The Restricted Inomata-McKinley spinor-plane, homotopic deformations and the Lounesto classification

D. Beghetto¹, * R. J. Bueno Rogerio¹, † and C. H. Coronado Villalobos¹‡

¹Universidade Estadual Paulista (UNESP)
Faculdade de Engenharia, Guaratinguetá,
Departamento de Física e Química
12516-410, Guaratinguetá, SP, Brazil

We define a two-dimensional space called the spinor-plane, where all spinors that can be decomposed in terms of Restricted Inomata-McKinley (RIM) spinors reside, and describe some of its properties. Some interesting results concerning the construction of RIM-decomposable spinors emerge when we look at them by means of their spinor-plane representations. We show that, in particular, this space accommodates a bijective linear map between mass-dimension-one and Dirac spinor fields. As a highlight result, the spinor-plane enables us to construct homotopic equivalence relations, revealing an algebraic-topological link between these spinors. In the end, we develop a simple method that provides the categorization of RIM-decomposable spinors in the Lounesto classification, working by means of spinor-plane coordinates, which avoids the often hard work of analysing the bilinear covariant structures one by one.

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I. INTRODUCTION

The so called Inomata-McKinley spinors are a particular class of solutions of the non-linear Heisenberg equation [1]. A subclass of Inomata-McKinley spinors called restricted Inomata-McKinley (RIM) spinors was revealed to be useful in describing neutrino physics [2]. It is well known that free linear massive (or mass-less) Dirac fields can be represented as a combination of RIM-spinors [2]. Moreover, it was recently shown [3] that such Dirac spinors are necessarily type-1 in the so-called Lounesto classification, and that they are all non exotic spinors, i.e., the spacetime itself needs to have an underlying trivial topology in order to enable the very existence of RIM-spinors. Thus, the decomposition in terms of RIM-spinors itself is not allowed in a spacetime with non-trivial topology.

The eigenspinors of charge conjugation operator, or mass-dimension-one (MDO) spinors, compose a new set of spinors with an interesting and complex structure on its own [4, 5]. MDO spinors form a complete set of eigenspinors of the charge conjugation operator, C, however, they have dual helicity and can take positive (self-conjugated) and negative (anti-self-conjugated) eigenvalues of C, contrasting with the Majorana, which take only the positive value and carry single-helicity. From the physical point of view, such spinors are constructed to be “invisible” to other particles, once all the couplings with the fields of the Standard Model are not allowed, except for the Higgs boson, thus, becoming a natural candidate to describe dark matter [4].

The idea of mapping MDO and Dirac spinor fields is not new [6–8]. However, the works developed towards this proposal use MDO as being a type-5 spinor field within Lounesto classification, taking the bilinear covariants associated to this class as fundamental elements in the construction of the mapping. It is well known that MDO fields do not fulfill the requisites to fit in the Lounesto Classification, since their dual is defined in a different way than the usually, which is a presupposition in such classification. Then, a mapping that transcends the need of using the bilinear structures associated to the spinors would be welcome.

Topology is a very important field of study not only in Mathematics, but also in many areas of Physics. By means of its methods and concepts, Topology often allows the discovery and a deep understanding of several substantial aspects in condensed matter, cosmology and many other fields. Furthermore, in particular by Algebraic Topology tools, interesting connections between only apparently disconnected areas and results are often revealed, which makes this field of study so powerful and interesting.

In the present work, we construct a space called spinor-plane, which is a two-dimensional space with its elements being every spinor that can be written in terms of RIM-spinors. The study of this space leads to a better understanding

*Electronic address: dbeghetto@feg.unesp.br
†Electronic address: rodolforogerio@feg.unesp.br
‡Electronic address: ccoronado@feg.unesp.br
of properties and relations between these spinors, as we shall see. The fundamental concept, in Algebraic Topology, of homotopic applications reveals impressive in the study of the spinors in this plane. Moreover, by means of the spinor-plane, we provide a truthful and direct categorization of RIM-decomposable spinors in the so-called Lounesto classification of spinor fields. Also, we show that an easily constructible bijective map between Dirac and MDO spinors is a direct result of the properties of the spinor-plane, dealing only with their decompositions in terms of RIM-spinors.

This paper is organized as follows: A short elementary review on the concept of Homotopy, on the Lounesto classification and on RIM-spinors is presented in two separated Subsections in Section II. The decomposition of MDO spinors in terms of RIM-spinors is made in Section III. In Section IV we construct the two-dimensional space of all RIM-decomposable spinors and present some of its properties, with the main results being shown as two Lemmas. Strong results relating homotopy and RIM-decomposable spinors are condensed in two Theorems rigorously constructed in Section V. In Section VI we devote our attention to the bilinear covariants of the particular class of RIM-decomposable spinors that has its adjoint defined in the Dirac fashion, with the results presented as two Propositions. In the last Section we conclude.

II. ELEMENTARY REVIEW

This section is reserved for a small review on the introductory elements that are necessary for the study carried out in the scope of this paper.

A. Homotopic Applications

Let \( X, Y \) be topological spaces. Two continuous applications \( f, g : X \to Y \) are said to be homotopic when there exists a continuous application

\[
H : X \times [0,1] \to Y
\]

such that \( H(x,0) = f(x) \) and \( H(x,1) = g(x) \) for all \( x \in X \). In this case, \( H \) is called a Homotopy between \( f \) and \( g \), and we write \( f \simeq g \).

Notice that, for every fixed \( t \in [0,1] \), we can consider a continuous application \( H_t : X \to Y \) defined by \( H_t(x) = H(x,t) \). In this case, this is equivalent to construct a family of continuous applications \( (H_t)_{t \in [0,1]} \) defined in \( X \to Y \).

One can understand this parameter \( t \) as being a time variable. This way, it gives, for the homotopy \( H \), the interpretation of a continuous deformation process of the application \( f \) happening in an unity of time: in the initial instant \( t = 0 \) we have \( f \), and in the final instant \( t = 1 \) we have \( g \). Moreover, for all intermediary instants \( t \in (0,1) \), the applications \( H_t \) provide intermediary stages of the deformation process.

The relation \( f \simeq g \) is an equivalence relation in the set of the continuous applications defined in \( X \to Y \). The equivalence classes constructed by this relation are called Homotopy classes.

B. The Lounesto Classification

Let an arbitrary spinor field, namely \( \psi \), be a given spinor field belonging to a section of the vector bundle \( P_{Spin_{1,3}}(M) \times \rho \mathbb{C}^4 \) where \( \rho \) stands for the entire representation space \( D^{(1/2,0)} \oplus D^{(0,1/2)} \). The usual bilinear covariants associated to \( \psi \) reads

\[
\begin{align*}
A &= \bar{\psi}\psi, \text{ (scalar)} \\
B &= i\bar{\psi}\gamma_5\psi, \text{ (pseudo-scalar)} \\
J &= \bar{\psi}\gamma_\mu\psi \gamma^\mu, \text{ (vector)} \\
K &= \bar{\psi}\gamma_\mu\gamma_\nu\psi \gamma^\mu, \text{ (axial-vector)} \\
S &= \frac{1}{2}\bar{\psi}i\gamma_\mu\psi\gamma^\mu \wedge \gamma^\nu, \text{ (bi-vector)}
\end{align*}
\]
where \( \bar{\psi} \) stands for the adjoint spinor. In general grounds, it is always expected to associate given amounts to physical observables. Such bilinear forms obey the so-called Fierz-Pauli-Kofink (FPK) identities, given by \[ J^2 = A^2 + B^2, \]
\[ J_\mu K_\nu - K_\mu J_\nu = -BS_{\mu\nu} - \frac{A}{2}\epsilon_{\mu\nu\alpha\beta}S^{\alpha\beta}, \]
\[ J_\mu K^\mu = 0, \]
\[ J^2 = -K^2. \]

So, the algebraic constraints presented in (2)-(6) reduce the possibilities of (only) six different spinor classes (for which \( J \) is always non-null), known as Lounesto Classification \[10]\:

1. \( A \neq 0, \ B \neq 0; \)
2. \( A \neq 0, \ B = 0; \)
3. \( A = 0, \ B \neq 0; \)
4. \( A = 0 = B, \ K \neq 0, \ S \neq 0; \)
5. \( A = 0 = B, \ K = 0, \ S \neq 0; \)
6. \( A = 0 = B, \ K \neq 0, \ S = 0. \)

with classes 1, 2 and 3 satisfying \( K, S \neq 0 \). The spinors belonging to the first three classes are called regular spinors while classes 4, 5 and 6 are labelled as singular spinors \[4, 8, 11-13\]. Spinors describing fermions in field theory belong to classes 1, 2 and 3 (namely Dirac spinors).

As recently was shown in \[14\], due to the adjoint structure of the MDO fermions \[5\], it is extremely necessary to deform the usual Clifford algebra in order to ascertain the right observance of the FPK identities, regarding MDO spinor fields.

C. A short overview on the non-linear Heisenberg theory formalism

The non-linear Heisenberg equation of motion is easily obtained by varying the action with respect to the spinor field, constructed by \[15, 16\]

\[ \mathcal{L} = \frac{i}{2} \bar{\psi}^H \gamma^\mu \partial_\mu \psi^H - \frac{i}{2} \partial_\mu \bar{\psi}^H \gamma^\mu \psi^H - sJ_\mu J^\mu, \]

thus, non-linear Heisenberg equation reads\[1\] \[2\]

\[ i\gamma^\mu \partial_\mu \psi^H - 2s(A + iB\gamma^5)\psi^H = 0, \]

where \( s \) stands for a constant which has dimension of \((\text{length})^2\) and the physical amounts \( A \) and \( B \) are given in terms of the usual bilinear covariants associated with Heisenberg spinor, given by (2) and (3), respectively. The Heisenberg spinor can be represented by a line in a two-dimensional plane \( \pi \), where each axis is represented by the left-hand and right-hand spinors \[2\]. In such a way that, the Heisenberg spinor can be portrayed as the following identity

\[ \psi^H = \psi^H_L + \psi^H_R, \]

in other words,

\[ \psi^H = \frac{1}{2}(\mathbb{1} + \gamma^5)\psi^H + \frac{1}{2}(\mathbb{1} - \gamma^5)\psi^H. \]

\[1\] The fundamental field equations must be non-linear in order to represent interaction. The masses of the particles should be a consequence of this interaction \[17\].
A particular class of solutions of the Heisenberg equation (12) is given by
\[ \partial_\mu \psi = (aJ_\mu + bK_\mu \gamma^5) \psi, \] (15)
with \( a, b \in \mathbb{C} \) of dimensionality (length)\(^2\), \( J_\mu \) and \( K_\mu \) are covariant and irrotational currents. A \( \psi \) that satisfies the condition (15) also satisfies the Heisenberg equation of motion if \( a \) and \( b \) are such that \( 2s = i(a - b) \) [2] and shall be called as RIM (restricted Inomata-McKinley) spinor. As recently was shown in [3] every Dirac spinor written in terms of RIM spinors belongs to the class 1 within Lounesto Classification. In order that (15) be integrable, the constants \( a \) and \( b \) must obey the constraint \( \text{Re}(a) = \text{Re}(b) \).

Hence, we are able to define \( J^2 = J_\mu J^\mu \) and consequently
\[ J_\mu = \partial_\mu S, \] (16)
where
\[ S = \frac{1}{(a + \bar{a})} \ln \sqrt{J^2}, \] (17)
represents a scalar, and similarly we can write
\[ K_\mu = \partial_\mu R, \] (18)
with
\[ R = \frac{1}{(b - \bar{b})} \ln \left( \frac{A - iB}{\sqrt{J^2}} \right), \] (19)
also being a scalar\(^2\). From (15), we obtain for the left-hand and right-hand Heisenberg spinors
\[ \partial_\mu \psi^H_L = (aJ_\mu + bK_\mu) \psi^H_L, \] (20)
\[ \partial_\mu \psi^H_R = (aJ_\mu - bK_\mu) \psi^H_R. \] (21)

Thus, to complete the program to be accomplished in the scope of this work, one is able to write an arbitrary spinor field, \( \psi \), in terms of a \( \pi \)-plane decomposed Heisenberg spinor
\[ \psi = e^F \psi^H_L + e^G \psi^H_R, \] (22)
and then, looking towards to write a linear theory in terms of a non-linear theory, one analyses the properties encoded on the functions \( F \) and \( G \) in order to the spinor (22) satisfy the Dirac equation. This is the prescription used in reference [2] to write Dirac spinors in terms of RIM-spinors. We will follow this idea in the next Section in order to also write MDO spinors in terms of RIM-spinors.

### III. MASS-DIMENSION-ONE FERMIONS AND RIM-SPINORS

Analogously as developed in [2], we analyse the possibility to write a MDO fermionic field [5] in terms of the non-linear Heisenberg spinors. All the discussion is based on two fundamental equations, the non-linear Heisenberg equation and the \textit{Dirac-like} equation for MDO fermions [4], which reads
\[ (i\gamma^\mu \partial_\mu \Xi \pm m \mathbb{1}) \chi^S/A_h(x) = 0, \] (23)
where the subscript \( h \) stands for the helicity \( h = \{ \pm, \mp \} \) and the operator \( \Xi \) in its matricial form is given by [14]
\[ \Xi = \begin{pmatrix}
\frac{ip \sin \theta}{m} & -i(E + p \cos \theta)e^{-i\phi} \\
\frac{i(E - p \cos \theta)e^{i\phi}}{m} & -\frac{ip \sin \theta}{m} \\
0 & 0 \\
0 & 0 \\
\frac{-ip \sin \theta}{m} & -i(E - p \cos \theta)e^{-i\phi} \\
\frac{i(E + p \cos \theta)e^{i\phi}}{m} & \frac{ip \sin \theta}{m}
\end{pmatrix}, \] (24)

\(^2\) In order to make the notation compact, we define \( \sqrt{J^2} \equiv J \)
where \( p = |p| \). Then, we obtain the identity
\[
\lambda^S_{\lambda h} = \frac{1}{2}(\mathbb{1} + \gamma^5)\lambda^S_{\lambda h} + \frac{1}{2}(\mathbb{1} - \gamma^5)\lambda^S_{\lambda h}.
\] (25)

Explicitly the left- and right-handed components,
\[
\lambda^S_{\lambda h} = \frac{1}{2}(\mathbb{1} - \gamma_5)\lambda^S_{\lambda h},
\]
\[
\lambda^S_{\lambda h} = \frac{1}{2}(\mathbb{1} + \gamma_5)\lambda^S_{\lambda h}.
\]

We are now able to initiate the process to reach the decomposition (or representation) of the MDO spinors in terms of RIM-spinors, following the ideas of the Subsection II C. Firstly, one can write
\[
\lambda^S_{\lambda h} = e^F\psi^H_{\lambda h} + e^G\psi^H_{\lambda h},
\]
and, consequently, for the left- and right-handed components, we obtain
\[
\lambda^S_{\lambda h} = e^F\psi^H_{\lambda h},
\]
\[
\lambda^S_{\lambda h} = e^G\psi^H_{\lambda h}.
\]

The symbol “\( \sim \)” over \( F \) and \( G \), although commonly used to represent the dual of \( \lambda \), is here simply to denote the functions related to \( \lambda \) in the attempt to RIM-decompose such a spinor, and do not have any relation to the dual of the field.

Following the program, the next step is to find the explicity form of \( \tilde{F} \) and \( \tilde{G} \) in order that \( \lambda^S_{\lambda h} \) satisfies (23). Following the same akin reasoning presented in [2] but now for the MDO spinors, we note that the \( x_\mu \) coordinates themselves are functions of \( S \) and \( R \), so the derivative can be written, following the chain-rule, as
\[
\partial_\mu = \partial_\mu S \frac{\partial}{\partial S} + \partial_\mu R \frac{\partial}{\partial R}.
\] (31)

Taking into account the relations in equations (17) and (19), we are able to write (31) in this fashion
\[
\partial_\mu = J_\mu \frac{\partial}{\partial S} + K_\mu \frac{\partial}{\partial R},
\] (32)

therefore, one obtains
\[
\partial_\mu \lambda^S_{\lambda h} = \left( \frac{\partial}{\partial S} J_\mu + \frac{\partial}{\partial R} K_\mu \right) \lambda^S_{\lambda h} + (aJ_\mu + bK_\mu)\lambda^S_{\lambda h},
\] (33)

and
\[
\partial_\mu \lambda^S_{\lambda h} = \left( \frac{\partial}{\partial S} J_\mu + \frac{\partial}{\partial R} K_\mu \right) \lambda^S_{\lambda h} + (aJ_\mu - bK_\mu)\lambda^S_{\lambda h}.
\] (34)

Taking advantage of the *Dirac-like* equation, we multiply the equations (33) and (34) by \( i\gamma^\mu \), then, using the fact that \( \Xi^2 = \mathbb{1} \) and \( [\Xi, \gamma^\mu p_\mu] = 0 \), we have
\[
i\gamma^\mu \partial_\mu \lambda^S_{\lambda h} = i(A - iB)\left( \frac{\partial}{\partial S} \tilde{F} - \frac{\partial}{\partial R} \tilde{F} \right) \lambda^S_{\lambda h} + i(A + iB)\left( \frac{\partial}{\partial S} \tilde{G} - \frac{\partial}{\partial R} \tilde{G} \right) \lambda^S_{\lambda h}.
\] (35)

Using the relations\(^3\) (A5)-(A8), one obtains the following set of equations:
\[
\begin{bmatrix}
(A - iB) \left( \frac{\partial}{\partial S} \tilde{F} - \frac{\partial}{\partial R} \tilde{F} \right) + (a - b) \mathbb{1} & i \lambda^S_{\lambda h} - i m \Xi_1 \lambda^S_{\lambda h} = 0,
\end{bmatrix}
\]
\[
\begin{bmatrix}
(A + iB) \left( \frac{\partial}{\partial S} \tilde{F} + \frac{\partial}{\partial R} \tilde{F} \right) + (a - b) \mathbb{1} & i \lambda^S_{\lambda h} - i m \Xi_2 \lambda^S_{\lambda h} = 0.
\end{bmatrix}
\] (36) (37)

\(^3\) For more informations, please, check the appendix.
At this stage, we freely summarized the notation and rewrite \((24)\) as it follows

\[
\Xi = \begin{pmatrix} \Xi_1 & 0_{2 \times 2} \\ 0_{2 \times 2} & \Xi_2 \end{pmatrix}.
\]  

(38)

After a bit of straightforward calculation, the solutions for \(\tilde{F} (S, R)\) and \(\tilde{G} (S, R)\) functions are given by

\[
\tilde{F} \pm (S, R) = -2i \sigma R \pm \frac{p \sin \theta (A + iB)e^{-2(a + \bar{a})S}}{2(a + \bar{a})},
\]

(39)

\[
\tilde{G} \pm (S, R) = +2i \sigma R \pm \frac{p \sin \theta (A - iB)e^{-2(a + \bar{a})S}}{2(a + \bar{a})}.
\]

(40)

Note that

\[
A + iB = \frac{J^2}{A - iB},
\]

(41)

and from \((17)\), we have

\[
J^2 = e^{2(a + \bar{a})S}.
\]

(42)

Therefore,

\[
e^{\tilde{F}_-} = \exp \left[ -2i \sigma R - \frac{1}{2} \frac{p \sin \theta}{(a + \bar{a})(A - iB)} \right],
\]

(43)

\[
e^{\tilde{G}_-} = \exp \left[ +2i \sigma R - \frac{1}{2} \frac{p \sin \theta}{(a + \bar{a})(A + iB)} \right],
\]

(44)

then, with \(\vartheta \equiv e^{2i \sigma R}\), we have

\[
e^{\tilde{F}_-} = \frac{1}{\vartheta} \exp \left[ -\frac{1}{2} \frac{p \sin \theta}{(a + \bar{a})(A - iB)} \right],
\]

(45)

\[
e^{\tilde{G}_-} = \vartheta \exp \left[ -\frac{1}{2} \frac{p \sin \theta}{(a + \bar{a})(A + iB)} \right].
\]

(46)

Following an analogue prescription, we can write

\[
e^{\tilde{F}_ \pm} = \frac{1}{\vartheta} \exp \left[ \pm \frac{1}{2} \frac{p \sin \theta}{(a + \bar{a})(A - iB)} \right],
\]

(47)

\[
e^{\tilde{G}_ \pm} = \vartheta \exp \left[ \pm \frac{1}{2} \frac{p \sin \theta}{(a + \bar{a})(A + iB)} \right].
\]

(48)

In this manner, we finally write the MDO spinors in terms of RIM-spinors

\[
\lambda = \frac{1}{\vartheta} \exp \left[ \pm \frac{p \sin \theta}{2(a + \bar{a})(A - iB)} \right] \psi_L^H + \vartheta \exp \left[ \pm \frac{p \sin \theta}{2(a + \bar{a})(A + iB)} \right] \psi_R^H,
\]

(49)

or, one is able to write the last expression in the fashion (replacing \(p\) to \(m\))

\[
\lambda = \left( \sqrt{\frac{J}{A - iB}} \right)^\rho \exp \left[ \pm \frac{m \sin \theta}{4 \text{Re}(a)(A - iB)} \right] \psi_L^H + \left( \sqrt{\frac{A - iB}{J}} \right)^\rho \exp \left[ \pm \frac{m \sin \theta}{4 \text{Re}(a)(A + iB)} \right] \psi_R^H,
\]

(50)

where we have defined \(\rho \equiv \frac{\text{Im}(a) - \text{Im}(b)}{\text{Im}(m)} = \frac{2a_{-}^2}{\text{Im}(b)}.\) Note that we omitted the upper index \(S/A\) due to the fact that such spinors differs from a global phase. As a net result we reach that the MDO fields can be freely represented as a combination of RIM-spinors which satisfy the non-linear Heisenberg equation.
IV. TWO-DIMENSIONAL SPINOR-SPACES: THE SPINOR-PLANE

We start this Section giving the definition of the spaces in which we will work on.

Definition 1. We denote by $\Pi^H$ the two-dimensional space whose the set $B = \{\Psi^L_H, \Psi^R_H\}$ (namely, the left- and right-handed components of the RIM-spinor $\Psi^H$) forms a basis. Analogously, we denote the spaces $\Pi^D$ (with basis $D = \{\Psi^L_D, \Psi^R_D\}$ being formed by the components of the Dirac-RIM spinor) and $\Pi^M$ (with basis formed by the MDO-RIM components $M = \{\lambda_L, \lambda_R\}$). These spaces will be called spinor-planes.

In order to achieve better organization, let us record that we can write Dirac spinors $\Psi^D$ and MDO spinors $\lambda$ in the $\Pi^H$ space, via basis $B$, as

$$\Psi^D = \exp\left[\frac{iM}{(a + \bar{a})J}\right] J^{2\sigma} \left(\sqrt{\frac{J}{A - iB}} \Psi^H_L + \sqrt{\frac{A - iB}{J}} \Psi^H_R\right), \quad (51)$$

$$\lambda = \exp\left[\pm \frac{m \sin \theta}{4\text{Re}(a)(A - iB)}\right] \left(\sqrt{\frac{J}{A - iB}}\right)^\rho \Psi^H_L + \exp\left[\pm \frac{m \sin \theta}{4\text{Re}(a)(A + iB)}\right] \left(\sqrt{\frac{A - iB}{J}}\right)^\rho \Psi^H_R, \quad (52)$$

with $J^{2\sigma} = \exp\{[2is - \frac{1}{2}(b - \bar{b})] S\} = \exp\left[-i \frac{\text{Im}(a)}{2\text{Re}(a)} \ln J\right]$. Now we will set the following notations for these complex numbers, for the sake of clarity:

$$\alpha \equiv \exp\left[\frac{iM}{(a + \bar{a})J}\right], \quad (53)$$

$$\beta \equiv J^{2\sigma}, \quad (54)$$

$$\delta \equiv \sqrt{\frac{J}{A - iB}}, \quad (55)$$

$$\epsilon \equiv \left(\sqrt{\frac{J}{A - iB}}\right)^\rho, \quad (56)$$

$$\omega \equiv \exp\left[\pm \frac{m \sin \theta}{4\text{Re}(a)(A - iB)}\right]. \quad (57)$$

$$\zeta \equiv \exp\left[\pm \frac{m \sin \theta}{4\text{Re}(a)(A + iB)}\right]. \quad (58)$$

In this fashion, one can denote the left- and right-handed components of the fields as

$$\Psi^D_L = \alpha \beta \delta \Psi^H_L, \quad (59)$$

$$\Psi^D_R = \alpha \beta \delta^{-1} \Psi^H_R, \quad (60)$$

$$\lambda_L = \epsilon \omega \Psi^H_L, \quad (61)$$

$$\lambda_R = \epsilon^{-1} \zeta \Psi^H_R, \quad (62)$$

which leads to

$$\lambda_L = \chi_1 \Psi^D_L, \quad (63)$$

$$\lambda_R = \chi_2 \Psi^D_R, \quad (64)$$

$$\Psi^D_L = \chi_1^{-1} \lambda_L, \quad (65)$$

$$\Psi^D_R = \chi_2^{-1} \lambda_R, \quad (66)$$

with the coefficients defined as $\chi_1 \equiv \epsilon \omega \delta^{-1} \beta^1 \alpha^{-1}$ and $\chi_2 \equiv \epsilon^{-1} \zeta \delta \beta^1 \alpha^{-1}$ being obviously invertible. These coefficients and their inverses are the tools that map Dirac-RIM spinors into MDO-RIM spinors and vice-versa. After some straightforward calculations, one achieves an explicit form of those complex coefficients as...
\[ 
\chi_1 = \left( \sqrt{\frac{J}{A - iB}} \right)^{\rho^{-1}} \exp \left\{ \frac{1}{2\text{Re}(a)} \left[ \pm \frac{m \sin \theta}{2(A - iB)} - i \left( \text{Im}(a) \ln J + \frac{M}{J} \right) \right] \right\}, 
\]
\[ 
\chi_1^{-1} = \left( \sqrt{\frac{A - iB}{J}} \right)^{\rho^{-1}} \exp \left\{ \frac{1}{2\text{Re}(a)} \left[ \pm \frac{m \sin \theta}{2(A - iB)} + i \left( \text{Im}(a) \ln J + \frac{M}{J} \right) \right] \right\}, 
\]
\[ 
\chi_2 = \left( \sqrt{\frac{A - iB}{J}} \right)^{\rho^{-1}} \exp \left\{ \frac{1}{2\text{Re}(a)} \left[ \pm \frac{m \sin \theta}{2(A + iB)} - i \left( \text{Im}(a) \ln J + \frac{M}{J} \right) \right] \right\}, 
\]
\[ 
\chi_2^{-1} = \left( \sqrt{\frac{J}{A - iB}} \right)^{\rho^{-1}} \exp \left\{ \frac{1}{2\text{Re}(a)} \left[ \pm \frac{m \sin \theta}{2(A + iB)} + i \left( \text{Im}(a) \ln J + \frac{M}{J} \right) \right] \right\}. 
\]

This way, one can obtain

\[ 
\lambda = \frac{1}{2} \left[ \chi_1(\mathbb{1} + \gamma^5) + \chi_2(\mathbb{1} - \gamma^5) \right] \Psi^D, 
\]
\[ 
\Psi^D = \frac{1}{2} \left[ \chi_1^{-1}(\mathbb{1} + \gamma^5) + \chi_2^{-1}(\mathbb{1} - \gamma^5) \right] \lambda. 
\]

If we define the matrices \( M \equiv \frac{1}{2} \left[ \chi_1(\mathbb{1} + \gamma^5) + \chi_2(\mathbb{1} - \gamma^5) \right] \) and \( N \equiv \frac{1}{2} \left[ \chi_1^{-1}(\mathbb{1} + \gamma^5) + \chi_2^{-1}(\mathbb{1} - \gamma^5) \right] \), it easily verifies that \( MN = NM = \mathbb{1} \), i.e., \( N = M^{-1} \). Then, we have just proved the following:

**Lemma 1.** Let \( \varphi_D \in \Pi^D \) and \( \varphi_\lambda \in \Pi^M \). There exists a linear isomorphism \( M : \Pi^D \to \Pi^M \), given by means of a matricial operator \( M = \frac{1}{2} \left[ \chi_1(\mathbb{1} + \gamma^5) + \chi_2(\mathbb{1} - \gamma^5) \right] \), such that

\[ 
\varphi_\lambda = M \varphi_D, 
\]
\[ 
\varphi_D = M^{-1} \varphi_\lambda. 
\]

**Lemma 1** shows a linear bijective (algebraic) map between special classes of MDO and Dirac fields, when both are decomposable in terms of RIM-spinors.

Note that an analogue procedure can be done between all the other combinations of the spinor-spaces. Thus, using \( (v, w)_A \) as a notation for the coordinates of a given spinor in a basis \( A \) of a spinor-space \( \Pi^A \), for \( A \in \{ B, D, M \} \), one can represent \( \Psi^H, \Psi^D \) and \( \lambda \) as

\[ 
\Psi^H = (1, 1)_B = (\alpha^{-1} \beta^{-1} \delta^{-1}, \alpha^{-1} \beta^{-1} \delta)_{\Pi^D} = (\epsilon^{-1} \omega^{-1}, \epsilon \zeta^{-1})_{\Pi^M}, 
\]
\[ 
\Psi^D = (\alpha \beta \delta, \alpha \beta \delta^{-1})_B = (1, 1)_D = (\chi_1, \chi_2)_{\Pi^M}, 
\]
\[ 
\lambda = (\epsilon \omega, \epsilon^{-1} \zeta)_B = (1, 1)_D = (\chi_1, \chi_2)_{\Pi^D}. 
\]

Precisely, the construction of the (invertible) operators \( L : \Pi^H \to \Pi^D \) and \( Q : \Pi^H \to \Pi^M \) leads to matricial representations given by

\[ 
L = \frac{1}{2} \left[ (\alpha \beta \delta)(\mathbb{1} + \gamma^5) + (\alpha \beta \delta^{-1})(\mathbb{1} - \gamma^5) \right], 
\]
\[ 
Q = \frac{1}{2} \left[ (\epsilon \omega)(\mathbb{1} + \gamma^5) + (\omega^{-1} \zeta)(\mathbb{1} - \gamma^5) \right], 
\]

such that

\[ 
\Psi^D = L \Psi^H, 
\]
\[ 
\Psi^H = L^{-1} \Psi^D, 
\]
\[ 
\lambda = Q \Psi^H, 
\]
\[ 
\Psi^H = Q^{-1} \lambda. 
\]

Then, we can state the following:
Lemma 2. Suppose the existence of a spinor-plane $\Pi^S$ with basis formed by left- and right-handed components of a given spinor $\psi = \psi_L + \psi_R$. If $\psi$ can be decomposed in terms of at least one of $\Psi^H$, $\Psi^D$ or $\lambda$ components with both coefficients non vanishing (in other words, the decomposition is invertible), then it can be written in terms of any of those spinors, i.e., $\Pi^S \cong \Pi^H \cong \Pi^D \cong \Pi^M$.

Proof. It is trivial, using the results of Lemma 1 and Equations (80 - 83).

Note that Lemma 1 is a corollary of Lemma 2.

Another fact that is worthwhile to mention is that $M$, $Q$ and $L$ as shown in Lemma 1 and Equations (78) and (79) are all diagonal (as, obviously, their inverses). This is because of the nature of the chirality projector operators, and we can define:

Definition 2. We define $\mathfrak{M}$ as being the space of all matricial operators $R$ such that $\psi = R\varphi$, with $\psi, \varphi$ being spinors that may be decomposed in terms of RIM-spinors. The space $\mathfrak{M}$ has the set of projector operators $\{\frac{1}{2}(1 + \gamma^5), \frac{1}{2}(1 - \gamma^5)\}$ as basis, working with complex coefficients to form elements of $\mathfrak{M}$, i.e.,

$$\forall R \in \mathfrak{M}, \exists c_1, c_2 \in \mathbb{C} : R = c_1 \frac{1}{2}(1 + \gamma^5) + c_2 \frac{1}{2}(1 - \gamma^5).$$

(84)

Explicitely, $R = \text{diag}(c_2, c_2, c_1, c_1)$.

It should be clear that, when $c_1, c_2 \neq 0$, every $R \in \mathfrak{M}$ is invertible, with $\text{diag}(c_2^{-1}, c_2^{-1}, c_1^{-1}, c_1^{-1}) = R^{-1} \in \mathfrak{M}$.

Finally, given the aspect of all those spinor-planes, we can understand them as being, in fact, exactly the same space, with the matrices $M, L, Q$ and their inverses being change-of-basis matrix operators between the basis $\mathcal{B}, \mathcal{D}$ and $\mathcal{M}$, with this being valid for every matrix $R \in \mathfrak{M}$ with other basis of the spinor-plane. This way, we can understand the space $\mathfrak{M}$ as being the space of all change-of-basis matrix operators in the spinor-plane. Then, we have found a two-dimensional space of all spinors that may be decomposed in terms of RIM-spinors (given its left- and right-handed components to form a basis on this space), equipped with a space of change-of-basis matrix operators.

V. THE s-SPACE, THE SPINOR-PLANE AND HOMOTOPIC FUNCTIONS

The reference [2] analyses carefully the domain of parameters $a$ and $b$, in order to avoid singularities on the potentials $S$ and $R$. Writing the complex numbers $a = a_0 e^{i\phi_1}$ and $b = b_0 e^{i\phi_2}$ in their polar forms, one is able to separate all the possible values for these complex numbers into (only) six disjoint domains:

$$\Omega_1 \equiv W_1 \otimes Z_1,$$
$$\Omega_2 \equiv W_4 \otimes Z_1,$$
$$\Omega_3 \equiv W_4 \otimes Z_1,$$
$$\Omega_4 \equiv W_2 \otimes Z_2,$$
$$\Omega_5 \equiv W_3 \otimes Z_2,$$
$$\Omega_6 \equiv W_3 \otimes Z_3,$$

in which the intervals are defined as $W_1 = (0, \frac{\pi}{2})$, $W_2 = (\frac{\pi}{2}, \pi)$, $W_3 = (\pi, \frac{3\pi}{2})$, $W_4 = (\frac{3\pi}{2}, 2\pi)$ for $\phi_1$, with an analogue definition for $Z_1, Z_2, Z_3$ and $Z_4$ as intervals of $\phi_2$. The point here is that for different choices of $a$ and $b$ in those six domains, one can construct different spinor configurations. Then, in order to make more clear our explanations, we define the s-space:

Definition 3. Let $\Omega \equiv \bigcup_{i=1}^{6} \Omega_i$ be the space of all the feasible choices of parameters $(\phi_1, \phi_2)$ for $a = a(\phi_1)$ and $b = b(\phi_2)$ that define the Heisenberg constant $s = \frac{i(a-b)}{2}$ for the RIM solution (15) of the the Heisenberg equation (12). We will call $\Omega$ the s-space.

Now we are able to introduce another interpretation for the two-dimensional spinor space, as we are dealing in this work: fix a basis on this space, say $\mathcal{B}$, so we are in the “RIM-copy” of the spinor-plane. In this copy, the spinor $\Psi^H$ is a linear function: accurately, it is the identity function $y_H(x) = x$. Yet in this copy of the plane, we have $\Psi^D$ given by the function $y_D(x) = (\frac{a-b}{i}) x$, and $\lambda$ given by $y_\lambda(x) = (\frac{a-b}{i})^{-1} \omega^{-1} \zeta x$. For both Dirac and MDO cases, we have the variable $x$ being defined via the s-space $\Omega$, i.e., $x = x(a(\phi_1), b(\phi_2))$, and also $y = y(a(\phi_1), b(\phi_2))$. But once a pair $(\phi_1, \phi_2) \in \Omega$ is fixed, all coordinates on the spinor-plane for every spinor is a pair $(x, y(x))$ in every basis.
In other words, Dirac, MDO and RIM spinors (depending on which basis we are working on the spinor-plane) are implicit functions of \(a\) and \(b\) (or, via s-space, of \(\phi_1\) and \(\phi_2\)), i.e., behave like functions of the type

\[
\varphi_{\mathcal{S}} : \Omega \longrightarrow \Pi^H
\]

\[
(\phi_1, \phi_2) \mapsto (f_1, f_2)_{\mathcal{S}},
\]

with \(f_1, f_2\) complex functions of the pair \((\phi_1, \phi_2) \in \Omega\). In a similar way, we can define \(\varphi_{\mathcal{D}} : \Omega \rightarrow \Pi^D\) and \(\varphi_{\mathcal{M}} : \Omega \rightarrow \Pi^M\).

Of course, it is also valid for every spinor in the spinor-plane.

It should be clear that both \(\Psi^D\) and \(\lambda\) are linear functions (also the identity function) when represented in their “own copies” of the spinor-plane (i.e., when they are written in terms of the basis \(\mathcal{D}\) and \(\mathcal{M}\) respectively). In fact, it is true for every possible spinor \(\psi\) as described in Lemma 2. Following this idea, one can think on the basis change being a deformation of the points (which are functions) on the spinor-plane, leading us to the attempt of construction of a homotopy on this space. Before initiate this, we need first to note that each point on the spinor-plane (in any fixed basis) can be written as \((x, y(x))\), with \(y : \mathbb{C} \rightarrow \mathbb{C}\). Notice that every \(y = y(x)\) is a function on topological spaces, once \(x(\phi_1, \phi_2)\) is set.

For us to begin the construction of the homotopy \(H\), let, for instance, the Dirac spinor \(\Psi^D\) be represented as \((x, f(x))_{\mathcal{D}} = (x, g(x))_{\mathcal{B}}\). Then, we know that \(f(x) = x\) and \(g(x) = (A - JB)x\). Now we need to find a continuous application \(H : \mathbb{C} \times [0, 1] \rightarrow \mathbb{C}\) such that \(H(x, 0) = f(x)\) and \(H(x, 1) = g(x)\) for all \(x\). Defining

\[
H(x, t) = (1 - t)f(x) + tg(x) = \left[1 + t \left(\frac{A - iB}{J}\right) - 1\right]x,
\]

we see that it satisfies the conditions, and a remarkable result comes out: for each fixed value \(j \in (0, 1)\), the function \(H(x, j) \equiv H_j(x)\) induces a new representation for \(\Psi^D\) as a pair \((x, H_j(x))_{\mathcal{A}} \in \Pi^A\), which corresponds to an intermediate copy \(\Pi^A\) of the spinor-plane or, equivalently, it gives to the spinor-plane a basis \(\mathcal{A} = \{\Psi^A_L, \Psi^A_R\}\) corresponding to the definition of an intermediate spinor \(\Psi^A\). Noticing this fact, and remembering the result of Lemma 2, we can state the following:

**Theorem 1.** Let \(f = f(x)\) and \(g = g(x)\) be functions such that \((x, f(x))_{\mathcal{A}_0}\) and \((x, g(x))_{\mathcal{A}_1}\) represent the same spinor on the spinor-plane by different basis \(\mathcal{A}_0\) and \(\mathcal{A}_1\). Then it is possible to construct a homotopy \(H(x, t)\) between \(f\) and \(g\) that defines an infinite family of spinors \(\Psi^A\) (that can be decomposed in terms of RIM-spinors), with each spinor being represented by the identity function if the basis \(\mathcal{A}_j = \{\Psi^A_L, \Psi^A_R\}\) is used (or equivalently, each spinor is represented by \((x, x)_{\mathcal{A}_j}\) for each fixed \(t = j \in [0, 1]\).

It is clear that one could construct all the spinors in the spinor-plane by just simply choosing a pair of complex numbers \((c_1, c_2)\) and writing down, for instance, \(\psi = c_1\psi^H_L + c_2\psi^H_R = (c_1, c_2)_{\mathcal{B}}\), but using the result of the Theorem 1 one can deform continuously the functions-coordinates between two specific spinors, instead of just choose, without any criteria, complex numbers as being the coordinates. In other words, this method provides a family of spinors which is related to each other by functions belonging to the same homotopy class.

Moreover, we can state another result, now concerning on homotopy and spinors on a fixed basis:

**Theorem 2.** There exists a homotopic equivalence relation between any two spinors \(\psi\) and \(\varphi\) that can be written in terms of RIM-spinors.

**Proof.** Looking at Equation (85), what happens is that for every pair \((\phi_1, \phi_2) \in \Omega\) we fix a complex number \(x = f_1\), and then another complex number \(f_2 = f_2(f_1) = f_2(x)\) is determined to form the pair \((f_1, f_2)\) which represents a spinor in a certain basis of the spinor-plane. Therefore, with the s-space \(\Omega\) acting as a support-space to construct the set of all allowed complex numbers \(x\), we can understand each spinor \(\psi\) itself as an application

\[
\psi : \mathbb{C} \longrightarrow \mathbb{C}^2
\]

\[
x \mapsto (x, y(x)).
\]

---

\(^4\) This is guaranteed by Lemma 2.

\(^5\) i.e., \(\Pi^D \ni \psi = (x, x)_S\) in the spinor-plane.

\(^6\) With these numbers depending on the values on the s-space.
Then, we can construct a homotopy $G_A : \mathbb{C} \times [0,1] \rightarrow \mathbb{C}^2$ between two spinors $\psi = (x,y_\psi)_A$ and $\varphi = (x,y_\varphi)_A$ in a fixed basis $A$, given by

$$G_A(x,t) = (x,(1-t)y_\psi + ty_\varphi).$$  \hfill(88)

Clearly, in a fixed basis $A$, we have $G_A(x,0) = (x,y_\psi) = \psi$ and $G_A(x,1) = (x,y_\varphi) = \varphi$. Thereupon, $G_A$ makes explicit an equivalence relation between the spinors $\psi$ and $\varphi$ themselves.

Again, we could simply construct “by hand” spinors in the spinor-plane defining points with the help of s-space $\Omega$, but the comprehensive result of Theorem 2 allows us to obtain representations $(x,y(x))$ of spinors, in a given basis, that are intermediary deformations of two known (homotopic related) spinors, i.e., one can think of equivalence homotopy classes of spinors.

We will discuss these two Theorems more deeply in the last Section.

VI. ON THE DIRAC DUAL, BILINEAR COVARIANTS AND LOUNESTO CLASSIFICATION

A. RIM-spinors and bilinear covariants

It is well known [2, 3] that RIM-spinors $\Psi^H$ are necessarily regular spinors, otherwise it would be possible to have $A = 0 = B$ (with $A = \bar{\Psi}^H \Psi^H$ and $B = i\bar{\Psi}^H \gamma^5 \Psi^H$) and then the Heisenberg non-linear equation would reduce to the ordinary linear Dirac equation. Because of that, it seems that the possibility to have only one of the bilinears $A = 0$ or $B = 0$ (i.e., type-2 and type-3 RIM-spinors) is perfectly feasible, since it remains intact the non-linear aspect of the Heisenberg equation. However,

**Lemma 3.** The RIM-spinors $\Psi^H$ are necessarily type-1 in Lounesto classification (i.e., $A,B \neq 0$).

**Proof.** Firstly, notice that $R = (b - \bar{b})^{-1} \ln (A + \bar{b} B)$, which means, in particular, that $(b - \bar{b}) \neq 0$. Now, in reference [2], it is claimed that $J_\mu$ and $K_\mu$ constitute a basis for vectors constructed by the derivative $\partial_\mu$ operating on functionals of $\Psi^H$, and one has the following equations (which are valid for every $\mu$):

$$\partial_\mu A = (a + \bar{a})A J_\mu + i(b - \bar{b})B K_\mu,$$
$$\partial_\mu B = (a + \bar{a})B J_\mu + i(b - \bar{b})A K_\mu. \hfill (89)$$

Suppose that $A = 0$ and $B \neq 0$. Then, Equation (89) gives $0 = i(b - \bar{b})B K_\mu$, which is a contradiction, since $\Psi^H$ is a regular spinor and we cannot have $K_\mu = 0$, $\forall \mu$. Thus, $\Psi^H$ cannot be type-3. Moreover, if we suppose that $A \neq 0$ and $B = 0$, then Equation (90) provides $0 = i(b - \bar{b})A K_\mu$, an analogue contradiction, and we conclude that $\Psi^H$ cannot be type-2, by the same reason as before. Therefore, we conclude that $A,B \neq 0$ and $\Psi^H$ is a regular type-1 spinor.  \hfill $\blacksquare$

We can extract more informations about $A$ and $B$ from the explicit form of the scalar $R$. In fact, we can note that $(A - iB) \neq 0$ and $J \equiv \sqrt{J^2} \neq 0$. Then, remembering that $J^2 = (A - iB)(A + iB)$, we also conclude that $(A + iB) \neq 0$. Now, let us represent a RIM-spinor as $\Psi^H = (\Psi_{11} \quad \Psi_{12} \quad \Psi_{21} \quad \Psi_{22})^T$. Let us define

$$A_1 \equiv \Psi^*_{21} \Psi_{11} + \Psi^*_{22} \Psi_{12}, \hfill (91)$$
$$A_2 \equiv \Psi^*_{11} \Psi_{21} + \Psi^*_{12} \Psi_{22}, \hfill (92)$$

with $r^*$ denoting the complex conjugate of $r \in \mathbb{C}$. Then it is straightforward to see that

$$A = A_1 + A_2, \hfill (93)$$
$$B = i(-A_1 + A_2). \hfill (94)$$

With this in hands, since $A,B \neq 0$, we conclude that $A_1 \neq \pm A_2$. Besides, $A + iB = 2A_1 \neq 0 \Rightarrow A_1 \neq 0$, and $A - iB = 2A_2 \neq 0 \Rightarrow A_2 \neq 0$.

These conditions will turn into strong constraints on RIM-decomposable bilinear covariants.
B. RIM-decomposable spinors and bilinear covariants

Let $\psi = \psi_L + \psi_R$ be a spinor that can be decomposed in terms of RIM-spinors, so there exists a matrix $R \in \mathbb{R}$ such that $\psi = R\Psi^H$. In this case, we can write $R = \text{diag}(r_1, r_1, r_2, r_2)$ with decomposition $\psi = r_1\Psi_L^H + r_2\Psi_R^H$. Suppose that its dual is constructed in the Dirac fashion $\bar{\psi} = \psi^\gamma\gamma^0$. Let us represent the bilinear covariants associated to $\psi$ as $A_\psi, B_\psi, J_\psi, K_\psi, S_\psi$.

We want to categorize all RIM-decomposable spinors in the Lounesto classification. In order to do that, initially we need to know the conditions for $J_\psi \neq 0$, because it is an imposition in the aforementioned classification. Since $\psi = R\Psi^H$, we can write

$$J_\psi^\mu = (\Psi^H)^{\dagger} R^{\dagger} \gamma^\mu R \Psi^H. \quad (95)$$

Representing $\Psi^H = (\Psi_1 \Psi_2)^T$ with $\Psi_j \equiv (\Psi_{j1} \Psi_{j2})^T$ for $j = \{1, 2\}$, we obtain

$$J_\psi^0 = |\Psi_1|^2|r_1|^2 + |\Psi_2|^2|r_2|^2, \quad (96)$$
$$J_\psi^1 = -|r_1|^2(\Psi_{11}^*\Psi_{11} + \Psi_{12}^*\Psi_{12}) + |r_2|^2(\Psi_{21}^*\Psi_{21} + \Psi_{22}^*\Psi_{22}), \quad (97)$$
$$J_\psi^2 = i\left[-|r_1|^2(\Psi_{12}^*\Psi_{11} - \Psi_{11}^*\Psi_{12}) + |r_2|^2(\Psi_{22}^*\Psi_{21} - \Psi_{21}^*\Psi_{22})\right], \quad (98)$$
$$J_\psi^3 = -|r_1|^2(|\Psi_{11}|^2 - |\Psi_{12}|^2) + |r_2|^2(|\Psi_{21}|^2 - |\Psi_{22}|^2). \quad (99)$$

One has to look for the conditions that lead to $J_\psi^\mu = 0, \forall \mu \in \{0, 1, 2, 3\}$, simultaneously. These conditions will form the exactly conditions that we have to avoid. We have to verify the components $J_\psi^\mu$ one by one. Thus, as a start, in order to reach $J_\psi^0 = 0$, one finds three options:

(i) $r_1 \neq 0, r_2 = 0$ and $|\Psi_1|^2 = 0$ (which, by symmetry, is equivalent to $r_2 \neq 0, r_1 = 0$ and $|\Psi_2|^2 = 0$).

(ii) $|\Psi_1|^2 = 0 = |\Psi_2|^2$.

(iii) $r_1 = r_2 = 0$.

Obviously, $r_1 = 0 = r_2$ is not an allowed option, as it leads to $\psi = 0$ with all bilinear covariants vanishing, which is not interesting. Note that the option (ii) leads to $\Psi^H = 0 = \bar{\psi}$, then we descart it. Now we have to analyse the case of option (i). In fact, one can easily verifies that condition (i) simultaneously vanishes Equations (96-99), i.e.,

$$J_\psi = 0 \Leftrightarrow (i). \quad (100)$$

Then, we conclude that we have to avoid condition (i).

The scalar $A_\psi = \bar{\psi}\psi$ and the pseudo-scalar $B_\psi = i\bar{\psi}\gamma^5\psi$ can both be written in terms of the four components of $\Psi^H$, as

$$A_\psi = (r_1r_2^*)(\Psi_{21}^\dagger\Psi_{11} + \Psi_{22}^\dagger\Psi_{12}) + (r_1^*r_2)(\Psi_{11}^\dagger\Psi_{21} + \Psi_{12}^\dagger\Psi_{22}), \quad (101)$$
$$B_\psi = i [-r_1^*r_2^*(\Psi_{21}^\dagger\Psi_{11} + \Psi_{22}^\dagger\Psi_{12}) + (r_1^*r_2)(\Psi_{11}^\dagger\Psi_{21} + \Psi_{12}^\dagger\Psi_{22})]. \quad (102)$$

For the particular case of $r_1, r_2 \in \mathbb{R}$ (in other words, if $R$ is real), we have the interesting fact $A_\psi = (r_1r_2^*)A$ and $B_\psi = (r_1r_2^*)B$, i.e., $A_\psi \propto A$ and $B_\psi \propto B$, and we have that $\psi = r_1\Psi_L^H + r_2\Psi_R^H$ is always a type-1 spinor when $r_1, r_2 \in \mathbb{R} - \{0\}$.

In order to have $\psi$ a RIM-decomposable spinor, we have two options$^7$: $r_1, r_2 \neq 0$ or $r_1 \neq 0, r_2 = 0$ or $r_2 \neq 0, r_1 = 0$. Now, note that the second option cannot happen with $|\Psi_1|^2 = 0 = (|\Psi_2|^2)^2$ occuring, since it would lead to condition (i). Then, for the sake of clarity, we will separate our study in two cases. On the case $r_1 \neq 0, r_2 = 0$, we will show that

---

$^7$ Remember that it is equivalent to $r_2 \neq 0$ and $r_1 = 0$. 

Lemma 4. For a RIM-decomposable spinor $\psi$ such that $\bar{\psi} = \psi^\dagger \gamma^0$, we have

$$J_\psi \neq 0 \Rightarrow (K_\psi \neq 0 \text{ and } S_\psi = 0),$$

(103)
everytime the conditions $r_1 \neq 0$ and $r_2 = 0$ (or, equivalently, $r_2 \neq 0$ and $r_1 = 0$) are satisfied.

Proof. We will look for the conditions to make $K_\psi = 0$ and $S_\psi = 0$ in this case. Firstly, let us analyse $K_\psi$. Analogously to what was made to reach Equations (96-99), we obtain

$$K^0_\psi = |\Psi_1|^2 r_1 - |\Psi_2|^2 r_2|^2.$$  

(104)

Now, suppose $r_2 = 0$ (then, $r_1 \neq 0$). Thus, in order to have $K^0_\psi = |\Psi_1|^2 r_1|^2 = 0$, one must have $|\Psi_1|^2 = 0$. But it would lead to the condition (i). Therefore, by relation (100), in this case we cannot have $K_\psi = 0$.

Now, let us analyse $S_\psi$. In the same fashion, we can write

$$S^{01} = -i [(r_2^* r_1)(\Psi_{22}^\dagger \Psi_{11} + \Psi_{12}^\dagger \Psi_{21}) - (r_1^* r_2)(\Psi_{12}^\dagger \Psi_{21} + \Psi_{11}^\dagger \Psi_{22})],$$

(105)

$$S^{02} = (r_2^* r_1)(\Psi_{22}^\dagger \Psi_{11} - \Psi_{21}^\dagger \Psi_{12}) - (r_1^* r_2)(\Psi_{12}^\dagger \Psi_{21} - \Psi_{11}^\dagger \Psi_{22}),$$

(106)

$$S^{03} = -i [(r_2^* r_1)(\Psi_{22}^\dagger \Psi_{11} - \Psi_{21}^\dagger \Psi_{12}) - (r_1^* r_2)(-\Psi_{11}^\dagger \Psi_{21} + \Psi_{12}^\dagger \Psi_{22})],$$

(107)

$$S^{12} = (r_2^* r_1)(\Psi_{21}^\dagger \Psi_{11} - \Psi_{22}^\dagger \Psi_{12}) + (r_1^* r_2)(\Psi_{12}^\dagger \Psi_{21} - \Psi_{11}^\dagger \Psi_{22}),$$

(108)

$$S^{13} = -i [(r_2^* r_1)(\Psi_{22}^\dagger \Psi_{11} - \Psi_{21}^\dagger \Psi_{12}) + (r_1^* r_2)(\Psi_{12}^\dagger \Psi_{21} - \Psi_{11}^\dagger \Psi_{22})],$$

(109)

$$S^{23} = (r_2^* r_1)(\Psi_{22}^\dagger \Psi_{11} + \Psi_{12}^\dagger \Psi_{21}) + (r_1^* r_2)(\Psi_{12}^\dagger \Psi_{21} + \Psi_{11}^\dagger \Psi_{22}).$$

(110)

Again, without loss of generalization, suppose $r_2 = 0$ (and, so, $r_1 \neq 0$). In this case, it is obvious that we always have $S^{01\nu} = 0$, i.e., $S^\psi = 0$, and it does not depend on any condition for the components $\Psi_{ij}$ whatsoever.

Summarizing, what we have found is that $J_\psi \neq 0 \Rightarrow K_\psi \neq 0$ and $J_\psi \neq 0 \Rightarrow S_\psi = 0$, when $r_1 \neq 0$ and $r_2 = 0$, which can be written as $J_\psi \neq 0 \Rightarrow (K_\psi \neq 0 \text{ and } S_\psi = 0)$. This ends the proof. \qed

Now, note that, if $r_1 \neq 0$ and $r_2 = 0$, then $A_\psi = 0 = B_\psi$: in other words, in this case we are dealing necessarily with singular spinors. But, using Lemma 4, we have that in order to classify these spinors on the spinor-plane by the Louesto classification, if $r_1 \neq 0$ and $r_2 = 0$, then we are dealing with type-6 singular spinors (see Subsection II.B).

What about other situations that could lead to singular RIM-decomposable spinors? Let us see. In fact, the other option left is to have $r_1, r_2 \neq 0$. Note that, looking at the definitions (91) and (92) and Equations (101) and (102), it is straightforward to see that we can write

$$A_\psi = (r_1 r_2^*) A_1 + (r_1^* r_2) A_2,$$

(111)

$$B_\psi = i [-(r_1 r_2^*) A_1 + (r_1^* r_2) A_2].$$

(112)

Then, as a last try, if we choose $r_1, r_2 \neq 0$, we see that we cannot have simultaneously $A_\psi = 0 = B_\psi$, thus $\psi$ is a regular spinor, and $K_\psi, S_\psi \neq 0$. Indeed, if we impose $A_\psi = 0 = B_\psi$ with $r_1, r_2 \neq 0$, then $(r_1 r_2^*) A_1 = -(r_1^* r_2) A_2$ and $(r_1^* r_2^*) A_1 = +(r_1 r_2) A_2$, which has no solution once we know that $A_1, A_2 \neq 0$. Note that this implies that we cannot have type-4 and type-5 singular RIM-decomposable spinors at all, since this exhausts all possibilities for $r_1$ and $r_2$ in the construction of a non-null $\psi$. Moreover, we have shown here that a decomposition leading to a singular spinor needs to satisfy $r_1 \neq 0$ and $r_2 = 0$, and having $r_1 \neq 0$ and $r_2 = 0$ is sufficient in order to have a decomposition leading to a singular spinor. Therefore, we can state the following:

Proposition 1. Suppose $\psi = R \Psi^H$, with $R = \text{diag}(r_1, r_1, r_2, r_2) \in \mathfrak{M}$ such that $\psi = r_1 \Psi_L^H + r_2 \Psi_R^H$, satisfying $J_\psi \neq 0$ and $\bar{\psi} = \psi^\dagger \gamma^0$. Then, the statements below are equivalent:

(i) $\psi$ is a singular spinor.

(ii) $\psi$ is a type-6 spinor.

(iii) $r_1 = 0$ or $r_2 = 0$ (but not both).

(iv) $\psi$ is projected only in $(0, \frac{1}{2})$ or $(\frac{1}{2}, 0)$ representation, i.e., $\psi \propto \Psi_L^H$ or $\psi \propto \Psi_R^H$. 
We have realized, thus, that once we are setting \( r_1, r_2 \neq 0 \) we are dealing with regular spinors. In this case, we have other three options to verify: type-1 \( (A_\psi, B_\psi \neq 0) \), type-2 \( (A_\psi \neq 0 \) and \( B_\psi = 0 \)) and type-3 \( (A_\psi = 0 \) and \( B_\psi \neq 0 \)). Since we know that \( \Psi^H \) itself and the Dirac RIM-decomposable field \( \Psi^D \) are both type-1 \[3\], we only have to check the other two possibilities. First, if we set \( A_\psi = 0 \), then \((r_1 r_2^*) A_1 = -(r_1 r_2^*) A_2 = 0\), and we can write \( B_\psi = 2i(r_1 r_2^*) A_2 = -2i(r_1 r_2^*) A_1 \neq 0\), then it is possible to have type-2 RIM-decomposable spinors. Analogously, if we set \( B_\psi = 0 \), then \((r_1 r_2^*) A_1 = (r_1 r_2^*) A_2 \neq 0\), and it leads to \( A_\psi = 2i(r_1 r_2^*) A_2 = 2i(r_1 r_2^*) A_1 \neq 0\), which means that it is also possible to have type-3 RIM-decomposable spinors. With this in hands, we state that

**Proposition 2.** Suppose \( \psi = R\Psi^H = r_1 \Psi^H_L + r_2 \Psi^H_R \), with \( R = \text{diag}(r_1, r_1, r_2, r_2) \in \mathcal{M} \), satisfying \( J_\psi \neq 0 \) and \( \ddot{\psi} = \psi^\dagger \gamma^0 \). Then, \( \psi \) is a regular spinor if, and only if, \( r_1, r_2 \neq 0 \). Yet, in this case, we have that:

(i) \( \psi \) is a type-1 spinor if, and only if, \( A \neq -iB \left( \frac{r_1 r_2^* + r_1^* r_2}{r_1 r_2^* + r_1^* r_2} \right) \).

(ii) \( \psi \) is a type-2 spinor if, and only if, \( (r_1 r_2^*)^2 \neq (r_1^* r_2)^2 \) and \( A = -iB \left( \frac{r_1 r_2^* + r_1^* r_2}{r_1 r_2^* - r_1^* r_2} \right) \).

(iii) \( \psi \) is a type-3 spinor if, and only if, \( (r_1 r_2^*)^2 \neq (r_1^* r_2)^2 \) and \( A = -iB \left( \frac{r_1 r_2^* - r_1^* r_2}{r_1 r_2^* + r_1^* r_2} \right) \).

**Proof.** Firstly, we have already seen that, in the hypothesis of this Proposition, the condition \( r_1, r_2 \neq 0 \) is necessary and sufficient for \( \psi \) to be a regular spinor: in fact, this particular result can be understood as a corollary of Proposition 1. Now, noticing that \( A_1 = \frac{A_\psi + B_\psi}{2} \) and \( A_2 = \frac{A_\psi - B_\psi}{2} \), one is able to write

\[
A_\psi = \frac{1}{2} \left[ A(r_1 r_2^* + r_1^* r_2) + iB(r_1 r_2^* - r_1^* r_2) \right],
\]

\[
B_\psi = -i \left[ A(r_1 r_2^* - r_1^* r_2) + iB(r_1 r_2^* + r_1^* r_2) \right].
\]

We know that \( A_1, A_2 \neq 0 \). Yet, one cannot reach \((r_1 r_2^* - r_1^* r_2) = 0 = (r_1 r_2^* + r_1^* r_2)\) with \( r_1, r_2 \neq 0 \): in fact, it would lead to \( A_\psi = 0 = B_\psi \), an unattainable case here, as we have seen.

Then, in order to reach \( B_\psi = 0 \), we need to have \( A(r_1 r_2^* - r_1^* r_2) = iB(r_1 r_2^* + r_1^* r_2) = 0 \). Moreover, one cannot have \((r_1 r_2^* + r_1^* r_2) = 0 \) or \((r_1 r_2^* - r_1^* r_2) = 0 \) isolated, because it will never make \( B_\psi = 0 \); in other words, \((r_1 r_2^*)^2 \neq (r_1^* r_2)^2 \).

Thus, the only option left is \( A(r_1 r_2^* - r_1^* r_2) = -iB(r_1 r_2^* + r_1^* r_2) \), which implies that \( A = -iB \left( \frac{r_1 r_2^* + r_1^* r_2}{r_1 r_2^* - r_1^* r_2} \right) \). In this case, it is guaranteed that \( A_\psi \neq 0 \). This proves item (ii).

Now, if one wants to have \( A_\psi = 0 \), for analogue reasons as the case above, we have \((r_1 r_2^*)^2 \neq (r_1^* r_2)^2 \) and \( A(r_1 r_2^* - r_1^* r_2) = -iB(r_1 r_2^* + r_1^* r_2) \). It leads to \( A = -iB \left( \frac{r_1 r_2^* + r_1^* r_2}{r_1 r_2^* - r_1^* r_2} \right) \) (with \( B_\psi \neq 0 \) guaranteed), which proves item (iii).

So far, we have seen that the only way to vanish \( A_\psi \) or \( B_\psi \) without vanish both at the same time is to have \( A(r_1 r_2^* - r_1^* r_2) = -iB(r_1 r_2^* + r_1^* r_2) \). Then, we conclude that we cannot have these conditions valid in order to keep both \( A_\psi, B_\psi \neq 0 \), i.e., having \( A \neq -iB \left( \frac{r_1 r_2^* + r_1^* r_2}{r_1 r_2^* - r_1^* r_2} \right) \) is equivalent to say that \( \psi \) can only be type-1, proving item (i).

Indeed, \( \Psi^H \) has \( r_1 = r_2 = 1 \), and Proposition 2 trivially confirms that \( \Psi^H \) is type-1, with \( A_{\Psi^H} = A \) and \( B_{\Psi^H} = B \) being easily obtained by Equations (113) and (114), as expected. As another example, for the Dirac spinor \( \Psi^D \), we have \( r_1 = \alpha \beta \delta \) and \( r_2 = \alpha \beta \delta^{-1} \) as defined in Section IV, and one can verify that \( \frac{r_1 r_2^* + r_1^* r_2}{r_1 r_2^* - r_1^* r_2} = -A \neq A \), and \( \frac{r_1 r_2^* + r_1^* r_2}{r_1 r_2^* - r_1^* r_2} = i \left( \frac{2 \beta}{\alpha \beta \delta} \right) \Rightarrow -iB \left( \frac{r_1 r_2^* + r_1^* r_2}{r_1 r_2^* - r_1^* r_2} \right) = B \neq A \), which confirms that \( \Psi^D \) is also type-1.

Summarizing the results of this Section, Propositions 1 and 2 provide an easy method to separate all spinors \( \psi \) allowed in the spinor-plane\[8\] in the Louesto classification by just looking at their coefficients \((r_1, r_2)\) in the “RIM-copy” \( \Pi^H \): if both coefficients are non vanishing, then the spinor is regular (with an easy way to verify if it is type-1,

\[8\] With dual defined as \( \ddot{\psi} = \psi^\dagger \gamma^0 \).
type-2 or type-3: simply divide the sum of the coefficients by the difference - and the difference by the sum - and multiply by \(-iB\) in order to verify if it is equal to \(A\), while if one (and only one) of the coefficients is zero then it is a singular type-6 spinor, with no need to the often hard work of construction of all the bilinear covariants. As we cannot have \(r_1 = 0 = r_2\), all feasible cases are contemplated.

VII. FINAL REMARKS

The second main result of the reference [3], concerning on exotic spinor fields, allows us to state that all spinors belonging to the spinor-plane that has a dynamic equation are not exotic spinors, i.e., the underlying topology of the space-time \(M\) of which these spinors may emerge is trivial, in the sense that it has a trivial fundamental group \(\pi_1(M) = 0\). In particular, this spinor-plane accommodates a bijective linear map between special classes (i.e., both being RIM-decomposable and, therefore, non exotic) of MDO and Dirac spinors. This mapping is quite natural, as it uses RIM-spinors as a fundamental element making the mediation between Dirac and MDO fields. Although this mapping has some constraints imposed in the fields themselves (they had to be RIM-decomposable), one does not have to work with the bilinear covariants, which is often a hard situation to deal with when we study MDO spinors, since they do not necessarily fit in the usual Lounesto classification. Therefore, the mapping developed here transcends the problem of Lounesto classification of MDO spinors.

Among the outcomes of this work, we emphasize that Theorem 1 is an exhaustive result: it gives not only the possibility to write down explicitly all possible spinors that can be decomposed by RIM-spinors, by giving the left- and right-handed components of each of them, but it also makes explicit an equivalence relation (via the homotopy \(H\)) between all the functions that represent spinor-plane coordinates. On the other hand, the Theorem 2 is another robust result, providing a way to deform spinors in the spinor-plane, enabling the composition and the eventual classification of equivalence classes of homotopic spinors via the homotopies \(G_A\).

The two Theorems are related, in the sense that both treat of the subject of writing down explicit forms of representing the spinors that can be written in terms of RIM-spinors, by showing a homotopic equivalence relation. The main difference between them, which indeed complements each other, lies on the fact that Theorem 1 provides a method to obtain the left- and right-handed components of the spinors by constructing basis for the spinor-plane, while Theorem 2 supplies a way to obtain points (the spinors themselves) in a given fixed basis. In other words, while in Theorem 1 we are continuously deforming the spinor-plane itself (obtaining new basis for the space), in Theorem 2 we are continuously deforming the points of the plane in a fixed basis (obtaining intermediate spinors between two fixed ones).

The understanding of the very nature of spinors is a field of study under development, which is as significant in Physics as in Mathematics. Theorems 1 and 2 may be a new way to look at the construction of spinors, opening the possibility to the discovery of interesting relations via Algebraic Topology by means of the study of homotopy, which is perhaps the most important idea in this branch of Mathematics.

Propositions 1 and 2 facilitate the categorization of RIM-decomposable spinors \(\psi\), that has \(\bar{\psi} = \psi \gamma^0\), in the Lounesto classification: they provide a complete and easy way to determine how these spinors are classified in the Lounesto classification when their dual is defined in the Dirac fashion. In fact, they connect the coefficients of their decomposition (or, in other words, their coordinates on the spinor-plane given in the basis \(\mathcal{B} = \{\psi_L^H, \psi_R^H\}\) directly with the Lounesto classification, avoiding the construction of all bilinear covariants and the often laborious process of check which of them are null and which ones are not. In particular, Proposition 2 is a generalization of the Lemma 1 in reference [3], which states that every Dirac spinor decomposable in terms of RIM-spinors is type-1 in the Lounesto classification.

It is worthwhile to make clear that the core of all results of this work is in the RIM-decomposition itself, in the sense that the major element that links all Lemmas\(^9\). Propositions and Theorems presented here is the pair of coefficients of the decomposition (or, in other words, the coordinates in the spinor-plane) of a given spinor in terms of RIM-spinors. Following this idea, one can study spinor properties in a very similar way if a given spinor is decomposable in terms of another. Thus, this work provides a working protocol that can be useful in other cases of the theory of spinors field of study.

With regard to direct physical applications, this work provides an algebraic-topological method of construction of any possible spinor field allowed in the Spinor Theory of Gravity (STG), which is a theory of gravitation built via a class of solutions of the linearized Einstein equations of General Relativity constructed from RIM-spinors [16, 18], i.e., a gravitation theory with RIM-spinors playing a fundamental role. Moreover, since bilinear covariants are associated

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\(^9\) The only one which is not related directly to the RIM-decomposition is Lemma 3, but it is about the RIM-spinor itself.
with physical observables, we developed a way to easily verify the possible couplings of a particle associated to a given spinor in this theory, by means of their coefficients in the RIM-decomposition.

Further results concerning questions about more properties related to the homotopies in the spinor-plane are under investigation. Moreover, the behaviour of MDO spinors and their bilinear covariants in this space is also a topic under study.

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Appendix A: Mass-dimension-one fields and the Fierz-Pauli-Kofink Identities

As it can be seen in [14], it is possible to build the basis vectors for the mass-dimension-one spinor’s case using the usual Clifford algebra. For any element $\Gamma$ belonging to such algebra, the FPK relation reads

$$\overline{\lambda_h} \Gamma \gamma_\mu \lambda_h = (\overline{\lambda_h} \Gamma \lambda_h) \lambda_h - (\overline{\lambda_h} \Gamma \gamma_5 \lambda_h) \gamma_5 \lambda_h,$$

where $\Gamma \in \{1, \gamma_5, \gamma_\mu, \Xi \gamma_\mu \Xi\}$. From the above relation we obtain the following:

$$J^2 = A^2 + B^2,$$

and we also have

$$\overline{\lambda_h} \Xi \gamma_5 \gamma_\mu \Xi \lambda_h \gamma^\mu \lambda_h = (\overline{\lambda_h} \Xi \gamma_5 \Xi \lambda_h) \lambda_h - (\overline{\lambda_h} \Xi^2 \lambda_h) \gamma_5 \lambda_h,$$

$$= (\overline{\lambda_h} \gamma_5 \lambda_h) \lambda_h - (\overline{\lambda_h} \lambda_h) \gamma_5 \lambda_h.$$

(A3)

Note that

$$[\Xi, \gamma_5] = 0, \quad \{\gamma_\mu, \gamma_5\} = 0 \quad \text{and} \quad \Xi^2 = 1,$$

with such relations at hands, one is able to write

$$(\overline{\lambda_h} \Xi \gamma_5 \gamma_\mu \Xi \lambda_h) \gamma^\mu \gamma^5 \lambda_h = - (\overline{\lambda_h} \Xi \gamma_5 \gamma_\mu \Xi \gamma_5 \lambda_h) \gamma^\mu \lambda_h,$$

$$= (\overline{\lambda_h} \Xi \gamma_\mu \Xi \lambda_h) \gamma^\mu \lambda_h.$$  

(A4)

Finally using relations (A3) and (A4), we obtain

$$J_\mu \gamma^\mu \lambda_L = (A - iB) \lambda_R,$$

$$J_\mu \gamma^\mu \lambda_R = (A + iB) \lambda_L,$$

$$K_\mu \gamma^\mu \lambda_L = -(A - iB) \lambda_R,$$

$$K_\mu \gamma^\mu \lambda_R = (A + iB) \lambda_L.$$  

(A5)  (A6)  (A7)  (A8)

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