Stability of Anisotropic Cylinder with Zero Expansion

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Abstract

We study the dynamical instability of anisotropic collapsing cylinder with the expansion-free condition, which generates vacuum cavity within fluid distribution. The perturbation scheme is applied to distinguish Newtonian, post-Newtonian and post-post Newtonian terms, which are used for constructing dynamical equation at Newtonian and post-Newtonian regimes. We analyze the role of pressure anisotropy and energy density inhomogeneity on the stability of collapsing cylinder. It turns out that stability of the cylinder depends upon these physical properties of the fluid, not on the stiffness of the fluid.

Keywords: Relativistic fluid; Pressure anisotropy; Stability.

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1 Introduction

The study of relativistic anisotropic stars is important due to various applications in astrophysical scenarios. The existence of anisotropy in the star models is justified from physical phenomena like solid core (Kippenhahn

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and Weigert 1990), phase transition (Sokolov 1980), mixture of two fluids (Letelier 1980), slow rotation (Herrera and Santos 1995) and pion condensation (Sawyer 1972). Some authors (Lobo 2006; Sharma and Maharaj 2007; Thirukkanesh and Maharaj 2008; Sharif and Bhatti 2012) have also investigated the physical significance of charged self-gravitating configurations with anisotropic pressures.

It is believed that the expansion scalar characterizes the motion of fluid. Since it describes small change in the volume of the matter with respect to proper time, so it has a significant role in the evolution of the relativistic systems. In this scenario, Skripkin (1960) presented an expansion-free model for the non-dissipative isotropic fluid. Herrera et al. (2008) generalized this to anisotropic and dissipative fluid by defining the motion of expansion-free fluid with radial velocity in two different ways. The study of such models, where the expansion-free condition necessarily generates a vacuum cavity within the fluid distribution has been of particular interest. The evolution of cavities has also been shown under purely areal evolution condition (Herrera et al. 2010). The same authors (Herrera et al. 2009) found that Skripkin model for non-dissipative perfect fluid with constant energy density is inconsistent with junction conditions and concluded that in general for such a fluid distribution the expansion-free condition requires inhomogeneous energy density. Di Prisco et al. (2011) studied the spherically symmetric distribution of anisotropic fluid with inhomogeneous energy density and found some exact analytic models under expansion-free condition.

In general relativity (GR), the stability of self-gravitating objects against perturbation is an important issue. The existence of any static configuration has been constrained through its stability against perturbations. Some people (Ivanov 2002; Dev and Gleiser 2003; Mak and Harko 2003) have investigated the role of anisotropy for the stability of physical objects. Also, different physical properties of fluid such as heat conduction, radiation density and viscosity would increase or decrease the instability range of the system (Herrera et al. 1989; Chan et al. 1989; 1993; 1994). Horvat et al. (2011) explored the stability of anisotropic configurations under radial perturbations. Sharif and Kausar (2011) have studied dynamical instability of the expansion-free fluid in $f(R)$ gravity. Herrera et al. (2012a) found that instability of the expansion-free fluid depends exclusively on energy density inhomogeneity and pressure anisotropy. In recent papers, we have also explored the problem of dynamical instability of cylindrical (Sharif and Azam 2012a), spherical (Sharif and Azam 2012b) and thin-shell wormholes (Sharif
Recent contributions (Di Prisco et al. 2007; Sharif and Bashir 2012; Kalam et al. 2012; Hossein et al. 2012) indicate substantial importance of anisotropic pressure on the dynamical evolution of relativistic objects. The role of cylindrical gravitational waves in GR allows to study the dynamics of cylindrical geometry. For instance, investigation of naked singularity (Nolan 2002), emission of gravitational radiation (Sharif and Ahmad 2007) from cylindrical gravitational collapse, gravitational collapse of hollow cylinder (Nakao et al. 2009) and structure scalars for cylindrically symmetric metric with dissipative anisotropic fluid (Herrera et al. 2012b). Recently, Sharif and Yousaf (2012) have found some exact analytical expansion-free cylindrical symmetric models with anisotropic fluid. This shows that anisotropy is important in astrophysical process.

In this paper, we investigate the dynamical instability of cylindrical symmetric spacetime with expansion-free condition. The paper has the following format. Section 2 deals with properties of kinematical variables of the fluid, Einstein field equations and junction conditions. In section 3, we perturb the field equations, dynamical equations and the mass function. Section 4 describe the dynamical instability of expansion-free fluid at Newtonian and post Newtonian (pN) regimes. We summarize the results in the last section.

2 Matter Distribution of Collapsing Cylinder

In this section, we describe some basic kinematical variables associated with the collapsing cylinder in the comoving coordinate system. For this purpose, we consider a cylindrical geometry in the interior region given as (Sharif and Azam 2012a)

\[ ds^2 = -A^2(t, r)dt^2 + B^2(t, r)dr^2 + C^2(t, r)d\theta^2 + dz^2, \tag{1} \]

where \(-\infty < t < \infty, 0 \leq r < \infty, -\infty < z < \infty, 0 \leq \theta \leq 2\pi\). Here we assume the fluid to be locally anisotropic. The energy-momentum tensor for such a fluid has the form (Sharif and Yousaf 2012)

\[ T^{-}_{\alpha\beta} = (\mu + p_r)v_\alpha v_\beta + p_r g_{\alpha\beta} + (p_z - p_r)s_\alpha s_\beta + (p_\theta - p_r)k_\alpha k_\beta, \tag{2} \]

where \(\mu, p_r, p_\theta, p_z\) are the energy density and the principal stresses. The unitary vectors \(v^\alpha, k^\alpha\) and \(s_\alpha\) satisfy the following relations

\[ v^\alpha v_\alpha = -1, \quad s^\alpha s_\alpha = k^\alpha k_\alpha = 1, \quad s^\alpha k_\alpha = v^\alpha k_\alpha = v^\alpha s_\alpha = 0. \]
We define these quantities in comoving coordinates as
\[ v_\alpha = -A\delta_\alpha^0, \quad k_\alpha = C\delta_\alpha^2, \quad s_\alpha = \delta_\alpha^3. \quad (3) \]

The kinematical variables associated with the nonrotating fluid distributions are the expansion scalar \( \Theta \), the four acceleration \( a_\alpha \) and the shear tensor \( \sigma_{\alpha\beta} \) defined as follows, respectively
\[ \Theta = v'_\alpha, \quad a_\alpha = v_{\alpha;\beta}v^\beta, \quad \sigma_{\alpha\beta} = v_{(\alpha;\beta)} + a_{(\alpha}v_{\beta)} - \frac{1}{3}\Theta(g_{\alpha\beta} + v_\alpha v_\beta). \quad (4) \]
The corresponding expansion scalar and non-vanishing components of four acceleration and shear tensor are as follows
\[ \Theta = \frac{1}{A}\left(\dot{B}\frac{\dot{C}}{B} + \dot{C}\frac{\dot{B}}{C}\right), \quad a_1 = \frac{A'}{A}, \quad a^2 = a^0a_\alpha = \left(\frac{A'}{AB}\right)^2, \]
\[ \sigma_{11} = \frac{B^2}{3A}\left(2\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \quad \sigma_{22} = -\frac{C^2}{3A}\left(\frac{\dot{B}}{B} - 2\frac{\dot{C}}{C}\right), \quad \sigma_{33} = -\frac{1}{3A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right). \quad (5) \]

Here dot and prime mean differentiation with respect to \( t \) and \( r \), respectively.

### 2.1 Einstein Field Equations

The Einstein field equations for Eq.(1) yield
\[ \kappa\mu A^2 = \left(\frac{A}{B}\right)^2\left(\frac{B'C'}{BC} - \frac{C''}{C}\right) + \frac{\dot{B}\dot{C}}{BC}, \quad (6) \]
\[ 0 = \frac{\dot{C}}{C} - \frac{\dot{C}A'}{CA} - \frac{\dot{B}C'}{BC}, \quad (7) \]
\[ \kappa p_r B^2 = \left(\frac{B}{A}\right)^2\left(\frac{\dot{A}\dot{C}}{AC} - \frac{\dot{C}}{C}\right) + \frac{A'C''}{AC}, \quad (8) \]
\[ \kappa p_\varphi C^2 = \left(\frac{C^2}{AB}\right)\left(\frac{A''}{B} - \frac{\ddot{B}}{A} + \frac{\dot{A}'B'}{A^2} - \frac{A'B'}{B^2}\right), \quad (9) \]
\[ \kappa p_z = \frac{A''}{AB^2} - \frac{\ddot{B}}{A^2B} + \frac{\dot{A}'B'}{A^3B} - \frac{A'B'}{AB^3} + \frac{\dot{A}'C'}{A^2B} - \frac{\dot{C}}{A^2C} - \frac{\dot{C}B'}{B^3C} + \frac{C''}{B^2C} + \frac{A'C''}{AB^2C} - \frac{\dot{B}'C'}{A^2BC}. \quad (10) \]
The mass function proposed by Thorne (1965) in the form of gravitational C-energy per unit specific length of the cylinder is defined as

\[ E = \frac{1}{8}(1 - l^{-2}\nabla^\beta \tilde{r} \nabla_{\beta} \tilde{r}), \]  

satisfying the following relations

\[ \rho^2 = \eta(\theta)a^a \eta_{\theta}, \quad l^2 = \eta(z)a^a \eta_z, \quad \tilde{r} = \rho l, \]

where \( l, \rho, \tilde{r}, \eta_{\theta} \) and \( \eta_z \) mean specific length, circumference radius, areal radius and the Killing vectors, respectively for the cylindrical geometry. Thus the specific energy (Poisson 2004) of the collapsing cylinder in the interior region can be written as

\[ m(t, r) = El = \frac{l}{8} \left[ 1 + \left( \frac{\dot{C}}{A} \right)^2 - \left( \frac{C''}{B} \right)^2 \right]. \]

The conservation of energy-momentum tensor (\( T^{-\alpha\beta}_{\gamma\delta} = 0 \)) yield the following dynamical equations

\[ \dot{\mu} + (\mu + p_r) \frac{\dot{B}}{B} + (\mu + p_\theta) \frac{\dot{C}}{C} = 0, \]

\[ p'_r + (\mu + p_r) \frac{A'}{A} + (p_r - p_\theta) \frac{C''}{C} = 0. \]

Equation (13) with the expansion scalar becomes

\[ \dot{\mu} + A \mu \Theta + \frac{\dot{B}}{B} p_r + \frac{\dot{C}}{C} p_\theta = 0. \]

### 2.2 The Exterior Spacetime and Junction Conditions

Junction conditions are required to ensure the correct behaviour of an exterior spacetime source. The outcome of junction conditions provides the basic information about the pressure on the boundary surface and total energy entrapped inside the boundary surface. In case of cylindrical symmetry, there are two possible exteriors namely the Levi-Civita (Levi-Civita 1917) metric (static case) and Einstein-Rosen (Einstein and Rosen 1937) metric (non-static case). Here, we take a timelike 3D hypersurface \( \Sigma \) which divides
the Riemannian spacetime into two regions interior $M^-$ and exterior $M^+$ each containing $\Sigma$ as a part of the boundary. The junction conditions join these two regions into one across the surface of discontinuity. We consider two cylindrical regions: the interior region defined by Eq. (1) and the exterior region in the retarded time coordinate $\nu$ is given by static cylindrical black hole (Chao-Guang 1995)

$$ds^2_+ = \left(\frac{2M}{R}\right) dv^2 - 2d\nu dR + R^2(d\theta^2 + \gamma^2 dz^2),$$  \hspace{1cm} (17)

where $\gamma$ is a constant and has the dimension of $\frac{1}{r}$ and $M$ is the mass associated with the exterior geometry.

For the smooth matching of adiabatic cylindrical solution to the static cylindrical solution, we consider the Darmois conditions (1927). The continuity of the first and second fundamental forms yields the following results on the $\Sigma^{(e)}$ (detail is given in Sharif and Azam 2012a)

$$\frac{dt}{d\tau} \Sigma^{(e)} = A(t, r)^{-1}, \quad C(t, r) \Sigma^{(e)} = R(\nu) = \frac{1}{\gamma},$$ \hspace{1cm} (18)

$$\left(\frac{dv}{d\tau}\right)^{-2} \Sigma^{(e)} = \left(\frac{-2M}{R} + 2\frac{dR}{d\nu}\right), \quad m - M \Sigma^{(e)} = \frac{l}{8}, \quad p_r \Sigma^{(e)} = 0.$$ \hspace{1cm} (19)

The above equations show conditions for the smooth matching of interior and exterior regions at the boundary $\Sigma^{(e)}$. The formation of internal vacuum cavity due to the expansion-free condition within the fluid distribution yields the following conditions

$$m(t, r) \Sigma^{(i)} = 0, \quad p_r \Sigma^{(i)} = 0,$$ \hspace{1cm} (20)

where Minkowski spacetime within cavity is matched to the matter distribution at the internal hypersurface $\Sigma^{(i)}$.

3 The Perturbation Scheme

It is well established that any physical model is subject to $\mu > 0$, $\mu > p$ and stability of perturbed modes. In fact, perturbations are small deviations from static background spacetime caused by external forces. Here, we use the perturbation scheme (Sharif and Azam 2012a, 2012b) to perturb the
field equations, dynamical equations, mass function and expansion scalar up to first order in $\varepsilon$. Initially, we assume that the given fluid has radial dependence (hydrostatic equilibrium form). Afterwards, all these functions depend upon $t$. Thus, the metric and material functions has the following form

$$A(t, r) = A_0(r) + \varepsilon T(t)a(r),$$  \hspace{1cm} (21)

$$B(t, r) = B_0(r) + \varepsilon T(t)b(r),$$  \hspace{1cm} (22)

$$C(t, r) = C_0(r) + \varepsilon T(t)c(r),$$  \hspace{1cm} (23)

$$\mu(t, r) = \mu_0(r) + \varepsilon \bar{\mu}(t, r),$$  \hspace{1cm} (24)

$$p_r(t, r) = p_{r0}(r) + \varepsilon \bar{p}_r(t, r),$$  \hspace{1cm} (25)

$$p_\theta(t, r) = p_{\theta0}(r) + \varepsilon \bar{p}_\theta(t, r),$$  \hspace{1cm} (26)

$$m(t, r) = m_0(r) + \varepsilon \bar{m}(t, r),$$  \hspace{1cm} (27)

$$\Theta(t, r) = \varepsilon \bar{\Theta}(t, r),$$  \hspace{1cm} (28)

where $0 < \varepsilon \ll 1$. We choose $C_0(r) = r$ as the radial coordinate. Using Eqs. (21)-(26), we have static configuration of the field equations as follows

$$\kappa \mu_0 = \frac{1}{B_0^2} \left( \frac{1}{r} \frac{B'_0}{B_0} \right),$$ \hspace{1cm} (29)

$$\kappa p_{r0} = \frac{1}{B_0^2} \left( \frac{1}{r} \frac{A'_0}{A_0} \right),$$ \hspace{1cm} (30)

$$\kappa p_{\theta0} = \frac{1}{A_0 B_0} \left( \frac{A''_0}{B_0} - \frac{A'_0' B'_0}{B_0^2} \right).$$ \hspace{1cm} (31)

The corresponding perturbed field equations becomes

$$\kappa \bar{\mu} = \frac{T}{B_0^2} \left[ \frac{B'_0}{B_0} \left( \frac{\bar{c}}{r} \right)' - \frac{1}{r} \left( \frac{b}{B_0} \right)' - \frac{\bar{c}''}{r} \right] - 2 \frac{\kappa b}{B_0} \mu_0 T,$$ \hspace{1cm} (32)

$$0 = 2 \frac{T}{A_0 B_0} \left[ \frac{\bar{c}}{r} - \frac{b}{r B_0} - \left( \frac{A'_0}{A_0} - \frac{1}{r} \frac{\bar{c}}{r} \right) \right],$$ \hspace{1cm} (33)

$$\kappa \bar{p}_r = - \frac{T}{A_0^2} \left( \frac{\bar{c}}{r} \right)' + \frac{T}{A_0 B_0} \left[ \frac{A'_0}{A_0} \left( \frac{\bar{c}}{r} \right)' + \frac{1}{r} \left( \frac{a}{A_0} \right)' \right] - 2 \frac{\kappa b}{B_0} p_{r0} T,$$ \hspace{1cm} (34)

$$\kappa \bar{p}_\theta = - \frac{T}{A_0^2} \left( \frac{b}{B_0} \right)' + \frac{T}{A_0 B_0} \left[ a'' \frac{A_0 B'_0}{B_0^2} - \frac{A_0 B'_0}{B_0^2} \left( \frac{a}{A_0} \right)' - \frac{a A''_0}{A_0 B_0} \right] - 2 \frac{\kappa b}{B_0} p_{\theta0} T.$$ \hspace{1cm} (35)
The static and perturbed configurations of the dynamical equations are

\[ p_r' + (\mu_0 + p_{r0}) \frac{A_0'}{A_0} + (p_{r0} - p_{\theta 0}) \frac{1}{r} = 0, \]  
\[ \tilde{\mu} + (\mu_0 + p_{r0}) \frac{b}{B_0} \tilde{T} + (\mu_0 + p_{\theta 0}) \frac{\tilde{c}'}{r} \tilde{T} = 0, \]  
\[ \bar{p}_r' + (\mu_0 + p_{r0}) \left( \frac{a}{A_0} \right)' + (\bar{\mu} + \bar{p}_r) \left( \frac{A_0'}{A_0} \right) + (p_{r0} - p_{\theta 0}) \left( \frac{\tilde{c}'}{r} \right)' + \frac{(\bar{p}_r - \bar{p}_0)}{r} = 0. \]

Integration of Eq. (37) yields

\[ \bar{\mu} = -(\mu_0 + p_{r0}) \frac{b}{B_0} T - (\mu_0 + p_{\theta 0}) \frac{\tilde{c}'}{r} T. \]  

The unperturbed and perturbed configuration of the mass function give

\[ m_0 = \frac{l}{8} \left[ 1 - \frac{1}{B_0^2} \right], \quad \bar{m} = -\frac{lT}{4B_0^2} \left[ \tilde{c}' - \frac{b}{B_0} \right]. \]  

The junction condition (19) with Eq. (25) provides the following relations

\[ p_{r0} \Sigma^{(e)} = 0, \quad \bar{p}_r \Sigma^{(e)} = 0. \]  

Using these results in Eq. (34), it follows that

\[ \tilde{T} - \alpha T \Sigma^{(e)} = 0, \]  

where

\[ \alpha(r) = \left( \frac{A_0}{B_0} \right)^2 \left[ A_0' \left( \frac{\tilde{c}'}{r} \right)' + \frac{1}{r} \left( \frac{a}{A_0} \right) \right] \left( \frac{r}{\tilde{c}} \right). \]

The general solution of the above equation becomes

\[ T(t) = c_1 \exp(\sqrt{\alpha_{\Sigma^{(e)}}} t) + c_2 \exp(-\sqrt{\alpha_{\Sigma^{(e)}}} t), \]

where \( c_1 \) and \( c_2 \) are arbitrary constants. Here, we are interested to find a solution which is real and shows a static system that starts collapsing at \( t = -\infty \) when \( T(-\infty) = 0 \). We assume all the radial functions on the hypersurface to be positive such that \( \alpha_{\Sigma^{(e)}} > 0 \). The corresponding solution of Eq. (42) has been obtained by setting \( c_1 = -1 \) and \( c_2 = 0 \)

\[ T(t) = -\exp(\sqrt{\alpha_{\Sigma^{(e)}}} t). \]
4 Expansion-free Condition and Dynamical Instability

In this section, we obtain an expansion-free dynamical equation and investigate the role of anisotropy on the dynamical instability of collapsing cylinder. For the sake of instability conditions at Newtonian and pN, we need to distinguish Newtonian, pN and post-post Newtonian (ppN) terms appearing in the dynamical equation. This has been done by converting the dynamical equation into c.g.s units. Inserting the value of $B_0^2$ from Eq.(40) in (29) and (30), it follows that

\[
\frac{B'_0}{B_0} = \frac{\kappa l \mu_0 r}{l - 8m_0}, \quad \frac{A'_0}{A_0} = \frac{\kappa l p_{r0} r}{l - 8m_0}.
\]  

(45)

Using the value of \(\frac{A'_0}{A_0}\) from the above equation in Eq.(36), the dynamical equation in relativistic units becomes

\[
p'_{r0} = -\left[\frac{\kappa l p_{r0} r}{l - 8m_0}\right] (\mu_0 + p_{r0} - \frac{p_{r0} - p_{\theta 0}}{r}).
\]  

(46)

and in c.g.s. units, it implies that

\[
p'_{r0} = -G\left[\frac{\kappa l p_{r0} r}{l - 8Gc^{-2}m_0}\right] (\mu_0 + c^{-2} p_{r0} - \frac{p_{r0} - p_{\theta 0}}{r}).
\]  

(47)

Expanding the above equation up to order $c^{-4}$, we obtain terms with order $c^0$, $c^{-2}$ and $c^{-4}$ corresponding to Newtonian, pN and ppN order terms given by

\[
p'_{r0} = -\kappa G r p_{r0} \mu_0 + \frac{(p_{r0} - p_{\theta 0})}{r} - \frac{\kappa G r p_{r0}}{c^2} \left(p_{r0} + \frac{8Gm_0 \mu_0}{l}\right)
- \frac{\kappa G r p_{r0} r}{c^4} \left(\frac{8Gm_0 p_{r0}}{l} + \frac{64G^2 m_0^2 \mu_0}{l}\right).
\]  

(48)

The perturbed form of the expansion scalar with Eq.(28) is

\[
\bar{\Theta} = \frac{T}{A_0} \left(\frac{b}{B_0} + \frac{\bar{c}}{r}\right).
\]  

(49)
Applying the expansion-free condition ($\bar{\Theta} = 0$), it follows that

$$\frac{b}{B_0} = -\frac{\bar{c}}{r}.$$  \hfill (50)

We would like to mention here that the expansion-free fluids evolve without being compressed (Herrera et al. 2008) and halt the formation of apparent horizons resulting a naked singularity (Joshi et al. 2002). Under the expansion-free condition, Eq.(33) becomes

$$2\frac{\dot{T}}{r^3B_0}\left(\frac{r^2\bar{c}}{A_0}\right)' = 0 \implies \bar{c} = k_1\frac{A_0}{r^2},$$  \hfill (51)

where $k_1$ is a constant of integration. We relate $\bar{p}_r$ and $\bar{\mu}$ through adiabatic index $\Gamma$ (Herrera et al. 1989)

$$\bar{p}_r = \Gamma \frac{p_{r0}}{\mu_0 + p_{\theta0}} \bar{\mu},$$  \hfill (52)

where $\Gamma$ measures the stiffness of the fluid and is assumed to be constant throughout the instability analysis. Using Eq.(50), the perturbed energy density $\bar{\mu}$ takes the form

$$\bar{\mu} = (p_{r0} - p_{\theta0})\frac{\bar{T}}{r}.$$  \hfill (53)

This shows the relevance of static background anisotropy with the perturbed energy density, which can be seen through Eq.(16) with the expansion-free condition. Inserting this value of $\bar{\mu}$ in Eq.(52), it follows

$$\bar{p}_r = 2\Gamma\frac{p_{r0}}{\mu_0 + p_{r0}}(p_{r0} - p_{\theta0})\frac{\bar{T}}{r}\bar{c}.$$  \hfill (54)

We note that $\bar{p}_r$ and $\bar{\mu}A_0'$ are of ppN order terms, hence discard these terms in the below stability analysis. From Eq.(36), we obtain

$$\frac{A_0'}{A_0} = -\frac{1}{\mu_0 + p_{r0}} \left[ p_{r0}' + \frac{(p_{r0} - p_{\theta0})}{r} \right].$$  \hfill (55)

Using the expansion-free condition and value of $\bar{p}_\theta$ from Eq.(38) along with Eq.(11) in (35), the dynamical expansion-free equation leads to

$$\kappa(\mu_0 + p_{r0})r \left( \frac{a}{A_0} \right)' + \kappa(p_{r0} - p_{\theta0})r \left( \frac{\bar{c}}{r} \right)' - \frac{\alpha_{\Sigma(c)}}{A_0^2} \frac{\bar{c}}{r} - \frac{2\kappa\bar{c}}{r} p_{\theta0}$$

$$- \frac{1}{A_0B_0} \left[ \frac{a''}{B_0} - \frac{A_0 B_0'}{B_0 B_0} \left( \frac{a}{A_0} \right)' + A_0' \left( \frac{\bar{c}}{r} \right)' - \frac{a}{A_0} \frac{A_0'}{B_0} \right] = 0.$$  \hfill (56)
The expressions for $\frac{A''_0}{A_0 B_0}$ and $(\frac{A}{A_0})'$ from Eqs. (51) and (54) can be written as

$$
\frac{A''_0}{A_0 B_0^2} = \frac{A'_0 B'_0}{A_0 B_0^3} + \kappa p_{\theta 0},
$$

and

$$
(\frac{a}{A_0})' = r \left[ \alpha_{\Sigma e} \left( \frac{\bar{c}}{r} \right) \left( \frac{B_0}{A_0} \right)^2 - \frac{A'_0}{A_0} \left( \frac{\bar{c}}{r} \right)' - 2 \kappa B_0^2 B_{\theta 0} \frac{\bar{c}}{r} \right].
$$

Inserting Eqs. (51), (57) and (58) in (56), it yields

$$
\kappa (\mu_0 + p_{\theta 0}) \left[ k_1 \frac{\alpha_{\Sigma e}}{r} \left( \frac{B_0^2}{A_0} \right) + 3 k_1 \frac{A_0}{r^2} \left( \frac{A'_0}{A_0} \right) - 2 k_1 A_0 B_0^2 \frac{\kappa p_{\theta 0}}{r} \right] + k_1 \kappa (p_{\theta 0} - p_{\theta 0}) A_0 \left( \frac{A'_0}{A_0} \right) \left( \frac{B_0}{A_0} \right) - \frac{\alpha_{\Sigma e}}{r} \frac{k_1}{A_0} \left( \frac{A'_0}{A_0} \right) - 2 k_1 \kappa p_{\theta 0} \frac{A_0}{r^2} - \frac{a''}{A_0 B_0^2} \\
+ \frac{1}{B_0^2} \frac{B'_0}{B_0} \left[ k_1 \frac{\alpha_{\Sigma e}}{r^3} \left( \frac{B_0^2}{A_0} \right) + 3 k_1 \frac{A_0}{r^4} \left( \frac{A'_0}{A_0} \right) - 2 k_1 A_0 B_0^2 \frac{\kappa p_{\theta 0}}{r^3} \right] \\
+ \frac{a}{A_0} \left\{ \frac{1}{B^2} \left( \frac{A'_0}{A_0} \right) \left( \frac{B'_0}{B_0} \right) + \kappa p_{\theta 0} \right\} = 0,
$$

(59)

where we have neglected the ppN terms $\bar{p}_{\alpha} \frac{A'_0}{A_0}$ and $\bar{p}_r$. Notice that the terms in Eq. (59) generally depend upon the radial function. This shows the relevance of areal radius in the onset of stability of collapsing fluid.

In order to have instability conditions, we define the following choice of radial functions $a = a_0 + a_1 r$, where $a_0, a_1$ are arbitrary positive constants. With these choices of radial functions and metric functions $A_0 = 1 - \frac{G m_0}{c^2 r}$, $B_0 = 1 + \frac{G m_0}{c^2 r}$, the dynamical equation (59) at pN approximation becomes

$$
\kappa \mu_0 \left( 1 + \frac{3 m_0}{r} \right) \frac{k_1 \alpha_{\Sigma e}}{r} - \frac{3 k_1 \kappa}{r^2} \left( \frac{1 - m_0}{r} \right) \left( \frac{p'_{\theta 0} + p_{\theta 0} - p_{\theta 0}}{r} \right) \\
- \frac{2 k_1 \kappa^2 p_{\theta 0} \mu_0}{r} \left( 1 + \frac{m_0}{r} \right) - \frac{k_1 \kappa (p_{\theta 0} - p_{\theta 0})}{r^2} \left( 1 - \frac{m_0}{r} \right) \left\{ \frac{p'_{\theta 0}}{\mu_0} + \frac{p_{\theta 0} - p_{\theta 0}}{r \mu_0} \right\} \\
+ \frac{3}{r} \right\} \frac{k_1 \alpha_{\Sigma e}}{r^3} \left( 1 + \frac{m_0}{r} + \frac{m_0^2}{r^2} \right) - \frac{2 k_1 \kappa p_{\theta 0}}{r^3} \left( 1 - \frac{m_0}{r} \right) + \frac{k_1 \alpha_{\Sigma e} \kappa \mu_0}{r}.
$$
\[ \times \left(1 + \frac{m_0}{r} + \frac{8m_0}{l}\right) - \frac{3k_1\kappa}{r^2} \left(1 - \frac{3m_0}{r} + \frac{8m_0}{l}\right) \left(\frac{p'_{r_0} + p_{r_0} - p_{\theta 0}}{r}\right) \\
- \frac{2k_1\kappa^2 \mu_0 p_{\theta 0}}{r} \left(1 - \frac{m_0}{r} + \frac{8m_0}{l}\right) - \kappa r(a_0 + a_1 r) \left(1 - \frac{m_0}{r} + \frac{8m_0}{l}\right) \\
\times \left(\frac{p'_{r_0} + p_{r_0} - p_{\theta 0}}{r\mu_0} - \frac{k_1}{r^3} \left(1 - \frac{3m_0}{r}\right) \left(\frac{p'_{r_0} + p_{r_0} - p_{\theta 0}}{\mu_0 r}\right) \right) \\
\times \left\{ \left(\frac{p'_{r_0}}{\mu_0} + \frac{p_{r_0} - p_{\theta 0}}{r\mu_0}\right) + \frac{3}{r} \right\} + \kappa p_{\theta 0}(a_0 + a_1 r) \left(1 + \frac{m_0}{r}\right) = 0, \quad (60) \]

where we assume \( G = 1 = c \) in the stability analysis. We assume \( \mu_0 \gg p_{r_0},\ \mu_0 \gg p_{\theta 0} \) and neglect the ppN order terms to obtain instability conditions at N approximation as

\[ -\kappa (6k_1 + (a_0 + a_1 r)r^3) r|p'_{r_0}| - k_1^2 (9p_{r_0} - 7p_{\theta 0}) - \kappa r^3 (a_0 + a_1 r) \]
\[ \times \left(p_{r_0} - 2p_{\theta 0}\right) - k_1 \alpha_{\Sigma(i)} + 2k_1 \alpha_{\Sigma(i)} \mu_0 r^2 - k_1 \alpha_{\Sigma(i)} \frac{m_0}{r} = 0. \quad (61) \]

Using the fact that the radial pressure is decreasing during the expansion-free collapse, i.e., \( p'_{r_0} < 0 \) and Eq.\( (13) \), the above equation implies

\[ \kappa (6k_1 + (a_0 + a_1 r)r^3) r|p'_{r_0}| = k_1 \alpha_{\Sigma(i)} + k_1 \kappa(9p_{r_0} - 7p_{\theta 0}) \]
\[ + \kappa r^3 (a_0 + a_1 r)(p_{r_0} - 2p_{\theta 0}) + 2k_1 \alpha_{\Sigma(i)} \left(\frac{l\kappa}{8r} \int_{r_{\Sigma(i)}}^{r} \mu_0 r dr - \kappa \mu_0 r^2\right) \quad (62) \]

We are interested to find the instability of the expansion-free fluid at N approximation. For this purpose, we require all the terms in the above equation positive. Since the quantities \( a_0, a_1 \) are positive constants, thus the instability is subject to the positivity of each term on the right hand side of (62). This has been assured through \( p_{r_0} > 2p_{\theta 0} \) and \( 9p_{r_0} > 11p_{\theta 0} \). For the positivity of the last term, we consider power law solution for the energy density, i.e., \( \mu_0 = \zeta r^n \), where \( \zeta > 0 \) is a constant and \( n \) has range in the interval \((\infty, \infty)\). The last term of the above equation for \( n \neq -2 \) implies

\[ \frac{l\kappa}{8r} \int_{r_{\Sigma(i)}}^{r} \mu_0 r dr - \kappa \mu_0 r^2 = \frac{l\kappa \zeta}{8r(n + 2)} \left(r^{n+2} - \frac{8(n + 2)}{l} r^{n+3} - r^{n+2}_{\Sigma(i)}\right), \quad (63) \]

which reduces to \( r(1 - \frac{8r(n+2)}{l}) > r_{\Sigma(i)} \). For \( n = -2 \), it gives

\[ \frac{l\kappa}{8r} \int_{r_{\Sigma(i)}}^{r} \mu_0 r dr - \kappa \mu_0 r^2 = \frac{l\kappa \zeta}{8r} \left[ \log \left(\frac{r}{r_{\Sigma(i)}}\right) - \frac{8r}{7}\right], \quad (64) \]
yielding $re^{\frac{2n}{r}} > r_{\Sigma(i)}$. These two inequalities describe the ranges of instability of the expansion-free fluid, which generally depends upon the given value of $n$ and specific length of the cylinder.

5 Summary

In this paper, we have explored the issue of dynamical instability of expansion-free fluid with cylindrical symmetry. We have assumed collapsing cylinder with inhomogeneous energy density and locally anisotropic pressure, which is compatible with expansion-free models. The matching conditions at the internal hypersurface (separating fluid distribution to cavity) and external hypersurface (dividing the vacuum solution to fluid) have been formulated. We have used perturbation scheme to distinguish the Newtonian, pN and ppN terms.

The relevance of expansion-free models in the study of astrophysical objects stems from the fact that these models may be helpful in the evolution of cosmic voids. Voids are underdense regions that fill 40 − 50% volume of the universe. The application of such a study would be seen in the cosmological phenomena where evolution of cavity in the given fluid distribution has been the subject of interest (Randall et al. 2011). Generally, the adiabatic index determines the dynamical instability of self-gravitating objects. For instance, the isotropic cylinder and sphere are unstable for $\Gamma < 1$ and $\Gamma < \frac{4}{3}$, respectively (Sharif and Azam 2012a; Chandrasekhar 1964). We have shown that Eq.(62) is independent of the adiabatic index, which supports the fact that expansion-free collapse continues without contraction of the fluid. Hence, the adiabatic index is irrelevant in this study. In our case, we have found that the stability of the fluid is affected by the local anisotropy of pressure and energy density.

Recently, we have explored the dynamical instability of expansion-free fluid with spherically symmetry and found that the dynamical instability is subject to $pr_0 > \frac{2}{5}p_{\perp 0}$ and has range in the interval $(-2, 2)$ (Sharif and Azam 2012b). In this work, we have found that the instabilities of Eq.(62) at Newtonian regime correspond to the constraints $pr_0 > 2\rho_{\theta 0}$, $9pr_0 > 7\rho_{\theta 0}$, $r(1 - \frac{8r(n+2)}{l})^{\frac{1}{n+2}} > r_{\Sigma(i)}$ and $re^{\frac{2n}{r}} > r_{\Sigma(i)}$. This shows that the instability of the fluid has no particular range and depends upon the value of $n$ and the specific length of the cylinder.

Moreover, these inequalities describe the instability ranges of expansion-
free fluid as well as the associated cavity with the fluid. The violation of any of the inequality would lead to diminish the instability. From these results, it is evident that physical properties of fluid, i.e., pressure anisotropy and energy density play a vital role in the onset of dynamical instability, which supports the fact that these quantities are important in studying the dynamics of self-gravitating objects. Finally, we remark that the stability analysis of the expansion-free fluid would be the same at pN approximation, where the relativistic correction terms $\frac{m}{\alpha}$ have been taken into account.

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