An open FRW model in loop quantum cosmology

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Abstract
An open FRW model in loop quantum cosmology is considered. The left and right invariant vector fields and holonomies along them are studied. It is shown that in the hyperbolic geometry of \( k = -1 \) it is possible to construct a suitable loop which provides us with the quantum scalar constraint originally introduced by Vandersloot [19]. The quantum scalar constraint operator with negative cosmological constant is proved to be essentially self-adjoint.

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1. Introduction

Loop quantum cosmology (LQC) is a novel approach to the quantum theory of cosmology [8–10]. The loop quantum gravity (LQG) [2, 17, 18] inspired quantization of the symmetry reduced models which are the test field for the full theory. It also provides some very interesting results, like quantum geometry effects and the absence of singularity [11]. During recent years there has been progress in the area of LQC [1, 4, 11–13]. Although parts of the isotropic (\( k = 0 \)) and homogeneous sectors of quantum theory are well understood there is still a problem with Bianchi class B models such as the open Friedmann–Robertson–Walker (FRW) model (so-called \( k = -1 \))\(^1\). One of the reasons for that is the well-known problem concerning the Hamiltonian formulation of class B models. However, one can derive that the isotropic \( k = -1 \) model in terms of real Ashtekar variables has correct Hamiltonian formulation (see [19] and references therein). The second problem comes from the geometric difficulty. It is not clear how to introduce a loop suitable for the quantization purposes. Although there has been recent progress in the FRW hyperbolic model [19] a potential gap arose. The object which was quantized in [19] was the holonomy along an open curve. However, such a holonomy should not be considered as the components of the curvature. Moreover, the holonomies considered in [19] were used with respect to the \( \gamma K = A - \Gamma \) variables rather than to the \( A \) connection (there is nothing wrong with it logically,\(^1\) Strictly speaking the isotropic \( k = -1 \) model is derived from anisotropic Bianchi V.

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but it makes the relation to the full theory obscure). However, the quantum theory described in [19] does not suffer from singularities when gravity is coupled to a homogeneous massless scalar field and has a correct classical limit as well. In that sense the theory of quantum \( k = -1 \) model is correct and provides new quantum gravitational effects. One can conclude that [19] is a major candidate to replace the old classical \( k = -1 \) model with the new quantum one. We show in the present paper how this model can become conceptually closer to the full (LQG) theory by introducing a suitable loop, which leads to the same scalar constraint as in [19]. Using a new loop in quantum theory has one more advantage. The implementation of ‘improved dynamics’ introduced by Ashtekar et al [6] is direct and natural for what is missing in [19]. However, our results are valid again only for the \( \gamma K \) holonomies.

This paper is organized as follows. In section 2, we briefly discuss the classical hyperbolic geometry of the \( k = -1 \) model as well as its Hamiltonian formulation. In section 3, the quantum scalar constraint is constructed in detail by introduction of a suitable loop. Section 4 describes the properties of the scalar constraint operator with negative cosmological constant.

2. Classical theory

2.1. FRW models

The well-known isotropic and homogeneous sector of general relativity can be considered as three so-called FRW models, where the metric tensor has a form

\[
g = -N^2(t) \, dt^2 + a(t)^2 [ (1 - kr)^2 \, dr^2 + r^2 \, d\Omega^2 ].
\] (2.1)

Due to the large number of symmetries there is only one gravitational degree of freedom, the scale factor \( a(t) \) which is a function of arbitrarily chosen time coordinate \( t \). \( N(t) \) is referred to as a lapse function and does not enter the equations of motion as a dynamical variable. \( k \) is a number which can take only three values. Each of the values of \( k \) corresponds to a different intrinsic curvature of spatial manifold \( \Sigma \). Spatially flat, closed and open universes are described by \( k = 0, +1, -1 \), respectively. Einstein equations for (2.1) describe the dynamics of the scale factor

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho_{\text{matter}}.
\] (2.2)

This equation describes evolution of the universe filled by matter density \( \rho_{\text{matter}} \).

2.2. The \( k = -1 \) geometry

It is well known [20] that the spatial part of (2.1) can be written in terms of left invariant 1-forms as

\[
\tilde{q} = a^2(t) \delta_{ij} \, \omega^i_a \omega^j_b \, dx^a \, dx^b := a^2(t) q_{ab} \, dx^a \, dx^b,
\] (2.3)

where \( i, j = 1, 2, 3 \). These 1-forms \( \omega^i_a \) satisfy the Maurer–Cartan equation

\[
\partial_a \omega^i_b = -\frac{1}{2} C^i_{jk} \omega^j_a \omega^k_b,
\] (2.4)

where for \( k = -1 \) structure constants are given by

\[
C^i_{ij} = \delta^i_j \delta_{j1} - \delta^i_j \delta_{i1}.
\] (2.5)

The same structure constants occur in the algebra of left invariant vector fields\(^2\) on \( \Sigma \) (it is the well-known Bianchi class V algebra)

\[
[\epsilon_i, \epsilon_j] = C^k_{ij} \epsilon_k.
\] (2.6)

\(^2\) Left invariant 1-forms are dual to left invariant vector fields: \( \omega^i = \omega^i_a \partial_a \) where \( \omega^i_a = \omega^i_a \, dx^a \) and \( \epsilon_i = \epsilon_i^a \partial_a \).
Left invariant 1-forms and vector fields can be written in some coordinates $x^a$ as
\begin{align}
\alpha_1 &= \partial_1, \quad \omega^1 = dx^1, \\
\alpha_2 &= e^{-x^1} \partial_2, \quad \omega^2 = e^{x^1} dx^2, \\
\alpha_3 &= e^{x^1} \partial_3, \quad \omega^3 = e^{-x^1} dx^3.
\end{align}
(2.7)

One can check that equations (2.6) and (2.4) are satisfied by (2.7). Note that $C^i_{ij} \neq 0$. Such algebras belong to the class B in the Bianchi classification.

2.3. Classical dynamics

The canonical quantization of full general relativity, as well as the symmetry reduced models, is based on their Hamiltonian formulation [2, 18]. In terms of Ashtekar variables the full Hamiltonian for GR is a sum of constraints
\begin{equation}
H_{gr}^{\text{tot}} = \int_{\Sigma} d^3 x (N^i G_i + N_0 C_0 + N h_{ac}), \tag{2.8}
\end{equation}
where
\begin{equation}
C_0 = E_i b^{ab}, \quad G_i = D_a E^a_i := \partial_a E^a_i + \epsilon_{ijk} A^j_k E^a_l
\end{equation}
(2.9)
are called diffeomorphism and Gauss constraints, respectively. $F = dA + \frac{i}{2}[A, A]$ is a curvature of Ashtekar connection (2.11). The most complicated scalar constraint has a form
\begin{equation}
H_{gr} := \int_{\Sigma} d^3 x N(x) h_{ac}
\end{equation}
\begin{equation}
= \frac{1}{16 \pi G} \int_{\Sigma} d^3 x N(x) \left( \frac{E^a_i E^b_j}{\sqrt{\det E}} e^{ij}_k F^k_{ab} - 2(1 + \gamma^2) \frac{E^a_i E^b_j}{\sqrt{\det E}} K^i_{ja} K^j_{kb} \right). \tag{2.10}
\end{equation}
The Ashtekar variables $(A, E)$ are constructed from the triads (see [2, 18] for details) in the following way:
\begin{equation}
A^i_a = \Gamma^i_a + \gamma K^i_a \quad E^a_i = \sqrt{|\det q|} e^a_i, \tag{2.11}
\end{equation}
where $q_{ab} = \delta_{ij} e^i_a e^j_b$, and $\det q$ stands for the determinant of spatial metric $q_{ab}$. $A$ and $E$ take values in $su(2)$ algebra and $su(2)^*$ dual algebra, respectively.

In the case of the symmetry reduced model (for the case $k = -1$ see (2.3)) the above equations simplify dramatically. (2.11) reduce to
\begin{equation}
A^i_a = -\epsilon^{ij} \rho_{ab} e_{ij} + \bar{\epsilon}^a \omega^a \quad E^a_i = \tilde{\rho} \sqrt{|\det q|} \epsilon^a_i, \tag{2.12}
\end{equation}
where $\bar{\epsilon} = \gamma \bar{a}$ and $\tilde{\rho} = a^2$. Note that connection $A$ is not diagonal. This is a very different situation than $k = 0$ and $k = 1$. One can check that Gauss and diffeomorphism constraints in variables (2.12) are satisfied automatically. The only non-trivial one is the scalar constraint. From (2.10) and (2.12) we get
\begin{equation}
H_{gr} = -\frac{3V_0}{8 \pi G y^2} \sqrt{|\tilde{p}|} (\bar{\epsilon}^2 - \gamma^2), \tag{2.13}
\end{equation}
where $\bar{\epsilon}$ and $\tilde{p}$ are canonically conjugated $[\bar{\epsilon}, \tilde{p}] = \frac{8 \pi G y}{V_0}$ and $N(t) = 1$. $V_0$ is a volume of elementary cell (see [6, 7, 15] for details). In the presence of matter (in the isotropic and homogeneous case) the term $H_{\text{mat}} = V_0 |\tilde{p}|^{1/2} \rho_{\text{mat}}$ is added to the gravitational scalar constraint
\begin{equation}
H_{gr}^{\text{tot}} = -\frac{3V_0}{8 \pi G y^2} \sqrt{|\tilde{p}|} (\bar{\epsilon}^2 - \gamma^2) + V_0 |\tilde{p}|^{1/2} \rho_{\text{mat}}. \tag{2.14}
\end{equation}
If $H_{gr}^{\text{tot}}$ is constrained to vanish, one can easily check that the Friedmann equation (2.2) is recovered (for $k = -1$). We showed, followed then by [19], that indeed the isotropic Bianchi V (class B) model has correct Hamiltonian formulation.
3. Quantum theory

3.1. Kinematics

Quantum kinematics in the full LQR is based on the classical Poisson bracket algebra between holonomy along an edge \( h_u[A] \) and fluxes (\( E \) smeared on 2-surface) \( E(S) \) [2, 18]. In the isotropic and homogeneous models \( k = 0, 1 \) holonomies are reduced to the so-called almost periodic functions \( \sum \xi_j e^{\xi_j} \). Classical algebra \( \{ \hat{p}, e^{i\xi c^2} \} \) is then easy to quantize and quantum theory is placed in the Bohr compactification of a real line [1, 4, 7, 15]. In the case of \( k = -1 \) the situation is more complicated [19]. Because the \( A \) connection is no longer diagonal the classical Poisson bracket algebra of the scale factor \( p \) with holonomies along the symmetry directions fails to be almost periodic functions, as well as holonomies. This means that one cannot construct the quantum algebra in the same Hilbert space (Bohr compactification of a real line). One of the possibilities is to abandon the \( A \) variable and use the connection \( \gamma K_{ia} \) (which for the \( k = -1 \) is diagonal) [19]. Then holonomies along left invariant vector fields \( e^{-} a_i \partial^a \) for the form

\[
\hat{h}_i^{(\mu)} = \mathcal{P} \exp \left( \int_0^\mu ds \gamma K^a_i (e^a) \tau_k \right) = e^{i\mu \tau_i} = 1 \cos \frac{\mu \tilde{c}}{2} + 2 \tau_i \sin \frac{\mu \tilde{c}}{2}, \tag{3.1}
\]

where \( \mu \) is the length of an edge. Now it is easy to build quantum algebra of basic operators. The quantum version of the classical Poisson bracket \( \{ \hat{p}, e^{i\xi c^2} \} = -i \frac{\mu}{8\pi G} \) is as follows:

\[
[\hat{p}, e^{i\xi c^2}] = \frac{8\pi G h}{6} e^{i\xi c^2}. \tag{3.2}
\]

These operators act on vectors from the Hilbert space \( \mathcal{H}^{gr} = L^2(\mathcal{R}_{Bohr}, d\mu_{Bohr}) \). Eigenstates of \( \hat{p} \) constitute an orthonormal basis \( \langle \mu' | \mu \rangle = \delta_{\mu' \mu} \) in \( \mathcal{H}^{gr} \)

\[
\hat{p} | \mu \rangle = \mu \frac{8\pi \gamma}{6} | \mu \rangle. \tag{3.3}
\]

where we denoted \( G h := \frac{8\pi \gamma}{6} \). The spectrum of \( \hat{p} \) is then discrete. Each state from \( \mathcal{H}^{gr} \) can be decomposed in the \( | \mu \rangle \) basis as a countable sum \( | \psi \rangle = \sum | \mu \rangle \psi(\mu) | \mu \rangle \). The norm of \( | \psi \rangle \) is then defined as

\[
\langle \psi | \psi \rangle = \sum_{\mu} \hat{\psi}(\mu) \psi(\mu). \tag{3.4}
\]

Using the classical relation between the physical volume of the elementary cell and the scale factor \( V = |p|^{3/2} \) one can easily construct the volume operator which is also diagonal in the \( | \mu \rangle \) basis

\[
\hat{V} | \mu \rangle = | \mu |^{3/2} \left( \frac{8\pi \gamma}{6} \right)^{3/2} \hat{p} | \mu \rangle. \tag{3.5}
\]

The holonomy matrix element operator (3.1) acts as translations in \( | \mu \rangle \) basis

\[
\hat{e}^{i\xi c^2} | \mu \rangle = | \mu + \tilde{c} \rangle. \tag{3.6}
\]

3.2. The loop—preparation

The formula for the scalar constraint (2.10) is simplified for symmetry reduced \( k = -1 \) model to

\[
H^{gr} = -\frac{1}{16\pi G \gamma^2} \int d^3x N(t) \frac{E_j^a E_j^b}{\sqrt{|\det E|}} \epsilon^{ijk} (\Lambda^{jk}_{ab} - \gamma^2 \Omega^{jk}_{ab}). \tag{3.7}
\]
where the part of the curvature proportional to the $r^2$ does not have any dynamical degrees of freedom ($\Omega^{ab}_{\mu} = 2\partial_{[a} \Gamma^{b]}_{\mu} + \hat{c}_{ij} \Gamma^{ab}_{\mu} - \hat{c}_{ik} \Gamma^{ib}_{\mu}$, where $\Gamma$ is defined in (2.12)). The curvature 2-form of the $A - \Gamma = \gamma K$ connection reads

$$\Lambda^{k}_{ab} = \partial_{[a} \gamma K^{b]}_{k} + \hat{c}_{ij} \gamma^{2} K^{i}_{a} K^{j}_{b} = \left(-C_{i,j}^{k} \gamma + \hat{c}^{2} \epsilon_{ijk} \gamma \right) \omega^{l}_{a} \omega^{l}_{b},$$

(3.8)

where $C_{ij}^{k}$ are defined in (2.5). Naively, we could introduce the loop such that its holonomy gives two components, one proportional to structure constants $C_{ij}^{k}$ of symmetry algebra and the second proportional to structure constants of $su(2)$ algebra $\epsilon_{ij}^{k}$. However, putting equation (3.8) into the scalar constraint (3.7) we find

$$E_{a}^{b} E_{b}^{c} \epsilon^{ij}_{k} \Lambda^{k}_{ab} = E_{a}^{b} E_{b}^{c} \epsilon^{ij}_{k} \epsilon_{lm}^{k} \gamma^{2} K_{a}^{i} K_{b}^{m}.$$  

(3.9)

The term in curvature $\Lambda$ proportional to $C_{ij}^{k}$ vanishes. The only term proportional to $\hat{c}^{2} \epsilon_{ij}^{k}$ contributes to the classical Hamiltonian which generates dynamics. Let us denote

$$\Lambda_{\text{eff}k}^{ab} = \epsilon_{ij}^{k} \hat{c}^{2} \omega_{a}^{l} \omega_{b}^{l}.$$  

(3.10)

Note that in (3.8) there are only three possibilities for each value of indices $i$, $j$ and $k$: $C_{ij}^{k} = 0$ and $\epsilon_{ij}^{k} \neq 0$, or $C_{ij}^{k} \neq 0$ and $\epsilon_{ij}^{k} = 0$, or $C_{ij}^{k} = 0$ and $\epsilon_{ij}^{k} = 0$. There are no such $i$, $j$, $k$ for which $C_{ij}^{k}$ and $\epsilon_{ij}^{k}$ contribute at the same ‘time’. This is a very different situation to the $k = +1$ model, where the terms proportional to $\hat{c}$ and $\hat{c}^{2}$ contribute to one and the same component of the curvature 2-form. Moreover, from (3.9) it is clear that the term proportional to $C_{ij}^{k}$ drops out in the scalar constraint. It is enough when we find the loop corresponding only to the $\hat{c}^{2}$ term in the curvature (3.8), namely to (3.10).

### 3.3. The loop

Let us now construct such a loop. We use techniques developed in [7, 15]. The idea is to use the fact that left invariant vectors commute with the right invariant ones. The left invariant fields are defined in (2.7) and the right invariant vector fields have the form [20]

$$\eta_{1} = \partial_{x_{1}} - x^{2} \partial_{x_{2}} - x^{3} \partial_{x_{3}}, \quad \eta_{2} = \partial_{x_{2}}, \quad \eta_{3} = \partial_{x_{3}}.$$  

(3.11)

It is easy to show that $[\eta_{i}, \eta_{j}] = 0$ for every $i$ and $j$. From the geometric interpretation of a Lie bracket of two vector fields it is clear that an arbitrary pair of left and right invariant vector fields define a closed curve. Moreover, integral curves of those fields define, in a natural way, a surface spanned on the loop. In order to define coordinates on this surface we use a well-known fact that every vector field on a given manifold generates a one-parameter group of diffeomorphisms $\phi(t)$ which maps a given point on a manifold $x_{0}$ to $\tilde{x}(t)$

$$\phi^{(\gamma)}(\tilde{x}_{0}) = \tilde{x}(t).$$

(3.12)

If the given vector field has a form $V = f^{a}(x) \partial_{y}^{a}$ (where $f^{a}(x)$ are components in given coordinates) such a one-parameter group can be derived from the following condition:

$$f^{a}(x) = \frac{dx^{a}}{dt}.$$  

(3.13)

Let us consider now an arbitrary point $x_{0} = (x_{0}^{1}, x_{0}^{2}, x_{0}^{3})$ on $\Sigma$ in the coordinate chart given by (2.7) and (3.11). The one-parameter diffeomorphism generated by vector fields $\eta_{2}$ and $\eta_{3}$ can be written as

$$\phi^{(\eta_{2})}(x_{0}) = (x_{0}^{1}, re^{-x_{0}^{2}} + x_{0}^{2}, x_{0}^{3}) \quad \phi^{(\eta_{3})}(x_{0}) = (x_{0}^{1}, x_{0}^{2}, s + x_{0}^{3}).$$

(3.14)
The holonomy (with respect to $\gamma K$) along the left invariant vector fields $\gamma e_i$ is simple to calculate (3.1). What about the holonomy along the right invariant fields? If we start from some point $\bar{x}_0$ on $\Sigma$, using the formula (2.7) and (3.11) we will find that the holonomy along $\gamma h_3$ has a form

$$h(\gamma h_3) = \exp \left( se^{i\mu}e^{i\tau_3} \right).$$

(3.15)

Such a holonomy depends on a starting point $x_0$ (note that the length of integral curve of $\gamma h_3$ with respect to the background metric $h(\gamma h_3) = \int \sqrt{g_{ab}} \gamma h_3^a \gamma h_3^b = se^{i\mu}$).

Now, the loop is defined as follows: we start the holonomy around the loop from an arbitrary point $\bar{x}_0$ on $\Sigma$. Using (3.14) we get

1. from $(x_0^1, x_0^2, x_0^3)$ we move along $\gamma e_2$ to the point $(x_0^1, t e^{-i\mu} + x_0^2, x_0^3)$,
2. from $(x_0^1, t e^{-i\mu} + x_0^2, x_0^3)$ we move along $\gamma h_3$ to the point $(x_0^1, te^{-i\mu} + x_0^2, s + x_0^3)$,
3. from $(x_0^1, te^{-i\mu} + x_0^2, s + x_0^3)$ we move to the point $(x_0^1, x_0^2, s + x_0^3)$ along $\gamma e_2$ but in the opposite direction than in (1) and
4. from $(x_0^1, x_0^2, s + x_0^3)$ we move to the starting point $(x_0^1, x_0^2, x_0^3)$ along $\gamma h_3$, but in the opposite direction than in (2).

What about the area of the surface spanned by $\gamma e_2$ and $\gamma h_3$? The determinant of a metric tensor pulled back to the surface depends on the point of $\Sigma$, $h := \det(h_{ab}) = e^{i\mu}$

and the area (with respect to the background metric) is

$$Ar = \int dt \int ds \cdot \sqrt{h} = ts e^{i\mu}.$$ 

(3.17)

If we take the length of an integral curve generated by $\gamma e_2$ to be equal to $\mu$, we can always choose such $s$ in (3.17) and (3.15) that $se^{i\mu} = \mu$. The physical area of the surface can be constrained to be minimal $Ar_{\text{eff}} = \bar{p}\mu^2 = \Delta$ (see [6, 15] for details). Keeping this in mind and using (3.15), (3.1) and $\mu$ condition we get a holonomy around the loop

$$h^{(\mu)}_{23} = e^{-i \mu \tau_3} e^{-i \mu \tau_2} e^{i \mu \tau_1} e^{i \mu \tau_3}.$$ 

(3.18)

As in [6, 15], shrinking the loop to zero we get the curvature 2-form

$$\omega_2 \omega_3 \Lambda_{\text{eff}}^{ab} = -2 \lim_{\mu \to 0} \frac{\mu h_{23}}{V_0^{2/3}} \mu^2 \left[ \frac{\sin^2(\mu \tau_3)}{V_0^{2/3}} \right] \delta_3^a.$$ 

(3.19)

Due to homogeneity and isotropy $\omega_2 \omega_3 \Lambda_{\text{eff}}^{ab}$ determines the $\Lambda_{\text{eff}}^{ab}$ completely

$$\Lambda_{\text{eff}}^{ab} = \lim_{\mu \to 0} \frac{\sin^2(\mu \tau_3)}{V_0^{2/3}} \frac{\sin^2(\mu \tau_2)}{\mu^2}.$$ 

(3.20)

Note that when we shrink our loop to a point $\mu \to 0$ we recover the important part of curvature 2-form (3.10) and this is all we need. Since $\sin \mu \tau_3$, as well as $\mu \tau_2$, is a well-defined operator in kinematical Hilbert space, the curvature(3.20) corresponds to a well-defined operator in $\mathcal{H}^{\gamma} = L^2(\mathcal{R}_{\text{Bohr}}, \tilde{d}\mu_{\text{Bohr}})$.

3.4. Quantum dynamics

Using results from the previous section we can write the classical scalar constraint regularized and rescaled by a factor of $16\pi G$

$$C_{\text{reg}}^{\gamma} = -\frac{6}{\sqrt{p}} \left( \frac{\sin^2(\mu \tau_3)}{\mu^2} - V_0^{2/3} \gamma^2 \right).$$

(3.21)
where we have used the rescaled \( c \) and \( p \) variables. While the term
\[
\sin \bar{\mu} c = \frac{1}{2i} (\exp(i \bar{\mu} c) - \exp(-i \bar{\mu} c))
\] (3.22)
corresponds to the well-defined operator in \( \mathcal{H}^{gr} = L^2(\mathcal{R}_{\text{Bohr}}, d\mu_{\text{Bohr}}) \) (i.e. translations in volume \( v = K \text{sgn}(\mu)|\mu|^{3/2}, \) see [6, 15] for details) the \( \sqrt{p} \) is quantized from the classical expression in the spirit of the full LQG in the following manner:
\[
\text{sgn}(p)\sqrt{|p|} = \frac{4}{3\kappa\gamma\bar{\mu}} \sum_k \text{Tr}(h_k^{(p)} \{ h_k^{(p)}^{-1}, V \} \tau_k).
\] (3.23)

When we put equations (3.21), (3.22) and (3.23) together we obtain the \( \hat{C}_{gr} \) operator. Its action on state \( |\psi\rangle = \sum_{\nu} \psi(\nu) |\nu\rangle \) is given by
\[
\hat{C}_{gr} \psi(\nu) = f(+)\psi(\nu + 4) + f(0)\psi(\nu) + f(-)\psi(\nu - 4),
\] (3.24)
where the functions \( f(\pm) \) are defined as
\[
f(+) = 27/16 \sqrt{8\pi/6\gamma^3} kl_{|\nu + 2||\nu + 1| - |\nu + 3||},
\]
\[
f(-) = f(+)(\nu - 4),
\] (3.25)
\[
f(0) = -f(+) - f(-) + A(\nu)
\]
and \( A(\nu) \) is
\[
A(\nu) = 3\sqrt{8\pi/6} kl_{|\nu|^{1/3} ||\nu + 1| - |\nu - 1||}.
\] (3.26)

This way we have found the same scalar constraint operator as that in [19]! It is then possible to interpret the Vandersloot [19] operator in the spirit of the full (LQG) theory: the curvature 2-form is replaced by the holonomy along a closed curve in the crucial scalar constraint operator. However, the holonomy used in the present paper and in [19] is considered as a function of \( \gamma K \) rather than \( A \) variable.

4. Properties of the quantum scalar constraint operator—universe with negative cosmological constant

The (3.24) operator defined in the previous section has the following properties:

- It is densely defined in \( \mathcal{H}^{gr} = L^2(\mathcal{R}_{\text{Bohr}}, d\mu_{\text{Bohr}}) \) with the domain
\[
\mathcal{D} = \left\{ |\psi\rangle \in \mathcal{H}^{gr} : |\psi\rangle = \sum_{i=1}^n a_i |v\rangle, a_i \in C, n \in N \right\},
\] (4.1)
where \( |v\rangle \) is volume eigenstate.

- The operator \( \hat{C}_{gr} \) preserves every subspace \( \mathcal{H}_\epsilon \) of \( \mathcal{H}^{gr} \)
\[
\mathcal{H}_\epsilon = \text{Span}\{\epsilon + 4n) \in \mathcal{H}^{gr}, n \in N, \}
\] (4.2)

where \( \epsilon \) is an arbitrary real number. We have then the following decomposition into orthogonal subspaces:
\[
\mathcal{H}^{gr} = \bigoplus_\epsilon \mathcal{H}_\epsilon.
\] (4.3)

- \( \hat{C}_{gr} \) is symmetric with respect to scalar product
\[
\langle \psi | \phi \rangle = \sum_v \bar{\psi}(v)\phi(v).
\] (4.4)
4.1. Negative cosmological constant

The classical expression for the cosmological constant has a form $C_{\Lambda} = 2 \text{sgn}(\Lambda)|p|^{3/2}|\Lambda|$ (do not confuse $\Lambda$ with curvature in (3.7)) and its contribution to the scalar constraint is of the following form:

$$
C'_{\text{gr}} = -\frac{6}{7} \sqrt{p} (c^2 - V_0^2/\gamma^2) + 2 \text{sgn}(\Lambda)|p|^{3/2}|\Lambda|.
$$

(4.5)

Because the volume operator $\hat{V} = |\hat{p}|^{3/2}$ (3.5) is known, it is simple to define the $\hat{C}'_{\text{gr}}$ operator

$$
\hat{C}'_{\text{gr}} \psi(v) = \hat{C}_{\text{gr}} \psi(v) + 2 \text{sgn}(\Lambda)|\Lambda| \left(\frac{8\pi \gamma}{6}\right) \frac{r_0^3}{\mu K} |v| \psi(v),
$$

(4.6)

where we used the spectrum of the volume operator in terms of $\nu$.

Theorem: The operator $\hat{C}'_{\text{gr}}$ defined in the domain $D$ is essentially self-adjoint.

Proof. If we rewrite the $\hat{C}'_{\text{gr}}$ in the following form:

$$
\hat{C}'_{\text{gr}} = \hat{C} + \hat{C}_0,
$$

(4.7)

where $\hat{C}_0$ is essentially self-adjoint, then in order to prove the theorem it is enough to show that

$$
\|\hat{C}_0 \psi\|^2 \leq \|\hat{C}_0 \psi\|^2 + \beta \|\psi\|^2
$$

(4.8)

for each $\psi \in D$ and some constant $\beta$ ([16] V.4.6). The action of (4.6) can be written as

$$
\hat{C}'_{\text{gr}} \psi(v) = \hat{C} \psi(v) + \hat{C}_0 \psi(v),
$$

(4.9)

where

$$
\hat{C} \psi(v) = f_{(+)}(v) \psi(v + 4) + f_{(-)}(v) \psi(v - 4)
$$

$$
\hat{C}_0 \psi(v) = \left(-f_{(-)}(v) + f_{(+)}(v) + A(v) - 2|\Lambda| \left(\frac{8\pi \gamma}{6}\right) \frac{r_0^3}{\mu K} |v|\right) \psi(v).
$$

(4.10)

$\hat{C}_0$ is a multiplication operator so it is obviously essentially self-adjoint. For the norm of the $\hat{C}$ operator the following inequality holds:

$$
\|\hat{C} \psi\|^2 = \|f_{(+)} U_4 + f_{(-)} U_{-4} \psi\|^2 \leq 2 \langle \psi \left(f_{(+)}^2 + f_{(-)}^2\right) |\psi\rangle,
$$

(4.11)

where $U_{\pm 4}$ is a unitary translation operator in $v$ representation defined by $\exp(±2i\mu c)$ (see [6, 15] for details). The (4.11) was derived from the inequality $\|u + w\|^2 \leq 2\|u\|^2 + 2\|w\|^2$. To conclude, condition (4.8) is enough to show that $C_0^2$ (from (4.10)) can be written as follows:

$$
C_0^2 = 2f_{(+)}^2 + 2f_{(-)}^2 + f_1 + f_0,
$$

(4.12)

where $f_1 > 0$ is a function coming from the square of (4.10) and $f_0 > 0$ is a bounded function which we can always add and it does not change the self-adjointness of $\hat{C}'_{\text{gr}}$. \qed

5. Conclusions

In this paper, we have found a nice analogue of square used in [6]. Because of the non-commuting character of the left invariant fields $\psi_c$ in the hyperbolic $k = -1$ geometry, the loop was constructed using both left and right invariant fields as in [7, 15]. The important feature of this loop is the very natural implementation of the so-called $\hat{\mu}$ condition (i.e. the
physical area of the loop is constrained to be minimal and equal to the quantum of area [3] which leads to improved dynamics [6]). Perhaps it seems surprising that our quantum loop leads to exactly the same operator as introduced by Vandersloot in [19]. This comes from the fact that the trace of holonomy around our closed curve (3.18) is precisely the same as the trace of holonomy around the curve generated by each pair \( e_i, e_j \) for \( i \neq j \) which is not closed as was pointed out in p 8 of [19]. Because our two scalar constraint operators are exactly the same, the correct semi-classical limit of the quantum theory numerically established in [19] is completely insensitive with respect to our results. Moreover, from the point of view of quantum theory there are no differences between the Vandersloot model and ours. There are differences in initial concepts, but they lead to the same quantum theory. However, assumptions presented in this paper are more natural from the full theory point of view.

In section 4, we have found an essentially self-adjoint operator corresponding to the scalar constraint with negative cosmological constant, but what is the situation when \( \Lambda = 0? \) What about the more physical case of the positive cosmological constant? Unfortunately, the theorem described in ([16] V.4.6) cannot be applied to that case due to the fact that inequality (4.8) no longer holds for \( \Lambda \geq 0 \). Moreover, a similar problem arises in the \( k = 0 \) and \( k = 1 \) models with positive cosmological constant when one wants to apply the above theorem. We hope that future investigations will provide an answer to the question about self-adjoint extensions of scalar constraint operators with \( \Lambda > 0 \).

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