Analytic $p$-adic Banach spaces and the fundamental lemma of Colmez and Fontaine

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Abstract

This article gives a new proof of the fundamental lemma of the “weakly admissible implies admissible” theorem of Colmez-Fontaine that describes the semi-stable $p$-adic representations. To this end, we introduce the category of spectral Banach spaces, which are $p$-adic Banach spaces with a $\mathbb{C}_p$-algebra of analytic functions, and the subcategory of effective Banach-Colmez spaces. The fundamental lemma states the surjectivity of certain analytic maps and describes their kernel. It is proven by an explicit count of solutions of the equations defining these maps. It is equivalent to the existence of functions of dimension and height of effective Banach-Colmez spaces.

Introduction

The “weakly admissible implies admissible” theorem of Colmez and Fontaine [CF00, Théorème A] states that there exists an equivalence of categories between the semi-stable $p$-adic representations and an explicitly described category of filtered $(\varphi, N)$-modules.

This theorem rests on a “fundamental lemma” [CF00, 2.1], of an analytic nature: let $U = \{x \in B^+_{\text{cris}}, \varphi(x) = px\}$, and for $h \geq 1$, let $Y = U^h \times_{\mathbb{C}_p} \mathbb{C}_p$, where $\mathbb{C}_p \to \mathbb{C}_p^h$ is any $\mathbb{C}_p$-linear map, and $U^h \to \mathbb{C}_p^h$ is the restriction of the reduction map $\theta : B^h_{\text{cris}} \to \mathbb{C}_p$. The fundamental lemma states that any map $f : Y \to \mathbb{C}_p$ of the form $f(u_1, \ldots, u_h) = u_1v_1 + \ldots + u_hv_h$ (for any $v_i \in B_{\text{cris}}$ such that $f$ does map $Y$ to $\mathbb{C}_p$) is either surjective, or has a $\mathbb{Q}_p$-finite-dimensional image.

This lemma was later improved by Colmez [Col02, 6.11], who proved that in the case where the map $f$ is surjective, its kernel has dimension $h$ over $\mathbb{Q}_p$. His proof uses sheaves of vector spaces over a suitable category of $\mathbb{C}_p$-Banach algebras.

This article gives an independent proof of the strong version of the fundamental lemma. To this aim, we introduce the new category of spectral Banach spaces and we see the $\mathbb{Q}_p$-vector spaces spaces and linear maps involved in the
Theorem as objects and morphisms of this category. The fundamental lemma is then closely related to the structure of the subcategory of effective Banach-Colmez spaces. The spectral Banach spaces are also interesting objects by themselves since, for example, they give a framework for the Fargues-Fontaine theory [FF11, §8].

The spectral Banach spaces are Banach spaces, plus an extra analytic structure provided by a $\mathbb{C}_p$-Banach algebra of analytic functions. This category naturally contains all finite-dimensional vector spaces over $\mathbb{Q}_p$ or $\mathbb{C}_p$. The objects of the full subcategory of effective Banach-Colmez spaces are the extensions of finite-dimensional $\mathbb{C}_p$-vector spaces by finite-dimensional $\mathbb{Q}_p$-vector spaces. This is for example the case of the spaces

$$E_{d,h} = \{ x \in B_{\text{ cris}}^+, \varphi^h(x) = p^d x \} \quad \text{for } 0 \leq d \leq h. \quad (0.0.1)$$

These objects are fundamental to the proof of the fundamental lemma, as it reduces to the two following facts:

- any effective Banach-Colmez space of dimension one is isomorphic to the direct sum of $E_{1,h}$ and a finite-dimensional $\mathbb{Q}_p$-vector space;

- for all $f_0, \ldots, f_{h-1} \in \mathbb{C}_p$, the map $E_{1,h} \to \mathbb{C}_p, x \mapsto f_0 \vartheta(x) + \cdots + f_{h-1} \vartheta(\varphi^{h-1}(x))$ (where $\vartheta : B_{\text{ cris}}^+ \to \mathbb{C}_p$ is the reduction morphism), is either zero, or surjective with a kernel of dimension $h$ over $\mathbb{Q}_p$.

The fundamental lemma is also interpreted as the existence of natural functions of dimension and height on the category of effective Banach-Colmez spaces.

The first part of this article describes the category of spectral Banach spaces and explains the analytic structure on some usual objects of $p$-adic Hodge theory. The second part proves the fundamental lemma. It first establishes the main properties of the objects $E_{d,h}$, then the structure theorem of effective Banach-Colmez spaces and the fundamental lemma for $E_{1,h}$ as described above.

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Notations

Throughout this document, we use the following notations from [Fon94]: $p$ is a prime number, $\mathbb{Q}_p$ is the field of $p$-adic integers, and $\mathbb{C}_p$ is the completed algebraic closure of $\mathbb{Q}_p$. For any integer $h$, $\mathbb{Q}_{p^h}$ is the unique unramified extension of $\mathbb{Q}_p$ of degree $h$. We write $\varphi$ for the absolute Frobenius automorphism and $[\cdot]$ for the Teichmüller lift. We define $\mathbb{Z}_p(1)$ as the Tate module of the multiplicative group $\mathbb{C}_p^\times$: its elements are families $(\varepsilon_n)$ such that $\varepsilon_{n+1} = \varepsilon_n$ and $\varepsilon_0 = 1$.

For any $p$-adic ring $B$, $\mathcal{O}_B$ is the ring of integers of $B$. The multiplication map by $p$ on $\mathcal{O}_{\mathbb{C}_p}/p$ is a ring homomorphism; we define the ring $R$ as the projective limit of $\mathcal{O}_{\mathbb{C}_p}$ for this map. The projection map $R \to \mathcal{O}_{\mathbb{C}_p}/p$ extends to a natural map $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_p}$, where $W(R)$ is the ring of Witt vectors
with coefficients in $R$. Let $A_{\text{cris}}$ be the $p$-adic completion of the divided power hull of $W(R)$, relative to the canonical divided powers on the ideal $\ker \theta$; also define $B^+_\text{cris} = A_{\text{cris}}[\frac{1}{p}]$. It is a discrete valuation ring, with quotient ring $\mathbb{C}_p$ and maximal ideal generated by an element $t$ such that $\varphi(t) = p \cdot t$.

For any two integers $d \geq 0, h \geq 1$, we define the slope space

$$E_{d,h} = \ker (\varphi^h \circ p^d : B^+_\text{cris} \to B^+_\text{cris}).$$

(0.0.2)

For any non-negative rational $\alpha = \frac{d}{h}$, with $d, h$ coprime, we also write $E_{\alpha} = E_{d,h}$; in particular, $E_d = E_{d,1}$. The slope spaces, and in particular $E_{1,h}$, play a central role in the proofs given below. These spaces also appear in [FF11] under the name $B_{E_{\alpha}} = p^d$. They are the graded quotients of the structure ring of the Fargues-Fontaine curve [FF11, 9.1, 10.1].

## 1 Spectral Banach spaces

For the convenience of the reader, we give a self-contained presentation of spectral Banach spaces. Some results are only stated; complete proofs can be found in [Phû09].

### 1.1 Spectral affine varieties

Let $p$ be a prime number. Let $\mathbb{Q}_p$ be the field of $p$-adic numbers, $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$, and $\mathcal{O}_{\mathbb{C}_p}$ be the ring of integers of $\mathbb{C}_p$.

We consider the categories of Banach spaces (and algebras) over $\mathbb{Q}_p$ as topological spaces, that is, up to equivalence of norm. A lattice of a Banach space $E$ over $\mathbb{Q}_p$ is a closed subgroup $\mathcal{E}$ such that the canonical map $\mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to E$ is an isomorphism.

**Definition 1.1.1.** Let $A$ be a topological $\mathbb{C}_p$-algebra. The spectrum $\text{Sp} A$ of $A$ is the set of all continuous $\mathbb{C}_p$-algebra morphisms from $A$ to $\mathbb{C}_p$.

By the Gelfand transform, any element $a$ of a topological $\mathbb{C}_p$-algebra $A$ may be seen as a function on $S = \text{Sp} A$ by defining $a(s) = s(a) \in \mathbb{C}_p$ for $s \in \text{Sp} A$. The spectrum is endowed with the weak topology, i.e. the coarsest topology for which all elements of $a$ are continuous on $S$.

To each open set $\Omega \subset S$ and each $a \in A$, we attach the semi-norm on $A$ defined by

$$\|a\|_{\Omega} = \sup \{|a(s)|, s \in \Omega \} \in [0, +\infty].$$

(1.1.2)

The set $\Omega$ is bounded if there exists an open set $A \subset A$ and a constant $M < +\infty$ such that $\|a\|_{\Omega} < M$ for all $a \in A$.

**Definition 1.1.3.** A topological $\mathbb{C}_p$-algebra $A$ is pro-spectral if $\text{Sp} A$ is non-empty, $\text{Sp} A$ is the reunion of all bounded open sets $\Omega$, and the topology on $A$ is defined by the family of semi-norms $\|\cdot\|_{\Omega}$. It is spectral if $S = \text{Sp} A$ is bounded and $\|\cdot\|_{S}$ is a norm defining the topology on $A$.  

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If $A$ is a pro-spectral $\mathbb{C}_p$-algebra, then the natural morphism from $A$ to the algebra of continuous functions on $\text{Sp} A$ is injective. In particular, $A$ is reduced. Conversely, if $A$ is reduced and topologically of finite type over $\mathbb{C}_p$, then it is spectral [FvdP04, 3.4.9].

For example, the set $\mathbb{C}_p\{X\}$ of formal series with radius of convergence $\geq 1$ is a spectral algebra, with spectrum homeomorphic to the closed ball $\mathcal{O}_{\mathbb{C}_p}$ [FvdP04, II.4.4]. The set of formal series with infinite radius of convergence is a pro-spectral algebra, with spectrum homeomorphic to $\mathbb{C}_p$.

**Definition 1.1.4.** The category of affine spectral varieties (over $\mathbb{C}_p$) is the opposite category to that of pro-spectral algebras (and continuous $\mathbb{C}_p$-algebra morphisms).

We shall only consider affine spectral varieties, and consequently omit the word “affine”.

The forgetful functor from the category of spectral varieties to that of topological spaces is faithful; therefore, we may see a spectral variety as a topological space with some extra structure. A morphism of spectral varieties will also be called analytic. For any pro-spectral $\mathbb{C}_p$-algebra $A$, we write $\text{Sp} A$ for the spectral variety attached to $A$.

Let $S = \text{Sp} A$ be a spectral variety. We say that $S$ is

- **bounded** if $A$ is spectral;
- **étale** if $S$ is a locally profinite topological space, and $A = C^0(S, \mathbb{C}_p)$;
- **connected** if $A$ has no idempotent elements apart from zero and one;
- **rigid** if $A$ is topologically of finite type over $\mathbb{C}_p$.

There is a natural functor from the category of locally profinite topological spaces to that of spectral varieties. It is fully faithful and left-adjoint to the forgetful functor: all continuous applications from an étale spectral variety to any spectral variety are analytic [Plü09, 1.4.4]. The only analytic morphisms from a connected spectral variety to an étale variety are the constant maps [Plü09, 1.6.10].

A morphism of spectral varieties is **surjective** if the underlying continuous function is surjective. This implies that the corresponding $\mathbb{C}_p$-algebra morphism is an (always injective) isometry for the spectral norm [Plü09, 1.6.1].

The category of spectral varieties has projective limits for families of surjective morphisms indexed by $\mathbb{N}$, corresponding to completed unions of pro-spectral algebras; it also has finite fibre products, corresponding to completed tensor products of pro-spectral algebras [Plü09, 1.6.5,1.6.7]. These constructions are compatible with the underlying topological spaces.

### 1.2 Spectral groups and Banach spaces

**Definition 1.2.1.** An (affine, commutative) spectral group is a group object in the category of affine spectral varieties.
Since all groups considered will be affine and commutative, these adjectives will henceforth be omitted.

The spectral groups inherit from the spectral varieties the existence of finite fibre products and projective limits of countable surjective morphisms. Since finite fibre products exist, this category has kernels. We say that a short sequence of spectral groups is exact if the underlying group sequence is exact.

A prorigid spectral group is a projective limit of a countable family of rigid spectral groups and surjective spectral group morphisms. Prorigid spectral groups admit the following two properties ([Plû09, 3.4.4 and 3.4.9]):

**Proposition 1.2.2 (Global inversion).** Let \( f : G \to H \) be a morphism of prorigid spectral groups. If \( f \) is bijective, then it is an (analytic) isomorphism.

**Proposition 1.2.3 (Connected-étale sequence).** Let \( G \) be a prorigid spectral group. Then \( G \) admits a biggest étale quotient \( \pi_0(G) \) and a biggest connected subgroup \( G^0 \), and there exists a canonical exact sequence

\[
0 \to G^0 \to G \to \pi_0(G) \to 0.
\]

**Definition 1.2.4.** An effective spectral Banach space is a spectral group \( E \) that is a Banach space and such that the multiplication map \( (\times \frac{1}{p}) : E \to E \) is analytic.

If \( E \) is an effective spectral Banach space, then any lattice \( E \) of \( E \) is a spectral group. An effective spectral Banach space \( E \) is prorigid if there exists a lattice \( E \subset E \) that is prorigid as a spectral group. In this case, in view of the global inversion theorem, the condition of analyticity of the multiplication by \( \frac{1}{p} \) is automatically satisfied. Any prorigid Banach space admits a connected-étale sequence; moreover, this sequence splits (non-canonically) ([Plû09, 3.5.5]).

**Examples 1.2.5.** Some examples of effective spectral Banach spaces and spectral groups are:

(i) The étale effective spectral Banach spaces are exactly the finite-dimensional \( \mathbb{Q}_p \)-vector spaces \( V \), as well as the set \( c_0(\mathbb{Q}_p) \) of all convergent sequences ([BGR84, 2.8.2/2]).

(ii) Any finite-dimensional \( \mathbb{C}_p \)-vector space \( L = \mathbb{C}_p^d \) has a canonical structure as a (connected, prorigid) spectral Banach space, where the analytic functions are the everywhere convergent formal series in \( d \) variables ([Plû09, 3.3.3]).

(iii) Since multiplication by \( p \) on \( \mathcal{O}_{\mathbb{C}_p} \) is analytic and surjective, the projective limit \( R \) is a spectral group. Its ring structure and the canonical ring morphism \( R \to \mathcal{O}_{\mathbb{C}_p}/p \) are analytic ([Plû09, 4.2.1]).

(iv) The ring of Witt vectors \( W(R) \) is spectral, as well as the \( W(R) \)-module of Witt bivectors

\[
BW(R) = \left\{ (x_n)_{n \in \mathbb{Z}}, \lim_{n \to \infty} v_R(x_n) > 0 \right\}.
\]

(1.2.6)
Moreover, the ring morphism \( \theta : BW(R) \to \mathbb{C}_p \) and Frobenius \( \varphi : BW(R) \to BW(R) \) are analytic [Plû09, 4.3.4].

The ring \( B^+_{\text{cris}} \) does not have a (canonical) analytic structure. However, there exists a canonical injection \( \eta : BW(R) \hookrightarrow B^+_{\text{cris}} \), which maps the bivector \((x_n)_{n \in \mathbb{Z}} \) to \( \sum p^{-n} x_p^n \); this map is easily seen to be \( W(R) \)-linear and continuous.

Proposition 1.2.7 ([Plû09, 4.4.3]). Let \( d \leq h \) be two integers such that \( 0 \leq d \leq h, h \geq 1 \).

(i) \( E_{d,h} \) is the set of all bivectors \((x_n)_{n \in \mathbb{Z}} \) in \( BW(R) \) satisfying the periodicity condition \( x_{n-d} = x_n^{p^h-d} \) for all \( n \in \mathbb{Z} \).

(ii) The analytic structure given by this homeomorphism between \( E_{d,h} \) and \( m^d \) makes \( E_{d,h} \) an effective spectral Banach subspace of \( BW(R) \).

(iii) The Frobenius morphism \( \varphi : E_{d,h} \to E_{d,h} \) and the reduction morphism \( \theta : E_{d,h} \to \mathbb{C}_p \) are analytic.

(iv) For any \( c \in \mathbb{Q}_p \), the multiplication map by \( c \) on \( E_{d,h} \) is analytic; for any \( d + d' \leq h \), the multiplication map \( E_{d,h} \times E_{d',h} \to E_{d+d',h} \) is analytic.

2 Effective Banach-Colmez spaces

**Definition 2.0.1.** The category of **effective Banach-Colmez spaces** is the full subcategory of all effective spectral Banach spaces \( E \) that insert in an analytic short exact sequence

\[
0 \longrightarrow V \longrightarrow E \longrightarrow L \longrightarrow 0,
\]

with \( V \) being a finite-dimensional \( \mathbb{Q}_p \)-vector space and \( L \) being a finite-dimensional \( \mathbb{C}_p \)-vector space. Such a short exact sequence is a **presentation** of \( E \).

The integers \( h = \dim_{\mathbb{Q}_p} V \) and \( d = \dim_{\mathbb{Q}_p} L \) are called the **height** and **dimension** of this presentation.

Although effective Banach-Colmez spaces seem to be a very specific case of spectral Banach spaces, the logarithmic exponential sequence proves that the group of \( p \)-division points of any rigid group has a natural structure as a Banach-Colmez space [FF11, 7.32].

The integers \( d \) and \( h \) do **a priori** depend on the choice of the presentation. The strong version of the fundamental lemma is equivalent to the fact, which we prove in this section, that these integers actually depend only on the effective Banach-Colmez space \( E \) itself.

We start this section by precisng the structure of some Banach-Colmez spaces via \( p \)-divisible groups; we then compute the extension group \( \text{Ext}^1(L, V) \), which in turns allow to state a structure theorem for one-dimensional Banach-Colmez spaces. This allows us to prove the “fundamental lemma” by reducing to the case where \( E = E_{1/h} \).
2.1 $p$-divisible groups and effective Banach-Colmez spaces

Let $k$ be a perfect subfield of $\mathbb{F}_p$ and $\Gamma$ be a $p$-divisible group [Dem72, II.11] over $k$. There exists an anti-equivalence of categories between $p$-divisible groups over $k$ and Dieudonné modules [Dem72, III.8] over the ring of Witt vectors $W(k)$.

A Dieudonné module is a free module of finite type $M$ over $W(k)$, with a semilinear map $\varphi : M \to M$ such that $\varphi(M) \supset pM$. We write $M(\Gamma)$ for the Dieudonné module associated with a $p$-divisible group $\Gamma$.

**Proposition 2.1.1.** Let $M$ be a Dieudonné module.

(i) For any continuous morphism $f : M \to B_{\text{cris}}^+$ commuting with $\varphi$, the image of $f$ is contained in $BW(R)$.

(ii) The group $E(M) = \text{Hom}_{W(k),\varphi}(M, B_{\text{cris}}^+)$ has a canonical structure as a spectral Banach space.

**Proof.** (i) Up to a finite extension $k'$ of the field $k$, $M$ is isomorphic ([Dem72, IV.4]) to a direct sum of modules of the form $M_{d,h} = W(k)[\varphi]/(\varphi^h - p^d)$. Since $E(M \otimes_k k') = E(M)$, it is enough to prove this when $M = M_{d,h}$. The condition $\varphi(M) \supset pM$ then means that $d \leq h$, and thus by 1.2.7(ii), the image of $f$ lies in $E_{d,h} \subset BW(R)$.

(ii) Let $h$ be the rank of the free module $M$ over $W(k)$. Then $E(M)$ is the kernel of the map $F : x \mapsto x \circ \varphi - \varphi \circ x$ from $\text{Hom}_{W(k)}(M, BW(R)) = BW(R)^h$ to itself. The map $F$ is analytic by 1.2.7(iii) and therefore its kernel is spectral. $\triangleright$

**Proposition 2.1.2.** Let $\Gamma$ be a $p$-divisible group over $k$, of dimension $d$ and height $h$, and let $M$ be the Dieudonné module of $\Gamma$. Let $R$ be the complete, perfect ring of characteristic $p$ defined in 1.2.5(iii). Then the group $\Gamma(R)$ has a canonical structure as a spectral Banach space, and it is canonically isomorphic to the space $E(M)$ defined in Prop. 2.1.1.

**Proof.** Let $d$ be the dimension of $\Gamma$. Since $\Gamma$ is smooth, by [Dem72, II.10,II.11], the choice of a basis of the tangent space to $\Gamma$ defines an isomorphism between the affine algebra of $\Gamma$ and the formal series algebra $k[[x_1, \ldots, x_d]]$. Such an isomorphism also identifies the set $\Gamma(R)$ with $m_R^d$. Moreover, any other choice of coordinates amounts to a $k$-linear automorphism of $m_R^d$, which is always analytic. We thus define the analytic structure on $\Gamma(R)$ by transport from $m_R^d$.

By definition of the Dieudonné functor [Fou77, III.1.2], we know that $\Gamma(R) = \text{Hom}_{W(k),\varphi}(M, CW(R))$. Since $\Gamma(R)$ is $p$-divisible, there exists a group isomorphism

$$\Gamma(R) = \text{Hom}_{W(k),\varphi}(M, CW(R)) = \text{Hom}_{W(k),\varphi}(M, BW(R)). \tag{2.1.3}$$

By 1.2.7(i), there exists a natural group isomorphism $\eta : E(M) \to \Gamma(R)$.

Let $(e_1, \ldots, e_h)$ be a $W(k)$-basis of $M$ such that $(e_1, \ldots, e_d)$ is a basis of $M/\varphi(M)$. Identifying $E(M)$ with a subspace of $BW(R)^h$ through this choice of coordinates, the map $\eta$ is then $(x_1, \ldots, x_h) \mapsto (x_{1,0}, \ldots, x_{h,0})$, where $(x_{i,0})$ is
the zero component of the Witt bivector \( x_i \). This map is therefore analytic. Since both spaces are prorigid, by the global inversion theorem, \( \eta \) is an analytic isomorphism. \( \triangleright \)

**Proposition 2.1.4.** Let \( B \) be a complete, perfect \( k \)-algebra. There exists a natural bijection between the set of continuous algebra morphisms

\[
\text{Hom}_{W(k), \text{cont}}(W(B), \mathcal{O}_{C_p}) \cong \text{Hom}_{k, \text{cont}}(B, R).
\]

**Proof.** Let \( f : B \to R \) be a continuous \( k \)-algebra morphism. Since \( B \) is perfect, there exists a unique sequence of maps \( f_n : B \to R \) such that \( f_0 = f \) and \( f_{n+1}^p = f_n \).

Let \( x = (x_n)_{n \geq 0} \in W(B) \). Then the series \( \sum p^n f_n(x_n) \) converges to some value \( g(x) \in \mathcal{O}_{C_p} \), and the map \( g : W(B) \to \mathcal{O}_{C_p} \) is a continuous \( W(k) \)-algebra morphism. We define a map \( \Theta : \text{Hom}(W(B), \mathcal{O}_{C_p}) \to \text{Hom}(B, R) \) by defining \( \Theta(f) = g \).

Conversely, given a continuous morphism \( g : W(R) \to \mathcal{O}_{C_p} \) and \( x \in B \), let \( x_n = p^n[x]p^{-n} \) be the Witt vector with \( n \)-th coordinate equal to \( x \), and define \( f(x) = (g(x_n))_{n \geq 0} \). We see that the map \( g \mapsto f \) is the inverse of \( \Theta \). \( \triangleright \)

**Proposition 2.1.5.** Let \( \Gamma \) be a connected \( p \)-divisible group over \( k \) and \( G \) be a smooth formal group over \( W(k) \) such that \( G \otimes_{W(k)} k = \Gamma \). Then the following hold:

(i) The group \( E(M) \) is canonically isomorphic to \( \text{Hom}(\mathbb{Q}_p, G(\mathcal{O}_{C_p})) \).

(ii) Let \( t_G \) be the tangent space to \( G \) at zero and \( T(\Gamma) \) be the Tate module \( \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p \Gamma) \). Then there exists an analytic exact sequence

\[
0 \to T(\Gamma)(\overline{k}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to E(M) \to t_G(\mathcal{O}_{C_p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.
\]

In particular, \( E(M) \) is an effective Banach-Colmez space.

**Proof.** (i) Let \( A \) and \( B \) be the affine algebras of \( G \) and \( \Gamma \), and \( [p]_A : A \to A \), \( [p]_B : B \to B \) be the multiplication by \( p \) in the corresponding groups. Let \( A' \) and \( B' \) be the projective limits of \( A \) and \( B \) for the maps \( [p]_A \) and \( [p]_B \). Since \( \Gamma \) is connected, \( [p]_B \) is cofinal to the Frobenius map of \( B \), and \( B' \) is therefore canonically isomorphic to the completion of the radical closure of \( B \). The reduction map \( A \to B \) then extends to a map \( A' \to B' \), whose reduction modulo \( p \) is an isomorphism \( A'/pA' \cong B' \). Since \( A' \) is \( p \)-adically separated and complete and \( B' \) is perfect, \( A' \) is isomorphic to the ring of Witt vectors \( W(B') \). Moreover, \( A' \) represents, by construction, the group functor \( U(G) = \text{Hom}(\mathbb{Q}_p, G) \).

By Proposition 2.1.4, there exists a bijection between the sets \( \text{Hom}(A', \mathcal{O}_{C_p}) = U(G)(\mathcal{O}_{C_p}) \) and \( \text{Hom}(B', R) = \Gamma(R) \). Moreover, this bijection being functorial in \( \Gamma \), it is actually a group isomorphism. Therefore, the three group functors \( U(G)(\mathcal{O}_{C_p}), \Gamma(R) \) and (by 2.1.2) \( E(M) \) are functorially isomorphic. In particular, the first of those three groups is independent of the choice of the lift \( G \).
The exact sequence in 2.1.5(ii) is the logarithmic exact sequence from [Tat67, §4], tensored with \( \mathbb{Q}_p \). Since \( T(\Gamma)(\mathcal{F}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) is étale, the left morphism is analytic; the right morphism is the formal group logarithm of \( G \) and, therefore, is analytic. \(<\)

Let \( M_{d,h} = W(k)[\varphi]/(\varphi^h - p^d) \). Then \( M_{d,h} \) is a Dieudonné module and \( E(M_{d,h}) = E_{d,h} \). By 2.1.2, there exists a canonical analytic structure on \( E_{d,h} = E(M_{d,h}) \). Moreover, the choice of a lift of the \( p \)-divisible group associated to \( M_{d,h} \) with height \( h \) and dimension \( d \).

**Proposition 2.1.6.** Let \( M \) be a Dieudonné module. Then the group of analytical linear applications from \( E(M) \) to \( \mathbb{C}_p \) is canonically isomorphic to \( M \otimes_{W(k)} \mathbb{C}_p \).

See [Plon9, 5.2.7] for a complete proof.

**Sketch of proof.** The map \( F : M \otimes_{W(k)} \mathbb{C}_p \to \text{Hom}_{\text{an}}(M, \mathbb{C}_p) \) associates to an element \( x \otimes \lambda \) the analytic function

\[
E(M) \quad \rightarrow \quad \mathbb{C}_p \\
(f : M \to B_{\text{cris}}^+) \quad \mapsto \quad \lambda \cdot \theta(f(x))
\]

To prove that \( F \) is bijective, we may, as in Prop. 2.1.1, assume that there exist integers \( 0 \leq d \leq h \) such that \( M = M_{d,h} \). In this case, we identify \( M_{d,h} \otimes \mathbb{C}_p \) to \( \mathbb{C}_p \), and thus \( F(\lambda_0, \ldots, \lambda_{h-1}) = \lambda_0 \theta + \ldots + \lambda_{h-1} \theta \circ \varphi^{h-1} \). This map is injective.

Finally, the surjectivity of \( F \) may be proven using an explicit computation of the analytic structure of \( E(M) \) as a projective limit of rigid Banach spaces associated to particular submodules of \( M \), \(<\)

For any effective Banach-Colmez space \( E \), let \( E^\vee_{\mathbb{C}_p} \) be the \( \mathbb{C}_p \)-vector space of analytic morphisms from \( E \) to \( \mathbb{C}_p \), and \( E^\vee_p \) for the dual \( \mathbb{C}_p \)-vector space to \( E^\vee_{\mathbb{C}_p} \). The biduality morphism \( \iota : E \to E^\vee_{\mathbb{C}_p} \) is called the **vector hull** of \( E \). We see by taking coordinates that any morphism from \( E \) to a \( \mathbb{C}_p \)-vector space factorizes uniquely through \( \iota \). In particular, according to proposition 2.1.6, for any \( 0 \leq d \leq h \), the space \( (E_{d,h})_{\mathbb{C}_p} \) is canonically isomorphic to \( \mathbb{C}_p^h \), with the hull map being given by \( \iota(x) = (\theta(\varphi^r(x)))_{r=0, \ldots, h-1} \).

### 2.2 The universal extension of \( \mathbb{C}_p \) by \( \mathbb{Q}_p \)

The Banach-Colmez space \( E_1 = \{ x \in B_{\text{cris}}^+, \varphi(x) = x \} \) has the canonical presentation [Fon94, 5.3.7.(ii)]

\[
0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow E_1 \xrightarrow{\theta} \mathbb{C}_p \longrightarrow 0, \tag{2.2.1}
\]

where \( \theta \) is the restriction to \( E_1 \) of the reduction map \( \theta : B_{\text{cris}}^+ \to \mathbb{C}_p \).

This extension is universal in the following way. Let \( V \) be a \( h \)-dimensional \( \mathbb{Q}_p \)-vector space, \( L \) be a \( d \)-dimensional \( \mathbb{C}_p \)-vector space, and \( f : L \to V \otimes \mathbb{C}_p \),
be a $\mathbb{C}_p$-linear application. We may then form the fibre product $E(f)$ in the following diagram, where the second line is (2.2.1) tensored with $V$:

$$
\begin{array}{c}
0 & \rightarrow & V(1) & \rightarrow & E(f) & \rightarrow & L & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & V(1) & \rightarrow & V \otimes_{\mathbb{Q}_p} E_1 & \rightarrow & V \otimes_{\mathbb{Q}_p} \mathbb{C}_p & \rightarrow & 0 \\
\end{array}
$$

(2.2.2)

Then $E(f)$ is an effective spectral Banach space, and (2.2.2) is a presentation of $E(f)$ as an effective Banach-Colmez space. The Banach space $E(f)$ is connected if $f(L) \cap V = V$ (or equivalently, if the transpose map $^t f : V^\vee_{\mathbb{Q}_p} \rightarrow L^\vee_{\mathbb{C}_p}$ is injective); it is étale if $L = 0$.

**Lemma 2.2.3.** Define a $p$-extractable element of $\mathbb{C}_p\{X\}$ as an element that admits a $p^n$-th root in $\mathbb{C}_p\{X\}$ for all $m$.

Let $g \in \mathbb{C}_p\{X\}$ be such that $g(0) = 1$ and $g(x)^p / g(px)$ is $p$-extractable. Then $g$ is the product of an exponential function and a $p$-extractable element.

**Proof.** Write $g(x)^p / g(px) = u_m(x)^{p^m}$ for all $m \in \mathbb{N}$. Then, for any $n$, the product $v_n(x) = \prod_{m \geq 0} u_{n+m+1}(p^m x)$ converges, and $v_n \in \mathbb{C}_p\{X\}$. Moreover, $v_{n+1}^{p^n} = v_n$ by construction, so that $v_0$ is $p$-extractable. Let $w = g / v_0$.

Then from the relations $v_n(x) = u_{n+1}(x)v_{n+1}(px)$ and $w = v_n^{p^n} / g$ we deduce $w(x)^p = w(px)$. Taking the logarithmic derivative of this relation yields $(w^p / w)(x) = (w^p / w)(px)$, so that $w^p(x) = w(x)w^p(x)$ for all $x$. Therefore $w$ is an exponential function. $	riangle$

**Proposition 2.2.4.** Any analytic extension of a $d$-dimensional $\mathbb{C}_p$-vector space $L$ by a $h$-dimensional $\mathbb{Q}_p$-vector space $V$ is of the form (2.2.2).

**Proof.** Let $(u_i)$ be a basis of $L$. Then the fibre product $E_i = E \times_L (\mathbb{C}_p u_i)$ is an analytic extension of $\mathbb{C}_p$ by $V$, and proving the proposition for all $E_i$ proves it for $E$. Therefore, we may assume that $L = \mathbb{C}_p$. Moreover, let $(f_j)$ be a basis of the dual of $V$. Then, for each $j$, the push-out $E^j = E + V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ has an analytic structure, its affine algebra being the quotient of that of $E$ by the closed ideal of the functions that are zero on $\text{Ker} f_j$. Therefore, the extension $0 \rightarrow \mathbb{Q}_p \rightarrow E^j \rightarrow L \rightarrow 0$ is analytic, and proving the result for all $E^j$ proves it for $E$.

We thus see that it suffices to prove the case where $L = \mathbb{C}_p$ and $V = \mathbb{Q}_p$; by multiplication by a suitable power of $p$, we may even reduce to the case of extensions of $\mathcal{O}_{\mathbb{C}_p} = \text{Sp} \mathbb{C}_p\{X\}$ by the spectral group $\mathbb{Z}_p$.

Let $S = \text{Sp} A$ be such an extension. The diagram $\mathbb{Z}_p(1) \rightarrow S \rightarrow \mathcal{O}_{\mathbb{C}_p}$ corresponds to continuous algebra morphisms $\mathbb{C}_p\{X\} \hookrightarrow A \twoheadrightarrow \mathcal{O}(\mathbb{Z}_p(1), \mathbb{C}_p)$. Let $\varepsilon = (\varepsilon_n)_{n \geq 0}$ be a topological generator of $\mathbb{Z}_p(1)$, and define $i_n : \mathbb{Z}_p^\rightarrow \rightarrow \mathbb{C}_p$ by $i_n(m) = \varepsilon_n^m$. Then $A$ is topologically generated over $\mathbb{C}_p\{X\}$ by elements $f_n$ such that $\rho(f_n) = i_n$. Multiplying $f_n$ by elements of $\mathbb{C}_p\{X\}$, we can assume that $f_{n+1}^{p^n} \equiv f_n \pmod{p}$. Therefore, for all $n$, the sequence $(f_{n+m})^{p^m}$ converges
to some element $g_n$ of $A$. The family $(g_n)$ again topologically generates $A$; moreover $g_{n+1}^p = g_n$ by construction, while $g_0 \in \mathbb{C}_p \{X\}$, $g_0(0) = 1$, and $g_0$ takes its values in $\mathcal{O}_\mathbb{C}^\times$.

For any $n$, $g_n^p$ and $g_n(px)$ coincide on $\mathbb{Z}_p(1)$, so that $\rho(g_n^p/g_n \circ [p]) = 1$ and $u_n = g_n(x)^p / g_n(px) \in \mathbb{C}_p \{X\}$. By construction, $g_0(x)^p / g_0(px) = u_0(x)$ and $u_0$ is extractable as defined in Lemma 2.2.3. Therefore there exists a sequence $v_n$ in $\mathbb{C}_p \{X\}$ and $a \in \mathcal{O}_C$ with $v_p(a) \geq 1/(p-1)$ such that $g_0(x) = \exp(a x) v_n$. Replacing $g_n$ by $v_n g_n$, we may assume that $g_0(x) = \exp(a x)$.

A point $s \in \text{Sp} A$ is thus determined by the values $s_n = g_n(s) \in \mathbb{C}_p$, satisfying the condition $v_p(s_0) \geq v_p(a)$ and $s_{n+1} = s_n$. Therefore, $\text{Sp} A$ is homeomorphic to a closed ball of $\mathfrak{m}_R$. The map $\text{Sp} A \to B^{+}_{\text{cris}}, s \mapsto \log[s]$ is then an analytic isomorphism between $\text{Sp} A$ and a lattice of $E_1$. <

2.3 Structure of effective Banach-Colmez spaces of dimension one

The following results give a full description of the category of Banach-Colmez spaces “of dimension one”, i.e. having a presentation $0 \to V \to E \to \mathbb{C}_p \to 0$. Namely, for these spaces, there exists an integer $h$ such that the connected-étale sequence is of the form

$$0 \to E_{1/h} \to E \to \pi_0(E) \to 0; \tag{2.3.1}$$

since this sequence splits, the space $E$ is (non-canonically) isomorphic to $E_{1/h} \oplus \pi_0(E)$, with $\pi_0(E)$ being a finite-dimensional $\mathbb{Q}_p$-vector space.

Proposition 2.3.2. Let $E$ be an effective Banach-Colmez space having a presentation of dimension one. Assume that $E$ is connected and not isomorphic to $\mathbb{C}_p$. Then there exists an integer $h \geq 1$ such that $E$ is isomorphic to $E_{1,h}$.

The proof uses the following lemma.

Lemma 2.3.3. Let $V$ be a $h$-dimensional $\mathbb{Q}_p$-vector subspace of $\mathbb{C}_p$. Write $\lambda : V \to \mathbb{C}_p$ the canonical injection and $\iota : E_{1/h} \to \mathbb{C}_p^h$ the vector hull of $E_{1/h}$. Then the map $\iota \otimes \lambda : E_{1/h} \otimes \mathbb{Q}_p V \to \mathbb{C}_p^h$ is surjective, and its kernel is a $h$-dimensional vector space over $\mathbb{Q}_p^h$.

Proof. The strategy involves the following steps:

(i) write $f$ as a left $D_{1/h}$-linear map, where $D_{1/h}$ is the division algebra over $\mathbb{Q}_p$ with Brauer invariant $1/h$;

(ii) compute lattices $\mathcal{E}$ of $E_{1/h}$ and $S$ of $\mathbb{C}_p^h$ such that $f(\mathcal{E}) \subset S$, and thus reduce the problem modulo $p$;

(iii) prove that the reduced map $\overline{f} : \mathcal{E}/\pi \mathcal{E} \to S/\pi S$ (where $\pi$ is a uniformizer of $D_{1/h}$) is surjective;

---

$^1$A different way to reach the same result is to consider instead the cocycle $\frac{g_0(x+y)}{g_0(x)g_0(y)}$ [Laz75, II.6.1][Pl609, 5.5.4]; we find it simpler to use its multiplication-by-$p$ analogues instead.
(iv) count the elements of $\text{Ker } f$ and thus proving that its is one-dimensional over $\mathbb{F}_{p^h}$.

The points (iii) and (iv) are proven here only in the special case where $V$ has a basis consisting of elements $(\lambda_i)$ with $v_p(\lambda_i) = 0$. The full proof, similar in spirit but somewhat longer, is detailed in [Pl009, 6.2.3].

**Step (i).** The division algebra $D_{1/h}$, having Brauer invariant $1/h$ over $\mathbb{Q}_p$, is the non-commutative algebra generated over $\mathbb{Q}_p$ by a uniformizer $\pi$ satisfying the relations $\pi^h = p$ and, for all $x \in \mathbb{Q}_p$, $\pi \cdot x = \varphi(x) \cdot \pi$.

The division algebra $D_{1/h}$ acts on $E_{1/h}$ with $\pi$ acting by the Frobenius morphism $\varphi$, and on $\mathbb{C}_p^h$ by $\pi(x_0, \ldots, x_{h-1}) = (x_1, \ldots, x_{h-1}, px_0)$

Let $(\lambda_1, \ldots, \lambda_h)$ be a basis of $V$; then the map $\iota \otimes \lambda$ may be written as

$$f : (x_1, \ldots, x_h) \mapsto \left(\sum_{i=1}^h \lambda_i \varphi^r(x_i)\right)_{r=0, \ldots, h-1}.$$  \hspace{1cm} (2.3.4)

From this, we immediately see that $f(\varphi(x)) = \pi(f(x))$; therefore, the map $f$ is $D_{1/h}$-linear.

**Step (ii).** By selecting an appropriate basis of $V$ over $\mathbb{Q}_p$, and up to multiplication by powers of $p$, we may assume that the $\lambda_i$ are sorted by increasing $p$-adic valuation, and that $v_p(\lambda_i) \in [0, 1 [$. Finally, the reductions of all the $\lambda_i$ with same $p$-adic valuation modulo the appropriate ideal of $\mathcal{O}_{\mathbb{C}_p}$ are linearly independent over $\mathbb{F}_p$.

For any $r \in \mathbb{R}$, define

$$g(r) = \min \left\{ v_p(\varphi(x)), \ x \in R, v_R(x) \geq r \right\} = \min \left\{ p^{nh} \rho - n, n \in \mathbb{Z} \right\}.$$  \hspace{1cm} (2.3.5)

Then $g : \mathbb{R} \to \mathbb{R}$ is a strictly monotonous function, piece-wise affine, with slope $p^{nh}$ on intervals of width $p^{-nh}$. This means that it is possible to find elements $\rho_1, \ldots, \rho_h \in \mathbb{R}$ such that, for all $r = 0, \ldots, h - 1$, the quantity $\tau_r = g(p^r \rho_i) + v(\lambda_i)$ does not depend on $i \in \{1, \ldots, h\}$. Therefore $f(\mathcal{E}) \subset \mathcal{S}$, where $\mathcal{E}$ and $\mathcal{S}$ are the lattices of $E_{1/h}^h$ and $\mathbb{C}_p^h$ defined by

$$\mathcal{E} = \left\{ (x_i)_{i=1, \ldots, h} \in E_{1/h}^h, v_R(x_i) \geq \rho_i \right\},$$

$$\mathcal{S} = \left\{ (y_r)_{r=0, \ldots, h-1} \in \mathbb{C}_p^h, v_p(y_r) \geq \tau_r \right\}. \hspace{1cm} (2.3.6)$$

The maximal order $D_{1/h}$ of $D_{1/h}$ is generated by $\mathbb{Z}_{p^h}$ and $\pi$; it is separated an complete for the $p$-adic topology and $D_{1/h} / \pi D_{1/h} = \mathbb{F}_{p^h}$. Both $\mathcal{E}$ and $\mathcal{S}$ are stable under the action of $D_{1/h}$, and the map $f$ reduces to a $\mathbb{F}_{p^h}$-linear map $\overline{f} : \mathcal{E} / \varphi(\mathcal{E}) \to \mathcal{S} / \varphi(\mathcal{S})$. The surjectivity of $\overline{f}$ is equivalent to that of $\overline{f}$, and the dimension (finite or not) of $\text{Ker } f$ over $D_{1/h}$ is equal to the dimension of $\text{Ker } \overline{f}$ over $\mathbb{F}_{p^h}$.

**Step (iii).** For any $x \in \mathcal{E}$, only a finite number of terms of the series $f(x)$ do not belong to $\pi(\mathcal{S})$; therefore, the Lemma reduces to counting the solutions of a system of polynomials over $m_R$. 

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We address here only the case where all \( v_i \) are zero. In this case, the images \( \lambda_i \) of the \( \lambda \) in \( \mathbb{F}_p \) are \( \mathbb{F}_p \)-linearly independent, and we may write

\[
\mathcal{E} = \left\{ x \in R^h, v_R(x_i) \geq \frac{1}{p^h - 1} \right\}, \quad \mathcal{S} = \bigoplus_{r=0}^{h-1} p^r \mathcal{O}_{C_p}.
\] (2.3.7)

Let \( x = (x_1, \ldots, x_h) \in \mathcal{E} \); then

\[
\tilde{f}(x) = \left( \sum_{i=1}^{h} \lambda_i \sum_{n \in \mathbb{Z}} p^{-n} \theta^i \left( [x^{p^{h+r}}] \right) \right)_{r=0, \ldots, h-1};
\] (2.3.8)

looking at the \( r \)-th component, the only non-zero terms of the series modulo \( \pi \mathcal{S} \) are: for \( r = 0 \), the terms with \( n \in \{0, 1\} \); for \( r = h - 1 \), the terms with \( n \in \{-1, 0\} \); for all other \( r \), only the term with \( n = 0 \).

For all \( z \in \mathcal{O}_{C_p} \), let \( \tilde{z} \) be an element of \( R \) such that \( \tilde{z}^{(0)} = z \). For \( b = (b_0, \ldots, b_{h-1}) \in \mathcal{S}/\pi \mathcal{S} \), the equation \( \tilde{f}(x_1, \ldots, x_h) = b \) is then equivalent to

\[
\begin{align*}
\sum \lambda_i \left( x_i + \tilde{p}^{-1} x_i^h \right) &= \tilde{b}_0, \\
\sum \lambda_i x_i^{p^r} &= \tilde{b}_r \quad \text{for} \ r = 1, \ldots, h - 2, \\
\sum \lambda_i \left( \tilde{p}^{h-1} x_i^{p^{-1}} + x_i^{h-1} \right) &= \tilde{b}_{h-1}.
\end{align*}
\] (2.3.9)

Since \( R \) is a perfect field of characteristic \( p \), the change of variables \( x_i = \tilde{p}^{1/(p^h-1)} y_i^p \), \( \lambda_i = \mu_i^{p^h} \) yields the equivalent equations in the variables \( y_i \in R \), for some constants \( c_0, \ldots, c_{h-1} \in \mathcal{R} \):

\[
\begin{align*}
\sum \mu_i^{p^{h-r-1}} y_i &= c_r \quad \text{for} \ r = 1, \ldots, h - 2, \\
\sum \mu_i^{p^{h-1}} (y_i + y_i^p) &= c_0, \\
\sum \mu_i^{p^h} (\varpi y_i + y_i^p) &= c_{h-1},
\end{align*}
\] (2.3.10)

where \( \varpi \) is an element of \( R \) such that \( v_R(\varpi) = -1 \). Since the \( \lambda_i \) (mod \( p \)) are \( \mathbb{F}_p \)-linearly independent, we can eliminate \( h - 2 \) variables using the linear equations, thus reducing to two equations in \((z_1, z_2)\) of the following form:

\[
\begin{align*}
z_i^{p^h} + \alpha_1 z_1 + \alpha_2 z_2 &= c, \\
z_2^{p^h} + \varpi \beta_1 z_1 + \varpi \beta_2 z_2 &= c',
\end{align*}
\] (2.3.11)

where \( \alpha_1 \beta_2 - \alpha_2 \beta_1 \) is a unit in \( R \). This has exactly \( p^{2h} \) solutions in \( \text{Frac} \ R \), all of which are integral over \( R \) and therefore belong to \( R \). Therefore \( \tilde{f} \) is surjective.

Step (iv). It remains to compute the dimension of the kernel of \( \tilde{f} \). Equation 2.3.9 has exactly \( p^{2h} \) solutions in \( \mathcal{E} \); this must be divided by the number solutions in \( \varphi(\mathcal{E}) \), which are the solutions such that \( v_R(x_i) \geq p/(p^h - 1) \).
Let $ξ ∈ R$ such that $ξ^{p^h−1} = ω$; then the change of variables $z_i = ξ w_i$ linearizes the second equation of (2.3.10). Therefore this system has exactly $p^h$ solutions, so that the kernel of $f$ has dimension 1 over $F_{p^h}$.

Finally, since $f$ is the reduction of $f$ modulo the maximal ideal, $f$ is surjective and its kernel is a line over the division algebra $D_{1/h}$. ◁

**Proof of Proposition 2.3.2.** The presentation of $E$ of dimension one corresponds by 2.2.4 to a $C_p$-linear map $f : C_p → V_{Q_p} ⊗ _{Q_p} C_p$ whose transpose $λ : V → C_p$ is injective. Lemma 2.3.3 shows that $ι ⊗ λ : E_{1,h} ⊗ V → C_p$ is injective with kernel of dimension one over the division algebra $D_{1/h}$. This means that $Ker(ι ⊗ λ)$ is generated by an element $a = (a_1, . . . , a_h)$, with $a_i ∈ E_{1/h}$.

Define $u : Q_p^{p^h} → E_{1/h}^h$ as the linear map $u(x_0, . . . , x_{h−1}) = ∑ x_i φ'(a)$, and $δ = det u = det(φ'(a_i))_{r,i} ∈ B_{cris}$. Then $δ$ is also the determinant of the map $D_{1/h} → Ker(ι ⊗ λ), c → c · a$. Since $a$ is a generator of $Ker(ι ⊗ λ)$, we have $δ ≠ 0$.

The construction (2.2.2) makes $E$ a sub-space of $E_1 ⊗ V = E_1^h$. Let $t$ be a topological generator of $Z_p(1) ⊂ E_1$. For any $x = (x_1, . . . , x_h) ∈ E_1^h$, define $g(x) = t^{−1} ∑ a_i x_i$. The map $g$ is analytic by construction and verifies $φ^h(g(x)) = pg(x)$. It only remains to show that $g(E) ⊂ B_{cris}^+$, and that $g : E → E_{1,h}$ is an isomorphism.

Let $x = (x_1, . . . , x_h) ∈ E ⊂ E_1^h$ and $y = g(x)$. Then $t y = ∑ a_i x_i$, and therefore for any $r ≥ 0$, $θ(t φ'(y)) = p^{−r} θ(φ'(t y)) = 0$, which means that $t φ'(y) ∈ Fil^r B_{cris}^+$. By [Fon94, 5.3.7], this implies that $y ∈ B_{cris}^+$. Since $φ'(t^{−1} x_i) = t^{−1} x_i$, the vector $(φ'(y))$ is the product of the matrix $(φ'(a_i))$ with the vector $(t^{−1} x_i)$. From $δ ≠ 0$ we thus deduce that $g$ is injective. Since $g$ is also a morphism of presentations of effective Banach-Colmez spaces between $E$ and $E_{1/h}$, it is easy to check that it is surjective, and therefore an analytic isomorphism. ◁

**Corollary 2.3.12.** Let $E$ be a Banach-Colmez space having a presentation $0 → V → E → C_p → 0$. Then there exists an integer $h$ such that the connected-étale sequence for $E$ is isomorphic to

$$0 → E_{1/h} → E → π_0(E) → 0.$$ 

**Proof.** This can be directly deduced from applying the proposition 2.3.2 to the connected component of $E$. ◁

### 2.4 Dimension and height

The original version of the fundamental lemma ([CF00, 2.1]) examines a map $f : Y → C_p$, where the object $Y$ is built as a fibre product of $C_p$ and $E_1^h$ in the same way as in 2.2.4 and is therefore an effective Banach-Colmez space, and the map $f$ is analytic. Therefore, it can be deduced from the theorem 2.4.1.

Moreover, 2.4.1 is also equivalent to the strong version of the fundamental lemma [Col02, 6.11]: the category of Banach-Colmez spaces is equivalent to that
**Theorem 2.4.1.** Let $0 \to V \to E \to \mathbb{C}_p \to 0$ be a presentation of an effective Banach-Colmez space with dimension one and height $h \geq 0$, and $f : E \to \mathbb{C}_p$ be any analytic morphism. Then either

(i) $f(E)$ is a finite-dimensional $\mathbb{Q}_p$-vector space with dimension at most $h$, or
(ii) $f$ is surjective and $f(E)$ is finite-dimensional over $\mathbb{Q}_p$ with dimension $h$.

By Proposition 2.3.2, it is enough to prove this when $E = E_{1/h}$. Since this spectral space is connected, the fundamental lemma then takes the shorter form below.

**Proposition 2.4.2.** Let $f : E_{1/h} \to \mathbb{C}_p$ be a nonzero analytic morphism. Then $f$ is surjective and $\dim_{\mathbb{Q}_p} \ker f = h$.

**Proof.** By 2.1.6, there exist $f_0, \ldots, f_{h-1} \in \mathbb{C}_p$ such that $f = f_0 \varphi + \ldots + f_{h-1} \varphi^{h-1}$. Replacing $f$ by $f \circ \varphi^j$, we may assume that $v_p(f_0) = 0$ and $v_p(f_i) \geq -\frac{n}{h}$.

By 1.2.7(i), we may identify $E_{1/h}$ with the set of all bivectors $(x_n)_{n \in \mathbb{Z}} \in {\mathbb{B}}W(R)$ with the periodicity condition $x_{n-1} = x_{n}^{p^{h-1}}$; this isomorphism to $m_R$ is analytic by 1.2.7(ii), so we may identify $E_{1/h}$ with $m_R$ and write

$$f(x) = \sum_{i \in \mathbb{Z}} f_i \varphi([x^i]) \quad (2.4.3)$$

where the sequence $(f_i)$ is extended over $\mathbb{Z}$ by $f_{i+h} = p^{-1}f_i$. Since this application is $\mathbb{Q}_p$-linear, we need only to prove that its image contains a ball of $\mathbb{C}_p$.

We use here an extension of the theory of Newton polygons to the ring $R$. Define the formal series $f^+$ and the polynomial $P$, both with coefficients in $\mathbb{C}_p$, as $f^+(x) = \sum_{n \geq 0} f_n x^{p^n}$ and $P(x) = f_0 x + \ldots + f_{h-1} x^{p^{h-1}}$. They verify the functional equation $f^+(x) = P(x) + \frac{1}{p} f^+(x^{p})$. We recall that the *Newton polygon* of $f$ [Rob00] is the inferior convex hull of the set $\{(p^n, v_p(u_n)), n \in \mathbb{N}\}$, and that its slopes are the valuations of the zeroes of $f$. Since $v_p(f_i) \geq -\frac{n}{h}$ and $v_p(f_0) = 0$, it has the vertices $(p^{nh}, -nh)$ for $n \in \mathbb{N}$.

Define $\rho = \frac{1}{h(p-1)}$ and let $b \in \mathcal{O}_{\mathbb{C}_p}$ such that $v_p(b) \geq \rho$. By the theory of Newton polygons, the equation $f^+(x) = b$ has exactly one solution $x_0 \in \mathbb{C}_p$ such that $v(x_0) = \rho$. Starting from $x_0$, we recursively define a sequence $(x_i)$ such that $f^+(x_i) = p^{-i}b$ and $\lim \inf v_p(x_i^{p^i} - x_i) - v_p(x_i) > 0$. Assume that $x_i$ is known, and let $y \in \mathbb{C}_p$ such that $y^{p^i} = x_i$. Then $f^+(y) = P(y) + \frac{1}{p} f^+(y^p) = P(y) + p^{-(i+1)}b$. Let $x_{i+1} = y + t$; then $f^+(x_{i+1}) = p^{-(i+1)}b$ if, and only if,

$$f^+(y + t) = f^+(y) - P(y). \quad (2.4.4)$$

The Newton polygon for this equation in $t$ shows that there exists exactly $q$ solutions $t$ such that $v_p(t) \geq q^{-1} \rho$. Choose any of them and define $x_{i+1} = y + t$. 

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*of “Espaces Vectoriels de dimension finie”, which are sheaves of finite-dimensional vector spaces over certain Banach algebras over $\mathbb{C}_p$.***
Then the sequence \((x_i)\) satisfies the condition \(v_p(x_{i+1}^q - x_i) \geq \frac{q-1}{q} p\). Thus, for all \(n\), the sequence \((x_{n+m}^q)_{m \geq 0}\) converges to an element \(x'_n\) of \(\mathcal{O}_{\mathbb{C}_p}\); and the sequence \((x'_n)_{n \in \mathbb{N}}\) is an element \(x'\) of \(\mathfrak{m}_R\) such that \(f(x') = b\).

It remains to count the dimension of the kernel of \(f\). This is given by the number of solutions when solving (2.4.4) for \(t\); since this number is exactly \(q = p^h\), the kernel is of dimension \(h\) over \(\mathbb{Q}_p\). \(\triangleleft\)

As a corollary of the fundamental lemma, we obtain the existence of natural functions of dimension and height on the category of effective Banach-Colmez spaces. These functions are additive over short exact sequences. Moreover, for any \(p\)-divisible group \(\Gamma\), the dimension and height of \(\Gamma\) as a \(p\)-divisible group are equal to the dimension and height of the Banach-Colmez space \(\Gamma(R)\).

**Corollary 2.4.5.** Let \(E\) be an effective Banach-Colmez space.

(i) There exists integers \(d = \dim E\) and \(h = \text{ht} E\) such that, for any presentation \(0 \to V \to E \to L \to 0\), we have \(d = \dim_{\mathbb{C}_p} L\) and \(d = \dim_{\mathbb{Q}_p} V\).

(ii) Let \(0 \to E' \to E \to E'' \to 0\) be an exact sequence of spectral Banach-Colmez spaces. Then \(\dim(E) = \dim(E') + \dim(E'')\) and \(\text{ht}(E) = \text{ht}(E') + \text{ht}(E'')\).

These integers are called the **dimension** and **height** of \(E\).

**Proof.** We prove (ii) first. Let \(0 \to E' \to E \to E'' \to 0\) be an exact sequence of effective Banach-Colmez spaces having the presentations \(0 \to V \to E \to L \to 0\), \(0 \to V' \to E' \to L' \to 0\), and \(0 \to V'' \to E'' \to L'' \to 0\).

Let \((u_i)\), \((u''_i)\) be bases of \(L\) and \(L''\) respectively. By replacing \(E''\) by \(E'' \times_{L''} (\mathbb{C}_p u''_i)\) and \(E\) by \(E \times_{L} (\mathbb{C}_p u_i)\), we may assume that \(L = L' = \mathbb{C}_p\). We may then directly deduce (ii) from applying the fundamental lemma to the composite map \(E \to E'' \to L''\).

Finally, (i) is the special case of (ii) where \(E' = 0\). \(\triangleleft\)

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