Schur multiplier norms for Loewner matrices

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Abstract

We study upper bounds on the Schur multiplier norm of Loewner matrices for concave and convex functions. These bounds then immediately lead to upper bounds on the ratio of Schatten $q$-norms of commutators $|| [A, f(B)] ||_q / || [A, B] ||_q$. We also consider operator monotone functions, for which sharper bounds are obtained.

Key words: Commutator, Schur multiplier norm, Loewner matrix, concave function, convex function

1 Introduction

The main impetus behind the work presented in this paper was to find good upper bounds on the ratio

$$|| [A, f(B)] ||_q / || [A, B] ||_q$$

(1)

in terms of the spectrum of $B$, where $A$ is a general $n \times n$ matrix, $B$ is a Hermitian $n \times n$ matrix, $[A, B]$ is the commutator $[A, B] = AB - BA$, $f$ is a given function $f : \mathbb{R} \to \mathbb{R}$, and $|| \cdot ||_q$ is the Schatten $q$-norm.

It is not hard to see that this problem immediately reduces to the problem of finding good upper bounds on the Schur multiplier $q$-norms of the Loewner matrix of $f$ in $B$. This will be shown in detail in Section 2. The bulk of this paper is devoted to obtaining such bounds.

We will restrict attention to two classes of functions $f$: first the functions that are operator monotone on an interval containing the spectrum of $B$ (see

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Theorem 1 in Section 3, and then in more generality the concave and convex functions (see Theorem 4 in Section 5).

Davies [6] has considered a similar question for the function $x \mapsto |x|$ but he was looking for a universal bound independent of $B$. He found that for all Schatten norms except the trace norm and operator norm, for all bounded operators $A$ and all self-adjoint operators $B$ in the Schatten $q$-class

$$|| [A, |B|] ||_q/|| [A, B] ||_q \leq 2(1 + \gamma_q)$$

where

$$\gamma_q = c \min(q, q/(q - 1))$$

and $c \geq 1$ is an absolute constant. For the trace norm and operator norm no finite constants $\gamma_1, \gamma_\infty$ exist. For the Frobenius norm he found the sharper bound

$$|| [A, |B|] ||_2 \leq || [A, B] ||_2.$$

In Section 6 we apply the main theorem of Section 5 to obtain bounds on $|| [A, |B|] ||/|| [A, B] ||$ in terms of the number of positive and negative eigenvalues of $B$.

2 Schur multiplier norms

I begin by showing that finding a sharp upper bound on the ratio (1) amounts to computing the Schur multiplier norm (induced by a Schatten norm) of a Loewner matrix.

Let $L$ and $A$ be two matrices of the same dimension, then their entrywise product is denoted by $L \circ A$, i.e. $(L \circ A)_{ij} = L_{ij}A_{ij}$. This product is known alternatively as the Schur product (or Hadamard product). The linear operator $S_L : A \mapsto L \circ A$ is called the Schur multiplier operator. Any norm $||| \cdot |||$ on $A$ induces a norm on $S_L$, which we’ll also denote by $|||S_L|||$, defined by

$$|||S_L||| = \sup_A \frac{|||L \circ A|||}{|||A|||}.$$  

We will be interested in particular in the Schatten $q$-norms, which are defined as $||A||_q = (\text{Tr}(A^*A)^{q/2})^{1/q}$. These include the trace norm ($q = 1$), the Frobenius norm ($q = 2$) and the operator norm $||A||$ (the limit of $q \to \infty$). The corresponding induced norms for $S_L$ are defined as

$$||S_L||_q = \sup_A \frac{||L \circ A||_q}{||A||_q}.$$
A basic property of any Schur multiplier norm is its self-duality. If $||| \cdot |||_D$ is the dual norm of $||| \cdot |||$, then $||| S_L ||| = ||| S_L |||_D$. In particular, $||| S_L |||_q = ||| S_L |||_{q'}$, where $1/q' = 1 - 1/q$. This can be proven easily using a standard duality argument. For example, [8] gives a proof for the operator norm and its dual, the trace norm, but the proof works for any other norm.

The importance of Schur multiplier norms for the problem considered in this paper follows from the following proposition:

**Proposition 1** Let $A$ be any matrix, and let $B$ be Hermitian with eigenvalues $b_i$. Let $L$ be the Loewner matrix of $f$ at $B$:

$$L_{ij} := \begin{cases} \frac{f(b_i) - f(b_j)}{b_i - b_j}, & b_i \neq b_j \\ f'(b_i), & b_i = b_j. \end{cases}$$

Then

$$||| [A, f(B)] ||| \leq ||| S_L ||| ||| [A, B] |||.$$  \hfill (2)

**Proof.** Working in the eigenbasis of $B$, the commutators can be expressed in terms of the Schur product as follows:

$$[A, B] = A \circ (b_i - b_j)^n_{i,j=1}, \quad [A, f(B)] = A \circ (f(b_i) - f(b_j))^n_{i,j=1}.$$ Consider now the Loewner matrix $L$ of the proposition. It is easy to see that this can be expressed in terms of $L$ as

$$[A, f(B)] = [A, B] \circ L = S_L([A, B]).$$

Hence, the norms of both commutators are related by

$$||| [A, f(B)] ||| \leq ||| S_L ||| ||| [A, B] |||.$$  \hfill \Box

For the Schatten 2-norm (Frobenius norm), the induced Schur multiplier norm is easily calculated:

$$||| S_L |||_2 = \max_A \frac{||| L \circ A |||_2}{||| A |||_2}$$

$$= \max_A \left( \frac{\sum_{i,j} |L_{ij}|^2 |A_{ij}|^2}{\sum_{i,j} |A_{ij}|^2} \right)^{1/2}$$

$$= \max_{i,j} |L_{ij}|.$$  \hfill (3)
Computing Schur multiplier norms for other norms than the 2-norm is in general very difficult, and the fact that all entries of $L$ are in a certain range by no means implies that $||S_L||$ should be in that range. Indeed, when $L$ is upper triangular with all entries above the diagonal equal to 1, and all others 0, its Schur multiplier norm is $O(\log n)$ \[3\].

Using complex interpolation, bounds for general Schatten $q$-norms can be derived from bounds for the 1-norm and the 2-norm. Indeed, by a direct application of Theorem 5.2 in Chapter 3 of [7], for any $1 \leq q \leq 2$ we have

$$||S_L||_q \leq ||S_L||_{1}^{2-q}||S_L||_{2}^{q-1}.$$  \(4\)

### 3 Operator monotone functions

The first and easiest class of functions treated here are the functions that are operator monotone on a given interval $I$.

**Theorem 1** Let $f$ be an operator monotone function on the interval $I$. Let $B$ be an $n \times n$ Hermitian matrix with spectrum in $I$. Let $L$ be the Loewner matrix of $f$ at $B$. Then, for all Schatten $q$-norms,

$$||S_L||_q \leq f'(\lambda_{\min}(B)).$$  \(5\)

Note that here $f'$ is always non-negative over $I$.

**Proof.** If $f$ is operator monotone, then its Loewner matrix $L$ is a positive semidefinite matrix. By a theorem of Schur (see [1], section 1.4), $S_L$ is then a completely positive map and $||S_L||$ (and hence $||S_L||_1$) is equal to max$_i L_{ii}$. In the present case, this number is equal to max$_i f'(b_i)$. By the concavity of operator monotone functions, this maximum is equal to $f'(\min_i b_i)$.

For the Schatten 2-norm, we already found that $||S_L||_2 = \max_{i,j} |L_{ij}|$. Again, in the present case max$_i |L_{ii}| = f'(\min_i b_i)$, which proves the inequality for the Frobenius norm.

Finally, using the complex interpolation bound \[3\], these two results imply that \(5\) holds for all Schatten norms. Indeed, for any $1 \leq q \leq 2$,

$$||S_L||_{q'} = ||S_L||_q \leq ||S_L||_{1}^{2-q}||S_L||_{2}^{q-1} \leq f'(\min_i b_i).$$

\[\Box\]
4 The numerical radius and its dual norm

In this section I obtain an intermediary result needed in the next section, which may be of independent interest.

The numerical radius is defined as

\[ w(A) = \sup_x \frac{|\langle Ax | x \rangle|}{||x||^2}. \]

This is a norm, and its dual norm is [2]

\[ ||Y||_{w^*} = \sup_X \frac{|\text{Tr} Y^* X|}{w(X)} = \sup \{ |\text{Tr} Y^* X| : w(X) \leq 1 \}, \]

which I will call the \(w^*\) norm here. The unit ball of the \(w^*\) norm is the absolute convex hull of the matrices of the form \(xx^*\) with \(x \in \mathbb{C}^n\) and \(||x|| = 1\); i.e. it is the set of matrices \(\sum_i \lambda_i x_i x_i^*\) for which \(\sum_i |\lambda_i| \leq 1\) and \(||x_i|| = 1\). This includes but is not limited to the normal matrices with trace norm not exceeding 1.

In general, the numerical radius never exceeds the spectral norm, \(w(X) \leq ||X||\). Likewise, the \(w^*\) norm is bounded below by the trace norm. Indeed,

\[ ||Y||_{w^*} = \sup_X \frac{|\text{Tr} Y^* X|}{w(X)} \geq \sup_X \frac{|\text{Tr} Y^* X|}{||X||} = ||Y||_1. \]

For normal matrices \(X\), the numerical radius is equal to the spectral norm: \(w(Y) = ||Y||\). Here we show that for normal matrices the \(w^*\) norm is equal to the trace norm.

**Theorem 2** If \(Y\) is normal, then \(||Y||_{w^*} = ||Y||_1\).

**Proof.** By a theorem of Ando [1], a matrix \(X\) has numerical radius at most one if and only if there exist contractions \(W\) and \(Z\), where \(Z\) is Hermitian, such that

\[ X = (I + Z)^{1/2} W (I - Z)^{1/2}. \]

The definition of the \(w^*\) norm can therefore be rewritten as

\[ ||Y||_{w^*} = \sup_X \{ |\text{Tr} Y^* X| : w(X) \leq 1 \} \]

\[ = \sup_{W,Z} \{ |\text{Tr} Y^* (I + Z)^{1/2} W (I - Z)^{1/2}| : Z = Z^*, ||Z|| \leq 1, ||W|| \leq 1 \} \]

\[ = \sup_Z \left\{ \sup_W \{ |\text{Tr} W (I - Z)^{1/2} Y^* (I + Z)^{1/2}| : ||W|| \leq 1 \} : Z = Z^*, ||Z|| \leq 1 \right\} \]

\[ = \sup_Z \left\{ ||(I - Z)^{1/2} Y^* (I + Z)^{1/2}||_1 : Z = Z^*, ||Z|| \leq 1 \right\}. \]
Since $Y$ is normal, it has a unitary spectral decomposition $Y = \sum_{j=1}^{n} \lambda_j u_j u_j^*$, with $\{u_j\}_{j=1}^{n}$ an orthonormal basis of $\mathbb{C}^n$. Hence,

$$||(I - Z)^{1/2}Y^*(I + Z)^{1/2}||_1 \leq \sum_j |\lambda_j| \ ||(I - Z)^{1/2}u_j u_j^*(I + Z)^{1/2}||_1.$$  

Noting that for any Hermitian contraction $Z$

$$||(I - Z)^{1/2}u_j u_j^*(I + Z)^{1/2}||_1 = \langle (I - Z^2)^{1/2}u_j | u_j \rangle 
\leq ||(I - Z^2)^{1/2}||_1 \leq 1,$$
we find

$$||(I - Z)^{1/2}Y^*(I + Z)^{1/2}||_1 \leq \sum_j |\lambda_j| = ||Y||_1,$$
and therefore

$$||Y||_{w^*} \leq ||Y||_1.$$  

\[\Box\]

**Corollary 1** For $n \times n$ Hermitian $L$,

$$||S_L|| = \max_{x \in \mathbb{C}^n} \{||L \circ xx^*||_1 : ||x|| \leq 1\}.$$  

**Proof.** By Corollary 3 in [2], if $L$ is Hermitian, $||S_L||$ is equal to $||S_L||_w$, the Schur multiplier norm of $S_L$ induced by the numerical radius:

$$||S_L||_w := \sup_{X} \frac{w(L \circ X)}{w(X)}.$$  

By Lemma 1 in [2], $||S_L||_w \leq 1$ if and only if for all vectors $x \in \mathbb{C}^n$,

$$||L \circ xx^*||_{w^*} \leq ||x||^2.$$  

If $L$ is Hermitian, then so is $L \circ xx^*$, so that by Theorem [2]

$$||L \circ xx^*||_{w^*} = ||L \circ xx^*||_1 = ||L \circ xx^*||_1.$$  

\[\Box\]

### 5 Concave and convex functions

In this section I consider the extension of Theorem [1] to the concave and convex functions. For these functions the Loewner matrix $L$ is no longer positive semidefinite in general. However, it satisfies a number of monotonicity
properties that will be useful in deriving upper bounds. Let, as before, $B$ a Hermitian $n \times n$ matrix with eigenvalues $(b_j)_{j=1}^{n}$ sorted in non-decreasing order, $b_1 \leq b_2 \leq \ldots \leq b_n$, and denote the Loewner matrix of $f$ at $B$ by $L$. The entries of $L$ are

$$L_{ij} := \begin{cases} \frac{f(b_i)-f(b_j)}{b_i-b_j}, & b_i \neq b_j \\ f'(b_i), & b_i = b_j. \end{cases}$$

For concave $f$ and non-decreasing $b$, these elements satisfy the following relations:

$$\begin{align*}
(R) : & \quad L_{ij} = L_{ji}; \\
& \quad \text{for } i \leq j < k, \ L_{ij} \geq L_{ik}; \\
& \quad \text{for } j < k \leq i, \ L_{ji} \geq L_{ki}.
\end{align*}$$

As a consequence, for $i < j$, $L_{ii} \geq L_{jj}$, and for all $i$ and $j$, $L_{11} \geq L_{ij} \geq L_{nn}$.

The case of the Frobenius norm is again very simple.

**Theorem 3** Let $B$ be a Hermitian $n \times n$ matrix with eigenvalues $(b_j)_{j=1}^{n}$ sorted in non-decreasing order, $b_1 \leq b_2 \leq \ldots \leq b_n$. Let $f$ be a function that is concave or convex on the interval $[b_1, b_n]$. Let $L$ be the Loewner matrix of $f$ at $B$. Then

$$||S_L||_2 \leq \max(|f'(b_1)|, |f'(b_n)|). \tag{6}$$

**Proof.** By (3), the upper bound is given by $\max_{i,j} |L_{ij}|$. For concave $f$, the properties (R) of $L$ imply that $\max_{i,j} |L_{ij}| = \max(|L_{11}|, |L_{nn}|)$. Since $L_{ii} = f'(b_i)$ this proves inequality (6). For convex $f$, simply replace $f$ by $-f$ and note that both sides of the inequality are invariant under this sign change. \(\square\)

For the Schur multiplier trace norm (operator norm) I start with a technical proposition about certain standardised monotonously increasing concave functions, as the general case follows easily from this case.

**Proposition 2** Let $B$ be a Hermitian $n \times n$ matrix with eigenvalues $(b_j)_{j=1}^{n}$ sorted in non-decreasing order, $b_1 \leq b_2 \leq \ldots \leq b_n$. Let $g$ be a function that is concave on the interval $[b_1, b_n]$, and for which $g'(b_1) = 1$ and $g'(b_n) = 0$. Let $K$ be the Loewner matrix of $g$ at $B$. Then

$$||S_K||_1 = ||S_K|| \leq 1 + \phi^{-1} \sum_{j=1}^{n} (1 - g'(b_j)), \tag{7}$$

where $\phi$ is the Golden Ratio, $\phi = (1 + \sqrt{5})/2 \approx 1.618$.

Note that the interpolation relation (4) can again be used to obtain bounds for general Schatten norms.
Proof. The matrix $K$ satisfies conditions (R), and $K_{11} = 1$ and $K_{nn} = 0$. From this I will derive an upper bound on $||S_K||$ in terms of the diagonal elements $k_j = K_{jj}$.

By Corollary 4, the Schur multiplier norm of $K$ can be characterised as

$$||S_K||_1 = ||S_K|| = \max_{x \in \mathbb{C}^n} \{||K \circ (xx^*)||_1 : ||x|| = 1\}.$$ 

We can find an upper bound on the trace norm of any matrix $A$ by partitioning $A$ as the block matrix

$$A = \begin{pmatrix} B & b \\ b^T & a \end{pmatrix},$$

where $B$ is the upper left $(n - 1) \times (n - 1)$ submatrix of $A$, $a = A_{nn}$ and $b$ is the $(n - 1)$-dimensional vector consisting of the first $(n - 1)$ entries of the last column of $A$. By a result of Bhatia and Kittaneh [5], the trace norm of $A$ can be bounded above by the sum of the trace norms of the four blocks, i.e.

$$||A||_1 = ||B||_1 + 2||b|| + |a|.$$

When we apply this to the matrix $K \circ (xx^*)$, we have $a = K_{nn}|x_n|^2 = 0$, $b_i = \pi_n x_i K_{in}$ and $B_{ij} = K_{ij} x_i \pi_j$, for $i, j = 1, \ldots, n - 1$.

Since the vector $x$ is normalised, the norm of the subvector of its first $n - 1$ entries is equal to $\sqrt{1 - |x_n|^2}$. Introducing the $(n - 1)$-dimensional normalised vector $y_i = x_i / \sqrt{1 - |x_n|^2}$, for $i = 1, \ldots, n - 1$, and partitioning $K$ conformally with $A$ as

$$K = \begin{pmatrix} Z & u \\ u^T & 0 \end{pmatrix},$$

we get $b = \pi_n \sqrt{1 - |x_n|^2} (y \circ u)$ and $B = (1 - |x_n|^2) (Z \circ (yy^*))$. Hence

$$||K \circ (xx^*)||_1 \leq (1 - |x_n|^2) ||Z \circ (yy^*)||_1 + 2|x_n| \sqrt{1 - |x_n|^2} ||y \circ u||.$$

As the maximisation over $x$ reduces to a maximisation over $|x_n|$ and over $y$, we obtain

$$||S_K|| \leq \max_{0 \leq x \leq 1} (1 - x^2) ||S_Z|| + 2x \sqrt{1 - x^2} \max_{y} \{||y \circ u|| : ||y|| \leq 1\}.$$ 

The maximisation $\max_y \{||y \circ u|| : ||y|| \leq 1\}$ yields $\max_i u_i$, which because of (R) is equal to $K_{1n}$ and therefore bounded above by 1. Furthermore, substituting $a = ||S_Z||$ and $x = \cos \theta$, the remaining maximisation is

$$\max_{0 \leq \theta \leq \pi/2} a(1 - \cos 2\theta)/2 + \sin 2\theta,$$

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which is the monotonously increasing function
\[ v(a) := a/2 + \sqrt{1 + (a/2)^2}. \]

This gives our second relation:
\[ \|S_K\| \leq v(\|S_Z\|). \] (8)

Let us write \( Z \) in terms of a matrix \( K' \) with upper left element 1 and lower right element 0: \( Z = k_{n-1}J + (1 - k_{n-1})K' \), where \( J \) is the \( n \times n \) matrix with \( J_{ij} = 1 \). Note that \( K' \) is a matrix that still obeys (R) but for which \( k'_{n-1} = 0 \) and \( k'_1 = 1 \), i.e. it has the same characteristics as the matrix \( K \) we started out with. The diagonal elements of \( K' \) in terms of those of \( K \) are given by
\[ k'_j := \frac{k_j - k_{n-1}}{1 - k_{n-1}}. \] (9)

By convexity of the Schur multiplier norm and the fact that \( \|S_J\| = 1 \), we have
\[ \|S_Z\| \leq k_{n-1} + (1 - k_{n-1})\|S_{K'}\|, \]
so that, by (8),
\[ \|S_K\| \leq v(k_{n-1} + (1 - k_{n-1})\|S_{K'}\|) \] (10)

The two relations (9) and (10) allow to find an easily computable upper bound on \( S_K \) via a recursion process. This process stops after \( n \) steps, as for a scalar \( \|S_a\| = |a| \). In the recursion, we need in succession the elements \( k_{n-1}, k'_{n-2}, k''_{n-3}, \ldots, k^{(m)}_{n-m-1} \), which I’ll abbreviate by \( a_m \), for \( m = 0, \ldots, n-2 \). Calculating it through, an explicit formula for the elements is
\[ a_0 = k_{n-1} \]
and, for \( m = 1, \ldots, n-2 \),
\[ a_m = k^{(m)}_{n-m-1} = \frac{k_{n-m-1} - k_{n-m}}{1 - k_{n-m}}. \]

The last element in this sequence is (since \( k_1 = 1 \))
\[ a_{n-2} = \frac{k_1 - k_2}{1 - k_2} = 1. \]

Then, denoting \( \|S_{K^{(m)}_n}\| \) by \( s_m \),
\[ s_m \leq v(a_m + (1 - a_m)s_{m+1}), \quad s_{n-2} = 1. \]
Defining \( t_m = s_m - 1 \) and
\[
b_m = 1 - a_m = \frac{1 - k_{n-m-1}}{1 - k_{n-m}},
\]
we have
\[
t_m \leq v(1 + b_{m+1}) - 1, \quad t_{n-2} = 0.
\]
It is easily verified that \( v(1 + x) - 1 \leq 1/\phi + x \), where \( \phi \) is the Golden Ratio. Thus
\[
t_m \leq b_m t_{m+1} + 1/\phi, \quad t_{n-2} = 0,
\]
whence
\[
t_0 \leq \phi^{-1}(1 + b_0 + b_0 b_1 + \ldots + b_0 b_1 \cdots b_{n-3}).
\]
It is immediately checked that \( b_0 b_1 \cdots b_j = 1 - k_{n-j-1} \), for \( j = 0, \ldots, n-3 \) and \( k_1 = 1, k_n = 0 \), so that
\[
t_0 \leq \phi^{-1} \sum_{j=1}^{n} (1 - k_j).
\]
This finally yields \( \|S_K\| \leq s_0 \leq 1 + \phi^{-1} \sum_{j=1}^{n} (1 - k_j) \). As \( K_{ii} = g'(b_i) \), the inequality of the proposition follows.

**Corollary 2** Let \( B \) be a Hermitian \( n \times n \) matrix with eigenvalues \( (b_j)_{j=1}^{n} \) sorted in non-decreasing order, \( b_1 \leq b_2 \leq \ldots \leq b_n \). Let \( h \) be a function that is concave on the interval \([b_1, b_n]\), and for which \( h'(b_1) = 0 \) and \( h'(b_n) = -1 \). Let \( K \) be the Loewner matrix of \( h \) at \( B \). Then
\[
\|S_K\| \leq 1 + \phi^{-1} \sum_{j=1}^{n} (1 + h'(b_j)).
\]  \hspace{1cm} (11)

**Proof.** This follows immediately from Proposition 2 with the matrix \( B \) replaced by \( B' = b_1 + b_n - B \) and defining \( h(x) = g(b_1 + b_n - x) \), so that \( h'(b_j) = -g'(b_1 + b_n - b_j) = -g'(b'_j) \).

We can now state and prove the main result of this paper.

**Theorem 4** Let \( B \) be a Hermitian \( n \times n \) matrix with eigenvalues \( (b_j)_{j=1}^{n} \) sorted in non-decreasing order, \( b_1 \leq b_2 \leq \ldots \leq b_n \). Let \( f \) be a function that is concave on the interval \([b_1, b_n]\). Let \( L \) be the Loewner matrix of \( f \) at \( B \). Then
\[
\|S_L\| \leq (\alpha - \beta) + \min \left( |\beta| + \phi^{-1} \sum_{j=1}^{n} (\alpha - f'(b_j)) \right),
\]
\[
|\alpha| + \phi^{-1} \sum_{j=1}^{n} (f'(b_j) - \beta),
\]
where $\alpha = f'(b_1)$ and $\beta = f'(b_n)$. For any function that is convex on the interval $[b_1, b_n]$, 
\[
||S_L|| \leq (\beta - \alpha) + \min \left( |\beta| + \phi^{-1} \sum_{j=1}^{n} (f'(b_j) - \alpha), \right.
\]
\[
\left. |\alpha| + \phi^{-1} \sum_{j=1}^{n} (\beta - f'(b_j)) \right).
\]

Proof. General concave functions $f$ can be mapped to the standardised functions $g$ and $h$ of Proposition 2 and Corollary 2. Note that $\alpha := f'(b_1) \geq f'(b_j) \geq f'(b_n) =: \beta$.

First we write
\[
f(x) = \beta x + (\alpha - \beta) g(x).
\]
Then
\[
(\alpha - \beta) g'(x) = f'(x) - \beta.
\]
Letting $L$ and $K$ be the Loewner matrices of $f$ and $g$, respectively, at $B$, 
\[
L = \beta J + (\alpha - \beta) K,
\]
where $J$ is the matrix all of whose entries are 1. As $||S_J|| = 1$,
\[
||S_L|| \leq |\beta| + (\alpha - \beta) ||S_K||
\]
\[
\leq |\beta| + (\alpha - \beta) \left( 1 + \phi^{-1} \sum_{j=1}^{n} (1 - g'(b_j)) \right)
\]
\[
= |\beta| + (\alpha - \beta) + \phi^{-1} \sum_{j=1}^{n} ((\alpha - \beta) - (f'(b_j) - \beta))
\]
\[
= |\beta| + (\alpha - \beta) + \phi^{-1} \sum_{j=1}^{n} (\alpha - f'(b_j)).
\]

We can also write
\[
f(x) = \alpha x + (\alpha - \beta) h(x).
\]
and obtain in a similar way
\[
||S_L|| \leq |\alpha| + (\alpha - \beta) + \phi^{-1} \sum_{j=1}^{n} (f'(b_j) - \beta).
\]

Taking the minimum of both bounds yields the bound of the corollary.
For convex $f$ we just replace $f$ by $-f$ and apply the result for concave functions. Since now $\alpha := f'(b_1) \leq f'(b_j) \leq f'(b_n) =: \beta$, an appropriate sign change has to be applied to the bound. \hfill \Box

When the spectrum of $B$ is not known, but it is known that $b_1 \leq B \leq b_n$, weaker bounds follow readily from this Theorem:

**Corollary 3** Let $B$ be a Hermitian $n \times n$ matrix bounded as $b_1 \leq B \leq b_n$. Let $f$ be a function that is either concave or convex on the interval $[b_1, b_n]$. Let $L$ be the Loewner matrix of $f$ at $B$. Then

$$||S_L|| \leq |\alpha - \beta|(1 + (n-1)\phi^{-1}) + \min(|\beta|, |\alpha|),$$

where $\alpha = f'(b_1)$ and $\beta = f'(b_n)$.

6 Examples

As a first application, we consider the function $f(x) = |x|$.

**Theorem 5** Let $B$ be a Hermitian $n \times n$ matrix with $r$ positive eigenvalues. Let $L$ be the Loewner matrix of the function $f(x) = |x|$ at $B$. Then, for $1 \leq r < n$,

$$||S_L|| \leq 3 + 2\phi^{-1}\min(r, n-r).$$

If $r$ is 0 or $n$, $||S_L||$ is 1.

Proof. For $1 \leq r < n$, $\alpha = f'(b_1) = 1$, $\beta = f'(b_n) = -1$, $f'(b_j) = 1$ for $r$ values of $j$, and $f'(b_j) = -1$ for $n-r$ values of $j$. The bound follows by simple calculation. \hfill \Box

Since the bounds only depend on the diagonal elements of the Loewner matrix, they are not expected to be sharp for specific functions. For the absolute value function, for example, it is known that in the $d = 2$ case the norm ratio lies between the values 1 and $\sqrt{2}$, whereas the Theorem gives the bound $3 + 2/\phi$ for $r = 1$.

For our second example, consider the following corollary of the main theorem. Let $C$ be a Hermitian matrix with spectrum $c_1 \leq c_2 \leq \ldots \leq c_n$. By putting $B = g(C)$ and $h = f \circ g$, we find:

**Corollary 4** For all $n \times n$ matrices $A$ and for any monotonously increasing function $g$ and any function $h$ such that $f = h \circ g^{-1}$ is concave,
\[ \frac{\| [A, h(C)] \|_1}{\| [A, g(C)] \|_1} \leq (\alpha - \beta) + \min \left( |\beta| + \phi^{-1} \sum_{j=1}^{n} (\alpha - f'(g(c_j))), |\alpha| + \phi^{-1} \sum_{j=1}^{n} (f'(g(c_j)) - \beta) \right), \]

where \( \alpha = f'(g(c_1)) \) and \( \beta = f'(g(c_n)) \).

Consider the functions \( h(x) = \log x \) and \( g(x) = \log(x) - \log(1 - x) \). Thus, \( f(x) = x - \log(1 + e^x) \), which is monotonously increasing and concave, and \( (f' \circ g)(x) = 1 - x \). The bound of the corollary then simplifies to

\[ c_n - c_1 + \min \left( 1 - c_n + \phi^{-1}(1 - nc_1), 1 - c_1 + \phi^{-1}(nc_n - 1) \right), \]

As \( c_1 \geq 0 \), this quantity is bounded above by \( 1 + \phi^{-1} = \phi \). We have therefore proven:

**Corollary 5** For any \( A \) and for any positive semidefinite \( C \) with \( \text{Tr} C = 1 \),

\[ \| [A, \log(C)] \|_1 \leq \phi \| [A, \log(C) - \log(\mathbb{I} - C)] \|_1. \]  

(12)

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