On a relation between intrinsic and extrinsic Dirichlet forms for interacting particle systems

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Abstract

In this paper we extend the result obtained in [AKR98a] (see also [AKR96a]) on the representation of the intrinsic pre-Dirichlet form \( E_{\pi_\sigma}^\Gamma \) of the Poisson measure \( \pi_\sigma \) in terms of the extrinsic one \( E_{\pi_\sigma, H_X}^{P_{\pi_\sigma}} \). More precisely, replacing \( \pi_\sigma \) by a Gibbs measure \( \mu \) on the configuration space \( \Gamma_X \) we derive a relation between the intrinsic pre-Dirichlet form \( E_{\mu}^\Gamma \) of the measure \( \mu \) and the extrinsic one \( E_{\mu, H_X}^{P_\mu} \). As a consequence we prove the closability of \( E_{\mu}^\Gamma \) on \( L^2(\Gamma_X, \mu) \) under very general assumptions on the interaction potential of the Gibbs measures \( \mu \).

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1 Introduction

In the recent papers \[AKR96a\], \[AKR96b\], \[AKR98a\], and \[AKR98b\] analysis and geometry on configuration spaces \(\Gamma_X\) over a Riemannian manifold \(X\), i.e,

\[\Gamma_X := \{ \gamma \subset X \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset X \},\]

was developed. One of the consequences of the discussed approach was a description of the well-known equilibrium process on configuration spaces as the Brownian motion associated with a Dirichlet form of the Poisson measure \(\pi_\sigma\) with intensity measure \(\sigma\) on \(B(X)\). This form is canonically associated with the introduced geometry on configuration spaces and is called *intrinsic Dirichlet form* of the measure \(\pi_\sigma\).

On the other hand there is a well-known realization (canonical isomorphism) of the Hilbert space \(L^2(\Gamma_X, \pi_\sigma)\) and the corresponding Fock space

\[\text{Exp}L^2(X, \sigma) := \bigoplus_{n=0}^{\infty} \text{Exp}_n L^2(X, \sigma),\]

where \(\text{Exp}_n L^2(X, \sigma)\) denotes the \(n\)-fold symmetric tensor product of \(L^2(X, \sigma)\) and \(\text{Exp}_0 L^2(X, \sigma) := \mathbb{C}\). This isomorphism produces natural operations in \(L^2(\Gamma_X, \pi_\sigma)\) as images of the standard Fock space operators, see e.g., \[KSSU98\] and references therein. In particular, we can consider the image of the annihilation operator from the Fock space as a natural version of a “gradient” operator in \(L^2(\Gamma_X, \pi_\sigma)\). The differentiable structure in \(L^2(\Gamma_X, \pi_\sigma)\) which appears in this way we consider as *external* because it is produced via transportation from the Fock space.

As was shown in \[AKR98a\], Section 5] the intrinsic Dirichlet form of the measure \(\pi_\sigma\) can be represented also in terms of the external Dirichlet form \(\mathcal{E}_{\pi_\sigma, H_X}\) with coefficient \(H_X^\sigma\) (the Dirichlet operator associated with \(\sigma\) on \(X\)) which uses this external differentiable structure, i.e.,

\[\int_{\Gamma} \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma) \rangle_{\pi_\sigma} d\pi_\sigma(\gamma) = \int_{\Gamma} \langle \nabla^P F(\gamma), H_X^\sigma \nabla^P G(\gamma) \rangle_{L^2(X, \sigma)} d\pi_\sigma(\gamma).\]

As a result we have a full spectral description of the corresponding Dirichlet operator \(H^\Gamma_{\pi_\sigma}\) which is the generator of the equilibrium process on \(\Gamma_X\).

If we change the Poisson measure \(\pi_\sigma\) to a Gibbs measure \(\mu\) on the configuration space \(\Gamma_X\) which describes the equilibrium of interacting particle
systems, the corresponding intrinsic Dirichlet form can still be used for constructing the corresponding stochastic dynamics (cf. [AKR98b, Section 5]) or for constructing a quantum infinite particle Hamiltonian in models of quantum fields theory, see [AKR99].

The aim of this paper is to show that even for the interacting case there is a transparent relation between the intrinsic Dirichlet form and the extrinsic one, see Theorem 5.1. The proof is based on the Nguyen-Zessin characterization of Gibbs measure (cf. [NZ79, Theorem 2] or Proposition 5.2 below) which on a heuristic level can be considered as a consequence of the Mecke identity (cf. [Mec67, Satz 3.1]), see Remark 5.3 below for more details.

As a consequence of the mentioned relation we prove the closability of the pre-Dirichlet form \((\mathcal{E}_\mu^\Gamma, \mathcal{F}_{\mathcal{C}^\infty_b}(\mathcal{D}, \Gamma))\) on \(L^2(\Gamma_X, \mu)\), where \(\mu\) is a tempered grand canonical Gibbs measure, see Section 2 for this notion. It turns out that this result is obtained as a “lifting” of the closable Dirichlet forms on \(X\). We would like to emphasize that we achieve this result under a general condition (see (6.2) below) on the potential \(\Phi\) which is not covered by condition (A.6) in [Osa96]. The closability is crucial (for physical reasons, see [AKR99], and) for applying the general theory of Dirichlet forms including the construction of a corresponding diffusion process (cf. [MR92]) which models an infinite particle system with (possibly) very singular interactions (cf. [AKR98b]).

Another motivation for deriving Theorem 5.1 is to use this result for studying spectral properties of Hamiltonians of intrinsic Dirichlet forms associated with Gibbs measures. This will be implemented in a forthcoming paper.

## 2 Preliminaries and Framework

In this section we describe some facts about probability measures on configuration spaces which are necessary later on.

Let \(X\) be a connected, oriented \(C^\infty\) (non-compact) Riemannian manifold. For each point \(x \in X\), the tangent space to \(X\) at \(x\) will be denoted by \(T_xX\); and the tangent bundle will be denoted by \(TX = \bigcup_{x \in X} T_xX\). The Riemannian metric on \(X\) associates to each point \(x \in X\) an inner product on \(T_xX\) which we denote by \(\langle \cdot, \cdot \rangle_{T_xX}\) and the associated norm will be denoted by \(|\cdot|_{T_xX}\). Let \(m\) denote the volume element.

\(\mathcal{O}(X)\) is defined as the family of all open subsets of \(X\) and \(\mathcal{B}(X)\) denotes the corresponding Borel \(\sigma\)-algebra. \(\mathcal{O}_c(X)\) and \(\mathcal{B}_c(X)\) denote the systems of
all elements in $\mathcal{O}(X)$, $\mathcal{B}(X)$ respectively, which have compact closures.

Let $\Gamma := \Gamma_X$ be the set of all locally finite subsets in $X$:

$$\Gamma_X := \{ \gamma \subset X \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset X \}.$$  

We will identify $\gamma$ with the positive integer-valued measure $\sum_{x \in \gamma} \varepsilon_x$. Then for any $\varphi \in C_0(X)$ we have a functional $\Gamma \ni \gamma \mapsto \langle \varphi, \gamma \rangle = \sum_{x \in \gamma} \varphi(x) \in \mathbb{R}$. Here $C_0(X)$ is the set of all real-valued continuous functions on $X$ with compact support. The space $\Gamma$ is endowed with the vague topology. Let $\mathcal{B}(\Gamma)$ denote the corresponding Borel $\sigma$-algebra. For $\Lambda \subset X$ we sometimes use the shorthand $\gamma \Lambda$ for $\gamma \cap \Lambda$.

For any $B \in \mathcal{B}(X)$ we define, as usual, $\Gamma \ni \gamma \mapsto N_B(\gamma) := |\gamma(B)| \in \mathbb{Z}_+ \cup \{+\infty\}$. Then $\mathcal{B}(\Gamma) = \sigma(\{ N_B | B \in \mathcal{B}_c(X), B \subset A \})$.

Let $d\sigma(x) = \rho(x)d\mu(x)$, where $\rho > 0$ $\mu$-a.e. such that $\rho \in H^{1,2}_{loc}(X)$ (the Sobolev space of order 1 in $L^2(X, \mu)$) and $\rho \notin L^1(X, \mu)$. We recall that the Poisson measure $\pi_\sigma$ (with intensity measure $\sigma$) on $(\Gamma, \mathcal{B}(\Gamma))$ is defined via its Laplace transform by

$$\int_\Gamma e^{\langle \gamma, \varphi \rangle} d\pi_\sigma(\gamma) = \exp \left( \int_X (e^{\varphi(x)} - 1) d\sigma(x) \right), \quad \varphi \in C_0(X),$$

see e.g. [AKR98a], [GV68], and [Shi94]. Let us mention that if $\rho \in L^1(X, \mu)$, then we have a finite intensity measure $\sigma$ on $X$, and in this case the corresponding measure $\pi_\sigma$ will be concentrated on finite configurations. The latter can be considered as a degenerated case which can be reduced to finite dimensional analysis on every subset of $n$-particle configurations.

Let us briefly recall the definition of grand canonical Gibbs measures on $(\Gamma, \mathcal{B}(\Gamma))$. We adopt the notation in [AKR98b], and refer the interested reader to the beautiful work by C. Preston, [Pre79], but also [Pre76], and [Geo79].

A function $\Phi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ will be called a potential iff for all $\Lambda \in \mathcal{B}_c(X)$ we have $\Phi(\emptyset) = 0$, $\Phi = \mathbb{1}_{\{N_\Lambda < \infty\}}\Phi$, and $\gamma \mapsto \Phi(\gamma \Lambda)$ is $\mathcal{B}_A(\Gamma)$-measurable.

For $\Lambda \in \mathcal{B}_c(X)$ the conditional energy $E^{\Phi}_\Lambda : \Gamma \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$E^{\Phi}_\Lambda(\gamma) := \begin{cases} \sum_{\gamma' \subset \gamma, \gamma'(\Lambda) > 0} \Phi(\gamma') & \text{if } \sum_{\gamma' \subset \gamma, \gamma'(\Lambda) > 0} |\Phi(\gamma')| < \infty, \\ +\infty & \text{otherwise}, \end{cases}$$

(2.2)
where the sum of the empty set is defined to be zero.

Later on we will use conditional energies which satisfy an additional assumption, namely, the stability condition, i.e., there exists \( B \geq 0 \) such that for any \( \Lambda \in \mathcal{B}_c(X) \) and for all \( \gamma \in \Gamma_{\Lambda} \)

\[
E^\Phi(\gamma) \geq -B|\gamma|.
\]

**Definition 2.1** For any \( \Lambda \in \mathcal{O}_c(X) \) define for \( \gamma \in \Gamma \) the measure \( \Pi^\Phi_{\Lambda}(\gamma, \cdot) \) by

\[
\Pi^\sigma_{\Lambda}(\gamma, \Delta) := \mathbb{1}_{\{Z_{\Lambda}^\sigma < \infty\}}(\gamma)[Z_{\Lambda}^\sigma(\gamma)]^{-1} \int_{\Gamma} \mathbb{1}_\Delta(\gamma_{X\setminus\Lambda} + \gamma') \exp[-E^\Phi(\gamma_{X\setminus\Lambda} + \gamma')] d\pi(\gamma'), \quad \Delta \in \mathcal{B}(\Gamma),
\]

where

\[
Z_{\Lambda}^\sigma(\gamma) := \int_{\Gamma} \exp[-E^\Phi(\gamma_{X\setminus\Lambda} + \gamma')] d\pi(\gamma').
\]

A probability measure \( \mu \) on \((\Gamma, \mathcal{B}(\Gamma))\) is called grand canonical Gibbs measure with interaction potential \( \Phi \) if for all \( \Lambda \in \mathcal{O}_c(X) \)

\[
\mu \Pi^\Phi_{\Lambda} = \mu. \tag{2.5}
\]

Let \( \mathcal{G}_{gc}(\sigma, \Phi) \) denote the set of all such probability measures \( \mu \).

**Remark 2.2** 1. It is well-known that \((\Pi^\sigma_{\Lambda})_{\Lambda \in \mathcal{O}_c(X)}\) is a \((\mathcal{B}_{X\setminus\Lambda}(\Gamma))_{\Lambda \in \mathcal{O}_c(X)}\)-specification in the sense of [Pre76, Section 6] or [Pre79].

2. For any \( \gamma \in \Gamma \) the measure \( \mu \Pi^\Phi_{\Lambda} \) in (2.5) is defined by

\[
(\mu \Pi^\sigma_{\Lambda})(\Delta) := \int_{\Gamma} d\mu(\gamma) \Pi^\sigma_{\Lambda}(\gamma, \Delta), \quad \Delta \in \mathcal{B}(\Gamma) \tag{2.6}
\]

and (2.3) are called Dobrushin-Landford-Ruelle (DLR) equations.
3 Intrinsic geometry on Poisson space

We recall some results to be used below from [AKR98a], [AKR96b] to which we refer for the corresponding proofs and more details.

A homeomorphism \( \psi : X \to X \) defines a transformation of \( \Gamma \) by

\[
\Gamma \ni \gamma \mapsto \psi(\gamma) = \{ \psi(x) | x \in \gamma \} = \sum_{x \in \gamma} \epsilon_{\psi(x)}.
\]

Any vector field \( v \in V_0(X) \) (i.e., the set of all smooth vector fields on \( X \) with compact support) defines (via the exponential mapping) a one-parameter group \( \psi^t, t \in \mathbb{R}, \) of diffeomorphisms of \( X. \)

**Definition 3.1** For \( F : \Gamma \to \mathbb{R} \) we define the directional derivative along the vector field \( v \) as (provided the right hand side exists)

\[
(\nabla^\Gamma_v F)(\gamma) := \frac{d}{dt} F(\psi^t_\gamma(\gamma))|_{t=0}.
\]

This definition applies to \( F \) in the following class \( \mathcal{F}C_0^\infty(D, \Gamma) \) of so-called smooth cylinder functions. Let \( D := C_0^\infty(X) \) (the set of all smooth functions on \( X \) with compact support). We define \( \mathcal{F}C_0^\infty(D, \Gamma) \) as the set of all functions on \( \Gamma \) of the form

\[
F(\gamma) = g_F(\langle \gamma, \varphi_1 \rangle, \ldots, \langle \gamma, \varphi_N \rangle), \quad \gamma \in \Gamma,
\]

where \( \varphi_1, \ldots, \varphi_N \in D \) and \( g_F \) is from \( C_0^\infty(\mathbb{R}^N). \) Clearly, \( \mathcal{F}C_0^\infty(D, \Gamma) \) is dense in \( L^2(\pi_\sigma) := L^2(\Gamma, \pi_\sigma). \) For any \( F \in \mathcal{F}C_0^\infty(D, \Gamma) \) we have

\[
(\nabla^\Gamma_v F)(\gamma) = \sum_{i=1}^{N} \frac{\partial g_F}{\partial s_i}(\langle \gamma, \varphi_1 \rangle, \ldots, \langle \gamma, \varphi_N \rangle) \langle \gamma, \nabla^X \varphi_i \rangle, \quad (3.2)
\]

where \( x \mapsto (\nabla^X \varphi_i)(x) = \langle \nabla^X \varphi_i(x), v(x) \rangle_{TX} \) is the usual directional derivative on \( X \) along the vector field \( v \) and \( \nabla^X \) denotes the gradient on \( X. \)

The logarithmic derivative of the measure \( \sigma \) is given by the vector field \( \beta^\sigma := \nabla^X \log \rho = \nabla^X \rho / \rho \) (where \( \beta^\sigma = 0 \) on \( \{ \rho = 0 \} \)). Then the logarithmic derivative of \( \sigma \) along \( v \) is the function \( x \mapsto \beta^\sigma_v(x) = \langle \beta^\sigma(x), v(x) \rangle_{T_x X} + \text{div}^X v(x), \) where \( \text{div}^X \) denotes the divergence on \( X \) w.r.t. the volume element \( m. \) Analogously, we define \( \text{div}^X_\sigma \) as the divergence on \( X \) w.r.t. \( \sigma, \) i.e., \( \text{div}^X_\sigma \) is the dual operator on \( L^2(\sigma) := L^2(X, \sigma) \) of \( \nabla^X. \)
Definition 3.2 For any \( v \in V_0(X) \) we define the logarithmic derivative of \( \pi_{\sigma} \) along \( v \) as the following function on \( \Gamma \):

\[
\Gamma \ni \gamma \mapsto B^{\pi_{\sigma}}_v(\gamma) := \int_X [\langle \beta_{\sigma}^v(x), v(x) \rangle_{T_x X} + \text{div}^X v(x)] d\gamma(x). \tag{3.3}
\]

Theorem 3.3 For all \( F,G \in FC^\infty_b(D,\Gamma) \) and any \( v \in V_0(X) \) the following integration by parts formula for \( \pi_{\sigma} \) holds:

\[
\int_\Gamma \nabla^\Gamma_v FGd\pi_{\sigma} = -\int_\Gamma F \nabla^\Gamma_v Gd\pi_{\sigma} - \int_\Gamma FGB^{\pi_{\sigma}}_v d\pi_{\sigma}, \tag{3.4}
\]

or \((\nabla^\Gamma_v)^* = -\nabla^\Gamma_v - B^{\pi_{\sigma}}_v\), as an operator equality on the domain \( FC^\infty_b(D,\Gamma) \) in \( L^2(\pi_{\sigma}) \).

Definition 3.4 We introduce the tangent space \( T_\gamma \Gamma \) to the configuration space \( \Gamma \) at the point \( \gamma \in \Gamma \) as the Hilbert space of \( \gamma \)-square-integrable sections \( V : X \to TX \) with scalar product \( \langle V^1, V^2 \rangle_{T_\gamma \Gamma} = \int_X \langle V^1(x), V^2(x) \rangle_{T_x X} d\gamma(x) \), \( V^1, V^2 \in T_\gamma \Gamma = L^2(X \to TX; \gamma) \). The corresponding tangent bundle is denoted by \( T\Gamma \).

The intrinsic gradient of a function \( F \in FC^\infty_b(D,\Gamma) \) is a mapping \( \Gamma \ni \gamma \mapsto (\nabla^\Gamma v)(\gamma) \in T_\gamma \Gamma \) such that \( \langle \nabla^\Gamma v F, \gamma \rangle_{T_\gamma \Gamma} = F(v) \) for any \( v \in V_0(X) \). Furthermore, by (3.2), if \( F \) is given by (3.1), we have for \( \gamma \in \Gamma \), \( x \in X \)

\[
(\nabla^\Gamma F)(\gamma; x) = \sum_{i=1}^N \frac{\partial g_{\Gamma}}{\partial s_i}(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle) \nabla^X \varphi_i(x). \tag{3.5}
\]

Definition 3.5 For a measurable vector field \( V : \Gamma \to TT \) the divergence \( \text{div}^\Gamma_{\pi_{\sigma}} V \) is defined via the duality relation for all \( F \in FC^\infty_b(D,\Gamma) \) by

\[
\int_\Gamma \langle V_\gamma, \nabla^\Gamma F(\gamma) \rangle_{T_\gamma \Gamma} d\pi_{\sigma}(\gamma) = -\int_\Gamma F(\gamma)(\text{div}^\Gamma_{\pi_{\sigma}} V)(\gamma) d\pi_{\sigma}(\gamma), \tag{3.6}
\]

provided it exists (i.e., provided

\[
F \mapsto \int_\Gamma \langle V_\gamma, \nabla^\Gamma F(\gamma) \rangle_{T_\gamma \Gamma} d\pi_{\sigma}(\gamma)
\]

is continuous on \( L^2(\pi_{\sigma}) \)).
Proposition 3.6 For any vector field \( V = G v \), where \( G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \), \( v \in V_0(X) \) we have

\[
(\text{div}^\Gamma_{\pi_\sigma} V)(\gamma) = \langle (\nabla^\Gamma G)(\gamma), v \rangle_{\Gamma \pi_\sigma} + G(\gamma)B^\pi_\sigma(\gamma). \tag{3.7}
\]

For any \( F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \) we introduce the \textit{pre-Dirichlet form} which is generated by the intrinsic gradient \( \nabla^\Gamma \) as

\[
\mathcal{E}^\Gamma_{\pi_\sigma}(F, G) = \int_{\Gamma} \langle (\nabla^\Gamma F)(\gamma), (\nabla^\Gamma G)(\gamma) \rangle_{\Gamma \pi_\sigma} d\pi_\sigma(\gamma). \tag{3.8}
\]

We will also need the \textit{classical pre-Dirichlet form} for the intensity measure \( \sigma \) which is given as \( \mathcal{E}^X_{\sigma}(\varphi, \psi) = \int_X \langle \nabla^X \varphi, \nabla^X \psi \rangle_{TX} d\sigma \) for any \( \varphi, \psi \in \mathcal{D} \). This form is associated with the \textit{Dirichlet operator} \( H^X_{\sigma} \) which is given on \( \mathcal{D} \) by

\[
H^X_{\sigma} \varphi(\gamma) := -\triangle^X \varphi(\gamma) - \langle \beta^\sigma(\gamma) \nabla^X \varphi(\gamma), \nabla^X \psi(\gamma) \rangle_{\tau_\gamma} \text{ and which satisfies } \mathcal{E}^X_{\sigma}(\varphi, \psi) = (H^X_{\sigma} \varphi, \psi)_{L^2(\sigma)}, \ \varphi, \psi \in \mathcal{D}, \text{ see e.g. } [BK92] \text{ and } [MR92].
\]

For any \( F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \), \( (\nabla^\Gamma \nabla^\Gamma F)(\gamma, x, y) \in T_{\gamma \pi_\sigma} \otimes T_{\gamma \pi_\sigma} \) and we can define the \( \Gamma \)-\textit{Laplacian} \( (\triangle^\Gamma F)(\gamma) := \text{Tr}(\nabla^\Gamma \nabla^\Gamma F)(\gamma) \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \). We introduce a differential operator in \( L^2(\pi_\sigma) \) on the domain \( \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \) by the formula

\[
(H^\Gamma_{\pi_\sigma} F)(\gamma) = -\triangle^\Gamma F(\gamma) - \langle \text{div}^X_{\sigma}(\nabla^\Gamma F)(\gamma, \cdot), \gamma \rangle. \tag{3.9}
\]

Theorem 3.7 The operator \( H^\Gamma_{\pi_\sigma} \) is associated with the intrinsic Dirichlet form \( \mathcal{E}^\Gamma_{\pi_\sigma} \), i.e.,

\[
\mathcal{E}^\Gamma_{\pi_\sigma}(F, G) = (H^\Gamma_{\pi_\sigma} F, G)_{L^2(\pi_\sigma)}, \tag{3.10}
\]

or \( H^\Gamma_{\pi_\sigma} = -\text{div}^\Gamma_{\pi_\sigma} \nabla^\Gamma \) on \( \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \). We call \( H^\Gamma_{\pi_\sigma} \) the intrinsic Dirichlet operator of the measure \( \pi_\sigma \).

4 Extrinsic geometry on Poisson space

We recall the extrinsic geometry on \( L^2(\pi_\sigma) \) based on the isomorphism with the Fock space. Our approach is based on [KSS97], but we should also mention [BLL95], [IK88], [KSSU98], [NV95], [Pri95] for related considerations and references therein. For proofs of the results stated below in this section, we refer to [AKR98a, Sect. 5].

Let us define another “gradient” on functions \( F : \Gamma \rightarrow \mathbb{R} \). This gradient \( \nabla^P \) has specific useful properties on Poissonian spaces. We will call \( \nabla^P \) the
Poissonian gradient. To this end we consider as the tangent space to \( \Gamma \) at any point \( \gamma \in \Gamma \) the same space \( L^2(\sigma) \) and define a mapping \( \mathcal{F}C_\sigma^\infty(\mathcal{D}, \Gamma) \ni F \mapsto \nabla^P F \in L^2(\pi_\sigma) \otimes L^2(\sigma) \) by
\[
(\nabla^P F)(\gamma, x) := F(\gamma + \varepsilon_x) - F(\gamma), \quad \gamma \in \Gamma, \; x \in X.
\]
We stress that the transformation \( \Gamma \ni \gamma \mapsto \gamma + \varepsilon_x \in \Gamma \) is \( \pi_\sigma \)-a.e. well-defined because \( \pi_\sigma(\{ \gamma \in \Gamma | x \in \gamma \}) = 0 \) for any \( x \in X \). The directional derivative is then defined as
\[
(\nabla^P \varphi F)(\gamma) = (\nabla^P F(\gamma, \cdot), \varphi)_{L^2(\sigma)} = \int_X [F(\gamma + \varepsilon_x) - F(\gamma)] \varphi(x) d\sigma(x), \quad \varphi \in \mathcal{D}.
\]
The Poissonian gradient \( \nabla^P \) yields (via a corresponding “integration by parts” formula) an orthogonal system of Charlier polynomials on the Poisson space \( \Gamma, \mathcal{B}(\Gamma), \pi_\sigma \).

For any \( n \in \mathbb{N} \) and all \( \varphi \in \mathcal{D} \) we introduce a function in \( L^2(\pi_\sigma) \) by
\[
Q_n^{\pi_\sigma} (\gamma; \varphi^{\otimes n}) := ((\nabla^P \varphi)^{\ast n} 1)(\gamma),
\]
and define \( Q_0^{\pi_\sigma} := 1 \). Due to the kernel theorem [BK95, Chap. 1] these functions have the representation \( Q_n^{\pi_\sigma} (\gamma; \varphi^{\otimes n}) = \langle Q_n^{\pi_\sigma} (\gamma), \varphi^{\otimes n} \rangle \), with generalized symmetric kernels \( \Gamma \ni \gamma \mapsto Q_n^{\pi_\sigma} (\gamma) \in \text{Exp}_n \mathcal{D}' \), \( n \in \mathbb{N} \). Here and below by \( \text{Exp}_n E \) we denote the \( n \)-th symmetric tensor power of a linear space \( E \). Then for any smooth kernel \( \varphi^{(n)} \in \text{Exp}_n \mathcal{D}^{\otimes n} \) we introduce the function \( Q_n^{\pi_\sigma} (\gamma; \varphi^{(n)}) := \langle Q_n^{\pi_\sigma} (\gamma), \varphi^{(n)} \rangle \) such that for all \( \varphi^{(n)} \in \text{Exp}_n \mathcal{D}^{\otimes n} \), \( \psi^{(m)} \in \text{Exp}_m \mathcal{D}^{\otimes m} \)
\[
\int_{\Gamma} Q_n^{\pi_\sigma} (\gamma; \varphi^{(n)}) Q_m^{\pi_\sigma} (\gamma; \psi^{(m)}) d\pi_\sigma(\gamma) = \delta_{nm} n! \langle \varphi^{(n)}, \psi^{(m)} \rangle_{L^2(\sigma^{\otimes n})}.
\]
Hence \( (4.3) \) extends to the case of kernels from the so-called \( n \)-particle Fock space \( \text{Exp}_n L^2(\sigma) \), \( n \in \mathbb{N} \), and we set \( \text{Exp}_0 L^2(\sigma) := \mathbb{R} \).

As usual the symmetric Fock space over the Hilbert space \( L^2(\sigma) \) is defined as \( \text{Exp} L^2(\sigma) := \oplus_{n=0}^{\infty} \text{Exp}_n L^2(\sigma) \), see e.g. [BK95] and [HKPS93]. The square of the norm of a vector \( (f^{(n)})_{n=0}^{\infty} \in \text{Exp} L^2(\sigma) \) is given by \( \sum_{n=0}^{\infty} n! \| f^{(n)} \|_{L^2(\sigma^{\otimes n})}^2 \).

For any \( F \in L^2(\pi_\sigma) \) there exists such a Fock vector, so that we have the following chaos decomposition
\[
F(\gamma) = \sum_{n=0}^{\infty} Q_n^{\pi_\sigma} (\gamma; f^{(n)}),
\]

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and moreover \( \|F\|_{L^2(\pi_\sigma)}^2 = \sum_{n=0}^{\infty} n! \|f^{(n)}\|_{L^2(\sigma^{\otimes n})}^2 \). And vice versa, by (4.3) any Fock vector generates a function from \( L^2(\pi_\sigma) \). This produces an isomorphism between \( L^2(\pi_\sigma) \) and \( \text{Exp}L^2(\sigma) \).

**Remark 4.1** In probability theory the functions \( Q^\pi_\sigma n(\gamma; f^{(n)}) \) are called the \( n \)-multiple stochastic integrals of \( f^{(n)} \) with respect to the compensated Poisson process generated by the Poisson measure \( \pi_\sigma \), see e.g. [NV95].

There is an alternative approach to the chaos decomposition on the Poisson space which uses the concept of generalized Appell systems, see e.g. [KSS97].

The following proposition shows that the operators \( \nabla^P_\varphi \) and \( \nabla^{P*}_\varphi \) play the role of the annihilation resp. creation operators in the Fock space \( \text{Exp}L^2(\sigma) \).

**Proposition 4.2** For all \( \varphi, \psi \in \mathcal{D}, n \in \mathbb{N} \) the following formulas hold

\[
\nabla^P_\psi Q^\pi_\varphi n(\gamma; \varphi^{\otimes n}) = n(\varphi, \psi)_{L^2(\sigma)} Q^\pi_\varphi n-1(\gamma; \varphi^{\otimes (n-1)})
\]

(4.6)

\[
\nabla^{P*}_\psi Q^\pi_\varphi n(\gamma; \varphi^{\otimes n}) = Q^\pi_\varphi n+1(\gamma; \varphi^{\otimes n} \hat{\otimes} \psi), \; \gamma \in \Gamma,
\]

(4.7)

where \( \varphi^{\otimes n} \hat{\otimes} \psi \) means the symmetric tensor product of \( \varphi^{\otimes n} \) and \( \psi \).

Next we give an explicit expression for the adjoint of the Poissonian gradient \( \nabla^{P*} \).

**Proposition 4.3** For any function \( F \in L^1(\pi_\sigma) \otimes L^1(\sigma) \) we have \( F \in \mathcal{D}(\nabla^{P*}) \) and the following equality holds

\[
(\nabla^{P*} F)(\gamma) = \int_X F(\gamma - \varepsilon_x, x) d\gamma(x) - \int_X F(\gamma, x) d\sigma(x), \; \gamma \in \Gamma,
\]

(4.8)

provided the right hand side of (4.8) is in \( L^2(\pi_\sigma) \).

**Proof.** For \( X = \mathbb{R}^d \) this proposition was proved in [KSSU98]. Let \( G \in \mathcal{F}_{C_b^\infty}(\mathcal{D}, \Gamma) \) be given. Then an application of (4.1) gives

\[
(\nabla^P G, F)_{L^2(\pi_\sigma) \otimes L^2(\sigma)} = \int_X \int_{\Gamma} G(\gamma + \varepsilon_x) F(\gamma, x) d\pi_\sigma(\gamma) d\sigma(x) - \int_X \int_{\Gamma} G(\gamma) F(\gamma, x) d\pi_\sigma(\gamma) d\sigma(x).
\]

(4.9)
Now we use the Mecke identity, see e.g., [Mec67, Satz 3.1]

\[ \int_{\Gamma} \left( \int_{X} h(\gamma, x) d\gamma(x) \right) d\pi_{\sigma}(\gamma) = \int_{X} \int_{\Gamma} h(\gamma + \varepsilon x, x) d\pi_{\sigma}(\gamma) d\sigma(x), \quad (4.10) \]

where \( h \) is any non-negative, \( B(\Gamma) \times B(X) \)-measurable function. By (4.10) the right hand side of (4.9) transforms into

\[ \int_{\Gamma} G(\gamma) \left[ \int_{X} F(\gamma - \varepsilon x, x) d\gamma(x) - \int_{X} F(\gamma, x) d\sigma(x) \right] d\pi_{\sigma}(\gamma), \]

which proves the proposition.

For any contraction \( B \) in \( L^2(\sigma) \) it is possible to define an operator \( \text{Exp}B \) as a contraction in \( \text{Exp}L^2(\sigma) \) which in any \( n \)-particle subspace \( \text{Exp}_nL^2(\sigma) \) is given by \( B \otimes \cdots \otimes B \) (\( n \) times). For any positive self-adjoint operator \( A \) in \( L^2(\sigma) \) (with \( \mathcal{D} \subset D(A) \)) we have a contraction semigroup \( e^{-tA}, t \geq 0 \), hence it is possible to introduce the second quantization operator \( d\text{Exp}A \) as the generator of the semigroup \( \text{Exp}(e^{-tA}), t \geq 0 \), i.e., \( \text{Exp}(e^{-tA}) = \exp(-td\text{Exp}A) \), see e.g., [RS75]. We denote by \( H^P_A \) the image of the operator \( d\text{Exp}A \) in the Poisson space \( L^2(\pi_{\sigma}) \) under the described isomorphism.

**Proposition 4.4** Let \( \mathcal{D} \subset D(A) \). Then the symmetric bilinear form corresponding to the operator \( H^P_A \) has the following form

\[ (H^P_A F, G)_{L^2(\pi_{\sigma})} = \int_{\Gamma} (\nabla^P F(\gamma), A\nabla^P G(\gamma)) d\pi_{\sigma}(\gamma), \quad F, G \in \mathcal{F} \mathcal{C}^{\infty}_b(\mathcal{D}, \Gamma). \]

The right hand side of (4.11) is called the “Poissonian pre-Dirichlet form” with coefficient operator \( A \) and is denoted by \( \mathcal{E}^P_{\pi_{\sigma},A} \).

Let us consider the special case of the second quantization operator \( d\text{Exp}A \), where the one-particle operator \( A \) coincides with the Dirichlet operator \( H^X_{\sigma} \) generated by the measure \( \sigma \) on \( X \). Then we have the following theorem which relates the intrinsic Dirichlet operator \( H^I_{\pi_{\sigma}} \) and the operator \( H^P_{H^X_{\sigma}} \).
Theorem 4.5 \( H_{\pi_\sigma}^\Gamma = H_{\mathcal{H}^\Gamma_{\#}}^P \) on \( \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}, \Gamma) \). In particular, for all \( F, G \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}, \Gamma) \)

\[
\int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma) \rangle_{T_\Gamma, \Gamma} d\pi_\sigma(\gamma) = \int_\Gamma \langle \nabla^P F(\gamma), H_{\mathcal{H}^\Gamma_{\#}}^P G(\gamma) \rangle_{L^2(\sigma)} d\pi_\sigma(\gamma) = \int_\Gamma \int_X \langle \nabla^X \nabla^P F(\gamma, x), \nabla^X \nabla^P G(\gamma, x) \rangle_{T_\Gamma X} d\sigma(x) d\pi_\sigma(\gamma). \tag{4.12}
\]

Remark 4.6 Theorem \ref{thm:4.5} gives full information about the spectrum of the intrinsic Dirichlet operator \( H_{\pi_\sigma}^\Gamma \) on the Poisson space in terms of the underlying Dirichlet operator \( H_{\sigma}^{\mathcal{H}^\Gamma_{\#}}^X \) coming from the intensity measure \( \sigma \). In the following section we obtain an analogue of Theorem \ref{thm:4.5} for the class of measures \( \mathcal{G}_1^{gc}(\sigma, \Phi) \) (see \ref{thm:6.3} below) which is the aim of this paper.

5 Relation between intrinsic and extrinsic Dirichlet forms

Here we consider the class of measures \( \mathcal{G}_1^{gc}(\sigma, \Phi) \) consisting of all \( \mu \in \mathcal{G}_1^{gc}(\sigma, \Phi) \) such that

\[
\int_\Gamma \gamma(K) d\mu(\gamma) < \infty \text{ for all compact } K \subset X.
\]

We define for any \( \mu \in \mathcal{G}_1^{gc}(\sigma, \Phi) \) the pre-Dirichlet form \( \mathcal{E}^\Gamma_\mu \) by

\[
\mathcal{E}^\Gamma_\mu(F, G) := \int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma) \rangle_{T_\Gamma, \Gamma} d\mu(\gamma), \ F, G \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{D}, \Gamma). \tag{5.1}
\]

After all our preparations we are now going to prove an analogue of \eqref{eq:4.12} for \( \mu \in \mathcal{G}_1^{gc}(\sigma, \Phi) \). We would like to emphasize that the corresponding formula \eqref{eq:5.2} is not obtained from \eqref{eq:4.12} by just replacing \( \pi_\sigma \) by \( \mu \in \mathcal{G}_1^{gc}(\sigma, \Phi) \). The essential difference is, in addition, an extra factor involving the conditional energy \( E_\Lambda^\Phi \).
Theorem 5.1 For any $\mu \in \mathcal{G}^{1}_{ge}(\sigma, \Phi)$, we have for all $F, G \in \mathcal{F}C_{b}^\infty(\mathcal{D}, \Gamma)$

$$
\mathcal{E}_{\mu}^{F}(F, G) = \int_{\Gamma} \langle \nabla^{\Gamma} F(\gamma), \nabla^{\Gamma} G(\gamma) \rangle_{T_{\gamma} \Gamma} d\mu(\gamma) = \int_{\Gamma} \int_{X} \langle \nabla^{\Gamma} \nabla^{P} F(\gamma, x), \nabla^{\Gamma} \nabla^{P} G(\gamma, x) \rangle_{T_{x} X} e^{-E_{\sigma}^{x}(\gamma + \varepsilon_{x})} d\sigma(x) d\mu(\gamma) \tag{5.2}
$$

Proof. Let us take any $F \in \mathcal{F}C_{b}^\infty(\mathcal{D}, \Gamma)$ of the form (3.1). Then given $\gamma \in \Gamma$ and $x \in X$ (4.1) implies that

$$
\nabla^{\Gamma} \nabla^{P} F(\gamma, x) = \nabla^{\Gamma} F(\gamma + \varepsilon_{x}) = \sum_{i=1}^{N} \frac{\partial g_{F}}{\partial s_{i}}(\langle \varphi_{1}, \gamma \rangle + \varphi_{1}(x), \ldots, \langle \varphi_{N}, \gamma \rangle + \varphi_{N}(x)) \nabla^{\Gamma} \varphi_{i}(x).
$$

Let us define $\hat{F}_{i}(\gamma) := \frac{\partial g_{F}}{\partial s_{i}}(\langle \varphi_{1}, \gamma \rangle, \ldots, \langle \varphi_{N}, \gamma \rangle)$, $i = 1, \ldots, N$. Obviously, it is enough to prove the equality (5.2) for $F = G$. Thus, inserting the result of $\nabla^{\Gamma} \nabla^{P} F(\gamma, x)$ into the right hand side of (5.2) we obtain

$$
\int_{\Gamma} \int_{X} \sum_{i,j=1}^{N} \langle \nabla^{\Gamma} \varphi_{i}(x), \nabla^{\Gamma} \varphi_{j}(x) \rangle_{T_{x} X} \hat{F}_{i}(\gamma) \hat{F}_{j}(\gamma + \varepsilon_{x}) e^{-E_{\sigma}^{x}(\gamma + \varepsilon_{x})} d\sigma(x) d\mu(\gamma) \tag{5.3}
$$

Then we need the following useful proposition which generalizes the Mecke identity to measures in $\mathcal{G}^{1}_{ge}(\sigma, \Phi)$, see [NZ79], [MMW79].

Proposition 5.2 Let $h : \Gamma \times X \rightarrow \mathbb{R}_+$ be $\mathcal{B}(\Gamma) \times \mathcal{B}(X)$-measurable, and let $\mu \in \mathcal{G}^{1}_{ge}(\sigma, \Phi)$. Then we have

$$
\int_{\Gamma} \int_{X} h(\gamma, x) d\gamma(x) d\mu(\gamma) = \int_{\Gamma} \int_{X} h(\gamma + \varepsilon_{x}, x) e^{-E_{\sigma}^{x}(\gamma + \varepsilon_{x})} d\mu(\gamma) d\sigma(x). \tag{5.4}
$$

Using this proposition we transform (5.3) into

$$
\int_{\Gamma} \sum_{i,j=1}^{N} \hat{F}_{i}(\gamma) \hat{F}_{j}(\gamma) \langle \langle \nabla^{\Gamma} \varphi_{i}(\cdot), \nabla^{\Gamma} \varphi_{j}(\cdot) \rangle_{T_{X} \Gamma}, \gamma \rangle d\mu(\gamma).
$$

On the other hand using (3.5) we obtain

$$
\langle \nabla^{\Gamma} F(\gamma), \nabla^{\Gamma} G(\gamma) \rangle_{T_{\gamma} \Gamma} = \sum_{i,j=1}^{N} \hat{F}_{i}(\gamma) \hat{F}_{j}(\gamma) \langle \langle \nabla^{\Gamma} \varphi_{i}(\cdot), \nabla^{\Gamma} \varphi_{j}(\cdot) \rangle_{T_{X} \Gamma}, \gamma \rangle.
$$
Therefore the equality on the dense $\mathcal{F}_b^\infty(\mathcal{D},\Gamma)$ is valid which proves the theorem. 

\begin{remark}
For the reader’s convenience let us give a heuristic proof of the Nguyen-Zessin characterization of Gibbs measures in (5.4) which really is a consequence of the Mecke identity (cf. (4.10)). Indeed, let us write (heuristically)

$$d\mu(\gamma) = \frac{1}{Z^\sigma,\Phi} e^{-E^\Phi(\gamma)} d\pi_\sigma(\gamma).$$

Then the function $E^\Phi(x)(\gamma + \varepsilon_x) = E^\Phi(\gamma + \varepsilon_x) - E^\Phi(\gamma)$ informally is the variation of the potential energy $E^\Phi(\gamma)$ when we add to the configuration $\gamma$ an additional point $x \in X$. Using this representation we have

$$\int_X \int_{\Gamma} h(\gamma + \varepsilon_x, x) e^{-E^\Phi(x)(\gamma + \varepsilon_x)} d\mu(\gamma) d\sigma(x) = (Z^\sigma,\Phi)^{-1} \int_X \int_{\Gamma} h(\gamma + \varepsilon_x, x) e^{-E^\Phi(x)(\gamma + \varepsilon_x)} d\pi_\sigma(\gamma) d\sigma(x).$$

Then we use the Mecke identity to transform the right hand side of the above equality into

$$(Z^\sigma,\Phi)^{-1} \int_{\Gamma} \left( \int_X h(\gamma, x) e^{-E^\Phi(\gamma)} d\gamma(x) \right) d\pi_\sigma(\gamma) = \int_{\Gamma} \left( \int_X h(\gamma, x) d\gamma(x) \right) d\mu(\gamma).$$

The rigorous proof in [NZ79] is obtained as a formalization of the heuristic computations above.

\section{Closability of intrinsic Dirichlet forms}

In this section we will prove the closability of the intrinsic Dirichlet form $(\mathcal{E}_\mu^\Gamma, \mathcal{F}_b^\infty(\mathcal{D},\Gamma))$ on $L^2(\mu) := L^2(\Gamma, \mu)$ for all $\mu \in G^1_{\text{gc}}(\sigma, \Phi)$, using the integral representation (5.2) in Theorem 5.1. The closability of $(\mathcal{E}_\mu^\Gamma, \mathcal{F}_b^\infty(\mathcal{D},\Gamma))$ over $\Gamma$ is implied by the closability of an appropriate family of pre-Dirichlet forms over $X$. Let us describe this more precisely.

We define new intensity measures on $X$ by $d\sigma_\gamma(x) := \rho_\gamma(x) dm(x)$, where

$$\rho_\gamma(x) := e^{-E^\Phi(x)(\gamma + \varepsilon_x)} \rho(x), \ x \in X, \gamma \in \Gamma \quad (6.1)$$
It was shown in [AR90, Theorem 5.3] (in the case $X = \mathbb{R}^d$) that the components of the Dirichlet form $(E_{\sigma,\gamma}^X, D_{\sigma,\gamma})$ corresponding to the measure $\sigma_\gamma$ are closable on $L^2(\mathbb{R}^d, \sigma_\gamma)$ if and only if $\sigma_\gamma$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ and the Radon-Nikodym derivative satisfies some condition, see (6.2) below for details. This result allows us to prove the closability of $(E_{\mu,\Gamma}^\Gamma, FC^\infty_b(\mathcal{D}, \Gamma))$ on $L^2(\mu)$. Let us first recall the above mentioned result.

**Theorem 6.1** (cf. [AR90, Theorem 5.3]) Let $\nu$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), d \in \mathbb{N}$ and let $D^\nu$ denote the $\nu$-classes determined by $D$. Then the forms $(E_{\nu,i,\nu}^\nu, D^\nu)$ defined by

$$E_{\nu,i}(u,v) := \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} d\nu, \ u,v \in D,$$

are well-defined and closable on $L^2(\mathbb{R}^d, \nu)$ for $1 \leq i \leq d$ if and only if $\nu$ is absolutely continuous with respect to Lebesgue measure $\lambda^d$ on $\mathbb{R}^d$, and the Radon-Nikodym derivative $\rho = d\nu/d\lambda^d$ satisfies the condition:

for any $1 \leq i \leq d$ and $\lambda^{d-1}$-a.e. $x \in \left\{ y \in \mathbb{R}^{d-1} \mid \int_\mathbb{R} \rho_{y}^{(i)}(s) d\lambda^1(s) > 0 \right\}$, $\rho_{x}^{(i)} = 0 \lambda^1$-a.e. on $\mathbb{R}\setminus R(\rho_{x}^{(i)})$, where $\rho_{x}^{(i)}(s) := \rho(x_1, \ldots, x_{i-1}, s, x_i, \ldots, x_d)$, $s \in \mathbb{R}$, if $x = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$, and where

$$R(\rho_{x}^{(i)}) := \left\{ t \in \mathbb{R} \mid \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\rho_{x}^{(i)}}(s) ds < \infty \text{ for some } \varepsilon > 0 \right\}.$$ (6.3)

There is an obvious generalization of Theorem 6.1 to the case where a Riemannian manifold $X$ is replacing $\mathbb{R}^d$, to be formulated in terms of local charts. Since here we are only interested in the “if part” of Theorem 6.1, we now recall a slightly weaker sufficient condition for closability in the general case where $X$ is a manifold as before.

**Theorem 6.2** Suppose $\sigma_1 = \rho_1 \cdot m$, where $\rho_1 : X \to \mathbb{R}_+$ is $\mathcal{B}(X)$-measurable such that

$$\rho_1 = 0 \text{ } m\text{-a.e. on } X \setminus \left\{ x \in X \mid \int_{\Lambda_x} \frac{1}{\rho_1} dm < \infty \text{ for some open neighbourhood } \Lambda_x \text{ of } x \right\}.$$ (6.4)
Then \((\mathcal{E}_{\sigma_1}^X, \mathcal{D}^{\sigma_1})\) defined by
\[
\mathcal{E}_{\sigma_1}^X(u, v) := \int_X \left\langle \nabla^X u(x), \nabla^X v(x) \right\rangle_{T_x X} d\sigma_1(x); \; u, v \in \mathcal{D},
\]
is closable on \(L^2(\sigma_1)\).

The proof is a straightforward adaptation of the line of arguments in [MR92, Chap. II, Subsection 2a] (see also [ABR89, Theorem 4.2] for details). We emphasize that (6.4) e.g. always holds, if \(\rho_1\) is lower semicontinuous, and that neither \(\nu\) in Theorem 6.1 nor \(\sigma_1\) in Theorem 6.2 is required to have full support, so e.g. \(\rho_1\) is not necessarily strictly positive \(m\)-a.e. on \(X\).

We are now ready to prove the closability of \((\mathcal{E}_{\mu}^\Gamma, FC^\infty_b(D, \Gamma))\) on \(L^2(\mu)\) under the above assumption.

**Theorem 6.3** Let \(\mu \in G^1_{gc}(\sigma, \Phi)\). Suppose that for \(\mu\)-a.e. \(\gamma \in \Gamma\) the function \(\rho_\gamma\) defined in (6.1) satisfies (6.4) (resp. (6.2) in case \(X = \mathbb{R}^d\)). Then the form \((\mathcal{E}_\mu^\Gamma, FC^\infty_b(D, \Gamma))\) is closable on \(L^2(\mu)\).

**Proof.** Let \((F_n)_{n \in \mathbb{N}}\) be a sequence in \(FC^\infty_b(D, \Gamma)\) such that \(F_n \to 0, n \to \infty\) in \(L^2(\mu)\) and
\[
\mathcal{E}_\mu^\Gamma(F_n - F_m, F_n - F_m) \xrightarrow{n,m \to \infty} 0. \tag{6.5}
\]
We have to show that
\[
\mathcal{E}_\mu^\Gamma(F_{n_k}, F_{n_k}) \xrightarrow{k \to \infty} 0 \tag{6.6}
\]
for some subsequence \((n_k)_{k \in \mathbb{N}}\). Let \((n_k)_{k \in \mathbb{N}}\) be a subsequence such that
\[
\left(\int_{\Gamma} F_{n_k}^2 d\mu\right)^{1/2} + \mathcal{E}_\mu^\Gamma(F_{n_{k+1}} - F_{n_k}, F_{n_{k+1}} - F_{n_k})^{1/2} \leq \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.
\]
Then
\[
\infty > \sum_{k=1}^{\infty} \mathcal{E}_\mu^\Gamma(F_{n_{k+1}} - F_{n_k}, F_{n_{k+1}} - F_{n_k})^{1/2}
\]
\[
\geq \sum_{k=1}^{\infty} \int_{\Gamma} \left(\int_X |\nabla^X \nabla^P(F_{n_{k+1}} - F_{n_k})(x, \gamma)|^2_{T_x X} e^{-E_\mu^\Phi(\gamma + \varepsilon_k)} d\sigma(x)\right)^{1/2} d\mu(\gamma)
\]
\[
= \int_{\Gamma} \sum_{k=1}^{\infty} \left(\int_X |\nabla^X \nabla^P(F_{n_{k+1}} - F_{n_k})(x, \gamma)|^2_{T_x X} \rho_\gamma(x) dm(x)\right)^{1/2} d\mu(\gamma),
\]
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where we used Theorem 5.1 and (6.1). From the last expression we obtain that
\[
\sum_{k=1}^{\infty} \mathcal{E}_{\sigma, \gamma}^{X}(u^{(\gamma)}_{n_{k+1}} - u^{(\gamma)}_{n_k}, u^{(\gamma)}_{n_{k+1}} - u^{(\gamma)}_{n_k})^{1/2}
= \sum_{k=1}^{\infty} \left( \int_{X} |\nabla^{X} \nabla^{P}(F^{n_{k+1}} - F^{n_k})(x, \gamma)|^{2} \tau_{X} \rho_{\gamma}(x)dm(x) \right)^{1/2}
< \infty \quad \text{for } \mu\text{-a.e. } \gamma \in \Gamma,
\] (6.7)
where for \( k \in \mathbb{N}, \gamma \in \Gamma \),
\[
u^{(\gamma)}_{n_k}(x) := F^{n_k}(\gamma + \varepsilon x) - F^{n_k}(\gamma), \quad x \in X.
\]
Note that \( \nu^{(\gamma)}_{n_k} \in \mathcal{D} \). (6.7) implies that for \( \mu\text{-a.e. } \gamma \in \Gamma \)
\[
\mathcal{E}_{\sigma, \gamma}^{X}(u^{(\gamma)}_{n_k} - u^{(\gamma)}_{n_l}, u^{(\gamma)}_{n_k} - u^{(\gamma)}_{n_l}) \rightarrow \mathcal{E}_{\sigma, \gamma}^{X}(0, 0) \quad k, l \rightarrow \infty.
\] (6.8)
Let \( \Lambda \subset \mathcal{O}_{c}(X) \).

**Claim 1:** For \( \mu\text{-a.e. } \gamma \in \Gamma \)
\[
\int_{X} (u^{(\gamma)}_{n_k}(x))^{2} \Pi_{\Lambda}(x)d\sigma_{\gamma}(x) \rightarrow 0.
\]
To prove Claim 1 we first note that for \( \mu\text{-a.e. } \gamma \in \Gamma \)
\[
\sigma_{\gamma}(\Lambda) < \infty,
\]
as follows immediately from (5.4) (taking \( h(\gamma, x) := \Pi_{\Lambda}(x) \) for \( x \in X, \gamma \in \Gamma \)),
since \( \mu \in \mathcal{G}_{\text{ge}}^{1} (\sigma, \Phi) \). Therefore, for \( \mu\text{-a.e. } \gamma \in \Gamma \)
\[
\int_{X} F^{2}_{n_k}(\gamma) \Pi_{\Lambda}(x)d\sigma_{\gamma}(x) = F^{2}_{n_k}(\gamma)\sigma_{\gamma}(\Lambda) \rightarrow 0.
\] (6.9)
Furthermore, by (5.4)
\[
\int_{\Gamma} \int_{X} F^{2}_{n_k}(\gamma + \varepsilon x) \Pi_{\Lambda}(x)d\sigma_{\gamma}(x)(1 + \gamma(\Lambda))^{-1}d\mu(\gamma)
= \int_{\Gamma} F^{2}_{n_k}(\gamma) \int_{X} \frac{\Pi_{\Lambda}(x)}{1 + \gamma(\Lambda) - \Pi_{\Lambda}(x)}(dx)d\mu(\gamma)
\leq \int_{\Gamma} F^{2}_{n_k}(\gamma)d\mu(\gamma) < \frac{1}{2k}.
\]
because the integral w.r.t. $\gamma$ is dominated by 1 for all $\gamma \in \Gamma$. Hence

\[
\infty > \sum_{k=1}^{\infty} \left( \int_{\Gamma} \int_{X} F_{n_k}^2 (\gamma + \varepsilon_x) \mathbb{I}_{\Lambda}(x) d\sigma_\gamma(x) (1 + \gamma(\Lambda))^{-1} d\mu(\gamma) \right)^{1/2}
\]

\[
\geq \sum_{k=1}^{\infty} \left( \int_{\Gamma} \int_{X} F_{n_k}^2 (\gamma + \varepsilon_x) \mathbb{I}_{\Lambda}(x) d\sigma_\gamma(x) \right)^{1/2} (1 + \gamma(\Lambda))^{-1} d\mu(\gamma).
\]

Therefore, for $\mu$-a.e. $\gamma \in \Gamma$

\[
\int_{X} F_{n_k}^2 (\gamma + \varepsilon_x) \mathbb{I}_{\Lambda}(x) d\sigma_\gamma(x) \rightarrow 0 \quad (6.10)
\]

Then Claim 1 follows by (6.9) and (6.10).

**Claim 2:** For $\mu$-a.e. $\gamma \in \Gamma$

\[
|\nabla X u_{n_k}^{(\gamma)}|_{TX} \rightarrow 0 \quad \sigma_\gamma-a.e.
\]

To prove Claim 2 we first note that clearly (6.7) implies that for $\mu$-a.e. $\gamma \in \Gamma$

\[
\mathcal{E}_{\mathbb{I}_{\Lambda}\sigma_\gamma}(u_{n_k}^{(\gamma)} - u_{n_l}^{(\gamma)}, u_{n_k}^{(\gamma)} - u_{n_l}^{(\gamma)}) \rightarrow 0 \quad (6.11)
\]

Hence we can apply Theorem 3.2 (resp. 6.1) to $\rho_1 := \mathbb{I}_{\Lambda}\rho_\gamma$ and conclude by Claim 1 and (6.11) that for $\mu$-a.e. $\gamma \in \Gamma$

\[
\mathcal{E}_{\mathbb{I}_{\Lambda}\sigma_\gamma}(u_{n_k}^{(\gamma)}, u_{n_k}^{(\gamma)}) \rightarrow 0,
\]

hence by (6.7)

\[
\mathbb{I}_{\Lambda}|\nabla X u_{n_k}^{(\gamma)}|_{TX} \rightarrow 0 \quad \sigma_\gamma-a.e.
\]

Since $\Lambda$ was arbitrary, Claim 2 is proven.

From Claim 2 we now easily deduce (6.6) by (5.2) and Fatou’s Lemma as follows:

\[
\mathcal{E}_{\mu}(F_{n_k}, F_{n_k}) \leq \int_{\Gamma} \liminf_{l \to \infty} \int_{X} |\nabla X (u_{n_k}^{(\gamma)} - u_{n_l}^{(\gamma)})|_{TX}^2 d\sigma_\gamma(x) d\mu(\gamma)
\]

\[
\leq \liminf_{l \to \infty} \mathcal{E}_{\mu}(F_{n_k} - F_{n_l}, F_{n_k} - F_{n_l}),
\]

which by (6.3) can be made arbitrarily small for $k$ large enough. ■
Remark 6.4 The above method to prove closability of pre-Dirichlet forms on configuration spaces $\Gamma_X$ extends immediately to the case where $X$ is replaced by an infinite dimensional “manifold” such as the loop space (cf. [MR97]).

Example 6.5 Let $X = \mathbb{R}^d$ with the Euclidean metric and $\sigma := z \cdot m$, $z \in (0, \infty)$. A pair potential is a $\mathcal{B}(\mathbb{R}^d)$-measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that $\phi(-x) = \phi(x)$. Any pair potential $\phi$ defines a potential $\Phi = \Phi_\phi$ in the sense of Section 2 as follows: we set $\Phi(\gamma) := 0$, $|\gamma| \neq 2$ and $\Phi(\gamma) := \phi(x-y)$ for $\gamma = \{x,y\} \subset \mathbb{R}^d$. For such pair potentials $\phi$ the condition in Theorem 6.3 ensuring closability of $(E_{\Gamma}^\mu, FC^\infty(D, \Gamma))$ on $L^2(\mu)$ for $\mu \in \mathcal{G}_{ge}^1(\sigma, \Phi)$ can be now easily formulated as follows: for $\mu$-a.e. $\gamma \in \Gamma$ and $m$-a.e. $x \in \{y \in \mathbb{R}^d | \sum_{y' \in \gamma \setminus \{y\}} |\phi(y-y')| < \infty\}$ it holds that $\int_{V_x} e^{\sum_{y' \in \gamma \setminus \{y\}} \phi(y-y')} m(dy) < \infty$ for some open neighbourhood $V_x$ of $x$. This condition trivially holds e.g. if $\text{supp}\phi$ is compact, $\{\phi < \infty\}$ is open, and $\phi^+ \in L^\infty_{\text{loc}}(\{\phi < \infty\}; m)$. If even $\mu \in \mathcal{G}_{ge}^1(z, \phi)$ and $\phi$ satisfies the assumptions in Proposition 7.1, then it suffices to merely assume that $\{\phi < \infty\}$ is open and $\phi^+ \in L^\infty_{\text{loc}}(\{\phi < \infty\}; m)$. This follows by an elementary consideration.

Remark 6.6 1. We emphasize that Example 6.5 generalizes the closability result in [Osa96], though an a-priori bigger domain for $E_{\Gamma}^\mu$ is considered there. However, Theorems 6.1-6.3 are also valid for this bigger domain. The proofs are exactly the same.

2. We also like to emphasize that similarly to Example 6.5 one proves the closability of $(E_{\Gamma}^\mu, FC^\infty(D, \Gamma))$ (or with a larger domain in [Osa96]) on $L^2(\mu)$ for $\mu \in \mathcal{G}_{ge}^1(\sigma, \Phi)$ in the case of multi-body potentials $\phi$.

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References

[ABR89] S. Albeverio, J. Brasche, and M. Röckner. Dirichlet forms and generalized Schrödinger operators. In H. Holden and A. Jensen, editors, *Schrödinger Operators*, volume 345 of *Lecture Notes in Phys.*, pages 1–42, Berlin, 1989. Springer-Verlag.

[AKR96a] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Canonical Dirichlet operator and distorted Brownian motion on Poisson spaces. *C. R. Acad. Sci., Paris Sé. I Math.*, 323:1179–1184, 1996.

[AKR96b] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Differential geometry of Poisson spaces. *C. R. Acad. Sci. Paris Sé. I Math.*, 323:1129–1134, 1996.

[AKR98a] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces. *J. Funct. Anal.*, 154:444–500, 1998.

[AKR98b] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces: The Gibbsian case. *J. Funct. Anal.*, 157:242–291, 1998.

[AKR99] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Diffeomorphism groups and current algebras: configuration spaces analysis in quantum theory. *Rev. Math. Phys.*, 11(97-073):1–23, 1999.

[AR90] S. Albeverio and M. Röckner. Classical Dirichlet forms on topological vector spaces - closability and a Cameron-Martin formula. *J. Func. Anal.*, 88:395–436, 1990.

[BK95] Yu. M. Berezansky and Yu. G. Kondratiev. *Spectral Methods in Infinite-Dimensional Analysis*. Kluwer Academic Publishers, Dordrecht, 1995.

[BLL95] Yu. M. Berezansky, V. O. Livinsky, and E. W. Lytvynov. A generalization of Gaussian white noise analysis. *Methods of Functional Analysis and Topology*, 1(1):28–55, 1995.
[Geo79] O. Georgii. *Canonical Gibbs Measures*, volume 760 of *Lectures Notes in Math.* Springer-Verlag, Berlin, 1979.

[GV68] I. M. Gel’fand and N. Ya. Vilenkin. *Generalized Functions*, volume 4. Academic Press, New York and London, 1968.

[HKPS93] T. Hida, H. H. Kuo, J. Potthoff, and L. Streit. *White Noise. An Infinite Dimensional Calculus.* Kluwer, Dordrecht, 1993.

[IK88] Y. Ito and I. Kubo. Calculus on Gaussian and Poisson white noises. *Nagoya Math. J.*, 111:41–84, 1988.

[KSS97] Yu. G. Kondratiev, J. L. Silva, and L. Streit. Generalized Appell systems. *Methods of Functional Analysis and Topology*, 3(3):28–61, 1997.

[KSSU98] Yu. G. Kondratiev, J. L. Silva, L. Streit, and F. G. Us. Analysis on Poisson and Gamma spaces. *Infinite Dimensional Analysis, Quantum Probabilities and Related Topics*, 1(1):91–117, 1998.

[Mec67] J. Mecke. Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen. *Z. Wahrsch. verw. Gebiete*, 9:36–58, 1967.

[MMW79] K. Matthes, J. Mecke, and W. Warmuth. Bemerkungen zu einer Arbeit von Nguyen Xuan Xanh und Hans Zessin. *Math. Nachr.*, 88:117–127, 1979.

[MR92] Z.-M. Ma and M. Röckner. *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*. Springer-Verlag, Berlin, 1992.

[MR97] Z.-M. Ma and M. Röckner. Construction of diffusions on configuration spaces. Preprint, 1997.

[NV95] D. Nualart and J. Vives. A duality formula on the Poisson space and some applications. In *Seminar on Stochastic Analysis, Random Fields and Applications*, volume 36, pages 205–213, Basel, 1995. Birkhäuser.

[NZ79] X. X. Nguyen and H. Zessin. Integral and differential characterizations of the Gibbs process. *Math. Nachr.*, 88:105–115, 1979.
[Osa96] H. Osada. Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions. *Comm. Math. Phys.*, 176(1):117–131, 1996.

[Osa98] H. Osada. Interacting Brownian motions with measurable potentials. *Japan Academy*, 74, Ser. A(1), 1998.

[Pre76] C. J. Preston. *Random Fields*, volume 534 of *Lectures Notes in Math.* Springer-Verlag, Berlin Heidelberg New York, 1976.

[Pre79] J. C. Preston. Canonical and microcanonical Gibbs states. *Z. Wahrsch. verw. Gebiete*, 46:125–158, 1979.

[Pri95] N. Privault. A transfer principle from Wiener to Poisson space and applications. *J. Funct. Anal.*, 132:335–360, 1995.

[RS75] M. Reed and B. Simon. *Methods of Modern Mathematical Physics*. Academic Press, New York and London, 1975.

[Shi94] H. Shimomura. Poisson measures on the configuration space and unitary representations of the group of diffeomorphism. *J. Math. Kyoto Univ.*, 34:599–614, 1994.