The Numerical Approach to the Fisher’s Equation via Trigonometric Cubic B-spline Collocation Method

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April 26, 2016

Abstract

In this study, we set up a numerical technique to get approximate solutions of Fisher’s equation which is one of the most important model equation in population biology. We integrate the equation fully by using combination of the trigonometric cubic B-spline functions for space variable and Crank-Nicolson for the time integration. Numerical results have been presented to show the accuracy of the current algorithm. We have seen that the proposed technique is a good alternative to some existing techniques for getting solutions of the Fisher’s equation.

Keywords: Finite element method, Collocation method, Trigonometric cubic B-spline, Crank-Nicolson method, Fisher’s equation.

1 Introduction

One of the most interesting equation in physical phenomena is reaction-diffusion equation. We focus on one of the special case of the reaction-diffusion given by

\[ U_t = \lambda U_{xx} + \beta U(1 - U), \quad -\infty < x < \infty \]  

where \( \beta \) is a real parameter. This equation is known as a Fisher’s equation (FE) which was introduced by Fisher [1] to describe the kinetic advancing rate of an advantageous gene. In a large number of biological and chemical phenomena, the reaction term is represented by \( \beta U(1 - U) \), where \( \beta > 0 \) can be dependent of the space variable. FE represents the evolution of the population due to the two competing physical processes and changes of interaction of diffusion and nonlinear reaction can be observed.

The trigonometric cubic B-spline (TCB) functions have been started to adapt the numerical methods for obtaining the solutions of the differential equations although use of
these are not widespread as the approximate functions in the numerical methods. The numerical methods for solving a type of ordinary differential equations with trigonometric quadratic and cubic splines are given by A. Nikolis in the papers. The linear two-point boundary value problems of order two are solved using TCB interpolation method[3]. The another numerical method employed the TCB is set up to solve a class of linear two-point singular boundary value problems in the study[6]. Very recently a collocation finite difference scheme based on TCB is developed for the numerical solution of a one-dimensional hyperbolic equation (wave equation) with non-local conservation condition[7]. A new two-time level implicit technique based on the TCB is proposed for the approximate solution of the nonclassical diffusion problem with nonlocal boundary condition in the study[8].

The numerical approaches for types of the Fisher equation have been given to model some events, which can not be exhibited by theoretical means. Although various numerical techniques are employed for getting numerical solutions of the Fisher equation, we only document the spline-based numerical methods. Orthogonal cubic spline collocation method is constructed for the extended Fisher-Kolmogorov equation[13]. The Galerkin method constructed using quartic B-splines and the Crank-Nicolson method is employed for the numerical solution of FE[14]. A numerical solution of the extended Fisher-Kolmogorov equation by using the quintic B-spline collocation scheme is proposed[15]. FE is fully discretized by the finite element method based on the quadratic B-spline Galerkin technique in space and by use of the Crank-Nicolson method for the integration of the obtained matrix ordinary differential equations in space[15]. Both the cubic B-spline collocation method and the usual finite difference technique is applied for getting the solutions of the well-known Fisher’s equation which was discretized in space and time respectively[17]. The Cubic B-spline quasi-interpolation is presented, by using the derivative of the quasi-interpolation to approximate the spatial derivative of the dependent variable and a low order forward difference to approximate the time derivative of the dependent variable for finding solutions of the Burgers–Fisher equation[18]. Combination of the quintic B-spline based collocation method and crank-Nicolson technique is designed to discretize the Fisher’s equation fully[20]. Crank-Nicolson method is used for time discretization. A collocation method based on modified cubic B-splines for spatial variable and derivatives is applied to the Fisher’s equation to produce a system of first-order ordinary differential equations, Which was solved by SSP-RK54 scheme[19]. The spline approximation has been used in spatial direction together with the finite difference approximation in time have been set up to gave numerical solution of the Fisher equation[21]. The exponential cubic B-spline collocation method is constructed to obtain numerical solutions of the Fisher equation[23].

The purpose of this studying is to obtain the numerical solutions of the Fisher’s equation via the trigonometric cubic B-spline collocation algorithm and research the solvability of the reaction-diffusion type equations by the B-spline finite element methods. Crank-Nicolson formulas and trigonometric cubic B-spline functions are used for time and space discretization respectively. Over the uniform mesh, Crank-Nicolson formulas are employed for time discretization whereas Rubin and Graves[9] technique is used for the lineariza-
Consider a uniform partition of the problem domain \([a = x_0, b = x_N]\) at the knots \(x_i, i = 0, \ldots, N\) with mesh spacing \(h = (b - a)/N\). On this partition together with additional knots \(x_{-1}, x_0, x_{N+1}, x_{N+2}, x_{N+3}\) outside the problem domain, \(CTB_i(x)\) can be defined as

\[
CTB_i(x) = \frac{1}{\theta} \begin{cases} 
\omega^3(x_{i-2}), & x \in [x_{i-2}, x_{i-1}] \\
\omega(x_{i-2})\phi(x_i) + \phi(x_{i+1})\omega(x_{i-1}) + \phi(x_{i+2})\omega^2(x_{i-1}), & x \in [x_{i-1}, x_i] \\
\omega(x_{i-2})\phi^2(x_{i+1}) + \phi(x_{i+2})\omega(x_{i-1})\phi(x_{i+1}) + \phi(x_{i+2})\omega(x_i), & x \in [x_i, x_{i+1}] \\
\phi^3(x_{i+2}), & x \in [x_{i+1}, x_{i+2}] \\
0, & \text{otherwise}
\end{cases}
\]

where \(\omega(x_i) = \sin\left(\frac{x-x_i}{2}\right), \phi(x_i) = \sin\left(\frac{x-x_i}{2}\right), \theta = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)\).

\(CTB_i(x)\) are twice continuously differentiable piecewise trigonometric cubic B-spline on the interval \([a, b]\). The iterative formula

\[
T_i^k(x) = \frac{\sin\left(\frac{x-x_{i-1}}{2}\right)}{\sin\left(\frac{x_{i+k-1}-x_{i-1}}{2}\right)}T_{i-1}^{k-1}(x) + \frac{\sin\left(\frac{x_{i+k-1}-x}{2}\right)}{\sin\left(\frac{x_{i+k-1}-x_{i+1}}{2}\right)}T_{i+1}^{k-1}(x), \quad k = 2, 3, 4, \ldots
\]

(6)

gives the cubic B-spline trigonometric functions starting with the CTB-splines of order 1

\[
T_i^1(x) = \begin{cases} 
1, & x \in [x_i, x_{i+1}] \\
0, & \text{otherwise}
\end{cases}
\]

Each \(CTB_i(x)\) is twice continuously differentiable and the values of \(CTB_i(x), CTB'_i(x)\) and \(CTB''_i(x)\) at the knots \(x_i\)'s can be computed from Eq.(6) as
Table 1: Values of \( B_i(x) \) and its principle two derivatives at the knot points

| \( i \) | \( T_i(x_k) \) | \( T'_i(x_k) \) | \( T''_i(x_k) \) |
|---|---|---|---|
| -1 | 0 | 0 | 0 |
| 0 | \( \sin^2(\frac{h}{2}) \csc(h) \csc(\frac{3h}{2}) \) | \( \frac{3}{4} \csc(\frac{3h}{2}) \) | \( \frac{3(1+3\cos(h))\csc^2(\frac{3h}{2})}{16(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} \) |
| 1 | \( \frac{2}{1+2\cos(h)} \) | 0 | \( \frac{-3\cot^2(\frac{h}{2})}{2+4\cos(h)} \) |
| 2 | \( \sin^2(\frac{h}{2}) \csc(h) \csc(\frac{3h}{2}) \) | \( -\frac{3}{4} \csc(\frac{3h}{2}) \) | \( \frac{3(1+3\cos(h))\csc^2(\frac{3h}{2})}{16(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} \) |
| 3 | 0 | 0 | 0 |

\( CTB_i(x) \), \( i = -1, \ldots, N+1 \) are a basis for the trigonometric spline space. An approximate solution \( U_N \) to the unknown \( U \) is written in terms of the expansion of the CTB as

\[
U_N(x, t) = \sum_{i=-1}^{N+1} \delta_i CTB_i(x) \tag{7}
\]

where \( \delta_i \) are time dependent parameters to be determined from the collocation points \( x_i, i = 0, \ldots, N \) and the boundary and initial conditions. The nodal values \( U \) and its first and second derivatives at the knots can be found from the (7) as

\[
U_i = \alpha_1 \delta_{i-1} + \alpha_2 \delta_i + \alpha_1 \delta_{i+1}
\]
\[
U'_i = \beta_1 \delta_{i-1} + \beta_2 \delta_{i+1}
\]
\[
U''_i = \gamma_1 \delta_{i-1} + \gamma_2 \delta_{i+1} + \gamma_1 \delta_{i+1}
\tag{8}
\]

\[
\alpha_1 = \sin^2(\frac{h}{2}) \csc(h) \csc(\frac{3h}{2}) \quad \alpha_2 = \frac{2}{1+2\cos(h)}
\]
\[
\beta_1 = -\frac{3}{4} \csc(\frac{3h}{2}) \quad \beta_2 = \frac{3}{4} \csc(\frac{3h}{2})
\]
\[
\gamma_1 = \frac{3((1+3\cos(h))\csc^2(\frac{3h}{2}))}{16(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} \quad \gamma_2 = -\frac{3\cot^2(\frac{h}{2})}{2+4\cos(h)}
\]

The Crank–Nicolson scheme is used to discretize time variables of the unknown \( U \) in the Fisher’s equation so that one obtain the time discretized form of the equation as

\[
\frac{U^{n+1} - U^n}{\Delta t} = \lambda \left( \frac{(U_{xx})^{n+1} + (U_{xx})^n}{2} \right) + \beta \frac{U^{n+1} + U^n}{2} - \beta (UU)^{n+1} + (UU)^n \tag{9}
\]

where \( U^{n+1} = U(x, t) \) is the solution of the equation at the \((n+1)\)th time level. Here \( t^{n+1} = t^n + \Delta t \), and \( \Delta t \) is the time step, superscripts denote \( n \)th time level, \( t^n = n\Delta t \). The nonlinear term \((U^2)^{n+1}\) in Eq. (9) may be linearized by using the following term [9]:

\[
(U^2)^{n+1} = 2U^n U^{n+1} - (U^n)^2 \tag{10}
\]

we get

\[
\frac{2}{\Delta t} U^{n+1} - \lambda U^{n+1}_{xx} - \beta U^{n+1} + 2\beta K U^{n+1} = \frac{2}{\Delta t} U^n + \lambda U^n_{xx} + \beta U^n \tag{11}
\]
Substitution of (7) into (11) leads to the fully-discretized equation:

\[
\chi_1 \delta_{m-1}^{n+1} + \chi_2 \delta_m^{n+1} + \chi_3 \delta_{m+1}^{n+1} = \chi_3 \delta_m^{n} + \chi_4 \delta_m^{n} + \chi_3 \delta_{m+1}^{n}
\]

where

\[
\chi_1 = \left( \frac{2}{\Delta t} - \beta + 2\beta K \right) \alpha_1 - \lambda \gamma_1
\]

\[
\chi_2 = \left( \frac{2}{\Delta t} - \beta + 2\beta K \right) \alpha_2 - \lambda \gamma_2
\]

\[
\chi_3 = \left( \frac{2}{\Delta t} + \beta \right) \alpha_1 + \lambda \gamma_1
\]

\[
\chi_4 = \left( \frac{2}{\Delta t} + \beta \right) \alpha_2 + \lambda \gamma_2
\]

\[K = \alpha_1 \delta_{i-1} + \delta_i + \alpha_1 \delta_{i+1}\]

\[\alpha_1 = \sin^2 \left( \frac{h}{2} \right) \csc(h) \csc \left( \frac{3h}{2} \right) \quad \alpha_2 = \frac{2}{1 + 2 \cos(h)}
\]

\[\gamma_1 = \frac{3((1 + 3 \cos(h)) \csc^2 \left( \frac{h}{2} \right))}{16(2 \cos \left( \frac{h}{2} \right) + \cos \left( \frac{3h}{2} \right))} \quad \gamma_2 = \frac{-3 \cot^2 \left( \frac{h}{2} \right)}{2 + 4 \cos(h)}
\]

The system consist of \(N+1\) linear equation in \(N+3\) unknown parameters \(d^{n+1} = (\delta_{-1}^{n+1}, \delta_0^{n+1}, \ldots, \delta_{N+1}^{n+1})\). To make solvable the system, boundary conditions \(\sigma_1 = U_0, \sigma_2 = U_n\) are used to find two additional linear equations:

\[\delta_{-1} = \frac{1}{\alpha_1}(U_0 - \alpha_2 \delta_0 - \alpha_3 \delta_1), \quad \delta_{N+1} = \frac{1}{\alpha_3}(U_n - \alpha_1 \delta_{N-1} - \alpha_2 \delta_N).\]

(13) can be used to eliminate \(\delta_{-1}, \delta_{N+1}\) from the system (12) which then becomes the solvable matrix equation for the unknown \(\delta_{-1}^{n+1}, \ldots, \delta_{N}^{n+1}\). A tridiagonal system of equation can be solved with tribanded Thomas algorithm.

Initial parameters \(\delta_{-1}^0, \delta_0^0, \ldots, \delta_{N+1}^0\) can be determined from the initial condition and first space derivative of the initial conditions at the boundaries as the following:

1. \(U_N(x_i, 0) = U(x_i, 0), i = 0, \ldots, N\)
2. \((U_x)_N(x_0, 0) = U'(x_0)(U_x)_N(x_N, 0) = U'(x_N)\).

3 Numerical tests

Numerical method described in the previous section is tested on three problems for getting solution of the Fisher’s equation and on one problem for getting the solution of the Fisher equation in order to demonstrate the robustness and numerical accuracy.
The discrete error norms $L_\infty$ 

$$L_\infty = |U - U_N|_\infty = \max_j |U_j - (U_N)^n_j|$$

is used to measure error between the analytical and numerical solutions. The relative error 

$$\text{Relative error} = \sqrt{\frac{\sum_{j=1}^{N} |(U_N)^{n+1}_j - (U_N)^n_j|^2}{\sum_{j=1}^{N} |(U_N)^n_j|^2}}$$

is used when the analytical solutions does not exist.

(a) Analytical solution of the Fisher equation 

$$u(x, t) = \left(1 + \exp\left(\sqrt{\frac{\beta}{6}}x - \frac{5\beta}{6}t\right)\right)^{-2}$$

is used in the numerical studies [10, 11] together with the nonlocal conditions. Over the interval $[a, b] = [-0.2, 0.8]$, the calculation is done with the number of knots $N = 40$ with time step $\Delta t = 0.0001$ for different values of $\beta = 2000, 5000$ and 10000 to compare results with other works [17]. The computed results are represented graphically for some times seen in Figs 1-3.
Fig. 1: solutions for $\beta = 2000, N = 40$

Fig. 2: solutions for $\beta = 5000, N = 40$

Fig. 3: solutions for $\beta = 10000, N = 40$

For $N = 64$, $\Delta t = 0.000005$, $\beta = 10000$, on the interval $[-0.2, 1.06]$ results are documented in Table 2 to compare with Ref [15], in terms of $L_\infty$ norm, at different time steps.

| Method | $t = 0.0005$ | $t = 0.0015$ | $t = 0.0025$ | $t = 0.0035$ |
|--------|--------------|--------------|--------------|--------------|
| Present | $1.02 \times 10^{-2}$ | $1.49 \times 10^{-1}$ | $3.24 \times 10^{-1}$ | $4.78 \times 10^{-1}$ |
| [15]   | $2.55 \times 10^{-3}$ | $1.62 \times 10^{-2}$ | $8.65 \times 10^{-2}$ | $6.98 \times 10^{-2}$ |

(b) Secondly, the initial pulse profile

$$u(x, 0) = \text{sech}^2(10x)$$

is chosen as the initial condition for our first numerical experiment. Under the boundary conditions [1] we obtained the solutions. In numerical calculations, the constants in Eq. (1) are selected $\lambda = 0.1$ and $\beta = 1$. For the discretization of space and time, space/time increments $h = 0.005$ and $\Delta t = 0.05$ are used. Then, the algorithm is run up to time $t = 40$ over the domain $[-50, 50]$. In short period Figure 4 is drawn for $[a, b] = [-2, 2]$ at
times \( t = 0.1, 0.2, 0.3, 0.4 \) and 0.5 in which we see that the diffusion is more dominant than the reaction. Figure 5 is drawn for \([a, b] = [-6, 6]\) at \( t = 0, 1, 2, 3, 4, \) and 5. After the concentration reached the lowest level, it start to increase up to a level \( u = 1\), so that the reaction are observed over the the diffusion. Finally, in Figure 6, solutions are depicted at \( t = 5, 10, 15, 20, 25, 30, 35 \) and 40 in the space interval \([a, b] = [-50, 50]\). As time advance more, the concentration return to initial form, then flatten through both sides with sharp lateral fronts. Thus solutions looks like bell-shape with flat top. We see that the diffusion is totally efficient in advance time.

Fig. 4: Solutions at early times. with at \( t = 0.1, 0.2, 0.3, 0.4, 0.5\).

Fig. 5: Short-time behavior. with at \( t = 0, 1, 2, 3, 4, 5\).

Fig. 6: Long-time behavior.
with at \( t = 0, 5, 10, 15, 20, 25, 30, 35, .40\).

In the Table 3 errors at some different times for the second test problem is shown.

**Table 3:** Relative errors at some different times for the second test problem b for \( N = 64 \)

Parameters: \( \lambda = 0.1, \beta = 1, \Delta t = 0.05 \) and \( x \in [-50, 50] \)

| Method | \( t = 5 \) | \( t = 10 \) | \( t = 15 \) | \( t = 20 \) | \( t = 40 \) |
|--------|-------------|-------------|-------------|-------------|-------------|
| Present| \( 1.383 \times 10^{-2} \) | \( 7.834 \times 10^{-3} \) | \( 6.029 \times 10^{-3} \) | \( 5.066 \times 10^{-3} \) | \( 3.416 \times 10^{-3} \) |
| [15]   | \( 1.386 \times 10^{-2} \) | \( 7.860 \times 10^{-3} \) | \( 6.054 \times 10^{-3} \) | \( 5.090 \times 10^{-3} \) | \( 3.434 \times 10^{-3} \) |
4 Conclusion

The collocation methods with trigonometric B-spline functions is made up to find solutions the Fisher’s equation. We have shown that method is capable of producing solutions of the Fisher’s equation fairly. The method can be used as an alternative to the more usual associate B-spline collocation and Galerkin methods.

Acknowledgement: This paper have been presented at the International Conference on Natural Science and Engineering, 2016, Kilis, Turkey.

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