Plane waves from double extended spacetimes

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Abstract

We study exact string backgrounds (WZW models) generated by nonsemisimple algebras which are obtained as double extensions of generic D–dimensional semisimple algebras. We prove that a suitable change of coordinates always exists which reduces these backgrounds to be the product of the nontrivial background associated to the original algebra and two dimensional Minkowski. However, under suitable contraction, the algebra reduces to a Nappi–Witten algebra and the corresponding spacetime geometry, no more factorized, can be interpreted as the Penrose limit of the original background. For both configurations we construct D–brane solutions and prove that all the branes survive the Penrose limit. Therefore, the limit procedure can be used to extract informations about Nappi–Witten plane wave backgrounds in arbitrary dimensions.

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1 Introduction

One of the main topics in string theory is the construction of exact backgrounds, that is two dimensional $\sigma$-models which are conformally invariant at the quantum level and at all orders in the $\alpha'$ expansion.

A distinguished class of exact backgrounds is given by the WZW models on group manifolds where the vanishing of the $\beta$ functions at all orders is ensured by the affine Sugawara construction [1]. For a WZW model associated to a given algebra, the Sugawara construction exists if and only if the algebra possesses an ad–invariant, symmetric and non-degenerate metric. In the case of a semisimple algebra this is the Killing metric and all the renormalization effects give simply a correction to the metric. Generalizations to nonsemisimple cases have been extensively studied. In [2] a WZW model based on the central extension of the 2d Poincaré algebra was constructed (NW model). This construction was then extended to more general nonsemisimple cases [3]–[7], while the corresponding generalized Sugawara quantization was completely analyzed in [10] for the whole class of nonsemisimple algebras which admit an invariant metric, i.e. algebras obtained as double extensions of abelian or semisimple algebras [12].

A second class of string backgrounds can be obtained by means of the Penrose limit ([13],[14]). The two classes partially overlap since generalized NW models can be obtained as Penrose limits of suitable geometries [15].

In the present paper we investigate the spacetime geometries which arise from WZW models associated to the abelian double extension of a generic semisimple $D$–dimensional Lie algebra. We first give a general proof that the extended algebra can be always reduced to the direct sum of the original algebra and a bidimensional abelian algebra $^1$. The corresponding spacetime geometry is then in some sense trivial since it reduces to the product of the original spacetime with two dimensional Minkowski. However, what makes these constructions interesting is that by taking a suitable Inönü-Wigner contraction [11] of the extended algebra, the new algebra which emerges is a Nappi–Witten like algebra. Therefore, the geometry described by the corresponding sigma model is no more trivial, being a $(D + 2)$–dimensional Nappi–Witten background. We show that it is the Penrose limit of the original model associated to the nonsemisimple algebra.

An interesting question which emerges is whether in the process of contracting the algebra (or equivalently of taking the Penrose limit on the corresponding sigma model) information is lost. To answer this question we study brane configurations in both cases and prove that all brane solutions we find in the contracted model correspond to the Penrose limit of brane solutions of the double extended original model. Therefore, all the information goes safely through the limit.

The plan of the paper is the following. In Section 2 we recall some basic facts about one dimensional double extensions and give the general proof of the fact that double extensions of semisimple Lie algebras are somehow trivial. Our proof can be easily generalized to the case considered in [10]. Next we show how to perform a suitable Inönü-Wigner contraction of the double extended algebra to obtain a nontrivial generalized NW algebra. In Section

$^1$This fact was shown in a more abstract way in [10] for the general case of nonreductive algebras.
3 we construct the corresponding WZW model and show that the contraction actually corresponds to a Penrose limit on the corresponding string background. We implement the affine Sugawara construction and compute the central charge of the model. In Section 4 we construct brane configurations for both models, the one associated to the double extended algebra and the one corresponding to the contracted algebra. In particular, we prove that the contraction can be used to extract all possible informations about the limit background. Many technical details are collected in two Appendices.

2 The double extended algebra and its contraction

We consider a D-dimensional Lie algebra $\mathcal{A}$ with generators $\tau_i, i = 1, \ldots, D$ satisfying

$$\left[\tau_i, \tau_j\right] = f^{k}_{ij} \tau_k . \quad (2.1)$$

Its one dimensional double extension is obtained by adding the new generators $H$ and $H^*$ such that

$$\left[\tau_i, \tau_j\right] = f^{k}_{ij} \tau_k + f_{ij} H^*$$

$$\left[H, \tau_i\right] = f^j_i \tau_j$$

while $H^*$ is an element of the center. Here $f_{ij}$ are antisymmetric matrices constrained by the Jacobi identities

$$f_{ij} f_{kl} + f_{jl} f_{ki} + f_{li} f_{kj} = 0 . \quad (2.3)$$

Defining the new generators $\tau := H, \tau_* := H^*$ the previous algebra can be written as

$$\left[\tau_I, \tau_J\right] = f^K_{IJ} \tau_K$$

where $I, J, K = 1, \ldots, D, \cdot, \cdot$. The indices are highered and lowered by the bi-invariant metric

$$\Omega_{IJ} = \begin{pmatrix} aK_{ij} & 0 & 0 \\ 0 & b & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where $K_{ij}$ being an invertible ad-invariant metric for $\mathcal{A}$. If $\mathcal{A}$ is semisimple one can take $K_{ij} = h_{ij}$ where $h_{ij}$ is the Killing metric of the semisimple algebra. We note that the constant $b$ in the metric is totally arbitrary and, at the classical level, it could be consistently set to zero.

The condition (2.3) has a nice interpretation. On the external algebra $\Lambda^*$ over $\mathcal{A}$ one can define the external derivative

$$\delta : \Lambda^* \longrightarrow \Lambda^*$$

(2.6)
as the operator which maps the $p$-form $\lambda$ into the $p+1$-form $\delta \lambda$ given by

$$\delta \lambda(v_1, \ldots, v_{p+1}) = \sum_{q<r=1}^{p+1} (-1)^{r+q} \lambda(v_1, \ldots, \hat{v}_q, \ldots, \hat{v}_r, \ldots, v_{p+1}, [v_q, v_r])$$  \hspace{1cm} (2.7)

where $v_i$ are vectors of $A$ and the hat means exclusion. The coefficients $f_{ij}$ define a $\mathbb{R}$-valued two form $\mathcal{F} := \frac{1}{2} f_{ij} \mu^i \wedge \mu^j$, where $\{\mu^i\}$ is the basis of $A^*$ dual to $\{\tau_i\}$. Therefore eq. (2.3) can be rewritten as

$$\delta \mathcal{F} = 0$$  \hspace{1cm} (2.8)

We concentrate on the case of $A$ being a semisimple Lie algebra: From the Whithead’s second lemma, the second cohomology class is trivial, $H^2(A) = 0$, and we can write

$$\mathcal{F} = \delta \lambda$$  \hspace{1cm} (2.9)

for a given 1-form $\lambda$. In components this condition reads

$$f_{ij} = f_{ij}^k \lambda_k$$  \hspace{1cm} (2.10)

Therefore, in the case of $A$ semisimple the constant matrices $f_{ij}$ entering the one-dimensional double extension are constrained to have the form (2.10). We will say that the double extension of a semisimple Lie algebra is polarized by the vector $\lambda_k$.

Since $f_{ij}$ are the coefficients of an exact two form there must exist a basis of the algebra which eliminates $f_{ij}$ in (2.2). This basis can be easily found: If we introduce the combinations

$$T_i := \tau_i + \lambda_i \lambda^*$$, \quad Z := H - \lambda^2 H^* - \lambda^i \tau_i$$, \quad Z^* := H^*$$  \hspace{1cm} (2.11)

where $\lambda^2 \equiv \lambda^i a K_{ij} \lambda^j$, the commutation rules reduce to

$$[T_i, T_j] = f^k_{ij} T_k$$  \hspace{1cm} (2.12)

with $Z$ and $Z^*$ both in the center. Under this redefinition the invariant metric transforms as $\Omega_b \longrightarrow \Omega_{\tilde{b}}$ with $\tilde{b} = b - \lambda^2$. We will call (2.11) the trivializing basis.

We have proven that the one dimensional double extension $D(A)$ of a semisimple Lie algebra always reduces to the direct sum of the original algebra and a bidimensional abelian algebra

$$D(A) = A \oplus \mathbb{R}^2$$  \hspace{1cm} (2.13)

The two sectors ($A$ and $\mathbb{R}^2$) are in fact orthogonal because of the particular structure of the invariant metric. As a consequence, the manifold realized via WZW construction will be the direct product of the semisimple group associated to $A$ and the two dimensional flat minkowskian spacetime

$$e^{D(A)} = e^A \otimes \mathbb{R}^{1,1}$$  \hspace{1cm} (2.14)
It is however interesting to consider the WZW construction corresponding to an algebra obtained as İnönü-Wigner contraction of (2.2). To this purpose we start from an ansatz for the metric slightly different from (2.5) in order to end up with a well-defined, invariant metric after the contraction. We consider the three–parameter family of invariant forms (for given $a$ and $b$ constant)

$$\Omega(\xi, \sigma, \rho) = \begin{pmatrix} \xi a K_{ij} & 0 & 0 \\ 0 & \sigma b & \rho \\ 0 & \rho & 0 \end{pmatrix}$$ (2.15)

and define the rescaled generators

$$P_i := \alpha \tau_i , \quad T := \alpha^2 H^* , \quad S := \xi H .$$ (2.16)

They generate the one–parameter family of algebras $\mathcal{A}_\alpha$ given by

$$[P_i, P_j] = \alpha f_{ij}^k P_k + f_{ij} T$$

$$[S, P_i] = f_{ij}^i P_j$$ (2.17)

In particular the r.h.s. of the commutators do not depend on $\xi$.

Before the contraction, the products between the elements of the basis give the nonvanishing results

$$(P_i, P_j) = a \xi \alpha^2 K_{ij} , \quad (S, S) = \sigma \xi^2 b , \quad (S, T) = \rho \xi \alpha^2 .$$ (2.18)

If we then choose

$$\xi = \frac{1}{\alpha^2} , \quad \sigma = \alpha^4 , \quad \rho = 1$$ (2.19)

we find that with respect to the new basis the product is well defined for $\alpha$ going to zero and, independently of the parameters, we have

$$(P_i, P_j) = a K_{ij} , \quad (S, S) = b , \quad (S, T) = 1 .$$ (2.20)

Therefore, taking the contraction $\alpha \to 0$ we obtain the algebra

$$[P_i, P_j] = f_{ij} T$$

$$[S, P_i] = \frac{1}{\alpha} K^{jk} f_{ik} P_j \equiv f_{ij}^i P_j$$ (2.21)

with $T$ central and invariant metric (2.5). This is a Nappi-Witten algebra, therefore no more trivializable.

We note that the change of basis (2.11) which trivializes the algebra $D(\mathcal{A})$ becomes singular in this limit according to the fact that the Nappi-Witten algebra is nonseparable. As we will see in Section 3, the relation between the trivial algebra (2.12) and the NW
algebra (2.21) through the contraction corresponds to the fact that the Penrose limit of Cartesian product spaces may generate nontrivial spacetimes. Before closing this Section we mention the fact that a slightly different contraction can be performed by starting with rescaled generators

\[ P'_i := \alpha \tau_i, \quad T' := \alpha^2 H^*, \quad S' := \xi (H - \frac{b}{2} H^*). \tag{2.22} \]

In the limit \( \alpha \to 0 \) we still obtain the algebra (2.21) but with metric (2.5) corresponding to \( b = 0 \).

### 3 The WZW model for the double extended algebra

We now construct the WZW model associated to the algebra (2.2). We parametrize the group elements as

\[ g = g_A e^{uH + vH^*}, \quad g_A = e^{\theta \tau_i}. \tag{3.1} \]

being \( g_A \) an element of the group \( e^A \). Using the general identity

\[ e^{-\theta} \partial_\alpha e^\theta = \int_0^1 e^{-x\theta} \partial_\alpha e^{x\theta} dx \tag{3.2} \]

we can compute the left current \( J := g^{-1} dg \) and find

\[ J = j^i \exp \left( uF^k_i \tau_k + (d\theta^i j^*_i + dv)H^* + duH \right) \tag{3.3} \]

where \( F \) is the matrix \( F^k_i := -f^k_{ij} \). The current \( j^i \) is the current of the unextended algebra \( j^i := d\theta^k j^i_k \), with

\[ j^i_k := \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \theta^{j_1} \ldots \theta^{j_n} f_{kj_1}^1 f_{kj_2}^2 \ldots f_{kj_{n-2}j_{n-1}}^{k_{n-1}} f_{kj_{n-1}j_n}^i, \tag{3.4} \]

whereas \( j^*_i \) is given by

\[ j^*_i := \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \theta^{j_1} \ldots \theta^{j_n} f_{ij_1}^1 f_{ij_2}^2 \ldots f_{ij_{n-2}j_{n-1}}^{k_{n-1}} f_{ij_{n-1}j_n}^i. \tag{3.5} \]

This current takes a relatively simple expression in the abelian case

\[ j^*_i = \frac{1}{2} \theta^i f_{ij} \tag{3.6} \]

and in the polarized case where it reduces to

\[ j^*_i = (j^i_j - \delta^i_j) \lambda_j. \tag{3.7} \]
The WZW action on group manifold is given by
\[
S = \frac{1}{4\pi} \int_{\partial \Sigma} d^2 \sigma \Omega_{IJ} j^I_\alpha j^J_\alpha + \frac{i}{12\pi} \int_{\Sigma} d^3 \sigma \Omega_{KLM} f_{IJ}^K \epsilon^{\alpha \beta \gamma} j^I_\alpha j^J_\beta j^L_\gamma
\]
(3.8)
where \( \Omega_{ij} \) is the metric (2.5) for the double extended algebra. Using the fact that \( F \cdot K \) is antisymmetric as a consequence of the invariance of \( K \), we find
\[
\Omega_{IJ} j^I_\alpha j^J_\alpha = a K_{ij} j_i^\alpha j_j^\alpha + b \partial_\alpha u \partial^\alpha u + 2 \partial_\alpha u (\partial^\alpha M^i j^*_i + \partial^\alpha v)
\]
(3.9)
which gives a sigma model with metric
\[
G_{ij} = a G^{(A)}_{ij}, \quad G_{uw} = 1, \quad G_{ui} = j^*_i,
\]
(3.10)
where \( G^{(A)}_{ij} = K_{lm} j^i_l j^m_j \) is the metric associated to the unextended algebra. In the same way, using the Jacobi identities, we find for the nonvanishing components of the simplectic structure
\[
B_{ij} = B^{(A)}_{ij} + \frac{u}{2} f_{lm} j^i_l j^m_j
\]
(3.11)
where locally \( dB^{(A)} = H^{(A)} \) with
\[
B^{(A)} = \frac{1}{2} B^{(A)}_{ij} j^i \wedge j^j, \quad H^{(A)} = \frac{1}{6} H^{(A)}_{ijk} j^i \wedge j^j \wedge j^k.
\]
(3.12)
The quantization of the model can be performed nonperturbatively by means of the Sugawara construction \[8\], \[9\], \[10\]. Given the level–k current algebra
\[
j_I(z) j_J(w) = - \frac{2k \Omega_{IJ}}{(z - w)^2} + f_{IJ} L j_L(w) + \text{reg.}
\]
(3.13)
the energy-momentum tensor takes the form
\[
T = L^{IJ} : j_I j_J :
\]
(3.14)
where \( L^{IJ} \) is the inverse of the matrix
\[
L_{IJ} = (-4k \Omega + h)_{IJ}
\]
(3.15)
and \( h_{IJ} \) is the Killing form of the double extended algebra
\[
h_{IJ} = -f_{IK} L f_{KL} K.
\]
(3.16)
In the polarized case, given the position (2.10) it takes the form

\[ h_{IJ} = \begin{pmatrix}
 h_{ij} & \lambda^k h_{ki} & 0 \\
 \lambda^i h_{lj} & \lambda^k h_{kl} & 0 \\
 0 & 0 & 0
\end{pmatrix} \]  

(3.17)

where \( h_{ij} \) is the Killing form of the algebra \( \mathcal{A} \). We note that, even if we were to start with a classical invariant metric (2.5) with \( b = 0 \), a nontrivial \( b \) would get produced by the quantization procedure.

Using the Sugawara construction we can compute the central charge as

\[ c = -4kL_{IJ}\Omega_{IJ}. \]

In our case we find

\[ c = D + \sum_{n=1}^{\infty} \text{Tr} \left( \frac{1}{4ak} K^{-1} h \right)^n \]

(3.18)

where \( D \) is the dimension of the double extended algebra, \( D = 2 + \dim \{ \mathcal{A} \} \). Given the particular structures of \( \Omega_{IJ} \) and \( h_{IJ} \) the central charge turns out to be independent of the parameter \( b \). In particular for the abelian case we have \( c = D \), whereas for \( \mathcal{A} \) a semisimple Lie algebra \( (K_{ij} = h_{ij}) \)

\[ c = D + \frac{D - 2}{4ak - 1} = c(\mathcal{A}) + 2. \]  

(3.19)

We now search for the coordinate transformation corresponding to the change of basis (2.11). In the new basis the generic element of the group (3.1) takes the form

\[ g = e^{\theta^i T_i} e^{u^\lambda T_\lambda} e^{(v - \lambda_k \theta^k) Z^*} = e^{\Psi^i(\theta^j, u)} T_i e^{u^\lambda Z^* + (v - \lambda_k \theta^k) Z^*} \]

(3.20)

where, using the Baker-Campbell-Hausdorff formula

\[ \Psi^i(\theta^j, u) = \theta^i + u^\lambda \lambda^j f^k_{jk} + \frac{u}{2} \theta^j \theta^k \lambda^l f^m_{kl} f^i_{jm} + \frac{u^2}{12} \theta^j \lambda^k \lambda^l f^m_{jk} f^i_{ml} + \cdots . \]  

(3.21)

In terms of the new coordinates

\[ \Psi^i = \Psi^i(\theta^j, u) \]

\[ \tilde{u} = u \]

\[ \tilde{v} = v - \lambda_k \theta^k \]

(3.22)

the sigma model can be easily found by following the previous calculations where we set \( \lambda = 0 \). In particular, the metric turns out to be a diagonal block matrix (in this case \( j^*_i = 0 \), see eqs. (3.7, 3.10)) and our solution completely factorizes as

\[ e^{A(\Psi^i)} \otimes \mathbb{R}^{1,1}(\tilde{u}, \tilde{v}) . \]  

(3.23)

Therefore, the spacetime geometries described by WZW models associated to double extended algebras are somehow trivial extensions of the spacetimes associated to the original semisimple algebra. However, as already mentioned, nontrivial backgrounds can arise by means of suitable Penrose limits. This will be the subject of the next Section.
4 The WZW model for the contracted algebra: The Penrose Limit

It is well–known [13] [15] [16] that the four NW₄ and six dimensional NW₆ plane wave backgrounds arise as Penrose limits of AdS₂×S₂ and AdS₃×S₃, respectively. In particular, for the second case, in [15] it has been shown that at the level of the algebras this limit can be interpreted as a group contraction, as a consequence of the existence of a null one–parameter subgroup corresponding to a null geodesic of the invariant metric.

We generalize this result to the whole class of models associated to double extended algebras. Starting from the original algebra (2.2) endowed with the metric (2.5) we define the inner product of two generators as

\[ \langle \tau_I, \tau_J \rangle = \Omega_{IJ} \]

Correspondingly, we see that there exist null one–dimensional subgroups generated by \( K \equiv (H - \frac{b}{2}H^*) \) and \( H^* \). Therefore, in analogy with the AdS₃×S₃ case, we might expect the contracted group to correspond to a WZW model which describes the Penrose limit of the original spacetime. In Section 1 we have discussed the Inönü–Wigner contraction of our original algebra. Starting from the rescaled generators (2.22) and taking the \( \alpha \to 0 \) limit amounts to perform the contraction along the null vector \( K \). On the other hand, the rescaled generators (2.16) correspond to a contraction along a vector which is a linear combination of \( K \) and \( H^* \). The two cases differ by the value of the parameter \( b \) appearing in the metric (2.5) which in the first case is zero, whereas in the second case is arbitrary.

In any case, the contraction of the original double extended algebra gives rise to a NW–like algebra which is known to correspond to a NW₃⁺₂ background. Being this background a plane wave it can be reasonably expected to be the Penrose limit of a nontrivial background.

We now elaborate on that. To prove that the contracted algebra actually corresponds to the WZW model in the Penrose limit, we need prove that the model constructed directly from the algebra (2.21) coincides with the Penrose limit of the model (3.8, 3.10, 3.11). The WZW model associated to the contracted algebra (2.21) is known [5], since the algebra is the \( (D + 2) \) dimensional generalization of the NW algebra. It has the general structure (3.8) where the invariant metric is of the form (2.5). By parametrizing the generic group element as

\[ g = e^{\Theta^a P_i e^{uS+vT}} \]

the corresponding sigma model describes a spacetime with metric

\[ G_{IJ} = \begin{pmatrix} G_{ij} & G_{iu} & 0 \\ G_{ui} & b & 1 \\ 0 & 1 & 0 \end{pmatrix} \]

We now consider the Penrose limit of the sigma model constructed in Section 2 for the original nonsemisimple algebra.

\[ ^{| ^2 \text{Since the two null vectors are not orthogonal the linear combination } K + \frac{b}{2}H^* \text{ which defines the direction of our contraction is not strictly a null vector.} \]
Given the group element (3.1) we perform the rescaling (2.16) of the generators
\[ g = e^{i\theta_i^\alpha P_i} e^{\alpha^2 u S + \frac{i}{2} T} \] (4.3)
and define the new coordinates
\[ \Theta^i = \frac{\theta^i}{\alpha} , \quad U = \alpha^2 u , \quad V = \frac{v}{\alpha^2} . \] (4.4)
Performing this change of coordinates in the original current \( J = g^{-1} dg \) we find
\[ J = \frac{1}{\alpha} j^i(\alpha \Theta) e^{\frac{1}{\alpha^2} U F^k_i P_k} + \left( \frac{1}{\alpha^2} j^i(\alpha \Theta) + dV \right) T + dUS \] (4.5)
where
\[ \frac{1}{\alpha} j^i(\alpha \Theta) = d\Theta^i + \frac{\alpha}{2} \Theta^k f^i_{jk} d\Theta^j + o(\alpha) \] (4.6)
and
\[ \frac{1}{\alpha^2} j^i(\alpha \Theta) = \frac{1}{2} \lambda^j_\Theta f^i_{jk} d\Theta^j + o(1) . \] (4.7)
These quantities have a well–defined limit \( \alpha \to 0. \) Therefore, taking this limit and computing the background metric according to the general prescription \( \Omega_{IJ} J^I_j J^J^\alpha = G_{IJ} \partial_\alpha X^I \partial^\alpha X^J , \quad X^I = (\Theta^i, U, V) , \) we find
\[ G_{ij} = aK_{ij} , \] (4.8)
\[ G_{UU} = b , \] (4.9)
\[ G_{UV} = 1 , \] (4.10)
\[ G_{Ui} = \frac{1}{2} \lambda^j_\Theta f^i_{jk} . \] (4.11)
This metric is of the form (1.2) so proving that the sigma model associated to the contracted algebra is the Penrose limit of the original sigma model.
To find the two form \( B \) let us recall that it comes out from the bulk term \( S[B] \sim \int H = \int dB . \) Now using the family of metrics with the chosen parameters, one has
\[ H = \frac{1}{6} f_{IJK} J^I \wedge J^J \wedge J^K = \frac{1}{6} f_{ijk} j^i \wedge j^j \wedge j^k + \frac{1}{2} f_{ij} j^i \wedge j^j \wedge du \]
\[ = \frac{1}{6} \alpha a K_{kl} f_{ij} j^i \wedge j^j \wedge \frac{1}{\alpha} \wedge \frac{1}{\alpha} + \frac{1}{2} \lambda^j_\Theta f^i_{jk} j^i \wedge j^j \wedge dU \] (4.12)
and taking the limit
\[ B_{ij} = \frac{U}{2} \lambda^j_\Theta f^k_{ij} . \] (4.13)
The equivalence between the contraction of the algebra and the Penrose limit of the corresponding background can be investigated also at the quantum level by studying how the Sugawara construction works under the limit \( \alpha \to 0 \). To this end we consider the family of models parametrized by \( \alpha \) and compute the corresponding central charge (see eq. (3.19))

\[
c_\alpha = D + \frac{\alpha^2(D-2)}{4ak - \alpha^2}.
\]

In the limit \( \alpha \to 0 \) we find \( c_\alpha \to D \) which is the correct central charge for the conformal sigma model associated to the contracted algebra \([2, 5]\). This proves that the limit is consistent also quantistically.

## 5 Boundary states and D-brane configurations

We can embed D–branes in a string background by determining boundary states which preserve conformal invariance. In the case of WZW solutions this translates into boundary conditions which have to be satisfied by the currents of the model. Following \([17]\) we impose the gluing conditions

\[
J_I(z) + M^J_L \bar{J}_J(\bar{z}) = 0
\]

where \( M_J^I \) are determined by requiring conformal invariance and current algebra to be preserved. If \( M \) is defined on the generators (2.4) as \( M(\tau_I) = \tau_J M^J_I \), then the previous conditions read \([17]\)

\[
M^T L M = L \tag{5.2}
\]

\[
[M(\tau_I), M(\tau_J)] = M([\tau_I, \tau_J]), \quad M^T \Omega M = \Omega \tag{5.3}
\]

where \( L \) is given in (3.15) and \( \Omega \) is the invariant metric of the extended algebra. The first condition follows from requiring conformal invariance, while the second one comes from imposing the invariance of the current algebra. We note that, given the particular form of the matrix \( L \), the condition (5.2) can be substituted by

\[
M^T h M = h \tag{5.4}
\]

where \( h \) is the Killing form.

The solutions to these equations fix the boundary states. They can be found for the cases of polarized algebras \([2.2, 2.10]\), trivial algebras \([2.12]\) and contracted algebras \([2.21]\). Details on the procedure for solving the equations in the three cases are given in the Appendices, while here we report only the results and discuss their physical interpretation. As shown in Appendix A, in the case of a semisimple algebra polarized by the vector \( \lambda \) the constraints have solution

\[
M^I_J = \begin{pmatrix}
N_i^j - \sigma \lambda^i \lambda_j & \lambda^k N_i^k - \nu \lambda^i & \sigma \lambda^i \\
\sigma \lambda_j & \nu & -\sigma \\
\lambda_k N_j^k - \mu \lambda_j & \gamma & \mu
\end{pmatrix}
\]

(5.5)
where $\nu, \sigma, \gamma, \mu$ are real constants satisfying the following equations
\begin{align}
  \sigma(\sigma \lambda^2 + \sigma b - 2\mu) &= 0 \ , \\
  1 + \nu \sigma \lambda^2 - \sigma \lambda N^i_j \lambda^j + \sigma b \nu + \sigma \gamma - \mu \nu &= 0 \ , \\
  \lambda^2(1 + \nu^2) - 2\nu \lambda N^i_j \lambda^j + b \nu^2 + 2\gamma \nu &= b \ .
\end{align}

The solutions to these equations will depend on a free parameter (for example $\nu$). The matrix $N^i_j$ has to be an isometry of the Killing metric
\[
N^i_l h_{lm} N^m_j = h_{ij}
\]
and can be realized as an element of the original semisimple Lie group in the adjoint representation
\[
N^i_j = \{ e^{\theta^k \sigma_k} \}^i_j, \quad \{ \sigma_k \}^i_j = f^i_{kj}
\]
Therefore the parameters $\theta^i$, together with the free parameter $\nu$, parametrize the moduli space of the solutions.

The equations (5.6–5.8) select two main classes of solutions:
- The one given by $\sigma = 0$ which we will call class $0$. In particular, in this class $\mu \nu = 1$.
- The one given by $\sigma \neq 0$ which we call class $\sigma$.

From the previous solutions it is easy to extract informations about the boundary states for the WZW model associated to the algebra $\mathcal{A}$ in the trivializing basis (2.11). In fact, performing the change of basis (2.11) amounts to consider the original algebra where we have set $\lambda = 0$. Since the solutions (5.5, 5.6–5.8) have a smooth limit for $\lambda \to 0$ the boundary states for the trivial case can be easily obtained
\[
M^I_J = \begin{pmatrix}
N^i_j & 0 & 0 \\
0 & \nu & -\sigma \\
0 & \gamma & \mu
\end{pmatrix}
\]
where $\nu, \sigma, \gamma, \mu$ satisfy
\begin{align}
  \sigma(\sigma b - 2\mu) &= 0 \ , \\
  1 + \sigma b \nu + \sigma \gamma - \mu \nu &= 0 \ , \\
  b \nu^2 + 2\gamma \nu &= b \ .
\end{align}

Finally, we consider boundary states for the model associated to the contracted algebra (2.21). In this case the solutions to the constraints (5.2, 5.3) read (see Appendix B for details)
\[
M_{(o)}^I_J = \begin{pmatrix}
N^i_j & r^i & 0 \\
0 & \nu & 0 \\
-\nu r_i N^i_j & -\frac{\nu}{2} r^2 & \nu
\end{pmatrix}
\]
where $N^i_j$ still satisfies (5.9) and the choice of the constants is restricted by the following equations

$$
\nu^2 = 1
$$

$$
f^*_{ij} = \nu N^i_k f^*_{jk}.
$$

(5.16)

Here, again, $f^*_{ij} = \lambda^k f^*_{ijk}$. As explained in Appendix B, the last equation requires $\lambda^k$ to be an eigenvector of $N$ with eigenvalue $\nu$ or $-\nu$. This can happen only for particular choices of $N$.

### 5.1 Brane solutions

If the previous solutions allow for a geometrical interpretation, they define D-brane configurations in the given background. In order to identify them, one has to translate the gluing conditions on the chiral currents into boundary conditions on the fields. As discussed in [18, 19, 22], the gluing conditions (5.1) coincide with boundary conditions for the WZW model on group manifold only for configurations near the identity. Therefore, solving (5.1) amounts to determine D–brane configurations in the group manifold passing through the identity.\(^3\)

We will concentrate on finding such configurations. To this end we parametrize the group elements as in (3.1). The chiral currents evaluated near the identity are then

$$
J(z) = g^{-1} \partial g = \partial \theta^i \tau_i + \partial u H + \partial v H^*
$$

$$
\bar{J}(\bar{z}) = -\bar{\partial} \bar{g}^{-1} = -\bar{\partial} \theta^i \tau_i - \bar{\partial} u H - \bar{\partial} v H^*
$$

(5.17)

which shows [17] that Neumann boundary conditions correspond to $J(z) = -\bar{J}(\bar{z})$, i.e. to positive eigenvalues of $M$, whereas Dirichlet conditions correspond to $J(z) = \bar{J}(\bar{z})$, i.e. to negative eigenvalues. In particular, if $(-1)^D \det M$ is positive we find odd dimensional D-branes, whereas D-branes are even dimensional in the opposite case.

The problem of determining D–brane configurations near the identity is therefore translated into the spectral problem for the matrix $M$\(^4\)

$$
v_J M^I_I = \omega v_I.
$$

(5.18)

From the condition (5.4) it follows $\det M = \pm 1$ and, as a consequence, the eigenvalues satisfy $|\omega| = 1$.

We solve eq. (5.18) for the different cases, polarized (2.2, 2.10), trivial (2.12) and contracted (2.21).

\(^3\)More general solutions, determining boundary states in a neighbourhood of a generic point $g$ would first require the translation of the conditions (5.1) from conditions involving algebra–valued quantities to conditions for group–valued quantities (see [18]).

\(^4\)We note that the matrix $M$ acts on the currents as a right multiplication so that we have to consider the spectral equation for left eigenvectors.
In the polarized case, given the particular structure (5.5) for $M$, we have
\[
\det(M^I_i - w^5^I_j) \equiv \det(N^i_j - w^δ^i_j) \left[ (w - \nu + \sigma \lambda^2)(w - \mu) + \gamma \sigma + \mu \sigma \lambda^2 - \mu \lambda^i \lambda^j \right] = 0. (5.19)
\]

Therefore, for both the 0 and $\sigma$ classes, $(D - 2)$ eigenvalues are determined by the eigenvalues of the isometry matrix $N$ of the invariant metric $h_{ij}$. From the condition (5.9) it follows that $\det N = \pm 1$ so that the eigenvalues $\xi$ of $N$ satisfy $|\xi| = 1$. If we call $\Xi_{i,\xi}$ the left eigenvector of $N$ corresponding to the generic eigenvalue $\xi$, the eigenvector for the matrix $M$ is
\[
v_\xi = (\Xi_{i,\xi}, \lambda^2 \Xi_{j,\xi}, 0). (5.20)
\]

The remaining eigenvalues depend on the specific class. We then determine them separately for the two classes.

**CLASS 0**

The extra eigenvalues are $w = \nu$ and $w = \mu = \frac{1}{\nu}$. In particular they have the same sign so determining two extra Neumann or two extra Dirichlet conditions. The corresponding eigenvectors are
\[
v_\nu = (0, \cdots, 0, 1, 0),
\]
\[
v_1 = (\lambda_i, \frac{\lambda^2 - b}{2}, -1)
\]

where we have used (5.8). The D-brane configurations corresponding to this class of solutions depend on the sign of $\det N$. If the matrix $N$ has an even number $2p$ of negative eigenvalues, the boundary conditions describe $(D - 1 - 2p)$-brane for $\nu > 0$ and $(D - 3 - 2p)$-brane for $\nu < 0$. Similarly, if $N$ has $2p+1$ negative eigenvalues the boundary configurations are $(D - 2 - 2p)$-brane for $\nu > 0$ and $(D - 4 - 2p)$-brane for $\nu < 0$.

**CLASS $\sigma$**

In this case the equation for the extra eigenvalues is
\[
w^2 + w[-\mu - \nu + \sigma \lambda^2] + \mu \nu + \gamma \sigma - \sigma \lambda^i \lambda^j N_{ij} = 0 \quad (5.23)
\]
which, using (5.6, 5.7) reduces to
\[
w^2 + w[-\mu - \nu + \sigma \lambda^2] - 1 = 0. \quad (5.24)
\]

This equation has two distinct real solutions of opposite signs which give a new Neumann and a new Dirichlet condition.

If $w_o, \; o = 1, 2$ are the solutions, then the corresponding eigenvectors are
\[
v_o = (\lambda_i, \frac{w_o + \sigma \lambda^2 - \mu}{\sigma}, -1). \quad (5.25)
\]
The two sets of D-branes which are described by these solutions correspond to the cases of even and odd number of negative eigenvalues for $N$. We have $(D - 2 - 2p)$-brane and $(D - 3 - 2p)$-brane geometries for even and odd number of negative eigenvalues, respectively.

Setting $\lambda = 0$ in the previous expressions we find the D–brane solutions for the model based on the trivializing basis (2.11). In particular, we note that the structure of the eigenvectors becomes

$$v_\xi = (\Xi_{\xi,i}, 0, 0),$$
$$v_o = (0, \cdots, 0, a, b) \quad (5.26)$$

for suitable constants $a, b$, and consequently, $\langle v_\xi, v_o \rangle = 0$.

We now consider D–brane solutions for the model associated to the contracted algebra. In this case the matrix of boundary conditions is given in (5.15) and its spectral equation becomes

$$\det \left( M_0 \delta_{IJ} - w \delta_{IJ} \right) = \det (N_{ij} - w \delta_{ij}) (w - \nu)^2 = 0. \quad (5.27)$$

We remind that in this case $\nu$ or $-\nu$ are required to be in the spectrum of $N$.

To find the eigenvectors we need discuss separately three cases: i) $\nu$ in the spectrum of $N$ with $\lambda$ the corresponding eigenvector; ii) $\nu$ in the spectrum of $N$ but $\lambda_j N^j_i = -\nu \lambda_i$; iii) $\nu$ not in the spectrum of $N$.

i) As first case we suppose $\nu$ to be eigenvalue of $N$ with degeneracy one and $\lambda_j N^j_i = \nu \lambda_i$. Then we have $(D - 3)$ eigenvalues $\xi \neq \nu, |\xi| = 1$ of the matrix $N$ plus the extra eigenvalue $\omega = \nu$ which will appear with degeneracy three (see eq. (5.27)). The first $(D - 2)$ eigenvectors are easy to find

$$v_\xi = \left( \Xi_{\xi,i}, \frac{\Xi_{\xi,j} r^j}{(\xi - \nu)}, 0 \right), \quad \xi \neq \nu,$$
$$v^{(1)} = (0, \cdots, 0, 1, 0). \quad (5.28)$$

We look for the two missing eigenvectors corresponding to the eigenvalue $\nu$. They will have necessarily the form $(Z_i, 0, c)$ where the unknowns $Z_i$ and $c$ satisfy

$$Z_j N^j_k - cr r_j N^j_k = \nu Z_k, \quad (5.29)$$
$$Z_i r^i - \nu \frac{c}{2} r^2 = 0 \quad (5.30)$$

as follows from the requirement to be eigenvectors of $M$ with eigenvalue $\nu$. The first equation is equivalent to

$$cr^i = -N^j_i Z^j + \nu Z^i. \quad (5.31)$$

This equation does not have solutions in general, unless a further constraint is satisfied. Precisely, since we have supposed $\lambda_j N^j_i = \nu \lambda_i$, it can be solved iff $r$ is orthogonal to $\lambda$. 
If this condition is satisfied, the system of equations (5.30) has solutions $Z_k = \lambda_k$ and $c = 0$ and the corresponding eigenvector reads $v_\nu^{(2)} = (\lambda_i, 0, 0)$. The other eigenvector is obtained by solving (5.31) on the space orthogonal to $\vec{\lambda}$. Here the solution exists uniquely. We can set for example $c = 1$ and $Z_i$ satisfying (5.31). Note that, as a consequence, (5.30) is automatically satisfied.

ii) When $\vec{\lambda}$ satisfies $\lambda_j N_{ij} = -\nu \lambda_i$ but $\nu$ is still in the spectrum of $N$ the two missing eigenvectors exist iff $\vec{r}$ is orthogonal to $\vec{\Xi}_\nu$, where $\vec{\Xi}_\nu N = \nu \vec{\Xi}_\nu$. They are given by $v_\nu^{(2)} = (\vec{\Xi}_\nu, 0, 0)$ and $v_\nu^{(3)} = (Z_i, 0, 1)$ with $\vec{Z}$ solving (5.31).

iii) Finally, we suppose $\nu$ not in the spectrum of $N$. Then $v_\xi$ and $v_\nu^{(1)}$ in (5.28) are $(D - 1)$ eigenvectors. The last one is $(Z_i, 0, 1)$, where $\vec{Z}$ is a solution of (5.31). Note that in this case $N_{ij} - \nu \delta_{ij}$ is invertible.

To summarize, we have found that the extra eigenvectors determined by eqs. (5.29, 5.30) always exist as far as $\nu$ is not in the spectrum of the matrix $N$, whereas in the opposite case they exist if and only if the vector $\vec{r}$ is orthogonal to the $\nu$–eigenvector.

## 5.2 D–branes in the Penrose Limit

The eigenvectors found above can be used to embed D–branes into the spacetime under consideration. For simplicity we call Neumann vectors the eigenvectors corresponding to positive eigenvalues. Given a boundary configuration $M$, we suppose that $(p + 1)$ Neumann vectors $v_\alpha$, $\alpha = 0, \ldots, p$ are present. Therefore, they select the tangent directions to a $Dp$–brane.

We consider the algebra generators

$$\sigma_\alpha = \sum_I v_{\alpha, I} \tau^I, \quad \alpha = 0, \ldots, p.$$  

(5.32)

Therefore, local coordinates $\chi_\alpha$, $\alpha = 0, \ldots, p$ for the brane are related to the spacetime coordinates $X^I = (\theta^i, u, v)$ through the equation

$$g = e^{\theta^i \tau_i} e^{u H + v H^*} = e^{\chi^\alpha \sigma_\alpha}$$  

(5.33)

which defines the embedding of the brane into the spacetime.

Using the Backer-Campbell-Hausdorff formula we find

$$g = e^{\phi_i(\theta^j, u) \tau_i + u H + (v + \psi(\theta^i, u))H^*}$$  

(5.34)

with

$$\phi_i(\theta^j, u) = \theta^i - \frac{u}{2} \theta^j f^i_j + \frac{u^2}{12} \theta^j f^k_j f^i_k + \frac{u}{12} \theta^j f^k_i f^i_k + \ldots$$

$$\psi(\theta^i, u) = -\frac{u}{12} \theta^j f^i_j f^k_i f^i_k + \ldots$$  

(5.35)

so that the embedding (5.33) becomes

$$\chi^\alpha v_\alpha^i = \phi_i(\theta^j, u),$$

15
This construction can be carried on for the models associated to the double extended algebra both in the original basis (2.2) and in the trivializing one (2.11), and for the contracted algebra (2.21). In particular, it is worth noting that in the trivializing case, given the particular structure (5.26) for the eigenvectors, D–brane solutions fall into two orthogonal classes: D–branes embedded in the D dimensional spacetime associated to the unextended algebra and D–branes in $\mathbb{R}^{1,1}$.

An interesting topic we are going to investigate concerns the behavior of the brane solutions under Penrose limit. In the previous subsection we have given brane solutions both for the model associated to the nonsemisimple algebra and for the model associated to its contraction. Since we have shown that the contracted model corresponds to a Penrose limit of the original one, the natural question which arises is whether the brane configurations corresponding to the contracted algebra can be all obtained as Penrose limit of the original configurations or part of them are lost in this limit.

Before entering the details of the discussion, we make a preliminary observation. If we rewrite the group element $g$ in terms of the rescaled basis $(P_i, S, T)$ and coordinates (4.4), then eqs. (5.36) become

\[
\begin{align*}
\frac{1}{\alpha^2} \chi^\alpha v^i_\alpha &= u \\
\frac{1}{\alpha^2} \chi^\alpha v^*_\alpha &= v + \psi(\theta^j, u) \\
\frac{1}{\alpha^2} \chi^\alpha v^i_\alpha &= \psi(\alpha\Theta^j, U) \\
\frac{1}{\alpha^2} \chi^\alpha v^*_\alpha &= V + \frac{1}{\alpha^2} \psi(\alpha\Theta^j, U)
\end{align*}
\]

(5.37)

where the right hand side has a finite limit for $\alpha \to 0$. Therefore, we can introduce new coordinates on the brane $Y^\alpha := \alpha^\zeta \chi^\alpha$ for some parameter $\zeta$ such that

\[
\lim_{\alpha \to 0} \left( \frac{v^i_\alpha}{\alpha^\zeta+1}, \frac{v^*_\alpha}{\alpha^\zeta-2}, \frac{v^*_\alpha}{\alpha^\zeta+2} \right) = \left( w^i_\alpha, w^*_\alpha, w^*_\alpha \right)
\]

(5.38)

is finite and gives an eigenvector for the $M_{(0)}$ boundary matrix in the Penrose limit configuration. This result seems to indicate that boundary configurations for the contracted model can be found as a limit of the configurations of the original model even if a priori we do not expect $M_{(0)}$ to be in general obtained as a limit of some $M$ of the original model. However, this is exactly the case as we are now going to prove in details.

We consider the constraints (5.3,5.4) for the one–parameter family of algebras $A_\alpha$ given in eq. (2.17). We are interested in studying the solutions to the constraints for finite $\alpha$ and compare the results obtained when $\alpha \to 0$ with the solutions (5.15–5.16) of the contracted case.
If we still set \( f_{ij} = f_{ij}^k \lambda_k \) the constraints from the first condition in (5.3) read

\[
0 = \lambda f_{kij} \frac{d^2}{d^2}
\]

\[
0 = \alpha f_{kij} \frac{d^2}{d^2} + \lambda f_{kij} \frac{d^2}{d^2} \tag{5.39}
\]

\[
M_i M^j \alpha f_{ij} = \alpha f_{ilm} M^*_k + \lambda f_{ilm} M^*_k \tag{5.40}
\]

\[
M^i M^j \alpha f_{ij} = \lambda f_{km} \frac{d^2}{d^2} \tag{5.41}
\]

\[
M^i M^j \alpha f_{ij} = 0 \tag{5.42}
\]

\[
M^i M^j \alpha f_{ij} = 0 \tag{5.43}
\]

\[
\alpha f_{ij} M^i M^j - \lambda f_{ij} (M^i M^j - M^*_m M^*_i) = \alpha M^k f_{lmj} + M^k f_{lmj} \tag{5.44}
\]

\[
\alpha f_{ij} M^i M^j - \lambda f_{ij} (M^i M^j - M^*_m M^*_i) = -M^k f_{hj} \lambda^k \tag{5.45}
\]

\[
\alpha f_{ij} M^i M^j - \lambda f_{ij} (M^i M^j - M^*_m M^*_i) = 0 \tag{5.46}
\]

\[
\alpha f_{ij} M^i M^j - \lambda f_{ij} (M^i M^j - M^*_m M^*_i) = 0 . \tag{5.47}
\]

Proceeding as in Appendix A we find

\[
M(\alpha)\frac{d^2}{d^2} = \left( \begin{array}{ccc}
N^i_j - \sigma \lambda^i \lambda_j & \frac{1}{\alpha} (\lambda^k N^1_k - \nu \lambda^i) & \frac{1}{\alpha} \sigma \lambda^i \\
\sigma \sigma \lambda_j & \nu & -\alpha^2 \sigma \\
\frac{1}{\alpha} (\lambda^k N^1_k - \mu \lambda_j) & \gamma & \mu
\end{array} \right) \tag{5.49}
\]

where the matrix \( N \) still satisfies (A.14, 5.3) and can be chosen as in (5.10). The constants appearing in \( M \) are constrained by the following equations

\[
\sigma (\sigma \lambda^2 + \alpha^2 \sigma b - 2\mu) = 0 , \tag{5.50}
\]

\[
\frac{1}{\alpha} \left[ 1 + \nu \sigma \lambda^2 - \sigma \lambda_1 N^1_j \lambda^j - \mu \lambda^j \right] + \alpha \sigma b \nu + \alpha \sigma \gamma = 0 , \tag{5.51}
\]

\[
\frac{1}{\alpha^2} \left[ \lambda^2 (1 + \nu^2) - 2 \nu \lambda^m N^1_m \lambda_k \right] + b \nu^2 + 2 \gamma \nu = b \tag{5.52}
\]

which come from the extra conditions in (5.3, 5.4).

In general the matrix \( M(\alpha) \) and the system of equations (5.50, 5.52) do not have a well-defined limit for \( \alpha \to 0 \). However we can expand the matrix elements in a power series of \( \alpha \) as follows

\[
N^i_j = N^i_j^{(0)} + \alpha N^i_j^{(1)} + \alpha^2 N^i_j^{(2)} + \ldots
\]

\[
\sigma = \sigma^{(0)} + \alpha \sigma^{(1)} + \alpha^2 \sigma^{(2)} + \ldots
\]

\[
\mu = \mu^{(0)} + \alpha \mu^{(1)} + \alpha^2 \mu^{(2)} + \ldots
\]

\[
\nu = \nu^{(0)} + \alpha \nu^{(1)} + \alpha^2 \nu^{(2)} + \ldots
\]

\[
\gamma = \gamma^{(0)} + \alpha \gamma^{(1)} + \alpha^2 \gamma^{(2)} + \ldots
\]

\[
\lambda^i = \lambda^i_j^{(0)} + \alpha \lambda^i_j^{(1)} + \alpha^2 \lambda^i_j^{(2)} + \ldots . \tag{5.53}
\]

For \( M^i \) to have a finite limit we find

\[
\lambda^k_{(0)} N^i_k^{(0)} - \nu \lambda^i_{(0)} = 0 \tag{5.54}
\]
where \( \nu^{2}_{(0)} = 1 \) as a consequence of \([5.9]\) at lowest order. It follows that \( M^{i} \) takes the form

\[
M^{i} \equiv r^{i} = N^{(1)}_{i} k^{l}_{(0)} - \nu^{(1)} k^{l}_{(0)} + N^{i}_{(1)} k^{l}_{(1)} - \nu^{(0)} k^{l}_{(1)} .
\] (5.55)

Similarly, for \( M^{*}_{j} \) to be well-defined in the limit we have

\[
\lambda^{(0)} k^{i}_{(0)} N^{i}_{(0)} j - \mu^{(0)} \lambda^{(0)} j = 0
\] (5.56)

so that

\[
M^{*}_{j} \equiv s_{j} = N^{(1)}_{i} k^{l}_{(0)} - \mu^{(1)} \lambda^{i}_{(0)} j + \lambda^{(1)} k^{i}_{(0)} N^{i}_{(0)} j - \mu^{(0)} \lambda^{i}_{(0)} j .
\] (5.57)

Note that the compatibility of \([5.54]\) with \([5.56]\) requires

\[
\mu^{(0)} = \nu^{(0)}.
\] (5.58)

Now we concentrate on equations \([5.50]\text{-}[5.52]\). Inserting the expansions \([5.53]\) in \([5.50]\) and taking the limit \( \alpha \to 0 \) we find

\[
\sigma^{(0)} = 0 \quad \text{or} \quad \sigma^{(0)} = 2\mu^{(0)}.
\] (5.59)

which implies \( \sigma^{(0)} = 0 \) or \( \sigma^{(0)} = 2\mu^{(0)} \). Using \([5.54, 5.56, 5.58]\) in \([5.51]\) one finds \( \mu^{(0)} \nu^{(0)} = 1 \) and the relation

\[
\mu^{(0)} \nu^{(1)} + \mu^{(1)} \nu^{(0)} = \sigma^{(0)} \nu^{(1)} + \sigma^{(0)} \nu^{(0)} - \sigma^{(0)} \nu^{(0)} i^{(1)} j^{(0)} i^{(0)} - \sigma^{(0)} \nu^{(0)} i^{(0)} j^{(0)} i^{(1)} = 0
\] (5.60)

which with \([A.14]\) at first order in \( \alpha \) can be used to show that

\[
s_{j} = -\nu^{(0)} \left[ N^{(0)} i^{(0)} j^{(0)} \right] h_{kl} r^{k}.
\] (5.61)

Finally we consider \([5.52]\). It has a finite limit if \( \lambda^{(0)} i^{(1)} j^{(0)} = 0 \) and the nontrivial part of the equation becomes

\[
2\lambda^{(1)} + \lambda^{(0)} \nu^{(1)} - 2\nu^{(0)} \lambda^{(0)} i^{(0)} j^{(0)} N^{i}_{(0)} l^{(0)} k^{(0)} - 2\nu^{(0)} \lambda^{(0)} i^{(0)} j^{(0)} N^{i}_{(0)} l^{(0)} k^{(0)}
\]

\[
-2\nu^{(0)} \lambda^{(0)} i^{(0)} j^{(0)} N^{i}_{(0)} l^{(0)} k^{(0)} + 2\gamma^{(0)} \nu^{(0)} = 0 .
\] (5.62)

From \([A.17]\) to second order in \( \alpha \) we find

\[
N^{i}_{(1)} l^{i}_{(1)} k^{l}_{(0)} h_{kl} + (N^{i}_{(0)} i^{(1)} l^{i}_{(0)} j^{(0)} + N^{i}_{(0)} i^{(0)} l^{i}_{(0)} j^{(0)}) h_{kl} = 0
\] (5.63)

which used in \( r^{2} := h_{ij} r^{i} r^{j} \) and then inserted in \([5.62]\) gives

\[
r^{2} + 2\gamma^{(0)} \nu^{(0)} = 0 .
\] (5.64)
Collecting all the results the final form for the matrix $M$ in the limit $\alpha \to 0$ is

$$M(\alpha \to 0)^{I \ J} = \begin{pmatrix} N^i_j - \sigma \lambda^i \lambda_j & r^i & 0 \\ 0 & \nu_{(0)} & 0 \\ s_j & \gamma_{(0)} & \nu_{(0)} \end{pmatrix}$$

(5.65)

where

$$N^i_j \lambda^j = \nu_{(0)} \lambda^i$$

(5.66)

$$2\gamma_{(0)}\nu_{(0)} + r^2 = 0$$

(5.67)

$$\nu_{(0)}^2 = 1$$

(5.68)

$$r^k(N^l_j h_{kt} - \sigma \lambda_k \lambda_j) + \nu_{(0)} s_j = 0$$

(5.69)

$$\sigma(\sigma \lambda^2 - 2\nu_{(0)}) = 0.$$ 

(5.70)

We have to compare this result with the solution (5.15–5.16) for the contracted algebra. For $\sigma = 0$ the two solutions coincide exactly. The case $\sigma \neq 0$ is also included since it can be traced back to the case $\sigma = 0$ by observing that if $N^i_j$ is an isometry then $N^i_j - \sigma \lambda^i \lambda_j$ is also an isometry whenever $\sigma \lambda^2 = 2\nu_{(0)}$. Note that $\vec{\lambda}$ is eigenvector of $M_{(0)}^{\ i \ j}$ with eigenvalue $\nu_{(0)}$ when $\sigma = 0$, whereas it corresponds to eigenvalue $-\nu_{(0)}$ when $\sigma \neq 0$. Given the arbitrariness of the vector $r^i$, we can conclude that the class of solutions obtained in the Penrose limit coincides with the class of solutions for the contracted algebra. Therefore the Penrose limit seems to carry on all the informations and nontrivial background configurations can be generated from the trivial ones by means of this limit.

In order to give further support to this statement we study the behavior of the eigenvectors under the limit.

We first consider the case $\sigma = 0$. For $\alpha$ finite, the eigenvectors of the matrix $M(\alpha)$ in (5.49) are

$$V_{\xi} = \left( \Xi_{\xi, i}, \frac{1}{\alpha} \lambda^i \Xi_{\xi, i}, 0 \right),$$

(5.71)

$$v_{\nu} = (0, \cdots, 0, 1, 0),$$

(5.72)

$$v_{1 \nu} = \left( \frac{\lambda^i}{\alpha}, x, -1 \right).$$

(5.73)

where $\Xi_\xi$ are eigenvectors of $N$ with eigenvalues $\xi$ and $x$ satisfies

$$(1 - \nu^2) \left( 2x - \frac{\lambda^2}{\alpha^2} + b \right) = 0.$$ 

(5.74)

We make the assumption for one of the $\xi$ eigenvalues to have the form

$$\tilde{\xi} = \nu + O(\alpha^2).$$

(5.75)

This includes both the cases $\nu$ in or not in the spectrum of $N$. 

19
We first concentrate on the eigenvectors $V_\xi$ with $\xi \neq \bar{\xi}$. In order to study their behavior under the limit $\alpha \to 0$ we expand the quantities appearing in $M$ as in (5.53) and similarly

$$\Xi^i_\alpha = \Xi^i_\alpha^{(0)} + \alpha \Xi^i_\alpha^{(1)} + \ldots ,$$  
(5.76)

$$\xi = \xi_\alpha^{(0)} + \alpha \xi_\alpha^{(1)} + \ldots .$$  
(5.77)

Note that $\lambda^{(0)}_i N^{(0)}_{ij} = \nu^{(0)}_i \lambda^{(0)}_j, \nu^{2}_{(0)} = 1$. From the condition for $\Xi_\xi$ to be an eigenvector of $N$, up to the first order in $\alpha$ we obtain

$$\Xi^{(0)}_i N^{(0)}_{ij} = \xi^{(0)}_i \Xi^{(0)}_j,$$  
(5.78)

$$\Xi^{(1)}_i N^{(0)}_{ij} + \Xi^{(0)}_i N^{(1)}_{ij} = \xi^{(1)}_i \Xi^{(0)}_j + \xi^{(0)}_i \Xi^{(1)}_j.$$  
(5.79)

In particular, being $\xi^{(0)}_\alpha \neq \nu^{(0)}_i, \Xi^{(0)}_j$ is orthogonal to $\lambda^{(0)}_i$ ($N$ is a unitary matrix). Using (5.79) and (5.55) we find

$$\frac{1}{\alpha} \lambda^i \Xi^i = \Xi^{(0)}_j \left( \lambda^{(0)}_j + \frac{\lambda^{(0)}_i N^{i}_{(1)j}}{\xi^{(0)}_j - \nu^{(0)}_j} \right) + o(1) = \frac{\Xi^{j}_{(0)}}{\xi^{(0)}_j - \nu^{(0)}_j} r_j + o(1).$$  
(5.80)

Therefore, in the limit we obtain the first set of eigenvectors in (5.28).

We then look for the remaining eigenvectors. The eigenvector (5.72), being independent of $\alpha$ survives the limit and coincides with the extra eigenvector in (5.28) corresponding to eigenvalue $\nu^{(0)}_i$. Now the question is whether in the limit other $\nu^{(0)}_i$ eigenvectors arise. In order to answer, we first note that when $\xi = \bar{\xi} (\bar{\xi}_\alpha^{(0)} = \nu^{(0)}_i, \bar{\xi}_\alpha^{(1)} = \nu^{(1)}_i)$ multiplying (5.79) by $\lambda^{(0)}_i$ we obtain the condition

$$\nu^{(1)}_i = 0.$$  
(5.81)

Inserted in (5.55) it says that $r^i$ has to be orthogonal to $\lambda^{(0)}_i$. This is exactly the condition we found in the model associated to the contracted algebra for the existence of extra $\nu^{(0)}_i$ eigenvectors.

To obtain these eigenvectors in the limit we start considering $V_\xi$. Since $\bar{\xi}_\alpha^{(0)} = \nu^{(0)}_i$ we can take $\Xi^{(0)}_i = \lambda^{(0)}_i$. As a consequence the eigenvector $V_\xi$ is generically divergent for $\alpha \to 0$. However, we can obtain a finite result by acting with the limit on the linear combination

$$\eta \equiv V_\xi = \frac{1}{\alpha} \lambda^2 \nu \nu.$$  
(5.82)

This is a $\nu$ eigenvector for $M(\alpha)$ up to terms $O(\alpha^2)$ and in the limit it gives rise to a $\nu^{(0)}_i$ eigenvector for the contracted matrix. By a further finite subtraction, the resulting eigenvector can be set into the form $v_\lambda = (\lambda^{(0)}_i, 0, 0)$. Finally we have the $1/\nu$ eigenvector (5.73) with $x$ solution of (5.74). From this equation expanded in powers of $\alpha$ we obtain again the condition (5.81) and the extra constraint.
\(\nu_{(2)} = 0\). We consider the linear combination \(v^\perp_1 - \frac{1}{\alpha} \eta\) with \(\eta\) given in (5.82). This is eigenvector of \(M(\alpha)\) with eigenvalue \(\nu_{(0)}\), up to \(O(\alpha)\) terms. After a suitable finite subtraction, in the limit it generates \((\lambda_{(1)j}^{(1)}, 0, -1)\) which is the last \(\nu_{(0)}\) eigenvector of the contracted matrix. In this case from (5.83) it follows

\[
\rho^i = \lambda_{(1)j}^{(1)} N_{(0)j}^i - \nu_{(0)} \lambda_{(1)}^i.
\] (5.83)

This is exactly the condition (5.81) with \(c = -1\) found in the previous subsection for the contracted case.

From this analysis we can conclude that in the case \(\sigma = 0\) we find a complete correspondence between the eigenvectors of \(M(\alpha)\) for \(\alpha \to 0\) and the eigenvectors of \(M_{(0)}\).

The case \(\sigma \neq 0\) can be treated in a similar manner and there are no problems to prove that the limit exists in any case and gives rise to the expected eigenvectors for the contracted matrix.

As remarked at the beginning of this section, the algebraic method for finding D–branes configurations holds only for D–branes passing through the identity of the group manifold \(G\). However the generic D–brane passing through a point \(g \in G\) can be obtained, as shown in [18], by a suitable “pull-back” to the origin of the gluing matrix which does not affect the Penrose limit process. Accordingly, our result holds for a generic D–brane. Therefore, we have proved that all the D–brane configurations of a \((D+2)\)–dimensional NW background can be obtained as Penrose limit of D–brane solutions of the background associated to the double extension of a semisimple D–dimensional algebra.

6 Conclusions

We have considered WZW models based on the double extension of a generic semisimple algebra. We have given a general proof that the corresponding background is simply the cartesian product between the group manifold associated to the original semisimple algebra and a bidimensional Minkowski spacetime. However, less trivial spacetime configurations can be obtained by taking a suitable Penrose limit. In this limit a generalized \((D+2)\)–dimensional Nappi–Witten background arises. We have shown that the Penrose limit corresponds at the level of the algebras to a suitable Inönü-Wigner contraction. In fact, the NW background can be realized as a WZW model associated to the double extension of an abelian algebra which is obtained as a contraction of our original extended algebra.

We have considered brane states which can live in these spacetime backgrounds. In particular, we have shown that non only the brane configurations of the double extended model survive the Penrose limit, but all the algebraically defined brane states living in the generalized NW background can be obtained as such a limit.

We have argued that the correspondence between the Penrose limit and the algebra contraction is consistent also at the quantum level. In fact, using the Sugawara construction,
we have computed the central charge of the one–parameter family of nonlinear sigma models associated to the family of algebras $A_\alpha$ and proved that in the limit $\alpha \to 0$ they generate the correct central charge of the NW sigma model. Our results give evidence that the procedures of contraction and quantization should commute, as indicated by the following diagram

\[
\text{Classical extended } WZW \xrightarrow{\text{Quantization}} \text{Quantum extended } WZW \\
\downarrow \text{Contraction} \hspace{3cm} \downarrow \text{Contraction} \\
\text{Classical } NW_D \text{ model} \xrightarrow{\text{Quantization}} \text{Quantum } NW_D \text{ model}
\]

We expect this correspondence to be very general. It might be useful to gain informations on Nappi–Witten backgrounds starting from the original double extended model. In particular, it could be used to construct the vertex operator algebras for the Nappi–Witten model via the Kac formalism ([20]) in order to reproduce the vertex operators found in [21], [22] and [23]. Possible generalizations of our results could be investigated for the cases of supersymmetric and/or noncommutative WZW models.

**Acknowledgments**

S. C. and G. O. would like to thank Bert Van Geemen for numerous helpful conversations. We also thank Dietmar Klemm and Giuseppe Berrino for discussions and Marco Rusconi for useful questions. This work was partially supported by INFN, COFIN prot. 2003023852_008 and the European Commission RTN program MRTN–CT–2004–005104 in which S. C. is associated to the University of Milano–Bicocca.
A Solution to the constraints in the polarized case

We solve the constraints \((5.3, 5.4)\) for \(A\) semisimple, with the position \((2.10)\). The first conditions in \((5.3)\) give rise to the following equations

\[
0 = \lambda^k f_{ki}^j M^*_j \quad (A.1)
\]

\[
0 = f_{ki}^j M^*_j + \lambda^j f_{kij} M^*_i \quad (A.2)
\]

\[
M^i_j M^j_m \lambda^k f_{ijk} = f_{lm}^k M^*_k + \lambda^k f_{lmk} M^*_m \quad (A.3)
\]

\[
M^i_j M^j_m \lambda^k f_{ijk} = \lambda^k f_{km}^h M^*_h \quad (A.4)
\]

\[
M^i_* M^j_* M^k f_{ijk} = 0 \quad (A.5)
\]

\[
M^i_* M^j_* \lambda^k f_{ijk} = 0 \quad (A.6)
\]

\[
f_{ij}^k M^i_* M^j_* = -\lambda^j f_{ij}^k (M^i_* M^j_* - M^j_* M^i_*) = M^k f_{lmj} + M^k \lambda^j f_{lmj} \quad (A.7)
\]

\[
f_{ij}^k M^i_* M^j_* - \lambda^j f_{ij}^k (M^i_* M^j_* - M^j_* M^i_*) = -M^k f_{hij} \lambda^h \quad (A.8)
\]

\[
f_{ij}^k M^i_* M^j_* - \lambda^j f_{ij}^k (M^i_* M^j_* - M^j_* M^i_*) = 0 \quad (A.9)
\]

\[
f_{ij}^k M^i_* M^j_* - \lambda^j f_{ij}^k (M^i_* M^j_* - M^j_* M^i_*) = 0 \quad (A.10)
\]

Eq. \((A.1)\) can be solved by setting \(M^*_j = \sigma \lambda^j\). Inserting into eq. \((A.2)\) we find \(M^*_i = -\sigma\). Similarly, equations \((A.5)\) and \((A.6)\) are satisfied by \(M^i_* = \epsilon \lambda^i\) and from \((A.9), (A.10)\) we obtain \(\epsilon = \sigma\).

If we define

\[
N^i_j \equiv M^i_j + \sigma \lambda^i \lambda_j \quad , \quad M^i_* \equiv \nu \quad , \quad M^i_* \equiv \mu \quad , \quad M^* \equiv \gamma
\]

the rest of equations \((A.3), (A.4), (A.7), (A.8)\) can be written as

\[
f_{lm}^k (M^*_m + \mu \lambda_j) = \lambda_k f_{lm}^* N^m_i N^i_m \quad (A.12)
\]

\[
\lambda^k f_{km}^j (M^*_j + \mu \lambda_j) = \lambda_k f_{km}^j i^i N^i_m M^i. \quad (A.13)
\]

\[
f_{ij}^k N^i_j N^j_m = N^k f_{lm}^* N^m_i N^i_m \quad (A.14)
\]

\[
f_{ij}^k N^i_j M^j_* = N^k f_{lm}^* h_i^j \lambda^h + \nu \lambda^h f_{hm}^k N^m_i. \quad (A.15)
\]

From the first equation, using eq. \((A.14)\) we find \(M^*_i = \lambda_k N^k_i - \mu \lambda_j\). Inserting this result into the second equation we obtain \(M^i_* = \lambda^k N^k_i + \eta \lambda^i\) and the last equation imposes \(\eta = -\nu\). Therefore the solution to \((A.1), (A.10)\) reads

\[
M^i_* = \begin{pmatrix}
N^i_j - \sigma \lambda^i \lambda_j & \lambda^k N^i_k - \nu \lambda^i & \sigma \lambda^i \\
\sigma \lambda_j & \nu & -\sigma \\
\lambda_k N^k_j - \mu \lambda_j & \gamma & \mu
\end{pmatrix}
\]

with \(N^i_j\) satisfying eq. \((A.14)\). On the matrix \(M\) we have still to impose the second constraint of \((5.3)\) and \((5.4)\). From \((5.4)\), using the explicit expression \((3.17)\) for the Killing metric we obtain as the only nontrivial condition

\[
N^i_j h^j_i N^m_j = h_i
\]
which implies $N^i_j$ to be invertible and to define the isometry group of the Killing metric $h_{ij}$. We can realize $N^i_j$ as an element of the original semisimple Lie group in the adjoint representation

$$N^i_j = \{e^{\theta^i}_j\}^i_j, \quad \{\sigma_k\}^i_j = f^i_{kj}.$$  \hspace{1cm} (A.18)

It is then easy to see that (A.14) is automatically satisfied.

Finally, the remaining equations in (5.3) give rise to the following conditions

$$\sigma(\sigma\lambda^2 + \sigma b - 2\mu) = 0$$  \hspace{1cm} (A.19)

$$1 + \nu \sigma \lambda^2 - \sigma \lambda_i N^i_j \lambda^j + \sigma b \nu + \sigma \gamma - \mu \nu = 0$$ \hspace{1cm} (A.20)

$$\lambda^2(1 + \nu^2) - 2\nu \lambda_i N^i_j \lambda^j + b \nu^2 + 2\gamma \nu = b$$ \hspace{1cm} (A.21)

which can be solved in terms of one free parameter.

## B Solution to the constraints in the contracted case

We now study the solutions to the constraints (5.3,5.4) in the case of the contracted algebra (2.21) where $f_{ij}$ is still given in terms of a vector $\lambda^k$ (see eq. (2.10)). In particular, we have $f_{ij}^* = \lambda^k f_{ijk}$ and $f_i^j = \lambda^k f_{ki}^j$.

For $\lambda \neq 0$ the corresponding equations are

$$M^i_j f_{ij}^j = 0$$ \hspace{1cm} (B.1)

$$f_{ij}^* M^*_{ij} = 0$$ \hspace{1cm} (B.2)

$$M^i_j M^j_k f_{ik}^* = f_{ij}^* M^*_{ij}$$ \hspace{1cm} (B.3)

$$M^i_j M^j_i f_{ik}^* = f_{ij}^j M^*_{ij}$$ \hspace{1cm} (B.4)

$$M^i_j M^j_i f_{ik}^* = 0$$ \hspace{1cm} (B.5)

$$M^i_j f_{ij}^* = 0$$ \hspace{1cm} (B.6)

$$M^i_j f_{ik}^j M^l_j - M^j_i f_{ik}^j M^l_i = f_{ij}^* M^*_{ij}$$ \hspace{1cm} (B.7)

$$M^i_j f_{ik}^j - M^j_i f_{ik}^j = f_{ij}^l M^l_i$$ \hspace{1cm} (B.8)

$$M^i_j f_{ik}^j - M^l_i M^l_j f_{kj}^* = 0$$ \hspace{1cm} (B.9)

$$M^i_j f_{ij}^k - M^l_i M^l_j f_{kj}^* = 0$$ \hspace{1cm} (B.10)

together with the isometry conditions

$$\Omega_{ij} = M^k_i M^l_j \Omega_{kl} + b M^*_{i} M^*_{j} + M^*_{i} M^l_j + M^l_i M^*_{j}$$ \hspace{1cm} (B.11)

$$0 = M^k_i M^l_j \Omega_{kl} + b M^*_{i} M^*_{j} + M^*_{i} M^l_j + M^l_i M^*_{j}$$ \hspace{1cm} (B.12)

$$0 = M^k_i M^l_j \Omega_{kl} + b M^*_{i} M^*_{j} + M^*_{i} M^l_j + M^l_i M^*_{j}$$ \hspace{1cm} (B.13)

$$b = M^k_i M^l_j \Omega_{kl} + b M^*_{i} M^*_{j} + 2 M^*_{i} M^*_{j}$$ \hspace{1cm} (B.14)

$$1 = M^k_i M^l_j \Omega_{kl} + b M^*_{i} M^*_{j} + M^*_{i} M^l_j + M^l_i M^*_{j}$$ \hspace{1cm} (B.15)

$$0 = M^k_i M^l_j \Omega_{kl} + b M^*_{i} M^*_{j} + 2 M^*_{i} M^*_{j}.$$ \hspace{1cm} (B.16)
Solving these equations as before one finds for the matrix $M$

$$M_{(0)} = \begin{pmatrix} N^i_j & r^i & 0 \\ 0 & \nu & 0 \\ s_j & -\frac{\nu}{2} r^2 & \nu \end{pmatrix} \quad (B.17)$$

where $N^i_j$ is still required to be an isometry of the metric and the following conditions must be satisfied

$$\nu^2 = 1 \quad (B.18)$$

$$\nu s_j = -r_i N^i_j \quad (B.19)$$

$$f_{ij}^* = \nu N^l_i N^k_j f_{lk}^* \quad (B.20)$$

Multiplying the last equation by $N^{-1}$ and exploiting the explicit realization for $N$, eq. (A.18), it is easy to see that the last condition can be satisfied only if $\lambda_j$, the vector in terms of which $f_{ij}^*$ is defined, is an eigenvector of $N$ with eigenvalue $\nu$ or $-\nu$. In fact, the last equation can be rewritten as

$$\lambda_k = \nu f_{ki}^* \tilde{f}_{ij} N^i_l \lambda_l \quad (B.21)$$

where $\tilde{f}_{ij}$ are the structure constants in the basis $\tilde{\tau}_i \equiv N^i_j \tau_j$, being $\tau_j$ the generators of the original semisimple algebra (see eq. (2.1)). We define the matrix $C_k^l = f_{ki}^* \tilde{f}_{ij}$ and consider a generic matrix $R$ of the form (A.18). Since the matrix $R$ is an isometry which also preserves the structure constants (see (A.14)) it is quite easy to show that the following chain of identities holds

$$R^s_h C_k^l = R^s_h f_{si}^* \tilde{f}_{ij} = f_{hi}^* R^i_s R^m_j \tilde{f}_{lm} = C_h^s R^h_k \quad (B.22)$$

Therefore the matrix $C$ commutes with all the elements of the group expressed in the adjoint representation. Being the group semisimple, this representation is irreducible so that $C$ must be proportional to the identity

$$C_k^l = \alpha \delta_k^l \quad (B.23)$$

Using the fact that the coefficients $f$ and $\tilde{f}$ generate two different basis of a D-dimensional subspace of the space of D × D antisymmetric matrices with scalar product generated by $h_{ij}$, it is easy to show that

$$\tilde{f}_{jk} = \alpha f_{jk} \quad (B.24)$$

with $\alpha = \pm 1$. Therefore eq. (B.21) reduces to $N^k_i \lambda^i = \pm \nu \lambda^k$. 

25
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