Continuous Cohomology and Ext-Groups

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Abstract

We prove that the continuous cohomology groups of an $l$-group $G$ with coefficients in a smooth $k$-representation $\pi$ of $G$ are isomorphic to the Ext-groups $\text{Ext}^i_G (\mathbb{1}, \pi)$ computed in the category of smooth $k$-representations of $G$. We apply this to show that if $\pi$ is a supersingular $\mathbb{F}_p$-representation of $\text{GL}_2(\mathbb{Q}_p)$, then the continuous cohomology of $\text{SL}_2(\mathbb{Q}_p)$ with values in $\pi$ vanishes.

Furthermore, we prove that the continuous cohomology groups of a $p$-adic reductive group $G$, with coefficients in an admissible unitary $\mathbb{Q}_p$-Banach space representation $\Pi$, are finite dimensional. We show that the continuous cohomology of $\text{SL}_2(\mathbb{Q}_p)$ with values in non-ordinary irreducible $\mathbb{Q}_p$-Banach space representations of $\text{GL}_2(\mathbb{Q}_p)$ vanishes.

1 Introduction

Let $k$ be a commutative ring with 1 and let $G$ be an $l$-group in the sense of [2], i.e. a topological group which has a fundamental system of neighborhoods of the unit element consisting of compact open subgroups.

A smooth $k$-representation of $G$ is a $k$-$G$-module $\pi$, such that the stabilizer $\text{Stab}_{G}(v)$ of any element $v \in \pi$ is open in $G$. Denote by $\text{Mod}^{\text{sm}}_G(k)$ the category of all smooth $k$-representations of $G$. We prove the following:

Theorem 1.1 (Corollary 4.4). For any $\pi \in \text{Mod}^{\text{sm}}_G(k)$, and any $i \geq 0$, we have isomorphisms:

$$\text{Ext}^i_G (\mathbb{1}, \pi) \cong H^i(G, \pi),$$

where $H^i(G, \pi)$ is the continuous cohomology group of $G$ with coefficients in $\pi$ and the Ext-group is computed in the category $\text{Mod}^{\text{sm}}_G(k)$.

The Ext-group in $\text{Mod}^{\text{sm}}_G(k)$ is well-defined, since the category has enough injectives by Proposition 2.1.1 in [6]. We prove Theorem 1.1 by showing that applying the functor of smooth vectors to the resolution of $\pi$ used to compute the continuous cohomology gives a resolution of $\pi$ in $\text{Mod}^{\text{sm}}_G(k)$. This question was raised by Emerton in [6, Section 2.2], who has proved this for compact groups ([6, Proposition 2.2.6]).

In Section 5 we apply our result to the group $G = \text{GL}_2(\mathbb{Q}_p)$. Let $|\cdot|$ be a norm on $\mathbb{Q}_p$, normalized so that $|p| = 1/p$. Then we can define a character $\varepsilon : \mathbb{Q}_p^\times \to \mathbb{Z}_p^\times$, by $x \mapsto x|x|$. Using Theorem 1.1 we prove the following:

Theorem 1.2 (Corollary 5.2). Let $k$ be a finite field of characteristic $p$ and let $\pi \in \text{Mod}^{\text{sm}}_{\text{GL}_2(\mathbb{Q}_p)}(k)$ be absolutely irreducible and not isomorphic to a twist by a character of $\mathbb{1}$, $\text{Sp}$ or $(\text{Ind}_B^G \alpha)_{\text{sm}}$, where $\alpha : B \to k^\times$ is defined by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \varepsilon(ad^{-1}) \mod p$. Then

$$H^i(\text{SL}_2(\mathbb{Q}_p), \pi) = 0,$$

for $i \geq 0$. 

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This includes for example all supersingular representations \( \pi \in \text{Mod}_{\text{Gen}}^{\text{sm}}(\mathbb{Q}_p)(k) \). The proof makes use of calculations of Ext-groups due to Paškūnas (\[10\]). This result is used by Colmez, Dospinescu and Nizioł in their forthcoming work.

In Section 6, we study the continuous cohomology groups of a \( p \)-adic reductive group with coefficients in an admissible unitary \( L \)-Banach space representation, where \( L/\mathbb{Q}_p \) is a finite extension. If \( \Pi \) is an admissible unitary \( L \)-Banach space representation of \( G \) and \( \Pi^0 \) is a \( G \)-invariant unit ball in \( \Pi \), then we show that (Proposition 6.5) for all \( i \geq 0 \) one has isomorphisms

\[
H^i(G, \Pi) \cong (\lim_n H^i(G, \Pi^0/\varpi^n\Pi^0))[1/\varpi],
\]

for \( \varpi \) a uniformizer of \( L \). The proof uses the Bruhat–Tits building of \( G \) to obtain a resolution of the trivial representation of \( G \) by compactly induced representations from compact-mod-center subgroups of \( G \). Such resolutions appear in the work of Schneider–Stuhler \[13\] and Casselman–Wigner \[4\]. Moreover, we deduce

**Theorem 1.3** (Corollary 6.6). Let \( \Pi \) be an admissible unitary \( L \)-Banach space representation of a \( p \)-adic reductive group \( G \), then \( H^i(G, \Pi) \) are finite dimensional over \( L \), for all \( i \geq 0 \).

In the case where \( G \) is a compact \( p \)-adic analytic group, these isomorphisms follow directly from \[5\].

As in Section 5 we apply these results to the group \( \text{GL}_2(\mathbb{Q}_p) \) and obtain a similar statement for Banach space representations:

**Theorem 1.4** (Proposition 6.11). Any absolutely irreducible admissible unitary \( L \)-Banach space representation \( \Pi \) of \( \text{GL}_2(\mathbb{Q}_p) \), which is not isomorphic to a twist by a unitary character of \( 1 \), \( \hat{\text{Sp}} \) or \( \text{Ind}_B^G \alpha \), has trivial continuous cohomology groups over \( \text{SL}_2(\mathbb{Q}_p) \), ie.

\[
H^i(\text{SL}_2(\mathbb{Q}_p), \Pi) = 0,
\]

for \( i \geq 0 \).

Here, \( \alpha : B \rightarrow L^\times \) is the representation of \( B \), defined by \( \alpha \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) \rightarrow \varepsilon(ad^{-1}) \).

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## 2 Continuous cohomology

Following \[1\], we define the continuous cohomology as follows: Let \( V \) be a topological \( G \)-module, ie. a topological abelian group \( V \) with a \( G \)-action such that \( G \) acts on \( V \) via group automorphisms and the map \( G \times V \rightarrow V \) is continuous. Then we can define the cochain complex

\[
C^n(G, V) := C(G^{n+1}, V) := \{ f : G^{n+1} \rightarrow V \text{ continuous} \},
\]

with differentials \( d^n : C^n(G, V) \rightarrow C^{n+1}(G, V) \), defined by

\[
d^n f(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \ldots, \widehat{g_i}, \ldots, g_{n+1}).
\]
We endow the spaces $C^n(G,V)$ with the compact-open topology. By defining a $G$-action via $(gf)(g_0, \ldots, g_n) = gf(g_1^{-1}g_0, \ldots, g_1^{-1}g_n)$, for $g, g_0, \ldots, g_n \in G$ and $f \in C^n(G,V)$, these will define topological $G$-modules. Furthermore, there is a continuous $G$-equivariant injection $V \hookrightarrow C(G,V)$, defined by $v \mapsto [g \mapsto v]$.

For a closed subgroup $H \leq G$ and a topological $G$-module $V$, define the induced representation to be

$$\text{Ind}_H^G V := \{ f \in C(G,V) \mid f(hg) = hf(g) \forall h \in H, \forall g \in G \},$$

with the subspace topology induced from $\text{Ind}_H^G V \subseteq C(G,V)$ and $G$-action $gf(g') := f(g'g)$, for $g, g' \in G$. Similarly, the compact induction is defined as

$$\text{c-Ind}_H^G V := \{ f \in \text{Ind}_H^G V \mid \text{the support of } f \text{ is compact modulo } H \}$$

with the same $G$-action.

**Lemma 2.1.** For a topological $G$-module $V$, there are homeomorphisms of $G$-modules

$$C^{n+1}(G,V) \cong C^0(G,C^n(G,V)) \text{ and } C^0(G,V) \cong \text{Ind}_1^G V,$$

for all $n \geq 0$.

**Proof.** For the first homeomorphism, see [3, X.3.4 Corollaire 2].

The assignment $f \mapsto [g \mapsto gf(g^{-1})]$ defines both maps $C^0(G,V) \to \text{Ind}_1^G V$ and its inverse. One can check that both maps are continuous and $G$-equivariant, hence the modules are homeomorphic.

**Lemma 2.2.** For two topological $G$-modules $V$ and $W$, one has the following isomorphism:

$$\text{Hom}^{\text{cts}}(V, \text{Ind}_1^G W) \cong \text{Hom}^{\text{cts}}(V, W),$$

where $\text{Hom}^{\text{cts}}(V, W)$ are continuous group homomorphisms and on the left hand side, we take continuous $G$-equivariant group homomorphisms.

**Proof.** [3, Lemma 2].

**Lemma 2.3.** The complex $0 \to V \to C^\bullet(G,V)$ is an exact complex of $G$-modules.

**Proof.** Let $C^\bullet$ be the complex with $C^{-1} = V$, $C^i = 0$, for $i \leq -2$ and $C^i = C^i(G,V)$ for $i \geq 0$. To prove the exactness of this complex, we construct a cochain homotopy $s^n : C^n \to C^{n-1}$, $n \in \mathbb{Z}$ between $id_{C^\bullet}$ and the zero map on $C^\bullet$. Define $(s^nf)(g_1, \ldots, g_n) := f(1, g_1, \ldots, g_n)$ for $f \in C^n(G,V)$, $n \geq 0$ and $s^n = 0$ for $n \leq -1$. Then $s^nf : G^n \to V$ is a continuous map, since it is the composition of the continuous maps $f$ and $\{1\} \times G^n \to G \times G^n$. Moreover, one can easily check that it satisfies $s^{n+1}d^n + d^{n-1}s^n = id_{C^n}$ for all $n$. Hence we have found the desired homotopy and the complex is exact.

**Definition 2.4.** The $i$th continuous cohomology group of $G$ with coefficients in $V$ is defined to be $H^i(G,V) := H^i(C^\bullet(G,V)^G)$.

**Remark 2.5.**

1. The $G$-modules $C(G,V)$ are acyclic for the continuous cohomology. Hence, the complex $0 \to V \to C^\bullet(G,V)$ is an acyclic resolution of $V$. (cf. [4, p. 201].)

2. If we equip a smooth representation $\pi \in \text{Mod}_{G}^{\text{sm}}(k)$, with the discrete topology, this will give a topological $G$-module and we can define the continuous cohomology groups $H^i(G,\pi)$ with coefficients in $\pi$. 

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3 Proof for compact open subgroups

Proposition 3.1 below can already be found in Section 2.2 of [6]. To make the article as self-contained as possible we decided to include a proof.

First, we consider the case of a compact $l$-group $K$ and a smooth representation $\pi \in \text{Mod}_{K}^\text{sm}(k)$ of $K$ equipped with the discrete topology, so that we can define $H^i(K, \pi)$ as above. Moreover, we denote by $1 \in \text{Mod}_{K}^\text{sm}(k)$ the free $k$-module of rank 1 with trivial $K$-action and $\text{Ext}^i_K(1, \pi)$ the Ext-group computed in $\text{Mod}_{K}^\text{sm}(k)$, ie. the $i$th right derived functor of $\text{Hom}_K(1, -)$ applied to $\pi$.

**Proposition 3.1.** One has isomorphisms

$$\text{Ext}^i_K(1, \pi) \cong H^i(K, \pi),$$

for all $i \geq 0$.

We split the proof into the following lemmas.

For a topological $K$-module $V$, we define the space of smooth functions $C_{\text{sm}}^p(K, V) \subseteq C(K, V)$, to be:

$$C_{\text{sm}}^p(K, V) := \{ f : K \to V \mid \exists H \leq K \text{ open, s.t. } hf = f \forall h \in H \}.$$

**Lemma 3.2.** If $V$ carries the discrete topology, then we have the following equality:

$$C(K, V) = C_{\text{sm}}^p(K, V).$$

**Proof.** The inclusion $C_{\text{sm}}^p(K, V) \subseteq C(K, V)$ is trivial. Conversely, if $f : K \to V$ is a continuous function, it is locally constant and, since $K$ is compact, the image of $f$ consists of finitely many elements $v_1, \ldots, v_n \in V$. Moreover, for each $k \in K$, there is an open subgroup $U_k \leq K$, such that $U_k k \subseteq f^{-1}(v_i)$ for some $i$. We get

$$K = \bigcup_{k \in K} U_k k = \bigcup_{i=1}^m U_{k_i} k_i.$$

Thus, taking $U := \bigcap_i U_{k_i} \cap \text{Stab}_K(f(k_i))$ is an open subgroup of $K$ with the property that for any $u \in U$, $uf = f$. ☐

**Lemma 3.3.** If the topology of $V$ is discrete, the compact-open topology on $C(K, V)$ is discrete.

**Proof.** Let $f : K \to V$ be an element in $C(K, V) = C_{\text{sm}}^p(K, V)$, ie. there exists a compact open subgroup $H \leq K$, such that $f(kh) = f(k)$ for all $h \in H$, $k \in K$. Since $K$ is compact, $K$ is the disjoint union of finitely many compact open cosets $k_i H$, $i = 1, \ldots, n$, on which $f$ is constant. Then $\{ f \} = \bigcap_{i=1}^n \Omega(k_i H, \{ f(k_i) \})$ is open as it is the finite intersection of the open sets

$$\Omega(k_i H, \{ f(k_i) \}) = \{ g \in C(K, V) \mid g(k_i H) \subseteq \{ f(k_i) \} \}$$

in $C(K, V)$. ☐
Lemma 3.4. If \( \pi \in \text{Mod}^\text{sm}_G(k) \) is a smooth representation of \( K \) equipped with the discrete topology, then \( 0 \to \pi \to C^*(K, \pi) \) is an acyclic resolution of \( \pi \) in the category of smooth representations \( \text{Mod}^\text{sm}_G(k) \).

Remark 3.5. If we say that a topological \( K \)-module is acyclic, we mean that it is acyclic for the continuous cohomology. Whereas acyclic in \( \text{Mod}^\text{sm}_G(k) \) means acyclic with respect to \( \text{Ext}^i_K(\mathbb{1}, -) \).

Proof. We know that the complex is exact and that \( C^n(K, \pi) \) is a smooth representation by the previous two lemmas. The representations \( C^n(K, \pi) \cong C(K, C^{n-1}(K, \pi)) \) are acyclic in the category \( \text{Mod}^\text{sm}_G(k) \). Indeed, since \( V := C^{n-1}(K, \pi) \) is again discrete, by Frobenius reciprocity, one has \( \text{Hom}_K(\mathbb{1}, C(K, V)) \cong \text{Hom}_k(k, V) \cong V \). Moreover, if one chooses a resolution \( V \to J^\bullet \) in the category of \( k \)-modules, applying \( C(K, -) \) gives a resolution of \( C(K, V) \) in \( \text{Mod}^\text{sm}_G(k) \). Hence, one has \( \text{Ext}^i_K(\mathbb{1}, C(K, V)) \cong \text{Ext}^i_k(k, V) \), which is zero for \( i > 0 \) and \( V \) for \( i = 0 \), as expected.

Proof of Proposition 3.4. By Lemma 3.4, we can compute the groups \( \text{Ext}^i_K(\mathbb{1}, \pi) \) by using the acyclic resolution above. More precisely, it is the \( i^{\text{th}} \) cohomology group of the complex

\[
\text{Hom}_K(\mathbb{1}, C^*(K, \pi)) \cong C^*(K, \pi)^K
\]

and this is the same as the continuous cohomology group \( H^i(K, \pi) \). This proves the Proposition.

4 The general case

If \( G \) is locally compact, but not compact, then the modules \( C^*(G, \pi) \) will not lie in \( \text{Mod}^\text{sm}_G(k) \). For example, if \( G = (\mathbb{Q}_p, +) \), then the indicator function on the set \( \bigcup_{n \geq 1} (p^{-n} + p^n \mathbb{Z}_p) \) is locally constant and hence continuous. But it is not smooth, since its stabilizer consists of \( x \in \mathbb{Q}_p \), such that \( x \in p^n \mathbb{Z}_p \) for all \( n \geq 1 \), and hence is zero, which is not open.

To conclude in this case, we will use the following functor from the category \( \text{Mod}_G(k) \) of \( k \)-representations of \( G \) to the category \( \text{Mod}^\text{sm}_G(k) \)

\[
(-)^{\text{sm}} : \text{Mod}_G(k) \to \text{Mod}^\text{sm}_G(k), \quad V \mapsto V^{\text{sm}} := \lim_K V^K,
\]

where the direct limit is taken over the directed family of compact open subgroups \( K \leq G \). Hence, \( V^{\text{sm}} \) can be seen as the subrepresentation of \( V \) consisting of the smooth vectors. Moreover, this functor is right-adjoint to the inclusion \( \text{Mod}^\text{sm}_G(k) \hookrightarrow \text{Mod}_G(k) \), since \( \text{Hom}_G(W, V) = \text{Hom}_G(W, V^{\text{sm}}) \) for a smooth representation \( W \in \text{Mod}^\text{sm}_G(k) \). This implies that the functor \((-)^{\text{sm}}\) preserves injective objects.

Lemma 4.1. For a compact open subgroup \( K \leq G \) and a smooth representation \( \pi \in \text{Mod}^\text{sm}_G(k) \), the \( G \)-modules \( C^n(G, \pi) \) are acyclic for the continuous cohomology \( H^*(K, -) \).

Moreover, the cohomology of the complex \( C^*(G, \pi)^K \) is \( H^*(K, \pi) \).
Proof. A proof is given in [4 Proposition 4 (a)]. The idea is as follows: since the quotient \( G/K \) is discrete, one has a continuous section \( s : G/K \to G \) of the natural projection \( G \to G/K \) with which one can define the \( K \)-invariant homeomorphism \( K \times G/K \to G \), \((k,gK) \mapsto s(gK)k\). Therefore, one gets an isomorphism of \( K \)-modules

\[
C(G, \pi) \cong \lim_{\text{direct limits}} \ C(K \times G/K, \pi) \cong C(K, C(G/K, \pi))
\]

which is acyclic as a \( K \)-module.

To conclude that the complex \( C^\bullet(G, \pi)^K \) computes the cohomology of \( K \), one also has to use that the resolution \( \pi \to C^\bullet(G, \pi) \) is a strong resolution of acyclic \( K \)-modules and hence can be used to compute the cohomology of \( K \) (\cite[Proposition 4.1]{4}).

Proposition 4.2. Let \( \pi \in \text{Mod}^\text{sm}_K(k) \) be a smooth \( k \)-representation of \( G \). Applying the functor \((-)^\text{sm} \) to the complex \( C^\bullet(G, \pi) \) in gives a resolution \( 0 \to \pi \to C^\bullet(G, \pi)^\text{sm} \) of \( \pi \) in \( \text{Mod}^\text{sm}_G(k) \).

Proof. We want to show that the cohomology \( H^i(C^\bullet(G, \pi)^\text{sm}) \) vanishes for \( i > 0 \). By definition, \( C^\bullet(G, \pi)^\text{sm} \) is the direct limit over all compact open subgroups \( K \leq G \) of \( K \)-fixed vectors \( C^\bullet(G, \pi)^K \). Since taking cohomology of a complex commutes with direct limits, we get

\[
H^i(C^\bullet(G, \pi)^\text{sm}) \cong \lim_{K} H^i(C^\bullet(G, \pi)^K).
\]

By Lemma 4.1 and Proposition 3.1, this is the same as \( \lim_{K} \text{Ext}^i_k(\mathbb{1}, \pi) \). Now, we fix an injective resolution \( I^\bullet \) of \( \pi \) in \( \text{Mod}^\text{sm}_G(k) \). The functor \( \text{c-Ind}_K^G(-) : \text{Mod}^\text{sm}_G(k) \to \text{Mod}^\text{sm}_G(k) \) is exact for compact open subgroups \( K \leq G \). By Frobenius reciprocity \( \text{Hom}_K(V, I) \cong \text{Hom}_G(\text{c-Ind}_K^G V, I) \), hence, the restrictions of \( I^\bullet \) to \( K \) are injective in \( \text{Mod}^\text{sm}_K(k) \). We can conclude by using again that cohomology commutes direct limits:

\[
H^i(C^\bullet(G, \pi)^\text{sm}) \cong \lim_{K} \text{Ext}^i_k(\mathbb{1}, \pi) = \lim_{K} H^i((I^\bullet)^K) \\
\cong H^i(\lim_{K} (I^\bullet)^K) = H^i((I^\bullet)^\text{sm}) \\
= H^i(I^\bullet) = 0
\]

Lemma 4.3. For all \( \pi, V \in \text{Mod}^\text{sm}_G(k) \), we have isomorphisms

\[
\text{Ext}^i_G(\pi, C^n(G, V)^\text{sm}) \cong \text{Ext}^i_k(\pi, C^{n-1}(G, V))
\]

where the Ext-group on the right hand side is computed in the category of \( k \)-modules.

In particular, \( \text{Ext}^i_G(\mathbb{1}, C^n(G, V)^\text{sm}) = 0 \) for all \( i \geq 1 \).

Proof. For a \( k \)-module \( M \), define \( \mathcal{F}(G, M) \in \text{Mod}_G(k) \) as follows:

\[
\mathcal{F}(G, M) = \{ f : G \to M \},
\]

with \( G \) acting via \( gf(g') := f(g'g) \).

Applying \((-)^\text{sm} \) gives the smooth subrepresentation

\[
\mathcal{F}(G, M)^\text{sm} = \{ f \in \mathcal{F}(G, M) \mid \exists H \leq G \text{ open, s. t. } hf = f \ \forall h \in H \}.
\]
For every $\pi \in \text{Mod}_G^{\text{sm}}(k)$, using Frobenius reciprocity, we get
\[
\text{Hom}_G(\pi, F(G, M)^{\text{sm}}) = \text{Hom}_G(\pi, F(G, M)) \cong \text{Hom}_k(\pi, M). \tag{2}
\]

Moreover, for $V \in \text{Mod}_G^{\text{sm}}(k)$, let $N := C^{n-1}(G, V)$, so that by Lemma 2.3 we have $C^n(G, V) \cong C(G, N) \cong \text{Ind}_I^G N$. This yields the inclusion
\[
C^n(G, V) \cong \text{Ind}_I^G N \subset F(G, N).
\]

Since for every compact open subgroup $K \leq G$ the quotient $G/K$ is discrete, every function from $G/K$ to $N$ is automatically continuous and thus we obtain:
\[
F(G, N)^K = F(G/K, N) = C(G/K, N) = (\text{Ind}_I^G N)^K,
\]
where the equality on the right comes from composing with the continuous projection $G \to G/K$. Taking the direct limit over all $K \leq G$ compact open yields
\[
F(G, N)^{\text{sm}} = \lim_{\longrightarrow \atop K \leq G} F(G, N)^K = \lim_{\longrightarrow \atop K \leq G} (\text{Ind}_I^G N)^K = (\text{Ind}_I^G N)^{\text{sm}}.
\]

Hence, we are reduced to showing that $\text{Ext}^j_G(\pi, F(G, M)^{\text{sm}}) \cong \text{Ext}^j_k(\pi, M)$, for every $k$-module $M$.

For this, we can choose an injective resolution $M \to I^\bullet$ by $k$-modules and apply the exact functor $F(G, -)^{\text{sm}}$ to obtain a resolution $F(G, M)^{\text{sm}} \to F(G, I^\bullet)^{\text{sm}}$ in $\text{Mod}_G^{\text{sm}}(k)$. Then the representations $F(G, I^m)$ are injective for all $m \geq 0$, by the isomorphism (2).

In conclusion,
\[
\text{Ext}^j_G(\pi, F(G, M)^{\text{sm}}) \cong H^j(\text{Hom}_G(\pi, F(G, I^\bullet)^{\text{sm}}))
\]
\[
\cong H^j(\text{Hom}_k(\pi, I^\bullet)) \cong \text{Ext}^j_k(\pi, M).
\]

Moreover, if $\pi = \mathbb{1}$, then $\text{Hom}_k(\mathbb{1}, I^\bullet) \cong I^\bullet$, thus
\[
\text{Ext}^j_G(\mathbb{1}, C^n(G, V)^{\text{sm}}) \cong \text{Ext}^j_k(\mathbb{1}, C^{n-1}(G, V)) = 0
\]
for all $i > 0$.

**Corollary 4.4.** Let $\pi \in \text{Mod}_G^{\text{sm}}(k)$. We have isomorphisms of cohomology groups
\[
\text{Ext}^j_G(\mathbb{1}, \pi) \cong H^j(G, \pi), \tag{3}
\]
for all $i \geq 0$.

**Proof.** Just note that $(C^\bullet(G, \pi)^{\text{sm}})^G \cong C^\bullet(G, \pi)^G$ and Lemma 2.3 $C^\bullet(G, \pi)^{\text{sm}}$ is an acyclic resolution of $\pi$ in $\text{Mod}_G^{\text{sm}}(k)$, hence $(C^\bullet(G, \pi)^{\text{sm}})^G$ computes $\text{Ext}^i_G(\mathbb{1}, \pi)$, whereas the cohomology of the right hand side is of course $H^i(G, \pi)$.

\[\square\]

## 5 \text{Mod } p \text{ representations of } \text{GL}_2(\mathbb{Q}_p)

From now on, let $G = \text{GL}_2(\mathbb{Q}_p)$, let $k$ be a finite field of characteristic $p$. Let $\mathbb{1} \in \text{Mod}_G^{\text{sm}}(k)$ be the trivial representation, $\text{Sp}$ be the Steinberg representation and let $\omega : \mathbb{Q}_p^\times \to k^\times$ be the character $\omega(x) = x|x| \pmod{p}$, where the absolute value $|\cdot|$ is normalized so that $|p| = 1/p$. This character induces a representation of the subgroup
$B \subseteq G$ of upper triangular matrices, $\alpha : B \to k^\times$, defined by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \omega(a)\omega(d)^{-1}$.

And we denote by $(\text{Ind}^G_B \alpha)_{\text{sm}}$ the induced representation, given by continuous functions $f : G \to k$ with $f(bg) = \alpha(b)f(g)$ for all $b \in B$ and $g \in G$.

We can use the isomorphism (\[10\]) to prove that the continuous cohomology group $H^i(\text{SL}_2(\mathbb{Q}_p), \pi)$ for some smooth irreducible representations $\pi$ is trivial.

**Proposition 5.1.** Let $\pi \in \text{Mod}^\text{adm}_G(k)$ be a smooth absolutely irreducible representation of $G$. Assume that for any character $\chi : \mathbb{Q}_p^\times \to k^\times$, $\pi \otimes \chi \circ \det$ is not isomorphic to $\mathbb{I}$, $\text{Sp}$ or $(\text{Ind}^G_B \alpha)_{\text{sm}}$, then one has

$$\text{Ext}^i_{G,\xi_\pi}(\text{Ind}^G_Z Z_{\text{SL}_2}(\mathbb{Q}_p) \xi_\pi, \pi) = 0,$$

for all $i \geq 1$, where the Ext-group is computed in the category $\text{Mod}^\text{adm}_{G,\pi}(k)$ of smooth representations of $G$ with central character $\xi_\pi$.

**Proof.** Note that $Z_{\text{SL}_2}(\mathbb{Q}_p)$ is of index 2 in $G$, so that the representation $\text{Ind}^G_Z Z_{\text{SL}_2}(\mathbb{Q}_p) \xi_\pi$ is 2-dimensional. After enlarging the field $k$, we may assume that the action of $\text{GL}_2(\mathbb{Q}_p)$ factors through an abelian quotient, so that there exist characters $\eta_1$ and $\eta_2$ and a short exact sequence of the form

$$0 \to \eta_1 \circ \det \to \text{Ind}^G_Z Z_{\text{SL}_2}(\mathbb{Q}_p) \xi_\pi \to \eta_2 \circ \det \to 0.$$

This induces the long exact sequence

$$\cdots \to \text{Ext}^i_{G,\xi_\pi}(\eta_2 \circ \det, \pi) \to \text{Ext}^i_{G,\xi_\pi}(\text{Ind}^G_Z Z_{\text{SL}_2}(\mathbb{Q}_p) \xi_\pi, \pi) \to \text{Ext}^i_{G,\xi_\pi}(\eta_1 \circ \det, \pi) \to \cdots$$

The representations $\eta_j \circ \det$, $j = 1, 2$, are clearly irreducible. Moreover, $\pi$ and $\eta_j \circ \det$ lie in the full subcategory $\text{Mod}^\text{adm}_{G,\xi_\pi}(k) = \text{Mod}^\text{fin}_{G,\xi_\pi}(k)$ of $\text{Mod}^\text{adm}_{G,\xi_\pi}(k)$ consisting of those representations that are locally admissible, or equivalently, that are locally of finite length (cf [\[10\] Section 5.4]). Moreover, by Proposition 5.17 of [\[10\]], we have isomorphisms $\text{Ext}^i_{G,\xi_\pi}(\eta_j \circ \det, \pi) \cong \text{Ext}^i_{G,\xi_\pi}(\eta_j \circ \det, \pi)$ and thus, we can compute the Ext-groups in the category $\text{Mod}^\text{fin}_{G,\xi_\pi}(k)$, which decomposes into a direct product of subcategories

$$\text{Mod}^\text{fin}_{G,\xi_\pi}(k) \cong \prod_B \text{Mod}^\text{fin}_{G,\xi_\pi}(k)_B$$

([\[10\] Proposition 5.34]). In particular, there are no extensions between representations lying in different blocks $B$. The blocks are described in Corollary 1.2 of [\[11\] and by assumption, $\pi$ does not lie in the same block of any character. Hence, $\text{Ext}^i_{G,\xi_\pi}(\eta_j \circ \det, \pi) = 0$ for all $i \geq 0$ and the claim follows.

**Corollary 5.2.** Under the assumptions of Proposition 5.1 we have

$$H^i(\text{SL}_2(\mathbb{Q}_p), \pi) = 0, \forall i \geq 0.$$

**Proof.** By Corollary 4.1.3 we know that $H^i(\text{SL}_2(\mathbb{Q}_p), \pi) \cong \text{Ext}^i_{\text{SL}_2(\mathbb{Q}_p)}(\mathbb{I}, \pi)$. We may assume that $Z \cap \text{SL}_2(\mathbb{Q}_p)$ acts trivially on $\pi$, since otherwise there are no non-trivial extensions between $\pi$ and $\mathbb{I}$. We therefore get

$$\text{Ext}^i_{\text{SL}_2(\mathbb{Q}_p)}(\mathbb{I}, \pi) \cong \text{Ext}^i_{\text{ZSL}_2(\mathbb{Q}_p),\xi_\pi}(\xi_\pi, \pi)$$

and, by Shapiro, this is the same as $\text{Ext}^i_{G,\xi_\pi}(\text{Ind}^G_Z \text{ZSL}_2(\mathbb{Q}_p) \xi_\pi, \pi)$, which is zero by Proposition 5.1.
6 Banach space representations

We want to apply the result to Banach space representations of a $p$-adic reductive group $G$. More precisely, we let $F/Q_p$ be a finite extension and $G$ a connected reductive group over $F$. Then take $G = G(F)$ to be the group of its $F$-rational points. Let $L$ be a finite extension of $Q_p$ with ring of integers $O$ and uniformizer $\varpi$. Denote $\text{Ban}_{G}^{\text{adm}}(L)$ the category of admissible unitary $L$-Banach space representations of $G$ in the sense of Schneider–Teitelbaum [14]. For $\Pi \in \text{Ban}_{G}^{\text{adm}}(L)$, let $\Pi^0$ be a $G$-invariant unit ball in $\Pi$. Then $\Pi^0$ is an open $O$-lattice in $\Pi$ and it is an admissible $G$-representation, in the sense that for every $n \geq 1$, the quotient $\Pi^0/\varpi^n\Pi^0$ is an admissible smooth representation in $\text{Mod}_G^{\text{sm}}(O/\varpi^n)$. Moreover, we have an isomorphism

$$\Pi \cong \left(\lim_{\rightarrow n} \Pi^0/\varpi^n\Pi^0\right)[1/\varpi]. \tag{4}$$

Since the representation $\Pi$ is a topological $G$-module, we can consider its continuous cohomology groups and we will show that the isomorphism (4) induces an isomorphism in cohomology:

$$H^i(G, \Pi) \cong \left(\lim_{\rightarrow n} H^i(G, \Pi^0/\varpi^n\Pi^0)\right)[1/\varpi]. \tag{5}$$

The isomorphism (4) was proved for compact $p$-adic analytic groups by Emerton in [3, Proposition 1.2.19 and 1.2.20]. The main problem extending his proof is, that in general, projective limits do not commute with taking cohomology. Therefore we will first prove some finiteness properties on the cohomology groups.

6.1 Finiteness conditions

Lemma 6.1. If $K$ is a compact $p$-adic analytic group, $\pi$ an admissible representation in $\text{Mod}_K^{\text{sm}}(O/\varpi^n)$, $n > 0$, then the continuous cohomology groups $H^i(K, \pi)$ are finitely generated $O/\varpi^n$-modules for all $i \geq 0$.

Proof. Let $\pi$ be admissible representation on an $O/\varpi^n$-module. Then its Pontryagin dual $\pi^\vee = \text{Hom}_{O}^{\text{cts}}(\pi, L/O)$ is a finitely generated $O/\varpi^n[K]$-module and since $K$ is compact $p$-adic analytic, the ring $O/\varpi^n[K]$ is Noetherian, by [9, V.2.2.4]. Hence, we can find a resolution $F_\bullet \to \pi^\vee$ by free $O/\varpi^n[K]$-modules of finite rank. Then, taking the dual, we will get an injective resolution $\pi \to (F_\bullet)^\vee$ of $\pi$ in $\text{Mod}_K^{\text{sm}}(O/\varpi^n)$ and the $K$-invariants $(F_i)^\vee^K$ are finitely generated as $O/\varpi^n$-modules. Then, since $O/\varpi^n[K]$ is Noetherian, using Corollary [4], we deduce that $H^i(K, \pi) \cong \text{Ext}_K^i(1, \pi) = H^i(((F_\bullet)^\vee)^K)$ is finitely generated over $O/\varpi^n$ for all $i$. \hfill $\square$

Lemma 6.2. Let $H$ be a topological group and let $\pi \in \text{Mod}_H^{\text{adm}}(O/\varpi^n)$ be an admissible representation of $H$. Assume that $H$ contains an open normal subgroup $N \leq H$ such that

- $H^i(N, \pi)$ is a finitely generated $O/\varpi^n$-module for all $i \geq 0$;
- the quotient group $H/N$ is either finite or a finitely generated abelian group.

Then the $O/\varpi^n$-modules $H^i(H, \pi)$ are finitely generated for all $i \geq 0$.

Proof. Again, by our main result [14] we can compute the continuous cohomology as the Ext group in the category $\text{Mod}_H^{\text{sm}}(O/\varpi^n)$ of smooth representations over $O/\varpi^n$. Since the functor of taking $H$-invariants is the composition of the functor

$$\text{Mod}_H^{\text{sm}}(O/\varpi^n) \xrightarrow{(-)^N} \text{Mod}_H^{\text{sm}}(O/\varpi^n) \xrightarrow{(-)^{H/N}} \text{Mod}(O/\varpi^n),$$

we can apply Lemma 6.1 to the finite group $H/N$ and obtain that $H^i(H/N, \pi)$ are finitely generated $O/\varpi^n$-modules. Then, by the snake lemma, the $H^i(H, \pi)$ are also finitely generated. \hfill $\square$
we obtain a Hochschild-Serre spectral sequence

\[ E_2^{p,q} = H^p(H/N, H^q(N, \pi)) \implies H^{p+q}(H, \pi). \]

Since this is a first quadrant spectral sequence, for fixed \((p, q)\), the limit term is \(E_\infty^{p,q} = E_2^{p,q}\) for some \(n\). Moreover, all of the modules \(H^q(H, \pi)\) are finitely generated and since the quotient group \(H/N\) is discrete, \(H^p(H/N, H^q(N, \pi))\) is the group cohomology of \(H^q(N, \pi)\). By assumption, the quotient group \(H/N\) is either finite or finitely generated and abelian. In both cases, the group ring \(O/\mathfrak{w}^n[H/N]\) is Noetherian and one can compute the cohomology groups \(H^p(H/N, H^q(N, \pi))\) using resolutions consisting of finitely generated \(O/\mathfrak{w}^n\)-modules. Therefore, all of the terms \(E_\infty^{n,q}\) are finitely generated as \(O/\mathfrak{w}^n\)-modules. By definition of the convergence of a spectral sequence, there is a finite filtration of \(H^i(H, \pi)\) such that the graded pieces are isomorphic to one of these finitely generated modules \(E_\infty^{p,q}\) and hence, \(H^i(H, \pi)\) itself is also a finitely generated \(O/\mathfrak{w}^n\)-module. \(\square\)

6.2 \(p\)-adic reductive groups

Let \(F\) be a finite field extension of \(\mathbb{Q}_p\), let \(G\) be a connected reductive group over \(F\) and \(G = G(F)\) its group of \(F\)-rational points. Following the notations of [15], we denote by \(X\) the reduced Bruhat–Tits building associated to \(G\) and for any \(q \geq 0\) we denote by \(X_q\) the set of all \(q\)-dimensional facets in \(X\). \(G\) acts transitively on the set of chambers of \(X\). Therefore, if we fix a chamber \(C\) and write \(C_q\) for the set of \(q\)-facets in \(C\), we can write \(X_q\) as union of finitely many orbits:

\[ X_q \cong \bigcup_{F \in C_q} G/P_F^\dagger, \]

where \(P_F^\dagger\) denotes the \(G\)-stabilizer of a facet \(F\).

In [15], the authors construct some compact open normal subgroups \(R_F\) of \(P_F^\dagger\). Moreover, they show that \(ZR_F\) is of finite index in \(P_F^\dagger\), where \(Z\) denotes the \(F\)-rational points of the connected center of \(G\). The authors construct projective resolutions of representations of \(G\) on \(\mathbb{C}\)-vector spaces which are generated by their \(R_F\)-fixed vectors by considering certain oriented functions with finite support on the set of oriented \(q\)-facets of \(X\). We proceed similarly in our situation to obtain an exact resolution of the trivial representation \(1 \in \text{Mod}_{\text{Gr}}^G(\mathcal{O})\).

By Proposition 11.7 and Theorem 11.16 in [1], \(X\) is a contractible space. In particular, by Corollary 2.11 in [7], the singular homology of \(X\) is trivial. More precisely, one has that \(H_0(X, \mathbb{Z}) = \mathbb{Z}\) and \(H_n(X, \mathbb{Z}) = 0\) for \(n > 0\). And since \(X\) is a \(\Delta\)-complex, the same equalities hold for the simplicial homology. Therefore, a complex that computes the simplicial homology of \(X\) is an exact resolution of \(\mathbb{Z}\) in the category of abelian groups.

We use this to get a resolution of the trivial representation \(1 \in \text{Mod}_{\text{Gr}}^G(\mathcal{O})\). Taking \(A\) to be the integers \(\mathbb{Z}\), the simplicial homology is the homology of the complex \(\Delta_n(X)\), where \(\Delta_n(X)\) is the free abelian group with basis given by the \(n\)-facets of \(X\). Considering the \(G\)-action on \(\Delta_n(X)\) induced from the action of \(G\) on the \(n\)-facets, one gets \(G\)-invariant isomorphisms

\[ \Delta_n(X) \cong \bigoplus_{F \in C_n} \text{c-Ind}_{P_F^\dagger}^{G} \mathbb{Z}. \]
Moreover, the differentials in the complex $\Delta_*(X)$ are $G$-equivariant and we get therefore an exact resolution of $Z$ in $\text{Mod}^\text{fin}_G(Z)$:
\[
\cdots \to \bigoplus_{F \in C_1} \text{c-Ind}^G_{F_F} Z \to \bigoplus_{F \in C_0} \text{c-Ind}^G_{F_F} Z \to Z \to 0. \tag{6}
\]

**Lemma 6.3.** Let $G$ be a $p$-adic reductive group, $\pi \in \text{Mod}^\text{adm}_G(\mathcal{O}/\varpi^n)$, then $H^i(G, \pi)$ is a finitely generated $\mathcal{O}/\varpi^n$-module.

**Proof.** Since $\pi$ is a smooth representation, we can apply Corollary 4.3 to get
\[
H^i(G, \pi) \cong \text{Ext}_G^i(I, \pi),
\]
where the right hand side is computed in the category of smooth representations of $G$ on $\mathcal{O}/\varpi^n$-modules. Since all terms in the resolution are free abelian groups, it will remain exact after tensoring with $\mathcal{O}/\varpi^n$ and we get a resolution of $I$ in $\text{Mod}^\text{fin}_G(\mathcal{O}/\varpi^n)$:
\[
\cdots \to \bigoplus_{F \in C_1} \text{c-Ind}^G_{F_F} I \to \bigoplus_{F \in C_0} \text{c-Ind}^G_{F_F} I \to I \to 0.
\]

Let $\pi \hookrightarrow I^\bullet$ be an injective resolution of $\pi$ in $\text{Mod}^\text{fin}_G(\mathcal{O}/\varpi^n)$. And consider the double complex
\[
C^{i,j} := \text{Hom}_G\left( \bigoplus_{F \in C_i} \text{c-Ind}^G_{F_F} I, I^j \right).
\]

Since the objects $I^j$ are injective, the complexes
\[
0 \to \text{Hom}_G(I, I^j) \to \text{Hom}_G\left( \bigoplus_{F \in C_i} \text{c-Ind}^G_{F_F} I, I^j \right)
\]
are exact. Therefore, the spectral sequence associated to the horizontal filtration of $C^{i,j}$ is given by $E_{2F}^{p,q} = H^p(\text{Hom}_G(I, I^q)) = \text{Ext}_G^q(I^p, \pi)$.

On the other hand, since the cohomology of the columns is
\[
H^q(\text{Hom}_G(\bigoplus_{F \in C_i} \text{c-Ind}^G_{F_F} I, I^\bullet)) = \text{Ext}_G^q(I^p, \pi) \cong \bigoplus_{F \in C_i} \text{Ext}_G^q(\text{c-Ind}^G_{F_F} I, \pi),
\]
the vertical filtration will give a spectral sequence
\[
E_{2,v}^{p,q} = H^p(\bigoplus_{F \in C_i} \text{Ext}_G^q(\text{c-Ind}^G_{F_F} I, \pi)) \Longrightarrow \text{Ext}_G^{p+q}(I, \pi).
\]

Therefore, the claim follows from the fact that $\text{Ext}_G^p(I, \pi)$ is a finitely generated $\mathcal{O}/\varpi^n$-module. Indeed, applying Lemma 6.2 to the groups $R_F \leq ZR_F$, implies that $\text{Ext}_G^p(\mathcal{O}/\varpi^n, \pi)$ is finitely generated and the same argument as in the Lemma applied to the pair $ZR_F \leq P^1_F$ gives the statement for $P^1_F$. \(\square\)

For an $\mathcal{O}$-module $M$, we say that it is $\varpi^\infty$-torsion, if every element in $M$ is annihilated by a power of $\varpi$.

**Lemma 6.4.** Let $G$ be a $p$-adic reductive group, $\Pi$ an admissible unitary Banach space representation of $G$ over $L$ and $\Pi^0$ a $G$-invariant unit ball in $\Pi$. Then we have
\[
H^i(G, \Pi/\Pi^0) \cong \lim_{\to n} H^i(G, \varpi^{-n}\Pi^0/\Pi^0).
\]

In particular, $H^i(G, \Pi/\Pi^0)$ is $\varpi^\infty$-torsion, ie. $H^i(G, \Pi/\Pi^0)[1/\varpi] = 0$. 

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The representation $\Pi/\Pi^0$ is discrete and $\varpi^\infty$-torsion. Moreover, it lies in the category $\text{Mod}^\text{sm}_G(O)$ and is isomorphic to the direct limit

$$\Pi/\Pi^0 \cong \lim_{n} \varpi^{-n}\Pi^0/\Pi^0.$$  

If $K \leq G$ is a compact open subgroup, the terms in the complex of continuous cochains $C^*(K,\Pi/\Pi^0)$ commute with the direct limit, since the image of any map in $C^n(K,\Pi/\Pi^0) \cong C^n(K,\lim_n \varpi^{-n}\Pi^0/\Pi^0)$ will be contained in the image of finitely many $\varpi^{-n}\Pi^0/\Pi^0$ in the limit. And since taking direct limits is exact in the category of $O$-modules (cf. [15, Theorem 2.6.15]), the cohomology groups $H^i(K,\Pi)$ commute with direct limits. This argumentation does not apply for a non-compact $G$. Instead, we proceed as in the proofs of the Lemmas 6.2 and 6.3 to reduce the proof to the compact situation.

To adapt the proofs, note that if $H$ is a discrete group, the continuous cohomology of $H$ is just group cohomology and if we assume that $H$ is either finite or finitely generated and abelian, then the group ring $\mathbb{Z}[H]$ is Noetherian and therefore, $H^i(H,\Pi) \cong \text{Ext}^i_{\mathbb{Z}[H]}(\mathbb{Z},\Pi)$ commutes with direct limits. Then argue as before: by using the resolution $\mathbb{R}$ and an injective resolution $\Pi/\Pi^0 \rightarrow I^\bullet$ in $\text{Mod}^\text{sm}_G(O)$, it is enough to prove that $H^i(P_F,\Pi)$ commutes with direct limits. This follows then from the fact that this is true for the compact open subgroup $R_F$, using as before the Hochschild-Serre spectral sequences associated to $R_F \leq ZR_F$, $ZR_F \leq P_F$.  

In particular, since each of the $H^i(G,\varpi^{-n}\Pi^0/\Pi^0)$ is $\varpi^\infty$-torsion, so is the direct limit $\lim_{n} H^i(G,\varpi^{-n}\Pi^0/\Pi^0) \cong H^i(G,\Pi/\Pi^0)$.  

**Proposition 6.5.** Let $G$ be a $p$-adic reductive group, $\Pi$ an admissible unitary Banach space representation of $G$ over $L$ and $\Pi^0$ a $G$-invariant unit ball in $\Pi$. Then we have

$$H^i(G,\Pi^0) \cong \lim_{n} H^i(G,\Pi^0/\varpi^n\Pi^0).$$

In particular,

$$H^i(G,\Pi) \cong (\lim_{n} H^i(G,\Pi^0/\varpi^n\Pi^0))[1/\varpi].$$

**Proof.** We consider the tower of cochain complexes $\cdots \rightarrow C_2^\bullet \rightarrow C_1^\bullet \rightarrow C_0^\bullet$, where $C_n^i := C^i(G,\Pi^0/\varpi^n\Pi^0)$ is the cochain complex computing the continuous cohomology of $\Pi^0/\varpi^n\Pi^0$. This projective system of cochain complexes satisfies the Mittag-Leffler condition, because each of the maps $C_n^i \rightarrow C_m^i$, $n \geq m$, is surjective. Indeed, since $\Pi^0/\varpi^n\Pi^0$ is discrete, the short exact sequence of representations

$$0 \rightarrow \varpi^n\Pi^0/\varpi^n\Pi^0 \rightarrow \Pi^0/\varpi^n\Pi^0 \rightarrow \Pi^0/\varpi^n\Pi^0 \rightarrow 0$$

induces a short exact sequence of complexes

$$0 \rightarrow C^i(G,\varpi^n\Pi^0/\varpi^n\Pi^0) \rightarrow C^i(G,\Pi^0/\varpi^n\Pi^0) \rightarrow C^i(G,\Pi^0/\varpi^n\Pi^0) \rightarrow 0,$$

which stays exact after taking $G$-invariants, since the module $C^i(G,\varpi^n\Pi^0/\varpi^n\Pi^0)$ is acyclic for the continuous cohomology for all $i$.

We can therefore apply Theorem 3.5.8 of [15], to get a short exact sequence

$$0 \rightarrow \lim_{n} H^{i-1}(C_n^\bullet) \rightarrow H^i(\lim_n C_n^\bullet) \rightarrow \lim_{n} H^i(C_n^\bullet) \rightarrow 0,$$

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where $\lim^{(1)}_n$ denotes the right derived functor of the projective limit functor. But this vanishes for finitely generated modules over complete Noetherian local rings (8 Theorem 1), so by Lemma 6.3 we get an isomorphism

$$H^i(\lim_n C_n^*) \to \lim_n H^i(C_n^*) = \lim_n H^i(G, \Pi^0/\varpi^n \Pi^0).$$

Since the inverse limit $\lim_n \Pi^0/\varpi^n \Pi^0$ can be seen as the inverse limit in the category of topological spaces, one has isomorphisms

$$\text{Hom}^{cts}(G^{i+1}, \lim_n \Pi^0/\varpi^n \Pi^0) \cong \lim_n \text{Hom}^{cts}(G^{i+1}, \Pi^0/\varpi^n \Pi^0),$$

inducing isomorphisms on the $G$-invariants

$$C^i(G, \lim_n \Pi^0/\varpi^n \Pi^0)^G \cong (\lim_n C^i(G, \Pi^0/\varpi^n \Pi^0))^G \cong \lim_n C^i(G, \Pi^0/\varpi^n \Pi^0)^G.$$

The second isomorphism follows again from the universal property of projective limits, since $\text{Hom}_G(1, \lim_n C^i(G, \Pi^0/\varpi^n \Pi^0)) \cong \lim_n \text{Hom}_G(1, C^i(G, \Pi^0/\varpi^n \Pi^0)).$ Therefore, the cohomology of $\lim_n C_n^*$ is in fact $H^i(G, \lim_n \Pi^0/\varpi^n \Pi^0) = H^i(G, \Pi^0).$

It remains to show that $H^i(G, \Pi) \cong H^i(G, \Pi^0)[1/\varpi]$. For this, consider the short exact sequence of $G$-modules

$$0 \to \Pi^0 \to \Pi \to \Pi/\Pi^0 \to 0.$$

The quotient $\Pi/\Pi^0$ is discrete, therefore we get a long exact sequence in cohomology

$$\ldots \to H^{i-1}(G, \Pi/\Pi^0) \to H^i(G, \Pi^0) \to H^i(G, \Pi) \to H^i(G, \Pi/\Pi^0) \to \ldots.$$

Since localization is exact, $H^i(G, \Pi/\Pi^0)$ is $\varpi$-torsion by Lemma 6.4 and $H^i(G, \Pi)$ is an $L$-vector space, we get

$$H^i(G, \Pi^0)[1/\varpi] \cong H^i(G, \Pi)[1/\varpi] \cong H^i(G, \Pi).$$

\[\square\]

**Corollary 6.6.** In the notation of Proposition 6.3, the $L$-vector spaces $H^i(G, \Pi)$ are finite dimensional, for all $i \geq 0$.

**Proof.** Since $H^i(G, \Pi) \cong H^i(G, \Pi^0)[1/\varpi]$, it suffices to show that $H^i(G, \Pi^0)$ is a finitely generated $O$-module. We have a short exact sequence

$$0 \longrightarrow \Pi^0 \longrightarrow \Pi \longrightarrow \Pi/\varpi \Pi^0 \longrightarrow 0,$$

which induces the long exact sequence of $O$-modules

$$\ldots \to H^i(G, \Pi^0) \longrightarrow H^i(G, \Pi) \longrightarrow H^i(G, \Pi/\varpi \Pi^0) \to \ldots.$$

In particular, the quotient $H^i(G, \Pi^0)/\varpi H^i(G, \Pi^0)$ can be embedded into the $O$-module $H^i(G, \Pi^0/\varpi \Pi^0)$, which is finite by Lemma 6.3. Moreover, $H^i(G, \Pi^0)$ is profinite by Lemma 6.3 and Proposition 6.5. Hence it is a compact $O$-module and the claim follows from the topological Nakayama’s lemma. \[\square\]
6.3 Banach space representations of $GL_2(\mathbb{Q}_p)$

We use Proposition 6.5 to prove an analogue of Corollary 5.2 for Banach space representations.

From now on, let $G = GL_2(\mathbb{Q}_p)$, let $\zeta : \mathbb{Q}_p^\times \to \mathbb{L}^\times$ be a unitary character and let $\text{Ban}_{G,\zeta}^\text{adm}(L)$ be the full subcategory of $\text{Ban}_{G}^\text{adm}(L)$ consisting of objects which have central character $\zeta$. This category does not have enough injectives or projectives, but we can consider the Yoneda Ext-groups in this category $\text{Ext}^i_{\text{Ban}_{G,\zeta}^\text{adm}(L)}(\Pi_1, \Pi_2)$.

We want to prove the following

**Proposition 6.7.** For $\Pi_1, \Pi_2 \in \text{Ban}_{G,\zeta}^\text{adm}(L)$, with $\Pi_1$ of finite length, one has

$$\text{Ext}^i_{\text{Ban}_{G,\zeta}^\text{adm}(L)}(\Pi_1, \Pi_2) \cong \left( \lim_{\to} \text{Ext}^i_{G,\zeta}(\Pi_1^{\Phi}/p^n, \Pi_2^{\Phi}/p^n) \right)[1/p],$$

where the right hand side is computed in the category $\text{Mod}_{G,\zeta}^\text{fin}(\mathcal{O})$.

By Proposition 5.34 and Proposition 5.36 of [10], we know that both categories $\text{Ban}_{G,\zeta}^\text{adm}(L)$ and $\text{Mod}_{G,\zeta}^\text{fin}(\mathcal{O})$ decompose into the direct sum, resp. product, of subcategories

$$\text{Ban}_{G,\zeta}^\text{adm}(L) \cong \bigoplus B \text{Ban}_{G,\zeta}^\text{adm}(L)_B,$$

$$\text{Mod}_{G,\zeta}^\text{fin}(\mathcal{O}) \cong \prod B \text{Mod}_{G,\zeta}^\text{fin}(\mathcal{O})_B.$$

In particular, there are no extensions between representations lying in different blocks and it is enough to show 77 for each block individually. Fix such a block $B$ and consider the category $\text{Mod}_{G,\zeta}^\text{fin}(\mathcal{O})_B$ and let $\mathcal{E}(\mathcal{O})$ be the category anti-equivalent to $\text{Mod}_{G,\zeta}^\text{fin}(\mathcal{O})_B$ via Pontryagin duality. Let $\pi_1, \ldots, \pi_n$ be representatives of the isomorphism classes of irreducible $k$-representations in $B$. Then the representation $\bigoplus_{i=1}^n \pi_i^\vee$ lies in $\mathcal{E}(\mathcal{O})$ and has a projective envelope $P$. As in Section 2 of [10], we define the ring $E = \text{End}_{\mathcal{E}(\mathcal{O})}(P)$ as the endomorphism ring of this projective envelope. We obtain an equivalence of categories

$$\mathcal{E}(\mathcal{O}) \overset{\sim}{\longrightarrow} \{\text{compact (right) } E\text{-modules}\},$$

$$M \mapsto \text{Hom}_{\mathcal{E}(\mathcal{O})}(P, M)$$

with inverse given by the completed tensor product $m \mapsto m \hat{\otimes}_E P$, for a compact $E$-module $m$.

The ring $E$ is finitely generated over its center, which is Noetherian (see Section 6.4 of [12]) and hence, our setup satisfies the assumptions of Section 4.2 in [10]. Following this section, we have a fully faithful functor

$$m : \text{Ban}_{G,\zeta}^\text{adm}(L)_B \to \text{Mod}_{E[1/p]}^\text{fg},$$

defined as follows: Let $\Pi \in \text{Ban}_{G,\zeta}^\text{adm}(L)_B$ and let $\Pi^0$ be a $G$-invariant unit ball of $\Pi$ then its Schikhof dual $(\Pi^0)^d := \text{Hom}_\mathcal{O}(\Pi^0, \mathcal{O})$ is an element of $\mathcal{E}(\mathcal{O})$, so by applying $\text{Hom}_{\mathcal{E}(\mathcal{O})}(P, \cdot)$, we obtain a compact $E$-module. We set

$$m(\Pi) := \text{Hom}_{\mathcal{E}(\mathcal{O})}(P, (\Pi^0)^d) \hat{\otimes}_E L.$$
This functor $m$ induces an isomorphism

$$\text{Ext}^i_{\text{Ban}^{\text{adm}}_{G,\xi}(L)_{\mathcal{B}}}(\Pi_1, \Pi_2) \cong \text{Ext}^i_{E[1/p]}(m(\Pi_2), m(\Pi_1)),$$

for $\Pi_1, \Pi_2 \in \text{Ban}^{\text{adm}}_{G,\xi}(L)_{\mathcal{B}},$ with $\Pi_1$ of finite length ([10, Corollary 6.4]). The Ext-group on the right hand side can then be expressed as projective limit in the following way:

Lemma 6.8. Let $m_1, m_2$ be finitely generated $E[1/p]$-modules. Then there exist $E$-stable $O$-lattices $m^0_1, m^0_2$ in $m_1$ and $m_2$ respectively, which are finitely generated as $E$-modules and we have

$$\text{Ext}^i_{E[1/p]}(m_2, m_1) \cong \left( \lim_{\longrightarrow} \text{Ext}^i_{E}(m^0_2/p^n, m^0_1/p^n) \right)[1/p],$$

where the groups $\text{Ext}^i_E$ are computed in the category of finitely generated $E$-modules.

Proof. We can choose a finite set of generators of the $E[1/p]$-module $m$, and let $m^0_i$ be the $E$-submodule of $m_i$ generated by those generators. Then this will be an $O$-lattice in $m_i$. The ring $E$ is in fact compact and is Noetherian, as it is finitely generated over its center, which is Noetherian ([12, Corollary 6.4]). Therefore, the finitely generated $E$-modules $m^0_i$ are compact, and since the quotient of $E$ by its Jacobson radical is a finite-dimensional vector space over the residue field of $O$, the $O$-modules $m^0_i$ are also compact and can be written as projective limit $m^0_i \cong \lim_{\longrightarrow} m^0_i/p^n$. We obtain

$$m_i \cong (\lim_{\longrightarrow} m^0_i/p^n) \otimes_O L.$$

Since the ring $E$ is Noetherian, we can compute the Ext-groups using resolutions by finitely generated modules, so that it commutes with the localization and projective limit and we obtain the stated isomorphism.

Note that since $E$ is Noetherian, the groups $\text{Ext}^i_{E}(m^0_2/p^n, m^0_1/p^n)$ computed in the category of finitely generated $E$-modules agree with the ones computed in the category of compact $E$-modules, which is anti-equivalent to $\text{Mod}^{\text{fin}}_{G,\xi}(O)_{\mathcal{B}}$. Hence, we get the following Lemma:

Lemma 6.9. In the notation of the previous Lemma, we have

$$\text{Ext}^i_{E}(m^0_2/p^n, m^0_1/p^n) \cong \text{Ext}^i_{G,\xi}(((m^0_1/p^n)\hat{\otimes}_E P)^\vee, ((m^0_2/p^n)\hat{\otimes}_E P)^\vee),$$

where the right hand side is computed in the category $\text{Mod}^{\text{fin}}_{G,\xi}(O)$.

Proof. Via the anti-equivalence between the category of compact $E$-modules and the category $\text{Mod}^{\text{fin}}_{G,\xi}(O)_{\mathcal{B}}$, the modules $m^0_i/p^n$ correspond to $((m^0_i/p^n)\hat{\otimes}_E P)^\vee$. The claim follows then from the decomposition of the category $\text{Mod}^{\text{fin}}_{G,\xi}(O)$ into blocks and the fact that the Ext-group of locally finite representations in $\text{Mod}^{\text{fin}}_{G,\xi}(O)$ is isomorphic to their Ext-group computed in the category of smooth representations $\text{Mod}^{\text{sm}}_{G,\xi}(O)$ (see Corollary 5.17 and 5.18 in [10]).

Lemma 6.10. Let $\Pi_1$ and $\Pi_2$ be admissible unitary Banach space representations in $\text{Ban}^{\text{adm}}_{G,\xi}(L)_{\mathcal{B}}, m_i := m(\Pi_i)$. Then we have

$$\text{Ext}^i_{G,\xi}(\Pi^0_1/p^n, \Pi^0_2/p^n) \cong \text{Ext}^i_{G,\xi}(((m^0_1/p^n)\hat{\otimes}_E P)^\vee, ((m^0_2/p^n)\hat{\otimes}_E P)^\vee).$$
Proof. This follows directly from the anti-equivalence of categories between \( \text{Mod}_{\hat{G}, \zeta}^{\text{fin}}(O)_R \) and compact \( E \)-modules.

Combining these Lemmas proves Proposition 6.7.

To state the next Proposition, we need to introduce the following notation:

Let \( \mathbf{1} \) be the trivial one-dimensional representation in \( \text{Ban}_{G}^{\text{adm}}(L) \), \( \hat{S} \) be universal unitary completion of the smooth Steinberg representation of \( G \) over \( L \). Let \( B \) be the subgroup of upper triangular matrices in \( G \). Let \( \hat{\alpha} : B \to L^\times \) be the representation of \( B \), defined by \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto ad^{-1}|ad^{-1}|. \) Then \( |\hat{\alpha}(b)| = 1 \) for all \( b \in B \) and we can define \( (\text{Ind}_{\hat{G}}^{\hat{G}} \hat{\alpha})_{\text{cts}} \) to be the induced representation, given by continuous functions \( f : G \to L \) with \( f(bg) = \hat{\alpha}(b)f(g) \) for all \( b \in B \) and \( g \in G \), with the supremum norm. This defines an admissible unitary Banach space representation of \( G \) (see Section 7 of [10] for more details).

**Proposition 6.11.** Let \( \Pi \in \text{Ban}_{G}^{\text{adm}}(L) \) be an absolutely irreducible admissible unitary Banach space representation of \( G \) and assume that it is not isomorphic to a twist by a unitary character of \( \mathbf{1} \), \( \hat{S} \) or \( (\text{Ind}_{\hat{G}}^{\hat{G}} \hat{\alpha})_{\text{cts}} \). Then, for all \( i \geq 0 \), we have

\[
H^i(\text{SL}_2(\mathbb{Q}_p), \Pi) = 0.
\]

**Proof.** We argue as in the proof of Corollary 5.2. Namely, by Proposition 6.5 and Corollary 4.4, we have

\[
H^i(\text{SL}_2(\mathbb{Q}_p), \Pi) \cong (\lim_n H^i(\text{SL}_2(\mathbb{Q}_p), \Pi^0/\varpi^n\Pi^0))[1/\varpi]
\]

\[
\cong (\lim_n \text{Ext}^i_{\text{SL}_2(\mathbb{Q}_p)}(\mathbf{1}, \Pi^0/\varpi^n\Pi^0))[1/\varpi].
\]

Without loss of generality, we may assume that \( Z \cap \text{SL}_2(\mathbb{Q}_p) \) acts trivially on \( \Pi \). Then we get

\[
H^i(\text{SL}_2(\mathbb{Q}_p), \Pi) \cong (\lim_n \text{Ext}^i_{\text{SL}_2(\mathbb{Q}_p), \zeta}(\zeta, \Pi^0/\varpi^n\Pi^0))[1/\varpi]
\]

\[
\cong (\lim_n \text{Ext}^i_{G, \zeta}(\text{Ind}_{Z \text{SL}_2(\mathbb{Q}_p)}^G \zeta, \Pi^0/\varpi^n\Pi^0))[1/\varpi]
\]

\[
\cong (\lim_n \text{Ext}^i_{G, \zeta}(\text{Ind}_{Z \text{SL}_2(\mathbb{Q}_p)}^G \zeta/\varpi^n, \Pi^0/\varpi^n\Pi^0))[1/\varpi],
\]

where in the bottom line, we consider the induced representation \( \text{Ind}_{Z \text{SL}_2(\mathbb{Q}_p)}^G \zeta \). Since \( Z \text{SL}_2(\mathbb{Q}_p) \) is of index 2 in \( G \) and the character \( \zeta \) is unitary, \( \text{Ind}_{Z \text{SL}_2(\mathbb{Q}_p)}^G \zeta \) is a 2-dimensional unitary Banach space representation of \( G \). Then we can apply Proposition 6.7 to get

\[
H^i(\text{SL}_2(\mathbb{Q}_p), \Pi) \cong \text{Ext}^i_{\text{Ban}_{G}^{\text{adm}}(L)}(\text{Ind}_{Z \text{SL}_2(\mathbb{Q}_p)}^G \zeta \otimes_O L, \Pi).
\]

Therefore, it suffices to prove that this Ext-group vanishes. We may replace \( L \) by a finite field extension so that we may assume without loss of generality, that \( \text{Ind}_{Z \text{SL}_2(\mathbb{Q}_p)}^G \zeta \otimes_O L \) is reducible. Since it is 2-dimensional, we then have a short exact sequence of the form

\[
0 \to \eta_1 \circ \det \to \text{Ind}_{Z \text{SL}_2(\mathbb{Q}_p)}^G \zeta \otimes_O L \to \eta_2 \circ \det \to 0.
\]
Therefore, it is enough to show that $\text{Ext}^i_{\text{Ban}^\text{adm}_{G,\zeta}(L)}(\eta \circ \det, \Pi) = 0$ for each $i \geq 0$. But by the decomposition

$$\text{Ban}^\text{adm}_{G,\zeta}(L) \cong \bigoplus_{\mathcal{B}} \text{Ban}^\text{adm}_{G,\zeta}(L)_{\mathcal{B}},$$

we can reduce to the case where $\Pi$ is a representation in the same block $\text{Ban}^\text{adm}_{G,\zeta}(L)_{\mathcal{B}}$ as $\eta \circ \det$. More precisely, this block is given by

$$\mathcal{B} = \{ \bar{\eta} \circ \det, \text{Sp} \otimes \bar{\eta} \circ \det, (\text{Ind}^G_F \alpha)^{ss} \otimes \bar{\eta} \circ \det \}.$$

And by [12, Section 6.2], the category $\text{Ban}^\text{adm}_{G,\zeta}(L)_{\mathcal{B}}$ decomposes as direct sum of subcategories

$$\text{Ban}^\text{adm}_{G,\zeta}(L)_{\mathcal{B}} \cong \bigoplus_{n \in \text{MaxSpec} Z[1/p]} \text{Ban}^\text{adm}_{G,\zeta}(L)_{\mathcal{B},n}.$$

Moreover, the irreducible objects of the subcategory $\text{Ban}^\text{adm}_{G,\zeta}(L)_{\mathcal{B},n}$ which contains $\eta \circ \det$ are described in Corollary 6.10 of [12] and contains precisely those representations that we excluded. In particular, there are no extensions between $\Pi$ and $\eta \circ \det$.

\[ \square \]

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