BRIDGELAND'S HALL ALGEBRAS AND HEISENBERG DOUBLES

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Abstract. Let $A$ be a finite dimensional hereditary algebra over a finite field, and let $m$ be a fixed integer such that $m = 0$ or $m > 2$. In the present paper, we first define an algebra $L_m(A)$ associated to $A$, called the $m$-periodic lattice algebra of $A$, and then prove that it is isomorphic to Bridgeland’s Hall algebra $\mathcal{DH}_m(A)$ of $m$-cyclic complexes over projective $A$-modules. Moreover, we show that there is an embedding of the Heisenberg double Hall algebra of $A$ into $\mathcal{DH}_m(A)$.

1. Introduction

The Hall algebra $\mathfrak{H}(A)$ of a finite dimensional algebra $A$ over a finite field was introduced by Ringel [6] in 1990. Ringel [5, 6] proved that if $A$ is representation-finite and hereditary, the twisted Hall algebra $\mathfrak{H}_v(A)$, called the Ringel–Hall algebra, is isomorphic to the positive part of the corresponding quantized enveloping algebra. By introducing a bialgebra structure on $\mathfrak{H}_v(A)$, Green [3] generalized Ringel’s work to an arbitrary finite dimensional hereditary algebra $A$ and showed that the composition subalgebra of $\mathfrak{H}_v(A)$ generated by simple $A$-modules gives a realization of the positive part of the quantized enveloping algebra associated with $A$.

In order to give a Hall algebra realization of the entire quantized enveloping algebra, one considers defining the Hall algebras of triangulated categories (for example, [4], [9], [10], [11]). Kapranov [4] defined an associative algebra, called the lattice algebra, for the bounded derived category of any hereditary algebra. By using the fibre products of model categories, Toën [9] defined an associative algebra, called the derived Hall algebra, for DG-enhanced triangulated categories. Later on, Xiao and Xu [11] generalized the definition of the derived Hall algebra to any triangulated category with some homological finiteness conditions. In [8], Sheng and Xu proved that for each hereditary algebra $A$ the lattice algebra of $A$ is isomorphic to its twisted and extended derived Hall algebra.

In 2013, for each finite dimensional hereditary algebra $A$ over a finite field, Bridgeland [1] defined an algebra associated to $A$, called Bridgeland’s Hall algebra of $A$, which is the Ringel–Hall algebra of 2-cyclic complexes over projective $A$-modules with some localization and reduction. He proved that the quantized enveloping algebra associated to

2010 Mathematics Subject Classification. 16G20, 17B20, 17B37.

Key words and phrases. $m$-periodic lattice algebra; Bridgeland’s Hall algebra; Heisenberg double.
the hereditary algebra $A$ can be embedded into Bridgeland’s Hall algebra of $A$. This
provides a realization of the full quantized enveloping algebra by Hall algebras. After-
wards, Yanagida [12] showed that Bridgeland’s Hall algebra of 2-cyclic complexes of any
finite dimensional hereditary algebra is isomorphic to the (reduced) Drinfeld double of
its extended Ringel–Hall algebra. Inspired by the work of Bridgeland, Chen and Deng
[2] considered Bridgeland’s Hall algebra $\mathcal{DH}_m(A)$ of $m$-cyclic complexes of a hereditary
algebra $A$ for each non-negative integer $m \neq 1$.

In this paper, let $A$ be a finite dimensional hereditary algebra over a finite field. For
any non-negative integer $m \neq 1$ or 2 we first define an $m$-periodic lattice algebra, and
then use it to give a characterization of the algebra structure on $\mathcal{DH}_m(A)$. Explicitly,
we show that Bridgeland’s Hall algebra $\mathcal{DH}_m(A)$ is isomorphic to the $m$-periodic lattice
algebra. As a byproduct, we show that there is an embedding of the Heisenberg double
Hall algebra of $A$ into $\mathcal{DH}_m(A)$.

Throughout the paper, let $m$ be a non-negative integer such that $m = 0$ or $m > 2$,
let $k$ be a fixed finite field with $q$ elements and set $v = \sqrt{q} \in \mathbb{C}$. Denote by $A$ a finite
dimensional $k$-algebra. We denote by $\text{mod} A$ and $D^b(A)$ the category of finite dimensional
(left) $A$-modules and its bounded derived category, respectively, and denote by $\mathcal{P} = \mathcal{P}_A$
the full subcategory of $\text{mod} A$ consisting of projective $A$-modules. Let $K(A)$ be the
Grothendieck group of $\text{mod} A$ and $\text{Iso}(A)$ the set of isoclasses (isomorphism classes) of
$A$-modules. For an $A$-module $M$, the class of $M$ in $K(A)$ is denoted by $\hat{M}$, and the
automorphism group of $M$ is denoted by $\text{Aut}(M)$. For a finite set $S$, we denote by $|S|$
its cardinality. We also write $a_M$ for $|\text{Aut}(M)|$. For a complex $M_\bullet$ of $A$-modules, its
homology is denoted by $H_*(M_\bullet)$. For a positive integer $m$, we denote the quotient ring
$\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}_m = \{0, 1, \ldots, m - 1\}$. By convention, $\mathbb{Z}_0 = \mathbb{Z}$.

2. Preliminaries

In this section, we summarize some necessary definitions and properties. All of the
materials can be found in [1], [2], [7] and [13].

2.1. $m$-cyclic complexes. Let $m$ be a positive integer. Given an additive category $\mathcal{A}$,
an $m$-cyclic complex $M_\bullet = (M_i, d_i)_{i \in \mathbb{Z}_m}$ over $\mathcal{A}$ consists of $m$ objects $M_i$ and $m$
morphisms $d_i : M_i \to M_{i+1}$ in $\mathcal{A}$ satisfying $d_{i+1}d_i = 0$ for all $i \in \mathbb{Z}_m$. Hence, each $m$-cyclic complex
$M_\bullet = (M_i, d_i)_{i \in \mathbb{Z}_m}$ can be diagrammed by

\[
\begin{array}{c}
M_{m-1} \\
\circlearrowleft
M_0 \\
\vdots
\end{array}
\]

\[
d_{m-1} \quad M_0 \quad d_0
\]

\[
M_{m-1} \quad M_i
\]
with \( d_{i+1}d_i = 0 \) for all \( i \in \mathbb{Z}_m \). A morphism \( f \) between two \( m \)-cyclic complexes \( M_\bullet = (M_i, d_i)_{i \in \mathbb{Z}_m} \) and \( N_\bullet = (N_i, c_i)_{i \in \mathbb{Z}_m} \) is given by \( m \) morphisms \( f_i : M_i \to N_i \) in \( A \) satisfying \( f_{i+1}d_i = c_if_i \) for all \( i \in \mathbb{Z}_m \).

Let \( f = (f_i)_{i \in \mathbb{Z}_m} \) and \( g = (g_i)_{i \in \mathbb{Z}_m} \) be two morphisms between \( m \)-cyclic complexes \( M_\bullet = (M_i, d_i)_{i \in \mathbb{Z}_m} \) and \( N_\bullet = (N_i, c_i)_{i \in \mathbb{Z}_m} \). We say that \( f \) is homotopic to \( g \) if there exist \( m \) morphisms \( s_i : M_i \to N_{i-1} \) in \( A \) such that \( f_i - g_i = s_{i+1}d_i + c_{i-1}s_i \) for all \( i \in \mathbb{Z}_m \). The category of \( m \)-cyclic complexes over \( A \) is denoted by \( C_m(A) \), and \( K_m(A) \) denotes the homotopy category of \( C_m(A) \) by identifying homotopic morphisms. As in usual complex categories, one can define quasi-isomorphisms in \( C_m(A) \) and \( K_m(A) \), and then get a triangulated category, denoted by \( D_m(A) \), by localizing \( K_m(A) \) with respect to the set of all quasi-isomorphisms. We write \( C_0(A), K_0(A) \) and \( D_0(A) \) for the category of bounded complexes, its homotopy category and derived category, respectively.

For each integer \( t \), there is a shift functor

\[
[t] : C_m(A) \to C_m(A), \quad M_\bullet \mapsto M_\bullet[t],
\]

where \( M_\bullet[t] = (X_i, f_i) \) is defined by

\[
X_i = M_{i+t}, \quad f_i = (-1)^td_{i+t}, \quad i \in \mathbb{Z}_m.
\]

Recall that \( A \) is a finite dimensional \( k \)-algebra throughout the paper. Applying the above construction to \( \mathcal{P} = \mathcal{P}_A \), we obtain the categories \( C_m(\mathcal{P}) \) and \( K_m(\mathcal{P}) \). In the sense of chain-wise exactness, \( C_m(\mathcal{P}) \) is a Frobenius exact category.

For an arbitrary homomorphism \( f : Q \to P \) of projective \( A \)-modules, define \( C_f = (X_i, d_i) \in C_m(\mathcal{P}) \) by

\[
X_i = \begin{cases} Q & i = m - 1; \\ P & i = 0; \\ 0 & \text{otherwise}, \end{cases} \quad d_i = \begin{cases} f & i = m - 1; \\ 0 & \text{otherwise}. \end{cases}
\]

So each projective \( A \)-module \( P \) determines an object \( K_P := C_{id_P} \) in \( C_m(\mathcal{P}) \).

The following lemma taken from [2] is important in the later calculations.

**Lemma 2.1.** ([2 Lem. 2.5]) If \( M_\bullet, N_\bullet \in C_m(\mathcal{P}) \), then there exists an isomorphism of vector spaces

\[
\text{Ext}^1_{C_m(\mathcal{P})}(N_\bullet, M_\bullet) \cong \text{Hom}_{K_m(\mathcal{P})}(N_\bullet, M_\bullet[1]).
\]

Let \( F \) be the natural covering functor from \( C_0(\text{mod } A) \) to \( C_m(\text{mod } A) \) with Galois group \( \langle [m] \rangle \). Let \( D^b(A)/[m] \) be the orbit category of \( D^b(A) \) with respect to the shift functor \([m]\).

**Lemma 2.2.** If \( A \) is of finite global dimension, then \( F \) induces a fully faithful functor

\[
F : D^b(A)/[m] \to K_m(\mathcal{P}).
\]
Proof. Since $A$ is of finite global dimension, we can equally well define $D^b(A)/[m]$ as the orbit category of $K^b(\mathcal{P})$. Then it is easy to see that $F$ induces a fully faithful functor $F : K^b(\mathcal{P})/\{m\} \rightarrow K_m(\mathcal{P})$. □

An $m$-cyclic complex $M_\bullet$ of $A$-modules is called acyclic if $H_*(M_\bullet) = 0$. For each $A$-module $P \in \mathcal{P}$, $K_P[r]$ for all $r \in \mathbb{Z}_m$ are acyclic. Clearly, for any complex $M_\bullet \in \mathcal{C}_m(\mathcal{P})$, $M_\bullet$ is acyclic if $M_\bullet \cong 0$ in $K_m(\mathcal{P})$. The following lemma gives a characterization of all acyclic complexes in $\mathcal{C}_m(\mathcal{P})$.

Lemma 2.3. ([1], Lem. 3.2; [13], Lem. 2.2) Suppose that $A$ is of finite global dimension. Then for each acyclic complex $M_\bullet \in \mathcal{C}_m(\mathcal{P})$, there exist objects $P_r \in \mathcal{P}$, $r \in \mathbb{Z}_m$, such that $M_\bullet \cong \bigoplus_{r \in \mathbb{Z}_m} K_{P_r}[r]$. Moreover, these projective $A$-modules $P_r$ are uniquely determined up to isomorphism.

2.2. Hall algebras. Given $A$-modules $L, M, N$, we define Ext$^1_A(M, N)_L$ to be the subset of Ext$^1_A(M, N)$, which consists of those equivalence classes of short exact sequences with middle term $L$. We define the Hall algebra $H(A)$ to be the vector space over $\mathbb{C}$ with basis $[M] \in \text{Iso} (A)$ and with associative multiplication defined by $[M] \cdot [N] = \sum_{[L] \in \text{Iso}(A)} F^L_{MN}[L]$, where $F^L_{MN} = \frac{|\text{Ext}^1_A(M, N)_L|}{|\text{Hom}_A(M, N)|} \cdot \frac{|\text{Aut} L|}{|\text{Aut} M| \cdot |\text{Aut} N|}$ and it is called the Ringel–Hall number associated to $A$-modules $L, M, N$.

From now on, we suppose that $A$ is hereditary. For $M, N \in \text{mod} A$, define $\langle M, N \rangle := \dim_k \text{Hom}_A(M, N) - \dim_k \text{Ext}^1_A(M, N)$. It induces a bilinear form $\langle \cdot, \cdot \rangle : K(A) \times K(A) \rightarrow \mathbb{Z}$, known as the Euler form. We also consider the symmetric Euler form $(\cdot, \cdot) : K(A) \times K(A) \rightarrow \mathbb{Z}$, defined by $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in K(A)$.

The twisted Hall algebra $H_v(A)$, called the Ringel–Hall algebra, is the same vector space as $H(A)$ but with twisted multiplication defined by $[M] \ast [N] = v^{(M, N)} \cdot [M] \diamond [N]$. We can form the extended Ringel–Hall algebra $H^e_v(A)$ by adjoining symbols $K_\alpha$ for $\alpha \in K(A)$ and imposing relations $K_\alpha \ast K_\beta = K_{\alpha + \beta}$, $K_\alpha \ast [M] = v^{(\alpha, M)} \cdot [M] \ast K_\alpha$. (2.1)
Definition 2.4. The twisted Hall algebra $\mathcal{H}_{tw}(\mathcal{C}_m(\mathcal{P}))$ of $\mathcal{C}_m(\mathcal{P})$ is the vector space over $\mathbb{C}$ with basis indexed by the isoclasses $[M_*]$ of objects in $\mathcal{C}_m(\mathcal{P})$, and with multiplication defined by

$$[M_*] \ast [N_*] = v(M_*, N_*) \cdot \sum_{[L_*]} \frac{\text{Ext}^1_{\mathcal{C}_m(\mathcal{P})}(M_*, N_*)_{L_*}}{\text{Hom}_{\mathcal{C}_m(\mathcal{P})}(M_*, N_*)} [L_*],$$

where $\langle M_*, N_* \rangle := \sum_{i \in \mathbb{Z}_m} \langle \hat{M}_i, \hat{N}_i \rangle$.

It is easy to see that $\mathcal{H}_{tw}(\mathcal{C}_m(\mathcal{P}))$ is an associative algebra (cf. [1, 2]). By some simple calculations, we have the following relations for the acyclic complexes $K_P[r]$, $r \in \mathbb{Z}_m$.

Lemma 2.5. ([13]) Let $M \in \text{mod } A$, $P, Q \in \mathcal{P}$ and $M_* \in \mathcal{C}_m(\mathcal{P})$. We have the following identities for each $r \in \mathbb{Z}_m$ in $\mathcal{H}_{tw}(\mathcal{C}_m(\mathcal{P}))$

$$[K_P[r]] \ast [M_*] = v^{(P, \hat{M}_m - \hat{M}_{m-1})}[K_P[r] \oplus M_*]; \quad (2.2)$$

$$[M_*] \ast [K_P[r]] = v^{-(M_m - M_{m-1}, \hat{P})}[K_P[r] \oplus M_*]; \quad (2.3)$$

$$[K_P[r]] \ast [M_*] = v^{(\hat{P}, M_m - M_{m-1})}[M_*] \ast [K_P[r]]. \quad (2.4)$$

In particular,

$$[K_P[r]] \ast [K_Q] = \begin{cases} v^{(P, Q)}[K_Q] \ast [K_P[r]] & r = 1, \\ v^{-(\hat{P}, Q)}[K_Q] \ast [K_P[r]] & r = m - 1, \\ [K_Q] \ast [K_P[r]] & \text{otherwise}; \end{cases} \quad (2.5)$$

$$[K_P[r]] \ast [C_M] = \begin{cases} v^{(\hat{P}, \hat{M})}[C_M] \ast [K_P[r]] & r = 0, \\ v^{(\hat{P}, \hat{Q}M)}[C_M] \ast [K_P[r]] & r = 1, \\ v^{-(\hat{P}, P_M)}[C_M] \ast [K_P[r]] & r = m - 1, \\ [C_M] \ast [K_P[r]] & \text{otherwise}. \end{cases} \quad (2.6)$$

Set $\mathcal{I} := \{v^n[M_*] \in \mathcal{C}_m(\mathcal{P}) \mid H_*(M_*) = 0, \ n \in \mathbb{Z}\}$. By Lemma 2.5, the set $\mathcal{I}$ satisfies the Ore conditions. So one considers the localization of $\mathcal{H}_{tw}(\mathcal{C}_m(\mathcal{P}))$ with respect to the set $\mathcal{I}$ (cf. [1, 2]).

Definition 2.6. The localized Hall algebra $\mathcal{DH}_m(A)$ of $A$, called Bridgeland’s Hall algebra, is the localization of $\mathcal{H}_{tw}(\mathcal{C}_m(\mathcal{P}))$ with respect to the set $\mathcal{I}$. Since $v$ is invertible, in symbols,

$$\mathcal{DH}_m(A) := \mathcal{H}_{tw}(\mathcal{C}_m(\mathcal{P}))[ [M_*]^{-1} \mid H_*(M_*) = 0 ].$$

For any $\alpha \in K(A)$ and $r \in \mathbb{Z}_m$, by writing $\alpha$ in the form $\alpha = \hat{P} - \hat{Q}$ for some $A$-modules $P, Q \in \mathcal{P}$, we define

$$K_{\alpha, r} := [K_P[r]] \ast [K_Q[r]]^{-1},$$
and it is easy to see that

\[ K_{\alpha,r} \ast K_{\beta,r} = K_{\alpha+\beta,r}. \tag{2.7} \]

We simply write \( K_\alpha \) for \( K_{\alpha,0} \). Note that the identities (2.4) – (2.6) continue to hold with the elements \([K_P[r]]\) and \([K_Q]\) replaced by \( K_{\alpha,r} \) and \( K_\beta \), respectively, for all \( \alpha, \beta \in K(A) \).

Each \( A \)-module \( M \) has a unique minimal projective resolution up to isomorphism of the form

\[ 0 \rightarrow \Omega_M \xrightarrow{\delta_M} P_M \xrightarrow{\varepsilon_M} M \rightarrow 0. \tag{2.8} \]

Given an \( A \)-module \( M \), we take a minimal projective resolution (2.8) of \( M \), and consider the corresponding \( m \)-cyclic complex \( C_M : = C_{\delta_M} \). Since the uniqueness of the minimal projective resolution up to isomorphism, the complex \( C_M \) is well-defined up to isomorphism.

By [1, 2], for each \( r \in \mathbb{Z}_m \), we define \( E_{M,r} : = v(\tilde{\Omega}_M, \tilde{M}) \cdot K_{-\tilde{\Omega}_M,r} \ast [C_M[r]] \in DH_m(A) \).

For each \( r \in \mathbb{Z}_m \), set \( e_{M,r} : = a^{-1}_M \cdot E_{M,r} \). We also simply write \( E_M \) and \( e_M \) for \( E_{M,0} \) and \( e_{M,0} \), respectively.

Let us reformulate a result from [2, Prop. 4.4].

**Proposition 2.7.** There is an embedding of algebras for each \( i \in \mathbb{Z}_m \)

\[ I_i : \mathcal{S}_v^e(A) \hookrightarrow DH_m(A), \ [M] \mapsto e_{M,i}; \ K_\alpha \mapsto K_{\alpha,i}. \tag{2.9} \]

Moreover, the multiplication map induces an isomorphism of vector spaces

\[ \bigotimes_{i \in \mathbb{Z}_m} \mathcal{S}_v^e(A) \rightarrow DH_m(A), \bigotimes_{i \in \mathbb{Z}_m} x_i \mapsto \prod_{i \in \mathbb{Z}_m} I_i(x_i). \tag{2.10} \]

**Remark 2.8.** If \( m = 0 \), then the tensor product in Proposition 2.7 is an infinite tensor. Here and elsewhere in this paper all infinite tensor products are understood in the restricted sense (cf. [4, Sec. 3.2]). \( \prod_{i \in \mathbb{Z}_m} a_i \) means the ordered product of the elements \( a_i \):

\[ \prod_{i \in \mathbb{Z}_m} a_i = \cdots a_i \cdot a_{i+1} \cdots . \]

3. **Heisenberg doubles**

Recall that \( A \) is a finite dimensional hereditary \( k \)-algebra. First of all, we give a counting symbol. For any fixed \( M, N, X, Y \in \text{mod} A \), we denote by \( W^{XY}_{MN} \) the set

\[ \{(f, g, h) \mid 0 \rightarrow X \xrightarrow{g} M \xrightarrow{f} N \xrightarrow{h} Y \rightarrow 0 \text{ is exact in } \text{mod} A\}, \]

and set

\[ \gamma^{XY}_{MN} := \frac{|W^{XY}_{MN}|}{a_M a_N}. \]

\[ ^1 \text{For each fixed } A \text{-module } M, \text{ we fix a minimal projective resolution (2.8) of } M \text{ using notations } P_M \text{ and } \Omega_M \text{ throughout the paper.} \]
Let Heis($A$) be the Heisenberg double of the extended Ringel–Hall algebra $\mathcal{S}_v(A)$ (cf. [4, Sec. 1.5]). By definition, Heis($A$) is an associative and unital $\mathbb{C}$-algebra generated by elements $Z^+_M$, $Z^-_M$ and $K_\alpha, K^-_\alpha$ with $[M] \in \text{Iso}(A)$ and $\alpha \in K(A)$, which are subject to the following relations:

\[ Z^+_M Z^+_N = v^{(M,N)} \sum_{[L]} F^L_{MN} Z^+_L, \quad K_\alpha Z^+_M = v^{(\alpha,M)} Z^+_M K_\alpha; \]
\[ Z^-_M Z^-_N = v^{(M,N)} \sum_{[L]} F^L_{MN} Z^-_L, \quad K^-_\alpha Z^-_M = v^{(\alpha,M)} Z^-_M K^-_\alpha; \]
\[ Z^-_M Z^+_N = \sum_{[X],[Y]} v^{(M-X,X-Y)} \gamma^{XY}_{MN} K_{M-X} Z^+_X Z^-_Y; \]
\[ K_\alpha Z^-_M = v^{-(\alpha,M)} Z^-_M K_\alpha, \quad K^-_\alpha Z^+_M = Z^+_M K^-_\alpha, \quad K_\alpha K^-_\beta = v^{-(\alpha,\beta)} K^-_\beta K_\alpha. \]

Clearly, Heis($A$) is naturally related to the subcategory $D^{[-1,0]}(A)$ of the derived category $D^b(A)$, which is consisting of two copies of $\text{mod } A$ inside $D^b(A)$ given by complexes concentrated in degrees 0 and −1. In other words, Heis($A$) gives rise to two copies of $\mathcal{S}_v(A)$ with Heisenberg double-type commutation relations (cf. [4, Prop. 1.5.3]). Moreover, Kapranov [4, Def. 3.1] introduced an associative and unital $\mathbb{C}$-algebra $L(A)$, called the lattice algebra of $A$, by taking not just two but infinitely many copies of $\mathcal{S}_v(A)$ and one copy of the group algebra $\mathbb{C}[K(A)]$, and by imposing Heisenberg double-like commutation relations between adjacent copies of $\mathcal{S}_v(A)$ and oscillator relations of the form $ab = \lambda_{ab} ba, \quad \lambda_{ab} \in \mathbb{R}$, between basis vectors of non-adjacent copies. In a similar manner to [4, Def. 3.1], we give the following definition.

**Definition 3.1.** The $m$-periodic lattice algebra $L_m(A)$ of $A$ is the associative and unital $\mathbb{C}$-algebra generated by the elements in \{\(Z^{(i)}_M\mid [M] \in \text{Iso}(A), i \in \mathbb{Z}_m\}\} and \{\(K^{(i)}_\alpha\mid \alpha \in K(A), i \in \mathbb{Z}_m\}\} with the following relations:

\[ K^{(i)}_\alpha K^{(i)}_\beta = K^{(i)}_{\alpha+\beta}, \quad K^{(i)}_\alpha K^{(j)}_\beta = \begin{cases} v^{(\alpha,\beta)} \cdot K^{(j)}_\beta K^{(i)}_\alpha & i = j + 1, \\ v^{(\alpha,\beta)} \cdot K^{(j)}_\beta K^{(i)}_\alpha & i = m - 1 + j, \\ K^{(j)}_\beta K^{(i)}_\alpha & \text{otherwise}; \end{cases} \]
\[ K^{(i)}_\alpha Z^{(j)}_M = \begin{cases} v^{(\alpha,M)} \cdot Z^{(j)}_M K^{(i)}_\alpha & i = j, \\ v^{-(\alpha,M)} \cdot Z^{(j)}_M K^{(i)}_\alpha & i = m - 1 + j, \\ Z^{(j)}_M K^{(i)}_\alpha & \text{otherwise}; \end{cases} \]
\[ Z^{(i)}_M Z^{(i)}_N = v^{(M,N)} \sum_{[L]} F^L_{MN} Z^{(i)}_L; \]
\[ Z^{(i+1)}_M Z^{(i)}_N = \sum_{[X],[Y]} v^{(M-X,X-Y)} \gamma^{XY}_{MN} K^{(i)}_{M-X} Z^{(i)}_{[Y]} Z^{(i+1)}_{[X]}; \]
\[
Z^{(i)}_{[M]} Z^{(j)}_{[N]} = Z^{(j)}_{[N]} Z^{(i)}_{[M]}, \quad i - j \neq 0, 1 \text{ or } m - 1. \tag{3.5}
\]

**Theorem 3.2.** Bridgeland’s Hall algebra \( \mathcal{DH}_m(A) \) is isomorphic to the \( m \)-periodic lattice algebra \( L_m(A) \).

**Proof.** First of all, we claim that there is a homomorphism of algebras
\[
\Phi : L_m(A) \rightarrow \mathcal{DH}_m(A)
\]
defined by \( \Phi(K^{(i)}_\alpha) = K_{\alpha,i} \) and \( \Phi(Z^{(i)}_{[M]}) = e_{M,i} \). That is to say, \( \Phi \) preserves the relations \( (3.1) - (3.4) \). Indeed, by the identities \( (2.7) \) and \( (2.8) \), clearly, the relation \( (3.1) \) is preserved. By the identities \( (2.5) \) and \( (2.6) \), for any \( M \in \text{mod } A \) and \( P \in \mathcal{P} \), it is easy to see that
\[
[K_P[r]] \ast (K_{-\Omega} \ast [C_M]) = \begin{cases} v^{(P,M)} \ast (K_{-\Omega} \ast [C_M]) \ast [K_P[r]] & r = 0, \\ v^{-(P,M)} \ast (K_{-\Omega} \ast [C_M]) \ast [K_P[r]] & r = m - 1, \\ (K_{-\Omega} \ast [C_M]) \ast [K_P[r]] & \text{otherwise}. \end{cases} \tag{3.6}
\]
Hence, the relation \( (3.2) \) is preserved. By Proposition \( 2.7 \), the relation \( (3.3) \) is preserved. Let us first consider the relation \( (3.5) \). Let \( M, N \in \text{mod } A \) and \( r \neq 0 \text{ or } 1 \text{ in } \mathbb{Z}_m \), then
\[
\text{Ext}^1_{C_m(\mathcal{P})}(C_M[r], C_N) \cong \text{Hom}_{K_m(\mathcal{P})}(C_M[r], C_N[1]) \\
\cong \text{Hom}_{K_m(\mathcal{P})}(C_M[r - 1], C_N) \\
= 0.
\tag{3.7}
\]
Similarly, for \( r \neq 0 \text{ or } m - 1 \text{ in } \mathbb{Z}_m \), we obtain that \( \text{Ext}^1_{C_m(\mathcal{P})}(C_N, C_M[r]) = 0 \). So it is easy to see that for \( r \neq 0, 1 \text{ or } m - 1 \), we have
\[
[C_M[r]] \ast [C_N] = [C_N] \ast [C_M[r]] = [C_M[r] \oplus C_N].
\]
Hence,
\[
(K_{-\Omega} \ast [C_M[r]]) \ast (K_{-\Omega} \ast [C_N]) = K_{-\Omega} \ast K_{-\Omega} \ast [C_M[r]] \ast [C_N] \\
= K_{-\Omega} \ast [C_N] \ast [C_M[r]] \\
= K_{-\Omega} \ast [C_N] \ast [C_M[r]] \\
= (K_{-\Omega} \ast [C_N]) \ast (K_{-\Omega} \ast [C_M[r]]). \tag{3.8}
\]
Namely, \( e_{M,r} \ast e_N = e_N \ast e_{M,r} \). So the relation \( (3.3) \) is preserved. We remove the proof of the relation \( (3.4) \) to the next section and prove that \( \Phi \) is an isomorphism right now.

For each \( i \in \mathbb{Z}_m \), let \( L_m^{(i)}(A) \) be the subalgebra of \( L_m(A) \) generated by the elements in \( \{Z^{(i)}_{[M]} \mid [M] \in \text{Iso } (A)\} \) and \( \{K^{(i)}_\alpha \mid \alpha \in K(A)\} \). Then, clearly, the multiplication map of \( L_m(A) \) induces an epimorphism of vector spaces
\[
\Theta : \bigotimes_{i \in \mathbb{Z}_m} L_m^{(i)}(A) \longrightarrow L_m(A).
\]
For each \( i \in \mathbb{Z}_m \), let \( \mathcal{H}^{(i)}(A) \) be the subalgebra of \( \mathcal{D} \mathcal{H}_m(A) \) generated by the elements in \( \{ e_{M,i} \mid [M] \in \text{Iso}(A) \} \) and \( \{ K_{\alpha,i} \mid \alpha \in K(A) \} \). Then, by Proposition 2.7, for each \( i \in \mathbb{Z}_m \), \( \Phi \) induces an algebra isomorphism

\[
\Phi_i : L_m^{(i)}(A) \rightarrow \mathcal{H}^{(i)}(A),
\]

moreover, there exists an isomorphism of vector spaces \( \Psi : \bigotimes_{i \in \mathbb{Z}_m} \mathcal{H}^{(i)}(A) \rightarrow \mathcal{D} \mathcal{H}_m(A) \).

Hence, we have the following commutative diagram

\[
\begin{array}{ccc}
\bigotimes_{i \in \mathbb{Z}_m} L_m^{(i)}(A) & \xrightarrow{\Theta} & L_m(A) \\
\bigotimes_{i \in \mathbb{Z}_m} \mathcal{H}^{(i)}(A) & \xrightarrow{\cong} & \mathcal{D} \mathcal{H}_m(A).
\end{array}
\]

Since \( \Phi \circ \Theta \) is an isomorphism, we obtain that \( \Theta \) is injective, and thus it is an isomorphism. So \( \Phi \) is an isomorphism. \( \square \)

Corollary 3.3. For each \( i \in \mathbb{Z}_m \), there exists an embedding of algebras

\[ J_i : \text{Heis}(A) \hookrightarrow \mathcal{D} \mathcal{H}_m(A), \quad Z^+_{[M]} \mapsto e_{M,i}, \quad Z^-_{[M]} \mapsto e_{M,i+1}, \quad K_\alpha \mapsto K_{\alpha,i}, \quad K^-_\alpha \mapsto K_{\alpha,i+1}. \]

Proof. For each \( i \in \mathbb{Z}_m \), clearly, the map

\[ J'_i : \text{Heis}(A) \hookrightarrow L_m(A), \quad Z^+_{[M]} \mapsto Z^+_{[M]}, \quad Z^-_{[M]} \mapsto Z^+_{[M]}, \quad K_\alpha \mapsto K^{(i)}_\alpha, \quad K^-_\alpha \mapsto K^{(i+1)}_\alpha \]

is an embedding of algebras. \( \square \)

4. The proof of the relation (3.4) in Theorem 3.2

Consider an extension of \( C_M[1] \) by \( C_N \)

\[ \eta : 0 \rightarrow C_N \rightarrow L_\bullet \rightarrow C_M[1] \rightarrow 0. \]

It induces a long exact sequence in homology

\[ H_{m-1}(C_N) \rightarrow H_{m-1}(L_\bullet) \rightarrow H_{m-1}(C_M[1]) \rightarrow H_0(C_N) \rightarrow H_0(L_\bullet) \rightarrow H_0(C_M[1]). \]

Clearly, \( H_{m-1}(C_Z) = H_0(C_Z[1]) = 0 \) and \( H_{m-1}(C_Z[1]) = H_0(C_Z) = Z \) for any \( Z \in \text{mod} A \). Hence, by writing

\[ L_\bullet = C_X[1] \oplus C_Y \oplus K_T \oplus K_W[1] \]

for some \( X, Y \in \text{mod} A \) and \( T, W \in \mathcal{P} \), we obtain an exact sequence of \( A \)-modules

\[ 0 \rightarrow X \rightarrow M \xrightarrow{\delta} N \rightarrow Y \rightarrow 0, \quad (4.1) \]
where $\delta$ is determined by the equivalence class of $\eta$ via the canonical isomorphisms
\[
\text{Ext}^1_{\mathcal{C}_m(\mathcal{P})}(C_M[1], C_N) \cong \text{Hom}_{K_m(\mathcal{P})}(C_M[1], C_N[1]) \cong \text{Hom}_A(M, N).
\]

Clearly, $\eta$ splits if and only if $\delta = 0$. By considering the kernels and cokernels of differentials in $C_N, C_M[1]$ and $L_\bullet$, we obtain that
\[
T \oplus P_Y \cong P_N, \ W \oplus \Omega_X \cong \Omega_M.
\]

It is easy to see that for any $\langle u \rangle \in \text{mod} A$ and $T, W \in \mathcal{P}$,
\[
|\text{Ext}^1_{\mathcal{C}_m(\mathcal{P})}(C_M[1], C_N)_{C_X[1]\oplus C_Y\oplus K_T\oplus K_W[1]}| = \frac{|W_{MN}^{XY}|}{a_X a_Y} = \frac{a_M a_N}{a_X a_Y} \gamma_{MN}^{XY}, \quad (4.3)
\]

Proof of Theorem 5.2 Let $M, N \in \text{mod} A$.

\[
[C_M[1]] * [C_N] = \sum_{[X], [Y]} |\text{Ext}^1_{\mathcal{C}_m(\mathcal{P})}(C_M[1], C_N)_{C_X[1]\oplus C_Y\oplus K_T\oplus K_W[1]}| \cdot [C_X[1] \oplus C_Y \oplus K_T \oplus K_W[1]].
\]

It is easy to see that $[K_T \oplus K_W[1]] * [C_X[1] \oplus C_Y] = v^{x_0} * [C_X[1] \oplus C_Y \oplus K_T \oplus K_W[1]]$, where $x_0 = \langle W, P_X + \Omega_X + \Omega_Y \rangle + \langle \hat{T}, \hat{P}_X + \hat{P}_Y + \hat{\Omega}_Y \rangle$ and $x_1 = \langle \hat{T}, \hat{P}_X + \hat{\Omega}_Y \rangle + \langle \hat{W}, \hat{\Omega}_X \rangle$.

So we obtain that
\[
[C_X[1] \oplus C_Y \oplus K_T \oplus K_W[1]] = v^{x_1 - x_0} \cdot [K_T \oplus K_W[1]] * [C_X[1] \oplus C_Y].
\]

Clearly, $[K_W[1]] * [K_T] = v^{(W, T)} \cdot [K_W[1] \oplus K_T]$, thus we get that
\[
[K_W[1] \oplus K_T] = v^{-(W, T)} \cdot [K_W[1]] * [K_T].
\]

Since $\text{Ext}^1_{\mathcal{C}_m(\mathcal{P})}(C_Y, C_X[1]) \cong \text{Hom}_{K_m(\mathcal{P})}(C_Y, C_X[2]) = 0$, we have
\[
[C_Y] * [C_X[1]] = \frac{v^{(\Omega_Y, P_X)}}{\text{Hom}_{\mathcal{C}_m(\mathcal{P})}(C_Y, C_X[1])} \cdot [C_X[1] \oplus C_Y] = v^{-(\Omega_Y, P_X)} \cdot [C_X[1] \oplus C_Y].
\]

Hence, $[C_X[1] \oplus C_Y] = v^{(\Omega_Y, P_X)} \cdot [C_Y] * [C_X[1]]$. Therefore,
\[
[C_M[1]] * [C_N] = \sum_{[X], [Y]} \frac{a_M a_N}{a_X a_Y} \gamma_{MN}^{XY} \cdot [K_{-\hat{\Omega}_M, 1} * [K_{-\hat{\Omega}_N} * [C_X[1]]] * [K_T] * [C_Y] * [C_X[1]],
\]

where $x_2 = \langle P_M, \Omega_N \rangle + 2x_1 - x_0 - \langle W, T \rangle - \langle \Omega_Y, P_X \rangle$. So,
\[
e_{M, 1} e_N = v^{(\Omega_M, M)} a_M^{-1} \cdot K_{-\hat{\Omega}_M, 1} * [C_M[1]]] * (v^{(\Omega_N, N)} a_N^{-1} \cdot K_{-\hat{\Omega}_N} * [C_N])
\]
\[
= v^{(\Omega_M, M) + (\Omega_N, N) - (P_M, \Omega_N)} a_M^{-1} a_N^{-1} \cdot K_{-\hat{\Omega}_M, 1} * K_{-\hat{\Omega}_N} * [C_M[1]] * [C_N]
\]
\[
= v^{x_2} \sum_{[X], [Y]} \frac{\gamma_{MN}^{XY}}{a_X a_Y} \cdot K_{-\hat{\Omega}_M, 1} * K_{-\hat{\Omega}_N} * [K_{-\hat{\Omega}_N}] * [K_T] * [C_Y] * [C_X[1]]
\]
\[
= v^{x_2} \sum_{[X], [Y]} \frac{\gamma_{MN}^{XY}}{a_X a_Y} \cdot K_{-\hat{\Omega}_N} * [C_Y] * K_{W - \hat{\Omega}_M, 1} * [C_X[1]]
\]
where $x_3 = \langle \Omega_M, M \rangle + \langle \Omega_N, N \rangle - (P_M, \Omega_N) + x_2$, $x_4 = x_3 + (\Omega_M, \Omega_N) + (\hat{W} - \hat{\Omega}_M, \hat{T}) + (\hat{W} - \hat{\Omega}_M, \hat{\Omega}_Y)$. Since $\hat{T} - \hat{\Omega}_N = -\hat{\Omega}_Y + \hat{M} - \hat{X}$ and $\hat{W} - \hat{\Omega}_M = -\hat{\Omega}_X$, we obtain that

$$e_{M,1}e_N = \sum_{[X],[Y]} v^{x_5, XY}_{MN} \cdot K_{\hat{M} - \hat{X}} \cdot e_Y \cdot e_{X,1},$$

where $x_5 = x_4 - \langle \hat{\Omega}_Y, \hat{Y} \rangle - \langle \hat{\Omega}_X, \hat{X} \rangle$.

Using the exact sequence (4.1), isomorphisms in (4.2) and the respective minimal projective resolutions (2.8) of $M, N, X, Y$, we obtain that

$$x_5 = (\Omega_M, M) + \langle \Omega_N, N \rangle - (P_M, \Omega_N) - \langle \Omega_N, P_M \rangle + (P_M, \Omega_N) + 2(\hat{P}_N - \hat{P}_Y, \hat{P}_X + \hat{\Omega}_Y)
+ 2(\hat{\Omega}_M - \hat{\Omega}_X, \hat{\Omega}_X) - \langle \hat{\Omega}_M - \hat{\Omega}_X, \hat{P}_X + \hat{\Omega}_X + \hat{\Omega}_Y \rangle - \langle \hat{P}_N - \hat{P}_Y, \hat{P}_X + \hat{P}_Y + \hat{\Omega}_Y \rangle
- (\hat{\Omega}_M - \hat{\Omega}_X, \hat{P}_N - \hat{P}_Y) + \langle \Omega_Y, P_X \rangle + \langle \Omega_M, \Omega_N \rangle + \langle \Omega_N, \Omega_M \rangle - \langle \Omega_X, \Omega_Y \rangle - \langle \Omega_Y, \Omega_X \rangle
- \langle \hat{\Omega}_X, \hat{P}_N - \hat{P}_Y \rangle - \langle \hat{P}_N - \hat{P}_Y, \hat{\Omega}_X \rangle - \langle \Omega_Y, Y \rangle - \langle \Omega_X, X \rangle
= (\Omega_M, M) + 2\Omega_X - \hat{\Omega}_X - \hat{P}_X - \hat{\Omega}_Y + \hat{P}_N + \hat{P}_Y + \hat{\Omega}_N)
+ (\Omega_N, \hat{N} - \hat{P}_M + \hat{\Omega}_M
+ \langle \hat{P}_N, 2\hat{P}_X + 2\hat{\Omega}_Y - \hat{X} - \hat{Y} \rangle + \langle \hat{P}_Y, -2\hat{P}_X - 2\hat{\Omega}_Y + \hat{P}_X + \hat{\Omega}_Y + \hat{\Omega}_Y + \hat{\Omega}_X \rangle
+ (\hat{\Omega}_X, -2\hat{\Omega}_X + \hat{\Omega}_Y + \hat{P}_X + \hat{\Omega}_X - \hat{P}_N - \hat{\Omega}_Y - \hat{X} + \hat{P}_Y) + \langle \hat{\Omega}_Y, \hat{P}_X - \hat{\Omega}_Y - \hat{Y} \rangle
= (\hat{\Omega}_M, \hat{M} - \hat{N} - \hat{X} + \hat{Y}) + \langle \hat{\Omega}_N, \hat{N} - \hat{M} \rangle + \langle \hat{P}_N, \hat{X} - \hat{Y} \rangle + \langle \hat{P}_Y, \hat{Y} - \hat{X} \rangle + \langle \hat{\Omega}_Y, \hat{X} - \hat{Y} \rangle
= (\hat{\Omega}_M - \hat{\Omega}_N, \hat{M} - \hat{N} + \hat{\Omega}_Y - \hat{P}_Y + \hat{P}_N - \hat{\Omega}_M, \hat{X} - \hat{Y})
= (\hat{\Omega}_M - \hat{\Omega}_N - \hat{Y} + \hat{P}_N - \hat{\Omega}_M, \hat{M} - \hat{N}) = \langle \hat{N} - \hat{Y}, \hat{M} - \hat{N} \rangle = \langle \hat{M} - \hat{X}, \hat{X} - \hat{Y} \rangle.$

Hence, for any $i \in \mathbb{Z}_m$,

$$e_{M,i+1}e_{N,i} = \sum_{[X],[Y]} v^{(\hat{M} - \hat{X}, \hat{X} - \hat{Y})/2}_{MN} \cdot K_{\hat{M} - \hat{X}, i} \cdot e_{Y,i} \cdot e_{X,i+1}.$$ Therefore, we complete the proof.

Acknowledgments

The author is grateful to Bangming Deng and Jie Sheng for their stimulating discussions and valuable comments. He also would like to thank the anonymous referee for modification suggestions.

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