Anomalous Commutators for Energy-Momentum Tensors

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Abstract

Anomalous contributions to the energy-momentum commutators are calculated for even dimensions, by using a non-perturbative approach that combines operator product expansion and Bjorken-Johnson-Low limit techniques. We first study the two dimensional case and give the covariant expression for the commutators. The expression in terms of light-cone coordinates is then calculated and found to be in perfect agreement with the results in the literature. The particular scenario of the light-cone frame is revisited using a reformulation of the BJL limit in such a frame. The arguments used for \( n = 2 \) are then generalized to the case of any even dimensional Minkowskian spacetime and it is shown that there are no anomalous contributions to the commutators for \( n \neq 2 \). These results are found to be valid for both fermionic and bosonic free fields. A generalization of the BJL-limit is later used to obtain double commutators of energy-momentum tensors and to study the Jacobi identity. The two dimensional case is studied and we find no existance of 3-cocycles in both the Abelian and non-Abelian case.

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I. Introduction

The evaluation of commutators and their algebra has been a subject of interest in field theory for many years. Several approaches have been proposed to carry out such a task, from the canonical to the perturbative one. The use of canonical commutators to evaluate current algebra relations has produced many results which can be explicitly measured. Still, in many instances, the canonical evaluation of current commutators is ill-defined as it was shown by Schwinger in [1]. In that case the evaluation of the equal time commutator $[J_a^0(x), J_b^0(y)]$ leads to the canonical result plus a term proportional to the gradient of the delta function $S_{ab}^{ij}(y)\partial^j\delta(\vec{x} - \vec{y})$, and possibly other higher derivatives of delta functions, which may be present. These gradient terms are called Schwinger terms. As canonical evaluation of equal time commutators present ambiguities, it is necessary to have an alternative way to define and calculate these commutators. Bjorken, Johnson and Low [2] proposed a definition that preserves all desirable features of the theory. The commutator is well defined and the results obtained coincide with the canonical ones whenever the latter are also well defined. As we will see in the next section, this definition relies on studying the limit of large energy transferred and therefore an operator product expansion (OPE) [3] is appropriate, allowing us to significatively reduce our calculations. The BJL-limit approach has already been used in the literature, but always within perturbation theory framework. These calculations turn out to be usually very lengthy and tedious. The need to consider many loop-diagrams, with ever increasing number of vertices, makes the study of anomalous commutators very hard when one increases the dimensionality of spacetime $n$. For instance, the study of anomalous commutators between vector and axial-vector currents in $n = 4$ dimensions involves the perturbative calculation of triangle, box and even pentagon diagrams [4]. For the case of energy-momentum commutators, the ones we are concerned with in this paper, the perturbative scheme is the same. The axial or vector-axial vertices found in the context of chiral anomalies (e.g. triangle diagram) are replaced by energy-momentum tensors, and one can start evaluating loop diagrams. However, instead of doing the perturbative calculation, an OPE will show to be more efficient and still valid for any value of the spacetime dimension.

The paper is structured in the following manner. In section II we describe the method in detail. Section III presents the procedure to calculate
in covariant form the anomalous energy-momentum contributions to the 2-dimensional commutators. The study is later on generalized in section IV to any even dimension. In section V we obtain the expression of the commutator using BJL and OPE techniques in the light-cone frame. Section VI is dedicated to the study of the Jacobi identity using the same techniques developed in previous sections.

II. Description of the method

In this section we will make use of a method previously used in [5] within the context of non-Abelian current algebra, to obtain the commutators of energy-momentum tensors. This method allows us to do the calculations without going through the lengthy steps of solving loop diagrams in perturbation theory. The method is consistent and has the advantage that can be used for both fermionic and bosonic models. Furthermore, results can easily be obtained for any value of the dimensionality of spacetime. This technique, which does not rely on perturbation theory, is based on the Bjorken-Johnson-Low (BJL) definition of equal time commutators and on the operator product expansion (OPE).

The BJL definition is totally general and arises from a time ordered product of two operators, say $A$ and $B$, and its representation in momentum space through a Fourier transform. The BJL prescription tells us that the equal time commutator of two operators is obtained from the high energy behavior of Green’s functions, as

$$\lim_{p^0 \to \infty} p^0 \int d^n x \, e^{ipx} \langle \alpha | T A(x/2) B(-x/2) | \beta \rangle =$$

$$i \int d^{n-1} x \, e^{-ip \cdot \vec{x}} \langle \alpha | [A(0, x/2), B(0, -\vec{x}/2)] | \beta \rangle. \quad (1)$$

Here $p^0$ stands for the energy component of the four momentum. In the BJL definition (1) above we have used the time ordered product $T$, which is not a Lorentz covariant object. In field theory one calculates (e.g. Feynman diagrams in perturbation theory) only the covariant object, denoted by $T^*$. The difference between $T$ and $T^*$ is local in position space, and corresponds to a covariant term involving delta functions of $x_0$ and its derivatives $\frac{\partial x_0}{\partial x_0}$. This extra term will take the form of a polynomial in $p^0$ when we go to momentum space. Therefore in eq.(1) we must drop all polynomials in $p^0$, 3
since they will not contribute to the covariant commutator. The point here is to make a Laurent expansion of the time ordered product and identify the residue of the $1/p^0$ term as the Fourier transform of the commutator.

Since we are interested in the large momentum transferred behavior, it is appropriate to express the singularities of the product of operators as a sum of non-singular local operators $[3]$,

$$
\int d^n x \ e^{ipx} \langle \alpha | T^* A(x/2) B(-x/2) | \beta \rangle = \sum_i c_i(p) \langle \alpha | O_i(0) | \beta \rangle,
$$

(2)

where these local operators $O_i$ are evaluated at $x = 0$. Taking the BJL limit in eq.(2) we find

$$
\int d^{n-1} x \ e^{-i\vec{p} \cdot \vec{x}} \langle \alpha | [A(0, \vec{x}/2), B(0, -\vec{x}/2)] | \beta \rangle = \sum_i \lim_{p^0 \to \infty} [-i p^0 c_i(p)] \langle \alpha | O_i(0) | \beta \rangle,
$$

(3)

where all terms in the coefficients $c_i(p)$ that grow as a power of $p^0$ must be dropped.

III. Commutators in two dimensions

The simultaneous use of OPE and BJL techniques to calculate commutators in a non-perturbative manner has been proven a useful way to evaluate different kind of such operators. This method avoids the tedious work usually associated with perturbative calculations.

In $[3]$ it was shown that these techniques combined together in the context of current commutators reproduce the results previously found in the literature $[4]$. It is our purpose now to examine the possible anomalous (non-canonical) contributions to the commutators of energy-momentum tensors in two dimensions. At this time we will only carry out our discussion in a Minkowskian spacetime with metric $\eta_{\alpha\beta}$ and signature $(+, -)$.

The two-point function for the energy-momentum tensor is given by

$$
O_{\mu\nu\rho\sigma}(p) = \int d^2 x \ e^{ipx} \langle \Omega | T^* (\theta_{\mu\nu}(x/2)\theta_{\rho\sigma}(-x/2)) | \Omega \rangle,
$$

(4)

where $\Omega$ denotes the vacuum state (the method holds for any two states $|\alpha\rangle$ and $|\beta\rangle$).
In an n-dimensional spacetime, simple dimensional analysis tells us that the canonical dimension of $O$ is $(\text{mass})^n$. This is straightforward if one recalls that $[\theta_{\alpha\beta}] = [\eta_{\alpha\beta}\mathcal{L}] = n$. In our present case $[O] = 2$, restricting the possible terms in the OPE to those of dimension equal or greater than two. To choose the terms to appear in the OPE, we pick those with the same dimensionality and symmetries (Lorentz covariance, gauge invariance, parity, etc.) as $O$. Out of all possible terms, we shall write only those that contribute when taking the BJL limit.

To illustrate our result below, we consider the case of the energy-momentum tensor for free fermions moving in a flat background, used by Alvarez-Gaumé and Witten in [4].

$$\theta_{\alpha\beta} = \frac{i}{4} \overline{\psi}(\gamma_{\alpha} \partial_{\beta} + \gamma_{\beta} \partial_{\alpha}) \psi,$$  \hspace{1cm} (5)

which is symmetric, $\theta_{\alpha\beta} = \theta_{\beta\alpha}$, and conserved, $\partial^\alpha \theta_{\alpha\beta} = 0$. Another possibility was studied by Guadagnini in [8] for the case of a free bosonic model. In that case, the traceless and symmetric energy-momentum tensor was given by

$$\theta_{\alpha\beta} = \partial_{\alpha} \phi^i \partial_{\beta} \phi^i - \eta_{\alpha\beta} \partial_{\lambda} \phi^i \partial_{\lambda} \phi^i.$$  \hspace{1cm} (6)

Since we are working with symmetric tensors in (5) and (6), we want our terms in the OPE of (4) to be symmetric when $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$, separately. From the symmetries of two-point functions we also need the following symmetry for $O$: $(\mu, \nu) \leftrightarrow (\rho, \sigma)$ and $p \leftrightarrow -p$. The OPE for contributing terms will be of the form

$$O_{\mu\nu\rho\sigma} = \frac{1}{p^2}[a_1 p_{\mu} p_{\nu} p_{\rho} p_{\sigma} \mathbf{1} + a_2 (p_{\mu} p_{\nu} \theta_{\rho\sigma} + p_{\rho} p_{\sigma} \theta_{\mu\nu}) + \frac{a_3}{p^2} p_{\mu} p_{\nu} p_{\rho} p_{\sigma} \Theta], \hspace{1cm} (7)$$

where $\Theta = \theta^\lambda_\lambda$ stands for the trace of the energy momentum tensor. There will be cases where $\theta_{\mu\nu}$ is traceless and hence the last term in the operator expansion will not appear. When considering interaction with the gravitational field there appears the so called trace anomaly, where $\Theta$ is proportional to the Ricci scalar $R$.

Applying the BJL limit to eq.(7)

$$\lim_{p^0 \to \infty} p^0 O_{\mu\nu\rho\sigma} = i \int dx e^{-i\vec{p} \cdot \vec{x}} [\theta_{\mu\nu}(0, \vec{x}/2), \theta_{\rho\sigma}(0, -\vec{x}/2)],$$
we find

\[
[\theta_{\mu\nu}(0, \vec{x}/2), \theta_{\rho\sigma}(0, -\vec{x}/2)] = a_1 \eta_{\mu(0} \eta_{\nu)i} \eta_{\rhoj} \eta_{\sigmak} \partial^i \partial^j \partial^k \delta(\vec{x})
- a_2 [\eta_{\mu(0} \eta_{\nu)i} \theta_{\rho\sigma} + \eta_{\rho(0} \eta_{\sigma)i} \theta_{\mu\nu}] \partial^i \delta(\vec{x})
- a_3 \eta_{\mu(0} \eta_{\nu)0} \eta_{\rho0} \eta_{\sigmai} \Theta \partial^i \delta(\vec{x}),
\]

(8)

where we have used the parentheses to denote symmetric permutation of second indices; i.e. \(\eta_{\mu(0} \eta_{\nu)i} = \eta_{\mu0} \eta_{\nu0} + \eta_{\mu i} \eta_{\nu 0}\). The above expression for the commutator is Lorentz covariant and the different components can be obtained by giving values zero and one to \(\mu, \nu, \rho\) and \(\sigma\). The equal time commutators are

\[
[\theta_{00}, \theta_{00}] = [\theta_{00}, \theta_{11}] = [\theta_{11}, \theta_{11}] = 0,
\]

and

\[
[\theta_{00}(0, \vec{x}/2), \theta_{01}(0, -\vec{x}/2)] = [a_2 \theta_{00} + a_3 \Theta] \delta'(\vec{x}),
[\theta_{01}(0, \vec{x}/2), \theta_{01}(0, -\vec{x}/2)] = 2a_2 \theta_{01} \delta'(\vec{x}),
[\theta_{01}(0, \vec{x}/2), \theta_{11}(0, -\vec{x}/2)] = a_2 \theta_{11} \delta'(\vec{x}) - a_1 \delta''(\vec{x}).
\]

(9)

At this time we are not interested in the concrete values of the multiplicative constants \(c_i\). These constants multiply the matrix elements \(\langle \alpha | O_i | \beta \rangle\), which in most cases can only be evaluated doing detailed calculations. In expression (8) we see that the only anomalous contribution takes place when we have one time-like index and the other three space-like indices (e.g. \([\theta_{01}, \theta_{11}]\)). The anomaly appears as a third derivative of the delta function. This result is similar to the Virasoro anomaly found for two dimensional world sheets in the context of superstring theory [9]. Also similar results are given in [7] using light-cone formalism, which we will review in section V.

**IV. Commutators in n dimensions**

The previous results can be generalized to the case of an n-dimensional spacetime, with \(n\) being even\(^1\). The following study will show that there are no anomalous contributions to the commutator of energy-momentum tensors

\(^1\)In the odd dimensional case, some study has been carried out for \(n=3\) in the context of current algebra [10].
for $n \geq 4$. For a term to contribute to $\mathcal{O}_{\mu\nu\rho\sigma}$ some conditions must be satisfied, i.e., it must have four Lorentz indices, its total canonical dimension must be $n$ in units of mass, and must be finite when multiplying it by $p^0$ and taking the limit of $p^0$ approaching infinity. We will construct terms made up with combinations of $1$, $\theta_{\mu\nu}$ and $\Theta$ operators, and study their behavior. Other higher order terms (e.g. $\Theta^p$, with $p \in \mathbb{N}$) could be constructed, but they will vanish when taking the BJL limit. The following are the possible terms:

i) The only operator appearing in the term is $1$. In $n \geq 2$ dimensions, the general expression for a term of such type will be

$$\frac{1}{p^{4-n}} p_\mu p_\nu p_\rho p_\sigma 1.$$  

Except for the case $n = 2$ which we studied in section III, this kind of term fails to give a finite contribution when $p^0$ approaches infinity.

ii) The only operator is $\theta_{\mu\nu}$. The expression for any value of $n$ will be of the form

$$\frac{p_\mu p_\nu}{p^2} \theta_{\rho\sigma},$$

which will only contribute to the commutator with a term like

$$\eta_{\mu(0}\eta_{\nu i)} \theta_{\rho\sigma} \partial^i \delta(\vec{x}). \quad (10)$$

iii) The only operator is $\Theta$. For any value of $n$, the term will be of the form

$$\frac{1}{p^4} p_\mu p_\nu p_\rho p_\sigma \Theta.$$  

In this case the commutator is proportional to

$$\eta_{\mu(0}\eta_{\nu 0}\eta_{\rho 0}\eta_{\sigma i)} \Theta \partial^i \delta(\vec{x}).$$

iv) Term made of products of the three operators: $1$, $\theta_{\mu\nu}$ and $\Theta$. The only possibility is
\[ \frac{1}{p^{n+2}} (p_\mu p_\nu \theta_{\rho\sigma} + p_\rho p_\sigma \theta_{\mu\nu}) \Theta, \]

which goes to zero as \( p^0 \to \infty \). A term with more products of energy-momentum tensors will require the existence of more momenta to compensate for the extra indices, some of which will be the same as one of the indices of the tensor, like in

\[ \frac{1}{p^{2n+4}} p_\mu p_\nu p_\rho p_\sigma \theta_{\rho\sigma} \theta^{\alpha\beta} \Theta. \]

This term will be zero because we are considering only conserved tensors. In any case, it will not contribute to the BJL limit, either.

We therefore see that there is no anomalous contribution to the commutators for \( n \geq 4 \), as we wanted to show.

V. Light-cone coordinates

In the study of two dimensional systems it is usual to work with light-cone (LC) coordinates. The LC picture is very physical and also provides a useful framework for calculations. Historically, it was LC quantization that first conclusively established that dual models were theories of strings [9].

We define the non-singular transformation

\[
x^- = \frac{1}{\sqrt{2}} (x^0 - x^1) \\
x^+ = \frac{1}{\sqrt{2}} (x^0 + x^1). \tag{11}
\]

In [11] a “canonical” formalism was proposed in which \( x^- \) was interpreted as the “time” variable \( x^0 \), and \( x^+ \) as the “space” variable \( x^1 \). As we will show in this section the identification has to be done carefully. The results in [3] are Lorentz covariant and valid for any inertial frame of reference. We encountered the anomaly coming as a third derivative of the delta function, in \([\theta_{01}, \theta_{11}]\). It could appear, if one is not careful, that with the “canonical” formalism proposed above, the anomaly should show up in the commutator \([\theta_{-+}, \theta_{++}]\), when we go to the light-cone frame. In order to derive the well-known anomalous commutator given in references [3, 8, 11], i.e. \([\theta_{++}, \theta_{++}]\),
we will combine an OPE similar to the one in (7) with the coefficients $c(p)$ written in terms of the light-cone coordinates, and a generalization of the BJL limit for the LC coordinates.

Under (11) the components of the metric become $\eta^{++} = \eta^{--} = 0$ and $\eta^{+-} = \eta^{-+} = 1$. If we denote by $x^c = (x^-, x^+)$ the position vector in the LC system, and $P$ the matrix to transform from $x$ to $x^c$, i.e., $x^c = Px$, then the energy-momentum tensor transforms according to $T^c = PTP^{-1}$. This implies the following relations for the components

$$\theta^{--} = \frac{1}{2}(\theta_{00} - 2\theta_{01} + \theta_{11})$$
$$\theta^{++} = \frac{1}{2}(\theta_{00} - 2\theta_{01} + \theta_{11})$$
$$\theta^{+-} = \frac{1}{2}(\theta_{00} + 2\theta_{01} + \theta_{11}).$$  \hspace{1cm} (12)

We can raise and lower indices with the help of the LC metric in the following way $x_{\pm} = x^\pm$. Other results can be obtained by simple algebraic manipulation

$$d^2x = dx^0 dx^1 = \sqrt{2} dx^- dx^+$$
$$px = p^- x^- + p^+ x^+$$
$$p^2 = 2p^- p^+, \hspace{1cm} (13)$$

or in case of working with fermionic models, one could find the relations

$$\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^1) = \gamma_{\pm}$$
$$\gamma^+ \gamma^+ = \gamma^- \gamma^- = 0$$
$$\{\gamma^+, \gamma^-\} = 2. \hspace{1cm} (14)$$

To evaluate the anomalous part of the commutators in the LC frame, we will need to modify the expression for the BJL limit. We can define the commutator in terms of LC coordinates as

$$\lim_{p_- \to \infty} p_- \int dx^+ dx^- e^{i(p^+ x^+ + p^- x^-)} \langle \Omega | T \left( \theta_{\alpha\beta}(x^-, x^+) \theta_{\epsilon\delta}(0) \right) | \Omega \rangle =$$
$$i \int dx^+ e^{ip^+ x^+} \langle \Omega | \left[ \theta_{\alpha\beta}(0, x^+), \theta_{\epsilon\delta}(0) \right] | \Omega \rangle. \hspace{1cm} (15)$$
Following the same steps as we did in the previous section, we now write the OPE in terms of LC coordinates\(^2\). We will only consider those terms that will contribute when taking the \(p_- \to \infty\) limit, at the end. Let us consider

\[
\frac{1}{p_- p_+} [b_1 p_\alpha p_\beta p_\epsilon p_\delta 1 + b_2 (p_\alpha p_\beta \theta_\epsilon \delta + p_\epsilon p_\delta \theta_\alpha \beta)],
\]

(16)
multiply it by \(p_-\) and finally take the limit of \(p_-\) approaching infinity. Note that when we take this limit in the covariant framework, the denominator behaves as \((p^0)^2\) whereas in the LC frame it goes like \(p_- p_+\). This causes the appearance of off-diagonal terms of the metric \(\eta_{-+}\) in the surviving coefficients, once the LC-BJL limit is taken. The final result is

\[
[\theta_{\alpha\beta}(0, x^+), \theta_{\epsilon\delta}(0)] = b_1 \eta_{-\alpha} \eta_{-\beta} \eta_{-\epsilon} \eta_{-\delta} \partial_3 \delta(x^+) \\
- b_2 (\eta_{-\alpha} \eta_{-\beta} \theta_{\epsilon\delta} + \eta_{-\epsilon} \eta_{-\delta} \theta_{\alpha\beta}) \partial_+ \delta(x^+).
\]

(17)
The anomalous part is the term proportional to the third derivative of the delta function. Since \(\eta_{++} = 1\), the only non-vanishing commutator appears for \(\alpha = \beta = \epsilon = \delta = +\). Therefore

\[
[\theta_{++}(0, x^+), \theta_{++}(0)]_{\text{Anom}} = b_1 \partial_3 \delta(x^+).
\]

(18)
This is in perfect agreement with the results found in the literature\([7, 9]\) when working with LC coordinates. If we were working with a non-traceless energy-momentum tensor, there will be an extra contribution of the form

\[-b_3 \eta_{\alpha(+} \eta_{\beta-} \eta_{\epsilon-} \eta_{\delta-} \Theta \partial_+ \delta(x^+),\]

where again, the parentheses mean symmetric summation with the second indices being switched around; in this case there appear four terms.

A particular scenario where one can find the result of eq. (18) is that concerning with Weyl spinors. The expression of the fermionic energy-momentum tensor considered in \([7]\) is the one given in eq. (5). Since these spinors satisfy the Weyl relation \(\gamma_5 \psi = -\psi\), it is straightforward to verify other relations like

\[
\gamma_- \psi = \partial_- \psi = 0.
\]

From this, it can be shown that

\[
\theta_{-+} = \theta_{++} = \theta_{-} = 0,
\]

(19)
\(^2\)Now the greek indices \(\alpha, \beta, \epsilon, \delta\) take values \(-, +\).
with $\theta_{++}$ being the only non-vanishing component.

Comparing (11) with the result obtained in (18) using LC coordinates, there appear to be some kind of contradiction if one is not careful. From (12) and (19), one finds

$$\theta_{00} = \theta_{11} = \theta_{01} = \theta_{10} = \frac{1}{2} \theta_{++},$$

which implies that $[\theta_{00}, \theta_{00}] = \frac{1}{4} [\theta_{++}, \theta_{++}] \sim \delta^{\mu}(x^+) \text{ from } (18)$, but $[\theta_{00}, \theta_{00}] = 0$, from (9)! The arguments of the delta functions in the Lorentz covariant frame are $(x^0, x^1)$ and in the LC system are $(x^-, x^+)$, so one needs to construct a symplectic form [11] to go from one to the other. The mere substitution using the expressions of (12) when going to the LC is not correct and a re-definition of the commutator by means of the BJL-limit, as in eq. (15), needs to be given in terms of the new coordinates. The results will then be consistent. It has been previously noticed that the LC formalism is not manifestly covariant, although it will be manifestly free of ghosts. It is clear that the effect of Lorentz transformation on the coordinates has to be quite subtle in the LC gauge since the choice of gauge is not Lorentz invariant [1].

VI. Jacobi Identity in two dimensions

We have seen in sections III through V, by making an OPE and taking the BJL limit, that the algebra of energy-momentum commutators presents anomalies. It is also interesting to check if these anomalies that appear in the single commutators will violate the Jacobi identity for double commutators, written as

$$[[[\theta_{\mu\nu}, \theta_{\rho\sigma}], \theta_{\lambda\tau}], \theta_{\lambda\tau}] + [[[\theta_{\rho\sigma}, \theta_{\lambda\tau}], \theta_{\mu\nu}], \theta_{\mu\nu}] + [[[\theta_{\lambda\tau}, \theta_{\mu\nu}], \theta_{\rho\sigma}], \theta_{\rho\sigma}] = 0.$$

The study of double commutators can be done in a similar manner to the single ones. In section II we saw how the BJL limit allows us to define commutators as the high energy behavior of correlation functions. It should be quite natural to look for a double high energy limit when interested in double commutators.

It is easy to verify the following relations

$$\frac{\partial}{\partial x_0} T A(x) B(y) C(z) = T(\dot{A}BC) + \delta(x_0 - y_0) T([A, B]_y C(z_0))$$
+ \delta(x_0 - z_0)T(B(y_0)[A,C](x_0)), \quad (20)
\frac{\partial}{\partial y_0} T(A(x)B(y)C(z)) = T(AB'C) + \delta(y_0 - x_0)T([B,A](y_0)C(z_0))
+ \delta(y_0 - z_0)T(A(x_0)[B,C](y_0)), \quad (21)

where the dot and prime denote derivatives with respect to $x_0$ and $y_0$ respectively. Following the same steps as in the derivation of the traditional BJL limit (ref.[6]), it can be shown that if we define

$$\mathcal{O}(p,q) = \int d^n x d^n y e^{i(px+qy)} \langle \alpha|TA(x)B(y)C(0)|\beta \rangle,$$

then the following identities hold

\begin{align*}
(I) &= \lim_{q_0 \to \infty} q_0 \lim_{p_0 \to \infty} p_0 \mathcal{O}(p,q) \\
&= \int d^{n-1} x \, d^{n-1} y \, e^{-i(\vec{p} \cdot \vec{x} + \vec{q} \cdot \vec{y})} \langle \alpha||[[A(0, \vec{x}), C(0)], B(0, \vec{y})]|\beta \rangle \\
(II) &= \lim_{p_0 \to \infty} p_0 \lim_{q_0 \to \infty} q_0 \mathcal{O}(p,q) \\
&= \int d^{n-1} x \, d^{n-1} y \, e^{-i(\vec{p} \cdot \vec{x} + \vec{q} \cdot \vec{y})} \langle \alpha||[[B(0, \vec{y}), C(0)], A(0, \vec{x})]|\beta \rangle \\
(III) &= \lim_{k_0 \to \infty} k_0 \lim_{p_0 \to \infty} p_0 \mathcal{O}(p,q) \\
&= \int d^{n-1} x \, d^{n-1} y \, e^{-i(\vec{p} \cdot \vec{x} + \vec{q} \cdot \vec{y})} \langle \alpha||[[A(0, \vec{x}), B(0, \vec{y})], C(0)]|\beta \rangle, \quad (22)
\end{align*}

where $k_0 = -(p_0 + q_0)$.

Thus the Jacobi identity will be given by $-(I) + (II) + (III)$, which is identically zero in absence of anomalies.

The behavior of gauge transformations in an anomalous gauge theory, as well as in a consistent gauge theory with Chern-Simons term, can be given a unified description in terms of cocycles. It is known that a 3-cocycle arises when a representation of a transformation group is non associative, and thus there is failure of the Jacobi identity [12]. The existance of these objects has been under investigation in quantum field theory for some time now. In the context of quantum mechanics 3-cocycles appear when translations are represented on configuration space $q$ in the presence of magnetic forces, specially a

\footnote{Using the translational invariance property of the Green’s functions, we choose $z = 0$ for simplicity.}
magnetic monopole. In that case one finds that 
\[ J(v^1, v^2, v^3) = \frac{e\hbar}{2m^3} \vec{\nabla} \cdot \vec{B}, \]
where \( v^i \) represent the components of the gauge invariant velocity operator. If \( \vec{\nabla} \cdot \vec{B} \neq 0 \), as in the case of a point monopole, the Jacobi identity fails \[13\].

Also a violation of the Jacobi identity appears in the quark model. When
the Schwinger term in the commutator between time and space components
of a current is a c-number, the Jacobi identity for triple commutators of
spatial current components must fail \[14\]. This fact has been verified in
perturbative BJL calculations \[15\]. Also non-perturbative calculations to
find 3-cocycles associated to non-Abelian gauge theories are under current
investigation \[16\]. There, the use of OPE techniques along with double BJL
limits to study the Jacobi identity, has proven to be an effective way to deal
with the current algebra of double commutators.

We would like to study the existence of 3-cocycles associated to Abelian
and Lie algebra valued energy momentum tensors. The study can be easily
restricted to the Abelian case once the non-Abelian one is known. Thus the
energy-momentum tensors will take values on a Lie algebra, \( \theta_{\mu\nu} = \theta^a_{\mu\nu} T_a \),
where the anti-hermitian representation matrices \( T^a \) satisfy the usual Lie
algebra relation \( [T^a, T^b] = f^{abc} T^c \), \( (T^a)^\dagger = -T^a \), and are normalized by
\( \text{Tr}(T^a T^b) = -\delta^{ab} \).

The definitions of eq. (22) are valid for any dimension, although we will
study here just the case of \( n = 2 \). From the symmetries of the energy-
momentum tensor and those of the new Green’s function itself
\[ O^{abc}_{\mu\rho\lambda\tau}(k_1, k_2) = \int d^2 x d^2 y e^{i(k_1 x + k_2 y)} \langle \Omega | T^a(\theta_{\mu\nu}^a(x) \theta_{\rho\sigma}^b(y) \theta_{\lambda\tau}^c(0)) | \Omega \rangle , \tag{23} \]
we find

i) Symmetry under the exchange \( \mu \leftrightarrow \nu, \rho \leftrightarrow \sigma \) or \( \lambda \leftrightarrow \tau \) separately.

ii) Symmetry under the exchange of pairs \( (\mu, \nu) \leftrightarrow (\rho, \sigma) \), \( a \leftrightarrow b \) and \( k_1 \leftrightarrow k_2 \).

iii) Symmetry under the exchange of \( (\mu, \nu) \leftrightarrow (\lambda, \tau) \), \( a \leftrightarrow c \), \( k_1 \leftrightarrow -(k_1 + k_2) \)
and \( k_2 \leftrightarrow k_2 \).

\[ \text{For a point monopole with strength } g \text{ located at } r_0 \text{ the divergence of } \vec{B} \text{ is } \vec{\nabla} \cdot \vec{B} = 4\pi g \delta(\vec{r} - \vec{r}_0). \]
iv) Symmetry under the exchange of \((\rho, \sigma) \leftrightarrow (\lambda, \tau), b \leftrightarrow c, k_1 \leftrightarrow k_1\) and \(k_2 \leftrightarrow -(k_1 + k_2)\).

From (23) we find the total canonical dimension of the operator to be \([O] = (\text{mass})^2\). To simplify the notation, we will not write all the symmetric partners for each term in the OPE. However, symmetries i) — iv) are understood to be necessary. In the operator expansion of (23), we assume parity to be a symmetry of the correlation function. Two possible candidate terms for the OPE, which contribute to the double limits are

\[
u^{abc} k_1 \mu k_2 \nu k_1 \lambda k_2 \tau \frac{1}{(k_1 + k_2)^2} \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) \textbf{1}, \tag{24}\]

and

\[
u^{abc} k_1 \mu k_2 \nu k_2 \rho k_1 \sigma k_1 \lambda k_2 \tau \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) \Theta, \tag{25}\]

with their properly symmetrized partners. In (24) the function \(\nu^{abc}\) is made of combinations of the Lie algebra totally antisymmetric and symmetric tensors \(f^{abc}\) and \(d^{abc}\). Therefore, we can write \(\nu^{abc} = u_1 f^{abc} + u_2 d^{abc}\). Similar arguments hold for \(\nu\) which will be of the form \(\text{Tr}(T^a T^b T^c T^d)\). Detailed calculations can be carried out to evaluate the three double commutators and for both terms we find the Jacobi identity not to be violated.

If the energy-momentum tensor we are dealing with is not traceless, then there is another possible term to appear in the OPE. This will be of the form

\[
u^{abc} k_1 \mu k_2 \nu k_2 \rho k_1 \sigma k_1 \lambda k_2 \tau \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} \right) \Theta, \tag{26}\]

and its symmetric partners, which satisfy relations i) — iv). The explicit evaluation of the double commutators in expression (22) leads again to the no violation of the Jacobi identity.

In two dimensions, as we mentioned in section IV, is quite convenient the use of LC coordinates. A straightforward generalization of the method shown above for the expression of double commutators, can also be obtained in terms of LC coordinates.

\[\text{VII. Conclusions}\]

In this paper we considered the BJL definition for commutators and applied it in conjunction with the operator product expansion. The method
could be applied both in the perturbative and non-perturbative regimes. We first derived the covariant expression for commutators of energy-momentum tensors in \( n = 2 \) dimensions, and found anomalous terms coming from third derivatives of delta functions. This anomalous result appeared again when using the BJL technique in the light-cone frame, and was in perfect agreement with those found in the literature. The method allowed us to calculate possible anomalous terms in other dimensions in a very straightforward way. In section IV, in particular, we found that there are no anomalous commutators in even dimensional Minkowskian spacetime for \( n \neq 2 \).

The method requires some additional knowledge about the behavior of the coefficient functions (which appear in the OPE) at large momentum transfer \( p \). In asymptotically free theories this is available via the renormalization group. The final results are expressed in terms of the residues of the coefficient functions (i.e., the constant multiplying the term behaving as \( 1/p^0 \)) and of the matrix elements of various local operators (the “condensates”). These quantities can be evaluated explicitly within perturbation theory; in the non-perturbative regime the condensates cannot be evaluated explicitly but can be used to parametrize the results. Another characteristic of the method is that the results are evaluated in terms of a set of unknown constants, the residues of the coefficient functions \( c_i \). For the applications we have considered this will not be a disadvantage, since we are just interested in the anomalous behavior of the energy-momentum commutators. The exact coefficients for these terms can only be found through detailed explicit calculation for each matrix element.

The existence of these anomalies at the single commutator level lead us to study the possibility of violations of the Jacobi identity. A generalization of the BJL limit technique to a double limit [16], allows us to find expressions for double commutators, and therefore be able to check whether or not the Jacobi identity holds. It vanishes at the canonical level, but when higher loop corrections are brought in, there is hope of generating 3-cocycles, as in other field theoretical contexts. From the terms studied in section VI, we conclude that there are no violations in the Jacobi identity of energy-momentum tensors in two dimensions, at the Abelian and non-Abelian level. This result concerns only with the non-interacting models (fermionic or bosonic) in a flat background metric. We have not studied the Jacobi identity for the case of \( n \neq 2 \), or the possibility of having mixed anomalies. These examples can be studied in the same way as we have done for \( n = 2 \), and the generalization
is straightforward.

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