COMPOSITIONS AND TENSOR PRODUCTS OF LINEAR MAPS BETWEEN MATRIX ALGEBRAS

SEUNG-HYEOK KYE

Abstract. In this semi-expository paper, we first explain key notions from current quantum information theory and criteria for them in a coherent way. These include separability/entanglement, Schmidt numbers of bi-partite states and block-positivity, together with various kinds of positive maps between matrix algebras like entanglement breaking maps, \( k \)-superpositive maps, completely positive maps, \( k \)-positive maps. We will begin with concrete examples of elementary positive maps given by \( x \mapsto s^* x s \), and use Choi matrices and duality to explain all the notions mentioned above. We also show that the Choi matrix can be defined free from coordinates. The above notions of positive maps give rise to mapping cones, whose dual cones are characterized in terms of compositions or tensor products of linear maps. Through the discussion, we exhibit an identity which connects tensor products and compositions of linear maps between matrix algebras through the Choi matrices. Using this identity, we show that the description of the dual cone with tensor products is possible only when the involving cones are mapping cones, and recover various known criteria with ampliation for the notions mentioned above. As another applications of the identity, we construct various mapping cones arising from ampliation and factorization, and provide several equivalent statements to PPT (positive partial transpose) square conjecture in terms of tensor products.

1. Introduction

Entanglement has been considered as one of the most important notions in current quantum information theory since its mathematical definition was given by Werner \[58\] for mixed states, which can be expressed as sums of tensor products of two matrices. Horodecki’s separability criterion \[25\] tells us that positive linear maps between matrices are essential to distinguish entanglement from separable states. Among positive maps, several classes like complete positive, \( k \)-positive and \( k \)-superpositive maps play important roles.

The notions of complete positivity and \( k \)-positivity had emerged in Stinespring’s representation theorem \[48\] in 1955, and examples distinguishing different \( k \)-positivities were found by Choi \[10\] and Tomiyama \[57\]. Motivated by Woronowicz’ work \[59\] utilizing the usual duality between mapping spaces and tensor products, the author \[18\] found the dual objects of \( k \)-positivity in the tensor products of matrix algebras, which are just Schmidt numbers in the current terminology. The notion of Schmidt

2020 Mathematics Subject Classification. 15A30, 81P15, 46L05, 46L07.
Key words and phrases. composition, tensor product, mapping cones, duality, Choi matrices, \( k \)-positive maps, \( k \)-superpositive maps, Schmidt numbers, entanglement, ampliation, factorization, PPT square conjecture.

partially supported by NRF-2020R1A2C1A01004587, Korea.
numbers had been introduced by Terhal and Horodecki [56] who showed that these numbers can be determined by $k$-positive maps. Bi-partite states of Schmidt number one are just separable states, and states which are not separable are called entangled. The counterparts of separable states in the mapping space through Choi matrices were introduced with various motivations [1, 26, 45] under the names; superpositive maps, entanglement breaking maps. This notion has been extended [14, 47] to $k$-superpositivity which is just the counterpart of Schmidt numbers at most $k$.

The first purpose of this note is to explain the above notions and various known criteria for them in a single framework. We will begin with $k$-superpositive maps which are nonnegative sums of the maps $x \mapsto s^*xs$ with matrices $s$ whose ranks are at most $k$. We use the duality and Choi matrices to define $k$-positivity and Schmidt number $k$, respectively. In this way, the original definition of $k$-positivity through ampliation is naturally recovered, and the Choi’s correspondence [11] between completely positivity of $\phi$ and the positivity of the Choi matrix $C_\phi$ is also obtained. The convex cones $\mathbb{SP}_k$ and $\mathbb{P}_k$ consisting of $k$-superpositive and $k$-positive maps, respectively, are mapping cones as well as convex cones. The notion of mapping cones was introduced by Størmer [51] to study the extension problem of positive maps. It is known [46, 52] that the dual cone of a mapping cone can be described in terms of composition and tensor product of linear maps.

It was shown in [19] that the description of the dual cones through composition is possible only when the involving cones are one-sided mapping cones. We will see that the description of dual cones through tensor products is possible only when the involving cones are (two-sided) mapping cones. To see this, we will exhibit the identity

\[(\phi_1 \otimes \phi_2)(C_\sigma) = C_{\phi_2 \circ \sigma \circ \phi_1^*}\]

which connects composition and tensor product of linear maps between matrix algebras through the Choi matrices. This identity will be used to recover various criteria through ampliation in a single framework. These include the characterizations of decomposable maps [50] and $k$-positive maps [18], together with separability criteria [25], characterizations of Schmidt numbers [56], entanglement breaking maps [26] and $k$-superpositive maps [14].

After we explain Choi matrices and duality with a bilinear pairing, we introduce basic notions including $k$-superpositivity, complete positivity, $k$-positivity for linear maps, together with Schmidt numbers and $k$-blockpositivity of tensor product of matrices in the next section. We also show that Choi matrices can be defined independent of standard matrix units to retain all the correspondences between linear maps and tensor products. We give in Section 3 a simple proof of the identity (1) and characterize mapping cones in terms of tensor products. This provides us various characterizations.
of $k$-positivity in terms of tensor products. In this section, we also introduce the mapping cones of decomposable maps and PPT maps. Section 4 will be devoted to recover various known criteria through ampliation using the identity (1).

In the remainder of the paper, we exhibit two more applications of the identity (1). Motivated by recent works [13] and [17], we suggest in Section 5 further constructions of mapping cones through ampliation and factorization. Those mapping cones are dual to each other, which may be considered as natural extensions of the duality between $k$-positivity and $k$-superpositivity. The identity (1) will be useful to deal with such mapping cones. We give one more application of the identity (1) in Section 6 to provide several equivalent claims to the PPT square conjecture in terms of tensor products. We close the paper with some questions in the final section.

This work is strongly motivated by the joint paper [19] with Erling Størmer and Mark Girard, and we will follow the notations in [19]. For examples, $M_A$ denotes the algebra of all complex matrices acting on the Hilbert space $\mathbb{C}^a$ with the dimension $a$. The author is grateful to Erling Størmer and Mark Girard for encouragement and comments on the draft, respectively. This is a revised version of the paper posted under the same title, which is rewritten to be self-contained for general audiences.

2. Convex cones in quantum information theory

Two main tools to explain various notions from quantum information theory are bilinear pairing to define the dual objects and the Choi matrices connecting linear maps and tensor products of matrices. For a given linear map $\phi : M_A \to M_B$, the Choi matrix $C_\phi \in M_A \otimes M_B$ is defined by

$$C_\phi = \sum_{i,j} |i\rangle\langle j| \otimes \phi(|i\rangle\langle j|) \in M_A \otimes M_B.$$ 

It is clear that $\phi \mapsto C_\phi$ is a linear isomorphism from the space $L(M_A, M_B)$ of all linear maps to $M_A \otimes M_B$. We denote by $H(M_A, M_B)$ the real space of all Hermiticity preserving maps.

With the bilinear pairing $\langle a, b \rangle = \text{Tr}(ab^t)$ between matrices, it is easily seen that the identity $\langle a \otimes b, C_\phi \rangle = \langle b, \phi(a) \rangle$ holds for $a \in M_A$, $b \in M_B$ and $\phi \in L(M_A, M_B)$. This is nothing but the usual bilinear pairing between dual of the tensor products and mapping spaces, and so, it is natural to define [18, 59] the bilinear pairing

$$\langle a \otimes b, \phi \rangle := \langle b, \phi(a) \rangle = \langle a \otimes b, C_\phi \rangle$$

between real space $(M_A \otimes M_B)^b$ of Hermitian matrices and the mapping space $H(M_A, M_B)$. We also define [46, 19] the bilinear pairing between mapping spaces by

$$\langle \phi, \psi \rangle := \langle C_\phi, C_\psi \rangle = \text{Tr}(C_\phi C_\psi^t)$$

for $\phi, \psi \in H(M_A, M_B)$. Suppose that $X$ and $Y$ are finite dimensional real vector space with a bilinear pairing $\langle \ , \ \rangle$ between them. For a subset $K \subset X$, the dual cone $K^\circ$ in
Y is defined by the set of all \( y \in Y \) satisfying \( \langle x, y \rangle \geq 0 \) for every \( x \in K \). It is easy to see that \( K^\circ \) is the smallest closed convex cone containing \( K \).

For a linear map \( \phi : M_A \to M_B \), the dual map \( \phi^* : M_B \to M_A \) is defined by
\[
\langle \phi^*(b), a \rangle = \langle b, \phi(a) \rangle, \quad a \in M_A, \ b \in M_B.
\]

It is easily seen that \( C_{\phi^*} \in M_B \otimes M_A \) is the flip of \( C_{\phi} \in M_A \otimes M_B \). It is also easily seen that the following identities
\[
(\phi \circ \psi)^* = \psi^* \circ \phi^*, \quad \langle \phi, \psi \rangle = \langle \phi^*, \psi^* \rangle,
\]
hold whenever the compositions and the bilinear pairings are defined.

We begin with the elementary positive map \( \text{Ad}_s : M_A \to M_B \) defined by
\[
\text{Ad}_s(x) = s^* x s, \quad x \in M_A,
\]
for an \( a \times b \) matrix \( s \in M_{A,B} \), or equivalently, a linear map \( s : \mathbb{C}^B \to \mathbb{C}^A \). It is well known \([49]\) that the map \( \text{Ad}_s \) generates an extreme ray of the convex cone of all positive maps from \( M_A \) to \( M_B \). It was also shown \([37]\) to generate an exposed ray. For a given \( k = 1, 2, \ldots \), we denote by \( \mathbb{S}P_k \) the convex cone generated by \( \text{Ad}_s \) with rank of \( s \leq k \), and call a map in \( \mathbb{S}P_k \) \( k \)-superpositive \([47]\). It is clear that the sequence \( \{\mathbb{S}P_k : k = 1, 2, \ldots \} \) is increasing, and their union coincides with the convex cone \( \mathbb{S}P_{\min\{a,b\}} \).

For a matrix \( s = \sum_{i,j} s_{ij} |i\rangle \langle j| \in M_{A,B} \), we define the vector \( \langle \bar{s} \rangle \) by
\[
\langle \bar{s} \rangle = \sum_{i,j} s_{ij} \langle i| \langle j| \in \mathbb{C}^A \otimes \mathbb{C}^B.
\]

Then it is straightforward to check the identity
\[
C_{\text{Ad}_s} = |\bar{s}\rangle \langle \bar{s}|
\]
holds. Indeed, we have
\[
C_{\text{Ad}_s} = \sum_{p,q} \langle p| \langle q| \otimes \left( \sum_{k,\ell} \bar{s}_{\ell k} |k\rangle \langle \ell| \right) \langle p| \langle q| \left( \sum_{i,j} s_{ij} \langle i| \langle j| \right)
\]
\[
= \left( \sum_{k,\ell} \sum_{p} \bar{s}_{\ell k} \langle p| \langle \ell| \right) \langle p| \left( \sum_{i,j} s_{ij} \sum_{q} \langle q| \langle i| \langle j| \right)
\]
\[
= \left( \sum_{k,\ell} \bar{s}_{\ell k} |k\rangle \langle \ell| \right) \left( \sum_{i,j} s_{ij} \langle i| \langle j| \right) = |ar{s}\rangle \langle \bar{s}|.
\]

We write \( C_K := \{C_\phi \in M_A \otimes M_B : \phi \in K\} \) for \( K \subset L(M_A, M_B) \). Because the rank of \( s \in M_{A,B} \) coincides with the Schmidt rank of \( \langle \bar{s} \rangle \in \mathbb{C}^A \otimes \mathbb{C}^B \), we see that the convex cone
\[
S_k := C_{\mathbb{S}P_k} \subset M_A \otimes M_B
\]
consists of all (unnormalized) states in the convex cone \( \mathcal{P}_{AB} := (M_A \otimes M_B)^+ \) which are convex sums of rank one projections onto vectors whose Schmidt ranks are at most \( k \). A state in \( S_k \setminus S_{k-1} \) is called to have Schmidt number \( k \). It is clear that \( S_{\min\{a,b\}} = \mathcal{P}_{AB} \).
For any $|\zeta\rangle = \sum_{i=1}^k |i\rangle|y_i\rangle \in \mathbb{C}^A \otimes \mathbb{C}^B$ whose Schmidt rank is at most $k$, we have $|\zeta\rangle\langle\zeta| = \sum_{i,j=1}^k |x_i\rangle\langle x_j| \otimes |y_i\rangle\langle y_j|$, and so, we have

$$
\langle|\zeta\rangle\langle\zeta|, \phi\rangle = \sum_{i,j=1}^k \langle|y_i\rangle\langle y_j|, \phi(|x_i\rangle\langle x_j|)\rangle
= \left\langle \sum_{i,j=1}^k |i\rangle\langle j| \otimes |y_i\rangle\langle y_j|, \sum_{i,j=1}^k |i\rangle\langle j| \otimes \phi(|x_i\rangle\langle x_j|) \right\rangle.
$$

Putting $|\xi\rangle = \sum_{i=1}^k |i\rangle|x_i\rangle \in \mathbb{C}^A \otimes \mathbb{C}^B$, we see that

$$
\langle|\xi\rangle\langle\xi|, \phi\rangle = \langle \eta\rangle\langle\eta|, (\text{id}_k \otimes \phi)(|\xi\rangle\langle\xi|).
$$

Therefore, we see that a map $\phi : M_A \to M_B$ belongs to $\mathcal{S}_k^\circ$ with respect to the bilinear pairing (2), or equivalently belongs to $\mathbb{S} \mathbb{P}_k^\circ$ with respect to (3) if and only if the map $\text{id}_k \otimes \phi$ from $M_k \otimes M_A$ into $M_k \otimes M_B$ is positive. We call such a map $\phi$ $k$-positive, and denote by $\mathbb{P}_k$ the convex cone of all $k$-positive maps. In short, we have seen that two mapping spaces $\mathbb{S} \mathbb{P}_k$ and $\mathbb{P}_k$ are dual to each other, that is, we have $\mathbb{S} \mathbb{P}_k^\circ = \mathbb{P}_k$ and $\mathbb{P}_k^\circ = \mathbb{S} \mathbb{P}_k$. The duality $\mathbb{P}_1 = \mathcal{S}_1^\circ$ between mapping spaces and tensor products is already implicit in [59], and $\mathbb{P}_k = \mathcal{S}_k^\circ$ was found in [18]. On the other hand, $\mathcal{S}_1 = \mathbb{P}_1^\circ$ was shown in [25] to get a criterion for separability.

Hermitian matrices in the convex cone $\mathcal{B} \mathbb{P}_k := C_{\mathbb{P}_k} \subset M_A \otimes M_B$ are called $k$-blockpositive. Then we see that $\mathcal{B} \mathbb{P}_k$ and $\mathbb{S} \mathbb{P}_k$ are dual to each other with respect to (2), and it follows that $\rho \in M_A \otimes M_B$ is $k$-blockpositive if and if $\langle |\xi\rangle \rho |\xi\rangle \geq 0$ for every $|\xi\rangle \in \mathbb{C}^A \otimes \mathbb{C}^B$ with Schmidt rank at most $k$. Because $\mathbb{S} \mathbb{P}_k$ is increasing when $k$ increases, $\mathcal{S}_k$ and $\mathcal{B} \mathbb{P}_k$ are increasing and decreasing, respectively. Because $\mathcal{S}_{\min(a,b)} = \mathcal{P}_{AB}$ is self-dual, we see that $\phi$ is $\min\{a, b\}$-superpositive if and only if it is $\min\{a, b\}$-positive if and only if $C_\phi$ is positive. A linear map $\phi$ is called completely positive if $\text{id}_k \otimes \phi$ is positive for every $k = 1, 2, \ldots$, and we denote by $C_\mathbb{P}_{AB}$ the convex cone of all completely positive maps from $M_A$ to $M_B$. Because $\text{id}_k \otimes \text{Ad}_s = \text{Ad}_k \otimes s$ is positive, we have the following:

**Theorem 2.1.** [11] For a linear map $\phi : M_A \to M_B$, the following are equivalent:

(i) $\phi$ is completely positive,

(ii) $\phi$ is $\min\{a, b\}$-positive,

(iii) $\phi$ is $\min\{a, b\}$-superpositive,

(iv) $C_\phi$ is positive.

Statement (iii) of Theorem 2.1 tells us that every completely positive map is a nonnegative sum of $\text{Ad}_s$’s. This is called a Kraus decomposition [31]. We summarize our discussion in the following diagram:
Bi-partite states in $S_1$ are called separable, and a state in $P_{AB}$ is called entangled when it is not separable.

We note that Choi matrices play central roles throughout the discussion. In the remainder of this section, we show that Choi matrices may be defined independent of the choice of coordinate systems. The Choi matrix of a linear map between matrix algebras has been considered in 1967 by de Pillis [16] who showed that a linear map $\phi$ preserves Hermiticity if and only if the Choi matrix $C_\phi$ is self-adjoint. The Choi matrices of positive maps and completely positive maps have been considered by Jamiołkowski [29] and Choi [11], respectively. The correspondence $\phi \mapsto C_\phi$ is now called the Jamiołkowski–Choi isomorphism. This isomorphism has been extended for various infinite dimensional cases [23, 24, 35, 36, 54], and multi-linear maps between matrix algebras [20, 21, 33]. It should be noted that the correspondences in the diagram (8) may be broken [40] when we change the standard basis $e_{ij}$ by another basis of matrix algebras in the definition $C_\phi = \sum e_{ij} \otimes \phi(e_{ij})$.

Now, we note that the Choi matrix $C_\phi \in M_A \otimes M_B$ of a linear map $\phi : M_A \to M_B$ is given by

$$C_\phi = \sum_{i,j} e_{ij} \otimes \phi(e_{ij}) = \sum_{i,j} (\text{id}_A \otimes \phi)(|i\rangle\langle j| \otimes |i\rangle\langle j|) = (\text{id}_A \otimes \phi)(|\omega\rangle\langle \omega|),$$

with $|\omega\rangle = \sum_i |i\rangle_i |i\rangle \in \mathbb{C}^A \otimes \mathbb{C}^A$. In order to replace $|\omega\rangle$ by another vector, we fix a matrix $s = \sum_{i,j} s_{ij} |i\rangle \langle j| \in M_A$ with the maximum rank, and replace $|\omega\rangle$ by $|\tilde{s}\rangle \in \mathbb{C}^A \otimes \mathbb{C}^A$ given by (5). In other words, we define

$$C^s_\phi = (\text{id}_A \otimes \phi)(|\tilde{s}\rangle\langle \tilde{s}|) \in M_A \otimes M_B,$$

for a linear map $\phi$ from $M_A$ to $M_B$. Note that $|\tilde{s}\rangle$ has the maximal Schmidt rank. When $s$ is the identity matrix, this gives rise to the usual Choi matrix. We note that $C^s_\phi = (\text{id}_A \otimes \phi)(C_{\text{Ad}_s})$ by (6), and so we get the following identity

$$C^s_\phi = (\text{id}_A \otimes \phi)(C_{\text{Ad}_s}) = (\text{id}_A \otimes \phi) \circ (\text{id}_A \circ \text{Ad}_s)(|\omega\rangle\langle \omega|) = C_{\phi \circ \text{Ad}_s}.$$

We also define the bilinear pairing on the real vector space $H(M_A, M_B)$ by

$$\langle \phi, \psi \rangle^s := \langle C^s_\phi, C^s_\psi \rangle,$$

for $\phi, \psi \in H(M_A, M_B)$, following the current definition of Choi matrices.

The statement (i) of the following Theorem 2.2 tells us that the dual object of the mapping cone $\mathbb{S}P_k$ is independent of the choice the matrix $s$. We also see that
the notions of Schmidt number $k$ and $k$-blockpositivity, as the Choi matrices of $k$-superpositive maps and $k$-positive maps, respectively, do not depend on the choice of $s \in M_A$.

**Theorem 2.2.** For $k = 1, 2, \ldots, \phi \in H(M_A, M_B)$ and $s \in M_A$ with the maximal rank, we have the following:

(i) $\langle \phi, \psi \rangle \geq 0$ for every $\psi \in S^{\phi}_k$ if and only if $\langle \phi, \psi \rangle^s \geq 0$ for every $\psi \in S^{\phi}_k$,

(ii) $\{C_{\phi} : \phi \in S^{\phi}_k\} = \{C_{\phi}^s : \phi \in S^{\phi}_k\}$,

(iii) $\{C_{\phi} : \phi \in S^{\phi}_k\} = \{C_{\phi}^s : \phi \in S^{\phi}_k\}$.

**Proof.** We first note that $Ad_s^s = Ad_s$. In fact, we have

$$\langle Ad_s^s(y), x \rangle = \langle y, Ad_s(x) \rangle = \langle y, s^*sx \rangle = \langle s^*ys^t, x \rangle = \langle Ad_s(y), x \rangle,$$

for $x \in M_A$ and $y \in M_B$. It follows that

$$\langle \phi, \psi \rangle^s = \langle \phi \circ Ad_s, \psi \circ Ad_s \rangle = \langle \phi, \psi \circ Ad_s \circ Ad_s^t \rangle = \langle \phi, \psi \circ Ad_s^t \rangle,$$

by (10). Therefore, we see that $\langle \phi, \psi \rangle^s \geq 0$ holds for every $\psi \in S^{\phi}_k$ if and only if $\langle \phi, \psi \circ Ad_s^t \rangle \geq 0$ for every $\psi \in S^{\phi}_k$ if and only if $\langle \phi, \sigma \rangle \geq 0$ for every $\sigma \in S^{\phi}_k \circ Ad_s^t$.

Because $s^t$ is nonsingular, we have $S^{\phi}_k = S^{\phi}_k \circ \{Ad_s^t\}$ for $k = 1, 2, \ldots$, and this proves (i). We also have

$$\{C_{\phi}^s : \phi \in S^{\phi}_k\} = \{C_{\phi \circ Ad_s} : \phi \in S^{\phi}_k\} = \{C_{\phi} : \phi \in S^{\phi}_k \circ Ad_s\},$$

by (10). This proves (ii) since $S^{\phi}_k = S^{\phi}_k \circ Ad_s$ by non-singularity of $s$. For (iii), it suffices to show $S^{\phi}_k \circ Ad_s$, or equivalently, $\phi \in S^{\phi}_k$ if and only if $\phi \circ Ad_s^{-1} \in S^{\phi}_k$. Indeed, we have

$$\phi \circ Ad_s^{-1} \in S^{\phi}_k \iff \langle \phi \circ Ad_s^{-1}, \psi \rangle \geq 0 \text{ for every } \psi \in S^{\phi}_k$$

$$\iff \langle \phi \circ Ad_s^{-1}, \psi \circ Ad_s^{-1} \rangle \geq 0 \text{ for every } \psi \in S^{\phi}_k$$

$$\iff \langle \phi, \psi \rangle^{s^{-1}} \geq 0 \text{ for every } \psi \in S^{\phi}_k$$

$$\iff \phi \in S^{\phi}_k,$$

by (i). This completes the proof. \qed

After the author had posted the first version of this paper, he considered more general situations to replace $|\omega\rangle\langle \omega|$ in (9) by arbitrary $\Sigma \in M_A \otimes M_A$, and defined

$$C_\phi^\Sigma = (id_A \otimes \phi)(\Sigma) \in M_A \otimes M_B.$$

It was shown in [34] that the Choi’s correspondence between completely positivity of $\phi$ and positivity of $C_\phi^\Sigma$ is retained if and only if $\Sigma$ is a positive rank one matrix whose range vector has the full Schmidt rank, that is, $\Sigma = |\bar{s}\rangle\langle \bar{s}|$ for a matrix $s$ of full rank.
3. Mapping cones and tensor products of linear maps

A closed convex cone $K$ of positive linear maps in $H(M_A, M_B)$ is called a right mapping cone [19] if $K \circ \mathbb{C}P_{AA} \subset K$ holds, where $K_1 \circ K_2$ denotes the set of all $\phi_1 \circ \phi_2$ with $\phi_i \in K_i$ for $i = 1, 2$. In fact, this is the case if and only if $K \circ \mathbb{C}P_{AA} = K$ holds, since $id_A \in \mathbb{C}P_{AA}$. Left mapping cones are defined similarly. A closed convex cone $K$ is also called a mapping cone when $\mathbb{C}P_{BB} \circ K \circ \mathbb{C}P_{AA} \subset K$ holds. A convex cone is a mapping cone if and only if it is both left and right mapping cone.

Suppose that $K$ is a convex cone with $SP_1 \subset K \subset P_1$. Then we see that $K$ is a right mapping cone if and only if $\langle \phi \circ \sigma, \psi \rangle \geq 0$ holds for every $\phi \in K$, $\sigma \in \mathbb{C}P_{AA}$ and $\psi \in K^\circ$ if and only if $\langle \phi, \psi \circ \sigma^* \rangle \geq 0$ holds for every $\phi \in K$, $\sigma \in \mathbb{C}P_{AA}$ and $\psi \in K^\circ$ if and only if $K^\circ$ is a right mapping cone, since $\mathbb{C}P_{AA}$ is self-dual. Similarly, we also see that $K$ is a right mapping cone if and only if $K^* = \{\phi^* : \phi \in K\}$ is a left mapping cone. It is clear that $SP_k$ is a mapping cone, and so $P_k$ is also a mapping cone.

When $K$ is a mapping cone in $H(M_A, M_A)$ with minor additional assumptions, it was shown in [16 52] that the following statements for a linear map $\phi$ are equivalent:

(MC1) $\phi \in K^\circ$,
(MC2) $\phi \circ \psi$ is completely positive for every $\psi \in K$,
(MC3) $\psi \otimes \phi$ is positive for every $\psi \in K$,
(MC4) $(\psi \otimes \phi)(C_{id}) \geq 0$ for every $\psi \in K$.

We may use the identity (1) to see that the equivalence (MC1) $\iff$ (MC2) implies that $K$ is a left mapping cone. In fact, we see that $\phi \in (\mathbb{C}P_{BB} \circ K)^\circ$ if and only if $\langle \phi, \sigma \circ \psi \rangle \geq 0$ for every $\psi \in K$ and $\sigma \in \mathbb{C}P_{BB}$ if and only if $\langle \phi \circ \psi^*, \sigma \rangle \geq 0$ for every $\psi \in K$ and $\sigma \in \mathbb{C}P_{BB}$ if and only if $\phi \circ \psi^* \in \mathbb{C}P_{BB}$ for every $\psi \in K$, and so we have

$$(\mathbb{C}P_{BB} \circ K)^\circ = \{\phi \in H(M_A, M_B) : \phi \circ \psi^* \in \mathbb{C}P_{BB} \text{ for every } \psi \in K\},$$

for a convex cone $K \subset H(M_A, M_B)$. The dual cone $(K \circ \mathbb{C}P_{AA})^\circ$ can be handled in the same way, and we have the following:

**Theorem 3.1.** [19] Suppose that $K$ is a closed convex cone of positive maps in $H(M_A, M_B)$. Then we have the following:

(i) $K$ is a left mapping cone if and only if

$$K^\circ = \{\phi \in H(M_A, M_B) : \phi \circ \psi^* \in \mathbb{C}P_{BB} \text{ for every } \psi \in K\},$$

(ii) $K$ is a right mapping cone if and only if

$$K^\circ = \{\phi \in H(M_A, M_B) : \psi^* \circ \phi \in \mathbb{C}P_{AA} \text{ for every } \psi \in K\}.$$

In order to investigate the relations between [MC2], [MC3] and [MC4], we need the identity (1) which relates compositions and tensor products of linear maps. We suppose that $\phi_i : M_{A_i} \rightarrow M_{B_i}$ is a linear map between matrix algebras $M_{A_i}$ and $M_{B_i}$, for $i = 1, 2$. Note that every element in the tensor product $M_{A_1} \otimes M_{A_2}$ is expressed as the Choi matrix $C_\sigma$ of a linear map $\sigma : M_{A_1} \rightarrow M_{A_2}$, and the composition $\phi_2 \circ \sigma \circ \phi_1^*$
maps $M_{B_1}$ into $M_{B_2}$. In this circumstance, we will show that the identity (1) holds. See Figure 1.

To prove the identity (1), we take $b_i \in M_{B_i}$ with $i = 1, 2$. Then we have

$$
\langle b_1 \otimes b_2, C_{\phi_2 \circ \sigma \circ \phi_1} \rangle_{B_1 B_2} = \langle b_2, \phi_2(\sigma(\phi_1^*(b_1))) \rangle_{B_2} \\
= \langle \sigma^*(\phi_1^*(b_2)), \phi_1^*(b_1) \rangle_{A_1} \\
= \sum_{i,j} \langle \sigma^*(\phi_1^*(b_2)), e_{i,j}^A \rangle_{A_1} \langle \phi_1^*(b_1), e_{i,j}^A \rangle_{A_1} \\
= \sum_{i,j} \langle b_2, \phi_2(\sigma(e_{i,j}^A)) \rangle_{B_2} \langle b_1, \phi_1(e_{i,j}^A) \rangle_{B_1} \\
= \sum_{i,j} \langle b_1 \otimes b_2, \phi_1(e_{i,j}^A) \otimes \phi_2(\sigma(e_{i,j}^A)) \rangle_{B_1 B_2},
$$

where $\{e_{i,j}^A\}$ denotes the matrix units of $A$. Therefore, it follows that

$$
C_{\phi_2 \circ \sigma \circ \phi_1} = \sum_{i,j} \phi_1(e_{i,j}^A) \otimes \phi_2(\sigma(e_{i,j}^A)) \\
= \sum_{i,j} (\phi_1 \otimes \phi_2)(e_{i,j}^A \otimes \sigma(e_{i,j}^A)) \\
= (\phi_1 \otimes \phi_2)(C_{\sigma}),
$$

as it was required. If we plug $\phi_1 = \id_A$, $\phi_2 = \phi : M_A \to M_B$ and $\sigma = \Ad_x$ in (1), then we recover the identity (11). In case when $A_1 = A_2$, $B_1 = B_2$ and $\sigma : M_A \to M_A$ is the identity map, we have

$$
C_{\phi_2 \circ \phi_1} = (\phi_1 \otimes \phi_2)(C_{\id})
$$

which recovers the relation (12) of [19] and suggests relations between [MC2] and [MC4].

In order to investigate the conditions [MC3] and [MC4], we consider the following diagrams

$$
\begin{array}{ccc}
M_A & \xrightarrow{\phi} & M_B \\
| \tau | & & | \id_A | \\
M_A & \xrightarrow{\psi} & M_B
\end{array}
\quad
\begin{array}{ccc}
M_A & \xrightarrow{\phi} & M_B \\
| \id_A | & & | \id_B | \\
M_B & \xrightarrow{\psi^*} & M_A
\end{array}
\quad
\begin{array}{ccc}
M_B & \xrightarrow{\phi^*} & M_A \\
| \id_B | & & | \id_B | \\
M_B & \xrightarrow{\psi^*} & M_A
\end{array}
$$

in Figure 1. Applying the identity (1), we have the following identities

$$
\langle \sigma^* \circ \psi \circ \tau, \phi \rangle = \langle \psi \circ \tau, \sigma \circ \phi \rangle = \langle \psi \circ \tau \circ \phi^*, \sigma \rangle = \langle \phi \otimes \psi \rangle (C_{\tau}), (C_{\sigma}), \\
\langle \phi, \sigma^* \circ \psi \rangle = \langle \phi^*, \psi^* \circ \sigma \rangle = \langle \psi \circ \phi^*, \sigma \rangle = \langle \phi \otimes \psi \rangle (C_{\id_A}), (C_{\sigma}), \\
\langle \phi, \psi \circ \sigma \rangle = \langle \psi^* \circ \phi, \sigma \rangle = \langle \phi^* \otimes \psi^* \rangle (C_{\id_B}), (C_{\sigma}).
$$

Figure 1. The map $\phi_1 \otimes \phi_2$ sends $C_{\sigma}$ to $C_{\phi_2 \circ \sigma \circ \phi_1^*}$. 
The first line tells us that $\phi \in (\mathbb{CP}_{BB} \circ K \circ \mathbb{CP}_{AA})^\circ$ if and only if $\phi \otimes \psi$ sends positive matrices to positive matrices for every $\psi \in K$, that is, $\phi \otimes \psi$ is a positive map for every $\psi \in K$. In other words, we have

$$\mathbb{CP}_{BB} \circ K \circ \mathbb{CP}_{AA})^\circ = \{\phi \in H(M_A, M_B) : \phi \otimes \psi \text{ is positive for every } \psi \in K\},$$

for a convex cone $K \subset H(M_A, M_B)$. The exactly same argument with the second and third identities may be applied to dual cones $(\mathbb{CP}_{BB} \circ K)^\circ$ and $(K \circ \mathbb{CP}_{AA})^\circ$, to get the following:

$$(\mathbb{CP}_{BB} \circ K)^\circ = \{\phi \in H(M_A, M_B) : (\phi \otimes \psi)(C_{id_A}) \in \mathcal{P}_{BB} \text{ for every } \psi \in K\},$$

$$(K \circ \mathbb{CP}_{AA})^\circ = \{\phi \in H(M_A, M_B) : (\phi^* \otimes \psi^*)(C_{id_B}) \in \mathcal{P}_{AA} \text{ for every } \psi \in K\}.$$ 

Therefore, we have the following:

**Theorem 3.2.** For a closed convex cone $K$ of positive maps from $M_A$ into $M_B$, we have the following:

(i) $K$ is a mapping cone if and only if

$$K^\circ = \{\phi \in H(M_A, M_B) : \phi \otimes \psi \text{ is positive for every } \psi \in K\},$$

(ii) $K$ is a left mapping cone if and only if

$$K^\circ = \{\phi \in H(M_A, M_B) : (\phi \otimes \psi)(C_{id_A}) \in \mathcal{P}_{BB} \text{ for every } \psi \in K\},$$

(iii) $K$ is a right mapping cone if and only if

$$K^\circ = \{\phi \in H(M_A, M_B) : (\phi^* \otimes \psi^*)(C_{id_B}) \in \mathcal{P}_{AA} \text{ for every } \psi \in K\}.$$ 

Therefore, we see that $\phi$ is $k$-positive, that is, $id_k \otimes \phi$ is positive if and only if $\psi \otimes \phi$ is positive for every $\psi \in \mathbb{SP}_k$. Since $id_k$ is a typical example of $\mathbb{SP}_k$, one may suspect if $id_k$ in the definition of $k$-positivity may be replaced by another $k$-superpositive map. When we fix a matrix $s$ with rank $k$, it is easy to see that $Ad_s \otimes \phi$ is positive if and only if $id_k \otimes \phi$ is positive using singular value decomposition of $s$. This is also due to the fact that $\mathbb{SP}_k$ is singly generated as a mapping cone.

**Proposition 3.3.** Let $s \in M_{A,B}$ be of rank $k$. Then we have $(\mathbb{CP} \circ \{Ad_s\} \circ \mathbb{CP})^\circ \subseteq \mathbb{SP}_k$.

**Proof.** We first note $\mathbb{CP} \circ \{Ad_s\} \circ \mathbb{CP} \subseteq \mathbb{SP}_k$, which implies $(\mathbb{CP} \circ \{Ad_s\} \circ \mathbb{CP})^\circ \subseteq \mathbb{SP}_k$. For the reverse inclusion, it suffices to show that $Ad_s \in \mathbb{CP} \circ \{Ad_s\} \circ \mathbb{CP}$ whenever rank of $a = \ell \leq k$, because every map in $\mathbb{SP}_k$ is the sum of such maps. Write rank of $a = \ell \leq k$. By singular value decomposition, we can take $v_1 : \mathbb{C}^k \rightarrow \mathbb{C}^B$ and $v_2 : \mathbb{C}^k \rightarrow \mathbb{C}^A$ such that

$$s = v_2 d_1 v_1^*, \quad v_1^* v_1 = v_2^* v_2 = id_{\mathbb{C}^k},$$

where $d_1$ is a $k \times k$ diagonal matrix with positive real diagonal entries. We also take $v_3 : \mathbb{C}^\ell \rightarrow \mathbb{C}^B$ and $v_4 : \mathbb{C}^\ell \rightarrow \mathbb{C}^A$ such that

$$a = v_4 d_2 v_3^*, \quad v_3^* v_3 = v_4^* v_4 = id_{\mathbb{C}^\ell},$$
where \(d_2\) is an \(\ell \times \ell\) diagonal matrix with positive real diagonal entries. Because \(\ell \leq k\), we may take \(w : \mathbb{C}^{\ell} \to \mathbb{C}^k\) so that \(d_2 = w^*d_1w\). Then we have

\[
a = v_4d_2v_3^* = v_4w^*d_1wv_3^* = (v_4w^*v_3^*)s(v_1wv_3^*).
\]

Therefore, we see that \(A_{d,a} = A_{d,v_1wv_3^*} \circ A_{d,v_4w^*v_3^*}\) belongs to \(\mathbb{C}P \circ \{A_{d}\} \circ \mathbb{C}P\).

Now, we fix \(s \in M_{AB}\) of rank \(k\). By Proposition 3.3, we see that \(\phi \in \mathbb{S}P_k\) if and only if \(\phi \in (\mathbb{C}P \circ \{A_{d}\} \circ \mathbb{C}P)^{\circ}\). We apply the first identity of (11) to the convex cone generated by \(\{A_{d}\}\), to conclude that \(\phi \in (\mathbb{C}P \circ \{A_{d}\} \circ \mathbb{C}P)^{\circ}\) holds if and only if \(\phi \otimes A_{d}\) is positive, or equivalently, \(A_{d} \otimes \phi\) is positive.

One more important example of a positive map is the transpose map \(t\). If \(K\) is a mapping cone then \(\{\phi \circ t : \phi \in K\}\) is also a mapping cone. Especially,

\[
\mathbb{C}CP := \mathbb{C}P \circ t
\]

is a mapping cone whose element is called *completely copositive*. If \(K_1\) and \(K_2\) are mapping cones then their convex hull \(K_1 \vee K_2\) and intersection \(K_1 \wedge K_2\) are also mapping cones. In this way, we have mapping cones

\[
\mathbb{D}EC := \mathbb{C}P \vee \mathbb{C}CP, \quad \mathbb{P}PT := \mathbb{C}P \wedge \mathbb{C}CP.
\]

The *partial transpose* in \(M_A \otimes M_B\) is defined by \((a \otimes b)^\Gamma = a^t \otimes b\). Because \(C_{\phi \circ t} = (C_{\phi})^\Gamma\), we see that \(\phi \in \mathbb{P}PT\) if and only if both \(C_{\phi}\) and \(C_{\phi}^\Gamma\) are positive. A state \(\varrho\) belongs to the convex cone

\[
\mathbb{P}PT := \mathbb{C}PP = \mathbb{C}CP \wedge C_{\mathbb{C}CP}
\]

if and only if \(\varrho^\Gamma\) is positive, and such states are called *of positive partial transpose* (PPT). We also denote

\[
\mathbb{D}EC := C_{\mathbb{D}EC}.
\]

It is clear that \(\mathbb{D}EC\) and \(\mathbb{P}PT\) are dual to each other, and \(\mathbb{D}EC \subset \mathbb{P}1\). By duality, we also have \(\mathbb{S}1 \subset \mathbb{P}PT\), that is, every separable state is of PPT [12, 41]. Maps in \(\mathbb{D}EC\) and \(\mathbb{P}PT\) are called *decomposable maps* and *PPT maps*, respectively. Now, we summarize as follows:

\[
L(M_A, M_B) : \quad \mathbb{S}P_1 \subset \mathbb{P}PT \subset \mathbb{C}P_{AB} \subset \mathbb{D}EC \subset \mathbb{P}_1
\]

\[
M_A \otimes M_B : \quad \mathbb{S}1 \subset \mathbb{P}PT \subset \mathbb{P}_{AB} \subset \mathbb{D}EC \subset \mathbb{B}P_1
\]

(13)

Comparing two chains of mapping cones in the diagrams (8) and (13), it is natural to ask if there are any inclusion relations between \(\mathbb{D}EC\) and \(\mathbb{P}k\), or equivalently those between \(\mathbb{P}PT\) and \(\mathbb{S}k\). In case of \((a, b) = (2, 2)\), it is known [49] that

\[
\mathbb{P}_1[M_2, M_2] = \mathbb{D}EC[M_2, M_2]
\]
with the obvious notations, that is, every positive map between \( M_2 \) is decomposable. See also [3, 53] for another proofs. It was also shown in [59] that
\[
\mathbb{P}_1[M_2, M_3] = \text{DEC}[M_2, M_3], \quad \mathbb{P}_1[M_3, M_2] = \text{DEC}[M_3, M_2]
\]

together with \( \mathbb{P}_1[M_2, M_4] \not\supseteq \text{DEC}[M_2, M_4] \). The first example of an indecomposable positive map was found in [11] when \((a, b) = (3, 3)\). In this case, it had been known that \( \mathbb{P}_2 \subset \text{DEC} \) holds for a special class of positive maps [9], and the dual claim \( \text{PPT} \subset S_2 \) in \( M_3 \otimes M_3 \) was also conjectured in [44]. See a survey article [32]. The relation
\[
\mathbb{P}_2[M_3, M_3] \subset \text{DEC}[M_3, M_3]
\]

was shown in [60] to be true. It is now known [4] that a 2-positive map need not to be decomposable in general, or equivalently, the Schmidt number of a PPT state may exceed two. In fact, it is known [5, 7, 28] that the Schmidt number of a PPT state may be arbitrary large. The relation
\[
\mathbb{P}_{n-1}[M_n, M_n] \subset \text{DEC}[M_n, M_n]
\]

is conjectured in [13], in connection with the PPT square conjecture which will be explained in Section 6.

4. Criteria using ampliation

Størmer [50] showed that a map \( \phi : M_A \to M_B \) is decomposable if and only if \( \text{id}_k \otimes \phi \) sends every PPT matrix to a positive matrix for every \( k = 1, 2, \ldots \). Motivated by this results, the author [18] showed that \( \phi \) is \( k \)-positive if and only if \( \phi \otimes \text{id}_B \) sends \( S_k \) to positive matrices. Ampliation is also useful to characterize properties of states in \( M_A \otimes M_B \). It was shown by Horodecki’s [25] that a state \( \varrho \in \mathcal{P}_{AB} \) is separable if and only if \( (\text{id}_A \otimes \phi)(\varrho) \in \mathcal{P}_{AA} \) for every positive map \( \phi : M_B \to M_A \). It is also shown by Terhal and Horodecki [56] that \( \varrho \in \mathcal{P}_{AA} \) has Schmidt number at most \( k \) if and only if \( (\text{id}_A \otimes \phi)(\varrho) \in \mathcal{P}_{AA} \) for every \( k \)-positive map \( \phi : M_A \to M_A \). Furthermore, it was shown in [26] that \( C_{\phi} \) is separable if and only if \( \text{id}_A \otimes \phi \) sends every states to a separable states. In this section, we use the identity [11] in a systematic way to recover the above criteria using ampliation.

We first investigate the role of the ampliation maps \( \text{id}_A \otimes \phi \) and \( \phi \otimes \text{id}_B \). To do this, we consider the following diagrams

\[
\begin{array}{ccc}
M_A & \xrightarrow{\text{id}_A} & M_A \\
\downarrow^{\sigma^*} & & \downarrow^{\psi} \\
M_A & \xrightarrow{\phi} & M_B \\
\end{array}
\quad
\begin{array}{ccc}
M_A & \xrightarrow{\phi} & M_B \\
\downarrow^{\psi} & & \downarrow^{\psi} \\
M_B & \xrightarrow{\text{id}_B} & M_B \\
\end{array}
\]

in Figure 1, to get the identities
\[
\langle \phi, \psi \circ \sigma \rangle = \langle \phi \circ \sigma^*, \psi \rangle = \langle C_{\phi \circ \sigma^*}, C_{\psi} \rangle = \langle (\text{id}_A \otimes \phi)(C_{\sigma^*}), C_{\psi} \rangle,
\]
\[
\langle \phi, \sigma \circ \psi \rangle = \langle \psi \circ \phi^*, \sigma^* \rangle = \langle C_{\psi \circ \phi}, C_{\sigma} \rangle = \langle (\phi \otimes \text{id}_B)(C_{\psi}), C_{\sigma} \rangle.
\]
From the first line, we see that $\phi \in (K \circ \mathbb{C}P_{AA})^\circ$ if and only if the ampliation $\text{id}_A \otimes \phi$ sends $\mathcal{P}_{AA}$ to $C_{K^\circ}$. We also see that $\phi \in (\mathbb{C}P_{BB} \circ K)^\circ$ if and only if $\phi \otimes \text{id}_B$ sends $C_K$ to $\mathcal{P}_{BB}$ from the second line. In other words, we have

$$
(K \circ \mathbb{C}P_{AA})^\circ = \{ \phi \in H(M_A, M_B) : (\text{id}_A \otimes \phi)(\varrho) \in C_{K^\circ} \text{ for every } \varrho \in \mathcal{P}_{AA} \},
$$

$$
(\mathbb{C}P_{BB} \circ K)^\circ = \{ \phi \in H(M_A, M_B) : (\phi \otimes \text{id}_B)(\varrho) \in \mathcal{P}_{BB} \text{ for every } \varrho \in C_K \}.
$$

Therefore, we have the following:

**Theorem 4.1.** [19] For a closed convex cone $K \subset H(M_A, M_B)$, the following are equivalent:

(i) $K$ is a right mapping cone,

(ii) $\phi \in K^\circ$ if and only if $\text{id}_A \otimes \phi$ sends positive matrices into $C_{K^\circ}$.

The following are also equivalent:

(iii) $K$ is a left mapping cone,

(iv) $\phi \in K^\circ$ if and only if $\phi \otimes \text{id}_B$ sends $C_K$ to positive matrices.

The statement (ii) of Theorem 4.1 with $K^\circ = \mathbb{S}P_1$ shows the following result which recovers the definition of entanglement breaking maps:

**Corollary 4.2.** [20] A linear map $\phi : M_A \to M_B$ is 1-superpositive if and only if $\text{id}_A \otimes \phi$ sends every state in $M_A \otimes M_B$ to a separable state.

On the other hands, the statement (iv) with $K^\circ = \mathbb{D}EC$ and $K^\circ = \mathbb{P}_k$ gives rise to the following:

**Corollary 4.3.** [50] [18] For $\phi : M_A \to M_B$, we have the following:

(i) $\phi$ is decomposable if and only if $\phi \otimes \text{id}_B$ sends PPT states to positive matrices,

(ii) $\phi$ is $k$-positive if and only if $\phi \otimes \text{id}_B$ sends states with Schmidt numbers at most $k$ to positive matrices.

In order to know the image of $C_\phi$ under ampliation maps, we replace $\sigma$ by $\phi$ in **Figure 1** to consider the following diagrams:

$$
\begin{array}{ccc}
M_A & \xrightarrow{\text{id}_A} & M_A \\
\downarrow \phi & & \downarrow \phi \\
M_B & \xrightarrow{\psi^*} & M_A \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
M_A & \xrightarrow{\psi} & M_B \\
\downarrow \phi & & \downarrow \phi \\
M_B & \xrightarrow{\text{id}_B} & M_B \\
\end{array}
$$

together with the identities

$$
\langle \phi, \psi \circ \sigma \rangle = \langle \psi^* \circ \phi, \sigma \rangle = \langle (\text{id}_A \otimes \psi^*) (C_\phi), C_\sigma \rangle,
$$

$$
\langle \phi, \sigma \circ \psi \rangle = \langle \phi \circ \psi^*, \sigma \rangle = \langle (\psi \otimes \text{id}_B) (C_\phi), C_\sigma \rangle.
$$

Therefore, we have

$$
(K \circ \mathbb{C}P_{AA})^\circ = \{ \phi \in H(M_A, M_B) : (\text{id}_A \otimes \psi^*) (C_\phi) \in \mathcal{P}_{AA} \text{ for every } \psi \in K \},
$$

$$
(\mathbb{C}P_{BB} \circ K)^\circ = \{ \phi \in H(M_A, M_B) : (\psi \otimes \text{id}_B)(C_\phi) \in \mathcal{P}_{BB} \text{ for every } \psi \in K \}.
$$
Theorem 4.4. [19] For a closed convex cone $K \subset H(M_A, M_B)$, the following are equivalent:

(i) $K$ is a right mapping cone,
(ii) $\phi \in K^\circ$ if and only if $(\text{id}_A \otimes \psi^*)(C_\phi) \geq 0$ for every $\psi \in K$.

Furthermore, the following are also equivalent:

(iii) $K$ is a left mapping cone,
(iv) $\phi \in K^\circ$ if and only if $(\psi \otimes \text{id}_B)(C_\phi) \geq 0$ for every $\psi \in K$.

We take the convex cone $K = \mathbb{S}^k_+$ to get the following characterization of Schmidt numbers of states. This gives rise to the separability criterion when $k = 1$.

Corollary 4.5. [25, 56] A state $\varrho \in M_A \otimes M_B$ belongs to $\mathcal{S}_k$ if and only if $(\text{id}_A \otimes \psi)(\varrho) \geq 0$ for every $k$-positive map $\psi : M_B \rightarrow M_A$ if and only if $(\psi \otimes \text{id}_B)(\varrho) \geq 0$ for every $k$-positive map $\psi : M_A \rightarrow M_B$.

In order to know what happens to the image of $C_{\text{id}_A}$ under the ampliation $\text{id}_A \otimes \phi$, we put two identity maps in Figure 1 as follows:

$$M_A \xrightarrow{\text{id}_A} M_A \xrightarrow{\phi} M_B$$

Then we get the identity $(\text{id}_A \otimes \phi)(C_{\text{id}_A}) = C_\phi$, and have the following:

Proposition 4.6. A linear map $\phi : M_A \rightarrow M_B$ belongs to a convex cone $K$ if and only if $(\text{id}_A \otimes \phi)(C_{\text{id}_A})$ belongs to $C_K$.

It is worthwhile to collect results for $K^\circ = \mathbb{S}^k_+$ to get equivalent conditions for $k$-superpositivity of a map $\phi : M_A \rightarrow M_B$, or Schmidt number of the corresponding state $C_\phi \in M_A \otimes M_B$ as follows:

Corollary 4.7. For $\phi \in H(M_A, M_B)$, the following are equivalent:

(i) $\phi$ is $k$-superpositive, that is, $\phi = \sum_i \text{Ad}_{s_i}$ with rank $s_i \leq k$,
(ii) $C_\phi$ belongs to $\mathcal{S}_k$, that is, has the Schmidt numbers $\leq k$,
(iii) $\psi^* \circ \phi$ is completely positive for every $k$-positive map $\psi : M_A \rightarrow M_B$,
(iv) $\phi \circ \psi^*$ is completely positive for every $k$-positive map $\psi : M_A \rightarrow M_B$,
(v) $\text{id}_A \otimes \phi$ sends every state to a state with Schmidt number $\leq k$,
(vi) $\phi \otimes \text{id}_B$ sends every $k$-blockpositive matrix to a positive matrix,
(vii) $(\text{id}_A \otimes \psi)(C_\phi) \geq 0$ for every $k$-positive map $\psi : M_B \rightarrow M_A$,
(viii) $(\psi \otimes \text{id}_B)(C_\phi) \geq 0$ for every $k$-positive map $\psi : M_A \rightarrow M_B$,
(ix) $(\phi \otimes \psi)(C_{\text{id}_A}) \geq 0$ for every $k$-positive map $\psi : M_A \rightarrow M_B$,
(x) $(\phi^* \otimes \psi^*)(C_{\text{id}_A}) \geq 0$ for every $k$-positive map $\psi : M_A \rightarrow M_B$,
(xi) $\phi \otimes \psi$ is positive for every $k$-positive map $\psi : M_A \rightarrow M_B$,
(xii) $(\text{id}_A \otimes \phi)(C_{\text{id}_A})$ is a state with Schmidt number $\leq k$. 

14
If $s = |\xi\rangle\langle\eta|$ is of rank one matrix, then we have

$$\text{Ad}_s(a) = |\eta\rangle\langle a|\xi\rangle\langle\eta| = \langle\xi|a|\xi\rangle|\eta\rangle\langle\eta|. $$

Therefore, we see that $\phi$ is 1-superpositive if and only if $\phi$ is of the following form

$$\phi(a) = \sum_k \langle a, v_k\rangle u_k$$

with positive matrices $u_k$ and $v_k$. This is called the Holevo form [22].

In the paper [26], a map $\phi$ was called entanglement breaking when $\phi$ satisfies the the condition (v) with $k = 1$, and it was shown that conditions (i), (iii), (iv) and (xii) are equivalent to (v) together with the Holevo form. On the other hand, a map was called in [1] superpositive when its Choi matrix is separable. Equivalent conditions (ii), (iii) and (v) for $k$-superpositive maps were given in [14].

5. Mapping cones arising from ampliation and factorizations

A linear map $\phi : M_A \to M_B$ is called $k$-entanglement breaking [13] if its ampliation map $\text{id}_k \otimes \phi : M_k \otimes M_A \to M_k \otimes M_B$ sends $(M_k \otimes M_A)^+$ into $S_1$. The convex cone $\text{EB}_k$ of all $k$-entanglement breaking maps is also a mapping cone [17]. Comparing with the mapping cones $P_k$ and $SP_k$, the inclusion relations can be summarized in Figure 2. Looking for mapping cones inside of the triangle in Figure 2, it is natural to consider the convex cone

(17) $A_k[K, L] := \{ \phi \in H(M_A, M_B) : (\text{id}_k \otimes \phi)(C_K) \subset C_L \},$

for $k = 1, 2, \ldots$, closed convex cones $K \subset H(M_k, M_A)$ and $L \subset H(M_k, M_B)$. Then we have

$$P_k = A_k[CP, CP], \quad \text{EB}_k = A_k[CP, SP].$$

The first identity of (15) also shows the following relation

(18) $(K \circ CP_{AA})^\circ = A_a[CP, K^\circ].$

Especially, we see that every mapping cone $K \subset H(M_A, M_B)$ can be expressed by

$$K = (K^\circ \circ CP_{AA})^\circ = A_a[CP, K].$$

This is nothing but Theorem [11] (ii). On other hand, the first identity of (16) may be written as $\langle \phi \circ \sigma^*, \psi \rangle = \langle (\text{id}_A \otimes \psi^*)(C_\phi), C_\sigma \rangle$, from which we have

(19) $A_a[K, CP]^* = (K \circ CP_{AA})^\circ.$
As special cases of the identity (I), we have the identities

\[(\text{id}_k \otimes \phi)(C_\sigma) = C_{\phi \circ \sigma}, \quad (\phi \otimes \text{id}_k)(C_{\sigma^*}) = C_{\sigma^* \circ \phi^*},\]

for \(\sigma \in H(M_k, M_A)\) and \(\phi \in H(M_A, M_B)\). These identities imply the following:

**Proposition 5.1.** Suppose that \(K\) and \(L\) are closed convex cones in \(H(M_k, M_A)\) and \(H(M_k, M_B)\), respectively. For \(\phi \in H(M_A, M_B)\), the following are equivalent:

(i) \(\phi \in A_k[K, L]\), that is, \(\text{id}_k \otimes \phi\) sends \(C_K\) into \(C_L\),

(ii) For every \(\sigma \in K\), we have \(\phi \circ \sigma \in L\),

(iii) For every \(\sigma \in K\), we have \(\sigma^* \circ \phi^* \in L^*\),

(iv) \(\phi \otimes \text{id}_k\) sends \(C_{K^*}\) into \(C_{L^*}\).

The equivalence between (i) and (iv) tells us that both the left and right ampliations basically give rise to the same class of maps, in most cases. Note that \(K^* \subset H(M_A, M_k)\) and \(L^* \subset H(M_B, M_k)\). It is clear that \(A_k[K, L]\) is a closed convex cone. The identities in (18) and (19) can be extended in much more general situations as follows:

**Proposition 5.2.** For closed convex cones \(K \subset H(M_k, M_A)\) and \(L \subset H(M_k, M_B)\), we have the following identities

\[A_k[K, L] = (L^\circ \circ K^*)^\circ, \quad A_k[K, L]^* = A_k[L^\circ, K^\circ].\]

**Proof.** Suppose that \(\phi \in A_k[K, L]\). Then for arbitrary given \(\psi \in L^\circ\) and \(\sigma \in K\), we have

\[\langle \psi \circ \sigma^*, \phi \rangle = \langle \psi, \phi \circ \sigma \rangle \geq 0,\]

since \(\phi \circ \sigma \in L\). This implies that \(\phi \in (L^\circ \circ K^*)^\circ\), and we get \(A_k[K, L] \subset (L^\circ \circ K^*)^\circ\). For the reverse inclusion, suppose that \(\phi \in (L^\circ \circ K^*)^\circ\) and \(\sigma \in K\). Then for every \(\psi \in L^\circ\), we have

\[\langle \phi \circ \sigma, \psi \rangle = \langle \phi, \psi \circ \sigma^* \rangle \geq 0,\]

and so we have \(\phi \circ \sigma \in L\). By Proposition 5.1 (ii), we have \(\phi \in A_k[K, L]\). This shows \((L^\circ \circ K^*)^\circ \subset A_k[K, L]\), and completes the proof of the first identity. By the first identity, we have

\[A_k[L^\circ, K^\circ]^* = (K \circ L^\circ)^{**} = (K \circ L^\circ)^{**} = (L^\circ \circ K^*)^\circ = A_k[K, L],\]

which shows the second identity. \(\Box\)

Motivated by Proposition 5.2, we consider the factorization properties. A linear map \(\phi : M_A \to M_B\) is called *factorized through* \(M_k\) when there exists \(\sigma : M_A \to M_k\) and \(\tau : M_k \to M_B\) such that \(\phi = \tau \circ \sigma\).
For a given natural number $k = 1, 2, \ldots$, and closed convex cones $K \subset H(M_k, M_A)$, $L \subset H(M_k, M_B)$, we define

$$F_k[K, L] = \{\tau \circ \sigma^*: \sigma \in K, \tau \in L\}^\circ.$$

So, maps in $F_k[K, L]$ are finite sums of linear maps which are factorized through $M_k$ as compositions $\tau \circ \sigma^*$ with $\sigma \in K$ and $\tau \in L$. By Proposition 5.2, we have the following:

**Theorem 5.3.** For a given natural number $k = 1, 2, \ldots$, and closed convex cones $K \subset H(M_k, M_A)$, $L \subset H(M_k, M_B)$, we have

$$A_k[K, L]^\circ = F_k[K, L]^\circ.$$

It is easily seen that the relation $F_k[\mathbb{C}P, \mathbb{C}P] = \mathbb{S}P_k$ holds. This recovers $\mathbb{S}P_k^\circ = A_k[\mathbb{C}P, \mathbb{C}P] = F_k$ by Theorem 5.3. We also have $\mathbb{E}B_k^r = F_k[\mathbb{C}P, P_1]$, which recovers [17, Theorem 3.13].

Now, we look for conditions with which $A_k[K, L]$ and/or $F_k[K, L]$ are mapping cones. To do this, we use the condition (ii) of Proposition 5.1

**Lemma 5.4.** For closed convex cones $K \subset H(M_k, M_A)$ and $L \subset H(M_k, M_B)$, we have the following:

(i) If $\mathbb{C}P_B \circ L \subset L$ then $\mathbb{C}P_B \circ A_k[K, L] \subset A_k[K, L]$.

(ii) If $\mathbb{C}P_A \circ K \subset K$ then $A_k[K, L] \circ \mathbb{C}P_A \subset A_k[K, L]$.

**Proof.** Suppose that $\mathbb{C}P_B \circ L \subset L$ holds, and take $\tau \in \mathbb{C}P_B$ and $\phi \in A_k[K, L]$. For any $\sigma \in K$, we have $\phi \circ \sigma \in L$ by the condition (ii) of Proposition 5.1. This implies that $(\tau \circ \phi) \circ \sigma = \tau \circ (\phi \circ \sigma) \in \mathbb{C}P_B \circ L \subset L$. Therefore, we have $\tau \circ \phi \in A_k[K, L]$ by (ii) of Proposition 5.1.

For (ii), suppose $\mathbb{C}P_A \circ K \subset K$, and take $\phi \in A_k[K, L]$ and $\tau \in \mathbb{C}P_A$. Then, for every $\sigma \in K$, we have $\tau \circ \sigma \in \mathbb{C}P_A \circ K \subset K$, which implies $(\phi \circ \tau) \circ \sigma = \phi \circ (\tau \circ \sigma) \in L$. Therefore, we have $\phi \circ \tau \in A_k[K, L]$. □

If both convex cones $K \subset H(M_k, M_A)$ and $L \subset H(M_k, M_B)$ consist of positive maps then it is clear that $F_k(K, L)$ also consists of positive maps. Since the dual cone of a mapping cone is also a mapping cone, we have the following:

**Theorem 5.5.** Suppose that both $K$ and $L$ are left mapping cones. Then the convex cones $A_k[K, L]$ and $F_k[K, L]^\circ$ are mapping cones.

Therefore, we see that $\mathbb{E}B_k$ is a mapping cone, as it was shown in [17]. The relation (18) recovers the fact that $(K \circ \mathbb{C}P)^\circ$ is a mapping cone whenever $K$ is a left mapping cone. It is clear that $F_k[K, L]$ is increasing with respect to the both variables $K$ and $L$. On the other hand, $A_k[K, L]$ are increasing with respect to the variable $L$ and decreasing with respect to $K$. We consider the case of $K = \mathbb{C}P$ to get various mapping cones.
\begin{figure}
\centering
\begin{tikzpicture}[scale=0.8]
  \node (A) {$\mathbb{SP}_n = K_{n,n} = \mathbb{P}_n = \mathbb{CP}$};
  \node (B) [below of=A] {$\mathbb{SP}_{n-1} = K_{n,n-1} \rightarrow K_{n-1,n-1} = \mathbb{P}_{n-1}$};
  \node (C) [below of=B] {$\cdots$};
  \node (D) [below of=C] {$\mathbb{SP}_2 = K_{n,2} \rightarrow K_{n-1,2} \rightarrow \cdots \rightarrow K_{3,2} \rightarrow K_{2,2} = \mathbb{P}_2$};
  \node (E) [below of=D] {$\mathbb{SP}_1 = K_{n,1} \rightarrow K_{n-1,1} \rightarrow \cdots \rightarrow K_{3,1} \rightarrow K_{2,1} \rightarrow K_{1,1} = \mathbb{P}_1$};
  \node (F) [below of=E] {$\mathbb{EB}_k \mathrel{\parallel} \mathbb{EB}_{k-1} \mathrel{\parallel} \mathbb{EB}_3 \mathrel{\parallel} \mathbb{EB}_2 \mathrel{\parallel} \mathbb{EB}_1$};
  \draw[->] (A) -- (B);
  \draw[->] (B) -- (C);
  \draw[->] (C) -- (D);
  \draw[->] (D) -- (E);
  \draw[->] (E) -- (F);
\end{tikzpicture}
\caption{Figure 3.}
\end{figure}

We fix $M_A = M_B = M_n$, $K = \mathbb{CP}$ and consider the cases of $L = \mathbb{SP}_\ell$ or $L = \mathbb{P}_\ell$ with $\ell = 1, 2, \ldots, n$. If $\ell \geq k$, then we have $\mathbb{SP}_k = \mathbb{SP}_\ell = \mathbb{CP} = \mathbb{P}_\ell = \mathbb{P}_k$ in $H(M_k, M_n)$. Therefore, all the mapping cones in the following inclusions

$$A_k[\mathbb{CP}, \mathbb{SP}_k] \subset A_k[\mathbb{CP}, \mathbb{SP}_\ell] \subset A_k[\mathbb{CP}, \mathbb{CP}] \subset A_k[\mathbb{CP}, \mathbb{P}_\ell] \subset A_k[\mathbb{CP}, \mathbb{P}_k]$$

coincide with $\mathbb{P}_k$. We also see that all the inclusions

$$\mathbb{P}_k = A_n[\mathbb{CP}, \mathbb{P}_k] \subset A_{\ell}[\mathbb{CP}, \mathbb{P}_k] \subset A_k[\mathbb{CP}, \mathbb{P}_k] = \mathbb{P}_k$$

become identities. In short, we have $A_k[\mathbb{CP}, \mathbb{P}_\ell] = \mathbb{P}_{\min(k, \ell)}$. The inclusion relations among all the remaining mapping cones are summarized in Figure 3 with the notations

$$K_{k,\ell} := A_k[\mathbb{CP}, \mathbb{SP}_\ell] = \{\phi \in H(M_A, M_B) : (\text{id}_k \otimes \phi)(\mathcal{P}) \subset S_\ell\},$$

for $k, \ell = 1, 2, \ldots, n$ with $k \geq \ell$.

In order to distinguish $k$-positivities, Tomiyama [57] considered the linear map $\phi_\lambda : M_n \rightarrow M_n$ given by

$$\phi_\lambda(x) = \lambda \text{Tr}(x) I_n - x, \quad x \in M_n,$$

with the parameter $\lambda \geq 1$, and showed that $\phi_\lambda$ is $k$-positive if and only if $\lambda \geq k$. See also [55]. The map $\phi_{n-1}$ was the example of Choi [10] to distinguish $n$-positivity and $(n-1)$-positivity. It was shown in [6, 17] that $\phi_k$ is even $k$-entanglement breaking as well as $k$-positive. Therefore, we have $\mathbb{EB}_k \subset \mathbb{P}_{k+1}$. The following tells us that there is no more possible inclusion relations in Figure 3.

\begin{proposition}
Suppose that $k, \ell, p, q = 1, 2, \ldots, n$ with $k \geq \ell$ and $p \geq q$. Then $K_{k,\ell} \subset K_{p, q}$ if and only if the following two conditions are satisfied:

(i) $k \geq p$,

(ii) $\ell \leq q$ or $\ell > q = p$.
\end{proposition}
Proof. The ‘if’ part is clear from Figure 3. Now, we suppose that $K_{k,\ell} \subset K_{p,q}$ holds. If $k < p$ then we have $\mathbb{EB}_k = K_{k,1} \subset K_{k,\ell} \subset K_{p,q} \subset \mathbb{P}_p \subset \mathbb{P}_{k+1}$, to get a contradiction. This shows that $k \geq p$.

To prove the condition (ii), we take $s = \sum_{i=1}^{\ell} e_{i,i} \in M_n$, and $\varphi = \sum_{i,j=1}^{p} e_{i,j} \otimes e_{i,j} \in M_p \otimes M_n$. Then we have $\text{Ad}_s \in \mathbb{SP}_\ell = K_{n,\ell} \subset K_{k,\ell} \subset K_{p,q}$ and $\varphi$ is positive in $M_p \otimes M_n$. Therefore, we see that

$$\min\{\ell,p\} \sum_{i,j=1}^{\min\{\ell,p\}} e_{i,j} \otimes e_{i,j} = (\text{id}_p \otimes \text{Ad}_s)(\varphi) \in M_p \otimes M_n$$

belongs to $S_q \subset M_p \otimes M_n$. This implies $\min\{\ell,p\} \leq q$, and gives rise to the condition (ii), since $p \geq q$. □

In the case of $n = 3$, we note that $\phi$ belongs to the mapping cone $K_{2,1} = \mathbb{EB}_2$ if and only if the map $\text{id}_2 \otimes \phi : M_2 \otimes M_3 \to M_2 \otimes M_3$ send $\mathbb{P}$ into $S_1$. Since $S_1 = \mathbb{PPT}$ in $M_2 \otimes M_3$ [27, 59], we see [13] that $\phi \in K_{2,1}$ if and only if $\phi$ is both 2-positive and 2-copositive, that is, $\phi \circ t$ is 2-positive. See [9] for parameterized examples of linear maps between $M_3$ which are both 2-positive and 2-copositive but not completely positive.

6. PPT SQUARE CONJECTURE

The PPT square conjecture claims that the composition of two PPT maps is 1-superpositive, that is, entanglement breaking, as it was proposed by Christandl in [43]. This conjecture which can be expressed by

$$\text{PPT} \circ \text{PPT} \subset \mathbb{SP}_1,$$

is supported by several results: If $\phi$ is a unital or trace preserving PPT map then $d(\phi^k, \mathbb{SP}_1)$ tends to 0 as $k \to \infty$ [30]; if $\phi$ is a unital PPT map then $\phi^n \in \mathbb{SP}_1$ for a positive integer $n$ [12]; the conjecture is true when $\phi$ is a PPT map between $3 \times 3$ matrices [8, 13]. See also [15]. It was also shown in [13] that the conjecture [14] implies that a finite iteration of a PPT map is entanglement breaking.

Equivalent claims and possibility to find counterexamples also have been discussed. It was shown in [38] that the existence of a nontrivial tensor-stable positive map implies the negation of the conjecture. Recall that a map $\phi$ is called tensor-stable positive [39] if $\phi^\otimes n$ is positive for every $n = 1, 2, \ldots$, and note that completely positive or completely copositive maps are trivially tensor-stable positive. It was also shown in [13] that the conjecture is true if and only if $P_1 \circ \text{PPT} \subset \text{DEC}$ if and only if $\text{PPT} \otimes \text{PPT}(\mathbb{P}) \subset S_1$ holds. In [19], several equivalent claims to the PPT square conjecture are found through composition as follows:

Proposition 6.1. [19] The following statements are equivalent:

(i) $\text{PPT} \circ \text{PPT} \subset \mathbb{SP}_1$.
(ii) $\text{PPT} \circ P_1 \subset \text{DEC}$.
(iii) $P_1 \circ \text{PPT} \subset \text{DEC}$. 

19
Here, equivalence between (i) and (iv) follows, since PPT is a mapping cone. The other claims are easy consequences of the identities in (4). For example, the identity

$$\langle \phi_1 \circ \sigma \circ \phi_2, \psi \rangle = \langle \sigma, \phi_1^* \circ \psi \circ \phi_2^* \rangle$$

with $\psi \in \mathbb{P}_1$, $\phi_i \in \text{PPT}$ and $\sigma \in \text{CP}$ proves (iv) $\iff$ (vi). Claims in Proposition 6.1 may be translated into those in terms of tensor products. By the identity (1), we have the relation

$$(20) \quad C_{K_3 \circ K_2 \circ K_1^*} = (K_1 \otimes K_3)(C_{K_2})$$

for any convex cones $K_1$, $K_2$ and $K_3$, whenever the above expression is meaningful. Furthermore, we also have

$$(21) \quad C_{K_3 \circ K_1^*} = (K_1 \otimes K_3)(C_{\text{id}_A})$$

whenever $K_1, K_3 \subseteq H(M_{A_i}, M_{B_i})$. Therefore, we see that $\text{PPT} \circ \text{PPT} \subseteq \text{SP}_1$ if and only if $(\text{PPT} \otimes \text{PPT})(C_{\text{id}_A}) \subseteq C_{\text{SP}_1}$. In this way, we use (20) and (21) to translate the statements Proposition 6.1 to those in terms of tensor products of linear maps.

**Proposition 6.2.** The following are equivalent to the PPT square conjecture:

(i) $(\text{PPT} \otimes \text{PPT})(C_{\text{id}_A}) \subseteq S_1$,
(ii) $(\text{P}_1 \otimes \text{PPT})(C_{\text{id}_A}) \subseteq \text{DEC}$,
(iii) $(\text{PPT} \otimes \text{P}_1)(C_{\text{id}_A}) \subseteq \text{DEC}$,
(iv) $(\text{PPT} \otimes \text{PPT})(\mathbb{P}) \subseteq S_1$,
(v) $(\text{PPT} \otimes \text{PPT})(\text{DEC}) \subseteq S_1$,
(vi) $(\text{PPT} \otimes \text{PPT})(\text{BP}_1) \subseteq \mathbb{P}$,
(vii) $(\text{PPT} \otimes \text{PPT})(\text{BP}_1) \subseteq \text{PPT}$.

The equivalent statement (iv) observed in [13] claims that if $\phi$ and $\psi$ are PPT maps then $\phi \otimes \psi$ is an entanglement annihilating map. This is true for maps between $3 \times 3$ matrices, since the PPT square conjecture is true in this case [8, 13].

Suppose that $\phi_i \in H(M_{A_i}, M_{B_i})$ for $i = 1, 2$, $\phi_3 \in H(M_{A_1}, M_{A_2})$ and $\phi_4 \in H(M_{B_1}, M_{B_2})$. Then we have the identity

$$\langle (\phi_1 \otimes \phi_2)(C_{\phi_3}), C_{\phi_4} \rangle = \langle \phi_2 \circ \phi_3 \circ \phi_1^*, \phi_4 \rangle$$

$$= \langle \phi_4 \circ \phi_1 \circ \phi_3^*, \phi_2 \rangle$$

$$= \langle (\phi_3 \otimes \phi_4)(C_{\phi_1}), C_{\phi_2} \rangle.$$

Hence, we see that the following equivalence

$$(K_1 \otimes K_2)(C_{K_3}) \subseteq C_{K_4} \iff (K_3 \otimes K_1^*)(C_{K_1}) \subseteq C_{K_2}^\circ$$
Proposition 6.3. The following are equivalent to the PPT square conjecture:

(i) \((\mathrm{id}_A \otimes P_1)(\text{PPT}) \subseteq \mathcal{D}\mathcal{E}\mathcal{C}\),
(ii) \((\mathrm{id}_A \otimes \text{PPT})(\mathcal{B}P_1) \subseteq \mathcal{D}\mathcal{E}\mathcal{C}\),
(iii) \((\mathrm{id}_A \otimes \text{PPT})(\text{PPT}) \subseteq \mathcal{S}_1\),
(iv) \((\mathcal{C}P \otimes P_1)(\text{PPT}) \subseteq \mathcal{D}\mathcal{E}\mathcal{C}\),
(v) \((\text{DEC} \otimes P_1)(\text{PPT}) \subseteq \mathcal{D}\mathcal{E}\mathcal{C}\),
(vi) \((P_1 \otimes \text{CP})(\text{PPT}) \subseteq \mathcal{D}\mathcal{E}\mathcal{C}\),
(vii) \((P_1 \otimes \text{DEC})(\text{PPT}) \subseteq \mathcal{D}\mathcal{E}\mathcal{C}\).

We note that the maps in Proposition 6.3 are linear maps from \(M_A \otimes M_B\) into itself, while maps in Proposition 6.2 were maps from \(M_A \otimes M_A\) into \(M_B \otimes M_B\). Equivalent claims of the conjecture through ampliation as in (i), (ii) and (iii) of Proposition 6.3 can be also obtained from Theorem 5.3. In fact, \(K_1 \circ K_2 \subseteq K_3\) holds if and only if \(F_a(K_1, K_2) \subseteq K_3\) if and only if \(K_3^\circ \subseteq A_a[K_1, K_2]\) by Theorem 5.3. Therefore, we have

\(K_1 \circ K_2 \subseteq K_3 \iff (\mathrm{id}_A \otimes K_3^\circ)(C_{K_1}) \subseteq C_{K_2}^\circ\).

For an example, we have \(\text{PPT} \circ \text{PPT} \subseteq \mathcal{S}_1\) if and only if \((\mathrm{id} \otimes P_1)(\text{PPT}) \subseteq \mathcal{D}\mathcal{E}\mathcal{C}\). In this way, we see that the first three claims in Proposition 6.1 are equivalent to those in Proposition 6.3.

As for linear maps between \(3 \times 3\) matrices, it was shown in [2] recently that the following relation

\(\text{PPT} \otimes \text{PPT}(\mathcal{B}P_1) \subseteq \mathcal{S}_1\)

holds. Comparing with Proposition 6.2 (iv), (v) (vi) and (vii), this is seemingly stronger than the PPT square conjecture. We also note that this is equivalent to the claims

\(\text{PPT} \circ P_1 \circ \text{PPT} \subseteq \mathcal{S}P_1,\) and \((P_1 \otimes P_1)(\text{PPT}) \subseteq \mathcal{D}\mathcal{E}\mathcal{C}\),

respectively. It would be interesting to know if these are equivalent to the PPT square conjecture in general.

7. Discussion

In this article, we began with the concrete examples of positive maps of the form \(x \mapsto s^*xs\) to introduce the convex cones in [8] which appear as key notions in the current quantum information theory. Main tools were Choi matrices and duality arising from the bilinear pairing. Such convex cones are mapping cones whose dual cones can be described in terms composition and tensor products of linear maps.

Because \(\mathcal{S}P_k\) is singly generated as a mapping cone, elements of the dual cone can be described by the tensor product with a single map \(\mathrm{id}_k\). Especially, we have \(\phi \in \mathcal{S}P_k^\circ\) if and only if \(\mathrm{id}_k \otimes \phi \in \mathcal{S}P_1^\circ\), and see that the ampliation connects \(\mathcal{S}P_k^\circ\) and \(\mathcal{S}P_1^\circ\). A
natural question arises: Does there exist an operation which connects $\mathbb{SP}_k$ and $\mathbb{SP}_1$? Such an operation would be very useful as criteria for $k$-superpositivity of maps and Schmidt numbers of bi-partite states, because various known separability criteria may be used for this purpose.

The identity (1) played essential roles through the whole discussion. It is also natural to look for analogous identities for arbitrary iterations of tensor products and compositions. Suppose that $\phi_i : M_{A_i} \to M_{B_i}$ is a linear map for $i = 1, 2, \ldots, n$. Then we have the linear map

$$\phi_1 \otimes \cdots \otimes \phi_n : M_{A_1} \otimes \cdots \otimes M_{A_n} \to M_{B_1} \otimes \cdots \otimes M_{B_n}.$$ 

We can take a nontrivial bi-partition $S \sqcup T = \{1, 2, \ldots, n\}$ of systems, then matrices in the domain can be dealt as the Choi matrix of a linear map from $\otimes_{i \in S} M_{A_i}$ to $\otimes_{i \in T} M_{A_i}$. See [21] for more details. Then we may apply (1) to get the identity. Alternatively, one may use the Choi matrices of multi-linear maps from $M_{A_1} \times \cdots \times M_{A_{n-1}}$ to $M_{A_n}$ as it was considered in [33]. Then we use the similar calculation as in the proof of (1) to get an identity. In both methods, we cannot obtain identities involving arbitrary iterations of a map by composition. Such an identity may be useful to deal with tensor-stable positive maps.

References

[1] T. Ando, Cones and norms in the tensor product of matrix spaces, Linear Alg. Appl. 379 (2004), 3–41.
[2] G. Aubrun and A. Müller-Hermes, Annihilation entanglement between cones, Preprint, arXiv 2110.11825.
[3] G. Aubrun and S. J. Szarek, “Alice and Bob meet Banach: The Interface of Asymptotic Geometric Analysis and Quantum Information Theory”, Math. Surveys Monog. Vol 223, Amer. Math. Soc., 2017.
[4] B. V. R. Bhat and H. Osaka, A factorization property of positive maps on C*-algebras, Intern. J. Quantum Inform. 8 (2020), 52.
[5] D. Cariello, Inequalities for the Schmidt number of bipartite states, Lett. Math. Phis. 110 (2020), 827–833.
[6] S. Chen and E. Chitambar, Entanglement-breaking superchannels, Quantum 4 (2020), 299.
[7] L. Chen, Y. Yang and W.-S. Tang, Schmidt number of bipartite and multipartite states under local projections Quantum Inf. Process 16 (2017), 75.
[8] L. Chen, Y. Yang and W.-S. Tang, Positive-partial-transpose square conjecture for $n = 3$, Phys. Rev. A 99 (2019), 012337.
[9] S.-J. Cho, S.-H. Kye and S. G. Lee, Generalized Choi maps in 3-dimensional matrix algebras, Linear Alg. Appl. 171 (1992), 213–224.
[10] M.-D. Choi, Positive linear maps on C*-algebras, Canad. Math. J. 24 (1972), 520–529.
[11] M.-D. Choi, Completely positive linear maps on complex matrices, Linear Alg. Appl. 10 (1975), 285–290.
[12] M.-D. Choi, Positive linear maps, Operator Algebras and Applications (Kingston, 1980), pp. 583–590, Proc. Sympos. Pure Math. Vol 38. Part 2, Amer. Math. Soc., 1982.
[13] M. Christandl, A. Müller-Hermes and M. M. Wolf, When Do Composed Maps Become Entanglement Breaking? Ann. Henri Poincaré 20 (2019), 2295–2322.
[14] D. Chruściński and A. Kossakowski, On Partially Entanglement Breaking Channels, Open Sys. Information Dyn. 13 (2006), 17–26.
[15] B. Collins, Z. Yin and P. Zhong, The PPT-square conjecture holds generically for some classes of independent states, J. Phys. A: Math. Theor. 51 (2018), 425301.
[16] J. de Pillis, *Linear transformations which preserve Hermitian and positive semidefinite operators*, Pacific J. Math. **23** (1967), 129–137.

[17] R. Devendra, N. Mallick and K. Sumesh, *Mapping cone of k-entanglement breaking maps*, preprint. arXiv 2105.14991.

[18] M.-H. Eom and S.-H. Kye, *Duality for positive linear maps in matrix algebras*, Math. Scand. **86** (2000), 130–142.

[19] M. Girard, S.-H. Kye and E. Størmer, *Convex cones in mapping spaces between matrix algebras*, Linear Algebra Appl. **608** (2021), 248–269.

[20] K. H. Han and S.-H, Kye, *Various notions of positivity for bi-linear maps and applications to tri-partite entanglement*, J. Math. Phys. **57** (2016), 015205.

[21] K. H. Han and S.-H, Kye, *Construction of multi-qubit optimal genuine entanglement witnesses*, J. Phys. A: Math. Theor. **49** (2016), 175303.

[22] A. S. Holevo, *Quantum coding theorems*, Russian Math. Surveys **53** (1998), 1295–1331.

[23] A. S. Holevo, *Entropy gain and the Choi–Jamiolkowski correspondence for infinite dimensional quantum evolutions*, Theor. Math. Phys. **166** (2011), 123–138.

[24] A. S. Holevo, *The Choi–Jamiolkowski forms of quantum Gaussian channels*, J. Math. Phys. **52** (2011), 042202.

[25] M. Horodecki, P. Horodecki and R. Horodecki, *Separability of mixed states: necessary and sufficient conditions*, Phys. Lett. A **223** (1996), 1–8.

[26] M. Horodecki, P. W. Shor and M. B. Ruskai, *Entanglement breaking channels*, Rev. Math. Phys. **15** (2003), 629–641.

[27] P. Horodecki, *Separability criterion and inseparable mixed states with positive partial transposition*, Phys. Lett. A **232** (1997), 333–339.

[28] M. Huber, L. Lami, C. Lancien and A. Müller-Hermes, *High-Dimensional Entanglement in States with Positive Partial Transposition*, Phys. Rev. Lett. **121** (2018), 200503.

[29] A. Jamiołkowski, *Linear transformations which preserve trace and positive semidefinite operators*, Rep. Math. Phys. **3** (1972), 275–278.

[30] K. Kraus, *Operations and effects in the Hilbert space formulation of quantum theory*, Foundations of quantum mechanics and ordered linear spaces (Marburg, 1973), pp. 206–229. Lecture Notes in Phys., Vol. 29, Springer, 1974.

[31] S.-H. Kye, *Facial structures for various notions of positivity and applications to the theory of entanglement*, Rev. Math. Phys. **25** (2013), 1330002.

[32] S.-H. Kye, *Three-qubit entanglement witnesses with the full spanning properties*, J. Phys. A: Math. Theor. **48** (2015), 235303.

[33] S.-H. Kye, *Choi matrices revisited*, J. Math. Phys. **63** (2022), 092202.

[34] Y. Li and H.-K. Du, *Interpolations of entanglement breaking channels and equivalent conditions for completely positive maps*, J. Funct. Anal. **268** (2015), 3566–3599.

[35] A. Mülher-Hermes, *Decomposability of linear maps under tensor powers*, J. Math. Phys. **59** (2018), 102203.

[36] M. Rahaman, S. Jaques and V. I. Paulsen, *Composition of PPT Maps*, Quantum Inform. Comput. **18** (2018), 0472–0480.

[37] K. Kraus, *Operations and effects in the Hilbert space formulation of quantum theory*, Foundations of quantum mechanics and ordered linear spaces (Marburg, 1973), pp. 206–229. Lecture Notes in Phys., Vol. 29, Springer, 1974.

[38] A. Peres, *Separability Criterion for Density Matrices*, Phys. Rev. Lett. **77** (1996), 1413–1415.

[39] M. Rahaman, S. Jaques and V. I. Paulsen, *Eventually entanglement breaking maps*, J. Math. Phys. **59** (2018), 062201.

[40] V. I. Paulsen and F. Shultz, *Complete positivity of the map from a basis to its dual basis*, J. Math. Phys. **54** (2013), 072201.

[41] A. Peres, *Separability Criterion for Density Matrices*, Phys. Rev. Lett. **77** (1996), 1413–1415.

[42] M. Rahaman, S. Jaques and V. I. Paulsen, *Eventually entanglement breaking maps*, J. Math. Phys. **59** (2018), 062201.

[43] M. B. Ruskai, M. Junge, D. Kribs, P. Hayden, A. Winter, *Operator structures in quantum information theory*, Final Report, Banff International Research Station (2012).

[44] A. Sanpera, D. Bruß and M. Lewenstein, *Schmidt number witnesses and bound entanglement*, Phys. Rev. A **63** (2001), 050301.
[45] P. W. Shor, *Additivity of the classical capacity of entanglement-breaking quantum channels*, J. Math. Phys. **43** (2002), 4334–4340.

[46] L. Skowronek, *Cones with a mapping cone symmetry in the finite-dimensional case*, Linear Algebra Appl. **435** (2011), 361–370.

[47] L. Skowronek, E. Størmer, and K. Zyczkowski, *Cones of positive maps and their duality relations*, J. Math. Phys. **50** (2009), 062106.

[48] W. M. Stinespring, *Positive functions on C*-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211–216.

[49] E. Størmer, *Positive linear maps of operator algebras*, Acta Math. **110** (1963), 233–278.

[50] E. Størmer, *Decomposable positive maps on C*-algebras*, Proc. Amer. Math. Soc. **86** (1982), 402–404.

[51] E. Størmer, *Extension of positive maps into B(H)*, J. Funct. Anal. **66** (1986), 235-254.

[52] E. Størmer, *Tensor products of positive maps of matrix algebras*, Math. Scand. **111** (2012), 5–11.

[53] E. Størmer, *Positive Linear Maps of Operator Algebras*, Springer-Verlag, 2013.

[54] E. Størmer, *The analogue of Choi matrices for a class of linear maps on Von Neumann algebras*, Intern. J. Math. **26** (2016), 1550018.

[55] K. Tanahashi and J. Tomiyama, *On the geometry of positive maps in matrix algebras*, Math. Z. **184** (1983), 101–108.

[56] B. M. Terhal and P. Horodecki, *A Schmidt number for density matrices* Phys. Rev. A **61** (2000), 040301.

[57] J. Tomiyama, *On the geometry of positive maps in matrix algebras. II*, Linear Alg. Appl. **69** (1985), 169–177.

[58] R. F. Werner, *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, Phys. Rev. A. **40** (1989), 4277–4281.

[59] S. L. Woronowicz, *Positive maps of low dimensional matrix algebras*, Rep. Math. Phys. **10** (1976), 165–183.

[60] Y. Yang, D. H. Leung and W.-S. Tang, *All 2-positive linear maps from M_3(\mathbb{C}) to M_3(\mathbb{C}) are decomposable*, Linear Alg. Appl. **503** (2016), 233–247.

**Department of Mathematics and Institute of Mathematics, Seoul National University, Seoul 151-742, Korea**

*Email address: kye at snu.ac.kr*