Consensus analysis of multi-agent systems with general linear dynamics and switching topologies by non-monotonically decreasing Lyapunov function

Xieyan Zhanga,b, Linfeng Chenb and Yan Chena

aCollege of Information Science and Engineering, Hunan Normal University, Changsha City, People’s Republic of China; bHunan Provincial Key Laboratory of Intelligent Computing and Language Information Processing, Hunan Normal University, Changsha City, People’s Republic of China

ABSTRACT
This paper investigates consensus problems for multi-agent systems with general linear dynamics and switching topologies. In order to deal with the intricate interaction between dynamics of isolated agents and switching-disconnected topologies, the consensus analysis is performed by non-monotonically decreasing Lyapunov function. Particularly, the design of consensus laws is explored from the perspective of fast time-varying systems with two time scales. Sufficient conditions for achieving consensus are derived, which depend on not only feedback gains and connectivity of network topologies but also the speed of topology switching and the stability of individual agent. Therefore consensus laws are generalized in the sense that the dynamics of linear agents are allowed to be unstable and switching topologies are allowed to be jointly weakly connected and balanced. Finally, numerical simulations are provided to demonstrate the effectiveness of theoretical results.

ARTICLE HISTORY
Received 11 January 2019
Accepted 15 May 2019

KEYWORDS
Consensus; general linear multi-agent systems; switching topologies; non-monotonically decreasing Lyapunov function

1. Introduction
Over the last decades, distributed coordination of multi-agent systems (MASs) has received much attention from multidisciplinary researchers due to its wide applications (Cao, Yu, Ren, & Chen, 2013; Chen, 2018; Knorn, Chen, & Middleton, 2016; Olfati-Saber, Fax, & Murray, 2007; Qin, Ma, Shi, & Wang, 2017; Ye, Dong, Lu, & Zhang, 2018). As is well known, consensus, referring to the collective objective of reaching agreement about some variables of interests such as position, velocity, frequency and voltage etc., plays a key role in a distributed coordination of MASs. In determining the collective behaviour, three different factors are fundamental, namely, the distributed coupling laws, the self-dynamics and the network topologies. At the beginning of research works (Moreau, 2004, 2005; Olfati-Saber & Murray, 2004), the agent usually has no dynamics in the absence of information exchange between agents and it is the exchange of information only that determines the evolutions of states of agents. Subsequently, researchers began to address the consensus for multi-agent systems of a more general model, i.e. each agent is assumed to take general linear dynamics (Li, Duan, Chen, & Huang, 2010; Scardovi & Sepulchre, 2009; Zhang, Lewis, & Das, 2011). Different from integrator agents, the individual dynamics of the agent and the exchange of information among agents will affect the consensus behaviour. For network topologies, consensus problems of MASs under switching topologies are more complicated than the cases of fixed topology. Although a lot of efforts have been devoted to this issue, there are still some open problems. We aim to address consensus of MASs with general linear dynamics and switching topologies by basic static coupling law in this paper.

Consensus problems with time-dependent communication links have been considered for MASs without self-dynamics (Moreau, 2004, 2005; Olfati-Saber & Murray, 2004). In Moreau (2004), an intrinsic limitation has been pointed out that balance conditions in terms of constraints on the column sums of the coupling matrix need to be imposed when the sum of squares function is considered as a candidate-Lyapunov function, and a generalized conclusion of consensus has been obtained by contraction analysis with the aid of the Lyapunov function $V(x) = \max_i(x_i) - \min_i(x_i)$. For general linear MASs including unstable dynamics, consensus problems under switching-disconnected topologies are much more complex, because convergence analyses depend on the stability of isolated agents and the connectivity of network topologies. Regardless of the intrinsic limitation, the difficulty of applying Lyapunov method is that each switched sub-system may not be a convergent one. With non-increasing Lyapunov function, several sufficient condi-
tions of consensus for marginally stable agents have been obtained by Ni and Cheng (2010), Qin, Yu, and Gao (2014), Su and Huang (2012a, 2012b), Huang (2017) and Meng, Yang, Li, Ren, and Wu (2018). However, the condition, derivatives of Lyapunov functions are negative semidefinite along the solutions of dynamics, cannot be satisfied in the case of exponentially unstable dynamics and switching-disconnected topologies. As an extension of classic Lyapunov function method, non-monotonically decreasing Lyapunov function method (NMDLF method) (Aeyels & Peuteman, 1999; Zhou, 2016) is applicable to complex time-varying dynamics, especially for fast time-varying systems. It is worth noting that, in the works of Li, Liao, Lei, Huang, and Zhu (2013; Li, Liao, Huang, Zhu, & Liu, 2015) and Ni, Wang, and Xiong (2012), consensus and leader-following consensus problems of general linear MASs have been considered by their averaged systems with the help of NMDLF method. However, conditions under which NMDLF is feasible have not been discussed thoroughly.

In Yang, Meng, Shi, Hong, and Johansson (2016), Chen, Yu, Tan, and Zhu (2016) and Qin, Ma, Yu, and Wang (2018), contraction analysis has been used to deal with consensus problems among nonlinear dynamics. Particularly, a novel analysis framework in Qin et al. (2018) has been proposed from a unified algebraic and geometric perspective to revisit the consensus problems for both generic linear systems and Lipschitz-type nonlinear systems over switching directed topologies. However, the assumption on switching communication graph is not the weakest connectivity condition.

As another method, product properties of infinite row-stochastic matrices (Wolfowitz, 1963) are effective in dealing with the consensus of MASs with the union of network topologies having a spanning tree. By exploring these properties, Qin and Yu (2014), Qin, Gao, and Yu (2014) have succeeded in the consensus analysis for general linear MASs allowing exponentially unstable agents. But the cost is excluding a part of agents whose inputs cannot be decoupled. In addition, Scardovi and Sepulchre (2009), Lu and Liu (2017) have investigated the synchronization (consensus) of a network of linear agent systems. They have proposed dynamic controllers that decouple the interaction between dynamics of agents and network topologies so that dynamics can be transformed to the form of single-integrators, by which consensus can be ensured. For this kind of dynamic controller, however, network topologies need sufficiently strong connectivity because the designed controllers are incapable of dominating the instability of the individual agent. It is worthwhile to mention that, in addition, some cases of random switching topologies have been investigated in the view of Markovian process. For instance, You, Li, and Xie (2013) has shown that the effect of switching topologies on consensus is determined by the union of topologies associated with the positive recurrent states of the Markovian process. Dong, Bu, Wang, and Han (2018) has dealt with the distributed state estimation problem where a Markovian chain has been introduced to describe the features of switching topologies, and has verified that the estimation error dynamics performs well.

In this paper, we investigate the consensus problem of MASs when switching disconnected topologies and exponentially unstable agents are taken into account together. The difficulties are that the associated Laplacian matrices may have eigenvalue zero with algebraic multiplicity larger than one and each agent has exponentially unstable mode. By analysing the advantage and disadvantage of existing methods, a novel and elaborate solution has been worked out to the consensus problem with these difficulties, where the consensus analysis is performed by NMDLF method, which allows the positive definite Lyapunov function to increase in some finite time intervals but with a general tendency of decrease. Particularly, the design of consensus laws is explored from the perspective of fast time-varying systems with two time scales. Sufficient conditions for achieving consensus depend on not only feedback gains and connectivity of network topologies but also the speed of topology switching and the stability of individual agent.

The remainder of this paper is organized as follows. Some preliminaries and problem formulations are provided in Section 2. The main result is worked out in Section 3. Numerical simulations are presented to verify the theoretical results in Section 4. Finally, conclusions are given in Section 5.

The following notations are used throughout this paper. Let \( \mathbb{R} \) and \( \mathbb{Z} \) denote the sets of real numbers and nonnegative integers, respectively. \( \mathbb{R}^n \) and \( \mathbb{R}^{m \times n} \) denote \( n \)-dimensional real vector space and \( m \times n \) real matrix space, respectively. \( \mathbb{I}_N \) denotes the set \( \{1, 2, \ldots, N\} \). \( \| \cdot \| \) denotes the Euclidean norm. The superscript ‘\(^T\)’ stands for matrix transposition and \( P > 0 \) means that \( P \) is a symmetric and positive definite matrix. Adjacent matrix and system matrix will be denoted by the same symbol \( A \), if they are not ambiguous in context. \( I_n \) and \( I \) denote an \( n \times n \) identity matrix and an identity matrix with appropriate dimensions, respectively. \( \lfloor a/b \rfloor \) denotes the maximum integer not greater than \( a/b \).

2. Preliminaries and problem formulations

2.1. Graph theory

We use a directed graph \( G = (V, E) \) to model the network topology of MASs with \( N \) agents, where
\[ V = \{v_1, v_2, \ldots, v_N\} \] is the node set and \( E \subseteq V \times V \) is the edge set. An edge \( e_{ij} \in E \) in \( G \) denotes that agents \( i \) can receive information from \( j \). The adjacent matrix \( A = [a_{ij}] \in \mathbb{R}^{N \times N} \) associated with \( G \) is defined by \( a_{ij} = 1 \) if \( e_{ij} \subseteq V \times V \), and \( a_{ij} = 0 \) otherwise, as well \( a_{ii} = 0 \) for all \( i \in \mathbb{N} \). The Laplacian matrix \( L = [l_{ij}] \in \mathbb{R}^{N \times N} \) is defined as \( l_{ij} = -a_{ij}, l_{ii} = \sum_{j=1}^{N} a_{ij}, j \neq i \). The neighbour set of the \( i \)th agent is denoted by \( N_i = \{i | a_{ij} = 1, i \neq j\} \). A directed graph is called strongly connected if any two distinct nodes of the graph can be connected by a directed path, while it is called weakly connected if replacing all of its directed edges with undirected edges produces a connected graph. A directed graph is called balanced if and only if \( t^T L = 0 \).

### 2.2. Problem formulations

This paper investigates the consensus problem of MASs under switching topologies, and the dynamics of the \( i \)th agent is described by

\[ \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i \in \mathbb{N}, \quad t \geq 0 \tag{1} \]

where \( x_i \in \mathbb{R}^n \) and \( u_i \in \mathbb{R}^m \) are the state and the control law of agent \( i \), and \( A \) and \( B \) are real constant matrices with appropriate dimensions and satisfy the following assumption.

**Assumption 2.1:** \((A, B)\) is stabilizable and the state matrix \( A \) may have eigenvalues in the open right-half plane.

The objective of this paper is to analyse and design control laws, which can be applied to a very general setting that allows exponentially unstable dynamical systems and switching-disconnected topologies, such that the \( N \) agents in (1) achieve consensus in the sense of \( \lim_{t \to \infty} |x_i(t) - x_j(t)| = 0 \) for any initial states.

### 3. Main results

In this section, we aim to prove that consensus can be achieved by exploring the properties of switching topologies and using the NMDLDF method.

First of all, we need to model the time sequences under time-scale \( s \) of topology switching which will be differentiated from the time-scale \( t \) of states of agents. Consider a time-scale \( s \) of directed dynamic topology denoted by \( G_{\sigma(s), s} \subseteq [0, +\infty) \) with an infinite sequence of nonempty, bounded and contiguous time intervals \([s_k, s_{k+1}), k \in \mathbb{Z} \) with \( s_{k+1} - s_k \leq T \) for some constant \( T > 0 \). Suppose that, in each time interval \([s_k, s_{k+1}), k \in \mathbb{Z} \), there is a sequence of non-overlapping subintervals \([s_k^0, s_k^1), \ldots, [s_k^{j-1}, s_k^j), \ldots, [s_k^{m-1}, s_k^m) \) with \( s_k^0 = s_k^m = s_k \leq s_k^{j+1} - s_k^j \leq T, \) for some integer \( m_k \geq 1 \) and a given constant \( \tau > 0 \), such that the interaction topology does not change during each of these subintervals. That is, during each subinterval \([s_k^j, s_k^{j+1})\), the graph \( G_{\sigma(s), s} \subseteq [s_k^j, s_k^{j+1}) \) is fixed with constant Laplacian matrix \( L_{s_k^j} \) where \( \sigma(s) : [0, +\infty) \to \mathbb{P} \) is a switching signal. Since \( N \), the number of agents, is finite, there is only finite number, say \( \hat{N} \), of different types of switching topologies and \( \mathbb{P} = \{1, \ldots, \hat{N}\} \). Therefore, denote \( \hat{G} = \{G_1, \ldots, G_{\hat{N}}\} \) as the set of all possible interaction topologies among the \( N \) agents.

**Assumption 3.1:** The graph \( G(s) \) is jointly weakly connected and balanced across all time intervals \([s_k, s_{k+1}), k \in \mathbb{Z} \) with \( s_{k+1} - s_k \leq T \) for some constant \( T > 0 \).

The basic distributed state feedback law is considered as follows:

\[ u_i(t) = K \sum_{j \in N_i(s)} a_{ij}(s)(x_j(t) - x_i(t)), \quad i \in \mathbb{N} \tag{2} \]

where \( N_i(s) = \{j | a_{ij}(s) = 1, i \neq j\} \) is the neighbour set of the \( i \)th agent with time-scale \( s \) of topology switching and \( a_{ij}(s) \), the \((i, j)\)-th entry of the adjacency matrix \( A \) associated with \( G_{\sigma(s)} \), evolves along the time-scale \( s \). Correspondingly, \( x_i(t) \) evolves along the time-scale \( t \). The relationship between time-scales \( s \) and \( t \) is established by parameter \( \epsilon = t/s > 0 \), which indicates the speed of variation in \( G_{\sigma(s)} \), \( s \in [0, +\infty) \). When \( 0 < \epsilon < 1 \), the system (1) with (2) can be considered as a fast time-varying system. In addition, \( K \in \mathbb{R}^{m \times n} \) is the feedback gain matrix.

Based on the above statements, we introduce the following error variables

\[ e_i(t) = x_i(t) - \frac{1}{N} \sum_{j=1}^{N} x_j(t), \quad i \in \mathbb{N} \tag{3} \]

For dynamics (1) under control law (2) with Assumption 3.1, it is easily to show that

\[ \dot{e}_i(t) = \frac{1}{N}(1^T \otimes I_N)\dot{x}_i(t), \quad i \in \mathbb{N} \]

\[ = A\dot{x}_i(t) + BK \sum_{j \in N_i(s)} a_{ij}(s)(x_j(t) - x_i(t)) \]

\[ - \frac{1}{N}(1^T \otimes I_N)(I_N \otimes A + L_{\sigma(s)} \otimes BK)x(t) \]
Let $A(t)$ be piecewise continuous in $t$ and elements of $A(t)$ be bounded. That is, there is a constant $\bar{A} > 0$ such that $||A(t)|| \leq \bar{A}$. Let $x(t)$ be the solution of (6) at $t$ with any initial state $x_\ast$ at $t_\ast$.

\[
\dot{x}(t) = A(t)x(t), \quad x(t_\ast) = x_\ast \tag{6}
\]

Then $||x(t)|| \leq ||x(t_\ast)||e^{\bar{A}(t-t_\ast)}$, $\forall t \in [t_\ast, t_\ast + T]$, $\forall 0 < T < +\infty$.

Remark 3.1: Referring to Khalil, H. (2001), Lemma 3.1 is a simple and general result, which estimates the bound of solution of (6) on compact time intervals by using Gronwall-Bellman inequality. The proof is omitted here since it can be easily verified.

Remark 3.2: Classical Lyapunov theorems require that Lyapunov functions decrease monotonically along the solution of dynamics, which are not feasible to test the stability of some complex time-varying systems. However, it is worth noting that the Lyapunov direct method is based on the simple mathematical fact that if a scalar function is bounded from below and decreasing, the function has a limit as time $t$ approaches infinity. The basic idea to test stability in a more general framework is allowing the positive definite function to increase in some finite time intervals but with a general tendency of decrease. In other words, it is unnecessary for Lyapunov functions to decrease monotonically, which is the main idea of NMDLF method given by the following lemma.

Lemma 3.2: Let $V : \mathbb{R}^q \times [0, +\infty) \rightarrow \mathbb{R}$ is a quadratic positive definite function. If there exist an infinite sequence of nonempty, bounded and contiguous time intervals $[t_k, t_{k+1})$, $k \in \mathbb{Z}$ with $t_{k+1} - t_k \leq T$ for some constant $T > 0$, and a constant $\nu > 0$, such that

\[
V(x(t_{k+1}), t_{k+1}) - V(x(t_k)), t_k \leq -\nu ||x(t_k)||^2 < 0, \forall k \in \mathbb{Z} \tag{7}
\]

Here $x(t_k)$ is the solution of (6) at $t_k$, $\forall k \in \mathbb{Z}$. Then, the equilibrium $x(t) = 0$ of (6) with any initial state $x(0)$ is exponentially stable.

Remark 3.3: Lemma 3.2 originates from the result of Theorem 1 proposed in Aeyels and Peuteman (1999). Relatively, the cases of linear time-varying systems were further studied by Zhou (2016). The idea of allowing indefinite time derivatives of Lyapunov functions is appropriate for dealing with the consensus problem with time-varying disconnected topologies. However, it is in general difficult to resolve Lyapunov inequalities or differential Lyapunov inequalities for time-dependent Lyapunov function. It is worthwhile to mention that the condition in Aeyels and Peuteman (1999) was directly used to obtain some conclusion in the view of averaged systems by Li et al. (2013, 2015) and Ni et al. (2012). Nevertheless, this condition should be worked out so that consensus under switching topologies can be ensured, which will be elaborated in this section.

The main result of this paper is demonstrated as follows.

Theorem 3.1: Consider MAS (1) with Assumption 2.1. If switching topologies satisfy Assumption 3.1 and there are $P > 0$ and $\bar{\epsilon} > 0$ such that

\[
PA + A^T P - 2\mu PBB^T P < -\mu I \tag{8}
\]

\[
\mu > M[T/\tau ]([[(\theta + 1)^2 - 1]
\]

where $\mu < \min\{\lambda_{ij}|k \in \mathbb{Z}, i = 2, \ldots, N\}$, $\theta = \tilde{A}e\tilde{A}^T (1 - e^{[T/\tau ]\tilde{A}e\tilde{A}^T} / (1 - e^{[T/\tau ]\tilde{A}e\tilde{A}^T})$, $\tilde{A}$ is a Lipschitz constant of the right-hand side function in (5), and $M$ is a bound of $||L_N \otimes (PA + A^T P) \otimes (L_{\sigma_i} + L_{\sigma_i}^T) \otimes BB^T||$.

Then there exists a feedback gain matrix $K = -B^T P$ such that MAS (1) can reach consensus by using control law (2).

Proof: Choose the following Lyapunov function candidate

\[
V(t) = e^T(t)(L_N \otimes P)e(t) \tag{10}
\]

where $e(t) = [e_1^T(t), \ldots, e_N^T(t)]^T$ is the solution of (5).

Taking the values of $V(t)$ at the infinite sequence $t_0, \ldots, t_k, \ldots$ with $t_k = \bar{\epsilon}s_k, k \in \mathbb{Z}$ from the view of time...
scale \( t \), we can get that

\[
V(t_{k+1}) - V(t_k) = \sum_{l=0}^{m_k-1} \left[ V(t_{k+1}^l) - V(t_k^l) \right]
\]

\[
= \sum_{l=0}^{m_k-1} \int_{t_k^l}^{t_{k+1}^l} \dot{V}(t)dt
\]

(11)

where partitioning the time interval \([t_k, t_{k+1}]\) and rewriting the difference of \(V(t)\) between the instants \(t_k\) and \(t_{k+1}\) into the sum of integrals are guided by the fact that \(V(t)\) is continuously differentiable at any time except for switching instants.

Differentiating \(V(t)\) along the trajectory of (5) between \(t_k^l\) and \(t_{k+1}^l\), \(l = 0, \ldots, m_k - 1\), we can get that

\[
V(t_{k+1}) - V(t_k) = \sum_{l=0}^{m_k-1} \int_{t_k^l}^{t_{k+1}^l} e^T(t_k) [I_N \otimes (PA + AT P)] \dot{e}(t)dt
\]

\[
-(L_{\sigma(s)} + L_{\sigma(s)}^T) \otimes PBB^T P e(t_k) dt
\]

(12)

where we have substituted \(K = -B^T P\) to get this equation. Then, by some manipulations, (12) can be described as follows.

\[
V(t_{k+1}) - V(t_k) = \sum_{l=0}^{m_k-1} \left\{ \int_{t_k^l}^{t_{k+1}^l} e^T(t) [I_N \otimes (PA + AT P)] \dot{e}(t)dt \right\}
\]

\[
-(L_{\sigma(s)} + L_{\sigma(s)}^T) \otimes PBB^T P e(t_k) dt
\]

(13)

According to Lemma 3.1, we know that

\[
||e(t_k^l)|| \leq \bar{A}_e e^{(l+1)\bar{A}_e T} ||e(t_k^l)||
\]

(20)

\[
||e(t) - e(t_k^l)|| \leq \bar{A}_e T e^{(l+1)\bar{A}_e T} ||e(t_k^l)||
\]

(21)

where \(\bar{A}\) is a Lipschitz constant of the right-hand side function in (5) and available by computation.

Thus,

\[
||e(t) - e(t_k^l)|| \leq \bar{A}_e T e^{(l+1)\bar{A}_e T} ||e(t_k^l)||
\]

(22)
such that

\[
A_{\epsilon} T e^{A_{\epsilon} T} \left[ \frac{1 - e^{(T/\tau)A_{\epsilon} T}}{1 - e^{A_{\epsilon} T}} \right] ||e(t_k)|| \leq \frac{\hat{A}_{\epsilon} T \hat{e}^{A_{\epsilon} T} m_k (1 - e^{m_{kr}A_{\epsilon} T})}{1 - e^{A_{\epsilon} T}} ||e(t_k)|| 
\]

we can obtain that

\[
V(t_{k+1}) - V(t_k) 
\leq e^T(t_k) e \left[ I_N \otimes (PA + A^T P) \right] e(t_k) 
- \sum_{l=0}^{m_k-1} \int_{t_k}^{t_{k+1}} ||e(t) - e(t_k)||^2 dt 
+ \mu T ||e(t_k)||^2 
= e^T(t_k) e \left[ I_N \otimes (PA + A^T P) \right] e(t_k) 
+ \pm T e \left[ I_N \otimes (PA + A^T P) \right] e(t_k) 
= \xi^T(t_k) e \left[ I_N \otimes (PA + A^T P) - 2\lambda_k \otimes \Phi_k \right] \xi(t_k) 
+ \pm T \xi^T(t_k) e \left[ I_N \otimes (PA + A^T P) - 2\lambda_k \otimes \Phi_k \right] \xi(t_k) 
= -\mu ||e(t_k)||^2 + \pm T ||e(t_k)||^2 
< 0 
\]

where (26) is obtained from (19), (24) and (25), (28) is obtained from $\xi_1(t_k) = \left( \left( I_N \otimes \sqrt{N} \right) \otimes I_n \right) e(t_k) \equiv 0$, and (29) is obtained by choosing $\mu < \min(\lambda_i, i = 2, \ldots, N)$, which is feasible as the set of all possible interaction topologies is finite. Thus, according to (8) and (9), we know (31) and (32) hold. Therefore, by Lemma 3.2, $e(t)$ is exponentially stable, and the consensus of MAS (1) is achieved.

**Remark 3.4:** Normally, classic Lyapunov function requires that there is a continuously differentiable positive definite function $V(t)$ so that $V(t)$ is negative definite. However it is not feasible to some dynamics whose states diverge from the equilibrium for some limited time intervals but will be asymptotically stable. Although Ni and Cheng (2010), Qin, Yu, et al. (2014), Su and Huang (2012a, 12b), Huang (2017) and Meng et al. (2018) only require $V(t)$ is negative semi-definite for the consensus problems of linear MASs, the eigenvalues of the state matrix of $A$ of each agent has to lie in the closed left half-plane. Compared with them, the idea of NMDLF allows the positive definite Lyapunov function to increase in some finite time intervals but with a general tendency of decrease, which can be depicted by Figure 1.
Following this idea, Lemma 3.2 is used to prove the consensus of MASs in Theorem 3.1 under the restrictions of disconnected switching topologies and unstable dynamics. In order to work out the conditions of Lemma 3.2, the time interval \([t_k, t_{k+1}]\) is partitioned into non-overlapping subintervals \([t_k^0, t_k^1], \ldots, [t_k^{m_k}, t_k^{m_k+1}], \ldots, [t_k^{m_k-1}, t_k^0]\) with \(t_k^0 = t_k^n = t_{k+1}\), for some integer \(m_k \geq 0\). The interaction topology does not change during each of such subintervals, where \(V(t)\) is continuously differentiable. Therefore we can jump into each subintervals and inspect the derivative of Lyapunov function.

In addition, the averaged systems were used to prove the consensus of MASs with exponentially unstable agents. While compared with Li et al. (2013, 2015) and Ni et al. (2012), it is not necessary to get the average system during the produce of controller design, and most importantly, explicit conditions that guarantee the Lyapunov function non-monotonically decreases are solved. Actually, it is almost impossible to work out problems by one solution without any cost. To highlight the differences, comparisons between the NMDLF in this paper and those related methods are shown in Table 1.

### 4. Numerical simulations

In this section, theoretical results will be verified by numerical simulations. Consider a multi-agent system with six agents, whose dynamics are dominated by following \(A\) and \(B\).

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -1 & a \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

where the eigenvalues of \(A\) are \(\lambda_1(A) = a, \lambda_2(A) = i\) and \(\lambda_3(A) = -i\). The initial states of agents are \(x_1(t_0) = [1, 2, 3]^T\), \(x_2(t_0) = [4, 5, 6]^T\), \(x_3(t_0) = [7, 8, 9]^T\), \(x_4(t_0) = [10, 11, 12]^T\), \(x_5(t_0) = [13, 14, 15]^T\) and \(x_6(t_0) = [16, 17, 18]^T\).

The process of topology switching is depicted by Figure 2, where the interaction topology switches along the loop formed by six disconnected topologies with \(s_{k+1} - s_k \leq 0.06\). Obviously, the union of the sub-graphs is weak connected and balanced.

| Methods | Major techniques | Advantages | Disadvantages |
|---------|-----------------|------------|--------------|
| Non-increasing Lyapunov function | Generalized Babalat’s Lemma (Qi, Yu, et al., 2014) | • Applicable to switching-disconnected topologies; • Applicable to switching-disconnected topologies; • Directed graph | • Undirected; • Constraints on state matrix A • Constraints on state matrix A |
| Non-monotonically decreasing Lyapunov function method | Applying averaged systems | • No constraints on state matrix A | • Averaged system is needed; |
| Contraction analysis | Graph and system theories | • No constraints on state matrix A; • Consensus explicitly depends on feedback gains, connectivity, switching speed, and stability of agents | • The union of network topologies is balanced |
| Infinite row-stochastic matrices | SIA matrices; Transition matrix | • Applicable to linear and nonlinear systems • Directed graph | • Connectivity condition is not the weakest |
| Decoupling interaction | Dynamical controllers; Change of variable | • No constraints on state matrix A • Directed graph | • Input matrix is invertible; • Unstable states are out of control |

**Figure 1.** The idea of NMDLF.
To demonstrate the unstable cases, two cases with $a = 0$ and $a = 0.1$ are simulated. By calculations, $\mu$ is chosen as 0.4, and feedback gain matrices $K$ are respectively obtained as follows:

$$K_{a=0} = [-0.9851, -1.1075, -2.0978]$$  \hspace{1cm} (34)

$$K_{a=0.1} = [-1.1734, -1.2319, -2.2871]$$  \hspace{1cm} (35)

The open-loop dynamics of the agents with $a = 0$ are critically stable. All state components of agents evolve in the form of oscillation and the consensus errors between agent 1 and the others, shown in Figure 3, are converging to zero under control law (2), which confirm the consensus of all agents.

On the other hand, the open-loop dynamics of the agents with $a = 0.1$ are exponentially unstable. There is one component of states of each agent that evolves rapidly in exponential divergence form, meanwhile, consensus errors between agent 1 and the others are converging to the zero under control law (2) as shown in Figure 4, which confirm the consensus of all state of agents.
5. Conclusions

The consensus analysis and design are worked out in this paper for general linear MASs under dynamic topology by using the NMDLF method. It is shown that the consensus of general linear MASs, including exponential unstable agents, can be achieved by basic static coupling laws when switching topologies are jointly weakly connected and balanced, and the switching speed satisfies some explicit conditions. Also, numerical examples including critically stable and exponential unstable cases are given to demonstrate the effectiveness of theoretical results. However, some global information on the graph is needed for the gain of static coupling laws, which is the limitation especially for the case of switching topologies. Under these circumstance, adaptive consensus protocols are preferred to distributed control of MASs. Future work will focus on the cases of adaptive coupling laws as well as time delays.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The authors would like to thank anonymous reviewers and the support of China Scholarship Council, funds of Hunan Province of China (2016J6099, 17C0958), Hunan Pro vincial Science and Technology Project Foundation (2018TP1018), and Hunan Normal University (2018078).

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