Fermions on the electroweak string∗

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Abstract

We construct a simple class of exact solutions of the electroweak theory including the naked \(Z\)-string and fermion fields. It consists in the \(Z\)-string configuration \((\phi, Z_\theta)\), the time and \(z\) components of the neutral gauge bosons \((Z_{0,3}, A_{0,3})\) and a fermion condensate (lepton or quark) zero mode. The \(Z\)-string is not altered (no feed back from the rest of fields on the \(Z\)-string) while fermion condensates are zero modes of the Dirac equation in the presence of the \(Z\)-string background (no feed back from the time and \(z\) components of the neutral gauge bosons on the fermion fields). For the case of the \(n\)-vortex \(Z\)-string the number of zero modes found for charged leptons and quarks is (according to previous results by Jackiw and Rossi) equal to \(|n|\), while for (massless) neutrinos is \(|n| − 1\). The presence of fermion fields in its core make the obtained configuration a superconducting string, but their presence (as well as that of \(Z_{0,3}, A_{0,3}\)) does not enhance the stability of the \(Z\)-string.

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It is well known that cosmic strings are originated in spontaneously broken gauge theories when the vacuum manifold is not simply connected. Strings originating at mass scales $\Lambda$ close to the Planck scale $M_{Pl}$ can yield (and be detected by) gravitational effects: gravitational lensing, seeds for galaxy formation, millisecond pulsar timing perturbations, etc. On the other hand, strings originating at $\Lambda \ll M_{Pl}$ have negligible gravitational effects and, correspondingly, cannot be detected through their gravitational interactions. In particular, the Nielsen-Olesen vortex solution of the abelian Higgs model can be embedded into the $SU(2)_L \times U(1)_Y$ electroweak theory (and then $\Lambda \sim G_F^{-1/2}$). This vortex of $Z$-particles is known as the (naked) $Z$–string. As stated above $Z$–strings have negligible gravitational interactions and experimental detection seems problematic, though they have been proposed as candidates to trigger baryogenesis at the electroweak phase transition (no matter what the order is). However their dynamical stability (not guaranteed by topological arguments) only holds for unrealistic values of $\sin^2 \theta_W$, a mechanism to stabilize the $Z$–string being still missing.

Another class of cosmic strings with non-gravitational effects were proposed by Witten. They are called superconducting strings because they have superconducting charge carriers (either charged bosons or fermions) with expectation values in the core of the string. Superconducting strings can yield observable effects even for $\Lambda \ll M_{Pl}$ (e.g. for $\Lambda \sim G_F^{-1/2}$). In particular superconducting strings with Fermi charge carriers can arise if there are normalizable fermion zero-modes bounded to the string.

In this paper we prove that the $Z$–string is superconducting, with leptons and quarks being the charge carriers. In particular, we will embed the naked $Z$-string into a field configuration with fermion fields, the time and z component of the electromagnetic, $A$, and $Z$ fields and the (unperturbed) $Z$–string. Fermion condensates are zero modes of the Dirac equation in the presence of the $Z$-string background. We have constructed solutions where fermions are either charged leptons, quarks or neutral leptons (neutrinos). As for the former (charged fermions) we have followed the analysis of Ref. [9]. For the latter (neutrinos) we have performed a similar analysis and found that neutrinos can be bounded at the string core by the weak interactions. We have found that the $Z$-string configuration is unaltered by the presence of both the fermion condensate and the time and z components of the electromagnetic and $Z$ fields. A complete numerical analysis is also presented, including the profiles for fermion densities and field configurations and mean radii for the different bound states.

A similar analysis has been recently performed in [10], where a solution of the electroweak theory with a single lepton family is constructed. The authors of Ref. [10] claim their solution is approximate since they put $A_0 = A_3 = Z_0 = Z_3 = 0$. However we have explicitly shown that the fields $A_0$, $A_3$, $Z_0$ and $Z_3$ are non–zero in the presence of fermionic densities and that indeed there is no feedback from them on the fermion zero modes. It is also explicitly proven in [10] that in the absence of the fields $A_0$, $A_3$, $Z_0$ and $Z_3$ the presence of the fermion condensate does not alter the stability properties.

\footnote{See, however, [4] for a recent proposal.}
of the $Z$-string. We have proven by symmetry arguments that this is also the case when dealing with the total solution containing non-zero values for $A_0$, $A_3$, $Z_0$ and $Z_3$. We have also shown the above features remain unchanged when zero-modes are either charged (leptons or quarks) or neutral (neutrinos) fermions.

2. We will consider the case of just one fermion (i.e. lepton or quark) species. Let $\Psi = \begin{bmatrix} \psi^+_L \\ \psi^-_L \end{bmatrix}$ be the left-handed fermionic doublet and $\psi^+_R$, $\psi^-_R$ their right-handed partners. The relevant lagrangian density is therefore:

$$
\mathcal{L} = -\frac{1}{4} W_{\mu\nu}^a W^{\mu\nu^a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + |D_{\lambda} \Phi|^2 - \lambda(\Phi^\dagger \Phi - \eta^2)^2 \\
+ i\bar{\psi} \not{\partial} \psi + i\bar{\psi}_R \not{\partial} \psi^+_R + i\bar{\psi}^-_R \not{\partial} \psi^-_R \\
- h_+ (\bar{\Phi} \psi^+_R + \text{h.c.}) - h_- (\bar{\Phi} \psi^+_R + \text{h.c.})
$$

(1)

where $W_{\mu\nu}^a = \partial_\mu W_{\nu}^a - \partial_\nu W_{\mu}^a + g_\epsilon^{a\kappa\lambda} W_{\kappa}^0 W_{\lambda}^\mu$, $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, $D_\mu = \partial_\mu + igT^a W_{\mu}^a + ig' Y B_{\mu}$, $T^a$ being the corresponding $SU(2)_L$ generator and $Y$ the $U(1)_Y$ hypercharge. $\Phi = \begin{bmatrix} \phi^+ \\ \phi \end{bmatrix}$ is the Higgs doublet and $\Phi \equiv i\sigma^2 \Phi^*$. The Euler-Lagrange equations are:

$$
D_\mu D^\mu \Phi = -2\lambda(\Phi^\dagger \Phi - \eta^2)\Phi - h_+ (\bar{\Psi} \psi^+_R i\sigma^2) - h_- (\bar{\Psi} \psi^-_R \Phi) \\
(D_\nu W^{\mu\nu})^a = -i\frac{g}{2}(\Phi^\dagger \sigma^a (D^\mu \Phi) - (D^\mu \Phi)^\dagger \sigma^a \Phi) - \frac{g}{2} \bar{\Psi} \gamma^\mu \sigma^a \Psi \\
D_\mu B^{\mu\nu} = -i\frac{g'}{2}(\Phi^\dagger (D^\mu \Phi) - (D^\mu \Phi)^\dagger \Phi) \\
-g'y_L \bar{\Psi} \gamma^\mu \Psi - g'y_R \bar{\psi}^+_R \gamma^\mu \psi^+_R - g'y_R \bar{\psi}^-_R \gamma^\mu \psi^-_R
$$

(2)

To solve eqs. (3) it is necessary to make an ansatz on the symmetry of the solution. Our aim is to find some configuration that could be interpreted as the $Z$-string plus a fermion condensate in its core. Then, our starting point will be to generalize the $Z$-string solution keeping, if possible, their symmetries. In particular, concerning global symmetries, the $Z$-string is invariant under the $Z_2$ parity given by the global $U(2)$ transformation on the bosonic fields $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The even fields under this parity

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2The +/- superscript refers to the up/down component, i.e. $\nu_\ell/u$ for leptons and $u/d$ for quarks. The absence of right-handed neutrino in the Standard Model implies, in our notation, $\psi^+_R = 0$, $h_+ = 0$ for leptons.
are: \( \begin{bmatrix} 0 \\ \phi \end{bmatrix}, W^3_\mu, B_\mu \). There are several possible ways to extend this symmetry to the fermionic fields. For example, we can choose one fermion to be odd and the second one, that will be designed by \( \psi_L, \psi_R \), to be even. It is clear from (2) that we can consistently fix to zero all the odd fields. Thus, the solutions under this ansatz are equivalent to those in the reduced \( U(1) \times U(1) \) model \(^4\), spontaneously broken to the \( U(1)_{\text{em}} \) with \( A_\mu \) and \( Z_\mu \) the corresponding gauge bosons.

The relevant field equations are obtained directly from (2) by replacing \( \Phi \to \phi, \Psi \to \psi_L^\dagger \); \( W^{1,2}, \phi^+ \to 0 \). In terms of the mass eigenstates gauge bosons, \( A_\mu = \sin \theta_W W^3_\mu + \cos \theta_W B_\mu; Z_\mu = \cos \theta_W W^3_\mu - \sin \theta_W B_\mu \), the covariant derivatives are:

\[
D_\mu \phi = (\partial_\mu + iq_H Z_\mu) \phi
\]

\[
D_\mu \psi_L = (\partial_\mu + iq_L Z_\mu + iq A_\mu) \psi_L
\]

\[
D_\mu \psi_R = (\partial_\mu + iq_R Z_\mu + iq A_\mu) \psi_R
\]

\( q \) being the electric charge of the corresponding field and \( q_{H,L,R} \) the eigenvalues for the Higgs boson and left and right fermions, respectively, of the \( Z \)-charge, defined in our notation as \( Q^Z = \frac{e}{\sin^2 \theta_W - \sin^2 \theta_W q/e} (T_3 - \sin^2 \theta_W q/e) \), where \( T_3 \) is the third component of the weak isospin, equal to 0 for singlets and \( \pm 1/2 \) for the doublet components.

In particular, for the gauge bosons we have

\[
\Box Z^\mu - \partial^\mu \partial^\nu Z_{\nu} =iq_H \left( \phi^\dagger (D^\mu \phi) - (D^\mu \phi)^\dagger \phi \right) + j^\mu_Z
\]

\[
\Box A^\mu - \partial^\mu \partial^\nu A_{\nu} = j^\mu_A
\]

where the fermionic currents on the right hand side of (4) are defined as

\[
j^\mu_Z = q_L \bar{\psi}_L \gamma^\mu \psi_L + q_R \bar{\psi}_R \gamma^\mu \psi_R
\]

\[
j^\mu_A = q \left( \bar{\psi}_L \gamma^\mu \psi_L + \bar{\psi}_R \gamma^\mu \psi_R \right)
\]

as deduced from eq. (3).

A general, static, \( z \)-independent ansatz still invariant under the combined action of rotation around the \( z \)-axis and the suitable gauge transformation is, in cylindrical coordinates \(^5\)

\[
\phi = f(r)e^{-in\theta}
\]

\[
Z^\alpha = Z^\alpha(r)
\]

\[
A^\alpha = A^\alpha(r)
\]

\(^3\)From here on, and for notational simplicity, we will drop the \( \pm \) superscript from even fields.

\(^4\)Of course, the question of the stability should be addressed in the whole \( SU(2) \times U(1) \) model.

\(^5\)Here \( \alpha = 0, 3, r, \theta \) and we will write the equations in the gauge \( Z^r = A^r = 0 \)
for bosonic fields and
\[
\begin{align*}
\psi_L(r, \theta) &= \frac{1}{\sqrt{2}} \begin{bmatrix} \psi_{L1}(r) e^{-im\theta} \\ \psi_{L2}(r) e^{-i(m-1)\theta} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
\psi_R(r, \theta) &= \frac{1}{\sqrt{2}} \begin{bmatrix} i\psi_{R1}(r) e^{-i(m+n)\theta} \\ i\psi_{R2}(r) e^{-i(m+n-1)\theta} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{align*}
\]
for fermions\footnote{The $i$ factor in the parametrization of $\psi_R$ is a matter of convention, and $\psi_{L1}(r), \psi_{L2}(r), \psi_{R1}(r), \psi_{R2}(r)$ are general complex functions. Notice also that, since we will look for zero modes, we have already fixed the fermion energy $\omega$ to zero and no prefactor $e^{-i\omega t}$ appears in (3).}, where we are using the Dirac representation:
\[
\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
with $\sigma^i$ the Pauli matrices and $\sigma^0 = 1$. Notice that an enlarged gauge configuration, compared to the electroweak string, is expected in general due to the presence of the fermionic currents that will act as sources.

Using the ansatz (3,4), the $\theta$–dependence cancels in the Euler–Lagrange equations. Defining $\psi^i_L = (\psi_{L1}, \psi_{L2}), \psi^i_R = (\psi_{R1}, \psi_{R2})$ we get
\[
f'' = -\frac{1}{r} f' + \left[ \frac{2i}{r} q_H Z_\theta - q_L \psi^\dagger L \sigma L \psi L - q_R \psi^\dagger R \sigma R \psi R \right]
\]
\[
Z''_0 = -\frac{1}{r} Z'_0 + 2q_H^2 |f|^2 Z_0 - q_L \psi^\dagger L \sigma L \psi L - q_R \psi^\dagger R \sigma R \psi R
\]
\[
Z''_3 = -\frac{1}{r} Z'_3 + 2q_H^2 |f|^2 Z_3 - q_L \psi^\dagger L \sigma L \psi L + q_R \psi^\dagger R \sigma R \psi R
\]
\[
Z''_0 = -\frac{1}{r} Z'_0 + 2q_H^2 |f|^2 Z_0 - 2q_H^2 |f|^2 + \frac{1}{r} Z_0 - q_L \psi^\dagger L \sigma L \psi L + q_R \psi^\dagger R \sigma R \psi R
\]
\[
A''_0 = -\frac{1}{r} A'_0 - q \psi^\dagger L \sigma L \psi L - q \psi^\dagger R \sigma R \psi R
\]
\[
A''_3 = -\frac{1}{r} A'_3 - q \psi^\dagger L \sigma L \psi L + q \psi^\dagger R \sigma R \psi R
\]
\[
A''_0 = -\frac{1}{r} A'_0 + \frac{1}{2} A_0 - q \psi^\dagger L \sigma L \psi L + q \psi^\dagger R \sigma R \psi R
\]
\[
0 = q \psi^\dagger L \sigma L \psi L - q \psi^\dagger R \sigma R \psi R
\]
where the last equations of (10) and (11) are constraints corresponding to the gauge conditions $Z^r = A^r = 0$, and
\[
\sigma^1 \psi^i_L' = \{ \frac{1}{r} M_L - i q_L \sigma^2 Z_\theta + i q_L \sigma^0 Z_0 - i q_L \sigma^3 Z_3 - i q_L^2 A_0 + i q_L \sigma^0 A_0 - i q_L \sigma^3 A_3 \} \psi_L - h f^\pm \sigma^0 \psi_R
\]
\[
\sigma^1 \psi^i_R' = \{ \frac{1}{r} M_R - i q_R \sigma^2 Z_\theta - i q_R \sigma^0 Z_0 - i q_R \sigma^3 Z_3 - i q_R^2 A_0 - i q_R \sigma^0 A_0 - i q_R \sigma^3 A_3 \} \psi_R - h f^\pm \sigma^0 \psi_L
\]
where the prime denotes the derivative with respect to the cylindrical radius $r$, and
\[
M_L = \begin{pmatrix} 0 & m - 1 \\ -m & 0 \end{pmatrix}, \quad M_R = \begin{pmatrix} 0 & m + n - 1 \\ -(m + n) & 0 \end{pmatrix}
\]
the ± sign corresponding to the case where the even fermionic fields are \( \psi^+ L, \psi^+_R \) and \( f^+ = f^*, f^- = f \).

Before going ahead with our ansatz, we will review some features of the vortex-fermion system. Jackiw and Rossi showed in Ref. [9] that this system has normalizable zero-modes if the fermions get their mass through their coupling to the scalar field. In fact, the number of these zero modes depends on the winding number of the vortex by an index theorem [11]. These zero modes are transverse [7], i.e. they are eigenstates of \( \gamma^0 \gamma^3 \),

\[
\gamma^0 \gamma^3 \psi = \kappa \psi, \quad \kappa^2 = 1,
\]

which is the operator on the fermions associated to the parity operation \((t, z) \to (-t, -z)\). Let us see how these zero modes are in our case. Notice that, as we remarked before, the gauge field configuration of the naked \( Z \)-string must be enlarged in the presence of this fermionic density. In particular, the time and \( z \) components will be different from zero.

Suppose that (14) also holds in our case and let us see whether it is consistent with eqs. (9). Then

\[
j^0_A = \kappa j^3_A
\]

and

\[
A^0 = \kappa A^3
\]

if the boundary conditions also obey this relation. In the same way,

\[
j^0_Z = \kappa j^3_Z
\]

then

\[
\frac{d^2}{dr^2}(Z^0 - \kappa Z^3) = -\frac{1}{r} \frac{dr}{dr}(Z^0 - \kappa Z^3) + 2 q_H |f|^2 (Z^0 - \kappa Z^3)
\]

and again

\[
Z^0 = \kappa Z^3
\]

for appropriate boundary conditions. Then,

\[
(\gamma^0 Z_0 + \gamma^3 Z_3) \psi_{L,R}(r, \theta) = 0
\]

\[
(\gamma^0 A_0 + \gamma^3 A_3) \psi_{L,R}(r, \theta) = 0
\]

where \( \psi_{L,R}(r, \theta) \) are the spinors defined in the ansatz (6). Eqs. (20) and (19) show that the fermionic zero modes (14) produce no feedback on the naked \( Z \)-string. In other words, the ansatz (6) is consistent with the existence of fermionic zero modes since odd quantities under the operator \( \gamma^0 \gamma^3 \) vanish.

Notice that the coefficients in the equations are real, and therefore the phases of \( f(r), \psi_L(r), \psi_R(r) \) are constant. We can use the global \( U(1) \times U(1) \) to bring them to zero. It is easy to check that condition (14) leads to the ansatz

\[
\psi_L = \begin{pmatrix} \psi_{L1} \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ \psi_{R2} \end{pmatrix}
\]
for \( \kappa = -1 \), and
\[
\psi_L = \begin{pmatrix} 0 \\ \psi_{L2} \end{pmatrix}, \quad \psi_R = \begin{pmatrix} \psi_{R1} \\ 0 \end{pmatrix}
\] (22)
for \( \kappa = 1 \). Using now the ansatz (21) or (22), and \( f(r) \) real, the constraints in (11) and (12), corresponding to the gauge conditions \( Z^r = A^r = 0 \), are trivially satisfied.

The conclusions of this discussion can be summarized as follows:

- It is possible to choose consistently fermions (both left and right components) in one \( \kappa \)-sector. These configurations correspond to zero-modes.
- In this case, the back reaction on the fields in the undressed electroweak string \((\phi, Z_0)\) is exactly zero.
- The equations for the fermions are just given by the Dirac equation in the electroweak string background.
- The time and \( z \) components of the gauge fields are different from zero.

As for the problem of the dynamical stability of this string configuration, it holds for the same values of physical parameters as in the case of the \( Z \)-string. To see that, let us just consider, on top of the ansatz (11) and (12), the same kind of dangerous perturbations \( \delta W^\pm_i(r, \theta) \) \((i = 1, 2)\), \( \delta \phi^\pm(r, \theta) \) that destabilize the \( Z \)-string. These perturbations are odd in our language. Let us focus, first, on fermionic terms. As fermions always appear in cubic terms with one boson, it is clear that this kind of perturbations decouple. This was explicitly proven in Ref. \([10]\). It is also straightforward to see that these perturbations decouple in the new bosonic terms induced in the energy by \( Z_{0,3}, A_{0,3} \). In fact the only relevant terms consistent with our ansatz and Lorentz covariance are

\[
\delta W^\pm_i(r, \theta) \delta W^\mp_j(r, \theta) \delta^{ij} \times \{ \alpha_W [A_0(r)^2 - A_3(r)^2] + \beta_W [Z_0(r)^2 - Z_3(r)^2] + \gamma_W [A_0(r)Z_0(r) - A_3(r)Z_3(r)] \}
\] (23)

and

\[
\delta \phi^\pm(r, \theta) \delta \phi^\mp(r, \theta) \times \{ \alpha_\phi [A_0(r)^2 - A_3(r)^2] + \beta_\phi [Z_0(r)^2 - Z_3(r)^2] + \gamma_\phi [A_0(r)Z_0(r) - A_3(r)Z_3(r)] \}
\] (24)

where the coefficients \( \alpha_{W,\phi}, \beta_{W,\phi} \) and \( \gamma_{W,\phi} \) are some dimensionless combinations of the gauge coupling constants. Using now the conditions (16) and (19), one can easily check that the terms (23) and (24) do vanish, as anticipated. As we see, the requirement of simplicity that made this configuration tractable is too strong to leave any room for stability improvements.

3. We have found a consistent ansatz with zero energy, but we still have to see if the equations admit a non–trivial, normalizable solution. In the bosonic sector we take
\( A_\theta(r) \equiv 0 \), and, for the rest of the fields, the boundary conditions
\[
f(0) = 0, \ f(\infty) = \eta
\]

\[
q_H Z_\theta \equiv \frac{v(r)}{r}, \ v(0) = 0, \ v(\infty) = -1
\]

(25)

\[
A'_{0,3}(0) = Z'_{0,3}(0) = A'_{0,3}(\infty) = Z'_{0,3}(\infty) = 0
\]

For the case of fermions getting their masses through the coupling to the Higgs field we follow the work of Jackiw and Rossi [9]. The relevant equations are:

\[
\begin{bmatrix}
\psi_{L1} \\
\psi_{R2}
\end{bmatrix}' = \left\{ \frac{1}{r} \begin{pmatrix}
-m & 0 \\
0 & m + n - 1
\end{pmatrix} + Z_\theta \begin{pmatrix}
q_L & 0 \\
0 & -q_R
\end{pmatrix} \right\} \begin{bmatrix}
\psi_{L1} \\
\psi_{R2}
\end{bmatrix} \\
-h \pm f \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{bmatrix}
\psi_{L1} \\
\psi_{R2}
\end{bmatrix}
\]

(26)

in the \( \kappa = -1 \) sector and

\[
\begin{bmatrix}
\psi_{L2} \\
\psi_{R1}
\end{bmatrix}' = \left\{ \frac{1}{r} \begin{pmatrix}
m - 1 & 0 \\
0 & -(m + n)
\end{pmatrix} + Z_\theta \begin{pmatrix}
-q_L & 0 \\
0 & q_R
\end{pmatrix} \right\} \begin{bmatrix}
\psi_{L2} \\
\psi_{R1}
\end{bmatrix} \\
-h \pm f \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{bmatrix}
\psi_{L2} \\
\psi_{R1}
\end{bmatrix}
\]

(27)

for \( \kappa = 1 \).

The behaviour of \( \psi_L, \psi_R \) at large \( r \) values is controlled by the coupling to the Higgs field, because the gauge field vanishes there. In general, the asymptotic solution will be, modulo some polynomial prefactors, a combination of two exponential functions, one increasing and another decreasing with \( r \). The requirement of normalizability translates into the condition \( \psi_L|_\infty = \psi_R|_\infty \).

For small \( r \), we have again two equations coupled by the Higgs term (the gauge term is negligible compared to the angular term for wound fermions). By imposing consistency of the equations and regularity of the spinors at the origin, we get that just some values of \( m \) (the parameter that controls the fermion winding) are selected once \( n \) is fixed. For these values, the behaviour of the two fermion fields near \( r = 0 \) is controlled by the centrifugal term coming from the angular momentum, and the Higgs term is negligible. The selected regions in the \( n - m \) plane for the two \( \kappa \) sectors and the two \( \psi^\pm \) fermions are shown in Fig. 1. Notice that the specific values of the fermionic gauge couplings are irrelevant in the above considerations on the number of zero modes. In fact, for the \( n \)-string there are \(|n|\) zero-modes.

A quick glance at Fig. 1 shows that if we try to put both \( \psi^\pm \) fermions in the same \( \kappa \) sector 7, the solution cannot be normalized. Therefore, if we are looking for

7In this case also \( W^{0,3}_\pm \) is generated, playing the same rôle as \( (A, Z)^{0,3} \), and the ansatz (3) would be trivially enlarged.
configurations involving both $\kappa$ sectors, such as those with fermion energy different from zero, the bosonic field ansatz must be enlarged. Of course, it would be very interesting to find an exact normalizable solution also in this case, but the lost of the symmetry makes that task almost impossible.

So far, we have studied the normalizability of the solution just for massive fermions. The rôle of the Higgs boson was crucial there, and we can expect a very different situation for massless fermions (neutrinos). The interaction of the neutrinos with the $Z$-string is purely gauge and is governed by:

$$\psi'_{L1} = \left(-\frac{m}{r} + q_{\nu}Z_{\theta}\right)\psi_{L1} \quad (28)$$

for $\kappa = -1$ and

$$\psi'_{L2} = \left(\frac{m-1}{r} - q_{\nu}Z_{\theta}\right)\psi_{L2} \quad (29)$$

when $\kappa = 1$. The behaviour of the neutrino field near $r = 0$ is controlled by the angular term, and if we impose $\psi$ to be regular at the origin we get:

$$-m \geq 0 \quad \text{for } \kappa = -1$$
$$m - 1 \geq 0 \quad \text{for } \kappa = 1$$

Due to the absence of the Higgs term, $\psi$ is not an exponential function of $r$ at large values, but goes like $r^{\alpha}$, where $\alpha$ is some integer. In particular, for $\kappa = -1$, $\psi_{L1} \sim r^{-(m+n)}$. Then, for $(m + n) > 1$, the neutrino distribution is normalizable. If we ask for a localized configuration, i.e. finite $\langle r \rangle$, $\langle r^2 \rangle$ and assume $m$ integer, i.e. periodic fermions when winding around the string, we get:

$$m + n > 2 \quad \text{for } \kappa = -1$$
$$m + n < -1 \quad \text{for } \kappa = 1$$

We have found that there are normalizable fermionic configurations with zero energy even in the case of massless particles. It means that the gauge interaction can be efficient enough to keep the neutrino near the string core. Let us illustrate this situation by using an analogous quantum mechanical system. From Eq. (28) it follows:

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} + V_{\text{eff}}\right)\psi_{L1} = 0 \quad (30)$$

where (for $\kappa=-1$)

$$V_{\text{eff}}(r) = \frac{1}{2} \left\{ \frac{m(m+1)}{r^2} + q_{\nu} \frac{dZ_{\theta}(r)}{dr} + q_{\nu}^2Z_{\theta}^2(r) - 2mq_{\nu}\frac{Z_{\theta}(r)}{r} \right\} \quad (31)$$

This is the Schrödinger equation for a particle of unit mass under the action of the potential described by $V_{\text{eff}}$. In Fig. 2 we have drawn the shape of this effective
potential as a function of the distance \( r \) to the string axis. We have worked out two cases: one for \( n = 2, m = 0 \) (i.e., without centrifugal term) and the second one for \( n = 4, m = -2 \) (with a centrifugal term). Notice the existence of a centrifugal barrier when \( m \neq -1, 0 \). For large \( r \) values, \( V_{\text{eff}} \sim r^2 \). For the relevant values of \( n \) and \( m \), the structure of the potential can be described by the existence of an absolute minimum with \( V_{\text{min}} < 0 \) and a barrier with \( V_{\text{max}} > 0 \) that decreases as \( 1/r^2 \) for large \( r \). As \( V_{\text{eff}}(\infty) = 0 \), we could consider the possibility of tunnelling of the neutrino zero mode to the large \( r \) region, thus dissociating the bound state. To estimate this probability, we can use the WKB approximation \(^{12} \) assuming that the potential vanishes for \( r > R \), with \( R \) large enough, and take the limit \( R \to \infty \). It is easy to see that this probability is \( \propto R^{-2\sqrt{(n+m)(n+m+1)}} \) and then goes to zero \(^9 \). We have also drawn in the same picture the effective potential for a case in which the distribution is normalizable but \( \langle r \rangle \) is infinite.

4. We have solved numerically the field equations for the neutrino and for massive fermions from the third generation. In Fig. 3a we have plotted fermionic density per unit of string length for several cases. They include the neutrino configuration for \( n = 2, m = 0 \) and a fermion with \( \tilde{M} = M_{\text{top}} \) in the limit \( q_L, q_R \to 0 \). For the neutrino, the interaction is purely gauge, whereas for the other illustrating fermion the interaction is only through the Higgs coupling. We have also drawn the fermionic density for the top quark with the same \( n, m \) values. Notice that, as one could expect, the density shape for the top quark almost coincides with the corresponding one in the limit of zero charges. In all cases, the typical mean radius is of the order of the string radius. We have also calculated the \( Z_0 \) field component corresponding to these configurations. The results are gathered in Fig. 3b. We have repeated this analysis for the bottom quark and the tau lepton. The corresponding profiles are shown in Fig. 4a,b. Notice that the density function goes to zero for large \( r \) much more slowly than in the case of the top quark. This is because the behaviour of the wave function for massive fermions, as we said before, is described by \( e^{-M_f r} \), with \( M_f \) the fermion mass. In Table 1 we list the mean and root-mean-squared radii corresponding to the configurations shown in Figs. 3a,4a. Finally, in Fig. 5a,b we show the radial electric field produced by the charged fermionic distributions. The asymptotic form of this field is given by the Gauss law, \( E_r \sim q r^{-1} \) with \( q \) the charge of the fermion.

5. In summary, we have constructed in this paper a simple class of exact solutions of the electroweak theory, which consists in the \( Z \)-string configuration, the gauge fields \( A_0, A_3, Z_0 \) and \( Z_3 \), and fermion zero modes bounded at the string core. We have explicitly worked out the cases where fermions are charged leptons or quarks, and neutral

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\(^8\)All the dimensional values are expressed in units of the corresponding power of \( \langle \phi \rangle \).

\(^9\)In a realistic situation, a natural value for \( R \) could be provided by the radius of the string loop, or by the typical distance between strings.
leptons (neutrinos). In the cases where zero modes are charged fermions, moving under the action of the electromagnetic field, the $Z$-string becomes superconducting. In our solution there is no feedback from gauge and fermion fields on the $Z$-string configuration. As for the stability problem our solution does not add anything new with respect to the case of the $Z$-string in the absence of fermion zero modes: either the $Z$-string configuration is extended to a stable configuration including other bosonic fields, or its stability is topologically guaranteed by the presence of an extra spontaneously broken (gauge) symmetry, remnant of the theory at some high scale [13]. In both cases the presence of fermion zero modes would make the string configuration superconducting and therefore detectable by non-gravitational interactions.

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| Particle | n | m | $\langle r \rangle$ | $\sqrt{\langle r^2 \rangle}$ |
|----------|---|---|-----------------|-----------------|
| neutrino | 2 | 0 | 4.45            | ---             |
| top      | 1 | 0 | 1.18            | 1.37            |
| bottom   | -1| 0 | 16.39           | 23.12           |
| tau      | -1| 0 | 50.04           | 69.57           |

Table 1: Mean and root-mean-square radii for fermionic configurations shown in Fig. 4a,b

**Figure captions**

**Fig. 1** Allowed regions in the $(n, m)$ plane for $\kappa = 1$ ($\kappa = -1$), circles (squares), and $\psi^+$ ($\psi^-$), open (black), fermions.

**Fig. 2** Plots of the effective potential for the $\psi_{L1}$ neutrino component, as defined in Eq. (31), for the indicated values of $(n, m)$.

**Fig. 3** a) Fermionic densities for the top zero mode $n = 1, m = 0$ and for the neutrino configuration with $n = 2, m = 0$. For illustration, the density for a massive top like fermion in the limit $q_{L,R} = 0$ is also shown (dashed line). In all cases, $\kappa = -1$. b) $Z_0$ component of the gauge field generated by the configurations shown in (a).

**Fig. 4** a) Fermionic densities for tau and bottom fermionic zero modes. In both cases, $n = -1, m = 0$ and $\kappa = -1$. b) $Z_0$ component of the gauge field generated by the configurations shown in (a).

**Fig. 5** a) Electric field $E_r$ generated by the top quark configuration shown in Fig. 3a. b) Electric field $E_r$ generated by the fermionic configurations shown in Fig. 4a.
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Fig. 2
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Fig. 3a
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http://arxiv.org/ps/hep-ph/9411411v1
Fig. 3b
Fig. 4b
Fig. 5b