Integral functionals on $L^p$-spaces: infima over sub-level sets

BIAGIO RICCERI

Dedicated to Professor Alfonso Villani, with esteem and friendship, on his sixtieth birthday

Abstract: In this paper, we establish the following result:

Let $(T, F, \mu)$ be a $\sigma$-finite measure space, let $Y$ be a reflexive real Banach space, and let $\varphi, \psi : Y \rightarrow \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals such that

$$\inf_{y \in Y} \frac{\min\{\varphi(y), \psi(y)\}}{1 + \|y\|^p} > -\infty$$

for some $p > 0$. Moreover, assume that $\varphi$ has no global minima, while $\varphi + \lambda \psi$ is coercive and has a unique global minimum for each $\lambda > 0$.

Then, for each $\gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\}$, with $\gamma \geq 0$, and for each $r > \inf_Y \psi$, if we put

$$V_{\gamma, r} = \left\{ u \in L^p(T, Y) : \int_T \gamma(t)\psi(u(t))d\mu \leq r \int_T \gamma(t)d\mu \right\},$$

we have

$$\inf_{u \in V_{\gamma, r}} \int_T \gamma(t)\varphi(u(t))d\mu = \inf_{\psi^{-1}(r)} \varphi \int_T \gamma(t)d\mu.$$

Key words: Integral functional; $L^p$-space; coercivity; uniqueness; global minimum.

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Here and in the sequel, $(T, F, \mu)$ ($\mu(T) > 0$) is a $\sigma$-finite measure space, $Y$ is a reflexive real Banach space and $\varphi, \psi : Y \rightarrow \mathbb{R}$ are two sequentially weakly lower semicontinuous functionals such that

$$\inf_{y \in Y} \frac{\min\{\varphi(y), \psi(y)\}}{1 + \|y\|^p} > -\infty$$

for some $p > 0$.

For each $\lambda \in [0, \infty]$, we denote by $M_\lambda$ the set of all global minima of $\varphi + \lambda \psi$ or the empty set according to whether $\lambda < +\infty$ or $\lambda = +\infty$. We adopt the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

Moreover, $a, b$ are two fixed numbers in $[0, +\infty]$, with $a < b$, and $\alpha, \beta$ are the numbers so defined:

$$\alpha = \max \left\{ \inf_Y \psi, \sup_{M_b} \psi \right\},$$
\[ \beta = \min \left\{ \sup_Y \psi, \inf M_\alpha \psi \right\}. \]

As usual, \( L^p(T, Y) \) denotes the space of all \( \mu \)-strongly measurable functions \( u : T \to Y \) such that
\[ \int_T \|u(t)\|^p d\mu < +\infty. \]
A functional \( P : Y \to \mathbb{R} \) is said to be coercive provided
\[ \lim_{\|y\| \to +\infty} P(y) = +\infty. \]
The aim of this paper is to establish the following result:

**THEOREM 1.** Assume that the functional \( \varphi + \lambda \psi \) is coercive and has a unique global minimum for each \( \lambda \in ]a, b[. \) Assume also that
\[ \alpha < \beta. \]
Then, for each \( \gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\} \), with \( \gamma \geq 0 \), and for each \( r \in ]a, b[ \), if we put
\[ V_{\gamma, r} = \left\{ u \in L^p(T, Y) : \int_T \gamma(t)\psi(u(t))d\mu \leq r \int_T \gamma(t)d\mu \right\}, \]
we have
\[ \inf_{u \in V_{\gamma, r}} \int_T \gamma(t)\varphi(u(t))d\mu = \inf_{\psi^{-1}(r)} \varphi \int_T \gamma(t)d\mu. \] (2)

The proof of Theorem 1 is entirely based on the following result that we have established in [5]:

**THEOREM A.** Under the assumptions of Theorem 1, for each \( r \in ]a, b[ \), there exists \( \lambda_r \in ]a, b[ \) such that the unique global minimum of \( \varphi + \lambda_r \psi \) lies in \( \psi^{-1}(r) \).

**Proof of Theorem 1.** First, we also assume that
\[ \varphi(0) = \psi(0) = 0. \]
Actually, once we prove the theorem under this additional assumption, the general version is obtained applying the particular version to the functions \( \varphi - \varphi(0) \) and \( \psi - \psi(0) \). Next, observe that the functionals \( \varphi \) and \( \psi \) are Borel (in the weak topology, and so in the strong one too). This implies that, for each \( u \in L^p(T, Y) \), the functions \( \varphi \circ u \) and \( \psi \circ u \) are \( \mu \)-measurable. On the other hand, in view of (1), for some \( c > 0 \), we have
\[ -c\gamma(t)(1 + \|u(t)\|^p) \leq \gamma(t)\min\{\varphi(u(t)), \psi(u(t))\} \]
for all \( t \in T. \) Since \( \gamma \in L^\infty(T) \cap L^1(T) \), the function \( t \to -\gamma(t)(1 + \|u(t)\|^p) \) lies in \( L^1(T) \), and so the integrals \( \int_T \gamma(t)\varphi(u(t))d\mu \) and \( \int_T \gamma(t)\psi(u(t))d\mu \) exist and belong to \( ]-\infty, +\infty[ \).

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For each \( \lambda \in [a, b] \), denote by \( \hat{y}_\lambda \) the unique global minimum in \( Y \) of the functional \( \varphi + \lambda \psi \).

By Theorem A, there exists \( \lambda_r \in [a, b] \) such that

\[
\psi(\hat{y}_\lambda) = r.
\]

So, we have

\[
\varphi(\hat{y}_\lambda) + \lambda_r r \leq \varphi(y) + \lambda_r \psi(y)
\]

for all \( y \in Y \). From this, it clearly follows that

\[
\varphi(\hat{y}_\lambda) = \inf_{\psi^{-1}(r)} \varphi.
\] (3)

Likewise, for each \( u \in L^p(T, Y) \), it follows that

\[
(\varphi(\hat{y}_\lambda) + \lambda_r r) \int_T \gamma(t)d\mu \leq \int_T (\gamma(t)(\varphi(u(t)) + \lambda_r \psi(u(t))))d\mu.
\]

Therefore, for each \( u \in V_{\gamma, r} \), we have

\[
\varphi(\hat{y}_\lambda) \int_T \gamma(t)d\mu \leq \int_T \gamma(t)\varphi(u(t))d\mu,
\]

and hence

\[
\varphi(\hat{y}_\lambda) \int_T \gamma(t)d\mu \leq \inf_{u \in V_{\gamma, r}} \int_T \gamma(t)\varphi(u(t))d\mu.
\] (4)

In view of (3), to get (2), we have to show that

\[
\varphi(\hat{y}_\lambda) \int_T \gamma(t)d\mu = \inf_{u \in V_{\gamma, r}} \int_T \gamma(t)\varphi(u(t))d\mu.
\] (5)

Arguing by contradiction, assume that (5) does not hold. So, in view of (4), we would have

\[
\varphi(\hat{y}_\lambda) \int_T \gamma(t)d\mu < \inf_{u \in V_{\gamma, r}} \int_T \gamma(t)\varphi(u(t))d\mu.
\] (6)

From (6), in turn, as \((T, \mathcal{F}, \mu)\) is \(\sigma\)-finite, it would follow the existence of \( \tilde{T} \in \mathcal{F} \), with \( \mu(\tilde{T}) < +\infty \), such that

\[
\varphi(\hat{y}_\lambda) \int_T \gamma(t)d\mu < \inf_{u \in V_{\gamma, r}} \int_T \gamma(t)\varphi(u(t))d\mu.
\] (7)

Now, consider the function \( \hat{u} : T \to Y \) defined by

\[
\hat{u}(t) = \begin{cases} 
\hat{y}_\lambda & \text{if } x \in \tilde{T} \\
0 & \text{if } x \in T \setminus \tilde{T}.
\end{cases}
\]
Clearly, \( \hat{u} \in L^p(T, Y) \). We also have
\[
\int_T \gamma(t)\psi(\hat{u}(t))d\mu = \int_T \gamma(t)\psi(\hat{u}(t))d\mu \leq r \int_T \gamma(t)d\mu
\]
and so \( \hat{u} \in V_{\gamma, r} \). But
\[
\int_T \gamma(t)\varphi(\hat{u}(t))d\mu = \varphi(\hat{y}_{\lambda_1}) \int_T \gamma(t)d\mu
\]
and this contradicts (7). The proof is complete. \( \triangle \)

**REMARK 1.** - In general, the conclusion of Theorem 1 is no longer true if, for some \( \lambda \in [a, b] \), the function \( \varphi + \lambda \psi \) has more than one global minimum. A simple example (with \( a = 0 \) and \( b = +\infty \)) is provided by taking \( Y = \mathbb{R} \),
\[
\varphi(y) = \begin{cases} 
  y^2 & \text{if } y \leq 1 \\
  2 - y & \text{if } y > 1 
\end{cases}
\]
and
\[
\psi(y) = y^2.
\]
So, \( \varphi \) is unbounded below and \( \varphi + \lambda \psi \) is coercive for all \( \lambda > 0 \). Clearly, we have \( \alpha = 0 \) and \( \beta = +\infty \). However, for \( r = 1 \), (2) is not satisfied, since \( 0 \in V_{\gamma, r} \), \( \int_T \gamma(t)\varphi(0)d\mu = 0 \), while \( \inf_{\psi^{-1}(1)} \varphi = 1 \).

**REMARK 2.** - At present, we do not know if the conclusion of Theorem 1 holds without the coercivity assumption on \( \varphi + \lambda \psi \).

We now consider a series of consequences of Theorem 1.

First, we want to state explicitly the form that Theorem 1 assumes when \( T = \mathbb{N} \), \( \mathcal{F} \) is the power set of \( \mathbb{N} \) and
\[
\mu(A) = \text{card}(A)
\]
for all \( A \subseteq \mathbb{N} \).

Denote by \( l_p(Y) \) the space of all sequences \( \{u_n\} \) in \( Y \) such that
\[
\sum_{n=1}^{\infty} \|u_n\|^p < +\infty.
\]

**THEOREM 2.** - Let \( \varphi, \psi \) satisfy the assumptions of Theorem 1.

Then, for each sequence \( \{a_n\} \in l_1(\mathbb{R})\setminus\{0\} \), with \( \inf_{n \in \mathbb{N}} a_n \geq 0 \), and for each \( r \in ]\alpha, \beta[ \), if we put
\[
V_{\{a_n\}, r} = \left\{ \{u_n\} \in l_p(Y) : \sum_{n=1}^{\infty} a_n \psi(u_n) \leq r \sum_{n=1}^{\infty} a_n \right\},
\]

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we have
\[ \inf_{\{u_n\} \in V_{\{a_n\}, c}} \sum_{n=1}^{\infty} a_n \varphi(u_n) = \inf_{\psi^{-1}(r)} \varphi \sum_{n=1}^{\infty} a_n. \]

The next two results deal with consequences of Theorem 1 in the case where \( \varphi \in Y^* \setminus \{0\} \).

**THEOREM 3.** - Let \( y \to \|y\|^q \) be strictly convex for some \( q > 1 \) and let \( \varphi \) be non-zero, continuous and linear. Moreover, let \( \eta : [0, +\infty[ \to \mathbb{R} \) be an increasing strictly convex function.

Then, for each \( \gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\} \), with \( \gamma \geq 0 \), and for each \( r > \eta(0) \) and \( p \geq 1 \), if we put
\[ V_{\gamma, r} = \left\{ u \in L^p(T, Y) : \int_T \gamma(t) \eta(\|u(t)\|^q) d\mu \leq r \int_T \gamma(t) d\mu \right\}, \]
we have
\[ \inf_{u \in V_{\gamma, r}} \int_T \gamma(t) \varphi(u(t)) d\mu = -\|\varphi\|_{Y^*} (\eta^{-1}(r))^\frac{1}{q} \int_T \gamma(t) d\mu. \]

**PROOF.** By the assumptions made on \( \eta \), the functional \( y \to \eta(\|y\|^q) \) is strictly convex and, for some \( m, c > 0 \), one has
\[ \eta(t) \geq mt - c \]
for all \( t \geq 0 \). As a consequence, for each \( \lambda > 0 \), the functional \( y \to \varphi(y) + \lambda \eta(\|y\|^q) \) is coercive and has a unique global minimum in \( X \). At this point, the conclusion follows directly from Theorem 1, applied taking \( a = 0, b = +\infty, \psi(y) = \eta(\|y\|^q) \) and observing that (1) holds for each \( p \geq 1 \) and that \( \alpha = \eta(0), \beta = +\infty \).

**THEOREM 4.** - Let \( \varphi \) be non-zero, continuous and linear and let \( \psi \) be \( C^1 \) with
\[ \lim_{\|y\| \to +\infty} \frac{\psi(y)}{\|y\|} = +\infty. \]  
(8)

Finally, assume that, for each \( \mu < 0 \), the equation
\[ \psi'(y) = \mu \varphi \]  
(9)

has a unique solution in \( Y \) or even at most two when \( \dim(Y) < \infty \).

Then, for each \( p \geq 1 \), the conclusion of Theorem 1 holds with any \( r > \inf_{Y} \psi \).

**PROOF.** In view of (8), the functional \( \varphi + \lambda \psi \) is coercive for each \( \lambda > 0 \). Let \( \hat{x} \) be a global minimum of this functional. Then, \( \hat{x} \) satisfies (9) with \( \mu = -\lambda^{-1} \). So, when \( \dim(Y) = \infty \), the uniqueness of \( \hat{x} \) follows from an assumption directly. Now, assume that \( \dim(Y) < \infty \). In this case, \( \varphi + \lambda \psi \) satisfies the Palais-Smale condition. As a consequence, if \( \varphi + \lambda \psi \) was admitting two global minima, then, thanks to Corollary 1 of [3], (9) would have at least three solutions for \( \mu = -\lambda^{-1} \), against an assumption. Now, we can apply Theorem 1, with \( p \geq 1, a = 0, b = +\infty \), observing that \( \alpha = \inf_{Y} \psi \) and \( \beta = +\infty \).

\[ \triangle \]
Here is a consequence of Theorem 1 in the case when $Y$ is a Hilbert space and $\varphi$ has a Lipschitzian derivative:

**THEOREM 5.** Let $Y$ be a Hilbert space, let $\varphi$ be $C^1$ and let $\varphi'$ be Lipschitzian, with Lipschitz constant $L > 0$. Assume that $\varphi'(0) \neq 0$. Set

$$S = \{ y \in Y : \varphi'(y) + Ly = 0 \}$$

and

$$\rho = \inf_{y \in S} \|y\|^2 .$$

Then, for each $\gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\}$, with $\gamma \geq 0$, and for each $r \in ]0, \rho[ \), $p \geq 2$, if we put

$$V_{\gamma,r} = \left\{ u \in L^p(T,Y) : \int_T \gamma(t) \|u(t)\|^2 d\mu \leq r \int_T \gamma(t) d\mu \right\} ,$$

we have

$$\inf_{u \in V_{\gamma,r}} \int_T \gamma(t) \varphi(u(t)) d\mu = \inf_{\|y\|^2 = r} \varphi(y) \int_T \gamma(t) d\mu .$$

**PROOF.** Note that the functional $y \mapsto \varphi(y) + \frac{\lambda}{2} \|y\|^2$ is convex if $\lambda = L$, while it is strictly convex and coercive if $\lambda > L$ (see, for instance, Proposition 2.2 of [6]). So, this functional has a unique global minimum if $\lambda > L$, while the set of its global minima coincides with $S$ if $\lambda = L$. At this point, the conclusion is obtained applying Theorem 1 with

$$\psi(y) = \frac{\|y\|^2}{2}$$

for all $y \in Y$ and

$$a = L, b = +\infty ,$$

taking into account that (1) is satisfied for each $p \geq 2$ since $\varphi'$ is Lipschitzian and observing that $\alpha = 0$ and $\beta = \frac{\rho}{2}$.

**REMARK 3.** Note that Theorem 5 is an extension of Theorem 1 of [7].

In the next result, we will apply Theorem 1 taking as $Y$ the usual Sobolev space $W^{1,q}_0(\Omega)$ with the usual norm

$$\left( \int_\Omega |\nabla v(x)|^q dx \right)^{\frac{1}{q}},$$

where $\Omega$ is bounded domain in $\mathbb{R}^n$ ($n \geq 3$) with smooth boundary and $q > 1$.

Moreover, if $u \in L^p(T, W^{1,q}_0(\Omega))$ we will write $u(t,x)$ instead of $u(t)(x)$. That is, we will identify $u$ with the function $\omega : T \times \Omega \rightarrow \mathbb{R}$ defined by

$$\omega(t,x) = u(t)(x)$$

for all $(t,x) \in T \times \Omega$. 

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THEOREM 6. - Let \( f : \mathbb{R} \rightarrow [0, +\infty[ \) be a continuous function, with \( f(0) = 0 \) and \( \liminf_{\xi \rightarrow +\infty} f(\xi) > 0 \), such that \( \xi \rightarrow \frac{f(\xi)}{\xi^{q-1}} \) is decreasing in \([0, +\infty[ \) and

\[
\lim_{|\xi| \rightarrow +\infty} \frac{f(\xi)}{|\xi|^{q-1}} = 0
\]

for some \( q > 1 \).

Then, for each \( \gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\} \), with \( \gamma \geq 0 \), and each \( r > 0 \), \( p \geq q \), if we put

\[
V_{\gamma,r} = \left\{ u \in L^p(T, W^{1,q}_0(\Omega)) : \int_T \gamma(t) \left( \int_\Omega |\nabla u(t, x)|^q dx \right) d\mu \leq r \int_T \gamma(t) d\mu \right\} ,
\]

we have

\[
\sup_{u \in V_{\gamma,r}} \int_T \gamma(t) \left( \int_\Omega F(u(t, x)) dx \right) d\mu = \sup_{v \in W^{1,q}_0(\Omega), \int_\Omega |\nabla v(x)|^q dx = r} \int_\Omega F(v(x)) dx \int_T \gamma(t) d\mu ,
\]

where

\[
F(\xi) = \int_0^\xi f(s) ds
\]

for all \( \xi \in \mathbb{R} \).

PROOF. We are going to apply Theorem 1 taking \( Y = W^{1,q}_0(\Omega) \) and

\[
\varphi(v) = -\int_\Omega F(v(x)) dx ,
\]

\[
\psi(v) = \int_\Omega |\nabla v(x)|^q dx
\]

for all \( v \in W^{1,q}_0(\Omega) \). Due to (10), by classical results, \( \varphi \) is sequentially weakly continuous in \( W^{1,q}_0(\Omega) \), (1) is satisfied for any \( p \geq q \), and, for each \( \lambda > 0 \), the functional \( \varphi + \lambda \psi \) is \( C^1 \), coercive and satisfies the Palais-Smale condition. Moreover, since \( f \geq 0 \), its non-zero critical points are strictly positive in \( \Omega \) ([1], [8]). Moreover, since the function \( \xi \rightarrow \frac{f(\xi)}{\xi^{q-1}} \) is decreasing in \([0, +\infty[ \), Proposition 4.2 of [2] ensures that, for each \( \lambda > 0 \), there exists at most one strictly positive critical point of \( \varphi + \lambda \psi \). As a consequence, we infer that, for each \( \lambda > 0 \), the functional \( \varphi + \lambda \psi \) has a unique global minimum in \( W^{1,q}_0(\Omega) \), since otherwise, in view of Corollary 1 of [3], it would have at least three critical points. Hence, we are allowed to apply Theorem 1 with \( a = 0 \) and \( b = +\infty \). Clearly, we have \( \alpha = 0 \) and \( \beta = +\infty \) (since \( \lim_{\xi \rightarrow +\infty} F(\xi) = +\infty \) and hence \( \varphi \) is unbounded below). The proof is complete. \( \triangle \)

The next application of Theorem 1 concerns a Jensen-like inequality in \( L^p \)-spaces.

THEOREM 7. - Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function, differentiable in \([0, +\infty[ \), with \( \sup_{[\delta, \infty[} f \leq 0 \). Assume that, for some \( \delta \geq 0 \), the function \( y \rightarrow \delta |y|^p - f(y) \) has no global minima in \( \mathbb{R} \),

\[
\limsup_{y \rightarrow +\infty} \frac{f(y)}{y^p} = \delta
\]

(11)
and the function
\[ y \to \frac{f'(y)}{y^{p-1}} \]
is injective in \([0, +\infty[.\]

Then, for each \( \gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\} \), with \( \gamma \geq 0 \), one has
\[ \int_T \gamma(t)f(u(t))d\mu \leq f \left( \left( \frac{\int_T \gamma(t)|u(t)|^p d\mu}{\int_T \gamma(t) d\mu} \right)^{\frac{1}{p}} \right) \int_T \gamma(t) d\mu , \]
for all \( u \in L^p(T) \).

**PROOF.** We are going to apply Theorem 1 with \( Y = \mathbb{R} \), \( \varphi(y) = -f(y) \), \( \psi(y) = |y|^p \) and \( a = \delta \), \( b = +\infty \). Fix \( \lambda > \delta \). From (11), we clearly infer that \( \varphi + \lambda \psi \) is coercive. We now show that this function has a unique global minimum. Arguing by contradiction, assume that \( y_1, y_2 \in \mathbb{R} \) are two distinct global minima of \( \varphi + \lambda \psi \). We can suppose that \( y_1 < y_2 \). Since \( \varphi(y) + \lambda \psi(y) > 0 \) for all \( y < 0 \) and \( \varphi(0) + \lambda \psi(0) = 0 \), it would follow that \( y_1 \geq 0 \). By the Rolle theorem, there would be \( y_3 \in [y_1, y_2[ \) such that
\[ p\lambda y_3^{p-1} = f'(y_3) . \]

As a consequence, we would have
\[ \frac{f'(y_2)}{y_2^{p-1}} = \frac{f'(y_3)}{y_3^{p-1}} , \]
contrary to the assumption that the function \( y \to \frac{f'(y)}{y^{p-1}} \) is injective in \([0, +\infty[. \) So, we are allowed to apply Theorem 1, observing that \( \alpha = 0 \) and \( \beta = +\infty \). Let \( u \in L^p(T) \setminus \{0\} \). Put
\[ r = \frac{\int_T \gamma(t)|u(t)|^p d\mu}{\int_T \gamma(t) d\mu} . \]

Clearly, we have
\[ \inf_{\psi^{-1}(r)} \varphi = -f \left( \left( \frac{\int_T \gamma(t)|u(t)|^p d\mu}{\int_T \gamma(t) d\mu} \right)^{\frac{1}{p}} \right) \]
and hence, since \( u \in V_{\gamma,r} \), it follows
\[ \int_T \gamma(t)f(u(t))d\mu \leq f \left( \left( \frac{\int_T \gamma(t)|u(t)|^p d\mu}{\int_T \gamma(t) d\mu} \right)^{\frac{1}{p}} \right) \int_T \gamma(t) d\mu , \]
as claimed. \( \triangle \)

**REMARK 4.** - The class of functions \( f \) satisfying the assumptions of Theorem 7 is quite broad. For instance, a typical function in that class is
\[ f(y) = a_0 \log(1 + (y^+)^p) + \sum_{i=1}^k a_i (y^+)^{q_i} \]
where \( y^+ = \max\{y, 0\} \), \( a_i \) \((i = 0, \ldots, k)\) are \( k + 1 \) non-negative numbers, with \( \sum_{i=0}^{k} a_i > 0 \), and \( q_i \) \((i = 1, \ldots, k)\) are \( k \) positive numbers less than \( p \).

As a consequence of this remark, we get, for instance, the following

**COROLLARY 1.** - For each \( \gamma \in L^\infty(T) \cap L^1(T) \setminus \{0\} \), with \( \gamma \geq 0 \), one has

\[
\int_T \gamma(t) \log(1 + (u(t))^p) d\mu \leq \log \left( 1 + \left[ \frac{\int_T \gamma(t)(u(t))^p d\mu}{\int_T \gamma(t) d\mu} \right] \right) \int_T \gamma(t) d\mu \tag{12}
\]

for all \( u \in L^p(T) \) with \( u \geq 0 \).

Assume that \( \mu(T) = \gamma = 1 \). It is worth noticing that, in this case, (12) can be obtained by the classical Jensen inequality only when \( p = 1 \). In fact, when \( p > 1 \), the function \( t \to \log(1 + t^p) \) is neither concave nor convex in \([0, +\infty[\). While, when \( p < 1 \), the use of the Jensen inequality would provide

\[
\int_T \log(1 + (u(t))^p) d\mu \leq \log \left( 1 + \left( \int_T u(t) d\mu \right)^p \right). \]

Note that this latter inequality is weaker than (12) since

\[
\int_T (u(t))^p d\mu \leq \left( \int_T u(t) d\mu \right)^p.
\]

The final result is an application of Theorem 1 to quasi-linear equations.

So, in the sequel, \( \Omega \subseteq \mathbb{R}^n \) is a bounded domain with smooth boundary and \( p > 1 \).

If \( n \geq p \), we denote by \( \mathcal{A} \) the class of all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
\sup_{y \in \mathbb{R}} \frac{|f(y)|}{1 + |y|^s} < +\infty,
\]

where \( 0 < s < \frac{pn-n+p}{n-p} \) if \( p < n \) and \( 0 < s < +\infty \) if \( p = n \). While, when \( n < p \), \( \mathcal{A} \) stands for the class of all continuous functions \( f : \mathbb{R} \to \mathbb{R} \). Given \( f \in \mathcal{A} \), consider the following Dirichlet problem

\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2} \nabla u) = f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

\((P_f)\)

Let us recall that a weak solution of \((P_f)\) is any \( u \in W_0^{1,p}(\Omega) \) such that

\[
\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx - \int_{\Omega} f(u(x))v(x) dx = 0
\]

for all \( v \in W_0^{1,p}(\Omega) \).
Moreover, $\lambda_{1,p}$ denotes the principal eigenvalue of the problem

\[
\begin{aligned}
-\text{div}(|\nabla u|^{p-2}\nabla u) &= \lambda|u|^{p-2}u & \text{in } \Omega \\
u &= 0 & \text{on } \partial\Omega.
\end{aligned}
\]

We have

\[
\lambda_{1,p} = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u(x)|^p \, dx}{\int_{\Omega} |u(x)|^p \, dx}.
\]

Also, let us recall the following consequence of the variational principle established in [4]:

**THEOREM B.** - Let $X$ be a reflexive real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals, with $\Phi(0) = \Psi(0) = 0$, and with $\Psi$ also coercive and continuous.

Then, for each $\sigma > \inf_X \Psi$ and each $\lambda$ satisfying

\[
\lambda > \frac{-\inf_{\Psi^{-1}([\sigma, +\infty])} \Phi}{\sigma},
\]

the functional $\lambda\Psi + \Phi$ has a local minimum belonging to $\Psi^{-1}([-\infty, \sigma])$.

The final result is as follows:

**THEOREM 8.** - Let $f \in A$, with $f \geq 0$, and let $F(y) = \int_0^y F(t) \, dt$ for all $y \in \mathbb{R}$.

Assume that:

(a$_1$) $\lim_{y \to 0^+} \frac{F(y)}{y^p} = +\infty$ ;

(a$_2$) $\delta := \limsup_{y \to +\infty} \frac{F(y)}{y^p} < +\infty$ ;

(a$_3$) the function $y \to \delta y^p - F(y)$ has no global minima in $[0, +\infty[$ ;

(a$_4$) for each $\lambda > p\delta$, the equation $\lambda y^{p-1} = f(y)$ has at most two solutions in $[0, +\infty[$.

Under such hypotheses, for each $\rho > 0$ and each $\nu \in [0, 1]$ satisfying

\[
\nu < \frac{\lambda_{1,p}\rho^p}{pF(\rho)},
\]

the problem

\[
\begin{aligned}
-\text{div}(|\nabla u|^{p-2}\nabla u) &= \nu f(u) & \text{in } \Omega \\
u &= 0 & \text{on } \partial\Omega
\end{aligned}
\]

has a positive weak solution satisfying

\[
\int_{\Omega} |\nabla u(x)|^p \, dx < \rho^p \lambda_{1,p} \text{meas}(\Omega).
\]
PROOF. Fix $\rho$ and $\nu$ as above. Since $f \geq 0$, by classical results ([1], [8]), the positive weak solutions of the problem are exactly the non-zero critical points in $W^{1,p}_0(\Omega)$ of the energy functional

$$u \to \frac{1}{p} \int_{\Omega} |\nabla u(x)|^p \, dx - \nu \int_{\Omega} F(u(x)) \, dx.$$  

We are going to apply Theorem 1 taking $Y = \mathbb{R}$, $\phi(y) = -\nu F(y)$, $\psi(y) = |y|^p$, $a = \delta$ and $b = +\infty$. Note that $\phi$ is non-negative in $]-\infty, 0]$. So, (1) is satisfied in view of $(a_2)$. Fix $\lambda > \delta$. From $(a_2)$ again, it follows that $\phi + \lambda \psi$ is coercive. Arguing by contradiction, assume that $\phi + \lambda \psi$ has two global minima, say $y_1, y_2$, with $y_1 < y_2$. Differently from Theorem 7, this time we are assuming $(a_1)$ from which it follows that

$$\inf_{[0, +\infty[} (\phi + \lambda \psi) < 0 .$$  

This fact implies that $y_1 > 0$. As a consequence, the equation

$$p \lambda y^{p-1} = \nu f(y)$$  

would admit the solutions $y_1, y_2$ and a third one in $]y_1, y_2[$ given by the Rolle theorem. But, this contradicts $(a_4)$. Hence, the function $\phi + \lambda \psi$ has a unique global minimum. Further, note that $\alpha = 0$ and, in view of $(a_3)$, $\beta = +\infty$. Then, if we put

$$V_{\rho} = \left\{ u \in L^p(\Omega) : \int_{\Omega} |u(x)|^p \, dx \leq \rho^p \text{meas}(\Omega) \right\} ,$$  

Theorem 1 ensures that

$$\sup_{u \in V_{\rho}} \int_{\Omega} F(u(x)) \, dx = F(\rho)\text{meas}(\Omega) . \quad (14)$$  

On the other hand, setting

$$B_{\rho} = \left\{ u \in W^{1,p}_0(\Omega) : \int_{\Omega} |\nabla u(x)|^p \, dx \leq \rho^p \lambda_{1,p} \text{meas}(\Omega) \right\} ,$$  

we have

$$B_{\rho} \subseteq V_{\rho} .$$  

Consequently

$$\sup_{u \in B_{\rho}} \int_{\Omega} F(u(x)) \, dx \leq \sup_{u \in V_{\rho}} \int_{\Omega} F(u(x)) \, dx . \quad (15)$$  

Now, if we put

$$\sigma = \rho^p \lambda_{1,p} \text{meas}(\Omega) ,$$  

in view of (13), (14) and (15), we have

$$\sup_{u \in W^{1,p}_0(\Omega) : \int_{\Omega} |\nabla u(x)|^p \, dx \leq \sigma} \int_{\Omega} \nu F(u(x)) \, dx < \frac{\sigma}{p} .$$  

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At this point, we can apply Theorem B taking $X = W^{1,p}_0(\Omega)$, $\Psi(u) = \int_{\Omega} |\nabla u(x)|^p dx$ and $\Phi(u) = -\nu \int_{\Omega} F(u(x)) \, dx$. Hence, the energy functional has a local minimum $u$ (which is therefore a solution of the problem) such that

$$\int_{\Omega} |\nabla u(x)|^p dx < \rho^p \lambda_{1,p} \text{meas}(\Omega) .$$

To show that $u \neq 0$, we finally remark that 0 is not a local minimum of the energy functional. Indeed, by a classical result, there is a bounded and positive function $v \in W^{1,p}_0(\Omega)$ such that

$$\int_{\Omega} |\nabla v(x)|^p dx = \lambda_{1,p} \int_{\Omega} |v(x)|^p dx .$$

By $(a_1)$, there is $\theta > 0$ such that

$$F(y) > \frac{\lambda_{1,p}}{\nu^p} y^p$$

for all $y \in]0, \theta[$. Hence, for each $\eta \in ]0, \sup_{\Omega} v[$, we have

$$\nu \int_{\Omega} F(\eta v(x)) \, dx > \frac{\lambda_{1,p}}{p} \int_{\Omega} |\eta v(x)|^p dx = \frac{1}{p} \int_{\Omega} |\nabla \eta v(x)|^p dx .$$

This shows that the energy functional takes negative values in each ball of $W^{1,p}_0(\Omega)$ centered at 0 and so 0 is not a local minimum for it. The proof is complete. \triangle

Note the following corollary of Theorem 8 (for the uniqueness, consider again Proposition 4.2 of [2]):

**COROLLARY 2.** - For each $\nu \in]0, 1[$, the unique positive weak solution of the problem

$$\begin{cases}
-\text{div}(|\nabla u| \nabla u) = \nu u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

satisfies the inequality

$$\int_{\Omega} |\nabla u(x)|^3 dx \leq \frac{27 \text{meas}(\Omega)}{8 \lambda_{1,3}^2} \nu^3 .$$
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Department of Mathematics
University of Catania
Viale A. Doria 6
95125 Catania
Italy
*e-mail address*: ricceri@dmi.unict.it