HOMOLOGICAL DIMENSION OF SOLVABLE GROUPS

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ABSTRACT. In this paper we prove that the homological dimension of an elementary amenable group over an arbitrary commutative coefficient ring is either infinite or equal to the Hirsch length of the group. Established theory gives simple group theoretical criteria for finiteness of homological dimension and so we can infer complete information about this invariant for elementary amenable groups. Stammbach proved the special case of solvable groups over coefficient fields of characteristic zero in an important paper dating from 1970.

1. STATEMENT OF RESULT

We calculate the homological dimension of an elementary amenable group relative to an arbitrary coefficient ring. Throughout the paper, coefficient ring means any non-zero commutative ring. We write $\text{hd}_k(G)$ for the homological dimension of the group $G$ over the coefficient ring $k$. When it makes sense, we write $h(G)$ for the Hirsch length of $G$. Hillman established the working definition of Hirsch length for elementary amenable groups in [10].

Theorem A. Let $G$ be an elementary amenable group and let $k$ be a coefficient ring. If $\text{hd}_k(G)$ is finite then $\text{hd}_k(G) = h(G)$.

This answers a question of Bridson and the first author [5, Conjecture I.1]. Theorem A says that the homological dimension of $G$ is either equal to the Hirsch length or is infinite. Since the below Proposition (which is well known) describes necessary and sufficient conditions for finiteness it follows that we know the homological dimensions of elementary amenable groups.

Proposition. Let $G$ be an elementary amenable group and let $k$ be a coefficient ring. Then $\text{hd}_k(G)$ is finite if and only if the following two conditions hold.

(i) $G$ has no $k$-torsion (meaning that the orders of elements of finite order in $G$ are invertible in $k$).

(ii) $h(G) < \infty$.

As observed by Fel’dman [8] one can draw the following conclusion for cohomological dimension.

Corollary A. If $G$ is a countable elementary amenable group with no $k$-torsion and with finite Hirsch length then $h(G) \leq \text{cd}_k(G) \leq h(G) + 1$.
Proof. Over any ring, a countably generated flat module has projective dimension at most one. From this, one can deduce that the inequalities \( \text{hd}_k(G) \leq \text{cd}_k(G) \leq \text{hd}_k(G) + 1 \) hold for any countable group \( G \) with \( \text{hd}_k(G) < \infty \). Note that the inequality \( h(G) \leq \text{cd}_2(G) \) is proved in [11] Lemma 1.10 and Theorem 1.11, building on the analysis of Hillman and Linnell [12]. □

Subsidiary Results. Along the way, we have found the need of two subsidiary results which may be of independent interest. The first is a technical splitting theorem:

**Theorem B.** Suppose that \( Q \) is a group with subgroups \( L, M, P \) so that the following conditions hold:

(i) \( M \) and \( P \) are normal in \( Q \).

(ii) \( LM \subseteq P \).

(iii) \( M \) is a Mal'cev complete nilpotent group of finite Hirsch length.

(iv) either \( L = 1 \) and \( P/M \) is nilpotent,

or \( LM = P \) and \( L \) is nilpotent.

Then \( Q \) has a subgroup \( Q_0 \) such that \( L \subseteq Q_0 \), \( P \cap Q_0 \) is nilpotent, and \( Q_0M = Q \). If moreover \( Q/M \) and \( L \) are finitely generated, then \( Q_0 \) can be taken to be finitely generated too.

The second is a consequence of the Bieri–Strebel theory of solvable groups. Crucial to the statement is the concept of a group that is locally of type \( \text{FP}_\infty \) meaning that every finite subset of the group is contained in a subgroup of type \( \text{FP}_\infty \). As we shall see later, an elementary amenable group which is locally of type \( \text{FP}_\infty \) and which has Hirsch length \( n < \infty \) always has the property that all finitely generated subgroups of Hirsch length \( n \) are of type \( \text{FP}_\infty \) and this is a reason why the property has prominent role. See also Definition 2.8 below and the remarks following it.

**Theorem C.** Let \( G \) be a group with a Mal'cev complete nilpotent normal subgroup \( N \) of finite Hirsch length such that \( G/N \) is free abelian of finite rank. Let \( x_1, \ldots, x_n \) be a sequence of elements of \( N \) whose natural images \( v_1, \ldots, v_n \) in the rational vector space \( V := N/[N, N] \otimes \mathbb{Q} \) form a basis. Each element \( g \in G \) acts by conjugation on \( N \) and therefore has an induced action on \( V \) and with respect to the basis \( v_1, \ldots, v_n \) this action is given by a matrix. That is to say, there is a representation \( \rho : G \to \text{GL}_n(\mathbb{Q}) \).

Suppose that \( X \) is a subset of \( G \) such that the matrices \( \rho(x) \) belong to \( \text{GL}_n(\mathbb{Z}) \) for all \( x \in X \) and such that \( G = \langle X \cup N \rangle \). Then \( G \) is locally of type \( \text{FP}_\infty \).

Organization of the paper. In the next section we include some background material about solvable groups of finite rank, Hirsch lengths, constructible groups, and inverse duality groups. Material on nilpotent groups and their Mal'cev completions is contained in Section 3. Some reductions for Theorem A are made in Section 4. Of special note, Lemma 4.5 implies that if \( G \) is an elementary amenable group with \( \text{hd}_k(G) = h(G) < \infty \) then for any subnormal subgroup \( H \) of \( G \) we also have \( \text{hd}_k(H) = h(H) < \infty \). Much of the technical drive in this paper concerns embedding groups as subnormal subgroups of nicer groups so that this Lemma can be used.

Theorem B is proved in Section 5 using classical cohomological vanishing. Theorem C is explained in Section 6, and Theorem A is proved in a special case in Section 7. The proof of Theorem A is completed using some further applications of Theorem B in Section 8. In broad outline the idea is to embed the original group,
by using Theorem B, into a group satisfying the hypotheses of Theorem C. Then we can use established theory of solvable groups of type $\text{FP}_\infty$ to prove Theorem A.

In the next two short Sections 6 and 7 we explain Theorem C and then go on to use it in combination with Theorem B to obtain an important special case of Theorem A.

Finally, Theorem B is applied twice more in Section 8 to reduce Theorem A to the case considered in the preceding section.

2. Background Material and Historical Remarks

Recall that the class of elementary amenable groups is the smallest class of groups containing all finite and all abelian groups, that is also closed under group extensions and directed unions. In his seminal paper [25] dating from 1929, von Neumann introduced the concept of amenability and related it to the Banach–Tarski paradox. He also showed that all elementary amenable groups are amenable. He coined the expression ‘$G$ ist messbar’, while the modern terminology ‘$G$ is amenable’ can be traced to more recent work of Day (see [7], for example), perhaps derived from the anagram mean-able: the amenable groups are precisely those groups for which the Banach space of bounded $\mathbb{R}$-valued functions admits an invariant mean.

Hirsch Length and Homological Dimension. The connection between Hirsch length and homological dimension of solvable groups was established by Stammbach who proved that $\text{hd}_K(G) = h(G)$ whenever $G$ is solvable and $K$ is a field of characteristic zero: his elegant calculation uses exterior powers of abelian groups [25]. This work, published in 1970 was quickly followed by important work of Fel’dman, and amongst other things Fel’dman makes a claim that Stammbach’s characteristic zero calculation can be extended to positive characteristic. However, this claim cannot be substantiated in the way that Fel’dman proposes. Bieri [3] gives a detailed account of Stammbach’s result but makes no comment how a calculation of homological dimension of solvable groups in positive characteristic might proceed. In fact, modules witnessing homological dimension in positive characteristic are significantly more complicated than those used by Stammbach. Complication of some kind is unavoidable in the light of [5, Lemma I.6].

The Hillman–Linnell Theorem. In the solvable case, the following result combines work of Mal’cev, Gruenberg and Robinson. We refer the reader to [20, §5.2] for commentary and proof. Hillman and Linnell [12] extended this to the larger class of elementary amenable groups, and for this general case we refer the reader to [11, Theorem 1.9]. Wehrfritz has given an alternative short and explicit account of this result in [27].

Lemma 2.1. Let $G$ be an elementary amenable group. If $h(G)$ is finite then $G$ has characteristic subgroups $T \subset N \subset H \subset G$ such that the following hold.

(i) $T$ is the unique largest normal locally finite subgroup of $G$
(ii) $N/T$ is the Fitting subgroup of $G/T$ and it is torsion-free and nilpotent
(iii) $H/N$ is a finite rank free abelian group
(iv) $G/H$ is finite.
The above conditions uniquely determine the subgroups $T$ and $N$. However, if $G/N$ has a non-trivial finite normal subgroup, there is not necessarily a natural condition specifying $H$ uniquely. □

Another useful way of understanding Hirsch length is as follows.

**Lemma 2.2.** Let $G$ be an elementary amenable group. Then $G$ has finite Hirsch length if and only if there is a series $1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G_n = G$ in which the factors are either cyclic or locally finite. Moreover, when these conditions hold then $G$ has finitely generated subgroups with the same Hirsch length.

**Proof.** We comment only on the last point. Suppose that $1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G_n = G$ is a series with cyclic or locally finite factors. Let $J = \{j; \ G_j/G_{j-1} \cong \mathbb{Z}\}$ and for $j \in J$ choose $g_j$ to be a generator of $G_j$ modulo $G_{j-1}$. Then the subgroup $\langle g_j; j \in J \rangle$ is finitely generated of the same Hirsch length as $G$. □

**Cohomological Dimension and Constructible Groups.** Calculations of cohomological dimension for solvable groups are harder. The theory is well developed in characteristic zero and it is known that the elementary amenable groups which satisfy $cd_2(G) = h(G) < \infty$ are precisely the torsion-free virtually solvable groups that are constructible (constructable) in the sense of Baumslag and Bieri [2]. A version of this fact was conjectured by Gildenhuys and Strebel [9] and proved by the first author [15]. Subsequently this led to a proof that elementary amenable groups of type $\text{FP}_\infty$ over $\mathbb{Z}$ are constructible [16] and to the construction of classifying spaces for proper actions for such groups, see [15, 18, 19]. (Results in [16, 19] also apply considerably beyond the elementary amenable case.)

**Inverse Duality Groups.** Key results of Fel'dman [8] are covered in Bieri’s notes and are used by Brown and Geoghegan [6] to establish the following fundamental result.

**Theorem 2.3** (The Inverse Duality Theorem). Let $G$ be a constructible elementary amenable group (that is a group with a subgroup of finite index that can be built up from the trivial group with a finite number of ascending \(\text{HHN}\)-extensions). If $G$ is torsion-free then $G$ is an inverse duality group. For such a group, it holds that $\text{hd}_k(G) = \text{cd}_k(G) = h(G) < \infty$ for all coefficient rings $k$.

We refer the reader to Bieri’s notes for an explanation of cohomological duality and in particular the notion of inverse duality group. The inverse duality theorem holds for a wider class of groups that can be described in terms of fundamental groups of graphs of groups.

**Baer’s class of polyniminimax groups: A short survey.** A group $G$ is called polyniminimax if it has a series $1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$ in which the factors are cyclic, quasicyclic or finite. Following Baer’s original work [1] the term polyniminimax has usually been abbreviated to minimax. The central role of polyniminimax groups is illustrated by the following classical results.

**Theorem 2.4** (Robinson [23], Kropholler [14]). Let $G$ be a finitely generated virtually solvable group. Then the following are equivalent.

(i) $G$ is polyniminimax.

(ii) $G$ has finite Prüfer rank.

(iii) $G$ has finite abelian section rank.
(iv) $G$ has no lamplighter sections.

The following structural information is well explained by Robinson \cite{Robinson21, Robinson22}.

**Theorem 2.5.** Let $G$ be a polyminimax group. Then the following are equivalent.

(i) $G$ is $\mathbb{Q}$-linear (i.e. isomorphic to a subgroup of $\text{GL}_n(\mathbb{Q})$ for some $n$).
(ii) $G$ is residually finite.
(iii) $G$ is virtually torsion-free.
(iv) $G$ has no quasicyclic subgroups.

In general the finite residual $R$ of $G$ is a direct product of finitely many quasicyclic groups. More recently it has been shown that every finitely generated polyminimax group is isomorphic to a quotient of a torsion-free polyminimax group, see \cite{Guralnick17}. In a slightly different direction, it is known that all finitely generated polyminimax groups are boundedly generated \cite{Dyer14, Proposition 1} and it is a difficult open question whether the converse holds within the class of elementary amenable groups.

**The Commensurated Subgroup Lemma.** Two subgroups $H$ and $K$ of a group $G$ are said to be commensurate if $H \cap K$ has finite index in both $H$ and $K$. The set $\text{Comm}_G(H) := \{ g \in G; \ g^{-1}Hg \text{ and } H \text{ are commensurate} \}$ is a subgroup of $G$ called the commensurator. A subgroup is commensurated when its commensurator equals $G$ (just as a subgroup is normal when its normalizer equals $G$).

A simple interconnection between commensurability and Hirsch length stems from the well known observation that if $H \leq G$ are polycyclic-by-finite groups then $h(H) \leq h(G)$ with equality if and only if $H$ has finite index in $G$. This has a generalization in terms of the invariant $h^*(G)$ of a polyminimax group $G$ defined to be the number of infinite terms in any cyclic/quasicyclic/finite series. It is an elementary variation that if $H \leq G$ are polyminimax then $h^*(H) \leq h^*(G)$ with equality if and only if $H$ has finite index in $G$. In general, for a subgroup $H$ of an elementary amenable group $G$ we have $h(H) \leq h(G)$ but usually equality of Hirsch lengths here cannot be expected to imply finite index of the subgroup. However the following important result goes a little deeper.

**Lemma 2.6.** Let $G$ be an elementary amenable group of finite Hirsch length that is locally polyminimax. Then $G$ has finitely generated subgroups of the same Hirsch length and all such subgroups are commensurate with one another and are commensurated.

**Proof.** Suppose first that $G$ is a finitely generated polyminimax group and that $H$ is a subgroup of the same Hirsch length. We prove by induction on $h^*(G)$ that $H$ has finite index in $G$. If $h^*(G) = 0$ then $G$ is finite and there is nothing to prove. If $h^*(G) > 0$ then $G$ has an infinite normal abelian subgroup $A$. Then $h^*(G/A) < h^*(G)$ and so by induction $HA$ has finite index in $G$. Replacing $G$ by $HA$ we may assume that $G = HA$. Now $H \cap A$ is normalized by both $A$ and $H$, so it is normal in $G$. We may replace $G$ by $G/H \cap A$ and assume that $G$ is the semidirect product of $A$ by $H$. Since $H$ and $G$ have the same Hirsch length it follows that $A$ is locally finite and since $A$ is finitely generated as a normal subgroup of $G$ (because $G$ is finitely generated and $A$ is complemented) it follows that $A$ is finite. Hence $H$ has finite index in $G$. 

In general, using Lemma 2.2 we can choose a finitely generated subgroup $H$ of $G$ with $h(H) = h(G)$. If $F$ is any finite subset of $G$ then

$$h(G) = h(H) \leq h(H \cup F) \leq h(G)$$

so $h(H) = h(H \cup F)$ and applying the above argument to the subgroup $H$ of the finitely generated group $H \cup F$ we deduce that $H$ has finite index in $H \cup F$. Now if $K$ is any other finitely generated subgroup of $G$ of the same Hirsch length then we can apply this argument to $(H \cup K)$ to deduce that $H$ and also $K$ both have finite index in $(H \cup K)$. In particular it follows that $H$ and $K$ are commensurate.

Keeping the same subgroup $H$ as in the preceding paragraph, let $g$ be any element of $G$. Then $g^{-1}Hg$ is another finitely generated subgroup of the same Hirsch length as $G$ and hence by the above, $H$ and $g^{-1}Hg$ are commensurate. It follows that $H$ is commensurated and the proof is complete.

**Corollary 2.7.** Let $G$ be a locally polyminimax group of finite Hirsch length. Then $\sup \{h^*(H) : H$ a finitely generated subgroup of $G\}$ is finite. Moreover, if $H$ is a finitely generated subgroup of $G$ then $h(H) = h(G)$ if and only if $h^*(H) = h^*(G)$.

In view of this it can be useful to introduce the invariant $h^*(G) := h^*(G) - h(G)$ in case $h$ and $h^*$ are defined. For a polyminimax group this difference is equal to the number of quasicyclic sections in any cyclic/quasicyclic/finite series for $G$.

The above applies well in the case of elementary amenable groups that are locally of type $\mathsf{FP}_\infty$. This is perhaps a good moment to review the use of the term *locally*.

**Definition 2.8.** If $\mathcal{X}$ is a class of groups or a group-theoretical property then by a *locally $\mathcal{X}$-group* we mean a group all of whose finite subsets are contained in $\mathcal{X}$-subgroups.

When $\mathcal{X}$ is a subgroup closed class or property then locally $\mathcal{X}$ groups are exactly those groups whose finitely generated subgroups belong to $\mathcal{X}$. For example, locally finite groups are groups all of whose finitely generated subgroups are finite. Locally $\mathsf{FP}_\infty$ groups are groups all of whose finitely generated subgroups are contained in subgroups that have type $\mathsf{FP}_\infty$. But note that a locally $\mathsf{FP}_\infty$ group can have finitely generated subgroups that are not of type $\mathsf{FP}_\infty$.

**Corollary 2.9.** Let $G$ be an elementary amenable group that is locally of type $\mathsf{FP}_\infty$ and which has finite Hirsch length. Then all finitely generated subgroups that have the same Hirsch length as $G$ are of type $\mathsf{FP}_\infty$ and are mutually commensurate and commensurated in $G$. □

### 3. Nilpotent Groups

We write $\gamma_i(G)$ for the $i$th term of the lower central series of $G$. It is defined inductively by $\gamma_1(G) = G$ and then by taking commutators: $\gamma_{i+1}(G) = [\gamma_i(G), G]$. We use a *right-handed* convention for commutators, namely; for group elements $x$ and $y$, we write $[x, y] := x^{-1}y^{-1}xy$. The same convention and notations are used in [24 Chapter 5] where the general theory is also developed. The *isolator series* $\overline{\gamma}_i(G)$ may be defined by $\overline{\gamma}_i(G) := \{ g \in G; g^m \in \gamma_i(G) \text{ for some } m \in \mathbb{N} \}$. The lower central series and the isolator series are both central series comprising fully invariant subgroups. A group is nilpotent if there is a $c \geq 0$ such that $\gamma_{c+1}(G) = 1$ and the least $c$ for which this holds is the *class*. A torsion-free group is nilpotent of class
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$\leq c$ if and only if $\mathfrak{p}_{c+1}(G) = 1$. An account of the theory of the isolator series can be found in [20 §2.3].

**Definition 3.1.** Let $\pi$ be a set of primes. We shall say that a nilpotent group is $\pi$-divisible if it is $p$-divisible for all $p$ in $\pi$ or equivalently when the function $g \mapsto g^n$ is surjective for every $\pi$-number $n$. We shall say that a torsion-free nilpotent group is $\pi$-complete when it is $\pi$-divisible. For such a group the function $g \mapsto g^n$ is bijective for every $\pi$-number $n$. A torsion-free nilpotent group is called Mal’cev complete if it is $\Pi$-complete where $\Pi$ is the set of all primes. A nilpotent group is Mal’cev complete precisely when the function $g \mapsto g^n$ is bijective for every natural number $n$.

The free nilpotent group on $d$ generators of class $c$ is defined to be $F/\gamma_{c+1}(F)$ where $F$ is the free group on $d$ generators. So long as $d \geq 2$, this group does indeed have class $c$. Of course if $d \leq 1$ then it is cyclic. Note that there is no particular reason for $d$ to be finite: the theory makes sense for any cardinal number $d$. The class $c$ is always a non-negative integer.

Every nilpotent group $G$ has a Mal’cev completion denoted by $G^Q$ which is uniquely determined, [20 2.1.1]. The theory is further developed in [20 §2.1]. There is a natural map from $G$ to its Mal’cev completion and this is injective precisely when $G$ is torsion-free. The free nilpotent groups are torsion-free and so embed into their Mal’cev completions. More generally if $\pi$ is any set of primes there is a $\pi$-completion of a nilpotent group which is $\pi$-divisible. These completions are characterized by the universal property that maps into $\pi$-divisible nilpotent groups extend uniquely to the $\pi$-completions.

**Definition 3.2.** By a free $\pi$-complete (resp. Mal’cev complete) group on $d$ generators of class $c$ we shall mean the $\pi$-completion (resp. Mal’cev completion) of the free nilpotent group on $d$ generators of class $c$. Whenever we use the terms $\pi$-complete or Mal’cev complete it is to be understood that the group in question is nilpotent.

**Lemma 3.3.** Let $G$ be a free Mal’cev complete group of class $c$. Let $H \to K$ be any surjective homomorphism where $H$ is Mal’cev complete of class at most $c$. Then every homomorphism from $G$ to $K$ factors through $H$.

*Proof.* Let $d$ be the dimension of the rational vector space $G/\gamma_2(G)$. Let $F$ be a free nilpotent group of class $c$ on $d$ generators. Now, we can identify $G$ with $F^Q$. The universal property of $F$ ensures that the composite map $F \to F^Q = G \to K$ factors through $H$. On passing to Mal’cev completion we have an induced map $F^Q = G \to H^Q = H$ as required. □

The following Lemma is required for the proof of an important result about Mal’cev complete nilpotent groups, namely Proposition 3.8 below.

**Lemma 3.4.** Let $G$ be a nilpotent group and let $H$ be a subgroup such that $H\gamma_2(G) = G$. Then $H = G$.

*Proof.* Since $G$ is nilpotent, $H$ is subnormal and so we may replace $H$ by its normal closure and without loss of generality we may assume that $H$ is normal. The quotient $G/H$ is therefore both perfect and nilpotent which implies $H = G$. □
Definition 3.5. For any group $G$ and any non-zero rational numbers $m$ we say that an automorphism $\phi$ of $G$ is $m$-powering provided that there are (non-zero) integers $p$ and $q$ such that for all $g \in G$,
\[(g^q)^\phi \in g^p \gamma_2(G) \text{ and } m = p/q.\]

By a powering automorphism we mean an automorphism that is $m$-powering for some $m$. When $G$ is a group such that $\gamma_2(G) < G$ then a powering automorphism is $m$-powering for a uniquely determined rational number $m$. Note that this uniqueness property therefore holds when $G$ is a non-trivial Mal'cev complete group.

Lemma 3.6. Let $G$ be a group. Let $\phi$ be an $m$-powering automorphism of $G$ where $m$ is a natural number. Then for each $j$, $\phi$ induces the scalar automorphism $m^j$ on the lower central factor $\gamma_j(G)/\gamma_{j+1}(G)$. Moreover, if $G$ is Mal'cev complete then the same conclusion holds on allowing $m$ to be any rational number.

Proof. This follows from the iterated commutator formula
\[\left[ g_1^m, \ldots, g_n^m \right] \equiv \left[ g_1, \ldots, g_n \right]^{m^j} \mod \gamma_{j+1}(G)\]
which holds for any group when $m$ is a natural number and which makes sense and holds for any Mal'cev complete group when $m$ is a rational number. 

Let $N$ be a group with $\gamma_2(N) < N$. Let $U$ denote the set of 1-powering automorphisms of $N$ and let $V$ denote the set of all powering automorphisms of $N$. Then $U$ and $V$ are subgroups of the automorphism group of $N$ and the function $V \to \mathbb{Q}^\times$ given by $\phi \mapsto m$ where $m$ is the unique natural number for which $\phi$ is $m$-powering is a homomorphism whose kernel is $U$.

For each $j$ and each automorphism $\phi$ of $N$, let $^j\phi$ denote the induced automorphism of $N/\gamma_{j+1}(N)$. The assignment $\phi \mapsto ^j\phi$ determines a homomorphism
\[U \to \text{Aut}(N/\gamma_{j+1}(N))\]
and we write $U_j$ for the kernel of this homomorphism. In this way we obtain a descending chain
\[U = U_1 \geq U_2 \geq U_3 \ldots\]
of normal subgroups of $U$. With this notation we have:

Lemma 3.7. For each $j$, $U_j$ is a normal subgroup of $U_{j+1}$ and the factor group $U_j/U_{j+1}$ is abelian and isomorphic to a subgroup of
\[\text{hom}(N/\gamma_2(N), \gamma_j(N)/\gamma_{j+1}(N)).\]

If $m$ is a natural number and $\phi$ is an $m$-powering automorphism of $N$ then the action of $\phi$ by conjugation on $U$ induces scalar multiplication by $m^{j+1}$ on the factor $U_j/U_{j+1}$. Moreover if $N$ is nilpotent then $U$ is also nilpotent and of class one less than the class of $N$.

The free Mal'cev complete groups admit an abundance of powering automorphisms, and this fact is vital for our arguments.

For a group $G$ and a subgroup $H$, let $\text{Aut}(G;H)$ denote the set of those automorphisms of $G$ which restrict to automorphisms of $H$. This is a subgroup of $\text{Aut}(G)$ and the restriction map affords a homomorphism $\text{Aut}(G;H) \to \text{Aut}(H)$. If $H$ is normal in $G$ then elements of $\text{Aut}(G;H)$ naturally induce automorphisms of $G/H$ and there is a homomorphism $\text{Aut}(G;H) \to \text{Aut}(G/H)$.
Proposition 3.8. Let \( N \) be a Mal’cev complete nilpotent group. Then there exist a free Mal’cev complete group \( \hat{N} \) of the same class as \( N \) and a homomorphism \( \pi: \hat{N} \to N \) such that the following hold:

(i) The induced map \( \pi: \hat{N}/\gamma_2(\hat{N}) \to N/\gamma_2(N) \) is an isomorphism.
(ii) The natural map \( \text{Aut}(\hat{N}; \text{Ker} \pi) \to \text{Aut}(N) \) is surjective and its kernel is contained in the subgroup \( U(\text{Aut}(\hat{N})) \) of \( 1 \)-powering automorphisms of \( \hat{N} \).

Proof. We include a sketch of the proof of the surjectivity in (ii) and leave the rest of the proof as an exercise. For the surjectivity, let \( \alpha \) be an automorphism of \( N \) and let \( F \) be a free nilpotent group of the same class as \( N \) on \( \dim_{\mathbb{Q}}(N/\gamma_2(N)) \) generators. Then we can choose a factorization of the composite \( F \to N \xrightarrow{\alpha} N \) through \( \hat{N} := F^{\mathbb{Q}} \) to obtain a commutative diagram

\[
\begin{array}{ccc}
F & \longrightarrow & N \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
\hat{N} & \longrightarrow & N \\
\end{array}
\]

Here, \( \alpha \) and \( \alpha' \) induce the same map on abelianization and when we extend \( \alpha' \) to \( F^{\mathbb{Q}} = \hat{N} \) we obtain an endomorphism \( \hat{\alpha} \) of \( \hat{N} \) that induces an injective and therefore surjective endomorphism of \( \hat{N}/\gamma_2(\hat{N}) \). Using Lemma 3.4 we can deduce that \( \hat{\alpha} \) induces a surjective and hence also injective endomorphism of \( N \). Therefore \( \hat{\alpha} \) is an automorphism of \( \hat{N} \) fitting into the following commutative diagram:

\[
\begin{array}{ccc}
\hat{N} & \longrightarrow & N \\
\downarrow{\hat{\alpha}} & & \downarrow{\alpha} \\
\hat{N} & \longrightarrow & N \\
\end{array}
\]

This shows that \( \alpha \) lies in the image of the map \( \text{Aut}(\hat{N}; \text{Ker} \pi) \to \text{Aut}(N) \). \( \square \)

4. Reductions for the Proof of Theorem A

Locally \( \text{FP}_\infty \) groups have an important role to play in our arguments. Before returning to this class we need to mention the key inequalities of a more elementary nature.

Lemma 4.1 ([5], Theorem 1.2). Let \( G \) be an elementary amenable group. Then the following are equivalent.

(i) \( \text{hd}_k(G) < \infty \).
(ii) \( h(G) < \infty \) and \( G \) has no \( k \)-torsion.

When these conditions hold, we have \( \frac{h(G)}{2} \leq \text{hd}_k(G) \leq h(G) \). \( \square \)

We refer the reader to [3] for the standard theory summarized in the following proposition.

Proposition 4.2. The following hold for any group \( G \) and any coefficient ring \( k \) such that \( \text{hd}_k(G) < \infty \).

(i) If \( H \) is a subgroup of \( G \) then \( \text{hd}_k(G) \geq \text{hd}_k(H) \).
(ii) If \( H \) has finite index in \( G \) then \( \text{hd}_k(G) = \text{hd}_k(H) \).
(iii) If \( T \) is a normal locally finite subgroup of \( G \) then \( \text{hd}_k(G/T) = \text{hd}_k(G) \).
(iv) If $H$ is a subgroup with the property that $|[H \cup F] : H| < \infty$ for all finite subsets of $G$ then $\text{hd}_k(G) = \text{hd}_k(H)$. □

We now return to the required properties of groups that are locally of type $\text{FP}_\infty$. The following two results are fundamental to our approach to Theorem A.

**Proposition 4.3.** Let $G$ be an elementary amenable group that is locally of type $\text{FP}_\infty$. If $\text{hd}_k(G) < \infty$ then $\text{hd}_k(G) = h(G)$.

**Proof.** Assume that $\text{hd}_k(G)$ is finite. Then $G$ has no $k$-torsion and $h(G)$ is finite by Lemma 3.1. Let $H$ be a finitely generated subgroup of $G$ of the same Hirsch length. Then $H$ is of type $\text{FP}_\infty$ by Corollary 2.9. Therefore $H$ is constructible and is virtually an inverse duality group over $\mathbb{Z}$. It follows that $h(G) = h(H) = \text{hd}_k(H) \leq \text{hd}_k(G)$. Lemma 4.1 gives the reverse inequality $\text{hd}_k(G) \leq h(G)$ and we deduce that $\text{hd}_k(G) = h(G)$. □

**Proposition 4.4.** Let $G$ be a group with a torsion-free nilpotent normal subgroup $N$ such that $G/N$ is abelian. Then the following are equivalent.

(i) $G$ is locally of type $\text{FP}_\infty$.
(ii) $G$ is locally constructible.
(iii) $G/[N,N]$ is locally of type $\text{FP}_\infty$.

This proposition can be deduced from the structural information in either of the papers [41,5].

Finally in this section, we record a simple spectral sequence argument which can be used to show that the validity of the conclusions of Theorem A are inherited by subnormal subgroups.

**Lemma 4.5.** Let $G$ be an elementary amenable group with $\text{hd}_k(G) = h(G) < \infty$ and let $N$ be a normal subgroup of $G$ such that $\text{hd}_k(G/N) < \infty$. Then $\text{hd}_k(G/N) = h(G/N)$ and $\text{hd}_k(N) = h(N)$. Furthermore, if $N$ is any normal subgroup of $G$, the conclusion $\text{hd}_k(N) = h(N)$ holds whether or not $\text{hd}_k(G/N) < \infty$.

**Proof.** By the Lyndon–Hochshild–Serre spectral sequence we have

$$\text{hd}_k(G) \leq \text{hd}_k(G/N) + \text{hd}_k(N).$$

By Proposition 4.2(i) we know $\text{hd}_k(N) < \infty$ and so Lemma 4.1 gives

$$\text{hd}_k(G/N) \leq h(G/N)$$

and $\text{hd}_k(N) \leq h(N)$.

Putting the pieces together we have

$$h(G) = \text{hd}_k(G) \leq \text{hd}_k(G/N) + \text{hd}_k(N) \leq h(G/N) + h(N) = h(G)$$

which gives the desired result in case $\text{hd}_k(G/N) < \infty$. In general, $G/N$ may not have finite homological dimension, but it is an elementary amenable group of finite Hirsch length and by Lemma 2.1 there exist subgroups $N \triangleleft G_1$ such that $N \leq N_1$, $G_1$ has finite index in $G$, $N_1/N$ is locally finite, and $G_1/N_1$ is torsion-free. By Proposition 4.2(iv), $\text{hd}_k(N) = \text{hd}_k(N_1)$ and by 4.2(ii), $\text{hd}_k(G) = \text{hd}_k(G_1)$. Note also that $N$ and $N_1$ have the same Hirsch length.

Now we can apply the above argument to the situation that $N_1$ is a normal subgroup of $G_1$ because $G_1/N_1$ being torsion-free of finite Hirsch length does indeed have finite homological dimension. This shows that $\text{hd}_k(N_1) = h(N_1)$, (and that $\text{hd}_k(G_1/N_1) = h(G_1/N_1)$ but we do not need this). The final statement of our lemma now follows by combining with the equalities $\text{hd}_k(N) = \text{hd}_k(N_1)$ and $h(N) = h(N_1)$. □
As remarked at the end of Section 1 this has the following consequence.

**Corollary 4.6.** Let $G$ be an elementary amenable group with $hd_k(G) = h(G) < \infty$. If $H$ is a subnormal subgroup of $G$ then $hd_k(H) = h(H) < \infty$.

Of course the corollary is also a consequence of Theorem A, but its role in the proof of Theorem A is significant. In the light of this, following some basic reductions the proof of Theorem A is mainly concerned with embedding a group as a subnormal subgroup of a locally FP$_\infty$ group.

5. **Cohomological Vanishing Results, Splittings, and Theorem B**

We shall need the following key vanishing result of Robinson in the special case when $S = \mathbb{Q}$ and $\dim_\mathbb{Q} M < \infty$.

**Theorem 5.1** (20 10.3.1 and 10.3.2). Let $S$ be a commutative ring, $G$ a nilpotent group, and $M$ an SG-module. If either
- $M$ is noetherian and $M_G = 0$, or
- $M$ is artinian and $M^G = 0$,

then $H^n(G, M) = H_n(G, M) = 0$ for every $n$. \(\Box\)

**The Proof of Theorem B.** We are now in a position to prove the technical splitting theorem that is required for our solution to the homological dimension calculation.

**Notational Remark.** In the following proof we write $A^B$ to indicate the set of $B$-fixed points in $A$ in case $B$ is a group acting on $A$. We use the right-handed convention for conjugation: $x^g = y^{-1}xy$ and note that $[x, y] = x^{-1}y^{-1}xy$. If $K$ is a normal abelian subgroup of $G$ then a derivation $\delta : G \to K$ is a function satisfying $\delta(gg') = (\delta g)^g (\delta g')$. An inner derivation is a derivation $\delta$ for which there exists $x \in K$ such that $\delta g = [g, x]$ for all $g$.

**Proof of Theorem B.** We proceed by induction on $h := h(M)$. If $h = 0$ then $M$ is trivial and we set $Q_0 := Q$: note that (iv) implies $P$ is nilpotent in this case and the result is immediate. Suppose now that $h > 0$. Choose $K$ to be a non-trivial divisible subgroup of $M$ of least possible Hirsch length subject to being normalized by $Q$. Note that $K$ must be abelian and therefore can be viewed as a $\mathbb{Q}Q$-module. As such $K$ is irreducible and the action of $Q$ descends to and action of $Q/K$. By induction, there is a subgroup $Q_1$ of $Q$ such that
- $LK \subseteq P \cap Q_1$.
- $(P \cap Q_1)/K$ is nilpotent, and
- $Q_1M = Q$.

At an important later step in this proof we shall want to restrict the action of $Q$ to $M$ and view $K$ as a $\mathbb{Q}M$-module. The choice of $K$ ensures that $K$ lies in the centre of $M$. We write $P_1 := P \cap Q_1$. As $MP_1 = P \cap Q_1M = P \cap Q = P$ and $K^M = K$ we deduce $K^P = K^{F^h}$.

The fact that $P$ is normal in $Q$ implies that $K^P$ is a $\mathbb{Q}Q$-submodule of $K$ thus the irreducibility of $K$ implies that either $K^P = K^{F^h} = 0$ or $K^P = K^{F^h} = K$. In the latter case, as $P_1/K$ is nilpotent we deduce that also $P_1$ is and we only have to take $Q_0 := Q_1$. Then $Q_0 \cap P = Q_1 \cap P = P_1$ is nilpotent, $L \subseteq Q_0$ and $Q_0M = Q_1M = Q$. If we have the extra hypothesis on the finite generation of $Q/M$ and $L$ then, in the inductive step we also deduce that $Q_1/K$ is finitely generated. Choose a set of lifts
X to \( Q_1 \) of a finite generating system of \( Q_1/K \) and a finite generating system \( Y \) of \( L \) and let now \( Q_0 \) be the subgroup of \( Q_1 \) generated by \( X \cup Y \). Then \( Q_0 \) is finitely generated and \( L \leq Q_0 \). Also, \( Q_0 \leq Q_1 \) thus \( Q_0 \cap P \leq Q_1 \cap P = P_1 \) is nilpotent. Finally, by construction \( Q_0K = Q_1 \) thus

\[
Q_0M = Q_0KM = Q_1M = Q.
\]

So we may assume now that \( K^P = K^{P_1} = 0 \). Then Theorem 5.1 shows that

\[
H^*(P_1/K, K) = 0 \quad (\ast)
\]

Using the spectral sequence \( H^*(Q_1/P_1, H^*(P_1/K, K)) \Rightarrow H^*(Q_1/K, K) \) with \( (\ast) \), we find that \( H^*(Q_1/K, K) \) also vanishes. In particular \( H^2(Q_1/K, K) = 0 \) and there is a splitting: there exists a group \( Q_0 \) of \( Q_1 \) such that \( Q_0K = Q_1 \) and \( Q_0 \cap K = \{1\} \). Note that \( P_1 \cap Q_0 \) is a complement to \( K \) in \( P_1 \) and is therefore nilpotent. Moreover, \( Q_0M = Q_1KM = Q \). In the case when \( L = 1 \) there is nothing else to prove.

So we assume now \( LM = P \). Then using again that \( M \) centralizes \( K \), it follows that \( K^P = K^{L} \). Therefore

\[
H^*(L, K) = 0. \quad (\dagger)
\]

At this point, we need to establish the conclusion \( L \subseteq Q_0 \) only, and while this conclusion may not be true of the \( Q_0 \) we are currently entertaining, we can replace \( Q_0 \) by a conjugate subgroup to achieve the goal. Here is the reasoning. Every element \( g \) of \( Q_1 \) is uniquely expressible in the form \( y_\ell k_\ell \) with \( y_\ell \in \bar{Q}_0 \) and \( k_\ell \in K \). The function \( g \mapsto k_\ell \) is a derivation from \( Q_1 \) to \( K \). If we restrict this derivation to \( L \), the vanishing \( (\dagger) \) of first cohomology says this derivation is inner. In other words there is an element \( x \in K \) such that \( k_\ell = [x, l] \) for all \( l \in L \). Thus \( l^\ell = l[l, x] \) belongs to \( Q_0 \) for all \( l \in L \). Hence we can replace \( Q_0 \) by \( Q_0^{\dagger} \) to ensure that \( L \subseteq Q_0 \).

To finish, note that if we have the extra hypothesis on finite generation then \( Q_1/K = Q_0K/K \cong Q_0 \) is finitely generated.

\[\square\]

6. Theorem C

Theorem C is vital for our arguments but is really a relatively straightforward consequence of the Bieri–Strebel theory. Let \( G \) and \( N \) be as in the statement. Let \( H \) be a finitely generated subgroup of \( G \) with the same Hirsch length as \( G \) and so that \( G = HN \). As shown in Lemma 2.6, all such subgroups are commensurate. Let \( A \) denote the abelian group \( H \cap N/H \cap \gamma_2(N) \). The quotient group \( Q := G/N \) acts on \( A \) and the Bieri–Strebel invariants \( \text{dis} \Sigma_A(Q) \) and \( \Sigma_A^c(Q) \) are defined as in \cite{BieriStrebel}. According to \cite{BieriStrebel} §1.12) the following are equivalent for an element \( q \in Q \):

- \( \Sigma_A^c \subset \mathcal{R}_q \)
- \( \text{dis} \Sigma_A^c(Q) \subset \mathcal{H}_q \), and
- \( q \) is integral with respect to \( A \).

Here, \( \mathcal{R}_q \) denotes the closed hemisphere \( \{[v]; v(q) \geq 0\} \). The third condition here means that there is a monic polynomial \( q \) with integer coefficients that annihilates \( A \). This is immediate when \( q \) is an element whose action on \( A \) (or \( N/\gamma_2(N) \)) is represented by an integer matrix. Theorem C now follows from the main results of \cite{BieriStrebel}. To see this we need only observe that the characteristic polynomial of an integer matrix \( M \) corresponding to a group element \( q \in Q \) annihilates \( A \) and so such a \( q \in Q \) is integral with respect to \( A \). The result follows by \cite{BieriStrebel} Theorem A(iii)].
7. A special case of Theorem A

Our goal in this section is to prove Theorem A in the special case when $G$ has the following structure:

- $G$ is a semidirect product $S \rtimes N$ where
- $N$ is a free nilpotent Mal'cev complete group of finite Hirsch length,
- $S$ is a subgroup of Aut($N$) whose derived subgroup $\gamma_2(S)$ is contained in the group $U$ of 1-powering automorphisms of $N$,
- $S$ is finitely generated and nilpotent.

**Proof that Theorem A holds in this special case.** Fix a basis $v_1, \ldots, v_n$ of the rational vector space $N/\gamma_2(N)$. With respect to this basis, each automorphism of $N/\gamma_2(N)$ is represented by a rational matrix in $\text{GL}_n(\mathbb{Q})$. Since $S$ is finitely generated, there is a natural number $m$ such that the matrices for the automorphisms that $S$ induces on $N/\gamma_2(N)$ all belong to $\text{GL}_n(\mathbb{Z}[1/m])$. Let $\phi$ be an $m$-powering automorphism of $N$ and consider the group $W := \langle U \cup \{ \phi \} \rangle$. Set $Q := SW$ and $P := SU$. Observe that $U$ is normal in $W$ and $W/U$ is cyclic and central in $Q/U$. Therefore the group $P$ is normal in $Q$. We now apply Theorem B with these groups $Q$ and $P$ and with $M := U$, and $L := S$. The output is a subgroup, which we denote by $Q_1$, of $Q$ such that $Q = Q_1 U$, $S \leq Q_1$, and $Q_1 \cap P$ is nilpotent. Now all subgroups of a nilpotent group are subnormal and so $S$ is subnormal in $Q_1 \cap P = Q_1 \cap SU$. Therefore $S$ is subnormal in $Q_1$ and it follows that $S \rtimes N$ is subnormal in $Q_1 \rtimes N$. By Corollary 4.6 we have reduced to showing that

$$\text{hd}_k(Q_1 \rtimes N) = h(Q_1 \rtimes N).$$

The result follows from Proposition 4.3 because, as we shall now see, the group $Q_1 \rtimes N$ is locally of type $\text{FP}_\infty$. The group $Q_1$ contains an $m$-powering automorphism $\theta$ (equal to $\phi$ modulo $U$) and if $X$ is any finite set of generators for the automorphisms of $N/\gamma_2(N)$ induced by $Q$ then the elements $\theta X \cup \{ \theta \}$ act by integer entry matrices. Moreover, the same is true on the abelianization of $U$, by Proposition 3.4. The group $M := Q_1 \cap U \rtimes N$ is a Mal'cev complete nilpotent normal subgroup of $Q_1 \rtimes N$ and we can apply Theorem C to this setup to deduce that $Q_1 \rtimes N$ is locally $\text{FP}_\infty$, so that Proposition 4.3 applies. 

\[\square\]

8. Proof of Theorem A

Let $G$ be a group with $\text{hd}_k(G) < \infty$. By Lemma 4.1 $G$ has finite Hirsch length. Therefore by Lemma 2.1 $G$ has a locally finite normal subgroup $T$ such that $G/T$ is torsion free nilpotent-by-free abelian of finite rank-by-finite. We know that

- $\text{hd}_k(G) = \text{hd}_k(G/T)$ for any locally finite normal subgroup $T$ of $G$, and
- $\text{hd}_k(G) = \text{hd}_k(H)$ for any subgroup $H$ of finite index in $G$,

(see Proposition 4.2). Obviously, the same happens for the Hirsch length and therefore we can replace $G$ by a section which has a torsion-free nilpotent normal subgroup $E$ such that $A := G/E$ is free abelian of finite rank. We can embed $G$ into a
larger group $\hat{G}$ fitting into the commutative diagram

\[
\begin{array}{ccc}
E & \longrightarrow & C \\
\downarrow & & \downarrow \\
E^Q & \longrightarrow & \hat{G} \\
\end{array}
\]

For the details of this construction see \cite{13} Proposition 1.1. We then have $\text{hd}_K(G) = \text{hd}_K(\hat{G})$ and $h(G) = h(\hat{G})$ thus we can replace $G$ by $\hat{G}$ and assume that $E$ is already Malcev complete.

Using Theorem B with $Q := P := G$, $M := E$ and $L := 1$ we may find a finitely generated nilpotent (thus polycyclic) subgroup $H$ of $G$ so that $G = HE$. Consider now the semidirect product $H \ltimes E$. There is a surjective map $H \ltimes E \twoheadrightarrow G = HE$ sending $(h,e)$ to $he$ so an application of Lemma \cite{13} implies that we can reduce the problem to the group $H \ltimes E$. Now, let $C_H(E)$ be the kernel of $h$ of the conjugacy action on $E$. This subgroup is normal in $H \ltimes E$ and the quotient map is $H \ltimes E \twoheadrightarrow H/C_H(E) \ltimes E$. Put $A = H/C_H(E)$. As the group $H$ is polycyclic, so is $C_H(E)$ thus we have that

\[
\text{hd}_K(H \ltimes E) = \text{hd}_K(A \ltimes E) + \text{hd}_K(C_H(E))
\]

which together with the fact that $\text{hd}_K(C_H(E)) = h(C_H(E))$ because it is polycyclic implies that we can further reduce the problem to the group $A \ltimes E$. As $A$ acts faithfully on $E$, we may see it as a subgroup of the group $\text{Aut}(E)$.

Using Proposition \cite{3} we choose a free Malcev complete group $\hat{E}$ with the same class so that there is a surjective homomorphism $\pi : \hat{E} \rightarrow E$ that induces an isomorphism $\hat{E}/\gamma_2(\hat{E}) \cong E/\gamma_2(E)$. There is an induced epimorphism

\[
\hat{\pi} : \text{Aut}(\hat{E};\text{Ker}\pi) \rightarrow \text{Aut}(E)
\]

whose kernel $Z$ is contained in the subgroup $U(\text{Aut}(\hat{E}))$ of 1-powering automorphisms. Note that $Z$ is both Malcev complete and nilpotent.

Let $T$ denote the preimage of $A$ under the map $\hat{\pi}$. At this point we may apply again Theorem B with $T$ playing the role of both $P$ and $Q$, with $M := Z$ and with $L := \{1\}$ and we deduce that there is some $S \leq T$ finitely generated and nilpotent such that $SZ = T$. We can now consider the semidirect product $S \ltimes E$ which collapses naturally via $\pi$ and $\hat{\pi}$ onto $A \ltimes E$. By Lemma \cite{3} it suffices to prove that

\[
\text{hd}_k(S \ltimes \hat{E}) = h(S \ltimes \hat{E}).
\]

We have reduced to the special case that was considered in preceding section and Theorem A follows.

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