ON WEAK EQUIVALENCES OF GRADINGS

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Abstract. When one studies the structure (e.g. graded ideals, graded subspaces, radicals,...) or graded polynomial identities of graded algebras, the grading group itself does not play an important role, but can be replaced by any other group that realizes the same grading. Here we come to the notion of weak equivalence of gradings: two gradings are weakly equivalent if there exists an isomorphism between the graded algebras that maps each graded component onto a graded component. The following question arises naturally: when a group grading on a finite dimensional algebra is weakly equivalent to a grading by a finite group? It turns out that this question can be reformulated purely group theoretically in terms of the universal group of the grading. Namely, a grading is weakly equivalent to a grading by a finite group if and only if the universal group of the grading is residually finite with respect to a special subset of the grading group. The same is true for all the coarsenings of the grading if and only if the universal group of the grading is hereditarily residually finite with respect to the same subset. We show that if \( n \geq 353 \), then on the full matrix algebra \( M_n(F) \) there exists an elementary group grading that is not weakly equivalent to any grading by a finite (semi)group, and if \( n \leq 3 \), then any elementary grading on \( M_n(F) \) is weakly equivalent to an elementary grading by a finite group. In addition, we study categories and functors related to the notion of weak equivalence of gradings. In particular, we introduce an oplax 2-functor that assigns to each grading its support and show that the universal grading group functor has neither left nor right adjoint.

1. Introduction

When studying graded algebras, one has to determine, when two graded algebras can be considered “the same” or equivalent. Recall that \( \Gamma: A = \bigoplus_{s \in S} A^{(s)} \) is a grading on an algebra \( A \) over a field \( F \) by a (semi)group \( S \) if \( A^{(s)}A^{(t)} \subseteq A^{(st)} \) for all \( s, t \in S \). Then we say that \( S \) is the grading (semi)group of \( \Gamma \) and the algebra \( A \) is graded by \( S \).

Let

\[
\Gamma_1: A = \bigoplus_{s \in S} A^{(s)}, \quad \Gamma_2: B = \bigoplus_{t \in T} B^{(t)}
\]

be two gradings where \( S \) and \( T \) are (semi)groups and \( A \) and \( B \) are algebras.

The most restrictive case is when we require that both grading (semi)groups coincide:

**Definition 1.1** (e.g. [11 Definition 1.15]). The gradings (1.1) are isomorphic if \( S = T \) and there exists an isomorphism \( \varphi: A \cong B \) of algebras such that \( \varphi(A^{(s)}) = B^{(s)} \) for all \( s \in S \). In this case we say that \( A \) and \( B \) are graded isomorphic.

In some cases, such as in [16], less restrictive requirements are more suitable.

**Definition 1.2** (16 Definition 2.3]). The gradings (1.1) are equivalent if there exists an isomorphism \( \varphi: A \rightarrow B \) of algebras and an isomorphism \( \psi: S \rightarrow T \) of (semi)groups such that \( \varphi(A^{(s)}) = B^{(\psi(s))} \) for all \( s \in S \).

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Remark 1.3. The notion of graded equivalence was considered by Yu. A. Bahturin, S. K. Seghal, and M. V. Zaicev in [6, Remark after Definition 3]. In the paper of V. Mazorchuk and K. Zhao [22] it appears under the name of graded isomorphism. A. Elduque and M. V. Kochetov refer to this notion as a weak isomorphism of gradings [11, Section 3.1]. More on differences in the terminology in graded algebras can be found in [16, §2.7].

If one studies the graded structure of a graded algebra or its graded polynomial identities [1, 2, 5, 12, 14], then it is not really important by elements of which (semi)group the graded components are indexed. A replacement of the grading (semi)group leaves both graded subspaces and graded ideals graded. In the case of graded polynomial identities reindexing the graded components leads only to renaming the variables. (It is important to notice however that graded-simple algebras graded by semigroups which are not groups can have a structure quite different from group graded graded-simple algebras [15].) Here we come naturally to the notion of weak equivalence of gradings.

Definition 1.4. The gradings (1.1) are weakly equivalent, if there exists an isomorphism $\varphi: A \cong B$ of algebras such that for every $s \in S$ with $A(s) \neq 0$ there exists $t \in T$ such that $\varphi(A(s)) = B(t)$.

Remark 1.5. This notion appears in [11, Definition 1.14] under the name of equivalence. We have decided to add here the adjective “weak” in order to avoid confusion with Definition 1.2.

Obviously, if gradings are isomorphic, then they are equivalent and if they are equivalent then they are also weakly equivalent. It is important to notice that none of the converse is true. However, if gradings (1.1) are weakly equivalent and $\varphi: A \cong B$ is the corresponding isomorphism of algebras, then $\Gamma_3: A = \bigoplus_{t \in T} \varphi^{-1}(B(t))$ is a grading on $A$ isomorphic to $\Gamma_2$ and the grading $\Gamma_3$ is obtained from $\Gamma_1$ just by reindexing the homogeneous components. Therefore, when gradings (1.1) are weakly equivalent, we say that $\Gamma_1$ can be regraded by $T$. If $A = B$ and $\varphi$ in Definition 1.4 is the identity map, we say that $\Gamma_1$ and $\Gamma_2$ are realizations of the same grading on $A$ as, respectively, an $S$- and a $T$-grading.

For a grading $\Gamma: A = \bigoplus_{s \in S} A(s)$, we denote by $\text{supp} \Gamma := \{s \in S \mid A(s) \neq 0\}$ its support. Note that $\Gamma$ is obviously equivalent to $\Gamma_0: A = \bigoplus_{s \in S_1} A(s)$ where $S_1$ is a subsemigroup of $S$ generated by $\text{supp} \Gamma$. (If $S$ is a group, we can consider instead the subgroup generated by $\text{supp} \Gamma$.)

Remark 1.6. Each weak equivalence between gradings $\Gamma_1$ and $\Gamma_2$ induces a bijection $\text{supp} \Gamma_1 \cong \text{supp} \Gamma_2$.

As we have already mentioned above, for many applications it is not important which particular grading among weakly equivalent ones we consider. Thus, if it is possible, one can try to regrade a semigroup grading by a group or even a finite group. The situation, when the latter is possible, is very convenient since the algebra graded by a finite group $G$ is an $FG$-comodule algebra and, in turn, an $(FG)^*$-module algebra where $FG$ is the group algebra of $G$, which a Hopf algebra, and $(FG)^*$ is its dual. In this case one can use the techniques of Hopf algebra actions instead of working with a grading directly (see e.g. [13]). Therefore, the following question arises naturally:

Question. Is it possible to regrade any grading of a finite dimensional algebra by a finite group?
that a group is *residually finite* if the intersection of its normal subgroups of finite index is trivial and *locally residually finite* if every its finitely generated subgroup is residually finite.

In 1996 M. V. Clase, E. Jespers, and A. Del Río [9, Example 2] (see also [10, Example 1.5]) gave an example of a group graded ring with finite support that cannot be regraded by a finite (semi)group. Despite the fact that they constructed a ring, not an algebra, it is obvious how to make an analogous example of an algebra over a field. However all these examples would have non-trivial nilpotent ideals. Until now it was unclear whether a finite dimensional semi-simple algebra could have a grading which cannot be regraded by a finite group.

In Theorem 5.5 below we show that there exist even elementary gradings (see Definition 2.1) on the full matrix algebras $M_n(F)$ (where $F$ is a field) that are not weakly equivalent to gradings by finite groups. This suggests the following problem:

**Problem 1.7.** Determine the set $\Omega$ of the numbers $n \in \mathbb{N}$ such that any elementary grading on $M_n(F)$ can be regraded by a finite group.

In Section 5 we prove the following theorem:

**Theorem 1.8.** The set $\Omega$ defined in Problem 1.7 is of the form \{ $n \in \mathbb{N} \mid 1 \leq n \leq n_0$ \} for some $3 \leq n_0 \leq 352$. In particular, for every $n \geq 353$ there exists an elementary grading on $M_n(F)$ that cannot be regraded by any finite group.

In Section 3 we provide a criterion for two group gradings on graded-simple algebras to be weakly equivalent and give an example of two twisted group algebras of the same group which are isomorphic as algebras, but whose standard gradings are not weakly equivalent.

In Section 4 we recall the definition of the universal group of a grading introduced in 1989 by J. Patera and H. Zassenhaus [24] and prove that any finitely presented group $G$ can be a universal group of an elementary grading $\Gamma$ on a full matrix algebra and moreover any finite subset of $G$ can be included in $\text{supp}\Gamma$ (Theorem 4.3). This result is used in the proof of Theorem 4.8. We conclude the section showing that the question whether a given grading is regradable by a finite group and, in particular, Problem 1.7 can be reformulated in a purely group theoretical way (see Theorem 4.10 and Problem 4.13).

In Section 5 we prove Corollary 5.6 where we show that for every $n \geq 353$ there exists an elementary grading on $M_n(F)$ that cannot be regraded by any finite group, and Theorem 5.8 where we show that if $n \leq 3$, then any elementary grading on $M_n(F)$ is weakly equivalent to an elementary grading by a finite group. Together this proves Theorem 1.8.

Categories and functors related to graded algebras have been studied extensively (see e.g. [23]) and several important pairs of adjoint functors have been noticed (see Section 6.1 for a brief summary). Each category consisted of algebras graded by a fixed group, i.e. isomorphisms in those categories coincided with isomorphisms of gradings. In order to obtain the proper categorical framework for weak equivalences, one has to deal with the category of algebras graded by any groups where the morphisms are all homomorphisms of algebras that map each graded component into some graded component. Here the induced partial maps on the grading groups come into play naturally. In Section 6 we show that the assignment of the support to a grading leads to an oplax 2-functor. In order to get an ordinary functor $R$ from the category of graded algebras to the category of groups which assigns to each grading its universal group, we restrict the sets of morphisms in the category of graded algebras to the sets of graded injective homomorphisms. We discuss the category obtained and show that the functor $R$ has neither left nor right adjoints (Propositions 6.7 and 6.8) and in order to force it to have a left adjoint we have to restrict our consideration very much, e.g. to group algebras of groups that do not have non-trivial one dimensional representations. In that case we even get an isomorphism of categories (Proposition 6.9).
2. Preliminaries

In this section we recall some basic facts about graded algebras.

Let $S$ be a semigroup and let $\Gamma: A = \bigoplus_{s \in S} A^{(s)}$ be a grading. The subspaces $A^{(s)}$ are called homogeneous components of $\Gamma$ and nonzero elements of $A^{(s)}$ are called homogeneous with respect to $\Gamma$. A subspace $V$ of $A$ is graded if $V = \bigoplus_{s \in S} (V \cap A^{(s)})$. If $A$ does not contain graded two-sided ideals, i.e. two-sided ideals that are graded subspaces, then $A$ is called graded-simple.

2.1. Lower dimensional group cohomology and graded division algebras. By the Bahturin — Seghal — Zaicev Theorem (Theorem 2.3 below) twisted group algebras, which are graded division algebras, play a crucial role in the classification of graded-simple algebras.

Let $A = \bigoplus_{g \in G} A^{(g)}$ be a $G$-graded algebra for some group $G$. If for every $g \in G$ all nonzero elements of $A^{(g)}$ are invertible, then $A$ is called a graded division algebra.

Finite dimensional $G$-graded division algebras over an algebraically closed field are described by the elements of the second cohomology groups of finite subgroups $H \subseteq G$ with coefficients in the multiplicative group of the base field.

Let $G$ be a group and let $F$ be a field. Denote by $F^\times$ the multiplicative group of $F$. Throughout the article we consider only trivial group actions on $F^\times$. In this case the first cohomology group $H^1(G, F^\times)$ is isomorphic to the group $Z^1(G, F^\times)$ of $1$-cocycles which in turn coincides with the group Hom$(G, F^\times)$ of group homomorphisms $G \to F^\times$ with the pointwise multiplication.

Recall that a function $\sigma: G \times G \to F^\times$ is a $2$-cocycle if $\sigma(u, v)\sigma(uw, v) = \sigma(u, vw)\sigma(v, w)$ for all $u, v, w \in G$. The set $Z^2(G, F^\times)$ of $2$-cocycles is an abelian group with respect to the pointwise multiplication. The subgroup $B^2(G, F^\times) \subseteq Z^2(G, F^\times)$ of $2$-coboundaries consists of all $2$-cocycles $\sigma$ for which there exists a map $\tau: G \to F^\times$ such that we have $\sigma(g, h) = \tau(g)\tau(h)\tau(gh)^{-1}$ for all $g, h \in G$. The factor group $H^2(G, F^\times) := Z^2(G, F^\times)/B^2(G, F^\times)$ is called the second cohomology group of $G$ with coefficients in $F^\times$. Denote by $[\sigma]$ the cohomology class of $\sigma \in Z^2(G, F^\times)$ in $H^2(G, F^\times)$.

Let $\sigma \in Z^2(G, F^\times)$. The twisted group algebra $F^\sigma G$ is the associative algebra with the formal basis $(u_g)_{g \in G}$ and the multiplication $u_gu_h = \sigma(g, h)u_{gh}$ for all $g, h \in G$. For trivial $\sigma$, i.e. when $\sigma(g, h) = 1$ for all $g, h \in G$, the twisted group algebra $F^\sigma G$ is the ordinary group algebra $FG$. Each twisted group algebra $F^\sigma G$ has the standard grading $F^\sigma G = \bigoplus_{g \in G} F^g G^{(g)}$ where $F^\sigma G^{(g)} = Fu_g$. Two twisted group algebras $F^{\sigma_1} G$ and $F^{\sigma_2} G$ are graded isomorphic if and only if $[\sigma_1] = [\sigma_2]$. (See e.g. [11] Theorem 2.13.)

Note that twisted group algebras are graded division algebras. In fact, the component $B^{(1G)}$ of an arbitrary graded division algebra $B$, that corresponds to the neutral element $1_G$ of the grading group $G$, is an ordinary division algebra. Thus, if the base field $F$ is algebraically closed and dim $B$ is finite, we have $B^{(1G)} = F1_B$ and $B \cong F^\sigma H$ for some finite subgroup $H \subseteq G$ and a $2$-cocycle $\sigma \in Z^2(H, F^\times)$. (See [11] Theorem 2.13 for the details.)

Suppose there exists a homomorphism $\varphi: F^\sigma G \to F$ of unital algebras. Then $\varphi(u_g)\varphi(u_h) = \varphi(u_gu_h) = \sigma(g, h)\varphi(u_{gh})$ and $\sigma(g, h) = \varphi(u_g)\varphi(u_h)\varphi(u_{gh})^{-1}$ for all $g, h \in G$, i.e. $\sigma$ is cohomologous to the trivial $2$-cocycle. Consequently, if $[\sigma]$ is non-trivial, then $F^\sigma G$ does not have one dimensional unital modules.

Recall that if $G$ is finite and char $F \nmid |G|$, then $F^\sigma G$ is semisimple. (The proof is completely analogous to the case of an ordinary group algebra, see e.g. [17] Theorem 1.4.1.) Therefore, if $G$ is finite, char $F \nmid |G|$, and the field $F$ is algebraically closed, the Artin–Wedderburn Theorem implies that $F^\sigma G$ is isomorphic to the direct sum of full matrix algebras $M_k(F)$. In the case $[\sigma]$ is non-trivial, the observation in the previous paragraph shows that $k \geq 2$ for
all $M_k(F)$. Unlike ordinary group algebras $FG$ of non-trivial groups, twisted group algebras $F\sigma G$ can be simple. (See e.g. [11, Theorem 2.15].)

Let $G$ be an abelian group and $\sigma \in Z^2(G, F^\times)$. Then in $F\sigma G$ we have $u_g u_h = \beta(g, h) u_h u_g$ where $\beta(g, h) := \sigma(g, h) \sigma(h, g)^{-1}$, $g, h \in G$, is the alternating bicharacter corresponding to $\sigma$. Recall that a function $\beta: G \times G \to F^\times$ is an alternating bicharacter if it is multiplicative in each variable and $\beta(g, g) = 1$ for all $g \in G$. It is easy to see that $\beta$ depends only on the cohomology class $[\sigma] \in H^2(G, F^\times)$ and not on the particular 2-cocycle $\sigma$.

If $G$ is finitely generated, then $G \cong (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_m\mathbb{Z})$ for some $m, n_i \in \mathbb{Z}_+$. In this case, in order to define an alternating bicharacter $\beta: G \times G \to F^\times$, it is necessary and sufficient to define the values $\beta(g_i, g_j)$ where

$$\beta(g_i, g_j)^{n_i} = \beta(g_i, g_j)^{n_j} = \beta(g_i, g_j) \beta(g_j, g_i) = 1$$

for all $1 \leq i, j \leq m$ and $g_i$ are generators of the cyclic components of $G$.

Given an alternating bicharacter $\beta: G \times G \to F^\times$, it is easy to define an algebra which is graded isomorphic to a twisted group algebra $F\sigma G$ with $[\sigma]$ corresponding to $\beta$:

$$u_{g_1}^{k_1} \cdots u_{g_m}^{k_m} u_{g_1}^{l_1} \cdots u_{g_m}^{l_m} = \left( \prod_{1 \leq i < j \leq m} \beta(g_j, g_i)^{k_j l_i} \right) u_{g_1}^{k_1 + l_1} \cdots u_{g_m}^{k_m + l_m}.$$  

Similar arguments show that if $\sigma_1, \sigma_2 \in Z^2(G, F^\times)$ have equal alternating bicharacters, then $F\sigma_1 G$ and $F\sigma_2 G$ are graded isomorphic, and $[\sigma_1] = [\sigma_2]$.

### 2.2. Elementary gradings and classification of graded-simple algebras.

**Definition 2.1.** Let $F$ be a field, $G$ be a group, let $n \in \mathbb{N}$, and let $(g_1, \ldots, g_n)$ be an $n$-tuple of elements of $G$. Define a grading on $M_n(F)$ by making each matrix unit $e_{ij}$ a $g_i g_j^{-1}$-homogeneous element. This grading is called the elementary $G$-grading defined by $(g_1, \ldots, g_n)$.

**Remark 2.2.** Note that such a grading is uniquely determined by defining the $G$-degrees of $e_{i, i+1}$, $1 \leq i \leq n-1$. If $G$ is an arbitrary group and $(h_1, \ldots, h_{n-1})$ is an arbitrary $(n-1)$-tuple of elements of $G$, then the elementary grading with $e_{i, i+1} \in M_n(F)^{(h_i)}$ can be defined by $(g_1, \ldots, g_n)$ where $g_i = \prod_{j=i}^{n-1} h_j$.

Let $n \in \mathbb{N}$, let $G$ be a group, let $\gamma = (g_1, \ldots, g_n)$ where $g_i \in G$, let $H \subseteq G$ be a finite subgroup, and let $\sigma \in Z^2(H, F^\times).$ Denote by $M(\gamma, \sigma)$ the algebra $M_n(F) \otimes_F F_\sigma H$ endowed with the grading where $e_{ij} \otimes u_h$ belongs to the $g_i h g_j^{-1}$-component.

**Theorem 2.3** (Bahturin — Seghal — Zaicev, see e.g. [8, Theorem 3] or [11, Corollary 2.22]). Let $A$ be a finite dimensional graded-simple $G$-graded algebra over an algebraically closed field $F$ where $G$ is a group. Then $A$ is graded isomorphic to $M(\gamma, \sigma)$ for some $n \in \mathbb{N}$, $\gamma = (g_1, \ldots, g_n)$ where $g_i \in G$, a finite subgroup $H \subseteq G$, and a 2-cocycle $\sigma \in Z^2(H, F^\times)$.

A criterion for two such gradings to be isomorphic can be found, e.g., in [3, Lemma 1.3, Proposition 3.1] or [11, Corollary 2.22]. Necessary and sufficient conditions for two graded-simple algebras to be graded equivalent were proven in [16, Theorem 2.20]. In the next section we study weak equivalences of gradings on two graded-simple algebras.

### 3. Weak equivalences of graded-simple algebras

In this section we prove a criterion for a weak equivalence of graded-simple algebras inspired by [11, Proposition 2.33], we present families of gradings for which the notions of
equivalence and weak equivalence coincide, and give an example of two twisted group algebras of the same abelian group that are isomorphic as ordinary algebras, but not graded weakly equivalent.

Let $A = \bigoplus_{g \in G} A^{(g)}$ be an algebra graded by a group $G$. A vector space $W = \bigoplus_{g \in G} W^{(g)}$ is a graded left $A$-module if $W$ is a left $A$-module and for each $g, h \in G$ we have $A^{(g)}W^{(h)} \subseteq W^{(gh)}$. Graded right modules are defined analogously.

Fix some $n \in \mathbb{N}$, a group $G$, an $n$-tuple $\gamma = (g_1, \ldots, g_n)$ where $g_i \in G$, a finite subgroup $H \subseteq G$, and $\sigma \in Z^2(H, F^x)$.

Consider the $G$-graded vector space $V$ with the basis $v_{ih}$, $1 \leq i \leq n$, $h \in H$, such that $v_{ih}$ is a homogeneous element of degree $g_i h$. Then $V$ is a graded left $M(\gamma, \sigma)$-module and a graded right $F^x H$-module with $(e_{jk} \otimes u_g)v_{ih} := \delta_{ik}\sigma(g, h)v_{j, gh}$ and $v_{ih}u_g := \sigma(h, g)v_{i, hg}$ for all $1 \leq i, j, k \leq n$ and $g, h \in H$ where $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j; \\ 1 & \text{if } i = j \end{cases}$ and $(u_g)_{g \in G}$ is the standard basis in $F^x H$. Note that $(av)u = a(vu)$ for all $a \in M(\gamma, \sigma)$, $v \in V$, and $u \in F^x H$.

**Lemma 3.1.** $V$ is an irreducible left $M(\gamma, \sigma)$-module.

**Proof.** Suppose $W$ is a non-trivial graded $M(\gamma, \sigma)$-submodule of $V$. Take non-zero $v = \bigoplus_{h \in H} \alpha_{ih}v_{ih} \in W$. If $\alpha_{ih} \neq 0$ for some $1 \leq i \leq n$ and $h_0 \in H$, then $e_i \otimes u_{1h_0}v = \bigoplus_{h \in H} \alpha_{ih}v_{ih} \in W$ is again non-zero. Since $W$ is a graded subspace and for fixed $i$ and different $h$ the elements $v_{ih}$ belong to different graded components, we get $\alpha_{ih}v_{ih} \in W$ and $v_{ih} \in W$. Hence $v_{ih} = (e_{ij} \otimes u_{hh_0^{-1}})v_{ih} \in W$ for all $1 \leq j \leq n$, $h \in H$. Therefore $W = V$. \hfill \Box

For a graded vector space $W = \bigoplus_{g \in G} W^{(g)}$ and an element $h \in G$ denote by $W[h]$ the same vector space endowed with the grading $W = \bigoplus_{g \in G} W^{(g)}$ where $W^{(g)} := W^{(gh^{-1})}$.

Note that $M(\gamma, \sigma)$ is the direct sum of graded left ideals $I_j = \bigoplus_{h \in H} F(e_{ij} \otimes u_h), 1 \leq j \leq n$, and each $I_j$ is isomorphic to $V^{[y_j]^{-1}}$ as a graded left $M(\gamma, \sigma)$-module via $e_{ij} \otimes h \mapsto v_{ih}$, $1 \leq i \leq n, h \in H$.

Recall that if $\alpha: H_1 \to H_2$ is a homomorphism of groups and $\sigma \in Z^2(H_2, F^x)$, then the function $\sigma(\alpha(g), \alpha(h))$ of arguments $g, h \in H_1$ is a 2-cocycle on $H_1$. We denote the cohomology class of this 2-cocycle by $[\sigma(\alpha(\cdot), \alpha(\cdot))]$.

Let $G_1, G_2$ be groups and let $n_1, n_2 \in \mathbb{N}$. Fix tuples $\gamma_i = (g_{i1}, \ldots, g_{in_i})$ where $g_{ij} \in G_i$, finite subgroups $H_i \subseteq G_i$, and 2-cocycles $\sigma_i \in Z^2(H_i, F^x)$, $i = 1, 2$. We say that the gradings on $M(\gamma_1, \sigma_1)$ and $M(\gamma_2, \sigma_2)$ satisfy Condition (*) if $n_1 = n_2$, there exist a group isomorphism $\alpha: H_1 \xrightarrow{\sim} H_2$, a permutation $\pi \in S_{n_1}$, and elements $t_i \in H_2, 1 \leq i \leq n_1$, such that $[\sigma_1] = [\sigma_2(\alpha(\cdot), \alpha(\cdot))]$ and the following condition holds: for every $1 \leq i, j, k, l \leq n_1$ and $h_1, h_2 \in H_1$ we have

$$g_{2, \pi(i)}t_{i}\alpha(h_1)t_{j}^{-1}g_{2, \pi(j)}^{-1} = g_{2, \pi(k)}t_{k}\alpha(h_2)t_{\ell}^{-1}g_{2, \pi(\ell)}^{-1}$$

if and only if

$$g_{1i}h_1g_{1j}^{-1} = g_{1k}h_2g_{1\ell}^{-1}.$$

**Theorem 3.2.** The gradings on $M(\gamma_1, \sigma_1)$ and $M(\gamma_2, \sigma_2)$ are weakly equivalent if and only they satisfy Condition (*). If $M(\gamma_1, \sigma_1)$ and $M(\gamma_2, \sigma_2)$ satisfy Condition (*), the algebra isomorphism $\varphi: M(\gamma_1, \sigma_1) \xrightarrow{\sim} M(\gamma_2, \sigma_2)$ implementing this weak equivalence can be defined e.g. by $\varphi(e_{ij} \otimes x_h) = e_{\pi(i), \pi(j)} \otimes y_{t_i}(h)t_j^{-1}$ where $(x_h)_{h \in H_1}$ is the formal basis in $F^{\sigma_1}H_1$ and $(y_i)_{i \in H_2}$ is the formal basis in $F^{\sigma_2}H_2$.

**Proof.** Suppose the gradings on $M(\gamma_1, \sigma_1)$ and $M(\gamma_2, \sigma_2)$ are weakly equivalent. Denote by $\varphi$ an isomorphism of algebras $M(\gamma_1, \sigma_1) \xrightarrow{\sim} M(\gamma_2, \sigma_2)$ that corresponds to the weak equivalence.
Construct the graded $M(\gamma_i, \sigma_i)$-modules $V_i$, $i = 1, 2$, as above. These modules are irreducible by Lemma 3.1.

Note that $M(\gamma_1, \sigma_1)$ is acting on $V_2$ via $a \cdot v = \varphi(a)v$ for $a \in M(\gamma_1, \sigma_1)$ and $v \in V_2$, and $V_2$ does not contain any non-trivial $G_2$-graded submodules. Define for $M(\gamma_1, \sigma_1)$ the minimal graded left ideals $I_j$ as above. Then for each $j$ either $I_j \cdot V_2 = 0$ or $I_j \cdot V_2 = V_2$. Since $M(\gamma_1, \sigma_1) \cdot V \neq 0$, we obtain that $V_2 = I_j \cdot v$ for some homogeneous $v \in V_2$ and some $j$ and $V_2$ is isomorphic to $I_j$ as a left $M(\gamma_1, \sigma_1)$-module. Moreover, this isomorphism maps each nonzero $G_1$-graded component onto some $G_2$-graded component. In fact, since $I_j \cong V_1^{[g,1]}$, there exists a linear isomorphism $\psi: V_1 \cong V_2$ such that $\varphi(a)\psi(v) = \psi(\varphi(a)v)$ for all $a \in M(\gamma_1, \sigma_1)$ and $v \in V_1$ and for each $g \in G_1$ with $V_1^{(g)} \neq 0$ there exists $\rho(g) \in G_2$ such that $\psi \left( V_1^{(g)} \right) = V_2^{(\rho(g))}$.

It is not difficult to check that $F^\alpha H_i$ is isomorphic to $\text{End}_{M(\gamma_i, \sigma_i)} V_i$ as an algebra through its action on $V_i$ from the right. Hence there exists an algebra isomorphism $\tau: F^\alpha H_1 \cong F^\alpha H_2$ such that $\psi(v)\tau(u) = \psi(vu)$ where $v \in V_1$ and $u \in F^\alpha H_1$. If $V_2^{(g)} \neq 0$ and $h \in H_1$, then

$$V_2^{(g)} \tau(x_h) = \psi \left( V_1^{(\rho^{-1}(g))} \right) \tau(x_h) = \psi \left( V_2^{(\rho^{-1}(g))} x_h \right) = \psi \left( V_1^{(\rho^{-1}(g)h)} \right) = V_2^{(\rho(\rho^{-1}(g)h))}.$$ (3.1)

However, $\tau(x_h) = \sum_{t \in H_2} \alpha_t y_t$ for some $\alpha_t \in F$. Since the sum $\bigoplus_{t \in H_2} V_2^{(g)} y_t = \bigoplus_{t \in H_2} V_2^{(g)}$ is direct,

$$\tau(x_h) = \lambda(h) y_{\alpha(h)}$$ (3.2)

for some $\lambda(h) \in F^\times$ and $\alpha(h) \in H_2$. In particular, $\alpha$ is a group isomorphism. Now

$$\sigma_1(h_1, h_2) \lambda(h_1 h_2) y_{\alpha(h_1, h_2)} = \sigma_1(h_1, h_2) \tau(x_{h_1, h_2}) = \tau(x_{h_1}) \tau(x_{h_2}) = \sigma_2(\alpha(h_1), \alpha(h_2)) \lambda(h_1) \lambda(h_2) y_{\alpha(h_1) \alpha(h_2)},$$

implies $[\sigma_1] = [\sigma_2(\alpha(\cdot), \alpha(\cdot))]$.

Note that $\dim V_1 = n_1 |H_1| = n_2 |H_2|$. Now $H_1 \cong H_2$ implies $n_1 = n_2$.

Equalities (3.1) and (3.2) imply $\rho(gh) = \rho(g)\alpha(h)$ for all $g \in G_1$ with $V_1^{(g)} \neq 0$ and all $h \in H_1$.

Since $\rho$ is a bijection between $\bigcup_{i=1}^{n_1} g_iH_1$ and $\bigcup_{i=1}^{n_2} g_2H_2$ which maps left cosets of $H_1$ onto left cosets of $H_2$, there exists a permutation $\pi \in S_{n_1}$ and elements $t_i \in H_2$ such that $\rho(g_i) = g_2 \pi(t_i)$, $1 \leq i \leq n_1$.

Denote by $v_{ih}$ the elements of the standard basis in $V_1$ defined before the theorem. Now $\varphi(e_{ij} \otimes x_h) \psi(v_{ij,1h_1}) = \psi(v_{ih})$. Note that $\deg \psi(v_{ij,1h_1}) = \rho(g_{ij}) = g_2 \pi(t_j)$ and $\deg \psi(v_{ih}) = \rho(g_{ih}) = g_2 \pi(t_i \alpha(h))$. Hence $\deg \varphi(e_{ij} \otimes x_h) = g_2 \pi(t_i \alpha(h))^{-1} g_2^{-1} \pi(t_j)$ for all $1 \leq i, j \leq n_1$.

Since $\varphi$ is a graded map and $\deg(e_{ij} \otimes x_h) = g_{1i} h_{1j}^{-1}$, we get the first part of the theorem.

The converse is trivial. □

The proposition below is verified directly.

**Proposition 3.3.** An elementary grading on a full matrix algebra can be weakly equivalent only to a grading isomorphic to an elementary grading on a full matrix algebra.

Let $G$ be a group. We say that a $G$-grading of an algebra $A$ is connected if the support of this grading generates the group $G$. Recall that a grading $A = \bigoplus_{g \in G} A^{(g)}$ is called strong if $A^{(g_1)} A^{(g_2)} = A^{(g_1 g_2)}$ for any $g_1, g_2 \in G$ and a grading is called nondegenerate if the product of a finite number of non-zero homogeneous components is again non-zero.
It is easy to see that if $A^{(g)} \neq 0$ for at least one $g \in G$ and $A$ is strongly graded, then $A^{(gh^{-1})} A^{(h)} = A^{(g)} \neq 0$ implies $A^{(h)} \neq 0$ for all $h \in G$. In particular, a strong grading is connected.

Here we introduce the notion of a strongly connected grading which is weaker than the notion of a connected nondegenerate grading and a strong grading.

**Definition 3.4.** A connected grading $\Gamma: A = \bigoplus_{g \in G} A^{(g)}$ is strongly connected if $A^{(g)} A^{(h)} \neq 0$ for all $g, h \in \text{supp} \ 1$.

**Lemma 3.5.** Weakly equivalent strongly connected gradings of finite dimensional algebras are equivalent.

**Proof.** Let $\Gamma_1: A = \bigoplus_{g \in G} A^{(g)}$ be a strongly connected grading. We claim that $\text{supp} \ 1$ coincides with $G$ itself. Take arbitrary $g = \prod_{i=1}^{r} g_i^{k_i}$ where $g_1, g_2, \ldots, g_r \in \text{supp} \ 1$ and $k_1, k_2, \ldots, k_r \in \mathbb{N}$. Now, using the strongly connectedness condition, we get by induction on $r$ that $A^{(g)} \neq 0$ and therefore $\text{supp} \ 1$ is closed under multiplication. Since $A$ is finite dimensional, $\text{supp} \ 1$ is finite subset of a group, which is closed under multiplication. Hence $\text{supp} \ 1$ is a group and $\text{supp} \ 1 = G$ since $\ 1$ is connected.

Let $\Gamma_2: B = \bigoplus_{h \in H} B^{(h)}$ be a strongly connected grading which is weakly equivalent to $\Gamma_1$ with the associated isomorphism $\phi: A \rightarrow B$. By the arguments above, $\text{supp} \ 2 = H$. Therefore, the natural bijection between the supports of the gradings is a map $\psi: G \rightarrow H$ between the grading groups. We claim that $\psi$ is a group isomorphism. Indeed, let $g, g' \in G$. Then $A^{(g)}$ and $A^{(g')}$ are both non-zero. Suppose $\varphi\left(A^{(g)}\right) = B^{(h)}$, $\varphi\left(A^{(g')}\right) = B^{(h')}$. Then $\psi(g) = h$ and $\psi(g') = h'$. Since $\psi$ is an isomorphism of algebras, we have

$$
\varphi\left(A^{(g)} A^{(g')}\right) = \varphi\left(A^{(g)}\right) \varphi\left(A^{(g')}\right) = B^{(h)} B^{(h')} \subseteq B^{(hh')}.
$$

On the other hand,

$$
\varphi\left(A^{(g)} A^{(g')}\right) \subseteq \varphi\left(A^{(gg')}\right).
$$

Consequently, since by the strongly connectedness condition $A^{(g)} A^{(g')} \neq 0$, we get $\varphi\left(A^{(gg')}\right) = B^{(hh')}$ and therefore $\psi(gg') = hh'$. Hence $\psi$ is indeed a group isomorphism.

We conclude by noticing that

$$
\varphi\left(A^{(g)}\right) = B^{(\psi(g))}
$$

and therefore $\Gamma_1$ and $\Gamma_2$ are equivalent. □

**Corollary 3.6.** In the following cases the weak equivalence of gradings of finite dimensional algebras implies the equivalence of gradings:

1. the standard gradings on twisted group algebras;
2. strong gradings;
3. nondegenerate gradings.

Based on Corollary 3.6 it is natural to ask when an isomorphism of twisted group rings implies a graded equivalence of the twisted group rings. It is clear that two isomorphic twisted group rings may be not graded equivalent. A simple example for that is the group algebras $\mathbb{C}C_4$ and $\mathbb{C}(C_2 \times C_2)$ (by $C_n$ we denote the cyclic group of order $n$). However, can this phenomenon happen for two twisted group algebras of the same group? It turns out that, provided the group $G$ is abelian, if $\mathbb{C}^n G$ and $\mathbb{C}^n G$ are isomorphic and simple, then they are graded equivalent [18, Theorem 18, [16, Proposition 2.4 (2)] (see the description of finite abelian groups of central type e.g. in [11, Theorem 2.15]). Nonetheless, for a non-abelian group $G$ it can happen that $\mathbb{C}^n G$ and $\mathbb{C}^n G$ are isomorphic and simple, but they are not graded equivalent. Rather complicated examples for that can be found in [16, §3.5]. However, if we relax the simplicity condition to the graded simplicity (twisted group algebras
are always graded-simple), examples even for abelian groups can be constructed and they are much simpler than those in [16, §3.5].

**Example 3.7.** Let 
\[ G = C_4 \times C_2 \times C_2 = \langle x \rangle \times \langle y \rangle \times \langle z \rangle. \]

Recall that in order to define a cohomology class for a finitely generated abelian group, it is enough to determine the values of the corresponding alternating bicharacter (see Section 2.1). Define two non-cohomologous classes \([\sigma], [\rho] \in H^2(G, \mathbb{C} \times \mathbb{C})\) as follows:

\[ [\sigma] : \alpha(x, y) = -1, \alpha(x, z) = 1, \alpha(y, z) = 1, \]
\[ [\rho] : \beta(x, y) = 1, \beta(x, z) = 1, \beta(y, z) = -1. \]

Here \(\alpha\) and \(\beta\) are the alternating bicharacters corresponding to \([\sigma]\) and \([\rho]\), respectively.

Let 
\[ C^\sigma G = \bigoplus_{i=1}^{k} M_{m_i}(\mathbb{C}), \quad C^\rho G = \bigoplus_{j=1}^{t} M_{n_j}(\mathbb{C}) \]
be the corresponding Artin–Wedderburn decompositions. Since \([\sigma]\) and \([\rho]\) are nontrivial, we have \(m_i, n_j > 1\) for any \(1 \leq i \leq k, 1 \leq j \leq t\). On the other hand, since the centers of both \(C^\sigma G\) and \(C^\rho G\) have dimensions greater than 1, by a simple calculation, we get
\[ C^\sigma G \cong C^\rho G \cong \bigoplus_{i=1}^{4} M_2(\mathbb{C}). \]

However, there is a homogenous element of order 4 in the center of \(C^\rho G\) while there is no such homogenous element in the center of \(C^\sigma G\). Therefore, these algebras are not graded equivalent and hence by Corollary 3.6 they are also not weakly equivalent.

### 4. Group-theoretical approach

Each group grading on an algebra can be realized as a \(G\)-grading for many different groups \(G\), however it turns out that there is one distinguished group among them [11, Definition 1.17], [24].

**Definition 4.1.** Let \(\Gamma\) be a group grading on an algebra \(A\). Suppose that \(\Gamma\) admits a realization as a \(G^\Gamma\)-grading for some group \(G^\Gamma\). Denote by \(\kappa^\Gamma\) the corresponding embedding \(\text{supp} \Gamma \hookrightarrow G^\Gamma\). We say that \((G^\Gamma, \kappa^\Gamma)\) is the universal group of the grading \(\Gamma\) if for any realization of \(\Gamma\) as a grading by a group \(G\) with \(\psi: \text{supp} \Gamma \hookrightarrow G\) there exists a unique homomorphism \(\varphi: G^\Gamma \to G\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{supp} \Gamma & \xrightarrow{\psi} & G \\
\kappa^\Gamma & \downarrow{\varphi} & \downarrow{\varphi} \\
& G^\Gamma &
\end{array}
\]

The observation below is a direct consequence of the definition.

**Proposition 4.2.** A grading \(\Gamma\) admits an infinite group turning \(\Gamma\) into a connected grading if and only if \(G^\Gamma\) is infinite.

In the definition above the universal group of a grading \(\Gamma\) is a pair \((G^\Gamma, \kappa^\Gamma)\). Theorem 4.3 below shows, in particular, that the first component of this pair can be an arbitrary finitely presented group. Furthermore, we can choose \(\Gamma\) to be an elementary grading on a full matrix algebra. The possibility to include a subset \(V\) to the support will be used later, in the proof of Theorem 5.5.
Theorem 4.3. Let $F$ be a field, let $G$ be a finitely presented group and let $V \subseteq G$ be a finite subset (possibly empty). Then for some $n \in \mathbb{N}$, depending only on the presentation of $G$ and the elements of $V$, there exists an elementary grading $\Gamma$ on $M_n(F)$ such that $G_{\Gamma} \cong G$ and $V \subseteq \text{supp} \Gamma$.

Proof. Suppose $G \cong \mathcal{F}(X)/N$ where $\mathcal{F}(X)$ is the free group on a finite set of generators $X = \{x_1, x_2, \ldots, x_\ell\}$ and $N$ is the normal closure of a finite set of words $w_1, \ldots, w_m$. Let $V = \{g_1, \ldots, g_s\} \subseteq G$. Choose $w_{m+1}, \ldots, w_{m+s} \in \mathcal{F}(X)$ such that $g_i = \bar{w}_{m+i}$, $1 \leq i \leq s$, where by $\bar{u}$ we denote the image of $u \in \mathcal{F}(X)$ in $G$. Suppose $w_i = y_{i1} \cdots y_{ik_i}$ where $y_{ij} \in X \cup X^{-1}$, $1 \leq i \leq m + s$.

Denote $n = \ell + 1 + \sum_{i=1}^{m+s} k_i$.

Let

$$
\Gamma: M_n(F) = \bigoplus_{g \in G} M_n(F)^{(g)}
$$

be the elementary $G$-grading defined as follows:

$$
e_{r,r+1} \in M_n(F)^{(g_{ij})} \text{ if } r = k_1 + \cdots + k_{i-1} + j, \ 1 \leq j \leq k_i, \ 1 \leq i \leq m + s
$$

and

$$
e_{r,r+1} \in M_n(F)^{(\bar{y}_i)} \text{ if } r = \sum_{i=1}^{m+s} k_i + j, \ 1 \leq j \leq \ell
$$

(the corresponding elementary grading exists by Remark 2.2).

Note that

$$
\sum_{i=1}^{m+s} r_{ij} = \prod_{j=1}^{k_{m+r}} \sum_{i=1}^{m+s} r_{ij} + 1 \in M_n(F)\left(\prod_{j=1}^{m+s} g_{m+r,j}\right) = M_n(F)^{(g_r)}
$$

is nonzero. Hence $g_r \in \text{supp} \Gamma$ for each $1 \leq r \leq s$ and $V \subseteq \text{supp} \Gamma$.

Now we claim that $G_{\Gamma} \cong G$.

Suppose that $\Gamma$ is realized as a grading by a group $H$. Then there exists an injective map $\psi: \text{supp} \Gamma \hookrightarrow H$ defined by $M_n(F)^{(g)} \subseteq M_n(F)^{(\psi(g))}$. We have

$$
\psi(g_1) \psi(g_2) = \psi(g_1) \psi(g_2)
$$

for any $g_1, g_2 \in G$ such that $M_n(F)^{(g_1)} M_n(F)^{(g_2)} \neq 0$. Since $M_n(F)$ is a unital algebra, we have $\psi(1_G) = 1_H$.

For every $x \in X$ we have $\bar{x}, \bar{x}^{-1} \in \text{supp} \Gamma$. Thus elements $\psi(\bar{x})$ and $\psi(\bar{x}^{-1}) = \psi(\bar{x})^{-1}$ are defined for all $x \in X$. By induction,

$$
\psi(y_{i1}) \cdots \psi(y_{ik_i}) = \psi(y_{i1} \cdots y_{ik_i}) = \psi(\bar{w}_i) = 1_H
$$

for all $1 \leq i \leq m$. Hence the elements $\psi(\bar{x})$ satisfy the relations of $G$. Therefore there exists a homomorphism $\varphi: G \rightarrow H$ such that $\varphi(\bar{x}) = \psi(\bar{x})$ for each $x \in X$. Since the set $\{\bar{x} | x \in X\}$ generates $G$, such a homomorphism is unique.

Now we have to prove that $\varphi|_{\text{supp} \Gamma} = \psi$. Every element $g$ of $\text{supp} \Gamma$ corresponds to a matrix unit $e_{ij}$ where either $i > j$ and $e_{ij} = e_{i,j-1}e_{i-1,j-2} \cdots e_{j+1,j}$, or $i < j$ and $e_{ij} = e_{i,j+1}e_{i+1,j} \cdots e_{j-1,j}$, or $i = j$ and $g = 1_G$. Since for every $1 \leq i, j \leq n$, such that $|i-j| = 1$, we have $e_{ij} \in M_n(F)^{(\bar{x})}$ for some $x \in X \cup X^{-1}$ and $\varphi(x) = \psi(x)$ for $x \in X \cup X^{-1}$, the induction on $|i-j|$ using (4.1) shows that $\varphi(g) = \psi(g)$. Hence $G \cong G_{\Gamma}$. \hfill $\square$

Remark 4.4. For each grading $\Gamma$ one can define a category $C_{\Gamma}$ where the objects are all pairs $(G, \psi)$ such that $G$ is a group and $\Gamma$ can be realized as a $G$-grading with $\psi: \text{supp} \Gamma \hookrightarrow G$ being the embedding of the support. In this category the set of morphisms between $(G_1, \psi_1)$ and $(G_2, \psi_2)$ is the set of functions $f: \text{supp} \Gamma_1 \rightarrow \text{supp} \Gamma_2$ such that $f|_{\text{supp} \Gamma_1} = \psi_1$ and $f(\text{supp} \Gamma_1) \subseteq \text{supp} \Gamma_2$.
and $(G_2, \psi_2)$ consists of all group homomorphisms $f : G_1 \to G_2$ such that the diagram below is commutative:

\[
\begin{array}{ccc}
\text{supp } \Gamma & \xrightarrow{\psi_1} & G_1 \\
& f \downarrow & \\
& \text{supp } \Gamma & \xrightarrow{\psi_2} G_2
\end{array}
\]

Then $(G_\Gamma, \pi_\Gamma)$ is the initial object of $\mathcal{C}_\Gamma$.

It is easy to see that if $\Gamma : A = \bigoplus_{g \in G} A^{(g)}$ is a grading, where $G$ is a group, then $G_\Gamma \cong \mathcal{F}(\text{supp } \Gamma)/N$ where $N$ is the normal closure of the words $g h t^{-1}$ for pairs $g, h \in \text{supp } \Gamma$ such that $A^{(g)} A^{(h)} \neq 0$ where $t \in \text{supp } \Gamma$ is defined by $A^{(g)} A^{(h)} \subseteq A^{(t)}$.

**Definition 4.5.** Let $\Gamma_1 : A = \bigoplus_{g \in G} A^{(g)}$ and $\Gamma_2 : A = \bigoplus_{h \in H} A^{(h)}$ be two gradings where $G$ and $H$ are groups and $A$ is an algebra. We say that $\Gamma_2$ is coarser than $\Gamma_1$, if for every $g \in G$ with $A^{(g)} \neq 0$ there exists $h \in H$ such that $A^{(g)} \subseteq A^{(h)}$. In this case $\Gamma_2$ is called a coarsening of $\Gamma_1$ and $\Gamma_1$ is called a refinement of $\Gamma_2$. Denote by $\pi_{\Gamma_1 \to \Gamma_2} : \Gamma_1 \to \Gamma_2$, the homomorphism defined by $\pi_{\Gamma_1 \to \Gamma_2}(\pi_\Gamma(g)) = \pi_\Gamma(h)$ for $g \in \text{supp } \Gamma_1$ and $h \in \text{supp } \Gamma_2$ such that $A^{(g)} \subseteq A^{(h)}$.

**Notation.** For a subset $U$ of a group $G$, denote by $\text{diff } U = \{uw^{-1} \mid u, v \in U, u \neq v\}$.

**Lemma 4.6.** Let $\Gamma_2 : A = \bigoplus_{h \in H} A^{(h)}$ be a coarsening of $\Gamma_1 : A = \bigoplus_{g \in G} A^{(g)}$. Let

\[
W := \{ \pi_{\Gamma_1}(g_1) \pi_{\Gamma_1}(g_2)^{-1} \mid g_1, g_2 \in \text{supp } \Gamma_1 \text{ such that } A^{(g_1)} \oplus A^{(g_2)} \subseteq A^{(h)} \text{ for some } h \in \text{supp } \Gamma_2 \}.
\]

Denote by $Q \triangleleft_G \Gamma_1$ the normal closure of $W$. Then $\ker \pi_{\Gamma_1 \to \Gamma_2} = Q$. In addition, $Q \cap \ker \pi_\Gamma(\text{supp } \Gamma_1) = W$.

**Proof.** Obviously, $Q \subseteq \ker \pi_{\Gamma_1 \to \Gamma_2}$.

Let $\pi_{\Gamma_1} : \mathcal{F}(\text{supp } \Gamma_1) \to \Gamma_1$ and $\pi_{\Gamma_2} : \mathcal{F}(\text{supp } \Gamma_2) \to \Gamma_2$ be the natural surjective homomorphisms. Denote by $\varphi : \mathcal{F}(\text{supp } \Gamma_1) \to \mathcal{F}(\text{supp } \Gamma_2)$ the surjective homomorphism defined by $\varphi(g) = h$ for $g \in \text{supp } \Gamma_1$ and $h \in \text{supp } \Gamma_2$ such that $A^{(g)} \subseteq A^{(h)}$. Then the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{F}(\text{supp } \Gamma_1) & \xrightarrow{\pi_{\Gamma_1}} & \Gamma_1 \\
\downarrow{\varphi} & & \downarrow{\pi_{\Gamma_1 \to \Gamma_2}} \\
\mathcal{F}(\text{supp } \Gamma_2) & \xrightarrow{\pi_{\Gamma_2}} & \Gamma_2
\end{array}
\]

Note that $\ker \varphi$ coincides with the normal closure in $\mathcal{F}(\text{supp } \Gamma_1)$ of all elements $g_1 g_2^{-1}$ where $g_1, g_2 \in \text{supp } \Gamma_1$ and $A^{(g_1)} \oplus A^{(g_2)} \subseteq A^{(h)}$ for some $h \in \text{supp } \Gamma_2$. Hence $\pi_{\Gamma_1}(\ker \varphi) = Q$.

Suppose $\pi_{\Gamma_1 \to \Gamma_2} \pi_{\Gamma_1}(w) = 1_{\Gamma_2}$ for some $w \in \mathcal{F}(\text{supp } \Gamma_1)$. Then (4.2) implies $\varphi(w) \in \ker \pi_{\Gamma_2}$. Therefore $\varphi(w)$ belongs to the normal closure of the words $h_1 h_2 u^{-1}$ for $h_1, h_2, u \in \text{supp } \Gamma_2$ with $0 \neq A^{(h_1)} A^{(h_2)} \subseteq A^{(u)}$. However the last inclusion holds if and only if $0 \neq A^{(g_1)} A^{(g_2)} \subseteq A^{(t)}$ for some $g_1, g_2, t \in \text{supp } \Gamma_1$ such that $A^{(g_1)} \subseteq A^{(h_1)}$, $A^{(g_2)} \subseteq A^{(h_2)}$. Hence we can rewrite $w = w_0 w_1$ where $w_0 \in \ker \varphi$ and $w_1 \in \ker \pi_{\Gamma_1}$. In particular, $\pi_{\Gamma_1}(w) = \pi_{\Gamma_1}(w_0) \in Q$. Since $\pi_{\Gamma_1}$ is surjective, we get $\ker \pi_{\Gamma_1 \to \Gamma_2} = Q$. Together with the obvious equality $\ker \pi_{\Gamma_1 \to \Gamma_2} \cap \ker \pi_{\Gamma_1}(\text{supp } \Gamma_1) = W$ this implies $Q \cap \ker \pi_{\Gamma_1}(\text{supp } \Gamma_1) = W$. \square

**Lemma 4.7.** Let $\Gamma_1 : A = \bigoplus_{g \in G} A^{(g)}$ be a grading by a group $G$. Then for each subset $W \subseteq \ker \pi_{\Gamma_1}(\text{supp } \Gamma_1)$ there exists a coarsening $\Gamma_2 : A = \bigoplus_{h \in H} A^{(h)}$ of $\Gamma_1$ such that $\ker \pi_{\Gamma_1 \to \Gamma_2} = Q$ where $Q$ is the normal closure of $W$ in $G_{\Gamma_1}$.
Proof. Let \( \pi_1 : F(\supp \Gamma_1) \to G_1 \) and \( \pi : G_1 \to G_1/\bar{Q} \) be the natural surjective homomorphisms. Consider the grading \( \Gamma_2 : A = \bigoplus_{u \in G_1/\bar{Q}} A^{(u)} \) where \( A^{(u)} := \bigoplus_{\pi_i \in \supp \Gamma_1, \ A^{(u)}}. \) We claim that \( G_1/\bar{Q} \) is the universal group of the grading \( \Gamma_2. \) If \( \Gamma_2 \) can be realized as a grading by a group \( H \) and \( \psi : \supp \Gamma_2 \to H \) is the corresponding embedding of the support, then there exists a unique homomorphism \( \phi : F(\supp \Gamma_1) \to H \) such that \( \phi(g) = \psi(\pi(\mathcal{X}_{\Gamma_1}(g))). \) Note that \( \ker(\pi_{\Gamma_1}) \) is the normal closure in \( F(\supp \Gamma_1) \) of:

1. The words \( g h t^{-1} \) for all \( g, h \in \supp \Gamma_1 \) such that \( A^{(g)} A^{(h)} \neq 0 \) where \( t \in \supp \Gamma_1 \) is defined by \( A^{(g)} A^{(h)} \subseteq A^{(t)}; \)
2. The words \( g_1 g_2^{-1} \) for all \( \mathcal{X}_{\Gamma_1}(g_1) \mathcal{X}_{\Gamma_1}(g_2)^{-1} \in W. \)

Hence \( \ker(\pi_{\Gamma_1}) \subseteq \ker \phi \) and there exists a homomorphism \( \bar{\phi} : G_1/\bar{Q} \to H \) such that \( \bar{\phi}(\bar{u}) = \psi(\bar{u}) \) for all \( \bar{u} \in \supp \Gamma_2. \) Since \( G_1/\bar{Q} \) is generated by \( \supp \Gamma_2, \) the homomorphism \( G_1/\bar{Q} \to H \) with this property is unique, \( G_1/\bar{Q} \) is the universal group of the grading \( \Gamma_2 \) and \( \pi_{\Gamma_1} \to \pi_{\Gamma_2} \) can be identified with \( \pi. \) \( \square \)

Remark 4.8. Note that the inclusion \( W \subseteq Q \cap \text{diff} \mathcal{X}_{\Gamma}(\supp \Gamma_1) \) in Lemma 4.7 can be strict.

Definition 4.9. Let \( G \) be a group and let \( W \) be a subset of \( G. \) We say that \( G \) is residually finite with respect to \( W \) if there exists a normal subgroup \( N \triangleleft G \) of finite index such that \( W \cap N = \varnothing. \) We say that \( G \) is hereditarily residually finite with respect to \( W \) if for the normal closure \( N \triangleleft G \) of any subset of \( W \) there exists a normal subgroup \( N \triangleleft G \) of finite index such that \( N_1 \subseteq N \) and \( W \cap N = W \cap N_1. \)

Theorem 4.10. Let \( G \) be a group, \( A \) be an algebra, and let \( \Gamma : A = \bigoplus_{g \in G} A^{(g)} \) be a \( G \)-grading on \( A. \) Then

1. \( \Gamma \) is weakly equivalent to a grading by a finite group if and only if \( G_\Gamma \) is residually finite with respect to \( \text{diff} \ \mathcal{X}_{\Gamma}(\supp \Gamma); \)
2. \( \Gamma \) and all its coarsenings are weakly equivalent to a grading by a finite group if and only if \( G_\Gamma \) is hereditarily residually finite with respect to \( \text{diff} \ \mathcal{X}_{\Gamma}(\supp \Gamma). \)

Proof. The grading \( \Gamma \) can be realized by any factor group of \( G_\Gamma \) that does not glue the elements of the support, i.e. distinct elements of the support have distinct images in that factor group. Then if \( G_\Gamma \) is residually finite with respect to \( \text{diff} \ \mathcal{X}_{\Gamma}(\supp \Gamma), \) there exists a finite factor group with this property. Conversely, if \( \Gamma \) admits a realization as a grading by a finite group \( G, \) then the subgroup of \( G \) generated by the support is a finite factor group of \( G_\Gamma \) that does not glue the elements of the support. Hence \( G_\Gamma \) is residually finite with respect to \( \text{diff} \ \mathcal{X}_{\Gamma}(\supp \Gamma) \) and the first part of the theorem is proved.

Suppose \( G_\Gamma \) is hereditarily residually finite with respect to \( \text{diff} \ \mathcal{X}_{\Gamma}(\supp \Gamma). \) By Lemma 4.6, for any coarsening \( \Gamma_1 \) of \( \Gamma \) there exists a normal subgroup \( Q \) which is the normal closure of

\[
Q \cap \text{diff} \ \mathcal{X}_{\Gamma}(\supp \Gamma) = \{ \mathcal{X}_{\Gamma_1}(g_1) \mathcal{X}_{\Gamma_1}(g_2)^{-1} | g_1, g_2 \in \supp \Gamma_1 \}
\]

such that \( G_{\Gamma_1} \cong G_\Gamma/\bar{Q}. \) Since \( G_\Gamma \) is hereditarily residually finite with respect to \( \text{diff} \ \mathcal{X}_{\Gamma}(\supp \Gamma), \) there exists a normal subgroup \( N \triangleleft G_\Gamma \) of finite index such that \( Q \subseteq N \) and

\[
N \cap \text{diff} \ \mathcal{X}_{\Gamma}(\supp \Gamma) = Q \cap \text{diff} \ \mathcal{X}_{\Gamma}(\supp \Gamma). \quad (4.3)
\]

Let \( N = \pi_{\Gamma \to \Gamma_1}(N). \) Suppose \( \mathcal{X}_{\Gamma_1}(h_1) \mathcal{X}_{\Gamma_1}(h_2)^{-1} \in N \) for some \( h_1, h_2 \in \supp \Gamma_1. \) Using the isomorphism \( G_\Gamma/N \cong G_{\Gamma_1}/\bar{N}, \) we get \( \mathcal{X}_{\Gamma}(g_1) \mathcal{X}_{\Gamma}(g_2)^{-1} \in N \) for all \( g_1, g_2 \in \supp \Gamma_1 \) such that \( A^{(g_1)} \subseteq A^{(h_1)} \) and \( A^{(g_2)} \subseteq A^{(h_2)}. \) Now (4.3) implies \( \mathcal{X}_{\Gamma_1}(g_1) \mathcal{X}_{\Gamma_1}(g_2)^{-1} \in Q, \ \mathcal{X}_{\Gamma_1}(h_1) = \mathcal{X}_{\Gamma_1}(h_2) \)
and \( h_1 = h_2 \). Hence \( \bar{N} \) does not glue the elements of the support of \( \Gamma_1 \). Since \(|G_{\Gamma_1}/\bar{N}| = |G_{\Gamma_1}/N| < +\infty\), the grading \( \Gamma_1 \) admits the finite grading group \( G_{\Gamma_1}/\bar{N} \).

Suppose that every coarsening \( \Gamma_1 \) of \( \Gamma \) admits a finite grading group. We claim that \( G_{\Gamma} \) is hereditarily residually finite with respect to \( \text{diff} \, G_{\Gamma}(\text{supp} \Gamma) \). Indeed, let \( N_1 \triangleleft G_{\Gamma} \) be a normal closure of a subset of \( \text{diff} \, G_{\Gamma}(\text{supp} \Gamma) \). By Lemma 4.7, there exists a grading \( \Gamma_1 \) such that \( G_{\Gamma_1} \cong G_{\Gamma}/N_1 \). Since \( \Gamma_1 \) admits a finite grading group, there exists a normal subgroup \( Q \triangleleft G_{\Gamma_1} \) of finite index such that \( G_{\Gamma_1}(h_1)G_{\Gamma_1}(h_2)^{-1} \notin Q \) for all \( h_1, h_2 \in \text{supp} \Gamma_1 \), \( h_1 \neq h_2 \). Let \( N := \pi_{\Gamma \rightarrow \Gamma_1}^{-1}(Q) \). Then \(|G_{\Gamma_1}/\bar{N}| = |G_{\Gamma_1}/Q| < +\infty\), \( N \supseteq \text{ker} \pi_{\Gamma \rightarrow \Gamma_1} = N_1 \), and

\[ N \cap \text{diff} \, G_{\Gamma}(\text{supp} \Gamma) \supseteq N_1 \cap \text{diff} \, G_{\Gamma}(\text{supp} \Gamma). \]

Suppose \( G_{\Gamma}(g_1)G_{\Gamma}(g_2)^{-1} \in N \) for some \( g_1, g_2 \in \text{supp} \Gamma \). Take \( h_1, h_2 \in \text{supp} \Gamma_1 \) such that \( A^{(g_1)} \subseteq A^{(h_1)} \) and \( A^{(g_2)} \subseteq A^{(h_2)} \). Then \( G_{\Gamma_1}(h_1)G_{\Gamma_1}(h_2)^{-1} \in Q \) and \( h_1 = h_2 \). Thus \( G_{\Gamma}(g_1)G_{\Gamma}(g_2)^{-1} \in N_1 \) and

\[ N \cap \text{diff} \, G_{\Gamma}(\text{supp} \Gamma) = N_1 \cap \text{diff} \, G_{\Gamma}(\text{supp} \Gamma). \]

As a consequence, \( G_{\Gamma} \) is hereditarily residually finite with respect to \( \text{diff} \, G_{\Gamma}(\text{supp} \Gamma) \). \( \square \)

The proposition below could be obtained as a consequence of Theorem 4.10, however we give a separate proof because of its easiness.

**Proposition 4.11.** Let \( \Gamma : A = \bigoplus_{g \in G} A^{(g)} \) be a grading of a finite dimensional algebra \( A \) by an abelian group \( G \). Then \( \Gamma \) is weakly equivalent to a grading by a finite group.

**Proof.** We can replace \( G \) with its subgroup generated by \( \text{supp} \Gamma \). Since \( \text{supp} \Gamma \) is finite, without loss of generality we may assume that \( G \) is a finitely generated abelian group. Then \( G \) is a direct product of free and primary cyclic groups. Replacing free cyclic groups with cyclic groups of a large enough order (see Example 4.12), we get a finite grading group. \( \square \)

**Example 4.12.** Let \( n \in \mathbb{N} \) and let \( \Gamma \) be the elementary \( \mathbb{Z} \)-grading on \( M_n(F) \) defined by the \( n \)-tuple \((1, 2, \ldots, n)\), i.e. \( e_{ij} \in M_n(F)^{(i-j)} \). Then

\[ \text{supp} \Gamma = \{-(n-1), -(n-2), \ldots, -1, 0, 1, 2, \ldots, n-1\} \]

and \( \Gamma \) is equivalent to the elementary \( \mathbb{Z}/(2n\mathbb{Z}) \)-grading defined by the \( n \)-tuple \((\bar{1}, \bar{2}, \ldots, \bar{n})\), i.e. \( e_{ij} \in M_n(F)^{(i-j)} \).

Consider the grading \( \Gamma_0 \) on \( M_n(F) \) by the free group \( \mathcal{F}(x_1, \ldots, x_{n-1}) \) such that \( e_{r,r+1} \in M_n(F)^{(x_r)} \) for \( 1 \leq r \leq n-1 \), i.e. defined by the \( n \)-tuple \((x_1x_2 \cdots x_{n-1}, x_2x_3 \cdots x_{n-1}, \ldots, x_{n-1})\). Note the neutral element component of \( \Gamma_0 \) is the linear span of matrix units \( e_{ii} \), \( 1 \leq i \leq n \), and is \( n \)-dimensional, and all the rest components are 1-dimensional. Since for each elementary grading the diagonal matrix units \( e_{ii} \) belong to the neutral element component of the grading and all matrix units are homogeneous, every elementary grading on \( M_n(F) \) is a coarsening of \( \Gamma_0 \). Since \( \mathcal{F}(x_1, \ldots, x_{n-1}) \) is free and all its free generators belong to \( \text{supp} \Gamma_0 \), \( G_{\Gamma_0} \cong \mathcal{F}(x_1, \ldots, x_{n-1}) \). Note also that

\[ \text{supp} \Gamma_0 = \{x_i x_{i+1} \cdots x_{j} \mid 1 \leq i \leq j \leq n-1\} \cup \{x_{j}^{-1} x_{j-1}^{-1} \cdots x^{-1}_1 \mid 1 \leq i \leq j \leq n-1\} \cup \{1\}. \]

Thus by Theorem 4.10 Problem 1.7 is equivalent to Problem 4.13 below:

**Problem 4.13.** Determine the set \( \Omega \) of the numbers \( n \in \mathbb{N} \) such that the group \( \mathcal{F}(x_1, \ldots, x_{n-1}) \) is hereditarily residually finite with respect to \( \text{diff} \, W_n \) where

\[ W_n = \{x_i x_{i+1} \cdots x_j \mid 1 \leq i \leq j \leq n-1\} \cup \{x_{j}^{-1} x_{j-1}^{-1} \cdots x^{-1}_1 \mid 1 \leq i \leq j \leq n-1\} \cup \{1\}. \quad (4.4) \]
5. Regrading full matrix algebras by finite groups

In this section we prove Theorem 1.8 that deals with Problems 1.7 and 4.13.

In the lemma below we use the idea of [9, Section 4] and show, in particular, that all semigroup regradings of elementary group gradings on $M_n(F)$ can be reduced to group regradings.

**Lemma 5.1.** Let $F$ be a field and let $n \in \mathbb{N}$. If $\Gamma: M_n(F) = \bigoplus_{t \in T} M_n(F)^{(t)}$ is a grading on $M_n(F)$ by a semigroup $T$ such that all $e_{ij}$ are homogeneous elements and there exists an element $e \in T$ such that all $e_{ii} \in M_n(F)^{(e)}$, then $e^2 = e$ and $\text{supp} \Gamma \subseteq U(eTe)$ where $U(eTe)$ is the group of invertible elements of the monoid $eTe$.

**Proof.** $e^2_1 = e_{11}$ implies $e^2 = e$. Since the identity matrix belongs to $M_n(F)^{(e)}$, we obtain $\text{supp} \Gamma \subseteq U(eTe)$. Now $e_{ij}e_{ji} = e_{ii}$ implies $\text{supp} \Gamma \subseteq U(eTe)$. □

The lemma below and its corollary show that if for some $m \in \mathbb{N}$ we have $m \not\in \Omega$ (see the definition of $\Omega$ in Problem 1.7), then $n \not\in \Omega$ for all $n \geq m$.

**Lemma 5.2.** Let $\Gamma: A = \bigoplus_{t \in T} A^{(t)}$ be a grading by a (semi)group $T$ on an algebra $A$ over a field $F$ and let $B$ be a graded subalgebra. Suppose that the grading on $B$ cannot be regraded by a finite (semi)group. Then $\Gamma$ cannot be regraded by a finite (semi)group either.

**Proof.** Each regrading on $\Gamma$ induces a regrading of the grading on $B$. Therefore, if it were possible to regrade $\Gamma$ by a finite (semi)group, the same would be possible for the grading on $B$. However, the latter is impossible. □

**Corollary 5.3.** If for some $m \in \mathbb{N}$ and a group $G$ there exists an elementary $G$-grading on $M_m(F)$ that is not weakly equivalent to a grading by a finite (semi)group, then an elementary $G$-grading with this property exists on $M_n(F)$ for every $n \geq m$.

**Proof.** Suppose this elementary $G$-grading on $M_m(F)$ can be realized by an $m$-tuple $(g_1, g_2, \ldots, g_m)$. Consider the elementary $G$-grading $\Gamma$ on $M_n(F)$ defined by the $n$-tuple $(g_1, g_2, \ldots, g_m, y_m, \ldots, y_m)$. The algebra $M_n(F)$ becomes a graded subalgebra of $M_m(F)$.

Therefore, if $\Gamma$ were weakly equivalent to a grading by a finite (semi)group, then it would be possible to reindex the graded components of $M_m(F)$ by elements of a finite group and the original $G$-grading on $M_m(F)$ had this property too. Hence $\Gamma$ is not weakly equivalent to any grading by a finite (semi)group. □

Since Problems 1.7 and 4.13 are equivalent, we immediately get

**Corollary 5.4.** If $\mathcal{F}(x_1, \ldots, x_{m-1})$ is not hereditarily residually finite with respect to diff $W_m$ for some $m \in \mathbb{N}$, then $\mathcal{F}(x_1, \ldots, x_{n-1})$ is not hereditarily residually finite with respect to diff $W_n$ for all $n \geq m$. (See the definition of $W_n$ in [4.4].)

Recall that a group is residually finite if the intersection of its normal subgroups of finite index is trivial.

**Theorem 5.5.** Let $F$ be a field and let $G$ be a finitely presented group which is not residually finite. (For example, $G$ is a finitely presented infinite simple group, see [18].) Then there exists an elementary $G$-grading on a full matrix algebra which is not weakly equivalent to any $H$-grading for any finite (semi)group $H$.

**Proof.** Let $g_0 \neq 1_G$ be an element that belongs to the intersection of all normal subgroups of $G$ of finite index. (In particular, if $G$ is simple, we take an arbitrary element $g_0 \neq 1_G$.)

By Theorem 4.13 there exists a $G$-grading $\Gamma$ on $M_n(F)$ for some $n \in \mathbb{N}$ such that $G_{\Gamma} \cong G$ and $g_0 \in \text{supp} \Gamma$. □
Suppose that $\Gamma$ is weakly equivalent to a grading by a finite semigroup $H$ and $\psi: \text{supp}\, \Gamma \to H$ is the corresponding embedding of the support. By Lemma 5.1 we may assume that $H$ is a group. Since $G_\Gamma \cong G$, there exists a unique homomorphism $\varphi: G \to H$ such that $\varphi|_{\text{supp}\, \Gamma} = \psi$. However $H$ is finite, $\ker \varphi$ is of finite index, and therefore $\varphi(g_0) = 1_H$. Since $g_0 \neq 1_G$ and $g_0, 1_G \in \text{supp}\, \Gamma$, this $H$-grading cannot be equivalent to $\Gamma$. □

**Corollary 5.6.** If $n \geq 353$, then $M_n(F)$ admits a grading by an infinite group that is not weakly equivalent to any grading by a finite (semi)group.

*Proof.* Consider Thompson’s finitely presented infinite simple group $G_{2,1}$ (see e.g. [18, Section 8]). We can take $g_0$ to be any of its generators which all are anyway in the support of the grading constructed in Theorem 4.3 for $G = G_{2,1}$. Therefore we may assume there that $V$ is empty. Summing up the lengths of the defining relators, we obtain that $n$ in the proof of Theorem 4.3 equals 353. Now we apply Corollary 5.3. □

Corollary 5.6 implies the upper bound in Theorem 1.8.

Recall that an algebra $M(\gamma, \sigma)$ (see the definition in Section 2.2), where $\gamma$ is an $n$-tuple of group elements, contains a graded subalgebra which is graded isomorphic to $M_n(F)$ with the elementary grading determined by $\gamma$, namely, the subalgebra that is the linear span of $e_{ij} \otimes u_{ij}$, $1 \leq i, j \leq n$. Therefore, using Corollary 5.6 and Lemma 5.2 one obtain that, for every $n \geq 353$, every finite subgroup $H \subseteq G_{2,1}$, and every $\sigma \in Z^2(H, F^\times)$, there exists an $n$-tuple $\gamma$ of elements of $G_{2,1}$ such that the standard grading on $M(\gamma, \sigma)$ is not weakly equivalent to a grading by a finite (semi)group.

Now we present a class of elementary gradings that are weakly equivalent to gradings by finite groups.

**Theorem 5.7.** Let $G$ be a group, let $F$ be a field, let $n \in \mathbb{N}$, and let $(g_1, \ldots, g_n)$ be an $n$-tuple of elements of $G$ such that $g_ig_j^{-1} = g_ig_k^{-1}$ if and only if either $i = k$, or $i = j$. Then the elementary grading on $M_n(F)$ defined by $(g_1, \ldots, g_n)$ is weakly equivalent to the elementary $S_{n+1}$-grading defined by $(\gamma_1, \ldots, \gamma_n)$ where $S_{n+1}$ is the symmetric group acting on $\{1, 2, \ldots, n+1\}$ and $\gamma_i = (1, i+1) \in S_{n+1}$ is the transposition switching 1 and $i+1$. The same grading on $M_n(F)$ is weakly equivalent to the elementary $\mathbb{Z}/(2^{n+1}\mathbb{Z})$-grading defined by $(\bar{2}, \bar{2}^2, \bar{2}^3, \ldots, \bar{2}^n)$.

*Proof.* In order to prove the first part of the theorem, it suffices to prove that $\gamma_i\gamma_j^{-1} = \gamma_k\gamma_{\ell}^{-1}$ if and only if either $\{i = k\}$ or $\{i = j\}$ or $\{j = \ell\}$ or $\{i = \ell\}$. However, if $i = j$, then $\gamma_i\gamma_j^{-1}$ is the identity permutation and if $i \neq j$, then $\gamma_i\gamma_j^{-1} = (1, j+1, i+1)$ (a 3-cycle).

In order to prove the second part of the theorem, we notice that $\bar{2}^i - \bar{2}^j = \bar{2}^k - \bar{2}^\ell$ if and only if $2^i - 2^j = 2^k - 2^\ell$ if and only if either $\{i = k\}$ or $\{i = \ell\}$. Indeed, dividing the equality $2^i - 2^j = 2^k - 2^\ell$ by $2^{\min(i, j, k, \ell)}$, we see that the both sides of it must be zero. □

In Theorem 5.7 we have constructed an elementary $G$-grading on $M_n(F)$ that is not weakly equivalent to a grading by a finite group. However, this grading is a coarsening of the elementary grading by the free group $F(z_1, \ldots, z_{n-1})$ that corresponds to $n$-tuple $(1, z_1, \ldots, z_{n-1})$ (this grading was considered in [8, Proposition 4.11], [16, Lemma 4.5]) and which is by Theorem 5.7 is weakly equivalent to a grading by a finite group. In other words, there exist gradings that can be regraded by a finite group, but some of their coarsenings cannot.

**Theorem 5.8.** Let $\Gamma$ be an elementary $G$-grading on the full matrix algebra $M_n(F)$ where $n \leq 3$, $F$ is a field, and $G$ is a group. Then $\Gamma$ is weakly equivalent to a grading by a finite group.
Remark 5.9. If $F$ is algebraically closed, then the theorem holds for all gradings on $M_n(F)$, $n \leq 3$, not necessarily elementary ones. Indeed, by Theorem 2.3 any grading on $M_n(F)$ is isomorphic to the standard grading on $M(\gamma, \sigma)$ for some $\gamma$ and $\sigma$. Comparing the dimensions, we obtain that either $M(\gamma, \sigma)$ is a twisted group algebra of a finite group, i.e. there is nothing to prove, or $\sigma$ is a cocycle of the trivial group and the grading $\Gamma$ is isomorphic to an elementary one.

Proof of Theorem 5.8. If $n = 1$, then $M_n(F) = F$ and it can be regraded by the trivial group. Therefore, we may assume that $n = 2, 3$. Recall that $e_{ij} \in M_n(F)^{(g_i, g_j)}$ where $(g_1, \ldots, g_n)$ is the $n$-tuple defining the elementary grading. Consider the $n \times n$ matrix $A$ where the $(i,j)$th entry is $g_i g_j^{-1}$. It is clear that this matrix completely determines the grading. Theorem 5.2 implies that if two elementary gradings $\Gamma_1$ and $\Gamma_2$ have matrices $A$ and $B$ such that two entries in $A$ coincide if and only if the corresponding entries in $B$ coincide, then $\Gamma_1$ and $\Gamma_2$ are weakly equivalent.

In the case $n = 2$, we get $A = \left( \begin{array}{cc} 1 & g \\ g^{-1} & 1 \end{array} \right)$ where $1 = 1_G$. Here we have two cases: $g = g^{-1}$ and $g \neq g^{-1}$. In the case $g = g^{-1}$ we can regrade $M_2(F)$ by $\mathbb{Z}/2\mathbb{Z}$ (the elementary grading is defined by the couple $(0, 1)$). In the case $g \neq g^{-1}$ we can regrade $M_2(F)$ by $\mathbb{Z}/3\mathbb{Z}$ (the elementary grading is again defined by the couple $(0, 1)$).

Consider now the case $n = 3$. The matrix with the entries $g_i g_j^{-1}$ is of the form $A = \left( \begin{array}{ccc} 1 & g & gh \\ g^{-1} & 1 & h \\ h^{-1} g^{-1} & h^{-1} & 1 \end{array} \right)$ and the same grading can be defined by the triple $(gh, h, 1)$. If $g \in \{1, h, h^{-1}, gh, h^{-1} g^{-1}\}$ or $h \in \{1, g, g^{-1}, gh, h^{-1} g^{-1}\}$, then the subgroup of $G$ generated by $g$ and $h$ is abelian and by Proposition 4.11 the algebra $M_3(F)$ can be regraded by a finite group. Therefore, below we assume that $|\{g, h, gh\}| = 3, 1 \notin \{g, h, gh\}$, and $\{g, h, gh\} \cap \{g^{-1}, h^{-1}, h^{-1} g^{-1}\} \neq \emptyset$ only if $g = g^{-1}, h = h^{-1}, \text{or } gh = h^{-1} g^{-1}$.

We have the following cases:

1. $\{g, h, gh\} \cap \{g^{-1}, h^{-1}, h^{-1} g^{-1}\} = \emptyset$. Here we can apply Theorem 5.4 since all the entries, except the diagonal ones, are different.

2. $gh = h^{-1} g^{-1}$, but $\{g, h\} \cap \{g^{-1}, h^{-1}\} = \emptyset$. Here $\Gamma$ is weakly equivalent to the elementary $\mathbb{Z}/6\mathbb{Z}$-grading defined by $g \mapsto 1$ and $h \mapsto 2$, i.e. by the triple $(3, 2, 0)$.

3. $gh = h^{-1} g^{-1}$, $g = g^{-1}$, but $h \neq h^{-1}$. Here $\Gamma$ is weakly equivalent to the elementary $S_3$-grading defined by $g \mapsto (12)$ and $h \mapsto (13)$.

4. $gh = h^{-1} g^{-1}$, $g \neq g^{-1}$, $h = h^{-1}$. Here $\Gamma$ is weakly equivalent to the elementary $S_3$-grading defined by $g \mapsto (123)$ and $h \mapsto (13)$.

5. $gh = h^{-1} g^{-1}$, $g = g^{-1}$, $h = h^{-1}$. Here $\Gamma$ is weakly equivalent to the elementary $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$-grading defined by $g \mapsto (0, 1)$ and $h \mapsto (1, 0)$.

6. $gh \neq h^{-1} g^{-1}$, $g = g^{-1}$, $h \neq h^{-1}$. Here $\Gamma$ is weakly equivalent to the elementary $\mathbb{Z}/6\mathbb{Z}$-grading defined by $g \mapsto 3$ and $h \mapsto 1$.

7. $gh \neq h^{-1} g^{-1}$, $g \neq g^{-1}$, $h = h^{-1}$. This case is treated analogously.

8. $gh \neq h^{-1} g^{-1}$, $g = g^{-1}$, $h = h^{-1}$. Here $\Gamma$ is weakly equivalent to the elementary $S_3$-grading defined by $g \mapsto (12)$, $h \mapsto (13)$.

All the cases have been considered and $\Gamma$ is weakly equivalent to an elementary grading by a finite group.

Remark 5.10. The elementary $S_3$-grading on $M_3(F)$ defined above by $g \mapsto (12)$, $h \mapsto (13)$ is not weakly equivalent to any of the gradings by abelian groups since $(12)(13) \neq (13)(12)$.

Proof of Theorem 1.8. Theorem 1.8 immediately follows from Corollaries 5.3, 5.6 and Theorem 5.8.
6. Categories related to graded algebras and homomorphisms

If \( A \) and \( B \) are objects in a category \( \mathcal{A} \), we denote the set of morphisms \( A \to B \) by \( \mathcal{A}(A, B) \).

6.1. Change of the grading group and the free-forgetful adjunction.

We first recall two classical examples of adjunctions in the categories related to graded algebras.

Let \( \text{Alg}_F^G \) be the category of algebras over a field \( F \) graded by a group \( G \). The morphisms in \( \text{Alg}_F^G \) are all the homomorphisms of algebras

\[
\psi: A = \bigoplus_{g \in G} A^{(g)} \to B = \bigoplus_{g \in G} B^{(g)}
\]

such that \( \psi(A^{(g)}) \subseteq B^{(g)} \) for all \( g \in G \). If \( \varphi: G \to H \) is a homomorphism of groups, then we have a functor \( \text{U}_\varphi: \text{Alg}_F^G \to \text{Alg}_F^H \) that assigns to a \( G \)-grading \( A = \bigoplus_{g \in G} A^{(g)} \) on an algebra \( A \) the \( H \)-grading

\[
A^{(h)} = \bigoplus_{g \in G, \varphi(g) = h} A^{(g)}
\]

and does not change the homomorphisms. This functor admits a right adjoint functor \( \text{K}_\varphi: \text{Alg}_F^H \to \text{Alg}_F^G \) where for \( B = \bigoplus_{h \in H} B^{(h)} \) we have \( \text{K}_\varphi(B) = \bigoplus_{g \in G} (\text{K}_\varphi(B))^{(g)} \) with \( (\text{K}_\varphi(B))^{(g)} = \{(g, b) \mid b \in B^{(\varphi(g))}\} \) (the vector space structure on \( (\text{K}_\varphi(B))^{(g)} \) is induced from those on \( B^{(\varphi(g))} \)) and the multiplication \( (g_1, a)(g_2, b) := (g_1g_2, ab) \) for \( g_1, g_2 \in G \), \( a \in B^{(\varphi(g_2))} \), \( b \in B^{(\varphi(g_1))} \). If \( \psi \in \text{Alg}_F^H(B_1, B_2) \), then \( \text{K}_\varphi(\psi)(g, b) := (g, \psi(b)) \) where \( g \in G \), \( b \in B_1^{(\varphi(g))} \).

We have a natural bijection

\[
\text{Alg}_F^H(\text{U}_\varphi(A), B) \to \text{Alg}_F^G(A, \text{K}_\varphi(B))
\]

where \( A \in \text{Alg}_F^G \), \( B \in \text{Alg}_F^H \).

Another example of an adjunction is the free-forgetful one. It is especially important for the theory of graded polynomial identities [1, 2, 5, 12, 14].

Let \( G \) be a group and let \( \text{Set}^*_G \) be the category whose objects are sets \( X \), containing a distinguished element 0, with a fixed decomposition \( X = \{0\} \sqcup \bigcup_{g \in G} X^{(g)} \) into a disjoint union. Morphisms between \( X = \{0\} \sqcup \bigcup_{g \in G} X^{(g)} \) and \( Y = \{0\} \sqcup \bigcup_{g \in G} Y^{(g)} \) are maps \( \varphi: X \to Y \) such that \( \varphi(0) = 0 \) and \( \varphi(X^{(g)}) \subseteq \{0\} \sqcup Y^{(g)} \) for all \( g \in G \). We have an obvious forgetful functor \( U: \text{Alg}_F^G \to \text{Set}^*_G \) that assigns to each graded algebra \( A \) the object \( U(A) = \{0\} \sqcup \bigcup_{g \in G} (A^{(g)} \setminus \{0\}) \). This functor has a left adjoint functor \( F: \text{Set}^*_G \to \text{Alg}_F^G \) that assigns to \( X = \{0\} \sqcup \bigcup_{g \in G} X^{(g)} \in \text{Set}^*_G \) the free associative algebra \( F(X \setminus \{0\}) \) on the set \( X \setminus \{0\} \) which is endowed with the grading defined by \( x_1 \cdots x_n \in F(X \setminus \{0\})^{(g_1 \cdots g_n)} \) for \( x_i \in X^{(g_i)} \), \( 1 \leq i \leq n \). Here we have a natural bijection

\[
\text{Alg}_F^G(F(X \setminus \{0\}), A) \to \text{Set}^*_G(X, U(A))
\]

where \( A \in \text{Alg}_F^G \), \( X \in \text{Set}^*_G \).

First of all, we notice that both examples deal with the categories \( \text{Alg}_F^G \) where for each category the grading group is fixed, i.e. the notion of isomorphism in \( \text{Alg}_F^G \) coincides with the notion of graded isomorphism. Second, the functor \( U_\varphi \) corresponds to a regrading, but of a very special kind, namely, the regrading induced by the homomorphism \( \varphi \). For this reason below we consider the category \( \text{GrAlg}_F \) of algebras over the field \( F \) graded by arbitrary groups, in which the notion of isomorphism will coincide with the notion of weak equivalence of gradings.
6.2. An oplax 2-functor from $\text{GrAlg}_F$. Recall that a homomorphism $\psi: A \to B$ between graded algebras $A = \bigoplus_{g \in G} A^{(g)}$ and $B = \bigoplus_{h \in H} B^{(h)}$ is graded if for every $g \in G$ there exists $h \in H$ such that $\psi(A^{(g)}) \subseteq B^{(h)}$.

Any graded homomorphism of graded algebras induces a map between subsets of the supports of the gradings. This gives rise to several functors which we study in the sections below.

Let $\text{GrAlg}_F$ be the category where the objects are group gradings on (not necessarily unital) algebras over a field $F$ and morphisms are graded homomorphisms between the corresponding algebras.

Consider also the category $\mathcal{C}$ where:

- the objects are triples $(G, S, P)$ where $G$ is a group, $S \subseteq G$ is a subset, and $P \subseteq S \times S$;
- the morphisms $(G_1, S_1, P_1) \to (G_2, S_2, P_2)$ are triples $(\psi, R, Q)$ where $R \subseteq S_1$, $Q \subseteq P_1 \cap (R \times R)$, and $\psi: R \to S_2$ is a map such that $\psi(g)\psi(h) = \psi(gh)$ for all $(g, h) \in Q$;
- the identity morphism for $(G, S, P)$ is $(\text{id}_S, S, P)$;
- if $(\psi_1, R_1, Q_1): (G_1, S_1, P_1) \to (G_2, S_2, P_2)$ and $(\psi_2, R_2, Q_2): (G_2, S_2, P_2) \to (G_3, S_3, P_3)$, then

$$(\psi, R, Q) = (\psi_2, R_2, Q_2)(\psi_1, R_1, Q_1): (G_1, S_1, P_1) \to (G_3, S_3, P_3)$$

is defined by $R = \{g \in R_1 \mid \psi_1(g) \in R_2\}$, $Q = \{(g, h) \in Q_1 \mid (\psi_1(g), \psi_1(h)) \in Q_2\}$ and $\psi(g) = \psi_2(\psi_1(g))$ for $g \in R$.

In many cases it is useful to consider categories $\mathcal{A}$ enriched over other categories $\mathcal{B}$, i.e. such categories where the hom-objects $\mathcal{A}(a, b)$ are objects of $\mathcal{B}$ and the composition and the assignment of the identity morphism are morphisms in $\mathcal{B}$. (See the precise definition in [20, Section 1.2].)

Recall that each partially ordered set (or poset) $(M, \preceq)$ is a category where the objects are the elements $m \in M$ and if $m \preceq n$, then there exists a single morphism $m \to n$. If $m \not\preceq n$, then there is no morphism $m \to n$. Denote by $\text{Cat}$ the category of small categories. Since the notion of a $\mathcal{C}$-enriched category coincides with the notion of a 2-category, every category enriched over posets is a 2-category.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are 2-categories. Denote by $\bullet$ the bifunctors of the horizontal composition in $\mathcal{A}$ and $\mathcal{B}$.

An oplax 2-functor $D: \mathcal{A} \to \mathcal{B}$ consists of

1. a map that assigns for each object $a \in \mathcal{B}$ an object $Da \in \mathcal{B}$;
2. functors $D(a_1, a_2): \mathcal{A}(a_1, a_2) \to \mathcal{B}(Da_1, Da_2)$ between the hom-categories for each two objects $a_1, a_2 \in \mathcal{A}$;
3. 2-cells (morphisms in categories $\mathcal{B}(Da_1, Da_2)$) $\delta_a: D\text{id}_{a_1} \Rightarrow \text{id}_{Da_1}$ and transformations $\delta_{g, f}: D(g \bullet f) \Rightarrow Dg \bullet Df$ natural in $f \in \mathcal{A}(a_1, a_2)$ and $g \in \mathcal{A}(a_2, a_3)$ such that

(a) $(\text{id}_{Df} \bullet \delta_a)\delta_{f, \text{id}_{a_1}} = \text{id}_{Df}$ and $(\delta_a \bullet \text{id}_{Dg})\delta_{\text{id}_{a_1}, g} = \text{id}_{Dg}$ for every $a, a_1, a_2 \in \mathcal{A}$ where $f \in \mathcal{A}(a, a_1)$ and $g \in \mathcal{A}(a_2, a)$;

(b) $$(\delta_{h, g} \bullet D\text{id}_f)\delta_{h, g, f} = (D\text{id}_h \bullet \delta_{g, f})\delta_{h, g, f}$$

for all $f \in \mathcal{A}(a_1, a_2)$, $g \in \mathcal{A}(a_2, a_3)$, $h \in \mathcal{A}(a_3, a_4)$, $a_1, a_2, a_3, a_4 \in \mathcal{A}$.

Note that in the case of the category $\mathcal{C}$ defined above, there exists an ordering $\preceq$ on the sets of morphisms $\mathcal{C}((G_1, S_1, P_1), (G_2, S_2, P_2))$: we say that $(\psi_1, R_1, Q_1) \preceq (\psi_2, R_2, Q_2)$ if $R_1 \subseteq R_2$, $Q_1 \subseteq Q_2$, and $\psi_1 = \psi_2|_{R_1}$. This partial ordering turns $\mathcal{C}$ into a category enriched over posets and, in particular, a 2-category.
There exists an obvious map $L: \text{GrAlg}_F \to \mathcal{C}$ where for $\Gamma: A = \bigoplus_{g \in G} A^{(g)}$ we have

$$L(\Gamma) := (G, \text{supp} \Gamma, \{(g_1, g_2) \in G \times G \mid A^{(g_1)} A^{(g_2)} \neq 0\})$$

and for a graded morphism $\varphi: \Gamma \to \Gamma_1$, where $\Gamma_1: B = \bigoplus_{h \in H} B^{(h)}$, the triple $L(\varphi) = (\psi, R, Q)$ is defined by $R = \{(g \in G \mid \varphi(A^{(g)}) \neq 0\}$,

$$Q = \{(g_1, g_2) \in R \times R \mid \varphi(A^{(g_1)}) \varphi(A^{(g_2)}) \neq 0\},$$

and $\psi$ is defined by $\varphi(A^{(g)}) \subseteq B^{(\psi(g))}$ for $g \in R$.

Unfortunately, $L$ is not an ordinary functor: we have

$$L(\varphi_1)L(\varphi_2) \cong L(\varphi_1\varphi_2), \quad (6.1)$$

but the inequality $(6.1)$ can be strict:

**Example 6.1.** Let $A = A^{(0)} \oplus A^{(1)}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, where

$$A^{(0)} = F1_A, \quad A^{(1)} = Fa \oplus Fb, \quad a^2 = ab = ba = b^2 = 0.$$ 

Let $\varphi: A \to A$ be the graded homomorphism defined by $\varphi(1_A) = 1_A, \varphi(a) = b, \varphi(b) = 0$. Then

$$L(A) = (\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2 \setminus \{(1, 1)\})$$

and

$$L(\varphi) = (\text{id}_{\mathbb{Z}/2\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2 \setminus \{(1, 1)\}),$$

and $L(\varphi)^2 = L(\varphi)$, however

$$L(\varphi)^2 = (\text{id}_{\mathbb{Z}/2\mathbb{Z}}, \{0\}, \{(0, 0)\}) \not\cong L(\varphi)^2.$$ 

The inequality $(6.1)$ means that there is a 2-cell between $L(\varphi_1)L(\varphi_2)$ and $L(\varphi_1\varphi_2)$. This turns $L$ to an oplax 2-functor between $\text{GrAlg}_F$ and $\mathcal{C}$ if we treat the category $\text{GrAlg}_F$ as a 2-category with discrete hom-categories (i.e. the only 2-cells in $\text{GrAlg}_F$ are the identity 2-cells between morphisms). All the equalities in the definition of an oplax 2-functor hold, since in the categories which are posets all diagrams commute.

**6.3. The universal grading group functors.** In order to obtain an ordinary functor between 1-categories, we have to restrict the sets of possible morphisms. We call a graded homomorphism *graded injective* if its restriction to each graded component is an injective map.

**Example 6.2.** Let $\chi: G \to F^\times$ be a homomorphism of a group $G$ to the multiplicative group $F^\times$ of a field $F$, i.e. a linear character. Then the induced homomorphism $\tilde{\chi}: FG \to F$ where

$$\tilde{\chi}\left(\sum_{g \in G} \alpha_g u_g\right) := \sum_{g \in G} \alpha_g \chi(g),$$

is graded injective. (Here we denote by $u_g$ the elements of the standard basis in $FG$.)

Consider the category $\widetilde{\text{Alg}}_F$ where the objects are group gradings on (not necessarily unital) algebras over a field $F$ and morphisms are graded injective homomorphisms between the corresponding algebras.

Then we have an obvious functor $R: \widetilde{\text{Alg}}_F \to \text{Grp}$ where for $\Gamma: A = \bigoplus_{g \in G} A^{(g)}$ we have $R(\Gamma) := G\Gamma$ and for a graded injective morphism $\varphi: \Gamma \to \Gamma_1$, where $\Gamma_1: B = \bigoplus_{h \in H} B^{(h)}$, the group homomorphism $R(\varphi)$ is defined by $\varphi(A^{(g)}) \subseteq B^{(R(\varphi)(g))}$, $g \in G$.

One can also restrict his consideration to unital algebras. Denote by $\widetilde{\text{Alg}}^1_F$ the category where the objects are group gradings on unital algebras over a field $F$ and morphisms are unital graded injective homomorphisms between the corresponding algebras. Denote by $R_1$ the restriction of the functor $R$ to the category $\widetilde{\text{Alg}}^1_F$. 
We call $R$ and $R_1$ the universal grading group functors. When it is clear, with which grading an algebra $A$ is endowed, we identify $A$ with the grading on it and treat $A$ as an object of $\text{GrAlg}_F$ or $\text{GrAlg}^1_F$.

6.4. Products and coproducts in $\text{GrAlg}_F$ and $\text{GrAlg}^1_F$. In this section we show that the restriction of the set of morphisms to graded injective ones makes the categories $\text{GrAlg}_F$ and $\text{GrAlg}^1_F$ quite different from $\text{GrAlg}_F$. It can be shown that if $A = \bigoplus_{g \in G} A(g)$ and $B = \bigoplus_{h \in H} B(h)$ are two group graded algebras, then their product in $\text{GrAlg}_F$ equals their product in the category of algebras endowed with the induced $G \times H$-grading and their coproduct in $\text{GrAlg}_F$ is their coproduct in category of algebras endowed with the induced $G \ast H$-grading where $G \ast H$ the coproduct (or the free product) of the groups $G$ and $H$. (See the general definition of a product and a coproduct in a category e.g. in [21, Chapter III, Section 3.4].) The proposition below shows that products in $\text{GrAlg}_F$ and $\text{GrAlg}^1_F$ could be different from those in $\text{GrAlg}_F$.

Recall that by $(u_g)_{g \in G}$ we denote the standard basis in a group algebra $FG$.

**Proposition 6.3.** Let $G$ and $H$ be groups and let $F$ be a field. Then $F(F^x \times G \times H)$ is the product of $FG$ and $FH$ (with the standard gradings) in both categories $\text{GrAlg}_F$ and $\text{GrAlg}^1_F$.

**Proof.** Let $\pi_1: F(F^x \times G \times H) \rightarrow FG$ and $\pi_2: F(F^x \times G \times H) \rightarrow FH$ be the homomorphisms defined by $\pi_1(u_{(a,g,h)}) = \alpha u_g$ and $\pi_2(u_{(a,g,h)}) = u_h$ for $\alpha \in F^x$, $g \in G$, $h \in H$. Obviously, they are graded injective.

Suppose there exists a graded algebra $A$ and graded injective homomorphisms $\varphi_1: A \rightarrow FG$ and $\varphi_2: A \rightarrow FH$. We claim that there exists a unique graded injective homomorphism $\varphi: A \rightarrow F(F^x \times G \times H)$ such that the following diagram commutes:

\[
\begin{array}{ccc}
F(F^x \times G \times H) & \rightarrow & FG \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
FH & \xrightarrow{\varphi_1} & A \\
\varphi_1 \downarrow & & \downarrow \varphi_2 \\
\varphi_1 & \rightarrow & \varphi_2 \\
\end{array}
\]

First we notice that, since each graded component of $FG$ has dimension one and $\varphi_1$ is graded injective, each graded component of $A$ must have dimension at most one. Suppose now that the graded injective homomorphism $\varphi$ indeed exists. Let $a \in A$ be a homogeneous element. Then $\varphi(a)$ must be homogeneous too, i.e. $\varphi(a) = \alpha u_{(\beta,g,h)}$ for some scalars $\alpha, \beta \in F^x$ and group elements $g \in G$, $h \in H$. Then $\varphi(a) = \pi_1 \varphi(a) = \alpha \beta u_g$ and $\varphi_2(a) = \pi_2 \varphi(a) = \alpha u_h$, i.e. $\varphi(a)$ is uniquely determined by $\varphi_1(a)$ and $\varphi_2(a)$.

Given $\varphi_1$ and $\varphi_2$, the homomorphism $\varphi$ is defined as follows. If $\varphi_1(a) = \lambda u_g$ and $\varphi_1(a) = \mu u_h$, then $\varphi(a) = \mu u_{(\lambda/g,h)}$. \hfill $\square$

**Corollary 6.4.** If the field $F$ consists of more than 2 elements, then the functors $R$ and $R_1$ do not have left adjoints.

**Proof.** Each functor that has a left adjoint preserves limits (see e.g. [21, Chapter V, Section 5, Theorem 1]) and, in particular, products. However,

\[R(F(F^x \times G \times H)) = R_1(F(F^x \times G \times H)) = F^x \times G \times H \cong R(FG) \times R(FH) = G \times H\]

in the case when both groups $G$ and $H$ are finite. \hfill $\square$
In the next section we prove that fact for a field $F$ of any cardinality.

Below we show that coproducts in $\widetilde{GrAlg}_F$ and $\widetilde{GrAlg}_F^1$ do not always exist.

**Proposition 6.5.** Let $G$ and $H$ be groups and let $F$ be a field. Then the coproduct of $FG$ and $FH$ (with the standard gradings) in the category $\widetilde{GrAlg}_F$ does not exist.

**Proof.** Suppose $A$ is the coproduct of $FG$ and $FH$ in $\widetilde{GrAlg}_F$ and $i_1: FG \to A$ and $i_2: FH \to A$ are the corresponding morphisms. Let $\varphi_1: FG \to F(G \times H)$ and $\varphi_2: FH \to F(G \times H)$ be the natural embeddings. Then there must exist $\varphi: A \to F(G \times H)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
FH & \xrightarrow{i_1} & A \\
\downarrow{\varphi_1} & & \downarrow{\varphi} \\
F(G \times H) & \xrightarrow{i_2} & FG
\end{array}
$$

In particular,

$$
\varphi(i_1(u_g)i_2(u_h)) = u_{(g,1_H)}u_{(1_G,h)} = u_{(g,h)} \neq 0
$$

and $i_1(u_g)i_2(u_h) \neq 0$.

Now let $\psi_1: FG \to FG \oplus FH$ and $\psi_2: FH \to FG \oplus FH$ be the natural embeddings. Then there must exist $\psi: A \to FG \oplus FH$ such that the following diagram commutes:

$$
\begin{array}{ccc}
FH & \xrightarrow{i_1} & A \\
\downarrow{\psi_1} & & \downarrow{\psi} \\
FG \oplus FH & \xrightarrow{i_2} & FG
\end{array}
$$

In particular,

$$
\psi(i_1(u_g)i_2(u_h)) = (u_g,0)(0,u_h) = 0
$$

and we get a contradiction since $i_1(u_g)i_2(u_h) \neq 0$ is a homogeneous element. \qed

One can show that the coproduct of $FG$ and $FH$ in $\widetilde{GrAlg}_F^1$ equals $F(G \ast H)$.

**Proposition 6.6.** Let $F$ be a field and let $A_i = \langle 1, a_i \rangle_F$, $i = 1, 2$, be two two-dimensional algebras such that $a_i^2 = 0$ with the $\mathbb{Z}/2\mathbb{Z}$-grading defined by $a_i \in A_i^{(1)}$. Then the coproduct of $A_1$ and $A_2$ does not exist neither in the category $\widetilde{GrAlg}_F$, nor in the category $\widetilde{GrAlg}_F^1$.

**Proof.** Suppose $A$ is the coproduct of $A_1$ and $A_2$ and $i_j: A_j \to A$, $j = 1, 2$, are the corresponding morphisms. Let $A_0 = \langle 1, a_1, a_2 \rangle_F$ be the three-dimensional algebra defined by $a_1^2 = a_2^2 = a_1a_2 = a_2a_1 = 0$ with the $\mathbb{Z}/3\mathbb{Z}$-grading defined by $a_j \in A_0^{(j)}$, $j = 1, 2$. Let $\varphi_j: A_j \to A_0$ be the natural embeddings.
Then there must exist \( \varphi: A \to A_0 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\varphi} & A_2 \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow{\varphi_1} & A_2 \\
\end{array}
\]

In particular, \( \varphi(i_1(a_1)i_2(a_2)) = a_1a_2 = 0 \) and \( \varphi(i_1(a_1)i_2(a_2)) = 0 \) since both \( i_1(a_1) \) and \( i_2(a_2) \) are homogeneous elements and \( \varphi \) is graded injective.

Now let \( B = \langle 1, b_1, b_2, b_1b_2 \rangle_F \) be the four-dimensional algebra defined by \( b_1^2 = b_2^2 = b_2b_1 = 0 \) and the \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)-grading defined by \( b_1 \in B^{(1,0)}, b_2 \in B^{(0,1)} \). Let \( \psi_j: A_j \to B \), where \( j = 1, 2 \), be the embeddings defined by \( a_j \mapsto b_j \).

Then there must exist \( \psi: A \to B \) such that the following diagram commutes:

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\psi} & A_2 \\
\downarrow & & \downarrow \\
A & \xrightarrow{\psi_1} & B \\
\end{array}
\]

In particular, \( \psi(i_1(a_1)i_2(a_2)) = b_1b_2 \neq 0 \) and we get contradiction with \( i_1(a_1)i_2(a_2) = 0 \). \( \Box \)

6.5. Absence of adjoints for the universal grading group functors. In this section we show that, unlike functors considered in Section 6.1, the functors \( R \) and \( R_1 \) defined in Section 6.3 do not have neither left nor right adjoints.

**Proposition 6.7.** The functor \( R \) has neither left nor right adjoint.

**Proof.** Suppose \( K \) is left adjoint for \( R \). Then we have a natural bijection

\[
\text{GrAlg}_F(K(H), \Gamma) \rightarrow \text{Grp}(H, R(\Gamma)).
\]

We claim that for each group \( H \) the grading \( K(H) \) is a grading on the zero algebra. Suppose \( K(H): A = \bigoplus_{g \in G} A^g \) and \( A \neq 0 \). Let \( \Lambda \) be a set of indices such that \( |\Lambda| > |\text{Hom}(H, G_{K(H)})| \). Consider the direct sum \( \bigoplus_{\lambda \in \Lambda} A \) of copies of \( A \) where each copy retains its \( G \)-grading \( K(H) \). Denote by \( \Xi \) the resulting \( G \)-grading on \( \bigoplus_{\lambda \in \Lambda} A \). Then \( G_{\Xi} \cong G_{K(H)} \), however

\[
|\text{GrAlg}_F(K(H), \Xi)| \geq |\Lambda| > |\text{Hom}(H, G_{K(H)})| = |\text{Grp}(H, R(\Xi))|
\]

which contradicts the existence of the natural bijection. Hence for each group \( H \) the grading \( K(H) \) is a grading on the zero algebra. In particular, each set \( \text{GrAlg}_F(K(H), \Gamma) \) contains exactly one element.

If \( H \) is a group that admits a non-trivial automorphism and \( FH \) is its group algebra with the standard grading \( \Gamma \), then \( \text{Grp}(H, R(\Gamma)) = \text{Hom}(H, H) \) contains more than one element and we once again get a contradiction. Hence the left adjoint functor for \( R \) does not exist.

Suppose \( K \) is right adjoint for \( R \). Then we have a natural bijection

\[
\text{Grp}(R(\Gamma), H) \rightarrow \text{GrAlg}_F(\Gamma, K(H)).
\]

Considering projections of \( \bigoplus_{\lambda \in \Lambda} A \) on different components (now we assume \( |\Lambda| > |\text{Hom}(G_{K(H)}, H)| \)), we again prove that \( K(H) \) is a zero grading for every group \( H \), each \( \text{GrAlg}_F(\Gamma, K(H)) \) must contain at most one element, but \( \text{Grp}(R(\Gamma), H) \) can contain more
than one element. We get a contradiction and the right adjoint functor for $R$ does not exist.

**Proposition 6.8.** The functor $R_1$ has neither left nor right adjoint.

**Proof.** Suppose $K$ is left adjoint for $R_1$. Then we have a natural bijection

$$\text{GrAlg}_F^1(K(H), \Gamma) \to \text{Grp}(H, R_1(\Gamma)).$$

We claim that for any group $H$ the grading $K(H)$ is the grading on an algebra isomorphic to $F$.

Indeed, let $H$ be a group and let $K(H) : A = \bigoplus_{g \in G} A^{(g)}$. Denote by $\Upsilon$ the grading on $F$ by the trivial group. Then $\text{Grp}(H, R_1(\Upsilon))$ and $\text{GrAlg}_F^1(K(H), \Upsilon)$ both consist of one element. In particular, there exists a unital homomorphism $\varphi : A \to F$. Hence there exists an ideal $\ker \varphi \subseteq A$ of codimension 1.

Now denote by $\Xi$ the grading on $A$ by the trivial group.

If $\ker \varphi \neq 0$, then $\text{GrAlg}_F^1(K(H), \Xi)$ consists of at least two different elements: the identity map $A \to A$ and the composition of $\varphi$ and the embedding $F \cdot 1_A \to A$. Since $\text{Grp}(H, R_1(\Xi))$ consists of a single element, we get $\ker \varphi = 0$ and $A \cong F$.

In particular, $\text{GrAlg}_F^1(K(H), \Gamma)$ contains exactly one element. Again, considering a group $H$ that admits a non-trivial automorphism and its group algebra $FH$ with the standard grading $\Gamma$, we obtain that $\text{Grp}(H, R(\Gamma)) = \text{Hom}(H, H)$ contains more than one element and we get a contradiction. Hence the left adjoint functor for $R_1$ does not exist.

Suppose $K$ is right adjoint for $R_1$. Then we have a natural bijection

$$\text{Grp}(R_1(\Gamma), H) \to \text{GrAlg}_F^1(\Gamma, K(H)).$$

Note that the set $\text{Grp}(R_1(\Gamma), H)$ is always non-empty since it contains at least the homomorphism that maps everything to $1_H$. Fix a group $H$. Let $K(H) : A = \bigoplus_{g \in G} A^{(g)}$. Let $B$ be an $F$-algebra with the cardinality $|B|$ that is greater than the cardinality $|A^{(1)}|$. For example, $B = \text{End}_F V$ where $V$ is a vector space with a basis that has a cardinality greater than $|A^{(1)}|$. Let $\Gamma$ be the grading on $B$ by the trivial group. Then there exist no injective maps $B \to A^{(1)}$ and therefore the set $\text{GrAlg}_F^1(\Gamma, K(H))$ is empty. Again we get a contradiction and the right adjoint functor for $R$ does not exist.

The observations above show that, in order to get indeed an adjunction, one must restrict the category of algebras to the algebras that are well determined by their universal grading group.

### 6.6. Adjunction in the case of group algebras

Let $F$ be a field and let $\text{Grp}_F'$ be the category where the objects are groups $G$ that do not have non-trivial one dimensional representations (in other words, $H^1(G, F^\times) = 0$) and the morphisms are all group homomorphisms. Let $\text{GrAlg}_F'$ be the category where the objects are group algebras $FG$ of groups $G$ from $\text{Grp}_F$ with the standard grading and the morphisms are all non-zero graded algebra homomorphisms.

Let $U$ be the functor $\text{GrAlg}_F' \to \text{Grp}_F'$ defined by $U(FG) = G$ and $\varphi \left( FG_1^{(g)} \right) \subseteq FG_1^{(U(\varphi)(g))}$ for $\varphi : FG_1 \to FG_2$. Denote by $F$ the functor which associates to each group its group algebra over the field $F$.

**Proposition 6.9.** There exists a natural bijection $\theta_{G,A} : \text{GrAlg}_F'(FG, A) \to \text{Grp}_F'(G, U(A))$ where $G \in \text{Grp}_F$ and $A \in \text{GrAlg}_F'$. Furthermore, $FU(-) = 1_{\text{GrAlg}_F'}$ and $U(F-)$ is an isomorphism of categories $\text{Grp}_F'$ and $\text{GrAlg}_F'$.
Proof. Let \( \varphi: FG \rightarrow FH \) be a non-zero graded algebra homomorphism. Then \( \varphi(u_{g_0}) \neq 0 \) for some \( g_0 \in G \) and \( \varphi(u_{g_0}) = \varphi(u_{g_0}u_{g_0}^g)\varphi(u_g) \neq 0 \) implies \( \varphi(u_g) \neq 0 \) for all \( g \in G \). Hence \( \varphi \) is graded injective. Therefore \( \varphi \) is determined by group homomorphisms \( \psi: G \rightarrow H \) and \( \alpha: G \rightarrow F^\times \) such that \( \varphi(u_g) = \alpha(g)u_{\psi(g)} \) for \( g \in G \). Since \( G \) does not have non-trivial one dimensional representations, \( \alpha \) is trivial and we have the natural bijection \( \theta_{G,A} \). The equalities \( FU(-) \cong 1_{\text{GrpAlg}_F}^\times \) and \( U(F-) \cong 1_{\text{GrpF}_+}^\times \) are verified directly. \( \square \)

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