ON THE OPERATOR SPACE UMD PROPERTY FOR NONCOMMUTATIVE $L_p$-SPACES

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ABSTRACT. We study the operator space UMD property, introduced by Pisier in the context of noncommutative vector-valued $L_p$-spaces. It is unknown whether the property is independent of $p$ in this setting. We prove that for $1 < p, q < \infty$, the Schatten $q$-classes $S_q$ are OUMD$_p$. The proof relies on properties of the Haagerup tensor product and complex interpolation. Using ultraproduct techniques, we extend this result to a large class of noncommutative $L_q$-spaces. Namely, we show that if $\mathcal{M}$ is a QWEP von Neumann algebra (i.e., a quotient of a $C^*$-algebra with Lance’s weak expectation property) equipped with a normal, faithful tracial state $\tau$, then $L_q(\mathcal{M}, \tau)$ is OUMD$_p$ for $1 < p, q < \infty$.

1. Introduction

Probabilistic techniques are well-established powerful tools in the study of Fourier analysis of vector-valued functions. In particular, Banach spaces having the UMD property, that is, the property of unconditionality for martingale differences play an important role. Deep connections with the boundedness of certain singular integral operators, such as the Hilbert transform, were established through the work of Burkholder, McConnell and Bourgain. Namely, Burkholder and McConnell [9] proved that if a Banach space $B$ is UMD, then the Hilbert transform is a bounded operator on the vector-valued Lebesgue space $L_p([0,1];B)$, for $1 < p < \infty$. Later, Bourgain [5] showed that, conversely, the boundedness of the Hilbert transform on $L_p([0,1];B)$ ($1 < p < \infty$) implies that $B$ is UMD. Recall that the Banach space $B$ is UMD if, for $1 < p < \infty$, there exists a constant $\beta_p > 0$ such that

$$
\left\| \sum_{n=1}^{k} \varepsilon_n dx_n \right\|_{L_p([0,1];B)} \leq \beta_p \left\| \sum_{n=1}^{k} dx_n \right\|_{L_p([0,1];B)},
$$

for all positive integers $k$, all sequences $\varepsilon = (\varepsilon_n)_{n=1}^{k}$ of numbers in $\{-1,1\}$ and all $B$-valued martingale difference sequences $dx = (dx_n)_{n=1}^{k}$. Equivalently, for all sequences $\varepsilon$ as above, the $\pm 1$ martingale transform $T_\varepsilon$ generated by $\varepsilon$, i.e., $T_\varepsilon \left( \sum_{n=1}^{k} dx_n \right) = \sum_{n=1}^{k} \varepsilon_n dx_n$, is a bounded operator on $L_p([0,1];B)$, with norm estimate

$$
\|T_\varepsilon : L_p([0,1];B) \to L_p([0,1];B)\| \leq \beta_p.
$$

The fact that the finiteness of $\beta_p$ for some $1 < p < \infty$ implies its finiteness for all such $p$ was first proved by Pisier; see Maurey [36]. Results of Burkholder [8] provided the first example of a UMD Banach space, namely, the real line $\mathbb{R}$. Other examples include the Schatten $p$-classes for
1 < p < \infty \) (Gutiérrez [22], Bourgain [6]), and the noncommutative \( L_p(M, \tau) \)-spaces associated with a von Neumann algebra \( M \), equipped with a normal, semifinite, faithful (abbreviated as n.s.f.) trace \( \tau \) (Berkson, Gillespie and Muhly [3]). We refer to Burkholder [10, 12] for more properties of UMD Banach spaces, connections to other topics and further references.

More recently, Pisier [48] developed a theory of noncommutative vector-valued Lebesgue spaces \( L_p(M; E) \) associated with a von Neumann algebra \( M \) with an n.s.f. trace. Two restrictions are required for the theory to be satisfactory: \( M \) has to be hyperfinite, and \( E \) equipped with an operator space structure, that is, a sequence of matrix norms \( \| \cdot \|_m \) defined on \( M_m(E) \) for each positive integer \( m \), such that for all \( x \in M_m(E), y \in M_n(E) \) and \( \alpha, \beta \in M_m(\mathbb{C}) \),

\[
\| x \oplus y \|_{m+n} = \max\{ \| x \|_m, \| y \|_n \}, \quad \| \alpha x \beta \|_m \leq \| \alpha \| \| x \|_m \| \beta \|.
\] (1.1)

We recall that the class of hyperfinite von Neumann algebras includes the algebra of bounded linear operators on a separable Hilbert space (in particular, matrix algebras), the classical \( L_\infty \)-spaces and group von Neumann algebras associated to amenable groups, and it is closed under von Neumann algebra tensor products. All the stability properties of the noncommutative Lebesgue spaces \( L_p(M; E) \) (e.g., duality) should be formulated in the category of operator spaces.

Noncommutative conditional expectations and martingales arise naturally in this setting. The \( L_p \)-theory of noncommutative martingales has achieved a rapid and considerable progress in recent years, see, e.g., Junge [25], Junge and Xu [29, 30, 31] and Randrianantoanina [51, 53]. Also, noncommutative \( BMO \) spaces were studied in [38, 40, 26]. The systematic investigation of various noncommutative martingale inequalities started from the seminal paper [47] of Pisier and Xu, where they introduced noncommutative Hardy spaces of martingales and proved the analogue of the Burkholder-Gundy square function inequalities. As a consequence, it follows that for \( 1 < p < \infty \), the \( \pm 1 \) martingale transform \( T_\varepsilon \) is a bounded operator on the noncommutative Lebesgue space \( L_p(M, \tau) \), associated with a von Neumann algebra \( M \) with an n.s.f. tracial state \( \tau \). This led naturally to formulating an appropriate notion of operator space \( \text{OUMD}_p \) property in this setting, as introduced by Pisier in [48], and to obtaining the first example of an operator space that is \( \text{OUMD}_p \) for \( 1 < p < \infty \), namely the complex plane \( \mathbb{C} \).

In this paper we study basic stability properties of \( \text{OUMD}_p \) operator spaces and consider some related questions formulated in [48]. Our paper is organized as follows. The construction of the noncommutative vector-valued Lebesgue spaces, as well as some of their stability properties (e.g. duality) are briefly discussed in the Preliminaries. Section 3 is devoted to the study of basic stability properties of \( \text{OUMD}_p \) operator spaces. Namely, we show that, as in the classical setting, the \( \text{OUMD}_p \) property is inherited by subspaces and quotients, and it is preserved under complex interpolation and by ultraproducts. If an operator space \( E \) is \( \text{OUMD}_p \), then its (standard) dual \( E^* \) is \( \text{OUMD}_{p'} \), where \( p' \) is the conjugate exponent of \( p \). Also, each matrix level \( M_m(E) \) is \( \text{OUMD}_p \). We end Section 3 with an example (based on a construction of Pisier from [48]) of a Hilbert space, subspace of some commutative \( C^* \)-algebra, which is \( \text{UMD} \) as a Banach space, but not \( \text{OUMD}_p \), for any \( 1 < p < \infty \). Section 4 contains our main results. It is unknown whether
the property is independent of $p$ in this setting. We provide the first non-trivial example of an operator space that is $OUMD_p$, independent of $p$, namely, we prove the following:

**Theorem 1.1.** Let $1 < p,q < \infty$. Then the Schatten $q$-class $S_q$ is $OUMD_p$.

The proof relies on properties of the Haagerup tensor product and complex interpolation, and it was inspired by a question of Z.-J. Ruan as to whether the column Hilbert space $C$ is $OUMD_p$ for some (all) $1 < p < \infty$. As an application of ultraproduct results due to Junge [25], it follows that a large class of noncommutative $L_q$-spaces are $OUMD_p$, independent of $p$. Namely, if $M$ is a QWEP von Neumann algebra (i.e., a quotient of a $C^*$-algebra with the weak expectation property of Lance [35], see Kirchberg [33]), equipped with an n.s.f. tracial state $\tau$, then $L_q(M,\tau)$ is $OUMD_p$ for $1 < p,q < \infty$. Furthermore, we show that the class of operator spaces which are $OUMD_p$, independent of $p$, contains all finite dimensional operator spaces, the vector-valued Schatten classes $S_u[S_v]$ for $1 < u,v < \infty$, as well as the noncommutative Lorentz spaces associated to a hyperfinite (and finite) von Neumann algebra. We end Section 4 with some intermediate results towards answering Z.-J. Ruan’s question, which remains open.

2. Preliminaries

We refer to Effros and Ruan [20] and Pisier [49] for details on operator spaces and completely bounded maps. We shall briefly recall some definitions. A (concrete) operator space on a Hilbert space $H$ is a norm closed linear subspace $E$ of $\mathcal{B}(H)$. For any positive integer $m$, the natural inclusion $M_m(E) \subseteq M_m(\mathcal{B}(H)) = \mathcal{B}(H^m)$ induces a norm $\| \cdot \|_m$ on $M_m(E)$. Ruan [54] gave an abstract characterization of operator spaces in terms of their matrix norms. Namely, an (abstract) operator space is a vector space $E$ equipped with matrix norms $\| \cdot \|_m$ on $M_m(E)$, for each positive integer $m$, satisfying axioms [11]. The morphisms in the category of operator spaces are completely bounded maps. Given a linear map between two operator spaces $\phi : E_0 \rightarrow E_1$, define $\phi_m : M_m(E_0) \rightarrow M_m(E_1)$ by $\phi_m([v_{ij}]) = [\phi(v_{ij})]$, for all $[v_{ij}]_{i,j=1}^m \in M_m(E_0)$. Let $\| \phi \|_{cb} = \sup\{\|\phi_m\| : m \in \mathbb{N}\}$. The map $\phi$ is called completely bounded if $\| \phi \|_{cb} < \infty$, and $\phi$ is called completely isometric if all $\phi_m$ are isometries. The space of all completely bounded maps from $E_0$ to $E_1$ is denoted by $\mathcal{CB}(E_0,E_1)$. Then $\mathcal{CB}(E_0,E_1)$ is an operator space with matrix norms defined by

$$M_m(\mathcal{CB}(E_0,E_1)) = \mathcal{CB}(E_0,M_m(E_1)),$$

for all positive integers $m$. The dual of an operator space $E_0$ is, again, an operator space $E^* = \mathcal{CB}(E,\mathbb{C})$. If $F$ is a closed subspace of $E$, then both $F$ and $E/F$ are operator spaces; $F$ is equipped with the induced operator space structure from $E$, while on $E/F$ the matrix norms are defined by $M_m(E/F) = M_m(E)/M_m(F)$ for all positive integers $m$. Let $(E_0,E_1)$ be a compatible couple of operator spaces. Recall the spaces

$$E_0 \oplus_p E_1 = \{x = (x_0,x_1) \in E_0 \oplus E_1 : \|x\|_p = (\|x_0\|_p^p + \|x_1\|_{E_1}^p)^{1/p} \}, \quad 1 \leq p < \infty,$$

$$E_0 \oplus_\infty E_1 = \{x = (x_0,x_1) \in E_0 \oplus E_1 : \|x\|_\infty = \max\{\|x_0\|_{E_0},\|x_1\|_{E_1}\} \}.$$
As noted in [46], \(E_0 \oplus_\infty E_1\) is an operator space with matrix norms defined by
\[
M_m(E_0 \oplus_\infty E_1) = M_m(E_0) \oplus_\infty M_m(E_1),
\]
for all positive integers \(m\), while the isometric embedding
\[
M_m(E_0 \oplus_1 E_1) \hookrightarrow \text{CB}(E_0^\infty \oplus E_1^*), M_m = M_m(E_0^\infty \oplus E_1^*) \equiv M_m((E_0 \oplus_1 E_1)^*)
\]
induce an operator space structure on \(E_0 \oplus_1 E_1\). For \(1 < p < \infty\), equip \(E_0 \oplus_p E_1\) with the operator space structure given by the isometric identification \(E_0 \oplus_p E_1 = l_p(\{E_0, E_1\})\), where \(l_p(\{E_0, E_1\})\) is the \(l_p\)-direct sum of \(E_0\) and \(E_1\). Furthermore, note that \(E_0 \cap E_1\) can be identified with the diagonal \(\Delta = \{(x, x) \in E_0 \oplus E_1\}\) of \(E_0 \oplus_\infty E_1\), while \(E_0 + E_1 = (E_0 \oplus_\infty E_1)/N\), where \(N = \{(x_0, x_1) \in E_0 \oplus E_1 : x_0 + x_1 = 0\}\). These identifications are then used to equip \(E_0 \cap E_1\) and, respectively, \(E_0 + E_1\) with appropriate operator space matrix norms. Following Pisier [46], for \(0 < \theta < 1\), we endow the interpolation space \([E_0, E_1]_\theta\) with a canonical operator space structure by defining for all positive integers \(m\),
\[
(2.2) \quad M_m([E_0, E_1]_\theta) = \{M_m(E_0), M_m(E_1)\}_\theta.
\]
Recall that the complex method of interpolation is an exact functor of exponent \(\theta\). Thus, if \((E_0, E_1)\) and \((F_0, F_1)\) are two compatible couples of operator spaces, and if a map \(u : E_0 + E_1 \to F_0 + F_1\) is completely bounded both from \(E_0\) to \(E_1\) and from \(F_0\) to \(F_1\), then \(u\) is completely bounded from \([E_0, E_1]_\theta\) to \([F_0, F_1]_\theta\), and, moreover, the following norm estimate holds:
\[
(2.3) \quad \|u : [E_0, E_1]_\theta \to [F_0, F_1]_\theta\|_{cb} \leq \|u : E_0 \to E_1\|^{1-\theta}_{cb} \|u : F_0 \to F_1\|^{\theta}_{cb}.
\]
Given operator spaces \(E \hookrightarrow \mathcal{B}(H)\) and \(F \hookrightarrow \mathcal{B}(K)\), the embedding \(E \otimes F \hookrightarrow \mathcal{B}(H \otimes_2 K)\) induces an operator space matrix norm \(\|\cdot\|_\vee\) on \(E \otimes F\). It is proved in [14] that this matrix norm is independent on the choice of the Hilbert spaces \(H\) and \(K\). The completion of \(E \otimes F\) with respect to the norm \(\|\cdot\|_\vee\) is called the injective tensor product of \(E\) and \(F\). The projective tensor product of \(E\) and \(F\) is defined such that the complete isometry
\[
(E \hat{\otimes} F)^* \cong \text{CB}(E, F^*)
\]
holds. Furthermore, the Haagerup tensor product of \(E\) and \(F\) is defined as the completion of \(E \otimes F\) with respect to the matrix norms
\[
\|u\|_{h,m} = \inf \{\|v\|_{\mathcal{M}} : u = v \otimes w, v \in M_{m,r}(E), w \in M_{r,m}(F), r \in \mathbb{N}\},
\]
where the element \(v \otimes w \in M_m(E \otimes F)\) is defined by \((v \otimes w)_{ij} = \sum_{k=1}^{m} v_{ik} \otimes w_{kj}\), for all \(1 \leq i, j \leq m\). The Haagerup tensor product is both injective and projective, associative, self-dual in the finite dimensional case (see [17]) and, in general, not commutative. Moreover, it behaves very nicely with respect to interpolation (see [34], [46]), namely,
Theorem 2.1. (Kouba) Let $(E_0, E_1)$ and $(F_0, F_1)$ be two compatible couples of operator spaces. Then $(E_0 \otimes^h F_0, E_1 \otimes^h F_1)$ is a compatible couple of operator spaces, and for all $0 < \theta < 1$ we have a complete isometry

$$[E_0 \otimes^h F_0, E_1 \otimes^h F_1]_\theta = [E_0, E_1]_\theta \otimes^h [F_0, F_1]_\theta.$$ 

We refer to the Appendix in [39] for a detailed proof, based on ideas of Pisier from [44].

The Schatten $p$-classes $S_p$ ($1 \leq p \leq \infty$) are non-commutative analogues of the Banach spaces $l_p$. We briefly recall the definition and discuss their operator space structure. If $m$ is a positive integer, denote by $S^m_\infty$ the space $M_m$, equipped with the norm $\| \cdot \|_\infty$ determined by its identification with $B(l^m_2)$. Also, we denote by $S^m_1$ the space $\{ \alpha \in M_m : \| \alpha \|_1 = \text{tr}((\alpha^* \alpha)^{1/2}) < \infty \}$. If $1 < p < \infty$, let $S^m_p = \{ \alpha \in M_m : \| \alpha \|^p \in S^m_1 \}$. It follows that, isometrically, $S^m_p = [S^m_\infty, S^m_1]_{1/p}$.

Note that the duality $M^*_m = S^m_1$ is given by the following parallel duality bracket

$$\langle [\beta_{ij}], [\alpha_{ij}] \rangle = \sum_{i,j=1}^m \beta_{ij} \alpha_{ij} = \text{tr}(\beta \alpha^t),$$

where $\alpha^t$ denotes the usual transposed of the matrix $\alpha$. Thus $S^m_1 = M^*_m$ has the operator space structure of the standard dual of $M_m$. Following Pisier [48], we equip $S^m_p$ with the operator space structure (2.2) obtained by interpolation. Respectively, in the infinite dimensional case, we denote by $S^\infty_\infty$ the space $K(l_2)$ of compact operators on $l_2$, equipped with the operator norm. Then $S^\infty_\infty$ carries a natural operator space structure. Let $S^\infty_1 = S^\infty_1$, equipped with the dual operator space structure. If $1 < p < \infty$, we have isometrically,

$$S^m_p = [S^\infty_\infty, S^\infty_1]_{1/p},$$

and we equip $S^m_p$ with the operator space structure (2.2) obtained by interpolation.

In the following, let $E$ be an operator space. Pisier [48] constructed by interpolation the non-commutative vector-valued Schatten $p$-classes $S^m_p[E], 1 \leq p \leq \infty$. For all $m \geq 1$, define

$$S^m_\infty[E] = S^m_\infty \otimes E, \quad S^\infty_\infty[E] = S^\infty_\infty \otimes E,$$

$$S^m_1[E] = S^m_1 \otimes E, \quad S^\infty_1[E] = S^\infty_1 \otimes E,$$

It turns out that $(S^\infty_\infty[E], S^\infty_1[E])$ (respectively, $(S^m_\infty[E], S^m_1[E])$) is a compatible couple for interpolation, and, for $1 < p < \infty$ and all positive integers $m$ we define

$$S^m_p[E] = [S^m_\infty[E], S^m_1[E]]_{1/p}, \quad S^m_p[E] = [S^\infty_\infty[E], S^\infty_1[E]]_{1/p}.$$

We equip $S^m_p[E]$ (respectively, $S^m_p[E]$) with the operator space structure (2.2). The noncommutative vector-valued Schatten $p$-classes can be expressed in terms of the Haagerup tensor product. Indeed, for $1 \leq p \leq \infty$ and all positive integers $m$, let

$$C^m_p = [C, R]_{1/p}, \quad C^m_p = [C^m, R^m]_{1/p},$$

$$R^m_p = [R, C]_{1/p}, \quad R^m_p = [R^m, C^m]_{1/p},$$

$$\| \cdot \|_1 = \text{tr}((\cdot^* \cdot)^{1/2})$$

where $\cdot^*$ denotes the usual transposed of the matrix $\cdot$. Thus $S^m_1 = M^*_m$ has the operator space structure of the standard dual of $M_m$. Following Pisier [48], we equip $S^m_p$ with the operator space structure (2.2) obtained by interpolation. Respectively, in the infinite dimensional case, we denote by $S^\infty_\infty$ the space $K(l_2)$ of compact operators on $l_2$, equipped with the operator norm. Then $S^\infty_\infty$ carries a natural operator space structure. Let $S^\infty_1 = S^\infty_1$, equipped with the dual operator space structure. If $1 < p < \infty$, we have isometrically,
where $C$ and $R$ denote, respectively, the column Hilbert space and the row Hilbert space. We should point out that we are using a different notation from the one in [48]. Namely, the space $C_p$ is denoted therein by $C \left( \frac{1}{p} \right)$ or $C[p]$ (respectively, $C_p^m$ is denoted by $C_m \left( \frac{1}{p} \right)$ or $C_m[p]$). Similarly, the space $R_p$ is denoted in [48] by $R \left( \frac{1}{p} \right)$ or $R[p]$ (respectively, $R_p^m$ is denoted by $R_m \left( \frac{1}{p} \right)$ or $R_m[p]$). Using Kouba’s interpolation result, Pisier (see [48]) proved that for $1 \leq p \leq \infty$ the following relations hold, completely isometrically,

\begin{equation}
S_p[E] = C_p \otimes^h E \otimes^h R_p, \quad S_p^m[E] = C_p^m \otimes^h E \otimes^h R_p^m.
\end{equation}

Let $\frac{1}{p} + \frac{1}{p'} = 1$. Under the parallel duality bracket (2.12) we have the following complete isometries

\begin{align*}
S_p[E]^* &= S_{p'}[E'^*], \\
S_p^m[E]^* &= S_{p'}^m[E'^*].
\end{align*}

Let $\mathcal{M}$ be a von Neumann algebra equipped with an n.f. tracial state $\tau$. For $1 \leq p < \infty$, the noncommutative $L_p(\mathcal{M}, \tau)$ space is defined as the closure of $\mathcal{M}$ with respect to the norm

$$
\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}.
$$

The trace $\tau$ induces a canonical contractive embedding $j : \mathcal{M} \to \mathcal{M}_*$ (where $\mathcal{M}_*$ denotes the unique predual of $\mathcal{M}$), given by

\begin{equation}
\langle j(x), y \rangle = \tau(xy).
\end{equation}

With this embedding, $(\mathcal{M}, \mathcal{M}_*)$ is a compatible couple for interpolation, and for $1 \leq p < \infty$ we have the isometry

$$
L_p(\mathcal{M}, \tau) = [\mathcal{M}, \mathcal{M}_*]_{\frac{1}{p}}.
$$

We now turn to the description of the appropriate operator space matrix norms on the noncommutative $L_p$-spaces. The space $L_\infty(\mathcal{M}) = \mathcal{M}$ carries a natural operator space structure, since $\mathcal{M}$ is a $C^*$-algebra. In order to describe the operator space structure on $L_1(\mathcal{M}, \tau)$ by keeping the trace duality pairing (2.10), we have to consider the opposite von Neumann algebra $\mathcal{M}^{\text{op}}$, as explained in [27]. Recall that $\mathcal{M}^{\text{op}} = \mathcal{M}$ as a vector space, but it is endowed instead with the reversed multiplication $x \circ y = yx$, for all $x, y \in \mathcal{M}$. The algebra $\mathcal{M}^{\text{op}}$ carries a natural operator space structure, with matrix norms defined for all positive integers $m$ by

\begin{equation}
\| x_{ij} \|_{M_m(\mathcal{M}^{\text{op}})} = \| x_{ji} \|_{M_m(\mathcal{M})}.
\end{equation}

Following Junge and Ruan [27], we define the operator space structure on $L_1(\mathcal{M}, \tau)$ by

\begin{equation}
L_1(\mathcal{M}, \tau) \cong (\mathcal{M}^{\text{op}})_*.
\end{equation}

The identification (2.12) is given by the complete isometry

$$
x \in L_1(\mathcal{M}, \tau) \mapsto \tau_x \in (\mathcal{M}^{\text{op}})_*,
$$

where $\tau_x(y) = \tau(xy)$, for all $y \in \mathcal{M}^{\text{op}}$. For $1 < p < \infty$ and all positive integers $m$ define

\begin{equation}
M_m(L_p(\mathcal{M}, \tau)) = [M_m(\mathcal{M}), M_m(L_1(\mathcal{M}, \tau))]_{\frac{1}{p}}.
\end{equation}
These matrix norms verify the Ruan axioms \(1.1\); hence they determine the natural operator space structure on \(L_p(\mathcal{M}, \tau)\). Furthermore, the following Fubini-type theorem holds isometrically,

\[
(2.14) \quad S_p^m[L_p(\mathcal{M}, \tau)] = L_p(M_m \otimes \mathcal{M}, \text{tr}_m \otimes \tau),
\]

for any positive integer \(m\) and \(1 \leq p \leq \infty\). Indeed, we have

\[
\|\sum_{i,j} x_{ij} y_{ij}\|_{S_p^m[L_1(\mathcal{M}, \tau)\| \leq \sup_{\|y_{ij}\|_{M_m(\mathcal{M}^{op})} \leq 1} \left\| \sum_{i,j} x_{ij} y_{ij} \right\| = \sup_{\|y_{ij}\|_{M_m(\mathcal{M}^{op})} \leq 1} \left\| \sum_{i,j} \tau(x_{ij} y_{ij}) \right\| = \sup_{\|z_{ij}\|_{M_m(\mathcal{M})} \leq 1} \left\| (\text{tr}_m \otimes \tau)([x_{ij}] \cdot [z_{ij}]) \right\| = \|\sum_{i,j} x_{ij} y_{ij}\|_{L_1(M_m \otimes \mathcal{M}, \text{tr}_m \otimes \tau)}.
\]

This proves \(2.14\) for \(p = 1\). For \(p = \infty\) the statement is clearly true. By Corollary 1.4 in [48], interpolation yields \(2.14\) for all \(1 \leq p \leq \infty\).

In the following assume, moreover, that \(\mathcal{M}\) is hyperfinite i.e., \(\mathcal{M}\) is the \(w^*\)-closure of an increasing net \((\mathcal{M}_n)_{n \geq 1}\) of finite dimensional von Neumann subalgebras. A celebrated theorem of Connes [13] establishes the connection between hyperfiniteness and injectivity. Namely, Connes proved that a von Neumann algebra \(\mathcal{M}\) is hyperfinite if and only if it is injective. Recall that a \(C^*\)-algebra \(A\) is called injective if and only if given \(C^*\)-algebras \(B\) and \(B_1\) such that \(B \subseteq B_1\) and a completely positive map \(\phi : B \to A\), then there exists a completely positive map \(\phi_1 : B_1 \to A\) such that \(\phi_1 | B = \phi\). A von Neumann algebra is called injective if it is injective as a \(C^*\)-algebra.

The following characterization of injective von Neumann algebras (see Torpe [57]) is very useful in applications. A von Neumann algebra \(\mathcal{M} \subseteq \mathcal{B}(H)\) is injective if and only if there exists a norm 1 projection \(E : \mathcal{B}(H) \to \mathcal{M}\), which is onto. The class of injective von Neumann algebras includes \(\mathcal{B}(H)\), and in particular, matrix algebras, the classical \(L_\infty\)-spaces, group von Neumann algebras associated to amenable groups, and it is closed under von Neumann algebra tensor products. Also, if \(\mathcal{M}\) is an injective von Neumann algebra and \(e\) is a projection in \(\mathcal{M}\), then \(e \mathcal{M} e\) is an injective von Neumann algebra.

Let \(E\) be an operator space. Following Pisier [48], define

\[
(2.15) \quad L_1(\mathcal{M}; E) = L_1(\mathcal{M}, \tau) \hat{\otimes} E,
\]

where the operator space structure of \(L_1(\mathcal{M}, \tau)\) is given by \(2.12\). Since \(\mathcal{M}\) is a finite and hyperfinite von Neumann algebra, it follows by results of Effros and Ruan (see [13] and [19]) that the canonical inclusion

\[
v : \mathcal{M} \hookrightarrow (\mathcal{M}^{op})_* \cong L_1(\mathcal{M}, \tau)
\]

is integral. Then, as explained in [48], the map \(v \otimes \text{Id}_E\) extends to a complete contraction

\[
(2.16) \quad \hat{v} : \mathcal{M} \hat{\otimes} E \hookrightarrow L_1(\mathcal{M}, \tau) \hat{\otimes} E.
\]

Therefore \((\mathcal{M} \hat{\otimes} E, L_1(\mathcal{M}; E))\) is a compatible couple for interpolation. For \(1 < p < \infty\), define

\[
(2.17) \quad L_p(\mathcal{M}; E) = [\mathcal{M} \hat{\otimes} E, L_1(\mathcal{M}; E)]_{\frac{1}{p}},
\]

and

\[
(2.18) \quad \mathcal{M} \hat{\otimes} E
\]
and equip $L_p(M; E)$ with the operator space matrix norms \((2.2)\) obtained by interpolation. The following Fubini theorem (see (3.6)’ of [18]) is a consequence of \((2.14)\). Let $1 \leq p \leq \infty$, then, for all von Neumann algebras $N$ with an n.f. tracial state $\phi$, we have the complete isometry
\[
L_p(M; L_p(N, \phi)) = L_p(M \otimes N, \tau \otimes \phi).
\]

We now discuss duality in the vector-valued setting. We show that if $M$ is finite dimensional, then the duality results hold under the trace duality pairing \((2.10)\), therefore the theory is consistent with the scalar-valued case.

**Proposition 2.2.** Let $1 \leq p < \infty$ and $p'$ be the conjugate exponent of $p$, i.e., $1/p + 1/p' = 1$. If $M$ is a finite dimensional von Neumann algebra equipped with an n.f. tracial state $\tau$, then the following complete isometry holds under the trace duality bracket \((2.10)\)
\[
(L_{p'}(M; E))^* = L_p(M^\text{op}; E^*)
\]

**Proof.** Since $M$ is finite dimensional, it follows that $L_{p'}(M; E) = S_{p^m} \oplus \ldots \oplus S_{p^m}$ for some positive integers $m_1, \ldots, m_k$. For simplicity of the argument, we will assume that
\[
L_{p'}(M; E) = S_{p^m}[E],
\]
where $m$ is a positive integer. We have
\[
\|([x_{ij}])(L_{p'}(M; E))^*) = \sup_{\|y_{ij}\|_{L_{p'}(M; E)} \leq 1} \{\|([x_{ij}], [y_{ij}])\| = \sup_{\|y_{ij}\|_{L_{p'}(M; E)} \leq 1} \left\{ \left\| \sum_{i,j=1}^m x_{ij}^* (y_{ij}) \right\| \right\}
\]
where the last equality follows from the following considerations. By \((2.11)\), the map $\psi_m : M \to M^\text{op}$, defined by $\psi_m([v_{ij}]_{i,j=1}^m) = [v_{ij}]_{i,j=1}^m$ is a complete isometry. Moreover, it extends to a complete isometry $\psi_m : L_1(M, \tau) \to L_1(M^\text{op}, \tau)$. By properties of the injective and projective tensor product, it follows that $\psi_m \otimes \text{Id}_{E^*}$ extends, respectively, to complete isometries
\[
M \otimes E^* \xrightarrow{\psi_m \otimes \text{Id}_{E^*}} M^\text{op} \otimes E^*,
\]
\[
L_1(M, \tau) \otimes E^* \xrightarrow{\psi_m \otimes \text{Id}_{E^*}} L_1(M^\text{op}, \tau) \otimes E^*.
\]
By \((2.17)\), interpolation with exponent $\theta = \frac{1}{p}$ shows that the map $\psi_m \otimes \text{Id}_{E^*} : L_p(M; E^*) \to L_{p'}(M^\text{op}; E^*)$ is a complete isometry, which completes the argument. \qed

### 3. Operator space OUMD$_p$: Definitions and Properties

Let $(M, \tau)$ be a von Neumann algebra equipped with an n.f. tracial state $\tau$. Let $(M_n)_{n \geq 1}$ be an increasing filtration of von Neumann subalgebras of $M$ such that $M = (\bigcup_n M_n)^{\text{\tiny$w^*$}}$. Given a positive integer $n$, there is a unique normal conditional expectation $E_n : M \to M_n$ such that
For all $a \in L_a(M_n), b \in L_r(M_n)$ and $x \in L_p(M)$, where $1/p = 1/r + 1/s$.

A non-commutative $L_p(M)$-martingale relative to the filtration $(M_n)_{n \geq 1}$ is a sequence $x = (x_n)_{n \geq 1}$ such that $x_n \in L_p(M)$ and $E_n(x_{n+1}) = x_n$, for all positive integers $n$. We say that $x$ is a bounded $L_p(M)$-martingale if $\|x\|_p = \sup_n \|x_n\|_p < \infty$. The difference sequence of $x$ is $dx = (dx_n)_{n \geq 1}$, where $dx_n = x_n - x_{n-1}$, with $x_0 = 0$. For $1 < p < \infty$, as a consequence of the uniform convexity of the space $L_p(M)$, we can and will identify the space of all bounded $L_p(M)$-martingales with $L_p(M)$ itself (see [37], Remark 1.3).

**Proposition 3.1.** Let $1 < p < \infty$. There exists $c_p > 0$, depending only on $p$, such that

$$\|T_\varepsilon : L_p(M, \tau) \to L_p(M, \tau)\|_{cb} \leq c_p,$$

where $\varepsilon$ denotes a sequence $(\varepsilon_n)_{n \geq 1}$ of numbers in $\{-1, 1\}$ and $T_\varepsilon$ is the $\pm 1$ martingale transform generated by $\varepsilon$, i.e., $T_\varepsilon \left( \sum_{n=1}^{k} dx_n \right) = \sum_{n=1}^{k} \varepsilon_n dx_n$, for all positive integers $k$ and all $L_p(M)$-martingale difference sequences $dx = (dx_n)_{n=1}^{k}$ relative to the filtration $(M_n)_{n \geq 1}$.

**Proof.** As a consequence of the Pisier-Xu noncommutative version of the Burkholder-Gundy square function inequalities, there exists a constant $c_p > 0$ such that

$$\|T_\varepsilon : L_p(M, \tau) \to L_p(M, \tau)\| \leq c_p.$$

By Lemma 1.7 of [38], we have

$$\|T_\varepsilon\|_{cb} = \sup_m \|\text{Id}_{S_p^m} \otimes T_\varepsilon : S_p^m[L_p(M, \tau)] \to S_p^m[L_p(M, \tau)]\|.$$

Let $m \geq 2$. We will prove that

$$\|\text{Id}_{M_m} \otimes T_\varepsilon : S_p^m[L_p(M, \tau)] \to S_p^m[L_p(M, \tau)]\| \leq c_p.$$

By Fubini’s theorem [214] we have the isometry $S_p^m[L_p(M, \tau)] = L_p(M_m \otimes M, \text{tr}_m \otimes \tau)$, where $\text{tr}_m$ is the standard normalized trace on $M_m$. Note that $(M_m \otimes M_n)_{n \geq 1}$ is a filtration of the algebra $M_m(M) = M_m \otimes M$. For all positive integers $n$, denote $\text{Id}_{M_m} \otimes E_n$ by $E_n$. Then $E_n : M_m(M) \to M_m(M_n)$ is the unique trace preserving conditional expectation onto $M_m(M_n)$.

Moreover, for all $x = [x_{ij}]_{i,j=1}^{m} \in M_m(L_p(M, \tau))$ and all positive integers $n$ we have

$$[\mathcal{E}_n(x_{ij})]_{i,j=1}^{m} = E_n(x).$$

By applying (3.20), together with (1.23) to the algebra $M_m(M)$ and its filtration $(M_m \otimes M_n)_{n \geq 1}$, we obtain for all positive integers $k$ and all sequences $\varepsilon = (\varepsilon_n)_{n \geq 1}$ of numbers in $\{-1, 1\}$

$$\left\| \sum_{n=1}^{k} (\text{Id}_{M_m} \otimes \varepsilon_n(\mathcal{E}_n - \mathcal{E}_{n-1}))(x) \right\|_p \leq c_p \left\| \sum_{n=1}^{k} (\text{Id}_{M_m} \otimes (\mathcal{E}_n - \mathcal{E}_{n-1}))(x) \right\|_p.$$
This shows that (3.22) holds and the proof is complete. □

**Remark 3.2.** Let $1 < p < \infty$ and $\varepsilon = (\varepsilon_n)_{n \geq 1}$ be a sequence of numbers in $\{-1, 1\}$. Then the $\pm 1$ martingale transform $T_\varepsilon$ generated by $\varepsilon$ is a self adjoint operator on $L_p(M, \tau)$, under the trace duality bracket (2.10). The proof of this fact is similar to the one in the classical setting, due to Burkholder [8]. The key point is that $(\bigcup_n L_p(M_n, \tau_n))^{-1} \rightarrow L_p(M, \tau)$. By (3.20), $T_\varepsilon$ is a bounded linear operator on $L_p(M, \tau)$. Therefore, it suffices to prove that the restriction of $T_\varepsilon$ to $L_p(M_n, \tau_n)$ is self adjoint, for all for all positive integers $n$. Indeed, if $x \in L_p(M_n, \tau_n)$, then $x = \sum_{k=1}^n d_k(x)$, where $d_k = \varepsilon_k \cdot \delta_{k-1}$, with $\delta_0 = 0$. Note that, for all positive integers $j$, $E_j$ is a self-adjoint operator on $L_p(M, \tau)$, since $E_j$ is the dual map of the canonical isometric embedding of $L_1(M_n, \tau)$ into $L_1(M, \tau)$. Therefore each $d_k$ is a self-adjoint operator with respect to the trace duality pairing (2.10). Since $T_\varepsilon(x) = \sum_{k=1}^n \varepsilon_k d_k(x)$, the conclusion follows.

For the remainder of this section we will assume, moreover, that $M$ is hyperfinite. Note that, consequently, each von Neumann subalgebra $M_n$ is hyperfinite, as well. Indeed, since $M$ is injective, there exists a norm 1 projection $E : B(H) \rightarrow M$, which is onto. Composing $E$ with the conditional expectation $\varepsilon_n : M \rightarrow M_n$ we obtain a norm 1 projection $\varepsilon_n \circ E : B(H) \rightarrow M_n$, which is onto. This ensures that $M_n$ is injective, or equivalently, by Connes’ theorem, $M_n$ is hyperfinite.

**Proposition 3.3.** Let $1 \leq p \leq \infty$. Then, for all positive integers $n$, the conditional expectation $\varepsilon_n : L_p(M, \tau) \rightarrow L_p(M_n, \tau_n)$ extends to a complete contraction

$$\varepsilon_n \otimes \text{Id}_E : L_p(M; E) \rightarrow L_p(M_n; E).$$

**Proof.** We first show that we have the complete contraction

$$\varepsilon_n \otimes \text{Id}_E : M \otimes E \rightarrow M_n \otimes E$$

By the injectivity property of the injective tensor product, it suffices to prove that the map $\varepsilon_n : M \rightarrow M_n$ is a complete contraction. Since $\|\varepsilon_n\| = 1$, this is equivalent to showing that $\varepsilon_n$ is completely positive (see Paulsen [42]). This is, indeed, the case, since for all $m \geq 1$, the map $\text{Id}_{M_m} \otimes \varepsilon_n : M_m \otimes M \rightarrow M_m \otimes M_m$ is the unique trace-preserving conditional expectation onto $M_m \otimes M_m$, hence it is positive. We now show that we have the complete contraction

$$\varepsilon_n \otimes \text{Id}_E : L_1(M, \tau) \otimes E \rightarrow L_1(M_n, \tau_n) \otimes E$$

Since the projective tensor product is projective, it suffices to prove that the map $\varepsilon_n : L_1(M, \tau) \rightarrow L_1(M_n, \tau_n)$ is a complete contraction. Let us denote for the moment this map by $u_n$, in order to avoid confusion. Note that, under the trace duality bracket (2.10), the dual map $u_n^*$ is exactly $\varepsilon_n : M \rightarrow M_n$. Indeed, for $x \in L_1(M, \tau)$ and $y \in L_1(M_n, \tau_n)$ we have

$$\langle u_n(x), y \rangle = \langle x, \varepsilon_n(y) \rangle.$$
It follows that $\|u_n\|_{cb} = \|u_n^*\|_{cb} = \|\xi_n\|_{cb} \leq 1$, which proves the assertion. By (3.25) and (3.26), interpolation with exponent $\theta = 1/p$ yields the conclusion for $1 < p < \infty$. \hfill \Box

**Remark 3.4.** In view of Proposition 3.3 we can consider vector-valued noncommutative martingales in this setting. Note that, as in the scalar-valued case, any $L_p(\mathcal{M};E)$-martingale with respect to the filtration $(\mathcal{M}_n)_{n \geq 1}$ can be approximated by finite $L_p(\mathcal{M};E)$-martingales (with respect to the same filtration). Therefore, it suffices to consider finite martingales only.

**Definition 3.5.** Let $E$ be an operator space and $1 < p < \infty$. We say that $E$ is *OUMD$_p$ with respect to the filtration $(\mathcal{M}_n)_{n \geq 1}$ if there exists a constant $c_p > 0$ such that

\begin{equation}
\left\| \sum_{n=1}^{k} \xi_n dx_n \right\|_{L_p(\mathcal{M};E)} \leq c_p \left\| \sum_{n=1}^{k} dx_n \right\|_{L_p(\mathcal{M};E)}
\end{equation}

for all positive integers $k$, all sequences $\xi = (\xi_n)_{n=1}^{k}$ of numbers in $\{-1,1\}$ and all martingale difference sequences $dx = (d_n)_{n=1}^{k} \subset L_p(\mathcal{M};E)$, relative to the filtration $(\mathcal{M}_n)_{n \geq 1}$.

If this holds for all hyperfinite von Neumann algebras $\mathcal{M}$, equipped with an n.f. tracial state $\tau$, and all filtrations of $\mathcal{M}$, we say that $E$ is *OUMD$_p$.

**Remark 3.6.** By Definition 3.5, $E$ is *OUMD$_p$ with respect to the filtration $(\mathcal{M}_n)_{n \geq 1}$ if and only if there exists a constant $c_p > 0$, depending on $p$ and $E$, such that

\begin{equation}
\left\| T_\xi \otimes \text{Id}_E : L_p(\mathcal{M};E) \to L_p(\mathcal{M};E) \right\| \leq c_p,
\end{equation}

for all finite sequences $\xi = (\xi_n)_{n \geq 1}$ of numbers in $\{-1,1\}$. Martingale differences are considered with respect to the filtration $(\mathcal{M}_n)_{n \geq 1}$. Note that a priori the constant $c_p$ might also depend on the algebra $\mathcal{M}$ and on the filtration.

**Lemma 3.7.** If $E$ is *OUMD$_p$, there exists $c_p > 0$, depending only on $p$ and $E$, such that

\begin{equation}
\left\| T_\xi \otimes \text{Id}_E : L_p(\mathcal{M};E) \to L_p(\mathcal{M};E) \right\| \leq c_p,
\end{equation}

for all hyperfinite von Neumann algebras $(\mathcal{M},\tau)$, equipped with an n.f. tracial state $\tau$, all filtrations of $\mathcal{M}$ and all finite sequences $\xi = (\xi_n)_{n \geq 1}$ of numbers in $\{-1,1\}$.

**Proof.** Assume that there exists a sequence $(\mathcal{M}_k, \tau_k)_{k \geq 1}$ of hyperfinite von Neumann algebras, equipped with n.f. tracial states $\tau_k$, such that the sequence of corresponding constants $c_p^{(k)}$ converges to $\infty$. We will show that this leads to a contradiction. Let $\mathcal{M} = \bigoplus_k \mathcal{M}_k$ be the direct sum of the algebras $\mathcal{M}_k$. Note that $\mathcal{M}$ is a hyperfinite von Neumann algebra and that we can define an n.f. tracial state $\tau$ on $\mathcal{M}$ by

$$\tau((x_k)_{k \geq 1}) = \sum_{k \geq 1} \frac{1}{2^k} \tau_k(x_k).$$

For each positive integer $k$, there is an increasing net of finite dimensional algebras $(\mathcal{M}_k^{(n)})_{n \geq 1}$ whose union generates $\mathcal{M}_k$ in the $w^*$-topology. For a finite sequence $\xi = (\xi_n)_{n \geq 1}$ of numbers
in \{-1,1\}, let \( T^k_\varepsilon \) denote the vector-valued \( \pm 1 \) martingale transform generated by \( \varepsilon \), where martingale differences are considered with respect to the algebra \( \mathcal{M}_k \) and its filtration \( (\mathcal{M}_k(n))_{n \geq 1} \). For all positive integers \( n \), let
\[
\mathcal{M}(n) = \bigoplus_k \mathcal{M}_k(n).
\]
Then \( (\mathcal{M}(n))_{n \geq 1} \) is a filtration of \( \mathcal{M} \). Let \( T_\varepsilon \) denote the \( \pm 1 \) martingale transform generated by \( \varepsilon \), where martingales are considered with respect to the algebra \( \mathcal{M} \) and its filtration \( (\mathcal{M}(n))_{n \geq 1} \). By our assumption it follows that for all positive integers \( k \), there exists \( y_k \in L_p(\mathcal{M}_k; E) \) such that \( \| y_k \|_{L_p(\mathcal{M}_k; E)} = 1 \) and
\[
\| (T^k_\varepsilon \otimes \text{Id}_E)(y_k) \|_{L_p(\mathcal{M}_k; E)} \geq c_p^{(k)}.
\]
Let \( Y_k = (0, \ldots, 0, y_k, 0, \ldots) \). It follows that
\[
\| (T^k_\varepsilon \otimes \text{Id}_E)(Y_k) \|_{L_p(\mathcal{M}_k; E)} = \left( \frac{1}{2^k} \tau_k \left( \| (T^k_\varepsilon \otimes \text{Id}_E)(y_k) \|^p \right) \right)^{1/p} = \frac{1}{2^{k/p}} \| (T^k_\varepsilon \otimes \text{Id}_E)(y_k) \|_{L_p(\mathcal{M}_k; E)} \geq \frac{1}{2^{k/p}} c_p^{(k)}.
\]
Moreover, we have
\[
\| Y_k \|_{L_p(\mathcal{M}; E)} = \left( \frac{1}{2^k} \tau_k (|y_k|^p) \right)^{1/p} = \frac{1}{2^{k/p}} \| y_k \|_{L_p(\mathcal{M}_k; E)} = \frac{1}{2^{k/p}} c_p^{(k)}.
\]
This shows that \( \| T_\varepsilon \otimes \text{Id}_E : L_p(\mathcal{M}; E) \to L_p(\mathcal{M}; E) \| \geq c_p^{(k)} \), for all positive integers \( k \). This implies that \( \| T_\varepsilon \otimes \text{Id}_E : L_p(\mathcal{M}; E) \to L_p(\mathcal{M}; E) \| = \infty \), which contradicts the assumption that \( E \) is \( \text{OUMD}_p \).

We denote by \( c_p(E) \) the smallest constant \( c_p > 0 \) satisfying (3.30).

Remark 3.8. A similar argument as in the proof of Proposition 3.1 shows that an operator space \( E \) is \( \text{OUMD}_p \), for some \( 1 < p < \infty \) if and only if
\[
(3.31) \quad \| T_\varepsilon \otimes \text{Id}_E : L_p(\mathcal{M}; E) \to L_p(\mathcal{M}; E) \|_{cb} \leq c_p(E),
\]
for all hyperfinite von Neumann algebras \( (\mathcal{M}, \tau) \) with an n.f. tracial state \( \tau \), all filtrations of \( \mathcal{M} \) and all finite sequences \( \varepsilon = (\varepsilon_n)_{n \geq 1} \) of numbers in \( \{-1,1\} \).

Remark 3.9. If we assume that \( \mathcal{M} \) is a commutative von Neumann algebra in the Definition 3.5, then we recover the classical notion of a \( \text{UMD} \) Banach space. In particular, if an operator space \( E \) is \( \text{OUMD}_p \), for some \( 1 < p < \infty \), then \( E \) is \( \text{UMD} \) (as a Banach space) and, moreover,
\[
\beta_p(\mathbb{C}) \leq \beta_p(E) \leq c_p(E).
\]
It was proved by Burkholder [11] that \( \beta_p(\mathbb{C}) = p^* - 1 \), where \( p^* = \max\{p, p/(p - 1)\} \).
Remark 3.10. By Proposition 3.11 and Remark 3.8, the operator space \( E = \mathbb{C} \) is \( OUMD_p \), for \( 1 < p < \infty \). Moreover, as proved by Randrianantoanina [51], the corresponding constant \( c_p(\mathbb{C}) \) has the same optimal order of growth as \( \beta_p(\mathbb{C}) \). This estimate follows directly by interpolation from a striking result proved in [51], namely the fact that the noncommutative \( \pm 1 \) martingale transforms are of weak type \((1, 1)\).

Remark 3.11. \( OUMD_p \) is a local property, i.e., if there exists an increasing sequence of closed subspaces \( (E_k)_{k \geq 1} \) of \( E \), whose union generates \( E \), such that each \( E_k \) is \( OUMD_p \) and the corresponding constants satisfy \( \sup_k c_p(E_k) < \infty \), then \( E \) is \( OUMD_p \).

The following proposition summarizes some of the properties of \( OUMD_p \) operator spaces. We refer to Pisier [49, 48] for background on the ultraproduct theory for operator spaces.

Proposition 3.12. Let \( 1 < p < \infty \) and \( E \) be an operator space.

1. If \( E \) is \( OUMD_p \) and \( F \) is an operator space completely isomorphic to \( E \), then \( F \) is \( OUMD_p \).
2. If \( E \) is \( OUMD_p \) and \( F \) is a closed subspace of \( E \), then both \( F \) and \( E/F \) are \( OUMD_p \).
3. If \( E \) is \( OUMD_p \), then \( L_p(\mathcal{N}; E) \) is \( OUMD_p \), for all hyperfinite von Neumann algebras \( \mathcal{N} \), equipped with an n.f. tracial state \( \tau \).
4. If \( E \) is \( OUMD_p \), then \( M_m(E) \) is \( OUMD_p \), for all positive integers \( m \).
5. If \( E \) is \( OUMD_p \), then its dual \( E^* \) is \( OUMD_{p'} \), where \( 1/p + 1/p' = 1 \).
6. Let \( I \) be an index set and \( (E_i)_{i \in I} \) be a family of \( OUMD_p \) operator spaces. Assume that the corresponding \( OUMD_p \) constants satisfy \( \sup_i c_p(E_i) < \infty \). Let \( \mathcal{U} \) be an ultrafilter on \( I \). Then the ultraproduct \( \hat{E} = (E_i)_{\mathcal{U}} \) is \( OUMD_p \).
7. If \( E \) and \( F \) are \( OUMD_p \), then \( E \oplus_q F \) is \( OUMD_p \), for \( 1 < q < \infty \).
8. Let \( 1 < q, s < \infty \) and \( 0 < \theta < 1 \) be such that \( (1 - \theta)/q + \theta/s = 1/p \). If \( (E_0, E_1) \) is a compatible couple of operator spaces such that \( E_0 \) is \( OUMD_q \) and \( E_1 \) is \( OUMD_s \), then \( E_\theta = [E_0, E_1]_\theta \) is \( OUMD_p \).

Proof. Let \( c_p(E) \) denote the \( OUMD_p \) constant of \( E \). Throughout the proof, let \( (\mathcal{M}, \tau) \) be a hyperfinite von Neumann algebra with an n.f. tracial state \( \tau \), and \( (\mathcal{M}_n)_{n \geq 1} \) a filtration of \( \mathcal{M} \). Martingale differences will be considered with respect to this filtration. Let \( \varepsilon = (\varepsilon_n)_{n \geq 1} \) be a finite sequence of numbers in \( \{-1, 1\} \) and denote by \( T_\varepsilon \) the corresponding \( \pm 1 \) martingale transform. For each statement, we will prove the boundedness of the appropriate vector-valued martingale transform.

1. Let \( u \) be a complete isomorphism \( u : E \to F \). Then, as observed in (3.1) of [48], the map \( \text{Id}_{L_p(\mathcal{M}, \tau)} \otimes u \) extends to a complete isomorphism \( \tilde{u} : L_p(\mathcal{M}; E) \to L_p(\mathcal{M}; F) \). Hence, there exists a constant \( c'_p(F) \), such that \( \| T_\varepsilon \otimes \text{Id}_F : L_p(\mathcal{M}; F) \to L_p(\mathcal{M}; F) \|_{cb} \leq c'_p(F) \). By Remark 3.8, this implies that \( F \) is \( OUMD_p \).

2. By (3.4) of [48], the space \( L_p(\mathcal{M}; F) \) can be identified with a closed subspace of \( L_p(\mathcal{M}; E) \),
and we have a complete isometry \( L_p(M; E/F) \cong L_p(M; E)/L_p(M; F) \). Hence
\[
\| T_\varepsilon \otimes \text{Id}_F : L_p(M; F) \to L_p(M; F) \|_{cb} \leq c_p(E),
\]
\[
\| T_\varepsilon \otimes \text{Id}_{E/F} : L_p(M; E/F) \to L_p(M; E/F) \|_{cb} \leq c_p(E).
\]
Therefore \( F \) and \( E/F \) are both OUMD\(_p\). Moreover, the corresponding OUMD\(_p\) constants satisfy
\[
(3.32) \quad c_p(F) \leq c_p(E), \quad c_p(E/F) \leq c_p(E).
\]
(3) An application of Fubini’s theorem \([2.18]\) yields the complete isometry
\[
(3.33) \quad L_p(M; L_p(N; E)) \cong L_p(M \otimes N; E).
\]
The assertion follows from the fact that \( E \) is OUMD\(_p\) with respect to the hyperfinite algebra \( M \otimes N \) and its filtration \((M_n \otimes N)_n \geq 1\). Moreover, the following estimate holds
\[
(3.34) \quad c_p(L_p(N; E)) \leq c_p(E).
\]
(4) We prove that \( M_m(E) = M_m \otimes E \) is completely isomorphic to \( S^m_p[E] = L_p(M_m; E) \). Equivalently, we show that \( d_{cb}(M_m(E), L_p(M_m; E)) < \infty \), where the c.b.-Banach-Mazur distance between two operator spaces \( E_0 \) and \( E_1 \) is defined by
\[
d_{cb}(E_0, E_1) = \inf\{\|\phi\|_{cb}\|\phi^{-1}\|_{cb} : \phi : E_0 \to E_1 \text{ is an isomorphism}\}.
\]
Recall, by \([2.16]\) the complete contraction \( \tilde{\varepsilon} : M_m(E) \hookrightarrow L_1(M_m) \otimes E \), induced by the canonical inclusion \( \varepsilon : M \hookrightarrow L_1(M, \tau) \). By interpolation it follows that
\[
(3.35) \quad \| \tilde{\varepsilon} : M_m(E) \to L_p(M_m; E) \|_{cb} \leq 1.
\]
It remains to show that \( \tilde{\varepsilon}^{-1} \in CB(L_p(M_m; E), M_m(E)) \). By results of Effros and Ruan \([20]\), the map \( \tilde{\varepsilon}^{-1} : L_1(M_m) \otimes E \to M_m \otimes E \) is completely bounded if and only if the map \( \varepsilon^{-1} : L_1(M_m) \to M_m \) is nuclear, and, moreover,
\[
(3.36) \quad \| \varepsilon^{-1} : L_1(M_m) \otimes E \to M_m \otimes E \|_{cb} \leq \nu(\varepsilon^{-1}),
\]
where \( \nu(\varepsilon^{-1}) \) denotes the nuclear norm of \( \varepsilon^{-1} \). Let \{\( e_{i,j}, e_{i,j}^* \)\}_{1 \leq i, j \leq m} \in M_m \times M_m^* \) be an Auerbach basis for \( M_m \). Then
\[
\nu(\varepsilon^{-1}) \leq \left\| \sum_{i,j=1}^m e_{i,j}^*(e_{i,j}) \right\| \leq \sum_{i,j=1}^m \left\| e_{i,j}^*(e_{i,j}) \right\| \leq m^2.
\]
From \([5.35]\) and the fact that \( \tilde{\varepsilon}^{-1} \) is a complete contraction on \( M_m(E) \), we obtain by interpolation
\[
(3.37) \quad \| \tilde{\varepsilon}^{-1} : L_p(M_m; E) \to M_m(E) \|_{cb} \leq m^{\frac{2}{p}}.
\]
This implies that
\[
d_{cb}(M_m(E), L_p(M_m; E)) \leq m^{\frac{2}{p}} < \infty.
\]
The assertion follows as a consequence of Items (1) and (3).

(5) Since $\mathcal{M}$ is hyperfinite, there exists a filtration $(\mathcal{N}_\alpha)_{\alpha \geq 1}$ of finite dimensional subalgebras whose union generates $\mathcal{M}$. For $\alpha \geq 1$, denote by $\tilde{\mathcal{E}}_\alpha$ the unique trace-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}_\alpha$. By Remark 3.2 in [18] (see also Theorem 3.4. therein) it follows that $(\bigcup_\alpha L_p(\mathcal{N}_\alpha; E^*))_{\alpha}^{-\parallel \cdot \parallel_p} = L_p(\mathcal{M}; E^*)$. This implies that

$$\| T_\varepsilon \otimes \text{Id}_{E^*} \|_{cb} \leq \sup_{\alpha} \| \tilde{\mathcal{E}}_\alpha(T_\varepsilon \otimes \text{Id}_{E^*})\tilde{\mathcal{E}}_\alpha : L_p(\mathcal{N}_\alpha; E^*) \to L_{p'}(\mathcal{N}_\alpha; E^*) \|_{cb}.$$  

The following complete isometry holds under trace duality (see Proposition 2.2):

$$L_{p'}(\mathcal{N}_\alpha; E^*) = (L_p(\mathcal{N}_\alpha^{\text{op}}; E))^*.$$  

Furthermore, since $(\bigcup_\alpha L_p(\mathcal{N}_\alpha^{\text{op}}; E))_{\alpha}^{-\parallel \cdot \parallel_p} = L_p(\mathcal{M}^{\text{op}}; E)$ and $E$ is $OUMD_p$ with respect to $\mathcal{M}^{\text{op}}$, it follows that

$$\| T_\varepsilon \otimes \text{Id}_E \|_{cb} \leq \sup_{\alpha} \| \tilde{\mathcal{E}}_\alpha(T_\varepsilon \otimes \text{Id}_E)\tilde{\mathcal{E}}_\alpha : L_p(\mathcal{N}_\alpha^{\text{op}}; E) \to L_p(\mathcal{N}_\alpha^{\text{op}}; E) \|_{cb} \leq c_p(E).$$

By passing to the dual and using the fact that the conditional expectations $\tilde{\mathcal{E}}_\alpha$ and $T_\varepsilon$ are self-dual under trace duality (see Remark 3.2), we obtain

$$\| \tilde{\mathcal{E}}_\alpha(T_\varepsilon \otimes \text{Id}_E)^*\tilde{\mathcal{E}}_\alpha : (L_p(\mathcal{N}_\alpha^{\text{op}}; E))^* \to (L_p(\mathcal{N}_\alpha^{\text{op}}; E))^* \|_{cb} \leq c_p(E).$$

Applying (3.39) together with (3.38), it follows that

$$\| T_\varepsilon \otimes \text{Id}_{E^*} : L_{p'}(\mathcal{M}; E^*) \to L_{p'}(\mathcal{M}; E^*) \|_{cb} \leq c_p(E).$$

Hence $E^*$ is $OUMD_{p'}$, and we have the following estimate for the corresponding constant

$$c_{p'}(E^*) \leq c_p(E).$$

(6) With same notation as in Item (5), we have the following estimate

$$\| T_\varepsilon \otimes \text{Id}_{\hat{E}} \|_{cb} \leq \sup_{\alpha} \| \tilde{\mathcal{E}}_\alpha(T_\varepsilon \otimes \text{Id}_{\hat{E}})\tilde{\mathcal{E}}_\alpha : L_p(\mathcal{N}_\alpha; \hat{E}) \to L_p(\mathcal{N}_\alpha; \hat{E}) \|_{cb}.$$  

For all $\alpha \geq 1$, let $m_\alpha = \dim(\mathcal{N}_\alpha)$. It follows that for $i \in I$,

$$L_p(\mathcal{N}_\alpha; E_i) = S_p^{m_\alpha}[E_i], \quad L_p(\mathcal{N}_\alpha; \hat{E}) = S_p^{m_\alpha}[\hat{E}].$$

Furthermore, by Lemma 5.4 in [18], there is a complete isometry

$$\phi_\alpha : S_p^{m_\alpha}[\hat{E}] \to (S_p^{m_\alpha}[E_i])_{\hat{U}}.$$  

By assumption, for all $i \in I$ the operator space $E_i$ is $OUMD_p$. Hence, by (3.41) the map $\tilde{\mathcal{E}}_\alpha(T_\varepsilon \otimes \text{Id}_{E_i})\tilde{\mathcal{E}}_\alpha : S_p^{m_\alpha}[E_i] \to S_p^{m_\alpha}[E_i]$ is completely bounded. Moreover, since $\tilde{\mathcal{E}}_\alpha$ is a complete contraction, we obtain the estimate

$$\| \tilde{\mathcal{E}}_\alpha(T_\varepsilon \otimes \text{Id}_{E_i})\tilde{\mathcal{E}}_\alpha \|_{cb} \leq \| T_\varepsilon \otimes \text{Id}_{E_i} \|_{cb} \leq c_p(E_i).$$

Define a map $\psi_\alpha$ by

$$\psi_\alpha = \left( \tilde{\mathcal{E}}_\alpha(T_\varepsilon \otimes \text{Id}_{E_i})\tilde{\mathcal{E}}_\alpha \right)_{\hat{U}}.$$
By Proposition 10.3.2 in [20] it follows that \( \psi_\alpha \in CB( (S^m_p [ E_i ])_U, (S^m_p [ E_i ])_U ) \) and, moreover, \( \| \psi_\alpha \| _{cb} \leq \sup_{i} c_p (E_i) \). Therefore, we obtain the following commuting diagram of completely bounded maps

\[
\begin{array}{ccc}
S^m_p [ E ] & \xrightarrow{\tilde{E}_\alpha \otimes \text{Id} \tilde{E}} & S^m_p [ \hat{E} ] \\
\downarrow \phi_\alpha & & \downarrow \phi_{\alpha}^{-1} \\
(S^m_p [ E_i ])_U & \xrightarrow{\psi_\alpha} & (S^m_p [ E_i ])_U
\end{array}
\]

We deduce that

\[
\| \tilde{E}_\alpha (T_\varepsilon \otimes \text{Id} \tilde{E}) \tilde{E}_\alpha \| _{cb} \leq \| \phi_{\alpha}^{-1} \circ \psi_\alpha \circ \phi_\alpha \| _{cb} \leq \| \phi_{\alpha}^{-1} \| _{cb} \| \psi_\alpha \| _{cb} \| \phi_\alpha \| _{cb} \leq \sup_i c_p (E_i) .
\]

By (3.46), this yields the conclusion. Moreover, the \( OUMD_p \) constant of the ultraproduct \( \hat{E} \) satisfies \( c_p (\hat{E}) \leq \sup_i c_p (E_i) \). By general results on ultraproducts, it follows that

\[
c_p (\hat{E}) \leq \lim_U c_p (E_i) .
\]

(7) We first show that \( E \oplus_p F \) is \( OUMD_p \) with respect to \( M \) and the filtration \( (M_n)_{n \geq 1} \). By Remark 3.11 it is enough to prove that there exists \( c_p (E, F) > 0 \) so that for all \( m \geq 1 \),

\[
\| T_\varepsilon \otimes \text{Id} E \oplus_p F : S^m_p [ E \oplus_p F ] \to S^m_p [ E \oplus_p F ] \| _{cb} \leq c_p (E, F) .
\]

Recall that by (2.9) in [18] we have the complete isometry

\[
S^m_p [ E \oplus_p F ] = S^m_p [ E ] \oplus_p S^m_p [ F ] .
\]

It follows that

\[
\| T_\varepsilon \otimes \text{Id} E \oplus_p F : S^m_p [ E \oplus_p F ] \to S^m_p [ E \oplus_p F ] \| _{cb} \leq [(c_p (E))^p + (c_p (F))^p]^{\frac{1}{p}} ,
\]

where \( c_p (E) \) and \( c_p (F) \) are the \( OUMD_p \) constants of \( E \), respectively \( F \). Hence (3.46) is proved. Furthermore, note that if \( 1 < q < \infty \), then

\[
d_{cb} (E \oplus_p F, E \oplus_q F) \leq 2 .
\]

The assertion follows now from Item (1). Moreover, from the proof we obtain the estimate

\[
c_p (E \oplus_q F) \leq [(c_p (E))^p + (c_p (F))^p]^{\frac{1}{p}} .
\]

(8) The statement immediately by interpolation from Remark 3.8 using the following completely isometric identity (see (3.5) in [18])

\[
[L_q (M; E_0), L_q (M; E_1)] = [L_p (M; E_0), L_p (M; E_0)] .
\]

Moreover, the corresponding constants satisfy the estimate

\[
(3.47) \quad c_p (E_0) \leq [c_q (E_0)]^{1-\theta} [c_q (E_1)]^\theta .
\]

This completes the proof. \( \square \)
Example 3.13. If $1 < p < \infty$, then the matrix algebras $M_m$ are $OUMD_p$, for all positive integers $m$. This follows immediately from Remark 3.10 and (4) in Proposition 3.12.

The following lemma provides a necessary condition for an operator space $E$ to be $OUMD_p$.

Lemma 3.14. Let $1 < p < \infty$, and $E$ an operator space. If $E$ is $OUMD_p$, then $S_p[E]$ is $UMD$ (as a Banach space).

Proof. Since $UMD$ is a local property, it suffices to show that for all positive integers $m$, the Banach space $S_p^m[E]$ is $UMD$. We will prove that there exists a constant $k_p > 0$, independent of $m$, such that

\[ ||T_\varepsilon \otimes \text{Id}_{S_p^m[E]} : L_p([0,1], F, \mu); S_p^m[E]) \to L_p([0,1], F, \mu); S_p^m[E]) || \leq k_p. \] (3.48)

Here $\mu$ is the Lebesgue measure on $[0,1]$, $F$ a $\sigma$-algebra of subsets of $[0,1]$, and $(F_n)_{n \geq 1}$ a filtration of $F$. Martingale differences are considered with respect to this filtration. By Fubini’s theorem, we have the complete isometry

\[ L_p([0,1]; S_p^m[E]) = L_p([0,1]; L_p(M_m; E)) = L_p(L_\infty([0,1]); L_p(M_m; E)) = L_p(L_\infty([0,1]) \otimes M_m; E) \] (3.49)

Note that $(L_\infty([0,1], F_n, \mu) \otimes M_m)_{n \geq 1}$ is a filtration of the hyperfinite von Neumann algebra $L_\infty([0,1]) \otimes M_m$. Moreover, for all positive integers $n$, the unique trace-preserving conditional expectation onto the subalgebra $L_\infty([0,1], F_n) \otimes M_m = \mathcal{E}_n = \mathcal{E}_n \otimes \text{Id}_{M_m}$, where $\mathcal{E}_n = E(\cdot | F_n)$. Let $k_p = c_p(E)$, where $c_p(E)$ is the $OUMD_p$ constant of $E$. The fact that $E$ is $OUMD_p$ with respect to the filtration $(L_\infty([0,1], F_n, \mu) \otimes M_m)_{n \geq 1}$ of $L_\infty([0,1]) \otimes M_m$, together with (2.19) yields (3.48), and the proof is complete. □

Example 3.15. There exists a Hilbert space, subspace of some commutative $C^*$-algebra which is $UMD$ (as a Banach space), but not $OUMD_p$, for any $1 < p < \infty$.

Let $E = \min(l_2)$. Then $E$ is a Hilbert space, hence it is $UMD$ (as a Banach space). Assume that $E$ is $OUMD_p$, for some $1 < p < \infty$. We will show that this leads to a contradiction. Following Pisier’s argument in [18], let $(\mathcal{M}, \tau)$ be the hyperfinite $II_1$ factor equipped with the canonical “dyadic filtration” $(\mathcal{M}_n)_{n \geq 1}$, where $\mathcal{M}_n = M_2 \otimes \ldots \otimes M_2$ ($n$ times). By (3) in Proposition 3.12 it follows that $L_p(\mathcal{M}; E)$ is $OUMD_p$; hence it is $UMD$ (as a Banach space). However, by Proposition 4.3 in [18], $L_p(\mathcal{M}; E)$ contains a subspace isomorphic to $c_0$. This contradicts the fact that $L_p(\mathcal{M}; E)$ is $UMD$ (as a Banach space), since $c_0$ is not $UMD$ (being non-reflexive). Therefore $E$ is not $OUMD_p$, for any $1 < p < \infty$. 17
4. Main results

As a consequence of Proposition 3.1 together with Fubini’s theorem, we immediately obtain the following

Proposition 4.1. Let $1 < p < \infty$. Then $L_p(N, \phi)$ is $OUMD_p$, for every von Neumann algebra $N$ equipped with an n.s.f. tracial state $\phi$.

Remark 4.2. Let $1 < p < \infty$ and $m$ be a positive integer. It follows that $S_p = L_p(B(l_2), \text{tr})$ and, respectively, $S^m_p = L_p(M_m, \text{Tr}_m)$ are $OUMD_p$. Here tr denotes the usual trace on $B(l_2)$, while $\text{Tr}_m$ denotes the non-normalized trace on $M_m$. Moreover, the corresponding $OUMD_p$ constants satisfy $\sup_m c_p(S^m_p) \leq c_p(S_p) < \infty$.

Proposition 4.3. The operator Hilbert space $OH$ is $OUMD_p$, for $1 < p < \infty$.

Proof. Recall that by Corollary 2.6 in [46], $OH$ can be obtained by complex interpolation between the column Hilbert space $C$ and the row Hilbert space $R$, namely,

\begin{equation}
OH = [C, R]^{1/2}.
\end{equation}

By the reiteration theorem for the complex method ([2], Theorem 4.6.1), it follows from (2.7) and (2.8) that for $1 < p < \infty$ we have

\begin{equation}
OH = [C_p, R_p]^{1/2}.
\end{equation}

Note that we can view $C_p$ as the column space of $S_p$ and, respectively, we can view $R_p$ as the row space of $S_p$. Hence, by (2) in Proposition 3.12 and Remark 4.2, it follows that both $C_p$ and $R_p$ are $OUMD_p$. Therefore, by (4.51), a further application of (8) in Proposition 3.12 yields the conclusion.

Corollary 4.4. Every finite dimensional operator space $E$ is $OUMD_p$, for $1 < p < \infty$.

Proof. Let $E$ be an $n$-dimensional operator space. By Corollary 9.3 of [46], there is an isomorphism $\phi : OH_n \to E$, such that $\|\phi\|_{cb} \leq \sqrt{n}$. Thus the c.b.Banach-Mazur distance between $E$ and $OH_n$ satisfies

\begin{equation}
d_{cb}(E, OH_n) \leq \sqrt{n}.
\end{equation}

Hence $E$ is completely isomorphic to $OH_n$. The conclusion follows by Proposition 1.3 (applied in the finite dimensional case) and (1) in Proposition 3.12.

Theorem 4.5. If $1 < u, p < \infty$, then $C_u$ is $OUMD_p$. 

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Proof. Since $OUMD_p$ is a local property (see Remark 3.11), it suffices to show that $C_m^u$ is $OUMD_p$, for all positive integers $m$, and
\[ \sup_m c_p(C_m^u) < \infty. \]

Let $(\mathcal{M}, \tau)$ be a hyperfinite von Neumann algebra with an n.f. tracial state $\tau$ and $(\mathcal{M}_n)_{n \geq 1}$ a filtration of $\mathcal{M}$. By Remark 4.2, $S_m^p$ is $OUMD_p$ with respect to the filtration $(\mathcal{M}_n)_{n \geq 1}$ of $\mathcal{M}$. By (2) in Proposition 3.12, the same holds for $C_m^p$ and $R_m^p$, as subspaces of $S_m^p$. We claim that there exist $0 < \theta < 1$ and $1 < q, s < \infty$ such that the following complete isometry holds
\[ [L_q(\mathcal{M}; C_m^p), L_s(\mathcal{M}; R_m^p)]_{1-\theta} = L_p(\mathcal{M}; C_m^u). \]  
(4.52)

By applying (3.5) in [48], this reduces to showing the existence of $0 < \theta < 1$ and $1 < q, s < \infty$ such that the following two relations hold
\[ \frac{1}{p} = \frac{1 - \theta}{q} + \frac{\theta}{s}, \]
(4.53)
\[ C_m^u = [C_m^p, R_m^p]_{\theta}. \]
(4.54)

Let $s'$ denote the conjugate exponent of $s$. Then we have
\[ R_m^s = [R_m^p, C_m^p]_{1-s} = [C_m^p, R_m^p]_{1-\frac{1}{s}} = C_m^{s'}, \]
Therefore, relations (4.53) and (4.54) are equivalent to
\[ \begin{cases} \frac{1}{p} = \frac{1 - \theta}{q} + \frac{\theta}{s}, \\ \frac{1}{u} = \frac{1 - \theta}{q} + \frac{\theta}{s'}. \end{cases} \]
Equivalently,
\[ \frac{1}{p} + \frac{1}{u} - \theta = \frac{2(1 - \theta)}{q} \quad \text{and} \quad \frac{1}{p} - \frac{1}{u} + \theta = \frac{2\theta}{s}. \]

Thus, we have to show that there exists $0 < \theta < 1$ such that
\[ \frac{1}{p} + \frac{1}{u} - \theta > 0, \quad 2(1 - \theta) > \frac{1}{p} + \frac{1}{u} - \theta, \quad \frac{1}{p} - \frac{1}{u} + \theta > 0, \quad 2\theta > \frac{1}{p} - \frac{1}{u} + \theta. \]

These conditions are equivalent to
\[ \left| \frac{1}{p} - \frac{1}{u} \right| < \theta < \min \left\{ \frac{1}{p} + \frac{1}{u} - 2 - \left( \frac{1}{p} + \frac{1}{u} \right) \right\}. \]  
(4.55)

Since $1 < p, u < \infty$, the following relations hold
\[ \left| \frac{1}{p} - \frac{1}{u} \right| < \max \left\{ \frac{1}{p}, \frac{1}{u} \right\} < 1, \]
(4.56)
\[ \min \left\{ \frac{1}{p} + \frac{1}{u} - 2 - \left( \frac{1}{p} + \frac{1}{u} \right) \right\} > 0. \]  
(4.57)
Since $1 < p, u$ we also have
\[
\frac{1}{p} - \frac{1}{u} < 2 - \(\frac{1}{p} + \frac{1}{u}\)
\quad \text{and} \quad
\frac{1}{u} - \frac{1}{p} < 2 - \(\frac{1}{p} + \frac{1}{u}\).
\]
Therefore, we see that
\[
0 < \left| \frac{1}{p} - \frac{1}{u} \right| < \min\left\{ \frac{1}{p} + \frac{1}{u}, 2 - \(\frac{1}{p} + \frac{1}{u}\) \right\},
\]
which implies the existence of some $0 < \theta < 1$ satisfying (4.55). Further, set
\[
q = \frac{2(1-\theta)}{\frac{1}{p} + \frac{1}{u} - \theta} \quad \text{and} \quad
s = \frac{2\theta}{\frac{1}{p} - \frac{1}{u} + \theta}.
\]
The above relations ensure that $1 < q, s < \infty$ and therefore the claim is proved. Since $C^m_u$ is $OUMD_q$ and $R^m_u$ is $OUMD_s$, it follows by interpolation from (4.52) that $C^m_u$ is $OUMD_p$ with respect to the filtration $(\mathcal{M}_n)_{n \geq 1}$ of $(\mathcal{M}, \tau)$. This argument implies that $C^m_u$ is $OUMD_p$. Moreover, using (4.47) and (4.32), we obtain from the proof the following estimates for the corresponding $OUMD_p$ constants
\[
c_p(C^m_u) \leq [c_q(C^m_q)]^{1-\theta}[c_s(R^m_s)]^\theta \leq [c_q(S_q)]^{1-\theta}[c_s(S_s)]^\theta.
\]
This implies that $\sup_m c_p(C^m_u) < \infty$, and the conclusion follows. \qed

**Remark 4.6.** Let $1 < u, p < \infty$. By (2.7) it follows from the equivalence theorem for the complex method (Theorem 4.3.1 in [2]) that
\[
(C_u)^* = R_u' \quad \text{and} \quad (R_u)^* = C_u',
\]
where $\frac{1}{u} + \frac{1}{u'} = 1$. Therefore, by Theorem 4.5 and (5) in Proposition 3.12 it follows that $R_u$ is $OUMD_p$.

**Proposition 4.7.** Let $1 < u, p < \infty$. Then the spaces $C_p \otimes h C_u$, $C_u \otimes h R_p$, $C_p \otimes h R_u$, $R_u \otimes h R_p$ are all $OUMD_p$.

**Proof.** Let $m$ be a positive integer, $(\mathcal{M}, \tau)$ a hyperfinite von Neumann algebra equipped with an n.s.f. tracial state $\tau$ and $(\mathcal{M}_n)_{n \geq 1}$ a filtration of $\mathcal{M}$. By Theorem 4.5 it follows that, in particular, $C^m_u$ is $OUMD_p$ with respect to the filtration $(\mathcal{M}_n \otimes M_m)_{n \geq 1}$ of $(\mathcal{M} \otimes M_m, \tau \otimes \text{tr}_m)$. Therefore,
\[
\| T_\varepsilon \otimes \text{Id}_{C^m_u} : L_p(\mathcal{M} \otimes M_m; C^m_u) \to L_p(\mathcal{M} \otimes M_m; C^m_u) \|_{cb} \leq c_p(C^m_u) \leq c_p(C_u) < \infty.
\]
Fubini’s theorem and (2.9) yield the complete isometries
\[
L_p(\mathcal{M} \otimes M_m; C^m_u) = L_p(\mathcal{M}; L_p(M_m; C^m_u))
= L_p(\mathcal{M}; S^m_p[C^m_u])
= L_p(\mathcal{M}; C^m_p \otimes h C^m_u \otimes h R^m_p).
\]
By the injectivity of the Haagerup tensor product it follows that

\[ L_p(\mathcal{M}; C^m_p \otimes^h C^m_u) \subseteq L_p(\mathcal{M}; C^m_p \otimes^h C^m_u \otimes^h R^m_p); \]
\[ L_p(\mathcal{M}; C^m_u \otimes^h R^m_p) \subseteq L_p(\mathcal{M}; C^m_u \otimes^h C^m_u \otimes^h R^m_p). \]

By (4.59), it follows that \( C^m_p \otimes^h C^m_u \) and \( C^m_u \otimes^h R^m_p \) are both \( OUMD_p \) with respect to the filtration \( (\mathcal{M}_n)_{n \geq 1} \) of \( (\mathcal{M}, \tau) \). Since this filtration was arbitrarily chosen, this argument shows that \( C^m_p \otimes^h C^m_u \) and \( C^m_u \otimes^h R^m_p \) are both \( OUMD_p \). Moreover, from the proof we obtain the following estimates for the corresponding constants

\[ c_p(C^m_p \otimes^h C^m_u), c_p(C^m_p \otimes^h R^m_p) \leq c_p(C^m_u) \leq c_p(C_u) < \infty. \]

A similar argument applied to the space \( R^m_u \), which is \( OUMD_p \) as a subspace of \( R_u \) (see Remark 4.6), shows that \( C^m_p \otimes^h R^m_u \) and \( R^m_u \otimes^h R^m_p \) are both \( OUMD_p \). Moreover,

\[ c_p(C^m_p \otimes^h R^m_u), c_p(R^m_u \otimes^h R^m_p) \leq c_p(R^m_u) \leq c_p(R_u) < \infty. \]

Since \( m \) is arbitrarily chosen, the conclusion follows by density and the fact that \( OUMD_p \) is a local property.

**Proposition 4.8.** Let \( 1 < u, p < \infty \). Then the spaces \( R^p \otimes^h C_u, C_u \otimes^h C_p, R^p \otimes^h R_u \) and \( R_u \otimes^h C_p \) are all \( OUMD_p \).

**Proof.** Let \( m \) be a positive integer. By Proposition 1.7 the operator spaces \( C^m_p \otimes^h C^m_u', C^m_u \otimes^h R^m_p, C^m_p \otimes^h R^m_u \) and \( R^m_u \otimes^h R^m_p \) are all \( OUMD_p' \), where \( \frac{1}{p'} + \frac{1}{u'} = 1 \) and \( \frac{1}{u} + \frac{1}{u'} = 1 \). Hence, by (5) in Proposition 3.12 the dual spaces are \( OUMD_p \). By (4.8) and the self-duality of the Haagerup tensor product in the finite dimensional case (see 17), we obtain the complete isometries

\[ (C^m_p \otimes^h C^m_u')^* = R^p \otimes^h R^m_u, \quad (C^m_u \otimes^h R^m_p')^* = R^m_u \otimes^h C^m_p, \]

respectively,

\[ (C^m_p \otimes^h R^m_u')^* = R^p \otimes^h R^m_u, \quad (R^m_u \otimes^h R^m_p')^* = C^m_u \otimes^h C^m_p. \]

Moreover, from (4.60), (4.61) and (3.42) we obtain the following estimates for the corresponding \( OUMD_p \) constants

\[ c_p(R^m_u \otimes^h C^m_p), c_p(C^m_u \otimes^h C^m_p) \leq c_p'(C^m_u) \leq c_p'(C_u) < \infty, \]
\[ c_p(R^m_u \otimes^h R^m_p), c_p(C^m_u \otimes^h R^m_p) \leq c_p'(R^m_u) \leq c_p'(R_u) < \infty. \]

The conclusion follows from the fact that \( OUMD_p \) is a local property.

**Proposition 4.9.** Let \( 1 < p < \infty \). If \( 1 < q < \infty \) satisfies \( \frac{2p}{p+1} < q < 2p \), then \( S_q \) is \( OUMD_p \).
Proof. We first show that there exist $0 < \theta < 1$ and $1 < u, v < \infty$ such that

\begin{align}
\begin{aligned}
[C_u, C_p]_\theta &= C_q \\
[R_p, R_v]_\theta &= R_q.
\end{aligned}
\end{align}

This is equivalent to showing that there exist $0 < \theta < 1$ and $1 < u, v < \infty$ such that the following relations hold

\begin{align}
\begin{aligned}
\frac{1}{q} &= \frac{1 - \theta}{u} + \frac{\theta}{p}, \\
\frac{1}{q} &= \frac{1 - \theta}{p} + \frac{\theta}{v}.
\end{aligned}
\end{align}

(4.63)

(4.64)

By the assumption on $q$ we have $\frac{2}{q} < 1 + \frac{1}{p}$. An easy computation shows that

$$p' \left( \frac{1}{q} - \frac{1}{p} \right) < \frac{p'}{q'},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. This ensures the existence of some $0 < \theta < 1$ such that

\begin{align}
\begin{aligned}
\left| p' \left( \frac{1}{q} - \frac{1}{p} \right) \right| &< \theta < \min \left\{ 1, \frac{p'}{q'} \right\}.
\end{aligned}
\end{align}

(4.65)

By (4.65) it follows that $\frac{1}{q} - \frac{\theta}{p} < 1 - \theta$. This implies the existence of some $1 < u < \infty$ such that (4.63) holds. Furthermore, (4.65) also shows that $\frac{1}{q} < 1 - \frac{\theta}{p} + \theta$. This implies the existence of some $1 < v < \infty$ such that (4.64) holds. Therefore the claim is proved.

An application of Kouba’s interpolation result (Theorem 2.1) yields the complete isometry

\begin{align}
\begin{aligned}
[C_u \otimes R_p, C_p \otimes h \otimes [R_p, R_v]] &\quad = \quad [C_u, C_p]_\theta \otimes h [R_p, R_v]_\theta \\
&\quad = \quad C_q \otimes h R_q = S_q.
\end{aligned}
\end{align}

(4.66)

By Proposition 4.7 both spaces $C_u \otimes R_p$ and $C_p \otimes h R_v$ are $OUMD_p$. Therefore, by (8) in Proposition 3.12 it follows by interpolation from (4.66) that $S_q$ is $OUMD_p$. □

Proposition 4.10. If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then $S_{p'}$ is $OUMD_p$.

Proof. Case 1: $2 \leq p$. We claim that there exist $0 < \theta < 1$ and $1 < u, v < \infty$ such that

\begin{align}
\begin{aligned}
[C_u, C_p]_\theta &= C_{p'} \\
[R_p, R_v]_\theta &= R_{p'}.
\end{aligned}
\end{align}

(4.67)

Equivalently,

\begin{align}
\begin{aligned}
\frac{1}{p'} &= \frac{1 - \theta}{u} + \frac{\theta}{p}, \\
\frac{1}{p'} &= \frac{1 - \theta}{p} + \frac{\theta}{v}.
\end{aligned}
\end{align}

(4.68)

(4.69)

Note that (4.68) is equivalent to

\begin{align}
\begin{aligned}
\frac{1}{p} = \frac{1 - \theta}{u'} + \frac{\theta}{p'} \iff \frac{1 + \theta}{p} = \frac{1 - \theta}{u'} + \theta,
\end{aligned}
\end{align}

(4.70)
where $\frac{1}{u} + \frac{1}{u'} = 1$. This yields a restriction upon $\theta$ as follows

$$\theta < \frac{1+\theta}{p} < 1 \iff \frac{1}{p-1} < \theta < p - 1.$$  \hspace{1cm} (4.71)

Note that since $2 < p < \infty$, we have $0 < \frac{1}{p-1} < 1 < p - 1$. Therefore, the above relations yield the restriction

$$\frac{1}{p-1} < \theta < 1.$$  \hspace{1cm} (4.72)

Choose $0 < \theta < 1$ such that (4.72) holds. Then we find $1 < u' < \infty$ by solving (4.70), and let $u$ be the conjugate exponent of $u'$. Furthermore, note that (4.69) is equivalent to

$$1 - \frac{1}{p} = \frac{1-\theta}{p} + \frac{\theta}{v} \iff 1 - \frac{2-\theta}{p} = \frac{\theta}{v}.$$  \hspace{1cm} (4.73)

This implies that

$$0 < \frac{2-\theta}{p} < 1 \iff p > 2 - \theta,$$  \hspace{1cm} (4.74)

which is obviously true, since by assumption $p > 2 > 2 - \theta$. Therefore, we can solve (4.73) to find $v$. Our claim is completely proved. By Kouba’s interpolation result and (4.67) we get

$$[C_u \otimes^h R_p, C_p \otimes^h R_v] = C_{p'} \otimes^h R_{p'} = S_{p'}.$$  \hspace{1cm} (4.75)

By Proposition 4.10 both spaces $C_u \otimes^h R_p$ and $C_p \otimes^h R_v$ are $OUMD_p$. Hence, by (8) in Proposition 3.12 it follows by interpolation that $S_{p'}$ is $OUMD_p$.

Case 2: $1 < p < 2$. The assertion follows by duality. Indeed, since $2 < p' < \infty$, it follows by Case 1 that $S_{(p')'} = S_p$ is $OUMD_{p'}$. By (5) in Proposition 3.12 this implies that $S_{p'}$ is $OUMD_p$. \hfill \Box

**Lemma 4.11.** If $1 < q < \infty$, then $S_q$ is $OUMD_2$.

**Proof.** By Remark 4.12, $S_q$ is $OUMD_q$. Also, by Proposition 4.10, $S_q$ is $OUMD_{q'}$, where $\frac{1}{q} + \frac{1}{q'} = 1$. By (8) in Proposition 3.12 interpolation with exponent $\theta = \frac{1}{2}$ yields the conclusion. \hfill \Box

**Theorem 4.12.** If $1 < p, q < \infty$, then $S_q$ is $OUMD_p$.

**Proof.** Case 1: $2 \leq q < \infty$. We will first show that if $p \geq 2$, then $S_q$ is $OUMD_p$. Indeed, if $p \geq q$, then we have $\frac{2p}{p+1} < 2 \leq q < 2p$. Thus, by Proposition 4.9 it follows that $S_q$ is $OUMD_p$. The case $p = 2$ follows from Lemma 4.11. On the other hand, if $2 < p < q$, then there exists $0 < \theta < 1$ such that

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}.$$  \hspace{1cm} (4.76)

By Lemma 4.11, $S_q$ is $OUMD_2$. Also, by Remark 4.12 $S_q$ is $OUMD_q$. Interpolating with exponent $\theta$ given by (4.76), it follows from (8) in Proposition 3.12 that $S_q$ is $OUMD_p$, which
proves the claim. In particular, as a consequence it follows by duality that $S_2$ is $OMD_u$, for all $1 < u < \infty$. Hence the case $q = 2$ is completely proved.

Furthermore, note that for $2 < q < \infty$ we have $1 < q' < 2$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Assume that $q' < p < 2$. There exists $0 < \eta < 1$ such that

$$\frac{1}{p} = \frac{1 - \eta}{q'} + \frac{\eta}{2}.$$  

(4.77)

By Proposition 4.10 $S_q$ is $OMD_{q'}$. Also, by Lemma 4.11 $S_q$ is $OMD_2$. Thus, by (8) in Proposition 3.12 interpolation with exponent $\theta$ given by (4.77) shows that $S_q$ is $OMD_p$. To summarize, we have proved so far that if $p \geq q'$, then $S_q$ is $OMD_p$. It remains to analyze the case when $1 < p < q'$. In this case we have $q' < 2 < p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$. By what we proved above, it follows that $S_q$ is $OMD_{p'}$. By Proposition 4.10 $S_p$ is $OMD_{p'}$. Note that $p < q' < q$, hence there exists $0 < \xi < 1$ such that

$$\frac{1}{q'} = \frac{1 - \xi}{p} + \frac{\xi}{q}.$$  

(4.78)

By (8) in Proposition 3.12 interpolation with exponent $\theta$ given by (4.78) implies that $S_{q'}$ is $OMD_{p'}$. By duality, an application of (5) in Proposition 3.12 shows that $S_q$ is $OMD_p$.

Case 2: $1 < q < 2$. Note that $2 \leq q' < \infty$. Thus, for $1 < p < \infty$, it follows by Case 1 that $S_{q'}$ is $OMD_{p'}$, where $p'$ is the conjugate exponent of $p$. By duality, (5) in Proposition 3.12 implies that $S_q$ is $OMD_p$. This completes the proof. ☐

**Proposition 4.13.** If $1 < p, q, u < \infty$, then $S_q[S_u]$ is $OMD_p$.

**Proof.** We first show that $S_q[S_u]$ is $OMD_{u'}$, where $\frac{1}{u} + \frac{1}{u'} = 1$. As showed in the proof of Theorem 4.5 there exist $0 < \theta < 1$ and $1 < v, s < \infty$, such that the following relations hold

$$\frac{1}{q} = \frac{1 - \theta}{v} + \frac{\theta}{s},$$  

(4.79)

$$\frac{1}{u} = \frac{1 - \theta}{v} + \frac{\theta}{s'}.$$  

(4.80)

Here $s'$ denotes the conjugate exponent of $s$. It follows that $S_q = [S_v, S_s]_\theta$ and, respectively $S_u = [S_v, S_s']_\theta$. An application of Corollary 1.4 in [48] shows that

$$S_q[S_u] = [S_v[S_v], S_s[S_s']]_\theta.$$  

(4.81)

Let $v'$ be the conjugate exponent of $v$. By Proposition 4.12 it follows that $S_v[S_v] \simeq S_v(N \times N)$ is $OMD_{v'}$. Also, since $S_{s'}$ is $OMD_s$, it follows by (3) in Proposition 3.12 that for all $m \geq 1$, $S_{s'}^m[S_s]$ is $OMD_s$, and moreover, the corresponding constants satisfy $c_s(S_{s'}^m[S_s]) \leq c_s(S_{s'})$. Since $OMD_{s'}$ is a local property, this implies that $S_s[S_{s'}]$ is $OMD_s$. Furthermore, from (4.80) we easily deduce that

$$\frac{1}{u'} = \frac{1 - \theta}{v'} + \frac{\theta}{s}.$$  

(4.82)

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Therefore, by (4.81) and (4.82) it follows by interpolation that $S_q[S_u]$ is $OUMD_q$. Using the fact that $S_u$ is $OUMD_q$, a similar argument as above shows that $S_q[S_u]$ is $OUMD_q$. By interpolation it follows that $S_q[S_u]$ is $OUMD_p$, for $\min\{u',q\} \leq p \leq \max\{u',q\}$.

Next, assume that $\max\{u',q,2\} < p < \infty$. This implies that $p > \min\{2,u,q\}$. Let $1 < t_0 < \min\{2,u,q\} < p < t_1 < \infty$. There exists $0 < \eta < 1$ such that

$$\frac{1}{q} = \frac{1-\eta}{t_1} + \frac{\eta}{t_0}.$$  

Let $2 < w < \infty$ be such that

$$\frac{1}{u} = \frac{1-\eta}{w} + \frac{\eta}{t_0}.$$  

By Corollary 1.4 of [48] it follows that

$$S_q[S_u] = [S_{t_1}[S_w],S_{t_0}[S_{t_0}]]_{\eta}.$$  

Let $u'$ denote the conjugate of $w$. Note that $u' < 2 < p < t_1$. Hence, as justified above, $S_{t_1}[S_w]$ is $OUMD_p$, while by Proposition 3.12, $S_{t_0}[S_{t_0}] \simeq S_0(\mathbb{N} \times \mathbb{N})$ is $OUMD_p$. Therefore, by (4.85) it follows by interpolation that $S_q[S_u]$ is $OUMD_p$. Furthermore, since $\max\{u',q\} \leq \max\{u',q,2\}$ an application of (8) in Proposition 3.12 shows that $S_q[S_u]$ is $OUMD_p$, for $\max\{u',q\} \leq p < \infty$. To summarize, so far we have showed that $S_q[S_u]$ is $OUMD_p$, for $\min\{u',q\} \leq p < \infty$. It remains to analyze the case when $1 < p < \min\{u',q\}$. This implies that $\min\{u,q'\} < p' < \infty$, where $q'$ is the conjugate exponent of $q$. By what we proved above it follows that $S_{q'}[S_{u'}]$ is $OUMD_{q'}$. By Corollary 1.8 of [48], $(S_{q'}[S_{u'}])^* = S_q[S_u]$. Hence, by (5) in Proposition 3.12 it follows that $S_q[S_u]$ is $OUMD_p$. This completes the proof. \hfill $\Box$

**Corollary 4.14.** Let $1 < p,q,u < \infty$. Then the spaces $C_u \otimes^h R_q$, $C_u \otimes^h C_q$, $R_u \otimes^h C_q$, and $R_u \otimes^h R_q$ are all $OUMD_p$.

**Proof.** By the injectivity and associativity properties of the Haagerup tensor product, we obtain

$$C_u \otimes^h R_q \subseteq C_u \otimes^h C_q \otimes^h R_q \otimes^h R_u = S_u[S_q].$$

By (2) of Proposition 3.12 and Proposition 4.13, it follows that $C_u \otimes^h R_q$ is $OUMD_p$. A similar argument applies for the spaces $C_u \otimes^h C_q$, $R_u \otimes^h C_q$, and $R_u \otimes^h R_q$. \hfill $\Box$

Recall that a $C^*$-algebra $A$ has the weak expectation property (WEP) of Lance [35], if for the universal representation $A \subset A^{**} \subset \mathcal{B}(H)$ there exists a contraction $P : \mathcal{B}(H) \to A^{**}$ such that $P|A = \text{Id}_A$. A $C^*$-algebra $B$ is said to be QWEP if it is a quotient of a WEP $C^*$-algebra; more precisely, there exists a $C^*$-algebra $A$ with the WEP and a closed two-sided ideal $I$ such that $B = A/I$. It is a long standing problem whether every $C^*$-algebra is QWEP (see Kirchberg [33] for many equivalent formulations). Note that an injective von Neumann algebra has the WEP, and, therefore, it is QWEP. Also, it was proved by Wassermann [58] that for $n \geq 2$, $VN(\mathbb{F}_n)$ is QWEP, where $\mathbb{F}_n$ is the free group on $n$ generators.
Proposition 4.15. Let \((\mathcal{M}, \tau)\) be a QWEP von Neumann algebra equipped with an n.f. tracial state \(\tau\). Then \(L_q(\mathcal{M}, \tau)\) is OUMD\(p\), for \(1 < p, q < \infty\).

Proof. By results of Junge [25], it follows that \(L_q(\mathcal{M}, \tau)\) is completely contractively complemented in an ultrapower \((S_q)\mathcal{U}\). Therefore, combining Theorem 4.12 with (6) and (2) of Proposition 3.12, we obtain the conclusion. \(\square\)

Remark 4.16. As a corollary, it follows that if \(G\) is an amenable group (in which case \(VN(G)\) is an injective von Neumann algebra), or \(G = \mathbb{F}_n(n \geq 2)\), then \(L_q(VN(G), \tau)\) is OUMD\(p\), for \(1 < p, q < \infty\).

Corollary 4.17. Let \(1 < p, q, u < \infty\). If \((\mathcal{M}, \tau)\) is hyperfinite von Neumann algebra and \((\mathcal{N}, \phi)\) is a QWEP von Neumann algebra equipped with n.f. tracial states \(\tau\) and \(\phi\), respectively, then \(L_q(\mathcal{M}; L_u(\mathcal{N}, \phi))\) is OUMD\(p\).

Proof. Let \(m \geq 1\). Using Junge’s results from [25], together with the injectivity and associativity properties of the Haagerup tensor product, we obtain a completely contractive inclusion

\[
S_q^m[L_u(\mathcal{N}, \phi)] \hookrightarrow S_q^m[(S_u)\mathcal{U}].
\]

By Lemma 5.4 of [28], we have the complete isometry

\[
S_q^m[(S_u)\mathcal{U}] = (S_q^m[S_u])\mathcal{U}.
\]

Therefore, by Proposition 4.13, together with (6), (1) and (2) of Proposition 3.12 it follows that \(S_q^m[L_u(\mathcal{N}, \phi)]\) is OUMD\(p\) and, moreover,

\[
c_p(S_q^m[L_u(\mathcal{N}, \phi)]) \leq c_p(S_q[S_u]).
\]

Since the spaces \(S_q^m[L_u(\mathcal{N}, \phi)]\) are dense in \(L_q(\mathcal{M}; L_u(\mathcal{N}, \phi))\) and OUMD\(p\) is a local property, the conclusion follows. \(\square\)

We end this section with a discussion about the operator space \(UMD\) property for the noncommutative Lorentz spaces \(L_{q,s}(\mathcal{M}, \tau)\) associated to a semifinite von Neumann algebra. We briefly recall some definitions. Let \(x\) be a \(\tau\)-measurable operator affiliated with \((\mathcal{M}, \tau)\). Following Fack and Kosaki [21], the \(t\)-th singular number of \(x\) is

\[
\mu_t(x) = \inf\{\|xe\| : e \text{ is a projection in } \mathcal{M}, \tau(1-e) \leq t\}.
\]

Following the general scheme of symmetric operator spaces associated to \((\mathcal{M}, \tau)\) and a rearrangement invariant Banach function space developed in [15] and [16], the noncommutative Lorentz spaces are defined as

\[
L_{q,s}(\mathcal{M}, \tau) = \{x \in L_0(\mathcal{M}) : \mu(x) \in L_{q,s}((0, \infty), m)\}, \quad \|x\|_{L_{q,s}(\mathcal{M}, \tau)} = \|\mu(x)\|_{p,q}.
\]
We refer the reader to Randrianantoanina \footnote{52} and Xu \footnote{59} \footnote{60} for more details on the noncommutative Lorentz spaces.

**Proposition 4.18.** Let \((\mathcal{M}, \tau)\) be a hyperfinite von Neumann algebra, equipped with an n.s.f. tracial state \(\tau\). Then \(L_{q,s}(\mathcal{M}, \tau)\) is OUMD\(_p\), for \(1 < q, s, p < \infty\).

**Proof.** There exists \(0 < \theta < 1\) and \(1 < q_1, q_2 < \infty\), such that \(\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}\). The formula describing the K-functional for the couple \((\mathcal{M}, L_1(\mathcal{M}, \tau))\) \footnote{50}, Corollary 2.3 together with the reiteration result proved in \footnote{60}, Theorem 5.4(ii) yield the complete isometry

\[
L_{q,s}(\mathcal{M}, \tau) = [L_{q_1}(\mathcal{M}, \tau), L_{q_2}(\mathcal{M}, \tau)]_{\theta,s}.
\]

We refer to Xu’s paper \footnote{60} for details on the operator space structure of the real interpolation space \([L_{q_1}(\mathcal{M}, \tau), L_{q_2}(\mathcal{M}, \tau)]_{\theta,s}\). In particular, it is proved in \footnote{60} that the following completely isometric embedding holds

\[
[L_{q_1}(\mathcal{M}, \tau), L_{q_2}(\mathcal{M}, \tau)]_{\theta,s} \hookrightarrow l_s(\{L_{q_1}(\mathcal{M}, \tau) + s 2^{-k}L_{q_2}(\mathcal{M}, \tau)\}_k ; 2^{-k\theta}),
\]

where \(L_{q_1}(\mathcal{M}, \tau) + s 2^{-k}L_{q_2}(\mathcal{M}, \tau)\) is defined as a quotient of \(L_{q_1}(\mathcal{M}, \tau) \oplus_s 2^{-k}L_{q_2}(\mathcal{M}, \tau)\), for \(k \geq 1\). Then, by (2.6)’ of \footnote{38}, the space \(l_s(\{L_{q_1}(\mathcal{M}, \tau) + s 2^{-k}L_{q_2}(\mathcal{M}, \tau)\}_k ; 2^{-k\theta})\) is a quotient of the operator space \(l_s(\{L_{q_1}(\mathcal{M}, \tau) \oplus 2^{-k}L_{q_2}(\mathcal{M}, \tau)\}_k ; 2^{-k\theta})\). Therefore, by (2) of Proposition \footnote{3.12} it suffices to show that \(l_s(\{L_{q_1}(\mathcal{M}) \oplus 2^{-k}L_{q_2}(\mathcal{M})\}_k ; 2^{-k\theta})\) is OUMD\(_p\). Note that we have the complete isometry

\[
l_s(\{L_{q_1}(\mathcal{M}) \oplus 2^{-k}L_{q_2}(\mathcal{M})\}_k ; 2^{-k\theta}) = l_s(L_{q_1}(\mathcal{M})) \oplus l_s(\{2^{-k}L_{q_2}(\mathcal{M})\}_k ; 2^{-k\theta})\).
\]

The space \(l_s(L_{q_1}(\mathcal{M}, \tau))\) is completely isometric to the subspace of \(S_s[L_{q_1}(\mathcal{M}, \tau)]\) formed by all the diagonal matrices (see \footnote{38}); moreover, the usual projection onto this subspace is a complete contraction. Since \(\mathcal{M}\) is hyperfinite, it follows by Corollary \footnote{4.17} that \(S_s[L_{q_1}(\mathcal{M}, \tau)]\) is OUMD\(_p\). Thus, by (1) of Proposition \footnote{3.12} it follows that \(l_s(L_{q_1}(\mathcal{M}, \tau))\) is OUMD\(_p\). Similar arguments show that \(l_s(\{2^{-k}L_{q_2}(\mathcal{M}, \tau)\}_k ; 2^{-k\theta})\) is OUMD\(_p\), as well. The conclusion follows from (7) of Proposition \footnote{3.12}. \(\square\)

**Question:** It was a question of Zhong-Jin Ruan whether the column Hilbert space \(C\) is OUMD\(_p\) for some (all) \(1 < p < \infty\). By Lemma \footnote{5.14} if \(C\) is OUMD\(_p\) for some \(1 < p < \infty\), then \(S_p[C]\) is UMD as a Banach space. Using the characterization of superreflexivity in terms of ultraproducts (see Heinrich \footnote{23}), together with results of Junge and Sherman from \footnote{28}, we were able to prove that \(S_p[C]\) is a superreflexive Banach space. During discussions initiated by Gilles Pisier, Timur Oikhberg found a surprisingly short proof of this result, which we present below. Recall that by \footnote{2.7} we have \(C_2 = [C, R]_{\frac{1}{2}} = [R, C]_{\frac{1}{2}} = R_2\). By \footnote{2.9} and Kouba’s theorem it follows that

\[
S_2[C] = [C, R]_{\frac{1}{2}} \otimes^h C \otimes^h [C, R]_{\frac{1}{2}} = [C \otimes^h C \otimes^h C, R \otimes^h C \otimes^h R]_{\frac{1}{2}}.
\]
Note that as a Banach space, $C \otimes h C \otimes h C$ is isometric to a Hilbert space, hence it is superreflexive. Pisier (see [43]) proved that, given a compatible couple of Banach spaces $B_0, B_1$, if one of them is superreflexive, then for $0 < \theta < 1$ the interpolation space $[B_0, B_1]_\theta$ is superreflexive, as well. Therefore, by (4.91) we conclude that $S_2[C]$ is a superreflexive Banach space. For $1 < p < \infty$, by (2.6) we have the (complete) isometry $$S_p[C] = [S_\infty[C], S_2[C]]_{\frac{1}{2p}},$$

A further application of Pisier’s result mentioned above yields the conclusion.

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