : \rho(x) = 0\}$. Then $\rho \circ F$ is a boundary defining function for $\partial \Omega$. Recall that the outer unit normals to $\partial \Omega$ and $\partial \tilde{\Omega}$ are defined as

$$\nu := -\frac{\nabla \rho}{|\nabla \rho|}|_{\partial \Omega} \quad \text{and} \quad \tilde{\nu} := -\frac{\nabla (\rho \circ F)}{|\nabla (\rho \circ F)|}|_{\partial \tilde{\Omega}},$$

respectively.

Appendix B. Identification of the Impedance Map $Z_{\sigma,\mu}^\omega$ with a Pseudodifferential Operator

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^\infty$ boundary. Here, we assume that $\sigma, \mu \in C^\infty(\overline{\Omega})$ such that $\sigma \geq \sigma_0$ and $\mu \geq \mu_0$ for some constants $\sigma_0, \mu_0 > 0$. The aim of this appendix is to show that the impedance map $Z_{\sigma,\mu}^\omega$ is a pseudodifferential operator of order 1 following [36]. Then we study the connection of $Z_{\sigma,\mu}^\omega$ to notion of apparent resistivity.

We work in a small neighborhood of a point $p \in \partial \Omega$. Without loss of generality, we fix a coordinate system near $p$ such that that $p = 0$ and that $\Omega \subset \{x \in \mathbb{R}^3 : x_3 > 0\}$ and $\partial \Omega \subset \{x \in \mathbb{R}^3 : x_3 = 0\}$ near $p$. As explained in Section 3, the equation (1) is equivalent to

$$L_{\sigma,\mu}(x,D)H := -\Delta H - \nabla(\nabla \beta \cdot H) - \nabla \alpha \times \nabla \times H - i\omega \sigma \mu H = 0 \quad \text{in} \quad \Omega,$$

$$\nabla \cdot (\mu H) = 0 \quad \text{in} \quad \Omega,$$

where $\alpha = \log \sigma$ and $\beta = \log \mu$. We use the notation $D_j = -i\partial \beta_j$, $j = 1,2,3$, $x' = (x_1,x_2)$ and $D' = (D_1,D_2)$. Then the equation (72) can we rewritten as

$$L_{\sigma,\mu}(x,D)H = (D_3^2 - D' - iM(x,D') - iN(x)D_3 - (\nabla \nabla \beta + i\omega \sigma \mu)H = 0 \quad \text{in} \quad \Omega,$$

where $D' = \partial_1^2 + \partial_1^2$ and

$$M = \begin{pmatrix} \partial_1 \beta \partial_1 D_1 - \partial_2 \alpha D_2 & \partial_2 \beta + \partial_2 \alpha \partial_1 D_1 & \partial_3 \beta + \partial_3 \alpha \partial_1 D_1 \\ \partial_1 \beta + \partial_1 \alpha D_2 & \partial_2 \beta + \partial_2 \alpha D_1 & \partial_3 \beta + \partial_3 \alpha D_1 \\ 0 & -\partial_2 \alpha D_2 - \partial_3 \beta D_1 \end{pmatrix}, \quad N = \begin{pmatrix} -\partial_3 \alpha & 0 & 0 \\ 0 & -\partial_3 \alpha & 0 \\ 0 & -\partial_3 \beta & -\partial_3 \beta \end{pmatrix}.$$ 

Suppose that $H \in H^{1}_{\text{div}}(\Omega)$ solves (72) and (73). Then the impedance map $Z_{\sigma,\mu}^\omega$ near $p$, in the above mentioned coordinates, has the form

$$Z_{\sigma,\mu}^\omega = \begin{pmatrix} (\nu \times H)_{|\partial \Omega} = (H_2, H_3, 0)_{|x_3 = 0} \circ \sigma^{-1}(\nabla H_1 - \partial_1 H_2, \partial_1 H_3, -\partial_2 H_3, 0)_{|x_3 = 0} \\ (\nu \times \sigma^{-1} \nabla \times H)_{|\partial \Omega} \end{pmatrix}.$$ 

since $\nu = (0,0,-1)$. To deal with the appearances of $\partial_3 H_1$ and $\partial_3 H_3$, $j = 1,3$, in the above expression, we need the following two results, which can be proven exactly as in Proposition 1 and Proposition 2 of [36].

Proposition B.1. There is a $3 \times 3$ matrix-valued pseudodifferential operator $B = B(x,D')$ of order 1 in $x'$, depending smoothly on $x_3$, such that

$$L_{\sigma,\mu}(x,D) \equiv (D_3 - iN(x) - iB(x,D'))(D_3 + iB(x,D')).$$

The principal symbol of $B(x,D')$ is $-\xi_j^{\alpha}$. Here and in what follows $\equiv$ denotes an equality modulo a smoothing operator and $\xi_j$ is the dual variable to $D_j$. 


Proposition B.2. If \( H \in H^1_{D(x)}(\Omega) \) solves (72), then \( \partial_3 H|_{\partial \Omega} = BH|_{\partial \Omega} \).

It follows from Proposition B.2 that
\[
\partial_3 H_j = B_{j1} H_1 + B_{j2} H_2 + B_{j3} H_3, \quad j = 1, 2, 3.
\]
Also, the equation (73) can be written as
\[
\partial_3 H_3 + \partial_3 \beta H_3 = -(\partial_1 \beta + \partial_1) H_1 - (\partial_2 \beta + \partial_2) H_2.
\]
Combining these two, we obtain
\[
(B_{33} + \partial_3 \beta) H_3 = -(B_{31} + \partial_1 \beta + \partial_1) H_1 - (B_{32} + \partial_2 \beta + \partial_2) H_2.
\]
We write \( J(x, D') = B_{33}(x, D') + \partial_3 \beta(x) \). Then
\[
J H_3 = -(B_{31} + \partial_1 \beta + \partial_1) H_1 - (B_{32} + \partial_2 \beta + \partial_2) H_2.
\]
Let \( K(x, D') \) be a pseudodifferential operator of order \(-1\) in \( x'\), depending smoothly on \( x_3 \), such that \( KJ = \text{Id} \), where \( \text{Id} \) is the identity operator. Clearly, the principal symbol of \( J(x, D') \) is \( -|\xi'|^2 \). To calculate the principal symbol of \( K(x, D') \), we use the identity \( KJ = \text{Id} \) at the principal symbol level. It follows then from [22, Corollary B.32] and [22, Corollary B.32] that the principal symbol of \( K(x, D') \) is \( -|\xi'|^{-1} \).

Using (75) and (76), one can show that \( Z_{\omega, \mu}^\omega \) is a \( 2 \times 2 \) matrix-valued pseudodifferential operator of order \( 1 \) such that
\[
\begin{align*}
(Z_{\omega, \mu}^\omega)_{j1} & = \sigma^{-1} B_{j2} + \sigma^{-1} (\partial_j - B_{j3}) \circ K \circ (B_{32} + \partial_2 \beta + \partial_2), \\
(Z_{\omega, \mu}^\omega)_{j2} & = -\sigma^{-1} B_{j1} - \sigma^{-1} (\partial_j - B_{j3}) \circ K \circ (B_{32} + \partial_2 \beta + \partial_2)
\end{align*}
\]
for \( j = 1, 2 \). From this, one can show that the principal symbol of \( Z_{\omega, \mu}^\omega \) is
\[
\frac{1}{\sigma(x', 0)|\xi'|^2} \begin{pmatrix}
\xi_1 \xi_2 \\
-\xi_1^2
\end{pmatrix}
\]
The expression for the impedance map can be used to define apparent resistivity, as in [23, 50, 52].

Remark B.3. As one can see, the principal symbol of \( Z_{\omega, \mu}^\omega \) determines \( \sigma|_{\partial \Omega} \) and all its tangential derivatives on \( \partial \Omega \). Studying the asymptotic expansion of the full symbol of \( Z_{\omega, \mu}^\omega \) as in [36], one can determine \( \partial^j \sigma|_{\partial \Omega} \) and \( \partial^j \mu|_{\partial \Omega} \) for all multi-indices \( \gamma \). To that end, one needs to perform a more detailed analysis of the asymptotic expansions of the full symbols of \( B \) constructed in Proposition B.1, and \( K \).

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