Four-boson scale near a Feshbach resonance

M. T. Yamashita\textsuperscript{1}, Lauro Tomio\textsuperscript{2}, A. Delfino\textsuperscript{3} and T. Frederico\textsuperscript{4}

\textsuperscript{1} Universidade Estadual Paulista, 18409-010, Itapeva, SP, Brazil
\textsuperscript{2} Instituto de Física Teórica, Universidade Estadual Paulista, 01405-900, São Paulo, SP, Brazil
\textsuperscript{3} Instituto de Física, Universidade Federal Fluminense, 24210-900, Niterói, RJ, Brazil
\textsuperscript{4} Departamento de Física, Instituto Tecnológico de Aeronáutica, Centro Técnico Aeroespacial, 12228-900 São José dos Campos, SP, Brazil

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Abstract. – We show that an independent four-body momentum scale $\mu(4)$ drives the tetramer binding energy for fixed trimer energy (or three-body scale $\mu(3)$) and large scattering length ($a$). The three- and four-body forces from the one-channel reduction of the atomic interaction near a Feshbach resonance disentangle $\mu(4)$ and $\mu(3)$. The four-body independent scale is also manifested through a family of Tjon-lines, with slope given by $\mu(4)/\mu(3)$ for $a^{-1} = 0$. There is the possibility of a new renormalization group limit cycle due to the new scale.

Recent progress in experimental techniques, creating tunable few-body interactions in trapped ultracold gases \cite{1}, is allowing the investigation of few-body quantum phenomena, with long wave-length triatomic properties being currently probed. In recent experiments \cite{2,3} it was confirmed the universal properties coming from the Efimov physics \cite{4} occurring for large two-body scattering lengths \cite{5–9}. So far experimentally, it was discovered that magnetic tunable molecular interactions in traps allow the formation of complex molecules like Cesium tetramers (Cs\textsubscript{4}) near the scattering threshold \cite{10}. One can foresee that the physics of four-atom systems in ultracold gases will be probed in the future with tunable interactions.

Near a Feshbach resonance the two-atom scattering length, $a$, can vary from very large negative values to positive values, allowing virtual or weakly-bound dimers. The scattering length is large in respect to the atom-atom interaction range ($r_0$), driving to the use of concepts developed for short range interactions and halo states \cite{11}. In the limit of large $a$, the interaction can be taken as of zero range \cite{12}. The appearance of Thomas-Efimov states in three-boson systems is controlled by the ratio $r_0/|a| \rightarrow 0$ \cite{4,13–15}. In this exact limit, it is observed an infinite sequence of three-body bound states \cite{16,17}, identified \cite{15,18} with an underlying renormalization group limit cycle \cite{19}. The collapse of the three-boson system when the two-body interaction range goes to zero for fixed $a$ demands one three-body scale to stabilize the system.

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In the nuclear physics context, it was found that the $^4\text{He}$ energy is determined by the triton one, which is known as the Tjon-line [20]. Once the triton energy is fitted to its experimental value the Tjon-line gives the observed $^4\text{He}$ binding. That result could be particular to the nuclear potentials models used in the calculations and may not be valid near a Feshbach resonance in atomic systems.

In this letter we show that a new parameter/scale governs the properties of four-atom systems near a Feshbach resonance. We also discuss that the three and four-body energies can be disentangled due to the presence of few-body forces from the one-channel reduction. First, we discuss the atomic interaction near the Feshbach resonance and its one-channel reduction. Second, we solve the four-boson problem for a zero-range interaction with two and three-boson binding energies fixed to investigate the necessity of a new scale to determine the four-boson binding energy.

The accepted wisdom is that near a Feshbach resonance only the two-body scale is moved through the large variations in the atom-atom scattering length. However, surprisingly enough, there are three and four-body potentials induced close to a Feshbach resonance, which can disentangle the corresponding scales and justify a thoroughly discussion of the independence of these scales for trapped atoms. To understand the origin of the induced interactions, we start with the Feshbach decomposition in $P$ and $Q$ spaces ($P + Q = 1$), with the $Q-$space representing the pair in the potential well where it is bound, while the $P$-space represents the pair in the lower potential well where the continuum channel of the ultracold regime happens. Three- and four-body effective potentials appear when the pair of particles forms the bound state in the $Q-$space when interacting with spectators. To make it concrete, we chose an example for a three-body system $(ijk)$ where a pair $(ij)$ forms a bound state in the well corresponding to the $Q$-space. The Hamiltonian is $H = PHP + QHQ + P v_{ij} Q + Q v_{ij}^i P$, where the transition potential between $P$ and $Q$ spaces is $v$. To simplify the discussion it is assumed that the channels Hamiltonians are $PHP = H_0 + V_{ik} + V_{ij}$ and $QHQ = H_0 + V_{ik} + V_{jk} + V_{ij}^Q$. $H_0$ is the kinetic energy operator. The gap $\Delta E = \lim_{r \to \infty} (V_{ij}^Q(r) - V_{ij}(r)) > 0$ gives the difference in the asymptotic values of two potential wells. The potential $V_{ij}$ is the non-resonant part of the interaction between $i$ and $j$, while $V_{ij}^Q$ is the second potential well where the Feshbach resonance lies. The effective Hamiltonian acting in the $P$-component of the wave function is

$$H_{eff} = H_0 + \sum_{r<s} V_{rs} + P v_{ij} Q G_{QQ}(E) Q v_{ij}^i P,$$

where the resolvent $G_{QQ}(E) = [E - H_0 - V_{ij}^Q - V_{ik} - V_{jk}]^{-1}$. The last term in the effective Hamiltonian carries the resonant term of the potential between particles $(ij)$ and the induced three-body potential, as can be understood from the decomposition of the $Q-$channel resolvent as $G_{QQ}(E) = [E - H_0 + V_{ij}^Q]^{-1} [1 + (V_{ik} + V_{jk}) G_{QQ}(E)]$. The resonant interaction between particles $ij$ is given by $P v_{ij} Q [E - H_0 - V_{ij}^Q]^{-1} Q v_{ij}^i P$, when it is accounted the dominance of the projection over the bound state of the potential $V_{ij}^Q$ near the scattering threshold. The three-body connected part of the remaining term corresponds to a three-body interaction, with intensity clearly depending on the position of the Feshbach resonance. Such, reasonings are easily extended to four particles. Therefore, in principle not only the scattering length can be tuned near a Feshbach resonance but also the three and four-body binding energies.

Turning off the excitation of the Feshbach resonance in the presence of spectator particles, the formalism is reduced to the one discussed in Ref. [21].

In short, near a Feshbach resonance the induced one-channel three and four boson forces
can drive independently the corresponding physical scales. Therefore four-body observables, like four-boson recombination rates or atom-trimer or dimer-dimer scattering lengths can exhibit correlations not constrained by one low-energy s-wave three-boson observable and $a$, if a four-boson scale really exists.

We begin our discussion of the relevant scales in the few-boson system reminding the form of the regulated trimer bound-state integral equation, in units of $\hbar = m = 1$ ($m$ is the boson mass), which is written as

$$f(q) = \frac{\pi^{-1}}{-a^{-1} + \sqrt{|E_3| + \frac{3}{4}q^2}} \int_0^\infty k^2 dk \int_{-1}^1 dz \left[ \frac{1}{E_3 + q^2 + k^2 + qkz} - \frac{1}{\mu(3)^2 + q^2 + k^2 + qkz} \right].$$

where the second term in Eq. (2) brings the physical scale, $\mu(3) \sim r_0^{-1}$, to the three-boson system [22] producing a finite ground-state energy and avoiding the Thomas collapse [13].

The sensitivity of three-boson s-wave observables to the short-range part of the interaction in weakly bound systems is parameterized through the value of the trimer binding energy which corresponds to the regularization scale $\mu(3)$. The s-wave three-boson observables are strongly correlated to the trimer energy and scales in the general universal form [16, 23]:

$$O_3(E, E_3, a) = |E_3|^\eta F_3 \left( \frac{E}{E_3}, a \sqrt{|E_3|} \right),$$

where $O_3$ can represent an observable at an energy $E$ or an excited trimer energy (the dependence on $E$ does not appear in this last case). The exponent $\eta$ gives the correct dimension to $O_3$. Eq. (2) is renormalization group (RG) invariant with its kernel being a solution of a non-relativistic Callan-Symanzik differential equation [24] as function of a sliding $\mu(3)$. In that way $E_3$ and three-body observables are independent of the subtraction point (see Ref. [23] for a discussion of the RG invariance in three-body systems).

In a four-boson system with short-range interactions one may be tempted to think in an independent momentum scale, which is still under discussion [25, 26]. In the hyperspherical expansion method used in Ref. [11, 25] it is enough to fix the scattering length, effective range and shape parameter in the limit of zero-range interaction to stabilizes the N-boson system. On the other hand within effective field framework it is enough a repulsive three-body force that stabilizes the trimer to allow finite values of the tetramer binding energy [26].

In our strategy to verify the necessity of a new four-boson scale, the dimer binding energy is set to zero while the trimer ground-state energy is kept fixed. Within these assumptions, to allow finite results for the tetramer energy, the Faddeev-Yakubovsky (FY) equations [27] should be regularized with the insertion of momentum cutoffs or subtraction scales. From our results for the binding energies in the limit of a zero-range interaction, it emerges the necessity of a new scale in the four-boson system, which depends crucially on the four-body scale, the subtraction point $\mu_4^2$. The existence of an independent four body scale can be in principle checked by measuring s-wave observables for three and four atoms near a Feshbach resonance. If the three and four-body scales can be made truly independent due to multi-atom forces, one can distinguish from the case that such scales are equal considering the correlation between two observables.

In the next, we show the systematical inclusion of the three and four-body subtraction points in the unregulated FY equations. First, it is natural to require $\mu(3)$ as the subtraction scale of the embedded three-boson system in the four-body one, otherwise the threshold for the four-boson energy would be different from the ground-state trimer energy. Second, the
Fig. 1 – Jacobi coordinates for the Faddeev-Yakubovsky amplitudes. $K$-type, left-side, and $H$-type, right side.

The dynamics of a four-boson system can be described in terms of the Faddeev-Yakubovsky amplitudes [27], $K$ and $H$, represented schematically in Fig. 1. The amplitude $K_{ij,k}$ corresponds to the partition in a three-body subsystem formed by particles, $ijk$, and the spectator one, $l$, while $H_{ij,kl}$ describes the partition in two subsystems formed by the pairs $ij$ and $kl$. There are 18 independent FY amplitudes, 12 $K$-type and 6 $H$-type. The bound state FY amplitudes satisfy:

$$
|K_{ij,k}\rangle = G_0 t_{ij}(E - E_{ij,k} - E_i)|K_{ik,j}\rangle + |K_{jk,i}\rangle + |H_{kl,i}\rangle,
$$

$$
|H_{ij,kl}\rangle = G_0 t_{ij}(E - E_{ij,kl} - E_{kl})|K_{kl,i}\rangle + |K_{ij,k}\rangle + |H_{ij,k}\rangle,
$$

where the four-body free resolvent is $G_0 = (E - H_0)^{-1}$ ($H_0$ is the kinetic energy operator) and $t_{ij}$ is the two-body T-matrix. The kinetic energies for the Jacobi momenta for identical particles of unit mass are $E_{ij,k} = \frac{2}{3}q_{ij,k}^2$, $E_i = \frac{2}{3}q_i^2$, $E_{ij,kl} = \frac{2}{3}q_{ij,kl}^2$, $E_{kl} = q_{kl}^2$. The total wave function of the four-body system is given by the sum $|\Psi\rangle = \sum_{i<j}|F_{ij}\rangle$, where the Faddeev components $|F_{ij}\rangle = G_0 V_{ij}|\Psi\rangle$ can be written in terms of the FY amplitudes as $|F_{ij}\rangle = |K_{ij,k}\rangle + |K_{ij,k}\rangle + |H_{ij,k}\rangle$.

The renormalized two-body T-matrix for a zero-range interaction is given by $t_{ij}(x) = |\chi_{ij}\rangle \tau(x)|\chi_{ij}\rangle$ with $\tau^{-1}(x) = 2\pi^2 [a^{-1} - \sqrt{-x}]$ and $\langle \tilde{q}_{ij} | \chi_{ij} \rangle = 1$. The structure of the FY equations is simplified for the zero-range interaction once the separation $|K_{ij,k}\rangle = G_0 |\chi_{ij}\rangle |K_{ij,k}\rangle$ and $|H_{ij,kl}\rangle = G_0 |\chi_{ij}\rangle |H_{ij,k}\rangle$ is recognized in Eqs. (4) and (5). The FY reduced amplitudes, $\mathcal{K}$ and $\mathcal{H}$, satisfy a coupled set of integral equations which need regularization. One recognizes that the resolvent of the immersed three-boson subsystem should carry the regularization scale $\mu(3)$ to avoid the Thomas collapse of the system. This is done in analogy with Eq. (2), where the momentum integration is regularized by a subtracted Green’s function in a subtraction point $\mu_2^2$.

In the l.h.s. of Eq. (3) one sees the correspondence with the three-boson equation written as $\tau^{-1}(e_2^1)|\mathcal{K}_{ij,k}\rangle - G_{ij,k}^{(3)}|\mathcal{K}_{ik,j}\rangle - G_{ij,j,k}^{(3)}|\mathcal{K}_{jk,i}\rangle = 0$. However, other terms are
present in the four-body equations which require regularization.

We introduce the scale \( \mu_4 \) such that the four-body free Green’s function \( G_0(E_4) \) is substituted by \( G_0(E_4) - \mu_4^2 G_0 \) in a direct generalization of Eq. (2), as suggested by [22]:

\[
\tau^{-1}(e_{ij,k}^{(1,3)}) | \tilde{K}_{ij,k}^{(3)} = G_{ij,k}^{(3)} | \tilde{K}_{ik,j}^{(3)} = G_{ij,jk}^{(3)} | \tilde{K}_{jk,k}^{(3)} = \frac{1}{2} \left[ \langle K_{ij,k}^{(3)} | H_{ik,j}^{(3)} + | K_{ik,j}^{(3)} + | K_{jk,i}^{(3)} + | K_{ik,j}^{(3)} + | K_{jk,i}^{(3)} \right],
\]

(6)

\[
\tau^{-1}(e_{ij,k}^{(2,2)}) | H_{ij,k} = G_{ij,k}^{(4)} \left[ | K_{ik,j}^{(4)} + | K_{jk,i}^{(4)} \right],
\]

(7)

where \( e_2^{(1,3)} = E(E_{ij,k} - E_i) \) and \( e_2^{(2,2)} = E(E_{ij,k} - E_k) \) are the two-body subsystem energies in the (1,3) and (2,2) partitions, respectively. The projected Green’s function operators are \( G_{ij,k}^{(N)} := \langle \chi_{ij} | G_0^{(N)} | \chi_{ik} \rangle \) with \( N \) equal 3 or 4, with the subtracted Green’s functions given by \( G_0^{(3)} = [E - H_0]^{-1} - [-\mu_3^2(I - H_0)]^{-1} \) and \( G_0^{(4)} = [E - H_0]^{-1} - [-\mu_4^2(I - H_0)]^{-1} \). Maintaining the two- and the three-body scales fixed, the four-body scale can be moved and its consequences for the tetramer binding investigated.

The four-boson integral equations for the reduced FY amplitudes are derived from the projection in Eqs. (6) and (7) in Jacobi relative momenta. The ground state of the four-boson system has total angular momentum zero, therefore the spectator functions depends only on the modulus of the momenta. The integral equations are analogous to the ones shown in Ref. [26] (without three-body forces), but with the four-body Green’s function substituted by their subtracted form as given in Eqs. (6) and (7). The numerical procedure followed Ref. [28].

The momentum scales in Eqs. (6) and (7) are \( \mu^{-1} \), \( \mu_3 \), and \( \mu_4 \), consequently the tetramer binding energy for \( \mu^{-1} = 0 \) depends on the momentum scales as

\[
E_4 = E_3 \left( \mu_4 / \mu_3 \right).
\]

(8)

In Table I, we present results for the tetramer ground state binding energy for \( |a| = \infty \) and considering a fixed \( \mu_3 \), such that we take \( \mu_3 = \mu_4 \) as our momentum unit. From the numerical solution of Eqs. (6) and (7), we obtain \( E_4 / \mu_3^2 \) for different values of \( \mu_4 / \mu_3 \). The trimer binding energy is fixed and given by the solution of Eq. (2), \( E_3 = -0.00927 \mu_3^2 \) [23]. We clearly observe in this table the dependence of \( E_4 \) on \( \mu_4 \) (for fixed \( \mu_3 \)). As \( \mu_4 / \mu_3 \) is varied from 1 to 20, the ratio \( E_4 / E_3 \) changes from 5 to about 78 (considering two digits).

In order to have \( E_4 \) independent on the regularization scale \( \mu_4 \), one should require that for \( \mu_4 / \mu_3 >> 1 \) the four-body scale would vanish as a physical scale. This is not observing in our results.

In view of the scaling relation Eq. (5), the results presented in the table are valid for any \( \mu_3 \). Using that \( E_3 = -0.00927 \mu_3^2 \) in Eq. (6), we obtain a family of Tjon lines, where each one corresponds to a given value of \( \mu_4 / \mu_3 \). From this result arises quite naturally the main claim of our work: the existence of a four-body scale, which is manifested through a family of Tjon-lines. The line slope is given by the ratio \( \mu_4 / \mu_3 \).

For equal three- and four-body scales, i.e., \( \mu_4 = \mu_3 = 1 \), the tetramer binding energy is \( E_4 \approx 5 E_3 \). The Tjon line [20] is \( E_4 \approx 4.72(E_1 + 2.48) \) MeV (\( E_0 \) is the \( ^4 \)He binding energy and \( E_1 \) the triton one), with the slope approached by our calculation. Also, it was recently found in Ref. [26] that \( E_4 \) scales as \( \sim 5 E_3 \), for equal bosons interacting through a zero-range two-body force, with a repulsive three-body potential needed to stabilize the trimer ground state energy against collapse. Our result for \( \mu_4 = \mu_3 \) agrees with both Refs. [20] and [26]. The physical reason for this agreement is explained in the next two paragraphs.

In the nuclear case, the nucleon-nucleon interaction is strongly repulsive at short distances and therefore the probability to have four nucleons simultaneously inside a volume \( \sim r_0^4 \) is
Table I – Four-boson binding energy for $E_2 = 0$ and fixed $\mu(3)$. The three-boson energy is $E_3 = -0.00927\mu^2(3)$ [23].

| $\mu(4)/\mu(3)$ | 1   | 2   | 4   | 7   | 10  | 15  | 20  |
|------------------|-----|-----|-----|-----|-----|-----|-----|
| $E_4/E_3$        | 5.0 | 7.8 | 13  | 22  | 29  | 51  | 78  |
| $E_4/\mu^2(3)$  | -0.046 | -0.072 | -0.124 | -0.20 | -0.27 | -0.47 | -0.72 |

quite small. For other potentials, like separable ones, parameterizing the nuclear force range, a similar result is found [29], even without a short range repulsion due to the non-locality of the interaction. In these cases, presumably the four-nucleon scale itself has much less opportunity to be evidenced, i.e., the three- and four-body scales correspond to the same short-range physics, which in our calculation come from $\mu(4) = \mu(3)$.

In Ref. [26], a three-body force is used to stabilize the shallowest three-body state, against the variation of the cut-off. Their results show that the Tjon line is reproduced when the three-body force is repulsive. Within this picture, the contribution of the three-body interaction in a configuration where the four bosons interact simultaneously is about four times more repulsive than having three-bosons in the force range and the other outside this range. Therefore, the dynamics of the short-range part of the interaction is manifested in the four-boson system mainly through small three-boson configurations, this should also qualitatively corresponds to the particular case $\mu(4) = \mu(3)$.

Also, we should note that, in Ref. [30], considering the three-boson problem near a narrow Feshbach resonance, Petrov concludes that three-body observables depend only on the resonance width (which implies in an effective range of the order of $\mu(3)^{-1}$) and the scattering length. One may suspect that his conclusion can also have consequences to four and more particle observables. However, we would like to stress that, within our present coupled-channel formalism, three and more body forces may appear, giving more freedom to the three and four atom systems. If one sticks to the Petrov’s assumption, than the three and four body scales are equal ($\mu(3) = \mu(4)$) and no independent behavior can be verified in practice.

The general scaling form of s-wave three-boson observables, Eq. (3), can be generalized to four-bosons with the dynamics parameterized by the scattering length, trimer and tetramer binding energies. A s-wave four-boson observable will be strongly correlated to $a$, $E_3$ and $E_4$:

$$O_4(E, E_4, E_3, a) = |E_4|^\eta F_4 \left( \frac{E}{E_4}, \frac{E_4}{E_3}, a\sqrt{|E_4|} \right),$$

where $O_4$ represents either an observable at energy $E$ or an excited tetramer energy (the dependence on $E$ does not appear in the last case). The exponent $\eta$ gives the correct dimension to $O_4$. There is the possibility of a new renormalization group limit cycle due to the new scale, and for $\mu(4) \rightarrow \infty$ one may speculate that the correlation given by Eq. (9) would approach a universal function.

In summary, by studying a four-boson system with a zero-range interaction, we conclude that a four-body scale is necessary when the two and three-boson scales are fixed. Extrapolating this result to more particles, we suggest that one new experimental information is required for each new particle added to the system near a Feshbach resonance, to include effects from many-body forces from the one-channel reduction. In the particular case, when only two-body forces are present, all scales are equal and can be for example associated with the effective range. However, as more accuracy is needed to describe the physics near a Feshbach resonance a distinction between 3 and 4 body scales must be introduced.
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REFERENCES

[1] Cornish S. L., et al., *Phys. Rev. Lett.*, 85 (2000) 1795; Roberts J. L., et al., *Phys. Rev. Lett.*, 86 (2001) 4211.
[2] Weber T., et al., *Phys. Rev. Lett.*, 91 (2003) 123201.
[3] Kraemer T., et al., *Nature*, 440 (2006) 315.
[4] Efimov V. M., *Phys. Lett. B*, 33 (1970) 563; *Comments Nucl. Part. Phys.*, 19 (1990) 271.
[5] Nielsen E. and Macek J. H., *Phys. Rev. Lett.*, 83 (1999) 563; *Comments Nucl. Part. Phys.*, 19 (1990) 271.
[6] Bedaque P. F., Braaten E. and Hammer H.-W., *Phys. Rev. Lett.*, 85 (2000) 908.
[7] Braaten E. and Hammer H.-W., *Phys. Rev. Lett.*, 87 (2001) 160407.
[8] Sorensen O., Fedorov D. V. and Jensen A. S., *Phys. Rev. Lett.*, 89 (2002) 173002.
[9] Braaten E., Hammer H.-W. and Kusunoki M., *Phys. Rev. Lett.*, 90 (2003) 170402.
[10] C. Chin, et al., *Phys. Rev. Lett.*, 94 (2005) 123201.
[11] Jackiw R., *M. A. B. Beg Memorial Volume*, edited by Ali A. and Hoodbhoy P. (World Scientific, Singapore) 1991.