IRREDUCIBLE COMPONENTS OF TWO-ROW SPRINGER FIBERS AND
NAKAJIMA QUIVER VARIETIES

MEE SEONG IM, CHUN-JU LAI, AND ARIK WILBERT

Abstract. We give an explicit description of the irreducible components of two-row Springer fibers in type A as closed subvarieties in certain Nakajima quiver varieties in terms of quiver representations. By taking invariants under a variety automorphism, we obtain an explicit algebraic description of the irreducible components of two-row Springer fibers of classical type. As a consequence, we discover relations on isotropic flags that describe the irreducible components.

1. Introduction

1.1. Background and summary. Quiver varieties were used by Nakajima in [Nak94, Nak98] to provide a geometric construction of the universal enveloping algebra for symmetrizable Kac-Moody Lie algebras altogether with their integrable highest weight modules. It was shown by Nakajima that the cotangent bundle of partial flag varieties can be realized as a quiver variety. He also conjectured that the Slodowy varieties, i.e., resolutions of slices to the adjoint orbits in the nilpotent cone, can be realized as quiver varieties. This conjecture was proved by Maffei, [Maf05, Theorem 8], thereby establishing a precise connection between quiver varieties and flag varieties.

Aside from quiver varieties, the second main object of study in this article is the Springer fiber, which plays a crucial role in the geometric construction of representations of Weyl groups (cf. [Spr76, Spr78]). In general, Springer fibers are singular and decompose into many irreducible components.

The first goal of this article is to study the irreducible components of two-row Springer fibers in type A from the point of view of quiver varieties using Maffei’s isomorphism. More precisely, since Springer fibers are naturally contained in the Slodowy variety, it makes sense to describe the image of the Springer fiber as a subvariety of the quiver variety under Maffei’s isomorphism. We achieve this goal by explicitly describing the quiver representations which represent points in the Springer fiber under Maffei’s isomorphism (see Theorem 3.8). Our proof relies on an explicit description of the irreducible components in terms of flags obtained by Stroppel–Webster, [SW12], based on earlier work by Fung, [Fung03].

Moreover, when restricted to the Springer fiber, Maffei’s isomorphism (which in general is not very explicit) actually becomes explicit (cf. Lemma 3.2), which allows for a translation of the results of Stroppel–Webster to the quiver variety, see Proposition 3.7. It would be interesting to investigate if the relations on the quiver variety of the irreducible components can be generalized to nilpotent endomorphisms with more than two Jordan blocks. For these nilpotent endomorphisms, obtaining an understanding of the precise geometric and combinatorial structure of the irreducible components remains an open problem.

The second goal of this article is to generalize the results in type A to all classical types. Our focus will be on two-row Springer fibers of type D. In fact, any two-row Springer fiber of type C is isomorphic to a two-row Springer fiber of type D, [Wil15, Li19]. Moreover, by the classification of nilpotent orbits, [Wil37, Ger61], there are no two-row Springer fibers of type B. Hence, it is enough to treat the type D case. Work of Henderson–Licata [HL14] and Li [Li19] shows that the type D Slodowy variety can be realized as a fixed-point subvariety of a type A quiver variety under a suitable variety automorphism. These fixed-point subvarieties play an important role in developing the geometric representation theory of symmetric pairs, [Li19]. Our second main result (see Theorem 6.1) is to explicitly compute the fixed points contained in the subvarieties of the quiver variety corresponding to the irreducible components...
in type A. By sending the result through Maffei’s isomorphism we obtain explicit algebraic relations that describe the irreducible components of the type D two-row Springer fiber in terms of isotropic flags. This generalizes the results by Stroppel–Webster. Before our manuscript, for two-row Springer fibers of type D, the only explicit algebraic construction of components required an intricate inductive procedure, see [ES16a, §6], based on [Spa82, vL89]. Other than that, only topological models were available, [ES16a, Wil18].

1.2. Type A: Two-row Springer fibers and quiver varieties. Let \( \mu : \tilde{\mathcal{N}} \to \mathcal{N} \) be the Springer resolution for nilpotent cone \( \mathcal{N} \) of \( \mathfrak{gl}_n(\mathbb{C}) \). For any element \( x \in \mathcal{N} \), one can associate a Slodowy slice \( S_x \), a Slodowy variety \( \tilde{S}_x \), and a Springer fiber \( B_x \) with the relations below:

\[
\begin{align*}
\xymatrix{ & B_x & \ar[l] \mid \mu \mid \ar[r] & \tilde{S}_x & \ar[l] \mid \mu \mid \ar[r] & \mathcal{N} \\
\{x\} & \ar[l] \mid \mu \mid \ar[r] & S_x & \ar[l] \mid \mu \mid \ar[r] & \mathcal{N} }
\end{align*}
\]

It is conjectured by Nakajima in [Nak94] and then proved by Maffei [Maf05, Theorem 8] that there is an isomorphism \( \tilde{\varphi} : M(d, v) \to \tilde{S}_x \) between the Slodowy variety and certain Nakajima quiver variety \( M(d, v) \), which is realized by a geometric invariant theory (GIT) quotient of certain quiver representation space \( \Lambda^+ (d, v) \), which consists of collections \( ((\Lambda_i)_i, (\Delta_i)_i, (\Gamma_i)_i)_i \) of linear maps, subject to certain stability and admissibility conditions.

For a nilpotent endomorphism \( x \) of Jordan type \( (n-k, k) \), it is well-known (cf. [Fun03, §7]) that the irreducible components \( \{ K^a \}_a \) of the Springer fiber \( B_x \) are parametrized by the so-called cup diagrams.

The purpose of this article is to give an explicit description (of the irreducible components) of the Springer fibers via the embedding into the corresponding Nakajima quiver varieties. Their relations are depicted as below:

\[
\begin{align*}
\Lambda^+ (d, v) \xrightarrow{p} M(d, v) & \xrightarrow{\tilde{\varphi}} \tilde{S}_x \\
\tilde{\varphi}^{-1}(B_x) & \xrightarrow{\tilde{\varphi}} B_x. \\
\tilde{\varphi}^{-1}(K^a) & \xrightarrow{\tilde{\varphi}} K^a
\end{align*}
\]

Each irreducible component \( K^a \) consists of pairs \( (x, F_a) \) for some complete flag \( F_a = (0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_n = \mathbb{C}^n) \) subject to certain relations imposed from the configurations of cups and rays in a cup diagram.

Our first result is to give explicit and new relations on the quiver representation side that correspond to the cup/ray relations in [SW12, Proposition 7] as below:

- **Cup relation** for \( i \neq j \) \( \cup \) \( F_j = x^{-\frac{1}{2}(j-i+1)} F_{i-1} \)
- \( \ker B_{i-1} B_i \cdots B_{j-1} = \ker A_{j-1} A_{j-2} \cdots A_{i+1} = \ker A_{j-1} A_{j-2} \cdots A_{i+1} \),

- **Ray relation** for \( i \) \( \mid \) \( F_i = x^{-\frac{1}{2}(i-\rho(i))} \prod_{k=1}^{n-k-1} \Gamma_{n-k} = 0 \)

where \( \rho(i) \in \mathbb{Z}_{>0} \) counts the number of rays (including itself) to the left of \( i \), and \( c(i) = \frac{i - \rho(i)}{2} \) is the total number of cups to the left of \( i \). In other words, we have constructed a subvariety \( M^a \subset M(d, v) \) inside the quiver variety using the above relations, whose points correspond to exactly one irreducible component of the Springer fiber, and we prove:
**Theorem A** (Theorem 3.8). For any cup diagram $a$, the Maffei–Nakajima isomorphism $\tilde{\varphi} : M(d, v) \to S_x$ between quiver variety and Slodowy variety restricts to an isomorphism $M^a \to K^a$ between irreducible components.

1.3. **Springer fibers of classical type.** For any type $\Phi = B, C$ or $D$, let $\mu_\Phi : N^\Phi \to N^\Phi$ be the Springer resolution for the nilpotent cone $N^\Phi$ of the Lie algebra $\mathfrak{g}^\Phi$ of type $\Phi$. For each $x \in N^\Phi \subset N$, one associates a Slodowy slice $S_x^\Phi$, a Slodowy variety $\tilde{S}_x^\Phi$, and a Springer fiber $B_x^\Phi$ with the relations below:

\[
B_x^\Phi = \mu_\Phi^{-1}(x) \quad \tilde{S}_x^\Phi = \mu_\Phi^{-1}(S_x^\Phi) \quad N^\Phi
\]

In the following, we would like to study the components of Springer fibers $B_x^\Phi$ associated with a nilpotent endomorphism $x \in N^\Phi$ of Jordan type $(n-k, k)$, using a generalized cup diagram approach. As mentioned in Section 1.3 it suffices to study the type D Springer fibers of two Jordan blocks. The type D cup diagrams are similar to the type A ones but with two new ingredients – marked cups and marked rays, which arise naturally in the process of folding a centro-symmetric type A cup diagram (see [ES16b, LS13]). For example, 

\[ \begin{array}{ccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\end{array} \Rightarrow \begin{array}{cccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \]

Here the cups and rays may carry a mark (i.e., ■) subject to some conditions. Unlike in type A, no explicit relations describing the flags in a given component were known for type D. In this paper, we discover explicit relations on the flags described by the cups and rays of a type D diagram. In order to achieve this, we study the process of taking fixed-points under a certain automorphism on the quiver varieties, which naturally corresponds to a folding of a type A cup diagram.

1.4. **Involutive quiver varieties.** For a Nakajima quiver variety $M(d, v)$ of type A, an automorphism $\theta$ on the type A Dynkin diagram induces a variety automorphism $\Theta$ which also depends on some non-degenerate bilinear form. These fixed-point subvarieties $M(d, v)\Theta$ appeared in [HL14] (see [Li19] for a more general construction of such subvarieties). It is shown in [HL14] that under the Maffei-Nakajima isomorphism, $M(d, v)\Theta$ encodes the type D Slodowy varieties.

We can now apply this folding technique to our subvariety $M^a \subset M(d, v)$ and then obtain a fixed-point subvariety $(M^a)\Theta \subset (M(d, v))\Theta$ associated with a type D cup diagram.

**Theorem B** (Theorem 6.1). For any type A cup diagram $a$ that is centro-symmetric, the Maffei-Nakajima isomorphism $\tilde{\varphi}$ restricts to an isomorphism $\hat{M}(a)\Theta \to \bigcup \hat{K}^a$, where $\hat{a}$ runs over all type D cup diagrams which unfold to $a$ in the sense of Algorithm 4.3. Moreover,

\[ M(d, v)\Theta \simeq \bigcup \hat{a} \hat{K}^a. \]

As an application, we obtain the relations on the isotopic flags imposed by marked cups and marked rays:

- **Marked cup relation for** $\uparrow$ $\downarrow$ $\leftarrow x^{1/2} F_i = x^{1/4+1} F_j$,

- **Marked ray relation for** $\uparrow$ $\Rightarrow F_i = \{ \langle e_1, \ldots, e_{i-2} \rangle, e_{i} (i+1), f_1, \ldots, f_{1/2(i-1)} \}$ if $n = 2k$,
  
  $\langle e_1, \ldots, e_{i-c(i)-1}, f_1, \ldots, f_{c(i)}, f_{c(i)+1} + e_{i-c(i)} \rangle$ if $n > 2k$. 

\[ \left\{ \begin{array}{ll}
\end{array} \right. \]
Acknowledgment. M.S.I. would like to thank the University of Georgia for organizing the Southeast Lie Theory Workshop X, where our collaborative research group initially came together. The authors also thank the AGANT group at the University of Georgia for supporting this project. This project was initiated as a part of the Summer Collaborators Program 2019 at the School of Mathematics at the Institute for Advanced Study (IAS). The authors thank the IAS for providing an excellent working environment. We are grateful for useful discussions with Jiuzu Zhu during the early stages of this project. We thank Anthony Henderson, Yiqiang Li, and Hiraku Nakajima for helpful clarifications. A.W. was also partially supported by the ARC Discover Grant DP160104912 “Subtle Symmetries and the Refined Monster”.

2. Springer fibers and Nakajima quiver varieties of type A

2.1. Springer fibers and Slodowy varieties. Fix integers $n, k$ such that $n \geq 1$ and $n - k \geq k \geq 0$. Let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{gl}_n(\mathbb{C})$. Let $G = GL_n(\mathbb{C})$. One may parametrize the $G$-orbits in $\mathcal{N}$ using partitions of $n$ by associating a nilpotent endomorphism to the list of the dimensions of the Jordan blocks.

Denote the complete flag variety in $\mathbb{C}^n$ by

$$\mathcal{B} = \{ (F_0 = 0 = F_0 \subset F_1 \subset \ldots \subset F_n = \mathbb{C}^n) \mid \dim F_i = i \text{ for all } 1 \leq i \leq n \}.$$ (2.1)

We denote the Springer resolution by $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$, $(u, F_\bullet) \mapsto u$, where

$$\widetilde{\mathcal{N}} = T^* \mathcal{B} \cong \{(u, F_\bullet) \in \mathcal{N} \times \mathcal{B} \mid u(F_i) \subseteq F_{i-1} \text{ for all } i \}. $$ (2.2)

We denote the Springer fiber of $x \in \mathcal{N}$ by

$$\mathcal{B}_x = \mu^{-1}(x).$$ (2.3)

For each nilpotent element $x \in \mathcal{N}$, we denote the Slodowy transversal slice of (the $G$-orbit of) $x$ by the variety

$$\mathcal{S}_x = \{ u \in \mathcal{N} \mid [u - x, y] = 0 \},$$ (2.4)

where $(x, y, h)$ is a $\mathfrak{sl}_2$-triple in $\mathcal{N}$ (here $(0, 0, 0)$ is also considered an $\mathfrak{sl}_2$-triple). We then denote the Slodowy variety associated to $x$ by

$$\widetilde{\mathcal{S}}_x = \mu^{-1}(\mathcal{S}_x) = \{(u, F_\bullet) \in \mathcal{N} \times \mathcal{B} \mid [u - x, y] = 0, \ u(F_i) \subseteq F_{i-1} \text{ for all } i \}.$$ (2.5)

In particular, $x \in \mathcal{S}_x$ and hence $\mathcal{B}_x = \mu^{-1}(\mathcal{S}_x) = \widetilde{\mathcal{S}}_x$.

From now on, let $n, k$ be non-negative integers such that $n - k \geq k$. If $x$ is of Jordan type $(n - k, k)$, we write

$$\mathcal{B}_{n-k,k} := \mathcal{B}_x, \quad \mathcal{S}_{n-k,k} := \mathcal{S}_x, \quad \widetilde{\mathcal{S}}_{n-k,k} := \widetilde{\mathcal{S}}_x.$$ (2.6)

2.2. Irreducible components of Springer fibers of two-rows. A cup diagram is a non-intersecting arrangement of cups and rays below a horizontal axis connecting a subset of $n$ vertices on the axis, and we identify any cup diagram $a$ with the collection of sets of endpoints of cups as below:

$$a \equiv \{(i_t, j_t) \subseteq \{1, \ldots, n\} \mid 1 \leq t \leq k \} \text{ for some } k \leq \left\lfloor \frac{n}{2} \right\rfloor.$$ (2.7)

We use a rectangular region (i.e., the dashed border in a diagram as below) that contains all the cups to represent the cup diagram. Note that a ray is now a through-strand in this presentation but we still call it a ray.

The irreducible components of the Springer fiber $\mathcal{B}_{n-k,k}$ can be labeled by the set $\mathcal{B}_{n-k,k}$ of all cup diagram on $n$ vertices with $k$ cups and $n - 2k$ rays. For example, when $n = 3$ we have

$$\mathcal{B}_{3,0} = \begin{cases} \{1, 2, 3\} \end{cases}, \quad \mathcal{B}_{2,1} = \begin{cases} \{1, 2, 3\}, \{1, 2, 3\} \end{cases}.$$ (2.8)

We denote by $K^a$ the irreducible component in $\mathcal{B}_{n-k,k}$ associated to the cup diagram $a \in \mathcal{B}_{n-k,k}$.1

1Algebraic Geometry, Algebra, and Number Theory.
**Proposition 2.1.** Let $x \in \mathcal{N}$ with Jordan type $(n-k,k)$. We fix a basis $\{e_i, f_j \mid 1 \leq i \leq n-k, 1 \leq j \leq k\}$ of $\mathbb{C}^n$ such that on which $x$ acts by

$$f_k \mapsto f_{k-1} \mapsto \cdots \mapsto f_1 \mapsto 0, \quad e_{n-k} \mapsto e_{n-k-1} \mapsto \cdots \mapsto e_1 \mapsto 0.$$ 

(a) There exists a bijection between the irreducible components of the Springer fiber $B_{n-k,k}$ and the set $B_{n-k,k}$ of cup diagrams on $n$ vertices with $k$ cups.

(b) The irreducible component $K^a \subseteq B_{n-k,k}$ consists of the pairs $(x, F_*) \in B_{n-k,k}$ which satisfy the following conditions imposed by the cup diagram $a$:

(i) For $i < j$, $\{i,j\}$ is a cup in $a$ if and only if

$$F_j = x^{-\frac{1}{2}(j-i+1)}F_{i-1},$$

where $x^{-1}$ denotes the preimage of a space under the endomorphism $x$.

(ii) If vertex $i$ is connected to a ray, let $\rho(i) \in \mathbb{Z}_{>0}$ be the number of rays to the left of $i$ (including itself). Then $F_i = F_{i-1} \oplus \langle e_{\frac{1}{2}(i+\rho(i))} \rangle$, or equivalently,

$$F_i = x^{-\frac{1}{2}(i-i\rho(i))}(x^{n-k-i\rho(i)}F_n).$$

**Proof.** See [Spa76] and [Var79] for part (a). For part (b), we refer to [SW12 Proposition 7] (see also [Fuc03 Theorem 5.2]).

**Remark 2.2.** The converse statement for 2.1 (ii)(b) is not true, i.e., if $F_i = F_{i-1} \oplus \langle e_{\frac{1}{2}(i+\rho(i))} \rangle$ then $i$ can still be connected by a cup.

2.3. **Quiver representations.** Now we follow [Maf05] to realize Nakajima’s quiver variety as equivalence classes of semistable orbits of a certain quiver representation space, which is equivalent to the usual proj construction in GIT. Let $S(d,v)$ be the quiver representation space of type $A$ (see [Maf05 Defn. 5]) with respect to dimension vectors $d = (\dim D_i), v = (\dim V_i)$. In this article we do not need the most general $S(d,v)$, and hence we will be focusing on certain special cases which we will elaborate below.

For $k < n-k$, let $S_{n-k,k}$ be the quiver representation space in Figure 1.

**Figure 1.** Quiver representations in $S_{n-k,k}$ and dimension vectors.

| $D_i$ | 0 | 0 | ... | 1 | 0 | ... | 1 | ... | 0 | 0 |
|-------|---|---|-----|---|---|-----|---|-----|---|---|
| $V_i$ | 1 | 2 | ... | $k$ | $k$ | ... | $k$ | ... | 2 | 1 |

**Remark 2.3.** Throughout this article, any space with an ineligible subscript is understood as a zero space (e.g., $V_0 = \{0\}, V_n = \{0\}$). Any linear map with an ineligible subscript is understood as a zero map (e.g., $A_{n-1} : V_{n-1} \rightarrow \{0\}, B_0 : V_1 \rightarrow \{0\}$).

In other words, $S_{n-k,k}$ can be identified as the space of quadruples $(A, B, \Gamma, \Delta)$ of collections of linear maps of the following form:

$$(A = (A_i : V_i \rightarrow V_{i+1})_{i=1}^{n-2}, \quad B = (B_i : V_{i+1} \rightarrow V_i)_{i=1}^{n-2}, \quad \Gamma = (\Gamma_i : D_i \rightarrow V_i)_{i=1}^{n-1}, \quad \Delta = (\Delta_i : V_i \rightarrow D_i)_{i=1}^{n-1})$$.
with dimension vectors $d = (\dim D_i)_i$ and $v = (\dim V_i)_i$ given by

$$
dim V_i = \begin{cases} 
  i & \text{if } i \leq k, \\
  k & \text{if } k \leq i \leq n - k, \\
  n - i & \text{if } i \geq n - k,
\end{cases} \quad \dim D_i = \begin{cases} 
  1 & \text{if } i = k, n - k, \\
  0 & \text{otherwise}.
\end{cases}
$$

For the special case $S_{k,k}$ when $n = 2k$, we use instead the quiver representations of the form in Figure 2 below.

**Figure 2.** Quiver representations in $S_{k,k}$ and the dimension vectors.

$$
dim D_i \begin{array}{cccccccc}
 0 & 0 & \ldots & 0 & 2 & 0 & \ldots & 0 & 0 \\
\end{array}
$$

$$
\begin{array}{ccccccccc}
V_1 & \overset{A_1}{\rightarrow} & V_2 & \overset{A_2}{\rightarrow} & \cdots & V_{k-1} & \overset{A_k}{\rightarrow} & V_k & \overset{\Gamma_1}{\rightarrow} \cdots \\
B_1 & & B_2 & & \cdots & B_k & & B_{k+1} & & \cdots \\
\end{array}
$$

$$
dim V_i \begin{array}{cccccccc}
 1 & 2 & \ldots & k - 1 & k & k - 1 & \ldots & 2 & 1 \\
\end{array}
$$

Following Nakajima, an element $(A, B, \Gamma, \Delta) \in S(d, v)$ is called admissible if the ADHM equations are satisfied. Equivalently, for all $1 \leq i \leq n - 1$,

$$
B_i A_i = A_{i-1} B_{i-1} + \Gamma_i \Delta_i. \quad (2.9)
$$

An admissible element is called stable if, for each collection $U = (U_i \subseteq V_i)_i$ of subspaces satisfying that

$$
\text{Im } \Gamma_i \subseteq U_i, \quad A_i(U_i) \subseteq U_{i+1}, \quad B_i(U_{i+1}) \subseteq U_i \quad \text{for all } i, \quad (2.10)
$$

it follows that $U_i = V_i$ for all $i$. We will use the following equivalent notion of stability due to Maffei:

**Lemma 2.4** ([Ma05, Lemmas 14, 2]). An admissible element $(A, B, \Gamma, \Delta) \in S(d, v)$ is stable if and only if, for all $1 \leq i \leq n - 1$,

$$
\text{Im } A_{i-1} + \sum_{j \geq i} \text{Im } \Gamma_{j \rightarrow i} = V_i, \quad (2.11)
$$

where it is understood that $A_0 = 0$, and that $\Gamma_{j \rightarrow i}$, for $j \geq i$, is the natural composition from $D_j$ to $V_i$, i.e.,

$$
\Gamma_{j \rightarrow i} = B_i \cdots B_{j-1} \Gamma_j. \quad (2.12)
$$

Denote the subspace (which we call the stable locus) of $S(d, v)$ consisting of elements that are admissible (i.e., the ADHM equations (2.9) are satisfied) and stable by

$$
\Lambda^+(d, v) = \{(A, B, \Gamma, \Delta) \in S(d, v) \mid (2.9), (2.11)\}. \quad (2.13)
$$

We denote by $\Lambda_{n-k,k}$ the set of admissible representations in $S_{n-k,k}$, and $\Lambda^+_{n-k,k}$ as its stable locus.

### 2.4. Nakajima quiver varieties

Let $V = \prod_i V_i$, $D = \prod_i D_i$. Now we define on any quiver representation $(A, B, \Gamma, \Delta)$ an action of $\text{GL}(V) = \prod_i \text{GL}(V_i)$ by

$$
g \cdot (A, B, \Gamma, \Delta) = ((g_{i+1} A_i g_i^{-1}), (g_i B_i g_{i+1}^{-1}), (g_i \Gamma_i), (\Delta_i g_i^{-1})), \quad g = (g_i)_i \in \text{GL}(V). \quad (2.14)
$$

We denote the Nakajima quiver variety as stable $\text{GL}(V)$-orbits on $S(d, v)$ satisfying pre-projective conditions, i.e.,

$$
M(d, v) := \Lambda^+(d, v)/\text{GL}(V). \quad (2.15)
$$
Denote also by $M_{n-k,k}$ the Nakajima quiver variety for $S_{n-k,k}$. We also call the projection onto the moduli space of the $GL(V)$-orbits by

$$p_{d,v} : \Lambda^+(d,v) \to M(d,v).$$

(2.16)

Denote also by $p_{n-k,k}$ for the projection onto $M_{n-k,k}$.

It is first proved in [Nak94, Thm. 7.2] that there is an explicit isomorphism $M(d,v) \to \tilde{S}_2$ for certain $d,v,x$ using a different stability condition. Here we recall a variant due to Maffei that suits our need.

**Proposition 2.5** ([Maf05, Lemma 15]). If $\dim D_i = 0$ for all $i$ unless $i = 1$, then the assignment below defines an isomorphism $\tilde{\varphi} = \tilde{\varphi}(d,v) : M(d,v) \simeq \tilde{S}_2$:

$$p_{d,v}(A,B,\Gamma,\Delta) \mapsto (\Delta_1 \Gamma_1, (0 \subset \ker \Gamma_1 \subset \ker \Gamma_{1\to 2} \subset \ldots \subset \ker \Gamma_{1\to n})),$$

(2.17)

where $x = \Delta_1 \Gamma_1$.

In general, Proposition 2.5 does not apply to all $M_{n-k,k}$ for $n \geq 3$. Our next step is to describe an explicit isomorphism $\tilde{\varphi}_{n-k,k} : M_{n-k,k} \simeq \tilde{S}_{n-k,k}$ due to Maffei in Proposition 2.8.

2.5. **Maffei’s isomorphism.** Following [Maf05], we utilize a modified quiver representation space $\tilde{S}_{n-k,k}$ as in Figure 3 below, for each $S_{n-k,k}$:

**Figure 3. Modified quiver representations in $\tilde{S}_{n-k,k}$.**

Here the vector spaces $(\tilde{D}_i, \tilde{V}_i)$ are given by

$$\tilde{D}_1 = D'_0, \quad \tilde{V}_i = V_i \oplus D'_i,$$

(2.18)

where

$$D'_i = \begin{cases} 
\{e_1, \ldots, e_{n-k-i}, f_1, \ldots, f_{k-i}\} & \text{if } i \leq k - 1, \\
\{e_1, \ldots, e_{n-k-i}\} & \text{if } k \leq i \leq n - k - 1, \\
\{0\} & \text{if } n - k \leq i \leq -1.
\end{cases}$$

(2.19)

Note that we utilize the following identification with the spaces $D^{(h)}_j$ in [Maf05]:

$$\langle e_i \rangle \equiv D^{(i)}_{n-k}, \quad \langle f_i \rangle \equiv D^{(i)}_k \quad \text{if } n > 2k,$$

$$\langle e_i, f_i \rangle \equiv D^{(i)}_k \quad \text{if } n = 2k.$$

(2.20)

Denote by $\tilde{d} = (\dim \tilde{D}_i), \tilde{v} = (\dim \tilde{V}_i)$ the dimension vectors. The advantage of manipulating over such modified quivers is that Proposition 2.5 applies, and hence it produces an isomorphism between the Nakajima quiver variety $M(d, \tilde{v})$ and the Slodowy variety $\tilde{S}(\tilde{d}, \tilde{v})$ for the dimension vectors $\tilde{d}$ and $\tilde{v}$.
Now we identify the linear maps $\tilde{A}_i, \tilde{B}_i, \tilde{\Gamma}_i, \tilde{\Delta}_i$ as block matrices in light of [Ma05] (9). For example, we have

$$\tilde{\Gamma}_1 = \begin{pmatrix} V_1 \left( f_b & \cdots & e_b \\ T_{1,0} & \cdots & T_{1,e} \\ \vdots & \vdots & \vdots \\ e_a & T_{1,0} & \cdots & T_{1,e} \end{pmatrix} \right) \tilde{\Delta}_1 = \begin{pmatrix} V_1 \left( f_b & \cdots & e_b \\ g_a & \cdots & g_e \\ \vdots & \vdots & \vdots \\ e_a & \cdots & e_e \end{pmatrix} \right)$$

\[(2.21)\]

with respect to the basis vectors indicated above and to the left of each matrix. In other words, the variables $A, B, S, T$ are certain linear maps with domains and codomains specified as below, for $\phi, \psi \in \{e, f\}$:

$$A_i : V_i \rightarrow V_{i+1}, \quad B_i : V_i \rightarrow V_{i+1},$$

$$S_{i,\phi,a} : V_{i+1} \rightarrow \langle \phi \rangle, \quad S_{i,\psi,a} : \langle \psi \rangle \rightarrow \langle \phi \rangle, \quad T_{i,\phi,a} : V_i \rightarrow \langle \phi \rangle, \quad T_{i,\psi,a} : \langle \psi \rangle \rightarrow \langle \phi \rangle.$$ \[(2.23)\]

A definition of the transversal element can be found in [Ma05] Defn. 16. In our context it is convenient to rewrite the definition as below: let $\pi_{D_i'}$ be the projection onto $D_i'$ (recall (2.19)), let $\bar{A} = \tilde{\Gamma}_1, \bar{B} = \tilde{\Delta}_1$, and let $(x_i, y_i, [x_i, y_i])$ be the fixed $\mathfrak{sl}_2$-triple on $\mathfrak{sl}(D_i')$ uniquely determined by

$$x_i(e_h) = \begin{cases} e_{h-1} & \text{if } 1 < h \leq n - k - i, \\
0 & \text{otherwise,} \end{cases} \quad y_i(e_h) = \begin{cases} h(n - k - i - h)e_{h+1} & \text{if } 1 \leq h < n - k - i, \\
0 & \text{otherwise,} \end{cases}$$

\[(2.24)\]

An admissible quadruple $(\bar{A}, \bar{B}, \bar{\Gamma}, \bar{\Delta})$ in $\bar{S}_{n-k}$ is called transversal if the following conditions hold, for $0 \leq i \leq n - 2$:

$$\pi_{D_i'} \bar{B}_i \bar{A}_i = x_i, y_i,$$ \[(2.25)\]

\[(2.26)\]

Denote the subspace in $\bar{S}_{n-k}$ consisting of transversal (hence admissible) and stable elements by

$$\bar{S}_+^{n-k} = \{(\bar{A}, \bar{B}, \bar{\Gamma}, \bar{\Delta}) \in \bar{S}_{n-k} | \quad (2.25), (2.26), (2.28), (2.29) \},$$ \[(2.27)\]

where the relations other than the transversal ones are

$$(\text{admissibility}) \quad \bar{B}_i \bar{A}_i = \bar{A}_{i-1} \bar{B}_{i-1} + \bar{\Gamma}_i \bar{\Delta}_i \quad \text{for } 1 \leq i \leq n - 1,$$ \[(2.28)\]
(stability) \[ \text{Im} \, \tilde{A}_{i-1} + \sum_{j \geq i} \text{Im} \, \tilde{\Gamma}_{j-i} = \tilde{V}_i \quad \text{for} \ 1 \leq i \leq n-1, \] (2.29)

where \( \tilde{\Gamma}_{j-i} \) is defined similarly as (2.12).

**Remark 2.6.** The system of equations (2.26) is not the easiest to work with. For example, it implies that the map \( \tilde{\Gamma}_1 \) must be of the following form:

\[
\tilde{\Gamma}_1 = \begin{pmatrix}
e_1 & e_2 & \ldots & \ldots & \ldots & e_{n-k} & f_1 & \ldots & \ldots & f_k \\
T_{0,1} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,1} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,e,1} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,e,1} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,e,n-k} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,e,n-k} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,f,1} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,f,1} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,f,n-k} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,f,n-k} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,f,n-k} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,f,n-k} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,f,n-k} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,f,n-k} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,f,n-k} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,f,n-k} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,f,n-k} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,f,n-k} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,f,n-k} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,f,n-k} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,f,n-k} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,f,n-k} & \ldots & \ldots & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
T_{0,f,n-k} & 0 & \ldots & \ldots & \ldots & 0 & T_{0,f,n-k} & \ldots & \ldots & 0 \end{pmatrix} \quad (2.30)
\]

One can then solve for all the unknown variables \( T_{0,e,1} \) using stability and admissibility conditions, together with the first transversality condition (2.25). In general the solutions are very involved (see [Ma05], Lemma 18). We will use the proposition below to show that the solutions are actually very simple in our setup.

**Proposition 2.7.** Let \( (A, B, \Gamma, \Delta) \in \Lambda_{n-k,k} \).

(a) There is a unique element \( (\tilde{A}, \tilde{B}, \tilde{\Gamma}, \tilde{\Delta}) \) \( \in \tilde{\Sigma}_{n-k,k} \) such that
\[
\tilde{A}_i = A_i, \quad \tilde{B}_i = B_i, \quad \tilde{\Gamma}_k = T_{k-1,V}^{f_{1}}, \quad \tilde{\Gamma}_{n-k} = T_{n-k-1,V}^{e_{1}}, \quad \tilde{\Delta}_k = S_{k-1,f,1}^{V}, \quad \tilde{\Delta}_{n-k} = S_{n-k-1,e,1}^{V}. \quad (2.31)
\]

(b) The assignment in part (a) restricts to a \( GL(V) \)-equivariant isomorphism \( \Phi : \Lambda_{n-k,k} \rightarrow \tilde{\Sigma}_{n-k,k} \).

(c) The map \( \Phi \) induces an isomorphism \( \Phi_M : M_{n-k,k} \cong p_{n-k,k}(\tilde{\Sigma}_{n-k,k}) \).

**Proof.** This is a special case of [Ma05], Lemmas 18, 19. \( \square \)

Now we are in a position to define Maffei’s isomorphism \( \tilde{\varphi} : M_{n-k,k} \rightarrow \tilde{S}_{n-k,k} \). Recall \( \Lambda^+(-,-) \) from (2.13), \( p_{d,v} \) from (2.16), \( M(-,-) \) from (2.15), \( \tilde{\varphi}(\tilde{d}, \tilde{v}) \) from Proposition 2.5, \( \Sigma_{n-k,k}^+ \) from (2.27). Finally, \( \tilde{\varphi}_{n-k,k} \) is defined such that the lower right corner of the diagram below commute:

\[
\begin{align*}
\Lambda^+(-,-) & \xrightarrow{\tilde{\varphi}} M(-,-) \xrightarrow{\tilde{\varphi}} \tilde{S}(-,-) \\
\tilde{\varphi}_{n-k,k} & \xrightarrow{\sim} p_{n-k,k}(\Sigma_{n-k,k}^+) \xrightarrow{\sim} \tilde{\varphi} \circ p(\Sigma_{n-k,k}^+) \\
\Phi & \xrightarrow{\sim} \Phi_M \xrightarrow{\sim} \tilde{\varphi}_{n-k,k} \xrightarrow{\sim} \tilde{S}_{n-k,k}
\end{align*}
\quad (2.32)
\]

**Proposition 2.8.** There map \( \tilde{\varphi}_{n-k,k} : M_{n-k,k} \rightarrow \tilde{S}_{n-k,k} \) defined above is an isomorphism of algebraic varieties.

**Proof.** This is a special case of [Ma05], Thm. 8 by setting \( N = n, r = (1, \ldots, 1) \), and \( x \in \mathcal{N} \) of Jordan type \( (n - k, k) \). \( \square \)
3. Components of Springer fibers of type A

For each cup \( a \in B_{n-k,k} \), our strategy to single out the irreducible component \( K^a \subset B_x \subset \mathcal{S}_{n-k,k} \) requires the following ingredients:

1. construction of a subset \( \mathcal{T}^a \subset \mathcal{T}_{n-k,k}^+ \) such that \( \bar{p}(\mathcal{T}^a) \simeq K^a \).
2. construction of a subset \( \Lambda^a \subset \Lambda_{n-k,k}^+ \) so that \( \Phi(\Lambda^a) \simeq \mathcal{T}^a \), which implies \( \Phi_M(p(\Lambda^a)) \simeq \bar{p}(\mathcal{T}^a) \simeq K^a \).

In light of (2.32), we have

\[
\begin{align*}
\Lambda^a \xrightarrow[p]{} \bar{p}(\mathcal{T}^a) & \xrightarrow[\Phi_M]{} K^a.
\end{align*}
\]

Recall that a cup diagram is uniquely determined by the configuration of its cups; that is, once the placement of the cups has been decided, rays emanate from the rest of the nodes. Hence, for our construction of \( \mathcal{T}^a \) and \( \Lambda^a \), we use only the information about the cups. For completeness we also give a characterization for the ray relation on the quiver representation side.

3.1. Irreducible components via quiver representations. Given a cup diagram \( a = \{\{i_t, j_t\}\}_{t \in B_{n-k,k}} \), we assume that \( i_t < j_t \) for all \( t \) and then denote the set of all vertices connected to the left (resp. right) endpoint of a cup in \( a \) by

\[
V^a_l = \{i_t \mid 1 \leq t \leq k\}, \quad V^a_r = \{j_t \mid 1 \leq t \leq k\}.
\]

Define the endpoint-swapping map by

\[
\sigma : V^a_l \to V^a_r, \quad i_t \mapsto j_t.
\]

Given \( i \in V^a_l \), denote the “size” of the cup \( \{i, \sigma(i)\} \) by

\[
\delta(i) = \frac{1}{2}(\sigma(i) - i + 1).
\]

For instance, a minimal cup connecting neighboring vertices has size \( \delta(i) = \frac{(i+1) - i + 1}{2} = 1 \), and a cup containing a single minimal cup nested inside has size 2.

For short we set, for \( 0 \leq p < q \leq n \),

\[
\bar{B}_{q-p} = \bar{B}_p \bar{B}_{p+1} \cdots \bar{B}_{q-1} : \tilde{V}_q \to \tilde{V}_p, \quad \bar{A}_{p-q} = \bar{A}_{q-1} \bar{A}_{q-2} \cdots \bar{A}_p : \tilde{V}_p \to \tilde{V}_q.
\]

Now we define

\[
\mathcal{T}^a = \{(\bar{A}, \bar{B}, \bar{\Gamma}, \bar{\Delta}) \in \mathcal{T}_{n-k,k}^+ \mid \ker \bar{B}_{i+\delta(i)-1 \to i-1} = \ker \bar{A}_{\sigma(i)-\delta(i) \to \sigma(i)} \text{ for all } i \in V^a_l\}.
\]

The kernel condition in (3.6) can be visualized in Figure 4 below:

**Figure 4.** The paths in the kernel condition of (3.6).
Note that for a minimal cup connecting neighboring vertices $i$ and $i + 1$ the relations in (3.6) take the simple form

\[
\ker \Delta_1 = \ker A_1 \quad \text{if } i = 1;
\]
\[
\ker B_{i-1} = \ker A_i \quad \text{if } 2 \leq i \leq n - 2;
\]
\[
\ker B_{n-2} = \tilde{V}_{n-1} = C \quad \text{if } i = n - 1.
\] (3.7)

**Proposition 3.1.** For $a \in B_{n-k,k}$, we have an equality $\tilde{p}(\Sigma^a) = \tilde{\varphi}^{-1}(K^a)$.

**Proof.** Thanks to Proposition [2.5] it suffices to show that, for any $i \in V_i^a$ and $(A, B, \Gamma, \Delta) \in \Sigma^a$, the kernel condition

\[
\ker \tilde{B}_{i+\delta(i)-1 \to i-1} = \ker \tilde{A}_{\sigma(i)-\delta(i) \to \sigma(i)}
\] (3.8)
is equivalent to the Fung/Stroppel–Webster cup relation (see Proposition [2.1] (b)(i))

\[
(\Delta_1 \tilde{\Gamma}_1)^{-\delta(i)} \ker \tilde{\Gamma}_1 \to i-1 = \ker \tilde{\Gamma}_1 \to \sigma(i).
\] (3.9)

Note that the left-hand side of (3.9) can be rewritten as follows:

\[
(\Delta_1 \tilde{\Gamma}_1)^{-\delta(i)} (\ker \tilde{\Gamma}_1 \to i-1) = (\ker \tilde{\Gamma}_1 \to i-1) (\Delta_1 \tilde{\Gamma}_1)^{-\delta(i)}
\]
\[
= (\ker \tilde{\Gamma}_1 \to i-1) (\tilde{A}_1 \tilde{A}_1)^{-\delta(i)} \tilde{\Gamma}_1
\]
\[
= (\ker \tilde{B}_{i+\delta(i)-1 \to i-1} \tilde{A}_1 \tilde{A}_1)^{-\delta(i)},
\]
where the second equality follows from applying $\delta(i)$ times the admissibility condition $\tilde{B}_1 \tilde{A}_1 = \tilde{\Gamma}_1 \tilde{\Delta}_1$; while the third equality follows from applying the admissibility condition $\tilde{A}_1 \tilde{B}_t \tilde{A}_t = \tilde{B}_t \tilde{A}_t$ repeatedly from $t = 2$ to $t = i + \delta(i) - 1 = \sigma(i) - \delta(i)$. Therefore, the cup relation (3.9) is equivalent to another kernel condition below

\[
\ker \tilde{B}_{i+\delta(i)-1 \to i-1} \tilde{A}_1 \tilde{A}_1 \to \sigma(i)
\] (3.10)

In particular, the kernels are equal for the two maps in Figure 5 given by dashed and solid arrows, respectively:

**Figure 5.** The kernel condition that is equivalent to (3.9).

\[
\begin{array}{c}
\tilde{D}_{i-1} \\
\tilde{\Gamma}_1 \\
\tilde{V}_1 \xrightarrow{\tilde{A}_1} \tilde{V}_2 \xrightarrow{\tilde{A}_2} \cdots \xrightarrow{\tilde{A}_{\sigma(i)-\delta(i)}} \tilde{V}_{\sigma(i)-\delta(i)} \xrightarrow{\tilde{A}_{\sigma(i)-\delta(i)-1}} \cdots \xrightarrow{\tilde{A}_{\sigma(i)}} \tilde{V}_{\sigma(i)}
\end{array}
\]

\[
\tilde{V}_{i-1} \xleftrightarrow{\tilde{B}_i} \tilde{V}_{i+\delta(i)-2} \xleftrightarrow{\tilde{B}_i} \cdots \xleftrightarrow{\tilde{B}_i} \tilde{V}_{i+\delta(i)-1}
\]

Note that (3.8) evidently implies (3.10). By the stability conditions on $\tilde{V}_1, \ldots, \tilde{V}_{i+\delta(i)-1}$ we see that the maps $\tilde{\Gamma}_1, \tilde{A}_1, \ldots, \tilde{A}_{i+\delta(i)-2}$ are all surjective, and so is its composition $\tilde{\Gamma}_1 \to i+\delta(i)-1$. Thus, (3.10) implies (3.8), and we are done. □
3.2. Springer fibers via quiver representations. For $1 \leq i \leq n-1$, we define an isomorphic copy of $D'_i$ (see (3.19)) with a shift of index by $t$ as

$$D'_i[t] = \langle f_{i+t} \mid f_i \in D'_i \rangle \oplus \langle e_{i+t} \mid e_i \in D'_i \rangle.$$  \hspace{1cm} (3.11)

By a slight abuse of notation, we denote by $\Delta_{\to i}$ the assignment given by

$$e_a \mapsto \Gamma_{n-k\to i} (e) \in V_i, \quad f_b \mapsto \Gamma_{k\to i} (f) \in V_i.$$  \hspace{1cm} (3.12)

We define the obvious composition by

$$\Delta_{j\to i} = \begin{cases} \Delta_i B_i \cdots B_{j-1} & \text{if } j \geq i, \\ \Delta_i A_i \cdots A_j & \text{if } j \leq i, \end{cases}$$  \hspace{1cm} (3.13)

and then define $\Delta_{i\to j}$ similarly.

Lemma 3.2. If $(\tilde{A}, \tilde{B}, \tilde{\Gamma}, \tilde{\Delta}) \in \mathcal{T}^+_{n-k,k}$ and $\Phi^{-1}(\tilde{A}, \tilde{B}, \tilde{\Gamma}, \tilde{\Delta}) = (A, B, \Gamma, \Delta) \in \mathcal{A}^+_{n-k,k}$, then $(\tilde{A}, \tilde{B}, \tilde{\Gamma}, \tilde{\Delta})$ must be of the form, for $1 \leq i \leq n-2$:

$$\tilde{\Gamma}_i = \begin{array}{c} V_i \\ D'_i \end{array} \left( \begin{array}{cc} D'_0 \setminus D'_1[1] & D'_1[1] \\ 0 & I_{t-2} \end{array} \right), \quad \tilde{\Delta}_i = \begin{array}{c} V_i \\ D'_i \setminus D'_1[1] \end{array} \left( \begin{array}{cc} 0 & D'_1[1] \\ \Delta_{i\to i} & 0 \end{array} \right).$$  \hspace{1cm} (3.14)

Moreover, the following equation hold, for $2 \leq i \leq n-1$,

$$\Delta_{i \to \Gamma_{n-k,k}} = 0.$$  \hspace{1cm} (3.16)

Proof. By Proposition 2.7 we know that $(\tilde{A}, \tilde{B}, \tilde{\Gamma}, \tilde{\Delta})$ must be the unique element in $\mathcal{T}^+_{n-k,k}$ that satisfies (2.31), which indeed are satisfied from the construction. What remains to show is that the formulas above do define an element in $\mathcal{T}^+_{n-k,k}$.

From construction, we have

$$\tilde{\Gamma}_i \tilde{\Delta}_i = \begin{array}{c} V_i \\ D'_i \setminus D'_2[1] \end{array} \left( \begin{array}{cc} 0 & D'_2[1] \\ 0 & I \end{array} \right), \quad \tilde{\Delta}_i \tilde{\Gamma}_i = \begin{array}{c} V_i \\ D'_i \setminus D'_1[1] \end{array} \left( \begin{array}{cc} 0 & D'_1[1] \\ \Delta_{i \to \Gamma_{n-k,k}} & 0 \end{array} \right).$$  \hspace{1cm} (3.17)

$$\tilde{A}_i \tilde{B}_i = \begin{array}{c} V_i \\ D'_{i+1} \setminus D'_{i+2}[1] \end{array} \left( \begin{array}{cc} A_i B_i & D'_{i+2}[1] \\ 0 & I \end{array} \right).$$  \hspace{1cm} (3.18)

$$\tilde{B}_i \tilde{A}_i = \begin{array}{c} V_i \\ D'_i \setminus D'_{i+1}[1] \end{array} \left( \begin{array}{cc} B_i A_i & D'_{i+1}[1] \\ 0 & I \end{array} \right).$$  \hspace{1cm} (3.19)

where $I$ represents the identity map of appropriate rank. The admissibility conditions (2.28) follow from comparing the entries in (3.18) – (3.19) together with the original admissibility condition (2.11) and (3.16).

By the original stability condition (2.29), the blocks in the first row for $\tilde{A}_i$ has full rank, and hence (2.29) follows.
From (3.19) we see that
\[ \pi D_i \widetilde{B}_i A_i \bigg|_{D_i'} - x_i = D_{i+1}' \begin{pmatrix} D_{i+1}' \setminus D_{i+1}'[1] & D_{i+1}'[1] \\ \Delta_{i+1} \rightarrow \Gamma \rightarrow i+1 & 0 \end{pmatrix}, \tag{3.20} \]
and thus (2.25) holds. Finally, a straightforward verification such as (2.30) shows that (2.26) are satisfied.

We can now combine Lemma 3.2 and the Nakajima-Maffei isomorphism to describe the complete flag assigned to each quiver representation.

**Corollary 3.3.** If \((x, F_*) = \varphi_{n-k,k}(p_{n-k,k}(A, B, \Gamma, \Delta))\) for some \((A, B, \Gamma, \Delta) \in \Lambda^+_{n-k,k}\), then
\[ F_i = \ker V_i \left( A_{i \rightarrow 0} \Gamma_{\rightarrow 1} \cdots A_{i \rightarrow t} \Gamma_{\rightarrow t} \cdots \Gamma_{\rightarrow i} \right), \]
where \(D_{t-1}'\), for \(1 \leq t \leq i\), is the space (depending on \(i\)) described below:
\[ D_{t-1}' = D_{t-1}'[t-1] - D_{t}'[t] = \begin{cases} \langle e_t, f_t \rangle & \text{if } t \leq k, \\
\{0\} & \text{if } k + 1 \leq t \leq n - k, \\
\{0\} & \text{otherwise.} \end{cases} \tag{3.21} \]

**Proof.** By Proposition 2.4 we know that the spaces \(F_i\) are determined by the kernels of the maps \(\widetilde{\Gamma}_{1 \rightarrow i}\). The assertion follows from a direct computation of \(\widetilde{\Gamma}_{1 \rightarrow i}\) using Lemma 3.2 which is,
\[ \widetilde{\Gamma}_{1 \rightarrow i} = V_i \begin{pmatrix} D_0'' & \cdots & D_{t-1}'' & \cdots & D_{t-1}'' & D_{t-1}'[i] \\ A_{1 \rightarrow 0} \Gamma_{\rightarrow 1} & \cdots & A_{i \rightarrow 0} \Gamma_{\rightarrow t} & \cdots & \Gamma_{\rightarrow i} & 0 \\ 0 & \cdots & 0 & \cdots & I \end{pmatrix}, \tag{3.22} \]

**Example 3.4.** The quiver representations in \(S_{2,2}\) are described as below:
\[ D_2 = \langle e, f \rangle \]
\[ \Gamma_2 \begin{cases} A_1 \rightarrow B_1 \\ A_2 \rightarrow B_2 \end{cases} \]
\[ V_1 = \mathbb{C} \quad V_2 = \mathbb{C}^2 \quad V_3 = \mathbb{C}. \]

Let \(\varphi_{2,2}^{-1}(x, F_*) = p_{2,2}(A, B, \Gamma, \Delta) \in M_{2,2}\). By Corollary 3.3 the isotropic flag \(F_*\) is described by,
\[ F_1 = \ker V_1 \begin{pmatrix} \langle f_1, e_1 \rangle \\ B_1 \Gamma_2 \end{pmatrix}, \]
\[ F_2 = \ker V_2 \begin{pmatrix} \langle f_1, e_1 \rangle & \langle f_2, e_2 \rangle \\ A_1 B_1 \Gamma_2 & \Gamma_2 \end{pmatrix}, \]
\[ F_3 = \ker V_3 \begin{pmatrix} \langle f_1, e_1 \rangle & \langle f_2, e_2 \rangle \\ A_2 A_1 B_1 \Gamma_2 & A_2 \Gamma_2 \end{pmatrix}. \]

In particular, if \(A_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = B_2, A_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} = B_1, \Gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \Delta = 0, \) then
\[ F_1 = \ker \begin{pmatrix} 0 & 1 \end{pmatrix} = \langle f_1 \rangle, \quad F_2 = \ker \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \langle f_1, e_1 - f_2 \rangle, \quad F_3 = \ker \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \langle f_1, e_1, f_2 \rangle. \]
The quiver representations in $S_{3,1}$ are described as below:

$$D_1 = \langle f \rangle, \quad D_3 = \langle e \rangle$$

$$\Delta_1 \left\langle \begin{array}{c} \Gamma_1 \\ A_1 \\ B_1 \end{array} \right\rangle \quad \Delta_3 \left\langle \begin{array}{c} \Gamma_3 \\ A_2 \\ B_2 \end{array} \right\rangle$$

$$V_1 = C \xrightarrow{A_1} V_2 = C \xrightarrow{A_2} V_3 = C$$

If $\tilde{\varphi}^{-1}(x, F\ast) = p_{3,1}(A, B, \Gamma, \Delta) \in M_{3,1}$, then the flag $F\ast$ can be described by

$$F_1 = \ker V_1 (\Gamma_1 B_1 B_2 \Gamma_3),$$

$$F_2 = \ker V_2 (A_1 \Gamma_1 A_1 B_1 B_2 \Gamma_3 B_2 \Gamma_3),$$

$$F_3 = \ker V_3 (A_2 A_1 \Gamma_1 A_2 A_1 B_1 B_2 \Gamma_3 A_2 B_2 \Gamma_3 \Gamma_3).$$

In particular, if $A_1 = 1, A_2 = 0, B_1 = 0, B_2 = 1, \Gamma_1 = 1, \Gamma_3 = 1, \Delta_1 = 0 = \Delta_3$, then

$$F_1 = \ker (1 \ 0) = \langle e_1 \rangle, \quad F_2 = \ker (1 \ 0 \ 1) = \langle e_1, f_1 - e_2 \rangle, \quad F_3 = \ker (0 \ 0 \ 0 \ 1) = \langle e_1, f_1, e_2 \rangle.$$

**Remark 3.5.** The previous example demonstrates that Corollary 3.3 provides an efficient way to compute the corresponding complete flag in the Slodowy variety, while generally it is very implicit to apply Maffei’s isomorphism $\tilde{\varphi}$. It can also be seen that $\ker \tilde{\Gamma}_{1 \to i}$ is not necessary a direct sum of the kernels of the blocks.

Define $\Lambda_{n-k,k}^{B}$ to be the subset of $\Lambda_{n-k,k}^{+}$ so that

$$p_{n-k,k}(\Lambda_{n-k,k}^{B}) = \tilde{\varphi}^{-1}(B_{n-k,k}).$$

In other words, $\Lambda_{n-k,k}^{B}$ is the incarnation of the Springer fiber we wish to study via the corresponding Nakajima quiver variety, which is characterized first by Lusztig [Lus91 Prop. 14.2(a)], [Lus98 Lemma 2.22]. Below we demonstrate an elementary proof in the two-row case using Lemma 3.2, which is stated without proof in [Mal05 Rmk. 24].

**Corollary 3.6.** If $x \in \mathcal{N}$ has Jordan type $(n-k, k)$, then

$$\Lambda_{n-k,k}^{B} = \{(A, B, \Gamma, \Delta) \in \Lambda_{n-k,k}^{+} \mid \Delta = 0\}.$$

**Proof.** If $(A, B, \Gamma, \Delta) \in \Lambda_{n-k,k}^{B}$ then $\Phi(A, B, \Gamma, \Delta) = (\tilde{A}, \tilde{B}, \tilde{\Gamma}, \tilde{\Delta})$ is exactly of the form described in Lemma 3.2, in addition to the fact that $\Delta_{1 \to 1} = x_0$ thanks to Proposition 2.5. Hence, we can extend (3.10) by including

$$\Delta_{1 \to 1} = 0. \quad (3.25)$$

We can now show by induction on $i$ that for $j \in \{k, n-k\}$ and $0 \leq i \leq j$,

$$\Delta_{i \to j} = 0, \quad (3.26)$$

which implies that $\Delta = 0$.

For the base case $i = 0$, we use the stability condition (2.11) for $V_1$, which is, $\Im \Gamma_{1 \to 1} = V_1$. Therefore, by (3.25) we have

$$0 = \Im \Delta_{1 \to 1} = \Delta_{1 \to 1}(V_1).$$

(3.27)

For the inductive step, we will deal with two cases: $i < k$ or $k \leq i < n-k$, assuming (3.26) holds for $0, 1, \ldots, i-1$. For the first case $i < k$, we use the stability condition (2.11) for $V_i$, i.e.,

$$\Im A_{i-1} + \Im \Gamma_{k \to i} + \Im \Gamma_{n-k \to i} = V_i.$$

(3.28)
Now by the inductive hypothesis we see that
\[ 0 = \Delta_{i-1 \to j} = \Delta_{i \to j} A_{i-1}. \] (3.29)

By (3.28) we have, for \( j \in \{k, n - k\}, \)
\[ \Delta_{i \to j}(V_i) = \Delta_{i \to j}(\text{Im} A_{i-1} + \text{Im} \Gamma_{n-k \to i} + \text{Im} \Gamma_{n-k \to i}) = \text{Im} \Delta_{i \to j} A_{i-1} + \text{Im} \Delta_{i \to j} \Gamma_{n-k \to i} + \text{Im} \Delta_{i \to j} \Gamma_{n-k \to i} = 0, \] (3.30)
where the last equality follows from (3.29) and (3.16). Thus \( \Delta_k = 0. \)

For the second case \( k \leq i \leq n - k \), we have \( \text{Im} A_{i-1} + \text{Im} \Gamma_{n-k \to i} = V_i \). Similarly,
\[ \Delta_{i \to n-k}(V_i) = \text{Im} \Delta_{i \to n-k} A_{i-1} + \text{Im} \Delta_{i \to n-k} \Gamma_{n-k \to i} = 0, \] (3.31)
which leads to \( \Delta_{n-k} = 0. \) We are done. \( \square \)

### 3.3. Maffei’s immersion and components of Springer fibers.

Now we can prove the final piece of the main theorem using Lemma 3.2. Given a cup diagram \( a \in B_{n-k,k} \), define
\[ \Lambda^a = \{(A, B, \Gamma, \Delta) \in B_{n-k,k}^+ | \ker B_{i+\delta(i)-1 \to i-1} = \ker A_{\sigma(i)-\delta(i) \to \sigma(i)} \ \text{for all} \ i \in V_i^a \}, \] (3.32)
where the maps \( A_{i \to j}, B_{j \to i} \) are defined similarly as their tilde versions in (3.5).

**Proposition 3.7.** For \( a \in B_{n-k,k} \), we have an equality \( \Phi(\Lambda^a) = \Sigma^a \).

**Proof.** Let \( \Phi^{-1}(\bar{A}, \bar{B}, \bar{\Gamma}, \bar{\Delta}) = (A, B, \Gamma, \Delta) \in \Lambda^a \). From Corollary 3.6 we see that \( \Delta \) must be zero. The proposition follows as long as we show that \( \ker \bar{B}_{i+\delta(i)-1 \to i-1} = \ker \bar{A}_{\sigma(i)-\delta(i) \to \sigma(i)} \) is equivalent to \( \ker \bar{B}_{i+\delta(i)-1 \to i-1} = \ker \bar{A}_{\sigma(i)-\delta(i) \to \sigma(i)} \). For simplicity, let us use the shorthand \( a = i - 1 < b = i + \delta(i) - 1 = \sigma(i) - \delta(i) < c = \sigma(i) \).

Using Lemma 3.2, we obtain that, for \( a < b < c \),
\[ \bar{B}_{b \to a} = V_a \begin{pmatrix} V_b & D'_b \\ D'_b & 0 \end{pmatrix}, \] (3.33)
\[ \bar{A}_{b \to c} = V_c \begin{pmatrix} V_b & D''_b & \cdots & D''_b & \cdots & D''_{c-1} & D''_c[c-b] \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}, \] (3.34)
where \( D''_t \) is the space (depending on fixed \( b < c \)) described below:
\[ D''_t = D'_t[t-b] \setminus D'_{t+1}[t-b+1] = \begin{cases} \langle e_{t-b+1}, f_{t-b+1} \rangle & \text{if } t \leq k, \\ \langle e_{t-b+1} \rangle & \text{if } k+1 \leq t \leq n-k, \\ \{0\} & \text{otherwise.} \end{cases} \] (3.35)

Since \( \bar{B}_{b \to a} \) acts as an identity on \( D'_b \), its kernel must lie in \( V_b \). Moreover, \( \ker \bar{B}_{b \to a} = \ker B_{b \to a} \).

Assume that \( \ker \bar{B}_{b \to a} = \ker \bar{A}_{b \to c} \). It follows that \( \ker \bar{A}_{b \to c} \subset V_b \). In other words, \( D'_b \setminus D'_{c-1}[c-b] \) must not lie in the kernel, and hence \( \ker \bar{A}_{b \to c} = \ker \bar{A}_{b \to c} = \ker \bar{B}_{b \to a} = \ker B_{b \to a} \).
On the other hand, assuming \( \ker B_{b \to a} = \ker A_{b \to c} \), we need to show that \( D'_b \setminus D'_{c-1} [c-b] \not\subseteq \ker \bar{A}_{b \to c} \).

In other words, for \( b \leq t \leq c-1 \), any composition of maps of the following form must be nonzero:

\[
\begin{align*}
V_{t+1} & \xleftarrow{A_{t+1 \to c}} V_c & & \xrightarrow{B_{c \to t+1}} V_{t+1} \\
V_c & \xrightarrow{A_{t \to c}} V_k & & \xrightarrow{B_{c \to t}} V_{t+1} \\
V_k & \xrightarrow{A_{t \to c}} V_{n-k} & & \xrightarrow{B_{c \to t}} V_{n-k} \\
V_{t+1} & \xrightarrow{A_{t+1 \to c}} V_c & & \xrightarrow{B_{c \to t+1}} V_{t+1} \\
V_c & \xrightarrow{A_{t \to c}} V_k & & \xrightarrow{B_{c \to t}} V_{t+1} \\
V_k & \xrightarrow{A_{t \to c}} V_{n-k} & & \xrightarrow{B_{c \to t}} V_{n-k}
\end{align*}
\]

(3.36)

if \( t + 1 \leq k \), \hspace{1cm} if \( t + 1 \neq n - k \).

Note that the spaces \( D'_b \) are nonzero only when \( t \leq n - k \), and hence the maps \( A_{t+1 \to c} \Gamma \to t+1 \) in (3.36) only exist when \( c \leq n - k \). Therefore, the proposition is proved for \( c > n - k \), and we may assume now \( c \leq n - k \).

We first prove (3.36) for the base case \( t = c - 1 \) by contradiction. Our strategy is to construct nonzero vectors \( v_i \in V_i \cap \text{Im} A_{1 \to i} \) for \( 1 \leq i \leq c \). If this claim holds, then by admissibility conditions from \( V_1 \) to \( V_i \), we have

\[
B_{i-1}(v_i) = B_{i-1} A_{1 \to i}(v_1) = A_{i-1 \to i-1} B_1 A_i(v_1) = 0,
\]

and hence there is a nonzero vector \( v_b \in \ker B_{b \to a} \). On the other hand, \( v_b \not\in \ker A_{b \to c} \) since \( A_{b \to c}(v_b) = v_c \neq 0 \), a contradiction. Now we prove the claim. Suppose that \( \Gamma_{n-k-\to c} = 0 \).

\[
\Gamma_{n-k-\to i} = B_{c \to i} \Gamma_{n-k-\to c} = 0, \quad \text{for all} \quad i \leq c.
\]

By the stability condition on \( V_1 \), we have

\[
V_1 = \text{Im} \Gamma_{k \to 1} + \text{Im} \Gamma_{n-k-\to 1} = \Gamma_{k \to 1}(f),
\]

and hence the vector \( \phi_i = \Gamma_{k \to i}(f) \in V_i \) are all nonzero for \( i \leq k \). Define \( v_i = A_{1 \to i}(\phi_1) \) for all \( i \). The stability condition on \( V_{2} \) now reads

\[
V_2 = \text{Im} A_1 + \text{Im} \Gamma_{k \to 2} + \text{Im} \Gamma_{n-k-\to 2} = \langle A_1(\phi_1) \rangle + \langle \phi_2 \rangle.
\]

Since \( V_2 \) is 2-dimensional, the vector \( v_2 = A_1(\phi_1) \) must be nonzero. An easy induction shows that, for \( 2 \leq i \leq k \), the vector \( v_i \) is nonzero. For \( k + 1 \leq i \leq c \), the stability condition on \( V_i \) is then

\[
V_i = \text{Im} A_{i-1} + \text{Im} \Gamma_{n-k-\to i} = A_{i-1}(V_{i-1}).
\]

Since both \( \dim V_i = \dim V_{i-1} = k \), the map \( A_{i-1} \) is of full rank, and hence \( v_i \neq 0 \) for \( k + 1 \leq i \leq c \). Therefore, we have seen that the assumption that \( \Gamma_{n-k-\to c} = 0 \) leads to a contradiction, and hence \( \Gamma_{n-k-\to c} \neq 0 \). A similar argument shows that \( \Gamma_{k \to c} \neq 0 \). The base case is proved.

Next, we are to show that (3.36) holds for \( b \leq t < c - 1 \). We write for short \( h = c - t - 1 \) to denote the size of the “hook” in the map \( A_{t+1 \to c} \Gamma_{n-k-t+1} \). For example, as shown in the figure below, the maps \( \Gamma_{n-k} \) have hook size 0, the maps \( A_{c-1} \Gamma_{n-k-1} \) have hook size 1, and so on:

\[
\begin{align*}
V_{c} & \xrightarrow{B_{c \to c-1}} V_{c-1} & \xrightarrow{B_{c \to c-1}} V_{c} \\
V_{j} & \xrightarrow{A_{c-1}} V_{c} & \xrightarrow{A_{c-1}} V_{c}
\end{align*}
\]

\[
\begin{align*}
\Gamma_{j} & \xrightarrow{D_{j}} \Gamma_{j} & \xrightarrow{D_{j}} \Gamma_{j} \\
V_{c} & \xrightarrow{V_{c}} V_{j} & \xrightarrow{V_{c}} V_{j}
\end{align*}
\]

\[
\begin{align*}
& h = 0 & h = 1 & h = 2 \\
V_{c} & \xrightarrow{B_{c \to c-1}} V_{c-1} & \xrightarrow{B_{c \to c-1}} V_{c} & \xrightarrow{B_{c \to c-1}} V_{c}
\end{align*}
\]

Note that \( h \) is strictly less than the size \( c - b \) of the cup. Our strategy is to construct nonzero vectors \( v_i \in V_i \cap \text{Im} A_{h+1 \to i} \) for \( h + 1 \leq i \leq c \). If this claim holds, then by admissibility conditions from \( V_{h+1} \) to \( V_i \) and an induction on \( i \), we have

\[
B_{i \to i - h-1}(v_i) = B_{i \to i - h-1} A_{i-1}(v_i) = A_{i-2} B_{i-1 \to i - h-2}(v_{i-1}) = 0.
\]

(3.42)
Note that the initial case holds since $B_{h+1-0}(v_h+1) = 0$. Hence, there is a nonzero vector $v_h \in \ker B_{b+h-1} \subset \ker B_{b+a}$. On the other hand, $v_h \not\in \ker A_{b+c}$ since $A_{b+c}(v_h) = v_c \neq 0$, a contradiction. We can now prove the claim. Suppose first that $A_{i+1-c} \Gamma_{n-k-i+1} = 0$. By the admissibility conditions from $V_{i+2}$ to $V_{n-k+h-1}$, we have

$$0 = A_{t+1-c} \Gamma_{n-k-t+1} = B_{n-k+h-c} A_{n-k-n-k+h} \Gamma_{n-k},$$

which can be visualized from the figure below by equating the two maps $D_{n-k} \rightarrow V_c$ represented by composing solid arrows and dashed arrows, respectively:

![Diagram](image)

For all $1 \leq i \leq t + 1$, we will show now any map $D_{n-k} \rightarrow V_i$ with exactly a size $h$ “hook” is zero. Precisely speaking, the admissibility conditions from $V_{i+1}$ to $V_{n-k+h-1}$ imply that

$$A_{i-i+h} \Gamma_{n-k-i} = B_{n-k+h-i} A_{n-k-n-k+h} \Gamma_{n-k},$$

which can be visualized as the figure below:

![Diagram](image)

It follows that $A_{i+i+h} \Gamma_{n-k-i} = 0$ since it is a composition of a zero map in (3.43). Since the hook size $h$ is less than the cup size $c - b$, which is less or equal to the total number $k$ of cups, we have $h < k$ and so

$$\dim V_h = h, \quad \dim V_{h+1} = h + 1.$$  

(3.45)

We claim that

$$\text{Im} \Gamma_{k-h+1} \neq 0.$$  

(3.46)

By the stability condition on $V_1$, we have

$$V_1 = \text{Im} \Gamma_{k+1} + \text{Im} \Gamma_{n-k+1}.$$  

(3.47)

For the dimension reason, either $\Gamma_{k+1}$ or $\Gamma_{n-k+1}$ is nonzero. If $\Gamma_{k+1} \neq 0$ then $\Gamma_{k-h+1} \neq 0$, and the claim follows. If $\Gamma_{n-k+1} \neq 0$, we define, for $1 \leq i \leq l \leq n - k$,

$$\epsilon_i = \epsilon^{(i)}_l = \Gamma_{n-k-i} (c) \neq 0, \quad \epsilon^{(i)}_l = A_{i-i+l}(\epsilon_i).$$  

(3.48)

An easy induction shows that if $\epsilon^{(i)}_l = 0$ for some $1 \leq i \leq l \leq h$ then $\Gamma_{k-l} \neq 0$, which proves the claim. So we now assume that $\epsilon^{(i)}_l \neq 0$ for all $1 \leq i \leq l \leq h$. Note that

$$\epsilon^{(1)}_1 = A_{1+1} \Gamma_{n-k+1}(e) = 0,$$

and hence rank $A_h = h - 1$. Now the stability condition on $V_{h+1}$ implies that

$$\dim V_{h+1} = \text{rank} A_h + \text{rank} \Gamma_{k-h+1} + \text{rank} \Gamma_{n-k-h+1}.$$  

(3.50)

and hence rank $\Gamma_{k-h+1} = 1$. The claim (3.46) is proved. Moreover, the vectors $\phi_i = \Gamma_{k-i} (f) \in V_i$ are all nonzero for $h + 1 \leq i \leq k$. Define $v_i = A_{h+1-i} (\phi_{h+1})$ for all $i > h$.

The stability condition on $V_{h+2}$ now reads

$$V_{h+2} = \begin{cases} \text{Im} A_{h+1} + \text{Im} \Gamma_{k-h+2} + \text{Im} \Gamma_{n-k-h+2} & \text{if } h + 2 \leq k, \\ \text{Im} A_{h+1} + \text{Im} \Gamma_{n-k-h+2} & \text{if } h + 2 > k. \end{cases}$$  

(3.51)
In either case, a dimension argument similar to the one given in the base case \( t = c - 1 \) shows that the vector \( v_{h+2} = A_{h+1}(\phi_{h+1}) \) must be nonzero since
\[
\epsilon_l^{(i)} = A_{i-1} \Gamma_{n-k-i} = 0 \quad \text{for} \quad l \geq i + h. \tag{3.52}
\]
For \( h + 2 \leq i \leq c \), a dimension argument using \((3.51)\) shows that \( v_i \neq 0 \), which leads to a contradiction, and hence \( A_{t+1} \rightarrow \Gamma_{n-k-t+1} = 0 \). A similar argument shows that \( A_{t+1} \rightarrow \Gamma_{k-t+1} \neq 0 \). The proposition is proved. \( \square \)

**Theorem 3.8.** Recall \( \Lambda^a \) from \((3.32)\). For any cup diagram \( a \in B_{n-k,k} \), we have an equality
\[
p_{n-k,k}(\Lambda^a) = \tilde{\varphi}^{-1}(K^a). \tag{3.53}
\]
As a consequence, \( \tilde{\varphi}^{-1}(B_{n-k,k}) = \bigcup_{a \in B_{n-k,k}} p_{n-k,k}(\Lambda^a) \).

**Proof.** We have
\[
\tilde{\varphi}^{-1}(K^a) = \tilde{\varphi}(\mathbb{T}^a) \quad \text{by Proposition 3.1}
\]
\[
= p_{n-k,k}(\Phi^{-1}(\mathbb{T}^a)) \quad \text{by Proposition 2.7}(b)(c) \tag{3.54}
\]
\[
= p_{n-k,k}(\Lambda^a) \quad \text{by Proposition 3.7} \quad \square
\]

### 3.4. The ray condition.
For completeness, in this section we characterize the ray condition \( F_i = x^{\frac{1}{2}(\rho(i)-\rho(i))}(x^{n-k-\rho(i)}F_n) \), on the quiver representation side.

**Proposition 3.9.** Let \( (x,F_{\bullet}) = \tilde{\varphi}(A,B,\Gamma,0) \in K^a \subset \tilde{S}_{n-k,k} \). Then the ray condition is equivalent to
\[
\begin{align*}
B_i A_i &= 0 \quad \text{if} \ c(i) \geq 1, \\
\Gamma_{n-k-i} &= 0 \quad \text{if} \ c(i) = 0,
\end{align*}
\tag{3.55}
\]
where \( c(i) = \frac{i-\rho(i)}{2} \) is the total number of cups to the left of \( i \).

**Proof.** Write \( \rho = \rho(i) \) and \( c = c(i) \) for short. By Corollary \((3.3)\), the ray relation is equivalent to that
\[
\ker(A_{n-k-\rho+1} \rightarrow n \Gamma_{n-k-\rho+1} \cdots A_{n-k-\rho} \rightarrow n \Gamma_{n-k-\rho}) = \ker(A_{c+1} \rightarrow \Gamma_{c+1} \cdots \Gamma_{\rho}). \tag{3.56}
\]
Note that there are \( \rho \) blocks on the left hand side of \((3.56)\) and each block is a zero map; while there are \( i - c = \rho + c \geq \rho \) blocks on the right hand side. Hence, the defining relations, by an elementary case-by-case analysis, are
\[
\begin{align*}
A_{c+\rho} \rightarrow n \Gamma_{n-k-\rho} &= 0, \quad A_{c+\rho} \rightarrow \Gamma_{n-k-\rho+1} \neq 0 \quad \text{if} \ c \geq 1, \\
A_{c+\rho} \rightarrow n \Gamma_{k-n-\rho} &= 0 \quad \text{if} \ c = 0.
\end{align*}
\tag{3.57}
\]
Note that by definition, \( \rho = 0 \) when \( c = 0 \). We are done. \( \square \)

### 4. Springer fibers for classical types

#### 4.1. Springer fibers and Slodowy varieties of type D.
From now on, let \( n = 2m \) be an even positive integer. Let \( \mathfrak{so}_{2m}(\mathbb{C}) \) be the Lie algebra of type \( D_m \). The type D nilpotent cone \( N^D = N \cap \mathfrak{so}_{2m}(\mathbb{C}) \) admits an action of the orthogonal group \( O_n(\mathbb{C}) \) by conjugation. It is known \((\text{VI}337)\) that the \( O_n(\mathbb{C}) \)-orbits of \( N^D \) are in bijection with the subset of all partitions of \( 2m \) whose even parts occur even times, i.e., the set
\[
\pi^D = \{ \lambda = (\lambda_i)_i : n = 2m \ | \ #\{i \ | \ \lambda_i = j\} \in 2\mathbb{Z} \ \text{for all} \ j \in 2\mathbb{Z} \}. \tag{4.1}
\]
Now we restrict ourselves to the two-row case, i.e., each orbit is parametrized by partitions of \( n = 2m \) of the form:
\[
(n-k,k) = (m,m), \quad \text{or} \quad (n-k,k) \in (2\mathbb{Z} + 1)^2. \tag{4.2}
\]
For each partition \((n - k, k)\) of the form as in \((4.2)\), we define a non-degenerate symmetric bilinear form \(\beta = \beta(n - k, k)\) whose associated matrix, under the ordered basis \(e_1, \ldots, e_{n-k}, f_1, \ldots, f_k\) of \(\mathbb{C}^n\), is

\[
M = \begin{cases} 
\begin{pmatrix} \langle e_i \rangle & \langle f_i \rangle \\ \langle f_i \rangle & J_m \end{pmatrix} & \text{if } n-k = k, \\
\begin{pmatrix} \langle e_i \rangle & \langle f_i \rangle \\ J_{n-k} & 0 \end{pmatrix} & \text{if } n-k > k,
\end{cases}
\]  

where \(J_m = \begin{pmatrix} -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix}\). \((4.3)\)

For each subspace \(W\) of \(\mathbb{C}^n\) we let \(W^\perp\) be its orthogonal complement in \(\mathbb{C}^n\) with respect to \(\beta\). \(W\) is called isotropic if \(W \subseteq W^\perp\). Denote the orthogonal Lie algebra corresponding to \(\beta\) by

\[
\mathfrak{so}_{2m}(\mathbb{C}; \beta) = \{X \in \mathfrak{gl}_n \mid MX = -X^t M\}. \quad (4.4)
\]

As a result, any \(x \in \text{End}(\mathbb{C}^n)\) of Jordan type as in \((4.2)\) sits inside \(\mathfrak{so}_{2m}(\mathbb{C}; \beta)\). Let \(\mathcal{B}^D = \mathcal{B}^D(\beta)\) be the flag variety of \(O_{2m}(\mathbb{C})\) with respect to \(\beta\), namely,

\[
\mathcal{B}^D = \{F_i \in \mathcal{B} \mid F_i = F_{2m-i} \text{ for all } i\}. \quad (4.5)
\]

**Remark 4.1.** It is intentional to use \(O_{2m}(\mathbb{C})\) rather than the special orthogonal group so that \(\mathcal{B}^D\) has two connected components.

Let \(\mathcal{N}^D\) be the cotangent bundle of \(\mathcal{B}^D\). Explicitly, we have

\[
\mathcal{N}^D = T^* \mathcal{B}^D = \{(u, F_i) \in \mathcal{N}^D \times \mathcal{B}^D \mid u(F_i) \subseteq F_{i-1} \text{ for all } i\}. \quad (4.6)
\]

The type D Springer resolution \(\mu_D : \mathcal{N}^D \rightarrow \mathcal{N}^D\) is given by \((u, F_\bullet) \mapsto u\). Given \(x \in \mathcal{N}^D\), the associated Springer fiber of type D is defined as the subvariety

\[
\mathcal{B}^D_x = \{F_i \in \mathcal{B}^D \mid x F_i \subseteq F_{i-1} \text{ for all } i\}. \quad (4.7)
\]

The type D Springer fiber only depends (up to isomorphism) on the \(O_n\)-orbit containing \(x\). For \(x \in \mathcal{N}^D\), we denote the type D Slodowy transversal slice of \((O_n\text{-orbit of } x)\) by the variety

\[
\mathcal{S}^D = \{u \in \mathcal{N}^D \mid [u - x, y] = 0\}, \quad (4.8)
\]

where \((x, y, h)\) is a \(\mathfrak{sl}_2\)-triple in \(\mathcal{N}^D\). Let the type D Slodowy variety associated to \(x\) be defined as

\[
\mathcal{S}_x^D = \mu_D^{-1}(\mathcal{S}^D_x) = \{(u, F_\bullet) \in \mathcal{N}^D \times \mathcal{B}^D \mid [u - x, y] = 0, \ u(F_i) \subseteq F_{i-1} \text{ for all } i\}. \quad (4.9)
\]

If \(x \in \mathcal{N}^D\) is of Jordan type \((n - k, k)\), we write

\[
\mathcal{B}^D_{n-k,k} := \mathcal{B}^D_x, \quad \mathcal{S}^D_{n-k,k} := \mathcal{S}^D_x, \quad \mathcal{S}^D_{n-k,k} := \mathcal{S}^D_x. \quad (4.10)
\]

**4.2. Irreducible components of type D Springer fibers.** In order to describe the irreducible components of \(\mathcal{B}^D_{n-k,k}\), we define the notion of a marked cup diagram.

**Definition 4.2.** A marked cup diagram is a cup diagram (see \((4.2)\)) in which each cup or ray may be decorated with a single marker satisfying the following rules:

\((M1)\) The vertices on the axis are labeled by \(1, 2, \ldots, m\).

\((M2)\) A marker can be connected to the right border of the rectangular region by a path which does not intersect any other cup or ray.

Given a cup diagram \(\hat{a} \in \mathcal{B}^D_{n-k,k}\), denote the sets of all vertices connected to the left (resp. right) endpoint of a marked cup in \(\hat{a}\) by \(X^\hat{a}_l\) (resp. \(X^\hat{a}_r\)); while the sets of vertices connected to the left (resp. right) endpoint of an unmarked cup in \(\hat{a}\) are denoted by \(V^\hat{a}_l\) (resp. \(V^\hat{a}_r\)). The endpoint-swapping map \(\sigma\) and the size formula \(\delta\) naturally extend to \(X^\hat{a}_l \sqcup V^\hat{a}_r\). A marked cup diagram on \(m\) vertices unfolds into a cup diagram on \(n = 2m\) vertices in the following sense.
Algorithm 4.3. Given a marked cup diagram \( \hat{a} \), we produce a cup diagram \( f(\hat{a}) \in B_{n-k,k} \) for some \( k \) as below:

1. Produce the mirror image of \( \hat{a} \) with respect to its right border. Relabel the \( 2m \) vertices on the top border of an unfolded diagram by \( 1, 2, \ldots, n \), from left to right.
2. Replace marked rays and cups using the rules below. Let \( f(\hat{a}) \) be the end result.
   - (a) If vertex \( i \) is connected to a ray that is accessible from the axis of reflection, then replace the two rays connected to vertices \( i \) and \( 2m + 1 - i \) by a cup connecting the two vertices.
   - (b) If vertex \( i \) and \( j \) (\( i, j \leq m \)) are connected by a marked cup, then replace the two marked cups connecting \( i, j \) and \( 2m + 1 - i, 2m + 1 - j \) respectively by two cups connecting \( i, 2m + 1 - j \) and \( 2m + 1 - i, j \), respectively.

We write \( B^{D}_{n-k,k} \) to denote the set of all marked cup diagrams on \( m \) vertices with exactly \( \left\lfloor \frac{k}{2} \right\rfloor \) cups. Note that for a fixed \( n = 2m \) and any \( k' \in \mathbb{Z}_{\geq 0} \), there is a unique \( k \) such that \( \left\lfloor \frac{k}{2} \right\rfloor = k' \) and that \( (n-k,k) \) is a partition as in (4.2).

Example 4.4. We present in below how to unfold all six marked cup diagrams in \( B^{D}_{3,3} \) having \( \left\lfloor \frac{3}{2} \right\rfloor = 1 \) cup:

| \( \hat{a} \) | \( f(\hat{a}) \) |
|---|---|
| ![Diagram 1] | ![Diagram 2] |
| ![Diagram 3] | ![Diagram 4] |

Here we use dashed line as the right border to emphasize that it is the axis of reflection, as well as that the markers must be accessible from the dashed line by a path that does not intersect any other rays or cups.

Proposition 4.5. There exists a bijection between the irreducible components of the Springer fiber \( B^{D}_{n-k,k} \) and the set \( B^{D}_{n-k,k} \) of marked cup diagrams.

Proof. It follows from combining [ES16a, Lemma 5.12], [Spa82, II.9.8] and [vL89, Lemmas 3.2.3, 3.3.3].

For each marked cup diagram \( \hat{a} \in B^{D}_{n-k,k} \), we denote by \( K\hat{a} \) the corresponding component in the type D Springer fiber \( B^{D}_{n-k,k} \).

Example 4.6. Let \( n = 4, k = 2 \). We fix a basis \( \{e_1, e_2, f_1, f_2\} \) of \( \mathbb{C}^4 \) so that \( x \) is determined by \( e_2 \mapsto e_1 \mapsto 0, f_2 \mapsto f_1 \mapsto 0 \). Define

\[
\hat{a}_{12} = \quad a_{13} = \quad \boxed{1 \ 2 \ 3 \ 4}.
\]

The corresponding marked cup diagrams are

\[
\hat{a}_{12} = \quad \hat{a}_{13} = \quad \boxed{1 \ 2}.
\]
The irreducible components in \( B_{2,2} \) are

\[
K^{\alpha_{13}} = \{(x, F_\bullet) \mid x^{-1}F_0 = F_2, x^{-1}F_2 = F_4, \dim F_i = i\}
\]
\[
= \{(x, 0) \subset \langle e_1 + f_1, e_1 + f_1 \rangle \subset \langle e_1, f_1 \rangle \subset \langle e_1, f_1, e_2 + f_2 \rangle \subset \mathbb{C}^4 \}\}
\]
\[
K^{\alpha_{12}} = \{(x, F_\bullet) \mid x^{-2}F_0 = F_4, x^{-1}F_1 = F_3, \dim F_i = i\}
\]
\[
= \{(x, F_\bullet) \mid x^{-1}F_1 = F_3, \dim F_i = i\}.
\]

By imposing the isotropic condition with respect to the matrix

\[
\begin{pmatrix}
e_1 & e_2 & f_1 & f_2 \\
e_1 & 0 & 0 & f_1 \\
e_2 & 0 & -1 & 0 \\
f_1 & 0 & -1 & 0 \\
f_2 & 1 & 0 & 0
\end{pmatrix}
\]

we obtain the two components in \( B_{2,2}^D \) as isotropic flags:

\[
K^{\alpha_{13}} = \{(x, 0) \subset \langle e_1 + f_1, e_1 + f_1 \rangle \subset \langle e_1, f_1 \rangle \subset \langle e_1, f_1, e_2 + f_2 \rangle \subset \mathbb{C}^4 \}\} \in B_{2,2}^D
\]
\[
= \{(x, F_\bullet) \in B_{2,2}^D \mid x^{-1}F_2 = F_4\};
\]

\[
K^{\alpha_{12}} = \{(x, 0) \subset \langle e_1 + f_1, e_1 + f_1 \rangle \subset \langle e_1, f_1 \rangle \subset \langle e_1, f_1, e_2 + f_2 \rangle \subset \mathbb{C}^4 \}\} \in B_{2,2}^D
\]
\[
= \{(x, F_\bullet) \in B_{2,2}^D \mid x^{-1}F_2 = F_3\}. \tag{4.11}
\]

We denote the component \((4.11)\) by \( K^{\alpha_{13}} \) simply because its relation coincides with the cup relation in type A and \( \hat{\alpha}_{13} \) is a type A cup; while we denote \((4.12)\) by \( K^{\alpha_{12}} \) since the relation \( x^{-1}F_2 = F_3 \) is a type D phenomenon and should correspond to the marked cup.

5. Fixed point subvarieties of quiver varieties

5.1. Automorphisms on quiver varieties. In literature the fixed point subvarieties of Nakajima’s quiver varieties are studied first by Henderson–Licata in [HL14] for the type A quivers associated with an explicit involution. For quivers associated with the symmetric pairs (or Satake diagrams), the corresponding fixed-point subvarieties are studied by Li in [Li19]. While it is difficult to compare the two automorphisms in an explicit way, both automorphism restrict to an isomorphism between the fixed-point subvariety and a Schubert variety of type D which is isomorphic to one of type C.

In [HL14], Henderson–Licata considers any diagram automorphism \( \theta \) which is an admissible automorphism in the sense of Lusztig, see [Lus93, §12.1.1]. For type A quiver varieties, such a diagram automorphism defines a variety automorphism \( \Theta = \Theta(\theta, \sigma_k) : M_{n-k,k} \to M_{n-k,k} \), which also depend on the choice of an involution \( \sigma_k : D_k \to D_k \). With a suitable choice of \( \sigma_k \), there is an isomorphism \( M_{n-k,k}^{\Theta} \simeq S_{n-k,k}^D \sqcup S_{n-k,k}^D \).

On the other hand, Li has constructed in [Li19] a family of automorphisms which works in a more general scenario. The automorphism \( \sigma = a \circ S_\omega \circ \tau \) is composed of a diagram automorphism \( a \), a reflection functor \( S_\omega \) for some Weyl group element \( \omega \), and an isomorphism \( \tau \) which deals with the so-called formed spaces which account for the orthogonality in our setup. By choosing \( a = 1, \omega = w_0 \), which is the longest element, he exhibits an isomorphism \( M_{n-k,k}^{\sigma} \simeq S_{n-k,k}^D \sqcup S_{n-k,k}^D \).

Moreover, in both cases the isomorphisms restrict to isomorphisms between the Springer fibers. In this section we investigate to what extent they preserve the components of Springer fibers.

5.2. Henderson–Licata’s fixed point subvarieties. We use identifications \( V_i \equiv V_{n-i} \) for all \( i \) and \( D_i \equiv D_{n-i} \) for all \( i \neq k \) (they are referred as isomorphisms \( \varphi_i \) and \( \sigma_i \) in [HL14, §3.2]). For all \( k \), let
$\sigma_k$ be the automorphism on $D_k$ determined by

$$\sigma_k(e) = e, \quad \sigma_k(f) = \begin{cases} f & \text{if } k \in 2\mathbb{Z}, \\ -f & \text{if } k \in 2\mathbb{Z} + 1. \end{cases}$$

Define the $\Theta$-action by

$$\Theta(\Delta_i) = \Delta_{n-i}, \quad \Theta(\Gamma_i) = \begin{cases} \Gamma_{n-i} & \text{if } i \neq k, \\ \Gamma_k \circ \sigma_k^{-1} & \text{if } i = k, \end{cases} \quad (5.1)$$

$$\Theta(A_i) = B_{n-1-i}, \quad \Theta(B_i) = A_{n-1-i}.$$ (5.2)

**Remark 5.1.** By [HL14 §3.2], for all $s \in \Lambda_{n-k,k}^+$, $\Theta$ sends the orbit containing $s$ to the orbit containing $\Theta(s)$, or, $\Theta(p(s)) = p(\Theta(s))$, where $p$ is the projection map in (2.16). Therefore, an element $[s] = [(A, B, \Gamma, 0)] \in M_{n-k,k}^R$ is fixed under $\Theta$ if and only if there exists a $g = (g_i) \in \text{GL}(V)$ such that

$$A_i = g_{i+1}B_{n-1-i}g_i^{-1}, \quad B_i = g_iA_{n-1-i}g_{i+1}, \quad \Gamma_i = \begin{cases} g_i\Gamma_{n-i} & \text{if } i \neq k, \\ g_k\Gamma_k\sigma_k & \text{if } i = k. \end{cases} \quad (5.2)$$

**Proposition 5.2.** Let $\Theta$ be defined as in (5.1), and let $n, k$ be such that $(n-k, k)$ is a type D partition (see (4.2)), then there is an isomorphism $M_{n-k,k}^\Theta \simeq \tilde{S}_{n-k,k}^D \sqcup \tilde{S}_{n-k,k}^D$.

**Proof.** See [HL14 (5.2), Thm. 5.3].

**Remark 5.3.** In [HL14 Thm. 5.3], it is also given an isomorphism for the type C Slodowy’s variety if the condition [HL14 (5.1)] is satisfied. Here, we only discuss type D result.

5.3. **Two Jordan blocks of equal size.** With type D in mind, we need $n = 2k$. Furthermore, thanks to Corollary 3.6, the action of $\Theta$ on the quiver representations in $\Lambda_{k,k}^R$ can be visualized as:

In this section we demonstrate how we obtained the marked cup relation by working out a few non-trivial examples.

**Example 5.4.** Let $n = 4 = 2k$. Consider $\Lambda_{2,2}^R = \{(A, B, \Gamma, \Delta) \in \Lambda_{2,2}^+ \mid \Delta = 0\}$, where the quiver representations are described below:

(5.3)

Define

$$a_{12} = \begin{array}{c} \includegraphics{a12} \end{array}, \quad a_{13} = \begin{array}{c} \includegraphics{a13} \end{array}, \quad \hat{a}_{12} = \begin{array}{c} \includegraphics{a12_h} \end{array}, \quad \hat{a}_{13} = \begin{array}{c} \includegraphics{a13_h} \end{array}.$$ (5.4)

By (3.3), we have

$$\Lambda^{a_{12}} = \{(A, B, \Gamma, 0) \in \Lambda_{2,2}^+ \mid \ker B_1 = \ker A_2\}, \quad (5.4)$$

$$\Lambda^{a_{13}} = \{(A, B, \Gamma, 0) \in \Lambda_{2,2}^+ \mid A_1 = 0, B_2 = 0\}. \quad (5.5)$$
By Remark 5.1 the fixed-points in $(M^{a_{12}})^\Theta$ and $(M^{a_{13}})^\Theta$ are described below: $\{s\} \in (M^{a_{12}})^\Theta$ if and only if
\begin{equation}
\begin{aligned}
s &= (A, B, \Gamma, 0) \in \Lambda_{2,2}^+, \quad \ker B_1 = \ker A_2, \text{ there is a } g = (g_i) \in \text{GL}(V) \text{ such that } \\
\Gamma_2 &= g_2\Gamma_2, \quad A_1 = g_2B_2g_1^{-1}, \quad A_2 = g_3B_1g_2^{-1}, \quad B_1 = g_1A_2g_2^{-1}, \quad B_2 = g_2A_1g_3^{-1},
\end{aligned}
\tag{5.6}
\end{equation}
while $\{s\} \in (M^{a_{13}})^\Theta$ if and only if
\begin{equation}
\begin{aligned}
s &= (A, B, \Gamma, 0) \in \Lambda_{2,2}^+, \quad \ker B_1 = \ker A_2, \text{ there is a } g = (g_i) \in \text{GL}(V) \text{ such that } \\
\Gamma_2 &= g_2\Gamma_2, \quad A_1 = 0 = B_2, \quad A_2 = g_3B_1g_2^{-1}, \quad B_1 = g_1A_2g_2^{-1}.
\end{aligned}
\tag{5.7}
\end{equation}
Using Proposition 2.5 the component $K^{a_{13}} = \tilde{\varphi}(p((A^{a_{11}})^\Theta))$ consists of pairs $(x, F_\bullet)$, where $F_i = \ker \Gamma_{1-i}$, and
\begin{equation}
x = \tilde{\Delta}_1\tilde{\Gamma}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{equation}
From Example 3.4 the isotropic flag $F_\bullet$ is described by,
\begin{equation}
\begin{aligned}
F_1 &= \ker (B_1\Gamma_2), \quad F_2 = \ker (A_1B_1\Gamma_2\Gamma_2), \quad F_3 = \ker (A_2A_1B_1\Gamma_2\Gamma_2).
\end{aligned}
\tag{5.8}
\end{equation}
Since $F_1$ is a 1-dimensional space which is killed by $x$, it must be spanned by a nonzero vector of the form $\lambda e_2 + \mu f_1$ for some $\lambda, \mu \in \mathbb{C}$. For $F_3$, we have $\ker A_2A_1B_1\Gamma_2 = \langle e, f \rangle$ since the admissibility conditions imply that
\begin{equation}
A_2(A_1B_1)\Gamma_2 = (A_2B_2)A_2\Gamma_2 = 0.
\end{equation}
Since the first block is a zero map, the space $\ker \langle 0, A_2\Gamma_2 \rangle$ is a direct sum $\langle e_1, f_1 \rangle \oplus \iota_2(\ker A_2\Gamma_2)$, where $\iota_t$ is the subscript-decorating linear map sending $e, f$ to $e_t, f_t$, respectively. Since $A_2 = g_3B_1g_2^{-1}$ and $\Gamma_2 = g_2\Gamma_2$, we have
\begin{equation}
\ker A_2\Gamma_2 = \ker g_3B_1\Gamma_2 = \ker B_1\Gamma_2 = \langle \lambda e + \mu f \rangle,
\end{equation}
and hence $F_3 = \langle e_1, f_1, \lambda e_2 + \mu f_2 \rangle$.
For $F_2$, there are two cases: $A_2 = B_2 = 0$ if $p_{2,2}(A, B, \Gamma, 0) \in (M^{a_{11}})^\Theta$, or $A_2 = g_2B_2g_3^{-1} \neq 0$ if $p_{2,2}(A, B, \Gamma, 0) \in (M^{a_{12}})^\Theta \setminus (M^{a_{13}})^\Theta$. If $A_1 = 0$, then $\ker A_1B_1\Gamma_2 = \langle e, f \rangle$ and hence $F_2 = \langle e_1, f_1 \rangle$.
If $A_1 \neq 0$, then $A_1$ is injective since its domain is one-dimensional, and hence
\begin{equation}
\ker A_1B_1\Gamma_2 = \ker B_1\Gamma_2 = \ker (g_1A_2g_2^{-1})g_2\Gamma_2 = \ker A_2\Gamma_2 = \langle \lambda e + \mu f \rangle.
\end{equation}
Moreover, we have $\ker \Gamma_2 = \ker A_1B_1\Gamma_2$ since $\ker \Gamma_2 \subset \ker A_1B_1\Gamma_2$ and $\dim \ker \Gamma_2 = \dim \ker A_1B_1\Gamma_2 = 1$. Therefore, $F_2 = \langle \lambda e_1 + \mu f_1, \lambda e_2 + \mu f_2 \rangle$.
Comparing this example with Example 4.6 we see that
\begin{equation}
\tilde{\varphi}((M^{a_{11}})^\Theta) = K^{a_{13}}, \quad \tilde{\varphi}((M^{a_{12}})^\Theta) = K^{a_{13}} \sqcup K^{a_{12}}.
\end{equation}
Here we label $K^{a_{13}}$ by a unmarked cup since it is characterized by
\begin{equation}
A_1 = B_2 = 0 \iff \ker B_2 = \ker A_3,
\end{equation}
which is the type A cup relation connecting the vertices 2 and 3, while $K^{a_{12}}$ is characterized by the relation
\begin{equation}
A_1, B_2 \neq 0 \iff \ker B_1\Gamma_2 = \ker A_1B_1\Gamma_2,
\end{equation}
which should be the type D relation for a marked cup of size 1.
5.4. **Two Jordan blocks of unequal sizes.** Thanks to Corollary 3.6, the quiver representations in $\Lambda_{n-k,k}^B$ are of the form

$$\begin{array}{cccccccc}
\dim D_i & 0 & 0 & \cdots & 1 & 0 & \cdots & 1 & \cdots & 0 & 0 \\
\end{array}$$

$$\begin{array}{cccccccc}
\dim V_i & 1 & 2 & \cdots & k & k & \cdots & k & \cdots & 2 & 1 \\
\end{array}$$

In below we demonstrate how the leftmost ray is characterized by working out the case $(3,1)$.

**Example 5.5.** Let $n = 4, k = 1$. Consider $\Lambda_{3,1}^B = \{(A, B, \Gamma, \Delta) \in \Lambda_{3,1}^+ \mid \Delta = 0\}$, where the quiver representations are described below (with fixed basis elements $v_1, w_1, v_3, e, f$):

$$\begin{align*}
\langle f \rangle & \quad \langle e \rangle \\
C & \quad C
\end{align*}$$

$$\begin{array}{ccccccc}
& A_1 & & A_2 & & & \\
\gamma_1 & C & B_1 & C & B_1 & C &
\end{array}$$

$$\begin{array}{ccccccc}
& A_1 & & A_2 & & & \\
\gamma_2 & C & B_2 & C & B_2 & C &
\end{array}$$

Define

$$a_1 = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}, \quad a_2 = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}, \quad a_3 = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}$$

By (3.32), we have

$$\begin{align*}
\Lambda^{a_1} & = \{(A, B, \Gamma, 0) \in \Lambda_{3,1}^+ \mid 0 = A_1\}, \\
\Lambda^{a_2} & = \{(A, B, \Gamma, 0) \in \Lambda_{3,1}^+ \mid \ker B_1 = \ker A_2\}, \\
\Lambda^{a_3} & = \{(A, B, \Gamma, 0) \in \Lambda_{3,1}^+ \mid B_2 = 0\}.
\end{align*}$$

By (5.2), the fixed-point subvarieties $(M^{a_1})^\Theta$ are described below: $[s] \in (M^{a_1})^\Theta = (M^{a_3})^\Theta$ if and only if

$$s = (A, B, \Gamma, 0) \in \Lambda_{3,1}^+, \quad A_1 = 0 = B_2, \text{ there is a } g = (g_i)_i \in \GL(V) \text{ such that}$$

$$\Gamma_1 = g_1 \Gamma_3, \quad \Gamma_3 = g_3 \Gamma_1, \quad A_2 = g_3 B_1 g_2^{-1}, \quad B_1 = g_1 A_2 g_2^{-1}.$$  

(5.19)

Since $A_1 = 0 = B_2$, the stability condition at $V_2$ is never satisfied and hence $(M^{a_1})^\Theta = \varnothing = (M^{a_3})^\Theta$. Meanwhile, $[s] \in (M^{a_2})^\Theta$ if and only if

$$s = (A, B, \Gamma, 0) \in \Lambda_{3,1}^+, \quad \ker B_1 = \ker A_2, \text{ there is a } g = (g_i)_i \in \GL(V) \text{ such that}$$

$$\Gamma_1 = g_1 \Gamma_3, \quad \Gamma_3 = g_3 \Gamma_1, \quad A_1 = g_2 B_2 g_1^{-1}, \quad A_2 = g_3 B_1 g_2^{-1}, \quad B_1 = g_1 A_2 g_2^{-1}, \quad B_2 = g_2 A_1 g_3^{-1}.$$  

(5.20)

From Example 3.4, the isotropic flag $F_\bullet$ is described by,

$$\begin{align*}
F_1 & = \ker(\Gamma_1 | B_1 B_2 \Gamma_3), \\
F_2 & = \ker(A_1 \Gamma_1 | A_1 B_1 B_2 \Gamma_3 | B_2 \Gamma_3), \\
F_3 & = \ker(A_2 A_1 \Gamma_1 | A_2 A_1 B_1 B_2 \Gamma_3 | A_2 B_2 \Gamma_3 | \Gamma_3).
\end{align*}$$

(5.21)

For $F_1$, by the stability condition on $V_1$, we have $\Gamma_1$ is surjective and so $\ker \Gamma_1 = 0$, while the cup relation $\ker B_1 = \ker A_2$ implies that

$$\ker B_1 B_2 \Gamma_3 = \ker A_2 B_2 \Gamma_3 = \ker 0 = \langle e \rangle,$$

(5.22)
and hence $F_1 = \langle e_1 \rangle$. For $F_3$, by a similar argument we see that $\ker A_2 A_1 \Gamma_1 = \ker B_1 A_1 \Gamma_1 = \ker 0 = \langle f \rangle$, $\ker \Gamma_3 = 0$, and hence $F_3 = \langle f_1, e_1, e_2 \rangle$.

For $F_2$, we have

$$\ker A_1 \Gamma_1 = \ker (g_2 B_2 g_1^{-1}) g_1 \Gamma_3 = \ker B_2 \Gamma_3. \quad (5.23)$$

By the dimension reason, $\ker A_1 \Gamma_1 = \ker B_2 \Gamma_3 = 0$, while $\ker (A_1 \Gamma_1 | B_2 \Gamma_3)$ is 1-dimensional, and is spanned by the vector $f + \lambda e$ for some $\lambda \neq 0$.

By (5.20), we have

$$\Gamma_1 = g_1 \Gamma_3 = g_1 g_3 \Gamma_1, \quad \Gamma_3 = g_3 \Gamma_1 = g_3 g_1 \Gamma_3, \quad (5.24)$$

and hence $g_1, g_3 \in \mathbb{C}^\times$ are inverses to each other. Moreover,

$$A_1 = g_2 B_2 g_1^{-1} = g_2 (g_2 A_1 g_3^{-1}) g_1^{-1} = g_2^2 A_1, \quad (5.25)$$

and hence $g_2 \in \mathbb{C}^\times$ is an involution, or $g_2 = \pm 1$. Then

$$0 = A_1 \Gamma_1 (f) + \lambda B_2 \Gamma_3 (e) = g_2 B_2 \Gamma_3 (e) + \lambda B_2 \Gamma_3 (e). \quad (5.26)$$

That is, $-\lambda$ is an eigenvalue of $g_2$ and so $\lambda \in \{ \pm 1 \}$. Therefore, $\varphi((M^\dagger)\Theta)$ splits into the following two connected components:

$$\{(0 \subset \langle e_1 \rangle \subset \langle e_1, f_1 + e_2 \rangle \subset \langle f_1, e_1, e_2 \rangle \subset \mathbb{C}^4)\}, \quad (5.27)$$

6. COMPONENTS OF SPRINGER FIBERS OF TYPE D

6.1. The branching rule. Given a type D cup diagram $\dot{a} \in B_{n-k,k}$, if $i$ is the left endpoint of a cup, i.e., $i \in V_i^{\dot{a}} \cup X_i^{\dot{a}}$, set

$$m(i) = i + \delta(i) - 1 = \left\lfloor \frac{i + \sigma(i)}{2} \right\rfloor, \quad \dot{m}(i) = \left\lfloor \frac{\sigma(i) + 1}{2} \right\rfloor + \left\lfloor \frac{i}{2} \right\rfloor - 1. \quad (6.1)$$

Let $\Lambda^{\dot{a}}$ be the subset of $\Lambda_{n-k,k}^B$ consisting of quadruples $(A, B, \Gamma, 0)$ satisfying the following relations, if $n - k = k$:

$$\begin{align*}
\ker B_{m(i) - i - 1} & = \ker A_{m(i) - \sigma(i)} \quad \text{for all } i \in V_i^a, \\
\ker B_{\dot{m}(i) - i} & = \ker A_{\dot{m}(i) - \sigma(i)} \quad \text{for all } i \in X_i^a, \\
A_{\frac{i + 1}{2} \rightarrow \Gamma_k} & = A_{i + 1 \rightarrow \Gamma_k} + \sigma_k \quad \text{if } i \text{ is connected to a marked ray,} \\
A_{\frac{i + 1}{2} \rightarrow \Gamma_k} & = -A_{i + 1 \rightarrow \Gamma_k - \sigma_k} \quad \text{if } i \text{ is connected to an unmarked ray,}
\end{align*} \quad (6.2)$$

while when $n - k > k$, the last two relations in (6.2) are replaced by the following:

$$\begin{align*}
A_{k \rightarrow k + \rho(i) - 1} & = -B_{n-k \rightarrow k + \rho(i) - 1} \Gamma_{n-k} \quad \text{if } i \text{ is connected to a marked ray,} \\
A_{k \rightarrow k + \rho(i) - 1} & = B_{n-k \rightarrow k + \rho(i) - 1} \Gamma_{n-k} \Gamma_{n-k} \quad \text{if } i \text{ is connected to an unmarked ray.}
\end{align*} \quad (6.3)$$

We further define

$$M^{\dot{a}} = p_{n-k,k}(\Lambda^{\dot{a}}), \quad K^{\dot{a}} = \varphi(M^{\dot{a}}). \quad (6.4)$$

The rest of the section is dedicated to the proof of the following theorem:

**Theorem 6.1.** Let $a \in B_{n-k,k}$ be a type A cup diagram.

(a) If $a$ is symmetric (with respect to the axis of reflection), then

$$(M^a)\Theta = \bigsqcup_{\dot{a}} M^{\dot{a}},$$

where $\dot{a}$ runs over all type D cup diagrams in $B_{n-k,k}^D$ which unfold to $a$ in the sense of Algorithm 4.3

(b) If $a$ is not symmetric, then $(M^a)^\Theta \subseteq (M^b)^\Theta$ for some symmetric $b \in B_{n-k,k}$.

As a consequence,

$$M_{n-k,k}^\Theta = \bigcup_{\dot{a} \in B_{n-k,k}^D} M^{\dot{a}}.$$
Now we define a set $\tilde{B}_{n-k,k}$ containing $B_{n-k,k}$ such that the cups and rays in any $a \in \tilde{B}_{n-k,k}$ can be decorated by markers. Note that $M^a$ also makes sense (cf. (6.2) – (6.3)) for $a \in \tilde{B}_{n-k,k}$.

**Lemma 6.2.** Let $a \in \tilde{B}_{n-k,k}$. If there is a marked cup, unmarked cup, or marked ray that does not cross the axis of reflection, then the relation it imposed in $(M^a)\Theta$ is equivalent to the relation imposed by its mirror image.

**Proof.** Assume that there is a (unmarked) cup connecting vertices $i, j$ on the left half of the diagram.

The cup relation is $\ker A_{m(i)\to j} = \ker B_{m(i)\to i-1}$. Using Remark 5.1, we obtain

$$\ker A_{m(i)\to j} = \ker A_{j-1}A_{j-2}\cdots A_{m(i)}$$

On the other hand, we obtain

$$\ker B_{m(i)\to i-1} = \ker B_{i-1}B_{i}\cdots B_{m(i)-1}$$

Hence, this cup relation is equivalent to another cup relation $\ker A_{n-m(i)\to n-i+1} = \ker B_{n-m(i)\to n-j}$ corresponding to the cup connecting vertices $n-j+1, n-i+1$, which is the mirror image of the original cup. Similarly, a marked cup relation is equivalent to the marked cup relation for its mirror image; so is the relation imposed by the leftmost ray.

**Corollary 6.3.**

(a) If $a \in \tilde{B}_{n-k,k}$ is symmetric (with respect to the axis of reflection) and has no cups that cross the axis, then $(M^a)\Theta = M^a$, where $\hat{a} \in B_{n-k,k}$ is the type D cup diagram obtained by cropping the right half of $a$.

(b) If $a \in B_{n-k,k}$ is not symmetric, then $(M^a)\Theta \subseteq (M^b)\Theta$ for some symmetric $b \in B_{n-k,k}$.

**Proof.** Part (a) follows immediately from Lemma 6.2. For Part (b), pick up to $\left\lfloor \frac{r}{2} \right\rfloor$ cups $\{\{i_t, j_t\} | 1 \leq t \leq r\}$ so that these cups lie entirely on one half of $a$, and they are not mirror images of each other. By Lemma 6.2, they impose relations that are equivalent to the ones imposed by the cups $\{\{n+1-j_t, n+1-i_t\} | 1 \leq t \leq r\}$ assuming (5.2) holds. Note that the remaining vertices $\{1, \ldots, n\} \setminus \{i_t, j_t, n+1-i_t, n+1-j_t\}$ show up in pairs. One can then define $b \in B_{n-k,k}$ using the 2r cups altogether with $k-2r$ symmetric cups connecting the remaining vertices from center to outside. Finally, using (5.2), the symmetric cups added impose only relations on $g_m$ and hence $(M^a)\Theta \subseteq (M^b)\Theta$.

**Example 6.4.** Let $a = \begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3 \quad 4
\end{array}
\end{array} \in B_{3,1}$. Since $k=1$, we cannot pick cups $\{i_t, j_t\}$ otherwise we have $2r > k$. Hence, $b = \begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3 \quad 4
\end{array}
\end{array} \in B_{3,1}$ and indeed we have $(M^a)\Theta = \emptyset \subseteq (M^b)\Theta$ as in Example 5.5.

If $a = \{1,4\}, \{2,3\}, \{5,6\} \in B_{3,3}$, we can choose $r = 1$, $\{i_1, j_1\} = \{2,3\}$ with mirror image $\{4,5\}$. Finally, we add enough cups to make sure $b \in B_{3,3}$ and so $b = \{2,3\}, \{4,5\}, \{1,6\}$.

**Algorithm 6.5.** Given $a \in \tilde{B}_{n-k,k}$ that has cups crossing the axis of reflection, in the following we demonstrate how to produce two diagram $a', a^- \in \tilde{B}_{n-k,k}$.

(1) If $a$ has exactly one cup connecting that crosses the axis of reflection, set $a'$ to be the diagram obtained from $a$ by replacing the cup by two (unmarked) rays connected to the endpoints of
the cup, respectively; while \( a^- \) is obtained similarly but with markers.

(2) Otherwise, set \( a' \) to be the diagram obtained from \( a \) by replacing the innermost two (nested) cups that cross the axis by two side-by-side cups without markers; while \( a^- \) is obtained similarly but with markers.

**Lemma 6.6.** If \( a \in \bar{B}_{n-k,k} \) has cups that cross the axis of reflection, then \((M^a)^\Theta = (M^{a'})^\Theta \cup (M^{a^-})^\Theta\).

**Proof.** It suffices to show that \((M^a)^\Theta \setminus (M^{a'})^\Theta = (M^{a^-})^\Theta\). There are two cases to be done – if \( a \) has at least two cups that cross the axis, or exactly one such cup.

Assume that \( a \) has exactly one cup crossing the axis, say the cup represented by \( \{i + 1, n - i\} \) for some \( i < m \). Note that the other cups lying entirely on one side of the diagram show up in pairs: if \( \{j, \sigma(j)\} \) represents a cup in \( a \) that lies entirely on one side, then \( \{n + 1 - \sigma(j), n + 1 - j\} \) represents its counterpart in \( a \), and the relations imposed by the two cups, respectively, are equivalent due to Remark 5.1.

Let \( (x, F_*) = \tilde{\varphi}(A, B, \Gamma, 0) \). By Proposition 5.2, the flag \( F_* \) is isotropic, and hence is determined by the above cup relations modulo \( F_i \) (and its counterpart \( F_{n-i} \)), which is, by Corollary 3.3, \( F_i = \ker(A_{1 \rightarrow i} \Gamma_{i \rightarrow 1} \cdots |A_{i-1} \Gamma_{i-1 \rightarrow i} \Gamma_{i \rightarrow i}) \). Note that cup relation for \( \{i + 1, n - i\}, \ker A_{m \rightarrow n-i} = \ker B_{m \rightarrow i}, \) is equivalent to

\[
\ker A_{m \rightarrow n-i} = \ker A_{m \rightarrow n-i} g_{m}^{-1}.
\] (6.7)

On the other hand, by Corollary 3.3 we have

\[
F_i = \ker(A_{1 \rightarrow i} \Gamma_{i \rightarrow 1} \cdots |A_{i-1} \Gamma_{i-1 \rightarrow i} \Gamma_{i \rightarrow i}).
\] (6.8)

By Algorithm 6.5 it remains to prove that \((M^a)^\Theta\) splits into two exclusive cases described by

\[
\begin{cases}
\Gamma_{k \rightarrow k+1} = \pm \Gamma_{k \rightarrow k+1} \sigma_k & \text{if } n-k = k, \\
A_{k \rightarrow k+p-1} \Gamma_{k} = \pm B_{n-k \rightarrow k+p-1} \Gamma_{n-k} & \text{if } n-k > k.
\end{cases}
\] (6.9)

For the case \( n-k = k = m \), note that in \( a \) there is no cups and hence \( F_i \) contains \( <e_t, f_t \mid 1 \leq t \leq \frac{i-1}{2}> \). It remains to investigate the possible vectors that span the 1-dimensional kernel of \( A_{\frac{i+1}{2} \rightarrow i} \Gamma_{k \rightarrow \frac{i+1}{2}} \). If \( \lambda e + \mu f \in \ker A_{\frac{i+1}{2} \rightarrow i} \Gamma_{k \rightarrow \frac{i+1}{2}} \), then by Remark 5.1 we have

\[
\lambda e + \mu f \in \ker A_{i-1} \cdots A_{\frac{i+1}{2}} B_{\frac{i+1}{2} \Gamma_k} = \ker g_{i} B_{n-i} \cdots B_{n-1-\frac{i+1}{2}} A_{n-1-\frac{i+1}{2} \Gamma_k} \sigma_k
\] (6.10)

and thus, by combining the admissibility conditions and the cup relation, we have

\[
\lambda e - \mu f \in \ker B_{n-\frac{i+1}{2} \rightarrow n-i} A_{k \rightarrow n-i} \Gamma_{k \rightarrow \frac{i+1}{2}} \cdots \Gamma_k = \ker A_{m \rightarrow n-i} A_{s \rightarrow m} \Gamma_{k \rightarrow s}
\]
(6.11)

\[
= \ker B_{m \rightarrow s} A_{s \rightarrow m} \Gamma_{k \rightarrow s} = \ker B_{n-s} A_{k \rightarrow s} \Gamma_k
\]

\[
= \ker A_{\frac{i+1}{2} \rightarrow i} \Gamma_{k \rightarrow \frac{i+1}{2}}.
\]
Therefore, it splits into two cases \( \lambda = 0 \) or \( \mu = 0 \), which correspond to whether \( A_{i+1} \to \Gamma_{k-i+\frac{i+1}{2}} \) is equal to \( \pm A_{i+1} \to \Gamma_{k-i+\frac{i+1}{2}} \).

For the case \( n - k > k \), a similar analysis reduces the discussion to investigate the 1-dimensional kernel of the map

\[
(A_k \to c+1 \Gamma_k | B_{n-k \to -c} \Gamma_{n-k}) : \langle f_{c+1}, e_{i-c} \rangle \to V_i = C^k.
\]  

(6.12)

A simpler deduction (due to the lack of the non-trivial involution \( \sigma_k \)) leads to that \( A_{k \to c+1} \Gamma_k = \pm B_{n-k \to -c} \Gamma_{n-k} \) (cf. Example 5.5).

Assume now \( a \) has more than one symmetric cup. We pick the two innermost nested cups connecting \( j+1, n-j \), and \( i+1, n-i \), respectively, such that \( i < j < m \). Write for short \( p = \lfloor \frac{i+j}{2} \rfloor > q = \lfloor \frac{i+j}{2} \rfloor \).

Note that \( F_{j+1} = \ker \left( \left( A_{1 \to j} \Gamma_{n-1} | \cdots | \Gamma_{j+1} \right) \right) \). We have \( \dim \ker A_{p+q \to j+1} \Gamma_{p+q} \neq 2 \), otherwise \( 2(p+q) \leq \dim F_{j+1} = j+1 + \{2p-1, 2p-2\} \), which is absurd.

Hence, there are two exclusive cases: whether \( \dim \ker A_{p+q \to j+1} \Gamma_{p+q} = 0 \) or 1. By a tedious analysis using admissibility conditions and the cups relations in \( a \), one recover the second condition in (6.2) assuming \( \dim \ker A_{p+q \to j+1} \Gamma_{p+q} = 1 \), and vice versa. \( \square \)

6.2. Two Jordan blocks of equal sizes. We have the following result which generalizes [Fun03] and [SW12] to type D (see also Proposition 2.1).

Theorem 6.7. Let \( x \in N^D \) be a nilpotent element of Jordan type \((k, k)\) as in (1.2) and \( \hat{a} \in B^D_{k,k} \). Then \( \tilde{\varphi}(M^\hat{a}) \subseteq B^D_{k,k} \) consists of the pairs \((x, F_x)\) which satisfy the following conditions imposed by the diagram \( \hat{a} \):

(i) If vertices \( i < j \leq m \) are connected by a cup without a marker, then

\[
x^{-\frac{1}{2}(j-i+1)} F_{i-1}.
\]

(ii) If vertices \( i < j \leq m \) are connected by a marked cup, then

\[
x^{\frac{1}{2}(j-i+1)} F_{i-1}.
\]

(iii) If vertex \( i \) is connected to a marked ray, then

\[
\langle e_1, \ldots, e_{\frac{1}{2}(i+1)}, f_1, \ldots, f_{\frac{1}{2}(i-1)} \rangle.
\]

(iv) If vertex \( i \) is connected to a ray without a marker, then

\[
\langle e_1, \ldots, e_{\frac{1}{2}(i-1)}, f_1, \ldots, f_{\frac{1}{2}(i+1)} \rangle.
\]

Proof. Part (i) is exactly the type A result. For part (ii), write \( p = \lfloor \frac{i+j}{2} \rfloor > q = \lfloor \frac{i+j}{2} \rfloor \) for short. According to the construction, in \( \hat{a} \) there are no rays to the right of any marked cup and hence

\[
(i, j) = \begin{cases} (2q, 2p-1) & \text{if } m \in 2\mathbb{Z} + 1, \\ (2q+1, 2p) & \text{if } m \in 2\mathbb{Z}. \end{cases}
\]  

(6.13)

Hence, the second relation in (6.2) now reads

\[
\ker B_{p+q \to i} = \ker A_{p+q \to j}. \tag{6.14}
\]

On the other hand, by Corollary 3.3 we have

\[
x^p F_j = \ker (A_{p+1} \to j \Gamma_{p+1} | \cdots | \Gamma_{j}),
\]

\[
x^q F_i = \ker (A_{q+1} \to i \Gamma_{q+1} | \cdots | \Gamma_{i}). \tag{6.15}
\]

which are equal by combining (6.14) and admissibility conditions.

Parts (iii) and (iv) follow from that \( \Gamma_{k \to i+1} = \pm \Gamma_{k \to i+\frac{i+1}{2}} \sigma_k \) is equivalent to \( \ker \Gamma_{k \to i+1} \to \Gamma_{k \to i+\frac{i+1}{2}} = \langle e \rangle \) or \( \langle f \rangle \) due to our choice of \( \sigma_k \). \( \square \)
Example 6.8. Let \((x, F_\bullet) \in \mathcal{C}(M_{\tilde{a}})\), where \(\tilde{a} \in B_{5,3}^D\) is the cup diagram below:

\[ 1 \ 2 \ 3 \ 4 \ 5 \]

Since vertices 1, 2 are connected by a cup without a marker, we have

\[ F_2 = x^{-1}F_0 = \langle e_1, f_1 \rangle. \]

Next, vertex 3 is connected to a marked ray, so

\[ F_3 = \langle e_1, e_2, f_1 \rangle. \]

Finally, vertices 4, 5 are connected by a marked cup, and thus

\[ x^3F_5 = x^2F_4. \]

6.3. Two Jordan blocks of unequal sizes. For this section we assume \(n - k > k\). Recall that \(\rho(i) \in \mathbb{Z}_{\geq 0}\) counts the number of rays (including itself) to the left of \(i\), and \(c(i) = \frac{i - \rho(i)}{2}\) is the total number of cups to the left of \(i\).

Theorem 6.9. Let \(x \in N^D\) be a nilpotent element of Jordan type \((n-k, k)\) as in (4.2) and \(\tilde{a} \in B_{n-k,k}^D\). Then \(\mathcal{C}(M_{\tilde{a}}) \subseteq B_{n-k,k}^D\) consists of the pairs \((x, F_\bullet)\) which satisfy the following conditions imposed by the diagram \(\tilde{a}\):

(i) If vertices \(i < j \leq m\) are connected by a cup without a marker, then

\[ F_j = x^{-\frac{1}{2}(j-i+1)}F_{i-1}. \]

(ii) If vertices \(i < j \leq m\) are connected by a marked cup, then

\[ x^{\frac{j-i}{2}}F_j = x^{\frac{j}{2}}F_i. \]

(iii) If vertex \(i\) is connected to a marked ray, then

\[ F_i = \langle e_1, \ldots, e_{i-c(i)-1}, f_1, \ldots, f_{c(i)}, f_{c(i)+1} + e_{i-c(i)} \rangle. \]

(iv) If vertex \(i\) is connected to the rightmost ray without a marker, then

\[ F_i = \langle e_1, \ldots, e_{i-c(i)-1}, f_1, \ldots, f_{c(i)}, f_{c(i)+1} - e_{i-c(i)} \rangle. \]

(v) If vertex \(i\) is connected to an unmarked ray that is not the rightmost, then

\[ F_i = \langle e_1, \ldots, e_{i-c(i)}, f_1, \ldots, f_{c(i)} \rangle. \]

Proof. It suffices to prove part (ii) - (iv) regarding the new relations for the rightmost ray. Write \(\rho(i), c = c(i) = \frac{i - \rho(i)}{2}\) for short. By Corollary 3.3 we have \(F_i = \ker(A_{1-i} \Gamma_1 \cdots \Gamma_{i-1})\), and we are to prove that

\[ \ker(A_{c+1-i} \Gamma_{k-c+1}) = 0 = \ker(A_{i-c-i} \Gamma_{n-k-i-c}), \]

\[ \ker(A_{c+1-i} \Gamma_{k-c+1} | A_{i-c-i} \Gamma_{n-k-i-c}) = \mathbb{C}. \] (6.16)

The former line follows from the admissibility conditions and (5.2), while the latter line follows from \(A_{c+1-i} \Gamma_{k-c+1} = \pm A_{i-c-i} \Gamma_{n-k-i-c}\), which is a consequence of (6.3). Therefore, the additional basis vector is \(f_{c+1} \pm e_{i-c}\).

Example 6.10. Let \((x, F_\bullet) \in \mathcal{C}(M_{\tilde{a}})\), where \(\tilde{a} \in B_{5,3}^D\) is the cup diagram below:

\[ 1 \ 2 \ 3 \ 4 \]

Firstly, vertex 1 is connected to an unmarked ray that is not the rightmost, so

\[ F_1 = \langle e_1 \rangle. \]
Secondly, although vertex 2 is connected to a ray without a marker, we use a different rule since it is the rightmost ray. Hence,

\[ F_2 = \langle e_1, f_1 - e_2 \rangle. \]

Finally, vertices 2, 3 are connected by a cup without a marker, we obtain that

\[ F_3 = x^{-1}F_1 = \langle e_1, e_2, f_1 \rangle. \]

REFERENCES

[ES16a] M. Ehrig and C. Stroppel, 2-row Springer fibres and Khovanov diagram algebras for type D, Canadian J. Math. 68 (2016), no. 6, 1285–1333.

[ES16b] Diagrammatic description for the categories of perverse sheaves on isotropic Grassmannians, Selecta Math. (N.S.) 22 (2016), no. 3, 14551536.

[Fun03] F.Y.C. Fung, On the topology of components of some Springer fibers and their relation to Kazhdan-Lusztig theory, Adv. Math. 178 (2003), no. 2, 244–276.

[Ger61] M. Gerstenhaber, Dominance over the classical groups, Ann. of Math. 74 (1961), no. 3, 532–569.

[HL14] A. Henderson and A. Licata, Diagram automorphisms of quiver varieties, Adv. Math. 267 (2014), 225–276.

[Li19] Y. Li, Quiver varieties and symmetric pairs, Represent. Theory 23 (2019), 1–56.

[LS13] T. Lejczyk and C. Stroppel, A graphical description of \((D_n, A_{n-1})\) Kazhdan-Lusztig polynomials, Glasgow Math. J. 55 (2013), no. 2, 313–340.

[Lus91] G. Lusztig, Quivers, perverse sheaves, and quantized enveloping algebras, J. Amer. Math. Soc. 4 (1991), no. 2, 365–421.

[Lus93] , Introduction to quantum groups, Progress in Mathematics, vol. 110, Birkhäuser Boston, Inc., Boston, MA, 1993.

[Lus98] , On quiver varieties, Adv. Math. 136 (1998), no. 1, 141–182.

[Maf05] A. Maffei, Quiver varieties of type A, Comment. Math. Helv. 80 (2005), no. 1, 1–27.

[Nak94] H. Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras, Duke Math. J. 76 (1994), no. 2, 365–416.

[Nak98] , Quiver varieties and Kac-Moody algebras, Duke Math. J. 91 (1998), no. 3, 515–560.

[Nak17] , Introduction to quiver varieties – for ring and representation theorists, Proceedings of the 49th Symposium on Ring Theory and Representation Theory, Symp. Ring Theory Represent. Theory Organ. Comm., Shimane, 2017, pp. 96–114.

[Spa76] N. Spaltenstein, The fixed point set of a unipotent transformation on the flag manifold, Nederl. Akad. Wetensch. Proc. Ser. A, vol. 79, 1976, pp. 452–456.

[Spa82] , Classes Unipotentes et Sous-groupes de Borel, Lecture Notes in Mathematics, vol. 946, Springer-Verlag, 1982.

[Spr76] T. A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, Invent. Math. 36 (1976), 173–207.

[Spr78] , A construction of representations of Weyl groups, Invent. Math. 44 (1978), 279–293.

[SW12] C. Stroppel and B. Webster, 2-block Springer fibers: convolution algebras and coherent sheaves, Comment. Math. Helv. 87 (2012), 477–520.

[Var79] J.A. Vargas, Fixed points under the action of unipotent elements of \(SL_n\) in the flag variety, Bol. Soc. Mat. Mexicana 24 (1979), no. 1, 1–14.

[vL89] M. van Leeuwen, A Robinson-Schensted algorithm in the geometry of flags for classical groups, PhD thesis, Rijksuniversiteit Utrecht, 1989.

[Wil18] A. Wilbert, Topology of two-row Springer fibers for the even orthogonal and symplectic group, Trans. Amer. Math. Soc. 370 (2018), 2707–2737.