Higher derivatives of the end-point map of a control-linear system via adapted coordinates*

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Abstract

We study the end-point map of a control-linear system in a neighborhood of an arbitrarily chosen trajectory. In particular, we want to calculate the \(k\)-th order derivative of this map in a given direction. A priori it is a solution of a quite complicated ODE depending on all derivatives of order less or equal \(k\). We prove that there exists a special coordinate system adapted to the geometry of the problem, which changes the system of ODEs describing all derivatives of the end-point map up to order \(k\) to equations of a control-affine (non-autonomous control-linear) system, with the direction of derivation playing the role of the new control. As an application we study controllability criteria for this system, obtaining first and second-order necessary optimality conditions of sub-Riemannian geodesics. In particular, for the case of an abnormal minimizer we can interpret Goh conditions as non-controllability conditions of this control-affine system for \(k = 2\). We make a hypothesis that for higher \(k\)’s its non-controllability corresponds to recently obtained higher-order analogs of the Goh conditions [Boarotto, Monti, Palmurella, 2020], [Boarotto, Monti, a Socionovo, 2022].

1 Introduction

The problem Local properties of the end-point map around a trajectory can be quite significant in some aspects of control theory. They play an important role in phenomena such as local controllability, abnormality, and optimality of trajectories. Let us name a few examples. For optimal control problems openness of the (extended, i.e. including the costs) end-point map around a trajectory excludes the possibility that this trajectory is optimal. Therefore non-openness of the end-point map gives necessary conditions for optimality. By using the standard version of the Open Mapping Theorem (OMT) we get first-order conditions for optimality, i.e equations of the Pontryagin Maximum Principle. These conditions are also fundamental for classifying extremals of the optimal control problem as normal or abnormal [Jur97, AS04]. In some specific situations, like the study of abnormal sub-Riemannian geodesics, the above conditions are, however, insufficient. Therefore one needs to use more specific versions of the OMT. A degree-two version is a basis of [AS96, Jó´z23], and

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recently there has been an attempt of using Sussmann’s version of the OMT [Sus03] to study sub-Riemannian
optimality conditions of any order [BMP20, BMS22].
For a control system on a manifold \( M \), whose set of admissible controls \( \Omega \) forms a vector space, perhaps the
most natural idea to study the time \( t \)-end-point map \( \text{End}^k : \Omega \to M \) is by calculating its Taylor expansion. In
this paper we consider such an expansion for a control-linear system
\[
\dot{q}(t) = \sum_{i=1}^{l} u_i(t) X_i(q(t)),
\]
where \( u = (u_1, \ldots, u_l) \in \Omega \subseteq \text{Meas}([0, T], \mathbb{R}^l) \) is the control, and \( X_i \), for \( i = 1, 2, \ldots, l \) are linearly
independent vector fields on \( M \). Our goal is to calculate the \( k \)-th order derivative of the end-point map at a
given control \( u \in \Omega \) in the direction of a control \( \Delta u \in T_u(\Omega) \cong \Omega \), i.e.
\[
\frac{d}{ds} \bigg|_{s=0} \text{End}^k[u + s \cdot \Delta u].
\]
At this point, it is worth to remark, that as \( \text{End}^k \) is a manifold-valued map, such a derivative usually has no
geometric sense, although it can be calculated in any chosen local coordinate system. This issue can be easily
resolved by using the geometric notion of a \( k \)-jet, rather than a coordinate-dependent notion of a \( k \)-th derivative.
We comment on this matter in Remark 1.1

Our results First of all, in Lemma 2.1 we were able to derive a hierarchy of ODEs, linear in both \( u \) and \( \Delta u \),
describing the desired derivatives in a local coordinate system. Our main result is, however, an observation
that there exists a time-dependent family of diffeomorphisms (depending on the given control \( u \) and the vector
fields \( X_i \)) which transforms the differential equations describing the set of derivatives
\[
(\text{End}^1, \text{End}^2, \ldots, \text{End}^k),
\]
i.e. the \( k \)-jet of the curve \( s \mapsto \text{End}^k(u + s \cdot \Delta u) \), into a system of ODEs for new variables \( (q^{(1)}, q^{(2)}, \ldots, q^{(k)}) \),
which is linear in \( \Delta u \), and has no explicit dependence of the initial control \( u \). These transformations are
described by a family of multi-linear maps – their construction and properties are stated as Theorem 3.2. For
reasons that will be clarified in the next paragraph, we call these newly constructed variables – adapted coordi-
nates. As a corollary, we formulate Theorem 3.5 which states that \((q^{(1)}(t, \Delta u), q^{(2)}(t, \Delta u), \ldots, q^{(k)}(t, \Delta u))\)
– the solutions of the considered ODE system in adapted coordinates – are trajectories of a non-autonomous
control-linear system (or, as we prefer to see it, a control-affine system in variables \( (t, q^{(1)}, \ldots, q^{(k)}) \) with the
drift being simply \( \partial_k \lambda \) with \( \Delta u \) playing the role of a linear control. Moreover, this system of ODEs is naturally
graded. The above results, i.e. Lemma 2.1 Theorem 3.2 and Theorem 3.5 are further generalized as, respectively,
Lemma 5.1 Theorem 5.2 and Theorem 5.3 to describe \( k \)-jets of curves \( s \mapsto \text{End}^k(u + s \cdot \Delta u^{(1)} + s^2 \cdot \Delta u^{(2)} + \ldots + s^k \cdot \Delta u^{(k)}) \),
which seem to be more promising from the point of view of applications.

The idea behind adapted coordinates is actually quite easy to explain. Note that after fixing a control \( u \), system
\( X \) evolves according to a time-dependent vector field \( X_u(t, q) = \sum_{i=1}^{l} u_i(t) X_i(q) \). Under mild assumption
this evolution produces a time-dependent flow \( X_u^{\text{TM}} \) acting on \( M \) (being a groupoid rather than a group). Now
the tangent map of this flow acts on \( TM \) and, in particular, it gives a distinguished evolution in the tangent
spaces \( T_{q(t)}M \) along a given trajectory of the system \( q(t) \). Thus after fixing a basis \((e_1(0), \ldots, e_n(0))\) of
\( T_{q(0)}M \), the flow spreads it into a family of bases \((e_1(t), \ldots, e_n(t))\), each for every particular tangent space
\( T_{q(t)}M \). Adapted coordinates \( q^{(1)}(t, \Delta u) \) are precisely the coordinates of the tangent vector \( \text{End}^k(\Delta u) \)
in this new basis. The reason why the evolution equation for \( t \mapsto \text{End}^k(\Delta u) \) looks simpler in these new
coordinates is because the natural evolution by the flow of the control vector field \( X_u(t, q) \) was included in
their construction. In other words, the new coordinates are adapted to the chosen evolution of the system.
Higher-order adapted coordinates \( (q^{(1)}, \ldots, q^{(k)}) \) are constructed analogously, by considering the action of the
flow of the control vector field on \( k \)-jets. Thus one should think that \((q^{(1)}, \ldots, q^{(k)})\) are coordinates in \( T^k_{q(t)}M \)
– the space of \( k \)-jets of curves passing through the base point \( q(t) \) – naturally „preferred” by the evolution on \( M \)
given by the field \( X_u(t, q) \). We clarify this intuition in Lemma 3.4. The presence of a natural graded structure
on \( T^kM \) [Sau89, GR12] is the reason why we observe the presence of the natural grading in the ODE systems.
We will denote the space of all \( k \)-jets on \( M \),More specifically, in this paper we will consider the Taylor expansion of the end-point map, which after identifying the assertion of Agrachev-Sarychev Index Lemma \([AS96]\) as non-controllability conditions for a certain control system involving first and second derivatives of the end-point map. However, by Theorem 3.5, in adapted coordinates \((t,q^{(1)},q^{(2)})\) this system is a control-affine system, and we may address the question of its controllability using the results of Sussmann and Jurdjevic \([SJ72]\). It turns out that the criteria for non-controllability give precisely the Goh conditions \([Goh66]\) (see Lemmas 4.6 and 4.7). It seems to us that such an interpretation of Goh conditions was not present in the literature so far, not to mention that (once the Agrachev-Sarychev Index Lemma is known) the proof does not require making any estimates. This observation is a basis of a Hypothesis 5.4 (backup-ed by some calculations) that higher-order Goh conditions introduced recently in \([BMP20,BMS22]\) are a consequence of non-controllability of a control-affine system described by Theorem 3.3. We also refer to a publication of one of us \([Jóź23]\) for further applications of adapted coordinates in sub-Riemannian geometry. Here these were used to study the geometry of second-order approximation of the end-point map around a minimizing trajectory.

A remark about jets

**Remark 1.1.** In general, there is a sense to speak about higher derivatives only for maps valued in a vector space. Indeed, consider the end-point map \( \text{End}^d: \Omega \to M \) at \( u \in \Omega \). Let \( \phi: M \supset U \to \mathbb{R}^n \) be local coordinates on \( M \) around \( p = \text{End}^d(u) \). We may then calculate the Taylor expansion of \( \phi \circ \text{End}^d: \Omega \to \mathbb{R}^n \). but then, say, the \( k \)-th order term will not transform well when passing to a new coordinate system. In other words, \( D_u^k \text{End}^d(\Delta u) \) is not a well-defined geometric object for \( k \geq 2 \).

There are essentially two ways to deal with this problem. The first one is to restrict our attention to a situation in which a higher derivative makes sense. This strategy has been used in some Agrachev’s works — see for example \([AS94]\), where the second derivative \( D_u^2 \text{End}^d \) is defined on \( \ker D_u \text{End}^d \) and takes values in \( \text{coker} D_u \text{End}^d = T_{\text{End}^d(u)} M/\text{Im} D_u \text{End}^d \). In a moment we shall explain why it is so. Similar constructions are present, for instance, in \([BMP20,BMS22]\).

In this paper we prefer a different approach. Instead of being interested in each particular term of the Taylor expansion, we want to consider the whole series up to a term of a given order \( k \). This can be formalized in the language of jets as follows (cf. \([Sau89]\)). We say that two curves \( \gamma, \gamma': (-\varepsilon, \varepsilon) \to M \) passing through \( p = \gamma(0) = \gamma'(0) \in M \) are tangent up to order \( k \) at \( p \), if in some (and thus any) coordinate system \( \phi: U \to \mathbb{R}^n \) around \( p \) we have for all \( m = 1,2,\ldots,k \):

\[
\frac{d^m}{ds^m}\big|_{s=0}\phi(\gamma(s)) = \frac{d^m}{ds^m}\big|_{s=0}\phi(\gamma'(s)).
\]

The relation of the \( k \)-th order tangency is an equivalence relation and its equivalence classes are called \( k \)-jets. We will denote the space of all \( k \)-jets on \( M \) by \( T^k M \). The assignment of a \( k \)-jet to its base point \( [\gamma]\mapsto p = \gamma(0) \) makes \( T^k M \to M \) a locally trivial bundle. In fact, this is an example of a graded bundle in the sense of Grabowski and Rotkiewicz \([GR12]\), meaning that the fibers \( T^p M \) are endowed with a canonical action of the multiplicative reals. Actually, the structure of \( T^k M \) is much more specific, namely natural projections \([\gamma]\mapsto [\gamma]_{(i-k)}(\gamma(0)) \) give rise to the tower of fibrations \( T^k M \to T^{k-1} M \to \ldots \to T^2 M \to TM \to M \), where the first level is a vector bundle and each higher level is an affine bundle modeled on \( TM \) (see \([Sau89]\)).

More specifically, in this paper we will consider the Taylor expansion of the end-point map, which after identifying a manifold \( M \) with \( \mathbb{R}^n \) by a choice of local coordinates may be written as follows:

\[
\text{End}^d[u + s\Delta u] = \text{End}^d[u] + s \cdot D_u \text{End}^d[\Delta u] + s^2 \cdot \frac{1}{2!} D_u^2 \text{End}^d[\Delta u] + \ldots + s^k \cdot \frac{1}{k!} D_u^k \text{End}^d[\Delta u] + o(s^k).
\]
As we mentioned earlier, individual terms $D^i_k \text{End}^i[\Delta u]$ have no geometric meaning for $i \geq 2$, yet their whole collection $(D_u \text{End}^i[\Delta u], D_u^2 \text{End}^i[\Delta u], \ldots, D_u^n \text{End}^i[\Delta u])$, encoding the $k$-jet of a curve $s \mapsto \text{End}^i[u + s \Delta u]$, does. This can be to some extent seen at the level of ODEs (2.2) which describe the time evolution of this $k$-jet: equation for $D^i_u \text{End}^i[\Delta u]$ contains terms depending on lower-order derivatives, hence there is no way to separate $D^i_u \text{End}[\Delta u]$ from the rest of the collection $(D_u \text{End}^i[\Delta u], D_u^2 \text{End}^i[\Delta u], \ldots, D_u^n \text{End}^i[\Delta u])$.

Finally, note that the concept of a jet allows to explain the understanding of second (and higher – see [BMP20]) derivatives in the spirit of Agrachev. Namely, take a curve $u_s = u + s \cdot u^{(1)} + s^2 \cdot u^{(2)} + o(s^2)$ in $\Omega$, and let us calculate the second Taylor expansion of $\text{End}(u_s)$ in some local coordinate system:

\begin{equation}
\text{End}(u + s \cdot u^{(1)} + s^2 \cdot u^{(2)}) \Rightarrow \text{End}(u) + s \cdot D_u \text{End}\left[u^{(1)}\right] + s^2 \left(D_u \text{End}\left[u^{(2)}\right] + \frac{1}{2} D_u^2 \text{End}\left[u^{(1)}, u^{(1)}\right]\right) + o(s^2).
\end{equation}

Thus to extract the $D^2_u \text{End}\left[u^{(1)}, u^{(1)}\right]$-term from the whole $2$-jet $(D_u \text{End}\left[u^{(1)}\right], D_u \text{End}\left[u^{(2)}\right] + D_u^2 \text{End}\left[u^{(1)}, u^{(1)}\right])$ one needs first to assume that $D_u \text{End}\left[u^{(1)}\right] = 0$ and then quotient out the $D_u \text{End}\left[u^{(2)}\right]$-term, precisely as in the Agrachev’s approach (where the second derivative is defined for $u^{(1)} \in \ker D_u \text{End}$, and takes values in $\text{coker} D_u \text{End}$).

**Notation: graded multi-indexes and the polynomial expansion of a composition of maps**

**Remark 1.2** (Notation convention). Consider a multi-index $\alpha = (a_1, a_2, \ldots, a_k)$ and let us introduce the following notation:

\begin{align*}
|\alpha| &= \sum_{i=1}^{k} a_i \text{ is the absolute value of } \alpha, \\
w(\alpha) &= \sum_{i=1}^{k} i \cdot a_i \text{ is the weight of } \alpha, \\
\alpha! &= a_1! \cdot a_2! \cdot \ldots \cdot a_k!.
\end{align*}

It is convenient to think that $\alpha$ is graded, with $a_i$ being of weight $i$. Moreover, for an $|\alpha|$-linear map $\Phi^{[\alpha]}$ and $b = (b^{(1)}, b^{(2)}, \ldots, b^{(k)})$ we define

\[ \Phi^{[\alpha]}[b^{\alpha}] := \Phi^{[\alpha]}[b^{(1)}, b^{(2)}, \ldots, b^{(k)}] = \prod_{a_i}^{a_1} \frac{1}{a_i!} \cdot D^{[\alpha]}_{\gamma^{(0)}} f [\gamma^{\alpha}] . \]

Throughout this paper we understand gradings and weights according to [GR12].

The above convention allows for an elegant description of the polynomial expansion of a composition of a function and a curve.

**Lemma 1.3** (Faa di Bruno). Consider a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$, whose polynomial expansion at $s = 0$ reads as

\[ \gamma(s) = \gamma(0) + s \cdot \gamma^{(1)} + s^2 \cdot \gamma^{(2)} + \ldots + s^k \cdot \gamma^{(k)} + o(s^k), \]

and let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then their composition $F := f(\gamma) : (-\varepsilon, \varepsilon) \to \mathbb{R}$ expands as

\[ F(s) = F(0) + s \cdot F^{(1)} + s^2 \cdot F^{(2)} + \ldots + s^k \cdot F^{(k)} + o(s^k), \]

where

\[ F^{(m)} = \sum_{\alpha, w(\alpha) = m} \frac{1}{\alpha!} \cdot D^{[\alpha]}_{\gamma^{(0)}} f [\gamma^{\alpha}] \]

with $\gamma = (\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(k)})$.

We leave the proof as an exercise.
2 Higher derivatives of the end-point map

A control-linear system Consider a control-linear system \((\Lambda)\) on an \(n\)-dimensional manifold \(M\) given by a rank-\(l\) distribution \(\mathcal{D}_p = \text{span}_\mathbb{R}\{X_1(p), \ldots, X_l(p)\} \subset T_p M\), for all \(p \in M\):

\[
(\Lambda) \quad \dot{q}(t) = \sum_{i=1}^l u_i(t) X_i(q(t)); \quad \text{with the initial condition } q(0) = q_0,
\]

where \(u(t) = (u_1(t), \ldots, u_l(t)) \in \mathbb{R}^l\) is the control. We assume that \(u\) belongs to some space \(\Omega\). Unless specified differently, we will assume only that \(\Omega\) is a vector subspace of the space of measurable maps \(\text{Meas}([0, T], \mathbb{R}^l)\).

The end-point map For each \(t \in [0, T]\) denote by

\[
\text{End}^l : \Omega \longrightarrow M; \quad \text{End}^l : u \mapsto q(t) \text{ satisfying } (\Lambda)
\]

the family of the end-point maps related with \((\Lambda)\). Our goal is to calculate \(D^k_{\omega} \text{End}^l[\Delta u]\) – the \(k\)-th derivative of \(\text{End}^l\) at a given control \(u\) in the direction of \(\Delta u \in T_u \Omega \simeq \Omega\), for each \(k = 1, 2, 3, \ldots\).

To do this let us consider a family of solutions \(q_s(t)\) of \((\Lambda)\) corresponding to the controls \(u + s \cdot \Delta u\), i.e.

\[
q_s(t) = \text{End}^l[u + s \Delta u] = \text{End}^l[u] + s \cdot \text{End}^l[\Delta u] + s^2 \cdot \frac{1}{2!} \text{End}^l[\Delta^2 u] + \ldots + s^k \cdot \frac{1}{k!} \text{End}^l[\Delta^k u] + o(s^k) = q(t) + s \cdot b^{(1)}(t, \Delta u) + s^2 \cdot b^{(2)}(t, \Delta u) + \ldots + s^k \cdot b^{(k)}(t, \Delta u) + o(s^k),
\]

where \(b^{(m)}(t, \Delta u) := \frac{1}{m!} \partial_s^m \Big|_{s=0} q_s(t) = \frac{1}{m!} D^m_{\omega} \text{End}^l[\Delta u]\).

Actually, in light of Remark [14] there is a sense to speak about higher derivatives \(D^k_{\omega} \text{End}^l[\Delta u]\) only if we locally identify \(M\) with \(\mathbb{R}^n\) by means of a particular choice of local coordinates. Therefore, while reading the remaining part of this article the Reader should remember that the ODEs characterizing curves \(b^{(m)}(t, \Delta u)\) are derived in a particular coordinate frame. However, in spite of that, the whole \(k\)-jet of the curve \(s \mapsto \text{End}^l[u + s \cdot \Delta u]\) at \(s = 0\), i.e.

\[
\dot{q}(t) + s \cdot b^{(1)}(t, \Delta u) + s^2 \cdot b^{(2)}(t, \Delta u) + \ldots + s^k \cdot b^{(k)}(t, \Delta u) + o(s^k)
\]

is a well-defined geometric notion. We shall refer to it as a \(k\)-variation of \(\text{End}^l\) at \(u\) in the direction of \(\Delta u\).

As \(\dot{b}^{(m)}(t, \Delta u) = \frac{1}{m!} \partial_s^m \big|_{s=0} q_s(t)\), for each \(m = 1, 2, \ldots, k\) curves \(b^{(m)}(t, \Delta u)\) are solutions of a system of ODEs obtained by an \(m\)-fold differentiation of \((2.1)\) at \(s = 0\). The first few of these equations are easy to calculate:

\[
\begin{align*}
\dot{b}^{(1)}(t, \Delta u) &= \sum_i u_i \left( D_{\dot{q}(t)} X_i[b^{(1)}] \right) + \sum_i \Delta u_i X_i \big|_{\dot{q}(t)} \\
\dot{b}^{(2)}(t, \Delta u) &= \sum_i u_i \left( \frac{1}{2!} D^2_{\dot{q}(t)} X_i[b^{(2)}] \right) + \sum_i \Delta u_i D_{\dot{q}(t)} X_i[b^{(1)}] \\
\dot{b}^{(3)}(t, \Delta u) &= \sum_i u_i \left( \frac{1}{3!} D^3_{\dot{q}(t)} X_i[b^{(3)}] \right) + \sum_i \Delta u_i \left( \frac{1}{2!} D^2_{\dot{q}(t)} X_i[b^{(2)}] \right) + \sum_i \Delta u_i \left( \frac{1}{3!} D^3_{\dot{q}(t)} X_i[b^{(1)}] \right) \\
&+ \sum_i \Delta u_i \left( D_{\dot{q}(t)} X_i[b^{(2)}] \right) + \sum_i \Delta u_i \left( \frac{1}{2!} D^2_{\dot{q}(t)} X_i[b^{(1)}] \right).
\end{align*}
\]
Here \( q(t) = q_0(t) \) is a trajectory of (\ref{eq:control}) corresponding to the control \( u \). A clear inductive pattern begins to be seen. Moreover, the above equations are naturally graded after putting \( \deg(b^{(m)}) = m \) and \( \deg(\Delta u_i) = 1 \). In general, we have the following result

**Lemma 2.1** (the general form of \( b^{(m)}(t, \Delta u)s \)). Consider the \( k \)-jet of the curve \( s \mapsto \text{End}^l[u + s \cdot \Delta u] \) at \( s = 0 \) given in local coordinates by

\[
\text{End}^l(u + s \cdot \Delta u) = q(t) + s \cdot b^{(1)}(t, \Delta u) + s^2 \cdot b^{(2)}(t, \Delta u) + \ldots s^k \cdot b^{(k)}(t, \Delta u) + o(s^k).
\]

Then curves \( b^{(m)}(t, \Delta u) \) are subject to the following system of ODEs:

\[
\dot{b}^{(m)}(t, \Delta u) = \sum_i u_i \left( \sum_{\alpha, \omega(\alpha) = m} \frac{1}{\alpha!} \cdot D_{q(t)}^{|\alpha|} X_i [\overline{\dot{b}}(t, \Delta u)^{\alpha}] + \sum_{\beta, \omega(\beta) = m-1} \frac{1}{\beta!} \cdot D_{q(t)}^{|\beta|} X_i [\overline{\dot{b}}(t, \Delta u)^{\beta}] \right)
\]

where we use the notation introduced on page 2, i.e. the summation is taken over all multi-indices \( \alpha = (a_1, a_2, \ldots) \) of a given weight \( \omega(\alpha) := \sum_i i \cdot a_i \) equal \( m \) and \( m - 1 \), respectively. \( D_{q(t)}^{|\alpha|} X_i \) denotes the \( |\alpha| \)-th derivative of \( X_i \) at \( q(t) \) understood as a \( |\alpha| \)-linear map, and \( \overline{\dot{b}}(t, \Delta u) = (b^{(1)}(t, \Delta u), b^{(2)}(t, \Delta u), \ldots, b^{(k)}(t, \Delta u)) \).

**Proof.** The \( m \)-fold differentiation of (\ref{eq:ODE}) gives us

\[
\frac{d}{dt} \left( \partial_s^m q_s(t) \right) = \partial_s^m (q_s(t)) = \partial_s^m \left( \sum_i (u_i(t) + s \cdot \Delta u_i(t)) X_i(q_s(t)) \right) = 
\sum_i u_i(t) \partial_s^m \left( X_i(q_s(t)) \right) + \sum_i \Delta u_i(t) \partial_s^m \left( s \cdot X_i(q_s(t)) \right) = 
\sum_i u_i(t) \partial_s^m \left( X_i(q_s(t)) \right) + \sum_i \Delta u_i(t) \left( m \cdot \partial_s^{m-1} \left( X_i(q_s(t)) \right) + s \cdot \partial_s^m \left( X_i(q_s(t)) \right) \right)
\]

Now considering the above equality at \( s = 0 \), since \( \partial_s^m \big|_0 q_s(t) = m! \cdot b^{(m)}(t) \) we get

\[
m! \cdot \dot{b}^{(m)}(t) = \sum_i u_i(t) \partial_s^m \big|_0 \left( X_i(q_s(t)) \right) + m \sum_i \Delta u_i(t) \partial_s^{m-1} \big|_0 \left( X_i(q_s(t)) \right)
\]

Finally \( q_s(t) \) expands as \( q_s(t) = q(t) + s \cdot b^{(1)}(t) + s^2 \cdot b^{(2)}(t) + \ldots s^k \cdot b^{(k)}(t) + o(s^k) \) so, by the results of Lemma 1.3

\[
\partial_s^m \big|_0 \left( X_i(q_s(t)) \right) = m! \sum_{\alpha, \omega(\alpha) = m} \frac{1}{\alpha!} \cdot D_{q(t)}^{|\alpha|} X_i [\overline{\dot{b}}(t)^{\alpha}] \quad \text{and} \quad
\partial_s^{m-1} \big|_0 \left( X_i(q_s(t)) \right) = (m-1)! \sum_{\beta, \omega(\beta) = m-1} \frac{1}{\beta!} \cdot D_{q(t)}^{|\beta|} X_i [\overline{\dot{b}}(t)^{\beta}]
\]

Thus \( \ref{eq:ODE} \) holds. \( \square \)

Let us comment briefly on the question of the existence of solutions of the system \( \ref{eq:ODE} \). On the one hand, we may argue that by a general result of existence, uniqueness, and parameter-regularity of ODEs in the sense of Carathéodory – see for example [BP04] – curves \( b^{(m)}(t, \Delta u) \) should be well defined for all \( t \) for which \( q(t) \) satisfying \( \ref{eq:control} \) makes sense. We may, however, see this fact from a more technical perspective. Note namely, that if the \((m-1)\)-jet \((b^{(1)}(t, \Delta u), \ldots, b^{(m-1)}(t, \Delta u))\) is known, then by \( \ref{eq:ODE} \), the equation for \( b^{(m)}(t, \Delta u) \) takes the form of an affine ODE

\[
\dot{b}^{(m)}(t) = \sum_i u_i(t) D_{q(t)} X_i [\overline{b}^{(m)}(t)] + f(t),
\]
where $f(t)$ is measurable. Thus, whenever one finds a fundamental solution of the homogeneous part of this equation,

$$
\dot{b} = \sum_i u_i(t) D_{q(t)} X_i[b],
$$

$b^{(m)}(t, \Delta u)$ can be derived by the standard Cauchy formula. Yet the fundamental solution of the homogeneous part is simply the tangent map of the time-dependent flow of the control vector field $X_u(t, q) = \sum_i u_i(t) X_i(q)$. Thus it is indeed well-defined whenever the flow of $X_u(t, q)$ is well-defined. For a more detailed discussion of time-dependent flows and their tangent maps see [JRT16 Sec. 2].

**Example 2.2** (Generalized Martinet system, part 1.). After [BMP20], consider a control system in $\mathbb{R}^3 \ni (x_1, x_2, x_3)$ given by a pair of vector fields

$$
X_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad X_2 = (1 - x_1) \cdot \frac{\partial}{\partial x_2} + x_1 \cdot \frac{\partial}{\partial x_3},
$$

where $p > 1$ is an integer. Let $u(t) \equiv (0, 1)$ be the control corresponding to a trajectory $q(t) = (0, t, 0)$, and take another control $\Delta u(t) = (\Delta u_1(t), \Delta u_2(t))$. Let us describe curves $b^{(m)}(t, \Delta u)$ for $m = 1, 2, \ldots, p, p + 1$ in this setting.

We have $D_{q(t)}^m X_1 \equiv 0$ for every $m = 1, 2, \ldots$, while only non-trivial derivatives of $X_2$ along the considered trajectory are

$$
D_{q(t)}^\tau X_2[\cdot, \ldots, \cdot] = -\langle d x_1, \cdot \rangle \cdot \frac{\partial}{\partial x_2} \quad \text{and} \quad D_{q(t)}^\tau X_2[\underbrace{\cdot, \ldots, \cdot}_p] = p!\langle d x_1, \cdot \rangle^p \cdot \frac{\partial}{\partial x_3}.
$$

It follows that equations (2.2) look as follows

$$
\dot{b}^{(1)}(t) = -b^{(1)}(t) \cdot \frac{\partial}{\partial x_2} + \Delta u_1(t) \cdot \frac{\partial}{\partial x_1} + \Delta u_2(t) \cdot \frac{\partial}{\partial x_2}
$$

$$
\dot{b}^{(2)}(t) = -b^{(2)}(t) \cdot \frac{\partial}{\partial x_2} - \Delta u_2(t) \cdot b^{(1)}(t) \cdot \frac{\partial}{\partial x_2}
$$

$$
\ldots
$$

$$
\dot{b}^{(m)}(t) = -b^{(m)}(t) \cdot \frac{\partial}{\partial x_2} - \Delta u_2(t) \cdot b^{(m-1)}(t) \cdot \frac{\partial}{\partial x_2}
$$

$$
\ldots
$$

$$
\dot{b}^{(p)}(t) = -b^{(p)}(t) \cdot \frac{\partial}{\partial x_2} - \Delta u_2(t) \cdot b^{(p-1)}(t) \cdot \frac{\partial}{\partial x_2} + \left(b^{(1)}(t)\right)^p \cdot \frac{\partial}{\partial x_3}
$$

$$
\dot{b}^{(p+1)}(t) = -b^{(p+1)}(t) \cdot \frac{\partial}{\partial x_2} - \Delta u_2(t) \cdot b^{(p)}(t) \cdot \frac{\partial}{\partial x_2} + p \left(b^{(1)}(t)\right)^{p-1} b^{(2)}(t) \cdot \frac{\partial}{\partial x_3} + \Delta u_2(t) \left(b^{(1)}(t)\right)^p \cdot \frac{\partial}{\partial x_3}.
$$

As we see the first appearance of the $\frac{\partial}{\partial x_3}$ direction happens in degree $p$.

Since the initial values are $b^{(m)}(0) = 0$, for every $m = 1, 2, \ldots, p + 1$ and $i = 1, 2, 3$, it is easy to solve the above system:

$$
b^{(1)}_1(t) = \int_0^t \Delta u_1(\tau) \, d \tau
$$

$$
b^{(1)}_2(t) = \int_0^t \Delta u_2(\tau) \, d \tau - \int_0^t b^{(1)}_1(\tau) \, d \tau = \int_0^t \Delta u_2(\tau) \, d \tau - \left[\int_0^\tau \Delta u_1(s) \, d s\right] \, d \tau
$$

$$
b^{(2)}_2(t) = -\int_0^t \Delta u_2(\tau) \cdot b^{(1)}_1(\tau) \, d \tau = -\int_0^t \Delta u_2(\tau) \cdot \left[\int_0^\tau \Delta u_1(s) \, d s\right] \, d \tau
$$

$$
b^{(p)}_3(t) = \int_0^t \left(b^{(1)}_1(\tau)\right)^p \, d \tau = \int_0^t \left(\int_0^\tau \Delta u_1(s) \, d s\right)^p \, d \tau
$$

$$
b^{(p+1)}_3(t) = \int_0^t \Delta u_2(\tau) \cdot \left(b^{(1)}_1(\tau)\right)^p \, d \tau = \int_0^t \Delta u_2(\tau) \cdot \left(\int_0^\tau \Delta u_1(s) \, d s\right)^p \, d \tau,
$$

with all other components $b^{(m)}_i(t)$, for $m = 1, 2, \ldots, p + 1$, equal to zero.
3 Added coordinates

The idea of adapted coordinates Recall the expansion

$$\text{End}^k((u + s \cdot \Delta u) = q(t) + s \cdot b(1)(t, \Delta u) + s^2 \cdot b(2)(t, \Delta u) + \ldots$$

(with \(b^{(m)}(t, \Delta u)\)'s as in Lemma 2.1) which describes the derivatives of the end-point maps \(\text{End}^k\) at \(u\) in the direction \(\Delta u\). As we have seen, the formulas for \(b^{(m)}(t, \Delta u)\)'s are quite complicated. Our goal is to introduce, for each \(m = 1, 2, \ldots\), a family of \(\mathbb{R}^n\)-valued maps \(q^{(m)}(t, \Delta u) = \left( q^{(m)}_a(t, \Delta u) \right)_{a=1,2,\ldots,n}\), which satisfy the following two properties:

(i) there is 1-1 correspondence between \((b^{(1)}(t, \Delta u), \ldots, b^{(k)}(t, \Delta u))\) and \((q^{(1)}(t, \Delta u), \ldots, q^{(k)}(t, \Delta u))\) for each natural \(k\);

(ii) the derivatives \(q^{(m)}(t, \Delta u)\) are linear in \(\Delta u\).

It is therefore justified to treat \((q^{(1)}, \ldots, q^{(k)}) \in \mathbb{R}^{n-k}\) as coordinates on the space of \(k\)-jets \(T_{q(t)}^k M \ni (b^{(1)}, \ldots, b^{(k)})\)'s. We will call them adapted coordinates, meaning that they are specially adapted to the geometry of the control system \(\text{Adapted coordinates in degrees 1, 2, and 3 :} \end{equation}

Adapted coordinates in degrees 1, 2, and 3 Let us study a few examples of a low degree before passing to the general case. For each \(a = 1, 2, \ldots, n\) define:

\[
q_a^{(1)}(t, \Delta u) := \Phi_a^{(1)}(t) \left[ b^{(1)}(t, \Delta u) \right]
\]

\[
q_a^{(2)}(t, \Delta u) := \Phi_a^{(1)}(t) \left[ b^{(2)}(t, \Delta u) \right] + \frac{1}{2!} \Phi_a^{(2)}(t) \left[ b^{(1)}(t, \Delta u), b^{(1)}(t, \Delta u) \right]
\]

\[
q_a^{(3)}(t, \Delta u) := \Phi_a^{(1)}(t) \left[ b^{(3)}(t, \Delta u) \right] + \Phi_a^{(2)}(t) \left[ b^{(2)}(t, \Delta u), b^{(1)}(t, \Delta u) \right] + \frac{1}{3!} \Phi_a^{(3)}(t) \left[ b^{(1)}(t, \Delta u), b^{(1)}(t, \Delta u), b^{(1)}(t, \Delta u) \right]
\]

We postulate \(\{ \Phi_a^{(1)}(0) | a = 1, 2, \ldots, n\}\) to be a fixed basis of \(T_{q(0)}^1 M\), and set \(\Phi_a^{(k)}(0) = 0\) for \(k = 2, k = 3, \) and each \(a = 1, 2, \ldots, n\). Now the evolution of \(q_a^{(1)}(t)\) reads as

\[
q_a^{(1)}(t, \Delta u) = \Phi_a^{(1)} \left[ b^{(1)} \right] + \Phi_a^{(1)} \left[ b^{(1)} \right] = \Phi_a^{(1)} \left[ b^{(1)} \right] + \Phi_a^{(1)} \left[ \sum_i u_i DX_i \left[ b^{(1)} \right] \right] + \Phi_a^{(1)} \left[ \sum_i \Delta u_i X_i \right] = 0 \quad \text{assuming } a \text{ to satisfy } \Phi_a^{(1)}
\]

Therefore we postulate the following evolution of \(\Phi_a^{(1)}(t)\):

\[
\dot{\Phi}_a^{(1)}(t)[b] + \sum_i u_i \cdot \Phi_a^{(1)}(t) \left[ DX_i [b] \right] = 0 \quad \text{for every } b \in T_{q(t)} M.
\]
Clearly, the above evolution is dual to the evolution on $T_{q(t)} M$ by the flow of the time-dependent vector field $q \mapsto \sum_i u_i(t) X_i(q)$. This guarantees that for each $t \in [0, T]$ covectors $\Phi_a^{(1)}(t)$ form a basis of $T_{q(t)} M$. In consequence, the procedure of constructing maps $q^{(k)}(t)$ from $b^{(k)}(t)$'s will be reversible for each $k = 1, 2, \ldots$.

In degree two we have:

$$
\dot{q}_a^{(2)}(t, \Delta u) = \Phi_a^{(1)}[b^{(2)}] + \Phi_a^{(2)}[b^{(2)}] + \frac{1}{2!} \Phi_a^{(2)}[b^{(1)}, b^{(1)}] + \Phi_a^{(3)}[b^{(3)}] + \sum u_i \cdot \Phi_a^{(1)}[D X_i[b^{(2)}]] + \sum \Delta u_i \{ \Phi_a^{(1)}[D X_i[b^{(1)}]] + \Phi_a^{(2)}[X_i, b^{(1)}] \}
$$

Therefore we postulate

$$
\dot{\Phi}_a^{(2)}[b, b] + \sum u_i \cdot \left\{ 2\Phi_a^{(2)}[D X_i[b], b] + \Phi_a^{(1)}[D^2 X_i[b, b]] \right\} = 0 
$$

for every $b \in T_{q(t)} M$.

If the above holds then, since $\Phi_a^{(2)}$ is symmetric 2-linear, for every $b, b' \in T_{q(t)} M$ we have:

$$
(3.5) \quad \dot{\Phi}_a^{(2)}[b, b'] + \sum u_i \cdot \left\{ \Phi_a^{(1)}[D^2 X_i[b, b']] + \Phi_a^{(2)}[D X_i[b], b'] + \Phi_a^{(2)}[D X_i[b'], b] \right\} = 0
$$

Analogously in degree 3:

$$
\dot{q}_a^{(3)}(t) = \Phi_a^{(1)}[b^{(3)}] + \Phi_a^{(2)}[b^{(2)}, b^{(1)}] + \Phi_a^{(3)}[b^{(3)}] + \sum u_i \cdot \Phi_a^{(1)}[D X_i[b^{(2)}]] + \sum \Delta u_i \{ \Phi_a^{(1)}[D^2 X_i[b^{(2)}, b^{(1)}]] + D X_i[b^{(1)}], b^{(2)}] \}
$$

Therefore we postulate that for every $b \in T_{q(t)} M$

$$
(3.6) \quad \dot{\Phi}_a^{(3)}[b, b, b] + \sum u_i \cdot \left\{ 3\Phi_a^{(3)}[D X_i[b], b, b] + 3\Phi_a^{(2)}[D^2 X_i[b, b], b] + \Phi_a^{(1)}[D^3 X_i[b, b, b]] \right\} = 0
$$
Summing up, under assumptions (3.4)–(3.6), our new coordinates (3.1)–(3.3), evolve as follows:

\[
\begin{align*}
q^{(1)}_a &= \sum_i \Delta u_i \Phi^{(1)}_a [X_i] \\
q^{(2)}_a &= \sum_i \Delta u_i \left\{ \Phi^{(1)}_a [DX_i [b^{(1)}]] + \Phi^{(2)}_a [X_i, b^{(1)}] \right\} \\
q^{(3)}_a &= \sum_i \Delta u_i \left\{ \Phi^{(1)}_a [DX_i [b^{(2)}]] + \Phi^{(1)}_a [DX_i [b^{(1)}], b^{(1)}] + \Phi^{(2)}_a [X_i, b^{(2)}] + \frac{1}{2!} \Phi^{(3)}_a [X_i, b^{(1)}], \right\} \\
\end{align*}
\]

**The general case** The pattern observed for terms up to order 3, continues in all degrees. We may thus generalize the transformations (3.1)–(3.3) (satisfying conditions (3.4)–(3.6)) in the following definition.

**Definition 3.1.** Consider a trajectory \( q(t) \) of a control-linear system (A) corresponding to the control \( u \in \Omega \). Choose a local coordinate system on \( M \) and for each \( m = 1, 2, \ldots \) define a time-dependent family of symmetric \( m \)-linear maps

\[
\Phi^{(m)}(t) = \left( \Phi^{(m)}_a(t) \right)_{a=1,2, \ldots, n} : T_{q(t)} M \times \cdots \times T_{q(t)} M \to \mathbb{R}^n
\]

by setting the following conditions:

- we set \( \{ \Phi^{(1)}_a(0) \}_{a=1,2, \ldots, n} \) to be a basis of \( T^*_{q(0)}M \), while for \( m = 2, 3, \ldots \) maps \( \Phi^{(m)}(0) \) can be arbitrary,

- \( \Phi^{(m)}(t) \)'s are subject to the following ODEs:

\[
(3.7) \quad \dot{\Phi}^{(m)}(t) [b, \ldots, b] + \sum_i u_i(t) \left( \sum_{s=1}^{m} \binom{m}{s} \Phi^{(m-s+1)}(t) \left[ \sum_{s=1}^{m} \binom{m}{s} \Phi^{(m-s+1)}(t) \left[ D^{s}_{q(t)}X_i [b, \ldots, b], b, \ldots, b \right] \right] \right) = 0.
\]

For each \( k = 1, 2, \ldots \) and \( t \in [0, T] \) the assignment

\[
\Psi^{(k)}(t) : T^k_{q(t)} M \ni (b^{(1)}, \ldots, b^{(k)}) \mapsto (q^{(1)}, \ldots, q^{(k)}) \in \mathbb{R}^{n \cdot k},
\]

given by (within the notation convention of Rem. [1.2]

\[
(3.8) \quad q^{(m)} = \left( q^{(m)}_a \right)_{a=1,2, \ldots, n} := \sum_{\alpha, w(\alpha) = m} \frac{1}{\alpha!} \cdot \Phi^{(\alpha)}(t) [\vec{b}^\alpha]
\]

will be called the **transformation of adapted coordinates** of degree \( k \) at time \( t \).

We summarize basic properties of the above notion in the following result.

**Theorem 3.2.** For every \( t \in [0, T] \), and every \( k \), the transformation of adapted coordinates \( \Psi^{(k)}(t) \) does not depend on the choice of a local coordinate system. Further it is an isomorphism between the fibre \( T^k_{q(t)} M \) and \( \mathbb{R}^{n \cdot k} \) respecting the natural graded structures in which \( \deg (b^{(i)}) = \deg (q^{(i)}) = i \).

For any control \( \Delta u \in T_u \Omega \simeq \Omega \) consider the k-jet of the curve \( s \mapsto \text{End}^k [u + s \cdot \Delta u] \) at \( s = 0 \) given in local coordinates by

\[
\text{End}^k [u + s \cdot \Delta u] := \text{End}^k [u] + s \cdot b^{(1)}(t, \Delta u) + s^2 \cdot b^{(2)}(t, \Delta u) + \ldots + s^k \cdot b^{(k)}(t, \Delta u) + o(s^k).
\]

Let \( (q^{(1)}(t, \Delta u), \ldots, q^{(k)}(t, \Delta u)) \) be the image of the k-jet \( (b^{(1)}(t, \Delta u), \ldots, b^{(k)}(t, \Delta u)) \) under the transformation of adapted coordinates \( \Psi^{(k)}(t) \). That is, for \( m = 1, 2, \ldots, k \) we have

\[
(3.9) \quad q^{(m)}(t, \Delta u) := \sum_{\alpha, w(\alpha) = m} \frac{1}{\alpha!} \cdot \Phi^{(\alpha)}(t) [\vec{b}(t, \Delta u)^\alpha].
\]
Then, curves $q^{(m)}(t, \Delta u)$ satisfy the following system of ODEs:

\[(3.10)\]

\[
\dot{q}^{(m)}(t, \Delta u) = \sum_i \Delta u_i(t) \cdot \left\{ \sum_{\alpha, \beta, w(\alpha)+w(\beta)=m-1} \frac{1}{\alpha!} \cdot \frac{1}{\beta!} \cdot \Phi^{|\alpha|+1}(t) \left[ D_{q(t)} X_i \tilde{b}(t, \Delta u)^\alpha, \tilde{b}(t, \Delta u)^\beta \right] \right\}
\]

**Proof.** The fact that the map $\Psi^{(k)}(t)$ is well-defined will be proved in the next paragraph (Lemma 3.4), where we shall give a geometric interpretation of this construction.

Let us address the problem of reversibility. Note that for each $t \in [0, T]$ the family $\{\Phi_a^{(1)}(t)\}_{a=1, \ldots, n}$ is a basis of $T_{q(t)} M$. Indeed this is easily seen from (3.4), which easily implies that

\[
\Phi_a^{(1)}(t)[\dot{b}^{(1)}(t)] = \text{const}
\]

whenever $b^{(1)}(t)$ is subject to $\dot{b}^{(1)}(t) = \sum_i u_i(t) \cdot D_{q(t)} X_i [b^{(1)}(t)]$. The latter is the evolution in $TM$ induced by the flow of the (time-dependent) control vector field $q \mapsto \sum_i u_i(t) X_i(q)$. As this flow consists of diffeomorphisms (see [11, Sec. 2]), then the evolution of $b^{(1)}(t)$ is reversible, and hence so is the evolution of $\Phi^{(1)}(t) := \left( \Phi_a^{(1)}(t) \right)_{a=1, 2, \ldots, n}$.

We conclude that the assignment

\[
T_{q(t)} M \ni b^{(1)} \longmapsto \left( q^{(1)} = \Phi^{(1)}(t)[b^{(1)}] \right) \in \mathbb{R}^n
\]

is reversible for each $t \in [0, T]$. Similarly, so is

\[
T_{q(t)}^2 M \ni (b^{(1)}, b^{(2)}) \longmapsto \left( q^{(1)}, q^{(2)} = \Phi^{(1)}(t)[b^{(2)}] + \Phi^{(2)}(t)[b^{(1)}, b^{(1)}] \right) \in \mathbb{R}^{2n},
\]

as we may express $b^{(1)}$ in terms of $q^{(1)}$ and then reverse $\Phi^{(1)}(t)$ to get $b^{(2)}$ from $q^{(2)}$.

In higher degrees an analogous argument proves that formulas (3.8) define isomorphisms between the space of $k$-jets of curves in $M$ at $q(t)$ and the space $\mathbb{R}^{k-n}$

\[
T_{q(t)}^k M \ni (b^{(1)}, \ldots, b^{(k)}) \longmapsto (q^{(1)}, \ldots, q^{(k)}) \in \mathbb{R}^{k-n}.
\]

Finally note that formulas (3.8) respect the gradings given by $\text{deg}(b^{(m)}) = \text{deg}(q^{(m)}) = m$.

Now let us prove the remaining part of the assertion. Differentiation of (3.9) gives us

\[
q^{(m)}(t, \Delta u) = \sum_{\alpha, w(\alpha)=m} \Phi^{(|\alpha|)}[\tilde{b}(t, \Delta u)^\alpha] + \sum_{\alpha, w(\alpha)=m} \sum_{l \leq m} \frac{1}{\alpha!} \cdot a_l \cdot \Phi^{(|\alpha|)}[\dot{b}^{(l)}, \tilde{b}(t, \Delta u)^{\alpha-1(l)}],
\]

where naturally $\alpha - 1_l = (a_1, \ldots, a_{l-1}, a_l - 1, a_{l+1}, \ldots, a_k)$. Now note that by (3.7) the derivatives $\Phi^{(|\alpha|)}$ are linear in $u_i$’s, while by (2.2) the derivatives $\dot{b}^{(l)}$ have a part linear in $u_i$’s and a part linear in $\Delta u_i$’s. Therefore, $q^{(m)}$ splits into a part linear in $u_i$’s and a part linear in $\Delta u_i$’s, i.e

\[
q^{(m)}(t, \Delta u) = \sum_i u_i \cdot A_i + \sum_i \Delta u_i \cdot B_i,
\]

where $A_i$ and $B_i$ do not depend on neither $u_i$’s, nor $\Delta u_i$’s. We will now calculate these two parts separately, proving that the first one is zero, while the second one is precisely the right-hand side of (3.10). This will end the proof.

Let us begin with the second part. Since $\dot{b}^{(m)}(t)$ do not depend on $\Delta u_i$’s, only the derivatives $\dot{b}^{(l)}$ contribute
Now let us calculate the part of $\dot{q}(m)(t, \Delta u)$ linear in $u_i$’s. Proceeding as in the previous part of the proof we arrive at

$$\sum_i \Delta u_i \cdot A_i = \sum_{\alpha, \omega(\alpha) = m} \Phi(\alpha) \left[ \tilde{b}(t, \Delta u)^\alpha \right] + \sum_{\alpha, \omega(\alpha) = m} \sum_{l \leq m} \frac{1}{\alpha!} \cdot a_l \cdot \Phi(\alpha) \left[ \text{part of } \dot{b}(t) \text{ linear in } u_i's, \tilde{b}(t, \Delta u)^{\alpha - 1} \right]$$

$$= \sum_{\alpha, \omega(\alpha) = m} \Phi(\alpha) \left[ \tilde{b}(t)^\alpha \right] + \sum_{\alpha, \omega(\alpha) = m} \sum_{l \leq m} \frac{1}{\alpha!} \cdot a_l \cdot \Phi(\alpha) \left[ \sum_i \Delta u_i \cdot \left\{ \sum_{\beta, \omega(\beta) = l} \frac{1}{\beta!} \cdot D[\beta]_{q(t)} X_i [\tilde{b}(t)^\beta], \tilde{b}(t)^{\alpha - 1} \right\} \right]$$

$$= \sum_{\alpha, \omega(\alpha) = m} \Phi(\alpha) \left[ \tilde{b}(t)^\alpha \right] + \sum_{i} \sum_{\alpha, \omega(\alpha) = m} \frac{1}{\alpha!} \cdot a_l \cdot \Phi(\alpha) \left[ \sum_{\beta, \omega(\beta) = l} \frac{1}{\beta!} \cdot D[\beta]_{q(t)} X_i [\tilde{b}(t)^\beta], \tilde{b}(t)^{\alpha - 1} - l \right]$$

Now observe that a triple $(\alpha, l, \beta)$ where $\omega(\alpha) = m$, $a_l > 0$ and $\omega(\beta) = l - 1$ uniquely determines a pair of multi-indexes $(\gamma = \alpha - 1_l, \beta)$ satisfying $w(\gamma) + w(\beta) = (m - l) + l - 1 = m - 1$. Thus we may change the summation order in the expression above to obtain

$$\sum_i \Delta u_i \cdot B_i = \sum_i \Delta u_i \cdot \left\{ \sum_{\gamma, \beta, \omega(\gamma) + \omega(\beta) = m - 1} \frac{1}{\gamma!} \cdot \Phi(\gamma + 1) \left[ D[\beta]_{q(t)} X_i [\tilde{b}(t)^\beta], \tilde{b}(t)^{\gamma} \right] \right\}$$

in agreement with (3.10).

Now let us calculate the part of $\dot{q}(m)(t, \Delta u)$ linear in $u_i$’s. Proceeding as in the previous part of the proof we arrive at

$$\sum_i \Delta u_i \cdot B_i = \sum_i \Delta u_i \cdot \left\{ \sum_{\gamma, \beta, \omega(\gamma) + \omega(\beta) = m - 1} \frac{1}{\gamma!} \cdot \Phi(\gamma + 1) \left[ D[\beta]_{q(t)} X_i [\tilde{b}(t)^\beta], \tilde{b}(t)^{\gamma} \right] \right\}$$

where the changes of summation order in the last passage are made analogously as before. We would like to show that the above equals to zero. To see this note that formula (3.7) describes the evolution of a $m$-linear map $\Phi(m)$. Due to the symmetry of $\Phi(m)$ we also have

$$-\Phi(m)[v_1, \ldots, v_m] = \sum_i u_i \cdot \left\{ \sum_{s \in \Sigma_m} \frac{1}{s!(m - s)!} \Phi(m - s + 1) \left[ D[q(t)] X_i [v_{\sigma(1)}, \ldots, v_{\sigma(s)}], v_{\sigma(s + 1)}, \ldots, v_{\sigma(m)} \right] \right\}$$

for every $m$-tuple of vectors $v_1, v_2, \ldots, v_m \in T_q M$. Hence

$$\sum_{\alpha, \omega(\alpha) = m} \frac{1}{\alpha!} \cdot \Phi(\alpha) [\tilde{b}(t)^\alpha] =$$

$$= \sum_{\alpha, \omega(\alpha) = m} \sum_{i} \frac{1}{\alpha!} \cdot \sum_{s \in \Sigma_m} \frac{1}{s!(\alpha - s)!} \Phi(\alpha - s + 1) \left[ D[q(t)] X_i [b^{(\sigma(1))}, \ldots, b^{(\sigma(s))}], b^{(\sigma(s + 1))}, \ldots, b^{(\sigma(\alpha))}] \right]$$

Due to the fact that $\Phi_a(\alpha - s + 1)$ and $D[q(t)] X_i$ are multi-linear, we may identify two permutations that have the same $s$ initial elements. The number of such permutations for a given $\alpha$ is precisely $s!(\alpha - s)!$. It follows
that we can identify the above triple sum over \((\alpha, \sigma, s)\) as a sum over sub-divisions of the multi-index \(\alpha\) into a sum \(\alpha = \beta + \gamma\) (so \(s = |\beta|, |\alpha| - s = |\gamma|\), and \(w(\alpha) = w(\beta) + w(\gamma)\)). The latter are however taken with multiplicity \(\binom{\alpha}{\beta} = \prod_i \binom{\alpha_i}{\beta_i}\), as \(\binom{\alpha}{\beta}\) is a number of ways an ordered sequence of \(\alpha_i\) elements \((b_1, \ldots, b_i)\) can be divided between in two groups of \(\beta_i\) and \(\gamma_i = \alpha_i - \beta_i\) elements. Since \(\alpha! = \binom{\alpha}{\beta} \cdot \beta! \cdot \gamma!\). We arrive at

\[
\sum_{\alpha, w(\alpha) = m} \Phi^{(\alpha)}(b(t, \Delta u)^\alpha) = -\sum_i u_i \left\{ \sum_{\gamma, \beta, w(\gamma) + w(\beta) = m} \frac{1}{|\beta|} \frac{1}{|\gamma|} \Phi^{(\gamma+1)}(D_{\alpha(t)} X, b(t, \Delta u)^\beta), b(t, \Delta u)^\gamma) \right\},
\]

proving that \(\sum_i u_i \cdot A_i \equiv 0\). This ends the proof.

**Example 3.3** (Generalized Martinet system, part 2.). Let us apply the above theory to Example 2.2. Equation (3.24) reads as

\[
\Phi^{(1)}(t)[\dot{x}] = -\Phi^{(1)}(t) [D_{\dot{q}}(t) X][\dot{x}] = \langle dx_1, \dot{x} \rangle \cdot \Phi^{(1)}(t)[\partial x_2].
\]

For a natural choice of initial value \(\Phi^{(1)}(0) = \text{Id}_{\mathbb{R}^3}\) this implies that \(\Phi^{(1)}(t)\) is the following linear isomorphism on \(\mathbb{R}^3\)

\[
\Phi^{(1)}(t)[\partial x_1] = \partial x_1 + t \cdot \partial x_2, \quad \Phi^{(1)}(t)[\partial x_2] = \partial x_2 \quad \text{and} \quad \Phi^{(1)}(t)[\partial x_3] = \partial x_3.
\]

In higher degrees \(\Phi^{(m)}(t) \equiv 0\), for \(m = 2, 3, \ldots, p-1, p+1, \ldots\), with the sole exception of a single component of \(\Phi^{(p)}(t)\), namely

\[
\Phi^{(p)}(t)[\partial x_1, \ldots, \partial x_p] = -p \cdot t \cdot \partial x_3.
\]

We can now use formula (3.8) defining the transformation of adapted coordinates to arrive at

\[
q^{(1)}(t) = \Phi^{(1)}(t) \left[ b^{(1)}(t) \right]
\]

\[
q^{(2)}(t) = \Phi^{(1)}(t) \left[ b^{(2)}(t) \right] + \Phi^{(2)}(t) \left[ b^{(1)}(t), b^{(1)}(t) \right] = \Phi^{(1)}(t) \left[ b^{(2)}(t) \right]
\]

\[
\ldots
\]

\[
q^{(m)}(t) = \Phi^{(1)}(t) \left[ b^{(m)}(t) \right]
\]

\[
\ldots
\]

\[
q^{(p)}(t) = \Phi^{(1)}(t) \left[ b^{(p)}(t) \right] + \frac{1}{p!} \Phi^{(p)}(t) \left[ b^{(1)}(t), \ldots, b^{(1)}(t) \right] = \Phi^{(1)}(t) \left[ b^{(p)}(t) \right] - t \left( b^{(1)}(t) \right)^p \cdot \partial x_3
\]

\[
q^{(p+1)}(t) = \Phi^{(1)}(t) \left[ b^{(p+1)}(t) \right] + \frac{1}{(p-1)!} \Phi^{(p)}(t) \left[ b^{(1)}(t), \ldots, b^{(1)}(t), b^{(2)}(t) \right] = \Phi^{(1)}(t) \left[ b^{(p+1)}(t) \right] - p \cdot t \left( b^{(1)}(t) \right)^{p-1} b^{(2)}(t) \cdot \partial x_3
\]

We can differentiate the above formulas, or use evolution equations (3.10) to get evolution equations for \(q^{(m)}\)'s which have a bit simpler form then those for \(b^{(m)}\)'s

\[
q^{(1)}(t) = \Delta u_1(t) \cdot (\partial x_1 + t \cdot \partial x_2) + \Delta u_2(t) \cdot \partial x_2
\]

\[
q^{(2)}(t) = -\Delta u_2(t) \cdot b^{(1)}(t) \cdot \partial x_2
\]

\[
\ldots
\]

\[
q^{(m)}(t) = -\Delta u_2(t) \cdot b^{(m-1)}(t) \cdot \partial x_2
\]

\[
\ldots
\]

\[
q^{(p)}(t) = -\Delta u_2(t) \cdot b^{(p-1)}(t) \cdot \partial x_2 - \Delta u_1(t) \cdot p \cdot t \left( b^{(1)}(t) \right)^{p-1} \cdot \partial x_3
\]

\[
q^{(p+1)}(t) = -\Delta u_2(t) \cdot b^{(p)}(t) \cdot \partial x_2 + \Delta u_2(t) \left( b^{(1)}(t) \right)^p \cdot \partial x_3 + \Delta u_1(t) \cdot p(p-1) \cdot t \left( b^{(1)}(t) \right)^{p-2} b^{(2)}(t) \cdot \partial x_3
\]
The solutions are
\[
q_1^{(1)}(t) = b_1^{(1)}(t) = \int_0^t \Delta u_1(\tau) \, d\tau
\]
\[
q_2^{(1)}(t) = \int_0^t \Delta u_2(\tau) + \int_0^t \tau \cdot \Delta u_1(\tau) \, d\tau
\]
\[
q_2^{(2)}(t) = -\int_0^t \Delta u_2(\tau) \cdot b_1^{(1)}(\tau) \, d\tau = -\int_0^t \Delta u_2(\tau) \cdot \left( \int_0^\tau \Delta u_1(s) \, ds \right) \, d\tau
\]
\[
q_3^{(p)}(t) = -t \left( b_1^{(1)}(t) \right)^p = -t \cdot \left( \int_0^t \Delta u_1(\tau) \, d\tau \right)^p
\]
\[
q_3^{(p+1)}(t) = \int_0^t \Delta u_2(\tau) \left( b_1^{(1)}(\tau) \right)^p \, d\tau = \int_0^t \Delta u_2(\tau) \cdot \left( \int_0^\tau \Delta u_1(s) \, ds \right)^p \, d\tau,
\]
with all other components \(q_i^{(m)}(t)\), for \(m = 1, 2, \ldots, p + 1\), equal to zero.

**Geometric interpretation of the adapted coordinates**

So far the construction of adapted coordinates may be seen as a computational trick which helps to simplify the evolution equations for curves \(b^{(m)}(t, \Delta u)\). However, as we shall see in this paragraph, it has a natural geometric interpretation.

Consider \(q(t)\) a trajectory of the control-linear system \((\mathbb{A})\) corresponding to the control \(u \in \Omega\) and initial condition \(q(0) = q_0\). Let now \(q_s(t)\) be a family of solutions of \((\mathbb{A})\) corresponding to the same control \(u\) and a smooth curve of initial conditions \(s \mapsto q_s(0)\), passing through \(q_0 = q_0(0)\) (so that \(q_s(0) = q(0)\)). It turns out that the assignment \(q_s(0) \rightarrow q_s(t)\) gives rise to a well-defined map on \(k\)-jets at \(s = 0\):
\[
T^k_{q_0} M \ni [q_s(0)]_{\sim k} \mapsto [q_s(t)]_{\sim k} \in T^k_{q(t)} M.
\]
We shall denote this map – the \(k\)-th tangent lift of the flow of the control vector field \(X_u\) – by \(T^k X^t_{q(t)}\). It describes the natural action on \(k\)-jets of the evolution (flow) of the control system \((\mathbb{A})\) for a fixed control \(u\). The following results explain the relation of \(T^k X^t_{u}\) with the transformation of adapted coordinates.

**Lemma 3.4.** Let \(q(t)\) be a trajectory of the control-linear system \((\mathbb{A})\) corresponding to a control \(u \in \Omega\). Consider a 1-parameter family of \(k\)-jets \(\bar{b}(t) = (b^{(1)}(t), \ldots, b^{(k)}(t)) \in T^k_{q(t)} M\). Then
\[
\bar{b}(t) = T^k X^t_{q(t)}[\bar{b}(0)] \quad \text{for every } t \in [0, T] \text{ if and only if }
\]
\[
\bar{b}(t) = (b^{(1)}(t), \ldots, b^{(k)}(t)) \text{ is constant under the transformation of adapted coordinates } \Psi^{(k)}(t).
\]
In other words
\[
\left( \Psi^{(k)}(t) \right)^{-1} [q^{(1)}(t), \ldots, q^{(k)}(t)] = \bar{b}(t) = T^k X^t_{q(t)}[\bar{b}(0)],
\]
where \((q^{(1)}, \ldots, q^{(k)}) = \Psi^{(k)}(0)[\bar{b}(0)]\). In particular, the construction of the transformation of adapted coordinates does not depend on the choice of the local coordinate system on \(M\).

**Proof.** Choose a local coordinate system on \(M\) and let \(q_s(t)\) be a family of solutions of \((\mathbb{A})\) as described at the beginning of this paragraph. Now
\[
\dot{q}_s(t) = \sum_i u_i(t) X_i(q_s(t)) \quad \text{with the initial condition } q_s(0),
\]

hence \(\bar{b}(t) = (b^{(1)}(t), \ldots, b^{(k)}(t))\) – the \(k\)-jet of \(s \mapsto q_s(t)\) at \(s = 0\), which by definition equals to \(T^k X^t_{u}[\bar{b}(0)]\) – satisfies an ODE obtained by a repetitive differentiation of the above equation with respect to \(s\). This will, however, be a special case of an ODE obtained by an analogous differentiation of equation (2.1) if we take \(\Delta u = 0\). As we know from Lemma 2.1 the latter equation is just (2.2), and so \(\bar{b}(t)\) is subject to (2.2) for \(\Delta u = 0\). Now, by the results of Theorem 2.2 after an application of the transformation of adapted coordinates, the curve \(\tilde{q}(t) = (q^{(1)}(t), \ldots, q^{(k)}(t)) = \Psi^{(k)}(t)[\bar{b}(t)]\) would satisfy equation (3.10) with \(\Delta u = 0\). Thus \(\tilde{q}(t) = 0\) which ends the proof.

\(\square\)

This is a simple consequence of the theorem about the regularity of the dependence of a solution of an ODE (in the sense of Carathéodory) on the initial condition.
Getting rid of $b^{(m)}$’s Our initial motivation behind the construction of adapted coordinates was to simplify the evolution equations for the polynomial expansion of $s \mapsto \text{End}^k [u + s \cdot \Delta u]$. As a result, we got the formula (3.10). Note, however, that as $q^{(m)}(t, \Delta u)$’s and $b^{(m)}(t, \Delta u)$’s are related by means of the transformation of adapted coordinates which is invertible, it is possible to express the right-hand side of (3.10) in terms of $q^{(m)}(t, \Delta u)$’s only. Thus we can interpret equation (3.10) as a time-dependent control-linear system in variables $(q^{(1)}, \ldots, q^{(k)})$ (or to view it differently a control-affine system in variables $(t, q^{(1)}, \ldots, q^{(k)})$), where $\Delta u \in \Omega$ plays the role of a control. This observation is summarised as follows.

Theorem 3.5. Consider a trajectory $q(t)$ of a control-linear system $\bar{A}$ corresponding to the control $u \in \Omega$ and choose $k \in \mathbb{N}$. Then there exists an affine distribution $D^{(k)} = \partial_t + \text{span}_\mathbb{R} \{ Y_1^{(k)}, \ldots, Y_l^{(k)} \}$ on $\mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \ni (t, q^{(1)}, \ldots, q^{(k)})$ with the following properties:

(i) consider a natural graded space structure on $\mathbb{R}^{1+kn}$ by setting $\deg(t) = 0$ and $\deg(q^{(m)}_a) = m$. Then for every $i = 1, 2, \ldots, l$ the field $Y_i^{(k)}$ is a homogeneous vector field of degree $-1$

(ii) for every control $\Delta u \in T_q \Omega \ni \Omega$ the curve $p^{(k)}(t) := (t, q^{(1)}(t, \Delta u), \ldots, q^{(k)}(t, \Delta u))$ - where $(q^{(1)}(t, \Delta u), \ldots, q^{(k)}(t, \Delta u))$ is the image of the $k$-jet of $s \mapsto \text{End}^k [u + s \cdot \Delta u]$ at $s = 0$ under the transformation of adapted coordinates $\Psi^{(k)}(t)$ considered in Theorem 3.2 is a trajectory of the control-affine system defined by $D^{(k)}$ for the control $\Delta u$, that is

$$\Sigma^{(k)}(\Delta u) \quad \hat{p}^{(k)}(t) = \partial_t + \sum_i \Delta u_i(t) \cdot Y_i^{(k)}(\hat{p}^{(k)}(t)) \quad \text{with } \hat{p}^{(k)}(0) = (0, 0, \ldots, 0).$$

Moreover, for each $k = 2, 3, \ldots$, the field $Y_i^{(k)}$ projects to $Y_i^{(k-1)}$ under the mapping

$$\mathbb{R}^{1+nk} \ni (t, q^{(1)}, \ldots, q^{(k-1)}, q^{(k)}) \mapsto (t, q^{(1)}, \ldots, q^{(k-1)}) \in \mathbb{R}^{1+n(k-1)}.$$

Proof. Fix a natural $k$ and consider evolution equations (3.10). They are of the form

$$q^{(m)}(t, \Delta u) = \sum_i \Delta u_i(t) \cdot \tilde{Y}_i^{(m)}(t, \tilde{b}(t, \Delta u)),$$

where $m = 1, 2, \ldots, k$ and $\tilde{Y}_i^{(m)}(t, \tilde{b}(t, \Delta u))$ is polynomial in $\tilde{b}(t, \Delta u) := (b^{(1)}(t, \Delta u), \ldots, b^{(k)}(t, \Delta u))$ of graded degree $m - 1$. Now we may substitute $\tilde{b}(t, \Delta u)$ with $\tilde{q}(t, \Delta u) := (q^{(1)}(t, \Delta u), \ldots, q^{(k)}(t, \Delta u))$ using the inverse of the transformation of adapted coordinates $\Psi^{(k)}(t)$. Note that the latter transformation intertwines the natural graded structures in the space of $b^{(m)}$’s and $q^{(m)}$’s. Thus we get

$$q^{(m)}(t, \Delta u) = \sum_i \Delta u_i(t) \cdot Y_i^{(m)}(t, \tilde{q}(t, \Delta u)),$$

where $Y^{(m)}(t, \tilde{q}) = \tilde{Y}_i^{(m)}(t, \Psi^{(k)}(t)^{-1}[\tilde{q}])$ is polynomial in $\tilde{q}$ of graded degree $m - 1$. \qed

Let us see how this result looks in low degrees. Consider the linear isomorphism $\Phi^{(1)}(t) : T_q M \rightarrow \mathbb{R}^n$ and denote its inverse by $A(t) : \mathbb{R}^n \rightarrow T_q M$. Then

$$b^{(1)}(t, \Delta u) = A(t)[q^{(1)}(t, \Delta u)] \quad \text{and} \quad b^{(2)}(t, \Delta u) = A(t)[q^{(2)}(t, \Delta u)] - \frac{1}{2!} A(t)\Phi^{(2)}[Aq^{(1)}(t, \Delta u), Aq^{(1)}(t, \Delta u)],$$

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and hence equations \( \Sigma^{(k)}(\Delta u) \) look as follows

\[
\dot{q}^{(1)} = \sum_{i} \Delta u_{i} \cdot \Phi^{(1)}[X_{i}]
\]

\[
\dot{q}^{(2)} = \sum_{i} \Delta u_{i} \cdot \left\{ \Phi^{(1)} \left[ DX_{i}[Aq^{(1)}] \right] + \Phi^{(2)} \left[ X_{i}, Aq^{(1)} \right] \right\}
\]

\[
\dot{q}^{(3)} = \sum_{i} \Delta u_{i} \cdot \left\{ \Phi^{(1)} \left[ DX_{i}[Aq^{(2)}] \right] - \frac{1}{2!} \Phi^{(1)} \left[ A\Phi^{(2)}[Aq^{(1)}], Aq^{(1)} \right] + \frac{1}{2!} \Phi^{(1)} \left[ D^{2}X_{i}[Aq^{(1)}], Aq^{(1)} \right] + \Phi^{(2)} \left[ DX_{i}[Aq^{(1)}], Aq^{(1)} \right] + \Phi^{(2)} \left[ X_{i}, Aq^{(2)} \right] - \frac{1}{2!} \Phi^{(2)} \left[ X_{i}, A\Phi^{(2)}[Aq^{(1)}], Aq^{(1)} \right] + \frac{1}{2!} \Phi^{(2)} \left[ X_{i}, Aq^{(1)}, Aq^{(1)} \right] \right\}
\]

Unfortunately, we were unable to derive exact formulas for \( \dot{q}^{(m)} \)'s in every degree, as they become very complicated with the increasing \( m \).

**Example 3.6 (Generalized Martinet system, part 3.).** In the previous Example 3.5 we derived adapted coordinates and their evolution equations for a generalized Martinet system. In that particular situation it is easy to reverse the transformation of adapted coordinates to arrive at

\[
b_{1}^{(m)}(t) = q_{1}^{(m)} \quad \text{for } m = 1, 2, \ldots, p, p + 1;
\]

\[
b_{2}^{(m)}(t) = q_{2}^{(m)}(t) - t \cdot q_{1}^{(m)}(t) \quad \text{for } m = 1, 2, \ldots, p, p + 1; \text{ and}
\]

\[
b_{3}^{(m)}(t) = q_{3}^{(m)}(t) \quad \text{for } m = 1, 2, \ldots, p - 1; \text{ and}
\]

\[
b_{3}^{(p)}(t) = q_{3}^{(p)}(t) + t \left( q_{1}^{(1)}(t) \right)^{p} - q_{3}^{(p+1)}(t) = q_{3}^{(p+1)}(t) + t \cdot p \left( q_{1}^{(1)}(t) \right)^{p-1} q_{1}^{(2)}(t).
\]

Thus evolution equations become

\[
\dot{q}^{(1)}(t) = \Delta u_{1}(t) \cdot (\partial_{x_{1}} + t \cdot \partial_{x_{2}}) + \Delta u_{2}(t) \cdot \partial_{x_{2}}
\]

\[
\dot{q}^{(2)}(t) = -\Delta u_{2}(t) \cdot q_{1}^{(1)}(t) \cdot \partial_{x_{2}}
\]

\[
\dot{q}^{(3)}(t) = -\Delta u_{2}(t) \cdot q_{1}^{(2)}(t) \cdot \partial_{x_{2}} - \Delta u_{1}(t) \cdot p \cdot t \left( q_{1}^{(1)}(t) \right)^{p-1} \cdot \partial_{x_{3}}
\]

\[
\dot{q}^{(p+1)}(t) = -\Delta u_{2}(t) \cdot q_{1}^{(p)}(t) \cdot \partial_{x_{2}} - \Delta u_{1}(t) \cdot p(p - 1) \cdot t \left( q_{1}^{(1)}(t) \right)^{p-2} q_{1}^{(2)} \cdot \partial_{x_{3}} + \Delta u_{2}(t) \cdot p \left( q_{1}^{(1)} \right)^{p} \partial_{x_{3}}.
\]

Therefore the fields \( Y_{1}^{(p+1)} \) and \( Y_{2}^{(p+1)} \) described in Theorem 3.5 are

\[
Y_{1}^{(p)} = \partial_{q_{1}} + t \cdot \partial_{q_{2}} - p \cdot t \left( q_{1}^{(1)} \right)^{p-1} \cdot \partial_{q_{3}} - p(p - 1) \cdot t \left( q_{1}^{(1)} \right)^{p-2} q_{1}^{(2)} \cdot \partial_{q_{3}} \quad \text{and}
\]

\[
Y_{2}^{(p)} = \partial_{q_{2}} - \sum_{m=1}^{p+1} q_{1}^{(m-1)} \cdot \partial_{q_{2}^{(m)}} + p \left( q_{1}^{(1)} \right)^{p} \cdot \partial_{q_{3}^{(p+1)}}.
\]

## 4 Examples and applications

### 4.1 Adapted coordinates for invariant systems on Lie groups

**The geometric setting** Let us consider a special situation of a control-linear system \( \Lambda \) with \( M = G \) being a Lie group, and \( D \) being a left-invariant distribution on \( G \), determined by an \( l \)-dimensional subspace
coordinates in the above setting. The comparison of the above system (4.2) with general formulas (2.2) reveals that $m > b$.

By (3.10), we conclude that for each $m \in \mathbb{R}$, curves $b^{(m)}(t)$ satisfy the following system of ODEs:

$$\begin{align*}
\dot{b}^{(1)}(t) &= b^{(1)}(t) \circ u(t) + g(t) \circ \Delta u(t) \\
\dot{b}^{(2)}(t) &= b^{(2)}(t) \circ u(t) + 2 \cdot b^{(1)}(t) \circ \Delta u(t) \\
&\quad \vdots \\
\dot{b}^{(k)}(t) &= b^{(k)}(t) \circ u(t) + k \cdot b^{(k-1)}(t) \circ \Delta u(t)
\end{align*}$$

(4.2)

**Calculation of adapted coordinates**

Now we would like to apply Theorem 3.2 and construct adapted coordinates in the above setting. The comparison of the above system (4.2) with general formulas (2.2) reveals that $D_{g(t)}X_i[b] = b \circ e_i$, and $D_{g(t)}^{m}X_i \equiv 0$ for $m > 1$. Hence, by the results of Theorem 3.2 also $\Phi_\alpha^{(m)}(t) \equiv 0$ for all $m > 1$. On the other hand, covectors $\Phi_\alpha^{(1)}(t)$ should satisfy

$$0 = \dot{\Phi}_\alpha^{(1)}(t)[b] + \Phi_\alpha^{(1)} \left( \sum_i u_i(t) D_{g(t)}X_i[b] \right) = \dot{\Phi}_\alpha^{(1)}(t)[b] + \Phi_\alpha^{(1)}[b \circ u(t)],$$

with $\Phi_\alpha^{(1)}(0) := \xi_\alpha$ being a fixed basis of $T^*_e G = \mathfrak{g}^*$. Let us identify $\mathfrak{g}$ with $\mathbb{R}^n$ by means of

$$I : \mathfrak{g} \ni b \mapsto \langle \langle \xi_1, b \rangle, \ldots, \langle \xi_n, b \rangle \rangle \in \mathbb{R}^n.$$ 

Now a simple calculation shows that $\Phi^{(1)}(t)[b] := I^{-1} \left( \Phi_1^{(1)}(t)[b], \ldots, \Phi_n^{(1)}(t)[b] \right)$ is just

$$\Phi^{(1)}(t)[b] = b \circ g(t)^{-1}.$$

By (3.10), we conclude that for each $m = 1, 2, 3, \ldots$ the adapted coordinates $q^{(m)}(t) = \Phi^{(1)}(t) \left[ b^{(m)}(t) \right] = b^{(m)}(t) \circ g(t)^{-1} \in \mathfrak{g}$ satisfy

$$q^{(m)}(t) = m \sum_i \Delta u_i(t) \Phi^{(1)}(t) \left[ D_{g(t)}X_i \left[ b^{(m-1)}(t) \right] \right] = m \Phi^{(1)}(t) \left[ b^{(m-1)}(t) \circ \Delta u(t) \right] = m b^{(m-1)}(t) \circ \Delta u(t) \circ g(t)^{-1}.$$ 

In this particular situation, we may easily get rid of $b^{(m)}$’s to arrive at the following system of ODEs:

$$\begin{align*}
\dot{q}^{(1)}(t) &= g(t) \circ \Delta u(t) \circ g(t)^{-1} = Ad_{g(t)} \Delta u(t) \\
\dot{q}^{(2)}(t) &= 2 b^{(1)}(t) \circ \Delta u(t) \circ g(t)^{-1} = 2 q^{(1)}(t) \circ g(t) \circ \Delta u(t) \circ g(t)^{-1} = 2 q^{(1)}(t) \circ Ad_{g(t)} \Delta u(t) \\
&\quad \vdots \\
\dot{q}^{(k)}(t) &= k q^{(k-1)}(t) \circ Ad_{g(t)} \Delta u(t).
\end{align*}$$

(4.3)
This leads to the following set of solutions

\[
\begin{align*}
    b^{(1)}(t) &= \left[ \int_0^t \operatorname{Ad}_{g(t)} \Delta u(t_1) \, dt_1 \right] \circ g(t) \\
    b^{(2)}(t) &= 2 \left[ \int_0^t q^{(1)}(t_1) \circ \operatorname{Ad}_{g(t_1)} \Delta u(t_1) \, dt_1 \right] \circ g(t) \\
    &= 2 \left[ \int_0^t \int_0^{t_1} \operatorname{Ad}_{g(t_2)} \Delta u(t_2) \, dt_2 \circ \operatorname{Ad}_{g(t_1)} \Delta u(t_1) \, dt_1 \right] \circ g(t) \\
    &\quad \cdots \\
    b^{(k)}(t) &= k! \left[ \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \operatorname{Ad}_{g(t_k)} \Delta u(t_k) \circ \cdots \circ \operatorname{Ad}_{g(t_1)} \Delta u(t_1) \, dt_k \cdots dt_1 \right] \circ g(t),
\end{align*}
\]

which are in perfect agreement with the results of [LD21], where the same formulas were obtained by a different method.

**Remark about general Lie groups** Finally, we would like to comment about the geometric sense of the formulas (4.4) on a general Lie group, where, at the level of the Lie algebra \( \mathfrak{g} \), there is no obvious analogue of the matrix multiplication \( \circ \). To do this observe that our basic equation (4.1) reads as

\[
g(t) = F(g(t), u(t)),
\]

where \( F : G \times \mathfrak{g} \to TG \) is the left trivialization of \( TG \), i.e. \( F(g, a) = T\mathfrak{g}[a] \), with \( L_g : G \to G \) given by \( L_g(h) = gh \). For our purposes, it will be convenient to treat \( F \) as a restriction of the map \( Tm : TG \times TG \to TG \) – the tangent map to the group multiplication \( m : G \times G \to G \) – to \( G \times T_eG \subset TG \times TG \), where on the first leg we embed \( G \) into \( TG \) as the zero section. Now system (4.2) describes the \( k \)-jet of (4.5), i.e. it is constructed by feeding \( T^kF : T^kG \times T^k\mathfrak{g} \to T^kG \) with the \( k \)-th jet \((g, b^{(1)}, \ldots, b^{(k)}) \in T^kG \), and \((u, \Delta u, 0, \ldots, 0) \in T^k\mathfrak{g} \equiv \mathfrak{g} \times \cdots \times \mathfrak{g} \), and composing the result with the canonical flip \( \kappa : T^kTG \simeq T^{k+1}G \) to get a vector tangent to \( T^kG \). Remembering that \( F \) is a restriction of \( Tm \), we conclude that, up to the canonical identification \( \kappa : T^kTG \simeq T^{k+1}G \), the multiplication in the system (4.2) is the multiplication \( T^kTm : T^kTG \times T^kTG \to T^kTG \) applied to a particular pair of \( k \)-jets embedded in \( T^kTG \). At this point, it is worth to remark, that \( T^kTG \) with the multiplication \( T^kTm \) is actually a Lie group – see [KMS93] 37.16.

Finally, in the setting of Lie groups we pass to adapted coordinates by the right translation by \( g(t)^{-1} \) acting on the \( k \)-jet \((q, b^{(1)}, \ldots, b^{(k)}) \), i.e

\[
(e, q^{(1)}, \ldots, q^{(k)}) = T^kR_{g(t)^{-1}}(e, b^{(1)}, \ldots, b^{(k)}).
\]

Now since in the multiplication we have \( m(R_g f, h) = m(fg, h) = m(f, gh) = m(f, L_gh) \), setting \((g, b^{(1)}, \ldots, b^{(k)}) = T^kR_{g(t)}(e, q^{(1)}, \ldots, q^{(k)}) \) in (4.2) does not changes the multiplication \( T^kF \subset T^kTm \), but translates to applying \( \operatorname{Ad}_{g(t)} \) to the element \( \Delta u(t) \) and vanishing the \( u(t) \)-part because of the evolution equations.

Summing up, the multiplication on the right-hand side of the system (4.3) is the multiplication \( T^kTm \) of an element \((e, q^{(1)}, \ldots, q^{(k)}) \in T^kG \subset T^kTG \) by an element \((0, \operatorname{Ad}_{g(t)} \Delta u(t), 0, \ldots, 0) \in \mathfrak{g}^{k+1} \equiv T^k\mathfrak{g} \subset T^kTG \). This, however, does not mean that we have a multiplication between elements of \( \mathfrak{g} \). In general, there is no canonical identification of \( T^kG \) with \( \mathfrak{g}^k \) (such an identification requires a choice of local coordinates – see Remark [1.1] or, as happens with matrix groups, an embedding of \( G \) into \( M_N(\mathbb{R}) \)). Thus the resulting formulas (4.4) have only a local sense.

### 4.2 Optimality conditions in sub-Riemannian geometry

In Sec. 3 we were able to associate a control-affine system \( (\Sigma^{(k)}(\Delta u)) \) with the problem of calculating the derivatives (or, more precisely, the jet-expansion) of the end-point map at a given trajectory of a control-linear
system (A). The results of Sussmann and Jurjevic [SJ72] give simple criteria for local non-controllability of a control system at a given point. It turns out that for $k = 1$ and $k = 2$ these criteria are closely related to necessary optimality conditions in sub-Riemannian geometry. Below we shall explain this relation.

**Sub-Riemannian geodesic problem** Consider the control-linear system (A) with $\Omega = L^2([0,T],\mathbb{R}^l)$. For a trajectory $q(t)$, which corresponds to a control $u \in L^2([0,T],\mathbb{R}^l)$, we define its energy at $t$ as

$$E^t(u) := \frac{1}{2} \int_0^t \sum_i u_i(\tau)^2 \, d\tau.$$ 

Given $T > 0$, and a pair of points $q_0, q_1 \in M$, the question is to find a trajectory of (A) which joins $q_0 = q(0)$ with $q_1 = q(T)$ while minimizing the energy $E^T$. Solutions of such a problem are called minimizing sub-Riemannian geodesics – see [Mon06] Rif14 for more details.

There is a fundamental relation between optimality in the above sense, and the properties of the extended end-point map $\text{End}^t : \Omega \rightarrow M \times \mathbb{R}$; defined by $\text{End}^t(u) := (\text{End}^t(u), E^t(u))$.

Fact 4.1. If $\text{End}^T$ is open at $u$, then the corresponding trajectory cannot be a sub-Riemannian geodesics.

All the known optimality criteria for sub-Riemannian geodesics, with the sole exception of [HLD16], depend on the above result. They use some version of a (higher-order) open mapping theorem (see [Sus03] for a set of general results of this type) linking the properties of (first or higher) derivatives of the end-point map $\text{End}^T$ with the openness of $\text{End}^T$ itself. In what follows we shall revise some of these classical results using the construction of adapted coordinates from Theorems 3.2 and 3.3.

**Sub-Riemannian optimality conditions of degree one** It follows easily from the standard open mapping theorem that if $\text{Im} D_u \text{End}^T$ (which is of course a vector space) equals $T_{\{q(T), E^T(u)\}}(M \times \mathbb{R})$, then $\text{End}^T$ is open at $u$, and hence the related trajectory $q(t)$ cannot be optimal. Thus an obvious necessary condition for optimality is that $\text{Im} D_u \text{End}^T$ should be a proper vector subspace in the tangent space $T_{\{q(T), E^T(u)\}}(M \times \mathbb{R})$. Curves satisfying this condition are called extremals. Among all extremals we distinguish a subclass of abnormal trajectories.

**Definition 4.2.** A trajectory $q(t)$ of (A) corresponding to a control $u \in L^2([0,T],\mathbb{R}^l)$ is called abnormal if $\text{Im} D_u \text{End}^T$ is a proper vector subspace of the tangent space $T_{\{q(T)\}}M$. In particular, an abnormal trajectory is an extremal.

Below we will concentrate solely on this class of curves as they are the most interesting (and mysterious) type of extremals [Mon06]. The basic reason for this is that an abnormal trajectory is an extremal regardless of the properties of the energy functional, thus being abnormal is a geometric property of the control system itself.

**Lemma 4.3.** Let $q(t)$, with $t \in [0,T]$, be an abnormal SR trajectory corresponding to the control $u \in L^2([0,T],\mathbb{R}^l)$. Then the related control-affine system $(\Sigma^{(1)})$ in $\mathbb{R} \times \mathbb{R}^n \ni (t, q^{(1)})$ described in Thm 3.3 is not controllable at any point $(t, 0)$. In particular, there exist an $n$-tuple $(\phi^1, \ldots, \phi^n) \in \mathbb{R}^n \setminus \{0\}$ such that the covector $\phi(t) = \sum_\alpha \phi^\alpha \Phi^{(1)}_{\alpha}(t) \in T_{q(t)}^*M$ satisfies the following condition

$$\langle \phi(t), X_i(q(t)) \rangle = 0 \text{ for all } i = 1, 2, \ldots, l \text{ and all } t \in [0,T].$$

Observe that due to (3.3), the covector $\phi(t)$ described above is a Pontryagin covector, i.e. its evolution is compatible with the evolution of (A) corresponding to $u$: 

$$\langle \dot{\phi}(t), b \rangle + \langle \phi(t), \sum_i u_i(t) \cdot D_{q(t)}X_i[b] \rangle = 0 \text{ for every } b \in T_{q(t)}M.$$ 

**Proof.** By definition, since $q(t)$ is an abnormal trajectory, we know that $\text{Im} D_u \text{End}^T = \{b^{(1)}(T, \Delta u) \mid \Delta u \in L^2([0,T],\mathbb{R}^l)\}$ is a proper vector subspace of $T_{q(T)}M$. Passing to the adapted coordinates $q^{(1)} \in \mathbb{R}^n$ – which
in degree one involves a linear transformation – we conclude that the space \( V := \{ q^{(1)}(T, \Delta u) \mid \Delta u \in L^2([0, T], \mathbb{R}^l) \} \) is a proper vector subspace of \( \mathbb{R}^n \). However, by the results of Thm\[3,5\] space \( V \) is just \( \mathcal{R}_T \) – the reachable set at time \( T \) of the control system
\[
\dot{q}^{(1)}(t) = \sum_i \Delta u_i(t) \cdot Y^i_q(t) = \sum_i \Delta u_i(t) \cdot \Phi^{(1)}(t)[X_i(q(t))]; \quad q^{(1)}(0) = 0.
\]

Clearly, by extending a control \( \Delta u \in L^2([0, T], \mathbb{R}^l) \) by zero on \([\tau, T] \), also \( \mathcal{R}_\tau \subset \mathcal{R}_T = V \) for every \( \tau \in [0, T] \). We conclude that the fields \( Y^i_q(t) \) must be tangent to \( V \) for all \( t \in [0, T] \). Hence, each non-zero covector \( \phi = (\phi^1, \ldots, \phi^n) \in (\mathbb{R}^n)^* \simeq \mathbb{R}^n \) annihilating \( V \), will satisfy \( 0 = \langle \phi, Y^i_q(t) \rangle = \langle \phi, \Phi^{(1)}(t)[X_i(q(t))] \rangle \) for every \( i = 1, 2, \ldots, l \) and every \( t \in [0, T] \). This ends the proof.

**Optimality conditions of degree two**  A basis of our considerations in this part is the following result

**Theorem 4.4** (Agrachew-Sarychew Index Lemma \[AS96\]). Let \( q(t) \), with \( t \in [0, T] \) be an abnormal minimizing sub-Riemannian geodesics corresponding to the control \( u \in L^2([0, T], \mathbb{R}^l) \), and denote by \( r \) the codimension of \( \text{Im} \, D_u \text{End}^T \) in \( T_{q(T)} \mathcal{M} \). Then there exists a covector \( \phi_0 \in T^*_{q(T)} \mathcal{M} \) with the following properties

- \( \phi_0 \) annihilates the image \( \text{Im} \, D_u \text{End}^T \)
- the negative index of the quadratic map
  \[
  \langle \phi_0, D^2_u \text{End}^T \rangle \big|_{\ker D_u \text{End}^T} : \ker D_u \text{End}^T \rightarrow \mathbb{R}
  \]
  is at most \( r - 1 \).

Above we treat a covector \( \phi_0 \in T^* \mathcal{M} \), as an element \( \phi_0 + 0 \cdot d \tau \in T^*(\mathcal{M} \times \mathbb{R}) \) – hence its action on \( T\mathbb{R} \)-vectors is trivial.

The above result is stated in terms of the extended end-point map \( \text{End}^T \). It is however easy to reformulate it in the language of the standard end-point map \( \text{End}^T \) for a moderate price of rising the index by one.

**Proposition 4.5** (modification of the Agrachew-Sarychew Index Lemma). Let \( q(t) \), with \( t \in [0, T] \), be an abnormal minimizing sub-Riemannian geodesics corresponding to the control \( u \in L^2([0, T], \mathbb{R}^l) \), and denote by \( r \) the codimension of \( \text{Im} \, D_u \text{End}^T \) in \( T_{q(T)} \mathcal{M} \). Then there exists a covector \( \phi_0 \in T^*_{q(T)} \mathcal{M} \) with the following properties

- \( \phi_0 \) annihilates the image \( \text{Im} \, D_u \text{End}^T \)
- the negative index of the quadratic map
  \[
  \langle \phi_0, D^2_u \text{End}^T \rangle \big|_{\ker D_u \text{End}^T} : \ker D_u \text{End}^T \rightarrow \mathbb{R}
  \]
  is at most \( r \).

**Proof.** Let \( \phi_0 \) be as in the assertion of Theorem 4.4. Since \( \phi_0 \) annihilates the \( T\mathbb{R} \)-direction, then clearly
\[
\langle \phi_0, D^2_u \text{End}^T \rangle = \langle \phi_0, D^2_u \text{End}^T \rangle.
\]

Now as \( \ker D_u \text{End}^T = \ker D_u \text{End}^T \cap \ker D_u E^T \) and \( E^T \) is an \( \mathbb{R} \)-valued map, the space \( \ker D_u \text{End}^T \) is a subspace of \( \ker D_u \text{End}^T \) of codimension less or equal 1. Thus if \( \langle \phi_0, D^2_u \text{End}^T \rangle \) would be negatively defined on some \( r + 1 \) dimensional vector subspace \( W \subset \ker D_u \text{End}^T \), then \( W' = W \cap \ker D_u E^T \subset \ker D_u \text{End}^T \) would be a vector subspace of dimension at least \( r \) on which \( \langle \phi_0, D^2_u \text{End}^T \rangle = \langle \phi_0, D^2_u \text{End}^T \rangle \) is negatively defined. This contradicts the assertion of Theorem 4.4. \( \square \)

Under a further assumption that the index mentioned in Prop. 4.5 is actually zero, results of Theorem 3.5 allow to derive the following Goh conditions.
Lemma 4.6. Let \( q(t) \) be a sub-Riemannian trajectory corresponding to a control \( u \in L^2([0,T], \mathbb{R}^l) \). Assume that a covector \( \phi_0 \in T_q^{*}(T)M \) satisfies the conditions

(i) \( \phi_0 \) annihilates the image \( \text{Im} D_u \text{End}^T \)

(ii) the negative index of the quadratic map

\[
\langle \phi_0, D_u^2 \text{End}^T \rangle |_{\ker D_u \text{End}^T} : \ker D_u \text{End}^T \rightarrow \mathbb{R}
\]

is zero.

Then there exists an \( n \)-tuple \( (\phi^1, \ldots, \phi^n) \in \mathbb{R}^n \setminus \{0\} \) such that the Pontryagin covector \( \phi(t) = \sum_i \phi^i \Phi_0^{(1)}(t) \in T_q^{*}(T)M \) satisfies the conditions

\[
\langle \phi(t), X_i(q(t)) \rangle = 0 \quad \text{and} \quad \langle \phi(t), [X_i, X_j](q(t)) \rangle = 0 \quad \text{for all} \quad i, j = 1, 2, \ldots, l \quad \text{and all} \quad t \in [0,T].
\]

Proof. By assumption, for every controls \( \Delta u^{(1)}, \Delta u^{(2)} \in L^2([0,T], \mathbb{R}^l) \) such that \( D_u \text{End}^T [\Delta u^{(1)}] = b^{(1)}(T, \Delta u^{(1)}) = 0 \) we have

\[
\langle \phi_0, \frac{1}{2} D_u^2 \text{End}^T [\Delta u^{(1)}] \rangle = b^{(2)}(T, \Delta u^{(1)}) \geq 0 \quad \text{and} \quad \langle \phi_0, D_u \text{End}^T [\Delta u^{(2)}] \rangle = b^{(1)}(T, \Delta u^{(2)}) = 0.
\]

Now observe that, since \( b^{(1)}(T, \Delta u^{(1)}) = 0 \), when passing to adapted coordinates formulas (3.1) and (3.2) give us

\[
\begin{align*}
q^{(2)}(T, \Delta u^{(1)}) &= \Phi^{(1)}(T) \left[ b^{(2)}(T, \Delta u^{(1)}) \right] \quad \text{and} \quad q^{(1)}(T, \Delta u^{(1)}) = \Phi^{(1)}(T) \left[ b^{(1)}(T, \Delta u^{(2)}) \right].
\end{align*}
\]

Hence for a non-zero covector \( \phi_0 = \Phi^{(1)}(T)^* \phi_0 \in \mathbb{R}^n \), we have \( \langle \phi_0, q^{(2)}(T, \Delta u^{(1)}) \rangle \geq 0 \) and \( \langle \phi_0, q^{(1)}(T, \Delta u^{(2)}) \rangle = 0 \), and thus

\[
\langle \phi_0, q^{(2)}(T, \Delta u^{(1)}) + q^{(1)}(T, \Delta u^{(2)}) \rangle \geq 0 \quad \text{whenever} \quad q^{(1)}(T, \Delta u^{(1)}) = \Phi^{(1)}(T) \left[ b^{(1)}(T, \Delta u^{(1)}) \right] = 0.
\]

In light of Theorem 3.5 this translates as a condition that the control system

\[
\begin{align*}
\dot{x}^{(1)} &= \sum_i \Delta u_i^{(1)}(t) \cdot Y_i^{(1)}(t) \\
\dot{x}^{(2)} &= \sum_i \Delta u_i^{(2)}(t) \cdot Y_i^{(2)}(t, x^{(1)}) + \sum_i \Delta u_i^{(2)}(t) \cdot Y_i^{(1)}(t)
\end{align*}
\]

(4.6)

i.e. \( x^{(1)}(t) = q^{(1)}(t, \Delta u^{(1)}) \) and \( x^{(2)}(t) = q^{(2)}(t, \Delta u^{(1)}) + q^{(1)}(t, \Delta u^{(2)}) \); is not controllable (a system in \( \mathbb{R}^n \times \mathbb{R}^n \), but also in \( V \times \mathbb{R}^n \), where \( V := \{ q^{(1)}(T, \Delta u) \mid \Delta u \in L^2([0,T], \mathbb{R}^l) \} \) is the reachable set of coordinates \( x^{(1)} \) as in the proof of Lemma 4.3 at time \( T \) at the point \( (x^{(1)} = 0, x^{(2)} = 0) \). In particular, by the criteria of local controllability of Jurdjevic-Sussmann [172], the control fields

\[
Z_i^{(2)}(t, x^{(1)}) := (Y_i^{(1)}(t), Y_i^{(2)}(t, x^{(1)})) \quad \text{and} \quad Z_i^{(1)} := (0, Y_i^{(1)}(t))
\]

and their Lie brackets calculated at \( (T, x^{(1)} = 0, x^{(2)} = 0) \) are contained in a proper linear subspace \( W \subset V \times \mathbb{R}^n \). Note also, that the reachable set of the discussed control system at any time \( T' \in [0,T] \) at the point \( (x^{(1)} = 0, x^{(2)} = 0) \) is naturally contained in the analogous reachable set at \( T \) (by simply extending the controls by zero on \( [T', T] \)). Thus we may assume that \( W \) is spanned by the control vector fields and their brackets calculated at \( (T', x^{(1)} = 0, x^{(2)} = 0) \) for all \( T' \in [0,T] \), and further, as the system \( \dot{x}^{(1)} = \sum_i \Delta u_i^{(1)} \cdot Y_i^{(1)}(t) \) is controllable in \( V \subset \mathbb{R}^n \), we may assume that \( W = V \times V' \), where \( V' \subset \mathbb{R}^n \) is a proper vector subspace.

Finally, a simple calculation shows that at time \( t \)

\[
[Z_i^{(2)}, Z_j^{(2)}] = (0, \Phi^{(1)}(t) \left[ [X_i, X_j](q(t)) \right]),
\]

hence, in particular, for every \( t \in [0,T] \) fields \( (0, \Phi^{(1)}(t) \left[ [X_i, X_j](q(t)) \right]) \) and \( (0, Y_i^{(1)}(t)) = (0, \Phi^{(1)}(t) \left[ X_i(q(t)) \right]) \) belong to \( V \times V' \). Taking \( (\phi^1, \ldots, \phi^n) \in \mathbb{R}^n \setminus \{0\} \) to be any covector annihilating \( V' \) we get the assertion. \( \square \)
Finally, we may refer to the recent results of [16223] where it is proved that it is possible to put the Agrachew-Sarychew index to zero by dividing the curve into a finite number of pieces.

Lemma 4.7 ([16223]). Let \( q(t) \), with \( t \in [0,T] \) be an abnormal minimizing sub-Riemannian geodesics corresponding to the control \( u(t) \), and denote by \( r \) the codimension of \( \text{Im} D_u \text{End}^I \) in \( T_q(T) M \). Then there exists at most \( r \) points \( 0 = \tau_1 < \tau_2 < \ldots < \tau_s \leq T \) such that on every subinterval \( [a,b] \subset (\tau_i, \tau_{i+1}) \) for \( i = 1,2, \ldots, s - 1 \) the trajectory \( q(t) \) with \( t \in [a,b] \) satisfies the assumptions of Lemma 4.6. In particular, on each of the pieces the assertion of Lemma 4.6 holds. Thus we get Goh conditions on each piece (perhaps for different Pontryagin covectors).

Actually, the tool of adapted coordinates can be used to give a more refined picture of the second-order optimality conditions in sub-Riemannian geometry. We refer to [16223] for more details.

5 General polynomial variations of the end-point map

Motivations Curves \( b^{(k)}(t, \Delta u) \) (and their counterparts \( q^{(k)}(t, \Delta u) \) in adapted coordinates) are obtained by studying reactions of the end-point map \( \text{End}^I \) to changes of the control argument of the form \( s \mapsto u + s \cdot \Delta u \). It is, however, most natural to study such reactions for a more general family of controls, i.e.

\[
(5.1) \quad s \mapsto u + s \cdot \Delta u^{(1)} + s^2 \cdot \Delta u^{(2)} + \ldots + s^k \cdot \Delta u^{(k)}.
\]

Geometrically this corresponds to studying the natural lift of the end-point map to the space of \( k \)-jets, i.e. \( T^k \text{End}^I : T^k_u \Omega \longrightarrow T^k_{q(t)} M \).

Let us observe that such a generalization looks promising from the point of view of applications. First of all, recall formula (4.1), which allowed us to understand Agrachev’s approach to the study of second derivatives. Further note that in our interpretation of sub-Riemannian optimality conditions of degree two in Lemma 4.6, it turned out that the non-controllability of the system (4.6) played a crucial role. This system is clearly not constructed from a variation of the end-point map related with the control shift \( s \mapsto u + s \cdot \Delta u \), as it involves two controls \( \Delta u^{(1)}, \Delta u^{(2)} \) instead of a single control \( \Delta u \). However, it is not difficult to guess that it corresponds to variation by the family of controls \( s \mapsto u + s \cdot \Delta u^{(1)} + s^2 \cdot \Delta u^{(2)} \) undergoing a transformation of adapted coordinates.

Polynomial variations of the end-point map Family of controls (5.1) is encoded by a \( k \)-tuple \( \Delta u := (\Delta u^{(1)}, \Delta u^{(2)}, \ldots, \Delta u^{(k)}) \in \Omega^k \), establishing a canonical isomorphism between the fibre of the \( k \)-jet bundle \( T^k_u \Omega \) and \( \Omega^k \) (true for every vector space). To simplify the notation, let us denote

\[
s \circ \Delta u := s \cdot \Delta u^{(1)} + s^2 \Delta u^{(2)} + \ldots + s^k \cdot \Delta u^{(k)}.
\]

By a \( k \)-variation of \( \text{End}^I \) at \( u \) in the direction of \( \Delta u \) we shall understand the \( k \)-jet of \( s \mapsto \text{End}^I [u + s \circ \Delta u] \). Analogously to our previous considerations, in a local coordinate system it can be encoded by a family of curves \( b^{(i)}(t, \Delta u) \), for \( i = 1,2, \ldots, k, \)

\[
\text{End}^I [u + s \circ \Delta u] \overset{\text{loc}}{=} q(t) + s \cdot b^{(1)}(t, \Delta u) + s^2 \cdot b^{(2)}(t, \Delta u) + \ldots + s^k \cdot b^{(k)}(t, \Delta u) + o(s^k),
\]

where \( b^{(m)}(t, \Delta u) \overset{\text{loc}}{=} \frac{1}{m!} \cdot \frac{\partial^m}{\partial s^m} \text{End}^I [u + s \circ \Delta u] \).

Let us remark that using Lemma 1.3 it is possible to express curves \( b^{(m)}(t, \Delta u) \) in terms of the derivatives
Let \( b \) denote \( \Delta u \) evaluated on various controls \( \Delta u^{(s)} \) forming the \( k \)-tuple \( \Delta u \). Indeed

\[
\begin{align*}
\text{End}^i[u + s \circ \Delta u] & \overset{\text{loc}}{=} \\
\text{End}^i[u] + D_u \text{End}^i[s \circ \Delta u] + \frac{1}{2!} D_u^2 \text{End}^i[s \circ \Delta u] + \ldots + \frac{1}{k!} D_u^k \text{End}^i[s \circ \Delta u] + o(s^k) = \\
\text{End}^i[u] + s \cdot D_u \text{End}^i[\Delta u^{(1)}] + s^2 \cdot \left( \frac{1}{2!} D_u^2 \text{End}^i[\Delta u^{(1)}, \Delta u^{(1)}] + D_u \text{End}^i[\Delta u^{(2)}] \right) + \ldots \\
+ s^k \cdot \left( \frac{1}{k!} D_u^k \text{End}^i[\Delta u^{(1)}, \ldots, \Delta u^{(1)}] + \frac{1}{(k-1)!} D_u^{k-1} \text{End}^i[\Delta u^{(2)}, \Delta u^{(1)}, \ldots, \Delta u^{(1)}] + \ldots \\
+ \frac{1}{2!} D_u^2 \text{End}^i[\Delta u^{(1)}] + D_u \text{End}^i[\Delta u^{(k)}] \right) + o(s^k) =
\end{align*}
\]

Leading to

\[
\begin{align*}
b^{(1)}(t, \Delta u) &= D_u \text{End}^i[\Delta u^{(1)}] \\
b^{(2)}(t, \Delta u) &= 2D_u \text{End}^i[\Delta u^{(2)}] + D_u^2 \text{End}^i[\Delta u^{(1)}, \Delta u^{(1)}] \\
& \ldots \\
b^{(k)}(t, \Delta u) &= k! \cdot D_u \text{End}^i[\Delta u^{(k)}] + \frac{k!}{2!} D_u^2 \text{End}^i[\Delta u^{(1)}] + \ldots + D_u^k \text{End}^i[\Delta u^{(1)}, \ldots, \Delta u^{(1)}].
\end{align*}
\]

**Description of curves** \( b^{(m)}(t, \Delta u) \)  It is possible to generalize our previous results about \( k \)-variations in the direction of \( \Delta u \) to describe curves \( b^{(m)}(t, \Delta u) \).

**Lemma 5.1** (the general form of \( b^{(m)}(t, \Delta u) \)). Consider a \( k \)-tuple \( du = (\Delta u^{(1)}, \ldots, \Delta u^{(k)}) \) \( \in \Omega^k \approx \mathbb{T}_u^k \Omega \). Let the \( k \)-jet of the curve \( s \mapsto \text{End}^i[u + s \circ \Delta u] \) at \( s = 0 \) be given in local coordinates by

\[
\begin{align*}
\text{End}^i[u + s \circ \Delta u] & \overset{\text{loc}}{=} q(t) \cdot b^{(1)}(t, \Delta u) + s^2 \cdot b^{(2)}(t, \Delta u) + \ldots + s^k \cdot b^{(k)}(t, \Delta u) + o(s^k).
\end{align*}
\]

Then curves \( b^{(m)}(t, \Delta u) \) are subject to the following system of ODEs:

\[
(5.2) \quad b^{(m)}(t, \Delta u) = \sum_{r=0}^{m} \sum_{i} \Delta u_i^{(r)}(t) \left( \sum_{\alpha, \omega(\alpha) = m-r} \frac{1}{\alpha!} \cdot D^{|\alpha|}_q \cdot X_i[b(t, \Delta u)^{\alpha}] \right),
\]

where we use the notation introduced on page 4 and we denote \( \Delta u^{(0)} = u \).

**Proof.** It is enough to repeat the proof of Lemma 2.1. Let \( q_s(t) \) be a solution of

\[
q_s(t) = \sum_i \left( u_i(t) + s \cdot \Delta u_i^{(1)}(t) + \ldots + s^k \cdot \Delta u_i^{(k)}(t) \right) X_i(q_s(t)).
\]

Then analogously as before

\[
\frac{\partial^m}{\partial s^m} \bigg|_0 [X_i(q_s(t))] = m! \cdot \left( \sum_{\alpha, \omega(\alpha) = m} \frac{1}{\alpha!} \cdot D^{|\alpha|}_q \cdot X_i[b(t, \Delta u)^{\alpha}] \right).
\]

Further, note that for \( r > m \) we have \( \frac{\partial^m}{\partial s^m} \bigg|_0 [s^r \cdot X_i(q_s(t))] = 0 \), while for \( r \leq m \)

\[
\frac{\partial^m}{\partial s^m} \bigg|_0 [s^r \cdot X_i(q_s(t))] = r! \cdot \left( \frac{m}{r} \right) \cdot \frac{\partial^{m-r}}{\partial s^{m-r}} [X_i(q_s(t))] = m! \cdot \left( \sum_{\alpha, \omega(\alpha) = m-r} \frac{1}{\alpha!} \cdot D^{|\alpha|}_q \cdot X_i[b(t, \Delta u)^{\alpha}] \right).
\]
Now for \( m \leq k \) we have

\[
\hat{b}^{(m)}(t, \Delta u) = \frac{1}{m!} \cdot \frac{d}{dt} \left( \hat{b}^{(m)}_0 |_{q_s(t)} \right) = \frac{1}{m!} \cdot \partial_s^m \left|_0 \right. (q_s(t)) = \]

\[
\frac{1}{m!} \cdot \partial_s^m \left|_0 \right. \left\{ \sum_i \left( u_i(t) + s \Delta u_i^{(1)}(t) + s^2 \Delta u_i^{(2)}(t) + \ldots + s^k \Delta u_i^{(k)}(t) \right) \right\} = \]

\[
\frac{1}{m!} \cdot \sum_i \left( \sum_{r=0}^{m} \frac{1}{\alpha!} \cdot D_{q(t)}^{[\alpha]} X_i \left[ \hat{b}(t, \Delta u)^{\alpha} \right] \right) \cdot \Delta u_i^{(r)}(t) = \]

This ends the proof. \( \square \)

**Adapted coordinates for polynomial \( k \)-variations** It is interesting to see how the \( k \)-variation of \( \text{End}^t \) in the direction of \( \Delta \bar{u} \) looks in adapted coordinates.

**Theorem 5.2.** Consider a \( k \)-tuple \( \Delta \bar{u} = (\Delta u^{(1)}, \ldots, \Delta u^{(k)}) \in \Omega^k \simeq T_x^k \Omega \). Let the \( k \)-jet of the curve \( s \mapsto \text{End}^t[u + s \circ \Delta \bar{u}] \) at \( s = 0 \) be given in local coordinates by

\[
\text{End}^t(u + s \circ \Delta \bar{u}) \xrightarrow{\text{loc}} q(t) + s \cdot b^{(1)}(t, \Delta \bar{u}) + s^2 \cdot b^{(2)}(t, \Delta \bar{u}) + \ldots + s^k \cdot b^{(k)}(t, \Delta \bar{u}) + o(s^k). \]

Let \( q^{(1)}(t, \Delta \bar{u}), \ldots, q^{(k)}(t, \Delta \bar{u}) \) be the image of the \( k \)-jet \( b^{(1)}(t, \Delta \bar{u}), \ldots, b^{(k)}(t, \Delta \bar{u}) \) under the transformation of adapted coordinates \( \Psi(t, \Delta \bar{u}) \). That is, for any \( m = 1, 2, \ldots k \) we have

\[
q^{(m)}(t, \Delta \bar{u}) := \sum_{\alpha, \alpha = m} \frac{1}{\alpha!} \cdot \Phi^{(\alpha)}(\beta) \left[ \hat{b}(t, \Delta \bar{u})^{\alpha} \right].
\]

Then, curves \( q^{(m)}(t, \Delta \bar{u}) \) satisfy the following system of ODEs:

\[
q^{(m)}(t, \Delta \bar{u}) = \sum_{r=1}^{m} \sum_{i} \Delta u_i^{(r)}(t) \cdot \sum_{\alpha, \beta, \alpha + \beta = m-r} \frac{1}{\beta!} \cdot \Phi^{(\alpha+1)}(\beta) \left[ D_{q(t)}^{[\beta]} X_i \left[ \hat{b}(t, \Delta \bar{u})^{\beta} \right] \right].
\]

By expressing \( b^{(m)}(t, \Delta \bar{u}) \)s in \( 5.4 \) via \( q^{(m)}(t, \Delta \bar{u}) \)s (i.e. using the inverse of the transformation of adapted coordinates) we get an immediate corollary, generalizing Theorem 5.5.

**Theorem 5.3.** Consider a trajectory \( q(t) \) of a control-linear system \( \Sigma \) corresponding to the control \( u \in \Omega \) and choose \( k \in \mathbb{N} \). Then for every \( k \)-tuple \( \Delta u = (\Delta u^{(1)}, \ldots, \Delta u^{(k)}) \in \Omega^k \simeq T^k_x \Omega \) the curve

\[
p^{(k)}(t) := (t, q^{(1)}(t, \Delta \bar{u}), \ldots, q^{(k)}(t, \Delta \bar{u})),
\]

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where \((q^{(1)}(t, \Delta u), \ldots, q^{(k)}(t, \Delta u))\) is the image under the transformation of adapted coordinates \(\Psi^{(k)}(t)\) of the \(k\)-jet of \(s \mapsto \text{End}^t[u + s \circ \Delta u]\) at \(s = 0\), is a trajectory of the control-affine system

\[
(5.5) \quad p^{(k)}(t) = \partial_t + \sum_{r=1}^{k} \sum_i \Delta u^{(r)}_i(t) \cdot Y^{(k-r+1)}(p^{(k)}(t)) \quad \text{with} \quad p^{(k)}(0) = (0, 0, \ldots, 0).
\]

Here, for \(m = 1, 2, \ldots, k\) fields \(Y^{(m)}_i\) are defined precisely as in the assertion of Theorem 3.5.

Let us now proceed with the proof of the former theorem.

**Proof of Theorem 3.2** Again we roughly repeat steps from the proof of Theorem 3.2. Since \(q^{(m)}(t, \Delta u)\) is given by (5.3), we have

\[
q^{(m)}(t, \Delta u) := \sum_{\alpha, w(\alpha) = m} \frac{1}{\alpha!} \cdot \Phi^{(|\alpha|)}(t)[\tilde{b}(t, \Delta u)^{\alpha}] + \sum_{\alpha, w(\alpha) = m} \sum_{l \leq m} \frac{1}{\alpha!} \cdot a_{l} \cdot \Phi^{(|\alpha|)}(t)[\tilde{b}^{(l)}(t, \Delta u)^{\alpha-1}]
\]

which has the form

\[
q^{(m)}(t, \Delta u) = \sum_i u_i \cdot A_i + \sum_{r=1}^{m} \sum_i \Delta u^{(r)}_i \cdot B^{(r)}_i
\]

Like in the proof of Theorem 3.2

\[
\sum_i u_i \cdot A_i = \sum_{\alpha, w(\alpha) = j} \Phi^{(|\alpha|)}[\tilde{b}(t)^{\alpha}] + \sum_i \left\{ \sum_{\gamma, \beta, w(\gamma) + w(\beta) = j} \frac{1}{\gamma!} \cdot \Phi^{(|\gamma|+1)}[D^{[\beta]}_{q(t)} X_i[\tilde{b}(t)^{\beta}], \tilde{b}(t)^{\gamma}] \right\} = 0.
\]

On the other hand

\[
\sum_i \Delta u^{(r)}_i \cdot B^{(r)}_i = \sum_{\alpha, w(\alpha) = m} \sum_{l \leq m} \frac{1}{\alpha!} \cdot a_{l} \cdot \Phi^{(|\alpha|)}[\text{part of } \tilde{b}^{(l)}\text{ linear in } \Delta u^{(r)}_i\text{, } \tilde{b}(t)^{\alpha-1}]] = \sum_{\alpha, w(\alpha) = m} \sum_{l \leq m} \frac{1}{\alpha!} \cdot a_{l} \cdot \Phi^{(|\alpha|)} \left[ \sum_i \Delta u^{(r)}_i \cdot \left\{ \sum_{\beta, w(\beta) = l-r} \frac{1}{\beta!} \cdot D^{[\beta]}_{q(t)} X_i[\tilde{b}(t)^{\beta}], \tilde{b}(t)^{\alpha-1}]] \right\} = \sum_i \Delta u^{(r)}_i \cdot \left\{ \sum_{\alpha, w(\alpha) = m} \sum_{l \leq m} \frac{1}{(\alpha - 1)!} \cdot \sum_{\beta, w(\beta) = l-r} \frac{1}{\beta!} \cdot \Phi^{(|\alpha - 1|+1)}[D^{[\beta]}_{q(t)} X_i[\tilde{b}(t)^{\beta}], \tilde{b}(t)^{\alpha-1}]] \right\}
\]

Now observe that a triple \((\alpha, l, \beta)\) where \(w(\alpha) = j, a_{l} > 0\) and \(w(\beta) = l - r\) uniquely determines a pair of multi-indexes \((\gamma = \alpha - 1, \beta)\) satisfying \(w(\gamma) + w(\beta) = (m - l) + l - r = m - r\). Thus we may change the summation order in the expression above to obtain

\[
\sum_i \Delta u^{(r)}_i \cdot B^{(r)}_i = \sum_i \Delta u^{(r)}_i \cdot \left\{ \sum_{\gamma, \beta, w(\gamma) + w(\beta) = m-r} \frac{1}{\gamma!} \cdot \frac{1}{\beta!} \cdot \Phi^{(|\gamma|+1)}[D^{[\beta]}_{q(t)} X_i[\tilde{b}(t)^{\beta}], \tilde{b}(t)^{\gamma}] \right\}. \]

This ends the proof.

**Example** Consider 3-variation in the direction of \(\Delta u = (\Delta u^{(1)}, \Delta u^{(2)}, \Delta u^{(3)})\). In that situation, it is quite easy to express curves \(q^{(m)}(t, \Delta u)\) for \(m = 1, 2, 3\) in terms of curves \(q^{(m)}(t, \Delta u^{(i)})\). It turns out that

\[
q^{(1)}(t, \Delta u) = q^{(1)}(t, \Delta u^{(1)})
\]

\[
q^{(2)}(t, \Delta u) = q^{(2)}(t, \Delta u^{(1)}) + q^{(1)}(t, \Delta u^{(2)})
\]

\[
q^{(3)}(t, \Delta u) = q^{(3)}(t, \Delta u^{(1)}) + 2q^{(2)}(t, \Delta u^{(1)}, \Delta u^{(2)}) + q^{(1)}(t, \Delta u^{(3)})
\]

Here symbol \(q^{(2)}(t, \Delta u^{(1)}, \Delta u^{(2)})\) denotes the symmetrization

\[
2q^{(2)}(t, \Delta u^{(1)}, \Delta u^{(2)}) := q^{(2)}(t, \Delta u^{(1)} + \Delta u^{(2)}) - q^{(2)}(t, \Delta u^{(1)}) - q^{(2)}(t, \Delta u^{(2)}).
\]
A hypothesis related with Goh-conditions By the results of Theorem 5.3, system (5.5) is a control system in \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (t, q^{(1)}, q^{(2)}, \ldots, q^{(k)}) \) controlled by a family of vector fields

\[
Z_i^{(k)} = (Y_i^{(1)}(t), Y_i^{(2)}(t, p^{(1)}), \ldots, Y_i^{(k)}(t, p^{(1)}, \ldots, p^{(k-1)})), \\
Z_i^{(k-1)} = (0, Y_i^{(1)}(t), \ldots, Y_i^{(k-1)}(t, p^{(1)}, \ldots, p^{(k-2)})), \\
\ldots \\
Z_i^{(2)} = (0, \ldots, 0, Y_i^{(1)}(t), Y_i^{(2)}(t, p^{(1)})), \\
Z_i^{(1)} = (0, \ldots, 0, Y_i^{(1)}(t))
\]

for \( m = 1, 2, \ldots, k \). Based on the proof of Lemma 4.6, we state the following

**Hypothesis 5.4.** Within the setting of Subsection 4.2, consider a sub-Riemannian abnormal extremal corresponding to a control \( u \in L^2([0, T], \mathbb{R}^l) \). If the assumptions of [BMS22, Thm 1.1] are satisfied then system (5.5) is not-controllable at zero.

If the above hypothesis would be true, we could get higher-order Goh conditions from Sussmann–Jurdjevic’s non-controllability conditions in essentially the same way as was done in the proof of Lemma 4.6. (Perhaps some other conditions could also be computable following the lines of [Agr23].) To back this up, after quite heavy calculations, it is possible to check that for all possible indices \( i, j, s \in \{1, 2, \ldots, l\} \) we have

\[
Z_i^{(1)} = (0, 0, \Phi^{(1)}(t)[X_i]), \\
[Z_i^{(2)}, Z_j^{(2)}] = (0, 0, \Phi^{(1)}(t)[X_i, X_j]) \quad \text{and} \\
[Z_i^{(3)}, [Z_j^{(3)}, Z_s^{(3)}]] = (0, 0, \Phi^{(1)}(t)[X_i, [X_j, X_s]] + \\
(0, 0, \Phi^{(1)}(t) \left[ A(t) \Phi^{(2)}(t)(X_i, X_j - X_s) \right] + \\
(0, 0, \Phi^{(1)}(t) \left[ DX_s[A(t) \Phi^{(2)}(t)(X_i, X_j)] - DX_j[A(t) \Phi^{(2)}(t)(X_i, X_s)] \right],
\]

where \( A(t) \) denotes the inverse of the map \( \Phi^{(1)}(t) \). Hence the existence of the covector \( (0, 0, \psi_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) annihilating all these vector fields will imply the existence of a Pontryagin covector (cf. Lemma 4.3 and 4.6) \( \phi(t) := \Phi^{(1)}(t)^*\psi_0 \) such that \( \langle \phi(t), X_i \rangle \) and \( \langle \phi(t), [X_i, X_j] \rangle \) vanish. To get conditions \( \langle \phi(t), [X_i, [X_j, X_k]] \rangle = 0 \), we should additionally assume that \( \psi_0 \) annihilates the remaining two terms in the triple Lie bracket. We believe that this condition will follow from the assumptions of [BMS22, Thm 1.1], i.e. the extremal being of corank 1 and vanishing of the second (intrinsic) derivative of the end-point map at \( u \). We shall study this topic in more detail in a future publication.

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