Uniqueness, Lipschitz stability and reconstruction for the inverse optical tomography problem

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Abstract. In this paper, we consider the inverse problem of recovering a diffusion \( \sigma \) and absorption coefficients \( q \) in steady-state optical tomography problem from the Neumann-to-Dirichlet map. We first prove a Global uniqueness and Lipschitz stability estimate for the absorption parameter provided that the diffusion \( \sigma \) is known. Then, we prove a Lipschitz stability result for simultaneous recovery of \( \sigma \) and \( q \). In both cases the parameters belong to a known finite subspace with a priori known bounds. The proofs relies on a monotonicity result combined with the techniques of localized potentials. To numerically solve the inverse problem, we propose a Kohn-Vogelius-type cost functional over a class of admissible parameters subject to two boundary value problems. The reformulation of the minimization problem via the Neumann-to-Dirichlet operator allows us to obtain the optimality conditions by using the Fréchet differentiability of this operator and its inverse. The reconstruction is then performed by means of an iterative algorithm based on a quasi-Newton method. Finally, we illustrate some numerical results.

Keywords: Optical tomography, Inverse problem, Uniqueness, Lipschitz stability, Monotonicity, Localized potentials.

1. Introduction

In this paper, we consider the inverse problem of recovering the parameters \( \sigma(x) \) and \( q(x) \) in the elliptic partial differential equation
\[
- \nabla \cdot (\sigma \nabla u) + q u = 0 \quad \text{in } \Omega,
\]
from the knowledge of all possible Cauchy data on the boundary \( \partial \Omega, \sigma \partial_\nu u|_{\partial \Omega}, u|_{\partial \Omega} \).

Problem [1] can be viewed as steady-state diffusion optical tomography, where light propagation is modeled by a diffusion approximation and the excitation frequency is set to zero. Here \( u \) represents the density of photons, \( \sigma \) the diffuse coefficient and \( q \) the optical absorption. This problem arises in medical imaging and in geophysics, for example, in reflection seismology assuming a description in terms of time-harmonic
scalar waves. For a full description of optical tomography, we refer the reader to the topical reviews of Arridge \cite{11} and Gibson, Hebben and Arridge \cite{2}.

The inverse problem of recovering $q$ from the knowledge of the Dirichlet to Neumann map was first introduced (in a slightly different setting) by Calder'\`on in \cite{3}. The uniqueness issue was treated by Sylvester and Uhlmann in \cite{4}. For more recent result on uniqueness, we refer the reader to \cite{5}. A log-type stability estimate was derived by Alessandrini in \cite{6}. As shown by Mandache \cite{7}, this log-type estimate is optimal. Thus for arbitrary potentials $q$, Lipschitz stability cannot hold. Motivated by this, and following analogous results in electrical impedance tomography and elasticity \cite{8,9,10}, here we will study the question whether the coefficient $q$ can be uniquely and stably reconstructed.

As mentioned in \cite{11}, the inverse problem of simultaneous reconstruction of $\sigma$ and $q$ is in general not uniquely solvable, i.e., it is not possible to uniquely determine both $\sigma$ and $q$ from boundary data of $u$ provided that $\sigma$ and $q$ are smooth. The reason is that a diffusion coefficient can be transformed into an absorption coefficient by setting

$$v := \sqrt{\sigma} u$$

which transforms equation (1) into

$$-\Delta v + cv = 0, \quad c = \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}} + \frac{q}{\sigma}.$$ 

If $\sigma = 1$ in a neighborhood of $\partial \Omega$, then the boundary values remain unchanged. Hence, boundary measurements can only contain information about $c$, from which one cannot extract $\sigma$ and $q$. Despite this negative theoretical result, a prominent result by Harrach \cite{12} demonstrates that uniqueness holds for piecewise constant diffusion and piecewise analytic absorption coefficients. The author proves that under this condition both parameters are simultaneously uniquely determined by knowledge of all possible pairs of Neumann and Dirichlet boundary values $\sigma \partial_{\nu} |_S, u |_S$, of solutions $u$ of (1), and $S$ is a non-empty subset of $\partial \Omega$.

In this paper we prove a global uniqueness and Lipschitz stability for the inverse problem of recovering $q$ then we prove a Lipschitz stability estimate for the inverse problem of recovering $\sigma$ and $q$ simultaneously. The proof rely on a monotonicity estimates combined with the techniques of of localized potentials.

The idea of using monotonicity and localized potentials method has lead to a several results for inverse coefficient problems; see for instance \cite{13,14,15,16,17,18,19,20}. Together with the recent results \cite{21,9,8,10}, this work shows that this idea can also be used to prove Uniqueness and Lipschitz stability results.

Lipschitz stability estimates for inverse and ill-posed problems are usually based on constructive approaches involving Carleman estimates or quantitative estimates of unique continuation \cite{22,23,24,25,26,27,28}. For some applications these constructive approaches also allowed to quantify the asymptotic behavior of the Lipschitz constant; see for instance \cite{29}.
Our approach on proving Lipschitz stability is relatively simple compared to previous works. The main tools are: standard (non quantitative) unique continuation, the monotonicity result and the method of localized potentials.

For the numerical solution, we reformulate the inverse problem into a minimization problem using a Kohn-Vogelius functional, and use a quasi-Newton method which employs the analytic gradient of the cost function and the approximation of the inverse Hessian is updated by BFGS scheme [30]. Let us stress that this numerical part does not build on the theoretical results but rather approaches the problem from a heuristic numerical side to demonstrate that useful numerical reconstructions are indeed possible. It remains a challenging open task how to unite the theoretical and numerical approaches in order to find rigorously justified reconstruction methods that work well in practically relevant settings.

The paper is organized as follows. In section 2, we introduce the forward, the Neumann-to-Dirichlet operator and the inverse problem. Section 3 and 4 contain the main theoretical tools for this work. Section 3 is devoted to the reconstruction of the absorption coefficient assuming that the diffusion coefficient is known. We formulate our main theoretical results: a global uniqueness result and a Lipschitz stability estimate. We show a monotonicity relation and we prove a Runge approximation result. Then we deduce the existence of localized potentials and prove the global uniqueness and Lipschitz stability estimate. Section 4 is concerned with the reconstruction of the diffusion and the absorption coefficients simultaneously. We first show a monotonicity result between the diffusion and absorption coefficients and the Neumann-to-Dirichlet operator and prove the existence of localized potentials. Then, we prove the Lipschitz stability estimate. In section 5, we introduce the minimization problem, and we compute the first order optimality condition. In the last section, satisfactory numerical results for two-dimensional problem are presented.

2. Problem formulation

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$), be a bounded domain with smooth boundary $\partial \Omega$. For $\sigma, q \in L^\infty_+(\Omega)$, where $L^\infty_+$ denotes the subset of $L^\infty$-functions with positive essential infima, we consider the following problem with Neumann boundary data $g \in L^2(\partial \Omega)$:

$$\begin{align*}
\begin{cases}
-\nabla \cdot (\sigma \nabla u) + qu &= 0 \quad \text{in } \Omega, \\
\sigma \partial_\nu u &= g \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}$$

(2)

where $\nu$ is the unit normal vector to $\partial \Omega$. The weak formulation of problem (2) reads

$$\int_{\Omega} \sigma \nabla u \cdot \nabla w \, dx + \int_{\Omega} quw \, dx = \int_{\partial \Omega} gw \, ds \quad \text{for all } w \in H^1(\Omega).$$

(3)

Using the Riesz representation theorem (or the Lax-Milgram-Theorem), it is easily seen that (3) is uniquely solvable and that the solution depends continuously on $g \in L^2(\partial \Omega)$ and $\sigma, q \in L^\infty_+(\Omega)$. When dealing with different source coefficients or Neumann data, we also denote the solution by $u_{(g)}^{(\sigma, q)}$. 

We denote by $\Lambda(\sigma,q)$ the so-called Neumann-to-Dirichlet map:

$$\Lambda(\sigma,q) : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega)$$

$$g \longmapsto u|_{\partial\Omega},$$

Thus the inverse problem we are concerned with is the following:

Find the parameters $\sigma,q$ from the knowledge of the map $\Lambda(\sigma,q)$. \hfill (4)

We will consider diffusion and absorption parameters that are a priori known to belong to a finite dimensional set of piecewise-analytic functions and that are bounded from above and below by a priori known constants. To that end, we first define piecewise-analyticity as in [8, Definition 2.1]

**Definition 1.** (a) A subset $\Gamma \subseteq \partial\Omega$ of the boundary of an open set $\Omega \subset \mathbb{R}^n$ is called a smooth boundary piece if it is a $C^\infty$-surface and $\Omega$ lies on one side of it, i.e. if for each $z \in \Gamma$ there exists a ball $B(z)$ and function $\gamma \in C^\infty(\mathbb{R}^{n-1}, \mathbb{R})$ such that

$$\Gamma = \partial\Omega \cap B(z) = \{ x \in B(z) : x_n = \gamma(x_1, \ldots, x_{n-1}) \},$$

$$\Omega \cap B(z) = \{ x \in B(z) : x_n > \gamma(x_1, \ldots, x_{n-1}) \}.$$

(b) $\Omega$ is said to have smooth boundary if $\partial\Omega$ is a union of smooth boundary pieces. $\Omega$ is said to have piecewise smooth boundary if $\partial\Omega$ is a countable union of the closures of smooth boundary pieces.

(c) A function $\varphi \in L^\infty(\Omega)$ is called piecewise constant if there exists finitely many pairwise disjoint subdomains $\Omega_1, \ldots, \Omega_N \subset \Omega$ with piecewise smooth boundaries, such that $\Omega = \Omega_1 \cup \ldots \cup \Omega_N$ and $\varphi|_{\Omega_i}$ is constant, $i = 1, \ldots, N$.

(d) A function $\varphi \in L^\infty(\Omega)$ is called piecewise analytic if there exist finitely many pairwise disjoint subdomains $\Omega_1, \ldots, \Omega_N \subset \Omega$ with piecewise smooth boundaries, such that $\Omega = \Omega_1 \cup \ldots \cup \Omega_N$, and $\varphi|_{\Omega_i}$ has an extension which is (real-)analytic in a neighborhood of $\overline{\Omega_i}$, $i = 1, \ldots, N$.

As mentioned in [8], it is not clear whether the sum of two piecewise-analytic functions is always piecewise-analytic, i.e. whether the set of piecewise-analytic functions is a vector space. However, this can be guaranteed with a slightly stronger definition of piecewise analyticity (see [31, lemma 1]). Therefore, we make the following definition.

**Definition 2.** A set $\mathcal{F} \subseteq L^\infty(\Omega)$ is called a finite-dimensional subset of piecewise-analytic functions if its linear span

$$\text{span} \mathcal{F} = \left\{ \sum_{j=1}^k \lambda_j f_j : k \in \mathbb{N}, \lambda_j \in \mathbb{R}, f_j \in \mathcal{F} \right\} \subseteq L^\infty(\Omega)$$

contains only piecewise-analytic functions and $\dim(\text{span} \mathcal{F}) < \infty$. 
3. Recovery of the absorption

In this section, we assume that $\sigma = \sigma_0 \chi_{\Omega \setminus \omega} + \sigma_1 \chi_{\omega}$, and $q = q \chi_{\omega}$, where $\sigma_0, \sigma_1$ are positive constants and $\omega \Subset \Omega$. We aim to recover the absorption parameter $q \in L^\infty_+ (\omega)$ from the NtD operator

$$\Lambda(q) : L^2(\partial \Omega) \to L^2(\partial \Omega) : g \mapsto u|_{\partial \Omega}.$$  

provided that $\sigma$ is known.

Given a finite-dimensional subset $\mathcal{F}$ of piecewise analytic functions and two constants $b > a > 0$, we denote the set $\mathcal{F}_{[a,b]} := \{ q \in \mathcal{F} : a \leq q(x) \leq b, \text{ for all } x \in \omega \}$.

Throughout this paper, the domain $\omega$, the finite-dimensional subset $\mathcal{F}$ and the bounds $b > a > 0$ are fixed, and the constants in the Lipschitz stability results will depend on them. Our first results show Uniqueness and Lipschitz stability for the inverse absorption problem in $\mathcal{F}_{[a,b]}$ when the complete infinite-dimensional NtD-operator is measured. We will show a monotonicity result:

$$q_1 \leq q_2 \implies \Lambda(q_1) \geq \Lambda(q_2) \text{ in the sense of quadratic forms},$$

and using monotonicity and localized potentials, we deduce the following uniqueness and stability result for determining $q$ from $\Lambda(q)$.

**Theorem 1** (Uniqueness). For $q_1, q_2 \in L^\infty_+ (\omega)$ that are piecewise analytic,

$$\Lambda(q_1) = \Lambda(q_2) \text{ if and only if } q_1 = q_2.$$

**Theorem 2** (Lipschitz stability). There exists a constant $C > 0$ such that

$$\|q_1 - q_2\|_{L^\infty(\omega)} \leq C\|\Lambda(q_1) - \Lambda(q_2)\|_{L(L^2(\partial \Omega))}, \quad \text{for all } q_1, q_2 \in \mathcal{F}_{[a,b]}.$$

3.1. Runge approximation and uniqueness

We first note the following unique continuation property. For every open connected subset $\mathcal{O} \subset \Omega$, only the trivial solution of

$$-\text{div}(\sigma \nabla u) + qu = 0 \text{ in } \mathcal{O},$$

vanishes on an open subset of $\mathcal{O}$ or possesses zero Cauchy data on a smooth, open part of $\partial \mathcal{O}$. When $\sigma$ is Lipschitz and $q$ is bounded, this property is proven in Miranda [32, Thm. 19, II]. It can be extended to the case of piecewise analytic $\sigma$ and $q$ by sequentially solving Cauchy problems (see [33]).

We will deduce the uniqueness theorem from the following Runge approximation result.

**Theorem 3** (Runge approximation). Let $q \in L^\infty_+ (\omega)$ be piecewise analytic. For all $f \in L^2(\omega)$ there exists a sequence $(g_n)_{n \in \mathbb{N}} \subset L^2(\partial \Omega)$ such that the corresponding solutions $u^{(g_n)}$ of (2) with boundary data $g_n$, $n \in \mathbb{N}$, fulfill

$$u^{(g_n)}|_{\omega} \to f \quad \text{in } L^2(\omega).$$
Proof. We introduce the operator
$$A : L^2(\omega) \to L^2(\partial \Omega), \quad f \mapsto Af := v|_{\partial \Omega},$$
where $v \in H^1(\Omega)$ solves
$$\int_{\Omega} \sigma \nabla v \cdot \nabla w \, dx + \int_{\omega} qw \, ds = \int_{\Omega} f w \, dx \quad \text{for all } w \in H^1(\Omega). \tag{5}$$
Let $g \in L^2(\partial \Omega)$ and $u \in H^1(\Omega)$ be the corresponding solution of problem (2). Then the adjoint operator of $A$ is characterized by
$$\int_{\omega} (A^* g) f \, ds = \int_{\partial \Omega} (Af) g \, ds = \int_{\Omega} \sigma \nabla u \cdot \nabla v \, dx + \int_{\omega} qw \, ds$$
$$= \int_{\omega} f u \, ds, \quad \text{for all } f \in L^2(\omega), \tag{6}$$
which shows that $A^* : L^2(\partial \Omega) \to L^2(\omega)$ fulfills $A^* g = u|_{\omega}$. The assertion follows if we can show that $A^*$ has dense range, which is equivalent to $A$ being injective.

To prove this, let $v|_{\partial \Omega} = Af = 0$ with $v \in H^1(\Omega)$ solving (5). Since (5) also implies that $\sigma \partial_v v|_{\partial \Omega} = 0$, and $\Omega \setminus \omega$ is connected, it follows by unique continuation that $v|_{\Omega \setminus \omega} = 0$ and thus $v^+|_{\partial \omega} = 0$. Since $v \in H^1(\Omega)$ this also implies that $v^-|_{\partial \omega} = 0$, and together with (5) we obtain that $v|_{\omega} \in H^1(\omega)$ solves
$$- \nabla \cdot (\sigma \nabla v) + q v = 0 \quad \text{in } \omega$$
with homogeneous Dirichlet boundary data $v|_{\partial \omega} = 0$. Hence, $v|_{\omega} = 0$, so that $v = 0$ almost everywhere in $\Omega$. From (5) it then follows that $\int_{\omega} f w \, ds = 0$ for all $w \in H^1(\Omega)$ and thus $f = 0$.

Proof of Theorem 4. For absorption parameters $q_1, q_2 \in L^\infty_+(\omega)$ and Neumann data $g, h \in L^2(\partial \Omega)$ we denote the corresponding solutions of (2) by $u^1_1, u^1_2, u^2_1$, and $u^2_2$ respectively.

The variational formulation (3) yields the orthogonality relation
$$\int_{\partial \Omega} h (\Lambda(q_2) - \Lambda(q_1)) g \, ds$$
$$= \int_{\partial \Omega} h \Lambda(q_2) g \, ds - \int_{\partial \Omega} g \Lambda(q_1) h \, ds = \int_{\partial \Omega} hu_2^2 \, ds - \int_{\partial \Omega} gu_1^h \, ds$$
$$= \int_{\Omega} \sigma \nabla u_1^h \cdot \nabla u_2^h \, dx + \int_{\omega} q_1 u_1^h u_2^h \, dx - \left( \int_{\Omega} \sigma \nabla u_2^h \cdot \nabla u_1^h \, dx + \int_{\omega} q_2 u_2^h u_1^h \, dx \right) \tag{7}$$
$$= \int_{\omega} (q_1 - q_2) u_1^h u_2^h \, dx.$$}

This shows that $\Lambda(q_1) = \Lambda(q_2)$ implies that
$$\int_{\omega} (q_1 - q_2) u_1^h u_2^h \, dx = 0, \quad \text{for all } g, h \in L^2(\partial \Omega).$$

Using the Runge approximation result in theorem 3, this yields that $(q_1 - q_2) u_1^h = 0$ (a.e.) in $\omega$ for all $h \in L^2(\partial \Omega)$, and using theorem 3 again, this implies $q_1 = q_2$. \qed
Theorem 2 will be proven in the following subsection.

3.2. Monotonicity, localized potentials and Lipschitz stability

To prove the Lipschitz stability result in Theorem 2, we first show a monotonicity estimate between the absorption coefficient and the Neumann-to-Dirichlet operator, and deduce the existence of localized potentials from the Runge approximation result.

**Lemma 1** (Monotonicity estimate). Let \( q_1, q_2 \in L^\infty_+(\omega) \) be two absorption parameters, let \( g \in L^2(\partial\Omega) \) be an applied boundary current, and let \( u_2 := u^g_{q_2} \in H^1(\Omega) \) solve (8) for the boundary current \( g \) and the absorption parameter \( q_2 \). Then

\[
\int_\Omega (q_1 - q_2)u^2_2 \, dx \geq \int_{\partial\Omega} g (\Lambda(q_2) - \Lambda(q_1)) \, g \, ds \geq \int_\Omega \left( q_2 - \frac{q_2}{q_1} \right) u^2_2 \, dx. \tag{8}
\]

**Proof.** Let \( u_1 := u^g_{q_1} \in H^1(\Omega) \). From the variational equation, we deduce

\[
\int_\Omega \sigma \nabla u_1 \cdot \nabla u_2 \, dx + \int_\omega q_1 u_1 u_2 \, dx = \int_{\partial\Omega} g \Lambda(q_2) g \, ds = \int_\Omega |\nabla u_2|^2 \, dx + \int_\omega q_2 u^2_2 \, dx.
\]

Thus

\[
\int_\Omega |\nabla (u_1 - u_2)|^2 \, dx + \int_\omega q_1 (u_1 - u_2)^2 \, dx = \int_\Omega |\nabla u_1|^2 \, dx + \int_\omega q_1 u^2_1 \, dx + \int_\Omega |\nabla u_2|^2 \, dx + \int_\omega q_1 u^2_2 \, dx - 2 \int_\Omega |\nabla u_2|^2 \, dx - 2 \int_\omega q_2 u^2_2 \, dx = \int_{\partial\Omega} g \Lambda(q_1) g \, ds - \int_{\partial\Omega} g \Lambda(q_2) g \, ds + \int_\omega (q_1 - q_2) u^2_2 \, dx.
\]

Since the left-hand side is nonnegative, the first asserted inequality follows. Interchanging \( q_1 \) and \( q_2 \), we get

\[
\int_{\partial\Omega} g \Lambda(q_2) g \, ds - \int_{\partial\Omega} g \Lambda(q_1) g \, ds = \int_\Omega |\nabla (u_2 - u_1)|^2 \, dx + \int_\omega q_2 (u_2 - u_1)^2 \, dx - \int_\omega (q_2 - q_1) u^2_1 \, dx
\]

\[
= \int_\Omega |\nabla (u_2 - u_1)|^2 \, dx + \int_\omega \left( q_2 u^2_2 - 2q_2 u_1 u_2 + q_1 u^2_1 \right) \, dx
\]

\[
= \int_\Omega |\nabla (u_2 - u_1)|^2 \, dx + \int_\omega q_1 \left( u_1 - \frac{q_2}{q_1} u_2 \right)^2 \, ds + \int_\omega \left( q_2 - \frac{q_2}{q_1} \right) u^2_2 \, dx.
\]

Since the first two integrals on the right-hand side are nonnegative, the second asserted inequality follows. \( \square \)

Note that we call Lemma 1 a monotonicity estimate because of the following corollary:

**Corollary 1** (Monotonicity). For two absorption parameters \( q_1, q_2 \in L^\infty_+(\omega) \)

\( q_1 \leq q_2 \) implies \( \Lambda(q_1) \geq \Lambda(q_2) \) in the sense of quadratic forms. \( \tag{9} \)
Let us stress, however, that Lemma 1 holds for any \( q_1, q_2 \in L^\infty_+ (\omega) \) and does not require \( q_1 \leq q_2 \) or \( q_1 \geq q_2 \).

The existence of localized potentials follows from the Runge approximation property as in [9, Lemma 4.3].

Lemma 2 (Localized potentials). Let \( q \in L^\infty_+ (\omega) \) be piecewise analytic, and let \( \mathcal{O} \subseteq \omega \) be a subset with positive boundary measure. Then there exists a sequence \( (g_n)_{n \in \mathbb{N}} \subset L^2 (\partial \Omega) \) such that the corresponding solutions \( u^{(g_n)} \) of (2) fulfill
\[
\lim_{n \to \infty} \int_{\mathcal{O}} |u^{(g_n)}|^2 \, ds = \infty \quad \text{and} \quad \lim_{n \to \infty} \int_{\omega \setminus \mathcal{O}} |u^{(g_n)}|^2 \, ds = 0.
\]

Proof. Using the Runge approximation property in Theorem 3 we find a sequence \( \tilde{g}_n \in L^2 (\partial \Omega) \) so that the corresponding solutions \( u^{(\tilde{g}_n)} \) fulfill
\[
u^{(\tilde{g}_n)} \big|_{\omega} \to \chi_{\mathcal{O}} \left( \int_{\mathcal{O}} dx \right)^{1/2} \quad \text{in} \quad L^2 (\omega).
\]

Hence
\[
\lim_{n \to \infty} \int_{\mathcal{O}} |u^{(\tilde{g}_n)}|^2 \, dx = 1 \quad \text{and} \quad \lim_{n \to \infty} \int_{\omega \setminus \mathcal{O}} |u^{(\tilde{g}_n)}|^2 \, dx = 0,
\]
so that
\[
g_n := \frac{\tilde{g}_n}{\left( \int_{\omega \setminus \mathcal{O}} \tilde{g}_n^2 \, dx \right)^{1/4}},
\]
has the desired property
\[
\lim_{n \to \infty} \int_{\mathcal{O}} |u^{(g_n)}|^2 \, dx = \lim_{n \to \infty} \frac{\int_{\mathcal{O}} |u^{(\tilde{g}_n)}|^2 \, dx}{\left( \int_{\omega \setminus \mathcal{O}} |u^{(\tilde{g}_n)}|^2 \, dx \right)^{1/2}} = \infty,
\]
\[
\lim_{n \to \infty} \int_{\omega \setminus \mathcal{O}} |u^{(g_n)}|^2 \, dx = \lim_{n \to \infty} \left( \int_{\omega \setminus \mathcal{O}} |u^{(\tilde{g}_n)}|^2 \, dx \right)^{1/2} = 0.
\]

Now, we are ready to proof Theorem 2.

Proof of Theorem 2. Let \( \mathcal{F} \subset L^\infty (\omega) \) be a finite dimensional subspace of piecewise analytic functions, \( b > a > 0 \), and
\[
q_1, q_2 \in \mathcal{F}_{[a,b]} = \{ q \in \mathcal{F} : \quad a \leq q(x) \leq b \ \text{for all} \ x \in \omega \}.
\]

For the ease of notation, we write in the following
\[
\|q_1 - q_2\| := \|q_1 - q_2\|_{L^\infty (\Omega)} \quad \text{and} \quad \|g\| := \|g\|_{L^2 (\partial \Omega)}.
\]
Since $\Lambda(q_1)$ and $\Lambda(q_2)$ are self-adjoint, we have that
\[
\|\Lambda(q_2) - \Lambda(q_1)\|_s = \sup_{|g| = 1} \left| \int_{\partial \Omega} g (\Lambda(q_2) - \Lambda(q_1)) g \, ds \right|
\]
\[
= \sup_{|g| = 1} \max \left\{ \int_{\partial \Omega} g (\Lambda(q_2) - \Lambda(q_1)) g \, ds, \int_{\partial \Omega} g (\Lambda(q_1) - \Lambda(q_2)) g \, ds \right\}.
\]
Using the first inequality in the monotonicity relation (8) in Lemma 1 in its original form, and with $q_1$ and $q_2$ interchanged, we obtain for all $g \in L^2(\partial \Omega)$
\[
\int_{\partial \Omega} g (\Lambda(q_2) - \Lambda(q_1)) g \, ds \geq \int_{\omega} (q_1 - q_2) |u^{(q_1)}|_2^2,
\]
\[
\int_{\partial \Omega} g (\Lambda(q_1) - \Lambda(q_2)) g \, ds \geq \int_{\omega} (q_2 - q_1) |u^{(q_2)}|_2^2,
\]
where $u^{(q_1)}$, $u^{(q_2)} \in H^1(\Omega)$ denote the solutions of (2) with Neumann data $g$ and absorption parameter $q_1$ and $q_2$, resp. Hence, for $q_1 \neq q_2$, we have
\[
\frac{\|\Lambda(q_2) - \Lambda(q_1)\|_s}{\|q_1 - q_2\|} \geq \sup_{|g| = 1} \phi \left( g, \frac{q_1 - q_2}{\|q_1 - q_2\|_{L^\infty(\omega)}}, q_1, q_2 \right),
\]
where (for $g \in L^2(\partial \Omega)$, $\zeta \in \mathcal{F}$, and $\kappa_1, \kappa_2 \in \mathcal{F}_{[a,b]}$)
\[
\phi (g, \zeta, \kappa_1, \kappa_2) := \max \left\{ \int_{\omega} \zeta |u^{(q_1)}|_2^2 \, dx, \int_{\omega} (-\zeta) |u^{(q_2)}|_2^2 \, dx \right\}.
\]
Introduce the compact set
\[
\mathcal{C} = \{ \zeta \in \text{span } \mathcal{F} : \|\zeta\|_{L^\infty(\omega)} = 1 \}.
\]
Then, we have
\[
\frac{\|\Lambda(q_2) - \Lambda(q_1)\|_s}{\|q_1 - q_2\|} \geq \sup_{|g| = 1} \phi (g, \zeta, \kappa_1, \kappa_2)
\]
\[
\geq \inf_{\zeta \in \mathcal{C}} \sup_{\kappa_1, \kappa_2 \in \mathcal{F}_{[a,b]}} \|g\| = 1 \phi (g, \zeta, \kappa_1, \kappa_2).
\]
The assertion of Theorem 2 follows if we can show that the right hand side of (26) is positive. Since $\phi$ is continuous, the function
\[
(\zeta, \kappa_1, \kappa_2) \mapsto \sup_{|g| = 1} \phi (g, \zeta, \kappa_1, \kappa_2)
\]
is semi-lower continuous, so that it attains its minimum on the compact set $\mathcal{C} \times \mathcal{F}_{[a,b]} \times \mathcal{F}_{[a,b]}$. Hence, to prove Theorem 5, it suffices to show that
\[
\sup_{|g| = 1} \phi (g, \zeta, \kappa_1, \kappa_2) > 0 \quad \text{for all } (\zeta, \kappa_1, \kappa_2) \in \mathcal{C} \times \mathcal{F}_{[a,b]} \times \mathcal{F}_{[a,b]}.
\]
To show this, let $(\zeta, \kappa_1, \kappa_2) \in \mathcal{C} \times \mathcal{F}_{[a,b]} \times \mathcal{F}_{[a,b]}$. Since $\|\zeta\|_{L^\infty(\omega)} = 1$, there exists a subset $\mathcal{O} \subseteq \omega$ with positive measure and $0 < \Theta < 1$ such that either

(a) $\zeta(x) \geq \Theta$ for all $x \in \mathcal{O}$, or (b) $-\zeta(x) \geq \Theta$ for all $x \in \mathcal{O}$.
In case (a), we use the localized potentials sequence in Lemma 4 to obtain a boundary current \( \hat{g} \in L^2(\partial \Omega) \) with
\[
\int_{\Omega} |u^{(g)}_{\kappa_1}|^2 \, ds \geq \frac{1}{\Theta} \quad \text{and} \quad \int_{\Omega} |u^{(g)}_{\kappa_1}|^2 \, ds \leq \frac{1}{2},
\]
so that (using again \( \|\zeta\|_{L^\infty(\omega)} = 1 \))
\[
\phi(\hat{g}, \zeta, \kappa_1, \kappa_2) \geq \int_\omega \zeta |u^{(g)}_{\kappa_1}|^2 \, dx \geq \Theta \int_\Omega |u^{(g)}_{\kappa_1}|^2 \, dx - \int_{\Omega\setminus\Omega} |u^{(g)}_{\kappa_1}|^2 \, dx \geq \frac{1}{2}.
\]
In case (b), we can analogously use a localized potentials sequence for \( \kappa_2 \), and find \( \hat{g} \in L^2(\partial \Omega) \) with
\[
\phi(\hat{g}, \zeta, \kappa_1, \kappa_2) \geq \int_\omega (-\zeta) |u^{(g)}_{\kappa_2}|^2 \, dx \geq \Theta \int_\Omega |u^{(g)}_{\kappa_2}|^2 \, dx - \int_{\Omega\setminus\Omega} |u^{(g)}_{\kappa_2}|^2 \, dx \geq \frac{1}{2}.
\]
Hence, in both cases,
\[
\sup_{\|g\|=1} \phi(g, \zeta, \kappa_1, \kappa_2) \geq \phi \left( \frac{\hat{g}}{\|\hat{g}\|}, \zeta, \kappa_1, \kappa_2 \right) = \frac{1}{\|\hat{g}\|^2} \phi(\hat{g}, \zeta, \kappa_1, \kappa_2) > 0,
\]
so that Theorem 2 is proven.

4. Simultaneous recovery of diffusion and absorption

The inverse problem of recovering \( \sigma \) and \( q \) simultaneously is known to be an ill-posed problem and stability results can only be obtained under a-priori assumptions.

For our problem, we will prove a stability result under the assumption that the coefficients belong to an a-priori known finite-dimensional subspace, that upper and lower bounds are a-priori known, and that a de definiteness condition holds.

As in the last section the main tools to prove the stability are the monotonicity and the existence of localized potentials, which are the subject of the following subsection.

4.1. Monotonicity and localized potentials

**Lemma 3** (Monotonicity). Let \( \sigma_1, \sigma_2, q_1, q_2 \in L^\infty(\Omega) \). Then
\[
\int_\Omega [(\sigma_2 - \sigma_1)|\nabla u_1|^2 + (q_2 - q_1)u_1^2] \, dx \geq \langle g, (\Lambda(\sigma_1, q_1) - \Lambda(\sigma_2, q_2)) g \rangle
\]
\[
\quad \geq \int_\Omega [(\sigma_2 - \sigma_1)|\nabla u_2|^2 + (q_2 - q_1)u_2^2] \, dx, \tag{13}
\]
\[
\langle g, (\Lambda(\sigma_1, q_1) - \Lambda(\sigma_2, q_2)) g \rangle \geq \int_\Omega \left[ \left( \frac{\sigma_1 - \sigma_2}{\sigma_2} \right) |\nabla u_1|^2 + \left( q_1 - \frac{q_1^2}{q_2} \right) u_1^2 \right] \, dx
\]
\[
\quad = \int_\Omega \left[ \frac{\sigma_1}{\sigma_2} (\sigma_2 - \sigma_1)|\nabla u_1|^2 + \frac{q_1}{q_2} (q_2 - q_1)u_1^2 \right] \, dx, \tag{14}
\]
for all \( g \in L^2(\partial \Omega) \) where \( u_1, u_2 \in H^1(\Omega) \) are the solutions of (2) with Neumann boundary data \( g \) on \( \partial \Omega \), and coefficients (\( \sigma_1, q_1 \), resp., (\( \sigma_2, q_2 \)).
Proof. The proof of (13) is given in [12, Lemma 4.1]. Following the proof of Lemma 1, we can easily deduce (14).

Theorem 4 (Localized potentials). Let $\sigma, q \in L_+^\infty(\Omega)$ that are piecewise analytic and $D \Subset \Omega$ be non empty open set such that $\Omega \setminus \overline{D}$ is connected. Let $B$ be a subdomain of $D$ with smooth boundary $\partial B$. Then there exists a sequence $(g_n)_{n \in \mathbb{N}} \subset L^2(\Omega)$, such that the corresponding solutions $(u(g_n))_{n \in \mathbb{N}}$ of (2) fulfill

\[
\lim_{n \to \infty} \|u^{(g_n)}\|_{L^2(B)}^2 = \infty,
\]
\[
\lim_{n \to \infty} \|u^{(g_n)}\|_{H^1(D \setminus \overline{B})}^2 = 0,
\]
\[
\lim_{n \to \infty} \|u^{(g_n)}\|_{L^2(\partial B)}^2 = 0,
\]
\[
\lim_{n \to \infty} \|\nabla u^{(g_n)}\|_{L^2(B)}^2 = \infty.
\]

Proof. This proof is based on the UCP for Cauchy data. First, we define the virtual measurement operators $A_j (j = 1, 2)$ by

\[ A_1 : L^2(B) \to L^2(\partial \Omega), \quad F \mapsto v|_{\partial \Omega}, \]

where $v \in H^1(\Omega)$ solves

\[
\int_{\Omega} \sigma \nabla v \cdot \nabla w \, dx + \int_{\Omega} qv w \, dx = \int_{B} F w \, dx \quad \text{for all } w \in H^1(\Omega),
\]

\[ A_2 : H^1(D \setminus \overline{B})' \to L^2(\partial \Omega), \quad G \mapsto v|_{\partial \Omega}, \]

where $v \in H^1(\Omega)$ solves

\[
\int_{\Omega} \sigma \nabla v \cdot \nabla w \, dx + \int_{\Omega} qv w \, dx = \langle G, w \rangle_{D \setminus \overline{B}} \quad \text{for all } w \in H^1(\Omega).
\]

Here $\langle \ldots \rangle_{D \setminus \overline{B}}$ denotes the dual pairing on $H^1(D \setminus \overline{B})' \times H^1(D \setminus \overline{B})$. First, we show that the dual operators $A'_1$ and $A'_2$ are given by

\[ A'_1 : L^2(\partial \Omega) \to L^2(B) : g \mapsto A'_1 g = u|_B, \]

\[ A'_2 : L^2(\partial \Omega) \to H^1(D \setminus \overline{B}) : g \mapsto A'_2 g = u|_{D \setminus \overline{B}}. \]

Let $F \in L^2(\Omega)$, $g \in L^2(\partial \Omega)$, $u, v \in H^1(\Omega)$ solve (2) and (19), respectively. Then,

\[
\int_{\Omega} F A'_1 g \, dx = \int_{\partial \Omega} g A_1 F \, ds = \int_{\Omega} \sigma \nabla v \cdot \nabla u \, dx + \int_{\Omega} qv u \, dx = \int_{B} F u \, dx.
\]

Let $G \in H^1(\Omega)$, $g \in L^2(\partial \Omega)$, $u, v \in H^1(\Omega)$ solve (2) and (20), respectively. Then,

\[
\int_{\Omega} G A'_2 g \, dx = \int_{\partial \Omega} g A_2 G \, ds = \int_{\Omega} \sigma \nabla v \cdot \nabla u \, dx + \int_{\Omega} qv u \, dx = \langle G, u \rangle_{D \setminus \overline{B}}.
\]

Next, we will prove that

\[ \mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\} \quad \text{and} \quad \mathcal{R}(A_1) \neq \{0\}. \]
Uniqueness, Lipschitz stability and reconstruction

Let \( \varphi \in \mathcal{R}(A_1) \cap \mathcal{R}(A_2) \). Then there exist \( v_1, v_2 \in H^1(\Omega) \) such that \( v_1|_{\partial \Omega} = v_2|_{\partial \Omega} = \varphi \), and
\[
\int_\Omega \sigma \nabla v_j \cdot \nabla w \, dx + \int_\Omega q v_j w \, dx = 0
\]
for all \( w \in H^1(\Omega) \) with \( \text{supp}(w) \subset \overline{\Omega} \setminus \overline{D} \). Hence,
\[
\text{div}(\sigma \nabla v_j) + q v_j = 0 \quad \text{in } \Omega \setminus \overline{D},
\]
and \( (\sigma \partial_{n} v_1)|_{\partial \Omega} = (\sigma \partial_{n} v_2)|_{\partial \Omega} = 0 \). The unique continuation principle for Cauchy data yields that \( v_1 = v_2 \) in \( \Omega \setminus \overline{D} \). Hence \( v := v_1 \chi_{D\setminus \overline{D}} + v_2 \chi_{\Omega \setminus (D\setminus \overline{D})} \in H^1(\Omega) \) and satisfies
\[
\begin{aligned}
&\text{div}(\sigma \nabla v) + q v = 0 \quad \text{in } \Omega, \\
&\sigma \partial_{n} v = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
It follows that \( v = 0 \) and thus \( \varphi = v|_{\partial \Omega} = 0 \), and consequently \( \mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\} \).

Next, we will prove that \( \mathcal{R}(A_1) \neq \{0\} \). We first prove the injectivity of the dual operator \( A'_1 \). Let \( g \in L^2(\partial \Omega) \) be such that \( A'_1 g = u|_D = 0 \). By the unique continuation principal, we conclude that \( u = 0 \) in \( \Omega \). This means that \( g = \sigma \partial_{n} u|_{\partial \Omega} = 0 \), which proves that \( A'_1 \) is injective. Hence \( A_1 \) has a dense range, i.e., \( \mathcal{R}(A_1) = L^2(\partial \Omega) \).

A fortiori \( \mathcal{R}(A_1) \neq \{0\} \), which together with \( \mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\} \), implies the range non inclusion \( \mathcal{R}(A_1) \nsubseteq \mathcal{R}(A_2) \). Using [34 Corollary 2.6], it follows that there exists a sequence \( \{g_n\}_{n \in \mathbb{N}} \subset L^2(\partial \Omega) \) such that
\[
\lim_{n \to \infty} \|A'_1 g_n\|_{L^2(B)}^2 = \lim_{n \to \infty} \|u^{(g_n)}\|_{L^2(B)}^2 = \infty,
\]
and
\[
\lim_{n \to \infty} \|A'_2 g_n\|_{H^1(D\setminus \overline{D})}^2 = \lim_{n \to \infty} \|u^{(g_n)}\|_{H^1(D\setminus \overline{D})}^2 = 0.
\]
(21) i.e. (15) and (16) hold. Also (17), holds from (21). Since
\[
\|u^{(g_n)}\|_{L^2(B)} \leq C \left( \|u^{(g_n)}\|_{L^2(\partial B)} + \|\nabla u^{(g_n)}\|_{L^2(B)} \right),
\]
where \( C > 0 \) is a constant, this also imply (18). \( \square \)

Let \( \mathcal{G} \) be a finite dimensional subset of piecewise analytic functions. We consider four constants \( 0 < c_1 \leq c_2 \) and \( 0 < c_3 \leq c_4 \) which are the lower and upper bounds of the parameters and define the set
\[
\mathcal{G}_{[c_1, c_2] \times [c_3, c_4]} = \{ (\sigma, q) \in \mathcal{G} : \quad c_1 \leq \sigma(x) \leq c_2, \quad c_3 \leq q(x) \leq c_4 \quad \text{for all } x \in \Omega \},
\]
In the following main result of this paper, the domain \( \Omega \), the finite-dimensional subset \( \mathcal{G} \) and the bounds \( 0 < c_1 \leq c_2 \) and \( 0 < c_3 \leq c_4 \) are fixed, and the constant in the Lipschitz stability result will depend on them.
Theorem 5 (Lipschitz stability). There exists a positive constant $C > 0$ such that for all $(\sigma_1, q_1), (\sigma_2, q_2) \in \mathcal{G}_{[c_1, c_2] \times [c_3, c_4]}$ with either

(a) $\sigma_1 \leq \sigma_2$ and $q_1 \leq q_2$ or
(b) $\sigma_1 \geq \sigma_2$ and $q_1 \geq q_2$,

we have

$$d_{\Omega}((\sigma_1, q_1), (\sigma_2, q_2)) := \max \left(\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)}, \|q_1 - q_2\|_{L^\infty(\Omega)}\right) \leq C\|\Lambda(\sigma_1) - \Lambda(\sigma_2)\|_\star.$$  \hfill (22)

Here $\|\|_\star$ is the natural norm of $\|\|_{L^2(\partial\Omega)}$.

Proof. For the sake of brevity, we write $\|\|_\star$ for $\|\|_{L^2(\partial\Omega)}$. We start with the reformulation of the right-hand side of estimate (22). Since $\Lambda(\sigma_1, q_1)$ and $\Lambda(\sigma_2, q_2)$ are self-adjoint, we have that

$$\|\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1)\|_\star = \sup_{\|g\|_2 = 1} \{\langle g, \Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1) \rangle g \} = \sup_{\|g\|_2 = 1} \max \{\langle g, (\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1)) g \rangle, \langle g, (\Lambda(\sigma_1, q_1) - \Lambda(\sigma_2, q_2)) g \rangle\}.$$  

Next, we apply both inequalities in the monotonicity relation (8) in Lemma 3 in order to obtain lower bounds for the corresponding integrals. We thus obtain for all $g \in L^2(\partial\Omega)$

$$\langle g, (\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1)) g \rangle \geq \int_{\Omega} (\sigma_1 - \sigma_2) |\nabla u_{\sigma_1, q_1}\|^2 dx + \int_{\Omega} (q_1 - q_2) |u_{\sigma_1, q_1}\|^2 dx \tag{23}$$

and

$$\langle g, (\Lambda(\sigma_1, q_1) - \Lambda(\sigma_2, q_2)) g \rangle \geq \int_{\Omega} (\sigma_2 - \sigma_1) |\nabla u_{\sigma_2, q_2}\|^2 dx + \int_{\Omega} (q_2 - q_1) |u_{\sigma_2, q_2}\|^2 dx \tag{24}$$

where $u_{\sigma_1, q_1}, u_{\sigma_2, q_2} \in H^1(\Omega)$ denote the solutions of (2) with Neumann data $g$ and parameters $(\sigma_1, q_1)$ and $(\sigma_2, q_2)$, respectively. Based on the estimates (23) and (24), we obtain for $(\sigma_1, q_1) \neq (\sigma_2, q_2)$

$$\frac{\|\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1)\|_\star}{d_{\Omega}((\sigma_1, q_1), (\sigma_2, q_2))} \geq \sup_{\|g\|_2 = 1} \Phi \left(g, \frac{\sigma_1 - \sigma_2}{d_{\Omega}((\sigma_1, q_1), (\sigma_2, q_2))}, \frac{q_1 - q_2}{d_{\Omega}((\sigma_1, q_1), (\sigma_2, q_2))}, (\sigma_1, q_1), (\sigma_2, q_2)\right), \tag{25}$$

and define for $g \in L^2(\partial\Omega)$, $(\zeta_1, \zeta_2) \in \mathcal{G}$, and $(\kappa_1, \tau_1), (\kappa_2, \tau_2) \in \mathcal{G}_{[c_1, c_2] \times [c_3, c_4]}$ the function

$$\Phi (g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) := \max \left(\Psi (g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1)), \Psi (g, (-\zeta_1, -\zeta_2, (\kappa_2, \tau_2))\right),$$

with

$$\Psi (g, (\beta, \gamma), (\kappa, \tau)) := \int_{\Omega} \beta |\nabla u_{(\kappa, \tau)}|^2 dx + \int_{\Omega} \gamma |u_{(\kappa, \tau)}|^2 dx.$$
We introduce the compact sets
\[ \mathcal{K}_+ = \left\{ (\zeta_1, \zeta_2) \in \text{span} \mathcal{G} : \zeta_1, \zeta_2 \geq 0 \text{ and } \max \left( \|\zeta_1\|_{L^\infty(\Omega)}, \|\zeta_2\|_{L^\infty(\Omega)} \right) = 1 \right\}, \]
\[ \mathcal{K}_- = \left\{ (\zeta_1, \zeta_2) \in \text{span} \mathcal{G} : \zeta_1, \zeta_2 \leq 0 \text{ and } \max \left( \|\zeta_1\|_{L^\infty(\Omega)}, \|\zeta_2\|_{L^\infty(\Omega)} \right) = 1 \right\}, \]
and denote \( \mathcal{K} := \mathcal{K}_+ \cup \mathcal{K}_- \). Then using that either assumption (a) or assumption (b) is fulfilled, we can rewrite (25) as
\[ \frac{\|\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1)\|_*}{d_{\Omega}(\sigma_1, q_1, \sigma_2, q_2)} \geq \inf_{(\zeta_1, \zeta_2) \in \mathcal{K}} \sup_{\|g\| = 1} \Phi (g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)). \tag{26} \]

The assertion of Theorem 5 follows if we can show that the right-hand side of (26) is positive. Since \( \Phi \) is continuous, we can conclude that the function
\[ ((\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) \mapsto \sup_{\|g\| = 1} \Phi (g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)), \]
is semi-lower continuous, so that it attains its minimum on the compact set \( \mathcal{K} \times \mathcal{G}_{[c_1, c_2] \times [c_3, c_4]} \times \mathcal{G}_{[c_1, c_2] \times [c_3, c_4]} \). Hence, to prove Theorem 5 it suffices to show that
\[ \sup_{\|g\| = 1} \Phi (g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) > 0, \tag{27} \]
for all \( ((\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) \in \mathcal{K} \times \mathcal{G}_{[c_1, c_2] \times [c_3, c_4]} \times \mathcal{G}_{[c_1, c_2] \times [c_3, c_4]} \).

In order to prove that (27) holds true, let \( ((\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) \in \mathcal{K} \times \mathcal{G}_{[c_1, c_2] \times [c_3, c_4]} \times \mathcal{G}_{[c_1, c_2] \times [c_3, c_4]} \).

We first treat the case that \( (\zeta_1, \zeta_2) \in \mathcal{K}_+ \). Then there exist an open subset \( \emptyset \neq B \subset \Omega \) and a constant \( 0 < \delta < 1 \), such that either
(i) \( \zeta_1|_B \geq \delta \), and \( \zeta_2 \geq 0 \), or
(ii) \( \zeta_2|_B \geq \delta \), and \( \zeta_1 \geq 0 \).

We use the localized potentials sequence in Theorem 4 to obtain a boundary load \( \tilde{g} \in L^2(\partial \Omega) \) with
\[ \int_B |u^\tilde{g}_{(\zeta_1, \tau_1)}|^2 \, dx \geq \frac{1}{\delta} \quad \text{and} \quad \int_B |\nabla u^\tilde{g}_{(\zeta_1, \tau_1)}|^2 \, dx \geq \frac{1}{\delta}. \tag{28} \]
In case (i), this leads to
\[ \Phi (\tilde{g}, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) \geq \int_\Omega \zeta_1 |\nabla u^\tilde{g}_{(\zeta_1, \tau_1)}|^2 \, dx + \int_\Omega \zeta_2 |u^\tilde{g}_{(\zeta_1, \tau_1)}|^2 \, dx \]
\[ \geq \int_B \zeta_1 |\nabla u^\tilde{g}_{(\zeta_1, \tau_1)}|^2 \, dx \geq \delta \int_B |\nabla u^\tilde{g}_{(\zeta_1, \tau_1)}|^2 \, dx \geq 1, \]
and in case (ii), we have
\[ \Phi (\tilde{g}, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) \geq \int_\Omega \zeta_1 |\nabla u^\tilde{g}_{(\zeta_1, \tau_1)}|^2 \, dx + \int_\Omega \zeta_2 |u^\tilde{g}_{(\zeta_1, \tau_1)}|^2 \, dx \]
\[ \geq \int_B \zeta_2 |u^\tilde{g}_{(\zeta_1, \tau_1)}|^2 \, dx \geq \delta \int_B |u^\tilde{g}_{(\zeta_1, \tau_1)}|^2 \, dx \geq 1. \]
Hence, in both cases,
\[
\sup_{\|g\| = 1} \Phi(g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) \geq \Phi\left(\frac{\tilde{g}}{\|\tilde{g}\|}, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)\right)
\]
\[
= \frac{1}{\|\tilde{g}\|^2} \Phi(\tilde{g}, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) > 0.
\]
For \((\zeta_1, \zeta_2) \in \mathcal{K}_-\), we can analogously use a localized potentials sequence for \((\kappa_2, \tau_2)\), and prove that
\[
\sup_{\|g\| = 1} \Phi(g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) > 0,
\]
and the proof of Theorem 5 is completed. \(\square\)

**Remark 1.** All the results of section 3 and section 4 stay valid when the Neumann to Dirichlet operator \(\Lambda(\sigma, q)\) is extended to \(H^{-\frac{1}{2}}(\partial \Omega) \rightarrow H^{\frac{1}{2}}(\partial \Omega)\). On these spaces, it is easily shown that \(\Lambda(\sigma, q)\) is bijective, and its inverse is the Dirichlet-to-Neumann operator \(\Lambda_D(\sigma, q) : f \rightarrow u^{(f)}_{\sigma, q}|_{\partial \Omega}\), where \(u^{(f)}_{\sigma, q}\) solves
\[
\begin{cases}
- \nabla \cdot (\sigma \nabla u^{(f)}_{\sigma, q}) + qu^{(f)}_{\sigma, q} = 0 & \text{in } \Omega, \\
\sigma \partial_n u^{(f)}_{\sigma, q} = f & \text{on } \partial \Omega.
\end{cases}
\]

### 5. Numerical approach to solve the inverse problem

In this section, we are interested in the following inverse problem

\[
\text{Find } \sigma, q \text{ knowing measurements } f_k = \Lambda(\sigma, q) g_k, \ k = 1, \ldots K, \quad (29)
\]

where \(f_k \in L^2(\partial \Omega)\) is a measurement of the density of photons corresponding to the input flux \(g_k\), and \(K \in \mathbb{N}\) is the number of measurements.

To solve the inverse problem (29) numerically, we consider a minimization problem of a Kohn-Vogelius type functional:

\[
\min_{(\sigma, q) \in G} J(\sigma, q) = \sum_{k=1}^{K} \int_{\Omega} \left( \sigma |\nabla (u_1^{(g_k)} - u_2^{(g_k)})|^2 + q|u_1^{(g_k)} - u_2^{(g_k)}|^2 \right) dx
\]
\[
+ \frac{\rho}{2} \int_{\Omega} (\sigma^2 + q^2) dx. \quad (30)
\]

Here \(u_1^{(g_k)}\) and \(u_2^{(g_k)}\) solve the following problems:

\[
\begin{cases}
- \nabla \cdot (\sigma \nabla u_1^{(g_k)}) + qu_1^{(g_k)} = 0 & \text{in } \Omega, \\
\sigma \partial_n u_1^{(g_k)} = g_k & \text{on } \partial \Omega, \\
\end{cases}
\]
\[
\begin{cases}
- \nabla \cdot (\sigma \nabla u_2^{(g_k)}) + qu_2^{(g_k)} = 0 & \text{in } \Omega, \\
u_2^{(g_k)} = f_k & \text{on } \partial \Omega, \\
\end{cases} \quad (31)
\]

\[
\begin{cases}
- \nabla \cdot (\sigma \nabla u_2^{(g_k)}) + qu_2^{(g_k)} = 0 & \text{in } \Omega, \\
u_2^{(g_k)} = f_k & \text{on } \partial \Omega,
\end{cases} \quad (32)
\]
When dealing with reconstruction of the absorption parameter $q$ where $\sigma$ is assumed to be known, the minimization problem (30) is reduced to

$$\min_{q \in F[a, b]} J(q) = \sum_{k=1}^{K} \int_{\Omega} \left( \sigma|\nabla (u_{1}^{(g_{k})} - u_{2}^{(g_{k})})|^{2} + q|u_{1}^{(g_{k})} - u_{2}^{(g_{k})})|^{2} \right) dx + \frac{\rho}{2} \int_{\Omega} q^{2} dx. \quad (33)$$

**Theorem 6.** The functional $J : L^{\infty}_{+}(\Omega)^{2} \to \mathbb{R}$, defined in (30), is Fréchet differentiable, and its Fréchet derivative at $(\sigma, q) \in L^{\infty}_{+}(\Omega)^{2}$ in the direction $(\hat{\sigma}, \hat{q}) \in L^{\infty}_{+}(\Omega)^{2}$ is given by

$$J'(\sigma, q) (\hat{\sigma}, \hat{q}) = \sum_{k=1}^{K} \int_{\Omega} \left( \hat{\sigma} \left( |\nabla u_{2}^{(g_{k})}|^{2} - |\nabla u_{1}^{(g_{k})}|^{2} \right) + \hat{q} \left( (u_{2}^{(g_{k})})^{2} - (u_{1}^{(g_{k})})^{2} \right) \right) dx \quad + \rho \int_{\Omega} (\sigma \hat{\sigma} + q \hat{q}) dx. \quad (34)$$

We need the following lemma to prove Theorem 6.

**Lemma 4.** The non-linear operator

$$\Lambda(\sigma, q) : L^{\infty}_{+}(\Omega)^{2} \to \mathcal{L}(L^{2}(\partial \Omega)) \quad (\sigma, q) \to \Lambda(\sigma, q)$$

is Fréchet differentiable and its derivative

$$\Lambda' : L^{\infty}_{+}(\Omega)^{2} \to \mathcal{L}(L^{\infty}(\Omega)^{2}, \mathcal{L}(L^{2}(\partial \Omega)))$$

is given by the bilinear form

$$\int_{\partial \Omega} g(\Lambda'(\sigma, q)(\delta_{1}, \delta_{2})) h \, ds = -\int_{\Omega} \delta_{1} \nabla u_{\sigma, q}^{(g)} : \nabla u_{\sigma, q}^{(h)} \, dx - \int_{\Omega} \delta_{2} u_{\sigma, q}^{(g)} u_{\sigma, q}^{(h)} \, dx, \quad (35)$$

for all $\sigma, q \in L^{\infty}_{+}(\Omega), \delta_{1}, \delta_{2} \in L^{\infty}(\Omega), g, h \in L^{2}(\partial \Omega)$ where $u_{\sigma, q}^{(g)} \in H^{1}(\Omega)$ is solution of the problem [3].

**Proof.** It follows from the monotonicity relation [8] that for all sufficiently small $\delta_{1}, \delta_{2} \in L^{\infty}(\Omega)$ such that $\sigma + \delta_{1}, q + \delta_{2} \in L^{\infty}_{+}(\Omega)$

$$\int_{\Omega} \left( \delta_{1}|\nabla u_{\sigma, q}^{(g)}|^{2} + \delta_{2} (u_{\sigma, q}^{(g)})^{2} \right) dx \geq \int_{\partial \Omega} g \left( \Lambda(\sigma, q) - \Lambda(\sigma + \delta_{1}, q + \delta_{2}) \right) g \, ds$$

$$\geq \int_{\Omega} \left( \sigma - \frac{\sigma^{2}}{\sigma + \delta_{1}} \right) |\nabla u_{\sigma, q}^{(g)}|^{2} dx + \int_{\Omega} \left( q - \frac{q^{2}}{q + \delta_{2}} \right) (u_{\sigma, q}^{(g)})^{2} dx.$$ 

Thus

$$\|\Lambda(\sigma, q) - \Lambda(\sigma + \delta_{1}, q + \delta_{2}) - \Lambda'(\sigma, q)(\delta_{1}, \delta_{2})\|_{\mathcal{L}(L^{2}(\partial \Omega))} = \sup_{g \in L^{2}(\partial \Omega)} \left| \int_{\partial \Omega} g \left( \Lambda(\sigma, q) - \Lambda(\sigma + \delta_{1}, q + \delta_{2}) - \Lambda'(\sigma, q)(\delta_{1}, \delta_{2}) \right) g \, ds \right|$$

$$\leq \int_{\Omega} \left( \left( \frac{\delta_{1}^{2}}{\sigma + \delta_{1}} \right) |\nabla u_{\sigma, q}^{(g)}|^{2} + \left( \frac{\delta_{2}^{2}}{q + \delta_{2}} \right) (u_{\sigma, q}^{(g)})^{2} \right) dx = O \left( \|\delta_{1}, \delta_{2}\|_{\infty} \right). \quad (36)$$

This shows that $\Lambda$ is Fréchet differentiable, and its derivative is given by [35].
Proof of Theorem 6. From the definition of the functional $J$, and applying Green’s formula once, we have

$$J(\sigma, q) = \sum_{k=1}^{K} \int_{\Omega} \sigma |\nabla (u_1^{(g_k)}|^2 \, dx + \sum_{k=1}^{K} \int_{\Omega} q |u_1^{(g_k)}|^2 \, dx + \sum_{k=1}^{K} \int_{\Omega} \sigma |\nabla (u_2^{(g_k)}|^2 \, dx \\
+ \sum_{k=1}^{K} \int_{\Omega} |q u_2^{(g_k)}|^2 \, dx - 2 \sum_{k=1}^{K} \int_{\partial\Omega} g_k f_k \, ds + \frac{\rho}{2} \int_{\Omega} (\sigma^2 + q^2) \, dx \tag{37}$$

From Lemma 4, $\Lambda(\sigma, q)$ is Fréchet differentiable with

$$\langle g_k, \Lambda'(\sigma, q)(\tilde{\sigma}, \tilde{q}) g_k \rangle = - \int_{\Omega} (\tilde{\sigma} |\nabla u_1^{g_k}|^2 + \tilde{q} (u_1^{g_k})^2) \, dx$$

and

$$\langle (\Lambda_D(\sigma, q))^f(\tilde{\sigma}, \tilde{q}) f_k, f_k \rangle = \langle (\Lambda(\sigma, q)^{-1})'(\tilde{\sigma}, \tilde{q}) f_k, f_k \rangle \tag{38}$$

$$= \int_{\Omega} (\tilde{\sigma} |\nabla u_1^{g_k}|^2 + \tilde{q} (u_1^{g_k})^2) \, dx.$$

Since $\int_{\partial\Omega} g_k f_k \, ds$ is constant and $(\sigma, q) \to \int_{\Omega} (\sigma^2 + q^2) \, dx$ is Fréchet differentiable, we conclude that $J$ is Fréchet differentiable and its derivative is given by (34). \qed

Remark 2. Using the same techniques, we can prove that the functional $J$ is Fréchet differentiable and its derivative is given by:

$$J'(\tilde{q}) \tilde{q} = \sum_{k=1}^{K} \int_{\Omega} + \tilde{q} ((u_2^{g_k})^2 - (u_1^{g_k})^2) \, dx + \rho \int_{\Omega} \tilde{q} \, dx.$$

6. Implementation details and numerical examples

In this section we perform some numerical tests using noiseless and noisy data. When dealing with reconstruction with noise data, the choice of the regularization parameter $\rho$ in (29) is crucial. Usually, it is determined using a knowledge of the noise level by, e.g., the discrepancy principle. However, in practice, the noise level may be unknown, rendering such rules inapplicable. To overcome this issue, we propose a heuristic choice rule based on the following balancing principle [35]: Choose $\rho$ such that

$$(\beta - 1) \sum_{k=1}^{K} \int_{\Omega} \left( |\sigma |\nabla (u_1^{g_k}) - u_2^{(g_k)}|^2 + q |u_1^{(g_k)} - u_2^{(g_k)}|^2 \right) \, dx - \frac{\rho}{2} \int_{\Omega} (\sigma^2 + q^2) \, dx = 0. \tag{38}$$

The idea behind this principle is to balance the data fitting term with the penalty term and the weight $\beta > 1$ controls the trade-off between them. The choice rule does not require the knowledge of the noise level, and has been successfully applied to linear and non linear inverse problems [36, 37, 38, 39].
Wen dealing only with the reconstruction of $q$, the balancing equation (38) is reduced to
\[
(\beta - 1) \sum_{k=1}^{K} \int_{\Omega} \left( \sigma |\nabla (u_1^{(g_k)} - u_2^{(g_k)})|^2 + q|u_1^{(g_k)} - u_2^{(g_k)}|^2 \right) \, dx - \frac{\rho}{2} \int_{\Omega} q^2 \, dx = 0.
\] (39)

For our problem, we compute a solution $\rho^*$ to the balancing equation (38) or (39) by the fixed point algorithm proposed in [37, 38].

We consider the following setup for our numerical examples: The domain $\Omega$ under consideration is the two dimensional unit disk centered at the origin. We use a Delaunay triangular mesh and a standard finite element method with piecewise finite elements to numerically compute the states for our problem. The measurements $f_k$ are computed synthetically by solving the direct problem (2). To simulate noisy data, the measurements $f_k$ are corrupted by adding a normal Gaussian noise with mean zero and standard deviation $\epsilon \|f_k\|_\infty$ where $\epsilon$ is a parameter. To avoid the so called 'inverse crime', the inverse problem is solved using 1016 elements, while the data $f_k$ is computed with 4064 elements. For all the computations we have used Matlab R2018a.

6.1. Example 1

In the following numerical results, the diffusion coefficient $\sigma$ is assumed to be known, and is given by $\sigma = 1\chi_{\Omega \setminus \omega} + 2\chi_\omega$, where $\omega$ is the disk of radius $1/2$ centered at the origin. The exact absorption coefficient to be recovered is given by
\[
q^\dagger(x_1, x_2) = 1 + \cos(\pi x_1) \cos(\pi x_2) \chi_{\{\|\!(x_1, x_2)\!\|_\infty < 0.5\}}.
\]

We obtain measurements $f_k$ corresponding to the fluxes
\[
g_k = 10 + \sin(k\theta), \quad \theta \in [0, 2\pi], \quad k = 1, \ldots 5.
\]
and we reconstruct $q$ by minimizing the functional
\[
\mathcal{J}(q) = \sum_{k=1}^{5} \int_{\Omega} \left( \sigma |\nabla (u_1^{(g_k)} - u_2^{(g_k)})|^2 + q|u_1^{(g_k)} - u_2^{(g_k)}|^2 \right) \, dx + \frac{\rho}{2} \int_{\Omega} q^2 \, dx,
\]
in the space of piecewise constant functions on the FEM mesh.

Figure 1 shows the true and the reconstructed absorption parameter with free noise and without regularization. We obtain a good approximation result. In this test, we committed the so-called inverse crime of using the same forward solver (i.e., the same finite element mesh) for simulating the data $f_k$. Figure 2 shows the sensitivity of the reconstruction with respect to the initialization. Figure 3 shows the sensitivity of the reconstruction with respect to the noise of measurements.

6.2. Examples 2

In this example the exact parameters to be recovered are given by
\[
\sigma^\dagger(x) = 7\chi_{D_1} + 4\chi_{D_2} + 3\chi_{\Omega \setminus D_1 \cup D_2}, \quad q^\dagger(x) = 6\chi_{D_1} + 5\chi_{D_2} + 2\chi_{\Omega \setminus D_1 \cup D_2},
\]
Figure 1. On the left the true solution and the right the reconstructed solution for \( \epsilon = 0 \) and \( \rho = 0 \). In this case the initialization is given by \( q(x_1, x_2) = 1 + (|x_1| < 0.5)(|x_2| < 0.5) \).

Figure 2. On the left reconstruction with initialization \( q(x_1, x_2) = 1 + (|x_1| < 0.5)(|x_2| < 0.5) \) and on the right reconstruction with initialization \( q(x_1, x_2) = 1 \). In both cases \( \epsilon = 0 \) and \( \rho = 0 \).

Figure 3. On the left, reconstruction with \( \epsilon = 0.03 \) and \( \rho = 0.04221603 \), on the right reconstruction with \( \epsilon = 0.05 \) and \( \rho = 0.01505686 \). In both cases the initialization is given by \( q(x_1, x_2) = 1 + (|x_1| < 0.5)(|x_2| < 0.5) \).

where \( D_1 \) and \( D_2 \) are assumed to be known and are given by:

\[
D_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : (x_1 + 0.6)^2 + x_2^2 < 0.2^2 \right\},
\]
\[ D_2 = \{ (x_1, x_2) \in \mathbb{R}^2 : (x_1 - 0.6)^2 + x_2^2 < 0.2^2 \}. \]

We use one measurement \( f \) corresponding to the flux \( g(\theta) = \sin(\theta) + \cos(\theta), \theta \in [0, 2\pi] \) and we reconstruct \( \sigma, q \) by minimizing the function

\[
J(\sigma, q) = \int_{\Omega} \left( \sigma |\nabla (u_1^{(g)} - u_2^{(g)})|^2 + q |u_1^{(g)} - u_2^{(g)}|^2 \right) dx + \frac{\rho}{2} \int_{\Omega} (\sigma^2 + q^2) dx,
\]

in the space of piecewise constant functions on the FEM mesh.

Figure 4 shows the true parameters to be recovered. Figure 5 depicts the reconstruction of the diffusion and the absorption coefficients with \( \epsilon = 0 \) and \( \rho = 0 \). In this case the initialization is given by

\[
(\sigma(x), q(x)) = \left(1 \chi_{D_1} + 1 \chi_{D_2} + 1 \chi_{\Omega \setminus D_1 \cup D_2}, 1 \chi_{D_1} + 1 \chi_{D_2} + 1 \chi_{\Omega \setminus D_1 \cup D_2} \right).
\]

Figure 6 depicts the reconstruction of the parameters with \( \epsilon = 0.05 \) and \( \rho = 1.66636600 \times 10^{-7} \). In this test the initialization is set to

\[
(\sigma(x), q(x)) = \left(3 \chi_{D_1} + 3 \chi_{D_2} + 3 \chi_{\Omega \setminus D_1 \cup D_2}, 3 \chi_{D_1} + 3 \chi_{D_2} + 3 \chi_{\Omega \setminus D_1 \cup D_2} \right).
\]
Remark 3. In example 1 and example 2 the cost functional to be minimized is non convex, then it might have some local minima. Therefore the accuracy of the reconstruction depends on the initial guess as shown in figure 2. The numerical solutions represent reasonable approximations and are stable with respect to a small amount of noise as shown in figure 3 and figure 4.

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