Online Packing and Covering Framework with Convex Objectives

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Abstract

We consider online fractional covering problems with a convex objective, where the covering constraints arrive over time. Formally, we want to solve

$$\min \{ f(x) \mid Ax \geq 1, x \geq 0 \},$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is convex, and the constraint matrix $A_{m \times n}$ is non-negative. The rows of $A$ arrive online over time, and we wish to maintain a feasible solution $x$ at all times while only increasing coordinates of $x$. We also consider packing problems of the form

$$\max \{ c^\top y - g(\mu) \mid A^\top y \leq \mu, y \geq 0 \},$$

where $g$ is a convex function. In the online setting, variables $y$ and columns of $A^\top$ arrive over time, and we wish to maintain a non-decreasing solution $(y, \mu)$. These problems are dual to each other when $g = f^*$ the Fenchel dual of $f$.

We provide an online primal-dual framework for both classes of problems with competitive ratio depending on certain “monotonicity” and “smoothness” parameters of $f$; our results match or improve on guarantees for some special classes of functions $f$ considered previously.

Using this fractional solver with problem-dependent randomized rounding procedures, we obtain competitive algorithms for the following problems: online covering LPs minimizing $\ell_p$-norms of arbitrary packing constraints, set cover with multiple cost functions, capacity constrained facility location, capacitated multicast problem, set cover with set requests, and profit maximization with non-separable production costs. Some of these results are new and others provide a unified view of previous results, with matching or slightly worse competitive ratios.
1 Introduction

We consider the following class of fractional covering problems:

$$\min \{ f(x) : Ax \geq 1, x \geq 0 \}. \tag{1}$$

Above, $f : \mathbb{R}^n \to \mathbb{R}$ is a non-decreasing convex function and $A_{m \times n}$ is non-negative. (Observe that we can transform the more general constraints $Ax \geq b$ with all non-negative entries into this form by scaling the constraints.) The covering constraints $a_i^T x \geq 1$ arrive online over time, and must be satisfied upon arrival. We want to design an online algorithm that maintains a feasible fractional solution $x$, where $x$ is required to be non-decreasing over time.

We also consider the Fenchel dual of (1) which is the following packing problem:

$$\max \{ 1^Ty - f^\star(\mu) : A^Ty \leq \mu, y \geq 0 \}. \tag{2}$$

Here, the variables $y_i$ along with columns of $A^T$ (or, alternatively, rows of $A$) arrive over time, and the Fenchel dual is formally defined in (6); see, e.g., [Roc70] for background and properties. Let $d$ denote the row sparsity of the matrix $A$, i.e., the maximum number of non-zeroes in any row, and let $\nabla_{l} f(z)$ be the $l^{th}$ coordinate of the gradient of $f$ at point $z \in \mathbb{R}^n$.

This paper gives an online primal-dual algorithm for this pair of convex programs (1) and (2). This extends the widely-used online primal-dual framework for linear objective functions to the convex case. The competitive ratio is given as the ratio between the primal and dual objective functions.

- In the first algorithm, the primal variables $x$ and dual variables $\mu$ are monotonically non-decreasing, while the dual variables $y$ are allowed to both increase and decrease over time. The competitive ratio of this algorithm is:

$$\frac{\text{Dual}}{\text{Primal}} \geq \max_{c > 0} \left[ \min_z \left( \frac{1}{8 \log(1 + d)} \min_{\ell=1}^n \left\{ \frac{\nabla_{\ell} f(z)}{\nabla_{\ell} f(cz)} \right\} \right) - \max_z \left( \frac{z^T \nabla f(z) - f(z)}{f(cz)} \right) \right]. \tag{3}$$

- In the second algorithm, all variables—primal variables $x$ as well as dual variables $y, \mu$—are required to be monotonically non-decreasing. The competitive ratio is slightly worse in this case, given by:

$$\frac{\text{Dual}}{\text{Primal}} \geq \max_{c > 0} \left[ \min_z \left( \frac{1}{2 \log(1 + d\rho)} \min_{\ell=1}^n \left\{ \frac{\nabla_{\ell} f(z)}{\nabla_{\ell} f(cz)} \right\} \right) - \max_z \left( \frac{z^T \nabla f(z) - f(z)}{f(cz)} \right) \right]. \tag{4}$$

Observe that the difference from (3) is the additional parameter $\rho$, which is defined to be an upper bound on the maximum-to-minimum ratio of positive entries in any column of $A$.

The above expressions are difficult to parse because of their generality, so the first special case of interest is that of linear objectives. In this case $z^T \nabla f(z) = f(z)$, and also $\nabla f(z) = \nabla f(cz)$, hence

*However, for clarity of exposition we provide the ratio as Dual/Primal and not vice versa.*
the competitive ratios are $O(\log d)$ for monotone primal, and $O(\log(dp))$ for monotone primal and duals. Both of these competitive ratios are known to be best possible [BN09, GN14].

The applicability of our framework extends to a number of settings, most of which have been studied before in different works. We now outline some of these connections.

- **Mixed Covering and Packing LPs.** In this problem, covering constraints $Ax \geq 1$ arrive online. There are also $K$ “packing constraints” $\sum_{j=1}^{n} b_{kj} \cdot x_{j} \leq \lambda_{k}$, for $k \in [K]$, that are given upfront. The right hand sides $\lambda_{k}$ of these packing constraints are themselves variables, and the objective is to minimize the $\ell_{p}$-norm $(\sum_{k=1}^{K} \lambda_{k}^{p})^{1/p}$ of the “load vector” $\lambda = (\lambda_{1}, \ldots, \lambda_{K})$. All entries $a_{ij}$ and $b_{kj}$ are non-negative. Clearly, the objective function is a monotonically non-decreasing convex function.

  We obtain an $O(p \log d)$-competitive algorithm for this problem, where $d \leq n$ is the rowsparsity of matrix $A$. Prior to our work, [ABFP13] gave an $O(\log K \cdot \log(dK))$-competitive algorithm for the special case of $p = \log K$ (corresponding to $\|\lambda\|_{\infty}$, the makespan of the loads); here $\gamma$ and $\kappa$ are the maximum-to-minimum ratio of the entries in the covering and packing constraints.

- **Set Cover with Multiple Costs.** Here the offline input is a collection of $n$ sets $\{S_{j}\}_{j=1}^{n}$ over a universe $U$, and $K$ different linear cost functions $B_{k} : [n] \to \mathbb{R}_+$ for $k \in [K]$. Elements from $U$ arrive online and must be covered by some set upon arrival, where the decision to select a set into the solution is irrevocable. The goal is to maintain a set-cover that minimizes the $\ell_{p}$ norm of the $K$ cost functions. Combining our framework with a simple randomized rounding scheme gives an $O(\frac{p}{\log p} \log d \log |U|)$-competitive randomized online algorithm; here $d$ is the maximum number of sets containing any element. The special case of $K = 1$ (when $p = 1$ without loss of generality) is the online set-cover problem [AAA+09], for which the resulting $O(\log d \log |U|)$-competitive bound is tight, at least for randomized polynomial-time online algorithms [Kor05].

- **Capacity Constrained Facility Location (CCFL).** Here we are given $m$ potential facility locations, each with an opening cost $c_{i}$ and a capacity $u_{i}$. Now, $n$ clients arrive online, each client $j \in [n]$ having an assignment cost $a_{ij}$ and a demand/load $b_{ij}$ for each facility $i \in [m]$. The online algorithm must open facilities (paying the opening costs $c_{i}$) and assign each arriving client $j$ to some open facility $i$ (paying the assignment cost $a_{ij}$, and incurring a load $p_{ij}$ on facility $i$). The makespan of an assignment is the maximum load on any facility. The objective in CCFL is to minimize the sum of opening costs, assignment costs and the makespan. Using our framework, we obtain an $O(\log^{2} m)$-competitive fractional solution to a convex relaxation of CCFL. This is then rounded online to get an $O(\log^{2} m \log mn)$-competitive randomized online algorithm. This competitive ratio is worse by a logarithmic factor than the best result [ABFP13], but it follows easily from our general framework.

- **Capacitated Multicast Problem (CMC).** This is a common generalization of CCFL and the online multicast problem [AAA+06]. There are $m$ edge-disjoint rooted trees $T_{1}, \ldots, T_{m}$ corresponding to multicast trees in some network. Each tree $T_{i}$ has a capacity $u_{i}$, and each edge $e \in \cup_{i=1}^{m} T_{i}$ has an opening cost $c_{e}$. A sequence of $n$ clients arrive online, and each must be
assigned to one of these trees. Each client \( j \) has a tree-dependent load of \( p_{ij} \) for tree \( T_i \), and is connected to exactly one vertex \( \pi_{ij} \) in tree \( T_i \). Thus, if client \( j \) is assigned to tree \( T_i \) then the load of \( T_i \) increases by \( p_{ij} \), and all edges on the path in \( T_i \) from \( \pi_{ij} \) to its root must be opened. The objective is to minimize the total cost of opening the edges, subject to the capacity constraints that the total load on tree \( T_i \) is at most \( u_i \). Solving a natural fractional convex relaxation, and then applying a suitable randomized rounding to it, we get an \( O(\log^2 m \log mn) \)-competitive randomized online algorithm that violates each capacity by an \( O((d + \log^2 m) \log mn) \) factor; here \( d \) is the maximum depth of the trees \( \{T_i\}_{i=1}^m \). The capacitated multicast problem with depth \( d = 2 \) trees generalizes the CCFL problem, in which case we recover the above result for CCFL.

- **Online Set Cover with Set Requests (SCSR).** We are given a universe \( U \) of \( n \) resources, and a collection of \( m \) facilities, where each facility \( i \in [m] \) is specified by (i) a subset \( S_i \subseteq U \) of resources (ii) opening cost \( c_i \) and (iii) capacity \( u_i \). The resources and facilities are given up-front. Now, a sequence of \( k \) requests arrive over time. Each request \( j \in [k] \) requires some subset \( R_j \subseteq U \) of resources. The request has to be served by assigning it to some collection \( F_j \subseteq [m] \) of facilities whose sets collectively cover \( R_j \), i.e., \( R_j \subseteq \cup_{i \in F_j} S_i \). Note that these facilities have to be open, and we incur the cost of these facilities. Moreover, if a facility \( i \) is used to serve client \( j \), this contributes to the load of facility \( i \), and this total load must be at most the capacity \( u_i \). This problem was considered recently by Bhawalkar et al. [BGP14].

Using an approach identical to that for the CCFL problem, we get an \( O(\log^2 m \log mnk) \)-competitive randomized online algorithm that violates each capacity by an \( O((\log^2 m \log mnk)^2) \) factor. Again this factor is weaker than the best result by a logarithmic factor, but directly follows from our general framework.

- **Profit Maximization with Production Costs (PMPC).** This is an application of the dual packing problem \([2]\), in contrast to the above applications which are all applications of the primal covering problem.

Consider a seller with \( m \) items that can be produced and sold. The seller has a production cost function \( g : \mathbb{R}^m_+ \to \mathbb{R}_+ \) which is monotone, convex and satisfies some other technical conditions; the total cost incurred by the seller to produce \( \mu_j \) units of every item \( j \in [m] \) is given by \( g(\mu) \). There are \( n \) buyers who arrive online. Each buyer \( i \in [n] \) is interested in subsets of items (bundles) that belong to a set family \( \mathcal{S}_i \subseteq 2^{[m]} \). The value of buyer \( i \) for subset \( S \in \mathcal{S}_i \) is given by \( v_i(S) \), where \( v_i : \mathcal{S}_i \to \mathbb{R}_+ \) is her valuation function. If buyer \( i \) is allocated a bundle \( T \in \mathcal{S}_i \), she pays the seller her valuation \( v_i(T) \). (Observe: this is not an auction setting.) The goal in the PMPC problem is to produce items and allocate subsets to buyers so as to maximize the profit \( \sum_{i=1}^n v_i(T_i) - g(\mu) \), where \( T_i \in \mathcal{S}_i \) denotes the subset allocated to buyer \( i \) and \( \mu \in \mathbb{R}^m_+ \) is the total quantity of all items produced. As mentioned above, we consider a non-strategic setting, where the valuation of each buyer is known to the seller.

\footnote{An important difference from prior work on such problems \([GMMS11, HK14]\): in these works, each item \( j \) had a separate production cost function \( g_j(\mu_j) \), and \( g(\mu) := \sum_j g_j(\mu_j) \). We call this the *separable* case. Our techniques allow the production cost to be *non-separable* over items—e.g., we can handle \( g(\mu) = (\sum_{j=1}^m \mu_j)^2 \).}
Our main result here is for the fractional version of the problem where the allocation to each buyer $i$ is allowed to be any point in the convex hull of the set family $S_i$. We show that for a large class of valuation functions (e.g., supermodular, or weighted rank-functions of matroids) and production cost functions, our framework provides a polynomial time online algorithm: the precise competitive ratio is given by expression (4) with $f = g^*$. As a concrete example, suppose the production cost function is $g(\mu) = (\sum_{j=1}^m \mu_j)^p$ for some $p > 1$. In this case, we get an $O(q \log \beta)^q$-competitive algorithm, where $q > 1$ satisfies $\frac{1}{q} + \frac{1}{p} = 1$, and $\beta$ is the maximum-to-minimum ratio of the valuation functions $\{v_i\}$.

As the above list indicates, the framework to solve fractional convex programs is fairly versatile and gives good fractional results for a variety of problems. In some cases, solving the particular relaxation we consider and then rounding ends up being weaker than the best known results for that specific problems (by a logarithmic factor); we hope that further investigation into this problem will help close this gap.

**Bibliographic Note:** In independent and concurrent work, Azar et al. [ACP14] consider online covering problems with convex objectives—i.e., problem (1). They also obtain a competitive ratio that depends on properties of the function $f$, but their parameterization is somewhat different from ours. As an example, for online covering LPs minimizing the $\ell_p$-norm of packing constraints, they obtain an $O(p \log(d\kappa\gamma))$-competitive algorithm, whereas we obtain a tighter $O(p \log d)$ ratio.

### 1.1 Techniques and Paper Outline

In §2.1 we give the first general algorithm for the convex covering problem (1) maintaining monotone primal variables (but allowing dual variables to decrease). The main observation is simple, yet powerful: convex optimization problems with a function $f$ can be reduced to linear optimization using the gradient of the convex function $f$. In the process we end up also giving a cleaner algorithm and proof for linear optimization problems as well, significantly simplifying the previous algorithm from [GN14]. The resulting algorithm performs multiplicative increases on the primal variables; for the dual, it does an initial increase followed by a linear decrease after some point.

In §2.2 we give the second general algorithm, which is simpler. The primal updates are the same as above but we skip the dual decreases. This results in a worse competitive ratio, but the loss is necessary for any monotone primal-dual algorithm [BN09].

In §3 and §4 we deal with the various applications of our framework. The high-level idea in all of these is to suitably cast each application in the form of either (1) or (2). All, but the applications in §3 are for the convex covering problem (1). Some comments on the main ideas to watch out for:

- For applications to combinatorial problems we have to define the convex relaxation with some care in order to avoid bad integrality gaps. Moreover, some of our convex relaxations are motivated by the particular constraints we want to enforce when subsequently rounding.

- For some of the problems our convex relaxations have an exponential number of constraints. To get a polynomial running time, we use the natural “separation oracle” approach. Moreover, we relax the constraints by a constant factor, so that each call to the separation oracle gives us a “big” improvement, and hence there are only a few updates per request.
• For capacity constrained facility location (in §3.3), capacitated multicast problem (in §3.4), and set cover with set requests (in §3.5), na""ıve randomized rounding is bad, and hence the rounding schemes introduces correlations between opening facilities and assigning clients. These correlations also motivate the specific convex relaxations we consider for the problems.

In §4 we consider the problem of profit maximization with production costs, which after some simplifications can be cast as a convex packing program as in (2). We want allocations to be non-decreasing over time, so we use our second general primal-dual algorithm, which maintains monotone solutions. We also show how this problem can be solved efficiently for some special classes of valuation functions: supermodular and matroid-rank-functions. This convex program can also be (randomly) rounded online to get integral allocations with the same multiplicative competitive ratio, but with an extra additive term. The additive term depends only on the number $m$ of items and the cost function $g$; in particular it does not depend on $n$, the number of buyers.

We note that such an additive loss is necessary for our approach due to an integrality gap of the convex relaxation.

1.2 Related Work

This paper adds to the body of work in online primal-dual algorithms; see [BN07] for a survey of this area. This approach has been applied successfully to a large class of online problems: set cover [AAA+09], graph connectivity and cuts [AAA+06], caching [BBN12], auctions [HK14], scheduling [DH14], etc. Below we discuss in more detail only work that is directly relevant to us.

Online packing and covering linear programs were first considered by Buchbinder and Naor [BN09], where they obtained an $O(\log n)$-competitive algorithm for covering and an $O(\log(n \frac{d_{\max}}{d_{\min}}))$-competitive algorithm for packing. The competitive ratio for covering linear programs was improved to $O(\log d)$ by Gupta and Nagarajan [GN14], where $d \leq n$ is the maximum number of non-zero entries in any row.

Azar, Bhaskar, Fleischer, and Panigrahi [ABFP13] gave the first algorithm for online mixed packing and covering LPs, where the packing constraints are given upfront and covering constraints arrive online; the objective is to minimize the maximum violation of the packing constraints. Their algorithm had a competitive ratio of $O(\log K \cdot \log(bc\gamma))$, where $K$ is the number of packing constraints and $\gamma$ (resp. $\kappa$) denotes the maximum-to-minimum ratio of entries in the covering (resp. packing) constraints. Using our framework, this bound can be improved to $O(\log K \cdot \log d)$. This is also best possible as shown in [ABFP13].

The capacity constrained facility location problem was also introduced by Azar, Bhaskar, Fleischer, and Panigrahi [ABFP13], who gave an $O(\log m \log mn)$-competitive algorithm. Our result for this problem is worse by a log-factor, but has the advantage of following directly from our general framework. Moreover, our approach can be extended to the capacitated multicast problem, which is a generalization of CCFL to multi-level facility costs. The online multicast problem (without capacities) was considered by Alon et al. [AAA+06] where they obtained an $O(\log m \cdot \log n)$-competitive randomized algorithm.

The online set cover problem with set requests was considered recently by Bhawalkar, Gollapudi, and Panigrahi [BGPI14] who obtained an $O(\log m \log mnk)$-competitive algorithm where capacities
are violated by an $O(\log^2 m \log mnk )$ factor. The competitive ratio obtained through our approach is worse by a logarithmic factor in the cost guarantee. Still, we think this is useful, since it follows with almost no additional effort, given our online fractional framework and the CCFL rounding scheme. Our approach is also likely to be useful in other such generalizations.

The class of online maximization problems with production costs was introduced by Blum, Gupta, Mansour, and Sharma [BGMS11] and extended by Huang and Kim [HK14]. The key differences from our setting are: (i) these papers deal with an auction setting where the seller is not aware of the valuations of the buyers, whereas our setting is not strategic, and (ii) these papers are restricted to separable production costs, whereas we can handle much more general (non-separable) cost functions.

2 The General Framework

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative non-decreasing convex function. We assume that the function $f$ is continuous and differentiable, and satisfies the following monotonicity condition:

$$\forall x \geq x' \in \mathbb{R}^n, \quad \nabla f(x) \geq \nabla f(x')$$

(5)

Here, $x \geq x'$ means $x_i \geq x'_i$ for all $i \in [n]$.

We consider the online fractional covering problem [1] where the constraints in $A$ arrive online. Our algorithm is a primal-dual algorithm, which works with the following pair of convex programs:

$$\begin{align*}
(P) : \min & \quad f(x) \\
& \quad Ax \geq 1 \\
& \quad x \geq 0.
\end{align*}$$

$$\begin{align*}
(D) : \max & \quad \sum_{i=1}^{m} y_i - f^*(\mu) \\
& \quad y^T A \leq \mu^T \\
& \quad y \geq 0.
\end{align*}$$

Here $f^*$ is the Fenchel dual of $f$, which is defined as

$$f^*(\mu) = \sup_z \{\mu^T z - f(z)\}.$$  

(6)

(Observe that by scaling the rows of $A$ appropriately, we can transform any covering LP of the form $Ax \geq b$ into the form above.) The following duality is standard.

Lemma 2.1 (Weak duality). Let $x, (y, \mu)$ be feasible primal and dual solutions to $(P)$ and $(D)$ respectively. Then,

$$\text{Primal objective } = f(x) \geq \sum_{i=1}^{m} y_i - f^*(\mu) = \text{Dual objective.}$$

(7)

Proof.

$$\sum_{i=1}^{m} y_i = y^T 1 \leq y^T A x \leq \mu^T x = (\mu^T x - f(x)) + f(x) \leq f^*(\mu) + f(x).$$

Rearranging we get the desired.

$\square$
2.1 The Algorithm

The algorithm maintains a feasible primal $x$ and a feasible dual solution $y$ at each time.

**Fractional Algorithm:** At round $t$:

- Let $\tau$ be a continuous variable denoting the current time.
- While the new constraint is unsatisfied, i.e., $\sum_{j=1}^{n} a_{tj} x_j < 1$, increase $\tau$ at rate 1 and:
  - Change of primal variables:
    - For each $j$ with $a_{tj} > 0$, increase each $x_j$ at rate
      $$\frac{\partial x_j}{\partial \tau} = \frac{a_{tj} x_j + \frac{1}{d}}{\nabla_j f(x)}.\tag{8}$$
      Here $d$ is an upper bound on the row sparsity of the matrix. $\nabla_j f(x)$ is the $j^{th}$-coordinate of the gradient $\nabla f(x)$.
  - Change in dual variables:
    - Set $\mu = \nabla f(\delta x)$, where $\delta > 0$ is determined later.
    - Increase $y_i$ at rate $r = \frac{1}{\log(1+2d^2)} \cdot \min_{\ell=1}^{n} \left\{ \nabla_{j} f(\delta x) \right\}$.
    - If the dual constraint of variable $x_j$ is tight, that is, $\sum_{i=1}^{t} a_{ij} y_i = \mu_j$, then,
      * Let $m_j^* = \arg \max_{i=1}^{t} \{a_{ij} y_i > 0\}$.
      * Increase $y_{m_j^*}$ at rate $-\frac{a_{m_j^*}}{a_{m_j^*}} \cdot r$.

(Note that this change occurs only if $a_{tj}$ is strictly positive.)

We emphasize that the primal algorithm does not depend on the value $\delta$. The last step in the algorithm decreases certain dual variables; all other steps only increase primal and dual variables. For the analysis, we denote $x^\tau, y^\tau, \mu^\tau, r^\tau$ as the value of $x, y, \mu, r$ at time $\tau$, respectively.

**Observation 2.2.** For any $\delta > 0$, the following are maintained.

- The algorithm maintains a feasible monotonically non-decreasing primal solution.
- The algorithm maintains a feasible dual solution with non-decreasing $\mu_j$.

**Proof.** The first property follows by construction, since we only increase $x$ till reaching a feasible solution. For the second property, we observe that the dual variables $\mu$ are non-decreasing since $\nabla f(x)$ is non-decreasing. We prove that $y, \mu$ is feasible by induction over the execution of the algorithm. While processing constraint $t$, if $\sum_{i=1}^{t} a_{ij} y_i^\tau < \mu_j^\tau$ for column $j$ we are trivially satisfied. Suppose that during the processing of constraint $t$, we have $\sum_{i=1}^{t} a_{ij} y_i^\tau = \mu_j^\tau$ for some dual constraint.
\( j \) and time \( \tau \). Now the dual decrease part of the algorithm kicks in, and the rate of change in the left-hand side of the dual constraint is:

\[
\frac{d}{d\tau} \left( \sum_{i=1}^{t} a_{ij} y_i^\tau \right) = a_{ij} \cdot r^\tau - a_{m^*j} \cdot \frac{a_{ij}}{a_{m^*j}} \cdot r^\tau = 0
\]

\[
\square
\]

Before analyzing the competitive factor, let us first prove the following claim.

**Claim 2.3.** For a variable \( x_j \), let \( T_j = \{i \mid a_{ij} > 0\} \) and let \( S_j \) be any subset of \( T_j \). Then,

\[
x_j^\tau \geq \frac{1}{\max_{i \in S_j} \{a_{ij}\} \cdot d} \left( \exp \left( \frac{\ln \left( 1 + 2d^2 \right)}{\mu_j} \sum_{i \in S_j} a_{ij} y_i^\tau \right) - 1 \right)
\]

\[
(9)
\]

**Proof.** Let \( \tau(i) \) denote the value of \( \tau \) at the arrival of the \( i \)th primal constraint. We first note that the increase in the primal variables at any time \( \tau(i) \leq \tau \leq \tau(i+1) \) can be alternatively formulated by the following differential equation.

\[
\frac{\partial x_j}{\partial y_i} = \log \left( 1 + 2d^2 \right) \cdot \min_{l=1}^{n} \left\{ \frac{\nabla f(x)}{\nabla f(x)} \right\} \cdot a_{ij} x_j + \frac{1}{\nabla f(x)} \geq \log \left( 1 + 2d^2 \right) \cdot \frac{a_{ij} x_j + \frac{1}{\nabla f(x)} \cdot y_i^\tau}{x_j^\tau(i) + \frac{1}{a_{ij} \cdot d}}  
\]

\[
(10)
\]

By solving the latter equation we get for any \( \tau(i) \leq \tau \leq \tau(i+1) \),

\[
x_j^\tau + \frac{1}{a_{ij} \cdot d} \leq \exp \left( \frac{\ln \left( 1 + 2d^2 \right) \cdot a_{ij} y_i^\tau}{\nabla f(x) \cdot (x_j^\tau + \frac{1}{\nabla f(x)} \cdot x_j^\tau(i) + \frac{1}{a_{ij} \cdot d})} \right)  
\]

\[
(11)
\]

where we use the fact that \( \nabla f(x) \) is monotonically non-decreasing. Note that Inequality (11) is satisfied even when no decrease is performed on the dual variables, and such a decrease only effects the right handside of the inequality. For convenience, let us denote \( \tau(t+1) = \tau \) (the actual value of \( \tau(t+1) \) has not been revealed by the algorithm yet). Multiplying over all indices in \( S_j \) we get,

\[
\exp \left( \frac{\ln \left( 1 + 2d^2 \right)}{\mu_j} \sum_{i \in S_j} a_{ij} y_i^\tau \right) \leq \exp \left( \sum_{i \in S_j} \frac{\ln \left( 1 + 2d^2 \right)}{\nabla f(x) \cdot (x_j^\tau + \frac{1}{\nabla f(x)} \cdot x_j^\tau(i) + \frac{1}{a_{ij} \cdot d})} \cdot a_{ij} y_i^\tau(i+1) \right)  
\]

\[
(12)
\]

\[
\leq \prod_{i \in S_j} \frac{x_j^\tau(i+1) + \frac{1}{a_{ij} \cdot d}}{x_j^\tau(i) + \frac{1}{a_{ij} \cdot d}} \leq \prod_{i \in S_j} \frac{x_j^\tau(i+1) + \frac{1}{a_{ij} \cdot d}}{x_j^\tau(i) + \frac{1}{a_{ij} \cdot d}} \leq \prod_{i \in T_j} \frac{x_j^\tau(i+1) + \frac{1}{a_{ij} \cdot d}}{x_j^\tau(i) + \frac{1}{a_{ij} \cdot d}}  
\]

\[
(13)
\]

Inequality (12) follows as \( \mu_j^\tau = \nabla f(\delta x^\tau) \) and the value of \( \nabla f(\delta x) \) monotonically non-decreases in time. Inequality (13) follows by substituting (11) into (12). Inequality (14) follows as the value of
\( x_j \) monotonically non-decreases in time. Finally, the last equality is obtained using a telescopic sum and the fact that \( x_j \) increases only in rounds with \( a_{tj} > 0 \).

**Theorem 2.4.** The competitive ratio of the algorithm is:

\[
\min_z \left( \frac{\min_{t=1}^{n} \left\{ \nabla f(\delta z) \right\}}{4 \ln(1+2d^2)} \right) - \max_z \left( \frac{(\delta z)^\top \nabla f(\delta z) - f(\delta z)}{f(z)} \right),
\]

where \( \delta > 0 \) is the parameter chosen in the algorithm.

**Proof.** Consider the update when primal constraint \( t \) arrives and \( \tau \) is the current time. Let \( U(\tau) \) denote the set of tight dual constraints at time \( \tau \). That is, for every \( j \in U(\tau) \) we have \( a_{tj} > 0 \) and \( \sum_{i=1}^{t} a_{ij} y_i^\tau = \mu_j^\tau \). So \( |U(\tau)| \leq d \) the row-sparsity of \( A \). Moreover, let us define for every \( j \in U(\tau) \), \( S_j = \{ i | a_{ij} > 0, y_i^\tau > 0 \} \). Clearly, \( \sum_{i \in S_j} a_{ij} y_i^\tau = \sum_{i=1}^{t} a_{ij} y_i^\tau = \mu_j^\tau \), hence by Claim 2.3 and the fact that \( \sum_j a_{tj} x_j^\tau < 1 \), we get for every \( j \in U(\tau) \),

\[
\frac{1}{a_{tj}} > x_j^\tau \geq \frac{1}{\max_{i \in S_j} \{ a_{ij} \}} \cdot d \left( \exp \left( \ln(1+2d^2) \right) - 1 \right),
\]

and after simplifying we get \( \frac{a_{tj}}{a_{m^*j}} = \frac{a_{tj}}{\max_{i \in S_j} \{ a_{ij} \}} \leq \frac{1}{2d} \). As a result, we can bound the rate of change in the dual expression \( \sum_{i=1}^{t} y_i \) at any time \( \tau \):

\[
\frac{d}{d\tau} \left( \sum_{i=1}^{t} y_i \right) \geq r^\tau - \sum_{j \in U(\tau)} \frac{a_{tj}}{a_{m^*j}} \cdot r^\tau \geq r^\tau - \sum_{j \in U(\tau)} \frac{1}{2d} \geq \frac{1}{2} r^\tau,
\]

where the last inequality follows as \( |U(\tau)| \leq d \).

On the other hand, when processing constraint \( t \) during the execution of the algorithm, the rate of increase of the primal objective \( f \) is:

\[
\frac{df(x^\tau)}{d\tau} = \sum_j \nabla_j f(x^\tau) \frac{\partial x_j^\tau}{\partial \tau} = \sum_{j | a_{tj} > 0} \nabla_j f(x^\tau) \left( \frac{a_{tj} x_j^\tau + \frac{1}{d}}{\nabla_j f(x^\tau)} \right) = \sum_{j | a_{tj} > 0} \left( a_{tj} x_j^\tau + \frac{1}{d} \right) \leq 2.
\]

The final inequality uses the fact that the covering constraint is unsatisfied, and that \( d \) is at least the number of non-zeroes in the vector \( a_t \). From (16) and (17) we can now bound the following primal-dual ratio:

\[
\frac{d}{df(x^\tau)} \left( \sum_{i=1}^{t} y_i \right) \geq r^\tau - \frac{\min_{\ell=1}^{n} \left\{ \nabla f(\delta x^\ell) \right\}}{4 \ln(1+2d^2)}.
\]

Thus, if \( \overline{x} \) and \( \overline{y} \) are the final primal and dual solutions we get,

\[
\sum_{i=1}^{m} \overline{y}_i \geq \min_{x} \min_{\ell=1}^{n} \left\{ \nabla f(\delta x^\ell) \right\} \cdot f(\overline{x}).
\]
To complete the proof of Theorem 2.4, we use the following standard claim.

**Claim 2.5.** For any \( a \in \mathbb{R}^n \), we have \( f^*(\nabla f(a)) = a^\top \nabla f(a) - f(a) \).

**Proof.** By definition, \( f^*(\nabla f(a)) = \sup_x \{ x^\top \nabla f(a) - f(x) \} \). Note that \( x^\top \nabla f(a) - f(x) \) is concave as a function of \( x \). So a necessary and sufficient condition for optimality is:

\[ \nabla_i f(x) = \nabla_i f(a), \quad \forall i \in [n]. \]

Thus setting \( x = a \), we have \( f^*(\nabla f(a)) = a^\top \nabla f(a) - f(a) \).

Finally, we can attain the competitive ratio by a simple application of Claim 2.5 and Inequality (19) to the definition of the dual. Indeed,

\[ \text{Dual} = \sum_{i=1}^m y_i - f^*(\mu) \geq \left( \min_{x'} \left\{ \min_{\ell=1}^n \left\{ \frac{\nabla \ell f(\delta x')}{\nabla \ell f(x')} \right\} \right\} \right. \]

\[ \left. \frac{4 \ln(1 + 2d^2)}{f(x)} \right) \cdot f(x) \]

by Inequality (19), and using Claim 2.5 (with \( a = \delta \mu \)), we get

\[ \geq \left[ \min_z \left( \min_{\ell=1}^n \left\{ \frac{\nabla \ell f(\delta z)}{\nabla \ell f(z)} \right\} \right) \right. \]

\[ \left. \frac{4 \ln(1 + 2d^2)}{f(z)} \right] \cdot \text{Primal} \]

Hence the proof.

How to choose the value of \( \delta \)? If we set \( c = 1/\delta \) and optimize over \( c \), the competitive ratio is:

\[ \frac{\text{Dual}}{\text{Primal}} \geq \max_{c > 0} \left( \min_z \left( \min_{\ell=1}^n \left\{ \frac{\nabla \ell f(z)}{\nabla \ell f(cz)} \right\} \right) \right. \]

\[ \left. \frac{4 \ln(1 + 2d^2)}{f(z)} \right) \cdot \text{Primal} \]

This expression looks quite formidable, however it simply captures how sharply the function \( f \) changes locally. For special cases it gives us very simple expressions; e.g., for linear cost functions \( f(x) = c^\top x \) it gives us \( \text{Dual} \geq \text{Primal}/O(\log d) \). See (a) for several such examples of applications using this framework.

### 2.1.1 Online Minimization

In the general framework above, we maintained both the primal and dual solutions simultaneously. If our goal is to solve (1) online, i.e., to minimize the convex function \( f(x) \) subject to covering constraints arriving online, then the dual values can be determined with hindsight once the final value of the primal variables \( \overline{x} \) has been computed. In particular, we set \( \mu = \nabla f(\delta \overline{x}) \) once and for
all, and increase \( y \) at a constant rate

\[
\tau = \min_{\ell=1}^{n} \frac{\nabla f(\delta x)}{\nabla f(x)} \log (1 + 2d^2).
\]

These modifications can be easily plugged into the analysis above, allowing us to omit the minimization over \( x' \) in the competitive ratio. (Observe that the update for the primal variables remains the same).

**Corollary 2.6.** For online minimization, the competitive ratio of the algorithm is:

\[
\max_{c>0} \min_z \left( \frac{\min_{\ell=1}^{n} \{ \nabla f(z) \}}{4 \ln(1 + 2d^2)} - \frac{z^T \nabla f(z) - f(z)}{f(cz)} \right) \tag{21}
\]

### 2.2 Monotone Online Maximization

If our goal is to solve (2) and maximize a dual objective function subject to packing constraints, then indeed the above framework increases the dual variables \( \mu \), however the dual variables \( y \) can both increase and decrease. (Moreover, this potential decrease is essential for the competitive ratio to be independent of the magnitude of entries in the matrix \( A \) [GN14]). In settings where decrease in dual variables is not allowed, we need to slightly modify (and simplify) the online dual update in the algorithm by setting

\[
r = \frac{\min_{\ell=1}^{n} \{ \nabla f(\delta x) \}}{\log (1 + 2d^2)},
\]

where \( \rho \) is an upper bound on \( \max_{t} \{ a_{tj} \} / \min_{t,a_{tj}>0} a_{tj} \) for all \( 1 \leq j \leq n \). And we skip the last step which decreases duals. Here, application of Claim 2.3 at any round \( t \) and time \( \tau(t) \leq \tau \leq \tau(t+1) \) yields

\[
\frac{1}{a_{ij}} \geq x_j^\tau \geq \frac{1}{\max_{t=1}^{t} \{ a_{ij} \}} \cdot d \left( \exp \left( \frac{\ln (1 + d\rho)}{\mu_j^\tau} \sum_{i=1}^{t} a_{ij} y_i \right) - 1 \right), \tag{22}
\]

which implies \( \ln \left( 1 + d \cdot \frac{\max_{t=1}^{t} \{ a_{ij} \}}{a_{tj}} \right) / \ln (1 + d\rho) \geq \sum_{i=1}^{t} a_{ij} y_i \), and thus guarantees \( \sum_{i=1}^{t} a_{ij} y_i \leq \mu_j^\tau \).

**Corollary 2.7.** For online maximization, when decreasing dual variables is not allowed, the adjusted algorithm obtains the following competitive ratio:

\[
\max_{c>0} \min_z \left( \frac{\min_{\ell=1}^{n} \{ \nabla f(z) \}}{2 \ln(1 + pd)} - \frac{z^T \nabla f(z) - f(z)}{f(cz)} \right) \tag{23}
\]

This results in a worse competitive ratio, but having monotone duals is useful for two reasons: (a) in some settings we need monotone duals, as in the profit maximization application in Section 4, and (b) we get a simpler algorithm since we skip the third step of the online dual update (involving
the dual decrease).

3 Applications

We show how the general framework above can be used to give algorithms for several previously-studied as well as new problems. In contrast to previous papers where a primal-dual algorithm had to be tailored to each of these problems, we use the framework above to solve the underlying convex program, and then apply a suitable rounding algorithm to the fractional solution.

3.1 \( \ell_p \)-norm of Packing Constraints

We consider the problem of solving a mixed packing-covering linear program online, as defined by Azar et al. [ABFP13]. The covering constraints \( Ax \geq 1 \) arrive online, as in the above setting. There are also \( K \) “packing constraints” \( \sum_{j=1}^{n} b_{kj} \cdot x_j \leq \lambda_k \) for \( k \in [K] \) that are given up-front. The right sides \( \lambda_k \) of these packing constraints are themselves variables, and the objective is to minimize \( \sum_{k=1}^{K} \lambda_k^p \) or alternatively, \( \|\lambda\|_p = \sqrt[p]{\sum_{k=1}^{K} \lambda_k^p} \). All the entries in the constraint matrices \( A = (a_{ij}) \) and \( B = (b_{kj}) \) are non-negative.

**Theorem 3.1.** There is an \( O(p \log d) \)-competitive online algorithm for fractional covering with the objective of minimizing \( \ell_p \)-norm of multiple packing constraints.

**Proof.** In order to apply our framework to this problem, we seek to minimize the convex function

\[
 f(x) = \frac{1}{p} \|Bx\|_p^p = \frac{1}{p} \sum_{k=1}^{K} (B_k x)^p = \frac{1}{p} \sum_{k=1}^{K} \left( \sum_{j=1}^{n} b_{kj} \cdot x_j \right)^p.
\]

This is the \( p \)-power of the original objective; above \( B_k = (b_{k1}, \ldots, b_{kn}) \) is the \( k \)th packing constraint. To obtain the competitive ratio, observe that \( \nabla_j f(x) = \sum_{k=1}^{K} b_{kj} \cdot (P_k x)^{p-1} \). Thus, we have for all \( c > 0, x \in \mathbb{R}_n^+ \) and \( 1 \leq j \leq n:\)

\[
\frac{f(z)}{f(cz)} = (1/c)^p
\]

\[
\frac{\nabla_j f(z)}{\nabla_j f(cz)} = (1/c)^{p-1}
\]

\[
\frac{\sum_{j=1}^{n} z_j \cdot \nabla f(z_j)}{f(cz)} = \frac{\sum_{j=1}^{n} z_j \sum_{k=1}^{K} b_{kj} \cdot (B_k z)^{p-1}}{f(cz)} = p \cdot \frac{f(z)}{f(cz)} = p(1/c)^p.
\]

Substituting \( \delta = 1/c \) and plugging into (20) we get:

\[
\text{Dual} \geq \left( \frac{\delta^{p-1}}{4 \ln(1 + 2d^2)} - p\delta^p + \delta^p \right) \cdot \text{Primal} \tag{24}
\]

So the primal-dual ratio (as a function of \( \delta \)) is \( \frac{\text{Dual}}{\text{Primal}} \geq \delta^{p-1}/L - (p-1)\delta^p \) where \( L = 4 \ln(1 + 2d^2) \). This quantity is maximized when \( \delta = \frac{1}{pL} \), leading to a primal-dual ratio of \( 1/(pL)^p \). Taking the \( p^{th} \)
root of this quantity gives us that the $\ell_p$-norm of the primal is at most $pL = O(p \log d)$ times the optimum.

When $p = \Theta(\log m)$, the $\ell_p$ and $\ell_\infty$ norms are within constant factors of each other, we obtain the online mixed packing-covering LP (OMPC) problem studied by Azar et al. [ABFP13]. For this setting this gives an improved $O(\log d \cdot \log m)$-competitive ratio, where $d$ is the row-sparsity of the matrix $A$, and $m$ is the number of packing constraints. This competitive ratio is known to be tight [ABFP13, Theorem 1.2].

**Remark 3.2.** The above result also holds if function $f$ is the sum of distinct powers of linear functions, i.e. $f(x) = \sum_{k=1}^{K} (B_k x)^{p_k}$ where $p_1, \ldots, p_K \geq 1$ may be non-uniform. For this case, we obtain an $O(p \log d)$-competitive algorithm where $p = \max_{k=1}^{K} p_k$.

### 3.2 Online Set Cover with Multiple Costs

Consider the online set-cover problem [AAA+09] with $n$ sets $\{S_j\}_{j=1}^{n}$ over some ground set $U$. Apart from the set system, we are also given $K$ cost functions $B_k : [n] \to \mathbb{R}_+$ for $k \in [K]$. Elements from $U$ arrive online and must be covered by some set upon arrival; the decision to select a set into the solution is irrevocable. The goal is to maintain a set-cover that minimizes the $\ell_p$ norm of the $K$ cost functions. We use Theorem 3.1 along with a rounding scheme (similar to [GKP12]) to obtain:

**Theorem 3.3.** There is an $O\left(\frac{p^3}{\log p} \log d \log r\right)$-competitive randomized online algorithm for set cover minimizing the $\ell_p$-norm of multiple cost-functions. Here $d$ is the maximum number of sets containing any element, and $r = |U|$ is the number of elements.

**Proof.** We use the following convex relaxation. There is a variable $x_j$ for each set $j \in [n]$ which denotes whether this set is chosen.

$$
\min \ g(x) = \sum_{k=1}^{K} \left( \sum_{j=1}^{n} b_{kj} \cdot x_j \right)^p + \sum_{j=1}^{n} \left( \sum_{k=1}^{K} b_{kj}^p \right) \cdot x_j \\
\text{s.t.} \sum_{j: e \in S_j} x_j \geq 1, \quad \forall e \in U \\
x \geq 0.
$$

We can use our framework to solve this fractional convex covering problem online. Although the objective has a linear term in addition to the $p$-powers, we obtain an $O(p \log d)^p$-competitive algorithm as noted in Remark 3.2.

Let $C^*$ denote the $p$th power of the optimal objective of the given set cover instance. Then it is clear that the optimal objective of the above fractional relaxation is at most $2C^*$. Thus the objective of our fractional online solution $g(x) = O(p \log d)^p \cdot C^*$.

To get an integer solution, we use a simple online randomized rounding algorithm. For each set $j \in [n]$, define $X_j$ to be a $\{0, 1\}$-random variable with $\Pr[X_j = 1] = \min\{4p \log r \cdot x_j, 1\}$. This can easily be implemented online. It is easy to see by a Chernoff bound that for each element $e$, it is not covered with probability at most $\frac{1}{e^p}$. If an element $e$ is not covered by this rounding, we choose the
set minimizing $\min_{j=1}^n \{ \sum_{k=1}^K b_{k,j}^p : e \in S_j \}$; let $\pi \in [n]$ index this set and $C_\pi = \sum_{k=1}^K b_{k,\pi}^p$. Observe that $C_\pi \leq C^*$ for all $e \in U$.

To bound the $\ell_p$-norm of the cost, let $C_k = \sum_{j=1}^n b_{k,j} \cdot X_j$ be the cost of the randomly rounded solution under the $k^{th}$ cost function, and let $C := \sum_{k=1}^K C_k^p$. Also for each element $e \in U$, define:

- $D_{ek} = b_{k,\pi}$ for all $k \in [K]$ and $D_e = C_\pi$ if $e$ is not covered by the rounding.
- $D_{ek} = 0$ for all $k \in [K]$ and $D_e = 0$ otherwise.

Note that $D_e = \sum_{k=1}^K D_{ek}^p$. The $p^{th}$ power of the objective function is:

$$C = \sum_{k=1}^K \left( C_k + \sum_{e \in U} D_{ek} \right)^p \leq 2^p \sum_{k=1}^K C_k^p + 2p \sum_{k=1}^K \sum_{e \in U} D_{ek}^p \leq 2^p \cdot C + 2p \sum_{k=1}^K \sum_{e \in U} D_{ek}^p \tag{25}$$

We now bound $\mathbb{E}[C]$ using (25). Observe that $\mathbb{E}[C_k] \leq 4p \log r \cdot \sum_{j=1}^n b_{k,j} \cdot x_j$. Since each $C_k$ is the sum of independent non-negative random variables, we can bound $\mathbb{E}[C_k^p]$ using a concentration inequality involving $p^{th}$ moments [Lat97]:

$$\mathbb{E}[C_k^p] \leq K_p \cdot \left( \mathbb{E}[C_k]^p + \sum_{j=1}^n \mathbb{E}[b_{k,j}^p \cdot X_j^p] \right) \leq K_p \cdot \left( (4p \log r)^p \left( \sum_{j=1}^n b_{k,j} \cdot x_j \right)^p + 4p \log r \sum_{j=1}^n b_{k,j}^p \cdot x_j \right).$$

Above $K_p = O(p / \log p)^p$. By linearity of expectation,

$$\mathbb{E}[C] = \sum_{k=1}^K \mathbb{E}[C_k^p] \leq K_p (4p \log r)^p \sum_{k=1}^K \left( \sum_{j=1}^n b_{k,j} \cdot x_j \right)^p + \sum_{j=1}^n b_{k,j}^p \cdot x_j \right) = K_p (4p \log r)^p \cdot g(x).$$

Thus we have $\mathbb{E}[C] = O \left( \frac{p^3}{\log p} \log d \cdot \log r \right)^p \cdot C^*$.

Observe that $\mathbb{E} \left[ \sum_{e \in U} D_e \right] = \sum_{e \in U} \Pr[e \text{ uncovered}] \cdot C_\pi \leq r^{-2p} \cdot \sum_{e \in U} C^* = r^{1-2p} \cdot C^*$. Using these bounds in (25), we have $\mathbb{E}[C] \leq 2^p \cdot \mathbb{E}[C] + (2r)^p \sum_{e \in U} \mathbb{E}[D_e] = O \left( \frac{p^2}{\log p} \log d \cdot \log r \right)^p \cdot C^*$. \hfill \Box

### 3.3 Capacity-constrained Facility Location (CCFL)

In the Capacity-constrained Facility Location (CCFL) problem, there are $m$ potential facility locations each with an opening cost $c_i$ and a capacity $u_i$ that are given up-front. There are $n$ clients which arrive online. Each client $j \in [n]$ has, for each facility $i \in [m]$, an assignment cost $a_{ij}$ and a demand/load $p_{ij}$. The online algorithm needs to open facilities (paying the opening costs) and assign each arriving client $j$ to some open facility $i$ (paying the assignment cost $a_{ij}$, and incurring a load $p_{ij}$ on $i$). The makespan of an assignment is the maximum load on any facility. The objective in CCFL is to minimize the sum of opening costs, assignment costs and the makespan. An integer
programming formulation for this problem is the following:

$$\begin{align*}
\min & \sum_{i=1}^{m} c_i x_i + \sum_{i,j} a_{ij} y_{ij} + \max_{j=1}^{n} \sum_{i=1}^{m} p_{ij} \cdot y_{ij} \\
\text{s.t.} & \sum_{i \in S} x_i + \sum_{i \not\in S} y_{ij} \geq 1, \quad \forall j \in [n], \forall S \subseteq [m] \\
y, x & \in \{0, 1\}.
\end{align*}$$

In order to apply our framework to CCFL, we allow the variables to be fractional, and use the following objective function with $p = \Theta(\log m)$.

$$f(x, y) = \left( \sum_{i=1}^{m} c_i x_i \right)^p + \left( \sum_{i,j} a_{ij} y_{ij} \right)^p + \sum_{i=1}^{m} \left( \sum_{j=1}^{n} p_{ij} \cdot y_{ij} \right)^p.$$  

Note that $f(x, y)^{1/p}$ is within a constant factor of the original objective. We refer to the above convex program as the fractional CCFL problem.

**Theorem 3.4.** There is an $O(\log^2 m)$-competitive online algorithm for fractional CCFL.

**Proof.** To apply our framework to solving fractional CCFL, we need a few observations. Firstly, although the function $f$ is not fully known in advance, we know at any time the parts of $f$ that correspond to variables appearing in constraints revealed until then. It is easy to check that this suffices for our framework to apply.

Another issue is that there are an exponential number of covering constraints. This does not affect the $O(p \log d) = O(\log^2 m)$ competitive ratio we obtain through Theorem 3.1 since it is independent of the number of covering constraints. However, the running time will be exponential in the straightforward implementation. In order to obtain a polynomial running time, we relax the covering constraints to $\frac{1}{2}$ (instead of one). Upon arrival of client $j$, we add covering constraints based on the following procedure.

While there is some $S \subseteq [m]$ with $\left( \sum_{i \in S} x_i + \sum_{i \not\in S} y_{ij} < \frac{1}{2} \right)$, do:

1. Add constraint $\sum_{i \in S} x_i + \sum_{i \not\in S} y_{ij} \geq 1$, and
2. Update solution $(x, y)$ according to the algorithm of Theorem 3.1.

Note that given a current solution $(x, y)$, the set $S$ that minimizes $\sum_{i \in S} x_i + \sum_{i \not\in S} y_{ij}$ is $S = \{ i \in [m] \mid x_i < y_{ij} \}$; comparing this to $1/2$ gives us the desired separation oracle. The number of iterations of the new procedure (per client arrival) is at most $4m$, because $\sum_{i=1}^{m} (\min\{x_i, 1\} + \min\{y_{ij}, 1\})$ increases by at least $\frac{1}{2}$ in each iteration, and this sum is always between 0 and $2m$. Hence, at any time $(2x, 2y)$ is a feasible fractional solution, which satisfies all constraints.

**3.3.1 Rounding the Fractional Solution Online**

The online fractional solution can be rounded in an online fashion to obtain a randomized $O(\log^2 m \cdot \log mn)$-competitive algorithm. While this is worse by a log $m$ factor than the result in [ABFP13], it follows directly from our general algorithm.
We use a “guess and double” approach in the rounding. Let $M$ denote some upper bound on the optimal offline value. Upon arrival of a new client, our algorithm will succeed if $M$ is a correct upper bound. If the algorithm fails then $M$ is doubled and we repeat the updates. We start with $M$ being some known lower bound. A phase is a sequence of client arrivals for which $M$ remains the same. At any point in the algorithm, the only allowed facilities are $\{i \in [m] \mid c_i \leq M\}$ and the only allowed assignments are $\{(i, j) \mid i \in [m], j \in [n], p_{ij} \leq M\}$. We denote by $I_M$ the restricted instance which consists only of the clients that arrive in this phase and the above facilities and allowed assignments. When we progress from one phase to the next (i.e. $M$ is doubled), we reset all the $x, y$ variables to zero.

Define a modified objective as follows:
\[
g(x, y) = \left(\sum_i c_i \left(x_i + \frac{\sum_j p_{ij} \cdot y_{ij}}{M}\right)\right)^p + \left(\sum_{i,j} a_{ij} y_{ij}\right)^p + \sum_i \left(\sum_j p_{ij} \cdot y_{ij}\right)^p.
\]

Note that this depends on the guess $M$ and is fixed for a single phase. Below we focus on the restricted instance $I_M$. Unless specified otherwise, clients $j$ and facilities $i$ are only from $I_M$.

Consider the following convex program:
\[
\begin{align*}
\min \quad & g(x, y) \\
\text{s.t.} \quad & \sum_{i \in S} x_i + \sum_{i \notin S} y_{ij} \geq 1, \quad \forall j \in [n], S \subseteq [m] \\
& y, x \geq 0.
\end{align*}
\]

When a new client $h$ arrives, the algorithm first updates the fractional solution to ensure the covering-constraints of client $h$ up to a factor 2, as in Theorem 3.4. Now we have to do the rounding. To do this, first define the following modified variables:
\[
\overline{y}_{ij} = \min\{y_{ij}, x_i\}, \quad \forall i, j, \text{ and} \\
\overline{x}_i = \max\left\{x_i, \frac{\sum_j p_{ij} \cdot y_{ij}}{M}\right\}, \quad \forall i.
\]

By construction, the variables $(\overline{x}, \overline{y})$ clearly satisfy:
\[
\begin{align*}
\sum_j p_{ij} \cdot \overline{y}_{ij} &\leq M \cdot \overline{x}_i \quad \forall i \\
\sum_{i=1}^m \overline{y}_{ij} &\geq \frac{1}{2} \quad \forall j \\
\overline{y}_{ij} &\leq \overline{x}_i \quad \forall i, j
\end{align*}
\]

**Claim 3.5.** Suppose there exists an integral solution to the current CCFL instance having cost at most $M$. Then the following inequalities hold, where $\alpha = O(\log^2 m)$ is the competitive ratio in Theorem 3.4:
\[
\sum_i c_i \cdot \overline{x}_i \leq 4\alpha \cdot M
\]
\[
\sum_{i,j} a_{ij} \cdot \overline{y}_{ij} \leq 4\alpha \cdot M \quad (31)
\]
\[
\sum_j p_{ij} \cdot \overline{y}_{ij} \leq 4\alpha \cdot M \quad \forall i. \quad (32)
\]

**Proof.** Since the optimal integral value of the current CCFL instance is at most \(M\), the optimal CCFL value of the restricted instance \(I_M\) is also at most \(M\). That is, there is an integral assignment with opening cost \(\leq M\), assignment cost \(\leq M\), and maximum load \(\leq M\). So the optimal fractional value of program (26) is at most \((2^M) + M^p + m \cdot M^p \leq m(3M)^p\). Since the fractional algorithm in Theorem 3.4 is \(\alpha\)-competitive, we have:

\[
g(x, y) \leq \alpha^p \cdot m(3M)^p \leq (4\alpha M)^p,
\]

since \(m \leq (4/3)^p\) for \(p \geq \log_{4/3} m\). This implies:

\[
\sum_i c_i \cdot x_i \leq \sum_i c_i \left( x_i + \frac{\sum_j p_{ij} \cdot y_{ij}}{M} \right) \leq g(x, y)^{1/p}
\]
\[
\sum_{i,j} a_{ij} \cdot \overline{y}_{ij} \leq \sum_{i,j} a_{ij} y_{ij} \leq g(x, y)^{1/p}
\]
\[
\sum_j p_{ij} \cdot \overline{y}_{ij} \leq \sum_j p_{ij} \cdot y_{ij} \leq g(x, y)^{1/p}
\]

and \(g(x, y)^{1/p} \leq 4\alpha M\) proves all three claims. \(\square\)

Hence, after the fractional updates, we check whether the conditions in (30)-(32) are satisfied; if not, we end the phase and double \(M\) (knowing by Claim 3.5 that \(M\) is a lower bound on the CCFL instance so far), and start the next phase with the new client \(h\) and the new value of \(M\). So assume that after fractionally assigning \(h\), all the inequalities (27)-(32) hold for the current value \(M\). Now we perform randomized rounding as follows.

- For each \(i\), set \(X_i\) to 1 with probability \(\min\{4 \log(mn) \cdot \overline{x}_i, 1\}\). Let \(F_f = \{i : \overline{x}_i \geq \frac{1}{4 \log(mn)}\}\) denote the set of fixed facilities for which \(\Pr[X_i = 1] = 1\).

- For each \(i, j\), define \(Z_{ij}\) as follows:

\[
\Pr[Z_{ij} = 1] = \begin{cases} 
\min\{4 \log mn \cdot \overline{y}_{ij}, 1\} & \text{if } i \in F_f, \\
\frac{\overline{y}_{ij}}{4 \log mn} & \text{otherwise}.
\end{cases}
\]

All the above random variables are independent. Each client \(j\) is assigned to some facility \(i\) with \(X_i \cdot Z_{ij} = 1\); if there are multiple possible assignments, the algorithm breaks ties arbitrarily. (For the sake of analysis, we may imagine that the client is assigned to all facilities such that \(X_i \cdot Z_{ij} = 1\).) If client \(j\) is unassigned, we open the facility corresponding to \(\min_{i=1}^m (c_i + a_{ij} + p_{ij})\) and assign \(j\) to it (note that this minimum value is at most \(M\)); we will show that this event happens with low probability, so the effect on the objective will be small. We now analyze this rounding.
Claim 3.6. For any client \( j \), \( \Pr[\text{\textit{j not assigned}}] = \Pr[\sum_i X_i \cdot Z_{ij} = 0] < 1/n^2 \).

Proof. If \( i \in F_f \), then \( E[X_i Z_{ij}] = E[Z_{ij}] = \min\{4 \log mn \cdot \overline{y}_{ij}, 1\} \). Else, \( E[X_i Z_{ij}] = 4 \log mn \cdot \overline{y}_{ij} \geq 4 \log n \cdot \overline{y}_{ij} \). In either case,

\[
\Pr[\text{\textit{j not assigned}}] = \Pr[\sum_i X_i Z_{ij} = 0] = \prod_i (1 - E[X_i Z_{ij}]) \leq \exp\left(-4 \log n \sum_i \overline{y}_{ij}\right) < 1/n^2,
\]

where the last inequality is by (28). \( \square \)

Claim 3.7. For any facility \( i \in F_f \), we have \( \Pr[\text{\textit{load}} > 32 \alpha \log mn \cdot M] \leq 1/m^2 \).

Proof. For facility \( i \in F_f \), the load assigned to it is \( \sum_j p_{ij} \cdot Z_{ij} \). This is a sum of independent \([0, M]\)-bounded random variables (by definition of the restricted instance \( I_M \)), with expectation at most \( 4 \log mn \sum_j p_{ij} \cdot \overline{y}_{ij} \), which by (32) is at most \( 16 \alpha \log mn \cdot M \). The claim now follows by a Chernoff bound. \( \square \)

Claim 3.8. For any facility \( i \notin F_f \), we have \( \Pr[\text{\textit{load}} > 4 \log mn \cdot M | X_i = 1] \leq 1/m^2 \).

Proof. Fix \( i \notin F_f \) and condition on \( X_i = 1 \). The load assigned to \( i \) is \( \sum_j p_{ij} \cdot (Z_{ij}|X_i = 1) \), which is a sum of independent \([0, M]\)-bounded random variables (again by definition of the restricted instance). The expectation is at most \( \sum_j p_{ij} \cdot \overline{y}_{ij} \leq M \), by (27). The claim again follows by a Chernoff bound. \( \square \)

Claim 3.9. \( \Pr[\text{\textit{opening cost}} > 32 \alpha \log mn \cdot M] < 1/n^2 \).

Proof. The opening cost is \( \sum_i c_i \cdot X_i \) which is a sum of independent \([0, M]\)-bounded random variables, whose expectation is at most \( 4 \log mn \sum_i c_i \cdot \overline{x}_i \leq 16 \alpha \log mn \cdot M \) by (30). The claim now follows by a Chernoff bound. \( \square \)

Claim 3.10. \( E[\text{\textit{assignment cost}}] \leq 16 \alpha \log mn \cdot M \).

Proof. The assignment cost is \( \sum_i \sum_j a_{ij} \cdot X_i Z_{ij} \) which has mean at most \( 4 \log mn \sum_j a_{ij} \overline{y}_{ij} \leq 16 \alpha \log mn \cdot M \) by (31). \( \square \)

Combining the above claims, and using the fact that each element is uncovered with probability less than \( 1/m^2 \), we get:

Lemma 3.11. The expected sum of opening and assignment costs and makespan is \( O(\alpha \log mn) \cdot M \).

A standard doubling argument accounts for all the phases as follows. Let \( M^* \) denote the final value of the parameter \( M \) achieved by the algorithm. By Claim 3.5 we have \( OPT > M^*/2 \). On the other hand, the expected cost in any phase corresponding to \( M \) is at most \( O(\alpha \log mn) \cdot M \) by Lemma 3.11. This gives a geometric sum with total cost at most \( O(\alpha \log mn) \cdot (M^*/M + M^*/2 + \cdots) \leq O(\alpha \log mn) \cdot OPT \). This proves the following theorem.

Theorem 3.12. There is a randomized \( O(\log^2 m \log mn) \)-competitive ratio for CCFL.

Remark 3.13. We can use randomized rounding with alteration, as in [GNI14], to obtain a more nuanced \( O(\log^2 m \cdot \log mf) \)-competitive ratio, where \( f \leq n \) is the “machine degree” i.e. \( \max_{i \in [m]} |\{j : p_{ij} < \infty\}| \). We omit the details.
3.4 Capacitated Multicast Problem

We consider the online multicast problem \([\text{AAA}^+06]\) in the presence of capacities, which we call the Capacitated Multicast (CMC) problem. In this problem, there are \(m\) edge-disjoint rooted trees \(T_1, \ldots, T_m\) corresponding to multicast trees in some network. Each tree \(T_i\) has a capacity \(u_i\) which is the maximum load that can be assigned to it. Each edge \(e \in \bigcup_{i=1}^m T_i\) has an opening cost \(c_e\). A sequence of \(n\) clients arrive online, and each must be assigned to one of these trees. Each client \(j\) has a tree-dependent load of \(p_{ij}\) for tree \(T_i\), and is connected to vertex \(\pi_{ij}\) in tree \(T_i\). Thus, if client \(j\) is assigned to tree \(T_i\) then the load of \(T_i\) increases by \(p_{ij}\), and all edges on the path in \(T_i\) from \(\pi_{ij}\) to its root must be opened. The objective is to minimize the total cost of opening the edges, subject to the capacity constraints that the total load on tree \(T_i\) is at most \(u_i\).

The capacitated multicast problem generalizes the CCFL problem. Indeed, let each machine \(i \in [m]\) correspond to a two-level tree \(T_i\) with capacity \(u_i\), where tree \(T_i\) has a single edge \(r_i\) incident to the root, and \(n\) leaves corresponding to the clients. Edge \(r_i\) has opening cost \(c_i\), and the leaf edge corresponding to client \(j\) has opening cost \(a_{ij}\). The load of client \(j\) in tree \(T_i\) is \(p_{ij}\). It is easy to check that a feasible solution to this CMC problem instance corresponds precisely to a CCFL solution with precisely the same cost.

In this section, we generalize the solution from the previous section to give the following result:

**Theorem 3.14.** There is a randomized online algorithm that given any instance of the capacitated multicast problem on \(d\)-level trees, and a bound \(C\) on its optimal cost, computes a solution of cost \(O(\log^2 m \cdot \log mn) \cdot C\) with congestion \(O((d + \log^2 m) \cdot \log mn)\).

The congestion of a solution is the maximum (over all facilities) of the multiplicative factor by which the capacity is violated.

The proof of this theorem will occupy the rest of this section. The main idea is similar: we solve a convex programming relaxation of this problem in an online fashion, and show how to round the solution online as well. However, these will require some ideas over and above those used in the previous section.

First, the convex relaxation. It will be convenient to augment each tree \(T_i\) as follows. For each client \(j\) with \(p_{ij} \leq u_i\) (i.e., that can be feasibly assigned to \(T_i\)), we introduce a new leaf vertex \(v_{ij}\) connected to vertex \(\pi_{ij} \in T_i\) via an edge of zero cost. These new leaf vertices \(v_{ij}\) are assigned a vertex weight \(p_{v_{ij}} := p_{ij}\), whereas all the original vertices of the trees are given zero weight. To minimize extra notation, we refer to these augmented trees also as \(T_i\). Finally we merge the roots of these trees \(T_i\) into a single root vertex \(r\) to get a new tree \(T = (V, E)\). For client \(j\), let \(V_j = \{v_{ij} \mid i \in [m] \text{ s.t. } p_{ij} \leq u_i\}\) denote the leaves in \(T\) corresponding to client \(j\).

For any edge \(e \in E\), denote the subtree of \(T\) below edge \(e\) by \(T^e\). Observe that if \(e\) was in \(T_i\) then \(T^e\) is a subtree of the \(i^\text{th}\) tree \(T_i\). In this case, we use the notation \(j \in T^e\) to denote that \(v_{ij} \in T^e\). For each vertex \(v \in V \setminus \{r\}\), its parent in \(T\) is denoted \(\tau(v)\).

Our fractional relaxation has a variable \(x_e\) for each edge \(e \in E\). For brevity, we use \(y_{ij} := x_{(v_{ij}, \tau(v_{ij}))}\) to denote the variable for the edge connecting the leaf-node corresponding to client \(j\) in tree \(T_i\) to its parent. The \(x_e\) variables naturally denote the “opening” of edges, and the \(y_{ij}\) variables denote...
the assignment of clients to trees. The objective is the following convex function:

\[ g(x) = \left( \sum_{i=1}^{m} \sum_{e \in T_i} c_e \left( x_e + \frac{2}{u_i} \sum_{v \in T_e} p_{v, \tau(v)} \right) \right)^p + \sum_{i=1}^{m} \left( \frac{2C}{u_i} \cdot \sum_{v \in T_i} p_{v, \tau(v)} \right)^p. \]  

(33)

In the above expression, we choose \( p = \Theta(\log m) \). Using the facts that the weights \( p_v \) are defined only for the new leaf nodes, and that leaf edges are denoted by the \( y_{ij} \) variables, we can write the above expression equivalently as follows:

\[ g(x) = g(x, y) = \left( \sum_{i=1}^{m} \sum_{e \in T_i} c_e \left( x_e + \frac{2}{u_i} \sum_{j \in T_e} p_{ij} \cdot y_{ij} \right) \right)^p + \sum_{i=1}^{m} \left( \frac{2C}{u_i} \cdot \sum_{j \in T_e} p_{ij} \cdot y_{ij} \right)^p. \]  

(34)

We will solve the following convex covering program:

\[
\begin{align*}
\min & \quad g(x) \\
\text{s.t.} & \quad \sum_{e \in \delta(S)} x_e \geq 1, \quad \forall \ V_j \subseteq S \subseteq V \setminus \{r\}, \quad \forall j \in [n] \\
& \quad x \geq 0.
\end{align*}
\]

The constraints say that the min-cut between the root and the nodes in the set \( V_j \), which contains all the nodes corresponding to client \( j \) in the various trees, is at least 1—i.e., \( j \) is (fractionally) connected at least to unit extent. Much as in Section 3.3, we deal with the exponential number of covering constraints as follows: we relax the covering constraints to \( \frac{1}{2} \) (instead of one). Upon arrival of client \( j \), we add covering constraints based on the following procedure.

While there is some \( V_j \subseteq S \subseteq V(G) \setminus \{r\} \) with \( \left( \sum_{e \in \delta(S)} x_e < \frac{1}{2} \right) \), do:

Add constraint \( \sum_{e \in \delta(S)} x_e \geq 1 \), and update \((x, y)\) according to Theorem 3.1.

Note that given a current solution \( x \), one can find such a “violated constraint” (if there is one) by a minimum-cut subroutine, which takes polynomial time. The number of iterations of the above procedure is at most \( 2|E| \), because \( \sum_{e \in E} \min\{x_e, 1\} \) increases by at least \( \frac{1}{2} \) in each iteration, but it starts at 0 and stays at most \( |E| \). Moreover, twice the solution is always a feasible solution, which implies an \( O(\log^2 m) \)-competitive online algorithm for the fractional problem.

3.4.1 Rounding the Solution Online

For the online rounding, define some modified variables. For each client \( j \in [n] \), compute a unit-flow \( F_j \) from the set \( V_j \) to the root \( r \) in the tree \( T \) with edge-capacities \( 2x \); note that the fractional solution guarantees this flow exists. Let \( f_e^j \) be the amount of flow on edge \( e \) in \( F_j \), and define \( f_e := \max_j f_e^j \). Note that the \( f_e \) values are monotone non-decreasing as we go up the tree \( T \). Now set:

\[ \pi_e = \max \left\{ f_e, \frac{2}{u_i} \sum_{j \in T_e} p_{ij} \cdot y_{ij} \right\}, \quad \forall e \in T_i, \quad \forall i \in [m]. \]  

(35)
Also define \( \overline{y}_{ij} = \overline{\pi}_{(v_{ij}, \tau(v_{ij}))} \) for any client \( j \) and tree \( T_i \), to capture the assignment of clients of trees.

**Claim 3.15.** The variables \( \overline{x}_e \) are monotone non-decreasing up the tree \( T \).

*Proof.* The flow values \( f_e \) are monotone non-decreasing up the tree. Also, for any tree \( T_i \) and any edge \( e \in T_i \), the quantity \( f'_e := \frac{2}{u_i} \sum_{j \in T_e} p_{ij} \cdot y_{ij} \) is also monotone non-decreasing up the tree, since it is the sum of non-negative quantities over larger subtrees. Since tree \( T \) is obtained by merging the trees \( T_i \) at the root, the monotonicity of \( \overline{x}_e = \max \{ f_e, f'_e \} \) maintained. \( \square \)

Note that \((\overline{x}, \overline{y})\) clearly satisfies:

\[
\sum_{j \in T_e} p_{ij} \cdot \overline{y}_{ij} \leq \sum_{v \in T_e} p_v \cdot \overline{\pi}_{v, \tau(v)} \leq u_i \cdot \overline{\pi}_e \quad \forall e \in T, \forall i \in [m]. \tag{36}
\]

\[
\overline{\pi}_e \leq \overline{\pi}_{\tau(e)} \quad \forall e \in T. \tag{37}
\]

\[
\sum_{i=1}^m \overline{y}_{ij} \geq 1 \quad \forall j \in [n]. \tag{38}
\]

**Claim 3.16.** Assuming there exists an integral solution to the CMC problem instance having cost at most \( C \), the following inequalities hold with \( \alpha = O(\log^2 m) \):

\[
\sum_{e \in T} c_e \cdot \overline{\pi}_e \leq 4\alpha \cdot C \tag{40}
\]

\[
\sum_{j \in T_i} p_{ij} \cdot \overline{y}_{ij} = \sum_{v \in T_i} p_v \cdot \overline{\pi}_{v, \tau(v)} \leq 4\alpha \cdot u_i, \quad \forall i \in [m]. \tag{41}
\]

*Proof.* The optimal integral solution of the current CMC problem instance has cost most \( C \), hence the optimal fractional value of our convex covering problem is at most \( (3C)^p + m \cdot (2C)^p \leq (m + 1)(3C)^p \), and our \( \alpha \)-competitive algorithm ensures that \( g(x, y) \leq \alpha^p \cdot (m + 1)(3C)^p \leq (4\alpha C)^p \) for \( p \geq \log_{d/3}(m + 1) = \Theta(\log m) \). This, in turn, implies that

\[
\sum_{e \in T} c_e \cdot \overline{\pi}_e \leq \sum_{e \in T} c_e \left( \overline{x}_e + \frac{2}{u_i} \sum_{j \in T_e} p_{ij} \cdot y_{ij} \right) \leq g(x, y)^{1/p} \tag{42}
\]

\[
\frac{C}{u_i} \sum_{j \in T_i} p_{ij} \cdot \overline{y}_{ij} \leq \frac{2C}{u_i} \sum_{j \in T_i} p_{ij} \cdot y_{ij} \leq g(x, y)^{1/p} \tag{43}
\]

which proves the claim. \( \square \)

Having defined these convenient modified variables, the rounding proceeds as follows. For each tree \( T_i \), the edges \( F_i = \{ e \in T_i \mid \overline{x}_e \geq 1 \} \) form a rooted subtree, by the monotonicity of the \( \overline{x} \) values. We include the edges in \( F_i \) in the solution deterministically. For the rest of the edges, we perform the following experiment \( \beta := \Theta(d \cdot \log mn) \) times independently, and take the union of the edges picked.

For each tree \( T_i \), independently:
(i) For each edge \( e \in T \setminus F \), pick it independently with probability \( \frac{\tau(e)}{\tau(p(e))} \), where we use \( \tau(e) \) to denote the parent edge of \( e \). An edge \( e \) whose parent edge does not lie in \( T \setminus F \) is chosen with probability \( x_e \).

(ii) If the load for tree \( T \) exceeds \( 8(d + 4\alpha) \cdot u \), declare failure for all clients assigned to \( T \).

The rounding in step (i) is from Garg et al. [GKR00] and hence is often called the GKR-rounding; it can be implemented online using ideas from [AAA+06]. Note that there is some probability that for some client \( j \) we may declare failure for all \( \beta \) experiments. In that case we can choose the path in that tree \( T \) for client \( j \) which is cheapest subject to \( p_{ij} \leq u_i \).

A client \( j \) is assigned in tree \( T \) if all edges on the path from \( v_{ij} \) to \( r \) are picked in \( T \) during Step 1, and if we don’t declare failure in Step 2; a client is assigned if it is assigned in at least one tree.

We first show that there is a good probability of any client being assigned in one run of the random experiment above.

**Claim 3.17.** For any client \( j \), \( \Pr[j \text{ assigned in one run}] \geq \frac{1}{2} \).

**Proof.** It is easy to check that for any tree \( T \), \( \Pr[j \text{ assigned to } T \text{ in Step 1}] = \min\{y_{ij}, 1\} \). Since the random choices in different trees \( T \) are independent,

\[
\Pr[j \text{ not assigned to any tree in step 1}] = \prod_{i=1}^{m} (1 - \min\{y_{ij}, 1\}) \leq e^{-\sum_{i=1}^{m} y_{ij}} \leq e^{-d \cdot u_i} \leq \frac{1}{e}.
\]

Next, we claim that conditioned on \( j \) being assigned in tree \( T \) in Step 1 (i.e., on all edges on the path \( P_{ij} \) from the root of \( T \) to \( v_{ij} \) being chosen in the solution), the conditional probability it is rejected in Step 2 is at most \( \frac{1}{8} \), i.e.,

\[
\Pr[j \text{ rejected in step 2} \mid j \text{ assigned to } T \text{ in step 1}] \leq \frac{1}{8}.
\]  

This would imply that \( j \) is assigned in at least one tree with probability \( (1 - 1/e) \cdot \frac{1}{8} \), and survives rejection in that tree with probability \( 7/8 \), giving \( (1 - \frac{1}{e}) \cdot \frac{7}{8} \geq \frac{1}{2} \).

To prove (44), let edges \( e_1, \ldots, e_k \) be the edges of \( T \setminus F \) on path \( P_{ij} \) at increasing distance from the root; hence \( e_k = (\tau(v_{ij}), v_{ij}) \). For \( h = 1, \ldots, k \), define subtree \( S_h := T_{e_h} \setminus T_{e_{h+1}} \) which consists of all nodes whose path to the root first intersects with \( P_{ij} \) at the edge \( e_h \). By the properties of the GKR rounding, we have in Step 1:

\[
E[\text{load from } S_h \mid j \text{ assigned to } T \text{ in step 1}] = \sum_{t \in S_h} p_{it} \cdot \frac{\bar{y}_{it}}{x_{eh}} \leq \frac{1}{x_{eh}} \cdot \sum_{t \in T_{e_h}} p_{it} \cdot \bar{y}_{it} \leq \frac{u_i}{x_{eh}} \cdot x_{eh} \leq u_i.
\]

Summing the expression above for all \( h = 1, \ldots, k \),

\[
E[\text{load from } \cup_{h=1}^{k} S_h \mid j \text{ assigned to } T \text{ in step 1}] \leq k \cdot u_i \leq d \cdot u_i,
\]

since the tree has depth at most \( d \).
This bounds the expected load of those clients whose paths to the root share an edge with \( P_{ij} \) in tree \( T_i \). For any other client \( \ell \), the conditioning does not matter, and hence

\[
\Pr[\ell \text{ assigned to } T_i \mid j \text{ assigned to } T_i \text{ in step 1}] = \Pr[\ell \text{ assigned to } T_i] = \overline{y}_{\ell i}.
\]

So using (41),

\[
E[\text{load from } [n] \setminus j \cup \bigcup_{h=1}^{k} S_h \mid j \text{ assigned to } T_i \text{ in step 1}] \leq \sum_{\ell \in T_i} p_{i\ell} \cdot \overline{y}_{i\ell} \leq 4\alpha \cdot u_i
\]

Thus the total expected load from \([n] \setminus j\) conditioned on \( j \) being assigned to \( T_i \) in Step 1 is at most \((d + 4\alpha) \cdot u_i\). Markov’s inequality now implies (44), and hence the claim.

By Step 2 of the algorithm, we immediately have:

**Claim 3.18.** The load assigned to tree \( T_i \) is at most \( 8\beta(d + 4\alpha) \cdot u_i \), for each \( i \in [m] \).

**Claim 3.19.** The expected opening cost is at most \( 4\alpha \beta \cdot C \).

**Proof.** The expected cost of edges chosen in each of the \( \beta \) independent trials is at most \( \sum_e c_e \cdot \overline{x}_e \leq 4\alpha C \) using (40). Summing the cost over all trials gives the claim.

**Claim 3.20.** For any client \( j \), the probability that \( j \) is unassigned is at most \( \frac{1}{mn^2} \).

**Proof.** By Claim 3.17, the probability of \( j \) being unassigned in one trial is at most \( \frac{1}{2} \). Since there are \( \beta = \Theta(\log mn) \) independent trials, the claim follows.

**Proof of Theorem 3.14:** By Claims 3.18 and 3.19, we know that the cost and load of the solution is at most the claimed bounds. Moreover, we know that the probability of the client not being assigned to any of the trees is at most \( \frac{1}{mn^2} \). Since this will increase the load of some tree \( i \) by at most \( u_i \) and the cost by at most \( OPT \), and happens with probability at most \( \frac{1}{mn} \), this increases the expected cost and congestion by a negligible factor.

### 3.5 Set Cover with Set Requests

We consider here the online set cover with set requests (SCSR) problem first considere by Bhawalkar et al. [BGP14], which is defined as follows. We are given a universe \( U \) of \( n \) resources, and a collection of \( m \) facilities, where each facility \( i \in [m] \) is specified by (i) a subset \( S_i \subseteq U \) of resources (ii) opening cost \( c_i \) and (iii) capacity \( u_i \). The resources and facilities are given up-front. Now, a sequence of \( k \) requests arrive over time. Each request \( j \in [k] \) requires some subset \( R_j \subseteq U \) of resources. The request has to be served by assigning it to some collection \( F_j \subseteq [m] \) of facilities whose sets collectively cover \( R_j \), i.e., \( R_j \subseteq \bigcup_{i \in F_j} S_i \). Note that these facilities have to be open, and we incur the cost of these facilities. Moreover, if a facility \( i \) is used to serve client \( j \), this contributes to the load of facility \( i \), and this total load must be at most the capacity \( u_i \).

As in previous sections, we give an algorithm to compute a solution online which violates the capacity constraint by some factor. Our main result for this problem is the following:
Theorem 3.21. There is a randomized online algorithm that given any instance of the set cover problem and a bound $C$ on its optimal cost, computes a solution of cost $O(\log^2 m \cdot \log mnk) \cdot C$ with congestion $O(\log^2 m \log mnk)$.

The ideas — for both the convex relaxation and the rounding — are very similar to that for CCFL; hence we only sketch the main ideas here. For the fractional relaxation, there is a variable $x_i$ for each facility $i \in [m]$ denoting if the facility is opened. For each request $j \in [k]$ and facility $i$ there is a variable $y_{ij}$ that denotes if request $j$ is connected to facility $i$. We set $p = \Theta(\log m)$ and the objective is:

$$g(x, y) = \left( \sum_i c_i \left( x_i + \frac{\sum_j y_{ij}}{u_i} \right) \right)^p + C^p \cdot \sum_i \left( x_i + \frac{1}{u_i} \sum_j y_{ij} \right)^p.$$

We define the following convex covering program, where we use $F(\ell) := \{ i \in [m] \mid \ell \in S_i \}$ for each resource $\ell \in U$.

$$\min g(x, y)$$

$$\text{s.t. } \sum_{i \in F} x_i + \sum_{i \in F(\ell) \setminus T} y_{ij} \geq 1, \quad \forall T \subseteq F(\ell), \forall \ell \in R_j, \forall j \in [k],$$

$$y, x \geq 0.$$

We can solve this convex program in an online fashion, much as in Theorem 3.1. Now for the rounding: we maintain the following modified variables:

$$\overline{y}_{ij} = \min \{ y_{ij}, x_i \}, \quad \forall i, j.$$

$$\overline{x}_i = \max \left\{ x_i, \sum_j y_{ij} \frac{1}{u_i} \right\}, \quad \forall i.$$

Note that $(\overline{x}, \overline{y})$ clearly satisfy:

$$\sum_j \overline{y}_{ij} \leq u_i \cdot \overline{x}_i \quad \forall i. \quad (45)$$

$$\sum_{i \in F(\ell)} \overline{y}_{ij} \geq \frac{1}{2} \quad \forall \ell \in R_j, \forall j \in [k]. \quad (46)$$

$$\overline{y}_{ij} \leq \overline{x}_i \quad \forall i, j. \quad (47)$$

Claim 3.22. Assuming that there is an integral solution to the SCSR instance having cost at most $C$, the following inequalities hold for $\alpha = O(\log^2 m)$:

$$\sum_i c_i \cdot \overline{x}_i \leq 4\alpha \cdot C. \quad (48)$$

$$\overline{x}_i \leq 4\alpha, \quad \forall i. \quad (49)$$
The proof is similar to Claim 3.5 and omitted.

The final randomized rounding is the same as for CCFL. For each facility \( i \), set \( X_i \) to one with probability \( \min \{ 4 \log (mnk) \cdot x_i, 1 \} \). Let \( \mathcal{F} \) denote the set of fixed facilities, i.e. \( \Pr[X_i = 1] = 1 \). So \( \mathcal{F} = \{ i : x_i > \frac{1}{4 \log (mnk)} \} \). For each request \( i \) and facility \( j \), set \( Z_{ij} \) to one with probability:

\[
\Pr[Z_{ij} = 1] = \begin{cases} 
\min \{ 4 \log (mnk) \cdot y_{ij}, 1 \} & \text{if } i \in \mathcal{F}, \\
\frac{y_{ij}}{x_i} & \text{otherwise.}
\end{cases}
\]

All the above random variables are independent. Each request \( i \) gets connected to all facilities \( j \) with \( X_i \cdot Z_{ij} = 1 \). The analysis of the rounding is also identical to that of CCFL, and is omitted. This completes the proof of Theorem 3.21.

4 Profit maximization with non-separable production costs

In this section we consider a profit maximization problem (called PMPC) for a single seller with production costs for items. There are \( m \) items that the seller can produce and sell. The production levels are given by a vector \( \mu \in \mathbb{R}^m_+ \); the total cost incurred by the seller to produce \( \mu_j \) units of every item \( j \in [m] \) is \( g(\mu) \) for some production cost function \( g : \mathbb{R}^m_+ \to \mathbb{R}_+ \). In this work we allow for functions \( g \) which are convex and monotone in a certain sense\(^\dagger\). There are \( n \) buyers who arrive online. Each buyer \( i \in [n] \) is interested in certain subsets of items (a.k.a. bundles) which belong to some set family \( S_i \subseteq 2^{[m]} \). The extent of interest of buyer \( i \) for subset \( S \in S_i \) is given by \( v_i(S) \), where \( v_i : S_i \to \mathbb{R}_+ \) is her valuation function.

If buyer \( i \) is allocated a subset \( T \in S_i \) of items, he pays the seller his valuation \( v_i(T) \). Consider the optimization problem for the seller: he must produce some items and allocate bundles to buyers so as to maximize the profit \( \sum_{i=1}^n v_i(T_i) - g(\mu) \), where \( T_i \in S_i \) denotes the bundle allocated to buyer \( i \) and \( \mu = \sum_{i=1}^n \chi_{T_i} \in \mathbb{R}^m \) is the total quantity of all items produced. (Here \( \chi_S \in \{0,1\}^m \) is the characteristic function of the set \( S \).) Observe that in this paper we consider a non-strategic setting, where the valuation of each buyer is known to the seller; this differs from an auction setting, where the seller has to allocate items to buyers without knowledge of the true valuation, and the buyers may have an incentive to mis-report their true valuations.

This class of maximization problems with production costs was introduced by Blum et al. \cite{BlumGMST11} and more recently studied by Huang and Kim \cite{HuangK14}. Both these works dealt with the online auction setting, but in both works they considered a special case where the production costs were separable over items; i.e, where \( g(\mu) = \sum_j g_j(\mu_j) \) for some convex functions \( g_j(\cdot) \). In contrast, we can handle general production costs \( g(\cdot) \), but we do not consider the auction setting. Our main result is for the fractional version of the problem where the allocation to each buyer \( i \) is allowed to be any point in the convex hull of the \( S_i \). In particular, we want to solve following convex program in an online fashion:

\[
\max \sum_{i=1}^n \sum_{T \in S_i} v_i(T) \cdot y_{iT} - g(\mu) \quad (D)
\]

\(^\dagger\)The formal conditions on \( g \) appear in Assumption 4.1.
\[
\sum_{T \in S_i} y_{iT} \leq 1 \quad \forall i \in [n], \tag{50}
\]
\[
\sum_{i=1}^n \sum_{T \in S_i} 1_{j \in T} \cdot y_{iT} - \mu_j \leq 0 \quad \forall j \in [m], \tag{51}
\]
\[
y, \mu \geq 0. \tag{52}
\]

Note that this problem looks like the dual of the covering problems we have been studying in previous sections, and hence is suggestively called \((D)\). Consider the following “dual” program that gives an upper bound on the value of \((D)\).

\[
\text{minimize} \quad \sum_{i=1}^n u_i + g^*(x) \tag{P}
\]
\[
u_i + \sum_{j \in T} x_j \geq v_i(T) \quad \forall i \in [n], \forall T \in S_i, \tag{53}
\]
\[
u_i, x \geq 0. \tag{54}
\]

Again, to be consistent with our general framework, we refer to this minimization (covering) problem as the “primal” \((P)\).

Notice that this primal-dual pair falls into the general framework of Section 2 if we set

\[
f(u, x) := \sum_{i=1}^n u_i + g^*(x). \tag{55}
\]

Indeed, if we were to construct the Fenchel dual of \((P)\) as in Section 2 we would again arrive at \((D)\) after some simplification (using the fact that \(g^{**} = g\) for any convex function \(g\) with subgradients\(^5\) [Roc70]). In order to apply now our framework, we assume that \(f\) is continuous, differentiable and satisfies \(\nabla f(z) \geq \nabla f(z')\) for all \(z \geq z'\). This translates to the following assumptions on the production function \(g\):

**Assumption 4.1.** Function \(g^* : \mathbb{R}^m_+ \to \mathbb{R}_+\) (recall \(g^*(x) = \sup_{\mu} \{x^T \mu - g(\mu)\}\)) is monotone, convex, continuous, differentiable and has \(\nabla g^*(x) \geq \nabla g^*(x')\) for all \(x \geq x'\).

Since we require irrevocable allocations, we cannot use the primal-dual algorithm from Section 2.1 since that algorithm could decrease the dual variables \(y_{iT}\). Instead, we use the algorithm from Section 2.2 which ensures both primal and dual variables are monotonically raised. We can now use the competitive ratio from (23)—when \(g^*(0) = 0\) this ratio is at least

\[
\max_{c > 0} \left\{ \min_z \left( \frac{\min_{i=1}^n \left\{ \nabla_i g^*(z) \right\}}{2 \ln(1 + pd)} - \max_z \frac{z^T \nabla g^*(z) - g^*(z)}{g^*(cz)} \right) \right\} \tag{55}
\]

In this expression, recall that \(d\) is the row-sparsity of the covering constraints in \((P)\), i.e. \(d = \)

\(^5\)A subgradient of \(g : \mathbb{R}^m \to \mathbb{R}\) at \(u\) is a vector \(V_u \in \mathbb{R}^m\) such that \(g(w) \geq g(u) + V_u^T (w - u)\) for all \(w \in \mathbb{R}^m\).
$1 + \max_{T \in \mathcal{S}_i} |T|$. And the term $\rho$ is the ratio between the maximum and minimum (non-zero) valuations any player $i$ has for any set in $\mathcal{S}_i$. In other words, 
\[
\rho \leq R := \frac{\max \{v_i(T) : T \in \mathcal{S}_i, i \in [n]\}}{\min \{v_i(T) : T \in \mathcal{S}_i, v_i(T) > 0, i \in [n]\}}. \tag{56}
\]

4.1 An Efficient Algorithm for (D)

To solve the primal-dual convex programs using our general framework in polynomial time, we need access to the following oracle:

**Oracle:** Given vectors $(u, x)$, and an index $i$, find a set $T \in \mathcal{S}_i$ such that
\[
u_i + \left( \sum_{j \in T} x_j - v_i(T) \right) < 0, \tag{57}
\]
or else report that no such set exists.

Given such an oracle, we maintain a $(u, x)$ such that $(2u, 2x)$ is feasible for (D) as follows. When a new buyer $i$ arrives, we use the oracle on $(2u, 2x)$. While it returns a set $T \in \mathcal{S}_i$, we update $(u, x)$ to satisfy the constraint (53). Else we know that $(2u, 2x)$ is a feasible solution for (D). This scaling by a factor of 2 allows us to bound the number of iterations as follows: when buyer $i$ arrives, define $Q_i = \min \{u_i, V_{i, \max}^i\} + \sum_{j=1}^{m} \min \{x_j, V_{i, \max}^j\}$ where $V_{i, \max}^i = \max \{v_i(T) | T \in \mathcal{S}_i\}$. Note that $Q_i \leq (m + 1)V_{i, \max}^i$ and $Q_i$ increases by at least $V_{i, \min}^i/2$ in each iteration where $V_{i, \min}^i = \min \{v_i(T) | T \in \mathcal{S}_i, v_i(T) > 0\}$. So the number of iterations is at most $O(mR)$ where $R$ is defined in (56). This gives us a polynomial-time online algorithm if $R$ is polynomially bounded.

What properties do we need from the collection $\mathcal{S}_i$ and valuation functions $v_i$ such that can we implement the oracle efficiently? Here are some cases when this is possible:

- **Small $\mathcal{S}_i$.** If each $|\mathcal{S}_i|$ is polynomially bounded then we can solve (57) just by enumeration. An example is when each buyer is “single-minded” i.e. she wants exactly one bundle.

- **Supermodular valuations.** Here, buyer $i$ has $\mathcal{S}_i = 2^{[m]}$ and $v_i : 2^{[m]} \to \mathbb{R}_+$ is supermodular, i.e. $v_i(T_1) + v_i(T_2) \leq v_i(T_1 \cup T_2) + v_i(T_1 \cap T_2)$ for all $T_1, T_2 \subseteq [m]$. In this case, we can solve (57) using polynomial-time algorithms for submodular minimization [Sch03], since the expression inside the minimum is a linear function minus a supermodular function.

- **Matroid constrained valuations.** In this setting, each buyer $i$ has some value $v_{ij}$ for each item $j \in [m]$ and the feasible bundles $\mathcal{S}_i$ are independent sets of some matroid. Here we can solve (57) by maximizing a linear function over a matroid. This is because the minimization
\[
\min_{T \in \mathcal{S}_i} \left( \sum_{j \in T} x_j - v_i(T) \right) = \min_{T \in \mathcal{S}_i} \sum_{j \in T} (x_j - v_{ij}) = -\max_{T \in \mathcal{S}_i} (v_{ij} - x_j),
\]

\footnote{An alternative description of such valuation functions is to have $\mathcal{S}_i = 2^{[m]}$ and $v_i'(T) =$ maximum weight independent subset of $T$ (where each item $j$ has weight $v_{ij}$). Viewed this way, the buyer’s valuation is a weighted matroid rank function which is a special submodular function.}
can be done in polynomial time [Sch03].

4.2 Online Rounding

We now have a deterministic online algorithm for (D) with competitive ratio as given in (55). Moreover, this algorithm runs in polynomial time for many special cases. Here we show how the fractional online solution can be rounded to give integral allocations. We make the following additional assumption on the production costs.

Assumption 4.2. There is a constant $\beta > 1$ such that $g(a\mu) \leq a^\beta \cdot g(\mu)$ for all $a \in [0, 1)$, $\mu \in \mathbb{R}_+^m$.

Theorem 4.3. For any $\epsilon \in (0, 1)$ there is a randomized online algorithm for PMPC under Assumptions 4.1 and 4.2 that achieves expected profit at least $\left(1 + \epsilon\right)^2 \cdot \frac{OPT}{m} - g \left(1 + \frac{\epsilon}{2} \cdot \ell \cdot \mathbf{1}\right)$, where $OPT$ is the offline optimal profit, $\alpha$ is the fractional competitive ratio and $L = O\left(\log m\right)$.

Note that the additive error term $g \left(1 + \frac{\epsilon}{2} \cdot \ell \cdot \mathbf{1}\right)$ is independent of the number $n$ of buyers: it depends only on the number $m$ of items and the production function $g$. We also give an example below which shows that any rounding algorithm for (D) must incur some such additive error.

We now describe the rounding algorithm. Let $\epsilon \in (0, 1)$ be any value; set $a = \left(1 + \epsilon\right)^2 \cdot \frac{OPT}{m} - g \left(1 + \frac{\epsilon}{2} \cdot \ell \cdot \mathbf{1}\right)$. The rounding algorithm scales the fractional allocation $y$ by factor $a < 1$ and performs randomized rounding. Let $M \in \mathbb{Z}_+^m$ denote the (integral) quantities of different items produced at any point in the online rounding. Upon arrival of buyer $i$, the algorithm does the following.

1. Update fractional solution $(y, \mu)$ according to the fractional online algorithm.
2. If $M_j > (1 + \epsilon) a \mu_j + \frac{\epsilon}{2} \log m$ for any $j \in [m]$ then skip.
3. Else, allocate set $T \in S_i$ to buyer $i$ with probability $a \cdot y_{iT}$.

Claim 4.4. $\Pr[M_j > (1 + \epsilon) a \mu_j + \frac{\epsilon}{2} \log m] \leq \frac{1}{m^2}$ for all items $j \in [m]$ and $\epsilon \in (0, 1)$.

Proof. Fix $j \in [m]$ and $\epsilon \in (0, 1)$. Note that $M_j$ is the sum of independent $0 - 1$ random variables with $\mathbb{E}[M_j] \leq a \cdot \mu_j$. The claim now follows by Chernoff bound.

Below $\ell := \frac{\epsilon}{2} \log m + 1$ and $L := (1 + \frac{\epsilon}{2}) \cdot \ell = O\left(\log m\right)$.

Lemma 4.5. The expected objective of the integral allocation is at least

$$a \left(1 - \frac{1}{m}\right) \cdot \sum_{i=1}^n \sum_{T \in S_i} v_i(T) \cdot y_{iT} - a \cdot g(\mu) - g(L \cdot \mathbf{1}).$$

Proof. Note that the algorithm ensures (in step 2 above) that $M \leq (1 + \epsilon) a \cdot \mu + \ell \cdot \mathbf{1}$. So with probability one, the production cost is at most:

$$g((1 + \epsilon) a \cdot \mu + \ell \cdot \mathbf{1}) = g\left(\frac{1}{1 + \epsilon} \cdot (1 + \epsilon)^2 a \cdot \mu + \frac{\epsilon}{1 + \epsilon} \cdot (1 + \frac{1}{\epsilon}) \ell \cdot \mathbf{1}\right) \leq g\left((1 + \epsilon)^2 a \cdot \mu\right) + g\left((1 + \frac{1}{\epsilon}) \ell \cdot \mathbf{1}\right).$$
\[ \leq \left( (1 + \epsilon)^2 a \right)^{\beta} \cdot g(\mu) + g(L \cdot 1) = a \cdot g(\mu) + g(L \cdot 1). \]

The first inequality is by convexity of \( g \); the second inequality uses Assumption 4.2 and \((1 + \epsilon)^2 a < 1\); the last equality is by definition of \( a \).

By Claim 4.4, the probability that we skip some buyer \( i \) is at most \( \frac{1}{m} \). Thus the expected total value is at least \((1 - \frac{1}{m}) a \cdot \sum_{i=1}^{n} \sum_{T \in S_i} v_i(T) y_{iT} \). Subtracting the upper bound on the cost from the expected value, we obtain the lemma. \( \square \)

This completes the proof of Theorem 4.3.

**Integrality Gap.** We note that the additive error term is necessary for any algorithm based on the convex relaxation \([D]\). Consider a single buyer with \( S_1 = 2^{[m]} \) and \( v_1(T) = |T| \). Let \( g(\mu) = \sum_{j=1}^{m} \mu_j^2 \). The optimal integral allocation clearly has profit zero. However the fractional optimum is \( \Omega(m) \) due to the feasible solution with \( y_{iT} = 2^{-m} \) for all \( T \subseteq [m] \) and \( \mu_j = \frac{1}{2} \) for all \( j \in [m] \). Thus any algorithm using this relaxation incurs an additive error depending on \( m \).

**4.3 Examples of Production Costs**

Here we give two examples of production costs (satisfying Assumptions 4.1 and 4.2) to which our results apply. In each case, we first show the competitive ratio obtained for the fractional convex program, and then use the rounding algorithm to obtain an integral solution.

**Example 1.** Consider a seller who can produce items in \( K \) different factories, where the \( k \)'th factory produces in one hour of work \( p_{kj} \) units of item \( j \). The production cost is the sum of \( q \)th powers of the work hours of the \( K \) factories (specifically, we get a linear production cost for \( q = 1 \) and the \( q \)th power of makespan when \( q \geq \log K \)). This corresponds to the following function:

\[
g(\mu) = \min \left\{ \frac{1}{q} \sum_{k=1}^{K} z_k^q : \sum_{k=1}^{K} p_{kj} \cdot z_k \geq \mu_j, \forall j \in [m], z \geq 0 \right\}, \quad (58)
\]

We scale the objective by \( 1/q \) to get a more convenient form. The dual function is:

\[
g^*(x) = \frac{1}{p} \sum_{k=1}^{K} \left( \sum_{j=1}^{m} p_{kj} \cdot x_j \right)^p, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.
\]

Applying our framework (as Assumption 4.1 is satisfied), as in Section 3.1, we obtain an \( \alpha = O(p \log \rho d)^p \)-competitive fractional online algorithm, where \( \rho = R \) the maximum-to-minimum ratio of valuations and row-sparsity \( d \leq m + 1 \). Combined with Theorem 4.3 (note that Assumption 4.2 is satisfied with \( \beta = q \)), setting \( \epsilon = \frac{1}{2} \), we obtain:

**Corollary 4.6.** There is a randomized online algorithm for PMPC with cost function \((58)\) for \( q > 1 \) that achieves expected profit at least \((1 - \frac{1}{m}) \frac{\text{OPT}}{O(\log m \cdot 1)} - g(O(\log m) \cdot 1)\).

Note that \( g(O(\log m) \cdot 1) \leq K \cdot O \left( \frac{\log m}{p_{\min}} \right)^q \) where \( p_{\min} > 0 \) is the minimum positive entry in \( p_{kj} \).
Example 2. This deals with the dual of the above production cost. Suppose there are $K$ different linear cost functions: for $k \in [K]$ the $k^{th}$ cost function is given by $(c_{k1}, \ldots, c_{km})$ where $c_{kj}$ is the cost per unit of item $j \in [m]$. The production cost $g$ is defined to be the (scaled) sum of $p^{th}$ powers of these $K$ different costs:

$$g(\mu) = \frac{1}{p} \sum_{k=1}^{K} \left( \sum_{j=1}^{m} c_{kj} \cdot \mu_j \right)^p.$$  

This has dual:

$$g^*(x) = \min \left\{ \frac{1}{q} \sum_{k=1}^{K} z_k^q : \sum_{k=1}^{K} c_{kj} \cdot z_k \geq x_j, \forall j \in [m], z \geq 0 \right\}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$  

The primal program $(P)$ after eliminating variables $\{x_j\}_{j=1}^m$ is given below with its dual:

minimize $\sum_{i=1}^{n} u_i + \frac{1}{q} \sum_{k=1}^{K} z_k^q$ \hspace{1cm} $(P')$  

maximize $\sum_{i=1}^{n} \sum_{T \in S_i} v_i(T) \cdot y_{iT} - \frac{1}{p} \sum_{k=1}^{K} \lambda_k^p$ \hspace{1cm} $(D')$

$$u_i + \sum_{j \in T} c_{kj} \cdot z_k \geq v_i(T), \quad \forall i \in [n], \forall T \in S_i,$$  

$$\sum_{T \in S_i} y_{iT} \leq 1, \quad \forall i \in [n],$$  

$$u, z \geq 0.$$  

Note that the row-sparsity in $(P')$ is $d = K + 1$ which is incomparable to $m$. We obtain a solution to $(P')$ by setting $x = Cz$ from any solution $(u, z)$ to $(P)$ where $C_{m \times K}$ has $k^{th}$ column $(c_{k1}, \ldots, c_{km})$. We can apply our algorithm to the convex covering problem $(P')$ as $g'(\lambda) = \frac{1}{p} \sum_{k=1}^{K} \lambda_k^p$ satisfies Assumption 4.1. This algorithm maintains monotone feasible solutions $(u, z)$ to $(P')$ and $(y, \lambda)$ to $(D')$. However, to solve $(D)$ online we need to maintain variables $(y, \mu)$ which is different from the variables $(y, \lambda)$ in $(D')$. We maintain $y$ in $(D)$ to be the same as that in $(D')$. We set the production quantities $\mu_j = \sum_{i=1}^{n} \sum_{T \in S_i} 1_{j \in T} \cdot y_{iT}$ so that all constraints in $(D)$ are satisfied. Note that the dual variables $y$ (allocations) and $\mu$ (production quantities) are monotone increasing- so this is a valid online algorithm. In order to bound the objective in $(D)$ we use the feasible solution $(y, \lambda)$ to $(D')$. Note that for all $k \in [K]$:

$$c_k^T \mu = \sum_{j=1}^{m} c_{kj} \cdot \mu_j = \sum_{j=1}^{m} c_{kj} \sum_{i=1}^{n} 1_{j \in T} \cdot y_{iT} = \sum_{i=1}^{n} \sum_{T \in S_i} y_{iT} \sum_{j \in T} c_{kj} \leq \lambda_k.$$  

So the objective of $(y, \mu)$ in $(D)$ is at least that of $(y, \lambda)$ in $(D')$. Our general framework then implies a competitive ratio for the fractional problem of $\alpha = O(q \log \rho \cdot d) = O(q \log \rho)^q$ where

$$\rho = R \cdot \frac{K \cdot \max \{ c_{kj} : k \in [K], j \in [m] \}}{\min \{ c_{kj} : k \in [K], j \in [m] \}}.$$  

31
Above $R$ is the maximum-to-minimum ratio of valuations, and recall $d \leq K + 1$.

Combined with Theorem 4.3 ($\varepsilon = \frac{1}{2}$), we obtain:

**Corollary 4.7.** There is a randomized online algorithm for PMPC with cost function (59) for $p > 1$ that achieves expected profit at least $(1 - \frac{1}{m})\frac{OPT}{O(q\log \rho)} - g(O(\log m) \cdot 1)$.

Here $g(O(\log m) \cdot 1) \leq K \cdot O(m \log m \cdot c_{\text{max}})^p$ where $c_{\text{max}}$ is the maximum entry in $c_{kj}$s.

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