3D rectification with Visual Sphere perspective: an algebraic alternative for P4P pose estimation

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Abstract

Presented algorithm solves co-planar P4P problem of parallel lines viewed in perspective with algebraic equation. Introduction of visual sphere model extends this algorithm to exotic non-linear projections, where view angle can span to 180° and beyond; a hard-limit of rectilinear perspective used in planar homography. This algorithm can perform full 3D reconstruction of visible rectangle, including pose estimation, and camera orientation. It requires some camera-lens information like angle of view (for rectilinear projection) or a perspective map.

Figure 1: 3D rectification and pose estimation of $ABCD$ quad.

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Introduction

Finding three-dimensional orientation of a visible rectangle in perspective is a difficult task (e.g. ArUco marker pose estimation\(^3\)). It requires solving Perspective-4-Point problem by estimation of homography matrix,\(^5\) which result may be noisy.\(^6\) If one is not familiar with the topic or terms, Appendix on page 9 gives brief introduction. This paper solves mentioned issues by incorporating visual-sphere perspective model.\(^1\) Presented algorithm estimates pose matrix and camera position algebraically. This process is limited to co-planar parallel lines, as vanishing points are indicated by such. Through parallel-sides vanishing-points, pose matrix is calculated and from pose matrix, position of points and camera is reconstructed. This process of pose estimation through algebraic means was first introduced by F.A. VAN DEN HEUVEL.\(^4\) Presented method extends previous solution to views beyond 180°. It also provides some geometrical proof for that method.

1 Pose estimation with P4P problem

To find pose matrix \(P\) of four projected rectangle’s points \(ABCD\), multiple cross product algorithm can be used.\(^4\) Given each visible point has \(R^3\) direction-vector derived from perspective vector map \(G\) or focal-length/sensor-size or angle of view \(\Omega\) (AOV). Here, perspective vector map is preferred, as it can describe projections beyond 180° of view.

1.1 Picture coordinates to direction vector

![Rectilinear projection model of ABCD square, counting clockwise from top-left corner.](image)

Figure 2: Rectilinear projection model of \(ABCD\) square, counting clockwise from top-left corner.

To derive \(\vec{G} \in R^3\) vector component from points in picture plane coordinates \(\vec{f} \in [0, 1]^2\), in a simple rectilinear view, having AOV \(\Omega\) is sufficient. In case of perspective vector map, the \(\vec{G}\) is obtained from \(R^3\_\text{pixel}\) values, mapped to
Algorithm for a simple rectilinear-view direction vector $\vec{G} \in \mathbb{R}^3$, from texture coordinates $\vec{f} = [\vec{f}_s, \vec{f}_t]$. 

\[
\begin{pmatrix}
2\vec{f}_s - 1 \\
(2\vec{f}_t - 1) \div a \\
\cot \frac{\Omega}{2}
\end{pmatrix}, \quad \text{if horizontal}\ \Omega
\]

\[
\begin{pmatrix}
a(2\vec{f}_s - 1) \div \sqrt{a^2 + 1} \\
(2\vec{f}_t - 1) \div \sqrt{a^2 + 1} \\
\cot \frac{\Omega_d}{2}
\end{pmatrix}, \quad \text{if diagonal}\ \Omega
\]

\[
\begin{pmatrix}
a(2\vec{f}_s - 1) \\
2\vec{f}_t - 1 \\
\cot \frac{\Omega_v}{2}
\end{pmatrix}, \quad \text{if vertical}\ \Omega
\]

Algorithm for a simple rectilinear-view direction vector $\vec{G} \in \mathbb{R}^3$, from texture coordinates $\vec{f} = [\vec{f}_s, \vec{f}_t]$, where $a$ represents picture aspect-ratio and $\Omega$ is the angle of view (aka FOV). Maximum $\Omega_d < 180^\circ$. Same algorithm can be expressed as GLSL function.

```cpp
vec3 getDirection(vec2 tex_coord, float aspect, float fov, int fov_type) {
  vec3 direction = vec3(2.0 * tex_coord - 1.0, // map to range [-1,1]
    1.0 / tan(0.5 * radians(fov)),
    1.0 / tan(0.5 * radians(fov) )
  );
  switch (fov_type) {
    default: // horizontal FOV (type 1)
      direction.y /= aspect;
      break;
    case 2: // diagonal FOV (type 2)
      direction.xy /= length(vec2(aspect, 1.0));
    case 3: // vertical FOV (type 3) and type 2 final step
      direction.x *= aspect;
      break;
  }
  return direction;
}
```

Listing 1: Rectilinear view direction vector $\vec{G}$ function from texture coordinates $\vec{f}$ in GLSL.
1.2 Pose matrix estimation from direction vectors

Having camera-space direction vector of each projected rectangle corner, pose matrix $P$ can be estimated.

$$
\begin{align*}
\dot{X} & = ||A \times B|| \times ||C \times D|| \\
\dot{Y} & = ||A \times D|| \times ||C \times B|| \\
\dot{Z} & = \dot{X} \times \dot{Y} = \hat{N} \\
\dot{P} & = \begin{bmatrix}
\dot{X}_1 & \dot{X}_2 & \dot{X}_3 \\
\dot{Y}_1 & \dot{Y}_2 & \dot{Y}_3 \\
\dot{Z}_1 & \dot{Z}_2 & \dot{Z}_3
\end{bmatrix}
\end{align*}
$$

Vectors $\dot{X}$ and $\dot{Y}$ aim at two spherical vanishing points. Plane formed by these two vectors is parallel to the plane of projected figure (rectangle/square and parallelogram), therefore cross product is figure’s normal $\hat{N}$. Same algorithm can be expressed as GLSL function.

Listing 2: Pose $P$ matrix function in GLSL. Matrix $\text{quad}$ represents $4 \times \mathbb{R}^3$ direction-vector of projected rectangle.

```glsl
vec3 normal(vec3 A, vec3 B) {
    return normalize(cross(A, B));
}

mat3 getPose(mat4x3 quad) {
    mat3 Pose;
    Pose[0] = normal(normal(quad[0], quad[1]), normal(quad[2], quad[3]));
    Pose[1] = normal(normal(quad[0], quad[3]), normal(quad[2], quad[1]));
    Pose[2] = cross(Pose[0], Pose[1]);
    return Pose;
}
```

**Theorem 1.** Angle between $\dot{X}$ and $\dot{Y}$ is equivalent to the angle between corresponding sides of projected figure (rectangle/square and parallelogram).

Let us define here $\theta$ as an angle between vectors $\dot{X}, \dot{Y}$ and $\alpha$ as an angle between figure’s visible corresponding sides. For parallelogram angle $\theta = \alpha$ and for a rectangle $\theta, \alpha = 90^\circ$.

**Proof.** Vectors $\dot{X}, \dot{Y}$ point to vanishing points of parallelogram/rectangle sides. Therefore plane on which both vectors lay is parallel to the plane of projected figure. Pointing to same vanishing points makes vectors $\dot{X}, \dot{Y}$ similar to corresponding sides of projected figure, therefore having same angle in-between. ■
(a) $\vec{X}$ points to spherical vanishing point – a cross-section of great circles $\overline{AB}, \overline{CD}$. Vector $\vec{X}$ is derived from normalized cross product of two tangent vectors of great-circles.

(b) $\vec{Y}$ points to spherical vanishing point – a cross-section of great circles $\overline{AD}, \overline{CB}$. Vector $\vec{Y}$ is derived from normalized cross product of two tangent vectors of great-circles.

Figure 3: Spherical vanishing points as pose matrix components $P_1$ and $P_2$.

2 Visual space position reconstruction

Having pose matrix and fiducial marker dimensions, position of the camera in relation to marker points can be estimated (and vice-versa).

2.1 3D rectification by plane intersection

Having pose matrix of projected figure, normal vector of the figure’s plane can be extracted from third component of the matrix. Points can be extended to intersection point with the figure’s plane using fraction of dot products.

$$\vec{A'} = \frac{\vec{C} \cdot \vec{N}}{\vec{A} \cdot \vec{N}} \vec{A}$$

$$\vec{B'} = \frac{\vec{C} \cdot \vec{N}}{\vec{B} \cdot \vec{N}} \vec{B}$$

$$\vec{D'} = \frac{\vec{C} \cdot \vec{N}}{\vec{D} \cdot \vec{N}} \vec{D}$$

Equations for extension of points $\vec{A}, \vec{B}, \vec{D}$ to a plane with $\vec{C}$ in numerator as reference point laying on plane’s surface.

$$\frac{\vec{C} \cdot \vec{N}}{\vec{A} \cdot \vec{N}} \vec{A} = \pm \frac{|\vec{C}| |\vec{N}| \cos \gamma}{|\vec{A}| |\vec{N}| \cos \alpha} \vec{A} = \pm \frac{h_1}{h_2} \vec{A} = \pm \frac{|\vec{A'}|}{|\vec{A}|} \vec{A} = \vec{A'}$$

Sign and length of normal vector $\vec{N}$ cancels out, as well as length of vectors $\vec{A}, \vec{C}$, giving proportion of distances $h$ to the plane. This three-dimensional rectification by plane intersection can be expressed as GLSL function.
```c
vec3 toPlane(vec3 vector, vec3 normal, vec3 plane_pt) {
    return dot(plane_pt, normal)/dot(vector, normal)*vector;
}

mat4x3 toPlane(mat4x3 quad, mat3 Pose, vec3 plane_pt) {
    float numerator = dot(plane_pt, Pose[2]);
    for (int i=0; i<4; i++)
        quad[i] *= numerator/dot(quad[i], Pose[2]);
    return quad;
}
```

Listing 3: Functions for vector–plane intersection, in GLSL.

![Diagram](image_url)

Figure 4: 3D rectification of \(ABCD\) quad, where \(\mathbf{N} \equiv \mathbf{Z}\). Pose matrix components are \(\mathbf{X} = \|\mathbf{A} \times \mathbf{B}\| \times ||\mathbf{C} \times \mathbf{D}||\), \(\mathbf{Y} = ||\mathbf{A} \times \mathbf{D}\| \times ||\mathbf{C} \times \mathbf{B}||\) and \(\mathbf{Z} = \mathbf{X} \times \mathbf{Y}\).

*Remark.* Method for obtaining normal vector of a parallelogram from cross products was first described by F.A. VAN DEN HEUVEL.

### 2.2 Full 3D reconstruction of marker position

Having rectified points \(\mathbf{A}', \mathbf{B}', \mathbf{C}, \mathbf{D}'\) of fiducial marker of known dimensions, camera-space 3D position can be reconstructed by simple scalar multiplication.

\[
u = \frac{a}{|\mathbf{B}' - \mathbf{A}'|} = \frac{b}{|\mathbf{C} - \mathbf{B}'|} = \frac{c}{|\mathbf{D}' - \mathbf{C}|} = \frac{d}{|\mathbf{A}' - \mathbf{D}'|}
\]

\[
\mathbf{A}'' = u\mathbf{A}',
\]

\[
\mathbf{B}'' = u\mathbf{B}',
\]

\[
\mathbf{C}' = u\mathbf{C},
\]

\[
\mathbf{D}'' = u\mathbf{D}'.
\]

(10)
Where $u$ is the vector scalar of rectified projection points, to known size and $a, b, c, d$ are fiducial marker sides length. Vectors $\vec{A}''', \vec{B}''', \vec{C}''', \vec{D}''$ represent reconstructed position of fiducial marker in camera-space.

```glsl
1 mat4x3 reconstructByWidth(mat4x3 figure, float width) {
2 return width/length(figure[2] - figure[3])*figure;
3 }
4 mat4x3 reconstructByHeight(mat4x3 figure, float height) {
5 return height/length(figure[0] - figure[3])*figure;
6 }
7
Listing 4: Fiducial marker points position reconstruction function in GLSL, where matrix \( \text{figure} \) represents rectified figure in camera space.
```

2.3 Full 3D reconstruction of camera position

Camera orientation in relation to the fiducial marker can be obtained from pose matrix $P$ and reconstructed points.

\[
\vec{O} = -\begin{bmatrix} \vec{B}''' \\ \vec{C}''' \\ \vec{D}''' \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}
\]

(15)

Where $\vec{D}'''$ is the fiducial marker origin point (left bottom corner of the marker) and $\vec{O}$ represents camera position in marker space. Same process can be expressed as GLSL function.

```glsl
1 vec3 getCameraPos(mat3 Pose, vec3 marker_origin_pt) {
2 return -marker_origin_pt*Pose;
3 }
4
Listing 5: Camera position and orientation reconstruction function in GLSL.
```

3 Appendix

Pose estimation is commonly used in computer vision for reproducing physical space from two-dimensional symbolic picture. It usually incorporates use of fiducial markers, like color points in movie special effects or binary square fiducial markers like ArUco markers for other purposes. While movie special-effects focus on PnP problem, more constrained fiducial environment provides simpler and more repeatable methods for 3D reconstruction. In case of square markers, number of corner points, perpendicularity of edges and opposite parallelism can be treated as a fiducial feature. In order to benefit from such constants, basic principles must be altered, like perspective projection model. Two vanishing points of a rectangle visible in perspective will point to two component vectors of pose matrix. But in case of rectilinear projection, vanishing point position can easily approach infinity, when one of its edges approaches parallel position with the projection plane. Such enormous numbers are undesirable.
in computational geometry, they are prone to reach precision limits. Another
school of thought about vanishing points is through visual-sphere perspective,\(^1\)
where vanishing point is formed by the intersection of great circles. Spherical
perspective geometry can be defined through euclidean normalized \(\mathbb{R}^3\) vectors,
which avoids spherical coordinate \(\mathbb{R}^2\) system and simplifies calculations.

3.1 On Perspective-\(n\)-Point problem

*Problem.* What is the smallest number of points for complete pose estimation?
Let us consider only a case, where back facing does not occur and image cannot
be mirrored.

*Example.* Photographing binary fiducial markers on solid planar surfaces.

*Theorem 2.* Minimum number of points for pose estimation is four with addi-
tional fiducial cue. If only simple points are considered, fiducial cue becomes the
fifth point.

**P3P** problem for projected equilateral triangle in perspective has four possible
normal vectors with three possible symmetry rotations giving total of twelve
possible pose matrices (see figure 5). Symmetry rotation can be solved with
additional cues enabling point sorting.

![Figure 5: Visualization of P3P problem for projected equilateral triangle](image)

**P4P** problem for projected square in perspective has single possible normal
direction and four possible symmetry rotations, giving total of four possible pose
matrices. Sorting visible points with additional fiducial cues can limit possible
symmetry rotations. Such cues can be derived from fiducial markers, using
color,\(^2\) shape, size or other unique features or known conditions.

**P5P** problem for projected square in perspective with additional point on one
of the square’s sides has single possible pose matrix. In such case, fifth point
is used as a fiducial marker for sorting other four points. Fifth point in such configuration can be extracted by measurement of collinearity, or in case of visual sphere perspective, coplanarity.

Remark. In visual sphere perspective, $[-1, 1]^3$ points $\hat{A}, \hat{B}, \hat{C}$ belong to a single great circle if $\hat{A} \times \hat{B} \cdot \hat{C} = 0$, as cosine of 90° is equal zero.\textsuperscript{a}

4 Conclusion

We proven that orientation matrix (a pose matrix) can be directly evaluated from spherical vanishing points for coplanar parallel lines. Also three-dimensional position of visible surface can be reconstructed in a direct, finite way. Such process could be easily integrated into a hardware solution for pose estimation and position reconstruction of binary-square markers. Close relation with Visual Sphere Perspective extends use of the algorithm to wide-angle camera lens, like fish-eye lens, which parameters are beyond limits of rectilinear planar projection. This is an ideal solution for controlled computer-vision environments.

\textsuperscript{a}algorithm known as triple product
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