Stability results for projective modules over Rees algebras

Ravi A. Rao and Husney Parvez Sarwar

School of Mathematics, T.I.F.R. Mumbai, Navy Nagar, Mumbai - 400005, India;

ravi@math.tifr.res.in, mathparvez@gmail.com

Abstract

We provide a class of commutative Noetherian domains $R$ of dimension $d$ such that every finitely generated projective $R$-module $P$ of rank $d$ splits off a free summand of rank one. On this class, we also show that $P$ is cancellative. At the end we give some applications to the number of generators of a module over the Rees algebras.

Key Words: Rees algebras, projective modules, free summand, cancellation.

Mathematics Subject Classification 2010: Primary: 13C10, 19A13; Secondary: 13A30.

1 Introduction

Let $R$ be a commutative Noetherian ring of Krull dimension $d$. A classical result of Serre [20] says that every finitely generated projective $R$-module $P$ of rank $> d$ splits off a free summand. This is the best possible result in general as it is evidenced from the well-known example of “the tangent bundle over real algebraic sphere of dimension two”. Therefore the question “splitting off a free summand” becomes subtle when rank$(P) = d$. If $R$ is a reduced affine algebra over an algebraically closed field, then for a rank $d$ projective $R$-module $P$, M.P. Murthy [13] defined an obstruction class $c_d(P)$ in the group $F^dK_0(A)$. Further assuming $F^dK_0(A)$ has no $(d-1)!$ torsion, he proved that $c_d(P) = 0$ if and only if $P$ splits off a free summand of rank one.

For a commutative Noetherian ring $R$ of dimension $d$, Bhatwadekar–Raja Sridharan [3, 4] defined an obstruction group called Euler class group, denoted by $E_d(R)$. Assume $Q \subset R$, then given a projective $R$-module of rank $d$, Bhatwadekar–Raja Sridharan defined an obstruction class $e_d(P)$ and proved $e_d(P) = 0$ in $E_d(R)$ if and only if $P$ splits off a free summand of rank one. Later, for a smooth scheme $X$ of dimension $n$, Barge–Morel [1] defined the Chow–Witt group $\tilde{CH}^j(X)$ ($j \geq 0$) and associated to each vector bundle $E$ of rank $n$ with trivial determinant an Euler class $\tilde{c}_n(E)$ in $\tilde{CH}^n(X)$. Let $A$ be a smooth affine domain of dimension $n$ and $P$ a finitely generated projective $A$-module of rank $n$. Then it was proved that $\tilde{c}_n(P) = 0$ if and only if $P \cong Q \oplus A$ for $n = 2$ in [1] (see also [8]), $n = 3$ in [7] and $n \geq 4$ in [12].

A recent result of Marco Schlichting [19] proved a similar kind of result for a commutative Noetherian ring $R$ of dimension $d$ all of whose residue fields are infinite. Precisely, given a rank $d$ oriented projective $R$-module $P$, he defined a class $e(P)$ in $H^d_{\text{Zar}}(R, \mathcal{K}^M_d)$ such that $e(P) = 0$ if and only if $P$ splits off a free summand of rank one.
One of the aims of this article is to provide a class of examples of commutative Noetherian rings $R$ of dimension $d$ such that every rank $d$ projective $R$-module splits off a free summand of rank one. We prove the following:

**Theorem 1.1** Let $R$ be a commutative Noetherian domain of dimension $d - 1$ ($d \geq 1$) and $I$ an ideal of $R$. Define $A := R[It]$ or $R[It, t^{-1}]$ (note that $\dim(A) \leq d$). Let $P$ be a projective $A$-module of rank $d$. Then $P \sim Q \oplus A$ for some projective $A$-module $Q$.

In particular, if $Q \subset A$, then the obstruction class $e_d(P)$ defined by Bhatwadekar–Raja Sridharan [4] is zero in $E^d(A)$. Also, in the view of Schlichting’s result [19], if we assume that all residue fields of $A$ are infinite, then $e(P)$ defined by Schlichting is zero in $H^d_{\text{zar}}(A, K^M_d)$. For $A = R[t]$ and for birational overrings of $R[t]$, a similar type of result is proved by Plumstead [14] and Rao ([10], [17]) respectively.

In this direction, a parallel problem is “the cancellation problem”. Let $P$ be a projective module over a commutative Noetherian ring $R$ of dimension $d$ such that $\text{rank}(P) > d$. Then Bass [2] proved that $P$ is cancellative i.e. $P \oplus Q \sim P' \oplus Q \Rightarrow P \sim P'$. Again this is the best possible result in general as it is evidenced by the same well-known example “tangent bundle over the real algebraic sphere of dimension two”. However Suslin (22) proved that if $R$ is an affine algebra of dimension $d$ over an algebraically closed field, then every projective $R$-module of rank $d$ is cancellative. We enlarge the class of rings by proving the following result.

**Theorem 1.2** Let $R$ be a commutative Noetherian domain of dimension $d - 1$ ($d \geq 1$) and $I$ an ideal of $R$. Define $A := R[It]$ or $R[It, t^{-1}]$. Then every finitely generated projective $A$-module of rank $d$ is cancellative.

For $A = R[t]$ and for birational overrings of $R[t]$, a similar type of result is proved by Plumstead [14] and Rao ([16], [17]) respectively.

The following result follows from our result Theorem 1.2 and a result of Wiemers [24, Theorem].

**Corollary 1.3** Let $R$ be a commutative Noetherian domain of dimension $d - 1$ ($d \geq 1$) such that $1/d! \in R$ and $I$ an ideal of $R$. Define $A := R[It]$ or $R[It, t^{-1}]$. Then every finitely generated projective $A[X_1, \ldots, X_n]$-module of rank $d$ is cancellative.

As an application of our result, we prove the following result.

**Theorem 1.4** Let $R$ be a commutative Noetherian domain of dimension $d$ and $I$ an ideal of $R$. Let $M$ be a finitely generated module over $A := R[It]$ or $R[It, t^{-1}]$. Then $M$ is generated by $e(M) := \text{Supp} \left( \mu_p(M) + \dim(A/p) \right)$ elements.

Let $A$ be a domain of dimension $n$, $R = A[X_1, \ldots, X_m]$ and $I$ the ideal of $R$ generated by $(X_1, \ldots, X_m)$. Then $R[It] = A[X_1, \ldots, X_m, X_1t, \ldots, X_mt]$. Note that in this case $R[It]$ becomes a monoid algebra $A[M]$, where $M$ is the monoid generated by $(X_1, \ldots, X_m, X_1t, \ldots, X_mt)$. In this case
Gubeladze [10] conjectured that every projective $A[M]$-module of rank $> n$ splits off a free summand of rank one. In Theorem [1.1] we have verified it affirmatively but our rank-dimension condition is not optimal. We note that the second author and Keshari [11] studied the problem of existence of unimodular elements over monoid algebras. But their results do not cover the above monoid algebra.

**Acknowledgement:** We would like to thank the referee for carefully reading the paper and some useful comments.

## 2 Notations, Rees algebras and some properties

*Throughout the paper, we assume that all the modules are finitely generated.*

Let $A$ be a commutative ring and $Q$ a $A$-module. We say $p \in Q$ is unimodular if the order ideal $O_Q(p) = \{ \phi(p) | \phi \in Q^* = \text{Hom}(Q, A) \}$ equals $A$. Let $p \in \text{Spec}(A)$. An element $q \in Q$ is said to be a basic element of $Q$ at $p$ if $q \not\in pQp$. We say $q$ is a basic element of $Q$ if it is a basic element of $Q$ at every prime ideal of $A$. Let $\mu_p(Q)$ denote the minimum number of generators of $Q|_p$ over $A_p$. Let $P$ be a projective $A$-module. We say $P$ is cancellative if $P \oplus Q \cong P' \oplus Q \Rightarrow P \cong P'$ for all projective $A$-modules $P'$ and $Q$.

The set of all unimodular elements in $Q$ is denoted by $U_m(Q)$. We write $E_n(A)$ for the group generated by the set of all $n \times n$ elementary matrices over $A$ and $U_m(A)$ for $U_m(A^n)$. We denote by $\text{Aut}_A(Q)$, the group of all $A$-automorphisms of $Q$.

For an ideal $J$ of $A$, we denote by $E(A \oplus Q, J)$, the subgroup of $\text{Aut}_A(A \oplus Q)$ generated by the automorphisms $\Delta_{a\phi} = (\begin{smallmatrix} 1 & a \phi \\ 0 & id_Q \end{smallmatrix})$ and $\Gamma_q = (\begin{smallmatrix} 1 & 0 \\ q & id_Q \end{smallmatrix})$ with $a \in J$, $\phi \in Q^*$ and $q \in JQ$. Further, we shall write $E(A \oplus Q)$ for $E(A \oplus Q, A)$. We denote by $U_m(A \oplus Q, J)$, the set of all $(a, q) \in U_m(A \oplus Q)$ with $a \in J$ and $q \in JQ$.

**Generalized dimension:** Let $R$ be a commutative ring and $S \subset \text{Spec}(R)$. Let $\delta : S \to \mathbb{N} \cup \{0\}$ be a function. Define a partial order on $S$ as $p << q$ if $p \subset q$ and $\delta(p) > \delta(q)$. We say that $\delta$ is a generalized dimension function on $S$ if for any ideal $I$ of $R$, $V(I) \cap S$ has only a finite number of minimal elements with respect to $<<$. We say that $R$ has the generalized dimension $d$ if $d = \max_{p \in \text{Spec}(R)} \delta(p)$. The notion of the generalized dimension was introduced by Plumstead in [14].

For example, the standard dimension function $\delta(p) = \text{coheight}(p) := \text{dim}(R/p)$ is a generalized dimension function. Thus, the generalized dimension of $R$ is $\leq$ the Krull dimension of $R$. Observe that if $s \in R$ is such that $R/(s)$ and $R_s$ have the generalized dimension $\leq d$, then the generalized dimension of $R \leq d$. Indeed, if $\delta_1$ and $\delta_2$ are generalized dimension functions on $R/sR$ and $R_s$ respectively with $\delta_1 \leq d$. Then we define $\delta : \text{Spec}(R) \to \mathbb{N} \cup \{0\}$ as follows $\delta(p) = \delta_1(p)$ if $s \in p$ and $\delta(p) = \delta_2(p)$ if $s \notin p$. Now clearly $\delta$ is a generalized dimension function on $R$ with $\delta \leq d$.

We note down an example of Plumstead [14] Example 4) where he has given a ring having generalized dimension $< \text{Krull dimension}$. Take $A := R[X]$, where $R$ is a ring having an element $s \in \text{rad}(R)$ with $\text{dim}(R/sR) < \text{dim}(R)$. Then Plumstead [14] proved that the generalized dimension of $A < \text{dim}(A)$. 


The Rees algebras and the extended Rees algebras: Let $R$ be a commutative Noetherian ring of dimension $d$ and $I$ an ideal of $R$. Then the algebra

$$R[I] := \{ \sum_{i=0}^{n} a_i t^i : n \in \mathbb{N}, a_i \in I^i \} = \oplus_{n \geq 0} I^n t^n$$

is called the Rees algebra of $R$ with respect to $I$. Sometimes it is also called the blow-up algebra. These algebras arise naturally in the process of blowing-up a variety along a subvariety. In the case of an affine variety $V(I) \subset \text{Spec}(R)$, the blowing-up is the natural map from $\text{Proj}(R[I]) \to \text{Spec}(R)$. It is a fact that dimension of $R[I]$ is not contained in any minimal primes of $R$, then the dimension of $R[I] = d + 1$ (see [23, Theorem 1.3]). Further if $I$ is not contained in $\text{Spec}(R)$, then the dimension of $R[I]$ is $d + 1$ (see [23, Theorem 1.3]). For further properties of $R[I]$, we refer the reader to [23].

One defines the extended Rees algebra $R[I, t^{-1}]$ with respect to an ideal $I$ of $R$ as a subring of $R[t, t^{-1}]$ as follows

$$R[I, t^{-1}] = \{ \sum_{i=-n}^{n} a_i t^i : n \in \mathbb{N}, a_i \in I^i \} = \oplus_{n \in \mathbb{Z}} I^n t^n,$$

where $I^n = R$ for $n \leq 0$. It is easy to observe that $R[I, t^{-1}]$ is birational to $R[I]$, hence the dimension of $R[I, t^{-1}] = d + 1$.

3 Splitting off and cancellation results for Rees algebras

Lemma 3.1 Let $R$ be a commutative ring, $S \subset R$ be a multiplicative subset and $I \subset R$ an ideal. Then $S^{-1}(R[I]) = S^{-1}R[(S^{-1}I)t]$.

Proof By definition $S^{-1}(R[I]) = S^{-1}R \oplus S^{-1}(IR)t \oplus S^{-1}(IR)2t^2 \oplus \cdots$ and

$$S^{-1}R[(S^{-1}I)t] = S^{-1}R \oplus S^{-1}I(S^{-1}R)t \oplus (S^{-1}I(S^{-1}R))2t^2 \oplus \cdots.$$ Since localization commutes with direct sums, we have $S^{-1}(R[I]) = S^{-1}R[(S^{-1}I)t]$. ■

Lemma 3.2 Let $A$ be a commutative Noetherian ring of dimension $d \geq 1$ and let $s$ be a non-zero-divisor of $A$. Then the generalized dimension of $A_{1+sA}$ is $\leq d - 1$.

Proof Let $\mathcal{P}_1$ be the set of all primes of $A_{1+sA}$ which contains $s$ and $\mathcal{P}_2 := \{ p \in \text{Spec}(A_{1+sA}) : ht(p) < d \}$. We claim that

$$\text{Spec}(A_{1+sA}) = \mathcal{P}_1 \cup \mathcal{P}_2.$$ Let $p \in \text{Spec}(A_{1+sA})$ be a prime ideal of height $d$. Hence $p$ is a maximal ideal of $A_{1+sA}$. We claim that $s \in p$. Suppose not, then $p + (s) = (1)$. This implies that there exists $p \in p$ such that $ap = 1 + sb$. But $1 + sb$ is a unit in $A_{1+sA}$ which is a contradiction to the fact $1 + sb = pa \in p$. This establishes the claim. Now following [13, Example 2], we get that the generalized dimension of $A_{1+sA}$ is $\leq d - 1$. ■
Lemma 3.3 (Plumstead [14]) Let $A$ be a commutative Noetherian ring of generalized dimension $d$ and $P$ a projective $A$-module of rank $\geq d + 1$. Then

1. $P$ has a unimodular element. More generally, if $M$ is a finitely generated $A$-module such that $\mu_p(M) \geq d$ for all $p \in \text{Spec}(A)$, then $M$ has a basic element.

2. $P$ is cancellative. In fact $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.

Proof This is an observation made by Plumstead in [14, page 1421, paragraph 4] except part (2) second statement. For this, let $(a, p) \in \text{Um}(A \oplus P)$. By Eisenbud–Evans Theorem (see the version in [14, §1]), there exists an element $q \in P$ such that $p + aq$ is a unimodular element. Hence there exists $\psi \in P^*$ such that $\psi(p + aq) = 1$. Now we have $\Gamma_{-aq} \Delta \psi \Delta \Gamma_q(a, p)^t = (1, 0)^t$ ($v^t$=transpose of the vector $v$). This finishes the proof.

Lemma 3.4 Let $R$ be a commutative Noetherian ring of dimension $d$ and let $A := R[It]$ or $R[It, t^{-1}]$. Let $s$ be a non-zero-divisor of $R$ and $P$ a projective $A_{s+As}$-module of rank $\geq d + 1$. Then

1. $P$ has a unimodular element. More generally, if $M$ is a finitely generated $A$-module such that $\mu_p(M) \geq d$ for all $p \in \text{Spec}(A)$, then $M$ has a basic element.

2. $P$ is cancellative. More generally, $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.

Proof This is a direct consequence of Lemma 3.2 and Lemma 3.3.

Theorem 3.5 Let $R$ be a commutative Noetherian domain of dimension $d$ and $I$ an ideal of $R$.

1. Let $P$ be a projective module over the Rees algebra $R[It]$ such that the rank of $P$ is $> d$. Then $P$ has a unimodular element.

2. Let $Q$ be a projective module over the extended Rees algebra $R[It, t^{-1}]$ such that the rank of $Q$ is $> d$. Then $Q$ has a unimodular element.

Proof (1) Let $A = R[It]$. Since $R$ is a domain, so is $R[t]$. Since $A$ is a subring of $R[t]$, we conclude that $A$ is a domain. If $I = (0)$, then $A = R$. In this case, the result follows from a classical result of Serre [20]. If $I = (1)$, then $A = R[t]$. In this case, the result follows from [14, Corollary 4]. So, we assume that $(0) \neq I \neq (1)$. We already noted that $\dim(A) \leq d + 1$. In fact here $\dim(A) = d + 1$ since $I \neq 0$. Hence in the view of a classical result of Serre [20], we only have to consider the case when the rank of $P$ is $d + 1$.

Let $S$ be the set of all non-zero-divisors of $R$. Then by Lemma 3.1 we have $S^{-1}R[It] \cong R'[t]$, where $R'$ is a field. Therefore $S^{-1}P$ is a free $S^{-1}A$-module. Since $P$ is finitely generated, there exists $s \in S$ such that $P_s$ is a free $A_s$-module.

Let $a$ be a non-zero non-unit element of $I$. Let us consider the following commutative diagram of rings
Lemma 3.6 (Criterion for cancellation) Let $R$ be a commutative Noetherian ring of dimension $d$ and $P$ a projective $R$-module of rank $n$. Let $s,t \in R$ such that $sR + tR = R$. Let $(b,p) \in \text{Um}(R \oplus P)$. Assume that there exist $\sigma_1 \in \text{Aut}(R_s \oplus P_s)$, $\sigma_2 \in \text{Aut}(R_t \oplus P_t)$ such that $(b,p)_s \sigma_1 = (1,0)$, $((b,p)_t) \sigma_2 = (1,0)$ respectively.

1. If we define $\eta = (\sigma_1)^{-1} \sigma_2$, then $\eta = \left( \begin{smallmatrix} 1 & 0 \\ s & \theta \end{smallmatrix} \right)$, where $\theta \in \text{Aut}(P_s)$.

2. Further if $\eta = (\eta_2)(\eta_1)_s$, where $\eta_2 = \left( \begin{smallmatrix} 1 & 0 \\ s & \theta \end{smallmatrix} \right) \in \text{Aut}(R_s \oplus P_s)$ and $\eta_1 = \left( \begin{smallmatrix} 1 & 0 \\ \ast & \theta \end{smallmatrix} \right) \in \text{Aut}(R_t \oplus P_t)$, then there exists $\phi \in \text{Aut}(R \oplus P)$ such that $(b,p) \phi = (1,0)$.
Proof For (1), we observe that \((1,0)\eta = (1,0)\). Hence it is easy to see that \(\eta = (1_0 0)\), where \(\theta \in \text{Aut}(P_{st})\). Consider the following Cartesian square:

\[
\begin{array}{c}
R \xrightarrow{p_1} R_s \\
\downarrow p_2 \quad \quad \downarrow j_1 \\
R_t \xrightarrow{j_2} R_{st}.
\end{array}
\]

We set \(\sigma_s = \sigma_1\eta_2\) and \(\sigma_t = \sigma_2\eta_1^{-1}\). Then we observe that \((b,p)\sigma_s = (1,0), (b,p)\sigma_t = (1,0)\) and \((\sigma_s)_t = (\sigma_t)_s\) (recall that \(\eta = (\sigma_1)^{-1}(\sigma_2)_s = (\eta_2)_t(\eta_1)_s\)). Now by a standard patching argument, we will get an automorphism \(\phi \in \text{Aut}(R \oplus P)\) such that \((b,p)\phi = (1,0)\). Now we explain how we get the automorphism \(\phi\). The following is a commutative diagram of modules where the front square and the back square are Cartesian.

\[
\begin{array}{c}
R \oplus P \xrightarrow{\phi} R_s \oplus P_s \\
R \oplus P \quad \quad \downarrow \phi_s \\
\downarrow R_t \oplus P_t \quad \quad \downarrow \phi_{st} \\
R_s \oplus P_s \quad \quad \downarrow \phi_s \\
R_{st} \oplus P_{st} \quad \quad \downarrow \phi_{st} \\
R_t \oplus P_t \quad \quad \downarrow \phi_t \\

\end{array}
\]

We get the homomorphism \(\phi\) by the universal property of the Cartesian square. We observe that \(\phi\) is locally an isomorphism and it sends \((b,p)\) locally to \((1,0)\). Hence we conclude that \(\phi\) is an isomorphism and it sends \((b,p)\) to \((1,0)\).

Lemma 3.7 Continuing with the notations as in Lemma 3.6, we further assume that \(P_s\) is free. Then we can write \(\eta = (\eta_2)_s(\eta_1)_s\), where \(\eta_2 = (\frac{1}{\eta_2} 0) \in \text{Aut}(R_s \oplus P_s)\) and \(\eta_1 = (\frac{1}{\eta_1} 0) \in \text{Aut}(R_t \oplus P_t)\) i.e. \((1,0)\eta_i = (1,0)\) for \(i = 1,2\).

Proof By [15] Corollary 3.2, we can write \(\eta = (\eta_1)_s(\eta_2)_s\) such that \(\eta_1 \in \text{Aut}(R_s \oplus P_s)\) and \(\eta_2 \in \text{Aut}(R_t \oplus P_t)\). Since \((1,0)\eta = (1,0)\), the explicit computations of \(\eta_1, \eta_2\) in the proof of [15] Proposition 3.1 can be modified suitably to get \(\eta = (\eta_1)_s(\eta_2)_s\) such that \((1,0)\eta_1 = (1,0)\) and \((1,0)\eta_2 = (1,0)\). Hence we have \(\eta_1 = (\frac{1}{\eta_1} 0) \in \text{Aut}(R_s \oplus P_s)\) and \(\eta_2 = (\frac{1}{\eta_2} 0) \in \text{Aut}(R_t \oplus P_t)\). Alternatively, one can use Quillen splitting lemma to get the required splitting as follows. Consider the elementary group \(E(R_{st}[Z] \oplus P_{st}[Z])\), where \(Z\) is a variable. It is easy to see that we can find \(\alpha(Z) \in E(R_{st}[Z] \oplus P_{st}[Z])\) with the properties \((1,0)\alpha(Z) = (1,0), \alpha(1) = \eta, \text{and } \alpha(0) = Id\). By Quillen’s splitting lemma [15] Theorem 1, paragraph 2, for \(g = (s)^N\) with large \(N\), we have

\[
\alpha(Z) = (\alpha(Z)\alpha(gZ)^{-1})\alpha(gZ)_s.
\]
with \( \alpha(Z)\alpha(gZ)^{-1} \in \text{Aut}(R_s[Z] \oplus P_s[Z]) \), \( \alpha(gZ) \in \text{Aut}(R_t[Z] \oplus P_t[Z]) \). Note that \( (1,0)\alpha(gZ) = (1,0) \); hence \( (1,0){(\alpha(Z)\alpha(gZ)^{-1}}) = (1,0) \). Specializing \( Z = 1 \) gives the required splitting of \( \eta = (\eta_1)_{i(\eta_2)} \), with \( \eta_1 \in \text{Aut}(R_t \oplus P_t), \eta_2 \in \text{Aut}(R_s \oplus P_s) \) such that \( (1,0)\eta_i = (1,0) \) for \( i = 1, 2 \).

**Theorem 3.8** Let \( R \) be a commutative Noetherian domain of dimension \( d \) and \( I \) an ideal of \( R \).

1. Let \( P \) be a projective module over the Rees algebra \( R[I[t] \) such that \( \text{rank}(P) > d \). Let \( (b,p) \in \text{Um}(R[I[t])] \oplus P) \). Then there exists \( \sigma \in \text{Aut}(R[I[t]] \oplus P) \) such that \( (b,p)\sigma = (1,0) \).

2. Let \( P \) be a projective module over the extended Rees algebra \( R[I[t,t^{-1}] \) such that \( \text{rank}(P) > d \). Let \( (b,p) \in \text{Um}(R[I,t^{-1}] \oplus P) \). Then there exists \( \sigma \in \text{Aut}(R[I,t^{-1}] \oplus P) \) such that \( (b,p)\sigma = (1,0) \).

**Proof** Let \( A = R[I[t] \). Since \( R \) is a domain, so is \( R[t] \). Since \( A \) is a subring of \( R[t] \), we conclude that \( A \) is a domain. If \( I = (0) \), then \( A = R \). In this case, the result follows from a result of Bass [2]. If \( I = (1) \), then \( A = R[t] \). In this case, the result follows from [14 Corollary 2]. So, we assume that \( (0) \neq I \neq (1) \). Note that in this case \( \dim(A) = d + 1 \). Hence in the view of a classical result of Bass [2], we only have to consider the case when the rank of \( P \) is \( d + 1 \).

Let \( S \) be the set of all non-zero-divisors of \( R \). Then using Lemma 5.1 we have \( S^{-1}R[I[t] \cong R'[t] \) where \( R' \) is a field. Therefore \( S^{-1}P \) is a free \( S^{-1}A \)-module. Since \( P \) is finitely generated, there exists \( s \in S \) such that \( P_s \) is a free \( A_s \)-module.

Let \( (b,p) \in \text{Um}(R[I[t]] \oplus P) \) and \( a \) a non-zero non-unit element of \( I \). We have already observed the following Cartesian square of rings in the proof of Theorem 5.3.

\[
\begin{array}{ccc}
R[I[t] & \xrightarrow{p_1} & R[I[t]]_{as} \cong R_{as}[t] \\
\downarrow{p_2} & & \downarrow{p_1} \\
R[I[t]]_{1+as}R[I[t]] & \xrightarrow{j_2} & (R_{as}[t])_{1+as}R[I[t]] \\
\end{array}
\]

Let \( T := 1 + asR[I[t] \). Since \( P_{as} \) is free, using a result of Suslin [21 Theorem 2.6], there exists \( \sigma_1 \in \text{E}(R[I[t]]_{as} \oplus P_{as}) \) such that \( ((b,p)_{as})\sigma_1 = (1,0) \) for \( d \geq 1 \). For \( d = 0, R \) becomes a field. Also in this case it is easy to see that there exists \( \sigma_1 \in \text{E}(R[I[t]]_{as} \oplus P_{as}) \) such that \( ((b,p)_{as})\sigma_1 = (1,0) \). By Lemma 3.4 there exists \( \sigma_2 \in \text{E}(R[I[t]]_{r} \oplus P_{r}) \) such that \( ((b,p)_{r})\sigma_2 = (1,0) \). Let \( \eta = (\sigma_1)^{-1}(\sigma_2)_{as} \). Since \( (1,0)\eta = (1,0), \eta = \left( \begin{smallmatrix} 1 & 0 \\ 0 & \theta \end{smallmatrix} \right) \), where \( \theta \in \text{GL}_{d+1}(B) \), where \( B = (R_{as}[t])_{1+as}R[I[t]] \). In fact \( \theta \in \text{SL}_{d+1}(B) \), since \( \eta \) is elementary.

By Lemma 3.7 we have \( \eta = (\eta_1)_{(\eta_2)}_{as} \), with \( \eta_1 \in \text{Aut}(R[I[t]]_{r} \oplus P_{r}), \eta_2 \in \text{Aut}(R_{as}[t] \oplus P_{as}) \) such that \( (1,0)\eta_i = (1,0) \) for \( i = 1, 2 \). Now by Lemma 5.5 we will get an automorphism \( \phi \in \text{Aut}(R[I[t]] \oplus P) \) such that \( (b,p)\phi = (1,0) \). This completes the proof of (1).

(2) The proof of the second part is a verbatim copy of the first part. But one needs to use the cancellation result for the Laurent extensions (see [21 Theorem 7.2]).
4 Applications

The following is the Eisenbud–Evans estimate on the number of generators of a module over the (extended) Rees algebras.

**Theorem 4.1** Let $R$ be a commutative Noetherian domain of dimension $d$ and $I$ an ideal of $R$. Let $M$ be a non-zero finitely generated module over $A := R[I]$ or $R[It,t^{-1}]$. Then $M$ is generated by $e(M) := \text{Supp}\{\mu_p(M) + \dim(A/p)\}$ elements.

**Proof** Suppose $M$ is generated by $n$ elements such that $n > e(M) > d$. Then it is enough to prove that $M$ is generated by $(n - 1)$ elements. Since $M$ is generated by $n$ elements, we have the following surjective map $A^n \to M \to 0$. Let $K$ be the kernel of this map. Since $A$ is Noetherian, $K$ is finitely generated. Also, $K$ is a torsion-free module since $K$ is a submodule of $A^n$. Let $S$ be the set of all non-zero divisors of $R$. Then $S^{-1}A \cong k[t]$, where $k$ is a field. Since $S^{-1}K$ is a torsion-free module over the PID $k[t]$, by the structure theorem of finite generated modules over a PID, we get that $S^{-1}K$ is $S^{-1}A$-free. Since $K$ is finitely generated, there exists $s \in S$ such that $K_s$ is $A_s$-free.

Let $x_1$ be a basic element of $K_s$. We set $T = 1 + sA$. By Lemma 3.3, we get that $K_T$ contains a basic element $x_2$ which is a unimodular element in $A^n_T$. Now patching $x_1$ and $x_2$, we get a basic element $x$ in $K$ which is a unimodular element $x \in A^n$. Hence we can write $A^n \cong xA \oplus P$. Now we observe that locally $P$ surjects onto $M$. Hence $P$ surjects onto $M$ globally. Note that rank of $P \geq n - 1 > d$. Hence by the cancellation result Theorem 3.8, we have $P \cong A^{n-1}$. Therefore $M$ is generated by $n - 1$ elements. This finishes the proof.

The following is a $K_1$-analog of the above results.

**Theorem 4.2** Let $R$ be a commutative domain of dimension $d$ and $I \subset R$ an ideal of $R$. Then for $n \geq \max\{3,d+2\}$, the natural map $\phi : \text{GL}_n(R[I])/\text{E}_n(R[I]) \to K_1(R[I])$ is an isomorphism.

**Proof** Surjectivity follows from Theorem 3.8. We have to prove injectivity i.e. stably elementary is elementary. We grade $A := R[I]$ as $A = A_0 \oplus It \oplus I^2t^2 \oplus \cdots$, where $A_0 = R$ and $A_+ = It \oplus I^2t^2 \oplus \cdots$. Let $\alpha \in \text{GL}_n(A)$ which is stably elementary. Multiplying by elementary matrices, we can assume that $\alpha \in \text{GL}_n(A,A_+)$. Let $p \in \text{Spec}(R)$. Note that the generalized dimension of $R_p[I_p]\leq d$. By usual estimation, we get $\alpha_p \in \text{E}_n(R_p[I_p])$. Then by following Gubeladze [9] Proposition 7.3, we get $\alpha \in \text{E}_n(R[I])$. This completes the proof.

**References**

[1] J. Barge, F. Morel, Groupe de Chow des cycles orientés et classes d’Euler des fibrés vectoriels, CRAS Paris 330 (2000), 287–290.

[2] H. Bass, K-theory and stable algebra, Publ. Math. Inst. Hautes Études Sci. 22 (1964), 5–60.

[3] S.M. Bhatwadekar, Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain and a question of Nori, Invent. Math. 133 (1998), no. 1, 161–192.
[4] S.M. Bhatwadekar, Raja Sridharan, *The Euler class group of a Noetherian ring*, Compositio Math. **122** (2000) no. 2, 183–222.

[5] S.M. Bhatwadekar, H. Lindel, R.A. Rao, *The Bass-Murthy question: Serre dimension of Laurent polynomial extensions*, Invent. Math. **81** (1985) 189–203.

[6] S.M. Bhatwadekar, A. Roy, *Stability theorems for overrings of polynomial rings*, Invent. Math. **68** (1982) 117–127.

[7] J. Fasel and V. Srinivas, *Chow-Witt groups and Grothendieck-Witt groups of regular schemes*, Adv. Math. **221** (2009), no. 1, 302–329

[8] J. Fasel, *Groups de Chow-Witt*, Mém. Soc. Math. Fr. (N.S.) **113**, 2008.

[9] J. Gubeladze, *The elementary action on unimodular rows over a monoid ring*, J. Algebra **148** (1992) 135–161.

[10] J. Gubeladze, *K-Theory of affine toric varieties*, Homology, Homotopy and Appl. **1** (1999) 135-145.

[11] M.K. Keshari and H.P. Sarwar, *Serre dimension of monoid algebras*, Proc. Indian Acad. Sci. Math. Sci. **127** (2017), no. 2, 269–280.

[12] F. Morel, *A^1-Algebraic topology over a field*, Lecture Notes in Math., Volume 2052, Springer, Hiedelberg 2012.

[13] M.P. Murthy, *Zero cycles and projective modules*, Ann. of Math. (2) **140** (1994), no. 2, 405–434.

[14] B. Plumstead, *The conjectures of Eisenbud and Evans*, Amer. J. Math. **105** (1983) 1417–1433.

[15] D. Quillen. *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.

[16] R.A. Rao, *Stability theorems for overrings of polynomial rings. II.*, J. Algebra **78** (1982), no. 2, 437–444.

[17] R.A. Rao, *Patching techniques in Algebra – Stability theorems for overrings of polynomial rings and Extendability of Quadratic modules with sufficient Witt index*, Ph.D. Thesis, Bombay University.

[18] A. Roy, *Application of patching diagrams to some questions about projective modules*, J. Pure Appl. Algebra **24** (1982), no. 3, 313–319.

[19] M. Schlichting, *Euler class group, and the homology of elementary and special linear groups*, Adv. in Math. **320** (2017), 1–81.

[20] J-P. Serre, *Sur les modules projectifs*, Sem. Dubreil-Pisot **14** (1960-61), 1–16.

[21] A.A. Suslin, *On structure of special linear group over polynomial rings*, Math. of USSR, Isvestija **11** (No. 1-3) (1977), 221–238 (English translation).

[22] A.A. Suslin, *Stably free modules*, Math. U.S.S.R. Sbornik, **102(144)** (1977), 537–550.

[23] W. Vasconcelos, *Integral closure: Rees algebras, multiplicities, algorithms*, Springer Monographs in Mathematics. Springer-Verlag, Berlin 2005.

[24] A. Wiemers, *Cancellation properties of projective modules over Laurent polynomial rings*, J. Algebra **156** (1993), 108–124.