SUFFICIENT OPTIMALITY CONDITIONS AND MOND-WEIR DUALITY RESULTS FOR A FRACTIONAL MULTIOBJECTIVE OPTIMIZATION PROBLEM

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Abstract. In this work, we are concerned with a fractional multiobjective optimization problem \((P)\) involving set-valued maps. Based on necessary optimality conditions given by Gadhi et al. [14], using support functions, we derive sufficient optimality conditions for \((P)\), and we establish various duality results by associating the given problem with its Mond-Weir dual problem \((D)\). The main tools we exploit are convexificators and generalized convexities. Examples that illustrate our findings are also given.

1. Introduction. A set-valued optimization problem is a minimization problem where the objective map and/or the constraint maps are set-valued maps acting between abstract spaces. Set-valued optimization provides an important generalization and unification of scalar as well as vector optimization problems. Therefore, this discipline has attracted a great deal of attention in recent years. For instance economics, optimal control, differential inclusions, image processing and more are set-valued optimization problems. Before describing the approach adopted in the paper and giving the main results, let us first recall some researches related to set-valued optimization. We point out that a set-valued optimization problem was explicitly appeared in [3], where the notion of derivative defined in terms of tangent cones was employed by Corley. El Abdouni and Thibault [10] were the first to give optimality conditions in terms of Lagrange multipliers. In [1], Corely gave optimality conditions for convex and nonconvex multiobjective problems in terms of Clarke derivative. In [11], Gadhi established optimality conditions for a D.C. set-valued optimization problem in terms of the weak and strong subdifferentials of set valued maps introduced by Sawaragi and Tanino [21]. In [13], using the extremal principle developed by Mordukhovich [18], Gadhi and Jawhar proposed necessary optimality conditions for an extremal fractional optimization problem.
In 1994, Demyanov [8] introduced the concept of convexificators to generalize the notion of upper convex and lower concave approximations. This important tool was used to extend, unify and sharpen various results in nonsmooth analysis and optimization [4, 9, 15, 16]. In [15], Jeyakumar and Luc gave a revised version of convexificator by introducing the notion of a convexificator which is a closed set but not necessarily bounded or convex. In [4] the concepts of upper semiregular convexificators and lower semiregular convexificators were used to obtain necessary optimality conditions for an inequality constrained mathematical programming problem. In [17], upper convexificators have been used to detect necessary optimality conditions for an optimistic bilevel optimization problem with convex lower-level problem. Recently in [12], Gadhi et al. used the concept of convexificator to establish sufficient optimality conditions for a bilevel multiobjective optimization problem.

Let \( n, p, q, m \in \mathbb{N} \). Let \( \phi_j : \mathbb{R}^p \to \mathbb{R} \) and \( \phi_t : \mathbb{R}^p \to \mathbb{R} \), \( j \in J = \{1, ..., q\} \), \( t \in T = \{1, ..., m\} \), be given lower semicontinuous functions and let \( F_i : \mathbb{R}^p \Rightarrow \mathbb{R} \) and \( G_i : \mathbb{R}^p \Rightarrow \mathbb{R} \), \( i \in I = \{1, ..., n\} \), be given locally Lipschitz set-valued mappings such that

\[
F_i (C) = \bigcup_{x \in C} F_i (x) \quad \text{and} \quad G_i (C) = \bigcup_{x \in C} G_i (x)
\]

and \( y_i \geq 0, z_i > 0 \) for all \( i \) and all \( y_i \in F_i (x) \), \( z_i \in G_i (x) \), \( x \in C \), where

\[
C = \{ x \in \mathbb{R}^p : \phi_j (x) \leq 0, \phi_t (x) = 0, \quad j \in J, \ t \in T \}.
\]

In this paper we deal with the following fractional multiobjective optimization problem:

\[
(P) : \begin{align*}
\min_x & \quad H (x) \\
\text{subject to} & \quad x \in C
\end{align*}
\]

where

\[
h = (h_1, \ldots, h_n) \in H (x) \iff \forall i \in I, \exists y_i \in F_i (x), \ z_i \in G_i (x) \quad \text{such that} \quad h_i = \frac{y_i}{z_i}.
\]

Let \( \overline{y}_i \in F_i (\overline{x}) \), \( \overline{z}_i \in G_i (\overline{x}) \), \( \overline{h}_i = \frac{\overline{y}_i}{\overline{z}_i} \), \( i \in I \), and \( \overline{h} = (\overline{h}_1, \ldots, \overline{h}_n) \in H (\overline{x}) \). The pair \((\overline{x}, \overline{h})\) is said to be a weak Pareto minimal point with respect to \( \mathbb{R}_+^n \) of the problem \((P)\) if

\[
h - \overline{h} \notin - \text{int} \mathbb{R}_+^n, \quad \forall x \in C, \ \forall h \in H (x).
\]

Here, \( \text{int} \mathbb{R}_+^n \) denotes the interior of the nonnegative orthant \( \mathbb{R}_+^n \) of the \( n \)-dimensional space \( \mathbb{R}^n \).

In general, problem \((P)\) is nonconvex and the Kuhn-Tucker optimality conditions (see Theorem 3.2) established by Gadhi et al. [14] are only necessary. Under what assumptions, are the Kuhn-Tucker conditions also sufficient for the optimality of problem \((P)\)?

Based on necessary optimality conditions given by Gadhi et al. [14], using convexificators together with a generalized convexity, our approach consists of formulating the Mond Weir dual problem \((D)\) and establishing duality theorems for \((P)\) and \((D)\). Sufficient optimality conditions are also given.

The rest of the paper is organised in this way. In the second section we reviewed all the necessary ingredients that have shaped the direction of solving our problem. Section 3 and Section 4 are devoted to sufficient optimality conditions and duality Theorems.
2. Preliminaries. This section contains notions, definitions, and preliminaries that will be used throughout the paper.

Let $A$ be a nonempty subset of $\mathbb{R}^p$. The closure of $A$, the convex hull of $A$ and the convex cone generated by $A$ are denoted by $\text{cl } A$, $\text{co } A$ and $\text{cone } A$ respectively. The negative polar cone $A^-$ is defined by

$$A^- = \{v \in \mathbb{R}^p : \langle v, a \rangle \leq 0, \ \forall a \in A \}.$$  

Let $F : \mathbb{R}^p \rightrightarrows \mathbb{R}$ be a set-valued mapping from $\mathbb{R}^p$ into $\mathbb{R}$. The domain and the graph of $F$ are defined by

$$\text{dom}(F) = \{x \in \mathbb{R}^p : F(x) \neq \emptyset \},$$

and

$$\text{gr}(F) = \{(x, y) \in \mathbb{R}^p \times \mathbb{R} : y \in F(x)\},$$

respectively. The contingent cone $T(A, x)$ to $A$ at $x \in \text{cl } A$ is given by

$$T(A, x) = \{v \in \mathbb{R}^p : \exists t_n \downarrow 0 \text{ and } \exists v_n \rightharpoonup v \text{ such that } x + t_n v_n \in A \}.$$  

The set valued mapping $F$ is said to be locally Lipschitz at $x$ if there exists a neighborhood $U$ of $x$ such that for some constant $\alpha > 0$ and for all $x_1, x_2 \in U$ we have

$$F(x_1) \subset F(x_2) + \alpha \|x_1 - x_2\| \mathbb{B}$$

Here, $\mathbb{B}$ indicate the unit ball of $\mathbb{R}$.

We give now some definitions related to convexificators from Jeyakumar and Luc [15]. We begin by defining the upper Dini directional derivative as follows.

**Definition 2.1.** [15] Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a given function and let $x \in \mathbb{R}^p$ where $f(x)$ is finite. The lower and upper Dini directional derivatives of $f$ at $x$ in the direction of $v$ are defined, respectively, by

$$f^-_d(x, v) = \liminf_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}$$

and

$$f^+_d(x, v) = \limsup_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}.$$  

The upper Dini directional derivative may be finite as well as infinite. In particular if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz, then the upper Dini directional derivative is finite.

We provide the following definition from [15]. For more details, see [15, Definition 2.1].

**Definition 2.2.** [15] Let $f : \mathbb{R}^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be a given function. A closed set $\partial^* f(x) \subset \mathbb{R}^p$ is said to be an upper convexificator (UCF) of $f$ at $x \in \mathbb{R}^p$ if for each $v \in \mathbb{R}^p$, we have

$$f^-_d(x, v) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, v \rangle.$$  

Remark 1. [14] The functions $\sigma$ and $\xi$ are locally Lipschitz in $x$, and $k$ is a Lipschitz-constant for $\sigma$ and $\xi$ at $x$ if $k \in \mathbb{R}$ is a Lipschitz-constant for $F$ at $x$. See also [6, 7].
The following definition has been proposed by Dutta and Chandra. See [4, Definition 2.4] for more details.

**Definition 2.3.** [4] Let \( f : \mathbb{R}^p \rightarrow \mathbb{R} \cup \{+\infty\} \) be a given function. A closed set \( \partial^* f(x) \subset \mathbb{R}^p \) is said to be an upper semi-regular convexificator (USRCF) of \( f \) at \( x \in \mathbb{R}^p \) if for each \( v \in \mathbb{R}^p \), we have

\[
f^+_f(x, v) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, v \rangle.
\]

(1)

If equality holds in (1) then \( \partial^* f(x) \) is called an upper regular convexificator.

**Remark 2.** When \( f \) is a locally Lipschitz function, the Clarke [2], Michel-Penot [20] and Mordukhovich [19] subdifferentials are upper semi-regular convexificators of \( f \).

**Example 2.4.** Consider the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \), defined by

\[
f(x, y) = 2|x| - \frac{1}{3}|y|.
\]

Then, it can easily be verified that

\[
\partial^* f(0, 0) = \left\{ \left( 2, -\frac{1}{3} \right), \left( -2, \frac{1}{3} \right) \right\}
\]

is an upper semi-regular convexificator of \( f \) at \( (0, 0) \); whereas the Mordukhovich subdifferential of \( f \) at \( (0, 0) \) and the Clarke subdifferential of \( f \) at \( (0, 0) \) are respectively the sets

\[
\partial^M f(0, 0) = \left\{ \left( t, \frac{1}{3} \right) \in \mathbb{R}^2 : -2 \leq t \leq 2 \right\} \cup \left\{ \left( t, -\frac{1}{3} \right) \in \mathbb{R}^2 : -2 \leq t \leq 2 \right\}
\]

and

\[
\partial^C f(0, 0) = \operatorname{co} \left\{ \left( 2, -\frac{1}{3} \right), \left( -2, \frac{1}{3} \right), \left( 2, \frac{1}{3} \right), \left( -2, -\frac{1}{3} \right) \right\}.
\]

Observe that the upper semi-regular convexificator \( \partial^* f(0, 0) \) is strictly included in the Mordukhovich subdifferential \( \partial^M f(0, 0) \). More than that, the convex hull of \( \partial^* f(0, 0) \) is a proper subset of \( \partial^C f(0, 0) \).

3. **Sufficient optimality conditions.**

**Definition 3.1.** [14] Suppose that \( \varphi_j, j \in J, \) and \( \phi_t, t \in T, \) admit upper semi-regular convexificators \( \partial^* \varphi_j, j \in J, \) and \( \partial^* \phi_t, t \in T, \) at \( \tau \in C. \) We say that the nonsmooth Guignard constraint qualification (NGCQ) holds at \( \tau \) with respect to \( \partial^* \varphi_j (\tau), j \in J, \) and \( \partial^* \phi_t (\tau), t \in T, \) if

\[
[T(C, \tau)]^\perp \subseteq \overline{\operatorname{cone}} \theta(\tau),
\]

where

\[
\theta(\tau) = \left( \bigcup_{j \in J_0(\tau)} \partial^* \varphi_j (\tau) \right) \cup \left( \bigcup_{t \in T} \partial^* \phi_t (\tau) \right) \cup \left( \bigcup_{t \in T} \partial^* (-\phi_t) (\tau) \right)
\]

such that

\[
J_0(\tau) = \{ j \in J : \varphi_j (\tau) = 0 \}.
\]

The following theorem has been proved by Gadhi, Hamdaoui, El idrissi and Rahou [14]. It gives necessary optimality conditions for \( (P) \).
Theorem 3.2. [14] Let \( \overline{x} \in C \). Suppose that \( \varphi_j, \ j \in J, \phi_i, \ t \in T, \sigma_i, \ i \in I, \) and \( (-\xi_i), \ i \in I, \) admit bounded upper semi-regular convexificators \( \partial^* \varphi_j(\overline{x}), \ j \in J, \partial^* \phi_i(\overline{x}), \ t \in T, \partial^* \sigma_i(\overline{x}), \ i \in I, \) and \( \partial^* (-\xi_i)(\overline{x}), \ i \in I, \) Let \( (\overline{x},\overline{h}) \) be a weak local Pareto minimal point with respect to \( \mathbb{R}^n_+ \) of the problem \( P \) such that the nonsmooth Guignard constraint qualification (NGCQ) holds at \( \overline{x} \). Then, there exist scalars \( \alpha_i \geq 0, \ i \in I, \) such that \( i \in I \),

\[
0 \in \left[ \sum_{i \in I} \alpha_i \left[ \partial^* \sigma_i(\overline{x}) + \overline{h}_i \partial^* (-\xi_i)(\overline{x}) \right] \right] + cl \left( \sum_{j \in J_0(\overline{x})} \text{cone} \partial^* \varphi_j(\overline{x}) + \sum_{i \in I} \text{cone} \partial^* \phi_i(\overline{x}) + \sum_{i \in I} \text{cone} \partial^* (-\phi_i)(\overline{x}) \right),
\]

where

\[
\overline{h}_i \overline{x}_i = \overline{h}_i \xi_i(\overline{x}) \text{ and } \overline{y}_i = \sigma_i(\overline{x}),
\]

(3)

In order to give sufficient optimality conditions, we need additional assumptions on the data. The following notions have been introduced by Dutta and Chandra [5].

Definition 3.3. [5] Let \( f : \mathbb{R}^n \to \mathbb{R} \) and \( a \in \mathbb{R}^n \). We assume that \( f \) admits an upper semi-regular convexificator \( \partial^* f(a) \).

- \( f \) is said to be \( \partial^* \)-convex at \( a \) iff for all \( x \in \mathbb{R}^n \):
  \[ \langle \xi, x-a \rangle \leq f(x) - f(a), \text{ for all } \xi \in \partial^* f(a). \]

- \( f \) is said to be \( \partial^* \)-quasiconvex at \( a \) iff for all \( x \in \mathbb{R}^n \):
  \[ f(x) - f(a) \leq 0 \Rightarrow \langle \xi, x-a \rangle \leq 0, \text{ for all } \xi \in \partial^* f(a). \]

- \( f \) is said to be \( \partial^* \)-pseudoconvex at \( a \) iff for all \( x \in \mathbb{R}^n \):
  \[ f(x) - f(a) < 0 \Rightarrow \langle \xi, x-a \rangle < 0, \text{ for all } \xi \in \partial^* f(a). \]

- \( f \) is said to be \( \partial^* \)-quasilinear at \( a \) iff \( f \) and \( (-f) \) are both \( \partial^* \)-quasiconvex at \( a \).

Theorem 3.4. Let \( \overline{x} \) be a feasible point of \( P \). Assume that \( \varphi_j, \ j \in J_0(\overline{x}), \phi_i, \ t \in T, \sigma_i, \ i \in I, \) and \( (-\xi_i), \ i \in I, \) admit upper semi-regular convexificators \( \partial^* \varphi_j(\overline{x}), \ j \in J_0(\overline{x}), \partial^* \phi_i(\overline{x}), \ t \in T, \partial^* \sigma_i(\overline{x}), \ i \in I, \) and \( \partial^* (-\xi_i)(\overline{x}), \ i \in I, \) at \( \overline{x} \). Suppose that \( \sigma_i, \ i \in I, \) and \( (-\xi_i), \ i \in I, \) are \( \partial^* \)-convex at \( \overline{x} \), that \( \varphi_j, \ j \in J_0(\overline{x}), \) are \( \partial^* \)-quasiconvex at \( \overline{x} \), that \( \phi_i, \ t \in T, \) are \( \partial^* \)-quasilinear at \( \overline{x} \) and that both (2) and (3) are satisfied. Then \( (\overline{x},\overline{h}) \) is a weak Pareto minimal point with respect to \( \mathbb{R}^n_+ \) of the problem \( P \).

Proof. From (2), we can find \( \partial^*_t \in co \left( \partial^* \sigma_t(\overline{x}) + \overline{h}_t \partial^* (-\xi_i)(\overline{x}) \right), \ i \in I, \gamma_j^{(n)} \in \partial^* \varphi_j(\overline{x}), \gamma_j^{(n)} \geq 0, \ j \in J_0(\overline{x}), \mu_t^{(n)} \in \partial^* \phi_t(\overline{x}), \tau_t^{(n)} \in \partial^* (-\phi_t)(\overline{x}) \mu_t^{(n)} \geq 0 \) and \( \tau_t^{(n)} \geq 0, \ t \in T, \) such that

\[
\sum_{i \in I} \alpha_i \partial^*_t + \lim_{n \to +\infty} \left[ \sum_{j \in J_0(\overline{x})} \gamma_j^{(n)} \gamma_j^{* (n)} + \sum_{i \in I} \mu_t^{(n)} \mu_t^{* (n)} + \sum_{t \in T} \tau_t^{(n)} \tau_t^{* (n)} \right] = 0.
\]
Consequently,

\[- \sum_{i \in I} \alpha_i \theta^*_t = \lim_{n \to +\infty} \left[ \sum_{j \in J_0(x)} \gamma_j^{(n)} \gamma_j^{* (n)} + \sum_{t \in T} \mu_t^{(n)} \mu_t^{* (n)} + \sum_{t \in T} \tau_t^{(n)} \tau_t^{* (n)} \right]. \tag{4}\]

First of all, we have

\[\varphi_j (x) \leq 0 \text{ and } \phi_t (x) = 0, \forall t \in T, \forall j \in J, \forall x \in C.\]

- Since \( \bar{x} \in C \), we have

\[\varphi_j (x) \leq \varphi_j (\bar{x}) \text{ and } \phi_t (x) = \phi_t (\bar{x}), \forall t \in T, \forall j \in J_0(\bar{x}), \forall x \in C.\]

Thus,

\[\varphi_j (x) - \varphi_j (\bar{x}) \leq 0 \text{ and } \phi_t (x) - \phi_t (\bar{x}) = 0, \forall t \in T, \forall j \in J_0(\bar{x}), \forall x \in C.\]

- From the \( \partial^* \)-quasiconvexity of \( \varphi_j \), \( j \in J_0(\bar{x}) \), and \( \partial^* \)-quasilinearity of \( \phi_t \), \( t \in T \), we have

\[\langle \gamma_j^{* (n)}, x - \bar{x} \rangle \leq 0, \langle \mu_t^{* (n)}, x - \bar{x} \rangle \leq 0 \text{ and } \langle \tau_t^{* (n)}, x - \bar{x} \rangle \leq 0, \forall x \in C.\]

Then,

\[\left\langle \sum_{j \in J_0(\bar{x})} \gamma_j^{(n)} \gamma_j^{* (n)} + \sum_{t \in T} \mu_t^{(n)} \mu_t^{* (n)} + \sum_{t \in T} \tau_t^{(n)} \tau_t^{* (n)}, x - \bar{x} \right\rangle \leq 0, \forall x \in C.\]

Thus,

\[\lim_{n \to +\infty} \left\langle \sum_{j \in J_0(\bar{x})} \gamma_j^{(n)} \gamma_j^{* (n)} + \sum_{t \in T} \mu_t^{(n)} \mu_t^{* (n)} + \sum_{t \in T} \tau_t^{(n)} \tau_t^{* (n)}, x - \bar{x} \right\rangle \leq 0, \forall x \in C.\]

- By (4), we deduce

\[\left\langle \sum_{i \in I} \alpha_i \theta^*_t, x - \bar{x} \right\rangle \geq 0, \forall x \in C.\]

This implies that we can find \( i \in I \) such that \( \alpha_i \neq 0 \) and

\[\langle \alpha_i \theta^*_t, x - \bar{x} \rangle \geq 0, \forall x \in C.\] \tag{5}

- Since \( \sigma_i \) and \( (-\xi_i) \) are \( \partial^* \)-convex at \( \bar{x} \), we have

\[\sigma_i (x) \geq \sigma_i (\bar{x}) + \langle a_i^*, x - \bar{x} \rangle, \forall a_i^* \in \partial^* \sigma_i (\bar{x}), \forall x \in C.\]

and

\[(-\xi_i) (x) \geq (-\xi_i) (\bar{x}) + \langle b_i^*, x - \bar{x} \rangle, \forall b_i^* \in \partial^* (-\xi_i) (\bar{x}), \forall x \in C.\]

Then, for all \( x \in C \), \( a_i^* \in \partial^* \sigma_i (\bar{x}) \) and \( b_i^* \in \partial^* (-\xi_i) (\bar{x}) \), we have

\[(\sigma_i + \bar{\xi}_i (-\xi_i)) (x) \geq (\sigma_i + \bar{\xi}_i (-\xi_i)) (\bar{x}) + \langle a_i^* + \bar{b}_i^*, x - \bar{x} \rangle.\]

It follows that, for all \( x \in C \) and \( A_i^* \in co [\partial^* \sigma_i (\bar{x}) + \bar{\xi}_i \partial^* (-\xi_i) (\bar{x})] \), we have

\[(\sigma_i + \bar{\xi}_i (-\xi_i)) (x) \geq (\sigma_i + \bar{\xi}_i (-\xi_i)) (\bar{x}) + \langle A_i^*, x - \bar{x} \rangle.\]

as \( A_i^* \) can be expressed as a convex combination of finite elements of \( \partial^* \sigma_i (\bar{x}) + \bar{\xi}_i \partial^* (-\xi_i) (\bar{x}) \).
Example 3.5. Consider the following multiobjective optimization problem

$$\text{(P^*) : } \left\{ \begin{array}{l}
\min \ H(x) = (H_1(x), H_2(x)) \\
\text{Subject to : } x \in C
\end{array} \right.$$ 

where

$$C = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0 \text{ and } \exp(x_2) - 1 = 0 \}$$

and

$$F_1(x_1, x_2) = \{ y \in \mathbb{R} : |x_1| + |x_2| + \frac{1}{2} \leq y \leq (|x_1| + |x_2|)^2 + x_1^2 + 2 \},$$

$$F_2(x_1, x_2) = \{ y \in \mathbb{R} : x_1 + |x_2| \leq y \leq 2(|x_1| + |x_2|)^2 + 3 \},$$

$$G_1(x_1, x_2) = \{ z \in \mathbb{R} : x_1 + 2 \leq z \leq x_1 + x_2 + 3 \},$$

$$G_2(x_1, x_2) = \{ z \in \mathbb{R} : \frac{1}{3}x_2 + 1 \leq z \leq x_2 + 2 \}.$$ 

The set-valued mappings $H_1$ and $H_2$ are given by

$$H_1(x_1, x_2) = \left\{ h_1 \in \mathbb{R} : \frac{|x_1| + |x_2| + \frac{1}{2}}{x_1 + x_2 + 3} \leq h_1 \leq \frac{(|x_1| + |x_2|)^2 + x_1^2 + 2}{x_1 + 2} \right\},$$

and

$$H_2(x_1, x_2) = \left\{ h_2 \in \mathbb{R} : \frac{x_1 + |x_2|}{x_2 + 2} \leq h_2 \leq \frac{2(|x_1| + |x_2|)^2 + 3}{\frac{1}{3}x_2 + 1} \right\}.$$ 

Consequently,

$$\sigma_1(x_1, x_2) = |x_1| + |x_2| + \frac{1}{2}, \quad \sigma_2(x_1, x_2) = x_1 + |x_2|,$$

$$\xi_1(x_1, x_2) = x_1 + x_2 + 3, \quad \xi_2(x_1, x_2) = x_2 + 2,$$

$$\varphi_1(x_1, x_2) = -x_1, \quad \varphi_2(x_1, x_2) = -x_2 \quad \text{and} \quad \phi(x_1, x_2) = \exp(x_2) - 1.$$
• On the one hand, \( (0,0) \) is a feasible point of \((P^*)\). Notice that

\[
H_1(0,0) = \left\{ h_1 \in \mathbb{R} : \frac{1}{6} \leq h_1 \leq 1 \right\} = \left[ \frac{1}{6}, 1 \right]
\]

and

\[
H_2(0,0) = \{ h_2 \in \mathbb{R} : 0 \leq h_2 \leq 3 \} = [0, 3].
\]

Consequently,

\[
\bar{y}_1 = \frac{1}{2}, \quad z_1 = 3, \quad \bar{h}_1 = \frac{1}{6}, \quad y_2 = 0, \quad z_2 = 2 \text{ and } \bar{h}_2 = 0.
\]

The functions \( \sigma_1, \sigma_2, (-\xi_1), (-\xi_2), \varphi_1, \varphi_2, \phi \) and \((-\phi)\) admit the following sets as bounded upper semi regular convexificators at \((0,0)\).

\[
\partial^* \sigma_1(0,0) = \left\{ \left( \begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \\ 1 \end{array} \right) \right\},
\]
\[
\partial^* \sigma_2(0,0) = \left\{ \left( \begin{array}{c} 1 \\ 1 \\ -1 \\ -1 \end{array} \right) \right\},
\]
\[
\partial^* (-\xi_1)(0,0) = \left\{ \left( \begin{array}{c} -1 \\ -1 \end{array} \right) \right\},
\]
\[
\partial^* (-\xi_2)(0,0) = \partial^* \varphi_2(0,0) = \partial^* (-\phi)(0,0) = \left\{ \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \right\},
\]
\[
\partial^* \varphi_1(0,0) = \left\{ \left( \begin{array}{c} -1 \\ 0 \end{array} \right) \right\} \text{ and } \partial^* \phi(0,0) = \left\{ \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right\}.
\]

- The function \( \sigma_1 \) is \( \partial^* \)-convex at \((0,0)\). Indeed

\[
\sigma_1(x_1, x_2) - \sigma_1(0,0) = |x_1| + |x_2| \geq \langle a^*, (x_1, x_2) - (0,0) \rangle, \quad \forall a^* \in \partial^* \sigma_1(0,0).
\]

- The function \( \sigma_2 \) is \( \partial^* \)-convex at \((0,0)\). Indeed

\[
\sigma_2(x_1, x_2) - \sigma_2(0,0) = x_1 + |x_2| \geq \langle b^*, (x_1, x_2) - (0,0) \rangle, \quad \forall b^* \in \partial^* \sigma_2(0,0).
\]

- The function \((-\xi_1)\) is \( \partial^* \)-convex at \((0,0)\). Indeed

\[
(-\xi_1)(x_1, x_2) - (-\xi_1)(0,0) = -x_1 - x_2 \geq \langle -1, -1, (x_1, x_2) - (0,0) \rangle = -x_1 - x_2.
\]

- The function \((-\xi_2)\) is \( \partial^* \)-convex at \((0,0)\). Indeed

\[
(-\xi_2)(x_1, x_2) - (-\xi_2)(0,0) = -x_2 \geq \langle (0, -1), (x_1, x_2) - (0,0) \rangle = -x_2.
\]

- The function \( \varphi_1 \) is \( \partial^* \)-quasiconvex at \((0,0)\). Indeed

\[
\varphi_1(x_1, x_2) - \varphi_1(0,0) = -x_1 \leq 0,
\]

implies that

\[
\langle (-1, 0), (x_1, x_2) - (0,0) \rangle = -x_1 \leq 0,
\]

where \( \partial^* \varphi_1(0,0) = \{ (-1,0) \}. \)

- The function \( \varphi_2 \) is \( \partial^* \)-quasiconvex at \((0,0)\). Indeed

\[
\varphi_2(x_1, x_2) - \varphi_2(0,0) = -x_2 \leq 0,
\]

implies that

\[
\langle (0, -1), (x_1, x_2) - (0,0) \rangle = -x_2 \leq 0,
\]

where \( \partial^* \varphi_2(0,0) = \{ (0, -1) \}. \)

- The function \( \phi \) is then \( \partial^* \)-quasilinear at \((0,0)\).
* The function $\phi$ is $\partial^*$-quasiconvex at $(0,0)$. Indeed,

$$\phi(x_1,x_2) - \phi(0,0) = \exp(x_2) - 1 \leq 0,$$

implies that

$$\langle (0,1), (x_1,x_2) - (0,0) \rangle = x_2 \leq 0,$$

where $\partial^* \phi(0,0) = \{(0,1)\}$.

* The function $(-\phi)$ is $\partial^*$-quasiconvex at $(0,0)$. Indeed,

$$(-\phi)(x_1,x_2) - (-\phi)(0,0) = 1 - \exp(x_2) \leq 0,$$

implies that

$$\langle (0,-1), (x_1,x_2) - (0,0) \rangle = -x_2 \leq 0,$$

where $\partial^* (-\phi)(0,0) = \{(0,-1)\}$.

In the other hand, we have

$$T(C,(0,0)) = \mathbb{R}_+ \times \{0\}, \quad [T(C,(0,0))]^- = \mathbb{R}_- \times \mathbb{R}$$

and

$$\theta(0,0) = \{(-1,0),(0,-1),(0,1)\}.$$

Thus,

$$\text{cone} \ \theta(0,0) = \mathbb{R}_- \times \mathbb{R}$$

then,

$$[T(C,(0,0))]^- = \text{cone} \ \theta(0,0).$$

Hence, the Guignard constraint qualification holds at $\bar{x} = (0,0)$.

Finally, as

$$\left(\begin{array}{c}
\frac{5}{6} \\
-\frac{5}{6}
\end{array}\right) \in \text{co} \ \left(\partial^* \sigma_1(0,0) + \frac{1}{6} \times \partial^*(-\xi_1)(0,0)\right),$$

and

$$\left(\begin{array}{c}
1 \\
1
\end{array}\right) \in \text{co} \ \left(\partial^* \sigma_2(0,0) + 0 \times \partial^*(-\xi_2)(0,0)\right),$$

one has

$$0 \in \left[\text{cl} \left(\begin{array}{cc}
\frac{1}{3} & \frac{5}{18} \\
-\frac{1}{3} & -\frac{5}{6}
\end{array}\right) + \frac{2}{3} \left(\begin{array}{c}
1 \\
1
\end{array}\right) + \frac{1}{3} \left(\begin{array}{c}
0 \\
0
\end{array}\right) + \frac{7}{9} \left(\begin{array}{c}
0 \\
-1
\end{array}\right)\right].$$

One deduces that (2) is satisfied for $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$. Then, by Theorem 3.4, we conclude that $(\bar{x},\bar{h}) = ((0,0),\left(\frac{1}{6},0\right))$ is a weak Pareto minimal point of $(P^*)$ with respect to $\mathbb{R}_+^2$. 
4. Mond-Weir type dual. In this section, we introduce a Mond-Weir type dual problem for the set valued fractional programming problem characterized by convexificators and investigate weak and strong duality results.

Let \( u \in \mathbb{R}^p \). Suppose that \( \varphi_j, \phi_i, \sigma_i \) and \( (-\xi_i) \) admit bounded upper semi-regular convexificator \( \partial^* \varphi_j (u), \partial^* \phi_i (u), \partial^* \sigma_i (u) \) and \( \partial^* (-\xi_i) (u) \) et \( i \in I, j \in J, t \in T \).

The Mond-Weir dual problem associated to \((P)\) is as follows:

\[
(D) : \quad \max_{(u, \alpha)} h
\]
subject to

\[
0 \in \left[ +cl \left( \sum_{j \in J_0(\pi)} \text{cone } \partial^* \varphi_j (u) + \sum_{i \in I} \text{cone } \partial^* \phi_i (u) + \sum_{i \in T} \text{cone } \partial^* (-\phi_i) (u) \right) - \sum_{i \in I} \alpha_i \sigma_i (u) + h_i \partial^* (-\xi_i) (u) \right]
\]

\[
\sigma_i (u) + h_i (-\xi_i) (u) \geq 0, \quad \forall i \in I,
\]

\[
\varphi_j (u) \geq 0 \quad \text{and} \quad \phi_i (u) = 0, \quad \forall j \in J_0(\pi), \forall t \in T,
\]

\[
h = (h_1, ..., h_n) \in H(u),
\]

and

\[
\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_+, \quad \sum_{i \in I} \alpha_i = 1.
\]

A point \((u, \alpha)\) satisfying all the constraints of \((D)\) is called a feasible point of \((D)\).

Let \( K \) be the set of all feasible points of \((D)\).

**Definition 4.1.** Let \((\pi, \beta) \in K\) and let \( \overline{h} \in H(\pi) \). The point \((\pi, \beta, \overline{h})\) is said to be a weak Pareto maximal point of the problem \((D)\) if for any feasible point \((u, \alpha) \in K\), we have

\[
\overline{h} - h \notin - \text{int} \mathbb{R}^n_+, \quad \forall h \in H(u).
\]

**Theorem 4.2.** Let \( x \) be a feasible point of \((P)\) and let \((u, \alpha) \) be a feasible point of \((D)\). Suppose that \( \sigma_i \) and \( (-\xi_i), i \in I, \) are \( \partial^* \)-convex at \( u \), that \( \varphi_j, j \in J_0(\pi), \) is \( \partial^* \)-quasiconvex at \( u \) and that \( \phi_j, t \in T, \) is \( \partial^* \)-quasilinear at \( u \). Then, there exists \( s \in I \) such that

\[
h'_s \geq h_s, \quad \forall h_s \in H_s(u), \forall h'_s \in H_s(x).
\]

**Proof.** We proceed by contradiction. Suppose that there exists a feasible point \( x \) of \((P)\) and a feasible point \((u, \alpha) \) of \((D)\) such that for all \( i \in I \), we can find \( h_i \in H_i(u) \) and \( h'_i \in H_i(x) \) such that

\[
h_i - h'_i > 0.
\]

- Since \( h'_i \in H_i(x) \), there exist \( y_i \in F_i(x) \) and \( z_i \in G_i(x) \), such that

\[
h'_i = \frac{y_i}{z_i},
\]

- Combining (9) and (10), we get

\[
y_i - h_i z_i < 0.
\]

Consequently,

\[
\sigma_i (x) + h_i (-\xi_i) (x) < 0.
\]
Proof. Let \( \bar{x}_i \) be weak Pareto maximal point of \( \mathcal{P} \). Then, there exists \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n \setminus \{0\} \) such that \( (\bar{x}_i, \alpha) \) is a feasible point of \( \mathcal{D} \). Moreover, if \( \sigma_i \) and \( (-\xi_i) \) are \( \partial^* \)-convex at \( \bar{x}_i \), \( i \in I \), \( \varphi_j \) is \( \partial^* \)-quasiconvex at \( \bar{x}_i \), \( j \in J_0(\bar{x}_i) \), \( \mu_t \) and \( \tau_t \) have the nonsmooth Guignard constraint qualification \( \mathcal{NGCQ} \), and \( \varphi \) is \( \partial^* \)-quasilinear at \( \bar{x}_i \), \( t \in T \), then \( (\bar{x}_i, \alpha) \) is a weak Pareto maximal point of \( \mathcal{D} \).

Proof. Let \( \bar{x}_i \) be weak Pareto minimal point of \( \mathcal{P} \) such that \( (\bar{x}_i) \) holds at \( \bar{x}_i \) and \( \mathcal{NGCQ} \) holds at \( \bar{x}_i \).}

\begin{align*}
- & \text{Since } \sigma_i(u) + h_i(-\xi_i)(u) \geq 0 \\
\text{we obtain } & (\sigma_i + h_i(-\xi_i))(x) - (\sigma_i + h_i(-\xi_i))(u) < 0. \tag{11}
\end{align*}

Using the \( \partial^* \)-convexity of \( \sigma_i \) and \( (-\xi_i) \) at \( u \), we obtain

\begin{align*}
(\sigma_i + h_i(-\xi_i))(x) - (\sigma_i + h_i(-\xi_i))(u) \geq \langle \vartheta^*_i, x - u \rangle. \tag{13}
\end{align*}

Since \( \alpha \in \mathbb{R}_+^n \setminus \{0\} \), combining (11) with (13), we get

\begin{align*}
\left< \sum_{i \in I} \alpha_i \vartheta^*_i, x - u \right> < 0. \tag{14}
\end{align*}

- Since \( \varphi_j(x) \leq 0 \leq \varphi_j(u) \), \( \varphi_t(x) = 0 \leq \varphi_t(u) = 0 \) and \( (-\phi_t)(x) = 0 \leq (-\phi_t)(u) = 0 \), using the \( \partial^* \)-quasiconvexity of \( \varphi_j \), \( j \in J_0(\bar{x}_i) \), and the \( \partial^* \)-quasilinearity of \( \varphi_t \), \( t \in T \), we have

\begin{align*}
\sum_{j \in J_0(\bar{x}_i)} \gamma_j^{(n)} \gamma_j^{(n)} + \sum_{t \in T} \mu_t \mu_t^{(n)} + \sum_{t \in T} \tau_t \tau_t^{(n)} \leq 0.
\end{align*}

Consequently,

\begin{align*}
\lim_{n \to +\infty} \left[ \sum_{j \in J_0(\bar{x}_i)} \gamma_j^{(n)} \gamma_j^{(n)} + \sum_{t \in T} \mu_t \mu_t^{(n)} + \sum_{t \in T} \tau_t \tau_t^{(n)} \right] \leq 0. \tag{15}
\end{align*}

Combining (12) with (15), we obtain

\begin{align*}
\left< \sum_{i \in I} \alpha_i \vartheta^*_i, x - u \right> \geq 0
\end{align*}

which contradicts (14).
By Theorem 3.2, one can find \( \alpha \in \mathbb{R}^n_+ \backslash \{0\} \) such that (2) and (3) hold. Consequently \((\overline{x}, \alpha)\) is a feasible point of \((D)\).

We claim that \((\overline{x}, \alpha, h)\) is a weak Pareto maximal point of \((D)\). On contrary, suppose that there exists \((u, \omega) \in K \) and \( h \in H(u) \) such that
\[
\overline{h} - h \in -\text{int} \left( \mathbb{R}^n_+ \right).
\] (16)

Since \((u, \omega)\) is a feasible point of \((D)\) and \(\overline{x}\) is a feasible point of \((P)\), according to Theorem 4.2 one gets
\[
\overline{h} - h \notin -\text{int} \left( \mathbb{R}^n_+ \right),
\]
which contradicts (16). Hence \((\overline{x}, \alpha, h)\) is a weak Pareto maximal point of \((D)\).

The following example explains how to employ Theorem 4.2 and 4.3.

**Example 4.4.** Let us consider the problem \((P^*\)) treated in Example 3.5. The Mond-Weir dual problem associated to \((P^*)\) is as follows:

\[
(D^*) : \quad \mathbb{R}^2_+ - \max_{(u, \alpha)} h \quad \text{subject to:}
0 \in \left\{ \begin{array}{l}
\alpha_1 \cos \theta \sigma_1(u) + h_1 \partial^*(\xi_1)(u) + \alpha_2 \cos \theta \sigma_2(u) + h_2 \partial^*(\xi_2)(u) \\
+ cl \left( \text{cone} \partial^* \varphi_1(u) + \text{cone} \partial^* \varphi_2(u) + \text{cone} \partial^* \phi(u) + \text{cone} \partial^* (-\phi)(u) \right) - \left\{ \begin{array}{l}
\sigma_1(u) + h_1 (\xi_1)(u) \geq 0, \quad \sigma_2(u) + h_2 (\xi_2)(u) \geq 0, \\
\varphi_1(u) = -u_1 \geq 0, \quad \varphi_2(u) = -u_2 \geq 0, \quad \phi(u) = \exp(u_2) - 1 = 0, \\
h = (h_1, h_2) \in H(u), \quad u = (u_1, u_2), \quad \alpha = (\alpha_1, \alpha_2), \quad \text{with } \alpha_1 + \alpha_2 = 1.
\end{array} \right. \right. 
\]

As already mentioned, \(H_1\) and \(H_2\) are given by
\[
H_1(u_1, u_2) = \left\{ h_1 \in \mathbb{R} : \frac{|u_1| + |u_2| + \frac{1}{2}}{u_1 + u_2 + 3} \leq h_1 \leq \frac{(|u_1| + |u_2|)^2 + u_1^2 + 2}{u_1 + 2} \right\},
\]
and
\[
H_2(u_1, u_2) = \left\{ h_2 \in \mathbb{R} : \frac{u_1 + |u_2|}{u_2 + 2} \leq h_2 \leq \frac{2(|u_1| + |u_2|)^2 + 3}{3u_2 + 1} \right\}.
\]

Remark that
\[
H_1(-1, 0) = \left\{ h_1 \in \mathbb{R} : \frac{3}{4} \leq h_1 \leq 4 \right\} = \left[ \frac{3}{4}, 4 \right]
\]
and
\[
H_2(-1, 0) = \left\{ h_2 \in \mathbb{R} : -\frac{1}{2} \leq h_2 \leq 5 \right\} = \left[ -\frac{1}{2}, 5 \right].
\]

- **On the one hand,** \((-1, 0), (\frac{1}{4}, \frac{3}{4})\) is a feasible point of \((D^*)\), where
\(\varphi_1(-1, 0) = 1 \geq 0, \varphi_2(-1, 0) = 0 \geq 0\) and \(\phi(-1, 0) = 0\).
Since
\[
\partial^* \sigma_1 (-1, 0) = \left\{ \left( -1, -1 \right), \left( 1, -1 \right), \left( -1, 1 \right), \left( 1, 1 \right) \right\},
\]
\[
\partial^* \sigma_2 (-1, 0) = \left\{ \left( 1, 1 \right), \left( 1, -1 \right) \right\},
\]
\[
\partial^* \phi (-1, 0) = \left\{ \left( 0, 1 \right) \right\},
\]
\[
\partial^* (-\xi_1) (-1, 0) = \left\{ \left( -1, -1 \right) \right\},
\]
\[
\partial^* (-\xi_2) (-1, 0) = \partial^* \varphi_2 (-1, 0) = \partial^* (-\phi) (-1, 0) = \left\{ \left( 0, 1 \right) \right\}.
\]
are bounded upper semi regular convexificators of \( \sigma_1, \sigma_2, (-\xi_1), (-\xi_2), \varphi_1, \varphi_2, \) and \( \phi \) at \((-1, 0), \) we have
\[
\begin{pmatrix}
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
\frac{9}{16} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\
-1 & 0 & 0 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
-\frac{3}{4} \\
\frac{1}{2}
\end{pmatrix}
\leq \begin{pmatrix}
\partial^* \sigma_1 (-1, 0) + \frac{3}{4} \times \partial^* (-\xi_1) (-1, 0)
\end{pmatrix}
\]
\[
\begin{pmatrix}
\frac{1}{2}
\end{pmatrix}
\leq \begin{pmatrix}
\partial^* \sigma_2 (-1, 0) - \frac{1}{2} \times \partial^* (-\xi_2) (-1, 0)
\end{pmatrix}.
\]

- On the other hand, \( \sigma_1, \sigma_2, (-\xi_1) \) and \((-\xi_2)\) are \( \partial^* \) convex at \((-1, 0), \) \( \varphi_1 \) and \( \varphi_2 \) are \( \partial^* \) quasiconvex at \((-1, 0)\) and \( \phi \) is quasilinear at \((-1, 0).\)
- For any feasible point \((x_1, x_2) \in C\) and any feasible point \((u_1, u_2), \alpha) \in K, \) we have \((x_1, x_2) \in \mathbb{R}_+ \times \{0\}\) and \((u_1, u_2) \in \mathbb{R}_- \times \{0\}\). Consequently,
\[
h_2 - h_2' \leq \frac{u_1}{2} - \frac{x_1}{2} \leq 0,
\]
for all \( h_2 \in H_2(u_1, u_2) \) and \( h_2' \in H_2(x_1, x_2). \) One concludes that Theorem 4.2 holds for \((P^*)\) and \((D^*).\)
- Since \((\overline{\pi}, \overline{h}) = ((0, 0), (\frac{1}{2}, 0))\) is a weak Pareto minimal point of \((P^*)\) such that \((NGCQ)\) holds at \((0, 0)\) (see Example 3.5) and since \((\overline{\pi}, \alpha) = ((0, 0), (\frac{1}{2}, \frac{3}{2}))\) is a feasible point of \((D^*),\) using Theorem 4.3, we deduce that \((\overline{\pi}, \alpha, \overline{h})\) is a weak Pareto maximal point of \((D^*).\)

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