SPECTRUM OF SYK MODEL III:
LARGE DEVIATIONS AND CONCENTRATION OF MEASURES

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Abstract. In [4], we proved the almost sure convergence of eigenvalues of the SYK model, which can be viewed as a type of law of large numbers in probability theory; in [5], we proved that the linear statistic of eigenvalues satisfies the central limit theorem. In this article, we continue to study another important theorem in probability theory – the concentration of measure theorem, especially for the Gaussian SYK model. We will prove a large deviation principle (LDP) for the normalized empirical measure of eigenvalues when $q_n = 2$, in which case the eigenvalues can be expressed in terms of these of Gaussian random antisymmetric matrices. Such LDP result has its own independent interest in random matrix theory. For general $q_n \geq 3$, we can not prove the LDP, we will prove a concentration of measure theorem by estimating the Lipschitz norm of the Gaussian SYK model.

1. Introduction

In this article, we will study the large deviation principle and the concentration of measure theorem for the Gaussian SYK model, instead of the general SYK model considered in [4, 5].

The Gaussian SYK model is [3, 6, 11, 14, 16]

\begin{equation}
H = i^{q_n/2} \frac{1}{\sqrt{n_{q_n}}} \sum_{1 \leq i_1 < i_2 < \cdots < i_{q_n} \leq n} J_{i_1 i_2 \cdots i_{q_n}} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_{q_n}},
\end{equation}

where $n$ is an even integer, $J_{i_1 i_2 \cdots i_{q_n}}$ are independent identically distributed (i.i.d.) standard real Gaussian random variables with mean 0 and variance 1; $\psi_j$ are Majorana fermions satisfying the algebra

\begin{equation}
\{\psi_i, \psi_j\} := \psi_i \psi_j + \psi_j \psi_i = 2 \delta_{ij}, \quad 1 \leq i, j \leq n.
\end{equation}

By the representation of the Clifford algebra, $\psi_i$ can be represented by $L_n \times L_n$ Hermitian matrices with $L_n = 2^{n/2}$. Actually $\{\psi_i\}_{1 \leq i \leq n}$ can be generated by Pauli matrices iteratively [12]. Let $\lambda_i, 1 \leq i \leq L_n$ be the eigenvalues of $H$. One may check that $H$ is Hermitian by the anticommutative relation [3], thus $\lambda_i$ are real numbers. One of the main tasks in random matrix theory is to understand the following normalized empirical measure of eigenvalues of $H$

\begin{equation}
\rho_n(\lambda) := \frac{1}{L_n} \sum_i \delta_{\lambda_i}(\lambda).
\end{equation}

Let’s first summarize the main results in [4, 5]. Other than the standard Gaussian random variables, in [4], we consider the general cases where $J_{i_1 i_2 \cdots i_{q_n}}$ are i.i.d. random variables with mean 0 and variance 1, and the $k$-th moment of $|J_{i_1 i_2 \cdots i_{q_n}}|$
is uniformly bounded for any fixed \( k \). We proved that \( \rho_n \) converges to a probability measure \( \rho_\infty \) almost surely (or with probability 1) in the sense of distribution, and the limiting density \( \rho_\infty \) depends on the limit of the quotient \( q_n^2/n \). To be more precise, let \( 2 \leq q_n \leq n/2 \) be even, then \( \rho_\infty \) will be the standard Gaussian measure if \( q_n^2/n \to 0 \); \( \rho_\infty \) is the semicircle law if \( q_n^2/n \to \infty \); and \( \rho_\infty \) is related to the \( q \)-Hermite polynomial theory if \( q_n^2/n \to a \). The results can be extended to even \( q_n \geq n/2 \) immediately. One can also derive the results for \( q_n \) odd. The main result in [5] is that the linear statistic of eigenvalues satisfies the central limit theorem, which indicates the information about the 2-point correlation of the eigenvalues.

Regarding the spectral properties of the SYK model, we also refer to the numerical results in [7, 8, 9, 10].

In this article, we continue to study the spectrum of the Gaussian SYK model. We will prove a large deviation principle (LDP) for eigenvalues when \( q_n = 2 \) and a concentration of measure theorem for general \( q_n \geq 3 \).

Throughout the article, we always assume \( n \) is an even integer, \( J_{i_1 \cdots i_{2n}} \) are standard Gaussian random variables and \( q_n^2/n \) has a limit. In physics, people care especially when \( q_n \) is an even integer, but the model is still a good one in mathematics if \( q_n \) is odd. Our main results apply to both cases. Moreover, we only state and prove the main results for \( 0 < q_n \leq n/2 \), the results can be extended to \( q_n \geq n/2 \) immediately. This is because, as explained in [4], there is a symmetry between the systems with interaction of \( q_n \) fermions and \( n - q_n \) fermions.

### 1.1. Large deviations.

When \( q_n = 2 \), the SYK model reads

\[
H = \frac{i}{\sqrt{n}} \sum_{1 \leq i_1 < i_2 \leq n} J_{i_1 i_2} \psi_{i_1} \psi_{i_2}.
\]

Let

\[
J = (J_{ij})_{1 \leq i, j \leq n}, \quad J_{ji} := -J_{ij}
\]

be the real Gaussian antisymmetric matrices. This system is totally solvable in physics. If the eigenvalues of \( J \) are \( \pm i \mu_j \) where \( \mu_j \geq 0 \) for \( 1 \leq j \leq n/2 \), then all eigenvalues of \( H \) are given explicitly as [4, 14]:

\[
\left( \frac{n}{2} \right)^{-1/2} \sum_{j=1}^{n/2} \pm \mu_j.
\]

The normalized empirical measure defined in [3] reads

\[
\rho_n := \frac{1}{L_n} \sum_{a_1, \ldots, a_{n/2} \in \{\pm 1\}} \delta_{\left( \frac{n}{2} \right)^{-1/2} \sum_{j=1}^{n/2} a_j \mu_j}.
\]

Then \( \rho_n \) will tend to the standard Gaussian measure almost surely [4] and the linear statistic of these eigenvalues satisfies the central limit theorem [5]. In this article, we will further study its large deviation principle. We refer to [1] for the definition and basic properties of the LDP, and several well-known LDP results regarding the eigenvalues of random matrices.
To state our result, we need to introduce an auxiliary space. Let $X$ be a subspace of $l^\infty$,

$$X = \{(x_j)_{j=0}^\infty \in l^\infty | x_j \geq x_{j+1} \geq 0, \forall j \in \mathbb{Z}, j > 0, x_0 \geq \sum_{j=1}^{+\infty} x_j^2\},$$

where

$$l^\infty = \{(x_j)_{j=0}^\infty | x_j \in \mathbb{R}; \sup_{j \geq 0} |x_j| < +\infty\},$$

with the metric

$$d(x, y) = \sup_{j \geq 0} |x_j - y_j|,$$

for $x = (x_j)_{j=0}^\infty, y = (y_j)_{j=0}^\infty$. Then $(l^\infty, d)$ is a complete metric space. By Fatou’s Lemma, we know that $X$ is a closed subspace of $l^\infty$, thus $(X, d)$ is also a complete metric space (Polish space).

For $n$ even, let us define $\gamma_n \in X$ as $(\gamma_n)_j = (\frac{n}{2})^{-\frac{1}{2}} \mu_j$ for $1 \leq j \leq n/2$, $(\gamma_n)_0 = (\frac{n}{2})^{-1} \sum_{j=1}^{n/2} \mu_j^2$ and $(\gamma_n)_j = 0$ for $j > n/2$, i.e.,

$$\gamma_n = \left(\frac{n}{2}\right)^{-1} \sum_{j=1}^{n/2} \mu_j^2, \left(\frac{n}{2}\right)^{-\frac{1}{2}} \mu_1, \cdots, \left(\frac{n}{2}\right)^{-\frac{1}{2}} \mu_{n/2}, 0, \cdots\).$$

For $x = (x_j)_{j=0}^\infty \in X$, let

$$J(x) := x_0 - \sum_{j=1}^{+\infty} x_j^2$$

and

$$X_0 = \{x \in X | J(x) = 0\},$$

then we have

$$J(x) \geq 0 \text{ and } \gamma_n \in X_0.$$  

We first have the LDP of $(\gamma_n)_{n>0, n \in \mathbb{Z}}$ in this auxiliary space,

**Proposition 1.** Let $\pm \mu_j$ be eigenvalues of Gaussian antisymmetric matrices $J$ as in (5). Then the random measure $(\gamma_n)_{n>0, n \in \mathbb{Z}}$ defined in (11) satisfies the LDP in $(X, d)$ with speed $n^2/4$ and good rate function

$$I(x) = \begin{cases} x_0 - 1 - \ln J(x), & x \notin X_0; \\ +\infty, & x \in X_0. \end{cases}$$

We define $\ln 0 = -\infty$, then $I$ is lower semicontinuous by Fatou’s lemma. As $J(x) = x_0 - \sum_{j=1}^{+\infty} x_j^2 \leq x_0$, we have $I(x) = x_0 - 1 - \ln J(x) \geq J(x) - 1 - \ln J(x) \geq 0$.

If the equality holds, we must have $x_0 = J(x) = 1$ and $\sum_{j=1}^{+\infty} x_j^2 = 0$, i.e., $x_j = 0$ for $j > 0$; actually this is the only point where $I(x)$ achieves its minimum, i.e.,

$$I(x_{\min}) = 0, \ x_{\min} = (1, 0, \cdots).$$
Let \( M_1(\mathbb{R}) \) be the set of Borel probability measures on \( \mathbb{R} \) equipped with the bounded Lipschitz metric
\[
d_{BL}(\mu, \nu) = \sup |\langle \mu, f \rangle - \langle \nu, f \rangle|,
\]
where the supremum is subject to all 1-Lipschitz functions \( f : \mathbb{R} \to \mathbb{R} \), i.e.,
\[
|f(x) - f(y)| \leq |x - y| \quad \text{and} \quad |f(x)| \leq 1.
\]
Then \( (M_1(\mathbb{R}), d_{BL}) \) is a Polish space [1].

The LDP of the normalized empirical measure \( \bar{\varphi} \) in \( (M_1(\mathbb{R}), d_{BL}) \) will be induced by the LDP of \( (\gamma_n)_{n \geq 0, n \in \mathbb{Z}} \) in \( (X, d) \), where we need to construct a continuous and injective function
\[
\varphi : X \to M_1(\mathbb{R})
\]
such that \( \varphi(\gamma_n) = \rho_n \). By (17), the Fourier transform of \( \rho_n \) is
\[
\hat{\rho}_n(s) = \langle \rho_n(\lambda), e^{is\lambda} \rangle = \prod_{j=1}^{n/2} \cos \left( \frac{n}{2} s \lambda_j \right).
\]
If we define the Fourier transform of the measure \( \varphi \) as
\[
\hat{\varphi}(x)(s) = e^{-J(x)s^2/2} \prod_{j=1}^{+\infty} \cos sx_j, \quad x = (x_j)_{j=0}^{\infty} \in X,
\]
then by definition of \( \gamma_n \in X_0 \), we must have
\[
\varphi(\gamma_n) = \rho_n.
\]
In [13], we will further show that \( \varphi \) is a Borel probability measure, continuous and injective. Hence, by the Contraction Principle (cf. Theorem D.7 in [1]), we have

**Theorem 1.** The normalized empirical measure \( \rho_n \) of eigenvalues of the Gaussian SYK model for \( q_n = 2 \) satisfies the LDP in \( (M_1(\mathbb{R}), d_{BL}) \) with speed \( n^2/4 \) and good rate function \( \tilde{I} \) such that \( \tilde{I}(x) = I(\varphi^{-1}x) \) if \( x \in \varphi(X) \) and \( \tilde{I}(x) = +\infty \) if \( x \not\in \varphi(X) \), where \( I(x) \) is defined by (14).

As a remark, by (16), one can conclude that \( \tilde{I} \) will achieve its minimum at \( \varphi(x_{min}) \) where \( x_{min} = (1, 0, \cdots) \). By definition (20), the Fourier transform \( \varphi(x_{min})(s) \) is the Gaussian function, and thus \( \varphi(x_{min}) \) is the Gaussian distribution, which implies that \( \tilde{I} \) achieves its minimum at the Gaussian distribution.

1.2. **Concentration of measure theorem.** We can not derive the LDP for general \( q_n \geq 3 \), but we can prove a weaker version which is the concentration of measure theorem. The proof is based on the following classical Gaussian concentration of measure theorem [13]: Let \( (a_k)_{1 \leq k \leq N} \) be \( N \)-dimensional Gaussian random vectors, and let \( F : \mathbb{R}^N \to \mathbb{R} \) be Lipschitz with Lipschitz constant \( L \), then there are universal constants \( C, c > 0 \) such that for \( t > 0 \),
\[
\mathbb{P}(|F(a_1, \cdots, a_N) - \mathbb{E}F(a_1, \cdots, a_N)| > t) \leq Ce^{-ct^2/L^2}.
\]
We denote the set
\[
I_n = \{(i_1, i_2, \cdots, i_{q_n}) : 1 \leq i_1 < i_2 < \cdots < i_{q_n} \leq n\}.
\]
For any coordinate \( R = (i_1, \cdots, i_{q_n}) \in I_n \), we denote
\[
J_R := J_{i_1 \cdots i_{q_n}} \quad \text{and} \quad \Psi_R := \psi_{i_1} \cdots \psi_{i_{q_n}}.
\]
Then we can simply rewrite

\[ H = \frac{j[q_n/2]}{\sqrt{\binom{n}{q_n}}} \sum_{R \in I_n} J_R \Psi_R. \]

If we consider \( H \) as a function of the standard Gaussian random vectors \((J_R)_{R \in I_n}\), then we first have the following Lipschitz estimates.

**Lemma 1.** Let \( x := (J_R)_{R \in I_n} \in \mathbb{R}^{\binom{n}{q_n}} \) be the Gaussian random vector and \( \rho_n \) be the normalized empirical measure \((3)\). We consider the SYK model \( H := H(x) \) as a function of \( x \).

(a) Let \( f : \mathbb{R} \to \mathbb{R} \) be Lipschitz, then the map \( x \mapsto \langle f, \rho_n \rangle \) is \((n q_n)^{-1/2} \| f' \|_{L^\infty(\mathbb{R})}\)-Lipschitz;
(b) For any probability measure \( \rho \) on \( \mathbb{R} \), the map \( x \mapsto d_{BL}(\rho_n, \rho) \) is \((n q_n)^{-1/2}\)-Lipschitz.

Once we have the above Lipschitz estimates, by (22) of the classical concentration of measure theorem for Gaussian random vectors, we can prove

**Theorem 2.** Let \( \rho_n \) be the normalized empirical measure of the Gaussian SYK model for any \( 0 < q_n \leq n/2 \) as in \((3)\) and \( \rho_\infty \) be the limiting measure according to the limit \( q_n^2/n \) as we derived in \([4]\). Given \( a > 0 \), then there exists \( C(a) > 0 \) such that

\[ \mathbb{P}(d_{BL}(\rho_n, \rho_\infty) > a) \leq C e^{-c(a)\binom{n}{q_n}}, \]

where \( C \) is some universal constant.

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## 2. Large Deviation Principle for \( q_n = 2 \)

When \( q_n = 2 \), the system is totally solvable and all eigenvalues can be expressed in term of eigenvalues of Gaussian random antisymmetric matrices (see \([10]\)). In this section, we will prove the LDP for the normalized empirical measure \( \rho_n \) (which is defined in \([7]\)) of these eigenvalues. There are mainly two steps: we will first derive the LDP in an auxiliary space \((X, d)\), then we construct a continuous and injective map \( \varphi : X \to M_1(\mathbb{R}) \) which will induce the LDP in \((M_1(\mathbb{R}), d_{BL})\) by the Contraction Principle.

### 2.1. Some integral inequalities.

Let \( J \) be the real Gaussian antisymmetric matrices as in \([7]\). We assume the eigenvalues of \( J \) are \( \pm i \mu_j \) where \( \mu_j \geq 0 \) for \( 1 \leq j \leq n/2 \). Then the joint density of these eigenvalues is \([15]\)

\[ J_n(\mu) := \frac{1}{Z_n} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} 1(\mu_1 > \cdots > \mu_{n/2} > 0), \]

where

\[ \Delta(\mu) = \prod_{1 \leq i < j \leq n/2} (\mu_i^2 - \mu_j^2), \quad \mu := (\mu_1, \cdots, \mu_{n/2}) \]
is the Vandermonde determinant. By Selberg integrals, the normalization constant

\[ Z_n = (\pi/2)^{n/2 - 1} \prod_{j=0}^{n/2 - 1} (2j)! . \]

Given

\[ x := (x_1, \ldots, x_{n/2}), \]

let’s denote

\[ x_{>k} := (x_{k+1}, \ldots, x_{n/2}), \quad \Delta(x_{>k}) := \prod_{k<i<j \leq \frac{n}{2}} (x_i^2 - x_j^2) \]

and

\[ \Sigma_{n-2k} := \{ (x_{k+1}, \ldots, x_{n/2}) : x_{k+1} > \cdots > x_{n/2} > 0 \} \]

for \( 0 \leq k \leq n/2 \). Then for \( x \in \Sigma_n \), we have

\[ x = x_{>0}, \quad 0 < \Delta(x_{>k-1}) = \Delta(x_{>k}) \prod_{j=k+1}^{n/2} (x_k^2 - x_j^2) < x_k^{n-2k}\Delta(x_{>k}) \]

and

\[ 0 < \Delta(x) \leq \Delta(x_{>k}) \prod_{j=1}^{k} x_j^{n-2j}. \]

We will need several integral inequalities.

**Lemma 2.** If \( a, b < 1/2 \) and \( 0 \leq k \leq n/2 \), we have

\[ \mathbb{E} e^{a \sum_{j=1}^{k} \mu_j^2 + b \sum_{j=k+1}^{n/2} \mu_j^2} \leq 2^{nk} (1 - 2a)^{-k(n-k-1)/4} (1 - 2b)^{-n/4 - (2^k - k)(\frac{n}{2} - k)}. \]

When \( k = 1, b = 0, a = 1/4 \), we further have

\[ \int_{\Sigma_n} |\Delta(\mu)|^2 e^{-a \sum_{j=1}^{n/2} \mu_j^2/4} d\mu \leq 2^{2n} Z_n. \]

**Proof.** By definition we have

\[ \int_{\Sigma_n} |\Delta(\mu)|^2 e^{-a \sum_{j=1}^{n/2} \mu_j^2/4} d\mu = Z_n, \]

where \( d\mu \) is the Lebesgue measure. For \( a > 0 \), by changing of variables, we have

\[ \int_{\Sigma_n} |\Delta(\mu)|^2 e^{-a \sum_{j=1}^{n/2} \mu_j^2/2} d\mu = Z_n a^{-\frac{n}{4} - 2(n/2)} = Z_n a^{-\frac{n(n-1)}{4}}. \]
Therefore, let’s denote $m := n - 2k$, we have

$$
Z_n E e^\sum_{j=1}^k \mu_j^2 + b \sum_{j=k+1}^{n/2} \mu_j^2
= \int \left| \Delta(\mu) \right|^2 e^{\sum_{j=1}^k \mu_j^2 + b \sum_{j=k+1}^{n/2} \mu_j^2} \nu \, d\mu
\leq \int \left( \prod_{j=1}^k \mu_j^{2(n-2j)} \left| \Delta(\mu_k) \right|^2 e^{-(1-2a) \sum_{j=1}^k \mu_j^2} \right) \right) \cdot Z_{n-2k}(1-2b)^{-\frac{m(m-1)}{4}}
\leq (1-2a)^{-k(n-k-\frac{1}{2})} \left( \prod_{j=1}^k \frac{(2n-2j)(n-2j+1)}{(2j)} \right) \cdot Z_{n-2k}(1-2b)^{-\frac{m(m-1)}{4}}
\leq (1-2a)^{-k(n-k-\frac{1}{2})} (2n-2k) Z_n(1-2b)^{-\frac{m(m-1)}{4}}
$$

which further gives

$$
\mathbb{E} e^{\sum_{j=1}^k \mu_j^2 + b \sum_{j=k+1}^{n/2} \mu_j^2} \leq 2^{nk} (1-2a)^{-k(n-k-\frac{1}{2})} (1-2b)^{-\frac{m(m-1)}{4}}.
$$

For $k = 1$, $b = 0$, $a = 1/4$, we obtain

$$
\int \left| \Delta(\mu) \right|^2 e^\frac{\nu}{4} \sum_{j=1}^{n/2} \mu_j^2 \, d\mu \leq (1-2/4)^{-n-k/2} Z_n \leq 2^n Z_n,
$$

which completes the proof.

Let’s denote the subset

$$
\Sigma_{n,a,b} = \left\{ (x_1, \ldots, x_{n/2}) \in \Sigma_n : a \left( \frac{n}{2} \right) < \sum_{j=1}^{n/2} x_j^2 < b \left( \frac{n}{2} \right) \right\}.
$$

**Lemma 3.** For $0 < a < 1 < b$, we have,

$$
\int \Sigma_{n,a,b} \left| \Delta(\mu) \right|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2} \, d\mu \geq Z_n \left( 1 - (ae^{1-a})^{\frac{n(n-1)}{4}} - (be^{1-b})^{\frac{n(n-1)}{4}} \right).
$$
Proof. For $0 < a < 1 < b$, we have

$$\int_{\Sigma_n \setminus \Sigma_{n,0,b}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu - \int_{\Sigma_n \setminus \Sigma_{n,0,b}} |\Delta(\mu)|^2 e^{-b^{-1} \sum_{j=1}^{n/2} \mu_j^2/2 - (1-b^{-1})a(n/2)/2} d\mu$$

$$\leq \int_{\Sigma_n} |\Delta(\mu)|^2 e^{-b^{-1} \sum_{j=1}^{n/2} \mu_j^2/2 - (1-b^{-1})a(n/2)/2} d\mu$$

$$= b^{\frac{n(n-1)}{4}} \int_{\Sigma_n} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2 - (b-1)(n/2)/2} d\mu$$

$$= b^{\frac{n(n-1)}{2}} Z_n e^{-(b-1)\frac{n(n-1)}{2}} = Z_n(b e^{1-a})^\frac{n(n-1)}{2}.$$

Here, we used the fact that $1 - b^{-1} > 0$ and

$$-\sum_{j=1}^{n/2} \mu_j^2/2 = -b^{-1} \sum_{j=1}^{n/2} \mu_j^2/2 - (1-b^{-1}) \sum_{j=1}^{n/2} \mu_j^2/2$$

$$\leq -b^{-1} \sum_{j=1}^{n/2} \mu_j^2/2 - (1-b^{-1}) \binom{n}{2}/2$$

for $\mu \in \Sigma_n \setminus \Sigma_{n,0,b}$. Similarly,

$$\int_{\Sigma_{n,0,a}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu \leq \int_{\Sigma_n} |\Delta(\mu)|^2 e^{-a^{-1} \sum_{j=1}^{n/2} \mu_j^2/2 - (1-a^{-1})a(n/2)/2} d\mu$$

$$= a^{\frac{n(n-1)}{4}} Z_n e^{-(a-1)\frac{n(n-1)}{2}} = Z_n(a e^{1-a})^\frac{n(n-1)}{2}.$$

Therefore, we will finish the proof by observing the following identity,

$$Z_n = \int_{\Sigma_n} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu = \int_{\Sigma_n \setminus \Sigma_{n,0,b}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu$$

$$+ \int_{\Sigma_{n,a,b}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu + \int_{\Sigma_{n,0,a}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu.$$

For $\delta > 0$, let’s denote the subset

$$\Sigma_{n,>\delta} = \left\{ (x_1, \ldots, x_{n/2}) \in \Sigma_n : x_1^2 > \delta \binom{n}{2} \right\}$$

and

$$\Sigma_{n,a,b,\delta} = \Sigma_{n,a,b} \setminus \Sigma_{n,>\delta}.$$

We will use Lemmas 2 and 3 to prove
Lemma 4. We have the following estimates,
(a) If $0 < a < 1 < b$, $\delta > 0$, then
\[
\liminf_{n \to +\infty} \frac{1}{n^2} \ln \int_{\sum_{n,a,b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu \geq 0.
\]
(b) If $0 < a < b \leq 1$, $\delta > 0$, then
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \int_{\sum_{n,a,b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu \geq 1 - b + \ln b.
\]
(c) If $1 \leq a < b$, $\delta > 0$, then
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \int_{\sum_{n,a,b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu \geq 1 - a + \ln a.
\]
Proof. By (25), we first have
\[
\int_{\sum_{n,>\delta}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu \leq \int_{\sum_n} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu - e^{-\delta(z)/4} d\mu \leq 2^n Z_n e^{-\delta(z)/4}.
\]
Then, by Lemma 34 we further have
\[
\int_{\sum_{n,a,b,\delta}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu \\
\geq \int_{\sum_{n,a,b}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu - \int_{\sum_{n,>\delta}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu \\
\geq Z_n \left(1 - (ae^{1-a})^{\frac{n(n-1)}{4}} - (be^{1-b})^{\frac{n(n-1)}{4}} - 2^n e^{-\delta(z)/4}\right).
\]
Thus if $n$ is large enough, for every fixed $0 < a < 1 < b$, $\delta > 0$, using $0 < ae^{1-a} < 1$, $0 < be^{1-b} < 1$, we have
\[
(26) \quad \int_{\sum_{n,a,b,\delta}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu \geq Z_n / 2,
\]
which implies
\[
\liminf_{n \to +\infty} \frac{1}{n^2} \ln \int_{\sum_{n,a,b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu \geq 0,
\]
which finishes part (a).
For every fixed $a, b, \lambda, \delta > 0$ such that $0 < a < 1/\lambda < b$ (i.e., $0 < \lambda a < 1 < \lambda b$), if we change variables first and then apply (26), we will have
\[
\int_{\sum_{n,a,b,\delta}} |\Delta(\mu)|^2 e^{-\lambda \sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu = \lambda^{\frac{n(n-1)}{4}} \int_{\sum_{n,a,b,\delta}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \nu_j^2 / 2} d\mu \\
\geq Z_n \lambda^{\frac{n(n-1)}{4}} / 2.
\]
If $\lambda > 1$, we have
\[
\int_{\Sigma_{n,a,b,\delta}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu \geq e^{(\lambda-1)a(n)/2} \int_{\Sigma_{n,a,b,\delta}} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu \geq e^{(\lambda-1)a(n)/2} Z_n \lambda^{-\frac{n(n-1)}{2}}.
\]
Therefore, if $0 < a < b \leq 1$, $\delta > 0$, for every $\lambda \in (1/b, 1/a)$ which is greater than 1, the above arguments imply
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \int_{\Sigma_{n,a,b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu \geq (\lambda - 1)a - \ln \lambda.
\]
Letting $\lambda \to (1/a)-$, we have
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \int_{\Sigma_{n,a,b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu \geq 1 - a + \ln a.
\]
Notice that for every $0 \leq a < a' < b \leq 1$, we have
\[
\int_{\Sigma_{n,a,b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu \geq \int_{\Sigma_{n,a',b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu,
\]
and thus
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \int_{\Sigma_{n,a,b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu \\
\geq \liminf_{n \to +\infty} \frac{4}{n^2} \ln \int_{\Sigma_{n,a',b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu \geq 1 - a' + \ln a'.
\]
Letting $a' \to b-$, we have
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \int_{\Sigma_{n,a,b,\delta}} Z_n^{-1} |\Delta(\mu)|^2 e^{-\sum_{j=1}^{n/2} \mu_j^2/2} d\mu \geq 1 - b + \ln b,
\]
which finishes part (b). The proof of part (c) follows part (b) similarly and we omit the proof.

\section*{2.2. LDP in an auxiliary space.}
Let’s prove Proposition \[1] The whole proof is separated into three parts.

\subsection*{2.2.1. Lower and upper bounds.}
We will prove the following

\textbf{Lemma 5.}
\[
\lim_{\epsilon \to 0^+} \liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon) \geq -I(x),
\]
\[
\lim_{\epsilon \to 0^+} \limsup_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon) \leq -I(x),
\]
where $I(x)$ is given by \[16\].
Let’s first consider the lower bound. Given \( x = (x_j)_{j=0}^{\infty} \in X \), by definition we have \( \sum_{j=1}^{+\infty} x_j^2 \leq x_0 < +\infty \) and \( \lim_{\delta \to 0^+} \sum_{j=1}^{+\infty} \min(x_j^2, \delta) = 0 \) by monotone convergence theorem. For every \( \epsilon \in (0, 1) \), there exists \( k > 0 \) such that \( \sum_{j=k+1}^{+\infty} x_j^2 < \epsilon^2/2 \). Let’s take \( \delta \in (0, \epsilon) \) such that \( \sqrt{k\delta} < \sqrt{x_0 + \epsilon/2} - \sqrt{x_0} \), then we have

Lemma 6. Let \( y = (y_j)_{j=0}^{\infty} \in X_0 \), i.e., \( y_0 = \sum_{j=1}^{+\infty} y_j^2 \). If \( x_j < y_j < x_j + \delta \) for \( 1 \leq j \leq k \), \( y_{k+1} < \epsilon \) and \( a < \sum_{j=k+1}^{+\infty} y_j^2 < a + \epsilon^2/2 \), where \( a := x_0 - \sum_{j=1}^{+\infty} x_j^2 \geq 0 \), then \( d(x, y) < \epsilon \).

Proof. Since \( x_{k+1}^2 \leq \sum_{j=k+1}^{+\infty} x_j^2 < \epsilon^2/2 \), thus \( 0 \leq x_{k+1} < \epsilon \). By assumption \( 0 \leq y_{k+1} < \epsilon \), we have

\[
\sup_{j \geq k+1} |x_j - y_j| \leq \sup_{j \geq k+1} \max(x_j, y_j) \leq \max(x_{k+1}, y_{k+1}) < \epsilon,
\]

where we used the fact that the coordinate of \( x, y \in X \) is decreasing.

If we combine this with the assumption that \( |x_j - y_j| < \delta < \epsilon \) for \( 1 \leq j \leq k \), we must have

\[
(27) \quad d(x, y) = \sup_{j \geq 0} |x_j - y_j| \leq \max(|x_0 - y_0|, \epsilon).
\]

Notice that \( \sum_{j=1}^{k} x_j^2 < \sum_{j=1}^{k} y_j^2 < \sum_{j=1}^{k} (x_j + \delta)^2 \), that \( \sqrt{k\delta} < \sqrt{x_0 + \epsilon/2} - \sqrt{x_0} \), that

\[
0 < \sum_{j=1}^{k} (x_j + \delta)^2 - \sum_{j=1}^{k} x_j^2 = 2\delta \sum_{j=1}^{k} x_j + k\delta^2 \leq 2\delta \left( \sum_{j=1}^{k} x_j^2 \right) + k\delta^2
\]

\[
\leq 2\delta (kx_0)^{1/2} + k\delta^2 = (\sqrt{x_0 + \epsilon} + \sqrt{k\delta})^2 - x_0 < \epsilon/2,
\]

and that \( a < \sum_{j=k+1}^{+\infty} y_j^2 < a + \epsilon/2 \), we have \( \sum_{j=1}^{k} x_j^2 + a < \sum_{j=1}^{+\infty} y_j^2 = y_0 < \sum_{j=1}^{k} (x_j + \delta)^2 + a + \epsilon/2 < \sum_{j=1}^{k} x_j^2 + \epsilon/2 + a + \epsilon/2 \). We also have \( \sum_{j=k+1}^{+\infty} x_j^2 < \epsilon^2/2 < \epsilon/2 \) and

\[
\sum_{j=1}^{k} x_j^2 \leq \sum_{j=1}^{+\infty} x_j^2 = x_0 - a = \sum_{j=1}^{k} x_j^2 + \sum_{j=k+1}^{+\infty} x_j^2 < \sum_{j=1}^{k} x_j^2 + \epsilon/2,
\]

thus \( x_0 - \epsilon/2 < \sum_{j=1}^{k} x_j^2 + a < y_0 < \sum_{j=1}^{k} x_j^2 + a + \epsilon \leq x_0 + \epsilon \), i.e., \( |x_0 - y_0| < \epsilon \). This completes the proof by (27). \( \Box \)
Given \( x \in X \) and \( 0 < \delta/4 < \delta < \epsilon \) defined above, recall \( \gamma_n \in X_0 \), by Lemma \[\text{V}\] where we replace \( y \) by \( \gamma_n \), for \( n > 2k \), \( n/2 \in \mathbb{Z} \), we have

\[
\mathbb{P}(d(\gamma_n, x) < \epsilon) \geq \mathbb{P}
\left(x_j < \left(\frac{n}{2}\right) - \frac{\delta}{2}\right) \left(\frac{n}{2}\right) < x_j + \delta, \ \forall \ 1 \leq j \leq k;
\]

\[
\left(\frac{n}{2}\right) - \frac{\delta}{2}\mu_{k+1} < \delta/4; \ \left(\frac{n}{2}\right) < \sum_{j=k+1}^{n/2} \mu_j < \left(\frac{n}{2}\right).
\]

For \( n \) large enough, we have \( a(n/2) < (a + \epsilon/2)(n-2k) \). Let \( m := n - 2k \) again, and \( \delta_j := \frac{4k-j}{4k} \delta \in (\delta/2, \delta) \) for \( 1 \leq j \leq 2k \), then we have

\[
\mathbb{P}(d(\gamma_n, x) < \epsilon) \geq \mathbb{P}
\left(x_j + \delta_j < \left(\frac{m}{2}\right) - \frac{\delta}{2}\right) \left(\frac{m}{2}\right) < x_j + \delta_{2j-1}, \ \forall \ 1 \leq j \leq k;
\]

\[
\mu_{k+1} < \left(\frac{m}{2}\right) - \frac{\delta}{4}; \ \left(\frac{m}{2}\right) < \sum_{j=k+1}^{n/2} \mu_j < \left(\frac{m}{2}\right).
\]

We refer to (28) for the final expression.

By definition of \( X \), we have \( x_j \geq x_{j+1} \geq 0 \), thus if \( x_j + \delta_j < \left(\frac{m}{2}\right) - \frac{\delta}{4}\mu_j < x_j + \delta_{2j-1} \) for \( 1 \leq j \leq k \), we will have

\[
\mu_j - \mu_{j+1} > \left(\delta_{2j} - \delta_{2j+1}\right) \left(\frac{n}{2}\right) = \frac{\delta}{4k} \left(\frac{n}{2}\right),
\]

for \( 1 \leq j < k \) and

\[
\mu_k > \delta_{2k} \left(\frac{n}{2}\right) = \frac{\delta}{4k} \left(\frac{n}{2}\right).
\]

If \( \mu_{k+1} < \left(\frac{m}{2}\right) - \frac{\delta}{4} < \left(\frac{n}{2}\right) \), then we have

\[
\mu_k - \mu_{k+1} > \left(\frac{\delta}{2} \left(\frac{n}{2}\right) - \frac{\delta}{4} \left(\frac{n}{2}\right)\right) = \left(\frac{\delta}{4k} \left(\frac{n}{2}\right)\right) \geq \frac{\delta}{4k} \left(\frac{n}{2}\right).
\]

Therefore, for \( 1 \leq l \leq k \), we must have

\[
\frac{\Delta(\mu_l-l)}{\Delta(\mu_{>l})} = \prod_{j=l+1}^{n/2} \frac{\mu_i^2 - \mu_j^2}{\mu_i^2 - \mu_{i+1}^2} \geq \prod_{j=l+1}^{n/2} \left(\frac{\mu_i^2}{\mu_{i+1}^2}\right) \geq \left(\frac{\delta}{4k} \left(\frac{n}{2}\right)\right)^{n-2l},
\]

and hence,

\[
\frac{\Delta(\mu)}{\Delta(\mu_{>k})} = \prod_{l=1}^{k} \frac{\Delta(\mu_{>l-1})}{\Delta(\mu_{>l})} \geq \prod_{l=1}^{k} \left(\frac{\delta}{4k} \left(\frac{n}{2}\right)\right)^{n-2l} = \left(\frac{\delta}{4k} \left(\frac{n}{2}\right)\right)^{nk-k(k+1)}.
\]

By (28), we have

\[
\left(\frac{n}{2}\right) \sum_{j=1}^{k} \mu_j^2 \leq \sum_{j=1}^{k} \left(x_j + \delta\right)^2 \leq \sum_{j=1}^{k} x_j^2 + \epsilon/2 \leq \sum_{j=1}^{+\infty} x_j^2 + \epsilon/2 = x_0 - a + \epsilon/2.
\]
Therefore, for \( n \) large enough, we can further estimate (29) as

\[
Z_n \mathbb{P}(d(\gamma_n, x) < \epsilon) \geq \int_{\sum_{j=1}^{n/2} (\frac{\delta}{n/2})} \left( \frac{\delta}{4k} \right)^{2^n} \left( \frac{n}{2} \right)^{(n-1)/2} e^{\frac{1}{k} \sum_{j=1}^{n/2} \mu_j^2/2} n^{n/2} e^{-((x_0-a+\epsilon/2)/2)^2} d\mu
\]

\[
= \left( \frac{\delta}{4k} \right)^{2^n} \left( \frac{n}{2} \right)^{n-1} \prod_{j=1}^{n/2} \left( \frac{n}{2} \right) e^{-((x_0-a+\epsilon/2)/2)^2} Z_m
\]

\[
\times Z_m^{-1} \int_{\sum_{j=1}^{n/2} (\frac{\delta}{n/2})} \left| \Delta(\mu_k) \right|^2 e^{-\frac{1}{k} \sum_{j=1}^{n/2} \mu_j^2/2} d\mu_k.
\]

Here, we used the fact that \((\delta_{2j-1} - \delta_{2j}) (\frac{n}{2}) = \frac{\delta}{4k} (\frac{n}{2})^k \cdot \frac{\mu_j}{\mu_k}\). Since \( m = n - 2k \), \( Z_n = (\pi/2)^{\frac{n-2k}{2}} \prod_{j=0}^{n/2-1} (2j)! \), we have

\[
Z_n/Z_m = \prod_{j=1}^{k} (\pi/2)^{\frac{n-2k}{2}} \geq (\pi/2)^{\frac{k}{2}! (n-2k)!} \leq (\pi/2)^{\frac{n}{2}! k!}.
\]

Using \( n! \leq n^n \) and \((\frac{n}{2})! \geq n > 0\), we have \( Z_n/Z_m \leq (\pi/2)^{\frac{n}{2}!} n^{kn} \), thus

\[
Z_n/Z_m \leq (\pi/2)^{\frac{n}{2}!} n^{kn}.
\]

Therefore, we have

\[
\mathbb{P}(d(\gamma_n, x) < \epsilon) \geq (\frac{\delta}{4k})^{2^{n-k}(k+1)/2} \left( \frac{n}{2} \right)^{-k(k+1)/2} e^{-((x_0-a+\epsilon/2)/2)^2} n^{n/2} e^{-\frac{1}{k} \sum_{j=1}^{n/2} \mu_j^2/2} d\mu_k.
\]
Hence, we have
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon) \geq -(x_0 - a + \epsilon/2)
\]
\[
+ \liminf_{m \to +\infty} \frac{4}{m^2} \ln \int_{\sum_{m,a,2/4,a+2/3}^2} Z_m^{-1} |\Delta(\mu_{>k})|^2 e^{-\sum_{j=k+1}^{n/2} \mu_j^2/2} d\mu_{>k}.
\]
If \( a \geq 1 \), by part (c) of Lemma 4, we have
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon)
\geq -(x_0 - a + \epsilon/2) + 1 - (a + \epsilon/4) + \ln(a + \epsilon/4).
\]
Since for \( \epsilon' \in (0, \epsilon) \), we have \( \mathbb{P}(d(\gamma_n, x) < \epsilon) \geq \mathbb{P}(d(\gamma_n, x) < \epsilon') \), thus
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon) \geq \liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon')
\geq -(x_0 - a + \epsilon'/2) + 1 - (a + \epsilon'/4) + \ln(a + \epsilon'/4),
\]
letting \( \epsilon' \to 0^+ \), we obtain
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon)
\geq -(x_0 - a) + 1 - a + \ln a = -x_0 + 1 + \ln a.
\]
Similarly, if \( 0 \leq a < 1 \), then for \( 0 < \epsilon < 1 - a \), we have \( 0 < a + \epsilon/4 < a + \epsilon/2 < 1 \). Now by Lemma 4 again, we have
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon) \geq -(x_0 - a + \epsilon/2)
\]
\[
+ \liminf_{m \to +\infty} \frac{4}{m^2} \ln \int_{\sum_{m,a,2/4,a+2/3}^2} Z_m^{-1} |\Delta(\mu_{>k})|^2 e^{-\sum_{j=k+1}^{n/2} \mu_j^2/2} d\mu_{>k}
\geq -(x_0 - a + \epsilon/2) + 1 - (a + \epsilon/2) + \ln(a + \epsilon/2).
\]
If \( 0 < \epsilon < 1, \ 0 < \epsilon' < \min(1-a, \epsilon) \), then
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon) \geq \liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon')
\geq -(x_0 - a + \epsilon'/2) + 1 - (a + \epsilon'/2) + \ln(a + \epsilon'/2),
\]
letting \( \epsilon' \to 0^+ \), we obtain
\[
\liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon) \geq -x_0 + 1 + \ln a.
\]
Therefore, for \( x = (x_j)_{j=0}^\infty \in X, \ a = x_0 - \sum_{j=1}^{+\infty} x_j^2, \ \epsilon \in (0, 1) \), we always have the lower bound
\[
(30) \quad \liminf_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \epsilon) \geq -x_0 + 1 + \ln a,
\]
in the sense that \( \ln 0 = -\infty \). Recall the definition of \( a \) in Lemma 4 we have
\[
a = J(x) = x_0 - \sum_{j=1}^{+\infty} x_j^2 \geq 0.
\]
This implies the lower bound in Lemma 5 if we define \( I(x) := x_0 - 1 - \ln J(x) \).
Now we consider the upper bound. For $A, B \in \mathbb{R}$, $k \in \mathbb{Z}$, $k > 0$, let’s define

$$G(x) = (A - B) \sum_{j=1}^{k} x_j^2 + B x_0, \quad x = (x_j)_{j=0}^{\infty} \in X.$$ 

Then $G$ is continuous in $X$ and

$$G(x) = A \sum_{j=1}^{k} x_j^2 + B \sum_{j=k+1}^{+\infty} x_j^2 \text{ if } x \in X_0.$$ 

Now for every $\delta > 0$, there exists $\varepsilon \in (0, 1)$ depending only on $x, A, B, k, \delta$ such that $G(y) > G(x) - \delta$ for $y \in X$, $d(x, y) < \varepsilon$. By definition of $\gamma_n \in X_0$, we further have

$$G(\gamma_n) = \left(\frac{n}{2}\right)^{-1} A \sum_{j=1}^{k} \mu_j^2 + \left(\frac{n}{2}\right)^{-1} B \sum_{j=k+1}^{n/2} \mu_j^2.$$ 

If $A, B < 1/2$, by (24) in Lemma 2 we have

$$\mathbb{P}(d(\gamma_n, x) < \varepsilon) \leq \mathbb{P}(G(\gamma_n) > G(x) - \delta) \leq e^{-\left(\frac{n}{2}\right)(G(x) - \delta)} \mathbb{E} e^{\left(\frac{n}{2}\right)G(\gamma_n)}$$

$$= e^{-\left(\frac{n}{2}\right)(G(x) - \delta)} \mathbb{E} e A k + B \sum_{j=k+1}^{n/2} \mu_j^2$$

$$\leq e^{-\left(\frac{n}{2}\right)(G(x) - \delta)} 2 e^{k(1 - 2A)^{-(n-k-\frac{1}{2})} (1 - 2B)^{-(\frac{n-1}{2}) - k}},$$

which implies

$$\limsup_{n \to +\infty} \frac{1}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \varepsilon) \leq -2(G(x) - \delta) - \ln(1 - 2B)$$

$$= -2(A - B) \sum_{j=1}^{k} x_j^2 - 2B x_0 + 2\delta - \ln(1 - 2B).$$

As $\limsup_{n \to +\infty} \frac{1}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \varepsilon)$ is an increasing function of $\varepsilon$, for every $A < 1/2$, $B < 1/2$, $\delta > 0$, $k \in \mathbb{Z}$, $k > 0$, we have

$$\lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \varepsilon)$$

$$\leq -2(A - B) \sum_{j=1}^{k} x_j^2 - 2B x_0 + 2\delta - \ln(1 - 2B).$$

Letting $A \to (1/2)^-$, $k \to +\infty$, $\delta \to 0^+$, we have

$$\lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \varepsilon)$$

$$\leq - (1 - 2B) \sum_{j=1}^{+\infty} x_j^2 - 2B x_0 - \ln(1 - 2B)$$

$$= (1 - 2B)a - x_0 - \ln(1 - 2B).$$

Therefore, we can further find the upper bound of the last line by choosing $B = (1 - 1/a)/2$ if $a > 0$ and $B \to -\infty$ if $a = 0$ (i.e., $x \in X_0$), and thus we have

$$\lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n^2} \ln \mathbb{P}(d(\gamma_n, x) < \varepsilon) \leq 1 - x_0 + \ln a,$$

(31)
which finishes the upper bound in Lemma 5

2.2.2. Compactness. Let’s recall that the rate function $I(x)$ is good if its level sets \( \{ x | I(x) \leq t \} \) are compact. We first give the following compactness criterion,

**Lemma 7.** The level sets $A_t := \{ x = (x_j)_{j=0}^\infty \in X | x_0 \leq t \}$ are compact.

**Proof.** Since the function $F(x) = x_0$ is continuous in $X$, the level sets $A_t = \{ x = (x_j)_{j=0}^\infty \in X | x_0 \leq t \}$ are closed and $A_t = \emptyset$ for $t < 0$. If $t \geq 0$, given a sequence $\{ x^k = (x^k_j)_{j=0}^\infty \} \subset A_t$, we have $0 \leq x^k_0 \leq t$, $(x^k_1)^2 \leq \sum_{j=1}^{+\infty} (x^k_j)^2 \leq x_0 \leq t$ and $0 \leq x^k_j \leq t^\frac{1}{2}$ for $j \geq 1$. Now we can find a subsequence $\{ x^{(k)} = (x^{(k)}_j)_{j=0}^\infty \} \subset A_t$ and $x^{(0)} = (x^{(0)}_j)_{j=0}^\infty$ such that $\lim_{k \to +\infty} x^{(k)}_j = x^{(0)}_j$ for $j \geq 0$. By Fatou’s lemma and the definitions of $X$ and $A_t$, we have $x^{(0)} \in A_t$ and $\lim_{t \to +\infty} x^{(0)}_j = 0$. Now for $k, l \in \mathbb{Z}$, $k, l \geq 0$, we have $\sup_{j \geq l+1} |x^{(k)}_j - x^{(0)}_j| \leq \sup_{j \geq l+1} \max(x^{(k)}_j, x^{(0)}_j) \leq \max(x^{(k)}_{l+1}, x^{(0)}_{l+1}) \leq x^{(0)}_{l+1} + |x^{(k)}_{l+1} - x^{(0)}_{l+1}|$; for $0 \leq j \leq l$, we have $|x^{(k)}_j - x^{(0)}_j| \leq x^{(0)}_{l+1} + |x^{(k)}_j - x^{(0)}_j|$. Thus $d(x^{(k)}, x^{(0)}) = \sup_{j \geq 0} |x^{(k)}_j - x^{(0)}_j| \leq x^{(0)}_{l+1} + \max_{0 \leq j \leq l+1} |x^{(k)}_j - x^{(0)}_j|$ and

$$0 \leq \limsup_{k \to +\infty} d(x^{(k)}, x^{(0)}) \leq x^{(0)}_{l+1} + \limsup_{k \to +\infty} \max_{0 \leq j \leq l+1} |x^{(k)}_j - x^{(0)}_j|$$

$$= x^{(0)}_{l+1} + \max_{0 \leq j \leq l+1} \limsup_{k \to +\infty} |x^{(k)}_j - x^{(0)}_j| = x^{(0)}_{l+1}.$$

Letting $l \to +\infty$, we have $\limsup_{k \to +\infty} d(x^{(k)}, x^{(0)}) = 0$, which means $x^{(k)} \to x^{(0)}$ in $X$ and $A_t$ is compact. This completes the proof. \( \square \)

Since $I$ is lower semicontinuous, the level sets $\{ x | I(x) \leq t \}$ are closed. For $x = (x_j)_{j=0}^\infty$, we have $0 \leq J(x) \leq x_0$, and thus $I(x) = x_0 - 1 - \ln J(x) \geq x_0 - 1 - \ln x_0 = x_0/2 + (x_0/2 - 1 - \ln(x_0/2)) = \ln 2 \geq x_0/2 - \ln 2$. Thus if $I(x) \leq t$, then $x_0 \leq 2(t + \ln 2)$, which implies $\{ x | I(x) \leq t \} \subseteq A_{2(t + \ln 2)}$. By Lemma 7, $A_{2(t + \ln 2)}$ is compact, thus the level sets $\{ x | I(x) \leq t \}$ are compact. Therefore, the rate function $I(x)$ is good.

2.2.3. Exponential tightness. We say that the sequence $Y_1, Y_2, \cdots$ is exponentially tight if for any $E > 0$, there exists a compact set $K_E \subset X$ such that

$$\limsup_{n \to +\infty} \frac{1}{a_n} \ln \mathbb{P}(Y_n \notin K_E) < -E.$$  

Regarding the exponentially tight measures, we have (see Appendix D in [1]),

**Lemma 8.** Let $(Y_n)_{n \geq 0, n \in \mathbb{Z}}$ be a sequence of random variables taking values in some Polish space $V$. Suppose that it is exponentially tight. If there exists a lower semicontinuous function $I : V \to [0, +\infty]$, such that for all $x \in V$ the following estimates of small ball probabilities hold

$$\lim_{\epsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{a_n} \ln \mathbb{P}(Y_n \in B(x, \epsilon)) \leq -I(x),$$

$$\lim_{\epsilon \to 0^+} \liminf_{n \to +\infty} \frac{1}{a_n} \ln \mathbb{P}(Y_n \in B(x, \epsilon)) \geq -I(x).$$
Then \( (Y_n)_{n>0,n\in\mathbb{Z}} \) satisfies LDP with rate function \( I(x) \).

By Lemma 8 and the results in \([21,24]\) and \([22,23]\) Proposition 1 follows once we prove that the sequence of random variables \((\gamma_n)_{n>0,n\in\mathbb{Z}}\) is exponentially tight.

Since the function \( F(x) = x_0 \) is continuous in \( X \) and \( F(\gamma_n) = (\gamma_n)_{n=0} = \left( \frac{n}{2} \right)^{-1} \sum_{j=1}^{n/2} \mu_j^2 \), taking \( k = 0 \), \( b = 1/4 \) in \([23]\), we have
\[
\mathbb{P}(\gamma_n \notin A_t) = \mathbb{P}(F(\gamma_n) > t) \leq e^{-\left(\frac{t}{2}\right)/4} E e^t F(\gamma_n) \leq e^{-\left(\frac{t}{2}\right)/4} (1 - 2/4)^{-\frac{n}{2}}.
\]

Then
\[
\lim_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(\gamma_n \notin A_t) \leq -t/2 - \ln(1 - 2/4) = -t/2 + \ln 2.
\]

For any \( E > 0 \), let’s choose \( t = 2(E + 1) > 0 \) and \( K_E := A_t \subset X \), then
\[
\limsup_{n \to +\infty} \frac{4}{n^2} \ln \mathbb{P}(\gamma_n \notin K_E) \leq -t/2 + \ln 2 < -t/2 + 1 = -E.
\]

By Lemma 7, \( K_E \) is compact, and thus \((\gamma_n)_{n>0,n\in\mathbb{Z}}\) is exponentially tight. This will complete the proof of Proposition 1.

2.3. Proof of Theorem 1

As explained in \([1,2]\), let’s define the map \( \varphi : X \to M_1(\mathbb{R}) \) via its Fourier transform \([20]\), then by definition of \( \gamma_n \), we must have \( \varphi(\gamma_n) = \rho_n \). There are three more properties we need to prove. First, \( \varphi(x) \) is a Borel probability measure. In fact, \( \varphi(x) \) is the density of the random variable \( Y = a_0 + \sum_{j=1}^{+\infty} x_j a_j \), where \((a_j)_{j=0}^{+\infty}\) are independent random variables such that \( \mathbb{P}(a_j = 1) = \mathbb{P}(a_j = -1) = 1/2 \) for \( j > 0 \), and \( a_0 \) is a Gaussian random variable with mean 0 and variance \( J(x) \).

Secondly, the map \( \varphi \) is continuous. To show this, we need the following fundamental lemma which indicates that the pointwise convergence of the Fourier transform convergence implies the convergence in \((M_1(\mathbb{R}), d_{BL})\) \([11,2]\).

Lemma 9. If \( \mu_n, \mu \in M_1(\mathbb{R}) \), \( \lim_{n \to +\infty} \mu_n(s) = \mu(s) \) for every \( s \in \mathbb{R} \), then \( \mu_n \to \mu \) in \( M_1(\mathbb{R}) \), i.e. \( \lim_{n \to +\infty} d_{BL}(\mu_n, \mu) = 0 \).

Now, for \( x = (x_j)_{j=0}^{+\infty} \in X \), \( J(x) = x_0 - \sum_{j=1}^{+\infty} x_j^2 \), we have
\[
\mathbb{E} \mathbb{E} \varphi(x) = e^{-J(x)s^2/2} \prod_{j=1}^{+\infty} \cos s x_j = e^{-x_0 s^2/2} \prod_{j=1}^{+\infty} (e^{sx_j^2/2} \cos s x_j).
\]

For \( x_k = (x_j^k)_{j=0}^{+\infty} \in X \) such that \( x_k \to x^0 \) in \( X \), we have \( \lim_{k \to +\infty} x_j^k = x_j^0 \) for every fixed \( j \geq 0 \) and
\[
\lim_{k \to +\infty} e^{-x_j^k s^2/2} = e^{-x_j^0 s^2/2}, \quad \lim_{k \to +\infty} e^{sx_j^k s^2/2} \cos s x_j^k = e^{sx_j^0 s^2/2} \cos s x_j^0.
\]

Since \( \lim_{t \to 0} t^{-2} \ln \cos t = -1/2 \), then for every \( \delta > 0 \), there exists \( \epsilon \in (0, 1) \) such that \( |t^2/2 + \ln \cos t| \leq t^2/2 \delta \) for \( |t| < \epsilon \). Since \( \lim_{j \to +\infty} x_j^0 = 0 \), for every fixed \( s \in \mathbb{R} \), there exists \( l > 0 \) such that \( |sx_j^0| < \epsilon \), then there exists \( k_0 > 0 \) such that \( |sx_j^k| < \epsilon \), \( x_j^k < \epsilon \), ...
$x_0^0 + 1$ for $k > k_0$, thus $|sx_j^k| \leq |sx_{l'}^{l'}| < \epsilon$ for $j \geq l$, $k > k_0$, and for $k > k_0$, $l' > l$ we have

$$\ln \prod_{j=l}^{+\infty} \left( e^{(sx_j^k)^2/2} \cos sx_j^k \right) \leq \sum_{j=l}^{+\infty} \left| (sx_j^k)^2/2 + \ln \cos sx_j^k \right| \leq \sum_{j=l}^{+\infty} (sx_j^k)^2 \delta$$

$$\leq s^2 \delta \sum_{j=1}^{+\infty} (x_j^k)^2 \leq s^2 \delta x_0^0 \leq s^2 \delta (x_0^0 + 1),$$

which implies the uniform convergence of the infinite product

$$e^{-x_0^0 s^2/2} \prod_{j=1}^{+\infty} \left( e^{(sx_j^k)^2/2} \cos sx_j^k \right), \quad k \geq 0.$$

By (32), we have

$$\lim_{k \to +\infty} e^{-x_0^0 s^2/2} \prod_{j=1}^{+\infty} \left( e^{(sx_j^k)^2/2} \cos sx_j^k \right) = e^{-x_0^0 s^2/2} \prod_{j=1}^{+\infty} \left( e^{(sx_j^k)^2/2} \cos sx_j^k \right),$$

this gives $\lim_{k \to +\infty} \varphi(x_k^k)(s) = \varphi(x_0^0)(s)$ for every fixed $s \in \mathbb{R}$. Therefore, by Lemma 9 we conclude the continuity of $\varphi$.

The end, the map $\varphi$ is injective. In fact, the second moment of the probability measure $\varphi(x)$ reads

$$\langle \varphi(x), \lambda^2 \rangle = J(x) + \sum_{j=1}^{+\infty} x_j^2 = x_0,$$

thus $x_0$ can be determined uniquely by $\varphi(x)$. Now we prove that $x_j$ can be determined inductively by $\varphi(x)$. Let $f_0(s) = \hat{\varphi}(x)(s)$, then $x_1 = \pi/(2 \inf\{t > 0|f_0(t) = 0\})$ and $x_1 = 0$ if $f_0(t) \neq 0$ for all $t \in \mathbb{R}$. Once $f_{k-1}$ and $x_k$ are determined, let $f_k(s) = f_{k-1}(s)/\cos sx_k$ and extend $f_k$ to be a continuous function for $s \in \mathbb{R}$, then $x_{k+1} = \pi/(2 \inf\{t > 0|f_k(t) = 0\})$ and $x_{k+1} = 0$ if $f_k(t) \neq 0$ for all $t \in \mathbb{R}$. In this way, we can determine $x_j$, $j > 0$ only using $f_0$. Thus $\varphi$ is injective.

Now we can give the LDP of $\rho_n$ by the following Contraction Principle [1].

**Lemma 10.** Let $(Y_n)_{n>0,n \in \mathbb{Z}}$ be a sequence of random variables taking values in some Polish space $X$. Let $\varphi : X \to V$ be continuous and injective, $V$ is also a Polish space. If $(Y_n)_{n>0,n \in \mathbb{Z}}$ satisfies LDP with speed $a_n$, going to infinity with $n$, and rate function $\hat{I}$ which is good, then $(\varphi(Y_n))_{n>0,n \in \mathbb{Z}}$ satisfies LDP with speed $a_n$ and good rate function $\hat{I}$ such that $\hat{I}(x) = I(\varphi^{-1} x)$ if $x \in \varphi(X)$ and $\hat{I}(x) = +\infty$ if $x \notin \varphi(X)$.

By Lemma 10 and the continuity and injectivity of $\varphi$ we proved above, $(\rho_n)_{n>0,n \in \mathbb{Z}}$ will satisfy the LDP in $(M_1(\mathbb{R}), d_{BL})$ with speed $a_n = n^2/4$ and good rate function $\hat{I}(x)$ such that $\hat{I}(x) = I(\varphi^{-1} x)$ if $x \in \varphi(X)$ and $\hat{I}(x) = +\infty$ if $x \notin \varphi(X)$. This completes the proof of Theorem 11.

### 3. Concentration of measure theorem for $q_n \geq 3$

Now we discuss the concentration of measure theorem for $\rho_n$ (defined in 3) of eigenvalues of the Gaussian SYK model for general $q_n \geq 3$. 
3.1. Notations and basic properties. Let’s first recall some notations and basic properties in [4] regarding the Majorana fermions. For a set $A = \{i_1, i_2, \cdots, i_m\} \subseteq \{1, 2, \cdots, n\}$, $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, we denote

$$\Psi_A := \psi_{i_1} \cdots \psi_{i_m} \text{ and } \Psi_A := I \text{ if } A = \emptyset.$$ 

We will need the following properties,

1. Given a set $A \subseteq \{1, 2, \cdots, n\}$,

$$\text{Tr} \Psi_A = 0 \text{ and } \Psi_A \neq \pm I \text{ are always true for } A \neq \emptyset.$$

2. For $A, B \subseteq \{1, 2, \cdots, n\}$, then

$$\Psi_A = \pm \Psi_B \text{ if and only if } A = B.$$

3. $$\Psi_A \Psi_B = \pm \Psi_{A\Delta B} \text{ where } A\Delta B := (A \setminus B) \cup (B \setminus A).$$

3.2. Proof of Lemma 1

Proof. Recall the notation [23], we may consider the random matrices $H$ as functions $H(x)$ which maps $x := (J_R)_{R \in I_n} \in \mathbb{R}^{n \times n}$ to the space of $L_n \times L_n$ Hermitian matrices which is equipped with the Hilbert-Schmidt norm

$$\|A\|^2_{H.S.} := \text{Tr}(AA^*) = \text{Tr}(A^2).$$

For $x = (J_R)_{R \in I_n}, x' = (J'_R)_{R \in I_n} \in \mathbb{R}^{n \times n}$, let’s write $H := H(x)$ and $H' := H(x')$. Then

$$\|H - H'\|^2_{H.S.} = \text{Tr}((H - H')^2) = (-1)^{[q_n/2]} \begin{pmatrix} n \\ q_n \end{pmatrix} \sum_{R \in I_n} \sum_{R' \in I_n} (J_R - J'_R)(J_R' - J'_R') \text{Tr}(\Psi_R \Psi_{R'}).$$

By properties 1 2 3 above, if $R \neq R'$, then $R \Delta R' \neq \emptyset$ and $\text{Tr}(\Psi_R \Psi_{R'}) = \pm \text{Tr}(\Psi_{R \Delta R'}) = 0$. If $R = R'$, then by the anticommutative property [2], we must have $\Psi_R \Psi_{R'} = \Psi_R^2 = (-1)^{[q_n/2]}I$ and $\text{Tr}(\Psi_R \Psi_{R'}) = (-1)^{[q_n/2]}L_n$. It follows that

$$\|H - H'\|^2_{H.S.} = L_n \begin{pmatrix} n \\ q_n \end{pmatrix} \sum_{R \in I_n} (J_R - J'_R)^2 = L_n \begin{pmatrix} n \\ q_n \end{pmatrix} ^{-1} \|x - x'|^2,$$

and thus the map $x \mapsto H(x)$ is $L_n^{1/2} \begin{pmatrix} n \\ q_n \end{pmatrix} ^{-1/2}$-Lipschitz.

Now we consider the map $H \mapsto (f, \rho_n)$ where $f$ is Lipschitz. Let $(\lambda_j)_{1 \leq j \leq L_n}$ be eigenvalues of $H$ and $(\lambda'_j)_{1 \leq j \leq L_n}$ be eigenvalues of $H'$ such that $\lambda_j \geq \lambda_{j+1}$, $\lambda'_j \geq \lambda'_{j+1}$ for $1 \leq j < L_n$. By definition $(f, \rho_n) = L_n^{-1} \sum_{j=1}^{L_n} f(\lambda_j)$, we have

$$|\langle f, \rho_n \rangle - \langle f, \rho'_n \rangle| = L_n^{-1} \left| \sum_{j=1}^{L_n} (f(\lambda_j) - f(\lambda'_j)) \right| \\ \leq \frac{\|f'\|_{L^\infty(\mathbb{R})}}{L_n} \sum_{j=1}^{L_n} \left| \lambda_j - \lambda'_j \right| \leq \|f'\|_{L^\infty(\mathbb{R})} \sqrt{L_n} \sum_{j=1}^{L_n} \left| \lambda_j - \lambda'_j \right|^2,$$
where we used the fact that $f$ is Lipschitz. The Hoffman-Wielandt inequality \[1\] further yields

\[
\sqrt{L_n^{-1} \sum_{j=1}^{L_n} |\lambda_j - \lambda'_j|^2} \leq L_n^{-1/2} \|H - H'\|_{H.S.},
\]

thus the map $H \mapsto \langle f, \rho_n \rangle$ is $L_n^{-1/2} \|f'\|_{L^\infty(\mathbb{R})}$-Lipschitz. Therefore, if we combine \[33\] \[34\] \[35\], when $f$ is 1-Lipschitz, we have

\[
|\langle f, \rho_n \rangle - \langle f, \rho'_n \rangle| \leq L_n^{-1/2} \|H - H'\|_{H.S.}
\]

\[
= L_n^{-1/2} \sqrt{L_n \left( \frac{n}{q_n} \right)^{-1} \|x - x'\|} = \left( \frac{n}{q_n} \right)^{-1/2} \|x - x'\|
\]
i.e., the map $x \mapsto H \mapsto \langle f, \rho_n \rangle$ is $\left( \frac{n}{q_n} \right)^{-1/2} \|f'\|_{L^\infty(\mathbb{R})}$-Lipschitz, which finishes (a).

By triangle inequality and definition of the bounded Lipschitz metric \[17\], after taking supremum over all 1-Lipschitz functions, we further have

\[
|d_{BL}(\rho_n, \rho) - d_{BL}(\rho'_n, \rho)| \leq d_{BL}(\rho_n, \rho'_n) \leq \left( \frac{n}{q_n} \right)^{-1/2} \|x - x'\|
\]

this completes the proof of (b). \[\Box\]

3.3. Proof of Theorem \[2\] Now we are ready to prove Theorem \[2\]

Proof. By part (b) of Lemma \[1\] $d_{BL}(\rho_n, \rho_\infty)$ is $L$-Lipschitz with Lipschitz constant $L = \left( \frac{n}{q_n} \right)^{-1/2}$. Therefore, by the concentration of measure theorem for Gaussian vectors \[22\], for any $t > 0$, we will have

\[
\mathbb{P}(|d_{BL}(\rho_n, \rho_\infty) - \mathbb{E}d_{BL}(\rho_n, \rho_\infty)| > t) \leq C e^{-ct^2/L^2} = C e^{-c(\frac{n}{q_n})t^2}.
\]

For the empirical measure of eigenvalues $\rho_n$ defined in \[3\] and its limit $\rho_\infty$ as proved in \[4\], for any 1-Lipschitz function $f$, we have

\[
|\langle f, \rho_n \rangle - \langle f, \rho_\infty \rangle| \leq |\langle f, \rho_n \rangle| + |\langle f, \rho_\infty \rangle| \leq 2 \|f\|_{L^\infty} \leq 2.
\]

Thus $0 \leq d_{BL}(\rho_n, \rho_\infty) \leq 2$. The fact that $\rho_n \rightarrow \rho_\infty$ almost surely (which is one of the main results in \[4\]) implies

\[
\lim_{n \rightarrow +\infty} d_{BL}(\rho_n, \rho_\infty) = 0, \text{ a.s.}
\]

Therefore, by the dominated convergence theorem, we have

\[
\lim_{n \rightarrow +\infty} \mathbb{E}d_{BL}(\rho_n, \rho_\infty) = 0.
\]

Thus for every $a > 0$, there exists $N_0 = N_0(a) > 0$ such that $\mathbb{E}d_{BL}(\rho_n, \rho_\infty) \leq a/2$ for $n > N_0$. Thus if $n > N_0$, by \[36\], we have

\[
\mathbb{P}(d_{BL}(\rho_n, \rho_\infty) > a) \leq \mathbb{P}(|d_{BL}(\rho_n, \rho_\infty) - \mathbb{E}d_{BL}(\rho_n, \rho_\infty)| > a/2) \leq C e^{-c(\frac{n}{q_n})(a/2)^2}.
\]

If $n \leq N_0$, we have

\[
\mathbb{P}(d_{BL}(\rho_n, \rho_\infty) > a) \leq 1 = e e^{-1} \leq C e^{-c(\frac{n}{q_n})^2} \leq C e^{-c(\frac{n}{q_n})^2 - N_0}.
\]

Therefore, we always have

\[
\mathbb{P}(d_{BL}(\rho_n, \rho_\infty) > a) \leq C e^{-c(a)(\frac{n}{q_n})}
\]

for $c(a) = \min(c \cdot (a/2)^2, 2^{-N_0(a)})$, which completes the proof Theorem \[2\]. \[\Box\]
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References

[1] G. W. Anderson, A. Guionnet and O. Zeitouni, An introduction to random matrices. Cambridge Studies in Advanced Mathematics, 118. Cambridge University Press, Cambridge, 2010.

[2] P. Brémaud, Fourier Analysis and Stochastic Processes, Springer, Universitext, 2014.

[3] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher and M. Tezuka, Black Holes and Random Matrices, J. High Energ. Phys. (2017) 2017: 118.

[4] R. Feng, G. Tian and D. Wei, Spectrum of SYK model, arXiv:1801.10072.

[5] R. Feng, G. Tian and D. Wei, Spectrum of SYK model II: central limit theorem, preprint.

[6] W. Fu, D. Gaiotto, J. Maldacena and S. Sachdev, Supersymmetric Sachdev-Ye-Kitaev models, Phys. Rev. D, 95 (2017) 026009.

[7] A.M. Garcia-Garcia and J.J.M. Verbaarschot, Spectral and thermodynamic properties of the Sachdev-Ye-Kitaev model, Phys. Rev. D94 (2016) 126010.

[8] A.M. Garcia-Garcia and J.J.M. Verbaarschot, Analytical Spectral Density of the Sachdev-Ye-Kitaev Model at finite N, Phys. Rev. D96 (2017) 066012.

[9] A.M. Garcia-Garcia, Yiyang Jia and J.J.M. Verbaarschot, Exact moments of the Sachdev-Ye-Kitaev model up to order $1/N^2$, High Energ. Phys. (2018) 2018: 146.

[10] A.M. Garcia-Garcia, Yiyang Jia and J.J.M. Verbaarschot, Universality and Thouless energy in the supersymmetric Sachdev-Ye-Kitaev Model, arXiv: 1801.01071.

[11] A. Kitaev, Hidden correlations in the Hawking radiation and thermal noise, KITP seminar, 12 February 2015, [http://online.kitp.ucsb.edu/online/joint98/kitaev/](http://online.kitp.ucsb.edu/online/joint98/kitaev/)

[12] H. B Lawson and M-L Michelsohn, Spin Geometry, (PMS-38), Volume 38.

[13] M. Ledoux, The concentration of measure phenomenon, volume 89 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2001.

[14] J. Maldacena and D. Stanford, Remarks on the Sachdev-Ye-Kitaev model, Phys. Rev. D 94 (2016) 106002.

[15] M. Mehta, Random matrices, Academic Press, third edition, 2004.

[16] S. Sachdev and J. Ye, Gapless spin-fluid ground state in a random quantum Heisenberg magnet, Phys. Rev. Lett. 70 (1993) 3339-3342.

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