A higgledy-piggledy set of planes
based on the ABB-representation of linear sets

Lins Denaux  
Ghent University

Jozefien D’haeseleer  
Ghent University

Geertrui Van de Voorde  
University of Canterbury

Abstract

In this paper, we investigate the Andr´e/Bruck-Bose representation of certain \( \mathbb{F}_q \)-linear sets contained in a line of \( \text{PG}(2,q^t) \). We show that scattered \( \mathbb{F}_q \)-linear sets of rank 3 in \( \text{PG}(1,q^3) \) correspond to particular hyperbolic quadrics and that \( \mathbb{F}_q \)-linear clubs in \( \text{PG}(1,q^t) \) are linked to subspaces of a certain 2-design based on normal rational curves; this design extends the notion of a circumscribed bundle of conics. Finally, we use these results to construct optimal higgledy-piggledy sets of planes in \( \text{PG}(5,q) \).

Keywords: Andr´e/Bruck-Bose representation, linear set, club, scattered linear set, normal rational curve, circumscribed bundle, higgledy-piggledy set
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1 Introduction

1.1 Motivation and overview

Linear sets are particular point sets in a finite projective space. They are of interest in finite geometry, and have been studied in recent years through their connections with other topics such as blocking sets, and their applications in coding theory (see e.g. [24, 21, 25]). Linear sets generalise the concept of a subgeometry as it has been shown that every linear set is either a subgeometry or the projection of a subgeometry [22].

The Andr´e/Bruck-Bose representation is a way to represent the projective plane over the field \( \mathbb{F}_{q^t} \) with \( q^t \) elements, as an incidence structure defined over the subfield \( \mathbb{F}_q \). It is a natural question to study the ABB-representation of certain ‘nice’ sets in the plane, and this has previously been done for sets such as sublines and subplanes [27], (sub)conics [26] and Hermitian unitals [6]. As such, one can ask the same question about the ABB-representation of \( \mathbb{F}_q \)-linear sets; we will give a partial answer in this paper.

We will see that the ABB-representation of a certain type of linear set gives rise to an interesting point set which can be described by using a subspace of a design of certain normal rational curves. This design is a generalisation of a well-known design based on the conics of a circumscribed bundle of conics [3].

After having introduced the necessary background and definitions in Section 1.2, we will show in Section 2 how to construct this design in a geometric way, and use coordinates to show that the obtained design is, in fact, isomorphic to the design of points and lines in a projective space. In Sections 3 and 4, we will turn our attention towards the ABB-representation of clubs of rank \( k \) in \( \text{PG}(1,q^t) \) (Theorem 3.8) and scattered linear sets of rank 3 in \( \text{PG}(1,q^3) \) (Theorem 4.6), both tangent to the line at infinity \( \ell_{\infty} \).

In Section 5, we first provide the necessary background on higgledy-piggledy sets, and then use the results of Sections 3 and 4 to show the existence and give explicit constructions of sets of seven planes in \( \text{PG}(5,q) \) in higgledy-piggledy arrangement. This answers an open question of [12]. It was this link which provided the incentive to consider the problem of determining the ABB-representation of linear sets in \( \text{PG}(1,q^3) \).
1.2 Preliminaries

The topics introduced in the following subsections are interrelated; for more information, we refer to [21], [27] and [9], respectively.

1.2.1 Field reduction and Desarguesian spreads

It is well-known that the vector space $V(r, q^t)$ is isomorphic to $V(rt, q)$; this isomorphism translates to a correspondence between the associated projective spaces $PG(r−1, q^t)$ and $PG(rt−1, q)$. Every point of $PG(r−1, q^t)$ corresponds to a $t$-dimensional vector space over $F_{q^t}$, which is a $t$-dimensional subspace of dimension $s$ and hence, corresponds to a $(t−1)$-dimensional subspace of $PG(rt−1, q)$. In this way, the point set of $PG(r−1, q^t)$ gives rise to a set $D$ of $(t−1)$-dimensional subspaces of $PG(rt−1, q)$ partitioning the point set of $PG(rt−1, q)$, that is, they form a $(t−1)$-spread of $PG(rt−1, q)$. Any spread isomorphic to $D$ is called a Desarguesian $(t−1)$-spread. Similarly, a $(k−1)$-dimensional subspace of $PG(r−1, q^t)$ corresponds to a $(kt−1)$-dimensional subspace of $PG(rt−1, q)$, spanned by elements of $D$. More formally, we can define the field reduction map $F_{q, r, t}$ which maps a $(k−1)$-dimensional subspace of $PG(r−1, q^t)$ to its associated $(kt−1)$-dimensional subspace of $PG(rt−1, q)$. We will omit the subscript of $F_{q, r, t}$ if the field size and dimensions are clear. If $S$ is a point set, we use $F(S)$ to denote the union of the images of the points in $S$ under $F$.

1.2.2 The André/Bruck-Bose representation

André [2] and Bruck and Bose [8] independently derived a representation of a projective plane of order $q^t$ in the projective space $PG(2t, q)$. We refer to this correspondence as the André/Bruck-Bose representation or the ABB-representation. Let $H_{\infty}$ be a hyperplane in $PG(2t, q)$ and let $D$ be a $(t−1)$-spread in $H_{\infty}$. Let $P$ be the set of affine points (i.e. those of $PG(2t, q)$, not contained in $H_{\infty}$), together with the $q^t + 1$ spread elements of $D$. Let $L$ be the set of $t$-spaces in $PG(2t, q)$ meeting $H_{\infty}$ in an element of $D$, together with the hyperplane at infinity $H_{\infty}$. The incidence structure $(P, L, I)$, with $I$ the natural incidence relation, is isomorphic to a projective plane of order $q^t$, which is called the André/Bruck-Bose plane corresponding to the spread $D$. The André/Bruck-Bose plane corresponding to a spread $D$ is Desarguesian if and only if the spread $D$ is Desarguesian. Now consider $PG(2, q^t)$ and let $\ell_{\infty}$ be a designated line at infinity. Let $H_{\infty} = F(\ell_{\infty})$ be a $(2t−1)$-dimensional subspace of $PG(3t−1, q) = F(PG(2, q^t))$. Fix a $2t$-space $\mu$ through $H_{\infty}$. It is not hard to see that the André/Bruck-Bose representation of an affine point $P$ of $PG(2, q^t)$ in $\mu \cong PG(2t, q)$ is the point $F(P) \cap \mu$. We let $\phi$ denote the André/Bruck-Bose map on affine points:

$$\phi(P) := F(P) \cap \mu.$$

The ABB-representation of a point $Q \in \ell_{\infty}$ is the $(t−1)$-space $F(Q)$.

1.2.3 Indicator spaces and Desarguesian subspreads

Finally, we recall the construction of a spread as introduced by Segre [28]. Embed $\Lambda \simeq PG(rt−1, q)$ as a subgeometry of $\Lambda^* \simeq PG(rt−1, q^t)$. The subgroup of $\text{PTL}(rt, q^t)$ fixing $\Lambda$ pointwise is isomorphic to $\text{Aut}(F_{q^t}/F_q)$. Consider a generator $g$ of this group. One can prove that that there exists an $(r−1)$-space $\nu$ skew to the subgeometry $\Lambda$ and that a subspace of $PG(rt−1, q^t)$ of dimension $s$ is fixed by $g$ if and only if it intersects the subgeometry $\Lambda$ in a subspace of dimension $s$ (see [9]). Let $P$ be a point of $\nu$ and let $L(P)$ denote the $(t−1)$-dimensional subspace generated by the conjugates of $P$, i.e., $L(P) = \langle P, P^g, \ldots, P^{g^{t−1}} \rangle$. Then $L(P)$ is fixed by $g$ and hence it intersects $PG(rt−1, q)$ in a $(t−1)$-dimensional subspace. Repeating this for every point of $\nu$, one obtains a set $D$ of $(t−1)$-spaces of the subgeometry $\Gamma$ forming a spread. This spread $D$ can be shown to be a Desarguesian spread and $\{\nu, \nu^g, \ldots, \nu^{g^{t−1}}\}$,
is called the indicator set of $\mathcal{D}$. An indicator set is also called a set of director spaces [28]. It is known from [9] Theorem 6.1] that for any Desarguesian $(t-1)$-spread of $\text{PG}(rt - 1, q)$ there exist a unique indicator set in $\text{PG}(rt - 1, q')$.

In this paper, we will make use of a particular coordinate system describing a subgeometry $\pi \simeq \text{PG}(t - 1, q)$ in $\text{PG}(t - 1, q')$, and for each $s|t$, we will define an $(s-1)$-spread denoted by $\mathcal{D}_s$ of $\pi$. In the case that $s = t$, this ‘spread’ of $\pi$ is the subspace $\pi$ itself. To describe the set-up, let $\sigma$ denote the collineation of $\text{PG}(t - 1, q')$ which maps a point with homogeneous coordinates $(x_0, x_1, x_2, \ldots, x_{t-1})$, $x_i \in \mathbb{F}_q'$, not all zero, onto the point with homogeneous coordinates $(x_0^q, x_1^q, x_2^q, \ldots, x_{t-2}^q)$. The fixed points of $\sigma$ then form a subgeometry $\pi \simeq \text{PG}(t - 1, q)$, consisting of all points with homogeneous coordinates $(x, x^q, x^{q^2}, \ldots, x^{q^{t-1}})$ for $x \in \mathbb{F}_q'$. Let $R$ denote the point with coordinates $(1, 0, \ldots, 0)$, then we see that $R^s = (0, 1, \ldots, 0)$, $R^{s^2} = (0, 0, 1, \ldots, 0)$, $R^{s^{t-1}} = (0, 0, \ldots, 1)$. Given $R$, every positive divisor $s$ of $t$ induces a unique Desarguesian $(s-1)$-spread $\mathcal{D}_s$ of $\pi$: consider $\Lambda_s = \text{Fix}(\sigma^s) \simeq \text{PG}(t - 1, q^s)$ and let $\Pi = \langle R, R^s, R^{s^2}, \ldots, R^{s^{t-1}} \rangle \cap \Lambda_s$. Then $\{\Pi, \Pi^s, \ldots, \Pi^{s^{t-1}}\}$ is a set of director spaces for $\mathcal{D}_s$ in $\text{PG}(t - 1, q)$.

We denote the extension of an element $D$ of $\mathcal{D}_s$ to $\text{PG}(t - 1, q')$ by $\overline{D}$.

For ease of notation in the case $s = t$, we define the ‘spread’ $\mathcal{D}_t$ to be equal to $\pi$ and the indicator set of $\pi$ to be the point set $\{R, R^s, \ldots, R^{s^{t-1}}\}$.

**Definition 1.1.** Let

$$P_x := \left( \frac{1}{x}, \frac{1}{x^q}, \frac{1}{x^{q^2}}, \ldots, \frac{1}{x^{q^{t-1}}} \right)$$

denote the point of $\pi \simeq \text{PG}(t - 1, q)$ corresponding to $\frac{1}{x} \in \mathbb{F}_q'$.

Note that $P_x = P_y$ if and only if $x/y \in \mathbb{F}_q$. Furthermore, it is easy to see that $P_x$ is contained in the element $D$ of $\mathcal{D}_s$ spanned by the points $X, X^s, \ldots, X^{s^{t-1}}$ where $X$ is stabilised by $\sigma^s$ and given by $X = \left( \frac{1}{x}, 0, \ldots, \frac{1}{x^{q^i}}, 0, \ldots, \frac{1}{x^{q^{t-1}}}, 0, \ldots, 0 \right)$. Geometrically, the point $X$ is the intersection point of $\overline{D}$ with $\Pi$, where the latter is the director space defining the spread $\mathcal{D}_s$. It now easily follows that two different points $P_x$ and $P_y$ lie in the same element of $\mathcal{D}_s$ if and only if $x/y \in \mathbb{F}_q$.

1.2.4 Arcs and normal rational curves

For any $m \in \mathbb{N}$ and $k \geq 1$, an $m$-arc of $\text{PG}(k, q)$ is a set of $m$ points in general position, i.e. every $k + 1$ points of this point set span $\text{PG}(k, q)$.

**Definition 1.2.** Let $1 \leq k \leq q$. A normal rational curve in $\text{PG}(k, q)$ is a $(q+1)$-arc projectively equivalent to the $(q+1)$-arc corresponding to the coordinates

$$\{(0, 0, \ldots, 0, 1) \cup \{(1, t, t^2, t^3, \ldots, t^k) : t \in \mathbb{F}_q \} \}.$$ 

A point set $\mathcal{C}$ of $\text{PG}(n, q)$ is a normal rational curve of degree $k$ if and only if it is a normal rational curve in a $k$-dimensional subspace of $\text{PG}(n, q)$. Note that a normal rational curve of degree 1 is a line, while one of degree 2 is a non-degenerate conic.

**Result 1.3** ([17] Theorem 1.18]). Consider a $(k + 2)$-arc $\mathcal{A}$ in $\text{PG}(k - 1, q)$, $k + 1 \leq q$, then there exists a unique normal rational curve of degree $k - 1$ through all points of $\mathcal{A}$.

**Result 1.4** ([18] Lemma 27.5.2(i)]). Let $\mathcal{C}$ be a normal rational curve of degree $k - 1$ in $\text{PG}(k - 1, q)$, and let $P \in \mathcal{C}$. The projection of $\mathcal{C} \setminus \{P\}$ from $P$ onto a $(k - 2)$-space disjoint from $P$ is a point set of size $q$ contained in a normal rational curve of degree $k - 2$. If $k + 1 \leq q$, then this normal rational curve is unique.
1.2.5 The ABB-representation of sublines and subplanes

The ABB-representation of $F_q$-sublines and tangent subplanes of $PG(2,q^t)$ was studied in [27]. In this paper, we will make use of the following cases tackled there:

Result 1.5 ([27]). (a) The affine points of an $F_q$-subline in $PG(2,q^t)$ tangent to $\ell_\infty$ correspond to the points of an affine line in the ABB-representation and vice versa.

(b) Suppose that $q \geq t$ and $k \mid t$. Let $m$ be an $F_q$-subline of $PG(2,q^t)$ external to $\ell_\infty$ where the smallest subline containing $m$ and tangent to $\ell_\infty$ is an $F_k$-subline. Then the ABB-representation of $m$ is a set of points $C$ in $PG(2t,q)$ such that

1. $C$ is a normal rational curve of degree $k$ contained in a $k$-space intersecting $H_\infty$ in an element of $D_k$.
2. its $F_q$-extension $C^*$ to $PG(2t,q^t)$ intersects the indicator set $\{\Pi, \Pi^\sigma, \ldots, \Pi^\sigma^{k-1}\}$ of $D_k$ in $k$ conjugate points.

and vice versa, any set $C$ with those properties gives rise to the point set of an $F_q$-subline, external to $\ell_\infty$.

1.2.6 Linear sets

For a more thorough introduction to linear sets, we refer to [21, 24]. In this paper, we will only be concerned with linear sets on a projective line, and we will use the geometrical point of view on linear sets using Desarguesian spreads. Let $D$ be the Desarguesian spread in $PG(2t-1,q)$ obtained as the image of the field reduction map on points of $PG(1,q^t)$. Then a set $S$ in $PG(1,q^t)$ is an $F_q$-linear set of rank $k$ if and only if there is a $(k-1)$-dimensional subspace $\pi$ of $PG(2t-1,q)$ such that

$$\mathcal{F}(S) = \mathcal{B}(\pi),$$

where $\mathcal{B}(\pi)$ is the set of elements of $D$ meeting $\pi$ in at least a point.

Definition 1.6. We denote the $F_q$-linear set $S$ such that $\mathcal{F}(S) = \mathcal{B}(\pi)$ by $L_\pi$.

The weight of a point $P$ in $L_\pi$ is $w + 1$ if $w$ is the dimension of $\mathcal{F}(P) \cap \pi$. Note that the weight of a point in a linear set is only well-defined if we specify the subspace $\pi$ defining $L_\pi$.

In this article, we focus on scattered $F_q$-linear sets in $PG(1,q^3)$ and clubs in $PG(1,q^t)$. A scattered linear set of rank $k$ in $PG(1,q^t)$ is an $F_q$-linear set of rank $k$ consisting of $\frac{q^{k-1}}{q-1}$ points. We see that all the points of a scattered linear set have weight one. If $L_\pi$ is a scattered linear set, then the subspace $\pi$ is called scattered (with respect to the Desarguesian spread $D$). A $t$-club of rank $k$ is an $F_q$-linear set $L_\pi$ such that there is one point of weight $t$ and all other points have weight one; if $t = k - 1$, this set is simply called a club. The point of weight $t$ is called the head of the club. As for the weight of the points in the linear set, we see that the head of the club is only well-defined with respect to the subspace $\pi$.

We have the following result about the possible intersection of an $F_q$-linear set and an $F_q$-subline.

Result 1.7 ([20, Theorem 8]). An $F_q$-subline intersects an $F_q$-linear set of rank $k$ of $PG(1,q^t)$ in at most $k$ or precisely $q + 1$ points.

The following results on clubs and scattered linear sets on a projective line reveal some useful geometric properties. Note that the authors of [20] did not include the necessary condition that $q \geq 3$.

Result 1.8 ([20, Corollary 13 and 15, 29, Theorem 3.7.4]). Suppose that $q \geq 3$. 

(a) If \( S \) is a club of \( \text{PG}(1, q^t) \), \( S \not\cong \text{PG}(1, q^2) \), then through two distinct non-head points of \( S \), there exists exactly one \( \mathbb{F}_q \)-subline contained in \( S \), which necessarily contains the head of the club.

(b) If \( S \) is a scattered linear set of rank 3 of \( \text{PG}(1, q^3) \), then through two distinct points of \( S \), there are exactly two \( \mathbb{F}_q \)-sublines contained in \( S \).

(c) Let \( q \geq 5 \). Consider a scattered plane \( \pi \) with respect to the Desarguesian 2-spread \( \mathcal{D} \) in \( \text{PG}(5, q) \) and let \( r \in \pi \). Then there is exactly one plane \( \pi' \neq \pi \) through \( r \) such that \( \mathcal{B}(\pi) = \mathcal{B}(\pi') \).

2 Generalising the circumscribed bundle of conics

In order to characterise the ABB-representation of clubs, tangent to \( \ell_\infty \), we will introduce a block design \( \mathcal{H} \) embedded in \( \text{PG}(t - 1, q) \), where blocks are certain normal rational curves. In the particular case when \( t = 3 \), this design is known as the design arising from a circumscribed bundle of conics. In [3], the authors describe three types of projective bundles, which they define to be a collection of \( q^2 + q + 1 \) conics mutually intersecting in exactly one point. The circumscribed bundles are bundles in the classical algebraic sense: given three conics in the bundle defined by equations \( f = 0 \), \( g = 0 \), \( h = 0 \) where \( h \) is not an \( \mathbb{F}_q \)-linear combination of \( f \) and \( g \), every conic in the bundle is defined by \( \lambda f + \mu g + \nu h = 0 \) for some \( \lambda, \mu, \nu \in \mathbb{F}_q \).

We see that the design \( (\mathcal{P}, \mathcal{B}) \) where points \( \mathcal{P} \) are the points of \( \text{PG}(2, q) \), blocks \( \mathcal{B} \) are the conics of the projective bundle, and incidence is inherited, forms a projective plane. The circumscribed bundle consists of all conics in \( \text{PG}(2, q) \) whose extension to \( \text{PG}(2, q^3) \) contains three fixed conjugate points \( R, R', R'' \) spanning \( \text{PG}(2, q^3) \). It can be deduced from [20] that the projective plane constructed via the circumscribed bundle is the Desarguesian plane \( \text{PG}(2, q) \). The design here will be a natural generalisation of this construction; for \( t \) prime, its definition is straightforward but for \( t \) non-prime, extra care must be taken.

Let \( e_0, e_1, \ldots, e_{t-1} \) be the standard basis vectors of length \( t \) (with 1 in the \( (i + 1) \)-th position and zero elsewhere) and let \( (v) \) denote the projective point of \( \text{PG}(t - 1, q^t) \) with homogeneous coordinates given by \( v \).

Lemma 2.1. (Using the notations introduced in [1,2,3] Consider the points \( R^i = (e_i), i = 0, \ldots, t - 1 \), in \( \text{PG}(t - 1, q^t) \) and two points \( P_a \neq P_b \) in \( \pi \cong \text{PG}(t - 1, q) \). Let \( s \) be the smallest integer such that \( a/b \in \mathbb{F}_q^* \) and let \( D \) be the element of the Desarguesian \((s - 1)\)-spread \( \mathcal{D}_s \) containing \( P_a \) and \( P_b \). Then

1. there is a unique normal rational curve \( \mathcal{C}^a,b \) of degree \( s - 1 \) through \( P_a \) and \( P_b \), contained in \( \overline{D} \), and meeting the indicator spaces \( \{\Pi, \Pi', \ldots, \Pi^{s-1}\} \) in \( s \) conjugate points.

2. the points of \( \mathcal{C}^a,b \) are given by \( \{K^a,b_{u,v} : u, v \in \mathbb{F}_q^*\} \) where

\[
K^a,b_{u,v} := \left\langle \sum_{i=0}^{s-1} \prod_{j=0, j \neq i}^{s-1} (a^{q^i} u - b^{q^j} v) w_i \right\rangle;
\]

and the conjugate points are \( Q, Q'^1, \ldots, Q'^{s-1} \) where \( Q'^{s-1} = (w_i) \) with...
points of the form
and in

Note that \( K \) conjugate points

Proof. Recall that, given \( D \), the set of \( s \) conjugate points contained in both the indicator spaces and in \( D \) is fixed. As discussed in Section 2.3, it is easy to check that the coordinates corresponding to this set \( \{ Q, Q^*, ..., Q^{s-1} \} \) of conjugate points is given by the vectors in \( \Pi \). By Result 1.3 we know that there is a unique normal rational curve of degree \( s - 1 \) containing the \( s \) conjugate points and the points \( P_a \) and \( P_b \).

It is well-known (see e.g. [17, Example 1.17]) that \( C^{a,b} \) as given in the statement of the lemma defines a normal rational curve; the degree of this curve is \( d \) if the point set \( \{(a^i, b^j)|i = 0, ..., t - 1\} \) in \( PG(1, q^t) \) consists of \( d + 1 \) different points. Recall that \( s \) is the smallest integer such that \( a/b \in F_q \), and hence, \( s \) is the smallest integer for which \( (\frac{a}{b})^s = \frac{a}{b} \). This means that the point set \( \{(a^s, b^i)|i = 0, ..., t - 1\} \) consists of \( s \) different points, implying that the degree of \( C^{a,b} \) is indeed \( s - 1 \).

Now consider the point \( K_{0,1}^{a,b} = \langle (-1)^{s-1} \sum_{i=0}^{s-1} (\prod_{j=0, j\neq i}^{s-1} b^j) w_i \rangle \). By dividing by \( (-1)^{s-1} \prod_{j=0}^{s-1} b^j \), we find that this point has coordinates \( \langle \frac{1}{b}, \frac{1}{b^2}, ..., \frac{1}{b^{s-1}} \rangle \), and hence, is the point \( P_b \). Similarly, \( K_{1,0}^{a,b} \) is the point \( P_a \), and we see that \( C^{a,b} \) indeed passes through \( P_a \) and \( P_b \).

Note that \( K_{b^{i'}, a^{i'}}^{a,b} = \langle w_{i'} \rangle \), \( i' = 0, 1, ..., s - 1 \). In other words, \( C^{a,b} \) indeed contains the \( s \) conjugate points \( Q, Q^*, ..., Q^{s-1} \).

Finally, if \( u, v \in F_q \), and using that \( b/a \in F_q \), it can be checked that \( P_{u bv} = K_{u,v}^{a,b} \), and vice versa, if a point \( K_{u,v}^{a,b} \) lies in \( \Pi \), then it follows that \( u, v \in F_q \). This means that the \( q + 1 \) different points of the form \( P_{u bv} \), where \( u, v \in F_q \), are precisely those in \( C^{a,b} \cap \Pi \); the normal rational curve \( C^{a,b} \) meets \( \Pi \) in a normal rational curve of \( \Pi \).

\[ w_0 = a(0, 0, ..., 0, 1, 0, ..., 0) \]
\[ w_1 = a^q(0, 1, 0, ..., 0, 1, 0, ..., 0) \]
\[ \vdots \]
\[ w_{s-1} = a^{q^{s-1}}(0, 0, ..., 1) \]

3. \( C^{a,b} \) meets \( \Pi \) in \( q + 1 \) points, determined by the points \( P_{a bv} \) where \( u, v \in F_q \).

**Remark 2.2.** The fact that \( P_{a bv} \) defines a normal rational curve in the subgeometry \( \Pi \) as seen in Lemma 2.1 also follows by considering the cyclic model of \( PG(t - 1, q^t) \) (see e.g. [13]): it is well-known that the inverse of a line in this model is a normal rational curve. In Lemma 2.1 we have described the extension of this normal rational curve to \( PG(t - 1, q^t) \).

**Definition 2.3.** Consider a subgeometry \( \Pi \cong PG(t - 1, q^t) \) arising as the set of fixed points of a collineation \( \sigma \) of \( PG(t - 1, q^t) \), and let \( R \) be a point such that the points \( R, R^\sigma, R^{\sigma^2}, ..., R^{\sigma^{t-1}} \) span \( PG(t - 1, q^t) \). Consider the Desarguesian subspreads \( D_s \) for every \( 1 < s \leq t \), \( s \mid t \), as defined in Subsection 1.2.3. Let \( \mathcal{H} \) denote the following incidence structure:

- Points \( \mathcal{P} \) are the points of \( \Pi \);
- Let \( P \) and \( Q \) be two distinct points of \( \Pi \), and \( s \) be the smallest integer such that \( P, Q \) are contained in the same element of \( D_s \), say \( D \). Then the unique block through \( P \) and \( Q \) is the set of points of \( \Pi \) contained in the normal rational curve of degree \( s - 1 \) through \( P, Q \) and the intersection points of \( \overline{D} \) with the indicator spaces \( \Pi^0, \Pi^{\sigma}, ..., \Pi^{\sigma^{t-1}} \).
In the case \( t = 3 \), the above construction reproduces the design obtained from the circumscribed bundle of conics; we have \( q^2 + q + 1 \) points in \( \mathcal{H} \). Since \( t \) is prime, necessarily \( s = 3 \) for all pairs of points. Recall that a normal rational curve of degree 2 is a conic, and hence, the block through two points \( P \) and \( Q \) is simply the intersection of \( \text{PG}(2,q) \) with the unique conic through \( P, Q, R, R^a \) and \( R^{a^2} \). We see that indeed, these five points are in general position, and that the unique conic through these 5 points intersects \( \pi \) in a subconic.

In the following Lemma, we will use the axiom of Veblen-Young to deduce that the point-line incidence geometry \( \mathcal{H} \) is isomorphic to the point-line incidence geometry of a projective space, which is necessarily \( \text{PG}(t-1,q) \). Note that this approach does not reprove the case \( t = 3 \).

**Theorem 2.4.** Let \( t > 3 \). The incidence structure \( \mathcal{H} \) is a 2-(\( \theta_{t-1}, q + 1, 1 \)) design, isomorphic to the design of points and lines in \( \text{PG}(t-1,q) \).

**Proof.** The fact that \( \mathcal{H} \) determines a 2-(\( \theta_{t-1}, q + 1, 1 \)) design follows directly from Lemma 2.1 and the fact that there are \( \theta_{t-1} \) points in \( \text{PG}(t-1,q) \). In order to show that it is isomorphic to the design of points and lines in \( \text{PG}(t-1,q) \), we will verify that the Veblen-Young axiom holds in \( \mathcal{H} \). More precisely, we will show that if the block through two points \( A \) and \( B \) (denoted by \( AB \)) has a point in common with the block \( CD \), then the block \( AD \) has a point in common with the block \( BC \).

Let \( A = P_a, B = P_b, C = P_c \) and \( D = P_d \) be four different points of \( \pi \) and assume that there is a point \( P \) on \( AB \) and \( CD \). By Lemma 2.1, \( P = P_{au_0-bv_0} \) for some \( u_0, v_0 \in \mathbb{F}_q \). Similarly, \( P = P_{cu_1-dv_1} \) for some \( u_1, v_1 \in \mathbb{F}_q \). Since \( P = P_{au_0-bv_0} = P_{cu_1-dv_1} \), it follows that \( (au_0 - bv_0)/(cu_1 - dv_1) \in \mathbb{F}_q \), so there exists an element \( \lambda \in \mathbb{F}_q \) with

\[
au_0 - bv_0 = \lambda(cu_1 - dv_1),
\]

or equivalently,

\[
au_0 + \lambda dv_1 = bv_0 + \lambda cu_1.
\]

This implies that \( P_{au_0+\lambda dv_1} = P_{bv_0+\lambda cu_1} \). Since \( \lambda, u_0, v_0, u_1, v_1 \in \mathbb{F}_q \), the left hand side is a point of \( C^{a,d} \) in \( \pi \), and the right hand side is a point of \( C^{b,c} \) in \( \pi \). Hence, the blocks \( AD \) and \( BC \) have a point in common. \[Q.E.D.\]

It follows that \( \mathcal{H} \) admits subspaces, and that we can talk about the dimension of this subspace. To avoid confusing with subspaces of \( \text{PG}(n,q) \), we will denote subspaces of \( \mathcal{H} \) by \( \mathcal{H} \)-subspaces. These \( \mathcal{H} \)-subspaces will appear in the characterisation of the ABB-representation of a club, tangent to \( \ell_\infty \) and with head different from \( P_\infty \).

### 3 Tangent clubs of rank \( k \) in \( \text{PG}(1,q^t) \)

As in Subsection 1.2.2 we let \( \ell_\infty \) be the line of \( \text{PG}(2,q^t) \) such that the ABB-representation of \( \text{PG}(2,q^t) \) has \( H_\infty = F(\ell_\infty) \) as the hyperplane at infinity of \( \mu = \text{PG}(2t,q) \). In this section, we will consider the ABB-representation of a linear set contained in a line \( \ell \neq \ell_\infty \) of \( \text{PG}(2,q^t) \). We will denote \( P_\infty = \ell \cap \ell_\infty \) and the corresponding spread element by \( \pi_\infty = F(P_\infty) \). Let \( H \) be the \( t \)-space in \( \text{PG}(2t,q) \) through \( \pi_\infty \) containing all the points of \( \phi(\ell \setminus \{ P_\infty \}) \).

**Remark 3.1.** The different perspectives on linear sets lead to different possible approaches for studying their ABB-representation. The (affine part of) the ABB-representation of a linear set \( L_r \) on a projective line \( \text{PG}(1,q^t) \) can be seen as the intersection of the set \( \mathcal{B}(r) \) with a \( t \)-dimensional subspace containing a fixed spread element of \( \mathcal{D} \). Furthermore, since a linear set of rank 3 can be seen as the projection of a subplane, and the ABB-representation of tangent and secant subplanes is understood (see 27), in Theorem 1.10 we are looking to characterise the projection of certain normal rational scrolls. The two above approaches make it possible to give a description of the ABB-representation of a linear set; for example, the ABB-representation
of a scattered linear set of rank 3 tangent to the line at infinity is the projection of a normal rational scroll. However, we found these descriptions insufficient to be able to fully characterise the ABB-representation of the linear sets as done with the approach of our paper.

3.1 Counting clubs of PG(1, q^t)

In order to characterise the ABB-representation of clubs, we will count the number of different clubs with a fixed head. Note that we are not dealing with (in)-equivalence nor simplicity here; in general, clubs of rank t in PG(1, q^t) are equivalent but the same is not true for clubs of rank k < t (see e.g. [10] and [23]). Furthermore, in general, clubs are not necessarily simple: if B(π) = B(π′) is a club for two subspaces π and π′ sharing a point, then it is not true that necessarily π = π′, nor is the head of the club determined by the point set itself (this was already noted in [14]). However, if we specify the head of the club, we can show the following statement:

Lemma 3.2. Let L_π = L_{π′} be two clubs of rank k in PG(1, q^t) with head P (that is, π and π′ are (k-1)-dimensional spaces and π ∩ F(P) and π′ ∩ F(P) are (k-2)-dimensional). If there is a point r in π ∩ π′, and not in F(P), then π = π′. Hence, there are \( \frac{q^{k-2}}{q-1} \) subspaces π′ such that L_π = L_{π′} is a club with head P.

Proof. Let π and π′ be as in the statement of the lemma and assume that π ≠ π′. Then there exists a point s ∈ π, not in π′, nor in F(P); since B(π) = B(π′), it follows that B(s) intersects π′ in a point s′. The line through r and s meets F(P) in a point, as does the line through r and s′; hence, both define the unique \( \mathbb{F}_q \)-subline through \( F^{-1}(B(r)), F^{-1}(B(s)) \) and P in L_π. But there is a unique transversal line through r to the regulus defined by the elements B(r), B(s), F(P), a contradiction. Finally, it is well-known that the elementwise stabiliser of the Desarguesian spread D acts transitively on the points inside a spread element (see e.g. [21, Lemma 4.3]). Hence, for all \( \frac{q^{k-2}}{q-1} \) points u in B(r) we find a unique subspace π′′ through u with B(π′′) = B(π) and π′′ ∩ F(P) a (k-2)-dimensional space, so the statement follows.

3.2 Clubs with head P_∞

The characterisation of the ABB-representation of clubs with head P_∞ easily follows by using the different perspectives on linear sets.

Proposition 3.3. Suppose that q ≥ 3. A point set S of PG(1, q^t) is an \( \mathbb{F}_q \)-linear club of rank k with head P_∞ if and only if the ABB-representation of S \{P_∞\} is an affine (k-1)-space of \( \Pi \).

Proof. Let M be an affine point set contained in the line \( \ell \neq \ell_\infty \) of PG(2, q^t). Recall that the ABB-representation of M can be obtained from intersecting the image of M under the field reduction map with the subspace \( \mu \) of dimension 2t through \( H_\infty \), where H_\infty is the (2t-1)-dimensional space F(\ell_\infty). We denote the subspace F(\ell) ∩ \mu containing the ABB-representation of the affine points of \( \ell \) by M. The ABB-representation of M is the intersection of spread elements F(P), where P ∈ M, with \( \Pi \). We claim that if M is the affine point set of a club with head P_∞, the points of this intersection form a subspace and vice versa.

First note that if \( \nu \) is an affine (k-1)-space of \( \Pi \), and \( \bar{\nu} \) denotes its projective completion, trivially, B(\bar{\nu}) is the set of elements of the Desarguesian spread meeting a (k-1)-space and intersecting P_∞ in a (k-2)-space; that is, it defines a club of rank k with head P_∞.

Vice versa, suppose that M is the affine point set of a club with head P_∞ = \ell_\cap \ell_\infty. By definition, there is a (k-1)-dimensional subspace π contained in F(\ell) such that S = B(π), and furthermore, such that π meets H_\infty in a (k-2)-dimensional space. If π is a subspace of \( \Pi \), then we are done. Otherwise, let \( \nu \) be a point of \( \Pi \) lying in a spread element of B(π), different from F(P_∞) = π_∞, then by Lemma 3.2 there is a subspace π′ through \( \nu \) such B(π′) = B(π). Since π′ lies in \( \Pi \), we find that π′ is the intersection of B(π) with \( \Pi \) and the statement follows.
Let \( \binom{n}{k}_q \) denote the number of \((k-1)\)-dimensional subspaces of \( \PG(n-1,q) \), that is,

\[
\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}
\]

and let \( \theta_m \) be the number of points in \( \PG(m-1,q) \), that is,

\[
\theta_m = \frac{q^m - 1}{q - 1}.
\]

**Proposition 3.4.** There are \( q^{t-k+1} \binom{t}{k-1}_q \) clubs \( L_\pi \) of rank \( k \) with head \( P_\infty \).

**Proof.** There are \( \binom{t}{k-1}_q \) subspaces of dimension \( k-2 \) in \( \pi_\infty = \mathcal{F}(P_\infty) \), and each of them lies on \( \frac{a^{2t-k+1} - q^{t-k+1}}{q-1} \) subspaces of dimension \( k-1 \), not contained in \( \pi_\infty \). By Lemma 3.2, there are \( \theta_{t-1} \) of such \((k-1)\)-spaces \( \pi \) giving rise to the same club. Hence, we find that there are

\[
q^{t-k+1} \binom{t}{k-1}_q \theta_{t-1} = q^{t-k+1} \binom{t}{k-1}_q
\]

clubs with head \( P_\infty \).

\( \square \)

### 3.3 Clubs with head different from \( P_\infty \)

**Proposition 3.5.** Let \( H \) and \( P_\infty \) be two different points of \( \PG(1,q^t) \). Then there exist \( \binom{t}{k-1}_q \) clubs \( L_\pi \) through \( P_\infty \) with head \( H \), where \( \pi \) is a \((k-1)\)-space.

Furthermore, there are \( q^t \binom{t}{k-1}_q \) clubs \( L_\pi \), where \( \pi \) is a \((k-1)\)-space, containing \( P_\infty \), with head different from \( P_\infty \).

**Proof.** Let \( \gamma = \mathcal{F}(H) \). A \((k-2)\)-space \( g \) in \( \gamma \) and a point \( P \) in \( \pi_\infty \) span a \((k-1)\)-space \( \langle g, P \rangle \) which defines a club with head \( H \) and containing \( P_\infty \). By Lemma 3.2, every club with head \( H \) and containing \( P_\infty \) is defined by exactly \( \theta_{t-1} \) such \((k-1)\)-spaces, so the total number of clubs through a fixed head point \( H \neq P_\infty \) and containing \( P_\infty \) is

\[
\frac{\binom{t}{k-1}_q \theta_{t-1}}{\theta_{t-1}}.
\]

There are \( q^t \) choices for a point \( H \neq P_\infty \), and each subspace \( \pi \) defines a unique \( H \), so there are \( q^t \binom{t}{k-1}_q \) clubs \( L_\pi \), where \( \pi \) is a \((k-1)\)-space and the head is different from \( P_\infty \).

\( \square \)

**Proposition 3.6.** There exists \( q^t \binom{t}{k-1}_q \) cones in \( \Pi \) with vertex a point \( H \notin \pi_\infty \) and base a \((k-2)\)-dimensional subspace of the 2-design \( \mathcal{H} \).

**Proof.** From Theorem 2.4 it follows that the number of \((k-2)\)-dimensional subspaces of \( \mathcal{H} \) equals the number of \((k-2)\)-spaces in \( \PG(t-1,q) \), that is, \( \binom{t}{k-1}_q \). Furthermore, there are \( q^t \) points in \( \Pi \), not in \( \pi_\infty \), each of which defines a unique cone with vertex that point and base a \((k-2)\)-dimensional subspace of \( \mathcal{H} \).

\( \square \)

In order to characterise the ABB-representation of a club with head, different from the point at infinity, we need the following Lemma from [1].

**Lemma 3.7 ([1], Lemma 5.7).** Assume that \( S \) is a point set in \( \PG(n,q) \), \( q \geq 4 \), with the property that every line intersects \( S \) in \( 0,1 \) or \( q+1 \) points. Then there exists a hyperplane \( H \) in \( \PG(n,q) \) such that either \( S \subseteq H \) or \( S^c \subseteq H \), where \( S^c \) denotes the complement of \( S \) in \( \PG(n,q) \).
**Theorem 3.8.** A set \( S \) is an \( \mathbb{F}_q \)-linear club of rank \( k \) in \( \text{PG}(1, q^3) \) containing \( P_\infty \) and with head \( H \neq P_\infty \), if and only if \( \phi(S \setminus \{P_\infty\}) \), the ABB-representation of \( S \setminus \{P_\infty\} \) in \( \text{PG}(2, q) \), is the affine point set of a cone with vertex \( \phi(H) \) and base an \( \mathcal{H} \)-subspace of dimension \((k - 2)\) in \( \mathcal{F}(P_\infty) \) (the spread element corresponding to \( P_\infty \)).

**Proof.** Let \( S \) be an \( \mathbb{F}_q \)-linear club of rank \( k \) containing \( P_\infty \) and with head \( H \neq P_\infty \), and let \( \phi(H) \) be the ABB-representation of the head \( H \). Let \( Q \notin \{H, P_\infty\} \) be a point of \( S \). By Result (8.1), we know that the subline through \( H, Q, P_\infty \) is contained in \( S \). By Result (8.5), the ABB-representation of the points, different from \( P_\infty \), of this subline are the affine points of the line through \( \phi(H) \) and \( \phi(Q) \). In other words, the \( q^{-1} \) points of \( S \setminus \{H, P_\infty\} \) are contained in \( \frac{q^{k-1}-1}{q-1} \) lines through \( \phi(H) \), that is, they form a cone with vertex \( \phi(H) \). The projective completions of those lines meet \( \mathcal{F}(P_\infty) \) in a set \( \mathcal{K} \) of \( \frac{q^{k-1}-1}{q-1} \) points.

Let \( R_i, i = 1, 2 \), be two different points of \( \mathcal{K} \), and let \( Q_i \) be a point on the line through \( \phi(H) \) and \( R_i \), different from \( \phi(H) \) and \( R_i \). We have that \( Q_i = \phi(S_i) \) for some point \( S_i \in S \). Moreover, from Result (8.5), we know that the subline \( m \) through \( H, S_1, S_2 \) is contained in \( S \). Let \( s \) be the integer such that the smallest subline containing \( m \) and tangent to \( \ell_\infty \) is an \( \mathbb{F}_q \)-subline. Then by Result (8.6), we know that the affine points of this subline correspond to a normal rational curve \( C \) through \( \phi(H), Q_1, Q_2 \), contained in an \( s \)-space meeting \( \mathcal{F}(P_\infty) \) in an element \( D \) of \( \mathcal{D}_s \), whose \( \mathbb{F}_q \)-extension intersects the indicator set of \( \mathcal{D}_s \) in \( s \) conjugate points. Note that \( R_1, R_2 \) are contained in \( D \), and hence, \( D \) is the unique element of \( \mathcal{D}_s \) containing \( R_1, R_2 \).

By Result (14), the projection of the normal rational curve \( C \) from the point \( \phi(H) \in C \) onto \( H_\infty \) is contained in a normal rational curve; this curve is contained in \( \pi_\infty \), goes through \( R_1, R_2 \) and the extension contains the same points in \( H_\infty \) as \( C \) did. Hence, the block of the design \( \mathcal{H} \) through \( R_1, R_2 \) contains \( q \) points of \( \mathcal{K} \). It follows that \( \mathcal{K} \) is a point set meeting every block in \( 0, 1, q \) (or \( q + 1 \)) points. By Theorem (2.3), \( \mathcal{H} \) is isomorphic to the point-line design of \( \text{PG}(t - 1, q) \) so we may use Lemma (3.7) to conclude that \( \mathcal{K} \) or its complement must be contained in a hyperplane \( \mu \) of the design \( \mathcal{H} \). Since \( \frac{t - 1}{q - 1} - |\mathcal{K}| > \frac{q^{k-1}-1}{q-1} \), the latter possibility does not occur. We can repeat the same reasoning in the \((t - 2)\)-dimensional \( \mathcal{H} \)-subspace \( \mu \): all blocks of \( \mu \) meet \( \mathcal{K} \) in \( 0, 1, q \) or \( q + 1 \) points, and since \( \frac{t - 1}{q - 1} - |\mathcal{K}| > \frac{q^{k-2}-1}{q-1} \), \( \mathcal{K} \) is contained in a hyperplane of \( \mu \), that is, a \((t - 3)\)-dimensional \( \mathcal{H} \)-subspace. Continuing in this fashion, we conclude that \( \mathcal{K} \) is contained in a \((k - 2)\)-dimensional \( \mathcal{H} \)-subspace. Since \( |\mathcal{K}| = \frac{q^{k-1}-1}{q-1} \), equality holds.

Furthermore, by Propositions (3.6) and (3.5), the number of such cones equals the number of \( \mathbb{F}_q \)-linear club of rank \( k \) containing \( P_\infty \) and with head \( H \neq P_\infty \), and the theorem follows. \( \square \)

### 4 Tangent scattered linear sets of rank 3 in \( \text{PG}(1, q^3) \)

We continue to use the same notations as in the previous section, as introduced in Subsection 1.2.2.

**Proposition 4.1.** Suppose that \( q \geq 5 \). Let \( \mathcal{U} \) be a point set of \( \text{AG}(3, q) \) with the following three properties:

1. for each line \( \ell \) holds that \( |\ell \cap \mathcal{U}| \in \{0, 1, 2, q\} \),

2. through each point of \( \mathcal{U} \), there exist precisely two lines that are contained in \( \mathcal{U} \), and

3. \( |\mathcal{U}| = q^2 + q \).

Let \( \pi_\infty \) be the plane at infinity when embedding \( \text{AG}(3, q) \) in \( \text{PG}(3, q) \). Then \( \mathcal{U} \) is the affine part of a hyperbolic quadric in \( \text{PG}(3, q) \) that intersects \( \pi_\infty \) in a non-degenerate cone.

**Proof.** We claim that the intersection of a plane \( \sigma \) with \( \mathcal{U} \) is either a cap or the union of two distinct lines. First note that it impossible for \( \sigma \cap \mathcal{U} \) to contain two lines \( \ell_1, \ell_2 \) and a point
R ∈ U \ (ℓ_1 ∪ ℓ_2): in this case, since q ≥ 5, we find that there are at least 3 lines through R meeting ℓ_1 and ℓ_2 in distinct points, which forces those lines to be contained in U by Property 1., contradicting Property 2.

Suppose that σ ∩ U is not a cap, then there exists a line r in σ with at least three points of U. By Property 1., r is contained in U. By Property 2., there exists another line contained in U through each of the q points on r; let ℓ_1, . . . , ℓ_q denote those lines. They are necessarily pairwise disjoint since otherwise, we would find a plane with three lines of U. Hence let

\[ U \cap \{ℓ_1, . . . , ℓ_q\} \]

In this case, giving rise to \[ µ(ℓ_1, . . . , ℓ_q) \], \[ q \] 2q lines. Hence let \[ P \] be a point at infinity incident with a line ℓ_p ∈ µ(U). From (\textasteriskcentered), we have that there is precisely one line in µ(U), different from ℓ_p whose extension is P. Since there are \[ q^2 + q \] points in U, each on exactly 2 lines, we have that there are \( 2(q + 1) \) lines contained in U, giving rise to \( q + 1 \) points in \( π_{∞} \). Furthermore, it follows from the fact that there are no planes with more than 2 lines that there are no triangles in U. Hence, U is indeed a generalised quadrangle of order \( (q, 1) \) embedded in PG(3,q). Since it has \( q^2 + q \) affine points by Proposition 3, it meets \( π_{∞} \) in \( q + 1 \) points forming a non-degenerate conic.

**Lemma 4.2.** Suppose that \( q ≥ 5 \). If \( S ⊃ P_{∞} \) is a scattered linear set of rank 3 of PG(1,q^3), then the ABB-representation of \( S \setminus \{P_{∞}\} \) is the affine part of a hyperbolic quadric \( Q \) intersecting the plane \( π_{∞} \) in a non-degenerate conic. Furthermore, the extension of this conic contains the 3 conjugate points defining the spread element \( π_{∞} \).

**Proof.** Let \( S ⊃ P_{∞} \) be a point set of PG(1,q^3), which is a scattered linear set of rank 3 and let \( T \) be the ABB-representation of \( S \setminus \{P_{∞}\} \).

We see that the three conditions of Proposition \textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered hold for \( U = T \):

1. An affine line \( ℓ \in \Pi \) corresponds to a tangent subline of PG(1,q^3). Condition 1 follows from Result \textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered.

2. By Result \textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered we know that through every two distinct points \( P_1, P_2 \) of \( S \) there are precisely two \( F_q \)-sublines contained in \( S \). Let \( P_1 \) be the point at infinity \( P_{∞} \) and let \( P_2 \) be a random affine point in \( S \). Then we know that \( P_2 \) is contained in precisely two tangent \( F_q \)-sublines. Hence, we know by Result \textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered that \( ϕ(P_2) \) is contained in precisely two lines fully contained in \( T \).

3. The scattered linear set contains \( q^2 + q + 1 \) points, of which \( q^2 + q \) affine ones.

This implies that \( T \) is the affine point set of a hyperbolic quadric. Now consider \( Q \), the extension to \( F_{q'} \) of the projective completion of \( T \). By Proposition \textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered through two points of \( S \setminus \{P_{∞}\} \), there are two sublines contained in \( S \), at least one of which, say \( m \), does not contain \( P_{∞} \). By Result \textasteriskcentered\textasteriskcentered\textasteriskcentered\textasteriskcentered we know that the \( F_q \)-subline \( m \),
corresponds to a normal rational curve $C$ whose extension to $\mathbb{F}_{q^4}$ contains the 3 conjugate points defining the spread element $\pi_\infty$. Since $m \subseteq \mathcal{S}$, the extension of $C$ is contained in $\mathcal{Q}$, and hence, $\mathcal{Q}$ contains the 3 conjugate points defining $\pi_\infty$. \hfill \square

**Remark 4.3.** The first part of Lemma 4.3 can also be proven using the coordinate description of $\mathcal{B}(\pi)$, where $\pi$ is a scattered plane in $\text{PG}(5,q)$ with respect to the Desarguesian plane spread $\mathcal{D}$, derived in [19]. If we intersect the hypersurface, whose coordinates are explicitly described there, with a 3-dimensional subspace containing a spread element $S$ of $\mathcal{D}$, we find the union of a hyperbolic quadric with the points of $S$. To show that the extension of this hyperbolic quadric contains the 3 conjugate points, one could then use the coordinates for the indicator sets derived in [7].

**Proposition 4.4.** There exists $\frac{1}{2}q^3(q^3-1)$ hyperbolic quadrics $\mathcal{Q}$ in $\Pi$, intersecting the plane $\pi_\infty$ in a non-degenerate conic $C$ such that its $\mathbb{F}_{q^4}$-extension contains the 3 conjugate points generated by the spread element $\pi_\infty$.

**Proof.** We again use the fact that all non-degenerate conics in $\pi_\infty$, such that its extension contains three fixed conjugated points, together with all points in $\pi_\infty$ form a $2-(\theta_2, q+1, 1)$-design as shown in [3]. Hence, there are $\theta_2$ possibilities for choosing an appropriate conic in $\pi_\infty$. It is known that the total number of hyperbolic quadrics in $\Pi$ is $\frac{1}{2}q^4(q^2+1)(q^3-1)$, the number of non-degenerate conics contained in a fixed hyperbolic quadric is $\theta_3 - (q+1)^2 = q(q^3-1)$ and the number of non-degenerate conics in a solid is $\theta_3 q^2(q^3-1)$ [LS]. We can now perform a double counting to obtain that there exist

$$\frac{1}{2}q^4(q^2+1)(q^3-1)q(q^2-1) = \frac{1}{2}q^3(q-1)$$

hyperbolic quadrics containing a fixed non-degenerate conic. Hence, in total, there are $\frac{1}{2}q^3(q^3-1)$ hyperbolic quadrics $\mathcal{Q}$ in $\Pi$, intersecting the plane $\pi_\infty$ in a non-degenerate conic $C$ such that its $\mathbb{F}_{q^4}$-extension contains the 3 conjugate points generated by the spread element $\pi_\infty$. \hfill \square

**Proposition 4.5.** Let $q \geq 5$. There exists $\frac{1}{2}q^3(q^3-1)$ scattered linear sets of rank 3 in $\text{PG}(1,q^3)$ which contain $P_\infty$.

**Proof.** We will first count the number of scattered planes in $\text{PG}(5,q)$ with respect to the Desarguesian plane spread $\mathcal{D}$. There are $\left[\frac{3}{q}\right]$ planes in $\text{PG}(5,q)$, of which $q^3+1$ are elements of $\mathcal{D}$. Now consider triples $(S, L, \pi)$, where $S$ is an element of $\mathcal{D}$, $L$ is a line in $S$, and $\pi$ is a plane containing $L$, different from $S$. It easily follows that there are $(q^3+1)(q^2+q+1)(q^3+q^2+q)$ such triples, and since the choice of the plane $\pi$ defines $S$ and $L$ in a unique way, we find $(q^3+1)(q^2+q+1)(q^3+q^2+q)$ planes meeting some spread element in exactly a line. We conclude that there are $\left[\frac{3}{q}\right] - (q^3+1) - (q^3+1)(q^2+q+1)(q^3+q^2+q) = (q^3+1)q^2(q^3-1)$ scattered planes. Now count $(\pi, r, S)$ where $r$ is a point of the scattered plane $\pi$ such that $L_\pi$ is the scattered linear set $S$. On one hand, we have $(q^3+1)(q^3-1)$ scattered planes $\pi$ determining a unique linear set $S$, and $q^2+q+1$ points $r$. On the other hand, by Result [LS]c, we have that given $S$ and $r$, there are exactly 2 planes $\pi$ through $r$ with $L_\pi = S$. It follows that $|S|(q^2+q+1)^2 = (q^3+1)q^2(q^3-1)(q^2+q+1)$, and hence, $|S| = \frac{(q^3+1)q^2(q^3-1)}{2}$. The number of scattered linear sets through each of the $q^3+1$ points of $\text{PG}(1,q^3)$ is a constant, so there are $\frac{q^3(q^3-1)}{2}$ scattered linear sets through $P_\infty$. \hfill \square

**Theorem 4.6.** A set $S$ is the ABB-representation of the affine point set of a scattered linear set of rank 3 in $\text{PG}(1,q^3)$, containing $P_\infty$, if and only if it is the affine point set of a hyperbolic quadric intersecting the plane $\pi_\infty$ in a non-degenerate conic $C$ such that its $\mathbb{F}_{q^4}$-extension contains the 3 conjugate points generated by the spread element $\pi_\infty$. 

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Proof. Lemma 4.2 proves that the ABB-representation of the affine point set of a scattered linear set of rank 3 in $\mathrm{PG}(1,q^3)$, containing $P_\infty$ is a hyperbolic quadric intersecting the plane $\pi_\infty$ in a non-degenerate conic $C$ whose extension contains the 3 conjugate points generating the spreadelement $\pi_\infty$. For the other direction, it suffices to note that the number of such hyperbolic quadrics found in Proposition 4.4 is precisely the number of scattered linear sets containing $P_\infty$, counted in Proposition 4.5.

5 The optimal case of seven planes of $\mathrm{PG}(5,q)$ in higgledy-piggledy arrangement

In order to define higgledy-piggledy sets, we need the concept of a strong $k$-blocking set, which was introduced in [11, Definition 3.1]. They have also appeared in the literature under the terminology generator sets and cutting blocking sets.

Definition 5.1. Let $k \in \{0,1,\ldots,n-1\}$. A strong $k$-blocking set in $\mathrm{PG}(n,q)$ is a point set that meets every $(n-k)$-dimensional subspace $\kappa$ in a set of points spanning $\kappa$.

Definition 5.2. Let $k \in \{0,1,\ldots,n-1\}$ and suppose that $\mathcal{K}$ is a set of $k$-subspaces in $\mathrm{PG}(n,q)$. If the union of points contained in at least one subspace of $\mathcal{K}$ is a strong $k$-blocking set, then the elements of $\mathcal{K}$ are said to be in higgledy-piggledy arrangement and the set $\mathcal{K}$ itself is said to be a higgledy-piggledy set of $k$-subspaces.

The goal is to construct higgledy-piggledy sets of small size. The following particular cases follow from the known lower bounds (see [16], and [12] for a slight improvement):

Corollary 5.3. If $0 < k < n-1$ and $q \geq 7$, then a higgledy-piggledy set of $k$-subspaces

1. contains at least 4 elements if $n = 3$,
2. contains at least 6 elements if $n = 4$, and
3. contains at least 7 elements if $n = 5$.

The above lower bounds are sharp ([11] Theorem 3.7, Example 9), [5] Proposition 12), [4] Theorem 3.15), [12] Theorem 33 and 39, Corollary 34 and 35), except for the case $(n,k) = (5,2)$. Concerning the latter case, the author of [12] used the following construction to find 8 planes in higgledy-piggledy arrangement.

Corollary 5.4. Suppose that $\mathcal{P}$ is a point set of $\mathrm{PG}(1,q^3)$ that is not contained in any $\mathbb{F}_q$-linear set of rank at most 3. Then $\mathcal{F}(\mathcal{P})$ is a higgledy-piggledy set of pairwise disjoint planes in $\mathrm{PG}(5,q)$.

Proof. This is a special case of [12] Theorem 16].

Any higgledy-piggledy set of planes constructed in this way consists of disjoint planes; however, it is worth noting that this is not a restriction:

Proposition 5.5 ([12] Proposition 40]). If $q \geq 7$, then any seven planes of $\mathrm{PG}(5,q)$ in higgledy-piggledy arrangement are pairwise disjoint.

Using the results obtained in previous sections, we are able to show that the lower bound of Corollary 5.3 is sharp in the case $n = 5$:

Theorem 5.6. There exist seven planes of $\mathrm{PG}(5,q)$ in higgledy-piggledy arrangement.
Proof. If \( q \leq 5 \), we can easily verify the statement using a computer package such as GAP (see e.g. [12], Code Snippet 56). Hence, assume that \( q \geq 5 \) for the remainder of this proof. By Corollary 5.4, it is sufficient to pick 7 points in \( \mathrm{PG}(1, q^3) \) such that no linear set of rank at most 3 contains all these 7 points. First note that if 7 points are contained in a linear set of rank \(<3\), they are also contained in a linear set of rank 3. Hence, we only need to show that it is possible to pick 7 points, not contained in a linear set of rank 3.

Pick a point \( P_\infty \) in \( \mathrm{PG}(1, q^3) \). Then we know from Proposition 3.4 that there are \( q^3 + q^2 + q \) clubs with head \( P_\infty \), from Proposition 3.5 that there are \( q^3(q^2 + q + 1) \) clubs through \( P_\infty \) with head different from \( P_\infty \), and from Proposition 1.5 that there are \( \frac{q}{2} q^3(q^3 - 1) \) scattered linear sets containing \( P_\infty \).

We will count the set \( S = \{(P_1, P_2, P_3, P_4, P_5, P_6, L)\} \) where \( P_i \neq P_\infty \) are different points of \( \mathrm{PG}(1, q^3) \) and \( L \) is a linear set of rank 3 containing \( P_\infty \) and \( P_i, i = 1, \ldots, 6 \). We have that

\[
|S| = (q^3 + q^2 + q)c + q^3(q^2 + q + 1)c + \frac{1}{2} q^3(q^3 - 1)d,
\]

where \( c = q^3(q^2 - 1)(q^2 - 2)(q^2 - 3)(q^2 - 4)(q^2 - 5) \) is the number of ways to pick 6 different points different from \( P_\infty \) in a club through \( P_\infty \), and \( d = (q^2 + q)(q^2 + q - 1)(q^2 + q - 2)(q^2 + q - 3)(q^2 + q - 4)(q^2 + q - 5) \) is the number of ways to pick 6 points different from \( P_\infty \) in a scattered linear set through \( P_\infty \).

If all choices of 6 points \( P_1, \ldots, P_6 \) would be contained in at least one linear set of rank 3 through \( P_\infty \), then \( |S| \geq q^3(q^3 - 1)(q^3 - 2)(q^3 - 3)(q^3 - 4)(q^3 - 5) \), a contradiction for \( q \geq 3 \).

We will now use the results of this paper to explicitly construct a set of 7 planes in \( \mathrm{PG}(5, q) \) in higgledy-piggledy arrangement. We start by writing down explicit equations of the set of conics in \( \mathrm{PG}(2, q) \) containing 3 fixed conjugate points.

**Lemma 5.7.** Let \( \omega \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q \) be a generator of \( (\mathbb{F}_{q^3})^* \), satisfying \( \omega^3 + \lambda_1 \omega^2 + \lambda_2 \omega + \lambda_3 = 0 \). Then the conics in \( \mathrm{PG}(2, q) \) whose extension to \( \mathbb{F}_{q^3} \) contains the points \((1, \omega, \omega^2), (1, \omega^q, \omega^{2q}), (1, \omega^{q^2}, \omega^{2q^2})\) are given by

\[
g_{d,e,f}(X_0, X_1, X_2) := (\lambda_3 e - \lambda_1 \lambda_3 f)X_0^2 + (\lambda_2 e + (\lambda_3 - \lambda_1 \lambda_2) f)X_0X_1 + (\lambda_1 e + (\lambda_2 - \lambda_1^2) f - d)X_0X_2 + dX_1^2 + eX_1X_2 + fX_2^2 = 0,
\]

with \( d, e, f \in \mathbb{F}_q \) not all zero.

**Proof.** An arbitrary conic \( C \) in \( \mathrm{PG}(2, q) \) has equation \( aX_0^2 + bX_0X_2 + cX_0X_2 + dX_1^2 + eX_1X_2 + fX_2^2 = 0 \) where \( a, b, c, d, e, f \in \mathbb{F}_q \). Note that if \((1, \omega, \omega^2)\) lies on the extension of \( C \) to \( \mathrm{PG}(2, q^3) \), then \((1, \omega^q, \omega^{2q})\) and \((1, \omega^{q^2}, \omega^{2q^2})\) also lie on this extension. Expressing that \((1, \omega, \omega^2)\) lies on \( C \), using that \( \omega^4 = (\lambda_1^2 - \lambda_2^2)\omega^2 + (\lambda_1 \lambda_2 - \lambda_3)\omega + \lambda_1 \lambda_3 \), and that 1, \( \omega \), and 1, \( \omega \) are \( \mathbb{F}_q \)-independent, we find the following system of equations:

\[
a - \lambda_3 e + \lambda_1 \lambda_3 f = 0
\]
\[
b - \lambda_2 e + (\lambda_1 \lambda_2 - \lambda_3) f = 0
\]
\[
c - d - \lambda_1 e + (\lambda_1^2 - \lambda_2) f = 0.
\]

**Proposition 5.8.** Let \( P_i(x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, 1), i = 1, \ldots, 6 \) be six non-coplanar points contained in a non-degenerate elliptic quadric intersecting the plane \( \pi : X_3 = 0 \) in the conic \( X_0X_2 - X_1^2 = 0 \).

Consider the quadrics

\[
Q(d, e, f, u, v, w, t, X_0, X_1, X_2, X_3) := g_{d,e,f}(X_0, X_1, X_2) + X_3(uX_0 + vX_1 + wX_2 + tX_3) = 0.
\]

\footnote{In fact, using similar code, one can check that there exist in fact 6 planes of \( \mathrm{PG}(5, 3) \) and 5 planes of \( \mathrm{PG}(5, 2) \) in higgledy-piggledy arrangement.}
Let $A$ be the $(6 \times 7)$-matrix whose $i$-th row $(A)_i = [d, e, f, u, v, w, t]^T = Q(d, e, f, u, v, w, t, x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, 1)$.

If $rk(A) = 6$, then the points $P_1, \ldots, P_6$, together with $P_\infty$, are the ABB-representation of a set of seven points in $PG(1, q^3)$ such that, under field reduction, these seven points form a higgledy-piggledy set of $7$ planes in $PG(5, q)$. That is, $\{ F(\varphi^{-1}(P_i)) \mid 1 \leq i \leq 6 \} \cup F(P_\infty)$ is a set of seven planes in $PG(5, q)$ in higgledy-piggledy arrangement.

Proof. By Corollary 5.3, it is sufficient to construct a set of $7$ points in $PG(1, q^3)$ such that no linear set of rank at most $3$ contains all these $7$ points. Embed the line $L = PG(1, q^3)$ in $PG(2, q^3)$ and select one point $P_\infty$ on $L$. Let $\ell_\infty$ be a line of $PG(2, q^3)$ through $P_\infty$, different from $L$ and consider the ABB-representation of $PG(2, q^3)$ with $\ell_\infty$ as line at infinity. Then the set of points $F(P)$, with $P$ a point of $L$ different from $P_\infty$, defines a $3$-dimensional subspace $\Pi$. We coordinatise in such a way that the points in $\Pi$ have coordinates $(x_0, x_1, x_2, x_3)$ such that the points with $x_3 = 0$ are the points in the plane $\pi = F(P_\infty)$ and the three conjugate points defining $\pi$ are $(1, \omega, \omega^2), (1, \omega^2, \omega^3), (1, \omega^3, \omega^2)$. In view of Proposition 5.8, Theorem 5.5, and Theorem 4.6, we need to find six affine points of $\Pi$ such that these are not contained in a plane, nor a cone with vertex not in $\pi$ and base a conic whose extension contains the $3$ conjugate points, nor a hyperbolic quadric through such a conic. All (possibly degenerate) quadrics meeting in a conic of the form (2) are given by an equation of the form

$$f_{d,e,f}(X_0, X_1, X_2) + X_3(uX_0 + vX_1 + wX_2 + tX_3) = 0.$$  \hspace{1cm} (4)

So if we pick six points, contained in an elliptic quadric $E$ meeting $\pi$ in the conic $X_0X_2 - X_1^2 = 0$, we simply need to show that $E$ is the only quadric with equation of the form (4) through those $6$ points. This happens if and only if the homogeneous system of $6$ equations in the variables $d, e, f, u, v, w, t$ that arises from substituting the coordinates of the six points has a unique solution up to scalar multiple, which happens if and only if its coefficient matrix $A$ has $rk(A) = 6$.

In order to give an explicit construction of six such points and make the computations easier, we will restrict ourselves to those values of $q$ such that there is a primitive cubic polynomial of a particular form.

Theorem 5.9. \hspace{1cm} (a) Let $q$ be odd, $q \equiv 1$ (mod 3). Let $a$ be a non-square in $F_q$, where $a \neq \frac{1}{2}$. The six points $(1, 0, -a, 1), (1, 0, -a, -1), (1, 1, 1, -a, 1), (1, -1, -1, -a, 1), (1, 1, 1, -a, -1), (1, -1, 1, -a, -1)$ give rise to a higgledy-piggledy set of $7$ planes in $PG(5, q)$.

(b) Let $q$ be even such that there is an irreducible polynomial of the form $\omega^3 + \omega + 1 = 0$. Let $a \in F_q$ with $Tr(a) = 1, a \neq 1$. The six points $(1, 0, a, 1), (1, 1, a, 1), (a, 0, 1, 1), (a, 1, 1, 1), (1, a, a^2, 1), (a^2, a, 1, 1)$ give rise to a higgledy-piggledy set of $7$ planes in $PG(5, q)$.

Proof. \hspace{1cm} (a) Since $q \equiv 1$ (mod 3), there is an irreducible polynomial of the form $\omega^3 + \lambda = 0$. Using Lemma 5.7 we find that the quadrics of the form (3) become

$$\lambda eX_0^2 + \lambda fX_0X_1 - dX_0X_2 + dX_1^2 + eX_1X_2 + fX_2^2 + X_3(uX_0 + vX_1 + wX_2 + tX_3) = 0.$$  \hspace{1cm} (5)

It is easy to check that the given six points are not coplanar. Furthermore, they are contained in the elliptic quadric $E$ with equation $X_0X_2 - X_1^2 - aX_3^2 = 0$, which meets $\pi$
in the conic \( X_0X_2 - X_1^2 = 0 \). Substituting the 6 points into (3) yields a system \( \Xi \) of 6 homogeneous equations in \( d, e, f, u, v, w, t \) whose associated coefficient matrix is given by

\[
\begin{bmatrix}
  a & \lambda & a^2 & 1 & 0 & -a & 1 \\
  a & \lambda & a^2 & -1 & 0 & a & 1 \\
  a & \lambda + 1 - a & (1 - a)^2 + \lambda & 1 & 1 & 1 & -a & 1 \\
  a & \lambda + a - 1 & (1 - a)^2 - \lambda & 1 & -1 & -1 & 1 & 0 \\
  a & \lambda + 1 - a & (1 - a)^2 + \lambda & -1 & -1 & a & 1 & 1 \\
  a & \lambda + a - 1 & (1 - a)^2 - \lambda & 1 & 1 & 1 & a & 1 \\
\end{bmatrix}
\]

It can be checked that this matrix has full rank if and only if \( a(1 - a)(2a - 1) \neq 0 \). The statement follows from Proposition 5.8.

(b) Now assume that \( q \) is even and \( \omega^3 = \omega + 1 \). Using Lemma 5.7, we find that the equation for the quadrics (3) now becomes

\[
eX_0^2 + (e + f)X_0X_1 + (d + f)X_0X_2 + dX_1^2 + eX_1X_2 + fX_2^2 + X_3(uX_0 + vX_1 + wX_2 + tX_3) = 0.
\]

(6) The six given points are contained in the elliptic quadric \( E \) with equation \( X_0X_2 + X_1X_3 + aX_3^2 = 0 \), which meets \( \pi \) in \( X_0X_2 + X_1^2 = 0 \). Again, these points are not coplanar, and expressing that those six points lie on an equation of the form (7) yields a system \( \Xi \) in \( d, e, f, u, v, w, t \) with coefficient matrix

\[
\begin{bmatrix}
  a & 1 & a + a^2 & 1 & 0 & a & 1 \\
  1 + a & a & 1 + a + a^2 & 1 & 1 & a & 1 \\
  a & a^2 & a + 1 & a & 0 & 1 & 1 \\
  1 + a & a^2 + a + 1 & 1 & a & 1 & 1 & 1 \\
  0 & 1 + a + a^3 & a + a^2 + a^4 & 1 & a & a^2 & 1 \\
  0 & a^4 + a^3 + a & a^3 + a^2 + 1 & a^2 & a & 1 & 1 \\
\end{bmatrix}
\]

This matrix has full rank if and only if \( a(1 + a) \neq 0 \). Hence, since \( a \neq 0, 1 \), the statement follows from Proposition 5.8.

References

[1] S. Adriaensen and L. Denaux. Small weight codewords of projective geometric codes. J. Combin. Theory Ser. A, 180:Paper No. 105395, 34, 2021.

[2] J. André. Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe. Math. Z., 60:156–186, 1954.

[3] R. D. Baker, J. M. N. Brown, G. L. Ebert, and J. C. Fisher. Projective bundles. Bull. Belg. Math. Soc. Simon Stevin, 1(3):329–336, 1994. A tribute to J. A. Thas (Gent, 1994).

[4] D. Bartoli, A. Cossidente, G. Marino, and F. Pavese. On cutting blocking sets and their codes. Forum Math., 34(2):347–368, 2022.

[5] D. Bartoli, G. Kiss, S. Marcugini, and F. Pambianco. Resolving sets for higher dimensional projective spaces. Finite Fields Appl., 67:101723, 14, 2020.

[6] S. G. Barwick, L. R. A. Casse, and C. T. Quinn. The André/Bruck and Bose representation in PG(2h, q): unitals and Baer subplanes. Bull. Belg. Math. Soc. Simon Stevin, 7(2):173–197, 2000.
[7] S. G. Barwick and W. Jackson. Sublines and subplanes of PG(2, q^3) in the Bruck-Bose representation in PG(6, q). *Finite Fields Appl.*, 18(1):93–107, 2012.

[8] R. H. Bruck and R. C. Bose. The construction of translation planes from projective spaces. *J. Algebra*, 1:85–102, 1964.

[9] L. R. A. Casse and C. M. O'Keefe. Indicator sets for t-spreads of PG((s + 1)(t + 1) − 1, q). *Boll. Un. Mat. Ital. B (7)*, 4(1):13–33, 1990.

[10] B. Csajbók, G. Marino, and O. Polverino. Classes and equivalence of linear sets in PG(1, q^n). *J. Combin. Theory Ser. A*, 157:402–426, 2018.

[11] A. A. Davydov, M. Giulietti, S. Marcugini, and F. Pambianco. Linear nonbinary covering codes and saturating sets in projective spaces. *Adv. Math. Commun.*, 5(1):119–147, 2011.

[12] L. Denaux. Higgledy-Piggledy Sets in Projective Spaces of Small Dimension. *Electron. J. Combin.*, 29(3):Paper No. 3.29–, 2022.

[13] G. Faina, G. Kiss, S. Marcugini, and F. Pambianco. The cyclic model for PG(n, q) and a construction of arcs. *European J. Combin.*, 23(1):31–35, 2002.

[14] Sz. L. Fancsali and P. Sziklai. Description of the clubs. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 51:141–146 (2009), 2008.

[15] Sz. L. Fancsali and P. Sziklai. Lines in higgledy-piggledy arrangement. *Electron. J. Combin.*, 21(2):Paper 2.56, 15, 2014.

[16] Sz. L. Fancsali and P. Sziklai. Higgledy-piggledy subspaces and uniform subspace designs. *Des. Codes Cryptogr.*, 79(3):625–645, 2016.

[17] J. Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course.

[18] J. W. P. Hirschfeld and J. A. Thas. *General Galois geometries*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1991. Oxford Science Publications.

[19] M. Lavrauw, J. Sheekey, and C. Zanella. On embeddings of minimum dimension of PG(n, q) × PG(n, q). *Des. Codes Cryptogr.*, 74(2):427–440, 2015.

[20] M. Lavrauw and G. Van de Voorde. On linear sets on a projective line. *Des. Codes Cryptogr.*, 56(2-3):89–104, 2010.

[21] M. Lavrauw and G. Van de Voorde. Field reduction and linear sets in finite geometry. In *Topics in finite fields*, volume 632 of *Contemp. Math.*, pages 271–293. Amer. Math. Soc., Providence, RI, 2015.

[22] G. Lunardon and O. Polverino. Translation ovoids of orthogonal polar spaces. *Forum Math.*, 16(5):663–669, 2004.

[23] V. Napolitano, O. Polverino, P. Santonastaso, and F. Zullo. Clubs and their applications, arxiv: 2209.13339, 2022.

[24] O. Polverino. Linear sets in finite projective spaces. *Discrete Math.*, 310(22):3096–3107, 2010.

[25] O. Polverino and F. Zullo. Connections between scattered linear sets and MRD-codes. *Bull. Inst. Combin. Appl.*, 89:46–74, 2020.
[26] C. T. Quinn. The André/Bruck and Bose representation of conics in Baer subplanes of $\text{PG}(2, q^2)$. J. Geom., 74(1-2):123–138, 2002.

[27] S. Rottey, J. Sheekey, and G. Van de Voorde. Subgeometries in the André/Bruck-Bose representation. Finite Fields Appl., 35:115–138, 2015.

[28] B. Segre. Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane. Ann. Mat. Pura Appl. (4), 64:1–76, 1964.

[29] G. Van de Voorde. Blocking Sets in Finite Projective Spaces and Coding Theory. PhD thesis, Universiteit Gent, Belgium, 2010.

Lins Denaux & Jozefien D’haeseleer
Ghent University
Department of Mathematics: Analysis, Logic and Discrete Mathematics
Krijgslaan 281 – Building S8
9000 Ghent
BELGIUM
email: lins.denaux@ugent.be
e-mail: jozefien.dhaeseleer@ugent.be
website: https://users.ugent.be/~ldnaux
website: https://users.ugent.be/~jmdhaese

Geertrui Van de Voorde
University of Canterbury (Te Whare Wānanga o Waitaha)
School of Mathematics and Statistics
Private Bag 4800 – Erskine Building
Christchurch 8140
NEW ZEALAND
e-mail: geertrui.vandevoorde@canterbury.ac.nz