Reliability Analysis for a Repairable Load-Sharing Parallel System

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Abstract—This paper considers a repairable load-sharing parallel system with two different components which are distributed as exponential distribution. The availability function and reliability function of the system are calculated and some special cases are discussed. Finally, we derive the bayesian estimator of parameter for the system.

Keywords—load-sharing system; availability; reliability; Bayesian estimation

I. INTRODUCTION

Reliability is one of the most important properties of complex systems. It is a common practice to use redundancy techniques improving system reliability [1]. In most papers, when analyzing redundancy, independence is assumed among the components within system. In other words, it is assumed that the failure of a component does not affect the failure rates of other components in a system. However, in the real world, many systems are load-sharing, where the assumption of independence is no longer valid. When components in a system fail one by one, the total load of the system is redistributed among the surviving components, resulting in an increased load shared by each surviving component. For example, an airplane is provided with two engines, where two engines share the power load. When any of the engines within the system fails, the surviving engine has to shoulder extra power load to ensure the safety of aircraft. Many empirical studies of mechanical systems [3] have proved that the workload strongly affects the component failure rate. The key factor is that how the load is to be distributed among the surviving components. The rules of ‘equal’, ‘local’, ‘monotone’ are commonly used in load-sharing system. For more details about load-sharing rules, see Singh [4, 5].

In reliability literatures, originally, Daniels [6] considered the load-sharing model for studying the reliability of a fiber composite material and afterwards Coleman [7] used this model for analyzing the strength behavior of fiber bundles. Singh [5] has also analyzed the k-components load-sharing parallel system model with discrete set up. Deshpande [8] provided a family of distributions to develop the load-sharing system models. Kim and Kvam [9] developed the statistical methodology for estimating the parameters of multi-components load-sharing system model. Early load-sharing k-out-of-n: G system models was researched by Lin [10], where it is assumed that components are identically distributed with exponential distribution. Shao and Lamberson [11] considered the reliability and availability of a load sharing repairable k-out-of-n: G system. Pozsgai and Neher [12] defined a capacity flow model. Tang and Zhang [13] modeled a new method with different components. Recently, Yun and Cha [14] investigated a general load sharing parallel system with two components when the lifetimes of the components in the system are arbitrary continuous random variables. Zhang and Balakrishnan [15] obtained some stochastic properties and parameter estimation for a general load-sharing parallel system. They gave some sufficient conditions for the stochastic order between two systems and considered the optimal allocation problems of one load standby in a series system. Also the parameter estimation was discussed. All these studies focus on un-repairable load-sharing systems.

In this article, we will consider a repairable load-sharing parallel system with two different components distributed as exponential. Specifically, Section 2 introduces model and some notations. Section 3 will give system availability and reliability. In Section 4, Bayesian estimator of parameter is discussed.

II. MODEL AND NOTATIONS

Let $X_i$ ($i=1,2$) be the lifetime of component distributed as exponential with respective failure $\lambda_1$ and $\lambda_2$ under full load condition. Let $X_{i\rho}$ be the lifetime of component $i$ under partial load condition $\rho$, with failure $\lambda_{i\rho}$, respectively. Some necessary assumptions are given as follows:

1). Each component is either working or failed.

2). If component 1 (2) failed at first, component 2 (1) immediately shifts to the full load condition and its failure rate changes from $\lambda_{i\rho}$ ($\lambda_1$) to $\lambda_i$ ($\lambda_2$).

3). If a component failed, it must be detected and removed from the system immediately. Also it must be put in repair at once. The switch-over time is negligible.

4). The repair time for component $i$ is exponential distribution with hazard rate $\lambda_i$ ($i=1,2$). The repaired component is as good as new and immediately reconnected.
to the system.

5) The two components operate independently under partial load conditions.

As described by Zhang and Balakrishnan [15], by the accelerated life model ([16], [17]), it is reasonably assumed that, for all \( t \geq 0 \),

\[
F_r(t) = F_1(g_r(\alpha) t), \quad F_p(t) = F_2(g(1-\alpha) t)
\]

(1)

where the assignment function \( g(\alpha) \) satisfies with:

(a) \( g_r(0) = 0 \), \( g_r(1) = 1 \) (b) \( 0 \leq g_r(\alpha) \leq 1 \) (c) \( g(\alpha) \) is strictly increasing with respect to \( \alpha \). When the system starts its operation, the total load \( L \) is shared by two components with assignment proportions \( a \), \( 1-a \) \( (0 \leq \alpha \leq 1) \) respectively (see Figure 1).

![FIGURE I. LOAD-SHARING PARALLEL SYSTEM.](image)

When one component fails, another surviving component takes the full load \( L \) and continues operating. For convenience, it was assumed that \( L \) equals one in their work. Based on the equalities of (1), the relation of the failure rates of two components between two load conditions are

\[
\lambda_r = g_r(\alpha) \lambda_1, \quad \lambda_p(t) = g_r(1-\alpha) \lambda_2
\]

(2)

III. SYSTEM AVAILABILITY AND RELIABILITY

A. System Availability

In this section, we will investigate the availability (denoted by \( A(t) \)) of the system. We assume that two components are operational at time 0. Each has an initial failure rate \( \lambda_r, \lambda_p \) in equation (2). Firstly, the system states are defined as follow:

- **State 0**: Two components are operating under full load condition;
- **State 1**: Component 2 has failed, component 1 operates under full load condition;
- **State 2**: Component 1 has failed, component 2 operates under full load condition;
- **State 3**: Component 1 has failed and is being repaired. During the time of being repaired for component 1, Component 2 failed. However, the system can return to the working state 1 with repair rate \( \mu_1 \);
- **State 4**: Component 2 has failed and is being repaired. During the time of being repaired for component 2, Component 1 failed. However, the system can return to the working state 2 with repair rate \( \mu_2 \).

Clearly, the system state space \( E = \{0,1,2,3,4\} \), the success state space be noted \( S = \{0,1,2\} \) and the failed state space \( F = \{3,4\} \). If we use \( S(t) \) to describe the system state at time \( t \), then \( \{S(t), (t \geq 0)\} \) is continuous-time Markov process, due to the memory-less property of exponential distribution. Let \( P_i(t) = P[S(t) = i] \) be noted the system state probability, \( P_i(\Delta t) \) be the transit probability of the system from state \( i \) to state \( j \) \( (i, j = 0,1,2,3,4) \) in time interval \( (t, t + \Delta t) \). \( P_i(\Delta t) = P[S(t + \Delta t) = j | S(t) = i] \).

From the conditional probability, the transit probabilities matrix \( P_i(\Delta t) \) of the system state is given by

\[
P_i(t) = \begin{bmatrix}
    e^{-\lambda_1 \Delta t} & e^{-\lambda_2 \Delta t}(1-e^{-\lambda_1 \Delta t}) & e^{-\lambda_2 \Delta t}(1-e^{-\lambda_1 \Delta t}) & (1-e^{-\lambda_1 \Delta t}) & 0 \\
    e^{-\lambda_2 \Delta t}(1-e^{-\lambda_1 \Delta t}) & e^{\lambda_1 \Delta t} & e^{\lambda_2 \Delta t}(1-e^{-\lambda_1 \Delta t}) & (1-e^{-\lambda_2 \Delta t}) & 0 \\
    e^{-\lambda_2 \Delta t}(1-e^{-\lambda_1 \Delta t}) & e^{\lambda_2 \Delta t}(1-e^{-\lambda_1 \Delta t}) & e^{\lambda_2 \Delta t}(1-e^{-\lambda_1 \Delta t}) & e^{\lambda_2 \Delta t} & 0 \\
    0 & 0 & 0 & e^{\lambda_2 \Delta t} & 0 \\
    0 & 0 & 0 & 0 & e^{\lambda_2 \Delta t}
\end{bmatrix}
\]

Using the full probability formula, the probabilities in different state of system at time \( (t + \Delta t) \) can be represented as

\[
\begin{align*}
P_0(t + \Delta t) &= P_0(t) + P_0(\Delta t)P_0(t) + P_0(\Delta t)P_2(t) + P_3(\Delta t)P_0(t) \\
P_1(t + \Delta t) &= P_0(t) + P_0(\Delta t)P_0(t) + P_0(\Delta t)P_2(t) + P_3(\Delta t)P_0(t) \\
P_2(t + \Delta t) &= P_0(\Delta t)P_0(t) + P_2(\Delta t)P_0(t) + P_0(\Delta t)P_2(t) + P_4(\Delta t)P_2(t) \\
P_3(t + \Delta t) &= P_0(\Delta t)P_0(t) + P_2(\Delta t)P_0(t) + P_3(\Delta t)P_2(t) + P_4(\Delta t)P_2(t) \\
P_4(t + \Delta t) &= P_4(\Delta t)P_0(t) + P_4(\Delta t)P_2(t)
\end{align*}
\]

And hence the matrix differential equation is

\[
\frac{d}{dt}P_i(t) = \begin{bmatrix}
    -\lambda_1 & \lambda_1 & 0 & 0 & 0 \\
    \lambda_2 & -\lambda_2 & \lambda_2 & 0 & 0 \\
    0 & \lambda_1 & -\lambda_1 & \lambda_1 & 0 \\
    0 & 0 & \lambda_2 & -\lambda_2 & \lambda_2 \\
    0 & 0 & 0 & \lambda_2 & -\lambda_2
\end{bmatrix}P_i(t)
\]
where $S$ is called the matrix of the transition rates of system state as follow:

$S = \begin{bmatrix}
- (\lambda_2 + \lambda_3) & \lambda_2 & \lambda_3 & 0 & 0 \\
\mu_4 & - (\lambda_1 + \mu_3) & 0 & \lambda_1 & 0 \\
0 & \mu_3 & 0 & - (\lambda_2 + \mu_4) & \lambda_2 \\
0 & 0 & \mu_5 & 0 & 0 \\
0 & 0 & 0 & \mu_6 & 0 \\
\end{bmatrix}$

It is clear that the availability function $A(t) = P_0(t) + P_1(t) + P_2(t)$

To obtain the system state probability $P_i(t)$ ($i=0, 1, 2, 3, 4$), we take the Laplace transform for equation (3) and obtain $P_i(s)$, where $P_i(s) = \int_0^\infty e^{-st}P_i(t)dt$. Then using the inverse Laplace transform of $P_i(s)$, we can obtain the system state probability $P_i(t)$.

A. System Reliability Function

System reliability is the availability of the same system without repair and replacement when components failed. In this case, the system states are defined as follow:

State 0: Two components are operating under full load condition;
State 1: Component 2 has failed, component 1 operates under full load condition;
State 2: Component 1 has failed, component 2 operates under full load condition;
State 3: Component 1 and 2 have failed, the system failed.

The system state space $E = \{0, 1, 2, 3\}$, the success state space $S = \{0, 1, 2\}$ and the failed state space $F = \{3\}$. Then the transition rates matrix of system state reduces to

$S = \begin{bmatrix}
- (\lambda_1 + \lambda_2) & \lambda_1 & \lambda_2 & 0 \\
0 & - \lambda_1 & 0 & \lambda_1 \\
0 & 0 & - \lambda_2 & \lambda_2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$

Based on the transition rates matrix, we obtain the following differential equations

$P_0'(t) = - (\lambda_1 + \lambda_2)P_0(t)$
$P_1'(t) = \lambda_1P_0(t) - \lambda_2P_1(t)$
$P_2'(t) = \lambda_2P_1(t) - \lambda_2P_2(t)$

Taking the Laplace transforms of differential equation (7), we obtain the following equations
Its inverse Laplace transform of equation (8) is

\[
P_0^*(s) = \frac{1}{s + \lambda_{p_1} + \lambda_{p_2}} \\
P_1^*(s) = \frac{\lambda_{p_1}}{(s + \lambda_1)(s + \lambda_{p_1} + \lambda_{p_2})} \\
P_2^*(s) = \frac{\lambda_{p_2}}{(s + \lambda_2)(s + \lambda_{p_1} + \lambda_{p_2})}
\]

(8)

Hence, the reliability function of the non-repairable system is given by

\[
R(t) = P_0(t) + P_1(t) + P_2(t)
\]

(9)

If let \(g_i(a) = a\) that is, \(\lambda_1 = a\lambda_2\) and \(\lambda_{\lambda}(t) = (1 - a)\lambda_2\), then the reliability function in (9) turns to

\[
R(t) = \frac{1}{\lambda_2 - \lambda_1} \left[ e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t} \right],
\]

given \(\lambda_1 \neq \lambda_2\). And hence the mean lifetime of the system

\[
E(T) = \int_0^\infty R(t) dt = \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2}.
\]

In this case, the reliability function and mean lifetime of a non-repairable system is independent of load allocation rate \(a\). When the failure rates of two components are constants for a general parallel system, it is well-known that

\[
E(T) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}.
\]

Clearly, the mean lifetime of the load-sharing system is larger than that of parallel system with two components. Thus, from the view of mean lifetime, load-sharing system is better than normal parallel system.

IV. BAYESIAN STUDY OF PARAMETER

In order to study the behavior of load-sharing parameters in Bayesian setup [18, 19], we incorporate the prior belief of the investigator regarding the variations in parameters. As described by Zhang and Balakrishnan [15], we assume that the component lifetimes for \(n\) load-sharing systems are observable, and \(g_i(a) = g_i(1 - a) = g(a)\) for any \(a > 0\). Suppose the random variables \(X_{i1}\) and \(X_{i2}\) represent the lifetime of the two components under full load condition in \(i\)th \((i = 1, 2, \ldots, n)\) parallel system, and that the spacing \(T_{ij}\) are the times between the \(j\)th and \((j - 1)\)th failures for the \(i\)th system, \((j = 1, 2)\).

First, we assume that the two components are identically distributed as exponential with common parameter \(\lambda\). It has been shown by Zhang and Balakrishnan [15], that the likelihood function for the sample of \(n\) systems is

\[
L(\lambda | a, T) = \lambda^{2n} g^a(\lambda) \exp \left\{ -\lambda \sum_{j=1}^n (2g(a)t_{i1} + t_{i2}) \right\}
\]

(10)

where \(T = t_i; 1 < i < n, j = 1, 2\). Let the parameter \(a\) be known and the prior distribution of the \(\lambda\) be assumed as gamma density \(G(a, b)\) with probability density function:

\[
g(\lambda) = \frac{\lambda^{a-1} e^{-\lambda b}}{\Gamma(b)} \quad (a, b, \lambda > 0).
\]

(11)

In the view of equations (10) and (11), the joint distribution of \(\lambda\) and \(T\) is

\[
h(\lambda, T) = \frac{a^2 \lambda^{2n-1} \exp \left\{ -\lambda \sum_{j=1}^n (2g(a)t_{i1} + t_{i2} + a) \right\}}{\Gamma(b)}.
\]

Then, the marginal density function of sample is

\[
m(T) = \int_0^\infty h(T, \lambda) d\lambda = \frac{(2n + b - 1) a^2 \lambda^{2n-1} g(a)}{\Gamma(b) \left\{ \sum_{j=1}^n (2g(a)t_{i1} + t_{i2} + a) \right\}^{2n+b}}.
\]

Hence, the posterior distribution of \(\lambda\) is

\[
\pi(\lambda | T) = \frac{h(T, \lambda)}{m(T)} = \frac{(2n + b - 1) \lambda^{2n-1} \exp \left\{ -\lambda \sum_{j=1}^n (2g(a)t_{i1} + t_{i2} + a) \right\}}{\left\{ \sum_{j=1}^n (2g(a)t_{i1} + t_{i2} + a) \right\}^{2n+b}}.
\]

Now, under squared error loss function, the Bayesian estimates of \(\lambda\) is
\[ \dot{\lambda} = E(\lambda | T) = \int_0^\infty \lambda \pi(\lambda | T) d\lambda \]

\[
= \frac{(2n+b-1)!(2n+b)!}{\sum_{r=0}^{n} \left( (2a_{t_1} + t_2) + 2a_{t_1+b-1} \right)^{2n+r}}.
\]

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