Remarks on the Cross Norm Criterion for Separability

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Abstract

Recently in Reference [quant-ph/0202121] a computational criterion of separability induced by greatest cross norm is proposed by Rudolph. There, Rudolph conjectured that the new criterion is not weaker than positive partial transpose criterion for separability. We show that there exist counterexample to this claim, that is, proposed criterion is weaker than the positive partial transpose criterion.

Keywords: Quantum entanglement, Bell decomposable states, Greatest cross norm, Positive partial transpose

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1 Introduction

Entanglement as the most non classical features of quantum mechanics has been attracted much attention in the past decade. Though, non local characters of quantum mechanics is singled out in many decades ago [1, 2], but it has recently received considerable attention in connection with theory of quantum information [3, 4, 5]. Entanglement is usually arise from quantum correlations between separated subsystems which can not be created by local actions on each subsystems. By definition, a bipartite mixed state $\rho$ is said to be separable if it can be expressed as

$$\rho = \sum_i w_i \rho_i^{(1)} \otimes \rho_i^{(2)}, \quad w_i \geq 0, \quad \sum_i w_i = 1,$$

(1-1)

where $\rho_i^{(1)}$ and $\rho_i^{(2)}$ denote density matrices of subsystems 1 and 2, respectively. Otherwise the state is entangled.

The central tasks of quantum information theory is to characterize and quantify entangled states. A first attempt in characterization of entangled states has been made by Peres and Horodecki family [6, 7]. It was shown that a necessary condition for separability of a two partite system is that its partial transpose be positive. Horodecki family show that this condition is sufficient for separability of composite systems only for dimensions $2 \otimes 2$ and $2 \otimes 3$.

A new criterion for separability and also an entanglement measure for two partite systems based on greatest cross norm are introduced by Rudolph [8, 9, 10]. In an interesting paper [10] he obtained the values of greatest cross norm for some states such as Werner states and isotropic states. In [10] Rudolph also introduced a computational criterion for separability of mixed states induced by greatest cross norm and he could obtain the separability conditions for some states such as Werner states, isotropic states and 2-qubit Bell diagonal states. He showed that the new criterion completely characterizes separability properties of pure states, Bell decomposable states
and isotropic states in arbitrary dimension. He conjectured that new criterion is neither weaker nor stronger than positive partial transpose (PPT) criterion introduced by Peres and Horodeckis in [6, 7].

In this paper we introduce Bell decomposable states of 2 \( \otimes \) 3 systems and we show that there is state in this category that is entangled in the sense of PPT criterion but it is separable in the sense of new criterion introduced in [10], that is the new criterion is weaker than PPT criterion.

The paper is organized as follows. In section 2 we briefly review greatest cross norm criterion for separability of two partite systems. The criterion induced by greatest cross norm is also reviewed. Bell decomposable states in 2 \( \otimes \) 3 systems are introduced in section 3 and also PPT conditions for separability of these states is obtained. Finally we show that there exist state that is entangled in the sense of PPT criterion but satisfy new criterion for separability proposed by Rudolph.

## 2 Trace class norm criterion for separability and associated induced separability criterion

In this section we briefly review greatest cross norm for separability of two partite systems introduced by Rudolph in [8] and also the induced criterion introduced in [10].

Let us consider Hilbert spaces \( H_1 \) and \( H_2 \) associated with particles 1 and 2, respectively. One can show that the spaces \( T(H_1) \) and \( T(H_2) \) of trace class operators on \( H_1 \) and \( H_2 \) are Banach spaces once they equipped with the trace class norm \( .||_1^{(1)} \) and \( .||_1^{(2)} \), respectively. The algebraic tensor product \( T(H_1) \otimes_{alg} T(H_2) \) of \( T(H_1) \) and \( T(H_2) \) is defined as the set of all finite sums \( \sum_{i=1}^{n} u_i \otimes v_i \) where \( u_i \in T(H_1) \) and \( v_i \in T(H_2) \) for all \( i \).
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A cross norm on \( T(H_1) \otimes_{alg} T(H_2) \) is defined by (see [8, 10] and references therein)

\[
\| t \|_\gamma := \inf \left\{ \sum_{i=1}^{n} \| u_i \|_1 \| v_i \|_1 \mid t = \sum_{i=1}^{n} u_i \otimes v_i \right\}, \tag{2-2}
\]

where \( t \in T(H_1) \otimes_{alg} T(H_2) \) and the infimum is taken over all finite decompositions of \( t \) into elementary tensors. The norm majorizes any subcross on \( T(H_1) \otimes_{alg} T(H_2) \) (a norm on \( T(H_1) \otimes_{alg} T(H_2) \) is subcross norm if \( \| t_1 \otimes t_2 \| \leq \| t_1 \|_1 \| t_2 \|_1 \) for all \( t_1 \in T(H_1) \) and \( t_2 \in T(H_2) \) and it is cross norm if saturates the inequality for all \( t_1 \in T(H_1) \) and \( t_2 \in T(H_2) \)) is called greatest cross norm.

The greatest cross norm criterion proposed by Rudolph is defined as follows [8, 10]. Let \( H_1 \) and \( H_2 \) be finite dimensional Hilbert spaces and \( \rho \) be a density operator on \( H_1 \otimes H_2 \). The density matrix \( \rho \) is separable if and only if \( \| \rho \|_\gamma = 1 \). Rudolph in [10] determines greatest cross norm for some states such as Werner states and isotropic states. In addition in the second part of Ref. [10], Rudolph introduced a new necessary separability criterion for bipartite systems induced by the greatest cross norm on Hilbert-Schmidt space.

In the Hilbert-Schmidt space \( HS(H_1 \otimes H_2) \), the operators of the Hilbert space \( H_1 \otimes H_2 \) are regarded as vectors. This space is equipped with the Hilbert-Schmidt inner product defined by \( \langle T|T' \rangle = tr(T^\dagger T') \), where \( T \) and \( T' \) are two operators acting on space \( H_1 \otimes H_2 \). Let \( HS(H_1) \) and \( HS(H_2) \) denote Hilbert-Schmidt spaces corresponding to Hilbert spaces \( H_1 \) and \( H_2 \), respectively. It has been shown in [10] that there exist a one-to-one correspondence between Hilbert-Schmidt operators \( T \in HS(H_1 \otimes H_2) \) and Hilbert-Schmidt operators \( \mathcal{U}(T) : HS(H_1) \rightarrow HS(H_1) \). Alternatively we can define the trace class norm of \( \mathcal{U}(T) \) denoted by \( T(\mathcal{U}(T)) \).

Every state \( T \in HS(H_1 \otimes H_2) \) in the Hilbert-Schmidt space can be written as [10]

\[
T = \sum_i \lambda_i E_i \otimes F_i \tag{2-3}
\]
where \( \{\lambda_i\}_i \) are non-negative real numbers and \( \{E_i\}_i \) and \( \{F_i\}_i \) are orthonormal bases of \( HS(H_1) \) and \( HS(H_2) \) respectively [10]. Also the trace class norm of \( \mathcal{U}(T) \) is equal to \( \mathcal{T}(\mathcal{U}(T)) = \sum_i \lambda_i \).

Rudolph in [10] proposed its new criterion for separability in a proposition which is quoted below:

**Proposition 1** [10] Let \( H \) be a finite dimensional Hilbert space and \( \rho \in \mathcal{T}(H \otimes H) \) be a density operator. If \( \rho \) is separable then

\[
\mathcal{T}(\mathcal{U}(\rho)) \leq 1.
\] (2-4)

Based on the above criterion, Rudolph obtained separability conditions of some states such as Werner states, isotropic states and 2-qubit Bell diagonal states. He conjectured that the new criterion is neither weaker nor stronger than the Peres-Horodeckies PPT criterion for separability.

In the next section we present state that violates positive partial transpose criterion but satisfy the separability criterion given in Eq. (2-4).

### 3 Bell decomposable states of \( 2 \otimes 3 \) quantum systems

In this section we review Bell decomposable states of \( 2 \otimes 3 \) quantum systems. A Bell decomposable density matrix acting on \( 2 \otimes 3 \) Hilbert state can be defined by

\[
\rho = \sum_{i=1}^{6} p_i |\psi_i\rangle \langle \psi_i|,
\]

\[
0 \leq p_i \leq 1,
\]

\[
\sum_{i=1}^{6} p_i = 1,
\] (3-5)

where \( |\psi_i\rangle \) are Bell states in \( H^2 \otimes H^3 \cong H^6 \) Hilbert space, defined by:

\[
|\psi_1\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle),
\]

\[
|\psi_2\rangle = \frac{1}{\sqrt{2}}(|11\rangle - |22\rangle),
\]

\[
|\psi_3\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |23\rangle),
\]

\[
|\psi_4\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |23\rangle),
\] (3-6)
\[ |\psi_5\rangle = \frac{1}{\sqrt{2}}(|13\rangle + |21\rangle), \quad |\psi_6\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |21\rangle). \]

It is quite easy to see that the above states are orthogonal and thus span the Hilbert space of \(2 \otimes 3\) systems.

A necessary condition for separability is presented by Peres \[6\]. He show that the matrix obtained from from partial transpose of separable state must be positive. Horodeckies \[7\] have shown that Peres criterion provides sufficient condition for separability only for composite quantum systems of dimension \(2 \otimes 2\) and \(2 \otimes 3\). This implies that the state given in Eq. (3-5) is separable if and only if the following inequalities satisfy

\[
(p_1 + p_2)(p_3 + p_4) \geq (p_5 - p_6)^2, \quad (3-7)
\]

\[
(p_3 + p_4)(p_5 + p_6) \geq (p_1 - p_2)^2, \quad (3-8)
\]

\[
(p_5 + p_6)(p_1 + p_2) \geq (p_3 - p_4)^2. \quad (3-9)
\]

On the other hand expanding Eq. (3-5) in terms of canonical base \(|i\rangle \otimes |j\rangle\) we get

\[
\rho = \frac{1}{2}(p_1 + p_2) |11\rangle \langle 11| + (p_1 - p_2) |11\rangle \langle 22| + (p_1 - p_2) |22\rangle \langle 11|
+ (p_1 + p_2) |22\rangle \langle 22| + (p_3 + p_4) |12\rangle \langle 12| + (p_3 - p_4) |12\rangle \langle 23|
+ (p_3 - p_4) |23\rangle \langle 12| + (p_3 + p_4) |23\rangle \langle 23| + (p_5 + p_6) |13\rangle \langle 13|
+ (p_5 - p_6) |13\rangle \langle 21| + (p_5 - p_6) |21\rangle \langle 13| + (p_5 + p_6) |21\rangle \langle 21|. \quad (3-10)
\]

Alternatively if \(|E_{ij}\rangle \equiv |i\rangle \langle j|\) denotes corresponding bases in the Hilbert-Schmidt space then we have
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\[ U(\rho) = \frac{1}{2} (p_1 + p_2) |E_{11}\rangle \langle E_{11}| + (p_1 - p_2) |E_{12}\rangle \langle E_{12}| + (p_1 - p_2) |E_{21}\rangle \langle E_{21}| \\
+ (p_1 + p_2) |E_{22}\rangle \langle E_{22}| + (p_3 + p_4) |E_{11}\rangle \langle E_{22}| + (p_3 - p_4) |E_{12}\rangle \langle E_{22}| \\
+ (p_3 - p_4) |E_{21}\rangle \langle E_{32}| + (p_3 + p_4) |E_{22}\rangle \langle E_{32}| + (p_5 + p_6) |E_{11}\rangle \langle E_{22}| \\
+ (p_5 - p_6) |E_{12}\rangle \langle E_{31}| + (p_5 - p_6) |E_{21}\rangle \langle E_{13}| + (p_5 + p_6) |E_{22}\rangle \langle E_{22}|, \]

(3-11)

where \( |E_{ij}\rangle \) is used to denote \( E_{ij} \). Now we can easily evaluate the eigenvalues of \( 4 \times 4 \) matrix \( U(\rho)U^\dagger(\rho) \) which yields

\[ \lambda_1 = \lambda_2 = A, \quad \lambda_3 = B + C, \quad \lambda_4 = B - C, \]

(3-12)

where

\[ A = \frac{1}{4} ((p_1 - p_2)^2 + (p_3 - p_4)^2 + (p_5 - p_6)^2), \]

\[ B = \frac{1}{4} ((p_1 + p_2)^2 + (p_3 + p_4)^2 + (p_5 + p_6)^2), \]

(3-13)

\[ C = \frac{1}{4} ((p_1 + p_2)(p_3 + p_4) + (p_3 + p_4)(p_5 + p_6) + (p_5 + p_6)(p_1 + p_2)). \]

It is easy to see that all eigenvalues are non-negative. Now we can easily determine the separability criterion given in Eq. (2-4) as

\[ \sum_{i=1}^{4} \sqrt{\lambda_i} = 2\sqrt{A} + \sqrt{B + C} + \sqrt{B - C} \leq 1. \]

(3-14)

In the rest of this section we shall present a counterexample to the claim that the criterion given in Eq. (2-4) is not weaker than PPT criterion for separability. Let us consider a Bell decomposable state given by

\[ p_1 = 0.3, \quad p_2 = 0, \quad p_3 = 0.2, \quad p_4 = 0.1, \quad p_5 = 0.4, \quad p_6 = 0. \]

(3-15)

It is quite easy to see that the state given by Eq. (3-15) violates PPT criterion given in Eq. (3-7), so it is entangled state. On the other hand, it is separable in the sense of criterion given in Eq.
This implies that the new criterion induced by the greatest cross norm is weaker than the PPT criterion for separability.

4 Conclusion

We have provided a counterexample to show that the newly proposed criterion of separability (induced by greatest cross norm) is proposed by Rudolph is weaker than the positive partial transpose criterion, therefore, we have still a long way ahead to solve the long standing separability criterion in mixed quantum states.

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