Properties Of Heterotic Vacua
From Superpotentials

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Abstract

We study the superpotential for the heterotic string compactified on non-Kähler complex manifolds. We show that many of the geometrical properties of these manifolds can be understood from the proposed superpotential. In particular we give an estimate of the radial modulus of these manifolds. We also show, how the torsional constraints can be obtained from this superpotential.

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1. Introduction and Summary

The heterotic string theory compactified on a Calabi-Yau three-fold \[1\] preserving an \( \mathcal{N} = 1 \) supersymmetry in four dimensions, has had great success in explaining many interesting dynamics of nearly standard like models in string theory (for a very recent discussion see e.g. \[2\]). However, in the absence of rigid vacua, there are many uncontrolled moduli, that are not fixed, at least at the tree level. Very generally, these moduli originate from the topological data of the internal manifold. The Kähler structure moduli and the complex structure moduli, classified by the hodge numbers \( h^{1,1} \) and \( h^{2,1} \) of the internal manifold respectively, give rise to a number of massless scalars in four dimensions in addition to the scalars, that we get from other \( p \)-form fields. As a result, the predictive power of string theory and \( \mathcal{M} \)-theory is lost, as the expectation value of these moduli fields determines the coupling constants of the standard model. This pathology can be overcome, if we can lift these moduli, by giving masses to these scalars. Since the radius of the internal manifold is also a modulus, the size of this manifold cannot be determined in a conventional Calabi-Yau compactification. Considering now the fact that quantum effects favour the limit of infinite radius, the radius of the manifold will eventually become very big. This is the so called Dine-Seiberg runaway problem \[3\], which has been one of the most important problems of string theory for a long time. Luckily, in recent times we are making a remarkable progress in this direction, by considering string theory and \( \mathcal{M} \)-theory compactifications with non-vanishing expectation values for \( p \)-form fluxes. These compactifications are generalizations of the conventional Calabi-Yau compactifications.

In the context of the recent developments in non-perturbative string theory, compactifications with non-vanishing fluxes were first found in \[4\], where \( \mathcal{M} \)-theory compactifications on complex four-manifolds were described in detail. As was shown therein, compactifications with fluxes, which preserve an \( \mathcal{N} = 2 \) supersymmetry in three dimensions can be
obtained by considering a more general type of compactification on a warped background. The idea of having a warped background has appeared in many earlier papers, even before the recent developments in non-perturbative string theory were made, particularly in \[5\],\[6\],\[7\],\[8\] and \[9\]. However, in all these compactifications precise models describing these backgrounds were never computed very explicitly.

The warped backgrounds found in \[4\] were later extended in two different directions. The first interesting direction was pointed out by \[10\]. These authors showed, that by switching on four-form fluxes in \(\mathcal{M}\)-theory it is possible to generate a superpotential, which freezes many of the moduli appearing in these compactifications. In fact, all the complex structure moduli and some Kähler structure moduli are fixed in the process. This is an important progress, because as a result string theory vacua with reduced moduli were generated even at string tree level. The fact, that fluxes freeze the moduli fields at tree level is particularly attractive, as it becomes easier to perform concrete calculations.

The second direction, in which \[4\] was generalized, was taken in \[11\]. Choosing a particular four-manifold, which is also a particular \(\mathcal{F}\)-theory vacuum at constant coupling, it was shown in \[11\], that after a series of U-duality transformations, one can obtain a six-dimensional compactification of the \(SO(32)\) heterotic string theory. This is particularly interesting from the phenomenological point of view. This compactification is again a warped compactification, as one would expect, but the six manifold is now inherently non-Kähler. In the earlier compactifications of \[4\], the warped manifolds were non-Kähler but conformally Calabi-Yau. Similar conformally Calabi-Yau compactifications can also be studied in the context of Type II theories (see e.g. the third reference in \[8\] and references therein). By choosing different four-folds and then performing a series of U-duality transformations, many new heterotic and Type II compactifications on non-Kähler manifolds can be generated \[12\],\[13\],\[14\],\[15\],\[16\] and \[17\]. Such compactifications are fascinating both, from the physics point of view, as they have many properties in common with the standard model and from the mathematical point of view, as many mathematical aspects of these manifolds are still unexplored territory. Some of these new mathematical aspects have been pointed out recently in \[18\],\[19\],\[20\] and \[17\]. The concrete manifolds described in many of these papers are non-trivial \(T^2\) fibrations over a four-dimensional Calabi-Yau base \[14\],\[13\],\[20\] and \[17\]. Furthermore, these manifolds have a vanishing Euler number and a vanishing first Chern class \[20\],\[17\].

One immediate next step would be to combine these two directions, i.e. one would like to compute the form of the superpotential for compactifications of the heterotic string
on non-Kähler complex manifolds in order to understand, how the moduli fields get frozen in this type of compactifications, as this is a rather important question for particle phenomenology. The non-Kähler manifolds, that we consider support three-form fluxes, that are real (we call them $H$). These fluxes will generate a superpotential in the heterotic theory and as a consequence many of the moduli fields will be frozen. The form of this superpotential has been computed in $[16]$ and in $[17]$. It is the goal of this paper to study the properties of this superpotential and its effect on the moduli fields appearing in these compactifications. As one would expect from the above discussions, all the complex structure moduli and some of the Kähler structure moduli are fixed in the process. An important Kähler structure moduli, that is frozen at tree level is the radial modulus. This has been shown already in $[17]$ and we shall see this here in much more detail. An immediate consequence of this is now apparent: there is no Dine-Seiberg runaway behavior for the radius and therefore the notion of compactification makes perfect sense, as the internal manifold will have a definitive size. However, this is not enough. We have to see whether supergravity analysis is also valid in four-dimensions, so that explicit calculations can be done. This would imply, that the internal six-manifold should have a large overall volume. We will show that even though the $T^2$ fiber has a volume of order $\alpha'$, the base can be made large enough, so that the total volume is large. But there is a subtlety related to the topology change which in fact hinders any nice supergravity description for the kind of background that we study. We will discuss this important issue later.

This paper is organized as follows: In section 2 we give a brief review of the earlier works on non-Kähler spaces. We show, how these manifolds can be realized directly in the heterotic theory without using T-duality arguments to the Type IIB theory $[17]$. In section 3 we discuss the stabilization of the radial modulus. This aspect has been partially discussed in $[17]$. Here we will give a fuller picture of the potential, that fixes this modulus and give a numerical estimate for the radius. To obtain this estimate, we have made some simplifying assumptions. In section 4 we show, that the estimate done in section 3 is not too far from what we obtain in a more realistic scenario. In section 5 we calculate an additional contribution to the heterotic superpotential and show, how the torsional constraints can be derived from the complete superpotential. All the contributions to the superpotential, that we have computed to this point are perturbative. Section 6 is dedicated to discussions and conclusions. In particular, we discuss the origin of non-perturbative contributions to the superpotential and applications of this type of compactifications in other possible scenarios, such as cosmology.
2. Brief Review of Torsional Backgrounds

Here is a lightning review of earlier works on non-Kähler spaces [11], [13], [20], [17]. The readers are however advised to go through these references, as we shall constantly be referring to them. The non-Kähler manifolds, that we study here are all six-dimensional spaces of the form

\[ ds^2 = \Delta_1^2 \, ds_{\text{CY}}^2 + \Delta_2^2 \, |dz^3 + \alpha dz^1 + \beta dz^2|^2, \]

(2.1)

where \( \Delta_i = \Delta_i(|z_1|, |z_2|) \) are the warp factors and \( \alpha, \beta \) depend on \( z^i \) and \( \bar{z}^j \), the coordinates on the internal space. The four-dimensional Calabi-Yau base is described by \( z^1 \) and \( z^2 \). For the examples studied earlier in [11], [13], [17], these functions were

\[ \alpha = 2i \, z^2, \quad \beta = -(4 + 2i) \, z^1, \quad \Delta_1^2 \equiv \Delta_2^2 = c_o + \psi(|z^1|, |z^2|), \quad \Delta_2 = 1, \]

(2.2)

where \( c_o \) is a constant, and \( \psi \to 0 \) when size of the manifold becomes infinite. There is also a background three-form, which is real and anomaly free and serves as the torsion for the underlying space. The dilaton is not constant and is related to the warp factor. The background supports a modified connection \( \tilde{\omega} \) instead of the usual torsion-free connection \( \omega_o \). In fact, the torsion \( \mathcal{T} \) is proportional to

\[ \mathcal{T} = \omega_o - \tilde{\omega}, \]

(2.3)

which in particular is also the measure of the real three-form in this background because we demand \( \mathcal{T} \) to be covariant and the “contorsion” tensor to be identified with it [3], [17]. This identification of contorsion tensor to the heterotic three-form actually has roots in the sigma model description of the heterotic string propagating on these manifolds. Consider the heterotic string with a gauge bundle \( A_\mu \). If we define the contorsion tensor \( \kappa \) to satisfy

\[ \kappa_{\mu} = \omega_{o\mu}^{ab} \sigma^{ab} - A_\mu^{AB} T^{AB} + \mathcal{O}(\alpha'), \]

(2.4)

where \( \sigma, T \) are the generators of holonomy and gauge groups respectively, then the theory becomes anomaly free with an almost vanishing two-loop sigma model beta function.

There is however a subtlety here. The above identification, though appears so natural, creates a problem, which is the following. In the relation (2.4), if we have an exact equality, with vanishing terms of order \( \mathcal{O}(\alpha') \), then the two-loop beta function would cancel exactly. In that case we will have no warped solution and the manifold will tend to go back to the usual Calabi-Yau compactification. On the other hand, having an \( \mathcal{O}(\alpha') \) term would
mean, that we have a non-zero beta function and therefore, these compactifications are not solutions of the string equations of motions. Either way is disastrous, unless we find a way out, that could save the day.

The resolution to this problem comes from the fact, that these manifolds are in fact rigid and therefore, they do not have an arbitrary size. Thus, even though we allow (2.4) with non-vanishing terms at order $\alpha'$, the two-loop beta function can become zero only, when the manifold attains a definite size. For any arbitrary size the beta function is non-zero and therefore our manifolds are not a solution of the equations of motion (a similar argument goes through for all the other moduli). Happily, as shown in [17] all the complex structure moduli, some Kähler structure moduli, in particular, the radial modulus do get stabilized in these compactifications at tree level. The remaining Kähler moduli would also get stabilized, if we incorporate quantum effects (see [21]).

\[
\begin{align*}
\text{TYPE IIB} & \quad 2T + S \\
\Delta \left( \frac{K^3 \times T^2}{Z_2} \right) & \quad \Delta^{-1}(\text{MINK}) \\
H_{\text{NS}} + H_{\text{RR}} & \\
\text{HETEROtic} & \quad H + \text{TWIST}
\end{align*}
\]

**Fig. 1**: The mapping of the Type IIB model to the heterotic model via two T-dualities and one S-duality.

Before we go into discussing more details on these compactifications, we should point out the fact, that given a heterotic background, switching on a three-form will not, in general, convert this to a non-Kähler manifold. In fact, this is clear from the figure above.

We start with a warped background in the Type IIB theory on $K^3 \times T^2 / Z_2$ with fluxes, and through a set of U-duality transformations we get the non-Kähler manifold discussed in [11], [13], [17]. Observe, that in this process we actually have a topology change, because on the heterotic side we go from a torus $T^2$ to a fibered torus, having no one-cycle. Therefore,
we need more than a three-form background to fully realize the non-Kähler spaces. We will dwell on this issue in the next section.

There is also an $\mathcal{F}$-theory picture, from which the construction of the heterotic manifold is rather straightforward. This is $\mathcal{F}$-theory at a constant coupling, where the elliptic curves degenerate to

$$y^2 = x^3 + a\phi^2 x + \phi^3,$$

where $\phi = \phi(z)$ is an arbitrary polynomial of degree 4 and $z$ is the coordinate of the $P^1$ base. In fact, this is $\mathcal{F}$ theory on $K3 \times K3$, where one of the $K3$ has degenerated to the $Z_2$ orbifold point. Under suitable rescaling of the above curve (2.5), one can easily show, that at a given orbifold point we have [22]

$$Y^2 = X^3 + \alpha X z^2 + z^3,$$

where $X, Y$ can be derived by knowing $x, y, z$. From Tate’s algorithm we can see the appearance of a $D_4$ singularity at that point.

3. Superpotential and Radial Modulus

In a recent paper [17] we showed, how all the complex structure moduli, some Kähler structure moduli and in particular the radial modulus are determined at tree level by switching on three-form fluxes in compactifications of the heterotic string on non-Kähler complex six-dimensional manifolds. The basic idea is, that a superpotential is induced by the fluxes. This gives masses to most of the moduli. The superpotential takes the form

$$W_{het} = \int G \wedge \Omega,$$

where $G$ is a three-form and $\Omega$ is the holomorphic (3,0)-form of the internal six-dimensional manifold. In the following we would like to determine what $G$ is. In the usual case where there is no torsion, $G$ is the real three-form of the heterotic theory [23]. We still have the real three-form in the presence of torsion (the torsion is actually identified with this real form), but as discussed in [17], there is another choice for the three-form $G$ appearing in the above superpotential, that is needed for non-Kähler internal manifolds. This three-form $G$ is again anomaly free and gauge invariant and satisfies the equation

$$G = dB + \alpha' \left[ \Omega_3 \left( \omega_o - \frac{1}{2} \tilde{G} \right) - \Omega_3 (A) \right],$$

where $\omega_o$ is a polynomial and $\tilde{G}$ is the real form of the heterotic theory.
where $\Omega_3(A) = \text{Tr} (A \wedge F - \frac{1}{2} A \wedge A \wedge A)$ is the Chern-Simons term for the gauge field $A$ and $\Omega_3(\omega_o)$ is the Chern-Simons term for the torsion free spin-connection $\omega_o$ (the trace will now be in the fundamental representation whereas the trace above was for the adjoint representation), while $B$ is the usual two-form potential of the heterotic theory. We have also defined $\tilde{G}$ as the one-form created out of three-form $G$ using vielbeins $e_i^a$ as $\tilde{G}_{ab} = G_{ijk} e_i^a e_j^b$. We see, that $G$ appears on both sides of the above expression and therefore we need to solve iteratively this equation in order to determine $G$. For the case considered here we can do this order by order in $\alpha'$. The equation to be solved is

$$G + \frac{\alpha'}{2} \text{tr} \left( \omega_o \wedge R_{\tilde{G}} + \tilde{G} \wedge R_{\omega_o} - \frac{1}{2} \tilde{G} \wedge R_{\tilde{G}} \right) = dB + \alpha' (\Omega_3(\omega_o) - \Omega_3(A)), \quad (3.3)$$

where we have introduced the curvature polynomials $R_{\tilde{G}}$ and $R_{\omega_o}$ as

$$R_{\tilde{G}} = d\tilde{G} - \frac{1}{3} \tilde{G} \wedge \tilde{G}, \quad \text{and} \quad R_{\omega_o} = d\omega_o + \frac{2}{3} \omega_o \wedge \omega_o. \quad (3.4)$$

To the lowest order in $\alpha'$ we can ignore the contributions from $d\tilde{G}$ (3.3), because they are of $O(\alpha'^2)$. We will also ignore the contributions from $d\omega_o$, because they are higher derivatives in the vielbeins. The above formula reduces to the usual heterotic three-form equation in the absence of torsion and the superpotential becomes the superpotential computed in [23], as can be easily seen. Now if we denote the size of the internal manifold as $t$ (we shall take $t$ to be a function of all the spatial coordinates), we obtain from (3.3) a cubic equation, which takes the generic form

$$h^3 + ph + q = 0, \quad \text{with} \quad G_{ijk} = h C_{ijk}, \quad \text{and} \quad g_{ij} = t g_o^{ij}, \quad (3.5)$$

for every component of the three-form $G$. Here $C$ is a constant antisymmetric tensor in six-dimensions, whose contractions are done with respect to the metric $g_o^{ij}$. And $g_o^{ij}$ is chosen to be constant locally, so that we can ignore the twist of the fiber. Also, we can relax the condition on $C$ a little bit. What we actually require is, that $C$ should be at least anti-symmetric in two of its indices. However, for all the calculations below we will only use the complete antisymmetric part of $C$ in analogy to the torsion-free spin-connection

\[1\] In other words, $\alpha$ and $\beta$ in (2.1) are constants locally.

\[2\] An alternative way to think about this is to regard $\tilde{G}_{ij}^{ab}$ in the same way as $\omega_{ai}^{ab}$. Thus, we define $\tilde{G}_{ij}^{ab} = t^{-1} h C_{ij}^{ab}$ and, therefore $G_{[ijk]} = \tilde{G}_{[i}^{ab} e_{|a|j} e_{|b|k]} = h C_{[ijk]}$. Only the anti-symmetric part of $C$ will be relevant.
\( \omega_o \). Here we will consider again only the antisymmetric part, unless mentioned otherwise. Another point to note is the choice of metric in (3.5). Our assumptions for \( C \) and the metric can therefore be summarized as

\[
C_{[ijk]} = \epsilon_{ijk}, \quad \Delta_1 = \Delta_2 = 1, \quad (3.6)
\]

where \( \Delta_i \) are the warp factors in (2.1). This will simplify the calculations done below.

In the next section we will consider the case, where the warp-factor is introduced back as \( \Delta_1 = \Delta_2 = \Delta \). We will, however, not go in much details for the case \( \Delta_1 = \Delta, \Delta_2 = 1 \) which is a little subtle and needs a more detailed analysis than what will be presented here. The calculation in full generality is not too different from the simple example, that we are considering herein. We will be using the definitions of \( p, q \) and \( f \) as

\[
p = \frac{t^3}{\alpha'}, \quad q = -\frac{ft^3}{\alpha'}, \quad \text{and} \quad f = (dB + \alpha' \Omega_3(\omega_o) - \alpha' \Omega_3(A))_{ijk} \epsilon^{ijk}. \quad (3.7)
\]

The first equation in (3.5) has three roots. One of them is real and the other two are complex conjugates of each other. The real root appears in the supersymmetry transformation of the low energy effective action of the heterotic string and satisfies the torsional equations of \([6], [5], [13]\). But for the construction of the superpotential the real root is not enough, as we will explain in the next paragraph.

The real solution fails to cover many interesting aspects of the non-Kähler geometry. So for example, one particular important aspect of non-Kähler manifolds, that was studied in \([7]\) is topology change. We start with a complex three manifold of the form \( K3 \times T^2 \) in the heterotic theory and then switch on a three-form flux. The final picture is, that we get a non-Kähler complex three-fold, whose first Betti number, \( b_1 \), is zero. Therefore, a transition from \( b_1 = 2 \rightarrow b_1 = 0 \) (see fig 1) has been performed. In the usual perturbative analysis it is difficult to see, how such a transition could take place by switching on a torsion three-form\([8]\). One needs an additional non-trivial twist in the geometry to achieve such a topology change. Therefore, we need both: a three-form background (i.e torsion) and a twist. The twist is proportional to the antisymmetrized spin-connection, because that is the only gravitational degree of freedom generating such a change. We can combine

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\(^3\) We will absorb the traces of the holonomy matrices in the definition of \( t \) for simplicity (as in \([7]\)). However, we will soon consider the case, where we keep all these dependences explicitly.

\(^4\) We could also have \( b_1 = 2 \rightarrow b_1 = 1 \), but that would be for a slightly different choice of non-Kähler manifold. Details on this have been discussed in \([20], [7]\).
these two to form a complex three-form. This is precisely what we get by solving the cubic equation above!

However, an immediate question would be: why a complex three-form instead of a real one? Of course, we cannot have any arbitrary combination of three-forms, because this would be inconsistent with the dynamics of the heterotic theory, let alone the fact, that it will be anomalous. There is however, a deeper reason why we have a complex three-form. This is related to the fact, that the complex three-form is compatible with the T-dual Type IIB framework. In Type IIB theory we can have a complex superpotential given in terms of NS-NS and R-R three-forms \( H_{NS} \) and \( H_{RR} \) respectively as

\[
W = \int (H_{RR} + \phi H_{NS} + \frac{i}{g_s} H_{NS}) \wedge \Omega,
\]

where \( \phi \) is the axion and \( g_s \) is the Type IIB coupling constant related to the dilaton. For the simplest case, where we take a vanishing axion-dilaton, the Type IIB superpotential is given simply in terms of a complex three-form \( G_3 = H_{RR} + iH_{NS} \). We can go to the heterotic theory by making two T-dualities and an S-duality, as shown in \([11]\) and \([13]\). Then \( H_{RR} \) becomes the real heterotic three-form\(^5\) and \( H_{NS} \) becomes the spin-connection \([14]\). These two fields combine in the heterotic theory to give a complex superpotential. This is again, what we get from our cubic equation.

The skeptic reader might still ask, whether such a complex superpotential could be obtained directly from the supersymmetry transformation rules or equivalently, from the lagrangian of the heterotic theory. Since the whole heterotic dynamics can be described in the T-dual Type IIB framework, where there exist a complex three-form \( G_3 \), we could as well write our heterotic lagrangian in terms of \( G \), by combining the spin-connection part and the real three-form part. This is an obvious straightforward exercise, that can be easily performed.

We can be a bit more precise here. Let us consider the usual heterotic lagrangian. In terms of the conventions that we followed here, this is given as (we use the notations of \([24]\))

\[
S = \frac{1}{\kappa_{10}^2} \int d^{10}x \sqrt{g} e^{-2\phi} \left[ R + 4|\partial\phi|^2 - \frac{1}{2}|f|^2 + \frac{\kappa_{10}^2}{g_{10}^2} \text{Tr}|F|^2 + O(\alpha'^2) \right]
\]

\((3.9)\)

\(^5\) To see the Chern-Simons part of the real three-form \( \mathcal{H} \), one has to carefully study the singularities in the dual \( \mathcal{M} \)-theory setup. The localized fluxes at the singularities and the non-zero curvature at those points conspire precisely to give the Chern-Simons part. These calculations have been done in detail in \([17]\) and therefore we refer the reader to this paper for more details.
where $\kappa_{10}$ and $g_{10}$ are defined in [24], $F = dA + \text{Tr} A \wedge A$ and $f$ is given in (3.7). We can rewrite the above lagrangian alternatively, to all orders in $\alpha'$, as

$$
S = \int_{\mathcal{M}_6} e^{-2\phi} \left[ 2|G|^2 + \text{Tr}|F|^2 + \sum_{m,n,p} a_{mnp} G^m F^n R^p \right] - \frac{1}{\kappa_4^2} \int d^4 x \sqrt{g_4} e^{-2\phi} |\partial \phi|^2 + \ldots
$$

(3.10)

where the interaction terms are to be contracted properly to form scalars. The coefficient $a_{mnp}$ is a constant (upto powers of dilaton) and we have denoted the non-Kähler manifold as $\mathcal{M}_6$. Observe that in the above lagrangian we haven’t yet defined $G$. We require $G$ to satisfy the following conditions:

(a) It should be complex.

(b) It should be anomaly free and gauge invariant, and

(c) It should be locally represented as

$$
G = a (\mathcal{H} + ...) + ib (\omega_o + ...),
$$

(3.11)

where $\mathcal{H}$ is the real three-form of the heterotic theory (the real root of the anomaly equation), $a$ and $b$ are arbitrary constants and the dotted terms will be estimated soon.

In the following analysis we will show that the complex root of the cubic equation does satisfy all the above three conditions, modulo possible geometric terms (and hence gauge invariant) that could in principle contribute to the imaginary part of the three-form $G$. We will ignore these contributions for the time being and only mention them later.

Let us now study the solutions of (3.5) carefully. The three roots of the cubic equation (3.5) can be written in terms of $p$ and $q$. We define two variables $A$ and $B$, that are functions of $p, q$, such that the roots of the cubic equations are

$$
A + B, \quad -\frac{1}{2}(A + B) \pm i\frac{\sqrt{3}}{2}(A - B).
$$

(3.12)

The variables $A, B$ are defined in [17] and are real. Therefore, the real root of the cubic equation is $A + B$. This is in fact the heterotic three-form that appears in the lagrangian.

The series expansion of this three-form in terms of powers of $p, q$ can be written as

$$
A + B = -\frac{q}{p} + \frac{q^3}{p^4} - \frac{3q^5}{p^7} + \frac{12q^7}{p^{10}} - \frac{55q^9}{p^{13}} + O(q^{11}).
$$

(3.13)

\footnote{We thank Xiao Liu for discussion on this aspect.}
For the analysis done in [17], we have taken the definitions of \( p, q \) appearing in (3.7). But these are only valid, when we ignore the contributions coming from the spin-connection \( \omega_o \). When we take these into account, the equations do get a little more complicated, as we shall discuss in the next section. For the time being we shall discuss a simple toy example, where we take the spin-connection as \( \omega_o[mnp] = \omega_o \epsilon_{mnp} \). Without loss of generality, and keeping terms only to the linear order in \( \omega_o \), we change the definitions of \( p \) and \( q \) to

\[
p = \frac{t^3}{\alpha'} - c f \omega_o, \quad q = -\frac{ft^3}{\alpha'} + b f^2 \omega_o,
\]

(3.14)

where \( c \) and \( b \) are constants. The above form of \( p, q \) can be derived for the realistic case, where we consider all the components of \( G_{ijk} \) and we shall do this in section 4. Furthermore, we are also ignoring possible constant shifts in \( p, q \) for simplicity. As discussed in [17], the expansion in powers of \( \frac{1}{p} \) is still a reasonable thing to do, because this is a small quantity, as long as the size of the three manifold, \( t \), is a large number. The above expansion can actually be terminated at order \( \alpha' \), because we have ignored contributions from \( dG \) and \( d\omega_o \), as these contributions are higher orders in \( \alpha' \) and higher derivatives in \( t \) respectively.

Doing this and calling the real solution as \( H \), we obtain

\[
H = f - \frac{\alpha' f^3}{t^3} + \frac{\alpha' \omega_o(c - b)f^2}{t^3} + \mathcal{O}(\alpha'^2).
\]

(3.15)

We make two observations here. First, all the terms in the above expansion are dimensionally the same as \( f \). For the case that \( H \) is given by \( f \) (3.7), as in the usual Calabi-Yau compactifications, terms of \( \mathcal{O}(\frac{1}{t}) \to 0 \), which means, that the radius of the manifold goes off to infinity. This is precisely the Dine-Seiberg runaway problem [3], which does not appear in this type of compactifications. For compactifications on non-Kähler complex manifolds the real three-form gets modified to the value given in (3.15) and the size of the radius is then finite. Secondly, note that for the usual case when we have no \( G \) dependences in the Chern-Simons part of the three-form in (3.2), this would have remained unaffected by scalings of the metric (because the torsion-free spin connection \( \omega_{o\mu}^{ab} \) is unaffected by scalings of the vielbeins) and therefore the radius would not have been stabilised. In the presence of \( G \) in the Chern-Simons part of (3.2) the two sides of the equation (3.2) scale differently and therefore the radial modulus is fixed. This is one of the basic advantages of torsional backgrounds.
Let us now discuss the complex solutions. We see from the choice of the roots (3.12), that we need the expansion for \( A - B \). This is given in terms of the \( \frac{1}{p} \) expansion as

\[
(A - B) = \frac{2}{\sqrt{3}} \left[ \sqrt{\frac{p}{3}} + \frac{3}{8} \frac{q^2}{p^{5/2}} - \frac{105}{128} \frac{q^4}{p^{11/2}} + \frac{3003}{1024} \frac{q^6}{p^{17/2}} - \frac{415701}{32768} \frac{q^8}{p^{23/2}} \right] + O(q^{10}).
\]

(3.16)

As we see in (3.12), this is just a part of the complex roots, as there is a contribution from the real root \( \mathcal{H} \). Again, we will keep the expansion to order \( \alpha' \). We call the complex roots as \( G \), with \( G \) now given as

\[
G = -\frac{1}{2} \left( f - \frac{\alpha' f^3}{t^3} + \frac{\alpha' \omega_o (c - b) f^2}{t^3} \right) \pm i \left( \sqrt{\frac{t^3}{\alpha'}} + \frac{3 f^2}{8} \sqrt{\frac{\alpha'}{t^3}} - \frac{\omega_o c f}{2} \sqrt{\frac{\alpha'}{t^3}} \right) + O(\alpha'^3/2),
\]

(3.17)

where to this order we do not see the effect of the constant \( b \) in the imaginary part.\(^7\)

An immediate disconcerting thing about the above expansion might be the fact, that the spin connection \( \omega_o \) appears with a coefficient \( f \) in the imaginary part of \( G \), as this is not expected from T-duality arguments. However, this is an illusion. As has been shown in \([17]\) and as we shall see in a moment in more detail, the size of the internal manifold is fixed by the choice of background \( f \) (see equation 3.31 below). Using the relation between \( f \) and \( t \) in (3.17), we can show, that the complex root locally takes the form

\[
G = -\frac{\mathcal{H}}{2} \mp i (\beta \omega_o + ...),
\]

(3.18)

where \( \beta \) is a pure constant and the dotted terms involve contributions from the radial modulus \( t \), that in general could be functions of \( f \) as well as \( \omega_o \). The above equation (3.18) is, what we expected from the T-dual Type IIB framework, because under T and S dualities the three-form tensor fields \( H_{RR} \) and \( H_{NS} \) of the Type IIB theory transform into the real heterotic three-form \( \mathcal{H} \) and the spin-connection respectively\(^8\). However, T-duality rules are only derived to the lowest order in \( \alpha' \), which is why we have performed our calculations directly in the heterotic theory, instead of in the T-dual Type IIB theory. We shall nevertheless use T-duality arguments from time to time for comparison. Notice,

\(^7\) This constant starts affecting the expansion at the next order \( \sqrt{\frac{2\alpha'}{t^3}} \) as \( \frac{1}{8}(2.5c-2b)f^3 - \frac{105}{128} f^4 \).

\(^8\) An important point to note here is the following: Incorporating higher order polynomials into the cubic equation (by putting in the values of \( dG \) iteratively), the complex root will change. In that case the real part of \( G \) will shift from the value \( \mathcal{H} \) by additive factors as discussed in (3.11). To the lowest order in \( \alpha' \) (which gives us the cubic equation) we do not see the shift.
that we can write the imaginary part of (3.17) as an effective spin-connection $\omega_{\text{eff}}$. When we fix the radius of our manifold to a specific value, the effective spin connection locally takes the form

$$\omega_{\text{eff}} = \omega_o + \gamma |f| + O(\omega_o^2, |f|^2),$$

(3.19)

where $\gamma$ is a constant related to $\beta$ (we will soon give an estimate of this) and $|f|$ is the background expectation value of $f$ given in (3.7). This effective connection is not related to the modified connection $\tilde{\omega}$ for the non-Kähler manifolds and therefore shouldn’t be confused with it. In fact this could get contribution from other covariant terms briefly alluded to earlier. Also we shall henceforth write $\omega_{\text{eff}}$ as $\omega$, unless mentioned otherwise.

The complex three-form $G$, that we have computed in (3.18), is what one would have expected from naive T-duality arguments, except that there is an overall sign difference, an extra factor of one half and the constant $\beta$ in front of the spin connection, that originates from higher order $\alpha'$ effects and thus cannot be seen by T-duality arguments. The overall factor is not very important, because it can be easily absorbed into the definition of the holomorphic $(3,0)$ form $\Omega$. But the sign is important. In fact, the sign in the above equation is not difficult to explain. In the usual Type IIB picture there exists a perturbative action of S-duality, that changes both three-forms $H_{NS}$ and $H_{RR}$ by a sign, without changing any other fields. This is the $SL(2, \mathbb{Z})$ operation

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

(3.20)

After performing this operation, we are left with the Type IIB three-form $G_3 = -H_{RR} - iH_{NS}$, which under naive T and S dualities will give us precisely (3.18). The way we have derived this form of $G$, is only valid locally, because of the choice of the background (3.5) and (3.6). But this local form of $G$ is consistent with, what one would expect from T- and S-dualities, as we saw above. Taking these considerations into account, our result for the superpotential is of the form

$$W_{\text{het}} = \int (\mathcal{H} + i\beta \omega) \wedge \Omega.$$ 

(3.21)

For compactifications on manifolds without torsion, where the Dine-Seiberg runaway problem appears, the complex three-form field becomes $G = f \pm i\infty$ and therefore the imaginary part decouples from the path integral. In this case all the cubic roots give the same

\footnote{The modified connection, as derived earlier in \cite{17}, is $\tilde{\omega} = \omega_o - \frac{1}{2} \mathcal{H}$.}
result. However, in the presence of torsion there is a splitting and three different solutions appear. Notice also, that the complex roots do not satisfy the torsional constraints as expected (torsional constraints being real). This will be discussed in section 5 in detail.

**Fig. 2:** This is the form of the radial potential, taking the imaginary part of the three-form into account. Along the x-axis we have represented the radius $t$, along the y-axis the value of $f$ and along the z-axis the potential $V(t, f)$. We have also scaled down $V(t, f)$ by a factor of 500. Observe, that for a fixed value of $f$, the potential has a minimum.

Let us now discuss the radial modulus stabilization. This issue has already been addressed in some detail in [17]. However, in [17] the potential was computed solely from the imaginary part of the three-form. The result was shown to be approximately (see fig. 2)

$$t = 0.722 \left( \alpha' |f|^2 \right)^{1/3},$$

(3.22)

where $|f|$ is the expectation value of the background $f$ field. The above value would shift a bit, if we include higher order $\alpha'$ corrections to the three-form. If we now incorporate the contribution coming from the real part of the three-form, the potential for the radial modulus becomes

$$V(t) = \frac{t^3}{\alpha'} - \frac{2\alpha' f^4}{t^3} + \frac{7\alpha'^2 f^6}{t^6},$$

(3.23)
where we keep terms to order $\alpha'^2$. Observe, that the spin-connection dependent term cannot contribute to the potential. The above potential fixes the radius of our manifold to be (see fig. 3)

$$t = 1.288 \, (\alpha' |f|^2)^{1/3}, \quad (3.24)$$

which is a little larger, than the radius calculated in (3.22). As we shall mention below, the actual value for the radial modulus is smaller, than the result given in (3.24), as there are other effects, that we need to take into account. As an aside, it is interesting to note, that the real part of the three-form fixes the value of the radius to be proportional to $(\alpha' |f|^2)^{1/3}$, but gives an imaginary answer, when we go to the next order.

---

**Fig. 3**: The radial potential considering both the imaginary and the real part of the three-form. There is again a minimum, but now shifted to a slightly larger value. Here we have scaled down the potential $V(t, f)$ by a factor of 1000.

In deriving (3.24) we have not taken into account the fact, that the radius $t$ also depends on the representation of the holonomy group. This fact was partially alluded to in a previous paper [17]. If we call the original radius as $t_o$, then $t$ is related to $t_o$ by the following relation

$$t^3 = t_o^3 \, \text{tr}(M^{ab}M^{cd}M^{ef}) \, \alpha^{ab} \alpha^{cd} \alpha^{ef}, \quad (3.25)$$
where $M^{ab}$ are the representation of the holonomy group and $\alpha^{ab}$ depend on the background three-form field and vielbein, as we describe in the following (this has been discussed earlier in [17]). The one-form $\tilde{G}_{\mu}^{ab}$, which is relevant in the context of the heterotic theory, when constructing the Chern-Simons form (see (3.2)), is given in terms of vielbeins $e_a^\mu$ as

$$ \tilde{G}_{\mu}^{ab} dx^\mu = 2 G_{\mu \nu \rho} e^\nu [e^b|^\rho] dx^\mu \equiv t^{-1} h \alpha^{ab}, \quad (3.26) $$

where [...] denotes the anti-symmetrization over the $a,b$ indices. The factor of $t^{-1}$ comes from the usual scaling of vielbeins, when we extract out the radial part $\sqrt{t}$. For the sake of completeness we refer the reader to [17], where a detailed discussion of this and other related issues were presented. We are also assuming, that the traces of the holonomy matrices $M_{ab}$ are non-zero and real constants. So our analysis herein will only work, if the above two conditions are met. In fact, the second condition can be partially relaxed, as we demonstrate later. Taking all these into account, the cubic term in the anomaly relation (3.3) will contribute

$$ \text{tr} (\tilde{G} \wedge \tilde{G} \wedge \tilde{G}) = h^3 \text{Tr} (M^{ab} M^{cd} M^{ef}) \alpha^{ab} \wedge \alpha^{cd} \wedge \alpha^{ef} \equiv h^3 t^{-3}, \quad (3.27) $$

in accordance with (3.25). The traces of the holonomy matrices are in general non-zero. The special case, when these become zero was discussed in [17], where it was shown, that the radial modulus can still be stabilized and it’s value can be explicitly evaluated. Let us consider a simple toy example, where we choose the holonomy matrices as $\sigma^{ij}$ with $\sigma^i$ being the Pauli matrices, so that (we take $i,j,k = 1,2,3$)

$$ \sigma^{12} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}, \quad \sigma^{23} = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}, \quad \sigma^{31} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}. \quad (3.28) $$

We can now calculate the traces by taking into account the anti-symmetrization of $\alpha^{ij}$. There appear six sums, all of which are the same. The final result can be written in terms of $\alpha^{ij}$. For a manifold that is approximately flat (i.e a manifold, which has an orbifold base) one can show, that all the $\alpha^{ij}$ are numerically 1. This will give us the value of the radius of the six-manifold $t_o$ as

$$ t_o = 0.2184 t = 0.2812 (\alpha'|f|^2)^{1/3}. \quad (3.29) $$

In deriving this result we have been a little sloppy. We took the order $\alpha'^2$ term into account in (3.23). As we know, this term will receive corrections from $d\tilde{G}$ terms. Let us therefore
tentatively write the additional contributions to the potential as \( \frac{n\alpha'^2 f^6}{t^6} \), where \( n \) is an integer. If we assume \( n \) to be small, then the contribution to (3.24) will be (in units of \( (\alpha'|f|^2)^{1/3} \))

\[
t = 1.28763 + 0.02565n - 0.00185n^2 + 0.000183n^3 - 2.04 \times 10^{-5}n^4 + O(n^5), \tag{3.30}
\]

which shows, that the results (3.24) or (3.29) are reliable. Of course, the calculation done in this simple example is not for the holonomy of the manifolds we are interested in. So an important question would be whether the radius can be stabilized to a finite value, after we incorporate all the additional \( \alpha' \) dependences, as well as the dependence on the holonomy matrices. To answer this question, the full iterative answer for \( G \) has to be calculated. This is unfortunately very complicated. However, if we do an expansion in \( \alpha' \) (with the choice of roots in (3.12)), we can show, that the contribution from higher order terms to (3.24) slowly becomes smaller and smaller.

Let us remark, that we can determine the form of the constants \( \gamma \) and \( \beta \) appearing in this section. However, in order to do this, we need the value of the radius \( t \). Since the calculations done above are to leading order in \( \alpha' \), we will assume, that the radius is fixed (to all orders in \( \alpha' \)) in terms of the flux density as

\[
t = m (\alpha'|f|^2)^{1/3}, \tag{3.31}
\]

where \( m \) is a finite constant. The fact, that this could be greater than 1, can be seen by incorporating a few higher order corrections. The values of \( \gamma \) and \( \beta \), that we get using the above value of the radius are

\[
\gamma = -\frac{2}{c} \left( m^2 + \frac{3}{8} \right), \quad \beta = \frac{c}{2m^{3/2}}. \tag{3.32}
\]

Finally, let us remark, that in this section we have made many simplifying assumptions, in order to determine the value of the radial modulus. In the next section we will pursue a more detailed analysis, that can be directly related to the non-Kähler manifolds discussed in the literature and where no simplifying assumptions will be done.
4. Detailed Analysis of Radial Modulus Stabilization

In the previous section, even though we have done a precise calculation, our analysis is still incomplete, because we have used many simplifying assumptions. In this section we shall perform an analysis, that is valid for heterotic string compactifications on non-Kähler manifolds, without making any simplifying assumptions. Some aspects of this have been already discussed in [17].

To do this calculation, we will take all the components of the three-form $G$ into account. Since the real part of the three-form $H$ is either a (2,1), (1,2), (3,0) or (0,3) form, we shall denote the various components as

$$G_{1\bar{2}\bar{3}} = h_1, \quad G_{\bar{1}2\bar{3}} = h_2, \quad G_{1\bar{2}3} = h_3,$$

and their complex conjugates as $\bar{h}_i, i = 1, 2, 3$. Supersymmetry requires, that $h_3 = \bar{h}_3 = 0$, so in the final result these components will become zero. The reason of why we retain the components with legs along the $z_3$ and $\bar{z}_3$ directions, is because in the T-dual Type IIB picture these components are the ones, that survive the orientifold projection. We will denote analogously the components of the spin-connection tensor $\omega_{oi}$ as $\omega_{oi}$ and $\bar{\omega}_{oi}$ with $i = 1, 2, 3$. Let us now derive the equation for the three-form $G_{1\bar{2}\bar{3}}$. First, we will need the one-forms, that can be constructed from the three-forms appearing in (4.1), as described in (3.26). They are explicitly given as

$$\tilde{G}^1_{ab} = 2h_1 e^{2[a} e^{b]\bar{3}} + 2\bar{h}_2 e^{2[a} e^{b]\bar{3}} + 2h_3 e^{2[a} e^{b]\bar{3}},$$

$$\tilde{G}^2_{cd} = 2h_1 e^{3[c} e^{d]\bar{1}} + 2\bar{h}_2 e^{3[c} e^{d]\bar{1}} + 2h_3 e^{3[c} e^{d]\bar{3}},$$

$$\tilde{G}^3_{ef} = 2h_1 e^{1[e} e^{f]\bar{2}} + 2\bar{h}_2 e^{1[e} e^{f]\bar{2}} + 2h_3 e^{1[e} e^{f]\bar{2}}.$$  \hspace{1cm} (4.2)

Similar results can be written for the one-form spin-connection $\omega_{oi}^{ab}$. We will again be ignoring the $d\omega_o$ and $d\tilde{G}$ contributions, as we will work only to order $\alpha'$. We will however continue to assume, that $e^{a\mu} = t^{-1/2} e^{a\mu}$ and therefore the results will again be valid locally. The above considerations will tell us, that the equation satisfied by the component $G_{1\bar{2}\bar{3}}$ takes the form

$$G_{1\bar{2}\bar{3}} + \frac{\alpha'}{2} \text{tr} \left[ -\frac{1}{3} \omega_o \wedge \tilde{G} \wedge \tilde{G} + \frac{2}{3} \tilde{G} \wedge \omega_o \wedge \omega_o + \frac{1}{6} \tilde{G} \wedge \tilde{G} \wedge \tilde{G} \right]_{1\bar{2}\bar{3}} = f_{1\bar{2}\bar{3}},$$

\hspace{1cm} (4.3)

\footnote{In this section the spin-connection is the usual spin-connection $\omega_o$ and not the effective spin-connection $\omega$.}
where \( f_{ijk} \) is defined earlier in (3.7). For the non-Kähler metric studied in [11], [13], [20] and [17], the base manifold was an orbifold. As a result, the only non-trivial factors in the metric are the warp factors. This would mean, that up to powers of the warp factor all components of the spin-connection are the same. Let us therefore take

\[
\omega_{a}^{b} = f_{i}(\Delta) \omega_{o} e^{ab},
\]

where \( f_{i} \) needs to be worked out for every components individually and \( \varepsilon^{ab} \) is the anti-symmetric tensor. As we discussed in [17], if we consider the set of equations (4.2) to the leading order in \( \alpha' \), we can replace the complex three-forms \( h_{2}, h_{3} \) by their real parts. Therefore, up to proportionality constants, we can write the above three one-forms as \( h_{1} + \alpha_{1}, h_{1} + \alpha_{2}, h_{1} + \alpha_{3} \), where \( \alpha_{i} \) can be easily calculated from the vielbeins \( e_{a}^{o} \). This will transform the cubic equation (4.3) into

\[
\frac{h_{1} t^{3}}{\alpha'} - A_{1} \omega_{o}(h_{1} + \beta_{1})(h_{1} + \beta_{2}) + A_{2} \omega_{o}^{2}(h_{1} + \beta_{3}) + (h_{1} + \beta_{4})(h_{1} + \beta_{5})(h_{1} + \beta_{6}) = \frac{f t^{3}}{\alpha'},
\]

where it is an easy exercise to relate \( \alpha_{i} \) and \( f_{i}(\Delta) \) to \( \beta_{1}, A_{1} \) and \( A_{2} \). We have absorbed the traces of the holonomy matrices discussed in the previous section into the definition of \( t \). One can also check, that similar equations hold for all other components of \( G \). Therefore, the generic cubic equation, that we get is

\[
h_{1}^{3} + m h_{1}^{2} + n h_{1} + s = 0,
\]

where \( m, n \) and \( s \) are integers given in terms of \( f, t, \beta_{i} \) and \( A_{j} \) as

\[
m = \beta_{4} + \beta_{5} + \beta_{6} - A_{1} \omega_{o},
\]

\[
s = -\frac{f t^{3}}{\alpha'} + \beta_{4} \beta_{5} \beta_{6} - A_{1} \omega_{o} \beta_{1} \beta_{2} + A_{2} \omega_{o}^{2} \beta_{3},
\]

\[
n = \frac{t^{3}}{\alpha'} + \beta_{4} \beta_{5} + \beta_{5} \beta_{6} + \beta_{6} \beta_{4} - A_{1} \omega_{o}(\beta_{1} + \beta_{2}) + A_{2} \omega_{o}^{2}.
\]

11 We are ignoring an important subtlety here. The fiber doesn’t scale with the warp factor as we saw in (2.1). Therefore, the size of the six-manifold should be expressed in terms of \( r_{1} \) (the radius of the fiber) and \( r_{2} \) (the radius of the base). For the time being, we shall ignore this subtlety, as this doesn’t affect the final result. We will consider this towards the end of the paper. Therefore our analysis is done for the case \( \Delta_{1} = \Delta_{2} = \Delta \) and \( \alpha, \beta \) as constants locally, in (2.1).

12 For more details, see section 4.4b of [17].
There is a quadratic term in this equation, that can be removed by shifting the three-form $h_1$ by $h_1 = h_1 - \frac{m_3}{3}$. This will give us precisely the cubic equation (3.5), with $p$ and $q$ defined as

\begin{equation}
    p = \frac{t^3}{\alpha'} - A_3 \tilde{\omega}_o + O(\tilde{\omega}_o^2),
    \quad q = -\frac{ft^3}{\alpha'} + A_4 \tilde{\omega}_o + A_5 + O(\tilde{\omega}_o^2),
\end{equation}

where $\tilde{\omega}_o$ is a shifted spin-connection, that is introduced to absorb the constant term in $p$ and $A_i$ are constants determined by (4.7). To order $\tilde{\omega}_o$ the above expression precisely coincides with the simplified discussion presented in the previous section (see eq. (3.14) and identify $\tilde{\omega}_o$ with $\omega_o$ there). There is one difference though, that we would like to discuss in some detail. Notice, that in the expression for $q$ given above, there is a constant $A_5$, that did not appear in our analysis in the previous section. This constant is rather harmless, because adding a constant $l$ into the definition of $q$ in (3.14), will give an additional constants in the real and the imaginary parts of $G$ in (3.17), that have the form

\begin{equation}
    -\frac{l\alpha'}{t^3} + O(\alpha'^2), \quad \text{and} \quad 0 + O(\alpha'^3/2),
\end{equation}

respectively. The above shift appearing in the real root doesn’t change the final expression for $G$, because we can tune $l$ to scale smaller than $f^3$. To summarize, what we have just shown is, that we can trust the analysis done in the previous section, even if some simplifying assumptions were done.

There is one more point, that we would like to discuss, before we derive the torsional constraints from our superpotential in the next section. This has to do with the signs of $p$ and $q$ in the cubic equation (3.5). According to the conventions, that we have chosen, the signs of $p$ and $q$ are fixed. However, is an interesting question to ask, what happens, if we reverse the signs of $p$ and $q$. Observe, that transforming $q \rightarrow -q$ does not change much the real and complex solutions for $G$, as they still retain their original form, except for an overall sign change in the function (3.13). However, changing $p \rightarrow -p$ makes (3.16) pure imaginary and therefore all the roots of the cubic equation become real. One may now wonder, if this is consistent with dual Type IIB picture.

To see, that this is indeed so we need to remind ourselves, that the tensor field $H_{RR}$ of the Type IIB theory turns into the real root $H$ of the heterotic theory and the tensor field $H_{NS}$ turns into the spin-connection. Let us therefore make the transformation

\begin{equation}
    H_{RR} \rightarrow iH_{RR}, \quad \text{and} \quad H_{NS} \rightarrow iH_{NS},
\end{equation}

20
which shifts the $i$ in the superpotential to the $H_{RR}$ part, while giving a relative minus sign between the three-forms. Recall, that we are considering the case, when we have a vanishing axion-dilaton, as this is directly related to the superpotential of the heterotic theory. The above transformation (4.10) will, in turn, convert our cubic equation (3.5) into

$$G^3 - p G + iq = 0,$$  

(4.11)

where we have taken $G \rightarrow iG$ in (3.5). The roots of the above equation are, in fact, slightly different, because the real root (3.13) will become purely imaginary, while the $i$ in the complex root (3.17) will trade places. A simple way to see this, would be to go back to the series expansions (3.13) and (3.16) and write them as

$$A + B = \sum_{n=0}^{\infty} a_n \frac{(-1)^{n+1}|q|^{2n+1}}{p^{3n+1}},$$

$$A - B = \sqrt{p} + \sum_{n=1}^{\infty} b_n \frac{(-1)^{n+1}|q|^{2n}}{p^{3n-\frac{3}{2}}},$$  

(4.12)

where $a_n, b_n$ can be determined from (3.13) and (3.16). Of course, this expansion is not very meaningful beyond the first few orders of $p$ and $q$, but we shall still use these series, to illustrate the generic behavior (we believe that putting higher order $\alpha'$ corrections to the system will change the coefficients $a_n$ and $b_n$, without altering the $p, q$ behavior). If we change $p \rightarrow -p$ and $q \rightarrow iq$, it is easy to check from (4.12), that both $A \pm B$ become purely imaginary, as mentioned above. Let us now make the transformation $q \rightarrow iq$, which will give us

$$G^3 - p G - q = 0, \quad p > 0, \quad q > 0,$$  

(4.13)

whose roots are all real\textsuperscript{13}. This is the case alluded to above. The question now is to trace this back to the Type IIB theory.

On the heterotic side, following the expansions (4.12) and performing the transformation $q \rightarrow iq$, makes the real root $A+B$ pure imaginary. One should be slightly careful here, because $q$ defined in the expansions of $A$ and $B$ involve $\sqrt{q^2}$, which can have any sign. Therefore, $H$ becomes purely imaginary. Since $H$ is directly the T-dual of $H_{RR}$, this would imply, that $H_{RR}$ goes back to the real value. And therefore, since $H_{NS}$ is pure imaginary,

\textsuperscript{13} The three real roots are given by $2a \cos \frac{\theta}{3}$ and $-a (\cos \frac{\theta}{3} \pm \sqrt{3} \sin \frac{\theta}{3})$, where we have defined $a = \sqrt{\frac{p}{3}}$ and $\cos \theta = \frac{q}{2} \sqrt{\frac{27}{p^3}}$.\textsuperscript{21}
the factor of \(i\) in the three-form \(G\) transforms this into a purely real three-form, exactly as we expected from the heterotic side! In other words, to make a transition from (3.5) to the cubic equation (4.13) in the heterotic theory, we need to make the transformation

\[
H_{RR} \rightarrow -H_{RR}, \quad \text{and} \quad H_{NS} \rightarrow iH_{NS},
\]

in the Type IIB theory for the conventions, that we are following. This will lead to a real three-form in the Type IIB theory, exactly as we have in the heterotic theory.

This concludes our discussion regarding the form of the superpotential involving the three-form flux for compactifications of the heterotic string on non-Kähler complex six-dimensional manifolds. In the next section we shall see, that there is one more term in the superpotential, if the effect of the non-abelian gauge fields is taken into account. Using the complete superpotential, we shall derive the form of the torsional constraints.

5. Superpotential and Torsional Constraints

The goal of this section is to derive the constraints following from supersymmetry, that compactifications of the heterotic string on non-Kähler complex three-folds have to satisfy. We will do so, by using the F-terms and D-terms, which describe these sort of compactifications. Let us first describe the F-terms in full detail. Until now we have considered the superpotential, that is written in terms of the three-form \(G\). But there is another superpotential, coming from the heterotic gauge bundle, which we will compute in this section. This superpotential is distinct from the Chern-Simons term

\[
-\alpha' \int \Omega_3(\mathcal{A}) \wedge \Omega,
\]

that we have already described and is most easily understood in terms of the corresponding \(\mathcal{M}\)-theory superpotential. In \(\mathcal{M}\)-theory on a four-fold \(\mathcal{X}\), \(G_4\)-fluxes induce two superpotentials, which can be expressed in terms of the holomorphic \((4,0)\)-form \(\Omega_4\) and the Kähler form \(J\) of the Calabi-Yau four-fold [10]

\[
W_\mathcal{M} = \int_\mathcal{X} G_4 \wedge \Omega_4, \quad \text{and} \quad \tilde{W}_\mathcal{M} = \int_\mathcal{X} J \wedge J \wedge G_4.
\]
Demanding $W_M = DW_M = 0$ and $\hat{W}_M = D\hat{W}_M = 0$, reproduces the constraints for unbroken supersymmetry in Minkowski space derived in [1], which state, that the only non-vanishing component of $G_4$ is the $(2, 2)$ component, which has to be primitive.

The first expression (5.2), is the origin of the superpotential (3.1), that we have been discussing so far, because it contributes to the two bulk three-forms $H_{NS}$ and $H_{RR}$ of the Type IIB theory. Therefore, it contributes to the heterotic three-form $H$ and spin connection $\omega$. The second expression (5.3) gives rise to a superpotential for the gauge bundle in the Type IIB theory or consequently to a second superpotential in the heterotic theory. In order to see this, let us consider $\cal{M}$-theory on $T^4/I_4 \times T^4/I_4$, which was discussed earlier in the literature in [13] and [17]. We decompose the $\cal{M}$-theory flux in a localized and a non-localized part. The non-localized part is responsible for the heterotic superpotential (3.21). The localized part is a little more subtle and it takes the form

$$
\frac{G_4}{2\pi} = \sum_{i=1}^{4} F_i(z^1, z^2, \bar{z}^1, \bar{z}^2) \land \Theta_i(z^3, z^4, \bar{z}^3, \bar{z}^4).
$$

Here the index ‘$i$’ labels four fixed points and at each fixed point, there are four singularities. Also, $z^{1,2}$ are the coordinates of the first $T^4/I_4$ and $z^{3,4}$ correspond to the coordinates of the second $T^4/I_4$. The harmonic $(1,1)$-forms near the fixed points of the orbifold limit of K3 are denoted by $\Theta^i$. Inserting (5.4) into (5.3), we get a contribution to the Type IIB superpotential, which after integrating out $\Theta$ over a two-cycle is of the form

$$
\sum_i \int F_i \land J \land J.
$$

The integral over the two-cycle is bounded, since the $\Theta^i$’s are normalizable. The $(1,1)$-forms $F_i$ have an interpretation as gauge fields on the Type IIB side.

Two T-dualities and an S-duality will not modify the above expression of the superpotential. Therefore, we obtain besides (3.21) a second superpotential for the heterotic theory

$$
\hat{W}_{het} = \sum_i \int F_i \land J \land J,
$$

where $J$ is the fundamental $(1,1)$ form of the internal space. However, this is not the whole story yet. As in the case for compactifications of the heterotic string on a Calabi-Yau threefold, we will not only have $F$-terms but also $D$-terms. The explicit form of these $D$-terms

\footnote{More details can be extracted from sec 2.5 of [17].}
can be computed from the supersymmetry transformation of the four-dimensional gluino. This supersymmetry transformation gives us the following constraints on the non-abelian two-form of the heterotic theory

\[ F^i_{ab} = F^i_{\bar{a}\bar{b}} = 0, \quad (5.7) \]

and

\[ J^{ab} F^i_{ab} = 0, \quad (5.8) \]

as has been explained e.g. in [25]. The last equation is the well known Donaldson-Uhlenbeck-Yau (DUY) equation. These constraints can be derived from a D-term, appearing in the four-dimensional theory

\[ D^i = F^i_{mn} J^{mn} = \epsilon^{\dagger} F^i_{mn} \Gamma^{mn} \epsilon, \quad (5.9) \]

as supersymmetry demands \( D^i = 0 = F^i_{mn} \epsilon^{\dagger} \Gamma^{mn} \epsilon \). However, it turns out, that these constraints on the gauge bundle can also be derived from the superpotential (5.6). This is because, it has been shown in [25], that the following identity holds

\[ F^i_{ab} J^{ab} = \frac{1}{2} F \wedge J \wedge J, \quad (5.10) \]

so that the DUY equation is equivalent to the supersymmetry constraint \( \widehat{W}_{het} = 0 \), while \( D \widehat{W}_{het} = 0 \) imposes no additional constraint. From a different perspective notice, that primitivity of the \( M \)-theory flux translates on the heterotic side into the previous conditions for the localized fluxes. As we will see below, the gauge bundle is further restricted. This additional condition comes from the three-form part of the superpotential.

We now would like to use the above results to derive the torsional equations of [5],[6] and [9], which are required for supersymmetry, from the superpotential

\[ W_{het} = \int G \wedge \Omega, \quad (5.11) \]

involving the complex three-form. Notice, that we have related this superpotential directly to the T-dual of the Type IIB superpotential. In particular, this implies, that the three-form \( G \) in the heterotic theory should be imaginary self-dual, because the corresponding T-dual configuration in IIB is! This means, that one condition for unbroken supersymmetry for compactifications of the heterotic string to four-dimensional Minkowski space is

\[ \star_6 (\mathcal{H} + i \beta \omega) = i (\mathcal{H} + i \beta \omega), \quad (5.12) \]
where the Hodge $\star_6$-operator is defined with respect to the six-dimensional internal manifold. Observe, that this is the gauge invariant three-form, which satisfies the Bianchi identity with respect to the connection with torsion.

Comparing the real and the imaginary sides of the above equation, we obtain the condition

$$\mathcal{H} = \pm \star_6 \beta \omega,$$

where we have kept the sign ambiguity, to reflect the fact, that the real three-form can have either sign in this space. This is basically the content of the torsional equation, which we shall write now in the more familiar form appearing in [3], [4] and [5]. Our goal is to express this constraint in terms of the fundamental two form $J_{mn}$, where $m, n$ are spatial coordinates on the non-Kähler space. But before we do that, we need to carefully define the spin-connection $\omega$. Recall, that this spin connection is the effective spin-connection given in (3.19) and therefore we have to be slightly careful in defining it. From the form of (3.19) we can see, that it has a piece proportional to $\omega_o$ and a piece proportional to $|f|$ plus higher order corrections. Furthermore, being a three-form it is completely antisymmetric in all of its three space-time indices. In terms of the vielbein therefore, it should be an anti-symmetric combination of $e$ and $\partial e$ for dimensional reasons. Let us therefore write the complete antisymmetric part of $\omega$ as

$$\omega_{[nml]} = G_1 \eta_{ab} e^a_n \partial_m e^b_l + G_2 \epsilon_{ab} e^a_n \partial_m e^b_l, \quad (5.14)$$

where $a, b$ are internal indices and $G_{1,2}$ are, in general, functions of the warp factors. We should also view this form of the spin-connection only locally, as the vielbeins are defined locally, because there are no one-forms on our non-Kähler spaces. There would be fermion contributions to the above formula (as given in eq 17.12 of [26]), but we are ignoring them for the time being. The above relation will imply, that we can write (5.13) equivalently as

$$\mathcal{H}_{mnp} = \beta \sqrt{g} \epsilon_{mnp}^{qrs} \omega_{qrs}, \quad (5.15)$$

At this point we can use the specific background, that we have been taking all through, which is given by (3.5) and (3.6) to determine the possible values of $G_{1,2}$. First, it is easy to check, that the $G_1$ dependent term vanishes for this choice of background. Notice again,

Recall that this effective spin-connection can have contributions from purely covariant terms. The result below is when we take into account all the possible geometric and non-geometric terms.
that our analysis is valid only, if the metric is of this form locally\footnote{It is an interesting question to obtain the full global picture, by taking $\alpha$ and $\beta$ to be non constant in (2.1). In this case the simple analysis of the cubic equation will no longer be valid and we have to do a more precise evaluation. The ansatz for the spin-connection made in (4.4) will no longer be valid either. This will affect the complex part and the real part of the three-form (3.17). We hope to address this elsewhere \cite{[27]}.}.

In the presence of warp factors we would in principle expect the definition of the vielbeins to get modified. But, as we will soon see, the form of the vielbeins still remains as above, but now with a modified $t$.

Therefore, in the absence of warp factors we are left with the second term in (5.14) with a constant $G_2$. Introducing back the warp factors will not change this conclusion.

Now equation (5.15) has almost the form, in which the torsional equation appears in \cite{[1]}, \cite{[3]} and \cite{[9]} (see e.g. equation 3.49 of \cite{[9]}), but it is still formulated in terms of the spin-connection instead of the fundamental two-form. We are ignoring the warp factors, so we do not see the dilaton explicitly in the formula.

Let us rewrite (5.15) in terms of the fundamental two-form $J$. This is not difficult, as the term proportional to $G_2$ is simply $dJ$, where $J$ is the usual two-form of the manifold. The precise relation between $G_2$ and $dJ$ can be derived in the following way. Let us first define a covariantly constant orthogonal matrix $N$, such that $N^\top = N^{-1}$ and this would convert the $D_4$ spinor indices (world-sheet indices) to vector indices. More details on the sigma model description of the heterotic string on non-Kähler manifolds have appeared in section 2.4 of \cite{[17]}.

We will follow the notations of that section. This means

\begin{equation}
S^a = N^a_q S^q, \quad S^i = e^i_a S^a, \tag{5.16}
\end{equation}

where $S^p, p = 1, \ldots, 8$ is the world-sheet superpartner of $X^i$, describing the light cone coordinates\footnote{Recall, that we are imposing $S^\dot{q} = 0$, therefore only 8 components remain. The gamma matrix $\Gamma_{p\dot{q}}^a$ acts as triality coefficients, that relate the three inequivalent representations of $D_4$, i.e the vector and the two spinor representations.}.

Therefore $N$ is an $8 \times 8$ matrix described in more detail in \cite{[28]} and \cite{[6]}.

Following earlier work, we can choose the $N$ matrices as antisymmetric, such that the two-form is given by \cite{[6]}

\begin{equation}
J_{ij} = N_{ab} e^a_i e^b_j, \tag{5.17}
\end{equation}
Now we can use the epsilon tensor of the Hodge $\star$ to rewrite the right hand side of (5.15), involving the spin-connection $\omega_{qrs}$ in terms of derivatives acting on the vielbeins as

$$H_{mnp} = \sqrt{g} \beta G_2 \epsilon^{qrs}_{mnp} \epsilon_{ab} e^a_q \partial_r e^b_s,$$  \hspace{1cm} (5.18)

where we have ignored a factor of 6, as we are more interested in the functional dependences. In deriving this, we have used the explicit form of the spin-connection given in (5.14). Now equation (5.18) suggests, that the right hand side can be expressed in terms of the derivative of the two-form $J_{ij}$ appearing in (5.17). In fact, we can exploit the antisymmetry of $N$ to express this as

$$[\star dJ]_{mnp} = \sqrt{g} \epsilon^{qrs}_{mnp} N_{ab} \partial_q (e^a_r e^b_s) = \sqrt{g} \epsilon^{qrs}_{mnp} N_{ab} e^a_q \partial_r e^b_s,$$  \hspace{1cm} (5.19)

where there would again be a proportionality constant, that we are ignoring. From (5.18) and (5.19), we require $\beta G_2 \epsilon_{ab} = 2N_{ab}$. The constant $G_2$ can then be easily worked out from the known expressions of $N$ and $\beta$. Notice, that in the calculation above we have always been taking simple derivatives, while we should have taken covariant derivatives. The connection appearing in this covariant derivative should be the Christoffel connection and not the torsional connection. More details of this calculation are given in [9], so we refer the reader to this paper for further information. After taking this into account, the torsional equation, that we obtain from our superpotential (5.11) is

$$H_{mnp} = \sqrt{g} \epsilon_{mnpqrs} D^q J^{rs},$$  \hspace{1cm} (5.20)

where $D^q$ is the covariant derivative. This is consistent with the result of the earlier literature [5], [6] and [9], when we have no warp factor. What happens, when we introduce back the warp factor? To see this recall, that the warp factor in the heterotic theory is proportional to coupling constant, i.e $\Delta = e^{\phi}$, with $\phi$ being the heterotic dilaton. In fact, both $H$ and $J$ will scale in some particular way with the warp factor giving us the following dilaton dependence of the torsional equation:

$$H_{mnp} = \sqrt{g} e^{a\phi} \epsilon_{mnpqrs} D^q (e^{b\phi} J^{rs}),$$  \hspace{1cm} (5.21)

where $a$ and $b$ can be determined for our case when $\Delta_1 = \Delta_2 = \Delta$. This gives us precisely the result, we had been looking for. This equation has recently also been discussed in [24], where the values of $a, b$ in (5.21) were derived from the Killing spinor equations and not
using any superpotential. For our case, when we restrict ourselves to the more realistic scenario of (2.1), we can view the radial modulus $t \equiv t(x, y)$ with $x$, the coordinates of 4d Minkowski space-time and $y$, the coordinates of the internal non-Kähler space, as

$$t(x, y) = \tilde{t}(x) \Delta^2(y)$$

(5.22)

so that the warp factor is *absorbed* in the definition of $t$ itself and $\tilde{t}$ is the “usual” radial modulus that is independent of the coordinates of the internal manifold. This way of looking at things tells us the reason why in the presence of warp factors (at least for the conformal case) we expect our analysis to go through. In fact for the case where we have $\Delta_1 = \Delta, \Delta_2 = 1$, the above choice of $t$ will tell us that we can take (instead of $\Delta_1 = \Delta_2 = 1$) a slightly more involved case where $\Delta_1 = 1, \Delta_2 = \Delta^{-2}$. This analysis, now with $\alpha, \beta$ no longer constant in (2.1), will be dealt in [27]. However various indirect arguments suggest that the values of $a, b$ in (5.21) are given by $a = b = 2$. Therefore we can now write (5.21) in a condensed way as

$$\mathcal{H} = e^{2\phi} \ast_6 d(e^{-2\phi} J).$$

(5.23)

One can show, that this form of the torsional equation is identical to the conventional form

$$\mathcal{H} = i(\partial - \bar{\partial}) J,$$

(5.24)

when specified to the case of compactifications with $SU(3)$ structure. To show this, we will use the description of manifolds with $SU(3)$ structure suggested in [20]. In the notations of [20], the metric (2.1), (2.2) has the form

$$g = e^{2\phi} \pi^* g_{CY} + \rho \otimes \bar{\rho},$$

(5.25)

where $g_{CY}$ is the metric of the Calabi-Yau base, in our case it is K3 or its orbifold limit $T^4/I_4$. Also, $\phi$ is a function on the base Calabi-Yau because recall, that it is related to

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18 The analysis of [29] which uses Killing spinor equations a-la ref. [5] does not require any specific background. In fact even though we use some background to illustrate the torsional equation and the radial stabilisation, our analysis is completely general as the technique of cubic equation discussed above does not require any specific input. Furthermore, though not directly related to our interest here, many new examples of non-compact geometries are studied in [29] preserving different fractions of susy (see the third reference of [29]). We thank the authors of [29] for informing us about these interesting developments.

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the warp factor as $\Delta = e^\phi$ and $\Delta$ is a function of the base, for the examples studied in [11], [13] and [17]. Finally, the (1,0) form $\rho$ is such that

$$d\rho \equiv \sigma = \omega_P + i\omega_Q,$$

(5.26)

with the real (1,1) forms on the Calabi-Yau base defined as

$$\omega_P = 2 \ dz^2 \wedge d\bar{z}^1 - 2 \ dz^1 \wedge d\bar{z}^2,$$

$$\omega_Q = (1 + i) \ dz^2 \wedge d\bar{z}^1 - (1 - i) \ dz^1 \wedge d\bar{z}^2,$$

(5.27)

obeying the susy condition $^{19} \star_4 \omega_{P,Q} = -\omega_{P,Q}$. These (1,1) forms are basically related to the fibered metric defined in (2.1), as one can easily extract, by comparing (5.25) and (2.1). For the fundamental form we have

$$J = e^{2\phi} \pi^* J_{CY} + \frac{i}{2} \rho \wedge \bar{\rho},$$

$$dJ = 2e^{2\phi} d\phi \wedge \pi^* J_{CY} + \frac{i}{2} (\sigma \wedge \bar{\rho} - \bar{\sigma} \wedge \rho),$$

(5.28)

where we have used $dJ_{CY} = 0$. Now we can use the special properties of $\sigma$ and $\rho$ with respect to the Hodge star operations

$$\star_4 \sigma = -\sigma, \quad \star_2 \rho = i\bar{\rho}, \quad \star_2 \bar{\rho} = -i\rho,$$

(5.29)

to get the following set of relations

1. $i(\partial - \bar{\partial}) J = i \left[ (dJ)^{2,1} - (dJ)^{1,2} \right] = i(\partial - \bar{\partial}) e^{2\phi} \wedge \pi^* J_{CY} + \star_6 \frac{i}{2} d(\rho \wedge \bar{\rho}),$

2. $e^{2\phi} \star_6 d(e^{-2\phi} J) = - \star_6 \left( 2d\phi \wedge \frac{i}{2} \rho \wedge \bar{\rho} \right) + \star_6 d \left( \frac{i}{2} \rho \wedge \bar{\rho} \right).$

(5.30)

We can now use the distributive properties of the Hodge star, to write the first term in the second relation of (5.30) as

$$\star_4 (2 \ d\phi) \wedge \star_2 \left( \frac{i}{2} \rho \wedge \bar{\rho} \right) = \star_4 (2 \ d\phi) = i(\partial - \bar{\partial}) e^{2\phi} \wedge \pi^* J_{CY}.$$

(5.31)

Plugging (5.31) into (5.30), we can easily see that

$$i(\partial - \bar{\partial}) J = e^{2\phi} \star_6 d(e^{-2\phi} J),$$

(5.32)

$^{19}$ We take $\star$ to be an anti-linear operation.
thus proving the relation (5.24). This relation is, of course, reflection of the fact, that the two-form $J$ is $\mathcal{H}$-covariantly constant with respect to the modified connection, which includes the torsion. There is a factor of $\frac{1}{2}$ in (5.24) different from the result presented in [5]. This has already been shown in [17] to be a consequence of the choice of the real three-form in the connection as $\frac{1}{2}\mathcal{H}$, instead of just $\mathcal{H}$. It is also clear from the relation (5.23), that $dJ \neq 0$, so that the six-manifold is non-Kähler. This property is directly related to the fact, that the fibration given by the metric (5.25) is non-trivial. The (1,0) form $\rho$, corresponding to the $T^2$-fiber, is not (globally) closed, and there is a mixture between the fiber and the base coordinates. As a consequence, the right hand side in the second formula in (5.28) is non-zero and $dJ \neq 0$.

Another way to see this relation, is to use the fact, that the holomorphic $T^2$-fiber is a torsional cycle and is zero in the real homology of the six manifold [20] and [17]. It is similar to the one-cycle in the $\mathbb{RP}^2$ example pictured in fig. 4. Indeed, suppose $dJ = 0$. The integral $\int_{T^2} J$ over the fiber $T^2$ gives its volume and must be non-zero. On the other hand, an integral of any closed form over a torsional cycle is zero. Therefore, we conclude $dJ \neq 0$.

![Fig. 4: The torsional cycle $C_1$ on $\mathbb{RP}^2$ is $\mathbb{Z}_2$ in the integer homology and zero in the real homology.](image)

The above result (5.21) is in string frame. In Einstein frame we expect $N^{ab}$ to change in the following way

$$N^{ab} \rightarrow e^{g\phi} N^{ab}, \quad (5.33)$$

where $g$ is a constant and therefore (5.19) will pick up an additive piece proportional to $\star \partial^a \phi J^{rs}$ from the fact, that the metric in string frame is related to the metric in Einstein...
frame via the relation \( g_{\mu\nu}^{\text{string}} = e^{\frac{h}{2}} g_{\mu\nu}^{\text{Einstein}} \). This would imply, that the torsional equation in Einstein frame becomes

\[
\mathcal{H}_{mnp} = e^{h\phi} \sqrt{g} \epsilon_{mnpqrst} [D^q J^{rs} + c \partial^q \phi J^{rs}],
\]

where \( h \) and \( c \) are constants, that can be easily determined by carefully studying the transformation rules from the string frame to the Einstein frame. This is precisely the form, in which the torsional equations appear in [9]. Observe, that the superpotential analysis gave a very simple derivation of this relation. Furthermore, on our six-dimensional space the two form \( J \) satisfies: \( \star J = \frac{1}{2} J \wedge J \). This implies, that the Nijenhuis tensor vanishes [9].

From the above torsional equation it is now easy to extract the additional constraint on the gauge bundle. We have already shown, that the gauge bundle in this space has to satisfy the Donaldson-Uhlenbeck-Yau equation. The torsional equations derived above show, that there is a further constraint (alluded to earlier in [17]) given by

\[
\text{Tr } F \wedge F = \text{tr } R \wedge R - i\partial \bar{\partial} J,
\]

in addition to the ones presented in (5.7) and (5.8). The solutions of these equations and their phenomenological aspects will be discussed in a forthcoming paper [27]. Before we end this section, we make the following observations.

First, from the form of the torsional equation it is clear, that we cannot scale the fundamental two-form \( J_{mn} \) in an arbitrary way. This is, of course, related to the stabilization of the radial modulus for this manifold. The point to note is, that the torsional equation (which is basically the statement, that the two-form \( J \) is covariantly constant with respect to some torsional connection) implies, that the modified connection is contained in \( SU(3) \), because the first Chern class vanishes. As we discussed earlier, this modified connection appears, when we choose our contorsion tensor to be precisely the torsion (see the discussion section of [17]).

The second observation is related to the sizes of the base K3 and the fiber torus \( T^2 \) for our non-Kähler manifold\(^{20}\). In (3.24) we fixed the overall radius \( t \) of our manifold. If we call the radius of the fiber \( T^2 \) as \( r_1 \) and the radius of the base K3 as \( r_2 \), then we have the identity \( t^3 = r_1^2 r_2^3 \). Is it now possible to fix both \( r_1 \) and \( r_2 \), knowing \( t \)? In principle from the choice of the potential (3.23) we cannot fix both. But we can use T-duality arguments.

\(^{20}\) The following discussion arose from a conversation with M. M. Sheikh-Jabbari. We thank him for many helpful comments.
to fix the radius of the fiber. Indeed in [13] it was shown, that fixing the Type IIB coupling constant actually fixes the volume of the fiber $T^2$ to $\alpha'$ (see eq. 3.3 of [13]). This would imply, that for our case we have

$$r_1 = \sqrt{\alpha'}, \quad r_2 = \sqrt{|f|}, \quad (5.36)$$

for the radii of the fiber and the base respectively. The fact, that the fiber is stabilized at the value $\alpha'$ is not too surprising, because we have shown, that our model can be understood from T-duality rules. Since T-dual models have a self-dual radius at value $\alpha'$, we expect the same for our case. An important question, however is now, whether we can trust the supergravity analysis. In fact, for our case we can only provide an effective four-dimensional supergravity description as long as the total six-dimensional volume is a large quantity because of the inherent topology change. Since $|f|$, the flux density, can in principle be large, even though the total flux over a three-cycle $C^3$, which is $\int_{C^3} |f|$ is a fixed quantity, we can have a large sized six-manifold. Observe that the duality chasing arguments that we followed to derive the background doesn’t rely on the existence of a supergravity description because the type IIB solution that we took is an exact F-theory background. Unfortunately a direct confirmation of this is not possible in the heterotic theory because it will in principle be difficult to explain the topology change via supergravity analysis.

The third observation is related to the potential $V(t, f)$ for the radial modulus $t$. One can easily verify, that $V(t, f)$ has the form of the potential for a half harmonic oscillator, whose minima we computed in the previous section. This would imply, that the corresponding Schrödinger equation will only have wave-functions, that vanish around the minima of the potential. This in turn will determine the spectrum of radial fluctuations of our system. We can also use (3.23) to calculate the possible mass of the radion. This is basically given by the usual formula: $\frac{\partial^2 V}{\partial t^2}$, which can be explicitly determined for our case. In principle this can be a large quantity, because of the arguments given above.

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21 We thank G. Cardoso, G. Curio, G. Dall’Agata and D. Luest for correspondence on this issue. See also [30].
22 We thank S. Kachru for discussions on this aspect.
6. Discussions

In this paper we have shown, that the perturbative superpotential for the heterotic string theory compactified on a non-Kähler complex threefold computed in [16] and [17], contains an additional term, if the effect of non-abelian gauge fields is taken into account. To the order that we took in the anomaly equation, our proposal for the complete perturbative superpotential for heterotic theory compactified on non-Kähler manifold $\mathcal{M}_6$, when we consider non-trivial warping and also possible geometric terms in the complex three-form $G$, is given by (ignoring the warp factors)

$$ W = \int_{\mathcal{M}_6} \left[ \mathcal{H} + i dJ \right] \wedge \Omega + \int_{\mathcal{M}_6} F \wedge J \wedge J. \quad (6.1) $$

This form of superpotential\(^2\) presumably survives to all orders in $\alpha'$ when the full iterative solution is found. Here we give a brief sketch of the situation when we incorporate all the possible effects. It is easy to show that the generic form of the three-form will now be given by the expression

$$ G = (a \, \mathcal{H} + \ast_6 A) + i \, (dJ + B), \quad (6.2) $$

where $A$ and $B$ are generic functions of $\omega_\omega$, the torsion-free spin-connection, and $f$ defined earlier. In this form we expect $G$ to be anomaly free and gauge invariant, therefore, upto possible gauge invariant terms, this should solve the anomaly equation. For the case discussed above, i.e. when we considered to the first order in $\alpha'$ in the anomaly equation, we had a cubic equation where it was shown explicitly that $A = B = 0$ and therefore $G$ was given as

$$ G = a \, \mathcal{H} + i \, dJ, \quad (6.3) $$

ignoring the warp factors. Now in the presence of $A$ and $B$ the ISD condition on our background will imply (ignoring possible constants)

$$ \mathcal{H} = \ast_6 dJ + \ast_6 (B - A). \quad (6.4) $$

There could be two possibilities now: (a) The torsional constraint derived earlier and mentioned above is invariant to all orders in $\alpha'$. In that case we expect $A = B$; or (b) The torsional constraint receives correction. In that case the corrections would be proportional

\(^2\) The above form of the superpotential (without the gauge field contribution) has also been derived recently using an alternative method in an interesting paper [30].
to $\ast_6 (B - A)$. Now, there are ample evidences that suggest that the former is true and therefore $A$ should be equal to $B$ upto possible gauge invariant terms. Furthermore if we also demand that

$$A = B = i \ast_6 A$$

then both the torsional equation and the superpotential (6.1) will be exact to all orders in $\alpha'$. More details on this will appear in [27].

The superpotential (6.1) is a generalization of the superpotential for the heterotic string theory compactified on a Calabi-Yau threefold, first described in [23] and recently in more detail in [31] and [23]. An important consequence is, that due to the presence of this potential all the complex structure moduli and some Kähler structure moduli get frozen. Furthermore, we have shown, that the torsional constraint $\mathcal{H} = i(\partial - \bar{\partial})J$ first found in [5], [6] and [9] can be obtained from this superpotential, in a similar way, as the superpotential of [10] reproduces the supersymmetry constraints derived in [4]. This torsional constraint implies, that the overall size of the internal manifold is fixed. Indeed, the previous formula is not invariant under a rescaling of the Kähler form $J$, as the left hand side is non-zero and frozen to a specific value. A direct computation of the scalar potential for the radial modulus shows, that this potential does have a minimum. We have given an estimate for the value of the radius in terms of the density of the $\mathcal{H}$ flux.

There are many interesting directions for future research. Let us just mention a few. Notice, that the non-Kähler manifolds discussed in this paper all have a vanishing Euler characteristics. It will be interesting to construct non-Kähler complex manifolds with non-zero Euler characteristics. For this generalization, we should start with a manifold, which looks like $K3 \times Z$, where $Z$ is a two-dimensional manifold with non-zero Euler characteristics on the Type IIB side. This is the minimal requirement. Of course, we can even get a generic six-dimensional manifold $X$, which should then have the following properties in the absence of fluxes: (a) compact and complex with non-zero Euler characteristics, (b) there exists a four-fold, which is a non-trivial $T^2$ fibration over $X$ and most importantly (c) should have an orientifold setup in the Type IIB framework. More details on this will be addressed in a future publication [27].

$^{24}$ Alternative framework for freezing some of the moduli using asymmetric orientifolds or duality twists have been discussed in [32]. It will be interesting to find the connection between flux-induced stabilization and these techniques.
Another direction for research in the future is the following. It has been shown some time ago in [25], that there are no perturbative corrections to the superpotential for compactifications of the heterotic string to four dimensions, but nevertheless there can be non-perturbative corrections. For the compactifications considered herein, there are non-perturbative corrections coming from the dilaton and it would be interesting to compute their explicit form. More concretely, the non-perturbative effect, that is directly responsible for the case at hand is gaugino condensation, which has been studied in [33], [34] and [35] for compactifications of the heterotic string on a Calabi-Yau threefold. In fact, the key observation has already been made in the corresponding Type IIB framework in the presence of fluxes in [37], where the form of the superpotential in the presence of non-perturbative effects has been presented. We expect a similar picture emerges in the heterotic theory compactified on non-Kähler complex manifolds. A detailed discussion will be presented in [27], so we will be brief here.

As observed in [33] for ordinary Calabi-Yau compactifications, the gluino bilinear term $\text{tr} \, \bar{\chi} \Gamma_{\mu\nu\rho} \chi$ appears in the lagrangian together with the three-from $H_{\mu\nu\rho}$ as a perfect square. If we denote the gluino condensate as $\kappa_{\mu\nu\rho}$, then the non-perturbative contribution to the superpotential is expected to be

$$W \sim \int \kappa \, e^{i\alpha + \beta f(\phi)} \wedge \Omega,$$

where $\beta$ may depend on other fields but not on the dilaton $\phi$ and $f(\phi)$ is some exponential function of $\phi$. As in [33] we have kept a phase $e^{i\alpha}$ (see [33] for more details on this). The above potential will break supersymmetry, because the gluino condensate does. Some details of this analysis have been discussed in [33] and [35]. In particular, it was shown, that some combination of the ten-dimensional dilaton and the radial modulus is fixed by this potential. It will be interesting to apply this mechanism to the examples studied in this paper, which have a fixed radius of compactification in order to obtain a model with both the radius and the dilaton fixed in terms of the expectation values of the $H$ flux and the gluino condensate $\bar{\chi} \Gamma_{\mu\nu\rho} \chi$. This model with stabilized radius and dilaton could be very useful to study cosmological scenarios, especially inflation. The T-dual version of this model (in the Type IIB theory) has been shown to give an interesting inflationary model.

A more detailed study of the potential for the heterotic string compactified on a Calabi-Yau three-fold, taking into account gaugino condensation will appear in [36]. We thank the authors of this paper for informing us about their results prior to publication.
It is plausible, that we can use the same setup, but now for the non-Kähler manifolds considered herein, to construct a cosmological scenario. We would then have constructed a rigid model, that closely simulates some realistic phenomena of nature. If realized, this would be a major achievement.

Note Added: Recently there appeared an interesting paper which discusses the origin of the superpotential (6.1) from an alternative point of view. This paper also discusses the possibility of non-existence of a supergravity description directly in the heterotic theory, which we perfectly agree because of the inherent topology change. We also corrected some erroneous statements regarding supergravity description.

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26 We have been informed that some related work on inflationary scenario which extends the work of [37] and [39] is currently being pursued [40].
References

[1] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, “Vacuum Configurations For Superstrings,” Nucl. Phys. B 258, 46 (1985); A. Strominger and E. Witten, “New Manifolds For Superstring Compactification,” Commun. Math. Phys. 101, 341 (1985).

[2] M. R. Douglas, “The Statistics of String/M Theory Vacua”, hep-th/0303194.

[3] M. Dine and N. Seiberg, “Couplings and Scales in Superstring Models”, Phys. Rev. Lett. 55, 366 (1985).

[4] K. Becker and M. Becker, “M-Theory on Eight-Manifolds,”, Nucl. Phys. B477 (1996) 155, hep-th/9605053.

[5] A. Strominger, “Superstrings With Torsion”, Nucl. Phys. B274 (1986) 253.

[6] C. M. Hull, “Superstring Compactifications with Torsion and Space-Time Supersymmetry,” In Turin 1985, Proceedings, Superunification and Extra Dimensions, 347-375, 29p; “Sigma Model Beta Functions and String Compactifications,” Nucl. Phys. B 267, 266 (1986); “Compactifications of the Heterotic Superstring,” Phys. Lett. B 178, 357 (1986); “Lectures on Nonlinear Sigma Models and Strings,” Lectures given at Super Field Theories Workshop, Vancouver, Canada, Jul 25 - Aug 6, 1986.

[7] C. M. Hull and E. Witten, “Supersymmetric Sigma Models and the Heterotic String,” Phys. Lett. B 160, 398 (1985); A. Sen, “Local Gauge And Lorentz Invariance Of The Heterotic String Theory,” Phys. Lett. B 166, 300 (1986); “The Heterotic String In Arbitrary Background Field,” Phys. Rev. D 32, 2102 (1985); “Equations Of Motion For The Heterotic String Theory From The Conformal Invariance Of The Sigma Model,” Phys. Rev. Lett. 55, 1846 (1985).

[8] S. J. Gates, “Superspace Formulation Of New Nonlinear Sigma Models,” Nucl. Phys. B 238, 349 (1984); S. J. Gates, C. M. Hull and M. Rocek, “Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models,” Nucl. Phys. B 248, 157 (1984); S. J. Gates, S. Gukov and E. Witten, “Two-dimensional Supergravity Theories from Calabi-Yau Four-folds,” Nucl. Phys. B 584, 109 (2000), hep-th/0005120.

[9] B. de Wit, D. J. Smit and N. D. Hari Dass, “Residual Supersymmetry Of Compactified D = 10 Supergravity”, Nucl. Phys. B283 (1987) 165 (1987); N. D. Hari Dass, “A no-go theorem for de Sitter compactifications?,” Mod. Phys. Lett. A 17, 1001 (2002), hep-th/0205056.

[10] S. Gukov, C. Vafa and E. Witten, “CFT’s from Calabi-Yau Four-folds,” Nucl. Phys. B 584, 69 (2000) [Erratum-ibid. B 608, 477 (2001)], hep-th/9906070.

[11] K. Dasgupta, G. Rajesh and S. Sethi, “M theory, orientifolds and G-flux,” JHEP 9908, 023 (1999), hep-th/9908088.

[12] S. Giddings, S. Kachru and J. Polchinski, “Hierarchies From Fluxes in String Compactifications”, hep-th/0105097; S. Kachru, M. B. Schulz and S. Trivedi, “Moduli Stabilization from Fluxes in a Simple IIB Orientifold”, hep-th/0201028. A. R. Frey and
J. Polchinski, “$N = 3$ Warped Compactifications”, Phys. Rev. D65 (2002) 126009, hep-th/0201029.

[13] K. Becker and K. Dasgupta, “Heterotic Strings with Torsion,” hep-th/0209077.

[14] S. Kachru, M. B. Schulz, P. K. Tripathy and S. P. Trivedi, “New Supersymmetric String Compactifications,” hep-th/0211182.

[15] S. Gurrieri and A. Micu, “Type IIB theory on half-flat manifolds,” hep-th/0212278.

[16] P. K. Tripathy and S. P. Trivedi, Compactifications with Flux on K3 and Tori,” hep-th/0301139.

[17] K. Becker, M. Becker, K. Dasgupta and P. S. Green, “Compactifications of heterotic theory on non-Kähler complex manifolds. I,” hep-th/0301161.

[18] G. L. Cardoso, G. Curio, G. Dall’Agata, D. Lust, P. Manousselis and G. Zoupanos, “Non-Kähler String Backgrounds and their Five Torsion Classes,” hep-th/0211118.

[19] S. Gurrieri, J. Louis, A. Micu and D. Waldram, “Mirror Symmetry in Generalized Calabi-Yau Compactifications,” hep-th/0211102.

[20] E. Goldstein and S. Prokushkin, “Geometric Model for Complex non-Kähler Manifolds with SU(3) Structure,” hep-th/0212307.

[21] K. Becker, M. Becker, M. Haack and J. Louis, “Supersymmetry Breaking and α’-Corrections to Flux Induced Potentials”, JHEP 0206 (2002) 060, hep-th/0204254.

[22] C. Vafa, “Evidence for F-Theory,” Nucl. Phys. B 469, 403 (1996), hep-th/9602022.

[23] M. Becker and D. Constantin, “A Note on Flux Induced Superpotentials in String Theory “, hep-th/0210131.

[24] J. Polchinski, “String Theory. Vol. 2: Superstring Theory And Beyond”.

[25] E. Witten, “New Issues in Manifolds of SU(3) Holonomy, Nucl. Phys. B 268 (1986) 79.

[26] J. Bagger and J. Wess, “Supersymmetry and Supergravity,” Princeton University Press.

[27] K. Becker, M. Becker, K. Dasgupta, E. Goldstein, P. S. Green and S. Prokushkin “Work in Progress.”

[28] M. B. Green and J. H. Schwarz, “Superstring Interactions,” Nucl. Phys. B 218, 43 (1983).

[29] J. P. Gauntlett, N. w. Kim, D. Martelli and D. Waldram, “Fivebranes wrapped on SLAG three-cycles and related geometry,” JHEP 0111, 018 (2001) hep-th/0110034.

[30] J. P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, “G-structures and wrapped NS5-branes,” hep-th/0205050.

[31] J. P. Gauntlett, D. Martelli and D. Waldram, “Superstrings with intrinsic torsion,” hep-th/0302158.
[30] G. L. Cardoso, G. Curio, G. Dall’Agata and D. Luest, “BPS Action And Superpotential For Heterotic String Compactifications With Fluxes”, hep-th/0306088.

[31] K. Behrndt and S. Gukov, “Domain Walls and Superpotentials from M theory on Calabi-Yau Three Folds”, Nucl. Phys. B 580 (2000) 225, hep-th/0001082.

[32] E. Silverstein, “(A)dS backgrounds from asymmetric orientifolds,” hep-th/0106209; S. Hellerman, J. McGreevy and B. Williams, “Geometric constructions of nongeometric string theories,” [hep-th/0208174]; A. Dabholkar and C. Hull, “Duality twists, orbifolds, and fluxes,” hep-th/0210209.

[33] M. Dine, R. Rohm, N. Seiberg and E. Witten, “Gluino Condensation in Superstring Models”, Phys. Lett. B156, 55 (1985).

[34] R. Rohm and E. Witten, “The Antisymmetric Tensor Field In Superstring Theory,” Annals Phys. 170, 454 (1986).

[35] K. i. Maeda, “Attractor In A Superstring Model: The Einstein Theory, The Friedmann Universe And Inflation,” Phys. Rev. D 35, 471 (1987).

[36] S. Gukov, S. Kachru, X. Liu, L. McAllister, To Appear.

[37] S. Kachru, R. Kallosh, A. Linde, S. Trivedi, “de-Sitter Vacua in String Theory,” hep-th/0312420.

[38] R. Kallosh, “N = 2 Supersymmetry and de Sitter Space,” hep-th/0109168; C. Herdeiro, S. Hirano and R. Kallosh, “String Theory and Hybrid Inflation / Acceleration,” JHEP 0112, 027 (2001), hep-th/0110271.

[39] K. Dasgupta, C. Herdeiro, S. Hirano and R. Kallosh, “D3/D7 Inflationary Model and M-theory,” Phys. Rev. D 65, 126002 (2002), hep-th/0203019; K. Dasgupta, K. h. Oh, J. Park and R. Tatar, “Geometric Transition Versus Cascading Solution,” JHEP 0201, 031 (2002), hep-th/0110050.

[40] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAllister, S. Trivedi, Work in Progress.