Rainbow Pancyclicity in Graph Systems

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Abstract

Let \( G_1, \ldots, G_n \) be graphs on the same vertex set of size \( n \), each graph with minimum degree \( \delta(G_i) \geq n/2 \). A recent conjecture of Aharoni asserts that there exists a rainbow Hamiltonian cycle i.e. a cycle with edge set \( \{e_1, \ldots, e_n\} \) such that \( e_i \in E(G_i) \) for \( 1 \leq i \leq n \). This can be viewed as a rainbow version of the well-known Dirac theorem. In this paper, we prove this conjecture asymptotically by showing that for every \( \varepsilon > 0 \), there exists an integer \( N > 0 \), such that when \( n > N \) for any graphs \( G_1, \ldots, G_n \) on the same vertex set of size \( n \) with \( \delta(G_i) \geq (1/2 + \varepsilon)n \), there exists a rainbow Hamiltonian cycle. Our main tool is the absorption technique. Additionally, we prove that with \( \delta(G_i) \geq n + 1/2 \) for each \( i \), one can find rainbow cycles of length \( 3, \ldots, n - 1 \).

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1 Introduction

Let \( G_1, \ldots, G_t \) be \( t \) graphs on the same vertex set \( V \) of size \( n \) where \( t \) is a positive integer. We denote the edge set of \( G_i \) by \( E(G_i) \) and assume that each edge in \( E(G_i) \) is coloured by \( i \) for \( 1 \leq i \leq t \). Let \( S \) be an edge set that is a subset of \( \bigcup_{i=1}^{t} E(G_i) \) and we say \( S \) is rainbow if any pair of edges in \( S \) have distinct colours. Rainbow Hamiltonian cycles

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have been studied by many authors. An edge-coloured graph $G$ is $k$-bounded if no colour appears more than $k$ times. Erdős, Nešetřil and Rödl [7] studied the problem for which $k$ any $k$-bounded $K_n$ contains a rainbow Hamiltonian cycle and they showed that $k$ could be any constant. Hahn and Thomassen [8] demonstrated that $k$ could grow as fast as $n^{\frac{1}{3}}$ and conjectured that the growth of $k$ could in fact be linear. This was confirmed by Albert, Frieze and Reed [4]. A recent result from Coulson and Perarnau [6] further strengthened this by replacing the complete graph with any Dirac graph. More precisely, they proved that there exists $\mu > 0$ and positive integer $n_0$ such that if $n \geq n_0$ and $G$ is a $\mu n$-bounded edge-coloured graph on $n$ vertices with minimum degree $\delta(G) \geq \frac{n}{2}$, then $G$ contains a rainbow Hamiltonian cycle. For rainbow Hamiltonian cycles in graph systems, Aharoni et al. [3] recently gave the following elegant conjecture, which is a natural generalization of Dirac’s theorem to the case of graph systems:

**Conjecture 1.** Given graphs $G_1, \ldots, G_n$ on the same vertex set of size $n$, if each graph has minimum degree at least $\frac{n}{2}$, then there exists a rainbow Hamiltonian cycle.

There have been several papers studying other rainbow structures in graph systems. For example, a well-known conjecture of Aharoni and Berger [1] asserts that if $M_1, \ldots, M_n$ are $n$ matchings of size at least $n+1$ on the same vertex set $V = X \cup Y$ where $X$ and $Y$ are disjoint and all edges of $M_i$ are between $X$ and $Y$, then there exists a rainbow matching of size $n$. This conjecture generalizes the famous Brualdi-Stein Conjecture, which asserts that every $n \times n$ Latin square has a partial transversal of size $n - 1$. The Aharoni-Berger Conjecture has been confirmed asymptotically by Pokrovskiy [12]. For more details about this topic, see [14].

In this paper, we prove an asymptotic version of Conjecture 1:

**Theorem 2.** For every $\varepsilon > 0$, there exists an integer $N > 0$, such that when $n > N$ for any graphs $G_1, \ldots, G_n$ on the same vertex set of size $n$, each graph with minimum degree $\delta(G_i) \geq (\frac{1}{2} + \varepsilon)n$, there exists a rainbow Hamiltonian cycle.

After we submitted this paper, Joos and Kim in [10] proved Conjecture 1 using a method different from ours. Nevertheless, we believe that our approach is of independent interest and could be applied to attack similar problems in hypergraphs.

Furthermore, we show that given $n$ graphs $G_1, \ldots, G_n$ with $\delta(G_i) \geq \frac{n+1}{2}$ for $1 \leq i \leq n$, we can find rainbow cycles with all possible lengths except a Hamiltonian one:

**Theorem 3.** Given graphs $G_1, \ldots, G_n$ on the same vertex set of size $n$, each graph with minimum degree $\delta(G_i) \geq \frac{n+1}{2}$, there exist rainbow cycles of length $3, 4, \ldots, n - 1$.

Combining Theorem 3 with the result of Joos and Kim, we derive that any $G_1, \ldots, G_n$ satisfying the assumption of Theorem 3 indeed contain rainbow cycles of all possible lengths $3, \ldots, n$. The lower bound of Theorem 3 is tight because one can take $n$ copies of $K_{\frac{n}{2}}$ where $n$ is even and there does not exist any odd rainbow cycle in such a system.

The main tool behind the proof of Theorem 2 is the absorbing method that was introduced by Rödl, Ruciński and Szemerédi [13]. Here we apply a rainbow version of the approach of Lo [11] by constructing a short rainbow cycle $C$ such that for any rainbow
path $P = v_1 \cdots v_p$ disjoint from $C$ and a new colour $s$ where the colour set of $P$ is also disjoint from that of $C$, we can absorb $P$ into $C$. In other words, we replace some edge $u_iu_{i+1}$ of $C$ by a path $u_iPu_{i+1}$, where $u_iu_1$ is coloured with $s$ and $v_pv_{i+1}$ is coloured with the colour of $u_iu_{i+1}$ in $C$. Finally, we find a rainbow Hamiltonian path $P$ on $V(G) \setminus V(C)$ and absorb $P$ into $C$ by the property of $C$ and thus obtain a rainbow Hamiltonian cycle.

2 Preliminaries and Notation

Let $G_1, \ldots, G_n$ be $n$ graphs on the same vertex set $V$ where $|V| = n$. Let $\delta(G_i)$ be the minimum degree of each $G_i$ for $1 \leq i \leq n$. By our assumption, we identify this graph system with an edge-coloured multigraph $G$ where $E(G)$ is the disjoint union of $E(G_i)$ for $i \in [n]$ and each edge in $E(G_i)$ is coloured by $i$. For any subgraph $H$ of $G$, let $\text{Col}(H)$ be the set of colours used by the edges of $H$. For every vertex $v \in V(G)$ and any colour $c \in [n]$, let $N_c(v)$ be the set of neighbours of $v$ that are adjacent to $v$ by an edge coloured by $c$. Let $S$ be any subset of $V$, we denote $N_c(v) \cap S$ by $N_c(v,S)$ and $|N_c(v) \cap S|$ by $d_c(v,S)$. For each pair of vertices $v_1, v_2 \in V(G)$, let $\text{Col}(v_1,v_2)$ be the set of colours used for the edges between $v_1$ and $v_2$ (Col$(v_1,v_2)$ is empty if there are no edges between $v_1$ and $v_2$). We will use the following version of Chernoff’s bound [9].

**Lemma 4.** Let $X$ be a binomially distributed random variable and $0 < \varepsilon < \frac{3}{2}$, then

$$P(|X - E(X)| \geq \varepsilon E(X)) \leq 2e^{-\frac{2\varepsilon^2}{3}E(X)}.$$

We first prove the following useful lemma:

**Lemma 5.** Let $P = v_1 \cdots v_p$ be a rainbow path and let $c, c'$ be two colours not used on $P$. If $d_c(v_1, V(P)) + d_c(v_p, V(P)) \geq p$, then there is a rainbow cycle of length $p$.

**Proof.** If $\{c, c'\} \cap \text{Col}(v_1,v_p) \neq \emptyset$, then $C = v_1 \cdots v_pv_1$ is a rainbow cycle by choosing the colour of $v_1v_p$ to be an element in $\{c, c'\} \cap \text{Col}(v_1,v_p)$. So we assume that $\{c, c'\} \cap \text{Col}(v_1, v_p) = \emptyset$. Suppose that there exists no rainbow cycle of length $p$. For each vertex $v_i$ with $v_{i+1} \in N_c(v_1, V(P))$, where $2 \leq i \leq p - 2$, we get that $v_i \notin N_c(v_p, V(P))$ since otherwise the cycle $v_1v_2 \cdots v_{i-1}v_p v_{i-1} \cdots v_{i+1}v_1$ must be a rainbow cycle where the colours of $v_1v_{i+1}$ and $v_pv_i$ are chosen to be $c$ and $c'$. Thus we get $d_c(v_1, V(P)) + d_c(v_p, V(P)) \leq p - 2$, which implies $p \leq p - 1$, a contradiction. \hfill $\square$

Our first result shows that a rainbow Hamiltonian path exists under a slightly weaker condition than that of Conjecture 1:

**Proposition 6.** Given graphs $G_1, \ldots, G_n$ on the same vertex set $V$ of size $n$, where $\delta(G_i) \geq \frac{n-1}{2}$ for $i \in [n]$, then there exists a rainbow Hamiltonian path.

**Proof.** Suppose not, let $P = v_1 \cdots v_k$, where $k \leq n - 1$, be a rainbow path with the maximum length. Thus there exist at least two colours $c, c'$ that are not used by the edges in $P$. Now consider the neighbourhood $N_c(v_1)$ and $N_c(v_k)$, we have

$$d_c(v_1) + d_c(v_k) \geq \frac{n - 1}{2} + \frac{n - 1}{2} = n - 1.$$
For each vertex \( u \in V - V(P) \), we have \( u \notin N_c(v_1) \cup N_c(v_k) \) and otherwise we can extend \( P \) into a larger rainbow path, a contradiction. Thus we get \( N_c(v_1), N_c(v_k) \subseteq V(P) \). However, since \( |V(P)| \leq n - 1 \) and \( d_c(v_1, V(P)) + d_c(v_k, V(P)) \geq n - 1 \geq |V(P)| \), by Lemma 5 we get a rainbow cycle \( C \) of length \( k \). Suppose that the colour \( c'' \) is not used by this cycle. Since the monochromatic graph coloured by \( c'' \) is connected, at least one edge \( e_0 \) coloured by \( c'' \) is between \( V(C) \) and \( V - V(C) \). Therefore, \( V(C) \cup \{e_0\} \) contains a rainbow path with length \( k + 1 \), a contradiction.

The lower bound here is best possible. One can take \( n \) copies of \( K^2_{2 - 1, 2^+1} \) where \( n \) is even and there does not exist a rainbow Hamiltonian path since \( K^2_{2 - 1, 2^+1} \) does not contain a Hamiltonian path.

### 3 Proof of Theorem 3

In this section, we give a proof of Theorem 3. Let \( G = (V, \bigcup_{i=1}^{n} E(G_i)) \) be the edge-colored multigraph with \( G_i \) as the graph of color \( i \). We first find a rainbow cycle of length \( n - 1 \) by following a classical proof of Dirac theorem. Then we obtain a rainbow cycle of length \( n - 2 \) or \( n - 3 \) and use it to build cycles of other lengths.

#### Claim 7. \( G \) contains a rainbow cycle of length \( n - 1 \).

**Proof.** By Proposition 6, we first find a rainbow Hamiltonian path \( P = v_1v_2 \cdots v_n \). Without loss of generality, suppose the colour of edge \( v_iv_{i+1} \) is \( i \) for \( 1 \leq i \leq n - 1 \) and the only colour that does not appear in \( P \) is \( n \). Now consider the subpath \( P' = v_1v_2 \cdots v_{n-1} \). Since \( |N_{n-1}(v_1)\backslash\{v_n\}| \geq \frac{n-1}{2} \) and \( |N_n(v_{n-1})\backslash\{v_n\}| \geq \frac{n-1}{2} \), we get \( d_{n-1}(v_1, V(P')) + d_n(v_n, V(P')) \geq n - 1 = |V(P')| \). By Lemma 5, we can find a rainbow cycle of length \( n - 1 \).

#### Claim 8. \( G \) contains either a rainbow cycle of length \( n - 2 \) or a rainbow cycle of length \( n - 3 \).

**Proof.** Suppose that \( G \) neither contains a cycle of size \( n - 3 \) nor \( n - 2 \). By Proposition 6, we can find a rainbow path \( P_1 = v_1v_2 \cdots v_{n-3} \) whose order is \( n - 3 \) and, without loss of generality, suppose the colour of edge \( v_iv_{i+1} \) is \( i \) for \( 1 \leq i \leq n - 4 \) and the set of colours that are not used in \( P_1 \) is \( S = \{n - 3, n - 2, n - 1, n\} \). We can deduce that \( N_{n-1}(v_1) \cap N_n(v_{n-3}) \cap (V(G) \backslash V(P_1)) = \emptyset \) since otherwise we already find a rainbow cycle of length \( n - 2 \), a contradiction. Now we get \( d_{n-1}(v_1, V(P_1)) + d_n(v_{n-3}, V(P_1)) \geq n - 2 \geq |V(P_1)| \). By Lemma 5, we can find a rainbow cycle of length \( n - 3 \), a contradiction.

Let \( C = v_1 \cdots v_p \) be a rainbow cycle where \( p = n - 2 \) or \( n - 3 \). In the remainder of the proof, we let \( v_i = v_{i-p} \) for \( i > p \). We will use the following claim as a tool to analyse the structure of \( G \) when it does not contain rainbow cycles of all length 3, \ldots, \( p + 1 \).

#### Claim 9. Let \( c, c' \) be two colours not used on \( C \) and \( x \in V \setminus V(C) \). If \( d_c(x, V(C)) + d_{c'}(x, V(C)) \geq p \), then one of the following properties is true:
(1) there exist rainbow cycles of length 3, \ldots, p + 1;

(2) \( d_c(x, V(C)) + d_c(x, V(C)) = p \) and we can partition \( V(C) \) into disjoint sets \( S_1 \) and \( S_2 \), where \( S_1 = \{ v_i+j-2 : v_j \in N_c(x, V(C)) \} \) and \( S_2 = N_c(x, V(C)) \) for some \( 3 \leq i \leq p+1 \).

**Proof.** Suppose that \( d_c(x, V(C)) + d_c(x, V(C)) \geq p \) and there is no rainbow cycle of length \( i \) for some \( 3 \leq i \leq p+1 \), thus for each vertex \( v_j \in N_c(x, V(C)) \), we have \( v_i+j-2 \not\in N_c(x, V(C)) \) since otherwise the cycle \( xv_jv_{j+1} \cdots v_{j+i-2}x \) is a rainbow cycle of length \( i \) by choosing the colours of \( xv_j \) and \( xv_{j+i-2} \) to be \( c \) and \( c' \). Therefore, we get \( S_1 \cap S_2 = \emptyset \) by definition. However, since \( |S_1| + |S_2| \geq p \) and \( S_1 \cup S_2 \subseteq V(C) \) it follows that \( V(C) \) is partitioned into \( S_1 \) and \( S_2 \) and \( |S_1| + |S_2| = p \), which implies \( d_c(x, V(C)) + d_c(x, V(C)) = p \).

**Case 1. \( p = n - 2 \).**

Suppose \( V(G) \setminus V(C) = \{ v', v'' \} \) and the colours not used by \( C \) are \( n-1 \) and \( n \). Suppose that for some \( 3 \leq j \leq n-1 \), there does not exist a rainbow cycle of size \( j \) in \( G \). By Claim 9, we conclude that \( d_{n-1}(v', V(C)) + d_n(v', V(C)) \leq n - 2 \), which implies that \( d_{n-1}(v', v'') + d_n(v', v'') \geq n + 1 - (n - 2) = 3 \). This is a contradiction. Therefore, \( G \) contains rainbow cycles of all sizes \( 3, \ldots, n - 1 \).

**Case 2. \( p = n - 3 \).**

Suppose \( V(G) \setminus V(C) = \{ v'_1, v'_2, v'_3 \} \) and the colours not used by \( C \) are \( n-2, n-1 \) and \( n \). Suppose that for some \( 3 \leq i \leq n - 2 \), there does not exist a rainbow cycle of size \( i \) in \( G \). We know that \( d_{n-1}(v'_3, V(C)) + d_n(v'_3, V(C)) \geq 2(n+1) - 2 \geq n - 3 \), thus by Claim 9 we get \( d_{n-1}(v'_3, V(C)) + d_n(v'_3, V(C)) = n - 3 \). This implies that \( d_{n-1}(v'_3, \{ v'_1, v'_2 \}) = d_n(v'_3, \{ v'_1, v'_2 \}) = 2 \) and hence \( \{ n - 1, n \} \subseteq \mathrm{Col}(v'_3 v'_1) \cap \mathrm{Col}(v'_3 v'_2) \).

By symmetry we now suppose that \( \mathrm{Col}(v'_a v'_b) = \{ n-2, n-1, n \} \) for every \( 1 \leq a < b \leq 3 \). Let \( T_1 = \{ v_{j+i-3} : v_j \in N_{n-2}(v'_1, V(C)) \} \) and \( T_2 = N_{n-1}(v'_2, V(C)) \), by an analogy to the proof of Claim 9, we find that \( T_1 \cap T_2 = \emptyset \), otherwise suppose that \( v_{j+i-3} \in T_2 \) for some \( j \), we thus have \( v'_2 v'_1 v_j v_{j+i-3} \) is a rainbow cycle with length \( i \) by choosing the colours of \( v'_2 v'_1, v'_1 v_j \) and \( v_{j+i-3} v'_2 \) to be \( n, n - 2 \) and \( n - 1 \), which is a contradiction. We actually get:

\[
 n - 3 \leq 2 + \frac{n + 1}{2} - 2 - \frac{n + 1}{2} - 2 \leq d_{n-2}(v'_1, V(C)) + d_{n-1}(v'_2, V(C)) \leq |V(C)| = n - 3,
\]

thus all the inequalities above must be equalities and we get

\[
 |T_1| + d_{n-1}(v'_2, V(C)) = n - 3,
\]

which implies that \( V(C) \) is partitioned into \( T_1 \) and \( T_2 \). Since all colours and vertices are symmetric, the similar conclusion follows by considering \( T_1 \) and \( N_a(v'_b, V(C)) \) for any \( n - 1 \leq a \leq n \) and \( 2 \leq b \leq 3 \). Thus, we finally obtain \( T_2 = N_a(v'_b, V(C)) \) for any \( n - 1 \leq a \leq n \) and \( 2 \leq b \leq 3 \). Now let \( T'_1 = \{ v_{j+i-3} : v_j \in N_a(v'_1, V(C)) \} \) and recall that \( T_2 = N_{n-1}(v'_2, V(C)) \). Therefore, we reach the similar conclusion that \( T_2 = N_b(v'_b, V(C)) \) for any \( n - 2 \leq a \leq n - 1 \) and \( 2 \leq b \leq 3 \) by considering \( T'_1 \) and \( T_2 \). This implies...
$T_2 = N_a(v'_b, V(C))$ for any $n - 2 \leq a \leq n$ and $2 \leq b \leq 3$. By symmetry, we actually get $T_2 = N_a(v''_b, V(C))$ for any $n - 2 \leq a \leq n$ and $1 \leq b \leq 3$.

Now we claim that there exists some $v_{j_0} \in T_2$ such that $v_{j_0-1} \notin T_2$. Suppose not, for each $v_j \in T_2$ we have $v_{j-1} \in T_2$. This implies $T_2 = V(C)$ but this is impossible since $T_1 \neq \emptyset$. Now since $v_{j_0-1} \notin T_2$ we get $v_{j_0-1} \in T_1$ and there exists some $v_{j_1} \in N_{n-2}(v'_i, V(C))$ such that $j_0 - 1 = j_1 + i - 3$, which implies $j_0 = j_1 + i - 2$. However, in this case the cycle $v'_1v_{j_1}v_{j_1+1}\cdots v_{j_0}v'_1$ is a rainbow cycle of length $i$ by choosing the colours of $v'_1v_{j_1}$ and $v'_1v_{j_0}$ to be $n - 2$ and $n - 1$, a contradiction. This concludes the proof. 

4 Proof of Theorem 2

In this section, we give a proof of Theorem 2 by proving a rainbow type of absorbing lemma. By $0 < \alpha < \beta$ we mean that there exists an increasing function $f : \mathbb{R} \to \mathbb{R}$ such that the subsequent argument is valid for any $0 < \alpha < f(\beta)$. We first introduce the absorbing lemma for Theorem 2:

**Lemma 10.** Let $n, \mu, \varepsilon$ be such that $\frac{1}{n} \ll \mu \ll \varepsilon$. For any graphs $G_1, \ldots, G_n$ on the same vertex set of size $n$, each graph having minimum degree at least $(\frac{1}{n} + \varepsilon)n$, there exists a rainbow cycle $C$ with length at most $\mu n$ such that for every rainbow path $P$ with $V(P) \cap V(C) = \emptyset$ and $\text{Col}(P) \cap \text{Col}(C) = \emptyset$, if $s$ is a colour that is not used by $C$ and $P$, then there exists a rainbow cycle $C'$ with

(i) $V(C') = V(C) \cup V(P)$;

(ii) $\text{Col}(P \cup C) \cup \{s\} = \text{Col}(C')$.

Combining Lemma 10 and Proposition 6, we immediately reach a proof of Theorem 2 as follows:

**Proof of Theorem 2.** Let $C$ be the absorbing cycle we given in Lemma 10. Now for each $i$, let $G'_i = G_i - V(C)$, the subgraph of $G_i$ induced on $V(G_i) \setminus V(C)$. Then $\delta(G'_i) \geq \frac{1+\varepsilon}{2} |V(G'_i)|$. Let $W$ be the set of colours that do not appear on any edge of $C$. One can thus construct a rainbow Hamiltonian path $P_1 = v_0v_1 \cdots v_t$ using exactly $|W| - 1$ colours of $W$ by Proposition 6. Suppose $s_1$ is the unique colour in $W$ that is not used by $P_1$. By Lemma 10 there exists a rainbow cycle $C'$ with $V(C') = V(C) \cup V(P)$ and $\text{Col}(P \cup C) \cup \{s_1\} = \text{Col}(C')$, which implies $C'$ is a rainbow Hamiltonian cycle and thus we conclude the proof. 

In the remaining part of this section, we give a proof of Lemma 10. First we introduce some notation and basic results. For any pair of two not necessarily distinct vertices $x_1, x_2 \in V(G)$ and four distinct colours $1 \leq s, i, j, k \leq n$, we define the set of absorbing paths $A_{s,i,j,k}(x_1, x_2)$ to be the family of edge-coloured 3-paths $P$ that satisfy the following conditions:

(i) $P = v_1v_2v_3v_4$ where $\{v_1, v_2, v_3, v_4\} \cap \{x_1, x_2\} = \emptyset$;

(ii) the edges $v_1v_2, v_2v_3, v_3v_4$ are coloured respectively by $i, j, k$;

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(iii) \( s \in \text{Col}(x_1v_2) \) and \( j \in \text{Col}(x_2v_3) \).

For every path \( P \) in \( A_{s,i,j,k}(x_1, x_2) \), we say that \( P \) is an absorbing path for \((x_1, x_2)\) with colour pattern \((s, i, j, k)\). In practice, we always choose \( x_1 \) and \( x_2 \) to be two endpoints of some path \( Q \).

**Claim 11.** For each pair \((x_1, x_2)\) and four distinct colours \( s, i, j, k \), we have \( |A_{s,i,j,k}(x_1, x_2)| \geq \frac{cn^4}{8} \) when \( n \) is sufficiently large.

**Proof.** First, choose a vertex \( v_2 \in N_s(x_1) \setminus \{x_2\} \). Pick another vertex \( v_3 \in (N_i(v_2) \cap N_j(x_2)) \setminus \{x_1\} \). The total number of such \( v_2, v_3 \) is at least \( \left( \frac{n}{2} + \varepsilon n - 1 \right)(2\varepsilon n - 1) \). Now fix \( v_2 \) and \( v_3 \). Choose \( v_1 \in N_i(v_2) \cap \{x_1, x_2, v_3\} \). Choose another vertex \( v_4 \in N_k(v_3) \setminus \{x_1, x_2, v_1, v_2\} \). Note that the total number of such \( v_1, v_4 \) is at least \( \left( \frac{n}{2} + \varepsilon n - 3 \right)(\frac{n}{2} + \varepsilon n - 4) \) and hence we derive that there exist at least

\[
\left( \frac{n}{2} + \varepsilon n - 1 \right)(2\varepsilon n - 1) \left( \frac{n}{2} + \varepsilon n - 3 \right) \left( \frac{n}{2} + \varepsilon n - 4 \right) \geq \frac{\varepsilon n^4}{8}
\]

absorbing paths for \((x_1, x_2)\) when \( n \) is sufficiently large. \( \square \)

**Proof of Lemma 10.** Let \( \mu_1 = \mu / 5 \) and \( \ell \) be new constant such that \( [\mu_1 n] - 1 \leq \ell \leq [\mu_1 n] + 1 \) and \( \ell \) is divisible by 3. For simplicity, we assume that \( \ell = \mu_1 n \). We fix \( \ell / 3 \) groups of colours \( C_i = \{3i - 2, 3i - 1, 3i\} \) where \( i = 1, \ldots, \ell / 3 \). Let \( \mathcal{P}_{C_i} \) be the set of all the paths \( P = v_0v_1v_2v_3 \) in \( G \) where the colours of \( v_0v_1, v_1v_2, v_2v_3 \) are \( 3i - 2, 3i - 1, 3i \) for all \( 1 \leq i \leq \ell / 3 \).

Now consider a random set \( W \) by selecting an element from each \( \mathcal{P}_{C_i} \) independently where every element in \( \mathcal{P}_{C_i} \) is chosen with probability \( 1 / |\mathcal{P}_{C_i}| \). For any colour \( s \) and any pair \((x_1, x_2)\), set \( A_s(x_1, x_2) = \bigcup_{i=1}^{\ell/3} (A_{s,3i-2,3i-1,3i}(x_1, x_2) \cap W) \). Now for each \( i \in [\ell / 3] \), let the random variable \( X_i \) be the indicative variable of the event that \( W \cap A_{s,3i-2,3i-1,3i}(x_1, x_2) \neq \emptyset \). Hence we get \( |A_s(x_1, x_2)| = \sum_{i=1}^{\ell/3} X_i \) and all \( X_i \)'s are independent. Using Claim 11, we get

\[
P(X_i = 1) = \frac{|A_{s,3i-2,3i-1,3i}(x_1, x_2)|}{|\mathcal{P}_{C_i}|} \geq \frac{\varepsilon n^4}{8} / n^4 \geq \frac{\varepsilon}{8}
\]

for \( i \in [\ell / 3] \) and hence

\[
E(|A_s(x_1, x_2)|) = \sum_{i=1}^{\ell/3} E(X_i) \geq \frac{\varepsilon \ell}{24}.
\]

By Lemma 4 with \( \varepsilon = 1/2 \), we see that

\[
P \left( |A_s(x_1, x_2)| < \frac{\varepsilon \ell}{48} \right) \leq 2e^{-\frac{\varepsilon \ell}{384}} \leq 2e^{-\frac{\varepsilon n \mu}{10^n}}.
\]

Now let \( Y \) be the number of pairs of 3-paths in \( W \) that intersect with each other. For some distinct \( 1 \leq i, j \leq \ell / 3 \), let \( Y_{i,j} \) be the indicative variable of the event that the path we choose in \( A_{C_i} \) intersects with the path we choose in \( A_{C_j} \). Thus we have \( Y = \sum_{i,j} Y_{i,j} \).
We claim that the size of set \( \{ P_1, P_2 \} \mid P_1 \in A_{C_1}, P_2 \in A_{C_2}, P_1, P_2 \) are intersecting with each other\) is at most \( 16n^7 \) for fixed \( i, j \). Since the number of \( P_1 \) is at most \( n^4 \), and when \( P_1 \) is fixed, the number of \( P_2 \) that we can choose is at most \( 16n^7 \). Besides, it is obvious that when \( P_1, P_2 \) are fixed, the probability that we have chosen \( P_1, P_2 \) together is \( \frac{1}{|A(C_3)||A(C_4)|} \leq \frac{1}{(n^4/8)^2} \) because \( |A_{C_1}|, |A_{C_2}| \geq \frac{n^4}{8} \) when \( n \) is sufficiently large. Therefore, we get

\[
E(Y) \leq \left( \frac{\ell/3}{2} \right) \cdot 16 \cdot n^7 \cdot \frac{1}{(n^4/8)^2} \leq 10^3 \mu_1^2 n \leq \frac{\varepsilon \mu_1 n}{200}.
\]

Using Markov’s inequality, we get that

\[
P\left( Y \geq \frac{\varepsilon \mu_1 n}{100} \right) \leq \frac{1}{2}.
\]

Now choose sufficiently large \( n \) such that

\[
2n^3 e^{-\frac{\varepsilon \mu_1 n}{100}} + \frac{1}{2} < 1.
\]

Thus by the union bound, with positive possibility, for each \( s \) and any pair \( (x_1, x_2) \) we have (i) \( |A_s(x_1, x_2)| \geq \frac{\varepsilon \mu_1 n}{48} \), and (ii) \( Y < \frac{\varepsilon \mu_1 n}{100} \).

Fix such \( W \), we delete one 3-paths in each intersecting pair of \( W \). Suppose that the remaining path family is \( W' \). Thus \( W' \) is a family containing mutually disjoint 3-paths and for every \( s \) and any pair \( (x_1, x_2) \) we get that

\[
\left| \bigcup_{i=1}^{\ell/3} (A_{s,3i-2,3i-1,3i}(x_1, x_2) \cap W') \right| \geq \frac{\varepsilon \mu_1 n}{48} - \frac{\varepsilon \mu_1 n}{100} \geq \frac{\varepsilon \mu_1 n}{100}.
\]

Let \( W' = \{ P_1, \ldots, P_k \} \) be the path family we found before and let \( S \) be the set of the colours that do not appear in any path in \( W' \). Let \( V' = V(G) \setminus \bigcup_{i=1}^{\ell/3} V(P_i) \). Without loss of generality, we suppose that \( P_i = v_1^{(i)} v_2^{(i)} v_3^{(i)} v_4^{(i)} \) for \( 1 \leq i \leq t \). Now for \( P_1, P_2 \), it is obvious that we can find a vertex \( u_1 \in V' \) such that \( u_1 v_4^{(1)} \) and \( u_1 v_4^{(2)} \) are two edges coloured with distinct colours in \( S \). Delete these two colours from \( S \) and the vertex \( u_1 \) from \( V' \). Repeat the above process for the path pair \( \{ P_2, P_3 \}, \ldots, \{ P_t, P_1 \} \), and at last we find \( u_1, \ldots, u_t \) and a rainbow cycle \( C \) with size at most \( 5\ell = \mu n \) that contains all the vertices in \( \bigcup_{i=1}^{\ell/3} V(P_i) \) and those \( u_i \) where \( 1 \leq i \leq t \). For every rainbow path \( P \subseteq V(G) - C \) such that the colour set of \( P \) is disjoint with the colour set of \( C \), if \( x_1, x_2 \) are two endpoints of \( P \) and \( s \) is a colour that does not appear in \( C \) and \( P \), then the pair \( (x_1, x_2) \) has at least one absorbing path \( P_0 = u_1 u_2 u_3 u_4 \) in \( C \) with colour pattern \( (s, 3i - 1, 3i - 2, 3i) \) for some \( i \in [\ell/3] \) since \( \frac{\varepsilon \mu_1 n}{100} \geq 1 \) when \( n \) is sufficiently large. Therefore, we insert the path \( P \) into the cycle \( C \) to get a rainbow cycle \( C - u_2 u_3 \cup u_2 P u_3 \) where \( x_1 u_2 \) is coloured by \( s \) and \( x_2 u_3 \) is coloured by \( 3i - 2 \), which completes our proof. \( \square \)

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