A KINETIC ALFVÉN WAVE CASCADE SUBJECT TO COLLISIONLESS DAMPING CANNOT REACH ELECTRON SCALES IN THE SOLAR WIND AT 1 AU

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ABSTRACT

Turbulence in the solar wind is believed to generate an energy cascade that is supported primarily by Alfvén waves or Alfvénic fluctuations at MHD scales and by kinetic Alfvén waves (KAWs) at kinetic scales $k_\perp \rho_i \gtrsim 1$. Linear Landau damping of KAWs increases with increasing wavenumber and at some point the damping becomes so strong that the energy cascade is completely dissipated. A model of the energy cascade process that includes the effects of linear collisionless damping of KAWs and the associated compounding of this damping throughout the cascade process is used to determine the wavenumber where the energy cascade terminates. It is found that this wavenumber occurs approximately when $|\gamma/\omega| \simeq 0.25$, where $\omega(k)$ and $\gamma(k)$ are, respectively, the real frequency and damping rate of KAWs and the ratio $\gamma/\omega$ is evaluated in the limit as $k_\perp \gg k_\parallel$. For plasma parameters typical of high-speed solar wind streams at 1 AU, the model suggests that the KAW cascade in the solar wind is almost completely dissipated before reaching the wavenumber $k_\perp \rho_i \simeq 25$. Consequently, an energy cascade consisting solely of KAWs cannot reach scales on the order of the electron gyro-radius, $k_\perp \rho_e \sim 1$. This conclusion has important ramifications for the interpretation of solar wind magnetic field measurements. It implies that power-law spectra in the regime of electron scales must be supported by wave modes other than the KAW.

Key words: solar wind – turbulence – waves

Online-only material: color figures

1. INTRODUCTION

Recent papers by Schekochihin et al. (2009), Howes et al. (2008a), and Schekochihin et al. (2008) have described a scenario for turbulence in collisionless magnetized plasmas, such as the solar wind, that can be briefly described as follows. The turbulence at large scales, scales larger than the ion inertial length $c/\omega_i \rho_i$ and the thermal ion gyro-radius $\rho_i$, consists of an energetically dominant Alfvén wave cascade that transfers energy from large to small scales. At scales on the order of the proton gyro-radius, $k_\perp \rho_i \sim 1$, the wavevector spectrum of the turbulence is highly anisotropic with energy concentrated in wavevectors nearly perpendicular to the mean magnetic field $B_0$ so that $k_\perp \gg k_\parallel$. At $k_\perp \rho_i \sim 1$, two things happen. On the one hand, there may be some nonlinear effects that are not well understood. On the other hand, a significant fraction of the energy in the Alfvén wave cascade excites a kinetic Alfvén wave (KAW) cascade that carries the energy down to scales on the order of the thermal electron gyro-radius where the turbulence is finally dissipated by collisionless Landau damping.

Sahraoui et al. (2009) used the above scenario to interpret spacecraft measurements of solar wind turbulence in the kinetic range of scales $k_\perp \rho_i \gtrsim 1$. These authors used the linear KAW dispersion relation and damping rates to argue that the KAW cascade should reach electron scales before being strongly damped, noting that the relative damping rate $\gamma/\omega$ does not become of order unity until $k_\perp \rho_e \sim 1$, where $\rho_e$ is the thermal electron gyro-radius. The purpose of the present paper is to point out that this argument is incomplete because it does not take into account the compounding of the damping throughout the course of the cascade process. KAWs are damped by collisionless Landau and transit-time damping throughout the entire wavenumber range of their existence from $k_\perp \rho_i \simeq 1$ to $k_\perp \rho_i \gg 1$. If the energy cascade process is thought of as taking place in a sequence of discrete steps, then there is dissipation at each step in the sequence and the effects of damping are compounded with each step, analogous to the way compound interest works. When this compounding is taken into account, it is found that for typical solar wind plasma parameters near 1 AU the KAW cascade is dissipated before reaching electron scales. This conclusion has important ramifications for the interpretation of solar wind magnetic field measurements. It implies that power-law spectra in the regime of electron scales must be supported by wave modes other than the KAW.

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In astrophysical plasmas, the effects of collisionless damping on the turbulence spectrum have been studied primarily using quasilinear theory (see, for example, Melrose 1994) and phenomenological models in which the energy cascade process is modeled as a diffusion process in wavenumber space (for example, Li et al. 2001; Stawicki et al. 2001; Cranmer & van Ballegooijen 2003; Jiang et al. 2009; Matthaeus et al. 2009). Other approaches include weak turbulence theory (Yoon 2006, 2007; Galtier 2006; Chandran 2008) and gyro-kinetic theory (Howes et al. 2008b; Schekochihin et al. 2009). Here we adopt a somewhat simpler approach that, like the diffusion models, is based on an equation expressing the conservation of energy in wavenumber space. Generally speaking, the objective of any of these models is to describe the essential physics as accurately and economically as possible.

In this study, the effects of damping on the KAW cascade are modeled using two complimentary approaches. The first approach is heuristic and shows how this damping takes place through a sequence of steps in wavenumber space whereby energy is damped at each step before being transferred to higher wavenumbers. The second approach is based on an equation for the conservation of energy in wavenumber space and two different expressions for the energy cascade rate that are similar to Kolmogorov’s relation $\varepsilon = k E(k)/\tau$. This approach has much in common with the cascade model developed by Howes et al. (2008a) and allows both the energy cascade rate $\varepsilon$ and the energy spectrum $E(k)$ to be computed as functions of wavenumber throughout the inertial range and dissipation range. In this study,
the main objective is to determine the point in wavenumber space where the KAW cascade terminates and, therefore, the energy cascade rate \( \varepsilon \) is the quantity of primary interest. Assuming that \( k_\perp \gg k_\parallel \), analytic solutions for the energy cascade rate \( \varepsilon \) may be derived that are convenient for purposes of analysis and prediction. These analytic solutions are new and are presented here for the first time.

One of the principal assumptions of this work is that the damping rates of linear wave theory are applicable at kinetic scales, even though nonlinear interactions are still strong. Although this same assumption has been made before by many investigators (Leamon et al. 1998; Quataert 1998; Gary 1999; Leamon et al. 1999; Quataert & Gruzinov 1999; Li et al. 2001; Stawicki et al. 2001; Cramer & van Ballegooijen 2003; Howes et al. 2008a), there is no well-established criteria for its validity. This is, perhaps, the greatest source of uncertainty for the theory presented here. It is also easy to envision a scenario in which turbulence at MHD scales is dissipated via collisionless magnetic reconnection involving structures (reconnection sites) that span length scales from the ion inertial length to the electron inertial length (Drake & Shay 2007). To some extent, this is a possible alternative to the wave cascade and damping scenario described by Schekochihin et al. (2009) and others.

For the moment, adopting the KAW cascade and linear damping scenario and assuming that the wave damping rates of the linear Vlasov–Maxwell theory are valid in the kinetic regime, then for typical solar wind conditions at 1 AU the models derived here predict that the KAW cascade terminates at wavenumbers less than or equal to \( k_\perp \rho_i \simeq 25 \) which implies that the KAW cascade cannot reach electron scales in the solar wind at 1 AU. These results agree with and are supported by those obtained previously using the cascade model of Howes et al. (2008a) which show a transition to an exponentially decaying magnetic energy spectrum that decays rapidly at around the same wavenumber. On the contrary, the magnetic energy spectra in the gyro-kinetic simulations of Howes et al. (2008b) do not appear to show any deviations from power-law behavior in the range \( 1 \lesssim k_\perp \rho_i \lesssim 8 \) as there should be if kinetic damping is having a significant effect. However, because these scales are near the smallest spatial scales in the simulation where an artificial electron hypercollisionality takes effect, the simulation may not accurately describe the physics when \( k_\perp \rho_i \sim 8 \).

Note on terminology. The term “energy cascade” or “cascade,” for short, refers to a nonlinear energy transfer process that transfers energy from large to small scales or vice versa. This term usually refers to the inertial range where the effects of dissipation are negligible. Here, this term is also used to describe the nonlinear energy transfer in the dissipation range. This slight abuse of notation should cause no confusion.

2. SIMPLE HEURISTIC MODEL

We begin with some general considerations that apply to the entire paper. Consider a homogeneous proton–electron plasma with a constant uniform background magnetic field. The equilibrium state is charge neutral, free of macroscopic electric fields, free of electric currents, and characterized by isotropic Maxwell distribution functions for both particle species. It is assumed that the wave amplitudes at kinetic scales are small enough that linear wave theory adequately describes the collisionless damping process. For a given wavenumber \( \mathbf{k} \) (real-valued), the real and imaginary parts of the wave frequency \( \omega + i \gamma \) may be computed numerically using the hot plasma dispersion relation (Stix 1992, chapter 10). For \( k_\perp \gg k_\parallel \), the KAW is uniquely identified by means of its asymptotic dispersion relation \( \omega = k_\parallel v_A \) in the limit as \( k_\perp \rightarrow 0 \) with \( k_\perp/k_\parallel \) or the angle of propagation held constant.

In general, the damping is minimized for all wavenumbers in the range from \( k_\perp \rho_i \approx 1 \) to \( k_\perp \rho_i \approx 40 \) when the angle of propagation \( \theta \) is close to 90 degrees, that is, near perpendicular to \( \mathbf{B} \). It is important to note that for all angles in a sufficiently small neighborhood of 90 degrees, the ratio of the damping rate to the real frequency, \( \gamma/\omega \), is approximately independent of the angle \( \theta \) so that all propagation angles in this range yield the same damping per wave period. In practice, this means that the damping per wave period cannot be reduced further by going from 89.9 degrees to 89.99 degrees, for example. For simplicity, it will be assumed in this section that \( \theta \) is close, but not equal to 90 degrees. This range of near-perpendicular angles is physically relevant because as a consequence of three-wave interactions the energy cascade process creates a spectrum in which the inequality \( k_\perp \gg k_\parallel \) is increasingly well satisfied as the cascade progresses to higher wavenumbers.

Figure 1. Attenuation of the wave amplitude in one wave period \( \alpha(k) = \exp[2\pi \gamma(k)/\omega(k)] \). The propagation angle \( \theta = 89.9^\circ \) is shown in blue and \( \theta = 89.9^\circ \) is shown in red. (A color version of this figure is available in the online journal.)

Investigation of the damping of the KAW cascade begins with the plot in Figure 1 which shows the attenuation of the wave amplitude in one wave period, that is, \( \exp(2\pi \gamma(k)/\omega(k)) \) versus \( k_\perp \) for a fixed angle of propagation close to 90 degrees. As just mentioned, the curve for 89.9 degrees in Figure 1 applies for all angles sufficiently close to 90 degrees. Now recall the critical balance hypothesis of Goldreich & Sridhar (1995, 1997) which states that in the inertial range the cascade time is equal to the Alfvén wave period. Using this as a guide, we assume that the cascade time in the kinetic regime is equal to the wave period of the KAWs. Consider the sequence of wavenumbers \( k_n = 2^n k_0 \), where \( k_0 \rho_i = 1 \) and \( n = 0, 1, 2, \ldots \). In one cascade time, one wave period, the energy at scale \( k_n \) cascades to scale \( k_{n+1} \) but in the process that energy is damped by a factor \( \alpha_n^2 \), where \( \alpha_n = \exp[2\pi \gamma(k_n)/\omega(k_n)] \) is the attenuation of the wave amplitude in one wave period. If \( \varepsilon(k_n) \) is the energy cascade rate at wavenumber \( k_n \), that is, the rate at which energy reaches scale \( k_n \), then the rate at which energy reaches the next scale \( k_{n+1} \) is

\[
\varepsilon_{n+1} = \alpha_n^2 \varepsilon_n,
\]

where \( \varepsilon_n = \varepsilon(k_n) \). This is the change in the energy cascade rate caused by linear damping after just one step in the cascade.
process. Starting from \( n = 0 \), the energy cascade rate after \( n \) steps is equal to

\[
e_\alpha = (\alpha_0 \alpha_1 \alpha_2 \cdots \alpha_{n-1})^2 \varepsilon_0. \tag{2}\]

This describes how the energy of the waves is damped as it cascades through the kinetic range of scales. To compute the wavenumber dependence of \( \varepsilon(k) \) for the KAW cascade it is only necessary to compute the relative damping rate \( \gamma/\omega \) from the hot plasma dispersion relation and then use Equation (2).

By way of illustration, consider the damping rates for 89.9 degrees shown in Figure 1. For each wavenumber \( k_n = 2^n k_0 \), it is straightforward to read off the values of \( \alpha_n \) from the plot in Figure 1 and then compute the cascade rate \( \varepsilon_n \) from Equation (2). This yields the results in Table 1 which show that the energy cascade rate is reduced by less than 1% of its original value by the time the cascade reaches \( k_\perp \rho_i = 30 \). The plasma parameters used to compute the damping rates in Figure 1 are typical of high-speed solar wind streams at 1 AU. For these parameters, the electron gyro-radius occurs when \( k_\perp \rho_e \simeq 1 \) or, equivalently, \( k_\perp \rho_i \simeq 60 \), and the electron inertial scale occurs when \( k_\perp c/\omega_{pe} \simeq 1 \) or, equivalently, \( k_\perp \rho_e \simeq 30 \). This simple calculation indicates that the KAW cascade is likely to be dissipated before reaching electron scales. As we show below, a more detailed model yields results that are in good agreement with those in Table 1.

### Table 1

| \( n \) | \( k_\perp \rho_i \) | \( \alpha_n \) | \( \varepsilon_n/\varepsilon_0 \) |
|--------|--------------------|-------------|------------------|
| 0      | 0.5                | 0.95        | 100%             |
| 1      | 1                  | 0.87        | 90%              |
| 2      | 2                  | 0.84        | 68%              |
| 3      | 4                  | 0.73        | 48%              |
| 4      | 8                  | 0.52        | 25%              |
| 5      | 16                 | <0.40       | 7%               |
| 6      | 32                 | ...         | <1%              |

#### 3. MODEL BASED ON CONSERVATION OF ENERGY

The energy spectrum \( E(k) \) is defined such that \( E(k) dk \) is the energy per unit mass contained in the fluctuations for all wavenumbers between \( k \) and \( k + dk \). The total energy contained in the entire spectrum is

\[
E_{\text{tot}} = \int_0^\infty E(k) \, dk. \tag{3}\]

The cascade rate \( \varepsilon(k) \) is the average energy per unit mass that passes through wavenumber \( k \) per unit time. The energy flows from low to high wavenumbers. Consider a small interval from \( k \) to \( k + dk \) in wavenumber space as illustrated in Figure 2. Energy is flowing into the boundary \( k \) from the left and out of the boundary \( k + dk \) to the right. Conservation of energy requires that the energy flowing out per unit time is equal to the energy flowing in per unit time minus the rate at which energy is dissipated in the layer. Hence,

\[
\varepsilon(k + dk) - \varepsilon(k) = 2\gamma E(k) \, dk, \tag{4}\]

where \( \gamma < 0 \) is the linear damping rate obtained from the KAW dispersion relation. This yields

\[
\frac{d\varepsilon}{dk} = 2\gamma E(k), \tag{5}\]

which expresses the conservation of energy in the cascade process. Note that if \( \gamma = 0 \), then \( \varepsilon = \text{const} \) and the energy cascade rate is independent of \( k \).

Equation (5) gives one relation between \( \varepsilon \) and \( E \). A second is Kolmogorov’s relation

\[
\varepsilon(k) = \frac{kE(k)}{\tau}, \tag{6}\]

where \( \tau \) is the energy cascade time. Kolmogorov’s relation assumes there is no dissipation of energy during the cascade process so that all the energy at scale \( k \) cascades to higher wavenumbers. When dissipation is present, a fraction \( \alpha^2 \) of the total energy at scale \( k \) cascades to higher wavenumbers and a fraction \( 1 - \alpha^2 \) is dissipated. The energy that is dissipated does not cascade to higher wavenumbers. In this case, the relation (6) is replaced by

\[
\varepsilon(k) = \frac{\alpha^2 kE(k)}{\tau}, \tag{7}\]

where \( \alpha^2(k) \) is the energy attenuation factor in time \( \tau \).

It is clear from physical considerations that in the dissipation range the energy cascade rate \( \varepsilon \) must depend on the dissipation rate. Hence, Kolmogorov’s relation (6) that was initially postulated for the inertial range needs to be modified. What is not clear is what the precise functional form should be. Justification for the form (7) can be obtained by considering a simple shell model. Let \( k_n = 2^n k_0 \) denote a sequence of shells in wavenumber space, where \( n \) is an integer and \( k_0 \) is an arbitrary wavenumber. The energy contained in the interval between \( k_n \) and \( k_{n+1} \) is \( k_n E(k_n) \). In the absence of dissipation, this energy cascades to scale \( k_{n+1} \) in one cascade time \( \tau(k_n) \) so that the energy cascade rate is

\[
\varepsilon_n = \frac{k_n E(k_n)}{\tau_n}. \tag{8}\]

When there is no dissipation, \( \varepsilon_n = \text{const} \) and the energy cascade rate is independent of scale. When dissipation is present, the energy at scale \( k_n \) is partly dissipated before it can cascade to the next level. If the energy is attenuated by a factor \( \alpha^2 \) in time \( \tau_n \), then the energy cascade rate becomes

\[
\varepsilon_n = \frac{\alpha^2 k_n E(k_n)}{\tau_n}, \tag{9}\]

where \( \alpha \) may depend on \( k_n \). This is another way to derive Equation (7).
In general, Equations (6) and (7) should also contain a constant coefficient $A$ which is dimensionless. In hydrodynamic turbulence, it follows from dimensional analysis that this coefficient is of order unity. In plasma turbulence, there are other parameters in the problem which may cause this coefficient to differ from unity. For example, in the inertial range, one such parameter is the normalized cross-helicity $\sigma_c$. Thus, Equation (7) takes the final form

$$\epsilon(k) = A \frac{\omega(k)^2 E(k)}{\tau},$$  

(10)

where in many cases of practical interest the dimensionless constant $A$ is of order unity.

### 3.1. Complete Model

Assuming that critical balance holds for the KAW cascade, the energy cascade time is equal to the wave period, $\tau = 2\pi/\omega$, and the energy cascade rate (10) has the form

$$\epsilon(k) = \frac{\omega(k)}{2\pi} A \exp(4\pi \gamma/\omega) k E(k),$$  

(11)

where $\alpha = \exp(2\pi \gamma/\omega)$ is the attenuation factor for the wave amplitudes in one wave period and $\alpha^2 = \exp(4\pi \gamma/\omega)$ is the attenuation factor for the energy. When Equation (11) is substituted into the conservation law (5), it follows that

$$\frac{k}{d\epsilon}{\epsilon}{dk} = \frac{1}{4\pi} \frac{\gamma}{\omega} \exp(-4\pi \gamma/\omega)$$

or, equivalently,

$$d(\log \epsilon) = \frac{1}{4\pi} \frac{\gamma}{\omega} \exp(-4\pi \gamma/\omega).$$

(12)

(13)

When the exponential factor on the right-hand side is omitted, this equation is almost identical to Equation (8) in Howes et al. (2008a) with the source term $S$ in Howes et al. (2008a) set equal to zero. The one minor difference is that the damping rates $\gamma$ and $\omega$ in Equation (13) are obtained from the hot plasma dispersion relation (Stix 1992) whereas Howes et al. (2008a) employ the damping rates obtained from gyro-kinetic theory.

Because $\omega > 0$ and $\gamma < 0$, the exponential factor in Equation (13) causes $\epsilon(k)$ to decrease more rapidly than it would if the exponential factor were omitted. When $|4\pi \gamma/\omega| > 1$, the exponential drives $\epsilon$ toward zero at a faster than exponential rate as can be seen in the solutions presented below. At this point, the dissipation becomes so effective that the energy cascade is abruptly terminated. However, even if the exponential factor is omitted from Equation (13) which is equivalent to using Equation (6) in place of Equation (7), the wavenumber where the cascade rate $\epsilon$ goes to zero is still roughly of the same order of magnitude. Therefore, the presence of this exponential factor is not crucial for the conclusions reached in this study.

If $\gamma(k)$ and $\omega(k)$ are known from the hot plasma dispersion relation, then Equation (13) may be solved to find how $\epsilon$ depends on $k$. Once the solution for $\epsilon(k)$ is known, the energy spectrum $E(k)$ is obtained from Equation (11). In general, the ratio $\gamma/\omega$ depends on the angle of propagation of the waves which needs to be taken into account. However, when $k_\perp \gg k_\parallel$ or, equivalently, when the angle of propagation is sufficiently close to $\pi/2$, then the ratio $\gamma/\omega$ becomes approximately independent of angle. This property is used to derive an approximate analytic solution below. In cases where the angle is not sufficiently close to $\pi/2$ the angle dependence may be taken into account as follows. Assume that the energy in the wavevector spectrum is concentrated near the critical balance curve so that the angle of propagation $\theta$ at a given scale $k_\perp = k$ is determined by the critical balance relation

$$\cot(\theta) = \frac{k_\parallel}{k_\perp} = \frac{\delta v_\perp}{v_{\parallel,ph}},$$

(14)

where $\delta v_\perp$ is the electron velocity perturbation of the wave in the plane perpendicular to $B_0$ and $v_{\parallel,ph}$ is the parallel phase speed of the KAWs. Note that the parallel phase speed can be much larger than the Alfvén speed in the kinetic regime and this needs to be included in the critical balance relation (14).

According to the critical balance hypothesis, the longitudinal crossing time of two wavepackets $\tau_\parallel = k_\parallel/\delta v_\parallel$, where $v_\parallel$ is the parallel group speed. The parallel phase and group velocities are approximately equal as can be seen from the approximate dispersion relation $\omega \propto k_\parallel \sqrt{1 + (k_\perp/\lambda_\parallel)^2}$ (see, for example, Hollweg 1999; Crammer & van Ballegooijen 2003). In addition to Equation (14), the self-consistent calculation of the propagation angle $\theta$ requires the relation

$$k E(k) \simeq (\delta b_\perp)^2,$$

(15)

where the magnetic field perturbation $\delta b_\perp$ is measured in velocity units. For the plasma parameters of interest here, the energy density of KAWs is dominated by magnetic and thermal energy (i.e., density fluctuations) with an approximate equipartition between magnetic and thermal energy as discussed, for example, by Terry et al. (2001). These two contributions have been lumped together into a single term in Equation (15). Critical balance tightly couples the magnitude of the energy spectrum to the propagation angle through Equations (14) and (15). For KAWs, the ratio $\delta v_\perp/\delta b_\perp$ may be determined from the hot plasma dispersion relation using the relations $\delta v_\parallel = -i\omega_0 \chi_\parallel E$, $\delta b_\perp = -i\omega_0 \chi_\perp$, where $\chi_\parallel$ is the electron current density and $\chi_\perp$ is the electron susceptibility (Stix 1992). This completes the specification of the model which consists of Equations (11), (13)–(15).

### 3.2. Analytic Model

The above model can be significantly simplified when $k_\perp \gg k_\parallel$ so that the ratio $\gamma/\omega$ becomes approximately independent of the propagation angle $\theta$. Simulations of incompressible MHD turbulence have shown that the inequality $k_\perp \gg k_\parallel$ is usually well satisfied at the smallest inertial range scales and, therefore, the assumption $k_\perp \gg k_\parallel$ is justified in the kinetic regime. It is useful to adopt the approximate expressions for the ratio $\gamma/\omega$ derived by Howes et al. (2006) using gyro-kinetic theory. Using Equations (62) and (63) in Howes et al. (2006), one obtains

$$\frac{\gamma}{\omega} \simeq \frac{1}{\alpha} \rho_1,$$

(16)

where $\omega > 0$ and $\alpha$ is a constant given by

$$a = \frac{1}{2} \frac{[\beta_1 + (2/b)]^{1/2}}{\beta_1 \rho_e/m_1} \left( \frac{\pi T_e}{m_e} \right)^{1/2} \left( 1 - \frac{1 + b \beta_1}{2(1 + (b \beta_1/2)^2)} \right).$$

(17)

Here, $m_e$ is the electron mass, $m_1$ is the proton mass, $T_e$ is the equilibrium electron temperature, $T_i$ is the equilibrium proton
temperature, \( b = 1 + T_e/T_i \), and \( \beta_i = n_i k_B T_i/(B_0^2/2\mu_0) \), where \( k_B \) is Boltzmann’s constant, \( n_0 \) is the equilibrium particle number density, \( B_0 \) is the equilibrium magnetic field, and \( \mu_0 \) is the permeability of free space (SI units). Equation (16) is valid when \( k_{\perp}\rho_i \gg 1 \) and \( k_{\perp}\rho_i \ll 1 \). Using typical plasma parameters for the solar wind at 1 AU, Equation (16) was compared to the ratio \( \gamma/\omega \) obtained from the hot plasma dispersion relation and found to be an excellent approximation except when \( k_{\perp}\rho_i \sim 1 \) where Equation (16) underestimated the damping although it remained accurate to within a factor of 2 or so.

The substitution of Equation (16) into Equation (12) yields the equation

\[
\frac{d(\log \varepsilon)}{dk} = -\frac{4\pi a}{A} \exp(4\pi ak),
\]

where \( k = k_{\perp}\rho_i \). The solution is given by

\[
\varepsilon = \varepsilon_0 \exp \left[ -\frac{1}{A} (e^{4\pi ak} - e^{4\pi ak_0}) \right].
\]

(19)

If the exponential factor is omitted from Equation (13) or, equivalently, Equation (6) is used in place of Equation (7), then instead of Equation (18) one obtains

\[
\frac{d(\log \varepsilon)}{dk} = -\frac{4\pi a}{A}
\]

(20)

which has the solution

\[
\varepsilon = \varepsilon_0 \exp \left[ -\frac{4\pi a}{A} (k - k_0) \right].
\]

(21)

Equations (19) and (21) describe how the energy cascade rate changes as a result of collisionless damping. The associated energy spectrum can be obtained from Equation (11) if the propagation angle \( \theta(k) \) is known, since the propagation angle is necessary to compute the wave frequency \( \omega(k) \). Therefore, the energy spectrum can only be obtained by solving the complete model consisting of Equations (11), (13)–(15). This is not attempted here because the energy spectrum is not needed for the purpose of this study.

The theoretical model developed here may also be adapted to study hydrodynamic turbulence. In hydrodynamics, the damping rate is

\[
\frac{d(\log \varepsilon)}{dk} = -\frac{4\pi a}{k^2} \exp(4\pi ak),
\]

where \( k = k_{\perp}\rho_i \), \( \rho_i \) is the density, \( \chi \) is the magnetic permeability, and \( \nu \) is the kinematic viscosity, and the cascade time is the nonlinear eddy turnover time \( \tau = 1/\nu v \), where \( v = (kE)^{1/2} \). The model is based on Equation (5) and either Equation (6) or Equation (7) with \( \alpha = \exp(-v k^2 \tau) \). An analytic solution is only possible when Equation (6) is used. Calculations of the energy cascade rate versus wavenumber obtained using these hydrodynamic models show that the energy cascade rate approaches zero near the Kolmogorov scale \( \eta \sim (\nu^3/\varepsilon)^{1/4} \). Moreover, the model calculations in the range \( k\eta < 1 \) are in reasonable agreement with the cascade rate \( \varepsilon(k) \) obtained using the model spectrum in Pope (2000). In fact, the wavenumber \( k\eta \gtrsim 1 \) where the energy cascade terminates in Pope’s solution lies between the cutoff wavenumbers \( k\eta \gtrsim 0.8 \) and \( k\eta \gtrsim 1.3 \) for the two solutions obtained using either Equation (6) or Equation (7).

4. RESULTS

Results are now presented for typical high-speed solar wind in the ecliptic plane near 1 AU with the plasma parameters

\[
V_{sw} > 600 \text{ km s}^{-1}, \beta_i = 1, \beta_e = 1/2, \text{ and } v_{th,e}/c = 2 \times 10^{-4}.
\]

This choice of parameters is based on some of our own work, both published (Podesta 2009) and unpublished, and results published in the literature (Feldman et al. 1977; Newbury et al. 1998; Schwenn 2006). For these parameters and for highly oblique angles of propagation (80 degrees < \( \theta < 90 \) degrees), the damping of KAWs is negligible for small wavenumbers, \( k_{\perp}\rho_i \ll 1 \). KAW damping starts to become significant around \( k_{\perp}\rho_i \approx 1/2 \). The initial wavenumber in the model calculations is \( k_{\perp}\rho_i \approx 1/2 \). The results for the two different theoretical models, Equations (19) and (21), are shown in Figures 3 and 4.

The energy cascade rate \( \varepsilon \) shown in Figures 3 and 4 decreases to zero around \( k_{\perp}\rho_i \approx 10 \) and \( k_{\perp}\rho_i \gtrsim 25 \) respectively. When \( \varepsilon/\varepsilon_0 \ll 1 \), the energy cascade terminates. Even though the two models yield noticeably different results, the wavenumber where the cascade rate becomes negligibly small, \( \varepsilon/\varepsilon_0 \ll 1 \), is of the same order of magnitude in both Figures 3 and 4. Thus, the two models are roughly consistent with each other and show that the KAW energy cascade cannot continue beyond the wavenumber \( k_{\perp}\rho_i \sim 25 \) because the energy flux is almost completely damped at that point. This is the central conclusion of this study. The behavior of the solutions, when the parameter \( A \) is varied, is also shown in Figures 3 and 4. The model (19) that includes the factor \( \alpha^2 \) in Kolmogorov’s relation (7) is less sensitive to variations in the parameter \( A \) than the model (21) that is based on Kolmogorov’s relation (6). Taken together, the results in Figures 3 and 4 suggest that the KAW cascade will be completely damped by collisionless Landau damping before the energy can reach the scale of the electron gyro-radius;
the electron gyro-radius occurs at \( k_\perp \rho_i \simeq 60 \) for the plasma parameters used here.

The wavenumber where the KAW cascade terminates can be estimated from Equations (19) and (21). The wavenumber where \( \varepsilon/\varepsilon_0 \) reaches some small value, say, \( \varepsilon/\varepsilon_0 = \delta \) with \( \delta = 10^{-2} \), can be determined as follows. From Equation (19), assuming \( 4\pi k_0 a \ll 1 \), the wavenumber where \( \varepsilon/\varepsilon_0 = \delta \) is given by

\[
k_\perp \rho_i \simeq \frac{1}{4\pi a} \log[1 + A \log(1/\delta)], \tag{22}
\]

where \( a \) is defined in Equation (17). For \( A = 1 \) and \( \delta = 10^{-2} \), this yields \( k_\perp \rho_i \simeq 10 \). From the second solution (21), the wavenumber where \( \varepsilon/\varepsilon_0 = \delta \) is given by

\[
k_\perp \rho_i \simeq \frac{A}{4\pi a} \log(1/\delta). \tag{23}
\]

For \( A = 1 \) and \( \delta = 10^{-2} \), this yields \( k_\perp \rho_i \simeq 28 \). These formulae are very convenient for estimating the wavenumber where the KAW cascade terminates.

The termination point can also be expressed in terms of the ratio \( \gamma/\omega \). Using Equation (16), the condition (22) with \( A = 1 \) and \( \delta = 10^{-2} \) is equivalent to \( |\gamma/\omega| = 0.14 \). Similarly, the condition (23) with \( A = 1 \) and \( \delta = 10^{-2} \) is equivalent to \( |\gamma/\omega| = 0.37 \). The application of theoretical models like those in Section 3 to hydrodynamic turbulence suggests that the actual termination point is somewhere between these two model predictions, \( |\gamma/\omega| = 0.14 \) and \( |\gamma/\omega| = 0.37 \). Hence, it is reasonable to conclude that the cutoff for the KAW cascade occurs approximately when \( |\gamma/\omega| \sim 0.25 \). Note that for the heuristic model in Section 2, the cascade terminates as soon as

\[
(\pi k_0)^{1/2} \rho_i \leq 0.1 \omega \rho_i.
\]

For example, if the cascade time is reduced from one wave period to half a wave period, then the wavenumber where the KAW cascade terminates is roughly twice as large. And, if the cascade time is reduced to one quarter of a wave period, then the wavenumber where the KAW cascade terminates is roughly four times as large. Consequently, the results are sensitive to the cascade time which is not known precisely.

A simple expression for the wavenumber where the KAW cascade terminates is

\[
\gamma/\omega \approx 25. \text{ Hence, one must be careful when using}
\]

5. CONCLUSIONS

In this study, two different methods were employed to calculate the collisionless damping of the KAW cascade. The first method demonstrates the effect of compounding on wave dissipation during the energy cascade process as illustrated by the calculation in Table 1. For nearly perpendicular propagating KAWs under typical solar wind conditions at 1 AU, the ratio \( |\gamma/\omega| \) is usually small from \( k_\perp \rho_i = 1 \) to \( k_\perp \rho_i = 10 \) so that \( |\gamma/\omega| \ll 1 \). Nevertheless, because the wave damping is compounded at each stage of the energy cascade process, the energy cascade is damped more rapidly than might be expected from an examination of the wavenumber dependence of \( \gamma/\omega \). The simple heuristic model used to quantify this compounding effect assumes that all the energy is carried by KAWs, that the cascade time is equal to the wave period, and that damping of these waves is governed by the linear Vlasov–Maxwell dispersion relation. The second method based on Equation (13) yields quantitatively similar results even though the precise form of Kolmogorov’s relation is unknown. Thus, both methods support the main conclusion of this work, that a cascade consisting solely of KAWs cannot reach the electron gyro-scale in the solar wind at 1 AU. This conclusion is also supported by the somewhat different cascade model developed by Howes et al. (2008a).

The conclusions are, of course, subject to some uncertainty. For example, if the cascade time is reduced from one wave period to half a wave period, then the wavenumber where the KAW cascade terminates is roughly twice as large. And, if the cascade time is reduced to one quarter of a wave period, then the wavenumber where the KAW cascade terminates is roughly four times as large. Consequently, the results are sensitive to the cascade time which is not known precisely.

A simple expression for the wavenumber where the KAW cascade terminates has been obtained from the analytical solutions (19) and (21). For \( A = 1 \), the cutoff occurs approximately when \( |\gamma/\omega| \simeq 0.25 \). Hence, one must be careful when using
damping rates obtained from the linear Vlasov–Maxwell wave theory to estimate the stopping point of the cascade. As a consequence of the effects of compounding, the wavenumber where the cascade terminates occurs not when $|\gamma/\omega| \simeq 1$ but approximately when $|\gamma/\omega| \simeq 0.25$.

The conclusion reached in the study of Sahraoui et al. (2009) should be re-examined in light of this result. Sahraoui et al. (2009) have suggested that high-frequency magnetic field spectra in the solar wind may be caused by a KAW cascade from proton to electron scales. To support this idea, Sahraoui et al. (2009) compute dispersion curves from the hot plasma dispersion relation using the solar wind parameters in their approximation (16) used in the theoretical model. Note from Equations (16) and (17) that when $T_e = T_p$ and $\beta_p < 1$, if $k_{\perp} \rho_i$ is held fixed ($k_{\perp} \rho_i \gg 1$), then $|\gamma/\omega| \propto \beta_p^{-1/2}$ even though for a fixed value of $k_{\perp} c/\omega_p$ the ratio $|\gamma/\omega|$ is independent of $\beta_p$.

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