Co-Pulsing FDA Radar

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Target localization based on frequency diverse array (FDA) radar has lately garnered significant research interest. A linear frequency offset (FO) across FDA antennas yields a range–angle-dependent beampattern that allows for the joint estimation of range and direction of arrival. Prior works on the FDA largely focus on the one-dimensional linear array to estimate only azimuth angle and range while ignoring the elevation and Doppler velocity. However, in many applications, the latter two parameters are also essential for target localization. Furthermore, there is also an interest in radar systems that employ fewer measurements in temporal, Doppler, or spatial signal domains. We address these multiple challenges by proposing a coprime L-shaped FDA, wherein coprime FOs are applied across the elements of L-shaped coprime array, and each element transmits at a nonuniform coprime pulse repetition interval (C-Cube). This co-pulsing FDA yields a significantly large number of degrees of freedom (DoFs) for target localization in the range–azimuth–elevation–Doppler domain while also reducing the time on target and the transmit spectral usage. By exploiting these DoFs, we develop a C-Cube autopairing (CCing) algorithm, in which all the parameters are ipso facto paired during a joint estimation. We show that the C-Cube FDA requires at least $2\sqrt{Q} + 1$ antenna elements and $2\sqrt{Q} + 1$ pulses to guarantee the perfect recovery of $Q$ targets as against $Q + 1$ elements and $Q + 1$ pulses required by both the L-shaped uniform linear array and the L-shaped linear FO FDA with uniform pulsing. We derive Cramér–Rao bounds (CRBs) for joint angle–range–Doppler estimation errors for the C-Cube FDA and provide the conditions under which the CRBs exist. Numerical experiments with our CCing algorithm show great performance improvements in parameter recovery, wherein the C-Cube radar achieves at least 15% higher target hit rate with shorter dwell time than its uniform counterparts.

1. INTRODUCTION

During the past several decades, phased array antenna technology has progressed significantly [1], [2], [3] and found applications in diverse fields, such as radar, sonar, ultrasound, and acoustics [4], [5]. The ability of phased arrays to electronically steer a coherent beam toward boresight is useful for tracking weak targets and suppressing strong sidelobe interferences from other directions. A phased array antenna has only angle-dependent beampattern, and as a result, it is used to estimate the direction of arrival only [6]. To localize targets in both angle and range, beam steering should be achieved across the signal bandwidth leading to a complicated waveform design. As an alternative, recently, a new framework of frequency diverse array (FDA) has been proposed, wherein a small frequency offset (FO) to the carrier frequency is applied across the array elements [7], [8], resulting in range- and angle-dependent beampattern. This has been shown to yield a joint estimation of target angle and range parameters [9], [10], [11]. In FDA radars, spatial (DoA) and range resolutions are fundamentally limited by array aperture and maximum frequency increment.

The classical FDA literature has largely focused on a one-dimensional (1-D) uniform linear array (ULA) with linearly increasing uniform FO across the array elements. The properties of the 1-D FDA, such as the periodicity of the beampattern in range, angle, and time domains, were introduced in [12], [13], and [14], where the coupling relationship of the beampattern and beam steering was also derived. Later, for this uniform linear FDA, joint DoA and range estimation algorithms were investigated in [15], [16], [17], and [18]. In [15], a double-pulse method for range–angle localization was proposed by alternating the antenna between a phased array (zero offset) and the FDA (nonzero offset) in subsequent pulses. This approach first estimates the target DoAs using the traditional phased array configuration and then localizes the targets in the range domain using the FDA. To estimate range and angle at the same time, an unambiguous approach for joint estimation was devised by combining multiple-input multiple-output (MIMO) configuration with FDA [16]. This FDA-MIMO radar exploits degrees of freedom (DoFs) in the range and angle domains. Its estimation accuracy and computational complexity has been shown to improve in a bistatic configuration [18]. Later works have addressed the degraded beam-focusing ability of FDA-MIMO through approaches, such as transmit subaperture FDA radar [17]. Often an exceedingly large number of antennas are required to synthesize a given array aperture to unambiguously distinguish closely spaced targets. The resulting unacceptably huge size, cost, weight, and area have led to the development of sparse arrays, which leverage the presence of a limited number of targets in the scanned region. A uniform [19] or random [20] removal of elements from a filled array leads to grating lobes or increased sidelobe levels, thereby reducing
the spatial resolution and directivity. However, these issues are mitigated through the use of more structured sparse designs, such as coprime arrays [21], [22], [23], which provide a closed form of sensor positions and offer enhanced DoFs for parameter estimation.

In the context of FDA, the introduction of the FOs requires additional bandwidth. Therefore, sparse FDA solutions focus on reducing both spectrum utilization and aperture without any serious degradation in localization performance. Some early FDA works suggested using logarithmic [24], nonuniform [25], and random [26] offsets to optimize the available bandwidth for filled FDAs. In [27], both coprime arrays and coprime FOs were introduced for an FDA radar and Bayesian compressive sensing was used to jointly estimate angles and ranges. This approach required predefined spatial grids leading to a tradeoff between gridding error and computational complexity. This coprime FDA was improved in [28] through a doubly Toeplitz-based estimation, which incorporated coarray interpolation and off-grid estimation techniques. To mitigate the effect of missing elements or holes in the space–frequency coarray, Liu et al. [29] introduced a moving time-modulated coprime FDA, wherein the majority of holes in the coarray positions and FOs could be filled. Recently, coprime FDA-MIMO has also been investigated to further enhance the accuracy and resolution performance through additional DoFs in polarization [30] or unfolded structures [31]. In [32], a sparse variant of FDA-MIMO with linear offsets was optimized for an optimal antenna placement. Nearly all of the aforementioned sparse geometries and FO designs have been investigated for only 1-D arrays. Some 2-D planar FDA arrays were considered in [33] and [34] for retrieving both the azimuth and elevation angles, but these configurations are overly complex.

Sparsity may also be applied to the slow-time domain. The Doppler resolution is determined by the number of transmit pulses. Hence, a large number of transmit pulses or a longer slow-time duration yields high Doppler precision but adversely affects the ability of the radar to track targets in other directions. Some recent studies address this by sparsely transmitting pulses randomly along slow time at nonuniform pulse repetition frequency (PRF) [37], [38]. Since nonuniform random pulsing leads to high sidelobes in the Doppler spectrum [39], very recently, Xu et al. [40] have suggested co-pulsing, i.e., transmitting pulses at coprime pulse repetition interval (PRI) for a single-antenna radar. As for the FDA, in general, there is very limited literature on Doppler estimation. A joint range–angle–Doppler localization was proposed in [41], but the array was a nonsparsely filled FDA-MIMO system. Sparse pulse transmission for the FDA has remained unexamined so far.

Contrary to prior works, we address a multitude of these shortcomings via a simpler 2-D FDA structure in the form of an L-shaped array, which performs coprime sampling in spatial, spectral, and Doppler domains. While L-shaped configuration has been previously investigated for the ULA [35] and the coprime array [36] (both with uniform PRI), its application to the FDA and coprime pulsing remains unexplored. In this article, we develop the theory and localization method for the L-shaped coprime array with coprime FOs and coprime pulsing (C-Cube) that allows for the joint estimation of 2-D DoAs, range, and Doppler velocity. We exploit the difference coarray, frequency difference equivalence, and PRI difference equivalence to achieve a significantly increased number of DoFs in the space–time–Doppler domain, thereby saving resources in aperture, spectrum, and dwell time. Our C-Cube autoping parameter retrieval (CCing) algorithm is based on the singular value decomposition (SVD) of the concatenated covariance matrix, wherein the elevation and azimuth (range and Doppler) are embodied in the left (right) eigenvectors. The autoping property between left and right eigenvectors allows for an automatic pairing of the 2-D DOA with range and Doppler velocity.

In the process, we also obtain results for several other co-pulsing L-shaped arrays: ULA, coprime array, FDA with linear and coprime FOs, and coprime FDA with linear FOs. These are compared with their uniform pulsing counterparts in Table I. Our theoretical performance guarantees show that, to perfectly recover parameters of \( Q \) targets, C-Cube requires a minimum of \( 2\sqrt{Q+1} \) antenna elements and \( 2\sqrt{Q}+1 \) pulses. In comparison, the uniform-pulsing L-shaped ULA and the L-shaped linear FO FDA need at least \( Q+1 \) elements and \( Q+1 \) pulses. Note that the L-shaped coprime array has a similar requirement of pulses and antennas as C-Cube, but the former yields only an angle-dependent beampattern. Note that arrays—FDAs and otherwise—with similar element spacing and pulsing pattern have identical guarantees regarding the number of elements and pulses. But these arrays may still differ in FOs and thereby have differing spectrum consumption. Overall, fewer pulses, antennas, and transmit frequencies in C-Cube compared to its ULA counterparts imply that the power and interceptibility of the former radar are also lower.

The rest of this article is organized as follows. In the next section, we formulate the signal model of L-shaped C-Cube radar. We develop our CCing algorithm in Section III and derive the recovery guarantees for various co-pulsing L-shaped arrays in Section IV, where we also obtain Cramér–Rao bounds (CRBs) for joint angle–range–Doppler estimation using the C-Cube FDA. We discuss other related sparse configurations and alternatives in Section V. We validate our methods through numerical experiments in Section VI. Finally, Section VII concludes this article.

Throughout this article, we reserve boldface lowercase, boldface uppercase, and calligraphic letters for vectors, matrices, and index sets, respectively. We denote the transpose, conjugate, and Hermitian by \(( \cdot )^T\), \(( \cdot )^*\), and \(( \cdot )^H\), respectively; \( \otimes \), \( \odot \), and \( \Diamond \) denote the Kronecker, Khatri–Rao, and Hadamard products, respectively; and \( \text{vec}(\cdot) \) is the vectorization operator that turns a matrix into a vector by stacking all the columns on top of another. The notation \( \lfloor \cdot \rfloor \) indicates the greatest integer smaller than or equal to the argument; \( \langle \cdot \rangle \) denotes the phase of its argument; \( \text{diag}(\cdot) \) denotes a diagonal matrix uses the elements of
TABLE I
Comparison With Prior Art

| Arraya | FO | Spectrumb | Beampatternc | PRI | Antennasd | Pulsesd |
|--------|----|-----------|--------------|-----|-----------|---------|
| L-shaped ULA (U-U) [35] | None | B | Angle | Uniform | Q + 1 | Q + 1 |
| L-shaped coprime array (C-U) [36] | None | B | Angle | Uniform | 2√Q + 1 − 1 | Q + 1 |

This article:

| L-shaped ULA (U-C) | None | B | Angle | Coprime | Q + 1 | 2√Q + 1 − 1 |
| L-shaped coprime array (C-C) | None | B | Angle | CoPrime | Q + 1 | 2√Q + 1 − 1 |
| L-shaped FDA (U-Cube) | Linear | B + (Ps−1)Δf | Angle-range | CoPrime | Q + 1 | 2√Q + 1 − 1 |
| L-shaped FDA (UUC) | Linear | B + (Ps−1)Δf | Angle-range | CoPrime | Q + 1 | 2√Q + 1 − 1 |
| L-shaped FDA (UCU) | Coprime | B + ξPs−1Δf | Angle-range | CoPrime | Q + 1 | 2√Q + 1 − 1 |
| L-shaped FDA (UCUC) | Coprime | B + (Ps−1)Δf | Angle-range | CoPrime | Q + 1 | 2√Q + 1 − 1 |
| L-shaped coprime FDA (C-Cube) | Coprime | B + ξPs−1Δf | Angle-range | CoPrime | Q + 1 | 2√Q + 1 − 1 |
| L-shaped coprime FDA (CCU) | Coprime | B + ξPs−1Δf | Angle-range | CoPrime | Q + 1 | 2√Q + 1 − 1 |

aThe first, second, and third letters in the abbreviations denote the array structure (ULA or coprime), FO (uniformly linear or coprime), and PRI (uniform or coprime), respectively; “-” is used to indicate the absence of FOs.
bCumulative transmit bandwidth of all array elements, where B is the bandwidth of transmit time-limited pulse, Ps is the number of sensors along each axis, Δf is the fundamental frequency increment, and ξPs−1 for coprime arrays denotes the maximum value in the coprime set S1 \cup S2.
cParameters that the antenna beam pattern depends upon.
dNumber of antennas in the same physical aperture, where Q is the number of targets.
eNumber of pulses in the same CPI.

Fig. 1. Illustration of L-shaped FDAs with identical physical aperture and CPI duration. (a) Uniform L-shaped array with uniform FO and uniform PRI. (b) Coprime L-shaped array with coprime FO and coprime PRI. In (b), along each axis, the sensor positions are given by the set S1 \cup S2, where S1 corresponds to the black circles and S2 corresponds to the gray circles. The holes in the coarray are denoted by “×.” The transmit pulse sequence has been illustrated for a CPI corresponding to eight uniform PRIs.

II. SIGNAL MODEL

Consider an L-shaped coprime array with 2Ps − 1 sensors, consisting of two orthogonally placed linear coprime arrays such that there are Ps sensors each along the x- and z-axes, including a common reference sensor at the origin (see Fig. 1). Denote ξ = [ξ0d, ξ1d, ..., ξPs−1d]T as the positions of the array sensors along the x-axis, where d is regarded as the fundamental spatial (interelement) spacing set to d = λb/2 = c/(2fb), fb is the base carrier frequency, c is the speed of light, ξm ∈ S1 = {Mi, 0 ≤ i ≤ Ni−1, i ∈ N} ∪ S2 = {Nj, 1 ≤ j ≤ 2Mj−2, j ∈ N}, Mj and Nj are coprime integers, and Ms < Mj. Without loss of generality, 0 = ξ0 < ξ1 < ... < ξPs−1. It follows that Ps = Ns + 2Mj − 1.

A. Transmit Signal

Each element transmits a signal with an incrementally offset carrier frequency. That is, a continuous-wave signal transmitted from the mth, 0 ≤ m ≤ Ps − 1, element of the x-axis is

\[ s_{ξ_m}(t) = A_{ξ_m}e^{j2π f_b t}, \quad 0 ≤ t ≤ T_p, \quad m = 0, 1, \ldots, P_s - 1 \]  

where \( A_{ξ_m} \) is the amplitude, \( f_{ξ_m} = f_b + ξ_m Δf \) is the carrier frequency of the mth element, Δf is the fundamental frequency increment, \( T_p = 1/B \) is the radar pulse duration, and B is its bandwidth.

Similarly, along the z-axis, the positions of the array sensor are also denoted as \( ξ = [ξ_0d, ξ_1d, \ldots, ξ_{P_s−1}d]^T \). The transmit signal from the mth, 0 ≤ m ≤ Ps − 1, element of the z-axis is

\[ s_{ξ_m}(t) = A_{ξ_m}e^{j2π f_b t}, \quad 0 ≤ t ≤ T_p, \quad m = 0, 1, \ldots, P_s - 1 \]  

where \( A_{ξ_m} \) is the amplitude, \( f_{ξ_m} = f_b + ξ_m Δf \) is the carrier frequency of the mth element, Δf is the fundamental frequency increment, \( T_p = 1/B \) is the radar pulse duration, and B is its bandwidth.

A total of \( K = Ns + 2Mj - 1 \) pulses are transmitted during the coherent processing interval (CPI).
The $k$th, $0 \leq k \leq K - 1$, pulse starts at time instant $\eta_k T$, where $\eta_i \in S_1 = \{M_i, 0 \leq i \leq N_i - 1, i \in \mathbb{N}\} \cup S_2 = \{N_j, 1 \leq j \leq 2M_i - 1, j \in \mathbb{N}\}$ and $T$ is the fundamental PRI for the case of uniform pulsing. Assume $0 = \eta_0 < \eta_1 < \cdots < \eta_{k-1}$. The $k$th, $0 \leq k \leq K - 1$, echo is sampled at the rate $1/T_p$ in fast time $t_q = \eta_i T + l_T p$, where $l = 0, 1, \ldots, L - 1 = [T/T_p]$.

B. Operating Conditions

Assume that the radar scenario consists of $Q$ far-field point targets. The transmit pulses are reflected back by the $Q$ targets, and these echoes are captured by the radar receiver. Our goal is to recover the unknown parameter set $\gamma_q = \{t_q, \varphi_q, r_q, v_q\}_{q=1}^Q$, where $\varphi_q$ is the azimuth angle, $r_q$ is the range, and $v_q$ is the velocity of the $q$th target. We make the following assumptions about the target and radar parameters:

A1 “Narrowband platform”: Assume that the L-shaped FDA radar works in a narrowband environment, so the maximum increment across the L-shaped array satisfies $2\Delta f\eta_{k} \ll f_0$.

A2 “Constant reflectivities”: The radar cross section of each target is described by a Swerling-I model [42], resulting in reflection coefficients being constant across pulses.

A3 “Unambiguous DoA”: To ensure the array structure deprived of ambiguity, we have

$$\sin \theta_q \sin \varphi_q \neq \sin \theta_p \sin \varphi_p,$$

$$\sin \theta_q \cos \varphi_q \neq \sin \theta_p \cos \varphi_p$$

for $1 \leq p \neq q \leq Q$. (3)

A4 “No range or Doppler ambiguities”: The range–Doppler pairs $(r_q, v_q)_{q=1}^Q$ lie in the radar’s unambiguous region of delay–Doppler plane $[0, R_{\max}] \times [0, v_{\max}]$. $R_{\max} = \frac{c}{2\Delta f}$ is the maximum unambiguous range and $v_{\max} = \frac{c}{2K}\Delta f$ is the maximum unambiguous velocity, i.e., the time delays are no longer than the PRI and Doppler frequencies are up to the PRF.

A5 “Constant delays”: The modulation in frequency arising from a moving target appears as a frequency shift in the received signal. We consider this shift to be small over a CPI under the condition $v_q \ll c/(2B\Delta f)$ so that the delay is approximated to be constant. In this case, the Doppler shifts induced are small, allowing for the piecewise-constant approximation: $v_q T \approx v_q \eta_k T$, for $t \in [\eta_k T, \eta_{k+1} T]$.

A6 “Constant Doppler shifts”: The velocity change of a target over a CPI is small compared with the velocity resolution, where $\frac{dv_q}{dt} \ll \frac{c}{2 \Delta f (\eta_{k} - \eta_{k-1})^2}$ is satisfied.

A7 “Slow tangential velocities”: The angle change of a target over a CPI is small compared with the angle resolution, i.e., $\frac{d \theta_q}{dt} \eta_{k-1} T \ll \frac{C_{n_2}}{\sin \theta_q \eta_{k-1} T}$ and $\frac{d \eta_{k-1} T}{dt} \eta_{k-1} T \ll \frac{\sin \theta_q}{C_{n_2}} \frac{dv_q}{dt} \eta_{k-1} T$, where $C_{n_2}$ is a positive constant. This allows for constant DoAs during the CPI.

C. Received Signal

Consider the linear array along the x-axis. The two-way time delay of the $q$th signal received by the $m$th, $0 \leq m \leq P_i - 1$ element at the $k$th, $0 \leq k \leq K - 1$, pulse is $\tau_{q,k}(y_q) = \frac{2r_q v_q T + \eta_{q,k} T}{c} \sin \theta_q \sin \varphi_q = \frac{2r_q v_q T + \eta_{q,k} T}{c} \sin \theta_q \sin \varphi_q$. Then, the received signal of the $q$th target corresponding to the $m$th ($P_i - 1 \leq m \leq P_i - 1$) transmit element, $n_{th}$ $(0 \leq n < P_i - 1)$ receive element, and $k$th $(0 \leq k \leq K - 1)$ pulse is

$$x_{k,n,m}(t, y_q) = \rho_q(t)e^{j2\pi f_{in}(t - \tau_{q,k}(y_q))}$$

(4)

where $\rho_q(t, q = 1, \ldots, Q$, are complex scattering coefficients of the targets, modeled as uncorrelated random variables with $E[\rho_q^* \rho_q] = \sigma_q^2 \delta_{q,p}$. Our focus is on the DoF enhancement at the receive array. Hence, for simplicity, assume that the information about transmit array is embodied in $\rho_q(t, q = 1, \ldots, Q$ [27].

After demodulation and applying bandpass filtering, the baseband signal corresponding to (4) is

$$x_{k,n,m}(t, y_q) = \rho_q(t)e^{j2\pi f_{in}(t - \tau_{q,k}(y_q))}$$

$$= \rho_q(t)e^{j2\pi \left(\eta_{q,k} T + \frac{r_q v_q T}{c} \sin \theta_q \sin \varphi_q \right)}$$

(5)

Note that since the frequency increment is negligible compared to the base carrier frequency, the signal spectrum spreading effect arising from the frequency increment in spatial and Doppler frequency domains is not significant [43]. We approximate the phases in the second and fourth terms after the second equality in (5) as

$$-j2\pi\left(\eta_{q,k} T + \frac{r_q v_q T}{c} \sin \theta_q \sin \varphi_q \right) \approx -j2\pi\left(\frac{\eta_{q,k} T}{c} + \frac{r_q v_q T}{c} \sin \theta_q \sin \varphi_q \right) \approx -j2\pi\left(\frac{\eta_{q,k} T}{c} \right)$$

(6)

and

$$-j2\pi\left(\eta_{q,k} T + \frac{r_q v_q T}{c} \sin \theta_q \sin \varphi_q \right) \approx -j2\pi\left(\frac{r_q v_q T}{c} \sin \theta_q \sin \varphi_q \right) \approx -j2\pi\left(\frac{r_q v_q T}{c} \right) \eta_{q,k}$$

(7)

Thus, (5) becomes

$$x_{k,n,m}(t, y_q) \approx \rho_q(t)e^{-j2\pi \left(\frac{\eta_{q,k} T}{c} \right)} e^{-j2\pi\left(\frac{r_q v_q T}{c} \sin \theta_q \sin \varphi_q \right)} e^{-j2\pi\left(\frac{r_q v_q T}{c} \sin \theta_q \sin \varphi_q \right)}$$

(8)

where the term $\exp\left(-j2\pi r_q v_q T / c \sin \theta_q \sin \varphi_q \right)$ is a constant that can be absorbed into the scattering coefficients such that $\tilde{\rho}_q(t) = \rho_q(t)\exp\left(-j2\pi f_{in}(t) / c \sin \theta_q \sin \varphi_q \right)$.
For $Q$ far-field targets, the received signal is the superposition of echoes from all targets, i.e.,
\[
x_{k,n,m}(t_s) = \sum_{q=1}^{Q} x_{n,m,k}^q(t_s, \gamma_q) + n^i_{k,n,m}(t_s)
\]
\[
= \sum_{q=1}^{Q} \tilde{\rho}_q(t_s) e^{-j2\pi \frac{\sin \theta_q \sin \psi_q}{\lambda} t_s} e^{-j2\pi \frac{\lambda r_q}{c} t_s}
\times e^{-j2\pi \frac{\lambda r_q}{c} t_s} + n^i_{k,n,m}(t_s).
\]
where $n^i_{k,n,m}(t_s)$ is the additive uncorrelated zero mean Gaussian white noise sequence with variance $\sigma_n^2$. Stacking $x_{k,n,m}(t_s)$ for all $-(P_s - 1) \leq m \leq P_s - 1$, $0 \leq n \leq P_s - 1$, and $0 \leq k \leq K - 1$ yields a $K P_s(2P_s - 1) \times 1$ vector
\[
x(t_s) = \sum_{q=1}^{Q} \tilde{\rho}_q(t_s) e(q) \otimes b(q) \otimes a_i(\theta_q, \psi_q) + n^i(t_s)
\]
\[
= (C \otimes B \otimes A_i) \rho(t_s) + n^i(t_s)
\]
where $a_i(\theta_q, \psi_q) = [1, e^{-j2\pi \xi_1 \sin \theta_q \sin \psi_q}, \ldots, e^{-j2\pi \xi_n \sin \psi_q}] \in \mathbb{C}^P \times 1$, $c(q) = [e^{j2\pi \nu_1 / \lambda} e^{j2\pi \nu_2 / \lambda}, \ldots, e^{j2\pi \nu_{K-1} / \lambda}] \in \mathbb{C}^{K-1} \times 1$, and $b(q) = [1, e^{-j2\pi \nu_1 / \lambda}, \ldots, e^{-j2\pi \nu_{K-2} / \lambda}] \in \mathbb{C}^{K-1} \times 1$ are the steering vectors corresponding to $(\theta_q, \psi_q)$, $r_q$, and $q$, respectively. In addition, $A_i = [a_i(\theta_q, \psi_q)]$, $C = [c(q)]$, $B = [b(q)]$, and $\rho(t_s) = [(\rho_1(t_s), \ldots, \rho_Q(t_s))]^T$. It follows from (10) that the total number of lags of physical space–time–frequency array equals the product of lags in each dimension. For example, the structure in Fig. 1(b) yields 864 lags when only nonnegative region is considered.

D. Contiguous Space–Time–Frequency Coarray

The $K P_s(2P_s - 1) \times K P_s(2P_s - 1)$ covariance matrix of data matrix $x(t_s)$ is
\[
R_x = E[x(t_s)x^H(t_s)]
\]
\[
= (C \otimes B \otimes A_i) \rho(t_s) + n^i(t_s)
\]
\[
= (C \otimes B \otimes A_i)^H \hat{\Sigma}_n K_{P_s}(2P_s - 1)
\]
where $R_x = E[\rho(t_s) \rho^H(t_s)] = \text{diag}([\sigma_1^2, \ldots, \sigma_Q^2])$. Vectorizing the matrix $R_x$ produces $(K P_s(2P_s - 1))^2 \times 1$ vector
\[
r_x = \text{vec}(R_x)
\]
\[
= [(C \otimes B \otimes A_i)^* \otimes (C \otimes B \otimes A_i)] \rho^*
\]
\[
+ \sigma_n^2 \text{vec}(I_{K P_s(2P_s - 1)})
\]
\[
= K [(C^* \otimes C) \otimes (B^* \otimes B) \otimes (A_i^* \otimes A_i)] \rho^*
\]
\[
+ \sigma_n^2 \text{vec}(I_{K P_s(2P_s - 1)})
\]
(12)
where $\rho^* = [\sigma_1^2, \ldots, \sigma_Q^2]^T$ is a vector containing the diagonal elements of the matrix $R_x$, $K \in \mathbb{C}^{K P_s(2P_s - 1)^2 \times K P_s(2P_s - 1)^2}$ is a permutation matrix that has exactly one entry of value 1 in each row and each column and 0 elsewhere; the $(i, j)$th entry of $A_i^* \otimes A_i$ is $(e^{-j2\pi (\xi_i - \xi_j) \sin \theta_q \sin \psi_q}, 0 \leq i, j \leq P_s - 1, 1 \leq q \leq Q)$; the $(i, j)$th entry of $C^* \otimes C$ is $(e^{-j2\pi (\xi_i - \xi_j) \lambda / \lambda c}, 0 \leq i, j \leq P_s - 1, 1 \leq q \leq Q)$; and the $(i, j)$th entry of $B^* \otimes B$ is $(e^{-j2\pi \nu_i / \lambda}, 0 \leq i, j \leq K - 1, 1 \leq q \leq Q)$.

The C-Cube structure [see Fig. 1(b)] yields space–time–frequency difference coarray along the positive $x(z)$-axis, as shown in Fig. 2; the negative side is obtained by mirroring along each dimension. The number of nonnegative lags increases from 864 to 8100. As is the case with the 1-D corprime array [44], holes also exist in the space–time–frequency difference coarray from (12). Our retrieval algorithm is based on the framework of subspace-based method, where the rotational invariance property is exploited. Hence, we are only concerned with the contiguous space–time–frequency coarray, namely the subcoarray between the first positive hole and the first negative hole. Therefore, the holes in difference coarray mainly affect the aperture of the contiguous space–time–frequency coarray. The larger aperture the contiguous space–time–frequency coarray has, more DOFs and better performance it yields. The literature suggests appropriate hole filling methods [44], [45] that may be used for structures mentioned in this article. This may yield more efficient coarray structures.

Following [44], a contiguous space–time–frequency coarray is produced, where $A_i^* \otimes A_i$ has the contiguous elements from $e^{-j2\pi (\xi_i - \xi_j) \sin \theta_q \sin \psi_q}$ to $e^{-j2\pi (\xi_P - \xi_j) \sin \theta_q \sin \psi_q}$, $C^* \otimes C$ has the contiguous elements from $e^{-j2\pi \nu_i / \lambda c}$ to $e^{2j\pi \nu_i / \lambda c}$, and $B^* \otimes B$ has the contiguous elements from $e^{-j2\pi \nu_i / \lambda c}$ to $e^{2j\pi \nu_i / \lambda c}$. For the sake of simplicity, assume $L_x = M_s N_s + M_s - 1, L_s = M_s N_s + M_s - 1, L_r = M_s N_s + M_s - 1$. Picking up these continuous entries from $r_x$ yields a new vector
\[
\tilde{r}_x = (\tilde{C} \otimes \tilde{B} \otimes \tilde{A}_i) \rho^* + \sigma_n^2 \tilde{e}
\]
(13)
where

\[
\mathbf{\tilde{A}}_1 = \left( e^{-j\pi L_z \sin \theta_1 \sin \varphi_1} \cdots e^{-j\pi L_z \sin \theta_Q \sin \varphi_Q} \right) \in \mathbb{C}^{(2L_z+1) \times Q}
\]

\[
\mathbf{\tilde{C}} = \left( e^{-j\pi L_z \Delta f_{r1}/c} \cdots e^{-j\pi L_z \Delta f_{rQ}/c} \right) \in \mathbb{C}^{(2L_z+1) \times Q}
\]

\[
\mathbf{\tilde{B}} = \left( e^{-j\pi v_z T_{L_b}/\lambda_b} \cdots e^{-j\pi v_z T_{L_b}/\lambda_b} \right) \in \mathbb{C}^{(2L_z+1) \times Q}
\]

and \( \mathbf{\tilde{e}} \) is an all-zero vector except a 1 in the middle position.

Similarly, for the \( z \)-axis, the received signal model in matrix form follows from (10) as

\[
z(t_s) = \sum_{q=1}^{Q} \mathbf{\tilde{p}}_q(t_s) \mathbf{c}(r_q) \otimes \mathbf{\tilde{b}}(v_q) \otimes \mathbf{\tilde{a}}_q(\theta_q, \varphi_q) + \mathbf{n}'(t_s)
\]

\[
= (\mathbf{C} \otimes \mathbf{B} \otimes \mathbf{A}_C) \mathbf{\rho}(t_s) + \mathbf{n}'(t_s)
\]

where \( \mathbf{a}_q(\theta_q, \varphi_q) = [1, e^{-j\pi \xi_1 \sin \theta_q \cos \varphi_q}, \ldots, e^{-j\pi \xi_{Q-1} \sin \theta_q \cos \varphi_q}]^T \in \mathbb{C}^{Q \times 1} \) and \( \mathbf{A}_C = [\mathbf{a}_1(\theta_1, \varphi_1), \ldots, \mathbf{a}_Q(\theta_Q, \varphi_Q)] \).

Repeating the computations from (11) to (13) gives the contiguous space–time–frequency coarray along the \( z \)-axis as

\[
\mathbf{\tilde{r}}_z = (\mathbf{\tilde{C}} \otimes \mathbf{\tilde{B}} \otimes \mathbf{\tilde{A}}_z) \mathbf{\rho}_z + \sigma_n^2 \mathbf{\tilde{e}}
\]

where

\[
\mathbf{\tilde{A}}_z = \left( e^{-j\pi L_z \sin \theta_1 \cos \varphi_1} \cdots e^{-j\pi L_z \sin \theta_Q \cos \varphi_Q} \right) \in \mathbb{C}^{(2L_z+1) \times Q}
\]

and power spectrum. In [37], both tensor-completion and tensor orthogonal matched pursuit were used to recover range, Doppler velocity, and DoA with undersampled signals. However, the aforementioned algorithms cannot be directly generalized to an L-shaped array, which requires an additional pairing procedure.

The literature suggests a few recovery methods for L-shaped arrays, such as subspace algorithm with 2-D spatial smoothing [49], connection-matrix method [50], and sparse reconstruction for 1-bit DoA estimation [51]. However, nearly all of these works focus on passive sensing and estimate only DoA while leaving out the retrieval of range and Doppler velocity. To this end, we now introduce an autopairing multiparameter retrieval, which addresses these issues when no two targets have any identical parameter values. The proposed algorithm also works when one or more of parameters of targets are the same, but this is at the cost of some DoFs in the angle–range–Doppler domain. This autopairing is possible because, unlike prior works, our L-shaped array is also an FDA, wherein the beampattern is dependent on both range and DoA. Hence, despite separate estimation of these parameters, the FDA properties allow us to pair the parameters uniquely.

III. AUTOPAIRING PARAMETER RETRIEVAL

We focus on the autopairing procedure of unknown parameters \( \mathbf{\gamma}_q = [\theta_q, \varphi_q, r_q, v_q]^T \) \( q = 1 \) \( Q \). Therefore, the variance of noise \( \sigma_n^2 \) is assumed to be known \textit{a priori} that is absorbed into \( \mathbf{\tilde{r}}_z \) in (13) and \( \mathbf{\tilde{r}}_r \) in (18), respectively. Then, by rearranging the elements of \( \mathbf{\tilde{r}}_z \) and \( \mathbf{\tilde{r}}_r \), into matrices \( \mathbf{X} \in \mathbb{C}^{(2L_z+1) \times (2L_z+1)(2L_z+1)} \) and \( \mathbf{Z} \in \mathbb{C}^{(2L_z+1)(2L_z+1)(2L_z+1)} \), we have \( \mathbf{X} = \mathbf{\tilde{A}}_z \mathbf{\rho}_z (\mathbf{\tilde{C}} \otimes \mathbf{\tilde{B}})^T \) and \( \mathbf{Z} = \mathbf{\tilde{A}}_r \mathbf{\rho}_r (\mathbf{\tilde{C}} \otimes \mathbf{\tilde{B}})^T \), where \( \mathbf{\tilde{r}}_z = \text{vec}(\mathbf{X}) \) and \( \mathbf{\tilde{r}}_r = \text{vec}(\mathbf{Z}) \). Define the concatenated covariance matrix

\[
\mathbf{R}_{xz} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{\tilde{A}}_z^T \\ \mathbf{\tilde{A}}_r \end{bmatrix} \begin{bmatrix} \mathbf{\tilde{C}} \otimes \mathbf{\tilde{B}} \end{bmatrix}^T \in \mathbb{C}^{(4L_z+2) \times (2L_z+1)(2L_z+1)}.
\]

The SVD of \( \mathbf{R}_{xz} \) yields

\[
\mathbf{R}_{xz} = [\mathbf{U}_1 \mathbf{U}_2] \begin{bmatrix} \mathbf{A} & 0 \\ 0 & 0 \end{bmatrix} [\mathbf{V}_1 \mathbf{V}_2]^H = \mathbf{U}_1 \mathbf{A} \mathbf{V}_1^H
\]

where the columns of the matrix \([\mathbf{U}_1 \mathbf{U}_2]\) are the left singular vectors of \( \mathbf{R}_{xz} \) with \( \mathbf{U}_1 \) containing the vectors corresponding to the first \( Q \) singular values, \( \mathbf{A} \) is a \( Q \times Q \) diagonal matrix with the \( Q \) nonzero singular values of \( \mathbf{R}_{xz} \), and \([\mathbf{V}_1 \mathbf{V}_2]\) contains the right singular vectors of \( \mathbf{R}_{xz} \) with \( \mathbf{V}_1 \) containing the vectors corresponding to the first \( Q \) singular values. Assume that there exist two invertible \( Q \times Q \) matrices \( \mathbf{T}_L \) and \( \mathbf{T}_R \) such that

\[
\mathbf{U}_1 = \begin{bmatrix} \mathbf{U}_{11} \\ \mathbf{U}_{12} \end{bmatrix} = \begin{bmatrix} \mathbf{\tilde{A}}_x \\ \mathbf{\tilde{A}}_z \end{bmatrix} \mathbf{T}_L \in \mathbb{C}^{(4L_z+2) \times Q}
\]

\[
\mathbf{V}_1^H = \mathbf{T}_R (\mathbf{\tilde{C}} \otimes \mathbf{\tilde{B}})^T \in \mathbb{C}^{Q \times (2L_z+1)(2L_z+1)}
\]

where \( \mathbf{U}_{11} \) and \( \mathbf{U}_{12} \) denote the first and last \( 2L_z+1 \) rows of \( \mathbf{U}_1 \), respectively. Proposition 1 provides sufficient conditions for the existence of \( \mathbf{T}_L \) and \( \mathbf{T}_R \).
Proposition 1 Consider an L-shaped coprime array with \( P_s = N_s + 2M_s - 1 \) sensors along each axis, and let \( d \) be the fundamental spatial spacing. Each sensor transmits a total of \( K = N_s + 2M_s - 1 \) pulses at coprime intervals with \( T \) as the fundamental PRI. There are \( Q \) far-field targets. If

\[
\begin{align*}
C1 \quad &d \leq \frac{2M_s}{T}; \\
C2 \quad &T \leq \frac{T_{fo\nu_{\text{max}}}}{2}; \\
C3 \quad &|f - f_{\nu_{\text{lo}}}| \leq \frac{2M_s}{T}; \\
C4 \quad &L_s \geq M_sN_s + M_s - 1 > \frac{2M_s}{T}; \\
C5 \quad &L_t \geq M_sN_t + M_t - 1 > \frac{2M_s}{T};
\end{align*}
\]

then there exist invertible \( Q \times Q \) matrices \( T_L \) and \( T_R \) such that (22) and (23) hold.

Proof See Appendix A.

In order to recover all four parameters, namely, azimuth, elevation, range, and Doppler from \( \tilde{\tau}_L \) and \( \tilde{\tau}_R \), sparse reconstruction methods may be employed. Among prior works, the joint estimation of signal parameters via rotational invariant techniques (ESPRIT) in [52] examines only the left singular vectors of correlation matrix to extract 2-D information in a passive array. An analogous algorithm for 4-D parameter estimation entails excessive computational load. This is also true for various 4-D tensor-based sparse recovery methods proposed for a single-array sensing in [37]. In contrast, our proposed CCing algorithm exploits the fact that the 2-D DoA and range–Doppler information are embodied in the left and right singular vectors of the correlation matrix, respectively, thus avoiding a 4-D search with high computational complexity.

A. 2-D DoA Estimation

To obtain the rotational invariant structure of the array to estimate the unknown parameters, denote the first (last) 2\( L_s \) rows of \( \tilde{A}_s \) as \( \tilde{A}_{s1} (\tilde{A}_{s2}) \) and \( \tilde{A}_{s1} (\tilde{A}_{s2}) \), respectively. Note that

\[
\tilde{A}_{s2} = \tilde{A}_{s1} \Phi, \quad \tilde{A}_{s1} = \tilde{A}_{s2} \Psi
\]

where \( \Phi = \text{diag}[e^{i\pi \sin \theta_1 \sin \phi}, \ldots, e^{i\pi \sin \theta_1 \sin \phi}] \) and \( \Psi = \text{diag}[e^{i\pi \sin \theta_2 \cos \phi}, \ldots, e^{i\pi \sin \theta_2 \cos \phi}] \).

Denote \( U_{111} \) (\( U_{112} \)) as the first (last) 2\( L_s \) rows of \( U_{11} \) (\( U_{12} \)) as the first (last) 2\( L_s \) rows of \( U_{12} \). Based on (22) and (24), we have \( U_{111} = \tilde{A}_{s1} T_L \), \( U_{112} = \tilde{A}_{s2} T_L = \tilde{A}_{s1} \Phi T_L = U_{111} T_L^{-1} \Phi T_L \), \( U_{121} = \tilde{A}_{s1} T_L \), and \( U_{122} = \tilde{A}_{s2} T_L = \tilde{A}_{s2} \Psi T_L = U_{121} T_L^{-1} \Psi T_L \). Then

\[
\begin{align*}
U_{111} U_{112} &= T_L^{-1} \Phi T_L, \\
U_{112} U_{121} &= T_L^{-1} \Psi T_L.
\end{align*}
\]

Therefore, one could obtain \( \Phi \) and \( T_L \) using the eigenvalue decomposition of \( U_{11}^H U_{112} \), up to a permutation. Denote by \( \hat{\Phi} \) and \( \hat{T}_L \) the resulting matrices. We then compute \( \hat{\Psi} \) as

\[
\hat{\Psi} = \hat{T}_L \left( U_{121} U_{122} \right)^{-1}. \tag{27}
\]

Once \( \Psi \) and \( \Phi \) are found, the elevation and azimuth angle are

\[
\hat{\varphi}_q = \arctan \left( \frac{\omega \Phi_{qq}}{\omega \Psi_{qq}} \right), \quad \hat{\theta}_q = \arcsin \left( \frac{\omega \Phi_{qq}}{\pi \sin \varphi_q} \right). \tag{28}
\]

Clearly, the parameters \( \hat{\varphi}_q \) and \( \hat{\theta}_q \) are paired automatically.

B. Joint Range–Doppler Estimation

Next, we estimate the range \( r_q \) and velocity \( v_q \). Using (22), denote the first 2\( L_s \) rows of \( \hat{\C}_1 \) and the first 2\( L_t \) rows of \( \hat{\B}_1 \) by \( \hat{\C}_1 \) and \( \hat{\B}_1 \), respectively. The last 2\( L_s \) rows of \( \hat{\C} \) and the last 2\( L_t \) rows of \( \hat{\B} \) are similarly denoted by \( \hat{\C}_2 \) and \( \hat{\B}_2 \), respectively. Note the relationships \( \hat{\B}_2 = \hat{\B}_1 \Gamma \) and \( \hat{\C}_2 = \hat{\C}_1 \Omega \), where \( \Gamma = \text{diag}[e^{i\pi v_q T/\lambda_1}, \ldots, e^{i\pi v_q T/\lambda_s}] \) and \( \Omega = \text{diag}[e^{i\pi \Delta f r_q/\lambda_1}, \ldots, e^{i\pi \Delta f r_q/\lambda_s}] \).

Define

\[
\begin{align*}
V_{B1}^H &= \hat{T}_R \left( \hat{\C}_1 \hat{\B}_1 \right)^T, \\
V_{B2}^H &= \hat{T}_R \left( \hat{\C}_1 \hat{\B}_2 \right)^T = \hat{T}_R \left( \hat{\C}_1 \hat{\B}_1 \Gamma \right)^T \\
&= \hat{T}_R \hat{\Gamma}^T \left( \hat{\C}_1 \hat{\B}_1 \right)^T = \hat{T}_R \hat{\Gamma} \hat{T}_R^{-1} V_{B1}^H, \\
V_{C1}^H &= \hat{T}_R \left( \hat{\C}_1 \hat{\B}_1 \right)^T \\
V_{C2}^H &= \hat{T}_R \left( \hat{\C}_1 \hat{\B}_2 \right)^T = \hat{T}_R \left( \hat{\C}_1 \hat{\B}_1 \omega \right)^T = \hat{T}_R \hat{\Omega} \hat{T}_R^{-1} V_{C1}^H.
\end{align*}
\]

Then, based on (30) and (32), we have

\[
\begin{align*}
V_{B1}^H U_{B1}^H &= \hat{T}_R \hat{\Gamma}^T, \\
V_{C1}^H U_{C1}^H &= \hat{T}_R \hat{\Omega} \hat{T}_R^{-1}.
\end{align*}
\]

Performing an eigenvalue decomposition of \( V_{B1}^H U_{B1}^H \) produces the resulting matrices \( \hat{T}_R \) and \( \hat{\Gamma} \). Compute \( \hat{\Omega} \) as \( \hat{\Omega} = \hat{T}_R^{-1} \left( V_{C1}^H U_{C1}^H \right)^T \hat{T}_R \). Once \( \hat{\Gamma} \) and \( \hat{\Omega} \) are found, the velocity and range are obtained as

\[
\hat{v}_q = \frac{\omega \Gamma_{qq}}{4\pi \lambda_1 \Delta f / c}, \quad \hat{r}_q = \frac{\omega \Omega_{qq}}{4\pi \Delta f / c}. \tag{35}
\]

Clearly, the parameters \( \hat{v}_q \) and \( \hat{r}_q \) are paired automatically.

C. Pairing of 2-D DoA With Range–Doppler

The estimates \( (\hat{\theta}_q, \hat{\varphi}_q), q = 1, 2, \ldots, Q \) in (28) are not yet paired with \( (\hat{\tau}_L, \hat{\tau}_R) \), \( i = 1, 2, \ldots, Q \) in (35). This is because \( \hat{T}_R \) in the above computations does not show dependence on \( \hat{T}_L \). We now prove that it is possible to obtain \( \hat{T}_R \) from \( \hat{T}_L \), thereby leading to an ipso facto pairing between \( (\hat{\tau}_L, \hat{\tau}_q), q = 1, 2, \ldots, Q \) and \( (\hat{\tau}_L, \hat{\tau}_R), i = 1, 2, \ldots, Q \). By invoking (20)–(22), we have

\[
\begin{align*}
R_{xz} &= \begin{bmatrix} \hat{A}_s^T & \hat{A}_r^T, \hat{\C}_1 \hat{\B}_1 \end{bmatrix} \hat{T}_R \left( \hat{\C}_1 \hat{\B}_1 \right)^T \\
R_{xz} &= \left[ U_1 \hat{A} V_1^H \right] \hat{T}_L \hat{T}_R \left( \hat{\C}_1 \hat{\B}_1 \right)^T \hat{T}_R^{-1} \hat{\Gamma} \hat{T}_R^{-1} \hat{T}_L \left( \hat{\C}_1 \hat{\B}_1 \right)^T \hat{T}_R^{-1}. \tag{36}
\end{align*}
\]

It follows that

\[
\hat{T}_L \hat{T}_R = \hat{R}_\rho \tag{37}
\]
which implies that
\[ T_R = A^{-1}T_L^{-1}R_p. \] (38)

Substituting (37) into (33) and (34) yields
\[ V_{C_1}^H (V_{C_1}^H)^* T_R = (A^{-1}T_L^{-1}R_pR^*_p - T_L^{-1}A^{-1}T_L^{-1}TTL_A^* \]
and
\[ V_{C_2}^H (V_{C_2}^H)^* T_R = (A^{-1}T_L^{-1}R_pR^*_p - T_L^{-1}A^{-1}T_L^{-1}TTL_A^* \]
where \( \tilde{T} \triangleq R_p^*\Gamma - \Gamma R_p^* \) and \( \tilde{\Omega} \triangleq R_p^*\Omega R_p^* \). Because \( R_p \) is a real-valued diagonal matrix, we have \( \tilde{\Gamma}_{qq} = \Gamma_{qq} \) and \( \tilde{\Omega}_{ll} = \Omega_{ll} \). Rewriting (39) and (40) yields
\[ \tilde{T} = T_L A^T (V_{C_1}^H (V_{C_1}^H)^*)^{-1} A^{-1}T_L^{-1} \]
\[ \tilde{\Omega} = T_L A^T (V_{C_2}^H (V_{C_2}^H)^*)^{-1} A^{-1}T_L^{-1}. \]

The velocity and range that correspond to the estimated elevation angle and azimuth angle \( (\hat{\theta}_q, \hat{\varphi}_q) \) in (28) are now computed as
\[ \hat{v}_q = \frac{\tilde{\Gamma}_{qq}}{4\pi f \lambda}, \quad \hat{r}_q = \frac{\tilde{\Omega}_{qq}}{4\pi f \lambda}. \] (43)

Algorithm 1 summarizes these steps of our C.Cing algorithm.

IV. PERFORMANCE ANALYSES

The error in the joint estimation of angle–range–Doppler of the C-Cube FDA is characterized by lower error bounds such as CRB. From (10) and (17), we rearrange the entries of vector \( [x^T (t_s) z^T (t_s)]^T \), leading to \( 2KP_4 (2P_3 - 1) \times 1 \) vector
\[ y(t_s) \triangleq \left( C \odot B \odot \left[ A_{x} \ A_{z} \right] \right) \rho(t_s) + n(t_s) \]
\[ = (C \odot B \odot A_{xz}^*) \rho(t_s) + n(t_s) \] (44)
where \( A_{xz} = [A_{xz}^T A_{xz}^T]^T \) and \( t_k = 1, \ldots, L_r \). Denote the unknown parameters as \( y = [\theta \varphi \rho r]^T \), \( \theta = [\theta_1, \ldots, \theta_Q]^T \), \( \varphi = [\varphi_1, \ldots, \varphi_Q]^T \), \( r = [r_1, \ldots, r_Q]^T \), and \( v = [v_1, \ldots, v_Q]^T \). The data vectors \( y(t_s) \) are independent identically distributed Gaussian random variables, distributed as
\[ y(t_s) \sim N(0, R(y)) \] (45)
where
\[ \mathbf{R}(y) \triangleq (C \odot B \odot A_{xz}) \mathbf{R}_p (C \odot B \odot A_{xz})^H + \sigma_n^2 I_{2KP_4 (2P_3 - 1)}. \]
In the following, we assume that signal variances \( \sigma_n^2 \) and noise variance \( \sigma_n^2 \), which are defined in (11) and (9), respectively, are known \( a \ priori \).

Then, the entries of the Fisher information matrix (FIM) \( J(y) \) [53] are
\[ \frac{1}{L_r} [J(y)]_{m,n} = \text{vec}^H \left( \frac{\partial \mathbf{R}(y)}{\partial \gamma_m} \right) \mathbf{W} \text{vec} \left( \frac{\partial \mathbf{R}(y)}{\partial \gamma_n} \right) = \left( \frac{\partial \text{vec}(\mathbf{R}(y))}{\partial \gamma_m} \right)^H \mathbf{W} \left( \frac{\partial \text{vec}(\mathbf{R}(y))}{\partial \gamma_n} \right) \] (46)

Algorithm 1: C-Cube Autopairing Parameter Retrieval.

**Input:** Received signal \( \mathbf{r}_l (r_s) \) in the space–time–frequency coarray domain.

**Output:** \( \hat{r}_q, \hat{\theta}_q, \hat{\varphi}_q, q \).

1. Construct \( \mathbf{X}, \mathbf{Z} \in \mathbb{C}^{2L_r + 1 \times (2L_r + 1)(2L_r + 1)} \) such that \( \mathbf{v}_s = \text{vec}(\mathbf{X}), \mathbf{v}_r = \text{vec}(\mathbf{Z}) \).

2. \( \mathbf{R}_{XZ} \leftarrow [\mathbf{X}^T \mathbf{Z}^T]^T \).

3. \( \mathbf{U}_1 (V_1) \leftarrow Q \) columns from \( \mathbf{U} (\mathbf{V}) \) corresponding to the largest \( Q \) singular values of \( \mathbf{R}_{XZ} = \mathbf{U} \mathbf{A} \mathbf{V}^H \) (via SVD).

4. \( \mathbf{U}_{11} (U_{12}) \leftarrow (\text{the first (last)} \) \( 2L_r + 1 \) rows of \( \mathbf{U}_1 \).

5. \( \Psi \leftarrow \mathbf{T}_{L} [U_{12} U_{12}]^{-1}, \) where \( \mathbf{U}_{11} U_{12} = \mathbf{T}_{L}^{-1} \mathbf{F} \mathbf{T}_{L} \) (via eigenvalue decomposition).

6. \( \hat{\varphi}_q = \arctan \left( \frac{\hat{\phi}_{q1}}{\hat{\phi}_{q2}} \right), \quad \hat{\theta}_q = \arcsin \left( \frac{\hat{\phi}_{q2}}{\pi \sin \hat{\phi}_{q1}} \right), \quad q = 1, 2, \ldots, Q. \)

7. **Range–Doppler Estimation and Pairing with 2-D DoA:**

8. Compute \( \hat{\Gamma}, \hat{\Omega} \) via (41) and (42). Estimation based on \( \mathbf{T}_{L} \), which embodies the permutation order of 2-D DoA.

9. \( \hat{v}_q = \frac{\hat{\Gamma}_{qq}}{4\pi f \lambda}, \quad \hat{r}_q = \frac{\hat{\Omega}_{qq}}{4\pi f \lambda}, \quad q = 1, 2, \ldots, Q. \)

where \( \mathbf{W} = \mathbf{R}^{-1} (y) \otimes \mathbf{R}^{-1} (y) \) and \( \gamma_n \) is the \( n \)th entry of \( \gamma \), \( 1 \leq n \leq 4Q \). The CRB of the \( n \)th unknown parameter \( \gamma_n \) is

\[ \text{CRB}(\gamma_n) = \left[ J^{-1}(y) \right]_{n,n}. \] (47)

Vectorizing \( \mathbf{R}(\gamma) \) leads to
\[ \mathbf{r}_{xz} = \text{vec}(\mathbf{R}(\gamma)) = \mathbf{K}_{xz} \left[ \left( C^* \odot C \right) \left( B^* \odot B \right) \odot \left( A_{xz}^* \odot A_{xz} \right) \right] \mathbf{r}_p + \sigma_n^2 \text{vec} \left( I_{2KP_4 (2P_3 - 1)} \right) \] (48)
where \( \mathbf{K}_{xz} \in \mathbb{C}^{4K_2^2 P_4^2 (2P_3 - 1)} \) is a known permutation matrix.

Define
\[ \mathbf{V}_\theta \triangleq \frac{\partial \mathbf{r}_{xz}}{\partial \theta^T} = \mathbf{K}_{xz} \left[ \left( C^* \odot C \right) \left( B^* \odot B \right) \odot \left( A_{xz}^* \odot A_{xz} \right) \right] \mathbf{R}_p \]
\[ = \mathbf{K}_{xz} \left[ \left( C^* \odot C \right) \left( B^* \odot B \right) \odot \frac{\partial \left( A_{xz}^* \odot A_{xz} \right)}{\partial \theta^T} \right] \mathbf{R}_p \] (49)
It follows from the closed form of CRBs in (54)–(57) that the azimuth angle, elevation angle, range, and Doppler velocity interact with each other. This implies that the coupling among these parameters has an effect on the estimation performance.

In addition, we also derive the guarantees for recovering targets. In particular, we discuss conditions on the number of antennas and pulses required to retrieve unknown parameters of Q far-field targets based on the properties of the matrices T_L and T_R obtained from the concatenated covariance matrix R_{XZ} of measurements. This general result then leads to similar guarantees for other L-shaped co-pulsing FDAs and non-FDAs mentioned in Table I. Theorem 3 provides lower bounds on the number of antennas and pulses required by the C-CUBE radar to guarantee perfect recovery of the unknown parameter set \( \gamma_q = \{\theta_q, \varphi_q, r_q, v_q\}_{q=1}^Q \) from \( R_{XZ} \).

**Theorem 3 (C-CUBE: L-SHAPED COPRIME ARRAY WITH COPRIME FO AND COPRIME PULSING)** Consider an L-shaped coprime array with \( P_s = N_s + 2M_t - 1 \) sensors along each axis and coprime FOs. Each sensor transmits at coprime PRI with a total of \( K = N_t + 2M_t - 1 \) pulses in a CPI. The fundamental spatial spacing and fundamental PRI are \( d \) and \( T \), respectively. If \( C1\text{--}C5 \) hold, then the unknown parameter set \( \gamma_q = \{\theta_q, \varphi_q, r_q, v_q\}_{q=1}^Q \) of \( Q \) far-field targets is perfectly recovered from \( R_{XZ} \) with the lower bounds on the number of physical sensor elements and the number of transmit pulses as, respectively

\[
P_s > 2\sqrt{Q + 1} - 2 \quad \text{and} \quad K > 2\sqrt{Q + 1} - 2.
\] (58)

**Proof** It follows from Proposition 1 that \( Uu_{111} \) and \( Uu_{121} \) are both full column rank and, therefore, left-invertible under \( C1 \) and \( C4 \). Therefore, we can recover \( \Phi \) and \( \Psi \) uniquely based on (25) and (26) using the same permutation matrix \( \Gamma_T \). Then, \( C1 \) implies that both \( \angle \Phi_q = \pi \sin \theta_q \sin \varphi_q \) and \( \angle \Psi_q = \pi \sin \theta_q \cos \varphi_q, 1 \leq q \leq Q \) are unique. Consequently, the parameters \( \{\theta_q, \varphi_q\}, 1 \leq q \leq Q \), can also be recovered uniquely. Similarly, from Proposition 1, \( V_n \) and \( V_c \), are both full column rank under \( C2, C3, \) and \( C4 \). Thus, \( V_n^H \) and \( V_c^H \) are right invertible. Therefore, \( \Gamma \) and \( \Omega \) can be recovered uniquely based on (33) and (34) using the permutation matrix \( \Gamma_T \). In addition, under \( C2 \) and \( C3 \), both \( \angle \Omega_{qy} = 4\pi v_q T / \lambda_b \) and \( \angle \Omega_{qy} = 4\pi r_q \Delta f / c, 1 \leq q \leq Q \), are unique thereby guaranteeing unique recovery of the parameters \( \{r_q, v_q\}, 1 \leq q \leq Q \). Using the procedure outline in (37)–(42), \( \{r_q, v_q\}, 1 \leq q \leq Q \), are autopaired with \( \{\theta_q, \varphi_q\}, 1 \leq q \leq Q \). Therefore, \( \gamma_q = \{\theta_q, \varphi_q, r_q, v_q\}_{q=1}^Q \) can be perfectly recovered from \( R_{XZ} \) based on \( C1\text{--}C5 \).

Next, the inequality of arithmetic and geometric means [54] yields \( 2M_N + 2M_s = 2M_s(N_s + 1) \leq \frac{(2M_s + N_s + 1)^2}{4} = \frac{(P_s + 2)^2}{4} \), which gives \( M_s N_s + M_s - 1 \leq \frac{(P_s + 2)^2}{8} - 1 \). Combining this with \( C4 \) gives

\[
P_s > 2\sqrt{Q + 1} - 2.
\] (59)

**Proof** See Appendix B.

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The lower bound on physical pulses is obtained \textit{mutatis mutandis} as

\[ K > 2\sqrt{Q+1} - 2. \]  

(60)

Theorem 3 shows that, in C-CUBE radar, both antennas and pulses need to be at least only \( 2\sqrt{Q+1} - 1 \) to guarantee perfect recovery of \( Q \). In contrast, the L-shaped ULA with uniform pulsing (U-U) [35] requires only \( Q + 1 \) antennas in the same aperture for perfect recovery while transmitting \( Q + 1 \) pulses in the same CPI. The L-shaped coprime array with uniform pulsing (C-U) [36] requires the same number of pulses but only \( 2\sqrt{Q+1} - 1 \) antennas. Similarly, U-C and C-U (non-FDA) arrays may also be hypothesised (see Table I). As for L-shaped FDA that employs uniform array, FO, and PRI, i.e., U-Cube, the following Corollary 4 is a direct consequence of Theorem 3.

\textbf{Corollary 4 (U-CUBE: L-SHAPED ULA WITH UNIFORM FO AND UNIFORM PULSING)} Consider an L-shaped ULA with \( P_i = N_i + 2M_i - 1 \) sensors along each axis and uniform linear FOs. Each sensor transmits at uniform PRI with a total of \( K = N_i + 2M_i - 1 \) pulses in a CPI. The fundamental spatial spacing and fundamental PRI are \( d \) and \( T \), respectively. If \( C1-C3 \) hold and

\[ C6 \begin{align*}
    P_i & \cong N_i + 2M_i - 1 > Q; \\
    C7 \quad K & \cong N_i + 2M_i - 1 > Q;
\end{align*} \]

the unknown parameter set \( \gamma_q = \{\theta_q, \phi_q, r_q, \nu_q\} \) of \( Q \) far-field targets are perfectly recovered from \( R_{xz} \) with the lower bounds on the number of physical sensor elements and the number of transmit pulses as, respectively

\[ P_i > Q \quad \text{and} \quad K > Q. \]  

(61)

\textbf{Proof} Replace the virtual manifold matrices \( \tilde{A}_x, \tilde{A}_z, \tilde{C} \), and \( B \) with the corresponding physical manifold matrices \( A_x, A_z, C, \) and \( B \), respectively. Then, \textit{ceteris paribus}, the result follows from repeating the steps of the proof in Theorem 3.

As mentioned in Section II, we are concerned with the scenario that all the parameters of targets are distinct. It follows from Theorem 3 and Corollary 4 that, if the number of antennas along the \( x(z) \)-axis and pulses is fixed as \( P_i \) and \( K \), respectively, the DOFs of C-Cube may reach \( O(\min\{P_i, K\}) \), while the DOFs of uniform counterparts could be at most only \( O(\min\{P_i, K\}) \) if the number of targets is fixed, for large \( Q \), C-Cube clearly outperforms U-Cube in the required number of sensors and pulses. If the number of potential targets is no more than \( Q \), our proposed C-Cube radar transmits only \( 2\sqrt{Q+1} - 1 \) pulses using \( 2\sqrt{Q+1} - 1 \) antennas compared to the U-Cube that emits \( Q + 1 \) pulses within the same CPI and places \( Q + 1 \) antennas in the same aperture. Obviously, our system has lower power and pulse transmission rate, which reduces its interception probability when compared to U-Cube. For similar results for UUC, CCU, UCU, CUC, CUU, and UCC L-shaped FDAs, we refer the reader to Corollary 7 in Appendix C. Furthermore, the guarantees of non-FDA L-shaped arrays, such as U-U, U-C, C-U, and C-C, also follow from Theorem 3 as in Corollary 8 of Appendix C.

\section{RELATIONSHIP WITH OTHER CONFIGURATIONS}

Several sparse alternatives are possible for the L-shaped FDA, FOs, and pulsing. Although different sparse configurations could be adopted for the arrays, FOs, and pulses, the number of array elements must be the same as that of FOs. Here, we compare a few related configurations with the basic coprime structure of Fig. 1. Some of these sparse structures have been reported for 1-D, usually non-FDA, arrays. But their suitability for L-shaped FDAs remains unexamined.

\subsection{A. Alternative L-Shaped FDA Coarrays}

A straightforward sparse array is achieved by randomly removing the elements from the ULA FDA [32]. This may also be applied to an L-shaped FDA. It is capable of reducing the hardware costs and retaining a reasonable estimation performance. However, random arrays also lead to increased sidelobes [20]. Here, the DoFs are also restricted because a coarray is not exploited. To increase the spatial DoFs, it is pertinent to consider sparse coarray structures.

The mutual coupling matrix of \( 2Q+1 \)-element filled L-shaped array, including FDA, is a \( (2Q+1) \times (2Q+1) \) symmetric matrix \( H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \), where

\[ H_{x} = H_{z} = \begin{bmatrix} 1 & h_1 & \cdots & h_Q \\ h_1 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ h_Q & \cdots & \cdots & 1 \end{bmatrix} \in \mathbb{C}^{(Q+1) \times (Q+1)} \]  

(62)

is the self-coupling (symmetric and Toeplitz) matrix along each axis [55] and

\[ H_{xt} = H_{zt} = \begin{bmatrix} h_{1x} & h_{2x} & \cdots & h_{Qx} \\ h_{1z} & h_{2z} & \cdots & h_{Qz} \\ \vdots & \vdots & \ddots & \vdots \\ h_{Qx} & h_{Qy} & \cdots & h_{1x} \end{bmatrix} \in \mathbb{C}^{Q \times Q} \]  

(63)

where \( H_{xt} \) and \( H_{zt} \) are the cross-coupling matrices between the \( x \)- and \( z \)-axes. Recall the following definition.

\textbf{Definition 5 (Coupling Leakage [56])} The coupling leakage of an L-shaped array with a mutual coupling matrix \( H \) is defined as \( L_H = \frac{\|H - \text{diag}(H)\|_F}{\|H\|_F} \).

We set \( h_1 = 0.3e^{j\pi/3} \) and \( h_2 = \frac{1}{7}e^{-j(1-\pi/8)} \). Then, the coupling leakage is computed using Definition 5 for any given L-shaped array structure (see Table II).

Our adopted coprime array is a special case of compressed interelement spacing (CACIS) coprime array [22], which doubles the number of sensors in a constituting subarray. To mitigate the mutual coupling among array elements, the coprime array with displaced subarray (CADiS)
L-shaped FDA structure is proposed [22] [see Fig. 3(a)], allowing the minimum interelement spacing to be much larger than the typical half-wavelength requirement, while it occupies larger space for the same number of physical elements.

The DoFs are also enhanced in a nested array [57], which is a concatenation of two ULAs: the inner and outer with $N_1$ and $N_2$ elements spaced at $d_1$ and $d_2$, respectively, such that $d_2 = (N_1 + 1)d_1$. Fig. 3(b) shows nested array for an L-shaped structure. Note that setting $M = 1$ and $L = N + 1$ in the CADiS L-shaped FDA converts it to nested CADiS structure [22]. The super-nested L-shaped array [see Fig. 3(c)] retains the benefits of the nested array and also achieves reduced mutual coupling by redistributing the elements of the dense ULA portion of the nested array. Fig. 3(d) and (e) shows the L-shaped concatenated nested array (CNA) [58] and the generalized nested array (GNA) [59], respectively. While the former is a nested array concatenated with its mirror image, the latter enlarges the interelement spacing of two concatenated uniform linear subarrays with two flexible coprime factors. Both the CNA and the GNA enjoy the merit of higher DoFs using fewer elements. However, some of the physical elements are closely located in the CNA leading to severe mutual coupling. The GNA alleviates this problem at the expense of space. This is also the case with the multicoset sparse array [60] [see Fig. 3(f)], which is constructed by a collection of interleaved sparse uniform subarrays such that the elements are laid out in a periodic nonuniform pattern over the aperture.

Our classical coprime L-shaped structure is attractive because it offers a tradeoff between high DoFs and mild mutual coupling arising from smaller aperture (see Table II). It may be viewed as the most fundamental coarray from which many arrays in Fig. 3 are obtained. The receive processing steps of these alternative arrays are obtained by replacing $A_x$ ($A_z$) in (10) [(17)] with the manifold matrix of the corresponding sparse array along each axis.

### Table II

| Array                        | $\mathcal{L}_{H^2}$ |
|------------------------------|----------------------|
| L-shaped uniform FDA (U-Cube) | 0.76                 |
| L-shaped CADiS FDA           | 0.4302               |
| L-shaped nested FDA          | 0.6303               |
| L-shaped super-nested FDA    | 0.5342               |
| L-shaped CNA FDA             | 0.5859               |
| L-shaped GNA FDA             | 0.5969               |
| L-shaped multicoset FDA      | 0.5985               |
| L-shaped coprime FDA (C-Cube)| 0.5340               |

$^a$Computed for common physical aperture of 35 $d$ along each axis.

radar is usually fixed in practice, assume that the cumulative transmit bandwidth of the L-shaped array is the same and given by $B_c = 2(L_f - 1)\Delta f$. Define the spectrum occupancy rate as $\eta = B_c$. Smaller values of $\eta$ imply that more spectral resources are free to be used for other purposes [37].
The literature suggests alternatives such as logarithmic FOs [24], [61], time-dependent FOs (TDFOs) [62], and random FOs [26]. In each case, the coprime FO has lower frequency occupancy under the same cumulative transmit bandwidth. For example, the frequency fed to the $m$th element in logarithmic FO configuration is $f_m = f_b + \Delta f_m$, where the FO of the $m$th element with reference to the carrier frequency $f_b$ is

$$\Delta f_m = \begin{cases} 
\log(m+1)\Delta f, & m \geq 0 \\
-\log(-m+1)\Delta f, & m < 0.
\end{cases} \quad (64)$$

It follows that the interelement FO decreases with the increase in the absolute value of $m$. If $\Delta f_{m+1} - \Delta f_m \geq B$ holds for all $m$, then the spectrum occupancy rate of logarithmic FOs is the same as that of U-Cube, namely, the number of FOs is $2(L_f - 1)$ and the spectrum occupancy rate is

$$\eta_{LOGFO} = \frac{2L_f B}{2(L_f - 1)\Delta f} = \frac{B}{\Delta f} \quad (65)$$

In case of TDFOs, the transmit frequency of the $m$th element is $f_m = f_b + m\Delta f(t)$. Here, the transmit frequency is flexible, but the FOs must be controlled accurately in real time leading to operational difficulties. The number of FOs is at least $2\frac{(L_f - 1)\Delta f}{\max\Delta f(t)}$, with the spectrum occupancy rate

$$\eta_{TDFO} = \frac{2(L_f - 1)\Delta f B}{\max\Delta f(t)2(L_f - 1)\Delta f} = \frac{B}{\max\Delta f(t)} \quad (66)$$

With random sparse FOs [26], the carrier frequency of the $m$th element is $f_m = f_b + \xi_m\Delta f$, where $\xi_m$ is randomly distributed over the interval $[-(L_f - 1), L_f - 1]$. While random FOs may utilize the same spectrum, its recovery procedure is more complicated.

While not reported in the existing literature, analogous to some of the coarrays mentioned in Section V-A, concatenated nested FOs (CNFOs) and generalized nested FOs (GNFOs) are also possible to gain more DoFs and lower frequency occupancy for the same cumulative bandwidth. For CNFOs, the frequency transmitted from the $m$th element is $f_m = f_b + \Delta f_m$, where

$$\Delta f_m = \begin{cases} 
[m + 2N(1 - N)]\Delta f, & -(4N - 2) \leq m \\
[mN + (N - 1)^2]\Delta f, & -(3N - 2) \leq m \leq -N \\
m\Delta f, & -(N - 1) \leq m \leq N - 1 \\
[mN - (N - 1)^2]\Delta f, & N \leq m \leq 3N - 2 \\
[m + 2N(1 - N)]\Delta f, & 3N - 1 \leq m \leq 4N - 2
\end{cases} \quad (67)$$

and $N = \left\lfloor \frac{-1 + \sqrt{1 + 2(1 + L_f)}}{2} \right\rfloor$. Therefore, the number of FOs is at least $8N - 4$ resulting in the occupancy rate

$$\eta_{CNFO} = \frac{(8N - 4)B}{2(L_f - 1)\Delta f} = \frac{(4N - 2)B}{(L_f - 1)\Delta f} \quad (68)$$

Similarly, for GNFOs, we have the $m$th element transmit frequency $f_m = f_b + \Delta f_m$, where

$$\Delta f_m = \begin{cases} 
-\lceil N\alpha - (m + N)\beta \rceil \Delta f, & -(2N - 1) \leq m \leq -N \\
\max\Delta f, & -(N - 1) \leq m \leq N - 1 \\
\lceil N\alpha + (m - N)\beta \rceil \Delta f, & N \leq m \leq 2N - 1.
\end{cases} \quad (69)$$

Here, $\alpha$ and $\beta$ are coprime integers and $N = \left\lceil \frac{L_f + \beta - 1}{\alpha + \beta} \right\rceil$. Then, the number of FOs is $4N - 2$ and the spectrum occupancy rate is

$$\eta_{GNFO} = \frac{(4N - 2)B}{2(L_f - 1)\Delta f} = \frac{(2N - 1)B}{(L_f - 1)\Delta f}. \quad (70)$$

Fig. 4 compares the occupancy rates of coprime FOs, logarithmic FOs, CNFOs, and GNFOs. The coprime FO has the lowest frequency occupancy than other FOs. This excludes the probabilistic random FOs and TDFOs, which also depend on other radar parameters.

C. Alternatives to Coprime Pulsing

Some other nonuniform PRF sequences have been developed in [40], [63], and [64]. The random sparse pulsing eliminates the Doppler ambiguity and enhances electronic counter-countermeasure (ECCM) capability. However, it leads to high sidelobes and restricted DoFs in the Doppler domain [63]. Co-pulsing mitigates false Doppler peaks and saves useful dwell time. The nested pulsing suggested in [40] may be extended to L-shaped arrays to obtain high Doppler resolution under the difference co-pulse concept. Here, a CPI comprises two sparse uniform pulse trains that have $N_1$ pulses with PRI $T$ and $N_2$ pulses with PRI $(N_1 + 1)T$, respectively. The CPI of L-shaped array remains the same, i.e., $T_c = (L_p - 1)T$. Denote the dwell time by $T_d$ and define the dwell time occupancy rate as $\kappa = \frac{T_d}{T_c}$. From the ECCM perspective, lesser the dwell time occupancy rate, better the pulsing scheme. Following a similar analysis as in the previous subsection for co-FOs, the co-pulsing achieves the same Doppler resolution but with lower dwell time than the uniform PRI.
Note that coprime pulsing is the most fundamental co-pulsing scheme and other sequences could be obtained from coprime pulsing. For example, if the first and second uniform sparse pulse trains of coprime pulsing have \( N_1 = N \) pulses with PRI \( T_1 = MT \) and \( N_2 = 2M - 1 \) with PRI \( T_2 = NT \), respectively, then, *ceteris paribus* nested pulsing is obtained by setting \( T_1 = T \). Nested pulsing yields super-nested sequence with a rearrangement of the positions of pulses as explained for spatial domain in [56]. The concatenated nested pulsing is derived from nested pulsing and its mirror image placed in succession. From coprime pulsing, CADiS pulsing results when \( L > 0 \). Finally, reducing the number of pulses in the second pulse train of coprime pulsing yields CACIS pulsing.

VI. NUMERICAL EXPERIMENTS

We validated our co-pulsing FDA model and methods through extensive numerical experiments. Unless otherwise noted, we set coprime integers to \( M = 2 \) and \( N_{t} = 3 \) for coprime arrays and coprime FOs. Thus, the number of sensors along either \( x \)-axis or \( z \)-axis is \( P_t = N_t + 2M_t - 1 = 6 \). The total number of sensors in an \( L \)-shaped coprime array is \( 2P_t - 1 = 11 \). For coprime PRI, the coprime numbers are set to \( M = 2 \) and \( N_{	ext{t}} = 3 \). Hence, a total of \( K = N_t + 2M_t - 1 = 6 \) pulses are transmitted during the CPI with the fundamental PRI and pulse duration of \( T = 0.05 \) ms and \( T_p = 0.5 \) \( \mu \text{s} \), respectively. Since the range periodicity is \( R_u = \frac{\nu}{2\pi T} \), the frequency increment \( \Delta f \) needs to satisfy the boundary condition: \( R_u \geq R_{\text{max}} \), i.e., \( \Delta f \leq c/(2R_{\text{max}}) = 20 \) kHz \([33]\). Therefore, we set the base carrier frequency \( \nu_b \) = 1 GHz and \( \Delta f = 20 \) kHz. Thus, the maximum unambiguous range becomes 7.5 km, and the maximum unambiguous velocity is \( v_{\text{max}} = \frac{c}{2\pi} = \frac{\nu}{2\pi T} = 3000 \) m/s. The parameters of \( Q \) far-field targets are assumed to be in the scope of \( \phi_q \in (0^\circ, 90^\circ) \), \( \varphi_q \in (-70^\circ, 70^\circ) \), \( r_q \in (100, 5000) \) m, and \( v_q \in (10, 400) \) m/s. The receive signal-to-noise ratio (SNR) was computed as

\[
\text{SNR} = 10 \log_{10} \left( \frac{||x||^2 + ||z||^2}{2KP(2P_t - 1)\sigma_n^2} \right) \quad (71)
\]

where \( \sigma_n^2 \) is the additive noise variance. Unless otherwise stated, the reflectivities \( \sigma_s^2 \) of targets are assumed to be equal. Throughout all the experiments, we used our CCE algorithm for parameter recovery.

In the following, we present the target recovery during various experiments in the elevation–azimuth, elevation–range, elevation–Doppler, azimuth–range, azimuth–Doppler, and range–Doppler planes simultaneously. Here, a successful detection (blue cross) occurs when the estimated target is within one range cell, one azimuth bin, and one Doppler bin of the ground truth (red circle); otherwise, the estimated target is labeled as a false alarm (circle with dark fill). Note that, for the purposes of clear illustration, the markers have been magnified; the exact location of the markers should be taken as their geometric centers.

**DoF enhancement:** We first examine the ability of our proposed C-Cube radar to enhance the DoFs in angle–range–Doppler domains by exploiting the difference coarray, frequency difference equivalence, and PRI difference equivalence. The proposed method is compared with the U-Cube. In this comparison, we assume that all the parameters of \( Q \) targets are distinct, namely, \( \theta_q \neq \theta_p \), \( \varphi_q \neq \varphi_p \), \( r_q \neq r_p \), \( v_q \neq v_p \), \( 1 \leq p \neq q \leq Q \). Without loss of generality, we set \( M = M_t \) and \( N_t = N_t \). From C4–C7, U-Cube and C-Cube can detect a maximum of \( Q(\min(P_t, K)) = 6 \) and \( Q(\min(2L, 2L)) = 14 \) targets, respectively. When \( Q = 3 \), Figs. 5 and 6, respectively, show that both C-Cube and U-Cube perfectly recover all the target parameters. However, when the targets are nearly doubled with \( Q = 7 \), U-Cube (see Fig. 7) is unable to retrieve all the parameters, while C-Cube (see Fig. 8) successfully estimates the entire set of target parameters, clearly demonstrating the superior performance of the latter FDA. In addition, Fig. 9 shows

![Fig. 5. Target detection by U-Cube radar with \( Q = 3 \) and SNR = 10 dB in (a) elevation–azimuth, (b) elevation–range, (c) elevation–Doppler, (d) azimuth–range, (e) azimuth–Doppler, and (f) range–Doppler planes. The red circles (blue crosses) indicate ground truth (detected targets).](image-url)

![Fig. 6. (a)–(f) As in Fig. 5 but for the C-Cube radar.](image-url)

![Fig. 7. (a)–(f) As in Fig. 5, but for \( Q = 7 \), all of which could not be detected.](image-url)
that our proposed method also works when one or more target parameters are identical (as long as the conditions C1–C5 of Theorem 3 are satisfied).

Define Hit Rate \( H_r = \frac{1}{Q} \sum_{q=1}^{Q} \left[ |\epsilon_l - \hat{\epsilon}_q| \leq \epsilon_{\theta}, |\epsilon_q - \hat{\epsilon}_q| \leq \epsilon_{\nu}, |\nu_l - \hat{\nu}_q| \leq \epsilon_{\nu}, l \in [1, K] \right] \), where \( \epsilon_{\theta}, \epsilon_{\nu}, \epsilon_{\nu}, \) and \( \epsilon_{\nu} \) are single-bin tolerances in elevation, azimuth, range, and velocity direction, respectively. We averaged hit rates over all Monte Carlo realizations at SNR = 10 dB; Figs. 5 and 6 show a single such realization for U-Cube and C-Cube. The hit rate for C-Cube was 15% higher than U-Cube.

Statistical performance: We analyze the statistical performance of our proposed C-Cube FDA with UUU, UUC, UCU, UCC, CUU, CUC, and CUC. Since the number of uncorrelated targets that L-shaped ULA-based FDA can detect is restricted by the number of physical sensors, for a fair comparison, we set \( Q = 2 \) with \( \gamma_1 = [10^\circ, 5^\circ, 2000 \text{ m}, 75 \text{ m/s}] \) and \( \gamma_2 = [25^\circ, 20^\circ, 2500 \text{ m}, 100 \text{ m/s}] \). The number of samples in each PRI were \( L = \lfloor T / T_p \rfloor = 100 \). We benchmark the performance of various arrays using the root-mean-square error (RMSE) \( \text{RMSE} = \frac{1}{Q} \sum_{q=1}^{Q} \left[ \sqrt{\frac{1}{Q} \sum_{j=1}^{J} (\eta_{q,j} - \hat{\eta}_q)^2} \right] \), where \( J \) (set to 200) is the number of Monte Carlo simulations, \( \eta_{q,j} \) is the estimated parameter (elevation angle, azimuth angle, range, or Doppler velocity) of the \( q \)-th target in the \( j \)-th Monte Carlo simulation, and \( \hat{\eta}_q \) is the true value of the same parameter.

Fig. 10(a) and (b) shows the RMSE of, respectively, elevation and azimuth angles versus the received SNR that was varied from \(-15 \) to \(15 \) dB in the steps of \(3 \) dB. For comparison, we also plot the corresponding root of CRB (RCRB) values. The DoA estimation performance of all the methods improves with SNR. The C-Cube structure benefits from the larger aperture in the coarray domain than, say, U-Cube. Similarly, other structures such as UUC, UCU, UCC, and CUC achieve improved estimation of either DoA [see Fig. 10(a) and (b)], range [see Fig. 10(c)], or Doppler [see Fig. 10(d)] over U-Cube because of the coprime structure in their sensor positions, FOs, and/or PRIs. Clearly, the C-Cube with a simultaneous coprime structure in its sensor positions, FOs, and PRIs outperforms all the other arrays. While both of our proposed C-Cube and CUC methods are more robust to noise than other configurations, the C-Cube also offers savings in the spectrum usage when compared with CUC. The gap between the RCRB and RMSEs in Fig. 10 arises from the fact that the RCRB is based on the whole space–time–frequency coarray of Fig. 2. But the CCing algorithm adopts a more practical approach of the contiguous space–time–frequency coarray. In order to reduce the gap, approaches similar to [45] may be employed to fill the holes in the coarray of Fig. 2. Higher coprime integers and unequal reflectivities: Keeping other parameters the same as in Fig. 10, assume that there are \( Q = 5 \) targets with \( \gamma_1 = [10^\circ, 5^\circ, 2000 \text{ m}, 75 \text{ m/s}] \), \( \gamma_2 = [25^\circ, 20^\circ, 2500 \text{ m}, 100 \text{ m/s}] \), \( \gamma_3 = [55^\circ, 30^\circ, 3000 \text{ m}, 150 \text{ m/s}] \), \( \gamma_4 = [50^\circ, -20^\circ, 3500 \text{ m}, 200 \text{ m/s}] \), and \( \gamma_5 = [60^\circ, -60^\circ, 4500 \text{ m}, 280 \text{ m/s}] \), respectively. We choose the coprime integers as \( M_t = 3 \) and \( N_t = 5 \). We analyzed the statistical performance of our proposed C-Cube FDA for unequal target reflectivities \( \{|\sigma_q^2| \neq 1\} \). The mean value of target power sequence was fixed to \( \sigma_1^2 \), namely, \( \frac{1}{Q} \sum_{q=1}^{Q} \sigma_q^2 = \sigma_1^2 \). The difference among target powers is reflected by the standard deviation (s.d.) of target power sequence, i.e.,

\[
\text{s.d.} = \sqrt{\frac{1}{Q} \sum_{q=1}^{Q} (\sigma_q^2 - \sigma_1^2)^2 / Q}.
\]

It follows from Fig. 11 that the parameter estimation performance of different s.d. values improves with an increase in SNR. In addition, a larger s.d. results in performance degradation, especially in the low-SNR regime because some weak targets get buried in noise or occluded by strong targets.

Performance/complexity comparison: For U-Cube, we compared the performance of our proposed CCing algorithm with the joint ESPRIT [52] and sparse-reconstruction-based PARAFAC (SR-PARAFAC) tensor decomposition [65] algorithms. The target parameters and FDA radar settings were as in the first experiment but with \( Q = 3 \) and fixed SNR of 20 dB. The number of sensors along each axis is the same as the number of pulses; we varied them simultaneously in increments of 2. The CCing had superior parameter estimation than the other two algorithms (see Fig. 12). Furthermore, the CCing algorithm is an order faster than joint ESPRIT and SR-PARAFAC algorithms (see Fig. 13); note that, in addition, the latter are not applicable to C-Cube.
Fig. 10. RMSE of (a) elevation, (b) azimuth, (c) range, and (d) Doppler velocity versus SNR for various L-shaped FDAs with $M_s = 2$, $N_s = 3$, and $Q = 2$.

Fig. 11. RMSE of (a) elevation, (b) azimuth, (c) range, and (d) Doppler velocity versus SNR for various s.d. values in the case of C-Cube configuration.

Fig. 12. RMSE of (a) elevation, (b) azimuth, (c) range, and (d) Doppler velocity for various retrieval algorithms at SNR = 20 dB in the U-Cube radar.

Fig. 13. Runtime of our proposed CCing compared with the joint ESPRIT and SR-PARAFAC algorithms with respect to the number of physical array elements for the U-Cube radar at fixed SNR of 20 dB.

VII. CONCLUSION

In this article, we investigated several new L-shaped co-pulsing FDA radar systems with the goal of reducing resources, such as sensors, spectrum, and dwell time. Drawing on the advances in coarrays, our proposed C-Cube configuration adopts a coprime structure across all three design parameters—array geometry, FOs, and PRIs—and, hence, achieves a high number of DoFs compared to L-shaped ULA, L-shaped coprime array, and L-shaped FDAs. These complex systems invariably present challenges in the joint estimation of target parameters. To this end, our CCing retrieval algorithm allows for an automatic pairing of the 2-D DoA with range and Doppler velocity while also exhibiting faster execution times. Both the analytical and numerical results validate the effectiveness and superiority of our C-CUBE system over various other L-shaped co-pulsing FDAs.

While more complex 2-D coarrays exist as in circular and elliptical geometries [66], the L-shaped arrays are simpler to implement. There is a rich heritage of sparse arrays, where parameter recovery algorithms do not require sparse reconstruction, such as zero redundancy arrays (ZRAs) [67], minimum redundancy arrays (MRAs), and low redundancy
arrays (LRAs) [68], including array patterns that approach Leech’s bounds [69]. In particular, the MRA [70] removes array elements from ULA such that the resulting sparse array has all the possible interelement spacings of the full array. The ZRA, MRA, and LRA minimize aliasing from the grating lobes while maintaining a reasonably constant integrated sidelobe level. However, they are not feasible for sparsifying large arrays. In this context, our approach may be viewed as obtaining low complexity for arbitrarily large sparse arrays. As a future research direction, this work may be extended to develop L-shaped processing for higher DoF structures such as mirrored arrays [71], which comprise an ordinary linear array and a reflector.

APPENDIX A
PROOF OF PROPOSITION 1
A. Preliminaries to the Proof
In order to prove the Proposition, recall the definition of the Kruskal rank.

DEFINITION 6 (SEE [72]) The Kruskal rank 

\[ \text{Krank}(A) \]

of the matrix \( A \) is defined as the largest integer \( \kappa \) such that every \( \kappa \) columns of \( A \) are linearly independent.

B. Proof of the Proposition
We transform the L-shaped coprime array with \( P_t = N_s + M_w + 1 \) physical sensors along each axis into a virtual filled L-shaped array in the difference coarray domain using (12)–(14), where each axis has \( 2M_wN_s + 2M_w - 1 \) consecutive lags with the virtual sensor positions from \( (M_sN_1 - M_w - 1)d \) to \( (M_1N_1 + M_s - 1)d \). Both \( A_x \) and \( \tilde{A}_x \) are Vandermonde matrices with the size \( (2L_x + 1) \times Q \). Using (3) and (18), these matrices have distinct columns.

Hence, if C4 is satisfied, both \( A_x \) and \( \tilde{A}_x \) have full column rank equal to \( Q \). Ergo, the stacked version \( \begin{bmatrix} A_x & \tilde{A}_x \end{bmatrix}^T \) also has the full column rank \( Q \), namely

\[ \text{rank}\left(\begin{bmatrix} A_x & \tilde{A}_x \end{bmatrix}\right) = Q. \quad (73) \]

Similarly, if \( C2 \) and \( C3 \) hold, \( \tilde{C} \) and \( \tilde{B} \) are Vandermonde matrices with the size \( (2L_x + 1) \times Q \) and \( (2L_x + 1) \times Q \), respectively. Then, when \( C4 \) and \( C5 \) are satisfied, both \( \tilde{C} \) and \( \tilde{B} \) have full column rank, namely \( \text{rank}(\tilde{C}) = \text{rank}(\tilde{B}) = Q \).

Using this result and following the definition of the Kruskal rank, we have

\[ \text{Krank}(\tilde{C}) = \text{Krank}(\tilde{B}) = Q. \quad (74) \]

From [73], it follows that

\[ \text{Krank}(\tilde{C} \odot \tilde{B}) \geq \min(Q, \text{Krank}(\tilde{C}) + \text{Krank}(\tilde{B}) - 1). \quad (75) \]

Substituting (74) into (75) yields \( \text{Krank}(\tilde{C} \odot \tilde{B}) \geq Q \). The number of columns of \( \tilde{C} \odot \tilde{B} \) is \( Q \). So

\[ \text{Krank}(\tilde{C} \odot \tilde{B}) = \text{rank}(\tilde{C} \odot \tilde{B}) = Q. \quad (76) \]

In addition, \( R_{\rho} \) in (20) is an \( Q \times Q \) diagonal matrix, which is invertible, namely

\[ \text{rank}(R_{\rho}) = Q. \quad (77) \]

Combining (73), (76), and (77), it follows that

\[ \text{rank}(\tilde{C} \odot \tilde{B})^T R_{\rho} = Q. \quad \text{rank}\left(\begin{bmatrix} \tilde{A}_x \n \tilde{A}_x \end{bmatrix} R_{\rho} \right) = Q. \quad (78) \]

After SVD in (21), we obtain

\[ R_{\rho}^T U_2 = (\tilde{C} \odot \tilde{B})^T R_{\rho} \begin{bmatrix} \tilde{A}_x \n \tilde{A}_x \end{bmatrix} U_2 = 0. \quad (79) \]

Since \( (\tilde{C} \odot \tilde{B})^T R_{\rho} \) is full column rank, it follows that

\[ \begin{bmatrix} \tilde{A}_x \n \tilde{A}_x \end{bmatrix}^T U_2 = 0. \quad (80) \]

Because \( (\tilde{A}_x^H \tilde{A}_x^H) \) has the same size with \( U_1^H \), \( U_2 \) is the null space of \( (\tilde{A}_x^H \tilde{A}_x^H) \) and \( U_1^H \). This implies that

\[ \text{range}\left(\begin{bmatrix} \tilde{A}_x \n \tilde{A}_x \end{bmatrix}\right) = \text{range}(U_1). \quad (81) \]

Ergo, there exists an invertible \( Q \times Q \) matrix \( T_x \) such that (22) holds.

Similarly, following the conjugate transpose version of (21)

\[ R_{\rho}^T = [V_1 V_2] \begin{bmatrix} \Lambda & 0 
 0 & 0 \end{bmatrix}^H [U_1 U_2]^H = V_1 A_1 U_1^H \quad (82) \]

we have

\[ R_{\rho}^T V_2 = \begin{bmatrix} \tilde{A}_x \n \tilde{A}_x \end{bmatrix} R_{\rho} (\tilde{C} \odot \tilde{B})^T V_2 = 0. \quad (83) \]

Invoking (78) gives \( (\tilde{C} \odot \tilde{B})^T V_2 = 0 \). This implies that \( V_2 \) is the null space of \( (\tilde{C} \odot \tilde{B})^T \) and \( V_1^H \) so that

\[ \text{range}(\tilde{C} \odot \tilde{B})^T = \text{range}(V_1). \quad (84) \]

Thus, there exists an invertible \( Q \times Q \) matrix \( T_{\rho} \) such that (23) holds.

APPENDIX B
PROOF OF THEOREM 2
From (46), we have

\[ \frac{J(\varphi)}{L_t} = [V_\varphi V_\varphi V_\varphi V_\varphi V_\varphi]^H W [V_\varphi V_\varphi V_\varphi V_\varphi V_\varphi] \]

\[ = [D_\varphi D_\varphi]^H W [D_\varphi D_\varphi] \]

\[ = \begin{bmatrix} D_\varphi^H W D_\varphi & D_\varphi^H W D_\varphi 
 D_\varphi^H W D_\varphi & D_\varphi^H W D_\varphi \end{bmatrix} \quad (85) \]

Using the matrix inversion lemma [74], we have

\[ \text{CRB}(\theta, \varphi) = \frac{1}{L_t} \begin{bmatrix} D_\theta^H W D_\theta & D_\theta^H W D_\varphi 
 D_\theta^H W D_\varphi & D_\theta^H W D_\theta \end{bmatrix} \]

\[ = \frac{1}{L_t} \begin{bmatrix} [V_\theta V_\varphi]^H W_\varphi W_\varphi \end{bmatrix} \quad (86) \]

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Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
\[
= \frac{1}{L_r} \left[ M_{11} \ M_{12} \right]^{-1}
\]
where
\[
M_{11} = V_{\theta}^H W^{1/2} \Pi_{W^{1/2}D_{\theta}}^+ W^{1/2} V_{\theta} \tag{87a}
\]
\[
M_{12} = V_{\theta}^H W^{1/2} \Pi_{W^{1/2}D_{\theta}}^+ W^{1/2} V_{\phi} \tag{87b}
\]
\[
M_{21} = V_{\phi}^H W^{1/2} \Pi_{W^{1/2}D_{\phi}}^+ W^{1/2} V_{\theta} \tag{87c}
\]
and
\[
M_{22} = V_{\phi}^H W^{1/2} \Pi_{W^{1/2}D_{\phi}}^+ W^{1/2} V_{\phi} \tag{87d}
\]
The CRBs of \( \theta \) and \( \phi \) are, respectively,
\[
\text{CRB}(\theta) = \frac{1}{L_r} (M_{11} - M_{12} M_{22}^{-1} M_{21})^{-1}
\]
\[
= \frac{1}{L_r} \left[ V_{\theta}^H W^{1/2} \Pi_{W^{1/2}D_{\theta}}^+ \left( \Pi_{\Pi_{\Pi_{W^{1/2}D_{\theta}}^+ W^{1/2} V_{\phi}}} \right) \times \Pi_{\Pi_{W^{1/2}D_{\phi}}^+ W^{1/2} V_{\theta}} \right]^{-1} \tag{88}
\]
and
\[
\text{CRB}(\phi) = \frac{1}{L_r} (M_{22} - M_{21} M_{11}^{-1} M_{12})^{-1}
\]
\[
= \frac{1}{L_r} \left[ V_{\phi}^H W^{1/2} \Pi_{W^{1/2}D_{\phi}}^+ \left( \Pi_{\Pi_{\Pi_{W^{1/2}D_{\phi}}^+ W^{1/2} V_{\phi}}} \right) \times \Pi_{\Pi_{W^{1/2}D_{\phi}}^+ W^{1/2} V_{\phi}} \right]^{-1} \tag{89}
\]
Similarly
\[
\text{CRB}(r, \nu) = \frac{1}{L_r} \left[ D_r^H W_D R - D_r^H W_D L (D_r^H W_D L)^{-1} D_r^H W_D L \right]^{-1}
\]
\[
= \frac{1}{L_r} \left[ (V_r, V_r)^H W^{1/2} \Pi_{W^{1/2}D_{r}}^+ W^{1/2} [V_r, V_r] \right]^{-1}
\]
\[
= \frac{1}{L_r} \left[ N_{11} N_{12} \right]^{-1} \tag{90}
\]
where
\[
N_{11} = V_r^H W^{1/2} \Pi_{W^{1/2}D_{r}}^+ W^{1/2} V_{r} \tag{91a}
\]
\[
N_{12} = V_r^H W^{1/2} \Pi_{W^{1/2}D_{r}}^+ W^{1/2} V_{\nu} \tag{91b}
\]
\[
N_{21} = V_{\nu}^H W^{1/2} \Pi_{W^{1/2}D_{\nu}}^+ W^{1/2} V_{r} \tag{91c}
\]
and
\[
N_{22} = V_{\nu}^H W^{1/2} \Pi_{W^{1/2}D_{\nu}}^+ W^{1/2} V_{\nu} \tag{91d}
\]
Then, the CRBs of \( r \) and \( \nu \) are, respectively,
\[
\text{CRB}(r) = \frac{1}{L_r} \left( N_{11} - N_{12} N_{22}^{-1} N_{21} \right)^{-1}
\]
\[
= \frac{1}{L_r} \left[ V_r^H W^{1/2} \Pi_{W^{1/2}D_{r}}^+ \left( \Pi_{\Pi_{\Pi_{W^{1/2}D_{r}}^+ W^{1/2} V_{\nu}}} \right) \times \Pi_{\Pi_{W^{1/2}D_{\nu}}^+ W^{1/2} V_{r}} \right]^{-1} \tag{92}
\]
and
\[
\text{CRB}(\nu) = \frac{1}{L_r} \left( N_{22} - N_{21} N_{11}^{-1} N_{12} \right)^{-1}
\]
The joint CRBs exist if and only if the FIM \( J(\gamma) \) is nonsingular. From (45), the (positive definite) matrix \( R(\gamma) > 0 \), leading to \( R^{-1}(\gamma) > 0 \). Then, we have \( W = R^{-1}(\gamma) \otimes R^{-1}(\gamma) > 0 \), implying that \( W \) is nonsingular.

For the necessary condition, i.e., if the CRBs exist, namely the \( J(\gamma) \) is nonsingular, we have
\[
\text{rank}([D_{L} D_{R}]) = \text{rank}([D_{L} D_{R}] R^{1/2} W [D_{L} D_{R}])
\]
\[
= \text{rank}(J(\gamma)/L_r) = 4Q. \tag{94}
\]
For the sufficient condition, i.e., if \( \text{rank}([D_{L} D_{R}]) = 4Q \), then
\[
\text{rank}(J(\gamma)/L_r) = \text{rank}([D_{L} D_{R}] R^{1/2} W [D_{L} D_{R}])
\]
\[
= \text{rank}([D_{L} D_{R}])
\]
\[
= 4Q \tag{95}
\]
which implies that \( J(\gamma) \) is nonsingular. This concludes the proof.

APPENDIX C

RECOVERY GUARANTEES FOR OTHER L-SHAPED ARRAYS

COROLLARY 7 (UUC, UCU, UCC, CUU, CCU, AND CUC FDAS) Consider an L-shaped FDA with \( P_s = N_s + 2M_p - 1 \) sensors along each axis. Each sensor transmits a total of \( K = N_t + 2M_r - 1 \) pulses in a CPI. The fundamental spatial spacing and fundamental PRI are \( d \) and \( T \), respectively. If for UUC: \( C1–C3 \) and \( C5 \) and \( C6 \) hold; for UCU: \( C1–C3 \) and \( C6 \) and \( C7 \) hold; for CCU: \( C1–C3 \) and \( C5 \) and \( C6 \) hold; for CUC: \( C1–C4 \) and \( C6 \) and \( C7 \) hold; for CCU: \( C1–C4 \) and \( C6 \) and \( C7 \) hold; then the unknown parameter set \( \gamma_q = [\theta_q, \nu_q, r_q, v_q]_{q=1}^Q \) of \( Q \) far-field targets are perfectly recovered from \( R_{K1Z} \) with the lower bounds on the number of physical sensor elements and the number of transmit pulses as, respectively,

for UUC: \( P_s > Q \) and \( K > 2\sqrt{Q+T} - 2 \);
for UCU: \( P_s > Q \) and \( K > Q \);
for CCU: \( P_s > Q \) and \( K > 2\sqrt{Q+T} - 2 \);
for CUC: \( P_s > Q \) and \( K > Q \);
for CCU: \( P_s > Q \) and \( K > 2\sqrt{Q+T} - 2 \);

PROOF For UUC, replace the virtual manifold matrices \( \hat{X}_s \), \( \hat{X}_s \), and \( \hat{C} \) with the corresponding physical manifold matrices \( X_s \), \( X_s \), and \( C \), respectively. For UCU, replace the virtual manifold matrices \( \hat{X}_s \), \( \hat{X}_s \), and \( \hat{C} \) with the corresponding physical manifold matrices \( X_s \), \( X_s \), and \( B \), respectively. For UCC, replace the virtual manifold matrices \( \hat{A}_s \) and \( \hat{A}_s \) with the corresponding physical manifold matrices \( A_s \) and \( A_s \), respectively.
For CUU, replace the virtual manifold matrices \( \tilde{\mathbf{C}} \) and \( \tilde{\mathbf{B}} \) with the corresponding physical manifold matrices \( \mathbf{C} \) and \( \mathbf{B} \), respectively. For CCU, replace the virtual manifold matrix \( \tilde{\mathbf{B}} \) with the corresponding physical manifold matrix \( \mathbf{B} \). For CUC, replace the virtual manifold matrix \( \tilde{\mathbf{C}} \) with the corresponding physical manifold matrix \( \mathbf{C} \).

Then, ceteris paribus, the result follows from repeating the steps of the proof in Theorem 3.

**Corollary 8** (U-U, U-C, C-U, and C-C) Consider an L-shaped array with \( P_s = N_s + 2M_s - 1 \) sensors along each axis. Each sensor transmits a total of \( K = N_s + 2M_s - 1 \) pulses in a CPI. The fundamental spatial spacing and fundamental PRI are \( d \) and \( T \), respectively.

If for U-U: \( \mathbf{C}_1, \mathbf{C}_2 \), and \( \mathbf{C}_6, \mathbf{C}_7 \) hold; for U-C: \( \mathbf{C}_1, \mathbf{C}_2 \), and \( \mathbf{C}_5, \mathbf{C}_6 \) hold; for C-U: \( \mathbf{C}_1, \mathbf{C}_2 \), and \( \mathbf{C}_4, \mathbf{C}_7 \) hold; and for C-C: \( \mathbf{C}_1, \mathbf{C}_2 \), and \( \mathbf{C}_4, \mathbf{C}_5 \) hold; then the unknown parameter set \( \mathbf{y}_q = \{ \theta_q, \phi_q, v_q \}^Q_{q=1} \) of \( Q \) far-field targets are perfectly recovered from \( \mathbf{R}_{XZ} \) with the lower bounds on the number of physical sensor elements and the number of transmit pulses as, respectively.

- \( P_s > Q \) and \( K > Q \);
- \( P_s > Q \) and \( K > 2\sqrt{Q+T} - 2 \);
- \( P_s > 2\sqrt{Q+T} - 2 \) and \( K > Q \);
- \( P_s > 2\sqrt{Q+T} - 2 \) and \( K > 2\sqrt{Q+T} - 2 \).

**Proof** Since there are no FOs in the L-shaped arrays U-U, U-C, C-U, and C-C, the matrix \( \mathbf{B} \) does not exist in these cases. Hence, ceteris paribus, the result follows from repeating the steps of the proofs of Theorem 3, Corollary 4, and Corollary 7, without considering any matrix \( \mathbf{B} \) in the computations. The results of U-U, U-C, C-U, and C-C then follow from that of U-Cube, UUC, CUC, and C-Cube, respectively.

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