MV-algebras freely generated by finite Kleene algebras

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Abstract. If \( V \) and \( W \) are varieties of algebras such that any \( V \)-algebra \( A \) has a reduct \( U(A) \) in \( W \), there is a forgetful functor \( U: V \rightarrow W \) that acts by \( A \mapsto U(A) \) on objects, and identically on homomorphisms. This functor \( U \) always has a left adjoint \( F: W \rightarrow V \) by general considerations. One calls \( F(B) \) the \( V \)-algebra freely generated by the \( W \)-algebra \( B \). Two problems arise naturally in this broad setting. The description problem is to describe the structure of the \( V \)-algebra \( F(B) \) as explicitly as possible in terms of the structure of the \( W \)-algebra \( B \). The recognition problem is to find conditions on the structure of a given \( V \)-algebra \( A \) that are necessary and sufficient for the existence of a \( W \)-algebra \( B \) such that \( F(B) \cong A \). Building on and extending previous work on MV-algebras freely generated by finite distributive lattices, in this paper we provide solutions to the description and recognition problems in case \( V \) is the variety of MV-algebras, \( W \) is the variety of Kleene algebras, and \( B \) is finitely generated—equivalently, finite. The proofs rely heavily on the Davey–Werner natural duality for Kleene algebras, on the representation of finitely presented MV-algebras by compact rational polyhedra, and on the theory of bases of MV-algebras.

1. Introduction.

Consider a variety of algebras \( V \), and write \( \mathcal{F}_\kappa^V \) for the algebra in \( V \) freely generated by a set of cardinality \( \kappa \). Suppose further that \( W \) is a variety such that any \( V \)-algebra \( A \) has a reduct \( U(A) \) in \( W \). Then there exists a forgetful functor \( U: V \rightarrow W \) that acts by \( A \mapsto U(A) \) on objects, and identically on homomorphisms. This functor \( U \) always has a left adjoint, as follows. Let \( B \) be any \( W \)-algebra, and say its cardinality is \( \kappa = |B| \). Then \( B \) is isomorphic to a quotient \( \mathcal{F}_\kappa^W / \Theta \), for some congruence \( \Theta \subseteq \mathcal{F}_\kappa^W \times \mathcal{F}_\kappa^W \). Because of our assumption about \( V \) and \( W \), each \( W \)-term also is a \( V \)-term; hence, \( \Theta \) generates a uniquely determined congruence on \( \mathcal{F}_\kappa^V \). More formally, there is a unique \( W \)-homomorphism, \( u_\kappa: \mathcal{F}_\kappa^W \rightarrow U(\mathcal{F}_\kappa^V) \), that extends the set-theoretic bijection between free generators. Writing \( u_\kappa^2: (\mathcal{F}_\kappa^W)^2 \rightarrow U(\mathcal{F}_\kappa^V)^2 \) for the product map \((x, y) \mapsto (u_\kappa(x), u_\kappa(y))\), let \( \overline{\Theta}_\kappa \) denote the \( V \)-congruence on \( \mathcal{F}_\kappa^V \) generated by \( u_\kappa^2(\Theta) \). Then there is a unique \( W \)-homomorphism

\[
B \cong \mathcal{F}_\kappa^W / \Theta \xrightarrow{u_\kappa^2} U(\mathcal{F}_\kappa^V / \overline{\Theta}_\kappa)
\]
that extends the map that sends the $\Theta$-class of a free generator of $\mathcal{F}_W^\kappa$ to the $\widehat{u}_k^2(\Theta)$-class of the corresponding generator of $\mathcal{F}_V^\kappa$. (Here and throughout, $\cong$ denotes the existence of an isomorphism). Now $\eta_B$ can be shown to be a universal arrow from $B$ to the functor $U$, in the sense of [11, Definition on p. 55]: for any $V$-algebra $A$ and any $W$-homomorphism $f: B \to U(A)$, there exists a unique $V$-homomorphism $f': \mathcal{F}_\kappa^V/\widehat{u}_k^2(\Theta) \to A$ such that $f = U(f') \circ \eta_B$, i.e., such that the following diagram commutes.

\[
\begin{array}{c}
U(A) \\
\uparrow
\end{array}
\quad
\begin{array}{c}
U(f')
\downarrow
\end{array}
\quad
\begin{array}{c}
\eta_B
\downarrow
\end{array}
\quad
\begin{array}{c}
U(\mathcal{F}_\kappa^V/\widehat{u}_k^2(\Theta))
\end{array}
\quad
\begin{array}{c}
B
\end{array}
\]

By [11, Theorem IV.1.2], this universal property of $\eta_B$ uniquely determines a functor $F: W \to V$ that is a left adjoint of $U$. The action of $F$ on a $W$-algebra $B$ is just

\[F(B) = \mathcal{F}_\kappa^V/\widehat{u}_k^2(\Theta),\]

and one calls $F(B)$ the $V$-algebra freely generated by the $W$-algebra $B$; one also says that $F(B)$ is free over $B$. (This terminology notwithstanding, note that neither $u_\kappa$ nor $\eta_B$ need be injective, and that $B$ need not be isomorphic to a subalgebra of the $W$-reduct of $F(B)$. Take for $W$ the variety of groups, and for $V$ the variety of Abelian groups. Then $F(B)$ is the Abelianization of $B$, and $\eta_B$ is a quotient map that need not be into). As with all universal constructions, $F(B)$ is uniquely determined to within an isomorphism. Also, note that if $B$ is finitely presented, i.e., if $\kappa$ is a finite integer and $\Theta$ is finitely generated, then so is $F(B)$, because in this case $\widehat{u}_k^2(\Theta)$ is itself finitely generated by construction.

Given this broad setting, two questions arise naturally.

(I) The description problem. Given a $W$-algebra $B$, describe the structure of the $V$-algebra $F(B)$ as explicitly as possible in terms of the structure of $B$.

(II) The recognition problem. Given a $V$-algebra $A$, find conditions on the structure of $A$ that are necessary and sufficient for the existence of a $W$-algebra $B$ such that $F(B) \cong A$.

Let us point out that (II) is usually harder than (I), for quite general reasons. Indeed, (I) does not entail an existence question: the problem is to obtain a more transparent description of a given object, namely, (1). By contrast, (II) explicitly asks whether an object with a certain property exists.

This paper is devoted to solving problems (I) and (II) in case $V$ is the variety of MV-algebras, $W$ is the variety of Kleene algebras, and $B$ is finitely generated. When $W$ is, instead, the variety of distributive lattices, and $B$ is a finitely generated distributive lattice, a solution to (I) and (II) is given in [14]. Although our proofs are independent of [14], the techniques used here build upon those employed in that paper. It may also be of some interest to